Statistical Guarantees for Approximate Stationary Points of Simple Neural Networks

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Abstract

Since statistical guarantees for neural networks are usually restricted to global optima of intricate objective functions, it is not clear whether these theories really explain the performances of actual outputs of neural-network pipelines. The goal of this paper is, therefore, to bring statistical theory closer to practice. We develop statistical guarantees for simple neural networks that coincide up to logarithmic factors with the global optima but apply to stationary points and the points nearby. These results support the common notion that neural networks do not necessarily need to be optimized globally from a mathematical perspective. More generally, despite being limited to simple neural networks for now, our theories make a step forward in describing the practical properties of neural networks in mathematical terms.

1. Introduction

Statistical theories for deep learning usually apply to exact, global optima of certain objective functions (Bartlett 1998, Bartlett and Mendelson, 2002, Bauer and Kohler, 2019, Kohler and Langer, 2021, Lederer, 2020a, Taheri et al., 2021, Schmidt-Hieber, 2020). But those objective functions cannot be solved explicitly and are highly non-convex, so that in practice, exact, global optimization is—at least to date—out of reach, and we can expect only approximate stationary points from current algorithms (see Figure 1). In other words, it is unclear whether the known theories have any meaning for the outputs of actual deep-learning pipelines.

Also other parts of machine learning face optimization problems that are challenging to optimize globally and to full precision. Accordingly, some statistical insights have already been established. For example, Bien et al. (2018, 2019) solve a non-convex problem in linear regression in a “convex” way and develop statistical theories for their solution. Loh and Wainwright (2015); Loh (2017) develop statistical theory for stationary points in another regression setup under curvature assumptions. Elsener and van de Geer (2018) establish more general theories for stationary points also under curvature assumptions. Taheri et al. (2019) propose an algorithm and statistical theory for approximate solutions in a convex setting. But it is currently unclear how to extend these insights to deep learning—if at all possible.

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In this paper, we develop statistical guarantees for the stationary points of simple neural networks (Theorem 1) and for the points in the vicinity of a stationary point (Theorem 2). Strikingly, the statistical rates match the “theoretical” rates in Taheri et al. (2021) up to log-terms. Hence, although our theories here are limited to simple networks, our paper makes a significant step toward the mathematical understanding of practical neural-network pipelines.

**Motivation**  Let us briefly motivate our study in a very simple example. Consider regression data consisting of the two input-output pairs \((x_1, y_1) = (-1, 0)\) and \((x_2, y_2) = (1, 1)\). On the one hand, we can relate the inputs and outputs through a simple linear-regression model \(y_i = \omega^* x_i + u_i\) parameterized by \(\omega^* \in \mathbb{R}\). The least-squares estimate for \(\omega^*\) minimizes the objective function \(f_1: \omega \mapsto \omega^2 + (1 - \omega)^2\). The function \(f_1\) is benign; in particular, it is smooth and convex (see the left panel of Figure 2). On the other hand, we can also relate the inputs and outputs through a very simple linear neural-network model \(y_i = \gamma^* \theta^* x_i + u_i\) parametrized by \(\gamma^*, \theta^* \in \mathbb{R}\). The least-squares estimate for \(\gamma^*, \theta^*\) minimizes the objective function \(f_2 : (\gamma, \theta) \mapsto (\gamma \theta)^2 + (1 - \gamma \theta)^2\). The earlier estimate and the one here are equivalent from a statistical perspective: we can simply identify \(\omega\) with the product \(\gamma \theta\). But the reparametrization changes the computational properties: the function \(f_2\) is not benign anymore; in particular, it is not convex (see the right panel of Figure 2).

This simple example illustrates that going from linear regression and other standard models to neural networks naturally leads to computational challenges; in particular, the corresponding objective functions are usually non-convex, which means that optimization algorithms may easily get stuck in non-optimal stationary points. Thus, we argue that statistical guarantees in deep learning should concern stationary points rather than just global optima.

**Paper outline**  In Section 2, we state the statistical guarantees for the stationary points of the neural-network objectives (Theorem 1) and the statistical guarantees for approximations of a stationary point (Theorem 2). We provide some numerical observations in Section 3. In Section 4, we study the behavior of the Hessian matrix of the population risk (Proposition 3) and then employ tools from empirical-process theory to bound the absolute difference between the gradient vector of the prediction risk \(\nabla \text{risk}_X[\gamma, \Theta]\) and population risk \(\nabla \text{risk}[\gamma, \Theta]\) (Lemma 4); these are the main ingredients for proving our Theorem 1. In Section 5, we provide an overview of the related works. More technical results and detailed proofs are deferred to the Section 6. We conclude our paper in Section 7.

**Notations**  We use \(\text{vec}(\gamma, \Theta)\) to generate a vector of length \(\mathbb{R}^{w+w \cdot d}\) from a vector \(\gamma \in \mathbb{R}^w\) and a matrix \(\Theta \in \mathbb{R}^{w \times d}\) (for generating the vector, we first push the elements of \(\gamma\) and then elements of \(\Theta\) row by row). We collect first order partial derivatives of prediction risk \(\text{risk}_X[\gamma, \Theta]\) and population risk \(\text{risk}[\gamma, \Theta]\) with respect to the \(\beta := \text{vec}(\gamma, \Theta)\) in the gradient vectors \(\nabla \text{risk}_X[\gamma, \Theta] \in \mathbb{R}^{w+w \cdot d}\) and \(\nabla \text{risk}[\gamma, \Theta] \in \mathbb{R}^{w+w \cdot d}\), respectively. We use the notation \(\|\cdot\|\) for a general vector norm and \(\|\cdot\|\) for a general matrix norm. We also define \(\|\gamma\|_1 := \sum_{j=1}^w |\gamma_j|\) and \(\|\Theta\|_1 := \sum_{j=1}^w \sum_{k=1}^d |\theta_{jk}|\).
2. Main results

Consider inputs $x_1, \ldots, x_n \in \mathbb{R}^d$ and corresponding outputs $y_1, \ldots, y_n \in \mathbb{R}$ that are connected via

$$y_i = f[x_i] + u_i$$

for an unknown target function $f : \mathbb{R}^d \to \mathbb{R}$ and unknown stochastic noise $u_1, \ldots, u_n \in \mathbb{R}$. Deep learning is about using the available data to approximate the unknown target function $f$ by a neural network. In this paper, we consider shallow and linear neural networks of the form

$$x \mapsto \gamma^\top \Theta x,$$

where

$$(\gamma, \Theta) \in \mathcal{B} := \{ (\gamma, \Theta) \in \mathbb{R}^w \times \mathbb{R}^{w \times d} \}.$$  

We assume that the target function can be approximated by such a neural network in the first place: there is a pair $(\gamma^*, \Theta^*) \in \mathcal{B}$ such that $f[x] \approx \gamma^*^\top \Theta^* x$. In fact, to have a sharp focus on statistical guarantees rather than the approximation properties of neural networks, we simply assume that $|\gamma^*|_1, \|\Theta^*\|_1 \leq \sqrt{\log n}$ and $f[x] = \gamma^*^\top \Theta^* x$ for all $x \in \mathbb{R}^d$, that is, the target function is itself a neural network with reasonably small parameters \footnote{The bound \(\sqrt{\log n}\) here and on the next page is merely for convenience: it can be replaced by any fixed constant or another function that is increasing mildly in the sample size \(n\).}. Note that parametrization of neural networks is ambiguous: there are typically infinitely many pairs $(\gamma^*, \Theta^*) \in \mathcal{B}$ that satisfy those conditions—compare to Taheri et al. (2021, Proposition 1); for further reference, we define $\beta^* := \text{vec}(\gamma^*, \Theta^*)$ for a fixed but arbitrary such pair of parameters. Note that the concentration of this study is on linear neural networks. Although linear neural networks are not used in practice, theoretical understanding of them is important because, they may reveal some properties of non-linear models (see our Section 3), while theoretical study of them is more simple.

The network parameters in deep learning are usually estimated via standard statistical techniques, such as least-squares or logistic loss, complemented by some type of regularization, such as dropout (Hinton et al. 2012; Molchanov et al., 2017; Neklyudov et al., 2017; Salehinejad and Valaee, 2019), batch normalization (Ioffe and Szegedy, 2015), explicit regularization (Alvarez and Salzmann, 2016; Feng and Simon, 2017; Liu et al., 2015; Scardapane et al., 2017), or low-rank approximation (Denil et al., 2013; Denton et al., 2014; Jaderberg et al., 2014). To fix ideas, we focus on least-squares...
complemented by (elementwise) $\ell_1$-regularization:

$$
(\widehat{\gamma}, \widehat{\Theta}) \in \arg\min_{(\gamma, \Theta) \in \mathcal{B}} \left\{ \frac{1}{n} \sum_{i=1}^{n} (y_i - \gamma^\top \Theta x_i)^2 + r \|\gamma, \Theta\|_1 \right\},
$$

where $r \in [0, \infty)$ is a tuning parameter to be calibrated (see Sardy et al. (2020) for some theory insights). Such estimators are standard in machine learning and statistics (Lederer, 2022; Eldar and Kutyniok, 2012). As usual, we measure the (in-sample-)prediction risk by

$$
\risk_X[\gamma, \Theta] := \frac{1}{n} \sum_{i=1}^{n} (y_i - \gamma^\top \Theta x_i)^2
$$

with the expectation over a new sample $(x, y)$ (that has the same distribution as $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$). We call $\beta := \text{vec}(\widehat{\gamma}, \widehat{\Theta})$ a stationary point of the objective in (2) if it satisfies (Elsener and van de Geer, 2018; Equation 6); (Loh and Wainwright, 2015; Equation 5)

$$
(\nabla \risk_X(\widehat{\gamma}, \widehat{\Theta}))^\top (\beta - \widehat{\beta}) + r \mathbf{z}^\top (\beta - \widehat{\beta}) \geq 0 \quad \forall \beta = \text{vec}(\gamma, \Theta) \text{ with } (\gamma, \Theta) \in \mathcal{B}
$$

for appropriate $\mathbf{z} \in \partial \|\beta\|_1$ (where $\partial \|\beta\|_1$ is the subdifferential of the regularizer at $\widehat{\beta}$). We call a stationary point $\beta$ reasonable once $\|\widehat{\beta}\|_1, \|\Theta\|_1 \leq \sqrt{\log n}$—again to avoid unnecessary complication. Due to the ambiguity of neural networks, there are infinitely many equivalent stationary and reasonable stationary points; importantly, our guarantees hold for every (reasonable) stationary point and target $\beta^\star$.

We say that a network indexed by $(\widehat{\gamma}, \widehat{\Theta})$ generalizes well if

$$
\risk[\widehat{\gamma}, \widehat{\Theta}] \approx \risk[\gamma^\star, \Theta^\star],
$$

that is, the network generalizes essentially as well as the best network. In the following, we show that not only the “statistical” network indexed by $(\widehat{\gamma}, \widehat{\Theta})$ but also every “practical” network indexed by a reasonable stationary point $(\widehat{\gamma}, \widehat{\Theta})$ of the objective function in (2) generalizes well. For simplicity, we assume that $x_1, \ldots, x_n \in \mathcal{N}(0, I_{d \times d})$ are i.i.d. standard Gaussian random vectors and that $u_1, \ldots, u_n \in \mathcal{N}(0, \sigma^2)$ are centered and Gaussian random variables and independent of $x_i$. Moreover, we call the total number of parameters in the network $p := w + w \cdot d$ the problem’s effective dimension and

$$
r_{\text{orc}} := \nu (\log n)^{3/2} \sqrt{\frac{\log(np)}{n}}
$$

the oracle tuning parameter, where $\nu \in (0, \infty)$ is a constant that depends only on the distributions of the inputs and noises. It has been shown that $r_{\text{orc}}$ is indeed an optimal tuning parameter of (2) in some sense (Taheri et al. 2021).

We then get the following result for a new sample pair $(x, y)$ with the same distribution as $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$.

**Theorem 1 (Statistical Guarantees for All Reasonable Stationary Points)** Any reasonable stationary point $(\widehat{\gamma}, \widehat{\Theta})$ of the objective function in (2) with $r \geq r_{\text{orc}}$ satisfies the risk bound

$$
\risk[\widehat{\gamma}, \widehat{\Theta}] \leq \risk[\gamma^\star, \Theta^\star] + 5r \sqrt{\log n}
$$

with probability at least $1 - 1/2n$. If $r = r_{\text{orc}}$, the bound becomes

$$
\risk[\widehat{\gamma}, \widehat{\Theta}] \leq \risk[\gamma^\star, \Theta^\star] + \nu (\log n)^2 \sqrt{\frac{\log(np)}{n}}.
$$
The proof of Theorem 1 is given in Section 6.2.

Theorem 1 proves the fact that for properly chosen tuning parameter \( r \) and large enough sample sizes, any reasonable stationary point of (2) generalizes essentially as well as \( \beta^* \). Our results essentially have the same rates as the ones in the literature (Taheri et al. 2021, Theorem 3) but, in stark contrast to those results, apply to all reasonable stationary points (including saddle points) rather than to the global optimum of the objective function only.

The assumption of Gaussian noise can be relaxed readily by replacing the Gaussian-type results in the empirical-process derivations by more general results (Adamczak 2008; Lederer and van de Geer, 2014; van de Geer and Lederer, 2013; Wellner, 2017).

Note that in finite time, stationary points can be computed just approximately using gradient-based algorithms. Now we extend our results in Theorem 1 to the points that are close but not necessarily equal to a stationary points. We define a pair \((\tilde{\gamma}, \tilde{\Theta})\) as a \( \tau \)–approximate stationary point if it satisfies

\[
|risk_X[\tilde{\gamma}, \tilde{\Theta}] + r|\tilde{\beta}|1 - risk_X[\gamma^*, \Theta^*] - r|\beta^*|1| \leq \tau
\]  

for \( \tau \in [0, \infty) \). Employing gradient based algorithms (in finite time), we can expect to get close to a stationary point in the sense that \( \tilde{\beta} \approx \beta \) (Lei et al., 2019). Then also \( |\tilde{\beta}|1 \approx |\beta|1 \), which means that an approximation of a reasonable stationary point is also reasonable once \( \tau \) is small enough. Then, we extract statistical guarantees for every practical network indexed by an approximate-reasonable stationary as following:

**Theorem 2 (Statistical Guarantees for Approximate Stationary Points)** Suppose that \((\tilde{\gamma}, \tilde{\Theta})\) is a \( \tau \)–approximate stationary point and that the conditions of Theorem 1 are satisfied. Then, we have

\[
risk[\tilde{\gamma}, \tilde{\Theta}] \leq risk[\gamma^*, \Theta^*] + 8r\sqrt{\log n} + \tau
\]

with probability at least \( 1 - 1/n \). If \( r = r_{oc} \), the bound becomes

\[
risk[\tilde{\gamma}, \tilde{\Theta}] \leq risk[\gamma^*, \Theta^*] + \nu(\log n)^2 \sqrt{\frac{\log(np)}{n}} + \tau.
\]

The proof of Theorem 2 is given in Section 6.2.

The bounds match the earlier ones with only two small differences: 1. a summand \( \tau \) is added to our statistical bounds and 2. the factor 5 in Equation (5) is replaced by a factor of 8 in Equation (8). Theorem 2 might look like a trivial extension of Theorem 1, but seem to be a trivial observation, but the fact that Equation (7) involves the (in-sample-)prediction risk and the sparsity factors make the proof considerably more involved.

**3. Numerical observations**

We provide here some numerical observations to clarify theories of Section 2. We minimize a least-squares complemented by \( \ell_1 \)-regularization for shallow neural networks with 1. linear and 2. ReLU activation functions. We consider neural networks with \( d = w = 10 \), that are trained over 500 and tested over 300 data sample generated from a standard normal distribution and labeled by a sparse-target network (having the same structure as above) plus a Gaussian noise. Note that here, we train the networks in a finite time, that means, trained networks are just an approximation of a stationary point (due to the non-convexity). We report the relative training error and the relative test error for a potential global optima and an approximate stationary point for linear and ReLU networks in Table 1. Let note that the potential global optima and approximate stationary point are reached over multiple times of training on a fixed data set and assigned by the trained networks with the lowest and highest training error, respectively (we reference to Figure 3 in Section 6.2 for a graphical view of convergence in training). Due to the non-convexity of the objective function,
Table 1: relative training error and test error for trained neural networks (with \(d = w = 10\)) with linear and ReLU activations in a potential global optima and an approximate stationary point.

|                | Linear | ReLU  |
|----------------|--------|-------|
| Training Error | 1      | 1.15  |
| Test Error     | 1      | 1.0001|
| Approximate Stationary Point | 1.11 | 1.09 |

Global optimization is far reaching and so we call the trained network with the lowest training error just as a potential global optima. Results reveal that the test error for a potential global optima and an approximate stationary point are very close in both linear and ReLU networks.

These observations reveal that: First, global optimization for neural networks is far reaching even for very simple neural networks. Second, very practical outputs in deep learning (approximate stationary points) can still generalize well—for linear networks and beyond.

### 4. Technical results

In this section, we provide technical results that are essential for proving our Theorem 1 but might also be of interest by themselves.

**Additional notation** For vectors \(\beta = \text{vec}(\gamma, \Theta) \in \mathbb{R}^p\) and \(\alpha := (\alpha_1, \ldots, \alpha_w) \in \mathbb{R}^w\) with \(\alpha_j \neq 0\) for all \(j \in \{1, \ldots, w\}\), we define \(\beta_\alpha := \text{vec}(\gamma_\alpha, \Theta_\alpha) \in \mathbb{R}^p\) as a rescaled version of \(\beta\) with \((\gamma_\alpha)_j := \gamma_j \cdot \alpha_j\) and \((\Theta_\alpha)_{jk} := \theta_{jk} / \alpha_j\) for all \(j \in \{1, \ldots, w\}\) and \(k \in \{1, \ldots, d\}\). We tabulate the second order partial derivatives of \(\text{risk}[\gamma, \Theta] = \text{vec}(\gamma, \Theta)\) in a matrix called \(\nabla^2 \text{risk}[\gamma, \Theta] \in \mathbb{R}^{p \times p}\). We use \(e_{\text{min}}[\cdot]\) to generate the smallest eigenvalue of a matrix. We use the notation \(0\) to generate a vector of zeros.

