A COCOMPLETE BUT NOT COMPLETE
ABELIAN CATEGORY

JEREMY RICKARD

(Received 11 July 2019; accepted 28 August 2019; first published online 5 February 2020)

Abstract

An example of a cocomplete abelian category that is not complete is constructed.

2010 Mathematics subject classification: primary 18E10; secondary 18E15.
Keywords and phrases: abelian categories, cocompleteness.

1. Introduction

In this paper we answer a question that was posted on the internet site MathOverflow
by Simone Virili [4]. The question asked whether there is an abelian category that
is cocomplete but not complete. It seemed that there must be a standard example, or
an easy answer, but, despite receiving a fair amount of knowledgeable interest, after
many months the question still had no solution.

Recall that a complete category is one with all small limits, and that for an abelian
category, which by definition has kernels, completeness is equivalent to the existence
of all small products. Similarly, because of the existence of cokernels, cocompleteness
for abelian categories is equivalent to the existence of all small coproducts.

A common, but wrong, first reaction to the question is that there are easy natural
examples, with something such as the category of torsion abelian groups coming to
mind: an infinite direct sum of torsion groups is torsion, but an infinite direct product
of torsion groups may not be. Nevertheless, this category does have products: the
product of a set of groups is simply the torsion subgroup of their direct product. In
categorical terms, the category of torsion abelian groups is a coreflective subcategory
of the category of all abelian groups (which certainly has products), with the functor
sending a group to its torsion subgroup being right adjoint to the inclusion functor.

Similar, but more sophisticated, considerations doom many approaches to finding
an example.

Recall that an (AB5) category, in the terminology of Grothendieck [2], is
a cocomplete abelian category in which filtered colimits are exact, and that a
Grothendieck category is an (AB5) abelian category with a generator. One’s favourite cocomplete abelian category typically tends to be a Grothendieck category, such as a module category or a category of sheaves, but it is well known that every Grothendieck category is complete (for a proof see, for example, [3, Proposition 8.3.27]).

More generally, any locally presentable category is complete [1, Corollary 1.28], and most standard constructions of categories preserve local presentability.

2. The construction

First we shall fix a chain of fields \( \{ k_\alpha \mid \alpha \in \text{On} \} \) indexed by the ordinals such that \( k_\beta/k_\alpha \) is an infinite-degree field extension whenever \( \alpha < \beta \). For example, we could take a class \( \{ x_\alpha \mid \alpha \in \text{On} \} \) of variables indexed by \( \text{On} \) and let \( k_\alpha = \mathbb{Q}(X_\alpha) \) be the field of rational functions in the set of variables \( X_\alpha = \{ x_\gamma \mid \gamma < \alpha \} \).

We generally adopt the convention that categories are locally small (that is, the class of morphisms between two objects is always a set). However, we will start by defining a ‘category’ \( C \) which is not locally small.

An object \( V \) of \( C \) consists of a \( k_\alpha \)-vector space \( V_\alpha \) for each ordinal \( \alpha \), together with a \( k_\alpha \)-linear map \( v_{\alpha,\beta} : V_\alpha \to V_\beta \) for each pair \( \alpha < \beta \) of ordinals, such that \( v_{\alpha,\gamma} = v_{\beta,\gamma} v_{\alpha,\beta} \) whenever \( \alpha < \beta < \gamma \). (When we denote an object by an upper case letter such as \( V \), we will always use, without further comment, the corresponding lower case letter for the linear maps \( v_{\alpha,\beta} \).)

A morphism \( \varphi : V \to W \) of \( C \) consists of a \( k_\alpha \)-linear map \( \varphi_\alpha : V_\alpha \to W_\alpha \) for each ordinal \( \alpha \), such that \( \varphi_\beta v_{\alpha,\beta} = w_{\alpha,\beta} \varphi_\alpha \) whenever \( \alpha < \beta \). Composition is defined in the obvious way. If \( \theta : U \to V \) and \( \varphi : V \to W \) are morphisms, then \( (\varphi \theta)_\alpha = \varphi_\alpha \theta_\alpha \).

