Omnigenous toroidal plasma equilibria

Allen H Boozer
Columbia University, New York, NY 10027
ahb17@columbia.edu
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Omnigenousity is the weakest condition that ensures that successive turning points of collisionless trapped particles are on the same magnetic surface in toroidal plasmas and that particle trajectories have a maximum deviation from magnetic surfaces that is proportional to their gyroradius. Departures from exact omnigenousity cause excursions of trapped particles from their home flux surfaces that are independent of their gyroradii, which is generally inconsistent with adequate confinement for fusion power plants. Omneigeneity requires that the contours of constant magnetic field strength be unbounded in at least one of the two angular coordinates in magnetic surfaces and that there be a symmetry in the field-strength wells along the field lines. A discussion of omnigenousity naturally leads to a discussion of the constraints on plasma confinement that are required for fusion power plants. These include implications of the Fokker-Planck equation being of the advection-diffusion type, constraints from non-equilibrium thermodynamics that arise from the fusion requirement of long energy confinement times compared to like-particle collision times, and a constraint that arises when the plasma heating is electron heating as in a fusion power plant.

I. INTRODUCTION

The importance of omnigiousity comes from the paradoxical collisionality regime of fusion power plants that utilize magnetic confinement [1]. The smallness of the cross section for the fusion of deuterium (D) and tritium (T) compared to the Coulomb cross section requires the plasma confinement time be long compared to ion and electron collision times and sets the temperature $T$ at which power plants must operate. Practical limits on the power density from DT fusion, which scales approximately as $(nT)^2$, requires the density be sufficiently low that the mean-free-path of electrons and ions between collisions be approximately a thousand times longer than the size of the plasma.

Without collisions, passing particles, which have a velocity parallel to the magnetic field, $v_i$, that is never zero, require the magnetic field lines lie on toroidal surfaces for confinement. A particular magnetic surface is denoted by the magnetic flux $\psi$, called the toroidal flux, it encloses. The magnetic field can be written in the Clebsch form,

$$2\pi \vec{B} = \vec{\nabla} \psi \times \vec{\nabla} \theta_0,$$

(1)

where a poloidal angle $\theta$ can be defined so $\theta = \theta_0 + \iota(\psi)\varphi$ with $\iota$ the rotational transform and $\varphi$ a toroidal angle, Appendix A.

Collisionless trapped particles, which have turning points at which $v_i = 0$, can be difficult to confine in a toroidal plasma. Omneigeneity is the weakest condition that ensures that successive turning points of collisionless trapped particles are on the same magnetic surface and that particle trajectories have a maximum deviation from magnetic surfaces that is proportional to their gyroradius. When the maximum deviation of collisionless particles from magnetic surfaces scales as their gyroradius, their collisional confinement is sufficient for a fusion power plant.

The definition of omnigiousity is based on the longitudinal action $J$ of Northrop and Teller [2]:

$$\frac{\partial J}{\partial \theta_0} = 0, \text{ where}$$

$$J(\psi, \theta_0, u) = \int v_i d\ell. \quad (3)$$

The integral is over the differential distance $d\ell$ along a field line from one turning point to the next of a trapped particle of energy $u$. A particular magnetic field line is denoted by a constant $\psi$ and $\theta_0$. The departure from omnigiousity can be measured by the dimensionless quantity $(\partial J/\partial \theta_0)/J$ as discussed in Section IV B 1.

Section II General transport properties is a brief discussion of non-equilibrium thermodynamics, the implications of the Fokker-Planck equation being of the advection-diffusion form, effects when only electrons are directly heated, as in a burning plasma, and the importance of omnigiousity.

Section III Particle drift trajectories discusses the basic features of particle motion in the small-gyroradius limit, which is the relevant limit of fusion power plants. Well known results from the study of weak toroidal ripple in tokamaks are shown to imply that breaking action conservation near maxima and minima of the field strength can have remarkably little effect on transport.

Section IV Derivatives of the action expresses the bounce time, precession, and radial drift of trapped particles in terms of derivatives of the action $J$. These expressions are used to obtain formulas that
are integrals of the field strength along the magnetic field lines.

Section **V** Construction of omnigenous equilibria discusses the freedom to produce exact omnigenity on a surface based on the formulas derived in Section **IV**.

Section **VI** Effects of trapped particle drifts obtains the deviation of a particle from the flux surface on which its turning points are located in an omnigenous system. The bootstrap current is shown to be unchanged by the deviation of an omnigenous stellarator from quasi-symmetry.

Section **VII** Discussion reviews the results of the paper.

Appendix A Magnetic fields in toroidal plasmas is a short review of Boozer coordinates for toroidally confined plasmas.

Appendix B Ripple transport reviews tokamak ripple transport, which has important implications for confinement in quasi-symmetric and in omnigenous systems as discussed in Section **III**.

II. GENERAL TRANSPORT PROPERTIES

A. Near-Maxwellian requirement

A power plant based on the fusion of deuterium and tritium requires the confinement of a plasma for hundreds of collision times for the ions and approximately ten thousand collision times for the electrons. The implication is that the distribution functions \( f \) must be very close to local Maxwellians \( f_M \).

1. Non-equilibrium thermodynamics

Since the distribution functions of fusion plasmas must be close to Maxwellian, the theory of non-equilibrium thermodynamics has important implications for the solutions to the Fokker-Planck equation \( \frac{T}{2m} \int \frac{\partial \tilde{f}}{\partial v} \cdot \tilde{v} \frac{\partial \tilde{f}}{\partial v} f_m dv, \) Equation (4) in [1],

$$ \dot{s}_c = -\Gamma \frac{d\mu_c}{T} + \frac{d1}{T} \frac{\Gamma}{d\psi_t} $$

with \( \mu_c = T \ln(n/T^{3/2}) \) the chemical potential, \( \Gamma \equiv \dot{\psi_t} \cdot \nabla \psi \) the particle flux and \( \Gamma = \dot{\psi_t} \cdot \nabla \psi_t \) the heat flux. The quantities \(-d(\mu_c/T)/d\psi_t \) and \(d(1/T)/d\psi_t \) are called thermodynamic forces, and \( \Gamma_c \) and \( \Gamma \) are the conjugate fluxes. A more detailed discussion of the application of non-equilibrium thermodynamics and kinetic theory to plasma transport is given in [2].

Theory of non-equilibrium thermodynamics and kinetic theory [3] allows one to address the subtle question of when like-particle collisions cause particle transport. This question is important for determining the ambient electric field, which has a large effect on impurity transport. The basic answer is that like-particle collisions can produce particle transport only through the viscosity tensor, which in fusion plasmas is well approximated by the parallel viscosity. The details require too much space for this paper.

2. The Fokker-Planck equation as an advection-diffusion equation

Mathematically, the Fokker-Planck equation for the distribution function \( f \) is of the advection-diffusion type. The advective part is called the Vlasov equation, which hyperbolic in type with characteristics given by the trajectories of the particle Hamiltonians, \( H(\tilde{p}, \tilde{x}, t) \). The entropy density \( s = -\int f \ln f dv \) is an invariant of the Vlasov equation. The diffusive part of the Fokker-Planck equation is the collision operator, which in plasmas has two special properties. The collision operator (1) diffuses particles in velocity \( \tilde{v} \) but not spatially \( \tilde{x} \), which makes the entropy production per unit volume well defined, and (2) has no effect on local Maxwellians when \( T_e = T_i \) due to the conservation of energy and momentum by collisions.

The advective term in advective-diffusion equations characteristically has chaotic characteristics, which means infinitesimally separated characteristics separate exponentially even while remaining in a bounded volume of phase space. When the characteristics are chaotic, the effects of diffusion are exponentially amplified. This is true for mixing in fluids [4], such as temperature equilibration in a room, and in magnetic reconnection [5] as well as in the entropy production by solutions to the Fokker-Planck equation, Equation (4). Chaos causes an exponential increase in \( \dot{\tilde{f}}/\dot{\tilde{v}} \) until collisions become important.
Two features make solutions of the Fokker-Planck equation that are of relevance to fusion plasma atypical for advection diffusion equations. (1) Collisional diffusion vanishes when the distribution is a local Maxwellian. (2) The required long confinement times compared to collision times of the ions and electrons in fusion plasmas implies the departure from local Maxwellian distributions \( \bar{f} \) must be small. The diffusion operators for fluid mixing and magnetic reconnection, have only a trivial null space—a constant distribution—unlike the local-Maxwellian null space of the collision operator.

In stellarator fusion plasmas, the external magnetic field must be chosen to have the special property that the particle Hamiltonians \( H(\vec{p}, \vec{x}) \) have constants of the motion such that when the distribution functions for a confined plasma are written in terms of these constants of the motion the deviations \( \bar{f} \) from local Maxwellsians are small. The existence of these constants of the motion precludes the chaotic trajectories which exponentially enhance entropy production.

The absence of continuous symmetries was thought to preclude appropriate constants of motion and to be a fatal flaw for the use of stellarators for fusion power plants. It was, therefore, surprising when it was found that such constants of the motion theoretically exist for stellarators \([6]\) and even more remarkable that stellarators could be designed \([7]\) and operated that exploit this \([8]\).

The disproof of the fatal flaw of stellarators is of fundamental importance to the achievement of fusion energy \([9]\). Unlike in tokamaks, the properties of the confining magnetic field in stellarators can be dominated by the externally produced magnetic field. This produces cage-like robust confinement, unique reliability of computational optimization, and steady-state burning plasmas with zero external power input to the plasma. The absence of continuous spatial symmetries, which made the existence of constants of the motion subtle, implies that approximately ten times as many externally produced distributions of magnetic field are available for optimization than in a tokamak with toroidal asymmetry.

When the \( \bar{f} \) given by the constants of the motion can be made small, the neoclassical entropy production is given by Equation \( 4 \) and the transport can be obtained using Equation \( 5 \).

Even when the Vlasov equation deviation \( \bar{f} \) cannot be made small, the full Fokker-Planck equation can give an \( f \), which when substituted into Equation \( 1 \) gives sufficiently small neoclassical transport for fusion power. This is particularly true when the Vlasov \( \bar{f} \) is large in only small region of velocity space—such as near the limit of either deeply and barely trapped particles, which is discussed in Section \( III F \).

