SMALL SCALE CREATION FOR SOLUTIONS OF THE SQG EQUATION

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ABSTRACT. We construct examples of solutions to the conservative surface quasi-geostrophic (SQG) equation that must either exhibit infinite in time growth of derivatives or blow up in finite time.

1. Introduction

The SQG equation appears in atmospheric science, where it models evolution of the temperature on the surface of a planet and can be derived under a number of assumptions from a more complete system of 3D rotating Navier-Stokes equations coupled with temperature via Boussinesq approximation. In mathematical literature, the SQG equation first appeared in [3], where a number of parallels with the 3D Euler equation were drawn (see [16] for more details), and a possible singular scenario was presented. Since then, the SQG equation has attracted attention of many researchers, in part because it appears to be perhaps the simplest looking equation of fluid dynamics for which the global regularity vs finite time blow up question remains open. In particular, the uniformly closing front singular scenario proposed in [3] has been ruled out in [7, 8]. More generally, one can look at the SQG equation as one member of the family of modified surface quasi-geostrophic equations, given by

\[ \partial_t \omega + (u \cdot \nabla) \omega = 0, \quad u = \nabla \perp \left( -\Delta \right)^{-1+\alpha} \omega, \quad \omega(x,0) = \omega_0(x). \]  

(1.1)

When \( \alpha = 0 \), we obtain the 2D Euler equation in vorticity form; the case \( \alpha = 1/2 \) corresponds to the SQG equation. The range \( 0 < \alpha < 1/2 \) has been considered both in geophysical [11] and mathematical [14] literature. Moreover, more singular models with \( 1/2 < \alpha < 1 \) have been analyzed as well [2]. For the entire \( 0 < \alpha < 1 \) range, local regularity is known but the question whether smooth solutions can blow up in finite time remains open. The only example of singularity formation for modified SQG equations has been recently given in [15] for patch solutions in half-plane for small \( \alpha \). While this example is suggestive, its implications for the smooth case are not clear. In fact, surprisingly, there has been not a single example of smooth solutions to the SQG equation which exhibit infinite in time growth of derivatives. Even though there are many such examples for the 2D Euler equation (see e.g. [18, 17, 10, 14, 19]), the strongest to date example of growth in derivatives of the SQG equation is given in [13] and it involves only finite time growth. Part of the reason for this situation is that most of the...
2D Euler growth examples involve boundary; however the modified SQG equations have not been studied as much in the settings with boundary (see, however, [5, 10] for recent advances). Moreover, smooth initial data deteriorates immediately to only Hölder regular if the support of \( \omega_0 \) contains the boundary in the conservative modified SQG setting with natural no penetration boundary condition. The only Euler growth constructions that are done without boundaries in the periodic setting are due to Denisov [10] and Zlatos [19]. The example of Denisov involves superlinear growth and can be extended to the \( 0 < \alpha < 1/2 \) range in a straightforward manner. But it is not clear how to extend it to the SQG case since a key part of the argument relies on control of \( \|u\|_{L^\infty} \) by \( \|\omega\|_{L^p} \) for some \( p < \infty \). The example of Zlatos, on the other hand, leads to exponential growth of \( \nabla^2 \omega \) for smooth solutions, and relies on representation of the velocity \( u \) near origin (and under assumption of odd-odd symmetry) that goes back to [14] and is specific to the Euler equation. Namely, one can isolate the relatively explicit “main term” in the velocity \( u \) that is of log-Lipschitz nature and dominates the rest of the Biot-Savart law in certain regimes. In the modified SQG case, no such “main term” behavior is expected.

In this paper, our main goal is to prove the following theorem.

**Theorem 1.1.** Consider the modified SQG equations (1.1) in periodic setting. For all \( 0 < \alpha < 1 \), there exist initial data \( \omega_0 \) such that

\[
\sup_{t \leq T} \| \nabla^2 \omega(\cdot, t) \|_{L^\infty} \geq \exp(\gamma T),
\]

for all \( T > 0 \) and constant \( \gamma > 0 \) that may depend on \( \omega_0 \) and \( \alpha \). This constant can be made arbitrarily large by picking \( \omega_0 \) appropriately.

**Remark.** A mild adjustment of our proof yields examples with exponential in time growth of \( \|\omega\|_{C^{1,\gamma}} \) for all \( 1 > \gamma > 0 \) if \( \alpha \in (0, 1/2) \) and \( 1 > \gamma > 2\alpha - 1 \) if \( \alpha \in (1/2, 1) \). Our focus here is on proving (1.2), so we leave details of the extension to Hölder \( C^{1,\gamma} \) norms as an exercise for interested reader.

Note that we do not prove global regularity of the solutions in these examples - solutions that blow up in finite time will also satisfy (1.2).