Now, we study the behavior of the Hessian matrix in a rescaled network as following:

**Proposition 3 (Hessian Behavior)** Suppose that \((\gamma, \Theta) \in \mathcal{B}\) with \(\Theta \Theta^\top\) invertible. Let \(a := [(a^1)\top, (a^2)\top] \in \mathbb{R}^p\) be a vector with \(\|a\|_2 = 1\), \(a^1 \in \mathbb{R}^{w-d}\), and \(a^2 \in \mathbb{R}^w\). If \(a^1 = 0\) or \(a^2 = 0\), we have for all \(\alpha \in \mathbb{R}^w \setminus \{0\}\)

\[
a^\top \nabla^2 \text{risk}[\gamma_\alpha, \Theta_\alpha] a \geq 0;
\]

Otherwise, above inequality holds for all \(\alpha := (1/c, \ldots, 1/c) \in \mathbb{R}^w\) with \(c \in [1, \infty)\) such that

\[
c^2 \geq 2\|\gamma\|_2^2 \|a^1\|_2^2 + 4\|a^1\|_2 \|a^2\|_2 \|\gamma^\top \Theta - \gamma^* \Theta^*\|_2 \sqrt{\min\{\Theta \Theta^\top\}} \|a^2\|_2^2.
\]

The proof of Proposition 3 is given in Section 6.2.

Note that if \(a^1 = 0\) or \(a^2 = 0\), the quadratic product on the Hessian matrix (in a rescaled network with parameters \((\gamma_\alpha, \Theta_\alpha)\)) is non-negative for all \(\alpha\), otherwise, it is non-negative just for \(\alpha\) with large enough \(c\).

**Lemma 4 (Empirical Processes)** It holds for each reasonable stationary point \(\tilde{\beta} = \text{vec}(\tilde{\gamma}, \tilde{\Theta})\) of the objection function in (2) that

\[
\left| (\nabla \text{risk}_X[\tilde{\gamma}, \tilde{\Theta}] - \nabla \text{risk}[\gamma, \Theta])^\top (\beta^* - \tilde{\beta}) \right| \leq r_{\text{orc}} \|\beta^* - \tilde{\beta}\|_1 + \frac{r_{\text{orc}}}{2n} \frac{1}{n}
\]

with probability at least \(1 - 1/2n\).
The proof of Lemma 4 is given in Section 6.2.

The result above establishes a bound for the absolute difference between $\nabla \text{risk}_X[\tilde{\gamma}, \tilde{\Theta}]$ and $\nabla \text{risk}[\tilde{\gamma}, \tilde{\Theta}]$ for every reasonable stationary point $(\tilde{\gamma}, \tilde{\Theta}) \in B$. We employ the result above for choosing the optimal tuning parameter. Proposition 3 and Lemma 4 are the main ingredients for proving our Theorem 1.

5. Related literature

Statistical theories for neural networks have widely been studied (Bartlett, 1998; Bartlett and Mendelson, 2002; Bauer and Kohler, 2019; Beknazaryan, 2021; Kohler and Langer, 2021; Kohler et al., 2021; Lederer, 2020a; Ma et al., 2022; Taheri et al., 2021; Schmidt-Hieber, 2020), but due to the non-convexity of neural-network objectives, it is unclear whether these theories can be applied for the outputs of actual deep-learning pipelines.

Non-convex objectives are also of interest in other parts of machine learning: Bien et al. (2018, 2019) solve a non-convex problem in linear regression in a “convex” way and they develop statistical theories for their solution. Loh and Wainwright (2015, Theorems 1,2) extract statistical guarantees for stationary points of non-convex objectives (allowing for non-convexity in both loss and penalty functions) in a regression-type settings, under a so-called “restricted-strong convexity” condition over the empirical loss (see their Display (4)). Loh (2017, Theorem 1) studies the behavior of stationary points of penalized robust estimators in a linear-regression setting. They prove that under a local “restricted-strong convexity” condition, stationary points within the region of restricted curvature are statistically consistent with the target. Also Elsener and van de Geer (2018, Theorem 1) derive sharp oracle inequalities for stationary points of general non-convex objectives made by a non-convex loss plus a convex penalty, under a restrictive condition called “two point marginal condition” on the theoretical loss. They exemplify their bounds for simple models like robust regression and binary classification. Their condition is kinda similar to the restricted-strong convexity but on the theoretical loss (and not on the empirical loss). Unfortunately, the curvature assumptions in these papers are infeasible for neural network settings, which means that their approaches cannot be applied here.

Another interesting direction is studying optimization landscape of non-convex objectives in deep learning (Eftekhari, 2020; Hardt and Ma, 2016; Kawaguchi, 2016; Lederer, 2020c; Mei et al., 2018; Zhou and Liang, 2018; Zhang et al., 2016; Yun et al., 2017). Yun et al. (2017) study the optimization landscape of deep and linear neural networks. They extract necessary and sufficient conditions for a critical point to be the global optima of the least-squares loss under some assumptions (input dimensions upper bounded by the number of data examples, $XX^\top$ and $YY^\top$ have full rank). They prove that for wide enough networks, a critical point of the empirical risk is a global minimum, if and only if the product of all the parameter matrices is of full rank (Yun et al., 2017, Theorem 1). Their result actually provide an efficiently checkable test for global optimality and also is extended for non-linear networks. Kawaguchi (2016, Theorem 2.3) proves that for deep and linear neural networks and under some assumptions ($XX^\top$ and $XY^\top$ have full rank), every local minimum is a global minimum and every critical point that is not a global minimum is a saddle point. They also prove that the same results hold for nonlinear-neural networks but under unrealistic assumptions (Kawaguchi 2016, Corollary 3.2). Zhou and Liang (2018, Theorem 2) also prove that linear neural networks with least-squares loss have no spurious local minimum. In any case, while interesting, these works do not provide statistical insights into the problem.

6. More technical results and proofs

In this section, we establish five auxiliary results and then the proofs of all our claims.
6.1 More technical results

We first differentiate the empirical risk \( \text{risk}_X[\gamma, \Theta] \) with respect to the parameters \( \beta = \text{vec}(\gamma, \Theta) \). We use the indices \( j, k \) for the first-order partial derivatives and indices \( j', k' \) for the second-order partial derivatives.

**Lemma 5 (First- and Second-Order Partial Derivatives of the Empirical Risk)** It holds for each \( j, j' \in \{1, \ldots, w\} \) and \( k, k' \in \{1, \ldots, d\} \) that

\[
\frac{\partial}{\partial \gamma_j} \text{risk}_X[\gamma, \Theta] = -\frac{2}{n} \sum_{i=1}^{n} \left( (y_i - \gamma^\top \Theta x_i)(\Theta x_i)_j \right) \quad \text{and} \quad \frac{\partial}{\partial \theta_{jk}} \text{risk}_X[\gamma, \Theta] = -\frac{2}{n} \sum_{i=1}^{n} \left( (y_i - \gamma^\top \Theta x_i)\gamma_j(x_i)_k \right);
\]

and

\[
\frac{\partial^2}{\partial \gamma_j \partial \gamma_{j'}} \text{risk}_X[\gamma, \Theta] = \frac{2}{n} \sum_{i=1}^{n} \left( (\Theta x_i)_j(\Theta x_i)_{j'} - (y_i - \gamma^\top \Theta x_i)(x_i)_{k'} \right)
\]

and

\[
\frac{\partial^2}{\partial \gamma_j \partial \theta_{jk}} \text{risk}_X[\gamma, \Theta] = \frac{2}{n} \sum_{i=1}^{n} \left( (\Theta x_i)_j(\Theta x_i)_k - (y_i - \gamma^\top \Theta x_i)(x_i)_k \right),
\]

Moreover, if \( j' = j \), it holds that

\[
\frac{\partial^2}{\partial \theta_{jk} \partial \gamma_j} \text{risk}_X[\gamma, \Theta] = \frac{2}{n} \sum_{i=1}^{n} \left( (\Theta x_i)_j(\Theta x_i)_k \right) \quad \text{and} \quad \frac{\partial^2}{\partial \theta_{jk} \partial \gamma_{j'}} \text{risk}_X[\gamma, \Theta] = \frac{2}{n} \sum_{i=1}^{n} \left( (\Theta x_i)_{j'}(\Theta x_i)_k \right).
\]

These derivatives are basic tools for us given that we work with stationary points. The proof of Lemma 5 is given in Section 6.2.

The next result is essentially a population version of the partial derivatives in Lemma 5, that is, sums are replaced by expectations.

**Lemma 6 (First- and Second-Order Partial Derivatives of the Population Risk)** It holds for each \( j, j' \in \{1, \ldots, w\} \) and \( k, k' \in \{1, \ldots, d\} \) that

\[
\frac{\partial}{\partial \gamma_j} \text{risk}[\gamma, \Theta] = -2E_{(x,y)}[(y - \gamma^\top \Theta x)(\Theta x)_j] \quad \text{and} \quad \frac{\partial}{\partial \theta_{jk}} \text{risk}[\gamma, \Theta] = -2E_{(x,y)}[(y - \gamma^\top \Theta x)\gamma_j(x)_k];
\]

and

\[
\frac{\partial^2}{\partial \gamma_j \partial \gamma_{j'}} \text{risk}[\gamma, \Theta] = 2E_{(x,y)}[(\Theta x)_{j'}(\Theta x)_j] \quad \text{and} \quad \frac{\partial^2}{\partial \theta_{jk} \partial \gamma_{j'}} \text{risk}[\gamma, \Theta] = 2\gamma_j'\gamma_jE_{(x,y)}[(x)_{k'}(x)_k].
\]

Moreover, if \( j' = j \), it holds that

\[
\frac{\partial^2}{\partial \theta_{jk} \partial \gamma_j} \text{risk}[\gamma, \Theta] = 2E_{(x,y)}[\gamma_j(x)_k(\Theta x)_j] - (y - \gamma^\top \Theta x)(x)_k]
\]

and

\[
\frac{\partial^2}{\partial \gamma_j \partial \theta_{jk}} \text{risk}[\gamma, \Theta] = 2E_{(x,y)}[\gamma_j(x)_k(\Theta x)_j] - (y - \gamma^\top \Theta x)(x)_k],
\]

and if \( j' \neq j \), it holds that

\[
\frac{\partial^2}{\partial \gamma_{j'} \partial \theta_{jk}} \text{risk}[\gamma, \Theta] = 2\gamma_{j'}E_{(x,y)}[(x)_k(\Theta x)_{j'}] \quad \text{and} \quad \frac{\partial^2}{\partial \theta_{jk} \partial \gamma_{j'}} \text{risk}[\gamma, \Theta] = 2\gamma_{j'}E_{(x,y)}[(x)_{k'}(\Theta x)_j].
\]
We use these results in the proofs of Theorem 1 and Proposition 3.
We then derive a uniform bound on the absolute difference between $\nabla \text{risk}_X[\gamma, \Theta]$ and $\nabla \text{risk}[\gamma, \Theta]$. We use the notation $\|\Theta\|_\infty := \max_{j \in \{1, \ldots, w\}} \sum_{k=1}^{d} |\theta_{jk}|$.

Lemma 7 (Uniform Bound on the Difference Between $\nabla \text{risk}_X[\gamma, \Theta]$ and $\nabla \text{risk}[\gamma, \Theta]$) It holds for each $t, \eta, \epsilon \in (0, \infty)$ and $\beta \in C_{\eta, \epsilon} := \{\beta = \text{vec}(\gamma, \Theta) \in \mathbb{R}^p : \|\beta^* - \beta\|_1 \leq \eta \text{ and } \|\gamma^T \Theta - \gamma^* \Theta^*\|_1 \leq \epsilon\}$ that

$$
\sup_{\beta \in C_{\eta, \epsilon}} \left(\|\nabla \text{risk}_X[\gamma, \Theta] - \nabla \text{risk}[\gamma, \Theta]\|_1^\top (\beta^* - \beta)\right) \leq 2t(\eta + \max\{1, \gamma^\infty, \|\Theta^\infty\|\}) (1 + \epsilon)
$$

with probability at least $1 - 4d^2 \rho e^{-\kappa n \min(t^2/\nu^2, t/\nu)}$ with constants $\nu, \kappa \in (0, \infty)$ depending only on the distributions of the inputs and noises.

The proof of Lemma 7 is given in Section 6.2.

The set $C_{\eta, \epsilon}$ contains all parameters in a neighborhood of $\beta^*$; in particular, the bound applies to $\beta^* = \text{vec}(\gamma^*, \Theta^*)$ itself—without any further assumption on $\beta^*$. The lemma is the main ingredient of our proof of Lemma 4.

We derive a uniform bound on the absolute difference between $\text{risk}_X[\gamma, \Theta]$ and $\text{risk}[\gamma, \Theta]$.

Lemma 8 (Uniform Bound on the Difference Between $\text{risk}_X[\gamma, \Theta]$ and $\text{risk}[\gamma, \Theta]$) Suppose that $\sup_{(\gamma, \Theta) \in B} \|\gamma^T \Theta^* - \gamma^T \Theta\|_\infty \leq \epsilon'$ for an $\epsilon' \in (0, \infty)$. Then, we have for each $t \in [0, \infty)$ that

$$
\sup_{(\gamma, \Theta) \in B} |\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta]| \leq t(1 + 4\epsilon' + 4\sqrt{\epsilon'})
$$

with probability at least $1 - 18d^2 \rho e^{-\kappa n \min(t^2/\nu^2, t/\nu)}$, with constants $\nu, \kappa \in (0, \infty)$ depending only on the distributions of the inputs and noises.

Lemma 8 is the main ingredient of our proof of Theorem 2. The proof of Lemma 8 is given in Section 6.2.

We finally derive a lemma studying invertibility of the line segment between two matrices. This lemma is employed in the proof of Theorem 1.

Lemma 9 Let define $H(t) := (A + tC)(A + tC)^\top$ for $A, C \in \mathbb{R}^{w' \times d'}$ with $w' \leq d'$ and $t \in (0, 1)$, where $A$ has full (row) rank. Then, $H(t)$ is not invertible at most in finitely many $t \in (0, 1)$.

The proof of Lemma 9 is given in Section 6.2.

6.2 Proofs

Here, we provide the proofs of our main claims.

6.2.1 Proof of Theorem 1

Proof The proof approach is based on Taylor’s theorem and the definition of stationary points.

We start by defining some notations: We use the notation $\gamma^T \Theta_{A, \bar{x}} := \gamma^T |\Theta, A| \bar{x}$ to generate an extended network indexed by $(\gamma, \Theta_{A})$ with $\bar{x} := (x^\top, \bar{x}^\top)^\top \in \mathbb{R}^{d+w-1}$, $\bar{x} \sim \mathcal{N}(0, I_{w-1} \times w-1)$, and $A = [v_1, \ldots, v_{w-1}] \in \mathbb{R}^{w \times w-1}$, with $v_1, \ldots, v_{w-1} \in \mathbb{R}^{w}$, is a matrix whose columns are basis of $\mathbb{R}^{w-1}$ such that $\gamma^T v_1 = \cdots = \gamma^T v_{w-1} = 0$. It means, the input’s dimension of the network is extended from $d$ to $d + w - 1$ and so the inner-layer matrix need also to be extended from $\Theta \in \mathbb{R}^{w \times d}$ to $[\Theta, A] \in \mathbb{R}^{w \times (d+w-1)}$. We also use the notation $\gamma_{A}^T \Theta_{\alpha, A, \bar{x}}$ to make an extended network that is also rescaled across the layers by a suitable $\alpha$. Note that the notation $\Theta_{[\alpha, A]}$ is equivalent with $(\Theta_{A})_{\alpha}$, both means we rescale a matrix $\Theta_{A} \in \mathbb{R}^{w \times (d+w-1)}$ with a vector $\alpha \in \mathbb{R}^{w}$ (see more details about
rescaled networks in Section 4). Using the above definitions, it is easy to see that $\gamma_\alpha^T\Theta_{\alpha,A}\tilde{x} = \gamma^T\Theta x$, that means, the output of the extended and rescaled network is exactly the same as the original network (using the definition of $A$ and rescaled weights). In other words, we have a network that is first extended and then rescaled while the output of the network is still the same as the original one. We use the notation $\text{risk}[\gamma_{\alpha},\Theta_{\alpha,A}] := \mathbb{E}((y - \gamma_{\alpha}^T\Theta_{\alpha,A}\tilde{x})^2)$ to compute the population risk in an extended and rescaled network. We also define $p' := w + w \cdot (d + w - 1)$ as the effective dimension of the extended network.

