This paper deals with the existence, the uniqueness and an approximation scheme of the solution to sweeping processes perturbed by a continuous signal of finite $p$-variation with $p \in [1,3]$. It covers pathwise stochastic noises directed by a fractional Brownian motion of Hurst parameter greater than $1/3$.



**Contents**

1. Introduction 1
2. Preliminaries 4
2.1. Sweeping processes 4
2.2. Young’s integral, rough integral 7
3. Existence of solutions 10
4. Some uniqueness results 16
5. Approximation scheme 21
6. Sweeping processes perturbed by a stochastic noise directed by a fBm 24
References 26

**MSC2010 :** 34H05, 34K35, 60H10.

1. **Introduction**

Consider a multifunction $C : [0,T] \rightarrow \mathbb{R}^e$ with $e \in \mathbb{N}^*$. Roughly speaking, the Moreau sweeping process (see Moreau [20]) associated to $C$ is the path $X$, living in $C$, such that when it hits the frontier of $C$, a minimal force is applied to $X$ in order to keep it inside of $C$. Precisely, $X$ is the solution to the following differential inclusion:

$$
\left\{ \begin{array}{l}
-\frac{dDY}{dt}(t) \in N_{C(t)}(Y(t)) \ [DY]\text{-a.e.} \\
Y(0) = a \in C(0),
\end{array} \right.
$$

where $DY$ is the differential measure associated with the continuous function of bounded variation $Y$, $[DY]$ is its variation measure, and $N_{C(t)}(Y(t))$ is the normal cone of $C(t)$ at $Y(t)$. This problem has been deeply studied by many authors. For instance, the reader can refer to Moreau [20], Valadier [25] or Monteiro Marques [19].

Several authors studied some perturbed versions of Problem (1), in particular by a stochastic multiplicative noise in Itô’s calculus framework (see Revuz and Yor [22]). For instance, the reader can refer to Bernicot and Venel [3] or Castaing et al. [5]. On reflected diffusion processes, which perturbed sweeping processes with constant

**Key words and phrases.** Approximation scheme ; Fractional Brownian motion ; Rough differential equations ; Rough paths ; Skorokhod reflection problems ; Stochastic differential equations ; Sweeping processes.
Consider the perturbed Skorokhod problem

\[
\begin{aligned}
X(t) &= H(t) + Y(t) \\
H(t) &= \int_0^t f(X(s))dZ(s) \\
-\frac{dDY}{dt}(t) &\in N_{C(t)}(Y(t)) \text{ [DY]-a.e. with } Y(0) = a,
\end{aligned}
\]

where \( C(t) = C(t) - H(t), t \in [0,T] \) (thus \( N_{C(t)}(Y(t)) = N_{C(t)}(X(t)) \), \( Z : [0,T] \to \mathbb{R}^d \) is a continuous signal of finite \( p \)-variation with \( d \in \mathbb{N}^* \) and \( p \in [1,\infty] \), \( f \in \text{Lip}^p(\mathbb{R}^e,\mathcal{M}_{e,d}(\mathbb{R})) \) with \( \gamma > p \), and the integral against \( Z \) is taken in the sense of rough paths. On the rough integral, the reader can refer to Lyons [16], Friz and Hairer [9]. Throughout the paper, the multifunction \( C \) satisfies the following assumption.

**Assumption 1.1.** \( C \) is a convex compact valued multifunction, continuous for the Hausdorff distance, and there exists a continuous selection \( \gamma : [0,T] \to \mathbb{R}^e \) satisfying

\[
\overline{B}_e(\gamma(t), r) \subset \text{int}(C(t)) \quad \forall t \in [0,T],
\]

where \( \overline{B}_e(\gamma(t), r) \) denotes the closed ball of radius \( r \) centered at \( \gamma(t) \).

This assumption is equivalent to saying that \( C(t) \) has nonempty interior for every \( t \in [0,T] \), see [5] Lemma 2.2.

In Falkowski and Slomiński [5], when \( p \in [1,2] \) and \( C(t) \) is a cuboid of \( \mathbb{R}^e \) for every \( t \in [0,T] \), the authors proved the existence and uniqueness of the solution of Problem (2). Furthermore, several authors studied the existence and uniqueness of the solution for reflected rough differential equations. In [11], M. Besalú et al. proved the existence and uniqueness of the solution for delayed rough differential equations and non-negativity constraints. Recently, S. Aida gets the existence of solutions for a large class of reflected rough differential equations in [2] and [1]. Finally, in [9], A. Deya et al. proved the existence and uniqueness of the solution for 1-dimensional reflected rough differential equations. An interesting remark related to these references is that when \( C \) is not a cuboid, moving or not, it is a challenge to get the uniqueness of the solution for reflected rough differential equations and sweeping processes.

For \( p \in [1,3] \), the purpose of this paper is to prove the existence of solutions to Problem (2) when \( C \) satisfies Assumption [11] and a necessary and sufficient condition for uniqueness close to the monotonicity of the normal cone which allows to prove the uniqueness when \( p = 1 \) and there is an additive continuous signal of finite \( q \)-variation with \( q \in [1,3] \). In this last case, the convergence of an approximation scheme is also proved.

Section 2 deals with some preliminaries on sweeping processes and the rough integral. Section 3 is devoted to the existence of solutions to Problem (2) when \( Z \) is a moderately irregular signal (i.e. \( p \in [1,2] \)) and when \( Z \) is a rough signal (i.e. \( p \in [2,3] \)). Section 4 deals with some uniqueness results. The convergence of an approximation scheme based on Moreau’s catching up algorithm is proved in Section 5 when \( p = 1 \) and there is an additive continuous signal of finite \( q \)-variation with \( q \in [1,3] \). Finally, Section 6 deals with sweeping processes perturbed by a pathwise stochastic noise directed by a fractional Brownian motion of Hurst parameter greater than 1/3.
The following notations, definitions and properties are used throughout the paper.

**Notations and elementary properties:**

1. \( C_h(t) := C(t) - h(t) \) for every function \( h : [0, T] \to \mathbb{R}^e \).
2. \( N_C(x) \) is the normal cone of \( C \) at \( x \), for any closed convex subset \( C \) of \( \mathbb{R}^e \) and any \( x \in \mathbb{R}^e \) (recall that \( N_C(x) = \emptyset \) if \( x \notin C \)).
3. \( \Delta_T := \{(s, t) \in [0, T]^2 : s < t \} \) and \( \Delta_{s,t} := \{(u, v) \in [s, t]^2 : u < v \} \) for every \( (s, t) \in \Delta_T \).
4. For every function \( x \) from \([0, T] \) into \( \mathbb{R}^d \) and \( (s, t) \in \Delta_T \), \( x(s, t) := x(t) - x(s) \).
5. Consider \( (s, t) \in \Delta_T \). The vector space of continuous functions from \([s, t] \) into \( \mathbb{R}^d \) is denoted by \( C^0([s, t], \mathbb{R}^d) \) and equipped with the uniform norm \( \| \|_{\infty, s, t} \) defined by
   \[
   \|x\|_{\infty, s, t} := \sup_{u \in [s, t]} \|x(u)\|
   \]
   for every \( x \in C^0([s, t], \mathbb{R}^d) \), or the semi-norm \( \| \|_{0, s, t} \) defined by
   \[
   \|x\|_{0, s, t} := \sup_{u, v \in [s, t]} \|x(v) - x(u)\|
   \]
   for every \( x \in C^0([s, t], \mathbb{R}^d) \). Moreover, \( \| \|_{\infty, T} := \| \|_{\infty, 0, T} \), \( \| \|_{0, T} := \| \|_{0, 0, T} \) and
   \[
   C_0^0([s, t], \mathbb{R}^d) := \{ x \in C^0([s, t], \mathbb{R}^d) : x(0) = 0 \}. 
   \]
6. Consider \( (s, t) \in \Delta_T \). The set of all dissections of \([s, t] \) is denoted by \( \mathcal{D}_{[s, t]} \) and the set of all strictly increasing sequences \((s_n)_{n \in \mathbb{N}} \) of \([s, t] \) such that \( s_0 = s \) and \( \lim_{n} s_n = t \) is a denoted by \( \mathcal{D}_\infty([s, t]) \).
7. Consider \( (s, t) \in \Delta_T \). A function \( x : [s, t] \to \mathbb{R}^d \) has finite \( p \)-variation if and only if,
   \[
   \|x\|_{p, \text{var}, s, t} := \sup \left\{ \sum_{k=1}^{n-1} \|x(t_k, t_{k+1})\|^p \right\}^{1/p} : n \in \mathbb{N}^* \text{ and } (t_k)_{k \in [1, n]} \in \mathcal{D}_{[s, t]} \}
   \]
   \[
   < \infty. 
   \]
   Consider the vector space
   \[
   C^p\text{-var}([s, t], \mathbb{R}^d) := \{ x \in C^0([s, t], \mathbb{R}^d) : \|x\|_{p, \text{var}, s, t} < \infty \}. 
   \]
   The map \( \| \|_{p, \text{var}, s, t} \) is a semi-norm on \( C^p\text{-var}([s, t], \mathbb{R}^d) \).
   Moreover, \( \| \|_{p, \text{var}, 0, T} := \| \|_{p, \text{var}, 0, T} \).

**Remarks :**

a. For every \( q, r \in [1, \infty[ \) such that \( q \geq r \),
   \[
   \forall x \in C^q\text{-var}([s, t], \mathbb{R}^d), \|x\|_{q, \text{var}, s, t} \leq \|x\|_{r, \text{var}, s, t}. 
   \]
   In particular, any continuous function of bounded variation on \([s, t] \) belongs to \( C^q\text{-var}([s, t], \mathbb{R}^d) \) for every \( q \in [1, \infty[ \).

b. For every \( (s, t) \in \Delta_T \) and \( x \in C^1\text{-var}([s, t], \mathbb{R}) \),
   \[
   \|x\|_{1, \text{var}, s, t} = \int_s^t |Dx|, 
   \]
   where \(|Dx|\) is the variation measure of the differential measure \( Dx \) associated with \( x \).
8. The vector space of Lipschitz continuous maps from $\mathbb{R}^e$ into $\mathcal{M}_{e,d}(\mathbb{R})$ is denoted by $\text{Lip}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$ and equipped with the Lipschitz semi-norm $\| \cdot \|_{\text{Lip}}$ defined by

$$\| \varphi \|_{\text{Lip}} := \sup \left\{ \frac{|\varphi(y) - \varphi(x)|}{\|y - x\|} : x, y \in \mathbb{R}^e \text{ and } x \neq y \right\}$$

for every $\varphi \in \text{Lip}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$.

9. For every $\lambda \in \mathbb{R}$,

$$[\lambda] := \max\{n \in \mathbb{Z} : n < \lambda\}$$

and $\{\lambda\} := \lambda - [\lambda]$.

10. Consider $\gamma \in [1, \infty[$. A continuous map $\varphi : \mathbb{R}^e \to \mathcal{M}_{d,e}(\mathbb{R})$ is $\gamma$-Lipschitz in the sense of Stein if and only if,

$$\| \varphi \|_{\text{Lip}^\gamma} := \| D^{(\gamma)} \varphi \|_{\{\gamma\}-\text{Hö}} \vee \max\{\| D^k \varphi \|_{\infty} : k \in [0, [\gamma]]\}$$

< $\infty$.

Consider the vector space

$$\text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R})) := \{ \varphi \in C^0(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R})) : \| \varphi \|_{\text{Lip}^\gamma} < \infty \}.$$

The map $\| \cdot \|_{\text{Lip}^\gamma}$ is a norm on $\text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$.

Remarks:

a. If $\varphi \in \text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$, then $\varphi \in \text{Lip}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$.

b. If $\varphi \in C^{(\gamma+1)}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$ is bounded with bounded derivatives, then $\varphi \in \text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$.

2. Preliminaries

This section deals with some preliminaries on sweeping processes and the rough integral. The first subsection states some fundamental results on unperturbed sweeping processes coming from Moreau [20], Valadier [25] and Monteiro Marques [19]. A continuity result of Castaing et al. [31], which is the cornerstone of the proofs of Theorem 3.1 and Theorem 3.2, is also stated. The second subsection deals with the integration along rough paths. In this paper, definitions and propositions are stated as in Friz and Hairer [9], in accordance with M. Gubinelli’s approach (see Gubinelli [12]).

2.1. Sweeping processes. The following theorem, due to Monteiro Marques [17, 18, 19] using an estimation due to Valadier (see [4, 25]), states a sufficient condition of existence and uniqueness of the solution of the unperturbed sweeping process defined by Problem 1.1.

