Collision Probability in an In-Line Equipment Model under Erlang Distribution*

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SUMMARY Flat Panel Displays (FPDs) are manufactured using many pieces of different processing equipment arranged sequentially in a line. Although the constant inter-arrival time (i.e., the tact time) of glass substrates in the line should be kept as short as possible, the collision probability between glass substrates increases as tact time decreases. Since the glass substrate is expensive and fragile, collisions should be avoided. In this paper, we derive a closed form formula of the approximate collision probability for a model, in which the processing time on each piece of equipment is assumed to follow Erlang distribution. We also compare some numerical results of the closed form and computer simulation results of the collision probability.

key words: stochastic model, collision probability, Erlang distribution, closed form, approximation

1. Introduction

Reflecting the increasing demand for Flat Panel Displays (FPDs) such as LCDs, plasma display panels, etc., more effective methods for their manufacture are required. The production rate improves with technological advancements such as the rapid enlargement of glass substrates and the miniaturization of patterns. Accordingly, production lines have to be modified to accommodate such advancements, and new optimization problems to be solved continue to arise. Lately, an advanced system called Crystal Flow[9] has been introduced in the production line of FPDs. It targets a higher level of line control in the next-generation production processes as well as in existing lines.

The main flow of the FPD process is shown in Fig. 1[11]. Each piece of processing equipment in Fig. 1 has a specialized role, such as cleaning, coating, proximity exposing, developing, etching, resist stripping, etc., and those pieces of equipment are connected in-line. Most production lines adopt a simple strategy to feed glass substrate into the first piece of equipment at a constant inter-arrival time, which is called the tact time. This strategy is simple and enables us to accurately predict the number of products.

Due to solution foaming, chemicals, heat treating, etc., the processing time at each piece of equipment is uncertain and may vary according to the conditions at that time. If a sheet of substrate is sent to a piece of equipment that is still processing a previously sent sheet of substrate, that newly sent sheet cannot be processed on the piece of equipment. This phenomenon is called a collision between substrates. Since the glass substrate is expensive and fragile, collisions should be avoided as much as possible.

A collision-like phenomenon is called a blocking in a flow shop model in the field of scheduling theory, and is studied as an important factor to determine line efficiency. If the processing time is deterministic, many study results on blocking exist (see extensive survey in [3]). If the processing time is stochastic, study results are somewhat limited in comparison with their deterministic counterparts. For example, see [7] and [8], where the purpose is to minimize the expected makespan. On the other hand, in queuing theory, depending on the rule for processing blockings (blocked calls cleared, blocked calls delayed, etc.), previous work mainly focused on performance measures in the steady state. We can find, in fact, that a wide range of literature in the field of queuing theory has been investigated, for example in [4], [5] and [6]. In contrast, in this paper, given the number of jobs to be processed in the prescribed tact time span, we focus on the measure recently presented in [1], which is the probability that there will be at least one collision, called the collision probability. In a comparatively new manufacturing system such as one manufacturing FPDs, the evaluation item (i.e., collision probability) is the new focus of observation.

The tact time (i.e., the inter-arrival time of substrates at the first piece of equipment) should be minimized in order to maximize the production rate. This, however, increases the collision probability. Thus there is a trade-off between...
where two parameters $\lambda$ and $k$ are a positive real number and a positive integer, respectively. Its expectation and variance are given by $k/\lambda$ and $k/\lambda^2$, respectively. Therefore, under Erlang distribution, we can set the expectation and the variance independently of each other by setting parameters $\lambda$ and $k$ appropriately. In Fig. 2, five different probability density functions are plotted, where expectations of all cases are the same, but their variances decrease through cases (a) to (e). Some of these pdfs are bell-shaped, and the pdf of Erlang distribution takes a value of zero for $x < 0$. Thus, Erlang distribution is flexible enough to represent actual processing times.

