A REGULAR GONOSOMAL EVOLUTION OPERATOR WITH UNCOUNTABLE SET OF FIXED POINTS

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Abstract. In this paper we study dynamical systems generated by a gonosomal evolution operator of a bisexual population. We find explicitly all (uncountable set) of fixed points of the operator. It is shown that each fixed point has eigenvalues less or equal to 1. Moreover, we show that each trajectory converges to a fixed point, i.e. the operator is regular. There are uncountable family of invariant sets each of which consisting unique fixed point. Thus there is one-to-one correspondence between such invariant sets and the set of fixed points. Any trajectory started at a point of the invariant set converges to the corresponding fixed point.

1. Introduction

Population dynamics theory is important to a proper understanding of living populations at all levels. This is a well developed branch of mathematical biology, which has a history of more than two hundred years.

The book [3] contains a short history of applications of mathematics to solving various problems in population dynamics. For background and motivations of the theory of population dynamics see [1]-[16].

In this paper we consider a bisexual population which consists females partitioned into types indexed by \( \{1, 2, \ldots, n\} \) and the males partitioned into types indexed by \( \{1, 2, \ldots, \nu\} \) (see [8], [10], [14] for details).

Let \( \gamma_{ik,j}^{(f)} \) and \( \gamma_{ik,l}^{(m)} \) be inheritance coefficients defined as the probability that a female offspring is type \( j \) and, respectively, that a male offspring is of type \( l \), when the parental pair is \( ik \) \( (i, j = 1, \ldots, n; \text{and } k, l = 1, \ldots, \nu) \). These quantities satisfy the following

\[
\gamma_{ik,j}^{(f)} \geq 0, \quad \gamma_{ik,l}^{(m)} \geq 0,
\]

\[
\sum_{j=1}^{n} \gamma_{ik,j}^{(f)} + \sum_{l=1}^{\nu} \gamma_{ik,l}^{(m)} = 1, \quad \text{for all } i, k, j, l.
\]

Define \((n + \nu - 1)-\text{dimensional simplex}:

\[
S^{n+\nu-1} = \left\{ s = (x_1, \ldots, x_n, y_1, \ldots, y_\nu) \in \mathbb{R}^{n+\nu} : x_i \geq 0, y_j \geq 0, \sum_{i=1}^{n} x_i + \sum_{j=1}^{\nu} y_j = 1 \right\}.
\]

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Denote 
\[ O = \{ s \in S^{n+\nu-1} : (x_1, \ldots, x_n) = (0, \ldots, 0) \text{ or } (y_1, \ldots, y_\nu) = (0, \ldots, 0) \} . \]

\[ S^{n,\nu} = S^{n+\nu-1} \setminus O. \]

Following [15] define an evolution operator \( V : S^{n,\nu} \to S^{n,\nu} \) (which is called normalized gonosomal operator) as

\[
V \colon \begin{cases} 
  x'_j = \frac{\sum_{i,k=1}^{n,\nu} \gamma_{ik,j} x_i y_k}{(\sum_{i=1}^{n} x_i) (\sum_{j=1}^{\nu} y_j)}, & j = 1, \ldots, n \\
  y'_l = \frac{\sum_{i,k=1}^{n,\nu} \gamma_{ik,l} x_i y_k}{(\sum_{i=1}^{n} x_i) (\sum_{j=1}^{\nu} y_j)}, & l = 1, \ldots, \nu.
\end{cases}
\]

(1.2)

The main problem: For given operator \( V \) and initial point \( z^{(0)} \in S^{n,\nu} \) what ultimately happens with the trajectory \( z^{(m)} = V(z^{(m-1)}), \ m = 1, 2, \ldots \)? Does the limit \( \lim_{m \to \infty} z^{(m)} \) exist? If not what is the set of limit points of the sequence?

In general, this is very difficult problem. In book [10] several recently obtained results related to this main problem are given.

In this paper we consider the special case: \( n = \nu = 2 \) and the following coefficients:

\[
\begin{align*}
\gamma_{11,1}^{(f)} &= a & \gamma_{11,2}^{(f)} &= 0 & \gamma_{11,1}^{(m)} &= b & \gamma_{11,2}^{(m)} &= 0 \\
\gamma_{12,1}^{(f)} &= 0 & \gamma_{12,2}^{(f)} &= \sigma_1 & \gamma_{12,1}^{(m)} &= \sigma_2 & \gamma_{12,2}^{(m)} &= 0 \\
\gamma_{21,1}^{(f)} &= 0 & \gamma_{21,2}^{(f)} &= a & \gamma_{21,1}^{(m)} &= b & \gamma_{21,2}^{(m)} &= 0 \\
\gamma_{22,1}^{(f)} &= 0 & \gamma_{22,2}^{(f)} &= a & \gamma_{22,1}^{(m)} &= 0 & \gamma_{22,2}^{(m)} &= b.
\end{align*}
\]

(1.3)

Then corresponding evolution operator \( W : S^{2,2} \to S^{2,2} \) is

\[
W \colon \begin{cases} 
  x' = \frac{axu}{(x+y)(u+v)} \\
  y' = \frac{\sigma_1 xv + ayu + ayv}{(x+y)(u+v)} \\
  u' = \frac{\sigma_2 xv + bxu + byu}{(x+y)(u+v)} \\
  v' = \frac{byv}{(x+y)(u+v)},
\end{cases}
\]

(1.4)

where coefficients satisfy

\( a + b = \sigma_1 + \sigma_2 = 1, \ a, b, \sigma_1, \sigma_2 > 0. \)
Remark 1. From the probabilities (1.3) one can notice that type 1 of females (resp. type 2 of males) can be born only if both parents have type 1 (resp. 2). Type 2 of females (resp. type 1 of males) can not be born if both parents have type 1 (resp. 2).

