MULTI-OBJECTIVE LINEAR QUADRATIC TEAM OPTIMIZATION *

ATHER GATTAMI †

Abstract. In this paper, we consider linear quadratic team problems with an arbitrary number of quadratic constraints in both stochastic and deterministic settings. The team consists of players with different measurements about the state of nature. The objective of the team is to minimize a quadratic cost subject to additional finite number of quadratic constraints. We will first consider the Gaussian case, where the state of nature is assumed to have a Gaussian distribution, and show that the linear decisions are optimal and can be found by solving a semidefinite program. We then consider the problem of minimizing a quadratic objective for the worst case scenario, subject to an arbitrary number of deterministic quadratic constraints. We show that linear decisions can be found by solving a semidefinite program.

Key words. Team Decision Theory, Game Theory, Convex Optimization.

AMS subject classifications. 99J04, 49K04

1. Introduction. We consider the problem of distributed decision making with information constraints under linear quadratic settings. For instance, information constraints appear naturally when making decisions over networks. These problems can be formulated as team problems. The team problem is an optimization problem with several decision makers possessing different information aiming to optimize a common objective. Early results in [11] considered static team theory in stochastic settings and a more general framework was introduced by Radner [12], where existence and uniqueness of solutions were shown. Connections to dynamic team problems for control purposes were introduced in [9]. In [4], the team problem with two team members was solved. The solution cannot be easily extended to more than two players since it uses the fact that the two members have common information; a property that doesn’t necessarily hold for more than two players. Also, a nonlinear team problem with two team members was considered in [2], where one of the team members is assumed to have full information whereas the other member has only access to partial information about the state of the world. Related team problems with exponential cost criterion were considered in [10]. Optimizing team problems with respect to affine decisions in a minimax quadratic cost was shown to be equivalent to stochastic team problems with exponential cost, see [5]. The connection is not clear when the optimization is carried out over nonlinear decision functions. The deterministic version (minimizing the worst case scenario) of the linear quadratic team decision problem was solved in [8].

In this paper, we will consider both Gaussian and deterministic settings (worst case scenario) for team decision problems under additional quadratic constraints. It’s well-known that additional constraints, although convex, could give rise to complex optimization problems if the optimized variables are functions (as opposed to real numbers). For instance linear functions, that is functions of the form \( \mu(x) = Kx \) where \( K \) is a real matrix, are no longer optimal. We will illus-
trate this fact by the following example:

**Example 1.** For $x \in \mathbb{R}$, we want to minimize the objective function

$$|u|^2$$

subject to

$$|x - u|^2 \leq \gamma$$

Some Hilbert space theory shows that the optimal $u$ is given by

$$u = \mu(x) = (|x| - \sqrt{\gamma})x/|x| \text{ if } |x|^2 > \gamma,$$

and

$$u = \mu(x) = 0 \text{ otherwise.}$$

Obviously, the optimal $u$ is a nonlinear function of $x$.

Increasing the dimension of $x$, and adding constraints on the structure of $u$, for instance $x \in \mathbb{R}^N$ and $u = \mu(x) = (\mu(x_1), ..., \mu(x_N))$, certainly makes the constrained optimization more complicated. The example above shows that, in spite of having a convex optimization carried out over a Hilbert space, the optimal decision function is nonlinear. However, we show in the upcoming sections that multi-objective problems behave nicely when considering the expected values of the objectives in the Gaussian case, in the sense that linear decisions are optimal. For the deterministic counterpart which is not an optimization problem over a Hilbert space, we show how to find the linear optimal decisions by semidefinite programming. However, the optimality of the linear decisions remains an open question.

2. **Notation.** The following table gives a list of the notation we are going to use throughout the text:

| Symbol | Description |
|--------|-------------|
| $\mathbb{S}^n$ | The set of $n \times n$ symmetric matrices. |
| $\mathbb{S}_+^n$ | The set of $n \times n$ symmetric positive semidefinite matrices. |
| $\mathbb{S}^{++}_n$ | The set of $n \times n$ symmetric positive definite matrices. |
| $\mathcal{M}$ | The set of measurable functions. |
| $C$ | The set of functions $\mu : \mathbb{R}^p \to \mathbb{R}^m$ with $\mu(y) = (\mu_1^T(y_1), \mu_2^T(y_2), ..., \mu_N^T(y_N))^T$, $\mu_i : \mathbb{R}^{p_i} \to \mathbb{R}^{m_i}$, $\sum_i m_i = m$, $\sum_i p_i = p$. |
| $[A]_{ij}$ | The element of $A$ in position $(i, j)$. |
| $\succeq$ | $A \succeq B \iff A - B \in \mathbb{S}_+^n$. |
| $\succ$ | $A \succ B \iff A - B \in \mathbb{S}^{++}_n$. |
| $\otimes$ | The Kronecker binary operation between two matrices $A$ and $B$, $A \otimes B$. |
| $\text{Tr}$ | $\text{Tr}[A]$ is the trace of the matrix $A$. |
| $\mathcal{N}(m, X)$ | The set of Gaussian variables with mean $m$ and covariance $X$. |
3. Linear Quadratic Gaussian Team Theory. In this section we will review some classical results in stochastic team theory with new simpler proofs for the linear quadratic case, that first appeared in [6] and [7].

