Associated Stieltjes-Carlitz polynomials and a generalization of Heun’s differential equation.

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Abstract
The generating function of Stieltjes-Carlitz polynomials is a solution of Heun’s differential equation and using this relation Carlitz was the first to get exact closed forms for some Heun functions. Similarly the associated Stieltjes-Carlitz polynomials lead to a new differential equation which we call associated Heun. Thanks to the link with orthogonal polynomials we are able to deduce two integral relations connecting associated Heun functions with different parameters and to exhibit the set of associated Heun functions which generalize Carlitz’s. Part of these results were used by the author to derive the Stieltjes transform of the measure of orthogonality for the associated Stieltjes-Carlitz polynomials using asymptotic analysis; here we present a new derivation of this result.

1 Introduction
The Stieltjes-Carlitz polynomials are orthogonal polynomials with a three term recurrence relation
\[(\lambda_n + \mu_n - x)F_n(x) = \mu_{n+1}F_{n+1} + \lambda_{n-1}F_{n-1}, \quad n \geq 0,\]
\[F_{-1}(x) = 0, \quad F_0(x) = 1,\] (1)
with the rates
\[
\begin{align*}
\lambda_n &= k^2(2n + 1)^2, \\
\mu_n &= (2n)^2 \\
\lambda_n &= (2n + 1)^2, \\
\mu_n &= k^2(2n)^2
\end{align*}
\] (2)
and \(0 < k^2 < 1\). Their orthogonality measure, given for instance in [3, p. 194], was first derived by Stieltjes [13] using continued fraction techniques. Later on Carlitz [2] obtained a generating function for the \(F_n(x)\) from which he was able to derive anew the orthogonality measure. The most striking fact, which does not seem to have been realized by Carlitz himself, is that the generating functions he had obtained give a finite set of

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exact solutions of Heun’s differential equation. The complete list of Carlitz results were gathered in [14], and an integral transformation connecting Heun functions with different parameters was derived. This last tool, combined with Carlitz results, led to an enlarged set of Heun functions.

More recently we have obtained a generating function for the associated Stieltjes-Carlitz polynomials with rates

\[ \lambda_n = k^2(2n + 2c + 1)^2, \quad \mu_n = 4(n + c)^2 + \mu_{n0} \]  
\[ \lambda_n = (2n + 2c + 1)^2, \quad \mu_n = 4k^2(n + c)^2 + k^2\mu_{n0} \]  

This result, combined with asymptotic analysis, has led to the Stieltjes transform of the associated Stieltjes-Carlitz polynomials [17].

Our aim is to show that these results lead to a finite set of exact solutions of what could be called the “associated Heun” differential equation which emerges as an equation satisfied by the generating functions of the polynomials \( F_n(x) \) with the rates (3), (4).

This link with orthogonal polynomials is even more fruitful since it gives a convenient tool to derive two new integral transformations relating associated Heun functions with different parameters.

The plan of this communication is the following.

Section 2 is devoted to a short summary on Heun’s differential equation and its relation with orthogonal polynomials.

In section 3 we present the associated Heun differential equation and derive two integral connection relations for its solutions.

In section 4 we give a finite set of exact solutions of the associated Heun differential equation which generalize Carlitz results to non-vanishing \((c, \mu)\).

In section 5, in order to cross-check the Stieltjes transform obtained in [17], we present a completely different derivation which uses Karlin and MacGregor representation theorem and the link between orthogonal polynomials and birth and death processes. Solving the Kolmogorov equations leads to a Stieltjes transform which is in perfect agreement with asymptotic analysis.

## 2 Heun differential equation

This equation is the most general second order differential equation with four regular singular points

\[ w = 0, 1, 1/k^2, +\infty \quad 0 \leq k^2 \leq 1 \]

and is described by the array of parameters

\[ P = \{ \alpha, \beta; \gamma, \delta, \epsilon; s \} \quad \alpha + \beta = \gamma + \delta + \epsilon - 1. \]

The accessory parameter \( s \) is unconstrained. We shall follow the standard notations of [19, p.576],[4, p.57-62] for Heun’s differential equation

\[
w(1-w)(1-k^2w)D^2F + [\gamma(1-w)(1-k^2w) - \delta w (1-k^2w) - \epsilon k^2w(1-w)]DF \\
+ (\alpha \beta k^2 w + s)F = 0
\]

(5)
with $D = \frac{d}{dw}$.

