OPTIMAL $(r,\delta)$-LRCS FROM ZERO-DIMENSIONAL AFFINE VARIETY CODES AND THEIR SUBFIELD-SUBCODES

C. GALINDO, F. HERNANDO, AND H. MARTÍN-CRUZ

ABSTRACT. We introduce zero-dimensional affine variety codes (ZA VCs) which can be regarded as $(r,\delta)$-locally recoverable codes (LRCs). These codes come with a natural bound for their minimum distance and we determine those giving rise to $(r,\delta)$-optimal LRCs for that distance, which are in fact $(r,\delta)$-optimal. A large subfamily of ZA VCs admit subfield-subcodes with the same parameters of the optimal codes but over smaller supporting fields. This fact allows us to determine infinitely many sets of new $(r,\delta)$-optimal LRCs and their parameters.

INTRODUCTION

Locally recoverable (or repairable) codes (LRCs) were introduced in [17]. The aim was to consider error-correcting codes to treat the repair problem for large scale distributed and cloud storage systems. Thus an error-correcting code $C$ is named an LRC with locality $r$ whenever any symbol in $C$ can be recovered by accessing at most $r$ other symbols of $C$ (see, for instance, the introduction of [13] for details). The literature contains a good number of papers on this class of codes, some of them are [45, 24, 29, 27, 21, 25, 35]. An interesting family of LRCs are the so-called locally recoverable Reed-Solomon codes which were obtained in [38], they are a variation of Reed-Solomon codes introduced by the authors for recovering purposes. Locally recoverable Reed-Solomon codes were extended to LRCs over arbitrary curves and rational maps in [4]. Among the different classes of codes considered as good candidates for local recovering, cyclic codes and subfield-subcodes of cyclic codes play an important role, because the cyclic shifts of a recovery set again provide recovery sets [8, 18, 20, 39]. In [30] the author introduces a model of locally recoverable code that also includes local error detection, increasing the security of the recovery system.

There is a Singleton-like bound for LRCs with locality $r$ [17]. Codes attaining this bound are named optimal $r$-LRCs and interesting constructions of this class of codes can be found in [38] and [40] (see also [3, 4, 31, 32, 35]). When considering codes over the finite field $\mathbb{F}_q$, $q$ being a prime power, optimal $r$-LRCs can be obtained for all lengths $n \leq q$ [42] and a challenging question is to study how long these codes can be [19].

The fact that simultaneous multiple device failures may happen leads us to the concept of LRCs with locality $(r,\delta)$ (or $(r,\delta)$-LRCs). This class of codes were introduced in [33], see Definition 1.2 in this paper, and they also admit a Singleton-like bound [33], which we reproduce in Proposition 1.3. Codes attaining this bound are named optimal $(r,\delta)$-LRCs or, in this paper, simply optimal codes. Optimal codes have been studied in [8, 23, 27, 37, 21, 6, 10, 34], mainly coming from cyclic and constacyclic codes. A somewhat different way for obtaining LRCs with locality $(r,\delta)$ was started in [13], where the supporting codes were the so-called $f$-affine variety codes. This type of codes were introduced in [14] and they have a good behaviour for constructing quantum error-correcting codes [12, 14, 11].

Key words and phrases. Locally recoverable codes, zero-dimensional affine variety codes, subfield-subcodes.

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This article introduces a new class of error-correcting codes, which we name zero-di-

mensional affine variety codes (ZA VCs) and contains the set of \( J \)-affine variety codes (see Definition 2.1). They are evaluation codes obtained as the image of maps 

\[
ev_P : V_\Delta \subset \mathbb{F}_q[X_1, \ldots, X_m] / I \rightarrow \mathbb{F}_q^n, \quad \ev_P(f) = (f(\alpha_1), \ldots, f(\alpha_n)),
\]

where \( m \) is a positive integer larger than 1, \( P = P_1 \times \cdots \times P_m = \{\alpha_1, \ldots, \alpha_n\} \) a suitable subset of \( \mathbb{F}_q^n \) and \( I \) the vanishing ideal at \( P \) of \( \mathbb{F}_q[X_1, \ldots, X_m] \). This evaluation map is also used in [5] to define codes with variable locality and availability. ZAVCs come with a natural bound on their minimum distance which allows us to obtain many optimal \((r, \delta)\)-LRCs. In fact, we are able to get all ZAVCs providing optimal codes when considering the mentioned bound.

ZA VCs are related with and include the family of codes introduced in [2] whose evaluation map is the same as ZAVCs but their evaluation sets \( V_\Delta \) are only a subset of ours. This makes that the sets \( \Delta \) in [2] have specific shapes while ours can have arbitrary shapes and therefore we obtain many more optimal \((r, \delta)\)-LRCs (see Remark 3.17 for details).

We are interested in optimal \((r, \delta)\)-LRCs and the recent literature presents a number of results giving parameters of codes of this type [8, 37, 6, 41, 7, 10, 43, 44, 9, 26, 22]. The length of most of these codes is a multiple of \( r + \delta - 1 \leq q \) and, in this case, and for unbounded length and small size fields, their distances have restrictions being at most \( 3\delta \). Larger distances can be obtained when \( q^2 + q \) is a bound for the length. One must use different constructions to get these optimal codes, and a large size of the supporting field seems to make easier to find optimal codes [36].

ZA VCs are evaluation codes that evaluate polynomials in monomial ideals such that the set of exponents of their generators determines the dimension and a bound for the minimum distance (see Propositions 2.4 and 2.6). Our recovery procedure based on interpolation also makes easy to obtain the values \( r \) and \( \delta \) of ZAVCs regarded as LRCs (Proposition 2.7). Supported on these facts, we perform a complete study of optimal ZAVCs. Subsection 3.1 is devoted to bivariate codes and Subsection 3.2 to multivariate codes. Theorem 3.10 (respectively, Theorem 3.14) proves that codes given in Propositions 3.1, 3.2 and 3.3 (respectively, Propositions 3.12 and 3.13) give the optimal LRCs one can get with this type of codes (considering the known bound for the distance) and that they determine all the parameters of the optimal LRCs given by ZAVCs (under the mentioned bound). These parameters are grouped in Corollary 3.11 for the bivariate case and in Corollary 3.15 for the multivariate case. Thus, one gets a large family of optimal LRCs that can be constructed by a unique and simple procedure.

This family provides, on the one hand, the parameters of those LRCs over \( \mathbb{F}_q \) given in [7] whose lengths are of the form \( N(r + \delta - 1) \), where \( N \) can be written as a product of integers less than or equal to \( q \) and, on the other hand, the parameters of those LRCs in [26] with length less than or equal to \( q^2 + q \).

The above codes do not give new parameters but subfield-subcodes of many subfamilies of them do give. In fact, in Section 4 we prove that, considering suitable subfield-subcodes over subfields \( \mathbb{F}_{q'} \) of \( \mathbb{F}_q \), we get LRCs over \( \mathbb{F}_{q'} \) with the same parameters of the original codes over \( \mathbb{F}_q \). This fact allows us to prove Theorems 4.2 and 4.9 which determine new optimal \((r, \delta)\)-LRCs. Many good choices are given in Corollaries 4.4, 4.5, 4.11 and 4.12, while specific parameters are showed in Corollaries 4.7, 4.8, 4.13 and 4.14. Finally, in Example 4.6 and Tables 1 and 2, one can find some numerical examples of new optimal LRCs over small fields.

1. Locally recoverable codes

In this section we give a brief introduction to locally recoverable codes (LRCs). An LRC is an error-correcting code such that any erasure in a coordinate of a codeword can be recovered from a set of other few coordinates. Let \( q \) be a prime power and \( \mathbb{F}_q \) the finite field
with $q$ elements. Let $C$ be a linear code over $\mathbb{F}_q$ with parameters $[n, k, d]_q$. A coordinate $i \in \{1, \ldots, n\}$ is locally recoverable if there is a recovery set $R \subseteq \{1, \ldots, n\}$ with $i \notin R$ such that for any codeword $c = (c_1, \ldots, c_n) \in C$, an erasure in the coordinate $c_i$ of $c$ can be recovered from the coordinates of $c$ with indices in $R$. Set $\pi_R: \mathbb{F}_q^n \to \mathbb{F}_q^n$ the projection map on the coordinates of $R$, and for $x \in \mathbb{F}_q^n$, write $x_R := \pi_R(x)$ and $C[R] := \{\pi_R(c) \mid c \in C\}$. Then:

**Proposition 1.1.** A set $R \subseteq \{1, \ldots, n\}$ is a recovery set for a coordinate $i \notin R$ if and only if $d(C[R]) \geq 2$, where $R = R \cup \{i\}$ and $d$ stands for the minimum distance.

The locality of a coordinate is the smallest cardinality of a recovery set for that coordinate. An LRC with locality $r$ is a code such that every coordinate is locally recoverable and $r$ is the largest locality of its coordinates. The parameters and locality of an LRC satisfy the following Singleton-like inequality.

$$k + d + \left\lceil \frac{k}{r} \right\rceil \leq n + 2.$$  

When the equality holds, the code is called optimal $r$-LRC.

By Proposition 1.1, if $R$ is a recovery set for $i$, then $d(C[R]) \geq 2$ and thus only one erasure can be corrected (also only up to to one error can be detected). But erasures can also occur in $x_R$ and then we could not recover $x_i$. To correct more than one erasure we introduce the concept of locality $(r, \delta)$, also named $(r, \delta)$-locality.

**Definition 1.2.** A code $C$ is locally recoverable with locality $(r, \delta)$ if, for any coordinate $i$, there exists a set of coordinates $\overline{R} = \overline{R}(i) \subseteq \{1, \ldots, n\}$ such that:

1. $i \in \overline{R}$ and $\#\overline{R} \leq r + \delta - 1$; and
2. $d(C[\overline{R}]) \geq \delta$.

Such a set $\overline{R}$ is called an $(r, \delta)$-recovery set for $i$ and $C$ an $(r, \delta)$-LRC.

In this paper, we will always refer to this type of locality and sometimes, abusing the notation, we will talk about locality $r$ understanding locality $(r, \delta)$ for some $\delta$ inferred from the context. The second condition in Definition 1.2 allows us to correct an erasure at coordinate $i$ plus any other $\delta - 2$ erasures in $\overline{R}\backslash\{i\}$ by using the remaining $r$ coordinates (also it allows us to detect an error at coordinate $i$ plus any other $\delta - 2$ errors in $\overline{R}\backslash\{i\}$ and correct them one by one by using the remaining $r$ coordinates). Notice that, when $\delta \geq 2$ and $C$ is an LRC with locality $(r, \delta)$, the (original definition of) locality of $C$ is $\leq r$. In fact, any subset $R \subseteq \overline{R}$ such that $\#R = r$ and $i \notin R$ fulfills $d(C[(\overline{R} \cup \{i\})]) \geq 2$, so by Proposition 1.1 $R$ is a recovery set for the coordinate $i$. There is also a Singleton-like inequality for $(r, \delta)$-LRCs:

**Proposition 1.3.** [33] The parameters $[n, k, d]_q$ of an $(r, \delta)$-LRC, $C$, satisfy

$$k + d + \left\lceil \frac{k}{r} \right\rceil (\delta - 1) \leq n + 1. \quad (1.1)$$

In this paper, $C$ is called an optimal $(r, \delta)$-LRC (or simply, an optimal LRC) whenever equality holds in (1.1).

In the next section we define the linear codes we will use for local recovery.

2. Zero-dimensional affine variety codes

Let $m > 1$ be a positive integer and consider a family $\{P_j\}_{j=1}^m$ of subsets of $\mathbb{F}_q$, with cardinality larger than one. Set

$$P = P_1 \times \cdots \times P_m = [\alpha_1, \ldots, \alpha_n] \subseteq \mathbb{F}_q^m.$$  

We usually write $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{im})$. Consider the quotient ring

$$R = \mathbb{F}_q[X_1, \ldots, X_m]/I.$$  

OPTIMAL $(r, \delta)$-LRCS FROM ZERO-DIMENSIONAL AFFINE VARIETY CODES AND THEIR SUBFIELD-SUBCODES
where $I$ is the ideal of the polynomial ring in $m$ variables $F_q[x_1, \ldots, x_m]$ vanishing at $P$. Then, $I = \langle f_1(x_1), \ldots, f_m(x_m) \rangle$, where $f_j(x_j) = \prod_{p \in P} (x_j - \beta)$ and $\deg(f_j) = \# P_j = n_j \geq 2$ [28]. Let

$$E = \{0, 1, \ldots, n_1 - 1\} \times \cdots \times \{0, 1, \ldots, n_m - 1\}.$$ 

Given $f \in R$, $f$ denotes both the equivalence class in $\mathcal{R}$ and the unique polynomial in $F_q[x_1, \ldots, x_m]$ with degree in $x_j$ less than $n_j$, $1 \leq j \leq m$, representing $f$. Thus

$$f(x_1, \ldots, x_m) = \sum_{(e_1, \ldots, e_m) \in E} f_{e_1, \ldots, e_m} x_1^{e_1} \cdots x_m^{e_m},$$

with $f_{e_1, \ldots, e_m} \in F_q$. Set $\text{supp}(f) = \{(e_1, \ldots, e_m) \in E \mid f_{e_1, \ldots, e_m} \neq 0\}$. For each subset $\Delta \subseteq E$, define $V_\Delta := \{f \in R \mid \text{supp}(f) \subset \Delta\}$ and for each element $e = (e_1, \ldots, e_m) \in E$, denote $X^e = x_1^{e_1} \cdots x_m^{e_m}$. Then, $V_\Delta = \langle X^e \mid e \in \Delta \rangle$. The linear evaluation map

$$\text{ev}_P : R \to F_q^n, \quad \text{ev}_P(f) = (f(\alpha_1), \ldots, f(\alpha_n)),$$

gives rise to the following class of evaluation codes.

**Definition 2.1.** The zero-dimensional affine variety code (ZA VC) $C_P^P$ is the following vector subspace of $F_q^n$ over the finite field $F_q$:

$$C_P^P := \text{ev}_P(V_\Delta) = \langle \text{ev}_P(X^e) \mid e \in \Delta \rangle \subseteq F_q^n.$$ 

We say that the ZAVC $C_P^P$ is bivariate (respectively, multivariate) when $m = 2$ (respectively, $m > 2$).

ZA VCs extend $J$-affine variety codes introduced in [14]. Denoting $R_t \subseteq F_q$ the set of $t$-roots of unity for some $t \mid q - 1$, a $J$-affine variety code is a ZAVC where each $P_j$ is of the form $R_t$ or $R_t \cup \{0\}$.

We also introduce the following definition which will be useful in the next sections.

**Definition 2.2.** Two subsets $\Delta_1, \Delta_2$ of $E$ are pseudoisometric if there exists $v = (v_1, \ldots, v_m) \in \mathbb{Z}^m$ such that

$$\Delta_2 = v + \Delta_1 := \{(e_1 + v_1, \ldots, e_m + v_m) \mid (e_1, \ldots, e_m) \in \Delta_1\}.$$ 

In that case, we say that the codes $C_{\Delta_1}^P$ and $C_{\Delta_2}^P$ are pseudoisometric.