2. Model

We describe a formal model of the FPD production line. The following notations will be used:

- $M_1, M_2, \ldots, M_m$: $m$ different pieces of equipment in the line.
- $J_1, J_2, \ldots, J_n$: $n$ jobs to be processed.
- $T_{ij}^{(j)} (> 0)$: Processing time of job $J_i$ on piece of equipment $M_j$.
- $T_{\text{tact}} (> 0)$: Tact time, i.e., the time difference between the start time instants of $J_i$ and $J_{i+1}$ for all $1 \leq i \leq n-1$ at the entrance to the line.

The production model is illustrated in Fig. 3. With the same time interval, $T_{\text{tact}}$, jobs are successively fed into the line at the entrance. Every job is first processed on piece of equipment $M_1$. It is then automatically transported to the next piece of equipment $M_2$ after it has been finished on $M_1$. It is assumed, for simplicity, that the transportation time between pieces of equipment is nil. As soon as $M_2$ receives the job, it starts processing. In this manner, every job is processed on the pieces of equipment in the order $M_1, M_2, \ldots, M_m$, and then sent to the exit. Moreover, we assume that the processing time $T_{ij}^{(j)}$ on $M_j$ is a random variable that follows Erlang distribution with parameters $\lambda_j$ and $k_j$, and all $T_{ij}^{(j)} (1 \leq i \leq n, 1 \leq j \leq m)$ are independent of each other.

In the above model, a collision occurs if the next job arrives at $M_j$ while $M_j$ is still processing the current job. The following lemma on the collision condition between jobs was given in [1].

**Lemma 1:** Suppose that $T_{ij}^{(j)} = t_i^{(j)}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. For $n$ ($\geq 2$) jobs, there is no collision in the above production line of $m$ pieces of equipment if and only if

$$\sum_{j=1}^{l} t_{ij}^{(j)} \leq T_{\text{tact}} + \sum_{j=1}^{l-1} t_{i+1}^{(j)}$$

holds for all $1 \leq i \leq n-1$ and $1 \leq l \leq m$.

Lemma 1 is important for the analysis contained in this paper, therefore we explain the physical meaning of the inequality in this lemma. Since a collision can only occur between two consecutive jobs, we pay attention solely to them. For each $i$ and $l$, we compare the time when job $J_i$ is completed on piece of equipment $M_l$, with the time when job
$J_{i+1}$, which is next fed into the system, is completed on the previous piece of equipment $M_{i-1}$. If the former time value is less than or equal to the latter, then no collision occurs. Otherwise, a collision occurs. Note that the transportation time is assumed to be nil.

3. Collision Probability for $m = 1$

In this section, we derive the collision probability when there is one piece of equipment. By Lemma 1 with $m = 1$, the no-collision probability between consecutive jobs $J_i$ and $J_{i+1}$ becomes as follows:

\[
\Pr\left( x_i^{(1)} \leq T_{\text{tact}} : 1 \leq i \leq n - 1 \right) = (\Pr\left( T_i^{(1)} \leq T_{\text{tact}} \right))^{n-1} \quad \text{(since the $T_i^{(1)}$ are i.i.d.)}
\]

\[
= \left( \int_0^{T_{\text{tact}}} \lambda_1 t^{\lambda_1 - 1} e^{-\lambda_1 t} \, dt \right)^{n-1}
\]

\[
= \left( 1 - \sum_{k=0}^{n-1} \frac{(\lambda_1 T_{\text{tact}})^k}{k!} e^{-\lambda_1 T_{\text{tact}}} \right)^{n-1}.
\]

Note that the last expression is given by iteratively performing integration by parts. Therefore, the collision probability for $m = 1$ is given by the closed form

\[
1 - \left( 1 - \sum_{k=0}^{n-1} \frac{(\lambda_1 T_{\text{tact}})^k}{k!} e^{-\lambda_1 T_{\text{tact}}} \right)^{n-1}.
\]

