PATH INTEGRALS WITH KINETIC COUPLING POTENTIALS

Christian Grosche

II. Institut für Theoretische Physik
Universität Hamburg, Luruper Chaussee 149
22761 Hamburg, Germany

ABSTRACT

Path integral solutions with kinetic coupling potentials $\propto p_1p_2$ are evaluated. As examples I give a Morse oscillator, i.e., a model in molecular physics, and the double pendulum in the harmonic approximation. The former is solved by some well-known path integral techniques, whereas the latter by an affine transformation.
1 Introduction

Quantum mechanics is about physics on the atomic level, i.e., wave-functions and energy levels of atoms and molecules. The simplest quantum mechanical system is the harmonic oscillator with the characteristic $h(n + \frac{1}{2})$ spectrum \([\pi - \pi]\) which was also the first system solved by means of the path integral. The important Coulomb system, i.e., the hydrogen atom, was only solved later on by Duru and Kleinert \([\pi, \pi]\). In this Paper I want to demonstrate the technique of path integration for some particular models with kinetic coupling. The first example consists of the potential force of two molecules, where the anharmonic interaction force is described by the Morse potential with an additional kinetic coupling $\propto$ and $2$ referring to particle one and two. The second example is the double pendulum, a system which is coupled by a $\propto \dot{\varphi}_1 \dot{\varphi}_2$ term. In the harmonic approximation the double pendulum is described by two coupled oscillators. In fact, these two examples are not too difficult to solve, but they demonstrate in a nice way the applicability of path integration techniques for systems which are coupled by means of the kinetic energy. Such problems seem not to have been studied by path integrals yet, whereas systems which are coupled through only a coordinate dependence, i.e. $\propto x_1 x_2$, can be found in e.g. \([\pi]\).

2 Morse Oscillator with Kinetic Coupling

The first example is a model in molecular physics, which takes into account the anharmonicity of the inter-molecular potential (e.g., H$_2$O, H$_2$S, C$_2$H$_2$), by means of the Morse potential \([\pi] - [\pi]\)

$$V(x_1 - x_2) = \lambda \left(1 - \alpha e^{-\beta(x_1 - x_2 - r_0)}\right)^2,$$

where $x_1, x_2$ denote the position of the first and second particle one with masses $m_1, m_2, r_0$ the equilibrium position, and $\alpha, \beta, \lambda$ are positive parameters. $\hbar$ denotes Planck’s constant divided by $2\pi$. The Hamiltonian of this system for the two particles has the form (with the momentum operators $P_{1,2} = (\hbar / \partial \partial_{x_1,2})$

$$H = \frac{1}{2m_1} P_1^2 + \frac{1}{2m_2} P_2^2 + \kappa P_1 P_2 + \lambda \left(1 - \alpha e^{-\beta(x_1 - x_2 - r_0)}\right)^2.$$

The corresponding Hamiltonian path integral for the time evolution from $x_{1,2}(t')$ to $x_{1,2}(t'')$ in the time interval $T = t'' - t'$ can thus be written as \([\pi]\)

$$K(x_{1''}, x_{1'}, x_{2''}, x_{2'}; T) = \int_{x_1(t') = x_{1'}}^{x_1(t'') = x_{1''}} Dx_1(t) \int_{x_2(t') = x_{2'}}^{x_2(t'') = x_{2''}} Dx_2(t) \int DP_1(t) \int DP_2(t) \times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ P_1 \dot{x_1} + P_2 \dot{x_2} - \frac{1}{2m_1} P_1^2 - \frac{1}{2m_2} P_2^2 - \kappa P_1 P_2 - \lambda \left(1 - \alpha e^{-\beta(x_1 - x_2 - r_0)}\right)^2 \right] dt \right\}.$$

We can switch to center-of-mass $X$ and relative coordinates $x_r$ by means of

$$X = \mu_1 x_1 + \mu_2 x_2,$$

$$X_r = x_1 - x_2 - r_0,$$

$$P = P_1 + P_2,$$

$$P_r = \mu_2 P_1 - \mu_1 P_2,$$

$$\{4\}$$
where \((i = 1, 2)\) \(\mu_i = m_i/M\), \((\mu_1 + \mu_2 = 1)\), \(M = m_1 + m_2\) is the total mass, and \(\mu = m_1m_2/M\) is the reduced mass. Vice versa we have

\[
x_1 = X + \frac{m_2}{M}x_r, \quad x_2 = X - \frac{m_1}{M}x_r, \quad P_1 = P_r + \mu_1 P, \quad P_2 = \mu_2 P - P_r.
\]

