Topology and Duality in Abelian Lattice Theories

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Abstract

We show how to obtain the dual of any lattice model, with inhomogeneous local interactions, based on an arbitrary Abelian group in any dimension and on lattices with arbitrary topology. It is shown that in general the dual theory contains disorder loops on the generators of the cohomology group of a particular dimension. An explicit construction for altering the statistical sum to obtain a self-dual theory, when these obstructions exist, is also given. We discuss some applications of these results, particularly the existence of non-trivial self-dual 2-dimensional $\mathbb{Z}_N$ theories on the torus. In addition, we explicitly construct the $n$-point functions of plaquette variables for the $U(1)$ gauge theory on the 2-dimensional $g$-tori.

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It has been known for half a century \[1\] that duality is a powerful tool for analyzing statistical models \[2\]. In recent years, these ideas have been revived and have had impact in the study of supersymmetric gauge theories and superstring theory. The basic idea is that, given a statistical, field theoretical or string model, there may exist another representation of the model in which strong and weak coupling limits are interchanged. Most discussions of duality in statistical systems apply to spaces which have trivial topology. However, there are many instances of physical interest, such as regularized finite temperature gauge theory, where a statistical system is defined on a space with non-trivial topology and where some version of duality could be a useful tool. In this Letter we shall show that duality can indeed be implemented on a topologically non-trivial space. Our results apply to models with degrees of freedom in Abelian groups which live on \(k\)-cells (sites, links, plaquettes, etc.) of a lattice that is the triangulation of a \(d\)-dimensional smooth manifold. We restrict our attention to models with nearest-neighbor interactions.

The essence of our result is that Abelian duality on a lattice with non-trivial homology requires the appearance of disorder defects on cohomology cocycles of the lattice in either the original or the dual theory. These defects are similar to the familiar t’Hooft loops which are used to characterize the phase structure of gauge theories. The defects can be classified systematically and this allows one to identify self-dual models on spaces with non-trivial topology. For example, given a model that is self-dual on an infinite space, one can identify the modification of the statistical sum which makes the model self-dual when some of the dimensions are compactified. We will present as an example, a self-dual Ising model and a self-dual \(\mathbb{Z}_N\) spin model on the 2-torus. Also, we explicitly compute the \(n\)-point functions of plaquette variables for the \(U(1)\) gauge theory on the 2-dimensional \(g\)-tori.

To present our results in their most general form, we must introduce some terminology. We use the language of simplicial homology \[3, 4\]. Consider a lattice \(\Omega\) and associate to every \(k\)-dimensional cell of the lattice an oriented generator \(c_k^{(i)}\) where \(i\) indexes the various cells of dimension \(k\). These objects are used as generators of the \(k\)-chain group, denoted by \(C_k(\Omega, G)\),

\[
\sum_{i=1}^{N_k} g_i \ c_k^{(i)} = g \in C_k(\Omega, G) \quad , \quad g_i \in G
\]

Here \(G\) is an arbitrary Abelian group with group multiplication implemented through addition and \(N_k\) is the number of \(k\)-cells in the lattice \(\Omega\). An element \(g \in C_k(\Omega, G)\) is called a \(G\)-valued \(k\)-chain or simply a \(k\)-chain. Clearly \(C_k(\Omega, G) = \bigoplus_{i=1}^{N_k} G\).

Two homomorphisms, the boundary \(\partial\) and the coboundary \(\delta\), define the chain complexes \((C_*(\Omega, G), \partial)\) and \((C_*(\Omega, G), \delta)\) where \(C_*(\Omega, G) \equiv \bigoplus_{k=0}^{d} C_k(\Omega, G)\):

\[
\begin{array}{cccccccccccccccccccc}
0 \xrightarrow{\partial_{d+1}} C_d(\Omega, G) \xrightarrow{\partial_d} \ldots \xrightarrow{\partial_{k+2}} C_{k+1}(\Omega, G) \xrightarrow{\partial_k} C_k(\Omega, G) \xrightarrow{\partial_{k-1}} C_{k-1}(\Omega, G) \xrightarrow{\partial_{k-2}} \ldots C_0(\Omega, G) & \xrightarrow{\partial_0} 0 \\
0 \xleftarrow{\delta_d} C_d(\Omega, G) \xleftarrow{\delta_{d-1}} \ldots \xleftarrow{\delta_{k+1}} C_{k+1}(\Omega, G) \xleftarrow{\delta_k} C_k(\Omega, G) \xleftarrow{\delta_{k-1}} C_{k-1}(\Omega, G) \xleftarrow{\delta_{k-2}} \ldots C_0(\Omega, G) & \xleftarrow{\delta_0} 0
\end{array}
\]

