Integrals containing the infinite product $\prod_{n=0}^{\infty} \left[ 1 + \left( \frac{x}{n+b} \right)^3 \right]$

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We study several integrals that contain the infinite product $\prod_{n=0}^{\infty} \left[ 1 + \left( \frac{x}{n+b} \right)^3 \right]$ in the denominator of their integrand. These considerations lead to closed form evaluation

$$\int_{-\infty}^{\infty} \frac{dx}{(e^x + e^{-x} + e^{ix\sqrt{3}})^2} = \frac{1}{3}$$

and to some other formulas.

1. The infinite product

$$\prod_{n=0}^{\infty} \left[ 1 + \left( \frac{\alpha + \beta n}{n+\alpha} \right)^3 \right]$$

and more general products have been studied in the literature (see [1], ch. 16). In this paper we consider integrals of the form

$$\int_{0}^{\infty} P_b(x)f(x)dx,$$  \hspace{1cm} (1)

where

$$P_b(x) = \frac{1}{\prod_{k=0}^{\infty} \left( 1 + \frac{x^3}{(k+b)^3} \right)}.$$  \hspace{1cm} (2)

Several such integrals will be evaluated in closed form. However while others do not have a closed form will allow us to evaluate some integrals of elementary functions.

Note that the infinite product in (2) can be written in terms of Gamma functions [2]

$$P_b(x) = \frac{\Gamma(b+x)\Gamma(b+\omega x)\Gamma(b+x/\omega)}{\Gamma^3(b)}, \quad \omega = e^{2\pi i/3}.$$  \hspace{1cm} (2)

The notation $\omega = e^{2\pi i/3}$ for third root of unity will be used throughout the paper.

2. Consider the contour integral

$$\int_{C} P_b(z) \frac{dz}{z}.$$  \hspace{1cm} (3)

along the contour depicted in Fig.1. We assume that $b > 0$. The most interesting cases considered in this paper correspond to $b = 1$ and $b = 1/2$.

Inside the contour of integration, the integrand $h(z) = P_b(z)/z$ has simple poles at $z = -(k+b-1)/\omega$, $k \in \mathbb{N}$, with residues

$$\frac{(-1)^k}{(k-1)!} \frac{|\Gamma(b - \omega(k+b-1))|^2}{(k+b-1)\Gamma^3(b)},$$

and no poles on the contour of integration if we choose $R = N + b - 1/2$ for some large natural number $N$. Also $h(z)dz$ is symmetric under the change $z \rightarrow \omega z$, and as a consequence the integrals along straight lines cancel each other out. Let’s denote the integrals along $\Gamma_R$ and $C_\varepsilon$ as $I_R$ and $I_\varepsilon$ respectively. Then

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon = -\frac{2\pi i}{3}.$$
and (Appendix A)

$$ \lim_{R \to +\infty} I_R = 0. $$

Using residue theorem we get

$$ \sum_{n=0}^{\infty} \frac{(-1)^n |\Gamma(b - \omega(n + b))|^2}{n!} = \frac{1}{3} \Gamma^3(b). \quad (4) $$

The integral 3.985.1 from [3]

$$ \int_{-\infty}^{\infty} \frac{e^{iax}}{\cosh^\nu \beta x} dx = \frac{2^{\nu-1}}{\beta \Gamma(\nu)} \Gamma \left( \frac{\nu + ai}{2} + \frac{a}{2\beta} \right) \Gamma \left( \frac{\nu - ai}{2} - \frac{a}{2\beta} \right) \quad (5) $$

allows to write (4) as an integral of a hypergeometric function

$$ \int_{-\infty}^{\infty} \frac{e^{ib\sqrt{3}x}}{\cosh^3 x} \frac{_{2}F_{1}(b, 3b, b + 1; -\frac{e^{i\sqrt{3}x}}{2 \cosh x})}{2} dx = 2^{3b-1} \frac{b}{3} \Gamma^3(b). \quad (6) $$