The Hamiltonian path integral in center-of-mass and relative coordinates is therefore given by

\[
K(X'', X', x'', x'; T) = \int_{X(t')=X'}^{x_r(t')=x_r'} DX(t) \int_{x_r(t')=x_r'} Dx_r(t) \times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{d}{4D} \dot{X}^2 + \frac{a}{4D} \dot{x}_r^2 - 2 \frac{b}{4D} \dot{x}_r \dot{X} - \lambda \left( 1 - \alpha e^{-\beta x_r} \right)^2 \right] dt \right\},
\]

and I have used the abbreviations

\[
a = \frac{1}{2M} + \kappa \mu_2 \mu_2, \quad d = \frac{1}{2\mu} - \kappa, \quad b = \frac{\kappa}{2}(\mu_2 - \mu_1),
\]

and \(D = ad - b^2 = \frac{1}{4}(1/m_1m_2 - \kappa^2)\). The limiting case \(\kappa = 0\) is easily recovered. The lattice formulation of the path integral (5) is given in the usual way by

\[
K(X'', X', x'', x'; T) = \lim_{N \to \infty} \left( \frac{1}{4\sqrt{D} \pi \epsilon h} \right)^N \prod_{j=1}^{N-1} \int_{\mathbb{R}} dX_j \int_{\mathbb{R}} dx_{r_j} \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N} \left[ \frac{d}{4D} (\Delta X_j)^2 + \frac{a}{4D} (\Delta x_{r_j})^2 - 2 \frac{b}{2D} \Delta X_j \Delta x_{r_j} - \lambda \left( 1 - \alpha e^{-\beta x_{r_j}} \right)^2 \right] \right\}.
\]

Here we have taken as usual \(\epsilon = T/N = (t'' - t')/N\) in the time slicing, \(X_j = X(t_j), x_{r_j} = x_r(t_j), (j = 1, \ldots, N)\), \(\Delta X_j = X_j - X_{j-1}, \Delta x_{r_j} = x_{r_j} - x_{r_{j-1}}\), as \(N \to \infty, \epsilon \to 0\), \(T\) fixed, and \(X'' = X(t''), X' = X(t'), x'' = x_r(t''), x' = x_r(t')\). The path integral in the center-of-mass coordinate is coupled to the relative coordinate \(x_r\) via \(\dot{x}_r X\), thus the \(X\)-path integration is that of a free particle with a magnetic field, which is time dependent but not \(X\)-path dependent. The classical solution for the center-of-mass coordinate \(X\) is found to be

\[
X_{C1}(t) = \frac{X'' - X'}{T}(t - t') - \frac{d}{2D} \int_{t'}^{t} \dot{x}_r(t') dt,
\]

with corresponding classical action

\[
S_{C1}[X_{C1}] = \frac{d}{4D} \frac{(X'' - X')^2}{T} - \frac{b^2}{4dD} \int_{t'}^{t} \dot{x}_r^2(t') dt.
\]
We can therefore separate off the center-of-mass path integration $X$ according to [3, 11]

$$K(X'', X', x''_r, x'_r; T) = \sqrt{\frac{d}{4D\pi i\hbar T}} \exp \left( \frac{i}{\hbar} \frac{d}{4D} \frac{(X'' - X')^2}{T} \right) \times \int_{x_r(t') = x''_r} Dx_r(t) \exp \left\{ i \hbar \frac{1}{4d} \int_{t'} ^{t''} \left[ \frac{1}{4d} x_r^2 - \lambda \left( 1 - \alpha e^{-\beta x_r} \right)^2 \right] dt \right\}$$

(12)

$$= \int_{\mathbb{R}} \frac{dK}{2\pi} \exp \left( iK(X'' - X') - i\hbar K^2 \frac{1}{2M(1 - 2\mu\kappa)} T - \frac{i}{\hbar} \lambda T \right) \times \int_{x_r(t') = x''_r} Dx_r(t) \exp \left\{ i \hbar \frac{1}{4d} \int_{t'} ^{t''} \left[ \frac{1}{4d} x_r^2 - \lambda \alpha^2 \left( e^{-2\beta x_r} - \frac{2}{\alpha} e^{-\beta x_r} \right) \right] dt \right\} .$$

(13)

Again, the case $\kappa = 0$ gives complete decoupling (note $1/4d = \mu/2$ for $\kappa = 0$).

