Gröbner Basis Procedures for Testing Petri Nets

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October 26, 2018

Abstract
This paper contains introductory material on Petri nets and Gröbner basis theory and makes
some observations on the relation between the two areas. The aim of the paper is to show how
Gröbner basis procedures can be applied to the problem of reachability in Petri nets, and to give
details of an application to testing models of navigational systems.

1 Introduction

Petri nets are a graphical and mathematical modelling tool applicable to many systems. They may
be used for specifying information processing systems that are concurrent, asynchronous, distributed,
parallel, non-deterministic, and/or stochastic. Graphically, Petri nets are useful for illustrating and
describing systems, and tokens can simulate the dynamic and concurrent activities. Mathematically,
it is possible to set up models such as state equations and algebraic equations which govern the be-
behaviour of systems. Petri nets are understood by both practitioners and theoreticians and so provide
a powerful communication link between them. For example, engineers can show mathematicians how
to make practical and realistic models and mathematicians may be able to produce theories to make
the systems more methodical or efficient, which is in fact demonstrated by this collaborative paper.

The area of computer algebra called Gröbner basis theory includes the rewriting theory widely used in
computer science and provides methods for handling the rule systems defining various types of alge-
braic structure. It has been proved that it is not always possible to deduce all consequences of a system
of rules – when it is possible the levels of complexity involved quickly require the use of computers. In the commutative case computational Gröbner basis methods have has been successfully applied in theorem proving, robotics, integer programming, coding theory, signal processing, enzyme kinetics, experimental design, differential equations, and many others. All major computer algebra packages now include implementations of these procedures, and pocket calculator implementations will soon be available. A collection of recent papers on Gröbner basis research is [3].

In this paper we show how Gröbner basis procedures can be applied to reversible Petri nets to solve the reachability problem. This provides a practical test which can be useful in the design and analysis of Petri nets. In particular the examples show a practical application of the Gröbner basis methods to Petri nets modelling navigation systems. Further details of these mechatronic navigation systems can be found in [6]. Related algebraic research, and preliminaries to this paper may be found in [8].

2 Background to Gröbner Bases

We give a brief summary of the main results in commutative Gröbner basis theory that will be used in this paper. For a fuller introduction to the subject see [1, 5].

Let \( X \) be a set. Then the elements of \( X^\Delta \) are all power products of elements of \( X \), including an identity 1, with multiplication defined in the usual way. The commutativity condition is summarised by \( xy = yx \) for all \( x, y \in X \). Let \( K \) be a field (the field of rational numbers, \( \mathbb{Q} \) suffices for our work).

Then \( K[X^\Delta] \) is the ring of commutative polynomials

\[
f = k_1m_1 + \cdots + k_tm_t
\]

where \( k_1, \ldots, k_t \in K \) and \( m_1, \ldots, m_t \in X^\Delta \) with the operations of polynomial addition and polynomial multiplication defined in the usual way.

Consider a set of polynomials \( P \subseteq K[X^\Delta] \). We say that two polynomials \( f \) and \( g \) of \( K[X^\Delta] \) are \textit{equivalent modulo} \( P \) and write \( f =_P g \) if their difference can be expressed in terms of \( P \), i.e.

\[
f - g = u_1p_1 + \cdots + u_np_n
\]

for some \( p_1, \ldots, p_n \in P, u_1, \cdots, u_n \in K[X^\Delta] \).

In 1965 Bruno Buchberger invented the concept of a Gröbner basis [2]. Techniques of Gröbner basis theory enable us to decide whether or not \( f =_P g \) for given \( P, f, g \) in \( K[X^\Delta] \) as above.

Computation begins by specifying an ordering \( > \) on the power products (this must be a well-ordering, compatible with multiplication). This enables us to define reduction modulo a set of polynomials \( P \) – multiples of polynomials in \( P \) are subtracted from a given polynomial \( f \) in order to obtain successively smaller polynomials – the reduction is denoted \( \rightarrow_P \). The reflexive, symmetric, transitive closure of \( \rightarrow_P \) coincides with the congruence \( =_P \). If \( P \) is a Gröbner basis then \( \rightarrow_P \) is confluent, meaning that there is a unique irreducible element in each congruence class, obtainable from any other element by repeated reduction modulo \( P \). If \( P \) is not a Gröbner basis then it is always possible to use Buchberger’s
algorithm to obtain a set of polynomials $Q$ which is a Gröbner basis such that $=P$ coincides with $=Q$.