In the static team decision problem, one would like to solve

$$\min_{\mu} \mathbb{E} \left[ x^T \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} x \right]$$

subject to

$$y_i = C_i x + v_i$$

$$u_i = \mu_i(y_i)$$

for $$i = 1, \ldots, N.$$ (3.1)

Here, $$x$$ and $$v$$ are independent Gaussian variables taking values in $$\mathbb{R}^n$$ and $$\mathbb{R}^p$$, respectively, with $$x \sim \mathcal{N}(0, V_{xx})$$ and $$v \sim \mathcal{N}(0, V_{vv}).$$ Also, $$y_i$$ and $$u_i$$ will be stochastic variables taking values in $$\mathbb{R}^{p_i}, \mathbb{R}^{m_i},$$ respectively, and $$p_1 + \cdots + p_N = p.$$ We assume that

$$\begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \in S_{m+n}^+, \quad (3.2)$$

and $$Q_{uu} \in S_m^{++}, m = m_1 + \cdots + m_N.$$

If full state information about $$x$$ is available to each decision maker $$u_i,$$ the minimizing $$u$$ can be found easily by completion of squares. It is given by $$u = Lx,$$ where $$L$$ is the solution to

$$Q_{uu}L = -Q_{ux}. \quad (3.3)$$

Then, the cost function in (3.1) can be rewritten as

$$J(x, u) = \mathbb{E}\{x^T (Q_{xx} - L^T Q_{uu} L) x\} + \mathbb{E}\{(u - Lx)^T Q_{uu} (u - Lx)\}. \quad (3.4)$$

Minimizing the cost function $$J(x, u),$$ is equivalent to minimizing

$$\mathbb{E}\{(u - Lx)^T Q_{uu} (u - Lx)\},$$

since nothing can be done about $$\mathbb{E}\{x^T (Q_{xx} - L^T Q_{uu} L) x\}$$ (the cost when $$u$$ has full information).

The next theorem is due to Radner [12], but we give a different formulation and proof that is simpler, which relies on the structure of the linear quadratic Gaussian setting:

**Theorem 1.** Let $$x$$ and $$v_i$$ be Gaussian variables with zero mean, taking values in $$\mathbb{R}^n$$ and $$\mathbb{R}^{p_i},$$ respectively, with $$p_1 + \cdots + p_N = p.$$ Also, let $$u_i$$ be a stochastic variable taking values in $$\mathbb{R}^{m_i},$$ $$Q_{uu} \in S_m^+, m = m_1 + \cdots + m_N,$$ $$L \in \mathbb{R}^{m \times n}, C_i \in \mathbb{R}^{p_i \times n},$$ for $$i = 1, \ldots, N.$$ Then, the optimal decision $$\mu$$ to the optimization problem

$$\min_{\mu} \mathbb{E}\{(u - Lx)^T Q_{uu} (u - Lx)\}$$

subject to

$$y_i = C_i x + v_i$$

$$u_i = \mu_i(y_i)$$

for $$i = 1, \ldots, N.$$ (3.4)

is unique and linear in $$y.$$
Proof. Let $Z$ be the linear space of functions such that $z \in Z$ if $z_i$ is a linear transformation of $y_i$, that is $z_i = A_i y_i$ for some real matrix $A_i \in \mathbb{R}^{m_i \times p_i}$. Since $Q_{uu} > 0$, $Z$ is a linear space under the inner product
\[ \langle g, h \rangle = \mathbb{E}\{g^T Q_{uu} h\}, \]
and norm
\[ \|g\|^2 = \mathbb{E}\{g^T Q_{uu} g\}. \]
The optimization problem in (3.4) where we search for the linear optimal decision can be written as
\[ \min_{u \in Z} \|u - Lx\|^2 \tag{3.5} \]
Finding the best linear optimal decision $u^* \in Z$ to the above problem is equivalent to finding the shortest distance from the subspace $Z$ to the element $Lx$, where the minimizing $u^*$ is the projection of $Lx$ on $Z$, and hence unique. Also, since $\mu^*$ is the projection, we have
\[ 0 = \langle u^* - Lx, \mu \rangle = \mathbb{E}\{(u^* - Lx)^T Q_{uu} u\}, \]
for all $u \in Z$. In particular, for $f_i = (0, 0, \ldots, z_i, 0, \ldots, 0) \in Z$, we have
\[ \mathbb{E}\{(u^* - Lx)^T Q_{uu} f_i\} = \mathbb{E}\{(u^* - Lx)^T Q_{uu} | z_i \} = 0. \]
The Gaussian assumption implies that $[(u^* (y) - Lx)^T Q_{uu}]_i$ is independent of $z_i = A_i y_i$, for all linear transformations $A_i$. This gives in turn that $[(u^* - Lx)^T Q_{uu}]_i$ is independent of $y_i$. Hence, for any decision $\mu \in M \cap C$, linear or nonlinear, we have that
\[ \mathbb{E}(u^* - Lx)^T Q_{uu} \mu(y) = \sum_i \mathbb{E}\{(u^* - Lx)^T Q_{uu} | \mu_i(y_i) \} = 0, \]
and
\[ \mathbb{E}(\mu(y) - Lx)^T Q_{uu}(\mu(y) - Lx) \]
\[ = \mathbb{E}(u^* - Lx + \mu(y) - u^*)^T Q_{uu}(u^* - Lx + \mu(y) - u^*) \]
\[ = \mathbb{E}(u^* - Lx)^T Q_{uu}(u^* - Lx) + \mathbb{E}(\mu(y) - u^*)^T Q_{uu}(\mu(y) - u^*) \]
\[ + 2 \mathbb{E}(u^* - Lx)^T Q_{uu}(\mu(y) - u^*) \]
\[ = \mathbb{E}(u^* - Lx)^T Q_{uu}(u^* - Lx) + \mathbb{E}(\mu(y) - u^*)^T Q_{uu}(\mu(y) - u^*) \]
\[ \geq \mathbb{E}(u^* - Lx)^T Q_{uu}(u^* - Lx) \]
with equality if and only if $\mu(y) = u^*$. This concludes the proof.