The scenario involved is the same as that of [19], and its geometry goes back to the Bahouri-Chemin stationary singular cross example [1] for the 2D Euler equation. We work on \( \mathbb{T}^2 = [-\pi, \pi]^2 \) and consider solutions that are odd in both \( x_1 \) and \( x_2 \). Generalizing the bounds in [19] [14] we show that the contribution from the local part of the Biot-Savart law that involves integration over \( |y| \lesssim |x| \) region is small if \( |x| \) is small and there is control over \( \nabla^2 \omega \). We then show that the “medium” field contributions from the region \( |x| \lesssim |y| \lesssim 1 \) are near identical for both components of the fluid velocity \( u_1 \) and \( u_2 \), the result replacing the “main term” argument in the 2D Euler case. The growth is then obtained by taking initial data with additional degeneracy and tracing trajectories staying increasingly close to the separatrices.

### 2. Key estimates

In this section we prove several key estimates that we will need in the construction. Since we will be working with solutions that are odd in both \( x_1 \) and \( x_2 \), the Biot-Savart law for the modified SQG equation in the periodic setting is given by (we omit constants depending on \( \alpha \) and time dependence here for the sake of simplicity):

\[
u_1(x) = \int_0^\infty \int_0^\infty \left( \frac{x_2 - y_2}{|x - y|^{2+2\alpha}} - \frac{x_2 - y_2}{|x - y|^{2+2\alpha}} - \frac{x_2 + y_2}{|x + y|^{2+2\alpha}} + \frac{x_2 + y_2}{|x + y|^{2+2\alpha}} \right) \omega(y) dy_1 dy_2, \tag{2.1}
\]
Proof. Lemma 2.1. Assume that the solution remains smooth at times where these inequalities are derived. Understood in the principal value sense. So we estimate the expression in (2.4) by
\[ u_2(x) = -\int_0^\infty \int_0^\infty \left( \frac{x_1 - y_1}{|x-y|^{2+2\alpha}} - \frac{x_1 - y_1}{|x-y|^{2+2\alpha}} - \frac{x_1 + y_1}{|x-y|^{2+2\alpha}} + \frac{x_1 + y_1}{|x+y|^{2+2\alpha}} \right) \omega(y) \, dy_1 \, dy_2. \] (2.2)

Here \( \bar{x} = (-x_1, x_2) \), \( \bar{y} = (x_1, -x_2) \), and the function \( \omega \) is extended to the entire plane by periodicity. We will later see that the integral converges absolutely at infinity if \( \alpha > 0 \). Near the singularity \( x = y \), the convergence is understood in the principal value sense if \( \alpha \geq 1/2 \). In what follows, we will denote the kernels in the integrals (2.1), (2.2) by \( K_1(x,y) \) and \( K_2(x,y) \) respectively.

Let \( L \geq 1 \) be a constant that we will eventually choose to be large enough. The first estimate addresses the contribution of the near field \( y_1, y_2 \leq L|x| \) to the Biot-Savart law provided that we have control of \( \|\nabla^2 \omega\|_{L^\infty} \). All the inequalities we show in the rest of this section assume that the solution remains smooth at times where these inequalities are derived.

**Lemma 2.1.** Assume that \( \omega \) is odd with respect to both \( x_1 \) and \( x_2 \), periodic and smooth. Take \( L \geq 2 \), and suppose \( L|x| \leq 1 \). Denote
\[ u_j^{\text{near}}(x) = \int_{[0,L|x|^2]} K_j(x,y) \omega(y) \, dy. \]

Then we have
\[ |u_j^{\text{near}}(x)| \leq C x_j |x|^{2-2\alpha} L^{2-2\alpha} \|\nabla^2 \omega\|_{L^\infty}. \] (2.3)

**Proof.** Let us carry out the estimates for \( u_1 \) as the case of \( u_2 \) is similar. We need to control
\[ \left| \int_0^L \int_0^L \left( \frac{(x_2 - y_2)(|\bar{x} - y|^{2+2\alpha} - |x - y|^{2+2\alpha})}{|\bar{x} - y|^{2+2\alpha}|x - y|^{2+2\alpha}} - \frac{(x_2 + y_2)(|x + y|^{2+2\alpha} - |\bar{x} - y|^{2+2\alpha})}{|\bar{x} - y|^{2+2\alpha}|x + y|^{2+2\alpha}} \right) \omega(y) \, dy_1 \, dy_2 \right|. \] (2.4)

