1. Introduction

1.1. Physical background. The transport of pollutants in subsurface environments is a complex phenomenon modeled by advection-diffusion-reaction equations describing the evolution of contaminant concentrations in porous medium through various mechanical and chemical processes. In what follows, we will briefly describe the equations governing the transport of one or more contaminants through an fluid-saturated porous medium, i.e. a medium characterized by a partitioning of its total volume into a solid phase (solid matrix) and a void or pore space that is filled by one or more fluids. For a more detailed description, we refer to [5].

Let \( \Omega \subset \mathbb{R}^n \) be a domain occupied by a porous medium, let \( z = z(x,t) \) denote the solute concentration of one contaminant component in the fluid phase and assume that the flow is at steady state and that the transport is described by advection, molecular diffusion, mechanical dispersion and chemical reaction (adsorption) between a solute and the surrounding porous skeleton. More than often the adsorption, accumulation of a pollutant on the solid matrix at the fluid-solid interface, is in fact the main mechanism responsible for the contaminant transport in soil. Based on a continuum approach, the mass conservation for \( z \) can be written as

\[
\phi \frac{\partial z}{\partial t} + \rho \frac{\partial b_a}{\partial t} + \nabla \cdot (z \mathbf{V} - \phi \mathbf{D} \nabla z) = f, \tag{1.1}
\]

where \( \phi \in (0, 1) \) is the constant porosity, \( z \mathbf{V} \) denotes the advective water flux (\( \mathbf{V} \) is the Darcy velocity), \( \rho > 0 \) is the constant bulk density of the solid matrix, \( \mathbf{D} \) stands for the hydrodynamic dispersion matrix describing both the molecular diffusion and the mechanical dispersion between the solute and the surrounding porous medium, and the source or sink terms are denoted by \( f \). Moreover, an isotherm \( b_a = b_a(z) \) describes the concentration of contaminant adsorbed on the solid matrix through a reactive adsorption process at constant temperature, which is assumed here to be infinitely fast (equilibrium). The most commonly used nonlinear equilibrium isotherms [5, 28] are

\[
b_a(z) = K z^p \quad \text{Freundlich,}
\]

\[
b_a(z) = K \frac{z}{1 + z} \quad \text{Langmuir,}
\]
where $K > 0$ is constant and the Freundlich exponent $p$ is generally chosen in $(0, 1)$ (the smaller the $p$, the higher the adsorption at low concentrations).

The values $p \in (0, 1)$ make (1.1) singular at $z = 0$ because, at least formally, \( \partial_t b_a(z) = b'_a(z) \partial z \) and \( b'_a(0) = \infty \). The equation may thus exhibit finite speed of propagation of compactly supported initial solutions giving rise to free boundaries that separate the region where the solute concentration vanishes from that with positive concentration. This is in marked contrast with the behavior of solutions for Freundlich exponents $p \geq 1$ or Langmuir isotherms, since the equation becomes nonsingular and the information propagates with infinite speed as usual for uniformly parabolic equations.

Equation (1.1), complemented with suitable initial and boundary conditions, and all its variants arising from different equilibrium and non-equilibrium, linear or non-linear, isotherms, have attracted considerable attention over the last 20 years, both from an analysis and numerical simulation point of view, see, e.g., [1, 3, 4, 9, 10, 19, 26]. It is the equilibrium Freundlich isotherm, however, which makes the problem most challenging due to the degeneracy/singularity.

In the multi-species case (transport of several contaminants), $z = (z_1, \ldots, z_N)$, is a vector-valued function but the evolution of the concentration of each component is still described by (1.1). The adsorption process is now competitive (different species competing for the same adsorption sites), thus leading to a coupled system of PDEs, and the competitive adsorption process is modeled by a multi-component isotherm, see [1, 20, 24, 25, 29] for a review of competitive equilibrium adsorption modeling. The most common multicomponent adsorption isotherms $b_a(z) = (b_{a1}(z), \ldots, b_{aN}(z))$ are

\[
\begin{align*}
    b_{a1}(z) &= K_1 \left( \sum_{j=1}^{N} \alpha_{ij} z_j \right)^{p_i - 1} z_i & \text{Freundlich,} \\
    b_{ai}(z) &= K_i \frac{1}{1 + \sum_{j=1}^{N} \alpha_{ij} z_j} z_i & \text{Langmuir,}
\end{align*}
\]

where $K_i, \alpha_{ij}, p_i$ are non-negative parameters. The $\alpha_{ij}$ are dimensionless competition coefficients describing the inhibition of species $i$ to the adsorption of species $j$. By definition $\alpha_{ii} = 1$ and the values for $\alpha_{ij}$ vary normally between zero (no competition between species $i$ and $j$) and ten, cf. [18].

As a first approximation, and to simplify the subsequent mathematical analysis, we consider here $K_i = K > 0$, $\alpha_{ij} = 1$, and $p_i = p \in (0, 1)$. The resulting isotherms read as

\[
\begin{align*}
    b_a(z) &= K |z|_1^{p-1} z & \text{Freundlich,} \\
    b_a(z) &= \frac{K}{1 + |z|_1} z & \text{Langmuir,}
\end{align*}
\]

(1.2)

where $|z|_1 = \sum_{i=1}^{N} |z_i|$ denotes the usual $l^1(\mathbb{R}^N)$ norm. In fact, we shall handle slightly more general isotherms, encompassing both the Freundlich and Langmuir ones, in a unified framework (see the next section). Neglecting the hydrodynamical effects ($\mathbf{V} = 0$) and assuming that $\mathbf{D} = \text{diag}\{d, d, \ldots, d\}$ for some diffusion coefficient $d > 0$, the remaining constants in the multicomponent version of (1.1) can be eliminated by a simple redefinition of the variables. This leads to the model problem

\[
\partial_t b(z) = \Delta z + f,
\]

(1.3)

where the nonlinearity $b(z) = z + b_a(z)$ takes into account both the inertial and adsorption effects.
1.2. Structural assumptions. In their celebrated work [2], Alt and Luckhaus studied systems of elliptic-parabolic PDEs including the b-term as in (1.3) and also nonlinear $p$-Laplacian type diffusion, Stefan problems, and reaction terms $f = f(x, t, z)$. Their analysis requires however a particular monotone structure which restricts the b-term to be of the form $b = D_2 \varphi$ for some convex potential $\varphi$ satisfying certain structural assumptions. In our case, the dependence of the Freundlich and Langmuir isotherms (1.2) on $z$ through the $l_1(\mathbb{R}^N)$-norm precludes any such monotonicity and, therefore, the results of [2] seem to be of no use here.

Though system (1.3) is formally parabolic for non-negative solutions, it is readily checked that the ellipticity fails for general solutions due to this dependence on the $l_1(\mathbb{R}^N)$ norm. A direct approach by Galerkin approximation as in [2] would produce here approximate solutions whose componentwise sign cannot be controlled uniformly. Since ellipticity fails for sign-changing components, the sequence of projected solutions does not enjoy enough compactness, hence the method from [2] cannot be adapted.

Throughout the whole paper we assume that $(1.4)$

$$b(z) = B(|z|_1)z$$

for some $B : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\beta(r) := B(|r|)r \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$$

and

$$(H_1) \quad \beta(0) = 0, \quad \beta(\pm \infty) = \pm \infty, \quad \beta'(r) > 0 \text{ for } r \neq 0.$$ 

Note that this includes the Freundlich and Langmuir isotherms (1.2), allows for blow-up $D_2 b(0) \sim \infty$, and that by definition $|b(z)|_1 = \beta(|z|_1)$. Most importantly, this type of nonlinearity possesses a structure which will allow us to control each component $z_i$ in terms of the scalar quantity $|z|_1$ as discussed in the next section.

It is more convenient for our analysis to move the nonlinearity from the time derivative to the space derivative in (1.3) through the change of variables $u = b(z)$. Towards that end, we first observe that since $\beta$ is monotone increasing, the inverse

$$\Phi = \beta^{-1}$$

is well-defined and continuous with $\Phi(0) = 0$. This allows us to invert

$$b(z) = u \iff z = \Phi(|u|_1) \frac{u}{|u|_1}$$

and rewrite problem (1.3) in the equivalent form

$$(1.5) \quad \partial_t u = \Delta \left( \frac{\Phi(|u|_1)}{|u|_1} u \right) + f.$$ 

We shall refer to $u$ as density, whereas we shall speak of concentration when dealing with the original $z$ variable. Note that, in the scalar case and for nonnegative solutions, (1.5) is the celebrated Generalized Porous Media Equation (GPME in short) $\partial_t u = \Delta \Phi(u) + f$. As is classical by now [7, 27] for GPME, we further assume that $\Phi \in C^1(\mathbb{R})$ satisfies the structural condition

$$(H_2) \quad s \in \mathbb{R} : \quad 1 \leq \frac{s\Phi'(s)}{\Phi(s)} \leq \frac{1}{a},$$ 

for some structural constant $a \in (0, 1)$. We also make the more specific assumption

$$(H_3) \quad s \in \mathbb{R} : \quad \frac{s\Phi''(s)}{\Phi'(s)} \geq \frac{1}{a},$$ 

which does not appear in scalar problems and will only be used in energy considerations. It is well-known that the scalar GPME is degenerate at $u = 0$ if $\Phi'(0) = 0$. 

and strictly parabolic if \( \Phi'(0) > 0 \), which can be seen from the divergence form \( \Delta \Phi(u) = \text{div}(\Phi'(u) \nabla u) \). Note in particular that the structural lower bound in \((H_2)\) includes both the degenerate slow diffusion \( \Phi'(0) = 0 \) and the nondegenerate case \( \Phi'(0) > 0 \), but the assumption \( \Phi \in C^1(\mathbb{R}) \) excludes fast diffusion \( \Phi'(0) = +\infty \).

We shall deal with the degenerate and the nondegenerate diffusions simultaneously in a unified framework, except in Section 4 where we discuss the existence of free boundaries and consequently restrict ourselves to slow diffusions \( \Phi'(0) = 0 \). Note also that in view of \((H_2)\) the function \( \Phi(s)/s \) is continuous and nondecreasing in \( \mathbb{R} \) with at most algebraic growth

\[
0 < s_1 < s_2 : \quad \Phi(s_2)/s_2 \leq \frac{(s_2/s_1)^{1/2}}{\Phi(s_1)/s_1},
\]

all properties that will be crucial in the subsequent analysis just as in the standard theory for GPME. The structural assumptions \((H_1)-(H_2)-(H_3)\) are easily verified for the physical Freundlich isotherm when \( p \in (0,1) \), as well as for the Langmuir isotherm. With these structural assumptions the blowup \( D_2 b(0) \sim \infty \) corresponds now to slow diffusion \( \Phi'(0) = 0 \), but linear diffusion \( \Phi(s) = s \) is also allowed. In fact, the Freundlich isotherm \( b_f(z) = (1 + |z|^{p-1})z \) behaves like \( z \) for large \( |z|_1 \), hence \( \beta(r) \sim r \) and \( \Phi(s) = \beta^{-1}(s) \sim s \) for large \( r \gg 1 \Leftrightarrow s \gg 1 \). In other words, the system \( \partial_t b_f(z) = \Delta z + f \) behaves as \( N \) uncoupled linear heat equations \( \partial_t z \approx \Delta z + f \) for large \( |z|_1 \). From the physical point of view, roughly speaking, this means that for very large concentrations the porous rock matrix saturates and the adsorption phenomena become negligible compared to inertial effects.

