Is supernilpotence super nilpotence?

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Dedicated to Ralph Freese, Bill Lampe, and J. B. Nation.

Abstract. We show that the answer to the question in the title is: “Yes, for finite algebras”.

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1. Introduction

The word “supernilpotence”, with a specific meaning, entered the general commutator theory lexicon just over a decade ago [1]. The name suggests that some equation of the form

\[ \text{supernilpotence} = \text{nilpotence} + \varepsilon \]  \quad (1.1)

should be true, but no such equation has ever been shown to hold except in restricted settings. Equation (1.1) is meant to express that supernilpotence implies nilpotence, but that nilpotence does not always imply supernilpotence.

In this paper we establish that Equation (1.1) holds for finite algebras. The results were obtained in Fall 2017 and announced at the conference Algebra and Lattices in Hawaii in Spring 2018. The question of whether Equation (1.1) holds for infinite algebras was posed at that conference, and answered shortly afterwards by two groups of researchers. The first solution came from Matthew...
Moore and Andrew Moorhead, who constructed in [10], for any \( n > 1 \), an algebra \( A_n \) that is \( n \)-step supernilpotent, but not solvable of any degree, hence not nilpotent of any degree. The second solution came from Steven Weinell, who determined in [13] all possible higher commutator behaviors of simple algebras. His work shows that there is a simple algebra satisfying \([1,1,1] = 0\) and \([1,1] = 1\). The first guarantees that the algebra is 2-step supernilpotent, while the second guarantees that the algebra is neutral, which is stronger than saying it is not solvable (hence not nilpotent) of any degree.

Let us describe the context of this research briefly. The word “nilpotent” was introduced into mathematics in [12] to describe an element \( A \) of an associative algebra which satisfies \( \exists n \geq 2(A^n = 0) \). The group-theoretic concept of nilpotence was isolated in the paper [3], which studied finite groups with one Sylow \( p \)-subgroup for each \( p \), i.e. finite groups that factor as a product of groups of prime power order. In [2], the concept of “central nilpotence” of loops was studied. This definition of nilpotence for finite loops agrees with the commutator-theoretic definition for finite groups, but does not agree with the “prime power factorization into nilpotent factors” property from [3]. The difference in these approaches to nilpotence for finite loops is made clear in [14], where it is shown that a finite loop \( L \) has a prime power factorization into nilpotent factors if and only if \( L \) is centrally nilpotent and \( L \) has a nilpotent multiplication group. This result was extended in [7] to any variety of finite signature which satisfies a congruence identity: a finite algebra in such a variety has a prime power factorization into nilpotent factors if and only if it has a finite bound on the arity of its nontrivial commutator terms if and only if it is nilpotent in the sense of ordinary commutator theory and has a twin monoid that is a nilpotent group. The middle condition, having a finite bound on the arity of nontrivial commutator terms, was shown to be equivalent to supernilpotence for congruence permutable varieties in [1]. Altogether, these results show that, for congruence permutable varieties, a finite algebra of finite signature is supernilpotent if and only if it has a prime power factorization into nilpotent factors.

It is not difficult to show that these results extend verbatim from congruence permutable varieties to varieties that omit type 1. The reason for this is that if \( A \) is a finite supernilpotent algebra in a variety that omits type 1, then the subvariety generated by \( A \) satisfies \( \text{typ}\{V(a)\} \subseteq \{2\} \). (To see why this is so, read the remark after Lemma 2.1.) Therefore, if \( A \) is a finite supernilpotent algebra in a variety that omits type 1, the subvariety generated by \( A \) will be congruence permutable and one can cite the results that apply to congruence permutable varieties.

But the presence of type 1 in a variety makes the problem difficult. For example, it is easy to see that any finite algebra that has a finite bound on the essential arities of its term operations must be supernilpotent of type 1, but already in this special case it is not obvious that such algebras must be nilpotent. (See [8] for a proof of nilpotence in this case.)