**Proposition 2.1.** Assume that $C$ is a convex compact valued multifunction, continuous for the Hausdorff distance, and such that there exists $(x, r) \in \mathbb{R}^e \times [0, \infty[$ satisfying

$$\overline{C}_t(x, r) \subset C(t) ; \forall t \in [0, T].$$

Then Problem 1.1 has a unique continuous solution of finite 1-variation $y : [0, T] \to \mathbb{R}^e$ such that

$$\| y \|_{1-\text{var}, T} \leq l(r, \| a - x \|),$$

where $l : \mathbb{R}^+_t \to \mathbb{R}^+_l$ is the map defined by

$$l(s, S) := \max \left\{ 0, \frac{S^2 - s^2}{2s} \right\} \text{ if } e > 1$$

$$\max\{0, S - s\} \text{ if } e = 1$$

This proposition is a consequence of the two following ones. These two propositions are also used in Section 5.
Proposition 2.2. Under Assumption \[\text{[4]}\] a map \( y : [0, T] \to \mathbb{R}^e \) is a solution of Problem \([1]\) if it satisfies the two following conditions:

1. For every \( t \in [0, T] \), \( y(t) \in C(t) \).
2. For every \((s, t) \in \Delta_T \) and \( z \in \cap_{\tau \in [s, t]} C(\tau) \),
   \[
   (z, y(t) - y(s)) \geq \frac{1}{2} \|y(t)\|^2 - \|y(s)\|^2.
   \]

Proposition 2.3. Consider \( n \in \mathbb{N}^* \), \((t^0_n, \ldots, t^n_n)\) the dissection of \([0, T]\) of constant mesh \(T/n\) and the step function \(Y^n\) defined by

\[
\begin{align*}
Y^n_0 &:= a \\
Y^n_{k+1} &:= P_C(t^0_{k+1})(Y^n_k) ; k \in [0, n-1] \\
Y^n(t) &:= Y^n_k ; t \in [t^k_n, t^{k+1}_n], k \in [0, n-1].
\end{align*}
\]

(1) Under the conditions of Proposition \([3]\) on \(C\), \(\|Y^n\|_{1\text{-var}, T} \leq l(r, \|a - x\|)\).

(2) Under Assumption \([4]\) for every \( m \in n\mathbb{N}^* \) and \( t \in [0, T] \), there exist \( i \in [1, n] \) and \( j \in [1, m] \) such that \( t \in [t^i_{m-1}, t^i_m] \) and \( t \in [t^j_{m-1}, t^j_m] \) and
   \[
   \|Y^n(t) - Y^m(t)\|^2 \leq 2d_H(C(t^i_n), C(t^j_m))(\|Y^n\|_{1\text{-var}, T} + \|Y^m\|_{1\text{-var}, T}).
   \]

See Monteiro Marques \([19]\), Chapter 2 for the proofs of the three previous propositions.

Let \( h \) be a continuous function from \([0, T]\) into \( \mathbb{R}^e \) such that \( h(0) = 0 \). If it exists, a Skorokhod decomposition of \((C, a, h)\) is a couple \((v_h, w_h)\) such that:

\[
\begin{align*}
\text{(3)} \quad v_h(t) &= h(t) + w_h(t) \\
-\frac{dDw_h}{d|Dw_h|}(t) &\in N_{C_h(t)}(w_h(t)) \text{ a.e. with } w_h(0) = a,
\end{align*}
\]

where \( v_h \) and \( w_h \) are continuous, and \( w_h \) has bounded variation. Since \( N_{C_h(t)}(x) = \emptyset \) when \( x \notin C_h(t) \), the system \([3]\) implies that \( |Dw_h| \text{ a.e.}, w_h(t) \in C_h(t) \), that is, \( v_h(t) \in C(t) \). Under Assumption \([4]\) by Proposition \([2]\) together with Castaing et al. \([5]\) Lemma 2.2, \((C, a, h)\) has a unique Skorokhod decomposition \((v_h, w_h)\).

Theorem 2.4. Under Assumption \([4]\), if \((h_n)_{n \in \mathbb{N}}\) is a sequence of continuous functions from \([0, T]\) into \( \mathbb{R}^e \) which converges uniformly to \( h \in C^0([0, T], \mathbb{R}^e) \), then

\[
\sup_{n \in \mathbb{N}} \|w_{h_n}\|_{1\text{-var}, T} < \infty
\]

and

\[
(v_{h_n}, w_{h_n}) \xrightarrow{n \to \infty} (v_h, w_h).
\]

See Castaing et al. \([5]\) Theorem 2.3].

Under Assumption \([4]\) note that there exist \( R > 0, N \in \mathbb{N}^* \) and a dissection \((t_0, \ldots, t_N)\) of \([0, T]\) such that

\[
\text{(4)} \quad \overline{B}_e(\gamma(t_k), R) \subset C(u)
\]

for every \( k \in [0, N - 1] \) and \( u \in [t_k, t_{k+1}] \).

Proposition 2.5. Under Assumption \([4]\):

1. The map \((v, w)\) is continuous from \( C^0([0, T], \mathbb{R}^e) \) to \( C^0([0, T], \mathbb{R}^e) \times C^{1\text{-var}}([0, T], \mathbb{R}^e) \).
(2) Consider \((s,t) \in \Delta_T\) and \(\rho \in [0, R/2]\) where \(R\) is defined in (4). For every \(h \in C_0^0([s,t], \mathbb{R}^n)\) such that \(\|h\|_{0,s,t} \leq \rho\),
\[
\|w_h\|_{1\text{-}\text{var},s,t} \leq \mathfrak{M}(\rho)
\]
with
\[
\mathfrak{M}(\rho) := \frac{N}{2\rho} \sup_{u \in [0,T]} \sup_{x,y \in C(u)} \|x - y\|^2.
\]

Proof. Refer to Castaing et al. [5, Lemma 5.3] for a proof of the first point.

Let us insert \(s\) and \(t\) in the dissection \((t_0, \ldots, t_N)\) of \([0, T]\) and define \(k(s), k(t) \in [0, N + 2]\) by
\[
t_{k(s)} := s \quad \text{and} \quad t_{k(t)} := t.
\]
Consider \(k \in [k(s), k(t) - 1]\) and \(u \in [t_k, t_{k+1}]\).

On the one hand,
\[
\overline{B}_c(\gamma(t_k) - h(t_k), \rho) \subset B_c(\gamma(t_k) - h(u), R) \subset C_h(u).
\]
So,
\[
\overline{B}_c(\gamma(t_k) - v_h(t_k), \rho) \subset C(u) - h(t_k, u) - v_h(t_k).
\]
On the other hand,
\[
v_h(t_k, u) = h(t_k, u) + w_{h,t_k}(u)
\]
with
\[
w_{h,t_k}(u) := w_h(u) - w_h(t_k).
\]
Moreover,
\[
- \frac{dDw_h}{d|Dw_h|}(u) \in N_{C(u) - h(u)}(w_h(u)) \ |Dw_h|\text{-a.e.}
\]
and then,
\[
\begin{cases}
- \frac{dDw_{h,t_k}}{d|Dw_{h,t_k}|}(u) \in N_{C(u) - h(t_k, u) - v_h(t_k)}(w_{h,t_k}(u)) \ |Dw_{h,t_k}|\text{-a.e.} \\
w_{h,t_k}(t_k) = 0.
\end{cases}
\]
So, by Proposition 2.1
\[
\|w_h\|_{1\text{-}\text{var},t_k,t_{k+1}} = \|w_{h,t_k}\|_{1\text{-}\text{var},t_k,t_{k+1}} \\
\leq l(\rho, \|\gamma(t_k) - v_h(t_k)\|).
\]
Therefore,
\[
\|w_h\|_{1\text{-}\text{var},s,t} = \sum_{k = k(s)}^{k(t) - 1} \|w_h\|_{1\text{-}\text{var},t_k,t_{k+1}} \\
\leq N \sup_{u \in [s,t]} l(\rho, \|\gamma(u) - v_h(u)\|) \\
\leq \mathfrak{M}(\rho).
\]
\[\square\]
2.2. Young’s integral, rough integral. The first part of the subsection deals with the definition and some basic properties of Young’s integral which allow to integrate a map \( y \in C^{p\text{-var}}([0, T], \mathcal{M}_{e,d}([\mathbb{R}])) \) with respect to \( z \in C^{q\text{-var}}([0, T], \mathbb{R}^d) \) when \( q, r \in [1, \infty] \) and \( 1/q + 1/r > 1 \). The second part of the subsection deals with the rough integral which extends Young’s integral when the condition \( 1/q + 1/r > 1 \) is not satisfied anymore. The signal \( z \) has to be enhanced as a rough path.

**Definition 2.6.** A map \( \omega : \Delta_T \to \mathbb{R}_+ \) is a control function if and only if,

1. \( \omega \) is continuous,
2. \( \omega(s, s) = 0 \) for every \( s \in [0, T] \),
3. \( \omega \) is super-additive:

\[
\omega(s, u) + \omega(u, t) \leq \omega(s, t)
\]

for every \( s, t, u \in [0, T] \) such that \( s \leq u \leq t \).

**Example.** Let \( p \geq 1 \). For every \( z \in C^{p\text{-var}}([0, T], \mathbb{R}^d) \), the map

\[
\omega_{p, z} : (s, t) \in \Delta_T \mapsto \omega_{p, z}(s, t) := \|z\|_{p\text{-var}, s, t}^p
\]

is a control function.

**Proposition 2.7.** Let \( p \geq 1 \). Consider \( x \in C^0([0, T], \mathbb{R}^d) \) and a sequence \((x_n)_{n \in \mathbb{N}}\) of elements of \( C^{p\text{-var}}([0, T], \mathbb{R}^d) \) such that

\[
\lim_{n \to \infty} \|x_n - x\|_{\infty, T} = 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|x_n\|_{p\text{-var}, T} < \infty.
\]

Then \( x \in C^{p\text{-var}}([0, T], \mathbb{R}^d) \) and

\[
\lim_{n \to \infty} \|x_n - x\|_{(p+r)\text{-var}, T} = 0 ; \forall \epsilon > 0.
\]

See Friz and Victoir [11] Lemma 5.12 and Lemma 5.27 for a proof.

**Proposition 2.8.** (Young’s integral) Consider \( q, r \in [1, \infty] \) such that \( 1/q + 1/r > 1 \), and two maps \( y \in C^{q\text{-var}}([0, T], \mathcal{M}_{e,d}([\mathbb{R}])) \) and \( z \in C^{r\text{-var}}([0, T], \mathbb{R}^d) \). For every \( n \in \mathbb{N}^+ \) and \((t_k^n)_{k \in [1, n]} \in \mathcal{D}_{[0, T]} \), the limit

\[
\lim_{n \to \infty} \sum_{k=1}^{n-1} y(t_k^n) z(t_k^n, t_{k+1})
\]

exists and does not depend on the dissection \((t_k^n)_{k \in [1, n]}\). That limit is denoted by

\[
\int_0^T y(s) dz(s)
\]

and called Young’s integral of \( y \) with respect to \( z \) on \([0, T]\). Moreover, there exists a constant \( c(q, r) > 0 \), depending only on \( q \) and \( r \), such that for every \((s, t) \in \Delta_T \),

\[
\left\| \int_0^s y(s) dz(s) \right\|_{r\text{-var}, s, t} \leq c(q, r) \|z\|_{q\text{-var}, s, t} (\|y\|_{r\text{-var}, s, t} + \|y\|_{\infty, s, t}).
\]

See Lyons [10] Theorem 1.16, Lejay [14] Theorem 1] or Friz and Victoir [11] Theorem 6.8 for a proof.

**Proposition 2.9.** Consider \( q, r \in [1, \infty] \) such that \( 1/q + 1/r > 1 \), two maps \( y \in C^{q\text{-var}}([0, T], \mathcal{M}_{e,d}([\mathbb{R}])) \) and \( z \in C^{r\text{-var}}([0, T], \mathbb{R}^d) \), and a sequence \((y_n)_{n \in \mathbb{N}}\) of elements of \( C^{q\text{-var}}([0, T], \mathcal{M}_{e,d}([\mathbb{R}])) \) such that:

\[
\lim_{n \to \infty} \|y_n - y\|_{\infty, T} = 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|y_n\|_{r\text{-var}, T} < \infty.
\]

Then,

\[
\lim_{n \to \infty} \left\| \int_0^s y_n(s) dz(s) - \int_0^s y(s) dz(s) \right\|_{\infty, T} = 0.
\]
See Friz and Victoir [11] Proposition 6.12 for a proof.

Consider \( p \in [2,3] \) and let us define the rough integral for continuous functions of finite \( p \)-variation.

Remark. In the sequel, the reader has to keep in mind that:
1. \( \mathcal{M}_{e,d}(\mathbb{R}) \cong \mathbb{R}^e \otimes \mathbb{R}^d \).
2. \( \mathcal{M}_{d,1}(\mathbb{R}) \cong \mathcal{M}_{1,d}(\mathbb{R}) \cong \mathbb{R}^d \).
3. \( \mathcal{L}(\mathbb{R}^d, \mathcal{M}_{e,d}(\mathbb{R})) \cong \mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)) \cong \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^e) \).

Definition 2.10. Consider \( z \in C^{1-\text{var}}([0,T], \mathbb{R}^d) \). The step-2 signature of \( z \) is the map \( S_2(z) : \Delta_T \to \mathbb{R}^d \times \mathcal{M}_d(\mathbb{R}) \) defined by

\[
S_2(z)(s,t) := \left( z(s,t), \int_{s<u<v<t} dz(v) \otimes dz(u) \right)
\]

for every \((s,t) \in \Delta_T \).

Notation. \( \mathcal{G}_T(\mathbb{R}^d) := \{ S_2(z)(0,\cdot) : z \in C^{1-\text{var}}([0,T], \mathbb{R}^d) \} \).

