Allocating Stimulus Checks in Times of Crisis

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ABSTRACT
We study the problem of financial assistance (bailouts, stimulus payments, or subsidy allocations) in a network where individuals experience income shocks. These questions are pervasive both in policy domains and in the design of new Web-enabled forms of financial interaction. We build on the financial clearing framework of Eisenberg and Noe that allows the incorporation of a bailout policy that is based on discrete bailouts motivated by stimulus programs in both off-line and on-line settings. We show that optimally allocating such bailouts on a financial network in order to maximize a variety of social welfare objectives of this form is a computationally intractable problem. We develop approximation algorithms to optimize these objectives and establish guarantees for their approximation ratios. Then, we incorporate multiple fairness constraints in the optimization problems and study their boundedness. Finally, we apply our methodology to data, both in the context of a system of large financial institutions with real-world data, as well as in a realistic societal context with financial interactions between people and businesses for which we use semi-artificial data derived from mobility patterns. Our results suggest that the algorithms we develop and study have reasonable results in practice and outperform other network-based heuristics. We argue that the presented problem through the societal-level lens could assist policymakers in making informed decisions on issuing subsidies.

1 INTRODUCTION
The World Wide Web has enabled new forms of financial interaction — through widely-used services like Venmo; through new mechanisms for lending, funding, and philanthropy; and through the collection data in support of all these activities. In the process, Web applications have begun to encounter a set of problems that belong to a broader area concerned with the analysis of spending patterns and the corresponding provision of financial assistance. This is a problem also faced by governments around the world, as they are faced with the problem of saving entities from economic ruin that are subject to financial shocks.

In all of these applications, both on-line and off-line, a standard policy framework is stimulus checks, i.e. cash injections so that consumption is stimulated [2, 4, 5, 12, 13, 26, 34, 35, 38]. A cardinal question faced by decision-makers in any such setting is who gets the subsidies. Such questions have become increasingly salient during COVID-19; for example, in the US the CARES act [1] gives a stimulus check within a certain range of income and increases the payments proportionally to the number of dependents. Other countries such as, for example, New Zealand and Greece offer stimulus checks of fixed value to affected employees. A common pattern in these cases is that when somebody’s income is below a certain threshold or satisfies some criterion then the household is entitled to a stimulus check of fixed value.

An important limitation is that these rules ignore contagion effects through the financial network. For example, if a business defaults that may translate to job loss for the employees who in their turn may not be able to pay their rent etc. The subject of who to bailout in a financial crisis is a subject of controversy [7, 9, 11, 14, 16, 25, 39], namely the policymakers are faced with the following quandary: do we need to bailout large businesses whose saving from default would help maintain job positions but can also make them richer and stronger wrt. the rest of the population, or do we need to bailout individuals (or groups thereof) and small businesses so as they are able to afford their rent and groceries with the fear of letting bigger companies collapse?

Breadth of Impact. The problem in question can find application in many additional domains. Generally, the problem can be used to allocate discrete resources (bailouts) from a discrete set subject to a budget, where the network represents a general supply/demand network that is subject to shocks. For instance, in a ridesharing application, bailouts correspond to vehicles, nodes correspond to neighborhoods, shocks correspond e.g. to vehicle failures, and liabilities correspond to demand for vehicles between neighborhoods. Similar formulations can extend to other problems such as compute centers, and advertisement in the Web.

Contributions. We develop a theoretical framework for optimally allocating discrete bailouts on the well-studied Eisenberg-Noe (EN)
model [19] of contagion with the presence of income shocks, in order to maximize welfare objectives. We show that finding the optimal policy under the discrete bailouts scheme is an NP-Hard problem, develop approximation algorithms, and prove hardness-of-result results. Then we turn to a set of questions within the model that relate to fairness and equity considerations in the provision of bailouts. In particular, we optimize the same objectives as above subject to an algorithmic fairness constraint that regards stimulus distribution via the Gini Coefficient [22] (GC). We study three notions of fairness: (i) the classical GC that measures all-pairs inequality, (ii) a variant of the GC introduced in [31] adjusted to our model, (iii) a novel GC based on the attributes of each node in the financial network (e.g. minority status). We show that the Price of Fairness (PoF) can be unbounded in the discrete case and bounded in the fractional case.

We apply our algorithms on real-world banking datasets from [15] from bank stress-testing, and societal-granularity network topologies inferred by anonymized Web mobility data from the SafeGraph platform, the US Census, and the US Economic Census. We compare the results of the proposed approximation algorithms with other network-based heuristics on these datasets. We empirically study the incorporation of the fairness constraint both between all pairs of nodes as well as in a minority-related context on the semi-artificial data. Our code is released open-source.

Related Work. The EN model introduced in [19] models a financial network where each node has assets and liabilities both wrt. the internal network. Nodes in the EN model pay all their available funds proportionally to their creditors under the equilibrium. Variants of the EN model have studied shocks [23], study the dynamic version [6, 20], bailout scenario [3, 18, 24], and credit default swaps [37, 40]. Specifically, the work of [3] considers fractional intervention methods under budget constraints contrary to discrete interventions that our paper studies. Closing, the problem has connections to Influence Maximization (IM) [10, 28].

2 MODEL

Setup. The EN model considers a directed network of payments $G([n], E)$ with the following structure. The network has $n$ nodes and internal liabilities which are represented on the directed edges $E$. Each node $j \in [n]$ of the payment network has an external influx of assets $c_j \geq 0$ and external liabilities $b_j \geq 0$ which correspond to the node’s external exchanges. A direct way to think about external liabilities are taxes that natural and legal people owe to their governments. Between two nodes $(j,i) \in E$ there is a liability $p_{ji} \geq 0$ from $j$ to $i$, which denotes how much $j$ owes to $i$ (in terms of monetary units, e.g. USD). The initial wealth, or equity, of node $j$ is given by $w_j = c_j + \sum_{i:j-i} p_{ij}$ where $p_{ji} = b_j + \sum_{i:i-j} p_{ji}$ are the cumulative liabilities of node $j$. The network contains no isolated nodes, i.e. every node $j$ has $p_{ji} > 0$ (either external liabilities, or internal liabilities or both). With $\beta_j = (p_{ji} - b_j)/p_{ij}$ we refer to the financial connectivity of $j$, i.e. the fraction of total liability that $j$ owes within the network (not to be confused with graph connectivity; see also [23]). We define $\beta_{\text{min}} = \min_{j \in [n]} \beta_j$ and $\beta_{\text{max}} = \max_{j \in [n]} \beta_j$.

The assets of each node are disrupted by a random shock $X_j \in [0, c_j]$ [23]. Under the shock, each node is able to pay its creditors an amount $\beta_j \in [0, p_{ij}]$. If the node is solvent, then $\beta_j = p_{ij}$. Otherwise, the node defaults to its creditors, and rescales its liabilities in order to pay its creditors to a reduced rate, according to the amount of money it owes to them. The relative liabilities matrix $\bar{\Lambda}$ is defined to have entries $\bar{\Lambda}_{ij} = p_{ii}/p_{ji}$ whenever $p_{ij} > 0$ and $\bar{\Lambda}_{ij} = 0$ otherwise. A standard bailout strategy when the recipients have high granularity (i.e. individuals, small businesses) is to bailout an individual from a fixed set of resources. Thus, to (perhaps partially) avert the shocks a policymaker can issue stimulus checks of value $L_x$ to a subset of nodes subject to a budget constraint $\Lambda > 0$. The clearing payment vector $\bar{\rho}$ is given by the EN equilibrium equations

$$\bar{\rho} = p \land (A^T \bar{\rho} + c - X + L \odot \bar{z})$$

where $\bar{z} \in (0,1)^n$ is the decision variable and $\odot$ is the Hadamard product. The payments are subject to a budget constraint $L^T \bar{z} \leq \Lambda$. The set of default nodes is defined to be the set $D = \{ j : \bar{\rho}_j < p_{ij} \}$ whereas the set of solvent nodes is the set $R = \{ j : \bar{\rho}_j = p_{ij} \}$. [23] makes the following assumption about the uniqueness of $\bar{\rho}$:

ASSUMPTION 1 (UNIQUENESS OF CLEARING VECTOR [23]). If every $i \in [n]$ can access $j \in [n]$ with $j$ has $b_j > 0$, then $\bar{\rho}$ is unique.