4. Derivation of Collision Probability for $m = 2$

We derive an approximate collision probability for the case of $m = 2$ by considering only two consecutive jobs. The reason why we pay attention to two consecutive jobs is as follows: even if we consider $n$ jobs, a collision is the phenomenon which occurs between only two consecutive jobs. Therefore, we first pay attention to only two consecutive jobs, and then we derive the no-collision probability between them. After that, considering $n$ jobs, as the number of pairs of two consecutive jobs is $n - 1$ ($J_1$ and $J_2$, $J_2$ and $J_3$, ..., $J_{n-1}$ and $J_n$), we approximate the no-collision probability over all $n$ jobs using the $(n-1)$-th power of the above derived probability of two consecutive jobs.

For this, the following random variables are introduced. $X_1 = T_i^{(1)} - T_{\text{tact}}$, $X_2 = T_i^{(1)} + T_i^{(2)} - T_{\text{tact}}$. The random variable $X_1$ means the time difference between the time when job $J_i$ is completed on piece of equipment $M_1$ and the tact time. Similarly, the random variable $X_2$ means the time difference between the time when job $J_i$ is completed on piece of equipment $M_2$ and the time when job $J_{i+1}$ is completed on the previous piece of equipment $M_1$. Therefore, each $X_1 \leq 0$ and $X_2 \leq 0$ is the event that, under the assumption that there are only two consecutive jobs $J_i$ and $J_{i+1}$, there is no collision between $J_i$ and $J_{i+1}$ on piece of equipment $M_1$ and $M_2$, respectively. Next, we introduce the following event $E_i$ for values of $i$ from 1 to $n - 1$.

$E_i$: Event that, under the assumption that there are only two consecutive jobs $J_i$ and $J_{i+1}$, there is no collision between them.

By Lemma 1 for $m = 2$, the probability of event $E_i$ occurring is given by

\[
\Pr(E_i) = \Pr(T_i^{(1)} \leq T_{\text{tact}}, T_i^{(1)} + T_i^{(2)} \leq T_{\text{tact}} + T_{i+1})
\]

\[
= \Pr(X_1 \leq 0, X_2 \leq 0)
\]

\[
= \int_0^{T_{\text{tact}}} f(x_1, x_2) dx_1 dx_2, \quad S_i : x_1 \leq 0, x_2 \leq 0,
\]

where $f(x_1, x_2)$ is the joint probability density function of random variables $X_1$ and $X_2$, which are not independent of each other.

Variables are then transformed by $y_1 = x_1$ and $x_2 = x_2 - x_1$. Thus, $S_i$ is expressed as $S_2$ in terms of $y_1$ and $y_2$:

\[
S_2 : y_1 \leq 0, y_1 + y_2 \leq 0.
\]

The Jacobian $\mathcal{J}$ for $x_1 = y_1$ and $x_2 = y_1 + y_2$, which corresponds to $S_1$, is given by

\[
\mathcal{J} = \left| \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right| = 1.
\]

Therefore,

\[
\text{Eq. (3)} = \int_{S_2} g(y_1, y_2) \mathcal{J} \, dy_1 dy_2
\]

\[
= \int_{-\infty}^{0} dy_1 \int_{-y_1}^{\infty} g_1(y_1) g_2(y_2) dy_2,
\]

where $g(y_1, y_2)$ is the joint probability density function of random variables

\[
Y_1 = T_i^{(1)} - T_{\text{tact}},
\]

\[
Y_2 = T_i^{(2)} - T_{i+1},
\]

and $g_1(y_1)$ and $g_2(y_2)$ are the pdfs of $Y_1$ and $Y_2$, respectively. The random variable $Y_1$ means the time difference between the processing time of job $J_i$ on piece of equipment $M_1$ and the tact time. Similarly, the random variable $Y_2$ means the time difference between the processing time of job $J_i$ on piece of equipment $M_2$ and the processing time of job $J_{i+1}$ on the previous piece of equipment $M_1$. Note that the second equality in Eq. (4) holds since $Y_1$ and $Y_2$ are assumed to be independent of each other.