### 1.3. Contents and main results.

Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set with smooth boundary \( \partial \Omega \) and define \( Q_T = \Omega \times (0,T) \) and \( \Sigma_T = \partial \Omega \times (0,T) \) for fixed \( T > 0 \). For given boundary data \( z^D(x,t) = (z_1^D, \ldots, z_N^D)(x,t) \), initial condition \( z^0(x) = (z_1^0, \ldots, z_N^0)(x) \), and the resultant of forcing terms \( f(x,t) = (f_1, \ldots, f_N)(x,t) \), we consider the following two equivalent formulations, the first written for the original concentrations \( z = (z_1, \ldots, z_N) \)

\[
\begin{aligned}
\partial_t b(z) &= \Delta z + f \quad \text{in } Q_T \\
z(x,0) &= z^0(x) \quad \text{in } \Omega \\
z &= z^D \quad \text{in } \Sigma_T
\end{aligned}
\]

and the second one for the densities \( u = (u_1, \ldots, u_N) \)

\[
\begin{aligned}
\partial_t u &= \Delta \left( \frac{\Phi(|u|_1)}{|u|_1} u \right) + f \quad \text{in } Q_T \\
u(x,0) &= u^0(x) \quad \text{in } \Omega \\
\frac{\Phi(|u|_1)}{|u|_1} u &= z^D \quad \text{in } \Sigma_T
\end{aligned}
\]

where \( b(z) = u \Leftrightarrow z = \Phi(|u|_1)/|u|_1 \). As usual for the scalar GPME, the boundary conditions in the density formulation \((1.8)\) are enforced in terms of the physical concentration \( \frac{\Phi(|u|_1)}{|u|_1} u = b^{-1}(u) = z^D \) rather than in the density variable \( u = u^D = b(z^D) \). It is easy to see formally that non-negative data \( f_1, u_0^i, z^D_i \geq 0 \) should lead to non-negative solutions \( u_i \geq 0 \Leftrightarrow z_i \geq 0 \) and, therefore, we shall only deal with such non-negative data and solutions. This is of course consistent with the fact that \( z_i \) represent physical concentrations and should stay non-negative when time evolves. Summing the equations in \((1.8)\) we recognize that \( w = |u|_1 = u_1 + \ldots + u_N \) is a non-negative solution to

\[
\text{(GPME)} \quad w = |u|_1 : \quad \begin{cases}
\partial_t w = \Delta (\Phi(w)) + F & \text{in } Q_T, \\
w(x,0) = w^0(x) & \text{in } \Omega, \\
\Phi(w) = g^D & \text{in } \Sigma_T
\end{cases}
\]
with $F = f_1 + \ldots + f_N \geq 0$, $g^D = |\mathbf{z}^D|_1 \geq 0$, and $w^0 = |\mathbf{u}^0|_1 \geq 0$. Note again that the boundary data are written for $\Phi(w)$ rather than for $w$, as is common for the scalar GPME.

The initial data and inhomogeneity should satisfy
\begin{equation}
\forall i = 1 \ldots N : 0 \leq w^0_i \leq M \quad \text{and} \quad 0 \leq f_i \leq M \quad \text{a.e. } (x,t) \in Q_T
\end{equation}
for some finite $M > 0$. The boundary data will always be taken non-negative and bounded as well, but we shall sometimes assume the following. If $\gamma : \Omega \to \partial \Omega$ is the usual trace operator then there exists $\mathbf{Z}^D(x,t)$ such that
\begin{equation}
\mathbf{z}^D = \gamma(\mathbf{Z}^D) : 0 \leq Z^D_i \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \partial_t Z^D_i \in L^\infty(Q_T)
\end{equation}
(we shall indistinctly write $\mathbf{z}^D$ or $\mathbf{Z}^D$ both for the trace or the extension to $\Omega$).

Let us now first introduce our main theorem, which addresses existence, uniqueness, and regularity.

**Theorem 1.1.** Assume that $(H_2)$ holds. For any data $0 \leq w^0_i \leq M$, $0 \leq z^D_i \leq M$, and $0 \leq f_i \leq M$ there exists a unique non-negative bounded very weak solution $\mathbf{u}$ to (1.8). Moreover, $w = |\mathbf{u}|_1$ is the unique non-negative bounded very weak solution to (GPME) and there exist positive constants $\alpha = \alpha(a,n) \in (0,1)$ and $C = C(a,T,n,N,M)$ such that
\begin{equation}
\|\mathbf{u}\|_{C^{0,\alpha/2}(Q')} \leq C(1 + 1/d' + 1/\sqrt{\tau})
\end{equation}
holds in all parabolic subdomains $Q' = \Omega' \times (\tau,T)$ with $0 < \tau < T$, $\Omega' \subset \subset \Omega$, and $d' = \text{dist}(\overline{\Omega'}, \partial \Omega)$.

Assume in addition that $(H_3)$ holds and that the data satisfy (1.9)-(1.10). Then $w$ is a global weak energy solution to (GPME) and $\mathbf{u}$ is a local weak energy solution to (1.8) in the sense that
\begin{equation}
\|\nabla(\rho \mathbf{u}_i)\|_{L^2(Q_T')} \leq C(1 + 1/d') \quad \forall i = 1 \ldots N,
\end{equation}
where $\rho = \Phi(w)$ and $\Phi(|\mathbf{u}_i|)$, holds in any $Q_T' = \Omega' \times (0,T)$ with some constant $C = C(a,T,n,N,M) > 0$.

The notions of very weak and weak energy solutions will be introduced in Section 3. It is also worth stressing that if the initial and boundary data are compatible, then the local regularity (1.11) can be improved to global regularity up to the bottom and lateral boundaries, see Proposition 2.4 later on.

Let us immediately comment on the strategy of the proof of Theorem (1.1), which is classical for scalar problems but requires technical work for system (1.8): we first establish existence of positive classical solutions $\mathbf{u}^k$ for approximated positive data $w^{0,k}, \mathbf{z}^{D,k}, f^k$ and derive some a priori regularity and energy estimates. Taking $k \to \infty$ finally gives the desired solution $\mathbf{u} = \lim \mathbf{u}^k$, which inherits regularity and energy estimates from the previous ones. This strategy is possible given the particular $l^1(\mathbb{R}^N)$ structure of the system which yields a scalar equation (GPME) for $w = |\mathbf{u}|_1$. The classical GPME theory [7, 27] will thus provide a priori information for the scalar $w$, which will in turn yield useful estimates for the vector-valued unknown $\mathbf{u}$. This feature is particularly salient in the proof of Proposition 2.1 (existence of classical positive solutions) where we show that $\mathbf{u}$ can in fact be reconstructed a posteriori from the a priori knowledge of $w = |\mathbf{u}|_1$ only. The idea of controlling the vector-valued $\mathbf{u}$ by means of the scalar $w$ will be the cornerstone of our analysis, and will also appear in the study of the Hölder regularity and of
the free-boundaries later on in Section 4. In the same spirit and in view of (1.8), a variable of interest is clearly
\[(1.13) \quad \varrho = \Phi(w), \quad w = |u|_1.\]

In the theory of (possibly degenerate) scalar diffusion equations such as (GPME) the so-called pressure variable \(p = \Phi'(w)\) plays an important role, as can be seen from the divergence form \(\Delta \Phi(w) = \text{div}(\Phi'(w)\nabla w). \) Since our structural assumption \((H_2)\) bounds the ratio \(p/\varrho\) both away from zero and from above, the degeneracy of \(p\) in (GPME) should be comparable to the degeneracy of \(\varrho\) in (1.8). This again illustrates the importance of the \(L^1(\mathbb{R}^N)\) structure since some a priori information can be obtained from the classical GPME theory on account of \(p \sim \rho\).

The paper is organized as follows: In Section 2 we consider smooth positive data, construct corresponding smooth positive solutions to (1.8), and establish a priori energy as well as Hölder estimates. The Hölder estimates are based on the celebrated method of intrinsic scaling \([12, 13, 14]\), a standard technique at least for scalar problems. In Section 3 we consider more general data, introduce different notions of weak solutions, and prove Theorem 1.1. Suitably approximating the data we show existence of a unique weak solution to problem (1.8), which inherits Hölder regularity and energy estimates from the smooth positive solutions constructed in Section 2. Finally in Section 4 we impose the degeneracy condition \(\Phi'(0) = 0\) and consider the problem in the whole space without the forcing term and with compactly supported initial data. We show that the corresponding Cauchy problem is well posed and admits free-boundary solutions. Based on the classical theory of GPME we investigate the finite speed of propagation of the free-boundaries and the evolution and interaction of distinct compactly supported initial concentrations.

2. SMOOTH POSITIVE SOLUTIONS AND A PRIORI ESTIMATES

We will assume throughout this section that the data is smooth and (componentwise) positive. Solutions of (1.8) and (GPME) corresponding to such data are shown to be classical and positive and satisfy certain a priori energy and locally uniform Hölder estimates.

**Proposition 2.1 (Existence of positive classical solutions).** Assume that \(z^D\) and \(u^0\) are smooth and positive and \(f\) is smooth and non-negative. Moreover, let \(F := |f|_1\) and assume that
\[
0 < m = \min \left\{ \text{ess inf}_{\Omega_T} |u^0|_1, \quad \text{ess inf}_{\Sigma_T} |z^D|_1 \right\},
\]
\[
0 < M = \max \left\{ \text{ess sup}_{\Omega_T} |u^0|_1, \quad \text{ess sup}_{\Sigma_T} |z^D|_1, \quad \text{ess sup}_{\Omega_T} F \right\}.
\]

Then there exists a classical solution \(u \in C^{2,1}(\Omega \times (0,T)) \cap C^{2,1}(\Omega \times [0,T]) \cap C^\infty(Q_T)\) to (1.8) with \(u_i > 0\) on \(\partial \Omega_T\). Moreover, defining \(w = |u|_1 = u_1 + \ldots + u_N\), \(w \in C^{2,1}(\Omega_T) \cap C^\infty(Q_T)\) is a classical solution to (GPME) and
\[
0 < m \leq w(x,t) \leq M(1+T) \quad \text{in} \, \Omega_T.
\]

**Remark 2.1.** We do not impose any compatibility conditions on the initial and boundary data along the corners \(\partial \Omega \times \{t = 0\}\). Although this limits the boundary regularity it has no importance in the sequel. Note also that we could prove uniqueness of positive classical solutions at this stage. However since we will later
establish a stronger uniqueness result (within the class of non-negative very weak solutions) we postpone the uniqueness issue until then.

**Proof.** We will exploit the diagonal structure of the system by first showing existence of a classical solution \( w \) to (GPME), then reconstructing \( u \) by solving \( N \) independent linear parabolic equations for the \( u_i \)'s, and finally checking that \( w = |u^0|_1 \) as desired.

For smooth positive data let \( u^0 := |u^0|_1 > 0 \) in \( \overline{\Omega} \) and \( g^D := |z^D|_1 > 0 \) in \( \overline{\Omega} \). Write \( \Delta \Phi(w) = \text{div}(\Phi'(w)\nabla w) \) and observe that hypothesis \((H_2)\) implies that \( \Phi'(w) > 0 \) is bounded away from zero and from above as long as \( 0 < m \leq w \leq C \) so that equation (GPME) is uniformly parabolic for such values of \( w \). Therefore, after approximating \( \Phi \) by a globally Lipschitz function \( \Phi_{\varepsilon} \) such that \( \Phi_{\varepsilon}(0) = 0 \), \( \Phi'(s) = \Phi_{\varepsilon}'(s) \) for \( |s| \in (\varepsilon, 1/\varepsilon) \), and \( \Phi_{\varepsilon}' \geq c_{\varepsilon} > 0 \), well known results for quasilinear parabolic equations (cf. [21]) guarantee the existence of a positive classical solution \( w_{\varepsilon}(x,t) \) to the \( \varepsilon \)-problem. A standard comparison principle with \( 0 \leq F \leq M \) and \( 0 < m \leq w_{\varepsilon}, g^D \leq M \) shows moreover that

\[
0 < m \leq w_{\varepsilon} \leq \max \left\{ \|u^0\|_{L^\infty(\Omega)}, \|g^D\|_{L^\infty(\Sigma_T)} \right\} + T \|F\|_{L^\infty(Q_T)} \leq M(1 + T)
\]
as in our statement. In particular for \( \varepsilon > 0 \) small enough there holds \( \Phi_{\varepsilon}(w_{\varepsilon}) = \Phi(w_{\varepsilon}) \), so that \( w_{\varepsilon} \) is in fact a classical solution to the original problem. The argument is standard for scalar equations and we refer, e.g., to [7, 27] for more details.