We shall denote by $H_n(P, w)$ the unique solution of (5) which is analytic for $|w| < 1$ and is normalized according to

$$H_n(P, w = 0) = 1. \quad (6)$$

This equation, for arbitrary values of $s$, can be solved in terms of hypergeometric functions only for two values of the parameter $k^2$:

1) if $k^2 = 0$ the solution analytic around $w = 0$ is

$$2F_1 \left( \frac{r_+, r_-}{\gamma}; w \right) \quad r_\pm = a \pm \sqrt{a^2 + s} \quad a = \frac{\gamma + \delta - 1}{2}$$

2) if $k^2 = 1$ the solution analytic around $w = 0$ was shown in [12] to be

$$(1 - w)^r 2F_1 \left( \frac{r + \alpha, r + \beta}{\gamma}; w \right) \quad r = a + \sqrt{a^2 - \alpha\beta - s}, \quad a = \frac{\gamma - \alpha - \beta}{2}$$

For some particular values of the accessory parameter $s$ Heun’s functions degenerate into hypergeometric functions of the variable $R(w)$, where $R(w)$ is a polynomial of second degree in $w$. These values of $s$ are listed in [10].

In all what follows we shall not consider these particular cases.

Since $H_n(P, w)$ is analytic for $|w| < 1$ we can consider it as the generating function of the polynomials $F_n(P, s)$, with variable $s$, such that

$$H_n(P, w) = \sum_{n \geq 0} F_n(P, s) w^n \quad |w| < 1 \quad (7)$$

Relations (5,7) imply routinely the three term recurrence relation for the $F_n$

$$\begin{cases} (\lambda_n + \mu_n + \gamma_n - s - \alpha\beta k^2) F_n = \mu_{n+1} F_{n+1} + \lambda_{n-1} F_{n-1}, & n \geq 0 \\ F_{-1} = 0, & F_0 = 1 \end{cases} \quad (8)$$

with

$$\lambda_n = k^2(n + \alpha)(n + \beta), \quad \mu_n = n(n + \gamma - 1), \quad \gamma_n = (1 - k^2)\delta n.$$ 

This recurrence exhibits the polynomial character of $F_n$ with respect to either the variable $s$ or the more familiar $x = s + \alpha\beta k^2$.

One should observe on (8) that only for $\delta = 0$ do we have a true birth and death process with birth rate $\lambda_n$ and death rate $\mu_n$ (see [3] for an introduction). For $\delta \neq 0$ we have killing in the sense of Karlin and Tavaré [8][9] with rate $\gamma_n$.

The finite set of exact solutions of (5) obtained by Carlitz can be found in [14, p.692] as well as as the following integral transform, quoted here for convenience.

Let us define $P' = \{\alpha' = \gamma, \beta' = \beta; \gamma' = \alpha, \delta' = \delta + \gamma - \alpha, \epsilon' = \epsilon + \gamma - \alpha; s\}$, then provided that $\text{Re} \gamma > \text{Re} \alpha > 0$, we have

$$H_n(P, w) = \frac{1}{B(\alpha, \gamma - \alpha)} \int_0^1 dt \ t^{\alpha-1}(1 - t)^{\gamma - \alpha - 1} H_n(P'; wt) \quad (9)$$

for $w$ in $\mathbb{C} \setminus [1, +\infty[$. This relation, combined with the set of Carlitz solutions gives another finite set of solutions described in [14, p.693].
Let us turn ourselves to the associated polynomials with the recurrence

\[
(\lambda_n + \mu_n + \gamma_n - s - (\alpha + c)(\beta + c)k^2 + k^2\delta c) F_n = \mu_{n+1}F_{n+1} + \lambda_{n-1}F_{n-1}, \quad n \geq 0
\]

\[
F_{-1} = 0, \quad F_0 = 1
\]

with

\[
\begin{align*}
\lambda_n &= k^2(n + c + \alpha)(n + c + \beta) \\
\mu_n &= (n + c)(n + c + \gamma - 1) + \mu_0 \\
\gamma_n &= (1 - k^2)\delta(n + c)
\end{align*}
\]

Two new parameters appear: \(c\) which is an association parameter, and \(\mu\) which is a co-recursivity parameter. Clearly, for vanishing \((c, \mu)\) the recurrence (10) reduces to (8). The constant terms added to \(s\) were chosen for notational convenience reasons.

We define

\[
H_n(c, \mu, P; w) = \sum_{n \geq 0} F_n(c, \mu, P)w^n \quad |w| < 1 \tag{11}
\]

and from (10) it is easy to obtain

\[
w(1 - w) (1 - k^2 w)D^2 F + [(\gamma + 2c)(1 - w)(1 - k^2 w) - \delta w(1 - k^2 w) - \epsilon k^2 w(1 - w)]DF
\]

\[
+ \left[ (\alpha + c)(\beta + c)k^2 w + \frac{c(c + \gamma - 1)}{w}(1 - w) + s - \delta c \right] F = \frac{c(c + \gamma - 1)}{w} + \mu \tag{12}
\]

In view of its origin it is natural to call this equation the associated Heun differential equation. This departs from the usual terminology where associated differential equations refer to homogeneous extensions of a given differential equation whereas here we have an inhomogeneous extension.