Proposition 1. Let $x$ and $v_i$ be independent Gaussian variables taking values in $\mathbb{R}^n$ and $\mathbb{R}^{p_i}$, respectively with $x \sim N(0, V_{xx})$, $v \sim N(0, V_{vv})$. Also, let $u_i$ be a stochastic variable taking values in $\mathbb{R}^{m_i}$, $m = m_1 + \cdots + m_N$, $Q_{uu} \in \mathbb{S}^+_m$, $C_i \in \mathbb{R}^{p_i \times n}$, and $L = -Q_{uu}^{-1} Q_{ux}$. Set $y_i = C_i x + v_i$. Then, the optimal solution $K_1, \ldots, K_N$ to the optimization problem
\[ \min_{K_i} \mathbb{E}(u - Lx)^T Q_{uu}(u - Lx) \]
subject to $u_i = K_i y_i$ \tag{3.6}
for $i = 1, \ldots, N$. 

is the solution of the linear system of equations

$$
\sum_{j=1}^{N} [Q_{uu}]_{ij} K_j (C_j V_{xx} C_i^T + [V_{vv}]_{ji}) = -[Q_{ux}]_i V_{xx} C_i^T, \quad \text{for } i = 1, \ldots, N. \quad (3.7)
$$

**Proof.** Let $K = \text{diag}(K_1, \ldots, K_N)$ and $C = [C_1^T \cdots C_N^T]^T$. The problem of finding the optimal linear feedback law $u_i = K_i y_i$ can be written as

$$
\min_{K_i} \text{Tr}[E\{Q_{uu}(u - Lx)(u - Lx)^T\}]
$$

subject to $u = K(Cx + v)$

Now

$$
f(K) = \text{Tr} E\{Q_{uu}(u - Lx)(u - Lx)^T\} = \text{Tr} E\{Q_{uu}(KCx + Kv - Lx)(KCx + Kv - Lx)^T\}
$$

$$
= \text{Tr} E\{Q_{uu}K(Cxx^TC_i^T + vv^T)K^T - 2Q_{uu}Lxx^TC_i^TK^T + Q_{uu}Lxx^TL^T + 2Q_{uu}(KC - L)xv^TK^T\}
$$

$$
= \text{Tr} [Q_{uu}K(CV_{xx}C_i^T + V_{vv})K^T - 2Q_{uu}LV_{xx}C_i^TK^T + Q_{uu}LV_{xx}L^T]
$$

$$
= \text{Tr} \left[ \sum_{i,j=1}^{N} [Q_{uu}]_{ij} K_j (C_j V_{xx} C_i^T + [V_{vv}]_{ji}) K_i^T - 2 \sum_{i,j=1}^{N} [Q_{uu}]_{ij} L_j V_{xx} C_i^T K_i^T \right] + \text{Tr}[Q_{uu}LV_{xx}L^T].
$$

(3.9)

A minimizing $K$ is obtained by solving $\nabla_{K_i} f(K)$:

$$
0 = \nabla_{K_i} f(K) = 2 \sum_{j=1}^{N} [Q_{uu}]_{ij} K_j (C_j V_{xx} C_i^T + [V_{vv}]_{ji}) - 2 \sum_{j=1}^{N} [Q_{uu}]_{ij} L_j V_{xx} C_i^T.
$$

(3.10)

Since $Q_{uu}L = -Q_{ux}$, we get that

$$
\sum_{j=1}^{N} [Q_{uu}]_{ij} L_j V_{xx} C_i^T = -[Q_{ux}]_i V_{xx} C_i^T,
$$

and the equality in (3.10) is equivalent to

$$
\sum_{j=1}^{N} [Q_{uu}]_{ij} K_j (C_j V_{xx} C_i^T + [V_{vv}]_{ji}) = -[Q_{ux}]_i V_{xx} C_i^T,
$$

and the proof is complete. \[ \square \]

In general, separation does not hold for the static team problem when constraints on the information available for every decision maker $u_i$ are imposed. That is, the optimal decision is not given by $u_i = L\hat{x}_i$, where $\hat{x}_i$ is the optimal estimated value of $x$ by decision maker $i$. We show it by considering the following example.

**Example 2.** Consider the team problem

$$
\begin{align*}
\text{minimize} & \quad \mathbb{E} \begin{bmatrix} x \end{bmatrix}^T \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\
\text{subject to} & \quad y_i = C_i x + v_i \\
& \quad u_i = \mu_i(y_i) \\
& \quad \text{for } i = 1, \ldots, N
\end{align*}
$$
The data we will consider is:

\[ N = 2, \quad C_1 = C_2 = 1, \quad x \sim \mathcal{N}(0, 1), \quad v_1 \sim \mathcal{N}(0, 1), \quad v_2 \sim \mathcal{N}(0, 1) \]

\[ Q_{xx} = 1, \quad Q_{uu} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad Q_{xu} = Q_{ux}^T = -\begin{bmatrix} 1 & 1 \end{bmatrix} \]

The best decision with full information is given by

\[ u = -Q_{uu}^{-1}Q_{ux}x = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} x = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} x. \]

The optimal estimate of \( x \) of decision maker 1 is

\[ \hat{x}_1 = \mathbb{E}\{x|y_1\} = \frac{1}{2}y_1, \]

and of decision maker 2

\[ \hat{x}_2 = \mathbb{E}\{x|y_2\} = \frac{1}{2}y_2. \]

Hence, the decision where each decision maker combines the best deterministic decision with her best estimate of \( x \) is given by

\[ u_i = \frac{1}{3}\hat{x}_i = \frac{1}{6}y_i, \]

for \( i = 1, 2 \). This policy gives a cost equal to 0.611. However, solving the team problem yields \( K_1 = K_2 = \frac{6}{5} \), and hence the optimal team decision is given by

\[ u_i = \frac{1}{5}y_i. \]

The cost obtained from the team problem is 0.600. Clearly, separation does not hold in team decision problems.

