On stable and unstable behaviour of certain rotation segments

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Abstract

In this paper, we study non-wandering homeomorphisms of the two-dimensional torus homotopic to the identity, whose rotation sets are non-trivial segments from $(0,0)$ to some totally irrational point $(\alpha, \beta)$. We show that for any $r \geq 1$, this rotation set only appears for $C^r$ diffeomorphisms satisfying some degenerate conditions. And when such a rotation set does appear, assuming several natural conditions that are generically satisfied in the area-preserving world, we give a clearer description of its rotational behaviour. More precisely, the dynamics admits bounded deviation along the direction $-(\alpha, \beta)$ in the lift, and the rotation set is locked inside an arbitrarily small cone with respect to small $C^0$-perturbations of the dynamics. On the other hand, for any non-wandering homeomorphism $f$ with this kind of rotation set, we also present a perturbation scheme in order for the rotation set to be eaten by the rotation set of some nearby dynamics, in the sense that the later set has non-empty interior and contains the former one. These two flavours interplay and share the common goal of understanding the stability/instability properties of this kind of rotation set.

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(Some figures may appear in colour only in the online journal)
1. Introduction

The notion of a rotation number was introduced by Poincaré in order to gather information on the average ‘rotational’ linear speed of a dynamical system. Rotation theory is well understood only for circle homeomorphisms or endomorphisms, and it is still a great source of problems in two dimensional manifolds (annulus, torus, higher genus surfaces, and so on). Focussing only on the torus, there are two most important and related topics, namely, the shape of the rotation set and how it changes depending on the dynamics. In this paper, we will work on both topics, based on one specific type of rotation set. On the one hand, we try to understand the variation of a rotation set depending on the homeomorphism, under different regularities. On the other hand, we study how a certain shape of rotation set restricts the dynamics.

In general, with respect to the Hausdorff topology, the rotation set varies upper-semicontinuously. It is also known that if the rotation set of some $f$ has non-empty interior, then it is in fact continuous at $f$ (see [31, 32]). Moreover, for a $C^0$-open and dense subset of homeomorphisms, the rotation set is stable (i.e., it remains unchanged under small perturbations, see [16, 35]). Note that this is true both in the set of all homeomorphisms, and in the set of area-preserving ones.

Below, in order to state our main results, we will use some notations that are mostly standard, and postpone their precise definitions to section 2.
Our objective is to look at an interesting situation, where the rotation set is a segment connecting \((0,0)\) to some totally irrational point \((\alpha, \beta)\). We want to understand how it can be changed under sufficiently small \(C^0\) perturbations, and what properties should such a homeomorphism satisfy. An example with such a rotation set was first described in [17] by Handel, who attributed it to Katok. It is produced by a slowing-down procedure from a constant speed irrational flow. A smooth area-preserving example was obtained by Addas-Zanata and Tal in [5]. See also the paper [26] by Kwapisz and Mathison which shows some special ergodic properties for an explicit example.

In the \(C^0\) category, the more precise task for us is to study when and how the rotation set of a dynamical system is ready to grow. One of the first results on this subject appeared in [1], where it was proved that if the rotation set \(\rho(\tilde{f})\) contains a non-rational vector \((\alpha, \beta)\) as an extremal point, then for any supporting line \(r\) (i.e., a straight line containing \((\alpha, \beta)\), such that the rotation set avoids one connected component of its complement, denoted \(O\)), by certain arbitrarily small \(C^0\)-perturbations, the new rotation set will intersect \(O\) (see also [15] for a \(C^1\) version of this theorem). In this paper, we are able to go one step further in this direction.

**Theorem 1.1.** Let \(\tilde{f} \in \tilde{\text{Homeo}}_{0,nw}(\mathbb{T}^2)\), whose rotation set \(\rho(\tilde{f})\) is a segment from \((0,0)\) to some totally irrational point \((\alpha, \beta)\). For any \(\epsilon > 0\), there exists \(\tilde{g} \in \tilde{\text{Homeo}}_{0}(\mathbb{T}^2)\) with \(\text{dist}_{C^0}(\tilde{g}, \tilde{f}) < \epsilon\), such that, \(\rho(\tilde{g})\) has non-empty interior, and \(\rho(\tilde{f}) \setminus \{(0,0)\} \subset \text{Int}(\rho(\tilde{g}))\).

**Remarks.**

(a) If the map \(\tilde{f}\) from the above theorem preserves area, then the perturbed map \(\tilde{g}\) can also be chosen in the area-preserving world. This is a well-known fact (follows from lemma 3.3), but it deserves to be mentioned, as it is one of the only cases known to us on how to perturb a non-wandering homeomorphism and remain non-wandering.

(b) Another comment about the above theorem is related to what happens around \((0,0)\): if ‘not too degenerate’ maps \(f\) are considered, that is, if \(f \in K\) (see definition 2.22), then \((0,0)\) cannot become an interior point for the perturbed map by proposition 2.24.

It is interesting to ask if the same statement is also true in \(C^1\) topology. If we go on to consider \(C^r\) diffeomorphisms, \(r \geq 1\), clearer descriptions should be expected. In particular, we proved the following result, which suggests the non-genericity of the set of non-wandering diffeomorphisms with this special kind of rotation set. We say ‘suggest’, because it is not known how to perturb a non-wandering diffeomorphism and remain non-wandering (unless of course, in some particular cases, like area-preserving maps).

**Theorem 1.2.** Let \(\tilde{f} \in \tilde{\text{Diff}}_{0,nw}(\mathbb{T}^2)\) for any \(r \geq 1\). Assume that for every integer \(n > 0\), the linear part \(Df^n\) computed at each \(n\)-periodic point does not have 1 as an eigenvalue and there are no saddle-connections. Then, the rotation set \(\rho(\tilde{f})\) is not a segment from \((0,0)\) to some totally irrational point \((\alpha, \beta)\).

Note that the above conditions are satisfied by generic area-preserving \(C^r\) diffeomorphisms, for all \(r \geq 1\). Naturally, the next task is the following. How can we understand a typical non-wandering diffeomorphism \(f\) which does admit such a rotation set? This seems to be hard, because very little is known on the set of non-wandering homeomorphisms or diffeomorphisms of a surface. As we said above, unlike in the area-preserving case, there does not exist a method available to make a perturbation within the set of non-wandering homeomorphisms.

Nevertheless, in the area-preserving setting, if one wants to obtain more information on the non-generic diffeomorphisms, traditionally, one works with generic families. This inspires us to formulate nice conditions, which hold true in a broader set of diffeomorphisms. This
approach eventually helps us to detect properties, which general non-wandering homeomorphisms might satisfy. Along this direction, we obtain the following two results.

**Theorem 1.3.** Suppose, for any integer \( r \geq 1 \), \( f \in \text{Diff}^1_0(nw)(\mathbb{T}^2) \) satisfies certain natural conditions, namely \( f \in K' \) (see definition 2.22). Suppose also that \( \rho(\tilde{f}) \) for some lift \( \tilde{f} \) is a segment from \((0,0)\) to a totally irrational point \((\alpha,\beta)\). Then, \( f \) has (finitely many) fixed points (and no periodic point which is not fixed), all with 0 topological index, and the local dynamics around them is obtained by gluing exactly two hyperbolic sectors. The stable branch of any of these fixed points does not intersect the unstable branch of any other point. Moreover, in the plane, for each fixed point, its unstable and stable branches are bounded in the direction orthogonal to \((\alpha,\beta)\) and the unstable (resp. stable) branch goes to infinity following the vector \((\alpha,\beta)\) (resp. \(-{(\alpha,\beta)}\)). Finally, any stable or unstable branch of a fixed point is dense in the torus.

**Theorem 1.4.** Under the same hypotheses of the above theorem, for any non-zero integer vector \((a,b)\), \(a\) and \(b\) coprimes, there exists a simple closed curve \( \gamma \) in \( \mathbb{T}^2 \), with homological direction \((a,b)\), such that any connected component of the lift of \( \theta \) to the plane is a Brouwer line for \( f \). Moreover, for any straight line \( \gamma \) containing \((0,0)\) and avoiding \((\alpha,\beta)\), there exists \( \varepsilon > 0 \), such that, for any \( \tilde{g} \in \text{Homeo}_0(\mathbb{T}^2) \), with \( d_C(\tilde{f}, \tilde{g}) < \varepsilon \), \( \rho(\tilde{g}) \) is contained in the union of \( \gamma \) and one of the connected components of its complement, the one which contains \( \rho(\tilde{f}) \setminus \{(0,0)\} \).

The next corollary is a direct consequence of theorem 1.4. Nevertheless, we will actually prove the corollary before the theorem (see lemma 6.2).

**Corollary 1.5.** Let \( f \) satisfy the conditions in theorem 1.4. Then it has bounded deviation along the direction \(-{(\alpha,\beta)}\).

We also consider the unbounded deviation along the direction \((\alpha,\beta)\). The next theorem requires one more condition, the existence of an invariant foliation, which is clearly satisfied in the particular case when \( f \) is the time-one map of some flow. See its more precise statement in section 7.

**Theorem 1.6.** Consider any \( \tilde{f} \) as in theorem 1.4. If \( f \) preserves a \( C^0 \) foliation of \( \mathbb{T}^2 \), then \( \tilde{f} \) has unbounded deviation along \((\alpha,\beta)\).

This last result leads to the following interesting question.

**Question 1.** For \( \tilde{f} \) which lifts \( f \in \text{Homeo}_0(\mathbb{T}^2) \), suppose \( \rho(\tilde{f}) \) is the line segment from \((0,0)\) to a totally irrational \((\alpha,\beta)\). Is it true that \( f \) has unbounded deviation along the direction \((\alpha,\beta)\)?

To conclude, let us briefly describe the organization of this paper and the main ideas used in the proofs.

In section 2, we will summarise some notations and previous results that will be used along the text.

In section 3, we introduce a perturbation technique, which is very useful under the condition of unbounded deviations along some direction. The purpose is to find \( \varepsilon \)-pseudo periodic orbits, which can be ‘closed’ in order to become periodic orbits, so with rational rotation vector. The difficulty is that, a priori, the original method in [1] does not give enough information to locate the position of these rational numbers, except that they are outside the rotation set \( \rho(f) \).

In section 4, we focus on the case when the map has bounded deviations. We apply a result proved by Jäger (see [18]) in order to obtain a semi-conjugacy between the restriction of the dynamics to a certain minimal set and the rigid torus rotation. Then, we prove that whenever
one can perturb the rigid rotation, we can also perturb the original homeomorphism. Combining both results from sections 3 and 4, we complete the proof of theorem 1.1.

From section 5, we start working with diffeomorphisms. There, we prove theorem 1.2. There are three main ingredients in this proof. The first one belongs to the theory of prime ends rotation numbers. The second one is the so-called bounded disk lemma, firstly proved by Koropecki and Tal in [25]. The third one consists of certain properties of invariant branches at hyperbolic periodic saddles, mostly from Fernando Oliveira’s paper [34].

In section 6, we work with diffeomorphisms satisfying certain conditions, which in the area-preserving case, are generic in the complement of the set of maps which satisfy the hypotheses of theorem 1.2. See for instance [10, 11]. First, we collect several results describing dynamical properties of maps which satisfy the conditions in definition 2.22. These results together imply theorem 1.3 and are also an important part in the proof of existence of the Brouwer lines (theorem 1.4).

In section 7, we continue to study diffeomorphisms satisfying the conditions from section 6 and prove theorem 1.6.

**Notational remark.** There will be a small abuse of notation among the text below. For example, when we introduce integers, positive constants along the arguments, choices will be made differently in different subsections, sometimes with the same name. However, they will be consistent within one single subsection.

## 2. Preliminaries and previous results

The main purpose of this section is to fix notations and to recall some previous results for later use. In some cases, the formulation contains some minor variations from the reference, and we will present short proofs only stressing the differences. We will also show some elementary lemmas as well.

Note that some of the notations were already used in the statements of the theorems in the introduction.

### 2.1. Planar topology and dynamics

For any planar subset $M$, denote by $\text{Int}(M)$ the interior of $M$, and by $\partial M$ the boundary of $M$. The following property in planar topology will be used. We say a planar set $F$ separates the points $x$ and $y$ if they are in different connected components of $F^c$.

**Lemma 2.1** *(theorem 14.3 in chapter V of [33]).* Let $F$ be a closed subset of the plane $\mathbb{R}^2$, separating two points $x$ and $y$. Then some connected component of $F$ also separates $x$ and $y$.

Let $M$ denote a metric space, and consider a homeomorphism $f : M \to M$. For any starting point $x_0$, we often use subscript to denote the $f$-iterates of $x_0$, i.e., $x_n = f^n(x_0)$. For $\varepsilon > 0$, we call an $\varepsilon$-pseudo periodic orbit (with period $n$), for a finite sequence of points $\{x^{(0)}, x^{(1)}, \ldots, x^{(n-1)}\}$, with the following properties.

\[
\text{dist}(f(x^{(i)}), x^{(i+1)}) < \varepsilon, \quad \text{for any } i = 0, \ldots, n-2, \text{ and}, \quad \text{dist}(f(x^{(n-1)}), x^{(0)}) < \varepsilon. \quad (2.1)
\]

Moreover, any point in an $\varepsilon$-pseudo periodic orbit as above, will be referred as an $\varepsilon$-pseudo periodic point.
We say a point $p$ is $f$-recurrent, if there exists $n_k \to \infty$, such that $f^{n_k}(p) \to p$. The following is a standard fact for all non-wandering dynamical systems on compact metric spaces $M$. We provide a proof for completeness.

**Lemma 2.2.** Suppose $f : M \to M$ is non-wandering. Then, the set of $f$-recurrent points, denoted by $\mathcal{R}(f) \subset M$, is dense.

**Proof.** Pick any open disk $B = B_0$. It suffices to show $B \cap \mathcal{R}(f) \neq \emptyset$. Since $f$ is non-wandering, there is some $n_1$ such that $f^{-n_1}(B_0) \cap B_0 \neq \emptyset$. Then we can choose some small closed disk $B_1$, with radius smaller than 1, such that $B_1 \subset f^{-n_1}(B_0) \cap B_0$. Note that every point in $B_1$ will return to $B_0$ at iterate $n_1$.

Inductively, suppose we have found increasing integers $n_1 < \cdots < n_k$, and closed disks $\{B_i\}_{i=1}^k$, such that for all $i = 1, \ldots, k$, $B_i$ has diameter smaller than $\frac{1}{i}$, such that $B_i \subset \text{Int}(B_{i-1} \cap f^{-n_i}(B_{i-1}))$. Then, we can choose some $n_{k+1} > n_k$ such that $B_k \cap f^{-n_{k+1}}(B_k) \neq \emptyset$, and some closed disk $B_{k+1}$ with radius smaller than $\frac{1}{k+1}$, such that $B_{k+1} \subset \text{Int}(B_k \cap f^{-n_{k+1}}(B_k))$.

Now for all $k \geq 1$, $B_k$ consists of points that will return to $B_{k-1}$ at time $n_k$. Now $\bigcap_{k \geq 1} B_k$ is a singleton, say, $\{x^*\}$. It follows that $f^n(x^*) \to x^*$.

Denote by $\mathbb{T}^2$ the two-dimensional ‘flat’ torus, whose universal covering space is $\mathbb{R}^2$, and let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the natural projection. Let $\text{Homeo}_0(\mathbb{T}^2)$ denote the set of homeomorphisms of $\mathbb{T}^2$ homotopic to the identity. Then, denote by $\text{Homeo}_{0,\text{nonw}}(\mathbb{T}^2)$ and $\text{Homeo}_{0,\text{Leb}}(\mathbb{T}^2)$ the set of non-wandering and area-preserving homeomorphisms, respectively. Note that both are subsets of $\text{Homeo}_0(\mathbb{T}^2)$. We also denote by $\text{Homeo}_0(\mathbb{T}^2)$ (respectively, $\text{Homeo}_{0,\text{nonw}}(\mathbb{T}^2)$) the set of lifts of homeomorphisms from $\text{Homeo}_0(\mathbb{T}^2)$ (respectively, $\text{Homeo}_{0,\text{nonw}}(\mathbb{T}^2)$) to the plane. Similarly, for $r \geq 1$, or $r = \infty$, denote by $\text{Diff}_{0,\text{nonw}}(\mathbb{T}^2)$ (respectively $\text{Diff}_{0,\text{Leb}}(\mathbb{T}^2)$) the set of $C^r$ diffeomorphisms of $\mathbb{T}^2$, which are non-wandering (respectively, area-preserving) and homotopic to the identity. Also, the sets of their lifts are denoted by $\text{Diff}_{0,\text{nonw}}(\mathbb{T}^2)$ and $\text{Diff}_{0,\text{Leb}}(\mathbb{T}^2)$, respectively.

We say a subset $K \subset \mathbb{T}^2$ is essential if there exists a non-trivial homotopy class, such that for any representative loop $\beta$ of it, $\beta \cap K \neq \emptyset$. Otherwise, it is called inessential. In other words, a subset $K$ is inessential if and only if it is contained in an open topological disk in $\mathbb{T}^2$. In this case, its complement is called fully essential. We will need the following result. For more details about the above notations, see [23, 25].

**Lemma 2.3 (Theorem 6 in [23]).** Let $f \in \text{Homeo}_{0,\text{nonw}}(\mathbb{T}^2)$ and suppose that the set of fixed points is inessential. Then there exists $M > 0$, such that, for any $f$-invariant topological open disk $D$, each connected component of $\pi^{-1}(D)$ has diameter bounded from above by $M$.

