A NOTE ON KÄHLER-RICCI FLOW ON FANO THREEFOLDS

MINGHAO MIAO AND GANG TIAN

Abstract. In this note, we show that the solution of Kähler-Ricci flow on every Fano threefold from the family No.2.23 in the Mori-Mukai’s list develops type II singularity. In fact, we show that no Fano threefold from the family No.2.23 admits Kähler-Ricci soliton and the Gromov-Hausdorff limit of the Kähler-Ricci flow must be a singular Q-Fano variety. This gives new examples of Fano manifolds of the lowest dimension on which Kähler-Ricci flow develops type II singularity.

1. Introduction

Ricci flow, which was introduced by R. Hamilton in [Ham82], has many developments in geometry over the past 40 years. It preserves the Kählerian condition. When restricted to Kähler manifolds, it is referred to as Kähler-Ricci flow. Let $X$ be a Fano manifold, that is, a compact Kähler manifold with positive first Chern class, we consider the following normalized Kähler-Ricci flow:

$$\left\{ \begin{array}{l}
\frac{\partial \omega(t)}{\partial t} = -\text{Ric} (\omega(t)) + \omega(t) \\
\omega(0) = \omega_0
\end{array} \right.$$  

where $\omega_0$ and $\omega(t)$ denote the Kähler forms of initial Kähler metric $g_0$ and the solution $g(t)$ of Ricci flow, respectively.

It was shown in [Cao85] that when $\omega_0$ represents $2\pi c_1(X)$, equation (1) has a global solution $\omega(t)$ for all $t \geq 0$. Then it is natural to ask what is the limiting behavior of $\omega(t)$ as $t \to \infty$. Let us first recall the definition of Kähler-Ricci soliton, which is a generalization of Kähler-Einstein metrics. A Kähler metric $\omega$ is a (shrinking) Kähler-Ricci soliton if for some holomorphic vector field $\xi$ on Fano manifold $X$, we have:

$$\text{Ric}(\omega) - \omega = L_\xi \omega$$

where $L_\xi \omega$ is taking Lie derivative along $\xi$. When $\xi = 0$, it reduces to the Kähler-Einstein metric. It was proved in [TZ07], [TZ13], [TZZZ13] and [DS20] that if $X$ admits a Kähler-Ricci soliton $\omega_{KRS}$ and $\omega_0$ represents $2\pi c_1(X)$, then as $t$ goes to $\infty$, the solution $\omega(t)$ of the normalized Kähler-Ricci flow (1) converges to $\omega_{KRS}$ up to the action of the automorphism group of $X$. It remains to see what we can expect if $X$ does not admit any Kähler-Ricci soliton. In this situation, the complex structures will jump as $t \to \infty$ and it was conjectured in [Tia97], also referred to as Hamilton-Tian conjecture, that:

Conjecture 1.1. For any global solution $\omega(t)$ of (1) as above, any sequence $(X, \omega(t))$ along Kähler-Ricci flow contains a subsequence converging to a Q-Fano variety $(X_\infty, \omega_\infty)$ in the Gromov-Hausdorff topology, and $(X_\infty, \omega_\infty)$ admits a smooth shrinking Kähler-Ricci soliton outside the singular set $S$ of $X_\infty$ which is closed and of Hausdorff codimension at least 4. Moreover, this subsequence of $(X, \omega(t))$ converges locally to $(X_\infty, \omega_\infty)$ in the Cheeger-Gromov topology.

This conjecture has been first solved for dimension less than or equal to 3 ([TZ10]) and subsequently for higher dimensions ([Bam18], [CW20], [WZ21], [BLXZ21]) by using different methods. The key of these methods is to establish certain compactness for Kähler-Ricci flow which leads to the partial $C^0$-estimate along the flow. We would also like to mention that a generalized version of the Hamilton-Tian conjecture was proved in [Bam18] and an algebraic proof of the conjecture was given in [BLXZ21] by using the work of [HL20].