Under this assumption one can apply the (contraction) mapping $\Phi(\bar{\rho}) = p \land (A^T \bar{\rho} + c - X + L \odot \bar{z})$ to iteratively compute the equilibrium vector under an allocation $\bar{z}$. $\Phi$ is increasing, positive, and concave [19]. In this paper, we rely on a very realistic assumption, that every node is directly obliged to the external sector, i.e.

ASSUMPTION 2. $\|A^T\|_1 = \beta_{\text{max}} < 1$.

In the context of our paper, Ass. 2 is a reasonable assumption as we are able to model the external liabilities with a tax (e.g. income tax) or other fixed costs (e.g. rent/utilities payment) that is owed to an external entity (e.g. government, bank etc.). At equilibrium, the solvent nodes are $R = \{ j : \bar{\rho}_j = p_{ij} \}$ and the default nodes are $D = \{ j : \bar{\rho}_j < p_{ij} \} = [n] \setminus R$.

Optimization Formulation. The vector $\bar{\rho}$ can also be found via solving an optimization problem of the following form [19, p. 10]:

$$\max \mathbb{E}_{X \sim D} [f(\bar{\rho})]$$

s.t. $\bar{\rho} \leq A^T \bar{\rho} + c - X + L \odot \bar{z}, 0 \leq \bar{\rho} \leq p$  \hspace{1cm} (OptBailouts)

$L^T \bar{z} \leq \Lambda, \bar{z} \in (0,1)^n$. Where $f$ is a strictly increasing function of $\bar{\rho}$. We denote the overall vector as $\bar{z} = (\bar{\rho}, \bar{z})^T \in [0, p] \times (0,1)^n$. The former type of

\footnote{Source code: https://github.com/papachristoumarinos/financial-contagion.}
objective we study is a linear objective that is a sum (with positive coefficients) of the payments in equilibrium, and the latter one is the total number of solvent nodes. Maximizing such objectives is reasonable both from a societal and a technical viewpoint (see also [2, 3, 30, 33]). From a societal viewpoint, the maximization of a sum (with positive coefficients) of the total payments in the network results in the (assuming honest) nodes being incentivized to clear their liabilities for the benefit of society, and, thus, each node’s utility corresponds to the liabilities it is able to pay off. From a technical viewpoint, the positively-weighted linear objectives are known to yield clearing vectors, and the number of solvent nodes objective can also be slightly modified to yield a clearing vector.

The linear objective parametrized by a vector $v > 0$ is

$$f(\bar{p}) = v^T \bar{p},$$

(L-OBJ)

and the following linear objectives are of particular interest:

**Sum of Payments:** $f_{\text{SoP}}(\bar{p}) = 1^T \bar{p}$.  
(SoP)

**Sum of Internal Payments:** $f_{\text{SoIP}}(\bar{p}) = \beta^T \bar{p}$.  
(SoIP)

**Sum of Taxes:** $f_{\text{SoT}}(\bar{p}) = (1 - \beta)^T \bar{p}$.  
(SoT)

**Fractional Solvency:** $f_{\text{FS}}(\bar{p}) = \frac{p_3}{\sum_{j \in [n]} p_j}$.  
(FS)

The number of solvent nodes objective is given as:

**Absolute Solvency:** $f_{\text{AS}}(\bar{p}) = \sum_{j \in [n]} 1(p_j = p_j)$.  
(AS)

We present an example of the discrete bailout model below:

**Example 1** (see Fig. 2). Let $\Lambda = 1$, $L = 1$ and the network that has nodes $\{1, 2\}$, external assets $c = (\frac{3}{2}, 0)^T$, external liabilities $b = (\frac{1}{2}, 1)^T$ and internal liability matrix $P$ with $p_{ij} = 1(i = j = 1)$. A point mass shock $x = (1, 0)^T$ causes node 1 to default, which in turn causes node 2 to default, since the asset vector (after shock) is $(\frac{3}{2}, 0)^T - (1, 0)^T = (\frac{1}{2}, 0)^T$. Thus node 1 can pay $\frac{1}{2}$ to node 2 and $\frac{1}{2}$ to the ext. creditors and, in turn, node 2 can only pay $\frac{1}{2}$ to its ext. creditors. The optimal solution is to bailout node 1 with 1 unit of cash at which case the asset vector becomes $(\frac{1}{2}, 0)^T$ again. We have that $\text{OPT}_{\text{SoP}} = \frac{3}{2}$, $\text{OPT}_{\text{AS}} = 2$, $\text{OPT}_{\text{FS}} = 2$, $\text{OPT}_{\text{SoT}} = \frac{3}{2}$, and $\text{OPT}_{\text{SoIP}} = 1$.

Finally, we use the following Lemma as a tool in many of our proofs. Briefly, the Lemma states that if the external assets vector is increased (point-wise) then the corresponding clearing vector is (point-wise) greater than the previous one.

**Lemma 1 (Comparison Lemma).** Let $N_1 = (P, c_1, b)$ and $N_2 = (P, c_2, b)$ be two EN instances with $c_1 \geq c_2$ (w.l.o.g. with $X = 0$). Then for the clearing vectors $\bar{p}_1$ and $\bar{p}_2$ it holds that $\bar{p}_1 \geq \bar{p}_2$ and $v^T \bar{p}_1 \geq v^T \bar{p}_2$ for every vector $v > 0$. Moreover, if $D_1$ is the set of the default nodes of $N_1$, then the vector $\tilde{p} = 1_{D_1} \otimes (c_1 - c_2) \geq 0$ satisfies $\bar{p}_1 \geq \bar{p}_2 + \delta$, and subsequently $v^T (\bar{p}_1 - \bar{p}_2) \geq v^T \delta$.

**Fractional Relaxation.** The fractional relaxation (Rel-OPTBailouts) of (OptBailouts) allows $z_i$ to range in the continuous $[0, 1]$ interval, where the variables that refer to the relaxation are denoted by $\tilde{x} = (\tilde{p}, \tilde{z})$.

$$\max \mathbb{E}_{X \sim D} [f(\bar{p})]$$

s.t. $\tilde{p} \leq A^T \tilde{p} + c - X \otimes \tilde{z}, 0 \leq \tilde{p} \leq p$  
(Rel-OPTBailouts)

$LZ \leq \Lambda, 0 \leq \tilde{z} \leq 1$.

Given an optimal solution $OPT_f$ to (OptBailouts) and an optimal solution $OPT_{\text{Rel}}$ to (Rel-OPTBailouts) the optimality conditions impose that $OPT_{\text{Rel}} \geq OPT_f$ since the relaxed problem has a larger feasible region. A feasible solution of (OptBailouts) is denoted by $\text{SOL}_{\text{f}}$ and satisfies $\text{SOL}_{\text{f}} \leq OPT_f$. Finally, we denote a feasible solution by $\tilde{x}$ (resp. $\tilde{z}$), an optimal solution by $\tilde{x}^*$ (resp. $\tilde{z}^*$).