It is straightforward to check that \( C \) is an additive (in fact, \( k_0 \)-linear) category, but not locally small: for example, if \( V \) is the object with \( V_\alpha = k_\alpha \) for every \( \alpha \) and \( v_{\alpha,\beta} = 0 \) for all \( \alpha < \beta \), then a morphism \( \varphi : V \to V \) has \( v_\alpha : k_\alpha \to k_\alpha \) multiplication by some scalar \( \lambda_\alpha \in k_\alpha \) for some arbitrary choice of \( \{ \lambda_\alpha \mid \alpha \in \text{On} \} \), so the class of endomorphisms of \( V \) is a proper class.

**Proposition 2.1.** \( C \) is a (not locally small) abelian category with (small) products and coproducts in which (small) filtered colimits are exact.

**Proof.** It is straightforward to check that the obvious ‘pointwise’ constructions give kernels, cokernels, products and coproducts. For example, if \( \varphi : V \to W \) is a morphism, then the kernel of \( \varphi \) is the object \( U \) with \( U_\alpha = \text{ker} \varphi_\alpha \) the kernel of \( \varphi_\alpha : V_\alpha \to W_\alpha \) and \( u_{\alpha,\beta} : U_\alpha \to U_\beta \) the natural map between the kernels of \( \varphi_\alpha \) and \( \varphi_\beta \) induced by \( v_{\alpha,\beta} \).

Since all these constructions are ‘pointwise’, it is also straightforward to check that every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel (so that \( C \) is abelian), and that small filtered colimits are exact, as the verification reduces to the corresponding facts for the category of \( k_\alpha \)-vector spaces. \( \Box \)
The (locally small) category that we are really interested in is a full subcategory $G$ of $C$.

**Definition 2.2.** An object $V$ of $C$ is $\alpha$-grounded if, for every $\beta > \alpha$, $V_\beta$ is generated as a $k_\beta$-vector space by the image of $v_{_{\alpha,\beta}}$. The full subcategory of $C$ consisting of the $\alpha$-grounded objects is denoted by $\alpha$-$G$.

**Definition 2.3.** An object $V$ of $C$ is grounded if it is $\alpha$-grounded for some ordinal $\alpha$. The full subcategory of $C$ consisting of the grounded objects is denoted by $G$.

**Theorem 2.4.** $G$ is a (locally small) (AB5) abelian category that is not complete.

**Proof.** Let $V$ be an $\alpha$-grounded object. If $\varphi : V \rightarrow W$ is a morphism, then, for $\beta > \alpha$, $\varphi_\beta$ is determined by $\varphi_\alpha$ and so the class of morphisms $\varphi : V \rightarrow W$ is a set. Hence, $G$ is locally small.

Let $\varphi : V \rightarrow W$ be a morphism between $\alpha$-grounded objects. Clearly, the kernel and cokernel of $\varphi$ are also $\alpha$-grounded. Hence, $G$ is closed under kernels and cokernels in $C$ and so it is an exact abelian subcategory of $C$.

If $\{V^i \mid i \in I\}$ is a set of grounded objects, then there is some ordinal $\alpha$ so that every $V^i$ is $\alpha$-grounded. Then $\bigoplus_{i \in I} V^i$ is also $\alpha$-grounded. Thus, $G$ is cocomplete and the inclusion functor $G \hookrightarrow C$ preserves coproducts and hence all colimits. Exactness of filtered colimits in $G$ therefore follows from the same property for $C$. Thus, $G$ is a locally small (AB5) abelian category.

For an ordinal $\alpha$, let $M^\alpha$ be the object with

$$M^\alpha_\beta = \begin{cases} 0 & \text{if } \beta < \alpha, \\ k_\beta & \text{if } \beta \geq \alpha \end{cases}$$

and $m^\alpha_{\beta,\gamma} : k_\beta \rightarrow k_\gamma$, the inclusion map for $\alpha \leq \beta < \gamma$. Then $M^\alpha$ is $\alpha$-grounded.