In [TZ16], assuming the compactness mentioned above, the second author and Zhang established the partial $C^0$-estimate for the Kähler-Ricci flow and proved that the Gromov-Hausdorff limit of Kähler-Ricci flow is a normal projective variety whose singular set coincides with $S$. One can show that the Gromov-Hausdorff limit coincides with the algebro-geometric limit in this situation, that is, if $X_i$ converges to $X_\infty$ in
the Gromov-Hausdorff topology, then $X_i$ and $X_\infty$ can be realized as fibers in a flat family. In [CSW18], they describe a "two-step degeneration" picture of the above process and establish the uniqueness of Gromov-Hausdorff limit $X_\infty$. Han and Li further confirmed in [HL20a] that the Gromov-Hausdorff limit $X_\infty$ depends only on the algebraic structure of $X$ and is independent of the choice of the initial metric of the flow.

We now turn to a brief discussion of the singularity formation of Kähler-Ricci flow. We recall that a solution $\omega(t)$ of Kähler-Ricci flow is of type I if the curvature of $\omega(t)$ is uniformly bounded, otherwise, we call $\omega(t)$ a solution of type II. There was a folklore conjecture closely related to Conjecture 1.1 stating that the Gromov-Hausdorff limit $X_\infty$ is always smooth, i.e., the singular set $S$ is always empty. This is equivalent to saying that Kähler-Ricci flow has no type II solution. The folklore conjecture was first disproved in [LTZ18] by considering Fano compactifications of semisimple Lie groups. However, the lowest dimension among these group compactifications is 6. Naturally, one is led to wonder whether there exist examples of Fano manifold of lower dimensions on which Kähler-Ricci flow develops singularities of type II. Since the folklore conjecture holds for complex dimension two or less, a natural question is whether the lowest possibility is 3.

In this short note, we answer this question by giving a family of examples of Fano threefolds that have type II solutions for the Kähler-Ricci flow. Here is our main theorem:

**Theorem 1.1.** Any Fano threefold $X$ from family No. 2.23 in Mori-Mukai’s list has type II solutions of the normalized Kähler-Ricci flow. Namely, the Gromov-Hausdorff limit of $X$ along the Kähler-Ricci flow is a singular $\mathbb{Q}$-Fano variety.

Theorem 1.1 will be deduced from the following theorem:

**Theorem 1.2.** Any Fano threefold $X$ from the family No. 2.23 in the Mukai-Mori’s list does not admit Kähler-Ricci soliton.

This theorem will be proved by showing that the automorphism group of $X$ is finite (see Lemma 2.1) and $X$ is $K$-unstable (see Lemma 2.2). In [Del22], Delcroix constructed examples of $K$-unstable Fano manifolds by blowing up quadrics along lower dimensional linear subquadrics and showed that they do not admit any Kähler-Ricci soliton. The lowest dimension of these examples of Delcroix is 5, so he further asked for examples of Fano threefolds or fourfolds which are $K$-unstable and do not admit Kähler-Ricci soliton. The Theorem 1.2 gives an affirmative answer to Question 1.3 in [Del22].

**Question 1.1.** ([Del22], Question 1.3) Does there exist examples of Fano threefolds with no Kähler–Ricci solitons that are $K$-unstable? What about Fano fourfolds?

We leave the following question for future study:

**Question 1.2.** Suppose $X$ is a Fano threefold from the family No 2.23, can we find the Gromov-Hausdorff limit $X_\infty$ explicitly? Are members from the family 2.23 the only examples of Fano threefolds on which Kähler-Ricci flow has solutions of type II?

2. Family No 2.23 of Fano threefolds

Due to the work of Iskovskikh-Mori-Mukai (see [MMS82, IP99]), we know every smooth Fano threefold belongs to one of the 105 deformation families. Here is the description of the family No. 2.23 in the Mori-Mukai’s list of smooth Fano threefold (see [MMS82]): Any $X$ in this family is obtained by blowing up a smooth quadric $Q \subset \mathbb{P}^4$ along a smooth irreducible curve $C$ which is an intersection of $H \in |O_Q(1)|$ and $Q' \in |O_Q(2)|$, note that $C$ is an elliptic curve of degree 4. There are two subfamilies:

- Subfamily (a): the hypersurface $H \in |O_Q(1)|$ is smooth.
- Subfamily (b): the hypersurface $H \in |O_Q(1)|$ is singular, but its singular set has an empty intersection with $Q \cap Q'$.