**Decision version of (OptBailouts).** The decision version of the problem takes as input a network with liability matrix $P$, external assets $c$, internal assets $b$, a stimulus vector $L > 0$, a stimulus value a budget $\Lambda \geq 0$ to be used for the bailouts, a shock distribution $D$, and a lower bound $f^*$ and answers YES if and only if there exists a set of nodes which are bailed out with $L$ such that in equilibrium we have that $\mathbb{E}_{X \sim D}[f(\bar{p})] \geq f^*$.

## 3 APPROXIMATION ALGORITHMS

It is easy to prove that the problem of determining the optimal bailouts is NP-Hard by a reduction from the 3-Set-Cover problem [27]. Fig. 3 shows a visual proof of the construction of the reduction and the complete proof can be found in the full paper. To mitigate this problem, we develop approximation algorithms. Our first algorithm is based on randomized rounding. More specifically, we solve the relaxation problem given in (Rel-OPTBailouts) and then we apply a randomized rounding scheme by rounding the decision variables randomly with coin flips of bias equal to their optimal values. While the rounding scheme itself is straightforward, the analysis is more subtle, and shows that this algorithm achieves an approximation guarantee of $\frac{1}{\epsilon} - o(1)$ for every linear objective given by a vector $v > 0$ of coefficients with $\xi = \xi(v)$. Our second result is a greedy algorithm that always chooses the node with the maximum gain in welfare subject to respecting the budget constraints. This type of algorithm has been extensively used for
maximizing monotone submodular functions subject to cardinality constraints [32], such as influence maximization problems [28], outbreak detection [29], facility location problems [17], and many more, and has an approximation ratio equal to $1 - 1/e$. Although the family of linear objectives we analyze in this paper is not submodular in general, we are able to adapt the style of analysis using the properties of $\Phi$ and Lemma 1, as we describe in Sec. 3.2. The algorithm achieves an approximation guarantee of $1 - e^{-1/\beta_{\text{max}}}$ under a reasonable condition. In addition, the hill-climbing algorithm is efficient and performs very well, outperforming all the network-based heuristics. The rounding algorithm has a better worst-case approximation ratio in our theoretical bounds than the greedy algorithm, however it performs slightly worse than it empirically. Finally, note that optimal bailouts with the (AS) objective is inapproximable within a poly-time computable function of the input size. Fig. 4 shows a visual proof of the statement, which is again based on a reduction from the 3-Set-Cover problem. The complete proof can be found in the full paper.

**3.1 Randomized Rounding Approximation**

In this Section, we prove approximation guarantees based on randomized rounding. In brief, we solve the relaxation problem where the indicator variables $z_i$ are allowed to range in $[0, 1]$, as presented in (Rel-OptBailouts). If $\tilde{\mathbf{z}} = (\tilde{\mathbf{p}}, \tilde{\mathbf{z}})^{\top}$ is the optimal solution to the relaxation problem, we construct the corresponding rounded solution based on setting each rounded variable $Z_i$ equal to 1 with probability $\tilde{z}_i$, and 0 with probability $1 - \tilde{z}_i$.

We start by giving an approximation guarantee for the expected costs of the rounded solutions. When we condition upon a shock $X = x$ we overload the notation SOL, OPT, OPT_{Rel} with SOL(x), OPT(x), OPT_{Rel}(x), so that $\text{SOL} = \mathbb{E}_{X \sim \mathcal{D}} [\text{SOL}(X)]$ (resp. $\text{OPT} = \mathbb{E}_{X \sim \mathcal{D}} [\text{OPT}(X)]$, and $\text{OPT}_{\text{Rel}} = \mathbb{E}_{X \sim \mathcal{D}} [\text{OPT}_{\text{Rel}}(X)]$) (see App. 6):

**Theorem 1.** The following results hold for the expected costs under the rounded variables $Z$ given a fixed shock $X = x$, for a linear objective $f(\tilde{p}) = v^T \tilde{p}$ with $v > 0$ and $\xi = \max \min \geq 1:

$$
\mathbb{E}_{Z \sim \text{Ber}(\tilde{z})} \left[ \text{SOL}_{f}(x) \right] \geq \left( 1 - \frac{\beta_{\text{max}}}{\xi} \right) \cdot \text{OPT}_{f}(x).
$$

Using the fact that

$$
\mathbb{E}_{Z \sim \text{Ber}(\tilde{z})} \left[ \text{SOL}_{f} \right] = \sum_x \text{Pr}[X = x] \mathbb{E}_{Z \sim \text{Ber}(\tilde{z})} \left[ \text{SOL}_{f}(x) \right]
$$

(resp. $\mathbb{E}_{Z \sim \text{Ber}(\tilde{z})} \left[ \text{OPT}_{f} \right]$) we get an approximation ratio of $(1 - \beta_{\text{max}}/\xi)$ for (L-OBJ) on expectation.

The full paper also contains a runtime analysis of the above algorithm. We have to note a subtle point here: when we use an independent randomized rounding scheme when the entries of $L$ are different the variance of $L^T Z$ is at most $\frac{1}{2} \sum_{j \in [n]} L_j^2$ and therefore there are instances where meeting the constraints of the problem incurs a very high runtime. For instance if $L_j = 2^{j-1}$ and $Z_j = 1/2$ then essentially the random variable $L^T Z$ is the uniform distribution on the set $\{0, \ldots, 2^{n-1}\}$ and the runtime can in theory be prohibitive. While this problem does not show in our experimental evaluation later, the full paper contains an approach that mitigates the problem and gives the same approximation guarantee with a good runtime.

Briefly, we use the dependent rounding scheme of [41] (that is specially engineered to guarantee high concentration of $L^T Z$ for different entries of $L$) as an oracle to round an optimal solution to the LP.

The **integrity gap** $\gamma_f$ (see [42] for definition) of the problem can be used to quantify the worst case ratio between a fractional optimal solution and an integral optimal solution. It is easy to conclude that the integrity gap of every linear objective can become unbounded. In detail, we start with the complete network $K_n$, with $c_i = b_i = 1$ and $p_{ij} = 1$, and a point-mass shock of $X_i = \epsilon$ for some $\epsilon \in (0, 1)$. We also let $L = n/k \cdot 1$ and $\epsilon > 0$. In the fractional optimum we have that $Z_i = \frac{1}{\epsilon}$ for $k/n$ nodes, and the optimum value of $\|z\|_1 \cdot n$ is achieved.

\[\text{In fact, any rounding scheme such that } \mathbb{E} \left[ Z_i \right] \geq \frac{1}{\epsilon} \text{ would yield the same approximation ratio guarantee on expectation.}\]

\[\text{It is an open question whether this analysis is tight, as well as whether an algorithm with a better approximation guarantee can be found.}\]
When we choose to round to $k$ nodes, we do that uniformly (since $\bar{z}_i^j = k/n$) over all $k$-sets, that is the probability of each $k$-set being selected is $\frac{1}{\binom{n}{k}}$. Thus the value optimum is just the value of the optimum given that the values of any $k$-set are set to 1. The rounded nodes’ bailouts would suffice to avert the shock, so they will be truncated to have a value of $\varepsilon$, from the problem constraints (debts must only be paid in at most their full value). Since all nodes are marginally default (that holds for the bailed-out nodes with the truncated values as well), we have from (see (3) in App. 6) that $\left(1/n\right)\sum_{\nu\in[n]}p_\nu = n - n\varepsilon + k\varepsilon$ which implies that $\text{OPT}_f = v^T\bar{p} \leq \|\bar{v}\|_\infty \left[n(n - \varepsilon(n - k))\right]$. We thus have $\sigma_f \geq \frac{\varepsilon}{1 - \varepsilon(1-k/n)}$. So for $k = o(n)$ (small) and $n$ large the gap tends to $\frac{\varepsilon}{1-\varepsilon}$ which can be arbitrarily bad for $\varepsilon \to 1$.