For the first term under the integral, we need to address the singularity where integration is understood in the principal value sense. So we estimate the expression in (2.4) by
\[ \left| \text{P.V.} \int_0^L \int_0^2 \int_0^{2x_2} \frac{(x_2 - y_2)(|\bar{x} - y|^{2+2\alpha} - |x - y|^{2+2\alpha})}{|\bar{x} - y|^{2+2\alpha}|x - y|^{2+2\alpha}} \omega(y) \, dy_1 \, dy_2 \, dy_2 \right| + \left| \int_0^L \int_0^L \frac{(x_2 + y_2)(|x + y|^{2+2\alpha} - |\bar{x} - y|^{2+2\alpha})}{|\bar{x} - y|^{2+2\alpha}|x + y|^{2+2\alpha}} \left( |\omega(y_1, y_2)| + |\omega(y_1, y_2 + 2x_2)| \right) \, dy_1 \, dy_2 \right|. \] (2.5)

Here we changed variable \( y_2 \mapsto y_2 - 2x_2 \) in the remainder of the integral of the first term from (2.4). The contribution \( |\omega(y_1, y_2 + x_2)| \) in the second integral comes from the rest of this term; the region of integration after the change of variable is enlarged a little using that the integrand (2.5) has fixed sign. Now in the first integral in (2.5) we use that the kernel is odd and the region of integration is symmetric with respect to \( y_2 = x_2 \) line, and replace \( \omega(y_1, y_2, t) \) by \( \omega(y_1, y_2, t) - \omega(y_1, x_2, t) \). Note that
\[ |\omega(y_1, y_2, t) - \omega(y_1, x_2, t)| = |\partial_{x_2} \omega(y_1, z_2, t)(y_2 - x_2)| \leq |\partial_{x_2}^2 \omega(z_1, z_2, t)y_1(y_2 - x_2)|, \] (2.6)

where \( z_2 \in (y_2, x_2) \) and \( z_1 \in (0, y_1) \). We applied mean value theorem twice and used that \( \partial_{x_2} \omega(0, y_2, t) \equiv 0 \) for all times (since \( \omega(0, y_2, t) \equiv 0 \) due to oddness). Using (2.6), we can
estimate the first integral in \((2.5)\) by \(\|\nabla^2 \omega\|_{L^\infty}\) times the following expressions; here \(C\) is a
constant that may change from line to line and may depend only on \(\alpha\):

\[
C \int_0^{L|x|} dy_1 \int_0^{2x_2} dy_2 \frac{\Phi(y_2) \gamma_2(\gamma_2 + y_2)(\gamma_2 + y_2 + 2x_2)}{x + y|2 + 2\alpha|\bar{x} - y|2 + 2\alpha|} \leq Cx_1 \left( x_2^{2 - 2\alpha} + \int_0^{L|x|} dy_1 \int_0^{2x_2} dy_2 \frac{1}{|y|^{2\alpha}} \right) \leq Cx_1 (x_2^{2 - 2\alpha} + x_2 L^{1 - 2\alpha} |x|^{-2\alpha}) \leq C x_1 |x|^{2 - 2\alpha} L^{2 - 2\alpha}. \tag{2.7}
\]

Here in the first step we used mean value theorem and \(|y_2 - x_2| \leq |x - y|\), in the second step
\(y_1 \leq |\bar{x} - y|\), and in the third step split the region of integration and changed variable in the
long range part.

In the second integral in \((2.5)\), we bound \(\omega(y_1, y_2)\) and \(\omega(y_1, y_2 + 2x_2)\) using odd-odd structure
by \(\|\nabla^2 \omega\|_{L^\infty} y_1 y_2\) and \(\|\nabla^2 \omega\|_{L^\infty} y_1 (y_2 + 2x_2)\) respectively. We get that this integral does not exceed

\[
C \||\nabla^2 \omega\|_{L^\infty} \int_0^{L|x|} dy_1 \int_0^{L|x|} dy_2 \frac{\Phi(y_2) \gamma_2(\gamma_2 + y_2)(\gamma_2 + y_2 + 2x_2)}{x + y|2 + 2\alpha|\bar{x} - y|2 + 2\alpha|} \leq C \||\nabla^2 \omega\|_{L^\infty} x_1 |x|^{2 - 2\alpha} L^{2 - 2\alpha}, \tag{2.8}
\]

where in the first step we used the estimate for \(\omega\) and mean value theorem, and in the second step
\(y_2 + 2x_2 \leq 2|\bar{x} - y|\) and \(y_1 \leq |x + y|\). Combining \((2.7)\) and \((2.8)\), we get the result of the lemma.

The next result records an important property of the Biot-Savart law that makes contribution of the \(|x| \leq |y| \lesssim 1\) region of the central cell to \(u_1\) and \(u_2\) nearly identical when \(L\) is large.

