Multiple Shocks in a Driven Diffusive System with Two Species of Particles

Farhad H Jafarpour

Physics Department, Bu-Ali Sina University, Hamadan, Iran
Institute for Studies in Theoretical Physics and Mathematics (IPM), P.O. Box 19395-5531, Tehran, Iran

Abstract

A one-dimensional driven diffusive system with two types of particles and nearest neighbors interactions has been considered on a finite lattice with open boundaries. The particles can enter and leave the system from both ends of the lattice and there is also a probability for converting the particle type at the boundaries. We will show that on a special manifold in the parameters space multiple shocks evolve in the system for both species of particles which perform continuous time random walks on the lattice.

Key words: Shock, Driven Diffusive Systems, Matrix Product States
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1 Introduction

Recently much attention has been paid to the study of one-dimensional driven diffusive systems because these relatively simple systems show a variety of critical phenomena such as out-of-equilibrium phase transitions and spontaneous symmetry breaking [1,2]. Another remarkable feature of these systems is that under special circumstances, for example by imposing restrictions on microscopic reaction rates, shock waves evolve in the system in the steady state [3]. A simple model of this kind is the Totally Asymmetric Simple Exclusion Process (TASEP). The TASEP is a system of particles with hardcore exclusion interactions which hop only in one direction on a one-dimensional lattice with open boundaries. This diffusion model can describe hopping conductivity in ionic conductors, traffic flow and interface growth [1,2]. The TASEP shows...
rich non-equilibrium behaviors such as shock wave [4,5,6], boundary induced phase transition [7], and dynamical scaling in the universality class of the Kardar-Parisi-Zhang equation [8,9]. This model has been solved exactly using a so-called Matrix Product Formalism (MPF) [10]. According to this formalism the stationary state probability distribution function of the system can be written in terms of expectation values of products of two non-commuting matrices which are associated with the existence of particles and holes at each site of the lattice and satisfy a quadratic algebra.

Another model of this kind is Partially Asymmetric Simple Exclusion Process (PASEP) with open boundaries. In this model, which is defined on an integer lattice $Z$ of length $L$, each site of the lattice $i$ ($1 \leq i \leq L$) is either empty $\tau_i = 0$ or occupied by at most one particle $\tau_i = 1$. The system evolves according to a stochastic dynamical rule: during each infinitesimal time step $dt$ the transitions allowed for the bond $(i, i + 1)$ are $(10 \rightarrow 01)$ with rate 1 and $(01 \rightarrow 10)$ with rate $x$. The parameter $x$ is positive and measures the strength of the driving field. Without losing generality one can assume that $x < 1$. Particles are injected from the first and the last sites of the lattice with rates $(1 - x)\alpha$ and $(1 - x)\delta$. They also leave the lattice from both the first and last sites with rates $(1 - x)\gamma$ and $(1 - x)\beta$ respectively. It is known that the phase diagram of the PASEP has three different phases depending on $\alpha, \beta, \gamma, \delta$ and $x$: in the steady state it has a high-density phase, a low-density phase, and a maximal-current phase. Recent investigations show that for the PASEP with open boundaries (and even on an infinite lattice) a travelling shock with a step-like density profile might evolve in the system provided that microscopic reaction rates are tuned appropriately [11,12]. The density of particles on the left and the right hand sides of the shock position is a function of these rates. The shock position then performs a random walk in the bulk of the system and also reflects from the boundaries. There is also a possibility for the existence of $n$ consecutive shocks in the system. In an infinite system consecutive multiple shocks evolve according to $n$-particle dynamics [11]. By slightly different definition of the shock it has been shown that the same phenomenon takes place for the PASEP on a finite lattice with open boundaries [12]. The PASEP has also been studied using the MPF [13,14]. It has been shown that its quadratic algebra has an $n$-dimensional matrix representation [15,16] provided that exactly the same constraints necessary for the existence of $n$ consecutive shocks in the system hold [12].

Despite the remarkable work on the dynamics and structure of shocks in the systems with one species of particles, not much is known about the shocks in the systems with more than one species of particles. In [17] the authors have introduced a one-dimensional driven lattice gas with two types of particles and nearest neighbor hopping. Part of their work is devoted to the study of a single shock dynamics in the system. They have investigated the time evolution equation of a product shock measure and shown that the shock is stable and performs a random walk provided that some constraints are satisfied. In this paper we study the same model; however, we investigate the possibilities
for the existence of multiple shocks in the system. It turns out that by imposing some conditions on the microscopic reaction rates, stable multiple shocks evolve in the system. For the special case where only a single shock exists, our results converge to the ones obtained in [17]. We will also show that under these constraints the quadratic algebra of this model can be mapped to the one which appears for the PASEP.