3.2 Greedy Algorithm

We consider a family of greedy hill-climbing algorithms in order to find the optimal bailout set for a linear objective $f(p) = v^T\bar{p}$, where $v \succ 0$. These algorithms run in $k \leq \Lambda/\text{lim}_{\text{min}}$ steps, and at each step they pick the (feasible) element with the largest marginal gain until the budget constraint is violated, namely given a fixed shocked set $X = x$ where the current set of bailouts is $S_x$, with $S_0 = \emptyset$, we have $u_{t+1} \in \text{argmax}_{u \in [n], S_0 \text{ feasible}} \left\{v^T\bar{p}_{S_x \cup \{u\}} - v^T\bar{p}_{S_x}\right\}$. This algorithm resembles the hill-climbing nature of [32] for constrained monotone submodular maximization which guarantees an $(1 - 1/e)$-approximation ratio. Below we prove that under a Small-Bailout Regime our algorithms achieve an approximation ratio of $1 - e^{-1-\text{lim}_{\text{min}}}/\varepsilon - o(1)$. We first state the Small-Bailout Regime condition, that suggests that whenever a node is bailed out, it remains default after the bailout (however with a greater value). It is important to note here that the hill climbing family of algorithms we consider would still work if run on an instance that violates the Small-Bailout Regime, and, in practice, the greedy hill climbing algorithm is the best-performing one, however the theoretical guarantee would hold only when this condition holds. Again, given an approximation guarantee for the fixed shock $x$ we can later argue that this guarantee is achieved in expectation. We state our assumption

**CONDITION 1 (Small-Bailout Regime). Under the randomness of $D$ with probability 1, for every step $t$ the node $u_t$ selected by the algorithm yields a clearing vector $\bar{p}_{S_x}$ such that $\bar{p}_{S_x \cup u_t} \leq \bar{p}_{u_t}$.**

This condition says that every node $u_t$ we include at iteration $t$ does not get "saturated" in the context of the constraint $\bar{p}_{S_x \cup u_t} = \bar{p}_{u_t}$ holding and the default constraint being inactive. Under this condition, the approximation ratio of the greedy algorithm follows:

**THEOREM 2. Under C1, for every linear objective the greedy hill-climbing algorithm achieves an approximation guarantee (under $X \sim D$) of $\text{SOL}_f \geq \left(1 - e^{-1-\text{lim}_{\text{min}}}/\varepsilon\right) \cdot \text{OPT}_f$.**

**Expectation via Samples.** The evaluation of $\mathbb{E}_{X \sim D}[\cdot]$ is done via independently sampling a number of shocks $X_1, \ldots, X_m$, apply the approximation algorithm $m$ times and then calculate the empirical mean. In the full paper we prove that when we approximate the expectation via samples the approximation ratio reduces additively by $O(\varepsilon)$ where $\varepsilon$ is an accuracy parameter.

4 FAIRNESS

4.1 Fairness Metrics

We say that an allocation is fair across the nodes if the allocation (discrete or fractional) obeys the following property: the amount of bailouts that a certain node gets "does not differ a lot from its neighbors". For instance, the "neighborhood" may refer to measuring inequality among all pairs of nodes, nodes who have a specific property, and nodes in the actual network. All metrics have to be homogeneous, i.e. multiplying all the bailouts by a positive number should not affect their value. We consider the following constraints to incorporate fairness in our model:

**Gini Coefficient.** The Gini Coefficient [22] measures the fairness of the stimulus allocations between all pairs of nodes defined as

$$\text{GC}(\hat{x}) = \frac{\sum_{i,j \in [n]} |L_i \hat{x}_i - L_j \hat{x}_j|}{2n \sum_{i,j \in [n]} (L_i \hat{x}_i)}.$$  

It evaluates to 0 when the stimuli are equally distributed and is $1 - 1/n$ when one node gets all the bailout amount. A useful optimization constraint is making the Gini coefficient at most $g \in [0,1]$, or equivalently $\sum_{i,j \in [n]} |L_i \hat{x}_i - L_j \hat{x}_j| \leq 2n \sum_{i,j \in [n]} (L_i \hat{x}_j)$. We say that an allocation for which the Gini Coefficient is at most $g$ is a $g$-unfair allocation. We note that a possible disadvantage of this metric is that it does not take into account each individual node’s debts, i.e. it treats all nodes on an equalized basis. This issue is mitigated by the (Sp-GC) metric which is presented below.

**Property Gini Coefficient.** The collection of real-world data from SafeGraph and the US Census we present in Sec. 5 consists of attributes characterize nodes. One of the key attributes in these datasets is the *minority status* of the owner of a business, if such business participates at the network as a node, or the demographic characteristics of a group of people, for instance the fraction of people belonging to a minority group within a Census Block Group under which we want to impose fairness constraints. (That is, to measure the relative assistance between different groups, in an approximate way).

This type of data motivates the following metric: We introduce the Property Gini Coefficient (Prop-GC) in which nodes may have a property of interest (such as the demographic group in the SafeGraph or Census data) along which we want to apply an equity analysis. We model this by a *property vector* $q \in [0,1]^n$, where each element $q_j$ corresponds to the probability that node $j \in [n]$ has this property, as follows: We let $n_q = \sum_{j \in [n]} q_j$ and $n-n_q$ be the total weights of the (soft) bipartition. For every node $j \in [n]$ is inequality subject to being in the minority group is given as $\frac{1}{n-n_q} \sum_{j \in [n]} (1 - q_j) |L_i \hat{x}_i - L_j \hat{x}_j|$. The sum over all $j$ with weight $q_j$ give the numerator of the (Prop-GC). The denominator of the (Prop-GC) is $\sum_{j \in [n]} q_j L_i \hat{x}_j$:

$$\text{PGC}(\hat{x}; q) = \frac{\sum_{i,j \in [n]} q_j (1 - q_j) |L_i \hat{x}_i - L_j \hat{x}_j|}{2(n-n_q) \cdot \sum_{j \in [n]} q_j L_i \hat{x}_j},$$  

(Prop-GC)

Note that taking $q = \frac{1}{2} \cdot 1$ reduces (Prop-GC) to the conventional GC. Moreover, for $L = \ell \cdot 1$ and $q^T \hat{x} = 0$ we observe that (Prop-GC)
becomes unbounded since the denominator goes to zero. One case where this happens, and further justifies the correctness of the criterion is when $q$ and $\bar{z}$ are a 0/1 vector and where the entries of $q$ are 1 the entries of $\bar{z}$ are 0, and vice-versa, where the entries of $q$ are 0 the entries of $\bar{z}$ are 1, which corresponds to giving all the bailouts to the majority group. We say that an allocation which achieves a (Prop-GC) at most $q$ subject to a property $q$ is $(g,q)$-unfair. Note that the (Prop-GC) constraint can be combined with the (GC) constraint as follows: given $g_b, g_w \geq 0$ we seek to find an allocation that respects both between-fairness, i.e. $P((\bar{z} \odot q) \leq g_b)$ and within-fairness, i.e. $GC(\bar{z} \odot q) \leq g_w$ and $GC(\bar{z} (1-q)) \leq g_w$.