4. Team Decision Problems with Power Constraints. Consider the modified version of the optimization problem (3.1):

\[
\begin{align*}
\min_{\mu} & \quad \mathbb{E}\left[ x^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right] \\
\text{subject to} & \quad y_i = C_i x \\
& \quad u_i = \mu_i(y_i) \\
& \quad \gamma_i \geq \mathbb{E}\|\mu_i(y_i)\|^2 \\
& \quad \text{for } i = 1, \ldots, N.
\end{align*}
\]

(4.1)

The difference from Radner’s original formulation is that we have added power constraints to the decision functions, \( \gamma_i \geq \mathbb{E}\|\mu_i(y_i)\|^2 \).

In optimization (minimization) problems, you define the value to be infinite if there doesn’t exist any feasible decision variable that satisfy the constraints. Therefore, usually, one assumes that there is a feasible point, and hence the value must be finite. Existence conditions are hard to derive usually in spite
of problems might be convex. So in practice, you run the algorithm and either you get a finite number, or it goes indefinitely. Conditions where you a decide whether you have a feasible problem or not are of great interest of course. It’s a nontrivial problem that is outside the scope of this paper.

In the sequel, we will prove a more general theorem, where we consider power constraints on a set of quadratic forms in both the state $x$ and the decision function $\mu$.

**Theorem 2.** Let $x$ be a Gaussian variable with zero mean and given covariance matrix $X$, taking values in $\mathbb{R}^n$. Also, let \( \begin{bmatrix} Q_0 & S_0 \\ S_0^T & R_0 \end{bmatrix} \in \mathbb{S}_{++}^{m+n}, \quad R_0 \in \mathbb{S}_+^m, \) \( \begin{bmatrix} Q_j & S_j \\ S_j^T & R_j \end{bmatrix} \in \mathbb{S}^{m+n}, \) and $R_j \in \mathbb{S}_+^n$, for $j = 1, \ldots, M$. Assume that the optimization problem

\[
\min_{\mu \in \mathcal{C}} \mathbb{E} \left[ x^T \mu(x) \right] \begin{bmatrix} Q_0 & S_0 \\ S_0^T & R_0 \end{bmatrix} \left[ x \mu(x) \right] \\
\text{subject to} \quad \mathbb{E} \left[ x^T \mu(x) \right] \begin{bmatrix} Q_j & S_j \\ S_j^T & R_j \end{bmatrix} \left[ x \mu(x) \right] \leq \gamma_j \quad (4.2)
\]

is feasible. Then, linear decisions $\mu$ given by $\mu(x) = K(X)x$, with $K(X) \in \mathbb{K}$, are optimal.

**Proof.** Consider the expression

\[
\mathbb{E} \left[ x^T \mu(x) \right] \begin{bmatrix} Q_0 & S_0 \\ S_0^T & R_0 \end{bmatrix} \left[ x \mu(x) \right] + \sum_{j=1}^M \lambda_j \left( \mathbb{E} \left[ x^T \mu(x) \right] \begin{bmatrix} Q_j & S_j \\ S_j^T & R_j \end{bmatrix} \left[ x \mu(x) \right] - \gamma_j \right) .
\]

Take the expectation of a quadratic form with index $j$ to be larger than $\gamma_j$. Then, $\lambda_j \to \infty$ makes the value of the expression above infinite. On the other hand, if the expectation of a quadratic form with index $j$ is smaller than $\gamma_j$, then the maximizer $\lambda_j$ is optimal for $\lambda_j = 0$.

Now let $p^*$ be the optimal value of the optimization problem (4.2), and consider the objective function

\[
\begin{bmatrix} x^T & u \end{bmatrix} \begin{bmatrix} Q_0 & S_0 \\ S_0^T & R_0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = x^T (Q_0 - S_0 R_0^{-1} S_0^T) x + (u - R_0^{-1} S_0^T x)^T R_0 (u - R_0^{-1} S_0^T x).
\]

We have that $Q_0 - S_0 R_0^{-1} S_0^T \succeq 0$, since it’s the Schur complement of $R_0$ in the positive semi-definite matrix $\begin{bmatrix} Q_0 & S_0 \\ S_0^T & R_0 \end{bmatrix}$. Since $R_0 \succ 0$, a necessary condition for the objective function to be zero is that $u = R_0^{-1} S_0^T x$, and so $u$ must be linear (In order for $u$ to have the structure given by $\mathcal{C}$, $R_0^{-1} S_0^T$ must be in $\mathbb{K}$, to satisfy the information constraints).
Now assume that \( p^* > 0 \). We have

\[
\begin{align*}
p^* &= \min_{\mu \in \mathbb{C}} \max_{\lambda_j \in \mathbb{R}^+} E \left[ x \gamma \right]^T \left( \begin{bmatrix} Q_0 & S_0 \\ S_0^T & R_0 \end{bmatrix} \right) \left[ x \mu \right] + \sum_{j=1}^M \lambda_j \left( E \left[ x \mu \right] \left[ Q_j \right] \left[ S_j \right] \right) \left[ x \mu \right] - \gamma_j \\
&= \min_{\mu \in \mathbb{C}} \max_{\lambda_j \in \mathbb{R}^+} E \left[ x \gamma \right]^T \left( \begin{bmatrix} Q_0 & S_0 \\ S_0^T & R_0 \end{bmatrix} \right) \left[ x \mu \right] + \sum_{j=1}^M \lambda_j \left( E \left[ x \mu \right] \left[ Q_j \right] \left[ S_j \right] \right) \left[ x \mu \right] - \sum_{j=1}^M \lambda_j \gamma_j. \tag{4.3}
\end{align*}
\]