**Remark 2.3.** In this paper, we say that two codes are isometric if there exists a bijective mapping between them that preserves Hamming weights. For $(v_1, \ldots, v_n), (w_1, \ldots, w_n) \in F_q^n$, define their $\ast$-product as

$$(v_1, \ldots, v_n) \ast (w_1, \ldots, w_n) = (v_1 \cdot w_1, \ldots, v_n \cdot w_n).$$

Then $\text{ev}_P(fg) = \text{ev}_P(f) \ast \text{ev}_P(g)$ for all $f, g \in R$. Assume that $\Delta_1, \Delta_2 \subseteq E$ are pseudoisometric sets such that $\Delta_2 = v + \Delta_1$ and $v_j \neq 0$ for $1 \leq j \leq m$. For simplicity, suppose $v_j < 0, 1 \leq j \leq m_1$, and $v_j > 0, m_1 + 1 \leq j \leq m$. Consider

$$\Delta_2' = (-v_1, -v_2, \ldots, -v_{m_1}, 0, \ldots, 0) + \Delta_2$$

and

$$\Delta_1' = (0, \ldots, 0, v_{m_1+1}, \ldots, v_m) + \Delta_1,$$

and then $\Delta_2' = \Delta_1'$. Thus

$$V_{\Delta_2'} = \left\{ X_1^{-v_1} \cdots X_{m_1}^{-v_{m_1}} g \mid g \in V_{\Delta_2} \right\},$$

and the codewords in $C_{\Delta_2'}^P$ are of the form

$$\text{ev}_P(X_1^{-v_1} \cdots X_{m_1}^{-v_{m_1}} g) = \text{ev}_P(X_1^{-v_1} \cdots X_{m_1}^{-v_{m_1}}) \ast \text{ev}_P(g),$$

where $g \in V_{\Delta_2}$, so $\text{ev}_P(g) \in C_{\Delta_1}^P$. When $0 \notin P_j$ for all $1 \leq j \leq m$, we have just proved that $C_{\Delta_2'}^P$ and $C_{\Delta_1}^P$ are isometric codes. The same reasoning proves that $C_{\Delta_2'}^P$ and $C_{\Delta_1}^P$ are isometric.
Thus \( C_{\Delta}^p \) and \( C_{\Delta_2}^p \) are isometric and this happens even when \( v_j \) is always negative or positive or when some coordinates \( v_j \) are 0.

When \( 0 \in P_j \) for some index \( 1 \leq j \leq m \), \( C_{\Delta}^p \) and \( C_{\Delta_2}^p \) need not be isometric which explains why we speak of pseudoisometric codes.

Length, dimension and a bound for the minimum distance of a ZAVC, \( C_{\Delta}^p \), are provided in Propositions 2.4 and 2.6. Let us state Proposition 2.4 whose proof is straightforward.

**Proposition 2.4.** Keep the above notation. The length \( n \) and dimension \( k \) of a ZAVC, \( C_{\Delta}^p \), are
\[
      n = \prod_{j=1}^{m} (n_j - e_j).
\]

**Definition 2.5.** The distance of an exponent \( e \in E \) is defined to be \( d(e) := \prod_{j=1}^{m} (n_j - e_j) \).

The codes \( C_{\Delta}^p \) admit the following primary Feng-Rao bound [1, 15, 16].

**Proposition 2.6.** Let \( C_{\Delta}^p \) be a ZAVC and let \( d \) its minimum distance. Define \( d_0 = d_0(C_{\Delta}^p) := \min\{d(e) \mid e \in \Delta \} \). Then, \( d \geq d_0 \).

**Proof.** We restrict to the case \( m = 2 \) since a proof for the general case is analogous. Let \( c = ev_p(f) \in C_{\Delta}^p \) such that \( f \in V_\Delta \). Set \( k_1 = \deg_{X_1}(f) \) and \( k_2 = \deg_{X_2}(f) \) the degrees in \( X_1 \) and \( X_2 \) of \( f \), respectively. Thus, \((k_1, k_2) \in E \) and then \( k_j < n_j \) for \( j = 1, 2 \). The weight of \( c \) is the number of no-roots of \( f \) in \( P \), and we can give a bound on this weight from the number of roots of \( f \). Fix \( p_2 \in P_2 \), then \( f|_{X_2} \subseteq f(X_1, p_2) \) is a polynomial in \( X_1 \) of degree \( k_1 \) with at most \( k_1 \) roots in \( P_1 \), therefore \( f(X_1, p_2) \) has at most \( k_1 \) roots of the form \((X_1, p_2) \). This is true for any \( p_2 \in P_2 \), so we have at most \( k_1 n_2 \) roots of \( f \) of the above form. Repeat the procedure by fixing \( p_1 \in P_1 \) and we obtain at most \( k_2 n_1 \) roots of \( f \). Thus one has at most \( k_1 n_2 + k_2 n_1 \) roots, but we must remove those we have added twice \( (k_1 k_2) \) ones. Hence, the polynomial \( f \) has at most \( k_1 n_2 + k_2 n_1 - k_1 k_2 \) roots in \( P \) and therefore at least \( n_1 n_2 - (k_1 n_2 + k_2 n_1 - k_1 k_2) = (n_1 - k_1)(n_2 - k_2) \) no-roots in \( P \), thus the weight of \( c \), \( w(c) \), is at least \((n_1 - k_1)(n_2 - k_2) \geq \min\{d(e) \mid e \in \Delta \} = d_0 \). This concludes the proof because \( w(c) \geq d_0 \) for any \( c \in C_{\Delta}^p \).

Next proposition and its proof show how to regard ZAVCs as LRCs. For each \( 1 \leq j \leq m \), define the support of \( V_\Delta \) at \( X_j \) as
\[
      \text{Supp}_{X_j}(V_\Delta) := \{ e_j \in \{0, 1, \ldots, n_j - 1\} \mid \text{there exists a monomial } X_1^{e_1} \cdots X_j^{e_j} \cdots X_m^{e_m} \text{ in } V_\Delta \}.
\]

**Proposition 2.7.** Let \( C_{\Delta}^p \) be a ZAVC. Denote \( \mathcal{K}_j := \#\text{Supp}_{X_j}(V_\Delta) \). Then, for each \( 1 \leq l \leq m \) such that \( \mathcal{K}_j < n_l \), \( C_{\Delta}^p \) is locally recoverable with locality \((\mathcal{K}_j, n_j - \mathcal{K}_j + 1) \).

**Proof.** Let \( c = (c_1, \ldots, c_m) = ev_p(f) \in C_{\Delta}^p \) be a codeword whose \( i \)th coordinate \( c_i \) we desire to recover and let \( k_j \) be the maximum degree on \( X_j \) of the monomials in \( V_\Delta \). We know that \( \text{supp}(f) \subseteq \Delta \) and thus \( \deg_{X_j}(f) \leq k_j \) for all \( j = 1, \ldots, m \). Choose a variable \( X_l \) (we will interpolate with respect to it), write \( c_i = f(\alpha_i) = f(\alpha_{i_1}, \ldots, \alpha_{i_m}) \) and consider the following subset of \( P \):
\[
      \overline{R}_p = \{ \alpha_l \in P \mid \alpha_l = \alpha_j \text{ for all } j \in \{1, \ldots, m\} \setminus \{l\} \}
      = \{ (\alpha_{i_1}, \ldots, \alpha_{i_{l-1}}, x, \alpha_{i_{l+1}}, \ldots, \alpha_{i_m}) \mid x \in P_l \},
\]
whose cardinality is \( \#\overline{R}_p = n_l \). A polynomial in \( V_\Delta \) can be expressed as
\[
      f(X_1, \ldots, X_m) = \sum_{(e_1, \ldots, e_m) \in \Delta} f_{e_1, \ldots, e_m} X_1^{e_1} \cdots X_m^{e_m}
      = \sum_{h=0}^{k_l} f_h(X_1, \ldots, X_{l-1}, X_{l+1}, \ldots, X_m) X_l^{k_l} \in \mathbb{F}_q[X_1, \ldots, X_{l-1}, X_{l+1}, \ldots, X_m][X_l].
\]
Replacing each \( X_j, j \neq i \), by \( \alpha_{ij} \), we get a polynomial in \( X_i, g(X_i) \), with constant coefficients, of degree at most \( k_i \). So we can interpolate \( g \) by using \( k_i + 1 \) points in \( \overline{R}_p \) (since \( k_i \leq n_i - 1 \)) to obtain those coefficients. However, we have \( k_i + 1 - \mathcal{K}_i \) conditions
\[
f_h(X_1, \ldots, X_{i-1}, X_i+1, \ldots, X_m) = 0
\]
h \( \not\in \text{Supp}_{X_i}(V_\Delta) \), and then we only need \( \mathcal{K}_i \) points in \( \overline{R}_p \) to obtain the coefficients of \( g \). Recall that \( \alpha_i \in \overline{R}_p \) implies that \( \mathcal{K}_i < n_i \). Then, we can recover \( c_i \) by evaluating \( g \). Let
\[
\mathcal{R} = \{ t \in \{1, \ldots, n\} \mid \alpha_i \in \overline{R}_p \}.
\]
The set \( \mathcal{R} \) is a \( (\mathcal{K}_i, n_i - \mathcal{K}_i + 1) \)-recovery set since \( i \in \mathcal{R} \), \# \( \mathcal{R} = n_i \) and \( C(\mathcal{R}) \) is a Reed-Solomon code (and thus an MDS code) whose minimum distance is exactly \( n_i - \mathcal{K}_i + 1 \). Hence, \( C^p_{\Delta} \) is an LRC with locality \( (\mathcal{K}_i, n_i - \mathcal{K}_i + 1) \).

**Remark 2.8.** Let \( C^p_{\Delta} \) be a ZAVC with parameters \( [n, k, d]_q \) and locality \( (r, \delta) \). Then by Propositions 1.3 and 2.6 the following inequalities
\[
k + d_0 + \left( \left\lceil \frac{k}{r} \right\rceil - 1 \right) (\delta - 1) \leq k + d + \left( \left\lceil \frac{k}{r} \right\rceil - 1 \right) (\delta - 1) \leq n + 1
\]
hold.

**Definition 2.9.** The **defect** of the code \( C^p_{\Delta} \) is the integer
\[
D = D(C^p_{\Delta}) := n + 1 - k - d_0 - \left( \left\lceil \frac{k}{r} \right\rceil - 1 \right) (\delta - 1) \geq 0.
\]

**Definition 2.10.** A code \( C^p_{\Delta} \) is called \( d_0 \)-**optimal** whenever its defect \( D \) vanishes. That is, \( C^p_{\Delta} \) is optimal and \( d = d_0 \).

**Remarks 2.11.** The next facts will be useful in the following:

1. The locality \( (r, \delta) \) provided in Proposition 2.7 depends on the variable \( X_i \) we choose to interpolate, which allows us to make the best choice of \( X_i \).
2. A \( d_0 \)-optimal code is always optimal but a code that is not \( d_0 \)-optimal may be optimal.

## 3. Optimal Zero-Dimensional Affine Variety Codes

In this section, we determine the parameters of \( d_0 \)-optimal ZAVCs. We start with the bivariate case.

### 3.1. The case \( m = 2 \)

For simplicity let us denote \( X_1 \) by \( X \) and \( X_2 \) by \( Y \). We represent \( E \) as a grid where coordinates \( (i, j) \) correspond to an exponent \( e \) labelled with their distance (Definition 2.5). Figure 1 shows the grid representation of \( E \) in the case when \( n_1 = 10 \) and \( n_2 = 9 \).

We start by looking for sets \( \Delta \subseteq E \) such that the code \( C^p_{\Delta} \) is \( d_0 \)-optimal, that is, its parameters satisfy
\[
k + d_0 + \left( \left\lceil \frac{k}{r} \right\rceil - 1 \right) (\delta - 1) = n + 1.
\]

From now on, we use shaded regions to represent sets formed by the points in \( E \) inside that region. By rectangle we will always refer to a subset of \( E \) whose representation as shaded set is a rectangle. The first result in this subsection shows when codes \( C^p_{\Delta} \) where \( \Delta \) has the shape of a rectangle, are \( d_0 \)-optimal.

**Proposition 3.1.** Keep the above notation, where \( q \) is a prime power, \( m = 2 \) and \( n_1, n_2 \geq 2 \) are the cardinalities of \( P_1 \) and \( P_2 \). Consider the sets
\[
\Delta = \Delta_{i,j} := \{(e_1, e_2) \mid 0 \leq e_1 < i, 0 \leq e_2 < j\} \subseteq E = \{0, \ldots, n_1 - 1\} \times \{0, \ldots, n_2 - 1\}
\]
i = 0 and 0 ≤ j ≤ n_2 - 1, in which case \((r, \delta) = (1, n_1)\).

- 1 ≤ i ≤ n_1 - 2 and j = n_2 - 1, in which case \((r, \delta) = (i + 1, n_1 - i)\).

- 0 ≤ i ≤ n_1 - 1 and j = 0, in which case \((r, \delta) = (1, n_2)\).

- i = n_1 - 1 and 1 ≤ j ≤ n_2 - 2, in which case \((r, \delta) = (j + 1, n_2 - j)\).

Sets \(\Delta\) as above are denoted by \(\Delta_{i,j}^1\).

**Proof.** Clearly, \(k = (i + 1)(j + 1)\) and \(d_0 = (n_1 - i)(n_2 - j)\). By interpolating with respect to \(X\), \(r = i + 1\) and \(\delta - 1 = n_1 - i - 1\). Then,

\[
\begin{align*}
k + d_0 + \left( \left\lfloor \frac{k}{r} \right\rfloor - 1 \right) (\delta - 1) &= (i + 1)(j + 1) + (n_1 - i)(n_2 - j) + \left( \left\lfloor \frac{(i+1)(j+1)}{i+1} \right\rfloor - 1 \right) (n_1 - i - 1) \\
&= n_1n_2 + 1 + i(j + 1 - n_2),
\end{align*}
\]

and the code is \(d_0\)-optimal if and only if \(i = 0\) or \(j = n_2 - 1\). Note that when \(j = n_2 - 1\) and \(i = n_1 - 1\) one does not get an LRC.

The remaining LRCs are obtained by interpolating with respect to \(Y\). \(\square\)

In the sequel, we will perform the procedure of considering a subset \(\Delta \subseteq E\) (starting set) and adding or removing elements to obtain a new subset \(\Delta^* \subseteq E\) (resulting set). The expression **gaining** (or **losing**) \(x\) in a parameter refers to the fact that the resulting code \(C_{\Delta^*}^p\) has a larger (or smaller) value for that parameter in a quantity of \(x\) units. We can also say that the parameter **increases** (or **decreases**) in \(x\).
The sets $\Delta^*$ obtained by removing the least distance point on the $(n_2 - 1)$th row (or $(n_1 - 1)$th column) of a rectangle $\Delta_{i,j}^1$ with $j = n_2 - 1$ and $i \geq 1$ (or $i = n_1 - 1$ and $j \geq 1$) also provide $d_0$-optimal codes since the left-hand side (LHS) of (2.1) remains the same. Indeed, when removing the point we lose one unit in dimension but we gain one unit in the bound for the minimum distance and $r$, $\delta$ and $\left\lfloor \frac{\delta}{r} \right\rfloor$ do not change. The following result generalizes this situation.

![Figure 3. Sets $\Delta_{i,s}^2$ and $\Delta_{j,s}^{2,\sigma}$ in Proposition 3.2](image)

**Proposition 3.2.** With notation as in Proposition 3.1, consider the subsets of $E$

$$\Delta = \Delta_{i,s}^2 := \{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq n_2 - 2\} \cup \{(e_1, n_2 - 1) \mid 0 \leq e_1 \leq s\},$$

where $\max\{0, 2i - n_1\} \leq s < i \leq n_1 - 2$ (see Figure 3 (1)).

Then, the ZAVCs, $C^P_{\Delta}$, are $d_0$-optimal $(r, \delta) = (i + 1, n_1 - i)$-LRCs.

Analogously, the ZAVCs, $C^P_{\Delta}$, where

$$\Delta = \Delta_{j,s}^{2,\sigma} := \{(e_1, e_2) \mid 0 \leq e_1 \leq n_1 - 2, 0 \leq e_2 \leq j\} \cup \{(n_1 - 1, e_2) \mid 0 \leq e_2 \leq s\} \subseteq E,$$

max $\{0, 2j - n_2\} \leq s < j \leq n_2 - 2$ (see Figure 3 (2)) are $d_0$-optimal $(r, \delta) = (j + 1, n_2 - j)$-LRCs.