Definition 2.11. The geometric \( p \)-rough paths metric space \( \Omega_{p,T}(\mathbb{R}^d) \) is the closure of \( \mathcal{G}_T(\mathbb{R}^d) \) in \( C^{p-\text{var}}([0,T], \mathbb{R}^d) \times C^{p/2-\text{var}}([0,T], \mathcal{M}_d(\mathbb{R})) \).

Definition 2.12. For \( z \in C^{p-\text{var}}([0,T], \mathbb{R}^d) \), a map \( y \in C^{p-\text{var}}([0,T], \mathcal{M}_{e,d}(\mathbb{R})) \) is controlled by \( z \) if and only if there exists \( y' \in C^{p-\text{var}}([0,T], \mathcal{L}(\mathbb{R}^d, \mathcal{M}_{e,d}(\mathbb{R}))) \) such that

\[
y(s,t) = y'(s)z(s,t) + R_y(s,t) : \forall (s,t) \in \Delta_T
\]

with \( \|R_y\|_{p/2-\text{var},T} < \infty \). For fixed \( z \), the pairs \((y, y')\) as above define a vector space denoted by \( \mathcal{D}^p_z([0,T], \mathcal{M}_{e,d}(\mathbb{R})) \) and equipped with the semi-norm \( \|\cdot\|_{z,p/2,T} \) such that

\[
\|(y, y')\|_{z,p/2,T} := \|y'\|_{p-\text{var},T} + \|R_y\|_{p/2-\text{var},T}
\]

for every \((y, y') \in \mathcal{D}^p_z([0,T], \mathcal{M}_{e,d}(\mathbb{R})) \).

Theorem 2.13. (Rough integral) Consider \( z := (z, Z) \in \Omega_{p,T}(\mathbb{R}^d) \) and \((y, y') \in \mathcal{D}^p_z([0,T], \mathcal{M}_{e,d}(\mathbb{R})) \). For every \( n \in \mathbb{N}^* \) and \((t_k^n)_{k \in [1,n]} \in \mathcal{D}_{[0,T]} \), the limit

\[
\lim_{n \to +\infty} \sum_{k=1}^{n-1} (y(t_k^n)z(t_k^n, t_{k+1}^n) + y'(t_k^n)Z(t_k^n, t_{k+1}^n))
\]

exists and does not depend on the dissection \((t_k^n)_{k \in [1,n]} \). That limit is denoted by

\[
\int_0^T y(s)dz(s)
\]

and called rough integral of \( y \) with respect to \( z \) on \([0,T] \). Moreover,

1. There exists a constant \( c(p) > 0 \), depending only on \( p \), such that for every \((s,t) \in \Delta_T \),

\[
\left\| \int_s^t y(u)dz(u) - y(s)z(s,t) - y'(s)Z(s,t) \right\| \leq c(p)(\|z\|_{p-\text{var},s,t} \|R_y\|_{p/2-\text{var},s,t} + \|Z\|_{p/2-\text{var},s,t} \|y'\|_{p-\text{var},s,t}).
\]

2. The map

\[
(y, y') \mapsto \left( \int_0^T y(s)dz(s), y \right)
\]

is continuous from \( \mathcal{D}^p_z([0,T], \mathcal{M}_{e,d}(\mathbb{R})) \) into \( \mathcal{D}^p_z([0,T], \mathbb{R}^e) \).
Remark. For every $t$ on the other hand, also by Proposition 2.7, for any $\varepsilon > 0$ such that $p + \varepsilon \in [2, 3]$, 
\[
\lim_{n \to \infty} \| (y_n', R_{y_n}) - (y', R_y) \|_{z, (p + \varepsilon)/2, T} = 0.
\]
By Proposition 2.14, 
\[
y' \in C^{p, \text{var}}([0, T], \mathcal{L}(\mathbb{R}^d, \mathcal{M}_{e,d}(\mathbb{R}))) \text{ and } R_y \in C^{p/2, \text{var}}([0, T], \mathcal{M}_{e,d}(\mathbb{R})).
\]
So, $(y', y) \in \mathcal{D}_{z}^{p/2}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$.

On the other hand, also by Proposition 2.7 for any $\varepsilon > 0$ such that $p + \varepsilon \in [2, 3]$, 
\[
\lim_{n \to \infty} \left\| \int_0^t y_n(s) \, dz(s) - \int_0^t y(s) \, dz(s) \right\|_{z, (p + \varepsilon)/2, T} = 0.
\]
Therefore, in particular:
\[
\lim_{n \to \infty} \left\| \int_0^t y_n(s) \, dz(s) - \int_0^t y(s) \, dz(s) \right\|_{z, (p + \varepsilon)/2, T} = 0.
\]

\[\square\]

Proposition 2.15. Consider $z := (z, \bar{z}) \in C^{1,2}([0, T], \mathbb{R}^d)$, $(x, x') \in \mathcal{D}_{z}^{p/2}([0, T], \mathcal{L}(\mathbb{R}^d, \mathcal{M}_{e,d}(\mathbb{R})))$ and $\varphi \in \operatorname{Lip}^{-1}(\mathbb{R}^e, \mathbb{R}^d)$. The couple of maps $(\varphi(x), \varphi(x'))$, defined by
\[
\varphi(x)(t) := \varphi(x(t)) \quad \text{and} \quad \varphi(x')(t) := D\varphi(x(t))x'(t)
\]
for every $t \in [0, T]$, belongs to $\mathcal{D}_{z}^{p/2}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$.

Remark. By Theorem 2.13 and Proposition 2.14 together,
\[
\int_0^t \varphi(x(u)) \, dz(u)
\]
is defined. For every $(s, t) \in \Delta_T$, consider
\[
\mathcal{J}_{\varphi, 0, s, t} := \left\| \int_s^t \varphi(x(u)) \, dz(u) - \varphi(x(s))z(s, t) - D\varphi(x(s))x'(s)z(s, t) \right\|.
\]
For every \((s, t) \in \Delta_T\), since

\[
\|\varphi(x)\|_{p\text{-var},s,t} \leq \|\varphi\|_{\text{Lip}^{-1}} \|x\|_{p\text{-var},s,t},
\]

\[
\|\varphi(x')\|_{p\text{-var},s,t} \leq \|\varphi\|_{\text{Lip}^{-1}} (\|x'\|_{p\text{-var},s,t} + \|x'\|_{\infty,s,t} \|x\|_{p\text{-var},s,t}) \quad \text{and}
\]

\[
\|R\varphi(x)\|_{p/2\text{-var},s,t} \leq \|\varphi\|_{\text{Lip}^{-1}} (\|x\|_{p\text{-var},s,t} + \|R\varphi\|_{p/2\text{-var},s,t}),
\]

by Theorem 2.13,

\[
\mathcal{I}_{\varphi,z}(s, t) \leq c(p) \|\varphi\|_{\text{Lip}^{-1}} (\|x'\|_{p\text{-var},s,t} + \|x'\|_{\infty,s,t} \|x\|_{p\text{-var},s,t} + \|x\|_{p\text{-var},s,t} + \|R\varphi\|_{p/2\text{-var},s,t}) \omega_{\varphi,z}(s, t)^{1/p},
\]

where \(\omega_{\varphi,z} : \Delta_T \to \mathbb{R}^+\) is the control function defined by

\[
\omega_{\varphi,z}(u, v) := 2^{p-1}(\|z\|_{p\text{-var},u,v} + \|Z\|_{p/2\text{-var},u,v}) \quad \forall (u, v) \in \Delta_T.
\]

**Proposition 2.16.** Consider \(z := (z, Z) \in G\Omega_{p,T}(\mathbb{R}^d)\), \(\varphi \in \text{Lip}^{-1}(\mathbb{R}^c, \mathbb{R}^d)\), a continuous map

\[
(x, x') : [0, T] \to \mathbb{R}^c \times \mathcal{L}(\mathbb{R}^d, \mathbb{R}^c),
\]

and a sequence \((x_n, x_n')_{n \in \mathbb{N}}\) of elements of \(\mathcal{D}^{p/2}_z([0, T], \mathbb{R}^c)\) such that

\[
(x_n', Rz_n) \xrightarrow{d_{\infty,T}} (x', Rz) \quad \text{and} \quad \sup_{n \in \mathbb{N}} \| (x_n, x_n') \|_{z,p/2,T} < \infty.
\]

Then, \((\varphi(x), \varphi(x')) \in \mathcal{D}^{p/2}_z([0, T], \mathcal{M}_{c,d}(\mathbb{R}))\) and

\[
\lim_{n \to \infty} \left\| \int_0^T \varphi(x_n(s))dz(s) - \int_0^T \varphi(x(s))dz(s) \right\|_{\infty,T} = 0.
\]

**Proof.** Since

\[
(x_n', Rz_n) \xrightarrow{d_{\infty,T}} (x', Rz) \quad \text{and} \quad \sup_{n \in \mathbb{N}} \| (x_n, x_n') \|_{z,p/2,T} < \infty,
\]

by Friz and Hairer [11] Theorem 7.5 together with Proposition 2.14

\[
(\varphi(x_n'), R_{\varphi(z_n)}) \xrightarrow{d_{\infty,T}} (\varphi(x'), R_{\varphi(z)}) \quad \text{and} \quad \sup_{n \in \mathbb{N}} \| (\varphi(x_n), \varphi(x_n')) \|_{z,p/2,T} < \infty.
\]

So, by Proposition 2.13

\[
\lim_{n \to \infty} \left\| \int_0^T \varphi(x_n(s))dz(s) - \int_0^T \varphi(x(s))dz(s) \right\|_{\infty,T} = 0.
\]

\[\square\]

3. EXISTENCE OF SOLUTIONS

The existence of a solution to Problem (3) is established in Theorem 3.1 when \(p \in [1, 2]\), and in Theorem 3.2 when \(p \in [2, 3]\).

**Theorem 3.1.** Under Assumption 1.1, if \(p \in [1, 2]\), Problem (3) has at least one solution which belongs to \(C^{p\text{-var}}([0, T], \mathbb{R}^c)\).

**Proof.** Consider the discrete scheme

\[
\begin{aligned}
X_n(t) &= H_n(t) + Y_n(t) \\
H_n(t) &= \int_0^t \frac{f(X_{n-1}(s))}{dDY_n(s)} ds
\end{aligned}
\]

where \(dDY_n(t) \in N_{C_{H_n}}(t)(Y_n(t))|DY_n|\)-a.e. with \(Y_n(0) = a\).
for Problem (2), initialized by

\[
\frac{dX_0}{d\mathcal{T}_0}(t) \in N_{C(t)}(X_0(t)) \text{ } |DX_0|\text{-a.e.}
\]

\[X_0(0) = a.\]

Since the map \(\|Z\|_{p,\text{var},0,0}\) is continuous from \([0,T]\) into \(\mathbb{R}_+\), and since \(\|Z\|_{p,\text{var},0,0} = 0\), there exists \(\tau_0 \in [0,T]\) such that

\[
\|Z\|_{p,\text{var},\tau_0} \leq \frac{m}{c(p,p)\|f\|_{\text{Lip}}^m(m + M + 1)}.
\]

where \(m := R/2\) and \(M := \mathcal{M}(R/2)\) (see Proposition 2.5(2)). Let us show that for every \(n \in \mathbb{N}\),

\[
\begin{cases}
\|X_n\|_{p,\text{var},\tau_0} \leq m + M \\
\|H_n\|_{p,\text{var},\tau_0} \leq m \\
\|Y_n\|_{1,\text{var},\tau_0} \leq M.
\end{cases}
\]

By (6) together with Proposition 2.5

\[
\|X_0\|_{p,\text{var},\tau_0} \leq M.
\]

Assume that Condition (7) is satisfied for \(n \in \mathbb{N}\) arbitrarily chosen. By Proposition 2.8 and since \(\|Z\|_{p,\text{var},0,0}\) is an increasing map,

\[
\|H_{n+1}\|_{p,\text{var},\tau_0} \leq c(p,p)\|Z\|_{p,\text{var},\tau_0}(\|Df\|_\infty\|X_n\|_{p,\text{var},\tau_0} + \|f \circ X_n\|_\infty,\tau_0) \\
\leq \mu c(p,p)\|f\|_{\text{Lip}}^m(m + M + 1) \\
\leq m.
\]

Since \(Y_{n+1} = w_{H_{n+1}}\), by Proposition 2.5

\[
\|Y_{n+1}\|_{1,\text{var},\tau_0} \leq M.
\]

Therefore,

\[
\|X_{n+1}\|_{p,\text{var},\tau_0} \leq \|H_{n+1}\|_{p,\text{var},\tau_0} + \|Y_{n+1}\|_{p,\text{var},\tau_0} \leq m + M.
\]

By induction, (7) is satisfied for every \(n \in \mathbb{N}\).

For every \(t \in [0,T]\), the map \(\|Z\|_{p,\text{var},t}\) is continuous from \([t,T]\) into \(\mathbb{R}_+\) and \(\|Z\|_{p,\text{var},t,t} = 0\). Moreover, the constant \(\mu\) depends only on \(p, m, M\) and \(\|f\|_{\text{Lip}}^m\).