Once there exists a smooth positive solution \( w \) to (GPME), the pressure \( \rho = \frac{\Phi(w)}{w} \) becomes smooth in the interior, belongs to \( C^{2,1}(\overline{\Omega} \times (0,T)) \) \( \cap C^{2,1}(\Omega \times [0,T]) \), and the bounds

\[
0 < C_1 \leq \rho \leq C_2 \quad \text{in} \quad Q^T
\]
hold for some \( C_1, C_2 > 0 \) depending on \( a, m, M, T \) only. Then standard results [21] on linear parabolic equations allow us to solve

\[
\begin{aligned}
\partial_t u_i &= \Delta(g \rho u_i) + f_i = \text{div}(\rho \nabla u_i) + \text{div}(u_i \nabla \rho) + f_i \quad \text{in} \quad Q_T, \\
u_i(x,0) &= u^0_i(x) \quad \text{in} \quad \Sigma_T, \\
u_i(x) &= \frac{z_i^D}{e} \quad \text{in} \quad \Omega,
\end{aligned}
\]
for fixed \( i = 1, \ldots, N \) and show that \( u_i \in C^{2,1}(\overline{\Omega} \times (0,T)) \cap C^{2,1}(\Omega \times [0,T]) \cap C^\infty(Q_T) \) (up to the corners if the data are compatible). Indeed, from (2.2) it follows that the equation is uniformly parabolic and the boundary condition reads simply as \( u_i = \frac{z_i^D}{e} \) on \( \Sigma_T \). The assumptions on the data and the strong maximum principle ensure moreover that \( u_i > 0 \) in \( Q_T \).

Let now \( \bar{w} = |u^0|_1 \) and observe that \( \partial_t w = \Delta \Phi(w) + F = \Delta(g \rho w) + F \). Because \( u_i > 0 \) we can write \( \bar{w} = u_1 + \ldots + u_N \). Summing (2.1) over \( i = 1, \ldots, N \), we obtain \( \partial_t \bar{w} = \Delta(g \rho \bar{w}) + F \). In other words, \( \bar{w} \) is a positive classical solution to the same equation as \( w \) with the same initial and boundary data. By standard uniqueness argument for smooth positive solutions we conclude that \( w = \bar{w} \). In particular \( \rho = \frac{\Phi(w)}{w} = \frac{\Phi(\bar{w})}{\bar{w}} = \frac{\Phi(|u^0|_1)}{|u^0|_1} \) in (2.1) and the proof is complete. \( \square \)

**Proposition 2.2 (A priori energy estimates).** Assume that hypotheses \((H_1), (H_2)\) and \((H_3)\) hold, let \( u \in C^{2,1}(\overline{Q_T}) \) be a classical positive solution corresponding to smooth positive data and assume that

\[
\|u^0\|_{L^\infty(\Omega)} + \|z^D\|_{L^\infty(\Omega; H^1(\Omega))} + \|\partial_t z^D\|_{L^\infty(Q_T)} + \|f\|_{L^\infty(Q_T)} \leq M,
\]
for some \( M > 0 \). Then we have the bounds

\[
\|\nabla(g \rho w)\|_{L^2(Q_T)} \leq C,
\]
(2.3)
where \( w = |u|_1 \) and \( \varrho = \frac{\Phi(w)}{w} \), and

\[
(2.4) \quad \|\nabla (\varrho u_i)\|_{L^2(Q_T')} \leq C(1 + 1/d'), \quad \forall i = 1 \ldots N.
\]

Here, the constant \( C > 0 \) depends on \( a, T, n, N, M \) only, and \( Q_T' = \Omega' \times (0, T) \) with \( \Omega' \subset \subset \Omega \) and \( d' = \text{dist}(\overline{\Omega'}, \partial \Omega) > 0 \).

**Remark 2.2.** We were not able to establish energy estimates for \( \nabla (\varrho u_i) \) up to the boundary as the dependence on \( 1/d' \) in (2.4) shows. This will not be an issue later on since in Proposition 3.1 we shall prove uniqueness within the class of very weak solutions and estimate (2.4) as well as assumption \((H_3)\), which is only used in proving (2.4), can be dispensed with while considering very weak solutions. On the other hand, estimate (2.4) is sufficient for our purposes in Section 4 where the problem is considered in \( \mathbb{R}^n \) with compactly supported initial data.

Observe also that the validity of estimate (2.4) up to the boundary would directly yield (2.3) since \( w = |u|_1 = \sum_1 u_i \) for non-negative solutions.

**Proof.** We will first establish (2.3) for the scalar variable \( w \) and then show how the particular structure of system (1.8) allows us to derive (2.4) for each component \( u_i \).

We shall denote by \( C \) any positive constant depending, as in the statement, only on \( a, T, n, N, M \) whereas the primed constants \( C' \) are also allowed to depend on \( d' = \text{dist}(\overline{\Omega'}, \partial \Omega) \).

**Step 1.** The assumptions on \( u^0, z^D, f \) translate into similar properties for the data \( w^0 = |u^0|_1, g^D = |z^D|_1, F = |f|_1 \) so by the comparison principle for solutions to (GPME) we have

\[
0 \leq u_i \leq |u|_1 = w \leq C(M, N, T).
\]

Recalling that \( gw = \Phi(w) \), inequality (2.3) is nothing but the usual (global) energy estimate for the GPME leading to the usual concept of weak energy solutions. For smooth positive solutions, bound (2.3) is easily derived for \( \|\nabla \Phi(w)\|_{L^2(Q_T)} = \|\nabla (\varrho u_i)\|_{L^2(Q_T')} \) by taking \( \varphi = (\Phi(w) - g^D) \in L^2(0, T; H^1_0(\Omega)) \) as a test function in (GPME), cf. [7, 27] for further details.

**Step 2.** Since \( 0 \leq w \leq C \), the structural assumptions imply that

\[
0 \leq \frac{\Phi(w)}{w} \leq \frac{1}{a} \Phi(w) \leq C(a, M, T).
\]

The \( L^\infty(Q_T) \)-norm of any term involving \( u_i, w, \varrho \) can thus be bounded by a constant \( C = C(a, T, n, N, M) > 0 \) now fix \( i \in \{1, \ldots, N\} \) and choose a cutoff function \( \chi = \chi(x) \in C^\infty(\Omega) \) such that \( 0 \leq \chi \leq 1 \) in \( \Omega \), \( \chi \equiv 1 \) in \( \Omega' \) and \( |\nabla \chi| \leq 2/d' \) where \( \Omega' \subset \subset \Omega \) and \( d' = \text{dist}(\overline{\Omega'}, \partial \Omega) \). Multiplying the \( i \)-th equation in (1.8) by a test function \( \varphi = \chi^2 g u_i \), integrating over \( Q_T \) and by parts in the Laplacian term, we obtain

\[
\int_{Q_T} \chi^2 |\nabla (\varrho u_i)|^2 \, dx \, dt = -2 \int_{Q_T} \varrho u_i \chi \nabla \chi \cdot \nabla (\varrho u_i) \, dx \, dt
\]

\[
+ \int_{Q_T} \chi^2 \varrho u_i f_i \, dx \, dt - \int_{Q_T} \chi^2 \varrho u_i \partial_t u_i \, dx \, dt.
\]

Integrating the last term by parts in \( t \) and using \( 0 \leq u_i, \varrho \leq C \) to bound the limit terms at \( t = 0, T \), gives

\[
\int_{Q_T} \chi^2 |\nabla (\varrho u_i)|^2 \, dx \, dt
\]
\[ \leq 2 \| \partial_t \varrho \|_{L^2(Q_T)} \| \chi \nabla (\varrho u_i) \|_{L^2(Q_T)} + C + \left( C + \frac{1}{2} \int_{Q_T} \chi^2 u_i^2 \partial_t \varrho \, dx \, dt \right) \]

\[ \leq \frac{1}{2} \| \chi \nabla (\varrho u_i) \|_{L^2(Q_T)}^2 + 8 \| \varrho u_i \nabla \chi \|_{L^2(Q_T)}^2 + C + \frac{1}{2} \int_{Q_T} \chi^2 u_i^2 \partial_t \varrho \, dx \, dt, \]

where we have also taken into account that \( 0 \leq \chi^2 \varrho_i f_i \leq C \) and used Young’s inequality. Estimating \( \| \varrho u_i \nabla \chi \|_{L^2(Q_T)}^2 \leq C/(dt)^2 \leq C' \) then yields the bound

\[ (2.5) \]

\[ \| \chi \nabla (\varrho u_i) \|_{L^2(Q_T)}^2 \leq C' + \int_{Q_T} \chi^2 u_i^2 \partial_t \varrho \, dx \, dt. \]

We exploit now the structure of the system to control \( A \). Indeed, since \( \varrho = \frac{\Phi(w)}{w} \) one easily computes for smooth positive solutions

\[ \partial_t \varrho = \frac{d}{dw} \left( \frac{\Phi(w)}{w} \right) \partial_t w = \frac{w \Phi'(w) - \Phi(w)}{w^2} (\Delta (\varrho w) + F). \]

Thus integrating by parts gives

\[ A = \int_{Q_T} \chi^2 (\varrho u_i)^2 \frac{1}{(\varrho w)^2} (w \Phi'(w) - \Phi(w)) (\Delta (\varrho w) + F) \, dx \, dt \]

\[ = - \int_{Q_T} \nabla (\chi^2 (\varrho u_i)^2) \frac{1}{(\varrho w)^2} (w \Phi'(w) - \Phi(w)) \cdot \nabla (\varrho w) \, dx \, dt \]

\[ - \int_{Q_T} \chi^2 (\varrho u_i)^2 \frac{1}{(\varrho w)^2} (w \Phi'(w) - \Phi(w)) \cdot \nabla (\varrho w) \, dx \, dt \]

\[ - \int_{Q_T} \chi^2 (\varrho u_i)^2 \nabla \left( \frac{1}{(\varrho w)^2} \right) (w \Phi'(w) - \Phi(w)) \cdot \nabla (\varrho w) \, dx \, dt \]

\[ - \int_{Q_T} \chi^2 (\varrho u_i)^2 \frac{1}{(\varrho w)^2} \nabla (w \Phi'(w) - \Phi(w)) \cdot \nabla (\varrho w) \, dx \, dt \]

\[ + \int_{Q_T} \chi^2 (\varrho u_i)^2 \frac{1}{(\varrho w)^2} (w \Phi'(w) - \Phi(w)) F \, dx \, dt \]

\[ = A_1 + A_2 + A_3 + A_4 + B. \]

Observing that \( 0 \leq \varrho u_i \leq \varrho w = \Phi(w) \) and that the hypothesis \((H_2)\) implies that \( 0 \leq w \Phi'(w) - \Phi(w) \leq C(a) \Phi(w) \), we can control the first term as

\[ A_1 = -2 \int_{Q_T} \chi \nabla \chi \cdot (\varrho u_i)^2 \frac{1}{(\varrho w)^2} (w \Phi'(w) - \Phi(w)) \nabla (\varrho w) \, dx \, dt \]

\[ \leq C \| \nabla \chi \|_{L^2(Q_T)} \| \nabla (\varrho w) \|_{L^2(Q_T)} \leq C'. \]