For this operator $W$ and arbitrarily initial point $s^{(0)} \in S^{2,2}$, we will study the trajectory $\{s^{(m)}\}_{m=0}^{\infty}$, where

$$s^{(m)} = W^m(s^{(0)}) = W(W(...W(s^{(0)})...)).$$

2. Fixed points

A point $s$ is called a fixed point of the operator $W$ if $s = W(s)$. The set of all fixed points denoted by Fix($W$).

Let us find all the fixed points of $W$ given by (1.4), i.e. we solve the following system of equations for $(x, y, u, v)$:

$$\begin{align*}
  x(x+y)(u+v) &= axu, \\
  y(x+y)(u+v) &= \sigma_1 xv + ayu + ayv, \\
  u(x+y)(u+v) &= \sigma_2 xv + bxu + byu, \\
  v(x+y)(u+v) &= byv.
\end{align*}$$

(2.1)

If $x = 0$ then $y \neq 0$ and from the second equation of the system (2.1) we get $y = a$. In addition, the third and the fourth equations of the system (2.1) give $u + v = b$.

If $v = 0$ then $u \neq 0$ and from the third equation of the system (2.1) we get $u = b$. The second and the third equations of the system (2.1) give $x + y = a$.

If $xv \neq 0$ then we come to

$$\begin{align*}
  (x+y)(u+v) &= au, \\
  y(x+y)(u+v) &= \sigma_1 xv + ayu + ayv, \\
  u(x+y)(u+v) &= \sigma_2 xv + bxu + byu, \\
  (x+y)(u+v) &= by.
\end{align*}$$

(2.2)

The first and the second equations of the system (2.2) give

$$\sigma_1 x + ay = 0.$$ At the same time the third and the fourth equations of the system (2.2) give

$$\sigma_2 v + bu = 0.$$ Since $a, b, \sigma_1, \sigma_2 > 0$ then when we solve the last two equations we obtain $x = y = u = v = 0$, however this point is not in the space $S^{2,2}$. Thus the set of all fixed points of operator (1.4) is Fix($W$) = $F_{11} \cup F_{12}$, where

$$F_{11} = \{(0,a,u,v) : u + v = b, \quad u, v \in [0,b]\}$$

and

$$F_{12} = \{(x,y,b,0) : x + y = a, \quad x, y \in [0,a]\}.$$
Definition 1. A fixed point $s$ of the operator $W$ is called hyperbolic if its Jacobian $J$ at $s$ has no eigenvalues on the unit circle.

Definition 2. A hyperbolic fixed point $s$ is called:

i) attracting if all the eigenvalues of the Jacobi matrix $J(s)$ are less than 1 in absolute value;

ii) repelling if all the eigenvalues of the Jacobi matrix $J(s)$ are greater than 1 in absolute value;

iii) a saddle otherwise.

It is not hard to see that $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 1 - \frac{v}{b}$ and $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 1 - \frac{x}{a}$ are eigenvalues of the fixed points of the forms $F_{11}$ and $F_{12}$ respectively. By these definitions we see that all fixed points of the operator (1.4) are nonhyperbolic fixed points.

3. Limit set

Denote

$$\partial S^{2,2} = \{t = (x, y, u, v) \in S^{2,2} : xyuv = 0\}.$$

Take any initial point $t = (x, y, u, v) \in \partial S^{2,2}$. Consider the following subsets of $\partial S^{2,2}$.

$$E_1 = \{(x, y, u, v) \in \partial S^{2,2} : x = 0\},$$
$$E_2 = \{(x, y, u, v) \in \partial S^{2,2} : y = 0\},$$
$$E_3 = \{(x, y, u, v) \in \partial S^{2,2} : u = 0\},$$
$$E_4 = \{(x, y, u, v) \in \partial S^{2,2} : v = 0\}.$$

If $t = (x, y, u, v) \in E_1$ then $W(t) \in F_{11}$. If $t = (x, y, u, v) \in E_4$ then $W(t) \in F_{12}$.

When $t = (x, y, u, v) \in E_2$ then $W(t) \in E_4$ and $W^2(t) \in F_{12}$.

When $t = (x, y, u, v) \in E_3$ then $W(t) \in E_1$ and $W^2(t) \in F_{11}$.

Now we take any initial point $t = (x, y, u, v) \in S^{2,2} \setminus \partial S^{2,2}$.