Thus, given a set of polynomials $P \subseteq K[X^\Delta]$, the problem of deciding whether $f$ is equivalent to $g$ modulo $P$ for any $f, g \in K[X^\Delta]$ can always be determined by calculating a Gröbner basis $Q$. The polynomials are equivalent if and only if their difference $f - g$ reduces modulo $Q$ to zero.

We will not explain these calculations in any greater detail, but refer the reader to texts on Gröbner bases, such as [1, 5]. In the commutative case it is always possible to determine a Gröbner basis, but computers are usually required for all but the most basic problems. In our examples we use MAPLE and GAP3, with some Gröbner basis procedures implemented by the second author [8].

### 3 Petri Nets

A Petri net has two types of vertices: places (represented by circles) and transitions (represented by double lines). Edges exist only between places and transitions and are labelled with their weights. In modelling, places represent conditions and transitions represent events. A transition has input and output places, which represent preconditions and postconditions (respectively) of the event. A good introduction to the ideas of Petri nets is [12].

**Definition 3.1 (Petri Net)** A Petri net (without specific initial marking) is a quadruple $\mathcal{N} = (X, T, F, w)$ where: $X$ is a finite set (of places), $T$ is a finite set (of transitions), $F \subseteq (X \times T) \cup (T \times X)$ is a set of edges (flow relation) and $w : F \to \mathbb{N}$ is a weight function.

The state of a system is represented by the assignation of “tokens” to places in the net.

**Definition 3.2 (Marking)** A marking is a function $M : X \to \mathbb{N} \cup \{0\}$.

Dynamic behaviour is represented by changes in the state of the Petri net which is formalised by the concept of firing.

**Definition 3.3 (Firing Rule)**

i) A transition $t$ is enabled if each input place $x$ of $t$ is marked with at least $w(x, t)$ tokens.

ii) An enabled transition may or may not fire – depending on whether or not the relevant event occurs.

iii) Firing of an enabled transition $t$ removes $w(x, t)$ tokens from each input place $x$ of $t$ and adds $w(t, y)$ tokens to each output place $y$ of $t$.

Despite their apparent simplicity, Petri nets can be used to model complex situations – for some examples see [12]. One of the main problems in Petri net theory is reachability – the problem corresponds to deciding which situations (modelled by the net) are possible, given some sequence of events.
Definition 3.4 (Reachability) A marking \( M_1 \) is said to be \emph{reachable} from a marking \( M_2 \) in a net \( \mathcal{N} \), if there is a sequence of firings that transforms \( M_2 \) to \( M_1 \). Often a Petri net comes with a specified initial marking \( M_0 \). The \emph{reachability problem} for a Petri net \( \mathcal{N} \) with initial marking \( M_0 \) is: Given a marking \( M \) of \( \mathcal{N} \), is \( M \) reachable in \( \mathcal{N} \)?

For the type of Petri nets defined so far, reachability is decidable in exponential time and space [12].

Reversibility is a property of Petri nets corresponding to the potential for the device being modelled to be reset. For our applications it is essential that we can reset, therefore this property is vital.

Definition 3.5 (Reversibility) A Petri net \( \mathcal{N} \) is called \emph{reversible} if a marking \( M' \) is reachable from a marking \( M \) in \( \mathcal{N} \), then \( M \) is reachable from \( M' \).

Different definitions of reversibility exist. The definition we use is chosen for engineering rather than mathematical reasons as in [12]. The paper [4] by Caprotti, Perscha and Hong contains a result apparently similar to ours, but they use a different definition of reversibility, which is much more restrictive – perhaps this is appropriate for different applications.

In order to apply Gröbner basis techniques we use monomials to represent the markings (there is a one-to-one correspondence between monomials and markings), and so associate a transition with the difference between two monomials (input and output).

Definition 3.6 (Polynomial Associated with a Marking) Let \( \mathcal{N} = (X, T, \mathcal{F}, w) \) be a Petri net. To every marking \( M \) we will associate a polynomial

\[
\text{pol}(M) := \prod_{x} x^{M(x)},
\]

that is the formal product of elements of \( X \) raised to the power \( M(x) \) (the number of tokens held at the place \( x \)).

