Shape optimization for suppressing coherent structure of two-dimensional open cavity flow

Takashi NAKAZAWA*, Takashi MISAKA** and Clair POIGNARD***

* Center for Mathematical Modeling and Data Science, Osaka University
Osaka 560-8531, Japan
E-mail: nakazawa@sigmath.es.osaka-u.ac.jp

** National Institute of Advanced Industrial Science and Technology
1-1-1 Umezono, Tsukuba-shi, Ibaraki, 305-0045, Japan

*** Institut of Mathematics of Bordeaux, University of Bordeaux
351 Cours de la Libération 33405 Talence, France

Abstract
This paper presents an optimal design obtained as a shape optimization problem in a domain with a singular point. For shape optimization, the eigenvalue in Snapshot Proper Orthogonal Decomposition (Snapshot POD) is defined as a cost function. The main problems are a Non-stationary Navier–Stokes problem and eigenvalue problem of Snapshot POD. An objective functional is described using Lagrange multipliers and finite element method. Two-dimensional open cavity flow is adopted for an initial domain, where the domain includes a singular point. In this paper, two kinds of sensitivities assuming velocity vector in $H^1$ and $H^2$ are used. Using $H^1$ gradient method for domain deformation, all triangles over a mesh are deformed as the cost function decreases. Finally, eigenvalues of Snapshot POD are compared in the initial and optimal domains.

Keywords: Shape optimization problem, Snapshot Proper Orthogonal Decomposition, Two-dimensional open cavity

1. Introduction

This paper presents an optimal design obtained using a shape optimization problem in a domain with a singular point. The author earlier reported specific examination of construction of a shape optimization method with Snapshot Proper Orthogonal Decomposition (snapshot POD; Nakazawa, 2019), in which eigenvalues of Snapshot POD are defined as a cost function. Time average and time fluctuation parts of transient flows are suppressed directly in a two-dimensional cavity flow with an isolated disk, where the isolated disk is used as a design boundary. As described herein, the shape optimization problem suggested in work by Nakazawa (2019) is improved for a domain with a boundary with a singular point, which is used as a design boundary. Thereby, a two-dimensional open cavity flow investigated by Sipp et al. (2007) and Barbagallo et al. (2009) is chosen as an initial domain. Then a numerical calculation is performed. The particular history, background, and procedure of the suggested shape optimization problem are described below.

With the rapid development of computer technology and numerical methods, shape optimization based on computational fluid dynamics (CFD) is playing an important role in fluid mechanics and aerodynamics design. Shape optimization problems in fluid dynamics were first addressed by Pironneau (1973, 1974) for the respective domains in which the stationary Stokes and Navier–Stokes equations are defined. Subsequently, Haslinger and Mäkinen (2003), Mohammadi and Pironneau (2001), and Moubachir and Zolesio (2006) constructed fundamental frameworks of flow-field shape-optimization problems. Recently, many researchers are examining the topic with compressible flow or Non-Newtonian flow such as Plotnikov et al. (2012), Zhang et al. (2015, 2016), and Romero (2017).

The original motivation is explained below. Although flow stabilization presents the important challenge of choosing which research field best addresses flow control, few reports of the relevant literature describe flow stabilization by shape
optimization. Nakazawa (2016) reported that minimization and maximization problems of dissipation energy are solved in two-dimensional cavity flow, where the stationary Navier–Stokes problem is used as the main problem, and where the dissipation energy is used as the cost function. After shape optimization, linear stability analysis is conducted in the initial and the optimal domains. The critical Reynolds numbers are, respectively, decreasing and increasing. Next, for controlling flow stability more directly, Nakazawa and Azegami (2015) reported a pioneering shape optimization method used to stabilize the disturbances. The method is based on linear stability theory. Particularly, the real part of the leading eigenvalue is used as the cost function. The stationary Navier–Stokes problem and the eigenvalue problem of the linear stability analysis are cited as main problems for obtaining the cost function. However, the methods explained above are not available for the case in which the non-stationary boundary condition is defined because the stationary Navier–Stokes problem should be solved as described in an earlier report by Nakazawa (2016).

To address the challenge explained above, the author constructed a shape optimization method using Snapshot POD (Nakazawa, 2019), in which the eigenvalue in Snapshot POD is defined as the cost function. Such the Snapshot POD has the same numerical procedure as Primary Component Analysis (PCA) and play a role as one of Reduced Order Model (ROM) based method as well as Dynamic Mode Decomposition (DMD). By utilizing it, the number of modes of Snapshot POD (if contribution rates is larger than arbitrary large number as 0.99) is generally less than the number sampled from each time steps. On the other hand, in the view point of fluid dynamics, the time dependent flow is able to be decomposed into the time average part (the 1st primary component) and the fluctuation part (over the 2nd primary components) and we can define the cost function to distinguish both of parts. And more, under definition of Snapshot POD in this paper, the eigenvalue is meaning the $L^2$ norm of the velocity vectors for the time average part or the fluctuation part, and momentum energy with corresponding modes is decreasing by decreasing the cost function, as a result. Especially, by using the eigenvalues with the time fluctuation part as the cost function, time fluctuation part in the time dependent flow is directly controlled. In Nakazawa (2019), a time dependent flow driven by the time periodic boundary condition at low Reynolds number is suppressed by the shape optimization problem which Snapshot POD is introduced as mentioned above. A brief summary of the shape optimization problem is presented below.

The sum of eigenvalues in Snapshot POD is defined as the cost function. For this study, the non-stationary Navier–Stokes problem and the eigenvalue problem in Snapshot POD are used as the main problems. The main problems are transformed from strong forms to weak forms with trial functions based on a standard application of finite element method (FEM). The functional is described using Lagrange multipliers with FEM. Next, its first variation, which is the same as the material derivative, is derived to evaluate sensitivity using adjoint variable method, assuming velocity vector in $H^2$ for avoiding singularity of sensitivity to the greatest extent possible based on the Generalized J integral described by Ohtsuka (1981, 1985, 1986) and by Ohtsuka and Khludnev (2000, 2018), although assuming $H^1$, as Nakazawa (2016) did. The Generalized J integral is described below. In fact, throughout reshaping steps or in the case of initial domain with a crack and corner on the boundary, the stress concentration is known to appear on such a boundary. To date, the energy release rate is evaluated as the sensitivity in a cracked domain and which plays an important role in the field of fracture mechanics. The application of sensitivity to the energy release rate was studied originally as the Generalized J Integral in a series of works by Ohtsuka (1981, 1985, 1986) and by Ohtsuka and Khludnev (2000, 2018). Recently, Kimura (2008) summarized a derivation for shape derivative of the functional by the Generalized J Integral. Mathematical justification of the existence and derivation of some formulas in a general geometric situation was presented in that report. In this paper, sensitivities assuming velocity vector in $H^1$ and $H^2$ are called as Volume Integration Type (VIT) and Generalized J Integration Type (GJIT), and both numerical results are compared. After evaluating the sensitivity, an initial domain is reshaped iteratively to obtain an optimal domain. Then the $H^1$ gradient method (Azegami, 1986) is used for stable domain deformation.

For numerical demonstrations with FreeFEM++ (Hecht, 2012) for all numerical calculations, the shape optimization problem is addressed as explained below. Two-dimensional open cavity flow is adopted, as described by Sipp et al. (2007) and Barbagallo et al. (2009). After numerical calculations, the eigenvalues of Snapshot POD are compared in the initial domain and in the optimal domain.

