Symplectic Cuts and Projection Quantization

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Abstract

The recently proposed projection quantization, which is a method to quantize particular subspaces of systems with known quantum theory, is shown to yield a genuine quantization in several cases. This may be inferred from exact results established within symplectic cutting.

1 Introduction

Motivated by studying the phase space $S^1 \times \mathbb{R}^+$, which is defined to be the restriction of $T^*S^1$ to positive momentum, we recently proposed the projection quantization \cite{BojowaldStrobl}. The conditions for its applicability were formulated as:

1. The phase space $\mathcal{P}$, which is to be quantized, can be characterized as a submanifold of a phase space $\tilde{\mathcal{P}}$ via restriction by means of inequalities $f_i > 0$ for a set of functions $\{f_i\}$ on $\tilde{\mathcal{P}}$ with mutually vanishing Poisson brackets. Furthermore, for each $i$ the set on which the opposite inequality, $f_i < 0$, is fulfilled has to be nonempty. For simplicity we assume that $\mathcal{P}$ is connected.

2. A quantum realization of $\tilde{\mathcal{P}}$ is known in which the functions $f_i$ may be promoted to self-adjoint, simultaneously diagonalizable operators $\hat{f}_i$.

For elementary examples this quantization scheme has been shown to yield the expected results in Secs. II.E and III.B.2 of Ref. \cite{BojowaldStrobl}. The aim of the present paper is to show its validity for a larger class of phase spaces which are specified more precisely below.

To quantize $\mathcal{P}$ by means of projection quantization one starts from the given quantization of $\tilde{\mathcal{P}}$ with the operators $\hat{f}_i$ acting on the Hilbert space $\tilde{\mathcal{H}}$. These operators, being required to be self-adjoint and simultaneously diagonalizable, have mutually commuting spectral families, which can be used to construct a projector $P$ to the positive part of the...
spectra of all the \( \tilde{f}_i \). To that end, we need simultaneous diagonalizability of the operators and mere commutativity on a dense domain would not suffice. The Hilbert space for the quantization of \( \mathcal{P} \) is defined to be the projection \( \mathcal{H} := PH \). Moreover, the projection can be used to project operators on \( \tilde{\mathcal{H}} \) to operators on \( \mathcal{H} \) as quantized observables. Note, however, that adjointness properties of those operators are conserved only under certain conditions. E.g., the projection of a self-adjoint operator on \( \tilde{\mathcal{H}} \) is in general symmetric on \( \mathcal{H} \), but not necessarily self-adjoint (see Ref. [4] for details).

The condition that the set determined by \( f_i < 0 \) is nonempty is introduced to exclude systems like \( T^*(\mathbb{R}^2 \setminus \{0,0\}) \). Using \( x \) and \( y \) as coordinates of \( \mathbb{R}^2 \), this phase space can be viewed as subspace of \( T^*\mathbb{R}^2 \) subject to the condition \( x^2 + y^2 > 0 \). The set \( x^2 + y^2 < 0 \) is, of course, empty. Using a standard quantization of \( T^*\mathbb{R}^2 \), zero lies in the continuous part of the spectrum of a quantization of \( x^2 + y^2 \), implying that the projector to its positive part is the identity. There would, therefore, be no difference in the quantum theories of \( T^*(\mathbb{R}^2 \setminus \{0,0\}) \) and \( T^*\mathbb{R}^2 \). In particular, the \( \theta \)-angle, which for this phase space is of physical relevance as demonstrated by the Aharonov–Bohm experiment, cannot be obtained. Such a failure can, however, also occur if the requirements of projection quantization are fulfilled. For instance, we can change the above condition to \( x^2 + y^2 > a \) with some positive number \( a \). Then the projector of projection quantization will be nontrivial, but we will not obtain the \( \theta \)-angle. This behavior is generic if the circle action has fixed points.

The above condition on the functions \( f_i \) can be reformulated more precisely as requiring that zero be a regular value of all \( f_i \), i.e. that \( df_i \) is nonzero for each \( i \) on the pre-image of zero. (It is possible to weaken this condition, e.g. by demanding that the moment map of the torus action generated by all the \( f_i \) has zero as a regular value. Below we will, however, impose the conditions \( f_i > 0 \) in steps, which means that each of them is treated as a single constraint. More generally, one could use multiple cutting \([2]\) to deal with the complete torus action.) The set \( f_i^{-1}(0) \) for each fixed \( i \) is then a reducible splitting hypersurface of \( \tilde{\mathcal{P}} \), i.e. it is an oriented hypersurface of codimension one with a free action generated by the Hamiltonian vector field of \( f_i \) and it splits \( \tilde{\mathcal{P}} \) in two disjoint open pieces \( \tilde{\mathcal{P}}_+ = f_i^{-1}(0, \infty) \) and \( \tilde{\mathcal{P}}_- = f_i^{-1}(-\infty,0) \) such that its positive normal vectors point into \( \tilde{\mathcal{P}}_+ \) and negative normal vectors into \( \tilde{\mathcal{P}}_- \) (see Ref. [3]).