Note that Eq. (4) (i.e. $\Pr(E_i)$) is valid even if the processing time of each piece of equipment follows general distribution. However, Eq. (4) has a closed form formula if Erlang distribution is assumed. In the next section, such a closed form is derived.

Each pdf $g_1(y_1)$ ($j = 1, 2$) is determined only by parameters related to piece of equipment $M_j$. Therefore, $\Pr(E_1) = \Pr(E_2) = \cdots = \Pr(E_{n-1})$ holds. Note that two events, $E_i$ and $E_j$ ($i \neq j$), are not independent of each other. However, we approximate the no-collision probability over all $n$ jobs by $\Pr(E_i)^{n-1}$. In this case, the approximate collision probability for $m = 2$ is given by

\[
1 - \Pr(E_i)^{n-1}.
\]
5. Closed Form Formula for \( m = 2 \)

In this section, we derive a closed form formula of Eq. (4) (i.e. \( \Pr(E_i) \)) under Erlang distribution.

By Eq. (5), the pdf \( g_1(y_1) \) of \( Y_1 \) is obtained by translating the pdf of Erlang distribution with parameters \( \lambda_1 \) and \( k_1 \) by \(-T_{\text{tact}}\). Therefore, we have

\[
g_1(y_1) = \begin{cases} \frac{\lambda_1^{k_1}}{(k_1-1)!} e^{\lambda_1 y_1} & (y_1 \geq -T_{\text{tact}}), \\ 0 & (y_1 < -T_{\text{tact}}). \end{cases}
\]

By Eq. (6), \( Y_2 \) is the sum of two independent random variables \( T_{i+1}^{(2)} \) and \(-T_{i+1}^{(1)}\) with the following pdfs, \( h_1 \) and \( h_2 \), respectively. The pdf of \( T_{i+1}^{(2)} \) follows Erlang distribution with parameters \( \lambda_2 \) and \( k_2 \), i.e.,

\[
h_1(x) = \begin{cases} \frac{\lambda_2^{k_2}}{(k_2-1)!} x^{k_2-1} e^{-\lambda_2 x} & (x \geq 0), \\ 0 & (x < 0). \end{cases}
\]

The pdf of \(-T_{i+1}^{(1)}\) is the reflection of the pdf of \( T_{i+1}^{(1)} \) with parameters \( \lambda_1 \) and \( k_1 \) with respect to the line \( y = 0 \),

\[
h_2(y) = \begin{cases} \frac{\lambda_1^{k_1}}{(k_1-1)!} e^{\lambda_1 y} & (y > 0), \\ 0 & (y \leq 0). \end{cases}
\]

Then, the pdf \( g_2(y_2) \) of \( Y_2 \) is given by the convolution of \( h_1 \) and \( h_2 \).

Case 1: \( y_2 \geq 0 \)

\[
g_2(y_2) = \int_{-\infty}^{\infty} h_1(x) h_2(y_2 - x) \, dx
= \int_{0}^{\infty} \frac{\lambda_1^{k_1}}{(k_1-1)!} \frac{\lambda_2^{k_2}}{(k_2-1)!} x^{k_2-1} e^{\lambda_1 y_2 - \lambda_2 x} \, dx
= \frac{\lambda_1^{k_1} \lambda_2^{k_2}}{(k_2-1)! (\lambda_1 + \lambda_2)^{k_2+k_1-1}} \sum_{l=0}^{k_2-1} \frac{(k_1 + k_2 - 2l)!}{(k_2-1-l)!} y_2^l \left( \frac{(\lambda_1 + \lambda_2)^l}{\lambda_1 \lambda_2} \right)^h.
\]

