Research Article

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A random von Neumann theorem for uniformly distributed sequences of partitions

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Abstract: In this paper, we prove a theorem that confirms, under a supplementary condition, a conjecture concerning random permutations of sequences of partitions of the unit interval.

Keywords: number theory, discrepancy, uniformly distributed sequences of partitions, probability theory

MSC 2020: 11-xx, 40-xx, 60-xx

1 Introduction

The general study of uniformly distributed sequences of partitions was initiated in [1], inspired by a beautiful construction and result from the study by Kakutani [2]. The subject is closely related to the theory of uniformly distributed sequences, initiated in [3]. There are two classical references for the subject: [4] and [5].

Kakutani took the interval $I = [0, 1]$, a number $\alpha \in [0, 1]$ and divided the interval in proportion $\alpha : 1 - \alpha$. Then, he divided the longest interval of this partition in the same proportion and iterated the procedure dividing always the longest interval of the $n$th partition, so as to obtain a sequence of partitions of $[0, 1]$. If at a certain step there were two or more intervals of maximal length, they were divided simultaneously.

Kakutani proved that this sequence of partitions (denote it by $\{\alpha^n I\}$) is uniformly distributed, which means that if $\alpha^n I = \{0 < t^k_1 < t^k_2 < \cdots < t^k_{N_k} < 1\}$ is the $k$th partition, then

$$\lim_{k \to \infty} \frac{1}{N_k} \sum_{l=1}^{N_k} f(t^k_l) = \int_0^1 f(t) \, dt,$$

for every continuous function $f$.

In other words, the discrete measure concentrated in the points $t^k_l$ converges weakly to the Lebesgue measure on $[0, 1]$.

The construction has been generalized in [1]. Let $\rho$ be any non trivial finite partition of $I$.

In the first step, the longest interval(s) of $\rho$ is subdivided positively homothetically to $\rho$. The partition obtained in this manner is denoted by $\rho^2 I$. In the second step, the same procedure is repeated on the longest interval(s), operating with $\rho$ on $\rho^2 I$. Iteration of this procedure leads to a sequence of partitions $\{\rho^n I\}$.

If $\rho = \{[0, q], [\alpha, 1]\}$, one gets Kakutani’s sequence.

The following theorem includes the results of the study by Kakutani ([1], Theorem 2.7)).
Theorem 1. The sequence \( \{\rho^n\} \) is uniformly distributed.

There are interesting connections between the theory of u.d. sequences of partitions and u.d. sequences of points. This connection is far reaching in the construction of a significant subclass of \( \rho \)-refinements, the so-called LS-sequences. The subject was initiated by the present author in [6], and it is connected with the van der Corput sequences of points.

LS-sequences are constructed starting from the partition \( \rho_{LS} \) made of \( L + S \) intervals \((L \text{ and } S \text{ are positive integers})\) of length \( \beta \) and \( \beta^2 \), respectively, where \( \beta \) is the positive solution of the equation \( L\beta + S\beta^2 = 1 \).

The present paper is concerned with a result in the domain of uniformly distributed sequences of partitions, related to a proposition by von Neumann for uniformly sequences of points [7].

Theorem 2. If \( \{x_n\} \) is a dense sequence of points in \([0, 1]\), then there exists a rearrangement of these points, \( \{x_m\} \), which is uniformly distributed.

One of the consequences of von Neumann’s result is that there are many u.d. sequences of points.

We will now introduce the definitions we need.

Definitions. Given a partition \( \pi = \{[t_{i-1}, t_i], 1 \leq i \leq N\} \), we denote by \( l_i = t_i - t_{i-1} \) the length of its \( i \)-th interval.

The diameter of \( \pi \), denoted by \( L \), is equal to \( \max_{1 \leq i \leq N} l_i \).

Given a sequence of partitions \( \{\pi_k\} \), we say that it is dense if, denoted by \( L_k \) the diameter of \( \pi_k \), \( \lim_{k \to \infty} L_k = 0 \).