2.2. Misiurewicz–Ziemian rotation set

Consider $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$. The foundations of the rotation theory in the torus were mainly developed by Misiurewicz and Ziemian in [31]. There, the following notion of a rotation set $\rho(\tilde{f})$ appears:

$$\rho(\tilde{f}) := \left\{ v = \lim_{k \to \infty} \frac{1}{n_k} \left( \tilde{f}^{n_k}(\tilde{z}_k) - \tilde{z}_k \right), \quad n_k \to \infty, \tilde{z}_k \in \mathbb{R}^2, \text{ whenever the limit exists} \right\}. \quad (2.3)$$

**Remark 2.4.** For any $f \in \text{Homeo}_0(\mathbb{T}^2)$, distinct choices of the lifts $\tilde{f}$ give distincts $\rho(\tilde{f})$. However, with distinct choices, these rotation sets only differ from each other by integer translations. Moreover, for any $f$-invariant compact subset $K \subset \mathbb{T}^2$, let $\rho(\tilde{f}, K)$ denote the rotation
set of $\tilde{f}$ restricted to the lift of $K$. The definition is similar to that in (2.3). The only difference is that the points $\tilde{z}_k$ in the expression are only allowed to be chosen in $\pi^{-1}(K)$.

One can also define the point-wise rotation vector as follows. For any $z \in \mathbb{T}^2$,

$$\rho(\tilde{f}, z) := \lim_{n \to \infty} \frac{1}{n} (\tilde{f}^n(z) - z),$$

when the limit exists. \hfill (2.4)

Another important definition is as follows. Consider any $f$-invariant Borel probability measure $\mu$, and denote by $\rho(\tilde{f}) = \int_{\mathbb{T}^2} (\tilde{f}(x) - x) d\mu(x)$ the average rotation vector of the measure $\mu$ (note that the integrand in this expression is in fact a function on $\mathbb{T}^2$). Define

$$\rho_{\text{meas}}(\tilde{f}) := \{ \rho(\tilde{f}) | \mu \text{ is a } f\text{-invariant Borel probability measure} \}.$$ \hfill (2.5)

The following result gathers many important properties of these notions:

**Theorem 2.5 (see [31, 32]).** For any $\tilde{f} \in \tilde{\text{Homeo}}_0(\mathbb{T}^2)$, $\rho_{\text{meas}}(\tilde{f})$ equals $\rho(\tilde{f})$, which is a compact and convex subset of $\mathbb{R}^2$. Moreover, every extremal point of the rotation set can be realized as the average rotation vector of some ergodic measure $\mu$.

### 2.3. Bounded deviations

For any non-trivial vector $w \in \mathbb{R}^2$, denote by $\text{pr}_w$ the projection of a vector along the $w$ direction

$$\text{pr}_w : \mathbb{R}^2 \to \mathbb{R}, r \mapsto \langle r, w \rangle.$$ \hfill (2.6)

Next, we introduce the important notion of bounded deviations.

**Definition 2.6.** Fix a non-trivial vector $w \in \mathbb{R}^2$. We say that $\tilde{f}$ has bounded deviation along direction $w$ (from its rotation set $\rho(\tilde{f})$), if there exists $M > 0$, such that for any $n \geq 0$ and any $\tilde{x} \in \mathbb{R}^2$,

$$\text{pr}_w (\tilde{f}^n(\tilde{x}) - \tilde{x} - n\rho(\tilde{f})) \leq M.$$

\hfill (2.7)

**Remark 2.7.** Note that with respect to this definition, having bounded deviation along $w$ and $-w$ are two different conditions.

The next lemma will be used several times later. We omit its proof because it almost follows directly from the definitions. By invariant line we mean an $f$-invariant simple closed curve in $\mathbb{T}^2$, non-homotopically trivial. And an invariant strip $\tilde{A}$ is an open connected $f$-invariant subset of the plane that satisfies $\tilde{A} = \tilde{A} + (a, b)$ for some non-zero integer vector $(a, b)$. Also, $\tilde{A}$ is bounded in the direction orthogonal to $(a, b)$.

**Lemma 2.8.** Suppose $\tilde{f}$ admits bounded deviation along both directions $w$ and $-w$. Then $\rho(\tilde{f})$ is contained in the straight line through the origin, whose direction is perpendicular to $w$. In particular, the conclusion of the lemma holds when there exists an invariant line or an invariant strip whose direction is perpendicular to $w$.

The following statement essentially follows from the Gottschalk–Hedlund theorem. See also [18] for a somewhat more elementary proof.

**Lemma 2.9 (proposition A in [18]).** Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ preserve a minimal set $K \subset \mathbb{T}^2$. Suppose $\rho(f, K) = \{(\alpha, \beta)\}$ where $(\alpha, \beta)$ is totally irrational, and $\tilde{f}|_{\pi^{-1}(K)}$ has
bounded deviation along every direction. Then, there exists a continuous surjective map \( \phi : K \to \mathbb{T}^2 \), homotopic to the inclusion, satisfying that

\[
\phi \circ f|_K = R_{(\alpha, \beta)} \circ \phi,
\]

where \( R_{(\alpha, \beta)} \) is the rigid rotation on \( \mathbb{T}^2 \). Moreover, we can lift \( \phi \) to a semi-conjugacy \( \tilde{\phi} : \pi^{-1}(K) \to \mathbb{R}^2 \), for which every fibre has uniformly bounded diameter.

The following result establishes the bounded deviation property in the perpendicular direction when the rotation set is the special one we are interested in.

**Theorem 2.10 (main result of [38]).** Suppose \( \tilde{f} \in \text{Homeo}_0(\mathbb{T}^2) \), and \( \rho(\tilde{f}) \) is the segment from \((0,0)\) to a totally irrational point \((\alpha, \beta)\). Then \( \tilde{f} \) has bounded deviation along the perpendicular directions \((-\beta, \alpha)\) and \((\beta, -\alpha)\).

### 2.4. Some fundamental tools in topological dynamics

Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) denote an orientation-preserving homeomorphism. A properly embedded oriented line \( \gamma : \mathbb{R} \to \mathbb{R}^2 \) is called a Brouwer line, if \( f(\gamma(\mathbb{R})) \) and \( f^{-1}(\gamma(\mathbb{R})) \) belong to different connected components of the complement of \( \gamma(\mathbb{R}) \). We also abuse notation by writing \( \gamma = \gamma(\mathbb{R}) \). We call these components the right of \( \gamma \) and the left of \( \gamma \), and denote them by \( R(\gamma) \) and \( L(\gamma) \), respectively.

The following result is usually attributed to Brouwer (see also [8]), we refer to [12, proposition 1.3] for a very useful generalization. Here we state a weaker version, which is sufficient for our use.

**Lemma 2.11.** Suppose \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is an orientation-preserving homeomorphism. If there exists a topological open disk \( U \), such that \( f(U) \cap U = \emptyset \), and for some \( k \geq 2 \), \( f^k(U) \cap U \neq \emptyset \), then, if the fixed points are isolated, \( f \) admits a fixed point with positive topological index.

**Definition 2.12.** Let \( f : (U, p) \to (V, p) \) denote a local homeomorphism, where \( U \) and \( V \) are two open neighbourhoods of an isolated fixed point \( p \in \mathbb{R}^2 \). Choose a small disk \( D \subset U \), whose boundary is \( \beta = \partial D \). Define \( g : \beta \to S^1 \), such that \( g(x) = \frac{x - f(x)}{\|x - f(x)\|} \). The topological index of \( f \) at the point \( p \) is defined as the degree of the map \( g \), denoted as \( \text{Index}_f(p) \).

The following is a consequence of the classical result usually referred to as Lefschetz fixed-point formula. (See for example theorem 8.6.2 in [20].)

**Lemma 2.13.** Let \( f \in \text{Homeo}_0(\mathbb{T}^2) \) and assume all fixed points are isolated. Then

\[
\sum_{p \in \text{Fix}(f)} \text{Index}_f(p) = 0.
\]

### 2.5. Generic conditions

Recall \( \text{Diff}^r_{0, \text{tw}}(\mathbb{T}^2) \) is the set of non-wandering \( C^r \) diffeomorphisms of \( \mathbb{T}^2 \), which are homotopic to the identity, and \( \text{Diff}^r_{0, \text{Leb}}(\mathbb{T}^2) \subset \text{Diff}^r_{0, \text{tw}}(\mathbb{T}^2) \) is the subset of area-preserving ones. Below, we say \( p \) is a periodic saddle-like point for \( f \) if there is \( n > 0 \), such that \( p \) is a \( f^n \)-fixed point, and with respect to \( f^n \), the dynamics near \( p \) is obtained by gluing a finite number of topological saddle-sectors, see [11]. As usual, we denote by \( W^u(p) \) (respectively, \( W^s(p) \)) the union of \( p \) and all the unstable branches at \( p \) (respectively, the stable branches at \( p \)).
Lemma 2.18. For any $r \geqslant 1$ or $r = \infty$, define $G' \subset \text{Diff}_0^{r, \text{aw}}(T^2)$ to be the subset of diffeomorphisms $f$ satisfying the following two conditions:

(a) If $p \in T^2$ is an $n$-periodic point, then $Df^n(p)$ does not have 1 as an eigenvalue.
(b) $f$ does not have saddle connections.

Remark 2.15. By theorems 3 (c) and 9 of [36], for all $r \geqslant 1$, the set $G' \cap \text{Diff}_0^{r, \text{Lab}}(T^2)$ contains a residual subset of $\text{Diff}_0^{r, \text{Lab}}(T^2)$. Thus the above conditions are generic for area-preserving diffeomorphisms.

Definition 2.14. For any $r \geqslant 1$ or $r = \infty$, define $G' \subset \text{Diff}_0^{r, \text{aw}}(T^2)$ to be the subset of diffeomorphisms $f$ satisfying the following two conditions:

(a) If $p \in T^2$ is an $n$-periodic point, then $Df^n(p)$ does not have 1 as an eigenvalue.
(b) $f$ does not have saddle connections.

Definition 2.15. Let $f : S \to S$ be a $C^1$ diffeomorphism on a closed surface $S$. Let $p, q$ be two periodic saddle-like points. We say that $W^r(p)$ and $W^s(q)$ intersect at a point $w$ in a topologically transverse way, if there exists an open disk $B = B(w, \delta)$ with some radius $\delta > 0$ such that (denote by $\alpha$, respectively $\beta$, the connected component of $W^r(p) \cap B$, respectively $W^s(q) \cap B$, both containing the point $w$):

(a) $B \setminus \alpha = B_1 \sqcup B_2$, which is a disjoint union.
(b) $\beta \setminus \{w\} = \beta_1 \sqcup \beta_2$, which is a disjoint union and $\beta_1 \subset B_1 \cup \alpha$, $\beta_2 \subset B_2 \cup \alpha$, with $\beta_1 \not\subset \alpha$ and $\beta_2 \not\subset \alpha$. In other words, $\beta_1$ intersects $B_1$ and $\beta_2$ intersects $B_2$.

Remark 2.16. Let $f : S \to S$ be a $C^1$ diffeomorphism on a closed surface $S$. Let $p, q$ be two periodic saddle-like points. We say that $W^r(p)$ and $W^s(q)$ intersect at a point $w$ in a topologically transverse way, if there exists an open disk $B = B(w, \delta)$ with some radius $\delta > 0$ such that (denote by $\alpha$, respectively $\beta$, the connected component of $W^r(p) \cap B$, respectively $W^s(q) \cap B$, both containing the point $w$):

(a) $B \setminus \alpha = B_1 \sqcup B_2$, which is a disjoint union.
(b) $\beta \setminus \{w\} = \beta_1 \sqcup \beta_2$, which is a disjoint union and $\beta_1 \subset B_1 \cup \alpha$, $\beta_2 \subset B_2 \cup \alpha$, with $\beta_1 \not\subset \alpha$ and $\beta_2 \not\subset \alpha$. In other words, $\beta_1$ intersects $B_1$ and $\beta_2$ intersects $B_2$.

Remark 2.17. The radius of the disk $\delta > 0$ can not be taken arbitrarily small in general, because the connected component of $\alpha \cap \beta$ containing $w$ could be a non-trivial arc.

The following lemma will be useful for obtaining non-contractible periodic orbits, i.e., those orbits with non-zero rotation vectors.

Lemma 2.18 (lemma 0 in [3]). Suppose $\widetilde{f} \in \text{Diff}^r(T^2)$ has a hyperbolic periodic saddle point $\widetilde{q}$, and suppose $W^r(\widetilde{q})$ and $W^s(\widetilde{q}) + (a, b)$ intersect in a topologically transverse way, for some integer vector $(a, b)$. Then there exists some integer $N > 0$ such that the diffeomorphism $\tilde{g} := \tilde{f}^N - (a, b)$ admits a fixed point $\tilde{p}$. Thus, $\rho(\tilde{f}, \pi(\tilde{p})) = (\frac{a}{N}, \frac{b}{N})$.

Remark 2.19. The argument that proves the above lemma can also be applied when the periodic point $\widetilde{q}$ has topological index 0, and admits exactly one stable and one unstable branch, whose local dynamics is described in figure 1.

2.6. A broader class of (non-generic) diffeomorphisms

The following defnition appears as definition 1.6 on page 12 of [11]. Such a study is based on the important work of Takens (see theorem 4.6 in [37]). Although all the theory in [11] was stated for $C^r$ maps, for the results we consider below, there is no substantial difference in the $C^s$ case.

Definition 2.20. Let $f : (U, p) \to (W, p)$ be a local planar $C^r$-diffeomorphism with an isolated fixed point $p$. Assume all eigenvalues of $Df(p)$ belong to the unit circle and let $S$ be the semi-simple part of $Df(p)$ in its Jordan normal form. Then up to a $C^s$-coordinate change, there exists a formal $C^s$ vector field $X$, invariant under $S$, such that, the $r$-jet of $f$ and the $r$-jet of $S \circ X_1$ coincide at $p$, where $X_1$ is the time-1 map of the formal flow generated by $X$. We say $f$ is of Lojasiewicz type at $p$, if the following condition holds:

- There exists an integer $k \leqslant r$ and constants $C, \delta > 0$, such that, for any $z$ satisfying $\|z - p\| \leqslant \delta$, then

$$
\|X(x)\| \geqslant C\|z - p\|^k.
$$

(2.10)
Figure 1. The local dynamics around a fixed point.

The following result was essentially obtained in section 2 of [4].

**Lemma 2.21.** Assume \( f \in \text{Diff}_{0,\text{nw}}^r(\mathbb{T}^2) \) has an isolated fixed point \( p \) and the topological index of \( f \) at \( p \) is zero. Also suppose that if both eigenvalues of \( Df \) at \( p \) lie in the unit circle, then \( f \) is of Lojasiewicz type at \( p \). Then, there exists exactly one stable and one unstable branch at \( p \). Moreover, the local dynamics can be precisely described, see figure 1.

**Proof.** Consider the linear transformation \( Df(p) \), which has positive determinant. If 1 is not an eigenvalue of \( Df(p) \), then \( p \) is either an hyperbolic fixed saddle point, an elliptic fixed point (that is, both eigenvalues are in the unit circle and not real), or \(-1\) is an eigenvalue. In all the above possibilities, \( p \) has topological index equal to \(-1\) or 1. As the index at \( p \) is zero, the above cases do not happen. If the two eigenvalues of \( Df(p) \) are 1 and some \( a > 0 \) with \( a \neq 1 \), then as an application of centre manifold theory (see [9]), we get that \( p \) can be a topological saddle, a topological sink (or source), or a saddle-node. Since by assumption \( p \) has topological index 0, it must be a saddle-node. In this case, \( p \) has two saddle sectors, and one sector which is either contracting or expanding, a contradiction with the non-wandering condition. For more details, see proposition 6 from [4], as well as [9].

Thus, \( Df(p) \) must have both eigenvalues equal to 1. The rest of the proof follows exactly the same lines from the argument in section 2 of [4]. \( \square \)

**Definition 2.22.** For any \( r \geq 1 \) or \( r = \infty \), define \( K_r \subset \text{Diff}_{0,\text{nw}}^r(\mathbb{T}^2) \) to be the subset of diffeomorphisms \( f \) satisfying the following four conditions:

(a) For all \( n > 0 \), there are at most finitely many \( n \)-periodic points for \( f \).

(b) For any \( f \)-periodic point \( z \), of prime period \( n \), if 1 is an eigenvalue of \( Df^n(z) \), then it has multiplicity 2 and \( f^n \) is of Lojasiewicz type at \( z \). Moreover, in this case the Index\(_{Df^n}^{}(z)\) is zero, and so lemma 2.21 implies that the local dynamics near \( z \) is given by figure 1. For families of maps, this situation corresponds to the birth or death of periodic points.

(c) For any \( f \)-periodic point \( w \), of prime period \( n \), if \(-1\) is an eigenvalue of \( Df^n(w) \), then Index\(_{Df^n}^{}(z)\) is 1. For families, this situation corresponds to a period-doubling bifurcation.