In case of a single condition \( f > 0 \) which generates a free circle action on \( \tilde{\mathcal{P}} \) we can employ the symplectic cutting technique\([2\, 4]\) to reformulate it as a constraint \( \phi = 0 \) on an extended phase space \( \tilde{\mathcal{P}} \times \mathbb{C} \). Here, \( \mathbb{C} \) is endowed with the symplectic structure \( \omega_\mathbb{C} = -\frac{i}{2} dz \wedge d\overline{z} \). If we denote the circle action on \( \tilde{\mathcal{P}} \) by \( S^1 : p \mapsto e^{it} \cdot p = \exp(tX_f)p \) (\( X_f \) being the Hamiltonian vector field of \( f \)), we have the free circle action \( S^1 : (p,z) \mapsto (e^{it} \cdot p, e^{-it}z) \) on \( \tilde{\mathcal{P}} \times \mathbb{C} \) with momentum map \( \phi := f - \frac{1}{2}|z|^2 \). The cut space \( \tilde{\mathcal{P}}_{\text{cut}} \) is defined as the reduced phase space \( \phi^{-1}(0)/S^1 \) subject to the constraint \( \phi = 0 \). It contains both the reduced phase space \( \tilde{\mathcal{P}}_{\text{red}} \) of \( \tilde{\mathcal{P}} \) subject to the constraint \( f = 0 \) and the subspace \( \mathcal{P} = \tilde{\mathcal{P}}_+ \) we are interested in. Due to the fact that \( \tilde{\mathcal{P}}_{\text{cut}} \) is obtained by gluing \( \tilde{\mathcal{P}}_{\text{red}} \) into \( \tilde{\mathcal{P}}_+ \) it has in general topological properties different from \( \mathcal{P} \). E.g., if \( \tilde{\mathcal{P}} \) is a cylinder, which is rotated by the circle action,

\[ \text{We are grateful to A. Alekseev for bringing this method to our attention.} \]
then $\mathcal{P}$ is a half-cylinder, which is not simply-connected, whereas $\tilde{\mathcal{P}}_{\text{cut}}$ is simply-connected with the finite boundary compactified to a single point. Such a topological difference will change the quantum theory, and therefore we will use an alteration of the standard symplectic cutting which leads to a phase space not containing the reduced phase space $\tilde{\mathcal{P}}_{\text{red}}$.

Obviously, $\tilde{\mathcal{P}}_{\text{red}}$ appears in $\tilde{\mathcal{P}}_{\text{cut}}$ as reduction of $\tilde{\mathcal{P}} \times \{0\} \subset \tilde{\mathcal{P}} \times \mathbb{C}$. It can then easily be seen that the altered symplectic cutting starting from the phase space $\tilde{\mathcal{P}} \times \mathbb{C}^*$, $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$, leads to $\tilde{\mathcal{P}}_+ = \mathcal{P}$ as symplectic manifolds. We will use this symplectic cutting when dealing with circle actions in Section 2. Translating it into a Dirac quantization will enable us to prove that for circle actions projection quantization leads to results equivalent to a quantization starting directly from $\mathcal{P}$.

The main idea goes as follows: At least under specific conditions, it has been proven (for further details and references cf. Section 2.2) that the Dirac quantization of a constraint $\phi = 0$ generating a circle action yields the same result as the quantization of the respective reduced phase space (“quantization and reduction commute”). Starting from $\tilde{\mathcal{P}} \times \mathbb{C}^*$, the symplectic reduction with respect to $\phi = 0$ yields nothing but the phase space $\mathcal{P}$ we are interested in. So, in order to prove that projection quantization yields a genuine quantization of $\mathcal{P}$, it suffices to show that it yields a quantum theory equivalent to the one obtained in a Dirac quantization of $\tilde{\mathcal{P}} \times \mathbb{C}^*$. This, however, is quite easy to show.

Excision of the origin of $\mathbb{C}$ is necessary also in order to generalize this construction to actions of the real line with no closed orbits in Section 3.

In the Discussion we will discuss examples for these methods and also present an example with a single function $f$ which fulfills the assumptions of projection quantization but for which the above results do not apply. We will finally comment on some possible generalizations of these considerations.

2 Projection Quantization with Circle Actions

As said above, we will translate symplectic cutting into a Dirac quantization, i.e. we will start by quantizing the phase space $\tilde{\mathcal{P}} \times \mathbb{C}^*$ followed by imposing the constraint $\phi = 0$ at the quantum level. A necessary ingredient of this procedure is the quantization of $\mathbb{C}^*$ with its observable $\frac{1}{2}|z|^2$, which will be presented first in terms of geometric quantization (the quantization of $\tilde{\mathcal{P}}$ is assumed to be known).

