Chern classes and the periods of mirrors.

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Abstract. We show how Chern classes of a Calabi Yau hypersurface in a toric Fano manifold can be expressed in terms of the holomorphic at a maximal degeneracy point period of its mirror. We also consider the relation between the Chern classes and the periods of mirrors for complete intersections in Grassmanian Gr(2, 5).

0. Introduction.

A relation between Chern classes of 3-dimensional Calabi Yau complete intersections in a product of weighted projective spaces and periods of holomorphic 3-forms on the mirror was described in [HKTY] where the following “remarkable identities” were observed. Let \(X\) be a 3-dimensional Calabi-Yau complete intersection in a product of \(k\) weighted projective spaces. Let \(w_0(z_1, \ldots, z_k) = \sum_{n_i \geq 0} c(n_1, \ldots, n_k) z_1^{n_1} \ldots z_k^{n_k}\) be the period of a holomorphic 3-form on \(X\) which admits a holomorphic extension into the point with a maximally unipotent monodromy normalized so that it takes there the value 1. Then \(c(n_1, \ldots, n_k)\) is a product of \((l_j(n_1, \ldots, n_k)!)^{\pm 1}\) where \(l_j\) are linear forms with integer coefficients. Define \(c(\ldots, \rho_i, \ldots)\) replacing in \(c(n_1, \ldots, n_k)\) each \((l_j)\) by \(\Gamma(l_j(\ldots, \rho_i, \ldots) + 1)\). For \(i = 1, \ldots, k\) let \(J_i\) be the class of the pull back on \(X\) of the Kahler form of \(i\)-th projective space which product contains \(X\). Let \(K_{ijk}\) be the Yukawa coupling, \(\partial_{\rho_i} = \frac{\partial}{\partial \rho_i}, D_2 = \frac{1}{2} K_{ijk} \partial_{\rho_j} \partial_{\rho_k}, D_3 = -\frac{1}{6} K_{ijk} \partial_{\rho_i} \partial_{\rho_j} \partial_{\rho_k}\). Then (cf. (4.21) in [HKTY])

\[
\int_X c_2 \wedge J_i = -24 D^{(2)} c(\rho_1, \ldots, \rho_k)_{(0, \ldots, 0)} \tag{0.1}
\]

\[
\int_X c_3 = i \frac{2\pi^3}{\zeta(3)} D^{(3)} c(\rho_1, \ldots, \rho_k)_{(0, \ldots, 0)} \tag{0.2}
\]

The purpose of this note is to extend this relationship to arbitrary dimension. We shall consider the case of hypersurfaces in toric Fano manifolds. For these Calabi Yau manifolds, we show that certain linear combinations of Chern classes can be expressed in terms of the period of the mirror holomorphic at the maximal degeneracy point. These combinations form a Hirzebruch’s multiplicative sequence i.e. obtained by applying Hirzebruch construction [Hi] to the series \(\frac{1}{\Gamma(1+z)}\). This multiplicative \(\Gamma\)-sequence, in turn, determines the Chern classes of the manifold (cf. lemma below).

More precisely we prove the following. Let \(\Delta\) be a reflexive Fano polytope (cf. [B1],[B2]) of dimension \(d\) (in other words \(\Delta\) is a simplicial polytope with: (a) vertices