**Spatial Gini Coefficient.** To make the GC take into account net-allocations, whereas their discrete counterparts consider discrete allocations.

\[ \text{SGC}(\bar{z};A) = \frac{\sum_{(i,j) \in E} a_{ij}(L_i\bar{z}_j - L_j\bar{z}_i)}{2\sum_{j \in [n]} \bar{z}_j} \]  

(Prop-GC)

The aforementioned definition also appears in [31] where the graph is assumed to have unit weights. In our case, the role of the unweighted graph plays the relative liability matrix $A$. Since $A$ is stochastic the total weight of each row is $\bar{\beta}_i < 1$ and the contribution of edge $(i,j)$ is $a_{ij}$. Normalizing by the sum $\sum_{j \in [n]} \bar{\beta}_j L_j$ allows for comparing different population groups and different bailout magnitudes. When the bailouts are distributed equally the (Sp-GC) is 0. If $A = A^T$ and one node gets all the bailouts, then the (Sp-GC) is bounded by 1. We say that an allocation which achieves an (Sp-GC) of at most $q$ is $(g,A)$-unfair. We note here that unlike (GC), the (Sp-GC) metric takes into account each node’s debt, that is a node $i$ with a significant (compared to its neighbors) liability to node $j$, i.e. it has $a_{ij} = \bar{\beta}_j$, then this deviation will get a higher weight in the calculation of the coefficient compared to $j$’s deviation from the rest of its neighbors. In the full paper we prove that when $A = A^T$ the (Sp-GC) metric is connected with the conductance of $A$.

### 4.2 Problem Formulation & Price of Fairness

We formulate the following relaxations to the optimization problems involving the aforementioned fairness metrics for a target fairness metric upper bound $g \geq 0$. First, we consider the following optimization problem that extends (OptBailouts) by adding the GC-dependent constraints:

\[
\begin{align*}
\text{max} & \quad \mathbb{E}_{X \sim D} [f(\bar{p})] \\
\text{s.t.} & \quad (\bar{p}, \bar{z}) \in \text{(OptBailouts)} \text{ constraints,} \\
& \quad \sum_{j \in [n]} [L_i\bar{z}_j - L_j\bar{z}_i] \leq 2ngL^T\bar{z} \\
\end{align*}
\]

(GC-Problem)

\[
\begin{align*}
\text{max} & \quad \mathbb{E}_{X \sim D} [f(\bar{p})] \\
\text{s.t.} & \quad (\bar{p}, \bar{z}) \in \text{(Rel-OptBailouts)} \text{ constraints,} \\
& \quad \sum_{j \in [n]} [\bar{d}_{ij} - 2ngL^T\bar{z}] \\
& \quad \bar{d}_{ij} \geq 0, -\bar{d}_{ij} \leq L_i\bar{z}_j - L_j\bar{z}_i \leq \bar{d}_{ij}, \forall (i,j). \\
\end{align*}
\]

(Rel-GC)

The optimization problem relaxations can be formulated mutatis mutandis for (Prop-GC), (Sp-GC) for a fairness bound $g \geq 0$. The optimal fractional solutions to these problems consider fractional allocations whereas their discrete counterparts consider discrete allocations.

Then, a natural question we might ask is, “What is the maximum effect of these fairness constraints on the welfare objective function?”. We define the Price of Fairness (PoF) [8] to be

\[
\text{PoF} = \frac{\text{OPT sans Fairness}}{\text{OPT with Fairness}}
\]

It is evident that the since the value in the numerator refers to an optimization problem with a larger feasible region than the one in the denominator that always PoF $\geq 1$. At glance, a natural question arises: Do there exist instances for which PoF is unbounded/bounded for discrete/fractional allocations? The answer to this question gives the following Theorem (App. 6):

**Theorem 3 (PoF Boundedness).** (1) There exist finite instances where the PoF is unbounded for discrete bailouts, for all the constraints defined by (GC), (Prop-GC), and (Sp-GC), and any linear objective given by a vector of coefficients $\nu > 0$.

(2) For every increasing objective $f$ with $f(0) = 0, ||f||_\infty < \infty,$ and for every $g \geq 0$ the PoF for fractional bailouts is bounded (for all fairness metrics).

### 5 EXPERIMENTS

**Datasets.** In this section, we evaluate our methods on public datasets from two kinds of sources: high-level granularity data, among nodes corresponding to financial institutions in a country, financial institutions between countries, or financial interactions between different financial sectors of the same country; and lower granularity data, among nodes corresponding to anonymized groups of people defined by the US Census. In the latter case, the relevant public datasets are constrained through anonymization or aggregation due to the privacy considerations of the individuals. We use the following datasets:

**German Banks.** Datasets from 22 German banks from the work of [15] where the internal and external assets and liabilities of German Banks are reported. The dataset contains $n = 22$ with $m = 435$ edges, with a mean outdegree $d_{out} = 19.8$. The network structure can be described by the ER network $G(n = 22, p = 0.94)$. SafeGraph. Data generated based on mobility data from the SafeGraph platform during April 2020. The nodes in the financial network represent: (i) Points of Interest nodes (POI nodes) that represent various businesses categorized by their NAICS codes to categories (i.e. grocery stores, banks, gas stations etc.) and the Census Block Group $^{9}$ (CBG) they are located at; (ii) CBG nodes that represent a set of households in a certain location. The dataset is constructed by access to an initial pair of geographical coordinates (i.e. latitude and longitude) and a number $k_{\text{NN}}$ of neighboring CBGs. The POI nodes are determined to be the businesses that are located in the $k_{\text{NN}}$-nearest neighboring CBGs based on the Haversine distance metric. Each POI provides data about the CBGs of its unique visitors $^{10}$ and the dwell times. For the source of its visitors we estimate the number of people that come from each CBG. From the dwell times of devices that are available we determine the % of people that work on the POIs and the ones who visit the POIs. We

$^{9}$A CBG is a unit used by the US Census. It is the smallest geographical unit for which the bureau publishes sample data, i.e. data which is only collected from a fraction of all households and contains 600-3K people.

$^{10}$A unique visitor is a unique mobile device. We assume that each device represents a distinct person.
create a financial liability edge from the POI to the CBG node to indicate the payment of a liability (i.e. a wage) and for the latter category we create a liability edge from the CBG node to the POI representing some form of expense (e.g. groceries). The weights are multiplied accordingly to represent the set of people that interact with each POI, resulting in a bipartite network. Each CBG node is associated with multiple data from the US Census, as well as every POI node is associated with data from the US Economic Census. For each CBG node, we estimate the average size of households per CBG, the average income level and the % of people that belong to a minority group.

We use the above data to estimate the external assets and liabilities of the CBG nodes. For the bailouts of CBG nodes we calculate a custom bailout devised by the CARES act that considers as income the average income of the CBG and as size of household the average size of household multiplied by an estimate for the number of people in that CBG who interact with the POI nodes. Similarly, for the POI nodes we use data from the US Economic Census and NAICS to determine average wages, income and expenses. For the bailouts of the POI nodes we use loan data regarding loans that were given during April 2020 as part of the SBA Paycheck Protection Program (PPP) provided by SafeGraph, adjusted to the number of workers being present in the network and the span of one month. Moreover, the loan data included demographic characteristics about the businesses in question so we were able to determine (or estimate in the case of missing data) the minority status of a business, i.e. the probability of a certain business being a business with a minority owner. A complete description of the data generation process is presented in App. 6.

The full paper contains more experiments with data from the EBA 2011 stress test [23], as well as semi-artificial data from the social payments application Venmo. In a nutshell, the experimental results and conclusions are similar to the ones presented below and are omitted.