Introduce the following notations

$$\alpha = \frac{x}{x + y}, \quad \beta = \frac{v}{u + v}, \quad \alpha' = \frac{x'}{x' + y'}, \quad \beta' = \frac{v'}{u' + v'},$$

which yields the nonlinear dynamical system

$$V : \begin{cases}
\alpha' = \frac{\alpha(1 - \beta)}{1 + (p_1 - 1)\alpha\beta}, \\
\beta' = \frac{\beta(1 - \alpha)}{1 + (p_2 - 1)\alpha\beta}
\end{cases},$$

with the initial point $(\alpha(0), \beta(0)) \in \Delta$, where

$$\Delta := \{(\alpha, \beta) \in \mathbb{R}^2 : 0 \leq \alpha \leq 1, \ 0 \leq \beta \leq 1\} = [0, 1]^2,$$

and

$$p_1 = \frac{\sigma_1}{a}, \quad p_2 = \frac{\sigma_2}{b}.$$
There are three cases for \( p_1, p_2 \).

1. \( p_1 = p_2 = 1 \),
2. \( p_1 > 1 > p_2 > 0 \),
3. \( p_2 > 1 > p_1 > 0 \). \hfill (3.4)

In order to find the fixed points of the operator \([3.2] \) we solve the following system of equations for \((\alpha, \beta)\)

\[
\begin{align*}
\alpha(1 + (p_1 - 1)\alpha\beta) &= \alpha(1 - \beta), \\
\beta(1 + (p_2 - 1)\alpha\beta) &= \beta(1 - \alpha).
\end{align*}
\hfill (3.5)
\]

This system of equations gives us \( \alpha \cdot \beta = 0 \), that is \( s_1 = (\alpha, 0) \) and \( s_2 = (0, \beta) \) are fixed points for the operator \([3.2] \) where \( \alpha \geq 0, \beta \geq 0 \).

Using the system of equations \([3.2] \) we obtain

\[
\begin{align*}
\alpha^{(m+1)} &= \frac{\alpha^{(m)}(1 - \beta^{(m)})}{1 + (p_1 - 1)\alpha^{(m)}\beta^{(m)}}, \\
\beta^{(m+1)} &= \frac{\beta^{(m)}(1 - \alpha^{(m)})}{1 + (p_2 - 1)\alpha^{(m)}\beta^{(m)}}.
\end{align*}
\hfill (3.6)
\]

**Lemma 1.** For any initial point \((\alpha, \beta) \in [0, 1]^2\) it holds that

\[
0 \leq \alpha^{(m+1)} \leq \alpha^{(m)}, \quad 0 \leq \beta^{(m+1)} \leq \beta^{(m)}.
\]

In particular, the sequences \( \alpha^{(m)} = \frac{x^{(m)}}{x^{(m)} + y^{(m)}}, \quad m \geq 1 \) and \( \beta^{(m)} = \frac{y^{(m)}}{x^{(m)} + y^{(m)}}, \quad m \geq 1 \) are convergent.

**Proof.** Since \( V : [0, 1]^2 \to [0, 1]^2 \) and for any \( m \in \mathbb{N} \)

\[
\begin{align*}
1 + (p_1 - 1)\alpha^{(m)} &\in [\min\{1, p_1\}; \max\{1, p_1\}], \\
1 + (p_2 - 1)\beta^{(m)} &\in [\min\{1, p_2\}; \max\{1, p_2\}], \\
1 + (p_1 - 1)\alpha^{(m)}\beta^{(m)} &\in [\min\{1, p_1\}; \max\{1, p_1\}], \\
1 + (p_2 - 1)\alpha^{(m)}\beta^{(m)} &\in [\min\{1, p_2\}; \max\{1, p_2\}]
\end{align*}
\]

then it holds that

\[
\alpha^{(m+1)} - \alpha^{(m)} = \frac{-\alpha^{(m)}\beta^{(m)}(1 + (p_1 - 1)\alpha^{(m)})}{1 + (p_1 - 1)\alpha^{(m)}\beta^{(m)}} \leq 0,
\]

and that

\[
\beta^{(m+1)} - \beta^{(m)} = \frac{-\alpha^{(m)}\beta^{(m)}(1 + (p_2 - 1)\beta^{(m)})}{1 + (p_2 - 1)\alpha^{(m)}\beta^{(m)}} \leq 0.
\]

This completes the proof. \( \square \)
Theorem 1. For any initial point \((x, y, u, v) \in S^{2,2}\) the sequence
\[ W^m(x, y, u, v) = (x^{(m)}, y^{(m)}, u^{(m)}, v^{(m)}) \]
is convergent and
\[ \lim_{m \to \infty} x^{(m)} \cdot v^{(m)} = 0. \]

Proof. By Lemma \([1]\) all trajectories of the operator \((3.2)\) have a limit point and since the operator is continuous, each trajectory converges to a fixed point \(s_t = (a, 0)\) or \(s_2 = (0, \beta)\). Therefore we have always
\[ \alpha^{(m)} \cdot \beta^{(m)} = 0, \quad \text{as } m \to \infty. \]