Now, let start the proof by writing a second-order Taylor expansion of risk$[\gamma^*,\Theta^*_{\alpha,A'}]$ (the risk in an extended and rescaled version of the target with $\beta^*_{\alpha,A'} = \text{vec}(\gamma^*,\Theta^*_{\alpha,A'}) \in \mathbb{R}^{p'}$) around an extended and rescaled version of a reasonable stationary $\beta_{\alpha,A} = \text{vec}(\gamma_{\alpha},\Theta_{\alpha,A}) \in \mathbb{R}^p$ with suitable $\alpha \in \mathbb{R}^w$ and $A,A' \in \mathbb{R}^{w \times w-1}$ (we see later how to assign suitable value for $\alpha$) to get

$$
\text{risk}[\gamma^*,\Theta^*_{\alpha,A'}] = \text{risk}[\gamma_{\alpha},\Theta_{\alpha,A}] + \nabla\text{risk}[\gamma_{\alpha},\Theta_{\alpha,A}]^T(\beta^*_{\alpha,A'} - \beta_{\alpha,A}) + \frac{1}{2}(\beta^*_{\alpha,A'} - \beta_{\alpha,A})^T \nabla^2\text{risk}[\gamma_{\alpha} + t(\gamma^*_{\alpha} - \gamma_{\alpha}),\Theta_{\alpha,A} + t(\Theta^*_{\alpha,A'} - \Theta_{\alpha,A})](\beta^*_{\alpha,A'} - \beta_{\alpha,A})
$$

for some $t \in (0,1)$ (Bertsekas et al. 2003, Proposition 1.1.13.a), where we use the notation $\nabla\text{risk}[\gamma_{\alpha},\Theta_{\alpha,A}] \in \mathbb{R}^{p'}$ and $\nabla^2\text{risk}[\gamma_{\alpha},\Theta_{\alpha,A}] \in \mathbb{R}^{p' \times p'}$ to collect the first and second order partial derivatives of risk$[\gamma_{\alpha},\Theta_{\alpha,A}]$ with respect to the $\beta_{\alpha,A}$, respectively (note that we have no assumption on $(\gamma^*,\Theta^*_{\alpha,A'})$ nor $(\gamma_{\alpha},\Theta_{\alpha,A})$ to have bounded norms).

Then, we employ the property of extended and rescaled networks that is risk$[\gamma_{\alpha},\Theta_{\alpha,A}] = \text{risk}[\tilde{\gamma},\tilde{\Theta}]$ and risk$[\gamma^*,\Theta^*_{\alpha,A}] = \text{risk}[\gamma^*,\Theta^*]$, and use the shorthand notation

$$
m := (\beta^*_{\alpha,A'} - \beta_{\alpha,A})^T \nabla^2\text{risk}[\gamma_{\alpha} + t(\gamma^*_{\alpha} - \gamma_{\alpha}),\Theta_{\alpha,A} + t(\Theta^*_{\alpha,A'} - \Theta_{\alpha,A})](\beta^*_{\alpha,A'} - \beta_{\alpha,A})
$$

to obtain

$$
\text{risk}[\gamma^*,\Theta^*] = \text{risk}[\tilde{\gamma},\tilde{\Theta}] + \text{risk}[\gamma_{\alpha},\Theta_{\alpha,A}]^T(\beta^*_{\alpha,A'} - \beta_{\alpha,A}) + \frac{1}{2}m.
$$

Now, we are motivated to show that $\nabla\text{risk}[\gamma_{\alpha},\Theta_{\alpha,A}]^T(\beta^*_{\alpha,A'} - \beta_{\alpha,A}) = \nabla\text{risk}[\tilde{\gamma},\tilde{\Theta}]^T(\beta^* - \tilde{\beta})$. To do so, we use 1. our Lemma 6 (for an extended and rescaled network), 2. the property of extended and rescaled networks, 3. linearity of expectations, 4. our assumption on $\tilde{x}$ that makes the first expectation zero, 5. some rewriting, 6. linearity of expectations, 7. our assumption on $\tilde{x}$ (let recall that $\tilde{x} = (x^T,\tilde{x}^T)^T \in \mathbb{R}^{d+w-1}$ with $\tilde{x} \sim \mathcal{N}(0,I_{d+w-1})$ that are also independent of $x$) that makes the second expectation zero, and 8. some rewriting to obtain that

$$
\frac{\partial}{\partial(\gamma_{\alpha})_j}\text{risk}[\gamma_{\alpha},\Theta_{\alpha,A}] = -2\mathbb{E}_{(x,y)}\left[(y - \gamma_{\alpha}^T\Theta_{\alpha,A}\tilde{x})(\Theta_{\alpha,A}\tilde{x})_j\right]
$$

$$
= -2\mathbb{E}_{(x,y)}\left[(y - \gamma^T\tilde{\Theta}x)(\tilde{\Theta}_{\alpha,A}\tilde{x})_j\right]
$$

$$
= -2\mathbb{E}_{(x,y)}\left[y(\tilde{\Theta}_{\alpha,A}\tilde{x})_j\right] + 2\mathbb{E}_{(x,y)}\left[\gamma^T\tilde{\Theta}x)(\tilde{\Theta}_{\alpha,A}\tilde{x})_j\right]
$$

$$
= 2\mathbb{E}_{(x,y)}\left[\gamma^T\tilde{\Theta}x)(\tilde{\Theta}_{\alpha,A}\tilde{x})_j\right]
$$

$$
= 2\mathbb{E}_{(x,y)}\left[\tilde{\gamma}^T\tilde{\Theta}x\sum_{k=1}^{d+w-1}(\tilde{\Theta}_{\alpha,A})_{jk}(\tilde{x})_k\right]
$$

$$
= 2\mathbb{E}_{(x,y)}\left[\tilde{\gamma}^T\tilde{\Theta}x\sum_{k=1}^{d}(\tilde{\Theta}_{\alpha,A})_{jk}(\tilde{x})_k\right] + 2\mathbb{E}_{(x,y)}\left[\tilde{\gamma}^T\tilde{\Theta}x\sum_{k=d+1}^{d+w-1}(\tilde{\Theta}_{\alpha,A})_{jk}(\tilde{x})_k\right]
$$

$$
= 2\mathbb{E}_{(x,y)}\left[\tilde{\gamma}^T\tilde{\Theta}x\sum_{k=1}^{d}(\tilde{\Theta}_{\alpha,A})_{jk}(x)_k\right]
$$


\[ = 2\mathbb{E}_{(x,y)} \left[ (\gamma^T \tilde{\Theta} x)(\tilde{\Theta}_a x)_j \right]. \]

Then, we 1. imply our result above for all \( j \in \{1, \ldots, w\} \), 2. use the definition of rescaled parameters and linearity of expectations to cancel \( \alpha \)'s, and 3. use our results in Lemma 6 to obtain

\[
\left( \frac{\partial}{\partial \gamma} \text{risk}[\gamma_a, \tilde{\Theta}_{\gamma, A}] \right)^\top (\gamma^* - \gamma_a) = 2 \left( \mathbb{E}_{(x,y)} \left[ (\gamma^T \tilde{\Theta} x)(\tilde{\Theta}_a x) \right] \right)^\top (\gamma^* - \gamma_a) \\
= 2 \left( \mathbb{E}_{(x,y)} \left[ (\gamma^T \tilde{\Theta} x)(\tilde{\Theta}_a x) \right] \right)^\top (\gamma^* - \tilde{\gamma}) \\
= \left( \frac{\partial}{\partial \gamma} \text{risk}[\gamma, \tilde{\Theta}] \right)^\top (\gamma^* - \tilde{\gamma}).
\]

Implying a similar argument as above for all partial derivatives, we conclude that \( \nabla \text{risk}[\gamma_a, \tilde{\Theta}_{\gamma, A}]^\top (\beta^* - \beta_a, A) = \nabla \text{risk}[\gamma, \tilde{\Theta}]^\top (\beta^* - \beta) \) (we omit the detailed proof). Tabulating this observation in the earlier display we obtain

\[
\text{risk}[\gamma^*, \Theta^*] = \text{risk}[\gamma, \tilde{\Theta}] + \nabla \text{risk}[\gamma, \tilde{\Theta}]^\top (\beta^* - \beta) + \frac{1}{2} m.
\]

Rearranging the display above we obtain

\[-\nabla \text{risk}[\gamma, \tilde{\Theta}]^\top (\beta^* - \beta) = \text{risk}[\gamma, \tilde{\Theta}] - \text{risk}[\gamma^*, \Theta^*] + \frac{1}{2} m.\]

Now, let recall the definition of stationary points in (3) that implies

\[
\nabla \text{risk}_X[\gamma, \tilde{\Theta}]^\top (\beta^* - \beta) + r z^\top (\beta^* - \beta) \geq 0.
\]

We 1. rearrange above inequality and expand the bracket, 2. use Hölder’s inequality and the fact that \( z^\top \beta = |\beta|_1 \) (recall that \( z \in \partial |\beta|_1 \)), and 3. use \( |z|_\infty \leq 1 \) to obtain

\[
-\nabla \text{risk}_X[\gamma, \tilde{\Theta}]^\top (\beta^* - \beta) \leq r z^\top \beta^* - r \tilde{z}^\top \beta \\
\leq r \|z\|_\infty |\beta^*|_1 - r |\beta|_1 \\
\leq r |\beta^*|_1 - r |\beta|_1,
\]

which by rearranging implies

\[
\nabla \text{risk}_X[\gamma, \tilde{\Theta}]^\top (\beta^* - \beta) + r |\beta^*|_1 - r |\beta|_1 \geq 0.
\]

Display above reveals the positiveness of the terms in its left hand side and so we can obtain

\[
-\nabla \text{risk}[\gamma, \tilde{\Theta}]^\top (\beta^* - \beta) \leq -\nabla \text{risk}[\gamma, \tilde{\Theta}]^\top (\beta^* - \beta) + \nabla \text{risk}_X[\gamma, \tilde{\Theta}]^\top (\beta^* - \beta) + r |\beta^*|_1 - r |\beta|_1,
\]

that is,

\[
-\nabla \text{risk}[\gamma, \tilde{\Theta}]^\top (\beta^* - \beta) \leq (\nabla \text{risk}_X[\gamma, \tilde{\Theta}] - \nabla \text{risk}[\gamma, \tilde{\Theta}])^\top (\beta^* - \beta) + r |\beta^*|_1 - r |\beta|_1.
\]

Now, let use our display earlier (obtained by Taylor expansion) rewriting the left hand side of the display above as

\[
\text{risk}[\gamma, \tilde{\Theta}] - \text{risk}[\gamma^*, \Theta^*] + \frac{1}{2} m \leq (\nabla \text{risk}_X[\gamma, \tilde{\Theta}] - \nabla \text{risk}[\gamma, \tilde{\Theta}])^\top (\beta^* - \beta) + r |\beta^*|_1 - r |\beta|_1.
\]
By some rearranging on the display above we obtain

\[
\text{risk}[\gamma, \Theta] \leq \text{risk}[\gamma^*, \Theta^*] + r \|\beta^*\|_1 + (\nabla \text{risk}_X[\gamma, \Theta] - \nabla \text{risk}[\gamma, \Theta])^\top (\beta^* - \beta) - r\|\beta\|_1 - \frac{1}{2} m.
\]

For the right hand side of the inequality above we 1. get an absolute value of the third term, 2. add a zero-valued factor, 3. use triangle inequality, and 4. use our results in Lemma 4 to obtain

\[
\text{risk}[\gamma, \Theta] \leq \text{risk}[\gamma^*, \Theta^*] + r \|\beta^*\|_1 + \left| (\nabla \text{risk}_X[\gamma, \Theta] - \nabla \text{risk}[\gamma, \Theta])^\top (\beta^* - \beta) \right| - r\|\beta\|_1 - \frac{1}{2} m
\]

\[
= \text{risk}[\gamma^*, \Theta^*] + 2r \|\beta^*\|_1 + \left| (\nabla \text{risk}_X[\gamma, \Theta] - \nabla \text{risk}[\gamma, \Theta])^\top (\beta^* - \beta) \right| - r\|\beta\|_1 - \frac{1}{2} m.
\]

\[
\leq \text{risk}[\gamma^*, \Theta^*] + 2r \|\beta^*\|_1 + \left| (\nabla \text{risk}_X[\gamma, \Theta] - \nabla \text{risk}[\gamma, \Theta])^\top (\beta^* - \beta) \right| - r\|\beta\|_1 - \frac{1}{2} m
\]

\[
\leq \text{risk}[\gamma^*, \Theta^*] + 2r \|\beta^*\|_1 + r_{\text{occ}} \|\beta^* - \beta\|_1 + \frac{r_{\text{occ}}}{2n} - r\|\beta\|_1 - \frac{1}{2} m
\]

with probability at least \(1 - 1/2n\).

The third and fifth terms in the last inequality above can be canceled if we choose the tuning parameter large enough. Hence, we obtain

\[
\text{risk}[\gamma, \Theta] \leq \text{risk}[\gamma^*, \Theta^*] + 2r \|\beta^*\|_1 + \frac{r_{\text{occ}}}{2n} - \frac{1}{2} m
\]

for \(r \geq r_{\text{occ}}\).

The rest of the proof is analyzing the behavior of \(m\). Let rewrite \(m = \|\beta^* A - \bar{\beta}_{\alpha, A}\|^2\) \(m'\) with

\[
m' := \frac{(\beta_{\alpha, A'} - \bar{\beta}_{\alpha, A})^\top}{\|\beta_{\alpha, A'} - \bar{\beta}_{\alpha, A}\|^2} \nabla^2 \text{risk}[\gamma, \Theta] + t(\gamma^* - \bar{\gamma}_\alpha), \bar{\Theta}_{\alpha, A} + t(\Theta^* - \bar{\Theta}_{\alpha, A}) \frac{(\beta_{\alpha, A'} - \bar{\beta}_{\alpha, A})}{\|\beta_{\alpha, A'} - \bar{\beta}_{\alpha, A}\|^2}.
\]

Now, we are motivated to employ our results in Proposition 3. To do so, we need to make sure about invertibility of matrix \((\bar{\Theta}_{\alpha} + t(\Theta^* - \bar{\Theta}_{\alpha}))\). Using the definition of the extended networks, it is easy to see that \(\bar{\Theta}_{\alpha} + t(\Theta^* - \bar{\Theta}_{\alpha})\) have full row rank. Then, using Lemma 9 we obtain that the line segment between two matrices \(\bar{\Theta}_{\alpha} + t(\Theta^* - \bar{\Theta}_{\alpha})\) is not invertible at most in finitely many \(t\). It means, if we shift \(t\) by a tiny value \(\zeta = 0\) then, we can make sure that in the new point \(t' = t - \zeta\) the corresponding matrix is invertible, that is,

\[
m' := \frac{(\beta_{\alpha, A'} - \bar{\beta}_{\alpha, A})^\top}{\|\beta_{\alpha, A'} - \bar{\beta}_{\alpha, A}\|^2} \nabla^2 \text{risk}[\gamma, \Theta] + (t - \zeta + \zeta)(\gamma^* - \bar{\gamma}_\alpha), \bar{\Theta}_{\alpha, A} + (t - \zeta + \zeta)(\Theta^* - \bar{\Theta}_{\alpha, A}) \frac{(\beta_{\alpha, A'} - \bar{\beta}_{\alpha, A})}{\|\beta_{\alpha, A'} - \bar{\beta}_{\alpha, A}\|^2}.
\]

where the second equation is reached by assuming \(\zeta\) is very close to zero and so we can ignore the remaining terms. Then, we have \((\bar{\Theta}_{\alpha} + t'(\Theta^* - \bar{\Theta}_{\alpha}))\) as an invertible matrix.

Implying Proposition 3 (with \(a = (\beta_{\alpha, A'} - \bar{\beta}_{\alpha, A})/\|\beta_{\alpha, A'} - \bar{\beta}_{\alpha, A}\|^2\) and \(d + w - 1\) and \(p'\) as the dimension of the input and the effective dimension, respectively) we obtain that \(m' \in [0, \infty)\) for appropriate \(a\), that is, \(a\) with large enough \(c\). The observation that \(m' \in [0, \infty)\) together with the definition of \(m\) implies that \(m \in [0, \infty)\) as well.
Tabulating this observation to the display earlier together with our assumption on $\beta^*$ ($|\beta^*|_1 = |\gamma^*|_1 + \|\Theta^*\|_1 \leq 2\sqrt{\log n}$) and the fact that $1/n \leq 4\sqrt{\log n}$, we obtain for all $r \geq r_{orc}$ that

$$
\text{risk}[\gamma, \Theta] \leq \text{risk}[\gamma^*, \Theta^*] + 2r|\beta^*|_1 + \frac{r_{orc}}{2n} - \frac{1}{2}m
$$

$$
\leq \text{risk}[\gamma^*, \Theta^*] + 2r|\beta^*|_1 + \frac{r_{orc}}{2n}
$$

$$
\leq \text{risk}[\gamma^*, \Theta^*] + 5r\sqrt{\log n}
$$

with probability at least $1 - 1/2n$.

Second claim is a trivial consequence of the first claim by 1. using $r = r_{orc}$ and 2. absorbing the constant 5 in $\nu$ and simplifying to obtain

$$
\text{risk}[\gamma, \Theta] \leq \text{risk}[\gamma^*, \Theta^*] + \nu (\log n)^{3/2} \sqrt{\frac{\log(np)}{n}} \left(5\sqrt{\log n}\right)
$$

$$
= \text{risk}[\gamma^*, \Theta^*] + \nu (\log n)^2 \sqrt{\frac{\log(np)}{n}},
$$

with probability at least $1 - 1/2n$, which completes the proof.

\[\blacksquare\]

6.2.2 Proof of Theorem 2

Proof Main ingredients of the proof are the definition of $\tau-$approximate stationary point and our Lemma 8.