**Shocks.** For all of our experiments we assume that the shocks $X_j$ for each $j \in [n]$ are independent uniform in $[0, c_i]$.

**Heuristic Methods.** We use the following heuristic methods to allocate stimulus, where for each step we augment each set (based on
the criteria below) maximally wrt. the budget constraint: (i) Wealth Policy. We sort the nodes in increasing order of initial wealth \( w_i \); (ii) Out-degrees; (iii) PageRank [36]; (iv) Eigenvector Centrality [21]; (v) Random Permutation. Such policies are well known benchmarks and have been used on similar tasks such as Influence Maximization [28]. For all experiments we also report the upper bound of OPT, which is the corresponding relaxation optimum OPT_rel that is an upper bound to the true optimum.

5.1 Experimental Setup

Discrete Allocations. Firstly, for various values of the stimuli vector \( L \), either fixed or varying, we report the corresponding objective values averaged over multiple draws of shocks from the corresponding shock distribution, where we report both the empirical mean and standard deviation (std). We parametrize the available budget with a pair of parameters. The former parameter \( \ell \) parametrizes the budget increase rate, and the latter parameter \( k \) parametrizes the multiplicity of resources. We make assumptions about the bailouts that fall into two main categories: (i) fixed bailouts: where \( L = \ell \cdot 1 \) and \( \Lambda(k) = \ell \cdot k \) and the number of bailouts \( k \) varies along the x-axis of the plots. This is equivalent with bailing out at most \( k \) nodes on the network where every node gets a stimulus value of \( \ell \). (ii) variable bailouts: in the SafeGraph experiments we determine the bailouts as discussed in the start of Sec. 5 and App. 6. We assume that for each step \( k \) the budget increases by some amount \( \ell \) so, again, the available budget is \( \Lambda(k) = \ell \cdot k \). The results of these experiments for the various datasets are reported in Figs. 5, 6. In brief, the greedy algorithm outperforms all other algorithms in all settings. Then, the rounding algorithm comes second, outperforming the other heuristics. Finally, note that the Wealth Policy performs very poorly due the fact that it is independent of the contagion process.

Fairness. Secondly, we perform experiments where we impose fairness constraints. In Figs. 6(a), 6(b), 6(c) we study the value of the (SoT) objective where we run the unconstrained optimization problem (i.e. with a large upper bound), and, second, we restrict the (GC) to be at most 0.1. We report the relaxation optimum, the rounded solutions to the problem and the realized (GC) after the optimization. We study the behaviour with respect to the (GC) and (Prop-GC) constraints. For the SafeGraph and the German Banks data we use the fuzzy version of the (Prop-GC) where the weights represent the probability that a node is a minority node. In other words, we want to impose constraints between minority and non-minority groups. For the former dataset, the weights \( q \) are the minority scores for each CBG and business where we impute missing data with the MLEs of the existing data. For the latter dataset, the values of \( q \) are sampled i.i.d. from Beta(2, 5). We use a budget increase rate of \( 10^5 \) for SafeGraph, similar to the one reported in Fig. 5, and a bailout \( L = 10^5 \cdot 1 \) for German Banks.

Lastly (Fig. 6(d)), we plot the relation between the upper bound on the (Sp-GC) coefficient and the PoF for the German Banks Data for \( L = 10^5 \cdot 1 \).

6 DISCUSSION

Discrete Allocations. Fig. 5 shows the performance of the various algorithms on the German Banks and the SafeGraph datasets. We observe that in the the greedy and the LP algorithm outperform the other heuristics.

On the German Banks dataset, in the worst case, the greedy algorithm outperforms the LP-based one by \( \approx 15\% \), whereas the PageRank and centrality-based heuristics are outperformed by greedy by \( \approx 58\% \) in the worst case. Finally, we note that in both objectives the random permutation, wealths and outdegree heuristic perform badly. The significantly decreased performance of the wealth heuristic is justified by the fact that nodes which are important for the bailout process and have priority (i.e. lower wealth) are not well-connected. Similarly, one can argue about the other heuristics, i.e. in the case of the out-degree heuristic nodes with low equity may be well connected to the other nodes and thus giving them priority does not contribute substantially to the overall objective. That suggest that the uneven form of such curves represent the fact that these simple heuristics are not good candidates for this optimization problem. The results regarding (SoT) are similar, so we omit them.

For the SafeGraph data, the randomized rounding algorithm outperforms all benchmarks and is also very close to the greedy algorithm with a worst case difference of about 7.1% and being as far as approximately 40% from the other heuristics in the worst case in the (SoT) plot. In the (SoP) plot the differences are about 18% in the worst case, however the greedy algorithm quickly approaches the randomized rounding algorithm.

Finally, Fig. 5 suggests that bailouts and (financial connectivity, centrality, wealth) may have some correlation; but not very high. We believe such naïve policies have pitfalls, i.e. including bailing out big corporations that may not need bailouts, or bailing out individuals who are not “important” to the network.

Fairness. The German banks dataset (GC) case (Figs. 6(a), 6(c)) has predictable behaviour. To be more specific, as the number of bailouts \( k \) is small, the instance for which Target Ginī ≤ 0.1 has a lower relaxation optimum as well as rounded value for \( k \leq 6 \) and later approaches the unconstrained optimum (i.e. where Target Ginī ≤ 1.0). In the constrained case, the Ginī coefficient rises to its upper bound 0.1 until \( k = 6 \) and then starts to drop, that coincides with the objective value plots. The explanation for this phenomenon is rather simple: at first, when resources are scarce, selecting certain nodes on the network subject to these resources creates inequality which is mitigated by the constraint at the expense of the quality of the solution creating a worse PoF of about 1.2 ≥ 1. When \( k \) is large enough, i.e. \( k \geq 6 \), the available resources allow the constrained version to create a solution close to the unconstrained version (by “rebalancing” some \( z_i^* \) values) which is reflected on both objective values, and eventually the PoF reaches 1 when the two solutions eventually meet. Similarly, when we constrain the (Prop-GC) (Fig. 6(c)) the PoF is approximately 1.16 in the worst case \( k = 2 \) and approaches 1 at \( k = 6 \).

The PoF for the SafeGraph subject to the (Prop-GC) constraint data is approximately 1, meaning that all the resources have been equitably allocated between minority/non-minority groups subject to the respective constraint(s). Lastly, in Fig. 6(d) we observe that the PoF drops quickly to 1 in most cases and a decent trade-off between fairness and optimality can be achieved when \( q = 0.4 \).
A PROOFS

The full version of the paper contains the complete proofs on all of the arguments presented in this version.

Proof of Theorem 1. To prove Theorem 1 we first observe that

\[(1 - \beta)T_S p \leq I_S^T (I - A^T)p \leq I_S^T (c - x) + (L \otimes z)^T 1_S. \tag{3}\]

Then we, let \(\vec{z}^* = (\vec{p}, \vec{2}^*)\) be an optimal solution to the relaxation problem, and let \(j\) be a default node under this solution. Then \(j\) satisfies the following equation (due to absolute priority),

\[\vec{p}_j^* = \sum_{i \neq j} a_{ij}\vec{p}_i^* + \epsilon_j - x_j + L_j \vec{2}^*_j \geq \epsilon_j - x_j + L_j \vec{2}^*_j, \tag{4}\]

(res. for \(\vec{p}\)) where we have used the fact that the optimal solution is feasible and that \(\vec{p}^* \geq 0\), and \(a_{ij} \geq 0\).