These homomorphisms are defined by their actions on the generators \(c_k^{(i)}\) (we display the
dimension subscripts on $\partial_k$ and $\delta_k$ only when essential),

$$\partial c^{(i)}_k = \sum_{j=1}^{N_{k-1}} [c^{(i)}_k : c^{(j)}_{k-1}] c^{(j)}_{k-1}, \quad \delta c^{(i)}_k = \sum_{j=1}^{N_{k+1}} [c^{(j)}_{k+1} : c^{(i)}_k] c^{(j)}_{k+1}$$

where the incidence number is given by,

$$[c^{(i)}_k : c^{(j)}_{k-1}] = \begin{cases} \pm 1 & \text{if the } j \text{'th } (k-1)-\text{cell is contained in the } i \text{'th } k-\text{cell} \\ 0 & \text{otherwise} \end{cases}$$

The plus or minus sign reflects the relative orientation of the cells. The boundary (co-boundary) chains and the exact (co-exact) chains are defined as,

$$B_k(\Omega, G) = \text{Im } \partial_{k+1}, \quad B^k(\Omega, G) = \text{Im } \delta_{k-1}$$
$$Z_k(\Omega, G) = \text{ker } \partial_k, \quad Z^k(\Omega, G) = \text{ker } \delta_k$$

These sets inherit their group structure from the chain complex. The boundary and co-boundary operators are nilpotent: $\partial \partial = 0$ and $\delta \delta = 0$. The quotient groups,

$$H_k(\Omega, G) = Z_k(\Omega, G)/B_k(\Omega, G), \quad H^k(\Omega, G) = Z^k(\Omega, G)/B^k(\Omega, G)$$

are the homology and cohomology groups respectively. The elements of the homology group are those chains that have zero boundary and are themselves not the boundary of a higher dimensional chain, whilst the elements of the cohomology group are those chains that have zero coboundary and are themselves not the coboundary of a lower dimensional chain. The inner product on $C_k(\Omega, G)$ is defined by its action on the generators,

$$\langle c^{(i)}_k, c^{(j)}_l \rangle = \delta_{k,l} \delta^{i,j}$$

and is linear in both arguments. The coboundary operator and boundary operators are dual to each other with respect to this inner product,

$$\langle \partial c^{(i)}_k, c^{(j)}_{k-1} \rangle = \langle c^{(i)}_k, \delta c^{(j)}_{k-1} \rangle$$

Using this language, the models which we consider can be stated as follows: the degrees of freedom take values in $G$ and are defined on $(k-1)$-cells, and the Boltzmann weights are defined on the coboundary of a $(k-1)$-chain.,

$$Z = \sum_{g \in C_{k-1}(\Omega, G)} \prod_{i=1}^{N_k} B_i \left( (\delta g, c^{(i)}_k) \right)$$

Here, and throughout our discussion, we allow the Boltzmann weights, $B_i$, to differ on each $k$-cell. This generalization allows one to consider a large class of models, for example some of our results are easily generalized to random bond models. Also, it enables one to use the partition function as a generating function for certain correlators. Of particular interest is the correlator of a disorder operator and an anti-disorder operator. It is obtained from the partition function by shifting $\delta g$ by a chain, $f$, which has support on an assembly of adjacent
(d − k)-dimensional objects denoted by Γ. Γ is constructed by choosing two arbitrary (k − 1)-
cells on the dual lattice and connecting them by an arbitrary path of k-cells. It is then
re-interpreted on the original lattice by taking the lattice-dual of the path. The resulting
object is Γ and the chain f is explicitly given by the expression \( a \sum_{i \in \Gamma} c_d^{(i)} \) for some \( a \in G \).
Clearly, there are as many disorder operators as there are elements in the group \([5]\).

A familiar example of a spin system is the Ising model which, in our notation, has \( G = \mathbb{Z}_2 \)
and \( k = 1 \) so that for \( g \in C_0(\Omega, G) \),

\[
\langle \delta g, c_1^{(i)} \rangle = \sum_{i=1}^{N_G} g_i \langle \delta c_0^{(i)}, c_1^{(i)} \rangle = g_{i_1(l)} - g_{i_2(l)}
\]

where \( i_1(l) \) and \( i_2(l) \) are the end points of the link \( l \). The Boltzmann weight for aligned spins
( \( g_{i_1(l)} - g_{i_2(l)} = 0 \mod 2 \) ) is \( e^\beta \) and for anti-aligned spins ( \( g_{i_1(l)} - g_{i_2(l)} = 1 \mod 2 \) ) it is \( e^{-\beta} \)
so that,

\[
Z = \sum_{\{g_i=0,1\}} \prod_l \exp \{ \beta \left( 1 - 2 \left( g_{i_1(l)} - g_{i_2(l)} \mod 2 \right) \right) \}
\]

More general models of this kind include Abelian gauge theories, spin models in arbitrary
dimensions, plaquette models, etc.

