A BLACK–SCHOLES INEQUALITY: APPLICATIONS AND GENERALISATIONS

MICHAEL R. TEHRANCHI
UNIVERSITY OF CAMBRIDGE

Abstract. The space of call price functions has a natural noncommutative semigroup structure with an involution. A basic example is the Black–Scholes call price surface, from which an interesting inequality for Black–Scholes implied volatility is derived. The binary operation is compatible with the convex order, and therefore a one-parameter sub-semigroup can be identified with a peacock. It is shown that each such one-parameter semigroup corresponds to a unique log-concave probability density, providing a family of tractable call price surface parametrisations in the spirit of the Gatheral–Jacquier SVI surface. The key observation is an isomorphism linking an initial call price curve to the lift zonoid of the terminal price of the underlying asset.

1. Introduction

We define the Black–Scholes call price function \( C_{BS} : [0, \infty) \times [0, \infty) \to [0, 1] \) by the formula

\[
C_{BS}(\kappa, y) = \int_{-\infty}^{\infty} (e^{yz} - y^2/2 - \kappa)^+ \varphi(z) dz
= \begin{cases}
\Phi\left( -\log \frac{\kappa}{y} + \frac{y}{2} \right) - \kappa \Phi\left( -\log \frac{\kappa}{y} - \frac{y}{2} \right) & \text{if } y > 0, \kappa > 0, \\
(1 - \kappa)^+ & \text{if } y = 0, \\
1 & \text{if } \kappa = 0,
\end{cases}
\]

where \( \varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \) is the standard normal density and \( \Phi(x) = \int_{-\infty}^{x} \varphi(z) dz \) is its distribution function. Recall the financial context of this definition: a market with a zero-coupon bond of unit face value, maturity \( T \) and initial price \( B_{0,T} \); a stock whose the initial forward price for delivery date \( T \) is \( F_{0,T} \); and a European call option written on the stock with maturity \( T \) and strike price \( K \). In the Black–Scholes model, the initial price \( C_{0,T,K} \) of the call option is given by the formula

\[
C_{0,T,K} = B_{0,T} F_{0,T} C_{BS} \left( \frac{K}{F_{0,T}}, \sigma \sqrt{T} \right),
\]

where \( \sigma \) is the volatility of the stock price. In particular, the first argument of \( C_{BS} \) plays the role of the moneyness \( \kappa = K/F_{0,T} \) and the second argument plays the role of the total standard deviation \( y = \sigma \sqrt{T} \) of the terminal log stock price.

The starting point of this note is the following observation.

Date: February 13, 2017.

Keywords and phrases: semigroup with involution, implied volatility, peacock, lift zonoid, log-concavity.

Mathematics Subject Classification 2010: 60G44, 91G20, 60E15, 26A51, 52A21, 20M20.
Theorem 1.1. For $\kappa_1, \kappa_2 > 0$ and $y_1, y_2 > 0$ we have
$$C_{BS}(\kappa_1 \kappa_2, y_1 + y_2) \leq C_{BS}(\kappa_1, y_1) + \kappa_1 C_{BS}(\kappa_2, y_2)$$
with equality if and only if
$$-\frac{\log \kappa_1}{y_1} - \frac{y_1}{2} = -\frac{\log \kappa_2}{y_2} + \frac{y_2}{2}.$$

While it is fairly straightforward to prove Theorem 1.1 directly, the proof is omitted as it is a special case of Theorem 3.5 below. Indeed, the purpose of this note is to try to understand the fundamental principle that gives rise to such an inequality. As a hint of things to come, it is worth pointing out that the expression $y_1 + y_2$ appearing on the left-hand side of the inequality corresponds to the sum of the standard deviations – not the sum of the variances. From this observation, it may not be surprising to see that a key idea underpinning Theorem 1.1 is that of adding comonotonic – not independent – normal random variables. These vague comments will be made precise in Theorem 2.8 below.

Before proceeding, we re-express Theorem 1.1 in terms of the Black–Scholes implied total standard deviation function, defined for $\kappa > 0$ to be the inverse function $Y_{BS}(\kappa, \cdot) : [(1 - \kappa)^+, 1) \to [0, \infty)$ such that
$$y = Y_{BS}(\kappa, c) \iff C_{BS}(\kappa, y) = c.$$ In particular, the quantity $Y_{BS}(\kappa, c)$ denotes the implied total standard deviation of an option of moneyness $\kappa$ whose normalised price is $c$. We will find it notationally convenient to set $Y_{BS}(\kappa, c) = \infty$ for $c \geq 1$. With this notation, we have the following interesting reformulation which requires no proof:

Corollary 1.2. For all $\kappa_1, \kappa_2 > 0$ and $(1 - \kappa_i)^+ < c_i < 1$ for $i = 1, 2$, we have
$$Y_{BS}(\kappa_1, c_1) + Y_{BS}(\kappa_2, c_2) \leq Y_{BS}(\kappa_1 \kappa_2, c_1 + \kappa_1 c_2)$$
with equality if and only if
$$-\frac{\log \kappa_1}{y_1} - \frac{y_1}{2} = -\frac{\log \kappa_2}{y_2} + \frac{y_2}{2}.$$ 
where $y_i = Y_{BS}(\kappa_i, c_i)$ for $i = 1, 2$.

To add some context, we recall the following related bounds on the function $C_{BS}$ and $Y_{BS}$; see [16, Theorem 3.1].

Theorem 1.3. For all $\kappa > 0$, $y > 0$, and $0 < p < 1$ we have
$$C_{BS}(\kappa, y) \geq \Phi(\Phi^{-1}(p) + y) - pk$$
with equality if and only if
$$p = \Phi\left(-\frac{\log \kappa}{y} - \frac{y}{2}\right).$$
Equivalently, for all $\kappa > 0$, $(1 - \kappa)^+ < c < 1$ and $0 < p < 1$ we have
$$Y_{BS}(\kappa, c) \leq \Phi^{-1}(c + pk) - \Phi^{-1}(p)$$
where $\Phi^{-1}(u) = +\infty$ for $u \geq 1$. 

2
In [16], Theorem 1.3 was used to derive upper bounds on the implied total standard deviation function $Y_{BS}$ by selecting various values of $p$ to insert into the inequality.

The function $\Phi(\Phi^{-1}(\cdot) + y)$ has appeared elsewhere in various contexts. For instance, it is the value function for a problem of maximising the probability of hitting a target considered by Kulldorff [14, Theorem 6]. (Also see the book of Karatzas [11, Section 2.6].) Kulik & Tymoshkevych [13] observed, in the context of proving a certain log-Sobolev inequality, that the family of functions $(\Phi(\Phi^{-1}(\cdot) + y))_{y \geq 0}$ forms a semigroup under function composition. We will see that this semigroup property is the essential idea of our proof of Theorem 1.1 and its subsequent generalisations.

The rest of this note is arranged as follows. In section 2, we introduce a space of call price functions and explore some of its properties. In particular, we will see that it has a natural noncommutative semigroup structure with an involution. In section 3, it is shown that the binary operation is compatible with the convex order, and therefore a one-parameter subsemigroup of the space of call functions can be identified with a non-negative peacock. The main result of this article is that each one-parameter semigroup corresponds to a unique (up to translation) log-concave probability density, generalising the Black–Scholes call price surface and providing a family of reasonably tractable call surface parametrisations in the spirit of the SVI surface. In section 4, we provide the proofs of the main results. The key observation is an isomorphism linking a call price curve to the lift zonoid of the terminal price of the underlying asset.

2. The space of call prices: binary operation and involution

2.1. The space of all prices. The main focus of this note is to study the structure of the following family of functions

$$\mathcal{C} = \{C : [0, \infty) \to [0, 1] : \text{convex}, C(\kappa) \geq (1 - \kappa)^+ \text{ for all } \kappa \geq 0\}$$

An example of an element of $\mathcal{C}$ is the Black–Scholes call price function $C_{BS}(\cdot, y)$ for any $y \geq 0$. Indeed, we will shortly see that a general element of $\mathcal{C}$ can be interpreted as the family of normalised call prices written on a given stock, where the strike varies but the maturity date is fixed.

Before proceeding, let us make some observations concerning $\mathcal{C}$. Firstly, recall that a finite-valued convex function on $[0, \infty)$ has a well-defined right-hand derivative at each point taking values in $[-\infty, \infty)$. Furthermore, this right-hand derivative is non-decreasing and right-continuous. For $C \in \mathcal{C}$ we make the notational convention that $C'$ denotes this right-hand derivative:

$$C'(\kappa) = \lim_{\varepsilon \downarrow 0} \frac{C(\kappa + \varepsilon) - C(\kappa)}{\varepsilon} \text{ for all } \kappa \geq 0.$$  

(We note in passing that the left-hand derivative of $C$ is also well-defined on the open interval $(0, \infty)$, but we will not use it here and therefore do not introduce more notation.)

We now collect some basic facts:

**Proposition 2.1.** Suppose $C \in \mathcal{C}$ and let $C'$ be its right-hand derivative. Then

1. $C$ is continuous and $C(0) = 1$.
2. $C'(\kappa) \geq -1$ for all $\kappa \geq 0$.
3. $C$ is non-increasing.
4. There is a number $0 \leq C(\infty) \leq 1$ such that $C(\kappa) \to C(\infty)$ as $\kappa \uparrow \infty$.  


Proof. (1) Since convex functions are continuous in the interior of their domains, we need only check continuity at $\kappa = 0$. But note that by definition $(1 - \kappa)^+ \leq C(\kappa) \leq 1$ for all $\kappa \geq 0$.

(2) Since $C'$ is non-decreasing, we need only show $C'(0) \geq -1$. But we have
\[
\frac{C(\kappa) - C(0)}{\kappa} \geq \frac{(1 - \kappa)^+ - 1}{\kappa} \geq -1 \text{ for all } \kappa > 0,
\]
and setting $\kappa \downarrow 0$ proves the claim.

(3) By convexity we have for all $\kappa \geq 0$ and $\varepsilon > 0$ and $n \geq 1$ that
\[
C(\kappa + \varepsilon) - C(\kappa) \leq \frac{C(\kappa + n\varepsilon) - C(\kappa)}{n} \leq \frac{1}{n} \to 0 \quad \text{where we used the fact that } C(\kappa) \geq 0 \text{ and } C(\kappa + n\varepsilon) \leq 1.
\]
(4) This follows from (3) with $C(\infty) = \inf_{\kappa \geq 0} C(\kappa)$.

Elements of the set $\mathcal{C}$ can be given a probabilistic interpretation:

**Proposition 2.2.** The following are equivalent:

(1) $C \in \mathcal{C}$.

