The Method of the Kernel of the Evolution Equation in the Theory of Gravity

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Abstract—Covariant perturbation theory allows one to calculate the nonlocal kernel of the evolution equation on a spin Riemannian manifold. The proposed axiomatic definition of effective action introduces a universal scale parameter with the dimension of the square of the distance into the dimensionless mathematical theory. It is shown that this purely geometric result has the physical meaning of the action of field theory, including gravity. The two lowest tensor orders of this covariant functional are independent of the type of the spin group and are local; they reproduce the action of general relativity with a cosmological constant. The modern value of the universal distance scale can be determined by the measured Hubble constant. This scale parameter, considered a physical variable, makes it possible to construct the cosmological theory axiomatically.

Keywords: gravity theory, Dirac operator, evolution equation, effective action, Hubble constant, universal scale

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In the last two decades, the theory of geometric flows [1] has been actively developing in mathematics, one type of which is the Ricci flow [2]. The differential Ricci flow equation connects the geometric quantities of a Riemannian manifold, the metric, and the Ricci tensor, but does not contain covariant derivatives. Ricci flows were used by G. Perelman in proving the Poincaré conjecture [3], which considers manifolds of constant curvature and therefore does not require knowledge of the gradient form of the kernel of the evolutionary equation. This area of geometric analysis was laid down by the works of H. Ruse [4]. The subsequent fundamental contribution of J. Synge [5] was actively used in physics, in particular, in unsuccessful attempts to construct a quantum theory of gravity [6] and in successful algorithms for satellite orientation [7]. Unfortunately, these works were not in demand in geometry, and interest in them has arisen only recently [8].

The gradient form of the Ricci flow has long been known in physics [10] under the name of the “trace of the kernel of the heat equation.” It is important to understand that the evolutionary equation has nothing to do with the heat conduction equation and represents a new kind of differential equation. This paper proposes looking at the functional trace of the kernel of the evolutionary equation as a geometric object, discarding the erroneous idea of it as a “method” of quantum field theory, which has taken root since the 1970s. We will clarify the physical meaning and application of the covariant effective action calculated on the basis of the evolutionary kernel [9].

Let’s start with a historical note that the action of the theory of gravity (Einstein–Hilbert action) was axiomatically obtained by D. Hilbert [11] from the general principles of the theory of invariants that he created. Today this action is recognized as the basis of the general theory of relativity (theory of gravitation) by A. Einstein [12]. Attempts to link this geometric result with the rest of physics have never stopped and are known as unified field theory. The only consistent path of development of physical theory is in the geometrization of physics, begun by B. Riemann, W.C. Clifford, and A. Poincaré. The evolutionary equation is taking us down this path.

At the same time, the method of effective action allows us to naturally solve the problem of the appearance of a physical scale (dimension) in a dimensionless (mathematical) physical theory. It is shown below that the mathematically correct definition of the main functional of the physical theory, the covariant effective action, introduces into physics a universal scale parameter. This action takes on a geometric shape. This conclusion is achieved by the method of “covariant perturbation theory” [10], which is historically related to the operator analysis created by O. Heaviside [13].
In November 2018, the global metrology community adopted a resolution on changes in the system of physical units SI [14]. Since May 2019, the fundamental physical constants have precise fixed values, and the units of physical quantities are determined by these constants. Therefore, Planck’s constant became the constant that determines the unit of mass. With the introduction of two new physical constants by M. Planck in 1900 [15], their number then became equal to the number of physical units, giving rise to the possibility of choosing the values of the constants. If one of the constants is a gravitational constant and units are assigned to their values, then such a system is called the Planck system [16]. In such a system, the values of physical units, expressed in terms of traditional SI units, take on unusual meanings. It is believed that these values represent the limits where the known laws of physics cease to hold, but this is not the case. Instead of 1, any other numbers can be chosen that will give rise to arbitrarily different Planck values.

The new SI is built hierarchically, it has seven defining (“fundamental” as opposed to “derivatives”) constants, which, nevertheless, depend on each other (determined through others) [17]. The only constant, independent of any other, is the atomic frequency, which determines the unit of time, a second, as the reciprocal of the frequency expressed as an integer.

Moreover, the first two terms of this sum are local and we will assume that the functional \( W \) is specified for the spacetime dimension \( D < 6 \), but the covariant effective action below is calculated in four dimensions corresponding to the observed physical world.

We will define effective action axiomatically,

\[
W(L^2) \equiv \int \frac{ds}{s} \text{Tr}K(s),
\]

and we will assume that the functional \( W \) is specified up to an arbitrary factor, the value of which is found from the experiment. It is obvious that the integral over proper time (5) must have a lower limit, which takes arbitrary positive value, since the integrand does not exist for \( s = 0 \). After the substitution of solution (4) into definition (5) and integration over \( s \) we get dimensionless functional \( W(L^2) \), which clearly...
depends on the value of the proper time at the lower limit, \( t^2 \),

\[
-W(t^2) = \sum_{n=0}^{\infty} (t^2)^{(n-2)} W(n)(t^2).
\] (6)

Parameter \( t^2 \) makes real meaning in terms of physical observables. The covariant effective action (6) was calculated in [20], but here we are interested in two of its simplest terms, which were omitted in [10, 20],

\[
-W(t^2) = \int \text{d}x^4 g^{1/2}(x) \text{tr} \left[ \frac{1}{2} (\frac{t^2}{12})^2 + \frac{1}{2} \hat{P} + Q(\mathcal{R}^2) \right].
\] (7)

Although the effective action is calculated in Euclidean spacetime, its local terms (7) do not depend on the signature of the metric.