**Proof.** Let us see a proof for the case $\Delta = \Delta_{i,s}^2$. The remaining one is analogous. $\Delta$ is obtained by removing the $(i - s)$ least distance points of $\Delta_{i,n_2-1}^1$ on the $(n_2 - 1)$th row with $0 \leq s < i$ as long as the distance $d(s, n_2 - 1) \leq d(i, n_2 - 2)$. In fact, this last inequality is equivalent to $n_1 - s \leq 2(n_1 - i)$ and to $s \geq 2i - n_1$. Interpolating with respect to $X$, the parameters of the code $C^P_{\Delta}$ are $k = (i + 1)(n_2 - 1) + s + 1$, $d_0 = n_1 - s$, $r = i + 1$ and $\delta - 1 = n_1 - i - 1$, and therefore

$$k + d_0 + \left\lfloor \frac{k}{r} \right\rfloor (\delta - 1) = (i + 1)(n_2 - 1) + s + 1 + n_1 - s$$

$$+ \left\lfloor \frac{(i + 1)(n_2 - 1) + s + 1}{i + 1} \right\rfloor (n_1 - i - 1)$$

$$= n_1 n_2 + 1. \square$$

The following result completes our family of sets $\Delta$ corresponding to ZAVCs, where $m = 2$, giving rise to $d_0$-optimal $(r, \delta)$-LRCs.

**Proposition 3.3.** With notation as in Proposition 3.1, consider the family of subsets of $E$

$$\Delta = \Delta_{i,j}^3 := \{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq j - 1\} \cup \{(0, j)\},$$
where \( 1 \leq i \leq n_1 - 2 \) and \( \max \left\{ 1, \frac{i(n_2 + 1) - n_1}{i} \right\} \leq j \leq n_2 - 2 \) (see Figure 4 (1)).

Then, the ZAVCs, \( C_{\Delta}^P \), are \( d_0 \)-optimal \((r, \delta) = (i + 1, n_1 - i)\)-LRCs.

Analogously, the ZAVCs, \( C_{\Delta}^P \), where

\[
\Delta = \Delta_{i,j}^3, \quad \Delta = \Delta_{i,j}^{3,0} := \{(e_1, e_2) \mid 0 \leq e_1 \leq i - 1, 0 \leq e_2 \leq j \} \cup \{(i, 0)\} \subseteq E,
\]

\( 1 \leq j \leq n_2 - 2 \), and \( \max \left\{ 1, \frac{j(n_2 + 1) - n_2}{j} \right\} \leq i \leq n_1 - 2 \) (see Figure 4 (2)), are \( d_0 \)-optimal \((r, \delta) = (j + 1, n_2 - j)\)-LRCs.

**Proof.** As before, we only perform the proof for the case \( \Delta = \Delta_{i,j}^3, \Delta \) is obtained by removing the points \((e_1, j), 1 \leq e_1 \leq i, \) of a rectangle

\[
\Delta_{i,j} = \{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq j \}
\]

with \( 1 \leq i \leq n_1 - 2 \) and \( 1 \leq j \leq n_2 - 2 \) such that \( d(0, j) \leq d(i, j - 1) \). As a consequence, \( n_1(n_2 - j) \leq (n_1 - i)(n_2 - j + 1) \), which is equivalent to \( i \leq \frac{n_1}{n_2 - j + 1}, \) or \( j \geq \frac{i(n_2 + 1) - n_1}{i} \). In this case, we interpolate with respect to \( X \) and the parameters of the code \( C_{\Delta}^P \) are \( k = (i + 1)j + 1, \)

\[
d_0 = n_1(n_2 - j), \quad r = i + 1 \quad \text{and} \quad \delta - 1 = n_1 - i - 1.
\]

Thus,

\[
k + d_0 + \left( \left\lceil \frac{k}{r} \right\rceil - 1 \right) (\delta - 1) = (i + 1)j + 1 + n_1(n_2 - j) + \left( \left\lceil \frac{(i + 1)j + 1}{i + 1} \right\rceil - 1 \right)(n_1 - i - 1)
\]

\[
= n_1n_2 + 1.
\]

\( \square \)

Our forthcoming Theorem 3.10 will prove that the families of ZAVCs given in Propositions 3.1, 3.2 and 3.3 determine the parameters of all \( d_0 \)-optimal bivariate \((m = 2) \,(r, \delta)\)-LRCs. Before proving our theorem we need some previous definitions and results.

**Definition 3.4.** A set \( \Delta \subseteq E \) is said to be **optimal** if the code \( C_{\Delta}^P \) is \( d_0 \)-optimal, that is, \( D = 0, \)

\( D \) being the defect of \( C_{\Delta}^P \) given in Definition 2.9.

Let \( \Delta \) (respectively, \( \Delta^* \)) be a starting (respectively, resulting) set included in \( E \). We say that \( \Delta^* \) is obtained following a **natural order** if \( \Delta^* \) comes from \( \Delta \) by successively removing (respectively, adding) points of least distance in \( \Delta \) (respectively, largest distance in \( E \setminus \Delta \). Assume that \( [n, k, d_0]_q \) (respectively, \( [n^*, k^*, d_0^*]_q \)) are the parameters, and \((r, \delta)\) (respectively, \((r^*, \delta^*)\)) the locality of the code \( C_{\Delta}^P \) (respectively, \( C_{\Delta}^P^* \)). Then, we define **variation** (of the code \( C_{\Delta}^P \)) with respect to \( C_{\Delta}^P \) produced in the LHS of (2.1) as the value

\[
k^* + d_0^* + \left( \left\lceil \frac{k}{r^*} \right\rceil - 1 \right) (\delta^* - 1) - \left( \frac{k^*}{r^*} - 1 \right) (\delta^* - 1).
\]
Denote $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

**Lemma 3.5.** Let $\Delta$ be a subset of $E$. Set 
\[
\begin{align*}
    a &:= \min \{e_1 \mid (e_1, e_2) \in \Delta\} \quad \text{and} \quad b := \min \{e_2 \mid (e_1, e_2) \in \Delta\}.
\end{align*}
\]

Fix an index $l \in \{1, 2\}$ and construct the set $\Delta^*$ as follows:

1. Set $\Delta_0 := (-a, -b) + \Delta$. Define 
   \[
   T_0 := \{ t \in \{1, 2, \ldots, n_l - 1 \} \mid \text{there is no } e \in \Delta_0 \text{ such that } e_t = t \},
   \]
   \[
   c_0 := |T_0| \text{ and } t_0 := \min T_0.
   \]

2. For every $i = 1, \ldots, c_0$, define inductively
   \[
   \Delta_i := \{ e \in \Delta_{i-1} \mid e_t < t_{i-1} \} \cup \{ e - e_t \mid e \in \Delta_{i-1} \text{ with } e_t > t_{i-1} \},
   \]
   \[
   T_i := \{ t - 1 \mid t \in T_{i-1}\setminus \{t_{i-1} \} \} \text{ and } t_i := \min T_i.
   \]

3. Set $l' := j \in \{1, 2\}\setminus \{l\}$ and define
   \[
   M := \max \{e_1 \mid e_1 \in \{0, 1, \ldots, n_l - 1\} \mid e \in \Delta_m \}
   \]
   and
   \[
   v_M := \max \{e_1 \mid e_1 = M \text{ and } e \in \Delta_m \}.
   \]

Consider the set
\[
\Delta'_M = \Delta_m \cup \{ e \in E \setminus \Delta_m \mid e_t \leq M, e_l \leq v_M \}.
\]

4. For every $i = M - 1, M - 2, \ldots, 0$, let 
   \[
   v_i := \max \{e_1 \mid e_1 = i \text{ and } e \in \Delta'_{i+1} \}
   \]
and inductively set
\[
\begin{align*}
\Delta_i &= \Delta'_{i+1}, \\
\text{when } v_i &\leq v_{i+1}, \text{ and} \\
\Delta_i &= \Delta'_{i+1} \cup \{ e \in E \setminus \Delta'_{i+1} \mid e_t \leq i, e_l \leq v_l \}, \\
\text{when } v_i &> v_{i+1}.
\end{align*}
\]

Finally, $\Delta^* := \Delta_0$.

Then, the variation produced in the LHS of (2.1) is $\geq 0$, that is, $D(C^*_\Delta) \leq D(C^\Delta_P)$.

**Proof.** Let $(n, k, \geq d_0)_q$ and $(r, \delta)$ (respectively, $(n_0, k_0, \geq (d_0)_q)_q$ and $(r_0, \delta_0)$) be the parameters and locality of the code $C^\Delta_{\Delta_0}$ (respectively, $C^P_{\Delta_0}$). The ZAVCs $C^\Delta_{\Delta_0}$ and $C^P_{\Delta_0}$ are pseudometric, so $n = n_0$, $k = k_0$, $r = r_0$ and $\delta = \delta_0$, but $d_0 \leq (d_0)_q$ because distances of the elements in $E$ increase when one considers exponents going to the left and to the down. Let $(n_0, k_0, \geq (d_0)_q)_q$ be the parameters and $(r_0, \delta_0)$ the locality of $C^\Delta_{\Delta_0}$. Suppose $l = 1$ (the remaining case is analogous), Step (2) removes vertical segments in $E \setminus \Delta_0$ and then $\#\text{Supp}_X(V_{\Delta_0}) = \max \left\{ \deg_X(f) \mid f \in V_{\Delta_0} \right\} + 1$, thus $n = n_0$, $k = k_0$, $r = r_0$ and $\delta = \delta_0$ but $d_0 \leq (d_0)_q$. Let $(i, j) \in \Delta_0$. Since every element in $E$ inside the rectangle that $(i, j)$ sets from $(0, 0)$ has larger distance than $(i, j)$, it makes sense to include all of them in the new set $\Delta$ in order to increase the dimension of the code and thus decreasing the defect. We perform it on Step (4), so that if $(n^*, k^*, \geq (d^*_0)_q)_q$ are the parameters and $(r^*, \delta^*)$ the locality of $C^\Delta_{\Delta^*}$, then $n = n^*$, $k = k^*$, $d_0 \leq d^*_0$, $r = r^*$ and $\delta = \delta^*$. Therefore,
\[
\begin{align*}
    k + d_0 + \left\lfloor \frac{k^*}{r^*} \right\rfloor (\delta - 1) - \left\lfloor \frac{k^*}{r^*} \right\rfloor (d^* - 1) \leq 0.
\end{align*}
\]
\[\square\]

**Remark 3.6.** Figure 5 shows some simple cases where the procedure described in Lemma 3.5 (for $l = 1$) is applied.
Lemma 3.7. Keep the notation as in Proposition 3.3. Let $\Delta_1$ and $\Delta_2$ be two subsets of $E$ of the form $\Delta_i^3$ (respectively, $\Delta_{i,j}^{3,\sigma}$) for some indices $i$, $j$, and such that $\Delta_2$ is obtained by removing points of $\Delta_1$ following a natural order. Then, there is no $d_0$-optimal code $C_\Delta^\sigma$ such that:

1. $\Delta_2 \subseteq \Delta \subseteq \Delta_1$, and
2. $\Delta$ is not of the form $\Delta_{i,j}^3$ (respectively, $\Delta_{i,j}^{3,\sigma}$).

Proof: We perform the proof for the case when $\Delta_1$ and $\Delta_2$ are of the form $\Delta_{i,j}^3$. The proof for the remaining case is analogous. It suffices to assume that

$$\Delta_1 = \Delta_{i,j}^{3} = \{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq j-1\} \cup \{(0, j)\}$$

and

$$\Delta_2 = \Delta_{i,j-1}^{3} = \{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq j-2\} \cup \{(0, j-1)\},$$

where $1 \leq i \leq n_1 - 2$ and $\left\lceil \frac{(n_2+1)-n_1}{i} \right\rceil + 1 \leq j \leq n_2 - 2$. Thus, we have to prove that there is no optimal set $\Delta$ such that $\Delta_2 \subseteq \Delta \subseteq \Delta_1$. A toy example in the case $n_1 = 10$ and $n_2 = 9$ is showed in Figure 6.

We start by removing the point $(0, j)$ in $\Delta_1 ((0, 7)$ in the example). Then the LHS of (2.1) loses one unit in dimension and also $\delta - 1$ units (since $\left\lceil \frac{\Delta}{r} \right\rceil$ decreases one unit), whereas in the bound for the minimum distance we gain

$$d(i, j-1) - d(0, j) = (n_1 - r + 1)(n_2 - j + 1) - n_1(n_2 - j) = n_1 - (r - 1)(n_2 - j + 1).$$

Thus, the variation produced in the LHS of (2.1) is

$$n_1 - (r - 1)(n_2 - j + 1) - \delta = n_1 - (r - 1)(n_2 - j + 1) - (n_1 - r + 1) = -(r - 1)(n_2 - j) < 0$$

(because $r = i + 1 \geq 2$). In our example we lose $\delta = 8$ (units) and gain 4, with variation equal to $-4$. 

**Figure 5. Examples in Remark 3.6**
that $d(0, (1))$ and

\[(2), \text{ with the aim of making easier the understanding of the remaining summands of the LHS of (2.1)} \text{, and we lose } 1 \text{ (unit) in dimension plus the bound on the minimum distance increases } n_2 - j + 1 \text{ and the defect } D \text{ vanishes, holds when we remove the next } r - 1 \text{ points on the (j − 1)th row following a natural order. Then we get } \Delta_2 \text{ and our result is proved.} \]

**Lemma 3.8.** Keep the notation as in Proposition 3.2. Consider a set $S = \Delta_{j,0}^i$, where $1 \leq i \leq n_1 - 2$, and a set $S^\sigma = \Delta_{j,0}^{2,i}$, where $1 \leq j \leq n_2 - 2$. Then, there is no $d_{\sigma}$-optimal code $C_{\Delta}^P$, $\Delta$ being the resulting set of removing less than $i + 1$ points from $S$ (or less than $j + 1$ points from $S^\sigma$) following a natural order.

**Proof.** Consider the set $S = \{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq n_2 - 2 \} \cup \{(0, n_2 - 1)\}$, which we show in Figure 7. The proof for $S^\sigma$ is analogous and we omit it.

Recall from Proposition 3.3 that $d(0, n_2 - 1) \leq d(i, n_2 - 2)$ if and only if $i \leq \frac{n_1}{2}$ and in such case $S$ is optimal. We divide our proof in two cases according to $i \leq \frac{n_1}{2}$ or $i > \frac{n_1}{2}$. Figure 8 shows an example of the case $n_1 = 10$ and $n_2 = 9$ showing sets $S$, where $i = 2 \leq \frac{n_1}{2}$ (Figure 8 (1)) and $i = 7 > \frac{n_1}{2}$ (Figure 8 (2)), with the aim of making easier the understanding of the proof.

Assume $i \leq \frac{n_1}{2}$, following a natural order the first point to remove is $(0, n_2 - 1)$. Then, in the LHS of (2.1) we lose 1 (unit) in dimension plus $\delta - 1$ and gain

\[d(i, n_2 - 2) - d(0, n_2 - 1) = 2(n_1 - r + 1) = n_1 - n_2 + 2\]

in the bound for the minimum distance. Thus, the variation produced in the LHS of (2.1) is

\[n_1 - 2r + 2 - \delta = n_1 - 2r + 2 - (n_1 - r + 1) = -r + 1 < 0\]

(because $r = i + 1 \geq 2$).
Notice that for each point on the \((n_2 - 2)\)th row we remove (excepting \((0, n_2 - 2)\)), the values \(r, \delta\) and \(\left\lceil \frac{n}{2} \right\rceil\) do not change, we lose 1 in dimension and gain 2 in the distance of the exponents in that row. Keeping our procedure of removing points by following a natural order, if we were in the best situation, that is, the natural order was to remove points of least distance on the \((n_2 - 2)\)th row, then the quantity gained in the distances of the exponents would just be the quantity gained in the bound of the minimum distance, so in the LHS of (2.1) we would sum 1 every time we remove a point. Therefore, removing the next \(r - 1 = i\) points on the \((n_2 - 2)\)th row, we would obtain the set

\[
\{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq n_2 - 3\} \cup \{(0, n_2 - 2)\},
\]

which in the best situation would be optimal, and no optimal set is obtained by removing less than \(i + 1\) points from \(S\).