We start the proof using the definition of a $\tau-$approximate stationary point in (7) that implies

$$
\text{risk}_X[\gamma, \Theta] + r|\beta|_1 \leq \text{risk}_X[\gamma, \Theta] + r|\beta|_1 + \tau.
$$

We add zero-valued terms to the both sides of the inequality above to obtain

$$
\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta] + \text{risk}[\gamma, \Theta] + r|\beta|_1 \leq \text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta] + \text{risk}[\gamma, \Theta] + r|\beta|_1 + \tau.
$$

Then, we 1. rearrange the terms, get an absolute value of the two terms, and use the properties of absolute values, 2. get a supremum over the reasonable parameter space $B_{res}$ using our assumptions that $(\gamma, \Theta), (\gamma, \Theta) \in B_{res} := \{ (\gamma, \Theta) \in B : |\gamma|_1, \|\Theta\|_1 \leq \sqrt{\log n}\}$ (we use our assumption that the stationary is reasonable and our argument in the paragraph above Theorem 2 to reach that $(\gamma, \Theta)$ is reasonable as well), 3. simplify, and 4. leave a negative term to obtain

$$
\text{risk}[\gamma, \Theta] \leq \text{risk}[\gamma, \Theta] + \text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta] + \text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta] + r|\beta|_1 - r|\beta|_1 + \tau
$$

$$
\leq \text{risk}[\gamma, \Theta] + \sup_{(\gamma, \Theta) \in B_{res}} |\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta]| + \sup_{(\gamma, \Theta) \in B_{res}} |\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta]| + r|\beta|_1 - r|\beta|_1 + \tau
$$

$$
+ r|\beta|_1 - r|\beta|_1 + \tau
$$

$$
= \text{risk}[\gamma, \Theta] + 2 \sup_{(\gamma, \Theta) \in B_{res}} |\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta]| + r|\beta|_1 - r|\beta|_1 + \tau
$$

$$
\leq \text{risk}[\gamma, \Theta] + 2 \sup_{(\gamma, \Theta) \in B_{res}} |\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta]| + r|\beta|_1 + \tau.
$$

Then, we use 1. our result above, 2. Lemma 8 bounding the second term with $t = \nu \sqrt{\log(32nd^2)/kn}$ and $B = B_{res}$ (with probability at least $1 - 1/2n$), 3. the definition of $B_{res}$ to replace $\sup_{(\gamma, \Theta) \in B_{res}} \|\gamma^* \Theta^* - \gamma \Theta\|$.
\[ \gamma^T \Theta^2 \leq \sup_{(\gamma, \Theta) \in \mathcal{B}_{\nu}} 2 \| \gamma \|_1^2 \leq 2 (\log n)^2 =: \epsilon', \]

4. our Theorem 1 upper bounding the first term (for \( r \geq r_{orc} \) with probability at least 1 - 1/2n), 5. our assumption that stationary is reasonable, 6. simplifying, 7. an assumption that \( n \geq 3 \) (just for simplifying the terms), and 8. the assumption that \( r \geq r_{orc} \) and the definition of \( r_{orc} \) (note that for simplicity, we absorb all the constants in \( \nu \)) to obtain

\[
\begin{align*}
\text{risk}[\gamma, \Theta] & \leq \text{risk}[\gamma, \Theta] + 2 \sup_{(\gamma, \Theta) \in \mathcal{B}_{\nu}} |\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta]| + r |\beta|_1 + \tau \\
& \leq \text{risk}[\gamma, \Theta] + 2 \nu \sqrt{\frac{\log(32n^2)}{\kappa n}} (1 + 4 \epsilon' + 4 \sqrt{\epsilon'}) + r |\beta|_1 + \tau \\
& \leq \text{risk}[\gamma, \Theta] + 2 \nu \sqrt{\frac{\log(32n^2)}{\kappa n}} (1 + 8 (\log n)^2 + 8 \log n) + r |\beta|_1 + \tau \\
& \leq \text{risk}[\gamma, \Theta] + 5 r \sqrt{\log n} + 2 \nu \sqrt{\frac{\log(32n^2)}{\kappa n}} (1 + 8 (\log n)^2 + 8 \log n) + r |\beta|_1 + \tau \\
& = \text{risk}[\gamma, \Theta] + 7 r \sqrt{\log n} + 2 \nu \sqrt{\frac{\log(32n^2)}{\kappa n}} (1 + 8 (\log n)^2 + 8 \log n) + r |\beta|_1 + \tau \\
& \leq \text{risk}[\gamma, \Theta] + 8 r \sqrt{\log n} + \tau
\end{align*}
\]

with probability at least 1 - (1 + 1)/2n, which is obtained by the fact that if \( a \leq z_1 + z_2 \)

\[
\mathbb{P}(a \leq c_1 + c_2) \geq \mathbb{P}(z_1 + z_2 \leq c_1 + c_2) \\
= 1 - \mathbb{P}(z_1 + z_2 > c_1 + c_2) \\
\geq 1 - (\mathbb{P}(z_1 > c_1) + \mathbb{P}(z_2 > c_2)),
\]

where \( a, z_1, z_2 \) are random variables and \( c_1, c_2 \) are constants, as desired.

Second claim is a trivial consequence of the first claim by 1. using \( r = r_{orc} \) and 2. absorbing the constant 8 in \( \nu \) to obtain

\[
\begin{align*}
\text{risk}[\gamma, \Theta] & \leq \text{risk}[\gamma, \Theta] + \nu (\log n)^{3/2} \sqrt{\frac{\log(np)}{n}} (8 \sqrt{\log n}) + \tau \\
& = \text{risk}[\gamma, \Theta] + \nu (\log n)^2 \sqrt{\frac{\log(np)}{n}} + \tau,
\end{align*}
\]

with probability at least 1 - 1/n, which completes the proof.

6.2.3 Proof of Proposition 3

\textbf{Proof}

The proof is based on basic algebra and property of scaling weights across the layers in neural networks.
Let consider all the network parameters as a vector of length $p$ (recall that $p = w + w \cdot d$). Then, we can tabulate the second order partial derivatives of $\text{risk}[\gamma, \Theta]$ in a matrix called $\nabla^2 \text{risk}[\gamma, \Theta] \in \mathbb{R}^{p \times p}$ (for notational simplicity, we focus on $\nabla^2 \text{risk}[\gamma, \Theta]$ for the moment and then we move to $\nabla^2 \text{risk}[\gamma, \Theta]_{\alpha}$ at the end of the proof) of the form

$$\nabla^2 \text{risk}[\gamma, \Theta] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with $A \in \mathbb{R}^{(w-d) \times (w-d)}$, $B \in \mathbb{R}^{(w-d) \times w}$, $C \in \mathbb{R}^{w \times (w-d)}$, and $D \in \mathbb{R}^{w \times w}$, where

$$A_{(j'-1)d+k', (j-1)d+k} := \frac{\partial^2}{\partial \theta_{j'k'} \partial \theta_{jk}} \text{risk}[\gamma, \Theta],$$

$$B_{(j'-1)d+k', j} := \frac{\partial^2}{\partial \theta_{j'k'} \partial \gamma_j} \text{risk}[\gamma, \Theta],$$

$$C_{j', (j-1)d+k} := \frac{\partial^2}{\partial \gamma_j \partial \theta_{jk}} \text{risk}[\gamma, \Theta],$$

$$D_{j', j} := \frac{\partial^2}{\partial \gamma_j \partial \gamma_j} \text{risk}[\gamma, \Theta]$$

for $j, j' \in \{1, \ldots, w\}$ and $k, k' \in \{1, \ldots, d\}$.

Applying the block-wise structure of $\nabla^2 \text{risk}[\gamma, \Theta]$, we are motivated to analyze the behavior of

$$a^\top \nabla^2 \text{risk}[\gamma, \Theta]a = (a^1)^\top A a^1 + (a^2)^\top C a^1 + (a^1)^\top B a^2 + (a^2)^\top D a^2.$$

Note that $C = B^\top$ (see Lemma 6), so, we are left to analyze the behavior of

$$a^\top \nabla^2 \text{risk}[\gamma, \Theta]a = (a^1)^\top A a^1 + 2(a^2)^\top C a^1 + (a^2)^\top D a^2$$

for all $a \in \mathbb{R}^p$ with $\|a\|_2 = 1$.

We do the proof in steps: We start by going through the three terms in the right hand side of display above separately, to write them in a mathematically nice formulation (Steps 1:3). In Step 4, we sum up over the results calculated in Steps 1:3. Finally in Step 5, we use our result in Steps 1:4 to prove the main claims of the proposition.

**Step 1**: We show that for $a^1 \in \mathbb{R}^{w-d}$ and $A \in \mathbb{R}^{(w-d) \times (w-d)}$,

$$(a^1)^\top A a^1 = 2 \sum_{k=1}^{d} \left( \gamma^\top (a^1)^k \right)^2,$$

where we denote $(a^1)^k := ((a^1)_k, (a^1)_{d+k}, \ldots, (a^1)_{(w-1)d+k})^\top \in \mathbb{R}^w$ (as a sub-vector of $a^1$) for each $k \in \{1, \ldots, d\}$.

We start by writing matrix product in the form of sums and fill the entries of matrix $A$ with the corresponding values from the definition to get

$$(a^1)^\top A a^1 = \sum_{j=1}^{w} \sum_{k=1}^{d} \sum_{j'=1}^{w} \sum_{k'=1}^{d} \left( a^1_{(j'-1)d+k'} \frac{\partial^2}{\partial \theta_{j'k'} \partial \theta_{jk}} \text{risk}[\gamma, \Theta] (a^1)_{(j-1)d+k} \right).$$

By Lemma 6 we have

$$\frac{\partial^2}{\partial \theta_{j'k'} \partial \theta_{jk}} \text{risk}[\gamma, \Theta] = 2 \gamma_{j'j} \gamma_j \mathbb{E}_{(x,y)} [(x)_k (x)_{k'}],$$
which using our assumption on \( x \) (identity covariance matrix) implies

\[
\frac{\partial^2}{\partial \gamma_j \partial \gamma_j} \text{risk}[\gamma, \Theta] = 2 \gamma_j \gamma_j
\]

for \( k = k' \) and zero otherwise (for \( k \neq k' \)). We use 1. our display earlier, 2. the result above, 3. the linearity of sums, 4. some rewriting (using multinomial theorem), and 5. implying our notation \((a^1)^k\) for writing the sum in the form of product to obtain

\[
(a^1)^\top A a^1 = \sum_{j=1}^w \sum_{k=1}^d \left( \sum_{j'=1}^w \sum_{k'=1}^d \left( (a^1)_{(j'-1)d+k'} \frac{\partial^2}{\partial \gamma_{j'} \partial \gamma_{j}} \text{risk}[\gamma, \Theta] \right) a^1_{(j-1)d+k} \right)
\]

\[
= 2 \sum_{j=1}^w \sum_{k=1}^d \left( \sum_{j'=1}^w \left( (a^1)_{(j'-1)d+k} \gamma_{j'} \gamma_j \right) a^1_{(j-1)d+k} \right)
\]

\[
= 2 \sum_{j=1}^w \sum_{k=1}^d \sum_{j'=1}^w \left( (a^1)_{(j'-1)d+k} \gamma_{j'} \gamma_j a^1_{(j-1)d+k} \right)
\]

\[
= 2 \sum_{k=1}^d \left( \sum_{j=1}^w \left( (a^1)_{(j-1)d+k} \gamma_j \right)^2 \right)
\]

\[
= 2 \sum_{k=1}^d \left( \gamma^\top (a^1)^k \right)^2.
\]

**Step 2:** We prove that for \( a^2 \in \mathbb{R}^w \) and \( D \in \mathbb{R}^{w \times w} \),

\[
(a^2)^\top D a^2 = 2 \sum_{k=1}^d \left( (\Theta_{\cdot,k})^\top a^2 \right)^2,
\]

where \( \Theta_{\cdot,k} \) denotes the \( k \)-th column of \( \Theta \).

For each \( j, j' \in \{1, \ldots, w\} \), we use 1. the result of Lemma 6, 2. the definition of covariance, 3. the fact that \( \text{Cov}(\Theta x) = \Theta \text{Cov}(x) \Theta^\top \), 4. the assumption on \( x \) (identity covariance), and 5. rewriting to obtain

\[
\frac{\partial^2}{\partial \gamma_j \partial \gamma_j} \text{risk}[\gamma, \Theta] = 2 \mathbb{E}_{(x,y)} [(\Theta x)_{j'j} \Theta x)_{j'j}]
\]

\[
= 2 \text{Cov}(\Theta x)_{j'j}
\]

\[
= 2(\Theta \text{Cov}(x) \Theta^\top)_{j'j}
\]

\[
= 2(\Theta \Theta^\top)_{j'j}
\]

\[
= 2 \sum_{k=1}^d \theta_{j'k} \theta_{jk}.
\]

We use 1. the definition of sub-matrix \( D \) to write the matrix product in the form of a sum, 2. tabulating above result and using the linearity of sums, 3. some rewriting (using the multinomial theorem), and 4. writing the sum in the form of product to obtain

\[
(a^2)^\top D a^2 = \sum_{j=1}^w \sum_{j'=1}^w \left( (a^2)_{j'j} \frac{\partial^2}{\partial \gamma_{j'} \partial \gamma_j} \text{risk}[\gamma, \Theta](a^2)^j \right)
\]
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We use 1. the result of Lemma 6, 2. linearity of expectations, 3. linearity of expectations and our assumption on \(x\), 4. our assumption on \(1, \ldots, d\), and 5. linearity of expectations and our assumption on \(x\), and some rearranging to obtain

\[
(a^2)\top C a^1 = 2 \sum_{k=1}^{d} \left( \gamma^\top (a^1)^k \right) \left( (\Theta_{k})\top a^2 \right) + 2 \sum_{k=1}^{d} \left( \left( (\gamma^\top \Theta - \gamma^\top \Theta^*)_k - \mathbb{E}(x,y) [(y - \gamma^\top \Theta^* x)_k] \right) (a^2)^\top (a^1)^k \right).
\]

Expanding \( (a^2)\top C a^1 \) yields

\[
(a^2)\top C a^1 = \sum_{j=1}^{w} \sum_{k=1}^{d} \left( \sum_{j'=1}^{w} (a^2)_{j'} \frac{\partial^2}{\partial \gamma_j \partial \theta_{jk}} \text{risk}[\gamma, \Theta] \right) (a^1)_{(j-1)d+k}.
\]

Now, we need to consider two different cases:

**Case 1:** \((j \neq j')\)

We use 1. the result of Lemma 6, 2. writing matrix product in the form of a sum, 3. linearity of sums and expectations, and 4. our assumption on \(x\) to get for each \(j, j' \in \{1, \ldots, w\} \) and \(k \in \{1, \ldots, d\} \) with \(j \neq j'\) that

\[
\frac{\partial^2}{\partial \gamma_j \partial \theta_{jk}} \text{risk}[\gamma, \Theta] = 2 \gamma_j \mathbb{E}(x,y) [(x)_k (\Theta x)_{j'}] = 2 \gamma_j \mathbb{E}(x,y) [(x)_k \sum_{k'=1}^{d} (\theta_{j'k'}(x)_{k'})] = 2 \gamma_j \sum_{k'=1}^{d} (\theta_{j'k'} \mathbb{E}(x,y) [(x)_k (x)_{k'}]) = 2 \gamma_j \theta_{j'k}.
\]

**Case 2:** \((j = j')\)

We use 1. the result of Lemma 6, 2. linearity of expectations, 3. linearity of expectations and our assumption on \(x\) (same argument as above), 4. linearity of expectations, 5. linearity of expectations and our assumption on \(x\), 6. adding a zero-valued term, and 7. again linearity of expectations, our assumption on \(x\), and some rearranging to obtain

\[
\frac{\partial^2}{\partial \gamma_j \partial \theta_{jk}} \text{risk}[\gamma, \Theta] = 2 \mathbb{E}(x,y) [(\gamma_j (x)_{k} (\Theta x)_{j} - (y - \gamma^\top \Theta x) (x)_{k})] = 2 \mathbb{E}(x,y) [(\gamma_j (x)_{k} (\Theta x)_{j})] + 2 \mathbb{E}(x,y) [(\gamma^\top \Theta x) (x)_{k} - y(x)_{k}] = 2 \gamma_j \theta_{jk} + 2 \mathbb{E}(x,y) [(\gamma^\top \Theta x) (x)_{k} - y(x)_{k}]
\]
\begin{align*}
&= 2\gamma_j \theta_{j,k} + 2E_{(x,y)} \left[ (\gamma^\top \Theta x)(x)_k \right] - 2E_{(x,y)} [y(x)_k] \\
&= 2\gamma_j \theta_{j,k} + 2(\gamma^\top \Theta)_k - 2E_{(x,y)} [y(x)_k] \\
&= 2\gamma_j \theta_{j,k} + 2(\gamma^\top \Theta)_k - 2E_{(x,y)} \left[ (y + \gamma \gamma^\top \Theta^* x - \gamma^* \Theta^* x)(x)_k \right] \\
&= 2\gamma_j \theta_{j,k} + 2(\gamma^\top \Theta - \gamma^* \Theta^*)_k - 2E_{(x,y)} \left[ (y - \gamma^* \Theta^* x)(x)_k \right].
\end{align*}