2. Formulation of the Problem

2.1. Initial domain

Letting $\Omega_0$ be a fixed bounded Lipschitz domain in $\mathbb{R}^d$ ($d \in \mathbb{N}$), and letting $\Omega$ be an open subset of $\Omega_0$, with a position vector denoted as $x \in \mathbb{R}^d$, then a two-dimensional open cavity flow $\Omega$ is adopted as the initial domain. For $d = 2$, the initial domain is $\Omega \subset \Omega_0 \subset \mathbb{R}^2$ as

$$\Omega = \Omega_0 \setminus (\Omega_{m1} \cup \Omega_{m2}),$$

(1)
Below, the mapping derive an adjoint problem, we need choose either Stiefel manifold or Hilbert space, and this paper selects Stiefel manifold. POD and the non-stationary Navier–Stokes problem are analyzed respectively on Stiefel manifold and Hilbert space. To Navier–Stokes problem and an eigenvalue problem in Snapshot POD. By the way, the eigenvalue problem of Snapshot problem is defined as

\[
\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}
\]

2.2. Main problems

For a shape optimization problem considering Snapshot POD, this paper presents the main problems: a non-stationary Navier–Stokes problem and an eigenvalue problem in Snapshot POD. By the way, the eigenvalue problem of Snapshot POD and the non-stationary Navier–Stokes problem are analyzed respectively on Stiefel manifold and Hilbert space. To derive an adjoint problem, we need choose either Stiefel manifold or Hilbert space, and this paper selects Stiefel manifold by evaluating Reynolds Average Navier–Stokes problem with eigenvalues and POD basis obtained from Snapshot POD. Below, the mapping \( \phi \) of the position vector \( x \) from the initial domain \( \Omega \) to the optimal domain \( \phi(\Omega) \) is assumed as given. Furthermore, the flow of a viscous incompressible fluid is assumed to occupy a bounded domain \( \Omega \).

2.2.1. Non-stationary Navier–Stokes problem

Problem 1 (Non-stationary Navier–Stokes) Find \( (u, p) : \phi(\Omega) \times (0, T) \to \mathbb{R}^d \times \mathbb{R} \) such that

\[
\frac{Du}{Dt} = -\nabla p + \frac{1}{Re} \Delta u \quad \text{in } \phi(\Omega) \times (0, T),
\]

\[
\nabla \cdot u = 0 \quad \text{in } \phi(\Omega) \times (0, T),
\]

\[
u \cdot u = 0 \quad \text{on } \Gamma_{\text{in}} \cup \phi(\Gamma_{\text{wall}} \cup \Gamma_{\text{d2}}) \times (0, T),
\]

\[
u = [1, 0]^T \quad \text{on } \Gamma_{\text{in}} \times (0, T),
\]

\[
\frac{\partial u_k}{\partial y} \bigg|_{y = 0} = 0 \quad \text{and } u_y = 0 \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{top}} \times (0, T),
\]

\[
\frac{\partial u_y}{\partial x} \bigg|_{x = 0} = 0 \quad \text{and } u_x = 0 \quad \text{on } \Gamma_{\text{out}} \times (0, T),
\]

\[
u = u^0 \quad \text{in } \phi(\Omega) \text{ at } t = 0,
\]

and \( u^0 \) represents a stationary solution of the stationary Navier–Stokes problem, where the velocity vector represents \( u = [u_x, u_y]^T \). The definition of Reynolds number is the same as Sipp et al. (2007) and Barbagallo et al. (2009).

Letting \( (\omega, q) \) be adjoint variables with respect to the velocity and the pressure, and then by discretizing in the time direction with the finite difference method, a set of necessary variables is found as \( \zeta_1 = \{ u, p, w, q \} \), where hereinafter \( u = [u^m]_{m=1}^{N_t}, p = [p^m]_{m=1}^{N_t}, w = [w^m]_{m=1}^{N_t}, \) and \( q = [q^m]_{m=1}^{N_t} \). The variational form of the non-stationary Navier–Stokes problem is defined as

\[
L^s(\phi, \zeta_1) = -\frac{1}{m} \sum_{m=1}^{N_t} \left\{ \int_\Omega G^s_m (x, \zeta_1) \, dx \right\}
\]

\[
= 0, \quad \forall (w, q)
\]
by setting \( m = N_2 - N_1 + 1 \) with \( N_1 = \frac{T_1}{\Delta t} \) and \( N_2 = \frac{T_2}{\Delta t} \) for time step size \( \Delta t \), at time \( t = T_1, T_2 \). The density function \( G^n_t(x, \xi_1) \) is presented as
\[
G^n_t(x, \xi_1) = \frac{Du}{Di}w^n - p^n\nabla \cdot w^n - q^n\nabla \cdot u^n + \frac{1}{Re} \nabla (w^n)^T \nabla (w^n)^T.
\] (22)

2.2.2. Snapshot proper orthogonal decomposition

We define a Snapshot POD analysis from time \( t = T_1 \) to \( T_2 \), where a weight function is prepared to extract arbitrary primary components from all primary components.

The correlation coefficient matrix \( R(N_1, N_2, \tilde{u}, \hat{u}) \in \mathbb{R}^{m \times m} \) is composed as
\[
\tilde{u} = \left[ (u^{N_1})^T, \ldots, (u^n)^T, \ldots, (u^{N_2})^T \right] \in \mathbb{R}^{d \times m}
\] (23)
as
\[
R(N_1, N_2, \tilde{u}, \hat{u}) = \int_\Omega \tilde{u}^T \hat{u} \, dx.
\] (24)

Let eigenvalues and eigenvectors of \( R \in \mathbb{R}^m \) and \( \hat{u} \in \mathbb{R}^{m \times m} \),
\[
\omega = \left[ \omega^1, \ldots, \omega^i, \ldots, \omega^m \right], \, \omega^i \in \mathbb{R},
\] (25)
\[
\hat{u} = \left[ \hat{u}^1, \ldots, \hat{u}^i, \ldots, \hat{u}^m \right], \, \hat{u}^i \in \mathbb{R}^m,
\] (26)
\[
\Phi = \Phi_\omega \omega^{-\frac{1}{2}} \in \mathbb{R}^{d \times m},
\] (27)
\[
\Phi_\omega = \hat{u} \hat{u}^T \in \mathbb{R}^{d \times m},
\] (28)
\[
\omega^{-\frac{1}{2}} = \text{diag}(\{\omega^i\}^{-\frac{1}{2}})
\] (29)

where \( R(N_1, N_2, \tilde{u}, \hat{u}) \) is a positive-semidefinite matrix satisfying the eigenvalue \( 0 \leq \omega \), and where \( \Phi^i \) represents the POD basis for the \( i \)-th primary component as
\[
\Phi = \left[ \Phi^1, \ldots, \Phi^i, \ldots, \Phi^m \right] \in \mathbb{R}^{d \times m}.
\] (30)

Using the definitions, we define snapshot POD analysis as described below.

Problem 2 (Snapshot Proper Orthogonal Decomposition)

Let the solution \( u \) of Problem 1 be given. Find \( \omega \in \mathbb{R}^m \) and \( \tilde{u} \in \mathbb{R}^{m \times m} \) for \( N_1, N_2 \in \mathbb{N} \) such that
\[
\text{diag}(\omega) - R(N_1, N_2, \tilde{u}, \hat{u}) \tilde{u} = 0.
\] (31)

For the optimization problem, let \( \zeta_2 = [\omega, \hat{u}, \alpha, u] \in \mathbb{R}^{d \times d} \) be the set of necessary variables used in Problem 2, where \( \alpha \) is an adjoint variable for the eigenvectors \( \hat{u} \).
\[
\alpha = \left[ \alpha^1, \ldots, \alpha^i, \ldots, \alpha^m \right], \, \alpha^i \in \mathbb{R}^m.
\] (32)

For the shape optimization problem studied here, Snapshot POD is a main problem. It plays a role as a constraint function. Therefore, the following functional is defined as
\[
L_2(\Phi^i, \zeta_2) = -\frac{1}{m \times N_1} \sum_{i=1}^{m \times N_1} \sum_{j=1}^{m} G_2(x, \zeta_2).
\] (33)

where
\[
G_2(x, \zeta_2) = \alpha \cdot \left\{ \delta_{j-k} \left( \text{diag}(\omega) - R_{\tilde{u}} \right) \right\}.
\] (34)

Weight function \( \delta_{j-k} \) is intended to extract from the \( 1 - m \) th primary components to the \( j - k \) th primary components.