(2) There is a non-negative random variable $S$ with $E(S) \leq 1$ such that
\[
C(\kappa) = E[(S - \kappa)^+] + 1 - E(S) = 1 - E(S \land \kappa) \text{ for all } \kappa \geq 0.
\]
In this case $P(S > \kappa) = -C'(\kappa)$ for all $\kappa \geq 0$.

(3) There is a non-negative random variable $S^*$ with $E(S^*) \leq 1$ such that
\[
C(\kappa) = E[(1 - S^* \kappa)^+] = 1 - E[1 \land (S^* \kappa)] \text{ for all } \kappa \geq 0.
\]
In this case $P(S^* < 1/\kappa) = C(\kappa) - \kappa C'(\kappa)$ for all $\kappa > 0$.

The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are straightforward to verify. The implication (1) $\Rightarrow$ (2) is standard, often discussed in relation to the Breeden–Litzenberger formula. A proof can be found in the paper of Hirsh & Roynette [8, Proposition 2.1], among other places. The implication (1) $\Rightarrow$ (3) can be proven in a similar manner; alternatively, in Theorem 2.6 below we show the equivalence of (2) and (3).

Note that in Proposition 2.2 we have
\[
P(S > 0) = -C'(0) = E(S^*) \quad \text{and} \quad E(S) = 1 - C(\infty) = P(S^* > 0).
\]

Figure 1 plots the graph of a typical element $C \in \mathcal{C}$.

Given a function $C \in \mathcal{C}$, we will say that any random variable $S$ such that
\[
C(\kappa) = 1 - E(S \land \kappa) \text{ for all } \kappa \geq 0
\]
is a primal representation of $C$. Of course, all primal representations of $C$ have the same distribution. Similarly, any random variable $S^*$ such that
\[
C(\kappa) = 1 - E[1 \land (S^* \kappa)] \text{ for all } \kappa \geq 0
\]
is a dual representation of $C$.  


Given $C \in \mathcal{C}$, the relationship between distribution of a primal representation $S$ and a dual representation $S^*$ is given by
\[ P(S > \kappa) = E[S^* \mathbb{1}_{\{S^* < 1/\kappa\}}] \text{ for all } \kappa \geq 0, \]
or equivalently that
\[ E[\psi(S) \mathbb{1}_{\{S > 0\}}] = E[S^* \psi(1/S^*) \mathbb{1}_{\{S^* > 0\}}] \]
for all non-negative measurable $\psi$.

Note that
\[ C(\kappa) = P(S^* < 1/\kappa) - \kappa P(S > \kappa) \text{ for all } \kappa \geq 0. \]

To discuss the financial interpretation of the set $\mathcal{C}$, we first define two subsets by
\[ C_+ = \{ C \in \mathcal{C} : C'(0) = -1 \}. \]
and
\[ C_1 = \{ C \in \mathcal{C} : C(\infty) = 0 \}. \]

Given a $C \in \mathcal{C}$ suppose $S$ and $S^*$ are primal and dual representations. Note that if $C \in C_+$ then $P(S > 0) = E(S^*) = 1$, while if $C \in C_1$ then $P(S^* > 0) = E(S) = 1$. As an example, notice that for the Black–Scholes call function we have
\[ C_{BS}(\cdot, y) \in C_1 \cap C_+ \text{ for all } y \geq 0. \]

The financial interpretation of the quantity $C(\kappa)$ is easiest in the case when $C \in C_1$. Consider a market with a stock. Fix a maturity date $T > 0$ and suppose we choose monetary units such that the initial forward price of the stock for delivery date $T$ is $F_{0,T} = 1$. Now let $S = F_{T,T}$ model the time $T$ price of a stock. We assume there is no arbitrage during the one period between $t = 0$ and $t = T$, and hence there exists an equivalent measure (a $T$-forward measure) such that the forward price of a claim is just the expected value of its payout. In particular, for the stock itself we have $E(S) = 1$. The initial forward price of a call option of strike (equivalently, moneyness) $\kappa$ is given by the formula $C(\kappa) = E[(S - \kappa)^+]$.

There is alternative financial interpretation in the case where $C \in C_+$. Again, since $E(S^*) = 1$ we may suppose that the time $T$ price of a stock stock (expressed in units of
its forward price) is modelled by $S^*$ and that there is a fixed forward measure under which forward prices are computed by expectation. In particular, we may consider the quantity
\[ \kappa C(1/\kappa) = \mathbb{E}[(\kappa - S^*)^+] \]
as the forward price of a put option with strike $\kappa$.

Now, if $C$ is not in $C_+$ it is still possible to interpret the quantity $\kappa C(1/\kappa) = \mathbb{E}[(\kappa - S^*)^+]$ as a put price, but things are more subtle. If we let $\mathbb{P}$ be a fixed forward measure, then the forward price of the put is equal to the expected value of its payout, consistent with the no-arbitrage principle as before. However, in this case we have $\mathbb{E}(S^*) < 1$. The interpretation of this inequality in a complete market where the stock pays no dividend is that it possible to replicate one share of the stock at time $T$ by admissibly trading in the bond and the stock itself, such that the cost of the replicating portfolio is less than the initial price of one share of the stock! Nevertheless, such a bizarre situation is possible in certain continuous-time arbitrage-free markets exhibiting a bubble in the sense of Cox & Hobson in which the forward price $(F_{t,T})_{0 \leq t \leq T}$ of the underlying asset is a non-negative strictly local martingale.

Finally, if $C$ is not in $C_1$ we can still interpret $C(\kappa)$ as the price of a call option. In this case we have $\mathbb{E}(S) < 1$, and so the stock price has a bubble as described above. Furthermore, note that
\[ \mathbb{E}[(S - \kappa)^+] = \mathbb{E}(S) - \mathbb{E}(S \wedge \kappa) < C(\kappa) \]
so the call price also has bubble. However, writing $C(\kappa)$ by the formula
\[ C(\kappa) = 1 - \kappa + \mathbb{E}[(\kappa - S)^+] \]
we have the interpretation that the market prices the put option by expectation and then the call option by put-call parity.

2.2. The binary operation. We now introduce a binary operation $\bullet$ on $C$ defined by
\[ C_1 \bullet C_2(\kappa) = \inf_{\eta > 0} [C_1(\eta) + \eta C_2(\kappa/\eta)] \text{ for } \kappa \geq 0. \]
We caution that the operation $\bullet$ is not the well-known inf-convolution; however, we will see in section 4.2 that $\bullet$ is related to the inf-convolution via an exponential map.

Our interest in this operation is due to the observation that Theorem 1.1 amounts to the claim that for $y_1, y_2 \geq 0$ we have
\[ C_{BS}(\cdot, y_1) \bullet C_{BS}(\cdot, y_2) = C_{BS}(\cdot, y_1 + y_2). \]
It turns out that this operation has a natural financial interpretation, which we will give in Theorem 2.8 below.

A first result gives a probabilistic procedure for computing the binary operation.

**Proposition 2.3.** Let $S$ be a primal representation of $C_1 \in C$, and $S^*_2$ a dual representation of $C_2 \in C$. For all $\kappa > 0$ we have
\[ C_1 \bullet C_2(\kappa) = C_1(\eta) + \eta C_2(\kappa/\eta) \]
where $\eta \geq 0$ is such that
\[ \mathbb{P}(S_1 < \eta) \leq \mathbb{P}(S^*_2 \geq \eta/\kappa) \]
and
\[ \mathbb{P}(S_1 \leq \eta) \geq \mathbb{P}(S^*_2 > \eta/\kappa). \]
Proof. The convex function
\[ \eta \mapsto C_1(\eta) + \eta C_2(\kappa/\eta) = 1 - \mathbb{E}(\eta \wedge S_1) + \mathbb{E}[(\eta - \kappa S_2^*)] \]
is minimised when 0 is in its subdifferential, yielding the given inequalities. \( \Box \)

We now come to the key observation of this note. To state it, we distinguish two particular elements \( E, Z \in \mathcal{C} \) defined by
\[ E(\kappa) = (1 - \kappa)^+ \quad \text{and} \quad Z(\kappa) = 1 \quad \text{for all} \ \kappa \geq 0. \]
Note that the random variables representing \( E \) and \( Z \) are constant, with \( S = 1 = S^* \) representing \( E \) and \( S = 0 = S^* \) representing \( Z \). The following result shows that \( \mathcal{C} \) is a noncommutative semigroup with respect to \( \cdot \), where \( E \) is the identity element and \( Z \) is the absorbing element:

**Theorem 2.4.** For every \( C, C_1, C_2, C_3 \in \mathcal{C} \) we have

1. \( E \cdot C = C \cdot E = C \).
2. \( Z \cdot C = C \cdot Z = Z \).
3. \( C_1 \cdot C_2 \in \mathcal{C} \).
4. \( C_1 \cdot (C_2 \cdot C_3) = (C_1 \cdot C_2) \cdot C_3 \).

One could prove parts (1), (2) and (4) directly, but part (3) requires a little work. We postpone the proof until section 4.1.

The following result shows that the subsets \( \mathcal{C}^+ \) and \( \mathcal{C}_1 \) are closed with respect to the binary operation.

**Proposition 2.5.** Given \( C_1, C_2 \in \mathcal{C} \) we have

1. \( C_1 \cdot C_2 \in \mathcal{C}_1 \) if and only if both \( C_1 \in \mathcal{C}_1 \) and \( C_2 \in \mathcal{C}_1 \).
2. \( C_1 \cdot C_2 \in \mathcal{C}^+ \) if and only if both \( C_1 \in \mathcal{C}^+ \) and \( C_2 \in \mathcal{C}^+ \).

Again, the proof appears in section 4.1.

2.3. The involution. For \( C \in \mathcal{C} \), let \( C^*(0) = 1 \) and
\[ C^*(\kappa) = 1 - \kappa + \kappa C \left( \frac{1}{\kappa} \right) \quad \text{for all} \ \kappa > 0. \]

As an example, notice for the Black–Scholes call function we have
\[ C_{\text{BS}}(\cdot, y)^* = C_{\text{BS}}(\cdot, y) \quad \text{for all} \ y \geq 0 \]

by the classical put-call symmetry formula.

The function \( C^* \) is clearly related to the well-known perspective function of the convex function \( C \) defined by \( (\eta, \kappa) \mapsto \eta C(\kappa/\eta) \); see, for instance, the book of Boyd & Vanderberghe [4, Section 3.2.6]. We now show that the operation \( * \) is an involution compatible with the binary operation \( \cdot \).

**Proposition 2.6.** Given \( C, C_1, C_2 \in \mathcal{C} \) we have

1. \( C^* \in \mathcal{C} \)
2. \( (C^*)^* = C \).
3. \( (C_1 \cdot C_2)^* = C_2^* \cdot C_1^* \)
Proof. (1) By the implication (1) \(\Rightarrow\) (2) of Proposition 2.2 we have
\[
C(\kappa) = 1 - \mathbb{E}(S \land \kappa) \text{ for all } \kappa \geq 0.
\]
Then
\[
C^*(\kappa) = 1 - \kappa + \kappa C(1/\kappa) = \mathbb{E}[(1 - S\kappa)^+] \text{ for all } \kappa \geq 0.
\]
By the implication (3) \(\Rightarrow\) (1) of Proposition 2.2 we have \(C^* \in \mathcal{C}\).