**Proposition 2.2.** Let \(L\) be a parameter and \(x\) be such that \(|L|x| \leq 1\). Assume that \(\omega\) is odd
with respect to both \(x_1\) and \(x_2\), \(\omega(x) \geq 0\) in \([0, \pi]^2\), and is positive on a set of measure greater than \((L|x|)^2\). Let us define

\[
\begin{align*}
    u^\text{med}_j(x) & = \int_{[0, x]^2 \setminus [0, L|x|]^2} K_j(x, y) \omega(y) dy.
\end{align*}
\]

Then for all sufficiently large \(L \geq L_0 \geq 2\) and \(x\) such that \(|L|x| \leq 1\) we have that

\[
1 - B L^{-1} \leq \frac{u^\text{med}_1(x) x_2}{x_1 u^\text{med}_2(x)} \leq 1 + B L^{-1}, \tag{2.9}
\]

with some universal constant \(B\).

**Remark.** The threshold \(L_0\) is a universal constant - it does not depend on \(\omega\).

The condition \(|L|x| \leq 1\) is only intended to make sure that the region of integration in \(u^\text{med}_j\)
nontrivial. When applying this result, \(L\) will be chosen first, and \(x\) will be taken small enough later.
Proof. Observe that both the positivity of $\omega$ in $[0, \pi]^2$ and the measure of the set where it is positive is conserved by evolution, due to incompressibility and invariance of the region $[0, \pi]^2$ under trajectory map. This point is explained in more detail below after the proof of Lemma \ref{lem:A}

The bound \eqref{eq:2.9} follows from more informative pointwise bound for the Biot-Savart kernel. We will provide details for $K_1$; the case of $K_2$ is similar and can actually be inferred by symmetry. Bring the expression for $K_1(x, y)$ in \eqref{eq:2.10} to the common denominator. The numerator will be equal to

\[
(x_2 - y_2)(|\bar{x} - y|^{2+2\alpha} - |x - y|^{2+2\alpha})|\bar{x} - y|^{2+2\alpha}|x + y|^{2+2\alpha} - (x_2 + y_2)(|x + y|^{2+2\alpha} - |\bar{x} - y|^{2+2\alpha})|\bar{x} - y|^{2+2\alpha}|x - y|^{2+2\alpha}.
\]

Let us first collect the terms with $y_2$ factor, and use mean value theorem to represent them as

\[
-4(1 + \alpha)x_1y_1y_2((z_1^2 + (x_2 - y_2)^2)\alpha|\bar{x} - y|^{2+2\alpha}|x + y|^{2+2\alpha} + (z_2^2 + (x_2 + y_2)^2)\alpha|\bar{x} - y|^{2+2\alpha}|x - y|^{2+2\alpha}),
\]

where $z_1^2, z_2^2$ lie between $(x_1 - y_1)^2$ and $(x_1 + y_1)^2$. We can rewrite this expression as

\[
-4(1 + \alpha)x_1y_1y_2|y|^{4+6\alpha}\left(\frac{z_1^2 + (x_2 - y_2)^2}{|y|^2}\right)^{2+2\alpha}\times\left(\frac{(x_1 - y_1)^2 + (x_2 + y_2)^2}{|y|^2}\right)^{2+2\alpha}.
\]

Since $|y| \geq L|x|$, the terms with $y_2$ factor give us a contribution equal to

\[
-8(1 + \alpha)x_1y_1y_2|y|^{4+6\alpha}\left(1 + O(L^{-1})\right).
\]

Now let us consider the terms with $x_2$ factor. Here we get

\[
x_2\left((|\bar{x} - y|^{2+2\alpha} - |x - y|^{2+2\alpha})|\bar{x} - y|^{2+2\alpha}|x + y|^{2+2\alpha} - (|x + y|^{2+2\alpha} - |\bar{x} - y|^{2+2\alpha})|\bar{x} - y|^{2+2\alpha}|x - y|^{2+2\alpha}\right).
\]

Observe that

\[
(|x + y|^{2+2\alpha} - |\bar{x} - y|^{2+2\alpha}) - (|\bar{x} - y|^{2+2\alpha} - |x - y|^{2+2\alpha}) = 4(1 + \alpha)x_2y_2((x_1 + y_1)^2 + z_3^2)\alpha - ((x_1 - y_1)^2 + z_4^2)\alpha) = 16\alpha(1 + \alpha)x_1x_2y_1y_2(z_3^2 + z_4^2)\alpha^{-1},
\]

where $z_3^2 \in ((x_2 - y_2)^2, (x_2 + y_2)^2)$ and $z_4^2 \in ((x_1 - y_1)^2, (x_1 + y_1)^2)$. Also,