2.2.3. Reynolds Average Navier–Stokes problem

Problem 3 (Reynolds Average Navier–Stokes)

Find \( (\bar{u}, \bar{p}) : (\Phi(\Omega) \times (0, T) \rightarrow \mathbb{R}^d \times \mathbb{R} \) such that
\[
\nabla \cdot C = -\nabla \bar{p} + \frac{1}{Re} \Delta \bar{u} \quad \text{in} \quad \Phi(\Omega) \times (0, T),
\] (35)
\[
\nabla \cdot \bar{u} = 0 \quad \text{in} \quad \Phi(\Omega) \times (0, T),
\] (36)
\[
\bar{u} = 0 \quad \text{on} \quad \Gamma_{\text{di}} \cup \Phi(\Gamma_{\text{wall}} \cup \Gamma_{\text{di}}) \times (0, T),
\] (37)
\[
\bar{u} = [1, 0]^T \quad \text{on} \quad \Gamma_{\text{in}} \times (0, T),
\] (38)
\[
\frac{\partial \bar{u}}{\partial y} = 0 \quad \text{and} \quad \bar{v} = 0 \quad \text{on} \quad \Gamma_{\text{n}} \cup \Gamma_{\text{top}} \times (0, T),
\] (39)
\[
\frac{\partial \bar{u}}{\partial x} = 0 \quad \text{and} \quad \bar{v} = 0 \quad \text{on} \quad \Gamma_{\text{out}} \times (0, T).
\] (40)
where
\begin{equation}
C = C_1 + C_2, \quad (41)
\end{equation}
\begin{equation}
C_1 = -\bar{u}\bar{u}^T \quad \text{and} \quad C_2 = -\frac{1}{2} \sum_{j=1}^{m} \left[ \Phi_{\omega_j}^T (\Phi_{\omega_j})^T \right], \quad (42)
\end{equation}
and \( \bar{u} = \left[ \bar{u}_x, \bar{u}_y \right]^T \) and \( \bar{p} \) depict the time average velocity vector and time average pressure expressed as
\begin{equation}
\bar{u} = \frac{1}{m} \sum_{i=N_1}^{N_2} u_i, \quad \bar{p} = \frac{1}{m} \sum_{i=N_1}^{N_2} p_i. \quad (43)
\end{equation}

Appendix 7.1 presents details of the tensor for the time fluctuation term \( C_2 \).

Letting \( (\bar{\omega}, \bar{q}) \) be adjoint variables with respect to the velocity and the pressure, then by discretizing in the time direction with the finite difference method, a set of necessary variables is found as \( \bar{\zeta}_i = [\bar{u}, \bar{p}, \bar{\omega}, \bar{q}] \). The variational form of the Reynolds Average Navier–Stokes problem is defined as
\begin{equation}
\tilde{L}_1 (\phi_0, \bar{\zeta}_1) = -\int_{\Omega} \tilde{G}_1 (x, \bar{\zeta}_1) \, dx \\
= 0, \quad \forall (\bar{\omega}, \bar{q}). \quad (44)
\end{equation}
The density function \( G_1 (x, \bar{\zeta}_1) \) is
\begin{equation}
G_1 (x, \bar{\zeta}_1) = -\bar{\rho} \nabla \cdot \bar{\omega} - \bar{\eta} \nabla \cdot \bar{u} + \left\{ -C + \frac{1}{Re} \nabla \bar{u}^T \right\} : \nabla \bar{\omega}^T. \quad (45)
\end{equation}

3. Shape Optimization Problem

The shape optimization problem using Snapshot POD in the open cavity flow is constructed next, with the Lagrange function first defined to deduce the first variation. Next, based on the Kuhn–Tucker condition, the main and adjoint problems are solved to obtain the main and adjoint variables, which are substituted into the first variation to evaluate sensitivity for the shape optimization problem.

3.1. Lagrange function

We formulate the following minimization problem of the cost function \( f \) as
\begin{equation}
f(\phi_0, \omega) = \sum_{j=N_1}^{N_2} \delta_{j-k} \omega^j. \quad (46)
\end{equation}
By the way, from definition of the correlation coefficient matrix \( R(N_1, N_2, \bar{u}, \bar{u}) \in \mathbb{R}^{m \times m} \), the eigenvalue is meaning the \( L^2 \) norm of the velocity vectors for the time average part or the fluctuation part, and momentum energy with corresponding modes is decreasing by decreasing the cost function, as a result.

Problem 4 (Shape Optimization) After letting \( f(\phi_0, \omega) \) be defined as Eq. (46), we find \( \phi(\Omega) \) such that
\begin{equation}
\min_{\delta} \left\{ f(\phi_0, \omega) : \{ (u^n, p^n) \}_{n=N_1}^{N_2}, (\omega, \bar{u}) \right\}. \quad (47)
\end{equation}
By application of the Lagrange multiplier method, Lagrange function \( L \) for the shape optimization problem in this study can be expressed as
\begin{equation}
L(\phi_0, \zeta_1, \zeta_2) = f(\phi_0, \omega) + L_1 (\phi_0, \bar{\zeta}_1) + L_2 (\phi_0, \bar{\zeta}_2), \\
\approx f(\phi_0, \omega) + L_1 (\phi_0, \bar{\zeta}_1) + L_2 (\phi_0, \bar{\zeta}_2). \quad (48)
\end{equation}
The first variation of the above functional is
\begin{equation}
\lim_{\epsilon \to 0} \frac{L(\phi, \zeta(\phi(x))) - L(\phi_0, \zeta(x))}{\epsilon} = L(\phi_0, \zeta(x)) [\zeta'] + L(\phi_0, \zeta(x)) [\phi]. \quad (49)
\end{equation}
Please see more details about deriving the first variation in Appendix 7.2.
For sensitivity analysis, the adjoint variable method is used to derive a main problem and an adjoint problem by setting
\[ L(\phi_0, \zeta(x)) [\zeta'] = 0. \] (50)
After solving the main and the adjoint problems, sensitivity is evaluated by substituting the main and adjoint variables into eq. (50).

3.2. Main and adjoint problems

Based on the adjoint variable method, the main problems of Problem 4 are introduced into Problem 3 from \( L(\phi_0, \zeta(x)) [\hat{w}', \hat{q}'] = 0 \) and Problem 2 from \( L(\phi_0, \zeta(x)) [\alpha'] = 0 \). In addition, the adjoint problems of Problem 4 are induced from \( L(\phi_0, \zeta(x)) [\omega'] = 0, \ L(\phi_0, \zeta(x)) [\tilde{u}', \tilde{p}'] = 0 \). Such strong forms are given as presented below.

**Problem 5 (Adjoint Problem for \( \omega \))**

Given eigenfunction \( \hat{u} \) of Problem 2, then find \( \alpha \in \mathbb{R}^{m \times m} \) such that
\[ \hat{u} \alpha = I, \] (51)
and \( \hat{u}, \alpha \) are the unitary matrix from Problem 5. Therefore, \( \alpha \) is obtained as the inverse matrix or the transposed matrix of \( \hat{u} \)
\[ \alpha = \hat{u}^{-1} = \hat{u}^T. \] (52)

**Problem 6 (Adjoint Problem for \( \hat{u} \))**

Let the solution \( u \) of Problem 1 be given. Find \( \alpha^T \in \mathbb{R}^{m \times m} \) such that
\[ \text{diag} \omega \alpha^T = R(N_1, N_2, \hat{u}, \hat{u}) \alpha^T. \] (53)
In fact, solving Problem 6 is unnecessary because \( \alpha \) has already been obtained in Problem 5.