(2) \((C^*)^*(\kappa) = 1 - \kappa + \kappa C^*(1/\kappa) = 1 - \kappa + \kappa [1 - 1/\kappa + (1/\kappa)C(\kappa)] = C(\kappa)
\)

(3) Using the definitions, we have for \(\kappa > 0\) that
\[
C^* \cdot C^*(\kappa) = \inf_{\eta > 0} [C^*_2((\eta) + \eta C^*_1(\kappa/\eta)]
\]
\[
= 1 - \kappa + \kappa \inf_{\eta > 0} [C_1(\eta/\kappa) + (\eta/\kappa)C_2(1/\eta)]
\]
\[
= 1 - \kappa + \kappa(C_1 \cdot C_2)(1/\kappa)
\]
\[
= (C_1 \cdot C_2)^*(\kappa)
\]
\(\square\)

The proof of Proposition 2.6(1) shows that the involution simply swaps the primal and dual representation. We restate it for emphasis.

**Proposition 2.7.** If there are non-negative random variables \(S\) and \(S^*\) such that
\[
C(\kappa) = 1 - \mathbb{E}(S \land \kappa) = 1 - \mathbb{E}[1 \land (S^* \kappa)] \text{ for all } \kappa \geq 0
\]
then
\[
C^*(\kappa) = 1 - \mathbb{E}(S^* \land \kappa) = 1 - \mathbb{E}[1 \land (S\kappa)] \text{ for all } \kappa \geq 0.
\]
In particular, \(C \in \mathcal{C}_+\) if and only if \(C^* \in \mathcal{C}_1\).

2.4. **An interpretation.** We are now in a position to give a probabilistic interpretation of the binary operation \(\bullet\).

**Theorem 2.8.** Let \(S_1\) be a primal representation of \(C_1 \in \mathcal{C}\), and \(S_2^*\) a dual representation of \(C_2 \in \mathcal{C}\), where \(S_1\) and \(S_2^*\) are defined on the same space. Then we have
\[
C_1 \bullet C_2(\kappa) \geq 1 - \mathbb{E}[S_1 \land (S_2^* \kappa)] \text{ for all } \kappa \geq 0,
\]
with equality if \(S_1\) and \(S_2^*\) are countermonotonic.

**Proof.** Fix \(\kappa > 0\) and pick \(\eta \geq 0\) such that
\[
\mathbb{P}(S_1 < \eta) \leq \mathbb{P}(S_2^* \geq \eta/\kappa)
\]
and
\[
\mathbb{P}(S_1 \leq \eta) \geq \mathbb{P}(S_2^* > \eta/\kappa).
\]
Recalling that for real \(a, b\) we have
\[
(a + b)^+ \leq a^+ + b^+
\]
with equality if and only if \(ab \geq 0\) we have
\[
1 - \mathbb{E}[S_1 \land (S_2^* \kappa)] = \mathbb{E}[(S_1 - \kappa S_2^* )^+] + 1 - \mathbb{E}(S_1)
\]
\[
\leq \mathbb{E}[(S_1 - \eta)^+] + \mathbb{E}[(\eta - \kappa S_2^* )^+] + 1 - \mathbb{E}(S_1)
\]
\[
= C_1(\eta) + \eta C_2(\kappa/\eta)
\]
\[
= C_1 \cdot C_2(\kappa)
\]
by Theorem 2.3.
Now if $S_1$ and $S_2^*$ are countermonotonic, then
\[ \{ S_1 < \eta \} \subseteq \{ S_2^* \geq \eta/\kappa \} \]
and
\[ \{ S_1 \leq \eta \} \supseteq \{ S_2^* > \eta/\kappa \} . \]
In particular, we have $(S_1 - \eta)(\eta - \kappa S_2^*) \geq 0$ almost surely, and hence there is equality above. □

We now consider the Black–Scholes call function in light of Theorem 2.8. Let $Z$ be a standard normal random variable, and for any $y \in \mathbb{R}$ let
\[ S^{(y)} = e^{-yZ - y^2/2} \]
so that $S^{(y)}$ is both a primal and dual representation of $C_{BS}(\cdot, |y|)$. Since for $y_1, y_2 \geq 0$ the random variables $S^{(y_1)}$ and $S^{(-y_2)}$ are countermonotonic, the identity
\[ C_{BS}(\cdot, y_1) \bullet C_{BS}(\cdot, y_2) = C_{BS}(\cdot, y_1 + y_2) \]
can be proven by noting
\[ C_{BS}(\kappa, y_1 + y_2) = 1 - \mathbb{E}[S^{(y_1)} \wedge (S^{(-y_2)}\kappa)]. \]

We now give the financial interpretation of Theorem 2.8 in the case where $C_1 \in C_1$, or equivalently, $\mathbb{E}(S_1) = 1$. In this case we have
\[ C_1 \bullet C_2(\kappa) = \max_{S_1, S_2^*} \mathbb{E}[(S_1 - S_2^*\kappa)^+] \text{ for all } \kappa \geq 0, \]
where the maximum is taken over all primal representations $S_1$ of $C_1$ and dual representations $S_2^*$ of $C_2$ defined on the same probability space. In particular, the quantity $C_1 \bullet C_2(\kappa)$ gives the upper bound on the no-arbitrage price of an option to swap $\kappa$ shares of an asset with price $S_2$ for one share of another asset with price $S_1$, given all of the call prices of both assets. This interpretation is related to the upper bound on basket options found by Hobson, Laurence & Wang [9, Theorem 3.1].

We can also give another probabilistic interpretation of the binary operation as a kind of product of primal representations. Suppose $S_1$ is a primal representation of $C_1$ and $S_2^*$ is a dual representation of $C_2$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that $S_1$ and $S_2^*$ are countermonotonic. Then applying Theorem 2.8 we have
\[ C_1 \bullet C_2(\kappa) = 1 - \mathbb{E}[S_1 \wedge (\kappa S_2^*)] \]
\[ = 1 - \mathbb{E}[S_2^* \mathbb{1}_{\{S_2^* > 0\}}(S_1/S_2^*) \wedge \kappa] \]
\[ = 1 - \mathbb{E}[(S_1\tilde{S}_2) \wedge \kappa]. \]

where the last expectation is under the absolutely continuous probability measure $\tilde{\mathbb{P}}$ with density
\[ \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = S_2^* + \frac{(1 - \mathbb{E}(S_2^*))}{\mathbb{P}(S_2^* = 0)} \mathbb{1}_{\{S_2^* = 0\}}. \]
(with the convention that $0/0 = 0$) and where we define another random variable $\tilde{S}_2$ by
\[ \tilde{S}_2 = \mathbb{1}_{\{S_2^* > 0\}} \frac{1}{S_2^*}. \]
Note that the random variable $\tilde{S}_2$ is a primal representation of $C_2$ under the measure $\tilde{P}$. However, although the random variable $S_1$ is a primal representation of $C_1$ under the measure $P$, it is generally not a primal representation under the measure $\tilde{P}$.

3. One-parameter semigroups and peacocks

3.1. An ordering. We can introduce a partial order $\leq$ on $\mathcal{C}$ by

$$C_1 \leq C_2 \text{ if and only if } C_1(\kappa) \leq C_2(\kappa) \text{ for all } \kappa \geq 0.$$  

The operation $\bullet$ interacts well with this partial ordering:

**Theorem 3.1.** For any $C_1, C_2 \in \mathcal{C}$, we have $C_1 \leq C_1 \bullet C_2$ and $C_2 \leq C_1 \bullet C_2$.

We defer the proof to section 4.1.

The partial order can be given a useful probabilistic interpretation when restricted to the family $C_1$ of call functions $C$ with $C(\infty) = 0$ whose primal representation $S$ satisfies $E(S) = 1$.

The following is well-known; see, for instance, the book of Hirsh, Profeta, Roynette & Yor [7, Exercise 1.7].

**Theorem 3.2.** Given $C_1, C_2 \in C_1$ with primal representations $S_1, S_2$. Then the following are equivalent

1. $C_1 \leq C_2$
2. $S_1$ is dominated by $S_2$ in the convex order, that is, $E[\psi(S_1)] \leq E[\psi(S_1)]$ for all convex $\psi$ such that $\psi(S_1)$ is integrable.

We find it useful to recall the definition of a term popularised by Hirsh, Profeta, Roynette & Yor [7]:

**Definition 3.3.** A peacock is a family $(S_t)_{t \geq 0}$ of integrable random variables increasing in the convex order.

The term peacock is derived from the French acronym PCOC, Processus Croissant pour l’Ordre Convexe.

From Theorem 3.1 we see that if $(C(\cdot, t))_{t \geq 0}$ is a one-parameter sub-semigroup of $C_1$ where each function $C(\cdot, t)$ has a primal presentation $S_t$, then the family $(S_t)_{t \geq 0}$ is a peacock. Interest in peacocks in probability and financial mathematics is due to the following theorem of Kellerer [10]:

**Theorem 3.4.** A family $(S_t)_{t \geq 0}$ of random variables is a peacock if and only if there exists a filtered probability space on which a martingale $(\tilde{S}_t)_{t \geq 0}$ is defined such that $S_t \sim \tilde{S}_t$ for all $t \geq 0$.

See the paper [8] of Hirsh & Roynette for a recent proof.

3.2. One-parameter semigroups. We now study the family of sub-semigroups of $\mathcal{C}$ indexed by a single parameter $y \geq 0$. We will make use of the following notation. For a probability density function $f$, let

$$C_f(\kappa, y) = \int_{-\infty}^{\infty} (f(z + y) - \kappa f(z))^+ dz = 1 - \int_{-\infty}^{\infty} f(z + y) \wedge [\kappa f(z)] dz$$

for $y \in \mathbb{R}$ and $\kappa \geq 0$. Note that

$$C_{BS} = C_\varphi$$
where $\varphi$ is the standard normal density.

