Optimization method of acoustic measurement station placement for wide-area miss distance

Yulong Wang1, 2, Xiaoxi Lu1, Yanhong Liu1, Liang Zhang1 and Chuangfei Wei1

1Huayin Ordnance Test Center, Huayin, Shaanxi, 714200, China
2Email: 21268678@qq.com

Abstract. This paper focuses on the issue of how to select the optimal deployment scheme for the dispersal of miss distance during acoustic rendezvous measurements. Firstly, the measurement theory of acoustic rendezvous is introduced, and then the error model of acoustic rendezvous measurement is deduced and theoretical analysis is carried out. Then, an error evaluation index is established based on the dispersion characteristics of miss distance. Finally, simulation experiments are conducted for the miss distance obeying uniform distribution and normal distribution. Simulation experiments verify the correctness of the method in this paper, which has important guiding significance for the miss distance test.

1. Introduction
The measurement of miss distance of conventional weapons and artillery weapons when shooting at targets is an important parameter for measuring the performance of artillery weapon systems [1]. It is an important basis for evaluating the accuracy of weapon firing which directly affects the preparation and identification of weapon firing tables. Acoustic measurement is a major testing technique in engineering application. The forward intersection is mainly used for calculating the position of target so that to obtain the miss distance when acoustic measurement is adopted. In the forward intersection measurement, when two measurement stations are used and fixed in advance, the angle measurement accuracy of the measurement station is certain for the weapon. At this time, the measuring error of the miss distance depends on the triangle of intersection [2-4]. The optimal intersection angle which can result in the smallest measuring error has been pointed out, of which the accuracy is best when the station is properly arranged to reach the above intersection angle.

However, in actual measurement tests, when the measurement station is fixed, there is still a spread range for the miss distance of the same type of weapon. In general, different types of weapons have different dispersion characteristics. For example, some projectiles have large deviations in level, and some projectiles have large deviations in height. Therefore, it is necessary to choose an optimal rendezvous angle adopting different types of weapons so as to minimize the measuring error and optimize the system measuring accuracy [5-7].

This paper introduces the measuring principle of acoustic rendezvous, conducts theoretical derivation and analysis of measuring errors, establishes evaluation indexes, and performs simulation. According to the dispersion characteristics of different types of weapons such as normal distribution and uniform distribution, they were verified through simulation experiments to obtain the optimal station for different dispersal, which provide a theoretical basis for the optimal station layout scheme for forward intersection measurement in the test. In the analysis of this paper, the distance between the two acoustic measurement stations is constant, the angle measurement accuracy is known, and the
angle of intersection is changed by changing the miss distance position to perform the optimal station analysis.

2. Principles of acoustic intersection measurement
The measurement principle of acoustic rendezvous is as follows: As shown in Figure 1, in the coordinate system O-XY, A and B are acoustic measurement stations adopting the circle array in reference [4]. The connection between the two measurement stations A and B is called the baseline. $P$ is the target point of which coordinates are unknown. $\alpha$, $\beta$ is the azimuth measured by measurement stations $A$ and $B$, $\gamma = 180^\circ - \alpha - \beta$ is the intersection angle. According to the coordinates of the two measurement stations $A$ and $B$, and the azimuth angle $\alpha$, $\beta$, the coordinates of point $P$ can be obtained.

![Figure 1. The principles of acoustic intersection measurement.](image)

Use the cotangent formula to get the P-coordinate of the target point:

$$
\begin{align*}
    x_p &= (x_A \cot \beta + x_B \cot \alpha + y_B - y_A) \left( \cot \alpha + \cot \beta \right)^{-1} \\
    y_p &= (y_A \cot \beta + y_B \cot \alpha + x_A - x_B) \left( \cot \alpha + \cot \beta \right)^{-1}
\end{align*}
$$

(1)

3. Mathematical models

3.1. Error formula of acoustic rendezvous measurement
Using the law of error propagation and a similar proof method to the optical forward intersection in geodesy, the measuring error formula for the acoustic intersection can be obtained as follows [1]:

$$
m_p = S \cdot m \cdot \sqrt{\sin^2 \alpha + \sin^2 \beta \left( \rho \sin^2 \gamma \right)^{-1}}
$$

(2)

of which $S$ is the distance between the two measurement stations $A, B$, $m$ is the measurement accuracy of the measurement station, $\rho = 206265$ is the conversion value from radians to seconds. The definition of $\alpha, \beta, \gamma$ as shown in Figure 1.

