IONESCU’S THEOREM FOR HIGHER RANK GRAPHS

S. KALISZEWSKI, ADAM MORGAN, AND JOHN QUIGG

Abstract. We will define new constructions similar to the graph systems of correspondences described by Deaconu et al. We will use these to prove a version of Ionescu’s theorem for higher rank graphs. Afterwards we will examine the properties of these constructions further and make contact with Yeend’s topological k-graphs and the tensor groupoid valued product systems of Fowler and Sims.

1. Introduction

In [Ion07], Ionescu defines a natural correspondence associated to any Mauldin-Williams graph. A Mauldin-Williams graph is a directed graph with a compact metric space associated to each vertex and a contractive map associated to each edge (a more rigorous definition is presented below). These graphs generalize iterated function systems and have self-similar invariant sets. Ionescu proved that the Cuntz-Pimsner algebra of the correspondence associated to any Mauldin-Williams graph is isomorphic to the graph $C^*$-algebra of the underlying graph.

Here we prove an analogue for higher rank graphs. Our arguments make extensive use of the graph systems of correspondences construction presented in [DKPS10] and (we hope) provide an interesting application of their ideas. We also define some other systems similar to those [defined in] of [DKPS10] and briefly describe how all of these systems fit into what Fowler and Sims refer to in [FS02] as product systems taking values in tensor groupoids.

This paper is organized as follows. In Section 2 we will present a brief overview of some of the preliminaries on $C^*$-correspondences, graph systems of correspondences, and topological k-graph algebras. In Section 3 we will define two systems that closely resemble A-systems of correspondences which we will call A-systems of homomorphisms and A-systems of maps. The A-system of maps will be a generalization

Date: October 29, 2014.

2000 Mathematics Subject Classification. Primary 46L05.

Key words and phrases. higher-rank graph $C^*$-algebra, Mauldin-Williams graph.
of the notion of a Mauldin-Williams graph. After defining some more terminology, we prove some basic facts about these systems and how they relate to one another. In Section 4, we define a k-graph analog of Mauldin-Williams graphs and prove our main theorem which generalizes Ionescu's main result from [Ion07]. In Section 5, we prove that the Cuntz-Pimsner algebra of the correspondence associated to any Λ-system of maps can be realized as the graph algebra of a certain topological k-graph. In Section 6, we briefly describe how all of the various Λ-systems fit into the framework described by Fowler and Sims in [FS02]. In Section 7, we will examine the question of which Λ-systems of correspondences arise from the other types of Λ-systems described here. Finally, in Section 8, we show that, perhaps disappointingly, the higher-rank Mauldin-Williams graphs of Section 4 do not give rise to any new “higher-rank fractals”.

2. Preliminaries

2.1. Correspondences. For C*-algebras A and B, we usually want our A − B correspondences X to be nondegenerate in the sense that $A \cdot X = X$, equivalently, the left-module homomorphism $\varphi_A : A \to \mathcal{L}_B(X) = M(\mathcal{K}_B(X))$ is nondegenerate. If $\varphi : A \to M(B)$ is a nondegenerate homomorphism, the standard $A − B$ correspondence $\varphi B$ is B viewed as a Hilbert module in the usual way and equipped with the left A-module structure induced by $\varphi$. The identity $B$-correspondence is $\text{id}_B$.

An isomorphism of an $A − B$ correspondence X onto a $C − D$ correspondence Y is a triple $(\theta_A, \theta, \theta_B)$, where

- $\theta_A : A \to C$ and $\theta_B : B \to D$ are C*-isomorphisms, and
- $\theta : X \to Y$ is a linear bijection such that

$$\langle \theta(\xi), \theta(\eta) \rangle_D = \theta_B(\langle \xi, \eta \rangle_B) \quad \text{for all } \xi, \eta \in X;$$

$$\theta(a \cdot \xi \cdot b) = \theta_A(a) \cdot \theta(\xi) \cdot \theta_B(b) \quad \text{for all } a \in A, \xi \in X, b \in B.$$  

$\theta_A$ and $\theta_B$ are the left- and right-hand coefficient isomorphisms, respectively. When both X and Y are $A − B$ correspondences, we require, unless otherwise specified, that the coefficient isomorphisms be the identity maps, and we sometimes emphasize that we are making this assumption by saying that $\theta : X \to Y$ is an $A − B$ correspondence isomorphism.

Recall from [EKQR00, Proposition 2.3] that for two nondegenerate homomorphisms $\varphi, \psi : A \to M(B)$, the standard $A − B$ correspondences $\varphi B$ and $\psi B$ are isomorphic if and only if there is a unitary multiplier $u \in M(B)$ such that $\psi = \text{Ad} u \circ \varphi$ (the special case of
imprimitivity bimodules is essentially \cite[Proposition 3.1]{BGR77}). In particular, if $B$ is commutative then $\varphi B \cong \psi B$ if and only if $\varphi = \psi$.

2.2. $\Lambda$-systems. Throughout, $\Lambda$ will be a row-finite $k$-graph with no sources, so that the associated Cuntz-Krieger relations take the most elementary form. In \cite{DKPS10}, Deaconu, Kumjian, Pask, and Sims introduced $\Lambda$-systems of correspondences: we have a Banach bundle $X \to \Lambda$ with fibres $\{X_\lambda\}_{\lambda \in \Lambda}$ such that

1. for each $v \in \Lambda^0$, $X_v$ is a $C^*$-algebra;
2. for each $\lambda \in u\Lambda v$, $X_\lambda$ is an $X_u - X_v$ correspondence;
3. there is a partially-defined associative multiplication on $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$ that is compatible with the multiplication in $\Lambda$ via the bundle projection $X \to \Lambda$;
4. whenever $\lambda, \mu \not\in \Lambda^0$ and $s(\lambda) = r(\mu)$, $x \otimes y \mapsto xy : X_\lambda \otimes_{A_s(\lambda)} X_\mu \to X_{\lambda\mu}$ is an isomorphism of $A_{r(\lambda)} - A_{s(\mu)}$ correspondences;
5. the left and right module multiplications of the correspondences coincide with the multiplication from the $\Lambda$-system.

For a $\Lambda$-system $X$ of correspondences, we will write

$$\varphi_\lambda : X_u \to \mathcal{L}(X_\lambda) \quad \text{for } \lambda \in u\Lambda$$

for the left-module structure map. Note that the multiplication in $X$ induces $X_u - X_v$ correspondence isomorphisms $X_\lambda \otimes X_v \cong X_\lambda$ for all $\lambda \in u\Lambda v$, but only induces isomorphisms $X_u \otimes X_\lambda \cong X_\lambda$ if every correspondence $X_\lambda$ is nondegenerate.

Given $\Lambda$-systems $X$ and $Y$ of correspondences, a map $\theta : X \to Y$ is a $\Lambda$-system isomorphism if

1. for all $\lambda \in u\Lambda v$,
$$\theta_\lambda := \theta|_{X_\lambda} : X_\lambda \to Y_\lambda$$

is an isomorphism of correspondences with coefficient isomorphisms $\theta_u, \theta_v$;
2. for all $\lambda \in u\Lambda v, \mu \in v\Lambda w$,
$$\theta_\lambda(\xi)\theta_\mu(\eta) = \theta_{\lambda\mu}(\xi\eta) \quad \text{for all } \xi \in X_\lambda, \eta \in X_\mu.$$

Since the multiplication in the $\Lambda$-system induces the left and right module multiplications for the correspondences, in the above we can relax (1) to

1. for all $\lambda \in \Lambda v$, $\theta_\lambda : X_\lambda \to Y_\lambda$ is a linear bijection satisfying
$$\langle \theta_\lambda(\xi), \theta_\lambda(\eta) \rangle_{Y_v} = \theta_v(\langle \xi, \eta \rangle_{X_v}) \quad \text{for all } \xi, \eta \in X_\lambda,$$

because (2) takes care of the coefficient maps. We emphasize that we’re requiring that, for each $v \in \Lambda^0$, $\theta_v$ be the right-hand coefficient
isomorphism for every correspondence isomorphism \( \theta_\lambda \) with \( s(\lambda) = v \), and also the left-hand coefficient isomorphism for every correspondence isomorphism \( \theta_\lambda \) with \( r(\lambda) = v \). Thus, if \( X \) and \( Y \) are isomorphic \( \Lambda \)-systems of correspondences, then without loss of generality we may assume (if we wish) that \( X_v = Y_v \) and \( \theta_v = \text{id}_{X_v} \) for every vertex \( v \), so that \( \theta_\lambda : X_\lambda \to Y_\lambda \) is an \( X_u - X_v \) correspondence isomorphism whenever \( \lambda \in u \Lambda v \).

2.3. Topological \( k \)-graphs. Recall [Yee06] that a topological \( k \)-graph is a \( k \)-graph \( \Gamma \) equipped with a locally compact Hausdorff topology making the multiplication continuous and open, the range map continuous, the source map a local homeomorphism, and the degree functor \( d : \Gamma \to \mathbb{N}^k \) continuous. Carlsen, Larsen, Sims, and Vittadello show in [CLSV11, Proposition 5.9] that every topological \( k \)-graph \( \Gamma \) gives rise to a \( \mathbb{N}^k \)-system \( Z \) of correspondences over \( A := C_0(\Gamma^0) \) as follows: For each \( n \in \mathbb{N}^k \) let \( Z_n \) be the \( A \)-correspondence associated to the topological graph \( (\Gamma^n, \Gamma^n, s|_{\Gamma^n}, r|_{\Gamma^n}) \), so that \( Z_n \) is the completion of the pre-correspondence \( C_c(\Gamma^n) \) with operations

\[
(f \cdot \xi \cdot g)(\alpha) = f(r(\alpha))\xi(\alpha)g(s(\alpha))
\]

and

\[
\langle \xi, \eta \rangle_A(v) = \sum_{\alpha \in \Gamma^n v} \xi(\alpha)\eta(\alpha),
\]

for \( \xi, \eta \in C_c(\Gamma^n) \), \( f, g \in A \). Then for \( \xi \in C_c(\Gamma^n), \eta \in C_c(\Gamma^m) \) define \( \xi \eta \in C_c(\Gamma^{n+m}) \) by

\[
(\xi \eta)(\alpha \beta) = \xi(\alpha)\eta(\beta) \quad \text{for } \alpha \in \Gamma^n, \beta \in \Gamma^m, s(\alpha) = r(\beta).
\]

In [Yee06], Yeend defined \( C^*(\Gamma) \) using a groupoid model, but [CLSV11, Theorem 5.20] shows that \( C^*(\Gamma) \cong \mathcal{N}\mathcal{O}_Z \), where \( \mathcal{N}\mathcal{O}_Z \) is the Cuntz-Nica-Pimsner algebra of the product system \( Z \). The topological \( k \)-graphs we encounter in this paper will be nice enough that \( \mathcal{N}\mathcal{O}_Z \) will coincide with the Cuntz-Pimsner algebra \( \mathcal{O}_Z \).

