Ensemble Kalman Filter using Gaussian-Sum Predicted State Probability Density Functions

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Abstract

We propose to use Gaussian-sum predicted state probability density functions (PDFs) in the algorithm of the ensemble Kalman filter (EnKF) to enhance its filtering accuracy. We analyze the EnKF in terms of the moment-matched linearization for the nonlinear observation model and show that the filtering accuracy of the EnKF can be improved by using the Gaussian-sum predicted state PDFs. We numerically confirm the effectiveness of the new filters through simulations using benchmark filtering problems of the vector nonlinear growth model and the satellite reentry.

1 Introduction

Pursuing effective and efficient filters for nonlinear state-space models is an important research topic and several representative filters were already proposed. These filter can be categorized into three groups such as Gaussian, Gaussian and non-Gaussian, and non-Gaussian filters. The state PDFs are estimated as Gaussian in the Gaussian filters and the examples are the extended Kalman filter (EKF) [1], the unscented Kalman filter (UKF) [2], and the cubature Kalman filter (CKF) [3]. The state PDFs for the non-Gaussian filters are estimated as non-Gaussian and the examples are the particle filter (PFs) such as the Monte Carlo filter (MCF) [4], the bootstrap filter (BF) [5], the extended Kalman particle filter (EKPF) [6], the unscented Kalman particle filter (UKPF) [6], the extended Kalman multiple distribution filter (EKMDF) [7] [8], and the unscented Kalman multiple distribution filter (UKMDF) [7] [8].

On the other hand, the Gaussian and non-Gaussian filters alternately use Gaussian and non-Gaussian state PDFs and the examples are the ensemble Kalman filter (EnKF) [9] and the Gaussian particle filter (GPF) [10]. The advantage of the EnKF is that the particle (ensemble) weights remain the same as the filter proceeds and thus, the filter instability due to the severe particle degeneracy [11] such that all of the weights become zero never happens. Moreover, unlike the GPF, the particle re-sampling is not necessary during the filter execution. These are prominent features compared with the PFs, although the filtering accuracy of the EnKF is deteriorated because of the use of the Gaussian-assumed state PDFs. The filter stability is important, especially when applying a filter to real-world problems and this is the reason that the EnKF is often preferred.

In this paper, we first analyze the EnKF and show that the algorithm is an approximation of the linear optimal filter for a nonlinear observation model. In doing so, we introduce the linearization model called the moment-matched linearization (MML) and point out that the conditional expectations in the MML are estimated by the Dirac’s delta mixture predicted state PDFs in the EnKF algorithm. We then propose to use the Gaussian-sum predicted state PDFs for estimating these conditional expectations, which provides the better estimations, and show that this new algorithm is expected to enhance the overall filtering accuracy. We next evaluate our proposed approach using simulations based on benchmark filtering problems and confirm the effectiveness of the proposed algorithms.

Therefore, the contributions of this paper are summarized as

- The analysis of the EnKF based on the MML for the nonlinear observation model and the clarification of the error source inherent in the EnKF algorithm,
- the formulation of a new filtering algorithm using the Gaussian-sum predicted state PDFs for estimating the conditional expectations in the MML,
- and simulation evaluation using benchmark filtering problems of vector nonlinear growth model [12] and satellite reentry [13].

The rest of this paper is organized as follows. The filtering problem is formulated in Section II and the EnKF algorithm is revisited in Section III. In Section IV, we perform the analysis of the EnKF using the MML for the nonlinear observation model and reveal its source of error resulting from the use of the Dirac’s delta mixture predicted state PDFs. We then propose to use the Gaussian-sum predicted state PDFs for calculating the MML and express the new filtering algorithm in Section V. The performance of the new filters using the EKF, the UKF, and the CKF is numerically evaluated and compared with the EnKF by simulations in Section VI and finally, Section VII concludes this note with the basic findings.
2 Problem Statement

We focus on designing a sequential state estimation algorithm for the following nonlinear state-space model:

\[
\begin{align*}
  x_0 & \sim N(m_0, P_0) \quad (1a) \\
  x_k &= f_k(x_{k-1}) + \nu_k, \quad \text{where } k = 1, 2, \ldots, \nu_k \sim N(0, Q_k) \quad (1b) \\
  y_k &= h_k(x_k) + w_k, \quad \text{where } k = 1, 2, \ldots, w_k \sim N(0, R_k) \quad (1c)
\end{align*}
\]

Here, \( x_k \in \mathbb{R}^n \) and \( y_k \in \mathbb{R}^n \) are state and observation at \( k \) and \( a \sim N(b, c) \) denotes that \( a \) follows a Gaussian distribution with mean \( b \) and covariance matrix \( c \). Then, the filtering problem is to design an algorithm of estimating \( p(x_k | y_k) \) using the estimate of \( p(x_{k-1} | y_{k-1}) \), where \( y_k = \{y_1, y_2, \ldots, y_k\} \) and \( y_0 = \emptyset \). Here, \( p(x_k | y_k) \) is called the filtered state PDF at \( k \) and the filtered state estimate \( \hat{x}_k \) is calculated by the conditional expectation \( \hat{x}_k = E[x_k | y_k] = \int x_k p(x_k | y_k) dx_k \). The corresponding estimation error covariance matrix \( \hat{P}_k \) is calculated by \( E[(\hat{x}_k - \hat{x}_k)(\hat{x}_k - \hat{x}_k)^T | y_k] \).