Now, we 1. use our earlier expansion, 2. separate the innermost sum in two cases, 3. use the result above (Case 1 and Case 2), 4. do some rearranging, 5. use linearity of sums and some rewriting, and 6. write sums in the form of vector product and rearranging to obtain

\begin{align*}
(a^2)^\top C a^1 \\
&= \sum_{j=1}^{w} \sum_{k=1}^{d} \left( \sum_{j'=1}^{w} \left( (a^2)_j \frac{\partial^2 \text{risk}[\gamma, \Theta]}{\partial \gamma_{j'} \partial \theta_{j,k}} (a^1)(j-1)d+k \right) \right) \\
&= \sum_{j=1}^{w} \sum_{k=1}^{d} \left( \sum_{j'=1}^{w} \left( (a^2)_j \frac{\partial^2 \text{risk}[\gamma, \Theta]}{\partial \gamma_{j'} \partial \theta_{j,k}} (a^1)(j-1)d+k \right) \right) \\
&\quad + \sum_{j=1}^{w} \sum_{k=1}^{d} \left( (a^2)_j \frac{\partial^2 \text{risk}[\gamma, \Theta]}{\partial \gamma_{j} \partial \theta_{j,k}} (a^1)(j-1)d+k \right) \\
&= 2 \sum_{j=1}^{w} \sum_{k=1}^{d} \sum_{j'=1}^{w} \left( (a^2)_j \gamma_j \theta_{j,k} (a^1)(j-1)d+k \right) \\
&\quad + 2 \sum_{j=1}^{w} \sum_{k=1}^{d} \left( (a^2)_j \left( (\gamma^\top \Theta - \gamma^* \Theta^*)_k - E_{(x,y)} \left[ (y - \gamma^* \Theta^* x)(x)_k \right] \right) (a^1)(j-1)d+k \right) \\
&= 2 \sum_{k=1}^{d} \sum_{j'=1}^{w} (a^2)_j \theta_{j,k} \left( \sum_{j=1}^{w} \gamma_j (a^1)(j-1)d+k \right) \\
&\quad + 2 \sum_{k=1}^{d} \left( (\gamma^\top \Theta - \gamma^* \Theta^*)_k - E_{(x,y)} \left[ (y - \gamma^* \Theta^* x)(x)_k \right] \right) \left( \sum_{j=1}^{w} (a^2)_j (a^1)(j-1)d+k \right) \\
&= 2 \sum_{k=1}^{d} \left( (\gamma^\top (a^1)_k) \left( (\Theta, k)^\top (a^2) \right) \right) \\
&\quad + 2 \sum_{k=1}^{d} \left( (\gamma^\top \Theta - \gamma^* \Theta^*)_k - E_{(x,y)} \left[ (y - \gamma^* \Theta^* x)(x)_k \right] \right) \left( (a^2)^\top (a^1)_k \right).
\end{align*}
**Step 4:** We prove that for any \( a = [(a^1)^\top, (a^2)^\top]^\top \in \mathbb{R}^p \) and \( (\gamma, \theta) \in \mathcal{B} \), it holds that

\[
a^\top \nabla^2 \text{risk}[\gamma, \theta]a = 2 \sum_{k=1}^{d} \left( (\Theta_{\cdot,k})^\top (a^2) + \gamma^\top (a^1)^k \right)^2 + 4 \sum_{k=1}^{d} (\gamma^\top \theta - \gamma^\top \Theta^* k)(a^2)^\top (a^1)^k
\]

\[- 4 \sum_{k=1}^{d} \mathbb{E}_{(x,y)} \left[ (y - \gamma^\top \Theta^* x)(x)_k \right] (a^2)^\top (a^1)^k.\]

We use 1. the block-wise structure of the Hessian matrix and some rearranging, 2. our results in Steps 1:3, and 3. multinominal theorem to obtain

\[
a^\top \nabla^2 \text{risk}[\gamma, \theta]a = (a^1)^\top A a^1 + (a^2)^\top D a^2 + 2(a^2)^\top C a^1
\]

\[= 2 \sum_{k=1}^{d} \left( \gamma^\top (a^1)^k \right)^2 + 2 \sum_{k=1}^{d} \left( (\Theta_{\cdot,k})^\top a^2 \right)^2 + 4 \sum_{k=1}^{d} \left( \gamma^\top (a^1)^k \right) \left( (\Theta_{\cdot,k})^\top a^2 \right)
\]

\[+ 4 \sum_{k=1}^{d} \left( \left( \gamma^\top \theta - \gamma^\top \Theta^* k \right) - \mathbb{E}_{(x,y)} \left[ (y - \gamma^\top \Theta^* x)(x)_k \right] \right) (a^2)^\top (a^1)^k
\]

\[= 2 \sum_{k=1}^{d} \left( (\Theta_{\cdot,k})^\top a^2 + \gamma^\top (a^1)^k \right)^2 + 4 \sum_{k=1}^{d} \left( \gamma^\top \theta - \gamma^\top \Theta^* k \right) (a^2)^\top (a^1)^k
\]

\[- 4 \sum_{k=1}^{d} \mathbb{E}_{(x,y)} \left[ (y - \gamma^\top \Theta^* x)(x)_k \right] (a^2)^\top (a^1)^k.\]

**Step 5:** Now, we employ our result in Steps 1:4 to prove the main claims of the proposition.

**Claim 1:** \( (a^1 \neq 0 \text{ and } a^2 = 0) \)

We use 1. the block-wise structure of the Hessian, 2. the assumption that \( a^2 = 0 \), 3. our result in Step 1, and 4. the fact that sum of non-negative terms is also non-negative to obtain

\[a^\top \nabla^2 \text{risk}[\gamma, \theta]a = (a^1)^\top A a^1 + 2(a^2)^\top C a^1 + (a^2)^\top D a^2
\]

\[= (a^1)^\top A a^1
\]

\[= 2 \sum_{k=1}^{d} \left( \gamma^\top (a^1)^k \right)^2
\]

\[\geq 0.
\]

Above display can also reveal that for all \( \alpha \in \mathbb{R}^w \setminus \{0\} \) (moving to a scaled version of the parameters)

\[a^\top \nabla^2 \text{risk}[\gamma_\alpha, \theta_\alpha]a = 2 \sum_{k=1}^{d} \left( \gamma_\alpha^\top (a^1)^k \right)^2 \geq 0,
\]

as desired.

**Claim 2:** \( (a^1 = 0 \text{ and } a^2 \neq 0) \)

The proof is similar with Claim 1 and so we omit the proof.

**Claim 3:** \( (a^1 \neq 0 \text{ and } a^2 \neq 0) \)

We use our results in Step 4 together with getting an absolute value of the two last terms to obtain

\[a^\top \nabla^2 \text{risk}[\gamma, \theta]a = 2 \sum_{k=1}^{d} \left( (\Theta_{\cdot,k})^\top a^2 + \gamma^\top (a^1)^k \right)^2 + 4 \sum_{k=1}^{d} \left( \gamma^\top \theta - \gamma^\top \Theta^* k \right) (a^2)^\top (a^1)^k.
\]
\begin{align*}
-4 \sum_{k=1}^{d} \mathbb{E}_{(x,y)} \left[ (y - \gamma^* \Theta^* x)(x)_k \right] (a^2)^\top(a^1) \\
\geq 2 \sum_{k=1}^{d} \left( (\Theta_{:,k})^\top a^2 + \gamma^\top(a^1)_k \right)^2 - 4 \sum_{k=1}^{d} \left( \gamma^\top \Theta - \gamma^* \Theta^* \right) (a^2)^\top(a^1)_k \\
-4 \sum_{k=1}^{d} \mathbb{E}_{(x,y)} \left[ (y - \gamma^* \Theta^* x)(x)_k \right] (a^2)^\top(a^1)_k . 
\end{align*}

First let us concentrate on the second term of display above and 1. use the triangle inequality and properties of absolute values, 2. use Hölder inequality, 3. get a factor $|a^2|_2$ out of the summation, 4. use Cauchy–Schwarz inequality, and 5. some rewriting to obtain

\begin{align*}
4 \sum_{k=1}^{d} \left( \gamma^\top \Theta - \gamma^* \Theta^* \right)_k (a^2)^\top(a^1)_k & \leq 4 \sum_{k=1}^{d} \left( \gamma^\top \Theta - \gamma^* \Theta^* \right)_k \|a^2\|_2 \|a^1\|_2 \\
& \leq 4 \|a^2\|_2 \sum_{k=1}^{d} \left( \gamma^\top \Theta - \gamma^* \Theta^* \right)_k \|a^1\|_2 \\
& = 4 \|a^2\|_2 \sum_{k=1}^{d} \left( \gamma^\top \Theta - \gamma^* \Theta^* \right)_k \|a^1\|_2 \\
& \leq 4 \|a^1\|_2 \|a^2\|_2 \|\gamma^\top \Theta - \gamma^* \Theta^*\|_2 .
\end{align*}

Then, we use 1. our assumption that $y = \gamma^* \Theta^* x + u$, 2. independence of $u$ and $x$, and 3. our assumption that $\mathbb{E}[x] = 0$ (also we have $\mathbb{E}[u] = 0$) to obtain

\begin{align*}
4 \sum_{k=1}^{d} \mathbb{E}_{(x,y)} \left[ (y - \gamma^* \Theta^* x)(x)_k \right] (a^2)^\top(a^1)_k & = 4 \sum_{k=1}^{d} \mathbb{E}_{(x,y)} \left[ u(x)_k \right] (a^2)^\top(a^1)_k \\
& = 4 \sum_{k=1}^{d} \mathbb{E}_{(x,y)} \left[ u \right] \mathbb{E}_{(x,y)} \left[ (x)_k \right] (a^2)^\top(a^1)_k \\
& = 0 .
\end{align*}

Tabulating two observations above in the previous display we obtain

\begin{align*}
\mathbf{a}^\top \nabla^2 \text{risk}[\gamma, \Theta] \mathbf{a} & \geq 2 \sum_{k=1}^{d} \left( (\Theta_{:,k})^\top a^2 + \gamma^\top(a^1)_k \right) - 4 \|a^1\|_2 \|a^2\|_2 \|\gamma^\top \Theta - \gamma^* \Theta^*\|_2 .
\end{align*}

Now, let define for each $k \in \{1, \ldots, d\}$ that $A_k := (\Theta_{:,k})^\top a^2$, $B_k := \gamma^\top(a^1)_k$, and using the fact $(A_k + B_k)^2 \geq \frac{1}{2}(A_k)^2 - (B_k)^2$ to obtain

\begin{align*}
\mathbf{a}^\top \nabla^2 \text{risk}[\gamma, \Theta] \mathbf{a} \\
\geq 2 \sum_{k=1}^{d} (A_k + B_k)^2 - 4 \|a^1\|_2 \|a^2\|_2 \|\gamma^\top \Theta - \gamma^* \Theta^*\|_2 \\
\geq \sum_{k=1}^{d} (A_k)^2 - 2 \sum_{k=1}^{d} (B_k)^2 - 4 \|a^1\|_2 \|a^2\|_2 \|\gamma^\top \Theta - \gamma^* \Theta^*\|_2 .
\end{align*}
Now, we analyze the first two terms in the right hand side of the last inequality above. We use 1. the definition of $A_k$, 2. some rewritings, 3. the linearity of sums, 4. the definition of matrix product, 5. property of eigenvalues ($\epsilon_{\min}(\Theta\Theta^\top)$ denotes the smallest eigenvalue of $\Theta\Theta^\top$), and 6. the norm definition to obtain

$$
\sum_{k=1}^d (A_k)^2 = \sum_{k=1}^d ((\Theta_{..,k})^\top \mathbf{a}^2)^2 \\
= \sum_{k=1}^d (\mathbf{a}^2)^\top \Theta_{..,k}(\Theta_{..,k})^\top \mathbf{a}^2 \\
= (\mathbf{a}^2)^\top \left( \sum_{k=1}^d \Theta_{..,k}(\Theta_{..,k})^\top \right) \mathbf{a}^2 \\
= (\mathbf{a}^2)^\top \Theta \Theta^\top \mathbf{a}^2 \\
\geq \epsilon_{\min}(\Theta \Theta^\top)(\mathbf{a}^2)^\top \mathbf{a}^2 \\
= \epsilon_{\min}(\Theta \Theta^\top)|\mathbf{a}^2|^2_2.
$$

Also, using 1. the definition of $B_k$, 2. the Cauchy–Schwarz inequality, 3. the linearity of sums, and 4. the definition of norms we obtain

$$
2 \sum_{k=1}^d (B_k)^2 = 2 \sum_{k=1}^d (\gamma^\top (\mathbf{a}^1)^k)^2 \leq 2 \sum_{k=1}^d \|\gamma\|^2_2 (\mathbf{a}^1)^k_2^2 = 2 \|\gamma\|^2_2 \sum_{k=1}^d (\mathbf{a}^1)^k_2^2 = 2 \|\gamma\|^2_2 \|\mathbf{a}^1\|^2_2.
$$

Collecting two displays above together with the earlier one we obtain

$$
\mathbf{a}^\top \nabla^2 \text{risk}[\gamma, \Theta] \mathbf{a} \geq \epsilon_{\min}(\Theta \Theta^\top)|\mathbf{a}^2|^2_2 - 2 \|\gamma\|^2_2 \|\mathbf{a}^1\|^2_2 - 4 \|\mathbf{a}^1\|^2_2 \|\mathbf{a}^2\|^2_2 \|\gamma^\top \Theta - \gamma^* \Theta^*\|^2_2.
$$

Now, it is time to concentrate on the Hessian behavior of $\nabla^2 \text{risk}[\gamma_{\alpha}, \Theta_{\alpha}]$ (and not $\nabla^2 \text{risk}[\gamma, \Theta]$). We use the known fact in neural networks that weights can be rescaled across the layers once activations are nonnegative-homogeneous. It says for a neural network parameterized by $(\gamma, \Theta)$, there is another network with the same objective value such that the covariates of $\gamma$ are multiplied by the covariates of $\alpha$ and the covariates in each column of $\Theta$ are divided by the covariates of $\alpha$. We use this fact with $\alpha_j = 1/c$ for all $j \in \{1, \ldots, w\}$, which $c \in (1, \infty)$, together with the above result to analyse the behavior of Hessian in $(\gamma_{\alpha}, \Theta_{\alpha})$ and get

$$
\mathbf{a}^\top \nabla^2 \text{risk}[\gamma_{\alpha}, \Theta_{\alpha}] \mathbf{a} \geq \epsilon_{\min}(\Theta_{\alpha} \Theta_{\alpha}^\top)|\mathbf{a}^2|^2_2 - 2 \|\gamma_{\alpha}\|^2_2 \|\mathbf{a}^1\|^2_2 - 4 \|\mathbf{a}^1\|^2_2 \|\mathbf{a}^2\|^2_2 \|\gamma_{\alpha}^\top \Theta_{\alpha} - \gamma_{\alpha}^* \Theta^*\alpha\|^2_2 \\
= c^2 \epsilon_{\min}(\Theta \Theta^\top)|\mathbf{a}^2|^2_2 - \frac{2}{c^2} \|\gamma_{\alpha}\|^2_2 \|\mathbf{a}^1\|^2_2 - 4 \|\mathbf{a}^1\|^2_2 \|\mathbf{a}^2\|^2_2 \|\gamma_{\alpha}^\top \Theta - \gamma_{\alpha}^* \Theta^*\|^2_2,
$$

where for the last line we use factorizing and the definition of scaled parameters. Using above display, we can guarantee positive semidefinite Hessian once $c$ is selected large enough because the first term can dominate the other two terms. So, we use $c \in [1, \infty)$ and our assumption on $\Theta \Theta^\top$ to obtain that for

$$
c^2 \geq \frac{2 \|\gamma\|^2_2 \|\mathbf{a}^1\|^2_2 + 4 \|\mathbf{a}^1\|^2_2 \|\mathbf{a}^2\|^2_2 \|\gamma^\top \Theta - \gamma^* \Theta^*\|^2_2}{\epsilon_{\min}(\Theta \Theta^\top)|\mathbf{a}^2|^2_2},
$$

we can guarantee positive semidefinite Hessian, as desired. \[\blacksquare\]
6.2.4 Proof of Lemma 4

**Proof** The proof idea is inspired by Elsener and van de Geer (2018, Lemma 14) and main ingredients are our Lemma 7 and union bounds.

Let define \( \tilde{r}(t) := 2t \) for \( t \in (0, \infty) \), \( s_{c_{n,*}} := (\eta + \max\{\|\gamma\|_\infty, \|\Theta^*\|_\infty\})(1 + \epsilon) \), which is basically defined by parameters \( \epsilon \) and \( \eta \) of \( C_{n,\epsilon} \) (recall that \( C_{n,\epsilon} = \{ \beta = \text{vec}(\gamma, \Theta) \in \mathbb{R}^p : \|\beta - \beta\|_1 \leq \eta \) and \( \|\gamma^\top \Theta - \gamma^* \Theta^*\|_1 \leq \epsilon \} \)), and \( Z(\beta, \beta^*) \) as a function of two vectors \( \beta \) and \( \beta^* \) (with \( \beta = \text{vec}(\gamma, \Theta) \)) defined as

\[
Z(\beta, \beta^*) := \left| (\nabla \text{risk}_X[\gamma, \Theta] - \nabla \text{risk}[\gamma, \Theta])^\top (\beta^* - \beta) \right|
\]

Using Lemma 7 and notations above and with assuming \( \tilde{\beta} \in C_{n,\epsilon} \) (specific values of \( \epsilon \) and \( \eta \) be assigned at the end of the proof) we obtain for each \( t \in (0, \infty) \) that

\[
\mathbb{P}\left( Z(\tilde{\beta}, \beta^*) \geq \eta \tilde{r}(t)s_{c_{n,*}} \right) \leq \mathbb{P}\left( \sup_{\beta \in C_{n,\epsilon}} Z(\beta, \beta^*) \geq \eta \tilde{r}(t)s_{c_{n,*}} \right) \leq 4d^2p e^{-\kappa n \min\{t^2/\nu^2, t/\nu\}}
\]

with \( \nu, \kappa \in (0, \infty) \) constants depending only on the distributions of the inputs and noises.

