Curvature Varifolds with Orthogonal Boundary

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Abstract

We consider the class $S_m^\perp(\Omega)$ of $m$-dimensional surfaces in $\overline{\Omega} \subset \mathbb{R}^n$ which intersect $S = \partial \Omega$ orthogonally along the boundary. A piece of an affine $m$-plane in $S_m^\perp(\Omega)$ is called an orthogonal slice. We prove estimates for the area by the $L^p$ integral of the second fundamental form in three cases: first when $\Omega$ admits no orthogonal slices, second for $m = p = 2$ if all orthogonal slices are topological disks, and finally for all $\Omega$ if the surfaces are confined to a neighborhood of $S$. The orthogonality constraint has a weak formulation for curvature varifolds. We classify those varifolds of vanishing curvature. As an application, we prove for any $\Omega$ the existence of an orthogonal 2-varifold which minimizes the $L^2$ curvature in the integer rectifiable class.

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1 Introduction

For a bounded domain $\Omega \subset \mathbb{R}^n$ of class $C^2$, consider the class $S_m^\perp(\Omega)$ of compact $C^1$ immersed submanifolds $\Sigma \hookrightarrow \overline{\Omega}$ of dimension $m \leq n - 1$, which are $C^2$ in the interior and meet $S = \partial \Omega$ orthogonally along their boundary. Denoting by $\nu^S$ the unit normal interior to $\Omega$, the condition means that $\nu^S \in T\Sigma$ along $\partial \Sigma \hookrightarrow S$. A natural variational problem in the class $S_m^\perp(\Omega)$ is to minimize the $L^p$ curvature energy

$$E^p(\Sigma) = \int_\Sigma |A_\Sigma|^p d\mu_\Sigma \quad \text{where } 1 \leq p < \infty.$$ 

In [1] Alessandroni and the first author constructed critical points of the Willmore energy subject to the orthogonality constraint, but only in the class of surfaces with small prescribed area. Here in Section 5 we address the global problem in the case
when the area is not fixed a priori. Area bounds depending on the curvature energy then play a key role, they are also of independent interest. It turns out that the area of surfaces in $S^m_\perp(\Omega)$ has an upper bound in terms of the curvature energy, unless the domain $\Omega$ is special.

**Definition 1.1 (Orthogonal Slices).** Let $x_0 + P \subset \mathbb{R}^n$ be an affine $m$-plane. Any component $\Delta \neq \emptyset$ of $\text{int}(\Omega \cap (x_0 + P))$ will be called an $m$-slice of $\Omega$. $\Delta$ is an orthogonal $m$-slice if it meets $S$ orthogonally along its boundary, that is

$$\nu^S(x) \in P \quad \text{for all } x \in \partial \Delta.$$ 

We have the following estimate.

**Theorem 1 (Mass Bounds).** Assume that $\Omega \subset \mathbb{R}^n$ has no orthogonal $m$-slice. Then there is a constant $C < \infty$ depending on $n$, $m$, $p$ and $\Omega$, such that

$$|\Sigma| + |\partial \Sigma| \leq C \int_{\Sigma} |A_\Sigma|^p \, d\mu_\Sigma \quad \text{for any } \Sigma \in S^m_\perp(\Omega). \quad (1)$$

Moreover for $m = p = 2$, the Euler characteristic is also estimated by

$$|\chi(\Sigma)| \leq C \int_{\Sigma} |A_\Sigma|^2 \, d\mu_\Sigma \quad \text{for any } \Sigma \in S^2_\perp(\Omega). \quad (2)$$

We also obtain bounds for general $C^2$ domains $\Omega$ if we restrict to surfaces $\Sigma$ which are supported in a neighborhood of $S = \partial \Omega$. More precisely let $\varrho_S > 0$ be the supremum of all $\varrho > 0$ such that the map

$$S \times [0, \varrho) \to U^+_{\varrho}(S), \ (x, r) \mapsto x + r\nu^S(x),$$

is a diffeomorphism. Here $U^+_{\varrho}(S)$ denotes the one-sided tubular neighborhood of $S$.

**Theorem 2 (Bounds near the Boundary).** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^2$ with boundary $S = \partial \Omega$. Then for any $\delta \in (0, \varrho_S)$ there is a constant $C = C(n, m, p, \Omega, \delta) < \infty$ such that for $\Sigma \in S^m_\perp(\Omega)$

$$|\Sigma| + |\partial \Sigma| \leq C \int_{\Sigma} |A_\Sigma|^p \, d\mu_\Sigma \quad \text{whenever } \Sigma \subset \overline{U^+_{\varrho}(S)} \quad (3)$$

For $m = p = 2$ the Euler characteristic $\chi(\Sigma)$ has a corresponding bound.

A variant of Theorem 1 in the case $m = p = 2$ is as follows.

**Theorem 3 (Bounds for Surfaces).** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^2$. Assume that all orthogonal 2-slices of $\Omega$ are topological disks. Then there is a constant $C < \infty$ depending on $n$ and $\Omega$ such that

$$|\Sigma| + |\partial \Sigma| + |\chi(\Sigma)| \leq C \left( \int_{\Sigma} |A_\Sigma|^2 \, d\mu_\Sigma + \chi(\Sigma)^+ \right) \quad \text{for any } \Sigma \in S^2_\perp(\Omega). \quad (4)$$
For \( \Sigma \) connected we have \( \chi(\Sigma)^+ = 1 \) if \( \Sigma \) is a disk and \( \chi(\Sigma)^+ = 0 \) otherwise. We note that Theorem 3 applies to any convex \( C^2 \) domain. In Chapter 4 we give an example showing that the assumptions in both Theorems 1 and 3 cannot be dropped.

There are a number of papers dealing with curvature-mass relations for confined curves and surfaces. Müller and Röger consider closed 2-dimensional surfaces with the type of a sphere in a 3-dimensional domain, e.g. a ball \( \mathbb{B} \). We also refer to the work of Pozzetta \[17\] and the references therein. In a different direction, Colding and Minicozzi proved compactness results for sequences of embedded minimal surfaces in 3-manifolds, see for instance \[7\]. A problem which is more closely related to the present paper is studied by Mondino \[15\], proving mass bounds for \( m \)-varifolds in a Riemannian manifold in terms of their curvature. He applies the concept of curvature varifolds due to Hutchinson \[12\]. The mass bounds are obtained under the condition that there exist no varifolds with zero curvature. In the special case \( m = 2 \), it is actually sufficient to rule out smooth totally geodesic immersions, see \[4\]. In our case, we need to introduce a weak formulation of orthogonality along the boundary for general curvature varifolds. For this we build on the notion of varifolds with boundary, defined and studied by Mantegazza \[14\].

**Definition 1.2** (orthogonal boundary). Let \( V \) be an \( m \)-varifold in \( \mathbb{R}^n \) with curvature \( B \in L^1(V) \) and support in \( \Omega \), and let \( \Gamma \) be an \((m - 1)\)-varifold in \( S = \partial \Omega \). We say that \( V \) is orthogonal to \( S \) along \( \Gamma \) if for all \( \phi \in C^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \)

\[
\int_{\mathbb{R}^n \times G(m, n)} (D_P \phi \cdot B + \langle \text{tr } B, \phi \rangle + \langle D_x \phi, P \rangle) \, dV(x, P) \tag{5}
\]

\[
= -\int_{G_{m-1}(T S)} \langle \nu^S(x), \phi(x, \nu^S(x) \wedge Q) \rangle \, d\Gamma(x, Q).
\]

Here \( \nu^S(x) \wedge Q \) denotes the \( m \)-plane spanned by \( \nu^S(x) \) and \( Q \). The varifold \( \Gamma \) in the definition is uniquely determined, and the class of varifolds \( V \) satisfying the definition (for some \( \Gamma \)) is denoted by \( CV_{\perp}^m(\Omega) \).

In Section 2 we classify varifolds \( V \in CV_{\perp}^m(\Omega) \) having curvature \( B_V = 0 \); using disintegration we derive a formula to which only the slices orthogonal to \( \Omega \) contribute. In Section 3 we then consider the infimum

\[
\kappa^{m,p}(\Omega) = \inf \{ \| B_V \|_{L^p(V)} : V \in CV_{\perp}^m(\Omega), \, M(V) = 1 \}. \tag{6}
\]

We prove the following generalization of Theorem 1

**Theorem 4** (Varifold Bounds). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain of class \( C^2 \). If \( \Omega \) has no orthogonal \( m \)-slices, then \( \kappa^{m,p}(\Omega) > 0 \) for all \( p \in [1, \infty) \). Moreover for any \( V \in CV_{\perp}^m(\Omega) \) with associated boundary varifold \( \Gamma \), we have

\[
M(V) \leq \frac{1}{\kappa^{m,p}(\Omega)} \| B_V \|_{L^p(V)}^p, \tag{7}
\]

\[
M(\Gamma) \leq C \| B_V \|_{L^p(V)}^p \quad \text{where } C = C(n, m, p, \Omega). \tag{8}
\]
In the final section this result is applied to a variational problem.

**Theorem 5** (Existence of Minimizers). For a bounded domain $\Omega \subset \mathbb{R}^n$ of class $C^3$, let $\text{ICV}^2_\perp(\Omega)$ be the class of varifolds in $\text{CV}^2_\perp(\Omega)$ which are integer rectifiable. Then the following infimum is attained:

$$\kappa(\Omega) = \inf \left\{ \|B_V\|_{L^2(V)}^2 : V \in \text{ICV}^2_\perp(\Omega), V \neq 0 \right\}. \tag{9}$$

The fact that the mass is not prescribed a priori poses a key difficulty. But since any orthogonal 2-slice is a minimizer, we may restrict to the case when such slices do not exist. Then Theorem 4 applies to give an upper mass bound. The lower bound is proved by contradiction. We show that if a minimizing sequence $V_k$ would converge to zero then

$$\lim_{k \to \infty} \int_{\Omega} |B_{V_k}|^2 \, dV_k \geq 8\pi = \int_{S^2_+} |B_{S^2}|^2 \, d\mu_{S^2},$$

where $S^2_+$ denotes the upper half sphere. However, a comparison surface strictly below $8\pi$ is constructed in Lemma 5.1. We note that our proof of Theorem 5 would apply to $m$-varifolds with curvature in $L^m$, once the round $m$-sphere was known to minimize $L^m$ curvature among varifolds in $\mathbb{R}^n$.

In Section 6 we prove that any compact $C^2$ hypersurface $S \subset \mathbb{R}^n$ has a $C^2$ approximation by smooth surfaces $S_i$ which have no orthogonal $m$-slices. In particular the assumption of Theorem 1 and Theorem 4 is generically satisfied.

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## 2 Preliminaries

We derive in this section a weak formulation of the orthogonality constraint which applies to varifolds. For this we recall briefly the definition of curvature varifolds due to Hutchinson [12], see also Mantegazza [14].

For an open set $U \subset \mathbb{R}^n$, we consider first the class of $m$-dimensional, properly embedded submanifolds $\Sigma \subset U$. The Grassmannian $G(m, n)$ of $m$-dimensional subspaces of $\mathbb{R}^n$ will be identified with the set of orthogonal projections $P \in \mathbb{R}^{n \times n}$ of rank $m$. On $\Sigma$ we then have the associated Gauß map

$$P : \Sigma \to G(m, n) \subset \mathbb{R}^{n \times n}, \quad P(x) = T_x \Sigma. \tag{10}$$

The tangential and normal components of a vector $X \in \mathbb{R}^n$ are denoted by $PX = X^\top$ and $P^\perp X = X^\perp$. In the space $BL(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ of $\mathbb{R}^n$-valued bilinear forms, we define

$$B(x)(v, w) = (D_v P)(x)w \quad \text{for } x \in \Sigma, \text{ and } v, w \in \mathbb{R}^n. \tag{11}$$
The second fundamental form $A(x)$ is a component of $B(x)$. More precisely

$$B(x)(v, w) = D_v(Pw)(x) = A(x)(v, w) \quad \text{for } v, w \in T_x\Sigma. \tag{12}$$

Reversely, $B(x)$ can be expressed by $A(x)$: for a tangent frame $\tau_\alpha = \tau_\alpha(x)$, $1 \leq \alpha \leq m$, and fixed $v \in T_x\Sigma$, $w \in \mathbb{R}^n$ we compute

$$D_v P \cdot w = D_v(Pw)$$

$$= [D_v(Pw)]^\perp + \sum_{\alpha=1}^m \langle D_v(Pw), \tau_\alpha \rangle \tau_\alpha$$

$$= A(v, w^\top) - \sum_{\alpha=1}^m \langle D_v(P^\perp w), \tau_\alpha \rangle \tau_\alpha \quad \text{(as } D_vw = 0)$$

$$= A(v, w^\top) + \sum_{\alpha=1}^m \langle w^\perp, D_v \tau_\alpha \rangle \tau_\alpha.$$

By definition of the second fundamental form, we conclude

$$B(v, w) = A(v^\top, w^\top) + \sum_{\alpha=1}^m \langle A(v^\top, \tau_\alpha), w^\perp \rangle \tau_\alpha \quad \text{for all } v, w \in \mathbb{R}^n. \tag{13}$$

In particular $|B(v, w)|^2 = |A(v^\top, w^\top)|^2 + \sum_{\alpha=1}^m \langle A(v^\top, \tau_\alpha), w^\perp \rangle^2$. Choosing an orthonormal basis $v_k$ of $\mathbb{R}^n$ with $v_\alpha = \tau_\alpha$ for $\alpha = 1, \ldots, m$, we obtain

$$|B|^2 = \sum_{\alpha, \beta=1}^m |A(\tau_\alpha, \tau_\beta)|^2 + \sum_{k=m+1}^n \sum_{\alpha, \beta=1}^m \langle A(\tau_\alpha, \tau_\beta), v_k \rangle^2 = 2 |A|^2. \tag{14}$$

Furthermore, the mean curvature vector is the trace of $B$, i.e.

$$\text{tr } B = \sum_{k=1}^n B(v_k, v_k) = \sum_{\alpha=1}^m A(\tau_\alpha, \tau_\alpha) = H. \tag{15}$$

For completeness we note also the other traces,

$$\sum_{k=1}^n \langle B(v_k, w), v_k \rangle = \langle H, w \rangle \quad \text{and} \quad \sum_{k=1}^n \langle B(v, v_k), v_k \rangle = 0.$$

To derive the first variation formula on $\Sigma$, we now compute

$$\text{div}_\Sigma X^\top = \text{div}_\Sigma X + \langle H, X \rangle \quad \text{for any } X \in C^1(\Sigma, \mathbb{R}^n). \tag{16}$$

Substituting $X(x) = \phi(x, P(x))$ where $\phi \in C^1(\overline{\Omega} \times \mathbb{R}^{n \times n}, \mathbb{R}^n)$, we get

$$\text{div}_\Sigma X = \langle D_{\tau_\alpha} X, \tau_\alpha \rangle = \langle D_x \phi(x, P) \tau_\alpha, \tau_\alpha \rangle + \langle D_P \phi(x, P) D_{\tau_\alpha} P, \tau_\alpha \rangle.$$
Using the Hilbert-Schmidt scalar product, we can write
\[ \langle D_x \phi(x, P) \tau_\alpha, \tau_\alpha \rangle = \langle D_x \phi(x, P), P \rangle. \]

Furthermore, recalling the definition of $B$ we have
\[ \langle D_P \phi(x, P) D_{\tau_\alpha} P, \tau_\alpha \rangle = \langle D_P \phi(x, P) B(\tau_\alpha, \cdot), \tau_\alpha \rangle = \langle D_P \phi(x, P) B(e_i, \cdot), e_i \rangle = \partial_{P^k} \phi^j(x, P) B^k_{ij} =: D_P \phi(x, P) \cdot B. \]

Collecting terms we obtain from (16), for $X(x) = \phi(x, P(x))$,
\[ \text{div}_\Sigma X^T = D_P \phi(x, P) \cdot B + \langle \text{tr} B, \phi(x, P) \rangle + \langle D_x \phi(x, P), P \rangle. \tag{17} \]

Now associate to $\Sigma$ the $m$-varifold $V = V_\Sigma$ in $U$ with weight measure $\mu_\Sigma = \mathcal{H}^m \llcorner \Sigma$ and measure $V^x = \delta_{T_x \Sigma}$ on $G(m, n)$, for each $x \in \Sigma$. If $X(x) = \phi(x, P(x))$ has compact support in $\Sigma$, then
\[ \int_{\Sigma} \text{div}_\Sigma X^T \, d\mu_\Sigma = \int_{U \times G(m, n)} (D_P \phi(x, P) \cdot B + \langle \text{tr} B, \phi(x, P) \rangle + \langle D_x \phi(x, P), P \rangle) \, dV(x, P). \]

On the basis of this calculation Hutchinson gave the following definition.