3. Other \( \Lambda \)-systems

We introduce a few constructions that are similar to \( \Lambda \)-systems of correspondences:

**Definition 3.1.**  (1) A \( \Lambda \)-system of homomorphisms is a pair \((A, \varphi)\), where \( A \to \Lambda^0 \) is a \( C^* \)-bundle and for each \( \lambda \in u \Lambda v \) we have a nondegenerate homomorphism \( \varphi_\lambda : A_u \to M(A_v) \), such that

\[
\varphi_\mu \circ \varphi_\lambda = \varphi_{\lambda \mu} \quad \text{if } s(\lambda) = r(\mu)
\]

\[
\varphi_v = \text{id}_{A_v} \quad \text{for } v \in \Lambda^0,
\]
where we have canonically extended $\varphi_\mu$ to $M(A_v)$.

(2) A $\Lambda$-system of maps is a pair $(T, \sigma)$, where $T \to \Lambda^0$ is a bundle of locally compact Hausdorff spaces and for each $\lambda \in u\Lambda v$ we have a continuous map $\sigma_\lambda : T_v \to T_u$, such that

\[ \sigma_\lambda \circ \sigma_\mu = \sigma_{\lambda\mu} \text{ if } s(\lambda) = r(\mu) \]
\[ \sigma_v = \text{id}_{T_v} \text{ for } v \in \Lambda^0. \]

**Remark 3.2.**

(1) Note that we need to impose the nondegeneracy condition on the homomorphisms $\varphi_\lambda$ so that composition is defined.

(2) Thus, a $\Lambda$-system of homomorphisms is essentially a contravariant functor from $\Lambda$ to the category of $C^*$-algebras and nondegenerate homomorphisms into multiplier algebras, and a $\Lambda$-system of maps is essentially a (covariant) functor from $\Lambda$ to the category of locally compact Hausdorff spaces and continuous maps.

(3) Every $\Lambda$-system $(T, \sigma)$ of maps gives rise to a $\Lambda$-system $(A, \sigma^*)$ of homomorphisms, with

\[ A_v = C_0(T_v) \text{ for } v \in \Lambda^0 \]
\[ \sigma_\lambda^*(f) = f \circ \sigma_\lambda \text{ for } \lambda \in \Lambda, f \in A_r(\lambda). \]

(4) Every $\Lambda$-system $(A, \varphi)$ of homomorphisms gives rise to a $\Lambda$-system of correspondences: for $\lambda \in u\Lambda v$ let $X_\lambda$ be the standard $A_u - A_v$ correspondence $\varphi_\lambda A_v$.

**Definition 3.3.** We call a $\Lambda$-system of maps $(T, \sigma)$

(1) **proper** if each map $\sigma_\lambda : T_{s(\lambda)} \to T_{r(\lambda)}$ is proper (in the usual sense that inverse images of compact sets are compact), and

(2) **dense** if each map $\sigma_\lambda : T_{s(\lambda)} \to T_{r(\lambda)}$ has dense range.

**Definition 3.4.** We call a $C^*$-homomorphism $\varphi : A \to M(B)$ **proper** if it maps into $B$ (and we will also denote it by $\varphi : A \to B$).

**Remark 3.5.** A nondegenerate homomorphism $\varphi : A \to M(B)$ is proper in the above sense if and only if $\varphi$ takes one (hence every) bounded approximate identity for $A$ to an approximate identity for $B$. Also, if $\sigma : Y \to X$ is a continuous map, then $\sigma^* : C_0(X) \to M(C_0(Y))$ is automatically nondegenerate, and is proper if and only if $\sigma$ is proper.

**Definition 3.6.** Let $X$ an $A - B$ correspondence, with left module map $\varphi_A : A \to \mathcal{L}(X) = M(K(X))$. We call $X$ **proper, nondegenerate**, or **faithful** if $\varphi_A$ has the corresponding property.
Definition 3.7. We call a $\Lambda$-system $(A, \varphi)$ of homomorphisms proper or faithful if each homomorphism $\varphi_\lambda$ has the corresponding property.

Definition 3.8. We call a $\Lambda$-system $X$ of correspondences proper, nondegenerate, full, or faithful if each correspondence $X_\lambda$ has the corresponding property.

[DKPS10] call a $\Lambda$-system $X$ of correspondences regular if it is proper, nondegenerate, full, and faithful. However, we believe that the fidelity is too much to ask, both for aesthetic and practical reasons.

Let $X$ be a $\Lambda$-system of correspondences, and let $A = \bigoplus_{v \in \Lambda^0} X_v$ be the $c_0$-direct sum of $C^*$-algebras. Then each $X_\lambda$ may be regarded as an $A$-correspondence. For each $n \in \mathbb{N}^k$, [DKPS10] defines an $A$-correspondence $Y_n$ by

$$Y_n = \bigoplus_{\lambda \in \Lambda^n} X_\lambda,$$

and [DKPS10] Proposition 3.17] shows that $Y = Y_X = \bigcup_{n \in \mathbb{N}^k} Y_n$ is an $\mathbb{N}^k$-system (i.e., a product system over $\mathbb{N}^k$) of $A$-correspondences, and moreover if $X$ is regular then so is $Y$. We will identify $X_\lambda$ with its canonical image in $Y_{d(\lambda)}$, i.e., we will blur the distinction between the external and internal direct sums of the $A$-correspondences $\{X_\lambda : \lambda \in \Lambda^n\}$.

Definition 3.9. We call a $\Lambda$-system $(T, \sigma)$ of maps

1. $k$-dense if for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$,

$$T_v = \bigcup_{\lambda \in v \Lambda^n} \sigma_\lambda(T_{s(\lambda)}),$$

and

2. $k$-regular if it is proper and $k$-dense.

Here is a minor strengthening of $k$-density that we will find useful later:

Definition 3.10. A $\Lambda$-system of maps $(T, \sigma)$ is $k$-surjective if

$$T_v = \bigcup_{\lambda \in v \Lambda^n} \sigma_\lambda(T_{s(\lambda)}) \text{ for all } v \in \Lambda^0, n \in \mathbb{N}^k.$$

Definition 3.11. We call a $\Lambda$-system $(A, \varphi)$ of homomorphisms

1. $k$-faithful if for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$,

$$\bigcap_{\lambda \in v \Lambda^n} \ker \varphi_\lambda = \{0\},$$

and
(2) $k$-regular if it is proper and $k$-faithful.

**Definition 3.12.** We call a $\Lambda$-system $X$ of correspondences

1. $k$-faithful if the associated $N^k$-system $Y_X$ is faithful, and
2. $k$-regular if it is proper, nondegenerate, full, and $k$-faithful.

**Remark 3.13.**

(1) If $(T, \sigma)$ is a $\Lambda$-system of maps, then the associated $\Lambda$-system $(A, \sigma^*)$ of homomorphisms is:

- proper if and only if $(T, \sigma)$ is, and
- faithfull if and only if $(T, \sigma)$ is dense.

(2) If $(A, \varphi)$ is a $\Lambda$-system of homomorphisms, then the associated $\Lambda$-system $X$ of correspondences is:

- automatically nondegenerate and full, and
- proper or faithful if and only if $(A, \varphi)$ has the corresponding property.

(3) We have organized our definitions so that a $\Lambda$-system $X$ of correspondences is $k$-regular if and only if the associated $N^k$-system $Y_X$ is regular.

We will need the following variation on the above:

**Lemma 3.14.**

1. If $(T, \sigma)$ is a $\Lambda$-system of maps, then the associated $\Lambda$-system $(A, \sigma^*)$ of homomorphisms is $k$-faithful if and only if $(T, \sigma)$ is $k$-dense, and consequently is $k$-regular if and only if $(T, \sigma)$ is.

2. If $(A, \varphi)$ is a $\Lambda$-system of homomorphisms, then the associated $\Lambda$-system $X$ of correspondences is $k$-faithful if and only if $(A, \varphi)$ is, and consequently is $k$-regular if and only if $(X, \varphi)$ is.

**Proof.** (1). This is routine, but we present the details for completeness. First assume that $(T, \sigma)$ is not $k$-dense. We will show that $(A, \sigma^*)$ is not $k$-faithful. We can choose $v \in \Lambda^0$ and $n \in N^k$ such that $\bigcup_{\lambda \in v \Lambda^n} \lambda(T(s(\lambda)))$ is not dense in $T_v$. We will show that $\bigcap_{\lambda \in v \Lambda^n} \ker \sigma^*_\lambda \neq \{0\}$. We can choose a nonzero $f \in C_0(T_v)$ that vanishes on $\bigcup_{\lambda \in v \Lambda^n} \lambda(T(s(\lambda)))$. Then for all $\lambda \in v \Lambda^n$ and all $g \in C_0(T(s(\lambda)))$,

$$
\sigma^*_\lambda(f)g = (f \circ \sigma_\lambda)g = 0.
$$

Thus $f \in \bigcap_{\lambda \in v \Lambda^n} \ker \sigma^*_\lambda$.

Conversely, assume that $(A, \sigma^*)$ is not $k$-faithful. We will show that $(T, \sigma)$ is not $k$-dense. We can choose $v \in \Lambda^0$ and $n \in N^k$ such that $\bigcap_{\lambda \in v \Lambda^n} \ker \sigma^*_\lambda \neq \{0\}$. We will show that $\bigcup_{\lambda \in v \Lambda^n} \lambda(T(s(\lambda)))$ is not dense in $T_v$. Choose a nonzero $f \in \bigcap_{\lambda \in v \Lambda^n} \ker \sigma^*_\lambda$. Then choose a nonempty open set $U \subset T_v$ such that $f(t) \neq 0$ for all $t \in U$. We will show that

$$
U \cap \bigcup_{\lambda \in v \Lambda^n} \lambda(T(s(\lambda))) = \emptyset.
$$
Let \( t \in \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_s(\lambda)) \), and choose \( \lambda \in v\Lambda^n \) and \( s \in T_s(\lambda) \) such that \( t = \sigma_\lambda(s) \). Then choose \( g \in C_0(T_s(\lambda)) \) such that \( g(s) = 1 \). Since \( f \in \ker \sigma^*_\lambda \),

\[
0 = (\sigma^*_\lambda(f)g)(s) = f(\sigma_\lambda(s))g(s) = f(t),
\]

so \( t \not\in U \).