2.1. Automorphism groups of threefolds from family No. 2.23 are finite. The following result was already known in [CPS18], and we include a proof for the reader’s convenience. This proof is more direct than the original one.

**Lemma 2.1.** For each $X$ from family No. 2.23, the automorphism group of $X$ is a finite group.

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1More counterexamples on group compactifications were later found in [LPY22].
Proof. To begin with, we note that \( \text{Aut}(X) = \text{Aut}(Q, C) \). Following the proof in [CPS18], we consider the following exact sequence:

\[
0 \longrightarrow \text{Ker}_H \overset{i}{\longrightarrow} \text{Aut}(Q, C) \overset{r}{\longrightarrow} \text{Aut}(H, C),
\]

where

\[
\text{Aut}(Q, C) = \{ \phi \in \text{Aut}(Q) \mid \phi(C) = C \},
\]
\[
\text{Aut}(H, C) = \{ \phi \in \text{Aut}(H) \mid \phi(C) = C \},
\]
\[
\text{Ker}_H = \{ \phi \in \text{Aut}(Q, C) \mid \forall x \in H, \phi(x) = x \},
\]

\( r : \text{Aut}(Q, C) \rightarrow \text{Aut}(H, C), \phi \mapsto \phi|_H \) is the restriction map and \( i \) is the natural inclusion.

We first show \( r \) is well-defined. Observe that: (1) \( \text{Aut}(Q, C) \subset \text{Aut}(\mathbb{P}^4) = \text{PSL}(5, \mathbb{C}) \) which is a linear algebraic group; (2) There are a hyperplane \( \bar{H} \) and a quadratic \( \bar{Q} \) of \( \mathbb{P}^4 \) such that \( H = \bar{H} \cap Q \) and \( Q' = \bar{Q} \cap Q' \).

It follows from (1) that any \( \phi \in \text{Aut}(Q, C) \) maps hyperplane \( \bar{H} \) to another hyperplane \( \bar{H}' := \phi(\bar{H}) \). We want to show \( \bar{H}' \neq \bar{H} \). If not, we assume \( \bar{H}' = \bar{H} \), then \( S = \bar{H} \cap \bar{H}' \) is 2-dimensional subspace. Since a smooth quadric of dimension 3 contains no linear spaces of dimension strictly greater than 1 (see, for example, section 6.1 of [GH11]), we conclude that \( C' = S \cap \bar{Q}' \) is a curve of degree 2, in particular, it is a rational curve and different from \( C \). Note \( C = Q \cap \bar{Q}' \cap \bar{H} \cap \bar{H}' \subset C' \) which implies \( C = C' \), so we get a contradiction, consequently, we must have \( \bar{H} = \bar{H}' \), it follows that \( \phi|_H \) is well-defined.

Next, we claim both \( \text{Ker}_H \) and \( \text{Aut}(H, C) \) are finite groups. Then by the exactness, we conclude \( \text{Aut}(Q, C) \) is a finite group.

To show \( \text{Ker}_H \) is a finite group, after changing the coordinate, we have two cases to check:

- Case 1: If \( H \) is smooth, we can choose a coordinate such that \( H = V(x_0) \) and \( Q = V(x_0^2 + x_1^2 + \cdots + x_4^2) \);
- Case 2: If \( H \) is singular, we can choose a coordinate such that \( H = V(x_0) \) and \( Q = V(x_0^3 + x_1^2 + \cdots + x_4^2) \).

