Reflection identities of harmonic sums of weight four

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We consider the reflection identities for harmonic sums at weight four. We decompose a product of two harmonic sums with mixed pole structure into a linear combination of terms each having a pole at either negative or positive values of the argument. The pole decomposition demonstrates how the product of two simpler harmonic sums can build more complicated harmonic sums at higher weight. We list a minimal irreducible bilinear set of reflection identities at weight four which present the main result of the paper. We also discuss how other trilinear and quartic reflection identities can be easily constructed from our result with the use of well known shuffle relations for harmonic sums.

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This paper is dedicated to memory of Lev Lipatov
I. INTRODUCTION

In this paper we continue discussion of our previous study\(^1\) regarding the reflection identities of harmonic sums, where a product of two harmonic sums of argument \(z\) and \(-1 - z\) is expressed through a linear combination of other harmonic sums of the same arguments, i.e.

\[
S_{a_1,a_2,...}(z)S_{b_1,b_2,...}(-1 - z) = S_{c_1,c_2,...}(z) + ... + S_{d_1,d_2,...}(-1 - z) + ...
\]  

(1)

The reflection identities at weight two presented here are not new and were known long time ago in the context of functions related to the Euler Gamma function. To the best of our knowledge they appear the earliest in Chapter 20 of the book by Nielsen\(^2\) and then were related to the harmonic sums (see eqs.6.11-6.15 of the paper by Blumlein\(^3\)). At weight three they were recently calculated by the author\(^1\). This paper deals with weight four.

The harmonic sums have pole singularities at negative integers. The reflection identities present a pole separation for a product of two sums with mixed pole structure. We call those functional relations the reflection identities because the argument of the harmonic sums is reflected with respect to the point \(z + (-1 - z) = -\frac{1}{2}\). The reflection identities up to weight three were published in our previous study\(^1\) and here we present them at weight four.

The harmonic sums are defined through a nested summation with their argument being the upper limit in the outermost sum\(^4-7\)

\[
S_{a_1,a_2,...,a_k}(n) = \sum_{n \geq i_1 \geq i_2 \geq ... \geq i_k \geq 1} \frac{\text{sign}(a_1)^{i_1}}{i_1^{\lfloor a_1 \rfloor}} ... \frac{\text{sign}(a_k)^{i_k}}{i_k^{\lfloor a_k \rfloor}}
\]  

(2)

In this paper we consider the harmonic sums with only real integer values of \(a_i\), which build the alphabet of the possible negative and positive indices. In Eq. (2) \(k\) is the depth and \(w = \sum_{i=1}^{k} |a_i|\) is the weight of the harmonic sum \(S_{a_1,a_2,...,a_k}(n)\).

The indices of harmonic sums \(a_1, a_2, ..., a_k\) can be either positive or negative integers and label uniquely \(S_{a_1,a_2,...,a_k}(n)\) for any given weight. However there is no unique way of building the functional basis for a given weight because the harmonic sums are subject to so called shuffle relations, where a linear combination of \(S_{a_1,a_2,...,a_k}(n)\) with the same argument but all possible permutations of indices can be expressed through a non-linear combinations of harmonic sums at lower weight. There is also some freedom in choosing the irreducible minimal set of \(S_{a_1,a_2,...,a_k}(n)\) that builds those non-linear combinations. The shuffle relations
make a connection between the linear and non-linear combinations of the harmonic sums of the same argument. For example, the shuffle relation at depth two reads

\[ S_{a,b}(z) + S_{b,a}(z) = S_{a}(z)S_{b}(z) + S_{\text{sign}(a)\text{sign}(b)(|a|+|b|)}(z) \] (3)

The shuffle relations of the harmonic sums is closely connected to the shuffle algebra of the harmonic polylogarithms\(^7\).

There is another type of identity called the duplication identities where a combination of harmonic sums of argument \(n\) can be expressed through a harmonic sum of the argument \(2n\). The duplication identities introduce another freedom in choosing the functional basis.

In this paper we consider the analytic continuation of the harmonic sums to from positive integer values of the argument to the complex plane denoted by \(\bar{S}_{a_1,a_2,...}(z)\) (this notation was introduced by Kotikov and Velizhanin\(^8\)). The analytic continuation is done in terms of the Mellin transform of corresponding Harmonic Polylogarithms and was recently used by Gromov, Levkovich-Maslyuk and Sizov\(^9,10\) and Caron Huot and Herraren\(^11\) for expressing the eigenvalue Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation using the principle of Maximal Transcedentality\(^12\) in super Yang-Mills \(\mathcal{N} = 4\) field theory. We plan to use their results together with analysis done by one of the authors and collaborators\(^16,17\) to understand the general structure of the BFKL equation in QCD and beyond.

The Mellin transform allows to make the analytic continuation to the complex plane. For example, consider the harmonic sum

\[ S_{-1}(z) = \sum_{k=1}^{z} \frac{(-1)^k}{k} \] (4)

The corresponding Mellin transform reads

\[ \int_0^1 \frac{1}{1+x} x^z = (-1)^z (S_{-1}(z) + \ln 2) \] (5)

One can see that \(S_{-1}(z)\) on its own is not an analytic function because of the term \((-1)^z\) and we impose that we start from even integer values of the argument \(z\). In this case we define its analytic continuation from even positive integers to all positive integers through

\[ \bar{S}_{a_1,a_2,...}(z) = (-1)^z S_{-1}(z) + ((-1)^z - 1) \ln 2 \] (6)

and thus we can write

\[ \bar{S}_{a_1,a_2,...}(z) = \int_0^1 \frac{1}{1+x} x^z - \ln 2 \] (7)
This way we defined $S_{a_1,a_2,...}^+(z)$ using the Mellin transform of ratio function $\frac{1}{1+z}$. In more complicated cases of other harmonic sums one includes also Harmonic Polylogarithms on top of the ratio functions, but the general procedure is very similar and largely covered in a number of publications\cite{3,6,13-15}.

It is worth mentioning that there is another analytic continuation for the harmonic sum, from odd positive integer values of the argument, which is different for harmonic sums with at least one negative index and denoted by $\tilde{S}_{a_1,a_2,...}^-(z)$. Both analytic continuations are equally valid. Our goal is to find a closed expression of the BFKL eigenvalue for all possible values of anomalous dimension and conformal spin, so that we follow the notation of Gromov, Levkovich-Maslyuk and Sizov\cite{9}, and use $\tilde{S}_{a_1,a_2,...}^+(z)$ throughout the text. For simplicity of presentation in this paper we write everywhere $S_{a_1,a_2,...}(z)$ instead of $\tilde{S}_{a_1,a_2,...}^+(z)$.

As it was already mentioned there is no unique way in defining a minimal irreducible set of harmonic sums due to the functional relations between them. For example, one can use the shuffle relations and them the minimal irreducible basis would include quadratic terms $S_{a_1}(z)S_{a_2}(z)$ in place of either $S_{a_1,a_2}(z)$ or $S_{a_2,a_1}(z)$. It is convenient to use shuffle relations to remove from the minimal basis the harmonic sums with the first index being equal 1, because those are divergent as $z \to \infty$. Then, the remaining harmonic sums give transcendental constants at $z \to \infty$. Most of constants are reducible and one is free to choose an irreducible set of transcendental constants at any given weight. We use the set implemented in the HarmonicSums package. The irreducible constants are given by

$$C_1 = \{\log(2)\}$$

and

$$C_2 = \{\pi^2, \log^2(2)\}$$

and

$$C_3 = \{\pi^2 \log(2), \log^3(2), \zeta_3\}$$

as well as

$$C_4 = \{\pi^4, \pi^2 \log^2(2), \log^4(2), \text{Li}_4\left(\frac{1}{2}\right), \zeta_3 \log(2)\}$$

where $C_w$ stands for a minimal set of irreducible constants at given weight $w$. There is only one of those at $w = 1$, two at $w = 2$, three at $w = 3$ and five irreducible constants at weight $w = 4$. 

4
We choose to use a linear minimal set of the harmonic sums to represent our results. In this set we do not apply shuffle relations and thus all the terms of the basis are linear in $S_{a_1,a_2,...}(z)$. This choice is dictated mostly by a convenience and was also used by Caron Huot and Herraren\[11] on which we would like to rely in our future calculations. The minimal linear set of harmonic sums we use is as follows

$$B_1 = \{S_{-1}, S_1\}$$

and

$$B_2 = \{S_{-2}, S_2, S_{-1,1}, S_{1,-1}, S_{1,1}, S_{-1,-1}\}$$

and

$$B_3 = \{S_{-3}, S_3, S_{-2,1}, S_{-2,1}, S_{2,-1}, S_{2,1}, S_{-1,1,-1}, S_{-1,1,1}, S_{1,-2}, S_{1,2}, S_{1,-1,-1}, S_{1,-1,1},$$

$$S_{1,1,-1}, S_{1,1,1}, S_{-1,-2}, S_{-1,2}, S_{-1,-1,-1}, S_{-1,-1,1}\}$$

as well as

$$B_4 = \{S_{-4}, S_4, S_{-3,-1}, S_{-3,1}, S_{2,-2}, S_{2,1}, S_{3,-1}, S_{3,1}, S_{-2,-1,-1}, S_{-2,-1,1}, S_{-2,1,-1}, S_{-2,1,1},$$

$$S_{2,-1,-1}, S_{2,-1,1}, S_{2,1,-1}, S_{2,1,1}, S_{1,-1,-1,-1}, S_{1,-1,1,-1}, S_{1,-1,1,1}, S_{1,1,-3}, S_{1,3}, S_{1,-2,-1},$$

$$S_{1,-2,1}, S_{1,-1,-2}, S_{1,-1,2}, S_{1,1,-2}, S_{1,1,2}, S_{1,-2,-1}, S_{1,2,1}, S_{1,-1,-1,-1}, S_{1,-1,-1,1}, S_{1,-1,1,1},$$

$$S_{1,-1,1,1}, S_{1,1,-1,-1}, S_{1,1,1,-1}, S_{1,1,1,1}, S_{1,-1,3}, S_{1,-3}, S_{1,-2,-1}, S_{1,-2,1},$$

$$S_{-1,-1,-2}, S_{-1,1,-2}, S_{-1,1,2}, S_{-1,2,-1}, S_{1,2,1}, S_{-1,1,-1,-1}, S_{-1,1,-1,1}, S_{-1,1,1,-1}, S_{-1,1,1,1}, S_{-1,-1,-1,-1}, S_{-1,-1,1,-1}, S_{-1,1,-1,1}, S_{-1,1,-1,1,1}, S_{-1,1,1,-1}, S_{-1,1,1,1}, S_{-1,2,2}, S_{2,2}\}$$

A comprehensive discussion on harmonic sums, irreducible constants, functional identities and possible choice of the minimal set of functions at given weight is presented by J. Ablinger\[13]. In this paper we focus only at the reflection identities for harmonic sums at weight $w = 4$ analytically continued from even positive points to complex plane. In the next Section we discuss them in more details along the method we use in our calculations.

**II. REFLECTION IDENTITIES**

The reflection identities at weight $w = 4$ are obtained by taking a product of harmonic sums of argument $z$ at weight $w = 1$ and harmonic sums of argument $-1 - z$ at weight
$w = 3$, i.e. $B_1 \otimes B_3$, and also by taking a product of harmonic sums of argument $z$ and $-1 - z$ at weight $w = 2$, i.e. $B_2 \otimes \bar{B}_2$.

The number of basis harmonic sums in $B_1$, $B_2$ and $B_3$ is given by

$$\text{Length}(B_1) = 2, \quad \text{Length}(B_2) = 6, \quad \text{Length}(B_3) = 18$$

(16)

so that the number of elements in the products $B_1 \otimes \bar{B}_3$ and $B_2 \otimes \bar{B}_2$ reads

$$B_1 \otimes \bar{B}_3 = 2 \times 18 = 36$$

(17)

and

$$B_2 \otimes \bar{B}_2 = \frac{6 \times (6 - 1)}{2} + 6 = 21,$$

(18)

resulting in the total number of irreducible reflections identities at weight $w = 4$ being equal to $21 + 36 = 57$.