The purpose of this paper is to investigate the general case, when type 1 is present. We will first show that a congruence on a finite algebra that is
supernilpotent has local twin monoids that are nilpotent groups (cf. [5,6,8,9]). Then we will argue that any congruence on a finite algebra that has local twin monoids that are nilpotent groups must be nilpotent. This suffices to prove that supernilpotence implies nilpotence for finite algebras.

2. Supernilpotence for finite algebras

Our goal is to prove that, for a congruence $\beta$ of a finite algebra $A$, the higher commutator condition

$$[\beta, \ldots, \beta] = 0,$$

with $k + 1$ $\beta$’s implies the binary commutator conditions

$$(\beta)^{\ell} = 0 = (\beta)^{m}$$

for some $\ell$ and $m$. Here, for an arbitrary congruence $\theta$, we define $(\theta)^{1} = [\theta]^{1} = \theta, (\theta)^{\ell+1} = [\theta, (\theta)^{\ell}]$, and $[\theta, (\theta)^{m}] = [\theta]^{m}$. We will connect the higher commutator to the binary commutator through an intermediate property involving the action of the $\beta$-twin monoids on minimal sets of $A$. We recall the relevant definitions and notation now.

Given congruences $\alpha, \beta \in \text{Con}(A)$, the set $M(\alpha, \beta)$ of $\alpha, \beta$-matrices is

$$M(\alpha, \beta) = \left\{ \begin{bmatrix} f(a, b) & f(a, b') \\ f(a', b) & f(a', b') \end{bmatrix} \mid f(x, y) \in \text{Pol}(A), a \alpha a', b \beta b' \right\}.
$$

The relation $C(\alpha, \beta; \delta)$ holds if

$$m_{11} \equiv m_{12} \pmod{\delta} \quad \text{implies} \quad m_{21} \equiv m_{22} \pmod{\delta} \quad (2.1)$$

whenever $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \in M(\alpha, \beta)$. One could indicate the two-dimensional nature of an $\alpha, \beta$-matrix with one of the simple diagrams

$$f(a, b) \quad \beta \quad f(a, b') \quad m_{11} \quad m_{12} \quad \text{or} \quad f(a', b) \quad f(a', b') \quad m_{21} \quad m_{22} \quad \text{or just}$$

Implication (2.1) could be indicated by

$$\begin{array}{cc}
\begin{array}{c}
\delta \\
\delta
\end{array}
& \quad \Rightarrow \\
\begin{array}{c}
\delta \\
\delta
\end{array}
\end{array}$$

We define $[\alpha, \beta]$ to be the least $\delta$ such that $C(\alpha, \beta; \delta)$ holds. We say that $\beta$ is abelian (or 1-step supernilpotent), and we write $[\beta, \beta] = 0$ for this, exactly when $C(\beta, \beta; 0)$ holds.
Now we describe the ternary case using less detail than in the binary case. Given three congruences \( \alpha, \beta, \gamma \in \text{Con}(a) \), an \( \alpha, \beta, \gamma \)-matrix is an object that could be depicted

\[
\begin{array}{c}
f(a, b, c) \\
\alpha \\
f(a', b, c) \\
\gamma \\
f(a', b, c') \\
\beta \\
f(a, b, c')
\end{array}
\]

The relation \( C(\alpha, \beta, \gamma; \delta) \) is defined by an implication depicted

\[
\begin{array}{c}
\delta \\
\delta \\
\delta \\
\delta
\end{array}
\Rightarrow
\begin{array}{c}
\delta \\
\delta \\
\delta \\
\delta
\end{array}
\]

(We are asserting with this diagram that, for every \( \alpha, \beta, \gamma \)-matrix, the implication holds.)

We define \([\alpha, \beta, \gamma]\) to be the least \( \delta \) such that \( C(\alpha, \beta, \gamma; \delta) \) holds. We say that \( \beta \) is 2-step supernilpotent, and write \([\beta, \beta, \beta] = 0\), exactly when \( C(\beta, \beta, \beta; 0) \) holds.

In no detail at all, one can guess how the set of \( \alpha_1, \ldots, \alpha_{k+1} \)-matrices is defined, and how the implication \( C(\alpha_1, \ldots, \alpha_{k+1}; \delta) \) is defined. (Or, one can refer to [11, Definitions 2.1 and 2.8] to see one way the notational complexities in high dimensions might be handled. For this paper, we only need sufficient understanding of the higher commutator to understand the proof of Lemma 2.1.)