**Definition 2.1** (varifold curvature). Let $V$ be a varifold in $U \subset \mathbb{R}^n$. We say that $V$ has weak curvature $B \in L^1_{\text{loc}}(V)$, where $B(x, P) \in BL(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, if for any $\phi \in C^1(U \times \mathbb{R}^{n \times n}, \mathbb{R}^n)$ with compact support in $U$ we have the identity
\[ \int_{U \times G(m, n)} (D_P \phi(x, P) \cdot B + \langle \text{tr} B, \phi(x, P) \rangle + \langle D_x \phi(x, P), P \rangle) \, dV(x, P) = 0. \tag{18} \]

For test functions $\phi = \phi(x)$ the identity simplifies as follows, see Lemma 38.4 in [18],
\[ \delta V(\phi) = \int_{U \times G(m, n)} \langle D\phi(x), P \rangle \, dV(x, P) = -\int_{U \times G(m, n)} \langle \text{tr} B(x, P), \phi(x) \rangle \, dV(x, P) = -\int_U \left( \int_{G(m, n)} \text{tr} B(x, P) \, dV^x(P), \phi(x) \right) \, d\mu_V(x). \]

As $|\text{tr} B| \leq \sqrt{m} |B|$, we have for $K = \text{spt} \phi \times G(m, n)$
\[ |\delta V(\phi)| \leq \sqrt{m} \|B\|_{L^1(V,K)} \|\phi\|_{C^0(U)}. \]
Therefore Definition 2.1 implies that \( V \) has weak mean curvature in \( U \) given by

\[
H_V(x) = \int_{G(m,n)} \text{tr } B(x, P) \, dV^x(P) \in L^1_{\text{loc}}(\mu_V). \tag{19}
\]

Now assume that \( \Sigma \) is a compact \( C^1 \)-submanifold with boundary \( \partial \Sigma \), which is \( C^2 \) in the interior and with second fundamental form \( A_\Sigma \in L^p(\mu_\Sigma) \), \( p \geq 1 \). Denote by \( \eta \) the interior co-normal and by \( \sigma_{\partial \Sigma} \) the induced boundary measure. We approximate \( \Sigma \) with compact \( C^1 \)-subdomains \( \Sigma_\varepsilon \subset \text{int}(\Sigma) \) such that \( \partial \Sigma_\varepsilon \) goes to \( \partial \Sigma \) in the \( C^1 \)-topology. Notice that

\[
\int_{\Sigma_\varepsilon} \text{div}_\Sigma X^T \, d\mu_\Sigma = - \int_{\partial \Sigma_\varepsilon} \langle X, \eta_\varepsilon \rangle \, d\sigma_{\partial \Sigma_\varepsilon} \xrightarrow{\varepsilon \to 0} - \int_{\partial \Sigma} \langle X, \eta \rangle \, d\sigma_{\partial \Sigma}.
\]

Inserting (17) and passing to the limit \( \varepsilon \to 0 \), thereby using \( A_\Sigma \in L^p \), we obtain for \( V = V_\xi \) the formula of Mantegazza (see Section 3 in [14])

\[
\int_{U \times G(m,n)} (D_p \phi(x, P) \cdot B + \langle \text{tr } B, \phi(x, P) \rangle + \langle D_x \phi(x, P), P \rangle) \, dV(x, P) = - \int_{\partial \Sigma} \langle X, \eta \rangle \, d\sigma_{\partial \Sigma}.
\]

Our interest is in the case when \( \Sigma \subset \overline{\Omega} \) meets \( S = \partial \Omega \) orthogonally along the boundary. This is true if and only if the last equation holds true with \( \eta = \nu^S |_{\partial \Sigma} \). Motivated by this we derive the following weak formulation. We associate to \( \partial \Sigma \) the \((m-1)\)-varifold \( \Gamma \), i.e. \( \Gamma \) is a Radon measure on the bundle

\[
G_{m-1}(TS) = \{(x, Q) : x \in S, Q \in G(m-1, n), Q \subset T_x S\}.
\]

For \( p : G_{m-1}(TS) \to S, p(x, Q) = x \), we have the weight measure

\[
p_* \Gamma = \mathcal{H}^{m-1} \cdot \partial \Sigma =: \sigma_T,
\]

and the vertical measure \( \Gamma^x = \delta_{T_x(\partial \Sigma)} \) for \( x \in \partial \Sigma \). Thus for \( \psi \in C^0(G_{m-1}(TS)) \)

\[
\int_{\partial \Sigma} \psi(x, T_x(\partial \Sigma)) \, d\sigma_{\partial \Sigma}(x) = \int_S \int_{G_{m-1}(TS)} \psi(x, Q) \, d\Gamma^x(Q) \, d\sigma_T(x) = \Gamma(\psi).
\]

In order to state the orthogonality condition, we adopt the following notation: for \( Q \in G_{m-1}(\mathbb{R}^n) \) and \( v \in \mathbb{R}^n \) with \( v \notin Q \), we denote by \( v \wedge Q \in G_m(\mathbb{R}^n) \) the span of \( Q \) and \( v \). In particular if \( \Sigma \) meets \( S \) orthogonally then \( P(x) = \nu^S(x) \wedge T_x(\partial \Sigma) \), and for any \( \phi \in C^1(\overline{\Omega} \times \mathbb{R}^{n \times n}, \mathbb{R}^n) \) we can write putting \( X(x) = \phi(x, P(x)) \)

\[
\int_{\partial \Sigma} \langle \nu^S, X \rangle \, d\sigma_{\partial \Sigma} = \int_{\partial \Sigma} \langle \nu^S(x), \phi(x, \nu^S(x) \wedge T_x(\partial \Sigma)) \rangle \, d\sigma_T(x) = \int_{G_{m-1}(TS)} \langle \nu^S(x), \phi(x, \nu^S(x) \wedge Q) \rangle \, d\Gamma(x, Q).
\]

This suggests
Definition 2.2 (orthogonal boundary). Let $V$ be an $m$-varifold in $\mathbb{R}^n$ with curvature $B \in L^1(V)$ and support in $\overline{\Omega}$, and let $\Gamma$ be an $(m-1)$-varifold in $S = \partial \Omega$. We say that $V$ is orthogonal to $S$ along $\Gamma$ if for all $\phi \in C^1(\mathbb{R}^n \times \mathbb{R}^{n \times n}, \mathbb{R}^n)$

$$
\int_{\mathbb{R}^n \times G(m,n)} (D_P \phi \cdot B + \langle \text{tr} B, \phi \rangle + \langle D_x \phi, P \rangle) \, dV(x, P) = -\int_{G_{m-1}(TS)} \langle \nu^S(x), \phi(x, \nu^S(x) \wedge Q) \rangle \, d\Gamma(x, Q).
$$

The class of varifolds $V$ satisfying the definition (for some $\Gamma$) is denoted by $CV^m_{\perp}(\Omega)$.

Remark 2.3. A notion of orthogonality at the boundary can be defined also for varifolds $V$ which only have mean curvature in $L^1(\mu_V)$. For a given Radon measure $\sigma$ on $S$, one says that $V$ is orthogonal along $\sigma$ if for all $\phi \in C^0_c(\mathbb{R}^n, \mathbb{R}^n)$

$$
\delta V(\phi) = \int_{\mathbb{R}^n \times G(m,n)} \langle D \phi, P \rangle \, dV(x, P) = -\int_{\mathbb{R}^n} \langle H_V, \phi \rangle \, d\mu_V - \int_S \langle \nu_S^S(x), \phi(x) \rangle \, d\sigma(x).
$$

Definition 2.2 implies this property for $H_V$ is as in (19) and $\sigma = \sigma_\Gamma$, the weight measure of $\Gamma$. To employ test functions depending on $P$, Definition 2.2 will be needed.

Lemma 2.4. The boundary varifold $\Gamma$ in Definition 2.2 is unique if it exists.

Proof. Subtracting the identities, we get by Radon-Nikodym a signed measure $\Lambda$ on $G_{m-1}(TS)$ such that for all $\phi \in C^1_c(\mathbb{R}^n \times \mathbb{R}^{n \times n}, \mathbb{R}^n)$

$$
\int_{G_{m-1}(TS)} \langle \nu^S(x), \phi(x, \nu^S(x) \wedge Q) \rangle \, d\Lambda(x, Q) = 0.
$$

For given $\varphi \in C^0(G_{m-1}(TS))$ we define $\phi(x, P) = \varphi(x, P \cap T_x S) \nu^S(x)$ whenever $\nu^S(x) \in P$. Using a suitable extension we obtain a function $\phi \in C^0(\mathbb{R}^n \times \mathbb{R}^{n \times n}, \mathbb{R}^n)$, such that for all $Q \in G_{m-1}(T_x S)$

$$
\langle \nu^S(x), \phi(x, \nu^S(x) \wedge Q) \rangle = \varphi(x, (\nu^S(x) \wedge Q) \cap T_x S) = \varphi(x, Q).
$$

We conclude

$$
\int_{G_{m-1}(TS)} \varphi(x, Q) \, d\Lambda(x, Q) = 0 \quad \text{for all } \varphi \in C^0(G_{m-1}(TS)).
$$

By density this implies $\Lambda = 0$. \qed

Remark 2.5. If Definition 2.2 is valid with $V$ and $\Gamma$, then it holds true for $\lambda V$ and $\lambda \Gamma$ with the same $B$, for any $\lambda > 0$. However, one has the scaling property

$$
\|B\|_{L^p(\lambda V)} = \lambda^{\frac{n}{p}} \|B\|_{L^p(V)}, \quad (21)
$$

8
Lemma 2.6 (Comparable Masses). There exists a constant $C = C(n, \Omega) < \infty$ such that for any $V, \Gamma$ as in Definition 2.2 one has

\begin{align*}
M(V) &\leq C \left( M(\Gamma) + \| B \|_{L^1(V)} \right), \\
M(\Gamma) &\leq C \left( M(V) + \| B \|_{L^1(V)} \right).
\end{align*}

(22)  \hspace{1cm}  (23)

Proof. Taking $\phi(x) = x - x_0$ for $x_0 \in \Omega$ in Definition 2.2 we obtain

$mM(V) \leq \sqrt{m} \text{diam}(\Omega) \left( M(\Gamma) + \| B \|_{L^1(V)} \right)$.

For the reverse inequality, we take $\phi = \varphi \nabla d_S$, where $d_S$ is the signed distance from $S$ (positive in $\Omega$) and $0 \leq \varphi \leq 1$ is a cutoff function with $\varphi|_S = 1$ and support in $U_\delta(S)$, for appropriate $\delta = \delta(\Omega) \leq 1$. This yields

\begin{align*}
M(\Gamma) &= - \int_{G_{m-1}(TS)} \langle \nu^S, \varphi \nabla d_S \rangle \, d\Gamma \\
&= \int_{\Pi \times G(m,n)} \left( \langle \text{tr } B, \varphi \nabla d_S \rangle + \langle D_x(\varphi \nabla d_S), P \rangle \right) \, dV \\
&\leq C \left( \| B \|_{L^1(V)} + M(V) \right).
\end{align*}

Remark 2.7. By Young’s inequality we have for any $p \in (1, \infty)$

$\| B \|_{L^p(V)} \leq \frac{1}{p} \| B \|_{L^1(V)}^p \| B \|_{L^1(V)} + \frac{p-1}{p} \varepsilon^{p-1} M(V)$.

Therefore the inequalities (22), (23) hold with $\| B \|_{L^1(V)}$ replaced by $\| B \|_{L^p(V)}$, with a constant that depends additionally on $p$.

Lemma 2.8 (compactness). Let $V_k, \Gamma_k$ be varifolds as in Definition 2.2, such that $M(V_k) \leq C$ and for some $p \in (1, \infty]$\n
\begin{equation}
\| B_k \|_{L^p(V_k)} \leq \Lambda < \infty \quad \text{for all } k.
\end{equation}

(24)

After passing to a subsequence, one has $V_k \to V$ and $\Gamma_k \to \Gamma$ as varifolds in $\mathbb{R}^n$, and $V, \Gamma$ satisfy Definition 2.2 for some $B \in L^p(V)$ with $\| B \|_{L^p(V)} \leq \Lambda$.

Proof. From Lemma 2.6 we have

$M(\Gamma_k) \leq C \left( M(V_k) + \| B_k \|_{L^1(V_k)} \right) \leq C \left( M(V_k) + M(V_k)^{1-\frac{1}{p}} \| B_k \|_{L^p(V_k)} \right) \leq C$.

Thus after passing to a subsequence, $V_k \to V$ and $\Gamma_k \to \Gamma$ as varifolds in $\mathbb{R}^n$. Let $BL = BL(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, and introduce the functionals

$L_k : C^0_c(\mathbb{R}^n \times G(m,n), BL) \to \mathbb{R}, \ L_k(\psi) = \int_{\mathbb{R}^n \times G(m,n)} \langle B_k, \psi \rangle \, dV_k$. 

9
By Hölder’s inequality
\[ L_k(\psi) \leq \|B_k\|_{L^p(V_k)} \|\psi\|_{L^p(V_k)} \]
\[ \leq C \|B_k\|_{L^p(V_k)} M(V_k)^{1 - \frac{1}{p}} \|\psi\|_{C^0(\mathbb{R}^n \times G(m,n))} \]
\[ \leq C \|\psi\|_{C^0(\mathbb{R}^n \times G(m,n))}. \]

After passing to a subsequence, we have \( L_k \to L \) in \( C^0_0(\Omega \times G(m,n))' \), and
\[ L(\psi) \leq \Lambda \|\psi\|_{L^p(V_k)}. \]

As \( L^p(V_k)' = L^p(V) \) for any \( p \in (1, \infty] \), the functional \( L \) is represented by some \( B \in L^p(V) \). The lemma is proved.

An immediate consequence of Lemma 2.8 is the following

**Theorem 6** (Existence of Minimizers). Let \( \Omega \subset \mathbb{R}^n \) be a bounded \( C^2 \) domain. For any \( m \leq n - 1 \) and \( p \in (1, \infty] \), let
\[ \chi^{m,p}(\Omega) = \inf \{ \|B_V\|_{L^p(V)} : V \in CV^m_\perp(\Omega), M(V) = 1 \}. \] (25)
Then \( \chi^{m,p}(\Omega) < \infty \), and the infimum is attained.

**Remark 2.9.** In view of Remark 2.5 we have
\[ \|B_V\|_{L^p(V)} \geq \chi^{m,p}(\Omega) M(V) \quad \text{for all } V \in CV^m_\perp(\Omega). \] (26)

**Remark 2.10.** A corresponding existence result holds in the class \( ICV^m_\perp(\Omega) \) of integer rectifiable varifolds in \( CV^m_\perp(\Omega) \), again for prescribed mass \( M(V) \). This follows by Allard’s integral compactness theorem [2], see also Section 5.

### 3 The Case of Zero Curvature

In this section we assume that \( V \) is an \( m \)-varifold in \( \mathbb{R}^n \) with support in \( \overline{\Omega} \times G(m,n) \). We can then use test functions which are continuous on \( \mathbb{R}^n \), since \( \overline{\Omega} \) is compact. We now state from the appendix the following disintegration procedure for varifolds.

- Let \( \nu = (p_2)_* V \), where \( p_2 \) is the projection onto \( G(m,n) \). For any \( P \in G(m,n) \), up to a set \( E \subset G(m,n) \) with \( \nu(E) = 0 \), there exists a Radon measure \( \mu_P \) on \( \mathbb{R}^n \) with \( \mu_P(\mathbb{R}^n) = 1 \) such that for all \( \phi \in C^0(\mathbb{R}^n \times G(m,n)) \) one has
\[ \int_{\mathbb{R}^n \times G(m,n)} \phi(x, P) dV(x, P) = \int_{G(m,n)} \int_{\mathbb{R}^n} \phi(x, P) d\mu_P(x) d\nu(P). \]
• Put $\mu_P^\perp = (P^\perp)_* \mu_P$ for $P \in G(m, n) \setminus E$. Then for any $z \in P^\perp$, up to a set $E_P \subset P^\perp$ with $\mu_P^\perp(E_P) = 0$, there is a Radon measure $\mu_{P,z}$ on $z + P$ with $\mu_{P,z}(z + P) = 1$ such that for all $\varphi \in C^0(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi(x) d\mu_P(x) = \int_{P^\perp} \int_{z + P} \varphi(x) d\mu_{P,z}(x) d\mu_P^\perp(z).$$

Taking $\phi(x, P) = \chi_E(P)$ we see that $V(\mathbb{R}^n \times E) = \nu(E) = 0$. Likewise we compute for the set $A = \{(x, P) : P \notin E, P^\perp x \in E_P\}$

$$V(A) = \int_{G(m, n)} \int_{P^\perp} \int_{z + P} \chi_A(x, P) d\mu_{P,z}(x) d\mu_P^\perp(z) d\nu(P)$$

$$= \int_{G(m, n)} \int_{P^\perp} \chi_{E_P}(z) \mu_{P,z}(z + P) d\mu_P^\perp(z) d\nu(P) = 0.$$

**Remark 3.1.** We will assume without loss of generality that the $\mu_P$ have support in the compact set $K_P = \{x \in \mathbb{R}^n : (x, P) \in \text{spt } V\} \subset \overline{\Omega}$. In fact

$$\int_{G(m, n)} \int_{\mathbb{R}^n} \phi(x, P) \chi_{K_P}(x) d\mu_P(x) d\nu(P) = \int_{\mathbb{R}^n \times G(m, n)} \phi(x, P) \chi_{\text{spt } V}(x, P) dV(x, P)$$

$$= \int_{\mathbb{R}^n \times G(m, n)} \phi(x, P) dV(x, P).$$

Thus we can replace the $\mu_P$ by $\mu_P^\perp K_P$ without change. Likewise, we may assume that $\mu_{P,z}$, for each $P \in G(m, n)$ and $z \in P^\perp$, is supported in $K_{P,z} = \text{spt } \mu_P \cap (z + P)$. In particular, for $x \in z + P$ we then have the implications

$$x \in \text{spt } \mu_{P,z} \Rightarrow x \in \text{spt } \mu_P \Rightarrow (x, P) \in \text{spt } V.$$

**Lemma 3.2.** Let $V$ be an $m$-varifold in $\mathbb{R}^n$ with support in $\overline{\Omega}$, which is orthogonal to $S$ along $\Gamma$ and has curvature $B_V = 0$. Then for $\nu$-a.e. $P \in G(m, n)$ and $\mu_P^\perp$-a.e. $z \in P^\perp$

$$\int_{z + P} \langle D\varphi(x), P \rangle d\mu_{P,z}(x) = 0 \quad \text{for all } \varphi \in C^1(\mathbb{R}^n, \mathbb{R}^n) \text{ tangential on } S.$$

**Proof.** We use the ansatz $\phi(x, P) = \varphi(x)\psi(P)$ in the orthogonal boundary condition, see Definition 2.2. This gives, since $\varphi(x)$ is tangential by assumption

$$0 = \int_{\mathbb{R}^n \times G(m, n)} \langle D_x \phi(x, P), P \rangle dV(x, P)$$

$$= \int_{G(m, n)} \psi(P) \int_{\mathbb{R}^n} \langle D\varphi(x), P \rangle d\mu_P(x) d\nu(P).$$
Thus for any \( P \in G(m, n) \), up to a set \( E(\varphi) \) with measure \( \nu(E(\varphi)) = 0 \), we have
\[
\int_{\mathbb{R}^n} \langle D\varphi(x), P \rangle d\mu_P(x) = 0.
\]

We may also apply the above equation with \( \varphi(x) \) replaced by \( \sigma(P^\perp x)\varphi(x) \) (for a smooth function \( \sigma \) and \( \varphi \) exactly as before). We have
\[
D\tau(\sigma(P^\perp x)) = D\sigma(P^\perp x)P^\perp \tau = 0 \quad \text{for } \tau \in P.
\]

Thus we get
\[
0 = \int_{P^\perp} \sigma(z) \int_{z+P} \langle D\varphi(x), P \rangle d\mu_{P,z}(x) d\mu_P(z).
\]

For any \( z \in P^\perp \), up to a set \( E_P(\varphi) \) with \( \mu_P(E_P(\varphi)) = 0 \), we conclude
\[
0 = \int_{z+P} \langle D\varphi(x), P \rangle d\mu_{P,z}(x).
\]

This is the desired result, except that the null sets depend on the test function \( \varphi \). However, the set of \( C^1 \) functions \( \varphi \) which are tangential along \( \partial \Omega \) is separable. We can choose fixed null sets such that the identity holds for \( \varphi \) in a countable dense set, and hence for all \( \varphi \) by approximation. \( \square \)

**Lemma 3.3** (zero curvature). Let \( V \) be a varifold in \( \Omega \) with curvature \( B = 0 \). Then for \( \nu \)-a.e. \( P \in G(m, n) \) and \( \mu_P^\perp \)-a.e. \( z \in P^\perp \), the following holds: for any connected component \( U \) of \( \Omega \cap (z + P) \) one has
\[
\mu_{P,z} = \theta_U \mathcal{L}^m \quad \text{on } U, \quad \text{for a constant } \theta_U \geq 0.
\] (27)

The lemma follows from the well-known constancy lemma below, using test functions \( \chi(x) = \omega(x)\tau_\alpha \) for an orthonormal basis \( \tau_\alpha \) of \( P \).