So, since \([0,T]\) is compact, there exist \(N \in \mathbb{N}^+\) and \((\tau_k)_{k \in [0,N]} \in \mathcal{D}_{[\tau_0,T]}\) such that

\[
\|Z\|_{p,\text{var},\tau_k,\tau_{k+1}} \leq \mu ; \forall k \in [0, N - 1].
\]

Since for every \(n \in \mathbb{N}^+\) the maps

\[
(s,t) \in \Delta_T \mapsto \|X_n\|_{p,\text{var},s,t},
\]

\[
(s,t) \in \Delta_T \mapsto \|H_n\|_{p,\text{var},s,t}
\]

are control functions, recursively, the sequence \((H_n, X_n, Y_n)_{n \in \mathbb{N}}^*\) is bounded in \(C^{p,1}_\mathcal{T} : = C^{p,\text{var}}([0,T], \mathbb{R}^e) \times C^{p,\text{var}}([0,T], \mathbb{R}^e) \times C^{1,\text{var}}([0,T], \mathbb{R}^e).\)

By Proposition 2.8 for every \(n \in \mathbb{N}^+\) and \((s,t) \in \Delta_T,

\[
\|H_n(t) - H_n(s)\| \leq c(p,p) \left(\|Df\|_\infty \sup_{n \in \mathbb{N}} \|X_n\|_{p,\text{var},T} + \|f\|_\infty\right) \|Z\|_{p,\text{var},s,t}.
\]

Since \((s,t) \in \Delta_T \mapsto \|Z\|_{p,\text{var},s,t}\) is a continuous map such that \(\|Z\|_{p,\text{var},t,t} = 0\) for every \(t \in [0,T]\), \((H_n)_{n \in \mathbb{N}^+}\) is equicontinuous. Therefore, by Arzelà-Ascoli’s theorem together with Proposition 2.7, there exists an extraction \(\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*\) such that \((H_{\varphi(n)})_{n \in \mathbb{N}^*}\) converges uniformly to an element \(H\) of \(C^{p,\text{var}}([0,T], \mathbb{R}^e).\)
Consider the discrete scheme under Assumption 1.1, if Theorem 3.2.\(G\)
First of all, let us control the solution of the discrete scheme for \(n \in \mathbb{N}^*\), defined in the sequel.
Therefore, since \((Z, c)\) and the positive constants \(\tau\), there exists \(\|X(t)\|_{L^2} \leq C_{\tau}\) by (9) together with Proposition 2.7,\(\|X(t)\|_{L^2} \leq C_{\tau}\) converges uniformly to \(X\), by Theorem 2.4.
Moreover, since \((X_n(t))_{n \in \mathbb{N}^*}\) converges uniformly to \(H\) in \(C^0([0, T], \mathbb{R}^d)\),
\[\lim_{n \to \infty} H(\varphi_n(t)) - \left(\int_0^t f(X(s))dZ(s)\right) = 0.\]
Therefore, since \((H(\varphi_n))_{n \in \mathbb{N}^*}\) converges also to \(H\) in \(C^0([0, T], \mathbb{R}^d)\),
\[H(t) = \int_0^t f(X(s))dZ(s) ; \forall t \in [0, T].\]
\[\square\]
In the sequel, assume that there exists \(Z : [0, T] \to \mathcal{M}_d(\mathbb{R})\) such that \(Z := (Z, \overline{Z}) \in G\Omega_p, T(\mathbb{R}^d)\).

**Theorem 3.2.** Under Assumption 1.1, if \(p \in [2, 3],\) Problem (2) has at least one solution which belongs to \(C^p([0, T], \mathbb{R}^e)\).

**Proof.** Consider the discrete scheme
\[
\begin{cases}
X_n(t) = H_n(t) + Y_n(t) \\
H_n(t) = \int_0^t f(X_{n-1}(s))dZ(s) \\
\frac{dDY_n}{d|DY_n|}(t) \in N_{C^0_n}(Y_n(t)) |DY_n|-a.e. \text{ with } Y_n(0) = a
\end{cases}
\]
for Problem (2), initialized by
\[
\begin{cases}
\frac{dDX_0}{d|DX_0|}(t) \in N_{C^0}(X_0(t)) |DX_0|-a.e. \\
X_0(0) = a.
\end{cases}
\]
Since the map \(\omega_p, Z(0, )\) is continuous from \([0, T]\) into \(\mathbb{R}^+_\), and since \(\omega_p, Z(0, 0) = 0,\) there exists \(\tau_0 \in [0, T]\) such that
\[
\omega_p, Z(0, \tau_0) \leq \frac{m_C}{c(p, 1)^p \|f\|_{\text{Lip}^p}(M + 1)^p} \wedge \frac{m_C}{c_2 \vee c_6(1 + \mu_C + M_R + M_R^2)^p} \wedge \frac{1}{1 + \mu_C + M_R^2 + M_R^2},
\]
where \(m_C := R/2, M_C := \mathfrak{M}(R/2), \mu_C := m_C + M_C,\)
\[M_R := (c_1 M_C^{4/p}) \vee (c_5(\mu_C^p + 1)^{1/p})\]
and the positive constants \(c_1, c_2, c_3\) and \(c_6,\) depending only on \(p\) and \(\|f\|_{\text{Lip}^p},\) are defined in the sequel.

First of all, let us control the solution of the discrete scheme for \(n \in \{0, 1\}:\n- (n = 0) By (2) together with Proposition 2.5,
\[\|X_0\|_{1, \text{var}, \tau_0} \leq M_C.\]
• (n = 1) Since \(X_0 \in C^{1\text{-var}}([0, T], \mathbb{R}^e)\), by Proposition 2.8,
\[
\|H_1\|_{p\text{-var}, \tau_0} \leq c(p, 1) \omega_p Z(0, \tau_0)^{1/p} \|f\|_{\text{Lip}^p}(M_C + 1) \\
\leq m_C.
\]

Since \(Y_1 = wH_1\), by Proposition 2.5
\[
\|Y_1\|_{1\text{-var}, \tau_0} \leq M_C.
\]

Therefore,
\[
\|X_1\|_{p\text{-var}, \tau_0} \leq \|H_1\|_{p\text{-var}, \tau_0} + \|Y_1\|_{p\text{-var}, \tau_0} \leq \mu_C.
\]

Let us show that for every \(n \in \mathbb{N}\backslash\{0, 1\}\),
\[
(X_{n-1}, f(X_{n-2})) \in \mathcal{D}^{p/2}_Z([0, \tau_0], \mathbb{R}^e)
\]
and
\[
\begin{align*}
\|X_n\|_{p\text{-var}, \tau_0} & \leq \mu_C \\
\|H_n\|_{p\text{-var}, \tau_0} & \leq m_C \\
\|Y_n\|_{1\text{-var}, \tau_0} & \leq M_C \\
\|R_{X_n}\|_{p/2\text{-var}, \tau_0} & \leq M_R.
\end{align*}
\]

Set \(X'_1 := f(X_0)\). For every \((s, t) \in \Delta_{\tau_0}\),
\[
R_{X_1}(s, t) = X_1(s, t) - X'_1(s)Z(s, t)
\]
\[
= Y_1(s, t) + \int_s^t f(X_0(u))dZ(u) - f(X_0(s))Z(s, t).
\]

By Young-Love estimate (see Friz and Victoir [11] Theorem 6.8, or [7] Section 3.6 and the interesting historical notes pages 212-213), for every \((s, t) \in \Delta_{\tau_0}\),
\[
\|R_{X_1}(s, t)\| \leq \|Y_1\|_{p/2\text{-var}, s, t} + \frac{1}{1 - 2^{1-3/p}} \|f\|_{\text{Lip}^p} \|Z\|_{p\text{-var}, \tau_0} \|X_0\|_{p/2\text{-var}, s, t}.
\]

By super-additivity of the control functions \(\|Y_1\|_{p/2\text{-var}, .}\) and \(\|X_0\|_{p/2\text{-var}, .}\), there exists a constant \(c_1 > 0\), depending only on \(p\) and \(\|f\|_{\text{Lip}^p}\), such that
\[
\|R_{X_1}\|_{p/2\text{-var}, \tau_0} \leq c_1 (\|Y_1\|_{p/2\text{-var}, \tau_0} + \|Z\|_{p/2\text{-var}, \tau_0} \|X_0\|_{p/2\text{-var}, \tau_0})^{2/p} \\
\leq c_1 M_C(1 + \omega_p Z(0, \tau_0)^{1/2})^{2/p}.
\]

Then,
\[
\|R_{X_1}\|_{p/2\text{-var}, \tau_0} \leq c_1 M_C^{4/p} \leq M_R
\]
and
\[
(X_1, f(X_0)) \in \mathcal{D}^{p/2}_Z([0, \tau_0], \mathbb{R}^e).
\]

So, the rough integral
\[
H_2 := \int_0^1 f(X_1(u))dZ(u)
\]
is well defined. For every \((s, t) \in \Delta_T\),
\[
\|H_2(s, t)\| \leq \|f\|_{\text{Lip}^p} \|Z\|_{p\text{-var}, s, t} + \|f\|_{\text{Lip}^p}^2 \|Z\|_{p/2\text{-var}, s, t} + \mathcal{F}_{Z, X_1}(s, t) \\
\leq (\|f\|_{\text{Lip}^p} \vee \|f\|_{\text{Lip}^p}^2) \omega_p Z(s, t)^{1/p} \\
+ c(p) \|f\|_{\text{Lip}^p}(\|f\|_{\text{Lip}^p} + 1) \|X_0\|_{p\text{-var}, s, t} + \|X_1\|_{p\text{-var}, s, t} \\
+ \|X_1\|_{p/2\text{-var}, s, t} + \|R_{X_1}\|_{p/2\text{-var}, s, t} \omega_p Z(s, t)^{1/p} \\
\leq c_2(1 + \mu_C + M_R + \mu_C^2) \omega_p Z(s, t)^{1/p},
\]
where \(c_2 > 0\) is a constant depending only on \(p\) and \(\|f\|_{\text{Lip}^p}\). By super-additivity of the control function \(\omega_p Z\):
\[
\|H_2\|_{p\text{-var}, \tau_0} \leq c_2(1 + \mu_C + M_R + \mu_C^2) \omega_p Z(0, \tau_0)^{1/p} \\
\leq m_C.
\]
So, by Proposition 2.5
\[ \|Y_2\|_{p\text{-var}, \tau_0} \leq M_C \]
and
\[ \|X_2\|_{p\text{-var}, \tau_0} \leq \|H_2\|_{p\text{-var}, \tau_0} + \|Y_2\|_{p\text{-var}, \tau_0} \leq \mu_C. \]
Therefore, Conditions (10)-(11) hold true for \( n = 2 \).

Assume that Conditions (10)-(11) hold true until \( n \in \mathbb{N}\setminus\{0, 1\} \) arbitrarily chosen. Set \( X'_n := f(X_{n-1}) \). For every \((s, t) \in \Delta\tau_0\),
\[
R_{X_n}(s, t) = X_n(s, t) - X'_n(s)Z(s, t)
= Y_n(s, t) + \int_s^t f(X_{n-1}(u))dZ(u) - f(X_{n-1}(s))Z(s, t).
\]
So, for every \((s, t) \in \Delta\tau_0\),
\[
\|R_{X_n}(s, t)\| \leq \|Y_n(s, t)\| + \|DF(X_{n-1}(s))f(X_{n-2}(s))Z(s, t)\| + f_{\text{suff}}Z_{X_{n-1}}(s, t)
\leq \|Y_n\|_{p/2\text{-var}, s, t} + \|f\|_{2\text{Lip}}^2 \|Z\|_{p/2\text{-var}, s, t}
+ \|f\|_{p/2\text{var}, s, t} \|R_{f(X_{n-1})}\|_{p/2\text{var}, s, t}
+ \|Z\|_{p/2\text{var}, s, t} \|DF(X_{n-1})f(X_{n-2})\|_{p\text{-var}, s, t}
\leq \|Y_n\|_{p/2\text{-var}, s, t} + \|f\|_{2\text{Lip}}^2 \|Z\|_{p/2\text{-var}, s, t}
+ \|f\|_{p/2\text{var}, s, t} \|Z\|_{p\text{-var}, \tau_0} \omega_n(s, t)^{2/p} + M(n, \tau_0) \|Z\|_{p/2\text{-var}, s, t},
\]
where
\[
M(n, \tau_0) := \|f(X_{n-2})\|_{p\text{-var}, \tau_0} + \|f(X_{n-2})\|_{\infty, \tau_0} X_{n-1} \|_{\text{p-var, } \tau_0}
\leq \|f\|_{2\text{Lip}} \left( \|X_{n-2}\|_{p\text{-var}, \tau_0} + \|X_{n-1}\|_{\text{p-var, } \tau_0} \right)
\]
and \( \omega_n : \Delta\tau_0 \to \mathbb{R}^+ \) is the control function defined by
\[
\omega_n(u, v) := 2^{p/2-1} \left( \|X_{n-1}\|_{p\text{-var}, u, v} + \|R_{X_{n-1}}\|_{p/2\text{-var}, u, v} \right)
\]
for every \((u, v) \in \Delta\tau_0\). By super-additivity of the control functions
\[
\|Y_{n}\|_{p/2\text{-var}, s, t} \|Z\|_{p/2\text{-var}, s, t} \text{ and } \omega_n,
\]
there exist three constants \( c_3, c_4, c_5 > 0 \), depending only on \( p \) and \( \|f\|_{\text{Lip}}^\gamma \), such that
\[
\|R_{X_n}\|_{p/2\text{-var}, \tau_0} \leq c_3(\|Y_{n}\|_{p/2\text{-var}, \tau_0} \|Z\|_{p/2\text{-var}, \tau_0}^{p/2} \omega_n(0, \tau_0) + M(n, \tau_0) \|Z\|_{p/2\text{-var}, \tau_0}^{p/2})^{2/p}
\leq c_4(\|Y_{n}\|_{p/2\text{-var}, \tau_0} + (1 + \omega_n(0, \tau_0)) + M(n, \tau_0) \|Z\|_{p/2\text{-var}, \tau_0})^{1/p}
\leq c_5(\mu_C^p + (1 + \mu_C^p) \|R_{X_{n-1}}\|_{p/2\text{-var}, \tau_0} + \mu_{Z_{n}}^p \omega_n(0, \tau_0))^{1/p}
\leq c_5(\mu_C^p + (1 + \mu_C^p) + \mu_C^p \omega_n(0, \tau_0))^{1/p}.
\]
Then, \( \|R_{X_n}\|_{p/2\text{-var}, \tau_0} \leq c_5(\mu_C^p + 1)^{1/p} \leq M_R \) and
\[
(X_n, f(X_{n-1})) \in \mathcal{D}_{\mathbb{Z}}^{p/2}([0, \tau_0], \mathbb{R}^e).
\]
So, the rough integral
\[
H_{n+1} := \int_0^t f(X(s))dZ(s)
\]
is well defined. For every \((s,t) \in \Delta_T\),