In what follows we will assume that the density $f$ has support of the form $[L, R]$ and is continuous and positive on $(L, R)$, for some constants $-\infty \leq L < R \leq +\infty$. Now let $Z$ be random variable with density $f$. For each $y \in \mathbb{R}$, define a non-negative random variable by

$$S^{(y)}(y) = \frac{f(Z + y)}{f(Z)}.$$ 

Note that $S^{(y)}$ is well-defined since $L < Z < R$ almost surely, and hence $f(Z) > 0$ almost surely. Note also that

$$\mathbb{P}(S^{(y)} > 0) = \int_{L}^{L \vee (R - y)} f(z) dz$$

and

$$\mathbb{E}(S^{(y)}) = \int_{R \wedge (L + y)}^{R} f(z) dz$$

In this notation, we have

$$C_{f}(\kappa, y) = 1 - \mathbb{E}[S^{(y)} \wedge \kappa]$$

so that by Proposition 2.2 we have $C_{f}(\cdot, y) \in \mathcal{C}$ for all $y \geq 0$ and that $S^{(y)}$ is a representation of $C_{f}(\cdot, y)$ for $y \geq 0$. In particular, for $y > 0$ we have

$$C_{f}(\cdot, y) \in \mathcal{C}_{+} \text{ if } R = +\infty$$

and

$$C_{f}(\cdot, y) \in \mathcal{C}_{1} \text{ if } L = -\infty.$$ 

By changing variables, we find that a dual representation of $C_{f}(\cdot, y)$ is given by

$$S^{(y)}_{*} = \frac{f(Z - y)}{f(Z)} = S^{(-y)}$$

and therefore we can write

$$C_{f}(\kappa, y) = \mathbb{P}(S^{(-y)} < 1/\kappa) - \kappa \mathbb{P}(S^{(y)} > \kappa)$$

for all $y \in \mathbb{R}$ and $\kappa > 0$. Note that

$$C_{f}(\cdot, y)^{*} = C_{f}(\cdot, -y).$$

It is interesting to observe that the call price surface $C_{f}$ satisfies the put-call symmetry formula $C_{f}(\cdot, y)^{*} = C_{f}(\cdot, y)$ if the density $f$ is an even function.

We can be even more explicit for densities $f$ supported on all of $\mathbb{R}$ with the property that for all $y > 0$ $z \mapsto f(z + y)/f(z)$ is continuous and strictly decreasing and such that

$$\lim_{z \downarrow -\infty} \frac{f(z + y)}{f(z)} = +\infty \quad \text{and} \quad \lim_{z \uparrow +\infty} \frac{f(z + y)}{f(z)} = 0.$$ 

Note that if $y < 0$ then $z \mapsto f(z + y)/f(z)$ is strictly increasing with

$$\lim_{z \downarrow -\infty} \frac{f(z + y)}{f(z)} = 0 \quad \text{and} \quad \lim_{z \uparrow +\infty} \frac{f(z + y)}{f(z)} = \infty.$$ 

For any $y \in \mathbb{R}\setminus\{0\}$ and $\kappa > 0$ we set $d(\kappa, y)$ to be the unique solution to

$$\frac{f(d + y)}{f(d)} = \kappa.$$
From the definition of $d(\cdot, \cdot)$ we note the identity
\[ d(1/\kappa, -y) = d(\kappa, y) + y. \]
In this case we have the following formula:
\[ C_f(\kappa, y) = F(d(\kappa, y) + y) - \kappa F(d(\kappa, y)) \text{ for all } \kappa > 0, y > 0. \]
Note that the standard normal density $\varphi$ verifies the above hypotheses with
\[ d(\kappa, y) = -\frac{\log \kappa}{y} - \frac{y}{\kappa}, \]
in line with the usual Black–Scholes formula. We note in passing that the call price function $\phi$ satisfies a non-linear partial differential equation
\[ \frac{\partial C_f}{\partial y} = \kappa \hat{H} \left( -\frac{\partial C_f}{\partial \kappa} \right) = \hat{H} \left( C_f - \kappa \frac{\partial C_f}{\partial \kappa} \right) \]
where $\hat{H} = f \circ F^{-1}$. The significance of the function $\hat{H}$ will be explored in section 4.2.

We now present a family of one-parameter sub-semigroups of $C$.

**Theorem 3.5.** Let $f$ be a log-concave probability density function. Then
\[ C_f(\cdot, y_1) \bullet C_f(\cdot, y_2) = C_f(\cdot, y_1 + y_2) \text{ for all } y_1, y_2 \geq 0. \]

Note that Theorem 3.5 says for all $\kappa_1, \kappa_2 > 0$ and $y_1, y_2 > 0$, that
\[ C_f(\kappa_1 \kappa_2, y_1 + y_2) \leq C_f(\kappa_1, y_1) + \kappa_1 C_f(\kappa_2, y_2). \]
Furthermore if $f$ is supported on all of $\mathbb{R}$ and $z \mapsto f(z + y)/f(z)$ is continuous and decreases strictly from $+\infty$ to 0, then there is equality if and only if
\[ d(\kappa_1, y_1) = d(\kappa_2, y_2) + y_2 \]
in the notation introduced above. In particular, Theorem 3.5 implies Theorem 1.1

While Theorem 3.5 is not especially difficult to prove, we will offer two proofs with each highlighting a different perspective on the operation $\bullet$. The first is below and the second is in Section 4.

**Proof.** Letting $Z$ be a random variable with density $f$, note that $f(Z + y)/f(Z)$ is a primal representation of $C_f(\cdot, y)$. Note also that by log-concavity of $f$, when $y \geq 0$ the function $z \mapsto f(z + y)/f(z)$ is non-increasing. Similarly, $f(Z - y)/f(Z)$ is a dual representation of $C_f(\cdot, y)$ and $z \mapsto f(z - y)/f(z)$ is non-decreasing. In particular, the random variables $f(Z + y_1)/f(Z)$ and $f(Z - y_2)/f(Z)$ are countermonotonic, and hence by Theorem 2.8 we have
\[ C_f(\cdot, y_1) \bullet C_f(\cdot, y_2)(\kappa) = 1 - \int_{-\infty}^{\infty} f(z + y_1) \wedge [\kappa f(z - y_2)]dz. \]
The conclusion follows from changing variables in the integral on the right-hand side. \qed

Combining Theorems 3.5 and 3.1 yields the following tractable family of peacocks.

**Theorem 3.6.** Let $f$ be a log-concave density with support of the form $(-\infty, R]$, let be a random variable $Z$ have density $f$ and let $Y : [0, \infty) \rightarrow [0, \infty)$ be increasing. Set
\[ S_t = \frac{f(Z + Y(t))}{f(Z)} \text{ for } t \geq 0. \]
The family of random variables \((S_t)_{t \geq 0}\) is a peacock.

Note that we can recover the Black–Scholes model by setting the density to \(f = \varphi\) the standard normal density and the increasing function to

\[ Y(t) = \sigma \sqrt{t} \]

where \(\sigma\) is the volatility of the stock.

The upshot of Theorem 3.6 is that if we define a family of arbitrage-free implied volatility surface by

\[ (\kappa, t) \mapsto \frac{1}{\sqrt{t}} Y_{BS}(\kappa, C_f(\kappa, Y(t))). \]

Given \(f\) and \(Y\), the above formula is reasonably tractable, and could be seen to be in the same spirit as the SVI parametrisation of the implied volatility surface given by Gatheral & Jacquier [6].

Here is a concrete example. Let \(f(z) = e^{z-e^z}\) be a Gumbel density function with

\[ F(z) = \int_{-\infty}^{z} f(x) dx = 1 - e^{-e^z} \]

the corresponding to the distribution function. Clearly \(f\) is a log-concave density. Letting \(Z\) have the Gumbel distribution and setting

\[ S(y) = \frac{f(Z + y)}{f(Z)} = e^{y-e^{Z(e^y-1)}} = e^y U e^{y-1} \]

where \(U = 1 - F(Z)\) has the uniform distribution. By Theorem 3.6 the family \(((\tau + 1)U^\tau)_{\tau \geq 0}\) is a peacock, where \(Y(\tau) = \log(\tau + 1)\).

There are various techniques for constructing a martingale whose marginals match a given peacock, including appealing to Dupire’s formula. However, in this case, we can be more explicit. In fact, letting \(W\) be a standard two-dimensional Brownian motion and

\[ Z_t = \frac{1}{\sqrt{2}} \int_0^t e^{\frac{1}{2}(t-s)} dW_s \]

an application of Itô’s formula shows that

\[ S_t = e^{t-\|Z_t\|^2} \]

defines a martingale and standard properties of the \(\chi^2\) distribution show that \(S_t \sim S(t)\) for all \(t \geq 0\).

We have found one family \((C_f)_{\tau \geq 0}\) of one-parameter semigroups of \(C\) indexed by the set of log-concave densities \(f\) on \(\mathbb{R}\). However, there are other one-parameter semigroups which are not in this family. For instance, one example is the trivial semigroup consisting of the identity element \(C_{\text{triv}}(\cdot, y) = E\) for all \(y \geq 0\). Another is the null semigroup is given by \(C_{\text{null}}(\cdot, 0) = E\) and \(C_{\text{null}}(\cdot, y) = Z\) for \(y > 0\). The following theorem says that these examples exhaust the possibilities.
Theorem 3.7. Suppose

\[ C(\kappa, 0) = (1 - \kappa)^+ \] for all \( \kappa \geq 0 \)

and

\[ C(\cdot, y_1) \bullet C(\cdot, y_2) = C(\cdot, y_1 + y_2) \] for all \( y_1, y_2 \geq 0 \).

Then exactly one of the following holds true:

1. \( C(\kappa, y) = (1 - \kappa)^+ \) for all \( \kappa \geq 0, y > 0 \);
2. \( C(\kappa, y) = 1 \) for all \( \kappa \geq 0, y > 0 \);
3. \( C = C_f \) some log-concave density \( f \).

The proof appears in Section 4. Note that possibility (3) above interpolates between possibilities (1) and (2). Indeed, fix a log-concave density \( f \) and set

\[ f^{(r)}(z) = r f(rz) \] for all \( z \in \mathbb{R}, r > 0 \).

Then

\[ C_f^{(r)} \to C_{\text{triv}} \text{ as } r \downarrow 0 \]

and

\[ C_f^{(r)} \to C_{\text{null}} \text{ as } r \uparrow \infty. \]

4. An isomorphism and lift zonoids

4.1. The isomorphism. In this section, to help understand the binary operation \( \bullet \) on the space \( C \) we show that there is a nice isomorphism of \( C \) to another function space converts the somewhat complicated operation \( \bullet \) into simple function composition \( \circ \).

We introduce a transformation \( \hat{\cdot} \) on the space \( C \) which will be particularly useful. For \( C \in C \) we define a new function \( \hat{C} \) on \([0, 1]\) by the formula

\[ \hat{C}(p) = \inf_{\kappa \geq 0} [C(\kappa) + p\kappa] \] for \( 0 \leq p \leq 1 \).

We can read off some properties of the new function \( \hat{C} \) quickly.

Proposition 4.1. Fix \( C \in C \) with primal representation \( S \) and dual representation \( S^* \).
(1) \( \hat{C} \) is non-decreasing and concave.