**Lemma 3.4** (constancy). Let \( \mu \) be a Radon measure on an open domain \( U \subset \mathbb{R}^m \), and assume that for all \( \omega \in C^\infty_c(U) \)
\[
\int_U \partial_\alpha \omega(x) d\mu(x) = 0 \quad \text{for } \alpha = 1, \ldots, m.
\]

Then \( \mu = \theta \mathcal{L}^m \) on \( U \) for some \( \theta \in [0, \infty) \).

The proof involving mollification is standard (cf. also Theorem 26.27 in [18]).

**Lemma 3.5** (dichotomy). Let \( V \) be an \( m \)-varifold in \( \mathbb{R}^n \) with support in \( \overline{\Omega} \), which is orthogonal to \( S \) along \( \Gamma \) and has curvature \( B_V = 0 \). Then for any \( (x_0, P_0) \in \text{spt } V \) with \( x_0 \in S \) one has the alternative
\[
\nu^S(x_0) \in P_0 \quad \text{or} \quad P_0 \subset T_{x_0}S.
\] (28)
In the second case, there exists $\varrho > 0$ such that

\[ (x, P_0) \in \text{spt } V \quad \text{for all } x \in (x_0 + P_0) \cap B_\varrho(x_0). \]  

(29)

In particular, $(x_0 + P_0) \cap B_\varrho(x_0) \subset \overline{\Omega}$.

Proof. Let $(x_0, P_0) \in \text{spt } V$, $x_0 \in S$, and assume

\[ \nu^S(x_0) \notin P_0. \]  

(30)

By continuity, there exists $r_0 > 0$ such that for all $0 < \delta, \varrho \leq r_0$ we have

\[ \nu^S(x) \notin P \quad \text{for all } x \in S \cap B_{2\varrho}(x_0), \ P \in B_\delta(P_0). \]

It follows that

\[ \nu^S(x) \cap Q \notin B_\delta(P_0) \quad \text{for all } x \in S \cap B_{2\varrho}(x_0), \ Q \in G_{m-1}(T_x S). \]

Thus for all $\varphi \in C^1_c(B_{2\varrho}(x_0), \mathbb{R}^n)$ and all $\psi \in C^1_c(B_\delta(P_0))$, we have

\[
\int_S \langle \nu^S(x), \varphi(x) \rangle \int_{G_{m-1}(T_x S)} \psi(\nu^S(x) \cap Q) d\Gamma^x(Q) d\sigma_{T}(x) \\
= \int_{S \cap B_{2\varrho}(x_0)} \langle \nu^S(x), \varphi(x) \rangle \int_{G_{m-1}(T_x S)} \psi(\nu^S(x) \cap Q) d\Gamma^x(Q) d\sigma_{T}(x) = 0.
\]

Definition 2.2, with $B_V = 0$ and $\phi(x, P) = \varphi(x) \psi(P)$, now implies

\[ 0 = \int_{G(m,n)} \psi(P) \int_{\mathbb{R}^n} \langle P, D\varphi(x) \rangle d\mu_P(x) d\nu(P). \]

Thus for $\nu$-a.e. $P \in B_\delta(P_0)$ we obtain

\[ \int_{\mathbb{R}^n} \langle P, D\varphi(x) \rangle d\mu_P(x) = 0 \quad \text{for all } \varphi \in C^1_c(B_{2\varrho}(x_0), \mathbb{R}^n). \]

Fix such a $P \in B_\delta(P_0)$, and consider a test function $\varphi(x) = \chi(Px)\zeta(P^\perp x)$, where $\chi \in C^1_c(P \cap B_\varrho(Px_0), \mathbb{R}^n)$ and $\zeta \in C^1_c(P^\perp \cap B_\varrho(P^\perp x_0))$. Then

\[ \int_{P^\perp} \zeta(z) \int_{z+P} \text{div}_P \chi(Px) d\mu_{P,z}(x) d\mu_P^\perp(z) = 0. \]

Therefore we conclude, for $\mu_P^\perp$-a.e. $z \in P^\perp \cap B_\varrho(P^\perp x_0)$,

\[ 0 = \int_{z+P} \text{div}_P \chi(Px) d\mu_{P,z}(x) \]

\[ = \int_{z+P} \text{div}_P \chi(x - z) d\mu_{P,z}(x) \quad \text{for all } \chi \in C^1_c(P \cap B_\varrho(Px_0), \mathbb{R}^n). \]
Thus for \( \nu \text{-a.e. } P \in B_\delta(P_0) \) and \( \mu_\perp^\perp \text{-a.e. } z \in P^\perp \cap B_\theta(P^\perp x_0) \), we have
\[
\mu_{P,z} = \theta \mathcal{L}^m(z + P) \quad \text{on } z + P \cap B_\theta(Px_0) \quad \text{for some } \theta \in [0, \infty). \quad (31)
\]

We claim that for any \( y_0 \in P_0 \cap B_{\theta/2}(0) \), we have \( (x_0 + y_0, P_0) \in \text{spt } V \). Let \( B_\sigma(x_0 + y_0) \) be given, wlog with \( \sigma \leq \frac{\delta}{4} \). For \( x \in B_\sigma(x_0 + y_0) \) we have
\[
|Px - Px_0| \leq |P(x - (x_0 + y_0))| + |(P - P_0)y_0| + |y_0| \leq \frac{3\theta}{4} + \frac{\delta \theta}{2},
\]
and
\[
|P^\perp x - P^\perp x_0| \leq |P^\perp(x - (x_0 + y_0))| + |(P^\perp - P^\perp_0)y_0| \leq \frac{\theta}{4} + \frac{\delta \theta}{2}.
\]

Thus \( |P(x - x_0)|, |P^\perp(x - x_0)| < \theta \) for all \( x \in B_\sigma(x_0 + y_0) \), provided \( \delta < \frac{1}{4} \). Now we consider a ball \( B_\varepsilon(x_0 + P y_0) \) and estimate
\[
|(x_0 + y_0) - (x_0 + P y_0)| = |P^\perp y_0| = |(P^\perp - P^\perp_0)y_0| \leq \frac{\delta \theta}{2}.
\]

We choose \( \varepsilon = \frac{\sigma}{2} \), and conclude
\[
B_\varepsilon(x_0 + P y_0) \subset B_\sigma(x_0 + y_0) \quad \text{for any } \delta < \frac{\sigma}{\theta}.
\]

Now we compute for fixed \( P \in B_\delta(P_0) \) satisfying (31)
\[
\mu_P(B_\sigma(x_0 + y_0)) \geq \mu_P(B_\varepsilon(x_0 + P y_0)) \\
= \int_{P^\perp} \mu^\perp_P((z + P) \cap B_\varepsilon(x_0 + P y_0)) d\mu^\perp_P(z) \\
= \int_{P^\perp} \theta P^\perp \mathcal{L}^m((z + P) \cap B_\varepsilon(x_0 + P y_0)) d\mu^\perp_P(z) \\
= \int_{P^\perp} \theta P^\perp \mathcal{L}^m((z + P) \cap B_\varepsilon(x_0)) d\mu^\perp_P(z) \\
= \mu_P(B_\varepsilon(x_0)).
\]

Integrating over all \( P \in B_\delta(P_0) \) we conclude
\[
V(B_\sigma(x_0 + y_0) \times B_\delta(P_0)) = \int_{B_\delta(P_0)} \mu_P(B_\sigma(x_0 + y_0)) d\nu(P) \\
\geq \int_{B_\delta(P_0)} \mu_P(B_\varepsilon(x_0)) d\nu(P) \\
= V(B_\varepsilon(x_0) \times B_\delta(P_0)) > 0,
\]

since \( (x_0, P_0) \in \text{spt } V \). This proves \( (x_0 + y_0, P_0) \in \text{spt } V \) for any \( y_0 \in P_0 \cap B_{\theta/2}(0) \). As a direct consequence, it follows that \( (x_0 + P_0) \cap B_{\theta/2}(x_0) \subset \overline{\Omega} \).
Lemma 3.6. Let $V$ be an $m$-varifold in $\mathbb{R}^n$ with support in $\overline{\Omega}$, which is orthogonal to $S$ along $\Gamma$ and has curvature $B_v = 0$. Let $x_0 \in S$, and assume $(x_0, P_0) \in \text{spt} V$ where $P_0 \subset T_{x_0}S$. Then for $z_0 = P_0^\perp x_0$ we have near $x_0$, if $P_0, z_0$ is as in Lemma 3.2

$$\mu_{P_0, z_0} = \theta L^m(z_0 + P_0) \quad \text{for some constant } \theta \geq 0.$$

Proof. For any $\varphi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ which is tangential on $S$, we have by Lemma 3.2

$$\int_{z_0 + P_0} \langle D\varphi(x), P_0 \rangle d\mu_{P_0, z_0}(x) = 0. \tag{32}$$

Choose a neighborhood $U \times I$ of $x_0$ on which $\Omega$ has a subgraph representation

$$\Omega \cap (U \times I) = \{ (y, z) \in U \times I : z < u(y) \}.$$

After rotation we assume $x_0 = 0$, $T_{x_0}S = \mathbb{R}^{n-1}$ and $P_0 = \mathbb{R}^m \times \{0\}$ where $m \leq n - 1$. For any $\sigma \in C^1_c(U \times I)$ we consider the test function

$$\varphi(y, z) = \sigma(y, z) (e_\alpha + \partial_\alpha u(y)e_n) \quad \text{where } x = (y, z) \text{ and } 1 \leq \alpha \leq m.$$

By construction the vector field $\varphi$ is tangential along $\partial \Omega$. Furthermore

$$\langle D\varphi(x), P_0 \rangle = \sum_{\beta=1}^m \langle \partial_\beta \varphi(x), e_\beta \rangle = \partial_\alpha \sigma(x).$$

Therefore (32) implies

$$\int_{\mathbb{R}^m} \partial_\alpha \sigma(x) d\mu_{R^m, 0} = 0 \quad \text{where } (P_0, z_0) = (\mathbb{R}^m, 0).$$

Since any function with compact support in $\mathbb{R}^m \cap U$ can be extended to a function with compact support in $U \times I$, the claim follows from Lemma 3.4. \hfill \Box

Theorem 7 (Orthogonal Varifolds with Curvature Zero). Let $V$ be an $m$-varifold in $\mathbb{R}^n$ with support in $\overline{\Omega}$, which is orthogonal to $S = \partial \Omega$ along an $(m-1)$-varifold $\Gamma$ and has curvature $B_V = 0$. Then for all $\phi \in C^0(\mathbb{R}^n \times G(m, n))$ and $\varphi \in C^0(G_{m-1}(TS))$, respectively, the following representations hold:

$$V(\phi) = \int_{G(m,n)} \int_{P_\perp} \sum_{\Delta \in S^m_\perp(\Omega, z + P)} \theta_\Delta \int_{\Delta} \phi(x, P) d\mathcal{L}^m(x) d\mu_P^\perp(z) d\nu(P),$$

$$\Gamma(\varphi) = \int_{G(m,n)} \int_{P_\perp} \sum_{\Delta \in S^m_\perp(\Omega, z + P)} \theta_\Delta \int_{\partial \Delta} \varphi(x, P \cap T_xS) d\mathcal{H}^{m-1}(x) d\mu_P^\perp(z) d\nu(P).$$

Here $S^m_\perp(\Omega, z + P)$ is the finite, possibly empty set of orthogonal $m$-slices of $\Omega$ in the affine plane $z + P$, and the $\theta_\Delta \geq 0$ are constants.
Proof. Let $\Delta$ be any $m$-slice of $\Omega$ in an affine plane $z + P$, where $P \in G(m, n)$ and $z \in P^\perp$. By definition, for any $x \in \Delta$ there exists $\rho > 0$ such that

$$(z + P) \cap B_{\rho}(x) \subset \overline{\Omega}.$$ 

In particular if $x \in \Delta \cap S$ then $P \subset T_x S$. Lemma \ref{Lem3.3} and Lemma \ref{Lem3.6} imply that $\mu_{P,z} = \theta L^m 
(x + P)$ locally near any $x \in \Delta$. Since $\Delta$ is connected, we conclude

$$\mu_{P,z} \Delta = \theta \Delta L^m \Delta \quad \text{for some } \theta \geq 0.$$ 

Actually, Lemma \ref{Lem3.6} requires that $(x, P) \in \text{spt } V$, but otherwise the statement follows with $\theta_\Delta = 0$, see Remark \ref{Rem3.1}. Now assume that $\theta_\Delta > 0$, and consider a boundary point $x \in \partial \Delta \subset S$. We then have $(x, P) \in \text{spt } V$ again by Remark \ref{Rem3.1}. We claim that $\nu^S(x) \in P$. Otherwise Lemma \ref{Lem3.3} yields that $(x + P) \cap B_{\rho}(x) \subset \overline{\Omega}$ for some $\rho > 0$. But $\Delta \cap B_{\rho}(x)$ is nonempty, thus $(x + P) \cap B_{\rho}(x) \subset \Delta$ and $x \in \Delta$, a contradiction. This shows that if $\theta_\Delta > 0$ then $\nu^S(x) \in P$ for all $x \in \partial \Delta$, in other words $\Delta$ is an orthogonal $m$-slice. Along $\partial \Delta$ we have

$$\nabla(\text{dist}_S|_{z+P})(x) = P \nabla \text{dist}_S(x) = \nu^S(x) \neq 0,$$

therefore $\Delta$ is a bounded domain in $z + P$ of class $C^2$. We denote by $\Delta_{P,z}$ the union of the finitely many orthogonal slices $\Delta \subset z + P$ with $\theta_\Delta > 0$, and introduce the function

$$\theta_{P,z} : z + P \to [0, \infty), \quad \theta_{P,z}(x) = \theta_\Delta \text{ for } x \in \overline{\Delta}. $$

The measure $\tilde{\mu}_{P,z} = L^m \theta_{P,z}$ satisfies, for any $\varphi \in C^1(z + P, P)$,

$$\int_{z+P} \langle D^\perp \varphi(x), P \rangle \, d\tilde{\mu}_{P,z}(x) = \sum_{\Delta \in S^m_{\perp}(\Omega,z+P)} \theta_\Delta \int_\Delta \langle D^\perp \varphi(x), P \rangle \, dL^m(x)$$

$$= - \sum_{\Delta \in S^m_{\perp}(\Omega,z+P)} \theta_\Delta \int_{\partial \Delta} \langle \varphi(x), \nu^S(x) \rangle \, dH^{m-1}(x).$$

In other words we have, in the sense of distributions on $z + P$,

$$\nabla \tilde{\mu}_{P,z} = -\theta_{P,z}(H^{m-1} \cap \partial \Delta_{P,z}) \otimes \nu^S. \quad (33)$$

On functions $\varphi \in C^0_c(\mathbb{R}^n \times G(m, n))$, we now consider the $m$-varifold $\vec{V}$ given by

$$\vec{V}(\varphi) = \int_{G(m,n)} \int_{P^\perp} \int_{z+P} \varphi(x, P) \, d\tilde{\mu}_{P,z}(x) \, d\mu_P^\perp(z) \, d\nu(P). \quad (34)$$

For $\varphi \geq 0$ we can estimate

$$0 \leq \vec{V}(\varphi) = \int_{G(m,n)} \int_{P^\perp} \int_{\Delta_{P,z}} \varphi(x, P) \, d\mu_{P,z}(x) \, d\mu_P^\perp(z) \, d\nu(P)$$

$$\leq \int_{G(m,n)} \int_{P^\perp} \int_{z+P} \varphi(x, P) \, d\mu_{P,z}(x) \, d\mu_P^\perp(z) \, d\nu(P)$$

$$= V(\varphi).$$

16
In particular $\tilde{V}$ is continuous, in fact we have

$$|\tilde{V}(\varphi)| \leq \tilde{V}(|\varphi|) \leq V(|\varphi|) \leq \mathbf{M}(V) \|\varphi\|_{C^0(\mathbb{R}^n \times G(m,n))}.$$ 

It follows that $\Lambda(\varphi) := V(\varphi) - \tilde{V}(\varphi)$ is a Radon measure on $\mathbb{R}^n \times G(m,n)$. Moreover $\tilde{V}(\varphi) = V(\varphi)$ whenever $\varphi$ has compact support in $\Omega \times G(m,n)$. Therefore $\Lambda$ has support in $S \times G(m,n)$. Using (33) we further have for $\phi \in C^1(\mathbb{R}^n \times G(m,n), \mathbb{R}^n)$

$$\int_{\mathbb{R}^n \times G(m,n)} \langle D_x \phi(x, P), P \rangle \, d\tilde{V}(x, P) = \int_{G(m,n)} \int_{P \perp} \int_{z + P} \langle D_x \phi(x, P), P \rangle \, d\tilde{\mu}_{P_z}(x) \, d\mu_P^1(z) \, d\nu(P)$$

$$= - \int_{G(m,n)} \int_{P \perp} \int_{\partial D_{P_z}} \theta_{P_z}(x) \langle \nu^S(x), \phi(x, P) \rangle \, d\mathcal{H}^{m-1}(x) \, d\mu_P^1(z) \, d\nu(P).$$

On the other hand, $V$ satisfies the orthogonal boundary constraint

$$\int_{\mathbb{R}^n \times G(m,n)} \langle D_x \phi(x, P), P \rangle \, dV(x, P) = - \int_{G_{m-1}(TS)} \langle \nu^S(x), \phi(x, \nu^S(x) \wedge Q) \rangle \, d\Gamma(x, Q).$$