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belonging to an integral lattice \( M \) of rank \( d \); (b) the origin being the only intersection point of the interior of \( \Delta \) and the lattice \( M \); (c) such that for each \( (d-1) \)-dimensional face its vertices form a basis of \( M \) and (d) the equation of this face is \( l = -1 \) where \( l \) is a linear form on \( M_l \). The corresponding toric variety \( X_\Delta \) is a Fano manifold i.e. its first Chern class is ample and the hypersurface which is the zero locus of a section of the line bundle on \( X_\Delta \) corresponding to \( \Delta \) is a Calabi Yau hypersurfaces \( V_\Delta \). Let \( \Delta^* \subset M^* \otimes \mathbb{R} \) \((M^* = \text{Hom}(M, \mathbb{Z}))\) be the polar polytope and \( f_{\Delta^*} \) be a generic linear combination of characters of \( M \otimes \mathbb{C}^* \) which belong to \( \Delta^* \). Then a period of the affine hypersurface \( f_{\Delta^*} = 0 \) in \( \text{Hom}(M^*, \mathbb{C}^*) \) i.e. \( \int_\gamma \frac{1}{f_{\Delta^*}} \prod \frac{dx_i}{x_i} \) (where \( x_i \)'s are coordinates in \( \text{Hom}(M^*, \mathbb{C}^*) \)) corresponding to a choice of a basis in \( M \) and \( \gamma \) is a \( d \)-cycle in the complement to \( f_{\Delta^*} = 0 \) in \( \text{Hom}(M^*, \mathbb{C}^*) \) satisfies a system of Picard Fuchs PDE. In appropriate partial compactification of \( \text{Hom}(M^*, \mathbb{C}^*) \) (constructed in \([HLY]\)) there is a maximal degeneracy point i.e. such that this system has only one, up to a constant factor, solution which admits a holomorphic extension into this point. Near the maximal degeneracy point this period has form \( \Sigma \) i.e. such that this system has only one, up to a constant factor, solution which admits a holomorphic extension into this point. Near the maximal degeneracy point this period has form \( \Sigma \). Let \( \rho_1, ..., \rho_r \in \mathbb{N}^> \prod i_i l_i(n_1, ..., n_r)^{\pm 1} z_1^{n_1} \ldots z_r^{n_r} \) where \( l_i \) are linear forms with integer coefficients. Let \( c_{\rho_1, ..., \rho_r} = \Gamma(l_i(\rho_1, ..., \rho_r) + 1)^{\pm 1}. \) We shall identify the tangent space to the compactification of the moduli space of hypersurface \( X_{\Delta^*} \) at a maximal degeneracy point with \( H^2(X_{\Delta}, \mathbb{C}) \) and coordinates \( (t_1, ..., t_d) \) of \( H^2(X_{\Delta}, \mathbb{C}) \) in a basis \( J_1, ..., J_d \) with the coordinates \( (x_1, ..., x_r) \) in the partial compactification of the moduli space of hypersurfaces \( V_\Delta^* \) near a maximal degeneracy point. Let \( K_{i_1, ..., i_d} = \int_{X_{\Delta^*}} \Omega \wedge \frac{\partial^4 \Omega}{\partial x_{i_1} \ldots \partial x_{i_d}} \) be the \( d \)-point function corresponding to \( V_\Delta^* \) (cf. \([GMP], [BvS]\)). Its value at the maximal degeneracy point is normalized to be equal to \( \int_{V_\Delta} J_{i_1} \wedge ... \wedge J_{i_d} \) (with such normalization it yields the data of enumerative geometry of rational curves on \( X_\Delta \) cf. \([GPM]\)). Then we have:

**Theorem.** If the assumption (***) below is satisfied, the degree \( k \) polynomial \( Q_k(c_1, ..., c_k) \) in the Hirzebruch’s multiplicative sequence corresponding to the series \( \frac{1}{\Gamma(1+z)} \) of \( V_\Delta \) satisfies:

\[
\int_{X_{\Delta}} Q_k(c_1, ..., c_k) \wedge J_{i_1} \wedge ... \wedge J_{i_{n-k}} = \sum_{(j_1, ..., j_k, i_1, ..., i_{n-k})} \frac{1}{k!} \frac{\partial^k c(\rho_1, ..., \rho_k)}{\partial \rho_{j_1} \ldots \partial \rho_{j_k}} |_{\rho_{j_1}=0, ..., \rho_{j_k}=0, K_{j_1, ..., j_k, i_1, ..., i_{n-k}}} \tag{0.3}
\]

where the summation is over all permutations \( (j_1, ..., i_n) \) of \((1, .., n)\).

