G-CORKS & HEEGAARD FLOER HOMOLOGY

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ABSTRACT. In [6], Auckly-Kim-Melvin-Ruberman showed that for any finite subgroup \( G \) of \( SO(4) \) there exists a contractible smooth 4-manifold with an effective \( G \)-action on its boundary so that the twists associated to the non-trivial elements of \( G \) don’t extend to diffeomorphisms of the entire manifold. We give a different proof of this phenomenon using the Heegaard Floer theoretic argument in [3].

1. Introduction

A cork is a contractible smooth 4-manifold with an involution on its boundary that does not extend to a diffeomorphism of the entire manifold. The first example of a cork was given by Akbulut in [1]. Since then, other examples have been constructed by Akbulut-Yasui in [4], Akbulut-Yasui in [5], and Tange in [14, 15]. Corks can be used to detect exotic structures, see [1, 4, 5, 6, 7, 14, 15]. In fact, any two smooth structures on a closed simply-connected 4-manifold differ by a single cork twist, see [8, 12]. A cork twist removes an embedded cork and reglues it using the involution. The involution on the boundary of a cork can be regarded as a \( \mathbb{Z}_2 \)-action, so it is natural to ask if contractible smooth 4-manifolds with other kinds of effective group actions on the boundaries can also be used to detect exotic structures. A number of recent papers have answered this in the affirmative, constructing examples of \( G \)-corks, \( G \neq \mathbb{Z}_2 \), that embed inside closed smooth 4-manifolds so that removing and regluing using the \( |G| \) twists produces \( |G| \) distinct smooth structures, see [6, 10, 11, 2, 15]. A \( G \)-cork is a contractible smooth 4-manifold with an effective \( G \)-action on its boundary so that the twists associated to the non-trivial elements of \( G \) do not extend to diffeomorphisms of the entire manifold.

The purpose of this paper is to use the Heegaard Floer theoretic argument in [3] to give a different proof that the examples in [6] are in fact \( G \)-corks. These examples are defined as follows. Fix a finite subgroup \( G \) of \( SO(4) \). Let \( n = |G| \). Let \( \mathcal{W} \) be the Akbulut cork from [1] shown in Figure 1.

![Figure 1. The Akbulut cork](image)

There is an isotopy of \( S^3 \) that interchanges the two link components on the right side of Figure 1. This gives an involution \( \tau \) of \( \partial \mathcal{W} \). Let \( \mathcal{S} \) denote the boundary sum \( \sharp_n \mathcal{W} \). Note that its boundary \( \partial \mathcal{S} \) inherits an involution \( \sigma \). Define \( \mathcal{T} \) to be the boundary sum of \( B^4 \) with \( n \) copies of \( \mathcal{S} \), namely \( \mathcal{T} = B^4 \sharp (G \times \mathcal{S}) \). So that we get a well-defined action of \( G \) on \( \partial \mathcal{T} \), we assume that the \( n \) copies of \( \mathcal{S} \) are attached along 3-balls in \( \partial B^4 \) that form a principal orbit under the linear action of \( G \) on \( \partial B^4 \). Then we define the action of \( G \) on \( \partial \mathcal{T} \) to be the linear action on \( \partial B^4 \) and left multiplication
on the copies of \( \partial S \). To get the \( G \)-cork, we twist a copy \( S' \) of \( S \) in the interior of \( 1 \times S \subset \overline{T} \) by the involution \( \sigma \). Let \( T \) denote the resulting 4-manifold \( B^4 \cong \left( S' \cup_{\sigma} \left( (1 \times S) - S' \right) \right) \# ((G - 1) \times \Sigma) \).

Note that the action of \( G \) on \( \partial T \) descends to an action of \( G \) on \( \partial \overline{T} \). Then we have the following:

**Theorem 1.** \( T \) is a \( G \)-cork.

The Heegaard Floer theoretic argument also yields the following easy consequence:

**Theorem 2.** The \( G \)-action on \( \partial T \) induces an effective \( G \)-action on \( HF^+(-\partial T, s) \), where \( HF^+ \) is the plus version of Heegaard Floer homology and \( s \) is the unique \( \text{Spin}^c \) structure on \( \partial T \).

We assume the reader is familiar with the basics of Heegaard Floer homology for 3 and 4-manifolds, contact geometry, Stein structures, and Lefschetz fibrations. We use \( \mathbb{Z}_2 \) coefficients throughout to avoid ambiguity in sign.

