THE KAKEYA MAXIMAL OPERATOR ON THE VARIABLE LEBESGUE SPACES

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Abstract. We shall verify the Kakeya (Nikodym) maximal operator $K_N$, $N \gg 1$, is bounded on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^2)$ when the exponent function $p(\cdot)$ is $N$-modified locally log-Hölder continuous and log-Hölder continuous at infinity.

1. Introduction

The purpose of this paper is to investigate the boundedness of the Kakeya (Nikodym) maximal operator on the variable Lebesgue spaces. Given a measurable function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, we define the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ to be the set of measurable functions such that for some $\lambda > 0$,

$$
\rho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \, dx < \infty.
$$

$L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space when equipped with the norm

$$
\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\}.
$$

The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ generalizes the classical Lebesgue space $L^p(\mathbb{R}^n)$: if $p(\cdot) \equiv p_0$, then $L^{p(\cdot)}(\mathbb{R}^n) = L^{p_0}(\mathbb{R}^n)$. Variable Lebesgue spaces have been studied in the past twenty years (see [1, 3, 4, 6, 7, 8, 9, 13, 14, 15]). For a locally integrable function $f$ on $\mathbb{R}^n$ the Hardy-Littlewood maximal operator $M$ is defined by

$$
Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy,
$$

where we have used $Q$ to denote the family of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes and $\int_Q f(x) \, dx$ to denote the usual integral average of $f$ over $Q$. Let $\mathcal{P}(\mathbb{R}^n)$ be the class of all functions $p(\cdot)$ for which the Hardy-Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. By the classical Hardy-Littlewood maximal theorem, any constant function $p(\cdot) \equiv p_0$ with $1 < p_0 < \infty$ belongs to $\mathcal{P}(\mathbb{R}^n)$. In [7], L. Diening showed that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ if and only if there exists a positive constant $c$ such that for any family of pairwise disjoint cubes $\pi$ and any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$
\left\| \sum_{Q \in \pi} \frac{1}{|Q|} \int_Q |f(y)| \, dy \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},
$$

where $\chi_E$ stands for the characteristic function of a measurable set $E \subset \mathbb{R}^n$. This result implies, for example, that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ if and only if $p'(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, where $p'(x) = \frac{p(x)}{p(x) - 1}$. However, since this result is very general, some simple sufficient conditions for which $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ has

2010 Mathematics Subject Classification. 42B25, 46E30.

Key words and phrases. Nikodym maximal operator; Kakeya maximal operator; variable Lebesgue spaces.

The second author is supported by the FMSP program at Graduate School of Mathematical Sciences, the University of Tokyo, and Grant-in-Aid for Scientific Research (C) (No. 23540187), the Japan Society for the Promotion of Science.
been studied by many authors (see [6, 3, 14, 15]). In [5], D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer give a new and simpler proof of the boundedness of the Hardy-Littlewood maximal operator $M$ on variable Lebesgue space $L^{p(·)}(\mathbb{R}^n)$.

**Theorem 1.3.** We have the following theorem.

Let $N \gg 1$ and $0 < p_− < p_+ < \infty$. Suppose that $p(·)$ is bounded from $L^{p(·)}(\mathbb{R}^2)$ to $L^{p(·)}(\mathbb{R}^2)$ and that $p(·)$ is continuous. Then there exist positive constants $c$ such that

$$
K_N \|f\|_{L^{p(·)}(\mathbb{R}^2)} \leq CN^c.
$$

Thus, in the framework of the variable Lebesgue spaces, we are interested in a small positive constant $c$ such that $N^c$ bounds from above $\|K_N\|_{L^{p(·)}(\mathbb{R}^2) \rightarrow L^{p(·)}(\mathbb{R}^2)}$.

The main result of this paper is the following (Theorem 1.3). The technique of the proof of this theorem is due to [3], which is used the machinery of Calderón-Zygmund cubes. We apply this technique to the rectangles in $\mathcal{B}_N$. For the precise estimate we need the following notion.
The letter \( C \) will be used for constants that may change from one occurrence to another. Constants with subscripts, such as \( C_1, C_2 \), do not change in different occurrences.