\[
|x - y|^{2+2\alpha}|x + y|^{2+2\alpha} - |\bar{x} - y|^{2+2\alpha}|x - y|^{2+2\alpha} = |\bar{x} - y|^{2+2\alpha}|x + y|^{2+2\alpha} - |\bar{x} - y|^{2+2\alpha}|x - y|^{2+2\alpha} = 4(1 + \alpha)x_2y_2((x_1 - y_1)^2 + z_5^2)\alpha + |x - y|^{2+2\alpha}((x_1 + y_1)^2 + z_6^2)\alpha),
\]

where $z_5^2, z_6^2$ belong to $((x_2 - y_2)^2, (x_2 + y_2)^2)$. Running a straightforward computation on \eqref{eq:2.12} using \eqref{eq:2.13} and \eqref{eq:2.14}, we get that the $x_2$ terms are equal to

\[
-x_2|\bar{x} - y|^{2+2\alpha}|x + y|^{2+2\alpha}16\alpha(1 + \alpha)x_1x_2y_1y_2(z_3^2 + z_4^2)\alpha^{-1} + 16x_1y_1x_2y_2(1 + \alpha)^2\times(x_2^2 + (x_2 + y_2)^2)\alpha(|x + y|^{2+2\alpha}((x_1 - y_1)^2 + z_5^2)\alpha + |x - y|^{2+2\alpha}((x_1 + y_1)^2 + z_6^2)\alpha) = 16(1 + \alpha)x_1x_2^2y_1y_2(\alpha|y|^{2+6\alpha}(-1 + O(L^{-1})) + 2(1 + \alpha)|y|^{2+6\alpha}(1 + O(L^{-1}))) =
\]
Combining (2.11), (2.15) and using that $|x| \leq |y|/L$, we obtain that the numerator (2.10) is equal to

$$-8(1 + \alpha)x_1y_1y_2|y|^{4+6\alpha}(1 + O(L^{-1}))$$

in the region $y_1, y_2 \geq L|x|$. Taking into account the denominator, we get that in this region

$$K_1(x, y) = -8(1 + \alpha)x_1y_1y_2|y|^{-4-2\alpha}(1 + f_1(x, y)),$$

where $|f_1(x, y)| \leq AL^{-1}$ with some universal constant $A$. A similar argument (or just symmetry considerations) establishes that

$$K_2(x, y) = 8(1 + \alpha)x_2y_1y_2|y|^{-4-2\alpha}(1 + f_2(x, y)),$$

with $|f_2(x, y)| \leq AL^{-1}$. Thus

$$\frac{u_1^{med}(x) x_2}{x_1 u_2^{med}(x)} = \frac{\int_{[0, \pi)^2 \setminus [0, L|x|]^2} y_1y_2|y|^{-4-2\alpha}(1 + f_1(x, y)) \omega \, dy}{\int_{[0, \pi)^2 \setminus [0, L|x|]^2} y_1y_2|y|^{-4-2\alpha}(1 + f_2(x, y)) \omega \, dy}.$$  

Choose $L_0$ so that $AL_0^{-1} \leq \frac{1}{4}$. Note that given our assumption that $\omega \geq 0$ in $[0, \pi)^2$ we have

$$1 - AL^{-1} \leq \frac{\int_{[0, \pi)^2 \setminus [0, L|x|]^2} y_1y_2|y|^{-4-2\alpha}(1 + f_{1,2}(x, y)) \omega \, dy}{\int_{[0, \pi)^2 \setminus [0, L|x|]^2} y_1y_2|y|^{-4-2\alpha} \omega \, dy} \leq 1 + AL^{-1},$$

and the integral in denominator is not zero since support of $\omega$ in $[0, \pi)^2$ has measure larger than $[0, L|x|]^2$. Then a simple computation shows that (2.9) follows for every $L \geq L_0$ with a constant $B = 3A$. \hfill \Box

Now we need to estimate the contribution of all cells other than the central one.

**Lemma 2.3.** Suppose that $|x| \leq 1$. Define

$$u_j^{far}(x) = \int_{[0, \infty)^2 \setminus [0, \pi)^2} K_j(x, y) \omega(y) \, dy.$$  

Then

$$|u_j^{far}(x)| \leq C(\alpha) x_j \| \omega \|_{L^\infty}.$$  

**Proof.** Note that the estimates (2.16), (2.17) on the Biot-Savart kernels continue to apply when $|x| \leq 1$, and $y \in [0, \infty)^2 \setminus [0, \pi)^2$. Then we get that

$$|u_j^{far}(x)| \leq C x_j \int_{[0, \infty)^2 \setminus [0, \pi)^2} y_1y_2|y|^{-4-2\alpha} |\omega(y)| \, dy \leq C x_j \| \omega \|_{L^\infty} \int_1^\infty r^{-1-2\alpha} \, dr = C(\alpha) x_j \| \omega \|_{L^\infty}. \hfill \Box$$