The predicted state PDF at \( k \) defined as \( p(x_k | y_{k-1}) \) is calculated by

\[
p(x_k | y_{k-1}) = \int p(x_k | x_{k-1})p(x_{k-1} | y_{k-1}) dx_{k-1} \quad (2)
\]

and the predicted state estimate \( \hat{x}_k \) and its estimation error covariance matrix \( \hat{P}_k \) are defined as \( E[x_k | y_{k-1}] \) and \( E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T | y_{k-1}] \), respectively.

3 EnKF

The EnKF algorithm using \( M \) ensembles for Eqs. (1a) to (1c) is revisited in this section. Given

\[
p(x_{k-1} | y_{k-1}) = \frac{1}{M} \sum_{i=1}^M \delta(x_{k-1} - \hat{x}_{k-1}^{(i)}) \quad (3)
\]

here, \( \delta(x_{k-1} - \hat{x}_{k-1}^{(i)}) \) is the Dirac’s delta function centered at \( \hat{x}_{k-1}^{(i)} \) and \( p(x_0) = \frac{1}{M} \sum_{i=1}^M \delta(x_0 - \hat{x}_{1}^{(i)}) \), where \( \hat{x}_{1}^{(i)} \sim N(m_0, P_0) \) which denotes that \( \hat{x}_{1}^{(i)} \) is sampled from \( N(m_0, P_0) \), the EnKF algorithm is

- **Time update**
  \[
  \begin{align*}
  \hat{x}_k^{(i)} &= f_k(\hat{x}_{k-1}^{(i)}) + \nu_k^{(i)} \quad (4a) \\
  \hat{x}_k &= [\hat{x}_k^{(1)} \hat{x}_k^{(2)} \ldots \hat{x}_k^{(M)}]W, \quad (4b) \\
  \hat{P}_k &= [\hat{x}_k^{(1)} \hat{x}_k^{(2)} \ldots \hat{x}_k^{(M)}]W'[\hat{x}_k^{(1)} \hat{x}_k^{(2)} \ldots \hat{x}_k^{(M)}] \quad (4c)
  \end{align*}
  \]

where \( W = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \), \( W' = (I - [W W \cdots W]) \times \text{diag} \left( \frac{1}{M-1} \frac{1}{M-1} \cdots \frac{1}{M-1} \right) \times (I - [W W \cdots W]) \)^T.

- **Observation update**
  \[
  \begin{align*}
  \hat{x}_k &= \hat{x}_k + K_k^{M}(y_k - \bar{y}_k^{(i)}) \quad (5a) \\
  \hat{P}_k &= \hat{P}_k + [\hat{x}_k \hat{x}_k]W[\hat{x}_k \hat{x}_k]^T, \quad (5b) \\
  \bar{P}_k &= [\bar{x}_k \bar{x}_k]W[\bar{x}_k \bar{x}_k]^T, \quad (5c)
  \end{align*}
  \]

where

\[
\begin{align*}
  K_k^{M} &= \bar{P}_k^{M} (\bar{P}_k^{M})^{-1} \quad (5d) \\
  \bar{P}_k^{M} &= [\bar{x}_k \bar{x}_k]^T \quad (5e) \\
  \bar{y}_k^{(i)} &= h_k(\hat{x}_k^{(i)}) \quad (5f) \\
  \bar{y}_k^{(i)} &= h_k(\hat{x}_k^{(i)}) + w_k^{(i)} \quad (5g)
  \end{align*}
\]

Here, \( \nu_k^{(i)} \sim N(0, Q_k) \) and \( w_k^{(i)} \sim N(0, R_k) \).

4 Analysis of EnKF

The EnKF is an approximation for the linear optimal filter for Eq. (1c). The linear optimal filter is equivalent to the Kalman filter (KF) [14] for the following MML of Eq. (1c):

\[
y_k \approx E[h_k(x_k) | y_{k-1}] + E[(h_k(x_k) | y_{k-1})(x_k - \hat{x}_k)^T | y_{k-1}] \\
\times \bar{P}_k^{-1}(x_k - \hat{x}_k) + w_k \quad (6)
\]

Here, by using the notation \( H_k \equiv E[(h_k(x_k) - E[h_k(x_k) | y_{k-1}])(x_k - \hat{x}_k)^T | y_{k-1}] \bar{P}_k^{-1} \), \( w_k \sim N(0, P_{y_k} - H_k \bar{P}_k H_k^T) \), where \( P_{y_k} = E[(y_k - \bar{y}_k)(y_k - \bar{y}_k)^T | y_{k-1}] \) and \( \bar{y}_k = E[y_k | y_{k-1}] = E[h_k(x_k) | y_{k-1}] \). \( w_k \) is a newly introduced independent white Gaussian noise and recalling that the predicted observation estimate and its estimation error covariance matrix for Eq. (1c) were \( E[h_k(x_k) | y_{k-1}] \) and \( \bar{P}_{y_k} \), the MML in Eq. (6) exactly preserves the same first two predicted moments as those for the original nonlinear observation model. The MML corresponds to an extension of the statistical linearization [15].