3. Here we specialize $b$ in (6) so that the hypergeometric function can be written in terms of elementary functions. This happens when $b = 1 + 3n$ or $b = 1/2 + 3n$, where $n$ is a non-negative integer. Only the two cases with $n = 0$ are considered below:

Let $b = 1/2$, then the hypergeometric function becomes $\sqrt{\frac{2 \cosh x}{2 \cosh x + e^{i\sqrt{3}x}}}$, and we get

$$ \int_{-\infty}^{\infty} \frac{\text{sech} x \, e^{i\frac{\sqrt{3}x}{2}}}{\sqrt{e^x + e^{-x} + e^{i\sqrt{3}x}}} dx = \frac{\pi}{3}. \quad (7) $$

If $b = 1$ then the hypergeometric function becomes $\frac{2 \cosh x (4 \cosh x + e^{i\sqrt{3}x})}{(2 \cosh x + e^{i\sqrt{3}x})^2}$, and we get

$$ \int_{-\infty}^{\infty} \frac{e^{i\sqrt{3}x}}{\cosh^2 x} \frac{4 \cosh x + e^{i\sqrt{3}x}}{(2 \cosh x + e^{i\sqrt{3}x})^2} dx = \frac{2}{3}, $$

which due to $\frac{4 \cosh x + e^{i\sqrt{3}x}}{\cosh^2 x (2 \cosh x + e^{i\sqrt{3}x})} = -\frac{1}{(2 \cosh x + e^{i\sqrt{3}x})} + \frac{1}{\cosh^2 x}$ can be simplified further as

$$ \int_{-\infty}^{\infty} \frac{dx}{(e^x + e^{-x} + e^{i\sqrt{3}x})^2} = \frac{1}{3}. \quad (8) $$
It is interesting to note that there is another way to write the sum (4) with \( b = 1 \) as an integral
\[
\int_{-\infty}^{\infty} \frac{e^{i\sqrt{3}x} \cosh x}{(e^x + e^{-x} + e^{i\sqrt{3}x})^2} \, dx = \frac{1}{12}. \tag{9}
\]

One might observe how the \( 2\pi/3 \) rotation symmetry of the product \( \prod_{k=1}^{\infty} \left( 1 + \frac{x^3}{k^3} \right) \) manifests itself in (8) and (9): The set of roots of the equation \( e^x + e^{-x} + e^{i\sqrt{3}x} = 0 \) has the same \( 2\pi/3 \) rotation symmetry (see Appendix B).

4. The last integral in section 3 gives
\[
\int_{-\infty}^{\infty} \frac{e^{i\sqrt{3}x} \sinh x \, dx}{(2 \cosh x + e^{i\sqrt{3}x})^2} = \frac{i\sqrt{3}}{12}. \tag{10}
\]

5. After the substitution \( t = e^{2x} \) equation (8) becomes
\[
\int_{0}^{\infty} \frac{dt}{(1 + t + t^{\alpha})^2} = \frac{2}{3}, \quad \alpha = \frac{1 + i\sqrt{3}}{2}. \tag{11}
\]

while combining (9) and (10) we find two analogous representations

\[ \int_{0}^{\infty} \frac{t^{\alpha}dt}{(1 + t + t^{\alpha})^2} = \frac{\alpha}{3}, \tag{12} \]
\[ \int_{0}^{\infty} \frac{t^{\alpha-1}dt}{(1 + t + t^{\alpha})^2} = \frac{1}{3\alpha}. \tag{13} \]

(13) is related to (12) by complex conjugation and change of variable \( t \to 1/t \).