At any rate it reduces to Heun equation in the particular cases \((c = 0, \mu = 0)\) and \((c = 1 - \gamma, \mu = 0)\), according to the relations

\[
H_n(0, 0, P; w) = H_n(P; w)
\]

\[
H_n(1 - \gamma, 0, P; w) = H_n(\tilde{P}; w) \tag{13}
\]

with the array \(\tilde{P} = \{1 - \gamma + \alpha, 1 - \gamma + \beta; 2 - \gamma, \delta, \epsilon; s - (1 - \gamma)\delta\}\).

Let us now derive two integral transformations relating associated Heun functions. To do this we switch from the \(F_n\) to the \(G_n\) defined by

\[
G_0 = F_0, \quad G_n = \mu_1 \cdots \mu_n F_n = (1 + c)_n(\gamma + c)_n F_n \quad n \geq 1 \tag{14}
\]

whose recurrence

\[
(\lambda_n + \mu_n + \gamma_n - s - (\alpha + c)(\beta + c)k^2 + k^2\delta c) G_n = G_{n+1} + \lambda_{n-1}\mu_nG_{n-1}, \quad n \geq 0
\]

\[
G_{-1} = 0, \quad G_0 = 1 \tag{15}
\]
reveals that \((-1)^n G_n\) is monic in the variable \(x = s + k^2(\alpha + c)(\beta + c) - k^2\delta c\).

The basic technique to get an integral transform is to look for a mapping of the parameters \(P\) which leaves invariant the recurrence (15).

A first possibility is the mapping \(P'_\alpha\)

\[
\begin{align*}
\alpha' &= \gamma, \beta' = \beta \\
\gamma' &= \alpha, \delta' = \delta + \gamma - \alpha, \epsilon' = \epsilon + \gamma - \alpha \\
s' &= s, c' = c, \mu' = \mu
\end{align*}
\]

for which we have

\[G_n(P'_\alpha) = G_n(P) n \geq 0.\]

Using (14) gives

\[F_n(P) = \frac{(c + \alpha)_n}{(c + \gamma)_n} F_n(P'_\alpha).\]  \hspace{1cm} (16)

If \(\text{Re} \ \gamma > \text{Re} \ \alpha > -\text{Re} \ c\) we can write

\[\frac{(c + \alpha)_n}{(c + \gamma)_n} = \frac{1}{B(\gamma - \alpha, \alpha + c)} \int_0^1 dt \ t^{n+c+\alpha-1}(1-t)^{\gamma-\alpha-1}\]

and inserting this in (13), multiplying each term by \(w^n\) and summing \(n\) from zero to infinity gives

\[H_n(c, \mu, P, w) = \frac{1}{B(\gamma - \alpha, \alpha + c)} \int_0^1 dt \ t^{c+\alpha-1}(1-t)^{\gamma-\alpha-1} H_n(c, \mu, P''_\alpha; wt)\]  \hspace{1cm} (17)

valid for \(|w| < 1\). The term by term integration is allowed since the right hand side power series is absolutely and uniformly convergent for \(|w| \leq R < 1\). Analytic continuation extends this relation to \(\mathbb{C} \setminus [1, +\infty[\). Clearly for \(c = \mu = 0\) we recover (9) and there is another integral transformation \(P'_\beta\) obtained from \(P'_\alpha\) by the exchange of the couples \((\alpha, \alpha')\) and \((\beta, \beta')\).

A second possibility which leaves invariant the recurrence (15) is \(P''_\alpha\) with

\[
\begin{align*}
\alpha'' &= 2 - \alpha, \beta'' = \beta + 1 - \alpha \\
\gamma'' &= \gamma + 1 - \alpha, \delta'' = \delta + 1 - \alpha, \epsilon'' = \epsilon + 1 - \alpha \\
s'' &= s + (\alpha - 1)(\gamma + \delta - \alpha), c'' = c + \alpha - 1, \mu'' = \mu
\end{align*}
\]

which leads to

\[F_n(P) = \frac{(c + \alpha)_n}{(c + 1)_n} F_n(P''_\alpha).\]

The corresponding integral transform follows analogously to (17)

\[H_n(c, \mu, P, w) = \frac{1}{B(1-\alpha, c + \alpha)} \int_0^1 dt \ t^{c+\alpha-1}(1-t)^{-\alpha} H_n(c + \alpha - 1, \mu, P''_\alpha; wt)\]

and is valid for \(1 > \text{Re} \ \alpha > -\text{Re} \ c\) and \(w \in \mathbb{C} \setminus [1, +\infty[\).