In order to calculate the reflection identities at weight $w = 4$ we use the basis harmonic sums at $w = 4$ listed in Eq. (15) together with basis harmonic sums at lower weight listed in the work of J. Ablinger (12)-(14) multiplied by irreducible constants at corresponding weight listed in Eqs. (8)-(10). This should be supplemented by irreducible constants at weight $w = 4$ listed in Eqs. (11). The number of basis sums in $B_4$ equals

$$\text{Length}(B_4) = 54$$

(19)

so that the total number of terms in the expansion ansatz at $w = 4$

$$B_4 + B_3 \otimes C_1 + B_2 \otimes C_2 + B_1 \otimes C_3 + C_4$$

(20)

is given by

$$\text{Length}(\text{ANZ}_4) = 54 + 18 \times 1 + 6 \times 2 + 2 \times 3 + 5 = 95$$

(21)

The full expansion ansatz at $w = 4$ is given by

$$\text{ANZ}_4 = \left\{ \pi^4, \pi^2 \log^2(2), \log^4(2), \text{Li}_4 \left( \frac{1}{2} \right), S_{-4}, S_{-3} \log(2), \pi^2 S_{-2}, S_{-2} \log^2(2), \pi^2 S_{-1} \log(2), S_{-1} \log^3(2), \pi^2 S_1 \log(2), S_1 \log^3(2), \pi^2 S_2, S_2 \log^2(2), S_3 \log(2), S_4, S_{-3,-1}, S_{-3,1}, S_{-2,-2}, \log(2) S_{-2,-1}, \log(2) S_{-2,1}, S_{-2,2}, S_{-1,-3}, \log(2) S_{-1,-2}, \pi^2 S_{-1,-1}, \log^2(2) S_{-1,-1}, \pi^2 S_{-1,1}, \log^2(2) S_{-1,1}, \log(2) S_{-1,2}, S_{-1,3}, S_{1,-3}, \log(2) S_{1,-2}, \pi^2 S_{1,-1}, \log^2(2) S_{1,-1}, \pi^2 S_{1,1}, \right\}$$
The expansion of the product of two functions of argument \( w \) and argument \( -1 - z \) we search in terms of two sets of ANZ, one of argument \( z \) and another one of argument \( -1 - z \). The total number of elements in this expression equals \( 95 \times 2 - 5 = 185 \), where we remove redundant five constants at \( w = 4 \) because they are the same for both arguments. We fix the 185 free coefficients using pole expansion of the product \( s_{a_1 a_2}(z)s_{b_1 b_2}(-1 - z) \) around negative integers of \( z = -5, \ldots, -1 \) and expanding to the second order of the expansion parameter. It turns out that to fix all 185 free coefficients we need only expansion up to first order and we use the second order of the expansion to double check our results. We checked our results listed in the Appendix by a direct numerical calculation at the accuracy \( 10^{-10} \).

Below we give two examples of reflection identities, \( S_1(z)S_{2,1}(-1 - z) \) for harmonic sums with positive indices and \( S_{-1}(z)S_{-2,-1}(-1 - z) \) for harmonic sums with negative indices. All 57 irreducible reflection identities at weight four are listed in the Appendix. The two examples are

\[
S_1(z)S_{2,1}(-1 - z) = \frac{6\zeta_2^2}{5} - S_2(z)\zeta_2 + S_2(-1 - z)\zeta_2 + 2S_1(z)\zeta_3 - 2\zeta_3S_1(-1 - z) + S_{3,1}(z)
- S_{3,1}(-1 - z) = S_{2,1,1}(z) + S_{1,2,1}(-1 - z) + S_{2,1,1}(-1 - z)
\]

and

\[
S_{-1}(z)S_{-2,-1}(-1 - z) = -\frac{\ln^4(2)}{6} - S_{-2}(z)\ln^2(2) + S_2(z)\ln^2(2) + \zeta_2\ln^2(2)
- S_{-2}(-1 - z)\ln^2(2) + S_2(-1 - z)\ln^2(2) - S_{-3}(z)\ln(2)
+ S_3(z)\ln(2) - 6\zeta_3\ln(2) - S_{-3}(-1 - z)\ln(2) + S_3(-1 - z)\ln(2)
- 2S_{-2,-1}(z)\ln(2) - 2S_{-2,-1}(-1 - z)\ln(2) + \frac{33\zeta_2^2}{20} - 4\text{Li}_3\left(\frac{1}{2}\right)
\]
\[\begin{align*}
+ \frac{1}{2} S_{-2}(z) \zeta_2 + \frac{1}{4} S_{-1}(z) \zeta_3 + \frac{1}{2} \zeta_2 S_{-2}(-1 - z) + \frac{1}{4} \zeta_3 S_{-1}(-1 - z) \\
+ S_{3,-1}(z) + S_{3,-1}(-1 - z) - S_{-2,-1,-1}(z) - S_{-2,-1,-1}(-1 - z) \\
- S_{-1,-2,-1}(-1 - z)
\end{align*}\]  

(24)

One can see that the reflection identities for harmonic sums with negative indices are more complicated than those with only positive indices and this happens mostly due to appearance of constant \(\ln(2)\), which originates from sign alternating summation in \(S_{-1}(z)\) absent for positive indices.

In the present paper we consider only bilinear reflection identities expressing a product of two harmonic sums of argument \(z\) and \(-1 - z\) in terms of a linear combination of other sums of the same arguments. One can consider also trilinear and quartic identities, but whose reducible and form a linear combination of the bilinear identities presented in this paper. For example, we can consider a trilinear term \(S_1(z) S_1(-1 - z) S_2(-1 - z)\) and write it as

\[S_1(z) S_1(-1 - z) S_2(-1 - z) = S_1(z) S_{1,2}(-1 - z) + S_1(z) S_{2,1}(-1 - z) - S_1(z) S_3(-1 - z)\]  

(25)

where we used a shuffle identity from Eq. (3)

\[S_{1,2}(z) + S_{2,1}(z) - S_3(z) = S_1(z) S_2(z)\]  

(26)

The expression for \(S_1(z) S_{1,2}(-1 - z)\) is given in Eq. (55) and for \(S_1(z) S_2(-1 - z)\) in Eq. (47). Plugging those together with Eq. (23) into Eq. (25) we get

\[S_1(z) S_1(-1 - z) S_2(-1 - z) = -\zeta_2 S_{1,1}(-1 - z) - \tilde{S}_{1,3} - S_{2,2}(-1 - z) - S_{3,1}(-1 - z) + 2 S_{1,2}(-1 - z) + S_{1,2,1}(-1 - z) + S_{2,1,1}(-1 - z) + 2 \zeta_2 S_2(-1 - z) + \zeta_3 S_1(-1 - z) + \zeta_2 S_{1,1}(z) + S_{3,1}(z) - S_{1,2,1}(z) - S_{2,1,1}(z) + \frac{4 \zeta_2^3}{3} - \zeta_2 S_2(z) + 2 \zeta_3 S_1(z)\]  

(27)

In a similar way one can can build any trilinear or quartic reflection identity using shuffle relations for harmonic sums and the bilinear reflection identities listed in the Appendix of this paper. All possible shuffle relations required for the present discussion are available in the HarmonicSums package by J. Ablinger. Shuffle relation before and after analytic continuation of the harmonic sums to the complex plane are the same.
III. CONCLUSIONS

We discuss the reflection identities for harmonic sums of weight four. There are 57 irreducible bilinear identities listed in the Appendix. All other bilinear reflection identities are easily obtained by a trivial change of argument $z \leftrightarrow -1 - z$. The trilinear and quartic identities for a product of three and four harmonic sums are obtained from the identities listed in the Appendix using the shuffle relations for harmonic sums. In our analysis we use the linear basis for harmonic sums and limit ourselves to harmonic sums analytically continued from even integer values of the argument to the complex plane. The analytic continuation from odd integers is beyond the scope of the present study.

In deriving the reflection identities presented in this paper we used Harmonic Sums package by J. Ablinger\textsuperscript{13}, HPL package by D. Maitre\textsuperscript{18} and dedicated Mathematica package for pomeron NNLO eigenvalue by N. Gromov, F. Levkovich-Maslyuk and G. Sizov\textsuperscript{9}.

We expanded around positive and negative integer points the product of two harmonic sums $S_{a_1,a_2,\ldots}(z)S_{b_1,b_2,\ldots}(-1 - z)$ and the functional basis built of pure Harmonic Sums with constants of relevant weight listed in Ref.\textsuperscript{13}. Then we compared the coefficients of the irreducible constants of a given weight and solved the resulting set of coefficient equations. We used higher order expansion to cross check our results. The bilinear reflection identities presented here are derived from the pole expansion based on the Mellin transform and then checked them against the shuffle identities and numerical calculations of the corresponding harmonic sums on the complex plane.

We attach a Mathematica notebook with our results.

IV. ACKNOWLEDGEMENTS

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A. Appendix

Here we list below the irreducible reflection identities at weight \( w = 4 \). In all our expressions we used the linear minimal set of harmonic sums given in Eqs. (12)-(15).

We use a compact notation where \( s_{a_1,a_2,...} \) stands for \( S_{a_1,a_2,...}(z) \) whereas \( \bar{s}_{a_1,a_2,...} \) stands for \( S_{a_1,a_2,...}(-1-z) \).

The constants are also written in a compact and readable way \( \ln 2 = \ln 2 \simeq 0.693147 \) and \( \text{LiHalf}_4 = \text{Li}_4 \left( \frac{1}{2} \right) \simeq 0.517479 \). For example, in this notation Eq. (23) is written as Eq. (51).

All other bilinear reflection identities are obtained by a trivial change of argument \( z \leftrightarrow -1 - z \).

### 1. Reflection identities originating from \( B_1 \otimes B_3 \)

\[
s_{-1} \bar{s}_{-3} = \frac{7\zeta_2^2}{5} - \frac{s_2 \zeta_2}{2} - \frac{1}{2} \bar{s}_2 \zeta_2 - \ln_2 s_{-3} + \ln_2 s_3 - \frac{3 \ln_2 \zeta_3}{2} - \frac{3}{4} s_{-1} \zeta_3
\]
\[-\ln_2 \bar{s}_{-3} - \frac{3}{4} \zeta_3 \bar{s}_{-1} - \ln_2 \bar{s}_3 - s_{-3,-1} - \bar{s}_{-1,-3}
\]

(28)

\[
s_{-1} \bar{s}_3 = -\frac{3\zeta_2^2}{5} + \frac{1}{2} s_{-2} \zeta_2 - \frac{1}{2} \bar{s}_{-2} \zeta_2 - \ln_2 s_{-3} + \ln_2 s_3 - \frac{3 \ln_2 \zeta_3}{2}
\]
\[-\frac{3}{4} s_{-1} \zeta_3 - \ln_2 \bar{s}_{-3} - \frac{3}{4} \zeta_3 \bar{s}_{-1} - \ln_2 \bar{s}_3 + s_{3,-1} - \bar{s}_{1,-3}
\]

(29)

\[
s_{-1} \bar{s}_{-2,-1} = -\frac{\ln_4^4}{6} - s_{-2} \ln_2^2 + s_2 \ln_2^2 + \zeta_2 \ln_2^2 - \bar{s}_{-2} \ln_2^2 + \bar{s}_2 \ln_2^2 - s_{-3} \ln_2 + s_3 \ln_2
\]
\[-6 \zeta_3 \ln_2 - \bar{s}_{-3} \ln_2 + \bar{s}_3 \ln_2 - 2 s_{-2,-1} \ln_2 - 2 \bar{s}_{-2,-1} \ln_2 + \frac{33 \zeta_2^2}{20} - 4 \text{LiHalf}_4
\]
\[+ \frac{1}{2} \zeta_3 \bar{s}_2 + \frac{1}{4} s_{-1} \zeta_3 + \frac{1}{4} \zeta_2 \bar{s}_{-2} + \frac{1}{4} \zeta_3 \bar{s}_{-1} + s_{3,-1} + \bar{s}_{3,-1} + s_{-2,-1,-1}
\]
\[-\bar{s}_{-2,-1,-1} - \bar{s}_{-1,-2,-1}
\]

(30)

\[
s_{-1} \bar{s}_{-2,1} = -\frac{\ln_4^4}{12} - \frac{1}{2} \bar{s}_{-2} \ln_2^2 + \frac{1}{2} s_2 \ln_2^2 + \frac{1}{2} \zeta_2 \ln_2^2 - \frac{1}{2} \bar{s}_{-2} \ln_2^2 + \frac{1}{2} \bar{s}_2 \ln_2^2 - s_{-3} \ln_2 + s_3 \ln_2
\]
\[-\frac{1}{2} \bar{s}_{-1,-1} - \bar{s}_{-1,-2,-1}
\]