If \( \pi = \{[t_{i-1}, t_i], 1 \leq i \leq N\} \) is a partition, its random permutation is a partition \( \pi' = \{[s_{h-1}, s_h], 1 \leq h \leq N\} \) defined by the points \( s_h = \sum_{j=0}^{h} l_j \), for \( 0 \leq h \leq N \), where \( s_0 = 0 \) and the indices \( [i] \), are successively taken at random, with probability \( \frac{1}{N} \), from the set \( \{i : 1 \leq s \leq N\} \).

We will denote by \( \pi! \) the set of all the \( N! \) permutations of \( \pi \).

2 Main results

In a previous paper [8], we proved the following result.

Proposition 3. If \( \{\pi_n\} \) is a dense sequence of partitions, then there exists a sequence of partitions \( \{\sigma_n\} \), with \( \sigma_n \in \pi_n! \), which is uniformly distributed.

In the same paper, we made the following conjecture.

Conjecture. If \( \{\pi_n\} \) is a dense sequence of partitions and we select at random a partition \( \sigma_k \in \pi_k! \), then \( \{\sigma_k\} \) is uniformly distributed with probability 1.

We need some preliminary calculations.

Let \( q \in [0, 1[ \) and denote by \( N_k(q) \) the integer such that

\[
\frac{N_k(q)}{N_k} \leq q < \frac{N_k(q) + 1}{N_k}.
\]

Consider a sequence \( \{q_m\} \) of points, which is dense in \([0, 1]\). For later convenience, we will denote by \( N_k(m) \) the integer \( N_k(q_m) \).
Select at random from the $N_k$ intervals of $\pi_k$, with probability $\frac{1}{N_k}$. Denote by $\xi^k_i$ the length of the interval selected in the $i$th draw ($1 \leq i \leq N_k(m)$) and consider the random variable

$$\eta^m_k = \sum_{i=1}^{N_k(m)} \xi^k_i.$$

Obviously,

$$E(\eta^m_k) = \sum_{i=1}^{N_k(m)} E(\xi^k_i) = \sum_{i=1}^{N_k(m)} \frac{1}{N_k} = \frac{N_k(m)}{N_k},$$

hence,

$$|E(\eta^m_k) - q_m| \leq \frac{1}{N_k}.$$

It is easy to see that the second moment of $\eta^m_k$ is uniformly bounded for any sequence of partitions.

This, together with the independence of the $\eta^m_k$'s (for $k \in \mathbb{N}$), would allow us to apply the strong law of large numbers and to conclude that, when $k$ tends to infinity, the sequence $\eta^m_k$ tends to $q_m$ in the Cesàro mean (and nothing more, at least following this line of thought).

But this is not what we were looking for.

If we want to identify a class for which the conjecture is true, we have to make some assumptions on the sequence $\{\pi_k\}$.

A simple sufficient assumption is expressed as follows:

$$\sum_{i=1}^{\infty} L_i^2 < \infty.$$

**Theorem 4.** If the series of squares of diameters of $\{\pi_k\}$ is convergent, then its random permutations $\sigma_k$ are uniformly distributed with probability 1.

**Proof.** We have

$$\text{Var}(\eta^m_k) = E\left(\left(\sum_{i=1}^{N_k(m)} \xi^k_i - \sum_{i=1}^{N_k(m)} \frac{1}{N_k}\right)^2\right)$$

$$= E\left(\sum_{i=1}^{N_k(m)} (\xi^k_i)^2 - 2 \sum_{i=1}^{N_k(m)} \xi^k_i \frac{1}{N_k} + \sum_{i=1}^{N_k(m)} \frac{1}{N_k^2}\right)$$

$$= E\left(\sum_{i=1}^{N_k(m)} (\xi^k_i)^2\right) - 2 \sum_{i=1}^{N_k(m)} E\left(\xi^k_i \frac{N_k(m)}{N_k}\right) + E\left(\sum_{i=1}^{N_k(m)} \frac{1}{N_k}\right)$$