Now introduce \( \lambda_0 \) and the matrix

\[
\begin{bmatrix}
Q \\
S^T \\
R
\end{bmatrix} = \sum_{j=0}^M \lambda_j \begin{bmatrix}
Q_j \\
S_j^T \\
R_j
\end{bmatrix},
\]

and consider the minimax problem

\[
p_0 = \min_{\mu \in \mathbb{C}} \max_{\sum_{j=0}^M \lambda_j = 1} E \left[ x \gamma \right]^T \left( \sum_{j=1}^M \lambda_j \begin{bmatrix}
Q_j \\
S_j^T \\
R_j
\end{bmatrix} \right) \left[ x \mu \right] - \sum_{j=1}^M \lambda_j \gamma_j. \tag{4.4}
\]

Note that a maximizing \( \lambda_0 \) must be positive, since \( \lambda_0 = 0 \) implies that \( p_0 \leq 0 \), while \( \lambda_0 > 0 \) gives \( p_0 > 0 \). We can always recover the optimal solutions of (4.3) from that of (4.4) by dividing all variables by \( \lambda_0 \), that is \( p^* = p_0 / \lambda_0 \), \( \lambda_j \mapsto \lambda_j / \lambda_0 \), and \( \mu \mapsto \mu / \lambda_0 \). Now we have the obvious inequality (\( \min \max \{ \cdot \} \geq \max \min \{ \cdot \} \))

\[
p_0 \geq \max_{\lambda_j \geq 0} \min_{\mu \in \mathbb{C}} \sum_{j=0}^M \lambda_j \left( E \left[ x \gamma \right]^T \left( \sum_{j=1}^M \lambda_j \begin{bmatrix}
Q_j \\
S_j^T \\
R_j
\end{bmatrix} \right) \left[ x \mu \right] - \sum_{j=1}^M \lambda_j \gamma_j \right.
\]

For any fixed values of \( \lambda_j \), we have \( R > 0 \), so Theorem \([1]\) gives the equality

\[
\min_{\mu \in \mathbb{C}} E \left[ x \mu \right] \left[ Q \right] \left[ S^T \right] \left[ R \right] \left[ x \mu \right] = \min_{K \in \mathbb{K}} E \left[ x \mu \right] \left[ Q \right] \left[ S^T \right] \left[ R \right] \left[ x \right],
\]

where the minimizing \( K \) is unique. Thus,

\[
p_0 \geq \max_{\lambda_j \geq 0} \min_{\mu \in \mathbb{C}} E \left[ x \mu \right] \left[ Q \right] \left[ S^T \right] \left[ R \right] \left[ x \mu \right] = \max_{\sum_{j=0}^M \lambda_j = 1} \min_{\lambda_j \geq 0} \sum_{j=0}^M \lambda_j \left( E \left[ x \mu \right] \left[ Q_j \right] \left[ S_j \right] \right) \left[ x \mu \right] - \sum_{j=1}^M \lambda_j \gamma_j.
\]

The objective function is radially unbounded in \( K \) since \( R > 0 \). Hence, it can be restricted to a compact subset of \( \mathbb{K} \). Thus,
\[ p_0 \geq \max_{\lambda_j \geq 0, \sum_{j=0}^{M} \lambda_j = 1} \min_{K \in \mathbb{K}} \mathbb{E} \left[ x^T \mu(x) \right] \left( \sum_{j=0}^{M} \lambda_j \left[ Q_j \ S_j \ R_j \right] \right) \left[ x \right] - \sum_{j=1}^{M} \lambda_j \gamma_j \]

\[ = \min_{K \in \mathbb{K}} \max_{\lambda_j \geq 0, \sum_{j=0}^{M} \lambda_j = 1} \mathbb{E} \left[ x^T \left( \sum_{j=0}^{M} \lambda_j \left[ Q_j \ S_j \ R_j \right] \right) \left[ x \right] - \sum_{j=1}^{M} \lambda_j \gamma_j \right] \]

\[ \geq \min_{\mu \in \mathcal{C}} \max_{\lambda_j \geq 0, \sum_{j=0}^{M} \lambda_j = 1} \mathbb{E} \left[ x^T \left( \sum_{j=0}^{M} \lambda_j \left[ Q_j \ S_j \ R_j \right] \right) \left[ x \right] - \sum_{j=1}^{M} \lambda_j \gamma_j \right] \]

\[ = p_0, \]

where the equality is obtained by applying Proposition 2 in the Appendix, the second inequality follows from the fact that the set of linear decisions \( Kx, K \in \mathbb{K} \), is a subset of \( \mathcal{C} \), and the second equality follows from the definition of \( p_0 \). Hence, linear decisions are optimal, and the proof is complete. \( \square \)

**Remark:** Although Theorem 2 is stated and proved for \( y = x \) and \( u = \mu(y) = \mu(x) \), it extends easily to the case \( y = Cx \) for any matrix \( C \), which often is the case in applications.