(31)
\[-3\zeta_3 \ln^2 - s_{-2,-1} \ln_2 + s_{-2,1} \ln_2 - \bar{s}_{-2,-1} \ln_2 - \bar{s}_{-2,1} \ln_2 + \frac{61\zeta_4^2}{40} - 2\text{LiHalf}_4 \]
\[-\frac{s_2\zeta_2}{2} - \frac{5}{8}s_{-1}\zeta_3 - \frac{1}{2}\zeta_2\bar{s}_{-2} - \frac{5}{8}\zeta_3\bar{s}_{-1} - \frac{1}{2}\zeta_2\bar{s}_2 - s_{-3,-1} + \bar{s}_{3,1} + s_{-2,1,-1} \]
\[-\bar{s}_{-2,-1,1} - \bar{s}_{-1,-2,1} \]  
(31)

\[s_{-1}\bar{s}_{2,-1} = -\frac{\ln^4}{6} - s_{-2} \ln^2 + s_2 \ln^2 + \zeta_2 \ln^2 + \frac{1}{2} \bar{s}_{-2} \ln^2 - \bar{s}_2 \ln^2 - s_{-3} \ln_2 \]
\[+ s_3 \ln_2 + \bar{s}_{-3} \ln_2 - \bar{s}_3 \ln_2 + 2s_{-2,-1} \ln_2 - 2\bar{s}_{2,-1} \ln_2 + \frac{5\zeta_2^2}{4} \]
\[-4\text{LiHalf}_4 - \frac{s_2\zeta_2}{2} + \frac{1}{4}s_{-1}\zeta_3 + \frac{1}{4}\zeta_3\bar{s}_{-1} + \frac{1}{2}\zeta_2\bar{s}_2 - s_{-3,-1} \]
\[+\bar{s}_{-3,-1} + s_{2,-1,-1} - \bar{s}_{-1,2,-1} - \bar{s}_{2,-1,-1} \]  
(32)

\[s_{-1}\bar{s}_{2,1} = \frac{\ln^4}{6} - \frac{1}{2} s_{-2} \ln^2 + \frac{1}{2} s_2 \ln^2 - \zeta_2 \ln^2 + \frac{1}{2} \bar{s}_{-2} \ln^2 - \frac{1}{2} \bar{s}_2 \ln^2 \]
\[-s_{-3} \ln_2 + s_3 \ln_2 - \frac{9\zeta_3 \ln_2}{4} + s_{-2,1} \ln_2 - s_{2,1} \ln_2 - s_{-2,-1} \ln_2 - s_{2,1} \ln_2 - \frac{3\zeta_2^2}{4} \]
\[+4\text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 - \frac{5}{8}s_{-1}\zeta_3 - \frac{1}{2}\zeta_2\bar{s}_{-2} - \frac{5}{8}\zeta_3\bar{s}_{-1} - \frac{1}{2}\zeta_2\bar{s}_2 \]
\[+s_{3,+1} - \bar{s}_{-3,1} - s_{2,1,-1} - \bar{s}_{-1,2,1} - \bar{s}_{2,-1,1} \]  
(33)

\[s_{-1}\bar{s}_{-1,1,-1} = -\frac{7\ln^4}{24} - s_{-2} \ln^2 + s_2 \ln^2 + \frac{9}{4}\zeta_2 \ln^2 - \frac{1}{2} \bar{s}_{-2} \ln^2 + \frac{1}{2} \bar{s}_2 \ln^2 \]
\[-\frac{1}{2}\bar{s}_{-1,-1} \ln^2 + \frac{3}{2}\bar{s}_{-1,1} \ln^2 + \frac{1}{2} \bar{s}_{-1,-1} \ln_2 - \frac{1}{2} \bar{s}_{-1,1} \ln_2 - s_{-3} \ln_2 + s_3 \ln_2 - 3\zeta_3 \ln_2 \]
\[+\frac{3}{2}\zeta_2\bar{s}_{-1} \ln_2 - 2s_{-2,-1} \ln_2 - s_{-1,-2} \ln_2 + s_{-1,2} \ln_2 + \bar{s}_{-1,-2} \ln_2 \]
\[-\bar{s}_{-1,2} \ln_2 + 2s_{-1,1,-1} \ln_2 - 2\bar{s}_{-1,1,-1} \ln_2 + \frac{13\zeta_2^2}{40} - \text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 \]
\[+\frac{1}{4}s_{-1}\zeta_3 - \frac{1}{8}\zeta_3\bar{s}_{-1} - \frac{1}{2}\zeta_2\bar{s}_{-1,1} + s_{3,-1} + \frac{1}{2}\zeta_2\bar{s}_{-1,1} - s_{-2,-1,-1} \]
\[-s_{-1,-2,-1} + \bar{s}_{-1,-2,-1} + \bar{s}_{2,1,-1} + s_{-1,1,-1,-1} - 2\bar{s}_{-1,1,-1,-1} - \bar{s}_{-1,1,-1,-1} \]  
(34)

\[s_{-1}\bar{s}_{-1,1,1} = -\frac{\ln^4}{6} - \frac{1}{3}s_{-1} \ln^2 - \frac{1}{2}s_{-2} \ln^2 + \frac{3}{4}\zeta_2 \ln^2 - \frac{1}{2}s_{-1,-1} \ln^2 \]
\[+\frac{1}{2}s_{-1,1} \ln^2 + \frac{1}{2}\bar{s}_{-1,-1} \ln_2 - \frac{1}{2} \bar{s}_{-1,1} \ln_2 - s_{-3} \ln_2 + s_3 \ln_2 + s_{-1}\zeta_2 \ln_2 \]
\[
\begin{align*}
-\frac{7\zeta_3 \ln 2}{8} & - s_{-2,-1} \ln 2 + s_{-2,1} \ln 2 - s_{-1,-2} \ln 2 + s_{-1,2} \ln 2 + s_{-1,1,-1} \ln 2 \\
- s_{-1,1,1} \ln 2 - s_{-1,1,-1} \ln 2 - s_{-1,1,1} \ln 2 & - \frac{27\zeta_3^2}{40} + 4\text{LiH}_4 - \frac{s_2 \zeta_2}{2} \\
\frac{9}{8}s_{-1}\zeta_3 & - \zeta_3 s_{-1} - s_{-3,-1} + \frac{1}{2}\zeta_2 s_{-1,-1} - \frac{1}{2}\zeta_2 s_{-1,-1} - \frac{1}{2}\zeta_2 s_{-1,1} \\
+ s_{-2,1,-1} + s_{-1,2,-1} + s_{-1,-2,1} + s_{2,1,1} - s_{-1,1,1,-1} & - 2s_{-1,1,1,-1} - s_{-1,1,-1,1}
\end{align*}
\]

(35)

\[
\begin{align*}
 s_{-1} s_{1,-2} &= -\frac{\ln^4}{6} + \frac{3}{2}\zeta_2 \ln^2 - s_{-3} \ln 2 + s_3 \ln 2 + \frac{1}{2}s_{-1}\zeta_2 \ln 2 - \frac{1}{2}s_1\zeta_2 \ln 2 - \frac{15\zeta_3 \ln 2}{4} \\
&+ \frac{1}{2}\zeta_2 s_{-1} \ln 2 - \frac{1}{2}\zeta_2 s_1 \ln 2 + s_{1,-2} \ln 2 - s_{-1,2} \ln 2 - s_{1,-2} \ln 2 - 2s_{-1,1} \ln 2 + 2\zeta_2^2 \\
&- 4\text{LiH}_4 - \frac{s_2 \zeta_2}{2} - \frac{1}{8}s_{-1}\zeta_3 + \frac{5s_1\zeta_3}{8} + \frac{1}{2}\zeta_2 s_{-2} - \frac{1}{8}\zeta_3 s_{-1} + \frac{13}{8}\zeta_3 s_{1} \\
&- \frac{1}{2}\zeta_2 s_{2} - s_{-3,-1} - \frac{1}{2}\zeta_2 s_{1,-1} + s_{-2,-2} - \frac{1}{2}\zeta_2 s_{1,-1} + s_{1,-2,-1} \\
&- s_{-1,1,-2} - s_{1,-1,-2}
\end{align*}
\]

(36)

\[
\begin{align*}
 s_{-1} s_{1,2} &= -\frac{\ln^4}{12} + \zeta_2 \ln^2 - s_{-3} \ln 2 + s_3 \ln 2 + \frac{1}{2}s_{-1}\zeta_2 \ln 2 - \frac{1}{2}s_1\zeta_2 \ln 2 \\
&- \frac{15\zeta_3 \ln 2}{4} + \frac{1}{2}\zeta_2 s_{-1} \ln 2 - \frac{1}{2}\zeta_2 s_1 \ln 2 + s_{1,-2} \ln 2 - s_{1,2} \ln 2 \\
&- s_{1,-2} \ln 2 - s_{1,2} \ln 2 + \frac{7\zeta_3^2}{8} - 2\text{LiH}_4 + \frac{1}{2}s_{-2}\zeta_2 - s_{-1}\zeta_3 \\
&+ \frac{s_1\zeta_3}{4} - \zeta_3 s_{-1} - \zeta_3 s_{1} - \frac{1}{2}\zeta_2 s_{1,-1} + s_{3,-1} + s_{-2,2} \\
&- \frac{1}{2}\zeta_2 s_{1,-1} - s_{1,2,-1} - s_{-1,1,2} - s_{1,-1,2}
\end{align*}
\]

(37)

\[
\begin{align*}
 s_{-1} s_{1,-1,-1} &= -\frac{\ln^4}{8} - s_{-1,1} \ln^2 + s_{-1} \ln^2 + \frac{7}{6}s_1 \ln^2 - s_{-3} \ln^2 - s_{-1,2} \ln^2 \\
&+ s_2 \ln^2 - \frac{1}{2}\zeta_2 \ln^2 - s_{-2} \ln^2 + s_2 \ln^2 + 2s_{1,-1} \ln^2 - s_{-3} \ln^2 \\
&+ s_3 \ln^2 - s_1 \ln^2 + 3\zeta_3 \ln^2 - s_{1,-2,1} \ln^2 + s_{1,-2} \ln^2 - s_{1,2} \ln^2 \\
&- s_{1,-2,1} \ln^2 + s_{1,2} \ln^2 + 2s_{1,-1,1} \ln^2 - 2s_{1,-1,1} \ln^2 + 2s_1 \ln^2 - 3\zeta_3^2 \\
&+ 3\text{LiH}_4 + \frac{1}{2}s_{-2}\zeta_2 - \frac{1}{4}s_{-1}\zeta_3 + \frac{s_1\zeta_3}{2} - \frac{1}{4}\zeta_3 s_{-1} - \frac{3}{4}\zeta_3 s_{1} \\
&- \frac{1}{2}\zeta_2 s_{1,-1} + s_{3,-1} + \frac{1}{2}\zeta_2 s_{1,-1} - s_{-2,-1,-1} - s_{1,2,-1} \\
&- s_{-2,-1,1} + s_{1,2,-1} + s_{-1,-1,-1} - s_{-1,1,-1,-1} - 2s_{1,-1,-1,-1}
\end{align*}
\]

(38)
\[ s_{-1} \tilde{s}_{1,1,1} = -\frac{\ln^2 2}{12} - \frac{1}{6} s_{-1} \ln^2 2 + \frac{5}{6} s_1 \ln^2 2 - \frac{1}{6} \tilde{s}_{-1} \ln^2 2 + \frac{1}{6} \tilde{s}_1 \ln^2 2 \\
- \frac{1}{2} \tilde{s}_{-2} \ln^2 2 + \frac{1}{2} s_2 \ln^2 2 - \frac{1}{2} \zeta_2 \ln^2 2 - \frac{1}{2} \tilde{s}_{-2} \ln^2 2 + \frac{1}{2} \tilde{s}_2 \ln^2 2 \\
+ s_{1,-1} \ln^2 2 - s_{-3} \ln 2 + s_3 \ln 2 - s_1 \zeta_2 \ln 2 - \frac{19 \zeta_3 \ln 2}{8} \\
- \frac{3}{2} \zeta_2 \tilde{s}_1 \ln 2 - s_{-2,-1} \ln 2 + s_{-2,1} \ln 2 + s_{1,-2} \ln 2 - s_{1,2} \ln 2 \\
+ s_{1,-1,-1} \ln 2 - s_{1,1,1} \ln 2 - \tilde{s}_{1,-1,-1} \ln 2 - s_{1,-1,1} \ln 2 + \frac{49 \zeta_2^4}{20} \\
- 6 \text{LiHalf}_4 - \frac{s_2 \zeta_2}{2} + \frac{1}{8} s_{-1} \zeta_3 + \frac{3 s_1 \zeta_3}{4} + \frac{1}{2} \zeta_2 \tilde{s}_2 + \frac{1}{8} \zeta_3 \tilde{s}_{-1} \\
+ \frac{3}{2} \zeta_3 \tilde{s}_1 - \frac{1}{2} \zeta_2 \tilde{s}_2 - s_{-3,-1} - \frac{1}{2} \zeta_2 \tilde{s}_{1,-1} - \zeta_2 \tilde{s}_{1,-1} + s_{-2,1,-1} \\
+ s_{1,-2,-1} + \tilde{s}_{-2,-1} + \tilde{s}_{1,2,-1} - s_{1,-1,1,-1} \\
- \tilde{s}_{-1,1,1,-1} - 2 \tilde{s}_{1,-1,1,1} \tag{39} \]

\[ s_{-1} \tilde{s}_{1,1,1} = -\frac{\ln^2 2}{3} - \frac{1}{6} s_{-1} \ln^2 2 - \frac{1}{6} s_1 \ln^2 2 - \frac{1}{6} \tilde{s}_{-1} \ln^2 2 + \frac{1}{6} \tilde{s}_1 \ln^2 2 - s_{-2} \ln^2 2 \\
+ s_2 \ln^2 2 + \frac{11}{4} \zeta_2 \ln^2 2 + \frac{1}{2} \tilde{s}_2 \ln^2 2 - \frac{1}{2} \tilde{s}_2 \ln^2 2 + \frac{1}{2} s_{1,-1} \ln^2 2 \\
- \frac{3}{2} s_{1,1} \ln^2 2 + \frac{1}{2} s_{1,-1} \ln^2 2 - \frac{1}{2} \tilde{s}_{1,1} \ln^2 2 - s_{-3} \ln 2 + s_3 \ln 2 + \frac{1}{2} s_{-1} \zeta_2 \ln 2 \\
+ \frac{1}{2} s_1 \zeta_2 \ln 2 - 4 \zeta_3 \ln 2 + \frac{1}{2} \zeta_2 \tilde{s}_{-1} \ln 2 + \zeta_2 \tilde{s}_1 \ln 2 + s_{1,-2} \ln 2 - s_{1,2} \ln 2 \\
+ 2 s_{2,-1} \ln 2 + \tilde{s}_{1,-2} \ln 2 - \tilde{s}_{1,2} \ln 2 - 2 s_{1,1,-1} \ln 2 - 2 \tilde{s}_{1,1,-1} \ln 2 + \frac{8 \zeta_2^2}{5} \\
- 4 \text{LiHalf}_4 - \frac{s_2 \zeta_2}{2} - \frac{s_1 \zeta_3}{4} - \frac{1}{8} \zeta_3 \tilde{s}_1 - s_{-3,-1} + \frac{1}{2} \zeta_2 s_{1,1} + \frac{1}{2} \zeta_2 \tilde{s}_{1,1} \\
+ s_{1,-2,-1} + s_{2,-1,-1} + \tilde{s}_{-2,-1} + \tilde{s}_{1,-2,-1} - s_{1,1,-1,-1} \\
- \tilde{s}_{-1,1,1,-1} - \tilde{s}_{1,-1,1,-1} - \tilde{s}_{1,1,-1,1} \tag{40} \]