As mentioned, the linear optimal filter for Eq. (1c) is equivalent to the KF for which the conditional expectations in Eq. (6) are estimated with respect to the non-Gaussian predicted state PDF at \( k \). After estimating these expectations using the non-Gaussian predicted state PDF, the KF for Eq. (6) is obtained by disregarding the third and higher order moments of the
predicted state PDF as shown below.

\[
\dot{x}_k = \bar{x}_k + K_k(y_k - E[h_k(x_k)|Y_{k-1}] - E[h_k(x_k)|Y_{k-1}]), \quad (7a)
\]

\[
P_k = \bar{P}_k + K_kH_kP_k, \quad \text{where} \quad K_k = \bar{P}_kH_k^T (P_{y_k})^{-1}
\]

\[
= E[(x_k - \bar{x}_k)(h_k(x_k) - E[h_k(x_k)|Y_{k-1}] - E[h_k(x_k)|Y_{k-1}])^T |Y_{k-1}]
\times (P_{y_k})^{-1} \quad (7c)
\]

Indeed, the particle averages for Eqs. (5b) and (5c) approximate Eqs. (7a) to (7c) [16] and the EnKF is an approximation algorithm of the linear optimal filter for Eq. (1c).

Note that \(E[h_k(x_k)|Y_{k-1}], \ E[(x_k - \bar{x}_k)(h_k(x_k) - E[h_k(x_k)|Y_{k-1}])^T |Y_{k-1}]\), and \(P_{y_k}\) in Eqs. (7a) to (7c) are all estimated by \(\{\hat{x}_k^{(1)}, \hat{x}_k^{(2)}, ..., \hat{x}_k^{(M)}\}\) as shown in Eqs. (5a) to (5g) and these particles express the following non-Gaussian predicted state PDF at \(k\) for the EnKF:

\[
p(x_k|Y_{k-1}) \approx \frac{1}{M} \sum_{i=1}^{M} \delta(x_k - \hat{x}_k^{(i)}) \quad (8)
\]

Equation (8) is approximately derived from Eqs. (1b), (2), and (3), however, its exact expression is

\[
p(x_k|Y_{k-1}) = \frac{1}{M} \sum_{i=1}^{M} N(f_k(\hat{x}_{k-1}^{(i)}), Q_k)
\]

\[
= \frac{1}{M} \sum_{i=1}^{M} N(m_k^{(i)}, P_k^{(i)}) \quad (9)
\]

Here, \(m_k^{(i)} := f_k(\hat{x}_{k-1}^{(i)})\) and \(P_k^{(i)} := Q_k\), where \(a := b\) means the replacement of \(a\) by \(b\). Therefore, Equation (8) causes some estimation errors and the use of Eq. (9) for estimating the conditional expectations in Eqs. (7a) to (7c) contributes to improve the overall filtering accuracy. We formulate this new filtering algorithm in the next section.

5 EnKF using Gaussian-Sum Predicted state PDFs

This section describes the new EnKF algorithm for which the Gaussian-sum predicted state PDFs are used to estimate the conditional expectations \(E[h_k(x_k)|Y_{k-1}], E[(x_k - \bar{x}_k)(h_k(x_k) - E[h_k(x_k)|Y_{k-1}])^T |Y_{k-1}]\), and \(P_{y_k}\) in Eqs. (7a) to (7c). These conditional expectations are calculated as

\[
E[h_k(x_k)|Y_{k-1}] = \int h_k(x_k)p(x_k|Y_{k-1})dx_k \quad (10)
\]

\[
E[(x_k - \bar{x}_k)(h_k(x_k) - E[h_k(x_k)|Y_{k-1}])^T |Y_{k-1}]
\]

\[
= \int (x_k - \bar{x}_k)(h_k(x_k) - E[h_k(x_k)|Y_{k-1}])^T |Y_{k-1})p(x_k|Y_{k-1})dx_k
\]

\[
\bar{P}_{y_k} = \int (h_k(x_k) - E[h_k(x_k)|Y_{k-1}])
\times (h_k(x_k) - E[h_k(x_k)|Y_{k-1}])^T |Y_{k-1})p(x_k|Y_{k-1})dx_k
\]

+ \(R_k\) \quad (12)

Then, using Eq. (9), Equations (10) to (12) become

\[
E[h_k(x_k)|Y_{k-1}] = \frac{1}{M} \sum_{i=1}^{M} \int h_k(x_k)N(m_k^{(i)}, P_k^{(i)})dx_k
\]

\[
E[(x_k - \bar{x}_k)(h_k(x_k) - E[h_k(x_k)|Y_{k-1}])^T |Y_{k-1}]
\]

\[
= \frac{1}{M} \sum_{i=1}^{M} \int (x_k - \bar{x}_k)(h_k(x_k) - E[h_k(x_k)|Y_{k-1}])^T
\times N(m_k^{(i)}, P_k^{(i)})dx_k \quad (14)
\]

\[
\bar{P}_{y_k} = \frac{1}{M} \sum_{i=1}^{M} \int (h_k(x_k) - E[h_k(x_k)|Y_{k-1}])
\times (h_k(x_k) - E[h_k(x_k)|Y_{k-1}])^T
\times N(m_k^{(i)}, P_k^{(i)})dx_k + \bar{R}_k \quad (15)
\]