As a corollary we obtain an expression for the Chern classes in terms of the period and the \( d \)-point function of the mirror. Multiplicative \( \Gamma \)-sequence is discussed in section 1, the proof of the theorem in section 2 where also special cases of it are made explicit. In section 3 we discuss the simplest example of the theorem and relation between the periods on Chern classes for manifolds which mirror was constructed recently in \([BCKS]\) via conifold transitions.
1. Multiplicative $\Gamma$-sequence.

Let $Q(z) = \frac{1}{\Gamma(1+z)}$. Since $Q(0) = 1$ there is a well defined multiplicative sequence corresponding to this power series (cf. [Hi]). The latter associates with a complex manifold $M$ having Chern classes $c_i, i = 1, \ldots \dim M$ weighted homogeneous polynomials $Q_i(c_1, \ldots c_i)$ of degree $2i$ where $deg c_i = 2i$.

**Lemma.** The coefficient of $c_i$ in $Q_i$ is equal to $\zeta(i)$ for $i \neq 1$ and to the Euler constant $\gamma = \lim_{n \to \infty} \sum \frac{1}{n} - \ln n$ for $i = 1$. In particular it is non zero and hence $c_i$ is a the polynomial in $Q_j(c_1, \ldots c_i)$, ($j \leq i$).

**Proof.** The coefficient of $c_i$ in $Q_i(c_1, \ldots c_i)$ is $s_i$ where

$$1 - z \frac{d}{dz} \log Q(z) = \sum_{i=0}^{\infty} (-1)^i s_i z^i$$

(cf. [H] sect. 1.4). In the case of $\Gamma$-sequence we have (cf. [E] p. 45)

$$\log \Gamma(1+z) = -\gamma z + \sum_{i=2}^{\infty} (-1)^i \zeta(i) z^i / i$$

This yields the lemma.

It isn’t hard to work out explicitly the polynomials $Q_i$ (using for example the recurrence formulas from [LW]). For small $i$ we have the following ($\gamma$ is the Euler constant $\lim_{m \to \infty} \sum \frac{1}{n} - \ln n$):

$$Q_1(c_1) = \gamma c_1$$

$$Q_2(c_1, c_2) = -\frac{1}{2} \zeta(2)c_1^2 + \zeta(2)c_2 + \gamma^2 c_1^2$$

$$Q_3(c_1, c_2, c_3) = \zeta(3)c_3 - (\zeta(3) - \zeta(2)\gamma)c_1c_2 + \left(\frac{1}{3}\zeta(3) + \frac{1}{6}\gamma^3\right)c_1^3$$

$$Q_4(c_1, c_2, c_3, c_4) = \zeta(4)c_4 + \frac{1}{2}(\zeta(2)^2 - \zeta(4))c_2^2 + (-\zeta(4) + \zeta(3)\gamma)c_1c_3 +$$

$$\left(\zeta(4) - \zeta(3)\gamma + \frac{1}{2}\zeta(2)\gamma^2 - \frac{1}{2}\zeta(2)c_1^2c_2 + \left(-\frac{1}{4}\zeta(4) - \frac{1}{4}\zeta(2)\gamma^2 + \frac{1}{3}\zeta(3)\gamma + \frac{1}{8}\zeta(2)^2 + \frac{1}{24}\gamma^4\right)c_1^4$$

In particular for a Calabi Yau manifold we have the following polynomials:

$$Q_2(c_2) = \zeta(2)c_2, Q_3(c_2, c_3) = \zeta(3)c_3, Q_4(c_2, c_3, c_4) = \frac{1}{2}(\zeta(2)^2 - \zeta(4))c_2^2 + \zeta(4)c_4$$