\[ s_{-1} \tilde{s}_{1,1,1} = -\frac{\ln^2 2}{24} - \frac{1}{6} s_{-1} \ln^2 2 + \frac{1}{6} s_1 \ln^2 2 - \frac{1}{6} \tilde{s}_{-1} \ln^2 2 + \frac{1}{6} \tilde{s}_1 \ln^2 2 - \frac{1}{2} s_{-2} \ln^2 2 \\
+ \frac{1}{2} s_2 \ln^2 2 + \frac{1}{4} \zeta_2 \ln^2 2 + \tilde{s}_{1,-1} \ln^2 2 - \frac{1}{2} s_{1,1} \ln^2 2 + \frac{1}{2} \tilde{s}_{1,-1} \ln^2 2 - \frac{1}{2} \tilde{s}_{1,1} \ln^2 2 \\
- s_{-3} \ln 2 + s_3 \ln 2 + \frac{1}{2} s_{-1} \zeta_2 \ln 2 - \frac{1}{2} \tilde{s}_1 \zeta_2 \ln 2 - \frac{1}{8} \zeta_2 \ln 2 + \frac{1}{2} \zeta_2 \tilde{s}_{-1} \ln 2 \\
- \frac{1}{2} \zeta_2 \tilde{s}_1 \ln 2 + s_{1,-2} \ln 2 - s_{1,2} \ln 2 + s_{2,-1} \ln 2 - s_{2,1} \ln 2 - s_{1,1,-1} \ln 2 \]
\[
+ s_{1,1,1} \ln_2 - \bar{s}_{1,1,1} \ln_2 - \bar{s}_{1,1,1} \ln_2 - \frac{11}{40} + \text{LiHalf}_4 + \frac{1}{2} s_{-2} \zeta_2
\]

\[
- \frac{7}{8} s_{-1} \zeta_3 + \frac{s_{1,3}}{4} - \frac{7}{8} \zeta_3 \bar{s}_{-1} - \frac{1}{8} \zeta_3 \bar{s}_1 - \frac{1}{2} \zeta_2 s_{1,1} - s_{3,1} - \frac{1}{2} \zeta_2 \bar{s}_{1,1} - \frac{1}{2} \zeta_2 \bar{s}_{1,1} - s_{1,2,1} - \bar{s}_{2,1,1} + \bar{s}_{-2,1,1} + \bar{s}_{1,2,1} + s_{1,1,1,1}
\]

\[- \bar{s}_{1,1,1,1} - s_{1,1,1,1} - \bar{s}_{1,1,1,1} \tag{41}\]

\[
s_{-1,1,2} = \zeta_2 \ln_2^2 - s_{-3} \ln_2 + s_3 \ln_2 + s_{-1} \zeta_2 \ln_2 + \frac{3 \zeta_3 \ln_2}{2} - s_{-1,2} \ln_2
\]

\[
+ s_{-1,2} \ln_2 - \bar{s}_{1,1,1} \ln_2 = \bar{s}_{-1,1,2} \ln_2 - \bar{s}_{-1,1,2} \ln_2 - \frac{11}{40} + \frac{1}{2} s_{-2} \zeta_2 - \frac{3}{4} s_{-1} \zeta_3
\]

\[- \frac{1}{2} \zeta_2 \bar{s}_{-2} + \frac{3}{2} \zeta_3 \bar{s}_{-1} + \frac{1}{2} \zeta_2 \bar{s}_2 + \frac{1}{2} \zeta_2 s_{-1,1} + s_{3,1}
\]

\[- \frac{1}{2} \zeta_2 \bar{s}_{1,1} - \bar{s}_{2,1} - s_{-1,1,1} - 2 \bar{s}_{1,1,1,2} \tag{42}\]

\[
s_{-1,1,2} = \ln_2^3 - \zeta_2 \ln_2^2 - s_{-3} \ln_2 + s_3 \ln_2 + s_{-1} \zeta_2 \ln_2 + \frac{3 \zeta_3 \ln_2}{2}
\]

\[-s_{-1,2} \ln_2 + s_{-1,2} \ln_2 - \bar{s}_{1,1,1} \ln_2 - \bar{s}_{-1,1,2} \ln_2 - \frac{8 \zeta_2^2}{5}
\]

\[+ 8 \text{LiHalf}_4 - \frac{s_2 \zeta_2}{2} - \frac{5}{4} s_{-1} \zeta_3 - 2 \zeta_3 \bar{s}_{-1} - s_{-3,1}
\]

\[+ \frac{1}{2} \zeta_2 s_{-1,1} - \frac{1}{2} \zeta_2 \bar{s}_{1,1} - \bar{s}_{2,1} + s_{-1,1,1} - 2 \bar{s}_{1,1,1,2} \tag{43}\]

\[
s_{-1,1,1,1} = - \ln_2^4 - \frac{4}{3} s_{-1} \ln_2^2 - s_{-2} \ln_2 + s_2 \ln_2 + \bar{s}_{-2} \ln_2 - \bar{s}_2 \ln_2
\]

\[-2 s_{-1,1} \ln_2 - s_{-3} \ln_2 + s_3 \ln_2 + s_{-1} \zeta_2 \ln_2 + 2 \zeta_3 \ln_2
\]

\[-s_{-1,2} \ln_2 + s_{-1,2} \ln_2 + 2 s_{2,1} \ln_2 - \bar{s}_{-1,1,2} \ln_2
\]

\[+ \bar{s}_{-1,2} \ln_2 - 2 s_{-1,1,1} \ln_2 - 2 \bar{s}_{-1,1,1} \ln_2 - \frac{21 \zeta_2^2}{20}
\]

\[+ 4 \text{LiHalf}_4 - \frac{s_2 \zeta_2}{2} - \frac{3}{4} s_{-1} \zeta_3 - \zeta_3 \bar{s}_{-1} - s_{-3,1}
\]

\[+ \frac{1}{2} \zeta_2 s_{-1,1} + \frac{1}{2} \zeta_2 \bar{s}_{-1,1} + s_{-1,2,1} + s_{2,1,1}
\]

\[+ \bar{s}_{-1,2,1} + \bar{s}_{2,1,1} - s_{-1,1,1,1} - 3 \bar{s}_{1,1,1,1,1} \tag{44}\]
\[ s_{-1 \bar{s}_{-1,-1,1}} = -\frac{13 \ln^2 4}{24} - s_{-1} \ln^3 2 - \frac{1}{2} s_{-2} \ln^2 2 + \frac{1}{2} s_2 \ln^2 2 + \frac{1}{2} \bar{s}_{-2} \ln^2 2 - \frac{1}{2} \bar{s}_2 \ln^2 2 - s_{-1} \ln^2 2 - s_{-3} \ln 2 + s_3 \ln 2 + s_{-1} \zeta_2 \ln 2 + \frac{17 \zeta_3 \ln 2}{8} - \frac{3}{2} \zeta_2 \bar{s}_{-1} \ln 2 - s_{-1} \ln 2 + s_{-1,2} \ln 2 + s_{2,1} \ln 2 - s_{2,1} \ln 2 - s_{-1,-1,-1} \ln 2
\]
\[ + s_{-1,-1} \ln 2 - \bar{s}_{-1,-1,1} \ln 2 - \bar{s}_{-1,-1,1} \ln 2 - \frac{47 \zeta_2^2}{40} + \text{LiHalf} + \frac{1}{2} s_{-2} \bar{\zeta}_2 - \frac{5}{8} s_{-1} \zeta_3 - \frac{1}{2} \zeta_2 \bar{s}_{-2} + \frac{13}{8} \zeta_3 \bar{s}_{-1} + \frac{1}{2} \zeta_2 \bar{s}_{-1,1} + s_{3,-1} - \zeta_2 \bar{s}_{-1,1} - s_{-1,-2,1} - s_{2,1,-1} + \bar{s}_{1,2,1}
\]
\[ + \bar{s}_{2,-1,1} + s_{-1,1,-1} - 3 \bar{s}_{-1,1,-1,1} \quad (45) \]

\[ s_1 \bar{s}_{-3} = \ln^4 6 - \zeta_2 \ln^2 2 - \frac{37 \zeta_2^2}{20} + 4 \text{LiHalf}_4 + s_{-2} \zeta_2 - \frac{7}{4} \bar{s}_{-1} \zeta_3 - \frac{3 s_1 \zeta_3}{4} - \zeta_2 \bar{s}_{-2}
\]
\[ - \frac{7}{4} \zeta_3 \bar{s}_{-1} + \frac{3}{4} \zeta_3 \bar{s}_1 - s_{-3,1} + \bar{s}_{1,-3} \quad (46) \]

\[ s_1 \bar{s}_3 = \frac{8 \zeta_2^2}{5} - s_2 \zeta_2 - \bar{s}_2 \zeta_2 + s_1 \zeta_3 - \zeta_3 \bar{s}_1 + s_{3,1} + \bar{s}_{1,3} \quad (47) \]

\[ s_1 \bar{s}_{-2,-1} = -\frac{\ln^4 4}{4} - \frac{1}{2} \bar{s}_{-2} \ln^2 2 + \frac{1}{2} s_2 \ln^2 2 + \frac{9}{2} \zeta_2 \ln^2 2 - \frac{1}{2} \bar{s}_{-2} \ln^2 2 + \frac{1}{2} \bar{s}_2 \ln^2 2 + \frac{3}{2} \bar{s}_{-1} \zeta_2 \ln 2
\]
\[ + \frac{3}{2} s_1 \zeta_2 \ln 2 - 3 \zeta_3 \ln 2 - \bar{s}_{-3} \ln 2 + \frac{3}{2} \zeta_2 \bar{s}_{-1} \ln 2 - \frac{3}{2} \zeta_2 \bar{s}_1 \ln 2 + \bar{s}_3 \ln 2 - s_{-2,1} \ln 2
\]
\[ - s_{-2,1} \ln 2 - \bar{s}_{-2,1} \ln 2 + \bar{s}_{-2,1} \ln 2 + \frac{15 \zeta_2^2}{8} - 6 \text{LiHalf}_4 - \frac{1}{2} s_{-2} \zeta_2 - \frac{s_2 \zeta_2}{2}
\]
\[ - \frac{5 s_1 \zeta_3}{8} + \frac{5}{8} \zeta_3 \bar{s}_1 - \frac{1}{2} \zeta_2 \bar{s}_2 + s_{3,1} - \bar{s}_{-3,1} - s_{-2,1,1} - \bar{s}_{-2,1,1} - \bar{s}_{1,-2,1} \quad (48) \]

\[ s_1 \bar{s}_{-2,1} = \frac{\ln^4 4}{4} - \frac{3}{2} \zeta_2 \ln^2 2 - \frac{11 \zeta_2^2}{8} + 6 \text{LiHalf}_4 + s_{-2} \zeta_2 - \frac{21}{8} s_{-1} \zeta_3 - \frac{5 s_1 \zeta_3}{8}
\]
\[ + \zeta_2 \bar{s}_{-2} - \frac{21}{8} \zeta_3 \bar{s}_{-1} + \frac{5}{8} \zeta_3 \bar{s}_1 - s_{-3,1} - \bar{s}_{-3,1} - s_{-2,1,1} + s_{-2,1,1} + \bar{s}_{1,-2,1} \quad (49) \]
\[ s_1 \bar{s}_{2,-1} = -\frac{\ln^4}{12} - \frac{1}{2} s_2 \ln^2 + \left( \frac{1}{2} s_2 \ln^2 - \frac{5}{2} s_2 \ln^2 - \frac{1}{2} \bar{s}_2 \ln^2 - \frac{1}{2} s_1 \ln^2 - \frac{3}{2} s_{-1} \zeta_2 \ln_2 \ight. \\
- \frac{3}{2} s_1 \zeta_2 \ln_2 - \frac{9 \zeta_3 \ln_2}{4} + \bar{s}_3 \ln_2 - \frac{3}{2} \zeta_2 \bar{s}_{-1} \ln_2 + \frac{3}{2} \zeta_2 \bar{s}_1 \ln_2 - \bar{s}_3 \ln_2 + s_{-2,-1} \ln_2 \\
+ s_{2,1} \ln_2 - \bar{s}_{2,-1} \ln_2 + s_{2,1} \ln_2 + \frac{21 \zeta_2^2}{40} - 2 \text{LiHalf}_4 + \frac{1}{2} s_2 \zeta_2 + \frac{7}{8} s_{-1} \zeta_3 \\
+ \frac{s_1 \zeta_3}{4} - \frac{1}{2} \zeta_2 \bar{s}_{-2} + \frac{7}{8} \zeta_3 \bar{s}_{-1} - \frac{1}{4} \zeta_3 \bar{s}_1 - s_{-3,1} - \bar{s}_{3,-1} + s_{2,-1} \\
+ \bar{s}_{1,2,-1} + \bar{s}_{2,1,-1} \] (50)

