Mineral seenaw model with $S_4$ flavor symmetry

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We discuss a neutrino mass model based on the $S_4$ flavor symmetry within the minimal seesaw framework, in which only two right-handed neutrinos are introduced and transform as 2 under $S_4$. Although the model contains less free parameters compared to the typical seesaw models, it provides a successful description of the observed neutrino parameters, and in particular, a nearly tri-bimaximal mixing pattern can be naturally accommodated. In addition, the heavy right-handed neutrino masses are found to be non-degenerate, while only the normal hierarchical mass spectrum is compatible with experiments for light neutrinos.

I. INTRODUCTION

In view of the compelling experimental evidence on neutrino oscillations, the origin of neutrino masses and lepton flavor mixing emerges as one of the most fundamental issues in particle physics. Since neutrinos are massless particles in the standard model (SM) of particle physics, a broad class of models extended the SM have been proposed in order to accommodate massive neutrinos. The seesaw mechanism [1,2] turns out to be among the most attractive extensions of the SM in virtue of its natural explanation of tiny neutrino masses.

In the canonical type-I seesaw model, three heavy right-handed neutrinos are introduced besides the SM particle contents, while a Majorana mass term $M_R$ is assumed, which is not subjected to the scale of electroweak symmetry breaking scale, i.e., $\Lambda_{\text{EW}} \sim 100$ GeV. The light neutrino mass scale is then strongly suppressed with respect to $\Lambda_{\text{EW}}$ due to the heavy right-handed neutrino masses.

In general, the type-I seesaw model is pestered with too many model parameters, and therefore, fails to predict the lepton flavor mixing pattern as well as the light neutrino mass spectrum. For example, in the case of the simplest type-I seesaw model with three right-handed neutrinos, there are in total fifteen free parameters in the Dirac mass matrix together with three unknown mass eigenvalues of heavy Majorana neutrinos, whereas the light neutrino mass matrix contains only nine physical parameters, indicating a lack of valuable predictions. Note that, in the most economical type-I seesaw model, i.e., the minimal seesaw model (MSM) [3,12], one could introduce only two right-handed neutrinos, whereas the observed neutrino mass hierarchy and lepton flavor mixing can be well interpreted. Such a minimal extension of the SM greatly reduces the number of free parameters, and hence is very predictive. For instance, in the MSM, one of the light neutrinos should be massless since $M_R$ is of rank 2, which indicates that $\sum_i m_i \simeq 0.05$ eV in the normal hierarchy case while $\sum_i m_i \simeq 0.1$ eV in the inverted hierarchy case with $m_1$ being the light neutrino masses. In case that future cosmological observations set more stringent constraints on the summation of the light neutrino masses, the MSM would then be the most plausible underlying model.

Recently, plenty of models extended the gauge group with flavor symmetries are studied in order to understand the lepton flavor mixing. In particular, the experimentally favored tri-bimaximal mixing pattern [13,15] can be naturally realized in many flavor symmetry models. It is therefore interesting to investigate if the neutrino masses and mixing can be realized in the MSM based on certain flavor symmetries. Now that there are only two right-handed neutrinos in the MSM, the symmetry group $G_f$ should contain at least one two-dimensional representation, if two right-handed neutrinos are located in the same multiplet of $G_f$. In addition, a three-dimensional representation should be employed in order to accommodate three generations of charged leptons in a natural way. In this sense, the permutation group $S_4$ appears as an attractive candidate for the MSM, since it is one of the smallest discrete groups containing one-, two- and three-dimensional representations. Similar models of the $S_4$ flavor symmetry within the canonical seesaw framework have been intensively studied in the literature [16,40].

In this work, we consider the MSM based on the $S_4$ flavor symmetry. In particular, we shall show that our scheme is rather compact whereas it is compatible with the experimental observation, i.e., the tri-bimaximal mixing pattern could be easily accommodated. The remaining parts of the work is organized as follows: In Sec. II we present the main content of our model, and formulate the general expressions of the lepton mass matrices. One interesting example is given in order to show how the tri-bimaximal mixing is realized. The information on the Higgs potential is also briefly discussed. Then, in Sec. III we perform a detailed numerical analysis, and illustrate the main results obtained in the model. Finally, our conclusions are presented in Sec. IV.