No matter how well the external magnetic field is designed so \( H(\vec{p}, \vec{x}) \) has appropriate constants of the motion, the plasma itself can produce perturbations that cause the particle trajectories to be chaotic. When this occurs the Fokker-Planck equation becomes exponentially sensitive to the diffusivity of the collision operator, as is typical of solutions to the advection-diffusion equation. This sensitivity allows collisional entropy production, Equation \( 4 \) even in the limit as collision frequency \( \nu \) goes to zero. GyroBohm diffusion, \( D_{\nu B} \equiv (\rho_i/a)T/eB \), which approximates the commonly observed transport in toroidal plasmas \([3]\), is an example. The spatial scale of gradients is \( a \) and \( \rho_i \) is the ion gyroradius.

### B. Ion and electron transport

In fusion plasmas and in electron-cyclotron heated plasmas, the heating power goes into the electrons and only by equilibration heats the ions. When the external heating heats only the electrons, and the equilibration time, \( \tau_{eq} \propto T_e^{3/2} \), is long compared to the ion energy confinement time, \( \tau_{Ei} \), the ions are colder than the electrons, by the ratio \( T_i/T_e = \tau_{Ei}/\tau_{eq} \). The electron temperature can then rise to whatever level is required for their energy confinement time \( \tau_{Ee} \) to balance the input power. For heating from a DT fusion burn, the ion energy confinement time and the electron energy confinement can neither be significantly shorter than the plasma confinement time required for a DT burn \( \tau_{DT} \).

When the plasma heating is electron heating, the steady-state ion temperature \( T_i \) is

\[
\frac{T_i}{\tau_{Ei}} = \frac{T_e - T_i}{\tau_{eq}} \quad \text{or} \quad (6)
\]

\[
\frac{T_i}{T_e} = \frac{\tau_{Ei}}{\tau_{Ei} + \tau_{eq}} \quad \text{(7)}
\]

The steady-state electron temperature \( T_e \) depends on the heating power per particle \( p_h \):

\[
p_h = \left( \frac{1}{\tau_{Ee}} + \frac{1}{\tau_{eq}} \right) T_e - \frac{T_i}{\tau_{eq}} \quad \text{so} \quad (8)
\]

\[
= \left( \frac{1}{\tau_{Ee}} + \frac{1}{\tau_{Ei} + \tau_{eq}} \right) T_e \quad \text{and} \quad (9)
\]

\[
= \left( \frac{1}{\tau_{Ee}} + \frac{1}{\tau_{Ei} + \frac{\tau_{eq}}{\tau_{Ee} \tau_{Ei}}} \right) T_i \quad \text{(10)}
\]

When the power source is a DT fusion burn, the power input per particle is approximately \( p_{DT} = \)
When the excursions $\Delta \psi$ of a fraction of the particles is independent of their gyroradius, the effect on transport—particularly for high-energy $\alpha$ particles and for electrons at low collisionality—can be inconsistent with fusion power plants. In the low collisionality limit, the intrinsic electron transport becomes larger than the intrinsic ion transport by the ratio of their collision frequencies when $\Delta \psi$ is independent of the gyroradius.

When the intrinsic electron particle transport is more rapid than the ion, an electric potential $\phi(\psi)$ is required to confine the electrons. This has the advantage of expelling impurities. However, the maximum collisionality for this limit is lower than the collisionality in most designs for fusion power plants.

In stellarator power plant designs, the electrons are generally better confined than the ions, which implies a $\phi(\psi)$ that confines ions and impurities.

Careful design of stellarators is required to obtain a scaling of the excursions $\Delta \psi$ that is proportional to the gyroradius. The least constraining design principle that is consistent with this scaling is omnigeneity, which makes this concept of central importance to magnetic confinement fusion.

### III. PARTICLE DRIFT TRAJECTORIES

#### A. General theory for collisionless particles

In 1980, Boozer [10] showed that as $\rho \to 0$, the center of the circle about which a particle gyrates has a drift velocity

$$
\vec{v}_d = \frac{v_{||}}{B} \vec{H},
$$

$$
\vec{H} \equiv \vec{B} + \vec{\nabla} \times (\rho_{||} \vec{B}).
$$

The parallel gyroradius $\rho_{||} \equiv mv_{||}/B$ is a function of the position $\vec{x}$ of the particle and its two conserved quantities

$$
\mu \equiv \frac{mv_{||}^2}{2B} \quad \text{the magnetic moment, and}
$$

$$
u \equiv \frac{1}{2} mv_{||}^2 + \mu B + q\phi \quad \text{the energy.}
$$

The magnitude of the particle velocity parallel and perpendicular to $\vec{B}$ are $v_{||}$ and $v_{\perp}$. When the energy confinement time of a toroidal plasma is long compared to the collision times, the electric potential $\phi(\vec{x})$ has only small departures from being constant along magnetic field lines.

Energy conservation for particles, $du/dt = 0$, holds when the system is time independent. Magnetic moment conservation, $d\mu/dt = 0$, requires the

$$
T_i^2/(T_{DT} \tau_{DT}), \text{ where } T_{DT} \approx 10 \text{ keV is the optimal temperature for a burn. Then,}
$$

$$
\frac{T_i}{T_{DT}} = \left( \frac{1}{\tau_{Ee}} + \frac{1}{\tau_{Ei}} + \frac{\tau_{eq}}{\tau_{Ee} \tau_{Ei}} \right) \tau_{DT}. \quad (11)
$$

Equations (9) and (11) give demanding requirements for a DT burn when $\tau_{Ee} \gg \tau_{eq} \gg \tau_{Ei}$. Then $T_i^2/\tau_{eq} = p_i$ with $\tau_{eq} \propto T_i^{3/2}$, and the electron temperature rises until its energy confinement time becomes shorter than the equilibration time assuming the heating power is independent of the electron temperature. The ion temperature divided by the temperature needed for a fusion burn is then $T_i/T_{eq} \approx \tau_{eq}/\tau_{Ei}$, so the ion energy confinement time must be sufficient for a burn.

When $\tau_{eq} \ll \tau_{Ei}$, the ratio $T_i/T_{DT} \approx (T_{DT}/T_{eq})(1 + \tau_{eq}/\tau_{Ei})$, which given the expression for $\tau_{eq}$ implies $T_{Ee}$ must be longer than $\tau_{DT}$.

The equilibration time is given by $n \tau_{eq} \approx 0.5 T_i^{3/2}$, and the requirement for DT ignition was given in Equation (A6) of Reference [2] as $n T_{DT} \tau_{DT} \approx 3$. Densities are measured in units of $10^{20}/m^3$ and temperatures in units of $10\text{ keV}$. The equilibration time becomes longer than $\tau_{DT}$ for $T_i \gtrsim 30 \text{ keV}$.

#### C. Importance of omnigeneity

Although magnetically-confined fusion plasmas must be sufficiently collisional for the distribution functions to be near-Maxwellian, in another sense magnetically confined fusion plasmas must be almost collisionless. The mean free path between collisions for ions and electrons are both of order 10 km. As discussed in Section IIIA, it is the sensitivity of advection-diffusion equations to chaos when diffusion is small that makes the existence of constants of motion so important. Omnigeneity, $\partial J/\partial \theta = 0$, is the weakest general condition for obtaining an appropriate constant of the motion. The conservation of the action $J$ alone is generally inadequate.

Ideally, the distribution functions would be Maxwellsians with the number density $n$ and temperature $T$ functions of the toroidal magnetic flux $\psi$. But, collisionless particle trajectories make excursions $\Delta \psi$ away from their home $\psi$-surfaces. The $\Delta \psi$ of collisionless trajectories can either be proportional to the gyroradius, $\rho \equiv mv/qB$, of the particle or independent of the gyroradius as $\rho \to 0$.

When the excursions $\Delta \psi$ of all particles in a plasma of radius $a$ are proportional to the gyroradius the deviations from local Maxwellsians scale as $\vec{f} \propto \rho/a$, which is called near-Maxwellian. The confinement of near-Maxwellian plasmas is adequate for fusion power plants.
magnetic field experienced by the particle change little from one gyro-period to the next. This requires \( \rho_{||}/L \ll 1 \), where \( L \) is the scale of variations in the magnetic field in the direction along \( \vec{B} \).

For passing particles, particles that have a parallel velocity that is never zero, \( v_{||} \neq 0 \), confinement is generally assured by the existence of magnetic surfaces that are denoted by \( \psi \), the toroidal magnetic flux enclosed by the surface. Wherever \( v_{||} \neq 0 \) the particles follow the effective magnetic field \( \vec{H} \), which differs from \( \vec{B} \) by only a small perturbation. Consequently, passing particles, those that never have \( v_{||} = 0 \), are generally well confined.

\( \vec{B} \) is singular wherever \( v_{||} = 0 \), which are the turning points of trapped particles. Trapped particles bounce back and forth in the energy well produced by a variation in the field strength along the magnetic field strength, Equation (15). The drift velocity \( \vec{v}_d \) is not singular at \( v_{||} = 0 \) but need not vanish. The singularity of the effective magnetic field \( \vec{H} \) can cause trapped particles to drift across the magnetic field lines an arbitrarily large distance. For a trapped particle, the confinement of the magnetic field lines to constant-\( \psi \) surfaces is only relevant to its trajectory between turning points. The turning points need not remain on a fixed \( \psi \)-surface as is required for adequate confinement for fusion.

Trapped-particle confinement can be ensured by either a conserved canonical momentum, which is discussed in Section III A or the longitudinal action \( J \), which is discussed in Section III D. But, \( J \)-conserving trajectories can have sufficiently large excursions in \( \psi \) to strike the walls. In certain cases in which a short wavelength ripple breaks \( J \)-conservation, the trajectory excursions directly caused by the breaking can be small, Section III F.

Assuming the electric potential \( \phi \) is constant on \( \psi \)-surfaces, the magnetic field strength at which a particle has a turning point is

\[
B_t(\psi) = \frac{u - q\phi(\psi)}{\mu} \quad (16)
\]

\[
= \frac{mv^2}{2\mu}. \quad (17)
\]

Energy \( u \) and magnetic moment \( \mu \) conservation imply that \( B_t \) is constant of the motion, \( dB_t/dt = 0 \), when the particle remains within a \( \psi \)-surface.

The most elementary concept for trapped particle confinement is that small gyroradius trapped particles must always lie in a range of \( \psi \) such that

\[
B_{\max}(\psi) > B_t > B_{\min}(\psi). \quad (18)
\]

When \( B_t \) is greater than the maximum field strength on the surface, the particle becomes passing and is thereby well confined. When \( B_t \) is less than the minimum field strength on the surface, there is no place the particle can be located on the surface and have positive kinetic energy.