Suppose \( \phi \in \text{Ker}_H \subset \text{PGL}(5, \mathbb{C}) \), then by definition, for \( x = [0 : x_1 : x_2 : x_3 : x_4] \in H \), we get \( \phi(x) = x \), hence \( \phi \) must be of the form:

\[
B = \begin{pmatrix}
  a & b & c & d & e \\
  f & \lambda & 0 & 0 & 0 \\
  g & 0 & \lambda & 0 & 0 \\
h & 0 & 0 & \lambda & 0 \\
i & 0 & 0 & 0 & \lambda
\end{pmatrix} \in \text{PGL}(5, \mathbb{C})
\]

For Case 1, since \( \phi \in \text{Aut}(Q) \), there exists \( \mu \in \mathbb{C}^* \) such that \( B \cdot B^T = \mu \cdot I_5 \). Hence, we conclude:

\[
\begin{aligned}
a^2 + b^2 + c^2 + d^2 + e^2 &= \mu \\
af + b\lambda &= ag + c\lambda = ah + d\lambda = ai + e\lambda = 0 \\
f^2 + \lambda^2 &= h^2 + \lambda^2 = i^2 + \lambda^2 = \mu \\
gf &= fh = fi = gh = gi = hi = 0
\end{aligned}
\]

It is easy to show that \( b = c = d = e = f = g = h = i = 0, \ a^2 = \lambda^2 = \mu \). Therefore, we deduce \( \text{Ker}_H \cong \mathbb{Z}/2\mathbb{Z} \) in Case 1.

For Case 2, since \( \phi \in \text{Aut}(Q) \), we have \( B \cdot \begin{pmatrix}
  0 & \frac{1}{2} & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1
\end{pmatrix} \cdot B^T = \mu \cdot \begin{pmatrix}
  0 & \frac{1}{2} & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1
\end{pmatrix} \) for some \( \mu \in \mathbb{C}^* \), we arrive at:

\[
\begin{aligned}
ab + c^2 + d^2 + e^2 &= \lambda \cdot f = 0 \\
b\lambda &= \mu \\
\frac{1}{2}bg + c\lambda &= \frac{1}{2}bh + d\lambda = \frac{1}{2}bi + e\lambda = 0 \\
\lambda g &= \lambda h = \lambda i = 0
\end{aligned}
\]

Hence, \( b = c = d = e = f = g = h = i = 0 \) and \( a = \lambda, \lambda^2 = \mu \), so \( \text{Ker}_H \) = id in Case 2.

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2For \( H \), such a lifting \( \bar{H} \) is unique.
To show $\text{Aut}(H, C)$ is a finite group: after intersecting a hyperplane in $\mathbb{P}^4$, we can reduce to the case for quadric surfaces $Q_1, Q_2 \subset \mathbb{P}^3$ and let $C = Q_1 \cap Q_2$, and it suffices to show $\text{Aut}(Q_1, C)$ is a finite group. Since $C$ is smooth, then by the result of [Fuj16] (see page 36), there exists a coordinate such that $Q_1$ and $Q_2$ can be simultaneously diagonalized:

$$Q_1 = V(x_0^2 + x_1^2 + x_2^2 + x_3^2)$$
$$Q_2 = V(x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2)$$

where $1, \lambda_1, \lambda_2, \lambda_3$ are pairwise disjoint. Just denote $F_1 = x_0^2 + x_1^2 + x_2^2 + x_3^2$ and $F_2 = x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2$, and we can rewrite $Q_1 = V(F_1), Q_2 = V(F_2)$. Take any automorphism $\phi \in \text{Aut}(Q_1, C) \subset \text{Aut}(\mathbb{P}^3)$. We have $\phi \cdot Q_1 = Q_1$ and $\phi \cdot Q_2 = V(F)$ where $F$ is a homogeneous polynomial of degree two. Note that $C = V(F_1, F_2)$. From $C \subset \phi \cdot Q_2$, we have $(F) \subset \sqrt{(F_1, F_2)} = (F_1, F_2)$ since $C$ is reduced. Then there exists $c, d \in \mathbb{C}$ such that $F = c \cdot F_1 + d \cdot F_2$ because the degree of $F$ is two. We denote the matrix corresponding to automorphism $\phi$ by $P \in \text{PGL}(4, \mathbb{C})$. And let $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$, then we have matrix equations:

$$\begin{pmatrix} PP^T & I_4 \\ PBP^T & cI_4 + dB \end{pmatrix} = \begin{pmatrix} I_4 \\ B \end{pmatrix}$$

Note $\text{Tr}(PBPT) = \text{Tr}(B) = 4c + d \cdot \text{Tr}(B)$, then $c = \frac{4}{d} \text{Tr}(B)$.