We shall expand the Boltzmann weight in \([3]\) in terms of the characters, \( \chi_R(g) \) of the
irreducible representations \( R \in G^* \),

\[
B_i(g) = \sum_{R \in G^*} b_i(R) \chi_R(g), \quad b_i(R) = \frac{1}{|G|} \sum_{h \in G} \chi_R(h) B_i(h) \quad (4)
\]

where \( |G| \) is the order of the group. In the case of a continuous group the normalized sum over
group elements is replaced by the Haar integration measure. Since \( G \) is Abelian, \( G^* \) inherits
an Abelian group structure, where the product (taken to be addition) is implemented via the
tensor product of representations of \( G \). In particular: \( aR + bS = (\otimes_{i=1}^a R) \otimes (\otimes_{i=1}^b S) \) for
\( a, b \in \mathbb{Z} \) and \( R, S \in G^* \). This implies the following factorization properties for the characters,

\[
\begin{align*}
\chi_R(h_1 + h_2) &= \chi_R(h_1) \chi_R(h_2) & R \in G^* \text{ and } h_1, h_2 \in G \\
\chi_R(a h_1) &= \chi_{aR}(h_1) = \chi_{R^a}(h_1) & a \in \mathbb{Z}
\end{align*} \quad (5)
\]

We insert \([3]\) into the partition function \([3]\) to obtain,

\[
Z = \sum_{g \in C_{k-1}(\Omega, G)} \prod_{i=1}^{N_k} \sum_{r_i \in G^*} b_i(r_i) \chi_{r_i} \left( \langle \delta g, c_1^{(i)} \rangle \right)
= \sum_{g \in C_{k-1}(\Omega, G)} \sum_{r \in C_1(\Omega, G^*)} \prod_{i=1}^{N_k} \left\{ b_i \left( \langle r, c_1^{(i)} \rangle \right) \chi_{(r,c_1^{(i)})} \left( \langle \delta g, c_1^{(i)} \rangle \right) \right\} \quad (6)
\]

Here, the product over \( k \)-cells has been interchanged with the sum over group representations.
Every \( k \)-cell has been associated with a representation \( r_i \in G^* \) and this information is encoded
in the \( G^* \) valued \( k \)-chain, \( r = \sum_{i=1}^{N_k} r_i c_k^{(i)} \). Applying the factorization properties \([3]\) to the
product of characters in \( \otimes \) one finds,

\[
\prod_{i=1}^{N_k} \chi_{r, c_k^{(i)}} \left( \langle \delta g, c_k^{(i)} \rangle \right) = \prod_{i=1}^{N_k} \chi_{r, c_k^{(i)}} \left( \sum_{j=1}^{N_k-1} g_j \ [c_k^{(i)} : c_k^{(j)}] \right) = \prod_{i=1}^{N_k} \prod_{j=1}^{N_k-1} \chi_{r, c_k^{(i)}} (g_j \ [c_k^{(i)} : c_k^{(j)}])
\]

\[
= \prod_{j=1}^{N_k} \chi_{r, c_k^{(i)}} \sum_{i=1}^{N_k} g_j \ [c_k^{(i)} : c_k^{(j)}] \ (g_j)
\]

(7)

The representations in the subscript of \( \chi \) can be rewritten in a more transparent manner by noticing the following,

\[
\sum_{i=1}^{N_k} \langle r, c_k^{(i)} \ [c_k^{(i)} : c_k^{(j)}] = \langle r, \sum_{i=1}^{N_k} [c_k^{(i)} : c_k^{(j)}] c_k^{(i)} \rangle = \langle r, \delta c_k \rangle = \langle \delta r, c_k \rangle
\]

which follows directly from linearity of the inner product and the definition of the coboundary operator \( \partial \). It is now possible to sum expression (7) over \( g \),

\[
\sum_{g \in C_{k-1}(\Omega, G)} \prod_{j=1}^{N_k-1} \chi_{(\delta r, c_k^{(j)} \ [g_j)} = \prod_{j=1}^{N_k-1} \sum_{g_j \in G} \chi_{(\delta r, c_k^{(j)} \ [g_j)} = \prod_{j=1}^{N_k} \chi_{(\delta r, c_k^{(j)} \ [G)} = \prod_{j=1}^{N_k} \delta \left( \langle \delta r, c_k^{(j)} \rangle, 0 \right)
\]

(8)