In a view of \((1.4)\) and \((3.1)\) we get
\[
\begin{align*}
x^{(m+1)} &= \frac{a x^{(m)} y^{(m)}}{(x^{(m)} + y^{(m)})(u^{(m)} + v^{(m)})} = a\alpha^{(m)}(1 - \beta^{(m)}), \\
y^{(m+1)} &= \frac{\sigma_1 x^{(m)} y^{(m)} + ay^{(m)} u^{(m)} + ay^{(m)} v^{(m)}}{(x^{(m)} + y^{(m)})(u^{(m)} + v^{(m)})} = \sigma_1 \alpha^{(m)} \beta^{(m)} + a(1 - \alpha^{(m)}), \\
u^{(m+1)} &= \frac{\sigma_2 x^{(m)} y^{(m)} + bx^{(m)} u^{(m)} + by^{(m)} u^{(m)}}{(x^{(m)} + y^{(m)})(u^{(m)} + v^{(m)})} = \sigma_2 \alpha^{(m)} \beta^{(m)} + b(1 - \beta^{(m)}), \\
v^{(m+1)} &= \frac{by^{(m)} v^{(m)}}{(x^{(m)} + y^{(m)})(u^{(m)} + v^{(m)})} = b\beta^{(m)}(1 - \alpha^{(m)}). \tag{3.7}
\end{align*}
\]
This completes the proof. \(\square\)

Define the following sets:
\[
T_0 = \left\{ (x, y, u, v) \in S^{2,2} : \lim_{m \to \infty} x^{(m)} = \lim_{m \to \infty} y^{(m)} = 0 \right\},
\]
\[
T_1 = \left\{ (x, y, u, v) \in S^{2,2} : \lim_{m \to \infty} x^{(m)} = 0, \quad \lim_{m \to \infty} y^{(m)} \in (0, a] \right\},
\]
\[
T_2 = \left\{ (x, y, u, v) \in S^{2,2} : \lim_{m \to \infty} x^{(m)} = 0, \quad \lim_{m \to \infty} v^{(m)} \in (0, b] \right\}.
\]
If \(t = (x, y, u, v) \in T_0\), then
\[ \lim_{m \to \infty} \beta^{(m)} = \lim_{m \to \infty} \alpha^{(m)} = 0. \tag{3.8} \]
and \((3.7)\) shows that for any initial point \(t = (x, y, u, v) \in T_0\) for the trajectories of the operator \((1.4)\) we have
\[ W^m_t = (x^{(m)}, y^{(m)}, u^{(m)}, v^{(m)}) \to (0, a, b, 0) \quad \text{as } m \text{ tends to } \infty. \]
If \(t = (x, y, u, v) \in T_1\), then
\[ \lim_{m \to \infty} \beta^{(m)} = 0 \quad \text{and} \quad \lim_{m \to \infty} \alpha^{(m)} = \alpha_0 \in (0, 1]. \tag{3.9} \]
System of equations (3.7) shows that for any initial point \( t = (x, y, u, v) \in T_1 \) for the trajectories of the operator (1.4) we have

\[
W^m_1 = (x^{(m)}, y^{(m)}, u^{(m)}, v^{(m)}) \to \left( a\alpha_0, a(1 - \alpha_0), b, 0 \right) \in F_{12} \quad \text{as } m \text{ tends to } \infty.
\]

If \( t = (x, y, u, v) \in T_2 \), then

\[
\lim_{m \to \infty} \alpha^{(m)} = 0 \quad \text{and} \quad \lim_{m \to \infty} \beta^{(m)} = \beta_0 \in (0, 1].
\] (3.10)

System of equations (3.7) shows that for any initial point \( t = (x, y, u, v) \in T_2 \) for the trajectories of the operator (1.4) we have

\[
W^m_1 = (x^{(m)}, y^{(m)}, u^{(m)}, v^{(m)}) \to \left( 0, a, b(1 - \beta_0), b\beta_0 \right) \in F_{11} \quad \text{as } m \text{ tends to } \infty.
\]

Therefore we have the following

**Corollary 1.** For any initial point \( t = (x, y, u, v) \in S^{2,2} \) the \( \omega \)-limit set \( \omega(t) \) of the operator (1.4) consists a single point and

\[
\omega(t) \in \begin{cases} 
\{(0, a, b, 0)\} & \text{if } t = (x, y, u, v) \in T_0, \\
F_{12} & \text{if } t = (x, y, u, v) \in T_1, \\
F_{11} & \text{if } t = (x, y, u, v) \in T_2.
\end{cases}
\] (3.11)

**Definition 3.** An operator \( W \) is called regular if for any initial point \( s^{(0)} \in S^{2,2} \), the limit

\[
\lim_{m \to \infty} W^m(s^{(0)})
\]

exists.

The following is a corollary of Theorem 1

**Corollary 2.** The operator (1.4) is regular.

We would like to describe the sets \( T_0, T_1 \) and \( T_2 \) implicitly.