Definition 3.7 (Polynomial Associated with a Transition) Each transition \( t \) has an associated polynomial

\[
\text{pol}(t) := \prod_{x} x^{w(x,t)} - \prod_{y} y^{w(t,y)},
\]

that is the input required for the transition to be enabled minus the output resulting from a firing. We often write \( \text{pol}(t) = l - r \), to distinguish the two terms.

To represent the dynamic structure we must consider how the transition polynomials are related to polynomials of markings which enable them and how firings of transitions affect the polynomials of the markings. Suppose a marking \( M_i \) enables a transition \( t_i \). By the definitions it is clear that this corresponds to \( \text{pol}(M_i) \) being equal to \( u_i l_i \) where \( \text{pol}(t_i) = l_i - r_i \) and \( u_i \) is a power product in \( X^\Delta \). It then follows that if \( t_i \) fires, the resulting marking \( M_{i+1} \) will have polynomial \( \text{pol}(M_{i+1}) = \text{pol}(M_i) - u_i \text{pol}(t_i) = u_i r_i \).
Example 3.8 (Polynomials and the Firing Rule)

The diagrams above show three different states of a transition $t_3$ of a Petri net Example 3.13. The polynomial associated with the transition is $\text{pol}(t_3) = x_3x_6 - x_4$. The first marking $M_1$ does not enable $t_3$; this corresponds to the fact that $\text{pol}(M_1) = (x_6)^2$ is not a multiple of $x_3x_6$. The second marking $M_2$ does enable $t_3$, and $\text{pol}(M_2) = x_3(x_6)^2$. The marking resulting from the firing of $t_3$ after it has been enabled by $M_2$ is $M_3$. In terms of polynomials the firing is represented by $\text{pol}(M_3) = \text{pol}(M_2) - x_6\text{pol}(t_3) = x_4x_6$. A firing sequence is denoted by $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} M_n$ where the $M_i$ are markings and the $t_i$ are transitions (events) transforming $M_{i-1}$ into $M_i$. In terms of polynomials the above firing sequence gives the information $\text{pol}(M_n) = \text{pol}(M_0) - u_1\text{pol}(t_1) - u_2\text{pol}(t_2) - \cdots - u_n\text{pol}(t_n)$ for some $u_1, u_2, \ldots, u_n \in X^\Delta$.

Theorem 3.9 (Reachability and Equivalence of Polynomials)

Let $\mathcal{N}$ be a reversible Petri net with initial marking $M_0$. Define $P := \{\text{pol}(t) : t \in T\}$. Then a marking $M$ is reachable in $\mathcal{N}$ if and only if $\text{pol}(M_0) =_P \text{pol}(M)$.

Proof First suppose that $M$ is reachable. Then there is a firing sequence $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{n-1}} M_{n-1} \xrightarrow{t_n} M$. Therefore, as above, there exist $u_1, \ldots, u_n \in X^\Delta$ such that $\text{pol}(M_0) - \text{pol}(M) = u_1\text{pol}(t_1) + \cdots + u_n\text{pol}(t_n)$. Hence $\text{pol}(M_0) =_P \text{pol}(M)$.

For the converse, suppose $\text{pol}(M_0) =_P \text{pol}(M)$. Then

$$\text{pol}(M_0) = \text{pol}(M) \pm u_1\text{pol}(t_1) \pm \cdots \pm u_m\text{pol}(t_m).$$

The proof is by induction on $m$.

For the base step put $m = 0$ then $\text{pol}(M_0) = \text{pol}(M)$. The correspondence between markings and their associated polynomials is one-to-one, so here $M_0 = M$ and $M$ is clearly reachable.