2.1 Geometric Quantization of $\mathbb{C}^*$

The standard symplectic structure of $\mathbb{C}^*$ is given by $\omega = -\frac{i}{2}dz \wedge d\overline{z}$. Writing $z = q + ip_q$, we can see that the observable $\frac{1}{2}|z|^2 = \frac{1}{2}(q^2 + p_q^2)$, which we have to quantize, is the Hamiltonian of the harmonic oscillator (with removed origin $q = p_q = 0$). Introducing polar coordinates $z = re^{i\varphi}$ shows that this phase space is symplectomorphic to $S^1 \times \mathbb{R}^+$ with symplectic structure $\omega = rd\varphi \wedge dr = d\varphi \wedge dp$, where the momentum $p = \frac{1}{2}r^2 = \frac{1}{2}|z|^2$ is introduced. The group theoretical quantization of this phase space has been studied
in detail in Refs. [5, 6] (see also Ref. [1]; the phase space also plays an important role in quantum optics [4]) together with the quantization of the observable $p$. In fact, this example motivated the definition of projection quantization. However, the group theoretical quantization lead to a quantum ambiguity (which is expected because the phase space is not simply connected) parameterized by a parameter $k \in \mathbb{R}^+$ (stemming from the positive discrete series of the $so(2,1)$-representations), whereas projection quantization was seen to lead more naturally to a parameter $k \in (0,1]$. Therefore, we use here an independent geometric quantization to decide which domain to use for the parameter. Furthermore, we prefer geometric quantization in this context because our later argumentation will be based completely on this scheme.

Noting the similarity to the harmonic oscillator we can quantize our phase space along the lines of this example following Ref. [8] to which we refer for details (see also Refs. [9, 10, 11]; we use Simms’ quantization of the harmonic oscillator and not the more usual quantization using a holomorphic polarization because it is then straightforward to include the $\theta$-angle for $\mathbb{C}^*$. Differences to Simms’ treatment will only occur because of a possible $\theta$-angle and when choosing the metaplectic structure.

In polar coordinates the symplectic form is $\omega = rd\varphi \wedge dr = d\Theta$ with symplectic potential $\Theta = -\frac{1}{2}r^2 d\varphi + \hbar \partial_r \varphi$. Here the $\theta$-angle appears because the phase space is not simply connected. As polarization we choose the one generated by the Hamiltonian vector field $\frac{\partial}{\partial \varphi}$ of the observable $\frac{1}{2}r^2$. We use the canonical metaplectic structure associated with this polarization [14]. Using the trivial Hermitean line bundle, wave functions can be written as $\psi = f \cdot s \otimes \nu$ with a function $f: \mathbb{C}^* \to \mathbb{C}$, the unit section $s$ of the prequantum line bundle, and a constant half-form $\nu$ satisfying $L_{\frac{\partial}{\partial \varphi}} \nu = 0$. This leads to the polarization condition

$$\nabla_{\frac{\partial}{\partial \varphi}} f \cdot s = \frac{\partial f}{\partial \varphi} \cdot s + \frac{i}{\hbar} f \Theta \left( \frac{\partial}{\partial \varphi} \right) s = \left( \frac{\partial f}{\partial \varphi} - \frac{i}{2\hbar} r^2 f + i\theta f \right) s = 0$$

which has only distributional solutions\[^4\] proportional to

$$f_n(r, \varphi) = \delta \left( r - \sqrt{2\hbar(n + \theta)} \right) e^{in\varphi}, \quad n \in \mathbb{Z}.$$ 

Because the label $n$ is restricted to satisfy $n + \theta > 0$, we can restrict the parameters to lie in $\theta \in (0,1]$ and $n \in \mathbb{N}_0$ in order to obtain a family of inequivalent quantizations labeled by the parameter $\theta$. For each fixed $\theta$ we will denote the Hilbert space abstractly generated by all $f_n$ as $H_\theta$.

The observable $p = \frac{1}{2}r^2 = \frac{1}{2}|z|^2$ acts on polarized states just by multiplication

$$\frac{1}{2}r^2 f_n(r, \varphi) = \hbar(n + \theta) f_n(r, \varphi)$$

with spectrum $\{\hbar(n + \theta) : n \in \mathbb{N}_0\}$. Comparing with the spectrum for $p$ obtained within the methods of Refs. [1, 4], we see that $\theta \in (0,1]$ leads to results equivalent to projection quantization, whereas the group theoretical quantization leads to a larger class of inequivalent quantum realizations.

\[^2\]We thank H. Kastrup for this remark and related discussions.