Case 2: \( y_2 < 0 \)

\[
g_2(y_2) = \int_{-\infty}^{\infty} h_1(x) h_2(y_2 - x) \, dx
= \int_{0}^{\infty} \frac{\lambda_1^{k_1}}{(k_1-1)!} \frac{\lambda_2^{k_2}}{(k_2-1)!} x^{k_2-1} e^{\lambda_1 y_2 - \lambda_2 x} \, dx
= \frac{\lambda_1^{k_1} \lambda_2^{k_2}}{(k_2-1)! (\lambda_1 + \lambda_2)^{k_2+k_1-1}} \sum_{l=0}^{k_2-1} \frac{(k_1 + k_2 - 2l)!}{(k_2-1-l)!} y_2^l \left( \frac{(\lambda_1 + \lambda_2)^l}{\lambda_1 \lambda_2} \right)^h.
\]

Using Eqs. (8) – (10) we simplify Eq. (4) (i.e. \( \Pr(E_i) \)) in the following equation. Since \( y_1 \leq 0 \) holds in Eq. (4), we have

\[
\int_{-\infty}^{0} g_2(y_2) \, dy_2 + \int_{0}^{\infty} g_2(y_2) \, dy_2
\]

\[
= \lambda_1 k_1 \lambda_2 k_2 \frac{\lambda_1^{k_1} \lambda_2^{k_2}}{(k_2-1)! (\lambda_1 + \lambda_2)^{k_2+k_1-1}} \sum_{l=0}^{k_2-1} \frac{(k_1 + k_2 - 2l)!}{(k_2-1-l)!} y_2^l \left( \frac{(\lambda_1 + \lambda_2)^l}{\lambda_1 \lambda_2} \right)^h.
\]

Therefore, by denoting the upper expression as \( I \), we obtain

\[
\Pr(E_i) = \int_{-\infty}^{0} g_1(y_1) \cdot Idy_1
= \lambda_1 \lambda_2 \int_{-T_{\text{tact}}}^{0} g_1(y_1) \cdot Idy_1 + \int_{-T_{\text{tact}}}^{\infty} g_1(y_1) \cdot Idy_1
\]

\[
= \lambda_1 \lambda_2 k_1 k_2 \frac{\lambda_1^{k_1} \lambda_2^{k_2}}{(k_2-1)! (\lambda_1 + \lambda_2)^{k_2+k_1-1}} \sum_{l=0}^{k_2-1} \frac{(k_1 + k_2 - 2l)!}{(k_2-1-l)!} y_2^l \left( \frac{(\lambda_1 + \lambda_2)^l}{\lambda_1 \lambda_2} \right)^h
\]

where

\[
A = \lambda_1 \lambda_2 \int_{-\infty}^{0} g_1(y_1) \cdot Idy_1
= \lambda_1 \lambda_2 k_1 k_2 \frac{\lambda_1^{k_1} \lambda_2^{k_2}}{(k_2-1)! (\lambda_1 + \lambda_2)^{k_2+k_1-1}} \sum_{l=0}^{k_2-1} \frac{(k_1 + k_2 - 2l)!}{(k_2-1-l)!} y_2^l \left( \frac{(\lambda_1 + \lambda_2)^l}{\lambda_1 \lambda_2} \right)^h.
\]

\[
B = \lambda_1 \lambda_2 \int_{-T_{\text{tact}}}^{\infty} g_1(y_1) \cdot Idy_1
= \lambda_1 \lambda_2 k_1 k_2 \frac{\lambda_1^{k_1} \lambda_2^{k_2}}{(k_2-1)! (\lambda_1 + \lambda_2)^{k_2+k_1-1}} \sum_{l=0}^{k_2-1} \frac{(k_1 + k_2 - 2l)!}{(k_2-1-l)!} y_2^l \left( \frac{(\lambda_1 + \lambda_2)^l}{\lambda_1 \lambda_2} \right)^h.
\]