**Problem 7 (Adjoint Problem for \( (\tilde{u}, \tilde{p}) \))**

With \( \phi \) and the time-averaged solution \( (\tilde{u}, \tilde{p}) \) of Problem 1, and with the eigenvalue and the eigenfunction \( (\omega, \hat{u}) \) of Problem 2 with \( \hat{u} = \alpha^T \) as given, find \( (\tilde{w}, \tilde{q}) : \Omega \rightarrow \mathbb{R}^d \times \mathbb{R} \) such that
\[ (\nabla \tilde{u})^T \tilde{w} - (\tilde{u} \cdot \nabla) \tilde{w} + \nabla \tilde{q} - \frac{1}{\text{Re}} \Delta \tilde{w} + \tilde{A} = 0 \] in \( \phi(\Omega), \] (54)
\[ \nabla \cdot \tilde{w} = 0 \] in \( \phi(\Omega), \] (55)
\[ \tilde{w} = 0 \] on \( \Gamma_{d1} \cup \phi(\Gamma_{\text{wall}} \cup \Gamma_{d2}) \cup \Gamma_{\text{in}} \times (0, T), \] (56)
\[ \frac{\partial \tilde{w}}{\partial y} = 0 \] and \( \tilde{w}_y = 0 \) on \( \Gamma_{\text{n}} \cup \Gamma_{\text{top}}, \] (57)
\[ \frac{\partial \tilde{w}}{\partial x} = 0 \] and \( \tilde{w}_x = 0 \) on \( \Gamma_{\text{out}}, \) (58)
where \( \tilde{w} = [\tilde{w}_x, \tilde{w}_y]^T \) and
\[ \tilde{A} = 2 \sum_{m=1}^{m} \delta_{j-m} \alpha \Phi_{\omega}^T. \] (59)

3.3. Sensitivity of the shape optimization problem

Here, based on the \( H^1 \) gradient method (Azegami, 1986), we evaluate the sensitivity of the shape optimization problem as, with Strain tensor \( E \)
\[ \int_{\Omega} E(\varphi) : E(\varphi) dx = -L(\phi_0, \zeta(x)) [\varphi], \] (60)
and the initial domain is deformed as \( \phi(\Omega) = \phi(\Omega) + \epsilon \varphi_{\mu}(\Omega) \). In fact, the term \( L(\phi_0, \zeta(x)) [\varphi] \) depends on the function space for \( \hat{u}, \tilde{w} \) as shown in Appendix 7.2. Please see details of the domain reshaping method in Appendix 7.3. As mentioned at Appendix 7.2, the domain variation \( \varphi \) should be in \( W^{1,\infty} \) and it is not guaranteed that the sensitivity after substituting all main and adjoint variable, is in \( W^{1,\infty} \). Somehow it is needed to make the sensitivity to be in \( W^{1,\infty} \), but, FEM is discretizing by \( H^1 \) in space. Thereby, by using the \( H^1 \) gradient method, the sensitivity is in \( H^1 \subset W^{1,\infty} \). Many previous studies provides its effectiveness.
4. Numerical Schemes

The (P1-P1) element pair for the velocity and pressure is used to discretize all equations spatially, with Brezzi-Pitkaranta’s pressure-stabilization (Brezzi and Pitkaranta, 1984). For all numerical calculations, FreeFEM++ (Hecht, 2012) is used.

The stationary solution \( (u^0, p^0) \) is obtained to solve the stationary Navier–Stokes problem using the Newton–Raphson method. The non-stationary solution \( \{(u^n, p^n)\}_{n=1}^N \) is obtained to solve Problem 1 with the UMFPACK solver presented by Davis (2004) for every time step of \( n = 1 \) to \( N \). For the material derivative term, the second-order characteristic curve method is used. Using it, Notsu and Tabata (2014) proved its mathematical proof and numerical availability. After obtaining the non-stationary solution \( \{(u^n, p^n)\}_{n=1}^N \), the correlation coefficient matrix \( R \) is formed for snapshot POD. The eigenvalue problem for the matrix \( R \) is solved in Problem 2 using lapack solver.

Based on the theory of the optimization problem considered herein, the adjoint problem of Problem 7 is solved to obtain \( (\bar{\theta}, \bar{\phi}) \) with UMFPACK solver (Davis, 2004). After evaluating the sensitivity, for domain deformation, the \( H^1 \) gradient method is used with UMFPACK solver (Davis, 2004).

5. Numerical Calculations and Discussion

In this section, numerical calculations and discussion are described. Before presenting numerical results, the author validates the number of nodes \( N_{\text{nodes}} \) and the number of elements \( N_{\text{elements}} \) in subsection 5.1. Next, based on finite element meshes \( (N_{\text{nodes}}, N_{\text{elements}}) \) decided in 5.1, the suggested shape optimization problem is performed, where sensitivities assuming velocity vector in \( H^1 \) and \( H^2 \) are called as Volume Integration Type (VIT) and Generalized J Integration Type (GJIT), and both numerical results are compared.

5.1. Spatial and temporal discretization, adaptive mesh refinement

Two-dimensional open cavity flow includes some singular points or areas in which stress concentration appears, for example two connections between the Neumann boundary condition and the Dirichlet boundary condition, and more corners of the boundary. Thereby, Adaptive Mesh Refinement (AMR) is applied for increasing numerical accuracy. Details of AMR and comparisons of numerical results with Sipp et al. (2007) and Barbagallo et al. (2009) are presented in Appendix 7.4 and Appendix 7.5. Velocity and pressure are discretized in the spatial direction using finite element method, with respective nodes and elements of \( (N_{\text{nodes}}, N_{\text{elements}}) = (44225, 87286) \). For discretization in time, the time step size \( \Delta t = 0.001 \) is used to take time integrations of Problem 1 at \( Re = 7500 \). Fig. 1 and the solid line on Fig. 9 show the momentum energy and eigenvalues for Snapshot POD in the initial domain with AMR for \( (N_{\text{nodes}}, N_{\text{elements}}) = (44225, 87286) \) at \( Re = 7500 \).

Next, the author takes inner products \( \{\boldsymbol{\theta}(t)\}_{t=2}^5 \) between the POD basis \( \Phi^j \) and velocity \( u(t) \) shown in Fig. 2.

\[
\boldsymbol{\theta}(t) = \int_{\Omega} \Phi^j \cdot u(t) \, dt.
\] (61)

From Fig. 2, long numerical simulation gives us more asymptotic flow field, but it is leading to high calculation cost. Next, to decrease calculation cost, a dependency of eigenvalues on sampling period \( (N_1, N_2) = (9000, 10000) \) and \( (19000, 20000) \) is checked in Table. 1. From Table. 1, relative errors between both of sampling period are respectively 0.446\%, 0.404\%, 4.385\%, 5.085\%. Therefore this study samples velocity vectors from \( N_1 = 9000 \) to \( N_2 = 10000 \) for Snapshot POD, for every reshaping step.

As described in the next subsection, Volume Integration Type and Generalized J Integration Type are used to evaluate the sensitivity for comparing numerical results. Moreover, three cases are examined as better combinations of eigenvalues in Snapshot POD: only the time average part \( \delta_{1-1} \) (Case 1); the time average and fluctuation parts \( \delta_{1-m} \) (Case 2); and only the time fluctuation part \( \delta_{2-m} \) (Case 3). In all cases, the step size \( \epsilon = 0.5 \) and the terminal condition \( \beta = 10^{-5} \) mentioned in Section 7.3 are used and the domain is reshaped by the \( H^1 \) gradient method.

