On the Sum Necessary to Assure a Degree Sequence is Potentially $H$-Graphic

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May 5, 2014

Abstract

A sequence of nonnegative integers $\pi = (d_1, d_2, \ldots, d_n)$ is graphic if there is a (simple) graph $G$ with degree sequence $\pi$. In this case, $G$ is said to realize or be a realization of $\pi$. Degree sequence results in the literature generally fall into two classes: forcible problems, in which all realizations of a graphic sequence must have a given property, and potential problems, in which at least one realization of $\pi$ must have the given property.

Given a graph $H$, a graphic sequence $\pi$ is potentially $H$-graphic if there is some realization of $\pi$ that contains $H$ as a subgraph. In 1991, Erdős, Jacobson and Lehel posed the following question:

Determine the minimum integer $\sigma(H, n)$ such that every $n$-term graphic sequence with sum at least $\sigma(H, n)$ is potentially $H$-graphic.

As the sum of the terms of $\pi$ is twice the number of edges in any realization of $\pi$, the Erdős-Jacobson-Lehel problem can be viewed as a potential degree sequence relaxation of the (forcible) Turán problem, wherein one wishes to determine the maximum number of edges in a graph that contains no copy of $H$.

While the exact value of $\sigma(H, n)$ has been determined for a number of specific classes of graphs (including cliques, cycles, complete bigraphs and others), very little is known about the parameter for arbitrary $H$. In this paper, we determine $\sigma(H, n)$ asymptotically for all $H$, thereby providing an Erdős-Stone-Simonovits-type theorem for the Erdős-Jacobson-Lehel problem.

**Keywords:** Degree sequence, potentially $H$-graphic sequence

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1 Introduction

A sequence of nonnegative integers $\pi = (d_1, d_2, ..., d_n)$ is graphic if there is a (simple) graph $G$ of order $n$ having degree sequence $\pi$. In this case, $G$ is said to realize or be a realization of $\pi$, and we will write $\pi = \pi(G)$. If a sequence $\pi$ consists of the terms $d_1, \ldots, d_t$ having multiplicities $\mu_1, \ldots, \mu_t$, we may write $\pi = (d_1^{\mu_1}, \ldots, d_t^{\mu_t})$. Unless otherwise noted, throughout this paper all sequences are nonincreasing. Additionally, we let $\sigma(\pi)$ denote the sum of the terms of $\pi$.

The study of graphic sequences dates to the 1950s, and includes the characterization of graphic sequences by Havel [18] and Hakimi [16] and an independent characterization by Erdős and Gallai [9]. Subsequent research sought to describe those graphic sequences that are realized by graphs with certain desired properties. Such problems can be broadly classified into two types, first described as “forcible” problems and “potential” problems by A.R. Rao in [27]. In a forcible degree sequence problem, a specified graph property must exist in every realization of the degree sequence $\pi$, while in a potential degree sequence problem, the desired property must be found in at least one realization of $\pi$.

Results on forcible degree sequences are often stated as traditional problems in structural or extremal graph theory, where a necessary and/or sufficient condition is given in terms of the degrees of the vertices (or equivalently the number of edges) of a given graph (e.g. Dirac’s Theorem on hamiltonian graphs or the number of edges in a maximal planar graph). Two older, but exceptionally thorough surveys on forcible and potential problems are due to Hakimi and Schmeichel [17] and S.B. Rao [28].

A number of degree sequence analogues to classical problems in extremal graph theory appear throughout the literature, including potentially graphic sequence variants of Hadwiger’s Conjecture [7, 29], the Sauer-Spencer graph packing theorem [1], and the Erdős-Sós Conjecture [25].

Pertinent to our work here is the Turán Problem, one of the most well-established and central problems in extremal graph theory.

Problem 1 (The Turán Problem). Let $H$ be a graph and $n$ be a positive integer. Determine the minimum integer $\text{ex}(H, n)$ such that every graph of order $n$ with at least $\text{ex}(H, n) + 1$ edges contains $H$ as a subgraph.

We refer to $\text{ex}(H, n)$ as the extremal number or extremal function of $H$. Mantel [20] determined $\text{ex}(K_3, n)$ in 1907 and Turán [30] determined $\text{ex}(K_t, n)$ for all $t \geq 3$ in 1941, a result considered by many to mark the start of modern extremal graph theory. Outside of these results, the exact value of the extremal function is known for very few graphs (cf. [2, 14, 8]). In 1966, however, Erdős and Simonovits [11] extended previous work of Erdős and Stone [12] and determined $\text{ex}(H, n)$ asymptotically for arbitrary $H$. More precisely, this seminal theorem gives exact asymptotics for $\text{ex}(H, n)$ when $H$ is a nonbipartite graph.

Theorem 1 (The Erdős-Stone-Simonovits Theorem). If $H$ is a graph with chromatic number $\chi(H) \geq 2$, then

$$\text{ex}(H, n) = \left(1 - \frac{1}{\chi(H)-1}\right) \binom{n}{2} + o(n^2).$$
The focus of this paper is the following problem posed by Erdős, Jacobson and Lehel in 1991 [10]. A graphic sequence \( \pi \) is potentially \( H \)-graphic if there is some realization of \( \pi \) that contains \( H \) as a subgraph.

**Problem 2.** Determine \( \sigma(H, n) \), the minimum even integer such that every \( n \)-term graphic sequence \( \pi \) with \( \sigma(\pi) \geq \sigma(H, n) \) is potentially \( H \)-graphic.