We assume without lose of generality that \( 1/n \leq \eta \) and continue the proof in two different cases:

**Case 1:** \( (\|\tilde{\beta} - \beta^*\|_1 \leq 1/n) \)

In this case, we use 1. the fact that \( \|\tilde{\beta} - \beta^*\|_1 \tilde{r}(t)s_{c_{n,*}} \geq 0 \), 2. our assumption that \( 1/n \leq \eta \) and the definition of \( s_{c_{n,*}} \), and 3. our assumption that \( \|\tilde{\beta} - \beta^*\|_1 \leq 1/n \), which implies that \( \tilde{\beta} \in C_{1/n,\epsilon} \) and our argument above to obtain for each \( t \in (0, \infty) \) that

\[
\mathbb{P}\left( Z(\tilde{\beta}, \beta^*) \geq 2\|\tilde{\beta} - \beta^*\|_1 \tilde{r}(t)s_{c_{n,*}} + \frac{\tilde{r}(t)}{n}s_{c_{n,*}} \right) \leq \mathbb{P}\left( Z(\tilde{\beta}, \beta^*) \geq \frac{\tilde{r}(t)}{n}s_{c_{n,*}} \right)
\]

\[
\leq \mathbb{P}\left( Z(\tilde{\beta}, \beta^*) \geq \frac{\tilde{r}(t)}{n}s_{c_{1/n,\epsilon}} \right) \leq 4d^2p e^{-\kappa n \min\{t^2/\nu^2, t/\nu\}}
\]

**Case 2:** \( (1/n < \|\tilde{\beta} - \beta^*\|_1 \leq \eta) \)

In this case, we use 1. the fact that for mutually exclusive events \( H_1, \ldots, H_n \): \( \mathbb{P}(\cup_{i=1}^n H_i) = \sum_{i=1}^n \mathbb{P}(H_i) \), 2. lower bound of \( \|\tilde{\beta} - \beta^*\|_1 \), 3. the fact that \( \tilde{r}(t)s_{c_{n,*}} \geq 0 \) and removing the lower bound, 4. the fact that \( 2i/n \leq \eta \), and 5. the fact that \( \tilde{\beta} \in s_{c_{2i/n,\epsilon}} \) and our earlier argument to obtain for each \( t \in (0, \infty) \) that

\[
\mathbb{P}\left( Z(\tilde{\beta}, \beta^*) \geq 2\|\tilde{\beta} - \beta^*\|_1 \tilde{r}(t)s_{c_{n,*}} + \frac{\tilde{r}(t)}{n}s_{c_{n,*}} \quad \text{for} \quad \frac{1}{n} < \|\tilde{\beta} - \beta^*\|_1 \leq \eta \right)
\]

\[
= \sum_{i=0}^{[\log_2(n\eta)]-1} \mathbb{P}\left( Z(\tilde{\beta}, \beta^*) \geq 2\|\tilde{\beta} - \beta^*\|_1 \tilde{r}(t)s_{c_{n,*}} + \frac{\tilde{r}(t)}{n}s_{c_{n,*}} \quad \text{for} \quad \frac{2i}{n} < \|\tilde{\beta} - \beta^*\|_1 \leq \frac{2i+1}{n} \right)
\]

\[
\leq \sum_{i=0}^{[\log_2(n\eta)]-1} \mathbb{P}\left( Z(\tilde{\beta}, \beta^*) \geq \frac{2i+1}{n}\tilde{r}(t)s_{c_{n,*}} + \frac{\tilde{r}(t)}{n}s_{c_{n,*}} \quad \text{for} \quad \frac{2i}{n} < \|\tilde{\beta} - \beta^*\|_1 \leq \frac{2i+1}{n} \right)
\]

\[
\leq \sum_{i=0}^{[\log_2(n\eta)]-1} \mathbb{P}\left( Z(\tilde{\beta}, \beta^*) \geq \frac{2i+1}{n}\tilde{r}(t)s_{c_{2i/n,\epsilon}} \quad \text{for} \quad \|\tilde{\beta} - \beta^*\|_1 \leq \frac{2i+1}{n} \right)
\]

\[
\leq \sum_{i=0}^{[\log_2(n\eta)]-1} \mathbb{P}\left( Z(\tilde{\beta}, \beta^*) \geq \frac{2i+1}{n}\tilde{r}(t)s_{c_{2i+1/n,\epsilon}} \quad \text{for} \quad \|\tilde{\beta} - \beta^*\|_1 \leq \frac{2i+1}{n} \right)
\]

\[22\]
We collect all pieces of the proof (Case 1 and Case 2), set \( t = \nu \sqrt{\log (8nd^2p[\log_2(n\eta)])(\kappa n)} \) (we use the notation \( \log \) as natural logarithm), and use the union bounds to obtain (we also need to assume \( n \) is large enough to get rid of the min operator)

\[
\mathbb{P} \left( Z(\tilde{\beta}, \beta^*) \geq 2\|\tilde{\beta} - \beta^*\|_1 \right) \leq \frac{1}{2n}.
\]

Now, we use the results above and the definitions of \( Z(\tilde{\beta}, \beta^*) \) and \( \tilde{t}(t) \) to obtain

\[
\mathbb{P} \left( \left( |\nabla \text{risk}_X[\tilde{\gamma}, \tilde{\Theta}] - \nabla \text{risk}([\tilde{\gamma}, \tilde{\Theta}])|^{**} \right) (\beta^* - \tilde{\beta}) \right) \geq 4\nu s_{n, s} \|\tilde{\beta} - \beta^*\|_1 \frac{\log(8nd^2p[\log_2(n\eta)])}{\kappa n} \\
+ 2\nu s_{n, s} \sqrt{\log(8nd^2p[\log_2(n\eta)])} \\
\leq \frac{1}{2n}.
\]

Then, we use our assumption that the stationary point \( (\tilde{\gamma}, \tilde{\Theta}) \) is reasonable to obtain: \( |\tilde{\gamma}^T\tilde{\Theta} - \gamma^T\Theta^*|_1 \leq |\tilde{\gamma}^T\tilde{\Theta}|_1 + |\gamma^T\Theta^*|_1 \leq |\tilde{\gamma}|_1 \|\Theta\|_{\infty} + |\gamma^*|_1 \|\Theta^*\|_{\infty} \leq 2 \log n \) (using triangle inequality, \( \tilde{\gamma} \) is large enough to get rid of the min operator), \( |\tilde{\Theta} - \Theta^*|_1 \leq |\delta|_1 + |\delta^*|_1 \leq |\delta|_1 + |\delta^*|_1 + |\delta^*|_1 \leq 4 \sqrt{\log n} \) (using triangle inequality, our definition of norm, and our assumption on reasonable target and stationary), which means we can assign \( \epsilon = 2 \log n \) and \( \eta = 4 \sqrt{\log n} \) for \( n \geq 2 \) we can make sure that \( 1/n \leq \eta \) is satisfied.

Now, we plug in the values of \( \epsilon = 2 \log n \), \( \eta = 4 \sqrt{\log n} \), and \( s_{n, s} = (\log n) + \max\{\|\gamma^*\|_{\infty}, \|\Theta^*\|_{\infty}\}(1 + \epsilon) \leq (5 \sqrt{\log n})(1 + 2 \log n) \leq 15(\log n)^{3/2} \) for \( n \geq 2 \) to conclude that

\[
\mathbb{P} \left( \left( |\nabla \text{risk}_X[\tilde{\gamma}, \tilde{\Theta}] - \nabla \text{risk}([\tilde{\gamma}, \tilde{\Theta}])|^{**} \right) (\beta^* - \tilde{\beta}) \right) \\
\geq 30 \sqrt{\kappa n^2} \nu (\log n)^{3/2} \sqrt{\log(8nd^2p[\log_2(4n \sqrt{\log n})])} \\
\leq \frac{1}{2n}.
\]
Then, we use the fact that $d \leq p$ and simplifying display above to obtain
\[
\mathbb{P}\left(\left|\nabla_{\Theta} \text{risk}_{X}[\gamma, \Theta] - \nabla_{\Theta} \text{risk}_{\tilde{\gamma}}[\tilde{\Theta}]\right|^\top (\beta^* - \tilde{\beta})\right|
\geq \frac{180}{\sqrt{\kappa n}} \nu \|\tilde{\beta} - \beta^*\|_1 (\log n)^{3/2} \sqrt{\log(np)} + \frac{90}{\sqrt{\kappa n^3}} \nu (\log n)^{3/2} \sqrt{\log(np)}
\leq \frac{1}{2n}.
\]
We finally absorb all the constants $(180/\sqrt{\kappa})$ in $\nu$ and use the definition of $r_{orc}$ to complete the proof.

6.2.5 Proof of Lemma 5

Proof. The proof consists of basic algebra.

Claim 1: We use 1. the definition of $\text{risk}_{X}[\gamma, \Theta]$, 2. the chain rule, and 3. taking the derivatives to obtain
\[
\frac{\partial}{\partial \gamma_j} \text{risk}_{X}[\gamma, \Theta] = \frac{\partial}{\partial \gamma_j} \left( \frac{1}{n} \sum_{i=1}^{n} (y_i - \gamma^\top \Theta x_i)^2 \right)
= -\frac{2}{n} \sum_{i=1}^{n} \left( (y_i - \gamma^\top \Theta x_i) \frac{\partial}{\partial \gamma_j} (\gamma^\top \Theta x_i) \right)
= -\frac{2}{n} \sum_{i=1}^{n} \left( (y_i - \gamma^\top \Theta x_i) (\Theta x_i)_j \right),
\]
as desired.

Claim 2: We use 1. the definition of $\text{risk}_{X}[\gamma, \Theta]$, 2. the chain rule, and 3. taking the derivatives to obtain
\[
\frac{\partial}{\partial \theta_{jk}} \text{risk}_{X}[\gamma, \Theta] = \frac{\partial}{\partial \theta_{jk}} \left( \frac{1}{n} \sum_{i=1}^{n} (y_i - \gamma^\top \Theta x_i)^2 \right)
= -\frac{2}{n} \sum_{i=1}^{n} \left( (y_i - \gamma^\top \Theta x_i) \frac{\partial}{\partial \theta_{jk}} (\gamma^\top \Theta x_i) \right)
= -\frac{2}{n} \sum_{i=1}^{n} \left( (y_i - \gamma^\top \Theta x_i) (\Theta x_i)_j \right),
\]
as desired.

Claim 3: We 1. use Claim 1, and 2. remove the term with zero derivative and use the chain rule to obtain
\[
\frac{\partial^2}{\partial \gamma_j \partial \gamma_j} \text{risk}_{X}[\gamma, \Theta] = \frac{\partial}{\partial \gamma_j} \left( -\frac{2}{n} \sum_{i=1}^{n} (y_i - \gamma^\top \Theta x_i) (\Theta x_i)_j \right)
= \frac{2}{n} \sum_{i=1}^{n} ((\Theta x_i)_j) (\Theta x_i)_j,
\]
as desired.
**Claim 4:** We 1. use Claim 2, 2. remove the term with zero derivative, and 3. take the derivative of the bracket, and 4. do some rearranging to obtain

\[
\frac{\partial^2}{\partial \theta_{j'k'} \partial \theta_{jk}} \text{risk}_X[\gamma, \Theta] = \frac{\partial}{\partial \theta_{j'k'}} \left( -\frac{2}{n} \sum_{i=1}^{n} \left( (y_i - \gamma^\top \Theta x_i) \gamma_j(x_i)_k \right) \right)
\]

\[
= \frac{\partial}{\partial \theta_{j'k'}} \left( \frac{2}{n} \gamma_j \sum_{i=1}^{n} \left( (\gamma^\top \Theta x_i)(x_i)_k \right) \right)
\]

\[
= \frac{2}{n} \gamma_j \sum_{i=1}^{n} ((x_i)_k(x_i)_k)
\]

\[
= \frac{2}{n} \gamma_j \gamma_j \sum_{i=1}^{n} ((x_i)_k(x_i)_k),
\]

as desired.

**Claims 5 and 6:** We only show the results for \(\frac{\partial^2}{\partial \theta_{j'k'} \partial \gamma_j} \text{risk}_X[\gamma, \Theta]\). The results for \(\frac{\partial^2}{\partial \gamma_j \partial \theta_{jk}} \text{risk}_X[\gamma, \Theta]\) can be obtained using the same arguments.

We consider two cases:

**Case 1:** if \(j' = j\), we use 1. Claim 1, 2. the chain rule, and 3. taking the derivatives and simplifying to obtain

\[
\frac{\partial^2}{\partial \theta_{jk} \partial \gamma_j} \text{risk}_X[\gamma, \Theta] = \frac{\partial}{\partial \theta_{jk}} \left( -\frac{2}{n} \sum_{i=1}^{n} \left( (y_i - \gamma^\top \Theta x_i)(\Theta x_i)_j \right) \right)
\]

\[
= \frac{2}{n} \sum_{i=1}^{n} \left( (\Theta x_i)_j \frac{\partial}{\partial \theta_{jk}}(y_i - \gamma^\top \Theta x_i) + (y_i - \gamma^\top \Theta x_i) \frac{\partial}{\partial \theta_{jk}}(\Theta x_i)_j \right)
\]

\[
= \frac{2}{n} \sum_{i=1}^{n} \left( \gamma_j(x_i)_k(\Theta x_i)_j - (y_i - \gamma^\top \Theta x_i)(x_i)_k \right),
\]

as desired.

**Case 2:** if \(j' \neq j\), we use 1. Claim 1, 2. the chain rule, and 3. taking the derivatives and rearranging to obtain

\[
\frac{\partial^2}{\partial \theta_{j'k'} \partial \gamma_j} \text{risk}_X[\gamma, \Theta] = \frac{\partial}{\partial \theta_{j'k'}} \left( -\frac{2}{n} \sum_{i=1}^{n} \left( (y_i - \gamma^\top \Theta x_i)(\Theta x_i)_j \right) \right)
\]

\[
= \frac{2}{n} \sum_{i=1}^{n} \left( (\Theta x_i)_j \frac{\partial}{\partial \theta_{j'k'}}(y_i - \gamma^\top \Theta x_i) + (y_i - \gamma^\top \Theta x_i) \frac{\partial}{\partial \theta_{j'k'}}(\Theta x_i)_j \right)
\]

\[
= \frac{2}{n} \gamma_j \sum_{i=1}^{n} ((x_i)_k(x_i)_k),
\]

as desired.

6.2.6 **Proof of Lemma 6**

**Proof** The proof for this lemma follows the same steps as in Lemma 5, just sums are replaced by expectations and so we omit the proof.
6.2.7 Proof of Lemma 7

**Proof** We start the proof with Hölder’s inequality and the definition of $C_{\eta,\epsilon}$, which implies $\|\beta^* - \beta\|_1 \leq \eta$ for all $\beta \in C_{\eta,\epsilon}$ to obtain

$$
\sup_{\beta = \operatorname{vec}(\gamma, \Theta) \in C_{\eta,\epsilon}} \left\| \nabla \text{risk}_X [\gamma, \Theta] - \nabla \text{risk}[\gamma, \Theta] \right\|^\top (\beta^* - \beta) \\
\leq \sup_{\beta = \operatorname{vec}(\gamma, \Theta) \in C_{\eta,\epsilon}} \left( \left\| \nabla \text{risk}_X [\gamma, \Theta] - \nabla \text{risk}[\gamma, \Theta] \right\|_\infty \|\beta^* - \beta\|_1 \right) \\
\leq \eta \sup_{\beta = \operatorname{vec}(\gamma, \Theta) \in C_{\eta,\epsilon}} \left\| \nabla \text{risk}_X [\gamma, \Theta] - \nabla \text{risk}[\gamma, \Theta] \right\|_\infty .
$$

The rest of the proof is using our Lemma 5 and Bernstein’s inequality (Vershynin, 2018, Corollary 2.8.3) to find an upper bound for $\sup_{\beta = \operatorname{vec}(\gamma, \Theta) \in C_{\eta,\epsilon}} \left\| \nabla \text{risk}_X [\gamma, \Theta] - \nabla \text{risk}[\gamma, \Theta] \right\|_\infty$. Note that for simplifying the notation, we use $E[\cdot]$ as a shorthand notation of $E_{(x_1, y_1), \ldots, (x_n, y_n)}[\cdot]$ throughout this proof.