Unfortunately, the constraint of Equation (13) on the range of \( \psi \) that a charged particle can cover is too broad to be consistent the the confinement of a plasma that has distribution functions that are sufficiently close Maxwellian for a fusion power plant.

There are four related concepts for obtaining far better trapped particle confinement in toroidal plasma equilibria: isoaction, omnigenity, quasi-symmetry, and axisymmetry. All axisymmetric equilibria are quasi-symmetric, all quasi-symmetric equilibria are omnigenous, and all omnigenous equilibria are isoaction.

### B. Symmetry and quasi-symmetry

The most restrictive principle for obtaining trapped particle confinement in a toroidal plasma is axisymmetry, which means \( \vec{B} \) is symmetric in the toroidal angle \( \varphi \). The Hamiltonian for charged particle motion has a canonical momentum \( p_{\varphi} = mv_{\varphi} + qA_{\varphi} \), which is conserved when the magnetic and electric fields have no \( \varphi \) dependence. This constraint limits excursions of particles from their home surface to distance no greater than the gyroradius in the magnetic field component given by toroidal component of the vector potential, \( A_{\varphi} \), which is the poloidal field.

The toroidal symmetry must be broken for a magnetic field that is curl-free throughout a toroidal region to produce a poloidal magnetic field, which is the definition of a stellarator. Absence of toroidal symmetry implies the \( p_{\varphi} \) of the exact particle Hamiltonian is not conserved, so it cannot be used to ensure particle confinement in a stellarator.

When the gyroradius \( \rho \) of a particle is small, the center of the circle about which the particle gyrates moves with the drift velocity \( \vec{v}_d \), given by Equation (12) and this velocity is given by a Hamiltonian, called the drift Hamiltonian [11], which has a simple representation in Boozer coordinates, Appendix A.

The remarkable feature of the drift Hamiltonian is that if toroidal symmetry is broken but the magnetic field strength and the electric potential are functions of \( \psi \) and

\[
\zeta = N\varphi - M\theta \quad (19)
\]

with \( N \) and \( M \) mutually prime integers, then a canonical momentum of the drift Hamiltonian, \( H_d \), is conserved, \( dP_h/dt = 0 \).

\[
P_h = Np_{\theta} + Mp_{\varphi} \quad (20)
\]
The magnetic field is and will be assumed to be given in Boozer coordinates, Appendix A with \( \frac{d\psi_p}{d\psi} = \iota \), the rotational transform.

Stellarators that obey this symmetry \( B(\psi, \zeta) \) are called quasi-symmetric.

### C. Trapped particles and the action

The theory of trapped-particle confinement in toroidal plasmas is essentially the same as the theory of the radial confinement of particles in a mirror machine.

Hall and McNamara [12] introduced the concept of omnigeneity in 1975 as a way to control radial particle motions in mirror machines, where the magnetic field can be written in the Clebsch representation, Equation (1). Particles that are trapped in a magnetic mirror have three conserved quantities, \( \mu \) the magnetic moment, \( u \) the energy, and \( J(\psi, \theta_0, u) \) the longitudinal action, Equation (3) of Northrop and Teller [2]: \( \psi \) is the magnetic flux and \( \theta_0 \) is a polar angle.

#### 1. Action conservation

Action conservation, \( dJ/dt = 0 \), requires that the magnetic field experienced by the particle change little between successive bounces of a trapped particle. As will be shown, the typical time required from one bounce to the next of a trapped particle, \( \tau_0 \), is \( \approx L_p/(\sqrt{\epsilon} v) \), where \( L_p \) is the length of a period of the stellarator and \( \pm \epsilon \) is the fractional variation in the field strength. But, the bounce time \( \tau_0 \) becomes logarithmically infinite as the turning point of a trapped particle approaches a maximum of the magnetic field. The drift velocity \( \vec{v}_d \) in the magnetic field has the typical value \((\rho/R)v\) in both directions across a magnetic field line, where \( R \) is the major radius. The component of \( \vec{v}_d \) that is within the \( \psi \) surface is called the precession velocity, which gives frequency \( d\theta_0/dt = \omega_{pr} \) with which the Clebsch angle precesses. When \( \rho/L_p \ll 1 \), the action is accurately conserved except when a particle makes a sudden change in its turning points.

Action conservation is broken when the conserved field strength, \( B_1 \), at which a particle has a turning point goes from being less than to exceeding a local maximum of the field strength. The implication is that contours of constant field strength in a magnetic surface, Section III C 2 must have an infinite extent in one of the the angular coordinates, so a particle can precess forever and never have its turning point disappear. As noted by Cary and Shasharina [13, the magnetic field strength \( B(\psi, \eta) \) must be a periodic function of \( \eta = \zeta - g \), where \( \zeta = N\varphi - M\theta \), Equation (19), and \( g \) is a periodic function of \( \zeta \) and one of the angles, which can be \( \theta \) when \( N \neq 0 \) or \( \varphi \) when \( M \neq 0 \). It should be noted that when the only angular dependence of \( g \) is \( \zeta \), the field is quasi-symmetric.

#### 2. Contours of magnetic field strength

When the magnetic field strength within a magnetic surface is known in the form \( B(\theta, \varphi) \), the contours of constant field strength are defined by \( dB/ds = 0 \) and are given by the differential equations

\[
\frac{d\theta}{ds} = -\frac{\partial B}{\partial \varphi} \frac{1}{\sqrt{(\partial B/\partial \theta)^2 + (\partial B/\partial \varphi)^2}}
\]

\[
\frac{d\varphi}{ds} = +\frac{\partial B}{\partial \theta} \frac{1}{\sqrt{(\partial B/\partial \theta)^2 + (\partial B/\partial \varphi)^2}}
\]

To ensure the magnetic field strength at all turning points \( B_1 \) never passes through a local maximum of \( B \) as the particle precesses in \( \theta_0 = \theta - \iota \varphi \), these contours cannot close at finite values of both \( \theta \) and \( \varphi \), but must extend to infinity in one or both angles. In quasi-symmetry, \( B(\psi, N\varphi - M\theta) \), the angle \( \theta = \theta_s + Ns/\sqrt{M^2 + N^2} \) and the angle \( \varphi = \varphi_s - Ms/\sqrt{M^2 + N^2} \), so \( M\theta - N\varphi = M\theta_s - N\varphi_s \) is constant for \( 0 \leq s \leq \infty \).

### D. Omnigenous trajectories

When \( J(\psi, \theta_0, u) \) is conserved, \( dJ/dt = (\partial J/\partial \psi)(d\psi/dt) + (\partial J/\partial \theta_0)(d\theta_0/dt) = 0 \). Consequently, the radial excursion of a particle as it precesses in \( \theta_0 \) is

\[
\frac{\partial \psi}{\partial \theta_0} \equiv \psi'(\psi, \theta_0, u) \quad \text{with} \quad (25)
\]

\[
\psi'(\psi, \theta_0, u) = -\frac{\partial J}{\partial \psi}. \quad (26)
\]

When the excursion \( \Delta \psi \) from the home flux surface of a particle is small compared to the \( \psi \)-scale of the variation of \( J \), the excursion is the maximum minus the minimum value of the integral \( \int \psi' d\theta_0/2 \). The
motion of a particle is omnigenous when \( \partial J/\partial \theta_0 = 0 \), and only then is the scaling of radial excursions proportional to the gyroradius. Equation (27) implies the excursion of a particle from a constant-\( \psi \) surface is indeed proportional to its gyroradius \( \rho \) when the omnigenous constraint is satisfied.

When \( \rho/R << 1 \), where \( R \) is the major radius of the torus, omnigeneity requires:

- The contours of constant-\( B \) must be unbounded in at least one of the angles.
- The net drift in the \( \psi \) direction must be zero between successive turning points of trapped particles.

The importance of omnigeneity to toroidal plasmas was recognized and used in the design of the W7-X stellarator [14]. In 1997, Cary and Shasharina [13] pointed out constraints associated with omnigeneity. In particular, they showed that although an omnigenous field can be smooth it cannot be analytic, for the angular derivatives can not be continuous to all orders as required for analyticity. Some deviation from exact omnigeneity is acceptable in fusion systems. Both the design of quasi-omnigenous magnetic configurations and the adequacy of their confinement require extensive computations.

A close approximation to omnigeneity provides an important goal in codes that optimize non-axisymmetric systems for magnetic fusion, in particular stellarators.

E. Isoaction trajectories

Isoaction means the action is conserved, \( dJ/dt = 0 \), along the trajectory of a particle, but two successive turning points can lie on magnetic surfaces separated by an amount \( \delta \psi \), which is proportional to the gyroradius. The time reversal invariance of a time-independent drift Hamiltonian implies the particle on its return bounce will drift in the same direction, so its turning point over the full back and forth motion is \( 2\delta \psi \).

The drift away from flux surface, on which the turning point lies, accumulates until the precession, \( \omega_{pr} \equiv d\theta_0/dt \), carries the particle to a location in which \( \psi \) drift has the opposite sign. That the \( \psi \)-drift changes sign is implied by the Hamiltonian \( u(\psi, \theta_0, J) \). As noted in Section [IV] and proven in [1], the particle energy \( u(\psi, \theta_0, J) \) is a Hamiltonian with a canonical momentum \( q\psi \) and a canonical coordinate \( \theta_0/2\pi \).

Energy \( u = mv^2/2 + q\phi \) and \( \mu \) conservation imply the action can be written as

\[
J = mv \int \sqrt{1 - \frac{B}{B_t}} \, dl, \tag{27}
\]

where turning-point field \( B_t \) is defined in Equation (10). When the effect of the electric potential \( \phi(\psi) \) is negligible, as it is for particles of sufficiently high energy, \( B_t \), the turning point field strength, is a constant of the motion. Although the magnitude of action differs, the contours of constant action in \( \psi, \theta_0 \) space are themselves then independent of the energy, mass, and charge of the particle.

The electric potential in a plasma is typically \( |\phi| \sim T/e \), and the expression for the action is modified so \( B_t \) becomes a function of \( \psi \) and charge; \( B_t = (u - q\phi(\psi))/\mu \), rather than being a charge-independent constant of the motion, as when \( \phi = 0 \). The constant-\( J \) contours in \( \psi, \theta_0 \) depend on the energy/temperature ratio, \( u/T \), and on the sign of the charge but not on the particle mass. At equal temperatures, \( u/T \) is the same for electrons and hydrogenic ions taken from a Maxwellian. The magnetic field variation in most toroidal plasmas, \( 2\epsilon = \Delta B/B \), is significantly less than unity, which makes the energy at which the effects of \( \phi \) of the constant-action contours negligible when \( u >> T/e \).