The path integral in the second line of (13) is that of the Morse potential which can be found in the literature [3]. Setting

$$K(X'', X', x''_r, x'_r; T) = \int_{\mathbb{R}} \frac{dK}{2\pi} e^{iK(\beta'' - \beta')} K_K(x''_r, x'_r; T) ,$$

(14)

$$K_K(x''_r, x'_r; T) = \int_{\mathbb{R}} \frac{dE}{2\pi} e^{-iET/\hbar} G_K(x''_r, x'_r; E) ,$$

(15)

where $G_K$ is the (energy-dependent) Green function corresponding to the kernel $K_K$. We find by means of the space-time transformation (Duru–Kleinert transformation) [3, 4, 3] for the Green function

$$G_K(x''_r, x'_r; E) = \frac{i}{\hbar} \int _0 ^{\infty} dT e^{iET/\hbar} \exp \left(-i\hbar K^2 \frac{1}{2M(1 - 2\mu\kappa)} T - \frac{i}{\hbar} \lambda T \right) \times \int_{x_r(t') = x''_r} Dx_r(t) \exp \left\{ i \hbar \frac{1}{4d} \int_{t'} ^{t''} \left[ \frac{1}{4d} x_r^2 - \lambda \alpha^2 \left( e^{-2\beta x_r} - \frac{2}{\alpha} e^{-\beta x_r} \right) \right] dt \right\}$$

$$= \frac{\Gamma \left( \lambda + \eta - \xi \right)}{2dh^2 T(1 + 2\eta)} e^{-\beta(x'_r + x''_r)/2} W_{\xi,\eta} \left( 2\xi e^{\beta x_r} \right) M_{\xi,\eta} \left( 2\xi e^{\beta x_r} \right) .$$

(16)

Here I have abbreviated

$$\xi = \frac{1}{\hbar} \sqrt{\frac{\lambda}{d}} , \quad \eta = \frac{1}{\hbar} \sqrt{-\frac{E - E_0}{d}} , \quad E_0 = \frac{h^2 K^2}{2M} \frac{1}{1 - 2\mu\kappa} + \lambda ,$$

(17)

and $x_{r,\xi}$ denotes the larger/smaller of $x'_r, x''_r$. From the poles of $G_K$ the bound states are given by

$$E_n = \frac{h^2 K^2}{2M} \frac{1}{1 - 2\mu\kappa} - \left( \frac{1}{2\mu} - \kappa \right) h^2 \frac{\beta^2}{2}(n + \frac{1}{2})^2 + 2h\beta(n + \frac{1}{2}) \sqrt{\lambda \left( \frac{1}{2\mu} - \kappa \right)} .$$

(18)

The first term is just the continuous spectrum which is not relevant for the discrete spectrum. The bound state wave-functions are determined by the residua of the poles of $G_K$, and we find [3, 12, 13]

$$\Psi_{n,K}(X, x_r) = \frac{e^{iKX}}{\sqrt{2\pi}} \sqrt{\frac{\beta(2\xi - 2n - 1)n!(2\alpha\xi)^2 - 2n - 1}{\Gamma(2\xi - n)}} \times \exp \left( \left( \xi - n - \frac{1}{2} \right) \beta x_r - \alpha \xi e^{\beta x_r} \right) L_n^{(2\xi - 2n - 1)} \left( 2\alpha\xi e^{\beta x_r} \right) .$$

(19)
The continuous spectrum is determined by the cut in $G_K$ and I obtain
\[ \Psi_{k,K}(X, x_r) = \frac{e^{iKX}}{\sqrt{2\pi}} \sqrt{\frac{\beta k \sinh \pi k}{2\pi^2 \alpha \xi}} \Gamma(ik - \xi + \frac{1}{2}) e^{-\beta x_r} W_{\xi,ik} \left( 2\alpha \xi e^{\beta x_r} \right), \tag{20} \]
with the energy-spectrum
\[ E_k = \frac{\hbar^2 K^2}{2M} \frac{1 - \mu M \kappa}{1 - 2\mu \kappa} + \left( \frac{1}{2M} + \kappa \mu_2 \mu_2 \right) \hbar^2 k^2. \tag{21} \]

