Quantum White Noises and The Master Equation for Gaussian Reference States

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Abstract
We show that a basic quantum white noise process formally reproduces quantum stochastic calculus when the appropriate normal / chronological orderings are prescribed. By normal ordering techniques for integral equations and a generalization of the Araki-Woods representation, we derive the master and random Heisenberg equations for an arbitrary Gaussian state: this includes thermal and squeezed states.

1 Quantum White Noises
It is possible to develop a formal theory of quantum white noises which nevertheless provides a powerful insight into quantum stochastic processes. We present the “bare bones” of the theory: the hope is that the structure will be more apparent without the mathematical gore and viscera.

Essentially, we need a Hilbert space $\mathcal{H}_0$ to describe a system of interest. We postulate a family of pseudo-operators $\{a^+_t : t \geq 0\}$ called creation noises; a formally adjoint family $\{a^-_t : t \geq 0\}$ called annihilation noises and a vector $\Psi$ called the vacuum vector. The noises describe the environment and are assumed to act trivially on the observables of $\mathcal{H}_0$.

The first structural relations we need is

$$a^-_t \Psi = 0 \quad \text{(QWN1)}$$

which implies that the annihilator noises annihilate the vacuum. The second
relations are given by the commutation relations

\[ [a_t^-, a_s^+] = \kappa \delta_+ (t-s) + \kappa^* \delta_- (t-s) \]  \hspace{1cm} (QWN2)

Otherwise the creation noises all commute amongst themselves, as do the annihilation noises. Here \( \kappa = \frac{1}{2} \gamma + i\sigma \) is a complex number with \( \gamma > 0 \). We have introduced the functional kernels \( \delta_{\pm} \) having the action

\[ \int_{-\infty}^{\infty} f(s) \delta_{\pm} (t-s) := f(t^{\pm}) \]  \hspace{1cm} (1)

for any Riemann integrable function \( f \).

If the right-hand side of (QWN2) was just \( \gamma \delta (t-s) \), then we would have sufficient instructions to deal with integrals of the noises wrt. Schwartz functions; however introducing the \( \delta_{\pm} \)-functions we can formally consider integrals wrt. piecewise Schwartz functions as we now have a rule for what to do at discontinuities. In particular, we can consider integrals over simplices \( \{ t > t_1 > \cdots > t_n > 0 \} \). The objective is to use the commutation rule (QWN2) to convert integral expressions involving the postulated noises \( a_t^\pm \) to equivalent normal ordered expressions, so that we only encounter the integrals of the \( \delta_{\pm} \)-functions against Riemann integrable functions.

We remark that the commutation relations (QWN2) arose from considerations of Markovian limits of field operators \( a_t^\pm (\lambda) \) in the Heisenberg picture satisfying relations of the type

\[ [a_t^\pm (\lambda), a_s^\pm (\lambda)] = K_{t-s} (\lambda) \theta (t-s) + K_{t-s} (\lambda)^* \theta (s-t) \]

where \( \theta \) is the Heaviside function and the right-hand side is a Feynman propagator.

### 1.1 Quantum Stochastic Calculus

#### 1.1.1 Fundamental Stochastic Processes

For real square-integrable functions \( f = f(t) \), we define the following four fields

\[ A^{ij} (f) := \int_0^\infty [a_s^+]^i [a_s^-]^j f(s) \, ds, \quad i, j \in \{0, 1\} \]  \hspace{1cm} (2)

where on the right-hand side the superscript denotes a power, that is \( [a]^0 = 1 \), \( [a]^1 = a \).

With \( 1_{[0,t]} \) denoting the characteristic function of the time interval \([0,t]\), we define the four fundamental processes

\[ A_t^{ij} := A_t^{ij} (1_{[0,t]}) = \int_0^t [a_s^+]^i [a_s^-]^j \, ds, \quad i, j \in \{0, 1\} \]  \hspace{1cm} (3)
1.1.2 Testing Vectors

We denote by $F^{(n)}$ the Hilbert space spanned by the symmterized vectors $f_1 \hat{\otimes} \ldots \hat{\otimes} f_n := A^{10}(f_1) \ldots A^{10}(f_n) \Psi$. The Hilbert spaces $F^{(n)}$ are then naturally orthogonal for different $n$ and their direct sum is the (Bose) Fock space $F = \oplus_{n=0}^{\infty} F^{(n)}$. The exponential vector with test function $f \in L^2(\mathbb{R}^+)$ is defined to be

$$\varepsilon(f) := \sum_{n=0}^{\infty} \frac{1}{n!} \hat{\otimes}^n f.$$

(4)

Note that $\langle \varepsilon(f) | \varepsilon(g) \rangle = \exp \langle f | g \rangle$ and that $\Psi \equiv \varepsilon(0)$.