The final estimate we need is a lower bound on the absolute value of the velocity components $(-1)^j u_j$, $j = 1, 2$, near the origin provided certain assumptions on the structure of vorticity.
Lemma 2.4. There exists a constant $1 > \delta_0 > 0$ such that if $\omega(0) \geq 0$ on $[0, \pi)^2$ and that $\omega(0) = 1$ if $0 \leq \omega \leq \pi - \delta$. Then for all $x$ and $L \geq L_0$ such that $L|x| \leq \delta$, we have that

$$(-1)^j u_j^{med}(x, t) \geq cx_j \delta^{-\alpha}. \quad (2.19)$$

Proof. First, let us describe a soft consequence of the incompressibility and symmetries for properties of the solution $\omega(x, t)$ (similar to the arguments in [14]). Note that while the solution stays smooth, we have

$$\omega(x, t) = \omega_0(\Phi_t^{-1}(x)) \quad (2.20)$$

where $\Phi_t(x)$ is a smooth, invertible, measure preserving flow map defined by

$$\frac{d\Phi_t(x)}{dt} = u(\Phi_t(x), t), \quad \Phi_0(x) = x. \quad (2.21)$$

In addition, it is not hard to check that odd symmetry of $\omega$ with respect to both $x_1$ and $x_2$ and periodicity imply that $u_1$ is odd with respect to $x_1 = 0$ and $x_1 = \pm \pi$, and $u_2$ is odd with respect to $x_2 = 0$ and $x_2 = \pm \pi$. For this reason, the region $[0, \pi)^2$ is invariant under the flow map. The formula (2.20) and the assumptions on $\omega_0$ then yield that the measure of the set in $[0, \pi)^2$ where $\omega(x, t)$ is not equal to one does not exceed $4\pi \delta$ for all $t$.

Next, observe that a consequence of the bound (2.19) is that if $L \geq L_0$ and $L|x| \leq 1$, then for all $y_1, y_2 \geq L|x|$ we have

$$(-1)^j K_j(x, y) \geq C x_j y_1 y_2 |y|^{-4-2\alpha} \quad (2.22)$$

for some constant $C > 0$. Then

$$(-1)^j u_j^{med}(x, t) \geq C x_j \int_{[0, \pi]^2 \setminus [0, L|x|^2]}^{y_1 y_2} \frac{\omega(y, t)}{|y|^{4+2\alpha}} dy \geq
$$

$$Cx_j \int_{M \sqrt{\delta}}^{1} \frac{1}{r^{1+2\alpha}} dr \geq cx_j \delta^{-\alpha}$$

with some universal $c > 0$. The value of the constant $C$ here changes from expression to expression. In the second step we used that the measure of the set where $\omega(x, t) < 1$ in $[0, \pi)^2$ does not exceed $C \delta$. We get a lower bound if we cut out of the region of integration a sector of radius $M \sqrt{\delta}$ where the value of the kernel is largest; $M$ needs to be chosen sufficiently large but is a universal constant. \hfill $\Box$

3. Construction

The next lemma is parallel to the one shown in [19]. In the construction, we will consider the initial data that have an additional degeneracy condition on the derivatives in $x_1$ on vertical axis. This lemma establishes that this property is preserved for the solution while it stays smooth.

Lemma 3.1. Suppose that in addition to being odd in $x_1$ and $x_2$ and periodic, the initial data $\omega_0$ also satisfies $\partial_{x_1}^{j-1}\omega_0(0, x_2) = 0$ for all $x_2, j = 1, \ldots, n$. Then the solution $\omega(x, t)$, while it remains smooth, also satisfies $\partial_{x_1}^{j-1}\omega(0, x_2, t) = 0$. 
Remark. That all even derivatives of $\omega_0$ in $x_1$ also vanish on $x_2$ axis follows from odd symmetry.

Proof. Let us show the result for $n = 1$, the only case we use in the construction. It can be extended to arbitrary $n$ by inductive argument. Let us differentiate the equation for $\omega$ with respect to $x_1$:

$$\partial_t \partial_{x_1} \omega + \partial_{x_1} u_1 \partial_{x_1} \omega + u_1 \partial_{x_1}^2 \omega + \partial_{x_1} u_2 \partial_{x_2} \omega + u_2 \partial_{x_1 x_2}^2 \omega = 0.$$ 

Note that $u_1$ is odd in $x_1$ and $u_2$ is even in $x_1$. Then the third and fourth terms in the above equation vanish if $x_1 = 0$. Denoting $v(x_2, t) = \partial_{x_1} \omega(0, x_2, t)$, we get that $v$ satisfies a self contained equation on the line $(0, x_2):

$$\partial_t v + u_2 \partial_{x_2} v + \partial_{x_1} u_1 v = 0,$$

and $v(x_2, 0) = 0$ by assumption. Then $v(x_2, t)$ must stay zero while $\omega$ stays smooth. □