The second term is bounded similarly as follows

\[ A_2 = -2 \int_{Q_T} \chi^2 (\varrho u_i) \nabla (\varrho u_i) \cdot \frac{1}{(\varrho w)^2} (w \Phi'(w) - \Phi(w)) \nabla (\varrho w) \, dx \, dt \]

\[ \leq 2 \| \nabla (\varrho u_i) \|_{L^2(Q_T)} \left\| \chi \frac{\varrho u_i \Phi'(w) - \Phi(w)}{\varrho w} \nabla (\varrho w) \right\|_{L^2(Q_T)} \]

\[ \leq C \| \nabla (\varrho u_i) \|_{L^2(Q_T)} \| \nabla (\varrho w) \|_{L^2(Q_T)} \]
\[
\begin{align*}
&\leq \frac{1}{2} \|\nabla (gw_i)\|^2_{L^2(Q_T)} + C \|\nabla (gw)\|^2_{L^2(Q_T)} \\
&\leq \frac{1}{2} \|\nabla (gw_i)\|^2_{L^2(Q_T)} + C ,
\end{align*}
\]
where we have also used the Young’s inequality (the first term on the right-hand side will be reabsorbed into (2.5)). The third quantity is controlled as
\[
A_3 = 2 \int_{Q_T} \chi^2 (gw_i)^2 \frac{\nabla (gw)}{(gw)^3} \cdot (w\Phi'(w) - \Phi(w)) \nabla (gw) \, dx \, dt
\]
\[
= 2 \int_{Q_T} \chi^2 (gw_i)^2 \frac{w\Phi'(w) - \Phi(w)}{\Phi(w)} \nabla (gw)^2 \, dx \, dt \leq C \|\nabla (gw)\|^2_{L^2(Q_T)} \leq C .
\]
In the fourth term we write \(\nabla (gw) = \nabla \Phi(w) = \Phi'(w) \nabla w\) and use (H3) to get
\[
A_4 = -\int_{Q_T} \chi^2 (gw_i)^2 \frac{1}{(gw)^2} \nabla (w\Phi'(w) - \Phi(w)) \cdot \nabla (gw) \, dx \, dt
\]
\[
= -\int_{Q_T} \chi^2 (gw_i)^2 \frac{1}{(gw)^2} w\Phi''(w) \nabla w \cdot \nabla (gw) \, dx \, dt
\]
\[
= -\int_{Q_T} \chi^2 (gw_i)^2 \frac{1}{(gw)^2} w\Phi''(w) \Phi'(w) \nabla (gw)^2 \, dx \, dt
\]
\[
\leq \frac{1}{a} \|\nabla (gw)\|_{L^2(Q_T)} \leq C .
\]

**Remark 2.3.** Note that an upper bound for \(A_4\) is obtained here by using the lower bound (H3) for \(\frac{\Phi''(s)}{\Phi'(s)}\). If in particular \(\Phi''(s) \geq 0\) for all \(s \geq 0\), which is the typical case for the PME nonlinearity \(\Phi(s) = |s|^{m-1}s\) in the range \(m \geq 1\), then \(A_4 \leq 0\). This convexity condition is also valid for the Freundlich isotherm since \(\beta_f(r) = r + r^p\) is concave and thus \(\Phi_f = \beta_f^{-1}\) is convex.

For the last term we obtain
\[
B = \int_{Q_T} \chi^2 (gw_i)^2 \frac{1}{(gw)^2} (w\Phi'(w) - \Phi(w)) F \, dx \, dt \leq C(a) \int_{Q_T} \Phi(w) F \, dx \, dt \leq C.
\]
Plugging the above estimates back into (2.5) finally yields
\[
\|\nabla (gw_i)\|^2_{L^2(Q_T)} \leq \|\chi \nabla (gw_i)\|^2_{L^2(Q_T)} \leq C' .
\]
Keeping track of the dependence of the estimates on \(d' = \text{dist}(\Omega', \partial \Omega)\) and optimizing all inequalities, one easily sees that \(C' = C(1+1/d')\) with \(C = C(a, T, n, N, M)\) only and the proof is complete. \(\square\)

We will next address the regularity issue.

**Proposition 2.3.** Let \(u\) and \(M\) be as in Proposition 2.1. There exist \(a = a(a, n) \in (0, 1)\) and \(C = C(a, T, n, N, M) > 0\) such that the estimate
\[
\|u\|_{C^{\alpha} \cap L^2(Q_T')} \leq C(1+1/d' + 1/\sqrt{T}) ,
\]
holds for any parabolic subdomain \(Q' = \Omega' \times (\tau, T)\), where \(0 < \tau < T\), \(\Omega' \subset \subset \Omega\), and \(d' = \text{dist}(\Omega', \partial \Omega)\).

**Remark 2.4.** We would like to stress that our proof handles both the nondegenerate \(\Phi'(0) > 0\) and degenerate \(\Phi'(0) = 0\) cases in a unified framework.
Suppose that \( \sup_{Q^\mu} w \leq \mu \) in an intrinsic cylinder

\[ Q^\mu := B_r \times (-\tau^\mu_r, 0), \quad \tau^\mu_r := \frac{-\mu}{\Phi(\mu)} r^2, \]

and define

\[ \bar{Q}^\mu := B_{3r/4} \times (-\frac{3}{4} r^\mu + \frac{1}{2} r^\mu_r) \subset Q^\mu. \]

Now consider the following two alternatives

\begin{align}
(2.6) & \quad |\bar{Q}^\mu \cap \{ w \leq \mu/2 \}| \leq \delta |\bar{Q}^\mu|, \\
(2.7) & \quad |\bar{Q}^\mu \cap \{ w \leq \mu/2 \}| > \delta |\bar{Q}^\mu|,
\end{align}

where \( \delta \in (0, 1) \) is a small parameter to be fixed shortly. The first one is the nondegenerate alternative and the second is the degenerate alternative. We will analyze them separately.

**Step 2: Nondegenerate alternative 1.** Set

\[ r_j = \left( 1 + \frac{1}{4} j \frac{\mu}{r^2} \right) r, \quad k_j := \left( 1 + \frac{1}{4} j \frac{\mu}{r^2} \right) \mu, \quad w_j := (k_j - w)_+. \]

Set also

\[ \bar{Q}^j := B_{r/4 + r_j} \times \left( -\frac{1}{2} r^\mu - \tau^\mu_r \left( \frac{r_j}{r} \right)^2, -\frac{1}{2} r^\mu_r \right), \]

and let \( \phi_j \) be a cut-off function such that \( \phi_j \) is smooth, \( 0 \leq \phi_j \leq 1 \), \( \phi_j \) vanishes on the parabolic boundary \( \partial_0 \bar{Q}^j \) (bottom and lateral), is one on \( Q^{j+1} \) and \( |\nabla \phi_j|^2 + (\partial_t \phi_j^2)_+ \leq r^{-2} 16^j + 1 \). Since \( w \) solves the equation \( \partial_t w - \Delta \Phi(w) = F \) and \( F \geq 0 \), it is easy to check that \( w_j \) is a weak subsolution to the equation \( \partial_t w_j - \text{div}(\Phi'(w)\nabla w_j) = 0 \), and testing the latter with \( w_j \phi_j^2 \) leads to the Caccioppoli inequality

\[
\sup_{-\tau^\mu \leq t < 0} \int_{B_r} \frac{w_j^2 \phi_j^2}{\tau^\mu} \, dx + \int_{Q^\mu} \Phi'(w)|\nabla (w_j \phi_j)|^2 \, dx \, dt \\
\leq c \int_{Q^\mu} \left[ \Phi'(w) w_j^2 |\nabla \phi_j|^2 + w_j^2 (\partial_t \phi_j^2)_+ \right] \, dx \, dt.
\]

Setting

\[ \bar{w}_j := \begin{cases} 
  k_j - k_{j+1} & \text{if } w \leq k_{j+1} \\
  w_j & \text{if } w > k_{j+1}
\end{cases} \]

and recalling from (H2) that \( \Phi(s)/s \) is monotone non-decreasing with at most algebraic growth, we see that

\[
\frac{r^2}{\tau^\mu} |\nabla \bar{w}_j|^2 = \frac{\Phi(\mu)}{\mu} |\nabla \bar{w}_j|^2 \leq c(a) \frac{\Phi(k_{j+1})}{k_{j+1}} |\nabla \bar{w}_j|^2 \leq c(a) \Phi'(w) |\nabla w_j|^2
\]

since in the support of \( \nabla \bar{w}_j \) we have \( w \geq k_{j+1} \geq \mu/4 \). Similarly,

\[
\Phi'(w) w_j^2 |\nabla \phi_j|^2 \leq c(a) 16^j \frac{1}{r^2} \frac{\Phi(\mu)}{\mu} w_j^2 \leq \frac{c(a) 16^j}{\tau^\mu} \mu^2 \chi_{\{w < k_j\}}.
\]
Collecting estimates we arrive at
\[
\sup_{-\tau^*/2 < t < 0} \int_{B_{r/2}} \tilde{w}_j^2 \phi_j^2 \, dx + r^2 \int_{Q_r^+} |\nabla (\tilde{w}_j \phi_j)|^2 \, dx \, dt \leq c16^j \mu^2 \left( \frac{\tilde{Q}^j \cap \{w < k_j\}}{|\tilde{Q}^j|} \right)
\]
Next, the parabolic Sobolev embedding (see [13, Proposition 3.1, p.7]) gives us
\[
\int_{Q_r^+} (\tilde{w}_j \phi_j)^{2(1+2/n)} \, dx \, dt \leq c(n) \left( \sup_{-\tau^*/2 < t < 0} \int_{B_r} \tilde{w}_j^2 \phi_j^2 \, dx + r^2 \int_{Q_r^+} |\nabla (\tilde{w}_j \phi_j)|^2 \, dx \, dt \right)^{1+2/n}.
\]
Since
\[
(\tilde{w}_j \phi_j)^{2(1+2/n)} \geq 4^{-6j} \mu^{2(1+2/n)} \chi_{\tilde{Q}^j \cap \{w < k_j+1\}},
\]
we get
\[
E_{j+1} \leq c(a, n) 4^{8j} E_{j+2/n} \quad \text{with} \quad E_j := \frac{|\tilde{Q}^j \cap \{w < k_j\}|}{|\tilde{Q}^j|}.
\]
A standard iteration lemma on fast geometric convergence of series ([13, p.12]) shows that if \(E_0 \leq \varepsilon^{-n/2} 4^{-2n^2}\) then \(E_j\) tends to zero as \(j \to \infty\). Indeed, choosing \(\delta := \varepsilon^{-n/2} 4^{-2n^2}\), it follows from (2.6) that \(w \geq \mu/4\) in \(B_{r/2} \times (-\tau^*/2, 0)\).