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**2. Proofs of Theorems 1 & 2**

We prove Theorem 1 first. We start by equipping \( T \) with the handle decomposition in Figure 2.

![Handle decomposition of \( T \): There are \( n \) rows. The first row represents \( S' \cup_{\sigma} \left( (1 \times S) - S' \right) \). Each row after represents a copy of \( S \).](image)

By [4, Lemma 5.3], \( T \) can be given a Stein structure that extends the standard Stein structure on \( B^4 \). Then \( \partial T \) inherits a contact structure \( \xi \). Now fix \( g \in G, g \neq 1 \); by abuse of notation we view this as a diffeomorphism of \( \partial T \). We want to show that \( g \) does not extend to a diffeomorphism of \( T \). Let \( s \) denote the unique \( \text{Spin}^c \) structure on \( T \); we also write \( s \) for its restriction to \( \partial T \). By puncturing \( T \) in the interior we can view \( T \) as a cobordism from \( -\partial T \) to \( S^3 \). Then we get the cobordism map \( F_{T,s}^+: HF^+(-\partial T, s) \to HF^+(S^3) \), see [13] for details. The twist \( g \) induces a second cobordism map
Let \( c^+(\xi) \in HF^+(\partial T, s) \) denote the contact invariant associated to \( \xi \). Then \( F^+_{\pi,s}(c^+(\xi)) \neq 0 \), but \( F^+_{T,s} \circ g^*(c^+(\xi)) = 0 \).

**Proof.** First attach a 2-handle to \( \partial T \) along a trefoil with framing 1 as in Figure 3.

If we replace the dotted 1-handle linking the trefoil with a pair of 3-balls and put the trefoil in Legendrian position, then we see that the Thurston-Bennequin number of the tangle is 2, which is 1 more than the framing we started with. By Eliashberg’s criterion [9, Proposition 2.3], the Stein structure on \( T \) can be extended over the 2-handle. Let \( M \) denote the cobordism on \( \partial T \) induced by the 2-handle attachment. Then \( M \) inherits a Stein structure. By [3, Lemma 3.6], we can extend \( M \) to a concave symplectic filling \( V \) of \( (\partial T, \xi) \) so that the closed smooth 4-manifold \( X := T \cup V \) has \( b_2^+ > 1 \) and admits the structure of a relatively minimal Lefschetz fibration over \( S^2 \) with generic fiber of genus \( > 1 \). Furthermore, if \( t \) denotes the canonical \( \text{Spin}^c \) structure on \( X \) (and also its restriction to \( V \)), then by [3, Theorem 3.2 & Lemma 3.6],

\[
F_{X,t}^{\text{mix}}(\theta_{(-2)}) = \theta_{(0)}^+,
\]

\[
F_{V,t}^{\text{mix}}(\theta_{(-2)}) = c^+(\xi).
\]

Here \( F_{X,t}^{\text{mix}} \) is the mixed homomorphism \( HF^-(S^3) \to HF^+(S^3) \) obtained by puncturing \( X \) twice, with one puncture in the interior of \( V \) and the other puncture equal to the above puncture of \( T \), \( F_{V,t}^{\text{mix}} \) is the mixed homomorphism \( HF^-(S^3) \to HF^+(\partial T, s) \) obtained by puncturing \( V \) in the same location as above, \( \theta_{(-2)} \) is the generator of \( HF^-(S^3) \) with absolute grading \(-2\), and \( \theta_{(0)}^+ \) is the generator of \( HF^+(S^3) \) with absolute grading 0, see [13] for details. Putting this together, we get

\[
0 \neq \theta_{(0)}^+ = F_{X,t}^{\text{mix}}(\theta_{(-2)}) = F_{T,s}^+(\theta_{(-2)}) \circ F_{V,t}^{\text{mix}}(\theta_{(-2)}) = F_{T,s}^+(c^+(\xi)).
\]
All that remains to show is that $F^+_{\overline{\pi},\overline{s}} \circ g^* (c^+(\xi)) = 0$. Let $X'$ denote $T \cup gV$ obtained from $X = T \cup V$ by removing $T$ and regluing it with the diffeomorphism $g : \partial T \to \partial T$. In $X'$ we have $T \cup gV$ which admits the following handle decomposition:

![Handle Decomposition Diagram](image)

Figure 4. $T \cup gM$, where the second row represents $(\{g\} \times S) \cup 2$-handle