3 The Double Pendulum

As the next example we consider the double pendulum with two masses $m_1, m_2$ attached to two strings with lengths $l_1, l_2$. $g$ denotes the gravitational acceleration. Its Lagrangian is given by
\[ \mathcal{L} = \frac{m_1 + m_2}{2} l_1^2 \dot{\varphi}_1^2 + \frac{m_2}{2} l_2^2 \dot{\varphi}_2^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + (m_1 + m_2) g l_1 \cos \varphi_1 + m_2 g l_2 \cos \varphi_2. \tag{22} \]

In the harmonic approximation we have $(l_1 = l_2 = l, M = m_1 + m_2)$
\[ \mathcal{L} = \frac{M}{2} l^2 \dot{\varphi}_1^2 + \frac{m_2}{2} l^2 \dot{\varphi}_2^2 + m_2 l^2 \dot{\varphi}_1 \dot{\varphi}_2 - \frac{M}{2} g l \varphi_1^2 - \frac{m_2}{2} g l \varphi_2^2, \tag{23} \]
which I consider in the sequel. The path integral formulation for the latter case has the form
\[ K(\varphi''_1, \varphi'_1, \varphi''_2, \varphi'_2; T) = \int D\varphi_1(t) \int D\varphi_2(t) \times \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{M}{2} l^2 \dot{\varphi}_1^2 + \frac{m_2}{2} l^2 \dot{\varphi}_2^2 + m_2 l^2 \dot{\varphi}_1 \dot{\varphi}_2 - \frac{M}{2} g l \varphi_1^2 - \frac{m_2}{2} g l \varphi_2^2 \right] dt \right\}. \tag{24} \]

The path integral (24) is of the form of a path integral for a general quadratic Lagrangian. Its solution is given by, e.g. \cite{2,3,4,6,14}
\[ K(\varphi''_1, \varphi'_1, \varphi''_2, \varphi'_2; T) = \left( \frac{1}{2\pi i \hbar} \right)^{D/2} \det \left( - \frac{\partial^2 S_{\text{Cl}}(\varphi'', \varphi')}{\partial \varphi''^m \partial \varphi'^b} \right) \exp \left( \frac{i}{\hbar} S_{\text{Cl}}(\varphi'', \varphi') \right), \tag{25} \]
where I have denoted $\varphi = (\varphi_1, \varphi_2)$, and $S_{\text{Cl}}(\varphi'', \varphi')$ is the corresponding classical action of the system. The determinant appearing in (25) is called the Morette-Van Hove determinant \cite{2,14}.

In order to determine the classical action, one considers the classical system of the coupled differential equations and switches form the coordinates $(\varphi_1, \varphi_2)$ to normal coordinates $(\xi_1, \xi_2)$ by means of an affine transformation \cite{14}. This is done in the following way: We write
\[ \mathcal{L} = \frac{1}{2} \dot{\varphi} A \dot{\varphi} - \frac{1}{2} \varphi K \varphi, \tag{26} \]
where $A, K$ are $n \times n$ matrices describing the classical system with its potentials and couplings. The classical system is diagonalized by considering the eigenvalue equation
\[ \det(K - \omega^2 A) = 0, \tag{27} \]
with eigenvalues $\omega^2$. This is performed by a matrix $C$, the affine transformation. The corresponding differential equations in the normal coordinates $\xi = C \varphi$ $(k = 1, \ldots, n)$ are
\[ \ddot{\xi}_k + \omega_k^2 \xi_k = 0, \tag{28} \]
with the $\omega_k$ the eigenfrequencies corresponding to the kth component. In our case we have $n = 2$, and we find

$$\xi_1(t) = \sqrt{\frac{M}{2}}(1 - r) \left( \varphi_1(t) - r\varphi_2(t) \right), \quad \xi_2(t) = \sqrt{\frac{M}{2}}(1 + r) \left( \varphi_1(t) + r\varphi_2(t) \right),$$