**Step 3: Nondegenerate alternative 2.** We test \(\partial_t w = \text{div} (\Phi'(w) \nabla w) + F\) with \((1/w - A\mu) + \xi^2\), where \(\xi \in C_0^\infty(B_{r/2}), 0 \leq \xi \leq 1, \xi \equiv 1\) in \(B_{r/4}\) and \(|\nabla \xi| \leq 8/r\). Note that the chosen test function vanishes on \(B_{r/2} \times \{-\tau^*/2\}\) by Step 2. Taking advantage of \(-F \left(\frac{1}{w} - A\mu\right) + \xi^2 \leq 0\), straightforward manipulations then lead to
\[
\sup_{-\tau^*/2 < t < 0} \int_{B_{r/2}} \log \left( \frac{\mu/4}{w} \right) \xi^2 \, dx + \frac{\tau^*}{4} \int_{B_{r/2} \times (-\tau^*/2, 0)} \frac{\Phi'(w)}{w^2} |\nabla (w - \mu/4)|^2 \, dx \, dt
\]
\[
\leq \tau^* \int_{B_{r/2} \times (-\tau^*/2, 0)} \Phi'(w) |\nabla \xi|^2 \, dx \, dt + \int_{B_{r/2}} \xi^2 \, dx
\]
\[
\leq \tau^* \int_{B_{r/2} \times (-\tau^*/2, 0)} \frac{1}{w} |\nabla \xi|^2 \, dx \, dt + \int_{B_{r/2}} \xi^2 \, dx
\]
\[
\leq \tau^* \frac{1}{\mu} \int_{B_{r/2} \times (-\tau^*/2, 0)} |\nabla \xi|^2 \, dx \, dt + \int_{B_{r/2}} \xi^2 \, dx \leq \frac{65}{\alpha}
\]
where we used successively \((H_2)\), the monotonicity of \(\Phi(s)/s\) with \(w \leq \mu\), the definition \(\tau^* \Phi(w)/w = \tau^2\), and the cutoff function properties \(|\nabla \xi| \leq 8/r, \xi^2 \leq 1\). As a consequence, we readily obtain
\[
\sup_{-\tau^*/2 < t < 0} |B_{r/4} \cap \{w(\cdot, t) < 4^{-1-m}\mu\}| \leq \frac{1}{m} \frac{2^{265}}{a \log 4} |B_{r/4}|,
\]
and in particular
\[
\frac{|B_{r/4} \times (-\tau^*/2, 0) \cap \{w < 4^{-1-m}\mu\}|}{|B_{r/4} \times (-\tau^*/2, 0)|} \leq \frac{1}{m} \frac{2^{265}}{a \log 4},
\]
for any \(m \in \mathbb{N}\).

Next, redefine \(k_j := 4^{-m-1}(2^{-1} + 2^{-1-j})\mu\) and \(r_j = (2^{-3} + 2^{-3-j})r\), and set
\[
\tilde{Q}^j := \tilde{B}^j \times (-\tau^*/2, 0), \quad \tilde{B}^j := B_{r_j}(0).
\]
Choose $\xi_j \in C^{\infty}_c(\bar{B}^j)$ in such a way that $0 \leq \xi_j \leq 1$, $\xi_j = 1$ in $\bar{B}^{j+1}$ and $|\nabla \xi_j| \leq 2^{k_j}/r$. The Caccioppoli estimate then takes the form

$$
\sup_{-\tau^j/2 < t < 0} \int_{\bar{B}_j} \frac{w^2 \xi_j^2}{\tau^j} \, dx + \int_{Q_j} \Phi'(w)|\nabla (w_j \xi_j)|^2 \, dx \, dt \leq c \int_{Q_j} \Phi'(w) |\nabla (w_j \xi_j)|^2 \, dx \, dt,
$$

because $\xi_j$ is independent of time and the newly defined $w_j$ vanishes on the initial boundary of $\hat{Q}^j$ by Step 2. Since $s \mapsto \Phi(s)/s$ is a nondecreasing function and $1/\tau^j = \Phi(\mu)/\mu$, it follows that

$$
\frac{\Phi(k_j)}{k_j} \sup_{-\tau^j/2 < t < 0} \int_{\bar{B}_j} w^2 \xi_j^2 \, dx + \int_{Q_j} \Phi'(w)|\nabla (w_j \xi_j)|^2 \, dx \, dt \leq c \frac{4j}{\tau^2} \Phi(k_j) k_j E_j,
$$

where this time $E_j := [\hat{Q}^j \cap \{w_j > 0\}] / |\hat{Q}^j|$. Analogously to Step 2, we then arrive at $E_{j+1} \leq c(n,a) 4^j E_j^{1+2/n}$, and by choosing $m \equiv m(n,a)$ large enough, i.e.

$$
E_0 \leq \frac{1}{2^n 6^5} \frac{2^n a \log 4}{m} \leq \hat{c}^{-n/2} 4^{-2n^2},
$$

we conclude that $w \geq 4^{-m-2} \mu$ in $Q^{\mu}_{t/8}$.

**Step 4: Degenerate alternative.** Let us then analyze the occurrence of (2.7). For this, set $v = \mu/2 - (w - \mu/2) + \|F\|_{L^n(Q^\mu)}(t + \tau^j)$, which is a nonnegative weak supersolution to $\partial_t v - [\Phi(\mu)/\mu] \div (b(x,t)\nabla v) \geq 0$ in $Q^\mu_t$ with $b(x,t) := \mu \Phi'(w(x,t))/\Phi(\mu)$. By definition of $v$ and $\mu = \sup w$ we have in the support of $\nabla v$

$$
\nabla v(x,t) \neq 0 \quad \Rightarrow \quad \mu/2 \leq w \leq \mu \quad \Rightarrow \quad c(a)^{-1} \leq b(x,t) \leq c(a).
$$

Redefining $b$ to be one on $\{\nabla v(x,t) = 0\}$ and scaling $b$ and $v$ as $b(x,t) = b(rx, \tau^j t)$ and $v(x,t) = v(rx, \tau^j t)$, we see that $v$ is a weak supersolution to the equation $\partial_t v - \div (b \nabla v) = 0$ in $B_1 \times (-1,0)$ with measurable coefficient $b$ bounded uniformly from below and from above by positive constants depending only on $a$. By the weak Harnack principle we then obtain

$$
\int_{B_{3/4} \times (-3/4, -1/2)} v \, dx \leq c(a) \inf_{B_{1/2} \times (-1/4,0)} v.
$$

Scaling back to $v$ and recalling its definition in terms of $w$, we finally get by (2.7)

$$
\sup_{Q^\mu_{t/2}} w \leq \mu \left(1 - \frac{\delta}{2c}\right) + \tau^j \|F\|_{L^n(Q^\mu)} \left(1 - \frac{1}{8c}\right).
$$

**Step 5: Conclusion from alternatives.** Taking

$$
\sigma = \sigma(a,n) := \min \left\{4^{-m-2}, \frac{\delta}{4c}, \frac{1 - 16^{-a}}{2}\right\} \in (0,1),
$$

we deduce from the previous alternatives that

either $\inf_{Q^\mu_{t/8}} w \geq \sigma \mu$ or $\sup_{Q^\mu_{t/2}} w \leq (1 - 2\sigma) \mu + \tau^j \|F\|_{L^n(Q^\mu)}$

holds provided that $\sup_{Q^\mu_{t/2}} w \leq \mu$. Choose now any $R > 0$ such that $B_{R} \times (-R^2,0) \subset Q_T$ (after translation), let $R_j := 8^{-j} R$

$$
\mu_0 := 1 + \Phi^{-1}(\sigma^{-1} R^2 \|F\|_{L^n(B_{R} \times (-R^2,0))}) + \sup_{B_{R} \times (-R^2,0)} w,
$$

and then inductively

$$
\mu_{j+1} := (1 - 2\sigma) \mu_j + \frac{\mu_j R_j^2 \|F\|_{L^n(B_{R} \times (-R^2,0))}}{\Phi(\mu_j)}
$$
for \( j \geq 0 \). Clearly \( \mu_j \geq (1 - 2\sigma)^j \mu_0 \). Using the algebraic growth (1.6) leads to
\[
\frac{R_j^2 \|F\|_{L^\infty(B_R \times (-R^2, 0))}}{\Phi(\mu_j)} = \frac{\Phi(\mu_0)}{\Phi(\mu_j)} R_j^2 \|F\|_{L^\infty(B_R \times (-R^2, 0))} \leq \sigma, 
\]
where the last inequality follows from the definition of \( \sigma \) and from the bound
\[
\mu_0 \geq \Phi^{-1} (\sigma^{-1} R^2 \|F\|_{L^\infty}) \quad \Rightarrow \quad \Phi(\mu_0) \geq \sigma^{-1} R^2 \|F\|_{L^\infty}. 
\]
Similarly, we get
\[
\frac{\mu_{j+1}}{\Phi(\mu_{j+1})} \left( \frac{R_j}{8} \right)^2 = \frac{16}{\Phi(\mu_{j+1})} \frac{\mu_j}{\Phi(\mu_{j+1})} \left( \frac{R_j}{2} \right)^2 \leq \sigma, 
\]
and alternatives reduce to
\[
\inf_{Q_{R_j}^{\mu_j}} w \geq \sigma \mu_j \quad \text{or} \quad \sup_{Q_{R_j}^{\mu_j}} w \leq \mu_{j+1}, 
\]
provided that \( \sup_{Q_{R_j}^{\mu_j}} w \leq \mu_j \). Observe that \( \sup_{Q_{R_j}^{\mu_j}} w \leq \mu_0 \) by the definition of \( \mu_0 \).
Since we are considering positive solutions \( w > 0 \) and \( \mu_j \to 0 \) as \( j \to \infty \) the degenerate alternative can clearly occur at most a finite number of times, thus
\[
\sigma \mu_j \leq \inf_{Q_{R_j}^{\mu_j}} w \leq \sup_{Q_{R_j}^{\mu_j}} w \leq \mu_j \quad \text{for some finite} \ J.
\]
Since \( \sigma = \sigma(a, n) \) only, the algebraic growth (1.6) then readily implies that
\[
\frac{1}{c(a, n)} \frac{\Phi(\mu_j)}{\mu_j} \leq \frac{\Phi(w)}{w} \leq c(a, n) \frac{\Phi(\mu_j)}{\mu_j} \quad \text{in} \ Q_{R_j}^{\mu_j}/8.
\]
Scaling as in step 4 and writing \( \partial_t u_i = \Delta u_i + f_i = \text{div}(\varphi \nabla u_i) + (\ldots), \) where \( \varphi = \Phi(w)/w, \) we see that each scaled component \( \bar{u}_i \) solves in \( B_{R_j} \times (-1, 0) \) a uniformly parabolic linear equation \( \partial_t \bar{u}_i = \text{div}(\bar{\varphi} \nabla \bar{u}_i) + (\ldots) \) in divergence form with a measurable coefficient \( \bar{\varphi} \) satisfying \( c(a, n) \leq \bar{\varphi} \leq c(a, n)^{-1}. \) In particular, \( \bar{u}_i \) are Hölder continuous and satisfy a DeGiorgi-Nash-Moser oscillation estimate which, scaling back to \( u_i, \) takes the explicit form
\[
\text{osc}_{Q_{R_j}^{\mu_j}} u_i \leq c \beta^{2} \mu_j + c \frac{R_j}{\Phi(\mu_j)} (\theta R_j)^2 \|f_i\|_{L^\infty(B_{R_j} \times (-R^2, 0))}, 
\]
for some \( \beta = \beta(a, n), \) \( c = c(a, n) \) only and for all \( \theta \in (0, 1). \) Observing that
\[
0 \leq f_i \leq |f_i|_1 = F 
\]
and recalling estimate (2.8) which holds for all \( j \) and with \( \sigma = \sigma(a, n), \) we obtain
\[
\forall \theta \in (0, 1) : \quad \text{osc}_{Q_{R_j}^{\mu_j}} u_i \leq c(\theta^3 + \theta^2) \mu_j. 
\]
Setting
\[
\alpha = \alpha(a, n) := \min\{\beta, -\log(1 - \sigma)/\log 8, 2\} 
\]
and increasing the constant \( c \) by a factor depending only on \( n, a, \) we conclude that
\[
\text{osc}_{B_{r} \times (-r^2, 0)} u_i \leq c \left( \frac{r}{R} \right)^{\alpha} \left( \frac{\Phi(\mu_0)}{\mu_0} \right)^{\alpha/2} \mu_0 
\]
for all \( r \in (0, R_j]. \) This yields the desired interior Hölder continuity estimate after standard manipulations. \( \Box \)
Assuming regularity and compatibility from the data, the solution can be shown to be Hölder continuous up to the boundary.