Let $y = \sin^2 \alpha + \sin^2 \beta$, from $\alpha + \beta = 180^\circ - \gamma$, have $\beta = 180^\circ - \gamma - \alpha$, so have:

$$
y = \sin^2 \alpha + \sin^2 \left( \alpha + \gamma \right)
$$

(3)

For any constant $\gamma$, find the first derivative of $y$ against $\alpha$ and make the derivative equal to zero to get:

$$
\frac{dy}{d\alpha} = 2 \sin \alpha \cos \alpha + 2 \sin(\alpha + \gamma) \cos(\alpha + \gamma) = 0
$$

(4)

Which is:

$$
\sin(2\alpha) = -\sin(2(\alpha + \gamma)) = \sin(2\beta)
$$

(5)
So there are: $\alpha = \beta$ or $\alpha + \beta = 0.5\pi$. When $\alpha + \beta = 0.5\pi$ and $\gamma = 0.5\pi$, there is $m_p = S \cdot m \cdot \rho^{-1}$. That is, the error is constant at this time. The situation when $\alpha = \beta$ is analyzed below.

When $\alpha = \beta$ and $\alpha + \beta = 180^\circ - \gamma$, we can have $\alpha = 90^\circ - 0.5\gamma$. Substituting it into (2), we get:

$$m_p = \sqrt{2S \cdot m \cdot \rho \cdot \cos(0.5\gamma) \cdot (\sin^2 \gamma)^{-1}}$$

(6)

From $\frac{dm_p}{d\gamma} = \sqrt{2S \cdot m \cdot (-0.5\sin^2 \gamma \sin 0.5\gamma - 2\sin \gamma \cos 0.5\gamma \cos \gamma) \cdot (\rho \cdot \sin^4 \gamma)^{-1}} = 0$, we get:

$$\gamma = 109^\circ 28'16''$$

(7)

The $\gamma$ in formula (7) is exactly the optimal intersection angle.

Finding the second derivative of $y$ against $\alpha$ in equation (3), we get:

$$\frac{d^2y}{d\alpha^2} = 2 \cos(2\alpha) + 2 \cos(2(\alpha + \gamma))$$

(8)

Since $\alpha = \beta$ and $\alpha + \beta = 180^\circ - \gamma$, we have:

$$\alpha = 90 - 0.5\gamma$$

(9)

Substituting equation (9) into equation (8) gives:

$$\frac{d^2y}{d\alpha^2} = -2\cos \gamma - 2\cos \gamma = -4\cos \gamma$$

(10)

It is known from equation (10) that when $\gamma < 90^\circ$, there is a maximum value at this time; when $\gamma > 90^\circ$, there is a minimum value at this time. It can be known from these analyses that the measuring error is the smallest when the intersection angle $\gamma = 109^\circ 28'16''$ and the stations are arranged symmetrically. Obviously, the maximum value of the measuring error is infinite.

3.2. Analysis of theoretical results

In Figure 2, a schematic diagram of the measuring error of the acoustic intersection is shown, of which $A$, $B$ is the two measurement stations, $P_1$, $P_2$, $P_3$ are three target points respectively, $PO$ is the vertical bisector of the line segment $AB$, and $\angle AP_1B = \gamma_1 < 90^\circ$, $\angle AP_2B = \gamma_2 = 90^\circ$, $\angle AP_3B = \gamma_3 > 90^\circ$. What’s more, Figure 2 shows the large arcs passing through the points $A$, $B$ and $P_1$, the semi-circular arcs passing through $A$, $B$ and $P_2$, and the small arcs passing through $A$, $B$ and $P_3$. The large arc, semi-arc, and small arc here represent the cases where the intersection angle is less than $90^\circ$, equal to $90^\circ$, and greater than $90^\circ$.

