INTEGRALS OF MOTION, TURNING POINTS, AND INERTIAL POINTS FOR ORBITS AROUND SPHERICAL BLACK-HOLES

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ABSTRACT. A locally conserved angular quantity is obtained for non-circular orbits of a massive particle around a spherical black hole. This quantity yields the angle in the spatial plane of motion at which an orbit has either a turning point or an inertial point. It is an analogue of the angle of the globally conserved Laplace-Runge-Lenz vector and Hamilton’s binormal vector which exist in Newtonian gravity, with the important difference that the angle jumps at every turning point in the case of elliptic-like orbits that precess. This feature is shared by the angle of the locally conserved Laplace-Runge-Lenz vector and Hamilton’s binormal vector in Newtonian gravity with a cubic correction. The angular quantity is derived as part of obtaining a complete set of integrals of motion for the timelike geodesic equations in Schwarzschild spacetime. In addition, a locally conserved temporal quantity and a locally conserved proper-time quantity are obtained from the derivation. The properties of these (new) locally conserved quantities are explained and illustrated for each type of non-circular orbit. Their existence is shown to be related to a group of hidden dynamical symmetries of the timelike geodesic equations.

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1. INTRODUCTION

The orbits of a massive particle around a spherical black hole are described by the timelike geodesic equations in Schwarzschild spacetime. It is well known that these equations can be integrated by use of Killing vectors to obtain the geodesics explicitly [1, 2], with the motion of the particle lying in a timelike hyperplane that is rotationally isometric to the equatorial hyperplane. A complete exposition can be found in the comprehensive reference [3].

The purpose of the present work is to obtain in explicit form a complete set of locally conserved quantities for the equations of motion in the equatorial hyperplane. In terms of standard coordinates $(t, r, \phi)$ for the hyperplane metric $(ds)^2 = -\frac{r^2 M}{r} (dt)^2 + \frac{r}{r-2M} (dr)^2 + r^2 (d\phi)^2$, the locally conserved quantities consist of energy, angular momentum, an angular $\phi$-quantity, a temporal $t$-quantity, and a proper-time $\tau$-quantity. The latter three quantities are typically expressed implicitly through initial conditions $r(\tau_0) = r_0$, $\phi(\tau_0) = \phi_0$, $t(\tau_0) = t_0$ for the geodesic motion [3]. In contrast, the new contribution here will be to derive these quantities in a form motivated by the Laplace-Runge-Lenz (LRL) vector in Newtonian
gravity [4, 5], which involves geometrical properties of an orbit’s dynamics rather than initial conditions.

The resulting (new) conserved $\phi, t, \tau$-quantities are physically and mathematically important:

- they allow predicting future and past turning/inertial points on an orbit strictly in terms of values of dynamical variables at any point;
- they yield a geometrical description of the orbit in terms of a complete set of integrals of motion;
- they reveal a hidden symmetry structure in the equations of motion;
- they are useful for checking numerical integration schemes for computation of orbits.

Specifically, for non-circular orbits, the $\phi$-quantity yields the coordinate angle of radial lines that intersect the orbit at either a turning point, where the radial speed is zero, or an inertial point, where the radial acceleration is zero. The respective physical meaning of these points, with respect to inertial observers at spatial infinity, is that turning points are the local extrema of the radial distance of a particle in the orbit, namely periapsis and apoapsis points, and inertial points are the local extrema of the particle’s radial Doppler shift in the orbit. An important feature of the $\phi$-quantity is that it is only locally conserved because it undergoes a jump at every apsis point in the case of an elliptic-like orbit that precesses. For all other non-circular orbits, there is no jump in the $\phi$-quantity.

Thus, the $\phi$-quantity as defined by turning points is an analogue of the angular orientation of the Laplace-Runge-Lenz (LRL) vector for “revolving” orbits in Newtonian gravity with a cubic correction [6, 7]; and when it is defined by inertial points, the $\phi$-quantity is an analogue of a variant of the LRL vector known as Hamilton’s binormal vector. The LRL vector and its variant have been discussed in detail recently [8] in the context of locally conserved quantities for general central force motion, including the example of “revolving” orbits. The piecewise nature of the LRL vector has been well-studied in classical mechanics [9, 10, 11, 12].

The locally conserved $\tau$-quantity and $\tau$-quantity describe the coordinate time(s) and proper time(s) at which successive turning/inertial points are reached in a non-circular orbit. In contrast to the $\phi$-quantity, they jump at every apsis point for all orbits that possess more than one turning point (not just for precessing elliptic-like orbits).

Each of the locally conserved quantities can be associated to an underlying symmetry of the Lagrangian for geodesic motion via Noether’s theorem. As is well known [13, 14], energy and angular momentum arise from Killing vector symmetries of the equatorial metric. The additional three conserved quantities turn out to arise from hidden dynamical symmetries of the geodesic equations, which will be derived in an explicit form.

The paper is organized as follows.

In section 2, the timelike geodesic equations that describe particle orbits in the equatorial hyperplane of the Schwarzschild black hole spacetime are briefly reviewed. These equations are formulated as a first-order dynamical system, which will be useful for deriving the locally conserved quantities (integrals of motion).

In section 3, the complete set of five locally conserved quantities for orbits parameterized by proper time is derived. The method uses the explicit separability of the geodesic equations, combined with a geometrical argument for setting the zero-point values of the quantities by adapting recent work [8] on integrals of motion in general central force dynamics.
In section 4, the $\phi, t, \tau$-quantities are expressed in explicit form in terms of elementary and elliptic functions for all of the different types of non-circular orbits in Schwarzschild spacetime.

In section 5, the global properties of the $\phi, t, \tau$-quantities are discussed. The $\phi$-quantity is shown to be multi-valued on orbits that precess, whereas the temporal quantities are shown to be multi-valued on orbits that possess more than one turning point.

In section 6, the underlying hidden dynamical symmetries of the geodesic Lagrangian are obtained. These symmetries together with the Killing vector symmetries associated with energy and angular momentum are shown to form a six-dimensional symmetry group with a semi-direct product structure.

In section 7, some concluding remarks are given.

An appendix discusses the Newtonian limit of the $\phi, t, \tau$-quantities and explains their relationship with the LRL vector and the variant Hamilton’s vector in Newtonian gravity.

2. Equations of motion and integrals of motion

Throughout, geometrized units in which $c = 1$ and $G = 1$ will be used.

In the Schwarzschild black hole spacetime, the metric is given by the line element

$$
(ds)^2 = -\frac{r-2M}{r}(dt)^2 + \frac{r}{r-2M}(dr)^2 + r^2 \sin^2 \theta (d\phi)^2 + r^2 (d\theta)^2
$$

(2.1)

with the use of standard spherical coordinates $(t, r, \phi, \theta)$, where $M$ denotes the Komar mass of the spacetime, and $r = 2M$ is the spherical null surface constituting the horizon.

Consider a free massive particle moving on a timelike geodesic outside of the horizon. As is well-known [13, 14, 3], any timelike geodesic lies in a hyperplane $\Sigma$ that is isometric to the equatorial hypersurface $\theta = \frac{1}{2}\pi$ under a global spatial rotation. (In Ref. [3], this hyperplane is called the “invariant plane”.) So hereafter, without loss of generality, $\theta$ will be fixed to be $\frac{1}{2}\pi$ by spherical symmetry. Then the metric on the hyperplane containing the timelike geodesic is given by

$$
(ds)^2|_{\Sigma} = -\frac{r-2M}{r}(dt)^2 + \frac{r}{r-2M}(dr)^2 + r^2 (d\phi)^2.
$$

(2.2)

This metric has two Killing vectors $\partial_t$ and $\partial_\phi$.

The particle’s 4-velocity is expressed as

$$
u = \sigma \partial_t + \nu \partial_r + \omega \partial_\phi,
$$

(2.3)

with the notation

$$
\sigma = \frac{dt}{d\tau}, \quad \nu = \frac{dr}{d\tau}, \quad \omega = \frac{d\phi}{d\tau},
$$

(2.4)

where $\tau$ denotes proper time of the particle, as defined by

$$
g(u, u) = -1.
$$

(2.5)

The equation of motion of the particle is given by the timelike geodesic equation

$$
\frac{du}{d\tau} = 0.
$$

(2.6)
In component form, the timelike geodesic equation (2.6) and the proper time equation (2.5) are given by
\[
\frac{d\sigma}{d\tau} + \frac{2M}{r(r - 2M)} \sigma v = 0, \quad (2.7a)
\]
\[
\frac{dv}{d\tau} - \frac{M}{r(r - 2M)} v^2 + \frac{M(r - 2M)}{r^3} \sigma^2 - (r - 2M)\omega^2 = 0, \quad (2.7b)
\]
\[
\frac{d\omega}{d\tau} + \frac{2}{r} \omega v = 0, \quad (2.7c)
\]
and
\[
\frac{r - 2M}{r} \sigma^2 - \frac{r}{r - 2M} v^2 - r^2 \omega^2 = 1. \quad (2.8)
\]
These equations (2.7)–(2.8) for the 4-velocity components (2.4) comprise a coupled system of nonlinear second-order ODEs when expressed in terms of the dynamical variables \((t(\tau), r(\tau), \phi(\tau))\):
\[
\ddot{t} = -\frac{2Mi\dot{r}}{r(r - 2M)}, \quad (2.9a)
\]
\[
\ddot{r} = \frac{Mr^2}{r(r - 2M)} - \frac{M(r - 2M)i^2}{r^3} + (r - 2M)\dot{\phi}^2, \quad (2.9b)
\]
\[
\ddot{\phi} = -\frac{2\dot{\phi}i}{r}, \quad (2.9c)
\]
and
\[
\frac{r - 2M}{r} i^2 - \frac{r}{r - 2M} \dot{i}^2 - r^2 \dot{\phi}^2 = 1, \quad (2.9d)
\]
where a dot denotes differentiation with respect to \(\tau\). Note that the proper time equation can be used to simplify the radial acceleration equation:
\[
\ddot{r} = (r - 3M)\dot{\phi}^2 - \frac{M}{r^2}. \quad (2.10)
\]
The second-order ODEs (2.9) can be derived from the Euler-Lagrange equations of the geodesic Lagrangian \([13, 14, 3]\):
\[
\mathcal{L} = \frac{1}{2} g(u, u) = \frac{1}{2} \left( -\frac{r - 2M}{r} i^2 + \frac{r}{r - 2M} \dot{i}^2 + r^2 \dot{\phi}^2 \right). \quad (2.11)
\]
Each solution \((t(\tau), r(\tau), \phi(\tau))\) of this ODE system (2.9) describes a timelike geodesic, parameterized by proper time, in Schwarzschild spacetime.

An integral of motion of the timelike geodesic equations is a scalar function
\[
I(\tau, t, r, \phi, \sigma, v, \omega) \quad (2.12)
\]
that is (locally) constant with respect to proper time, \(\tau\), on (segments of) every timelike geodesic \((t(\tau), r(\tau), \phi(\tau))\). In particular, \(I\) is not necessarily a global constant on the entirety of a geodesic but may be merely piecewise constant and undergo finite jumps at certain points. This generality is essential for understanding the \(t, \tau\)-integrals of motion for orbits with more than one turning point as well as the \(\phi\)-integral of motion for precessing orbits.

Integrals of motion describe locally conserved quantities. In the literature on ODEs and classical mechanics, such quantities are sometimes called first integrals. However, in the
literature on (super)integrability, a first integral is typically meant to be a globally conserved (single-valued) quantity, such as energy and angular momentum. This distinction will be important for understanding the status of the $\phi, t, \tau$-quantities. In particular, their existence does not imply superintegrability, but they still are associated with dynamical symmetries in a local sense, as will be explained in section 6. See also Ref. \[15\] for a general discussion.

3. Complete set of integrals of motion

It is well known that the timelike geodesic equations (2.9) are separable in the sense that they can be integrated to obtain the geodesic motion $(t(\tau), r(\tau), \phi(\tau))$ in terms of initial conditions. See Ref. \[3\] for a comprehensive treatment.

For the equations in their first-order dynamical form (2.4) and (2.7), the goal here will be to derive a complete set of functionally independent integrals of motion (2.12). Since the proper time equation (2.8) provides a relation among the variables $r, \sigma, v, \omega$, one of them can be eliminated in terms of the others, without loss of generality in the function (2.12). It is convenient to eliminate

$$ v^2 = (1 - 2M/r)^2 \sigma^2 - (1 - 2M/r)(1 + r^2 \omega^2). \tag{3.1} $$

Then the resulting function

$$ I(t, r, \phi, \sigma, \omega) \tag{3.2} $$

satisfies

$$ 0 = \frac{dI}{d\tau} = I_t + I_r \sigma + I_r v + I_\phi \omega - I_\sigma \frac{2M \sigma v}{r(r - 2M)} - I_\omega \frac{2 \omega v}{r} \tag{3.3} $$

locally in $\tau$ for all timelike geodesics. This is a linear first-order PDE which can be turned into a system of ODEs via the method of characteristics [16]:

$$ \begin{align*}
    \frac{d\tau}{1} &= \frac{dt}{\sigma} = \frac{dr}{v} = \frac{d\phi}{\omega} = -\frac{r(r - 2M)d\sigma}{2M \sigma v} = -\frac{rd\omega}{2 \omega v}. \tag{3.4}
\end{align*} $$

In particular, each solution of this ODE system defines a characteristic curve in the space of variables $(\tau, t, r, \phi, \sigma, \omega)$, where the curve corresponds to a timelike geodesic with an arbitrary parameterization.

The process of solving the ODE system (3.4) amounts to integrating the timelike geodesic equations (2.4) and (2.7). From this point of view, integrals of motion correspond to constants of integration. A main contribution in this section will be to identify and explain the physical and mathematical meaning of the integrals of motion beyond the well known ones for energy and angular momentum. In particular, the derivation of the additional integrals of motion will be related to global features of the timelike geodesics, rather than to initial conditions as is done in standard treatments [3].