To conclude suppose now that \(i > \frac{n}{2}\). In this case the set \(S\) is not optimal. Indeed, the LHS of (2.1) for the code \(C_{1}^{P} \leq 1\) is

\[
n_1 n_2 + 1 + n_2 - 2i,
\]

which is less than \(n_1 n_2 + 1\). Removing the point with least distance we lose \(\delta\) units in the LHS of (2.1). Then, we need to add

\[
2i - n_1 + \delta = 2i - n_1 + (n_1 - i) = i
\]

units to the LHS of (2.1). These units should come from the sum \(k + d_0\) so that the resulting set is optimal. The same reasoning given in the above paragraph explains that the best situation for gaining units happens by removing points on the \((n_2 - 2)\)th row and again, by removing less than \(i\) more points from \(S\), we cannot get an optimal set, which concludes the proof. \(\square\)

**Example 3.9.** Let us apply Lemma 3.8 to the example showed in Figure 8, where \(n_1 = 10\) and \(n_2 = 9\). If \(i = 2 \leq \frac{n}{2}\), then \((r, \delta) = (3, 8)\) and we must show that the resulting set of removing less than 3 points from \(S\) following a natural order is not optimal. When we remove the first point, \((0, 8)\), then we lose \(\delta = 8\) and gain 6 in the LHS of (2.1), thus we lose 2. The next and last point to remove is \((2, 7)\). Here, we lose 1 unit in dimension and gain 2 in the bound for the minimum distance, but the resulting set \(\Delta\) is not optimal since \(D(C_{\Delta}^{P}) = 1\).

As for the case \(i = 7 \geq \frac{n}{2}\), we have \((r, \delta) = (8, 3)\) and the resulting set of removing less than 8 points from \(S\) following a natural order is not optimal. Indeed, when we remove the first point, \((7, 7)\), then we lose \(\delta = 3\) and gain 2 units in the LHS of (2.1), thus we lose 1. The next six points to remove (together with the defect of the resulting code) are \((6, 7) (D = 1), (7, 6) (D = 1), (0, 8) (D = 2), (5, 7) (D = 1), (4, 7) (D = 2)\) and \((6, 6) (D = 3)\) and no such a set is optimal.
Theorem 3.10. The families of ZAVCs given in Propositions 3.1, 3.2 and 3.3 determine the parameters of all bivariate (m = 2) ZAVCs which are d₀-optimal (r,δ)-LRCs. That is to say, if \( C_{\Delta}^p \) is a d₀-optimal LRC, then there exists a ZAVC, \( C_{\Delta}^p \), as in Propositions 3.1, 3.2 or 3.3 having the same parameters \( n, k, d, r \) and \( \delta \) as \( C_{\Delta}^p \).

Proof. We start by checking \( d_0 \)-optimality when interpolating with respect to \( X \). Unless we say otherwise, every process of removing (or adding points) of (or to) a subset in \( E \) is performed by following a natural order (see the definition below Definition 3.4). We also exemplify the proof for the case \( n_1 = 10 \) and \( n_2 = 9 \). Let us fix the locality \( r = i + 1 \) of the optimal sets \( \Delta \subseteq E \) that we want to find. Since we know that every set \( \Delta \) with \( i = 0 \) is optimal (Proposition 3.1) and we are not interested in those where \( i = n_1 - 1 \) as they do not provide LRCs, we assume \( 1 \leq i \leq r - 1 \leq n_1 - 2 \). By Lemma 3.5, it suffices start with the following set:

\[
\Delta_{i,n_2-1}^1 = \{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq n_2 - 1\}
\]

(see Figure 9 corresponding to our example where \( n_1 = 10, n_2 = 9 \)).

![Figure 9. \( \Delta_{i,n_2-1}^1 \)](image)

This set is optimal by Proposition 3.1 and the largest optimal set with \( r = i + 1 \). Proposition 3.2 proves that the \( i - s \) sets obtained from \( \Delta_{i,n_2-1}^1 \) by removing one by one until \( i - s \) points on the \((n_2 - 1)\)th row are also optimal for \( s = 0 \) if \( i \leq \frac{n_1}{2} \) and \( s = 2i - n_1 \) if \( i > \frac{n_1}{2} \). The last obtained set is

\[
\Delta_{i,0}^2 = \Delta_{i,n_2-1}^3 = \{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq n_2 - 2 \} \cup \{(0, n_2 - 1)\},
\]

when \( i \leq \frac{n_1}{2} \), and

\[
M_{2i}^i := \Delta_{i,2i-n_1}^2 = \{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq n_2 - 2 \} \cup \{(e_1, n_2 - 1) \mid 0 \leq e_1 \leq 2i - n_1\},
\]

if \( i > \frac{n_1}{2} \).

Figure 10 (1) shows \( \Delta_{2,0}^0 \) for the case \( n_1 = 10, n_2 = 9 \). Within the same case, one can see \( M_2^i \) in Figure 10 (2).

Now if \( i > \frac{n_1}{2} \), \( M_2^i \) is the least (with respect to inclusion) optimal set such that \( r = i + 1 \) considered in Propositions 3.1 and 3.2. With respect to \( i \leq \frac{n_1}{2} \), define

\[
M_1^i := \Delta_{i,b}^3 = \{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq b - 1 \} \cup \{(0, b)\},
\]

where \( b = \lceil \frac{(n_2 + 1) - n_2}{i} \rceil \) (Figure 11 shows \( M_1^i \) for the case \( n_1 = 10, n_2 = 9 \)) and the unique optimal sets \( \Delta \) such that \( M_1^i \subseteq \Delta \subseteq \Delta_{i,0}^2 \) are among those given in Proposition 3.3 (see Proposition 3.3 and Lemma 3.7). Then, \( M_1^i \) is the least optimal set such that \( r = i + 1 \) considered in Propositions 3.1, 3.2 and 3.3 when \( i \leq \frac{n_1}{2} \).

Therefore, it only remains to prove that under the above cases, there is no optimal set \( \Delta \) such that \( \Delta \subset M_1^i \) or \( \Delta \subset M_2^i \). Recall that the parameters corresponding to an optimal set must attain the bound

\[
k + d_0 + \left(\left\lceil \frac{k}{r} \right\rceil - 1\right)(\delta - 1) \leq n_1 n_2 + 1.
\]
Let us start with $M_2^i$. If we remove up to $s \,(= 2i - n_1)$ points, then $r$, $\delta$ and $\lceil \frac{k}{2} \rceil$ do not change. The first point to remove is $(s, n_2 - 1)$, but then we lose one unit in dimension which cannot be recovered by adding one unit in the bound for the minimum distance because $d(i, n_2 - 2) = d(s, n_2 - 1)$. Then, the variation produced in the LHS of (2.1) considering $M_2^i$ as the starting set is $\Delta = V(M_2^i) = -1$. In order to achieve $V = 0$, the best situation would hold when the process of removing points was from right to left on the $(n_2 - 1)$th row (without removing points from rows below with the same distance that the previous removed point) because that way, for every removed point we would lose one unit in dimension and gain one unit in the bound for the minimum distance. But even in those cases we would not get $V = 0$.

Suppose that, in our next step, we have removed the $s$ least distance points in $M_2^i$ obtaining the set

$$S^i = \{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq n_2 - 2 \} \cup \{(0, n_2 - 1)\},$$

which is not optimal by the reasoning just given (see Figure 12 for the case $n_1 = 10$, $n_2 = 9$ and $i = 7$).

Next we study what happens when removing points of either $M_2^i$ or $S^i$ according to our two cases $i \leq \frac{n_1}{2}$, $i > \frac{n_1}{2}$. Our reasoning differentiates each previous case in two new sub-cases. The first subcase corresponds to remove a multiple of $r$ points and the second one to delete an intermediate number of points. Since we are looking for sets $\Delta$ with value $r$ corresponding to their locality, the reasoning will finish when we delete the last remaining point in the $i$th column because if one performs a new step, the new code will have smaller locality.
Suppose that we remove $\lambda r$ points of the starting sets $M_1^i$ or $S^i$. Then, in the LHS of (2.1) we lose $\lambda r$ units (corresponding to the dimension) plus $\lambda (\delta - 1)$ units (because for every $r$ points removed, $\lceil \frac{k}{n} \rceil$ decreases one unit), and we gain at most $\lambda n_1$ in the bound for the minimum distance (because the best situation holds by deleting $(0, j)$ and every point at the right of $(0, j - 1)$, $j = b, \ldots, b - \lambda + 1$, for $M_1^i$, and $j = n_2 - 1, \ldots, n_2 - \lambda$ for $S^i$). Then, the variation produced in LHS of (2.1) would be smaller than or equal to

$$-\lambda r - \lambda (\delta - 1) + \lambda n_1 = -\lambda r - \lambda (n_1 - r) + \lambda n_1 = 0.$$  

Nevertheless, the mentioned gaining does not happen because for every $j < b$ (for $M_1^i$) and $j \leq n_2 - 1$ (for $S^i$) there are points in rows below with lower distance than $(0, j)$, at least the point $(i, j - 1)$. Thus, we cannot get the mentioned best situation and if we remove a multiple of $r$ points from $M_1^i$ or $S^i$, we do not find optimal sets. Figure 13 (1) (respectively, 13 (2)) shows what happens in our example with $n_1 = 10$, $n_2 = 9$ when removing $3 \cdot (3 = r)$ (respectively, $2 \cdot (8 = r)$) points from $M_1^i$ (respectively, $S^i$).

Suppose now that we remove an intermediate number of points, that is, $\lambda r - s$ ($1 \leq s < r$) points from $M_1^i$ or $S^i$. For this end, it suffices to start with the set

$$\mathcal{S}^i = \{(e_1, e_2) \mid 0 \leq e_1 \leq i, 0 \leq e_2 \leq n_2 - 2\} \cup \{(0, n_2 - 1)\},$$  

where we do not impose any restriction to the index $i$. Notice that $\mathcal{S}^i = S^i$ when $i > \frac{n_1}{2}$ and $M_1^i \subseteq \mathcal{S}^i$ otherwise. When $i \leq \frac{n_1}{2}$, there is no lose of generality if one considers $\mathcal{S}^i$ instead.

---

**Figure 12.** Set $S^i$ in Theorem 3.10

**Figure 13.** Resulting set (1) (respectively, (2)) when removing 9 (respectively, 16) points from $M_1^i$ (respectively, $S^i$) in Theorem 3.10
of $M'_1$ because the sets obtained by removing an intermediate number of points can be obtained either by removing $\lambda_0 r - s$ points from $M'_1$ or by removing $\lambda r - s$ points from $\overline{S}$, where 

$$\lambda = \lambda_0 + n_2 - 1 - b.$$ 

Then, removing points of $\overline{S}$ as mentioned, we obtain a set $S'$ whose bound on the minimum distance is $d_0 \leq n_1(\lambda + 1) - (\lambda + 1)$ which is not optimal. We prove it by contradiction. Assume that this set is optimal, that is, $k + d_0 = n_1 n_2 + 1$, where $t = \left\lceil \frac{k}{r} \right\rceil - 1(n_1 - r)$. Now, if we add to this set the next $(\lambda - 1)r$ points (following a natural order) without altering the locality $r$, we obtain a new set $S''$ that satisfies

$$n_1 n_2 + 1 \geq k + (\lambda - 1)r + d_0 + \Delta d + \left( \frac{k + (\lambda - 1)r}{r} \right) - 1(n_1 - r)$$

$$= k + (\lambda - 1)r + d_0 + \Delta d + t + (\lambda - 1)(n_1 - r) = k + d_0 + \Delta d + t + (\lambda - 1)n_1$$

$$= n_1 n_2 + 1 + \Delta d + (\lambda - 1)n_1 \geq n_1 n_2 + 1 + (\lambda - 1)n_1 + (\lambda - 1)n_1$$

$$= n_1 n_2 + 1,$$

where $\Delta d$ is the variation produced in the bound on the minimum distance when obtaining $S''$ from $S'$. Then $-(\lambda - 1)n_1 \leq \Delta d < 0$ and thus, $\Delta d = -(\lambda - 1)n_1$. Therefore, we get an optimal set $S''$ that could be also obtained by removing less than $r$ points of $S$, a contradiction by Lemma 3.8. To illustrate this last part of the proof, in Figures 14 (1), 15 (1) and 16 (1) (respectively, 14 (2), 15 (2) and 16 (2)) we show (within the example where $n_1 = 10$, $n_2 = 9$) the sets $\overline{S}$ (corresponding to the case $i \leq n_2$), $S'$ obtained by removing $\lambda r - s = 6 \cdot 3 - 2$ points from $\overline{S}$ and $S''$ obtained by adding $(\lambda - 1)r = (6 - 1)3$ points to $S'$ (respectively, $\overline{S}$ (corresponding to the case $i > n_2$)), $S'$ obtained by removing $\lambda r - s = 2 \cdot 8 - 5$ from $\overline{S}$ and $S''$ obtained by adding $(\lambda - 1)r = (2 - 1)8$ points to $S'$.

We have checked that fixed a locality $r$, the only $d_0$-optimal codes obtained by interpolating with respect to $X$ are of the type of those given in Propositions 3.1, 3.2 and 3.3. It is clear that one can perform the same reasoning by interpolating with respect to $Y$. This concludes the proof after noticing that although in the procedure of interpolating with respect to $X$ (respectively, $Y$) with starting set $\Delta$ and resulting set $\Delta^*$, $\Delta^*$ would have less defect when interpolating with respect to $Y$ (respectively, $X$), we would not find new optimal sets because this optimality would be discarded in the process of interpolating with respect to $Y$ (respectively, $X$).

As a consequence of Theorem 3.10, Corollary 3.11 determines the parameters and $(r, \delta)$-localities of the optimal $(r, \delta)$-LRCs we can obtain with the bound on the minimum distance.
Figure 15. Sets $S'$ obtained by removing points from $\overline{S}'$ in Theorem 3.10

Figure 16. Sets $S''$ obtained by adding points to $S'$ in Theorem 3.10

Notice that, in order not to repeat cases and since the variables play the same role, the parameters are written only with the notation we have used to interpolate with respect to $X$.

**Corollary 3.11.** Let $F_q$ be a finite field. For each pair $(n_1, n_2)$ of integers such that $2 \leq n_1, n_2 \leq q$, there exists an optimal $(r, \delta)$-LRC with length $n = n_1 n_2$, parameters $[n, k, d]_q$ and locality $(r, \delta)$ as follows:

1. $k = (i + 1)(j + 1), d = (n_1 - i)(n_2 - j)$, where
   - $i = 0$ and $0 \leq j \leq n_2 - 1$, being the locality $(r, \delta) = (1, n_1)$; or
   - $1 \leq i \leq n_1 - 2$ and $j = n_2 - 1$, being the locality $(r, \delta) = (i + 1, n_1 - i)$.