This paper is organized as follows. In section 2 we will briefly review the PASEP from the MPF point of view. In section 3 we will define the model and apply the MPF to study its steady state properties. In section 4 the conditions for the existence of multiple shocks in our model will be studied. The concluding remarks will be given in the last section.

2 The PASEP

In this section we will first review the steady state properties of the PASEP with open boundaries defined in section 1, because as we will see the steady state properties of our two-species model are closely related to those of the PASEP. The exact steady state properties of the PASEP for all values of the boundary and bulk parameters might be calculated by using the MPF [14] which involves the representation theory of a quadratic algebra equivalent to a $q$-deformed harmonic oscillator algebra [13]. According to the MPF the stationary probability distribution function of the PASEP is given by

$$P\{\{\tau_1, \cdots, \tau_L\}\} \propto \langle W| \prod_{i=1}^{L} (\tau_i D + (1 - \tau_i) E)|V\rangle$$

provided that the operators $D$ and $E$ besides the vectors $\langle W|$ and $|V\rangle$ satisfy the quadratic algebra [16]

$$DE - xED = (1 - x)(D + E)$$
$$\langle \beta D - \delta E|V\rangle = |V\rangle$$
$$\langle W| (\alpha E - \gamma D) = \langle W|.$$  (4)

The non-commuting operators $D$ and $E$ stand for the existence of a particle and a hole at each site of the lattice. It has been shown that the associated quadratic algebra (2-4) has exactly one $n$-dimensional irreducible representation for any finite $n$ provided that the following constraint is satisfied by the bulk and the boundary rates [15,16]

$$x^{1-n} = \kappa_+ (\alpha, \gamma) \kappa_+ (\beta, \delta)$$  (5)
Fig. 1. A typical Bernoulli shock measure for \( n-1 \) consecutive shocks for the PASEP. The density of particles at the shock positions \( k_i \) is 1.

The evolution of product shock measures in the PASEP has been studied both for open boundary condition and on an infinite lattice. In the following we study the dynamics of travelling shocks in the PASEP with open boundaries briefly. We define a Bernoulli measure for \( n-1 \) consecutive shocks as a product measure with density 1 at the shock positions \( k_i \) and intermediate densities \( \rho_i \) between sites \( k_{i-1} \) and \( k_i \). This product measure can be written as follows

\[
|k_1, k_2, \ldots, k_{n-1}\rangle = |\rho_1\rangle \otimes |1\rangle \otimes |\rho_2\rangle \otimes |1\rangle \otimes \cdots \otimes |1\rangle \otimes |\rho_n\rangle \otimes |L-k_{n-1}\rangle.
\]  

(7)

in which we have defined

\[
\kappa_+(u, v) = \frac{-u + v + 1 + \sqrt{(u - v - 1)^2 + 4uv}}{2u}.
\]  

(6)

Typical \( n \)-dimensional representations for (2-4) are given in [15] and [16]. The finite dimensional Fock representations allow us to drive exact results for the PASEP on some special curves of the phase diagram. Theses curves pass through the high-density and the low-density phases but not the maximal-current phase.

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\[
|k_1, k_2, \ldots, k_{n-1}\rangle = |\rho_1\rangle \otimes |1\rangle \otimes |\rho_2\rangle \otimes |1\rangle \otimes \cdots \otimes |1\rangle \otimes |\rho_n\rangle \otimes |L-k_{n-1}\rangle.
\]  

(7)

in which \( 0 < \rho_1 < \rho_2 < \cdots < \rho_n < 1 \). A typical shock measure is plotted in Fig. 1. Now the question is that how (7), as an initial configuration, evolves in time. This is given by the master equation. It is known [11,12] that for the PASEP with open boundaries the time evolution equation of the Bernoulli shock measure (7) has a closed form and is similar to the time evolution equation of \( n-1 \) random walkers on a finite lattice provided that we have

\[
\rho_1 = \frac{1}{1 + \kappa_+(\alpha, \gamma)}
\]  

(8)

\[
\rho_n = \frac{\kappa_+(\beta, \delta)}{1 + \kappa_+(\beta, \delta)}
\]  

(9)
and that the consecutive densities are related through

\[ \frac{\rho_{i+1}(1 - \rho_i)}{\rho_i (1 - \rho_{i+1})} = \frac{1}{x} \quad (10) \]

in which \( i = 1, \cdots, n - 1 \). By iteration (10) leads to

\[ \frac{\rho_n(1 - \rho_1)}{\rho_1 (1 - \rho_n)} = x^{1-n}. \quad (11) \]