\[
C = \lambda_1 \lambda_2 \int_{0}^{\infty} g_2(y_2) \, dy_2
= \lambda_1 \lambda_2 k_1 k_2 \frac{\lambda_1^{k_1} \lambda_2^{k_2}}{(k_2-1)! (\lambda_1 + \lambda_2)^{k_2+k_1-1}} \sum_{l=0}^{k_2-1} \frac{(k_1 + k_2 - 2l)!}{(k_2-1-l)!} y_2^l \left( \frac{(\lambda_1 + \lambda_2)^l}{\lambda_1 \lambda_2} \right)^h.
\]

\[
D = \lambda_1 \lambda_2 \int_{0}^{\infty} g_2(y_2) \, dy_2
= \lambda_1 \lambda_2 k_1 k_2 \frac{\lambda_1^{k_1} \lambda_2^{k_2}}{(k_2-1)! (\lambda_1 + \lambda_2)^{k_2+k_1-1}} \sum_{l=0}^{k_2-1} \frac{(k_1 + k_2 - 2l)!}{(k_2-1-l)!} y_2^l \left( \frac{(\lambda_1 + \lambda_2)^l}{\lambda_1 \lambda_2} \right)^h.
\]

\[
E = \lambda_1 \lambda_2 \int_{0}^{\infty} g_2(y_2) \, dy_2
= \lambda_1 \lambda_2 k_1 k_2 \frac{\lambda_1^{k_1} \lambda_2^{k_2}}{(k_2-1)! (\lambda_1 + \lambda_2)^{k_2+k_1-1}} \sum_{l=0}^{k_2-1} \frac{(k_1 + k_2 - 2l)!}{(k_2-1-l)!} y_2^l \left( \frac{(\lambda_1 + \lambda_2)^l}{\lambda_1 \lambda_2} \right)^h.
\]
\[
Pr(E_i) = \Pr \left( \sum_{j=1}^{l} T_i^{(j)} \leq T_{\text{tact}} + \sum_{j=1}^{l-1} T_{i+1}^{(j)} : 1 \leq l \leq m \right)
\]

\[
= \Pr (X_l \leq 0 : 1 \leq l \leq m)
\]

\[
= \int \cdots \int S_1 f(x_1, x_2, \ldots, x_m) dx_1 dx_2 \cdots dx_m,
\]

\[S_1 : x_1 \leq 0 \text{ for all } 1 \leq l \leq m, \quad (12)\]

where \( f(x_1, x_2, \ldots, x_m) \) is the joint probability density function of random variables \( X_l \) for all \( 1 \leq l \leq m \).

These variables are transformed by \( y_1 = x_1, y_j = x_j - x_{j-1} \), and \( S_1' \) is expressed as \( S_2' \):

\[
S_2' : \sum_{i=1}^{j} y_i \leq 0 \quad \text{for all } 1 \leq j \leq m.
\]

The Jacobian \( \mathcal{J} \) for \( x_i = \sum_{j=1}^{i} y_j \) for all \( 1 \leq i \leq m \), which corresponds to \( S_1' \), is given by

\[
\mathcal{J} = \left| \begin{array}{cccc}
1 & 0 & 0 & \cdots \\
1 & 1 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots
\end{array} \right| = 1.
\]

Therefore, Eq. (12) becomes as follows:

\[
\int \cdots \int S_2 g(y_1, y_2, \ldots, y_m) \left| \mathcal{J} \right| dy_1 dy_2 \cdots dy_m
\]

\[
= \int_{-\infty}^{0} dy_1 \int_{-\infty}^{y_1} dy_2 \cdots \int_{-\infty}^{y_{j-1}} g_1(y_1) g_2(y_2) \cdots g_m(y_m) dy_m,
\]

\[(13)\]

where \( g(y_1, y_2, \ldots, y_m) \) is the joint probability density function of random variables

\[
Y_1 = T_i^{(1)} - T_{\text{tact}},
\]

\[
Y_j = T_i^{(j)} - T_{i+1}^{(j-1)} \quad \text{for } 2 \leq j \leq m,
\]

and each \( g_j(y_j) \) is the pdf of \( Y_j \). The random variable \( Y_j \) means the time difference between the processing time of job \( J_i \) on piece of equipment \( M_i \) and the processing time of job \( J_{i+1} \) on the previous piece of equipment \( M_{j-1} \). Note that the equality in Eq. (13) holds because we assume all \( Y_j \) are independent of each other.