\[
\|H_{n+1}(s,t)\| \leq \|f\|_{\text{Lip}^p} \|Z\|_{p\text{-var},s,t} + \|f\|_{\text{Lip}^p}^2 \|Z\|_{p/2\text{-var},s,t} + 3f,\mathbb{Z},X_n(s,t)
\]

\[
\leq (\|f\|_{\text{Lip}^p} \vee \|f\|_{\text{Lip}^p}^2) \omega_p,\mathbb{Z}(s,t)^{1/p}
+ c(p) \|f\|_{\text{Lip}^p}(1 \vee \|f\|_{\text{Lip}^p})(\|X_{n-1}\|_{p\text{-var},s,t} + \|X_n\|_{p\text{-var},s,t}
+ \|X_n\|_{p\text{-var},s,t}^2 + \|R_{X_n}\|_{p/2\text{-var},s,t}) \omega_p,\mathbb{Z}(s,t)^{1/p}
\]

\[
\leq c_0(1 + \mu_C + M_R + \mu_C^2) \omega_p,\mathbb{Z}(s,t)^{1/p},
\]

where \(c_0 > 0\) is a constant depending only on \(p\) and \(\|f\|_{\text{Lip}^p}\). By super-additivity of the control function \(\omega_p,\mathbb{Z}\):

\[
\|H_{n+1}\|_{p\text{-var},\tau_0} \leq c_0(1 + \mu_C + M_R + \mu_C^2) \omega_p,\mathbb{Z}(0,\tau_0)^{1/p}
\leq m_C.
\]

So, by Proposition \ref{prop:superadditivity}:

\[
\|Y_{n+1}\|_{p\text{-var},\tau_0} \leq M_C
\]

and

\[
\|X_{n+1}\|_{p\text{-var},\tau_0} \leq \|H_{n+1}\|_{p\text{-var},\tau_0} + \|Y_{n+1}\|_{p\text{-var},\tau_0} \leq M_C.
\]

By induction, Conditions \ref{cond:1} - \ref{cond:2} are satisfied for every \(n \in \mathbb{N} \setminus \{0,1\}\). As in the proof of Theorem \ref{thm:uniform_convergence}, the sequence \((H_n,X_n,Y_n)_{n \in \mathbb{N} \setminus \{0,1\}}\) is bounded in \(\mathcal{E}_T^{1,1}\). In addition, the sequence \((R_{X_n})_{n \in \mathbb{N} \setminus \{0,1\}}\) is bounded in \(C^{p/2\text{-var}}([0,T],\mathbb{R}^c)\).

For every \(n \in \mathbb{N} \setminus \{0,1\}\) and \((s,t) \in \Delta_T\),

\[
\|H_n(s,t)\| \leq (\|f\|_{\text{Lip}^p} \vee \|f\|_{\text{Lip}^p}^2)(\sup_{n \in \mathbb{N}} \|X_{n-2}\|_{p\text{-var},T} + \|X_{n-1}\|_{p\text{-var},T}
+ \sup_{n \in \mathbb{N}} \|X_{n-1}\|_{p\text{-var},T} + \sup_{n \in \mathbb{N}} \|R_{X_{n-1}}\|_{p/2\text{-var},T}) \omega_p,\mathbb{Z}(s,t)^{1/p},
\]

Since \(\omega_p,\mathbb{Z}\) is a control function, \((H_n)_{n \in \mathbb{N} \setminus \{0,1\}}\) is equicontinuous. Therefore, by Arzelà-Ascoli’s theorem together with Proposition \ref{prop:uniform_convergence} there exists an extraction \(\varphi : \mathbb{N} \setminus \{0,1\} \to \mathbb{N} \setminus \{0,1\}\) such that \((H_{\varphi(n)})_{n \in \mathbb{N} \setminus \{0,1\}}\) converges uniformly to an element \(H\) of \(C^{p\text{-var}}([0,T],\mathbb{R}^c)\).

Since \((H_{\varphi(n)})_{n \in \mathbb{N} \setminus \{0,1\}}\) converges uniformly to \(H\), by Theorem \ref{thm:uniform_convergence} the sequence \((X_{\varphi(n)},Y_{\varphi(n)})_{n \in \mathbb{N} \setminus \{0,1\}}\) converges uniformly to \((X,Y) := (\varphi_H,\varphi_W)\). So, for every \(t \in [0,T]\),

\[
\begin{aligned}
X(t) &= H(t) + Y(t) \\
\frac{dY}{dt}(t) &\in N_{C_H(t)}(Y(t)) \text{ if } |DY|\text{-a.e. with } Y(0) = a,
\end{aligned}
\]

and by Proposition \ref{prop:existence}

\[
X \in C^{p\text{-var}}([0,T],\mathbb{R}^c) \text{ and } Y \in C^{1\text{-var}}([0,T],\mathbb{R}^c).
\]

Denoting \(X' := f(X)\), \(X'\) (resp. \(R_X\)) is the uniform limit of \((X_{\varphi(n)})_{n \in \mathbb{N} \setminus \{0,1\}}\) (resp. \((R_{X_{\varphi(n)}})_{n \in \mathbb{N} \setminus \{0,1\}}\)). So, by Proposition \ref{prop:uniform_convergence}

\[
\lim_{n \to \infty} \left\| H_{\varphi(n)} - \int_0^t f(X(s))d\mathbb{Z}(s) \right\|_{\infty,T} = 0.
\]

Therefore, since \((H_{\varphi(n)})_{n \in \mathbb{N}}\) converges also to \(H\) in \(C^0([0,T],\mathbb{R}^c)\),

\[
H(t) = \int_0^t f(X(s))d\mathbb{Z}(s) \quad \forall t \in [0,T].
\]

\(\square\)
4. Some uniqueness results

When \( p = 1 \) and there is an additive continuous signal of finite \( q \)-variation with \( q \in [1, 3] \), the uniqueness of the solution to Problem (2) is established in Proposition 4.1 below. Proposition 4.2 and Proposition 4.3 provide necessary and sufficient conditions for uniqueness of the solution when \( p \in [1, 2] \) and \( p \in [2, 3] \) respectively. These conditions are close to the monotonicity of the normal cone which allows to prove the uniqueness when \( p = 1 \) (see Proposition 4.1).

**Proposition 4.1.** Assume that \( p = 1 \) and consider the Skorokhod problem

\[
\begin{align*}
X(t) &= H(t) + Y(t) \\
H(t) &= \int_0^t f(X(s))dZ(s) + W(t) \\
&- \frac{dDY}{dDY}(t) \in N_{Ch_t}(Y(t)) \text{ a.e. with } Y(0) = a,
\end{align*}
\]

where \( W \in C^{q-\text{var}}([0, T], \mathbb{R}^n) \) with \( q \in [1, 3] \). Under Assumption 1.1, Problem (12) has a unique solution which belongs to \( C^{q-\text{var}}([0, T], \mathbb{R}^n) \).

**Proof.** Consider two solutions \((X, Y)\) and \((X^*, Y^*)\) of Problem (2) on \([0, T]\). Since \((s,t) \in \Delta_T \mapsto \|Z\|_{1-\text{var}, s,t}\) is a control function, there exists \( n \in \mathbb{N}^* \) and \((\tau_k)_{k \in [0,n]} \) in \( \mathcal{D}_{[0,T]} \) such that

\[
\|Z\|_{1-\text{var}, \tau_k, \tau_{k+1}} \leq M := \frac{1}{4\|f\|_{\text{Lip}^q}}; \quad \forall k \in [0,n-1].
\]

For every \( t \in [0, \tau_1] \),

\[
\|X(t) - X^*(t)\|^2 = \|H(t) - H^*(t)\|^2 + 2\int_0^t \langle Y(s) - Y^*(s), d(Y - Y^*)(s) \rangle \\
+ 2\int_0^t \langle H(t) - H^*(t), d(Y - Y^*)(s) \rangle \\
\leq m_1(\tau_1)^2 + 2m_2(t) + 2m_3(t),
\]

with \( m_1(\tau_1) := \|H - H^*\|_{\infty, \tau_1} \),

\[
m_2(t) := \int_0^t \langle X(s) - X^*(s), d(Y - Y^*)(s) \rangle,
\]

and

\[
m_3(t) := \int_0^t \langle H(t) - H^*(t) - (H(s) - H^*(s)), d(Y - Y^*)(s) \rangle.
\]

Consider \( t \in [0, \tau_1] \). By Friz and Victoir [11], Proposition 2.2:

\[
\|H(t) - H^*(t)\| = \left\| \int_0^t (f(X(s)) - f(X^*(s)))dZ(s) \right\| \\
\leq \|f\|_{\text{Lip}^q}\|X - X^*\|_{\infty, \tau_1}\|Z\|_{1-\text{var}, \tau_1}.
\]

So,

\[
m_1(\tau_1) \leq \frac{1}{4}\|X - X^*\|_{\infty, \tau_1}.
\]

Since the map \( x \in C(t) \mapsto N_{Ch_t}(x) \) is monotone, \( m_2(t) \leq 0 \). By the integration by parts formula,

\[
m_3(t) = \int_0^t \langle Y(s) - Y^*(s), d(H - H^*)(s) \rangle \\
= \int_0^t \langle X(s) - X^*(s) - (H(s) - H^*(s)), (f(X(s)) - f(X^*(s)))dZ(s) \rangle.
\]
Therefore, by Friz and Victoir [11], Proposition 2.2 and Inequality (13),

\[ m_3(t) \leq \|DF\|_\infty \|X - X^*\|_{\infty, \tau_1} (\|X - X^*\|_{\infty, \tau_1} + \|H - H^*\|_{\infty, \tau_1}) \|Z\|_{1, \text{var}, \tau_1} \]

\[ \leq \|f\|_{\text{Lip}} \|Z\|_{1, \text{var}, \tau_1} (1 + \|f\|_{\text{Lip}} \|Z\|_{1, \text{var}, \tau_1}) \|X - X^*\|^2_{\infty, \tau_1} \]

\[ \leq 5/16 \|X - X^*\|^2_{\infty, \tau_1}. \]

Therefore,

\[ \|X - X^*\|^2_{\infty, \tau_1} \leq \frac{11}{16} \|X - X^*\|^2_{\infty, \tau_1}. \]

Necessarily, \((X, Y) = (X^*, Y^*)\) on \([0, \tau_1]\).

For \(k \in [0, n - 1]\), assume that \((X, Y) = (X^*, Y^*)\) on \([0, \tau_k]\). By Equation (13) and exactly the same ideas as on \([0, \tau_1]\):

\[ \|X - X^*\|^2_{\infty, \tau_k, \tau_{k+1}} \leq \frac{11}{16} \|X - X^*\|^2_{\infty, \tau_k, \tau_{k+1}}. \]

So, \((X, Y) = (X^*, Y^*)\) on \([0, \tau_{k+1}]\). Recursively, Problem (2) has a unique solution on \([0, T]\).

\[ \square \]

**Remark.** The cornerstone of the proof of Proposition (14) is that

\[ \int_0^t \langle X(s) - X^*(s), d(Y - Y^*)(s) \rangle \leq 0 \quad \forall t \in [0, T]. \]

Thanks to the monotonicity of the map \(x \in C(t) \mapsto N_{C(t)}(x) (t \in [0, T])\), Inequality (15) is true. When \(p \in [1, 3]\), it is not possible to get inequalities involving only the uniform norm of \(X - X^*\). In that case, the construction of the Young/rough integral suggests to use ideas similar to those of the proof of Proposition (14) but using the \(p\)-variation norm of \(X - X^*\).