We use our result in Lemma 5 and i.i.d assumption on the data, 2. Equation (1) and our assumption that $\theta = \gamma^\top \Theta^\star x$, zero-mean noise, linearity of expectations, and factorizing, 3. the definition of sup-norm, triangle inequality, and Hölder’s inequality, 4. the definition of $C_{\eta,\epsilon}$, which implies $\|\gamma^\top \Theta^* - \gamma^\top \Theta\|_1 \leq \epsilon$, 5. adding a zero-valued term and rewriting, and 6. the triangle inequality and the definition of $C_{\eta,\epsilon}$, which implies $\|\gamma - \gamma^*\|_1 \leq \|\beta - \beta^*\|_1 \leq \eta$, to obtain for each $j \in \{1, \ldots, w\}$ and $k \in \{1, \ldots, d\}$ that

$$
\left| \frac{\partial}{\partial \theta_{jk}} \text{risk}_X [\gamma, \Theta] - \frac{\partial}{\partial \theta_{jk}} \text{risk}[\gamma, \Theta] \right| \\
= -2 \frac{n}{n} \sum_{i=1}^n (y_i - \gamma^\top \Theta x_i) \gamma_j(x_i)_k + E \left[ \frac{2}{n} \sum_{i=1}^n (y_i - \gamma^\top \Theta x_i) \gamma_j(x_i)_k \right] \\
= 2 |\gamma_j| \left[ \frac{1}{n} \sum_{i=1}^n u_i(x_i)_k + (\gamma^\top \Theta^* - \gamma^\top \Theta)(x_i(x_i)_k - E[x_i(x_i)_k]) \right] \\
\leq 2 |\gamma| \left( \frac{1}{n} \sum_{i=1}^n u_i(x_i)_k \right) + \|\gamma^\top \Theta - \gamma^\top \Theta^*\|_1 \left[ \frac{1}{n} \sum_{i=1}^n (E[x_i(x_i)_k] - x_i(x_i)_k) \right] \|_\infty \\
\leq 2 |\gamma| \left( \frac{1}{n} \sum_{i=1}^n u_i(x_i)_k \right) + \epsilon \left[ \frac{1}{n} \sum_{i=1}^n (E[x_i(x_i)_k] - x_i(x_i)_k) \right] \|_\infty \\
= 2 |\gamma - \gamma^\star \|_\infty \left( \frac{1}{n} \sum_{i=1}^n u_i(x_i)_k \right) + \epsilon \left[ \frac{1}{n} \sum_{i=1}^n (x_i(x_i)_k - E[x_i(x_i)_k]) \right] \|_\infty \\
\leq 2 (\eta + |\gamma^\star\|_\infty \left( \frac{1}{n} \sum_{i=1}^n u_i(x_i)_k \right) + \epsilon \left[ \frac{1}{n} \sum_{i=1}^n (x_i(x_i)_k - E[x_i(x_i)_k]) \right] \|_\infty \right).
$$

We continue to work on the absolute value and sup-norm term in the last inequality above separately. For each $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, d\}$, we use our assumptions on $x_i$ and $u_i$ to obtain that $z_i := u_i(x_i)_k$ are independent and sub-exponential random variables with zero-mean (Vershynin, 2018, Lemma 2.7.7) and so, we can employ Bernstein’s inequality in Vershynin (2018, Corollary 2.8.3) to obtain for each $t \in [0, \infty)$ that

$$
\mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^n u_i(x_i)_k \right| \geq t \right) \leq 2 e^{-\kappa \min\{t^2/\nu^2, t/\nu\} n}
$$

with $\kappa \in (0, \infty)$ an absolute constant and $\nu := \max_{i \in \{1, \ldots, n\}} \|u_i(x_i)_k\|_{\psi_1} \in (0, \infty)$ a constant that depends on the distributions of $x$ and $u$ (for a sub-exponential random variable $z$, we define $\|z\|_{\psi_1} := \inf\{q \in [0, \infty) : Ee^{\|z\|/q} \leq 2\}$).
Now we study the behavior of the sup-norm term in the last inequality of the earlier display. Let rewrite the sup-norm in the form of a max as

\[ \left\| \frac{1}{n} \sum_{i=1}^{n} (x_i(x_i)_k - E[(x_i(x_i)_k)] \right\|_\infty = \max_{k' \in \{1, \ldots, d\}} \left\| \frac{1}{n} \sum_{i=1}^{n} ((x_i)_k(x_i)_k - E[(x_i)_k(x_i)_k]) \right\| \]

Following the same argument as earlier and for each \( i \in \{1, \ldots, n\} \) and \( k, k' \in \{1, \ldots, d\} \), we use our assumption on \( x_i \) to obtain that \( z'_i := (x_i)_k(x_i)_k - E[(x_i)_k(x_i)_k] \) are independent sub-exponential random variables with zero-mean and again we can employ Bernstein’s inequality (Vershynin, 2018, Corollary 2.8.3) to obtain for each \( t' \in [0, \infty) \) that

\[ \mathbb{P} \left( \left\| \frac{1}{n} \sum_{i=1}^{n} ((x_i)_k(x_i)_k - E[(x_i)_k(x_i)_k]) \right\| \geq t' \right) \leq 2e^{-\kappa' \min\{t^2/\nu^2, t'/\nu'\} n} \]

with \( \kappa' \in (0, \infty) \) an absolute constant and \( \nu' := \max_{i \in \{1, \ldots, n\}} \left\| (x_i)_k(x_i)_k - E[(x_i)_k(x_i)_k] \right\|_{\psi_1} \in (0, \infty) \) a constant that depends on the distribution of \( x \).

Then, we use our result above together with the fact that if \( \mathbb{P}(|b_i| \geq t) \leq a \) holds for all \( i \in \{1, \ldots, p\} \), then we also have \( \mathbb{P}(\max_{i \in \{1, \ldots, p\}} |b_i| \geq t) \leq pa \) to obtain

\[ \mathbb{P} \left( \max_{k' \in \{1, \ldots, d\}} \left\| \frac{1}{n} \sum_{i=1}^{n} ((x_i)_k(x_i)_k - E[(x_i)_k(x_i)_k]) \right\| \geq t' \right) \leq 2de^{-\kappa' \min\{t^2/\nu^2, t'/\nu'\} n} \]

Collecting all pieces above together with considering \( t = t' \), we obtain for each \( j \in \{1, \ldots, w\} \) and \( k \in \{1, \ldots, d\} \) that

\[ \left| \frac{\partial}{\partial \mathbf{\gamma}} \text{risk}_\mathbf{X} [\mathbf{\gamma}, \mathbf{\Theta}] - \frac{\partial}{\partial \mathbf{\psi}} \text{risk}[\mathbf{\gamma}, \mathbf{\Theta}] \right| \leq 2t(n + \|\mathbf{\gamma}^*\|_\infty)(1 + \epsilon) \]

with probability at least \( 1 - 2e^{-\kappa \min\{t^2/\nu^2, t'/\nu'\} n} - 2de^{-\kappa' \min\{t^2/\nu^2, t'/\nu'\} n} \), which is obtained using the fact that

\[ \mathbb{P}(A + bD \leq t + bt) = 1 - \mathbb{P}(A + bD > t + bt) \geq 1 - \mathbb{P}(A > t) - \mathbb{P}(D > t) \]

for any \( b \in (0, \infty) \) and \( t \in \mathbb{R} \).

Then, we follow the same argument as earlier and use 1. our result in Lemma 5 and i.i.d assumption on the data, 2. the properties of absolute values and linearity of expectations, 3. some rewriting, 4. Hölder’s inequality, 5. Equation (1) and our assumptions that \( f[\mathbf{x}] = \mathbf{\gamma}^\top \mathbf{\Theta} \mathbf{x} \), zero-mean noise, and definition of sup-norm, 6. triangle inequality, compatible norms (for a matrix \( A \in \mathbb{R}^{d \times d} \), we define \( \|A\|_{\infty,1} := \max_{k \in \{1, \ldots, d\}} \| \sum_{k'=1}^{d} \| A_{k', k} \| \), and the definition of \( \mathcal{C}_{\eta, \epsilon} \), which implies \( \| \mathbf{\gamma}^\top \mathbf{\Theta}^* - \mathbf{\gamma}^\top \mathbf{\Theta} \| \leq \epsilon \), 7. adding a zero-valued term, 8. the triangle inequality and the definition of \( \mathcal{C}_{\eta, \epsilon} \), which implies \( \| \mathbf{\Theta} - \mathbf{\Theta}^* \|_1 \leq \| \mathbf{\Theta} - \mathbf{\Theta}^* \|_1 \leq \eta \) to obtain for each \( j \in \{1, \ldots, w\} \) that

\[ \left| \frac{\partial}{\partial \mathbf{\gamma}_j} \text{risk}_\mathbf{X} [\mathbf{\gamma}, \mathbf{\Theta}] - \frac{\partial}{\partial \mathbf{\psi}_j} \text{risk}[\mathbf{\gamma}, \mathbf{\Theta}] \right| = \left| -\frac{2}{n} \sum_{i=1}^{n} ((y_i - \mathbf{\gamma}^\top \mathbf{\Theta} x_i)(\mathbf{\Theta} x_i)_j) + E \left[ \frac{2}{n} \sum_{i=1}^{n} ((y_i - \mathbf{\gamma}^\top \mathbf{\Theta} x_i)(\mathbf{\Theta} x_i)_j) \right] \right| = \frac{2}{n} \sum_{i=1}^{n} ((y_i - \mathbf{\gamma}^\top \mathbf{\Theta} x_i)(\mathbf{\Theta} x_i)_j - E[(y_i - \mathbf{\gamma}^\top \mathbf{\Theta} x_i)(\mathbf{\Theta} x_i)_j]) \]
Then, we use the same argument as earlier to treat the sup-norm terms above (we use our assumptions on $x_i$ and $u_i$ and application of Bernstein’s inequality) to obtain that

$$\left| \frac{\partial}{\partial \gamma_j} \text{risk}_X[\gamma, \Theta] - \frac{\partial}{\partial \gamma_j} \text{risk}[\gamma, \Theta] \right| \leq 2(\eta + \|\Theta^*\|_{\infty})(1 + \epsilon)$$

with probability at least $1 - 2de^{-\kappa \min\{t^2/\nu^2, t/\nu\}n} - 2d^2e^{-\kappa' \min\{t^2/\nu^2, t/\nu\}n}$ ($\kappa, \nu, \kappa', \nu'$ are constants depending only on the distributions of the inputs and the noises).

Collecting all the pieces above, we obtain that for each $i \in \{1, \ldots, p\}$ the corresponding gradient difference is bounded

$$\left| (\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta])_i \right| \leq 2t(\eta + \max\{|\gamma^*|_{\infty}, \|\Theta^*\|_{\infty}\})(1 + \epsilon)$$

with probability at least $1 - 4d^2e^{-\kappa_{u,x} \min\{t^2/(\nu_{u,x})^2, t/\nu_{u,x}\}n}$ with $\nu_{u,x} := \max\{\nu, \nu'\}$ and $\kappa_{u,x} := \min\{\kappa, \kappa'\}$ ($\nu_{u,x}$ and $\kappa_{u,x}$ are constants depending only on the distributions of the inputs and noises).

Now we use 1. the definition of sup-norm and 2. our results above together with our earlier argument about implying max operator (note that the gradient vector is of dimension $p$) to obtain for each $t \in [0, \infty)$ that

$$\sup_{\beta = \text{vec}((\gamma, \Theta)) \in C_{\gamma, \Theta}} \|\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta]\|_{\infty} = \sup_{\beta = \text{vec}((\gamma, \Theta)) \in C_{\gamma, \Theta}} \max_{i \in \{1, \ldots, p\}} \left| (\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta])_i \right|$$

$$\leq 2t(\eta + \max\{|\gamma^*|_{\infty}, \|\Theta^*\|_{\infty}\})(1 + \epsilon)$$

with probability at least $1 - 4d^2pe^{-\kappa_{u,x} \min\{t^2/(\nu_{u,x})^2, t/\nu_{u,x}\}n}$.

Collecting all the pieces of the proof, we obtain for each $t \in [0, \infty)$ that

$$\sup_{\beta = \text{vec}((\gamma, \Theta)) \in C_{\gamma, \Theta}} \left| (\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta])^T(\beta^* - \beta) \right|$$

$$\leq \eta \sup_{\beta = \text{vec}((\gamma, \Theta)) \in C_{\gamma, \Theta}} \|\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta]\|_{\infty}$$

$$\leq 2t(\eta + \max\{|\gamma^*|_{\infty}, \|\Theta^*\|_{\infty}\})(1 + \epsilon)$$

with probability at least $1 - 4d^2pe^{-\kappa_{u,x} \min\{t^2/(\nu_{u,x})^2, t/\nu_{u,x}\}n}$, where for the ease of notations we replace $\kappa_{u,x}$ and $\nu_{u,x}$ with $\nu$ and $\kappa$ (constants depending only on the distributions of the inputs and noises) in the statement of the lemma.

$\blacksquare$
6.2.8 Proof of Lemma 8

**Proof**  Main ingredients of the proof are symmetrization of probabilities (van de Geer, 2016, Lemma 16.1) and Bernstein’s inequality (Vershynin, 2018, Corollary 2.8.3).

We note that for simplifying the notations, we use $E[\cdot]$ as a shorthand notation of $E_{(x,y)\sim (x_n,y_n)}[\cdot]$ throughout this proof.

Let start the proof and use 1. the definition of risk $\gamma,\Theta$ and risk $\gamma,\Theta$, 2. the i.i.d assumption on the data and that $y_i = \gamma^\top \Theta^* x_i + u_i$, 3. expanding the squared-terms and rearranging, and 4. the triangle inequality to obtain

$$\sup_{(\gamma,\Theta) \in B} \left| \text{risk}_x[\gamma,\Theta] - \text{risk}[\gamma,\Theta] \right|$$

$$= \sup_{(\gamma,\Theta) \in B} \left| \frac{1}{n} \sum_{i=1}^{n} \left( (y_i - \gamma^\top \Theta x_i)^2 - E_{(x,y)} \left[ (y - \gamma^\top \Theta x)^2 \right] \right) \right|$$

$$= \sup_{(\gamma,\Theta) \in B} \left| \frac{1}{n} \sum_{i=1}^{n} \left( (\gamma^\top \Theta^* x_i + u_i - \gamma^\top \Theta x_i)^2 - E \left[ (\gamma^\top \Theta^* x_i + u_i - \gamma^\top \Theta x_i)^2 \right] \right) \right|$$

$$= \sup_{(\gamma,\Theta) \in B} \left| \frac{1}{n} \sum_{i=1}^{n} \left( (\gamma^\top \Theta^* x_i - \gamma^\top \Theta x_i)^2 - E \left[ (\gamma^\top \Theta^* x_i - \gamma^\top \Theta x_i)^2 \right] \right) \right|$$

$$+ 2 \left( (\gamma^\top \Theta^* x_i - \gamma^\top \Theta x_i) u_i - E \left[ (\gamma^\top \Theta^* x_i - \gamma^\top \Theta x_i) u_i \right] \right) + \left( u_i^2 - E[u_i^2] \right).$$

$$\leq \sup_{(\gamma,\Theta) \in B} \left| \frac{1}{n} \sum_{i=1}^{n} \left( (\gamma^\top \Theta^* x_i - \gamma^\top \Theta x_i)^2 - E \left[ (\gamma^\top \Theta^* x_i - \gamma^\top \Theta x_i)^2 \right] \right) \right|$$

$$+ 2 \sup_{(\gamma,\Theta) \in B} \left| \frac{1}{n} \sum_{i=1}^{n} \left( (\gamma^\top \Theta^* x_i - \gamma^\top \Theta x_i) u_i - E \left[ (\gamma^\top \Theta^* x_i - \gamma^\top \Theta x_i) u_i \right] \right) \right|$$

$$+ \left| \frac{1}{n} \sum_{i=1}^{n} u_i^2 - E[u_i^2] \right|. $$

Now, we continue to work on each term in the last inequality above separately in steps:

**Step 1:** Using Vershynin (2018, Corollary 2.8.3) together with our assumption on noise, which implies the squared of Gaussian noise is sub-exponential, we obtain for each $\tilde{t} \in [0, \infty)$ that

$$\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} \left( u_i^2 - E[u_i^2] \right) \geq \tilde{t} \right) \leq 2 e^{-\kappa \min\{\tilde{t}, \tilde{t}/\nu^2, \tilde{t}/\nu \} n},$$

where $\kappa, \nu \in (0, \infty)$ are constants depending only on the distribution of the noise (let note that our constants $\kappa$ and $\nu$ may change from line to line in this proof, but they constantly depend just on the distribution of the inputs or noise or both).

**Step 2:** We now prepare the application of van de Geer (2016 Lemma 16.1). Let 1. define $R^2$ and 2. use Hölder’s inequality and factorizing to obtain

$$R^2 := \sup_{(\gamma,\Theta) \in B} \frac{1}{n} \sum_{i=1}^{n} E \left[ (\gamma^\top \Theta^* x_i - \gamma^\top \Theta x_i)^4 \right]$$

$$\leq \sup_{(\gamma,\Theta) \in B} \left\| \gamma^\top \Theta^* - \gamma^\top \Theta \right\|_1^4 \frac{1}{n} \sum_{i=1}^{n} E \left[ \|x_i\|_\infty^4 \right].$$

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We also employ some linear algebra together with compatible norms (for a matrix \( A \in \mathbb{R}^{d \times d} \), we define \( \| A \|_{\infty,1} := \max_{k \in \{1, \ldots, d\}} \sum_{k'=1}^{d} |A_{k',k}| \)) to obtain

\[
\left| \frac{1}{n} \sum_{i=1}^{n} \zeta_i (\gamma^T \Theta^* x_i - \gamma^T \Theta x_i)^2 \right| = \frac{1}{n} \sum_{i=1}^{n} (\gamma^T \Theta^* x_i - \gamma^T \Theta x_i) \zeta_i \left( (\gamma^T \Theta^* x_i - \gamma^T \Theta x_i)^T \right) \\
= \frac{1}{n} \sum_{i=1}^{n} (\gamma^T \Theta^* - \gamma^T \Theta) x_i \zeta_i x_i^T (\gamma^T \Theta^* - \gamma^T \Theta) \\
\leq \| (\gamma^T \Theta^* - \gamma^T \Theta)^2 \|_{\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} \zeta_i x_i x_i^T \right\|_{\infty,1}.
\]

Then, we use 1. symmetrization of probabilities (van de Geer, 2016, Lemma 16.1) with \( \mathcal{R} \) as defined earlier, 2. the display above, 3. our assumption that \( \sup_{(\gamma, \Theta) \in \mathcal{B}} \| (\gamma^T \Theta^* - \gamma^T \Theta)^2 \|_{\infty} \leq c' \) and some rearranging, 4. the definition of \( \ell_{\infty,1} \)-norm for a matrix above, 5. the fact that if \( \mathbb{P}(|b_i| \geq t) \leq a \) holds for all \( i \in \{1, \ldots, d\} \), then we also have \( \mathbb{P}(\max_{i \in \{1, \ldots, d\}} |b_i| \geq t) \leq da \) (for \( k \in \{1, \ldots, d\} \)), 6. the fact that for a vector \( a \in \mathbb{R}^d \), \( \mathbb{P}(\sum_{i=1}^{d} |a_i| \geq t) \leq d \max_{k \in \{1, \ldots, d\}} \mathbb{P}(|a_k| \geq t) \), and 7. our assumption on \( x \) (to get rid of max term) together with Vershynin (2018, Corollary 2.8.3) to obtain for each \( t \in [0, \infty) \) that

\[
\mathbb{P} \left( \sup_{(\gamma, \Theta) \in \mathcal{B}} \left| \frac{1}{n} \sum_{i=1}^{n} \left( (\gamma^T \Theta^* x_i - \gamma^T \Theta x_i)^2 - \mathbb{E} \left[ (\gamma^T \Theta^* x_i - \gamma^T \Theta x_i)^2 \right] \right) \right| \geq 4 \mathcal{R} \sqrt{\frac{2t}{n}} \right) \\
\leq 4 \mathbb{P} \left( \sup_{(\gamma, \Theta) \in \mathcal{B}} \left| \frac{1}{n} \sum_{i=1}^{n} \zeta_i (\gamma^T \Theta^* x_i - \gamma^T \Theta x_i)^2 \right| \geq \mathcal{R} \sqrt{\frac{2t}{n}} \right) \\
\leq 4 \mathbb{P} \left( \sup_{(\gamma, \Theta) \in \mathcal{B}} \left\| (\gamma^T \Theta^* - \gamma^T \Theta)^2 \right\|_{\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} \zeta_i x_i x_i^T \right\|_{\infty,1} \geq \mathcal{R} \sqrt{\frac{2t}{n}} \right) \\
\leq 4 \mathbb{P} \left( \left\| \frac{1}{n} \sum_{i=1}^{n} \zeta_i x_i x_i^T \right\|_{\infty,1} \geq \mathcal{R} \sqrt{\frac{2t}{n}} \right) \\
\leq 4 \mathbb{P} \left( \max_{k \in \{1, \ldots, d\}} \sum_{k'=1}^{d} \left| \frac{1}{n} \sum_{i=1}^{n} \zeta_i (x_i)_k (x_i)_{k'} \right| \geq \frac{\mathcal{R}}{c} \sqrt{\frac{2t}{n}} \right) \\
\leq 4d \mathbb{P} \left( \sum_{k'=1}^{d} \left| \frac{1}{n} \sum_{i=1}^{n} \zeta_i (x_i)_{k'} (x_i)_{k'} \right| \geq \frac{\mathcal{R}}{c} \sqrt{\frac{2t}{n}} \right) \\
\leq 4d^2 \max_{k' \in \{1, \ldots, d\}} \mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n} \zeta_i (x_i)_{k'} (x_i)_{k'} \right| \geq \frac{\mathcal{R}}{c} \sqrt{\frac{2t}{n}} \right) =: t'' \right) \\
\leq 8d^2 e^{-\kappa \min \{t'''/\nu^2, t''/\nu\} n}.
\]

where \( \kappa, \nu \in (0, \infty) \) are constants depending only on the distribution of the inputs.