The integral in Eq. (14) is

\[
E^N[(x_kh_k(x_k))^T] - \bar{x}_kE^N[h_k(x_k)^T]
\]

\[
- m_k^{(i)}E[h_k(x_k)|Y_{k-1}] + \bar{x}_kE[h_k(x_k)|Y_{k-1}]^T
\]

\[
= E^N[(x_kh_k(x_k))^T] - m_k^{(i)}E^N[h_k(x_k)^T]
\]

\[
+ m_k^{(i)}E[h_k(x_k)^T] - \bar{x}_kE[h_k(x_k)^T]
\]

\[
- m_k^{(i)}E[h_k(x_k)|Y_{k-1}]^T + \bar{x}_kE[h_k(x_k)|Y_{k-1}]^T
\]

\[
= E^N[(x_k - m_k^{(i)})(h_k(x_k) - E[h_k(x_k)])^T]
\]

\[
+ (m_k^{(i)} - \bar{x}_k)(E[h_k(x_k) - E[h_k(x_k)|Y_{k-1}])]^T
\]

Here, \(E^N[\cdot] = \int \cdot N(m_k^{(i)}, P_k^{(i)})dx_k\).

Similarly, the integral in Eq. (15) is

\[
E^N[(h_k(x_k)h_k(x_k))^T] - E[h_k(x_k)|Y_{k-1}]E^N[h_k(x_k)^T]
\]

\[
- E[h_k(x_k)]E[h_k(x_k)|Y_{k-1}]^T
\]

\[
+ E[h_k(x_k)|Y_{k-1}]E[h_k(x_k)|Y_{k-1}]^T
\]

\[
= E^N[(h_k(x_k)h_k(x_k))^T] - E[h_k(x_k)]E[h_k(x_k)^T]
\]

\[
+ E[h_k(x_k)|Y_{k-1}]E[h_k(x_k)|Y_{k-1}]^T
\]

\[
= E^N[(h_k(x_k) - E[h_k(x_k)])^T(h_k(x_k) - E[h_k(x_k)])^T]
\]

\[
+ (E[h_k(x_k)] - E[h_k(x_k)|Y_{k-1}])^T
\]

\[
\times (E[h_k(x_k)] - E[h_k(x_k)|Y_{k-1}])^T
\]
Therefore, for each Gaussian PDF, performing the following

\[ E^N[h_k(x_k)] = \int h_k(x_k)N(m_k^{(i)}, P_k^{(i)})dx_k \]  

(16)

\[ E^N[(x_k - m_k^{(i)})(h_k(x_k) - E^N[h_k(x_k)])^T] = \int (x_k - m_k^{(i)})(h_k(x_k) - E^N[h_k(x_k)])^T \times N(m_k^{(i)}, P_k^{(i)})dx_k \]  

(17)

\[ E^N[(h_k(x_k) - E^N[h_k(x_k)])(h_k(x_k) - E^N[h_k(x_k)])^T] = \int (h_k(x_k) - E^N[h_k(x_k)])(h_k(x_k) - E^N[h_k(x_k)])^T \times N(m_k^{(i)}, P_k^{(i)})dx_k \]  

(18)

and reminding that \( \bar{x}_k = \frac{1}{\sqrt{N}} \sum_{i=1}^{M} m_k^{(i)} \), Equations (13) to (15) are calculated. Actual calculations for Eqs. (16) to (18) are based on the Gaussian filters such as the EKF, the UKF, and the CKF and for example, Equations (16) to (18) estimated by the EKF are

\[ E^N[h_k(x_k)] = h_k(m_k^{(i)}) \]  

(19)

\[ E^N[(x_k - m_k^{(i)})(h_k(x_k) - E^N[h_k(x_k)])^T] = P_k^{(i)}(\tilde{H}_k^{(i)})^T \]  

(20)

\[ E^N[(h_k(x_k) - E^N[h_k(x_k)])(h_k(x_k) - E^N[h_k(x_k)])^T] = \tilde{H}_k^{(i)}P_k^{(i)}(\tilde{H}_k^{(i)})^T \]  

(21)

Here, \( \tilde{H}_k^{(i)} \) is the Jacobian matrix of \( h_k(x_k) \) evaluated at \( m_k^{(i)} \).

On the other hand, Equations (16) to (18) estimated by the UKF are

\[ E^N[h_k(x_k)] = [h_k(x_{k,i}^{(1)}) h_k(x_{k,i}^{(2)}) \cdots h_k(x_{k,i}^{(2n+1)})]W_s \]  

(22)

\[ E^N[(x_k - m_k^{(i)})(h_k(x_k) - E^N[h_k(x_k)])^T] = [m_k^{(i)} \chi_{k,i}^{(1)} \cdots \chi_{k,i}^{(2n+1)}]W' \]  

(23)

\[ E^N[(h_k(x_k) - E^N[h_k(x_k)])(h_k(x_k) - E^N[h_k(x_k)])^T] = [h_k(h_{k,i}^{(1)}) h_k(h_{k,i}^{(2)}) \cdots h_k(h_{k,i}^{(2n+1)})]W_s' \]  

(24)