2. Proof of Theorem 1.3

The following argument is due to T. Kopaliani [12] (see also [11]). Recall that the conjugate function \( p'(x) \) is defined by \( \frac{1}{p'(x)} + \frac{1}{p(x)} = 1 \). The following generalized Hölder inequality and a duality relation can be found in [13]:

\[
\int_{\mathbb{R}^2} |f(x)g(x)| \, dx \leq 2\|f\|_{p()}\|g\|_{p'()},
\]

\[
\|f\|_{p()} \leq \sup_{\|g\|_{p'(\cdot)} \leq 1} \int_{\mathbb{R}^2} |f(x)g(x)| \, dx.
\]

Suppose that \( K_N \) is bounded from \( L^{p(\cdot)}(\mathbb{R}^2) \) to \( L^{p(\cdot)}(\mathbb{R}^2) \). Then for every rectangle \( R \in \mathcal{B}_N \) we have

\[
\|K_N\|_{L^{p(\cdot)} \to L^{p(\cdot)}} \geq \|K_N f\|_{p()} \geq \left\| \int_R f(y) \, d\chi_R \right\|_{p(\cdot)} = \int_R f(y) \, d\|\chi_R\|_{p(\cdot)}
\]

for all nonnegative \( f \) with \( \|f\|_{p(\cdot)} \leq 1 \). Taking supremum all such \( f \), we have

\[
\|K_N\|_{L^{p(\cdot)} \to L^{p(\cdot)}} \geq 1 \left\| \chi_R \right\|_{p'(\cdot)} \|\chi_R\|_{p(\cdot)}
\]

for all \( R \in \mathcal{B}_N \), where \( |R| \) denotes the area of the rectangle \( R \).

Suppose that \( p(\cdot) \) is continuous and is not constant. Then we can find two closed squares \( B_1 \) and \( B_2 \) in \( \mathbb{R}^2 \) with \( |B_1|, |B_2| < 1 \), such that

\[
p_+(B_1) < p_-(B_2).
\]
Without loss of generality we may assume that
\[ B_1 = [0, s] \times [0, s] \text{ and } B_2 = [0, s] \times [t - s, t] \text{ for some } t > s > 0. \]

We take \( N \) with \( t/N < s \) and let \( R = [0, t/N] \times [0, t] \). Then we have \( R \in \mathcal{B}_N \) and
\[ |R \cap B_1| = |R \cap B_2| = \frac{st}{N}. \]

Observe now that the following embeddings hold:
\[ L^{p(\cdot)}(B_2) \hookrightarrow L^{p^{-1}(B_2)}(B_2), \]
\[ L^{p(\cdot)}(B_1) \hookrightarrow L^{(p_+(B_1))'}(B_1), \]
where \( \frac{1}{(p_+(B_1))} + \frac{1}{p_+(B_1)} = 1 \). It follows that
\[ \frac{1}{|R|} \|\chi_R\|_{p(\cdot)} \|\chi_{R \cap B_2}\|_{L^{p(\cdot)}(B_2)} \|\chi_{R \cap B_1}\|_{L^{p(\cdot)}(B_1)} \geq \frac{1}{|R|} \|\chi_{R \cap B_2}\|_{L^{p^{-1}(B_2)}(B_2)} \|\chi_{R \cap B_1}\|_{L^{(p_+(B_1))'}(B_1)} \]
\[ = |R|^{-1} \cdot |R \cap B_2|^{-1} \cdot |R \cap B_1| \cdot t^{-2} \cdot (st)^{1 + \frac{1}{p_+(B_2)} - \frac{1}{p_+(B_1)}} \cdot N^{cN} \]
where we have used \( |B_1|, |B_2| < 1 \). Since by (2.2) \( p_+(B_1) - \frac{1}{p_+(B_2)} > 0 \), we conclude by (2.1) that \( \|K_N\|_{L^{p(\cdot)} \rightarrow L^{p(\cdot)}} \) has a lower bound \( N^\varepsilon \) with \( \varepsilon > 0 \).

3. Proof of Theorem 1.5

In what follows we shall prove Theorem 1.5. We need two lemmas.