This second integral relation is a genuinely new result, since it changes the value of the association parameter from \(c\) to \(c + \alpha - 1\). For this reason it could not appear in the previous analyses where \(c = 0\). Here too the interchange of the couples \((\alpha, \alpha'')\) and \((\beta, \beta'')\) leads to another mapping \(P''_\beta\).


\section{Exact solutions of the associated Heun equation}

Before giving the set of associated Heun functions which generalize Carlitz ones we shall explain, on the first of them, how they can be constructed.

We first make the change of function

\[ G = w^c F \]

(18)

which brings (12) to

\[
\begin{align*}
&w(1 - w)(1 - k^2 w) D^2 G + [\gamma(1 - w)(1 - k^2 w) - \delta w(1 - k^2 w) - \epsilon k^2 w(1 - w)] D G \\
&\quad + [s + k^2 c(c + \epsilon + \gamma - 1) + \alpha \beta k^2 w] G = c(c + \gamma - 1) w^{c-1} + \mu w^c.
\end{align*}
\]

The first exact solution will correspond to the parameters

\[ P = \{\alpha = 0, \beta = \frac{1}{2}; \gamma = \frac{1}{2}, \delta = \frac{1}{2}, \epsilon = \frac{1}{2}; s = \sigma - k^2 c^2\} \quad c > 0. \]

The change of variable

\[ \sqrt{w} = sn(\theta; k^2) \]

(19)

(in what follows, concerning elliptic functions we stick to the notations of [19]; here \( \sqrt{w} \) is the square root which is positive for real positive \( w \)) reduces the differential equation to

\[ \partial^2_\theta G + 4\sigma G = 2c(2c - 1)(sn^2 \theta)^{c-1} + 4\mu(sn^2 \theta)^c = J(\theta) \]

Its solution is

\[ G(\theta) = \int_0^\theta du \frac{\sin 2\sqrt{\sigma}(\theta - u)}{2\sqrt{\sigma}} J(u) \]

(20)

and is meaningful provided that \( c > 1/2 \). In order to extend this integral representation to \( c > 0 \) an integration by parts of \( (sn^2 u)^{c-1} \) is needed with the final result

\[
\begin{align*}
&w^c Hn(c, \mu, P; \sigma - k^2 c^2; w) = \int_0^{\theta(w)} du \cos 2\sqrt{\sigma}(\theta(w) - u)2c(sn^2 u)^{c-1/2} cn u dn u \\
&\quad + \int_0^{\theta(w)} du \frac{\sin 2\sqrt{\sigma}(\theta(w) - u)}{2\sqrt{\sigma}} \left[ 4(c^2 + c^2 k^2 + \mu)(sn^2 u)^c - 2c(2c + 1)k^2(sn^2 u)^{c+1} \right].
\end{align*}
\]

(21)

The variable \( \theta(w) \) is obtained through the inversion of relation (19)

\[ \theta(w) = \int_0^{\sqrt{w}} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}. \]

It is analytic for \( |w| < 1 \); its analytic extension to the whole complex \( w \) plane has been described in [1, p.122] and is analytic but for the branch points \( w = 1, 1/k^2 \).

We take for \( (sn^2 u)^{c-1/2} \) and \( w^c \) one and the same principal branch in order to secure the analyticity of \( Hn(w) \) for \( |w| < 1 \).

It is lengthy, even if straightforward, to check that (21) is indeed a solution of (12) in the complex plane deprived with the points \( w = 1, 1/k^2 \) and that the normalization condition (14) does hold.

As mentioned in the previous section, this associated Heun function should reduce to a Heun function either if \((c = \mu = 0)\) or if \((c = 1/2, \mu = 0)\). In the first case a limiting procedure which makes use of

\[
\lim_{c \to 0} \int_0^\theta du \ f(u) \ 2c(\text{sn}^2 u)^{c-1/2} = f(0)
\]
gives

\[
\lim_{c \to 0} H_n(c, \mu, P; \sigma; w) = \cos 2\sqrt{\sigma}(w) + \frac{\mu}{\sigma} \left(1 - \cos 2\sqrt{\sigma}(w)\right).
\]

In the second limiting case, using relation (20) we get

\[
\lim_{c \to 1/2} \sqrt{\text{w}} H_n \left(c, \mu, P; \sigma - \frac{k^2}{4}; w\right) = \sin \frac{2\sqrt{\sigma}(w)}{\sqrt{\sigma}} + 4\mu \int_0^{\theta(w)} du \ \frac{\sin \frac{2\sqrt{\sigma}(\theta(w) - u)}{\sqrt{\sigma}}}{\frac{2\sqrt{\sigma}}{2\sqrt{\sigma}}} \ \text{sn} \ u
\]