2. Periods for the mirrors of hypersurfaces in toric varieties.

We shall use the construction of the maximal degeneracy point and notations from [HLY]. Let $M$ be a lattice $\mathbb{Z}^d$ and $N$ denotes its dual. A cone in $N \otimes \mathbb{R}$ is called large if it is generated by finitely many vectors from $N$ and its dimension is equal to $\text{rk} N$. A large
A fan $\Sigma$ in $N \otimes \mathbb{R}$ is called regular if all its large cones are regular. We consider the fan $\Sigma$ corresponding to a Fano polytope $\Delta \subset M \otimes \mathbb{R}$. The intersections of the cones from $\Sigma$ and the polytope $\Delta^*$ polar to $\Delta$ form a (maximal, cf. [HLY]) triangulation of the latter.

As was mentioned the corresponding toric variety $X_{\Delta}$ is a non-singular Fano manifold.

Let $\Sigma(1)$ be the set of primitive elements on the edges of the fan $\Sigma$. Since $\Delta$ is a Fano polytope the following property (*) (assumed in [HLY], cf. 4.7) is satisfied: $\Sigma(1)$ forms a (maximal, cf. [HLY]) triangulation of the latter.

Let $\Sigma$ be a Fano polytope the following property (*) (assumed in [HLY], cf. 4.7) is satisfied: $\Sigma(1)$ forms a (maximal, cf. [HLY]) triangulation of the latter. Moreover the fan $\Sigma$ admits a natural refinement called the secondary fan $S\Sigma$ (cf. [GKZ]). Moreover the fan $S\Sigma$ admits a natural refinement called the Grobner fan (cf. [HLY] section 4). We shall assume that:

\[ (** \) the cone $C(\Sigma)$ in $L_{\Sigma}$ generated by $\mu_1, \ldots, \mu_p$ is a cone of both the secondary and Grobner fans and that the generators of the edges of this cone form a basis of the integral lattice of $L_{\Sigma}$.\]

The space $PL(\Sigma)$ of piecewise linear functions on $N \otimes \mathbb{R}$ linear on each cone of $\Sigma$ can be identified with a subspace of $N^*$. Moreover $L_{\Sigma}^* = N^*/M \otimes \mathbb{R} = PL(\Sigma)/M \otimes \mathbb{R}$ and the latter is canonically isomorphic to $H^2(X_{\Delta}, \mathbb{R})$. Lefschetz theorem identifies this group with $H^2(V_{\Delta}, \mathbb{R})$ if $d \geq 3$. In this identification the Kahler cones of $X_{\Delta}$ and $V_{\Delta}$ are identified with $C(\Sigma)$.

According to [HLY] a consequence of (***) is that in the partial compactification of $C^* \times C^* / Hom(N, C^*)$, given by the fan consisting of the cones in the closure of the Kahler cone, the point corresponding to the cone $C(\Sigma)$ is a maximal degeneracy point for the GKZ system:

\[
\sum_{l_\mu > 0} (\frac{\partial}{\partial a_\mu})^{l_\mu} - \sum_{l_\mu < 0} (\frac{\partial}{\partial a_\mu})^{-l_\mu} \Pi(a) = 0 (l \in L_{\Sigma}), (\sum_{\mu} < u, \bar{\mu} > a_\mu \frac{\partial}{\partial a_\mu} - < u, \beta >) \Pi(a) = 0
\]

(2.1)

where $\beta = (-1, 0, \ldots, 0)$ and $u \in M \otimes \mathbb{R}$. In other words there is only one solution of (2.1) which admits a holomorphic extension in a neighborhood of the point of compactification corresponding to the cone $C(\Sigma)$. Moreover the period:

\[
\Pi(\gamma) = \int_\gamma \frac{1}{f_{\Sigma}} \prod_i \frac{dX_i}{X_i}, \quad f_{\Sigma} = \sum_{\mu} X_\mu
\]

(2.2)