6. If we apply the approach of section 2 to
\[
\int_{C} P_b(z) \, dz,
\]
instead, then the integrals over straight lines no longer cancel out. However, there is nevertheless a simplification: in this case the sum analogous to (4) reduces to an integral of elementary function for all \( b \) so that in this case we find the transformation
\[
\int_{0}^{\infty} \frac{dx}{\prod_{k=0}^{\infty} \left( 1 + \frac{x^3}{(k+b)^3} \right)} = \frac{4\pi \Gamma(3b)}{\Gamma^3(b) \sqrt{3}} \int_{-\infty}^{\infty} \frac{e^{ix\sqrt{3}} \, dx}{(e^x + e^{-x} + e^{ix\sqrt{3}})^{3b}}. \tag{14}
\]
7. In principle integrals (7) and (8) can be written in terms of real-valued functions by calculating the real part of the integrand. The resulting formulas are cumbersome and therefore omitted. However there is another way to get a compact integral of a real valued function, at least for (8). First of all, all the roots of the function \( e^{z} + e^{-z} + e^{iz\sqrt{3}} \) lie on the three rays \( z = ir\alpha^k, \ r > 0, \ (k = 0, 1, 2) \) (see Appendix A). If one bends the contour of integration so that it never crosses these zeroes then the integral (8) will not change. Since the integrand decreases exponentially when \( z \to \infty \), \( 0 < \arg z < \pi/6 \) or \( 5\pi/6 < \arg z < \pi \) we have

\[
\frac{1}{\beta} \int_0^\infty \frac{dx}{\left(e^{-x/\beta} + e^{x/\beta} + e^{-ix\sqrt{3}/\beta}\right)^2} + \beta \int_0^\infty \frac{dx}{\left(e^{x/\beta} + e^{-x/\beta} + e^{ix\sqrt{3}/\beta}\right)^2} = \frac{1}{3}, \quad \beta = e^{\pi i/6},
\]

and after elementary simplifications

\[
\int_0^\infty \frac{e^{x\sqrt{3}} \cos \left(\frac{\pi}{6} x\right)}{(2\cos x + e^{x\sqrt{3}})^2} dx = \frac{1}{6}. \quad (15)
\]

Similarly, for the case \( b = 1 \) of (14)

\[
\int_0^\infty \frac{dx}{\left(1 + \frac{x^2}{\beta^2}\right) \left(1 + \frac{x^2}{\beta^2}\right) \left(1 + \frac{x^2}{\beta^2}\right) \cdots} = 8\pi \int_0^\infty \frac{e^{\sqrt{3}} x}{(2\cos x + e^{\sqrt{3}})^3} dx. \quad (16)
\]

8. It turns out that (8) has a parametric extension. Consider the contour integral

\[
\int_{C'} P_1(z) \frac{e^{ax} dz}{z}, \quad (17)
\]

where the contour \( C' \) is a circle of radius \( R = N + 1/2 \) for large natural \( N \). Since \( |P_1(z)| \) decreases exponentially with \( N \) on the circle \( C' \) (Appendix A), (17) will be zero in the limit \( N \to \infty \) for sufficiently small \( |a| \). Therefore the sum of residues of the integrand over three sets of simple poles \( z = -ke^{2\pi i j/3}, \ k \in \mathbb{N}, \ (j = 0, 1, 2) \) plus a simple pole at the origin, will be 0 according to residue theorem. As a result one will obtain three sums similar to (4) and then convert them to integrals of the type (8). However there is a trick that allows to avoid these calculations. Note that the factor \( e^{ax} \) in (17) will introduce additional factors \( \exp \left(-ake^{2\pi i j/3}\right) \) in the sum over residues. When converted to an integral via (4) these factors have the effect of multiplying \( e^{ix\sqrt{3}} \) by \( \exp \left(-ae^{2\pi i j/3}\right) \):

\[
\sum_{j=1}^{3} \int_{-\infty}^{\infty} \frac{dx}{\left(e^x + e^{-x} + \exp \left(-ae^{2\pi i j/3}\right) e^{ix\sqrt{3}}\right)^2} = 1,
\]

or equivalently

\[
\int_{-\infty}^{\infty} \frac{dx}{\left(e^x + e^{-x} + e^{ax+ix\sqrt{3}}\right)^2} + \int_{-\infty}^{\infty} \frac{e^a dx}{\left(e^{ax} + e^{-x} + e^{ix\sqrt{3}}\right)^2} + \int_{-\infty}^{\infty} \frac{e^a dx}{\left(e^{ax} + e^{-x} + e^{-ix\sqrt{3}}\right)^2} = 1, \quad (18)
\]

where \( |a| \) is sufficiently small.