Suppose that $W$ is any object of $C$. A morphism $\varphi : M^\alpha \rightarrow W$ is determined by $\varphi_\alpha : M^\alpha_\alpha = k_\alpha \rightarrow W_\alpha$, so $C(M^\alpha, W) \cong W_\alpha$ and $M^\alpha$ represents the functor $W \mapsto W_\alpha$ from $C$ to $k_\alpha$-vector spaces. Also, if $\alpha < \beta$, then the obvious morphism $\varphi : M^\beta \rightarrow M^\alpha$, with $\varphi_\gamma$ the identity map $k_\gamma \rightarrow k_\gamma$ for $\gamma \geq \beta$, induces a commutative square

$$
\begin{array}{ccc}
C(M^\alpha, W) & \Rightarrow & W_\alpha \\
\downarrow_{C(\varphi, W)} & & \downarrow_{w_{\alpha,\beta}} \\
C(M^\beta, W) & \Rightarrow & W_\beta
\end{array}
$$

We shall show that in $G$ there is no product of a countably infinite collection of copies of $M^\alpha$, so $G$ is not complete.

Suppose that $P$ is the product (in $G$) of countably many copies of $M^\alpha$. Then, for any ordinal $\beta \geq \alpha$,

$$G(M^\beta, P) \cong G(M^\beta, M^\alpha)^\mathbb{N} \cong (M^\beta_{\beta})^\mathbb{N} \cong k^\mathbb{N}_\beta$$

as $k_\beta$-vector spaces. But, if $\alpha < \beta < \gamma$, then $k^\mathbb{N}_\beta$ is not generated as a $k_\gamma$-vector space by $k^\mathbb{N}_\beta$, since $k_\gamma/k_\beta$ is an infinite field extension. So, $P$ cannot be $\beta$-grounded for any $\beta$. □
Remark 2.5. We have already noted that products do exist in $C$, with the obvious pointwise construction. However, the product of $\alpha$-grounded objects need not be $\beta$-grounded for any $\beta$.

Products also exist in $\alpha\cdot G$, since although the pointwise product may not be $\alpha$-grounded, $\alpha\cdot G$ is a coreflective subcategory of $C$. We can $\alpha$-ground an object $V$ of $C$ by replacing $V_\beta$ by $v_{\alpha, \beta}(V_\alpha)$ for $\beta > \alpha$, and the product of a set of objects in $\alpha\cdot G$ is obtained by $\alpha$-grounding the product in $C$.

Remark 2.6. Each category $\alpha\cdot G$ is a cocomplete and complete abelian category (in fact, a Grothendieck category, with $\bigoplus_{\beta \leq \alpha} M_\beta$ a generator). The inclusion functors $\alpha\cdot G \to \beta\cdot G$ are exact and preserve coproducts, but do not preserve products, which explains why their union $G$ has coproducts but does not have products (or at least not in an obvious way). Thanks are due to Zhen Lin Low for making this observation in a comment on MathOverflow [4].

Remark 2.7. Of course, an example of a complete abelian category that is not cocomplete can be constructed by taking the opposite category of $G$.

Acknowledgement

I would like to thank MathOverflow and its community for introducing me to this and many other interesting questions, and for their comments on this question in particular.

References

[1] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, London Mathematical Society Lecture Note Series, 189 (Cambridge University Press, Cambridge, 1994).

[2] A. Grothendieck, ‘Sur quelques points d’algèbre homologique’, *Tohoku Math. J. (2)* 9 (1957), 119–221.

[3] M. Kashiwara and P. Schapira, *Categories and Sheaves*, Grundlehren der mathematischen Wissenschaften, 332 (Springer, Berlin, 2006).

[4] S. Virili, ‘Cocomplete but not complete abelian category’, MathOverflow (2012), https://mathoverflow.net/q/112574.

Jeremy Rickard, School of Mathematics, University of Bristol, Bristol BS8 1TW, UK

e-mail: j.rickard@bristol.ac.uk