We take a test function of the form $\phi(x, P) = \varphi(x)$ where $\varphi|_S = 0$. Then

$$0 = \int_{\mathbb{R}^n \times G(m,n)} \langle D \varphi(x, P), P \rangle \, dV(x, P) - \int_{\mathbb{R}^n \times G(m,n)} \langle D \varphi(x, P), P \rangle \, d\tilde{V}(x, P)$$

$$= \int_{S \times G(m,n)} \langle D \varphi(x, P), P \rangle \, d\Lambda(x, P)$$

$$= \int_{S \times G(m,n)} \langle D \varphi(x) \nu^S(x), \nu^S(x) \rangle \, d\Lambda(x, P)$$

$$= \int_S \Lambda^x(G(m,n)) \langle D \varphi(x) \nu^S(x), \nu^S(x) \rangle \, d\mu_A(x),$$

where $\mu_A, \Lambda^x$ are the measures arising from the disintegration of $\Lambda$, cf. Theorem \ref{disintegration}. Taking now $\varphi(x) = \text{dist}_S(x) \nu^S(x)$ yields $D \varphi(x) \nu^S(x) = \nu^S(x)$ along $S$. We conclude

$$0 = \int_S \Lambda^x(G(m,n)) \, d\mu_A(x) = \Lambda(\mathbb{R}^n \times G(m,n)).$$

This shows $\Lambda = 0$ and hence $V = \tilde{V}$, which is the desired representation for $V$. Finally, we claim that the boundary varifold $\Gamma$ in Definition \ref{boundary_varifold} has the representation

$$\Gamma(\varphi) = \int_{G(m,n)} \int_{P \perp} \int_{\partial D_{P_z}} \theta_{P_z}(x) \varphi(x, P \cap T_x S) \, d\mathcal{H}^{m-1}(x) \, d\mu_P^1(z) \, d\nu(P).$$

First note that $P \cap T_x S \in G_{m-1}(T_x S)$ as $\nu^S(x) \in P$ for $x \in \partial D_{P_z}$. For given $\varphi \in C^1(G_{m-1}(TS))$ we consider as test function $\phi(x, P) = \varphi(x, P \cap T_x S) \nu^S(x)$, suitably
extended to a $C^1$ function on $\mathbb{R}^n \times G(m, n)$. From $V = \tilde{V}$ and $B_V = 0$ we get

\[
\int_{G_{m-1}(TS)} \varphi(x, Q) d\Gamma(x, Q)
= \int_{G_{m-1}(TS)} \langle \nu^S(x), \phi(x, \nu^S(x) \wedge Q) \rangle d\Gamma(x, Q)
= -\int_{\mathbb{R}^n \times G(m, n)} \langle D_x \phi(x, P), P \rangle dV(x, P)
= \int_{G(m, n)} \int_{\partial P, z} \theta_{P, z}(x) \langle \nu^S(x), \phi(x, P) \rangle dH^{m-1}(x) d\mu_P^+(z) d\nu(P)
= \int_{G(m, n)} \int_{\partial P, z} \theta_{P, z}(x) \varphi(x, P \cap T_x S) dH^{m-1}(x) d\mu_P^+(z) d\nu(P).
\]

\[
4 \text{ Mass and Topology Bounds}
\]

In this section Theorem 7 is applied to prove the bounds stated in the introduction.

**Proof of Theorem 4.** We first prove that $\kappa_{m,p}(\Omega) > 0$. As $\kappa_{m,p}(\Omega) \geq \kappa_{m,1}(\Omega)$ by definition, it suffices to consider the case $p = 1$. Assume by contradiction that there is a sequence $V_k \in CV^m_\perp(\Omega)$ with $M(V_k) = 1$, such that $\|B_{V_k}\|_{L^1(V_k)} \to 0$. For the associated boundary varifolds $\Gamma_k$, we have by Lemma 2.6

\[
M(\Gamma_k) \leq C(M(V_k) + \|B_{V_k}\|_{L^1(V_k)}) \leq C
\]

where $C = C(n, \Omega)$.

Thus we can assume that $V_k \to V$ where again $M(V) = 1$, and also $\Gamma_k \to \Gamma$ as varifolds. By passing to the limit in (20) we get

\[
\int_{\mathbb{R}^n \times G(m, n)} \langle D_x \phi, P \rangle dV(x, P) = -\int_{G_{m-1}(TS)} \langle \nu^S(x), \phi(x, \nu^S(x) \wedge Q) \rangle d\Gamma(x, Q).
\]

This means that $V$ is orthogonal to $S$ along $\Gamma$ and has curvature zero. But $\Omega$ has no orthogonal $m$-slices by assumption, and hence Theorem 7 implies that $V = 0$, a contradiction. For given nonzero $V \in CV^m_\perp(\Omega)$, let $V' = \lambda V$ where $\lambda = M(V)^{-1}$, thus $M(V') = 1$. By definition of $\varphi_{m,p}(\Omega)$ and (21)

\[
\|B_V\|_{L^p(V')} = M(V) \|B_{V'}\|_{L^p(V')} \geq M(V) \varphi_{m,p}(\Omega).
\]

The estimate for $\Gamma$ follows from Lemma 2.6 and Remark 2.7.

\[
4 \text{ Mass and Topology Bounds}
\]

**Proof of Theorem 7.** As discussed in Section 2 any $\Sigma \in S^m_{\perp}(\Omega)$ with $\int_{\Sigma} |A_{\Sigma}|^p d\mu_{\Sigma} < \infty$ induces a curvature varifold in $CV^m_\perp(\Omega)$. Hence the mass bound (11) follows from
Theorem 4. Now we turn to the special case when $m = 2$ and $p = 2$. Let $\Sigma \in S^2_\perp(\Omega)$ with $\int_{\Sigma} |A_\Sigma|^2 \, d\mu_\Sigma < \infty$. Using $\eta_{\partial \Sigma} = \nu^S |_{\partial \Sigma}$ we have by the Gauss-Bonnet Theorem and Meusnier’s Theorem

$$2\pi \chi(\Sigma) = \int_{\Sigma} K_\Sigma \, d\mu_\Sigma + \int_{\partial \Sigma} h^S(\tau, \tau) \, d\sigma_{\partial \Sigma}. \quad (35)$$

This is immediate if $\Sigma$ is of class $C^2$ up to the boundary. For general $\Sigma \in S^2_\perp(\Omega)$ one uses an approximation, see also the calculation before (2.10) in [1]. Now (35) yields

$$2\pi |\chi(\Sigma)| \leq \frac{1}{2} \int_{\Sigma} |A_\Sigma|^2 \, d\mu_\Sigma + \|h^S\|_{C^0(\Sigma)} |\partial \Sigma|.$$

The claimed estimate for $\chi(\Sigma)$ follows easily from this and equation (1).

**Proof of Theorem 2.** To prove claim (3) it is sufficient to bound $|\Sigma|$, since then $|\partial \Sigma|$ is estimated by Lemma 2.6. Assume by contradiction that there is a sequence $\Sigma_k \in S^m_\perp(\Omega)$ with $\Sigma_k \subset U_\delta(S)$, such that

$$\int_{\Sigma_k} |A_{\Sigma_k}|^p \, d\mu_{\Sigma_k} < \frac{1}{k} |\Sigma_k|.$$

Passing to $V_k = \Sigma_k/|\Sigma_k| \in CV^m(\Omega)$ we have $\mathbf{M}(V_k) = 1$, and (21) yields

$$\|B_{V_k}\|_{L^p(V_k)}^p = 2^{p/2} \|A_{V_k}\|_{L^p(V_k)}^p/|\Sigma_k| < 2^{p/2} \frac{1}{k} \to 0.$$

$V_k$ satisfies Definition 2.2 with boundary $\Gamma_k = \partial \Sigma_k/|\Sigma_k|$. As $\mathbf{M}(\Gamma_k) \leq C$ by Lemma 2.6 we may assume that $V_k \to V$ and $\Gamma_k \to \Gamma$ as varifolds. It follows that $\mathbf{M}(V) = 1$, $V$ is orthogonal to $S$ and has curvature zero. Moreover

$$\text{spt} \mu_V \subset \overline{U_\delta(S)}. \quad (36)$$

Let $\Delta$ be any orthogonal $m$-slice of $\Omega$. Choose $p \in \Delta$ such that $\varrho = \text{dist}(p, \partial \Delta)$ is maximal. Then there exist at least two points $x \in \partial \Delta$ such that $p = x + \varrho \nu^S(x)$, and thus the normal injectivity radius of $S$ satisfies $\varrho_S \leq \varrho$. We get for $\Omega_\delta = \Omega \setminus U_\delta(S)$

$$|\Delta \cap \Omega_\delta| \geq c(m)(\varrho_S - \delta)^m > 0.$$

On the other hand, by comparing with a ball containing $\Omega$, we have $|\Delta| \leq C$ for $C = C(m, \Omega)$. From the representation in Theorem 7 we now get

$$\mu_V(\Omega_\delta) = \int_{G(m,n)} \int_{P^\perp} \sum_{\Delta \in S^m_\perp(\Omega_\delta + P)} \theta_\Delta |\Delta \cap \Omega_\delta| \, d\mu^\perp_P(z) \, dv(P)$$

$$\geq \int_{G(m,n)} \int_{P^\perp} \sum_{\Delta \in S^m_\perp(\Omega_\delta + P)} \theta_\Delta |\Delta| \, d\mu^\perp_P(z) \, dv(P)$$

$$= \int_{G(m,n)} \int_{P^\perp} \sum_{\Delta \in S^m_\perp(\Omega_\delta + P)} \theta_\Delta |\Delta| \, d\mu^\perp_P(z) \, dv(P)$$

This contradicts (36) and proves claim (3). The bound on the Euler characteristic in the case $m = p = 2$ follows as in the proof of Theorem 1. $\square$
Example 4.1. If Ω is a round ball or an \( n \)-dimensional ellipsoid, then any orthogonal slice \( \Delta \) contains the origin. Adapting the arguments yields the following: for any \( \varepsilon > 0 \) there is a constant \( C = C(n, m, p, \Omega, \varepsilon) < \infty \) such that for any \( \Sigma \in S^m_{\perp}(\Omega) \)

\[
|\Sigma| + |\partial \Sigma| \leq C \int_{\Sigma} |A_{\Sigma}|^p \, d\mu_{\Sigma} \quad \text{whenever } \Sigma \subset \Omega \setminus B_{\varepsilon}(0).
\]

Proof of Theorem \([\blacksquare] \) It suffices to prove the bound for \( |\Sigma| \). Assume by contradiction that there are immersed surfaces \( \Sigma_k \in S^2_{\perp}(\Omega) \) such that

\[
\int_{\Sigma_k} |A_{\Sigma_k}|^2 \, d\mu_{\Sigma_k} + \chi(\Sigma_k)^+ < \frac{1}{k} |\Sigma_k|.
\]

The sequence \( V_k = \Sigma_k/|\Sigma_k| \in C\mathcal{V}^2_{\perp}(\Omega) \) has norm \( M(V_k) = 1 \) and boundary \( \Gamma_k = \partial \Sigma_k/|\Sigma_k| \). After passing to a subsequence we have \( V_k \to V \) and \( \Gamma_k \to \Gamma \) as varifolds, where Definition \([\blacksquare]\) holds in the limit with curvature \( B_V = 0 \). Now by \((33)\) we have, denoting by \( \tau_{\partial \Sigma_k} \) the unit tangent vector of \( \partial \Sigma_k \),

\[
\int_{\partial \Sigma_k} h^S(\tau_{\partial \Sigma_k}, \tau_{\partial \Sigma_k}) \, d\sigma_{\partial \Sigma_k} \leq 2\pi \chi(\Sigma_k)^+ - \int_{\Sigma_k} K_{\Sigma_k} \, d\mu_{\Sigma_k}.
\]

We get a well-defined function \( \varphi : G_1(TS) \to \mathbb{R} \) by putting \( \varphi(x, Q) = h^S(x)(\tau, \tau) \) where \( \tau \in Q \) with \( |\tau| = 1 \). By the varifold convergence \( \Gamma_k \to \Gamma \), we have

\[
\int_{G_1(TS)} \varphi(x, Q) \, d\Gamma(x, Q) = \lim_{k \to \infty} \int_{G_1(TS)} \varphi(x, Q) \, d\Gamma_k(x, Q)
\]

\[
= \lim_{k \to \infty} \frac{1}{|\Sigma_k|} \int_{\partial \Sigma_k} h^S(\tau_{\partial \Sigma_k}, \tau_{\partial \Sigma_k}) \, d\sigma_{\partial \Sigma_k}
\]

\[
\leq \lim_{k \to \infty} \frac{1}{|\Sigma_k|} \left( 2\pi \chi(\Sigma_k)^+ - \int_{\Sigma_k} K_{\Sigma_k} \, d\mu_{\Sigma_k} \right)
\]

\[
= 0.
\]

On the other hand Theorem \([\blacksquare]\) yields the representation

\[
\Gamma(\varphi) = \int_{G(2, m)} \int_{\Delta \in S^2_{\perp}(\Omega, \varepsilon + P)} \theta_{\Delta} \int_{\partial \Delta} \varphi(x, P \cap T_x S) \, d\mathcal{H}^1(x) \, d\mu^p_P(z) \, d\nu(P).
\]

Since \( \Delta \) is an orthogonal slice, we have by the computation above

\[
\varphi(x, P \cap T_x S) = h^S(x)(\tau_{\partial \Delta}(x), \tau_{\partial \Delta}(x)) = \kappa_{\partial \Delta}(x).
\]

Here \( \Delta \) is a domain in the plane \( P \) and \( \kappa_{\partial \Delta} \) is the Euclidean curvature of its boundary. Now by assumption each \( \Delta \) is a topological disk, therefore by the Hopf Umlaufsatz \([\blacksquare]\)

\[
\int_{\partial \Delta} \varphi(x, P \cap T_x S) \, d\mathcal{H}^1(x) = \int_{\partial \Delta} \kappa_{\partial \Delta}(x) \, d\mathcal{H}^1(x) = 2\pi.
\]
Inserting we obtain
\[ \Gamma(\varphi) = 2\pi \int_{\mathbb{R}^n} \int_{P^\perp} \sum_{\Delta \in S^2_{\perp}(\Omega, z + P)} \theta_{\Delta} \mu_{\perp}^1(z) \, d\nu(P). \]

But this yields, since \(|\Delta| \leq C(n, \Omega)|\),
\[ 1 = M(V) = \int_{G(2,n)} \int_{P^\perp} \sum_{\Delta \in S^2_{\perp}(\Omega, z + P)} \theta_{\Delta} |\Delta| \leq C(n, \Omega) \Gamma(\varphi) \leq 0. \]

\[ \square \]

**Example 4.2.** The assumptions in Theorem 1 and Theorem 3 cannot be dropped. Let \( \Omega = \{(x, y, z) \in \mathbb{R}^3 : 1 < \sqrt{x^2 + y^2} < 2, -1 < z < 1\} \).

If \( V_k \) is the varifold given by the annulus \( A = \{(x, y, 0) : 1 < \sqrt{x^2 + y^2} < 2\} \) with multiplicity \( k \), then \( V_k \) has zero curvature and orthogonal boundary in \( \Omega \), but the mass \( M(V_k) = 3k\pi \) goes to infinity.

By a slight modification one also gets a counterexample of disk-type surfaces with bounded curvature and orthogonal boundary in a smooth domain. Namely one can add to the previous example two copies of the vertical strip \( \{(x, 0, z) : 1 < x < 2, 0 < z < 1\} \), thereby obtaining a surface with orthogonal boundary which can be parametrized on a rectangle. A suitable approximation yields a smooth example.

## 5 Minimizers with Orthogonal Boundary

**Proof of Theorem 5.** If \( \Omega \) has an orthogonal slice \( \Delta \), then it is a minimizer with curvature zero and the theorem follows. Otherwise let \( V_k \in CV^2_{\perp}(\Omega) \) be a minimizing sequence. By Theorem 4 we then have
\[ M(V_k) + M(\Gamma_k) \leq C \| B_{V_k} \|^2_{L^2(\Omega)} \quad \text{where} \quad C = C(n, \Omega). \quad (37) \]

Lemma 2.8 yields that after passing to a subsequence, \( V_k \to V \) and \( \Gamma_k \to \Gamma \) as varifolds in \( \mathbb{R}^n \), where \( V, \Gamma \) satisfy Definition 2.7 and hence \( V \in CV^2_{\perp}(\Omega) \), and further
\[ \| B_V \|^2_{L^2(\Omega)} \leq \liminf_{k \to \infty} \| B_{V_k} \|^2_{L^2(\Omega_k)}. \]

By Remark 2.3, we have for any \( \phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n) \)
\[ \delta V_k(\phi) = -\int_{\mathbb{R}^n} \langle H_{V_k}, \phi \rangle \, d\mu_{V_k} - \int_{\Gamma} \langle \nu^S(x), \phi(x) \rangle \, d\sigma_{\Gamma_k}. \]
Here $H_{V_k}(x) = \text{tr} B_{V_k}(x, T_x \mu_{V_k})$ and $\sigma_{\Gamma_k}$ is the weight measure of $\Gamma_k$. In particular

$$|\delta V_k(\phi)| \leq \left( \| H_{V_k} \|_{L^2(V_k)} M(V_k)^{\frac{1}{2}} + M(\Gamma_k) \right) \| \phi \|_{C^0(\mathbb{R}^n)} \leq C \| \phi \|_{C^0(\mathbb{R}^n)}.$$

Now Allard’s integral compactness [2] implies $V \in \text{ICV}_\perp^2(\Omega)$. All that remains to show is that $V \neq 0$, i.e. we need a lower bound

$$\liminf_{k \to \infty} M(V_k) > 0. \quad (38)$$

This is achieved by the following three lemmas. Namely, if we had $M(V_k) \to 0$, then Lemma 5.2 and Lemma 5.3 would imply

$$\liminf_{k \to \infty} \| B_{V_k} \|^2_{L^2(V_k)} \geq 8\pi.$$

But then $V_k$ is not a minimizing sequence by Lemma 5.1, a contradiction. \hfill \square