The \( g_1(y_1) \) is given by Eq. (8), and \( g_j(y_j) \) for all \( 2 \leq j \leq m \) are given by the convolution of two pdfs of \( T_i^{(j)} \) and \( -T_{i+1}^{(j-1)} \). Since the derivation of Eq. (13) into a closed form formula is similar to the one shown in the previous section, we have omitted the details. Therefore, the approximate collision probability \( 1 - Pr(E_i)^{m-1} \) is also given in a closed form formula.

We close this section by the following simple observation that holds for the general distribution of processing time. From Eq. (12),
Collision probability is less than or equal to \( \alpha \). In this section, \( \alpha \) is set to one. In Fig. 4, \( T_{\text{tact}} \) is about 1.49 times the expectation of the processing time.

We know that \( \Pr(E_i) \) is an increasing function of \( T_{\text{tact}} \). This naturally implies that \( 1 - \Pr(E_i)^{n-1} \) is a decreasing function of \( T_{\text{tact}} \).

Proposition 1: The approximate collision probability \( 1 - \Pr(E_i)^{n-1} \) is monotonically decreasing with tact time for the general distribution of processing time.

7. Numerical Results and Computer Simulation

Based on the above formulae, we obtain some numerical results by using MATHEMATICA [12]. For our computation in this section, the number of jobs is set to \( n = 1,000 \), and the parameters of the Erlang distributions are set so that the expectation of the processing time on each piece of equipment becomes equal to one (i.e., \( k = \lambda \) in Eq. (1)).

Firstly, we compute the collision probability for \( m = 1 \), i.e. Eq. (2), as shown in Fig. 4, where the horizontal axis denotes the tact time \( T_{\text{tact}} \). It shows that, as the tact time passes a certain threshold, the collision probability decreases rapidly, clearly exhibiting the trade-off between the tact time and the collision probability.

We consider the optimization problem presented in [1], which minimizes the tact time by including the collision probability as part of the input. Therefore, we denote by \( T^*_{\text{tact}} \) the minimum tact time under the condition that the collision probability is less than or equal to \( \alpha \% \) (0 \( \leq \alpha \leq 100 \)). In this section, \( \alpha \) is set to one. In Fig. 4, \( T^*_{\text{tact}} \) is about 1.49, which is 1.49 times the expectation of the processing time.

Note that the variance of the Erlang distribution for Fig. 4 is \( k_1/\lambda_1^2 = 0.01 \). We also test smaller variances, 0.001, 0.0001, and the results are shown in Fig. 5 and Fig. 6, respectively. They show that \( T^*_{\text{tact}} \) for these variances are about 1.15 and 1.05, respectively, indicating that \( T^*_{\text{tact}} \) decreases as the variance of processing time decreases.

Next, using Eq. (11) we compute the approximate collision probability Eq. (7) for \( m = 2 \). The numerical results are shown in Fig. 7, Fig. 8, and Fig. 9, for variances 0.2, 0.1 and 0.0667, respectively. The resulting \( T^*_{\text{tact}} \) are about

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**Fig. 4** Collision probability when \( m = 1, k_1 = 100, \lambda_1 = 100 \) (variance = 0.01), and \( n = 1,000 \).

**Fig. 5** Collision probability when \( m = 1, k_1 = 100, \lambda_1 = 100 \) (variance = 0.001), and \( n = 1,000 \).

**Fig. 6** Collision probability when \( m = 1, k_1 = 10,000, \lambda_1 = 10,000 \) (variance = 0.0001), and \( n = 1,000 \).

**Fig. 7** Collision probability when \( m = 2, k_1 = 5, k_2 = 5, \lambda_1 = 5, \lambda_2 = 5 \) (variance = 0.2), and \( n = 1,000 \).