\[ s_1 \bar{s}_{2,1} = \frac{6 \zeta_2^2}{5} - s_2 \zeta_2 + \bar{s}_2 \zeta_2 + 2 s_1 \zeta_3 - 2 \zeta_2 s_1 + s_{3,1} - \bar{s}_{3,1} - s_{2,1,1} + \bar{s}_{1,2,1} + \bar{s}_{2,1,1} \] (51)

\[ s_1 \bar{s}_{-1,1,-1} = -\frac{\ln^4}{4} + \frac{1}{6} s_{-1} \ln^3 - \frac{1}{6} s_1 \ln^3 - \frac{1}{6} \bar{s}_{-1} \ln^3 + \frac{1}{6} \bar{s}_1 \ln^3 - \frac{1}{2} s_{-2} \ln^2 + \frac{1}{2} s_2 \ln^2 \\
+ \frac{3}{2} \zeta_2 \ln^2 - \frac{3}{2} s_{-2} \ln^2 + \frac{1}{2} \bar{s}_2 \ln^2 + s_{-1,1} \ln^2 + \frac{5}{2} s_{-1} \zeta_2 \ln_2 + \frac{1}{2} s_1 \zeta_2 \ln_2 \\
- \frac{17 \zeta_3 \ln_2}{8} - \zeta_2 \bar{s}_{-1} \ln_2 - \frac{1}{2} \zeta_2 \bar{s}_1 \ln_2 - s_{-2,-1} \ln_2 - s_{-2,1} \ln_2 + \bar{s}_{-1,-2} \ln_2 \\
- \bar{s}_{-1,2} \ln_2 + s_{-1,1,-1} \ln_2 + s_{-1,1,1} \ln_2 - \bar{s}_{-1,1,-1} \ln_2 + \bar{s}_{-1,1,1} \ln_2 + \frac{19 \zeta_2^2}{20} \\
- 2 \text{LiHalf}_4 - \frac{1}{2} s_{-2} \zeta_2 - \frac{s_2 \zeta_2}{2} - \frac{3}{4} s_{-1} \zeta_3 + \frac{s_1 \zeta_3}{8} + \frac{1}{4} \zeta_3 \bar{s}_{-1} - \frac{1}{8} \zeta_3 \bar{s}_1 \\
+ \frac{1}{2} \zeta_2 s_{-1,-1} + \frac{1}{4} \zeta_2 s_{-1,1} + s_{3,1} - \frac{1}{2} \zeta_2 \bar{s}_{-1,-1} - s_{-2,-1,1} - s_{-1,-2,1} \\
- \bar{s}_{-2,1,-1} - \bar{s}_{-1,2,-1} + s_{-1,1,-1,1} + 2 \bar{s}_{-1,1,1,1} + \bar{s}_{1,-1,1,1} \] (52)

\[ s_1 \bar{s}_{-1,1,1} = -\frac{\ln^4}{8} - \frac{1}{6} s_{-1} \ln^3 - \frac{1}{6} s_1 \ln^3 - \frac{1}{6} \bar{s}_{-1} \ln^3 + \frac{1}{6} \bar{s}_1 \ln^3 + \frac{1}{2} s_{-1} \zeta_2 \ln_2 \\
+ \frac{s_1 \zeta_2}{2} + \frac{1}{2} \zeta_2 \bar{s}_{-1} \ln_2 - \frac{1}{2} \zeta_2 \bar{s}_1 \ln_2 - \frac{1}{2} s_{-1} \zeta_2 \ln_2 + \frac{13 \zeta_2^2}{20} + 3 \text{LiHalf}_4 - s_{-2} \zeta_2 \\
+ \frac{23}{8} s_{-1} \zeta_3 - \frac{7 s_1 \zeta_3}{8} + \frac{1}{4} \zeta_3 \bar{s}_{-1} + \frac{7}{8} \zeta_3 \bar{s}_1 - s_{-3,1} - \zeta_2 \bar{s}_{-1,1} + \zeta_2 \bar{s}_{-1,1} + s_{-2,1,1} \\
-s_{-1,1} - \bar{s}_{-1,2,1} - s_{-1,1,1,1} + 2 \bar{s}_{-1,1,1,1} + \bar{s}_{1,-1,1,1} \] (53)

\[ s_1 s_{1,-2} = \frac{\ln^4}{6} + \frac{1}{2} \zeta_2 \ln^2 + \frac{3}{2} \zeta_{-1} \zeta_2 \ln_2 - \frac{3}{2} s_1 \zeta_2 \ln_2 + \frac{3}{2} \zeta_2 \bar{s}_{-1} \ln_2 + \frac{3}{2} \zeta_2 \bar{s}_1 \ln_2 \]
\[-\frac{21\zeta_2^2}{10} + 4\text{LiHalf}_4 + s_{-2}\zeta_2 - \frac{7}{4}s_{-1}\zeta_3 + \frac{s_1\zeta_3}{2} + \frac{1}{2}\zeta_2\tilde{s}_{-2} - \frac{7}{4}\zeta_3\tilde{s}_{-1} + \frac{1}{4}\zeta_3\tilde{s}_1 - \frac{1}{2}\zeta_2\tilde{s}_{1,-1} - \frac{3}{2}\zeta_2s_{1,1} - \frac{1}{2}\zeta_2\tilde{s}_1,1 + \frac{1}{2}\zeta_2\tilde{s}_1,1 - \frac{3}{2}\zeta_2s_{1,1} - \frac{3}{2}\zeta_2\tilde{s}_{1,-1} + \frac{1}{2}\zeta_2\tilde{s}_1,1
\]

\[-s_{2,-2} + s_{1,-2,1} + 2\tilde{s}_{1,1,-2}\]

(54)

\[s_1\tilde{s}_{1,2} = \frac{6\zeta_2^2}{5} - s_2\zeta_2 + s_{1,1}\zeta_2 - \tilde{s}_{1,1}\zeta_2 + s_1\zeta_3 + 2\zeta_3\tilde{s}_1 + s_{3,1} - \tilde{s}_{2,2} - s_{1,2,1} + 2\tilde{s}_{1,1,2}\]

(55)

\[s_1\tilde{s}_{1,-1,-1} = \frac{\ln^4}{3} + s_1\ln^3 - \frac{1}{2}s_{-2}\ln^2 + \frac{1}{2}s_2\ln^2 + \frac{11}{4}\zeta_2\ln^2 - s_{-2}\ln^2 + s_2\ln^2
\]

\[+ \frac{3}{2}s_{1,-1}\ln^2 + \frac{1}{2}s_{1,1}\ln^2 + \frac{1}{2}s_{1,-1}\ln^2 - \frac{1}{2}s_{1,1}\ln^2 + \frac{1}{2}s_{1,1}\ln^2 + \frac{3}{2}\zeta s_{1,1}\ln^2
\]

\[-s_{2,1}\ln^2 - \tilde{s}_{1,2}\ln^2 + s_{1,2}\ln^2 + s_{1,-1,1}\ln^2 + s_{1,1,1}\ln^2 - \tilde{s}_{1,-1,1}\ln^2
\]

\[+ \tilde{s}_{1,-1,1}\ln^2 + \frac{19s^2}{40} + \frac{1}{2}s_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} - \frac{5s_1\zeta_3}{8} + \frac{1}{2}\zeta\tilde{s}_{1,1}
\]

\[+ \frac{1}{2}\zeta_2s_{1,1} + s_{3,1} - \frac{1}{2}\zeta_2\tilde{s}_{1,1} - s_{2,-1,1} - s_{1,2,1} - \tilde{s}_{1,-2,1} - \tilde{s}_{2,-1,1}
\]

\[+ s_{1,-1,1,1} + \tilde{s}_{1,-1,1} + 2\tilde{s}_{1,1,1,-1}\]

(56)

\[s_1\tilde{s}_{1,-1,1} = \frac{11\ln^4}{24} + \frac{2}{3}s_1\ln^3 + \frac{3}{4}\zeta_2\ln^2 - \frac{1}{2}\tilde{s}_{-2}\ln^2 + \frac{1}{2}\tilde{s}_{2}\ln^2 + \frac{1}{2}s_{1,1}\ln^2
\]

\[+ \frac{1}{2}s_{1,1}\ln^2 + \frac{1}{2}\tilde{s}_{1,-1}\ln^2 - \frac{1}{2}s_{1,1}\ln^2 + \frac{1}{2}\tilde{s}_{-1}\zeta_2\ln^2 - \frac{3}{2}s_1\zeta_2\ln^2
\]

\[+ \frac{1}{2}\zeta_2\tilde{s}_{-1}\ln^2 + \zeta s_{1}\ln^2 - \frac{87\zeta^2}{40} + 5\text{LiHalf}_4 + s_{-2}\zeta_2 - \frac{13}{8}s_{-1}\zeta_3
\]

\[+ \frac{s_1\zeta_3}{8} + \frac{1}{2}\zeta_2\tilde{s}_{-2} - \frac{13}{8}\zeta_3\tilde{s}_{-1} - \frac{1}{2}\zeta_2\tilde{s}_{2} - s_{-3,1} - \frac{3}{2}\zeta_2s_{1,1}
\]

\[+ \frac{1}{2}\zeta s_{1,1} + \frac{1}{2}\zeta_2\tilde{s}_{1,-1} + \zeta_2\tilde{s}_{1,1} + s_{2,-1,1} + s_{1,-2,1} - \tilde{s}_{1,-2,1}
\]

\[-\tilde{s}_{2,-1,1} - s_{1,-1,1,1} + \tilde{s}_{1,-1,1,1} + 2\tilde{s}_{1,1,1,1}\]

(57)

\[s_1\tilde{s}_{1,1,-1} = -\frac{\ln^4}{8} - \frac{1}{3}s_1\ln^3 - \frac{1}{2}s_{-2}\ln^2 + \frac{1}{2}s_2\ln^2 - \frac{3}{2}\zeta_2\ln^2 + \frac{1}{2}\tilde{s}_{-2}\ln^2
\]

\[-\frac{1}{2}\tilde{s}_{2}\ln^2 - s_{1,1}\ln^2 - s_{3,1}\zeta_2\ln^2 - \frac{25\zeta_3\ln^2}{8} - \frac{3}{2}\zeta_2\tilde{s}_1\ln^2
\]

17
\[\begin{align*}
&+ s_{2,-1} \ln s_2 + s_{2,1} \ln s_2 + \bar{s}_{1,-2} \ln s_2 - \bar{s}_{1,2} \ln s_2 - \bar{s}_{1,1,-1} \ln s_2 - \bar{s}_{1,1,1} \ln s_2 \\
&- \bar{s}_{1,1,-1} \ln s_2 + \bar{s}_{1,1,1} \ln s_2 + \frac{13\zeta_2^2}{40} - \text{LiHalf}_4 + \frac{1}{2} s_{-2} \zeta_2 + \frac{s_2 \zeta_2}{2} \\
&- \frac{1}{8} s_{-1} \zeta_3 + \frac{3 s_1 \zeta_3}{4} - \frac{1}{8} \zeta_3 \bar{s}_{-1} + \frac{1}{4} \zeta_3 \bar{s}_1 - s_{-3,1} - \frac{1}{2} \zeta_2 \bar{s}_{1,-1} \\
&- \frac{1}{2} \zeta_2 \bar{s}_{1,1} - \frac{1}{2} \zeta_2 \bar{s}_{1,-1} + s_{1,-2,1} + s_{2,-1,1} - \bar{s}_{1,2,-1} \\
&- \bar{s}_{2,1,-1} - s_{1,1,-1,1} + 3 \bar{s}_{1,1,1,1} \\
&\text{(58)}
\end{align*}\]

\[\begin{align*}
&\frac{8 \zeta_2^2}{5} - s_2 \zeta_2 + s_{1,1} \zeta_2 + \bar{s}_{1,1} \zeta_2 + 2 s_1 \zeta_3 + \zeta_3 \bar{s}_1 + s_{3,1} - s_{1,2,1} \\
&- s_{2,1,1} - \bar{s}_{1,2,1} - \bar{s}_{2,1,1} + s_{1,1,1,1} + 3 \bar{s}_{1,1,1,1} \\
&\text{(59)}
\end{align*}\]