A congruence \( \beta \in \text{Con}(A) \) is \( k \)-supernilpotent or \( k \)-step supernilpotent if \( C(\beta, \ldots, \beta; 0) \) holds (with \( k+1 \) instances of \( \beta \)). We also write \([\beta, \ldots, \beta] = 0\) (with \( k+1 \) instances of \( \beta \)).

If \( \beta \in \text{Con}(A) \), then a \( k \)-dimensional \( \beta \)-snag, or a \( \beta^{[k]} \)-snag, is a pair \((0, 1) \in A^2, 0 \neq 1\), for which there is a \( k \)-dimensional matrix in \( M(\beta, \ldots, \beta) \) where \( 2^k - 1 \) entries have value 0 and the remaining entry has value 1.

Recall from [4, Definition 7.1] that, in tame congruence theory, a \( 2 \)-snag is a pair \((0, 1), 0 \neq 1\), for which there is a binary polynomial \( x \land y \) whose restriction to \( \{0, 1\} \) is the meet operation: \( 0 \land 0 = 0 \land 1 = 1 \land 0 = 0 \) and \( 1 \land 1 = 1 \). If \((0, 1) \in \beta\) is a \( 2 \)-snag, then the matrix

\[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
0 \land 0 & 0 \land 1 \\
1 \land 0 & 1 \land 1
\end{bmatrix} \in M(\beta, \beta)
\]
witnesses that $(0, 1)$ is a $\beta^{[2]}$-snag. Moreover, using the polynomial $x_1 \wedge x_2 \wedge \cdots \wedge x_k$ (parenthesized in any way) in place of $x \wedge y$, one can show that if $(0, 1) \in \beta$ is a 2-snag, then it is a $\beta^{[k]}$-snag for any $k$.

**Lemma 2.1.** If

- $A$ is a finite algebra,
- $\beta \in \text{Con}(A)$, and
- $\beta$ is $k$-supernilpotent,

then $A$ has no $\beta^{[k+1]}$-snags.

**Proof.** Any matrix witnessing that $(0, 1)$ is a $\beta^{[k+1]}$-snag also witnesses that $C(\beta, \ldots, \beta; 0)$ (with $k+1$ $\beta$’s) fails. □

Lemma 2.1 is already strong enough to show that supernilpotent congruences on finite algebras are solvable, since (by the lemma and the remarks preceding it) supernilpotence implies the absence of 2-snags, which implies solvability according to [4, Theorem 7.2].

A subset $U$ of an algebra $A$ is a **neighborhood** if it is the image of some idempotent unary polynomial. (That is, $U = e(A)$ for some $e(x) \in \text{Pol}_1(A)$ satisfying $e(e(x)) = e(x)$ on $A$.) Two unary polynomials $f(x), g(x) \in \text{Pol}_1(A)$ are $\beta$-twins if there is a polynomial $h(x, y) = :h_y(x)$ and $\beta$-related parameter sequences $a, b$ such that $f(x) = h_a(x)$ and $g(x) = h_b(x)$ on $A$. The $\beta$-twin monoid on $U$, $T_{\beta}(A, U)$, is the monoid of self-maps $f|U : U \to U$ induced by all unary polynomials $f$ of $A$ satisfying

1. $f(A) \subseteq U$, and
2. $f$ is a $\beta$-twin of some unary polynomial $g$ whose restriction to $U$ is the identity function on $U$.

It is explained in [7, Lemma 2.2] why, when $A$ is finite, there is a single polynomial $s_y(x)$ and a single tuple $a$ such that $T_{\beta}(A, U)$ consists entirely of the functions of the form $s_b(x)|U$ where $b \beta a$. We call $s$ a **generic polynomial** for $T_{\beta}(A, U)$.

**Lemma 2.2.** If

- $A$ is a finite algebra,
- $\beta \in \text{Con}(A)$, and
- $A$ has no $\beta^{[k+1]}$-snags,

then for any neighborhood $U$ of $A$, $T_{\beta}(A, U)$ acts on $U$ as a nilpotent group of permutations, with nilpotence class at most $k$.