Mathematically, the ODE system (3.4) constitutes a first-order dynamical system for the six variables $(t(\tau), r(\tau), \phi(\tau), \sigma(\tau), v(\tau), \omega(\tau))$, where the proper time equation (2.8) provides a constraint among these variables. Hence there are five unconstrained degrees of freedom in the solutions describing timelike geodesics parameterized by proper time. As a consequence, the ODE system (3.4) will have five functionally independent integrals of motion. Four of these will have no explicit dependence on $\tau$, while the fifth will involve $\tau$ explicitly. The proper time equation (2.8) itself can be also viewed as an integral of motion.

An integral of motion $I(t, r, \phi, \sigma, \omega)$, with $I_\tau = 0$, defines a local constant of motion. When it is evaluated on any given timelike geodesic, $I(t(\tau), r(\tau), \phi(\tau), \sigma(\tau), \omega(\tau))$ is either a global
constant on the entirety of the geodesic or a piecewise constant which undergoes finite jumps at certain points, depending on the nature of the geodesic.

The four constants of motion determined by the ODE system (3.4) will physically describe energy, angular momentum, plus an angular $\phi$-quantity and a temporal $t$-quantity. The remaining integral of motion will physically describe a proper time $\tau$-quantity. This set of five locally conserved quantities will now be derived for all non-circular timelike geodesics. Their global properties, specifically whether they are globally conserved or only piecewise conserved, will be discussed in section 5 for each type of geodesic. A discussion of the physical meaning of the angular $\phi$-quantity and the temporal $t, \tau$-quantities and their status as analogues of LRL-type quantities also will be given.

3.1. Derivation. The ODE system (3.4) can be arranged into a triangular form in which the ODEs successively become separable and hence can be integrated. First, $\frac{dr}{v} = -\frac{r d\omega}{2 \omega v}$ is separable, which yields the integral of motion

$$I_1 = \omega r^2.$$  \hfill (3.5)

Likewise, $\frac{dr}{v} = -\frac{r(r - 2M)d\sigma}{2M \sigma v}$ is separable, and hence it yields the integral of motion

$$I_2 = \frac{r - 2M}{r} \sigma.$$  \hfill (3.6)

Then the relation (3.1) can be rewritten in terms of the integrals of motion $I_1$ and $I_2$, giving

$$v^2 = I_2^2 - \frac{r - 2M}{r^3} I_1^2 - \frac{r - 2M}{r}$$  \hfill (3.7)

which expresses $v$ as a function only of $r$. Next, through this expression along with $I_1$, $\frac{dr}{v} = \frac{d\Phi}{\omega} = \frac{r^2 d\Phi}{I_1}$ is separable. This yields the integral of motion

$$I_3 = \Phi - I_1 \int \frac{dr}{r^2 v},$$  \hfill (3.8)

provided that $v$ is not identically zero. Similarly, with the use of $I_2$, $\frac{dr}{v} = \frac{dt}{\sigma} = \frac{(r - 2M)dt}{r I_2}$ is separable, thus yielding one more integral of motion

$$I_4 = t - I_2 \int \frac{r dr}{(r - 2M)v},$$  \hfill (3.9)

again provided that $v$ is not identically zero. Note that $v \equiv 0$ corresponds to a circular timelike geodesic.

The four integrals of motion (3.5), (3.6), (3.8), (3.9) do not contain $\tau$ explicitly, and therefore they are local constants of motion for non-circular timelike geodesics. A fifth integral of motion, which will involve $\tau$ explicitly, comes from integrating $\frac{d\tau}{1} = \frac{dr}{v}$. This yields

$$I_5 = \tau - \int \frac{dr}{v}.$$  \hfill (3.10)

The proper time equation (2.8) can be viewed as being a sixth integral of motion which serves as a constraint.
This set of six integrals of motion (3.5), (3.6), (3.8), (3.9), (3.10), and (2.8) is complete for non-circular timelike geodesics. In particular, the timelike geodesic equations in first-order form (2.4), (2.7) for \((t(\tau), r(\tau), \phi(\tau), \sigma(\tau), v(\tau), \omega(\tau))\), or equivalently in second-order form (2.9) for \((t(\tau), r(\tau), \phi(\tau))\), have a total of six dynamical degrees of freedom related by one constraint.

3.2. Normalization (Zero-Point Values). In general, an arbitrary constant can be added to any unconstrained integral of motion. This freedom physically represents the choice of a zero-point value for the integral of motion. The quantities \(I_1, I_2, I_3,\) and \(I_4\), which respectively have physical units of angular momentum per unit mass, energy per unit mass, radians, and time, as well as the quantity \(I_5\) which also has physical units of time, will be written as

\[
\begin{align*}
I_1 &= L + L_0, \\
I_2 &= E + E_0, \\
I_3 &= \Phi + \Phi_0, \\
I_4 &= T + T_0, \\
I_5 &= T + T_0,
\end{align*}
\]

where \(L, E, \Phi, T\) and \(T\) will denote the normalized quantities and where \(L_0, E_0, \Phi_0, T_0\) and \(T_0\) denote the zero-point constants. The question of how to choose physically meaningful zero-point values for each of these locally conserved quantities (3.11) will now be discussed.

It is simplest to begin with \(I_1\) and \(I_2\), since they have a well-known direct relation to the energy and momentum of a freely moving massive particle as measured by stationary observers in the hyperplane \(\Sigma\) in which the geodesic motion takes place.

The four-momentum of the particle is given by

\[
p = mu = m\sigma \partial_t + mv \partial_r + m\omega \partial_\phi
\]

where \(m\) is the rest mass of the particle. In terms of the orthonormal frame

\[
\hat{e}_t = \sqrt{\frac{r}{r - 2M}} \partial_t, \quad \hat{e}_r = \sqrt{\frac{r - 2M}{r}} \partial_r, \quad \hat{e}_\phi = \frac{1}{r} \partial_\phi
\]

for stationary observers located at any point \((t, r, \phi)\) on the geodesic, the energy, radial momentum, and \(\phi\)-momentum measured by these observers are respectively given by

\[
\hat{E} = -g(\hat{e}_t, p) = m\sqrt{\frac{r - 2M}{r}} \sigma, \\
\hat{p}^r = g(\hat{e}_r, p) = m\sqrt{\frac{r}{r - 2M}} v, \\
\hat{p}^\phi = g(\hat{e}_\phi, p) = mr\omega.
\]

These local energy and momentum expressions can be used straightforwardly to fix physically meaningful values of \(L_0\) and \(E_0\). This is usually implicit in considering the integration constants that arise when the timelike geodesic equations are being integrated, since energy and angular momentum are well known physical quantities [13, 14, 3]. Nevertheless, the formulation that will be presented here will be very helpful as a guide for understanding how...
to find physically useful values for the zero-point constants $\Phi_0$ and $T_0$, without resorting to initial conditions for a geodesic.

A physically meaningful value of $L_0$ comes from expressing $\hat{p}^\phi$ in terms of the integral of motion $I_1 = \omega r^2 = L + L_0$:

$$\hat{p}^\phi = \frac{m}{r}(L + L_0). \quad (3.17)$$

This yields

$$L = \frac{r \hat{p}^\phi}{m} - L_0, \quad (3.18)$$

which has units of angular momentum per unit mass. Consequently, a physically sensible zero-point is to have $L = 0$ when $\omega = 0$, namely $L$ should vanish when the geodesic motion of the particle is purely radial. This implies $L_0 = 0$, which thereby can be used to normalize $I_1$ for all timelike geodesics. Hence, with this choice of zero-point, the normalized integral of motion becomes

$$I_1 = L = r^2 \omega = \frac{r \hat{p}^\phi}{m} \quad (3.19)$$

which is the angular momentum per unit mass of the particle as measured by stationary observers at any point $(t(\tau), r(\tau), \phi(\tau))$ on the timelike geodesic.

Similarly for $E_0$, it is useful to express $\hat{E}$ in terms of the integral of motion $I_2 = \frac{r - 2M}{r} \sigma = E + E_0$:

$$\hat{E} = m \sqrt{\frac{r}{r - 2M}} (E + E_0). \quad (3.20)$$

Rearranging to get $E$, this yields

$$E = \sqrt{\frac{r - 2M}{r}} \hat{E} = \frac{r - 2M}{m} - E_0 \quad (3.21)$$

which has units of energy per unit mass. A physically sensible zero-point is to take either $E = 1$ or $E = 0$ in the limit when $r$ goes to $\infty$, namely these two zero-point values respectively represent total mass-energy and kinetic plus potential energy, divided by the particle’s rest mass, when the geodesic motion of the particle is far from the horizon. Since they differ only by a constant, the choice amounts to making a convention. Here the total mass-energy will be chosen: $E = 1$, which implies $E_0 = 0$. This zero-point can be used to normalize $I_2$ for all timelike geodesics. Hence the normalized integral of motion becomes

$$I_2 = E = \frac{r - 2M}{r} \sigma = \sqrt{\frac{r - 2M}{r} \hat{E}} \quad (3.22)$$

which is the total energy per unit mass of the particle as measured by stationary observers at any point $(t(\tau), r(\tau), \phi(\tau))$ on the timelike geodesic. It is also the red-shifted energy per unit mass as measured by stationary observers at radial infinity \cite{13} \cite{14}.

Now, for the angular integral of motion (3.8) and the two temporal integrals of motion (3.9) and (3.10), a physically meaningful choice of zero-point will be introduced and explained. These three integrals turn out not to be directly related to local quantities measured by stationary observers, in contrast to the situation for the energy and angular momentum integrals. Instead they are related to global features of the timelike geodesic, as follows.
To begin, first rearrange \( I_3 = \Phi + \Phi_0 = \phi - I_1 \int \frac{dr}{r^2v} \) to get

\[
\Phi = \phi - L \int_{r_0}^{r} \frac{dr}{r^2v} \tag{3.23}
\]

with \( \Phi_0 \) having been absorbed into an integration constant \( r_0 \). Similarly arrange \( I_4 = T + T_0 = t - I_2 \int \frac{rdr}{(r-2M)v} \) as well as \( I_5 = \tau + \tau_0 = \tau - \int \frac{dr}{v} \), getting

\[
T = t - E \int_{r_0}^{r} \frac{rdr}{(r-2M)v} \tag{3.24}
\]

and

\[
\tau = \tau - \int_{r_0}^{r} \frac{dr}{v} \tag{3.25}
\]

In each of these integrals, \( v \) is given by the radial velocity expression (3.7) which is now rewritten via the relations (3.19) and (3.22):

\[
v = \text{sgn}(v) \sqrt{E^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right)} \tag{3.26}
\]

The choice of the integration constant \( r_0 \) in these integrals (3.23) and (3.24) corresponds to a choice of zero-point constants \( \Phi_0, T_0 \) and \( \tau_0 \). Note that \( r_0 \) will be taken to be the same constant in all three integrals.

Hence the question of how to choose a physically meaningful value for the zero-points becomes the question of finding a distinguished radial value \( r = r_0 \) directly from the radial velocity equation (3.26), without using any features that would be specific to a particular timelike geodesic, such as initial conditions. Two general possibilities consist of a turning point (TP), defined as a radial value \( r \) at which the radial velocity \( v \) vanishes, and an inertial point (IP), defined as a radial value \( r \) at which the radial acceleration \( \dot{v} \) vanishes.

On any timelike geodesic, with angular momentum (3.19) and energy (3.22), all turning points are given by the positive real roots \( r = r^* \) of the radial velocity equation (3.26) when \( v = 0 \), while all inertial points are given by the positive real roots \( r = r^* \) of the radial acceleration equation (2.9b) when \( \dot{v} = 0 \), with \( \dot{\phi} = \omega = L/r^2 \) from the relation (3.19). In particular, the turning point equation can be expressed as a cubic

\[
(E^2 - 1)r^3 + 2Mr^2 - L^2(r - 2M) = 0, \tag{3.27}
\]

and similarly the inertial point equation is a quadratic

\[
Mr^2 - L^2(r^* - 3M) = 0. \tag{3.28}
\]

The positive real roots of these two equations can be found as a by-product of the well-known classification [3] of the different types of timelike geodesics, which are summarized in section 4. This classification shows that at least one turning point exists whenever

\[
E \leq 1 \text{ and } L^2 < 16M^2 \tag{3.29}
\]

or

\[
E \leq E_{\text{circ}} \text{ and } L^2 \geq 16M^2 \tag{3.30}
\]
where $E_{\text{circ}}$ is the energy of the unstable or marginally stable circular orbit; and that at least one inertial point exists whenever

$$E \geq E_{\text{circ}} \text{ and } L^2 \geq 12M^2$$  (3.31)

where $E_{\text{circ}}$ is the energy of the stable or marginally stable circular orbit.

The classification of timelike geodesics also shows that neither a turning point nor an inertial point exists when

$$E > 1 \text{ and } L^2 < 12M^2.$$  (3.32)

Timelike geodesics in this case either plunge into the horizon or escape to infinity. An obvious physical suggestion for an alternative to turning points and inertial points would be $r_0 = 2M$ given by the radial location of the horizon. This choice will work for defining the zero-point of the angular $\phi$-quantity (3.23) and the temporal $\tau$-quantity (3.25), since the quadrature terms will be finite at $r = 2M$. However, the similar quadrature term in the temporal $t$-quantity (3.23) will be singular at $r = 2M$, which is a consequence of the well-known property that $t$ breaks down as a coordinate at the horizon. This issue could be avoided by going to coordinates that are non-singular across the horizon [13, 14, 3].

3.3. New locally conserved quantities. The preceding discussion establishes the following main results.

**Theorem 1.** For all non-circular timelike geodesics, the integrals of motion (3.11) yield five locally conserved quantities

$$L = r^2 \dot{\phi},$$  (3.33)

$$E = \frac{r - 2M}{r} t,$$  (3.34)

$$\Phi = \phi - L \int_{r_0}^{r} \frac{\sgn(v) \, dr}{r^2 \sqrt{E^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right)}} \mod 2\pi,$$  (3.35)

$$T = t - E \int_{r_0}^{r} \frac{\sgn(v) r \, dr}{(r - 2M) \sqrt{E^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right)}},$$  (3.36)

$$\mathcal{T} = \tau - \int_{r_0}^{r} \frac{\sgn(v) \, dr}{\sqrt{E^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right)}}.$$  (3.37)

where $r_0$ can be chosen to be either a turning point $r_\ast$ (when either condition (3.29) or condition (3.30) holds), an inertial point $r^\ast$ (when condition (3.31) holds), or the horizon-crossing point $2M$ (when condition (3.32) holds). Each of these five quantities (3.33)–(3.36) can be evaluated locally in terms of the values of $t(\tau), r(\tau), \phi(\tau), \dot{t}(\tau), \dot{\phi}(\tau)$ at any point parameterized by $\tau$ on a given geodesic. (The proper time equation (2.9d) determines the value of $\dot{r}(\tau)$ in terms of the other values.)