**Lemma 3.1.** Let \( N \gg 1 \). Suppose that \( p(\cdot) \) is \( N \)-modified locally log-Hölder continuous. Then, for any rectangle \( R \in \mathcal{B}_N \),
\[ |R|^{p_+(\cdot) - p_-(\cdot)} \leq N^{cN}. \]

**Proof.** When \( |R| \geq 1 \), there is nothing to prove. Suppose that \( |R| < 1 \). Since \( p(\cdot) \) is continuous, there exist \( x, y \in R \) such that \( p(x) = p_-(R) \) and \( p(y) = p_+(R) \). It follows that
\[ |R|^{p_+(\cdot) - p_-(\cdot)} = |R|^{p(y) - p(x)} \leq \left( \frac{|x - y|^2}{N} \right)^{\frac{1}{p(y) - p(x)}} \]
\[ = \exp \left\{ \left( \frac{1}{p(y)} - \frac{1}{p(x)} \right) \log \left( \frac{|x - y|^2}{N} \right) \right\} = \exp \left\{ \left( \frac{1}{p(x)} - \frac{1}{p(y)} \right) \log \left( \frac{N}{|x - y|^2} \right) \right\} \]
\[ \leq \exp \left\{ \log (N^{cN}) \right\} = N^{cN}, \]
where we have used \( |x - y| < \sqrt{N} \) and the \( N \)-modified local log-Hölder continuity of \( p(\cdot) \). \( \square \)

**Lemma 3.2 ([3, Lemma 2.4]).** Suppose that \( p(\cdot) \) is log-Hölder continuous at infinity. Let \( P(x) = (e + |x|)^{-M} \), \( M \geq 2 \). Then there exists a constant \( c \) depending on \( M \), \( p(\infty) \) and \( e_\infty \) such that given any set \( E \) and any function \( F \) such that \( 0 \leq F(y) \leq 1 \), \( y \in E \),
\[ \int_E F(y)^{p(y)} \, dy \leq c \int_E F(y)^{p(\infty)} \, dy + c \int_E P(y)^{p(\infty)} \, dy, \]
\[ \int_E F(y)^{p(\infty)} \, dy \leq c \int_E F(y)^{p(y)} \, dy + c \int_E P(y)^{p(\infty)} \, dy. \]
The estimate for $f$. We shall verify that, if $\lambda_1 = C_1^{p} - C_2$, then

\begin{equation}
\rho_p(\frac{T_k f_1}{\lambda_1}) = \int_{\mathbb{R}^2} \left( \frac{T_k f_1(x)}{\lambda_1} \right)^{p(x)} \, dx \leq C.
\end{equation}

It follows from H"older’s inequality that

\[
\rho_p(\frac{T_k f_1}{\lambda_1}) \\
= \sum_{Q \in \mathcal{D}_k} \int_Q \left( \frac{1}{\lambda_1} \right)^{p(x)} \left( \int_{R(Q)} f_1(y) \, dy \right)^{p(x)} \, dx \\
\leq \sum_{Q \in \mathcal{D}_k} \int_Q \left( \frac{1}{\lambda_1} \right)^{p(x)} \left( \int_{R(Q)} f_1(y) \frac{p(p-R(Q))}{p} \, dy \right)^{\frac{p}{p-R(Q)}} \, dx \\
= \sum_{Q \in \mathcal{D}_k} \int_Q \left( \frac{1}{\lambda_1} \right)^{p(x)} \left( \frac{1}{|R(Q)|} \right)^{\frac{p}{p-R(Q)}} \left( \int_{R(Q)} f_1(y) \frac{p(p-R(Q))}{p} \, dy \right)^{\frac{p}{p-R(Q)}} \, dx.
\]

There holds, for $|R(Q)| \geq 1$,

\[
\left( \frac{1}{C_1^{p}} \right)^{p(x)} \left( \frac{1}{|R(Q)|} \right)^{\frac{p(p-R(Q))}{p-R(Q)}} \leq \left( \frac{1}{|R(Q)|} \right)^{p}.
\]
where we have used $C_1 \geq 1$ and $\frac{p(x)}{p_{-}(R(Q))} \geq 1$. Also, there holds, for $|R(Q)| < 1$,

$$
\left( \frac{1}{C_1^p} \right)^{p(x)} \left( \frac{1}{|R(Q)|} \right)^{p_{-}(R(Q))} \leq \left( \frac{1}{|R(Q)|} \right)^{p_{-}(p(x))} \left( \frac{1}{|R(Q)|} \right)^{p_{-}(R(Q))} = \left( \frac{1}{|R(Q)|} \right)^{p_{-}(p(x)) - p_{-}} \left( \frac{1}{|R(Q)|} \right)^{p_{-}} \leq \left( \frac{1}{|R(Q)|} \right)^{p_{-}},
$$

where we have used

$$
C_1 \geq |R(Q)|^{\frac{1}{p_{+}(R(Q))} - \frac{1}{p_{-}(R(Q))}} \text{ and } \frac{p(x)}{p_{+}(R(Q))} \leq 1.
$$