**Proof.** The finite monoid $T_{\beta}(A, U)$ acts on $U$. If it contains a nonpermutation, then it contains an idempotent nonpermutation. If $T_{\beta}(A, U)$ contains an idempotent nonpermutation, then by iterating the generic polynomial $s_y(x)$ as a function of $x$ until it is idempotent in $x$ for every $y$ we obtain a polynomial $t_y(x)$ for which there are $\beta$-related parameter sequences $b, c$ such that

1. $t_y(t_y(x)) = t_y(x)$ and $t_y(A) \subseteq U$ for all parameter sequences $y$ from $A$,
2. $t_b(x) = x$ for $x \in U$, and
(iii) the restriction of $t_c(x)$ to $U$ is an idempotent nonpermutation of $U$. In particular, there are $u, v \in U$ such that $u \neq t_c(u) = v = t_c(v)$.

Notice that $(u, v) = (t_{b}(u), t_{c}(u)) \in \beta$, since $b \beta c$.

Let $p(x, y_1, \ldots, y_k) = (t_{y_k} \circ \cdots \circ t_{y_1})(x)$. It is not hard to see that the $(k + 1)$-dimensional matrix obtained from $p$ by making the $\beta$-related choices $x \in \{u, v\}$ and $y_i \in \{b, c\}$ witnesses that $(v, u)$ is a $[\beta^{k+1}]$-snag, since the value is $u$ if $x = u$ and $y_i = b$ for all $i$ and the value is $v$ otherwise. This shows that if $A$ has no $[\beta^{k+1}]$-snags, then we cannot have $u \neq v$ as above, hence $\text{Tw}_\beta(A) \cup U$ contains no nonpermutations. In this case, $\text{Tw}_\beta(A, U)$ acts on $U$ as a group of permutations.

Now suppose that $\text{Tw}_\beta(A, U)$ acts on $U$ as a group of permutations, but not as a group of permutations of nilpotence class at most $k$. There must exist permutations in $\text{Tw}_\beta(A, U)$ which fail to satisfy the group-theoretic identity

$$[x_1, \ldots, x_{k+1}] = 1,$$

which asserts $k$-step nilpotence. More fully, this means that there exist permutations $\gamma_1, \ldots, \gamma_{k+1} \in \text{Tw}_\beta(A, U)$ such that the permutation of $U$ represented by the group commutator $[\gamma_1, \ldots, \gamma_{k+1}]$ is not the identity function on $U$. To unravel this statement even further, there exist $u \neq v$ in $U$ such that $[\gamma_1, \ldots, \gamma_{k+1}](v) = u$. Since each $\gamma_i$ is the restriction to $U$ of a $\beta$-twin of a polynomial whose restriction to $U$ is the identity function on $U$, there must exist parameter sequences $b, c$, with $b \beta c$, such that for the generic polynomial $s_y(x)$ we have

(i) $s_b(x) = x$ on $U$

(ii) $s_{c_i}(x) = \gamma_i(x)$ on $U$.

$A$ has a polynomial $q(x, y_1, \ldots, y_{k+1})$ equal to $[s_{y_1}, \ldots, s_{y_{k+1}}](x)$. The restriction to $U$ of this polynomial satisfies

$$q(x, c_1, \ldots, c_{k+1}) = [\gamma_1, \ldots, \gamma_{k+1}](x),$$

which is a permutation of $U$ that maps $v$ to $u$. But any other specialization of $q(x, y_1, \ldots, y_{k+1})$ with $y_i \in \{b, c\}$ results in a polynomial which, as a function on $U$ in the variable $x$, is the identity permutation of $U$. Consider the $(k + 1)$-dimensional matrix obtained from $q$ by fixing $x = v$ and making the $\beta$-related choices $y_i \in \{b, c\}$. This matrix witnesses that $(v, u)$ is a $[\beta^{k+1}]$-snag, since the value is $u$ if each $y_i$ is chosen to be $c_i$, while the value is $v$ if any $y_i$ is chosen to be $b$. In the contrapositive form, if $A$ has no $[\beta^{k+1}]$-snags, then $\text{Tw}_\beta(A, U)$ is a group of permutations of $U$ of nilpotence class at most $k$. \hfill \Box