The \( i \)th random walker hops to the left and the right with rates \( L_i = \frac{1 - \rho_{i+1}}{1 - \rho_i} x \) and \( R_i = \frac{1 - \rho_i}{1 - \rho_{i+1}} \) respectively. Using (8) and (9) one can rewrite (11) in terms of \( \kappa_+ (\alpha, \gamma) \) and \( \kappa_+ (\beta, \delta) \) to see that it is simply the condition (5) i.e. the necessary condition for the existence of an \( n \)-dimensional representation for (2-4). One can now construct the stationary state of the PASEP in terms of a linear superposition of Bernoulli shock measures with \( n - 1 \) shocks; therefore, the \( n \)-dimensional representation of the quadratic algebra describes the stationary linear combination of shock measures with \( n - 1 \) consecutive shocks.

3 Two-species model with open boundaries

The model is defined on a finite lattice of length \( L \) with two species of particles. Each site of the lattice is either empty or occupied by a particle of type \( A \) or \( B \). We assume that in the bulk of the lattice the particles hop to the left and right according to the following reaction rules

\[
\text{Bulk:} \begin{cases} 
A0 \rightarrow 0A & \text{with rate } 1 \\
0A \rightarrow A0 & \text{with rate } x \\
B0 \rightarrow 0B & \text{with rate } 1 \\
0B \rightarrow B0 & \text{with rate } x \\
AB \rightarrow BA & \text{with rate } y \\
BA \rightarrow AB & \text{with rate } y.
\end{cases} \quad (12)
\]
At the boundaries the particles are injected and extracted. The particle type can also be converted there. The boundary processes are defined as follows

\[
\text{Boundaries: } \begin{cases} 
0 \to A \text{ with rate } \alpha_{l(r),A} \\
0 \to B \text{ with rate } \alpha_{l(r),B} \\
A \to 0 \text{ with rate } \beta_{l(r),A} \\
B \to 0 \text{ with rate } \beta_{l(r),B} \\
A \to B \text{ with rate } \gamma_{l(r),A} \\
B \to A \text{ with rate } \gamma_{l(r),B} \\
\end{cases}
\] (13)

in which the indexes \( l \) and \( r \) indicate the left and the right boundaries respectively. In the following we will investigate the steady state properties of this model using the MPF.

We assign the operators \( E, A \) and \( B \) to an empty site, an \( A \) particle and a \( B \) particle respectively. By applying the standard MPF [10] we find the following quadratic algebra for our model defined by (12) and (13)

\[
AE - xEA = \bar{a}E + (\bar{a} + \bar{b})A
\] (14)

\[
BE - xEB = \bar{b}E + (\bar{a} + \bar{b})B
\] (15)

\[
y(AB - BA) = \bar{a}B - \bar{b}A
\] (16)

for the bulk operators and also

\[
[(\beta_{rA} + \gamma_{rA})A - \alpha_{rA}E - \gamma_{rB}B - \bar{a}]|V\rangle = 0
\] (17)

\[
[(\beta_{rA} + \gamma_{rA})B - \alpha_{rB}E - \gamma_{rA}B - \bar{b}]|V\rangle = 0
\] (18)

\[
\langle W|[(\beta_{lA} + \gamma_{lA})A - \alpha_{lA}E - \gamma_{lB}B + \bar{a}] = 0
\] (19)

\[
\langle W|[(\beta_{lB} + \gamma_{lB})B - \alpha_{lB}E - \gamma_{lA}A + \bar{b}] = 0
\] (20)

for the boundary terms in which \( \bar{a} \) and \( \bar{b} \) are non-zero arbitrary numbers. We have looked for finite-dimensional representations of the algebra. We have found that a one-dimensional representation for the algebra exists in which the operators \( A, B \) and \( E \) are replaced by non-zero real number \( a, b \) and \( e \). This representation is associated with a uniform density for each type of particles on the lattice. Defining these densities as \( \rho_A := \frac{a}{a+b+e} \) and \( \rho_B := \frac{b}{a+b+e} \) the necessary conditions for the existence of a one-dimensional representation are

\[
\bar{\rho}_A = \alpha_{lA}(1 - \rho_A - \rho_B) - (\beta_{lA} + \gamma_{lA})\rho_A + \gamma_{lB}\rho_B
\] (21)

\[
= (\beta_{rA} + \gamma_{rA})\rho_A - \alpha_{rA}(1 - \rho_A - \rho_B) - \gamma_{rB}\rho_B
\] (22)