\[\begin{align*}
&s_{1 \bar{s}_{-1,-2}} = \frac{\ln^4}{6} - \frac{1}{2} \zeta_2 \ln s_2 + 2 s_{-1} \zeta_2 \ln s_2 - s_1 \zeta_2 \ln s_2 - \zeta_2 \bar{s}_{-1} \ln s_2 + \zeta_2 \bar{s}_1 \ln s_2 \\
&+ 4 \text{LiHalf}_4 - s_2 \zeta_2 - \frac{5}{8} s_{-1} \zeta_3 + \frac{13 s_1 \zeta_3}{8} - \frac{1}{2} \zeta_2 \bar{s}_{-2} + \frac{1}{8} \zeta_3 \bar{s}_{-1} \\
&- \frac{13}{8} \zeta_3 \bar{s}_1 + \frac{1}{2} \zeta_2 \bar{s}_2 + \frac{3}{2} \zeta_2 \bar{s}_{-1,-1} + \frac{1}{8} \zeta_2 \bar{s}_{1,-1} + s_{3,1} - \bar{s}_{-2,-2} \\
&\frac{3}{2} \zeta_2 \bar{s}_{-1,-1} + \frac{1}{2} \zeta_2 \bar{s}_{-1,-1} - s_{-1,-2,1} + \bar{s}_{-1,-1,-2} + \bar{s}_{1,-1,-2} \\
&\text{(60)}
\end{align*}\]

\[\begin{align*}
&s_{1 \bar{s}_{-1,2}} = \frac{\ln^4}{12} + \frac{3}{2} \zeta_2 \ln s_2 + 2 s_{-1} \zeta_2 \ln s_2 + \frac{1}{2} s_1 \zeta_2 \ln s_2 + \frac{1}{2} \zeta_2 \bar{s}_{-1} \ln s_2 - \frac{1}{2} \zeta_2 \bar{s}_1 \ln s_2 \\
&- \frac{9 \zeta_2^2}{8} + 2 \text{LiHalf}_4 + s_{-2} \zeta_2 - 2 s_{-1} \zeta_3 - s_1 \zeta_3 + \zeta_3 \bar{s}_{-1} + \zeta_3 \bar{s}_1 - s_{-3,1} \\
&- \zeta_2 \bar{s}_{-1,-1} - \bar{s}_{-2,-2} - \zeta_2 \bar{s}_{-1,1} + s_{-1,2,1} + \bar{s}_{-1,1,2} + \bar{s}_{1,-1,2} \\
&\text{(61)}
\end{align*}\]

\[\begin{align*}
&s_{1 \bar{s}_{-1,-1,-1}} = -\frac{5 \ln^4}{8} - \frac{7}{6} s_{-1} \ln s_2 - \frac{1}{6} s_1 \ln s_2 - \frac{1}{6} \bar{s}_{-1} \ln s_2 - \frac{1}{6} \bar{s}_{1} \ln s_2 - \frac{1}{2} \bar{s}_{-2} \ln s_2 \\
&+ \frac{1}{2} s_2 \ln s_2 - \frac{5}{4} \zeta_2 \ln s_2 + \bar{s}_{-2} \ln s_2 - \bar{s}_{2} \ln s_2 - \frac{3}{2} s_{-1,-1} \ln s_2 - \frac{1}{2} \bar{s}_{-1,-1} \ln s_2 \\
&+ \frac{1}{2} \bar{s}_{-1,-1} \ln s_2 + \frac{1}{2} \zeta_2 \bar{s}_1 \ln s_2 + s_{2,-1} \ln s_2 + s_{2,1} \ln s_2 - \bar{s}_{-1,-2} \ln s_2 + \bar{s}_{-1,-2} \ln s_2 \\
&- s_{-1,-1,-1} \ln s_2 - s_{-1,-1,1} \ln s_2 - s_{-1,-1,-1} \ln s_2 + \bar{s}_{-1,-1,1} \ln s_2 + \frac{3 \zeta_2^2}{40} \\
&\text{(62)}
\end{align*}\]
\[-\text{LiHalf}_4 + \frac{1}{2} s_{-2} \zeta_2 + \frac{s_2 \zeta_2}{2} + \frac{3}{8} s_{-1} \zeta_3 - \frac{s_1 \zeta_3}{4} + \frac{3}{4} \zeta_3 \bar{s}_1 + \frac{1}{4} \zeta_3 \bar{s}_1 \]

\[-s_{-3,1} - \frac{1}{2} \zeta_2 s_{-1,1} - \frac{1}{2} \zeta_2 s_{-1,1} - \frac{1}{2} \zeta_2 s_{-1,1} + s_{-1,2,1} + s_{-1,1,1} \]

\[-\bar{s}_{-2,-1,-1} - \bar{s}_{-1,-2,-1} - s_{-1,-1,1,1} + \bar{s}_{-1,-1,1,1} \]

\[+\bar{s}_{-1,1,-1,1} + \bar{s}_{1,-1,1,1} \]

(62)

\[s_1 \bar{s}_{-1,-1,1} = -\frac{\ln^2}{2} - \frac{5}{6} s_{-1} \ln^2 - \frac{1}{6} s_1 \ln^2 - \frac{1}{6} \bar{s}_{-1} \ln^2 - \frac{1}{6} \bar{s}_1 \ln^2 + \frac{3}{4} \zeta_2 \ln^2 \]

\[+\frac{1}{2} \bar{s}_{-2} \ln^2 - \frac{\zeta_2}{2} \bar{s}_{-2} \ln^2 - \frac{1}{2} s_{-1,-1} \ln^2 - \frac{1}{2} s_{-1,1} \ln^2 + \frac{1}{2} \bar{s}_{-1,-1} \ln^2 \]

\[-\frac{1}{2} \bar{s}_{-1,-1} \ln^2 + \frac{3}{2} s_{-1} \zeta_2 \ln^2 - \frac{1}{2} s_1 \zeta_2 \ln^2 + \zeta_2 \bar{s}_{-1} \ln^2 + \frac{1}{2} \zeta_2 \bar{s}_1 \ln^2 \]

\[+\frac{3 \zeta_2^2}{5} - s_2 \zeta_2 + \frac{1}{4} s_{-1} \zeta_3 + \frac{7 s_1 \zeta_3}{4} - \frac{1}{2} \zeta_2 \bar{s}_{-2} + \frac{1}{8} \zeta_3 \bar{s}_{-1} - \frac{7}{4} \zeta_3 \bar{s}_1 \]

\[+\frac{1}{2} \zeta_2 \bar{s}_2 + \frac{3}{2} \zeta_2 s_{-1,-1} + \frac{1}{2} \zeta_2 s_{-1,1} + s_{3,1} + \frac{1}{2} \zeta_2 \bar{s}_{-1,-1} + \frac{1}{2} \zeta_2 \bar{s}_{-1,1} \]

\[-s_{-1,-2,1} - s_{2,1,1} - \bar{s}_{-2,-1,1} - \bar{s}_{-1,-2,1} + s_{-1,-1,1,1} + \bar{s}_{-1,-1,1,1} \]

\[+\bar{s}_{-1,1,-1,1} + \bar{s}_{1,-1,1,1} \]

(63)
2. Reflection identities originating from $B_2 \otimes B_2$

\[ s_{-2\bar{s}-2} = \frac{8\zeta^2_2}{5} - s_2\zeta_2 - \bar{s}_2\zeta_2 + s_{-2,-2} + \bar{s}_{-2,-2} \]  \hspace{1cm} (64)

\[ s_{-2\bar{s}_2} = \frac{\ln^4_4}{3} - 2\zeta_2 \ln^2_2 - \frac{29\zeta^2_2}{10} + 8\text{LiHalf}_4 + \frac{1}{2} s_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} - \frac{7}{2} s_{-1}\zeta_3 - \frac{1}{2} \zeta_2\bar{s}_2 \\
-\frac{7}{2} \zeta_3\bar{s}_{-1} - \frac{1}{2} \zeta_2\bar{s}_2 - s_{-2,-2} + \bar{s}_{-2,-2} \]  \hspace{1cm} (65)

\[ s_{-2\bar{s}_{-1,1}} = \frac{\ln^4_4}{12} + \frac{1}{2} s_{-2} \ln^2_2 + \frac{1}{2} s_2 \ln^2_2 - \frac{1}{2} \bar{s}_{-2} \ln^2_2 + \frac{1}{2} \bar{s}_2 \ln^2_2 + \frac{3}{2} s_{-1}\zeta_3 \ln_2 - \frac{3}{2} \zeta_2\bar{s}_{-1} \ln_2 \\
-\frac{11\zeta^2_2}{40} + 2\text{LiHalf}_4 - s_2\zeta_2 - \frac{1}{8} s_{-1}\zeta_3 + \frac{1}{2} \zeta_2\bar{s}_{-2} + \frac{5}{8} \zeta_3\bar{s}_{-1} + \frac{1}{2} \zeta_2\bar{s}_2 \\
+ s_{-2,-2} + \frac{3}{2} \zeta_2\bar{s}_{-1,-1} - \frac{1}{2} \zeta_2\bar{s}_{-1,1} - \frac{1}{2} \zeta_2\bar{s}_{-1,1} - \bar{s}_{3,1} \\
s_{-1,1,-2} + \bar{s}_{-2,-1,1} + \bar{s}_{-1,-2,1} \]  \hspace{1cm} (66)

\[ s_{-2\bar{s}_{1,-1}} = -\frac{\ln^4_4}{4} - \frac{1}{2} s_{-2} \ln^2_2 - \frac{1}{2} s_2 \ln^2_2 + 4\zeta_2 \ln^2_2 + \frac{1}{2} \bar{s}_{-2} \ln^2_2 - \frac{1}{2} \bar{s}_2 \ln^2_2 \\
+ \frac{3}{2} s_{-1}\zeta_2 \ln_2 - s_1\zeta_2 \ln_2 - \frac{15\zeta_3 \ln_2}{4} - \bar{s}_{-3} \ln_2 + \frac{3}{2} \zeta_2\bar{s}_{-1} \ln_2 \\
+ s_3 \ln_2 - s_{1,-2} \ln_2 - s_{1,2} \ln_2 + \bar{s}_{1,-2} \ln_2 - \bar{s}_{1,2} \ln_2 + \frac{17\zeta^2_2}{8} \\
-6\text{LiHalf}_4 + \frac{1}{2} s_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} + \frac{13 s_1\zeta_3}{8} + \frac{5}{8} \zeta_3\bar{s}_1 - \frac{1}{2} \zeta_2\bar{s}_2 \\
+ s_{-2,-2} - \frac{1}{2} \zeta_2\bar{s}_{1,-1} - \bar{s}_{-3,-1} - \frac{1}{2} \zeta_2\bar{s}_{1,-1} - s_{1,-1,-2} \\
+ \bar{s}_{-2,1,-1} + \bar{s}_{-1,-2,1} \]  \hspace{1cm} (67)

\[ s_{-2\bar{s}_{1,1}} = \frac{\ln^4_4}{4} + \frac{3}{2} s_{-1}\zeta_2 \ln_2 - \frac{3}{2} s_1\zeta_2 \ln_2 + \frac{3}{2} \zeta_2\bar{s}_{-1} \ln_2 - \frac{3}{2} \zeta_2\bar{s}_1 \ln_2 - \frac{13\zeta^2_2}{8} \\
+ 6\text{LiHalf}_4 + \frac{1}{2} \ln_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} - \frac{21}{8} s_{-1}\zeta_3 + \frac{s_1\zeta_3}{8} + \zeta_2\bar{s}_{-2} \\
- \frac{21}{8} \zeta_3\bar{s}_{-1} + \frac{5}{8} \zeta_3\bar{s}_1 - \frac{3}{2} \zeta_2\bar{s}_{1,-1} + \frac{1}{2} \zeta_2\bar{s}_{1,1} - s_{2,-2} - \bar{s}_{-3,1} \\
- \frac{3}{2} \zeta_2\bar{s}_{1,-1} - \frac{1}{2} \zeta_2\bar{s}_{1,1} + s_{1,1,-2} + \bar{s}_{-2,1,1} + \bar{s}_{-1,2,1} \]  \hspace{1cm} (68)
\[ s_{-2 \bar{s},-1,-1} = \frac{\ln^4 2}{6} + s_{-2} \ln^2 2 + s_2 \ln^2 2 - \bar{s}_{-2} \ln^2 2 + \bar{s}_2 \ln^2 2 + s_{-1} \zeta_2 \ln 2 - \frac{3\zeta_3 \ln 2}{2} \\
+ \frac{23\zeta^2_2}{20} + 4 \text{LiHalf}_4 + \frac{1}{2} s_{-2} \zeta_2 - \frac{s_2 \zeta_2}{2} - \frac{5}{2} s_{-1} \zeta_3 - \frac{1}{2} \bar{s}_2 \bar{s} - 2 \\
- \frac{1}{4} \zeta_3 \bar{s}_{-1} + \frac{1}{2} \zeta_2 s_{-1,-1} - s_{-2,-2} - \frac{1}{2} \zeta_2 \bar{s}_{-1,-1} - \bar{s}_{3,-1} + s_{-1,-1,-2} \\
+ \bar{s}_{-2,-1,-1} + \bar{s}_{-1,-2,-1} \]

\[ s_{2 \bar{s}_2} = \frac{12\zeta^2_2}{5} - s_{2,2} - \bar{s}_{2,2} \]  