Collecting results above, we obtain for each \( t'' \in [0, \infty) \) that

\[
\mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n} \left( (\gamma^T \Theta^* x_i - \gamma^T \Theta x_i)^2 - \mathbb{E} \left[ (\gamma^T \Theta^* x_i - \gamma^T \Theta x_i)^2 \right] \right) \right| \geq 4 \epsilon t'' \right) \\
\leq 8d^2 e^{-\kappa \min \{t'''/\nu^2, t''/\nu\} n}.
\]
Step 3: Let define \((\mathcal{R}')^2\) and use Hölder’s inequality to obtain
\[
(\mathcal{R}')^2 := \sup_{(\gamma, \Theta) \in \mathcal{B}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\left(\left(\gamma^* \Theta^* x_i - \gamma \Theta x_i\right) u_i\right)^2\right]
\]
\[
\leq \sup_{(\gamma, \Theta) \in \mathcal{B}} \|\gamma^* \Theta^* - \gamma \Theta\|_1^2 \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[|x_i u_i|^\frac{2}{3}\right].
\]

Then, we use 1. symmetrization of probabilities (van de Geer, 2016, Lemma 16.1) with \(\mathcal{R}'\) defined as above, 2. Hölder’s inequality, 3. our assumption that \(\sup_{(\gamma, \Theta) \in \mathcal{B}} \|\gamma^* \Theta^* - \gamma \Theta\|_\infty \leq \epsilon'\), the fact that for a vector \(a \in \mathbb{R}^d\), \(\mathbb{P}(|a|_1 \geq t) \leq d \max_{i \in \{1, \ldots, d\}} \mathbb{P}(|a_i| \geq t) \leq d^2 \mathbb{P}(|a_i| \geq t)\), and the assumption on inputs (for \(k \in \{1, \ldots, d\}\)), and 4. Vershynin (2018, Corollary 2.8.3) together with our assumptions on the input and noise to obtain for each \(t' \in [0, \infty)\) that
\[
\mathbb{P}\left(\sup_{(\gamma, \Theta) \in \mathcal{B}} \frac{1}{n} \sum_{i=1}^{n} \left|\left(\gamma^* \Theta^* x_i - \gamma \Theta x_i\right) u_i - \mathbb{E}\left[\left(\gamma^* \Theta^* x_i - \gamma \Theta x_i\right) u_i\right]\right| \geq 4\mathcal{R}' \sqrt{\frac{2t'}{n}}\right)
\]
\[
\leq 4\mathbb{P}\left(\sup_{(\gamma, \Theta) \in \mathcal{B}} \frac{1}{n} \sum_{i=1}^{n} \zeta_i \left(\gamma^* \Theta^* x_i - \gamma \Theta x_i\right) u_i \geq \mathcal{R}' \sqrt{\frac{2t'}{n}}\right)
\]
\[
\leq 4\mathbb{P}\left(\sup_{(\gamma, \Theta) \in \mathcal{B}} \left\|\gamma^* \Theta^* - \gamma \Theta\right\|_{\infty} \frac{1}{n} \sum_{i=1}^{n} \zeta_i |x_i u_i| \geq \mathcal{R}' \sqrt{\frac{2t'}{n}}\right)
\]
\[
\leq 4d^2 \mathbb{P}\left(\frac{1}{n} \sum_{i=1}^{n} \zeta_i (x_i)_k u_i \geq \mathcal{R}' \sqrt{\frac{2t'}{\epsilon' n}}\right)
\]
\[
\leq 8d^2 e^{-\kappa \nu \min(t''', t''/\nu)} n,
\]

where \(\kappa, \nu \in (0, \infty)\) are constants depending only on the distributions of the inputs and noise.

Collecting results above we obtain that
\[
\mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^{n} \left(\gamma^* \Theta^* x_i - \gamma \Theta x_i\right) u_i - \mathbb{E}\left[\left(\gamma^* \Theta^* x_i - \gamma \Theta x_i\right) u_i\right]\right| \geq 4\sqrt{\epsilon' t'''}\right)
\]
\[
\leq 8d^2 e^{-\kappa \nu \min(t''', t''/\nu)} n,
\]

where \(\kappa, \nu \in (0, \infty)\) are constants depending only on the distributions of the inputs and noise.

Collecting all the pieces of the proof in steps 1.3, we obtain for each \(t \in [0, \infty)\) that
\[
\sup_{(\gamma, \Theta) \in \mathcal{B}} \left|\text{risk}_X[\gamma, \Theta] - \text{risk}[\gamma, \Theta]\right| \leq t \left(1 + 4\epsilon' + 4\sqrt{\epsilon'}\right)
\]
with probability at least \(1 - (2 + 8d^2 + 8d^2) e^{-\kappa \nu \min(t''', t''/\nu)} n\) or \(1 - 18d^2 e^{-\kappa \nu \min(t''/\nu)} n\) (using the assumption that \(d \geq 1\)), where we consider \(t = t = t'' = t'''\) and \(\kappa, \nu \in (0, \infty)\) are constants depending only on the distributions of the inputs and noise.

6.2.9 Proof of Lemma 9

Proof The proof follows just basic linear algebra.

Since \(H(t)\) is invertible exactly when \((A + tC)^\top\) have full (column) rank, we are left to study the rank of \((A + tC)^\top = A^\top + tC^\top\). To do so, we employ the Singular Value Decomposition (SVD) of \(A^\top \in \mathbb{R}^{d \times w'}\), that is, \(A^\top = UDV^\top\) with \(U \in \mathbb{R}^{d \times w'}\), \(V \in \mathbb{R}^{w' \times w'}\), and \(D \in \mathbb{R}^{w' \times w'}\) that \(U, V\)
Figure 3: Training error for trained neural networks (with \( d = w = 10 \)) with linear (left panel) and ReLU (right panel) activations in 5 different runs (allocated with different colors). Due to the non-convexity of neural networks, optimization algorithm may end up in different approximate stationary points.

are semi-orthogonal matrices and \( D \) has the same rank as \( A \), in this case full rank. Now, we are motivated to make a squared matrix as

\[
U^T(A^T + tC^T)V = U^T(UDV^T + tC^T)V = D + tU^T C^T V = tD(t^{-1}I_{w'} + D^{-1}U^T C^T V),
\]

where we used the SVD form of matrix \( A \), orthogonal property of \( U, V \), and some rewriting. Since matrices \( U \) and \( V \) have rank \( w \), for studying the rank of \( A^T + tC^T \) it is enough to study determinant of \( U^T(A^T + tC^T)V \). We then use our display above, properties of determinants for squared matrices, and characteristic polynomial to obtain

\[
\det(U^T(A^T + tC^T)V) = \det(D(t^{-1}I_{w'} + D^{-1}U^T C^T V)) = \det(D)\det(t^{-1}I_{w'} + D^{-1}U^T C^T V)
\]

Since, \( \det(D) \neq 0 \) and \( t \neq 0 \), then the \( t \) which \( H(t) \) is singular are the roots of \( p_Z(-t^{-1}) \), where \( Z = D^{-1}U^T C^T V \). Since the roots of \( p_Z \) are the eigenvalues of \( Z \), we have found that the only \( t \) for which \( H(t) \) fails to be invertible are the negative reciprocals of the (nonzero) eigenvalues of \( Z \). Since, any \( w' \times w' \) matrix has at most \( w' \) distinct eigenvalues, there are just finitely many \( t \) such that \( H(t) \) is not invertible, as desired.

\[\square\]

7. Discussion

This paper establishes statistical guarantees for approximate stationary points of regularized neural networks. Despite being limited to linear networks, the theory is a considerable step forward in three ways: First, previous empirical studies show that the optimization theory of linear-neural networks may exhibit some properties of non-linear models as well (Saxe et al., 2013), which means that our theories should also give some insights into non-linear networks (see also our Section 3). Second, several papers consider the existence or non-existence of spurious local minima, but importantly, our theory also applies to saddle points, which can be also hard to escape from in optimization. And third, our new statistical approach inspired by high-dimensional statistics will hopefully spark further progress in the mathematical understanding of deep learning.
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References

R. Adamczak. A tail inequality for suprema of unbounded empirical processes with applications to markov chains. *Electron. J. Probab.*, 13:1000–1034, 2008.

J. Alvarez and M. Salzmann. Learning the number of neurons in deep networks. In *Adv. Neural Inf. Process Syst.*, pages 2270–2278, 2016.

P. Bartlett. The sample complexity of pattern classification with neural networks: The size of the weights is more important than the size of the network. *IEEE Trans. Inform. Theory*, 44(2):525–536, 1998.

P. Bartlett and S. Mendelson. Rademacher and Gaussian complexities: Risk bounds and structural results. *J. Mach. Learn. Res.*, 3:463–482, 2002.

B. Bauer and M. Kohler. On deep learning as a remedy for the curse of dimensionality in nonparametric regression. *Ann. Statist.*, 47(4):2261–2285, 2019.

A. Beknazaryan. Neural networks with superexpressive activations and integer weights. *arXiv:2105.09917*, 2021.

D. Bertsekas, A. Nedic, and A. Ozdaglar. *Convex analysis and optimization*, volume 1. Athena Scientific, 2003.

J. Bien, I. Gaynanova, J. Lederer, and C. Müller. Non-convex global minimization and false discovery rate control for the trex. *J. Comput. Graph. Statist.*, 27(1):23–33, 2018.

J. Bien, I. Gaynanova, J. Lederer, and C. Müller. Prediction error bounds for linear regression with the trex. *Test*, 28(2):451–474, 2019.

M. Denil, B. Shakibi, L. Dinh, M. Ranzato, and N. De Freitas. Predicting parameters in deep learning. In *Adv. Neural Inf. Process Syst.*, pages 2148–2156, 2013.

E. Denton, W. Zaremba, J. Bruna, Y. LeCun, and R. Fergus. Exploiting linear structure within convolutional networks for efficient evaluation. In *Adv. Neural Inf. Process Syst.*, pages 1269–1277, 2014.

A. Eftekhari. Training linear neural networks: Non-local convergence and complexity results. In *ICML*, pages 2836–2847, 2020.

Y. Eldar and G. Kutyniok. *Compressed sensing: theory and applications*. Cambridge Univ. Press, 2012.

A. Elsener and S. van de Geer. Sharp oracle inequalities for stationary points of nonconvex penalized m-estimators. *IEEE Trans. Inform. Theory*, 65(3):1452–1472, 2018.

J. Feng and N. Simon. Sparse-input neural networks for high-dimensional nonparametric regression and classification. *arXiv:1711.07592*, 2017.

M. Hardt and T. Ma. Identity matters in deep learning. *arXiv:1611.04231*, 2016.

G. Hinton, L. Deng, D. Yu, G. Dahl, A. Mohamed, N. Jaitly, A. Senior, V. Vanhoucke, P. Nguyen, T. Sainath, and B. Kingsbury. Deep neural networks for acoustic modeling in speech recognition: The shared views of four research groups. *IEEE Signal Process. Mag.*, 29(6):82–97, 2012.
S. Ioffe and C. Szegedy. Batch normalization: Accelerating deep network training by reducing internal covariate shift. In ICML, pages 448–456. PMLR, 2015.

M. Jaderberg, A. Vedaldi, and A. Zisserman. Speeding up convolutional neural networks with low rank expansions. arXiv:1405.3866, 2014.

K. Kawaguchi. Deep learning without poor local minima. arXiv:1605.07110, 2016.

M. Kohler and S. Langer. On the rate of convergence of fully connected deep neural network regression estimates. Ann. Statist., 49(4):2231–2249, 2021.

M. Kohler, S. Langer, and U. Reif. Estimation of a regression function on a manifold by fully connected deep neural networks. arXiv:2107.09532, 2021.

J. Lederer. Risk bounds for robust deep learning. arXiv:2009.06202, 2020a.

J. Lederer. No spurious local minima: on the optimization landscapes of wide and deep neural networks. b, 2020b.

J. Lederer. Optimization landscapes of wide deep neural networks are benign. arXiv:2010.00885, 2020c.

J. Lederer. Fundamentals of High-Dimensional Statistics. Springer Texts in Statistics, 2022.

J. Lederer and S. van de Geer. New concentration inequalities for suprema of empirical processes. Bernoulli, 20(4):2020–2038, 2014.

Y. Lei, T. Hu, G. Li, and K. Tang. Stochastic gradient descent for nonconvex learning without bounded gradient assumptions. IEEE Trans. Neural Netw. Learn. Syst., 31(10):4394–4400, 2019.

B. Liu, M. Wang, H. Foroosh, M. Tappen, and M. Pensky. Sparse convolutional neural networks. In IEEE Int. Conf. Comput. Vis. Pattern Recognit., pages 806–814, 2015.

P. Loh. Statistical consistency and asymptotic normality for high-dimensional robust m-estimators. Ann. Statist., 45(2):866–896, 2017.

P. Loh and M. Wainwright. Regularized m-estimators with nonconvexity: Statistical and algorithmic theory for local optima. JMLR, 16(1):559–616, 2015.

X. Ma, S. Sardy, N. Hengartner, N. Bonenko, and Y. Lin. A phase transition for finding needles in nonlinear haystacks with lasso artificial neural networks. arXiv:2201.08652, 2022.

S. Mei, Y. Bai, and A. Montanari. The landscape of empirical risk for nonconvex losses. Ann. Statist., 46(6A):2747–2774, 2018.

D. Molchanov, A. Ashukha, and D. Vetrov. Variational dropout sparsifies deep neural networks. arXiv:1701.05369, 2017.

K. Neklyudov, D. Molchanov, A. Ashukha, and D. Vetrov. Structured bayesian pruning via log-normal multiplicative noise. In Adv. Neural Inf. Process Syst., pages 6775–6784, 2017.

H. Salehinejad and S. Valaee. Ising-dropout: a regularization method for training and compression of deep neural networks. In ICASSP, pages 3602–3606. IEEE, 2019.

S. Sardy, N. Hengartner, N. Bonenko, and Y. Lin. What needles do sparse neural networks find in nonlinear haystacks. arXiv:2006.04041, 2020.

A. Saxe, J. McClelland, and S. Ganguli. Dynamics of learning in deep linear neural networks. In Adv. Neural Inf. Process Syst., 2013.
S. Scardapane, D. Commiello, A. Hussain, and A. Uncini. Group sparse regularization for deep neural networks. *Neurocomputing*, 241:81–89, 2017.

J. Schmidt-Hieber. Nonparametric regression using deep neural networks with relu activation function. *Ann. Statist.*, 48(4):1875–1897, 2020.

M. Taheri, N. Lim, and J. Lederer. Balancing statistical and computational precision and applications to penalized linear regression with group sparsity. *arXiv:1609.07195*, 2019.

M. Taheri, F. Xie, and J. Lederer. Statistical guarantees for regularized neural networks. *Neural Networks*, 142:148–161, 2021.

S. van de Geer. *Estimation and testing under sparsity*. Springer, 2016.

S. van de Geer and J. Lederer. The Bernstein–Orlicz norm and deviation inequalities. *Probab. Theory Related Fields*, 157:225–250, 2013.

R. Vershynin. *High-dimensional probability: An introduction with applications in data science*. Cambridge Univ. Press, 2018.

J. Wellner. The Bennett-Orlicz norm. *Sankhya A*, 79(2):355–383, 2017.

C. Yun, S. Sra, and A. Jadbabaie. Global optimality conditions for deep neural networks. *arXiv:1707.02444*, 2017.

Y. Zhang, J. Lee, and M. Jordan. $\ell_1$-regularized neural networks are improperly learnable in polynomial time. In *ICML*, pages 993–1001, 2016.

Y. Zhou and Y. Liang. Critical points of linear neural networks: Analytical forms and landscape properties. In *ICLR*, 2018.