CONFORMAL SYMMETRIES OF THE YAMABE AND Paneitz
OPERATORS

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Abstract. We study first and second order conformal symmetries of the Yamabe Laplacian on a general pseudo-Riemannian manifold and of the Paneitz operator on Einstein spaces. We show that first order conformal symmetries of the Yamabe operator induce second order conformal symmetries. We show that on an Einstein space every conformal Killing vector field induces a conformal symmetry of the Paneitz operator and, conversely, every first order conformal symmetry of the Paneitz operator arises from a conformal Killing vector field. Possible extensions of our main theorems are illustrated by examples. We correct formulas in [3] and [11].

1. Introduction

In this paper, we study first and second order conformal symmetries of the Yamabe operator on an arbitrary pseudo-Riemannian manifold and of the Paneitz operator on Einstein spaces. Specifically, we generalize part of a theorem of Gover and Šilhan [7] by proving that any pair of first order conformal symmetries of the Yamabe operator naturally yields a second order conformal symmetry. Moreover, in the context of Einstein spaces, we show that every conformal Killing vector induces a first order conformal symmetry of the Paneitz operator, and each of its first order conformal symmetries arises from a conformal Killing vector; this is an analogue of a special case of the same theorem of Gover and Šilhan. Finally we consider examples suggesting the possibility of extending this result to non-Einstein spaces, and of proving the aforementioned second order result for the Paneitz operator.

More precisely, consider a pseudo-Riemannian manifold $(M,g)$ of dimension $n \geq 2$ and a local Lie group $G$ acting on $M$ by local diffeomorphisms. We denote by $\mathfrak{g}$ the Lie algebra of $G$, seen as a subalgebra of the Lie algebra $\mathcal{X}(M)$ of smooth vector fields on $M$.

In the case $G$ is a group of isometries, the corresponding vector fields on $M$, seen as a differential operators $D$ acting on scalar fields, must satisfy

\begin{equation}
[\Delta, D] = 0,
\end{equation}

where $\Delta = g^{ij} \nabla_i \nabla_j$ is the Laplacian on $M$. If $G$ is a group of conformal transformations, the corresponding (first order) differential operators $D$ will satisfy

\begin{equation}
[\Delta_Y, D + q] = r \Delta_Y,
\end{equation}

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where $\Delta_Y = \Delta - \frac{n-2}{4(n-1)}R$ is the Yamabe Laplacian, for scalar fields $q$ and $r$ completely determined by $D$.

More generally, a differential operator $D$ of any order satisfying (1) or (2) is called a symmetry of the Laplacian, or a conformal symmetry of the Yamabe Laplacian, respectively. One can further generalize these concepts. Given a differential operator $P$ on $M$ acting on scalar fields, a differential operator $D$ is a conformal symmetry of $P$ if there exists a differential operator $B$ such that

$$[P, D] = BP,$$

is an identity. Requiring that $B$ be identically $0$ gives the (exact) symmetries of $P$.

It is well known that the principal symbol $T$ of a symmetry of the Laplacian must satisfy the Killing equation (see for example [4]); in local coordinates, if $T = T^{i_1...i_k}$ then

$$T^{(i_1,...,i_k; i_{k+1})} = 0,$$

(4)

Second order symmetries of the Laplacian are of particular importance because of their relationship to separation of variables of the Hamilton-Jacobi and Helmholtz equations, see for example [13] and [9]. In [3] Carter characterized first and second order symmetries of the Yamabe Laplacian in a general pseudo-Riemannian manifold.

Second order conformal symmetries of the Laplacian and Yamabe Laplacian have been studied in relation to R-separation of variables of the Hamilton-Jacobi and Laplace equations in [1],[10], and to R-separation of variables of the (conformally invariant) Yamabe equation $\Delta_Y \Psi = 0$ on a Lorentzian 4-dimensional manifold in [11]. In [12] the authors give a characterization of second order conformal symmetries of the Yamabe Laplacian on a general pseudo-Riemannian manifold in terms of quantization maps.

Further, higher order conformal symmetries of the Laplacian have been studied by Eastwood [4] from a representation-theoretic perspective. Eastwood shows, in particular, that the principal symbol $T$ of a conformal symmetry of $\Delta$ must be a conformal Killing tensor. That is, $T$ must satisfy the conformal Killing equation

$$T^{(i_1,...,i_k; i_{k+1})} = g^{(i_1i_2}C_{i_3...i_{k+1})}.$$

(5)

Since the symmetrized tensor product $K$ of two conformal Killing vectors is a conformal Killing tensor, it is natural to ask whether $K$ gives rise to a second order conformal symmetry of $\Delta_Y$. One of our main results answers this question affirmatively:

**Theorem 4.4** Let $(M^n, g)$ be an arbitrary pseudo-Riemannian manifold. Let $X_1$ and $X_2$ be first order conformal symmetries of the Yamabe operator $\Delta_Y = \Delta - \frac{n-2}{4(n-1)}R$ on $M$. Then $S := \frac{1}{2}\{X_1, X_2\}$, where $\{,\}$ denotes the anti-commutator, is a second order conformal symmetry of $\Delta_Y$.