**Fig. 8** Collision probability when \( m = 2, k_1 = 10, k_2 = 10, \lambda_1 = 10, \lambda_2 = 10 \) (variance = 0.1), and \( n = 1,000 \).
Collision probability (CP) evaluated by simulation when $k_j = 1,000$, $\lambda_j = 1,000$ (variance = 0.001) for all $1 \leq j \leq m$, and $n = 1,000$. [\%]

| $m$  | 1.20 | 1.25 | 1.30 | 1.35 | 1.40 | 1.45 | 1.50 | 1.55 | 1.60 | 1.65 |
|------|------|------|------|------|------|------|------|------|------|------|
| 3    | 91.39| 19.27| 1.21 | 0.04 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 4    | 99.99| 77.83| 16.32| 1.52 | 0.09 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 5    | 100.0| 99.16| 57.61| 11.35| 1.29 | 0.11 | 0.01 | 0.00 | 0.00 | 0.00 |
| 6    | 100.0| 100.0| 91.23| 37.52| 7.19 | 0.98 | 0.10 | 0.01 | 0.00 | 0.00 |
| 7    | 100.0| 100.0| 99.40| 71.11| 22.36| 4.26 | 0.63 | 0.07 | 0.01 | 0.00 |
| 8    | 100.0| 100.0| 99.99| 92.39| 47.06| 12.72| 2.46 | 0.39 | 0.05 | 0.01 |
| 9    | 100.0| 100.0| 100.0| 98.95| 72.81| 27.97| 6.95 | 1.37 | 0.23 | 0.03 |

8. Conclusions

In the model presented in [1], it is assumed that the processing time of each piece of equipment is a random variable that follows a continuous probability distribution. In [1], two cases were specifically discussed, in which processing time followed a normal and an exponential distribution. In this paper, we have assumed, considering the features of actual processing times, that processing time follows an Erlang distribution, which is a new focus of observation. Moreover, since our model is simple, it has broad utility and the potential of having a wide domain of applicability in the field of mass production and not just in the manufacture of FPDs and semiconductors.

We presented a closed form formula of the approximate collision probability in the model comprising of $m$ pieces of equipment when the processing time of each piece of equipment follows an Erlang distribution. This closed form is an extension of the one presented in [1], which showed a closed form of the approximate collision probability when the processing time follows an exponential distribution. We have also shown numerical results, as well as computer simulation results. Our approximate formula has reasonable accuracy, as experimentally confirmed.

There are also cases where numerical results and simulation results are different. For example, since the approximate formula is expressed using the $(n-1)$-th power, when the number of jobs $n$ is very small and the value for the collision probability is relatively large, cases of small percentage differences between numerical results and simulation results exist. It could be considered as one possible area of future work to reveal this phenomenon mathematically from the viewpoint of guaranteeing accuracy.

Presenting a closed form for the case of general $m$ can be considered as another area of future work. Since such an expression may be very complicated, it can be assumed that some recursive formulae may be useful.
Additionally, it may become hard to obtain numerical values from the closed form depending on parameter conditions. For example, when comparing such values with simulation results in Table 3, overflow might occur trying to perform numerical computation for the closed form using MATHEMATICA. Solving this is a possible area of further work. Moreover, research related to the computable order for the parameters is also an area of future interest.

Finally, we discuss some differences between the model presented in this paper and an actual FPD production line. In a real FPD production line, in order to avoid a collision between substrates, a substrate will not be transported to the next piece of equipment if it is occupied. In this situation, the substrate may be kept at the current piece of equipment even if the required process on it has been completed. This is referred to as tandem queues with blocking. However, considering chemicals, heat treating, etc., it is not very acceptable to allow waiting on a piece of equipment because FPD quality could decay. As another solution for avoiding collisions, buffers could be made before each piece of equipment. Analyzing the collision probability with buffer space included is also a potential area of future research.

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