(2) \( \hat{C} \) is continuous and

\[
\hat{C}(0) = C(\infty) = 1 - \mathbb{E}(S) = \mathbb{P}(S^* = 0).
\]

(3) For \( 0 \leq p \leq 1 \) and \( K \geq 0 \) such that

\[
\mathbb{P}(S > \kappa) \leq p \leq \mathbb{P}(S \geq \kappa),
\]

we have

\[
\hat{C}(p) = C(\kappa) + p\kappa.
\]

(4) \( \min\{p \geq 0 : \hat{C}(p) = 1\} = -C'(0) = \mathbb{P}(S > 0) = \mathbb{E}(S^*). \)

(5) \( \hat{C}(p) \geq p \) for all \( 0 \leq p \leq 1 \).

Figure 2 plots the graph of a typical element \( \hat{C} \in \hat{C} \).

**Proof.** (1) The infimum of a family of concave and non-decreasing functions is again concave and non-decreasing.

(2) A concave function is function is continuous in the interior of its domain. Since \( \hat{C} \) is non-decreasing it is continuous at \( p = 1 \). We need only check continuity at \( p = 0 \). By definition we have

\[
\hat{C}(0) = \inf_{\kappa \geq 0} C(\kappa) = C(\infty).
\]

On the other hand, since \( \hat{C} \) is non-decreasing we have by definition

\[
\hat{C}(0) \leq \hat{C}(p) \leq C(\kappa) + \kappa p.
\]

Now sending first \( p \downarrow 0 \) and then \( \kappa \uparrow \infty \) in the above inequality proves the continuity of \( \hat{C} \).

(3) The convex function \( \kappa \mapsto C(\kappa) + p\kappa \) is minimised when \( 0 \) is contained in the subdifferential, which amounts the displayed inequality.

(4) By implication (3) with \( \kappa = 0 \), we have \( \hat{C}(p) = 1 \) for all \( p \geq \mathbb{P}(S > 0) \). On the other hand, suppose \( p < \mathbb{E}(S^*) \). Then by continuity, there exists a large enough \( N \) such that \( p < \mathbb{E}(S^* \wedge N) \). Hence

\[
\hat{C}(p) \leq C(1/N) + p/N = 1 + (p - \mathbb{E}(S^* \wedge N))/N < 1.
\]

(5) By concavity \( \hat{C}(p) \geq (1 - p)\hat{C}(0) + p\hat{C}(1) \). The conclusion follows from \( \hat{C}(0) \geq 0 \) and \( \hat{C}(1) = 1 \). \( \Box \)

We now show that \( ^* \) is a bijection:

**Theorem 4.2.** Suppose \( g : [0, 1] \to [0, 1] \) is continuous and concave with \( g(1) = 1 \). Let

\[
C(\kappa) = \max_{0 \leq p \leq 1} [g(p) - pk] \text{ for all } \kappa \geq 0.
\]

Then \( C \in C \) and \( g = \hat{C} \).

The above theorem is a variant of the Fenchel–Moreau theorem of convex analysis. We include a proof for completeness.
Proof. Note that $C$, being the maximum of a family of convex functions, is convex. We have for all $\kappa \geq 0$ the upper bound

$$C(\kappa) \leq \max_{0 \leq p \leq 1} g(p) = 1$$

with equality when $\kappa = 0$. We also have the lower bounds

$$C(\kappa) \geq g(p) - p\kappa \text{ for all } 0 \leq p \leq 1.$$ 

In particular, by plugging in $p = 0$, we have that

$$C(\kappa) \geq g(0) \geq 0$$

and, by plugging in $p = 1$, that

$$C(\kappa) \geq g(1) - \kappa = 1 - \kappa.$$ 

This shows that $C(\kappa) \geq (1 - \kappa)^+$ and hence $C \in \mathcal{C}$ as claimed.

Now the lower bound yields

$$g(p) \leq \hat{C}(p) \text{ for all } 0 \leq p \leq 1.$$ 

We need only show the reverse inequality. By the concavity of $g$ we have for all $0 < p, p_0 < 1$ that

$$g(p) \leq g(p_0) + g'(p_0)(p - p_0)$$

where $g'$ is the right-hand derivative of $g$. By the continuity of $g$, the above inequality also holds for $p = 0, 1$. Hence

$$C(\kappa) \leq \max_{0 \leq p \leq 1} [g(p_0) + g'(p_0)(p - p_0) - p\kappa]$$

$$= g(p_0) - g'(p_0)p_0 + (g'(p_0) - \kappa)^+$$

Again by the concavity of $g$ we have

$$\frac{g(p_0) - g(p_0 - \varepsilon)}{\varepsilon} \geq \frac{g(1) - g(p_0)}{1 - p_0}$$

for all $0 < \varepsilon < p_0$, and in particular, since $g(p_0) \leq 1 = g(1)$, we have

$$g'(p_0) \geq 0.$$ 

Therefore

$$\hat{g}(p) \leq \inf_{\kappa \geq 0} [g(p_0) - g'(p_0)p_0 + (g'(p_0) - \kappa)^+ + p\kappa]$$

$$= g(p_0) + g'(p_0)(p - p_0)$$

for all $0 \leq p \leq 1$. Setting $p = p_0$ we have $\hat{C}(p) \leq g(p)$ for $0 < p < 1$. The continuity of $\hat{C}$ and $g$ means the inequality also holds for $p = 0, 1$, completing the proof. □

The following theorem explains our interest in the bijection $\leq$: it converts the binary operation $\bullet$ to function composition $\circ$. A version of this result can be found in the book of Borwein & Vanderwerff [3, Exercise 2.4.31].

**Theorem 4.3.** For $C_1, C_2 \in \mathcal{C}$ we have

$$\hat{C}_1 \bullet \hat{C}_2 = \hat{C}_1 \circ \hat{C}_2$$
Proof. By the continuity of a function $C \in \mathcal{C}$ at $\kappa = 0$, we have the equivalent expression

$$\hat{C}(p) = \inf_{\kappa > 0} [C(\kappa) + p\kappa]$$

for $0 \leq p \leq 1$.

Hence for any $0 \leq p \leq 1$ we have

$$\hat{C}_1 \bullet \hat{C}_2(p) = \inf_{\kappa > 0} [\hat{C}_1 \bullet \hat{C}_2(\kappa) + p\kappa]$$

$$= \inf \{ \inf_{H > 0} [\hat{C}_1(H) + HC_2(\kappa/H)] + p\kappa \}$$

$$= \inf \{ \inf_{H > 0} \{ \inf_{\kappa > 0} [C_1(H) + H \inf_{\kappa > 0} [C_2(\kappa) + p\kappa]] \} + p\kappa \}$$

$$= \inf \{ \inf_{H > 0} [\hat{C}_1 \circ \hat{C}_2(p)] \}$$

$$= \hat{C}_1 \circ \hat{C}_2(p).$$

We are now ready to give the remaining proof of the results of Sections 2 and 3.

Proof of Theorem 2.4. (1) In light of Theorem 4.3 we need to show $\hat{E}(p) = p$ for all $0 \leq p \leq 1$. Indeed for all $0 \leq p \leq 1$ and $\kappa \geq 0$ we have

$$p(1 - \kappa) \leq p(1 - \kappa) + \kappa \leq (1 - \kappa) + \kappa$$

and hence

$$E(\kappa) + p\kappa \geq p,$$

with equality if $\kappa = 1$.

(2) Note that $\hat{Z}(p) = \inf_{\kappa > 0} [1 + p\kappa] = 1$ for all $0 \leq p \leq 1$. The claim follows since $\hat{C}(1) = 1$ for all $C \in \mathcal{C}$.

(3) Thanks to Theorem 4.3 we need only check that $h = \hat{C}_1 \circ \hat{C}_2$ is in $\hat{C}$. That $h$ is a continuous map $[0, 1] \to [0, 1]$ with $h(1) = 1$ is easy to check. That $h$ is concave follows from the computation: fix $0 \leq p_1, p_2 \leq 1$ and $0 \leq \lambda \leq 1$ and set $\mu = 1 - \lambda$. Since $\hat{C}_2$ is concave and $\hat{C}_1$ is non-decreasing we have

$$h(\lambda p_1 + \mu p_2) \geq \hat{C}_1(\lambda \hat{C}_2(p_1) + \mu \hat{C}_2(p_2))$$

$$\geq \lambda \hat{C}_1 \circ \hat{C}_2(p_1) + \mu \hat{C}_1 \circ \hat{C}_2(p_2)$$

where we used the concavity of $\hat{C}_1$ in the second line.

(4) The associativity of $\circ$ follows is inherited from the associativity of $\bullet$. □

Proof of Proposition 2.5. Recall that $C \in \mathcal{C}_1$ if and only if $\hat{C}(0) = 0$. We prove claim (1) by noting the inequality

$$\hat{C}_1 \circ \hat{C}_2(0) = \hat{C}_1 \bullet \hat{C}_2(0) \geq \max\{\hat{C}_1(0), \hat{C}_2(0)\}$$

and from which we conclude that $\hat{C}_1 \bullet \hat{C}_2 = 0$ if and only if $\hat{C}_1(0) = 0 = \hat{C}_2(0)$.

Claim (2) is proven combining claim (1) with the observations that $C \in \mathcal{C}_+$ if and only if $C^* \in \mathcal{C}_1$ and that $(C_1 \bullet C_2)^* = \hat{C}^*_2 \bullet \hat{C}^*_1$.

We can introduce a partial order $\leq$ on $\hat{C}$ by

$$\hat{C}_1 \leq \hat{C}_2 \text{ if and only if } \hat{C}_1(p) \leq \hat{C}_2(p) \text{ for all } 0 \leq p \leq 1.$$

The bijection $\hat{\bullet}$ interacts well with this partial ordering:

Proposition 4.4. For $C_1, C_2 \in \mathcal{C}$ we have $C_1 \leq C_2$ if and only if $\hat{C}_1 \leq \hat{C}_2$. □
Proof. Suppose $C_1 \leq C_2$. Then for any $p \in [0,1]$ and $\kappa \geq 0$ we have
\[
\hat{C}_1(p) \leq C_1(\kappa) + p\kappa \\
\leq C_2(\kappa) + p\kappa
\]
and taking the infimum over $\kappa$ yields $\hat{C}_1 \leq \hat{C}_2$.