The last equality follows due to the orthogonality of the characters, \(|G|^{-1} \sum_{g \in G} \chi_R(g) \chi_S(g) = \delta(R, S)\) for \( R, S \in G^* \). The partition function now depends on \( k \)-chains with constraints,

\[
Z = |G|^{N_k-1} \sum_{r \in C_k(\Omega, G^*)} \prod_{i=1}^{N_k} b_i \left( \langle r, c_k^{(i)} \rangle \right) \prod_{j=1}^{N_k-1} \delta \left( \langle \delta r, c_k^{(j)} \rangle, 0 \right)
\]

However, the constraints simply force \( r \) to be an exact chain \( \partial r = 0 \). This implies that \( r \) is the sum of a boundary chain and representatives of elements of the homology group under inclusion into the chain group,

\[
r = \partial r' + \sum_{a=1}^{A_k} h_a \gamma_a , \quad r' \in C_{k+1}(\Omega, G^*)
\]

(9)

where \( \{ \gamma_a : a = 1, \ldots, A_k \} \) are the generators of \( H_k(\Omega, G^*) \cong \bigoplus_{a=1}^{A_k} H_{k,a}(\Omega, G^*) \) and \( h_a \in H_{k,a}(\Omega, G^*) \). We will use \( h \in H_k(\Omega, G^*) \) to denote this inclusion, which is necessary so that addition in the argument of the Boltzmann weight is group multiplication in \( G^* \) and not in \( H_{k,a}(\Omega, G^*) \). Removing the constraints and inserting this form for \( r \) into the partition function one finds,

\[
Z = |G|^{N_k-1-d_{k+1}} \sum_{h \in H_k(\Omega, G^*)} \sum_{r \in C_{k+1}(\Omega, G^*)} \prod_{i=1}^{N_k} b_i \left( \langle h + \partial r, c_k^{(i)} \rangle \right)
\]

(10)

Here we have dropped the prime on the dummy index \( r' \), and \( d_k \equiv \dim \ker \partial_k \) which is required to prevent overcounting.

The final step in the duality transformation is to re-interpret \( c_k^{(i)} \) as elements on the dual lattice: so that we make the association \( c_{d-k}^{(i)} \leftrightarrow c_k^{(i)} \) where \( c_k^{(i)} \) live on the dual lattice \( \Omega^* \).
Under this association $\langle \partial c_k^{(i)}, c_l^{(j)} \rangle = \langle \delta c_{d-k}^{* (i)}, c_{d-l}^{* (j)} \rangle$ follows naturally. With this reinterpretation the dual partition function is given by the compact expression,

$$Z = |G|^N_{d-k+1} \frac{D_{d-k}^*}{D_{d-k} - 1} \sum_{h \in H^{d-k}(\Omega^*, G^*)} \sum_{r \in C_{d-k-1}(\Omega^*, G^*)} \prod_{i=1}^{N_{d-k}^*} b_i \left( \langle h + \delta r, c_{d-k}^{* (i)} \rangle \right)$$  \hspace{1cm} (11)

where $D_{d-k}^* = d_k$ and is the dimension of the kernel of $\delta_{d-k}$ on the dual lattice. Notice that $h$ is now an element in the $(d-k)$’th cohomology group of the dual lattice, which is isomorphic to the $k$’th homology group of the lattice.

We now see that, in general, the dual theory has additional disorder terms corresponding to the generators of the cohomology group of the dual lattice. By comparing (11) and (3), it is clear that, in order for (3) to be self-dual, $H_k(\Omega, G^*)$ has to be trivial. Of course, in addition, the lattice must be self-dual, $k$ must equal $d - k$ so that the original and dual degrees of freedom live on the same type of cells and $G$ must be isomorphic to $G^*$.\footnote{We interpret this duality in the general sense where the Boltzmann factors $B_i(g)$ for each group element $g$ can be regarded as independent coupling constants and the dual Boltzmann factors $b_i(r)$ for each $r \in G^* \cong G$ are the images of these constants under the duality transformation. In the normal sense of duality one requires a more specific manner in which $B(g)$ transforms into $b(r)$, in particular, one requires these to have the same functional form.}