We are now ready to prove Theorem 1.1

Proof of Theorem 1.1 Let us choose the initial data $\omega_0$ as follows. First, as we already discussed, $\omega_0$ is odd with respect to both $x_1$ and $x_2$, $1 \geq \omega_0(x) \geq 0$ in $[0, \pi]^2$ and it equals 1 in this region, apart from a strip of width $\leq \delta$ along the boundary. The parameter $\delta \leq \delta_0 < 1$ will be fixed later. We also require $\partial_{x_1} \omega_0(0, x_2) = 0$ for all $x_2$, a condition that is preserved for all times while the solution stays smooth by Lemma 3.1. Finally, we assume that in a small neighborhood of the origin of order $\sim \delta$ we have $\omega_0(x_1, x_2) = \delta^{-4} x_1^3 x_2$. Note that $\partial_{x_1 x_2}^2 \omega_0(0, x_2) = 0$ by oddness, so this is the “maximal” behavior of $\omega_0$ under our degeneracy condition.

Fix arbitrary $T \geq 1$; for small $T$ the result follows automatically as $\|\nabla^2 \omega(\cdot, t)\|_{L^\infty} \geq c\delta^{-2}$ for all times. Take $x_1^0 = e^{-T \delta^{-\alpha/2}}$ and $x_2^0 = (x_1^0)^\beta$ where $\beta$ is a parameter. In general, we will have three parameters in our construction: $\delta$, $L$ and $\beta$. The parameters $\beta$ and $L$ are chosen to satisfy

$$\beta = 5, \quad L \geq L_0 \geq 2, \quad 2\beta(2 + B)L^{-1} \leq 1, \quad \text{(3.1)}$$

where $L_0$ and $B$ are universal constants from Proposition 2.2. Throughout the construction, we will place constraints on $\delta$ that will be consistent; we will recap these requirements at the end of the argument. Note that

$$\omega_0(x_1^0, x_2^0) = \delta^{-4}(x_1^0)^3 \beta = \delta^{-4} e^{-(3 + \beta) T \delta^{-\alpha/2}}.$$ 

Consider the trajectory $(x_1(t), x_2(t))$ originating at $(x_1^0, x_2^0)$. We will be tracking this trajectory until either time reaches $T$, or $x_2(t)$ reaches $x_1^0$, or $\|\nabla^2 \omega(\cdot, t)\|_{L^\infty}$ becomes large enough to satisfy the lower bound we seek.

Let us denote

$$T_0 = \min \left( T, \min \{ t : x_2(t) = x_2^0 \}, \min \{ t : \|\nabla^2 \omega(\cdot, t)\|_{L^\infty} \geq \exp(cT) \} \right).$$

Observe that for all $t \leq T_0$, we have $x_2(t) \leq x_2^0$. Suppose that for some $0 \leq t_0 \leq T_0$, we have for the first time

$$|u_j^{\text{near}}(x(t_0), t_0)| + |u_j^{\text{far}}(x(t_0), t_0)| \geq L^{-1}(-1)^j u_j^{\text{med}}(x(t_0), t_0), \quad \text{(3.2)}$$

for either $j = 1$ or $j = 2$, where $L \geq L_0$ is to be fixed later (we include inequality as an option in (3.2) since we could have $t_0 = 0$). Note that we must have $x_1(t_0) \leq x_1^0$, since due to (2.19) we
have \( u_1(x(t), t) \leq 0 \) for \( t < t_0 \). Because of the estimates (2.3), (2.18) and (2.19), the inequality (3.2) implies that
\[
C|x(t_0)|^{2-2\alpha}L^{2-2\alpha}\|\nabla^2 \omega(\cdot, t_0)\|_{L^\infty} + C\|\omega\|_{L^\infty} \geq cL^{-1}\delta^{-\alpha}.
\] (3.3)
Application of (2.19) requires \( L|x(t_0)| \leq \delta \), which holds if
\[
2e^{-\delta^{-\alpha/2}}L \leq \delta.
\] (3.4)
Now suppose also that \( \delta \) is such that
\[
c\delta^{-\alpha/2} \geq 2CL^{3-2\alpha}
\] (3.5)
and
\[
c\delta^{-\alpha} \geq 2C\|\omega\|_{L^\infty}L.
\] (3.6)
Then (3.3) implies that
\[
\|\nabla^2 \omega(\cdot, t_0)\|_{L^\infty} \geq \delta^{-\alpha/2}|x(t_0)|^{-2+2\alpha} \geq \delta^{-\alpha/2}(x_0^0)^{-2+2\alpha} = \delta^{-\alpha/2}e^{(2-2\alpha)\delta^{-\alpha/2}T}.
\]
Thus the bound we seek is satisfied at \( t_0 \) and we are done. Therefore, from now on we can assume that for all \( t \leq t_0 \), we have
\[
|u_j^{near}(x(t), t)| + |u_j^{far}(x(t), t)| \leq L^{-1}(-1)^ju_j^{med}(x(t), t)
\] (3.7)
for \( j = 1, 2 \).