\[ s_{2 \bar{s}_{-1,1}} = \frac{\ln^4 2}{6} + \frac{1}{2} s_{-2} \ln^2 2 + \frac{1}{2} s_2 \ln^2 2 + \zeta_2 \ln^2 2 - \frac{1}{2} \bar{s}_{-2} \ln^2 2 + \frac{1}{2} \bar{s}_2 \ln^2 2 + \frac{3}{2} s_{-1} \zeta_2 \ln 2 \\
+ \frac{3}{2} \zeta_2 \bar{s}_{-1} \ln 2 - \frac{5\zeta^2_2}{4} + 4 \text{LiHalf}_4 - \frac{1}{2} s_{-2} \zeta_2 - \frac{s_2 \zeta_2}{2} - \frac{29}{8} s_{-1} \zeta_3 - \frac{1}{2} \zeta_2 \bar{s} - 2 \\
- \frac{5}{8} \zeta_3 \bar{s}_{-1} - \frac{1}{2} \zeta_2 \bar{s}_2 + s_{-2,2} + \zeta_2 s_{-1,1} + \bar{s}_{-3,1} + \zeta_2 \bar{s}_{-1,1} - s_{-1,1,2} \\
- \bar{s}_{-1,2,1} - \bar{s}_{2,-1,1} \]

\[ s_{2 \bar{s}_{1,-1}} = \frac{\ln^4 2}{12} - \frac{1}{2} s_{-2} \ln^2 2 - \frac{1}{2} s_2 \ln^2 2 - \zeta_2 \ln^2 2 + \frac{1}{2} \bar{s}_{-2} \ln^2 2 - \frac{1}{2} \bar{s}_2 \ln^2 2 + \frac{1}{2} s_{1} \zeta_2 \ln 2 - \frac{15\zeta_3 \ln 2}{4} \\
- \bar{s}_{-3} \ln 2 - \frac{3}{2} \zeta_2 \bar{s}_1 \ln 2 + \bar{s}_3 \ln 2 - s_{1,-2} \ln 2 - s_{1,2} \ln 2 + \bar{s}_{1,-2} \ln 2 - \bar{s}_{1,2} \ln 2 - \frac{11\zeta^2_2}{40} \\
+ 2 \text{LiHalf}_4 - \frac{7}{8} s_{-1} \zeta_3 - s_{1} \zeta_3 + \frac{1}{2} \zeta_2 \bar{s} - 2 - \frac{7}{8} \zeta_3 \bar{s}_{-1} + \frac{1}{4} \zeta_3 \bar{s}_1 + s_{-2,2} - \frac{1}{2} \zeta_2 s_{1,-1} \\
- \frac{1}{2} \zeta_2 \bar{s}_{1,-1} + \bar{s}_{3,-1} - s_{1,-1,2} - \bar{s}_{1,2,-1} - \bar{s}_{2,1,-1} \]

\[ s_{2 \bar{s}_{1,1}} = \frac{4\zeta^2_2}{5} - \bar{s}_2 \zeta_2 - s_{1,1} \zeta_2 + \bar{s}_{1,1} \zeta_2 + s_{1} \zeta_3 + 2 \zeta_3 \bar{s}_1 - s_{2,2} + \bar{s}_{3,1} + s_{1,1,2} \\
- \bar{s}_{1,2,1} - \bar{s}_{2,1,1} \]
\[s_{2\bar{s}-1,-1} = -\frac{\ln^4}{6} + s_{-2} \ln^2 + s_2 \ln^2 + 2\zeta_2 \ln^2 s_{-2} \ln^2 + \bar{s}_2 \ln^2 + s_{-1} \zeta_2 \ln^2 - \frac{3\zeta_3 \ln^2}{2} + s_{-3} \ln^2 - s_3 \ln^2 + s_{-1,-2} \ln^2 + s_{-1,2} \ln^2 + \bar{s}_{-1,-2} \ln^2 - \bar{s}_{-1,2} \ln^2 + \frac{7\zeta_2^2}{4} - 4\text{LiHalf}_4 + s_{-1} \zeta_3 + \frac{1}{4} \zeta_3 s_{-1} + \frac{1}{2} \zeta_2 \bar{s}_2 + \frac{1}{2} \zeta_2 s_{-1,-1} - s_{2,2} + \bar{s}_{-3,-1} - \frac{1}{2} \zeta_2 \bar{s}_{-1,-1} + s_{-1,-1,2} - \bar{s}_{-1,2,-1} - \bar{s}_{2,-1,-1} \]

\[s_{-1,1} \bar{s}_{-1,1} = -\frac{\ln^4}{4} - \frac{1}{2} s_{-2} \ln^2 + \frac{1}{2} s_2 \ln^2 - \frac{1}{2} \zeta_2 \ln^2 - \frac{1}{2} s_{-2} \ln^2 - \frac{1}{2} \bar{s}_2 \ln^2 + s_{-1,1} \ln^2 + s_{-1,-1} \ln^2 + s_{-1,1} \ln^2 - s_{1,1} \ln^2 + \zeta_3 \ln^2 - \frac{3\zeta_2^2}{2} - 6\text{LiHalf}_4 + \frac{1}{2} s_{-2} \zeta_2 + \frac{s_2 \zeta_2}{2} - \frac{3}{8} s_{-1} \zeta_2 \ln^2 - \frac{3}{2} \zeta_2 \ln^2 s_{-1} - s_{-2,-1} \ln^2 - \frac{19\zeta_2^2}{20} + 6\text{LiHalf}_4 + \frac{1}{2} s_{-2} \zeta_2 + \frac{s_2 \zeta_2}{2} - 3 \zeta_2 s_{-1,-1} + \frac{1}{2} \zeta_2 \bar{s}_2 - 2 \zeta_2 s_{-1,-1} - \zeta_2 s_{-1,-1} - s_{3,1} - 2 \zeta_2 \bar{s}_{-1,-1} - \zeta_2 s_{-1,-1} - s_{3,1} + s_{-2,-1,1} + 2 s_{-1,-2,1} + s_{2,1,1} + s_{-2,-1,1} + 2 \bar{s}_{-1,-2,1} + \bar{s}_{2,1,1} - 2 s_{-1,-1,1} - 2 \bar{s}_{-1,-1,1} - \bar{s}_{1,1,-1,1} \]

\[s_{-1,1} \bar{s}_{1,-1} = -\frac{\ln^4}{3} - \frac{2}{3} \ln^2 s_{-1} + \frac{2}{3} s_1 \ln^2 s_{-1} \ln^2 s_{-1} \ln^2 \frac{1}{3} \ln^2 + \frac{1}{3} \bar{s}_1 \ln^2 + \frac{1}{3} \bar{s}_1 \ln^2 - \frac{1}{2} \zeta_2 \ln^2 s_{-1,1} \ln^2 - \frac{1}{2} \zeta_2 \ln^2 s_{-1,1} \ln^2 + s_{1,-1,2} \ln^2 + s_{1,-2,1} \ln^2 + s_{-1,-1} \ln^2 + s_{-1,1} \ln^2 + s_{-1,1} \ln^2 + s_{-1,1} \ln^2 + s_{-1,1} \ln^2 + s_{-1,1} \ln^2 + s_{-1,1} \ln^2 + s_{-1,1} \ln^2 + s_{-1,1} \ln^2 + \frac{47\zeta_2^2}{40} + 4\text{LiHalf}_4 + s_{-2} \zeta_2 + s_{-1} \zeta_3 + \frac{3}{4} s_{-1} \zeta_3 + \frac{1}{4} \zeta_3 \bar{s}_1 + \frac{3}{4} \zeta_3 s_{-1} - \frac{1}{2} \zeta_2 \bar{s}_2 - \frac{1}{2} \zeta_2 s_{-1,-1} - \frac{1}{2} \zeta_2 s_{-1,-1} - s_{3,1} - \bar{s}_{-3,-1} + \frac{1}{2} \zeta_2 \bar{s}_{-1,-1} - \frac{1}{2} \zeta_2 \bar{s}_{-1,-1} + 2 s_{-2,-1,1} + s_{-1,-2,1} + s_{1,2,1} + 2 \bar{s}_{-2,1,-1} + \bar{s}_{-1,2,-1} + \bar{s}_{1,-2,-1} - s_{1,-1,1,-1} - 2 s_{1,-1,1,-1} - 2 \bar{s}_{1,-1,1,-1} - \bar{s}_{1,-1,1,-1} \]

\[s_{-1,1} \bar{s}_{1,1} = -\frac{\ln^4}{8} - \frac{1}{3} \ln^2 s_{-1} - \frac{1}{3} s_1 \ln^2 s_{-1} - \frac{1}{3} s_{-1} \ln^2 s_{-1} - \frac{1}{3} \bar{s}_1 \ln^2 s_{-1} - \frac{1}{3} \bar{s}_{-1} \ln^2 s_{-1} - \frac{1}{2} \ln^2 s_{-1} + \ln^2 s_{-1} + \frac{1}{2} \ln^2 s_{-1} + \frac{1}{2} \ln^2 s_{-1} \]
\[
\begin{align*}
&+ \frac{3}{4} \zeta_2 \ln^2 + \frac{1}{2} s_{1,-1} \ln^2 - \frac{1}{2} s_{1,1} \ln^2 + \frac{1}{2} \bar{s}_{1,-1} \ln^2 + \frac{1}{2} \bar{s}_{1,1} \ln^2 + s_{1} \zeta_2 \ln^2 \\
&+ \frac{1}{2} s_{1} \zeta_2 \ln^2 + \zeta_2 \bar{s}_{-1} \ln^2 - \zeta_2 \bar{s}_1 \ln^2 - \frac{39 \zeta_3^2}{40} + 3 \text{LiHalf}_4 - \frac{1}{2} s_{-2} \zeta_2 - \frac{s_2 \zeta_2}{2} \\
&+ \frac{1}{4} s_{-1} \zeta_3 + \frac{1}{4} \zeta_3 \ln^2 - \frac{1}{8} s_{-2} \ln^2 - \bar{s}_2 \ln^2 - \frac{7}{2} s_{-1} \zeta_2 \ln^2 - 2 \zeta_3 \ln^2 + s_{3} \ln^2 - \bar{s}_3 \ln^2 \\
&- s_{2,-1} \ln^2 - s_{2,1} \ln^2 + 2 \bar{s}_{-1,-2} \ln^2 - 2 \bar{s}_{-1,2} \ln^2 - \bar{s}_{2,-1} \ln^2 + s_{2,1} \ln^2 \\
&+ 2 s_{-1,-1,-1} \ln^2 + 2 s_{-1,-1,1} \ln^2 + 2 \bar{s}_{-1,-1,-1} \ln^2 + 2 \bar{s}_{-1,-1,1} \ln^2 - \frac{33 \zeta_3^2}{40} \\
&+ 3 \text{LiHalf}_4 - s_2 \zeta_2 - \frac{9}{4} s_{-1} \zeta_3 - \frac{1}{2} \zeta_2 \bar{s}_{-2} - \frac{3}{8} \zeta_3 \bar{s}_{-1} + s_{-3,1} + \frac{3}{2} \zeta_2 \bar{s}_{-1,-1} \\
&+ \frac{1}{2} \zeta_2 \bar{s}_{-1,-1} - \frac{1}{2} \zeta_2 \bar{s}_{-1,1} + \frac{\bar{s}}{2} \zeta_2 \bar{s}_{-1,1} - \bar{s}_{3,-1} - 2 \bar{s}_{1,2,1} - 2 s_{2,1,1} \\
&+ \bar{s}_{-2,-1,-1} + 2 \bar{s}_{-1,-2,-1} + \bar{s}_{2,1,-1} + 3 s_{-1,-1,1} \\
&- 2 \bar{s}_{-1,-1,1} - \bar{s}_{-1,1,1} \\
&\text{(77)}
\end{align*}
\]

\[
\begin{align*}
&s_{-1,1} \bar{s}_{-1,-1} = -\frac{11 \ln^4}{24} + \frac{11}{4} \zeta_2 \ln^2 - \frac{1}{2} \bar{s}_{-1,1} \ln^2 + \frac{1}{2} \bar{s}_{1,1} \ln^2 + \frac{1}{2} \bar{s}_{2,1} \ln^2 + \frac{1}{2} s_{-1,1} \ln^2 + \frac{1}{2} s_{1,1} \ln^2 \\
&+ s_{1,-1} \ln^2 - 2 s_{1,1} \ln^2 + s_{1,-1} \ln^2 - 2 \bar{s}_{1,1} \ln^2 - s_{-3} \ln^2 + s_3 \ln^2 \\
&+ s_{-1} \zeta_2 \ln^2 + \frac{3}{2} s_1 \zeta_2 \ln^2 - 8 \zeta_3 \ln^2 - \bar{s}_{3,-1} - s_{3,-1} + \frac{1}{2} \zeta_2 \bar{s}_{-1} \ln^2 \\
&+ 2 s_{-2,-1} \ln^2 - 4 s_{1,1,-1} \ln^2 - 4 \bar{s}_{1,1,-1} \ln^2 + \frac{12 \zeta_3^2}{5} - \frac{33 \zeta_3^2}{40} + 3 \text{LiHalf}_4 - \frac{s_2 \zeta_2}{2} \\
&- \frac{3 s_1 \zeta_3}{8} - \frac{3}{8} \zeta_3 \bar{s}_1 - \frac{1}{2} \zeta_2 \bar{s}_2 - s_{-3,-1} + \zeta_2 s_{1,1} - \bar{s}_{3,-1} + \zeta_2 \bar{s}_{1,1} \\
&+ s_{-2,1,-1} + 2 s_{1,-2,1} + s_{2,-1,-1} + \bar{s}_{-2,1,-1} + 2 \bar{s}_{1,-2,1} + \bar{s}_{2,-1,-1} \\
&- s_{1,-1,1,-1} - 2 s_{1,1,1,1} - \bar{s}_{1,-1,1,1} - 2 \bar{s}_{1,1,-1,1} \\
&\text{(78)}
\end{align*}
\]