In the classical mechanics and astronomy literature, a point $(t(\tau), r(\tau), \phi(\tau))$ at which $r$ is a local extremum on a given orbit is commonly called an *apsis*; a local minimum point is called a *periapsis*, and a local maximum point is called an *apoapsis*. Note that every apsis on an orbit yields a turning point, but the set of all turning points $r = r_\ast$ may include values of $r$ that do not occur on a given orbit, since turning points are determined just by the energy $E$ and the shape of the effective potential $V_{\text{eff}}(r)$ [14, 3]. The same consideration applies to inertial points $r = r^\ast$. Note that inertial points always come in pairs corresponding to $v$
being positive or negative on different parts of an orbit. Their physical meaning is that the radial Doppler shift for a particle in the orbit, as measured by observers at spatial infinity, is a local extremum.

The angular momentum $L$ and the energy $E$ are well known physical quantities which are globally constant on all timelike geodesics; namely, they are global constants of motion. In contrast, the angular quantity $\Phi$ and the temporal quantities $T$ and $\mathcal{T}$ are new. As will be shown in sections 4 and 5 they are constants of motion locally on non-circular timelike geodesics, whereas globally, $T$ and $\mathcal{T}$ will be multi-valued whenever the geodesic describes an orbit that has more than one apsis, and $\Phi$ will be multi-valued whenever the geodesic describes an orbit that precesses. Their analogues in Newtonian gravity with cubic corrections, which have been discussed recently in Ref. [8], are closely related to the well-known Newtonian LRL vector and the less familiar Hamilton’s vector [4, 5].

**Theorem 2.** For a given non-circular orbit, the angular integral of motion $\Phi$ is the coordinate angle (in the plane of the orbit) of the point $r = r_0$ on the orbit; the two temporal integrals of motion $T$ and $\mathcal{T}$ are respectively the coordinate time and the proper time at which these points are reached. When $r_0$ is a turning point, $\Phi$ represents an analogue of the angle of the Newtonian LRL vector; when $r_0$ is an inertial point, $\Phi$ represents an analogue of the angle of Hamilton’s vector. $\Phi$ has no Newtonian analogue when $r_0$ is a horizon-crossing point.

### 4. Evaluation of the Angular and Temporal Integrals of Motion for Non-circular Orbits

The angular integral of motion (3.35) and the two temporal integrals of motion (3.36) and (3.37) will now be evaluated with an arbitrary choice of $r_0$, similarly to the Newtonian limit shown in the appendix. The technical aspect will employ the approach used in Ref. [3] for integration of the timelike geodesic equations. One simplification is that the integrals will be parameterized explicitly in terms of the roots of the turning point cubic equation (3.27) in all cases. This reduces the algebraic complexity of the expressions for the integrals of motion, especially the two temporal integrals. (It is straightforward to express the results in terms of the eccentricity and latus rectum parameters in Ref. [3] which are sometimes used for comparisons with the Newtonian case and for post-Newtonian approximations.)

Global properties and physical meaning of these integrals of motion, for a universal choice of $r_0$ given by either a turning point $r_*$, an inertial point $r^*$, or a horizon-crossing point $2M$, will be discussed in the next section.

To begin, it will be convenient to introduce the reciprocal radial variable

$$u = \frac{2M}{r}. \quad (4.1)$$

This differs by the factor $2M$ compared to what is used in Ref. [3] and has the advantage that $u = 1$ corresponds to the horizon $r = 2M$. Hereafter, an overbar denotes a quantity or a variable divided by $2M$. In particular,

$$\bar{L} = \frac{L}{2M}. \quad (4.2)$$
With the change of variable (4.1), the angular integral of motion (3.35) is given by
\[ \Phi = \phi + \text{sgn}(vL)I^\theta(u; u_0) \mod 2\pi, \quad I^\Phi(u; u_0) = \int_{u_0}^{u} \frac{du}{\sqrt{Q(u)}}, \] (4.3)
and the temporal integrals of motion (3.36) and (3.37) are given by
\[ \mathcal{T} = \hat{t} + \frac{\text{sgn}(v)E}{|L|}I^T(u; u_0), \quad I^T(u; u_0) = \int_{u_0}^{u} \frac{du}{(1 - u)L^2\sqrt{Q(u)}}, \] (4.4)
\[ \mathcal{\bar{T}} = \bar{t} + \frac{\text{sgn}(v)}{|L|}I^{\bar{T}}(u; u_0), \quad I^{\bar{T}}(u; u_0) = \int_{u_0}^{u} \frac{du}{u^2\sqrt{Q(u)}}, \] (4.5)
where
\[ Q(u) = u^3 - u^2 + \bar{L}^{-2}u + (E^2 - 1)\bar{L}^{-2} = v^2/\bar{L}^2 \] (4.6)
is the radial velocity equation (3.26) expressed as a cubic polynomial in \( u \). For evaluating the quadratures, the root structure of \( Q(u) = 0 \) needs to be known. It is determined by the discriminant \( D \) of \( Q(u) \) and the discriminant \( D_0 \) of \( Q'(u) \):
\[ L^6D = \frac{1}{27}(4(L^2 - 3)^3 - L^2(27E^2 - 18 - 2L^2)^2) = 27L^2(E^2_+ - E^2)(E^2 - E^2_-), \] (4.7a)
\[ L^4D_0 = 4(L^2 - 3), \] (4.7b)
where
\[ E^2_\pm = \frac{2}{3} + \frac{2}{27}\bar{L}^2(1 \pm \sqrt{1 - 3/L^2}). \] (4.8)
As shown in Ref. [3], the three roots \( u_1, u_2, u_3 \) of the cubic equation \( Q(u) = 0 \) can be classified into the list of five cases shown in Table 1. In the cases where all of the roots are real, they will be ordered \( u_3 \geq u_2 \geq u_1 \); in the case where only one of the roots is real, it will be designated as \( u_1 \). The orbits that occur in each case are summarized in Tables 2 and 3 in the next section.

Note that the three roots are functions of the two parameters \( L^2 \) and \( E^2 \). From the cubic equation \( Q(u) = 0 \), the roots obey the relations
\[ u_1 + u_2 + u_3 = 1, \quad \text{sgn}(u_1) = \text{sgn}(1 - E^2). \] (4.9)
In the case when the roots are real, another useful relation is that they lie between the roots of the quadratic equation \( Q'(u) = 0 \):
\[ u_\pm = \frac{1}{3}(1 \pm \sqrt{1 - 3/L^2}). \] (4.10)

**Table 1.** Root structure of \( Q(u) = 0 \) and range of \( u \) for \( Q(u) \geq 0 \).

| Case | Discriminants | Root type | Range of \( u \) | \( E^2 \) | \( L^2 \) |
|------|---------------|-----------|-----------------|----------|---------|
| (1)  | \( D = 0, D_0 = 0 \) | \( u_1 = u_2 = u_3 > 0 \) | \( u \geq u_1 \) | \( \frac{2}{3} (= E^2_\pm) \) | 3 |
| (2a) | \( D = 0, D_0 > 0 \) | \( u_3 > u_2 = u_1 > 0 \) | \( u \geq u_3 \) | \( E^2_- \) | > 3 |
| (2b) | \( D = 0, D_0 > 0 \) | \( u_3 = u_2 > u_1 \) | \( u \geq u_2 \) or \( u_2 \geq u \geq \max(u_1, 0) \) | \( E^2_+ \) | > 3 |
| (3)  | \( D > 0 \) | \( u_3 > u_2 > u_1 \) | \( u \geq u_3 \) or \( u_2 \geq u \geq \max(u_1, 0) \) | \( > E^2_- \) and \( < E^2_+ \) | > 3 |
| (4)  | \( D < 0 \) | \( u_1, u_2 = \bar{u}_3 \) | \( u \geq \max(u_1, 0) \) | \( < E^2_- \) or \( > E^2_+ \) | > 3 |

|                  |                  |                  |                  |                  |        |
|------------------|------------------|------------------|------------------|------------------|--------|
| **TABLE 1**      | **ROOT STRUCTURE** | **OF** | **Q(U) = 0** | **AND RANGE OF U FOR Q(U) ≥ 0** |        |
| **CASE**         | **DISCRIMINANTS** | **ROOT TYPE** | **RANGE OF U** | **E2** | **L2** |
| **(1)**          | **D = 0, D0 = 0** | **U1 = U2 = U3 > 0** | **U ≥ U1** | **2/3 ( = E2±)** | **3**   |
| **(2a)**         | **D = 0, D0 > 0** | **U3 > U2 = U1 > 0** | **U ≥ U3** | **E2−** | **> 3** |
| **(2b)**         | **D = 0, D0 > 0** | **U3 = U2 > U1** | **U ≥ U2 or U2 ≥ U ≥ max(U1, 0)** | **E2+** | **> 3** |
| **(3)**          | **D > 0**          | **U3 > U2 > U1** | **U ≥ U3 or U2 ≥ U ≥ max(U1, 0)** | **> E2− and < E2+** | **> 3** |
| **(4)**          | **D < 0**          | **U1, U2 = û3** | **U ≥ max(U1, 0)** | **< E2− or > E2+** | **> 3** |
|                  |                  |                  |                  |                  | **L2 ≤ 3** |
For each of the five separate cases in Table 1, the roots as well as the quadratures will be presented next.

**Case (1)** The discriminant conditions $D = D_0 = 0$ yield

$$\bar{L}^2 = 3, \quad E^2 = \frac{8}{9}.\quad (4.11)$$

This gives a triple root:

$$u_1 = u_2 = u_3 = \frac{1}{3}.\quad (4.12)$$

The quadratures (4.3)–(4.5) are given in terms of elementary functions:

$$I^\Phi(u; u_0) = \left( \frac{-2}{\sqrt{u - u_1}} \right)|_u^{u_0}; \quad (4.13a)$$

$$I^T(u; u_0) = \left( \frac{-3}{\sqrt{u_1^3}} \arctan \left( \sqrt{\frac{u - u_1}{u_1}} \right) + \frac{1}{u_1 \sqrt{u - u_1}} \left( \frac{1}{u} - \frac{3}{u_1} \right) \right)|_u^{u_0}; \quad (4.13b)$$

$$I^T(u; u_0) = \left( \frac{2}{\sqrt{1 - u_1^3}} \arctanh \left( \frac{u - u_1}{1 - u_1} \right) - \frac{2u_1 + 3}{\sqrt{u_1^5}} \arctan \left( \sqrt{\frac{u - u_1}{u_1}} \right) \right. \right.$$

$$+ \left. \frac{1}{u_1 \sqrt{u - u_1}} \left( \frac{1}{u} - \frac{u_1 - 3}{u_1 (u_1 - 1)} \right) \right)|_u^{u_0}. \quad (4.13c)$$

**Case (2)** The discriminant conditions $D_0 \neq D = 0$ give two cases which are distinguished by the value of $E^2$.

**Subcase (2a)**

$$\bar{L}^2 > 3, \quad E^2 = E^2_\neq.\quad (4.14)$$

This yields a root structure consisting of a double root and a single root:

$$u_1 = u_2 = \frac{1}{3}(1 - \sqrt{1 - 3/L^2}) = \frac{1}{2}(1 - u_3), \quad u_3 = \frac{1}{3}(1 + 2\sqrt{1 - 3/L^2}). \quad (4.15)$$

They have the ranges

$$1 > u_3 > \frac{1}{3}, \quad \frac{1}{3} > u_1 = u_2 > 0.\quad (4.16)$$

The quadratures (4.3)–(4.5) are given by elementary functions:

$$I^\Phi(u; u_0) = \left( \frac{2}{\sqrt{u_3 - u_1}} \arctan \left( \sqrt{\frac{u - u_3}{u_3 - u_1}} \right) \right)|_u^{u_0}; \quad (4.17a)$$

$$I^T(u; u_0) = \left\{ \begin{array}{c}
- \frac{1}{u_1 \sqrt{u_3}} \left( \frac{2}{u_1} + \frac{1}{u_3} \right) \arctan \left( \sqrt{\frac{u - u_3}{u_3}} \right) \\
+ \frac{2}{u_1^2 \sqrt{u_3 - u_1}} \arctan \left( \sqrt{\frac{u - u_3}{u_3 - u_1}} - \sqrt{\frac{u - u_3}{u_1 u_3 u}} \right) \end{array} \right\} |_u^{u_0}. \quad (4.17b)$$
\[ I^T(u; u_0) = \left( -\frac{1}{u_1\sqrt{u_3}} \left( 2 + \frac{2}{u_1} + \frac{1}{u_3} \right) \arctan \left( \frac{u - u_3}{u_3} \right) \right. \]
\[ + \frac{2}{u_1^2(1 - u_1)\sqrt{u_3 - u_1}} \arctan \left( \frac{\sqrt{u - u_3}}{u_3 - u_1} \right) \] 
\[ + \frac{2}{(1-u_1)\sqrt{1-u_3}} \arctanh \left( \frac{\sqrt{u - u_3}}{1 - u_3} \right) \left( \frac{\sqrt{u - u_3}}{u_1u_3u} \right) \right|_{u_0}^{u}. \] 

(4.17c)

Subcase (2b)

\[ L^2 > 3, \quad E^2 = E_2^2. \] 

(4.18)

This yields a root structure consisting of a single root and a double root:

\[ u_1 = \frac{1}{3} \left( 1 - 2\sqrt{1 - 3/L^2} \right), \quad u_2 = u_3 = \frac{1}{3} \left( 1 + \sqrt{1 - 3/L^2} \right) = \frac{1}{2} (1 - u_1). \] 