For \(\mu = 0\), using the relations (13) we recover Carlitz results:

\[
\begin{align*}
H_n \left(0, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; s = \sigma; w\right) &= \cos(2\sqrt{\sigma}(w)) \\
\sqrt{\text{w}} H_n \left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; s = \sigma - \frac{1}{4} - \frac{k^2}{4}; w\right) &= \frac{\sin 2\sqrt{\sigma}(\theta(w))}{\sqrt{\sigma}}
\end{align*}
\]

In order to get the remaining set of exact solutions one has to change (18) into

\[
G = w^{c+\lambda}(1 - w)^{\mu}(1 - k^2 w)^{\nu} F
\]

where \(\lambda, \mu, \nu\) take the values 0 or 1/2. We get in this way seven more solutions to be listed below.

- \(P = \left\{\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{k^2}{2}; s = \sigma - \frac{1}{4} - \frac{k^2}{4}\right\}\)

\[
w^c \sqrt{1 - w} H_n(c, \mu, P; w) = \int_0^{\theta(w)} du \ \cos 2\sqrt{\sigma}(\theta(w) - u) \ 2c(\text{sn}^2 u)^{c-1/2} \text{dn} \ u \\
+4(k^2 c^2 + \mu) \int_0^{\theta(w)} du \ \frac{\sin 2\sqrt{\sigma}(\theta(w) - u)}{2\sqrt{\sigma}} \ (\text{sn}^2 u)^c \text{cn} \ u 
\]

which reduces for \((c = \mu = 0)\) and \((c = 1/2, \mu = 0)\) to

\[
\begin{align*}
\sqrt{1 - w} H_n \left(\frac{1}{2}; \frac{1}{2}, \frac{3}{2}, \frac{1}{2}; s = \sigma - \frac{1}{4}; w\right) &= \cos(2\sqrt{\sigma}(w)) \\
\sqrt{w(1 - w)} H_n \left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}, \frac{1}{2}; s = \sigma - 1 - \frac{k^2}{4}; w\right) &= \frac{\sin 2\sqrt{\sigma}(\theta(w))}{2\sqrt{\sigma}}
\end{align*}
\]

- \(P = \left\{\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{k^2}{2}; s = \sigma - k^2(c + \frac{1}{2})^2\right\}\)

\[
w^c \sqrt{1 - k^2 w} H_n(c, \mu, P; w) = \int_0^{\theta(w)} du \ \cos 2\sqrt{\sigma}(\theta(w) - u) \ 2c(\text{sn}^2 u)^c \text{cn} \ u \\
+4(c^2 + \mu) \int_0^{\theta(w)} du \ \frac{\sin 2\sqrt{\sigma}(\theta(w) - u)}{2\sqrt{\sigma}} \ (\text{sn}^2 u)^c \text{dn} \ u
\]
which reduces for \((c = \mu = 0)\) and \((c = 1/2, \mu = 0)\) to

\[
\begin{cases}
\sqrt{1 - k^2 w} H_n \left( \frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2} ; \sigma - \frac{k^2}{4} ; w \right) = \cos(2\sqrt{\sigma} \theta(w)) \\
\sqrt{w(1 - k^2 w)} H_n \left( \frac{3}{2}; \frac{3}{2}, \frac{3}{2}, \frac{3}{2} ; \sigma - 1 - k^2 ; w \right) = \frac{\sin 2\sqrt{\sigma} \theta(w)}{2\sqrt{\sigma}}
\end{cases}
\]

\(P = \left\{ 1, \frac{3}{2}, \frac{1}{2} \right\} \) \(s = \sigma - \frac{1}{4} - k^2(c + \frac{1}{2})^2\)

\[
\sqrt{(1 - w)(1 - k^2 w)} H_n(c, \mu, P; w) = \int_0^{\theta(w)} du \ \frac{\cos 2\sqrt{\sigma} \theta(w)}{2\sqrt{\sigma}} \ \frac{\sin 2\sqrt{\sigma} \theta(w) - u}{2\sqrt{\sigma}} F(u)
\]

which reduces for \((c = \mu = 0)\) and \((c = 1/2, \mu = 0)\) to

\[
\begin{cases}
\sqrt{(1 - w)(1 - k^2 w)} H_n \left( \frac{3}{2}; \frac{3}{2}, \frac{3}{2}, \frac{3}{2} ; \sigma - 1 - k^2 ; w \right) = \frac{\sin 2\sqrt{\sigma} \theta(w)}{2\sqrt{\sigma}}
\end{cases}
\]