\[^3\]This is to be expected for a polarization with compact leaves [14].
We complete this discussion with a remark on the metaplectic structure (see Ref. [10] for details). Due to $H^1(\mathbb{C}^*,\mathbb{Z}_2) = \mathbb{Z}_2$ there are two inequivalent metaplectic structures on the phase space. The structure different from the one used above can be obtained by restricting the canonical metaplectic structure of $T^*\mathbb{R}$ to the subspace $\mathbb{C}^* = T^*\mathbb{R}\setminus\{(0,0)\}$. As is well known from the harmonic oscillator, this leads to a metaplectic correction in the spectrum of the Hamiltonian providing the zero point energy. The spectrum is then $\{\hbar(n + \frac{1}{2})\}$. In the above quantization we chose the canonical metaplectic structure associated with the polarization $\frac{\partial}{\partial \varphi}$, which appears to be more natural when interpreting the phase space as $S^1 \times \mathbb{R}^+$ with the observable $p$. In this case there is no metaplectic correction.

2.2 Dirac Quantization and Symplectic Cuts

In the preceding subsection we have shown that geometrical and projection quantization lead to equivalent results for the phase space $\mathbb{C}^*$; in particular, both schemes yield the same domain for the $\theta$-angle. We will now extend this result to a larger class of phase spaces by using symplectic cutting. As in the classical framework (manipulating symplectic manifolds) the phase space $\mathbb{C}^*$ and its quantization play an important role.

Projection quantization is devised to the quantization of phase spaces $\mathcal{P}$ which are subspaces of a larger phase space $\tilde{\mathcal{P}}$ given by suitable conditions $f_i > 0$. It can be helpful for phase spaces $\mathcal{P}$ which are complicated to quantize explicitly, but which are embedded into a phase space $\tilde{\mathcal{P}}$ with a well understood quantization (e.g., $\tilde{\mathcal{P}}$ could be a cotangent bundle; the method is, however, not restricted to this case). One then starts from the known quantization of $\tilde{\mathcal{P}}$ and projects to a subspace of the quantum Hilbert space to obtain the Hilbert space of $\mathcal{P}$. The basic idea to prove that this will, under certain conditions, lead to the correct result is to use Dirac quantization of symplectic cutting. As main ingredient into this proof we use theorems which state the commutation of reduction and quantization which means that a quantization of the reduced phase space of a constrained system is equivalent to the space annihilated by all constraint operators represented on a Hilbert space quantizing the unreduced phase space of the system. Such theorems go back to a conjecture of Guillemin and Sternberg [12] in case of compact Kähler manifolds, and have recently been proved and extended using symplectic cutting (see, e.g. Refs. [13, 4, 3], in Ref. [14] the arguments are generalized to non–Kähler manifolds). Independently, this has been investigated in Refs. [15, 16] for not necessarily compact phase spaces which are physically more interesting. The main condition for commutation of reduction and quantization is that the action of the gauge group $G$ generated by the constraints preserves the structure used to quantize $\tilde{\mathcal{P}}$, namely the polarization and the metaplectic structure. Then one can project the quantization structure to the reduced phase space and establish the commutation of reduction and quantization in a controlled way. In the following we restrict our considerations to the class of phase spaces where the conditions for a commutation of reduction and quantization are fulfilled. The only requirement on the symplectic geometry of a particular phase space is that it has to permit a polarization and a metaplectic structure compatible with the constraints in the above sense.
In the present section we first treat phase spaces $\tilde{P}$ with a single constraint $f$ which generates a free circle action on $\tilde{P}$ with momentum map $f$ such that zero is a regular value of $f$, and will later generalize to torus actions. By assumption, furthermore, the quantum theory of $\tilde{P}$ and its Hilbert space $\tilde{H}$ are known. (If the quantization is not unique, we can use any of the inequivalent quantum realizations of $\hat{P}$.) In the preceding section we derived the Hilbert spaces $\mathcal{H}_\theta$ for the quantum theory of $\mathbb{C}^*$, which we use to construct the Hilbert space $\mathcal{H} \otimes \mathcal{H}_\theta$ of $\tilde{P} \times \mathbb{C}^*$ for arbitrary $\theta \in (0, 1]$. Together with the Hilbert space $\tilde{H}$ we assume to know a self-adjoint quantization $\hat{f}$ on $\tilde{H}$ of the function $f$. Combining this with the quantization of $\frac{1}{2}|z|^2$ acting on $\mathcal{H}_\theta$ we obtain the quantized constraint

$$\hat{\phi} = \hat{f} \otimes 1 - 1 \otimes \frac{1}{2}|z|^2$$

acting on $\tilde{H} \otimes \mathcal{H}_\theta$, which imposes symplectic cutting at the quantum level.