Through the theoretical analysis and schematic diagram of the measuring error rule of acoustic rendezvous above, the following results can be obtained: 1) When the rendezvous angle $\gamma < 90^\circ$, the error is the smallest; 2) When the rendezvous angle $\gamma < 90^\circ$, which is shown in Figure 2, on the arc of three points $A$, $P_1$, $B$, the error at the point $P_1$ is the largest, that is, when $P_1$ moving from point $A$ or point $B$, the error will decrease; 3) When the intersection angle $\gamma = 90^\circ$ and the target point passes through the circular arc of the three points $A$, $P_2$, $B$, the error is equal everywhere; 4) When the intersection angle $\gamma > 90^\circ$, for example, in Figure (2), the error at the point is the smallest when the target point is on the arc passing through the three points, that is, when $P_3$ moving to $A$ or $B$, the error will increase.
4. Evaluation methods

We have theoretically analyzed the errors of the acoustic intersection measurement above. However, in practical test, the target point to be measured is uncertain, but a certain spreading range is formed around a certain theoretical target point. Simply saying that the best intersection angle is a certain fixed angle does not have much meaning for the test, it is more necessary to know the magnitude of the error within a certain dispersion range. Therefore, error evaluation method is firstly established, and the simulation to obtain the contour map of the error distribution is applied. On that basis, we analyzes how to choose the optimal intersection angle according to the dispersion range of miss distance to find out the optimal station placement.

4.1. Error Evaluation Index

According to the theoretical analysis, we first find out the minimum and maximum error in a certain dispersion range. At the same time, due to the spread of weapons, the error at a certain point may be large, but the probability of occurrence at that point is so small that the effect of this error is very small as well. Therefore, the concept of average error is introduced. That is, if there are $N$ errors $m_i, i=1,2,\cdots,N$ and the corresponding occurrence probability is $p_i, i=1,2,\cdots,N$, and the average error is $\bar{\sigma} = \sum_{i=1}^{N} m_i p_i$. Therefore, there are the following three indicators and criteria for evaluating the pros and cons of the best rendezvous angle: 1) the minimum error: the best rendezvous angle with which the minimum error is small, is better than the best rendezvous angle when the minimum error is large; 2) the maximum error: the best rendezvous angle with which the maximum error is small, is better than the best rendezvous angle when the maximum error is large; 3) the average error: the best rendezvous angle with which the average error is small, is better than the best rendezvous angle when the average error is large.

4.2. Simulation results and analysis

The simulation conditions are as follows: Figure 3 shows the contour map of the error distribution. In the coordinate system $O-XY$, 1 # measuring station coordinates are (0m, 0m), and 2 # measuring station coordinates are (0m, 1000m). For every square ranging from $[0m, 1000m] \times [0m, 1000m]$,
take a point every 1 m to calculate the error, and the angle accuracy of the measurement station is 1 mil. The calculation error results are plotted as contour lines.

From Figure 3, it can be seen intuitively that when the intersection angle is too small or too large, the measuring error is relatively large. And when the optimal intersection angle is 109°28’16”, the error changes relatively gently in the direction of the parallel baseline, but changes sharply in the direction of the vertical baseline.

![Figure 3. Contour map of error distribution.](image)

Also the results can be obtained from the simulation error contour. When the intersection angle is 109°28’16”, the error is the smallest, which further validates the results of previous theoretical proofs. And we discover that the closer to the baseline, the more dramatic the error changes. It can be seen that the rendezvous angle during optimal station placement is mainly between 109°28’16” and 90°, so the following simulation experiments are based on this.

5. Simulation experiment analysis
The simulation experiments for the measuring error are shown for the cases where the dispersion obeys uniform distribution and Gaussian distribution.