3.1. **Case 1.** Let us have

\[
p_1 = p_2 = 1.
\]

Then operator (3.2) looks like:

\[
V_1 : \begin{cases} 
\alpha' = \alpha - \alpha\beta \\
\beta' = \beta - \alpha\beta
\end{cases}
\] (3.12)

where \((\alpha; \beta) \in \Delta.\n
s_1 = (\alpha, 0) \quad \text{and} \quad s_2 = (0, \beta) \quad \text{are non-hyperbolic fixed points of (3.12) with the eigenvalues} \lambda_1 = 1, \lambda_2 = 1 - \alpha \in [0, 1] \quad \text{and} \lambda_1 = 1, \lambda_2 = 1 - \beta \in [0, 1] \quad \text{respectively.} \n
We say the set \( E \) is invariant respect to the operator \( V \) if \( V(E) \subseteq E. \)
Lemma 2. The following sets
\[ M_0 = \{ (\alpha, \beta) \in [0,1]^2 : \beta = \alpha \} \]
\[ M_1 = \{ (\alpha, \beta) \in [0,1]^2 : \beta < \alpha \} \]
and
\[ M_2 = \{ (\alpha, \beta) \in [0,1]^2 : \beta > \alpha \} \]
are invariant sets respect to the operator (3.12).

Proof. Straightforward. □

We look for the invariant curves of the operator (3.12). Let \( \beta = g(\alpha) \) be an invariant curve then \( \beta' = g'(\alpha') \) and to find invariant curve leads to solve the following iterative functional equation
\[ f(\alpha)(\alpha - f(\alpha))(1 - \alpha) = \alpha(f(\alpha) - f(f(\alpha))) \] (3.13)
where \( f(\alpha) = \alpha(1 - g(\alpha)) \) which is not identically zero.

We solve (3.13) in the space \( C^\infty[0,1] \).

The equation (3.13) gives \( f(0) = 0 \). Moreover from \( f \in C^\infty[0,1] \) we get
\[ f(\alpha) = \sum_{k=1}^{\infty} c_k \alpha^k \] (3.14)
and
\[ f(f(\alpha)) = \sum_{k=1}^{\infty} c_k f^k(\alpha) = \sum_{k=1}^{\infty} d_k \alpha^k \] (3.15)
where
\[ d_k = \sum_{l=1}^{k} c_l \left( \sum_{i_1+i_2+...+i_l=k} c_{i_1} \cdot c_{i_2} \cdot ... \cdot c_{i_l} \right). \]

Theorem 2. The solutions of the functional equation (3.13) are
\[ f(\alpha) = \alpha \quad \text{and} \quad f(\alpha) = \theta \alpha - \alpha^2 \]
where \( \theta \) is an arbitrary constant.

In particular
\[ g(\alpha) = 0 \quad \text{and} \quad g(\alpha) = \alpha + 1 - \theta \]
are the only invariant curves of the operator (3.12).

Proof. Substituting (3.14) and (3.15) to the (3.13) we obtain
\[ \sum_{k=1}^{\infty} a_k \alpha^k \equiv \sum_{k=1}^{\infty} b_k \alpha^k, \] (3.16)
which is equivalently to
\[ a_k = b_k \quad \text{for all} \quad k = 1, 2, ... \] (3.17)
where
\[ b_k = \sum_{l=1}^{k} c_{l+1} \left( \sum_{i_1+i_2+...+i_l=k} c_{i_1} \cdot c_{i_2} \cdot ... \cdot c_{i_l} \right) \]
and
\[ a_k = \begin{cases} 
-1 - c_2 + c_1 & \text{if } k = 1, \\
c_{k+1} - c_k & \text{if } k = 2, 3, ... \end{cases} \quad (3.18) \]
From identity of (3.16) for \( k = 1 \) it holds that
\[(1 - c_1)(1 + c_2) = 0.\]
For \( k = 2 \) we see that
\[ c_3(1 - c_1^2) = c_2(1 + c_2). \]
For \( k = 3 \) we see that
\[ c_4(1 - c_1^3) = c_3(1 + c_2 + 2c_1c_2). \]
These last three equations imply that
\[ c_1 = 1, \ c_2 = 0, \ c_3 = 0 \text{ or } c_1 \text{ is arbitrary, } c_2 = -1, \ c_3 = 0. \]
Now we show by induction that \( c_k = 0 \) for all \( k = 3, 4, ... \).
Suppose \( c_k = 0 \) for all \( k = 3, 4, ... n \). Then putting \( k = n, \ k = n + 1 \) and \( k = n + 2 \) in (3.17) we get
\[ c_{n+1} - c_n = \sum_{l=1}^{n} c_{l+1} \left( \sum_{i_1+i_2+...+i_l=n} c_{i_1} \cdot c_{i_2} \cdot ... \cdot c_{i_l} \right) \]
\[ c_{n+2} - c_{n+1} = \sum_{l=1}^{n+1} c_{l+1} \left( \sum_{i_1+i_2+...+i_l=n+1} c_{i_1} \cdot c_{i_2} \cdot ... \cdot c_{i_l} \right) \]
and
\[ c_{n+3} - c_{n+2} = \sum_{l=1}^{n+2} c_{l+1} \left( \sum_{i_1+i_2+...+i_l=n+2} c_{i_1} \cdot c_{i_2} \cdot ... \cdot c_{i_l} \right). \]
These last three equations equivalent to
\[ c_{n+1} = c_{n+1}c_1^n \]
\[ c_{n+2}(1 - c_1^{n+1}) = c_{n+1}(1 + c_2 + nc_2c_1^{n-1}) \]
\[ c_{n+3}(1 - c_1^{n+2}) = c_{n+2} + c_2c_{n+2} + \frac{n!}{2!}c_{n+1}^2c_1^{n-2} + (n + 1)c_{n+2}c_2c_1^n \]
If \( c_1 \neq \pm 1 \) or \( c_1 = -1 \) and \( n \) is odd then (3.19) gives \( c_{n+1} = 0. \) If \( c_1 = 1 \) then (3.20) gives \( c_{n+1} = 0, \) otherwise if \( c_1 = -1 \) and \( n \) is even then from (3.20) and (3.21) we come to
\[ c_{n+1}[(1 + (1 - n)c_2)(1 + (n + 1)c_2) + n!c_2^2] = 0 \]
which shows again that \( c_{n+1} = 0. \) Thus for all \( k = 3, 4, ... \) we have \( c_k = 0. \) That is
\[ f(\alpha) = \alpha \quad \text{and} \quad f(\alpha) = \theta\alpha - \alpha^2 \]
are solutions of the iterative functional equation (3.13), where \( \theta \) is an arbitrary constant.
This completes the proof. □