(4.19)

They have the ranges

\[ \frac{1}{3} > u_1 > -\frac{1}{3}, \quad \frac{2}{3} > u_2 = u_3 > \frac{1}{3}. \] 

(4.20)

The quadratures (4.3)–(4.5) are given by elementary functions:

\[ I^\Phi(u; u_0) = \text{sgn}(u_2 - u) \left( \frac{2}{u_2 - u_1} \right. \] 
\[ \left. \arctan \left( \frac{\sqrt{u - u_1}}{u_1} \right) \right|_{u_0}^{u}; \quad (4.21a) \]

\[ I^T(u; u_0) = \begin{cases} 
\text{sgn}(u_2 - u) \left( \frac{1}{u_2\sqrt{|u_1|}} \left( \frac{1}{u_1} + \frac{2}{u_2} \right) \arctan \left( \frac{u - u_1}{u_1} \right) \right. \\
+ \frac{2}{u_2^2\sqrt{u_3 - u_1}} \arctanh \left( \frac{\sqrt{u - u_1}}{u_2 - u_1} \right) \\
+ \frac{1}{u_1u_2} \frac{\sqrt{u - u_1}}{u} \right|_{u_0}^{u}, \quad u_1 > 0 \\
\text{sgn}(u_2 - u) \left( \frac{1}{u_2\sqrt{|u_1|}} \left( \frac{1}{|u_1|} - \frac{2}{u_2} \right) \arctan \left( \frac{u - u_1}{|u_1|} \right) \right. \\
+ \frac{2}{u_2^2\sqrt{u_3 - u_1}} \arctanh \left( \frac{u - u_1}{u_2 - u_1} \right) \\
+ \frac{1}{u_1u_2} \frac{\sqrt{u - u_1}}{u} \right|_{u_0}^{u}, \quad u_1 < 0 \\
\text{sgn}(u_2 - u) \left( \frac{2}{\sqrt{u_2}} \arctanh \left( \frac{\sqrt{u}}{u_2} \right) - \frac{2}{u_2\sqrt{u}} \left( \frac{1}{u_2} + \frac{1}{3u} \right) \right), \quad u_1 = 0 
\end{cases} \] 

(4.21b)
\[ F^T(u; u_0) = \begin{cases} 
\text{sgn}(u_2 - u) \left( \frac{1}{u_2 \sqrt{u_1}} \left( 2 + \frac{2}{u_2} + \frac{1}{u_1} \right) \arctan \left( \frac{u - u_1}{u_1} \right) 
- \frac{2}{u_2^2 (u_2 - 1) \sqrt{u_2 - u_1}} \arctanh \left( \frac{u - u_1}{u_2 - u_1} \right) 
+ \frac{2}{(u_2 - 1) \sqrt{1 - u_1}} \arctanh \left( \frac{u - u_1}{1 - u_1} \right) 
+ \frac{1}{u_1 u_2} \left| u \right|_u^{u_0}, \quad u_1 > 0 \right) 
\right) 
\end{cases} \]

\( (4.21c) \)

**Case (3)** The discriminant condition \( D > 0 \) gives
\[ \bar{L}^2 > 3, \quad E_+^2 < E^2 < E_+^2. \]

This yields a root structure consisting of three distinct roots:
\[ u_{3-n} = \frac{1}{3} + \frac{2}{3} \sqrt{1 - 3/L^2} \cos \left( \frac{1}{3} \arccos \left( \frac{1 + 9(1 - \frac{3}{2} E^2)/L^2}{\sqrt{(1 - 3/L^2)^3}} \right) - \frac{2}{3} \pi n \right), \quad n = 0, 1, 2. \]

They have the ranges
\[ u_1 < \frac{2}{3} - u_+ < u_2 < u_+ < u_3, \quad \frac{1}{3} < u_+ < \frac{2}{3}. \]

There are two different cases for the quadratures (4.3)–(4.5), which are distinguished by the range of \( u \). Both cases involve the Jacobian elliptic functions
\[ \begin{align*}
F(\psi, k) &= \int_0^\psi \frac{1}{\sqrt{1 - k^2 \sin^2 \vartheta}} \, d\vartheta, \quad E(\psi, k) = \int_0^\psi \sqrt{1 - k^2 \sin^2 \vartheta} \, d\vartheta, \\
\Pi(m; \psi, k) &= \int_0^\psi \frac{1}{(1 - m \sin^2 \vartheta) \sqrt{1 - k^2 \sin^2 \vartheta}} \, d\vartheta,
\end{align*} \]

where \( 0 < k < 1 \). (See e.g. Ref. [17] for details about these functions.)
When $u \geq u_3$, the quadratures are given by

\[
I^\Phi(u; u_0) = \left. \frac{2}{\sqrt{u_3 - u_1}} F(\psi(u), k) \right|_{u_0}^u; \tag{4.26a}
\]

\[
I^T(u; u_0) = \left\{ \begin{array}{cl}
\frac{u_2 + u_3}{u_2^2 u_3 \sqrt{u_3 - u_1}} F(\psi(u), k) + \frac{\sqrt{u_3 - u_1}}{u_1 u_2 u_3} \Phi(\psi(u), k) \\
-\frac{u_2 + u_3}{u_2^2 u_3 \sqrt{u_3 - u_1}} \Pi(m; \psi(u), k) \\
-\frac{\sqrt{u - u_1 \sqrt{u - u_3}}}{u_1 u_3 \sqrt{u - u_2}} \right|_{u_0}^u, \quad u_1 \neq 0
\end{array} \right.
\tag{4.26b}
\]

\[
I^T(u; u_0) = \left\{ \begin{array}{cl}
\frac{2(u_2 + u_3)}{3u_2^2 \sqrt{u_3}} F(\psi(u), k) - \frac{4}{3u_2^2 \sqrt{u_3}} \Phi(\psi(u), k) \\
+ \frac{2(u_3 - u_2)}{3u_3 \sqrt{u}} \left( \frac{2u_2 + u_3}{u_2 u_3} + \frac{1}{u} \right) \sqrt{\frac{u - u_3}{u - u_2}} \right|_{u_0}^u, \quad u_1 = 0
\end{array} \right.
\tag{4.26c}
\]

where

\[
k = \sqrt{\frac{u_2 - u_1}{u_3 - u_1}}, \quad m = \frac{u_2}{u_3}, \quad n = \frac{1 - u_2}{1 - u_3}, \quad \sin \psi(u) = \sqrt{\frac{u - u_3}{u - u_2}}. \tag{4.27}
\]

When $u_2 \geq u \geq \max(0, u_1)$, the quadratures have the form

\[
I^\Phi(u; u_0) = \left. \frac{2}{\sqrt{u_3 - u_1}} F(\psi(u), k) \right|_{u_0}^u; \tag{4.28a}
\]

\[
I^T(u; u_0) = \left\{ \begin{array}{cl}
-\frac{1}{u_1 u_2 \sqrt{u_3 - u_1}} F(\psi(u), k) + \frac{\sqrt{u_3 - u_1}}{u_1 u_2 u_3} \Phi(\psi(u), k) \\
+ \frac{u_1 u_2 + u_1 u_3 + u_2 u_3}{u_1^2 u_2 u_3 \sqrt{u_3 - u_1}} \Pi(m; \psi(u), k) \\
+ \frac{\sqrt{u - u_1 \sqrt{u_3 - u_1}}}{u_1 u_2 u_3} \right|_{u_0}^u, \quad u_1 \neq 0
\end{array} \right.
\tag{4.28b}
\]

\[
I^T(u; u_0) = \left\{ \begin{array}{cl}
\frac{2(u_2 + 2u_3)}{3u_2^3 \sqrt{u_3}} F(\psi(u), k) - \frac{4}{3u_2^3 \sqrt{u_3}} \Phi(\psi(u), k) \\
- \frac{2 \sqrt{u_3 - u_1 \sqrt{u_3 - u_1}}}{3u_2 u_3 \sqrt{u}} \left( \frac{2}{u_2 u_3} + \frac{1}{u} \right) \left|_{u_0}^u, \quad u_1 = 0
\end{array} \right.
\tag{4.28c}
\]
\[ I^T(u; u_0) = \left\{ \begin{array}{ll}
\frac{2}{u_1\sqrt{u_3 - u_1}} \Pi(m; \psi(u), k) \\
\quad + \frac{2}{(1-u_1)\sqrt{u_3 - u_1}} \Pi(n; \psi(u), k) \bigg|_{u_0}^u , \quad u_1 \neq 0 \\
\frac{2}{\sqrt{u_3}} F(\psi(u), k) - \frac{2}{u_2\sqrt{u_3}} E(\psi(u), k) \\
\quad + \frac{2}{\sqrt{u_3}} \Pi(n; \psi(u), k) - \frac{2\sqrt{u_3 - u\sqrt{u_2 - u}}}{u_2u_3\sqrt{u}} \bigg|_{u_0}^u , \quad u_1 = 0
\end{array} \right. ; \quad (4.28c) \]

where

\[ k = \sqrt{\frac{u_2 - u_1}{u_3 - u_1}} , \quad m = 1 - \frac{u_2}{u_1} , \quad n = \frac{u_2 - u_1}{1 - u_1} , \quad \sin \psi(u) = \sqrt{\frac{u - u_1}{u_2 - u_1}} . \quad (4.29) \]

**Case (4)** The discriminant condition \( D < 0 \) splits into two cases

\[ \bar{L}^2 \leq 3, \quad 0 < E^2 < \infty \quad (4.30a) \]

and

\[ \bar{L}^2 > 3, \quad E^2 < E_+^2 \quad \text{or} \quad E^2 > E_+^2. \quad (4.30b) \]

In both cases, the roots have the same structure consisting of a real root and a pair of complex conjugate roots:

\[ u_1 = \frac{1}{3}(1 + q_+ + q_-) , \quad u_2 = \bar{u}_3 = \frac{1}{6}(2 - (q_+ + q_-) + i\sqrt{3}(q_+ - q_-)) , \quad (4.31) \]

where

\[ q_{\pm} = \sqrt[3]{1 + 9(1 - \frac{3}{2}E^2)/\bar{L}^2 \pm \sqrt{(1 + 9(1 - \frac{3}{2}E^2)/\bar{L}^2)^2 - (1 - 3/\bar{L}^2)^3}} . \quad (4.32) \]

They have the ranges

\[ u_1 < 1 , \quad \Re(u_2) = \Re(u_3) > 0. \quad (4.33) \]

The quadratures \((4.3)-(4.5)\) are given by

\[ I^\Phi(u; u_0) = \frac{-\sqrt{2}}{\sqrt{\alpha + \beta}} F(\psi(u), ik) \bigg|_{u_0}^u ; \quad (4.34a) \]
\[
I^\tau(u; u_0) = \begin{cases} 
\frac{\beta^2 + u_1(2 - 3u_1)}{2\sqrt{u_1^3 \sqrt{\beta^2 + u_1(1 - 2u_1)}}} \arctan \left( \frac{\sqrt{u - u_1} \sqrt{\beta^2 + u_1(1 - 2u_1)}}{\sqrt{u_1} \sqrt{\beta^2 + (u - 2u_1 - 1)(u - u_1)}} \right) \\
+ \frac{2}{u_1(\beta - u_1) \sqrt{\beta + 2\alpha}} F(\psi(u), ik) - \frac{\sqrt{\beta + 2\alpha}}{2u_1(\beta^2 + u_1(1 - 2u_1))} E(\psi(u), ik) \\
- \frac{\beta^2 + u_1(2 - 3u_1)}{2\sqrt{u_1^3 \sqrt{\beta^2 + u_1(1 - 2u_1)}}} \arctanh \left( \frac{\sqrt{u - u_1} \sqrt{\beta^2 + u_1(1 - 2u_1)}}{\sqrt{u_1} \sqrt{\beta^2 + (u - 2u_1 - 1)(u - u_1)}} \right) \\
+ \frac{2}{u_1(\beta - u_1) \sqrt{\beta + 2\alpha}} F(\psi(u), ik) - \frac{\sqrt{\beta + 2\alpha}}{2u_1(\beta^2 + u_1(1 - 2u_1))} E(\psi(u), ik) \\
\end{cases}
\]

\[
= \frac{2(\beta - 2)}{3\beta^3 \sqrt{2\beta - 1}} F(\psi(u), ik) + \frac{2\sqrt{2\beta - 1}}{3\beta^4} E(\psi(u), ik) \\
- \frac{2\sqrt{\beta^2 + u(u - 1)}}{3\beta^2 \sqrt{u}} \left( \frac{1}{u} + \frac{2}{\beta(u + \beta)} \right) \bigg|_{u_0}^u, \quad u_1 = 0
\]

(4.34b)
where

\[ k = \sqrt{\frac{\beta - \alpha}{\beta + \alpha}}, \quad m = \frac{(\beta - u_1)^2}{(\beta + u_1)^2}, \quad n = \frac{(\beta - u_1 + 1)^2}{(\beta + u_1 - 1)^2}, \quad \sin(\psi(u)) = \frac{\beta + u_1 - u}{\beta - u_1 + u}. \]  

(4.34d)

with

\[ \alpha = \frac{1}{2}(3u_1 - 1) = \frac{1}{2}(q_+ + q_-), \quad \beta^2 = |u_2|^2 + u_1(2u_1 - 1) = \alpha^2 + \frac{1}{12}(q_+ - q_-)^2. \]  

(4.34e)

5. **Physical properties of the angular and temporal conserved quantities**

The physical meaning of the angular integral of motion (4.3) and the two temporal integrals of motion (4.4) and (4.5) will now be discussed for all of the different types of timelike non-circular orbits. In each case, the choice of \( r_0 \) as either a turning point, an inertial point, or a horizon-crossing point will be considered, and the resulting properties of the integrals of motion as analogues of the LRL angle and LRL time in Newtonian gravity will be described.
To begin the discussion, a short summary of the well-known classification of orbits \[3\] is presented together with the possibilities for \(r_0\) for each type of orbit.