(d) There are no connections between saddle-like periodic points.
\(K' \setminus G'\) can be thought as a sort of set of typical diffeomorphisms in the complement of \(G'\). The definition can also be justified as follows. Let \(\mathcal{F}\) denote some one-parameter \(C^r\)-generic family of area-preserving diffeomorphisms,

\[\mathcal{F} := \{f_t\}_{t \in [a, b]} \subset \text{Diff}^r_{0, \text{Leb}}(\mathbb{T}^2), \quad \text{for some } r \geq 1. \tag{2.11}\]

The following statement is a combination of results from [3, 4]. The proofs were based on previous results contained in [11, 30].

**Lemma 2.23 (section 1.3.3 of [3] and section 2 of [4]).** For such a generic \(C^r\)-family \(\mathcal{F}\) as above, all maps \(f_t\) belong to \(K_r\); in particular, such a family never has saddle-like connections (tangencies are of course allowed), and with respect to periodic points, the only allowed degeneracies for a certain parameter \(t\) are, period-doubling bifurcations (item (c) above) and the one which appears in item (b), which are related to the birth or disappearance of periodic points.

The next result gives a perturbation consequence based on the local dynamics near fixed points. Let \(\text{Fix}(\tilde{f}) = \{z \in \mathbb{T}^2 : \forall \tilde{z} \in \pi^{-1}(z), \tilde{f}(\tilde{z}) = \tilde{z}\}\).

**Proposition 2.24 (proposition 9 of [4]).** Suppose \(\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)\). Assume \(\text{Fix}(\tilde{f})\) is finite and for any \(z_0 \in \text{Fix}(f)\), there exists a local continuous chart \(\psi : U \to \mathbb{R}^2\), such that, for any \(z \in (U \cap f^{-1}(U)) \setminus \{z_0\}\), \(p_1 \circ \psi \circ f(z) > p_1 \circ \psi(z)\). Then, there exists \(\varepsilon > 0\), such that for any \(\tilde{g} \in \text{Homeo}(\mathbb{T}^2)\) with \(\text{dist}_{C^0}(\tilde{g}, \tilde{f}) < \varepsilon\), \((0, 0) \notin \text{int}(\rho(\tilde{g}))\).

**Remark 2.25.** The above proposition clearly applies to the local dynamics described in figure 1. Later, this proposition will be used to establish properties of bounded deviation in the direction \((-\alpha, -\beta)\). A bit more precisely, working with maps from \(K'\), a perturbation consequence under unbounded deviation assumption will be obtained, and it will contradict proposition 2.24 (see section 6).

### 2.7 Prime ends rotation numbers

For an open topological disk \(D\) contained in some surface, one can attach an artificial circle, called the prime ends circle, denoted by \(b_{\text{PE}}(D)\). Moreover, the prime ends circle \(b_{\text{PE}}(D)\) can be topologized, such that the union \(D \cup b_{\text{PE}}(D)\) is homeomorphic to the standard closed unit disk in \(\mathbb{R}^2\). We call it the prime ends disk, and this procedure is referred to as prime ends compactification. This is the beginning of Carathéodory’s prime ends theory, and we refer to [21, 28, 29] for more details.

If \(f\) is a homeomorphism on the closure of an open topological disk \(D\) into itself, then \(f\) extends to the unit disk \(D \cup b_{\text{PE}}(D)\), so it induces a homeomorphism on the prime ends circle \(b_{\text{PE}}(D)\). Then, the dynamics restricted to this circle defines a rotation number, called the prime ends rotation number, and denoted \(\rho_{\text{PE}}(f, D)\). The following lemma is a combination of important results from several papers.

**Lemma 2.26.** Suppose \(\tilde{h}\) is a lift of some \(h \in \text{Diff}_{0, \text{aw}}(\mathbb{T}^2)\) satisfying the following properties:

(i) There are at most finitely many \(h^n\)-fixed points for all \(n \geq 1\);

(ii) For all \(n \geq 1\), and any \(h\)-periodic point \(p\), of period \(n\),

1. Either none of the eigenvalues of \(Dh^n(p)\) is equal to 1,
2. Or, if one of the eigenvalues of $Dh^n(p)$ is 1, then the topological index of $h^n$ at $p$ is zero and actually, both eigenvalues are equal to 1 (see the proof of lemma 2.21). Moreover, $p$ is of Lojasiewicz type for $h^n$.

Let $K \subset \mathbb{R}^2$ be an $h$-invariant continuum and let $O$ denote an $\tilde{h}$-invariant connected component of $K$. Write $\rho_{PE}(h, O)$ to denote the prime ends rotation number of $h$ restricted to $O$. Then the following statements are true:

(a) If $\rho_{PE}(\tilde{h}, O)$ is rational and $O$ is bounded, or $\rho_{PE}(\tilde{h}, O)$ is zero, then $\partial O$ contains only saddle-like $h$-periodic points, and connections between these saddle-like periodic points.

(b) If $O$ is not equal to $K$ and $\rho_{PE}(\tilde{h}, O)$ is irrational, then there is no $\tilde{h}$-periodic point in $\partial O$.

Sketch of the proof. Let us show item (a). Assume $\rho_{PE}(\tilde{h}, O) = \frac{p}{q}$ which is in reduced form. Then, as $h$ is non-wandering, a prime chain corresponding to a $q$-periodic prime end $\hat{z}$ has the property that each of its crosscuts must intersect its image under $\tilde{h}^q$, otherwise the corresponding cross sections would contain wandering domains, even in the torus (this argument goes back to Cartwright–Littlewood, see for instance proposition 2.1 of [14]). So, the principal set of $\hat{z}$ is made of $\tilde{h}^q$-fixed points, something that implies the first assertion of item (a), i.e., there exists some $\tilde{h}^q$-fixed point $z \in \partial O$.

The main results of [22] (see also theorem 1.2, corollary 1.3 and theorem 1.4 of the report [24] from ICM 2018), imply:

- If $q = 1$, then all $\tilde{h}$-periodic points $z \in \partial O$ are fixed and the eigenvalues of $D\tilde{h}(z)$ are both real and positive;
- If $O$ is bounded, for any value of $p/q$, all $\tilde{h}$-periodic points $z \in \partial O$ have prime period $q$ and the eigenvalues of $D\tilde{h}^q(z)$ are both real and positive;

So, as $h$ is non-wandering, a periodic point $z \in \partial O$ is either an hyperbolic saddle or both eigenvalues of $D\tilde{h}^q(z)$ are equal to 1 and $z$ has topological index 0. In this way, lemma 2.21 implies that $z$ is a saddle-like periodic point, either a hyperbolic saddle or a point with zero index and local dynamics as is figure 1.

With this local description, theorem 5.1 in [28] implies that the boundary $\partial O$ contains connections between saddle-like periodic points, as described above. We should remark that although in reference [24], most statements assume preservation of area and boundedness of $O$ as a planar subset, the arguments therein only use the fact that the dynamics is non-wandering restricted to a small neighbourhood of the compact set $K$. This completes the proof of item (a).

Item (b) is a direct consequence of theorem C of [21].

3. Perturbations for homeomorphisms with unbounded deviation

In this section and in the next, for any $w \in \mathbb{R}^2$, we will write $[w]$ to denote an integer two-vector which is the closest to $w$. Later when we see some displacement $\tilde{f}(x) - x$ which is very close to an integer two-vector, we will write $[\tilde{f}(x) - x]$ to express that integer two-vector. Also, a condition assumed in all the theorems proved here is unbounded deviation for a fixed direction, along which, we want our rotation set to grow.

**Theorem 3.1.** Let $\tilde{f} \in \text{Homeo}_{0,\text{aw}}(\mathbb{T}^2)$ whose rotation set $\rho(\tilde{f})$ is a line segment from $(0, 0)$ to some $(\alpha, \beta) \in \mathbb{R}^2$ which is totally irrational. Assume $\tilde{f}$ has unbounded deviation along the direction $(\alpha, \beta)$. Then $\tilde{f}$ can be $C^0$-approximated by $\tilde{g} \in \text{Homeo}_0(\mathbb{T}^2)$ such that $\rho(\tilde{g})$ has non-empty interior and
\[ \rho \tilde{f} \setminus \{(0,0)\} \subset \text{Int}(\rho(\bar{g})). \]  

When the rotation set is as above, we can also study a similar situation around the other endpoint, \((0,0)\).

**Theorem 3.2.** Let \(f\) and \(\rho f\) be as in theorem 3.1. Assume \(\tilde{f}\) admits unbounded deviation along \( - (\alpha, \beta) \). Then \( f\) can be \(C^0\)-approximated by \( \bar{g} \in \text{Homeo}_0(\mathbb{T}^2)\), such that \((0,0) \in \text{Int}(\rho(\bar{g}))\).

### 3.1. Some preparations

Given the totally irrational vector \((\alpha, \beta)\), define

\[ L_0 : y = \frac{\beta}{\alpha} x, \]  
\[ L_1 : \alpha x + \beta y = \alpha^2 + \beta^2, \]

which are straight lines along the directions \((\alpha, \beta)\), \((-\beta, \alpha)\), respectively, and intersecting at the point \((\alpha, \beta)\). Also define the four connected components of the complement of \(L_0 \cup L_1\) in \(\mathbb{R}^2\). See figure 2.

\[ \Delta_0 = \{ \tilde{z} \in \mathbb{R}^2 | \text{pr}_{(\alpha, \beta)}(\tilde{z}) > 0 \text{ and } \text{pr}_{(\alpha, \beta)}(\tilde{z} - (\alpha, \beta)) > 0 \}. \]  
\[ \Delta_1 = \{ \tilde{z} \in \mathbb{R}^2 | \text{pr}_{(\alpha, \beta)}(\tilde{z}) < 0 \text{ and } \text{pr}_{(\alpha, \beta)}(\tilde{z} - (\alpha, \beta)) < 0 \}. \]  
\[ \Omega_0 = \{ \tilde{z} \in \mathbb{R}^2 | \text{pr}_{(\alpha, \beta)}(\tilde{z}) < 0 \text{ and } \text{pr}_{(\alpha, \beta)}(\tilde{z} - (\alpha, \beta)) > 0 \}. \]  
\[ \Omega_1 = \{ \tilde{z} \in \mathbb{R}^2 | \text{pr}_{(\alpha, \beta)}(\tilde{z}) > 0 \text{ and } \text{pr}_{(\alpha, \beta)}(\tilde{z} - (\alpha, \beta)) < 0 \}. \]

**Lemma 3.3.** Let \(\{x(0), \ldots, x(n-1)\}\) be an \(\varepsilon\)-pseudo periodic orbit of some \(f \in \text{Homeo}_0(\mathbb{T}^2)\). Suppose it lifts to a (finite) \(\varepsilon\)-pseudo orbit segment for \(\tilde{f}\), starting at some lift \(\tilde{x}\) of \(x(0)\) and ending at some lift \(\tilde{x}'\) of \(x(n)\). Then, \(\tilde{f}\) can be \(C^0\)-perturbed to \(\bar{g} \in \text{Homeo}_0(\mathbb{T}^2)\), so that the lifted sequence is a real orbit segment for \(\bar{g}\). In particular, \(\tilde{x}' = \bar{g}^n(\tilde{x})\) and \(\bar{g}\) projects down to an \(\varepsilon\)-perturbation \(g\) of \(f\). Moreover, if \(f\) preserves area, then so does \(g\).

**Proof.** See [13]. For the area-preserving case, just note that \(g\) is obtained from \(f\) by a series of perturbations supported in finitely many disjoint disks. And it is well-known that these perturbations, which are the identity in the boundary of each disk, can be chosen as area-preserving themselves.

Recall that theorem 2.10 says that \(\tilde{f}\) has bounded deviation along perpendicular directions. Based on this, the following lemma gives small displacements along the perpendicular directions, for some chosen iterates.

**Lemma 3.4 (small displacement).** Let \(w\) be either \((-\beta, \alpha)\) or \((\beta, -\alpha)\). Then, for any \(\delta > 0\) and any \(\tilde{z}_0 \in \mathbb{R}^2\), there exists \(n_0\) such that,

\[ \text{pr}_w(\tilde{f}^n(\tilde{z}_0) - \tilde{z}_0) < \delta, \quad \text{for any } n > n_0. \]

**Proof.** Observing theorem 2.10, we can choose \(n_0 \geq 1\), with

\[ \text{pr}_w(\tilde{f}^{n_0}(\tilde{z}_0) - \tilde{z}_0) > \sup_{n \geq 1} \{ \text{pr}_w(\tilde{f}^n(\tilde{z}_0) - \tilde{z}_0) \} - \delta. \]

Then, for any \(n > n_0\), (3.8) follows immediately.
3.2. The irrational endpoint

**Proof of theorem 3.1.** Fix $\varepsilon > 0$. By theorem 1 of [1], for any ergodic measure with average rotation vector $(\alpha, \beta)$, around a typical point which is $f$-recurring, there is some point $y$ and a positive integer $n$, such that

$$\text{dist}_{T^2}(f^n(y), y) < \varepsilon. \quad (3.10)$$

$$\Pr_{(\alpha, \beta)}(\tilde{f}^n(y) - y \neq n(\alpha, \beta)) > 0, \quad (3.11)$$

where we use $\lfloor \tilde{f}^n(y) - y \rfloor$ to denote the nearest integer two-vector to the vector $\tilde{f}^n(y) - y$. Considering figure 2, the last estimate means that the vector $\frac{1}{n}(\tilde{f}^n(y) - y)$ lies in the union of regions $\Omega_0, \Delta_0$ and their common boundary. We aim to perturb the dynamics to make two such vectors to lie in $\Omega_0$ and $\Delta_0$, respectively.

Now let us look for a $6\varepsilon$-pseudo periodic orbit of $f$, lifting to a $6\varepsilon$-pseudo $\tilde{f}$-orbit segment starting at some $\tilde{z}'$ and ending at some $\tilde{z}''$, so that the rational vector

$$\frac{1}{n}(\tilde{z}'' - \tilde{z}') \in \Omega_0. \quad (3.12)$$

We assume the totally irrational $(\alpha, \beta)$ has norm 1 for simplicity. For any $K > 0$, define $R_K \subset T^2$ to be the set of points $x$ such that for at least $K$ choices of positive integers $n$, we have $\text{dist}_{T^2}(f^n(x), x) \leq \frac{1}{K}$ and $\Pr_{(\alpha, \beta)}(\tilde{f}^n(x) - x \neq n(\alpha, \beta)) \geq K$. We claim that $R_K$ is non-empty for any positive constant $K$. In fact, we can cover $T^2$ with $N$ disks (for some integer $N$), all with diameter $\frac{1}{K}$. Then by the assumption on unbounded deviation in the direction $(\alpha, \beta)$, it is not hard to find $\tilde{x}$, and integers $0 = m_0 < m_1 < \cdots < m_{KN}$, such that, for any $k = 1, \ldots, KN$,

$$\Pr_{(\alpha, \beta)}(\tilde{f}^{m_k}(\tilde{x}) - \tilde{f}^{m_{k-1}}(\tilde{x}) - (m_k - m_{k-1})(\alpha, \beta)) \geq K. \quad (3.13)$$

By the pigeonhole principle, for at least $K + 1$ choices of the indices among those $m_j$’s, the corresponding iterates of $x_j$ lie in one single disk with diameter no more than $\frac{1}{K}$. Clearly, between any two of these chosen ones, say $m_i < m_j$, we see

$$\Pr_{(\alpha, \beta)}(\tilde{f}^{m_j}(\tilde{x}) - \tilde{f}^{m_i}(\tilde{x}) - (m_j - m_i)(\alpha, \beta)) \geq K, \quad (3.14)$$

and in particular the claim follows by taking the first iterate among those.

Next we take an accumulation point $x^*$ of the sets $R_K$ as $K$ tends to infinity. Then we look at a sort of ‘skeleton’ in the torus, namely, the sequence of points

$$\{x^*, x^* - 4\varepsilon(-\beta, \alpha), x^* - 2 \cdot 4\varepsilon(-\beta, \alpha), \ldots, x^* - K_0 \cdot 4\varepsilon(-\beta, \alpha)\}, \quad (3.15)$$

where the number $K_0$ is chosen to be the least integer so that $x^* - (K_0 + 1) \cdot 4\varepsilon(-\beta, \alpha)$ is $\varepsilon/2$-close to $x^*$. Then we go on to choose orbit segments with starting points and ending points near these skeleton points.

By lemma 2.2, the point $x^* - 4\varepsilon(-\beta, \alpha)$ is approximated by an $f$-recurring point. Then, with the help of lemma 3.4, one can find a point $z_1$ which is some forward iterate of the recurrent point we just found, and a positive integer $n_1$, such that both $z_1$ and $f^{n_1}(z_1)$ are $\varepsilon$-close to $x^* - 4\varepsilon(-\beta, \alpha)$, and that

$$\Pr_{(-\beta, \alpha)}(\tilde{f}^n(z_1) - \tilde{z}_1) < \varepsilon. \quad (3.16)$$
Next, by choosing another $f$-recurrent point near the point $x^* - 2 \cdot 4\varepsilon (-\beta, \alpha)$ and then applying lemma 3.4, we can find another orbit segment satisfying similar estimates. In fact, we can inductively find $z_1, z_2, \ldots, z_{K_0}$ with disjoint orbits, and integers $n_1, n_2, \ldots, n_{K_0}$, such that, for any $i = 2, \ldots, K_0$, both $z_i$ and $f^{n_i}(z_i)$ are $\varepsilon$-close to $x^* - i \cdot 4\varepsilon (-\beta, \alpha)$, and that

$$\text{pr}(-\beta, \alpha) \left( \tilde{f}^{n_i}(\tilde{z}_i) - \tilde{z}_i - n_i(\alpha, \beta) \right) < \varepsilon.$$ (3.17)

Then we sum all the deviations (in the direction $(\alpha, \beta)$) created in each of the above orbit segments

$$M := \sum_{i=1}^{K_0} \text{pr}(-\beta, \alpha) \left( \tilde{f}^{n_i}(\tilde{z}_i) - \tilde{z}_i - n_i(\alpha, \beta) \right).$$ (3.18)

Note the jump from the ending point of one segment to the starting point of another creates a deviation in the direction $(\alpha, \beta)$ upper bounded by $2\varepsilon$. When we consider a point $x$ in $R_K$ with $K > |M| + 2\varepsilon(K_0 + 1)$, by definition, for at least $K$ positive integers, the corresponding iterates of $x$ all lie in a same disk of diameter at most $1/K$, and pairwise the deviation along the direction $(\alpha, \beta)$ is at least $K$. Moreover, the dynamics has bounded deviation along the perpendicular direction (see theorem 2.10). So when $K$ is sufficiently large, we can find two of these iterates so that along the direction $(-\beta, \alpha)$, the deviation is at most $\varepsilon$ (with a similar argument as in lemma 3.4).