(29)

where $r = \sqrt{m_2/M}$, $\omega_{1,2} = (Mg/m_1l)(1 \pm r)$. Vice versa we have

$$\varphi_1(t) = \frac{1}{l\sqrt{2M}} \left( \frac{\xi_1(t)}{\sqrt{1 - r}} + \frac{\xi_2(t)}{\sqrt{1 + r}} \right), \quad \varphi_2(t) = \frac{1}{l\sqrt{2M}} \left( \frac{-\xi_1(t)}{r\sqrt{1 - r}} + \frac{\xi_2(t)}{r\sqrt{1 + r}} \right).$$

(30)

Therefore we can write down the solution of the path integral (24) in the following form [1]–[4, 6]

$$K(\varphi', \varphi''; T) = \prod_{k=1,2} \sqrt{\frac{m_k\omega_k}{2\pi \hbar}} \sin \omega_k T \exp \left\{ \frac{i}{\hbar} \frac{m_1\omega_1}{2 \sin \omega_1 T} (1 - r) \left( (\varphi'_1 - r\varphi''_2)^2 + (\varphi'_1 - r\varphi'_2)^2 \right) + \frac{m_2\omega_2}{2 \sin \omega_2 T} (1 + r) \left( (\varphi'_1 + r\varphi'_2)^2 + (\varphi'_1 + r\varphi''_2)^2 \right) - \sqrt{1 - r^2} \left( (\varphi'_1 - r\varphi'_2)(\varphi''_1 - r\varphi''_2) + (\varphi'_1 + r\varphi'_2)(\varphi'_1 + r\varphi''_2) \right) \right\}$$

(31)

$$= \prod_{k=1,2} \sum_{n_k=0}^{\infty} \Psi_{n_k}(\varphi'_k)\Psi_{n_k}(\varphi''_k) e^{-i(\omega_k + 1/2)T},$$

(32)

The energy spectrum has the form

$$E_{n_1, n_2} = \hbar(\omega_1 n_1 + \omega_2 n_2 + 1) = \hbar \left[ \frac{Mg}{m_1l} \left( n_1 + n_2 + \sqrt{\frac{m_2}{M}} (n_1 - n_2) \right) + 1 \right],$$

(33)

and the wave-functions are given by ($\xi_k$ as in (29))

$$\Psi_{n_k}(\varphi_k) = \left( \frac{m_k\omega_k}{\pi \hbar} \right)^{1/4} \frac{1}{2^{n_k} n_k!} \exp \left( - \frac{m_k\omega_k}{2\hbar} \xi_k^2 \right) H_{n_k} \left( \sqrt{\frac{m_k\omega_k}{\hbar}} \xi_k \right).$$

(34)

The expansion (33) can be derived by means of the Hille–Hardy formula [3, 17].

4 Conclusions

The results clearly show the applicability of path integration techniques in systems with kinetic energy couplings. The first example of a Morse potential interaction, a model known in molecular physics, was transformed in center-of-mass $X$ and relative coordinates $x_r$, followed by the path integration of the center-of-mass coordinate, thus achieving a complete separation of variables in $X$ and $x_r$. This left the remaining path integral as a path integral $K_K$ of the Morse potential, and the corresponding Green function $G_K$ could be written down. The usual analysis of $G_K$ gave the wave-functions and the energy spectrum of the discrete and continuous spectrum, respectively. The bound state energy spectrum has the typical feature of the energy spectrum of a Morse
potential, however modified by the coupling constant $\kappa$ of the $\kappa P_1 P_2$-term. Let us note that the choice of the Morse potential for the interaction in the relative coordinate is but an example. We can take any potential $V(x_r) = V(|x_1 - x_2|)$. This has the consequence that in (13) the potential term $\lambda \alpha^2 (e^{-2\beta x_r} - (2/\alpha)e^{-\beta x_r})$ is replaced by $V(x_r)$, and the corresponding path integral is exactly solvable for any exactly solvable one-dimensional potential, modified by the mass-term $1/4d$.