The statement of Theorem 4.4 is not surprising at the level of principal symbols. However, the lower order terms impose additional conditions, which turn out to be identically satisfied. This “hereditary” behaviour of conformal symmetries of $\Delta_Y$ is not explicitly addressed in [12]. In the case of a locally conformally flat manifold the result in Theorem 4.4 follows from [7, Theorem 2.4].
Gover and Šilhan [7] studied higher order conformal symmetries of higher order GJMS operators [8] on locally conformally flat spaces. In particular, the authors establish that a (trace-free) conformal Killing tensor always gives rise to a conformal symmetry of every GJMS operator.

The fourth order GJMS operator is known as the Paneitz operator [14]. On an Einstein space $M$, the Paneitz operator is given by the expression [6]

$$
P = (\Delta - c_1 R)(\Delta - c_2 R),$$

where $c_1$ and $c_2$ are explicit constants depending on the dimension of $M$. In this paper we initiate the study of first and second order conformal symmetries of the Paneitz operator on Einstein manifolds, extending some of the known results on conformally flat manifolds [7] to a more general context. Our main result in this direction is

**Theorem 3.4** Let $(M^n, g)$ be an Einstein manifold of dimension $n \geq 3$. Then $N^k \nabla_k + Q$ is a conformal symmetry of the Paneitz operator if and only if $N^k \nabla_k$ is a conformal Killing vector and $Q = \frac{n-4}{2n} N^k : k$.

If $M$ is locally conformally flat, the conclusion of Theorem 3.4 follows from [7, Theorem 2.4].

This paper is organized as follows. In Section 2 we introduce conformally covariant differential operators, GJMS operators and, in particular, the Yamabe and Paneitz operators. In Section 3 we cover technical lemmas and prove Theorem 3.4. We provide examples suggesting Theorem 3.4 could be extended to a wider class of manifolds. In Section 4 we correct formulas given in [3] and [11] and further discuss second order conformal symmetries of Laplacian operators. We prove Theorem 3.4 and we explore the possibility of an analogous result for the Paneitz operator via various examples.

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## 2. Conformally covariant operators

A differential operator $P_g$ on a manifold $(M, g)$ is said to be conformally covariant if, under a conformal change of metric $g \mapsto \hat{g} := e^{2f} g$, $f \in C^\infty(M, \mathbb{R})$, it transforms according to

$$
P_{\hat{g}} = e^{-\gamma'} f \cdot P_g \cdot e^{\gamma f}$$

for some constants $\gamma$, $\gamma'$. The equation $P_{\hat{g}} \psi = 0$ is therefore conformally invariant in the sense that $P_g \psi = 0$ if and only if $P_{\hat{g}} e^{-\gamma f} \psi = 0$.

An important class of conformally covariant operators are the so-called GJMS operators introduced in [5]. The GJMS operator of order $2k$ transforms as

$$
P_{k, \hat{g}} = e^{-(\frac{n}{2} + k) f} \cdot P_{k, g} \cdot e^{(\frac{n}{2} - k) f}.$$  

Note that if $k = \frac{n}{2}$ the kernel of $P_{k, g}$ is preserved under any conformal change of metric.
In [6] Gover gives an explicit expression for all the GJMS operators of order $2k$ on pseudo-Riemannian Einstein spaces of dimension $n \geq 3$. Namely,

$$ P_k = \prod_{l=1}^{k} (\Delta - c_l R), $$

where $\Delta = g^{ij} \nabla_i \nabla_j$, $R$ is the scalar curvature, and $c_l = \frac{(n+2l-2)(n-2l)}{4n(n-1)}$.

In particular we have the Yamabe Laplacian

$$ \Delta_Y := P_1 = \Delta - \frac{n-2}{4(n-1)} R, $$

and the Paneitz operator

$$ \mathbb{P} := P_2 = (\Delta - c_1 R)(\Delta - c_2 R), $$

where $c_1 = \frac{n-2}{4(n-1)}$ and $c_2 = \frac{(n+2)(n-4)}{4n(n-1)}$.

### 3. First order conformal symmetries

Throughout this paper we will assume that $(M, g)$ is a smooth pseudo-Riemannian manifold of dimension $n$. We will adopt the following sign convention: given a vector field $v^k$, the Riemann curvature tensor is defined by

$$ \nabla_i \nabla_j v^k - \nabla_i \nabla_j v^k = R_{klij} v^l $$

and the Ricci tensor by

$$ R_{ij} = R^k_{ikj}. $$

In [3] Carter derives the necessary and sufficient conditions for $N^k \nabla_k$ to be a first order symmetry of $\Delta + U$, where $U$ is any scalar field. First order conformal symmetries can easily be derived and we do so in Lemma 3.2.

We will need the following general facts about conformal Killing vectors.