\[
\begin{align*}
&s_{1,-1} \bar{s}_{1,-1} = -\frac{\ln^4}{12} - \frac{1}{2} s_{-2} \ln^2 + \frac{1}{2} \bar{s}_2 \ln^2 + \frac{15}{2} \zeta_2 \ln^2 - \frac{1}{2} \bar{s}_{-2} \ln^2 + \frac{1}{2} \bar{s}_{2,1} \ln^2 \\
&+ s_{1,-1} \ln^2 - 2 s_{1,1} \ln^2 + \bar{s}_{1,-1} \ln^2 - 2 \bar{s}_{1,1} \ln^2 - s_{-3} \ln^2 + s_3 \ln^2 \\
&+ s_{-1} \zeta_2 \ln^2 + \frac{3}{2} \bar{s}_1 \zeta_2 \ln^2 - 8 \zeta_3 \ln^2 - \bar{s}_{3,-1} - s_{3,-1} + \zeta_2 \bar{s}_{-1} \ln^2 \\
&+ 2 s_{-2,-1} \ln^2 - 4 s_{1,1,-1} \ln^2 - 4 \bar{s}_{1,1,-1} \ln^2 + \frac{12 \zeta_3^2}{5} - \frac{33 \zeta_3^2}{40} + 3 \text{LiHalf}_4 - \frac{s_2 \zeta_2}{2} \\
&- \frac{3 s_1 \zeta_3}{8} - \frac{3}{8} \zeta_3 \bar{s}_1 - \frac{1}{2} \zeta_2 \bar{s}_2 - s_{-3,-1} + \zeta_2 s_{1,1} - \bar{s}_{3,-1} + \zeta_2 \bar{s}_{1,1} \\
&+ s_{-2,1,-1} + 2 s_{1,-2,1} + s_{2,-1,-1} + \bar{s}_{-2,1,-1} + 2 \bar{s}_{1,-2,1} + \bar{s}_{2,-1,-1} \\
&- s_{1,-1,1,-1} - 2 s_{1,1,1,1} - \bar{s}_{1,-1,1,1} - 2 \bar{s}_{1,1,-1,1} \\
&\text{(79)}
\end{align*}
\]
\[ s_{1,1}\bar{s}_{1,1} = \frac{\ln^4 2}{8} - \frac{1}{4} \zeta_2 \ln^2 2 - \frac{1}{2} \bar{s}_{-2} \ln^2 2 + \frac{1}{2} \bar{s}_2 \ln^2 2 + \frac{1}{2} s_{1,1} \ln^2 2 - \frac{1}{2} \bar{s}_{1,1} \ln^2 2 \\
- \frac{3}{2} s_{1,1} \ln^2 2 - s_{-3} \ln 2 + s_3 \ln 2 + \frac{1}{2} s_{-1} \zeta_2 \ln 2 - 2 s_1 \zeta_2 \ln 2 - 4 \zeta_3 \ln 2 + \frac{1}{2} \zeta_2 \bar{s}_{-1} \ln 2 \\
- \frac{5}{2} \zeta_2 \bar{s}_1 \ln 2 + 2 s_{1,-2} \ln 2 - 2 s_1 \ln 2 + s_{2,-1} \ln 2 - s_{2,1} \ln 2 + \bar{s}_{2,-1} \ln 2 + \bar{s}_{2,1} \ln 2 \\
- 2 s_{1,1} \ln 2 + 2 s_{1,1,1} \ln 2 - 2 s_{1,1,-1} \ln 2 - 2 \bar{s}_{1,1,1} \ln 2 + \frac{\zeta_2^2}{20} + \text{LiHalf}_4 + \frac{1}{2} s_{-2} \zeta_2 \\
- s_{-1} \zeta_3 + \frac{s_1 \zeta_3}{2} + \frac{1}{2} \zeta_2 \bar{s}_{-2} - \zeta_3 \bar{s}_{-1} + \frac{5}{8} \zeta_3 \bar{s}_1 + \frac{1}{2} \zeta_2 \bar{s}_2 - \zeta_2 \bar{s}_{1,-1} + s_{3,-1} - \bar{s}_{-3,1} \\
- \zeta_2 \bar{s}_{1,-1} - \zeta_2 \bar{s}_{1,1} - 2 s_{1,2,-1} - 2 s_{2,1,-1} + \bar{s}_{-2,1,1} + 2 \bar{s}_{1,-2,1} + \bar{s}_{2,-1,1} \\
+ 3 s_{1,1,1} - \bar{s}_{1,-1,1,1} - 2 \bar{s}_{1,1,-1,1} \quad (80) \]

\[ s_{1,-1} \bar{s}_{-1,-1} = - \frac{5 \ln^4 2}{24} - \frac{2}{3} s_{-1} \ln^3 2 + \frac{2}{3} s_1 \ln^3 2 - \frac{2}{3} \bar{s}_{-1} \ln^3 2 + \frac{2}{3} \bar{s}_1 \ln^3 2 - \frac{1}{2} s_{-2} \ln^2 2 + \frac{1}{2} s_2 \ln^2 2 \\
- \frac{9}{4} \zeta_2 \ln^2 2 + \frac{1}{2} s_{-1,-1} \ln^2 2 + \frac{1}{2} s_{-1,1} \ln^2 2 + 2 s_{1,-1} \ln 2 + \frac{1}{2} \bar{s}_{-1,-1} \ln 2 + \frac{3}{2} \bar{s}_1 \ln 2 + \bar{s}_{-3} \ln 2 + \frac{1}{2} \zeta_2 \bar{s}_{-1} \ln 2 \\
- s_{-3} \ln 2 + s_3 \ln 2 - s_{-1} \zeta_2 \ln 2 - \frac{3}{2} s_1 \zeta_2 \ln 2 + \bar{s}_{-3} \ln 2 + \frac{1}{2} \zeta_2 \bar{s}_{-1} \ln 2 \\
+ \frac{1}{2} \zeta_2 \bar{s}_1 \ln 2 - \bar{s}_3 \ln 2 - 2 s_{-2,-1} \ln 2 - s_{-1,-2} \ln 2 + s_{1,-2} \ln 2 + s_{1,-2} \ln 2 \\
- s_{1,2} \ln 2 + 2 \bar{s}_{-2,-1} \ln 2 + s_{-1,-2} \ln 2 - \bar{s}_{-1,-2} \ln 2 - \bar{s}_{1,-2} \ln 2 + \bar{s}_{1,2} \ln 2 \\
+ 2 s_{1,-1,-1} \ln 2 + 2 s_{1,-1,-1} \ln 2 - 2 \bar{s}_{1,-1,1} \ln 2 - 2 s_{1,-1,-1} \ln 2 - \frac{\zeta_2^2}{5} \\
+ \text{LiHalf}_4 + \frac{1}{2} s_{-2} \zeta_2 - \frac{1}{8} s_{-1} \zeta_3 + \frac{s_1 \zeta_3}{2} - \frac{1}{2} \zeta_2 \bar{s}_{-2} - \frac{1}{2} \zeta_3 \bar{s}_{-1} - \frac{3}{4} \zeta_3 \bar{s}_1 \\
- \frac{1}{2} \zeta_2 \bar{s}_{-1,-1} - \frac{1}{2} \zeta_2 \bar{s}_{1,-1} + s_{3,-1} + \frac{1}{2} \zeta_2 \bar{s}_{-1,1} + \frac{1}{2} \zeta_2 \bar{s}_{1,-1} - \bar{s}_{3,-1} \\
- s_{-2,-1,-1} - s_{-1,-2,-1} - s_{1,2,-1} - s_{2,1,-1} + 2 \bar{s}_{-2,-1,-1} + \bar{s}_{-1,-2,-1} \\
+ \bar{s}_{1,2,-1} + s_{-1,-1,-1} + s_{-1,1,-1} + s_{1,-1,-1} \\
- s_{-1,1,-1,-1} - 2 \bar{s}_{1,-1,-1,-1} \quad (81) \]

\[ s_{1,1} \bar{s}_{1,1} = \frac{12 \zeta_2^2}{5} - s_2 \zeta_2 - \bar{s}_2 \zeta_2 + 2 s_{1,1} \zeta_2 + 2 \bar{s}_{1,1} \zeta_2 + 3 s_1 \zeta_3 + 3 \zeta_3 \bar{s}_1 + s_{3,1} \\
+ \bar{s}_{3,1} - 2 s_{1,2,1} - 2 s_{2,1,1} - 2 \bar{s}_{1,2,1} - 2 \bar{s}_{2,1,1} + 3 s_{1,1,1,1} + 3 \bar{s}_{1,1,1,1} \quad (82) \]
\[ s_{1,1 \bar{s}-1,-1} = -\frac{\ln^4}{3} + \frac{2}{3}s_{1,1} \ln^3 - \frac{2}{3}s_{1,1} \ln^3 - \frac{1}{3} \bar{s}_{1,1} \ln^3 + \frac{1}{3} \bar{s}_{1,1} \ln^3 - \frac{1}{2} \bar{s}_{1,1} \ln^2 + \frac{1}{2} \bar{s}_{1,1} \ln^2 \\
+ \zeta_2 \ln^2 + \frac{1}{2} \bar{s}_{1,1} \ln^2 + \frac{1}{2} \bar{s}_{1,1} \ln^2 - \frac{1}{2} \bar{s}_{1,1} \ln^2 + \frac{1}{2} \bar{s}_{1,1} \ln^2 + \frac{3}{2} \bar{s}_{1,1} \ln^2 \\
+ \frac{1}{2} \bar{s}_{1,1} \ln^2 + \frac{1}{2} \bar{s}_{1,1} \ln^2 - \frac{1}{2} \bar{s}_{1,1} \ln^2 + \frac{1}{2} \bar{s}_{1,1} \ln^2 + \frac{1}{4} \zeta_2 \ln^2 + \bar{s}_{1,1} \ln^2 \\
+ s_1 \zeta_2 \ln^2 + \frac{1}{4} \zeta_2 \ln^2 + \bar{s}_{-1,1} \ln^2 - \bar{s}_{-1,1} \ln^2 - s_{-1,1} \ln^2 - s_{-1,1} \ln^2 + \bar{s}_{-1,1} \ln^2 \\
- \bar{s}_{-1,1} \ln^2 + \bar{s}_{-1,1} \ln^2 - \bar{s}_{-1,1} \ln^2 + \bar{s}_{-1,1} \ln^2 + \bar{s}_{-1,1} \ln^2 + \bar{s}_{-1,1} \ln^2 \\
+ \bar{s}_{-1,1} \ln^2 + s_{-1,1} \ln^2 + s_{-1,1} \ln^2 + s_{-1,1} \ln^2 + s_{-1,1} \ln^2 (83) \]

\[ s_{-1,1 \bar{s}-1,-1} = -\ln^4 + 2\zeta_2 \ln^2 + 2s_{-1,1} \ln^2 + 2s_{-1,1} \ln^2 + 2s_{-3,1} \ln^2 - s_3 \ln^2 \\
- s_1 \zeta_2 \ln^2 - 4\zeta_3 \ln^2 + \bar{s}_{-3,1} \ln^2 - \bar{s}_{3,1} \ln^2 - \bar{s}_{3,1} \ln^2 + 2s_{-3,1} \ln^2 \\
- 2s_{-3,1} \ln^2 - 2s_{-3,1} \ln^2 + 2s_{-3,1} \ln^2 - 2s_{-3,1} \ln^2 - 2s_{-3,1} \ln^2 \\
+ 4s_{-3,1,1} \ln^2 + 4s_{-3,1,1} \ln^2 + \frac{14\zeta_2}{5} - 8\text{LiHalf}_4 + \frac{s_2 \zeta_2}{2} \\
+ \frac{5}{4} \zeta_3 \ln^2 + \frac{5}{4} \zeta_3 \ln^2 + \frac{1}{2} \zeta_2 \ln^2 + s_{-3,1} - \zeta_2 s_{-1,1} + \bar{s}_{-3,1} \\
- \zeta_2 \bar{s}_{-1,1} - 2s_{-1,2,1} - s_{-1,2,1} - 2s_{-1,2,1} - 2s_{-1,2,1} - 2s_{-1,2,1} \\
+ 3s_{-1,1,1,1} - 3s_{-1,1,1,1} (84) \]

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