5.2. Numerical results for Volume Integration Type (VIT)

In all cases, the sum of eigenvalues of snapshot POD from \( i = 2 \) to \( m \), \( \omega^i_{j=2} \omega_i \) are not decreasing, as shown in Fig. 3(a). Fig. 5 depicts sensitivities for Case 1, Case 2, and Case 3 evaluated as Volume Integration Type (VIT) in the initial domain.
Fig. 1  Momentum energy in the initial domain with AMR for \((N_{\text{nodes}}, N_{\text{elements}}) = (44225, 87286)\) at \(\text{Re} = 7500\), (a) \(n \in [1, \cdots, 10000]\) and (b) \(n \in [9000, \cdots, 10000]\).

(a) the 2nd primary component for \(n \in [1, \cdots, 20000]\)

(b) the 3rd primary component for \(n \in [1, \cdots, 20000]\)

(c) the 4th primary component for \(n \in [1, \cdots, 20000]\)

(d) The 5th primary component for \(n \in [1, \cdots, 20000]\)

Fig. 2  Inner products between the POD basis and velocity every time step in the initial domain with AMR for \((N_{\text{nodes}}, N_{\text{elements}}) = (44225, 87286)\) at \(\text{Re} = 7500\).

Table 1  Comparison of eigenvalues for \(i = 2\) to 5 in the initial domain, depending on sampling period.

| Sampling Period \((N_1, N_2)\) | \(i = 2\) | \(i = 3\) | \(i = 4\) | \(i = 5\) |
|-----------------------------|--------|--------|--------|--------|
| (9000, 10000)               | 0.0986441 | 0.09616 | 0.0189 | 0.0186 |
| (19000, 20000)              | 0.0982812 | 0.09496 | 0.0193 | 0.0191 |
| Relative Error              | 0.446%   | 0.404% | 4.385% | 5.085% |
5.3. Numerical results for Generalized J Integration Type (GJIT)

Fig. 3(b) shows the sum of eigenvalues of snapshot POD from $i = 2$ to $m$, $\sum_{i=2}^{m} \omega_i$ for Case 1, Case 2, and Case 3.

In Case 1, the value is monotonically non-increasing until the eighth step. Thereafter, it is monotonically non-decreasing. Originally, this shape optimization problem does not guarantee that the cost function is convex, because Navier-Stokes problem which is one of a representative nonlinear equation, is used as the constraint function. And more, even if GJIT is used, Case 1 is unable to obtain the sensitivity evaluating precisely stress concentration near the point $\Gamma_{d1} \cap \Gamma_{wall}$, because time fluctuation part (in particular, over the 2nd primary component) involves complex flow behavior and generates stress concentration. In fact, Fig. 4 shows stream function in the initial domain for the 1st to the 4th primary components, and and stream function with the 1st primary component has smooth flow structure, and stream function with over the 2nd primary components involves complex flow structure around the corner. However, this paper does not study mathematical and physical aspects of this cost function’s behavior.

For Case 2 and Case 3, the values are monotonically non-increasing until the ninth step and the tenth step, respectively. They become asymptotic at about the twentieth step. Generally, in the case in which one defines a smooth boundary as a design boundary, a cost function is monotonically increasing or decreasing but it seems that such behavior of the value is monotonically non-increasing or non-decreasing. For that reason, huge shear and normal stresses appear near the point $\Gamma_{d1} \cap \Gamma_{wall}$. The situation is similarly singular point.

In fact, GJIT relies on the assumption that the velocity gradient and its adjoint variable are in $H^2$. Similarly, $H^1$ is assumed for Volume Integration Type (VIT). Therefore, GJIT is available to assume function regularity $C^1$ class higher than VIT. Sensitivity near point $\Gamma_{d1} \cap \Gamma_{wall}$ is evaluated more precisely as a result, even though $\sum_{i=2}^{m} \omega_i$ is not decreasing monotonically, rather it is monotonically non-increasing or non-decreasing. In fact, Fig. 6 depicts sensitivities for Case 1, Case 2, and Case 3 evaluated using GJIT in the initial domain. Especially, comparing Fig. 5(c) and Fig. 6(c), a sensitivity of the latter one can be evaluated precisely near the point $\Gamma_{d1} \cap \Gamma_{wall}$.

Fig. 7 and Fig. 8 represent sensitivity for Case 2 and Case 3 evaluated using GJIT in the optimal domains. A corner at $\Gamma_{d1} \cap \Gamma_{wall}$ in the initial domain becomes a smooth curve in the optimal domain as a result of the shape optimization problem.

6. Conclusions

As described herein, the author combines AMR with a shape optimization method for controlling the time fluctuation component of a transient flow, effectively based on Snapshot Proper Orthogonal Decomposition (Snapshot POD) and Reynolds Average Navier–Stokes equation (RANS). Particularly, the sum of eigenvalues in Snapshot POD is defined as the cost function. The non-stationary Navier–Stokes problem and the eigenvalue problem in Snapshot POD are used as main problems. The main problems are transformed from strong forms to weak forms with trial functions based on a standard framework of finite element method (FEM). The functional is described using the Lagrange multiplier method with FEM. Next, its material derivative is derived to evaluate sensitivity using adjoint variable method. Sensitivities of two kinds can be evaluated: VIT and GJIT. Respectively, velocity gradient and its adjoint variables are assumed in $H^1$ class and $H^2$ class. The initial domain is reshaped iteratively until the cost function satisfies the terminal condition, where
Fig. 4 Stream function in the initial domain.

Fig. 5 Sensitivity by VIT in the initial domain.
Fig. 6  Sensitivity by GJIT in the initial domain.
Fig. 7 Sensitivity for Case 2 by GJIT.

Fig. 8 Sensitivity for Case 3 by GJIT.
the $H^1$ gradient method is used for stable domain deformation. A two-dimensional open cavity flow is adopted for a numerical demonstration because the domain has the singular point on a corner at $\Gamma_{d1} \cap \Gamma_{wall}$. It clarifies the difference between VIT and GJIT.

Numerical results reveal that in the cases of evaluated sensitivities by VIT, the sums of eigenvalues of snapshot POD from $i = 2$ to $m$, $\sum_{i=2}^{m} \omega_i$ are not decreasing. Only in Case 2 and Case 3 were sensitivities evaluated by GJIT, with results showing that $\sum_{i=2}^{m} \omega_i$ becoming asymptotic after monotonically non-increasing or non-decreasing. It is considered that such behavior of $\sum_{i=2}^{m} \omega_i$ throughout reshaping steps results from the singular point. However, this paper is not discussing theoretically about this cost function’s behavior, and it is needed to analyze that the time fluctuation part’s eigenvalues are decreasing as a result of the shape optimization problem only in Case 2 and Case 3 by GJIT, in the view point of mathematical and physical aspects.

7. Appendix

7.1. Derivation of the tensor for time fluctuation flow

Regarding the tensor for the time fluctuation term $C_2$, the time fluctuation velocity $u_d(t)$ is represented as

$$u_d(t) = \sum_{i=2}^{m} \Phi_i \tau^i(t).$$  

(62)

Therein, $\tau^i(t)$ represents a time periodic function with unity amplitude and with frequency to each POD mode $i$. Finally, we are able to derive tensor $C_2$ with $\Phi_i$ on a POD basis $\Phi$ and eigenvalue $\omega$ as

$$C_2 = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} u_d(t)u_d(t)^T \, dt$$

$$= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \left( \sum_{i=2}^{m} \Phi_i \tau^i(t) \right) \left( \sum_{i=2}^{m} \Phi_i \tau^i(t) \right)^T \, dt$$

$$= \frac{2}{\pi} \int_{0}^{1/2\pi} \left( \sum_{i=2}^{m} \Phi_i \Phi_i^T \right) \left[ \tau^i(t)^2 \right] \, dt$$

$$= \frac{2}{\pi} \sum_{i=2}^{m} \left( \Phi_i \Phi_i^T \right) \left[ \int_{0}^{1/2\pi} \tau^i(t)^2 \, dt \right]$$

$$= \frac{1}{2} \sum_{i=2}^{m} \left[ \Phi_i \Phi_i^T \right]$$  

(63)

where one trigonometric identity engenders

$$\int_{0}^{1/2\pi} \tau^i(t)^2 \, dt = \frac{\pi}{4}.$$  

(64)

7.2. Domain variation

We consider domain deformation $\phi$ as $\Omega \rightarrow \phi(\Omega)$, where $\phi$ is the $\mathbb{R}^d$-valued function. For $|\varepsilon| \ll 1$, mapping $\phi$ is represented by $\phi = \phi_0 + \varepsilon \varphi$ in $W^{1,\infty}(\bar{\Omega}, \mathbb{R}^d)$. We designate it by the identity map $\phi_0(\Omega) = \Omega$ and the domain variation $\varphi$. 