Let $\tilde{F}_2 = F_2 - \frac{1}{\sqrt{2}} \text{Tr}(B) F_1$, and $\tilde{Q}_2 = V(\tilde{F}_2)$, then $\phi \cdot Q_1 = Q_1, \phi \cdot Q_2 = \tilde{Q}_2$. And $C = Q_1 \cap Q_2 = Q_1 \cap \tilde{Q}_2$. Namely, we finally reduce to find solutions for $P \in \text{GL}(4, \mathbb{C})$ satisfying:

$$\begin{pmatrix} PP^T & I_4 \\ PBP^T & B \end{pmatrix}$$

Since $1, \lambda_1, \lambda_2, \lambda_3$ are pairwise disjoint, we conclude that $P$ must be a diagonal matrix with diagonal entry $\pm 1$, which shows $\text{Aut}(H, C)$ is a finite group. 

\[ \square \]

2.2. Threefolds from family No.2.23 are K-unstable. In this subsection, we will show all members from family No. 2.23 is K-unstable by computing $\beta$ invariant (For a complete treatment of $\beta$ invariant, we refer our reader to section 3 of [Xu21] or chapter 1 of [ACC+21]). This result was already known by Fujita ([Fuj16], Lemma 9.9, see also [ACC+21] Lemma 3.7.4). For the reader’s convenience, we also include a proof here.

Lemma 2.2. For every Fano threefold $X$ from family No. 2.23, $X$ is divisorially unstable, namely, there exists a divisor $E$ on $X$ such that $\beta_X(E) < 0$. In particular, this implies $X$ is K-unstable, so $X$ does not admit Kähler-Einstein metric.

Proof. Let $A$ be a hyperplane section of $Q$, then by adjunction formula $K_Q \sim -3A$ and $K_X - \pi^*K_Q = E$, where $E$ is the exceptional divisor, so $-K_X = 3\pi^*A - E$. Let $\tilde{H}$ be the strict transform of $H$, then $\pi^*H = \tilde{H} + E \sim \pi^*A$. For $t \geq 0$, consider $-K_X - t \cdot \tilde{H} = (3 - t)\pi^*A + (t - 1)E$. Note for $0 \leq t \leq 3$, $-K_X - t \cdot \tilde{H}$ is a big divisor. To check nefness:

$$\begin{cases} (-K_X - t \cdot \tilde{H}) \cdot f = 1 - t \geq 0 \\ (-K_X - t \cdot \tilde{H}) \cdot \pi^*L = 3 - t \geq 0 \end{cases}$$