**Proposition 2.4.** Let \( u, M \) be as in Proposition 2.1 and \( \alpha = \alpha(a, n) \) as in Proposition 2.3. Assume further that the initial and boundary data are compatible and \( \beta \)-Hölder continuous with some \( \beta \in (0,1) \), i.e. there is \( U \in C^{\beta,\beta/2}(Q_T) \) such that \( u^0 = U(.,0) \) and \( x^D = \frac{\Phi(U)}{|U|} U \) in \( \Sigma_T \). Then \( u \in C^{\gamma,\gamma/2}(Q_T) \), with \( \gamma = \min(\alpha, \beta) \). Moreover, the Hölder norm of \( u \) depends only on \( a, n, N, M, T \) and on the \( \beta \)-Hölder norm of the data.

**Proof.** Our assumptions on the data turn into similar compatibility and regularity conditions for the scalar problem (GPME). A straightforward modification of our interior argument (see, e.g., [13, 27]) shows that \( w \) is \( C^{\gamma,\gamma/2} \) up to the boundary, which in turn yields the same regularity for \( u \) through the linear parabolic equation \( \partial_t u_i = \Delta(u_i) + f_i = \text{div}(\varrho \nabla u_i) + \text{div}(u_i \nabla \varrho) + f_i \). \( \square \)

### 3. Weak solutions

Let us first introduce different notions of solutions.

**Definition 1** (weak solutions).

(i) A non-negative function \( w \in L^\infty(Q_T) \) is called bounded very weak solution of (GPME) if the equality

\[
\int_{Q_T} \{w \partial_t \varphi + \Phi(w) \Delta \varphi + F \varphi \} \, dx \, dt = - \int_\Omega w^0(x) \varphi(x,0) \, dx + \int_{\Sigma_T} g_D \frac{\partial \varphi}{\partial \nu} \, dx \, dt
\]

holds for all \( \varphi \in C^{2,1}(Q_T) \) vanishing on \( \Sigma_T \) and in \( \Omega \times \{ t = T \} \).

(ii) A non-negative function \( w \in L^\infty(Q_T) \) is called a bounded weak energy solution of (GPME) if \( \Phi(w) \in L^2(0,T; H^1(\Omega)) \), the trace \( \gamma(\Phi(w)) = g_D \) in \( L^2(0,T; H^{1/2}(\partial \Omega)) \) and the equality

\[
\int_{Q_T} \{w \partial_t \varphi - \nabla \Phi(w) \cdot \nabla \varphi + F \varphi \} \, dx \, dt = - \int_\Omega w^0(x) \varphi(x,0) \, dx
\]

holds for all \( \varphi \in C^{2,1}(Q_T) \) vanishing on \( \Sigma_T \) and in \( \Omega \times \{ t = T \} \).

(iii) A function \( u = (u_1, \ldots, u_N) \in L^\infty(Q_T) \) is called a (non-negative) bounded very weak solution of (1.8) if \( u_i \geq 0 \) a.e. in \( Q_T \) and the equality

\[
\int_{Q_T} \{u_i \partial_t \varphi + \varrho u_i \Delta \varphi + f_i \varphi \} \, dx \, dt + \int_{\Sigma_T} u_i^0(x) \varphi(x,0) \, dx = \int_{\Sigma_T} \varrho_i \frac{\partial \varphi}{\partial \nu} \, dx \, dt,
\]

where \( \varrho = \frac{\Phi(u)}{|u|} \), holds for any \( i = 1, \ldots, N \) and for all \( \varphi \in C^{2,1}(Q_T) \) vanishing on \( \Sigma_T \) and in \( \Omega \times \{ t = T \} \).

(iv) A function \( u \in L^\infty(Q_T) \) is called a (non-negative) bounded weak energy solution of (1.8) if \( u_i \geq 0 \) a.e. in \( Q_T \), \( \varrho_i \in L^2(0,T; H^1(\Omega)) \), the trace \( \gamma(\varrho u_i) = z_i^D \) in \( L^2(0,T; H^{1/2}(\partial \Omega)) \), and the equality

\[
\int_{Q_T} \{w \partial_t \varphi - \nabla (\varrho u_i) \cdot \nabla \varphi + f_i \varphi \} \, dx \, dt = - \int_\Omega z_i^0(x) \varphi(x,0) \, dx,
\]

where \( \varrho = \frac{\Phi(u)}{|u|} \), holds for any \( i = 1, \ldots, N \) and for all \( \varphi \in C^{2,1}(Q_T) \) vanishing on \( \Sigma_T \) and in \( \Omega \times \{ t = T \} \).

The notion of very weak solutions preserves the diagonal structure of the system as shown in the following lemma.
Lemma 3.1. If \( u \) is a non-negative bounded very weak (resp. energy) solution of system (1.8) in the sense of Definition 1 then \( w = |u|_1 \) is a non-negative bounded very weak (resp. energy) solution to problem (GPME) in the sense of Definition 1.

Proof. Sum equalities (3.3) over \( i \) from 1 to \( N \) and observe that by definition 
\[
\sum_i \rho_i u_i = \rho \sum_i u_i = \rho |u|_1 = \Phi(w), \quad \sum_i f_i = F, \quad \sum_i u_i^0 = |u^0|_1 = w^0, \quad \text{and} \quad \sum_i z_i^D = |z^D|_1 = g^D.
\]

We will now address uniqueness. Note that the following proposition guarantees uniqueness also within the class of energy solutions since weak energy solutions are in particular very weak solutions.

Proposition 3.1 (Uniqueness). Given the non-negative and bounded data \( f, u^0, z^D \) there exists at most one non-negative bounded very weak solution to problem (1.8) in the sense of Definition 1.

Proof. Let \( u^1 \) and \( u^2 \) be two solutions to problem (1.8), corresponding to the same initial and boundary data. It follows from the previous lemma that \( w^1 = |u^1|_1 \) and \( w^2 = |u^2|_1 \) are both bounded very weak solutions to the same Cauchy-Dirichlet problem (GPME). A standard comparison result for such solutions [27, Theorem 6.5] provides uniqueness within this class. Thus \( w^1 = w^2 = w \) and, in particular, the pressures coincide, \( \rho^1 = \rho^2 = \rho = \frac{\Phi(w)}{|w|} \). Next, we use a duality proof, as in proving the comparison results for GMPE, to show that \( u^1 = u^2 \). In fact, the situation here is simpler because we already know that \( \rho^1 = \rho^2 \). For the sake of completeness, we nonetheless give the details.

Fixing any \( i \in \{1 \ldots N\} \), denote \( \tilde{u} = u_i^1 - u_i^2 \), and subtracting the weak formulation (3.3) satisfied by \( u^2 \) from that satisfied by \( u^1 \), we see that
\[
\int_{Q_T} \left\{ \tilde{u} \partial_t \varphi + \rho \tilde{u} \Delta \varphi \right\} \ dx \ dt = 0
\]
for all \( \varphi \in C^{2,1}(Q_T) \) vanishing on \( \Sigma_T \cup \{ t = T \} \). Fix some arbitrary \( \theta \in C^\infty_0(Q_T) \), choose \( \varepsilon > 0 \), and let \( \varphi_{\varepsilon} = \max(\rho, \varepsilon) \). Since \( u^1 \) and \( u^2 \) are bounded so is \( \rho = \rho_1 = \rho_2 \), and we can construct a smooth approximation \( \{ \varphi_{\varepsilon,k} \}_{k \in \mathbb{N}} \) to \( \varphi_{\varepsilon} \) such that \( \varepsilon \leq \varphi_{\varepsilon,k} \leq C \). For fixed \( \varepsilon, k \) we can then solve the approximate dual backward equation
\[
\begin{align*}
\partial_t \varphi + \varphi_{\varepsilon,k} \Delta \varphi &= \theta & \text{in } Q_T \\
\varphi &= 0 & \text{in } \Sigma_T \\
\varphi(., T) &= 0 & \text{in } \Omega.
\end{align*}
\]
for a unique \( \varphi = \varphi_{\varepsilon,k} \in C^{2,1}(Q_T) \cap C^\infty_0(Q_T) \). Since \( \varphi \) vanishes by construction on \( \Sigma_T \) and in \( \Omega \times \{ t = T \} \) it is admissible as a test function in (3.5). This gives
\[
\left| \int_{Q_T} \tilde{u} \theta \ dx \ dt \right| = \left| \int_{Q_T} \tilde{u} (\theta - \varphi_{\varepsilon,k}) \Delta \varphi \ dx \ dt \right| \\
\leq \left( \int_{Q_T} \tilde{u}^2 |\theta - \varphi_{\varepsilon,k}|^2 \ dx \ dt \right)^{1/2} \left( \int_{Q_T} \varphi_{\varepsilon,k} |\Delta \varphi|^2 \ dx \ dt \right)^{1/2} \\
\leq \frac{C}{\varepsilon^{1/2}} \| \theta - \varphi_{\varepsilon,k} \|_{L^2(Q_T)} \left( \int_{Q_T} \varphi_{\varepsilon,k} |\Delta \varphi|^2 \ dx \ dt \right)^{1/2},
\]
for a unique \( \varphi = \varphi_{\varepsilon,k} \in C^{2,1}(Q_T) \cap C^\infty_0(Q_T) \). Since \( \varphi \) vanishes by construction on \( \Sigma_T \) and in \( \Omega \times \{ t = T \} \) it is admissible as a test function in (3.5). This gives
\[
\left| \int_{Q_T} \tilde{u} \theta \ dx \ dt \right| = \left| \int_{Q_T} \tilde{u} (\theta - \varphi_{\varepsilon,k}) \Delta \varphi \ dx \ dt \right| \\
\leq \left( \int_{Q_T} \tilde{u}^2 |\theta - \varphi_{\varepsilon,k}|^2 \ dx \ dt \right)^{1/2} \left( \int_{Q_T} \varphi_{\varepsilon,k} |\Delta \varphi|^2 \ dx \ dt \right)^{1/2} \\
\leq \frac{C}{\varepsilon^{1/2}} \| \theta - \varphi_{\varepsilon,k} \|_{L^2(Q_T)} \left( \int_{Q_T} \varphi_{\varepsilon,k} |\Delta \varphi|^2 \ dx \ dt \right)^{1/2},
\]
because $\tilde{u} \in L^\infty$ and $\varrho_{x,k} \geq \varepsilon$. Since $\varphi$ is smooth, a straightforward computation shows that (cf. [27, Theorem 6.5])

$$\left( \int_{Q_T} \varrho_{x,k} |\Delta \varphi|^2 \, dx \, dt \right)^{1/2} \leq C \|\nabla \theta\|_{L^2(Q_T)}$$

for some $C > 0$ independent of $\varepsilon, k, \theta$. For fixed $\varepsilon > 0$ we can then choose $k$ large enough such that $|\varrho_{x} - \varrho_{x,k}|_{L^2(Q_T)} \leq \varepsilon$. By definition of the cutoff function $\varrho_{x}$, we have $0 \leq \varrho_{x} - \varrho \leq \varepsilon$. Hence $|\varrho - \varrho_{x,k}|_{L^2(Q_T)} \leq C_\varepsilon$, and we obtain

$$\left| \int_{Q_T} \tilde{u} \theta \, dx \, dt \right| \leq C_\varepsilon^{1/2} \|\nabla \theta\|_{L^2(Q_T)}.$$  

Because $\theta \in C_0^\infty(Q_T)$ was arbitrary and $\varepsilon$ was independent of $\theta$ we conclude letting $\varepsilon \to 0$ that $\tilde{u} = u_1^i - u_2^2 = 0$ a.e. in $Q_T$ and the proof is complete.