Fig. 9 Comparison of eigenvalues of Snapshot POD with the initial domain and the optimal domain of Case 2 and Case 3 with AMR for $(N_{nodes}, N_{elements}) = (44225, 87286)$ at $Re = 7500$. 

[DOI: 10.1299/jfst.2021jfst0002] © 2021 The Japan Society of Mechanical Engineers
For domain deformation, the Jacobi matrices are written as
\[
(\nabla \phi^T)^T = (\nabla \phi_0^T)^T + \epsilon (\nabla \varphi^T)^T + o(\epsilon^2).
\] (65)
The determinant is obtained as
\[
\det \left( (\nabla \phi^T)^T \right) = 1 + \epsilon \nabla \cdot \varphi + o(\epsilon^2).
\] (66)
Therefore,
\[
\phi(dx) = \det \left( (\nabla \phi^T)^T \right) dx = \left(1 + \epsilon \nabla \cdot \varphi + o(\epsilon^2)\right) dx.
\] (67)
Next, for such \( \zeta \), the scalar-valued function \( \zeta(\phi(x)) \) and the vector-valued function \( \nabla \zeta(\phi(x)) \) in \( \phi(\Omega) \) are deduced as
\[
\zeta(\phi(x)) = \zeta(x) + \epsilon \zeta(x) + o(\epsilon^2)
\] (68)
\[
\nabla \zeta(\phi(x)) = \nabla \zeta(x) + \epsilon \nabla \zeta(x) + o(\epsilon^2)
\] (69)
where \( (\cdot) \) and \( (\cdot)' \) represent the material derivative and the Fréchet derivative with respect to \( \zeta \). In fact, the material derivative of eq. (69) depends on the function space. In the cases of \( \zeta(\phi(x)) \in H^1(\phi(\Omega); \mathbb{R}^d) \) and \( \zeta(\phi(x)) \in H^2(\phi(\Omega); \mathbb{R}^d) \), we have, respectively,
\[
\nabla \zeta(\phi(x)) = \nabla \zeta(x) + \epsilon \nabla \zeta(x) + o(\epsilon^2),
\] (70)
and
\[
\nabla \zeta(\phi(x)) = \nabla \zeta(x) + \epsilon \nabla \zeta(x) + \varphi \Delta \zeta(x) + o(\epsilon^2).
\] (71)
In fact, this paper presents consideration of finite element method for numerical calculation. Also, \( \zeta(\phi(x)) \) should be in \( H^1(\phi(\Omega); \mathbb{R}^d) \). Therefore, Eq. (71) must be transformed as
\[
\nabla \zeta(\phi(x)) = \nabla \zeta(x) + \epsilon \left\{ \nabla \zeta'(x) - (\nabla \varphi^T)(\nabla \zeta(x)') \right\} + o(\epsilon^2).
\] (72)
The method of deriving eq. (72) is explained by Kimura (2008).

Finally, from eq. (67) and from eq. (68) and eq. (69), the functional in \( \phi(\Omega) \) can be rewritten as
\[
L(\phi, \zeta(\phi(x)))
= L(\phi_0, \zeta(x)) + \epsilon \left\{ L(\phi_0, \zeta(x)) \zeta' + L(\phi_0, \zeta(x)) \varphi \right\} + o(\epsilon^2),
\] (73)
and in the case of \( \zeta = [\zeta_1, \zeta_2] \),
\[
L(\phi_0, \zeta(x)) \zeta'
= L(\phi_0, \zeta(x)) \zeta_1' + L(\phi_0, \zeta(x)) \zeta_2'
= L(\phi_0, \zeta(x)) \left[ \bar{u}' \right] + L(\phi_0, \zeta(x)) \left[ \bar{\varphi}' \right]
+ L(\phi_0, \zeta(x)) \left[ \alpha' \right] + L(\phi_0, \zeta(x)) \left[ \bar{\alpha}' \right]
\] (74)
and for \( H^1(\phi(\Omega); \mathbb{R}^d) \)
\[
L(\phi_0, \zeta(x)) \varphi
= \int_\Omega \left\{ \left( C + \frac{1}{Re} \nabla \bar{u}^T \right) : \nabla \bar{u}^T - 2 \sum_{i=1}^{m} \delta_{j-i} \alpha \phi_0 \delta_{j-i} \phi_0^T \nabla \bar{u}^T \right\} + \left( 2 \varphi \cdot \nabla \bar{u} \right) - 2 \left( \sum_{i=1}^{m} \delta_{j-i} \alpha \phi_0 \delta_{j-i} \phi_0^T \right) \cdot (\varphi \cdot \nabla \bar{u}) \right\} dx,
\] (75)
and for $H^2(\phi(\Omega); \mathbb{R}^d)$

$$L(\phi_0, \zeta(x)) [\varphi]$$

$$= \int_{\Omega} \left( \left( C + \frac{1}{\text{Re}} \nabla \bar{u}^T \right) : \nabla \varphi - 2 \sum_{i=1}^{m} \delta_{j=1} \Phi_j^T \Phi_i \right) \nabla \cdot \varphi$$
$$- \left( 2 \varphi \cdot \nabla \bar{u}^T + \frac{1}{\text{Re}} \nabla \varphi^T (\bar{u})^T \right) : \nabla \bar{w}^T$$
$$- 2 \sum_{i=1}^{m} \delta_{j=1} \Phi_j^T \Phi_i \left( \varphi \cdot \nabla \bar{u} - \left( C + \frac{1}{\text{Re}} \nabla \bar{u}^T \right) : (\nabla \varphi^T)(\nabla \bar{w}^T) \right)$$
$$- \bar{q} tr \left[(\nabla \varphi^T)(\nabla \bar{w}^T)\right] - \bar{p} tr \left[(\nabla \varphi^T)(\nabla \bar{u}^T)\right] \right] dx, \quad \text{(76)}$$

As a result, the first variation of the functional is

$$\lim_{\epsilon \to 0} \frac{L(\phi, \zeta(\phi(x))) - L(\phi_0, \zeta(x))}{\epsilon} = L(\phi_0, \zeta(x)) [\zeta'] + L(\phi_0, \zeta(x)) [\varphi]. \quad \text{(77)}$$

For sensitivity analysis, the adjoint variable method is used to derive a main problem and an adjoint problem by setting

$$L(\phi_0, \zeta(x)) [\zeta'] = 0. \quad \text{(78)}$$

After solving the main and the adjoint problems, the sensitivity is evaluated by substituting the main and adjoint variables into eq. (76).

### 7.3. Domain Reshaping Method

We obtain the optimal domain numerically using the following iterating scheme, where an integer number $k \in \mathbb{N}$ denotes the iteration step, and where positive values $\epsilon, \beta \in \mathbb{R}$ represent arbitrary small parameters, $K \in \mathbb{N}$ denotes the final step number when the iteration scheme finishes.