5.1. **Types of timelike orbits and choices of \(r_0\).** The radial acceleration equation \([2.10]\) can be expressed in the physical form

\[
\ddot{r} = -\frac{M}{r^2} + \frac{L^2(r - 3M)}{r^4} = F_{\text{eff}} = -\frac{dV_{\text{eff}}}{dr}
\]

through the use of the angular momentum \([3.33]\). The effective radial force \(F_{\text{eff}}\) has the form of a central force for which the associated effective potential \(V_{\text{eff}}\) can be obtained directly by expressing the proper time equation \([3.26]\) in the oscillator form \([13]\)

\[
\frac{1}{2}v^2 + V_{\text{eff}} = \frac{1}{2}(E^2 - 1),
\]

where \(v\) is the radial velocity. The effective potential is given by

\[
V_{\text{eff}} = -\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} = \frac{1}{2}((1 - u)(1 + L^2u^2) - 1) = \frac{1}{2}(E^2 - 1 - Q(u)),
\]

where \(Q(u)\) is the cubic polynomial \([4.6]\). This potential has the following features \([13, 3]\):

- no extrema when \(L^2 < 12M^2\);
- an inflection when \(L^2 = 12M^2\), with \(V_{\text{eff}} = \frac{8}{9}\);
- a maximum and a minimum when \(L^2 > 12M^2\), with \(V_{\text{eff}} = E^2_+\) and \(V_{\text{eff}} = E^2_-\), respectively.

Here

\[
E^2_\pm = \frac{2}{3} + \frac{2}{27}L^2\left(1 \pm \sqrt{1 - 3/L^2}\right).\]

Tables 2 and 3 summarize all of the types of timelike non-circular orbits that arise from the shape of the effective potential \([5.3]\). For each type of orbit, the possibilities for \(u_0 = 1/r_0\) are listed. A horizon-crossing point (HP) refers to \(u = 1\), namely \(r = 2M\); an inertial point (IP) refers to

\[
u^* = u_\pm\]

as given by the roots of the quadratic equation \(Q'(u^*) = 0\) (cf \([3.28]\)); and a turning point (TP) refers to

\[
u^*_\pm = u_1, u_2, u_3\]

as given by the roots of the cubic equation \(Q(u^*) = 0\) (cf \([3.27]\)). Finally, a superscript indicates the multiplicity of choices.

### Table 2. Types of bounded non-circular orbits.

| Orbit type                      | \(\bar{L}^2\)          | \(E^2\)          | Root case | Range of \(u\) | Zero-point choices |
|---------------------------------|------------------------|-------------------|-----------|----------------|--------------------|
| horizon-crossing                | \(< 3\)                | \(< 1\)           | (4)       | \(\geq u_1\)  | TP, HP             |
|                                 | \(\geq 3\)             | \(< E^2_+\)       | (2a),(3)  | \(\geq u_3\)  | TP, HP             |
|                                 | \(\geq 3 \text{ and } < 4\) | \(\geq E^2_+\) \text{ and } \(< 1\) | (4)       | \(\geq u_1\)  | TP, IP             |
| asymptotic circular             | \(\geq 3\)             | \(E^2_+\)         | (1), (2b) | \(\geq u_3\)  | HP                 |
| horizon-crossing                | \(\geq 3 \text{ and } < 4\) | \(E^2_+\)         | (2b)      | \(\geq u_1\text{ and } < u_2\) | TP, IP             |
| asymptotic circular             | \(\geq 3 \text{ and } < 4\) | \(E^2_+\)         | (2b)      | \(\geq u_1\text{ and } < u_2\) | TP, IP             |
| elliptic-like                   | \(\geq 4\)             | \(> E^2_+\) \text{ and } \(< 1\) | (3)       | \(\geq u_1\text{ and } \leq u_2\) | TP^2, IP           |
|                                 | \(\geq 4\)             | \(> E^2_-\) \text{ and } \(< 1\) | (3)       | \(\geq u_1\text{ and } \leq u_2\) | TP^2, IP           |
Table 3. Types of unbounded orbits.

| Orbit type               | $L^2$ | $E^2$ | Root case | Range of $u$ | Zero-point choices |
|--------------------------|-------|-------|-----------|--------------|--------------------|
| horizon-crossing         | $< 3$ | $\geq 1$ | (4)       | $> 0$        | HP                 |
|                          | $\geq 3$ and $< 4$ | $> 1$ | (4)       | $> 0$        | IP, HP             |
|                          | $\geq 4$ | $> E^2_+$ | (4)       | $> 0$        | IP, HP             |
| asymptotic circular     | $4$   | $1$   | (2b)      | $> 0$ and $< u_2$ | IP                |
| parabolic-like           |       |       |           |              |                    |
| asymptotic circular     | $> 4$ | $E^2_+$ | (2b)      | $> 0$ and $< u_2$ | IP                |
| hyperbolic-like          | $> 4$ | $> 1$ and $< E^2_+$ | (3) | $> 0$ and $\leq u_2$ | TP, IP            |

5.2. Angular and temporal conserved quantities for non-circular orbits. For each orbit type, the explicit expressions for the angular conserved quantity (4.3) and the two temporal conserved quantities (4.4) and (4.5) will be stated here in terms of the quadratures (4.13), (4.17), (4.21), (4.26), (4.28), (4.34), with the various choice(s) of $u_0$ shown in tables 2 and 3. Their global nature, specifically whether they are global conserved quantities or only piecewise conserved quantities, will also be stated.

The results are presented in Table 4 for bounded non-circular orbits and Table 5 for unbounded orbits. Both tables are divided into separate cases for orbits that lie outside of the horizon and orbits that cross the horizon. Derivations and figures for each case will be given afterwards.

Orbits that do not cross the horizon possess at least one inertial point $r = r^* = 2M/u^*$, and consequently, this point provides a universal choice of $u_0 = u^* = 2M/r^*$. The resulting conserved quantities are analogues of the angle and the time corresponding to Hamilton’s binormal vector in Newtonian gravity. If the orbit is either bounded or not asymptotically circular, then at least one turning point $r = r_* = 2M/u_*$ exists, which can be used alternatively as a choice of $u_0 = u_* = 2M/r_*$. The resulting conserved quantities are analogues of the LRL angle and time in the Newtonian case.

Orbits that cross the horizon necessarily possess a horizon-crossing point. This provides a universal choice of $u_0 = 2M/r_0 = 1$. The resulting conserved quantities have no analog in the Newtonian case.

5.3. Global properties of angular and temporal conserved quantities. For any orbit, the angular and temporal integrals of motion (4.3), (4.4), (4.5) are locally constant on each part of the orbit that does not contain a turning point. This can be seen directly from their expressions: the mathematical condition when a jump can occur is if the factor sgn($v$) which multiplies the quadratures $I^\Phi(u;u_0), I^T(u;u_0), I^T(u;u_0)$ changes sign as the orbit goes through a turning point $r = r_* \neq 2M/u_0$.

Hence, these three integrals of motion will be globally constant for orbits with no turning points. Their global properties for orbits with a turning point crucially depend on whether or not the turning point undergoes precession. A turning point $r = r_*$ is said to precess if there are multiple angles $\phi = \phi_1, \phi_2, \ldots$ (mod $2\pi$) at which the orbit reaches $r = r_*$. In particular, the multiplicity may be finite or infinite.
Table 4. Conserved angular and temporal quantities
\[ \Phi = \phi + \text{sgn}(vL)I^\Phi(u; u_0), \bar{T} = \bar{t} + \frac{E \text{sgn}(v)}{|L|} I^T(u; u_0), \bar{T} = \bar{\tau} + \frac{\text{sgn}(v)}{|L|} I^T(u; u_0) \]

for **bounded non-circular orbits**.

| Orbit                   | \( I^\Phi(u; u_0) \) | \( u_0 \)   | \( \Phi \) | \( \bar{T}, \bar{T} \) | Physical Meaning                      |
|-------------------------|----------------------|------------|-------------|--------------------------|---------------------------------------|
| elliptic-like           | (4.28)               | TP \( u_2 \) | multi-val.  | multi-val.               | periapsis apoapsis Doppler max/min    |
|                         |                      | TP \( u_1 \) | multi-val.  | multi-val.               |                                       |
|                         |                      | IP \( u_- \) | multi-val.  |                         |                                       |
| asymptotic circular     | (4.21)               | TP \( u_1 \) | global     | global double-val.      | apoapsis Doppler max/min              |
| horizon-crossing        | (4.21), \( L^2 > 3 \) | IP \( u_- \) | global     | global double-val.      | horizon                              |
|                         | (4.13), \( \bar{L}^2 = 3 \) |            |            |                         |                                       |
| asymptotic circular     | (4.21), \( L^2 > 3 \) | HP 1       | global     | \( \infty \), global   | horizon                              |
|                        | (4.13), \( \bar{L}^2 = 3 \) |            |            |                         |                                       |
| horizon-crossing        | (4.34), \( E^2 < E_-^2 \) | HP 1       | global     | global double-val.      | apoapsis horizon                      |
|                        | (4.17), \( E^2 = E_-^2 \) |            |            |                         |                                       |
|                        | (4.26), \( E^2 < E^2 < E_+^2 \) |            |            |                         |                                       |
|                        | (4.34), \( E_+^2 < E^2 < 1 \) |            |            |                         |                                       |

**Proposition 1.** (i) The angular and temporal integrals of motion (4.3), (4.5), and (4.4) in the **TP** case \( u_0 = 2M/r_* \) are global conserved quantities on an orbit (namely, they are constant on the entirety of the orbit) iff the orbit has no turning points that precess. (ii) In the **IP** and **HP** cases \( u_0 = 2M/r^* \) and \( u_0 = 1 \), the angular and temporal integrals of motion (4.3), (4.5), and (4.4) are global conserved quantities iff the orbit has no turning points.

This result will be now be applied to determine the global nature of the angular and temporal integrals of motion (4.3), (4.5), and (4.4) for each of the different types of orbits listed in Tables 2 and 3.

The number of turning points for an orbit can be established through consideration of the equation

\[ \phi = \Phi - \text{sgn}(vL)I^\Phi(2M/r; 2M/r_0) \equiv f(r; \Phi, r_0) \quad (5.7) \]

which determines the local shape for all orbits. Specifically, the shape is defined by projecting the points \((t(\tau), r(\tau), \phi(\tau))\) on the orbit into the spacelike manifold coordinatized by \((r, \phi)\), with line element \((ds)^2 = \frac{r}{r - 2M} (dr)^2 + r^2(d\phi)^2\), where \(2M \leq r < \infty\), \(0 \leq \phi < 2\pi\). (This manifold can be identified with \(\Sigma/\mathbb{R}\) where \(\Sigma\) is the equatorial hyperplane \(\theta = \frac{1}{2}\pi\), and \(\mathbb{R}\) denotes the timelike line generated by the Killing vector \(\partial_t\).) The orbit shape equation (5.7) has an important reflection symmetry property which is related to existence of turning points, as will now be explained.
Table 5. Conserved angular and temporal quantities

\[ \Phi = \phi + \text{sgn}(v\bar{L})I^{\Phi}(u; u_0), \bar{T} = \bar{t} + \frac{E \text{sgn}(v)}{L} I^{\bar{T}}(u; u_0), \bar{T} = \bar{T} + \frac{\text{sgn}(v)}{L} I^{\bar{T}}(u; u_0) \]

for unbounded orbits.

| Orbit                  | \( I^\Phi(u; u_0) \) | \( u_0 \) | \( \Phi \) | \( \bar{T}, \bar{T} \) | Physical Meaning |
|------------------------|------------------------|----------|-----------|--------------------------|----------------|
| hyperbolic-like        | \( \text{(4.28)} \)   | TP. \( u_2 \) | global    | double-val.              | periapsis Doppler max/min |
| parabolic-like         | \( \text{(4.28)} \)   | IP. \( u_- \) | double-val.| global                    | Doppler max/min |
| asymptotic circular    | \( \text{(4.21)} \)   | IP. \( u_- \) | double-val.| Doppler max/min          | |
| asymptotic circular    | \( \text{(4.21)} \)   | IP. \( u_- \) | double-val.| Doppler max/min          | |
| horizon-crossing       | \( \text{(4.34)}, L^2 < 3 \) | HP. \( 1 \) | global    | \( \infty \), global   | horizon          |
|                        | \( \text{(4.34)}, L^2 \geq 3 \) | IP. \( u_\pm \) | global    | \( \infty \), global   | Doppler max/min |

For orbits that possess at least one turning point, the apses consist of the set of points \((r_*, \phi_*)\) where \(\phi_*\) is the angle \(\phi\) at which each turning point \(r = r_*\) occurs on the orbit. A radial apsis line refers to the radial line that connects the horizon to a given apsis on the orbit in the spatial surface \((r, \phi)\), namely \(\phi = \phi_*\) with \(2M \leq r \leq r_*\).

The local shape of an orbit near an apsis can be found, up to a global rotation, by taking \(r_0 = r_*\) and \(\Phi = \phi_*\) in the orbit shape equation \(\Phi = \phi_* - \text{sgn}(v\bar{L})I^{\Phi}(2M/r; 2M/r_*)\). Since the radial velocity \(v\) changes sign at the apsis, note that \(\phi - \phi_* = -\text{sgn}(\bar{L})I^{\Phi}(2M/r; 2M/r_*) \leftrightarrow -\text{(}\phi - \phi_*\text{)} = \text{sgn}(\bar{L})I^{\Phi}(2M/r; 2M/r_*)\). Hence, the orbit will be locally reflection symmetric with respect to the radial apsis line

\[
(r, \phi) \rightarrow (r, 2\phi_* - \phi) \tag{5.8}
\]

for \(r\) in some radial interval that has \(r_*\) being an endpoint.

Global information about the orbit shape can be obtained from the local reflection-symmetry \(\text{(5.8)}\) combined with knowledge of whether the orbit is bounded or unbounded, whether it crosses the horizon, and whether it possesses a circular asymptote.

Consider, first, unbounded orbits that do not cross the horizon and are not asymptotically circular. Clearly, these orbits must possess two asymptotes and at least one turning point which is the periapsis. Starting at one asymptote, \(r\) will decrease from \(\infty\) to \(r_*\) which is a local periapsis. The reflection-symmetry \(\text{(5.8)}\) will thus hold for \(r_* \leq r < \infty\). This implies that \(r\) will increase from \(r_*\) to \(\infty\), which is the other asymptote. Hence, the periapsis \(r = r_*\) is the only turning point of the orbit. Moreover, this point does not precess, namely there is only a single angle \(\phi_*\) at which \(r = r_*\) is reached on the orbit.