9. There is a similarity between (4) with \( b = 1 \) and the identity due to Ramanujan (4), p. 309)

\[
e^{ax} = -a \sum_{k=0}^{\infty} \frac{\Gamma \left(\frac{a+k(ci-b)}{2ci}\right) (-2ie^{-by} \sin cy)^k}{\Gamma \left(\frac{-a+k(ci-b)}{2ci} + 1\right) k!}.
\]
After equating the coefficients of $a^1$ in Taylor series expansion of both sides into powers of $a$ and transforming the Gamma function in the denominator via Euler’s reflection formula

$$y = \frac{1}{2\pi i c} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \Gamma\left(\frac{k}{2} + \frac{ikb}{2c}\right) \Gamma\left(\frac{k}{2} + \frac{ikb}{2c}\right) \sin \pi\left(\frac{k}{2} + \frac{ikb}{2c}\right) (-2ie^{-by} \sin cy)^k. \quad (19)$$

To make this similarity more exact we differentiate (19) with respect to $y$, divide by $(c \cot cy - b)$, repeat this procedure one more time and then set $c = 1, b = \sqrt{3}$

$$\frac{2\pi i \sin y}{\cos y - \sqrt{3} \sin y} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left| \Gamma(1 - \omega k) \right|^2 \sin \left(\pi ke^{\pi i/3}\right) \left(-2ie^{-\sqrt{3}y} \sin y\right)^k. \quad (20)$$

This series converges when $|2e^{-\sqrt{3}y} \sin y| < e^{-\pi/2}\sqrt{3}$. It will be convenient to use another variable $\alpha$ related to $y$ by

$$e^{-\alpha} = 2e^{-\sqrt{3}y} \sin y.$$

The condition that the series (20) converges now takes a very simple form $\Re \alpha > \frac{\pi}{2\sqrt{3}}$. In the following it will be assumed for simplicity that $\alpha > \frac{\pi}{2\sqrt{3}}$.

Is it possible that (20) leads to evaluation of integrals with infinite product $\prod_{k=1}^{\infty} \left(1 + z_k^3\right)$? Consider the contour integral

$$\int_C \frac{(-ie^{-\alpha})^{-\omega z} \sin \pi z}{z \prod_{k=1}^{\infty} \left(1 + z_k^3\right)} \, dz, \quad (21)$$

where $C$ is the contour in Fig.2. Due to the asymptotics

$$|(-ie^{-\alpha})^{-\omega z} \sin \pi z| \sim \frac{1}{2} \exp \left[-\left(\frac{\alpha}{2} + \frac{\pi \sqrt{3}}{4}\right)x + \left(\frac{5\pi}{4} - \frac{\alpha \sqrt{3}}{2}\right)y\right], \quad 0 < \arg z < \frac{2\pi}{3},$$

and the result of Appendix, the integral over circular arc $\Gamma_R$ vanishes in the limit $R \to \infty$ if

$$\begin{cases} -\left(\frac{\alpha}{2} + \frac{11\pi}{4\sqrt{3}}\right)x + \left(\frac{5\pi}{4} - \frac{\alpha \sqrt{3}}{2}\right)y < 0, & 0 < \arg z < \frac{\pi}{3}, \\ \left(\frac{\pi}{2\sqrt{3}} - \alpha\right)(x + y\sqrt{3}) < 0, & \frac{\pi}{3} < \arg z < \frac{2\pi}{3}. \end{cases}$$

When $\alpha > \frac{\pi}{2\sqrt{3}}$ these conditions are satisfied automatically.