The double pendulum in the harmonic approximation could be separated in terms of normal coordinates $\xi$ which are uniquely determined by the affine transformation $C$ in terms of the original coordinates $\varphi$. The path integral can be stated explicitly, and the wave-functions and the energy spectrum as well. The formulation in the normal coordinates is just the product of two one-dimensional harmonic oscillators, whereas the formulation in the original coordinates is somewhat more complicated, and features the interdependence of the two oscillations of the respective single pendula. Limiting cases, e.g. $m_1 \gg m_2$, are easily recovered yielding two (almost) independent oscillations ($m_2 l^2 g \ddot{\varphi}_1 \ddot{\varphi}_2$ becomes negligible). For the sake of simplicity I have considered only the case $l_1 = l_2 = l$. A generalization to $l_1 \neq l_2$ is easily done, and left to the reader. The generalization to higher dimensions is also straightforward; again we would obtain a product of $n$-dimensional harmonic oscillators in terms of normal coordinates, i.e., $\xi = Cq$, respectively $q = C^{-1}\xi$, with $C$ the corresponding affine transformation.

These two examples thus show in a nice way the possibility of solving path integrals for kinetic energy couplings. Other interesting cases would be a particle in rotating magnetic fields, an arrangement important for trapping electrons, and non-orthogonal coordinate systems in spaces of constant curvature [18]. This will be studied in subsequent investigations.

References

[1] Feynman, R.P.: Space-Time Approach to Non-Relativistic Quantum Mechanics. Rev. Mod. Phys. 20 (1948) 367–387.
[2] Feynman, R.P., Hibbs, A.: Quantum Mechanics and Path Integrals. McGraw Hill, New York, 1965.
[3] Grosche, C., Steiner, F.: Handbook of Feynman Path Integrals. Springer, Berlin, Heidelberg, 1998.
[4] Schulman, L.S.: Techniques and Applications of Path Integration. John Wiley & Sons, New York, 1981.
[5] Duru, I.H., Kleinert, H.: Solution of the Path Integral for the H-Atom. Phys. Lett. B 84 (1979) 185–188. Quantum Mechanics of H-Atoms from Path Integrals. Fortschr. Phys. 30 (1982) 401–435.
[6] Kleinert, H.: Path Integrals in Quantum Mechanics, Statistics and Polymer Physics. World Scientific, Singapore, 1995.
[7] Mills, I.M., Robinette, A.G.: On the Relationship of Normal Modes to Local Modes in Molecular Vibrations. Mol. Phys. 56 (1985) 743–765.
[8] Fan, H.-Yi, Klauder, J.R.: Eigenvectors of Two Particles’ Relative Position and Total Momentum. Phys. Rev. A 49 (1994) 704–707.
[9] Fan, H.-Yi, Xiong, Y.: Common Eigenstates of Two Particles’ Center-of-Mass Coordinates and Mass-Weighted Relative Momentum Phys. Rev. A 51 (1995) 3343–3346.
[10] Fan, H.-Yi, Chen, B.Z., Fun, Y.: Energy Levels for the Morse Oscillator with and Additional Kinetic Coupling Potential. Phys. Lett. A 213 (1996) 226–230.
[11] Grosjean, C.C., Goovaerts, M.J.: The Analytical Evaluation of One-Dimensional Gaussian Path-Integrals. J. Comput. Appl. Math. 21 (1988) 311–331.
[12] Duru, I.H.: Morse-Potential Green’s Function with Path Integrals. *Phys. Rev.* D 28 (1983) 2689–2692.

[13] Grosche, C.: The Path Integral on the Poincaré Upper Half-Plane With a Magnetic Field and for the Morse Potential. *Ann. Phys. (N. Y.)* 187 (1988) 110–134.

[14] Grosjean, C.C.: A General Formula for the Calculation of Gaussian Path-Integrals in Two and Three Euclidean Dimensions. *J. Comput. Appl. Math.* 23 (1988) 199–234.

[15] Choquard, Ph., Steiner, F.: The Story of Van Vleck’s and Morette-Van Hove’s Determinants. *Helv. Phys. Acta* 69 (1996) 636–654.

[16] Landau, L.D., Lifschitz, E.M.: *Lehrbuch der Theoretischen Physik* I (in German). Akademie Verlag, Berlin, 1979.

[17] Gradshteyn, I.S., Ryzhik, I.M.: *Table of Integrals, Series, and Products*. Academic Press, New York, 1980.

[18] Kalnins, E.G.: On the Separation of Variables for the Laplace Equation $\Delta \psi + K^2 \psi = 0$ in Two- and Three-Dimensional Minkowski Space. *SIAM J. Math. Anal.* 6 (1975) 340–374.