### 5. Computation of The Optimal Team Decisions

The optimization problem that we would like to solve when assuming linear decisions is

\[
\min_{\gamma_0, K \in \mathbb{K}} \gamma_0 \\
\text{subject to } \mathbb{E} \left[ x^T \left[ \begin{array}{c} x \\ KCx \end{array} \right], \left[ \begin{array}{cc} Q_j & S_j \\ S_j^T & R_j \end{array} \right], \left[ \begin{array}{c} x \\ KCx \end{array} \right] \right] \leq \gamma_j, \ j = 0, \ldots, M, \quad (5.1)
\]

\[ x \sim \mathcal{N}(0, X^2). \]

Note that we can write the constraints as

\[
\mathbb{E} \left[ x^T \left[ \begin{array}{c} x \\ KCx \end{array} \right], \left[ \begin{array}{cc} Q_j & S_j \\ S_j^T & R_j \end{array} \right], \left[ \begin{array}{c} x \\ KCx \end{array} \right] \right] = \mathbb{E} \left\{ \text{Tr} \left[ \begin{array}{c} x \end{array} \right]^T \left[ \begin{array}{cc} I & Q_j \\ & S_j^T \end{array} \right], \left[ \begin{array}{c} x \end{array} \right], \left[ \begin{array}{cc} I & \left[ S_j \right]^T \\ & R_j \end{array} \right], \left[ \begin{array}{c} I \end{array} \right] \right\}
\]

\[ = \text{Tr} \left[ \begin{array}{c} x \end{array} \right]^T \left[ \begin{array}{cc} I & Q_j \\ & S_j^T \end{array} \right], \left[ \begin{array}{c} x \end{array} \right], \left[ \begin{array}{cc} I & \left[ S_j \right]^T \\ & R_j \end{array} \right], \left[ \begin{array}{c} I \end{array} \right] \right] X, \quad (5.2)
\]

where we used that \( \mathbb{E} x x^T = X^2 \). Hence, we obtain a set of convex quadratic inequalities (convex since \( R_j \succ 0 \) for all \( j \))

\[ \text{Tr} X \left[ \begin{array}{c} I \end{array} \right]^T \left[ \begin{array}{cc} Q_j & S_j \\ S_j^T & R_j \end{array} \right], \left[ \begin{array}{c} I \end{array} \right] X \leq \gamma_j. \]

There are many existing computational methods to solve convex quadratic optimization problems (see [3]).

Alternatively, we can formulate the optimization problem as a set of linear matrix inequalities as follows. For simplicity, we will assume that \( R_j \succ 0 \) for all
Theorem 3. The team optimization problem (5.1) is equivalent to the semi-definite program

$$\begin{align*}
\min_{\gamma_0, K \in K} & \quad \gamma_0 \\
\text{subject to} & \quad \text{Tr} P_j \leq \gamma_j \\
& \quad 0 \preceq \begin{bmatrix}
P_j - XQ_jX - XS_j K C X - XC^T K^T S_j^T X & XC^T K^T R_j \\
R_j K C X & R_j
\end{bmatrix}
\end{align*}$$

(5.3)

Proof. Introduce the matrices $P_j \in S^n$, and write the given constraints as

$$\gamma_j \geq \text{Tr} P_j$$

$$P_j - X \begin{bmatrix}
I & Q_j \\
K C & S_j \\
S_j^T & R_j
\end{bmatrix} \begin{bmatrix}
I \\
K C
\end{bmatrix} X \succeq 0.$$  

(5.4)

Now we have that

$$0 \preceq X \begin{bmatrix}
I & Q_j \\
K C & S_j \\
S_j^T & R_j
\end{bmatrix} \begin{bmatrix}
I \\
K C
\end{bmatrix} X = P_j - XQ_jX - XS_j K C X - XC^T K^T S_j^T X - XC^T K^T R_j K C X.$$  

(5.5)

Since $R_j \succ 0$, the quadratic inequality above can be transformed to a linear matrix inequality using the Schur complement (3), which is given by

$$\begin{bmatrix}
P_j - XQ_jX - XS_j K C X - XC^T K^T S_j^T X & XC^T K^T R_j \\
R_j K C X & R_j
\end{bmatrix} \succeq 0.$$  

Hence, our optimization problem to be solved is given by

$$\begin{align*}
\min_{K \in K} & \quad \gamma_0 \\
\text{subject to} & \quad \text{Tr} P_j \leq \gamma_j \\
& \quad 0 \preceq \begin{bmatrix}
P_j - XQ_jX - XS_j K C X - XC^T K^T S_j^T X & XC^T K^T R_j \\
R_j K C X & R_j
\end{bmatrix}
\end{align*}$$

(5.6)

which proves our theorem. \qed

6. Minimax Team Theory. We considered the problem of static stochastic team decision in the previous sections. This section treats an analogous version for the deterministic (or worst case) problem. Although the problem formulation is very similar, the ideas of the solution are considerably different, and in a sense more difficult.

The deterministic problem considered is a quadratic game between a team of players and nature. Each player has limited information that could be different from the other players in the team. This game is formulated as a minimax problem, where the team is the minimizer and nature is the maximizer.
6.1. Deterministic Team Problems. Consider the following team decision problem

\[ \inf_{\mu} \sup_{x \neq 0} J(x, u) / \|x\|^2 \]

subject to \( y_i = C_i x \)
\( u_i = \mu_i(y_i) \)
for \( i = 1, \ldots, N \) (6.1)

where \( u_i \in \mathbb{R}^{m_i}, m = m_1 + \cdots + m_N, C_i \in \mathbb{R}^{p_i \times n} \).