**Lemma 2.3.** If

- $A$ is a finite algebra,
- $\beta \in \text{Con}(A)$,
- $\delta < \theta$ in $\text{Con}(A)$,
- $U \in \text{Min}_A(\delta, \theta)$,
- $N$ is a $\langle \delta, \theta \rangle$-trace of $U$, and
- $\text{Tw}_\beta(A, U)$ acts as a group of permutations on $U$,

then $C(\beta, N^2; \delta)$ holds.
Proof. This is proved in [6, Lemma 3.4] in the case $\beta = 1$, but the same argument holds for general $\beta$. Namely, the argument shows that if $C(\beta, N^2; \delta)$ fails, then $Tw_\beta(A, U)$ contains an idempotent nonpermutation. \qed

Now we recall from [5, Definition 4.12] the definition of a $\beta$-regular quotient $\langle \delta, \theta \rangle$ of type 1.

**Definition 2.4.** Let $A$ be a finite algebra with a tame quotient $\langle \delta, \theta \rangle$ of type 1, and let $\beta$ be a congruence of $A$. $\langle \delta, \theta \rangle$ is $\beta$-regular if whenever

- $U \in \text{Min}_A(\delta, \theta)$,
- $N$ is a trace of $U$,
- $H_{N, \beta}$ is the subgroup of $Tw_\beta(A, U)$ consisting of polynomials that map $N$ into itself,
- $p(x) \in H_{N, \beta}$ has a fixed point modulo $\delta$ on $N$ (i.e., $p(u) \equiv \delta u$ for some $u \in N$),

then $p(x)$ is the identity modulo $\delta$ on $N$ (i.e., $p(x) \equiv \delta x$ for all $x \in N$).

The preceding definition hints that there will be some group theory component to the next part of the proof. We isolate the fact from group theory that will be needed in our proof.

**Lemma 2.5.** If $G$ is finite group, $K$ is a core-free maximal subgroup of $G$, and $H \triangleleft G$ is a nilpotent normal subgroup, then $H$ is abelian and $H \cap K = \{1\}$.

Recall that the core of a subgroup $K$ of $G$ is the intersection of the $G$-conjugates of $K$. $K$ is core-free in $G$ if its core is trivial, equivalently $K$ is core-free if $\{1\}$ is the only subgroup of $K$ that is normal in $G$.

**Proof.** If $H \subseteq K$, then since $H \triangleleft G$ and $K$ is core-free we get $H = \{1\}$, so $H$ is abelian and $H \cap K = \{1\}$.

Assume $H \not\subseteq K$. Since $K$ is maximal in $G$ and $H \triangleleft G$, we derive that $HK = G$. Since the center of $H$ is characteristic in $H$ and $H \triangleleft G$, $Z(H)$ is normal in $G$. $Z(H)$ is nontrivial, since $H$ is nilpotent. $Z(H) \not\subseteq K$, since $K$ is core-free, so $G = Z(H)K$. Note that $Z(H) \cap K$ is normal in both $Z(H)$ (since $Z(H)$ is abelian) and $K$ (since $Z(H) \triangleleft G$), hence $Z(H) \cap K \triangleleft Z(H)K = G$, hence $Z(H) \cap K$ is contained in the core of $K$. This shows that $Z(H) \cap K = \{1\}$.