\[
= (1-x)(1 - \rho_A - \rho_B)\rho_A
\] (23)
\[ \bar{\rho}_B = \alpha_{IB}(1 - \rho_A - \rho_B) - (\beta_{IB} + \gamma_{IB})\rho_B + \gamma_{IA}\rho_A \]  
(24)

\[ = (\beta_{rB} + \gamma_{rB})\rho_B - \alpha_{rB}(1 - \rho_A - \rho_B) - \gamma_{rA}\rho_A \]  
(25)

\[ = (1 - x)(1 - \rho_A - \rho_B)\rho_B \]  
(26)

in which \( \bar{\rho}_A := \frac{\bar{a}}{\bar{a} + \bar{b}} \) and \( \bar{\rho}_B := \frac{\bar{b}}{\bar{a} + \bar{b}} \). We have also found that the algebra (14-20) has a finite-dimensional representations provided that the matrices \( A \) and \( B \) commute with each other but not with \( E \). This means that in the steady state the probability for finding configurations consisting of blocks of mixtures of \( A \) and \( B \) particles surrounded by holes does not depend on the exact position of the particles in each block, instead only the number of them in each block will be important. For instance one can consider the case \( A = rB \) in which \( r \) is a non-zero real number. This condition indicates that the density of \( A \) particles on each site is always \( r \) times the density of \( B \) particles on the same site. By defining a new operator \( D := A + B = (1 + \frac{1}{r})A \) associated with the total density of particles on each site of the lattice and choosing \( \bar{a} := \frac{r(1-x)}{1+r} \) it can easily be verified that (14) and (15) both converge to (2) while (16) gives \( \bar{b} = \frac{1-x}{1+r} \). By defining two new parameters \( \delta \) and \( \beta \) and imposing the following constraints on the right boundary rates

\[ \delta := \frac{1}{1-x}(\alpha_{rA} + \alpha_{rB}) = \frac{(1+r)}{r(1-x)}\alpha_{rA} \]  
(27)

\[ \beta := \frac{1}{1-x}(\beta_{rA} + \gamma_{rA} - \frac{\gamma_{rB}}{r}) = \frac{1}{1-x}(\beta_{rB} + \gamma_{rB} - r\gamma_{rA}) \]  
(28)

in which \( \alpha_{rA} = r\alpha_{rB} \) one can see that the equations (17) and (18) become identical to (3). On the other hand one can define two new parameters \( \alpha \) and \( \gamma \) and impose some constraints on the left boundary rates

\[ \alpha := \frac{1}{1-x}(\alpha_{IA} + \alpha_{IB}) = \frac{(1+r)}{r(1-x)}\alpha_{IA} \]  
(29)

\[ \gamma := \frac{1}{1-x}(\beta_{IA} + \gamma_{IA} - \frac{\gamma_{IB}}{r}) = \frac{1}{1-x}(\beta_{IB} + \gamma_{IB} - r\gamma_{IA}) \]  
(30)

in which \( \alpha_{IA} = r\alpha_{IB} \) to see that the equations (19) and (20) become identical to (4). Therefore, the operator \( D \) associated with the total density of particles of kind \( A \) and \( B \) on each site of the lattice besides the operator \( E \) satisfy the PASEP quadratic algebra, provided that the constraints (27-30) are satisfied. Now one can easily see that on an special manifold defined by the aforementioned constraints in the parameters space of our model an \( n \)-dimensional representation exists for the quadratic algebra (14-20) provided that (5) is also held. To conclude, we have shown that under some conditions the quadratic algebra of our two-species model defined by (12) and (13) has exactly one \( n \)-dimensional irreducible representation for any finite \( n \). Having
a finite-dimensional representation for the algebra one can easily calculate the physical quantities such as the mean density of particles at each site.

4 Dynamics of Multiple shocks

Now we investigate the shock formation and also the shock dynamics in our two-species model. Since the quadratic algebra of the model in terms of the total density of particles operator $D$ is exactly the one for the PASEP and therefore, has a finite-dimensional representation, one can conclude that multiple shocks might evolve in the system provided that the constraints (8), (9) and (10) obtained for the PASEP also hold for the total density of particles in our model defined as $\rho := \rho_A + \rho_B$. However, since the density of $A$ particles on each site is always $r$ times the density of $B$ particles at the same site it turns out that the multiple shocks structures exist for both densities $\rho_A$ and $\rho_B$ separately.