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II. THE MODEL

The discrete group $S_4$ is the permutation group of four distinct objects, which contains 24 group elements and 5 irreducible real representations. Among the five representations, two are one-dimensional ($\mathbf{1}_1$ and $\mathbf{1}_2$), one is two-dimensional ($\mathbf{2}$), and two are three dimensional ($\mathbf{3}_1$ and $\mathbf{3}_2$). The group properties, i.e., the Kronecker products and the Clebsch-Gordan coefficients, can be found in the appendices of Ref. [20].

The total symmetry of our model is then chosen to be

$$G = SU(3)_c \otimes SU(2) \otimes U(1)_Y \otimes S_4, \quad (1)$$

under which the lepton content in our model is placed as

$$L \sim (1, 2, -1)(\mathbf{3}_2), \quad (2)$$

$$\ell_R \sim (1, 1, -2)(\mathbf{3}_2), \quad (3)$$

$$\nu_R \sim (1, 1, 0)(\mathbf{2}), \quad (4)$$

Note that in Ref. [43], the minimal seesaw model is considered whereas the two right-handed neutrinos are assigned to the trivial representation of $S_4$. Furthermore, the Higgs assignments in our model are given by

$$\phi_0 \sim (1, 2, -1)(\mathbf{1}_1), \quad (5)$$

$$\phi_1, \phi_2 \sim (1, 2, -1)(\mathbf{2}), \quad (6)$$

$$\xi_1, \xi_2, \xi_3 \sim (1, 2, -1)(\mathbf{3}_1), \quad (7)$$

$$(\chi_1, \chi_2) \sim (1, 1, 0)(\mathbf{2}), \quad (8)$$

where the $SU(2)$ doublets Higgs fields are in analogy to these in Ref. [20], whereas an additional $SU(2)$ singlet Higgs $\chi$ is introduced. We will show later on that $\chi$ is crucial to ensure the corrected prediction on the neutrino mixing angles as well as the light neutrino masses. Note that we mainly focus our attention on the lepton flavor mixing, and hence do not include the quark sector in our discussions. A simple way to contain the quark mixing in our model is to make a naive assumption that all the quarks belong to the identity representation, i.e., $\mathbf{1}_1$, and then the quark flavor mixing and masses can be obtained via the standard Yukawa couplings to $\phi_0$.

By using the group algebra of $S_4$, we can write the invariant Yukawa couplings for leptons as

$$\mathcal{L} = \alpha_0 \left[ (L_1 e_R + L_2 \mu_R + L_3 \tau_R) \phi_0 \right] + \alpha_1 \left[ \sqrt{3} (L_2 \mu_R - L_3 \tau_R) \phi_1 \right. \\
+ (-2L_1 e_R + L_2 \mu_R + L_3 \tau_R) \phi_2 \left. \right] + \alpha_2 \left[ (L_2 \tau_R + L_3 \mu_R) \xi_1 + (L_1 \tau_R + L_3 e_R) \xi_2 + (L_1 \mu_R + L_2 e_R) \xi_3 \right. \\
+ \beta_0 \left[ \frac{2}{\sqrt{6}} \nu_R \xi_1 + \left( -\nu_R \xi_1 + \sqrt{3} \nu_R \xi_2 \right) \xi_2 + \left( -\nu_R \xi_1 - \sqrt{3} \nu_R \xi_3 \right) \xi_3 \right] \\
+ \beta_1 \left[ \nu_R \chi_1 + \left( \nu_R \chi_1 + \sqrt{3} \nu_R \chi_2 \right) \chi_2 + \left( -\nu_R \chi_1 - \sqrt{3} \nu_R \chi_3 \right) \chi_3 \right] \\
+ \frac{M}{2} \left[ \nu_R^\dagger \nu_R + \nu_R^\dagger \nu_R \right] + h.c., \quad (9)$$

where $\xi_1$ is the conjugate of $\xi_1$ related by $\xi_i \equiv i r_2 \xi_i^\dagger$, and a bare Majorana mass $M$ is included.