**Lemma 5.1.** For any bounded domain $\Omega \subset \mathbb{R}^n$ of class $C^3$ there exists a surface $\Sigma \in \text{S}_\perp^2(\Omega)$ with varifold curvature

$$\int_{\Sigma} |B_{\Sigma}|^2 \, d\mu_{\Sigma} < 8\pi. \quad (39)$$

**Proof.** Let $S \subset \mathbb{R}^n$ be given by a graph representation $y = u(z)$ where $z \in \mathbb{R}^{n-1}$ with $|z| \leq 1$, such that

$$u(0) = 0 \quad \text{and} \quad Du(0) = 0.$$

The upward unit normal of the graph is given by

$$\nu(z) = \frac{(-Du(z), 1)}{\sqrt{1 + |Du(z)|^2}}.$$

We identify $x \in \mathbb{R}^2$ with $(x, 0) \in \mathbb{R}^{n-1}$, and consider the 2-dimensional halfsphere

$$S_+ = \{(x, y) : x \in \mathbb{R}^2, y \geq 0, |x|^2 + y^2 = 1\} \subset \mathbb{R}^n.$$

We now define the map

$$f : S_+ \to \mathbb{R}^n, \ f(x, y) = (x, u(x)) + y \nu(x).$$

For $u(x) \equiv 0$ we have $f(x, y) = (x, 0) + y(0, 1) = (x, y)$. Now consider

$$u_\lambda(z) = \begin{cases} \frac{1}{\lambda} u(\lambda z) & \text{for } \lambda \neq 0, \\ 0 & \text{for } \lambda = 0. \end{cases}$$
Clearly \( u_\lambda (z) = -u_\lambda (-z) \), and \( Du_\lambda (z) = Du(\lambda z) \). Moreover \( u_\lambda \) is \( C^1 \) in both variables \((\lambda, z)\), in fact we have

\[
u_\lambda (z) = \frac{(-Du(\lambda z), 1)}{\sqrt{1 + |Du(\lambda z)|^2}}.
\]

We obtain the variation

\[
f_\lambda : S_+ \to \mathbb{R}^n, \quad f_\lambda (x, y) = (x, u_\lambda (x)) + y \nu_\lambda (x).
\]

Clearly \( u_0 (x) = 0 \), \( \nu_0 (x) = (0, 1) \) and \( f_0 (x, y) = (x, y) \). Now

\[
\frac{\partial}{\partial \lambda} Du(\lambda z) \cdot e_i |_{\lambda=0} = D^2 u(0)(z, e_i) = h^S(z, e_i).
\]

Here \( h^S \) is the second fundamental form of \( S \) at \((z, u(z)) = (0, 0)\). Next compute

\[
\frac{\partial}{\partial \lambda} u_\lambda (z) |_{\lambda=0} = \int_0^1 D^2 u(0)(tz, z) \, dt = \frac{1}{2} h^S(z, z),
\]

\[
\frac{\partial}{\partial \lambda} \nu_\lambda (z) |_{\lambda=0} = \left. \frac{\partial}{\partial \lambda} \frac{(-Du_\lambda (z), 1)}{(1 + |Du_\lambda (z)|^2)^{\frac{1}{2}}} \right|_{\lambda=0} = -\sum_{i=1}^{n-1} h^S(z, e_i) e_i = D\nu(0)z.
\]

In summary the velocity field of the variation is

\[
\phi(x, y) := \left. \frac{\partial}{\partial \lambda} f_\lambda (x, y) \right|_{\lambda=0} = \frac{1}{2} h^S(x, x)(0, 1) + y D\nu(0)x.
\]

The surface \( f(x, y) \) meets the graph of \( u(z) \) orthogonally along \( S^1 = \partial S_+ \subset \mathbb{R}^2 \). To see this, let \( x \in S^1 \) and choose a unit vector \( v \in \mathbb{R}^2 \), \( v \perp x \). At the boundary we have one tangent vector of \( f(x, y) \) given by

\[
\frac{d}{d\alpha} f(\cos \alpha x + \sin \alpha v, 0) |_{\alpha=0} = (v, Du(x)v).
\]

A second tangent vector is obtained by

\[
Df(x, 0) \cdot (0, 1) = \nu(x) = \frac{(-Du(x), 1)}{\sqrt{1 + |Du(x)|^2}}.
\]

The first vector is tangent to \( f|_{S^1} \), the second is orthogonal to the first and is normalized. Hence the second vector is the interior co-normal \( \eta \) along the boundary, and is equal to \( \nu(x) \), so that \( f(x, y) \) satisfies the orthogonal boundary constraint. Since the
same computation applies to \( f_\lambda \) the family \( f_\lambda(x, y) \) is admissible for comparison. We have the following first variation formula for the Willmore functional with boundary terms, see Theorem 1 in [1]:

\[
\frac{d}{d\lambda} W(f_\lambda)|_{\lambda=0} = \frac{1}{2} \int_{\Sigma} \langle \tilde{W}(f), \phi \rangle d\mu_g + \frac{1}{2} \int_{\partial\Sigma} \omega(\eta) ds_g.
\]

The boundary term is given by

\[
\omega(\eta) = 2\langle \phi, \nabla_\eta H \rangle - d\langle \phi, H \rangle(\eta) - \frac{1}{2} |H|^2 \langle \phi, \eta \rangle.
\]

In the case of \( S_+ \) we have \( \tilde{W} = 0, H(x, y) = -2(x, y) \) and \( \nabla H = 0 \). Thus

\[
\langle \phi, H \rangle = -h^S(x, x)y - 2y \langle D\nu(0)x, x \rangle = h^S(x, x)y.
\]

Moreover, \( \eta = (0, 1) \) along \( \partial S_+ \) and thus \(-d\langle \phi, H \rangle(\eta) = -h^S(x, x)\). For the third term on the boundary, we get directly \(-\frac{1}{2} |H|^2 \langle \phi, \eta \rangle = -h^S(x, x)\). We conclude that \( \omega(\eta) = -2h^S(x, x) \) and thus

\[
\frac{d}{d\lambda} W(f_\lambda)|_{\lambda=0} = - \int_{S^1} h^S(x, x) ds(x) = -2\pi \text{tr}_{\mathbb{R}^2} h^S.
\]

We may now assume that the 2-plane “\( \mathbb{R}^2 \)” in the definition of \( S_+ \) is spanned by the directions of the two largest principal curvatures of \( S \). Furthermore we may assume that our graph representation was chosen around a point where the sum of the two largest principal curvatures is positive. (Such point always exists due to the compactness of \( \Omega \)). In this case we have achieved

\[
\frac{d}{d\lambda} W(f_\lambda)|_{\lambda=0} < 0.
\]

On the other hand, for \( \lambda = 0 \) the integral of \( |A^\circ|^2 \) is zero and hence minimal, therefore its derivative vanishes. Using \( |A|^2 = |A^\circ|^2 + \frac{1}{2} |H|^2 \) we conclude

\[
\frac{d}{d\lambda} \int_{S^1} |A_{f_\lambda}|^2 d\mu_{f_\lambda}|_{\lambda=0} < 0.
\]

Thus for small \( \lambda > 0 \) we have comparison surfaces \( f_\lambda \) with curvature energy below the half sphere.

**Lemma 5.2.** Let \( V_k \in \text{ICV}(\Omega) \) be a sequence with \( \mu_{V_k} \to 0 \) locally in \( \Omega \). Assume there are points \( x_k \in \text{spt} \mu_{V_k} \) such that \( x_k \to x_0 \in \Omega \). Then

\[
\liminf_{k \to \infty} \|B_{V_k}\|_{L^2(V_k)}^2 \geq 16\pi = \|B_{S^2}\|_{L^2(S^2)}^2.
\]
Proof. Let $0 < \rho < \text{dist}(x_0, \partial \Omega)$. By the monotonicity formula, see equations (A.8) and (A.10) in [13], we have for sufficiently large $k$

$$1 \leq \frac{\mu_{V_k}(B_\rho(x_k))}{\pi \rho^2} + \frac{1}{16\pi} \int_{B_\rho(x_k)} |H_{V_k}|^2 d\mu_{V_k} + \frac{1}{2\pi \rho^2} \int_{B_\rho(x_k)} \langle H_{V_k}(x), x - x_0 \rangle d\mu_V$$

$$\leq \left( \frac{1}{16\pi} + \varepsilon \right) \int_{B_\rho(x_k)} |H_{V_k}|^2 d\mu_{V_k} + C(\varepsilon) \frac{\mu_{V}(B_\rho(x_k))}{\rho^2}.$$ 

Letting $k \to \infty$ and then $\varepsilon \downarrow 0$ we infer

$$\liminf_{k \to \infty} \int_{B_\rho(x_k)} |H_{V_k}|^2 d\mu_{V_k} \geq 16\pi.$$ 

From $H_V(x) = \text{tr} B(x, T_x \mu_V)$ we obviously have $|B_V|^2 \geq \frac{1}{2} |H_V|^2$, but this improves by the factor 2. Namely, Hutchinson showed the following relations for any $v \in \mathbb{R}^n$:

- $B(x, P)(v, P) \subset P^\perp$ and $B(x, P)(v, P^\perp) \subset P$ \hspace{1cm} ([12], Proposition 5.2.4(iii))
- $\langle B(v, \tau), \nu \rangle = \langle B(v, \nu), \tau \rangle$ for $\tau \in P, \nu \in P^\perp$ \hspace{1cm} ([12], Proposition 5.2.4(i)).

Moreover, Mantegazza proved in [14] Theorem 5.4 that $B(x, P)(P^\perp v, \cdot) \equiv 0$ for $P = T_x \mu$, if $V$ is integer rectifiable. Thus $H_V(x) = \text{tr} B^\perp(x, T_x \mu)$ and

$$\int_{\mathbb{R}^n} |H_V|^2 d\mu_V \leq 2 \int_{\mathbb{R}^n} |B_V^\perp|^2 d\mu_V = \int_{\mathbb{R}^n} |B_V|^2 d\mu_V.$$ \hspace{1cm} (41)

This finishes the proof of the lemma. \hfill \Box

**Lemma 5.3.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^2$, and let $V_k \in \text{ICV}^2(\Omega)$ be a sequence with $\mathcal{M}(V_k) \to 0$ and $\text{spt} V_k \subset U_{\delta_k}(S)$ where $\delta_k \to 0$. Then

$$\liminf_{k \to \infty} \int_{\mathbb{R}^n} |B_{V_k}|^2 dV_k \geq 8\pi = \int_{\mathbb{R}^2_+} |B_{S^2}|^2 dV_{S^2}. \hspace{1cm} (42)$$

The rest of this section is devoted to the proof of Lemma 5.3. A monotonicity formula for $m$-dimensional varifolds with free boundary and mean curvature in $L^p$, $p > m$, was proved by Grütter and Jost in [10], see also De Masi [8]. However, it seems unclear whether the argument of [10] extends to the critical case $p = m = 2$. Our strategy is to reflect the varifold across $S$. This approach is not possible in the setting of [10], because the estimate for the mean curvature of the reflected varifold involves the second fundamental form, see equation (58) below.

In the following, it will be convenient to pullback the geometry by the reflection map.
Definition 5.4. Let \( G = (g_{ij}) \in C^1(U, \mathbb{R}^{n \times n}) \) be a Riemannian metric on the open set \( U \subset \mathbb{R}^n \). For an \( m \)-varifold \( V \) on \( U \) we consider the functional

\[
V^g(\phi) = \int_{U \times G(m,n)} \phi(x, P) JG(x, P) dV(x, P) \quad \text{for} \ \phi \in C^0_c(U \times G(m, n)).
\]

Here \( JG(x, P) \) denotes the Riemannian volume element. With respect to a standard orthonormal basis \( \tau_\alpha \) of \( P \) where \( \alpha = 1, \ldots, m \), we have

\[
JG(x, P) = \left( \det g(x)(\tau_\alpha, \tau_\beta) \right)^{\frac{1}{2}} = \left( \det PG(x)|_P \right)^{\frac{1}{2}}.
\]

The Riemannian mass of \( V \) is given by

\[
M^g(V) = \int_{U \times G(m,n)} JG(x, P) dV(x, P).
\]

The first variation of \( V^g \) with respect to a vector field \( X \in C^1_c(U, \mathbb{R}^n) \) is defined by

\[
\delta V^g(X) = \frac{d}{dt} M^g((\varphi_t)_\sharp V)|_{t=0},
\]

where \( \varphi_t \) is the flow associated to \( X \).

Lemma 5.5 (Riemannian first variation). Let \( X \in C^1_c(U, \mathbb{R}^n) \) be a vector field with associated flow \( \varphi_t \). Assume that \( V \) is an integer rectifiable \( m \)-varifold in \( U \) with weak second fundamental form \( B \in L^1(V) \). Then we have the first variation formula

\[
\delta V^g(X) = - \int_U \langle G(x) H_g(x), X(x) \rangle JG(x, P(x)) d\mu_V(x),
\]

where \( \mu_V \) is the weight measure of \( V \) and \( P(x) = T_x V \) is the approximate tangent space. The vector \( H_g \) decomposes as \( H_g(x) = \sum_{i=1}^3 \phi_i(x, P(x)) \) where

\[
\phi_1(x, P)^i = \frac{1}{JG(x, P)} g^{lm} \frac{\partial}{\partial P_k} (JG Q_m)^i(x, P) B^k_{ij}(x, P),
\]

\[
\phi_2(x, P) = \frac{1}{JG(x, P)} G(x)^{-1} \frac{\partial}{\partial x^j} (JG Q)^T P e_j,
\]

\[
\phi_3(x, P)^i = \frac{1}{2} g(x)^{ij} \langle Q(x, P) G(x)^{-1} \partial_x G(x), P \rangle.
\]

Here \( Q(x, P) \) is the projection onto \( P \) with respect to the inner product \( g(x) \).

Proof. For a diffeomorphism \( \varphi \in C^1(U, U) \) the pushforward varifold is defined by

\[
\varphi_\sharp V(\psi) = \int_{U \times G(m,n)} \psi(\varphi(x), D\varphi(x)P) J\varphi(x, P) dV(x, P).
\]
Here by $D\varphi(x)P$ we mean the image subspace, and $J\varphi(x,P)$ is the standard Jacobian of $\varphi$ on $P$. The Riemannian mass of $\varphi_tV$ is then

$$M^g(\varphi_tV) = \int_{U \times G(m,n)} JG(y,Q)\,d(\varphi_tV)(y,Q)$$

$$= \int_{U \times G(m,n)} JG(\varphi(x),D\varphi(x)P)\,J\varphi(x,P)\,dV(x,P).$$

Now consider the identity maps $I : (P, g(x)) \to P$ and $J : D\varphi(x)P \to (D\varphi(x)P, g(\varphi(x)))$. Let $A(x) : P \to D\varphi(x)P$, $A(x)v = D\varphi(x)v$. We have

$$J^g\varphi(x,P) = \det((JA(x))\,JA(x))\\frac{1}{2}$$

$$= \det(J^*J)\\frac{1}{2} \det(A(x)^*A(x))\\frac{1}{2} \det(I^*I)\\frac{1}{2}$$

$$= \frac{JG(\varphi(x),D\varphi(x)P)}{JG(x,P)} J\varphi(x,P).$$

It follows that

$$M^g(\varphi_tV) = \int_{U \times G(m,n)} J^g\varphi(x,P)\,dV^g(x,P) \quad \text{where} \quad dV^g = JG\,dV.$$

Now assume that $\varphi = \varphi_t$ is the flow of a vector field $X \in C^1_c(U, \mathbb{R}^n)$. Choose an orthonormal basis $v_1, \ldots, v_m$ of $P$ with respect to $g(x)$, and calculate

$$\frac{\partial}{\partial t}J^g\varphi_t(x,P)\big|_{t=0} = \frac{\partial}{\partial t}\left(\det g(\varphi_t(x))(D\varphi_t(x)v_\alpha,D\varphi_t(x)v_\beta)\right)^{\frac{1}{2}}\big|_{t=0}$$

$$= g(x)(DX(x)v_\alpha,v_\alpha) + \frac{1}{2}Dg(x)(v_\alpha,v_\alpha)$$

$$= \text{tr} \left( Q(x,P)DX(x) \right) + \frac{1}{2} \text{tr} \left( Q(x,P)G(x)^{-1}DXG(x) \right)$$

$$= \langle Q(x,P)DX(x),P \rangle + \frac{1}{2} \langle Q(x,P)G(x)^{-1}DXG(x),P \rangle.$$

We write

$$JG(x,p)\langle Q(x,P)DX(x),P \rangle = \langle D_x(JG(x,P)Q(x,P)X(x)),P \rangle$$

$$- \sum_{j=1}^n \langle \frac{\partial}{\partial x^j}(JG(x,P)Q(x,P))\,X(x),Pe_j \rangle.$$

We now apply the curvature varifold property with the test vector field $Y(x,P) = JG(x,P)Q(x,P)X(x)$. Noting that $Y$ is $C^1$ and has compact support, we get

$$\int_{U \times G(m,n)} \langle D_x(JG(x,P)Q(x,P)X(x)),P \rangle\,dV(x,P) \quad (52)$$

$$= -\int_{U \times G(m,n)} (DPY \cdot B + \langle \text{tr} B, Y \rangle)\,dV(x,P).$$
Since $V$ is integer rectifiable, it has weak mean curvature

$$H^k_V(x) = \sum_{i=1}^n B^k_{ii}(x, P(x)) \in L^1(\mu_V),$$

where $P(x)$ is the projection onto $T_x\mu$ and $\mu_V$ is the weight measure. Now we use $H(x) \perp T_x\mu$-almost-everywhere, see Brakke [5, Chapter 5] or Mantegazza [14, Theorem 5.4]. As $Y(x, P(x)) \in T_x\mu$, the last term in (52) vanishes. We conclude

$$\delta V^g(X) = \frac{d}{dt} M^g((\varphi_t)_t)_{\mid t=0}$$

$$= - \int_{U \times G(m,n)} D_{P} Y \cdot B dV(x, P)$$

$$- \frac{1}{2} \int_{U \times G(m,n)} (Q(x, P) G(x)^{-1} D_X G(x), P) JG(x, P) dV(x, P).$$

Finally, we rewrite this in Riemannian form, that is

$$\delta V^g(X) = - \int_{U} \langle G(x) H_g(x), X(x) \rangle JG(x, P(x)) d\mu_V(x).$$

Now one readily checks this formula for $H_g$ as given in the statement of the Lemma. □

**Lemma 5.6 (local energy bound).** Assume that in addition to the assumptions of the previous lemma, we have

$$|G(x) - E_n| \leq \varepsilon \quad \text{and} \quad |DG(x)| \leq \Lambda.$$  (53)

Then we can estimate, for $P = P(x)$,

$$\|H_g(x)\|_g^2 JG(x, P) \leq |H(x)|^2 + C \varepsilon |B(x, P)|^2 + C(\varepsilon, \Lambda).$$  (54)