At a given energy, \( u \), the precession speed \( \sim \rho v/R \) and the constant action surfaces are similar for electrons and hydrogenic ions. Nevertheless, the implications for transport are very different because of the electron collision frequency is approximately two orders of magnitude larger than the ion. In power-plant-like plasmas, near-thermal electrons are too collisional to follow their constant-action contours to completion while ions can. When \( \omega_{pr} < v_e v_t \), the product of the two collision frequencies, the electrons are the better confined species.

Constant-action trajectories were studied in asymmetric mirrors [15] in 1963 and is closely related to the concept of banana drift transport [16]. Burby, MacKay, and Naik have studied the general conditions for avoiding changes in their trajectory type [17], which breaks action conservation, and call such equilibria isodrastic.

F. Failure of action conservation

The most difficult place to have action conservation is near the maxima and the minima of the magnetic field strength in the magnetic surfaces. When the magnetic field strength is an analytic function of the spatial coordinates, exact omnigeneity cannot be achieved near field maxima, Cary and Shasharina [13] and Section [V]. When short wavelength variations in the magnetic field strength are present, a different issue generally breaks not only omnigeneity but also action conservation near both the maxima and the minima of the field strength.
The archetypal example of the breaking of action conservation is the short wavelength variation in the magnetic field strength due to the \( N_c \approx 20 \) toroidal field coils of a tokamak. For a review, see [18]. The magnetic field strength in a tokamak can approximated as

\[
B = B_0 \left( 1 + \epsilon_\ell \cos \theta + \epsilon_c \cos(N_c \varphi) \right),
\]

where \( \epsilon_\ell = r/R \) is the inverse aspect ratio, which is orders of magnitude larger than \( \epsilon_c \), the field strength variation, or ripple, due to the \( N_c \) toroidal field coils.

All two-helicity expressions for the magnetic field strength have related transport properties [6], so any quasi-symmetry broken by a short wavelength perturbation behaves similarly to a tokamak with toroidal ripple. The effects of ripple on transport are considered in more detail in Appendix B just a summary will be given here.

Although \( \epsilon_c \ll \epsilon_\ell \), the existence of non-zero \( \epsilon_c \) is important to study the importance of the weak ripple regime, since action-breaking short wavelength ripple-like magnetic wells are difficult to avoid near maxima and minima of an otherwise quasi-symmetric or omnigenous magnetic field strength. Indeed, the existence of non-zero \( \epsilon_c \)-like effects limits the effort that should be placed on approximating omnigeneity near field maxima.

It should be noted that magnetic field ripple effects do not necessarily result from having space between the coils. For example, a helically symmetric stellarator can be supported by two, or even one, helical wires. A canonical momentum is then exactly conserved, and coil ripple does not cause enhanced neoclassical transport.

The design of coils that have minimal ripple transport while having the largest possible space between the coils for access is relatively unexplored. In quasi-helical symmetry, the basic concept is that the helical path of the coils should be that of the magnetic field strength, \( B(\psi, \theta - N_\varphi) \).

IV. DERIVATIVES OF THE ACTION

The derivatives of the action that require differentiation of the magnetic field are subtle because consistent coordinates must be used. Section 1V.A explains how this is done using Boozer coordinates, Appendix A which simplify transport calculations.

One derivative of the action does not involve the magnetic field. That derivative is with respect to the energy \( u \) and gives the time required for a trapped particle to bounce between its two turning points,

\[
\tau_b = \int \frac{\mathrm{d}t}{v_{||}} = \frac{\partial J}{\partial u} \tag{30}
\]

The action \( J \) is a function of \( (\psi, \theta_0, u) \), which implies the energy \( u \) can be written as a function \( u(\psi, \theta_0, J) \). As shown in Section VI.D.4 of [1], \( u(\psi, \theta_0, J) \) is a Hamiltonian for \( \mathrm{d}\psi/\mathrm{d}t \) and \( \mathrm{d}\theta_0/\mathrm{d}t \) with a canonical momentum \( \psi \) and a canonical coordinate \( \theta_0/2\pi \). This canonical form gives a constraint on the action-conserving drifts.

A. \( J \) in magnetic coordinates

When the magnetic field strength \( B(\psi, \theta, \varphi) \) is given in Boozer coordinates, the infinitesimal distance along the magnetic field lines \( \mathrm{d}l \) must be given in those coordinates as well. The position vector in Boozer coordinates is \( \vec{x}(\psi, \theta_0, \varphi) \). A magnetic field line in the field \( 2\pi \vec{B} = \vec{\nabla} \psi \times \vec{\nabla} \theta_0 \) has a fixed \( \psi \) and \( \theta_0 = \theta - \varphi \). Equation (A3) for \( \vec{B} \) and the orthogonality relations of general coordinates, Appendix A imply

\[
\frac{\mathrm{d}l}{\mathrm{d}\varphi} = \frac{\vec{B}}{B} \left( \frac{\partial \vec{B}}{\partial \varphi} \right)_{\psi(\theta_0)} \tag{32}
\]

\[
= \frac{\mu_0(G + iL)}{2\pi B}. \tag{33}
\]
Using Equation (27), the action \( J(\psi, \theta, u) \) can be written in magnetic coordinates as

\[
J = \sqrt{2mu} \frac{\mu_0 (G + iI)}{2\pi} \int \sqrt{1 - \frac{B}{B_t} \frac{d\varphi}{B_t} B}.
\]  

(34)

### B. Departure from omnigeneity

The most important derivative of the action for a discussion of omnigeneity is \( \partial J/\partial \theta_0 \), which must be zero when omnigeneity holds.

The derivative

\[
d\left( \sqrt{1 - \frac{B}{B_t} B} \right) \frac{d\theta_0}{dB} = - \frac{1 - \frac{B}{B_t} B}{B^2 \sqrt{1 - \frac{B}{B_t} B}}, \quad \text{so} \quad \frac{\partial J}{\partial \theta_0} = - \sqrt{2mu} \frac{\mu_0 (G + iI)}{2\pi} \int \frac{1 - \frac{B}{B_t} B}{B^2 \sqrt{1 - \frac{B}{B_t} B}} \frac{dB}{\theta_0} d\varphi.
\]

(35)

(36)

1. **Simple measure of departure**

A simple measure of the departure from exact omnigeneity on a magnetic surface for use in optimization calculations is the dimensionless quantity

\[
\frac{1}{J} \frac{\partial J}{\partial \theta_0} = - \frac{\int \frac{1 - \frac{B}{B_t} B}{B^2 \sqrt{1 - \frac{B}{B_t} B}} \frac{dB}{\theta_0} d\varphi}{\int \frac{1 - \frac{B}{B_t} B}{B^2 \sqrt{1 - \frac{B}{B_t} B}} d\varphi}.
\]

(37)

\( B_t \) is a constant, the magnetic field strength at the turning points of the particle, Equation (17), \( \theta_0 \) is a constant that gives the trajectory of the particle along a magnetic field line between bounces, and the \( \varphi \) integration is between two successive turning points. Given the field strength in Boozer coordinates \( B(\psi, \theta, \varphi) = B(\psi, \theta_0 + \varphi, \varphi) \), exact omnigeneity on the magnetic surface \( \psi \) requires the expression for \( \left( \partial J/\partial \theta_0 \right)/J \) of Equation (37) be zero for all \( B_t \) and \( \theta_0 \) such that \( B_t > B \). The integration can be started from an arbitrary \( \varphi \) location by choosing the initial \( \theta \) such that \( B_t > B \) and integrating forward and backward to the two \( \varphi \) locations where \( B = B_t \).

The constant-\( \varphi \) radial excursions are given by \( \left( \partial J/\partial \theta_0 \right)/\left( \partial J/\partial \psi \right) \) and this ratio has been used in optimization studies (19) by minimizing

\[
\gamma_\varphi = \frac{2}{\pi} \arctan \left( \frac{\partial J/\partial \theta_0}{\partial J/\partial \psi} \right).
\]

(38)

When the electric potential \( \phi \) satisfies \( d\phi/d\psi = 0 \), \( \partial J/\partial \psi \) generally passes through zero between deeply trapped and barely trapped particles. When \( d\phi/d\psi \neq 0 \), the derivative \( \partial J/\partial \psi \) is always zero for a certain kinetic energy for either electrons and ions.

Dividing \( \partial J/\partial \theta_0 \) by \( J \) corrects for the intrinsic smallness of \( J \) and its derivatives for deeply trapped particles without introducing the subtleties associated with the vanishing of \( \partial J/\partial \psi \) in particular parts of velocity space.

2. **\( S(\psi, \theta_0, B_t) \) as a measure**

Another measure of the departure from omnigeneity \( S(\psi, \theta_0, B_t) \) will be used in Sections V.A and V.B to construct omnigenous fields.

The integration variable in Equation (36) can be changed from \( d\varphi \) to \( dB/(\partial B/\partial \varphi) \). Then,

\[
\frac{\partial J}{\partial \theta_0} = \frac{\mu_0 (G + iI)}{2\pi N_p B_t} S(\theta_0, B_t), \quad \text{where} \quad (39)
\]

\[
S = -N_p B_t \int_{B_{min}}^{B_t} \frac{1 - \frac{B}{B_t} B}{B^2 \sqrt{1 - \frac{B}{B_t} B}} \frac{dB}{\partial B/\partial \varphi}.
\]

(40)

where \( N_p \) is the number of periods of the stellarator.