The eight particular cases for which either \((c = \mu = 0)\) or \((c = 1/2, \mu = 0)\) reproduce the results collected in the table [14, p.692]. There are four other solutions:

\(P = \left\{ 1, \frac{3}{2}, \frac{1}{2} \right\} \) \(s = \sigma - \frac{1}{4} - k^2(c + \frac{1}{2})^2\)

Let us define

\[
F(u) = 2c(2c + 1)(\text{sn}^2 u)^{c-1/2} + 4\mu(\text{sn}^2 u)^{c+1/2}
\]

then we have

\[
\sqrt{w} H_n(c, \mu; w) = \int_0^{\theta(w)} du \ \frac{\sin 2\sqrt{\sigma} \theta(w) - u}{2\sqrt{\sigma}} F(u)
\]

\(P = \left\{ 1, \frac{3}{2}, \frac{1}{2} \right\} \) \(s = \sigma - 1 - k^2(c + \frac{1}{2})^2\)

\[
\sqrt{w(1 - w)} H_n(c, \mu; w) = \int_0^{\theta(w)} du \ \frac{\sin 2\sqrt{\sigma} \theta(w) - u}{2\sqrt{\sigma}} F(u)\text{cn} u
\]

\(P = \left\{ 1, \frac{3}{2}, \frac{1}{2} \right\} \) \(s = \sigma - 1 - k^2(c + 1)^2\)

\[
\sqrt{w(1 - k^2 w)} H_n(c, \mu; w) = \int_0^{\theta(w)} du \ \frac{\sin 2\sqrt{\sigma} \theta(w) - u}{2\sqrt{\sigma}} F(u)\text{dn} u
\]

\(P = \left\{ \frac{3}{2}, \frac{1}{2} \right\} \) \(s = \sigma - 1 - k^2(c + 1)^2\)

\[
\sqrt{w(1 - w)(1 - k^2 w)} H_n(c, \mu; w) = \int_0^{\theta(w)} du \ \frac{\sin 2\sqrt{\sigma} \theta(w) - u}{2\sqrt{\sigma}} F(u)\text{cn} u \text{dn} u
\]
For these last cases the limiting cases \((c = 0, \mu = 0)\) and \((c = 1/2, \mu = 0)\) do not give anything new. Furthermore the reader can check that all the solutions given here are correctly normalized at \(w = 0\).

The results (22, 23) were first derived in [17] whilst all the other are new. Obviously this set of solutions can be further enlarged using the two integral transforms of section 3.

5 Associated Stieltjes-Carlitz polynomials

In [17] the Stieltjes transform of the orthogonality measure of the associated Stieltjes-Carlitz polynomials has been derived for the first time. The main tool used in this work is Markov theorem and asymptotic analysis. It is of some interest to check this result using a completely different approach and this is the aim of this section.

The strategy used here was already applied to the Stieltjes-Carlitz polynomials in [16] and provided for a new derivation of the orthogonality measures. Its generalization, described in [15] and in [16], led to a one parameter family of orthogonality measures for the indeterminate moment problem corresponding to the rates

\[
\lambda_n = (4n + 1)(4n + 2)^2(4n + 3), \quad \mu_n = (4n - 1)(4n)^2(4n + 1).
\]

What is basic in this approach is the connection between orthogonal polynomials and birth and death processes; the whole problem to get the orthogonality measure is reduced to the resolution of a linear partial differential equation.

Let us first recall that the first family of associated Stieltjes-Carlitz polynomials are defined by the recurrence relation

\[
(\lambda_n + \mu_n - x)F_n = \mu_{n+1}F_{n+1} + \lambda_{n-1}F_{n-1}, \quad n \geq 0
\]

\[
F_{-1} = 0, \quad F_0 = 1
\]

with the rates

\[
\lambda_n = k^2(2n + 2c + 1)^2, \quad \mu_n = 4(n + c)^2 + \mu_0 c \geq 0
\]

In order to obtain the Stieltjes transform of their orthogonality measure we shall relate them to the birth and death process whose Kolmogorov equation is

\[
\frac{d}{dt}P_{m,n}(t) = \lambda_{n-1}P_{m,n-1}(t) + \mu_{n+1}P_{m,n+1}(t) - (\lambda_n + \mu_n)P_{m,n}(t)
\]

\[
P_{m,n}(0) = \delta_{mn}.
\]