A similar argument shows that unbounded orbits that are asymptotically circular cannot possess any turning points. Likewise, unbounded orbits that cross the horizon cannot possess
any turning points, because the horizon can only be entered once. Hence, all of these orbits have no precession.

Consider, next, bounded orbits that are asymptotically circular. These orbits must possess at least one turning point which is the apoapsis. Starting outward near the asymptotic circle, \( r \) will increase until a local apoapsis \( r = r_* \) is reached. The reflection-symmetry \( (5.8) \) thereby holds for \( r_* \geq r > r_{\text{circ}} \), where \( r_{\text{circ}} \) is the location of the limiting circle. This implies that \( r \) will decrease from \( r_* \) to \( r_{\text{circ}} \) asymptotically. Therefore, the apoapsis \( r = r_* \) is the only turning point of the orbit, and this point does not precess.

A similar argument applies to bounded orbits that cross the horizon but are not asymptotically circular. Bounded orbits that are asymptotically circular and cross the horizon cannot possess any turning points, because the horizon can only be entered once. Therefore, all of these orbits have no precession.

The remaining type of bounded orbits to consider is elliptic-like orbits. These orbits will possess at least two turning points which are the apoapsis \( r = r_*^+ \) and the periapsis \( r = r_*^- \). The reflection-symmetry \( (5.8) \) thus can be applied to each segment of the orbit with \( r_*^- \leq r \leq r_*^+ \). This implies that the global orbit shape is obtained by successive composition of these segments such that the angles \( \phi_{\pm} \) at which \( r = r_*^{\pm} \) is reached either comprise a finite sequence or an infinite sequence, mod \( 2\pi \). The change in angle between two successive apoapses or periapses on the orbit is given by the integral

\[
\Delta \phi = \text{sgn}(\bar{L})2I^\phi(2M/r_*^+, 2M/r_*^-)
\]

where \( I^\phi \) is the quadrature \( (4.28a) \). The orbit is physically precessing if and only if \( \Delta \phi \neq 0 \) mod \( 2\pi \). A precessing orbit is periodic (closed) when \( \Delta \phi/(2\pi) \) is a rational number, and otherwise the orbit is non-periodic when \( \Delta \phi/(2\pi) \) is an irrational number. In all cases, the corresponding changes in proper time and coordinate time are given by the integrals

\[
\Delta \bar{\tau} = (2/\bar{L})I^\tau(2M/r_*^+, 2M/r_*^-), \quad \Delta \bar{t} = (2E/\bar{L})I^\tau(2M/r_*^+, 2M/r_*^-),
\]

where \( I^\tau \) and \( I^\tau \) are the quadratures \( (4.28b) \) and \( (4.28c) \).

The preceding discussion, combined with Proposition 1, establishes the following classification result for the global properties of the angular and temporal integrals of motion \( (4.3), (4.4) \). \( (4.5) \).

**Theorem 3.** (i) The cases in which the angular and temporal integrals of motion \( (4.3) \) and \( (4.4) \) are global constants of motion, and the proper-time integral of motion \( (4.5) \) is a global conserved quantity, consist of: for \( u_0 = 2M/r_* \) (TP case), unbounded orbits, and bounded orbits other than elliptic-like ones; for \( u_0 = 2M/r_*^* \) (IP case), unbounded orbits that are either asymptotically circular or horizon crossing; for \( u_0 = 1 \) (HP case), unbounded horizon-crossing orbits, and bounded horizon-crossing orbits that are asymptotically circular. (ii) In all other cases except elliptic-like non-circular orbits, the angular and temporal integrals of motion \( (4.3) \) and \( (4.4) \) are double-valued piecewise constants of motion, and the proper-time integral of motion \( (4.5) \) is a double-valued piecewise conserved quantity, which undergo a jump \( (5.9)-(5.10) \) at the apoapsis if \( u_0 = 2M/r_*^- \), or at the periapsis if \( u_0 = 2M/r_*^+ \). (iii) In the case of elliptic-like non-circular orbits, the angular and temporal integrals of motion \( (4.3) \) and \( (4.4) \) are multi-valued piecewise constants of motion, and the proper-time integral of motion \( (4.5) \) is a multi-valued piecewise conserved quantity, which undergo a jump \( (5.9)-(5.10) \) at every apoapsis if \( u_0 = 2M/r_*^- \), or at every periapsis if \( u_0 = 2M/r_*^+ \).
5.4. Figures: locally conserved angular quantity \( \Phi \). In all figures, a dashed line is the horizon, a dotted line is a circular orbit, a solid brown diamond is an inertial point, a solid blue box is a turning point, and a solid red circle is a horizon point. A black symbol denotes the \( r_0 \) point where \( \phi = \Phi \) is the angle at which the orbit reaches \( r = r_0 \).

First, the universal choice of \( r_0 \) given by an inertial point is illustrated for all orbits that lie outside of the horizon. The conserved quantity \( \Phi \) for these orbits is the analogue of the angle of Hamilton’s vector in Newtonian gravity. As seen from Tables 2 and 3, the bounded orbit types consist of elliptic-like and asymptotic circular, and the unbounded types consist of hyperbolic-like, parabolic-like, asymptotic circular hyperbolic-like, and asymptotic circular parabolic-like. These are shown in Figures 1 to 5. The elliptic-like orbit illustrates precession of the inertial point. In this case \( \Phi \) is multi-valued and thus describes a piecewise constant of motion. In the other five cases, \( \Phi \) is single-valued and therefore describes a global constant of motion.

![Figure 1. Precessing elliptic-like orbit: inertial point](image1.png)

The choice of an inertial point for \( r_0 \) can also be made in the case of horizon-crossing orbits that have \( \overline{L}^2 \geq 3 \), as seen from Tables 2 and 3. For these orbits, the conserved quantity \( \Phi \) is analogous to the angle of Hamilton’s vector in Newtonian gravity for unbounded orbits. In particular, \( \Phi \) is single-valued and therefore describes a global constant of motion. Figure 6 shows the case of unbounded orbits, and Figure 7 shows the case of bounded orbits.

Next, for elliptic-like, parabolic-like, hyperbolic-like orbits — which are the counterparts of Newtonian orbits — the choice of \( r_0 \) given by a turning point is illustrated. The resulting conserved quantity \( \Phi \) is the analogue of the angle of the LRL vector in Newtonian gravity. Figures 8 to 10 show the orbits and the turning points. The elliptic-like orbit illustrates precession of the turning point. In this case \( \Phi \) is multi-valued and thus describes a piecewise constant of motion. In the parabolic and hyperbolic cases, \( \Phi \) is single-valued and therefore describes a global constant of motion.

A turning point can also be chosen for \( r_0 \) in the case of bounded orbits that are either asymptotic circular or horizon-crossing. The conserved quantity \( \Phi \) is analogous to the angle
of the reflected LRL vector in Newtonian gravity, which points in the direction of the apoapsis instead of the periapsis. Figures 11 and 12 show the orbits and the turning points. For all of these orbits, $\Phi$ is single-valued and therefore describes a global constant of motion.

Finally, for all orbits that cross the horizon, a universal choice of $r_0$ given by a horizon point is shown in Figures 13 to 15. The conserved quantity $\Phi$ for these orbits has no analogue in Newtonian gravity. It is single-valued in the case of unbounded orbits and double-valued in the case of bounded orbits.
Each integral of motion corresponds to a generator of a local symmetry of the timelike geodesic equations (2.9) through Noether's theorem [23, 24] for variational symmetries. A local symmetry can be represented by a generator with the characteristic form [25]

$$X = P_t \partial_t + P_r \partial_r + P_\phi \partial_\phi$$

whose components $P_t, P_r, P_\phi$ are functions of $\tau, t, r, \phi, \dot{t}, \dot{r}, \dot{\phi}$. The condition for a generator to define a symmetry is that it infinitesimally leaves invariant the space of solutions

\[6.1\]
Figure 6. Horizon-crossing orbits: inertial point

Figure 7. Bounded horizon-crossing orbit: inertial point

\((t(\tau), r(\tau), \phi(\tau))\) of the geodesic equations, where the infinitesimal transformation of a solution is given by

\[(\delta t, \delta r, \delta \phi)|_E = X(t, r, \phi)|_E = (P^t, P^r, P^\phi)|_E\]  

with \(E\) denoting the solution space.

A generator \((6.1)\) is a \textit{variational symmetry} if (and only if) it infinitesimally leaves invariant the geodesic Lagrangian \((2.11)\) to within a total \(\tau\) derivative:

\[\text{pr}X(\mathcal{L}) = \frac{d}{d\tau} K\]  

for some function \(K\) of \(\tau, t, r, \phi, \dot{t}, \dot{r}, \dot{\phi}\), where \(\text{pr}X\) is the prolongation of \(X\) given by

\[\text{pr}X = X + \frac{d}{d\tau} P^t \partial_t + \frac{d}{d\tau} P^r \partial_r + \frac{d}{d\tau} P^\phi \partial_\phi\]  

(6.4)
which acts on the dynamical variables \((t, r, \phi)\) and their \(\tau\) derivatives \((\dot{t}, \dot{r}, \dot{\phi})\). Since variational symmetries leave invariant the extremals of the Lagrangian, every variational symmetry is symmetry of the equations of motion, \(\delta \mathcal{L}/\delta t = 0, \delta \mathcal{L}/\delta r = 0, \delta \mathcal{L}/\delta \phi = 0\).

Noether's theorem arises from the variational identity

\[
prX(\mathcal{L}) = (\delta \mathcal{L}/\delta t)P_t + (\delta \mathcal{L}/\delta r)P_r + (\delta \mathcal{L}/\delta \phi)P_\phi + \frac{d}{d\tau}(\mathcal{L}_t P_t + \mathcal{L}_r P_r + \mathcal{L}_\phi P_\phi) \tag{6.5}
\]

which holds for an arbitrary generator \((6.1)\). This identity \((6.5)\) combined with the variational symmetry condition \((6.3)\) yields

\[
\dot{I} = (\delta \mathcal{L}/\delta t)P_t + (\delta \mathcal{L}/\delta r)P_r + (\delta \mathcal{L}/\delta \phi)P_\phi, \quad I = K - (\mathcal{L}_t P_t + \mathcal{L}_r P_r + \mathcal{L}_\phi P_\phi) \tag{6.6}
\]
showing that $I$ is an integral of motion due to $\dot{I}|_{E} = 0$. Conversely, for any integral motion $I$, the determining equation

\[
\dot{I}|_{E} = I_t \dot{t} + I_r \dot{r} + I_\phi \dot{\phi} - I_t (\frac{2M \dot{r}}{r(r-2M)}) - I_\phi (\frac{2\dot{\phi} \dot{r}}{r}) \\
+ I_r (\frac{M \dot{r}^2}{r(r-2M)}) - \frac{M(r-2M)\dot{t}^2}{r^3} + (r-2M)\dot{\phi}^2 = 0
\]

(6.7)
can be combined with the chain rule to get
\[
\dot{I} = \frac{r - 2M}{r} (\delta L/\delta t) I_t - \frac{r}{r - 2M} (\delta L/\delta r) I_r - r^2 (\delta L/\delta \phi) I_\phi. \tag{6.8}
\]
Equating the relations (6.8) and (6.6) yields an explicit Noether correspondence between integrals of motion and variational symmetry generators.
Figure 14. Bounded horizon-crossing orbits: horizon points

Figure 15. Asymptotic circular horizon-crossing orbit: horizon point

Proposition 2. There is a one-to-one correspondence between integrals of motion $I$ and variational symmetry generators $X = P^t \partial_t + P^r \partial_r + P^\phi \partial_\phi$:

$$
P^t = \frac{r}{r - 2M} I_t, \quad P^r = -\frac{r - 2M}{r} I_r, \quad P^\phi = -\frac{1}{r^2} I_\phi. \quad (6.9)
$$
The angular momentum and energy integrals of motion (3.33) and (3.34) correspond to symmetries

\[ X_{(t)} = \partial_t, \quad X_{(\phi)} = -\partial_{\phi} \]  

which are Killing vectors of the hyperplane \( \Sigma \) in which the motion takes place.

The proper-time equation (2.9d), which can be viewed as an integral of motion, corresponds to the symmetry

\[ X_{(\tau)} = -\dot{t} \partial_t - \dot{r} \partial_r - \dot{\phi} \partial_{\phi} \]  

which generates a translation in \( \tau \) for solutions of the equations of motion.

These three symmetries (6.10)–(6.11) are easily seen to mutually commute and thus they comprise a three-dimensional abelian algebra. They can be shown to come from a point transformation group acting on the variables \( (\tau, t, r, \phi) \) such that the equations of motion are invariant.

The angular and temporal integrals of motion (3.35), (3.36), (3.37) are found to correspond to hidden symmetries of geodesic equations. To write down these symmetries in the simplest form, it is convenient to work in the enlarged space of variables \( (\tau, t, r, \phi) \) where an infinitesimal transformation of the form \( (\delta \tau, \delta t, \delta r, \delta \phi) = (f, f\dot{t}, f\dot{r}, f\dot{\phi}) \) for any function \( f(\tau, t, r, \phi) \) has a trivial action on solutions \( (t(\tau), r(\tau), \phi(\tau)) \). (Specifically, the induced infinitesimal transformation on a solution is given by \( \delta(t(\tau), r(\tau), \phi(\tau)) = (\delta t(\tau) - \dot{t}(\tau)\delta \tau, \delta r(\tau) - \dot{r}(\tau)\delta \tau, \delta \phi(\tau) - \phi(\tau)\delta \tau) = (0, 0, 0) \).) The underlying generator \( X_{\text{triv}} = f \frac{d}{d\tau} \) represents a trivial symmetry \[23, 24\]. The hidden symmetries can then be expressed as

\[ X_{(\Phi)} = -LY_{(\Phi)} \partial_{\tau} + \Phi E \partial_{t} - \Phi L \partial_{\phi}, \]  
\[ X_{(T)} = -EY_{(T)} \partial_{\tau} + T E \partial_{t} - T L \partial_{\phi}, \]  
\[ X_{(\tau)} = -Y_{(T)} \partial_{\tau} + T E \partial_{t} - T L \partial_{\phi}, \]  

modulo trivial symmetries, with

\[ Y_{(\Phi)} = \int_{r_0}^{r} \frac{(r - 2M) \, dr}{r^3 v^3}, \quad Y_{(T)} = \int_{r_0}^{r} \frac{dr}{v^3}, \quad Y_{(\tau)} = \int_{r_0}^{r} \frac{(r - 2M) \, dr}{r v^3}. \]  

Note that components in these generators (6.12) have essential nonlinear dependence on \( \dot{\phi} \) and \( \dot{t} \) through the expressions (3.33) and (3.34) for \( L \) and \( E \).