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Of course, the Ising model on an infinite plane satisfies all the requirements of self-duality. However, suppose one compactifies both directions so that the model now lives on a torus. Topology is the only obstruction to that model being self-dual. To see this explicitly we write down the partition function for the dual model. Firstly note that the generators of $H^1(\Omega, G)$ are the cocycles $h^1 = \sum_{t \in \sigma^1} c_1^{(t)}$ and $h^2 = \sum_{t \in \sigma^2} c_1^{(t)}$, where the sums are over all links in $\sigma^i$ shown in figure 1. These cocycles are the duals (in the sense of the inner product) to the cocycles $h^1 = \sum_{t \in \sigma^1} c_1^{(t)}$ and $h^2 = \sum_{t \in \sigma^2} c_1^{(t)}$, where the sums are over all links in $\sigma^i$ shown in figure 1. These cocycles are the duals (in the sense of the inner product) to the cocycles $h^1 = \sum_{t \in \sigma^1} c_1^{(t)}$ and $h^2 = \sum_{t \in \sigma^2} c_1^{(t)}$, where the sums are over all links in $\sigma^i$ shown in figure 1. These cocycles are the duals (in the sense of the inner product) to the cocycles $h^1 = \sum_{t \in \sigma^1} c_1^{(t)}$ and $h^2 = \sum_{t \in \sigma^2} c_1^{(t)}$, where the sums are over all links in $\sigma^i$ shown in figure 1. These cocycles are the duals (in the sense of the inner product) to the cocycles $h^1 = \sum_{t \in \sigma^1} c_1^{(t)}$ and $h^2 = \sum_{t \in \sigma^2} c_1^{(t)}$, where the sums are over all links in $\sigma^i$ shown in figure 1. These cocycles are the duals (in the sense of the inner product) to the
cycles $\gamma_i$ which generate the homology group. Using (11), the dual of the Ising model on the torus has partition function,

$$Z \propto \sum_{h_1, h_2 = 0, 1} \sum_{\{r_i, r_0 = 0, 1\}} \exp \left\{ \beta^* \left[ \sum_{l \in \sigma^1 \cup \sigma^2} \left( 1 - 2 \left( r_{i_1(l)} - r_{i_2(l)} \right) \mod 2 \right) \right] 
+ \sum_{l \in \sigma^1} \left( 1 - 2 \left( r_{i_1(l)} - r_{i_2(l)} + h_1 \mod 2 \right) \right) + \sum_{l \in \sigma^2} \left( 1 - 2 \left( r_{i_1(l)} - r_{i_2(l)} + h_2 \mod 2 \right) \right) \right\} \right\}$$

Here $\beta^* = -\frac{1}{2} \ln \tanh \beta$ is the well known dual coupling constant, also notice that $\sigma^1 \cap \sigma^2 = \emptyset$ thus all links are accounted for in the Boltzmann weight. The dual partition function is then the sum of four copies of the Ising models in which the couplings along $\sigma^i$ are taken to be $(\beta^*, \beta^*)$, $(-\beta^*, \beta^*)$, $(\beta^*, -\beta^*)$ and $(-\beta^*, -\beta^*)$. The extra terms have the interpretation of disorder defects [3, 5].

We have shown that the dual theory on a lattice with non-trivial cohomology contains extra topological terms. In order to formulate self-dual models in these cases it is necessary to begin with a theory that contains a subset of the cohomology generators. We shall see that under duality these extra terms cancel some of the topological terms that would appear in the dual theory. Such models have partition functions given by,

$$Z = \sum_{h \in \tilde{H}^k_{(A)}(\Omega, G)} \sum_{g \in C_{k-1}(\Omega, G)} \prod_{i=1}^{N_k} B_i \left( (\delta g + h, c_k^{(i)}) \right)$$

where $\tilde{H}^k_{(A)}(\Omega, G)$ is a subgroup of $\tilde{H}^k(\Omega, G)$ generated by a subset of the generators of the cohomology group, $\{\sigma^a : a \in A \subseteq \{1, \ldots, A_k\}\}$. An element $h \in \tilde{H}^k_{(A)}(\Omega, G)$ is written as, $h = \sum_{a \in A} h^a \sigma^a$ with $h^a \in H^{k,a}(\Omega, G) \rightarrow G$. As before, the inclusion is necessary so that addition in the argument of the Boltzmann weight is group multiplication in $G$ rather than in $H^{k,a}(\Omega, G)$.

Once again the first step in the duality transformation is to perform a character expansion of the Boltzmann weights,

$$Z = \sum_{r \in C_k(\Omega, G^*)} \sum_{h \in \tilde{H}^k_{(A)}(\Omega, G)} \sum_{g \in C_{k-1}(\Omega, G)} \prod_{i=1}^{N_k} b_i \left( (r, c_k^{(i)}) \right) \chi_{(r, c_k^{(i)})} \left( (\delta g + h, c_k^{(i)}) \right)$$

In the above we have encoded the representations that live on $k$-cells in the $k$-chain $r$, and introduced the character coefficients $b_i \left( (r, c_k^{(i)}) \right)$ as in equation (4). The properties of the characters [3] factorize the partition function,

$$Z = \sum_{r \in C_k(\Omega, G^*)} \prod_{i=1}^{N_k} b_i \left( (r, c_k^{(i)}) \right) \sum_{g \in C_{k-1}(\Omega, G)} \prod_{j=1}^{N_k} \chi_{(r, c_k^{(j)})} \left( (\delta g, c_k^{(j)}) \right) \sum_{h \in \tilde{H}^k_{(A)}(\Omega, G)} \prod_{i=1}^{N_k} \chi_{(r, c_k^{(i)})} \left( (h, c_k^{(i)}) \right)$$