where $f \subset E$ is a line in exceptional surface and $L$ is a line in quadric threefold $Q$ which does not intersect with $C$. Hence, $-K_X - t \cdot \tilde{H}$ is nef if and only if $0 \leq t \leq 1$. If $1 < t \leq 3$, Zariski decomposition of $-K_X - t \cdot \tilde{H}$ is given by $-K_X - t \tilde{H} = (3 - t) \cdot \pi^*A + (t - 1) \cdot E$ where $(3 - t) \cdot \pi^*A$ is the positive part and $(t - 1) \cdot E$ is
the negative part. By the formula in \[IP99\] (see Lemma 2.2.14), we have
\[
(-K_X - t \cdot \tilde{H})^3 = [(3 - t) \cdot \pi^* A + (t - 1) \cdot E]^3
\]
\[
= (3 - t)^3 \cdot A^3 + 3(3 - t)(t - 1)^2 \pi^* A \cdot (-\pi^* C + \deg(N_{C/Q})f) - (t - 1)^3 c_1(N_{C/Q})
\]
\[
= (3 - t)^3 \cdot A^3 - 3(3 - t)(t - 1)^2 A \cdot C - (t - 1)^3 (3A \cdot C)
\]
\[
= -2(t^3 + 3t^2 + 3t - 15)
\]
Thus,
\[
\text{vol}(-K_X - t \cdot \tilde{H}) = \begin{cases} 
-2(t^3 + 3t^2 + 3t - 15) & 0 \leq t \leq 1 \\
2(3 - t)^3 & 1 < t \leq 3
\end{cases}
\]
Therefore,
\[
\beta_X(\tilde{H}) = A_X(\tilde{H}) - S_X(\tilde{H})
\]
\[
= 1 - \frac{1}{30} \int_0^1 -2(t^3 + 3t^2 + 3t - 15) \, dt - \frac{1}{30} \int_1^3 2(3 - t)^3 \, dt
\]
\[
= -\frac{1}{12} < 0
\]
Hence \(X\) is divisorially unstable, so \(X\) is K-unstable by Fujita-Li’s valuation criterion (\[Li17\], \[Fuj16\]), then \(X\) does not admit Kähler-Einstein metric by Yau-Tian-Donaldson theorem.

**Remark 2.1.** The threefolds in Family No.2.23 provide more counterexamples to the following conjecture: *If a compact Fano manifold has no nontrivial holomorphic vector fields, then it admits a Kähler-Einstein metric.* The first counterexample was constructed by the second author in \[Tia97\] by deforming the Mukai-Umemura threefold.

### 3. Proof of Theorem 1.2 AND Theorem 1.1

We will first prove Theorem 1.2 which is restated as follows:

**Theorem 3.1.** For all Fano threefolds \(X\) from family No 2.23, \(X\) does not admit Kähler-Ricci soliton.

**Proof.** This theorem directly follows from Lemma 2.1 and Lemma 2.2. We prove by contradiction, assuming that \(X\) admits a Kähler-Ricci soliton, namely, there exists a Kähler metric \(\omega\) in Kähler class \(2\pi c_1(X)\) and a holomorphic vector field \(\xi\) such that \(\text{Ric}(\omega) - \omega = L_\xi \omega\). By Lemma 2.1 all holomorphic vector fields on \(X\) must be trivial, so \(\omega\) becomes a Kähler-Einstein metric, which contradicts that \(X\) does not admit Kähler-Einstein metric from Lemma 2.2. Therefore, we conclude that \(X\) does not admit any Kähler-Ricci soliton.

Next, we prove Theorem 1.1 which is restated as follows:

**Theorem 3.2.** Any Fano threefold \(X\) from Family No. 2.23 in Mori-Mukai’s list has type II solutions for the normalized Kähler-Ricci flow. Namely, the Gromov-Hausdorff limit of Kähler-Ricci flow on \(X\) is a singular \(\mathbb{Q}\)-Fano variety.

**Proof.** We prove by contradiction again, assuming that the Gromov-Hausdorff limit \(X_\infty\) is smooth, then there exists a smooth deformation \(\pi : X \to \Delta\) such that \(\pi\) is a smooth morphism and general fiber \(\pi^{-1}(t) \cong X\) for \(t \neq 0\) and \(\pi^{-1}(0) = X_\infty\). Since \(\pi\) is a smooth morphism, by Ehresmann’s theorem, we conclude \(b_2(X) = b_2(X_\infty) = 2\), \(b_3(X) = b_3(X_\infty) = 2\) and \(\text{Vol}(X) = \text{Vol}(X_\infty) = 30\), from the geography of smooth Fano threefolds, we conclude that \(X_\infty\) still belongs to Family No. 2.23. Since \(X_\infty\) is smooth, it has been known for long that \(X_\infty\) admits a Kahler-Ricci soliton, which contradicts with Theorem 3.1. Therefore, the limit \(X_\infty\) must be singular.
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