A. Charged lepton masses

In our model, the $S_4$ flavor symmetry is assumed to be spontaneously broken by the vacuum expectation values (VEVs) of Higgs scalars, i.e., $\langle \phi_i \rangle = v_i$, $\langle \xi_i \rangle = u_i$, and $\langle \chi_i \rangle = x_i$. One then arrives at the mass matrix of charged leptons as

$$M_\ell = \begin{pmatrix} a_0 - 2a_2 & b_3 & b_2 \\ b_3 & a_0 + \sqrt{3}a_1 + a_2 & b_1 \\ b_2 & b_1 & a_0 - \sqrt{3}a_1 + a_2 \end{pmatrix} \quad (10)$$

where we have defined $a_0 = \alpha_0 v_0$, $(a_1, a_2) = (\alpha_1 v_1, \alpha_1 v_2)$, and $b_i = \alpha_2 u_i$ (for $i = 1, 2, 3$). In general, all the parameters in the mass matrix can be complex, while in case of CP-conservation, there are totally six real parameters in $M_\ell$. For simplicity, we will take all the parameters to be real, but comment later on the most general case with CP-violating effects.

According to Eq. (10), the contributions from $\phi_i$ merely affect the diagonal entries, whereas $\xi_i$ appear in the off-diagonal elements. In the limit $a_i \gg b_i$, $M_\ell$ approximates to a nearly diagonal form, and the charged-lepton masses are solely determined by $a_i$. Note that this is indeed a very realistic scenario if the VEVs of $\xi_i$ are much smaller than those of $\phi_i$. Explicitly, the sum of the VEVs has to be equal to the electroweak scale, i.e., $\sum_i |V_{ei}|^2 \simeq (174 \text{ GeV})^2$. Since $\phi_0$ should also be responsible for the generation of the top-quark mass, one may reasonably take $v_0 \simeq 174$ GeV with all the other VEVs being much smaller than $v_0$. In our model, we assume that $v_1, v_2 \sim \text{GeV}$ and $u_1 \sim \text{MeV}$. As a result, the eigenvalues of $M_\ell$ are approximately given by $a_0 - 2a_2$, $a_0 + \sqrt{3}a_1 + a_2$, and $a_0 - \sqrt{3}a_1 + a_2$, respectively. Compared to the charged-lepton masses, one immediately obtains

$$a_0 \simeq \frac{1}{3} (m_e + m_\mu + m_\tau), \quad (11)$$

$$a_1 \simeq \frac{1}{\sqrt{3}} (m_\mu - m_\tau), \quad (12)$$

$$a_2 \simeq \frac{1}{6} (m_\mu + m_\tau - 2m_e). \quad (13)$$

In addition, the diagonalization matrix for $M_\ell$ is nearly an identity matrix, i.e., $V_\ell \simeq I$.

B. Neutrino mass matrix

Since there are only two right-handed neutrinos in the MSM, the Dirac mass of neutrinos is a $3 \times 2$ matrix, viz.

$$M_D = \begin{pmatrix} 2X_1 & 0 \\ -X_2 & \sqrt{3}X_2 \\ -X_3 & \sqrt{3}X_3 \end{pmatrix}, \quad (14)$$
where $X_i = \frac{\delta_m}{\sqrt{2}}$ for $i = 1, 2, 3$. The right-handed neutrino mass matrix in our model is given by

$$M_R = \begin{pmatrix} A + C & B \\ B & A - C \end{pmatrix},$$
(15)

where $A = M_1$, $B = \beta_1 x_1$, and $C = \beta_1 x_2$, respectively. In case of $M_R \gg M_D$, Eq. (15) leads to the masses of right-handed neutrinos as

$$M_{1,2} = A \pm \sqrt{B^2 + C^2}.$$
(16)

Note that, as aforementioned, the natural scale of $M_D$ relies on the VEVs $u_i$ implying $X_i \sim \mathcal{O}(\text{MeV})$. This in turn helps us to estimate that the right-handed neutrino masses should be around $\mathcal{O}(10^2)$ TeV, which turn out to be beyond the scope of forthcoming collider experiments. In case of $\text{right-handed neutrinos as } V_\epsilon \text{ linking the neutrino mass eigenstates with their flavor eigenstates is then given by }$

$$U = V_\epsilon V_\nu \simeq V_\nu,$$
(19)

where the last approximation follows since we have taken the charged-lepton mass matrix to be nearly diagonal.

In the standard (i.e., CKM-like) parametrization one has

$$U = R_{23} P_3 R_{13} P_1^{-1} P_{12} P_M,$$
(20)

where $R_{ij}$ correspond to the elementary rotations in the $ij = 23, 13, \text{ and } 12$ planes (parametrized in what follows by three mixing angles $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$), $P_1 = \text{diag}(1, 1, e^{i\delta_1})$, and $P_M = \text{diag}(e^{i\theta_1/2}, e^{i\theta_2/2}, 1)$ contain the Dirac and Majorana CP-violating phases, respectively.