(2). First assume that \((A,\varphi)\) is not \( k\)-faithful. We will show that \( X \) is not \( k\)-faithful. We can choose \( v \in \Lambda^0 \) and \( n \in \mathbb{N}_k \) such that \( \bigcap_{\lambda \in v\Lambda^n} \ker \varphi_\lambda \neq \{0\} \). We will show that the \( A \)-correspondence \( Y_n \) is not faithful. Choose a nonzero \( a \in A_v \) such that \( \varphi_\lambda(a) = 0 \) for all \( \lambda \in v\Lambda^n \). Let

\[
y = (x_\lambda) \in Y_n = \bigoplus_{\lambda \in \Lambda^n} X_\lambda.
\]

Then \( a \cdot y \) is the \( \Lambda^n \)-tuple \( (a \cdot x_\lambda) \), where for \( \lambda \in \Lambda^n \) we have

\[
a \cdot x_\lambda = \begin{cases} \varphi_\lambda(a)x_\lambda & \text{if } r(\lambda) = v \\ 0 & \text{if } r(\lambda) \neq v. \end{cases}
\]

Since \( \varphi_\lambda(a)x_\lambda = 0 \) for all \( \lambda \in v\Lambda^n \), \( x_\lambda \in X_\lambda = A_{s(\lambda)} \), we have \( a \cdot y = 0 \), and we have shown that \( Y_n \) is not faithful.

Conversely, assume that \( X \) is not \( k\)-faithful. We will show that \((A,\varphi)\) is not \( k\)-faithful. We can choose \( n \in \mathbb{N}_k \) such that the \( A \)-correspondence \( Y_n \) is not faithful, so we can find a nonzero \( a \in A \) such that \( a \cdot y = 0 \) for all \( y \in Y_n \). Then \( a = (a_v) \) is a \( \Lambda^0 \)-tuple with \( a_v \in A_v \) for each \( v \), so we can choose \( v \in \Lambda^0 \) such that \( a_v \neq 0 \). We will show that \( a_v \in \bigcap_{\lambda \in v\Lambda^n} \ker \varphi_\lambda \). Let \( \lambda \in v\Lambda^0 \) and \( b \in A_{s(\lambda)} \). Define a \( v\Lambda^n \)-tuple \( (x_\mu) \in Y_n \) by

\[
x_\mu = \begin{cases} b & \text{if } \mu = \lambda \\ 0 & \text{if } \mu \neq \lambda. \end{cases}
\]

Then

\[
\varphi_\lambda(a_v)b = \left( a_v \cdot (x_\lambda) \right)_\lambda = 0. \tag*{□}
\]

Remark 3.15. The argument in the last paragraph of the above proof is a routine adaptation of that used in [DKPS10, Proposition 3.1.7].

Our motivation for introducing the properties of \( k\)-density and \( k\)-fidelity is that the Mauldin-Williams graphs considered by Ionescu — where we have a 1-graph \( \Lambda \) whose 1-skeleton \( E \) is finite, a \( \Lambda \)-system \((T,\sigma)\) of maps in which each space \( T_v \) is a compact metric space and each map \( \sigma_\lambda \) is a (strict) contraction — are typically 1-dense in the above sense rather than dense. More precisely, a Mauldin-Williams graph \((T,\sigma)\) is dense (in our terminology) if and only if every map \( \sigma_e \)
(for \( e \in E^1 \)) is surjective, which is usually not the case, and 1-dense if and only if for all \( v \in E^0 \) we have
\[
\bigcup_{e \in vE^1} \sigma_e(T_{s(e)}) = T(v),
\]
which is always the case (after replacing the original spaces by an “invariant list”). Thus, since we want to consider a version of Ionescu’s theorem for \( k \)-graphs, we must allow the weakened property of \( k \)-fidelity (of Definition 3.12) rather than insisting upon fidelity.

[DKPS10, Definition 3.2.1] define a representation of a \( \Lambda \)-system \( X \) in a \( C^* \)-algebra \( B \) as a map \( \rho : X \to B \) such that

1. for each \( v \in \Lambda^0 \), \( \rho_v : X_v \to B \) is a \( C^* \)-homomorphism;
2. whenever \( \xi \in X_{\lambda}, \eta \in X_{\mu} \),
   \[
   \rho_{\lambda}(\xi)\rho_{\mu}(\eta) = \begin{cases} 
   \rho_{\lambda\mu}(\xi\eta) & \text{if } s(\lambda) = r(\mu) \\
   0 & \text{if otherwise.}
   \end{cases}
   \]
3. whenever \( \xi \in X_{\lambda}, \eta \in X_{\mu}, \) if \( d(\lambda) = d(\mu) \) then
   \[
   \rho_{\lambda}(\xi^*)\rho_{\mu}(\eta) = \begin{cases} 
   \rho_{s(\lambda)}(\langle \xi, \eta \rangle_{X_{s(\lambda)}}) & \text{if } \lambda = \mu \\
   0 & \text{if otherwise,}
   \end{cases}
   \]
and when \( X \) is regular [DKPS10] defines a representation \( \rho \) to be Cuntz-Pimsner covariant if for all \( v \in \Lambda^0, n \in \mathbb{N}^k \), and \( a \in X_v \),

4. \[
   \rho_v(a) = \sum_{\lambda \in v\Lambda^0} \rho(\lambda)(\varphi(\lambda)(a)),
\]
where \( \rho(\lambda) = \rho(1)_{\lambda} : \mathcal{K}(X_{\lambda}) \to B \) is the associated homomorphism. Then [DKPS10] defines a representation \( \rho \) to be universal if for every representation \( \tau : X \to C \) there is a unique \( C^* \)-homomorphism \( \Phi = \Phi_{\tau} : B \to C \) such that \( \Phi \circ \rho_{\lambda} = \tau_{\lambda} \) for all \( \lambda \in \Lambda \), and a Cuntz-Pimsner covariant representation to be universal if it satisfies the above universality property for all Cuntz-Pimsner covariant representations. Then they point out that, by the nondegeneracy that is included in the regularity assumption, (1)–(3) above can be replaced by the following set of conditions: each \( \rho_{\lambda} \) is a correspondence representation in \( B \), \( \rho \) is multiplicative whenever this makes sense, and \( \rho_u \) and \( \rho_v \) have orthogonal images for all \( u \neq v \in \Lambda^0 \).

For the \( \mathbb{N}^k \)-system \( Y = Y_X \) associated to a regular \( \Lambda \)-system \( X \), [DKPS10, Proposition 3.2.3] shows that there is a bijection between the representations \( \rho : X \to B \) and the representations \( \psi : Y \to B \).
such that

\[ \psi \circ \iota_\lambda = \rho_\lambda \quad \text{for all } \lambda \in \Lambda. \]

However, it is crucial for our results to note that the proof of [DKPS10, Proposition 3.2.3] only requires nondegeneracy of \( Y \), not of \( X \).

[DKPS10, Proposition 3.2.5] shows that if \( X \) is regular then a representation \( \rho : X \to B \) is Cuntz-Pimsner covariant if and only if the associated representation \( \psi : Y \to B \) is. We turn this result into a definition:

**Definition 3.16.** Let \( X \) be a \( k \)-regular \( \Lambda \)-system of correspondences, with associated \( \mathbb{N}^k \)-system \( Y \), and let \( \rho : X \to B \) be a representation of \( X \), with associated representation \( \psi : Y \to B \). We define \( \rho \) to be Cuntz-Pimsner covariant if \( \psi \) is, in other words

\[ \sum_{\lambda \in v\Lambda^\infty} \rho^{(\lambda)} \circ \varphi_\lambda = \rho_v \quad \text{for all } v \in \Lambda^0. \]

**Remark 3.17.** To reiterate, the only difference between Definition 3.16 and the definition of Cuntz-Pimsner covariance given in [DKPS10, Definition 3.2.1] is that in the latter the \( \Lambda \)-system \( X \) is required to be regular, while we only require \( k \)-regularity. In any event, [DKPS10, Definition 3.2.7] defines the \( \mathcal{C}^* \)-algebra of a regular \( \Lambda \)-system \( X \) to be the Cuntz-Pimsner algebra \( \mathcal{O}_Y \), and in [DKPS10, Corollary 3.2.6] they notice that the representation \( \rho^{j_Y} : X \to \mathcal{O}_Y \) is a universal Cuntz-Pimsner covariant representation, where \( j_Y : Y \to \mathcal{O}_Y \) is the universal Cuntz-Pimsner covariant representation.

We emphasize that, even though we only require the \( \Lambda \)-system \( X \) to be \( k \)-regular, the theory of [DKPS10] carries over with no problems, as we've indicated. They use the notation \( \mathcal{C}^*(A, X, \chi) \) for the \( \mathcal{C}^* \)-algebra of \( X \), but we'll write it as \( \mathcal{O}_X \). If \( \rho : X \to B \) is any Cuntz-Pimsner covariant representation, we'll write \( \Phi_\rho : \mathcal{O}_X \to B \) for the homomorphism whose existence is guaranteed by universality; [DKPS10] would write it as \( \Phi_{\rho, \pi,} \), because they write \( \pi \) for the restriction of \( \rho \) to the \( \mathcal{C}^* \)-bundle \( X|_{\Lambda^0} \) (and they write \( A \) for this \( \mathcal{C}^* \)-bundle, as well as for the section algebra \( \bigoplus_{v \in \Lambda^0} X_v \) — we reserve the name \( A \) for this latter \( \mathcal{C}^* \)-algebra).

Note that since we assume that \( \Lambda \) is row-finite with no sources, the infinite-path space \( \Lambda^\infty \) is locally compact Hausdorff, and is the disjoint union of the compact open subsets \( \{ v\Lambda^\infty \}_{v \in \Lambda^0} \). We get a \( \Lambda \)-system of maps \( (\Lambda^\infty, \tau) \), where for \( \lambda \in u\Lambda v \) the continuous map

\[ \tau_\lambda : v\Lambda^\infty \to u\Lambda^\infty \]
is defined by $\tau_\lambda(x) = \lambda x$. Moreover, this $\Lambda$-system is $k$-regular. This system has the following properties: if $\lambda \in u\Lambda v$ then $\tau_\lambda$ is a homeomorphism of $v\Lambda^\infty$ onto the compact open set $\lambda\Lambda^\infty \subset u\Lambda^\infty$, and consequently $\tau_\lambda^*$ is a surjection of $C(u\Lambda^\infty)$ onto $C(v\Lambda^\infty)$.