**Remark 3.1.** The above uniqueness proof does not really require $L^\infty(Q_T)$ bounds but merely that $u, g u \in L^p_{\text{loc}}(Q_T)$. In fact, scalar parabolic equations such as (GPME) benefit usually from smoothing properties that should allow one to extend the theory to $L^1$ data. Due to the coupled vectorial nature of the problem and the lack of space we shall not pursue this direction here.

Theorem 1.1 allows for the initial data $u^0$ to vanish identically in some ball $B_r(x_0) \subset \Omega$. As we will see in Section 4, this leads to free boundaries in the degenerate case $\Phi' = 0$. It will also become clear in the proof that the structural condition $(H_4)$ needs to be enforced only to get the estimate (1.12). In fact, this energy estimate plays no role whatsoever in the analysis so one may actually dispense with it.

**Proof of Theorem 1.1.** Uniqueness follows from Proposition 3.1. The existence argument is based on a “lifting” technique, classical for scalar GPME and working here one again thanks to the diagonal $l^1(\mathbb{R}^N)$ structure of the system.

We first lift and approximate the bounded non-negative data $u^{0,k}, f^k, z^D$ component-wise by smooth functions $u^{0,k}_i, f^k_i, z^D_i$ such that $\frac{1}{k} \leq u^{0,k}_i, f^k_i, z^D_i \leq C + \frac{1}{k}$ for some constant $C > 0$ depending only on the data, and

$$\|u^{0,k} - u^0\|_{L^1(\Omega)} + \|f^k - f\|_{L^1(Q_T)} + \|z^D - z^D\|_{L^1(\Sigma_T)} \to 0$$

as $k \to \infty$. By Proposition 2.1, given the smooth data $u^{0,k}, f^k, z^D$ there exists a positive classical solution $u^k$ to (1.8) which is bounded in $Q_T$ uniformly in $k$. By virtue of Proposition 2.3 ({$u^k$}) is also bounded in $C^{\alpha,\alpha/2}(Q')$ for any subdomain $Q'$ and for some $\alpha = \alpha(a,n) \in (0,1)$. By diagonal extraction we may then assume that $u^k \to u$ in $C_{\text{loc}}(Q_T)$ with the limit function $u$ satisfying the local $C^{\alpha,\alpha/2}$-estimate (1.11). In particular $u^k(x,t) \to u(x,t) \geq 0$ pointwise in $Q_T$. From the continuity of $\Phi$ with $\lim_{s \to 0} \frac{\phi(s)}{s} = \Phi'(0)$ it then follows that $\phi^k = \phi([u^k_i])_{[u^k_i]} \to \phi([u_i])_{[u_i]} = \varphi$ a.e. in $Q_T$. Since $u^k, \varphi^k$ are bounded uniformly in $L^\infty(Q_T)$ we conclude by dominated convergence that $u^k_i \to u_i$ and $\phi^k u^k_i \to g u_i$ in $L^p(Q_T)$ for all $p \in [1,\infty)$. Given that $u^k$ is a smooth positive solution and that for all $i = 1, \ldots, N$ it holds

$$\int_{Q_T} \left\{ u^k_i \partial_i \varphi + \phi^k u^k_i |\Delta \varphi + f^k_i \varphi| \right\} \, dx \, dt + \int_{\Omega} u^{0,k}_i(x) \varphi(x,0) \, dx = \int_{\Sigma_T} z^D_i \partial_i \varphi \, \frac{\partial \varphi}{\partial \nu} \, dx \, dt.$$  

The previous strong $L^p(Q_T)$ convergence and convergence of the data allow one to send $k \to \infty$ to obtain (3.3). Similarly by Lemma 3.1 we see that $w = \lim u^k = \lim |u^k| = |u|$ is a very weak solution to (GPME).
Regarding the energy estimates, if the data satisfy (1.10) and (1.9), then they can be approximated as before by smooth positive data satisfying in addition
\[
\|u^{0,k}\|_{L^\infty(Q_T)} + \|f^k\|_{L^\infty(Q_T)} + \|\nabla w^k\|_{L^\infty(Q_T)} + \|\nabla \partial_t u^{0,k}\|_{L^2(Q_T)} + \|\nabla z^{D,k}\|_{L^\infty(Q_T)} \leq C.
\]
By Proposition 2.2 we get
\[
\|\nabla (\phi^k w^k)\|_{L^2(Q_T)} \leq C
\]
and
\[
\|\nabla (\phi^k u^k_i)\|_{L^2(Q_T)} \leq C(1 + 1/d') \quad \forall i = 1 \ldots N
\]
uniformly in \(k\) for some \(C = C(a, n, N, T, M)\) only. Since \(\phi^k u^k_i, \phi^k w^k \to \rho u_i, \rho w\) we conclude that \(\nabla (\rho u_i), \nabla (\rho w)\) satisfy the same \(L^2\) bounds and the proof is complete. \(\square\)

4. Free boundaries

In this section we set \(Q = \mathbb{R}^n \times (0, \infty), Q^{-T} = \mathbb{R}^n \times (\tau, T), Q^T = \mathbb{R}^n \times (0, T),\) and consider the Cauchy Problem
\[
\begin{align*}
\partial_t u &= \Delta \left( \frac{\Phi(|u|)}{|u|} u \right) \quad \text{in } \mathbb{R}^n \times (0, \infty) \\
u(x, 0) &= u^0(x) \quad \text{in } \mathbb{R}^n
\end{align*}
\]
with a non-negative, bounded and compactly supported initial data \(u^0\). If the modulus of ellipticity \(\rho = \Phi(|u|)/|u|\) in (4.1) vanishes when \(|u|_1 = 0\) (the degenerate case), compactly supported solutions should evolve from compactly supported initial data. By analogy with scalar equations, the free boundary \(\Gamma(t) := \partial \text{supp } u(., t)\) should then propagate with finite speed in the sense that, at any \(x_0 \not\in \text{supp } u(., t_0)\) we should have \(x_0 \not\in \text{supp } u(., t_0 + h)\) for small enough \(h > 0\). Although this behaviour is well understood for scalar equations, the coupled nature of system (4.1) prevents us from just recalling known results. Instead, we will again resort to our central idea, based on the particular structure of the system, that controlling \(w = |u|_1\) controls each individual species \(u_i\). Indeed, \(0 \leq u_i \leq |u|_1 = w\), thus \(\text{supp } u_i \subset \text{supp } w\) and \(u\) will propagate with finite speed as long as \(w\) does. This will in turn be ensured by looking at the scalar problem
\[
\begin{align*}
\partial_t w &= \Delta \Phi(w) \quad \text{in } \mathbb{R}^n \times (0, \infty) \\
w(x, 0) &= w^0(x) \quad \text{in } \mathbb{R}^n
\end{align*}
\]
with a non-negative, bounded and compactly supported initial data \(w^0 = |u^0|_1,\).

Given that assumption (\(H_2\)) does not rule out nondegenerate diffusion (we may have \(\Phi'(0) > 0\)) for which the finite speed of propagation obviously fails, we need to impose an extra degeneracy condition. For the scalar Cauchy problem (4.2) this is normally done through replacing the structural condition (\(H_2\)) by the slow diffusion hypothesis
\[
(S_{ab}) \quad \forall s > 0 : \quad 1 < a \leq \frac{s \Phi'(s)}{\Phi(s)} \leq b
\]
for some constants \(1 < a \leq b\). One readily sees that condition \((S_{ab})\) implies \(\Phi'(0) = 0\) and the algebraic behaviour \(0 \leq \Phi(1)s^b \leq \Phi(s) \leq \Phi(1)s^a\) for small \(s\), which usually provides information on the finite speed of propagation in terms of \(a, b\). In the case of pure PME nonlinearity \(\Phi(s) = s^m\) the Cauchy problem (4.2) has been widely studied and the qualitative and quantitative theory of free boundaries is now well understood, see e.g. [16, 17] and references therein. Partial results \([8, 11]\) also hold for general nonlinearities \(\Phi(s)\). Note that \((S_{ab})\) holds for the Freundlich isotherm \(b_f(z) = (1 + |z|^{m-1})z\) when \(s \to 0\) but not when \(s \to \infty\). Given that the
degeneracy is essentially a local feature at the level sets \( \{ s \approx 0 \} \) and that we are only considering globally bounded solutions \( 0 \leq |u|_1 = w \leq M \), it will be enough for our purpose to assume

\[
(S'_M) \quad \forall M > 0, \forall s \in (0, M) : \quad 1 < a_M \leq \frac{s \Phi'(s)}{\Phi(s)} \leq b_M.
\]

This local version of \((S_{ab})\) is readily checked for the Freundlich isotherm, and the lower bound \(a_M \searrow 1\) when \(M \to \infty\).

**Remark 4.1.** In one space dimension and for scalar GPME \( \partial_t w = \Delta \Phi(w) \), Peletier [23] showed that a necessary and sufficient condition for finite speed of propagation and existence of free boundaries is \( \int_0^1 \frac{dr}{\Phi(r)} < \infty \), where \( \beta = \Phi^{-1} \). This is again true for the Freundlich isotherm and holds whenever \((S'_M)\) does, but for the ease of the exposition we do not seek optimality here.

We start our analysis with a standard statement:

**Proposition 4.1.** Assume that conditions \((H_1)\) and \((S'_M)\) hold, and let the initial datum \( 0 \leq w^0(x) \leq M \) be compactly supported in \( B_{R_0} \) for some \( R_0 > 0 \). Then the Cauchy problem \((4.2)\) admits a unique weak energy solution \( 0 \leq w(x,t) \leq M \) and \( \| \nabla \Phi(w) \|_{L^2(Q)} \leq C(R_0, M, n) \). Moreover, \( w(\cdot,t) \) is compactly supported for all \( t > 0 \), the free boundary \( \Gamma(t) = \partial \text{supp} \, w(\cdot,t) \) propagates with finite speed, and

\[
(4.3) \quad \forall t \geq 0 : \quad \text{supp} \, w(\cdot,t) \subseteq B_{R(t)} \quad \text{with} \quad R(t) := R_0 + C_1 t^\lambda,
\]

where \( C_1(M, R_0) > 0 \) and \( \lambda = \lambda(M, n) > 0 \).

**Proof.** Existence and uniqueness are proven in [27]. Since \( 0 \leq w^0 \leq M \), the comparison principle gives \( 0 \leq w \leq M \) in \( Q \). The \( L^2(Q) \)-bound for \( \nabla \Phi(w) \) easily follows from letting \( t \to \infty \) in the classical energy identity

\[
(4.4) \quad \int_{B_R} \Psi(w(t,x)) \, dx + \int_0^t \int_{B_R} |\nabla \Phi(w(x,\tau))|^2 \, dx \, d\tau = \int_{B_R} \Psi(w^0(x)) \, dx
\]

where \( \Psi(s) := \int_0^s \Phi(s') \, ds' \) (see [27] for details). Indeed with our assumptions \( w^0 \) is bounded and compactly supported hence \( \| \Psi(w^0) \|_{L^1(\mathbb{R}^n)} \leq \Psi(M) \text{meas}(B_{R_0}) = C(a, M, R_0, n) \).

As for the finite speed of propagation, our assumption \((S'_M)\) and boundedness \( 0 \leq w(x,t) \leq M \) allow us to appeal to [11, Theorems 1.2 and 1.6] to conclude that \((4.3)\) holds as desired.

We can now establish the corresponding result on the multicomponent Cauchy problem \((4.1)\).