2. $k = (i + 1)(n_2 - 1) + s + 1, d = n_1 - s$ and $(r, \delta) = (i + 1, n_1 - i)$, where
   $$\max\{0, 2i - n_1\} \leq s < i \leq n_1 - 2.$$

3. $k = (i + 1)j + 1, d = n_1(n_2 - j)$ and $(r, \delta) = (i + 1, n_1 - i)$, where $1 \leq i \leq n_1 - 2$ and
   $$\max\{1, \frac{m(n_2 - 1) - m_i}{i} \} \leq j \leq n_2 - 2.$$

### 3.2. The case $m \geq 3$.

In Subsection 3.1 we have studied bivariate codes $C^{P}_{\Delta}$, obtained sets $\Delta \subseteq \{0, 1, \ldots, n_1 - 1\} \times \{0, 1, \ldots, n_2 - 1\}$ giving rise to $d_0$-optimal LRCs and determined all the parameters of the $d_0$-optimal bivariate ZAVCs. We devote this subsection to the same purpose in the multivariate case. Thus $R = F_q[X_1, \ldots, X_m]_{/I}$, where $m \geq 3$ and $\Delta \subseteq \{0, 1, \ldots, n_1 - 1\} \times \cdots \times \{0, 1, \ldots, n_m - 1\}$. The forthcoming Propositions 3.12 and 3.13 are the analogue to Propositions 3.1 and 3.2 for multivariate ZAVCs and allow us to determine the parameters of the $d_0$-optimal LRCs of the type $C^{P}_{\Delta}$, $m \geq 3$. 
Proposition 3.12. Keep the notation as given at the beginning of Section 2. For each index
\(j_0 \in \{1, \ldots, m\}\), set \(i_j = n_j - 1\) for all \(j \in \{1, \ldots, m\} \setminus \{j_0\}\) and \(i_{j_0} \in \{0, 1, \ldots, n_{j_0} - 2\}\), and consider
\[
\Delta = \Delta^1_{i_1 \ldots i_m} := \{(e_1, \ldots, e_m) \mid 0 \leq e_j \leq i_j, \text{ for all } j = 1, \ldots, m\}.
\]
Then, the ZAVC, \(C^p_{\Delta^1}\), is a \(d_0\)-optimal LRC with locality \((r, \delta) = (i_{j_0} + 1, n_{j_0} - i_{j_0})\). Furthermore, \(\Delta^1_{i_1 \ldots i_m}\) are the unique sets of the form \(\Delta' = \{(e_1, \ldots, e_m) \mid 0 \leq e_j \leq l_j \text{ for all } j = 1, \ldots, m\}\), where \(0 \leq l_j \leq n_j - 1\), providing \(d_0\)-optimal LRCs.

Proof. We interpolate with respect to \(X_1\) (the proof is analogous if we interpolate with respect to any other variable). Consider a set \(\Delta'\) as in the statement.

We start by assuming that \(l_j = n_j - 1\) for \(m - 2\) indices \(j\). Without loss of generality suppose that \(l_j = n_j - 1\) for all \(j = 3, \ldots, m\). Then, the point that defines the bound on the minimum distance is \((l_1, l_2, n_3 - 1, \ldots, n_m - 1)\) and the parameters of this code give the following value for the RHS of (2.1):
\[
d_0 + k + \left(\left\lceil \frac{k}{r} \right\rceil - 1\right) (\delta - 1) = (n_1 - l_1)(n_2 - l_2) + (l_1 + 1)(l_2 + 1)(n_3 n_4 \ldots n_m)
\]
\[
= (n_1 - l_1)(n_2 - l_2) + n_1(l_2 + 1)n_3 n_4 \ldots n_m - (n_1 - l_1 - 1)
\]
\[
= n_1(l_2 + 1)n_3 n_4 \ldots n_m + (n_1 - l_1)(n_2 - l_2 - 1) + 1.
\]
Thus, the code is \(d_0\)-optimal if and only if \(l_2 = n_2 - 1\) (and \(l_1 \in \{0, 1, \ldots, n_1 - 2\}\) for being an LRC).

We conclude the proof after noticing that the same reasoning allows us to prove the proposition when the number of indices \(j\) in \(\Delta'\) such that \(l_j = n_j - 1\) is less than \(m - 2\).

Our next result shows that deleting, from a set \(\Delta^1_{i_1 \ldots i_m}\), a suitable number of least distance points on the line \(e_j = n_j - 1, j \neq j_0\), a \(d_0\)-optimal LRC is also obtained. This is because for each removed point we lose one unit in dimension but we gain one unit in the bound for the minimum distance and \(r, \delta\) and \(\left\lceil \frac{k}{r} \right\rceil\) do not change. As a consequence the LHS in (2.1) remains constant.

Proposition 3.13. Keep the notation as in Proposition 3.12. Define
\[
\Delta = \Delta^2_{i_1 \ldots i_m} := \Delta^1_{i_1 \ldots i_m} \setminus \{(n_1 - 1, \ldots, n_{j_0} - 1, e_{j_0}, n_{j_0} + 1 - 1, \ldots, n_m - 1) \mid 0 \leq e_{j_0} \leq i_{j_0}\},
\]
where \(s\) satisfies \(\max\{1, 2i_{j_0} - n_{j_0} + 1\} \leq s \leq i_{j_0} \leq n_{j_0} - 2\) or \(i_{j_0} = s = 0\).

Then the ZAVC, \(C^p_{\Delta}\), is a \(d_0\)-optimal LRC with locality \((r, \delta) = (i_{j_0} + 1, n_{j_0} - i_{j_0})\).

Proof. The distance, \(d(p)\), of the point
\[
p = (n_1 - 1, n_2 - 1, \ldots, n_{j_0 - 1} - 1, i_{j_0}, n_{j_0 + 1} - 1, \ldots, n_m - 1, n_m - 1)
\]
determines the bound for the minimum distance of the code \(C^p_{\Delta^1_{i_1 \ldots i_m}}\). We look for an index \(0 \leq s \leq i_{j_0}\) which gives the value \(i_{j_0} - s + 1\) corresponding to the number of points in \(\Delta^1_{i_1 \ldots i_m}\) that meet the line \(e_j = n_j - 1, j \neq j_0\), and have distance less than \(2(n_{j_0} - i_{j_0})\). The set \(\Delta\) candidate to be optimal is obtained by deleting from \(\Delta^1_{i_1 \ldots i_m}\) those points because \(2(n_{j_0} - i_{j_0})\) is the distance of any point in the set
\[
V = \{p - e_j \text{ for all } j \in \{1, \ldots, m\} \setminus \{j_0\}\},
\]
where \(e_j = (\delta_{j_1}, \ldots, \delta_{j_m})\), \(\delta_{ij}\) being the Kronecker delta, and \(V \subseteq \Delta^1_{i_1 \ldots i_m} \setminus \{p\}\). Thus, \(n_{j_0} - s < 2(n_{j_0} - i_{j_0})\), what is equivalent to \(s \geq 2i_{j_0} - n_{j_0} + 1\).

Therefore, in order to \(\Delta\) be a candidate for being optimal, \(s \geq \max(0, 2i_{j_0} - n_{j_0} + 1)\). The dimension of the code \(C^p_{\Delta}\) is
\[
k = n_1 n_2 \cdots n_{j_0 - 1}(i_{j_0} + 1)n_{j_0 + 1} \cdots n_m - (i_{j_0} - s + 1),
\]
and the bound on the minimum distance of $C^P_\Delta$ is given by the point with coordinates $e_j = n_j - 1$, $j \neq j_0$, $e_{j_0} = s - 1$ when $s \geq 1$ or by any point of $V$ when $s = 0$. Then $d_0 = n_{j_0} - s + 1$ for $s \geq 1$ and $d_0 = 2(n_{j_0} - i_{j_0})$ when $s = 0$. Moreover we interpolate with respect to $X_{j_0}$ (it is the only way to obtain an LRC), so $r = i_{j_0} + 1$ and $\delta - 1 = n_{j_0} - i_{j_0} - 1$. Thus, the value for $k + d_0 + \left[\left\lceil \frac{k}{s} \right\rceil - 1\right](\delta - 1)$ (the LHS of (2.1)) is
\[
\begin{align*}
&n_1 n_2 \ldots n_{j_0-1} (i_{j_0} + 1) n_{j_0+1} \ldots n_m - (i_{j_0} - s + 1) + n_{j_0} - s + 1 \\
&+ \left(\frac{n_1 n_2 \ldots n_{j_0-1} (i_{j_0} + 1) n_{j_0+1} \ldots n_m - (i_{j_0} - s + 1)}{i_{j_0} + 1}\right) - 1\cdot(n_{j_0} - i_{j_0} - 1) \\
&= n_1 n_2 \ldots n_m - i_{j_0} + n_{j_0} - (n_{j_0} - i_{j_0} - 1) = n_1 n_2 \ldots n_m + 1, \text{if } s \geq 1 \text{ and}
\end{align*}
\]
\[
\begin{align*}
&n_1 n_2 \ldots n_{j_0-1} (i_{j_0} + 1) n_{j_0+1} \ldots n_m - (i_{j_0} - s + 1) + 2(n_{j_0} - i_{j_0}) \\
&+ \left(\frac{n_1 n_2 \ldots n_{j_0-1} (i_{j_0} + 1) n_{j_0+1} \ldots n_m - (i_{j_0} - s + 1)}{i_{j_0} + 1}\right) - 1\cdot(n_{j_0} - i_{j_0} - 1) \\
&= n_1 n_2 \ldots n_m - i_{j_0} + 2(n_{j_0} - i_{j_0}) - 2(n_{j_0} - i_{j_0} - 1) = n_1 n_2 \ldots n_m + 1 - i_{j_0}, \text{otherwise,}
\end{align*}
\]
which proves that $\Delta$ is optimal and concludes the proof. \qed

**Theorem 3.14.** Keep the notation as in Section 2 and assume $m \geq 3$. Let $C^P_\Delta$ be a $d_0$-optimal LRC, then there exists a set $\Delta^* \subseteq E$ of the form given in Propositions 3.12 or 3.13 such that $C^P_\Delta$ has the same parameters $n, k, d, r$ and $\delta$ as $C^P_\Delta$.

**Proof.** The proof follows by a close reasoning to that given in Theorem 3.10. By a multivariate version of Lemma 3.5, one can start with a set $\Delta$ as that given in Proposition 3.12 and remove points in a natural order. The main difference with the case $m = 2$ is that when $m \geq 3$, we should delete points out of a plane, which enlarges the defect (Definition 2.9), giving rise to the two possibilities described in Propositions 3.12 and 3.13 for sets $\Delta$ such that $C^P_\Delta$ is $d_0$-optimal. \qed

**Corollary 3.15.** Let $\mathbb{F}_q$ be a finite field and consider an integer $m \geq 3$. For every $m$-tuple $(n_1, \ldots, n_m)$ of integers such that $2 \leq n_j \leq q$, $j \in \{1, \ldots, m\}$, there exists an optimal $(r, \delta)$-LRC with length $n = n_1 \cdots n_m$, parameters $[n, k, d, r, \delta]_q$ and locality $(r, \delta)$ as follows:

1. $k = n_1 \cdots n_{j_0-1}(i_{j_0} + 1)n_{j_0+1} \cdots n_m$, $d = n_{j_0} - i_{j_0}$ and $(r, \delta) = (i_{j_0} + 1, n_{j_0} - i_{j_0})$, where $i_{j_0} \in \{0, 1, \ldots, n_{j_0} - 2\}$.

2. $k = n_1 \cdots n_{j_0-1}(i_{j_0} + 1)n_{j_0+1} \cdots n_m - (i_{j_0} - s + 1)$, $d = n_{j_0} - s + 1$ and $(r, \delta) = (i_{j_0} + 1, n_{j_0} - i_{j_0})$, where

$$\max\{1, 2i_{j_0} - n_{j_0} + 1\} \leq s \leq i_{j_0} \leq n_{j_0} - 2.$$ 

3. $k = n_1 \cdots n_{j_0-1}n_{j_0+1} \cdots n_m - 1$, $d = 2n_{j_0}$ and $(r, \delta) = (1, n_{j_0})$.

**Remark 3.16.** Keep the notation as in Section 2, so let $m \geq 2$. Let $\mathbb{N}$ be the set of nonnegative integers and $\Delta$ be a subset of $E$ satisfying some of the conditions in Propositions 3.1, 3.2, 3.3, 3.12 or 3.13. Define $\Delta^* := \nu + \Delta$ for any $\nu \in \mathbb{N}^m$ such that $\Delta^* \subseteq E$. If $0 \notin P_j$ for all $1 \leq j \leq m$, then the ZAVC $C^P_{\Delta^*}$ is optimal with the same parameters and locality as $C^P_\Delta$. This result follows straightforwardly from Remark 2.3.

**Remark 3.17.** ZAVCs are related with and include the family of codes introduced in [2], codes whose evaluation map is the same as ZAVCs but their evaluation sets $V_\Delta$ are only a subset of those used for ZAVCs. Specifically, the codes in [2] are subcodes of affine cartesian codes (of order $d$), which the authors name quasi affine cartesian codes (of order $d$). The corresponding set $V_\Delta$ in [2] is the set of polynomials $f$ in $\mathbb{F}_q[X_1, \ldots, X_m]$ with total degree bounded by $d$ such that a fixed variable $X_i$ has degree $\deg_{X_i}(f) \leq d_i < n_i - 1$ (see [2, Definitions 2.2 and 2.3]). Therefore, while ZAVCs allow arbitrary sets $\Delta \subseteq E$, sets $\Delta$ of quasi affine
cartesian codes are of the form
\[ \Delta = \Delta_2 = \{(e_1, \ldots, e_m) \in E \mid e_1 + \cdots + e_m \leq d, e_i \leq d_i \}. \]

As a consequence we obtain many more \((r, \delta)\)-optimal LRCs than those given in [2, Corollary 4.2]. Thus, if we fix the locality \( r = d_1 + 1 \), then we obtain optimal sets giving optimal codes which are not considered in [2]. These are those of Proposition 3.1 for \( i = d_1 = 0, j < n_2 - 2, \) and \( i < n_1 - 2, j = d_1 = 0 \); those of Proposition 3.2 for \( s \leq d_1 - 2 \); those of Proposition 3.3; and those of Proposition 3.13 for \( j_0 = l, i_{j_0} = d_1 \geq 2 \) and \( \max(1, 2(d_1 - n_j) + 1) \leq s \leq d_1 - 1 \). Moreover we also give many more optimal LRCs, regarded as subfield-subcodes of ZAVCs, as we will explain in the next section.

4. Optimal subfield-subcodes

\( J \)-affine variety codes were introduced in [14] and they are a subclass of ZAVCs. We devote this section to prove that subfield-subcodes of some \( J \)-affine variety codes keep the parameters and \((r, \delta)\)-locality of the original ZAVCs they come from over small supporting fields, giving rise to new \((r, \delta)\)-LRCs. In fact, in this section we provide optimal LRCs whose length is a multiple of \( r + \delta - 1 > q \) (in some cases \( r + \delta - 1 > q + 1 \), \( r > 1 \) and \( \delta > 2 \) and their parameters cannot be found in the literature [8, 23, 27, 37, 21, 6, 41, 7, 10, 43, 44, 34, 9, 26, 22].

Subfield-subcodes of \( J \)-affine variety codes were also used in [13] to provide \((r, \delta)\)-LRCs, however the recovery procedure was different and the obtained codes were distinct of those in this section, most of them non-optimal.

4.1. Subfield-subcodes. In this subsection we recall some facts about subfield-subcodes which will be useful in the forthcoming subsections. We keep the notation as in Section 2. Assume that \( q = p^l \), where \( p \) is a prime number and \( l \geq 2 \). Pick a positive integer \( h \) such that \( h \mid l \) and regard \( \mathbb{F}_{p^h} \) as a subfield of \( \mathbb{F}_q = \mathbb{F}_{p^l} \). Consider a subset \( J \subseteq \{1, \ldots, m\} \) and assume that the polynomials \( f_j(X_j) \) generating the ideal \( I \) are of the form
\[ f_j(X_j) = X_j^{n_j} - 1, \]
for some \( n_j \mid q - 1 \) if \( j \in J \), and
\[ f_j(X_j) = X_j^{n_j} - X_j, \]
where \( n_j - 1 \mid q - 1 \), otherwise. Then, each set \( P_j \) is the set of \((n_j)\)th roots of unity if \( j \in J \) or the set of \((n_j - 1)\)th roots of unity together with 0 otherwise. The corresponding ZAVC is denoted by \( C^{P_j} \). As introduced in [14], \( C^{P_j} \) is a \( J \)-affine variety code.