1. **Step 1**: Set $k = 0$ and $\Omega^k, \zeta^k, \phi^k$.
2. **Step 2**: Define the functional $L(\phi_0^k, \zeta^k(x))$.
3. **Step 3**: Derive the first variation of the functional $L(\phi_0, \zeta(x)) [\zeta'] + L(\phi_0, \zeta(x)) [\varphi]$.
4. **Step 4**: Solve the main and adjoint problems from $L(\phi_0, \zeta(x)) [\zeta'] = 0$.
5. **Step 5**: Substitute the main and adjoint variables $\zeta$ into $L(\phi_0, \zeta(x)) [\varphi]$.
6. **Step 6**: Obtain the new domain $\phi^{k+1}(\Omega^{k+1}) = \phi_0(\Omega^k) + \epsilon \varphi^k_{H^1}(\Omega^k)$.
7. **Step 7**: Judge the convergence:
   - If terminal condition $|f^{k+1} - f_0| < \beta$ is satisfied for the cost function $f$, then stop.
   - Otherwise, replace $k + 1$ with $k$ and return to Step 2.

In fact, all the variables, functions, function spaces, and functionals used for this study depend on iteration step $k$ for domain deformation. Hereinafter, for convenience, iteration step $k$ is not described.

### 7.4. Adaptive Mesh Refinement

In this section, AMR distributed in FreeFem++ is summarized as explained below.

Letting $\eta(x)$ be a data function describing any physical state in the domain for finite element method, then a Taylor expansion of the data function $\eta(x)$ with respect to any interior point $x$ in an element over a mesh can be expressed as

$$\eta(x) = \eta(x_c + t \delta x)$$
$$= \eta(x_c) + \left( \nabla \eta \right)_{x=x_c} (t \delta x) + \frac{1}{2} (t \delta x)^T \left( \nabla^2 \eta \right)_{x=x_c} (t \delta x) + o \left( (t \delta x)^2 \right)$$
$$= \eta_h(x) + \frac{1}{2} (t \delta x)^T \left( \nabla^2 \eta \right)_{x=x_c} (t \delta x) + o \left( (t \delta x)^2 \right), \quad \text{(79)}$$

for $t \in [0, 1]$, where $x_c$ represents the position vector at a node of an element in a mesh, and where $\eta_h$ denotes the linear approximation for $\eta$. The interpolation error $e(x)$ at a displacement $t \delta x$ from node $x_c$ can be expressed as

$$e(x) = \int_0^1 |\eta(x) - \eta_h(x)| \, dt$$
\[
\approx \int_0^1 \frac{1}{2} (d\delta x)^T \left[ (\nabla \nabla \eta)_{|x=x} \right] (d\delta x) \, dt \\
\leq \frac{1}{2} (d\delta x)^T \nabla \nabla \eta_{|x=x} (d\delta x) \int_0^1 t^2 \, dt \\
= \frac{1}{6} (d\delta x)^T \nabla \nabla \eta_{|x=x} (d\delta x) \tag{80}
\]

As described herein, for increasing numerical accuracy on some singular points or areas where a stress concentration appears in two-dimensional open cavity flow, the Hessian matrix \( \nabla \nabla \eta = \nabla \left( u^0 \right)^T \) is used, where \( u^0 \) is the solution of the stationary Navier–Stokes problem obtained using Newton–Raphson method. Finally, the maximal interpolation error over a mesh is written as

\[
e(x) = \frac{1}{6} (d\delta x)^T \left| \nabla \left( u^0 \right)^T \right|_{x=x} (d\delta x). \tag{81}
\]

The author only works with one variable \( \eta \) for meeting the fixed error tolerance \( \epsilon_i \) that must be equidistributed over the mesh as

\[
\sup_{x \in \Omega(2)} |\eta(x) - \eta_i(x)| \leq \frac{1}{6} (d\delta x)^T \left| \nabla \left( u^0 \right)^T \right|_{x-x} (d\delta x). \tag{82}
\]

The procedure of AMR distributed in Freefem++ is the following for \( i = 1 \cdots N_{\text{nodes}} \), where \( N_{\text{nodes}} \in \mathbb{N} \) represents the number of the nodes.

Step 1 Set \( i = 1 \) and \( \epsilon_i \) arbitrarily.

Step 2 Let \( d_i \) stand for the length of the edge \( a_i \).

Step 3 Compare \( \epsilon_i \) and \( d_i \):

- If \( d_i > \epsilon_i \) and \( i \leq N_{\text{nodes}} \), then the edge \( a_i \) must be cut into two edge and return to Step 2.

- If \( d_i \leq \epsilon_i \) and \( i \leq N_{\text{nodes}} \), then replace \( i \) with \( i + 1 \) and return to Step 2.

-Otherwise stop.

Additional details related to the numerical procedure are reported elsewhere in the literature (Castro-Diaz and Hecht, 2006) along with a mathematical proof in continuous and discrete spaces of the domain (Alauzet et al., 2006).

7.5. Parameter Validation

The author validates finite element meshes with and without AMR, changing the number of nodes \( N_{\text{nodes}} \) and the number of elements \( N_{\text{elements}} \) and performing linear stability analysis, where Fig. 10 represents numerical demonstrations of finite element meshes with and without AMR.

![Finite element meshes with and without AMR](image)

(a) without AMR  
(b) with AMR

Fig. 10 Finite element meshes with and without AMR for \( (N_{\text{nodes}}, N_{\text{elements}}) = (3068, 5822) \).

For solving eigenvalue problems of linear stability analysis, Arnoldi method distributed in Freefem++ is applied, where \( 0 \pm 7.5i \) is set for the shift parameter. As reported by Barbagallo et al. (2009), \( 4.66 \pm 7.88i \) is obtained at \( \text{Re} = 7500 \). Table 2 shows eigenvalues with increasing \( (N_{\text{nodes}}, N_{\text{elements}}) \). Especially, it seems that the case of the initial domain with AMR approximates the result reported by Barbagallo et al. (2009) \( 4.66 \pm 7.88i \) for \( (N_{\text{nodes}}, N_{\text{elements}}) = (44225, 87286) \).

Next, the critical Reynolds number \( \text{Re}_c \) is investigated for \( (N_{\text{nodes}}, N_{\text{elements}}) = (44225, 87286) \). Sipp et al. (2007) presents \( \text{Re}_c = 4140 \), where eigenvalue is \( 0 \pm 7.5i \). Table 3 shows eigenvalue from \( \text{Re} = 4100 \) to \( 4180 \), which depicts almost \( 0 \pm 7.5i \) at \( \text{Re} = 4140 \) as in the result of Sipp et al. (2007). Moreover, Fig. 11(a) portrays a spectrum at \( \text{Re} = 4140 \).

Barbagallo et al. (2009) investigates eigenvalues at \( \text{Re} = 7500 \) and presents unstable modes of three kinds, in particular \( 0.890 \pm 10.9i, 0.729 \pm 13.8i, \) and \( 0.466 \pm 7.88i \). Numerical investigation of linear stability for \( (N_{\text{nodes}}, N_{\text{elements}}) = (44225, 87286) \) shows unstable modes \( 0.886 \pm 10.906i, 0.716 \pm 13.8044i, \) and \( 0.465 \pm 7.88766i \), presented in Fig. 11(b).

Finally, by performing Snapshot POD, eigenvalues of Snapshot POD are obtained every \( (N_{\text{nodes}}, N_{\text{elements}}) \), as shown in Table 4. From them, such eigenvalues are almost converging from \( (N_{\text{nodes}}, N_{\text{elements}}) = (44225, 87286) \).
Fig. 11  Eigenvalues in the initial domain with AMR for \((N_{\text{nodes}}, N_{\text{elements}}) = (44225,87286)\) at Re = 4140.