Let \(f(\vec{p}) = v^T \vec{p}\) for some \(v > 0\) (i.e. a strictly increasing linear function of \(\vec{p}\)). The rounded variables are sampled from \(\text{Be}(\vec{z}^*)\). So, for a fixed shock \(X = x\) under the randomness of \(z\) we have that \(\mathbb{E}[\text{SOL}_f(x)] = \sum_{z \in \{0, 1\}^n} \mathbb{P}(Z = z) \cdot \mathbb{E}[\text{SOL}_f(x)|Z = z]\). Conditioned on the event \((Z = \vec{z})\), we use the sets \(D_{\text{SOL}(\vec{z})}\) and \(R_{\text{SOL}(\vec{z})}\) to denote the sets of default and solvent nodes under the assignment \(z \in \{0, 1\}^n\). We therefore break the above sum as a partition over the solvent and default nodes: \(\mathbb{E}[\text{SOL}_f(x)|Z = \vec{z}] = \mathbb{E}\left[\sum_{j \in D_{\text{SOL}(\vec{z})}} v_j \vec{p}_j + \sum_{j \in R_{\text{SOL}(\vec{z})}} v_j \vec{p}_j | Z = \vec{z}\right].\)

Firstly, every solvent node \(j\) satisfies \(\vec{p}_j = p_j \geq \vec{p}_j^* \geq \frac{1 - \beta_{\max}}{\epsilon} \vec{p}_j^*\). Summing over the set of solvent nodes we get

\[\mathbb{E}\left[\sum_{j \in D_{\text{SOL}(\vec{z})}} v_j \vec{p}_j | Z = \vec{z}\right] \geq \frac{1 - \beta_{\max}}{\epsilon} \mathbb{E}\left[\sum_{j \in D_{\text{SOL}(\vec{z})}} v_j \vec{p}_j | Z = \vec{z}\right].\]

Secondly, for the default nodes we have from (4) (3) that

\[\mathbb{E}\left[\sum_{j \in D_{\text{SOL}(\vec{z})}} v_j (\epsilon_j - x_j + L_j \vec{2}^*_j) | Z = \vec{z}\right] \geq \mathbb{E}\left[\sum_{j \in \text{OPT}_f(x)} v_j \vec{p}_j | Z = \vec{z}\right].\]

\[\mathbb{E}\left[\sum_{j \in D_{\text{SOL}(\vec{z})}} v_j \vec{p}_j | Z = \vec{z}\right] \geq \mathbb{E}\left[\sum_{j \in D_{\text{SOL}(\vec{z})}} v_j (\epsilon_j - x_j + L_j \vec{2}^*_j) | Z = \vec{z}\right] \geq \mathbb{E}\left[\sum_{j \in \text{OPT}_f(x)} v_j \vec{p}_j | Z = \vec{z}\right].\]

where \(\zeta = v_{\max}/v_{\min}\) and \(v_{\max} \geq v_j\). Thus, since the LP bound is an upper bound to the optimal we get \(\mathbb{E}[\text{SOL}_f(x)] \geq \left(\frac{1 - \beta_{\max}}{\epsilon}\right) \cdot \mathbb{E}[\text{OPT}_f(x)].\)

Remark. The above proof can work with any rounding scheme for which \(\mathbb{E}[Z_i] \geq \vec{z}^*_i\) (see full paper).

Proof of Lemma 2. We first fix a shock \(X = x\). C1 helps us establish a lower bound on the marginal gain. We apply Lemma 1 with asset vectors \(c_1 = c - x + L \otimes 1_{S \cup \{u_i\}}\) and \(c_2 = c - x + L \otimes 1_{S}\). Therefore, under C1, the marginal gain at each iteration \(i \leq k\) where node \(u_i\) is chosen is at least \(v_{\min} L_{u_i}\). Next, we prove the following regarding bailing out two sets \(S, T\) with \(S \subseteq T\):

\[\|\vec{p}_T - \vec{p}_S\| = \|p \land (A^T \vec{p}_T + c - x + L \otimes 1_T) - p \land (A^T \vec{p}_S + c - x + L \otimes 1_S)\|_{l_1} \leq \|A^T \vec{p}_T + c - x + L \otimes 1_T - A^T \vec{p}_S - c - x + L \otimes 1_S\|_{l_1} \leq \|A^T\|_{l_1} \|\vec{p}_T - \vec{p}_S\|_{l_1} + 1 + L_1^T 1_{T \setminus S}\]

where we have used Lemma 1, the fixed point op. the non-expansion of \(p \land (-)\), norm consistency, \(1_T - 1_S = 1_{T \setminus S}\) for \(S \subseteq T\), and that \(\|A^T\|_{l_1} = \beta_{\max} \in (0, 1)\). Rearranging we get \(\|\vec{p}_T - \vec{p}_S\| \leq \|A^T\|_{l_1} \|\vec{p}_T - \vec{p}_S\|_{l_1} + 1 + L_1^T 1_{T \setminus S}\)

Therefore for a linear objective \(v^T \vec{p}\) with \(v > 0\) since the EN model would yield the same clearing vector \(\vec{p}\) [19] we have

\[v^T (\vec{p}_T - \vec{p}_S) \leq \max_{i \leq k} v_T (\vec{p}_T - \vec{p}_S) \leq \frac{v_{\max}}{\max_{e \in S\setminus S^*}} \sum_{i \leq k} L_{u_i}\]

Now, let \(S\) be a set and let \(S^*\) be the optimal bailout set. Let \(S^* \setminus S = (j_1, \ldots, j_q)\) for \(q \leq k\) we have

\[v_T \vec{p}_{S^*} \leq v_T \vec{p}_{S\setminus S^*}\]

\[= v_T \vec{p}_S + v_{\max} \sum_{i=1}^{q} v_T (\vec{p}_{S \setminus \{j_1, \ldots, j_{i-1}\}} - \vec{p}_{S \setminus \{j_1, \ldots, j_i\}})\]

\[\leq v_T \vec{p}_S + \sum_{i=1}^{q} \frac{v_{\max} L_{j_i}}{1 - \beta_{\max}}\]

\[\leq v_T \vec{p}_S + \frac{v_{\max} L_{\max}}{1 - \beta_{\max}} \max_{e \in S} v_T (\vec{p}_{S \setminus \{e\}} - v_T \vec{p}_S)\]

where we have used: monotonicity wrt. to the bailout set, \(S^* \subseteq S \cup S^*\), (5), and that \(\zeta = v_{\max}/v_{\min}\). Rearranging we get that \(\max_{e \in S, e \text{ feasible}} \{v_T \vec{p}_{S \setminus \{e\}} - v_T \vec{p}_S\} \geq \rho (v_T \vec{p}_{S^*} - v_T \vec{p}_S)\) for \(\rho = (1 - \beta_{\max})/\zeta\). Finally, we let \(S^k\) be the solution at set \(t\) and let \(k\) be the total number of iterations of the algorithm. Then, \(v_T \vec{p}_{S^k} \geq \rho v_T \vec{p}_{S^*} + (1 - \rho) v_T \vec{p}_{S_{k-1}} \geq \cdots \geq (1 - \zeta)^{\max_{e \in S} v_T (\vec{p}_{S \setminus \{e\}} - v_T \vec{p}_S)} \cdot v_T \vec{p}_{S^*}\) Taking expectations over \(X \sim \mathcal{D}\) in the above expression we get the desired result.