1.1.3 Quantum Stochastic Processes

Let $\mathcal{H}$ be a fixed Hilbert space, we wish to consider operators on the tensor product $\mathcal{H} \otimes F$. A family of operators $(X_t)_{t \geq 0}$ defined on a common domain $D \otimes \mathcal{E}(\mathcal{T})$ can be understood as a mapping from $D \times D \times \mathcal{T} \times \mathcal{T} \times \mathbb{R}^+ \to \mathbb{C}$: $(\phi, \psi, f, g, t) \mapsto \langle \phi \otimes \varepsilon(f) | X_t \psi \otimes \varepsilon(g) \rangle$.

1.1.4 Quantum Stochastic Integrals

Let $\left(X_t^{ij}\right)_{t \geq 0}$ be four adapted processes. The quantum stochastic integral having these processes as integrands is

$$X_t = \int_0^t ds \left[ a_s^+ \right]^i X_s^{ij} \left[ a_s^- \right]^j$$

(5)

where we use the Einstein convention that repeated indices are summed (over 0,1). We shall use the differential notation $dX_t = [a_t^+]^i X_t^{ij} [a_t^-]^j dt$ or even $\frac{dX_t}{dt} = [a_t^+]^i X_t^{ij} [a_t^-]^j$.

The key feature is that the noises appear in normal ordered form in the differentials. Suppose that the $\left(X_t^{ij}\right)_{t \geq 0}$ are defined on domain $D \otimes \mathcal{E}(\mathcal{R})$, then it follows that

$$\left\langle \phi \otimes \varepsilon(f) \frac{dX_t}{dt} \psi \otimes \varepsilon(g) \right\rangle = \left[f^* (t)\right]^i \left\langle \phi \otimes \varepsilon(f) X_t^{ij} \psi \otimes \varepsilon(g) \right\rangle \left[g(t)\right]^j$$

for all $\phi, \psi \in D$ and $f, g \in \mathcal{R}$. (The equivalence can be understood here as being almost everywhere.)
1.1.5 Quantum Itô’s Formula

Let $X_t$ and $Y_t$ be quantum stochastic integrals, then the product $X_tY_t$ may be brought to normal order using (QWN2). In differential terms we may write this as

$$d (X_tY_t) = (dX_t) Y_t + X_t (dY_t) = \left( dX_t \right) Y_t + X_t \left( dY_t \right)$$

where the Itô differentials are defined as

$$(\hat{d}X_t) Y_t := \left[ a^+ \right]^i \left( X_t \right) Y_t^j \left[ a^- \right]^j dt ;$$

$$X_t \left( \hat{d}Y_t \right) := \left[ a^+ \right]^k \left( X_t \right) Y_t^{kl} \left[ a^- \right]^l dt ;$$

The quantum Itô “table” corresponds to following relation for the fundamental processes:

$$(\hat{d}A_t^i) \left( \hat{d}A_t^j \right) = \gamma \left( \hat{d}A_t^{ij} \right).$$

1.1.6 Quantum Stochastic Differential Equations

The differential equation \( \frac{dX_t}{dt} = \left[ a^+ \right]^i \left[ a^- \right]^j X_t^{ij} \) with initial condition $X_0 = x_0$ (a bounded operator on $H_0$) corresponds to the differential equation system

$$\left\langle \phi \otimes \varepsilon (f) \left| \frac{dX_t}{dt} \right| \psi \otimes \varepsilon (g) \right\rangle = \left[ f^* (t) \right]^i \left\langle \phi \otimes \varepsilon (f) \left| X_t^{ij} \right| \psi \otimes \varepsilon (g) \right\rangle \left[ g (t) \right]^j$$

and in general one can show the existence and uniqueness of solution as a family of operators on $H_0 \otimes F$. The solution can be written as $X_t = x_0 + \int_0^t ds \left[ a^+ \right]^i \left[ a^- \right]^j X_s^{ij}$. In Hudson-Parthasarathy notation this is written as $X_t = x_0 + \int_0^t X_s^{ij} \otimes \hat{d}A_t^{ij}$, where the tensor product sign indicates the continuous tensor product decomposition $F = \Gamma \left( L^2 (0, t) \right) \otimes \Gamma \left( L^2 (t, \infty) \right)$.