\(P_{m,n}(t)\) is the probability of a population \(n\) at time \(t\) provided that it was \(m\) at time \(t = 0\). It is therefore positive and bounded by one. The link between (24) and (25) is provided by Karlin and McGregor representation theorem [6], [7]

\[
P_{m,n}(t) = \frac{1}{\pi_m} \int_0^\infty d\Psi(x)F_m(x)F_n(x)e^{-tx}
\]

with

\[
\pi_0 = 1, \quad \pi_m = \frac{\lambda_0 \cdots \lambda_{m-1}}{\mu_1 \cdots \mu_m}, \quad m = 1, 2, \ldots
\]
From this representation theorem it follows that a possible way to get $\Psi$ is to compute the Laplace transform of

$$P_{00}(t) = \int_0^\infty d\Psi(x) \ e^{-tx}$$

which we shall write

$$\tilde{P}_{00}(p) = \int_0^\infty dt \ e^{-pt} P_{00}(t).$$

For $\text{Re } p > 0$ this is nothing but

$$\tilde{P}_{00}(p) = \int_0^\infty d\Psi(x) \ \frac{p}{p + x}$$

closely related to the Stieltjes transform of the orthogonality measure since we have

$$\tilde{P}_{00}(p) = -S(-p).$$

In this approach no recourse to asymptotic analysis is needed to get $S(z)$: we just require $\tilde{P}_{00}(p)$. We shall describe in the following how this can be worked out just by solving linear partial differential equations.

As a first step we consider the change of basis $P_{m,n}(t) \rightarrow P_{mn}(t)$ such that

$$P_{mn}(t) = \frac{(1 + c)^n}{(1/2 + c)^n} P_{m,n}(t)$$

(26)

Kolmogorov equation becomes

$$\frac{d}{dt} P_{m,n}(t) = \tilde{\lambda}_{n-1} P_{m,n-1}(t) + \tilde{\mu}_{n+1} P_{m,n+1}(t) - (\lambda_n + \mu_n) P_{m,n}(t)$$

$$P_{m,n}(0) = \frac{(1 + c)^m}{(1/2 + c)^m} \delta_{mn}.$$  

(27)

with

$$\tilde{\lambda}_n = k^2(2n + 2c + 1)(2n + 2c + 2), \quad \tilde{\mu}_n = (2n + 2c - 1)(2n + 2c)$$

whilst $\lambda_n, \mu_n$ are given by (3).

In order to solve (27) we introduce the generating function

$$H_m(t, w) = \sqrt{1 - k^2 w} \sum_{n \geq 0} P_{mn}(t) \ w^{n+c} \quad |w| < 1.$$  

The Kolmogorov equation becomes a linear partial differential equation for $H_m$

$$\partial_t H_m(t, w) = \{4w(1 - w)(1 - k^2 w) \partial_w^2 + 2[(1 - w)(1 - k^2 w) - w(1 - k^2 w)$$

$$-k^2 w(1 - w)] \partial_w \} H_m(t, w) - \left(\mu + \frac{2c(2c-1)}{w}\right) w^c \sqrt{1 - k^2 w} P_{m0}(t)$$

(28)

with the boundary condition

$$H_m(0, w) = \frac{(1 + c)^m}{(1/2 + c)^m} w^{m+c} \sqrt{1 - k^2 w}.$$
It is convenient, from a notational point of view, to keep $P_{m0}(t)$; however this is related to the generating function $H_m(t, w)$ by

$$P_{m0}(t) = \lim_{w \to 0} w^{-c} H_m(t, w)$$

From now on we shall restrict ourselves to $m = 0$, and in order to simplify we change the variable to $w = \text{sn}^2(\theta, k^2)$ which maps $[0, 1]$ into $[0, K]$. Deleting the subscript $m = 0$ in $H_0$ we are led to

$$\partial_t H(t, \theta) = \partial^2_\theta H(t, \theta) - \left(\mu + \frac{2c(2c - 1)}{\text{sn}^2 \theta}\right) (\text{sn}^2 \theta)^c \text{dn} \theta P_{00}(t)$$

with

$$H(0, \theta) = \text{dn} \theta (\text{sn}^2 \theta)^c.$$

Let us stress that since the moment problem for the associated Stieltjes-Carlitz polynomials is determined the measure $\Psi$ is unique and therefore the solution of the Kolmogorov equations is unique [11].

In order to get it we extend $H(t, \theta)$, a priori defined for $\theta \in [0, K]$, to the interval $\theta \in [-K, +K]$ by using the symmetry $\theta \leftrightarrow -\theta$ of the equation and of the boundary value, and further to all real values of $\theta$. This last step is possible since and the boundary conditions are periodic with period $2K$.

We shall use a Laplace transform in the variable $t$

$$H(t, \theta) \rightarrow \tilde{H}(p, \theta) = \int_0^{+\infty} dt \ e^{-pt} H(t, \theta)$$

to solve equation [29]. We shall first examine what can be said on general grounds on $\tilde{H}(p, \theta)$.