**Lemma 3.18.** For each $u \in \Lambda^0$ and $\lambda \in u\Lambda$ let $p_\lambda = s_\lambda s_\lambda^*$, the set $D_u = \text{span}\{p_\lambda : \lambda \in u\Lambda\}$ is a unital commutative $C^*$-subalgebra of $C^*(\Lambda)$, with unit $p_u$, and the subalgebras $\{D_u\}_{u \in \Lambda^0}$ are pairwise orthogonal. Moreover, if $D$ denotes the commutative $C^*$-subalgebra $\bigoplus_{u \in \Lambda^0} D_u$ of $C^*(\Lambda)$, then there is an isomorphism $\theta : C_0(\Lambda^\infty) \to D$ that takes the characteristic function of $\lambda\Lambda^\infty = \{\lambda x : s(\lambda) = r(x)\}$ to $p_\lambda$ and $C(u\Lambda^\infty)$ to $D_u$. Finally, the diagram

$$
\begin{array}{ccc}
C(u\Lambda^\infty) & \xrightarrow{\tau_\lambda^*} & C(v\Lambda^\infty) \\
\downarrow{\theta} & & \downarrow{\theta} \\
D_u & \xrightarrow{\text{Ad} s_\lambda^*} & D_v
\end{array}
$$

commutes.

**Proof.** This is probably folklore, at least for directed graphs, and in any case is standard: truncation gives an inverse system $\{\Lambda^n, \tau_{m,n}\}$ of surjections among nonempty finite sets, whose inverse limit is $\Lambda^\infty$, and for each $n$ the commutative $C^*$-subalgebra $D^n := \text{span}\{p_\lambda : \lambda \in \Lambda^n\}$ of $C^*(\Lambda)$ has spectrum $\Lambda^n$, so the inductive limit $D = \text{span}\{D^n : n \in \mathbb{N}^k\}$ has spectrum $\Lambda^\infty$. □

**Definition 3.19.** Let $(T, \sigma)$ be a $\Lambda$-system of maps. A continuous map $\Phi : \Lambda^\infty \to T$ is intertwining if $\Phi \circ \tau_\lambda = \sigma_\lambda \circ \Phi$ for all $\lambda \in \Lambda$. We say $(T, \sigma)$ is self-similar if there is a surjective intertwining map $\Phi : \Lambda^\infty \to T$.

**Proposition 3.20.** Every self-similar $\Lambda$-system of maps $(T, \sigma)$ is $k$-surjective, and each space $T_v$ is compact.

**Proof.** First, $T_v$ is compact because the intertwining property and surjectivity of $\Phi$ imply that $T_v = \Phi(v\Lambda^\infty)$, which is a continuous image

\[\text{with } \tau_{m,n}(\lambda) = \lambda(0, n) \text{ for } \lambda \in \Lambda^m \text{ and } n \leq m\]
of the compact set $v\Lambda^\infty$. For the $k$-surjectivity, if $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ then

$$T_v = \Phi(v\Lambda^\infty)$$

$$= \Phi \left( \bigcup_{\lambda \in v \Lambda^n} \lambda \Lambda^\infty \right)$$

$$= \bigcup_{\lambda \in v \Lambda^n} \Phi(\lambda \Lambda^\infty)$$

$$= \bigcup_{\lambda \in v \Lambda^n} \sigma_\lambda(\Phi(s(\lambda)\Lambda^\infty))$$

$$= \bigcup_{\lambda \in v \Lambda^n} \sigma_\lambda(T_{s(\lambda)}).$$

□

**Definition 3.21.** Let $(T, \sigma)$ be a $\Lambda$-system of maps, and let $S \subset T$ be locally compact. For each $v \in \Lambda^0$ let $S_v = S \cap T_v$. Suppose that

$$\sigma_\lambda(S_v) \subset S_u$$

whenever $\lambda \in \Lambda^u$. Then $(S, \sigma|_S)$ is a $\Lambda$-subsystem of $(T, \sigma)$, where

$$(\sigma|_S)_\lambda = \sigma_\lambda|_{S_v}$$

for all $\lambda \in \Lambda_v$.

Note that our terminology makes sense: every $\Lambda$-subsystem is in fact a $\Lambda$-system.

**Proposition 3.22.** Let $(T, \sigma)$ be a $\Lambda$-system of maps, and let $\Phi : \Lambda^\infty \to T$ be an intertwining map. Put

$$T'_v = \Phi(v\Lambda^\infty) \text{ for each } v \in \Lambda^0$$

$$T' = \bigcup_{v \in \Lambda^0} T'_v.$$

Then $(T', \sigma|_{T'})$ is a self-similar $k$-surjective $\Lambda$-subsystem of $(T, \sigma)$, and each $T'_v$ is compact.

**Proof.** First of all, each $T'_v$ is compact since $v\Lambda^\infty$ is compact and $T_v$ is Hausdorff. Thus $T'$ is locally compact, since the sets $T_v$ are pairwise disjoint and open. For each $\lambda \in u\Lambda^u$ we have

$$\sigma_\lambda(T'_v) = \sigma_\lambda(\Phi(v\Lambda^\infty))$$

$$= \Phi(\lambda \Lambda^\infty)$$

$$\subset \Phi(u\Lambda^\infty)$$

$$= T'_u,$$
so \((T', \sigma_{|T'})\) is a \(\Lambda\)-subsystem of \((T, \sigma)\). It is self-similar because \(\Phi\) is intertwining and maps \(\Lambda^\infty\) onto \(T'\) by construction. Then by Proposition \ref{prop:3.20} \((T', \sigma_{|T'})\) is \(k\)-surjective. \hfill \hfill \Box

**Theorem 3.23.** Let \((T, \sigma)\) be a self-similar \(k\)-regular \(\Lambda\)-system of maps, and let \(X\) be the associated \(\Lambda\)-system of correspondences. Then 
\[
\mathcal{O}_X \cong C^*(\Lambda).
\]

*Proof.* Our strategy will be to find a Cuntz-Pimsner covariant representation \(\rho : X \to C^*(\Lambda)\) whose image contains the generators, and then apply the Gauge-Invariant Uniqueness Theorem. Recall that for \(\lambda \in u\Lambda v\), \(X_\lambda\) is the standard \(C_0(T_u) - C_0(T_v)\) correspondence \(\sigma^*_\lambda C_0(T_v)\).

Define \(\rho_\lambda : X_\lambda \to C^*(\Lambda)\) by
\[
\rho_\lambda(f) = s_\lambda \theta(\Phi^*(f)) \quad \text{for} \quad f \in C_0(T_v).
\]
Then \(\rho_\lambda\) is linear, and for \(f, g \in C_0(T_v)\) we have
\[
\rho_\lambda(f)^* \rho_\lambda(g) = \theta(\Phi^*(f)) s_\lambda^* s_\lambda \theta(\Phi^*(g)) = \theta(\Phi^*(f)) p_v \theta(\Phi^*(g)) = \theta(\Phi^*(fg)) = p_v(\langle f, g \rangle_{C(T_v)}).
\]

For \(\lambda \in \Lambda v, \mu \in v\Lambda w, f \in C_0(T_v),\) and \(h \in C_0(T_w)\) we have
\[
\rho_\lambda(f) \rho_\mu(h) = s_\lambda \theta(\Phi^*(f)) s_\mu \theta(\Phi^*(h)) = s_\lambda p_v \theta(\Phi^*(f)) s_\mu \theta(\Phi^*(h)) = s_\lambda s_\mu \Ad s_\mu^* \circ \theta(\Phi^*(f)) \theta(\Phi^*(h)) = s_\lambda s_\mu \theta(\Phi^*(f)) \theta(\Phi^*(h)) = s_\lambda \mu \theta(\Phi^*(f)) \theta(\Phi^*(h)) = s_\lambda \mu \theta(\Phi^*(\sigma^*_\mu(f)h)) = \rho_\lambda(\sigma^*_\mu(f)h) = \rho_\lambda(\sigma^*_\mu(f)h).
\]
It follows that \(\rho : X \to C^*(\Lambda)\) is a representation.

Next we show that \(\rho\) is Cuntz-Pimsner covariant. Let \(u \in \Lambda^0, n \in \mathbb{N}^k,\) and \(f \in X_u = C_0(T_u)\). We need to show that
\[
\rho_u(f) = \sum_{\lambda \in u\Lambda^u} \rho^{(\lambda)} \circ \varphi_\lambda(f),
\]
where
\[ \varphi_\lambda : C_0(T_u) \to \mathcal{K}(X_\Lambda) \]
is the left-module structure map. We need a little more information regarding the homomorphism
\[ \rho^{(\lambda)} = \rho^{(1)}_\lambda : \mathcal{K}(X_\Lambda) \to C^*(\Lambda). \]
For \( \lambda \in u\Lambda v \) we have \( X_\lambda = \sigma^*_\lambda C_0(T_v) \), so
\[ \mathcal{K}(X_\lambda) = C_0(T_v), \]
and for \( g, h \in C_0(T_v) \) the rank-one operator \( \theta_{g,h} \) is given by (left) multiplication by \( gh \). Thus
\[
\rho^{(\lambda)}(gh) = \rho_\lambda(g)\rho_\lambda(h)^* \\
= s_\lambda \theta \circ \Phi^*(g)\theta \circ \Phi^*(\overline{h})s_\lambda^* \\
= s_\lambda \circ \theta \circ \Phi^*(gh)s_\lambda^* \\
= \text{Ad} s_\lambda \circ \rho_v(gh).
\]
Since every function in \( C_0(T_v) \) can be factored as \( gh \), we conclude that the homomorphism \( \rho^{(\lambda)} \) coincides with
\[ \text{Ad} s_\lambda \circ \rho_v : C_0(T_v) \to C^*(\Lambda). \]
Also, \( \varphi_\lambda : C_0(T_u) \to \mathcal{K}(X_\Lambda) \) coincides with \( \sigma^*_\lambda : C_0(T_u) \to C_0(T_v) \) (note that \( \sigma^*_\lambda \) maps into \( C_0(T_v) \) because \( \sigma_\lambda \) is proper). Thus
\[
\sum_{\lambda \in u\Lambda v} \rho^{(\lambda)} \circ \varphi_\lambda(f) = \sum_{\lambda \in u\Lambda v} \text{Ad} s_\lambda \circ \theta \circ \Phi^* \circ \sigma^*_\lambda(f) \\
= \sum_{\lambda \in u\Lambda v} \text{Ad} s_\lambda \circ \theta \circ \tau^*_\lambda \circ \Phi^*(f) \\
= \sum_{\lambda \in u\Lambda v} \text{Ad} s_\lambda \circ \text{Ad} s^*_\lambda \circ \theta \circ \Phi^*(f) \\
= \sum_{\lambda \in u\Lambda v} \text{Ad} s_\lambda s^*_\lambda \circ \theta \circ \Phi^*(f) \\
= \sum_{\lambda \in u\Lambda v} p_\lambda \theta \circ \Phi^*(f) \\
= p_u \rho_u(f) \quad \text{(since} \sum_{\lambda \in u\Lambda v} p_\lambda = p_u \text{)} \\
= \rho_u(f),
\]
since \( \rho_u(C_0(T_u)) \subset D_u \) and \( p_u = 1_{D_u} \).
Therefore $\rho$ gives rise to a homomorphism $\Psi_\rho : O_X \to C^*(\Lambda)$ such that