Here, \( \chi_{k,i}^{(j)} (j = 1, 2, \cdots, 2n+1) \), \( W_s \), and \( W_s' \) are sigma points and the weights as follows:

\[ \chi_{k,i}^{(1)} = m_k^{(i)} \]

\[ \chi_{k,i}^{(j)} = \begin{cases} m_k^{(i)} + \sqrt{3} \sqrt{P_k^{(i)}}_{j-1} & (j = 2, \cdots, n+1) \\ m_k^{(i)} - \sqrt{3} \sqrt{P_k^{(i)}}_{j-n-1} & (j = n+2, \cdots, 2n+1) \end{cases} \]

\[ W_s = \left[ \left( 1 - \frac{n}{3} \right) \frac{1}{6} \frac{1}{6} \cdots \frac{1}{6} \right]^T \]

\[ W_s' = (I - [W_s W_s \cdots W_s]) \times \left( \left( 1 - \frac{n}{3} \right) \frac{1}{6} \frac{1}{6} \cdots \frac{1}{6} \right) \times (I - [W_s W_s \cdots W_s])^T \]

Here, \( \sqrt{P_k^{(i)}_{j-1}} \) denotes the \( (j-1) \)th column of the Cholesky matrix of \( P_k^{(i)} \). The estimates by the CKF are

\[ E^N[h_k(x_k)] = [h_k(x_{k,i}^{(1)}) h_k(x_{k,i}^{(2)}) \cdots h_k(x_{k,i}^{(2n)})]W_c \]  

(25)

\[ E^N[(x_k - m_k^{(i)})(h_k(x_k) - E^N[h_k(x_k)])^T] = [m_k^{(i)} \chi_{k,i}^{(2)} \cdots \chi_{k,i}^{(2n)}]W_c' \]  

(26)

\[ E^N[(h_k(x_k) - E^N[h_k(x_k)])(h_k(x_k) - E^N[h_k(x_k)])^T] = [h_k(h_{k,i}^{(1)}) h_k(h_{k,i}^{(2)}) \cdots h_k(h_{k,i}^{(2n)})]W_c' \]  

(27)

Here, \( \chi_{k,i}^{(j)} (j = 1, 2, \cdots, 2n) \), \( W_c \), and \( W_c' \) are cubature points and the weights as follows:

\[ \chi_{k-1}^{(j)} = \begin{cases} m_k^{(i)} + \sqrt{n} \sqrt{P_k^{(i)}_j} & (j = 1, \cdots, n) \\ m_k^{(i)} - \sqrt{n} \sqrt{P_k^{(i)}_{j-n}} & (j = n+1, \cdots, 2n) \end{cases} \]

\[ W_c = \left[ \frac{1}{2n} \frac{1}{2n} \cdots \frac{1}{2n} \right]^T \]

\[ W_c' = (I - [W_c W_c \cdots W_c]) \times \left( \frac{1}{2n} \frac{1}{2n} \cdots \frac{1}{2n} \right) \times (I - [W_c W_c \cdots W_c])^T \]

Using Eqs. (19) to (27), the conditional expectations in Eqs. (7a) to (7c) are estimated and the observation update of the new EnKF is formulated. Given Eq. (3), the new filter using the EKF in Eqs. (19) to (21) is summarized as

- **Time update (Eq. (9))**
- **Observation update (Eqs. (5a) to (5g)), where**

\[ x_k^{(i)} \sim N(m_k^{(i)}, P_k^{(i)}) \]

\[ \bar{x}_k = \frac{1}{M} \sum_{i=1}^{M} m_k^{(i)} \]

\[ E[h_k(x_k)|y_{k-1}] = \frac{1}{M} h_k(m_k^{(i)}) \]
\[
\begin{align*}
\bar{P}_{x,k}^M &= \frac{1}{M} \sum_{i=1}^{M} \left( P_k^{(i)} (\tilde{H}_k^{(i)})^T + (m_k^{(i)} - \hat{x}_k) \right) \\
\tilde{P}_{x,k}^M &= \frac{1}{M} \sum_{i=1}^{M} \left( \tilde{H}_k^{(i)} P_k^{(i)} (\tilde{H}_k^{(i)})^T + (h_k(m_k^{(i)}) - E[h_k(x_k)|Y_{k-1}]) \right) \\
&\quad \times (h_k(m_k^{(i)}) - E[h_k(x_k)|Y_{k-1}]) + R_k
\end{align*}
\]

In the similar way, the new filtering algorithms using the UKF in Eqs. (22) to (24) and the CKF in Eqs. (25) to (27) are also obtained. Compared with the EnKF, although the Gaussian filters such as the EKF, the UKF, and the CKF are incorporated, the conditional expectations in the MML are estimated more accurately using the Gaussian-sum predicted state PDFs and the overall filtering accuracy is expected to be improved. We next numerically evaluate the performance of the new filters using simulations based on the benchmark filtering problems.

6 Simulations

The filtering performance of the EnKF and the new filters using the EKF, the UKF, and the CKF are evaluated for the two benchmark filtering problems: vector nonlinear growth model [12] and satellite reentry [13]. We first explain these models and define the evaluation measures for the filtering accuracy. Then, the evaluation results for the filtering accuracy and calculation cost are provided.