Table 2  Linear stability analysis with and without AMR in the initial domain

| Without AMR ( Real,Imag ),\((N_{\text{nodes}},N_{\text{elements}})\) | With AMR ( Real,Imag ),\((N_{\text{nodes}},N_{\text{elements}})\) |
|---|---|
| (0.68125±7.81754),(3068,58222) | (0.45849±7.88511),(15520,30472) |
| (0.40933±7.80358),(8298,16074) | (0.45861±7.88797),(20373,40074) |
| (0.46679±7.84882),(15974,31218) | (0.46528±7.89157),(27219,53533) |
| (0.46205±7.87956),(25770,50602) | (0.46647±7.89419),(32895,64809) |
| (0.47340±7.91105),(38288,75430) | (0.46433±7.88541),(44225,87286) |
| (0.47094±7.98100),(53319,105284) | (0.46497±7.88421),(53079,104862) |

Table 3  Critical Reynolds number \(Re_c\) for \((N_{\text{nodes}}, N_{\text{elements}}) = (44225,87286)\)

| Reynolds number \(Re\) | Real | Imag ±7.4917 |
|---|---|---|
| 4100 | -0.007166 | ±7.4917 |
| 4120 | -0.003741 | ±7.4915 |
| 4140 | -0.000326 | ±7.4957 |
| 4160 | 0.004309 | ±7.5015 |
| 4180 | 0.007970 | ±7.5038 |

Table 4  Eigenvalues of Snapshot POD every \((N_{\text{nodes}}, N_{\text{elements}})\) in the initial domain with AMR

| \((N_{\text{nodes}}, N_{\text{elements}})\) | \(i = 2\) | \(i = 3\) | \(i = 4\) | \(i = 5\) | \(i = 6\) | \(i = 7\) |
|---|---|---|---|---|---|---|
| (15520,30472) | 0.123729 | 0.122086 | 0.048609 | 0.040498 | 0.026051 | 0.021824 |
| (20373,40074) | 0.105721 | 0.103434 | 0.021236 | 0.020547 | 0.008953 | 0.008417 |
| (27219,53533) | 0.108516 | 0.104436 | 0.021503 | 0.020207 | 0.008692 | 0.008423 |
| (32895,64809) | 0.103814 | 0.100078 | 0.019919 | 0.019378 | 0.011342 | 0.010566 |
| (44225,87286) | 0.097497 | 0.095360 | 0.018742 | 0.018345 | 0.015410 | 0.013604 |
| (53079,104862) | 0.098259 | 0.096072 | 0.018998 | 0.018982 | 0.014393 | 0.012881 |
7.6. Linear Stability in the Optimal Domain

The author investigates linear stability in the optimal domains for Case 2 and Case 3, shown as spectra in Fig. 12. Compared to results presented in Fig. 11(b), the shape optimization problem makes the disturbance more unstable than the initial domain.

![Spectra for (a) Case 2 and (b) Case 3 evaluated using Generalized J Integration Type in the optimal domains.](image)

Fig. 12 Spectra for (a) Case 2 and (b) Case 3 evaluated using Generalized J Integration Type in the optimal domains.

References

Alauzet, F., Loseille, A., Dervieux, A. and Frey, J.P, Multi-Dimensional Continuous Metric for Mesh Adaptation, In Proc. of 15th Int. Meshing Roundtable, Vol.15, (2006), pp.191-214.
Azegami, H. and Wu, Z., Domain Optimization Analysis in Linear Elastic Problems: Approach Using Traction Method, JSME Int. J. Ser. A, Vol.39, (1996) pp.272-278.
Barbagallo, A., Sipp, D., and Schmid, J.P., Closed-loop control of an open cavity flow using reduced-order models, J. Fluid Mech., Vol.641, (2009), pp.1-50.
Brezi, F., Pitkaranta, J.: On the stabilization of finite element approximations of the Stokes equations. In: Hackbusch, W. (ed.) Efficient Solutions of Elliptic Systems, pp. 11-19. Vieweg, Wiesbaden (1984).
Castro-Diaz, M.J., Hecht, F., Anisotropic Surface Mesh Generation, INRIA Res. Rep., Vol.2672, (2006), pp.1-31.
Davis, T., Algorithm 832: UMFPACK, an unsymmetric-pattern multifrontal method, ACM Trans. on Math. Software, Vol.30, (2004), pp.196-199.
Hecht, F, New development in FreeFem++, J. of Numerical Math., Vol.20, (2012), pp.251-265.
Haslinger, J. and Maksimenko, A.E., Introduction to Shape Optimization: Theory, Approximation, and Computation, SIAM, Philadelphia, 2003.
Kimura, M., Shape derivative of minimum potential energy: abstract theory and applications, Jindrich Nečas Center for Mathematical Modeling, Lecture notes Volume IV, Topics in Mathematical Modeling, (2008), pp.1-38.
Mohammadi, B. and Pironneau, O., Applied Shape Optimization for Fluids, Oxford University Press, 2001.
Moubachir, M. and Zolesio, J.P., Moving Shape Analysis and Control: Applications to Fluid Structure Interactions, Chapman and Hall / CRC Pure and Applied Mathematics. Boca Raton, 2006.
Nakazawa, T., Increasing the Critical Reynolds number by maximizing dissipation energy problem, Proceedings of the Fifth International Conference on Jets, Wakes and Separated Flows (ICJWSF2015), Editor Antonio Segalini, Chapter 77, (2016), pp.613-620.
Nakazawa, T. and Azegami, H., Shape Optimization Method improving Hydrodynamic Stability, Jpn. J. of Indust. and Appl. Math., Vol.33, (2015), pp.167-181.
Nakazawa, T., Optimal Design by Adaptive Mesh Refinement on Shape Optimization of Flow Fields Considering Proper Orthogonal Decomposition, Interdisciplinary Information Sciences, Vol.25, No. 2(2019), DOI: 10.4036/iis.2019.B.02.
Notsu, H. and Tabata, M., Error Estimates of a Stabilized Lagrange–Galerkin Scheme of Second-Order in Time for the Navier–Stokes Equations, Mathematical Fluid Dynamics, Present and Future, (2014), pp.497-530.
Ohtsuka, K, Generalized J-integral and three-dimensional fracture mechanics 1, Hiroshima Math. J., Vol.11, (1981), pp. 21-52.
Ohtsuka, K, Generalized J-integral and its application 1 – Basic theory –, Japan J. Appl. Math. J., Vol.2, (1985), pp.329-350.
Ohtsuka, K, Generalized J-integral and three-dimensional fracture mechanics 2, Hiroshima Math. J., Vol.16, (1986), pp.327-352.
Ohtsuka, K and Khludnev, A., Generalized J-integral method for sensitivity analysis of static shape design, Control and Cybernetics, Vol.29, (2000), pp.513-533.
Ohtsuka, K and Khludnev, A., Shape differentiability of Lagrangians and application to Stokes problem, SIAM J. Control Optim., Vol.56, (2018), pp.3668-3684.
Pironneau, O., On optimum profiles in Stokes flow, JFM, Vol.59, (1973), pp.117-128.
Pironneau, O., On optimum design in fluid mechanics, JFM, Vol.64, (1974), pp.97-110.
Plotnikov, P, and Sokolowski, J, Compressible Navier-Stokes Equations: Theory and Shape Optimization, Birkhaeuser, 2012.
Romero, S. J., Non-Newtonian laminar flow machine ro-tor design by using topology optimization, Struct. Multi-disc. Optim., Vol. 55, pp. 1711–1732, 2017.
Sipp, D. and Lebedev, A., Global stability of base and mean flows: a general approach and its applications to cylinder and open cavity flow, J. Fluid Mech., Vol.593, (2007), pp.333-358.
Zhang, B. and Liu, X.: Topology optimization study of arterial bypass configurations using the level set method, Struct. Multidisc. Optim., Vol. 51, pp 773–798, 2015.
Zhang, B., Liu, X.: Topology optimization design of non-Newtonian roller-type viscous micropump, Struct. Multi-disc. Optim., Vol. 53, pp. 409–424, 2016.