$(\gamma$, as above, is a $d = rkN$ cycle in the complement in $Hom(N, C^*)$ to the hypersurface $f_{\Sigma}(X, a) = 0$ is a solution of the system (2.1). Since we assume that $\Delta$ is Fano in fact any solution of (2.1) is a period (cf. [H]).
On the other hand we have the following series representation:

\[ a_0 \Pi_\gamma(a) = (2\pi i)^d \sum_{l_1, \ldots, l_p, \mu \geq 0, l_1, \ldots, l_p = 0} (-1)^{l_1 + \ldots + l_p} \frac{a_1^{l_1} \cdots a_p^{l_p}}{(l_1)! \cdots (l_p)!} \]  

(2.3)

The summation in the latter can be changed into summation over the Mori cone in \( L_A \) by assigning to \((l_1, \ldots, l_p)\) the vector \((l_0, l_1, \ldots, l_p) \in L_A \) where \( l_0 = -l_1 - \ldots - l_p \). If \( l^{(1)}, \ldots, l^{(p-d)} \) is a basis in the Mori cone then in corresponding canonical coordinates (cf. (3.1) \([HLY]\))

\[ x_k = (-1)^{l_0^{(k)}} a^{l^{(k)}} \]

we have (cf. (5.12) in \([HLY]\))

\[ a_0 \Pi_\alpha(x_1, \ldots, x_{p-d}) = \sum_{m_1, \ldots, m_{p-d}, \Sigma m_k l_0^{(k)} \leq 0} \frac{\Gamma(-\Sigma m_k l_0^{(k)} + 1)}{\Gamma(\Sigma m_k l_1^{(k)} + 1) \cdots \Gamma(\Sigma m_k l_p^{(k)} + 1)} x_1^{m_1}, \ldots, x_{m_{p-n}}^{m_{p-n}} \]  

(2.4)

Let \( J_i (i = 1, \ldots, p - d) \) be elements of \( L_A^* \) forming the basis dual to \( l^{(k)} \) and \( D_i, (i = 1, \ldots, p) \) be the cohomology classes in \( H^2(X_\Delta, \mathbb{Z}) \) dual to codimension one orbits of \( X_\Delta \). Since \( D_i \) correspond to the generators of one dimensional cones of \( \Sigma \) and under identification \( H^2(V_\Delta, \mathbb{Z}) \) correspond to \( \mu_i \) we have \( D_i = \Sigma_k J_k l_i^{(k)} \), \((i = 1, \ldots, p)\) and the total Chern class of the Calabi Yau hypersurface in \( X_\Delta \), which has as the the dual cohomology class \( D_1 + \ldots D_p \), is the sum of the terms of degree not exceeding \( d \) in the expansion of

\[ \frac{(1 + D_1) \cdots (1 + D_p)}{(1 + D_1 + \ldots + D_p)} \]  

(2.5)

Since in (2.4) one can view \( m_k \) as elements of \( L_A^* \) and hence identified them with \( J_k \)'s, it follows that the term \( Q_k(c_1, \ldots, c_k) \) of the degree \( k \) of \( \Gamma \)-sequence coincides with the term of degree \( k \) in the formal series

\[ C(J_1, \ldots J_{n-p}) = \frac{\Gamma(-\Sigma J_k l_0^{(k)} + 1)}{\Gamma(\Sigma J_k l_1^{(k)} + 1) \cdots \Gamma(\Sigma J_k l_p^{(k)} + 1)} \]  

(2.6)

The latter is equal to

\[ \sum_{j_1, \ldots, j_k} \frac{1}{k!} \frac{\partial^k C(J_1, \ldots, J_k)}{\partial J_{j_1} \cdots \partial J_{j_k}} J_{j_1} \cdots J_{j_k} \]  

(2.7)

The theorem follows from comparison (2.4) and (2.6).