For the induction step we assume that a marking $M'$ is reachable from $M_0$ if

$$\text{pol}(M_0) = \text{pol}(M') \pm u_1\text{pol}(t_1) \pm \cdots \pm u_m\text{pol}(t_{m-1}).$$
for a fixed $m$. Now suppose $M$ is a marking such that
\[ \text{pol}(M) = \text{pol}(M_0) \pm u_1\text{pol}(t_1) \pm \cdots \pm u_m\text{pol}(t_m). \]

Then for some $i \in \{1, \ldots, m\}$ either $\text{pol}(M_0) = u_i l_i$ or $\text{pol}(M_0) = u_i r_i$ where $\text{pol}(t_i) = l_i - r_i$. In the first case $\text{pol}(M_0) = u_i l_i$. Observe that $M_0$ enables $t_i$ and define a marking $M'$ by $M_0 \xrightarrow{t_i} M'$. Then
\[ \text{pol}(M') = \text{pol}(M) \pm u_1\text{pol}(t_1) \pm \cdots \pm u_i l_i \text{pol}(t_i-1) \pm u_{i+1} \text{pol}(t_{i+1}) \pm \cdots \pm u_m\text{pol}(t_m) \]
so, by assumption, $M$ is reachable from $M'$ and so $M$ is reachable from $M_0$. In the second case $\text{pol}(M_0) = u_i r_i$. There is a marking $M'$ such that $\text{pol}(M') = u_i r_i$ and
\[ \text{pol}(M') = \text{pol}(M) \pm u_1\text{pol}(t_1) \pm \cdots \pm u_i r_i \text{pol}(t_i-1) \pm u_{i+1} \text{pol}(t_{i+1}) \pm \cdots \pm u_m\text{pol}(t_m). \]
Now, $M$ is reachable from $M'$ by assumption and $M_0$ is reachable from $M'$ by a firing of $t_i$. By reversibility, therefore, $M'$ is reachable from $M_0$ and hence $M$ is reachable from $M_0$. \hfill \square

**Corollary 3.10 (Gröbner Bases Determine Reachability)**

Reachability in a reversible Petri net can be determined using a Gröbner basis.

**Proof** Let $K$ be a field. First observe that $P \subseteq K[X^\Delta]$. Let $Q$ be a Gröbner basis for $P$. Then $\text{pol}(M) = \text{pol}(M_0)$ if and only if there exists $p \in K[X^\Delta]$ such that $\text{pol}(M)$ and $\text{pol}(M_0)$ reduce to $p$ by $\rightarrow_Q$. \hfill \square

**Remark 3.11 (Catalogue of Reachable Markings)** Recall that Gröbner bases techniques use an ordering on the power products. There is a one-to-one correspondence between power products and markings. We can begin to catalogue the markings in increasing order. Given a Gröbner basis for the polynomials of the transitions of a Petri net it can be determined whether each marking is reachable: if the power product reduces to the same irreducible power product as the initial marking then it is reachable. In this way the Gröbner basis can be used to build up a list of reachable markings.

**Remark 3.12 (Testing for Reversibility in Petri Net Design)** The reversibility of a Petri net can be interpreted as the ability to reset the application it models. Whilst the reachability of a place, given an initial marking, can be determined by standard means, reversibility cannot be established directly.

Calculating a Gröbner basis for the Petri net makes the determination of reachable markings much more obvious, and unwanted markings can be immediately detected. There are two reasons why unwanted markings may occur. In the first case there is a basic error in the net which allows some firing sequence of marking which should be avoided; the Gröbner basis is effective in showing up these markings. The second type of problem occurs when marking supposed to be unreachable is found to be reachable, the implication here being that the net is not truly reversible. As reversibility is a
desirable property, the net can then be modified and retested.

In practical terms Gröbner bases have been shown by the authors to be useful in Petri net design – repeated testing by computing Gröbner bases shows up unintended effects or non-reversibility. Our examples are Petri nets designed by the first author to model software interfaces to hardware components of mobile robot navigation systems, and their development was helped in this way.

**Example 3.13 (Software Interface for Motors)** This Petri net represents the software interface between a user and the set of motors used to drive a mobile robot.

Here, once the motors have been initialised, the user may input the required speed and direction for each motor. This information is then interpreted and written to the relevant port, if there is also a token available in the “ready” place (3), to enable the “interpret speed and direction” transition $t_3$.