1.2 Quantum Stochastic Evolutions

A quantum stochastic evolution is a family $(J_t)_{t \geq 0}$ mapping from the bounded observables on $H_0$ to the bounded observables on $H_0 \otimes F$. We are particularly interested in those taking the form

$$J_t (X) = U_t^\dagger X U_t$$

where $U_t$ is a unitary, adapted process satisfying some linear qsde (a stochastic Schrödinger equation). In the rest of this section we establish a Wick’s theorem for working with such processes.

1.2.1 Normal-Ordered QSDE

Let $V_t$ be the solution to the qsde

$$\frac{dV_t}{dt} = L_{ij} \left[ a^+_i \right]^j V_t \left[ a^-_j \right] \quad V_0 = 1;$$

where $U_t$ is a unitary, adapted process satisfying some linear qsde (a stochastic Schrödinger equation). In the rest of this section we establish a Wick’s theorem for working with such processes.
where the $L_{ij}$ are bounded operators in $\mathcal{H}_0$. The associated integral equation is $V_t = 1 + \int_0^t dt_1 L_{ij} \left[a^+_{t_1}\right]^i V_{t_1} \left[a^-_{t_1}\right]^j$ which can be iterated to give the formal series

$$V_t = 1 + \sum_{n=1}^{\infty} \int_0^t dt_1 \ldots dt_n L_{i_1 j_1} \ldots L_{i_n j_n} \left[a^+_{t_1}\right]^{i_1} \ldots \left[a^+_{t_n}\right]^{i_n} \left[a^-_{t_n}\right]^{j_n} \ldots \left[a^-_{t_1}\right]^{j_1}$$

(9)

where $\tilde{N}$ is the normal ordering operation for the noise symbols $a^\pm_t$.

Necessary and sufficient conditions for unitary of $V_t$ are that

$$L_{ij} + L^\dagger_{ji} + \gamma L^\dagger_{1i} L_{1j} = 0.$$  

(11)

(Necessity is immediate from the isometry condition $d(U_t^\dagger U_t) = U_t^\dagger \hat{d}(U_t) + \hat{d}(U_t^\dagger) U_t + \hat{d}(U_t^\dagger) \hat{d}(U_t) = U_t^\dagger \left( L_{ij} + L^\dagger_{ji} + \gamma L^\dagger_{1i} L_{1j} \right) U_t \otimes \hat{d} A^2 = 0$, but suffices to establish co-isometry $d(U_t^\dagger U_t) = 0$.) Equivalently, we can require that

$$L_{11} = \frac{1}{\gamma} (W - 1), \quad L_{10} = L,$$

$$L_{01} = -L^\dagger W, \quad L_{00} = -\frac{1}{2} \gamma L^3 L - iH;$$

where $W$ is unitary, $H$ is self-adjoint and $L$ is arbitrary.

### 1.2.2 Time-Ordered QSDE

Let $U_t$ be the solution to the qdsd

$$\frac{dU_t}{dt} = i E_{ij} \left[a^+_{t}\right]^i \left[a^-_{t}\right]^j U_t, \quad U_0 = 1;$$

(12)

where the $E_{ij}$ are bounded operators in $\mathcal{H}_0$. Here we naturally interpret $\Upsilon_t = E_{ij} \left[a^+_{t}\right]^i \left[a^-_{t}\right]^j$ as a stochastic Hamiltonian. (For $\Upsilon_t$ to be Hermitian we would need $E_{11}$ and $E_{00}$ to be self-adjoint while $E_{10}^\dagger = E_{01}$.)