So, we have proved that
\[ \gamma_\theta = \{ (\alpha, \beta) \in [0,1]^2 : \beta = g(\alpha) = \alpha + 1 - \theta, \; \theta \in [0,2] \} \]
is one-parametric family of invariant curves.

Note that
\[
\bigcup_{\theta \in [0,1)} \gamma_\theta = M_2, \quad \bigcup_{\theta \in (1,2]} \gamma_\theta = M_1, \quad \gamma_1 = M_0
\]
and
\[ \gamma_{\theta_1} \cap \gamma_{\theta_2} = \emptyset \text{ for any } \theta_1 \neq \theta_2. \]
Thus it suffices to study the dynamical system on each invariant curve \( \gamma_\theta \). We have the following result (See Figure 1).

**Theorem 3.** The following assertions hold

(i) If \( \theta = 1 \) then for any initial point \( t = (\alpha, \beta) \in M_0 \), (i.e. \( \alpha = \beta \)) we have
\[
\lim_{m \to \infty} V_1^{(m)}(\alpha, \beta) = \lim_{m \to \infty} (\alpha^{(m)}, \beta^{(m)}) = (0; 0).
\]

(ii) If \( \theta \in (1,2] \) then for any initial point \( t = (\alpha, \beta) \in \gamma_\theta \) we have
\[
\lim_{m \to \infty} V_1^{(m)}(\alpha, \beta) = \lim_{m \to \infty} (\alpha^{(m)}, \beta^{(m)}) = (\theta - 1; 0).
\]

(iii) If \( \theta \in [0,1) \) then for any initial point \( t = (\alpha, \beta) \in \gamma_\theta \) we have
\[
\lim_{m \to \infty} V_1^{(m)}(\alpha, \beta) = \lim_{m \to \infty} (\alpha^{(m)}, \beta^{(m)}) = (0; 1 - \theta).
\]

![Figure 1. Dynamics of the operator (3.12) on the invariant lines \( \gamma_\theta \). The trajectory converges to the fixed point on the intersection of the line and the axes \( O\alpha \) or \( O\beta \).](image-url)
Going back to the old variables \((x, y, u, v)\), when \(p_1 = p_2 = 1\) we obtain \(\sigma_1 = a\), \(\sigma_2 = b\) and

\[
\Omega_\theta = \{(x, y, u, v) \in S^{2,2} : \frac{v}{u + v} = \frac{x}{x + y} + 1 - \theta\}
\]
is an invariant surface respect to the operator \((1.4)\) and it holds that

\[
\bigcup_{\theta \in [0,1)} \Omega_\theta = T_2 = \{(x, y, u, v) \in S^{2,2} : yv > xu\}, \quad \bigcup_{\theta \in (1,2]} \Omega_\theta = T_1 = \{(x, y, u, v) \in S^{2,2} : yv < xu\}, \quad \Omega_1 = T_0 = \{(x, y, u, v) \in S^{2,2} : yv = xu\}
\]
and

\(\Omega_{\theta_1} \cap \Omega_{\theta_2} = \emptyset\) for any \(\theta_1 \neq \theta_2\).

Thus it suffices to study the dynamical system on each invariant surfaces \(\Omega_\theta\). As a corollary of Theorem 3 we have the following

**Theorem 4.** The following assertions hold

(i) For any initial point \(t = (x, y, u, v) \in T_0\), we have

\[
\lim_{m \to \infty} W^{(m)}(x, y, u, v) = \lim_{m \to \infty} (x^{(m)}, y^{(m)}, u^{(m)}, v^{(m)}) = (0; a; b; 0).
\]

(ii) If \(\theta \in (1, 2]\) then for any initial point \(t = (x, y, u, v) \in \Omega_\theta\) the following holds

\[
\lim_{m \to \infty} W^{(m)}(x, y, u, v) = \lim_{m \to \infty} (x^{(m)}, y^{(m)}, u^{(m)}, v^{(m)}) = (a(\theta - 1); a(2 - \theta); b; 0).
\]

(iii) If \(\theta \in [0, 1)\) then for any initial point \(t = (x, y, u, v) \in \Omega_\theta\) the following holds

\[
\lim_{m \to \infty} W^{(m)}(x, y, u, v) = \lim_{m \to \infty} (x^{(m)}, y^{(m)}, u^{(m)}, v^{(m)}) = (0; a; b\theta; b(1 - \theta)).
\]

**Corollary 3.** The operator \((1.4)\) has infinitely many fixed points and for each such fixed point there is nonintersecting trajectories which converge to the fixed points.