In the next step, we choose $z_0$ and some iterate $f^{n_0}(z_0)$, in a way similar to the choices of $z_i$ and $f^{n_i}(z_i)$ above, with good deviation properties. Recall now that $x^*$ is an accumulation point of $R_K$. Therefore, by the above paragraph, we take a point $x \in R_K$, sufficiently close to $x^*$, with respect to some sufficiently large $K$. Then, sufficiently close to $x$, we can find two large iterates of $x$, namely, $z_0$ and $f^{n_0}(z_0)$ for some positive integer $n_0$, so that, eventually, both $z_0$ and $f^{n_0}(z_0)$ are $\varepsilon/2$-close to $x^*$, and moreover,

$$\text{pr}(-\beta, \alpha) \left( \tilde{f}^{n_0}(\tilde{z}_0) - \tilde{z}_0 \right) < \varepsilon,$$ and (3.19)
\[
\text{pr}_{(\alpha, \beta)} \left( \tilde{f}^{n}(\tilde{z}_0) - \tilde{z}_0 - n_0(\alpha, \beta) \right) > |M| + 2\varepsilon(K_0 + 1).
\] (3.20)

We stress again, that during the whole process, we can require that the whole orbits of the points \( z_i, \ i = 0, \ldots, K_0 \) are all pairwise disjoint. Then we write down the following \( K_0 + 1 \) point-wise disjoint \( f \)-orbit segments, namely

\[
\{ z_1, f(z_1), \ldots, f^{n_1-1}(z_1) \}, \ldots, \{ z_{K_0}, \ldots, f^{n_{K_0}-1}(z_{K_0}) \}, \{ z_0, \ldots, f^{n_0-1}(z_0) \}. \] (3.21)

They together form a \( 6\varepsilon \)-pseudo periodic orbit of period

\[
\ell = \sum_{j=0}^{K_0} n_j.
\] (3.22)

Expression (3.20) implies that the final deviation of the whole pseudo orbit along \((\alpha, \beta)\) is positive, because the sum of the deviations created in all the jumps is at most \(2\varepsilon(K_0 + 1)\), and the sum of the deviations created inside each of these segments except \( \{ z_0, \ldots, f^{n_0-1}(z_0) \} \) is upper bounded by \(|M|\). On the other hand, among these \( K_0 + 1 \) segments in (3.21), the way we jump between two consecutive orbit segments gives at least \(2\varepsilon\) deviation along the direction \((\beta, -\alpha)\), and within each segment the deviation along \((-\beta, \alpha)\) is at most \(\varepsilon\), by the estimates (3.16) and (3.17). So in the end we see positive deviation along \((\beta, -\alpha)\). It follows that this pseudo orbit sees a rotation vector in the region \( \Omega_0 \).

A similar argument can be done to obtain another \( 6\varepsilon \)-pseudo periodic orbit, containing some point \( y_0 \), which sees rotation vector in \( \Delta_0 \) with respect to \( f \). Moreover, we can choose these two pseudo orbits to be disjoint from each other. Then we apply lemma 3.3 twice to close these two \( 6\varepsilon \)-close to \( f \) in the \( C^0 \) topology, and admits two periodic points \( y_0 \) and \( z_0 \). By the above construction, it follows that \( \rho(\tilde{g}, y_0) \in \Delta_0 \) and \( \rho(\tilde{g}, z_0) \in \Omega_0 \).

Since \((0, 0)\) is an extremal point of \( \rho(f) \), by [13] \( f \) admits a fixed point \( p^* \), which lifts to a fixed point of \( \tilde{f} \). Now as we look back at the whole perturbation process above, we can choose all the orbit segments far from \( p^* \). This means that the perturbations can be made away from \( p^* \). Thus, the rotation set of \( \tilde{g} \) satisfies \( \rho(\tilde{g}) \supset \{(0, 0), \rho(\tilde{g}, z_0), \rho(\tilde{g}, y_0)\} \). Since any rotation set is convex (theorem 2.5), \( \rho(\tilde{g}) \) must have non-empty interior, and \( \text{Int}(\rho(\tilde{g})) \supset \rho(\tilde{f}) \setminus \{(0, 0)\} \), as we wanted to prove. As \( \varepsilon \) can be chosen to be arbitrarily small, the proof of theorem 3.1 is completed.

\[ \square \]

### 3.3. Origin as endpoint

**Proof of theorem 3.2.** This proof is similar to the above one. We fix \( \varepsilon > 0 \) now. First let us define two new regions as follows.

\[
\Pi_0 := \{ \tilde{z} \in \mathbb{R}^2 | \text{pr}_{(\alpha, \beta)}(\tilde{z}) < 0 \text{ and } \text{pr}_{(-\beta, \alpha)}(\tilde{z}) > 0 \}.
\] (3.23)

\[
\Pi_1 := \{ \tilde{z} \in \mathbb{R}^2 | \text{pr}_{(\alpha, \beta)}(\tilde{z}) < 0 \text{ and } \text{pr}_{(-\beta, \alpha)}(\tilde{z}) < 0 \}.
\] (3.24)

For any \( K > 0 \), define \( \mathcal{R}_K \) to be the set of points \( z \) such that for at least \( K \) choices of integers \( n > 0 \), the following holds.

\[
\text{dist}_{\mathbb{R}^2}(f^n(z), \tilde{z}) \leq \frac{1}{K}.
\] (3.25)

\[
\text{pr}_{(\alpha, \beta)}(f^n(\tilde{z}) - \tilde{z}) \leq -K.
\] (3.26)
With the condition of unbounded deviations in the direction $-(\alpha, \beta)$, by a similar argument as in previous subsection, we can show $\mathcal{R}'_K$ is non-empty for any $K > 0$.

By taking $K \geq \frac{1}{\varepsilon}$, we can find an $\varepsilon$-pseudo periodic point $y_0$ of period $n_\varepsilon$, which realizes a rotation vector in the union of the regions $\Pi_0, \Pi_1$ and their common boundary.

Then, very similar to the previous subsection, let $y'$ be an accumulation point of $\mathcal{R}'_K$ as $K$ tends to infinity. With the help of lemmas 2.2 and 3.4, as well as the definition of $\mathcal{R}'_K$, we can choose finitely many orbit segments, which altogether form a $6\varepsilon$-pseudo periodic orbit, lifting to a $6\varepsilon$-pseudo orbit for $f$, namely $\{\tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_{16}\}$, such that

$$v' = \frac{1}{n_\varepsilon}[^{\tilde{z}_{16}} - \tilde{z}_0] \in \Pi_1,$$  

(3.27)

where $[^{\tilde{z}_{16}} - \tilde{z}_0]$ denotes the nearest integer pair to the vector $^{\tilde{z}_{16}} - \tilde{z}_0$.

Thus we obtain a pseudo periodic orbit seeing rotation vector $v' \in \Pi_0$. A similar argument also gives a rotation vector $v'' \in \Pi_0$. Since $(\alpha, \beta)$ is an extremal point of $\rho(f)$, by theorem 2.5, there exists some ergodic $f$-invariant measure $\mu$ satisfying $\rho_\mu(f) = (\alpha, \beta)$. Moreover, for a $\mu$-typical point $x \in \mathbb{T}^2$, for some increasing integer sequence $n_j$, and any lift $\tilde{x}$,

$$\lim_{j \to \infty} f^{n_j}(x) = x.$$  

(3.28)

$$\lim_{j \to \infty} \frac{1}{n_j} \left( f^{n_j}(\tilde{x}) - \tilde{x} \right) = (\alpha, \beta).$$  

(3.29)

Then for sufficiently large $n_j$, $\{x, f(x), \ldots, f^{n_j-1}(x)\}$ forms an $\varepsilon$-pseudo periodic orbit, such that the vectors $v', v''$ and the rational vector

$$w = \frac{1}{n_j}[^{\tilde{f}^{n_j}(\tilde{x})} - \tilde{x}]$$  

(3.30)

span a triangle, which contains the origin $(0, 0)$ in its interior. Note that we can choose the three pseudo orbits to be pair-wise disjoint. Then, applying lemma 3.3 three times, we obtain a $6\varepsilon$-perturbation $\tilde{g}$ of $f$. The rotation set $\rho(\tilde{g})$ contains at least the three rational vectors $v', v''$ and $w$. By the convexity of the rotation set, $(0, 0)$ is contained in $\text{Int}(\rho(\tilde{g}))$. We have completed the proof. \hfill \square

4. Perturbations for homeomorphisms with bounded deviation

4.1. The totally irrational rigid rotation

We start by showing a simple perturbation result for the rigid rotation $R_{(\alpha, \beta)}$ on $\mathbb{T}^2$, where $(\alpha, \beta)$ is totally irrational.

**Proposition 4.1.** For any $\varepsilon > 0$, there exists $\tilde{g} \in \tilde{\text{Homeo}}_0(\mathbb{T}^2)$ with $\text{dist}_{\mathcal{C}}(\tilde{g}, R_{(\alpha, \beta)}) < \varepsilon$, such that, $\rho(\tilde{g})$ has interior, and $(\alpha, \beta) \in \text{Int}(\rho(\tilde{g}))$.

**Proof.** Since $(\alpha, \beta)$ is totally irrational, the rigid rotation $R_{(\alpha, \beta)}$ is minimal. For any $x_0 \in \mathbb{T}^2$ and for any small $\varepsilon > 0$, consider a small disk $B = B(x_0, \varepsilon)$, which is divided into four regions as follows.

$$\Delta_1(x_0, \varepsilon) := \{ x \in B(x_0, \varepsilon) | \text{pr}_{(\alpha, \beta)}(x - x_0) < 0, \text{pr}_{(-\beta, \alpha)}(x - x_0) < 0 \}.$$  

(4.1)

$$\Delta_2(x_0, \varepsilon) := \{ x \in B(x_0, \varepsilon) | \text{pr}_{(\alpha, \beta)}(x - x_0) < 0, \text{pr}_{(-\beta, \alpha)}(x - x_0) > 0 \}.$$  

(4.2)
\[ \Omega_1(x_0, \varepsilon) := \{ x \in B(x_0, \varepsilon) | \text{pr}_{(\alpha, \beta)}(x - x_0) < 0, \text{pr}_{(-\beta, \alpha)}(x - x_0) > 0 \}. \] (4.3)

\[ \Omega_0(x_0, \varepsilon) := \{ x \in B(x_0, \varepsilon) | \text{pr}_{(\alpha, \beta)}(x - x_0) > 0, \text{pr}_{(-\beta, \alpha)}(x - x_0) < 0 \}. \] (4.4)

By minimality of \( R_{(\alpha, \beta)} \), we can choose some integer \( n \), such that \( R^n_{(\alpha, \beta)}(x_0) \in \Delta_1(x_0, \varepsilon) \). Then, for proper choices of lifts \( \tilde{R} \) and \( \tilde{x}_0 \) of \( R_{(\alpha, \beta)} \) and \( x_0 \), respectively, \( \tilde{R}^n(\tilde{x}_0) \) is \( \varepsilon \)-close to \( \tilde{x}_0 + (a, b) \) for some \((a, b) \in \mathbb{Z}^2 \). Moreover, we can write \( v_1 = (\xi, \eta) \), and then clearly \( v_1 \in \Delta_0((\alpha, \beta), \frac{\pi}{a}) \).

In other words, we can find an \( \varepsilon \)-pseudo periodic orbit for the rigid rotation \( R_{(\alpha, \beta)} \), which sees a rational rotation vector in the region \( \Delta_0((\alpha, \beta), \frac{\pi}{a}) \). We argue in a similar way with respect to the other three regions. Then, we obtain four \( \varepsilon \)-pseudo periodic orbits for \( R_{(\alpha, \beta)} \), starting with \( x_0, y_0, z_0, w_0 \), respectively. We can also require these orbit segments to be point-wise disjoint. Then, applying lemma 3.3 four times, these four pseudo orbits can be closed via an \( \varepsilon \)-perturbation, which produces four periodic orbits. These periodic orbits will have four rational rotation vectors \( v_1, v_2, v_3, v_4 \), respectively, whose convex hull contains \((\alpha, \beta)\) in its interior. Therefore \((\alpha, \beta) \in \text{Int}(\rho(\mathcal{G})) \).

**Remark 4.2.** This proposition can be compared with theorem 1 in [19], which states the following. With respect to the \( C^r \) topology with \( r \) sufficiently high, due to the ‘KAM’ phenomenon, the \( C^r \)-perturbed rotation set either misses \((\alpha, \beta)\), or it equals \{ \((\alpha, \beta)\) \}, provided that \((\alpha, \beta)\) satisfies certain Diophantine conditions.

### 4.2. Bounded deviations

In this subsection, we assume that \( \tilde{f} \) has bounded deviation along the direction \((\alpha, \beta)\). We consider this case for the sake of completeness, but it is possible that it might not happen at all (cf theorem 1.6 and question 1).

**Theorem 4.3.** Suppose \( \tilde{f} \in \tilde{\text{Homeo}}_0(T^2) \), whose rotation set \( \rho(\tilde{f}) \) is the segment from \( (0, 0) \) to the totally irrational point \((\alpha, \beta)\). Assume \( \tilde{f} \) has bounded deviation along the direction \((\alpha, \beta)\). Then \( \tilde{f} \) can be \( C^0 \)-approximated by \( \tilde{g} \in \tilde{\text{Homeo}}_0(T^2) \), such that \( \rho(\tilde{g}) \) has interior, and \( \rho(f) \setminus \{(0, 0)\} \subset \text{Int}(\rho(\tilde{g})) \).

Assume for some \( M > 0 \), for any \( \tilde{x} \in \mathbb{R}^2 \) and any \( n \geq 1 \),

\[ \text{pr}_{(\alpha, \beta)}(f^n(\tilde{x}) - \tilde{x} - n(\alpha, \beta)) \leq M. \] (4.5)

**Definition 4.4.** Let \( \mathcal{M}_{(\alpha, \beta)} \) denote the set of ergodic \( f \)-invariant Borel probability measures, which have \((\alpha, \beta)\) as rotation vector. Then write

\[ \mathcal{S}_{(\alpha, \beta)} := \bigcup_{\mu \in \mathcal{M}_{(\alpha, \beta)}} \text{supp}(\mu), \] (4.6)

where \( \text{supp}(\mu) \) denotes the support of \( \mu \).

The following lemma, whose proof depends on Atkinson’s theorem on cocycles (see [6]), appears as lemma 6 of [2] or proposition 65 of [27].

**Lemma 4.5.** Suppose \( \tilde{f} \) satisfies condition (4.5). Then for any \( x \in \mathcal{S}_{(\alpha, \beta)} \) with a lift \( \tilde{x} \), and for any \( n \geq 1 \),

\[ \text{pr}_{(\alpha, \beta)}(f^n(\tilde{x}) - \tilde{x} - n(\alpha, \beta)) \geq -M. \] (4.7)
In particular, any invariant ergodic measure \( \mu \) such that \( \text{supp}(\mu) \subset S(\alpha, \beta) \) is contained in \( \mathcal{M}(\alpha, \beta) \).

**Proof of theorem 4.3.** Fix \( \varepsilon > 0 \). Choose any minimal set \( K \subset S(\alpha, \beta) \). Then \( \rho(\tilde{f}, K) = \{(\alpha, \beta)\} \). By assumption (4.5), theorem 2.10 and lemma 4.5, \( f_{x}^{-1}(K) \) has bounded deviation along every direction. So we can apply lemma 2.9 in order to find a semi-conjugacy \( \phi \) between \((K, f|_{K})\) and \((\mathbb{T}^{2}, R)\), where \( R = R_{(\alpha, \beta)} \) denotes the rigid rotation on \( \mathbb{T}^{2} \) by \( \alpha, \beta \).

There is a lift \( \tilde{\phi} \) of \( \phi \), which conjugates \( f_{x}^{-1}(K) \) and \( \tilde{R} \).

Note that the pre-image of every point under \( \tilde{\phi} \) has diameter uniformly bounded from above, say by a constant \( C_{0} > 0 \).