Consequently, each of these symmetries (6.12) describes a dynamical (non-point) symmetry. It is straightforward by a direct computation to show that they mutually commute and thus form a three-dimensional abelian algebra. (See Ref. [19, 20, 21, 22, 12, 8] for discussion of the analogous dynamical symmetries associated to the LRL vector for general central force motion in classical mechanics.)

A direct computation of the commutators between the point symmetries (6.10)–(6.11), (6.12) and the dynamical symmetries (6.12) shows that these two symmetry subalgebras commute. Hence, the following characterization of the full symmetry algebra is obtained.

**Theorem 4.** The set of symmetries (6.10), (6.11), (6.12) corresponding to the integrals of motion for the timelike geodesic equations comprise a six-dimensional abelian symmetry algebra.
7. Concluding remarks

The main results in this work can be viewed as being motivated by the similarity between the timelike geodesic equations in Schwarzschild spacetime and the equations of motion in general central force dynamics, which share invariance under the $SO(3)$ group of rotations. In particular, Schwarzschild spacetime and Newtonian spacetime share the same rotational Killing vectors as well as the same time translation Killing vector.

This shared symmetry structure can be understood as what ultimately gives rise to the locally conserved $\phi$-quantity and $t$-quantity presented in Theorem[1]. The locally conserved $\tau$-quantity arises from the difference between absolute Newtonian time and relativistic proper time.

The physical meaning and mathematical properties of these conserved quantities (integrals of motion) are likewise similar to the corresponding quantities that appear in central force dynamics. While the $\phi$-quantity is the direct analogue of the angle of either the LRL vector or Hamilton’s binormal vector in the plane of motion, which are well recognized as locally conserved vectors in the classical mechanics literature, the $t, \tau$-quantities are less widely known and only recently have been studied in detail [8, 15].

There are several open questions for future work. First, it would be worthwhile to extend the derivation of integrals of motion to the full (non-equatorial) geodesic equations. This should lead to finding an analogue of the LRL vector and Hamilton’s vector — not just their coordinate angles — for geodesics in Schwarzschild spacetime. Second, the integrals of motion can be studied in a Hamiltonian context, which will help to reveal the underlying hidden symmetry group for the full geodesic equations. Last, it would be interesting to explore generalizing the methods and the results from Schwarzschild spacetime to more general spacetimes with symmetry, such as non-static spherically symmetric spacetimes like the Friedman-Lemaitre-Robinson-Walker cosmology, and stationary axisymmetric spacetimes like the Kerr black hole.

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Appendix: Newtonian Limit

The integrals of motion (3.35), (3.36) and (3.37) will now be evaluated in the Newtonian limit to show that they respectively correspond to the angle and the time at which a particle reaches either an apsis point $r = r_0$ or an inertial point $r = r_0$ on a non-circular orbit under Newtonian gravity. Their explicit relationship to the Newtonian LRL vector and Hamilton’s binormal vector also will be described.

7.1. Newtonian LRL vector. To begin, recall the expression for the conserved LRL vector of a particle in a non-circular orbit in Newtonian gravity:

$$\vec{A}_{\text{Newt}} = m \vec{v} \times \vec{L} - m^2 \vec{M} \hat{r}$$

(7.1)
where \( \hat{r} \) is the unit radial vector, \( \vec{v} = \dot{\vec{r}} \) is the velocity vector, \( \vec{L} = m\vec{r} \times \vec{v} \) is the angular momentum vector, and \( m \) is the particle mass. As is well known, the orbit of the particle lies in a plane orthogonal to \( \vec{L} \). Let \((r, \phi)\) be polar coordinates in this plane, with associated unit vectors \( \hat{r} = (\cos \phi, \sin \phi) \) and \( \hat{\phi} = (-\sin \phi, \cos \phi) \). Then the LRL vector at any point \((r(t), \phi(t))\) on the orbit is given by

\[
\frac{1}{m^2} \vec{A}_{\text{Newt}} = \left( \frac{L^2}{r(t)} - M \right) \hat{r} + L v(t) \hat{\phi} = \frac{|\vec{A}_{\text{Newt}}|}{m^2} (\cos \varphi, \sin \varphi)
\]  

where

\[
\tan \varphi = \frac{\tan \phi(t) - \alpha(t)}{1 + \alpha(t) \tan \phi(t)}, \quad \alpha(t) = \frac{Lr(t)v(t)}{L^2 - Mr(t)}.
\]

It is straightforward to show that \( \dot{A} = 0 \) holds as a consequence of the Newtonian equations of motion of the particle, and hence \( \varphi \) (modulo \( 2\pi \)) is a constant of the motion. A variant of the LRL vector is

\[
\vec{B}_{\text{Newt}} = \vec{L} \times \vec{A}_{\text{Newt}} = \frac{L|\vec{A}_{\text{Newt}}|}{m^2} (-\sin \varphi, \cos \varphi)
\]

called Hamilton’s binormal vector \([5]\). Its associated angle is given by \( \varphi + \frac{\pi}{2} \).

Non-circular orbits with \( L \neq 0 \) are elliptic, parabolic, and hyperbolic, such that the origin \( r = 0 \) coincides with a focal point. These orbits are classified by their energy, which is \( -M^2/(2L^2) < E_{\text{Newt}} < 0 \) for elliptic orbits, \( E_{\text{Newt}} = 0 \) for parabolic orbits, and \( E_{\text{Newt}} > 0 \) for hyperbolic orbits. The LRL vector \( \vec{A} \) has the property that \([4, 5]\) it lies on the radial line from the origin to the periapsis point on the particle’s orbit. In particular, the angle of this apsis line is given by \( \varphi \). Similarly, the variant vector \( \vec{B} \) lies on the perpendicular bisector line of this radial apsis line through the origin.

### 7.2. Evaluation of the angular and temporal integrals of motion.

The Newtonian limit of the integrals of motion \((3.35), (3.36)\) and \((3.37)\) is obtained by taking \( r/(2M) \gg 1 \) and \( E - 1 \simeq E_{\text{Newt}} \ll 1 \) along with \( \tau \simeq t \). In this limit, the proper time equation \((5.2)\) reduces to the Newtonian energy equation

\[
E_{\text{Newt}} \simeq \frac{1}{2} v^2 + V_{\text{Newt}}
\]

where

\[
V_{\text{Newt}} = -\frac{M}{r} + \frac{L^2}{2r^2}
\]

is the Newtonian potential per unit mass. Similarly, the Newtonian limit of the turning point equation \((3.27)\) and the inertial point equation \((3.28)\) are respectively given by

\[
E_{\text{Newt}} r_*^3 + Mr_*^2 - \frac{1}{2} L^2 r_* \simeq 0, \quad r_* > 0,
\]

\[
Mr_*^2 - L^2 r_* \simeq 0, \quad r_* > 0.
\]

These roots provide a distinguished dynamical point \( r = r_0 \) which corresponds to defining a physically meaningful zero-point value for the integrals of motion \((3.35), (3.36)\) and \((3.37)\) in the Newtonian limit.
It is straightforward to see that the angular integral of motion (3.35) reduces to the quantity
\[ \Phi \simeq \phi - L \int_{r_0}^{r} \frac{\text{sgn}(v) \, dr}{r^2 \sqrt{2(E_{\text{Newt}} - V_{\text{Newt}})}} \mod 2\pi, \] (7.9)
and that the two temporal integrals of motion (3.36) and (3.37) both reduce to the quantity
\[ T \simeq T \simeq t - \int_{r_0}^{r} \frac{\text{sgn}(v) \, dr}{\sqrt{2(E_{\text{Newt}} - V_{\text{Newt}})}}. \] (7.10)

To proceed with evaluating these quantities, introduce the reciprocal radial variable
\[ u = M/r, \] (7.11)
which is analogous to the variable used in [3] for classifying orbits in Schwarzschild spacetime. Note that \( u \) lacks a factor of 2 compared with the notation in section 4, which will help to simplify the subsequent equations here. Hereafter, an overbar will denote a quantity or a variable divided by \( M \). Then the Newtonian integrals of motion (7.9) and (7.10) are given by
\[ \Phi_{\text{Newt}} = \phi + \bar{L} \int_{u_0}^{u} \frac{\text{sgn}(v) \, du}{\sqrt{Q(u)}} \mod 2\pi, \] (7.12)
\[ T_{\text{Newt}} = t + M \int_{u_0}^{u} \frac{\text{sgn}(v) \, du}{u^2 \sqrt{Q(u)}}, \] (7.13)
where
\[ Q(u) = v^2 = 2(E_{\text{Newt}} - V_{\text{Newt}}) = 2E_{\text{Newt}} + 2u - \bar{L}^2 u^2. \] (7.14)

Moreover, the Newtonian turning point equation (7.7) corresponds to \( Q(u) = 0 \) for \( u = u_+ > 0 \), namely
\[ 0 = Q(u_+) = 2E_{\text{Newt}} + 2u_+ - \bar{L}^2 u_+^2, \] (7.15)
while the Newtonian inertial point equation (7.8) corresponds to \( Q'(u) = 0 \) for \( u = u^* > 0 \), namely
\[ 0 = Q'(u^*) = 2(1 - \bar{L}^2 u^*). \] (7.16)

The quadratures appearing in the integrals of motion (7.12) and (7.13) are straightforward to evaluate by using the factorization
\[ Q(u) = \bar{L}^2 (u_+ - u)(u - u_-) \] (7.17)
in terms of the roots
\[ u_{\pm} = \frac{1}{\bar{L}^2} \left( 1 \pm \sqrt{1 + 2E_{\text{Newt}} \bar{L}^2} \right). \] (7.18)
The nature of these roots also determines the type of the orbit with a given value of energy \( E_{\text{Newt}} \) and angular momentum \( \bar{L} \). In particular, for elliptic, parabolic, and hyperbolic orbits, both roots are real and distinct. As a consequence, there is no need to break up the evaluation of the quadratures into cases given by the separate types of orbit.
When the roots (7.18) are real and distinct, the quadratures are explicitly given by

$$I_\text{Newt}^B(u; u_0) \equiv |\bar{L}| \int_{u_0}^{u} \frac{du}{\sqrt{Q(u)}} = \int_{u_0}^{u} \frac{du}{\sqrt{(u_+ - u)(u - u_-)}}$$

$$= \left(-2 \arctan \sqrt{\frac{u_+ - u}{u - u_-}}\right)\bigg|_{u_0}^{u},$$

(7.19)

and

$$I_\text{Newt}^T(u; u_0) \equiv |\bar{L}| \int_{u_0}^{u} \frac{du}{u^2 \sqrt{Q(u)}} = \int_{u_0}^{u} \frac{du}{u^2 \sqrt{(u_+ - u)(u - u_-)}}$$

$$= \left(\sqrt{\frac{(u - u_-)(u_+ - u)}{u_+ u_- u}} - \frac{u_+ + u_-}{\sqrt{u_+ u_-}} \arctan \sqrt{\frac{u_-(u_+ - u)}{u_+(u - u_-)}}\right)\bigg|_{u_0}^{u}.$$  

(7.20)

Here the tan function has the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$. To complete the evaluation of the integrals of motion, it is necessary to specify $u_0$ universally for all non-circular orbits. by choosing a turning point or an inertial point. In particular, the resulting integrals of motion turn out to be related to the Newtonian LRL vector (7.2) when a turning point is chosen, and alternatively to the variant vector (7.4) when an inertial point is chosen. More details can be found in Ref. [8].

A side remark is that these quadratures (7.19) and (7.20) are only applicable when $E_{\text{Newt}} > -M^2/(2L^2)$. Specifically, for $E_{\text{Newt}} = -M^2/(2L^2)$, the roots (7.18) are no longer distinct, $u_+ = u_- = 1/\bar{L}^2$, and hence $v^2 = Q(u) = -\bar{L}^2(u - u_\pm)^2$ must vanish identically (due to the opposite signs on the two sides). This implies $u = u_\pm = 1/\bar{L}^2$, corresponding to $r = L^2/M$ which is the radius of a circular orbit with angular momentum $L$ and energy $E_{\text{Newt}} = -M^2/(2L^2)$.

### 7.3. LRL quantities for elliptic orbits.

For an elliptic orbit, both roots (7.18) of $Q(u)$ are positive, and so the range of $u$ is $u_- \leq u \leq u_+$. Hence, each root is a turning point $u_* = u_\pm$ and $u_* = u_-$. There is a single inertial point $u^* = 1/\bar{L}^2$.

Every elliptic orbit with a given energy $-1/(2\bar{L}^2) < E_{\text{Newt}} < 0$ possesses a single apoapsis point, which has $r = r_*^+ = M/u_-$, and a single periapsis point, which has $r = r_-^* = M/u_+$. These apsis points comprise the two turning points in the Newtonian effective potential. The two points at an angular separation $\Delta \phi = \frac{1}{2} \pi$ between successive apsis points on the orbit have $r = r^* = L^2/M$ which coincides with the radius of a circular orbit having the same angular momentum and which comprises the inertial point in the Newtonian effective potential.