The places are labelled $x_1, \ldots, x_{11}$. There are eight transitions, and their polynomials are as follows:

- $pol(t_1) = x_1 - x_2 x_3$
- $pol(t_2) = x_2 - x_7$
- $pol(t_3) = x_3 x_6 - x_4$
- $pol(t_4) = x_4 - x_5$
- $pol(t_5) = x_7 - x_6$
- $pol(t_6) = x_5 - x_3 x_8$
- $pol(t_7) = x_3 x_8 - x_1$
- $pol(t_8) = x_8 - x_7$

The Gröbner basis for this set of polynomials – with respect to a degree-lexicographic ordering – is

$$\{x_4 - x_1, x_5 - x_1, x_6 - x_2, x_7 - x_2, x_8 - x_2, x_2 x_3 - x_1\}.$$  

The catalogue of markings reachable from an initial marking $x_1$ is quickly calculated to be:

$$\{x_1, x_4, x_5, x_2 x_3, x_3, x_6, x_3 x_7, x_3 x_8\}.$$
This catalogue can be examined by the Petri net designer who interprets the different states. When unexpected states appear in the catalogue it indicates an error, which generally signifies that the net is not reversible.

For Petri nets such as this to execute efficiently, it is essential that the user can confirm both the reachability and the reversibility of the net. For instance, should the place “done” (5) prove to be unreachable from an initial marking where the place “start” (1) held a token, this would show that no data would be written to the port in transition “write to port” ($t_4$), thus making the motors uncontrollable. If the net here was non-reversible, it would indicate that the motors could not be disabled, which in this situation is undesirable. Once the Petri net has been tested for such bugs, the user need only concern themselves with the simple functions executed within individual transitions, greatly decreasing the likelihood of a serious, or perhaps dangerous, failure of the robot.

4 Coloured Petri Nets

A coloured Petri net circulates tokens of more than one type. The transitions in the net are affected differently by different combinations of colours of tokens. An example of this is where tokens represent data signals. Incomplete or corrupt signals should be dealt with differently from complete signals, these two types of data would be represented by different colours of tokens (“pass” and “fail” in Example 4.3).

Recall that if $C$ is a set (of colours) then $C^\Delta$ is the set of all power products of elements of $C$. Essentially an element of $C^\Delta$ assigns a non-negative integer to each element of $C$. The definition of a coloured Petri net that we give uses this kind of notation, but is equivalent to that given by Murata in [12]. One element $m$ of $C^\Delta$ is said to be a multiple of another element $l$ if $m = ul$ for some $u \in C^\Delta$.

Definition 4.1 (Coloured Petri Net) A coloured Petri net is a quintuple $N_C = (X, T, C, F, w)$, where $X$ is a set of places, $T$ is a set of transitions, $C$ is a set of colours, $F \subseteq (X \times T) \cup (T \times X)$ is the flow relation and $w : F \to C^\Delta$. A marking in $N_C$ is a function $M : X \to C^\Delta$. The firing rule is as follows:

i) A transition $t$ is enabled if each input place $x$ of $t$ is marked with a multiple of $w(x, t)$.

ii) An enabled transition may or may not fire.

iii) A firing of an enabled transition $t$ deletes the power product $w(x, t)$ from the marking at each input place $x$, and appends the marking at each output place $y$ with the power product $w(t, y)$.

A coloured Petri net can in fact be considered as a structurally folded version of an ordinary Petri net if the number of colours is finite. Each place $x$ is unfolded into a set of places, one for each colour of token which $x$ may hold, and each transition $t$ is unfolded into a number of transitions, one for each way that $t$ may fire. It is immediate that the techniques discussed in the previous section may be applied to coloured Petri nets. In fact we can pass directly from the coloured Petri net to commutative polynomials in $K[(X \times C)^\Delta]$, where $K$ is a field. Elements of $(X \times C)^\Delta$ are written $(x_1, c_1) \cdots (x_n, c_n)$, where $x_1, \ldots, x_n \in X$, and $c_1, \ldots, c_n \in C$. We define $(x_i, c_i)(x_j, c_j) = (x_i, c_i c_j)$ when $x_i = x_j$. 

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Theorem 4.2 (Gröbner Bases for Coloured Petri Nets)

Let \( \mathbf{N}_C \) be a coloured Petri net. If \( M \) is a marking in \( \mathbf{N}_C \), then define the polynomial associated with the coloured marking to be \( \text{pol}(M) := \prod_X (x, M(x)) \). Similarly if \( t \) is a transition in \( \mathbf{N}_C \), then define the polynomial associated with the coloured transition to be \( \text{pol}(t) := \prod_X (x, w(x, t)) - \prod_Y (y, w(t, y)) \).