Recall the four regions defined from (4.1) to (4.4). Suppose \( \delta \) is a positive constant smaller than \( \varepsilon \), so that \( \text{dist}_{\mathbb{T}^{2}}(x, x') < \delta \) implies both \( \text{dist}_{\mathbb{T}^{2}}(\tilde{f}(x), \tilde{f}(x')) < \varepsilon \) and \( \text{dist}_{\mathbb{T}^{2}}(\phi \circ f^{2}(x), \phi \circ f^{2}(x')) < \varepsilon \).

Since \( R = R_{(\alpha, \beta)} \) is minimal, for some sufficiently large positive integer \( n \), and for a point \( x_{0} \in \mathbb{T}^{2} \) with its pre-image \( \gamma_{0} = \phi^{-1}(x_{0}) \), we have that
\[
R^{n}(x_{0}) \in \Omega_{1}(x_{0}, \varepsilon).
\]
\begin{equation}
d_{H}(f^{n}(\gamma_{0}), \gamma_{0}) < \delta / 2,
\end{equation}
where \( d_{H} \) denotes the Hausdorff distance among the space of compact subsets of \( \mathbb{T}^{2} \). Estimate (4.8) means the following. By choosing a lift \( \tilde{x}_{0} \) of \( x_{0} \), we write \( (a, b) = [R^{n}(\tilde{x}_{0}) - \tilde{x}_{0}] \) to denote the integer two-vector which is nearest to the vector \( \tilde{R}^{n}(\tilde{x}_{0}) - \tilde{x}_{0} \). Then we have
\[
v = \frac{[\tilde{R}^{n}(\tilde{x}_{0}) - \tilde{x}_{0}]}{n} = \left( \frac{a}{n}, \frac{b}{n} \right) \in \Omega_{0}.
\]
Equivalently (see figure 2),
\begin{equation}
pr_{(\alpha, \beta)}(v - (\alpha, \beta)) > 0.
\end{equation}
\begin{equation}
pr_{(\beta, \alpha)}(v - (\alpha, \beta)) < 0.
\end{equation}

Furthermore, we can in fact obtain infinitely many returning times, say, \( n_{1} < n_{2} < n_{3} < \ldots \), so that
\[
R^{n_{j}}(x_{0}) \in \Omega_{1}(R^{n_{j}}(x_{0}), \varepsilon) \quad \text{for every } j < k,
\]
\begin{equation}
d_{H}(f^{n_{j}}(\gamma_{0}), f^{n_{j-1}}(\gamma_{0})) < \delta / 2^{k} \quad \text{for all } k.
\end{equation}
Then we start from any fixed \( x_{*} \in \gamma_{0} \), and look at its iterates \( f^{n_{j}}(x_{*}) \). Since these iterates are always at distance smaller than \( \delta \) with \( \gamma_{0} \), by covering \( \gamma_{0} \) with finitely many disks of diameter at most \( \delta \), we can find two iterates returning to the same disk.

Suppose we find two such iterates \( f^{n_{j}}(x_{*}) \) and \( f^{n_{j}}(x_{*}) \), with \( n_{j} < n_{j} \). Then put \( y = f^{n_{j}}(x_{*}) \) and consider the segments \( \{y, f(y), \ldots, f^{n_{j}-n_{j}}(y)\} \), whose two endpoints have distance at most \( \delta \), and both are \( \delta \)-close to some point \( y_{*} \in \gamma_{0} \). By the choice of \( \delta \), we can change the two ending points in the above segment to the same point \( y_{*} \), obtaining the sequence \( \{y_{*}, f(y), f^{2}(y), \ldots, f^{n_{j}-n_{j}}(y)\} \). We claim it is an \( \varepsilon \)-pseudo periodic orbit. To see this, we can make two jumps with distances smaller than \( \varepsilon \). The first jump happens from \( f(y_{*}) \) to \( f(y) \), and the second jump happens from \( f^{n_{j}-n_{j}}(y) \) to \( y \).
As long as $2\varepsilon < 1$, this final pseudo orbit must see a rotation vector which is the same with the rigid rotation. More precisely, the lift of the $\varepsilon$-pseudo orbit above must start at some point $\tilde{p}$ and end at $\tilde{p} + (a, b)$, so that it sees the rotation vector $v = (\alpha, \beta) \in \Omega$. In a similar way, we can find another $\varepsilon$-pseudo $f$-periodic orbit which sees a rational rotation vector $u \in \Delta$. Moreover, as we explained before, there is also at least one $f$-fixed point $p^*$ which lifts to an $\tilde{f}$-fixed point $\tilde{p}^*$. Notethat we can choosethe two pseudo orbits and the fixed point to be pairwise disjoint. So, if we apply lemma 3.3 twice, we obtain an $\varepsilon$-perturbation $\tilde{g} \in \tilde{\text{Homeo}}_0(T^2)$, whose projection $g$ has at least three periodic orbits, $p^*, p_0, q_0$, whose rotation vectors, $\rho(\tilde{g}, p^*) = (0, 0)$, $\rho(\tilde{g}, p_0) = v$ and $\rho(\tilde{g}, q_0) = u$ are vertices of a triangle which contains $\rho(\tilde{f}) \setminus \{(0, 0)\}$ in its interior. By theorem 2.5, this triangle is clearly contained in $\rho(\tilde{g})$, something that finishes this proof. □

Proof of theorem 1.1. Theorem 1.1 follows immediately from theorems 3.1 and 4.3. □

5. Generic diffeomorphisms

In this section, we prove the following theorem.

Theorem 5.1 (theorem 1.2 restated). Let $f \in \mathcal{G}^r$. Then for any lift $\tilde{f}$, the rotation set $\rho(\tilde{f})$ cannot be a segment from $(0, 0)$ to a totally irrational point $(\alpha, \beta)$.

Proof. The proof of this theorem will go through the whole section. We assume the following conditions and arrive at a contradiction in the end of the proof.

\begin{align*}
f &\in \mathcal{G}^r. \quad (5.1) \\
\rho(\tilde{f}) &\text{is the segment from (0, 0) to the totally irrational point } (\alpha, \beta). \quad (5.2)
\end{align*}

Since $(\alpha, \beta)$ is an extremal point of $\rho(\tilde{f})$, by theorem 2.5, we can choose an $f$-recurrent point $z_*$, such that

$$
\rho(\tilde{f}, z_*) = \lim_{n \to \infty} \frac{1}{n} (\tilde{f}^n(z_*) - z_*) = (\alpha, \beta). \quad (5.3)
$$

Now, consider the family of all the $f$-invariant open topological disks. We can define a partial order among this family with respect to the usual inclusion relation. It is standard to check that with respect to this order, the family forms a partially ordered set, for which every chain has an upper bound. So by Zorn’s lemma, we conclude the existence of maximal elements. As $f$ is non-wandering, lemma 2.3 implies that there exists a constant $M > 0$, such that, every connected component $D$ of the lift of a maximal open $f$-invariant disk $D$ is $f$-invariant, and

$$
diam(\tilde{D}) < M. \quad (5.4)
$$

Also, if $D$ is an $f$-invariant maximal open topological disk, then it contains fixed points. This follows from a classical argument: pick some point $p \in D$. If $p$ is not fixed, then for a sufficiently small open ball $B$ centred at $p$, contained in $D$, we have $B$ is disjoint from $f(B)$ and $f^n(B)$ intersects $B$ for a sufficiently large $n > 0$ (there are no wandering points). And this implies the existence of a fixed point inside $D$, see lemma 2.11. Since for $f \in \mathcal{G}^r$, there are finitely many fixed points, it follows that there are at most finitely many maximal open $f$-invariant disks.

Note that $z_*$ is disjoint from the closure of the union of these finitely many maximal open $f$-invariant disks, because the orbit of any $\tilde{z}_* \in \pi^{-1}(z_*)$ is unbounded in $\mathbb{R}^2$. Thus, we can
choose some $\delta > 0$ such that, the open disk $B(z, \delta)$ is still disjoint from the closure of the union of these maximal open $f$-invariant disks. Define the set

$$U := \text{the connected component of } \bigcup_{n \in \mathbb{Z}} f^n(B(z, \delta)) \text{ which contains } z.$$  \hspace{1cm} (5.5)

The following lemma allows us to find a good hyperbolic saddle fixed point.

**Lemma 5.2.** There exists at least one fixed hyperbolic saddle point $Q^*$, which is contained in $U$.

**Proof.** Clearly $U$ is open. Since $f$ is non-wandering, $U$ is $f^{n^*}$-invariant for some $n^* > 0$. Recall the notions in subsection 2.1, and claim that $U$ is essential. Suppose otherwise and let $U_{\text{filled}}$ be the union of $U$ with all the connected components of the complement of $U$ which are contractible. This construction implies that $U_{\text{filled}}$ is an open disk and, by lemma 2.3 applied to $f^{n^*}$, all connected components of the lift of $U_{\text{filled}}$ to the plane have bounded diameter and are $f^{n^*}$-invariant (because $(0,0)$ is the only rational point contained in the rotation set). In particular, any lift of $z^*$ has bounded orbit, a contradiction with (5.3).

The next claim is that $U$ is in fact fully essential. Moreover, the following proposition holds:

**Proposition 5.3.** Under our hypothesis on the rotation set, if $U$ is a periodic essential connected, either open or closed subset of $\mathbb{T}^2$, then it is in fact fully essential (that is, when $U$ is open, it contains curves in infinitely many different homotopy classes, and when it is closed, its complement is the union of periodic open disks, none of them unbounded when lifted to the plane).

**Proof.** If $U$ is open and not fully essential, then all the homotopically non-trivial loops contained in $U$ are homotopic to each other. Fix one homotopically non-trivial loop $\gamma \subset U$ and choose connected components of their lifts, $\tilde{\gamma}$ and $\tilde{U}$, such that

$$\tilde{\gamma} \subset \tilde{U}.$$  \hspace{1cm} (5.6)

Clearly, there exists some integer vector $(a, b) \neq (0,0)$, such that $\tilde{\gamma} = \tilde{\gamma} + (a, b)$. Moreover, since by assumption $U$ is not fully essential, $\tilde{U} \cap (\tilde{U} + i(-b, a)) = \emptyset$ for any integer $i \neq 0$. Therefore, $\tilde{U}$ is contained in the strip bounded by $\tilde{\gamma} - (-b, a)$ and $\tilde{\gamma} + (-b, a)$.  \hspace{1cm} (5.7)

This is a contradiction with the shape of the rotation set, because the rotation vector of any point in $U$ (when it exists) must have rational slope $b/a$, see also lemma 2.8.

In case $U$ is closed and not fully essential, some connected component of its complement is open, periodic and essential. So by the above, this component is fully essential, a contradiction with the hypothesis that $U$ is itself essential. \hfill $\square$

So $U$ is fully essential. Moreover, if $U$ is not the whole torus, then any connected component of $(\overline{U})^c$ is a periodic open disk (the periodicity follows from the non-wandering hypothesis). In particular, by the choice of $\delta > 0$, each $f$-invariant maximal open disk (if any) is a connected component of $(\overline{U})^c$.

By assumption (5.1), every fixed point has non-zero topological index. In this case, the fixed point is also called a non-degenerate fixed point. In general there are two types of such points:

(a) $p \in \text{Fix}(f)$ has topological index 1.

(b) $p \in \text{Fix}(f)$ has topological index $-1$. In this case, $p$ is a hyperbolic saddle, and both eigenvalues of $Df(p)$ are positive real numbers, one larger than 1 and the other smaller.
Now by lemma 2.26, condition (b) of definition 2.14 implies that each \( f \)-invariant open disk \( D \) has prime ends rotation number \( \rho_{\text{WE}}(f, D) \not\in \mathbb{Q} \). In particular, such a prime ends rotation number is not zero, so the sum of the indices of fixed points contained in \( D \) is equal to 1. In this way, the sum of the indices of fixed points contained in the union of all the (finitely many) maximal \( f \)-invariant open disks is positive (or zero, in case \( \mathcal{U} \) is the whole torus). By the Lefschetz fixed point formula (lemma 2.13), the sum of the indices at all the fixed points is zero. And as \((0,0)\) is an extremal point of the rotation set, \( f \) must have fixed points (see [13]). So it follows that there exists at least one negatively indexed fixed point, denoted \( Q_* \), contained in the complement of the union of these maximal open \( f \)-invariant disks. Thus, \( Q_* \) is a fixed hyperbolic saddle point, which belongs to \( \mathcal{U} \), and we have finished the proof.

For a fixed hyperbolic saddle point \( Q_* \) (or a fixed saddle-like point), let an \emph{unstable branch} (respectively, \emph{stable branch}) at \( Q_* \) be one of the connected components of \( W^u(Q_*) \setminus \{ Q_* \} \) (respectively, one of the connected components of \( W^s(Q_*) \setminus \{ Q_* \} \)). Choose a lift \( \tilde{Q}_* \) of the hyperbolic saddle point \( Q_* \), which is fixed by \( f \). We can then lift the corresponding stable and unstable branches at \( Q_* \) to those branches at \( \tilde{Q}_* \).

Before the next result we present a remark.

\textbf{Remark 5.4.} In the next proposition, in case we start with the unstable and the stable branches at an index 0 saddle-like fixed point which does not belong to any \( f \)-invariant open disk, its conclusion holds with no changes. The arguments are the same, with a difference that, when we apply lemma 2.18, we actually need the statement for saddle-like fixed points, as was explained in remark 2.19. Note also, the conditions stated in lemma 2.26 are such that in both cases it can be applied.

\textbf{Proposition 5.5.} It is not possible that some unstable branch \( \tilde{\lambda}_u \) and some stable branch \( \tilde{\lambda}_s \) at the hyperbolic saddle \( \tilde{Q}_* \), intersect.

\textbf{Proof.} Suppose by contradiction that a stable branch \( \tilde{\lambda}_s \) at \( \tilde{Q}_* \) intersects an unstable branch \( \tilde{\lambda}_u \) at \( \tilde{Q}_* \). We can then choose an intersection point \( \tilde{w} \), such that, the arc along \( \tilde{\lambda}_u \) from \( \tilde{Q}_* \) to \( \tilde{w} \) and the arc along \( \tilde{\lambda}_s \) from \( \tilde{Q}_* \) to \( \tilde{w} \) are disjoint, except at their endpoints. It follows that, the union of these two arcs bounds a topological disk \( \tilde{D} \). Now we define

\[
\tilde{D}_{\text{sat}} = \bigcup_{n \in \mathbb{Z}} \tilde{f}^n(\tilde{D}).
\]

Note that \( \tilde{D}_{\text{sat}} \) is an open and connected \( \tilde{f} \)-invariant subset of the plane.

If there exists some integer vector \((a, b) \in \mathbb{Z}^2 \setminus \{(0,0)\}\), such that

\[
\tilde{D}_{\text{sat}} \cap \left( \tilde{D}_{\text{sat}} + (a, b) \right) \neq \emptyset,
\]

then either \( \tilde{\lambda}_u \) intersects \( \tilde{\lambda}_s + (a, b) \) topologically transversely, or \( \tilde{\lambda}_u \) intersects \( \tilde{\lambda}_s - (a, b) \) topologically transversely (see definition 2.16). In both cases, it follows from lemma 2.18 that there exists a periodic orbit \( p_0 \) whose rotation vector \( \rho(f, p_0) \) is non-zero and rational, which is a contradiction with assumption (5.2).

Thus, we are left with the case when \( \tilde{D}_{\text{sat}} \) does not intersect any of its non-trivial integer translations. As before, we consider the filled open set \( \text{Fill}(\tilde{D}_{\text{sat}}) \), which is given by the union of \( \tilde{D}_{\text{sat}} \) and all the bounded connected components of its complement. It is not hard to see that \( \text{Fill}(\tilde{D}_{\text{sat}}) \) is an open topological disk, which does not intersect any of its non-zero integer translations. Thus, we can consider its projection \( D_{\text{fill}} := \pi(\text{Fill}(\tilde{D}_{\text{sat}})) \), which is an \( f \)-invariant
open disk. By lemma 2.2, Fill($\tilde{D}_{sat}$) has bounded diameter. Since $f \in G^r$, in particular there are no saddle connections, lemma 2.26 says that the prime ends rotation number $\rho_{PE}(f, \text{Fill}(\tilde{D}_{sat}))$ is irrational, and so the boundary $\partial \text{Fill}(\tilde{D}_{sat})$ does not contain any periodic point. This implies that

$$\tilde{Q}_* \in \text{Fill}(\tilde{D}_{sat}).$$

(5.10)

This is a contradiction because $Q_*$ does not belong to a fixed open disk by lemma 5.2. So $\tilde{\lambda}_w$ does not intersect any stable branch at $\tilde{Q}_*$. □

The next lemma can also be applied to a more general setting. The following remark concerns this.

**Remark 5.6.** Similar to proposition 5.5, the same conclusion of lemma 5.7 holds for an index 0 saddle-like fixed point which does not belong to an $f$-invariant open disk. The proof under this hypothesis follows the same lines of the one below.

**Lemma 5.7.** Each stable or unstable branch at $\tilde{Q}_*$ is unbounded in $\mathbb{R}^2$.

**Proof.** For definiteness, fix any unstable branch $\tilde{\lambda}_w$ at $\tilde{Q}_*$, and assume by contradiction that it is bounded.