$$\Psi_\rho \circ \rho^X = \rho,$$

where $\rho^X : O_X \to C^*(\Lambda)$ is the universal Cuntz-Pimsner covariant representation. For each $v \in \Lambda^0$, the continuous map $\Phi : \Lambda^\infty \to T$ takes $v\Lambda^\infty$ into $T_v$, so $\Phi^*$ restricts to a nondegenerate homomorphism from $C_0(T_v)$ to $C(v\Lambda^\infty)$, and hence the homomorphism $\rho_v : C_0(T_v) \to D_v$ is nondegenerate. It follows that for each $\lambda \in \Lambda v$ the generator $s_\lambda$ is in the range of $\rho_\lambda : X_\lambda \to C^*(\Lambda)$. Thus $\Psi_\rho : O_X \to C^*(\Lambda)$ is surjective.

Finally, we appeal to the Gauge-Invariant Uniqueness Theorem [DKPS10, Theorem 3.3.1] to show that $\Psi_\rho$ is injective. Note that [DKPS10] assume that the $\Lambda$-system $X$ is regular, while we only assume that it is $k$-regular; as we have mentioned before, $k$-regularity is all that’s required to make the results of [DKPS10] true. First of all, for each $v \in \Lambda^0$, $\Phi$ maps $v\Lambda^0$ onto $T_v$, and it follows that $\rho_v : C_0(T_v) \to D_v$ is faithful. Thus the direct sum

$$\Psi_\rho|_\Lambda = \bigoplus_{v \in \Lambda^0} \rho_v : \bigoplus_{v \in \Lambda^0} C_0(T_v) \to \bigoplus_{v \in \Lambda^0} D_v \subset C^*(\Lambda)$$

is also faithful. Let $\gamma : T^k \to \text{Aut}C^*(\Lambda)$ be the gauge action. For $\lambda \in \Lambda^n v, f \in C_0(T_v)$, and $z \in T^k$,

$$\gamma_z \circ \rho_\lambda(f) = \gamma_z(s_\lambda \theta \circ \Phi^*(f))$$

$$= \gamma_z(s_\lambda) \rho_v(f) \quad \text{(since } \rho_v(f) \in D_v \subset C^*(\Lambda)\gamma)$$

$$= z^n s_\lambda \rho_v(f)$$

$$= z^n \rho_\lambda(f),$$

so by [DKPS10] Theorem 3.3.1] $\Psi_\rho$ is faithful. \hfill \Box

4. Mauldin-Williams $k$-graphs

We continue to let $\Lambda$ be a row-finite $k$-graph with no sources.

**Proposition 4.1.** Let $(T, \sigma)$ be a $\Lambda$-system of maps such that each $T_v$ is a complete metric space and, for some $c < 1$ and every $\lambda \in \Lambda$,

$$\delta_v(\sigma_\lambda(t), \sigma_\lambda(s)) \leq c^{d(\lambda)} \delta_v(t, s) \quad \text{for all } \lambda \in \Lambda, t, s \in T_{s(\lambda)},$$

where $\delta_v$ is the metric on $T_v$, $d$ is the degree functor, and $|n| = \sum_{i=1}^k n_i$ for $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$. Then there exists a unique $k$-surjective $\Lambda$-subsystem $(K, \psi)$ such that each $K_v$ is a bounded closed subset of $T_v$, and in fact each $K_v$ is compact.
Note that to check the hypothesis it suffices to show that each of the generating maps \( \sigma_\lambda \) for \( \lambda \in \Lambda^{e_i} \) has Lipschitz constant at most \( c \), where \( e_1, \ldots, e_k \) is the standard basis for \( \mathbb{N}^k \).

**Proof.** Let

\[
\mathcal{C} = \prod_{v \in \Lambda^0} \mathcal{C}(T_v),
\]

where for \( v \in \Lambda^0 \) we let \( \mathcal{C}(T_v) \) denote the set of bounded closed subsets of \( T_v \), which is complete under the Hausdorff metric. Since \( \Lambda^0 \) is countable, \( \mathcal{C} \) is a complete metric space. For each \( n \in \mathbb{N}^k \) define a map \( \tilde{\sigma}^n : \mathcal{C} \to \mathcal{C} \) by

\[
\tilde{\sigma}^n(C)_v = \bigcup_{\lambda \in v \Lambda^n} \sigma_\lambda(C_{s(\lambda)}).
\]

As in [MW88], \( \tilde{\sigma}^n \) is a contraction, and so has a unique fixed point in \( \mathcal{C} \). We need to know that the maps \( \{ \tilde{\sigma}^n \}_{n \in \mathbb{N}^k} \) all have the same fixed point, and it suffices to show that they commute. Let \( n, m \in \mathbb{N}^k \). Then for all \( C = (C_v)_{v \in \Lambda^0} \in \mathcal{C} \) and \( v \in \Lambda^0 \) we have

\[
\tilde{\sigma}^n \circ \tilde{\sigma}^m(C)_v = \tilde{\sigma}^n(\tilde{\sigma}^m(C)_v)
\]

\[
= \bigcup_{\lambda \in v \Lambda^n} \sigma_\lambda(\tilde{\sigma}^m(C_{s(\lambda)}))
\]

\[
= \bigcup_{\lambda \in v \Lambda^n} \sigma_\lambda \left( \bigcup_{\mu \in s(\lambda) \Lambda^m} \sigma_\mu(C_{s(\mu)}) \right)
\]

\[
= \bigcup_{\lambda \in v \Lambda^n} \bigcup_{\mu \in s(\lambda) \Lambda^m} \sigma_\lambda \circ \sigma_\mu(C_{s(\mu)})
\]

\[
= \bigcup_{\lambda \mu \in v \Lambda^n \Lambda^m} \sigma_{\lambda \mu}(C_{s(\lambda \mu)})
\]

\[
= \bigcup_{\alpha \in v \Lambda^{n+m}} \sigma_\alpha(C_{s(\alpha)}),
\]

which, by the factorization property of \( \Lambda \), coincides with

\[
\bigcup_{\mu \in v \Lambda^m} \bigcup_{\lambda \in s(\mu) \Lambda^n} \sigma_\mu \circ \sigma_\lambda(C_{s(\lambda \mu)}) = \tilde{\sigma}^m \tilde{\sigma}^n(C)_v.
\]

Letting \( (K_v)_{v \in \Lambda^0} \) be the unique common fixed point of \( \tilde{\sigma} \) on \( \mathcal{C} \), we see that, setting \( K = \bigcup_{v \in \Lambda^0} K_v \) and \( \psi = \sigma|_K \), the restriction \( (K, \psi) \) of \( (T, \sigma) \) is the unique \( k \)-surjective \( \Lambda \)-subsystem with bounded closed subsets \( K_v \).

To see that in fact every \( K_v \) is compact, play the same game with \( \mathcal{C}(T_v) \) replaced by the set of compact subsets of \( T_v \), again getting a
unique fixed point. But since the compact subsets are among the bounded closed subsets, the resulting \( \Lambda \)-subsystem must coincide with the one we found above, by uniqueness.

**Definition 4.2.** A Mauldin-Williams \( \Lambda \)-system is a \( k \)-surjective \( \Lambda \)-system of maps \( (T, \sigma) \) such that each \( T_v \) is a compact metric space and, for some \( c < 1 \), every \( \sigma_{\lambda} : T_{s(\lambda)} \to T_{r(\lambda)} \) is a contraction with Lipschitz constant at most \( c^{\|d(\lambda)\|} \).

**Proposition 4.3.** Every Mauldin-Williams \( \Lambda \)-system \( (T, \sigma) \) is self-similar, and if \( X \) is the associated \( \Lambda \)-system of correspondences then \( \mathcal{O}_X \cong C^*(\Lambda) \).

**Proof.** We adapt the technique of Ionescu [Ion07]. Let \( x \in v\Lambda^\infty \), so that \( x : \Omega_k \to \Lambda \) is a \( k \)-graph morphism. For each \( n \in \mathbb{N}^k \) let \( x(0, n) \) be the unique path \( \lambda \in \Lambda^n \) such that \( x = \lambda y \) for some (unique) \( y \in s(\lambda)\Lambda^\infty \).

By definition of Mauldin-Williams \( \Lambda \)-system, the range of each \( \sigma_{x(0, n)} \) has diameter at most \( c^{\|n\|} \). Thus by compactness there is a unique element \( \Phi(x) \in T_v \) such that

\[
\bigcap_{n \in \mathbb{N}^k} \sigma_{x(0, n)}(T_{s(x(0, n))}) = \{\Phi(x)\}.
\]

We get a map \( \Phi : \Lambda^\infty \to T \), which is continuous because for each \( x \in \Lambda^\infty \) the images under \( \Phi \) of the neighborhoods \( x(0, n)\Lambda^\infty \) of \( x \) have diameters shrinking to 0. By construction, it’s obvious that

\[
\Phi(\lambda x) = \sigma_{\lambda}(\Phi(x)) \quad \text{for all } \lambda \in \Lambda, x \in s(\lambda)\Lambda^\infty,
\]

so \( \Phi \) is intertwining.

We show that \( \Phi \) is surjective. Put \( T' = \Phi(\Lambda^\infty) \). By Proposition 3.22, \( (T', \sigma|_{T'}) \) is \( k \)-surjective with each \( T'_v \) compact, which implies that \( T' = T \) by the uniqueness in Proposition 4.1.

Finally, it now follows from Theorem 3.23 that \( \mathcal{O}_X \cong C^*(\Lambda) \). \( \square \)

**Remark 4.4.** It would be completely routine at this point to adapt Ionescu’s techniques to prove a higher-rank version his other “no-go theorem” [Ion07, Theorem 3.4], namely that there are no “noncommutative Mauldin-Williams \( \Lambda \)-systems” of maps.