**Corollary 1.** (cf. (0.1),(0.2),[HKTY].) Let \( X \) be a Calabi Yau hypersurface of dimension 3. Then

\[ \int_X c_2 \wedge J_i = \frac{3}{\pi^2} K_{ijk} \frac{\partial^2 c(0, \ldots, 0)}{\partial x_i \partial x_j \partial x_k} \]  

(2.8)

\[ \int_X c_3 = \frac{6}{\zeta(3)} K_{ijk} \frac{\partial^3 c(0, \ldots, 0)}{\partial x_i \partial x_j \partial x_k} \]  

(2.9)
2. Let $X$ be a Calabi-Yau hypersurface of dimension 4 in a non-singular toric Fano manifold. Then

$$\int_X c_2 \wedge J_i \wedge J_j = \frac{3}{\pi^2} K_{ijkl} \frac{\partial c^2(0, \ldots, 0)}{\partial \rho_i \partial \rho_j}, \quad \int_X c_3 \wedge J_i = \frac{6}{\zeta(3)} K_{ijkl} \frac{\partial^3 c(0, \ldots, 0)}{\partial x_i \partial x_j \partial x_k}$$

(2.10)

$$\int_X \left( \frac{1}{2} \zeta(2)^2 - \zeta(4) \right) c_2^2 + \zeta(4) c_4 = \frac{1}{24} K_{ijkl} \frac{\partial c^4(0, \ldots, 0)}{\partial \rho_i \partial \rho_j \partial \rho_k \partial \rho_k}$$

(2.11)

These identities are a consequences of (0.3) and identities (1.7).

3. Concluding remarks.

1. **Example.** For a hypersurface $V_d$ of degree $d$ in $\mathbb{P}^d$ the Chern polynomial is the sum of the terms degree less than $d + 1$ in the series $\frac{(1+h)^{d+1}}{1+3h}$ i.e. the $\Gamma$-sequence for $V_d$ is the sum of the terms of less than $d + 1$ in $\frac{\Gamma(1+dh)}{\Gamma(1+2dh)}$. (0.3) is a consequence of the fact that after replacing $h$ by $m \in \mathbb{Z}^+$ the latter becomes the coefficient of the holomorphic at the maximal degeneracy point period of the $d$-form on the mirror of $V_d$.

2. Let $X_{1,1,3}$ (resp. $X_{1,2,2}$) be a non-singular complete intersection of the Grassmanian $Gr(2,5)$ embedded via Plucker embedding in $\mathbb{P}^9$ and hypersurfaces of degrees $(1, 1, 3)$ (resp. $(1, 2, 2)$). It is shown in [BKCS] that holomorphic period of the mirror has presentation

$$\sum_m \left[ (3m)!m!^2 \sum_{r,s} \frac{1}{m!^2} \binom{m}{r} \binom{s}{r} \binom{m}{s}^2 \right] z^m$$

(3.1)

resp.

$$\sum_m \left[ m!(2m)!^2 \sum_{r,s} \frac{1}{m!^2} \binom{m}{r} \binom{s}{r} \binom{m}{s}^2 \right] z^m$$

(3.2)

Though the corresponding differential equations are not of hypergeometric type (since they have more than three singular points) and the ratio of coefficients is not a rational function of $m$, nevertheless the ratio of the coefficients of the series (3.1) and (3.2) is equal to the value of the ratio of Hizrebruch $\Gamma$ sequences for $X_{1,1,3}$ and $X_{1,2,2}$. Indeed the Chern polynomials of the these manifolds are restrictions on cohomology of corresponding manifolds of $\frac{c(Gr(2,5))}{(1+h)^2(1+3h)}$ and $\frac{c(Gr(2,5))}{(1+h)(1+2h)}$ respectively where $c(Gr(2,5))$ is the Chern polynomial of the Grassmanian and $h$ is the cohomology class of the hyperplane section i.e. the ratio of the $\Gamma$-sequences is $\frac{\Gamma(3h+1)\Gamma(h+1)}{\Gamma(2h+1)}$, which for after replacing $h$ by $m \in \mathbb{Z}^+$ is equal to the ration of the coefficients of the periods of mirrors of corresponding Calabi Yau manifolds.
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