So far we have established that $Z(H)$ is a normal complement to $K$. This shows that any $g \in G$ is uniquely representable as $g = zgkhg$ where $z_g \in Z(H)$ and $k_g \in K$. This uniqueness implies that $\varphi : H \rightarrow Z(H) : h \mapsto z_h$ is a function. Moreover, if $h \in H$, then in the representation $h = z_hk_h$, we have $k_h = z_h^{-1}h \in H$, so in fact $k_h \in H \cap K$. This is enough to establish that the function $\varphi : H \rightarrow Z(H) : h \mapsto z_h$ is a group homomorphism. To see this, assume that $h = z_hk_h$ and $h' = z_h'k_h'$. Then

$$hh' = z_hk_hz_h'k_h' = z_h(k_hz_h')k_h' \overset{?}{=} z_h(z_h'k_h)k_h' = (z_hz_h')(k_hk_h').$$

Here the equality $\overset{?}{=}$ is justified by the facts that $k_h \in H \cap K \subseteq H$ and $z_h' \in Z(H)$. Now the unique $Z(H)K$-representation of $hh'$ is both $z_hh'k_hh'$ and $(z_hz_h')(k_hk_h')$, so $z_hh' = z_hz_h'$, which is what it means for $\varphi$ to be a homomorphism.
By examination one sees that $\varphi$ is the identity on its image $Z(H)$, and that the kernel of $\varphi$ is $H \cap K$. Therefore $H \cap K$ is a normal complement to $Z(H)$ in $H$. Thus forces $H \cong Z(H) \times (H \cap K) \cong Z(H) \times H/Z(H)$. The rightmost side has smaller nilpotence degree than the leftmost unless $H = Z(H)$ and $H \cap K = \{1\}$, so we conclude that $H$ is abelian and $H \cap K = \{1\}$, as desired. \hfill \Box

\textbf{Lemma 2.6.} If

- $A$ is a finite algebra,
- $\beta \in \text{Con}(A)$,
- $\delta \prec \theta$ is a type-1 covering in $\text{Con}(A)$,
- $U \in \text{Min}_A(\delta, \theta)$,
- $N$ is a $\langle \delta, \theta \rangle$-trace of $U$, and
- $\text{Tw}_\beta(A, U)$ acts as a nilpotent group of permutations on $U$,

then $\langle \delta, \theta \rangle$ is $\beta$-regular.

\textbf{Proof.} Let $M = A|_N/\delta|_N$. By [4, Corollary 5.2 (1)], $M$ is a minimal algebra of type 1. By the definition of type 1 [4, Definition 4.10], $M$ is polynomially equivalent to a $G$-set. By [4, Lemma 2.4] and the fact that $\delta \prec \theta$, $M$ is simple.

Let $G$ be the group of polynomial permutations of $M$. Let $H = H_{N,\beta}/\delta$ be the subgroup of $G$ represented by $\beta$-twins of the identity. Note that $H < G$, since any conjugate of a twin of the identity is a twin of a conjugate of the identity, hence is a twin of the identity. Also note that $H$ is nilpotent, since it is a quotient of $H_{N,\beta}$, which is a subgroup of the nilpotent group $\text{Tw}_\beta(A, U)$. Finally let $K = K_{u/\delta}$ be the stabilizer of some point $u/\delta \in N/\delta = M$. To prove that $\langle \delta, \theta \rangle$ is $\beta$-regular we must show that any $\beta$-twin of the identity in $G$ (i.e., an element representing an element of $H$) which has some fixed point $u/\delta$ on $N/\delta = M$ (i.e., which lies in some $K = K_{u/\delta}$) must be the identity modulo $\delta$ (i.e., represents the identity element of $G$). In short, we want to prove that $H \cap K = \{1\}$.

As a first case, assume that $M$ is a discrete $G$-set (so $G$, $H$, $K$ are all trivial). Clearly $H \cap K = \{1\}$.

The remaining case is the one where $M$ is not discrete. Since $M$ is simple, $G$ acts primitively on $M$, and any 1-point stabilizer $K = K_{u/\delta}$ is a maximal subgroup of $G$. We are now in the situation of Lemma 2.5, so $H \cap K = \{1\}$, as desired. \hfill \Box

\textbf{Theorem 2.7.} Suppose that $A$ is a finite algebra, and $\beta \in \text{Con}(A)$. The following implications hold among the listed properties: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

1. $\beta$ is $k$-supernilpotent.
2. $A$ has no $\beta^{[k+1]}$-snags.
3. $\text{Tw}_\beta(A, U)$ acts as a nilpotent group of permutations, with nilpotence class at most $k$, on any minimal set $U$ of $A$.
4. $C(\beta, \theta; \delta)$ and $C(\theta, \beta; \delta)$ hold whenever $\delta \prec \theta$ in $\text{Con}(A)$.
5. $\beta$ is (both left and right) nilpotent.
Proof. The fact that (1) ⇒ (2) is proved in Lemma 2.1.