From these definitions we observe that a transition \( t \) in a coloured Petri net has an associated polynomial of the form \( \text{pol}(t) = l - r \) where \( l, r \in (X \times C)^\Delta \). The transition \( t \) is enabled by a marking \( M \) if \( \text{pol}(M) = ul \), for some \( u \in (X \times C)^\Delta \). If \( t \) fires then the new marking has associated polynomial \( ur \).

It follows that if we define \( P := \{ \text{pol}(t) : t \in T \} \) then a marking \( M \) is reachable if and only if \( \text{pol}(M) =_P \text{pol}(M_0) \). Therefore if \( Q \) is a Gröbner basis for \( P \) it is decidable whether or not \( M \) is reachable in \( \mathbf{N}_C \).

The results (and proofs) are naturally very similar to the results for standard Petri nets. The value is in the application – where it is more efficient to work with coloured nets it is appropriate to associate polynomials to these models directly.

Example 4.3 (Software Interface for Compass) The following Petri net shows the software interface to an external compass, where the compass provides data in the form of an ASCII string.

The states here are numbered, but two types of token: “pass” \((x)\) and “fail” \((y)\), circulate in the net. This Petri net is initialised with a single “pass” \((x)\) token at the “start” place (1) together with a “pass” \((x)\) and a “fail” \((y)\) token in each of the places “input” (18) and “continue” (19). The additional tokens at (18) and (19) provide the colouring essential for rigorous testing of this Petri net. For instance, when the “return data” \(t_3\) or \(t_20\) transition is fired, the colour of the token output to place “raw data ready” (3) depends solely on the colour of the token from place “input” (18). The transitions “read in” \(t_4\) or \(t_{21}\), “calculate checksum” \(t_5\) or \(t_{22}\) and “test” \(t_6\) or \(t_{23}\) will output a token matching the input token, having no effect on the colouring, but the transition “find bearing” \(t_7\) will only be enabled by a “pass” token, which represents a received ASCII string with a correct checksum, as determined in the “test” \(t_6\) or \(t_{23}\) transition. A “fail” token would instead enable the transition “data request” \(t_{16}\), which will provide a value using dead reckoning in place of the corrupted data.

Colouring of this net is helpful, as it ensures that only complete uncorrupt data is used. The Petri net of this example was constructed by repeated testing using Gröbner basis methods. We use \(x_i\) to denote a “pass” token at place \(i\), and \(y_i\) to denote a “fail” token at place \(i\). The initial marking is therefore associated with the monomial \(x_1x_{18}x_{19}y_{18}y_{19}\). The set \(P\) of polynomials associated with the transitions is as follows:

\[
\begin{align*}
&x_1 - x_2x_4, \quad x_5 - x_{12}, \quad x_2x_{18} - x_3x_{18}, \quad y_2y_{18} - y_3y_{18}, \quad x_3x_{13} - x_6, \quad y_3x_{13} - y_6, \quad x_6 - x_7, \quad y_6 - y_7, \\
&x_7 - x_8, \quad y_7 - y_8, \quad x_8 - x_{10}, \quad x_{12} - x_{13}, \quad x_{11} - x_2x_{14}, \quad x_{14}x_{19} - x_{15}x_{19}, \quad x_{14}y_{19} - y_{15}y_{19}, \quad y_{15} - x_{17}, \quad y_3x_{17} - x_{16}, \quad x_{2}x_{17} - x_{16}, \quad x_{15} - x_{12}, \quad x_4 - x_5, \quad y_8 - x_9, \quad x_9 - x_{11}, \quad x_{10} - x_{11}, \quad x_{16} - x_1.
\end{align*}
\]

Using MAPLE a Gröbner basis \(Q\) for \(P\) with respect to the order \(t\text{deg}\) has 47 rules

Given the initial marking \(x_1x_{18}x_{19}y_{18}y_{19}\), there are 11 reachable markings having five tokens and 32 reachable markings having six tokens. Examining the catalogue of reachable states and relating them to the situations they represent will confirm that the net will behave as the user would expect.
5 Further Considerations

5.1 Boundedness

Another interesting property is boundedness – the maximum number of tokens that may exist at a particular place or the maximum number of tokens that can exist in the entire net – given an initial marking. It is obvious to see how the catalogue may be used to check either type of boundedness, but more interesting to observe that certain information may be derived directly from the \((tdeg)\) Gröbner basis. If the Gröbner basis contains only polynomials \(l - r\) (assume \(l > r\)) such that \(l\) and \(r\) are power products of the same total degree then all reachable markings will have the same number of tokens. The least number of tokens possible is the degree of the reduced form of the polynomial associated with the initial marking. Regarding the polynomials \(l - r\) as reduction rules \(l \rightarrow r\) we can sometimes determine the most number of tokens possible by examining the degree-reducing rules to find what multiples of the reductum can be reduced to the same form as the initial marking (it was possible to do this with the 47 rule Gröbner basis obtained for our last example).