The integrand in Equation (40) can be rewritten using

\[
\left( \frac{\partial B}{\partial \psi} \right)_0 = - \left( \frac{\partial \varphi}{\partial \theta_0} \right)_B,
\]

(41)

\[
d\psi = \frac{\partial B}{\partial \psi} d\psi + \frac{\partial B}{\partial \theta_0} d\theta_0 + \frac{\partial B}{\partial \varphi} d\varphi.
\]

(42)

There is a subtlety in Equation (40) that must be addressed. At each value of \( B \) in the range \( B_t \geq B \geq B_{min} \) represents at least two locations along a magnetic field line. Regions in which \( \left( \partial B/\partial \varphi \right)_0 > 0 \) must be distinguished from those in which \( \left( \partial B/\partial \varphi \right)_0 < 0 \). This can be done by writing

\[
\left( \frac{\partial \varphi}{\partial \theta_0} \right)_B > \left( \frac{\partial \varphi}{\partial \theta_0} \right)_B \quad \text{and} \quad \left( \frac{\partial \varphi}{\partial \theta_0} \right)_B < \left( \frac{\partial \varphi}{\partial \theta_0} \right)_B.
\]

(43)

(44)

The dimensionless function \( S(\theta_0, B_t) \) can be rewritten as

\[
S = N_p B_t \int_{B_{min}}^{B_t} \frac{1 - \frac{B}{B_t} B}{B^2 \sqrt{1 - \frac{B}{B_t} B}} \frac{dB}{\partial B/\partial \varphi} B > \left( \frac{\partial \varphi}{\partial \theta_0} \right)_B \leq dB.
\]

(45)
Equation (45) defines a sufficient condition for exact omnigeneity on a magnetic surface $\psi$. Exact omnigeneity is obtained when the dimensionless quantity $S(\psi, \theta_0, B_t) = 0$ for every possible value for $B_t$ and $\theta_0$ as is the case when

$$\left(\frac{\partial \varphi}{\partial \theta_0}\right)_B = \left(\frac{\partial \varphi}{\partial \theta_0}\right)_{B<}$$

at each $(\theta_0, B)$. (46)

One can use the symmetry in the derivative of $\zeta \equiv N\varphi - M\theta$ instead $\varphi$;

$$\left(\frac{\partial \varphi}{\partial \theta_0}\right)_B = \frac{M}{N - iM} + \frac{\partial \zeta}{\partial \theta_0} B.$$ (47)

In quasi-symmetry, $(\partial \zeta/\partial \theta_0)_B = 0$, and Equation (46) is automatically satisfied.

C. The bounce time

The expression for the bounce time in terms of the action was obtained in Equation (51). Here, that expression will be expressed in terms of $B_t$ and $B$ and evaluated for deeply trapped particles.

$$\tau_b = \frac{\partial J}{\partial u}$$

$$= \frac{\mu_0 (G + i\lambda)}{2\pi} \int B \sqrt{\frac{2}{\pi}} \frac{d\varphi}{u - q\varphi - \mu B}.$$ (49)

$$= \frac{\mu_0 (G + i\lambda)}{N_p v B_t} D_\tau, \text{ where}$$

$$D_\tau \equiv \frac{B_t N_p}{2\pi} \int_{\varphi_1}^{\varphi_1} \frac{d\varphi}{B \sqrt{1 - \frac{B}{B_t}}}.$$ (50)

$\varphi_i$ is the turning point of the banana, $N_p$ is the number of periods of the stellarator, and $D_\tau$ is the dimensionless duration of the time between turning points.

The bounce time depends on the $\varphi$ dependence of the field strength along a field line. A simple assumption is $B = B_{max} - (B_{max} - B_{min})(1 + \cos N\varphi)/2$ with $N$ the number of periods. Even this assumption gives a complicated integral which is logarithmically singular as $B_t \to B_{max}$. Nevertheless, the bounce time near the field minimum can be easily calculated and is representative of the bounce time except for barely trapped particles. Near the minimum has the form $B = B_{min} + (B_{max} - B_{min})N^2\varphi^2/4$ and

$$\sqrt{1 - \frac{B}{B_t}} = N\varphi_t \sqrt{\frac{\epsilon}{2}} \sqrt{1 - \left(\frac{\varphi}{\varphi_t}\right)^2}, \text{ where}(52)$$

$$\epsilon \equiv \frac{B_{max} - B_{min}}{2B_t}. \quad \text{Since}$$

$$\int_1^\epsilon \frac{ds}{\sqrt{1 - s^2}} = \pi$$

$$\tau_b = \frac{\mu_0 (G + i\lambda)}{2\pi N_p v B_t}, \text{ so}$$

$$D_\tau \approx \frac{1}{2\epsilon}.$$ (56)

when $\epsilon << 1$. The major radius $R \equiv \mu_0 (G + i\lambda)/(2\pi B)$ and the length of a period of a stellarator is $L_p \equiv 2\pi R/N$, so the characteristic bounce time for trapped particles is

$$\tau_b = \frac{L_p}{\sqrt{2\epsilon v}}.$$ (57)

D. The precession frequency

The precession frequency, $\omega_{pr} = -2\pi(\partial J/\partial \psi)/q\tau_b$, is determined by the most complicated derivative of the action, $\partial J/\partial \psi$. This derivative has three terms: one proportional to $d(G + i\lambda)/d\psi$, another proportional to $d\varphi/d\psi$, and a third proportional to $\partial B/\partial \psi$. The first term, which is $(d\ln(G + i\lambda)/d\psi) J$, is usually not important in stellarators and will be ignored in the expressions given below for the sake of simplicity.

Differentiating Equation (51) for $J$ with respect to $\psi$ and ignoring $d(G + i\lambda)/d\psi$,

$$\frac{\partial J}{\partial \psi} = -q \frac{d\varphi}{d\psi} \tau_b$$

$$- \frac{\mu_0 (G + i\lambda)}{2\pi} mv \int \frac{1 - \frac{B}{B_t}}{B^2 \sqrt{1 - \frac{B}{B_t}}} \partial B \frac{d\varphi}{\partial \psi} d\varphi$$

$$= - \left(q \frac{d\varphi}{d\psi} + mv^2 \frac{\partial \ln(B)}{\partial \psi}\right) \tau_b,$$ (58)

where

$$\langle \frac{\partial \ln(B)}{\partial \psi} \rangle = \int \frac{1 - \frac{B}{B_t}}{B^2 \sqrt{1 - \frac{B}{B_t}}} \frac{\partial B}{\partial \psi} d\varphi$$

$$= \int \frac{d\varphi}{B \sqrt{1 - \frac{B}{B_t}}.$$ (60)

The precession frequency is then

$$\omega_{pr} = 2\pi \frac{d\varphi}{d\psi} + 2\pi \frac{mv^2}{q} \frac{\partial \ln(B)}{\partial \psi}.$$ (61)

In a thermal plasma, $mv^2/2$ is independent of the mass of the particle. When the electron and ion temperatures are equal, the electric and magnetic
The implication is that \( S(\psi, \theta_0, B_t) \) must be zero for all \( \theta_0 \) for a particle with turning points at \( B = B_t \) to have the distance in flux that the particle drifts, \( \Delta \psi \), be proportional to its gyroradius \( \rho \). (2) As discussed in Section \( \text{III C} \) when the field strength is written as \( B(\psi, \eta) \), the constant-\( \eta \) curves, which must be closed and obey periodicity constraints:

\[
\eta = \zeta - g(\theta, \eta), \quad (62)
\]

\[
\zeta = N \varphi - M \theta, \quad \text{and} \quad (63)
\]

\[
\theta_0 = \theta - \frac{\upsilon}{\eta} (M \theta + g(\theta, \eta) + \eta) \quad \text{(64)}
\]

since \( \theta = \theta_0 + \nu \varphi \). \( N \) and \( M \) are integers that have no integer factor other than unity in common, and \( g(\theta, \eta) \) must be periodic in its arguments. The form of Equation (62) assumes \( N \neq 0 \). If \( N = 0 \), then \( g \) must be chosen to depend on \( \varphi \) and \( \eta \) rather than \( \theta \) and \( \eta \) to have two independent variables.

The function \( (\partial \varphi / \partial \theta_0)_B = (\partial \varphi / \partial \theta_0)_\eta \), and

\[
\left( \frac{\partial \varphi}{\partial \theta_0} \right)_\eta = \frac{M}{N - \nu M} + \frac{1}{N - \nu M} \left( \frac{\partial \zeta}{\partial \theta_0} \right)_\eta \quad (65)
\]

\[
= \frac{M}{N - \nu M} + \frac{(\partial g/\partial \theta_0)_\eta}{N - \nu M} \left( \frac{\partial \theta}{\partial \theta_0} \right)_\eta \quad (66)
\]

When \( \nu / N \) and \( g / N \) are small, as they are in cases of interest, \( (\partial \eta/\partial \varphi)_B \approx N \). The implication is that the sign of \( (\partial B/\partial \varphi)_0 \) is determined by the sign of \( (\partial B/\partial \eta)_0 \). The symmetry required for omnigenicity, \( (\partial \varphi / \partial \theta_0)_B = (\partial \varphi / \partial \theta_0)_B < \), is equivalent to the symmetry \( (\partial \zeta / \partial \theta_0)_> = (\partial \zeta / \partial \theta_0)_< \) since \( M/(N - \nu M) \) is symmetric.

Equation (62) implies

\[
\left( \frac{\partial \zeta}{\partial \theta_0} \right)_\eta = \left( \frac{\partial g(\theta(\theta_0, \eta), \eta)}{\partial \theta} \right)_\eta, \quad \text{where} \quad (67)
\]

\[
\theta(\theta_0, \eta) = \frac{\theta_0 + \frac{\upsilon}{\eta} \eta + \frac{\nu}{N} g(\theta, \eta)}{1 - \frac{\nu}{N} M}. \quad (68)
\]

The function \( g(\theta, \eta) \) must be chosen so \( (\partial \zeta / \partial \theta_0)_\eta > = (\partial \zeta / \partial \theta_0)_\eta < \) to obtain the proper symmetry in \( \eta \). Subtleties of this choice will be illustrated in Sections \( \text{V A} \) and \( \text{V B} \) by the relatively simple case in which \( M = 0 \) and \( g \) is sufficiently small that only terms linear in \( g \) need to be retained.

The first term on the right-hand side of Equation (60) is the quasi-symmetric form for \( (\partial \varphi / \partial \theta_0)_\eta \); the second term gives the deviation of an omnigenous magnetic field from quasi-symmetry. This deviation has a \( \theta \)-average that is zero since

\[
0 = \oint \frac{\partial \varphi}{\partial \theta_0} d\theta = \frac{M}{N - \nu M} \oint \left( \frac{\partial \varphi}{\partial \theta_0} \right)_\eta \left( \frac{\partial \theta_0}{\partial \theta} \right)_\eta d\theta. (70)
\]

A. Omnigeneity near \( B_{min} \)

By an appropriate choice of the starting points of the angles \( (\theta, \varphi) \), a field strength minimum can be taken to be at \( \eta = 0 \). To simplify the discussion it will be assumed that \( M = 0 \) and that only terms that are linear in \( g \) need to be retained.