5.1. Dispersion subject to uniform distribution
Let the theoretical target point be \((x_0, y_0)\), and record it as \(G = [x_1, x_2] \times [y_1, y_2]\) to stand for a rectangular area centering on the theoretical target point, where \(x_0 = 0.5(x_2 + x_1)\) and \(S = (x_2 - x_1) \cdot (y_2 - y_1)\) are the area of the rectangle. Assuming that the actual target points obey the uniform distribution on the rectangle, the density function is as follows:

\[
 f(x, y) = \begin{cases} \frac{S^{-1}}{G}, & (x, y) \in G \\ 0, & (x, y) \notin G \end{cases} \tag{11}
\]

In the simulation, the rectangular areas centering on the points corresponding to 90° and 109°28’16” are respectively used for calculation under different spreading ranges. When the
intersection angle is 109° 28'16", the coordinate of their corresponding point is (354,500); when the intersection angle is 90°, the coordinate is (500,500). When taking the coordinate of the point corresponding to the intersection angle 109° 28'16" as the center, take $G = [54, 654] \times [200, 800]$; when taking the coordinates of the point corresponding to the intersection angle 90° as the center, take $G = [200, 800] \times [200, 800]$. The error evaluation indexes such as the minimum error, the maximum error, and the average error are simulated, and the corresponding results are obtained. Then, how to determine the optimal intersection angle is discussed. The results obtained are shown in Table 1.

| Spread in X direction | Spread in Y direction | Intersection angle is 109° 28'16" | Intersection angle is 90° |
|----------------------|----------------------|----------------------------------|--------------------------|
|                      |                      | Minimum error | Maximum error | Average error | Minimum error | Maximum error | Average error |
| 100                  | 100                  | 0.962         | 0.976         | 0.967         | 1.001         | 1.109         | 1.051         |
| 100                  | 300                  | 0.962         | 1.144         | 1.007         | 0.962         | 1.260         | 1.070         |
| 100                  | 500                  | 0.962         | 1.903         | 1.119         | 0.962         | 1.450         | 1.111         |
| 300                  | 100                  | 0.962         | 0.990         | 0.970         | 1.002         | 1.133         | 1.058         |
| 300                  | 300                  | 0.962         | 1.144         | 1.010         | 0.962         | 1.289         | 1.077         |
| 300                  | 500                  | 0.962         | 1.903         | 1.119         | 0.962         | 1.482         | 1.117         |
| 500                  | 100                  | 0.962         | 1.023         | 0.977         | 1.002         | 1.186         | 1.074         |
| 500                  | 500                  | 0.962         | 1.144         | 1.105         | 0.962         | 1.350         | 1.092         |
| 500                  | 500                  | 0.962         | 1.903         | 1.120         | 0.962         | 1.549         | 1.130         |

It is known from Table 1 that the minimum error with which the optimal intersection angle is 109° 28'16" and the minimum error with which the optimal intersection angle is 90° are basically equal. The maximum error increases as the spreading range increases, and when the optimal intersection angle is taken as 109° 28'16", the maximum error increases faster. Further, when the target points spread widely in the direction of the parallel baseline, the maximum error has a small change when the optimal intersection angle is taken as 109° 28'16". And when the target points are spread in the vertical baseline direction, the maximum error has a large change is smaller when the optimal intersection angle is taken as 109° 28'16". The variation law of the average error is the same as the maximum error.

5.2. Cases in which distribution follows a normal distribution

Assuming that the actual target points obey a two-dimensional independent normal distribution, there are:

$$f(x, y) = \left(2\pi\sigma_x\sigma_y\right)^{-1/2} \exp\left(-0.5\left[(x - \mu_x)^2\sigma_x^{-2} + (y - \mu_y)^2\sigma_y^{-2}\right]\right)$$  \hspace{1cm} (12)

In the simulation, the rectangular areas centered on the corresponding points of 90° and 109° 28'16" are respectively calculated under different spreading ranges. Similarly, The error evaluation indexes such as the minimum error, the maximum error, and the average error are simulated, and the corresponding results are obtained. Then, how to determine the optimal intersection angle is discussed. Taking the calculation area as $[200, 800] \times [200, 800]$, the result is as follows:
Table 2. Calculation results of errors in normal distribution (m).