The converse implication is proven as above, by appealing to Theorem 4.2. \hfill \Box

Now we can prove Theorem 3.1:

Proof of Theorem 3.1. Note that $\hat{C}_1$ is non-decreasing and $\hat{C}_2(p) \geq p$ for all $0 \leq p \leq 1$. Hence by Theorem 4.3 for any $0 \leq p \leq 1$ we have
\[
\hat{C}_1 \cdot \hat{C}_2(p) = \hat{C}_1 \circ \hat{C}_2(p) \\
\geq \hat{C}_1(p).
\]
We conclude that $C_1 \cdot C_2 \geq C_1$ by Proposition 4.4. Similarly, the inequality $\hat{C}_1(p) \geq p$ for all $0 \leq p \leq 1$ implies that $C_1 \cdot C_2 \geq C_2$. \hfill \Box

In preparation for reproving Theorem 3.5 and proving Theorem 3.7 we identify the image of the set of functions $C_f$ under the isomorphism $\hat{\cdot}$. For a parametric family $(C(\cdot, y))_{y \geq 0} \subseteq \mathcal{C}$ we will use the notation
\[
\hat{C}(p, y) = \hat{C}(\cdot, y)(p) \text{ for all } 0 \leq p \leq 1, y \geq 0.
\]

Theorem 4.5. Let $f$ be a log-concave probability density with support $[L, R]$ for $-\infty \leq L < R \leq +\infty$, and let
\[
F(x) = \int_{-\infty}^{x} f(z)dz
\]
be the cumulative distribution function. Let $F^{-1} : [0,1] \to [L, R]$ to be the inverse function. Then
\[
\hat{C}_f(p, y)(p) = F(F^{-1}(p) + y) \text{ for all } 0 \leq p \leq 1, y \geq 0.
\]

Proof. Fix $y \geq 0$. We first check that the identity holds for $p = 1$ since
\[
F(F^{-1}(1) + y) = F(R + y) = 1
\]
and for $p = 0$ since
\[
F(F^{-1}(0) + y) = F(L + y) = \int_{L}^{L+y} f(z)dz = C_f(\infty, y).
\]

Now let $p = F(z_0)$ for some $L < z_0 < R$, and set $\kappa_0 = f(z_0 + y)/f(z_0)$. For notational convenience, let $Z$ be a random variable with density $f$ and let
\[
S = \frac{f(Z + y)}{f(Z)},
\]
so that $S$ is a primal representation of $C_f(\cdot, y)$. Since $f$ is log-concave, the function $z \mapsto f(z + y)/f(z)$ is non-increasing, and hence
\[
\{S > \kappa_0\} \subseteq \{Z < z_0\} \subseteq \{Z \leq z_0\} \subseteq \{S \geq \kappa_0\}.
\]
By Proposition 4.1(3), we have
\[
\hat{C}_f(p, y) = C_f(\kappa_0, y) + p\kappa_0 \\
= 1 - \mathbb{E}[S \wedge \kappa_0] + \mathbb{P}(Z \leq z_0)\kappa_0 \\
= 1 - \mathbb{E}[S \mathbb{1}_{(Z>z_0)}] \\
= F(z_0 + y).
\]

Another proof of Theorem 3.7. Note that by Theorem 4.5 the family of functions \((\hat{C}_f(\cdot, y))_{y \geq 0}\) form a semigroup with respect to function composition. The result follows from applying Theorems and 4.2 and 4.3.

We now come to proof of Theorem 3.7.

Proof of Theorem 3.7. Before giving all the details, we give a quick outline. The key observation is that if a function \(C : [0, \infty) \times [0, \infty) \to [0, 1]\) satisfies the hypotheses of the theorem, then the conjugate function \(\hat{C} : [0, 1] \times [0, \infty) \to [0, 1]\) is such that

\[
\hat{C}(p, 0) = p \text{ for all } 0 \leq p \leq 1
\]

and satisfies the translation equation

\[
\hat{C}(\hat{C}(p, y_1), y_2) = \hat{C}(p, y_1 + y_2) \text{ for all } 0 \leq p \leq 1 \text{ and } y_1, y_2 \geq 0.
\]

The conclusion of the theorem is that there only three types of solutions to the above functional equation such that \(\hat{C}(\cdot, y) \in \hat{C}\) for all \(y > 0\):

1. \(\hat{C}(p, y) = p\) for all \(0 \leq p \leq 1\) and \(y > 0\),
2. \(\hat{C}(p, y) = 1\) for all \(0 \leq p \leq 1\) and \(y > 0\),
3. \(\hat{C}(p, y) = F(F^{-1}(p) + y)\) for all \(0 \leq p \leq 1\) and \(y > 0\) where \(F(z) = \int_{-\infty}^{z} f(x)dx\) and \(f\) is a log-concave probability density.

To rule out possibility (1), from now on we assume \(\hat{C}(p_0, y_0) > p_0\) for some \(0 \leq p_0 < 1\) and \(y_0 > 0\). We now show that this assumption implies that \(\hat{C}(p, y) > p\) for all \(0 < p < 1\) and \(y > 0\). By the concavity of \(\hat{C}(\cdot, y_0)\), we have

\[
\hat{C}(p, y_0) \geq \begin{cases} 
\frac{p}{p_0} \hat{C}(p_0, y_0) + \frac{p_0 - p}{p_0} \hat{C}(0, y_0) & \text{for } 0 < p < p_0 \\
\frac{1}{1-p_0} \hat{C}(p_0, y_0) + \frac{p - p_0}{1-p_0} \hat{C}(1, y_0) & \text{for } p_0 \leq p < 1
\end{cases}
\]

where have used \(\hat{C}(0, y_0) \geq 0\) and \(\hat{C}(1, y_0) = 1\). This shows that \(\hat{C}(p, y) > p\) for all \(0 < p < 1\) and \(y \geq y_0\) since \(\hat{C}(p, \cdot)\) is non-decreasing for each \(p\). Now using the translation equation, we have

\[
\hat{C}(\hat{C}(p, y_0/2), y_0/2) = \hat{C}(p, y_0) > p
\]

shows that there exists a \(0 \leq p_1 < 1\) such that \(\hat{C}(p_1, y_0/2) > p_1\) and hence \(\hat{C}(p, y_0/2) > p\) for all \(0 < p < 1\) as before. Iterating this argument shows that \(\hat{C}(p, y) > p\) for all \(0 < p < 1\) and \(y > 0\) as claimed. Note that we have shown that

\[
\hat{C}(p, y + \varepsilon) > \hat{C}(p, y) \text{ for all } \varepsilon > 0 \text{ and } 0 < p < 1, y \geq 0 \text{ such that } \hat{C}(p, y) < 1.
\]

by the functional equation.
We now show that the remaining possibilities are either (2) or (3) as above. In both cases, we will use the following lemma:

**Lemma 4.6.** Fix $0 \leq p_0 < 1$. Then for any $n \geq 1$ we have

$$\hat{C}(p, ny) \geq 1 - (1 - p) \left( \frac{1 - \hat{C}(p_0, y)}{1 - p_0} \right)^n \text{ for all } p_0 \leq p \leq 1, y \geq 0.$$  

*Proof of Lemma 4.6.* By concavity of $\hat{C}(\cdot, y)$ and that $\hat{C}(1, y) = 1$ for all $y \geq 0$ we have

$$\hat{C}(p, y) \geq \frac{1 - p}{1 - p_0} \hat{C}(p_0, y) + \frac{p - p_0}{1 - p_0} \text{ for } p_0 \leq p \leq 1.$$  

By the semigroup property we have for $n \geq 1$ that

$$\hat{C}(p, ny) = \hat{C}(C(p, (n - 1)y), y) \geq \frac{1 - \hat{C}(p_0, (n - 1)y_0)}{1 - p_0} \hat{C}(p_0, y_0) + \frac{\hat{C}(p, (n - 1)y_0) - p_0}{1 - p_0} \text{ for } p_0 \leq p \leq 1$$

since $\hat{C}(\cdot, y_0)$ is non-decreasing for all $y_0 \geq 0$ and hence $\hat{C}(p, (n - 1)y) \geq \hat{C}(p_0, (n - 1)y) \geq p_0$. The result follows by induction. 

We will for the moment assume that there exists $0 \leq p_0 < 1$ such that

\[ \inf_{y > 0} \hat{C}(p, y) > p_0 \]  

We now show that this implies possibility (2). Note that $\inf_{y > 0} \hat{C}(\cdot, y)$ is concave, and by the concavity argument above, we have

$$\inf_{y > 0} \hat{C}(p, y) > p \text{ for all } 0 < p < 1.$$  

Fix a $0 \leq p_0 < 1$. By our tentative assumption (*), there exists $\varepsilon > 0$ such that $\hat{C}(p_0, y) \geq p_0 + \varepsilon$ for all $y > 0$. By Lemma 4.6 we have for all $n \geq 1$ that

$$\hat{C}(p, y) \geq 1 - (1 - p) \left( \frac{1 - \varepsilon}{1 - p_0} \right)^n \text{ for all } p_0 \leq p \leq 1, y > 0.$$  

Sending $n \uparrow \infty$ shows $\hat{C}(p, y) = 1$ for all $p_0 \leq p \leq 1, y > 0$. Finally, since $p_0$ was arbitrary, we have shown that our assumption (*) implies case (2).

From now on we will assume that

\[ \inf_{y > 0} \hat{C}(p, y) = p \text{ for all } 0 \leq p \leq 1 \]  

We will appeal to the treatment of the translation equation appearing in Aczél’s book [11, Chapter 6.1] which will allow us to conclude that

$$\hat{C}(p, y) = F(F^{-1}(p) + y) \text{ for all } 0 \leq p \leq 1, y \geq 0$$  

for a strictly increasing, continuous distribution function $F$. Since $\hat{C}(\cdot, y)$ is concave by Proposition 4.1, we can use a result of Bobkov [2, Proposition A.1] to conclude that $F$ is differentiable with $F' = f \log$-concave.
We now show that the function $\hat{C}$ has enough regularity to apply Aczel’s technique for solving the translation equation. We first show that for fixed $p$ the function $\hat{C}(p, \cdot)$ is continuous. Let

$$\Delta(\varepsilon) = \sup_{0 \leq p \leq 1} [\hat{C}(p, \varepsilon) - p].$$

By the translation equation we have for all $0 \leq p \leq 1$ and $0 \leq \varepsilon \leq y$ that

$$\hat{C}(p, y) - \Delta(\varepsilon) \leq \hat{C}(p, y - \varepsilon) \leq \hat{C}(p, y + \varepsilon) \leq \hat{C}(p, y) + \Delta(\varepsilon).$$

Note that by Dini’s theorem assumption (**) implies the a priori stronger assumption that $\Delta(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, from which continuity of $\hat{C}(p, \cdot)$ follows.

Next, we show that for all $0 < p \leq 1$ we have $\hat{C}(p, y) \uparrow 1$ as $y \uparrow \infty$. Fix a $0 < p_0 < 1$ and $y_0 > 0$. We have already shown that since we are not in case (1) that $\hat{C}(p_0, y_0) > p_0$. By Lemma 4.6 we have

$$\hat{C}(p, ny_0) \geq 1 - (1 - \hat{C}(p_0, y_0)) \left(1 - p_0\right)^n \to 1 \text{ for all } p_0 \leq p \leq 1.$$ 

Since $p_0 > 0$ was arbitrary, the convergence holds for all $p > 0$.