In order to get the explicit expression of $U$, a fully diagonalization of $m_\nu$ is involved, and the results are rather tedious. However, since $m_\nu$ is of rank 2, there exists an eigenvector $k = (r_1 r_2, r_1 T$ satisfying $m_\nu k = 0$. In case that the light neutrino mass spectrum is inverted hierarchy (i.e., $m_2 > m_1 \gg m_3$), $k$ corresponds to the third column of $U$. Compared to Eq. (20), we obtain

$$\tan \theta_{23} = \frac{r_2}{r_1},$$
(21)

$$\tan \theta_{13} = \frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}.$$  
(22)

In view of the experimentally measured maximal atmospheric angle and small reactor mixing angle, the relation $r_1 \simeq r_2 \ll 1$ has to be fulfilled. The two non-vanishing masses are then approximately given by

$$m_1 \approx m_0 \left[ 1 - c_2 + \sqrt{(1 - c_2)^2 + \mathcal{O}(r_1, r_2)} \right],$$
(23)

$$m_2 \approx m_0 \left[ 1 - c_2 - \sqrt{(1 - c_2)^2 + \mathcal{O}(r_1, r_2)} \right].$$
(24)

No matter what value of $c_2$ one chooses, it is not possible to let the two masses to be nearly degenerate (i.e., $m_1 \approx m_2$), which is indeed required for the inverted mass hierarchy case. Therefore, by analyzing the eigenvector of $m_\nu$, we can conclude that the inverted light neutrino mass hierarchy is not compatible with the model.

Henceforth, we shall concentrate on the normal hierarchy case, namely $m_1 < m_2 \ll m_3$. Here, we show one interesting example, in which the tri-bimaximal mixing pattern (i.e., $\theta_{12} \approx 35.3^\circ$, $\theta_{23} = 45^\circ$ and $\theta_{13} = 0$) is predicted. Concretely, we make the assumptions that $r_1 = r_2 = 2$ and $\epsilon_1 = 0$. Equation (25) now reduces to

$$m_\nu = m_0 \begin{pmatrix} 2c_2 - 2 & 2(1 - c_2) & 2(1 - c_2) \\ -4(2 + c_2) & 4(1 + 2c_2) \end{pmatrix}.$$  
(25)

One observes from Eq. (25) that, with the assumptions above, a $\mu - \tau$ symmetry appears in $m_\nu$, which generally predicts a maximal atmospheric mixing angle, i.e., $\theta_{23} = 45^\circ$, and a vanishing $\theta_{13}$. It is then easy to prove that the diagonalization matrix of $m_\nu$ takes exactly the tri-bimaximal mixing form, i.e.,

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}} & \frac{1}{\sqrt{2}} \\ -\sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$
(26)

Note that, the realization of the TeV minimal seesaw model turns out to be more natural compared to the typical low-scale type-I seesaw model, since the light neutrino masses could be protected by certain underlying symmetries and hence do not suffer from large radiative corrections.
while the light neutrino masses are given by
\[ m_1 = 0, \quad m_2 = 6m_0(1 - \epsilon_2), \quad m_3 = 12m_0(1 + \epsilon_2). \]
Consequently, both the tri-bimaximal mixing and the normal neutrino mass spectrum \((m_1 < m_2 \ll m_3)\) are accommodated.

Furthermore, if we relax the assumptions on \(r_i\) and \(\epsilon_i\), a deviation from the tri-bimaximal mixing can be achieved. For example, in the case \(\epsilon_1 \neq 0\), the light neutrino mass matrix can be written as
\[
\begin{align*}
m_\nu &= m_0 U_{\text{TB}} \begin{pmatrix}
0 & 0 & 0 \\
0 & 6(\epsilon_2 - 1) & -6\sqrt{2}\epsilon_1 \\
0 & -12(1 + \epsilon_2)
\end{pmatrix} U^T_{\text{TB}} \\
&= U_{\text{TB}} R_{23}(\theta) \text{ diag}(0, m_2, m_3) R^T_{23}(\theta) U^T_{\text{TB}},
\end{align*}
\]
with
\[
\begin{align*}
m_2 &= 3m_0 \left(3 + \epsilon_2 - \sqrt{(1 + 3\epsilon_2)^2 + 8\epsilon_1^2}\right), \\
m_3 &= 3m_0 \left(3 + \epsilon_2 + \sqrt{(1 + 3\epsilon_2)^2 + 8\epsilon_1^2}\right),
\end{align*}
\]
and
\[
\sin 2\theta = \frac{2\sqrt{2}\epsilon_1}{\sqrt{(1 + 3\epsilon_2)^2 + 8\epsilon_1^2}}.
\]