Definition 4.1. The linear code \( S^{P_j}_\Lambda := C^{P_j} \cap \mathbb{F}_{p^h}^n \) is the subfield-subcode over the field \( \mathbb{F}_{p^h} \) of \( C^{P_j} \).

When \( j \notin J \), the evaluation of monomials containing \( X_j^0 \) or containing \( X_j^{n_j - 1} \) may be different (see [11] for details). This explains the difference on the powers on the variables when equipping \( E \) with the following structure which we will assume in the sequel. When \( j \notin J \) we then identify the set \( \{0, 1, \ldots, n_j - 1\} \) with the ring \( \mathbb{Z}/n_j \mathbb{Z} \). Otherwise, if \( j \in J \), we identify the set \( \{1, \ldots, n_j - 1\} \) with \( \mathbb{Z}/(n_j - 1) \mathbb{Z} \), and we extend the addition and multiplication in this ring to \( \{0, 1, \ldots, n_j - 1\} \), by setting \( 0 + e = e, 0 \cdot e = 0 \) for all \( e = 0, 1, \ldots, n_j - 1 \).

We call a set \( \Omega \subseteq E \) a cyclotomic set with respect to \( p^h \) if \( p^h \omega = \Omega \) for all \( \omega = (\omega_1, \ldots, \omega_m) \in \Omega \). Minimal cyclotomic sets are those of the form \( \Lambda = \{p^h e \mid i \geq 0\} \), for some element \( e \in E \). For each minimal cyclotomic set \( \Lambda \), denote by \( x \) the minimum element in \( \Lambda \) with respect to the lexicographic order and set \( \Lambda = \Lambda_x \). Hence, \( \Lambda_x = \{x, p^h x, \ldots, p^{h(\# \Lambda_x - 1)} x\} \). Fixed an index \( j \in \{1, \ldots, m\} \), we analogously define a set \( \Omega^j \subseteq \{0, 1, \ldots, n_j - 1\} \) or \( \Lambda^j \subseteq \{0, 1, \ldots, n_j - 1\} \) and call it (minimal) cyclotomic set in a single variable with respect to \( p^h \) if the above conditions are satisfied but changing \( \Omega \) by \( \Omega^j \) and \( E \) by \( \{0, 1, \ldots, n_j - 1\} \). Again, denoting by
x the minimum element in \( \Lambda^1 \), we set \( \Lambda^j = \Lambda^1_x \). For example, if \( m = 3 \), \( x = (1, 4, 5) \) and \( \Lambda_x = \{(1,4,5),(2,1,3),(4,2,6)\} \subseteq E \) is a minimal cyclotomic set with respect to 2, the corresponding minimal cyclotomic set in a single variable for \( j = 3 \) is the set \( \Lambda^3_3 = \{3,5,6\} \subseteq \{0,1,\ldots,n_j-1\} \). Note that the above example corresponds to the case \( p = 2, h = 1, l = 3, J = \emptyset \), and \( n_1 = n_2 = n_3 = 8 \).

Let \( X \) be the set of representatives of all minimal cyclotomic sets in \( E \). For any non-empty subset \( \Delta \subseteq E \), let \( X(\Delta) = \{x \in X \mid \Lambda_x \subseteq \Delta \} \). We say that the set \( \Delta \) is closed if

\[
\Delta = \bigcup_{x \in X(\Delta)} \Lambda_x,
\]

that is, it is a union of minimal cyclotomic sets.

Closed sets will be the key for obtaining optimal \((r, \delta)\)-LRCs from subfield-subcodes. To explain it, we recall, on the one hand, that if \( \Delta \) is a closed set, then \( \dim(C^{\text{RL}}_\Delta) = \#\Delta \) [13, Theorem 2.3]. On the other hand, the minimum distance of a subfield-subcode \( S^{\text{RL}}_\Delta \) admits the bound given in Proposition 2.6. Thus, if we choose \( \Delta \) to be closed, the code over \( F_p^{\text{RL}} \), \( S^{\text{RL}}_\Delta \), has the same parameters \( n \) and \( k \), the same bound for the minimum distance as \( C^{\text{RL}}_\Delta \) and also the same locality \((r, \delta)\) since the recovery method presented in Proposition 2.7 can also be applied.

### 4.2. Optimal \((r, \delta)\)-LRCs coming from subfield-subcodes of bivariate ZAVCs

In this subsection, we use some results in Section 3 and those in the above paragraph to provide some families of optimal \((r, \delta)\)-LRCs coming from subfield-subcodes of bivariate \( J \)-affine variety codes. We group our families in Theorem 4.2. For the proof we only need to use Corollary 3.11 and to prove that the involved sets \( \Delta \) are closed with respect to \( p^h \). Let \( R_{\leq i} \subseteq F_p^{\text{RL}} \) denote the set of \((i)\)th roots of unity.

**Theorem 4.2.** Keep the notation as in Section 2 and Subsection 4.1. Fix \( i \in \{1, 2\} \) and denote \( i' \) the unique element \( i' \in \{1, 2\} \setminus \{i\} \). Pick \( n_i > p^h \) and let \( 1 \leq z \leq n_i - 1 \) be a positive integer. Set

\[
\Omega^i := \Lambda^i_0 \cup \Lambda^i_1 \cup \cdots \cup \Lambda^i_{x_z},
\]

where \( \Lambda^i_j \subseteq \{0,1,\ldots,n_i-1\} \) is the minimal cyclotomic set in a single variable containing \( x \) and \( x_1,\ldots,x_z \neq 0 \). Let \( c_i := \#\Omega^i \) and \( m_i := \max \Omega^i \). Define

\[
\Delta_j := \Omega^j \times \{0,1,\ldots,j-1\} \cup \{(0,j)\},
\]

when \( i = 1 \), and

\[
\Delta_j := \{0,1,\ldots,j-1\} \times \Omega^2 \cup \{(j,0)\},
\]

otherwise. Assume \( c_i < n_i - 1 \), max \( \{1, n_i+1 - \frac{m}{m_i}\} \leq j \leq n_i - 1 \) if \( j < n_i - 1 \), also \( n_i > p^h + 1 \). Then, the following subfield-subcodes \( S^{\text{RL}}_\Delta \) over the field \( F_p^{\text{RL}} \) are optimal \((r, \delta)\)-LRCs:

1. \( P = P_1 \times P_2 \), where \( P_i = R_{n_i} \), \( P_i = R_{n_i-1} \cup \{0\} \) and \( n_i, n_i - 1 \mid q - 1 \); \( J = \{i\} \) and \( \Delta = \Delta_{n_i-1} \), in which case \( r(\delta) = (c_i, n_i - c_i + 1) \).
2. \( P = P_1 \times P_2 \), where \( P_i = R_{n_i-1} \cup \{0\} \), \( P_i = R_{n_i-1} \cup \{0\} \) and \( n_i - 1, n_i - 1 \mid q - 1 \); \( J = \emptyset \) and \( \Delta = \Delta_{n_i-1} \), in which case \( r(\delta) = (c_i, n_i - c_i + 1) \).
3. \( P = P_1 \times P_2 \), where \( P_i = R_{n_i} \), \( P_i = R_{n_i} \) and \( n_i \mid q - 1 \); \( n_i \mid p^h - 1 \); \( J = \{i, 1\} \) and \( \Delta = \Delta_j \), in which case \( r(\delta) = (c_i, n_i - c_i + 1) \).
4. \( P = P_1 \times P_2 \), where \( P_i = R_{n_i} \), \( P_i = R_{n_i-1} \cup \{0\} \) and \( n_i \mid q - 1 \); \( n_i \mid p^h - 1 \); \( J = \emptyset \) and \( \Delta = \Delta_{n_i-1} \), in which case \( r(\delta) = (c_i, n_i - c_i + 1) \).
5. \( P = P_1 \times P_2 \), where \( P_i = R_{n_i-1} \cup \{0\} \), \( P_i = R_{n_i} \), and \( n_i - 1 \mid q - 1 \); \( n_i \mid p^h - 1 \); \( J = \{i'\} \) and \( \Delta = \Delta_j \), in which case \( r(\delta) = (c_i, n_i - c_i + 1) \).
6. \( P = P_1 \times P_2 \), where \( P_i = R_{n_i-1} \cup \{0\} \), \( P_i = R_{n_i-1} \cup \{0\} \) and \( n_i - 1 \mid q - 1 \); \( n_i - 1 \mid p^h - 1 \); \( J = \emptyset \) and \( \Delta = \Delta_j \), in which case \( r(\delta) = (c_i, n_i - c_i + 1) \).
Proof. It suffices to prove that every set \( S \) is closed with respect to \( p^h \) and that \( C^{R/J}_\Delta \) is optimal.

We start by proving that the sets \( \Delta \) in (1) and (2) are closed. In a single variable, the minimal cyclotomic set containing 0 is always \( \{0\} \). In addition, when \( 0 \in P_l \) for any \( l \), the minimal cyclotomic set in a single variable \( \Lambda_{n_l - 1} = [0, 1, \ldots, n_l - 1] \) is the set \( \Lambda_{n_l - 1} = [0, 1, \ldots, n_l - 1] \).

Indeed, with the identification described in Subsection 4.1, it holds the following chain of equalities:

\[
p^h(n_l - 1) = (p^h - 1)(n_l - 1) + n_l - 1 = (p^h - 1)n_l + n_l - p^h = p^h - 1 + n_l - p^h = n_l - 1.
\]

Thus, for \( e \in E \) such that \( e_l = n_l - 1 \) and \( e_{l'} = 0 \), where \( l' \) is the unique element \( l' \in \{1, 2\} \), one has that \( \Lambda_{n_l} = \{e\} \) and \( \Omega^1 \times \{n_l - 1\} \) and \( \{n_l - 1\} \times \Omega^2 \) are closed. For every \( l = 1, 2 \), the set \( \{0, 1, \ldots, n_l - 1\} \) is closed and so, \( \Omega^1 \times [0, 1, \ldots, n_l - 1] \), \( [0, 1, \ldots, n_l - 1] \) and \( \{0, 1, \ldots, n_l - 1\} \times \Omega^2 \) and \( [0, 1, \ldots, n_l - 1] \times [0] \) are also closed. Sets \( \Delta \) in (1) and (2) are obtained from these sets either by considering

\[
((0) \times [0, 1, \ldots, n_l - 1]) \cup ([0, 1, \ldots, n_l - 1]) \setminus (\{0\} \times [0, 1, \ldots, n_l - 1])
\]

giving rise, again, to a closed set.

The fact that the remaining sets (3)-(6) are closed follows by a similar reasoning as above and by noticing that when \( P_l = R_{n_l} \), \( n_l \) \( p^h - 1 \) or \( P_l = R_{n_l - 1} \cup \{0\} \), \( n_l - 1 \) \( p^h - 1 \), one can identify \( p^h \) with 1 when computing minimal cyclotomic sets.

The condition \( j \geq n_{l'} + 1 - \frac{m_l}{m_{l'}} \) comes from imposing that the distance of the point with coordinates \( e_l = 0 \), \( e_{l'} = \) is less than or equal to that of the point given by \( e_l = m_l \), \( e_{l'} = j - 1 \). Thus, applying Corollary 3.11 (2) (respectively, 3.11 (3)) when \( j = n_{l'} - 1 \) (respectively, otherwise), the parameters of \( C^{R/J}_\Delta \) are

\[
n = n_1 n_2, \quad k = c_j + 1, \quad d = n_{l'} - j \quad \text{and} \quad (r, \delta) = (c_j, n_l - c_l + 1).
\]

Therefore, \( C^{R/J}_\Delta \) is optimal, which concludes the proof.

\[\square\]

Remark 4.3. Theorem 4.2 does not give an exhaustive list of the optimal \((r, \delta)\)-codes one can find from subfield-subcodes of ZAVCs. In fact, we impose the condition \( n_l > p^h \) (or \( n_l > p^h + 1 \)) and choose sets \( \Delta \) in order to get new families of \( q \)-ary optimal \((r, \delta)\)-LRCs whose lengths are a multiple of \( r + \delta - 1 > q \) (or \( r + \delta - 1 > q + 1 \)).

The following Corollaries 4.4 and 4.5 provide easy to get sets \( \Delta \), \( P \) and \( J \) giving rise to subfield-subcodes \( S^{R/J}_\Delta \) which are optimal \((r, \delta)\)-LRCs.

Corollary 4.4. Keep the notation as in Section 2, Subsubsection 4.1 and Theorem 4.2. Let \( 1 < s \leq \frac{1}{r} \) such that \( p^h - 1 \mid q - 1 \) (and, if \( s = 2 \), also \( p^h \geq 3 \)) and fix \( \{x_1, x_2, \ldots, x_s\} = \{1, 2, \ldots, z\} \). Then, the following subfield-subcodes \( S^{R/J}_\Delta \) over the field \( \mathbb{F}_{p^s} \) are optimal \((r, \delta)\)-LRCs:

1. \( P = P_1 \times P_2 \), where \( P_1 = R_{p^{h_{i-1}}} \), \( P_2 = R_{p^{h_{i-1}}} \cup \{0\} \), \( z < \frac{p^h}{2} \) and \( n_l - 1 \mid q - 1 \); \( J = \emptyset \) and \( \Delta = \Lambda_{n_l - 1} \) in which case \( (r, \delta) = (z + 1, p^h - z - 1) \).

2. \( P = P_1 \times P_2 \), where \( P_1 = R_{p^{h_{i-1}}} \cup \{0\} \), \( P_2 = R_{p^{h_{i-1}}} \cup \{0\} \), \( z \leq \frac{p^h}{2} \) and \( n_l - 1 \mid q - 1 \); \( J = \emptyset \) and \( \Delta = \Lambda_{n_l - 1} \) in which case \( (r, \delta) = (z + 1, p^h - z) \).

3. \( P = P_1 \times P_2 \), where \( P_1 = R_{p^{h_{i-1}}} \cup \{0\} \), \( P_2 = R_{p^{h_{i-1}}} \cup \{0\} \), \( z \leq \frac{p^h}{2} \) and \( n_l - 1 \mid p^h - 1 \); \( J = \{1, 2\} \) and \( \Delta = \Lambda_j \), where \( \max \{1, n_l - 1 - \frac{1}{2} \left( p^h - 1 \right) \} \leq j \leq n_l - 1 \), in which case \( (r, \delta) = (z + 1, p^h - z - 1) \).
(4) \( P = P_1 \times P_2 \), where \( P_i = R_{p^{n_i-1}}, P_1 = R_{n_1-1} \cup \{0\}, z < \frac{p^h}{2} \) and \( n_i - 1 \mid p^h - 1; \) \( J = \{i\} \) and \( \Delta = \Delta_j \), where \( \max \left\{ 1, n_i + 1 - \frac{p^h}{p^{\omega_i+1}} \right\} \leq j \leq n_i - 1, \) in which case \( (r, \delta) = (zs + 1, p^{hs} - zs - 1) \).

(5) \( P = P_1 \times P_2 \), where \( P_i = R_{p^{n_i-1}} \cup \{0\}, P_1 = R_{n_1}, z \leq \frac{p^h}{2} \) and \( n_i - 1 \mid p^h - 1; \) \( J = \{i'\} \) and \( \Delta = \Delta_j \), where \( \max \left\{ 1, n_i + 1 - \frac{p^h}{z} \right\} \leq j \leq n_i - 1, \) in which case \( (r, \delta) = (zs + 1, p^{hs} - zs) \).