With these preliminary observations, we can now solve the functional equation. Fix a $0 < p_0 < 1$. Let

$$R = \inf\{y \geq 0 : \hat{C}(p_0, y) = 1\}$$

and $F_0 = \hat{C}(p_0, \cdot)$. From above the function $F_0$ is a strictly increasing continuous function from $[0, R)$ onto $[p_0, 1)$. Therefore, it has strictly increasing continuous inverse function $F_0^{-1} : [p_0, 1) \to [0, R)$. The semigroup property implies

$$\hat{C}(F(y_1), y_2) = \hat{C}(\hat{C}(p_0, y_1), y_2)$$

$$= \hat{C}(p_0, y_1 + y_2)$$

$$= F(y_1 + y_2)$$

for $y_1, y_2 \geq 0$ and hence

$$\hat{C}(p, y) = F_0(F_0^{-1}(p) + y) \text{ for all } p_0 \leq p < 1, y \geq 0.$$ 

We now use this procedure inductively, by fixing a sequence $p_n \downarrow 0$ and setting $y_n$ such that

$$\hat{C}(p_n, y_n) = p_{n-1}.$$ 

Let

$$z_n = y_1 + \ldots + y_n$$

and let

$$F_n(x) = \hat{C}(p_n, z_n + x).$$
Note by the semigroup property

\[ F_n(R) = \hat{C}(p_n, y_n + z_n - 1 + x) = \hat{C}(C(p_n, y_n), z_{n-1} + x) = \hat{C}(p_{n-1}, z_{n-1} + x) = F_{n-1}(R) \]

for all \( x \geq -z_{n-1} \). But by the argument above we have that \( F_n : [-z_n, R) \to [p_n, 1) \) is strictly increasing and continuous so

\[ \hat{C}(p, y) = F_n(F_{n-1}^{-1}(p) + y) \]

for all \( 0 \leq p < 1, y \geq 0 \).

So let

\[ L = -\sup_n z_n \]

and

\[ F(x) = \begin{cases} F_n(x) & \text{if } x \geq -z_n \\ 0 & \text{if } x < L. \end{cases} \]

Note that \( F(-z_n) = p_n \to 0 \) so \( F \) is continuous. Also note

\[ F^{-1}(p) = F_{n-1}^{-1}(p) \]

if \( p_n \leq p < 1 \).

We have shown

\[ \hat{C}(p, y) = F(F^{-1}(p) + y) \]

for all \( 0 < p < 1, y \geq 0 \) for a strictly increasing, continuous distribution function \( F \) as desired. Appealing to Bobkov’s result [2] shows \( F \) is the distribution function of a log-concave density, completing the proof.

\[ \square \]

### 4.2. Infinitesimal generators and the inf-convolution.

Let \( f \) be a log-concave density with distribution function \( F \), and let

\[ \hat{C}(p, y) = F(F^{-1}(p) + y) \]

for all \( 0 \leq p \leq 1, y \geq 0 \).

The content of Theorem 3.7 is that, aside from the trivial and null semigroups, the only semigroups of \( \hat{C} \) with respect to composition are of the above form. The infinitesimal generator is given by

\[ \frac{\partial}{\partial y} C(p, y) \bigg|_{y=0} = f \circ F^{-1}(p) \]

for all \( 0 \leq p \leq 1 \), where we take the version of \( f \) which is continuous on its support \([L, R]\). Letting \( \hat{H} = f \circ F^{-1} \) we have

\[ \frac{\partial}{\partial y} \hat{C}(p, y) = \hat{H}[\hat{C}(p, y)] \]

for \( y \geq 0, \hat{C}(p, y) < 1 \).

Note that the above ordinary differential equation also holds for the trivial semigroup with \( \hat{H} = 0 \).

We now show that we can recover the semigroup from the function \( \hat{H} \). Let

\[ \mathcal{H} = \{ h : [0, 1] \to [0, \infty), \text{ concave } \}, \]

and pick a \( \hat{H} \in \mathcal{H} \). If \( \hat{H}(p) = 0 \) for all \( 0 \leq p < 1 \), the semigroup is trivial as noted above.
So suppose that \( \hat{H}(p_0) > 0 \) for some \( 0 < p_0 < 1 \). By concavity \( H(p) > 0 \) for all \( 0 < p < 1 \). Fix an arbitrary \( 0 < p_0 < 1 \), for instance \( p_0 = 1/2 \), and let

\[
G(p) = \int_{p_0}^{p} \frac{d\phi}{\hat{H}(\phi)}.
\]

Note that for \( 0 < p < 1 \) the integral is well-defined and finite as \( \hat{H} \) is positive and continuous by concavity. Let \( L = G(0) \) and \( R = G(1) \), and define a function \( F : [L, R] \to [0, 1] \) as the inverse function \( F = G^{-1} \), and extend \( F \) to all of \( \mathbb{R} \) by \( F(x) = 0 \) for \( x \leq L \) and \( F(x) = 1 \) for \( x \geq R \). Note that we can compute the derivative as

\[
F'(x) = \frac{1}{G' \circ G^{-1}(x)} = \hat{H}(F(x)) \text{ for all } x \in \mathbb{R}.
\]

Setting \( f = F' \), we have \( \hat{H} = f \circ F^{-1} \). Finally, by a result of Bobkov [2, Proposition A.1], the function \( f \) is log-concave since \( \hat{H} \) is concave by assumption.

We now would like to interpret the above discussion in terms of the semigroups of call price function \( C \) with respect to the binary operation \( \circ \). Note that the space \( \mathcal{H} \) introduced above is closed under addition. Furthermore, we have for every non-null semigroup \( \hat{C} \) that

\[
\hat{C}(\kappa, \varepsilon) \approx p + \varepsilon \hat{H}(p) \text{ for small } \varepsilon > 0.
\]

Recall that the inf-convolution of two functions \( f_1, f_2 : \mathbb{R} \to \mathbb{R} \) is defined by

\[
(f_1 \square f_2)(x) = \inf_{y \in \mathbb{R}} [f_1(x - y) + f_2(y)] \text{ for } x \in \mathbb{R}.
\]

The basic property of the inf-convolution (see [3, Exercise 2.3.15] for example) is that it becomes addition under conjugation: for a function \( f : \mathbb{R} \to \mathbb{R} \) define a new function \( \hat{f} \) by

\[
\hat{f}(p) = \inf_{x \in \mathbb{R}} [f(x) + xp] \text{ for } 0 \leq p \leq 1,
\]

so that

\[
\hat{f_1}(p) + \hat{f_2}(p) = \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} [f_1(x - y) + f_2(y) + xp]
\]

\[
= \inf_{z \in \mathbb{R}} [f_1(z) + zp] + \inf_{y \in \mathbb{R}} [f_2(y) + yp]
\]

\[
= \hat{f_1}(p) + \hat{f_2}(p),
\]

in analogy with Theorem 4.3. Since there is an exponential map lifting function addition + to function composition \( \circ \) in \( \hat{C} \), we can apply the isomorphism \( \hat{\circ} \) to conclude that there is an exponential map lifting inf-convolution \( \square \) to the binary operation \( \circ \) in \( C \).

To elaborate, since \( p = E(p) \) where \( E(\kappa) = (1 - \kappa)^+ \) is the identity element of \( C \), we expect that for small \( \varepsilon > 0 \) we have

\[
C(\kappa, \varepsilon) \approx [E \square H_\varepsilon](\kappa)
\]

\[
= H_\varepsilon(\kappa - 1)
\]

where \( H_\varepsilon = \varepsilon \hat{H} \), so that

\[
H_\varepsilon(x) = \varepsilon \hat{H}(x/\varepsilon)
\]

and

\[
H(x) = \sup_{0 \leq p \leq 1} [\hat{H}(p) - p\kappa] .
\]
To make this more precise, let

\[ H = \{ H : \mathbb{R} \rightarrow [0, \infty) \text{ convex with } 0 \leq H(x) - (-x)^+ \leq \text{const.} \} . \]

As the notation suggests, the operation \( \hat{\cdot} \) is a bijection between the sets \( \mathcal{H} \) and \( \hat{\mathcal{H}} \) which can be proven as in Theorem 4.2. In particular, the space \( \mathcal{H} \) can be identified with the generators of one-parameter semigroups in \( \mathcal{C} \) and hence is closed under inf-convolution.

For a function \( H \in \mathcal{H} \), boundedness and convexity show that \( H \) is non-increasing on \( [0, \infty) \) and that \( H(x) + x \) is non-decreasing \( (-\infty, 0] \). In particular, we have the existence of constants \( a, b \geq 0 \) such that

\[
H(x) \to a \text{ as } x \uparrow +\infty \\
H(x) + x \to b \text{ as } x \downarrow -\infty
\]

so by a suitable version of Proposition 2.2, for instance, [8, Proposition 2.1], we have that exist an integrable random variable \( X \) such that

\[
H(x) = a + \mathbb{E}([X - x]^+] \\
= b - \mathbb{E}(X \wedge x)
\]

from which we deduce

\[
\mathbb{E}(X) = b - a.
\]

The function \( H \) can be identified with the generator of a one-parameter semigroup \( (C(\cdot, y))_{y \geq 0} \) of \( \mathcal{C} \), or more probabilistically, the random variable \( X \) and one of the constants \( a \) or \( b \) is the generator of the family of primal representations \( (S(y))_{y \geq 0} \).

In the case where \( f \) is a log-concave density supported on \( [L, R] \) the generator is calculated as

\[
H(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} C(1 + \varepsilon x, \varepsilon) \\
= f(L) + \int_{L^+}^{R^-} (f'(z) - f(z)x)^+ dz \\
= f(R) - \int_{L^+}^{R^-} f'(z) \wedge [f(z)x] dz
\]

by the dominated convergence theorem. Letting \( Z \) be a random variable with density \( f \) and \( S(y) = f(Z+y)/f(Z) \), we have that the generating random variable is \( X = f'(Z)/f(Z) \) with constants \( a = f(L) \) and \( b = f(R) \).