### 5. The Associated Topological \( k \)-Graph

Let \( \Lambda \) be a row-finite \( k \)-graph with no sources, and let \( (T, \sigma) \) be a \( k \)-regular \( \Lambda \)-system of maps. We do not assume that \( (T, \sigma) \) is self-similar unless otherwise noted.

Let \( (T, \sigma) \) be a \( \Lambda \)-system of maps. We want to define a topological \( k \)-graph \( \Lambda \ast T \) as follows:
(1) $\Lambda^*T = \{(\lambda, t) \in \Lambda \times T : t \in T_{s(\lambda)}\}$;
(2) $s(\lambda, t) = (s(\lambda), t)$ and $r(\lambda, t) = (r(\lambda), \sigma_\lambda(t))$;
(3) if $s(\lambda) = r(\mu)$ and $t = \sigma_\mu(s)$, then $(\lambda, t)(\mu, s) = (\lambda\mu, s)$;
(4) $d(\lambda, t) = d(\lambda)$.

$\Lambda^*T$ has the relative topology from $\Lambda \times T$, and is the disjoint union of the open subsets $\{\lambda\} \times T_{s(\lambda)}$, each of which is a homeomorphic copy of $T_{s(\lambda)}$.

Proposition 5.1. The above operations make $\Lambda^*T$ into a topological $k$-graph.

Proof. This is routine. For instance, it’s completely routine to check that $\Lambda^*T$ is a small category and the map defined in (4) is a functor. Let’s check the unique factorization property: Let $(\lambda, t) \in \Lambda^*T$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$. Then we can uniquely write $\lambda = \mu\nu$ with $d(\mu) = m$ and $d(\nu) = n$. We have

$$(\lambda, t) = (\mu, \sigma_\nu(t))(\nu, t), \quad d(\mu, \sigma_\nu(t)) = m, \quad \text{and} \quad d(\nu, t) = n,$$

and $(\mu, \sigma_\nu(t))$ and $(\nu, t)$ are unique since $\mu$ and $\nu$ are. It’s immediate that the degrees match up and this factorization is unique.

The multiplication on the category $\Lambda^*T$ is continuous and open because it is in fact a local homeomorphism from the fibred product $(\Lambda^*T) \times (\Lambda^*T)$ to $\Lambda^*T$, which for each $(\lambda, \mu) \in \Lambda \times \Lambda$ with $s(\lambda) = r(\mu)$ maps the open subset

$$\left(\{\lambda\} \times T_{s(\lambda)}\right) \times \left(\{\mu\} \times T_{s(\mu)}\right) \cap \left((\Lambda^*T) \times (\Lambda^*T)\right)$$

bijectively onto the open subset $\{\lambda\mu\} \times T_{s(\mu)}$.

To see that the source map on $\Lambda^*T$ is a local homeomorphism, just note that it restricts to homeomorphisms

$$\{\lambda\} \times T_{s(\lambda)} \to \{s(\lambda)\} \times T_{s(\lambda)}.$$

Remark 5.2. One could reasonably regard a $\Lambda$-system of maps as an action of $\Lambda$ on the space $T = \bigsqcup_{v \in \Lambda^0} T_v$, and the topological $k$-graph $\Lambda^*T$ as the associated transformation $k$-graph.

Remark 5.3. If each $T_v$ is discrete and every map $\sigma_\lambda : T_{s(\lambda)} \to T_{r(\lambda)}$ is bijective, then the above $k$-graph $\Lambda^*T$ coincides with that of $[\text{PQR05}, \text{Proposition 3.3}]$, where the main point was that the coordinate projection $(\lambda, t) \mapsto \lambda$ is a model for coverings of the $k$-graph $\Lambda$.

Proposition 5.4. Let $(T, \sigma)$ be a $k$-regular $\Lambda$-system of maps, and let $(A, \varphi)$ be the associated $\Lambda$-system of homomorphisms, which in turn has an associated $\Lambda$-system $X$ of correspondences. Then

$$\mathcal{O}_X \cong C^*(\Lambda^*T),$$
where $\Lambda \ast T$ is the topological $k$-graph of Proposition 5.1.

**Proof.** Our strategy is to show that $O_X$ and $C^*(\Lambda \ast T)$ are isomorphic to the Cuntz-Pimsner algebras of isomorphic $\mathbb{N}^k$-systems of correspondences. Recall that $O_X \cong O_Y$, where $Y = Y_X$ is the $\mathbb{N}^k$ system associated to $X$. Thus for each $n \in \mathbb{N}^k$ we have

$$Y_n = \bigoplus_{\lambda \in \Lambda^n} X_{\lambda},$$

where $X_{\lambda}$ is the correspondence over $A = \bigoplus_{v \in \Lambda^0} A_v$ naturally associated (via identifying the $A_v$’s with direct summands in $A$) to the standard $A_{r(\lambda)} - A_{s(\lambda)}$ correspondence $\sigma^*_\lambda A_{s(\lambda)}$ determined by the homomorphism $\sigma^*_\lambda : A_{r(\lambda)} \to M(A_{s(\lambda)})$ given by composition with $\sigma_v : T_{s(v)} \to T_{r(v)}$.

On the other hand, by [CLSV11, Theorem 5.20] $C^*(\Lambda \ast T)$ is isomorphic to the Cuntz-Nica-Pimsner algebra $\mathcal{NO}Z$, where $Z$ is the $\mathbb{N}^k$-system of $C_0((\Lambda \ast T)^0)$-correspondences associated to the topological $k$-graph $\Lambda \ast T$. As we’ll show in this proof, the $\mathbb{N}^k$-systems $Z$ and $Y$ are isomorphic. Since the $\Lambda$-system $(T, \sigma)$ is $k$-regular, so is $Y$, and hence so is $Z$. In particular, since each pair in $\mathbb{N}^k$ has an upper bound, and $C_0((\Lambda \ast T)^0)$ maps injectively into the compacts on $Z_n$ for every $n \in \mathbb{N}^k$, it follows from [SY10, Corollary 5.2] that $\mathcal{NO}Z = \mathcal{O}Z$, because by [Fow02, Proposition 5.8] $Z$ is compactly aligned.

Let’s see what the $\Lambda$-system $Z$ looks like in this situation: for each $n \in \mathbb{N}^k$, the correspondence $Z_n$ over $C_0((\Lambda \ast T)^0)$ is a completion of $C_c((\Lambda \ast T)^n)$. We can safely identify $(\Lambda \ast T)^n$ with $T = \bigsqcup_{v \in \Lambda^0} T_v$, and hence $C_0((\Lambda \ast T)^0)$ with $A = \bigoplus_{v \in \Lambda^0} C_0(T_v)$, and in this way $Z_n$ becomes an $A$-correspondence. For $\xi, \eta \in C_c((\Lambda \ast T)^n) = C_c(\Lambda^n \ast T)$, the inner product is given by

$$\langle \xi, \eta \rangle_A(t) = \sum_{\lambda \in \Lambda^n v} \overline{\xi(\lambda, t)} \eta(\lambda, t), \quad t \in T_v, v \in \Lambda^0,$$

and the right and left module operations are given for $f \in A$ by

$$(\xi \cdot f)(\lambda, t) = \xi(\lambda, t) f(t)$$

$$(f \cdot \xi)(\lambda, t) = f(\sigma_\lambda(t)) \xi(\lambda, t).$$

Note that

$$(\Lambda \ast T)^n = \bigsqcup_{\lambda \in \Lambda^n} (\{\lambda\} \times T_{s(\lambda)}).$$

Thus for each $\lambda \in \Lambda^n v$ we have a natural inclusion map

$$C_c(\{\lambda\} \times T_v) \hookrightarrow Z_n.$$
and $Z_n$ is the closed span of these subspaces. Moreover, their closures form a pairwise orthogonal family of subcorrespondences of $Z_n$:
\[
Z_n(\lambda) = C_c(\{\lambda\} \times T_v) \quad \text{for } \lambda \in \Lambda^n v,
\]
and we see that
\[
Z_n = \bigoplus_{\lambda \in \Lambda^n} Z_n(\lambda)
\]
as $A$-correspondences.

We will obtain an isomorphism $\psi : Y \to Z$ of $\mathbb{N}^k$-systems by defining isomorphisms $\psi_n : Y_n \to Z_n$ of $A$-correspondences and then verifying that
\[
\psi_n(\xi)\psi_m(\eta) = \psi_{n+m}(\xi\eta) \quad \text{for all } (\xi, \eta) \in Y_n \times Y_m.
\]
By the above, to get an isomorphism $\psi_n : Y_n \to Z_n$ it suffices to get isomorphisms $\psi_{n,\lambda} : X_{\lambda} \to Z_n(\lambda)$ for each $\lambda \in \Lambda^n$. If $\lambda \in \Lambda^n v$ and

\[
\xi \in C_c(T_v) \subset X_{\lambda}
\]
define
\[
\psi(\xi) \in C_c(\{\lambda\} \times T_v) \subset Z_n(\lambda)
\]
by
\[
\psi(\xi)(\lambda, t) = \xi(t).
\]
Routine computations show that $\psi_{n,\lambda}$ is an isomorphism.

Now we check multiplicativity, and again it suffices to consider the fibres of the $\Lambda$-system $X$: if
\[
\xi \in X_{\lambda} \quad \text{for } \lambda \in \Lambda^n v
\]
\[
\eta \in X_{\mu} \quad \text{for } \mu \in v\Lambda^m
\]
then for $t \in T_{s(\mu)}$ we have
\[
(\psi_{n,\lambda}(\xi)\psi_{m,\mu}(\eta))(\lambda\mu, t) = \psi_{n,\lambda}(\xi)(\lambda, \sigma_{\mu}(t))\psi_{m,\mu}(\mu, t) = \xi(\sigma_{\mu}(t))\eta(t) = (\xi\eta)(t) = (\psi_{n+m,\lambda\mu}(\xi\eta))(\lambda\mu, t).
\]

6. The Tensor Groupoids

Recall that in [FS02] Fowler and Sims study what they call product systems taking values in a tensor groupoid. Their product systems are over semigroups, and here we want to consider the special cases related to our $\Lambda$-systems of homomorphisms or maps, where the $k$-graph $\Lambda$ has a single vertex, and so in particular is a monoid whose identity element is the unique vertex. Since we won’t need to do serious work with the concept, here we informally regard a tensor groupoid as a groupoid $\mathcal{G}$
with a “tensor” operation $X \otimes Y$ and an “identity” object $1_G$ such that
the “expected” redistributions of parentheses and canceling of tensoring
with the identity are implemented via given natural equivalences. As
defined in [FS02], a product system over a semigroup $S$ taking values in
a tensor groupoid $G$ is a family $\{X_s\}_{s \in S}$ of objects in $G$ together with
an associative family $\{\alpha_{s,t}\}_{s,t \in S}$ of isomorphisms
$$\alpha_{s,t} : X_s \otimes X_t \to X_{st},$$
and moreover if $S$ has an identity $e$ then $X_e = 1_G$ and $\alpha_{e,s}, \alpha_{s,e}$ are the
given isomorphisms $1_G \otimes X_s \cong X_s$ and $X_s \otimes 1_G \cong X_s$.