5.2 Use and Efficiency

Similarly to [4] we point out that although in general Gröbner basis computation can be lengthy, the type arising from Petri nets are not usually complex, involving only two-term polynomials with unitary coefficients. There is no problem, in any case with ordinary or coloured Petri nets, as commutative Gröbner bases can always be found, using a computer algebra package (e.g. MAPLE).

Although it is possible to make use of existing implementations of Buchberger’s Algorithm it would be practical to include the Gröbner basis procedures as part of the software in our mechatronic navigation systems. One aim of the research in [6] is to provide an easier way of safely programming a mobile robot. By using a Petri net to model the navigation system the C code controlling the robot is split into small pieces, corresponding to the transitions in the net. A transition can be programmed in a few lines, and code for a selection of alternative transitions could be provided in advance. The structure of the net corresponds to the structure of the executable program, and thus by replacing individual transitions in the net the whole program for controlling the mobile robot can be rewritten and retested with the minimum difficulty. The Gröbner basis tests would form an important part of the software, particularly in terms of safety. One example this work could be applied to would be an autonomous excavator. By using the Petri net representation, modifications to the control of the excavator could be made in the field, without the requirement for on site programming expertise. The Gröbner basis testing would provide a catalogue of reachable markings. If any undesirable (dangerous) states of the Petri net were shown to be reachable, this problem could be rectified by further alteration to the net until the model was shown to be satisfactory.

5.3 Streamed Petri Nets

We are interested in Petri nets that can model systems involving streams of data. Places will hold ordered lists of coloured tokens rather than unordered sets of tokens. This introduces a degree of noncommutativity into the Petri net. The Gröbner basis situation is more interesting here than with the ordinary Petri nets. Undecidability of the word problem [10] indicates the existence of streamed
Petri nets for which it is not possible to determine whether or not a state is reachable. The streamed models we have worked with store the streams of data as stacks or allow random access to any substream of data within a given stream. The problem with this is that the type of streamed Petri net suitable for our more advanced models is one whose transitions read data streams from the left and build them up on the right. This is a net to which we cannot yet apply Gröbner basis theory, but hope to investigate in future work.

5.4 Enhanced Petri Nets

Inhibitor arcs are the simplest extension to a basic Petri net. The inhibitor arc is represented by a line with a small circle at the end, equivalent to the $\text{NOT}$ in switching theory, and is used to prevent a transition from firing. If a transition $t$ has an inhibitor arc from a place $p$ then $t$ is enabled only when there are tokens in all of its ordinary input places and no tokens in the place $p$. The inhibitor arcs provide an alternative method of forcing a decision between two enabled transitions. These decisions can also be made randomly, or with the use of colours, but in this specific case, the inhibitor arc can give one transition priority over the other by preventing the second transition from firing. This method of decision making could be useful in any system where one function should be given priority over another. For instance, if a Petri net driving a mobile robot detected an obstruction, it would be important that it should stop, or alter the speed of the motors before attempting to read any sensors.

It is interesting to consider how the Gröbner basis methods could be extended to cover variations of the Petri net theory, especially when the results of the extensions are motivated by the requirement for testing modifications to navigation systems.

5.5 Linked Petri Nets

The motivation for our work has been the application to control systems of mobile robots, using the TRAMP philosophy (Toolkit for Rapid Autonomous Mobile Prototyping). It allows the analysis of different control components of a single mobile robot and it would be desirable for the Petri nets to be logically linked to provide a unified model of the control of the device. The analysis of the nets by Gröbner bases should then be extended to provide an analysis of the model as a whole. The problem of the subdivision of a large net into suitable components (objects) and the extension of local analyses of such components to global checks on reachability, safety etc, are examples of the well known local to global problem.

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