**Lemma 3.1.** Let $(M^n, g)$ be an arbitrary pseudo-Riemannian manifold and let $N = N^k \nabla_k$ be a conformal Killing vector. Then

(a) $N^i_{\ j, \ j} + N^j_{\ i, \ j} + \frac{n-2}{n} N^j_{\ i, \ j} = 0.$

(b) $(n-1)N^k_{\ ;k, \ j} + N^k_{\ ;j, \ k} R + \frac{n}{2} N^k R_{,k} = 0.$

**Proof.** We have $N^{(i, j)} = \frac{1}{n} N^{k}_{,k} g^{ij}$.

$$ N^i_{\ j, \ j} = 2N^{(i, j)}_{\ j} - N^i_{\ ;j, \ j} = 2N^{(i, j)}_{\ j} - N^j_{\ i, \ j} - N^j_{\ i, \ j} = \frac{2}{n} N^j_{\ i, \ j} - N^j_{\ i, \ j} - N^j R_{i, \ j} $$

$$ = \frac{2-n}{n} N^j_{\ i, \ j} - N^j R_{i, \ j}. $$


Since $R^{ij\cdot}_{\cdot} = \frac{1}{2} R^{\cdot}_{\cdot}$,
\[
\frac{2 - n}{n} N^{ij\cdot}_{\cdot} = N^{i\cdot}_{\cdot j} + (N^i R^j)_{\cdot j} = N^{i\cdot}_{\cdot j} + (N^i R^j)_{\cdot j} = N^{i\cdot}_{\cdot j} + 2(N^i R^j)_{\cdot j}
\]
\[
= N^{i\cdot}_{\cdot j} + 2N^j \cdot R^i_{\cdot} + 2N^j \cdot R^i_{\cdot} = N^{i\cdot}_{\cdot j} + \frac{2}{n} N^{k\cdot}_{\cdot kj} g^{ij} R_{ij} + N^j R_{ij}
\]
\[
= N^{i\cdot}_{\cdot j} + \frac{2}{n} N^{k\cdot}_{\cdot kj} R + N^j R_{ij}. \]

Lemma 3.2. Let $(M^n, g)$ be an arbitrary pseudo-Riemannian manifold and let $U$ be a scalar field on $M$. There exists a scalar field $r$ such that
\[
[g^{ij} \nabla_i \nabla_j + U, N^k \nabla_k + Q] = r(g^{ij} \nabla_i \nabla_j + U)
\]
if and only if

1. $N^{(i\cdot j)} = \frac{1}{n} N^{k\cdot}_{\cdot kj} g^{ij}$,
2. $Q^{\cdot j} = \frac{n-2}{2n} N^{k\cdot}_{\cdot kj}$,
3. $(n-2)N^{k\cdot}_{\cdot kj} = 2n(\frac{2}{n} N^{k\cdot}_{\cdot kj} U + N^k U_{\cdot k})$.

In such case $r = \frac{2}{n} N^{k\cdot}_{\cdot kj}$.

Proof. \[ [g^{ij} \nabla_i \nabla_j + U, N^k \nabla_k + Q] = 
= 2N^{(i\cdot j)} \nabla_i \nabla_j + (N^{i\cdot j} + N^j R^i_{\cdot j} + 2Q^{\cdot j}) \nabla_i + (Q^{\cdot j} - N^j U_{\cdot j})
= r g^{ij} \nabla_i \nabla_j + r U. \]

Equating same degree (symmetrized) covariant derivatives we conclude that $N^k \nabla_k$ must be a conformal Killing vector. Using (a) from Lemma 3.1, the rest of the conditions easily follow. \qed

Proposition 3.3. Let $(M^n, g)$ be an arbitrary pseudo-Riemannian manifold. Then $N^k \nabla_k + Q$ is a conformal symmetry of the Yamabe operator if and only if $N^k \nabla_k$ is a conformal Killing vector and $Q = \frac{n-4}{2n} N^{k\cdot}_{\cdot kj}$.

Proof. Taking $U = -c_1 R$ in Lemma 3.2, condition 3 rewrites as
\[
(n-1)N^{k\cdot}_{\cdot kj} = -(N^{k\cdot}_{\cdot kj} R + \frac{n}{2} N^k R_{\cdot k})
\]
which, by Lemma 3.1, is always satisfied by any conformal Killing vector. \qed

A result analogous to Proposition 3.3 holds for the Paneitz operator on Einstein spaces.

Theorem 3.4. Let $(M^n, g)$ be an Einstein manifold with $R_{ij} = c g_{ij}$ and $n \geq 3$. Then $N^k \nabla_k + Q$ is a conformal symmetry of the Paneitz operator if and only if $N^k \nabla_k$ is a conformal Killing vector and $Q = \frac{n-4}{2n} N^{k\cdot}_{\cdot kj}$. 

Proof. We want to solve

\[ [\mathbb{P}, N^k \nabla_k + Q] = r\mathbb{P}. \]

Since \( R = nc, \mathbb{P} = (\Delta - ncc_1)(\Delta - ncc_2) \). Let \( d_1 = (c_1 + c_2)n = \frac{n^2 - 2n - 4}{2(n-1)} \) and \( d_2 = c_1 c_2 n^2 = \frac{n(n+2)(n-2)(n-4)}{16(n-1)^2} \).