$$\leq E\left(\sum_{i=1}^{N_k} (\xi^k_i)^2\right) - 2 \sum_{i=1}^{N_k} E\left(\xi^k_i \frac{N_k}{N_k}\right) + \frac{N_k}{N_k^2}\right)$$

$$= E\left(\sum_{i=1}^{N_k} (\xi^k_i)^2\right) - 2 \frac{N_k}{N_k} \sum_{i=1}^{N_k} \xi^k_i + \sum_{i=1}^{N_k} \frac{1}{N_k}\right)$$

$$= E\left(\sum_{i=1}^{N_k} (\xi^k_i)^2\right) - 2 \frac{N_k}{N_k} \sum_{i=1}^{N_k} \xi^k_i + \sum_{i=1}^{N_k} \frac{1}{N_k}\right)$$

$$\leq E\left(\sum_{i=1}^{N_k} (\xi^k_i)^2\right) - 2 \frac{1}{N_k} \sum_{i=1}^{N_k} \xi^k_i + \sum_{i=1}^{N_k} \frac{1}{N_k}\right)$$

$$\leq E\left(\sum_{i=1}^{N_k} (\xi^k_i)^2\right) - 2 \frac{1}{N_k} \sum_{i=1}^{N_k} \xi^k_i + \sum_{i=1}^{N_k} \frac{1}{N_k}\right)$$

$$= E\left(\sum_{i=1}^{N_k} (\xi^k_i)^2\right) - 2 \frac{1}{N_k} \sum_{i=1}^{N_k} \xi^k_i + \sum_{i=1}^{N_k} \frac{1}{N_k}\right)$$

$$\leq E\left(\sum_{i=1}^{N_k} (\xi^k_i)^2\right) - 2 \frac{1}{N_k} \sum_{i=1}^{N_k} \xi^k_i + \sum_{i=1}^{N_k} \frac{1}{N_k}\right)$$

$$\leq E\left(\sum_{i=1}^{N_k} (\xi^k_i)^2\right) - 2 \frac{1}{N_k} \sum_{i=1}^{N_k} \xi^k_i + \sum_{i=1}^{N_k} \frac{1}{N_k}\right)$$

$$\leq E\left(\sum_{i=1}^{N_k} (\xi^k_i)^2\right) - 2 \frac{1}{N_k} \sum_{i=1}^{N_k} \xi^k_i + \sum_{i=1}^{N_k} \frac{1}{N_k}\right)$$

$$\leq E\left(\sum_{i=1}^{N_k} (\xi^k_i)^2\right) - 2 \frac{1}{N_k} \sum_{i=1}^{N_k} \xi^k_i + \sum_{i=1}^{N_k} \frac{1}{N_k}\right)$$

Apply now the Čebyšev inequality. By our assumption, we have, for every $\varepsilon > 0$ (and every $m$),

$$\sum_{k=1}^{\infty} P(|\eta^m_k - E(\eta^m_k)| > \varepsilon) \leq \sum_{n=1}^{\infty} \frac{\text{Var}(\eta^m_k)}{\varepsilon^2} < \infty.$$
Recalling that $E(n^m_k)$ tends to $q_m$ and applying the Borel-Cantelli lemma [9, Theorem 4.2.1], we obtain that

$$\lim_{k \to \infty} n^m_k = q_m$$

almost surely for every $m \in \mathbb{N}$.

The set $\{q_m\}$ is countable; therefore, the aforementioned limit holds almost surely for all the values of $m$ simultaneously.

Observe now that $\lim_{k \to \infty} \frac{N_k(q)}{N_k}$ is an increasing function of $q$. Therefore, it follows that, almost surely,

$$\lim_{k \to \infty} \frac{N_k(q)}{N_k} = q$$

for every $q \in [0, 1]$.

In other words, the empirical distribution function $F_k$ of $\sigma_k$ tends almost surely to the distribution function of the random variable $U$ uniformly distributed on $[0, 1]$.

On the other hand, convergence in distribution is known to be equivalent to weak convergence, so the desired conclusion follows. □

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