Proof of Theorem 3. Discrete Bailouts. We start by proving that there exist instances where the discrete PoF is unbounded:

General Instance. Suppose that we have the star network \(S_n\) with node 1 being in the center and nodes 2 through \(n - 1\) being the peripheral nodes. Let \(b = 1, c = n \cdot 1_\{1\}\), and \(p_i = 1\) for \(i \neq 1\). The bailouts are \(L = n \cdot 1\) and the total budget is \(\Lambda = n\). The network is hit with a point-mass shock \(X = c\). We will (w.l.o.g.) focus on
the (SoP) objective [It can be shown for any linear objective given by a vector of coefficients \( u > 0 \) the same result holds]. The unbounded objective has value \( 2n - 1 \) since bailing out the central node restores the network to its initial state. Let \( g = 0 \). We have the following results:

(GC) Solution. Subject to absolute equality (i.e. \( g = 0 \)), it should hold that for the bailout indicator variables \( z_j \) that \( \sum_{j \in E} |z_j(g) - \bar{z}_j| = 0 \). Thus \( \bar{z}_j(g) = 0 \), therefore the objective value is 0. That yields a PoF of \( \frac{2n - 1}{2} = \infty \).

(Prop-GC) Solution. Let \( q = 1 \). Again, similarly to the previous case, we get \( \bar{z}_j(g, q) = 0 \) for all \( j \). Thus the PoF is \( \infty \). (Sp-GC) Solution. We again get that for the bailout indicator variables \( z_j \) that \( \sum_{j \in E} |z_j(g, q) - \bar{z}_j| = 0 \). Thus \( \bar{z}_j(g, q) = 0 \), therefore the objective value is 0. That yields a PoF of \( \frac{2n - 1}{2} = \infty \).

Fractional Bailouts. We will prove the correctness of the Theorem for the (GC) metric since the other metrics have exactly the same proof. Fix some \( g \geq 0 \). The fractional PoF is equal to \( \frac{\mathcal{F}(p)(1)}{\mathcal{F}(p)(g)} \). Assume, for contradiction, that the PoF is unbounded. The following can happen:

\[
f(\mathcal{F}(p)(g)) \neq 0 \quad \text{and} \quad f(\mathcal{F}(p)(1)) \text{ is such that the limit of their ratio goes to infinity. This is a contradiction since } f \text{ can get values up to } ||f||_{\infty}. \text{ Thus, we arrive at a contradiction.}
\]

\[
f(\mathcal{F}(p)(g)) = 0. \text{ Since } f \text{ is strictly increasing with } f(0) = 0 \text{ then } \mathcal{F}(p)(g) = 0. \text{ We will show that } \mathcal{F}(p)(g) = 0 \text{ if and only if } x = c \text{ w.p. 1 and } \mathcal{F}(X) = 0. \text{ We remind here that there are no isolated nodes (to the internal or the external sector) and hence } p > 0. \text{ The } (\Rightarrow) \text{ direction is trivial since the fairness constraint is always satisfied (hence it does not affect the feasible region) and the fixed point operator is simply } \mathcal{F}(p)(g) = f \wedge (A^T p(g)), \text{ so } \mathcal{F}(0) = p \wedge 0 = 0, i.e. \mathcal{F}(p)(g) = 0. \text{ For the } (\Leftarrow) \text{ direction, since } p > 0 \text{ the only way for } \mathcal{F}(p)(g) \text{ to be 0 is } (i) \text{ } g = \Lambda = 0 \text{ which is impossible since by the definition of the problem } \Lambda > 0, \text{ and (ii) the relative liability non-homogeneous part is zero, } c - X + L \circ \mathcal{F}(g) = 0 \text{ (the fairness constraint is trivially satisfied). Since } c - X \geq 0, L > 0 \text{ and } \mathcal{F}(g) \geq 0, \text{ the only way for the equation to hold is } X = c \text{ with probability } 1 \text{ and } \mathcal{F}(g) = 0, \text{ yielding a contradiction. Therefore, } f(\mathcal{F}(p)(g)) > 0.
\]

\[B \text{ SAFEGRAPH DATA ADDENDUM}\]

We give the detailed steps we used and assumptions we relied on to build the SafeGraph data. We form a bipartite graph:

POI nodes. We construct the POI nodes by setting a location on Earth and looking at the kNN-nearby CBGs that contain POIs according to the Haversine distance. We choose the location Ithaca Commons (NY, US) location, and \( k_{\text{NN}} = 3 \).

CBG nodes. We use the Monthly Patterns data for the period between April and May 2020 to determine the CBGs that interact with the POIs. For each POI, we use the column visitor_home_cbg to list the CBGs and the total number of visitors from each CBG to each POI. These visitors correspond to the number of unique devices that interact with the POI. Moreover, we assume that each distinct device represents a different person. For each CBG node we use data from the US Census we also log demographic characteristics such as: (i) race, (ii) average income, (iii) size of households, (iv) unemployment rates. Using these data for each CBG we calculate: (i) the probability of belonging to a racial minority group (i.e. non-White), (ii) the average income, (iii) the average size of a household on this CBG, (iv) the probability of someone in the CBG being unemployed. The above network consists of 152 nodes (for our choice of parameters).

We use the bucketed dwell times to determine the % of people that are working in a business. More specifically, we consider a worker a device that has spent more than 240 min in the POI and we classify the person as a non-worker otherwise. Workers come from CBGs and are paid wages whereas non-workers come from CBGs and have expenses on the corresponding POI. For every CBG \( j \) let \( p_{\text{worker}} \) be this %.

For every CBG \( i \) that interacts with a POI \( j \) with \( n_{ij} \) people we add an edge \((j, i)\) referring to \( \Pi_{\text{worker}} = \left\lfloor n_{ij} \times p_{\text{worker}} \right\rfloor \) workers, and an edge \((i, j)\) referring to \( n_{ij} - \left\lfloor n_{ij} \times p_{\text{worker}} \right\rfloor \) non-worker nodes. We determine the weights of these edges (per unit) as follows: for the \((j, i)\) edge we take wage data by the US Ec. Census and NAICS and create a liability by the business equal to the monthly wage of an employee which we multiply by \( n_{ij} \). For the \((i, j)\) edge we use data from the Consumer Expenditure Survey conducted by the US Economic Census and add a liability regarding \( n_{ij} - \left\lfloor n_{ij} \times p_{\text{worker}} \right\rfloor \) equal to the average monthly expenditures due to the specific NAICS code of the business.

The number of workers for each POI \( j \) is \( n_{ij} \) workers (and the number of non-workers is defined respectively). Similarly we count the number of workers and number of non-workers for each CBG \( i \). We estimate the total number of households in the CBG that are related to interaction with the corresponding POIs We estimate the number of households as \( n_{i} \) households

\[
\text{avg. size of household at CBG } i.
\]

For each CBG \( i \) we use the average income of the CBG and the average size of the household to calculate the value of a bailout as it would be imposed by the CARES act (see [1]). The bailout is multiplied by the estimated number of households calculated previously.

For each POI \( j \) we have data from loans that were given on multiple businesses as parts of the COVID-19 PPP payments provided by SafeGraph. The loan value is normalized to span a month and to the true number of jobs reported in each business and then is multiplied by the number of workers at POI \( j \) in the network. We use the business owner racial attribute to determine minority/non-minority businesses. For each POI \( j \) we use the total assets earned annually from NAICS and normalize accordingly. From this amount we subtract the revenue due to nodes within the network (i.e. in-bound edges). The external liabilities of the POI are determined in a similar way. To ensure that \( A2 \) holds throughout the experiments we assert a negligible base amount for the external liabilities. For each CBG \( j \) the process is similar where for external assets US Census income data, as in POIs, but now normalized wrt. the est. number of households and the inbound connections from the POIs. We set the avg. expenses to \( \approx $63,000 \).