**Proof.** We consider the sets

$$\mathcal{G} = \{G \in \mathbb{R}^{n \times n} : G^T = G, |G - E_n| \leq \frac{1}{2}\},$$

$$\mathcal{M} = \{M \in L(\mathbb{R}^n, \mathbb{R}^{n \times n}) : |M| \leq \Lambda\}.$$

For given $G \in \mathcal{G}$ and $P \in G(m, n)$, we choose a standard orthonormal basis $\tau_1, \ldots, \tau_m$ of $P$ and define $g_{\alpha \beta} = \langle G\tau_\alpha, \tau_\beta \rangle$. The following quantities are well-defined:

$$J(G, P) = \left( \det g_{\alpha \beta} \right)^{\frac{1}{2}},$$

$$Q(G, P)e_j = g^{\alpha \beta} \langle G\tau_\beta, e_j \rangle \tau_\alpha.$$
To see that the functions are smooth, one can choose a smooth orthonormal basis \( \tau_\alpha(P) \) for \( P \) near a point \( P_0 \in G(m,n) \). Such a basis can be obtained by taking a basis \( \tau_\alpha \) for \( P_0 \) and applying the Gram-Schmidt process to the vectors \( P\tau_\alpha \). Putting \( G = G(x) \) and \( M = DG(x) \), one verifies that the three functions \( \phi_i(x, P) \) for \( P = P(x) \) have the following structure:

\[
\begin{align*}
\phi_1(x, P) &= f_1(G, P)^{ij} B_{ik}^j, \\
\phi_2(x, P) &= f_2(G, P) \ast M, \\
\phi_3(x, P) &= f_3(G, P) \ast M.
\end{align*}
\]

(Here, \( f(G, P) \ast M \) refers to a sum of products of terms of expressions which are smooth in \( G, P \) with entries of \( M \).) For \( G = E_n \) and \( M = 0 \) we compute precisely

\[
\phi_1^l = \delta_{lm} \frac{\partial P_i}{\partial P_k^l} B_{ij} = B_{ii} = H^l \quad \text{and} \quad \phi_2 = \phi_3 = 0.
\]

It follows that

\[
|H_g(x) - H(x)| \leq C |f_1(G, P) - f_1(E_n, P)| |B| + C \left( |f_2(G, P)| + |f_3(G, P)|\right) |M| \\
\leq C \varepsilon |B| + CA.
\]

We can now estimate further

\[
||H_g||_g^2 - |H|^2| \leq \left( 2|H| + C \varepsilon |B| + CA \right) \cdot \left( C \varepsilon |B| + CA \right) \\
\leq C(|B| + CA)(C \varepsilon |B| + CA) \\
\leq C \varepsilon |B|^2 + CA |B| + CA^2 \\
\leq C \varepsilon |B|^2 + C(\varepsilon) A^2.
\]

Finally we use \( \langle Gv, v \rangle \leq (1 + C \varepsilon) |v|^2 \) and conclude

\[
\|H_g\|_g^2 JG \leq (1 + C \varepsilon) |H_g|^2 \\
\leq (1 + C \varepsilon) (|H|^2 + C \varepsilon |B|^2 + C(\varepsilon) A^2) \\
\leq |H|^2 + C \varepsilon |B|^2 + C(\varepsilon) A^2.
\]

The lemma is proved. \( \square \)

We now turn to the problem of mapping a varifold by a diffeomorphism.

**Lemma 5.7.** Let \( \sigma \in C^2(U, \hat{U}) \) be a diffeomorphism of open sets \( U, \hat{U} \subset \mathbb{R}^n \), with pullback metric \( g_{ij} = \langle \partial_i \sigma, \partial_j \sigma \rangle \). Let \( V \) be an integer rectifiable varifold with compact support in \( U \), having second fundamental form \( B \in L^2(V) \). Then the pushforward varifold \( \hat{V} = \sigma_* V \) has weak mean curvature \( \hat{H} \in L^2(\hat{\mu}) \), for \( \hat{\mu} = \mu_{\hat{V}} \), given by

\[
\hat{H} \circ \sigma = D\sigma \cdot H_g.
\]

Moreover, we have for \( JG(x) = JG(x, T_x \mu_{\hat{V}}) \) and any Borel set \( E \subset U \)

\[
\int_{\sigma(E)} |\hat{H}(y)|^2 d\hat{\mu}(y) = \int_E \|H_g(x)\|_g^2 JG(x) d\mu(x).
\]

(55)

29
Proof. By definition of the varifold pushforward, we have
\[ \sigma^\# V(\eta) = \int_U \eta(\sigma(x), D\sigma(x)P)J\sigma(x, P) dV(x, P). \]

For an orthonormal basis \( \tau_\alpha \) of \( P \) we have \( \langle D\sigma(x)\tau_\alpha, D\sigma(x)\tau_\beta \rangle = g(x)(\tau_\alpha, \tau_\beta) \), thus
\[ M(\sigma^\# V) = \int_U J\sigma(x, P) dV(x, P) = \int_U JG(x, P) dV(x, P) = M^g(V). \]

Now for given \( Y \in C^1_c(\hat{U}, \mathbb{R}^n) \) we let \( X \in C^1_c(U, \mathbb{R}^n) \) be the \( \sigma \)-related vector field, i.e. \( Y \circ \sigma = D\sigma \cdot X \). Then we have \( \psi_t \circ \sigma = \sigma \circ \varphi_t \) for the related flows, and
\[ M((\psi_t)^\# (\sigma^\# V)) = M((\psi_t \circ \sigma)^\# V) = M((\sigma \circ \varphi_t)^\# V) = M^g((\varphi_t)^\# V). \]

Therefore we obtain
\[ \delta \hat{V}(Y) = \frac{d}{dt} M((\psi_t)^\# (\sigma^\# V))|_{t=0} = \frac{d}{dt} M^g((\varphi_t)^\# V)|_{t=0} = -\int_U \langle G(x)H_g(x), X(x) \rangle JG(x, P(x)) d\mu(x). \]

Transforming back to \( \sigma^\# V \) yields
\[ \int_U \langle G(x)H_g(x), X(x) \rangle JG(x, P(x)) d\mu(x) = \int_{U \times G(m,n)} \langle D\sigma(x)H_g(x), D\sigma(x)X(x) \rangle J\sigma(x, P) dV(x, P) \]
\[ = \int_{U \times G(m,n)} \langle D\sigma(\sigma^{-1}(y))H_g(\sigma^{-1}(y)), Y(y) \rangle|_{y=\sigma(x)} J\sigma(x, P) dV(x, P) \]
\[ = \int_{\hat{U} \times \mathbb{R}^n} \langle D\sigma(\sigma^{-1}(y))H_g(\sigma^{-1}(y)), Y(y) \rangle d(\sigma^\# V)(y, P). \]

We conclude
\[ \delta \hat{V}(Y) = -\int_{\hat{U}} \langle \hat{H}, Y \rangle d\hat{\mu} \quad \text{where} \quad \hat{H}(y) = D\sigma(x)H_g(x)|_{x=\sigma^{-1}(y)}. \quad (57) \]

Substituting \( X(x) = H_g(x) \) in the previous calculation, we see finally
\[ \int_U \|H_g(x)\|^2 JG(x) d\mu(x) = \int_{\hat{U}} |\hat{H}(y)|^2 d\hat{\mu}(y). \]

\[ \square \]
We now specialize to the situation where $\Omega \subset \mathbb{R}^n$ is a bounded $C^2$ domain with boundary $S = \partial \Omega$, and consider the reflection

$$\sigma : U \rightarrow U, \quad \sigma(x + rv^S(x)) = x - rv^S(x)$$

where $(x, r) \in S \times (-\delta, \delta)$.  \hfill (58)

Here $\delta > 0$ is chosen so small that the two-sided tubular neighborhood $U = U_\delta(S) = \{x + rv^S(x) : (x, r) \in S \times (-\delta, \delta)\}$ is open. Notice further that $\sigma(U) = U$. We put $g_{ij}(x) = (\partial_i \sigma(x), \partial_j \sigma(x))$ for all $x \in U$.

**Lemma 5.8.** For $x \in S$, $v \in \mathbb{R}^n$ we have, where $P_S$ is the projection onto $TS$,

$$g_{ij}(x) = \delta_{ij} \quad \text{and} \quad Dg_{ij}(x)v = 4\langle \nu^S(x), v \rangle h^S(x)(P_S e_i, P_S e_j). \quad (59)$$

**Proof.** First the $r$-derivative of (58) gives

$$D\sigma(x + rv^S(x))\nu^S(x) = -\nu^S(x).$$

Second let $\gamma(s)$ be a curve in $S$ with $\gamma(0) = x$ and $\gamma'(0) = v \in T_x S$. Then we compute, denoting by $W(x)$ the Weingarten map of $S$,

$$(\text{Id} - rW(x))v = \frac{d}{ds}(\gamma(s) - r\nu(\gamma(s)))|_{s=0}$$

$$= \frac{d}{ds}\sigma(\gamma(s) + rv^S(\gamma(s)))|_{s=0}$$

$$= D\sigma(x + rv^S(x))(\text{Id} + rW(x))v.$$

This implies

$$D\sigma(x + rv^S(x))v = (\text{Id} - rW(x))(\text{Id} + rW(x))^{-1}v \quad \text{for } v \in T_x S.$$  

For $r = 0$ we get as expected

$$D\sigma(x)v^S(x) = -\nu^S(x) \quad \text{and} \quad D\sigma(x)v = v.$$  

This implies $G(x) = D\sigma(x)^T D\sigma(x) = E_n$ for $x \in S$. Moreover it follows that

$$DG(x)v = 0 \quad \text{for } x \in S, \ v \in T_x S.$$  

We compute further

$$\frac{\partial}{\partial r} D\sigma(x + rv^S(x))\nu^S(x)|_{r=0} = \frac{\partial}{\partial r}(-\nu^S(x)) = 0.$$  

For the tangential derivative we get, again for $v \in T_x S$,

$$\frac{\partial}{\partial r} D\sigma(x + rv^S(x))v|_{r=0} = \frac{\partial}{\partial r}((\text{Id} - rW(x))(\text{Id} + rW(x))^{-1}v)|_{r=0}$$

$$= -2W(x)v.$$
Thus we have, again for \( v, w \in T_xS \),
\[
\begin{align*}
\frac{\partial}{\partial r} g(x + r\nu^S(x))(\nu^S(x), \nu^S(x)) |_{r=0} &= 0, \\
\frac{\partial}{\partial r} g(x + r\nu^S(x))(\nu^S(x), v) |_{r=0} &= 0, \\
\frac{\partial}{\partial r} g(x + r\nu^S(x))(v, w) |_{r=0} &= 4 h^S(x)(v, w).
\end{align*}
\]

The lemma is proved. \( \square \)

We now compute the vector \( H_g(x) \) as defined in Lemma \( \ref{lem:5.5} \) for points \( x \in S \).

**Lemma 5.9.** For \( x \in S \) and \( P = P(x) \) we have the formula
\[
H_g(x) = H(x) - 2 \Delta_P d_S(x) P^\perp \nu^S(x) + 4 \, P^\perp D \nabla d_S(x) \cdot P \nu^S.
\]
(60)

Here the notation \( \Delta_P d_S(x) = \text{tr}_P D^2 d_S(x) \) is used.

**Proof.** Let \( P \in G(m, n) \) be any fixed subspace, with Euclidean orthonormal basis \( \tau_\alpha \) for \( 1 \leq \alpha \leq m \). Put \( g_{\alpha\beta}(x) = g(x)(\tau_\alpha, \tau_\beta) \). At a point \( x \in S \) we have
\[
g_{\alpha\beta}(x) = \langle \tau_\alpha, \tau_\beta \rangle = \delta_{\alpha\beta},
\]
\[
\partial_x g_{\alpha\beta}(x) = 4 \langle \nu^S(x), e_k \rangle D^2 d_S(x)(\tau_\alpha, \tau_\beta).
\]

We then have
\[
\partial_x JG(x, P) = 2 \langle \nu^S(x), e_k \rangle D^2 d_S(\tau_\alpha, \tau_\alpha) = 2 \langle \nu^S(x), e_k \rangle \Delta_P d_S(x). \quad (61)
\]

Next we compute for \( G(x) = (g_{\alpha\beta}(x)) \),
\[
\langle \partial_x G, P \rangle = 4 \langle \nu^S(x), e_k \rangle h^S(x)(\tau_\alpha, \tau_\alpha) = 4 \langle \nu^S(x), e_k \rangle \Delta_P d_S(x). \quad (62)
\]

Now we address the \( x^k \) derivative of \( Q(x, P) \). Using again \( g_{\alpha\beta} = g(\tau_\alpha, \tau_\beta) \) we have
\[
Q(x, P)v = g^{\alpha\beta}(\tau_\beta, v)\tau_\alpha \quad \text{for any } v \in \mathbb{R}^n.
\]

Now we have, again for any \( v \in \mathbb{R}^n \),
\[
\partial_x g^{\alpha\beta} = -4 \langle \nu^S(x), e_k \rangle D^2 d_S(x)(\tau_\alpha, \tau_\beta),
\]
\[
\partial_x g(\tau, v) = 4 \langle \nu^S(x), e_k \rangle D^2 d_S(x)(\tau, v).
\]

Inserting we have
\[
\partial_x Q(x, P)v = -4 \langle \nu^S(x), e_k \rangle D^2 d_S(x)(\tau_\alpha, \tau_\beta)\langle \tau_\beta, v \rangle \tau_\alpha + 4 \langle \nu^S(x), e_k \rangle D^2 d_S(\tau_\alpha, v)\tau_\alpha,
\]
which yields
\[
\partial_x Q(x, P)v = 4 \langle \nu^S(x), e_k \rangle D^2 d_S(x)(\tau_\alpha, P^\perp v)\tau_\alpha + 4 \langle \nu^S(x), e_k \rangle P D \nabla d_S(x) P^\perp v. \quad (63)
\]

32
Now \( H_g(x) = \sum_{i=1}^{3} \phi_i(x, P(x)) \), where for \( g_{ij}(x) = \delta_{ij} \) the \( \phi_i \) are as follows:
\[
\phi_1(x, P)^t = \partial P_{kj}^i (JGQ^i_1)(x, P) B_{ij}^k(x, P),
\]
\[
\phi_2(x, P) = (\partial x^k JG(x, P)) P e_k + \partial x^k Q(x, P)^T Pe_k,
\]
\[
\phi_3(x, P)^t = -\frac{1}{2} (\partial_i G(x, P)).
\]

For \( \phi_3 \) we have directly
\[
\phi_3(x, P) = -2\langle \nu^S(x), e_k \rangle \Delta_P d_S(x)e_k = -2\Delta_P d_S(x) \nu^S(x).
\]

For \( \phi_2(x, P) \) we first calculate
\[
\langle \partial x^k Q(x, P)^T v, w \rangle = \langle v, \partial x^k Q(x, P)w \rangle
\]
\[
= 4\langle \nu^S(x), e_k \rangle \langle v, D^2 d_S(\tau_\alpha, P^\perp w) \rangle
\]
\[
= 4\langle \nu^S(x), e_k \rangle D^2 d_S(x)(Pv, P^\perp w).
\]

Inserting we compute further
\[
\phi_2(x, P) = 2\langle \nu^S(x), e_k \rangle \Delta_P d_S(x) Pe_k + 4\langle \nu^S(x), e_k \rangle D^2 d_S(x)(P \nu^S(x), P^\perp e_l)e_l
\]
\[
= 2\Delta_P d_S(x)P \nu^S(x) + 4 D^2 d_S(x)(P \nu^S(x), P^\perp e_l)e_l.
\]

To simplify further we use
\[
D^2 d_S(x)(P \nu^S(x), P^\perp e_l)e_l = (D\nabla d_S(x)P \nu^S(x), P^\perp e_l)e_l = P^\perp D\nabla d_S(x)P \nu^S(x).
\]

Thus we arrive at
\[
\phi_2(x, P) = 2\Delta_P d_S(x)P \nu^S(x) + 4 P^\perp D\nabla d_S(x)P \nu^S(x).
\]

It remains to compute \( \phi_1(x, P) \). This involves only a partial derivative with respect to the \( P \) variable, hence we can assume that \( x \in S \) is a fixed point, in particular we have for all \( P \) that \( JG(x, P) = 1 \) and \( Q(x, P) = P \). We then get easily
\[
\phi_1(x, P)^t = \frac{\partial P_{ji}^i}{\partial P_{kj}^j} B_{ij}^k(x, P) = \delta_{ij}\delta_{kl} B_{ij}^k(x, P) = B_{ii}^t.
\]

In other words we just have
\[
\phi_1(x, P(x)) = H(x).
\]

Collecting the three terms we finally have
\[
H_g(x) = H(x) - 2\Delta_P d_S(x)P^\perp \nu^S(x) + 4 P^\perp D\nabla d_S(x)P \nu^S(x).
\]
\[
(64)
\]
Proposition 5.10 (reflecting the varifold). Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^3$ domain, and let $V \in \text{ICV}^m(\Omega)$ be given with $\text{spt} \, \mu_V \subset U$. Here we suppose that $U = U_\delta(S)$ is a tubular neighborhood of $S = \partial \Omega$ such that the reflection $\sigma \in C^2(U, U)$ across $S$ is defined. Define

$$W(\phi) = V(\phi) + \sigma_\nu V(\phi) \quad \text{for} \ \phi \in C_0^0(\mathbb{R}^n \times G(m, n)).$$

(65)

Then the integer rectifiable varifold $W$ has the following properties:

(a) $\mu_W = 2\mu_V$ on $S$, and $T_x\mu_W = T_x\mu_V \subset T_x S$ whenever $T_x\mu_V$, $x \in S$, exists.