6.1 Vector nonlinear growth model

The state-space model is expressed as

\[
\begin{align*}
x_0 &= [0 \ 0]^T \\
x_k &= \begin{bmatrix} x_{k,1} \\ x_{k,2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} x_{k-1,1} + \frac{25 x_{k-1,1}^2 + 8 \cos(1.2 (k-1))}{x_{k-1,2}^2} \\ \frac{1}{2} x_{k-1,2} + \frac{25 x_{k-1,2}^2 + 8 \cos(1.2 (k-1))}{x_{k-1,1}^2} \end{bmatrix} + \nu_k \\
y_k &= \begin{bmatrix} y_{k,1} \\ y_{k,2} \end{bmatrix} = \begin{bmatrix} x_{k,1}^2 + x_{k,2}^2 \\ \frac{20}{x_{k,1}^2 - x_{k,2}^2} \end{bmatrix} + w_k
\end{align*}
\]

Here, \(x_{k,1}\) and \(x_{k,2}\) denote the first and second elements of \(x_k\) and the same applies to \(y_k\). \(\nu_k\) and \(w_k\) are independent white Gaussian noises and \(\nu_k \sim N(0, 0|0|^T, \text{diag}(1, 1))\) and \(w_k \sim N(0, 0|0|^T, \text{diag}(1, 1))\), respectively. \{\((x_1, y_1), (x_2, y_2), \ldots, (x_{100}, y_{100})\}\} was generated and the average mean absolute errors, that is, \(\hat{E}[|x_{k,1} - \hat{x}_{k,1}|]\) and \(\hat{E}[|x_{k,2} - \hat{x}_{k,2}|]\) estimated by 100 Monte Carlo runs and then, averaged over \(k = 1, 2, \ldots, 100\) were adopted as the evaluation measures for the filtering accuracy.

The initial condition for the filters was \(\hat{x}_0 = [0 \ 0]^T\) and \(\hat{P}_0 = \text{diag}(2, 2)\). The system and observation noise covariance matrices were the same as those for the simulation model. The results are shown in Table 1, where the new filters using the EKF, the UKF, and the CKF are abbreviated as NF(EKF), NF(UKF), and NF(CKF), respectively.

Table 1: Filtering errors of the EnKF, the NF(EKF), the NF(UKF), and the NF(CKF) with varying \(M\) from 10 to 300. For each filter, the upper and lower rows denote average mean absolute errors for the first and second elements of the state.

| \(M\) | EnKF | NF(EKF) | NF(UKF) | NF(CKF) |
|-------|------|---------|---------|---------|
| 10    | 4.65 | 3.07    | 2.98    | 3.04    |
| 20    | 2.88 | 2.45    | 2.41    | 2.41    |
| 30    | 2.45 | 2.37    | 2.36    | 2.35    |
| 40    | 2.41 | 2.36    | 2.35    | 2.37    |
| 50    | 2.33 | 2.3     | 2.28    | 2.28    |
| 68    | 2.35 | 2.29    | 2.28    | 2.29    |
| 86    | 2.3   | 2.3     | 2.29    | 2.3     |
| 100   | 2.32 | 2.31    | 2.31    | 2.31    |
| 178   | 2.3   | 2.31    | 2.31    | 2.31    |
| 291   | 2.29 | 2.29    | 2.29    | 2.29    |

From Table 1, the accuracy enhancement of the new filters is clearly observed for cases of using small particle numbers such as \(M = 10\) to \(M = 30\). For this problem, as the \(M\) became large, the improvement in the accuracy over the EnKF became small. Among the new filters, the use of the UKF or the CKF lead to the superior state estimation accuracy.

Table 2 summarizes the calculation cost for each filter. The costs are defined by average times in seconds required for the filters to complete one run. As shown in this table, the filtering speed for the new filters using large particle numbers was much slower than the EnKF because the banks of the EKFs, the UKFs, and the CKFs rendered the calculation costs expensive. On the other hand, we confirmed that the new filters using the relatively small numbers of particles retained the superior filtering speed of the EnKF. The stability of the new filters as well as for the EnKF was also confirmed.

Table 2: Calculation costs measured by average times in seconds required for completing one run.

| \(M\) | EnKF | NF(EKF) | NF(UKF) | NF(CKF) |
|-------|------|---------|---------|---------|
| 10    | 0.11 | 0.18    | 0.21    | 0.21    |
| 20    | 0.13 | 0.25    | 0.32    | 0.33    |
| 30    | 0.14 | 0.33    | 0.45    | 0.46    |
| 40    | 0.14 | 0.42    | 0.56    | 0.59    |
| 50    | 0.17 | 0.54    | 0.68    | 0.69    |
| 100   | 0.26 | 0.9     | 1.25    | 1.26    |
| 200   | 0.45 | 1.69    | 2.38    | 2.37    |
| 300   | 0.68 | 2.53    | 3.67    | 3.65    |
6.2 Satellite reentry