We see that by the definition of $f_1$

$$
\left( \int_{R(Q)} f_1(y) \frac{p_{-}(R(Q))}{p_{-}(y)} dy \right)^{p_{-}(p(x))} \leq \left( \int_{R(Q)} f_1(y)^{p(y)} dy \right)^{p_{-}(p(x))} \left( \int_{R(Q)} f_1(y)^{p(y)} dy \right)^{p_{-}} \leq \left( \int_{R(Q)} f_1(y)^{p(y)} dy \right)^{p_{-}} \leq \left( \int_{R(Q)} f_1(y)^{p(y)} dy \right)^{p_{-}} \leq 1.
$$

These yield

$$
\rho \left( \frac{T_k f_1}{\lambda_1} \right) \leq \sum_{Q \in D_k} \int_Q \left( \frac{1}{C_2} \right)^{p(x)} \left( \int_{R(Q)} f_1(y)^{\frac{p(y)}{p_{-}}} dy \right)^{p_{-}} dx.
$$

Therefore, since $R(Q) \supset Q$ and $\frac{\mu(x)}{p_{-}} \geq 1$,

$$
\rho \left( \frac{T_k f_1}{\lambda_1} \right) \leq \frac{1}{(\log N)^2} \int_{\mathbb{R}^2} K_N \left[ \frac{p(x)}{p_{-}} \right](x)^{p_{-}} dx \leq C \int_{\mathbb{R}^2} f_1(x)^{p(x)} dx \leq C,
$$

where we have used \textbf{[11]}.  

**The estimate for $f_2$.** We shall verify that, if $\lambda_2 = C_2$, then

$$
\rho \left( \frac{T_k f_2}{\lambda_2} \right) = \int_{\mathbb{R}^2} \left( \frac{T_k f_2(x)}{\lambda_2} \right)^{p(x)} dx \leq C.
$$

Since $f_2 \leq 1$, we immediately see that

$$
F = \frac{1}{\lambda_2} \int_{R(Q)} f_2(y) dy \leq 1.
$$
Therefore, by Lemma 3.2 with $P(x) = (e + |x|)^{-2}$,
\[
\rho_{p(\cdot)} \left( \frac{T_k f_2}{\lambda_2^2} \right) = \sum_{Q \in D_k} \int_Q \left( \frac{1}{\lambda_2^2} \int_{R(Q)} f_2(y) \, dy \right)^{p(x)} \, dx \\
\leq C \sum_{Q \in D_k} \int_Q \left( \frac{1}{\lambda_2^2} \int_{R(Q)} f_2(y) \, dy \right)^{p(\infty)} \, dx + C \sum_{Q \in D_k} \int_Q P(x)^{p(\infty)} \, dx.
\]
Since $p(\infty) \geq 2$ and the cubes $Q \in D_k$ are disjoint, we can immediately estimate the second term:
\[
\sum_{Q \in D_k} \int_Q P(x)^{p(\infty)} \, dx = \int_{\mathbb{R}^2} P(x)^{p(\infty)} \, dx \leq C.
\]
We shall estimate the first term. It follows that
\[
\sum_{Q \in D_k} \int_Q \left( \frac{1}{\lambda_2^2} \int_{R(Q)} f_2(y) \, dy \right)^{p(\infty)} \, dx \\
\leq \frac{1}{(\log N)^2} \sum_{Q \in D_k} \int_Q K_n f_2(x)^{p(\infty)} \, dx \\
\leq C \int_{\mathbb{R}^2} f_2(x)^{p(\infty)} \, dx,
\]
where we have used (11). Since $f_2 \leq 1$ we can apply Lemma 3.2 again,
\[
\int_{\mathbb{R}^2} f_2(x)^{p(\infty)} \, dx \leq C \int_{\mathbb{R}^2} f_2(x)^{p(x)} \, dx + C \int_{\mathbb{R}^2} P(x)^{p(\infty)} \, dx \leq C.
\]
Altogether, we obtain (32).

**Conclusion.** The estimates (31), (32) and Lemma 3.1 yield the theorem.

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