3.2. **Case 2.** Let we have

\[ p_1 > 1 > p_2 > 0, \]

or

\[ p_2 > 1 > p_1 > 0. \]

**Lemma 3.** The set

\[ M_1 = \{(\alpha, \beta) \in \Delta : \beta \geq \alpha\} \]
is an invariant set respect to the operator \((3.2)\) when \(p_1 > 1 > p_2 > 0\).

The set

\[ M_2 = \{(\alpha, \beta) \in \Delta : \beta \leq \alpha\} \]
is an invariant set respect to the operator \((3.2)\) when \(p_2 > 1 > p_1 > 0\).

**Proof.** Straightforward. \(\Box\)
In this case to find invariant curves for the operator (3.2) leads to solve the following iterative functional equation

\[ f(\alpha)(\alpha - f(\alpha))(1 - \alpha)[1 + (p_1 - 1)f(f(\alpha))] = \alpha(f(\alpha) - f(f(\alpha)))[1 + (p_2 - 1)\alpha + (p_1 - p_2)f(\alpha)] \] (3.22)

where \( f(\alpha) = \alpha(1 - g(\alpha)) \setminus [1 + (p_1 - 1)g(\alpha)] \) which is not identically zero.

As above when we search the solution of the last functional equation in the space \( C^\infty[0, 1] \) we get

\[ \sum_{k=1}^{\infty} a_k \alpha^k \equiv \sum_{k=1}^{\infty} b_k \alpha^k, \] (3.23)

which is equivalently to

\[ a_k = b_k \text{ for all } k = 1, 2, ... \] (3.24)

where

\[ a_k = \sum_{j=0}^{k} e_j q_{k-j}, \quad b_k = \sum_{j=0}^{k} n_j m_{k-j} \]

and

\[ e_j = \begin{cases} 1 - c_1 & \text{if } j = 0, \\ -\sum_{l=1}^{j} q_{l+1} \left( \sum_{i_1+i_2+...+i_l=j} c_{i_1} \cdot c_{i_2} \cdot ... \cdot c_{i_l} \right) & \text{if } j = 1, 2, ..., k. \end{cases} \] (3.25)

\[ q_j = \begin{cases} 1 & \text{if } j = 0, \\ (p_2 - 1) + (p_1 - p_2)c_1 & \text{if } j = 1, \\ (p_1 - p_2)c_j & \text{if } j = 2, 3, ..., k. \end{cases} \] (3.26)

\[ n_j = \begin{cases} 1 & \text{if } j = 0, \\ (p_1 - 1) \sum_{l=1}^{j} q_{l+1} \left( \sum_{i_1+i_2+...+i_l=j} c_{i_1} \cdot c_{i_2} \cdot ... \cdot c_{i_l} \right) & \text{if } j = 1, 2, ..., k. \end{cases} \] (3.27)

\[ m_j = \begin{cases} 1 - c_1 & \text{if } j = 0, \\ -1 + c_1 - c_2 & \text{if } j = 1, \\ c_j - c_{j+1} & \text{if } j = 2, 3, ..., k. \end{cases} \] (3.28)

Substituting (3.25), (3.26), (3.27), (3.28) to the (3.24) then when \( k \geq 3 \) we attain recurrence formula for \( c_k \).
Figure 2. Dynamics of the operator (3.2) on invariant concave curves for the case $p_1 > p_2 > 0$. The trajectory converges to the fixed point on the intersection of the invariant curve and the axes $O\alpha$ or $O\beta$.

\[ c_k(1-c_k^{k-1}) = \sum_{j=1}^{k-2} [(c_{k-j-1} - c_{k-j})(p_1 - 1)d_j + (p_1 - p_2)c_{k-j}d'_j] \]
\[ + c_{k-1} - (p_1 - 1)d_{k-2} + (1 - c_1)(p_1 - 1)d_{k-1}(1 - c_1)(p_1 - p_2)c_k \]
\[ + (p_2 - 1)d'_{k-2} - \sum_{l=1}^{k-1} c_{l+1}(\sum_{i_1+i_2+...+i_l=k} c_{i_1} \cdot c_{i_2} \cdot ... \cdot c_{i_l}) \]  

(3.29)

where $c_k$ is the coefficient at (3.14) and

\[ d_j = \sum_{l=1}^{j} c_l(\sum_{i_1+i_2+...+i_l=j} c_{i_1} \cdot c_{i_2} \cdot ... \cdot c_{i_l}), \quad d'_j = \sum_{l=1}^{j} c_{l+1}(\sum_{i_1+i_2+...+i_l=j} c_{i_1} \cdot c_{i_2} \cdot ... \cdot c_{i_l}). \]