The state-space model is expressed as

\[ m_0 = [6500.4 \ 349.14 \ -1.8093 \ -6.7967 \ 0.6932]^T \]

\[ P_0 = \text{diag}(10^{-6} \ 10^{-6} \ 10^{-6} \ 10^{-6} \ 0) \]

\[ x_k = \begin{bmatrix} r_{k,1} \\ r_{k,2} \\ v_{k,1} \\ v_{k,2} \\ c_k \end{bmatrix}^T \]

\[ = \begin{bmatrix} r_{k,1} - v_{k,1} \Delta \\ r_{k,2} - v_{k,2} \Delta \\ v_{k,1} + (D_{k-1} - 1 - G_{k-1} r_{k-1,2}) \Delta \\ v_{k,2} + (D_{k-1} - 1 - G_{k-1} r_{k-1,2}) \Delta \\ c_k \end{bmatrix} + v_k, \]

where

\[ D_{k-1} = -0.59783 \exp(c_{k-1}) \exp(\frac{6374 - r_{k-1}}{13.406}) v_{k-1}, \]

\[ r_{k,1} = \sqrt{r_{k-1,1}^2 + v_{k-1,2}^2} \]

\[ G_{k-1} = -3.986 \cdot 10^5 \frac{1}{r_{k-1}^3} \]

\[ y_k = \begin{bmatrix} d_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} \sqrt{(r_{k-1} - 6374)^2 + v_{k,2}^2} \\ \tan^{-1}\left(\frac{v_{k,2}}{r_{k,1} - 6374}\right) \end{bmatrix} + w_k \]

Here, \( r_{k,1}, r_{k,2}, v_{k,1}, \) and \( v_{k,2} \) are the elements of the two-dimensional position and velocity vectors of the satellite at \( k \) in a unit of km/kg/sec. \( c_k \) is a satellite shape parameter that affects the drag force parameter \( D_{k-1} \) acting on the satellite velocity. \( G_{k-1} \) is the gravity force parameter. \( \Delta \) is 0.05s. The observation data at \( k \) are distance \( d_k \) and angle \( \theta_k \) to the satellite measured from an earth station located at (6374, 0). \( v_k \) and \( w_k \) are independent white Gaussian noise with \( v_k \sim N(0 \ 0 \ 0 \ 0)^T \) \( \text{diag}(0.024064 \times 10^{-5} \ 2.4064 \times 10^{-5})) \) and \( w_k \sim N(0 \ 0 \ 0 \ 0)^T \) \( \text{diag}(10^{-2} \ 17 \ 10^{-3}) \), respectively. \( \{x_{1,1} \ y_{1,1} \ x_{2,1} \ y_{2,1} \ \cdots \ x_{4000} \ y_{4000}\} \) was generated, where \( \{y_1 \ y_2 \ y_5 \ \cdots \ y_{3999}\} = \emptyset \). Therefore, the simulation duration was 200s and the observation data was made every 0.1 second.

The evaluation measures were the average root mean square errors of the satellite position and parameter, that is, \( \sqrt{E[(r_{k,1} - \hat{r})^2 + (r_{k,1} - \hat{r})^2]} \) and \( \sqrt{E[(c_k - \hat{c}_k)^2]} \) estimated by 100 Monte Carlo runs and then, averaged over \( k = 2, 4, \ldots, 4000 \). The initial condition for the filters was \( x_0 = [6500.4 \ 349.14 \ -1.8093 \ -6.7967 \ 0]^T \) and \( P_0 = \text{diag}(10^{-6} \ 10^{-6} \ 10^{-6} \ 10^{-6} \ 1). \)

The system noise for the filters was \( v_k \) following \( N(0 \ 0 \ 0 \ 0)^T \) \( \text{diag}(0.024064 \times 10^{-5} \ 2.4064 \times 10^{-5} \ 10^{-6}) \) and the observation noise was the same that in the simulation model. Therefore, this simulation is for the simultaneous state and parameter estimation problem. Note that since \( \{y_1 \ y_2 \ y_5 \ \cdots \ y_{3999}\} = \emptyset \), the time updates of the filters were repeated twice until processing the new observation data. Therefore, for the new filters using the EKF, the UKF, and the CKF, after the time updates in Eq. (9), the subsequent time updates based on the time updates of the EKF, the UKF, and the CKF were necessary. For example, given \( \{\hat{x}^{(1)} \ y^{(2)} \ \cdots \ \hat{x}^{(M)} \} \), \( m_k^{(1)} \) and \( P_k^{(1)} \) were calculated according to Eq. (9) at \( k - 1 \) and then, \( m_k^{(1)} \) and \( P_k^{(1)} \) in Eq. (9) at \( k \) were obtained by employing the time update of the EKF, the UKF, or the CKF, respectively.

The average root mean square errors for satellite position and parameter are shown in Table 3. The results for the EnKF with \( M = 10 \) were not obtained due to the filter instability. As shown in this table, the use of the EnKF, the UKF, or the CKF did not produce significant differences for the new filters. The new filters using the small numbers of particles such as \( M = 30 \) and \( M = 50 \) scored the significantly higher state estimation accuracy than that for the EnKF. For example, 0.45 and 0.08 for the NF(EnKF) using \( M = 50 \) mean that the averaged estimation errors for the satellite position and parameter were 450m and 0.08, respectively, while those for the EnKF were 31.76km and 1.03.