| No | $\sigma_x$ | $\sigma_y$ | Minimum error | Maximum error | Average error | Minimum error | Maximum error | Average error |
|----|------------|------------|---------------|---------------|--------------|---------------|---------------|--------------|
| 1  | 50         | 50         | 0.962         | 0.997         | 0.965        | 0.985        | 1.146        | 1.049        |
| 2  | 50         | 100        | 0.962         | 1.144         | 0.975        | 0.962        | 1.264        | 1.054        |
| 3  | 50         | 150        | 0.962         | 1.584         | 0.994        | 0.962        | 1.404        | 1.063        |
| 4  | 100        | 50         | 0.962         | 1.005         | 0.966        | 0.985        | 1.168        | 1.051        |
| 5  | 100        | 100        | 0.962         | 1.144         | 0.976        | 0.962        | 1.289        | 1.056        |
| 6  | 100        | 150        | 0.962         | 1.584         | 0.994        | 0.962        | 1.430        | 1.064        |
| 7  | 150        | 50         | 0.962         | 1.031         | 0.967        | 0.985        | 1.207        | 1.054        |
| 8  | 150        | 100        | 0.962         | 1.144         | 0.977        | 0.962        | 1.331        | 1.059        |
| 9  | 150        | 150        | 0.962         | 1.584         | 0.996        | 0.962        | 1.476        | 1.068        |

Figure 4 displays the errors according to the serial number. The intersection angle $16'28''10^\circ$ is identified by the letter A and the intersection angle $90^\circ$ is identified by the letter B.

From Table 2 and Figure 4, the following conclusions are similar to those in the uniform distribution case. The minimum error when the optimal intersection angle is taken as $109^\circ28'16''$ is basically equal to the minimum error when the optimal intersection angle taken as $90^\circ$. The maximum error increases as the spreading range increases, and when the optimal intersection angle is taken as $109^\circ28'16''$, the maximum error increases faster. The maximum error changes are small when the target points spread in a large parallel baseline direction; the maximum error change are large when the target points spread in a large vertical baseline direction. The difference between the normal distribution case and the uniform distribution case is that the change law of the average error is the same as the maximum error when the distribution is uniform, but the average error in the normal distribution is basically unchanged, and the average error corresponding to the intersection angle taken as $109^\circ28'16''$ is significantly better than the average error corresponding to the intersection angle taken as $90^\circ$. 
6. Conclusions
In this paper, the measuring error of acoustic rendezvous is studied, and a large number of data simulations with computer are performed on the error distribution of different scattered miss distance quantities. The following conclusions are obtained: 1) When the minimum error is more concerned, the optimal intersection angle can be taken both as $109^\circ 28'16''$ or $90^\circ$; 2) The maximum error increases with the increase of the dispersion range, and the velocity of the maximum error increasing at the intersection angle taken as $109^\circ 28'16''$ is faster than the intersection angle taken as. Therefore, if the maximum error is more concerned, the optimal intersection angle is taken as. 3) When the target points scattering mainly along the parallel baseline, the error changes are small; when the target points scattering mainly on the vertical baseline direction, the error changes are large. 4) When the targets’ dispersion obeys a uniform distribution, the variation law of the average error is similar to the maximum error. 5) When the targets’ dispersion follows the normal distribution, the average error is basically unchanged, and the average error corresponding to the intersection angle of $109^\circ 28'16''$ is significantly better than the corresponding average error when the intersection angle is $90^\circ$.

By studying the contour map of the measuring error distribution of acoustic rendezvous, an optimal rendezvous measurement scheme was established. From the theoretical analysis and the simulation result, it is proved that the selection of the optimal station layout scheme can improve the test accuracy, which has practical guidance significance for the test work, and can provides a application foundation for the optimal station layout scheme in the intersection survey.

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