We were not able to solve these systems for the coefficients. Therefore the following is an open problem:

**Open problem.** Describe all solutions of the functional equation (3.22).

Numerical analysis shows that (see at the Figure 2 and Figure 3) in the cases $p_1 > 1 > p_2 > 0$ (resp. in the case $p_2 > 1 > p_1 > 0$) there are nonintersecting concave (resp. convex) invariant curves and the trajectory started on an invariant curve converges to the intersecting point of the invariant curve and the axes $O\alpha$ or $O\beta$. These numerical analysis
Figure 3. Dynamics of the operator (3.2) on invariant convex curves for the case $p_2 > 1 > p_1 > 0$. The trajectory converges to the fixed point on the intersection of the invariant curve and the axes $O\alpha$ or $O\beta$.

and the above considered particular cases allowed us to make the following

**Conjecture.** If $p_1 > 1 > p_2 > 0$ (or $p_2 > 1 > p_1 > 0$) then for each fixed point $p \in \text{Fix}(W)$ there exists unique invariant surface $\Gamma_p \subset S^{2,2}$, such that for any initial point $s^{(0)} \in \Gamma_p$ the limit of its trajectory (under operator (1.4)) converges to the fixed point $p$. Moreover,

$$\bigcup_{p \in \text{Fix}(W)} \Gamma_p = S^{2,2}.$$  

4. Conclusion

Let $s^{(0)} = (x, y, u, v) \in S^{2,2}$ be an initial state, i.e. the probability distribution on the set of female and male types.

The following are interpretations of our results:

- The set of all fixed points is subset of the boundary of $S^{2,2}$ means that at least one type of female or male in future of population will surely disappear.
- The existence of invariant curves (in particular lines) means that if states of the population initially satisfied a relation (described the invariant set) then the future of the population remains in the same relation.
- Regularity of the operator means that for any initial state of the population we can explicitly determine its limit (final) state.
• For any \( s^{(0)} \in T_0 \) as time goes to infinity the type 1 of female and type 2 of males will disappear (die).
• For any \( s^{(0)} \in T_1 \) as time goes to infinity the type 2 of males will disappear.
• For any \( s^{(0)} \in T_2 \) as time goes to infinity the type 1 of females will disappear.

References

[1] Absalamov A.T., Rozikov U.A. The Dynamics of Gonosomal Evolution Operators, Jour. Applied Nonlinear Dynamics. 9(2) (2020), 247–257.
[2] Absalamov A.T. The Global Attractiveness of the Fixed Point of a Gonosomal Evolution Operator. Discontinuity Nonlinearity and Complexity. 10(1) (2021), 143–149.
[3] Baca"er N. A short history of mathematical population dynamics. Springer-Verlag London, Ltd., London, 2011.
[4] Ganikhodzhaev R.N., Mukhamedov F.M. and Rozikov U.A. Quadratic stochastic operators and processes: results and open problems. Inf. Dim. Anal. Quant. Prob. Rel. Fields. 14(2), (2011), 279–335.
[5] Hardin A.J.M., Rozikov U.A. A quasi-strictly non-Volterra quadratic stochastic operator. Qualit. Theory Dyn. Syst. 18(3) (2019), 1013–1029.
[6] Kesten H. Quadratic transformations: A model for population growth, I, II, Adv. Appl. Probab. 2(2) (1970), 1–82; 179–228.
[7] Ladra M., Rozikov U.A. Evolution algebra of a bisexual population. Jour. Algebra. 378 (2013), 153–172.
[8] Lyubich Y.I. Mathematical structures in population genetics. Springer-Vergar, Berlin (1992)
[9] Reed M.L. Algebraic structure of genetic inheritance. Bull. Amer. Math. Soc. (N.S.) 34(2) (1997), 107–130.
[10] Rozikov U.A., Population dynamics: algebraic and probabilistic approach. World Sci. Publ. Singapore, 2020.
[11] Rozikov U.A., Usmonov J.B. Dynamics of a population with two equal dominated species. Qualit. Theory Dyn. Syst. 19(2) (2020), Paper No. 62, 19 pages.
[12] Rozikov U.A., Shoyimardonov S.K. Leslie’s prey-predator model in discrete time. Inter. Jour. Biomath. 13(6) (2020), 2050053, 25 pages.
[13] Rozikov U.A. Evolution operators and algebras of sex linked inheritance. Asia Pacific Math. Newsletter. 3(1) (2013), 6–11.
[14] Rozikov U.A., Zhamilov U.U. Volterra quadratic stochastic operators of bisexual population. Ukraine Math. Jour. 63(7) (2011), 985–998.
[15] Rozikov U.A., Varro R. Dynamical systems generated by a gonosomal evolution operator. Discontinuity, Nonlinearity and Complexity, 5 (2016), 173–185.
[16] Varro R. Gonosomal algebra. Jour. Algebra, 447 (2016), 1–30.

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