The calculation costs are summarized in Table 4, where the costs were defined by the average times in seconds required for processing one observation data. Since the observation interval was 0.1s, the NF(EnKF)

Table 3: Filtering errors of the EnKF, the NF(EnKF), the NF(UKF), and the NF(CKF) with varying \( M \) from 10 to 300. For each filter, the upper and lower rows denote average root mean square errors for satellite position and parameter, respectively.

| \( M \) | \( \text{EnKF} \) | \( \text{NF(EnKF)} \) | \( \text{NF(UKF)} \) | \( \text{NF(CKF)} \) |
|-------|------------|------------|------------|------------|
| 10    | 0.11       | 0.16       | 0.11       | 0.11       |
| 20    | 0.63       | 0.64       | 0.63       | 0.64       |
| 30    | 0.51       | 0.51       | 0.51       | 0.51       |
| 40    | 0.47       | 0.47       | 0.47       | 0.47       |
| 50    | 0.45       | 0.45       | 0.45       | 0.45       |
| 999   | 0.07       | 0.07       | 0.07       | 0.07       |
| 300   | 0.36       | 0.36       | 0.36       | 0.36       |
| 600   | 0.07       | 0.07       | 0.07       | 0.07       |

Table 4: Calculation costs measured by average times in seconds required for processing one observation data.

| \( M \) | \( \text{EnKF} \) | \( \text{NF(EnKF)} \) | \( \text{NF(UKF)} \) | \( \text{NF(CKF)} \) |
|-------|------------|------------|------------|------------|
| 10    | 0.005      | 0.005      | 0.005      | 0.005      |
| 20    | 0.009      | 0.009      | 0.009      | 0.009      |
| 30    | 0.009      | 0.009      | 0.009      | 0.009      |
| 40    | 0.014      | 0.014      | 0.014      | 0.014      |
| 50    | 0.017      | 0.017      | 0.017      | 0.017      |
| 999   | 0.017      | 0.017      | 0.017      | 0.017      |
| 300   | 0.017      | 0.017      | 0.017      | 0.017      |

and the NF(CKF) using \( M = 300 \) were found as not applicable to this specific problem. The other filters
were all applicable and the overall results showed the effectiveness of the new filters.

7 Conclusions

We analyzed the EnKF using the MML for the non-linear observation model and showed that the use of the Dirac’s delta mixture predicted state PDF for the estimation of the MML deteriorated the accuracy of the EnKF. We then proposed a new EnKF algorithm using the Gaussian-sum predicted state PDF for the estimation of the MML and investigated its effectiveness through simulations based on two benchmark filtering problems of the vector nonlinear growth model and satellite reentry.

From the results obtained, we confirmed that the new filters using the EKF, the UKF, and the CKF outperformed the EnKF in terms of the filtering accuracy, while enjoying the superior filtering speed and stability of the EnKF. The new EnKF algorithms also achieved the promising filtering accuracy even for cases of using the relatively small numbers of particles.

References

[1] L. A. McGee and S. F. Schmidt: Discovery of the Kalman filter as a practical tool for aerospace and industry, NASA Technical Memorandum 86847, pp. 1–21, 1985.

[2] S. J. Julier, J. K. Uhlmann, and H. F. Durrant-Whyte: A new approach for filtering nonlinear systems, the 1995 American Control Conference, pp. 1628–1632, 1995.

[3] I. Arasaratnam and S. Haykin: Cubature Kalman filters, IEEE Transactions on Automatic Control., vol. 54, no. 6, pp. 1254–1269, 2009.

[4] G. Kitagawa: Monte Carlo filter and smoother for non-Gaussian nonlinear state space models, Journal of Computational and Graphical Statistics., vol. 5, issue 1, pp. 1–25, 1996.

[5] N. J. Gordon, D. J. Salmond, and A. F. M. Smith: Novel approach to nonlinear/non-Gaussian Bayesian state estimation, IEE Proceedings F (Radar and Signal Processing), vol. 140, issue 2, pp. 107–113, 1993.

[6] R. van der Merwe, A. Doucet, N. de Freitas, and E. Wan: The unscented particle filter, Advances in Neural Information Processing Systems 13, pp. 563–569, 2000.

[7] M. Murata and K. Hiramatsu: Non-Gaussian filter for continuous-discrete models, IEEE Transactions on Automatic Control., vol. 64, issue 12, pp. 5260–5264, 2019.

[8] S. Yun and R. Zanetti: Sequential Monte Carlo filtering with Gaussian mixture sampling, Journal of Guidance, Control, and Dynamics., vol. 42, no. 9, pp. 2069–2077, 2019.

[9] G. Evensen: Sequential data assimilation with a nonlinear quasi-geostrophic model using Monte Carlo methods to forecast error statistics, Journal of Geophysical Research., vol. 99, no. C5, pp. 10,143–10,162, 1994.

[10] J. H. Kotecha and P. M. Djurić: Gaussian particle filtering, IEEE Transactions on Automatic Control., vol. 51, no. 10, pp. 2592–2601, 2003.

[11] A. Doucet, N. de Freitas, and N. Gordon: Sequential Monte Carlo Methods in Practice. Springer-Verlag, NY; 2001.

[12] B. Jia and M. Xin: Refined nonlinear Gaussian quadrature filter, the 2019 American Control Conference., pp. 5366–537, 2019.

[13] S. Julier and J. K. Uhlmann: Unscented filtering and nonlinear estimation, IEEE Proceedings, vol. 92, no 3, pp. 401–422, 2004.

[14] R. E. Kalman: A new approach to linear filtering and prediction problems, Transactions of the ASME - Journal of Basic Engineering., vol. 82, no. 1, pp. 35–45, 1960.

[15] A. H. Jazwinski: Stochastic Processes and Filtering Theory. Dover Publications, NY; 1970.

[16] M. Murata: On ensemble Kalman filter and smoother, the 49th ISCIIE International Symposium on Stochastic Systems Theory and Its Applications, pp. 15–20, 2017.