Iterating the associated integral equation leads to

$$U_t = 1 + \sum_{n=1}^{\infty} \left( -i \right)^n \int_{t > t_1 > \ldots > t_n > 0} dt_1 \ldots dt_n E_{i_1 j_1} \ldots E_{i_n j_n} \left[a^+_{t_1}\right]^{i_1} \ldots \left[a^+_{t_n}\right]^{i_n} \left[a^-_{t_n}\right]^{j_n} \ldots \left[a^-_{t_1}\right]^{j_1}$$

(13)

$$= \tilde{T} \exp \left\{ -i \int_0^t ds E_{ij} \left[a^+_{s}\right]^i \left[a^-_{s}\right]^j \right\},$$

(14)

where $\tilde{T}$ is the time ordering operation for the noise symbols $a^\pm_t$.  

5
1.2.3 Conversion From Time-Ordered to Normal-Ordered Forms

Using the commutation relations (QWN2) we can put the time-ordered expressions in (12) to normal order. The most efficient way of doing this is as follows:

\[
\left[a_i^{-}, U_t\right] = \left[a_i^{-}, 1 - i \int_{0}^{t} ds \, E_{ij} \left[a_s^{+}\right]^i \left[a_s^{-}\right]^j U_s\right] = -i \int_{0}^{t} ds \, E_{ij} \left[a_i^{-'} a_s^{+}\right] \left[a_s^{-}\right]^j U_s
\]

or \(a_i^{-} U_t - U_t a_i^{-} = -i E_{11} a_i^{-} U_t - i E_{10} U_t\). This implies the rewriting rule

\[
a_i^{-} U_t = (1 + i \kappa E_{11})^{-1} \left\{U_t a_i^{-} - i E_{10} U_t\right\}.
\]

Thus \(i E_{ij} \left[a_s^{+}\right]^i \left[a_s^{-}\right]^j U_t = i E_{01} \left[a_i^{+}\right]^i U_t + i E_{11} \left[a_i^{+}\right]^i (1 + i \kappa E_{11})^{-1} \left\{U_t a_i^{-} - i E_{10} U_t\right\} \). From this we deduce the following result:

**Theorem 1** Time-ordered and normal-ordered forms are related as

\[
\mathbf{T} \exp \left\{-i \int_{0}^{t} ds \, E_{ij} \left[a_s^{+}\right]^i \left[a_s^{-}\right]^j\right\} \equiv \mathbf{N} \exp \left\{\int_{0}^{t} ds \, L_{ij} \left[a_s^{+}\right]^i \left[a_s^{-}\right]^j\right\}
\]

where

\[
L_{11} = -i E_{11} (1 + i \kappa E_{11})^{-1}, \quad L_{10} = -i (1 + i \kappa E_{11})^{-1} E_{10},
\]

\[
L_{01} = -i E_{01} (1 + i \kappa E_{11})^{-1}, \quad L_{00} = -i E_{00} + \kappa E_{01} (1 + i \kappa E_{11})^{-1} E_{10}.
\]

2 Gaussian States

The above construction requires the existence of a vacuum state \(\Psi\); by application of creation fields we should be able to reconstruct a Hilbert-Fock space for which \(\Psi\) is cyclic. But what about non-vacuum states? We describe now the trick we shall use in order to consider more general states for the simple case of one bosonic degree of freedom.

Let \(a, a^\dagger\) satisfy the commutation relations \([a, a^\dagger] = 1\). A state \(\langle \cdot \rangle\) is said to be Gaussian or quasi-free if we have

\[
\langle \exp \left\{iz^* a + i a^\dagger\right\}\rangle = \exp \left\{\frac{1}{2} n (n + 1) z z^* + m^* z^2 + m z^* z + i z^* \alpha + i \alpha^* z\right\};
\]

in particular, \(\langle a\rangle = \alpha, \langle aa^\dagger\rangle = n + 1, \langle a a^\dagger\rangle = m\). From the observation that \(\langle (a + \lambda a^\dagger)^\dagger (a + \lambda a^\dagger)\rangle \geq 0\) it follows that the restriction \(|m|^2 \leq n (n + 1)\) must apply.
Now suppose that $a_1, a_{1\dagger}$ and $a_2, a_{2\dagger}$ are commuting pairs of Bose variables and let
\[ a = xa_1 + ya_{2\dagger} + za_2 + \alpha \] (19)
where $x, y, z, \alpha$ are complex numbers. The commutation relations are main-
tained if $|x|^2 - C|y|^2 + |z|^2 = 1$. Taking the vacuum state for both variables $a_i, a_{i\dagger}$ ($i = 1, 2$) then we can reconstruct the state $\langle \rangle$ if $|x|^2 + |z|^2 = n + 1$ and $yz = m$. That is
\[ x = \sqrt{n + 1 - \frac{|m|^2}{n}}, \quad y = \sqrt{n}, \quad z = \frac{m}{\sqrt{n}}. \] (20)