The first term in (7) is universal for any theory with an operator of the form (1), and the second is given by a specific form \( \hat{P} \). In modern physics, fundamental fields are described by massless spinors [21]; in a geometric description they can be considered as properties of a spin manifold [22, 23]. The covariant Dirac operator in the form (1) contains the Ricci curvature scalar with the coefficient \((-1/4)\) [24, 6, 23]. Then, in order to obtain the effective action of such a theory, as a result of the general form (7), one must make a substitution, \( \text{tr} \hat{P} = -\frac{1}{12} \text{tr} \hat{R} \) (in which the matrix trace operation turns the gauge field tensor to zero and makes the spin group dependence trivial, \( \text{tr} \hat{R} \)). Action (7) can be reduced to the form adopted in the general theory of relativity [12], multiplying by \( 12/t^2 \) that according to the main hypothesis of this work should not change the physical content of the action (we do not consider cosmological theories),

\[
\mathcal{W}(t^2) = \int \text{d}x^4 g^{1/2} \left\{ (6t^{-2} - R) + t^2 Q(\mathcal{R}^2) \right\}.
\] (8)

The first term of expression (8) is obviously interpreted as a “cosmological constant.” We emphasize that this is a conventional name, since we do not build a cosmological theory, and the scale parameter will be included in all equations of physical theory, because it is \( t^2 \) that specifies physical dimensions according to the hierarchical principle used in the New SI (2019) of physical units [17]. The second term in (8) has the form of the Einstein—Hilbert gravitational action with the correct sign.

You can find the value of the universal scale parameter, knowing the cosmological constant,

\[
\Lambda = 6/t^2.
\] (9)

The standard cosmological model [21] assumes the value \( \Lambda \approx 10^{-52} \text{ m}^{-2} \), but since \( \Lambda \) is determined by the Hubble constant, \( H_0 = 73.48 \pm 1.66 \text{ (km/s)/Mpc} \) with 1.66 (km/s)/Mpc [25], the physical dimension of which is frequency,

\[
H_0 = 2.38 \pm 0.05 \times 10^{-18} \text{ s}^{-1},
\] (10)

and then it is natural to use the value of the Hubble radius,

\[
l = c/H_0 = 1.26 \times 10^{26} \text{ m},
\] (11)

which is given by the observable \( H_0 \). Both \( l \) and \( \Lambda^{1/2} \) are equal in order of magnitude to the observed universe, as Paul Dirac proposed [12]. Setting the universal scale as the largest distance in nature in the absence of the smallest makes the theory of observed physical phenomena closed.

Since the proper time is a parameter with physical dimension, the scale \( t^2 \), obviously, can be considered a physical constant only when the evolution of the physical world as a whole (the universe) is not taken into account. Indeed, in the existing cosmological theories, the Hubble constant and the cosmological constant determined by it are considered variable quantities [21]. The variability of a universal scale parameter that hierarchically sets all the rest of the defining physical constants [14] makes them variables as well. This necessarily means that Dirac’s hypothesis about the change in Newton’s gravitational constant is true [26].

If we accept three propositions that follow convincingly from a variety of well-known mathematical and physical facts—(1) the evolutionary equation is the fundamental equation of physics; (2) SI (2019), the modern metrological system of physical units, self-consistently describes the structure of the observable physical world; and (3) cosmology, as a science of studying the evolution of the physical world, can be built on physical experiments carried out only in the local part of the universe—then all physical constants must change along with the evolution of the universe. The third postulate, however, cannot be experimentally confirmed, so any cosmological theory is only a scientific hypothesis.

The “derivation” of the value of the cosmological constant in the physical theory cannot exist; this quantity can only be measured. The action of the theory of gravity contains both the well-known terms of lower orders (8) and higher orders in the form of non-local tensor invariants [27], which are terms that modify the general theory of relativity.

After the end of the presented analysis (2016), we found in the literature that the idea of constructing a physical theory with a variable parameter of the cosmological constant type was proposed by Dirac [28]. Dirac modified the theory of H. Weyl and showed that the electromagnetic action and the cosmological constant arise in field theory from the requirement of invariance of the action with respect to an extended class of space-time transformations. Above, we based on the mathematical principle of the kernel of the evolutionary equation (2) as a fundamental equation of physics, which leads to a more general physical theory.
One of the problems that can be solved by this method is the axiomatic finding of the modified action of gravitational theory in order to experimentally test it.

A more important consequence, however, is the final construction of covariant electrodynamics with a scale parameter such as the cosmological constant. Axiomatically defined covariant effective action (7) is a functional of physical fields. It is found exclusively by means of geometric analysis and has nothing to do with quantum field theory. Since the effective action is expressed in terms of the tensors of the observed fields, when varying in the metric, it generates a nonlocal energy-momentum tensor [29], which makes it possible to solve equations of evolutionary problems with the initial conditions, in particular, radiation problems. Recall that this method was originally intended to solve the Schwinger problem of particle production by an electromagnetic field and the problem of Hawking radiation in black hole physics. However, mathematics is universal, so the kernel of the evolutionary equation is applicable both in cosmology and in the physics of the condensed state of matter [30]. Many problems are still awaiting solution.

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