Note that the trefoil, thought of in $X'$, gives rise to an embedded torus of self-intersection 1. By [3, Theorem 3.1], $X'$ does not have any basic classes, so for every Spin$^c$ structure $t'$ on $X'$, the mixed homomorphism $F^{\text{mix}}_{X',t'} : HF^-(S^3) \to HF^+(S^3)$ is identically zero. Separately, we have a homeomorphism $X \to X'$ that is the identity on $V$. Let $t'$ denote the Spin$^c$ structure on $X'$ that corresponds to the canonical Spin$^c$ structure $t$ on $X$. Note that $t'$ restricted to $V$ is the Spin$^c$ structure $t$. Then we have

$$0 = F^{\text{mix}}_{X',t'}(\theta_{(-2)}) = F^+_{\overline{\pi},\overline{s}} \circ g^* \circ F^{\text{mix}}_{V,t}(\theta_{(-2)}) = F^+_{\overline{\pi},\overline{s}} \circ g^* (c^+(\xi)).$$

Having proved the main technical lemma, we now finish off the proof of Theorem. Suppose to the contrary the diffeomorphism $g : \partial T \to \partial T$ extends to a diffeomorphism $\overline{g} : T \to T$. Then the diffeomorphism $g^{-1} : \partial T \to \partial T$ extends to the diffeomorphism $\overline{g}^{-1} : T \to T$. Note that $\overline{g}^{-1}$ is orientation-preserving. Let $C$ denote the cobordism from $-\partial T$ to $S^3$ obtained by stacking the cobordism $(-\partial T \times [0,\frac{1}{2}]) \cup_g (-\partial T \times [\frac{1}{2},1])$ from $-\partial T$ to $-\partial T$ on top of the punctured $T$ (with puncture denoted by $*$). Let $C'$ denote the cobordism from $-\partial T$ to $S^3$ obtained by stacking the identity cobordism $(-\partial T \times [0,\frac{1}{2}]) \cup_{id} (-\partial T \times [\frac{1}{2},1])$ from $-\partial T$ to $-\partial T$ on top of $T$ punctured at $\overline{g}^{-1}(\ast)$. The orientation-preserving diffeomorphism $\overline{g}^{-1} : T \to T$ gives us the following orientation-preserving diffeomorphism $\Theta : C \to C'$: on $-\partial T \times [0,\frac{1}{2}]$, we take $\Theta$ to be the identity; on $-\partial T \times [\frac{1}{2},1]$, we take $\Theta$ to be $g^{-1} \times id$; and on the punctured $T$, we take $\Theta$ to be $\overline{g}^{-1}$. Let $s_C$ denote the unique Spin$^c$ structure on $C$ and let $s_{C'}$ denote the unique Spin$^c$ structure on $C'$. We then get this commutative diagram:
\[ \begin{align*}
HF^+(-\partial T, s) & \xrightarrow{F^+_{C, SC}} HF^+(S^3) \\
& \downarrow (\Theta|_{\partial T})_* \\
HF^+(-\partial T', s) & \xrightarrow{F^+_{C', SC'}} HF^+(S^3).
\end{align*} \]

Note that
\[ F^+_{C, SC} = F^+_{C, SC} \circ g^*, \]
\[ F^+_{C', SC'} = F^+_{C', SC'} \]
\[ (\Theta|_{\partial T})_* = id. \]

From the lemma, \( F^+_{C, SC} (c^+ (\xi)) = 0 \) and \( F^+_{C', SC'} (c^+ (\xi)) \neq 0 \), but this contradicts the commutativity of the diagram. This concludes the proof of Theorem 1.

Remark. The above argument can also be used to show that the smooth 4-manifold \( T' \) obtained by starting with \( B^4 \natural (G \times W) \) and twisting a copy of \( W \) in \( 1 \times W \) is a \( G \)-cork. However it is not yet known if \( T' \) admits an embedding into a closed smooth 4-manifold so that removing and regluing using the \(|G|\) twists produces \(|G|\) distinct smooth structures.

We now prove Theorem 2. Define the action of \( G \) on \( HF^+(-\partial T, s) \) by: \( g \cdot x = g^*(x) \). To see that this is effective, we need to show that for any \( g \neq 1 \) there is an \( x \in HF^+(-\partial T, s) \) so that \( g^*(x) \neq x \). So fix \( g \neq 1 \). The above lemma implies that \( g^* (c^+ (\xi)) \neq c^+ (\xi) \). Hence we can take our \( x \) to be \( c^+ (\xi) \). This concludes the proof of Theorem 2.

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