Provided that quantization and reduction commute, the quantization of $\mathcal{P} = \tilde{P}_+$ is given by the kernel of $\hat{\phi}$. We know the spectrum of $\frac{1}{2}|z|^2$ from the preceding subsection, where we showed that it is discrete. Similarly, the spectrum of $\hat{f}$ is discrete: $f$ generates a Hamiltonian circle action on $\tilde{P}$ which, provided that $f$ is quantizable in the sense of geometric quantization, implies that $\hat{f}$ generates a unitary (possibly projective) $S^1$-representation on $\mathcal{H}$. This representation splits into sectors $\tilde{H}_{\theta_j}$ on which $e^{2\pi i \varphi} \in S^1$ has eigenvalues of the form $e^{2\pi i (n+\theta_j) \varphi}$, $\theta_j \in (0, 1]$, with integer values of $n$. In other words, in each of these sectors, $\hat{f}$ has as spectrum a subset of $\{n + \theta_j : n \in \mathbb{Z}\}$.

Let us first assume that there is only one $\theta_j = \theta'$, i.e. $\tilde{H} = \tilde{H}_{\theta'}$ (this is always the case if the algebra of observables is represented irreducibly on $\tilde{H}$, because the elements of the fundamental group of $\tilde{P}$ are required to commute with all observables; for a discussion cf, e.g. Ref. [17]). Then for $\theta \neq \theta'$ the kernel of $\hat{\phi}$ is trivial, whereas for $\theta = \theta'$ the constraint $\hat{\phi} = 0$ acting on $\psi \otimes f_n \in \tilde{H} \otimes \mathcal{H}_\theta$ takes the form $\hat{f} \psi = (n + \theta) \psi$ with $n \in \mathbb{N}_0$. This constraint projects precisely to those states of $\tilde{H}$ which are eigenstates of $\hat{f}$ with positive eigenvalues, leading exactly to the result of projection quantization.

If there are different values of $\theta_j$ in $\mathcal{H}$ (the algebra of observables is then represented reducibly), we have to match all the $\theta$-sectors separately by choosing an appropriate direct sum of $\mathcal{H}_\theta$ as quantization of $\mathbb{C}^*$.

These considerations can easily be extended to the case of more than one commuting conditions $f_j > 0$. We can reduce them one after another, not running into problems because their actions commute and therefore project to the cut spaces. If, moreover, the commutation assumption on quantization and reduction is fulfilled for the constraints $f_j = 0$ on $\tilde{P}$, it also holds for $\phi_j = 0$ on $\tilde{P} \times \mathbb{C}^*$. This is a consequence of the above construction, where we always took direct products of quantization data. They are conserved by the circle actions generated by $\phi_j$ provided the data on $\tilde{P}$ are conserved by the actions generated by $f_j$ (it is immediate to see that this is also fulfilled for the constraint $\frac{1}{2}|z|^2$ on $\mathbb{C}^*$ in the above quantization).

We thus may conclude
Theorem 1 (Projection Quantization with Circle Actions): Let the functions $f_j$ on a phase space $\tilde{\mathcal{P}}$, with zero being a regular value for each of them, generate mutually commuting circle actions. Assume further that their quantizations $\hat{f}_j$ on the Hilbert space $\hat{\mathcal{H}}$ generate mutually commuting unitary actions and that reduction commutes with quantization for each of them.

Then projection quantization applied to $\tilde{\mathcal{P}}$ with the conditions $f_j > 0$ yields a quantization of $\mathcal{P} = \tilde{\mathcal{P}}_+$. Let us emphasize that our general assumption of commutation of quantization and reduction has to be imposed only for the constraints $f_j$. As such, this is a statement about the reduced phase space $\tilde{\mathcal{P}}_{\text{red}}$ (characterized by $f_j = 0$) and its quantization. Its validity can in many cases be seen by employing the results collected in the references mentioned above or by checking it by hand in specific cases. As the construction shows, we can then for granted that reduction commutes with quantization also for each constraint $\phi_j$ on $\tilde{\mathcal{P}} \times \mathbb{C}^*$ associated with $f_j$. This allowed us to prove the desired result without requiring a condition (like the commutation assumption) for the constraints $\phi_j$ or the phase space $\tilde{\mathcal{P}}_+$. As already discussed in the Introduction, the quantization obtained using the present formulation of projection quantization is not always the most general one. In particular, if the fundamental group of $\tilde{\mathcal{P}}_+$ does not coincide with the one of $\tilde{\mathcal{P}}$ (there can be contractible loops in $\tilde{\mathcal{P}}$ which become noncontractible after imposing $f_i > 0$), some of the $\theta$–angles necessary for a general quantization of $\tilde{\mathcal{P}}_+$ may be missed. At least in some cases an adaption of the quantization scheme may accomodate for these additional $\theta$–angles; we intend to come back to this issue elsewhere. If, on the other hand, any two nonhomotopic loops in $\tilde{\mathcal{P}}_+$ are nonhomotopic also in $\tilde{\mathcal{P}}$, the $\theta$–angles in the most general quantization of $\tilde{\mathcal{P}}$ are sufficient to yield the most general quantum theory of $\tilde{\mathcal{P}}_+$. This is e.g. the case in the paradigmatic example of a cylinder cut to a half–cylinder.