The neutrino mixing angles are then modified to
\[
\begin{align*}
s_{12} &= \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} \sin^2 \frac{\theta}{2}, \\
s_{23} &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \sin \theta - \frac{2}{\sqrt{2}} \sin^2 \frac{\theta}{2}, \\
s_{13} &= \frac{1}{\sqrt{3}} \sin \theta.
\end{align*}
\]
According to the above equations, the deviations of \(\theta_{ij}\) from their exact tri-bimaximal values are correlated by \(\theta\), and in the limit \(\theta \rightarrow 0\) (or effectively \(\epsilon_1 \rightarrow 0\)), the exact tri-bimaximal mixing will be reproduced. Note that, the correction to \(s_{12}\) is proportional to \(\sin^2 \frac{\theta}{2}\), and thus is strongly suppressed for a small \(\theta\). Therefore, \(\theta_{12}\) is rather stable against \(\epsilon_1\) corrections \([18, 19]\).

\[\text{D. Higgs potential}\]

Now that the previous discussions rely on the VEVs of the scalar fields, we are coming to the question of the possible Higgs potential and its minima. Apart from the \(SU(2)\) singlets \(\chi_i\), our Higgs setup is essentially the same as the Higgs sector considered in Ref.\([20]\), where only \(SU(2)\) doublets are introduced. We thereby only show the Higgs potential parts involving \(\chi_i\), viz.,

\[
V_\chi = -\mu_1^2 (\chi_1^2 + \chi_2^2) + \mu_2 (3\chi_1^2\chi_2 - \chi_3^2) + \omega_1 (\chi_1^2 + \chi_2^2)^2 + \omega_2 [(\chi_1\chi_2 + \chi_2\chi_1) + (\chi_1^2 - \chi_2^2)]^2 \\
+ \rho_1 \left[ |\phi_0^0\phi_0^0 (\chi_1^2 - \chi_2^2) | + \rho_2 \left( |\phi_0^1\phi_1^0 |^2 + |\phi_0^1\phi_2^0 |^2 \right) + \rho_3 \left( |\phi_0^1\phi_1^1 + \phi_0^1\phi_2^2 + \text{h.c.} | \right) \\
+ \rho_4 \left[ |\phi_1^0\phi_1^0 (\chi_1\chi_2 + \chi_2\chi_1) + \phi_0^0\phi_2^0 (\chi_1^2 - \chi_2^2) + \text{h.c.} | \right) \\
+ \epsilon_1 (|\phi_1^0\phi_1 + \phi_2^0\phi_2 |^2 + \epsilon_2 (|\phi_1^1\phi_1^0 + \phi_2^1\phi_2^0 |^2 |\chi_1\chi_2 + \chi_2\chi_1 | + (|\phi_1^1\phi_1 + \phi_2^2\phi_2 |^2 + \epsilon_4 |\phi_1^1\phi_1 + \phi_2^1\phi_2 |^2 + \epsilon_5 |\phi_1^1\phi_2 + \phi_2^1\phi_1 |^2 \\
+ \epsilon_6 \left( |\phi_1^0\chi_2 + \phi_2^0\chi_2 |^2 + |\phi_1^1\phi_1 + \phi_2^1\chi_2 |^2 \right) \\
+ k_1 (|\xi_1^1\chi_1 + \xi_2^1\chi_2 + \xi_3^1\chi_3 | (\chi_1^2 + \chi_2^2) + k_2 \left[ \sqrt{3} (|\xi_1^2\xi_2 - \xi_3^1\xi_3 | (\chi_1\chi_2 + \chi_2\chi_1) + (|\xi_1^2\xi_2 + \xi_3^2\chi_1 - 2\xi_1^1\xi_1 | (\chi_1^2 - \chi_2^2) \\
+ k_3 \left( 4|\xi_1\chi_1 |^2 + |\sqrt{3}\xi_2\chi_1 + \xi_3\chi_2 |^2 + |\sqrt{3}\xi_3\chi_1 - \xi_3\chi_2 |^2 \right) + k_4 \left( 4|\xi_1\chi_1 |^2 + |\sqrt{3}\xi_2\chi_2 - \xi_2\chi_1 |^2 + |\sqrt{3}\xi_3\chi_2 + \xi_3\chi_1 |^2 \right) \\
+ k_5 \left( \sqrt{3} (|\xi_1^2\xi_2 - \xi_3^1\xi_3 | \chi_1 + (-2\xi_1^1\xi_1 + \xi_2^1\xi_2 + \xi_3^1\chi_3 | (\chi_1^2 + \chi_2^2) \right) \right) \right) \right) \right) \right) \right).
\]