(6) \( P = P_1 \times P_2 \), where \( P_i = R_{p^{n_i-1}} \cup \{0\}, P_1 = R_{n_1-1} \cup \{0\}, z \leq \frac{p^h}{2} \) and \( n_i - 1 \mid p^h - 1; \) \( J = \emptyset \) and \( \Delta = \Delta_j \), where \( \max \left\{ 1, n_i + 1 - \frac{p^h}{z} \right\} \leq j \leq n_i - 1, \) in which case \( (r, \delta) = (zs + 1, p^{hs} - zs) \).

Proof. The proof follows from Theorem 4.2 by noticing that in all cases \( n_i > p^h + 1, \) \( \Omega^i = \Lambda_0^i \cup \Lambda_1^i \cup \cdots \cup \Lambda_j^i \), where \( \Lambda_j^i = \{x, x p^h, x p^{2h}, \ldots, x p^h(s-1)\} \), because conditions \( z < \frac{p^h}{2} \) and \( z \leq \frac{p^h}{2} \) imply \( x p^h(s-1) \leq z p^h(s-1) < p^{hs} - 1 \). Therefore, \( c_i = zs + 1 \) and \( m_i = z p^{(s-1)h}, c_i < n_i - 1 \) and \( \max \left\{ 1, n_i + 1 - \frac{p^h}{m_i} \right\} \leq j \leq n_i - 1 \).

**Corollary 4.5.** Keep the notation as in Section 2, Subsection 4.1 and Theorem 4.2, but allowing \( z = 0 \). Let \( 1 < s \leq \frac{1}{h} \) be such that \( p^{hs} - 1 \mid q - 1 \) (and, if \( s = 2 \), also \( p^h \geq 3 \)). Let \( 1 \leq w \leq \left\lfloor \frac{(p^h-1)p^h}{2^{p^h-1}} \right\rfloor \) be a positive integer and pick \( a_1 < a_2 < \cdots < a_w \) positive integers which are multiples of \( \frac{p^h-1}{p^h-1} \).

Then, the following subfield-subcodes \( S^D_{p^h} \) over the field \( \mathbb{F}_{p^h} \) are optimal \((r, \delta)\)-LRCs:

1. \( P = P_1 \times P_2 \), where \( P_i = R_{p^{n_i-1}}, P_1 = R_{n_1-1} \cup \{0\}, a_w \leq \frac{p^{h-1}}{2} \) and \( n_i - 1 \mid q - 1 \) (and, if \( z \neq 0 \), also \( z < \frac{p^h}{2} \)); \( J = \{i\} \) and \( \Delta = \Delta_{n_1-1} \), in which case \( (r, \delta) = (zs + w + 1, p^{hs} - z - w - 1) \).

2. \( P = P_1 \times P_2 \), where \( P_i = R_{p^{n_i-1}} \cup \{0\}, P_1 = R_{n_1-1} \cup \{0\}, a_w \leq \frac{p^h}{2} \) and \( n_i - 1 \mid q - 1 \) (and, if \( z \neq 0 \), also \( z \leq \frac{p^h}{2} \)); \( J = \emptyset \) and \( \Delta = \Delta_{n_1-1} \), in which case \( (r, \delta) = (zs + w + 1, p^{hs} - z - w) \).

3. \( P = P_1 \times P_2 \), where \( P_i = R_{p^{n_i-1}} \cup \{0\}, P_1 = R_{n_1}, a_w \leq \frac{p^{h-1}}{2} \) and \( n_i - 1 \mid p^h - 1 \) (and, if \( z \neq 0 \), also \( z \leq \frac{p^h}{2} \)); \( J = \{1, 2\} \) and \( \Delta = \Delta_j \), where \( \max \left\{ 1, n_i + 1 - \frac{p^h-1}{a_w} \right\} \leq j \leq n_i - 1 \) (and, if \( z \neq 0 \), also \( n_i + 1 - \frac{p^h-1}{a_w} \) \( \leq j \)), in which case \( (r, \delta) = (zs + w + 1, p^{hs} - z - w - 1) \).

4. \( P = P_1 \times P_2 \), where \( P_i = R_{p^{n_i-1}} \cup \{0\}, P_1 = R_{n_1-1} \cup \{0\}, a_w \leq \frac{p^{h-1}}{2} \) and \( n_i - 1 \mid p^h - 1 \) (and, if \( z \neq 0 \), also \( z < \frac{p^h}{2} \)); \( J = \emptyset \) and \( \Delta = \Delta_j \), where \( \max \left\{ 1, n_i + 1 - \frac{p^h-1}{a_w} \right\} \leq j \leq n_i - 1 \) (and, if \( z \neq 0 \), also \( n_i + 1 - \frac{p^h-1}{a_w} \leq j \)), in which case \( (r, \delta) = (zs + w + 1, p^{hs} - z - w) \).

5. \( P = P_1 \times P_2 \), where \( P_i = R_{p^{h-1}} \cup \{0\}, P_1 = R_{n_1}, a_w \leq \frac{p^{h-1}}{2} \) and \( n_i - 1 \mid p^h - 1 \) (and, if \( z \neq 0 \), also \( z \leq \frac{p^h}{2} \)); \( J = \{i'\} \) and \( \Delta = \Delta_j \), where \( \max \left\{ 1, n_i + 1 - \frac{p^h-1}{a_w} \right\} \leq j \leq n_i - 1 \) (and, if \( z \neq 0 \), also \( n_i + 1 - \frac{p^h-1}{a_w} \leq j \)), in which case \( (r, \delta) = (zs + w + 1, p^{hs} - z - w - 1) \).

6. \( P = P_1 \times P_2 \), where \( P_i = R_{p^{h-1}} \cup \{0\}, P_1 = R_{n_1-1} \cup \{0\}, a_w \leq \frac{p^{h-1}}{2} \) and \( n_i - 1 \mid p^h - 1 \) (and, if \( z \neq 0 \), also \( z \leq \frac{p^h}{2} \)); \( J = \emptyset \) and \( \Delta = \Delta_j \), where \( \max \left\{ 1, n_i + 1 - \frac{p^h-1}{a_w} \right\} \leq j \leq n_i - 1 \) (and, if \( z \neq 0 \), also \( n_i + 1 - \frac{p^h-1}{a_w} \leq j \)), in which case \( (r, \delta) = (zs + w + 1, p^{hs} - zs - w) \).
Proof. The proof follows from Theorem 4.2. Indeed, in all cases \( n_j > p^h + 1 \) and the cardinality of the cyclotomic sets \( \Lambda_{a_1}, \ldots, \Lambda_{a_w} \) is one because the fact that each \( a_i, 1 \leq i \leq w \), is a multiple of \( \frac{p^h - 1}{p - 1} \) implies \( a_i p^h = a_t \mod p^h - 1 \) for all \( t \in \{1, \ldots, w\} \). Moreover, if \( z \neq 0 \), each cyclotomic set \( \Lambda^j, 1 \leq j \leq z \), has cardinality equal to \( s \) since the conditions \( z < \frac{p^h}{2} \) or \( z \leq \frac{p^h}{2} \) imply \( xp^{h(s-1)} < zp^{h(s-1)} - 1 \). Therefore, \( c_i = zs + w + 1 < n_i - 1 \) and
\[
\max \left\{ \frac{1}{m_i}, \frac{n_i - 1 - \frac{n_i}{m_i}}{1} \right\} \leq j \leq n_i - 1,
\]
because when \( z = 0, m_i = a_w \) and, otherwise, \( m_i \) is either \( a_w \) or \( zp^{(s-1)}h \).

Examples 4.6. Next we give new optimal LRCs obtained by applying Theorem 4.2 and Corollaries 4.4 and 4.5.

1. Consider \( q = 3^4, p^h = 3, i = z = x_1 = 1, n_1 = 20, n_2 = 2, \) then by Theorem 4.2 (1) one gets a \([40,6,20]_3 \) optimal \((5,16)\)-LRC.
2. Consider \( q = 2^h, p^h = 2, i = 2, z = x_1 = 1, n_1 = 10, n_2 = 8, \) then by Theorem 4.2 (2) one gets a \([63,37,8]_2 \) optimal \((4,5)\)-LRC.
3. Consider \( q = 5^2, p^h = 5, i = z = 1, x_1 = 3, n_1 = 12, n_2 = 5, \) then by Theorem 4.2 (4) one gets a \([60,5,36]_3 \) optimal \((2,11)\)-LRC.
4. Consider \( q = 3^4, p^h = 9, i = z = x_1 = 1, n_1 = 41, n_2 = 4, \) then by Theorem 4.2 (5) one gets a \([164,7,82]_9 \) optimal \((3,38)\)-LRC.
5. Consider \( q = 2^6, p^h = 4, i = z = x_1 = 2, x_2 = 7, n_1 = 4, n_2 = 22, \) then by Theorem 4.2 (6) one gets a \([88,16,22]_4 \) optimal \((5,18)\)-LRC.
6. Consider \( q = 4^h, p^h = 4, i = 2, z = 1, s = 2, n_1 = 6, n_2 = 15, \) then by Corollary 4.4 (1) one gets a \([90,16,15]_4 \) optimal \((3,13)\)-LRC.
7. Consider \( q = 2^6, p^h = 8, i = 1, z = 2, w = 2, a_1 = 9, a_2 = 18, s = 2, n_1 = 63, n_2 = 8, \) then by Corollary 4.5 (4) one gets a \([504,43,126]_8 \) optimal \((7,57)\)-LRC.

The sets \( \Delta \) introduced in Theorem 4.2 and used in the above examples are shown in Figure 17. We make explicit the decomposition of the set \( \Delta = \bigcup_{i=0}^{4} \{ \{0,1,4\} \times \{0,1,4\} \} \cup \{ \{5,0\} \} \) in Example 4.6 (6) as a union of minimal cyclotomic sets. Indeed, \( \Delta \) is the union of the following minimal cyclotomic sets:
\[
\begin{align*}
\Lambda(0,0) & = \{0,0\}, & \Lambda(1,0) & = \{1,0\}, \{4,0\}, \{2,0\}, \{3,0\}, \\
\Lambda(5,0) & = \{5,0\}, & \Lambda(0,1) & = \{0,1\}, \{0,4\}, \{1,1\}, \{1,4\}, \\
\Lambda(1,2) & = \{2,1\}, \{3,4\}, & \Lambda(3,1) & = \{3,1\}, \{2,4\}, \{4,1\}, \{4,4\}.
\end{align*}
\]

The next Corollaries 4.7 and 4.8 give parameters and \((r,\delta)\)-localities of the optimal \((r,\delta)\)-LRCs we have obtained in Corollaries 4.4 and 4.5.

Corollary 4.7. Let \( F_q \) be a finite field with \( q = p^h \), \( p \) being a prime number. Consider positive integers \( h \) and \( s \) such that \( h \mid |, 1 < s < \frac{1}{h} \) and \( p^{h+1} - 1 \mid q - 1 \) (and, if \( s = 2 \), also \( p^h \geq 3 \)). Consider also positive integers \( n_1, n_2, j \) and \( z \) satisfying some of the following conditions:

1. \( n_1 = p^{h-1}, n_2 - 1 \mid q - 1, j = n_2 - 1 \) and \( z < \frac{p^h}{2} \).
2. \( n_1 = p^{h+1}, n_2 + 1 \mid q - 1, j = n_2 - 1 \) and \( z \leq \frac{p^h}{2} \).
3. \( n_1 = p^{h+1}, n_2 \mid p^h - 1, z \leq \frac{p^h}{2} \) and \( \max \left\{ 1, n_2 + 1 - \frac{1}{z} \left(p^h - \frac{1}{p^{h+1}}\right) \right\} \leq j \leq n_2 - 1. \)
4. \( n_1 = p^{h+1}, n_2 \mid p^h - 1, z \leq \frac{p^h}{2} \) and \( \max \left\{ 1, n_2 + 1 - \frac{1}{z} \left(p^h - \frac{1}{p^{h+1}}\right) \right\} \leq j \leq n_2 - 1. \)
5. \( n_1 = p^{h+1}, n_2 \mid p^h, z \leq \frac{p^h}{2} \) and \( \max \left\{ 1, n_2 + 1 - \frac{p^h}{z} \right\} \leq j \leq n_2 - 1. \)
6. \( n_1 = p^{h+1}, n_2 \mid p^h - 1, z \leq \frac{p^h}{2} \) and \( \max \left\{ 1, n_2 + 1 - \frac{p^h}{z} \right\} \leq j \leq n_2 - 1. \)
Regard $\mathbb{F}_{p^h}$ as a subfield of $\mathbb{F}_q$. Then, there exists an optimal $(r, \delta)$-LRC over $\mathbb{F}_{p^h}$ with parameters $n = n_1 n_2$, $k = (zs + 1)j + 1$, $d = n_1 (n_2 - j)$ and $(r, \delta) = (zs + 1, n_1 - zs)$.

**Corollary 4.8.** Let $\mathbb{F}_q$ be a finite field with $q = p^l$, $p$ being a prime number. Consider positive integers $h$ and $s$ such that $h \mid l$, $1 < s \leq \frac{1}{2}$ and $p^{hs} - 1 \mid q - 1$ (and, if $s = 2$, also $p^h \geq 3$). Consider also positive integers $n_1$, $n_2$, $j$ and $w$ and a nonnegative integer $z$ satisfying some of the following conditions:

1. $n_1 = p^{hs} - 1$, $n_2 - 1 \mid q - 1$, $j = n_2 - 1$ and $w \leq \left\lfloor \frac{p^h - 1}{2} \right\rfloor$ (and, if $z \neq 0$, also $z < \frac{p^h}{2}$).
2. $n_1 = p^{hs}$, $n_2 - 1 \mid q - 1$, $j = n_2 - 1$ and $w \leq \left\lfloor \frac{(p^h - 1)p^{hs}}{2(p^{hs} - 1)} \right\rfloor$ (and, if $z \neq 0$, also $z \leq \frac{p^h}{2}$).
3. $n_1 = p^{hs} - 1$, $n_2 \mid p^h - 1$, $w \leq \left\lfloor \frac{p^h - 1}{2} \right\rfloor$ and $\max \{1, n_2 + 1 - \frac{1}{2} \left( p^h - \frac{1}{p^{hs} - 1} \right) \} \leq j \leq n_2 - 1$ (and, if $z \neq 0$, also $z < \frac{p^h}{2}$ and $n_i + 1 - \frac{1}{2} \left( p^h - \frac{1}{p^{hs} - 1} \right) \leq j$).
4. $n_1 = p^{hs} - 1$, $n_2 - 1 \mid p^h - 1$, $w \leq \left\lfloor \frac{p^h - 1}{2} \right\rfloor$ and $\max \{1, n_2 + 1 - \frac{1}{2} \left( p^h - \frac{1}{p^{hs} - 1} \right) \} \leq j \leq n_2 - 1$ (and, if $z \neq 0$, also $z < \frac{p^h}{2}$ and $n_i + 1 - \frac{1}{2} \left( p^h - \frac{1}{p^{hs} - 1} \right) \leq j$).

**Figure 17.** Sets $\Delta$ considered in Examples 4.6.
(5) \( n_1 = p^{h_1}, n_2 \mid p^h - 1, w \leq \left[ \frac{(p^h-1)p^h}{2(p^h-1)} \right] \text{ and } \max \left\{ 1, n_2 + 1 - \frac{p^h}{z} \right\} \leq j \leq n_2 - 1 \) (and, if \( z \neq 0, \text{ also } z \leq \frac{p^h}{2} \text{ and } n_1 + 1 - \frac{p^h}{z} \leq j \)).

(6) \( n_1 = p^{h_1}, n_2 - 1 \mid p^h - 1, w \leq \left[ \frac{(p^h-1)p^h}{2(p^h-1)} \right] \text{ and } \max \left\{ 1, n_2 + 1 - \frac{p^h}{z} \right\} \leq j \leq n_2 - 1 \) (and, if \( z \neq 0, \text{ also } z \leq \frac{p^h}{2} \text{ and } n_1 + 1 - \frac{p^h}{z} \leq j \)).