We have

\[ \mathbb{P} = g^{ij} g^{kl} \nabla_i \nabla_j \nabla_k \nabla_l - cd_1 g^{ij} \nabla_i \nabla_j + c^2 d_2 = g^{ij} g^{kl} \nabla_i \nabla_j \nabla_k \nabla_l - c(d_1 + \frac{2}{3}) g^{ij} \nabla_i \nabla_j + c^2 d_2. \]

Since,

\[ [g^{ij} g^{kl} \nabla_i \nabla_j \nabla_k \nabla_l, N^m \nabla_m + Q] = \]

\( 2(g^{ij} N^{i,j}_k + g^{ij} N^{i,kl} \nabla_i \nabla_j \nabla_k \nabla_l + 2(g^{ij} N^{i,kl}_k + 2N^{i,j}_k + c g^{ij} N^{i,j} + g^{ij} Q^{i,j}_k) \nabla_i \nabla_j \nabla_l + 2(N^{i,j}_k + 2N^{i,j}_k + 2Q^{i,j}_k \nabla_i \nabla_l + (N^{i,j}_k + 2N^{i,j}_k - 2N^{i,j}_k + 2Q^{i,j}_k + 2Q^{i,j}_k) \nabla_l + 4Q^{i,j}_k \nabla_l,\)

Then,

\[ [\mathbb{P}, N^m \nabla_m + Q] = [g^{ij} g^{kl} \nabla_i \nabla_j \nabla_k \nabla_l, N^m \nabla_m + Q] - cd_1 [g^{ij} \nabla_i \nabla_j, N^k \nabla_k + Q] \]

\[ = 4g^{ij} N^{i,j}_k \nabla_i \nabla_j \nabla_k \nabla_l + (2g^{ij} N^{i,j}_k + 4N^{i,j}_k) \nabla_i \nabla_j \nabla_l + (2N^{i,j}_k + 2N^{i,j}_k + \frac{2}{3} - 2d_1)c N^{i,j}_k - \frac{4}{3} g^{ij} N^{i,j}_k R^{i,j}_k + 2Q^{i,j}_k \nabla_l + cQ^{i,j}_k \nabla_l - cd_1 (N^{i,j}_k + cN^{i,j}_k) - 2cd_1 Q^{i,j}_k \nabla_l + Q^{i,j}_k \nabla_l - cd_1 Q^{i,j}_k \nabla_l \]

After symmetrizing derivatives we equate same-order terms in (8). The fourth order terms give

\[ 4g^{ij} N^{i,j}_k = r g^{ij} g^{kl}, \]

which holds if and only if \( N^k \nabla_k \) is a conformal Killing vector and \( r = \frac{4}{n} N^k ; k \). Indeed,

\[ 4g^{ij} N^{i,j}_k = \frac{2}{3} \left[ g^{ij} N^{(i,j)}_k + g^{ik} N^{(j)}_k + g^{il} N^{(i,j)}_k + g^{ik} N^{(i)}_k + g^{il} N^{(i,j)}_k + g^{kl} N^{(i,j)}_k \right] \]

If \( N \) is a conformal Killing vector (10) becomes

\[ \frac{4}{3} \frac{1}{n} N^k ; k \left[ g^{ij} g^{kl} + g^{ik} g^{jl} + g^{il} g^{jk} \right] = \frac{4}{n} N^k ; k g^{ij} g^{kl}. \]
Conversely, contracting the $i$ and $j$ indices in (9)
\[ \frac{2}{3} \left[ (n + 4)N^{(k,l)}_i + g^{kl}N^j_{,ij} \right] = r \frac{1}{3} (n + 2)g^{kl}, \]
and contracting the two remaining indices in (11) we obtain $r = \frac{4}{n}N^k_{;k}$. Substituting $r$ back into (11) gives
\[ (n + 4)N^{(k,l)}_i = \frac{1}{n} (2n + 4 - n)N^k_{;k}g^{kl} \]
so $N^{(k,l)}_i = \frac{1}{n} N^k_{;k} g^{kl}$.

Using that $N$ is a conformal Killing vector the third order terms become
\[ \frac{4 - n}{n} N^k_{;k} (i^j g^{ji}) + 2g^{(ij}Q_{;l)} = 0, \]
which holds if and only if $Q_{;l} = \frac{n - 4}{2n} N^k_{;k}.$

Considering the information obtained so far, the condition imposed by the second order terms becomes
\[ ((n-1)N^i_{;l;k} + ncN^k_{;k})g^{ij} = 0, \]
which holds for any conformal Killing vector by (b) in Lemma 3.1. Likewise, the equation satisfied by the first order terms reduces to
\[ (n-1)N^i_{;l;k} + ncN^k_{;k} = 0. \]

Finally, the 0th order terms give
\[ \frac{n - 4}{2n} (N^i_{;l;k}j - d_1 cN^i_{;l;k}) = c^2 d_2 n N^k_{;k}. \]

We have that
\[ N^i_{;l;k}j - d_1 cN^i_{;l;k} = \frac{-n}{n - 1} cN^i_{;l;k} + \frac{n(n^2 - 2n - 4)}{2(n - 1)^2} c^2 N^i_{;l} \]
\[ = \frac{n^2}{(n - 1)^2} c^2 N^i_{;l} + \frac{n(n^2 - 2n - 4)}{2(n - 1)^2} c^2 N^i_{;l} \]
\[ = \frac{n(n^2 - 4)}{2(n - 1)^2} c^2 N^i_{;l}. \]

Then,
\[ \frac{(n - 4) n(n^2 - 4)}{2n} \frac{2(n - 1)^2}{} c^2 N^i_{;l} = d_2 n \frac{c^2 N^i_{;l}}{4}. \]

\[ \square \]

**Remark 3.5.** Let $(M^n, g)$ be an Einstein manifold of dimension $n \geq 3$. The first order differential operator $N^k \nabla_k + Q$ commutes with the Paneitz operator if and only if $N^k \nabla_k$ is a Killing vector field and $Q$ is a constant.