2.3 Observables

Dirac quantization also contains a prescription to obtain observables on the physical Hilbert space $\mathcal{H}_1$, on which the constraints are solved, from those on the original Hilbert space $\mathcal{H}_0$. An operator $\mathcal{O}_0$ on $\mathcal{H}_0$ which is an observable in the strict sense, i.e. which commutes with the quantum constraint operator $\hat{\phi}$ and thus with the projector $P$ to $\mathcal{H}_1$, projects just to the same operator restricted to $\mathcal{H}_1$.

More generally, one can project any operator $\mathcal{O}_0$ on $\mathcal{H}_0$ to an operator $P \mathcal{O}_0 P$. The new operator annihilates the orthogonal complement of $\mathcal{H}_1$ and can, therefore, be reduced to an operator on this subspace. If we denote the inclusion of the subspace $\mathcal{H}_1$ into $\mathcal{H}_0$ as $\iota_0: \mathcal{H}_1 \hookrightarrow \mathcal{H}_0$ and the projection from $\mathcal{H}_0$ to $\mathcal{H}_1$ as $\pi_0: \mathcal{H}_0 \to \mathcal{H}_1$, we can write the final operator on $\mathcal{H}_1$ as $\mathcal{O}_1 = \pi_0 \circ \mathcal{O}_0 \circ \iota_0: \mathcal{H}_1 \to \mathcal{H}_1$.

In our case we have $\mathcal{H}_0 = \hat{\mathcal{H}} \otimes \mathcal{H}_\theta$ and $P$ is the projector to the kernel $\mathcal{H}$ of the constraint $\hat{\phi}$. The quantum theory on $\mathcal{H}$ is assumed to be known, and therefore we have observables $\mathcal{O}$ acting on this Hilbert space. They can be extended trivially to operators $\mathcal{O} \otimes \mathbb{I}$ on $\mathcal{H}_0$ and, using the procedure described above, projected to operators on $\mathcal{H}$.
This leads exactly to the definition of observables which has been given in Ref. [1] in the framework of projection quantization. Thus, Dirac quantization of symplectic cutting leads, under the conditions stated in the Theorem, to the same observables as projection quantization.

Using the projection of operators, we can associate an operator \( O \) on the Hilbert space \( \mathcal{H} \), which quantizes \( \mathcal{P} \), to each operator \( \hat{O} \) on \( \hat{\mathcal{H}} \). If \( \hat{O} \) is the quantization of a phase space function on \( \hat{\mathcal{P}} \), \( O \) can be regarded as quantization of the same phase space function restricted to \( \mathcal{P} \subset \hat{\mathcal{P}} \). However, even if \( \hat{O} \) is selfadjoint (or unitary), \( O \) will be selfadjoint (unitary) in general only if \( \hat{O} \) is an observable in the strict sense of Dirac quantization, i.e. if it commutes with the projector to the physical subspace. Otherwise, \( O \) is in general only hermitean (isometric). A more detailed discussion has been given in Ref. [1].

3 Projection Quantization with Line Actions

We now indicate how the results of the preceding section can be generalized to the case that the orbits of the action generated by \( f \) do not close but are homeomorphic to \( \mathbb{R} \). For technical reasons, we restrict our considerations to conditions \( f > 0 \) where \( f \) is chosen to be a coordinate on \( \hat{\mathcal{P}} \). First we have to adapt the symplectic cutting to line actions.

We replace \( \mathbb{C}^* \) used before by its universal covering space \( \tilde{\mathbb{C}}^* \) parameterized by \((r, x) \in \mathbb{R}^+ \times \mathbb{R} \) with covering map \((r, x) \mapsto re^{ix}\). The symplectic form is \( \omega = r dx \wedge dr \). The rest of symplectic cutting is as before with \( \phi := f - \frac{1}{2} r^2 \) generating a free action on \( \mathcal{P} \times \tilde{\mathbb{C}}^* \).

The subset \( \mathcal{P} = \hat{\mathcal{P}}|_{f>0} \) is symplectomorphic to \( \phi^{-1}(0)/\mathbb{R} \).