2.1 Generalized Araki-Woods Construction

Let $\mathfrak{h}$ be a fixed one-particle Hilbert space and let $j$ be a anti-linear conjugation
on $\mathfrak{h}$, that is $\langle j\phi | j\psi \rangle_{\mathfrak{h}} = \langle \psi | \phi \rangle_{\mathfrak{h}}$ for all $\phi, \psi \in \mathfrak{h}$. Denote by $A(\phi)$ the annihilator
on $\Gamma(\mathfrak{h})$ with test function $\phi$ (that is, $A(\phi) \varepsilon(\psi) = \langle \phi | \psi \rangle_{\mathfrak{h}} \varepsilon(\psi)$). A state $\langle \rangle$ on $\Gamma(\mathfrak{h})$ is said to be (mean-zero) Gaussian / quasi-free if there is a positive
operator $N \geq 0$; an operator $M$ with $|M|^2 \leq N(N + 1)$ and $[N, M] = 0$; and a
fixed anti-linear conjugation $j$ such that
\[ \langle \exp \{ iA(\phi) + iA^\dagger(\phi) \} \rangle = \exp \left\{ -\frac{1}{2} \langle \phi | (2N + 1) \phi \rangle_{\mathfrak{h}} - \frac{1}{2} \langle M\phi | j\phi \rangle_{\mathfrak{h}} - \frac{1}{2} \langle j\phi | M\phi \rangle_{\mathfrak{h}} \right\} \] (21)
for all $\phi \in \mathfrak{h}$. In particular, we have the expectations
\[ \langle A(\phi) A^\dagger(\psi) \rangle = \langle \phi | N\psi \rangle_{\mathfrak{h}}, \]
\[ \langle A(\phi) A(\psi) \rangle = \langle M\phi | j\psi \rangle_{\mathfrak{h}}, \]
\[ \langle A^\dagger(\phi) A^\dagger(\psi) \rangle = \langle j\phi | M\psi \rangle_{\mathfrak{h}}. \]

The state is said to be gauge-invariant when $M = 0$. The case where
$N = (1 - e^{-\beta H})^{-1}$ and $M = 0$ yields the familiar thermal state at inverse
temperature $\beta$ for the non-interacting Bose gas with second quantization of $H$
as Hamiltonian. The vacuum state is, of course $N = 0, M = 0$.

The standard procedure for treating thermal states of the interacting Bose
gas is to represent the canonical commutations (CCR) algebra on the tensor
product $\Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$ and realize state as a double Fock vacuum state. This
approach generalizes to the problem at hand. For clarity we label the Fock
spaces with subscripts 1 and 2 and consider the morphism from the CCR algebra
over $\Gamma(\mathfrak{h})$ to that over $\Gamma(\mathfrak{h})_1 \otimes \Gamma(\mathfrak{h})_2$ induced by
\[ A(\phi) \mapsto A_1(X\phi) \otimes 1_2 + 1_1 \otimes A_2^\dagger(jY\phi) + 1_1 \otimes A_2(Z\psi) \] (22)
where
\[ X = \sqrt{N + 1 - \frac{|M|^2}{N}}, \quad Y = \sqrt{N}, \quad Z = \frac{M}{\sqrt{N}}. \] (23)
Here we write $A_i(\psi)$ for annihilators on $\Gamma(\mathfrak{h})_i$ ($i = 1, 2$).
3 Gaussian Noise

Now let $a_1^\pm(t)$ and $a_2^\pm(t)$ be independent (commuting) copies of quantum white noises with respective vacua $\Psi_1$ and $\Psi_2$. Let $x,y,z$ be as in (18) and let $\alpha$ be arbitrary complex. We consider the quantum white noise(s) defined by

$$a_i^+ := xa_i^- (t) + ya_2^- (t) + za_2^- (t) + \alpha.$$  \hspace{1cm} (24)