**Remark 3.6.** Let $(M^n, g)$ be an Einstein manifold of dimension $n \geq 3$. There is a correspondence between the first order conformal symmetries of the Yamabe operator and the first order conformal symmetries of the Paneitz operator. Namely, $N^k \nabla_k + Q$ is a conformal symmetry of $\Delta_Y$ if and only if $N^k \nabla_k + \frac{n-4}{n-2} Q$ is a conformal symmetry of $\mathbb{P}$.
The case of a 2-dimensional manifold is quite special since the algebra of conformal symmetries is infinite dimensional. We can still establish a correspondence between conformal symmetries of $\Delta$ and $\Delta^2$.

**Corollary 3.7.** Let $(M^2, g)$ be 2-dimensional. There is a correspondence between the first order conformal symmetries of $\Delta$ and those of $\Delta^2$. Namely, $N^k\nabla_k$ is a conformal symmetry of $\Delta$ if and only if $N^k\nabla_k - \frac{1}{2}N^k;_k$ is a conformal symmetry of $\Delta^2$.

One could ask whether Theorem 3.4 can be extended to hold on a general pseudo-Riemannian manifold. We present examples in spaces that are neither Einstein nor locally conformally flat where the conclusion of Theorem 3.4 holds, thus suggesting an affirmative answer to this question.

**Example 3.8.** Consider $\mathbb{R}^4$ endowed with the *pp-wave* metric

$$g = \frac{x^2 - y^2}{u^4}du^2 - 2dudv + dx^2 + dy^2,$$

$$(u, v, x, y) = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4.$$

This metric is Lorentzian and Ricci-flat. Note that $g$ is not locally conformally flat since the Weyl tensor has non-zero components, for example $C_{1212} = -\frac{1}{u^4}$.

Let us conformally rescale the metric by $\hat{g} = e^{2f}g$ so as to obtain a metric that is not Einstein. For instance take, $f = f(v)$ a smooth function of $v$ only. We have $R_{\hat{g}} = 6e^{-2f(v)}\frac{x^2 - y^2}{u^4}(f''_v + f_{vv})$, so $\hat{g}$ is not Einstein as long as $f''_v + f_{vv} \neq 0$.

These metrics have maximal conformal group admitting seven conformal Killing vectors. One can check that all of them give rise to conformal symmetries of the Paneitz operator.

**Example 3.9.** Let $H_2$ be the 5-dimensional Heisenberg group and let $\Gamma \subset H_2$ be a uniform discrete subgroup as in [5]. Then $M = \Gamma \backslash H_2$ is a compact manifold. Let $(x_1, x_2, y_1, y_2, t)$ be local coordinates. We endow $M$ with the left-invariant metric

$$g = dx_1dx_1 + dx_2dx_2 + (1 - x_1^2)dy_1dy_1 - 2x_1x_2dy_1dy_2 + 2x_1dy_1dt + (1 - x_2^2)dy_2dy_2 + 2x_2dy_2dt - dt dt.$$

This metric is neither Einstein nor locally conformally flat. The frame $\{X_1, X_2, Y_1, Y_2, T\}$, where $X_i = \frac{\partial}{\partial x_i}, Y_i = \frac{\partial}{\partial y_i} + x_i\frac{\partial}{\partial t}, T = \frac{\partial}{\partial t}$, is orthonormal for this metric. The Laplacian on scalar fields is therefore given by

$$\Delta_g = X_1^2 + X_2^2 + Y_1^2 + Y_2^2 + T^2.$$

By [2] the Paneitz operator is given by

$$\mathbb{P}_g = \Delta_g^2 + \frac{1}{8}\Delta_g + 2T^2 - \frac{359}{2304}.$$
The metric $g$ does not admit conformal Killing vectors, other than exact Killing vectors. Consider instead $\hat{g} = e^{2f(x_1)}g$ where $f$ is a smooth function of $x_1$ only. Now, $R_{\hat{g}} = -e^{2f(x_1)}(12f_x^2 + 8fx_1x_x - 1)$, so $\hat{g}$ will not Einstein if $R_{\hat{g}}$ is not a constant.

The metrics $\hat{g}$ admits 4 conformal Killing vectors. The Paneitz operator in the metric $\hat{g}$ is given by

$$ \mathcal{P}_{\hat{g}} = e^{-\frac{f}{2}g_{xx}} \mathcal{P}_g e^{\frac{f}{2}g_{xx}} $$

and each of the conformal Killing vectors can be checked to induce a first order symmetry of $\mathcal{P}_{\hat{g}}$. For instance, let $L = -y_2\nabla x_1 - y_1\nabla x_2 + x_2\nabla y_1 + x_1\nabla y_2 + (-y_1y_2 + x_1x_2)\nabla t$. Then

$$ [\mathcal{P}_{\hat{g}}, L - \frac{1}{2}y_2f_{x_1}] = -4y_2f_{x_1} \cdot \mathcal{P}_{\hat{g}}. $$

4. Second order conformal symmetries

We now deal with second order differential operators on $M$ that satisfy

$$ [g^{ij}\nabla_i \nabla_j + U, \nabla_i K^{ij} \nabla_j + N^k \nabla_k + Q] = (C^k \nabla_k + D)(g^{ij}\nabla_i \nabla_j + U) $$

for some vector field $C = C^k \nabla_k$ and scalar field $D$.