Systems of homomorphisms. Let $A$ be a $C^*$-algebra, and $G$ be the
tensor groupoid whose objects are the nondegenerate homomorphisms
$\pi : A \to M(A)$, whose only morphisms are the identity morphisms on
objects, and with identity $1_G = \text{id}_A$. Define a tensor operation on $G$ by composition:
$$\pi_1 \otimes \pi_2 = \pi_2 \circ \pi_1,$$
where $\pi_2$ has been canonically extended to a strictly continuous unital
endomorphism of $M(A)$. Standard properties of composition show that
$G$ is indeed a tensor groupoid, in a trivial way: the tensor operation is
actually associative, and $1_G$ acts as an actual identity for tensoring, so
the axioms of [FS02] for a tensor groupoid are obviously satisfied.

Due to the special nature of this tensor groupoid $G$, a product system over $\mathbb{N}^k$ taking values in $G$, as in [FS02, Definition 1.1], is a homomorphism $n \mapsto \varphi_n$ from the additive monoid $\mathbb{N}^k$ into the monoid of non-
degenerate homomorphisms $A \to M(A)$ under composition, in other
words such a product system is precisely what we call in the current paper an $\mathbb{N}^k$-system of homomorphisms.

Systems of maps. Quite similarly to the above, let $T$ be a locally compact Hausdorff space, and $G$ be the tensor groupoid whose objects are the continuous maps $\sigma : X \to X$, whose only morphisms are the identity morphisms on objects, and with identity $1_G = \text{id}_X$. Define a tensor operation on $G$ by composition:
$$\sigma \otimes \psi = \sigma \circ \psi.$$
Again, $G$ is indeed a tensor groupoid, in a trivial way, because the tensor
operation is actually associative, and $1_G$ acts as an actual identity for tensoring.

A product system over $\mathbb{N}^k$ taking values in $G$, as in [FS02, Definition 1.1], is a homomorphism $n \mapsto \sigma_n$ from the additive monoid $\mathbb{N}^k$ into the monoid of continuous selfmaps of $X$ maps under composition,
in other words such a product system is precisely what we call in the current paper an $\mathbb{N}^k$-system of maps.

7. REVERSING THE PROCESSES

In Remark 3.2 we noted that every $\Lambda$-system of maps gives rise to a $\Lambda$-system of homomorphisms, and every $\Lambda$-system of homomorphisms gives rise to a $\Lambda$-system of correspondences. In this section we will investigate the extent to which these two processes are reversible.

**Question 7.1.** When is a given $\Lambda$-system of correspondences isomorphic to the one associated to a $\Lambda$-system of homomorphisms?

Investigating this question requires us to examine balanced tensor products of standard correspondences. First we observe without proof the following elementary fact.

**Lemma 7.2.** Let $\varphi : A \to M(B)$ and $\psi : B \to M(C)$ be nondegenerate homomorphisms. Then there is a unique $A - C$ correspondence isomorphism

$$\theta : \varphi B \otimes_B \psi C \xrightarrow{\cong} \psi \circ \varphi C$$

such that

$$\theta(b \otimes c) = \psi(b)c \quad \text{for } b \in B, c \in C.$$

We can analyze the question of whether a given $\Lambda$-system $X$ of correspondences is isomorphic to one coming from a $\Lambda$-system of homomorphisms in several steps:

First of all, without loss of generality we can look for a $\Lambda$-system of homomorphisms of the form $(A, \varphi)$.

Next, for each $\lambda \in u \Lambda v$ the $A_u - A_v$ correspondence $X_\lambda$ must be isomorphic to a standard one, more precisely there must exist a linear bijection

$$\theta_\lambda : X_\lambda \to A_v$$

and a nondegenerate homomorphism

$$\varphi_\lambda : A_u \to M(A_v)$$

such that

$$(7.1) \quad \theta_\lambda(\xi) \star \theta_\lambda(\eta) = \langle \xi, \eta \rangle_{A_v} \quad \text{for all } \xi, \eta \in X_\lambda$$

$$(7.2) \quad \theta_\lambda(a \cdot \xi \cdot b) = \varphi_\lambda(a) \theta_\lambda(\xi)b \quad \text{for all } a \in A_u, \xi \in X_\lambda, b \in A_v.$$

Moreover, whenever $\lambda \in u \Lambda v, \mu \in v \Lambda w$ we must have

$$\varphi_{\lambda \mu} A_w = X_{\lambda \mu} \cong X_\lambda \otimes_{A_v} X_\mu$$
\[
\varphi_{\lambda} A_v \otimes A_v \varphi_{\mu} A_w \\
\cong \varphi_{\mu} \varphi_{\lambda} A_w,
\]
so there exists a unitary multiplier \( U(\lambda, \mu) \in M(A_w) \) such that
\[
\varphi_{\mu} \circ \varphi_{\lambda} = \text{Ad} U(\lambda, \mu) \circ \varphi_{\lambda \mu}.
\]
The \( U(\lambda, \mu) \)'s satisfy a kind of “two-cocycle” identity coming from associativity of composition of the \( \varphi_{\lambda} \)'s.

Now, if this \( \Lambda \)-system of correspondences is isomorphic to one associated to a \( \Lambda \)-system \( (A, \psi) \) of homomorphisms, then for each \( \lambda \in u\Lambda v \) we must have an isomorphism \( \varphi_{\lambda} A_v \cong \psi_{\lambda} A_v \) of \( A_u - A_v \) correspondences, and so there must be a unitary multiplier \( W_\lambda \in M(A_v) \) such that
\[
\varphi_{\lambda} = \text{Ad} W_\lambda \circ \psi_{\lambda}.
\]
Since \( (A, \psi) \) is a \( \Lambda \)-system of homomorphisms, whenever \( \lambda \in u\Lambda v, \mu \in v\Lambda w \) we have
\[
\varphi_{\lambda \mu} = \text{Ad} W_{\lambda \mu} \circ \psi_{\lambda \mu} \\
= \text{Ad} W_{\lambda \mu} \circ \psi_{\mu} \circ \psi_{\lambda} \\
= \text{Ad} W_{\lambda \mu} \circ \text{Ad} W^*_{\mu} \circ \varphi_{\mu} \circ \text{Ad} W^*_{\lambda} \circ \varphi_{\lambda} \\
= \text{Ad} W_{\lambda \mu} W^*_{\mu} \varphi_{\mu} (W^*_{\lambda}) \circ \varphi_{\mu} \circ \varphi_{\lambda} \\
= \text{Ad} W_{\lambda \mu} W^*_{\mu} \varphi_{\mu} (W^*_{\lambda}) \text{Ad} U(\lambda, \mu) \circ \varphi_{\lambda \mu},
\]
so since the homomorphisms \( \varphi_{\lambda} \) are nondegenerate we see that, in the quotient group of the unitary multipliers of \( A_w \) modulo the central unitary multipliers, the cosets satisfy
\[
[U(\lambda, \mu)] = [\varphi_{\mu} (W_{\lambda}) W^*_{\mu} W^*_{\lambda}],
\]
giving a sort of cohomological obstruction (which we won’t make precise) to the \( \Lambda \)-system of correspondences being isomorphic to a one associated to a \( \Lambda \)-system \( (A, \psi) \) of homomorphisms.

Note that if all the \( C^* \)-algebras \( A_v \) are commutative, then none of the above unitary multipliers appear, so once we have \( \theta_{\lambda} \)'s and \( \varphi_{\lambda} \)'s satisfying \( \Box \) then the pair \( (A, \varphi) \) will automatically be a \( \Lambda \)-system of homomorphisms whose associated \( \Lambda \)-system of correspondences is isomorphic to \( X \). What makes this happen is the way in which the correspondences \( X_{\lambda} \) fit together. This is worth recording:

**Proposition 7.3.** Let \( X \) be a \( \Lambda \)-system of correspondences such that every \( A_v \) is commutative. Then \( X \) is isomorphic to the \( \Lambda \)-system associated to a \( \Lambda \)-system of homomorphisms if and only if, whenever \( \lambda \in u\Lambda v, X_{\lambda} \) is isomorphic to a standard \( A_u - A_v \) correspondence \( \varphi_{\lambda} A_v \).
Proposition 7.4. Let \((A, \varphi)\) be a \(\Lambda\)-system of homomorphisms such that every \(A_v\) is commutative, and for each \(v \in \Lambda^0\) let \(T_v\) be the maximal ideal space of \(A_v\). Then there is a unique \(\Lambda\)-system of maps \((T, \sigma)\) such that \((A, \varphi)\) is the associated \(\Lambda\)-system of homomorphisms.

On the other hand, every \(\Lambda\)-system of homomorphisms is uniquely isomorphic to the one associated to a \(\Lambda\)-system of maps, at least in the only circumstances where it makes sense:

Proof. This follows immediately from the duality between the category of commutative \(C^*\)-algebras and nondegenerate homomorphisms into multiplier algebras and the category of locally compact Hausdorff spaces and continuous maps. \(\square\)

8. No higher-rank fractals

In Proposition 3.22 we showed that every \(\Lambda\) system of maps \((T, \sigma)\) has a self-similar \(k\)-surjective \(\Lambda\)-subsystem \((T', \sigma|_{T'})\). The self-similar set \(T'\) is the part of the system that would generally be referred to as the “fractal”. It is natural to wonder whether the generalization to \(k\)-graphs presented here gives rise to any new fractals that could not have arisen from the corresponding constructions for 1-graphs. The answer to this question turns out to be “no” for reasons we will now explain. Throughout the following discussion, let \(p = (1, 1, \ldots, 1) \in \mathbb{N}^k\)

Definition 8.1. For a \(k\)-graph \(\Lambda\) we define the **diagonal 1-graph** \(E\) as follows:

\[
\begin{align*}
E^0 &= \Lambda^0 \\
E^1 &= \{e_\lambda : \lambda \in \Lambda, d(\lambda) = p\} \\
r(e_\lambda) &= r(\lambda) \\
s(e_\lambda) &= s(\lambda).
\end{align*}
\]

If \((T, \sigma)\) is a \(\Lambda\)-system of maps, then we define the **diagonal \(E\)-system** \((T, \rho)\) of \((T, \sigma)\) to be the \(E\)-system of maps such that \(\rho_{e_\lambda} = \sigma_\lambda\) for all \(e_\lambda \in E^1\). Finally, let \(\alpha : E^* \to \Lambda\) be the map defined by \(\alpha(e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}) = \lambda_1 \lambda_2 \cdots \lambda_n\).