In the following we will study the case where there is only a single shock in the system in details. This will be associated with the existence of a two-dimensional representation for the quadratic algebra. We define the total density of particles on the left- (right-) hand site of the shock position as $\rho_{l(r)} := \rho_{l(r)A} + \rho_{l(r)B}$. Since $A = rB$ we always have

\[
\frac{\rho_A}{\rho_B} = \frac{\rho_{rA}}{\rho_{rB}} = r 
\]

\[
\rho_{l(r)A} = \frac{r}{1 + r} \rho_{l(r)} \quad \text{(32)}
\]

\[
\rho_{l(r)B} = \frac{1}{1 + r} \rho_{l(r)} \quad \text{(33)}
\]

From our discussion in the previous section and using (11) we see that in order to have a single shock the total density of particles on different sides of the shock should satisfy the condition

\[
\frac{\rho_r (1 - \rho_l)}{\rho_l (1 - \rho_r)} = \frac{1}{x} \quad \text{(34)}
\]

On the other hand the total density of particles on each side of the shock should be obtained from (8) and (9)

\[
\rho_l = \frac{1}{1 + \kappa_+ (\alpha, \gamma)} \quad \text{(35)}
\]

\[
\rho_r = \frac{\kappa_+ (\beta, \delta)}{1 + \kappa_+ (\beta, \delta)} \quad \text{(36)}
\]
in which $\alpha$, $\beta$, $\gamma$ and $\delta$ should be replaced from (27-30). It is not difficult to verify that using (27-30) the condition (35) can also be written as

$$\alpha_{lA} = \rho_{lA}((1 - x) + \frac{\beta_{lA} + \gamma_{lA} - \gamma_{lB}}{1 - \rho_{lA} - \rho_{lB}})$$

and (36) as

$$\alpha_{rA} = \rho_{rA}(-(1 - x) + \frac{\beta_{rA} + \gamma_{rA} - \gamma_{rB}}{1 - \rho_{rA} - \rho_{rB}})$$

As we mentioned in section 2 the shock position hops to the left and to the right with the following rates:

$$L = \frac{1 - \rho_l}{1 - \rho_r} x$$

$$R = \frac{1 - \rho_r}{1 - \rho_l}.$$  

It is also known that for the PASEP with a single shock the shock position reflects from the left boundary with rate $\bar{L} = (1 - x)(\frac{\alpha}{\rho_l} + \rho_l)$ and also from the right boundary with rate $\bar{R} = (1 - x)(\frac{\alpha}{\rho_r} - \rho_r)$. For our two-species model they become

$$\bar{L} = \frac{\alpha_{rA} + \alpha_{rB}}{\rho_r} + (1 - x)\rho_l$$

$$\bar{R} = \frac{\alpha_{lA} + \alpha_{lB}}{\rho_l} - (1 - x)\rho_r.$$  

In [17] the authors have introduced a two-species model with open boundaries similar to the one studied here and investigated the time evolution of a single shock in the system by introducing a product shock measure. They have found that under exactly the same conditions which we introduced in (27)-(38) a stable single shock evolves in the system which performs a random walk on the lattice. However, in this paper we have proven that not only one, but also multiple stable shocks might evolve in the system under slightly different conditions. The expressions which we have obtained for the hopping rate of the shock positions in the bulk of the lattice and also the reflection rates from the boundaries are quit in agreement with those obtained in [17]. One
can easily check that the necessary conditions for having a stationary product measure with uniform densities for both kinds of particles obtained in [17] are exactly the conditions for the existence of a one-dimensional representation of the quadratic algebra (14-20) obtained in (21-26).

5 Concluding remarks

In this paper we have studied a two-species model with open boundaries and the conditions under which an invariant multiple shocks measure exists for it. We have found that from the MPF point of view the quadratic algebra of the model can be mapped to that obtained for the PASEP, if one defines the total density of particles, and also redefines the boundary conditions (the sum of the densities of particles at each site is defined as the total density of particles at that site). Then we have been able to introduce a finite-dimensional Fock representation for this quadratic algebra, provided that the microscopic rates satisfy some constraints. It terms of the total density of particles these constraints are nothing but the necessary conditions for the existence of multiple shocks in the system. The associated quadratic algebra of the model has also a one-dimensional representation under some restrictions which are the conditions for the existence of a uniform stationary product measure for the model. For the single-shock case our results obtained from MPF analysis agree with those obtained in [17] from study of the product shock measure dynamics. In [12] the authors have shown that for three different families of one-dimensional driven diffusive models with open boundaries, the stationary state distribution function can be written as a linear superposition of single shock measures, provided that some constraints are fulfilled. In [18] the author has studied the same models and shown that from the MPF point of view these constraints are the necessary conditions for the existence of two-dimensional representations for their quadratic algebras. The problem now worth studying is whether every finite-dimensional representation of the quadratic algebra of a given one-dimensional driven diffusive model with open boundaries is associated with expression of its stationary state distribution function in terms of superposition of product shock measures. This is still under investigation.

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