The fact that (2) ⇒ (3) is a consequence of Lemma 2.2.

Now we prove that (3) ⇒ (4). Assume (3). We first prove that β is left nilpotent, and then we prove (4).

Assume the sequence $\beta = (\beta)^1 \geq (\beta)^2 \geq \cdots$ stabilizes at $\theta$, that is that $\theta = \bigcap_i (\beta)^i$. If $\theta = 0$, then $\beta$ is left nilpotent and there is nothing more to prove at this point. Otherwise $\theta = (\beta)^N = (\beta)^{N+1} > 0$ for some $N$. Choose $\delta$ so that $\delta \prec \theta$ in this case.

We cannot have $C(\beta, \delta; \theta)$, else $[\beta, \theta] = (\beta)^{N+1} \leq \delta < \theta = (\beta)^N$, which contradicts the choice of $\theta$. But we do have $C(\beta, N^2; \delta)$ for some (any) $\langle \delta, \theta \rangle$-trace, according to Lemma 2.3. Using terminology from [5], the fact that $\beta$ centralizes a $\langle \delta, \theta \rangle$-trace $N$ but does not centralize the entire congruence quotient means that $\langle \delta, \theta \rangle$ is not $\beta$-coherent. From [5, Lemma 4.2], the type of $\langle \delta, \theta \rangle$ must be 1. From [5, Lemma 4.13], $\langle \delta, \theta \rangle$ cannot be $\beta$-regular. However, we proved in Lemma 2.6 that $\langle \delta, \theta \rangle$ is $\beta$-regular when it is of type 1. This contradicts our assumption that $\beta = (\beta)^1 \geq (\beta)^2 \geq \cdots$ stabilizes at some $\theta > 0$, so we may conclude that $\beta$ is left nilpotent.

Now we change notation and allow $\langle \delta, \theta \rangle$ to be an arbitrary prime quotient of $A$. It is shown in [5, Lemmas 3.1 and 3.2] that the following conditions are equivalent when $\beta \in \text{Con}(A)$ is left nilpotent and the type of the prime quotient $\langle \delta, \theta \rangle$ is not 1:

(i) $C(\beta, \theta; \delta)$,
(ii) $[\beta, \theta] \leq \delta$,
(iii) $C(\theta, \beta; \delta)$, and
(iv) $[\theta, \beta] \leq \delta$.

Moreover, we can add the following equivalent condition to this list when the type of $\langle \delta, \theta \rangle$ is not 1:

(v) $C(\beta, N^2; \delta)$ for some (any) $\langle \delta, \theta \rangle$-trace $N$.

The reason that this can be added is that $C(\beta, \theta; \delta) \Rightarrow C(\beta, N^2; \delta)$ always holds, since $N^2 \subseteq \theta$, while [5, Lemma 4.2] proves that the reverse implication $C(\beta, N^2; \delta) \Rightarrow C(\beta, \theta; \delta)$ can only fail when the type of $\langle \delta, \theta \rangle$ is 1. We proved in Lemma 2.3 that (v) holds under our assumption (3), so we derive here that each of (i)–(iv) from above hold when the type of $\langle \delta, \theta \rangle$ is not 1.

When the type of $\langle \delta, \theta \rangle$ is 1, we can argue the same way as long as $\langle \delta, \theta \rangle$ is $\beta$-regular, according to [5, Lemma 4.14]. Since we established $\beta$-regularity in Lemma 2.6, we are done.

Now it is easy to show that (4) ⇒ (5). From (4) it follows that if $\theta$ is any nonzero congruence of $A$, then $[\beta, \theta] < \theta$ and $[\theta, \beta] < \theta$, so by the finiteness of $\text{Con}(A)$ any mixed commutator expression involving enough $\beta$’s must equal zero. □

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