Now we can proceed similar to the case of a circle action by commuting quantization and reduction. There are, however, two differences: First, there is no \( \theta \)-angle and, second, the spectrum of \( \frac{1}{2} r^2 \) is continuous (we can use a quantization similar to that of \( \mathbb{C}^* \), but without the restriction \( n \in \mathbb{Z} \)). Therefore, zero will lie in the continuous part of the spectrum of the constraint \( \phi \), which leads to technical difficulties when projecting to its kernel. In the following we will use group averaging [18, 19] and assume \( f \) to be chosen as coordinate (otherwise the following calculations have to be adapted appropriately). In particular, the polarization chosen to quantize \( \hat{\mathcal{P}} \) contains the Hamiltonian vector field \( X_f \) of \( f \) and the symplectic potential \( \Theta_{\tilde{\mathcal{P}}} \) on \( \tilde{\mathcal{P}} \) is adapted to \( X_f \), i.e. \( \Theta_{\tilde{\mathcal{P}}}(X_f) = 0 \). In \( \tilde{\mathcal{C}}^* \) we choose the polarization generated by \( \frac{\partial}{\partial x} \) and symplectic potential \( \Theta = rxdx = xdp, \ p = \frac{1}{2} r^2 \). Quantum states of \( \tilde{\mathcal{P}} \times \tilde{\mathbb{C}}^* \) can then be represented as \( \psi(f, y) \chi(p) \), where \( \psi \) is a quantum state of \( \tilde{\mathcal{P}} \) depending on \( f \) and other continuous or discrete labels collected in \( y \).

The constraint \( \hat{\phi} \) generates the unitary \( \mathbb{R} \)-action

\[
\psi(f, y) \chi(p) \mapsto e^{it(f-p)} \psi(f, y) \chi(p).
\]

For group averaging we use test states which are smooth and of compact support. The rigging map \( \eta \) is then determined by

\[
\eta(\psi_1 \chi_1)[\psi_2 \chi_2] = \mu_y \left( \int_R df \int_{R^+} dp \nu(y, f) \int_R dt e^{it(f-p)} \psi_1(f, y) \chi_1(p) \overline{\psi_2(f, y) \chi_2(p)} \right)
\]
= \mu_y \left( \int_{\mathbb{R}^+} dp \nu(p,y) \psi_1(p,y) \chi_1(p) \psi_2(p,y) \chi_2(p) \right)

factoring without restriction the measure for polarized states of \( \tilde{\mathcal{P}} \) into \( \mu_y \int_{\mathbb{R}} df \nu(f,y) \).

This calculation demonstrates that the image of the rigging map coincides with the spectral projection to the positive part of the spectrum of \( f \) (the coordinate \( f \) is replaced by the positive coordinate \( p \)). Assuming commutation of quantization and reduction, we see that also for line actions projection quantization yields a quantization of \( \mathcal{P} = \tilde{\mathcal{P}}_+ \) where a coordinate is constrained to be positive:

**Theorem 2 (Projection Quantization with Line Actions):** Let \( f_j \) be coordinates generating line actions on a phase space \( \tilde{\mathcal{P}} \) which is equipped with a polarization containing the vector fields generated by the \( f_j \) and a symplectic potential adapted to the polarization. Assume further that reduction commutes with quantization for each of the \( f_j \) regarded as constraints.

Then projection quantization applied to \( \tilde{\mathcal{P}} \) with the conditions \( f_j > 0 \) yields a quantization of \( \mathcal{P} = \tilde{\mathcal{P}}_+ \).

4 Discussion

The result of this paper is a proof that projection quantization proposed in Ref. [1] leads to a correct quantization in particular cases. The results apply in case of circle and line actions generated by functions \( f_i \) on \( \tilde{\mathcal{P}} \), with the main assumption that reduction and quantization with respect to \( f_i = 0 \) commute (as has been proven for a fairly general class of systems, cf. the citations above).

The systems covered by our requirements provide further examples supplementary to those of Ref. [1] in which projection quantization leads to the correct results. These include e.g. the following class of simple systems: \( T^*(S^1)^n \) or \( T^*\mathbb{R}^n \) with some of the momenta or coordinates (for \( T^*\mathbb{R}^n \)) constrained to be positive. More generally, arbitrary linear combinations of the coordinates of \( T^*\mathbb{R}^n \) or linear combinations with integer coefficients of the momenta of \( T^*(S^1)^n \) can be constrained to be positive. But certainly many more examples are covered by the present considerations, e.g. in cases where a globally defined coordinate on a cotangent bundle is constrained to be positive and one uses the vertical polarization. For instance, the phase space \( T^*GL^+(n,\mathbb{R}) \), which is the cotangent bundle on the manifold of \( n \times n \)-matrices of positive determinant, can be treated along these lines if one embeds it into the cotangent bundle \( T^*M(n,\mathbb{R}) \cong T^*\mathbb{R}^{n^2} \) on the manifold of all \( n \times n \)-matrices via the restriction \( f := \det A > 0 \) for \( A \in M(n,\mathbb{R}) \). One then directly obtains a quantization of \( T^*GL^+(n,\mathbb{R}) \) on the space of square integrable functions on \( GL^+(n,\mathbb{R}) \subset \mathbb{R}^{n^2} \). The fundamental operators are projected to multiplication and derivative operators the latter of which are no longer self-adjoint (analogous to the phase space \( T^*\mathbb{R}^+ \cong T^*GL^+(1,\mathbb{R}) \)). This system has already been dealt with in a group theoretical quantization [20] leading to the same Hilbert space besides a large family of “degenerate” quantizations. This is similar to the phase space \( S^1 \times \mathbb{R}^+ \) where group theoretical quantization yields a larger class of
quantizations some of which have to be regarded as unphysical \[1\]. It is, however, not always possible to select the physical representations intrinsically from properties of the system, e.g. by demanding positive spectra of suitable operators, and here a comparison with other quantization schemes, as projection quantization, can help. Moreover, also in this case the projection quantization is much simpler to apply than the group theoretical one; and, according to the results of the present paper, the application of the projection approach is fully legitimate here: it yields a quantization equivalent to a genuine geometric quantization of the restricted space.