Firstly since the $P_{mn}$ are probabilities we have the bounds

$$0 \leq P_{mn}(t) \leq \frac{(1+c)_n}{(1/2 + c)_n} \quad n = 0, 1 \cdots \quad t \geq 0$$

which imply

$$0 \leq H(t, \theta) \leq \text{dn} \theta (\text{sn}^2 \theta)^c \text{F}_1 \left(\frac{1, 1 + c}{1/2 + c}; \text{sn}^2 \theta\right)$$

for any real $\theta$. From theorem 2.1 of [18, p.38] it follows that $\tilde{H}(p, \theta)$ is analytic in the domain Re $p > 0$ uniformly for real $\theta$.

Secondly the small time behaviour of the transition probabilities is given by

$$\lim_{t \to 0} \mathcal{P}_{mn}(t) = \delta_{mn}$$

from this and theorem 1 of [18, p.181] we conclude to

$$\lim_{p \to +\infty} \tilde{H}(p, \theta) = \lim_{t \to 0} H(t, \theta) = \text{dn} \theta (\text{sn}^2 \theta)^c.$$
Thirdly $\bar{H}(p, \theta)$ must be periodic in $\theta$, with period $2K$, and for $P_{00}(t)$ to exist it is necessary that
\begin{equation}
\lim_{\theta \to 0} H(t, \theta) = 0 \quad t \geq 0.
\end{equation}

Having stated the most useful properties of $\bar{H}(p, \theta)$ let us now take the Laplace transform of equation (29). We get
\begin{equation}
\frac{\partial^2}{\partial \theta^2} \bar{H}(p, \theta) - p \bar{H}(p, \theta) = -H(0, \theta) + A(\theta) \bar{P}_{00}(p) = J(p, \theta)
\end{equation}
with
\begin{equation}
A(\theta) = \left( \mu + \frac{2c(2c - 1)}{\text{sn}^2 \theta} \right) \text{dn} \left( \text{sn}^2 \theta \right)^c.
\end{equation}
This equation has for general solution even in $\theta$
\begin{equation}
\bar{H}(p, \theta) = e^{\sqrt{p} \theta} \int_{0}^{\theta} d\phi e^{-\sqrt{p} \phi} J(p, \phi) + (\theta \leftrightarrow -\theta) + C(p) \cosh(\sqrt{p} \theta)
\end{equation}
The integral over $\phi$ is convergent at 0 provided that we take $c > 1/2$.

The necessary condition (30) implies $C(p) = 0$ and the periodicity of $\bar{H}(p, \theta)$ in the variable $\theta$ implies
\begin{equation}
\int_{0}^{2K} d\phi e^{-\sqrt{p} \phi} J(p, \phi) = 0 \quad c > 1/2
\end{equation}
from which we deduce
\begin{equation}
\bar{P}_{00}(p) = \frac{\int_{0}^{2K} d\phi e^{-\sqrt{p} \phi} H(0, \phi)}{\int_{0}^{2K} d\phi e^{-\sqrt{p} \phi} A(\phi)}.
\end{equation}
Using the periodicity of $A$ and $\bar{H}$ brings this ratio to
\begin{equation}
\bar{P}_{00}(p) = \frac{\int_{0}^{K} d\phi \cosh \sqrt{p}(K - \phi) H(0, \phi)}{\int_{0}^{K} d\phi \cosh \sqrt{p}(K - \phi) A(\phi)}
\end{equation}
and the change of variable $p = -z$ gives eventually the Stieltjes transform
\begin{equation}
\int_{0}^{+\infty} d\Psi(z) = -\frac{\int_{0}^{K} du \cos \sqrt{z}(K - u) H(0, u)}{\int_{0}^{K} du \cos \sqrt{z}(K - u) A(u)}
\end{equation}
so that if we define
\begin{equation}
D(c, \mu; z) = \int_{0}^{K} du \cos \sqrt{z}(K - u) \left( 2c(2c - 1) + \mu \text{sn}^2 u \right) \text{dn} u \left( \frac{\text{sn}^2 u}{{\Gamma}(2c + 1)} \right)^{-1}
\end{equation}
we end up with
\begin{equation}
\int_{0}^{+\infty} d\Psi(s) = -\frac{D(c + 1, 0; s)}{D(c, \mu; s)} \quad c > 1/2
\end{equation}
in perfect agreement with the result derived in [17]. In this reference the polynomials with rates
\begin{equation}
\lambda_n = (2n + 2c + 1)^2, \quad \mu_n = 4k^2(n + c)^2 + \mu k^2 \delta_{n0} \quad 0 < k^2 < 1
\end{equation}
have been shown to follow from the result (31) using the transformation theory for Jacobian elliptic functions.
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