The same approach as in section 2 yields

$$\int_0^{\infty} \frac{(-ie^{-\alpha})^{-\omega x} \sin \pi x - (-ie^{-\alpha})^{-x/\omega} \sin \pi \omega x}{x \prod_{k=1}^{\infty} \left(1 + z_k^3\right)} \, dx$$

$$= 2\pi i \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \left| \Gamma(1 - \omega k) \right|^2 \sin \left(\pi ke^{\pi i/3}\right) (-ie^{-\alpha})^k$$

$$= \frac{4\pi^2 \sin y}{(\cos y - \sqrt{3} \sin y)^3}.$$

For real $\alpha$ one can decompose the function in the numerator of this integral into real and imaginary parts

$$(-ie^{-\alpha})^{-\omega x} \sin \pi x - (-ie^{-\alpha})^{-x/\omega} \sin \pi \omega x = f(x, \alpha) + ig(x, \alpha)$$
where
\[
f(x, \alpha) = \frac{1}{2} e^{-\sqrt{3} \pi x / 4 - \alpha x / 2} \left( e^{\sqrt{3} \pi x} \sin \left( \frac{\pi - 2\sqrt{3} \alpha}{4} x \right) + 2 \sin \left( \frac{2\sqrt{3} \alpha + 3\pi}{4} x \right) - \sin \left( \frac{2\sqrt{3} \alpha - 5\pi}{4} x \right) \right),
\]
\[
g(x, \alpha) = \frac{1}{2} e^{-\sqrt{3} \pi x / 4 - \alpha x / 2} \left( \cos \left( \frac{2\sqrt{3} \alpha - 5\pi}{4} x \right) - e^{\sqrt{3} \pi x} \cos \left( \frac{\pi - 2\sqrt{3} \alpha}{4} x \right) \right).
\]

As a result
\[
\int_0^\infty \frac{g(x, \alpha) \, dx}{x \prod_{k=1}^\infty \left( 1 + \frac{x^3}{k^3} \right)} = 0,
\]
(22)
\[
\int_0^\infty \frac{f(x, \alpha) \, dx}{x \prod_{k=1}^\infty \left( 1 + \frac{x^3}{k^3} \right)} = \frac{8\pi^2 \sin \frac{y}{\sqrt{3}}}{(\sqrt{3} \sin y - \cos y)^3},
\]
(23)

where \(y\) is the root of the equation \(2e^{-y\sqrt{3}} \sin y = e^{-\alpha}\) near \(y = 0\).

These formulas simplify when \(\alpha = \frac{5\pi}{2\sqrt{3}}\)
\[
\int_0^\infty \frac{1 - e^{\pi \sqrt{3} \cos \pi x} e^{-\frac{\pi}{2\sqrt{3}} x}}{x \left( 1 + \frac{x^3}{3!} \right) \left( 1 + \frac{x^3}{2^3} \right) \left( 1 + \frac{x^3}{2^3} \right) \ldots} \, dx = 0,
\]
(24)
\[
\int_0^\infty \frac{\sin \pi x \left( 4 \cos \pi x - e^{\pi \sqrt{3} x} \right) e^{-\frac{\pi}{2\sqrt{3}} x}}{x \left( 1 + \frac{x^3}{3!} \right) \left( 1 + \frac{x^3}{2^3} \right) \left( 1 + \frac{x^3}{2^3} \right) \ldots} \, dx = \frac{8\pi^2 \sin \frac{y}{\sqrt{3}}}{(\sqrt{3} \sin y - \cos y)^3},
\]
(25)

where \(y = 0.0054167536 \ldots\) is the root of the equation \(2e^{-y\sqrt{3}} \sin y = e^{-\frac{5\pi}{2\sqrt{3}}}\).

10. The hyperbolic log-trigonometric integral
\[
\text{Im} \int_0^\infty \frac{dt}{(it\sqrt{3} + \ln(2 \sinh t))^2} = 0,
\]
(26)

or in terms of real valued functions
\[
\int_0^\infty \frac{t \ln (2 \sinh t)}{\left[ 3t^2 + \ln^2 (2 \sinh t) \right]^2} \, dt = 0,
\]
(27)