In a probabilistic setting, uniqueness up to equality almost everywhere can be obtained for Brownian motion, with \(p > 2\), in the frame of Itô calculus, using the martingale property of stochastic integrals and Doob’s inequality, see [24, 15, 23] for a fixed convex set \(C\) and [3, 4] for a moving set.

The two following propositions show that when \(p \in [1, 3]\), there exist some conditions close to Inequality (15), ensuring the uniqueness of the solution to Problem (2).

**Proposition 4.2.** Consider \((s, t) \in \Delta_T, p \in [1, 2]\) and two solutions \((X, Y)\) and \((X^*, Y^*)\) to Problem (2) under Assumption (14). On \([s, t], (X, Y) = (X^*, Y^*)\) if and only if \(X(s) = X^*(s)\) and

\[ \int_u^t \langle X(u, r) - X^*(u, r), d(Y - Y^*)(r) \rangle \leq 0 \quad \forall (u, v) \in \Delta_{s,t}. \]

**Proof.** For the sake of simplicity, the proposition is proved on \([0, T]\) instead of \([s, t]\), with \((s, t) \in \Delta_T\).

First of all, if \((X, Y) = (X^*, Y^*)\) on \([s, t]\), then

\[ \int_u^t \langle X(u, r) - X^*(u, r), d(Y - Y^*)(r) \rangle = 0 \quad \forall (u, v) \in \Delta_{s,t}. \]

Now, let us prove that if \(X(s) = X^*(s)\) and Inequality (16) is true, then \((X, Y) = (X^*, Y^*)\).
For every \((s, t) \in \Delta_T\),
\[
\|X(s, t) - X^*(s, t)\|^2 = \|H(s, t) - H^*(s, t)\|^2 \\
+ 2 \int_s^t \langle Y(s, u) - Y^*(s, u), d(Y - Y^*)(u) \rangle \\
+ 2 \int_s^t \langle H(s, t) - H^*(s, t), d(Y - Y^*)(u) \rangle \\
= \|H(s, t) - H^*(s, t)\|^2 \\
+ 2 \int_s^t \langle X(s, u) - X^*(s, u), d(Y - Y^*)(u) \rangle \\
+ 2 \int_s^t \langle H(s, t) - H(s, u) - (H^*(s, t) - H^*(s, u)), d(Y - Y^*)(u) \rangle \\
\leq \|H - H^*\|^2_{p\text{-var}, s, t} + 2m(s, t)
\]
with
\[
m(s, t) := \int_s^t \langle H(u, t) - H^*(u, t), d(Y - Y^*)(u) \rangle.
\]
Let \((s, t) \in \Delta_T\) be arbitrarily chosen.

On the one hand,
\[
m(s, t) \leq 2e \cdot c(p, p)\|H - H^*\|_{p\text{-var}, s, t}\|Y - Y^*\|_{p\text{-var}, s, t}.
\]
So, there exists a constant \(c_1 > 0\), not depending on \(s\) and \(t\), such that
\[
\|X(s, t) - X^*(s, t)\|^p \leq (\|H - H^*\|^2_{p\text{-var}, s, t} + 2m(s, t))^{p/2} \\
\leq c_1(\|H - H^*\|^p_{p\text{-var}, s, t} \\
+ (\|H - H^*\|^p_{p\text{-var}, s, t})^{1/2}(\|Y - Y^*\|^p_{p\text{-var}, s, t})^{1/2}).
\]
Since \(1/2 + 1/2 = 1\), the right-hand side of the previous inequality defines a control function (see Friz and Victoir [11], Exercice 1.9), and then there exists a constant \(c_2 > 0\), not depending on \(s\) and \(t\), such that
\[
\|X - X^*\|^p_{p\text{-var}, s, t} \leq c_1(\|H - H^*\|^p_{p\text{-var}, s, t} + \|H - H^*\|_{p\text{-var}, s, t}^2\|Y - Y^*\|_{p\text{-var}, s, t}^2) \\
(17) \leq c_2(\|H - H^*\|^p_{p\text{-var}, s, t} + \|H - H^*\|_{p\text{-var}, s, t}^2\|X - X^*\|_{p\text{-var}, s, t}^2).
\]
The right-hand side of the previous inequality defines a control function.

On the other hand, since \(X(0) = X^*(0)\):
\[
\|H - H^*\|_{p\text{-var}, s, t} \leq c(p, p)\|Z\|_{p\text{-var}, s, t}(\|f \circ X - f \circ X^*\|_{p\text{-var}, s, t} \\
+ \|f(X(s)) - f(X(0)) - (f(X^*(s)) - f(X^*(0)))\|) \\
\leq 2c(p, p)\|Z\|_{p\text{-var}, s, t}\|f \circ X - f \circ X^*\|_{p\text{-var}, t}.
\]
Consider \((u, v) \in \Delta_t\) and
\[
\delta(u, v) := \|(f \circ X)(u, v) - (f \circ X^*)(u, v)\|.
\]
Applying Taylor’s formula to the map \(f\) between \(X(u, v)\) and \(X^*(u, v)\):
\[
\delta(u, v) \leq \left\| \int_0^1 Df(X(u + \theta X(u, v))(X(u, v) - X^*(u, v))d\theta \right\| \\
+ \left\| \int_0^1 (Df(X(u + \theta X(u, v)) - Df(X^*(u) + \theta X^*(u), v))X^*(u, v)d\theta \right\| \\
\leq \|f\|_{\text{Lip}}(\|X - X^*\|_{p\text{-var}, u, v} + 2\|X^*\|_{p\text{-var}, u, v}\|X - X^*\|_{p\text{-var}, t}).
\]
So, there exists a constant $c_3 > 0$, not depending on $t$, such that
\[
\|f \circ X - f \circ X^*\|_{p\text{-var},t} \leq c_3 \|X - X^*\|_{p\text{-var},t},
\]
and then there exists a constant $c_4 > 0$, not depending on $s$ and $t$, such that
\[
\|H - H^*\|_{p\text{-var},s,t} \leq c_4 \|Z\|_{p\text{-var},s,t} \|X - X^*\|_{p\text{-var},t}.
\]
By Equation (17) and Equation (18) together, there exists a constant $c_5 > 0$, not depending on $s$ and $t$, such that
\[
\|X - X^*\|_{p\text{-var},s,t} \leq c_5 \|Z\|_{p\text{-var},s,t} \|X - X^*\|_{p\text{-var},t}.
\]
Since $X^* = X$ and then there exists a constant $c$ depending on $s$, such that
\[
\|X - X^*\|_{p\text{-var},s,t} \leq c \|Z\|_{p\text{-var},s,t} \|X - X^*\|_{p\text{-var},t}.
\]
So, there exists a constant $c_6$ depending on $s$, such that
\[
\|X - X^*\|_{p\text{-var},s,t} \leq c_6 \|Z\|_{p\text{-var},s,t} \|X - X^*\|_{p\text{-var},t}.
\]
By Equation (17) and Equation (18) together, there exists a constant $c_7$, such that
\[
\|X - X^*\|_{p\text{-var},s,t} \leq c_7 \|Z\|_{p\text{-var},s,t} \|X - X^*\|_{p\text{-var},t}.
\]
Since $(u, v) \in \Delta_T \mapsto \|Z\|_{p\text{-var},u,v}$ is a control function, there exists $N \in \mathbb{N}$ and $(\tau_k)_{k \in [0, N]} \in D([0,T])$ such that
\[
\|Z\|_{p\text{-var},\tau_k,\tau_{k+1}} \leq \frac{1}{4c_7} ; \forall k \in [0, N-1].
\]
First,
\[
\|X - X^*\|_{p\text{-var},\tau_1} \leq c_5 \|Z\|_{p\text{-var},\tau_1} \|X - X^*\|_{p\text{-var},\tau_1} \leq \frac{1}{2} \|X - X^*\|_{p\text{-var},\tau_1}.
\]
So, $X = X^*$ on $[0, \tau_1]$. For $k \in [1, N-1]$, assume that $X = X^*$ on $[0, \tau_k]$. Then,
\[
\|X - X^*\|_{p\text{-var},\tau_k,\tau_{k+1}} = \|X - X^*\|_{p\text{-var},\tau_k,\tau_{k+1}} \leq \frac{1}{2} \|X - X^*\|_{p\text{-var},\tau_{k+1}}.
\]
So, $X = X^*$ on $[0, \tau_{k+1}]$. Recursively, $X = X^*$ on $[0, T]$. \qed

**Proposition 4.3.** Consider $(s,t) \in \Delta_T$, $p \in [2,3]$ and two solutions $(X,Y)$ and $(X^*,Y^*)$ to Problem (2) under Assumption (7). On $[s,t]$, $(X,Y) = (X^*,Y^*)$ if and only if $X(s) = X^*(s)$.

\[
\int_u^v \langle R_X(u,r) - R_X^*(u,r), d(Y - Y^*)(r) \rangle \leq 0 ; \forall (u,v) \in \Delta_{s,t}.
\]

**Proof.** For the sake of simplicity, the proposition is proved on $[0, T]$ instead of $[s,t]$ with $(s,t) \in \Delta_T$.

First of all, if $(X,Y) = (X^*,Y^*)$ on $[s,t]$, then
\[
\int_u^v \langle R_X(u,r) - R_X^*(u,r), d(Y - Y^*)(r) \rangle = 0 ; \forall (u,v) \in \Delta_{s,t}.
\]
Now, let us prove that if $X(s) = X^*(s)$ and Inequality (18) is true, then $(X,Y) = (X^*,Y^*)$.

There exists a constant $c_{1} > 0$ such that for every $(s,t) \in \Delta_T$,
\[
\|(X - X^*, (X - X^*))'\|_{Z,p/2,s,t} = \|f(X) - f(X^*)\|_{p\text{-var},s,t} + \|R_X - R_X^*\|_{p/2\text{-var},s,t} \leq c_1 \|R_X - R_X^*\|_{p/2\text{-var},s,t} + \|Z\|_{p\text{-var},s,t} \|(X - X^*, (X - X^*))'\|_{Z,p/2,s,t}.
\]
Let us find a suitable control function dominating
\[
(s,t) \in \Delta_T \mapsto \|R_X - R_X^*\|_{p/2\text{-var},s,t}.
\]
For every $(s, t) \in \Delta_T$,
\[
\| R_X(s, t) - R_X^*(s, t) \|^2 = \| R_H(s, t) - R_H^*(s, t) \|^2 \\
+ 2 \int_s^t \langle R_H(s, t) - R_H^*(s, t), d(Y - Y^*)(u) \rangle \\
+ 2 \int_s^t \langle R_H(s, u) - R_X(s, u), d(Y - Y^*)(u) \rangle \\
+ 2 \int_s^t \langle R_H(s, t) - R_H(s, u) - (R_H^*(s, t) - R_H^*(s, u)), d(Y - Y^*)(u) \rangle \\
\leq \| R_H - R_H^* \|_{p/2, \text{var}, s, t}^2 + 2m(s, t)
\]
with
\[
m(s, t) := \int_s^t \langle R_H(u, t) - R_H^*(u, t), d(Y - Y^*)(u) \rangle.
\]
Let $(s, t) \in \Delta_T$ be arbitrarily chosen.

On the one hand,
\[
m(s, t) \leq 2c \cdot c(p, p) \| R_H - R_H^* \|_{p/2, \text{var}, s, t} \| Y - Y^* \|_{p/2, \text{var}, s, t}.
\]
So, there exists a constant $c_2 > 0$, not depending on $s$ and $t$, such that
\[
\| R_X(s, t) - R_X^*(s, t) \|^{p/2} \leq (\| R_H - R_H^* \|_{p/2, \text{var}, s, t}^2 + 2m(s, t))^{p/4} \\
\leq c_2 (\| R_H - R_H^* \|_{p/2, \text{var}, s, t}^{p/4} \\
+ (\| R_H - R_H^* \|_{p/2, \text{var}, s, t}^{p/4})^{1/2} (\| Y - Y^* \|_{p/2, \text{var}, s, t}^{p/4})^{1/2}).
\]
Since $1/2 + 1/2 = 1$, the right-hand side of the previous inequality defines a control function (see Friz and Victoir [11], Exercise 1.10), and then there exists a constant $c_3 > 0$, not depending on $s$ and $t$, such that
\[
\| R_X - R_X^* \|_{p/2, \text{var}, s, t}^{p/4} \leq c_2 (\| R_H - R_H^* \|_{p/2, \text{var}, s, t}^{p/4} \\
+ (\| R_H - R_H^* \|_{p/2, \text{var}, s, t}^{p/4})^{1/2} (\| Y - Y^* \|_{p/2, \text{var}, s, t}^{p/4})^{1/2}) \\
\leq c_3 (\| R_H - R_H^* \|_{p/2, \text{var}, s, t}^{p/4} \\
+ (\| R_H - R_H^* \|_{p/2, \text{var}, s, t}^{p/4})^{1/2} (\| Y - Y^* \|_{p/2, \text{var}, s, t}^{p/4})^{1/2})
\]
(21)