In [3] Carter shows the following result.

**Proposition 4.1** (Carter [3]).

$$ [g^{ij}\nabla_i \nabla_j, \nabla_i K^{ij} \nabla_j] = 2K^{(ij; k)} \nabla_i \nabla_j \nabla_k + 3K^{(ij; k)} \nabla_i \nabla_j $$

$$ + [K^{(ij; k)} + \frac{1}{2}g_{kl}(K^{(kl; i)j} - K^{(kl; j)i}) - \frac{4}{3}K^{[i| \nabla_k; j]} \nabla_i. $$

The expression given in [3], however, contains a typographical mistake where the term

$$ K^{(ij; k)} \nabla_i $$

is missing. In [11] Kamran and McLenaghan derive the necessary and sufficient conditions for $\nabla_i K^{ij} \nabla_j$ to be a symmetry of $\Delta + U$. The omitted term in [3] is carried over to [11]. The correct expression, in a slightly more general context, is given in the following proposition.

**Proposition 4.2.** Let $(M^n, g)$ be an arbitrary pseudo-Riemannian manifold and $U$ a scalar field on $M$.

$$ [g^{ij}\nabla_i \nabla_j + U, \nabla_i K^{ij} \nabla_j + N^k \nabla_k + Q] = (C^k \nabla_k + D)(g^{ij}\nabla_i \nabla_j + U) $$

if and only if

1. $K^{(ij; k)} = \frac{1}{2}C^{(ij; k)},$

2. $C^{(i; j)} + 2N^{(i; j)} = \frac{1}{n}(C^{k; i} + 2N^{k; i})g^{ij},$

3. $K^{(ij; k)} = \frac{1}{2}g_{kl}(K^{(kl; i)j} - K^{(kl; j)i}) - \frac{4}{3}K^{[i| \nabla_k; j]} \nabla_i + N^{i; k} + N^k \nabla_k + 2Q^i - 2K^{ij} \nabla_j$  

$$ - C^iU - \frac{2}{3}C^k \nabla_k = 0, $$

4. $(K^{ij}U; i)j + N^k U; k + Q^i; i + C^k U; k - DU = 0,$

5. $D = \frac{1}{n}(C^{k; i} + 2N^{k; i}).$
Again, we may take $U = -\frac{(n-2)}{4(n-1)}R$ to obtain the second order conformal symmetries of the Yamabe operator. A characterization of such symmetries has also been given in [12].

**Remark 4.3.** Let $L$ and $M$ be conformal Killing vectors. Then the symmetrized tensor product $K^{ij} := L^i M^j$ is a conformal Killing tensor. Indeed, $K^{(ij;k)} = C^{(ij)k}$ where $C^i = \frac{1}{n}(L^k M^i + M^k L^i)$.

Since the property of being conformal Killing is inherited under symmetrized tensor products, a natural question is whether two first order conformal symmetries induce a second order conformal symmetry. Let $\{\cdot,\cdot\}$ denote the anti-commutator. Note that 

$$\frac{1}{2}\{L^i \nabla_i, M^j \nabla_j\} = \nabla_i K^{ij} \nabla_j - \frac{1}{2}(L^i M^j + M^i L^j) \nabla_i,$$ 

where $K^{ij} := L^i M^j$. We answer this question in the following theorem.

**Theorem 4.4.** Let $(M^n, g)$ be an arbitrary pseudo-Riemannian manifold and $U$ a scalar field on $M$. Suppose $L^k \nabla_k + A$ and $M^k \nabla_k + B$ are conformal symmetries of $\Delta + U$. Then 

$$S := \frac{1}{2}\{L^i \nabla_i + A, M^j \nabla_j + B\}$$ 

is not, in general, a second order symmetry of $\Delta + U$. However, $S + q$ is a second order symmetry if and only if $q$ is a scalar field satisfying 

1. $q^i;_i = \frac{1}{3}[2(K^{[i} R^{j]});_j + 2L^{[i} B^{j]};_j + 2M^{[i} A^{j]};_j + (BL^k + AM^k)R^i_k]$, 
2. $q^i;_i = 0$.

In such case, 

$$[\Delta + U, S + q] = (C^k \nabla_k + D)(\Delta + U),$$ 

where $C^k = \frac{2}{n}(L^k M^i + M^k L^i)$ and $D = \frac{1}{2}C^k;_k$.

**Proof.** Let 

$$K^{ij} = \frac{1}{2}(L^i M^j + L^j M^i),$$ 

$$N^k = -\frac{1}{n}(L^k M^j;_j + M^k L^j;_j),$$ 

$$Q = \frac{1}{2}(L^i B^j;_i + 2AB + M^i A;_i).$$

Then, $S = \nabla_i K^{ij} \nabla_j + N^k \nabla_k + Q$.