The sum over $g$ was performed previously and produced a delta function forcing $r$ to be an exact chain (see (7) and (8)). The sum over the cohomology elements will force additional
constraints on the representations. Using the factorization properties of the characters and the explicit representation $h = \sum_{a \in A} h^a \sigma^a$ one finds,

$$\chi_{(r,c_k^{(i)})}(\langle h,c_k^k \rangle) = \prod_{a \in A} \chi_{(r,c_k^{(i)})}(h^a \langle \sigma^a,c_k^k \rangle) = \prod_{a \in A} \chi_{(r,c_k^{(i)})}(\sigma^a,c_k^k)(h^a)$$

Performing the sum over $H^{k,a}$ and product over $k$-cells yields,

$$\sum_{h \in \bar{H}^{k,a}_k(\Omega,G)} \prod_{i=1}^{N_k} \chi_{(r,c_k^{(i)})}(\langle h,c_k^k \rangle) = \prod_{a \in A} \sum_{h^a \in H^{k,a}(\Omega,G)} \prod_{i=1}^{N_k} \chi_{(r,c_k^{(i)})}(\sigma^a,c_k^k)(h^a)$$

$$= \prod_{a \in A} \sum_{h^a \in H^{k,a}(\Omega,G)} \chi_{(r,c_k^{(i)})}(\langle h,c_k^k \rangle) = \prod_{a \in A} \sum_{h^a \in H^{k,a}(\Omega,G)} \chi_{(r,\sigma^a)}(h^a) \quad (13)$$

Now interpreting the inner product $\langle r, \sigma^a \rangle$ as a representation of $H^{k,a}(\Omega,G)$, which can be done since $H^{k,a}(\Omega,G)$ is a quotient subgroup of $G$, and performing the sum over $h^a$ forces these representations to be the trivial ones. Then (13) reduces to a product of delta functions, $\prod_{a \in A} |H^{k,a}(\Omega,G)| \delta(\langle r, \sigma^a \rangle, 0)$.

Putting this information together we obtain the very compact expression for the partition function,

$$Z = |G|^{N_{k-1} - d_{k+1}} \prod_{a' \in A} |H^{k,a'}(\Omega,G)| \sum_{r \in C_k^k(\Omega,G^*)} \left( \prod_{i=1}^{N_k} b_i(\langle r,c_k^i \rangle) \right) \delta(\partial r, 0) \prod_{a \in A} \delta(\langle r, \sigma^a \rangle, 0)$$

It is possible to solve the constraints and remove the delta functions. Since $r$ is forced to be exact, take it to be of the form (9). The other constraints then allows one to solve for some of the coefficients $h_b$. On lattices with no boundaries or torsion there exists an isomorphism between cocycles and cycles induced by the inner product. The generators can then be paired in the following way: $\langle \gamma_a, \sigma^a \rangle = 1$ for $a = 1, \ldots, A_k$ (the inner product is unity since one can always choose the canonical generators which intersect on only one cell; compare figure 1.) Restricting ourselves to lattices that have these properties and solving the other constraints leads to,

$$\langle \partial r', \sigma^a \rangle + \sum_{b=1}^{A_k} h_b(\gamma_b, \sigma^a) = 0 \quad \Rightarrow \quad h_a = -\langle \partial r', \sigma^a \rangle = -\langle r', \delta \sigma^a \rangle = 0 \quad \text{for } a \in A$$

the last equality follows since $\{\sigma^a\}$ are cocycles. Thus we see that the extra constraints force the coefficients of the cycles "dual" to the cocycles to vanish. Inserting these constraints into the partition function one obtains,

$$Z = |G|^{N_{k-1} - d_{k+1}} \prod_{a' \in A} |H^{k,a'}(\Omega,G)| \sum_{h \in \bar{H}^{k}_k(\Omega,G^*)} \sum_{r \in C_{k+1}(\Omega,G^*)} \prod_{i=1}^{N_k} b_i(\langle \partial r + h, c_k^{(i)} \rangle)$$

In the above $\bar{A}$ denotes the compliment of the set $A$, so that the homology subgroup is generated by those cycles that have no counterpart in the modified model (12). Now interpreting
this partition function on the dual lattice by making the associations mentioned earlier we find,

\[ Z = |G|^{N_{d-k+1}^* - D_{d-k-1}^*} \prod_{a' \in A} |H_{d-k,a'}(\Omega, G)| \sum_{h \in H_{d-k}^* (\Omega, G^*)} \sum_{r \in C_{d-k-1}(\Omega^*, G^*)}^{N_{d-k}^*} \prod_{i=1}^{N_{d-k}^*} b_i \left( ((\delta r + h), c_{d-k}^*) \right) \]