**Theorem 4.1** (Free Boundary solutions). Let conditions \((H_1)\), \((H_3)\), and \((S'_M)\) hold. Assume that \( u^0 \in L^\infty(\mathbb{R}^n) \) is componentwise non-negative with \( w^0 = |u^0|_1 \leq M \), and such that \( \text{supp} \, w^0 \subseteq B_{R_0} \) for some \( R_0 > 0 \). Then there exists a unique non-negative very weak solution \( u \in L^\infty(Q) \) to \((4.1)\). Moreover,

(i) \( w = |u|_1 \) is the unique weak energy solution to \((4.2)\), \( 0 \leq w \leq M \), and

\[ \| \nabla \Phi(w) \|_{L^2(Q)} \leq C(M, R_0) \]

(ii) \( u \) is a local energy solution to \((4.1)\) in the sense that for all \( T > 0 \) we have

\[ \forall i = 1, \ldots, N : \quad \| \nabla(gu_i) \|_{L^2(Q^T)} \leq C(M, R_0, T), \]

where \( g = \frac{\Phi(w)}{w} \).
(iii) $\text{supp } w(.,t)$ propagates with finite speed, and
\[ \forall t \geq 0, i = 1, \ldots, N: \quad \text{supp } u_i(.,t) \subseteq \text{supp } w(.,t) \subseteq B_{R(t)} \]
with $R(t) = R_0 + C_1 t^\lambda$ for some $C_1(M, R_0) > 0$ and $\lambda = \lambda(M, n) > 0$ only.

(iv) There is $\alpha = \alpha(M, n) \in (0,1)$ such that $u$ is $(\alpha, \alpha/2)$-Hölder continuous in any strip $Q^{\tau,T} = \mathbb{R}^n \times (\tau, T)$, $0 < \tau < T$, and
\[ \|u\|_{C^{\alpha/2}(Q^{\tau,T})} \leq C(1 + 1/\sqrt{T}). \]
for some $C(M, T, n, N) > 0$ only.

(v) If $\Phi$ is smooth in $\mathbb{R}^+$ then $u_i$ is smooth in $\{u_i > 0\} \cap \{t > 0\}$.

**Remark 4.2.** As in Theorem 1.1, the structural condition $(H_3)$ is only needed to get (ii) and can be relaxed by restricting (4.1) to very weak solutions instead of energy solutions. Moreover, as in Proposition 2.4, the Hölder regularity estimate (iv) can be extended up to $t = 0^+$ if we assume further $C^3(\mathbb{R}^n)$ regularity from $u^0$.

Note in particular that in (iii) we only claim that $w$ has finite speed of propagation but not that the individual species propagate with finite speed. In fact the support of each species can be discontinuous in time, see the discussion at the end of this section.

**Proof.** Arguing exactly as in the proof of Proposition 3.1 but choosing now test function $\theta \in C_0^\infty(\mathbb{R}^n \times (0, \infty))$ it is easy to show uniqueness within the class of very weak solutions. It is therefore enough to prove existence in finite time intervals $[0, T]$ for any fixed $T > 0$.

Given that the free boundary should a priori propagate with finite speed, solutions to the Cauchy problem in the whole space should agree with solutions of the Cauchy-Dirichlet problem in $B_R \times (0, T)$ with zero boundary conditions, as long as $R > 0$ is large enough so that the free boundary stays at a positive distance from $\partial B_R$ for all $t \leq T$. We should therefore be able to construct solutions to the Cauchy problem in $\mathbb{R}^n \times (0, T)$ by considering auxiliary Dirichlet problems in large balls and using Theorem 1.1.

From Proposition 4.1 we can define the unique solution $\bar{w}$ of the Cauchy problem (4.2) in $\mathbb{R}^n \times (0, \infty)$ with initial data $w^0 := |u^0|_1$. For any fixed $T > 0$ choose $R > 0$ large enough so that $R(T) \leq R/2$ and
\[ \forall t \leq T: \quad \text{supp } \bar{w}(.,t) \subseteq B_{R(T)} \subseteq B_{R/2}, \]
where $R(T) = R_0 + C_1 T^\lambda$ and the constants $C_1$ and $\lambda$ are as in Proposition 4.1. Next, let $u = u_R$ be the unique solution to problem (1.8) in $B_R \times (0, T)$ corresponding to the initial data $u^0$ and zero boundary values on $\partial B_R$ and given by Theorem 1.1. It also follows from Theorem 1.1 that $w := |u|_1$ is a weak solution to the corresponding Cauchy-Dirichlet problem in $B_R \times (0, T)$ with zero boundary values. Given the definition of $R$ we thus see that $\bar{w}$ remains at a distance $R/2$ away from $\partial B_R$. It is easy to see that the restriction $\bar{w}|_{B_R \times (0,T)}$ is also a weak solution to the same Cauchy-Dirichlet problem as $w$. By standard uniqueness theorem for weak solutions of (GPME) we conclude that $w = \bar{w}$ in $B_R \times (0, T)$. In particular
\[ \forall t \leq T, i = 1 \ldots N: \quad \text{supp } u_i(.,t) \subseteq \text{supp } w(.,t) = \text{supp } \bar{w}(.,t) \subseteq B_{R(t)} \]
where $R(t) = R_0 + C_1 t^\lambda$ as before, and the distance between $\text{supp } u$ and $\partial B_R$ is at least $R/2 > 0$ for all $t \leq T$. Extending $u$ and $w$ by zero outside of $B_{R_0}$ it is then a simple exercise to verify that these extensions satisfy the weak formulations of (4.1) and (4.2) in the whole space, whence existence of free-boundary solutions in $\mathbb{R}^n \times (0, T)$ for arbitrary $T > 0$. The energy estimate (i) and propagation properties (iii) immediately follow from the definition of $\bar{w}$ and Proposition 4.1.
For fixed $T > 0$ take now $R > 0$ large enough so that $u$ stays supported in $B_R$ for all $t \leq T$. Viewing $u$ as the unique solution to the Cauchy-Dirichlet problem in $B_{R+1}$ and taking $\Omega = B_{R+1}$, $\Omega' = B_R$ in Theorem 1.1 we have $d' = 1$ in (1.12), thus
\[
\|\nabla (gu_i)\|_{L^2(R^n \times (0,T))} = \|\nabla (gu_i)\|_{L^2(B_R \times (0,T))} \leq C(a, n, N, M, T)
\]
as claimed in (ii), with $a \equiv a(M)$ only depending on $a_M, b_M$ through (H$_2$).

Assertion (iv) is proven similarly by considering the Cauchy-Dirichlet problem in $B_{R+1}$, choosing $\Omega' = B_R \subset B_{R+1} = \Omega$ in Theorem 1.1 and taking $d' = 1$ in estimate (1.11).

To prove (v) we use a local bootstrap argument. If $w > 0$ in some $B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$, $t_0 > 0$, then in particular the pressure $p = \Phi'(w) > 0$ there. Since $w$ is Hölder continuous and $\Phi$ is smooth also $p$ is Hölder continuous. Moreover, $w$ solves a uniformly parabolic equation in divergence form: $\partial_tw = \Delta \Phi(w) = \text{div}(p
abla w)$. Hence $w \in C^{1+\beta}$ for some $\beta$. By bootstrapping we immediately see that $w$ is locally smooth. Consider now any species $u_i$ and observe that if $u_i > 0$ in $B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$ then $w = |u_i| \geq 0$ and also $p = \Phi(w)/w > 0$. Since $u_i$ solves $\partial_t u_i = \Delta (gu_i) = \text{div}(p\nabla u_i) + \text{div}(u_i \nabla g)$ with now smooth coefficients we conclude that $u_i$ is smooth and the proof is complete.

Although degenerate, problem (4.2) is nonetheless diffusive in nature and we expect that the information cannot propagate backwards as confirmed by the following result.

**Proposition 4.2** (Persistence property). Under the hypotheses of Theorem 4.1, the support $\Omega(t) := \{x : w(x,t) > 0\}$ of $w = |u|_1$ is non-contracting in time.

**Proof.** Dahlberg and Kenig [6] proved that nonnegative solutions to (4.2) satisfy certain Harnack inequality provided that $(S_{ab})$ holds globally in $s > 0$. In this case positivity of $w$ at $(x_0, t_0)$ implies positivity at $(x_0, t)$ for all later $t \geq t_0$ and the support is non-contracting. Since we are dealing here with bounded solutions $0 \leq |u|_1 = w \leq M$ and we enforced the local condition $(S_{ab}^M)$ we can essentially assume that $(S_{ab})$ holds, e.g. by extending $\Phi(s)$ outside of $[0, M]$ by some $C^1$ function $\tilde{\Phi}(s)$ satisfying $(S_{ab}^M)$ uniformly in $s > 0$. Our assertion then follows from the results in [6], see also [11, Corollary 1.5] for a precise statement.

We end this section with a formal discussion on a divide and rule behaviour of the system and a resulting finite/infinite speed of propagation for the individual species, which can be easily made rigorous.

Consider two sets of bounded and compactly supported initial data $\tilde{u}^0 = (u_{k}^0, \ldots, u_{k}^0)$ and $\tilde{a}^0 = (a_{k+1}^0, \ldots, a_{k}^0)$, consisting in two separate patches of $k$ and $N-k$ species, respectively, initially at a positive distance from each other. By Theorem 4.1(iii) the solutions $\tilde{u}(t,x)$ and $\tilde{a}(t,x)$ of the corresponding $k$ and $N-k$ dimensional Cauchy problem propagate with finite speed and thus stay at a positive distance, until their supports meet, possibly at some time $t = T$. It is then easy to check that, for $t \leq T$, $u = (\tilde{u}, \tilde{a})$ is the unique solution of the Cauchy problem with initial data $u^0 = (\tilde{u}^0, \tilde{a}^0)$. This is the aforementioned divide and rule behaviour: as long as the two supports do not meet, it is enough to solve two independent lower dimensional Cauchy problems. After the meeting time, the two patches start interacting, and the global system is not uncoupled anymore.

As a consequence, the supports of individual species may have a jump in time at $t = T$, even though the global support, $\text{supp } u = \text{supp } |u|_1 = \text{supp } w$, of the vector-valued solution propagates with finite speed as stated in Theorem 4.1. Consider, for example, two species $u = (u_1, u_2)$ with initial compact supports at a positive distance from each other. Assume that the supports meet for the first time at
$t = T$ and look like two tangent balls (thus only one species is present in each ball at that time). The persistence property implies that at $t = T^+$ the support of $w = |u_i| = u_1 + u_2$ should look like an 8-shaped domain with a thin aperture connecting the balls. Therefore, the diffusion coefficient $\rho = \Phi(w)$ becomes positive in the entire 8-shaped domain for $t > T$. Since each species solves globally the linear equation $\partial_t u_i = \text{div}(\rho \nabla u_i) + (\ldots)$, with support in either of the two balls at time $t = T$, the diffusion occurring across the aperture will ensure infinite speed of propagation between the two balls. Thus the support of $u_i$ should jump from only one ball at $t = T$ to the whole 8-shaped domain at $t = T^+$, but the global support $\text{supp } u_i = \text{supp } w = \text{supp } u_1 + u_2$ nonetheless propagates with finite speed.

This scenario is illustrated in figures 1 and 2 in the simpler one-dimensional setting (the numerical simulations were obtained by adapting the interface-tracking scheme from [22]).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{ figures.png}
\caption{Snapshot of $u_1(t, \cdot)$ (dashed line) and $u_2(t, \cdot)$ (solid line) depicted at different time instants. The supports meet at $t \approx 0.044$}
\end{figure}

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Figure 2. Supports in the \((x,t)\) plane: \(u_1\) (left), \(u_2\) (middle) and 
\(w = u_1 + u_2\) (right). Horizontal lines at \(t \approx 0.044\) correspond to
jumps in time when the supports meet for the first time, see also
Figure 1

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