Compared to the Higgs potential in Ref.\([20]\), there are in total 21 more parameters. So we are confident to arrive at the suitable minima of the Higgs potential, and the VEV structure described in the previous analysis can be easily satisfied. Furthermore, we did not discuss in detail the Higgs spectrum, which may result in the flavor
changing neutral currents as well as lepton flavor violating problems. However, such problems commonly occur in models with more than one Higgs doublets, and can be ignored if the flavor changing Higgs are all heavier than a few TeV.

III. NUMERICAL ILLUSTRATIONS

We proceed to the numerical illustrations. The input values for the neutrino parameters are taken from Ref. [50]. For example, in the normal hierarchy case, the mass-squared differences measured in atmospheric and solar neutrino experiments read

\[
\Delta m^2_{21} = (7.12 \sim 8.13) \times 10^{-5} \text{ eV}^2, \quad (38)
\]
\[
\Delta m^2_{31} = (2.18 \sim 2.73) \times 10^{-3} \text{ eV}^2, \quad (39)
\]
while the allowed ranges of three mixing angles are

\[
\sin^2 \theta_{12} = 0.27 \sim 0.37, \quad (40)
\]
\[
\sin^2 \theta_{23} = 0.39 \sim 0.64, \quad (41)
\]
\[
\sin^2 \theta_{13} < 0.04, \quad (42)
\]

FIG. 1: The allowed parameter regions in the \( r_1 - r_2 \) plane (upper plot) and the \( \epsilon_1 - \epsilon_2 \) plane (lower plot).

The allowed regions of the ratio \( M_1/M_2 \) with respect to \( \epsilon_1 \).

Note that, there are slightly differences between the fitted parameters in the inverted and normal hierarchies. In the normal hierarchy case, the above mass-squared differences correspond to the allowed range of the mass ratio \( 5.4 < m_3/m_2 < 5.8 \).

In our numerical analysis, we do not make any assumptions on the model parameters, and randomly choose the values of \( r_i, \epsilon_i \) and \( m_0 \). The predicted neutrino mixing angles and masses (in the normal hierarchy case) are then compared with Eqs. (38)-(42), while the allowed parameter spaces of \( r_i \) and \( \epsilon_i \) are shown in Fig. 1. From the upper plot, one observes that the allowed regions of \( r_1 \) and \( r_2 \) are symmetric, which is actually resulted from the \( \nu - \tau \) symmetry in the neutrino mass matrix. In addition, none of \( r_1 \) or \( r_2 \) can be zero, while \( r_1 \approx r_2 \approx 2 \) is quite favored according to the numerical results. In the lower plot, \( \epsilon_1 = 0 \) is allowed but \( \epsilon_2 = 0 \) is not, indicating that \( \chi \) is required in order to fit the experimental data. Furthermore, for a fixed value of \( \epsilon_1 \), there are two allowed regions for \( \epsilon_2 \) corresponding to \( \epsilon_2 > 1 \) and \( \epsilon_2 < 1 \), respectively.

Since the right-handed neutrino masses are also correlated to \( r_i \), we present in Fig. 2 the predicted mass ratio between two right-handed neutrinos. One reads from the figure that the mass ratio is generally larger than 2 showing that the resonant leptogenesis mechanism [51] may not simply apply to this model. \(^2\)

Now we turn to the special case with the assumption \( r_1 = r_2 = 2 \), namely, a \( \mu - \tau \) symmetry exists in the neutrino sector. The allowed parameter regions of \( \theta_{ij} \) and \( M_1/M_2 \) are illustrated in Fig. 3. As we expected, there exist strong correlations between three mixing angles according to the upper and middle plots. This is