Regard \( \mathbb{F}_{p^h} \) as a subfield of \( \mathbb{F}_q \). Then, there exists an optimal \((r, \delta)\)-LRC over \( \mathbb{F}_{p^h} \) with parameters \( n = n_1 n_2, k = (zs + w + 1) j + 1, d = n_1 (n_2 - j) \) and \((r, \delta) = (zs + w + 1, n_1 - zs - w)\).

Table 1 provides parameters of some new optimal \((r, \delta)\)-LRCs coming from subfield-subcodes deduced from Corollaries 4.7 and 4.8.

| Item in Corollary | \( p^h \) | \( s \) | \( q \) | \( n \) | \( k \) | \( d \) | \( r \) | \( \delta \) |
|------------------|---------------|-------|-------|------|------|------|------|------|
| 4.7 (3) (for \((z, j) = (2, 1)\)) | 9 | 2 | 81 | 320 = 80·4 | 6 | 240 | 5 | 76 |
| 4.7 (5) (for \((z, j) = (3, 2)\)) | 9 | 2 | 6561 | 324 = 81·4 | 15 | 162 | 7 | 75 |
| 4.7 (6) (for \((z, j) = (1, 2)\)) | 4 | 2 | 256 | 64 = 16·4 | 7 | 32 | 3 | 14 |
| 4.8 (2) (for \((z, w) = (2, 2)\)) | 5 | 2 | 625 | 425 = 25·17 | 113 | 25 | 7 | 19 |
| 4.8 (2) (for \((z, w) = (2, 1)\)) | 4 | 3 | 64 | 640 = 64·10 | 73 | 64 | 8 | 57 |
| 4.8 (2) (for \((z, w) = (2, 1)\)) | 4 | 2 | 16 | 64 = 16·4 | 19 | 16 | 6 | 11 |

Table 1. Optimal subfield-subcodes over \( \mathbb{F}_{p^h} \)

4.3. Optimal \((r, \delta)\)-LRCs coming from subfield-subcodes of multivariate ZAVCs. The corresponding version of Theorem 4.2 for the multivariate case is the following.

**Theorem 4.9.** Keep the notation as in Section 2 and Subsection 4.1. Fix \( j_0 \in \{1, \ldots, m\} \) such that \( n_j - 1 \mid q - 1 \) for all \( j \in \{1, \ldots, m \} \setminus \{ j_0 \} \), \( n_{j_0} \mid q - 1 \) (or \( n_{j_0} - 1 \mid q - 1 \)) and \( j_0 > p^h \). Let \( 1 \leq z \leq n_{j_0} - 1 \) be a positive integer. Set \( P_j = R_{n_j - 1} \cup \{0\} \) for all \( j \in \{1, \ldots, m\} \setminus \{ j_0 \} \), \( P_{j_0} = R_{n_{j_0}} \) (or \( P_{j_0} = R_{n_{j_0} - 1} \cup \{0\} \)) and define

\[
\Omega_x : = \Lambda_0^x \cup \Lambda_1^x \cup \cdots \cup \Lambda_{z-1}^x,
\]

where \( \Lambda_x^0 \subseteq \{0, 1, \ldots, n_{j_0} - 1\} \) is the minimal cyclotomic set in a single variable containing \( x \) and \( x_1, \ldots, x_z \neq 0 \). Let \( c_{j_0} : = \# \Omega_x^0 \) and \( m_{j_0} : = \max \Omega_x^0 \). Define

\[
\Delta^0 : = \{0, 1, \ldots, n_{j_0} - 1 \} \times \cdots \times \Omega_x^0 \times \cdots \times \{0, 1, \ldots, n_{m} - 1\}.
\]

If \( c_{j_0} < n_{j_0} - 1 \) and \( m_{j_0} \leq \frac{n_{j_0}}{2} \), then the subfield-subcode \( S_{\mathcal{A}}^{P_{j_0}} \) over the field \( \mathbb{F}_{p^h} \) with \( \Delta = \Delta^0 \setminus \{(n_1 - 1, \ldots, n_{j_0} - 1, e_{j_0}, n_{j_0+1} - 1, \ldots, n_{m} - 1) \mid 1 \leq e_{j_0} \leq m_{j_0}\} \) and \( J = \{ j_0 \} \) (or \( J = \emptyset \)) is an optimal \((r, \delta)\)-LRC with \((r, \delta) = (c_{j_0}, n_{j_0} - c_{j_0} + 1)\).

**Proof.** Taking into account the last paragraph of Subsection 4.1, we must prove that the set \( \Delta \) is closed with respect to \( p^h \) and that \( c_{j_0}^{\Delta} \) is optimal.

By the proof of Theorem 4.2, \( \Lambda_0 = \{0\} \) and \( \Lambda_{j_0 - 1} = \{n_j - 1\} \) for all \( j \in \{1, \ldots, m\} \setminus \{ j_0 \} \) are minimal cyclotomic sets in a single variable. Thus for those \( p \in E \) such that \( p_j = n_j - 1 \) for all \( j \in \{1, \ldots, m\} \setminus \{ j_0 \} \) and \( p_{j_0} = 0 \), one has that \( \Lambda_p = \{p\} \) and, then, the set

\[
\Delta : = \{n_1 - 1 \} \times \cdots \times \Omega_x^0 \times \cdots \times \{n_m - 1\}
\]

is closed with respect to \( p^h \). For every \( j \in \{1, \ldots, m\} \), the set \( \{0, 1, \ldots, n_j - 1\} \) is closed and thus \( \Delta^0 \) is closed. The set \( \Delta \) satisfies \( \Delta = \Delta^0 \setminus \Delta' \cup \Lambda_p \) and therefore it is closed with respect to \( p^h \).

We impose the condition \( m_{j_0} \leq \frac{n_{j_0}}{2} \) because this implies that the distance of the point \( p \) is less than or equal to that of any point with \( e_{j_0} = m_{j_0}, e_{j'} = n'_{j'} - 2, j' \in \{1, \ldots, m\} \setminus \{ j_0 \} \) and...
let $e_j = n_j - 1$, $j \in \{1, \ldots, m\}\setminus \{j_0, j'\}$. Thus, the parameters of the ZAVC, $C^{R_J}_\Delta$, are $n = n_1 \cdots n_m$, $k = n_1 \cdots n_j - 1, n_{j+1} \cdots n_m - (c_{j_0} - 1)$, $d = n_{j_0}$ and $(r, \delta) = (c_{j_0}, n_{j_0} - c_{j_0} + 1)$, corresponding with item (2) in Corollary 3.15. Therefore, $C^{R_J}_\Delta$ is optimal, which concludes the proof.

Remark 4.10. As in the case of bivariate codes (see Remark 4.3), we impose conditions in Theorem 4.9 for providing new families of optimal $(r, \delta)$-LRCs.

We conclude this paper with four results about new optimal LRCs and their parameters. Proofs are similar to those given in the bivariate case and they are omitted.

Corollary 4.11. Keep the notation as in Section 2, Subsection 4.1 and Theorem 4.9. Let $1 < s \leq \frac{l}{h}$ such that $p^{hs} - 1 | q - 1$ (and, if $s = 2$, also $p^h \geq 3$). Set $P_j = R_{n_{j-1}} \cup \{0\}$ with $n_j - 1 | q - 1$ for all $j \in \{1, \ldots, m\}\setminus \{j_0\}$. Fix $(x_1, x_2, \ldots, x_z) = (1, 2, \ldots, z)$. Then, the following subfield-subcodes $S^{R_J}_{\Delta}$ over the field $\mathbb{F}_{p^h}$ are optimal $(r, \delta)$-LRCs:

1. $P = P_1 \times \cdots \times P_{m-1}$, where $P_{j_0} = R_{p^{hs}-1}$, $z \leq \frac{p^h}{z}$ and $J = \{j_0\}$, in which case $(r, \delta) = (zs + 1, p^{hs} - zs - 1)$.

2. $P = P_1 \times \cdots \times P_{m-1}$, where $P_{j_0} = R_{p^{hs}-1} \cup \{0\}$, $z \leq \frac{p^h}{z}$ and $J = \emptyset$, in which case $(r, \delta) = (zs + 1, p^{hs} - zs)$.

Corollary 4.12. Keep the notation as in Section 2, Subsection 4.1 and Theorem 4.9 but allowing $z = 0$. Let $1 < s \leq \frac{l}{h}$ such that $p^{hs} - 1 | q - 1$ (and, if $s = 2$, also $p^h \geq 3$). Let $1 \leq w \leq \left\lfloor \frac{(p^h - 1)p^h}{p^{hs} - 1} \right\rfloor$ be a positive integer and pick $a_1 < a_2 < \cdots < a_w$ positive integers which are multiples of $\frac{p^{hs} - 1}{p^h - 1}$. Set

$$\Omega^j = \Lambda_{1}^j \cup \Lambda_{2}^j \cup \cdots \cup \Lambda_{a_1}^j \cup \cdots \Lambda_{a_w}^j$$

and $P_j = R_{n_{j-1}} \cup \{0\}$ with $n_j - 1 | q - 1$ for all $j \in \{1, \ldots, m\}\setminus \{j_0\}$. Then, the following subfield-subcodes $S^{R_J}_{\Delta}$ over the field $\mathbb{F}_{p^h}$ are optimal $(r, \delta)$-LRCs:

1. $P = P_1 \times \cdots \times P_{m-1}$, where $P_{j_0} = R_{p^{hs}-1}$, $a_w \leq \frac{p^{hs} - 1}{z}$ (and, if $z \neq 0$, also $z \leq \frac{p^h}{z}$) and $J = \{j_0\}$, in which case $(r, \delta) = (zs + w + 1, p^{hs} - zs - w - 1)$.

2. $P = P_1 \times \cdots \times P_{m-1}$, where $P_{j_0} = R_{p^{hs}-1} \cup \{0\}$, $a_w \leq \frac{p^h}{z}$ (and, if $z \neq 0$, also $z \leq \frac{p^h}{z}$) and $J = \emptyset$, in which case $(r, \delta) = (zs + w + 1, p^{hs} - zs - w)$.

The following Corollary 4.13 (respectively, 4.14) determines the parameters and $(r, \delta)$-localities of the optimal $(r, \delta)$-LRCs we have obtained in Corollary 4.11 (respectively, 4.12).

Corollary 4.13. Let $\mathbb{F}_q$ be a finite field with $q = p^l$, $p$ being a prime number. Consider positive integers $h$ and $s$ such that $h | l$, $1 < s \leq \frac{l}{h}$ and $p^{hs} - 1 | q - 1$ (and, if $s = 2$, also $p^h \geq 3$). Consider also positive integers $n_1, \ldots, n_m$, $j$ and $z$ satisfying $n_j - 1 | q - 1$, $j \in \{2, \ldots, m\}$, and some of the following conditions:

1. $n_1 = p^{hs} - 1$ and $z < \frac{p^h}{z}$.

2. $n_1 = p^{hs}$ and $z \leq \frac{p^h}{z}$.

Regard $\mathbb{F}_{p^h}$ as a subfield of $\mathbb{F}_q$. Then, there exists an optimal $(r, \delta)$-LRC over $\mathbb{F}_{p^h}$ with parameters $n = n_1 \cdots n_m$, $k = (zs + 1)n_2 \cdots n_m - zs$, $d = n_1$ and $(r, \delta) = (zs + 1, n_1 - zs)$.

Corollary 4.14. Let $\mathbb{F}_q$ be a finite field with $q = p^l$, $p$ being a prime number. Consider positive integers $h$ and $s$ such that $h | l$, $1 < s \leq \frac{l}{h}$ and $p^{hs} - 1 | q - 1$ (and, if $s = 2$, also $p^h \geq 3$). Consider also positive integers $n_1, \ldots, n_m$, $j$ and $w$ and a nonnegative integer $z$ satisfying $n_j - 1 | q - 1$, $j \in \{2, \ldots, m\}$, and some of the following conditions:
\[(1) \ n_1 = p^h - 1 \text{ and } w \leq \left\lceil \frac{p^h - 1}{2} \right\rceil \ (\text{and, if } z \neq 0, \text{ also } z < \frac{p^h}{2}). \]
\[(2) \ n_1 = p^h \text{ and } w \leq \left\lceil \frac{(p^h - 1)p^h}{2(p^m - 1)} \right\rceil \ (\text{and, if } z \neq 0, \text{ also } z \leq \frac{p^h}{2}). \]

Regard $\mathbb{F}_{p^h}$ as a subfield of $\mathbb{F}_q$. Then, there exists an optimal $(r, \delta)$-LRC over $\mathbb{F}_{p^h}$ with parameters $n = n_1 \cdots n_m$, $k = (zs + w + 1)n_2 \cdots n_m - (zs + w)$, $d = n_1$ and $(r, \delta) = (zs + w + 1, n_1 - zs - w)$.

We finish by giving in Table 2 parameters of new optimal $(r, \delta)$-LRCs coming from subfield-subcodes deduced from Corollaries 4.13 and 4.14.

| Item in Corollary | $p^h$ | $s$ | $q$ | $n$ | $k$ | $d$ | $r$ | $\delta$ |
|------------------|--------|-----|-----|-----|-----|-----|-----|--------|
| 4.13 (1) (for $(m, z) = (3, 1)$) | 4 | 2 | 16 | 360 = 15 \cdot 24 | 72 | 15 | 3 | 13 |
| 4.13 (2) (for $(m, z) = (3, 1)$) | 2 | 3 | 64 | 128 = 8 \cdot 4 \cdot 4 | 64 | 8 | 4 | 5 |
| 4.13 (2) (for $(m, z) = (3, 1)$) | 2 | 4 | 256 | 384 = 16 \cdot 4 \cdot 6 | 116 | 16 | 5 | 12 |
| 4.14 (1) (for $(m, z, w) = (3, 3, 3)$) | 7 | 2 | 49 | 960 = 48 \cdot 4 \cdot 5 | 191 | 48 | 10 | 39 |
| 4.14 (2) (for $(m, z, w) = (4, 2, 2)$) | 5 | 2 | 625 | 648 = 24 \cdot 3 \cdot 3 \cdot 3 | 183 | 24 | 7 | 18 |
| 4.14 (2) (for $(m, z, w) = (3, 1, 1)$) | 3 | 2 | 81 | 135 = 9 \cdot 5 \cdot 3 | 57 | 9 | 4 | 6 |

Table 2. Optimal $(r, \delta)$-subfield-subcodes over $\mathbb{F}_{p^h}$

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Email address, Carlos Galindo: galindo@uji.es
OPTIMAL $(r,\delta)$-LRCS FROM ZERO-DIMENSIONAL AFFINE VARIETY CODES AND THEIR SUBFIELD-SUBCODES

INSTITUTO UNIVERSITARIO DE MATEMÁTICAS Y APLICACIONES DE CASTELLÓN AND DEPARTAMENTO DE
MATEMÁTICAS, UNIVERSITAT JAUME I, CAMPUS DE RIU SEC., 12071 CASTELLÓ, SPAIN

Email address, Fernando Hernando: carrillof@uji.es

INSTITUTO UNIVERSITARIO DE MATEMÁTICAS Y APLICACIONES DE CASTELLÓN AND DEPARTAMENTO DE
MATEMÁTICAS, UNIVERSITAT JAUME I, CAMPUS DE RIU SEC., 12071 CASTELLÓ, SPAIN

Email address, Helena Martín-Cruz: martinh@uji.es

INSTITUTO UNIVERSITARIO DE MATEMÁTICAS Y APLICACIONES DE CASTELLÓN, UNIVERSITAT JAUME I, CAM-
PUS DE RIU SEC., 12071 CASTELLÓ, SPAIN