The first claim is that, the closure $\text{cl}(\tilde{\lambda}_w)$ must intersect all the other branches at $\tilde{Q}_*$. To see the claim, assume by contradiction that $\text{cl}(\lambda_w)$ does not intersect some branch $\lambda$. Then there exists some connected component $U$ of the complement of $\text{cl}(\lambda_w)$, containing $\lambda$. Since $\lambda$ is $\tilde{f}$-invariant, so is $\tilde{U}$. Note that $\tilde{Q}_* \in \partial \tilde{U}$, and it is in fact accessible through the branch $\tilde{\lambda}$, from the interior of $\tilde{U}$. Thus, the prime ends rotation number $\rho_{PE}(\tilde{f}, \tilde{U})$ must be equal to 0. And so, lemma 2.26 implies the existence of connections between saddle-like points, something that contradicts item (b) of definition 2.14.

The second claim is that, if $\tilde{\lambda}$ is any other branch at $\tilde{Q}_*$, then $\text{cl}(\tilde{\lambda}_w) \supset \tilde{\lambda}$. To prove this claim, note first that if $\tilde{\lambda}$ is another unstable branch, then $\tilde{\lambda}$ does not intersect $\tilde{\lambda}_w$. And if $\tilde{\lambda}$ is a stable branch, then by proposition 5.5, we again obtain that $\lambda$ does not intersect $\lambda_w$. The following argument is a variation of one due to Fernando Oliveira in the area-preserving case (see lemma 2 of [34]). We include it here for completeness.

Assume by contradiction that

$$\text{cl}(\tilde{\lambda}_w) \not\supset \tilde{\lambda}. \quad (5.11)$$

Since $\text{cl}(\tilde{\lambda}_w)$ is a connected $\tilde{f}$-invariant compact subset, there is a compact simple arc $\gamma$ contained in $\tilde{\lambda}$, such that $\text{cl}(\tilde{\lambda}_w) \cap \gamma$ consists of exactly the two endpoints of $\gamma$ (and $\gamma$ minus its endpoints is free under $\tilde{f}$). Then there are two possibilities:

(a) For all non-zero integer vectors $(m, n)$, $(\text{cl}(\tilde{\lambda}_w) \cup \gamma) \cap (\text{cl}(\tilde{\lambda}_w) + (m, n)) = \emptyset$.

(b) For some $(m_0, n_0) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, $(\text{cl}(\tilde{\lambda}_w) \cup \gamma) \cap (\text{cl}(\tilde{\lambda}_w) + (m_0, n_0)) \neq \emptyset$.

In case (a), we can find a bounded connected component of the complement of $\text{cl}(\tilde{\lambda}_w) \cup \gamma$, whose boundary contains $\gamma$. We denote it as $\tilde{O}$. Then look at the set $O = \pi(\tilde{O})$ in the torus. It is not hard to see that, $O$ is itself wandering or it contains a wandering domain for $f$, which is a contradiction.

More precisely, the following well-known proposition holds:

**Proposition 5.8.** Suppose $T : S \to S$ is an orientation preserving non-wandering $C^1$ diffeomorphism of an orientable surface $S$, $K \subset S$ is a $T$-invariant contractible continuum and $\sigma$ is a branch at a hyperbolic periodic saddle. If $\sigma \cap K \neq \emptyset$, then $\sigma$ is actually contained in $K.$
Proof. Considering a iterate of $T$, we can assume that both $K$ and $\sigma$ are $T$-invariant. By contradiction, assume $\sigma$ intersects $K$, but it is not contained in $K$. Then, in $(K \cup \sigma)^r$ there exists at least one (ininitely many, actually) connected component which is an open topological disk, denoted $O$, such that $\partial O$ intersects $\sigma$ in a single connected component. Either $O$ is wandering, or for some $n > 0$, $O \cap T^n(O) = \emptyset$ for all $i \in \{1, 2, \ldots, n-1\}$ and $T^n(O) \subset O$. In the second possibility, $O \setminus \overline{T^n(O)}$ is an open set, which is also wandering. This contradiction concludes the proof. □

In case (b), either $\overline{cl(\tilde{\lambda}_u)}$ intersects $\overline{cl(\tilde{\lambda}_n) + (m_0, n_0)}$, or as $\gamma$ is contained in a branch at $\tilde{Q}$, and $\overline{cl(\tilde{\lambda}_u) + (m_0, n_0)}$ is a closed $f$-invariant set, we get that $\tilde{Q}$ belongs to $\overline{cl(\tilde{\lambda}_u) + (m_0, n_0)}$. So it is always the case that $\overline{cl(\tilde{\lambda}_u)}$ intersects $\overline{cl(\tilde{\lambda}_n) + (m_0, n_0)}$. And if we define

$$\tilde{L} := \bigcup_{k \in \mathbb{Z}} \overline{cl(\tilde{\lambda}_n) + k(m_0, n_0)}$$

(5.12)

then $\tilde{L}$ is closed, connected, $\tilde{L} + (m_0, n_0) = \tilde{L}$ and it is bounded in the direction perpendicular to $(m_0, n_0)$. Moreover, $\tilde{f}(\tilde{L}) = \tilde{L}$. But this shows, see lemma 2.8, that $\rho(\tilde{f})$ must be contained in a line of rational slope (parallel to the vector $(m_0, n_0)$), which is a contradiction with assumption (5.2). This shows our second claim.

Note that we could have started with any stable branch $\tilde{\lambda}_s$ as well. So these two claims above show that, under the assumptions (5.1) and (5.2), if one branch at the hyperbolic fixed point $\tilde{Q}_s$ is bounded, then all four branches are bounded. Moreover, in this case, they all have the same closure. This result can be found in Mather’s paper [28] when area is preserved. Similarly, for the conclusion of proposition 5.9.

Now the final arguments follow exactly from Oliveira in [34], page 582, namely we get an intersection between a stable and an unstable branch. As this homoclinic intersection contradicts proposition 5.5, all four branches at $\tilde{Q}$ are unbounded.

To conclude, we present a sketch of Oliveira’s argument used above. Let us denote the four branches at $\tilde{Q}$ by $\tilde{\lambda}_{1,u}, \tilde{\lambda}_{2,u}, \tilde{\lambda}_{1,s}$ and $\tilde{\lambda}_{2,s}$. We are assuming that $\overline{cl(\tilde{\lambda}_i)}$ is bounded and equal to the same continuum for all $i \in \{1, 2\}$ and $j \in \{u, s\}$. This implies that there exist two branches, one stable and one unstable, and a local quadrant $\text{Quad}$ at $\tilde{Q}$ adjacent to at least one of them, such that both branches accumulate on $\tilde{Q}^*$ through $\text{Quad}$. In order to see this, assume $\tilde{\lambda}_{1,u}$ accumulates on some point in $\tilde{\lambda}_{1,s}$, for instance, through the first quadrant. So, $\tilde{\lambda}_{1,u}$ accumulates on $\tilde{Q}^*$ through the first quadrant. As $\tilde{\lambda}_{1,s}$ accumulates on the whole $\tilde{\lambda}_{1,u}$, it also accumulates on $\tilde{Q}^*$ through the first quadrant. So, a picture similar to one of the possibilities in figure 3 must happen. More precisely, there always exists a Jordan curve separating the local $\tilde{\lambda}_{1,s}$ from the hatched area of $\text{Quad}$. This hatched area is the set of points inside $\text{Quad}$ whose first positive iterate does not belong to $\text{Int}(\text{Quad})$. In other words, if one follows some stable branch at $\tilde{Q}^*$ starting at the saddle, until it hits the boundary of $\text{Quad}$, this first hitting point belongs to the boundary of the hatched area.

And it is easy to see that the stable branch can not accumulate on $\tilde{Q}^*$ through $\text{Quad}$ without intersecting the unstable branch: the only way it can enter $\text{Quad}$ is through the hatched area. And it has to intersect the unstable branch in order to reach that area. □

**Proposition 5.9.** The projections of the four branches are two by two disjoint and they have the same closure, denoted as follows

$$K = \overline{cl(\pi(\tilde{\lambda}_{1,u}))} = \overline{cl(\pi(\tilde{\lambda}_{2,u}))} = \overline{cl(\pi(\tilde{\lambda}_{1,s}))} = \overline{cl(\pi(\tilde{\lambda}_{2,s}))}.$$  

(5.13)

Moreover, each connected component of the complement of $K$ is a periodic open disk.
Proof. Let us consider one branch, for instance, $\tilde{\lambda}_{1,a}$, which is unbounded in $\mathbb{R}^2$ by lemma 5.7. Then, the closure of its projection, $\text{cl}(\pi(\tilde{\lambda}_{1,a}))$, is an essential subset of $\mathbb{T}^2$. By proposition 5.3, it is in fact fully essential. By item (a) of definition 2.14 and assumption (5.1), the fixed point set of $f^n$ is finite, for all $n \geq 1$. So, theorem 2.3 implies that every connected component of the lift of the complement of $\text{cl}(\pi(\tilde{\lambda}_{1,a}))$ is a bounded $\tilde{\lambda}$-periodic disk. Thus, if $\tilde{\lambda}$ is any branch (possibly the same branch), and since $\tilde{\lambda}$ is unbounded, then $\pi(\tilde{\lambda})$ intersects $\text{cl}(\pi(\tilde{\lambda}_{1,a}))$. As $\pi(\tilde{\lambda})$ can not intersect $\text{cl}(\pi(\tilde{\lambda}_{1,s}))$ by proposition 5.8, $\pi(\lambda) \subset \text{cl}(\pi(\tilde{\lambda}_{1,a}))$. Since we have chosen the two branches arbitrarily, the proof is over. □

By proposition 5.9, for the fixed saddle point $Q_*$ in the torus, each of its four branches accumulates on all the other three branches, as well as on itself. Now we need to recall the final arguments of the proof of theorem 2 in [34]. More precisely, from page 591 to page 594 of [34], the following result can be extracted:

Theorem (from the end of [34]). Let $Q_*$ be a fixed hyperbolic saddle for a $C^1$ diffeomorphism of the torus homotopic to the identity, such that all branches at $Q_*$ are unbounded when lifted to $\mathbb{R}^2$, and the closure of each branch in $\mathbb{T}^2$ accumulates on all the four branches. Then,

$$\left(\lambda_{1,u} \cup \lambda_{2,u}\right) \cap \left(\lambda_{1,s} \cup \lambda_{2,s}\right) \neq \emptyset \quad (5.14)$$

with a topologically transverse intersection.

So, either

$$\left(\tilde{\lambda}_{1,u} \cup \tilde{\lambda}_{2,u}\right) \cap \left(\tilde{\lambda}_{1,s} \cup \tilde{\lambda}_{2,s}\right) \neq \emptyset, \quad (5.15)$$

or

$$\left(\tilde{\lambda}_{1,u} \cup \tilde{\lambda}_{2,u}\right) \cap \left(\tilde{\lambda}_{1,s} \cup \tilde{\lambda}_{2,s} + (m,n)\right) \neq \emptyset, \quad \text{for some } (m,n) \in \mathbb{Z}^2 \setminus (0,0), \quad (5.16)$$

in both cases, with topologically transverse intersections.

The first is a contradiction with proposition 5.5, and for the second case, we can use a similar argument as in the proof of proposition 5.5, to create a non-contractible periodic orbit. This is a contradiction with the assumption on the shape of $p(f)$, i.e., (5.2).

To conclude, as we did in lemma 5.7, we present a sketch of the proof of the above theorem, contained in the aforementioned pages of [34].
First, note that for any local quadrant, $Quad$ at $Q_*$, both adjacent branches accumulate on $Q_*$ through it. Actually, a stronger statement is true: for any local quadrant $Quad$, all branches accumulate on $Q_*$ through $Quad$. In order to see this, remember that, as the closure of all branches is the same, if some branch does not enter some local quadrant $Quad$ at $Q_*$, then all branches avoid $Quad$. So the complement of the continuum given by the closure of any branch would have a connected component $U$ (a periodic topological disk) containing $Quad$. But this implies that $Q_* \in \partial U$, which is a contradiction with lemma 2.26.

So, without loss of generality, let $Quad_1$ be a local quadrant contained in the first quadrant and $\lambda_{1,u}$ and $\lambda_{1,s}$ be the branches adjacent to it. Follow $\lambda_{1,u}$ starting at $Q_*$ until the first time it reaches $Quad_1$, at a point $z \in \partial Quad_1$. This could happen so that this sub arc from $\lambda_{1,u}$ whose endpoints are $Q_*$ and $z$, united with a segment from $z$ to $Q_*$, is either a contractible loop, or not. If the loop is contractible, then it bounds a disk $D$ that separates the hatched area in the left case of figure 3 from a local part of $\lambda_{1,s}$. As the only way for $\lambda_{1,s}$ to enter $Quad$ is through the hatched area, there must be an intersection between $\lambda_{1,s}$ and $\lambda_{1,u}$. Although the picture is in the torus now, if the loop is contractible, the same argument used in the plane works.

We are left to consider the case when, for any of the four branches, whenever it returns to some adjacent local quadrant $Quad_i$ ($i = 1, 2, 3, 4$), it forms a non-contractible loop (as we said, $Quad_i$ is the first local quadrant and so on, rotating counterclockwise). The situation in the universal cover is as in figure 4.

There, we consider $\tilde{\lambda}_{1,s}$ and $\tilde{\lambda}_{1,u}$ starting at $\tilde{Q}_*$ until the first point each of them has in some connected component of $\pi^{-1}(\partial Quad_1)$ and then back to some integer translate of $\tilde{Q}_*$ through a segment (these integer translates of $\tilde{Q}_*$ are denoted $r(1, U)$ for $\lambda_{1,u}$ and $r(1, s)$ for $\lambda_{1,s}$). Our first claim is: with this construction, we get a ‘web’ on the plane, whose building blocks are all the integer translates of the curvilinear rectangle, denoted $Rect$ in figure 4.

In other words, either the theorem is proved, or we can assume that $r(1, U)$ and $r(1, s)$ are not parallel. To see this, without loss of generality assume $r(1, U)$ is horizontal. Denote by $\gamma_u$...
the simple arc from $\tilde{Q}_s$ to $\tilde{Q}_s + r(1, u)$ given by the union of a connected piece of $\tilde{\lambda}_{1,u}$ from $\tilde{Q}_s$ to the first point $\tilde{z}$ in $\pi^{-1}(\partial \text{Quad}_1)$ and a linear segment from $\tilde{z}$ to $\tilde{Q}_s + r(1, u)$. Clearly,

$$\Gamma_u = \bigcup_{i \in \mathbb{Z}} (\gamma_u + i.r(1, u))$$

is an horizontal line which separates the plane into two connected components, one above and one below.

And moreover, $\Gamma_u$ separates a local part of $\tilde{\lambda}_{1,s}$ from all the translates of the hatched area in figure 5, which is the entering area for $\tilde{\lambda}_{1,s}$ in $\pi^{-1}(\partial \text{Quad}_1)$. So, if $r(1, s)$ is parallel to $r(1, U)$, there must be a topologically transverse intersection between $\tilde{\lambda}_{1,s}$ and $\lambda_{1,u} + i.r(1, u)$ for some integer $i$.

Now, back to the case when $r(1, U)$ and $r(1, s)$ are not parallel, we know that $\tilde{\lambda}_{2,s}$ and $\tilde{\lambda}_{2,u}$ are also both unbounded, so they have to leave $\text{Rect}$. If they do not intersect $\tilde{\lambda}_{1,s}$ and $\tilde{\lambda}_{1,u}$, the only possibilities are, for $\tilde{\lambda}_{2,u}$, it leaves $\text{Rect}$ through the connected components of $\pi^{-1}(\partial \text{Quad}_1)$ that contain $\tilde{Q}_s + r(1, u)$ or $\tilde{Q}_s + r(1, u) + r(1, s)$. And $\tilde{\lambda}_{2,s}$ leaves $\text{Rect}$ through the connected components of $\pi^{-1}(\partial \text{Quad}_1)$ that contain $\tilde{Q}_s + r(1, s)$ or $\tilde{Q}_s + r(1, u) + r(1, s)$.

It is easy to see that, unless both $\tilde{\lambda}_{2,s}$ and $\tilde{\lambda}_{2,u}$ leave $\text{Rect}$ through the connected component of $\pi^{-1}(\partial \text{Quad}_1)$ that contains $\tilde{Q}_s + r(1, u) + r(1, s)$, the diagram in figure 4 implies that there must be an intersection between $\lambda_{2,s}$ and $\tilde{\lambda}_{2,u}$ and we are done.

So, assume this is the case. Now we fall in the situation described in figure 6.

In this new figure we constructed another topological open disk, called $\text{Rect'}$, defined in the following way:

**Definition of $\text{Rect'}$.** For each of the four branches $\tilde{\lambda}_{1,s}$, $\tilde{\lambda}_{1,u}$, $\tilde{\lambda}_{2,s}$ and $\tilde{\lambda}_{2,u}$, as before, we consider a connected arc starting at $\tilde{Q}_s$ and ending at the first point $\tilde{z}_{i,j}$ in some connected component of $\pi^{-1}(\partial \text{Quad}_1)$ (for $i \in \{1, 2\}$ and $j \in \{s, u\}$). To this arc, we add a linear segment whose endpoints are $\tilde{z}_{i,j}$ and $\tilde{Q}_s + r(i, j)$, where $r(2, s)$ and $r(2, U)$ are defined exactly as $r(1, s)$ and $r(1, U)$.