is also related to the infinite product in the title. Indeed
\[
\int_0^\infty \frac{\sin \pi x \, dx}{x \prod_{k=1}^\infty \left( 1 - \frac{x^3}{k^3} \right)} = \int_0^\infty \frac{\sin \pi x}{x} \frac{\Gamma(1-x)\Gamma(1-\omega x)\Gamma(1-x/\omega)}{\Gamma(1-x)} \, dx
\]
\[
= \pi \int_0^\infty \frac{\Gamma(1-\omega x)\Gamma(1-x/\omega)}{\Gamma(1+x)} \, dx
\]
\[
= \pi \int_0^\infty B(1-\omega x, 1-x/\omega) \, dx
\]
\[
= \pi \int_0^\infty dx \int_0^\infty \frac{t^{-\omega x}}{(1+t)^{x+1}} \, dt
\]
(28)
Changing the order of integration and calculating the integral over $x$ we get

$$
\int_0^\infty \frac{\sin \pi x}{x} \prod_{k=1}^{\infty} \left(1 - \frac{x^3}{k^3}\right) dx = -2\pi \int_0^\infty \frac{dt}{(it\sqrt{3} + \ln(2\sinh t))^2}.
$$

(29)

is the statement of the fact that the integral on the RHS of (29) is real. Of course by replacing in (28) $\omega$ with any complex number of unit argument one gets other integrals like (26).

It is known that Laplace transform of the digamma function leads to some log-trigonometric integrals [5–7] that contain the expression $x^2 + \ln^2(2e^{-a}\cos x)$ in the denominator. This expression should be compared to the expression $3t^2 + \ln^2(2\sinh t)$ in the denominator of (27).

Appendix A: Asymptotics of the product of gamma functions

Due to the asymptotic relation

$$
\ln \Gamma(z) = \left(z - \frac{1}{2}\right) \ln z - z + O(1), \quad |\arg z| < \pi,
$$

one has

$$
\ln \{\Gamma(b + z)\Gamma(b + \omega z)\Gamma(b + z/\omega)\} = 3 \left(b - \frac{1}{2}\right) \ln z - \frac{2\pi}{\sqrt{3}} x + O(1), \quad |\arg z| < \frac{\pi}{3}.
$$

From this it follows that

$$
|P_b(z)| = C|z|^{3b-3/2} . \begin{cases} 
  e^{-\frac{2\pi}{\sqrt{3}} x}, & 0 < \arg z < \frac{\pi}{3}, \\
  e^{\frac{2\pi}{\sqrt{3}} x - \pi y}, & \frac{\pi}{3} < \arg z < \frac{2\pi}{3},
\end{cases}
$$

where $z = x + iy$.

Appendix B: Roots of the equation $e^{i\sqrt{3}z} + 2\cosh z = 0$

The fact that the roots of the equation $e^{i\sqrt{3}z} + 2\cosh z = 0$ are symmetric under $z \rightarrow \omega z$ is easy to check directly.

Since $\frac{1}{\sqrt{3}}e^{-\pi\sqrt{3}/2} = 0.0329...$ is quite small the equation $e^{i\sqrt{3}z} + 2\cosh z = 0$ will have roots close to $\pi i \left(n + \frac{1}{2}\right)$, where $n$ is a non-negative integer. Below it is shown that these are the only roots in the upper half plane.

Let $f(z) = e^{i\sqrt{3}z}$, $g(z) = 2\cosh z$. Obviously, on the real axis $|f(z)| < |g(z)|$. Now consider $f(z)$ and $g(z)$ on the closed contour $C$ depicted in Fig.2

![Fig.2](image-url)
Here $\Gamma_R$ is a semicircle of radius $R = \pi N$ for some large natural number $N$. We have for $z = x + iy \in \Gamma_R$

$$|f(z)| = e^{-\sqrt{3}y} \leq 1,$$

$$|g(z)| = 2\sqrt{\sinh^2 x + \cos^2 \sqrt{\pi^2 N^2 - x^2}} \geq 2.$$

Thus $|f(z)| < |g(z)|$ on the contour $C$. According to Rouche’s theorem this means that the function $f(z) + g(z)$ has the same number of roots inside the contour $C$ as the function $g(z)$, as required.

This analysis shows that the roots of $e^{i\sqrt{3}z} + 2 \cosh z = 0$ are located on the three rays $z = i\omega^k$, $k = 0, 1, 2$.

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