\( J(x, u) \) is a quadratic cost given by

\[ J(x, u) = \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \]

where

\[ \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \in \mathbb{S}^{m+n}. \]

We will be interested in the case \( Q_{uu} \succ 0 \). The players \( u_1, \ldots, u_N \) make up a team, which plays against nature represented by the vector \( x \), using \( \mu \in \mathcal{S} \), that is

\[ \mu(Cx) = \begin{bmatrix} \mu_1(C_1x) \\ \vdots \\ \mu_N(C_Nx) \end{bmatrix}. \]

Theorem 4. If the value of the game (6.1) is equal to \( \gamma^* \), then there is a linear decision \( \mu(Cx) = KCx \), with \( K = diag(K_1, \ldots, K_N) \), achieving that value.

Proof. For a proof, consult [8]. \( \Box \)

6.2. Relation with The Stochastic Minimax Team Decision Problem.

Now consider the stochastic minimax team decision problem

\[ \min_K \max_{E[\|x\|^2] = 1} E \left\{ x^T \begin{bmatrix} I & Q_{xx} & Q_{xu} \\ K & Q_{ux} & Q_{uu} \end{bmatrix} \begin{bmatrix} I \\ K \\ x \end{bmatrix} \right\}. \]

Taking the expectation of the cost in the stochastic problem above yields the equivalent problem

\[ \min_K \max_{\text{Tr}X = 1} \text{Tr} \begin{bmatrix} I & Q_{xx} & Q_{xu} \\ K & Q_{ux} & Q_{uu} \end{bmatrix} \begin{bmatrix} I \\ K \\ X \end{bmatrix} \]

where \( X \) is a positive semi-definite matrix, and is the covariance matrix of \( x \), i.e. \( X = E \|x\|^2 \).

Hence, we see that the stochastic minimax team problem is equivalent to the deterministic minimax team problem, where nature maximizes with respect to all covariance matrices \( X \) of the stochastic variable \( x \) with variance \( E \|x\|^2 = E \|x\|^2 = \text{Tr} X = 1 \).

7. Deterministic Team Problems with Quadratic Constraints. Consider the team problem (6.1). An equivalent condition for the existence of a decision function \( \mu^* \in \mathcal{C} \) that achieves the value of the game \( \gamma^* \) is that

\[ \begin{bmatrix} x \\ \mu^*(Cx) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ \mu^*(Cx) \end{bmatrix} \leq \gamma^* \|x\|^2 \]
for all $x$, which is equivalent to
\[
\begin{bmatrix}
  x \\
  \mu^*(Cx)
\end{bmatrix}^T
\begin{bmatrix}
  Q - \gamma^* I & S \\
  S^T & R
\end{bmatrix}
\begin{bmatrix}
  x \\
  \mu^*(Cx)
\end{bmatrix} \leq 0
\]
for all $x$. This is an example of a power constraint. We could also have a set of power constraints that have to be mutually satisfied. For instance, in addition to the minimization of the worst case quadratic cost, we could have constraints on the induced norms of the decision functions
\[
\frac{\|\mu_i(C_i x)\|^2}{\|x\|^2} \leq \gamma_i \quad \text{for all } x \not= 0, \quad i = 1, \ldots, M,
\]
or equivalently given by the quadratic inequalities
\[
\|\mu_i(C_i x)\|^2 - \gamma_i \|x\|^2 \leq 0 \quad \text{for all } x, \quad i = 1, \ldots, M.
\]
Also, the team members could share a common power source, and the power is proportional to the squared norm of the decisions $\mu_i$:
\[
\sum_{i=1}^{M} \|\mu_i(C_i x)\|^2 - c\|x\|^2 \leq 0 \quad \text{for all } x,
\]
for some positive real number $c$.

It's not clear whether linear decisions are optimal, since the example given at the introduction indicates that, in deterministic settings, nonlinear decision are optimal. However, the next result shows how to obtain the linear optimal decisions by solving a semidefinite program.

**Theorem 5.** Let $C_i \in \mathbb{R}^{p_i \times n}$, for $i = 1, \ldots, N$. Let $\begin{bmatrix} Q_j & S_j \\ S_j^T & R_j \end{bmatrix} \in \mathbb{S}_+^{m \times n}$ for $j = 0, \ldots, M$, and $R_j \in \mathbb{S}_+^m$ for $0 = 1, \ldots, M$. Then, the set of quadratic matrix inequalities
\[
\begin{bmatrix}
  x \\
  KCx
\end{bmatrix}^T
\begin{bmatrix}
  Q_j & S_j \\
  S_j^T & R_j
\end{bmatrix}
\begin{bmatrix}
  x \\
  KCx
\end{bmatrix} \leq 0 \quad \forall x, \quad j = 0, \ldots, M,
\]
is equivalent to
\[
\begin{bmatrix}
  Q_j + S_j KC + C^T K^T S_j & C^T K^T R_j \\
  R_j KC & -R_j
\end{bmatrix} \preceq 0, \quad i = 0, \ldots, M.
\]