On the other hand, since $X(0) = X^*(0)$:
\[
\| f(X)'(s) - f(X^*)'(s) \| = \| Df(X(s)) f(X(s)) - Df(X(0)) f(X(0)) \\
- (Df(X^*(s)) f(X^*(s)) - Df(X^*(0)) f(X^*(0))) \| \\
\leq \| f(X)' - f(X^*)' \|_{p, \text{var}, s, t}.
\]
Then,
\[
\| R_H(s, t) - R_H^*(s, t) \| \leq 2 \| f(X)' - f(X^*)'(s, t) + (f(X)'(s) - f(X^*)'(s)) Z(s, t) \| \\
\leq c(p) (\| R_H - R_H^* \|_{p/2, \text{var}, s, t} \| Z \|_{p/2, \text{var}, s, t} \\
+ \| f(X)' - f(X^*)' \|_{p, \text{var}, s, t} \| Z \|_{p/2, \text{var}, s, t}) \\
+ \| f(X)' - f(X^*)' \|_{p, \text{var}, s, t} \| Z \|_{p/2, \text{var}, s, t}.
\]
So, with the same ideas as in P. Friz and M. Hairer [9, Theorem 8.4 p. 115], there exists a constant \( c_4 > 0 \), not depending on \( s \) and \( t \), such that
\[
\| R_H - R_{H\ast} \|_{p/2, \text{var}, s, t} \leq c_4 \|(X - X\ast ) (X - X\ast )'\|_{Z, p/2, t, \mathcal{W}_p, Z(s, t)^{1/p}}.
\]

By Equations (20), (21) and (22) together, there exists a constant \( c_5 > 0 \), not depending on \( s \) and \( t \), such that
\[
\|(X - X\ast ) (X - X\ast )'\|_{Z, p/2, s, t} \leq c_5 \|(X - X\ast ) (X - X\ast )'\|_{Z, p/2, t, \mathcal{W}_p, Z(s, t)^{1/(2p)}}.
\]

The conclusion of the proof is the same as in Proposition 4.2  \( \square \)

5. APPROXIMATION SCHEME

In Proposition 4.2 it has been proved that, under Assumption 1.1, Problem (2) has a unique solution \((X, Y)\) if \( p = 1 \) and if, moreover, there is an additive continuous signal of finite \( q \)-variation \( W \) with \( q \in [1, 3] \). This section deals with the convergence of the following approximation scheme for \( X \):

\[
\begin{aligned}
X^0_n &= \Phi(X^0) \\
X_{k+1}^n &= pC(t_{k+1})(X_k^n + f(X_k^n)(t^n_{k+1} - t^n_k) + W(t^n_{k+1}, t^n_{k})) ; k \in [0, n-1],
\end{aligned}
\]

where \( n \in \mathbb{N}^+ \) and \((t^n_k, \ldots, t^n_n)\) is the dissection of \([0, T]\) of constant mesh \( T/n \).

Consider the maps \( X^n, H^n \) and \( Y^n \) from \([0, T]\) into \( \mathbb{R}^\infty \) defined by \( X^n(t) := X^a_n \).

\[
H^n(t) := \sum_{i=0}^{k-1} f(X^n_i)(t^n_{i+1} - t^n_i) + f(X^n_k)(t - t^n_k) + W(t)
\]

and

\[
Y^n(t) := X^n(t) - H^n(t^n_k)
\]

for every \( k \in [0, n-1] \) and \( t \in [t^n_k, t^n_{k+1}] \).

\textbf{Lemma 5.1.} Under Assumption 1.1, one can extract a uniformly converging subsequence from any subsequence of \((H^n)_{n \in \mathbb{N}^+}\).

\textbf{Proof.} On the one hand, since \( C(t) \) is a bounded set for every \( t \in [0, T] \), \( C \) is continuous on \([0, T]\) for the Hausdorff distance and \( X^n([0, T]) \subset \cup_{t \in [0, T]} C(t) \) for every \( n \in \mathbb{N}^+ \) by construction,

\[
\sup_{n \in \mathbb{N}^+} \|X^n\|_{\infty, T} < \infty.
\]

On the other hand, consider \((s, t) \in \Delta_T \) and \( j, k \in [0, n] \) such that \( s < t^n_i \leq t \) for every \( i \in [j, k] \). Then,

\[
\|H^n(t) - H^n(s)\| = \left\| \sum_{i=0}^{k-1} f(X^n_i)(t^n_{i+1} - t^n_i) + f(X^n_k)(t - t^n_k) - \sum_{i=0}^{j-2} f(X^n_i)(t^n_{i+1} - t^n_i) - f(X^n_{j-1})(s - t^n_j) + W(s, t) \right\|
\]

\[
\leq \|x(s, t) := |t - s| \sup_{n \in \mathbb{N}^+} \|f \circ X^n\|_{\infty, T} + \|W\|_{\text{var}, s, t}.
\]

Since \((s, t) \in \Delta_T \mapsto x(s, t)\) is a continuous map such that \( x(t, t) = 0 \) for every \( t \in [0, T] \), \((H^n)_{n \in \mathbb{N}^+}\) is equicontinuous. Therefore, by Arzelà-Ascoli’s theorem, one can extract a uniformly converging subsequence from any subsequence of \((H^n)_{n \in \mathbb{N}^+} \).  \( \square \)
Lemma 5.2. Under Assumption 1.1, there exist $R > 0$ and $N \in \mathbb{N}^*$ such that
$$\sup_{n \in \mathbb{N}^*} \|Y^n\|_{1\text{-}var,T} \leq M(N, R)$$
with
$$M(N, R) := N R \left( \|\gamma\|_{\infty,T} + \sup_{n \in \mathbb{N}^*} \|X^n\|_{\infty,T} + \varphi(0, T) \right)^2.$$

Proof. On the one hand, since the map $(s, t) \in \Delta_T \mapsto \varphi(s, t)$ defined in the proof of Lemma 5.2 is continuous and satisfies $\varphi(t, t) = 0$ for every $t \in [0, T]$, by Assumption 1.1 there exist $R > 0$, $N \in \mathbb{N}^*$ and a dissection $(\tau_0, \ldots, \tau_N)$ of $[0, T]$ such that
$$\overline{B}_e(\gamma(\tau_i), R) \subset C(u)$$
for every $i \in [0, N - 1]$ and $u \in [\tau_i, \tau_{i+1}]$. Then,
$$\overline{B}_e(\gamma(\tau_i) - H^n(\tau_i), R/2) \subset \overline{B}_e(\gamma(\tau_i) - H^n(u), R) \subset C(u) - H^n(u)$$
for every $i \in [0, N - 1]$ and $u \in [\tau_i, \tau_{i+1}]$.

On the other hand, for every $k \in [1, n]$, $\gamma(\tau_i)$ is defined in the proof of Lemma 5.3. For every $(s, t) \in \Delta_T$, $\varphi(t, t) = 0$ for every $t \in [0, T]$, by Assumption 1.1 there exist $R > 0$, $N \in \mathbb{N}^*$ and a dissection $(\tau_0, \ldots, \tau_N)$ of $[0, T]$ such that
$$\overline{B}_e(\gamma(\tau_i), R) \subset C(u)$$
for every $i \in [0, N - 1]$ and $u \in [\tau_i, \tau_{i+1}]$. Then,
$$\overline{B}_e(\gamma(\tau_i) - H^n(\tau_i), R/2) \subset \overline{B}_e(\gamma(\tau_i) - H^n(u), R) \subset C(u) - H^n(u)$$
for every $i \in [0, N - 1]$ and $u \in [\tau_i, \tau_{i+1}]$.

So, for any $i \in [0, N - 1]$, by applying Proposition 5.3 to $Y^n$ on $[\tau_i, \tau_{i+1}]$,
$$\|Y^n\|_{1\text{-}var, \tau_i, \tau_{i+1}} \leq l(R/2, \|\gamma(\tau_i) - H^n(\tau_i) - Y^n(\tau_i)\|) \leq R^{-1}\|\gamma(\tau_i) - H^n(\tau_i) - Y^n(\tau_i)\|^2.$$

Since there exists $j \in [0, n]$ such that $Y^n(\tau_j) = Y^n(t^n_j)$,
$$\|Y^n\|_{1\text{-}var, \tau_i, \tau_{i+1}} \leq R^{-1}\|\gamma(\tau_i)\| + \|H^n(\tau_i) - H^n(t^n_j)\| + \|H^n(t^n_j) + Y^n(t^n_j)\| \leq R^{-1}\|\gamma(\tau_i)\| + \|\gamma(\tau_i) - t^n_j, \tau_i + \|X^n_j\|\|^2 \leq N^{-1}M(N, R).$$

Therefore,
$$\|Y^n\|_{1\text{-}var,T} = \sum_{i=0}^{N-1} \|Y^n\|_{1\text{-}var, \tau_i, \tau_{i+1}} \leq M(N, R).$$

Lemma 5.3. Under Assumption 1.1, for every $(s, t) \in \Delta_T$ and $z \in \cap_{\tau \in [s,t]}(C(\tau) - H^n(\tau))$,
$$\langle z, Y^n(t) - Y^n(s) \rangle \geq \frac{1}{2} \|Y^n(t)\|^2 - \|Y^n(s)\|^2.$$

Proof. Consider $(s, t) \in \Delta_T$. There exists a maximal interval $[j, k] \subset [0, n]$ such that
$$s < t^n_j \leq t; \forall i \in [j, k].$$
Consider $z \in \cap_{\tau \in [s,t]}(C(\tau) - H^n(\tau))$. In particular, for every $i \in [j, k]$, there exists $y_i \in C(t^n_i)$ such that $z = y_i - H^n(t^n_i)$. For every $i \in [j, k]$,
$$\langle z - Y^n(t^n_i), Y^n(t^n_i) - Y^n(t^n_{i-1}) \rangle = \langle y_i - H^n(t^n_i) - Y^n(t^n_i), Y^n(t^n_i) - Y^n(t^n_{i-1}) \rangle = \langle y_i - X^n_i, X^n_i - (X^n_{i-1} + H^n(t^n_{i-1}, t^n_i)) \rangle \geq 0$$
because
$$X^n_i = p_{C(t^n_i)}(X^n_{i-1} + H^n(t^n_{i-1}, t^n_i)).$$
Then,
\[
(z, Y^n(t) - Y^n(s)) = \langle z, Y^n(t^n_i) - Y^n(t^n_{i-1}) \rangle = \sum_{i=j}^{k} (z, Y^n(t^n_i) - Y^n(t^n_{i-1}))
\]
\[
\geq \sum_{i=j}^{k} \|Y^n(t^n_i), Y^n(t^n_i) - Y^n(t^n_{i-1})\|
\]
\[
\geq \frac{1}{2} \sum_{i=j}^{k} (\|Y^n(t^n_i)\|^2 - \|Y^n(t^n_{i-1})\|^2) = \frac{1}{2}(\|Y^n(t)\|^2 - \|Y^n(s)\|^2).
\]

Now, \(W\) is \(1/q\)-Hölder continuous from \([0, T]\) into \(\mathbb{R}^c\). Then, there exists a constant \(C_\varphi > 0\) such that
\[
\varphi(s, t) \leq C_\varphi |t - s|^{1/q}; \forall (s, t) \in \Delta_T.
\]

**Theorem 5.4.** Assume that \(C\) fulfills Assumption [14] and that there exists \((K, \alpha) \in [0, \infty] \times [0, 1]\) such that
\[
d_H(C(s), C(t)) \leq K|t - s|^{\alpha}; \forall (s, t) \in \Delta_T.
\]
Then, \((X^n, Y^n)_{n \in \mathbb{N}^*}\) converges uniformly to the unique solution \(X, Y\) to Problem [4].

**Proof.** Consider an extraction \(\psi : \mathbb{N}^* \to \mathbb{N}^*\) such that \((H^{\psi(n)}))_{n \in \mathbb{N}^*}\) is uniformly converging to a limit \(H^*\).

On the one hand, consider \(n \in \mathbb{N}^*\) such that \(T/n \in [0, 1], m \in n\mathbb{N}^*\) and \(t \in [0, T]\). By Proposition [2, 3] (2) together with Lemma [5, 2] there exist \(R > 0, N \in \mathbb{N}^*, i \in [1, n]\) and \(j \in [1, m]\) such that \(t \in [t^n_{i-1}, t^n_i], t \in [t^m_{j-1}, t^m_j]\) and
\[
\|Y^n(t) - Y^m(t)\|^2 \leq 2d_H(C(t^n_i), C(t^m_j)) \times (\|Y^n\|_{1-var, T} + \|Y^m\|_{1-var, T})
\]
\[
\leq 4M(N, R)(d_H(C(t^n_i), C(t^m_j)) + \|H^n(t^n_i) - H^m(t^m_j)\|)
\]
\[
\leq 4M(N, R)(K|t^n_i - t^m_j|^\alpha + \|H^n(t^n_i) - H^m(t^m_j)\|)
\]
\[
\leq 4M(N, R)((C_\varphi + K)|T/n|^\alpha + \|H^n - H^m\|_{\infty, T}).
\]
Consider \(\varepsilon > 0\). There exists \(N_\varepsilon \in \mathbb{N}^*\) such that for every \(n, m \in \mathbb{N}^* \cap [N_\varepsilon, \infty],
\[
|T/\psi(n)|^{\alpha/2}, |T/\psi(m)|^{\alpha/2} \leq \frac{\varepsilon}{16M(N, R)(C_\varphi + K) + 1}
\]
and
\[
\|H^{\psi(n)} - H^{\psi(m)}\|_{\infty, T} \leq \frac{\varepsilon}{16M(N, R)}.
\]
Consider \(n, m \in \mathbb{N}^* \cap [N_\varepsilon, \infty]\) and let \(p\) be the least common multiple of \(\psi(n)\) and \(\psi(m)\). By Inequality [20]:
\[
\|Y^{\psi(n)} - Y^{\psi(m)}\|_{\infty, T} \leq 2(\|Y^{\psi(n)} - Y^{\psi}\|_{\infty, T} + \|Y^{\psi(m)} - Y^{\psi}\|_{\infty, T})
\]
\[
\leq 8M(N, R)((C_\varphi + K)|T/\psi(n)| + \|H^{\psi(n)} - H^{\psi(1)}(p)\|_{\infty, T})
\]
\[
+ 8M(N, R)((C_\varphi + K)|T/\psi(m)| + \|H^{\psi(m)} - H^{\psi(1)}(p)\|_{\infty, T}).
\]
Since \(p \geq \psi(n)\) and \(p \geq \psi(m)\), \(\psi(1)(p) \geq n \cap m \geq N_\varepsilon\). Then, by [27] and [28] together:
\[
\|Y^{\psi(n)} - Y^{\psi(m)}\|_{\infty, T} \leq \varepsilon.
\]
Therefore, \((Y^{\psi(n)})_{n \in \mathbb{N}}\) is a uniformly converging sequence and by Equation (24), \((X^{\psi(n)})_{n \in \mathbb{N}}\) also. In the sequel, the limit of \((Y^{\psi(n)})_{n \in \mathbb{N}}\) (resp. \((X^{\psi(n)})_{n \in \mathbb{N}}\)) is denoted by \(Y^\ast\) (resp. \(X^\ast\)).