Next, suppose that \( T_0 = T \). Then due to (3.7) and (2.19) we have
\[
u_1(x(t), t) \leq -(1 - L^{-1})c\tau_1\delta^{-\alpha}
\]
for all \( t \leq T \). Also \( x_1(t) \leq x_1^0 \) for all \( t \leq T \). Therefore,
\[
x_1(T) \leq x_1^0e^{-\frac{\tau}{\delta^{\alpha}}}.
\]
On the other hand,
\[
\omega(x_1(T), x_2(T), T) = \omega_0(x_1^0, x_2^0) = \delta^{-4}e^{-(3+\beta)T\delta^{-\alpha/2}}.
\]
Since
\[
\omega(0, x_2(T), T) = \partial_{x_1}\omega(0, x_2(T), T) = 0,
\]
we obtain that
\[
\|\partial_{x_1}^2 \omega(\cdot, T)\|_{L^\infty} \geq 2\omega(x_1(T), x_2(T), T)x_1(T)^{-2} \geq \delta^{-4}e^{(\delta^{-\alpha/2}- (3+\beta))\delta^{-\alpha/2}T}.
\]
Taking \( \delta \) so that
\[
c\delta^{-\alpha/2} \geq 2(3 + \beta)
\] (3.8)
makes sure that the lower bound we seek holds in this case, too.

It remains to consider the case where \( T_0 < T \) and (3.7) holds for all \( t \leq T_0 \). Then \( x_2(T_0) = x_1^0 \). By (3.7), for all \( 0 \leq t \leq T_0 \) we have
\[
-\frac{u_1(x(t), t)}{x_1(t)} \geq -(1 - L^{-1})\frac{u_1^{med}(x(t), t)}{x_1(t)}.
\] (3.9)
By the estimate (2.9) of Proposition 2.2, we also have
\[- \frac{u_1^{med}(x(t), t)}{x_1(t)} \geq (1 - BL^{-1}) \frac{u_2^{med}(x(t), t)}{x_2(t)}. \]
(3.10)

Finally, by (3.1) again,
\[- \frac{u_2^{med}(x(t), t)}{x_2(t)} \geq (1 - L^{-1}) \frac{u_2(x(t), t)}{x_2(t)}. \]
(3.11)

Combining (3.9), (3.10) and (3.11), we get that
\[- \frac{u_1(x(t), t)}{x_1(t)} \geq (1 - (2 + B)L^{-1}) \frac{u_2(x(t), t)}{x_2(t)}. \]
(3.12)
due to our choice (3.1) of $L$. Therefore
\[
\frac{x_0^0}{x_1(T_0)} = e^{-\int_0^{T_0} \frac{u_1(x(t), t)}{x_1} dt} \geq e^{(1-(2+B)L^{-1}) \int_0^{T_0} \frac{u_2(x(t), t)}{x_2} dt} = \left( \frac{x_2(T_0)}{x_2^0} \right)^{1-(2+B)L^{-1}} = (x_1^0)^{(1-\beta)(1-(2+B)L^{-1})}.
\]

Here we used that $x_0^0 = (x_1^0)^\beta$. It follows that
\[
x_1(T_0) \leq (x_1^0)^\beta(1-(2+B)L^{-1})+(2+B)L^{-1}.
\]

Similarly to the previous case, this implies that
\[
\| \partial_{x_1}^2 \omega(\cdot, T_0) \|_{L^\infty} \geq 2 \omega(x_1(T_0), x_2(T_0), x_1(T_0)^{-2} \geq \delta^{-4}(x_1^0)^{3+\beta-2\beta(1-(2+B)L^{-1})} = \delta^{-4}(x_1^0)^{3-\beta+2\beta(2+B)L^{-1}} \geq \delta^{-4} e^{\delta^{-\alpha/2} \beta(2+B)L^{-1}T} \geq \delta^{-4} e^{\delta^{-\alpha/2}T},
\]
where in the last step we used (3.1). Finally, it remains to fix $\delta \leq \delta_0$ (where $\delta_0$ is a universal constant from Lemma 2.4) so that the conditions (3.4), (3.5), (3.6) and (3.8) are satisfied. $\square$

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