For example, the family of Black–Scholes call prices \( C_{BS} \) is generated by a standard normal random variable \( X \sim N(0, 1) \) and constants \( a = 0 = b \). The corresponding function \( H \) is

\[
H(x) = \mathbb{E}([X - x]^+] = \varphi(x) - x\Phi(-x)
\]

which is normalised call price function in the Bachelier model. The conjugate function \( \tilde{H} = \varphi \circ \Phi^{-1} \) is the Gaussian isoperimetric function.
4.3. **Lift zonoids.** Finally, to see why one might want to compute what the Legendre transform of a call price with respect to the strike parameter, we recall that the zonoid of an integrable random $d$-vector $X$ is the set
\[
Z_X = \{ \mathbb{E}[X g(X)] \text{ measurable } g : \mathbb{R}^d \to [0, 1] \} \subseteq \mathbb{R}^d,
\]
and that the lift zonoid of $X$ is the zonoid of the $(1 + d)$-vector $(1, X)$ given by
\[
\hat{Z}_X = \{ (\mathbb{E}[g(X)], \mathbb{E}[X g(X)]) \text{ measurable } g : \mathbb{R}^d \to [0, 1] \} \subseteq \mathbb{R}^{1+d}.
\]
The notion of lift zonoid was introduced in the paper of Koshevoy & Mosler [12].

In the case $d = 1$, the lift zonoid $\hat{Z}_X$ is a convex set contained in the rectangle
\[
[0, 1] \times [-m_-, m_+].
\]
where $m_{\pm} = \mathbb{E}(X^\pm)$. The precise shape of this set is intimately related to call and put prices as seen in the following theorem.

**Theorem 4.7.** Let $X$ be an integrable random variable. Its lift zonoid is given by
\[
\hat{Z}_X = \left\{ (p, q) : \sup_{x \in \mathbb{R}} \{ px - \mathbb{E}((x - X)^+) \} \leq q \leq \inf_{x \in \mathbb{R}} \{ px + \mathbb{E}((X - x)^+) \}, \ 0 \leq p \leq 1 \right\}.
\]
Note that if we let
\[
\Theta(x) = \mathbb{P}(X \geq x)
\]
then we have
\[
\mathbb{E}[(X - x)^+] = \int_x^\infty \Theta(\xi)d\xi
\]
by Fubini’s theorem. Also if we define the inverse function $\Theta^{-1}$ for $0 < p < 1$ by
\[
\Theta^{-1}(p) = \inf \{ x : \Theta(x) \geq p \}
\]
then by a result of Koshevoy & Mosler [12, Lemma 3.1] we have
\[
\hat{Z}_X = \left\{ (p, q) : \int_{1-p}^1 \Theta^{-1}(\phi)d\phi \leq q \leq \int_0^p \Theta^{-1}(\phi)d\phi, \ 0 \leq p \leq 1 \right\}.
\]
from which Theorem 4.7 can be proven by Young’s inequality. However since the result can be viewed as an application of the Neyman–Pearson lemma, we include a short proof for completeness.

**Proof.** For any measurable function $g$ valued in $[0, 1]$ and $x \in \mathbb{R}$ we have
\[
X g(X) \leq (X - x)^+ + x g(X)
\]
with equality when $g$ is such that
\[
1_{(x, \infty)} \leq g \leq 1_{[x, \infty)}.
\]
Now suppose $(p, q) \in \hat{Z}_Z$ so that $p = \mathbb{E}[g(X)]$ and $q = \mathbb{E}[X g(X)]$ for some $g$. Hence, computing expectations in the inequality above yields
\[
q \leq \mathbb{E}[(X - x)^+] + xp.
\]
with equality if
\[
\mathbb{P}(X > x) \leq p \leq \mathbb{P}(X \geq x).
\]
By replacing \(g\) with \(1 - g\), we see that \((p, q) \in \hat{Z}_X\) if and only if \((1 - p, \mathbb{E}(X) - q) \in \hat{Z}_X\), yielding the lower bound. □

We remark that the explicit connection between lift zonoids and the price of call options has been noted before, for instance in the paper of Mochanov & Schmutz \[15\]. In the setting of this paper, given \(C \in \mathcal{C}\) with represented by \(S\), the lift zonoid of \(S\) is given by the set

\[
\hat{Z}_S = \{(p, q) : 1 - \hat{C}(1 - p) \leq q \leq \mathbb{E}(S) - 1 + \hat{C}(p), \ 0 \leq p \leq 1\}
\]

We recall that a random vector \(X_1\) is dominated by \(X_2\) in the lift zonoid order if \(\hat{Z}_{X_1} \subseteq \hat{Z}_{X_2}\). Koshevoy & Mosler \[12, \text{Theorem 5.2}\] noticed that in the \(d = 1\) case, that the lift zonoid order is exactly the convex order. Our Proposition 4.4 can thus be seen as a special case.

We conclude this section by exploiting Theorem 4.7 to obtain an interesting identity:

**Theorem 4.8.** Given \(C \in \mathcal{C}\), let

\[
\hat{C}^{-1}(q) = \inf \{p \geq 0 : \hat{C}(p) \geq q\} \text{ for all } 0 \leq q \leq 1.
\]

Then

\[
\hat{C}^*(p) = 1 - \hat{C}^{-1}(1 - p) \text{ for all } 0 \leq p \leq 1.
\]

**Proof.** Let \(S\) be a primal representation and \(S^*\) be a dual representation of \(C\).

Note that for all \(0 \leq p \leq 1\) we have

\[
\hat{C}(p) - \hat{C}(0) = \sup \{\mathbb{E}[Sg(S)] : g : \mathbb{R} \rightarrow [0, 1] \text{ with } \mathbb{E}[g(S)] = p\}
\]

and hence for any \(0 \leq q \leq 1\) we have

\[
\hat{C}^{-1}(q) = \inf \{\mathbb{E}[g(S)] : g : \mathbb{R} \rightarrow [0, 1] \text{ with } \mathbb{E}[g(S)] = q - \hat{C}(0)\}
\]

\[
= 1 - \sup \{\mathbb{E}[g(S)] : g : \mathbb{R} \rightarrow [0, 1] \text{ with } \mathbb{E}[g(S)] = 1 - q\}
\]

\[
= \mathbb{P}(S > 0) - \sup \{\mathbb{E}[g(S)1_{S > 0}] : g : \mathbb{R} \rightarrow [0, 1] \text{ with } \mathbb{E}[g(S)1_{S > 0}] = 1 - q\}
\]

\[
= \mathbb{E}(S^*) - \sup \{\mathbb{E}\left[\left[S^*g(S^*)1_{S^* > 0}\right] : g : \mathbb{R} \rightarrow [0, 1] \text{ with } \mathbb{E}\left[g(S^*)1_{S^* > 0}\right] = 1 - q\}\}
\]

\[
= 1 - \hat{C}^*(1 - q)
\]

where we have used the observation that the optimal \(g\) in the final maximisation problem assigns zero weight to the event \(\{S^* = 0\}\). □

4.4. **An extension.** Let \(F\) be the distribution function of a log-concave density \(f\) which is supported on all of \(\mathbb{R}\), so that \(L = -\infty\) and \(R = +\infty\) in the notation of section \[3\]. Let

\[
\hat{C}_f(p, y) = F(F^{-1}(p) + y) \text{ for all } 0 \leq p \leq 1, y \in \mathbb{R}.
\]

By Theorem 4.5 we have

\[
\hat{C}_f(p, y) = C_f(\cdot, y)(p) \text{ for all } 0 \leq p \leq 1, y \geq 0.
\]

It is interesting to note that the family of functions \((\hat{C}_f(\cdot, y))_{y \in \mathbb{R}}\) is a **group** under function composition, not just a semigroup. Indeed, we have

\[
\hat{C}_f(\cdot, -y) = \hat{C}_f(\cdot, y)^{-1} \text{ for all } y \in \mathbb{R}.
\]

Note that \(\hat{C}_f(\cdot, y)\) is increasing for all \(y\), is concave if \(y \geq 0\) but is convex if \(y < 0\). In particular, when \(y < 0\) the function \(\hat{C}_f(\cdot, y)\) is not the concave conjugate of a call function.
in \( C \). Unfortunately, the probabilistic or financial interpretation of the inverse is not readily apparent.

For comparison, note that for \( y \geq 0 \) we have by Theorem 4.8 that

\[
\hat{C}_f(\cdot, -y)(p) = \hat{C}_f(\cdot, y)^*(p)
\]

\[= 1 - F(F^{-1}(1 - p) - y) \text{ for all } 0 \leq p \leq 1.\]

5. ACKNOWLEDGEMENT

I would like to thank the Cambridge Endowment for Research in Finance for their support. I would also like to thanks Thorsten Rheinländer for introducing me the notion of a lift zonoid. Finally, I would like to thank the participants of the London Mathematical Finance Seminar Series, where this work was presented.

REFERENCES

[1] J. Aczél. Lectures on Functional Equations and Their Applications. Mathematics in Science and Engineering 19. Academic Press. (1966)
[2] S. Bobkov. Extremal properties of half-spaces for log-concave distributions. Annals of Probability 24(1): 35–48. (1996)
[3] J.M. Borwein and J.D. Vanderwerff. Convex Functions: Constructions, Characterizations and Counterexamples. Encyclopedia of Mathematics and Its Applications 109. Cambridge University Press. (2010)
[4] S. Boyd and L. Vandenberghe. Convex Optimization. Cambridge University Press. (2004)
[5] A.M.G. Cox and D.G. Hobson. Local martingales, bubbles and option prices. Finance and Stochastics 9: 477–492. (2005)
[6] J. Gatheral and A. Jacquier. Arbitrage-free SVI volatility surfaces. Quantitative Finance 14(1): 59–71. (2014)
[7] F. Hirsh, Ch. Profeta, B. Roynette and M. Yor. Peacocks and Associated Martingales, with Explicit Constructions. Bocconi & Springer Series. (2011)
[8] F. Hirsh and B. Roynette. A new proof of Kellerers theorem. ESAIM: Probability and Statistics 16: 48–60. (2012)
[9] D. Hobson, P. Laurence, and T.-H. Wang. Static-arbitrage upper bounds for the prices of basket options. Quantitative Finance 5(4): 329–342. (2005)
[10] H.G. Kellerer. Markov-Komposition und eine Anwendung auf Martingale. Mathematische Annalen 198: 99–122. (1972)
[11] I. Karatzas. Lectures on the Mathematics of Finance. CRM Monograph Series, American Mathematical Society. (1997)
[12] G. Koshevoy and K. Mosler. Lift zonoids, random convex hulls and the variability of random vectors. Bernoulli 4: 377–399. (1998)
[13] A.M. Kulik and T.D. Tymoshkevych. Lift zonoid order and functional inequalities. Theory of Probability and Mathematical Statistics 89: 83–99. (2014)
[14] M. Kulldorff. Optimal control of favorable games with a time-limit. SIAM Journal on Control and Optimization 31(1): 52–69. (1993)
[15] I. Molchanov and M. Schmutz. Multivariate extension of put-call symmetry. SIAM Journal on Financial Mathematics 1(1): 396–426. (2010)
[16] M.R. Tehranchi. Uniform bounds on Black–Scholes implied volatility. SIAM Journal on Financial Mathematics 7(1): 893–916. (2016)

Statistical Laboratory, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK

E-mail address: m.tehranchi@statslab.cam.ac.uk