On the other hand, consider \((s,t) \in \Delta T\), \(z \in \cap_{\tau \in [s,t]} C(\tau)\) and \(\tau \in [s,t]\). By Lemma 5.3:

\[
(z - H^{\psi(n)}(\tau), Y^{\psi(n)}(t) - Y^{\psi(n)}(s)) \geq \frac{1}{2}(\|Y^{\psi(n)}(t)\|^2 - \|Y^{\psi(n)}(s)\|^2).
\]

So, when \(n\) goes to infinity:

\[
\langle z - H^\ast(\tau), Y^\ast(t) - Y^\ast(s) \rangle \geq \frac{1}{2}(\|Y^\ast(t)\|^2 - \|Y^\ast(s)\|^2).
\]

Therefore, by Proposition 2.2:

\[
-\frac{dDY^\ast}{dt}(t) \in NC(t) - H^\ast(t)(Y^\ast(t)) \ |DY^\ast|-a.e.
\]

Moreover, since \((X^{\psi(n)})_{n \in \mathbb{N}}\) is a sequence of step functions uniformly converging to \(X^\ast\), the definition of \((H^{\psi(n)})_{n \in \mathbb{N}}\) given by Equality (24) ensures that:

\[H^\ast(t) = \int_0^t f(X^\ast(s)) ds + W(t) \ ; \forall t \in [0,T].\]

Since the solution \((X,Y)\) to (2) is unique by Proposition 4.1, \((X^\ast,Y^\ast) = (X,Y)\) and \(H^\ast = X - Y\).

We have proved that, for each subsequence of \((X^n,Y^n)_{n \in \mathbb{N}}\), we can extract a further subsequence which converges uniformly to the solution \((X,Y)\). Thus \((X^n,Y^n)_{n \in \mathbb{N}}\) converges uniformly to \((X,Y)\).

\[\square\]

6. Sweeping processes perturbed by a stochastic noise directed by a fBm

First of all, let us recall the definition of fractional Brownian motion.

**Definition 6.1.** Let \((B(t))_{t \in [0,T]}\) be a \(d\)-dimensional centered Gaussian process. It is a fractional Brownian motion of Hurst parameter \(H \in [0,1]\) if and only if,

\[
\text{cov}(B_i(s), B_j(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H})\delta_{i,j}
\]

for every \((i,j) \in [1,d]^2\) and \((s,t) \in [0,T]^2\).

For more details on fractional Brownian motion, we refer the reader to Nualart [21] Chapter 5.

Let \(B := (B(t))_{t \in [0,T]}\) be a \(d\)-dimensional fractional Brownian motion of Hurst parameter \(H \in [1/3,1]\), defined on a probability space \((\Omega, \mathcal{A}, P)\).

By Garcia-Rodemich-Rumsey’s lemma (see Nualart [21] Lemma A.3.1]), the paths of \(B\) are \(\alpha\)-Hölder continuous for every \(\alpha \in [0,H]\). So, in particular, the paths of \(B\) are continuous and of finite \(p\)-variation for every \(p \in [1/H, \infty]\). By Friz and Victoir [11] Proposition 15.5 and Theorem 15.33], there exists an enhanced Gaussian process \(\mathbf{B}\) such that \(\mathbf{B}^{(1)} = B\).
Consider $b \in C^{[\alpha]+1}(\mathbb{R}^e)$, $\sigma \in C^{[\alpha]+1}(\mathbb{R}^e,M_{e,d}(\mathbb{R}))$ and the following sweeping process, perturbed by a pathwise stochastic noise directed by $B$:

$$
X(t) = H(t) + Y(t)
$$

(29) 

$$
H(t) = \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dB(s) - \frac{dDY}{dt}(t) \in N_{C_\text{tr}}(Y(t)) \text{-a.e. with } Y(0) = a.
$$

In the following, since $H$ can be deduced from $X$, and $Y$ from $X$ and $H$, we say that $X$ is a solution to Problem (29) if the corresponding triple $(X,H,Y)$ satisfies (29).

Let $W := (W(t))_{t \in [0,T]}$ be the stochastic process defined by

$$
W(t) := te_1 + \sum_{k=1}^d B_k(t)e_{k+1}; \forall t \in [0,T].
$$

By Friz and Victoir [11, Theorem 9.26], there exists a $G\Omega_{p,T}(\mathbb{R}^{d+1})$-valued enhanced stochastic process $W$ such that $W^{(1)} := W$. Consider also the map $f : \mathbb{R}^e \to M_{e,d+1}(\mathbb{R})$ defined by:

$$
f(x)(u,v) := b(x)u + \sigma(x)v; \forall x \in \mathbb{R}^e, \forall (u,v) \in \mathbb{R}^{d+1}.
$$

So, Problem (29) can be reformulated as follow:

$$
\begin{align*}
X(t) &= H(t) + Y(t) \\
H(t) &= \int_0^t f(X(s))dW(s) \\
- \frac{dDY}{dt}(t) &\in N_{C_\text{tr}}(Y(t)) \text{-a.e. with } Y(0) = a.
\end{align*}
$$

Therefore, the previous results of this paper apply to Problem (29).

**Theorem 6.2.** (Existence) Assume that, for every $t \in [0,T]$, $C(t)$ is a random set with convex compact values with nonempty interior, and that the paths of $C$ are continuous for the Hausdorff distance. Then Problem (29) has at least one solution, whose paths belong to $C^{p\text{-var}}([0,T],\mathbb{R}^e)$, for $p \in [1/H,\infty]$.

**Proof.** This is a direct pathwise application of Theorems 3.1 and 3.2. \qed

**Proposition 6.3.** (Existence and uniqueness for an additive fractional noise) Assume that, for every $t \in [0,T]$, $C(t)$ is a random set with convex compact values with nonempty interior, and that the paths of $C$ are continuous for the Hausdorff distance. If $\sigma$ is a constant map, then Problem (29) has a unique solution, whose paths belong to $C^{p\text{-var}}([0,T],\mathbb{R}^e)$, for $p \in [1/H,\infty]$.

**Proof.** This is a direct pathwise application of Theorem 3.1 Theorem 3.2 and Proposition 4.1. \qed

**Remark.** For instance, Proposition 6.3 ensures the existence and uniqueness of the solution to a multidimensional reflected fractional Ornstein-Uhlenbeck process.

**Proposition 6.4.** Assume that, for every $t \in [0,T]$, $C(t)$ is a random set with convex compact values with nonempty interior, and that the paths of $C$ are $\alpha$-Hölder continuous for the Hausdorff distance with $\alpha \in [0,1]$. If $\sigma$ is a constant map, then the sequence of processes $(X^n)_{n \in \mathbb{N}^*}$ defined by

$$
\begin{align*}
X^n_0 &:= a \\
X^k_{n+1} &:= \rho_{C((k+1)T/n)}(X^n_b + b(X^n_b)T/n + \sigma B(kT/n,(k+1)T/n)); k \in [0,n-1] \\
X^n(t) &:= X^n_b; t \in [kT/n,(k+1)T/n], k \in [0,n-1]
\end{align*}
$$


for every $n \in \mathbb{N}^*$ converges pathwise uniformly to the unique solution $X$ to Problem (29).

Proof. This is a direct pathwise application of Theorem 5.4. □

References

[1] S. Aida. Reflected Rough Differential Equations. Stochastic Processes and their Applications 125, 9, 3570-3595, 2015.

[2] S. Aida. Rough Differential Equations Containing Path-Dependent Bounded Variation Terms. ArXiv e-prints, August 2016.

[3] F. Bernicot and J. Venel. Stochastic Perturbation of Sweeping Process and a Convergence Result for an Associated Numerical Scheme. Journal of Differential Equations 251, 4-5, 1195-1224, 2011.

[4] C. Castaing. Sur une nouvelle classe d’équation d’évolution dans les espaces de Hilbert. Séminaire d’Analyse Convexe 13, 10, 28 pages, 1983. with an appendix from M. Valadier.

[5] C. Castaing, M. D. P. Monteiro Marques and P. Raynaud de Fitte. A Skorohod Problem Governed by a Closed Convex Moving Set. Journal of Convex Analysis 23, 2, 387-423, 2016.

[6] A. Deya, M. Gubinelli, M. Hofmanova and S. Tindel. One-Dimensional Reflected Rough Differential Equations. ArXiv e-prints, October 2016.

[7] R. M. Dudley and R. Norvaiša. Concrete Functional Calculus. Springer Monographs in Mathematics, Springer, New York, 2011.

[8] A. Falkowski and L. Słominski. Sweeping Processes with Stochastic Perturbations Generated by a Fractional Brownian Motion. ArXiv e-prints, May 2015.

[9] P. Friz and M. Hairer. A Course on Rough Paths with an Introduction to Regularity Structures. Universitext, Springer, Cham, 2014.

[10] P. Friz and A. Shekhar. General Rough integration, Levy Rough paths and a Levy–Kintchine type formula. The Annals of Probability 45, 4, 2707-2765, 2017.

[11] P. Friz and N. Victoir. Multidimensional Stochastic Processes as Rough Paths. Cambridge Studies in Advanced Mathematics 120, Cambridge University Press, 2010.

[12] M. Gubinelli. Controlling Rough Paths. Journal of Functional Analysis 216, 1, 86-140, 2004.

[13] W. Kang and K. Ramanan. A Dirichlet Process Characterization of a Class of Reflected Diffusions. The Annals of Probability 38, 3, 1062-1105, 2010.

[14] A. Lejay. Controlled Differential Equations as Young Integrals: A Simple Approach. Journal of Differential Equations 249, 8, 1777-1798, 2010.

[15] P.-L. Lions and A.-S. Sznitman. Stochastic Differential Equations with Reflecting Boundary Conditions. Comm. Pure Appl. Math. 37, 4, 511-537, 1984.

[16] T. Lyons, M. Caruana and T. Lévy. Differential Equations Driven by Rough Paths. Lecture Notes in Mathematics 1908, Springer, Berlin, 2007. Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6-24, 2004, with an introduction concerning the Summer School by Jean Picard.

[17] M. D. P. Monteiro Marques. Rafle par un convexe semi-continu inférieurement d’intérieur non vide en dimension finie. Séminaire d’analyse convexe 14, 6, 24 pages, 1984.

[18] M. D. P. Monteiro Marques. Rafle par un convexe continu d’intérieur non vide en dimension infinie. Séminaire d’analyse convexe 16, 4, 11 pages, 1986.

[19] M. D. P. Monteiro Marques. Differential Inclusions in Nonsmooth Mechanical Problems: Shocks and Dry Friction. Progress in Nonlinear Differential Equations and their Applications 9, Birkhäuser Verlag, Basel, 1993.

[20] J. J. Moreau. Solutions du processus de rafle au sens des mesures différentielles. Travaux Sém. Anal. Convexe 6, 1, 17, 1976.

[21] D. Nualart. The Malliavin Calculus and Related Topics. Second Edition. Probability and its Applications, Springer-Verlag, Berlin, 2006.

[22] D. Revuz and M. Yor. Continuous Martingales and Brownian Motion. Third Edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 293, Springer-Verlag, Berlin, 1999.

[23] Y. Saisho. Stochastic Differential Equations for Multidimensional Domain with Reflecting Boundary. Probability Theory and Related Fields 74, 3, 455-477, 1987.

[24] H. Tanaka. Stochastic Differential Equations with Reflecting Boundary Condition in Convex Regions. Hiroshima Math. J. 9, 1, 163-177, 1979.

[25] M. Valadier. Lipschitz Approximation of the Sweeping (or Moreau) Process. Journal of Differential Equations 88, 2, 248-264, 1990.