Proposition 8.2. The map \(i : \Lambda^\infty \to E^\infty\) defined by \(\alpha(i(x)(j,l)) = x(jp, lp)\) is a bijection and \(i^{-1}\) is continuous.

Proof. First we must show that this is well-defined. This just amounts to showing that \(\alpha\) is injective. To see this recall that if \(\alpha(e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}) = \lambda\) then \(\lambda = \lambda_1 \lambda_2 \cdots \lambda_n\) where each \(\lambda_i\) has degree \(p\) and hence \(d(\lambda_1 \lambda_2 \cdots \lambda_n) = np\). Since there is only one way to write \(np\)
as a sum of $p$'s, there is only one such decomposition of $\lambda$ (by unique factorization), so if $\alpha(e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_n}) = \alpha(e_{\gamma_1}e_{\gamma_2}\cdots e_{\gamma_n})$ we must have $\lambda_i = \gamma_i$ for all $i$.

Next, to show that $i$ is injective, suppose $i(x) = i(y)$ for $x, y \in \Lambda^\infty$. Then by definition we must have that $x(jp, lp) = y(jp, lp)$ for all $j, l \in \mathbb{N}$, and in particular we have that $x(0, jp) = y(0, jp)$ for all $j \in \mathbb{N}$. But since $\{jp\}_j$ is a cofinal increasing sequence in $\mathbb{N}^k$, $x$ and $y$ are uniquely determined by their values on the pairs $(0, jp)$ (see [KP, Remarks 2.2]) so we must have $x = y$.

Now, to show that $i$ is surjective, let $z \in E^\infty$. We wish to find an infinite path $x \in \Lambda^\infty$ such that $i(x) = z$. We will again make use of the fact that such an $x$ is uniquely determined by its values on $(0,.jp)$. Specifically, if $z(0, j) = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_j}$, then we let $x(0, jp) = \lambda_1\lambda_2\cdots\lambda_j$. Then $\alpha(i(x)(0, j)) = x(0, jp) = \lambda_1\lambda_2\cdots\lambda_j$ and $\alpha(z(0, j)) = \lambda_1\lambda_2\cdots\lambda_j$
so by the injectivity of $\alpha$ we have that $i(x)(0, j) = z(0, j)$ and since $i(x)$ and $z$ are uniquely determined by their values at $(0, j)$ we have that $i(x) = z$.

Finally, we need to show that $i^{-1}$ is continuous. We have

$$\alpha(i(x)(j, l)) = x(jp, lp) = \lambda_j\cdots\lambda_l,$$

where $\lambda_j\cdots\lambda_l$ is the unique decomposition of $x(jp, lp)$ into paths of degree $p$. Since $\alpha$ is injective, we get $i(x)(j, l) = e_{\lambda_j}\cdots e_{\lambda_l}$. Since this holds for all $(j, l)$ we must have that $i^{-1}(e_{\lambda_1}e_{\lambda_2}\cdots) = \lambda_1\lambda_2\cdots$ for all $e_{\lambda_1}e_{\lambda_2}\cdots \in E^\infty$. Recall that the topologies on $E^\infty$ and $\Lambda^\infty$ are generated by the collections $\{Z(P) : P \in E^*\}$ and $\{Z(\lambda) : \lambda \in \Lambda\}$ respectively where $Z(P) = \{Pz : z \in s(P)E^\infty\}$ and $Z(\lambda) = \{\lambda x : x \in s(\lambda)\Lambda^\infty\}$. Thus a net $\{\lambda_1^a\lambda_2^a\cdots\}_a \in \Lambda^\infty$ converges to $\lambda_1\lambda_2\cdots$ in $\Lambda^\infty$ if for all $n \in \mathbb{N}$ there is $a_0 \in A$ such that $\lambda_j^a = \lambda_j$ for all $j \leq n$ and $\alpha \geq a_0$, and similarly for nets in $E^\infty$. Now, suppose $\{e_{\lambda_1^a}\lambda_2^a\cdots\}_a \in A$ converges to $e_{\lambda_1}\lambda_2\cdots$ in $E^\infty$. Then for all $n \in \mathbb{N}$ there is $a_0 \in A$ such that $e_{\lambda_j^a} = e_{\lambda_j}$ for all $j \leq n$ and $\alpha \geq a_0$. Thus $\lambda_j^a = \lambda_j$ for all $j \leq n$ and $\alpha \geq a_0$, and we have shown that the net $\{i^{-1}(e_{\lambda_1^a}\lambda_2^a\cdots)\}_a \in A = \{\lambda_1^a\lambda_2^a\cdots\}_a \in A$ converges to $i^{-1}(e_{\lambda_1}\lambda_2\cdots) = \lambda_1\lambda_2\cdots$ in $\Lambda^\infty$. Therefore $i^{-1}$ is continuous.

**Proposition 8.3.** Let $(T, \sigma)$ be a $\Lambda$-system of maps and let $(T, \rho)$ be the diagonal $E$-system of $(T, \sigma)$. If $\Phi : \Lambda^\infty \to T$ is intertwining with respect to $(T, \sigma)$ then $\Phi \circ i^{-1} : E^\infty \to T$ is intertwining with respect to $(T, \rho)$.

**Proof.** We have:

$$\Phi \circ i^{-1} \circ \tau_\lambda(x) = \Phi(i^{-1}(e_\lambda x)) = \Phi(\lambda i^{-1}(x)) = \Phi \circ \tau_\lambda(i^{-1}(x)).$$
but since $\Phi$ is intertwining, this gives:

$$= \sigma_\lambda \circ \Phi(i^{-1}(x)) = \rho_{\lambda_c} \circ \Phi \circ i^{-1}(x).$$

Since $x$ was arbitrary, we have $(\Phi \circ i^{-1}) \circ \tau_\lambda = \rho_\lambda \circ (\Phi \circ i^{-1})$ so $\Phi \circ i^{-1}$ is intertwining with respect to $(T, \rho)$. \hfill \square

**Definition 8.4.** If $(T, \sigma)$ is a $\Lambda$ system of maps, $\Phi$ is an intertwining map, and $(T', \sigma|_{T'})$ is the self-similar $k$-surjective $\Lambda$-subsystem of Proposition 3.22, then we call $T'$ the attractor of $(T, \sigma, \Phi)$.

**Theorem 8.5.** Let $\Lambda$ be a $k$-graph. Suppose $(T, \sigma)$ is a $\Lambda$-system of maps, $\Phi$ is an intertwining map with respect to $(T, \sigma)$, and $T'$ is the attractor of $(T, \sigma, \Phi)$. Then there exist a 1-graph $E$ with $E^0 = \Lambda^0$, an $E$-system of maps $(T, \rho)$, and an intertwining map $\Psi$ with respect to $(T, \rho)$ such that if $T''$ is the attractor of $(T, \rho, \Psi)$ then $T'' = T'$.

**Proof.** Let $E$ be the diagonal 1-graph of $\Lambda$, $(T, \rho)$ be the diagonal $E$-system of $(T, \sigma)$, and $\Psi = \Phi \circ i^{-1}$. Proposition 8.3 shows that this is an intertwining map. For all $v \in \Lambda^0$ we have

$$T''_v = \Psi(vE^\infty) = \Phi(i^{-1}(vE^\infty)) = \Phi(v\Lambda^\infty) = T'_v,$$

and hence $T'' = T'$. \hfill \square

**References**

[BGR77] L. Brown, P. Green, and M. Rieffel, *Stable isomorphism and strong Morita equivalence of $C^*$-algebras*, Pacific J. Math. 71 (1977), 349–363.

[CLSV11] T. M. Carlsen, N. S. Larsen, A. Sims, and S. T. Vittadello, *Co-universal algebras associated to product systems, and gauge-invariant uniqueness theorems*, Proc. Lond. Math. Soc. (3) 103 (2011), no. 4, 563–600.

[DKPS10] V. Deaconu, A. Kumjian, D. Pask, and A. Sims, *Graphs of $C^*$-correspondences and Fell bundles*, Indiana Univ. Math. J. 59 (2010), no. 5, 1687–1735.

[EKQR00] S. Echterhoff, S. Kaliszewski, J. Quigg, and I. Raeburn, *Naturality and induced representations*, Bull. Austral. Math. Soc. 61 (2000), 415–438.

[Fow02] N. J. Fowler, *Discrete product systems of Hilbert bimodules*, Pacific J. Math. 204 (2002), no. 2, 335–375.

[FS02] N. J. Fowler and A. Sims, *Product systems over right-angled Artin semigroups*, Trans. Amer. Math. Soc. 354 (2002), 1487–1509.

[Ion07] M. Ionescu, *Operator algebras and Mauldin-Williams graphs*, Rocky Mountain J. Math. 37 (2007), no. 3, 829–849.

[KPQ13] S. Kaliszewski, N. Patani, and J. Quigg, *Obstructions to a general characterization of graph correspondences*, J. Austral. Math. Soc. 95 (2013), 169–188.

[KP] A. Kumjian and D. Pask, *Higher rank graph $C^*$-algebras*, New York J. Math. 6 (2000), 1–20.
[MW88] R. D. Mauldin and S. C. Williams, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc. **309** (1988), no. 2, 811–829.

[PQR05] D. Pask, J. Quigg, and I. Raeburn, *Coverings of k-graphs*, J. Algebra **289** (2005), no. 1, 161–191.

[SY10] A. Sims and T. Yeend, *C*-algebras associated to product systems of Hilbert bimodules, J. Operator Theory **64** (2010), no. 2, 349–376.

[Yee06] T. Yeend, *Topological higher-rank graphs and the C*-algebras of topological 1-graphs*, Operator theory, operator algebras, and applications, Contemp. Math., vol. 414, Amer. Math. Soc., Providence, RI, 2006, pp. 231–244.

**School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287**

*E-mail address*: kaliszewski@asu.edu

**School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287**

*E-mail address*: ammorgan@asu.edu

**School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287**

*E-mail address*: quigg@asu.edu