The generators of \( H_{d-k}^* (\Omega, G^*) \) are the cocycles on the dual lattice that are associated with the set of cycles \( \{ \gamma_b : b \in \mathcal{A} \} \) on the original lattice. In the case of the Ising model with one cocycle, \( \{\sigma^1\} \), added in at the start the constraints eliminate \( \{\gamma_1\} \), so that \( \{\gamma_2\} \) survives. The dual theory then has an extra term supported on the lattice-dual to \( \{\gamma_1\} \), denoted by \( \{\gamma^{\ast 1}\} \) however, that cocycle is \( \{\sigma^1\} \).

In order to illustrate the physical content of the ideas in this paper, we discuss the \( \mathbb{Z}_N \) model on the torus with topological terms in some detail. We will denote elements of \( \mathbb{Z}_N \) by \( n \), the representations of \( \mathbb{Z}_N \) are isomorphic to \( \mathbb{Z}_N \) and as such we label them by the integers \( r = 0, 1, \ldots, N - 1 \). The characters are given by \( \chi_r(n) = \exp(irn2\pi/N) \).

We discuss the most general case on the torus by choosing one of the subsets of the generators of the first Homology group: \( s_1 = \{\emptyset\}, s_2 = \{\sigma^1\}, s_3 = \{\sigma^2\} \) or \( s_4 = \{\sigma^1, \sigma^2\} \) (compare figure 1.) For a given choice of \( s_a \) define the model by the following partition function,

\[ Z(s_a, \beta) = \sum_{\{n_i\} \in s_a} \prod_{i \in s_a} B\left( n_{i_1} - n_{i_2} \mod N \right) \prod_{\sigma \in s_a} \sum_{\{h_\sigma=0\}} \prod_{l \in \sigma} B\left( n_{i_1} - n_{i_2} + h_\sigma \mod N \right) \]

for \( a = 1, 2, 3 \) or 4. In Villain form the Boltzmann factor is given by (equal on all links)

\[ B(n) = \sum_{j \in \mathbb{Z}} \exp \left\{ -\frac{\beta}{2} \left( \frac{n}{N} - j \right)^2 \right\} . \tag{14} \]

The coefficients \( b(r) \) of the character expansion are given by,

\[ b(r) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{j \in \mathbb{Z}} e^{-\frac{\beta}{2} \left( \frac{n}{N} - j \right)^2} e^{-i\frac{2\pi}{N}rn} = \frac{1}{N} \sum_{n \in \mathbb{Z}} e^{-\frac{\beta}{2}n^2} e^{-i\frac{2\pi}{N}rn} = \sqrt{\frac{2\pi}{\beta}} \sum_{m \in \mathbb{Z}} e^{-\frac{\beta}{2} \left( \frac{\pi}{\beta} - m \right)^2} \tag{15} \]

Here \( \beta^* = (2\pi)^2 N^2 \beta^{-1} \) is the dual coupling constant. Thus we see that \( b(r) \) has the same functional form as \( B(n) \). By applying the above discussed rules on the cancellation of the cycles we find the following transformation properties of \( Z(s_a, \beta) \) under the duality transformation,

\[ Z(s_a, \beta) = \left( \frac{4\pi}{\beta} \right)^N \prod_{b=1}^{4} M_{a,b} Z(s_b, \beta^*) \]

where the transformation matrix \( M_{a,b} \) is given by

\[ M = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \]
The matrix $M$ obeys $M^2 = 1$ as should be since applying the duality transformation twice gives the original model. A similar structure for the case of the homogeneous Ising model in a different approach was discussed in [7]. The appearance of cocycles in duality transformations was also appreciated by Rakowski and Sen [3], however they did not discuss self-duality.

Self-dual and anti-self-dual models can be constructed using the eigenvectors of $M$. Since $M^2 = 1$, the eigenvalues are either $+1$ or $-1$. In particular three eigenvectors have eigenvalue $1$ giving rise to the self dual models $Z(s_2, \beta)$, $Z(s_3, \beta)$ and $Z(s_1, \beta) + 2Z(s_4, \beta)$. The fourth eigenvector with eigenvalue $-1$ corresponds to the anti-self-dual model $-Z(s_1, \beta) + 2Z(s_4, \beta)$.

This strategy of choosing a complete set of cycles, computing the corresponding dual theories and finding eigenvectors of the transformation matrix $M$ is a method for finding self-dual theories which can easily be implemented for other models and in higher dimensions. In general, $M$ contains a single non-zero entry in each row. Since $M$ is non-singular (in fact $M^2 = 1$), this implies that every column also has only one non-zero entry.