\( M = 0 \) stellarators, which are called isodynamic [14] or equivalently quasi-poloidal [20], are a particularly interesting special case. The results derived for them are related to the properties of omnigenous stellarators in general.

Calculating to order \( \eta^2 \),

\[
g = g_0(\theta) + g_1(\theta) \eta + g_2(\theta) \eta^2 + \cdots \quad \text{(71)}
\]

\[
\theta = \theta_0 + \frac{\upsilon}{N} \eta + \frac{\nu}{N} g_0(\theta_0) \cdots \quad \text{(72)}
\]

\[
g = g_0(\theta_0) + g_1(\theta_0) \frac{\upsilon}{N} \eta + \frac{1}{2} g_2(\theta_0) \left( \frac{\upsilon}{N} \right)^2 \eta^2 + g_1(\theta_0) \eta + g_1(\theta_0) \frac{\nu}{N} \eta^2 + g_2(\theta_0) \eta^2. \quad \text{(73)}
\]

The required symmetry in \( (\partial \zeta / \partial \theta_0)_\eta \) implies \( (\partial g / \partial \theta_0)_\eta \) cannot contain any odd powers of \( \eta \). The implication is that

\[
g_1(\theta_0) = -\frac{\nu}{N} g_0''(\theta_0) \quad \text{and} \quad (74)
\]

\[
\left( \frac{\partial \zeta}{\partial \theta_0} \right)_\eta = g_0(\theta_0) + \left[ g_2(\theta_0) - \left( \frac{\nu}{N} \right)^2 g_0''(\theta_0) \right] \eta^2 \quad \text{(75)}
\]

through \( \eta^2 \) order. The functions \( g_0(\theta) \) and \( g_2(\theta) \) are arbitrary, but \( g_1(\theta) \) must be chosen to cancel a linear term in \( \eta \).

B. Omnigeneity near \( B_{max} \)

The discussion in Section \( \text{V A} \) of omnigeneity near a minimum of a magnetic field can be applied near a
field maximum as well in a mirror machine, but not in a torus. When \( \eta = 0 \) at a minimum, small values of \( \eta \), positive and negative, are the two sides of the well. When the magnetic field has only one well and \( B = B_0(1 - \epsilon_B \cos \eta) \), successive maxima are at \( \eta = -\pi, \pi, \) etc. Unlike in a mirror machine, \( \eta = -\pi \) and \( \eta = \pi \) are the same contour in a torus and must have the same deformation \( \partial g(\theta(\theta_0, \eta), \eta) / \partial \theta_0 \).

To illustrate the subtlety, as before let \( M = 0 \) and calculate only to linear order in \( g \). Equation (65) is then \( \theta = \theta_0 + (\epsilon/\eta) \eta \). The two sides of the same constant-\( B \) contour have \( \eta = \pi - \delta \eta \) and \( \eta = -\pi + \delta \eta \), or

\[
\frac{\partial g(\theta_0 + \frac{\epsilon}{\eta}(\pi - \delta \eta), \pi - \delta \eta)}{\partial \theta_0} = \frac{\partial g(\theta_0 - \frac{\epsilon}{\eta}(\pi - \delta \eta), -\pi + \delta \eta)}{\partial \theta_0} \quad (76)
\]

for all \( \theta_0 \) and \( \delta \eta \). The periodicity of \( g(\theta, \eta) \) in \( \theta \) makes this impossible when \( \epsilon \) is not an integer. Not only must \( g \) vanish at \( \eta = \pm \pi \) but also all its derivatives with respect to \( \delta \eta \).

The behavior at the field maximum is in effect the proof of Cary and Shasharina [13] that not only must the magnetic field have the quasi-symmetric form \( B_{\text{max}}(N \varphi - M \theta) \) at its maximum, but also any deviation from the quasi-symmetry from for values of \( \eta \) away from the maximum requires the magnetic field strength be non-analytic.

The lack of analyticity is not surprising given that magnetic field must have the exact quasi-symmetric form at the surface maximum for all values of \( \theta_0 \), but the extent of the implied deviations from omnigenicity are unclear as are the effects on transport at low collisionality. As discussed in Section III F small but short wavelength perturbations often prevent action conservation near maxima of the field strength but need not have an unacceptable effect on confinement.

Assuming action is conserved, Section VII D will examine the effects of known departures from omnigenicity on transport.

VI. EFFECTS OF TRAPPED PARTICLE DRIFTS

A. General relations

Equation (46) for exact omnigenicity can be understood using the explicit radial drift to obtain the deviation \( \delta \psi_b \) of particles from the \( \psi \) surfaces between turning points. The deviation \( \delta \psi_b \) is called the banana width because of the shape of the trajectories.

The drift velocity is given by Equation (12), and the drift velocity across a magnetic surface that encloses a toroidal flux \( \psi \) is

\[
\vec{v}_d \cdot \nabla \psi = \frac{mv^2}{qB^2} \vec{B} \times \left( \frac{v^2}{2} \kappa + \frac{v^2}{2v_B} \nabla \ln B \right) \cdot \nabla \psi, \quad (77)
\]

\[
(\vec{B} \times \kappa) \cdot \nabla \psi = (\vec{B} \cdot \nabla \ln B) \cdot \nabla \psi, \quad (78)
\]

\[
\vec{v}_d \cdot \nabla \psi = \frac{mv^2}{qB^2} \left(1 - \lambda B \right) (\vec{B} \times \nabla \ln B) \cdot \nabla \psi, \quad (79)
\]

where \( \kappa \equiv \vec{b} \cdot \nabla \vec{b} \) is the curvature of a magnetic field line. The electric potential has been assumed to be constant on the magnetic surfaces, \( \phi(\psi) \).

The first order in \( \rho \) change in the \( \psi \) position of a particle is given by \( \delta \psi = (\vec{v}_d \cdot \nabla \psi) dt \), where \( dt = d\ell / |\vec{v}| \). Since \( \vec{B} \cdot \nabla \vec{B} = B \partial B / \partial \vec{B} \) along a line, \( d\ell = (\vec{B} \cdot \nabla \vec{B}) d\vec{B} \) along the line. Consequently, the change in \( \psi \) as the field strength \( B(\psi, \theta, \varphi) \) changes on a particular magnetic field line \( (\psi, \theta_0) \) is

\[
\left( \frac{\partial \psi}{\partial B} \right)_{\theta_0} = \frac{mv \left(1 - \frac{\lambda B}{2} \right) (\vec{B} \times \nabla \ln B) \cdot \nabla \psi}{qB^2 \sqrt{1 - \lambda B}} \frac{\vec{B} \cdot \nabla \psi}{\vec{B} \cdot \nabla \vec{B}} \quad (80)
\]

where

\[
Y = \frac{(\vec{B} \times \nabla \vec{B}) \cdot \nabla \psi}{\vec{B} \cdot \nabla \vec{B}}, \quad (82)
\]

\[
\frac{\mu_0 I}{\nabla B} \left( \frac{\partial B}{\partial \theta} \right)_{\theta_0} = \mu_0 I + \frac{\mu_0 (G + iI)}{N - iM} \left( \frac{\partial \varphi}{\partial \theta} \right)_{\theta_0} = Y_{qs} + Y_o. \quad (84)
\]

Equation (81) implies the deviation of a particle from a \( \psi \) surface scales as \( mv/\sqrt{B} \), which is its gyroradius.

Using Equation (65) the part of \( Y \) that is equivalent to that in a quasi-symmetric system \( Y_{qs} \) and the additional part \( Y_o \) that arises from the system being omnigenous but not quasi-symmetric are

\[
Y_{qs} \equiv \frac{\mu_0 I + \frac{\mu_0 (G + iI) M}{N - iM}}{N - iM} \quad \text{and} \quad (86)
\]

\[
Y_o \equiv \frac{\mu_0 (G + iI)}{N - iM} \left( \frac{\partial \varphi}{\partial \theta} \right)_{\theta_0} \quad (87)
\]

The function \( Y \) was defined [21] by Plunk, Landreman, and Helander in 2019. The form of \( Y \) in Equation (82) is identical to the upsilon of Equation (59) in Hall and McNamara’s discussion of omnigenicity [12].
B. Banana orbits and bootstrap current

In 2009, Helander and Nührenberg showed that for the special case of current-free isodynamic stellarators, which means $M = 0$, the bootstrap current in perfect omnigeneity is zero. Here it is shown that the bootstrap current is given by the quasi-symmetric part of the field strength and not modified by the deviation of an omnigenous stellarator from quasi-symmetry.

The banana width of a trapped particle as it goes from its turning point at $B_t$ to $B_{min}$ in the direction in which $(\partial B/\partial \phi)_{\theta_0}$ is positive

\[
\delta \psi_{b>} \equiv \int_{B_{min}}^{B_t} \left( \frac{\partial \psi}{\partial B} \right)_{\theta_0} dB 
\]

\[
= \frac{mv}{qB_{min}^2} \sqrt{1 - \frac{B_{min}}{B_t}} (Y_s + Y_o) dB 
\]

The banana width is the sum of a part that is equivalent to that of a quasi-symmetric system $\delta \psi_{b,qs}$ and a part $\delta \psi_{b,o}$ that is due to the system being omnigenous but not quasi-symmetric. The two parts are

\[
\delta \psi_{b,qs} = -\mu_0 \left( I + \frac{MG + NI}{N - \I M} \right) \frac{mv}{qB_{min}} \sqrt{1 - \frac{B_{min}}{B_t}}; 
\]

\[
\delta \psi_{b,o} = -\frac{\mu_0 G + \I M}{qN_p B_t} S_>, 
\]

\[
S_> \equiv N_p B_t \int_{B_{min}}^{B_t} \frac{1}{B_{min}^2} \left( \frac{\partial \zeta}{\partial \theta_0} \right)_{B>} dB. 
\]

Equation (70) implies that the average of contribution of $\rho_{b,o}$ to the banana orbit width vanishes when integrated over the full range of $\theta_0$. Since the bootstrap current is due to barely trapped particles being systematically outside of their home flux surface on one leg of the trajectory and systematically inside on the other leg, the omnigenous contribution to the banana orbit does not contribute to the bootstrap current.

The bootstrap current in an omnigenous magnetic field equals the value that would have had had the field been quasi-symmetric, $B(\psi, N\phi - M\theta)$.