For each $i \in \{1, 2\}$ and $j \in \{s, u\}$, the union of the previous subarc contained in $\tilde{\lambda}_{i,j}$ with the linear segment from $\tilde{z}_{i,j}$ to $\tilde{Q}_s + r(i, j)$ is denoted $\tilde{\alpha}_{i,j}$. $\text{Rect'}$ is then, given by:

$$\text{Rec'} = \tilde{\alpha}_{1,u} \cup (\tilde{\alpha}_{2,u} + r(1, u)) \cup (\tilde{\alpha}_{1,u} + r(1, u) + r(1, s))$$

$$\cup (\tilde{\alpha}_{1,s} + r(1, u) + r(1, s)) \cup (\tilde{\alpha}_{2,s} + r(1, s)) \cup \tilde{\alpha}_{1,s}$$

**Figure 5.** The case when $r(1, u)$ and $r(1, s)$ are parallel.
As explained before, we are assuming that $r(2, U) = r(2, s) = r(1, U) + r(1, s)$, otherwise there already existed topologically transverse intersections between stable and unstable branches, and the theorem is proved.

Again, as $\tilde{\lambda}_s$ and $\tilde{\lambda}_u$ are both unbounded, they have to leave $\text{Rect}'$. If they do not intersect $\tilde{\lambda}_s$ and $\tilde{\lambda}_u$, from the position of the exits and entrances in $\text{Rect}'$, there must be an intersection between $\tilde{\lambda}_s$ and $\tilde{\lambda}_u$, (we are using the fact that unstable branches leave $\text{Rect}'$ through one of the exits and stable branches leave $\text{Rect}'$ through one of the entrances).

A final remark is that figures 3, 4 and 6 were taken from [34]: we just adapted them to our notation.

6. A broader class of (non generic) diffeomorphisms

In the proof of theorem 3.1, we keep the fixed points away from the support of the perturbation. Thus, the rotation set after the perturbation still contains the point $(0, 0)$. On the other hand, it seems possible that $(0, 0)$ will be ‘mode locked’ in the following sense. Possibly, for all sufficiently small perturbations, $(0, 0)$ is not contained in the interior of the perturbed rotation set. This intuition comes from the phenomenon called rational mode locking. The case when the rotation set has non-empty interior was treated in [4]. One of the theorems proved there states that rational mode locking happens under some conditions that are satisfied for generic one-parameter families.
This section has two objectives. First, we prove several results describing the dynamics of diffeomorphisms $f \in \mathcal{K}'$, which together, imply theorem 1.3. And then, using the previous results and some delicate topological arguments, we show the existence of lots of Brouwer lines in the universal cover, which are lifts of essential loops in the torus for all possible homotopy classes. In the end of the section we explain how theorem 1.4 in the introduction can be deduced from the existence of Brouwer lines.

We start by describing the dynamics in $\mathcal{K}'$. In this whole section, $f \in \mathcal{K}'$ and $\tilde{f}$ is a lift of $f$ whose rotation set is the segment from $(0,0)$ to a totally irrational point $(\alpha, \beta)$.

**Proposition 6.1.** Every periodic point $p$ is indeed a fixed point, with topological index 0, and the local dynamics around $p$ can be described explicitly as in lemma 2.21. In particular, $p$ admits exactly one stable and one unstable branch.

**Proof.** By the assumption on the shape of the rotation set $\rho(\tilde{f})$, all the periodic orbits of $f$ must be contractible, that is, they lift to periodic orbits of $\tilde{f}$. Suppose by contradiction that there exists a periodic point which is not fixed, or there exists a fixed point which does not have topological index 0. In the first case, by lemma 2.11, there exists some fixed point with positive topological index. On the other hand, it is easy to see from the definition of $\mathcal{K}'$ that the indices at fixed points can only assume one of the following values: $-1, 0$ or $1$. Observing lemma 2.13, as in both possibilities above there exists fixed points with non-zero index, there must always exist at least one fixed point which has topological index $-1$. And from the definition of $\mathcal{K}'$, periodic points with negative indexes are hyperbolic saddles. As there exists at least one fixed hyperbolic saddle point $Q$, lemma 5.2 implies that there exists a fixed hyperbolic saddle point $\tilde{Q}$, contained in $\overline{\mathcal{U}}$, where $\mathcal{U}$ is defined in expression (5.5). To see this, note that if $Q$ belongs to $\mathcal{U}$, then there is nothing to prove. And if it does not, then it belongs to some maximal open invariant disk (a connected component of the complement of $\mathcal{U}$), which has irrational prime ends rotation number (as before, because there are no connections between saddle-like points, see lemma 2.26). So, the sum of the indices of fixed points in the complement of $\overline{\mathcal{U}}$ is positive (it is 1 for each component). Therefore, as in the proof of lemma 5.2, there must be a saddle in $\overline{\mathcal{U}}$. As maps in $\mathcal{K}'$ have no connections between saddle-like periodic points, lemma 2.26 implies that proposition 5.5, lemma 5.7 and proposition 5.9 are also valid in this setting. So, from the last theorem of the previous section (the one taken from Oliveira’s paper), we get a contradiction with proposition 5.5. So every fixed point has topological index 0 and there are no other periodic points. Since $f \in \mathcal{K}'$, the local dynamics around a fixed point is given by lemma 2.21. □

The next lemma is corollary 1.5. Note that it depends on theorem 3.2.

**Lemma 6.2 (corollary 1.5 restated).** The lift $\tilde{f}$ has bounded deviation along the direction $-(\alpha, \beta)$. Equivalently, there exists $M > 0$, such that for any $\tilde{x}$ and $n \geq 1$,

$$pr_{-(\alpha,\beta)}(\tilde{f}^n(\tilde{x}) - \tilde{x}) \leq M.$$  \hfill (6.1)

**Proof.** By proposition 2.24, for any $\tilde{g} \in \text{Homeo}_{0}(\mathbb{T}^2)$ which is a sufficiently small perturbation of $\tilde{f}$, it is not possible that $\rho(\tilde{g})$ contains $(0,0)$ in its interior. Then by theorem 3.2, $\tilde{f}$ must have bounded deviation along $-(\alpha, \beta)$. □

**Lemma 6.3.** For any $\tilde{f}$-fixed point $\tilde{p}$, its stable and unstable branches are both unbounded. Their projections to the torus do not intersect. Moreover, the projection of each branch is dense in the torus.

**Proof.** A first observation is that there is no periodic open disk. If such a disk existed, then from our hypotheses, its prime ends rotation number would be irrational. So $f$ would have...
periodic points with positive index (see the end of the proof of lemma 5.2), something that is not allowed by proposition 6.1.

Fix some fixed point \( p \) in the plane that lifts \( p \). As there are no periodic open disks, the stable and the unstable branches at \( \tilde{p} \) do not intersect (see remark 5.4). So lemma 5.7 implies that both the stable and the unstable branches at \( \tilde{p} \) are unbounded (see remark 5.6).

Now we show that \( W'(p) \) is dense in \( \mathbb{T}^2 \) (a similar argument works for \( W^u(p) \)). Since the lift \( W'(\tilde{p}) \) is unbounded, the closure \( \overline{W}(p) \) must be an essential subset of \( \mathbb{T}^2 \). So, proposition 5.3 implies that it is fully essential.

In this way, each connected component of its complement is a periodic open disk. As there are none, \( W'(p) \) is dense in \( \mathbb{T}^2 \).

Finally, if the stable and unstable branches at \( p \) intersect, then for some \( \tilde{p} \) lift of \( p \), either its stable and unstable branches intersect, and we know they can not, or the unstable branch at \( \tilde{p} \) intersects (maybe in a tangency) the stable branch at \( \tilde{p} + (m, l) \) for some non-zero integer vector \((m, l)\). In this second case, there exists a Jordan curve in the plane, which is given by the union of two arcs: one contained in the unstable branch at \( \tilde{p} \) and the other contained in the stable branch at \( \tilde{p} + (m, l) \). As this Jordan curve bounds a disk and \( W'(p) \) is dense in \( \mathbb{T}^2 \), we get that the stable branch at some integer translate \( \tilde{p} + (m', l') \) has a topologically transverse intersection with the unstable branch at \( \tilde{p} \). From what we did above, \((m', l')\) can not be equal to \((0, 0)\). And if it is not \((0, 0)\), then lemma 2.18 and remark 2.19 give a contradiction. So, there are no homoclinic intersections in the torus. □

The next lemma shows that \( f \) also does not admit heteroclinic intersections.

**Lemma 6.4.** For any two fixed points \( p_1 \) and \( p_2 \), we have

\[
W^u(p_1) \cap W^s(p_2) = \emptyset.
\]  
(6.2)

**Proof.** Suppose \( f \) admits some fixed points \( p_1 \) and \( p_2 \), and

\[
W^u(p_1) \cap W^s(p_2) \neq \emptyset.
\]  
(6.3)

As there are no saddle-like connections, it is possible to find lifts \( W^u(\tilde{p}_1) \) and \( W^s(\tilde{p}_2) \) of these branches, such that their intersection is non-empty and there is some Jordan curve, which is the union of one sub-arc of \( W^u(\tilde{p}_1) \) and one sub-arc of \( W^s(\tilde{p}_2) \). Moreover, this Jordan curve bounds a topological disk, denoted \( \tilde{U} \), whose projection \( U = \pi(\tilde{U}) \) is a proper open subset of \( \mathbb{T}^2 \).

Since both \( W'(p_1) \) and \( W^s(p_2) \) are dense in \( \mathbb{T}^2 \), each of them must intersect \( U \). So, there are homoclinic intersections, which do not exist by the previous lemma. This contradiction ends the proof. □

The goal now is to show that both invariant branches at a fixed point tend to infinity. First we present some technical propositions.

**Proposition 6.5.** Let \( \tilde{z} \) be a point in \( \mathbb{R}^2 \). Then the \( \omega \)-limit set of \( \tilde{z} \), denoted \( \omega(\tilde{z}) \), is either a single fixed point or empty. And when it is a point, \( \tilde{z} \) belongs to the stable branch at that point.

**Proof.** Either \( \omega(\tilde{z}) \) is empty, a singleton, or it has more than one point. If it contains more than one point, then some \( \tilde{w} \) in \( \omega(\tilde{z}) \) is not fixed, because each fixed point is isolated. So \( \tilde{w} \) is contained in some disk \( U \), such that,

\[
\tilde{f}(U) \cap U = \emptyset,
\]  
(6.4)
And lemma 2.11 implies that \( \tilde{f} \) admits a fixed point with positive topological index, a contradiction with proposition 6.1.

If \( \omega(\tilde{z}) \) is a singleton, say, equal to \( \{\tilde{r}\} \), then \( \tilde{r} \) is necessarily a fixed point. So \( \tilde{z} \) belongs to the stable branch at the point \( \tilde{r} \) because it converges to \( \tilde{r} \) under positive iterates. \( \square \)

**Proposition 6.6.** Let \( \tilde{z} \) be a point in \( \mathbb{R}^2 \) such that \( \omega(\tilde{z}) \) is empty. Then given a compact subset \( K \) of \( \mathbb{R}^2 \), there exists \( \varepsilon > 0 \) and \( N > 0 \) such that \( f^n(B_{\varepsilon}(\tilde{z})) \cap K = \emptyset \), for all \( n \geq N \).

**Proof.** By contradiction, assume there exists a sequence \( n_i \to +\infty \) and \( \tilde{w}_i \to \tilde{z} \) such that \( f^{n_i}(\tilde{w}_i) \in K \) for all \( i \geq 0 \). As \( \omega(\tilde{z}) \) is empty, the sequence \( \{f^n(\tilde{z})\}_{n \geq 1} \) converges to infinity as \( n \to +\infty \). By theorem 2.10 and lemma 6.2, \( f \) has bounded deviation along the three directions \(-\alpha, \beta), (-\beta, \alpha) \) and \( (\beta, -\alpha) \). So the sequence \( f^n(\tilde{z}) \) converges to infinity along the direction \((\alpha, \beta)\).

In particular, for some large \( n_0 \),

\[
\inf_{\tilde{w} \in K} \left( \text{pr}_{(\alpha, \beta)}(f^{n_0}(\tilde{z}) - \tilde{w}) \right) > 2M + 1,
\]

where \( M > 0 \) comes from estimate (6.1).

Then there exists some small disk \( B \) centred at \( \tilde{z} \), such that, for every point \( \tilde{b} \in B \), the above estimate also holds true. Now, we can choose a sufficiently large \( i \), with \( \tilde{w}_i \in B \), and \( n_i > n_0 \). Thus,

\[
\text{pr}_{(\alpha, \beta)} \left( f^{n_0}(\tilde{w}_i) - f^{n_0}(\tilde{w}_i) \right) \\
\geq \inf_{\tilde{w} \in K} \left( \text{pr}_{(\alpha, \beta)}(f^{n_0}(\tilde{w}_i) - \tilde{w}) \right) \\
\geq 2M + 1.
\]

As this is a contradiction with lemma 6.2, the proof is completed. \( \square \)

**Lemma 6.7.** Let \( \tilde{p} \) be a fixed point (for \( \tilde{f} \)). Then both its stable and unstable branches intersect every compact set in a closed subset. More precisely, for the unstable branch \( W^u(\tilde{p}) \setminus \{\tilde{p}\} \) for example, if \( \lambda \subset W^u(\tilde{p}) \setminus \{\tilde{p}\} \) denotes the closure of a fundamental domain, then for any compact set \( K \), the set \( \{n \geq 1 | f^n(\lambda) \cap K \neq \emptyset \} \) is finite.

**Proof.** Consider the unstable branch \( W^u(\tilde{p}) \setminus \{\tilde{p}\} \) and choose a fundamental domain contained in it, whose closure we denote by \( \lambda \). Suppose by contradiction that there exists some compact set \( K \subset \mathbb{R}^2 \), an integer sequence \( n_i \to +\infty \), and a sequence \( \tilde{q}_i \in \lambda \), such that

\[
\tilde{f}^{n_i}(\tilde{q}_i) \in K.
\]

By extracting a subsequence if necessary, we can assume \( \tilde{q} \in \lambda \) is the limit point of the sequence \( \{\tilde{q}_i\}_{i \geq 1} \). Considering the \( \omega \)-limit set of \( \tilde{q} \), proposition 6.5 implies that either \( \omega(\tilde{q}) \) is empty or a singleton. In case \( \omega(\tilde{q}) \) is a singleton, say, equal to \( \{\tilde{r}\} \), then \( \tilde{r} \) is necessarily a fixed point and \( \tilde{q} \) belongs to its stable branch, that is, there is an heteroclinic point, a contradiction with lemma 6.4.

So, \( \omega(\tilde{q}) \) is empty and proposition 6.6 gives a contradiction with expression (6.10), something that finishes the proof. \( \square \)
**Proof of theorem 1.3.** The proof now follows easily from proposition 6.1, lemmas 6.3, 6.4, 6.7 and theorem 2.10.

**Theorem 6.8.** Let $\tilde{f}$ denote some lift of some $f \in K'$, and suppose $\rho(\tilde{f})$ is the line segment from $(0,0)$ to $(\alpha, \beta)$. Then, for any coprime integer pair $(a, b) \neq (0, 0)$, there exists a torus loop $\ell = \ell_{(a,b)}$, which can be lifted to an $f$-Brouwer line $\tilde{\ell}$, such that $\ell + (a, b) = \tilde{\ell}$.

**Proof.** Up to a change of coordinates and/or considering $f^{-1}$ if necessary, we reduce to the case when $(a, b) = (0, 1)$ and $\alpha > 0$.

**Proposition 6.9.** There exists an oriented properly embedded curve $\tilde{\gamma} \subset \mathbb{R}^2$, with the following properties.

(a) $\tilde{\gamma} + (0, 1) = \tilde{\gamma}$, and $\tilde{\gamma}$ is oriented in the direction $(0, 1)$.
(b) $\tilde{\gamma}$ does not contain any $\tilde{f}$-fixed points.
(c) Let $R(\tilde{\gamma})$ denote the unbounded complementary domain to the right of $\tilde{\gamma}$. For any $\tilde{f}$-fixed point contained in $R(\tilde{\gamma})$, its unstable branch does not intersect $\tilde{\gamma}$.
(d) Analogously, let $L(\tilde{\gamma})$ denote the unbounded complementary domain to the left of $\tilde{\gamma}$. For any $\tilde{f}$-fixed point contained in $L(\tilde{\gamma})$, its stable branch does not intersect $\tilde{\gamma}$.

**Proof.** First, we remember that there exist constants $M^* > 0$ given by theorem 2.10 and $M > 0$ defined in expression (6.1), such that for any fixed point $\tilde{p}$, $\tilde{p}$ is contained in the $M^*$-neighbourhood of a straight line of slope $\beta/\alpha$, containing $\tilde{p}$, and moreover, if $\tilde{\lambda}$ is a stable (or unstable) branch and $V$ is the vertical line through $\tilde{p}$, then $\tilde{\lambda}$ is contained in $\text{cl}(V_L) + (M \cos(\theta), 0)$ (or $\text{cl}(V_R) - (M \cos(\theta), 0)$), where $V = V_R \cup V_L$ and $\theta$ is the angle between the horizontal line and the vector $(\alpha, \beta)$.

Now, start with a vertical line $\ell$, oriented upwards, which does not contain any $\tilde{f}$-fixed point. The complement, $\ell^c$, consists of two unbounded connected components. Denote by $R(\ell)$ (respectively, $L(\ell)$) the right component (respectively, the left component). Let $O_-$ (respectively, $O_+$) be the set given by the union of the stable branches (respectively, unstable branches) of all the $\tilde{f}$-fixed points belonging to $L(\ell)$ (respectively, $R(\ell)$). We claim that both $O_-$ and $O_+$ are closed sets. The arguments are similar, so it suffices to prove the claim for $O_-.