If $u_0 = u_+$ is chosen in the angular and temporal integrals of motion (7.12) and (7.13), then the quadratures (7.19) and (7.20) become

$$I_\text{Newt}^B(u; u_+) = -2 \arctan \sqrt{\frac{u_+ - u}{u - u_-}} = -2 \arctan \left(\frac{\bar{L}|(u_+ - u)}{\sqrt{Q(u)}}\right),$$

(7.21)
\[ I_{\text{Newt}}^{T}(u; u_{+}) = \frac{\sqrt{(u - u_{-})(u_{+} - u)}}{u_{+} - u} - \frac{u_{-} + u_{+}}{\sqrt{u_{+} - u}} \arctan \sqrt{u_{-}(u_{+} - u)} + \frac{\sqrt{Q(u)}}{|L|u_{+} - u} - \frac{u_{-} + u_{+}}{\sqrt{u_{+} - u}} \arctan \left( \frac{|L|\sqrt{u_{-}(u_{+} - u)}}{\sqrt{u_{+}Q(u)}} \right). \] (7.22)

Both quadratures will vanish for \( u = u_{+} \), corresponding to the periapsis point, where
\[ r = r_{-} = M/u_{+} = \frac{M}{2|E_{\text{Newt}}|} \left( 1 - \sqrt{1 - 2|E_{\text{Newt}}|L^{2}} \right). \] (7.23)

Hence, using expressions (7.17) and (7.18), this yields
\[ \Phi_{\text{Newt}} = \phi + \text{sgn}(vL)I_{\text{Newt}}^{\phi}(M/r; M/r_{-}) = \phi - 2 \arctan \left( \frac{L(r - r_{-})}{r_{-}rv} \right) \mod 2\pi, \] (7.24)
\[ T_{\text{Newt}} = t + \frac{M^{2} \text{sgn}(v)}{|L|}I_{\text{Newt}}^{T}(M/r; M/r_{-}) = t + \frac{Mvr}{2|E_{\text{Newt}}|} - \frac{M}{\sqrt{2|E_{\text{Newt}}|^{3}}} \arctan \left( \frac{\sqrt{2|E_{\text{Newt}}|(r - r_{-})}}{rv} \right). \] (7.25)

At the periapsis point \((r(t_{*}), \phi(t_{*})) = (r_{-}, \phi_{-})\) on the orbit, \( \Phi_{\text{Newt}} = \phi_{-} \) and \( T_{\text{Newt}} = t_{*} \) hold, thereby showing that \( \Phi_{\text{Newt}} \) is the angle \( \phi_{-} \) of the periapsis line associated with the LRL vector, and that \( T_{\text{Newt}} \) is the time \( t_{*} \) at which the periapsis point is reached after \( t = 0 \), namely \((r(T_{\text{Newt}}), \phi(T_{\text{Newt}})) = (r_{-}, \Phi_{\text{Newt}})\). This conclusion can also be directly verified from the explicit expression for the LRL vector (7.2) through use of the identity \( 2 \arctan(x) = \arctan(2x/(1 - x^{2})) \). In particular, \( \Phi_{\text{Newt}} = \varphi \).

The LRL integrals of motion (7.24) and (7.25) have the following global properties. On any elliptic orbit, \( \Phi_{\text{Newt}} \) is locally constant and changes by \( \Delta \phi = 2\pi = 0 \mod 2\pi \) when the apoapsis point is reached, while \( T_{\text{Newt}} \) is locally constant and jumps by \( \Delta t = \pi M/\sqrt{2|E_{\text{Newt}}|^{3}} \) which is the period of the orbit. Thus, \( \Phi_{\text{Newt}} \) is single-valued, and \( T_{\text{Newt}} \) is multi-valued. Hence, \( \Phi_{\text{Newt}} \) is a global constant of motion.

Finally, instead of the turning point \( u_{0} = u_{+} \), suppose that the inertial point \( u_{0} = u^{*} = 1/L^{2} \) given by the root of equation (7.16) is chosen. Since \( I_{\text{Newt}}^{\phi}(u_{+}; u^{*}) = 2 \arctan \left( \frac{u_{+} - u^{*}}{u^{*} - u_{-}} \right) = 2 \arctan(1) = \frac{\pi}{2} \), the angular integral of motion (7.12) becomes the LRL angle plus \( \frac{\pi}{2} \), which is the angle of the radial line from the origin to one of the two inertial points on the orbit (depending on the signs of \( v \) and \( L \)). Likewise, the temporal integral of motion (7.13) becomes the time at which this inertial point is reached after \( t = 0 \). The global properties of these quantities \( \Phi_{\text{Newt}} \) and \( T_{\text{Newt}} \) are the same as in the LRL case. The conserved vector associated with them is Hamilton’s vector (7.4).

7.4. LRL quantities for hyperbolic and parabolic orbits. For a hyperbolic orbit, the roots (7.18) of \( Q(u) \) satisfy \( u_{+} > 0 > u_{-} \), and so the range of \( u \) is \( 0 < u \leq u_{+} \). Hence, there is a single turning point \( u_{*} = u_{+} \). There is also a single inertial point \( u^{*} = 1/L^{2} \). The turning point corresponds to the periapsis point on the orbit, which has \( r = r_{*} = M/u_{+} \).

If again \( u_{0} = u_{+} \) is chosen in the angular and temporal integrals of motion (7.12) and (7.13), then the quadrature (7.19) evaluates to the same expression (7.21) as in the elliptic.
case, whereas the quadrature (7.20) now evaluates to

$$
I^T_{\text{Newt}}(u; u_+) = \sqrt{\frac{|u - u_-|}{u_+ - u_-}} \arctanh \left( \frac{u_- + u_+}{|u_+|} \right) \sqrt{\frac{|u_-|(u_+ - u)}{u_+ u - u_-}}
$$

(7.26)

through the identity \( \arctan(ix) = i\arctanh(x) \). Both quadratures (7.26) and (7.22) will vanish for \( u = u_+ \), corresponding to the periapsis point, where

$$
r = r_* = M/u_+ = \frac{M}{2E_{\text{Newt}}} \left( \sqrt{1 + 2E_{\text{Newt}}L^2} - 1 \right).
$$

(7.27)

Hence, using expressions (7.17) and (7.18), this yields

$$
\Phi_{\text{Newt}} = \phi + \text{sgn}(vL) I^\Phi_{\text{Newt}}(M/r; M/r_*) = \phi - 2 \arctan \left( \frac{L(r - r_*)}{r_0 r v} \right) \mod 2\pi,
$$

(7.28)

$$
T_{\text{Newt}} = t + \frac{M^2 \text{sgn}(v)}{|L|} I^T_{\text{Newt}}(M/r; M/r_*)
$$

(7.29)

$$
= t + \frac{M}{2E_{\text{Newt}}} \frac{vr}{\sqrt{2E_{\text{Newt}}^3}} \arctan \left( \frac{\sqrt{E_{\text{Newt}}(r - r_*)}}{rv} \right).
$$

Similarly to the elliptic case, \( \Phi_{\text{Newt}} \) is the angle of the periapsis line associated with the LRL vector (7.2), and \( T_{\text{Newt}} \) is time at which the periapsis point is reached after \( t = 0 \), namely \( (r(T_{\text{Newt}}), \phi(T_{\text{Newt}})) = (r_*, \Phi_{\text{Newt}}) \). In particular, \( \Phi_{\text{Newt}} = \varphi \).

For a parabolic orbit, since \( E_{\text{Newt}} = 0 \), the roots (7.18) of \( Q(u) \) satisfy \( u_+ = 2/L^2 > u_- = 0 \). Hence, just as in the hyperbolic case, there is a single inertial point \( u_+ = u_+ \) which corresponds to the periapsis point on the orbit,

$$
r = r_* = 2L^2/M.
$$

(7.30)

Again, take \( u_0 = u_+ \) in the angular and temporal integrals of motion (7.12) and (7.13). Because \( u_- = 0 \), the evaluation of the quadratures (7.19) and (7.20) now simplifies and can be obtained either from the leading term in an asymptotic expansion as \( u_- \to 0 \), or by direct integration:

$$
I^\Phi_{\text{Newt}}(u; u_+) = \int_{u_0}^u \frac{du}{u(u_+ - u)} = -2 \arctan \left( \frac{u_+ - u}{u} \right),
$$

(7.31)

and

$$
I^T_{\text{Newt}}(u; u_+) = \int_{u_0}^u \frac{du}{u^2(u_+ - u)} = \frac{2(u_+ + 2u)\sqrt{u_+ - u}}{3u_+^2 \sqrt{u_+}^3}.
$$

(7.32)

Both of these quadratures will vanish for \( u = u_+ \), corresponding to the periapsis point (7.30). Hence, this yields

$$
\Phi_{\text{Newt}} = \phi + \text{sgn}(vL) I^\Phi_{\text{Newt}}(M/r; M/r_*) = \phi - 2 \arctan \left( \frac{rv}{L} \right) \mod 2\pi,
$$

(7.33)

$$
T_{\text{Newt}} = t + \frac{M^2 \text{sgn}(v)}{|L|} I^T_{\text{Newt}}(M/r; M/r_*) = t - \frac{vr(Mr + L^2)}{6L^2}.
$$

(7.34)
As before, $\Phi_{\text{Newt}}$ is the angle of the periapsis line associated with the LRL vector \((7.2)\), and $T_{\text{Newt}}$ is time at which the periapsis point is reached after $t = 0$, namely $(r(T_{\text{Newt}}), \phi(T_{\text{Newt}})) = (r_*, \Phi_{\text{Newt}})$ with $\Phi_{\text{Newt}} = \varphi$.

In both the hyperbolic and parabolic cases, $\Phi_{\text{Newt}}$ and $T_{\text{Newt}}$ are single-valued and globally constant. Hence, $\Phi_{\text{Newt}}$ is a global constant of motion for all hyperbolic and parabolic orbits.

Finally, suppose that instead the inertial point $u_0 = u^* = 1/\bar{L}^2$ is chosen. In both the hyperbolic and parabolic cases, $I_\Phi^{\Phi}(u_+; u^*) = 2 \arctan \sqrt{u_+ - u^*} / u^* - u_- = 2 \arctan(1) = \pi / 2$, and so the angular integral of motion becomes the LRL angle plus $\pi / 2$, which is the angle of the radial line from the origin to one of the two inertial points on the orbit (depending on the signs of $v$ and $L$). The temporal integral of motion likewise becomes the time at which this inertial point is reached after $t = 0$. Moreover, the two inertial points coincide with the radius of a circular orbit having the same angular momentum. Just as in the elliptic case, the conserved vector associated with the angular and temporal quantities for hyperbolic and parabolic orbits is Hamilton’s vector \((7.4)\).

**References**

[1] Y. Hagihara, Theory of the relativistic trajectories in a gravitational field of Schwarzschild, Jpn. J. Astron. Geophys. 8 (1931), 67–175.
[2] C.G. Darwin, The gravity field of a particle, Proc. Roy. Soc. A 249 (1959) 180–194; The gravity field of a particle II, Proc. Roy. Soc. A 263 (1961), 39–50.
[3] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press) 1983.
[4] H. Goldstein, C. Poole, J. Safko, *Classical Mechanics* (3rd ed.), (Addison Wesley) 2000.
[5] B. Cordani, *The Kepler Problem* (Birkhaeuser) 2003.
[6] S. Chandrasekhar, *Newton’s Principia for the Common Reader*, (Oxford University Press) 1995.
[7] D. Lynden-Bell, R.M. Lynden-Bell, On the shapes of Newton’s revolving orbits, Notes and Records of the Royal Society of London. 51(2) (1997), 195–198.
[8] S.C. Anco, T. Meadows, V. Pascuzzi, Some new aspects of first integrals and symmetries for central force dynamics, J. Math. Phys. 57 (2016) 062901.
[9] V.B. Serebrennikov, A.E. Shabad, Method of calculation of the spectrum of a centrally symmetric Hamiltonian on the basis of approximate $O_4$ and $SU_3$ symmetries, Theor. Math. Phys. 8, (1971) 644–653.
[10] L.H. Buch, H.H. Denman, Conserved and piecewise-conserved Runge vectors for the isotropic harmonic oscillator, Amer. J. Phys. 43 (1975) 1046–1048.
[11] A. Peres, A classical constant of motion with discontinuities, J. Phys. A: Math. Gen. 12 (1979), 1711–1713.
[12] P.G.L. Leach, G.P. Flessas, Generalisations of the Laplace–Runge–Lenz vector, J. Nonlin. Math. Phys. 10 (2003), 340–423.
[13] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation* (W.H. Freeman and Co.) 1973.
[14] R.M. Wald, *General Relativity* (The University of Chicago Press) 1984.
[15] S.C. Anco, A. Ballesteros, M. Gandarias, Global versus local (super)integrability of a nonlinear oscillator, Phys. Lett. A. 383 (2019), 801–807.
[16] F. John, *Partial Differential Equations* Applied Math. Sci. Volume 1 (Springer, New York) 1982.
[17] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*, Volume 55, National Bureau of Standards (1964).
[18] H. Bacry, J. Ruegg, J.-M. Souriau, Dynamical groups and spherical potentials in classical mechanics, Commun. Math. Phys. 3 (1966), 323–333.
[19] D.M. Fradkin, Existence of the dynamic symmetries $O_4$ and $SU_3$ for all classical central potential problems, Prog. Theor. Phys. 37 (1967), 798–812.
[20] N. Mukunda, Dynamical symmetries and classical mechanics, Phys. Rev. 155 (1967) 1383–1386.
[21] J.M. Lévy-Leblond, Conservation laws for gauge-invariant Lagrangians in classical mechanics, Amer. J. Phys. 39 (1971), 502–506.

[22] H. Rodgers, Symmetry transformations of the classical Kepler problem, J. Math. Phys. 14 (1973), 1125–1129.

[23] P.J. Olver, Applications of Lie Groups to Differential Equations, (Springer, New York) 1986.

[24] S.C. Anco, Generalization of Noether’s theorem in modern form to non-variational partial differential equations. In: Recent progress and Modern Challenges in Applied Mathematics, Modeling and Computational Science, 119–182, Fields Institute Communications, Volume 79, 2017.

[25] G. Bluman and S.C. Anco, Symmetry and Integration Methods for Differential Equations, Applied Math. Sci. Volume 154 (Springer, New York) 2002.