\(^2\) The right-handed neutrinos are degenerate in Ref. [20] since their masses are originated from a bare Majorana mass term, whereas in our model, due to contributions from \( \chi \), a mass splitting between \( M_1 \) and \( M_2 \) is included.
in good agreement with our analytical results aforementioned since the mixing angles are connected by a single parameter \( \theta \). The most severe constraint comes from \( \theta_{23} \), and its experimental allowed range can be fulfilled. As for \( \theta_{13} \), an upper bound \( \theta_{13} \lesssim 5^\circ \) can be obtained. As has been shown, \( \theta_{12} \) is confined to it’s tri-bimaximal mixing value, and rather stable compared to the two other mixing angles. In the particularly interesting limit \( \epsilon_1 = 0 \), the exact tri-bimaximal mixing pattern will be reproduced. Finally, from the lower plot, we also find that the right-handed neutrino mass spectrum should be hierarchical, e.g., \( M_1/M_2 \sim 3 \).

We stress that our discussions are based on the assumption of real Yukawa couplings as well as scalar VEVs, whereas in the most general situation, both of them could be complex. In the presence of CP-violating effects, the imaginary parts of the model parameters \( A, B, C \) and \( X \) could significantly change the predictions addressed here, and we therefore study it further. Since the neutrinoless double beta decay process rely on the Majorana feature of light neutrinos, we illustrate in Fig. 4 the allowed ranges of the effective mass, i.e., \( m_{ee} = \sum m_i U_{ei}^2 \), and \( \theta_{13} \) in the presence of CP violation. The model parameters are the same as those in Fig. 3 except that we allow them to be complex. One observes from the plot that any value of \( \theta_{13} \) satisfying the current experimental constraint can be achieved, whereas there exist strong constraints on \( |m_{ee}| \), in particular for a smaller \( \theta_{13} \), indicating potentially attractive signatures in future non-oscillation experiments.

![FIG. 3: The allowed parameter regions of \( \theta_{13} \) and \( M_1/M_2 \).](image)

![FIG. 4: The allowed parameter regions of \( |m_{ee}| \) and \( \theta_{13} \).](image)

**IV. CONCLUSION**

In this work, we presented a minimal seesaw model based on the discrete \( S_4 \) flavor symmetry. In our model, besides the SM fermion content, two right-handed neutrinos are introduced transforming as an \( S_4 \) doublet. The structure of the model is minimal in the sense that there are at most two massive light neutrinos which are indeed required to account for the observed solar and atmospheric neutrino oscillations. The number of model parameters are reduced greatly compared to the ordinary type-I seesaw, and thus allow us to make useful predictions on the neutrino parameters. After carefully exploring the parameter spaces, we found that the inverted neutrino mass hierarchy is ruled out whereas the nor-
nual hierarchy can be well accommodated in this framework. In particular, the tri-bimaximal mixing pattern can be naturally obtained from simple assumptions on the model parameters, while the deviation of three mixing angles from their exact tri-bimaximal mixing values are correlated by a single model parameter. In addition, the right-handed neutrinos feature a hierarchical mass spectrum, i.e., the ratio between right-handed neutrino masses is generally larger than 2.

Note that, in the current discussions, we have ignored the CP-violating effects, since there is yet no direct experimental information on leptonic CP violation. However, in the most general case, the CP-violating phases can be easily included since all the coefficients of Yukawa couplings as well as the VEVs could in principle be complex. In fact, the CP-violating effects are very crucial in order to explain the baryon asymmetry of the Universe via thermal leptogenesis mechanism in the seesaw model [22]. In addition, a dirac CP-violating phase may also be searched for at future long-baseline neutrino oscillation experiments.

Finally, we stress that the right-handed neutrinos may not be necessarily heavy, e.g., their masses could be located around keV scales. One may wonder that, in the mass range $M_i \sim \text{keV}$ (i.e., the right-handed neutrinos are sterilized), if the right-handed neutrinos could be viewed as warm dark matter so as to explain simultaneously the neutrino mass generation and the dark matter puzzle. Unfortunately, this is not possible in the current model, since the stability of keV right-handed neutrinos on the cosmic time scale requires the mixing between sterile and active neutrinos to be smaller than $10^{-4}$, which leads the mass scale of light neutrinos to be about $10^{-5} \text{ eV}$ [53]. Such tiny neutrino masses are in conflict with neutrino oscillation experiments. Possible variations extending the MSM may provide successful warm dark matter candidate, (e.g., an additional light right-handed neutrino transforming as a singlet under $S_4$), which are however beyond the scope of current work.

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