(b) $W$ has weak mean curvature $H_W \in L^2(\mu_W)$, given by

$$H_W(x) = \begin{cases} H_V(x) & \text{for } x \in \Omega, \\ (D\sigma \cdot H_g)(\sigma(x)) & \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega}, \\ \frac{1}{2}(H_V(x) + D\sigma(x)H_g(x)) & \text{for } x \in S. \end{cases}$$

(66)

Here $H_g$ is as in Lemma 5.3. For $x \in S$ we have precisely

$$H_W(x) = P^\top_S H_V(x) + \Delta_p d_S(x) \nu^S(x).$$

(67)

(c) Assume $|G - E_n| \leq \varepsilon$ and $|DG| \leq \Lambda$ on $U$. Then $W$ satisfies

$$\int_{\mathbb{R}^n} |H_W|^2 d\mu_W \leq 2 \int_{\Omega} |H_V|^2 d\mu_V + C\varepsilon \int_{\overline{\Omega}} |B_V|^2 d\mu_V + C(\Lambda, \varepsilon)\mathbf{M}(V).$$

(68)

Proof. Assuming $V = \nu(M, \theta)$ we have $\sigma_\nu V = \nu(\sigma(M), \theta \circ \sigma)$. Thus

$$W = \nu(M \cup \sigma(M), \theta + \theta \circ \sigma)$$

is integer rectifiable and $\mu_W = 2\mu_V$ on $S$. Now if $T_x\mu_V$ exists for $x \in S$, then $T_x\mu_V$ is contained in the halfspace $\langle \nu^S(x), \nu \rangle \geq 0$, and hence $T_x\mu_V \subset T_x S$. In general $T_{\sigma(x)} \mu_{\sigma_\nu V} = D\sigma(x)T_x \mu_V$, and hence $T_x \mu_{\sigma_\nu V} = T_x V = T_x W$ for $x \in S$.

We now turn to (b). In the definition of varifold with orthogonal boundary, we use a test function $\phi \in C^1_c(\mathbb{R}^n)$ to get the first variation formula

$$\delta V(\phi) = -\int_{\Omega} \langle H(x), \phi(x) \rangle \, d\mu_V(x) - \int_S \langle \nu^S(x), \phi(x) \rangle \, d\mu_T(x).$$

Here $H(x) = \sum_{i=1}^n B_{ii}(x, T_x \mu_V)$ and we used that $\Gamma^x(G_{m-1}(T_x S)) = 1$ by definition. We now repeat the computation of Lemma 5.3. Since here $\phi$ is not compactly supported in $\Omega$, we obtain additional boundary terms:

$$\delta V^g(\phi) = -\int_{\Omega} \langle G(x)H_g(x), \phi(x) \rangle \, JG(x, P(x)) \, d\mu_V(x)$$

$$-\int_S \langle \nu^S(x), \phi(x) \rangle \, d\mu_T(x) - \int_S \langle P(H(x), \phi(x) \rangle d\mu_V(x).$$

34
The last term will eventually be cancelled, however we do not know at present that the relation $H_V \perp T_{\mu_V}$ holds on $S$. Replacing $\phi$ by the $\sigma$-related field $\psi$, i.e. $\phi(\sigma(x)) = D\sigma(x)\psi(x)$, we get

$$\delta V^g(\psi) = -\int_{\Omega} \langle G(x)H_g(x), \psi(x) \rangle JG(x, P(x)) \, d\mu_V(x)$$

$$+ \int_S \langle \nu^S(x), \phi(x) \rangle \, d\mu_T(x) - \int_S \langle PH(x), \phi(x) \rangle \, d\mu_V(x)$$

$$= -\int_{\mathbb{R}^n \setminus \Omega} \langle D\sigma(\sigma(y)) \cdot H_g(\sigma(y)), \phi(y) \rangle \, d\mu_{\sigma_V}(y)$$

$$+ \int_S \langle \nu^S(x), \phi(x) \rangle \, d\mu_T(x) - \int_S \langle PH(x), \phi(x) \rangle \, d\mu_V(x).$$

Now $\delta(\sigma_V)(\phi) = \delta V^g(\psi)$. When adding the variations of $V$ and $\sigma_V$, the boundary terms involving $\Gamma$ cancel, and we obtain

$$\delta W(\phi) = -\int_{\mathbb{R}^n} \langle H_W, \phi \rangle \, d\mu_W,$$

where

$$H_W(x) = \begin{cases} 
H_V(x) & \text{for } x \in \Omega, \\
D\sigma(\sigma(x))H_g(\sigma(x)) & \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega} \\
\frac{1}{2}(H_V(x) + PH_V(x) + D\sigma(\sigma(x))H_g(\sigma(x))) & \text{for } x \in S.
\end{cases}$$

This proves that $W$ has weak mean curvature in $L^2(\mu_W)$. Now $P\nu^S = 0$ on $S$, thus Lemma 5.9 yields

$$H_g(x) = H_V(x) - 2\Delta_p d_S(x)\nu^S(x) \quad \mu_V\text{-a.e. on } S.$$ But $H_W(x) \perp T_{x\mu_W}$ for $\mu_W$-a.e. $x \in \mathbb{R}^n$ by Brakke’s theorem, therefore on $S$ we obtain

$$0 = P_{T_x\mu_W} H_W(x) = \frac{3}{2}PH_V(x) \quad \text{for } \mu_W\text{-a.e. } x \in S.$$ This shows that $H_V(x)$ is perpendicular to $T_{x\mu_V}$ $\mu_V$-a.e. also on $S$, and claim (b) is proved. Finally claim (c) follows from Lemma 5.6.

Proof of Lemma 5.3. We apply the reflection method to $V = V_k$. For $k$ large, the assumptions of statement (c) in Proposition 5.10 are satisfied. Using Willmore’s inequality [19], see also [13], we get

$$16\pi \leq \int_{\mathbb{R}^n} |H_{W_k}|^2 \, d\mu_{W_k}$$

$$\leq 2 \int_{\Omega} |H_{V_k}|^2 \, d\mu_{V_k} + C\varepsilon \int_{\Omega} |B_{V_k}|^2 \, d\mu_{V_k} + C(\Lambda, \varepsilon)M(V_k)$$

$$\leq (2 + C\varepsilon) \int_{\Omega} |B_{V_k}|^2 \, d\mu_{V_k} + C(\Lambda, \varepsilon)M(V_k),$$

35
The set of \( u \) of \( \partial \) plane intersects \( S \).

Theorem 8

\[
\nabla S = \epsilon \nabla g,
\]

for the derivative with respect to \( x \). For \( \Omega \subset \mathbb{R}^n \) be a bounded \( C^2 \) domain with boundary \( S = \partial \Omega \). For \( m \geq 2 \) the following holds:

1. There exist smooth surfaces \( S_i \) converging to \( S \) in \( C^2 \), which have no orthogonal \((m - 1)\)-slices, in particular the domains \( \Omega_i \) have no orthogonal \( m \)-slices.

2. The set of domains \( \Omega \) without orthogonal \( m \)-slices is open in the \( C^2 \) topology.

Claim (1) does not extend to \( m = 1 \), in fact for any \( x_0 \in S \) the line \( x_0 + \mathbb{R} v^S(x_0) \) intersects \( S \) orthogonally. More interestingly, the segment between points \( x_0, x_1 \in S \) with \( |x_0 - x_1| = \text{diam}(S) \) is orthogonal to \( S \).

The proof starts with a local analysis near a given orthogonal \((m - 1)\)-slice \( \Gamma_0 \subset S \) in the plane \( z_0 + P_0 \), where \( P_0 \in G(m, n) \), \( z_0 \in \mathbb{R}^n \) and \( P_0 z_0 = 0 \). For any \( x \in \Gamma_0 \) we have \( P_0 = T_x \Gamma_0 \oplus \mathbb{R} v^S(x) \) and thus \( T_x S = T_x \Gamma_0 \oplus P_0^\perp \). Our first goal is to construct a parametrization of all nearby \((m - 1)\)-slices of \( S \), whether orthogonal or not. For suitable \( \epsilon_0 > 0 \) we chose Fermi type coordinates

\[
E : \{ (x, v) \in \Gamma_0 \times P_0^\perp : |v| < \epsilon_0 \} \to U_{\epsilon_0}(\Gamma_0) \subset S,
\]

\( E \) is a \( C^1 \) diffeomorphism with \( DE(x, 0) = \text{Id}_{T_x S} \). Describing perturbations of \( S \) in direction of the normal \( v^S \) would lead to a loss of differentiability. Instead we choose a unit vector field \( \xi \in C^2(S, \mathbb{R}^n) \) with \( \langle \xi, v^S \rangle \geq \frac{1}{2} \). For \( u \in C^2(S) \) we consider the map

\[
g_u \in C^2(S, \mathbb{R}^n), g_u(x) = x + u(x) \xi(x).
\]

Writing \( \nabla \) for the derivative with respect to \( x \in S \), we get

\[
\nabla g_u(x) v = v + (du(x) v) \xi(x) + u(x) \nabla \xi(x) v.
\]

The set of \( u \in C^2(S) \) with \( \|u\|_{C^2(S)} < \delta_0 \) corresponds to a neighborhood of \( S \) in the space of surfaces with \( C^2 \) topology, sending \( u \) to \( S_u = g_u(S) \). The unit normal \( \nu_u \) along \( g_u \) is defined by the equation \( N(u, \nu_u) = 0 \) where

\[
N : C^2(S) \times C^1(S, \mathbb{R}^n) \to C^1(TS) \oplus C^1(S), N(u, \nu) = (\nabla g_u)^* \nu + \frac{1}{2}(|\nu|^2 - 1).
\]
Lemma 6.1. There exist $\nu_u = \nu^S$ for $u = 0$, and we compute

$$D_u[\nabla g_u](0) \varphi = \xi \otimes d\varphi + \varphi \nabla \xi,$$

$$D_uN(0, \nu^S) \varphi = (\langle \nu^S, \xi \rangle \text{grad} \varphi + \varphi(\nabla \xi)^* \nu^S) \oplus \{0\},$$

$$D_uN(0, \nu^S) \psi = P_{TS} \psi \oplus \langle \nu^S, \psi \rangle.$$

Evidently, $D_uN(0, \nu^S)$ is an isomorphism, hence $\nu : C^2(S) \to C^1(S, \mathbb{R}^n)$, $\nu(u) = \nu_u$, is of class $C^1$. Moreover at $u = 0$, its Fréchet derivative is

$$D\nu(0) \varphi = - (\langle \nu^S, \xi \rangle \text{grad} \varphi + \varphi(\nabla \xi)^* \nu^S).$$

(69)

To describe perturbations of the affine plane, we consider the manifold

$$M = \{(z, P) \in \mathbb{R}^n \times \mathbb{R}^{n \times n} : P^T = P = P^2, \text{tr} \, P = m, \, Pz = 0\}.$$

For $(z, P, u, v) \in M \times C^2(S) \times C^1(\Gamma_0, P_0^\perp)$ with $\|u\|_{C^2(S)} < \delta_0$ and $\|v\|_{C^0(\Gamma_0)} < \epsilon_0$ we define $F(z, P, u, v) \in C^1(\Gamma_0, P_0^\perp)$ by

$$F(z, P, u, v)(x) = P_0^\perp P^\perp (g_u(E(x, v)) - z)|_{v=v(x)}.$$

(70)

For $|P - P_0|^2 \leq \frac{1}{2}$ we have $|P_0^\perp P^\perp w|^2 \geq \frac{1}{2} |P^\perp w|^2$, which then implies

$$F(z, P, u, v) = 0 \iff g_u(E(x, v(x))) \in z + P \text{ for all } x \in \Gamma_0.$$

The function $F$ has derivatives

$$D_uF(z_0, P_0, 0, 0) \varphi = \varphi P_0^\perp \xi \quad \text{and} \quad D_vF(z_0, P_0, 0, 0)\eta = \eta.$$

(71)

Lemma 6.1. There exist $\delta_1 \in (0, \delta_0]$, $\epsilon_1 \in (0, \epsilon_0]$ and a $C^1$ map $\eta : U \to V$ where

$$U = \{(z, P, u) \in M \times C^2(S) : |z - z_0|, |P - P_0|, \|u\|_{C^2(S)} < \delta_1\},$$

$$V = \{v \in C^1(\Gamma_0, P_0^\perp) : \|v\|_{C^1(\Gamma_0)} < \epsilon_1\},$$

such that the following statements hold:

1. for all $(z, P, u) \in U$, $x \in \Gamma_0$ and $v \in P_0^\perp$ with $|v| < \epsilon_1$ we have

$$g_u(E(x, v)) \in z + P \iff v = \eta(z, P, u)(x).$$

(72)

2. The map $\gamma(z, P, u) : \Gamma_0 \to \mathbb{R}^n$, $\gamma(z, P, u)(x) = g_u(E(x, v(x)))$ for $v = \eta(z, P, u)$ is a $C^1$ parametrization of an $(m - 1)$-slice $\Gamma(z, P, u)$ of $S_u$ in the plane $z + P$.

3. $\eta(z_0, P_0, 0) = 0$ and $D_u\eta(z_0, P_0, 0) \varphi = -\varphi P_0^\perp \xi$. 

37
Proof. Applying the implicit function theorem, we obtain neighborhoods $U, V$ as claimed such that the equation $F(z, P, u, v) = 0$ is solved by a function $v = \eta(z, P, u),$ and hence $g_u(E(x, v(x))) \in z + P$ for all $x \in \Gamma_0.$ Now consider

$$\phi : U \times \Gamma_0 \times P_0^+ \to \mathbb{R}^n, \phi(z, P, u, x, v) = P_0^+ P (g_u(E(x, v)) - z).$$

Assume there are sequences $(z_k, P_k, u_k) \to (z_0, P_0, 0)$ and $x_k \in \Gamma_0,$ such that

$$\phi(z_k, P_k, u_k, x_k, v_k) = \phi(z_k, P_k, u_k, x_k, w_k) = 0,$$

where $v_k, w_k \in P_0^+$ with $v_k, w_k \to 0$ and $v_k \neq w_k.$ By compactness, we may assume that $x_k \to x_0 \in \Gamma_0.$ Then we obtain by subtracting

$$0 = \phi(z_k, P_k, u_k, x_k, w_k) - \phi(z_k, P_k, u_k, x_k, v_k)$$

$$= D_v \phi(z_0, P_0, 0, x_0, 0)(w_k - v_k)$$

$$+ \int_0^1 (D_v \phi(z_k, P_k, u_k, x_k, t w_k + (1 - t)v_k) - D_v \phi(z_0, P_0, 0, x_0, 0)) \cdot (w_k - v_k) \, dt$$

$$= w_k - v_k + o(|w_k - v_k|).$$

For $k \to \infty$ we get a contradiction, and the lemma follows. \hfill \qed

We now turn to orthogonal slices near $\Gamma_0.$ For $U$ as in Lemma 6.1 we define the map $G : U \to C^1(\Gamma_0, P_0^+)$ by

$$G(z, P, u)(x) = (P_0^+ P^+ \nu_u)(E(x, v))|_{v=\eta(z, P, u)(x)}. \quad (73)$$

We have $G(z, P, u) = 0$ if and only if $\gamma(z, P, u)$ parametrizes an orthogonal $(m - 1)$-slice of $S_u.$ The derivative $D_u G(z_0, P_0, 0)$ is the operator $L_0 : C^2(S) \to C^1(\Gamma_0, P_0^+)$ given by

$$L_0 \varphi = -P_0^+ \left( \nu_S^* \xi \right) \text{grad} \varphi + \varphi (\nabla \xi)^* \nu_S^* + \varphi \xi \right)|_{\Gamma_0}. \quad (74)$$

In Appendix B we construct a right inverse $R_0 : C^1(\Gamma_0, P_0^+ \to C^2(S)$ to $L_0.$ The space $W_0 = \text{im} \ R_0$ is closed in $C^2(S),$ and we have the direct sum decomposition

$$C^2(S) = \ker L_0 \oplus W_0, \varphi = \varphi - R_0(L_0 \varphi) \oplus R_0(L_0 \varphi). \quad (75)$$

Thus $L_0|_{W_0} : W_0 \to C^1(\Gamma_0, P_0^+)$ is an isomorphism.

Lemma 6.2. There exist $\delta_2 \in (0, \delta_1], \varepsilon_2 \in (0, \varepsilon_1]$ and a map $g_0 \in C^1(A_0, W_0)$ where

$$A_0 = \{(z, P, \chi) \in M \times \ker L_0 : |z - z_0|, |P - P_0|, \|\chi\|_{C^2(S)} < \delta_2\},$$

such that for any $u = \chi + w \in \ker L_0 \oplus W_0$ with $\|u\|_{C^2(S)} < \varepsilon_2$ we have the equivalence

$$G(z, P, u) = 0 \iff \|\chi\|_{C^2(S)} < \delta_2 \text{ and } w = g_0(z, P, \chi).$$
We define the family of functions

\[ C(\Gamma_0) = \{ u = \chi + g_0(z, P, \chi) : |z - z_0|, |P - P_0| \leq \frac{\delta_2}{2}, \|\chi\|_{C^2(S)} < \delta_2 \} \tag{76} \]

For \( \|u\| < \varepsilon_2 \) we have \( u \in C(\Gamma_0) \) if and only if the slice \( \Gamma(z, P, u) \) is orthogonal to \( S_u \), for some \((z, P) \in M\) with \( |z - z_0|, |P - P_0| \leq \frac{\delta_2}{2} \).

In the following we employ the Hausdorff distance \( d_H(\Gamma, \Gamma') \) for slices \( \Gamma, \Gamma' \).

**Lemma 6.3.** There is a constant \( \delta = \delta(\Gamma_0) > 0 \) such that whenever \( \Gamma \) is an orthogonal \((m - 1)\)-slice of \( S_u \) in \( z + P \) with \( d_H(\Gamma, \Gamma_0), \|u\|_{C^2(S)} < \delta \) then \( u \in C(\Gamma_0) \).

**Proof.** Let \( \Gamma_k \subset z_k + P_k \) be orthogonal \((m - 1)\)-slices of \( S_{u_k} \), where \( \text{dist}_H(\Gamma_0, \Gamma_k) \to 0 \) and \( \|u_k\|_{C^2(S)} \to 0 \). Then \( (z_k, P_k) \) (sub-)converges to some \((z, P) \in M\), where \( \Gamma_0 \subset z + P \) by Hausdorff convergence. We must have \((z, P) = (z_0, P_0)\), otherwise \( \Gamma_0 \) would be contained in an affine subspace of dimension at most \((m - 1)\). For \( k \) large we have \( \Gamma_k \subset U_{\varepsilon_1}(\Gamma_0) \). Lemma 6.1(1) now implies that \( \Gamma_k = \Gamma(z_k, P_k, u_k) \). We have \( \eta(z_k, P_k, u_k) \to 0 \) in \( C^1(\Gamma_0, P^\perp_0) \). Now decompose \( u_k = \chi_k + w_k \in \ker L_0 \oplus W_0 \). Then \( |z_k - z_0|, |P_k - P_0| \leq \frac{\delta_2}{2}, \|u_k\|_{C^2(S)} < \varepsilon_2 \) and \( \|\chi_k\|_{C^2(S)} < \delta_2 \) for \( k \) large. As \( \Gamma_k \) is an orthogonal slice, we have

\[
G(z_k, P_k, u_k)(x) = (P_k^\perp P_0^\perp u_k)(E(x, v))|_{v = \eta(z_k, P_k, u_k)} = 0 \quad \text{for all } x \in \Gamma_0.
\]

Therefore Lemma 6.2 and 6.3 yield \( u_k \in C(\Gamma_0) \). \( \square \)

This finishes the local analysis near \( \Gamma_0 \). The following lemma implies that the set \( \mathcal{O}(S) \) of orthogonal \((m - 1)\)-slices of \( S \), equipped with Hausdorff distance, becomes a compact metric space.