We have 

$$[g^{ij} \nabla_i \nabla_j + U, \nabla_i K^{ij} \nabla_j + N^k \nabla_k + Q] =$$ 

$$= 2K^{(ij;k)} \nabla_i \nabla_j \nabla_k + (3K^{(ij)k)} + 2N^{(i;j)} \nabla_i \nabla_j +$$ 

$$+ (K^{(ij)k)} + g_{kl}(K^{(kl;j)} - K^{(kl;j)}) \nabla_j - \frac{4}{3}(K^{p[i} R^{j]};_j + N^{i;j} + N^j R^i - 2K^{ij} U;j) +$$ 

$$+ 2Q^{(i)} \nabla_i - (K^{ij} U;j)_i - N^i U;j + Q^j;j,$$
Finally, the terms of order zero require

\[(\nabla_i \nabla_j + U, \nabla_i K^{ij} \nabla_j + N^k \nabla_k + Q') = -(2N^k \nabla_k + N^k) (g^{ij} \nabla_i \nabla_j + U)\]

where \(Q' = Q + q\).
The case of the Yamabe Laplacian is again special. As stated in the introduction, we are able to show that two conformal Killing vectors always induce a second order conformal symmetry of $\Delta_Y$.

**Theorem 4.5.** Let $(M^n, g)$ be an arbitrary pseudo-Riemannian manifold. If $L^k \nabla_k + A$ and $M^k \nabla_k + B$ are conformal symmetries of $\Delta_Y$, then $S := \frac{1}{2} \{ L^k \nabla_k + A, M^k \nabla_k + B \}$ is a second order conformal symmetry of $\Delta_Y$.

**Proof.** We will show that $q$, as defined in Theorem 4.4, is constant. First,

$$A_{ij} = \frac{n-2}{2n} L^k_{;k} g^{ij} = -\frac{1}{2} (L^i_{;k} L^j_{;k} + L^k_{;j} R^i_{;k} + L^k_{;i} R^j_{;k})$$

$$= -\frac{1}{2} (L^i_{;k} L^j_{;k} + L^p_{;k} R^i_{;k} R^j_{;k} - L^i_{;p} R^j_{;k} + L^k_{;i} R^j_{;k} + L^k_{;j} R^i_{;k})$$

$$= -\frac{1}{2} (L^i_{;k} L^j_{;k} + 2L^p_{;k} R^i_{;k} R^j_{;k} + L^p_{;k} R^i_{;k} R^j_{;k} - L^i_{;p} R^j_{;k} + L^k_{;i} R^j_{;k} + L^k_{;j} R^i_{;k})$$

$$= -\frac{1}{2} (L^i_{;k} L^j_{;k} + 2L^p_{;k} R^i_{;k} R^j_{;k} + L^k_{;i} R^j_{;k} - L^i_{;p} R^j_{;k} + L^k_{;j} R^i_{;k}).$$

Therefore,

$$A_{ij} = A_{ij} = -\frac{1}{2} (L^i_{;k} L^j_{;k} + 2L^p_{;k} R^i_{;k} R^j_{;k} + L^k_{;i} R^j_{;k} - L^i_{;p} R^j_{;k} + L^k_{;j} R^i_{;k})$$

$$= -\frac{1}{2} \left( \frac{1}{n(n-1)} L_{;k} k g^{ij} + \frac{1}{2(n-1)} L^k_{;k} R_{;k} g^{ij} + 2 \frac{1}{n} L^k_{;k} R_{;k} g^{ij} + L^k_{;i} R^j_{;k} - L^i_{;p} R^j_{;k} + L^k_{;j} R^i_{;k} \right)$$

Since

$$L^k_{;i} R^j_{;k} - L^i_{;k} R^j_{;k} = \frac{1}{2} (L^i_{;k} R^j_{;k} + L^k_{;i} R^j_{;k} - L^i_{;k} R^j_{;k} - L^j_{;k} R^i_{;k})$$

$$= \frac{1}{2} (2L^i_{;k} R^j_{;k} - 2L^k_{;i} R^j_{;k} - 2L^i_{;k} R^j_{;k} + 2L^k_{;j} R^i_{;k})$$

$$= L^k_{;j} R^i_{;k} - L^i_{;k} R^j_{;k},$$

we have

$$A_{ij} = -\frac{1}{2} \left( \frac{1}{n(n-1)} L_{;k} k R_{;j} g^{ij} + \frac{1}{2(n-1)} L^k_{;k} R_{;k} g^{ij} + \frac{1}{2(n-1)} L^k_{;k} R_{;k} g^{ij} + 2 \frac{1}{n} L^k_{;k} R_{;k} g^{ij} + L^k_{;j} R^i_{;k} + L^k_{;i} R^j_{;k} \right)$$