Choose a closed disk $B$ of radius 1 intersecting $O_-$. By theorem 2.10, lemmas 6.2 and 6.7, there are only finitely many $\tilde{f}$-fixed points in $L(\ell)$, whose stable branches intersects $B$. Moreover, the intersection of each such stable branch with $B$ is a closed set. So $O_- \cap B$ is closed. Therefore, $O_-$ is closed.

It is clear that $(O_-)^c$ has a connected component which is unbounded to the right. More precisely, this component contains some translated domain $R(\ell) + (M \cos(\theta), 0)$, where $M$ and $\theta$ are as above.

Suppose by contradiction that $(O_-)^c$ is not connected. Then there exists a connected component $C$, which is contained in $L(\ell) + (M \cos(\theta), 0)$. Observe that $C$ is open and its boundary is contained in $O_-$. Recall that the unstable branch of every $f$-fixed point is dense in $T^2$. So, it follows that, the unstable branch of some $f$-fixed point intersects $C$. Since this branch is unbounded to the right, it must intersect the boundary of $C$, which is a contradiction, because stable and unstable branches do not intersect. So, $(O_-)^c$ is connected. The same conclusion holds for $(O_+)^c$.

Now consider the one point compactification of the plane, identified with $S^2$. The two closed sets $O_-$ and $O_+$ lift to $\tilde{O}_-$ and $\tilde{O}_+$, respectively, which are connected closed subsets, because every stable and unstable branch lifts to some closed set containing the point $\infty \in S^2$. Clearly, $\tilde{O}_- \cap \tilde{O}_+ = \{ \infty \}$. By lemma 2.1, the complement of $\tilde{O}_- \cup \tilde{O}_+$ is an open connected subset of...
Figure 7. Stable branches of fixed points in $\mathcal{L}(\ell)$, unstable branches of fixed points in $\mathcal{R}(\ell)$ and the curve $\gamma$

$\mathbb{R}^2$. Note that, if a point $\tilde{z} \in (O_- \cup O_+)\gamma$, then $\tilde{z} + (0, k) \in (O_- \cup O_+)\gamma$ for any $k \in \mathbb{Z}$, because of the relations $O_- = O_- + (0, 1)$ and $O_+ = (O_+) + (0, 1)$.

So, we can choose an arc $\delta$ connecting $\tilde{z}$ and $\tilde{z} + (0, 1)$ such that

$$\delta \cap (O_- \cup O_+) = \emptyset$$

and

$$\delta \cap (\delta + (0, 1))$$

contains exactly one point.

Therefore, the union

$$\gamma := \bigcup_{i \in \mathbb{Z}} (\delta + (0, i))$$

is a properly embedded real line, which satisfies all four properties, see figure 7.

□

Lemma 6.10. Let $\gamma$ be obtained from proposition 6.9. For any fixed point $\tilde{q} \in \mathcal{R}(\gamma)$, there exists a small closed neighbourhood $M$ containing $\tilde{q}$, such that $\{f^n(M)\}_{n \geq 0} \subset \mathcal{R}(\gamma)$. In particular, the forward iterates of $M$ do not intersect $\gamma$. 

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Proof. Fix some fundamental domain \( \tilde{\lambda} \) of \( W^u(\tilde{q}) \) very close to \( \tilde{q} \), whose endpoints are \( \tilde{y} \) and \( \tilde{f}(\tilde{y}) \). As \( W^u(\tilde{q}) \) is unbounded to the right, there exists \( N > 0 \) such that \( \text{dist}(\tilde{f}^N(\tilde{\lambda}), \tilde{\gamma}) > 2M + 2C_{\tilde{f}} \), where \( M \) is the constant obtained in (6.1), and \( C_{\tilde{f}} \) is given by:

\[
C_{\tilde{f}} = \max_{\tilde{z} \in \mathbb{R}^2} \| \tilde{f}(\tilde{z}) - \tilde{z} \|.
\]  

(6.14)

Note that \( \bigcup_{n=0}^{\infty} \tilde{f}^n(\tilde{\lambda}) \) does not intersect \( \tilde{\gamma} \). We can choose a small open neighbourhood \( V \) of \( \tilde{\lambda} \), such that \( \bigcup_{n=0}^{N} \tilde{f}^n(V) \) is sufficiently close to \( \bigcup_{n=0}^{N} \tilde{f}^n(\tilde{\lambda}) \), so that it does not intersect \( \tilde{\gamma} \) and \( \text{dist}(\tilde{f}^N(V), \tilde{\gamma}) \) is also larger than \( 2M + 2C_{\tilde{f}} \). Observing lemma 6.2, we can also ensure that \( \tilde{f}^n(V) \) does not intersect \( \tilde{\gamma} \), for all \( n > N \). Finally, choose a small neighbourhood \( \tilde{M} \) of \( \tilde{q} \), such that, for every point in \( \tilde{M} \), either it belongs to the stable branch of \( \tilde{q} \), or it has a forward iterate belonging to \( V \) (see figure 8 for the choices of these neighbourhoods). In fact, \( \bigcup_{n=0}^{\infty} \tilde{f}^n(\tilde{M}) \subset \tilde{M} \cup (\bigcup_{n=0}^{\infty} \tilde{f}^n(V)) \). Therefore, all non-negative iterates of \( \tilde{M} \) can not leave \( \mathcal{R}(\tilde{\gamma}) \). \( \Box \)

Lemma 6.11. Let \( \tilde{\gamma} \) be the curve obtained in proposition 6.9. Then there exists a positive integer \( N \) such that,

\[
\tilde{f}^N(\tilde{\gamma}) \bigcap \tilde{\gamma} = \emptyset.
\]  

(6.15)

Proof. Suppose by contradiction that there exists some sequence of points \( \tilde{z}_n \in \tilde{\gamma} \) such that \( \tilde{f}^n(\tilde{z}_n) \in \tilde{\gamma} \). Noticing item (a) of proposition 6.9, we can choose all the points \( \tilde{z}_n \) in a compact fundamental domain of \( \tilde{\gamma} \), denoted \( K \). In particular, they have an accumulation point \( \tilde{z}_* \in K \). Up to extracting a subsequence, simply assume \( \tilde{z}_n \to \tilde{z}_* \).

Moreover, from theorem 2.10, there exists a constant \( M^* > 0 \) such that, for all integers \( n > 0 \), \( \tilde{f}^n(K) \cap \tilde{\gamma} \) is contained in the \( M^* \)-neighbourhood of \( K \) in \( \tilde{\gamma} \).

By theorem 2.10 and lemma 6.2, the forward orbit \( \{ \tilde{f}^n(\tilde{z}_*) \}_{n \geq 0} \) is bounded in three directions \((-\alpha, -\beta), (\beta, -\alpha), (-\beta, \alpha))\). Then, either \( \{ \tilde{f}^n(\tilde{z}_*) \}_{n \geq 0} \) is unbounded in the direction \((\alpha, \beta)\), or it is bounded. We seek contradictions in both cases.

If \( \{ \tilde{f}^n(\tilde{z}_*) \}_{n \geq 0} \) is bounded, then proposition 6.5 implies that \( \omega(\tilde{z}_*) \) must be a single fixed point. So \( \tilde{z}_* \) belongs to some stable branch \( W^s(\tilde{q}) \), for some fixed point \( \tilde{q} \in \mathcal{R}(\tilde{\gamma}) \). By
lemma 6.10, there exists a compact neighbourhood $\tilde{M}$ of $\tilde{q}$, whose forward iterates do not intersect $\tilde{\gamma}$. And there exists a positive integer $m_0$, such that $\bigcup_{n=0}^m f^{-k}(\tilde{M})$ contains some neighbourhood $N$ of $\tilde{z}_a$. This is a contradiction, because for all sufficiently large integers $m \geq m_0$, $\tilde{z}_m \in N$ and $f^m(\tilde{z}_m) \in \tilde{\gamma}$.

The other case is when the orbit of $\tilde{z}_a$ is unbounded, equivalently $\omega(\tilde{z}_a) = \emptyset$. Proposition 6.6 implies that there exists $\varepsilon > 0$ and $N > 0$ such that
\[ \tilde{f}^n(B_r(\tilde{z}_a)) \cap \text{cl}(M^* - \text{neighbourhood of } K) = \emptyset, \quad \text{for all } n \geq N. \]

As this contradicts the choice of the sequence $\tilde{z}_a \to \tilde{z}_a$, the lemma is proved. $\square$

For the oriented curve $\tilde{\gamma}$ from proposition 6.9 above, $\mathcal{R}(\tilde{\gamma})$ denotes the unbounded connected component of $(\tilde{\gamma})^f$ in the direction of $(\alpha, \beta)$. In lemma 6.11, we have obtained the integer $N$ such that $\tilde{f}^N(\tilde{\gamma}) \subset \mathcal{R}(\tilde{\gamma})$.

The following is a standard argument. Consider the finite union of curves,
\[ Q := \bigcup_{j=0}^{N-1} f^j(\tilde{\gamma}). \]  

(6.16)

Clearly, the complement of $Q$ has a component which is unbounded in the direction of $-(\alpha, \beta)$. If $\tilde{\ell}$ is the boundary of this component, then $f(\tilde{\ell}) \cap \tilde{\ell} = \emptyset$. This $\tilde{\ell}$ is clearly the lift of a vertical loop in the torus. And so the proof of theorem 6.8 is over. This construction appears in detail, for instance in proposition 3.1 of [7].

We close this section by restating and proving the remaining part of theorem 1.4.

**Theorem 6.12 (remains of theorem 1.4).** Let $f$ denote some lift of $f \in \mathcal{K}'$, and $\rho(\tilde{f})$ be the line segment from $(0,0)$ to $(\alpha, \beta)$. Let $\gamma$ be any straight line passing through $(0,0)$, which does not contain $\rho(\tilde{f})$. Then there exists $\varepsilon_0 > 0$ such that for any $\tilde{g} \in \text{Homeo}_0(\mathbb{T}^2)$, which is $C^0, \varepsilon_0$-close to $\tilde{f}$, the rotation set $\rho(\tilde{g})$ does not intersect the connected component of $\gamma^f$ which does not intersect $\rho(\tilde{f})$.

**Proof.** Choose two reduced integer vectors, $(a, b)$ and $(a', b')$, with the following properties.

(a) The two rays from $(0,0)$ in the directions $(a, b)$ and $(a', b')$ define a closed cone $C$ which contains the vector $(\alpha, \beta)$ in its interior.

(b) The interior of $C$ is contained in one of the connected components of $\gamma^f$.

By theorem 6.8, there are two $\tilde{f}$-Brouwer lines $\tilde{\ell}_1$ and $\tilde{\ell}_2$, such that $\tilde{\ell}_1 + (a, b) = \tilde{\ell}_1$, and $\tilde{\ell}_2 + (a', b') = \tilde{\ell}_2$. Since both $\tilde{\ell}_1$ and $\tilde{\ell}_2$ are lifts of simple closed curves in $\mathbb{T}^2$, there exists $\varepsilon_0$, such that, for any $\tilde{g} \in \text{Homeo}(\mathbb{T}^2)$, with $\text{dist}_{C^0}(\tilde{g}, \tilde{f}) < \varepsilon_0$, those two lines $\tilde{\ell}_1$ and $\tilde{\ell}_2$ are still Brouwer lines for $\tilde{g}$. This implies that $\rho(\tilde{g}) \subset C$. In particular, $\rho(\tilde{g}) \setminus \{(0,0)\}$ is contained in the connected component of $\gamma^f$ which contains $\rho(\tilde{f}) \setminus \{(0,0)\}$.

$\square$

7. Unbounded deviations

In this section, we show the following theorem.

**Theorem 7.1 (theorem 1.6 restated).** Suppose $\tilde{f}$ is a lift of some $f \in \mathcal{K}'$, and $\rho(\tilde{f})$ is a segment from $(0,0)$ to a totally irrational point $(\alpha, \beta)$. Assume further that $f$ preserves a foliation on $\mathbb{T}^2$. Then $\tilde{f}$ has unbounded deviation along the direction $(\alpha, \beta)$.  

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Proof. Let us assume by contradiction that there exists $M > 0$, such that
\[
\sup_{\tilde{x} \in \mathbb{R}^2, n \geq 1} \text{pr}_{(\alpha, \beta)} \left( \tilde{f}^n(\tilde{x}) - \tilde{x} - n(\alpha, \beta) \right) \leq M. \tag{7.1}
\]
Recalling definition 4.4, $S_{(\alpha, \beta)}$ is the closure of the union of the support of all the $f$-invariant ergodic probability measures whose average rotation vector for $\tilde{f}$ is $(\alpha, \beta)$. Then, lemma 4.5 shows that for any lift $\tilde{x} \in \mathbb{R}^2$ of some $x \in S_{(\alpha, \beta)}$, and any $n \geq 1$,
\[
\text{pr}_{(\alpha, \beta)} \left( \tilde{f}^n(\tilde{x}) - \tilde{x} - n(\alpha, \beta) \right) \geq -M. \tag{7.2}
\]
Clearly, for any fixed point $p$, there exists a small disk $B$ containing $p$, such that for any point in $B$, its $f$-iterates will remain close to $p$ for a long time, both in the future and in the past. Expression (7.2) implies immediately that, when $B$ is sufficiently small, the whole orbit of an arbitrary point in $S_{(\alpha, \beta)}$ does not intersect $B$.

Choose a fixed point $p$. Since $f$ preserves the foliation $\mathcal{F}$, then the leaf $\mathcal{F}(p)$ containing $p$ must be the union of $W^s(p)$ and $W^u(p)$. Choose a local leaf $L \subset \mathcal{F}(p)$, which connects some point $y \in W^s(p)$ and $y' \in W^u(p)$. Let $V$ be an open neighbourhood of $L$ such that for any local leaf in $V$, its forward and backward iterates under $f$ also intersect $V$. Choose two small arcs $\gamma$ and $\gamma'$, both contained in $V$, transverse to the local foliation restricted to $V$, such that the arc $\gamma$ connects $y$ to a point $x$ and $\gamma'$ connects $y'$ to a point $x'$. Moreover, $f(\gamma)$ and $f^{-1}(\gamma')$ are also both contained in $V$ and $x$ and $x'$ bound a local leaf $\theta^+$. It is also convenient to choose $\theta^+$ so that it belongs either to $W^s(p)$ or $W^u(p)$. This is possible because both branches are dense in $T^2$, see lemma 6.3. Denote the closed region bounded by $\gamma, \theta^+, \gamma', L$ as $K$ (see figure 9). Note that $K$ can be chosen arbitrarily close to $L$.

The claim is that, some local leaf in $K$, which is contained in $W^s(p)$ or $W^u(p)$, and different from $L$, must contain a fundamental domain of $W^s(p)$ or $W^u(p)$, that is, some sub-arc connecting a point and its image. Suppose by contradiction that this is not true.

Then one of the following cases must happen (see figure 9).

(a) $f(\theta^+)$ intersects $\theta^+$.
(b) $f(\theta^+)$ is above $\theta^+$.
(c) $f(\theta^+)$ is below $\theta^+$.

If case (a) happens, since $\theta^+$ belongs to $W^s(p)$ or $W^u(p)$, then it contains a fundamental domain of $W^s(p)$ or $W^u(p)$ and we are done.
Up to considering the backward dynamics and exchanging the roles of stable and unstable branch, we can simply assume $f(\theta^+)$ is below $\theta^+$. Then, $f(\gamma)$ is contained in $K$, provided the region is chosen sufficiently close to $L$.

Since $W^s(p)$ is dense, we can follow it until the first time it enters the region $K$. Denote by $z \in W^s(p)$ the first entering point ($z \in \gamma'$). The local leaf containing $z$, denoted $T$, intersects $\gamma$ at a point $q$. If $f^{-1}(z) \in T$, then we find the fundamental domain in $T$. If $f^{-1}(z) \notin T$, then $f(q) \in K$, and this contradicts the fact that $z$ is the first returning point to $K$ along $W^s(p)$.

So, there is a fundamental domain of $W^s(p)$ contained in some local leaf in $K$, other than $L$. Now we pick a lift $\tilde{p}$ of $p$, and consider the lifted leaf $\tilde{\gamma}$ containing $W^s(\tilde{p})$ and $W^u(\tilde{p})$. By previous paragraphs, there is some integer $(a, b)$ such that, the curves $\tilde{\gamma}$ and $\tilde{\gamma} + (a, b)$ bound an infinite strip $\tilde{H}$, whose union with these two curves covers $\mathbb{T}^2$. Moreover, we can find a small fundamental domain for $f$ restricted to $\tilde{H}$, namely $\tilde{K}' \subset \tilde{K}$, such that for any point $\tilde{z} \in \tilde{H}$ whose orbit is positively and negatively unbounded, its orbit must intersect $\tilde{K}'$ (see figure 10).

On the other hand, we can choose $K \subset B$, where the disk $B$ was obtained at the beginning of the proof. Therefore, $K' = \pi(\tilde{K}') \subset K \subset B$ intersects the orbit of any chosen point in $S(\alpha, \beta)$ (one whose orbit is unbounded both in the future and past). And this is a contradiction with the fact that $S(\alpha, \beta)$ avoids $B$. $\square$

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