As a second example, we show how one can use the inhomogeneous form of the partition function to compute correlators. In particular we compute some correlators of lattice gauge theories on 2-dimensional $g$-tori. In this case we choose $k = 2$ so that the degrees of freedom live on links of the lattice, and leave the group $G$ arbitrary for the moment. The partition function is,

$$Z = \sum_{\{g_i \in G\}} \prod_{p \in \Omega} B_p (\prod_{l \in p} g_l)$$

Here $l$ and $p$ labels the links and plaquettes of the lattice respectfully, and we have reverted to the multiplicative notation for group multiplication. Since our formalism allows for inhomogeneous Boltzmann weights it is possible to compute the expectation value of arbitrary $n$-point functions of plaquette variables in arbitrary representations. This includes eg. the filled Wilson loop. We alter the weights of the Boltzmann factors for the plaquettes in the support $\Gamma$ of our $n$-point function as follows:

$$B_p(g) \to \chi_{S_p}(g)B_p(g) = \sum_{R \in G^*} b_p(R)\chi_R(g)\chi_{S_p}(g) = \sum_{R \in G^*} b_p(R - S_p)\chi_R(g) \quad , \quad \text{for all } p \in \Gamma$$

This introduces a plaquette variable in representation $S_p$ for all plaquettes $p$ in $\Gamma$. The dual partition function is rather trivial since there are no cells of dimension negative one, and $H_0(\Omega, G^*) \cong G^*$. Using this fact, the $n$-point function is given by,

$$\left\langle \prod_{p \in \Gamma} \chi_{S_p}(g) \right\rangle = \frac{\sum_{h \in G^*} \prod_{i \in \Gamma^*} b_i(h) \prod_{i \in \Gamma} b_i(h - S_i^*)}{\sum_{h \in G^*} \prod_{i \in \Gamma} b_i(h)}$$

here $i$ labels the sites of the dual lattice, $\Gamma^*$ is the lattice-dual to $\Gamma$ and $S_i^* = S_p$. In the particular case of $G = U(1)$, and choosing the heat kernel action, $b_i(r) = (\sqrt{2\pi\beta})^{-1}e^{-r^2/2\beta}$, one can express the partition function in terms of $\theta$-functions [4],

$$\left\langle \prod_{p \in \Gamma} \chi_{S_p}(g) \right\rangle = \left( \prod_{i \in \Gamma^*} e^{-(S_i^*)^2/2\beta} \right) \frac{\sum_{h \in \mathbb{Z}} \exp \{-2(\beta)^{-1}N_0 h^2 + \beta^{-1}h \sum_{i \in \Gamma} S_i^*\}}{\sum_{h \in \mathbb{Z}} \exp \{-2(\beta)^{-1}N_0 h^2\}}$$

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For the (filled) Wilson loop, $\Gamma$ is a collection of adjacent plaquettes with all $S_p = 1$. The overall factor in (16) is given by $\exp \left( \frac{-1}{2\beta} \left( \sum_{i \in \Gamma^*} (S_i^*)^2 - \frac{1}{N_0^*} \left( \sum_{i \in \Gamma^*} S_i^* \right)^2 \right) \right) \frac{\theta_3 \left( \frac{2\beta}{N_0^*}; -\frac{1}{N_0^*} \sum_{i \in \Gamma} S_i^* \right)}{\theta_3 \left( \frac{2\beta}{N_0^*}; 0 \right)}$ (16)

These few examples by no means exhaust the applications of our formalism. A systematic study of its implications for confining gauge theories in higher dimensions is in progress.

References

[1] H.A. Kramers and G.H. Wannier, Phys. Rev. 60 (1941) 252.
[2] R. Savit, Rev. Mod. Phys. 52 (1980) 453.
[3] J. Munkres, Elements of Algebraic Topology, Addison-Wesley, Menlo Park 1984.
[4] K. Drühl and H. Wagner, Ann. Phys. 141 (1982) 225.
[5] E. Fradkin and L.P. Kadanoff, Nucl. Phys. 170 (1980) 1.
[6] L.P. Kadanoff and H. Ceva, Phys. Rev. B 3 (1971) 3918.
[7] A.I. Bugrij and V.N. Shadura, JEPT 82 (1996) 552; Phys.Rev. B55 (1997) 1045; Zh. Eksp. Teor. Fiz. 113 (1998) 240.
[8] M. Rakowski and S. Sen, Lett. Math. Phys. 42 (1997) 195; M. Rakowski, Phys. Rev. D52 (1995) 354.
[9] J. Spanier and K.B. Oldham, An Atlas of Functions, Hemisphere Publishing Corporation, New York 1987.