C. Systematic drift of trapped particles

The banana width of a trapped particle as it goes from its turning point at $B_t$ to $B_{min}$ in the direction in which $(\partial B/\partial \phi)_{\theta_0}$ is negative, $\delta \psi_{b<}$ is the same except the $S_>$ is replaced by

\[
S_< \equiv N_p B_t \int_{B_{min}}^{B_t} \frac{1}{B_{min}^2} \left( \frac{\partial \zeta}{\partial \theta_0} \right)_{B<} dB. 
\]

The turning points of a trapped particle are on the same $\psi$ surface only when $S_< = S_>$. The difference $\delta \psi_{b>} - \delta \psi_{b<}$ gives the radial drift per bounce, which is also given by Equation (15) for $S = S_> - S_<$ since the quasi-symmetric part of the banana motion always obeys the required symmetry.

D. Effects of departures from omnigeneity

The importance of departures from exact omnigeneity on the transport of plasma depends on (1) at what value of $B$ the departure occurs and (2) the collisionality. Here it will be assumed that the action is conserved exactly.

1. Collisionless trajectories

The effect of a departure from omnigeneity on collisionless trajectories is described by $\partial \psi/\partial \theta_0 = \psi'(\psi, \theta_0, u)$, where $\psi' = -\partial J/\partial \psi$, Equation (20).

Equation (89) for $\partial J/\partial \theta_0$ can be rewritten as

\[
\psi' = -\frac{mv^2}{q} \left( \frac{\partial \ln(B)}{\partial \theta_0} \right) \tau_0 \text{ with } \tau_0; \quad \psi' = -\frac{mv^2}{q} \left( \frac{\partial \ln(B)}{\partial \psi} \right) \frac{1}{B} \text{ with } \frac{1}{B} \psi_0. 
\]

Using Equation (59),

\[
\psi'(\psi, \theta_0, u) = -\frac{1}{q} \left( \frac{\partial \ln(B)}{\partial \theta_0} \right) \tau_0 + \left( \frac{\partial \ln(B)}{\partial \psi} \right) \frac{du}{d\psi}. 
\]

As discussed in Section [1111], the deviation $\Delta \psi$ of a particle from its home flux surface is given by half of maximum minus the minimum value of the integral $\int \psi' d\theta_0$ provided the precession, which is represented by the denominator of Equation (97), does not vanish.

The precession frequency is given by Equation (54). For particles with energies comparable to or below the thermal energy $mv_0^2/2 = 3T/2$, the derivative of electric potential, $d\phi/d\psi$, tends to dominate the precession while at higher energies the derivative $\partial \ln(B)/\partial \psi$ dominates.
The potential $\phi$ is generally set by the need to balance the pressure gradient of the poorer confined species, which makes $|q\phi| \approx T$.

The variation in the magnetic field strength is determined by the curvature $k$ of the magnetic field lines or the displacement of the magnetic field by the pressure. The field line curvature has different signs at different places on the magnetic surface, which means the precession produced by the magnetic field generally has different signs for deeply and barely trapped particles and passes through zero for some turning-point value. For thermal particles, the precession due to $d\ln(B)/d\psi$ is smaller by a factor $ak$ relative to the drift due to $d\phi/d\psi$. For high energy particles, $mv^2 >> T$ the precession due to $d\phi/d\psi$ is negligible.

When the denominator of Equation (97) vanishes, the excursions are especially large with $\delta\psi$ negligible.

2. Effects of collisions

The importance of collisionality depends on the mean free path $\Lambda_{mfp}$, the characteristic distance a particle goes before undergoing a 90° scatter within its $v^L/v$ cosine function. $\Lambda_{mfp}$ is similar for electrons and ions, approximately 10 km in a power plant, and enters results in two ways: (1) as the ratio of $\Lambda_{mfp}/L_p$, where $L_p$ is the length of a period and (2) as the ratio of $\rho \Lambda_{mfp}/L_p^2$. The gyroradius $\rho$ differs by the square root of the mass ratio $\rho_i/\rho_e = \sqrt{m_i/m_e}$, which is almost two orders of magnitude.

The collisionality of omnigenious plasmas depends on the ratio of the precession frequency to the effective collision frequency $\nu_{eff} = \nu_c/\sqrt{\epsilon_B}$, where $\nu_c$ is the collision frequency and $\epsilon_B = (B_{max} - B_{min})/(B_{max} + B_{min})$. The Fokker-Planck collision operator is diffusive so the time it takes to scatter through all trapped particle pitch angles is $\sqrt{\epsilon_B}/\nu_c$.

Simple step-size arguments imply that the diffusion coefficient of a particle in $\psi$ is given by

$$D \approx \left(\frac{d\psi}{dt}\right)^2 \frac{\nu_{eff}}{\nu_{eff}^2 + \omega_{pr}^2}$$  \hspace{1cm} (98)

$$\frac{d\psi}{dt} = \frac{2\pi}{q} \frac{mv^2 S}{D}, \text{ using}$$  \hspace{1cm} (99)

$$\frac{d\psi}{dt} = -\frac{2\pi}{q} \left(\frac{\partial u}{\partial \theta_0}\right)_{\psi,J} = \frac{2\pi}{q} \frac{\partial J/\partial \theta_0}{\partial J/\partial \psi}, \hspace{1cm} (100)$$

Units can be surprising; $mv^2/q$ has units of magnetic flux divided by time. The important quantity is $(D)$, an average over the magnetic surface of a Maxwellian distribution.

For ions or electrons, $\nu_{th} \approx 10^4(T/B)$ in meters-squared per second when the temperature $T$ is given in units of 10 keV and $B$ in Tesla, so $\omega_{pr} \approx 10^4(T/(Ba^2)$ with the minor radius in meters. The collision frequency for deuterium ions is $\nu_d \approx 51n/T^{3/2}$ with $n$ in units of $10^{20}m^{-3}$. The collision frequency for electrons is $\nu_e \approx 4 \times 10^4n/T^{3/2}$. Consequently,

$$\frac{\omega_{pr}}{\nu_d} \approx 200T^{1/2}/Bn^2; \hspace{1cm} (101)$$

$$\frac{\omega_{pr}}{\nu_e} \approx 2.5T^{3/2}/Bn^2. \hspace{1cm} (102)$$

VII. DISCUSSION

Adequate confinement of small gyroradius particles in toroidal magnetic configurations is obtained when the magnetic field satisfies the omnigenity constraint exactly. This constraint is simply stated in Boozer coordinates.

Two properties of a magnetic field determine whether it is omnigenous: (1) All of the constant $B$ contours on the magnetic surfaces must be unbounded in at least one of the angular coordinates. (2) The magnetic field strength must have a certain symmetry so that the two turning points of trapped particles lie on the same magnetic surface. This condition is satisfied when $(\partial \zeta/\partial \theta_0)\psi_B$ is the same function of $B$ when $\vec{B} \cdot \nabla B$ is positive and negative; $\zeta \equiv N\varphi - M\theta$.

In exact omnigenity, the field strength at maxima has the form $B_{max}(\psi, M\theta - N\varphi)$. When this form is analytically extended to apply to all $\theta$ and $\varphi$ and not just at the field maximum, the resulting field strength is quasi-symmetric, $B(\psi, M\theta - N\varphi)$. An implication is that exact omnigenity is not consistent with analyticity unless the field is exactly quasi-symmetric. This was shown by Cary and Shasharina [13], but they also showed analytic fields could be close to omnigenity but far from quasi-symmetry.

Since exact omnigenity is in principle not possible except in quasi-symmetry, it is important to study how exact omnigenity can be broken while minimizing the effect on confinement, which is the subject of Section VII D. The fundamental difficulty in obtaining exact omnigenity is at magnetic field maxima,
but as in the case of a tokamak with ripple, achieving action conservation close to a field maximum is extremely difficult but apparently not extremely important.

If the omnigeneous constraint were exactly satisfied, departures from Maxwellian distributions would scale as the gyroradius to system size. It is important to investigate whether omnigenity gives significant additional freedom in the design of stellarators that could be quasi-symmetric. This is in addition to the use of omnigenity to design stellarators of the W7-X type, in which $M = 0$, that do not have a quasi-symmetric limit.

An important starting point in an investigation of how accurately omnigeny can be achieved is to study a particular magnetic surface that contains a fixed toroidal flux $\psi$ in a curl-free magnetic field $[22]$. This surface should contain about half the flux that is expected to be enclosed by the outermost confining magnetic surface since this is where the temperature gradient tends to be largest in most models of fusion plasmas. The trajectory confinement properties near the magnetic axis of a toroidal plasma are irrelevant since analyticity requires that $\nabla p = 0$ there for quantities such as pressure, density, or temperature.

The different effects of like and unlike particle collisions are particularly important for ion transport. The constraints of non-equilibrium thermodynamics, Section II.A.1, hold when the confinement times of electrons and ions are long compared to their collision times $[3]$. Like-particle collisions can produce particle transport in exact omnigenity with small gyroradii only through the parallel viscosity. A further discussion of these constraints would make this paper too long.

As discussed in Section II.B, the energy confinement time of ions and of electrons must each be longer that the time required for a DT burn, $\tau_{DT}$ due the heating for DT fusion going to the electrons and only heating the bulk ions through ion-electron collisional equilibration.

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**Author Declarations**

The author has no conflicts to disclose.

**Data availability statement**

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

**Appendix A: Magnetic fields in toroidal plasmas**

The discussion of plasma confinement in plasmas confined on magnetic surfaces is simplified by the use of Boozer coordinates $[24]$. These coordinates always exist for scalar-pressure plasmas confined on nested toroidal magnetic surfaces and represent the magnetic field in the contra- and co-variant forms

$$2 \pi \hat{\mathbf{B}} = \nabla \psi \times \nabla \theta + \iota(\psi) \nabla \varphi \times \nabla \psi; \quad (A1)$$
$$= \mu_0 I(\psi) \nabla \theta + \mu_0 G(\psi) \nabla \varphi + \beta_\varphi \nabla \psi. \quad (A2)$$

$I(\psi)$ is the toroidal current within a volume that is enclosed by a magnetic surface that contains a toroidal magnetic flux $\psi$, $G(\psi)$ is the poloidal current that is outside that surface, which is the current that passes through the central hole of that torus, $\iota(\psi)$ is the rotational transform or average field-line twist on that surface, $\beta_\varphi(\psi, \theta, \varphi)$ is proportional to the pressure gradient, $\theta$ is a poloidal, and $\varphi$ is a toroidal angle.