**Lemma 6.4.** Let \( \Gamma_k \) be orthogonal \((m - 1)\)-slices for \( S_{u_k} \), where \( \|u_k\|_{C^2(S)} \to 0 \). After passing to a subsequence, the \( \Gamma_k \) converge to an orthogonal \((m - 1)\)-slice \( \Gamma_0 \neq \emptyset \) of \( S \) with respect to Hausdorff distance.

**Proof.** We assume that \( \Gamma_k \) is contained in \( z_k + P_k \) where \( P_k z_k = 0 \). Arguing as in the proof of Theorem 2 we get a point \( x_k \in z_k + P_k \) in \( \Omega_{u_k} \) with

\[
\text{dist}(x_k, S_{u_k}) \geq \rho_{S_{u_k}} \xrightarrow{k \to \infty} \rho_S > 0.
\]

We may assume \( P_k \to P_0 \) and \( x_k \to a \in \Omega \), which implies \( z_k = P_k^\perp x_k \to P_0^\perp a =: z_0 \).

Now consider a point \( y_0 = \lim_{k \to \infty} y_k \in S \) where \( y_k \in \Gamma_k \). Putting \( S_k = S_{u_k} \) we have \( v^S(y_0) = \lim_{k \to \infty} v^{S_k}(y_k) \in z_0 + P_0 \). Let \( A^{\Gamma_k} \) be the second fundamental form of \( \Gamma_k \) as submanifold of \( \mathbb{R}^n \). As \( \Gamma_k \) is totally geodesic in \( S_k \), we have

\[
A^{\Gamma_k}(y)(v, w) = A^{S_k}(v, w) \quad \text{for any } v, w \in T_y \Gamma_k.
\]

39
Hence the norms $\|A^\Gamma_k\|_{L^\infty}$ are bounded. Moreover, Lemma 2.6 implies a bound for the measures $|\Gamma_k|$. After passing to a subsequence, we can assume

$$\mathcal{H}^{m-1} \Gamma_k \to \gamma \quad \text{in} \quad C^0_c(\mathbb{R}^n)'.$$

We claim that $\Gamma_k$ converges to $\Gamma := \text{spt} \gamma$ in Hausdorff distance. First assume by contradiction that there exist \(x_k \in \Gamma \) with

$$\limsup_{k \to \infty} \text{dist}(x_k, \Gamma_k) \geq \delta > 0.$$

After passing to a subsequence, we get $x_k \to x \in \Gamma$ and $\text{dist}(x, \Gamma_k) \geq \delta > 0$, a contradiction. Secondly assume, again by contradiction, that there exist $x_k \in \Gamma_k$ with

$$\limsup_{k \to \infty} \text{dist}(x_k, \Gamma) \geq \delta > 0.$$

There is a $\rho > 0$ (the Langer radius) such that for any $k$ and any $x \in \Gamma_k$, the submanifold $\Gamma_k$ contains a graphical piece given by a function

$$h_k : T_x \Gamma_k \cap B_\rho(x) \to T_x \Gamma_k^\perp \quad \text{where} \quad |Dh_k| \leq 1,$$

see for instance Theorem 2.6 in [6]. In particular $|\Gamma_k \cap B_{\frac{\rho}{2}}(x_k)| \geq c > 0$ uniformly, again a contradiction. It remains to show that $\Gamma \subset S \cap (z_0 + P_0)$ is an $(m-1)$-dimensional manifold. This follows by representing the slices $\Gamma_k$ locally as graphs, using the implicit function theorem as before.

Proof of Theorem 8. By Lemma 6.4 the set $\mathcal{O}(S)$ has a finite covering by the Hausdorff distance balls

$$\mathcal{O}(S) = \bigcup_{i=1}^\ell B_{\delta_i}(\Gamma_i) \quad \text{for} \quad \delta_i = \delta(\Gamma_i) > 0 \quad \text{as in Lemma 6.3}.$$

We claim that if $S_u$ has an orthogonal $(m-1)$-slice $\Gamma$ for $\|u\|_{C^2(S)}$ small, then

$$u \in \bigcup_{i=1}^\ell C_i \quad \text{where} \quad C_i = C(\Gamma_i).$$

Namely, let $\Gamma'_k$ be orthogonal $(m-1)$-slices of $S_{u_k}$ where $u_k \to 0$ in $C^2(S)$. By Lemma 6.4 we can assume that the $\Gamma'_k$ converge to an orthogonal $(m-1)$-slice $\Gamma$ of $S$. Now $\Gamma \in B_{\delta_i}(\Gamma_i)$ for some $i \in \{1, \ldots, \ell\}$. It follows that $d_H(\Gamma'_k, \Gamma_i) < \delta_i$ for $k$ large. But then $u_k \in C_i$ by Lemma 6.3.

It is easy to see that $C_i \cap B_\rho(0)$ is closed in $B_\rho(0)$ for $\rho > 0$ small. For given $u \in B_\rho(0) \cap C_i$ we now construct a perturbation $u' \in U_i = B_\rho(0) \setminus C_i$. Decompose $u = \chi + w$ and choose a subspace $X \subset W_i$ of dimension dim $M+1$ with $w \in X$.  

40
Here $W_i$ is the space introduced in (75) for $\Gamma_i$. By Hahn-Banach there is a continuous projection $Q : W_i \to X$. Consider the $C^1$ map

$$\phi : M' = \{(z, P) \in M : |z - z_0|, |P - P_0| \leq \frac{\delta_2}{2}\} \to X, \phi(z, P) = Q(g_i(z, P, \chi)).$$

The image of $\phi$ is a Lebesgue null set in $X$. Thus there exists $w' \in X \cap B_\varepsilon(w)$ with $w' / \in Q(g_i(M', \chi))$, in particular $w' / \in g_i(M', \chi) \cap X$.

It follows that $u' = \chi + w' \in U_i$. Moreover, for those $j \in \{1, \ldots, \ell\}$ for which $u$ is already in $U_j$, we can arrange that also $u' \in U_j$ by choosing the perturbation sufficiently small. Repeating this step at most $\ell$ times yields a function $u_\rho \in \bigcap_{i=1}^\ell U_i$, which means that $S_{u_\rho}$ has no orthogonal slices. By approximation we even obtain a smooth surface with that property, applying Lemma 6.4. This finishes the proof of statement (1) in Theorem 8.

For statement (2) consider a sequence $\Omega_{u_k}$ with $\|u_k\|_{C^2(S)} \to 0$, where $\Omega_{u_k}$ has an orthogonal $m$-slice $\Delta_k$ in $z_k + P_k$. By Lemma 6.4 we may assume $(z_k, P_k) \to (z, P)$, and $\Gamma_k = \partial \Delta_k$ converges in Hausdorff distance to an orthogonal $(m-1)$-slice $\Gamma$ (possibly with several components) of $S$. Now put

$$\Delta = \left\{ x \in (z + P) \setminus \Gamma : x = \lim_{k \to \infty} x_k \text{ for some } x_k \in \Delta_k \right\}.$$

By the argument of Theorem 2 which was also used in Lemma 6.4, there are points $a_k \in \Delta_k$ with $\text{dist}(a_k, S) \geq \varrho_S > 0$, thus $\Delta \neq \emptyset$. Now if $x \in \Delta$ and $\varrho < \text{dist}(x, \Gamma)$, then $(z_k + P_k) \cap B_\varrho(x) \subset \Delta_k$ for $k$ large, hence $(z + P) \cap B_\varrho(x) \subset \Delta$. This shows that $\Delta$ is open and relatively closed in $(z + P) \setminus \Gamma$, and hence $\Delta$ is an orthogonal $m$-slice of $\Omega$. The proof of Theorem 8 is now complete.

7 Appendix

Appendix A. The following disintegration theorem is used in the paper. We refer to [18], Lemma 38.4, and [3], Theorem 5.3.1, for further information.

Theorem 9 (disintegration). Let $\gamma$ be a Radon measure on $X \times Y$ with compact support, where $X, Y$ are metric spaces, and denote by $\pi : X \times Y \to Y$ the projection map. Then $\beta = \pi_*\gamma$ is a Radon measure, and there is a $\beta$-a.e. uniquely determined family $(\alpha_y)_{y \in Y}$ of Radon probability measures on $X$, such that for any Borel function $\phi : X \times Y \to [0, \infty]$ one has

$$\int_{X \times Y} \phi \, d\gamma = \int_Y \int_X \phi(x, y) \, d\alpha_y(x) \, d\beta(y).$$

(77)
Appendix B. In this appendix, we construct the (bounded, linear) right inverse $R_0$ as announced in equation (74). We first construct $R_0$ on $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^p$.

**Lemma 7.1.** There exists a bounded operator $R_0 : C^0(\mathbb{R}^m, \mathbb{R}^p) \to C^1(\mathbb{R}^n)$ such that

$$R_0 \psi(x, 0) = 0 \quad \text{and} \quad \text{grad} (R_0 \psi)(x, 0) = (0, \psi(x)) \quad \text{for all } x \in \mathbb{R}^m.$$  

Moreover, $\|R_0 \psi\|_{C^{k+1}(\mathbb{R}^n)} \leq C_k \|\psi\|_{C^k(\mathbb{R}^m)}$ and $(R_0 \psi)(x, z) = 0$ for $\text{dist}(x, \text{spt} \psi) \geq |z|$.

**Proof.** Fix $\eta \in C^1_c(\mathbb{R}^m)$ with support in $B_1(0)$ and weight $\int_{\mathbb{R}^m} \eta(x) \, dx = 1$. For $\psi \in C^0(\mathbb{R}^m)$ we define $u \in C^0(\mathbb{R}^m \times \mathbb{R})$ by

$$u(x, z) = z (\eta_z \ast \psi)(x) \quad \text{where} \quad \eta_z \ast \psi(x) = \int_{\mathbb{R}^m} \eta(y) \psi(x - zy) \, dy.$$  

Clearly $u(x, 0) = 0$. For $z \neq 0$ we have, putting $\sigma_{m,z} = (\text{sign } z)^m$,

$$(\eta_z \ast \psi)(x) = \sigma_{m,z} \int_{\mathbb{R}^m} z^{-m} \eta \left( \frac{x - y}{z} \right) \psi(y) \, dy.$$  

Differentiating under the integral and transforming back yields

$$\partial_i u(x, z) = \int_{\mathbb{R}^m} \partial_i \eta(y) \psi(x - zy) \, dy \quad \text{for } i = 1, \ldots, m,$$

$$\partial_z u(x, z) = \int_{\mathbb{R}^m} \eta(y) \psi(x - zy) \, dy - \int_{\mathbb{R}^m} \left( m\eta(y) + \sum_{i=1}^m y_i \partial_i \eta(y) \right) \psi(x - zy) \, dy.$$  

Thus $Du \in C^0(\mathbb{R}^{m+1}, \mathbb{R}^{m+1})$ and $\|u\|_{C^1(\mathbb{R}^{m+1})} \leq C \|\psi\|_{C^0(\mathbb{R}^m)}$. Since $\eta$ has weight one while $m\eta + y \cdot D\eta = \text{div}(\eta(y)y)$ has weight zero, we conclude $\partial_z u(x, 0) = \psi(x)$. Now assume $\psi \in C^k(\mathbb{R}^m)$ for some $k \geq 1$. We have for any $\alpha \in \mathbb{N}_0^n$, $s \in \mathbb{N}_0$ with $|\alpha| + s = k$

$$\partial^\alpha (\partial_z)^s (\eta_z \ast \psi)(x, z) = \int_{\mathbb{R}^m} \sum_{\beta \in \mathbb{N}_0^m, |eta| = s} \eta_{\beta}(y) \partial^{\alpha + \beta} \psi(x - zy) \, dy, \quad \text{where}$$

$$\eta_{\beta}(y) = (-1)^{|\beta|} \frac{s!}{\beta!} \eta(y) y^\beta.$$  

For $s = 0$ we have $\partial^\alpha (z \eta_z \ast \psi) = z \partial^\alpha (\eta_z \ast \psi) = z \eta_z \ast \partial^\alpha \psi$, while for $s \geq 1$

$$\partial^\alpha (\partial_z)^s (z \eta_z \ast \psi) = z \partial^\alpha (\partial_z)^s (\eta_z \ast \psi) + s \partial^\alpha (\partial_z)^{s-1} (\eta_z \ast \psi)$$

$$= \sum_{\beta \in \mathbb{N}_0^m, |eta| = s} z (\eta_{\beta}) z \ast \partial^{\alpha + \beta} \psi + s \sum_{\beta \in \mathbb{N}_0^m, |eta| = s-1} (\eta_{\beta}) z \ast \partial^{\alpha + \beta} \psi.$$  

The second sum involves only derivatives of order at most $k - 1$. For the first sum we apply the argument above, with $\eta$ replaced by $\eta_{\beta}$. This shows $\partial^\alpha (\partial_z)^s (z \eta_z \ast \psi) \in C^1(\mathbb{R}^{m+1})$ and hence $z \eta_z \ast \psi \in C^{k+1}(\mathbb{R}^{m+1})$. To prove the theorem, we finally put
\[ u(x, z) = \sum_{j=1}^{p} u_j(x, z) \] where \( u_j(x, z) = \sum_{j=1}^{p} z_j \eta_{x_j} \ast \psi_j(x) \). The claim follows immediately defining \( R_0 \psi = u \).

Before we proceed to the general case we observe that if \( \text{spt}(\psi_j) \subset W \) for all \( j = 1, \ldots, p \) then for each \( \epsilon > 0 \) one may also achieve that \( \text{spt}(R_0 \psi) \subset B_{2\sqrt{p} \varepsilon}(W) \times (-\varepsilon, \epsilon)^p \). Indeed, let \( \alpha \in C_0^\infty((-\varepsilon, \epsilon)^p) \) be a fixed function such that \( \alpha \equiv 1 \) on \( (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})^p \) and look at \( \tilde{R}_0 : C^0(\mathbb{R}^m, \mathbb{R}^p) \to C^1(\mathbb{R}^n) \) defined via

\[ (\tilde{R}_0 \psi)(x, z) = \alpha(z)(R_0 \psi)(x, z). \]  

(78)

Clearly, \( \tilde{R}_0 \) has all the properties in the statement of the previous lemma. Moreover, \( (x, z) \in \text{spt}(\tilde{R}_0 \psi) \) implies \( z \in (-\varepsilon, \epsilon)^p \) and \( (x, z) \in \text{spt}(R_0 \psi) \). One infers \( |z| \leq \sqrt{p} \varepsilon \) and by the previous lemma this means \( \text{dist}(x, \text{spt}(\psi)) \leq \sqrt{p} \varepsilon \).

**Lemma 7.2.** Let \( N \) be an \( n \)-dimensional \( C^l \)-manifold and \( M \) be a compact \( m \)-dimensional \( C^l \)-submanifold without boundary, \( l \geq 1 \). Then there exists a bounded linear operator \( R_0 : C^0(M, (TM)^\perp) \to C^1(N) \) such that

\[ R_0 \psi|_M = 0 \quad \text{and} \quad \nabla R_0 \psi|_M = \psi. \]

Further \( \|R_0 \psi\|_{C^{k+1}(N)} \leq C_k(M, N)\|\psi\|_{C^k(M)} \) for all \( k = 0, \ldots, l-1 \).

**Proof of Lemma 7.2.** Let \( M \) and \( N \) be as in the statement and \( \psi \in C^{k-1}(M, (TM)^\perp) \) be arbitrary. It is possible to choose a finite collection of open sets \( U_1, \ldots, U_r \subset N \) covering \( M \) and maps \( \Phi_j \in C^l(U_j; \mathbb{R}^n) \) which are diffeomorphisms onto their images and satisfy \( \Phi_j(U_j \cap M) = \Phi_j(U_j) \cap \mathbb{R}^m \). Note in particular that \( D\Phi_j(x)(T_x M) = \mathbb{R}^m \). Let now \( \beta_1, \ldots, \beta_r \) be a partition of unity for \( M \) such that \( \beta_j \in C^l_c(M \cap U_j) \) for all \( j \). By openness of \( U_j \) it is possible to choose \( \varepsilon > 0 \) such that \( B_{2\sqrt{p} \varepsilon}(\Phi_j(\text{spt}(\beta_j))) \times (-\varepsilon, \epsilon)^p \subset \subset \Phi_j(U_j) \) for all \( j = 1, \ldots, r \). Next define \( w_j := \beta_j[(D\Phi_j)^*]^{-1}\psi \circ \Phi_j^{-1} \) and \( u : N \to \mathbb{R} \) to be

\[ u(x) = \sum_{j=1}^{r} \tilde{R}_0 w_j(\Phi_j(x)), \]

where \( \tilde{R}_0 \) is chosen as in (78) with \( \varepsilon > 0 \) as above, in particular one has \( \text{spt}(\tilde{R}_0 w_j) \subset \subset \Phi_j(U_j) \). We remark that for the well-definedness and \( C^{k+1} \)-smoothness of \( \tilde{R}_0 w_j \) we need to ensure that \( w_j \in C^k(\mathbb{R}^m, \mathbb{R}^p) \). This is due to the fact that \( D\Phi_j \in C^{l-1} \) and maps \( TM \) to \( \mathbb{R}^m \), which implies that \( (D\Phi_j)^* \) maps \( \mathbb{R}^p \) to \( (TM)^\perp \). One readily checks that \( u|_M = 0 \) and with the chain rule one obtains for \( x \in M \)

\[ \nabla u(x) = \sum_{j=1}^{r} D\Phi_j(x)^* (\nabla \tilde{R}_0 w_j)(\Phi_j(x)) \]

\[ = \sum_{j=1}^{r} D\Phi_j(x)^* (\beta_j(x)(D\Phi_j(x)^*)^{-1}\psi(x)) = \psi(x). \]

The claim follows now defining \( R_0 \psi = u \). From the explicit construction one easily deduces the desired operator norm estimates for \( R_0 \). □

43
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