**Proof.** We have the following chain of inequalities:
\[
\begin{bmatrix}
  x \\
  KCx
\end{bmatrix}^T
\begin{bmatrix}
  Q_j & S_j \\
  S_j^T & R_j
\end{bmatrix}
\begin{bmatrix}
  x \\
  KCx
\end{bmatrix} \leq 0
\]
Multi-Objective Linear Quadratic Team Optimization

\[
\begin{bmatrix}
I \\
KC
\end{bmatrix}^T
\begin{bmatrix}
Q_j & S_j \\
S_j^T & R_j
\end{bmatrix}
\begin{bmatrix}
I \\
KC
\end{bmatrix} \preceq 0
\]

\[
\downarrow
\]

\[
Q_j + S_j KC + C^T K S_j^T + C^T K R_j KC \preceq 0
\]

\[
\downarrow
\]

\[
A = \begin{bmatrix}
Q_j + S_j KC + C^T K S_j^T & C^T K^T R_j \\
R_j KC & -R_j
\end{bmatrix} \preceq 0,
\]

where the last equivalence follows from taking the Schur complement of \( R_j \) in \( A \) (see [3]). Hence, our optimization problem becomes

\[
\begin{bmatrix}
Q_j + S_j KC + C^T K S_j^T & C^T K^T R_j \\
R_j KC & -R_j
\end{bmatrix} \preceq 0, \quad i = 0, ..., M. \tag{7.3}
\]

This completes the proof. \( \square \)

8. Conclusions. We have studied multi-objective linear quadratic optimization of team decisions in both stochastic and deterministic settings. Constrained decision problems tend to have nonlinear optimal solutions. We have shown that for the Gaussian setting, linear decisions are in fact optimal, and we can find the linear optimal solutions by solving a semidefinite program. We then explore the problem of finding the linear optimal decisions for its deterministic counterpart and show that we can find the optimal solution by solving a semidefinite program. Future work will consider optimality of the linear decisions in the deterministic framework. Another problem of interest is an an \( S \)-procedure sort of a result, where we want to find decision function \( \mu \) such that the inequality \( J_0(\mu(x), x) \leq 0 \) is satisfied if \( J_1(\mu(x), x) \leq 0 \), where \( J_0, J_1 \) are some quadratic forms in \( \mu \) and \( x \). However, this is a much harder problem since the search for linear function \( \mu(x) \) is not a convex problem, and it’s not clear if it can be convexified.

9. Acknowledgements. The author is grateful for Prof. Anders Rantzer and Prof. Bo Bernhardsson for discussions on the topic. This work is supported by the Swedish Research Council.

REFERENCES

[1] T. Basar and G. J. Olnder. Dynamic Noncooperative Game Theory. SIAM, 1999.
[2] P. Bernhard and N. Hovakimyan. Nonlinear robust control and minimax team problems. International Journal of Robust and Nonlinear Control, 9(9):239–257, 1999.
[3] S. Boyd and L. Vandenberghe. Convex Optimization. Cambridge University Press, 2004.
[4] G. Didinsky and T. Basar. Minimax decentralized controllers for discrete-time linear systems. In 41st Conference on Decision and Control, pages 481–486, 2002.
A. Gattami. Distributed stochastic control: A team theoretic approach. In 17th International Symposium on Mathematical Theory of Networks and Systems, 2006.

[7] A. Gattami. Optimal Decisions with Limited Information. PhD thesis, Lund University, 2007.

[8] A. Gattami, B. Bernhardsson, and A. Rantzer. Robust team decision theory. IEEE Trans. Automatic Control, 57(3):794–798, march 2012.

[9] Y.-C. Ho and K.-C. Chu. Team decision theory and information structures in optimal control problems-part i. IEEE Trans. on Automatic Control, 17(1), 1972.

[10] J. Krainak, J. L. Speyer, and S. I. Marcus. Static team problems-part i. IEEE Trans. on Automatic Control, 27(4):839–848, 1982.

[11] J. Marschak. Elements for a theory of teams. Management Sci., 1:127–137, 1955.

[12] R. Radner. Team decision problems. Ann. Math. Statist., 33(3):857–881, 1962.

Appendix.

Game theory. Let $J = J(u, w)$ be a functional defined on a product vector space $U \times W$, to be minimized by $u \in U \subset U$ and maximized by $w \in W \subset W$, where $U$ and $W$ are the constrained sets. This defines a zero-sum game, with kernel $J$, in connection with which we can introduce two values, the upper value

$$J^* := \inf_{u \in U} \sup_{w \in W} J(u, w),$$

and the lower value

$$J^* := \sup_{w \in W} \inf_{u \in U} J(u, w).$$

Obviously, we have the inequality $J^* \geq J^*$. If $J^* = J^*$, then $J^*$ is called the value of the zero-sum game. Furthermore, if there exists a pair $(u^*, w^* \in W)$ such that

$$J(u^*, w^*) = J^*,$$

then the pair $(u^*, w^*)$ is called a (pure-strategy) saddle-point solution. In this case, we say that the game admits a saddle-point (in pure strategies). Such a saddle-point solution will equivalently satisfy the so-called pair of saddle-point inequalities:

$$J(u^*, w) \leq J(u^*, w^*) \leq J(u, w^*), \quad \forall u \in U, \forall w \in W.$$

Proposition 2. Consider a two-person zero-sum game on convex finite dimensional action sets $U_1 \times U_2$, defined by the continuous kernel $J(u_1, u_2)$. Suppose that $J(u_1, u_2)$ is strictly convex in $u_1$ and strictly concave in $u_2$. Suppose that either

(i) $U_1 \times U_2$ are closed and bounded, or

(ii) $U_i \subseteq \mathbb{R}^{m_i}$, $i = 1, 2$, and $J(u_1, u_2) \to \infty$ as $\|u_1\| \to \infty$, and $J(u_1, u_2) \to -\infty$ as $\|u_2\| \to \infty$.

Then, the game admits a unique pure-strategy saddle-point equilibrium.

Proof. See [1], pp. 177.

Remark. The assumption of strict convexity and concavity in Proposition 2 can be relaxed to only convexity and concavity, and a saddle-point exists in pure strategies, but it is not necessarily unique.