Another class of examples where the proofs of this paper can be used is provided, e.g., by phase spaces which are compact Kaehler manifolds. For additional conditions in this case we refer to the literature \[2, 3, 12, 13\].

We now briefly describe a system which is not covered by the results of the present article, but can nevertheless be dealt with using projection quantization. Let the phase space \( \mathcal{P} \) be the subspace of the product \( \tilde{\mathcal{P}} = T^*S^1 \times T^*S^1 \) subject to the condition \( f := p_1 - p_2^2 > 0 \) in terms of the usual coordinates \((\varphi_1, p_1)\) and \((\varphi_2, p_2)\) of the two cylinders. The action on \( \tilde{\mathcal{P}} \) generated by \( f \) has orbits winding around the torus \( S^1 \times S^1 \) parameterized by \((\varphi_1, \varphi_2)\) and they are closed if and only if \( p_2 \) is rational. Otherwise they are homeomorphic to \( \mathbb{R} \). We have thus neither of the cases of pure circle or line actions dealt with in the preceding sections.

Nevertheless, the assumptions of projection quantization as recapitulated in the Introduction are fulfilled and the method may be applied for a quantization: Each of the two cylinders in \( \tilde{\mathcal{P}} \) can be quantized in the usual way leading to quantum states of the form \( \psi_{n_1,n_2}(\varphi_1,\varphi_2) = e^{in_1\varphi_1}e^{in_2\varphi_2} \) (assuming for simplicity vanishing \( \theta \)-angles). Projection quantization then selects those states which have positive eigenvalues for \( \hat{f} \): \( \hat{f}\psi_{n_1,n_2} = (n_1 - n_2^2)\psi_{n_1,n_2} \) resulting in the condition \( n_2^2 < n_1 \) for quantum states of \( \mathcal{P} = \tilde{\mathcal{P}}_+ \).

Of course, it would be of interest to extend the proofs presented here for circle and line actions to a more general class of actions, including the above system. Already this relatively simple example of a phase space \( \mathcal{P} \) is complicated to quantize by standard methods, while projection quantization is almost trivial to apply.

These remarks also apply for imposing the condition of a nondegenerate metric of fixed signature in quantum gravity (see e.g. Ref. \[21\]). In this context one in addition has to take into account that one is dealing with a constrained system. The constraint \( \phi = 0 \), imposing the condition \( f > 0 \), then arises in addition to the usual constraints of the gravity system. Consistency leads to compatibility conditions between the original constraints and \( \phi \) (see also the remarks in Ref. \[1\]). As shown independently in Ref. \[22\], methods similar to those of projection quantization can also be used to solve constraints by “thickening” a constraint surface given, e.g., by \( f = 0 \) to a set given by \(-\epsilon < f < \epsilon\) (which corresponds to a twofold application of projection quantization).

As the preceding example demonstrates, projection quantization can be helpful for the quantization of physically interesting systems, even if the above theorems do not apply. Quantization schemes are justified usually by showing that they yield the expected results for standard test models. Some elementary systems have been dealt with on these grounds.
already in Ref. [1]. A large class of further systems is covered implicitly by the proofs presented in this article. So we can trust the method also for more complicated systems. Still, further tests and possibly adaptations of the method are of interest.

The main advantage of projection quantization is that it is extremely simple if it applies, i.e. if its assumptions are fulfilled. Its applicability, however, is smaller than that of a typical quantization procedure: The phase space of interest has to be embedded into a larger phase space with known quantization, where the kind of embedding is also restricted by the requirements on the functions $f_i$.

In this context, extensions of the symplectic cutting method to actions of nonabelian groups [2] can be of interest. This may lead to a generalization of projection quantization to noncommuting conditions $f_j$, which would allow a more general class of embeddings of the phase space.

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