$$= -\frac{1}{2} \left( \frac{1}{n(n-1)} L_{;k} k R_{;j} g^{ij} + \frac{1}{2(n-1)} L^k_{;k} R_{;k} g^{ij} + \frac{1}{2(n-1)} L^k_{;k} R_{;k} g^{ij} + 2 \frac{1}{n} L^k_{;k} R_{;k} g^{ij} + L^k_{;j} R^i_{;k} + L^k_{;i} R^j_{;k} \right).$$
Then,
\[
M^i A^{,j}_i - M^j A^{,i}_i = -\frac{1}{2} M^i L^j L^k R_{jk} - \frac{1}{4} M^j L^k R_{jk} + \frac{1}{n} M^j L^k R^i_{jk} + \frac{1}{2} M^j L^k R^i_{jk;k}
\]
\[
- \frac{1}{2} M^j L^k R_{jk} + \frac{1}{2} M^j L^k R^i_{j;k},
\]
so it follows that
\[
M^i A^{,j}_i - M^j A^{,i}_i + L^j B^{,j}_i - L^j B^{,i}_j =
\]
\[
- K^{ij;k} R_{jk} - \frac{1}{2} K^{ij} R_{ij} + \left(\frac{2 - n}{2n} + \frac{1}{2}\right) (M^j L^k ;k + L^j M^k ;k) R^i_{jk} + K^{jk} R^i_{jk;k}
\]
\[
+ \frac{1}{2} (M^j L^k ;j + L^j M^k ;j) R^i_{k}
\]
\[
= - K^{ij;k} R_{jk} - \frac{1}{2} K^{ij} R_{ij} + \frac{2 - n}{2n} (M^j L^k ;k + L^j M^k ;k) R^i_{jk} + K^{jk} R^i_{jk;k} + K^{jk} R^i_{j;k}
\]
\[
= -2(K^{k[j} R^{i]}_{k})_{,j} - (BL^k + AM^k) R^i_{k},
\]
as needed.

Next, we ask whether first order conformal symmetries of the Paneitz operator induce second order symmetries. This is much harder to obtain by direct computation. If \( M \) is locally conformally flat, a positive answer follows from [7]. The following examples suggest that first order conformal symmetries of \( \mathbb{P} \) would also induce second order conformal symmetries in geometries that are not locally conformally flat.

**Example 4.6.** Consider the conformally pp-wave metrics introduced in Example 3.8. Every pair of conformal Killing vectors can be checked to induce a second order symmetry of the Paneitz operator. For instance, consider the first order conformal symmetries

\[
[\mathbb{P} \hat{g}, L^k \nabla_k] = 2((x^2 + y^2) f_v + 2u) \cdot \mathbb{P} \hat{g} \quad \text{and} \quad [\mathbb{P} \hat{g}, M^k \nabla_k] = 4(2f_v + 1) \cdot \mathbb{P} \hat{g}.
\]

Let \( S := \frac{1}{2} \{ L^k \nabla_k, M^k \nabla_k \} \). Then

\[
[\mathbb{P} \hat{g}, S] = (C^k \nabla_k + \frac{1}{2} C^k ;k) \cdot \mathbb{P} \hat{g},
\]
where \( C^k = \frac{4}{n}(L^k M^j ;j + M^k L^j ;j) = 2((4v f_v + 2) L^k + ((x^2 + y^2) f_v + 2u) M^k), \frac{1}{2} C^k ;k = 4((8uv + 3x^2 + 3y^2) f_v + 4v(x^2 + y^2) f_v f_v + v(x^2 + y^2) f_v f_v + 4u).

**Example 4.7.** Consider \((M, \hat{g})\) from Example 3.9. Every pair of conformal Killing vectors induces a second order symmetry of the Paneitz operator. Consider for instance

\[
L = -y_1 \nabla_{x_1} + x_1 \nabla_{y_1} + \frac{1}{2}(x_1^2 - y_1^2) \nabla_t \quad \text{and} \quad M = -x_2 \nabla_{x_1} + x_1 \nabla_{x_2} - y_2 \nabla_{y_1} + y_1 \nabla_{y_2}.
\]

Then,

\[
[\mathbb{P} \hat{g}, L^k \nabla_k + \frac{1}{10} L^k ;k] = -4y_1 f_{x_1} \mathbb{P} \hat{g} \quad \text{and} \quad [\mathbb{P} \hat{g}, M^k \nabla_k + \frac{1}{10} M^k ;k] = -4x_2 f_{x_1} \mathbb{P} \hat{g}.
\]
Let $S := \frac{1}{2} \{ L^k \nabla_k + \frac{1}{10} L^k_{;k}, M^k \nabla_k + \frac{1}{10} M^k_{;k} \}$, We have

$$[\mathbb{P}_{\tilde{g}}, S] = (C^k \nabla_k + \frac{1}{2} C^k_{;k}) \cdot \mathbb{P}_{\tilde{g}}.$$ 

Here $C^k = \frac{4}{n} (L^k M^j_{;j} + M^k L^j_{;j}) = -4x_2 f_{x_1} L^k - 4y_1 f_{x_1} M^k; \frac{1}{2} C^k_{;k} = 2f_{x_1} y_2 + 20x_2 y_1 f_{x_1}^2 + 4x_2 y_1 f_{x_1 x_1}.$

5. Conclusions

A number of questions that naturally follow from this work remain to be settled. First, we described all the first order symmetries of the Paneitz operator on Einstein manifolds starting from the fact that, on such manifolds, the Paneitz operator can be written as a polynomial in the Laplacian with constant coefficients. It is not clear whether the Einstein condition is essential for the vector space of first order conformal symmetries to be isomorphic to the space of conformal Killing vectors, our examples suggest it is not. The nature of the interplay between curvature properties and the conditions for the existence of first order symmetries remains to be fully understood. Further, higher order GJMS operators might share an analogous behaviour on Einstein or perhaps more general spaces.

Second, we showed how first order conformal symmetries of the Yamabe operator induce second order symmetries and suggested this could also be true for the Paneitz operator. Again, it is not evident whether the curvature could be an obstruction for such a hereditary property to hold for the Paneitz operator. Higher order GJMS operator might also admit a similar hereditary property. We hope to investigate these questions in the future.

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