The field line representation of Equation (1) and Equation (A1) is simple, $\theta = \theta_0 + \iota(\psi) \varphi$. The field line denoted by $(\psi, \theta_0)$ passes through the point $(\psi, \theta, \varphi)$ at $\varphi = 0$. The covariant representation in $(\psi, \theta_0, \varphi)$ coordinates is

$$2 \pi \hat{\mathbf{B}} = \mu_0(G + \iota I) \nabla \varphi + \mu_0 I \nabla \theta_0 + B_\psi \nabla \psi. \quad (A3)$$

A number of the derivations use the theory of general coordinates, which is derived in a two-page appendix in either [1] or [3].

The position vector $\hat{x}(\psi, \theta_0, \varphi)$ gives the point in $(x, y, z)$ Cartesian coordinates associated with a
point \((\psi, \theta_0, \varphi)\) coordinates. Essential concepts are the Jacobian \(\mathcal{J}\), the orthogonality relations, and the dual relations:

\[
\mathcal{J} \equiv \left( \frac{\partial \vec{x}}{\partial \psi} \times \frac{\partial \vec{x}}{\partial \theta_0} \right) \cdot \frac{\partial \vec{x}}{\partial \varphi} = \frac{1}{\left( \nabla \psi \times \nabla \theta_0 \right) \cdot \nabla \varphi}, \tag{A4}
\]

\[
\frac{\partial \vec{x}}{\partial \psi} \cdot \nabla \psi = 1 \text{ for any coordinate } \psi, \tag{A6}
\]

\[
\frac{\partial \vec{x}}{\partial \psi} \times \frac{\partial \vec{x}}{\partial \theta_0} = \mathcal{J} \nabla \varphi, \quad \text{and} \tag{A7}
\]

\[
\nabla \psi \times \nabla \theta_0 = \frac{1}{\mathcal{J}} \frac{\partial \vec{x}}{\partial \varphi}. \tag{A9}
\]

Equations \(A6\) and \(A7\) are called the orthogonality relations, and Equations \(A8\) and \(A9\) are called the dual relations. These equations hold for any cyclic choice of the three coordinates.

Equation \(B1\), \(2\pi \vec{B} = \nabla \psi \times \nabla \theta_0\) implies

\[
\vec{B} = \frac{\vec{B} \cdot \nabla \varphi}{2\pi} \left( \frac{\partial \vec{x}}{\partial \varphi} \right) \tag{A10}
\]

\[
\mathcal{J} = \frac{1}{2\pi \vec{B} \cdot \nabla \varphi}, \tag{A11}
\]

\[
= \frac{\mu_0 (G + \iota t)}{(2\pi)^2 \vec{B}^2}, \tag{A12}
\]

which follows from the product of Equations \(A1\) and \(A2\). It should be noted that the Jacobian of \((\psi, \theta_0, \varphi)\) coordinates equals that of \((\psi, \theta, \varphi)\) coordinates.

Studies of particle confinement in toroidal plasmas essentially means the confinement of particles near particular \(\psi\) surfaces, so only the two in-surface coordinates arise in a subtle way. The third coordinate can be assumed to be \(\psi\), and omitting the explicit statement of this coordinate allows a simplification of the notation.

\[\text{Appendix B: Ripple transport}\]

Extensive studies have been made of the effect on plasma transport of the toroidal ripple due to the \(N_c\) toroidal field coils which pass through the major radius \(R_c\) on the outboard side of a tokamak. A 2017 review was given by Catto \[18\]. The magnetic field strength at a minor radius \(r\), which is small compared to the radius of the magnetic axis at \(R_c\), can approximated as

\[B = B_0 \left( 1 - \frac{\epsilon_t}{\epsilon_c} \cos \theta + \epsilon_c \cos(N_c \varphi) \right), \tag{B1}\]

where \(\epsilon_t = \frac{r(\psi)}{R}, \epsilon_c = (R + r \cos \theta)^N_c / R_c^{N_c}\), and \(r = \sqrt{\psi/(\pi B_0)}\). Note \(R_c\) must be significantly larger than \(R + r\) for the plasma plus surrounding blankets and shields to be enclosed by the toroidal field coils.

Any two-helicity expression for the magnetic field strength has similar transport properties \[8\], so any quasi-symmetry broken by short wavelength perturbation behaves similarly to a tokamak with ripple in the field strength.

Extrema of the field strength along a field line are at \((\partial B/\partial \varphi)_{\theta_0} = 0\), which requires \(\iota \epsilon_t \sin(\theta_0 + \iota \varphi) = N_c \epsilon_c \sin(N_c \varphi)\). Let

\[N_\ast \equiv \frac{\epsilon_t}{\epsilon_c}, \quad \text{and} \tag{B2}\]

\[\theta_c \equiv \frac{N_c}{N_\ast}. \tag{B3}\]

The loss of ripple-trapped particles is extreme and unacceptable unless \(\theta_c\) is much less than one, which is a constraint adopted in the standard discussion of ripple losses \[18\]. However, ripple-trapped particles exist for much smaller values of \(\theta_c\). The lack of emphasis on this small \(\theta_c\) regime indicates the perception of its unimportance.

Ripple extrema exist when the ripple can change the sign of

\[
\left( \frac{\partial^2 B/B_0}{\partial \varphi^2} \right)_{\theta_0} = \iota^2 \epsilon_t \cos \theta - N_c^2 \epsilon_c \cos(N_c \varphi). \tag{B4}\]

The term involving the coil ripple is bigger than the axisymmetric term when \(N_c^2 \epsilon_c > \iota^2 \epsilon_t\), or when

\[\theta_c > \frac{\iota}{N_c}. \tag{B5}\]

A typical number of toroidal field coils is \(N_c = 20\), and a typical rotational transform in the main part to a tokamak plasma is \(\iota \sim 1/2\).

An interesting part of design space to explore is

\[1 \gg \theta_c \gtrsim \frac{\iota}{N_c}. \tag{B6}\]

When \(1 \gg \theta_c\), the magnetic wells due to the toroidal field coil ripple arise only near the maximum and minimum of the axisymmetric magnetic field strength,

\[|\theta| \lesssim \theta_c \quad \text{and} \quad |\theta - \pi| \lesssim \theta_c. \tag{B7}\]

When \(\theta_c \ll 1\), the maximum radial drift or a ripple-trapped particle is \(\Delta r \lesssim |v_r|/v_{\theta_0} r \theta_c\), where the radial drift \(|v_r| \approx (\rho/R) v \sin \theta_c\) and the drift in the \(\theta_0\) direction \(|v_{\theta_0}| \approx (\rho/R) v\) when \(\theta_c \ll 1\). That is,

\[
\frac{\Delta r}{r} \lesssim \theta_c^2 \lesssim \frac{\iota^2}{N_c^2}. \tag{B8-9}\]
when \( \theta_c \) is as small as it can be and still have ripple-trapped particles.

Even when \( \theta_c \sim \nu/N_c \), particles can make many bounces in a ripple well. Equation (13) implies that Taylor expansion in \( B \) along a magnetic field line though second order plus energy conservation, \( mv^2/2 = mv^2/2 + \mu_B \), gives \( R^2\varphi^2 \approx v^2 N_c^2 \epsilon_c \). The bounce time \( \tau_{rb} \approx \frac{\delta \varphi}{\dot{\varphi}} \approx \frac{R}{v N_c^2 \sqrt{\epsilon_c}} \) (B10)

\[
\tau_{rb} \approx \frac{\delta \varphi}{\dot{\varphi}} \approx \frac{R}{v N_c^2 \sqrt{\epsilon_c}}
\]

Precession limits the time a particle stays ripple trapped

\[
\tau_{rt} \approx \frac{r \theta_c}{\pi} \approx \frac{r R N_c}{\rho \nu N_c^2} \approx \frac{a N_c^3}{\rho N_c^{3/2} \nu \epsilon_t}
\]

The number of bounces a ripple-trapped particle makes is

\[
N_{rb} \approx \frac{\tau_{rt}}{\tau_{rb}} \approx \frac{a N_c^3}{\rho N_c^{3/2} \nu \epsilon_t}
\]

when \( N_c \) is as small as it can be, \( N_c = N_c^2 / \epsilon_t \), for a fixed \( N_c \) and still have a ripple well.

In addition to the creation of ripple wells, the existence of the ripple has two effects no matter how small \( \epsilon_c \) may be: (1) banana-drift transport \([10, 11]\), which is the action-conserving effect on toroidally trapped particles and (2) the action-breaking of particles switching back and forth between being barely trapped to barely passing.

In quasi-axisymmetry, \( P_e \) is a constant of the motion,

\[
P_e = \frac{\mu_0(G + iI)}{2 \pi B} mv_\parallel - \frac{q\psi_p(\psi)}{2 \pi}.
\]

(B17)

Since \( \delta \psi_p = \pm 2\pi Br\delta r \) and \( R \approx \mu_0(G + iI)/(2\pi B) \), the deviation of a particle from a constant-\( r \) magnetic surface is

\[
\delta r = \frac{\delta v_\parallel}{v} \frac{\rho}{\epsilon_t}
\]

(B18)

In perfect axisymmetry, there are two types of barely passing particles. One type has \( v_\parallel > 0 \) and lies outside the magnetic surface on which \( v_\parallel \) most closely approaches zero as it passes the \( B \) maximum and deviates as much as \( \delta r = \sqrt{\epsilon_c}/\epsilon_t \) from the surface as it passes the magnetic field minimum. The other type of barely passing particle has \( v_\parallel < 0 \) and lies inside that surface by a distance as large as \( \delta r = -\sqrt{\epsilon_c}/\epsilon_t \). There is only one type of barely trapped particle which lies outside the surface on which its turning points are located as much as \( \delta r = \sqrt{\epsilon_c}/\epsilon_t \) on its \( v_\parallel > 0 \) leg and inside the surface when \( v_\parallel < 0 \). In other words, the trajectories of barely trapped particles have trajectories that half the time look like barely passing particles with \( v_\parallel > 0 \) and half the time like particles with \( v_\parallel < 0 \).

When \( \epsilon_c \) is so small that there are no ripple wells, precession causes particles to switch back and forth between passing and toroidally trapped. While toroidally trapped, particles undergo banana drift diffusion but not while they are passing. When ripple wells exist only near the field extrema, barely toroidally trapped particles can switch between being ripple trapped and passing. The overall effect on confinement would appear to be similar to banana drift transport, but careful numerical calculations should be carried out for clarification.

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