We consider the stochastic dynamics generated by the formal Hamiltonian

$$\mathcal{H}_t = Ca_1^+ + C^\dagger a_1^- + F$$ \hspace{1cm} (25)

where $C$ and self-adjoint $F$ are operators on $\mathcal{H}_0$. We are required to “normal order” the unitary process $U_t = \mathcal{T} \exp \left\{ -i \int_0^t ds \mathcal{H}_s \right\}$ for the given state and this means normal order the $a_1^\pm(t)$ and the $a_2^\pm(t)$. By using the same technique as in (13) we find that, for instance, $[a_1^- (t), U_t] = -i \int_0^t ds \ [a_1^- (t), \mathcal{H}_s] U_s = -ikxCU_t$. We can deduce the rewriting rules

$$a_1^- (t) U_t = U_t (a_1^- (t) - i kx C)$$

$$a_2^- (t) U_t = U_t (a_2^- (t) - i kx C - i ky C^\dagger).$$

The qse $\frac{dU_t}{dt} = -i \mathcal{Y}_t U_t$ can be then be rewritten as

$$\frac{dU_t}{dt} = -i :\mathcal{Y}_t U_t : -i CyU_t (-i k^* C - i ky C^\dagger)$$

$$-i Cx U_t (-i k^* C) - i C^\dagger z U_t (-i k^* C - i ky C^\dagger)$$

where $:\mathcal{Y}_t U_t :$ is the reordering of $\mathcal{Y}_t U_t$ placing all creators $a_1^+ (t)$ and $a_2^+ (t)$ to the left and all annihilators $a_1^- (t)$ and $a_2^- (t)$ to the right. Rearranging gives

$$\frac{dU_t}{dt} = -i C (x a_1^+ (t) U_t + y U_t a_2^- (t) + z^* a_2^+ (t) U_t + \alpha^* U_t)$$

$$-i C^\dagger (x U_t a_1^- (t) + y a_2^+ (t) U_t + z U_t a_2^- (t) + \alpha U_t)$$

$$- [i F + \kappa ((n+1) C^\dagger C + nCC^\dagger + m^* CC + mC^\dagger C^\dagger)] U_t.$$ \hspace{1cm} (26)

To obtain an equivalent Hudson-Parthasarathy qse, we introduce quantum Brownian motions $A_t = \int_0^t (xa_1^- (s) + ya_2^- (s) + za_2^- (s)) ds$ with the understanding that, for $R_t$ adapted, $R_t \otimes dA_t = (x R_t a_1^- (t) + y R_t a_2^- (t) + z R_t a_2^- (t))$ and $R_t \otimes dA_t = (x a_1^+ (t) R_t + y R_t a_2^- (t) + z R_t a_2^- (t))$. Then

$$dU_t = -i C U_t \otimes dA_t + C^\dagger U_t \otimes dA_t - GU_t \otimes dt$$ \hspace{1cm} (27)

where $G = i (F + \alpha^* C + \alpha C^\dagger) + \kappa ((n+1) C^\dagger C + nCC^\dagger + m^* CC + mC^\dagger C^\dagger)$. Note that the quantum Itó table will be

$$dA_t dA_t^\dagger = \gamma (n+1) dt; \hspace{0.5cm} dA_t^\dagger dA_t = \gamma n dt;$$

$$dA_t^\dagger dA_t^\dagger = \gamma m^* dt; \hspace{0.5cm} dA_t dA_t^\dagger = \gamma m dt.$$ \hspace{1cm} (28)
It is readily shown, either by normal ordering or by means of the quantum stochastic calculus, that the stochastic Heisenberg equation is then
\[ dJ_t (X) = -i J_t ([X, C^\dagger]) \otimes dA_t^\dagger - i J_t ([X, C]) \otimes dA_t + J_t (L (X)) \otimes dt \] (29)
where
\[ L (X) = \gamma \left\{ (n + 1) C^\dagger XC + nCXC^\dagger + mCXC + m^*C^\dagger XC^\dagger \right\} - XG - G^\dagger X. \] (30)

Finally the master equation is obtained by duality: \( \frac{d}{dt} \varrho = L' (\varrho) \) where \( tr \{ \varrho L (X) \} = tr \{ L' (\varrho) X \} \).

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