We will refer to \( \sigma(H, n) \) as the potential number or potential function of \( H \). As \( \sigma(\pi) \) is twice the number of edges in any realization of \( \pi \), the Erdős-Jacobson-Lehel problem can be viewed as a potential degree sequence relaxation of the Turán problem.

In [10], Erdős, Jacobson and Lehel conjectured that \( \sigma(K_t, n) = (t - 2)(2n - t + 1) + 2 \). The cases \( t = 3, 4 \) and \( 5 \) were proved separately (see respectively [10], [15] and [20], and [21]), and Li, Song and Luo [22] proved the conjecture true for \( t \geq 6 \) and \( n \geq \left(\frac{t}{2}\right) + 3 \). In addition to these results for complete graphs, the value of \( \sigma(H, n) \) has been determined exactly for a number of other specific graph families, including complete bipartite graphs [5, 24], disjoint unions of cliques [13], and the class of graphs with independence number two [14] (for a number of additional examples, we refer the reader to the references of [14]). Despite this, relatively little is known in general about the potential function for arbitrary \( H \). In this paper, we determine \( \sigma(H, n) \) asymptotically for all \( H \), thereby giving a potentially \( H \)-graphic sequence analogue to the Erdős-Stone-Simonovits Theorem.

## 2 Constructions and Statement of Main Result

We assume that \( H \) is an arbitrary graph of order \( k \) with at least one nontrivial component and furthermore that \( n \) is sufficiently large relative to \( k \). We let \( F < H \) denote that \( F \) is an induced subgraph of \( H \). For each \( i \in \{\alpha(H) + 1, \ldots, k\} \) let

\[
\nabla_i(H) = \min\{\Delta(F) : F < H, |V(F)| = i\},
\]

and consider the sequence

\[
\tilde{\pi}_i(H, n) = ((n - 1)^{k-i}, (k - i + \nabla_i(H) - 1)^{n-k+i}).
\]

If this sequence is not graphic, that is if \( n - k + i \) and \( \nabla_i(H) - 1 \) are both odd, we reduce the smallest term by one. To see that this yields a graphic sequence, we make two observations. First, \((\nabla_i(H) - 1)\)-regular graphs of order \( n - k + i \geq \nabla_i(H) \) exist whenever \( \nabla_i(H) - 1 \) and \( n - k + i \) are not both odd. If \( n - k + i \) and \( \nabla_i(H) - 1 \) are both odd, it is not difficult to show that the sequence \((\nabla_i(H) - 1)^{n-k+i-1}, \nabla_i(H) - 2) \) is graphic.

Our focus here is the asymptotic behavior of the potential function, and as such we consider the dominating term of \( \sigma(\tilde{\pi}_i(H, n)) \). Let

\[
\bar{\sigma}_i = (2(k - i) + \nabla_i(H) - 1).
\]

In [14], the first author and J. Schmitt conjectured the following.
Conjecture 1. Let $H$ be a graph, and let $\epsilon > 0$. There exists an $n_0 = n_0(\epsilon, H)$ such that for any $n > n_0$
\[ \sigma(H, n) \leq \max_{H' \subseteq H} (\bar{\sigma}_{\alpha(H')+1}(H') + \epsilon)n. \]

The condition that one must examine subgraphs of $H$ is necessary. As an example, for $t \geq 3$ let $H$ be obtained by subdividing one edge of $K_{1,t}$. Since $K_{1,t}$ is a subgraph of $H$, any sequence that is potentially $H$-graphic is necessarily potentially $K_{1,t}$-graphic. However, both graphs have independence number $t$ and $\bar{\sigma}_{t+1}(H) < \bar{\sigma}_{t+1}(K_{1,t})$.

We show that in fact one needs only examine somewhat large induced subgraphs of $H$. The following, which determines the asymptotics of the potential function precisely for arbitrary $H$, is the main result of this paper.

Theorem 2. Let $H$ be a graph and let $n$ be a positive integer. If $\bar{\sigma}(H)$ is the maximum of $\bar{\sigma}_i(H)$ for $\alpha(H) + 1 \leq i \leq |H|$, then
\[ \sigma(H, n) = \bar{\sigma}(H)n + o(n). \]

As was pointed out in [14], Conjecture 1 is correct for all graphs $H$ for which $\sigma(H, n)$ is known. Consequently, it is feasible that Theorem 2 is actually an affirmation of Conjecture 1. That is, for all $H$ it is possible that
\[ \bar{\sigma}(H) = \max_{H' \subseteq H} (\bar{\sigma}_{\alpha(H')}+1(H')), \]
however we are unable to either verify or disprove this at this time.

The proof of Theorem 2 is an immediate consequence of the following two results.

Theorem 3. If $H$ is a graph and $n$ is a positive integer, then for all $\alpha(H) + 1 \leq i \leq |H|$, $\sigma(H, n) \geq \sigma(\pi_i(H, n)) + 2$.

Theorem 4. Let $H$ be a graph, and let $\omega = \omega(n)$ be an increasing function that tends to infinity with $n$. There exists an $N = N(\omega, H)$ such that for any $n \geq N$
\[ \sigma(H, n) \leq \bar{\sigma}(H)n + \omega(n). \]

The proof of Theorem 4 relies on repeated use of the following result, which may be of independent interest. Let $H$ be a graph, and let $(h_1, \ldots, h_k)$ be the degree sequence of $H$. A graphic sequence $\pi = (d_1, \ldots, d_n)$ is degree sufficient for $H$ if $d_i \geq h_i$ for $1 \leq i \leq k$.

Theorem 5 (The Bounded Maximum Degree Theorem). Let $H$ be a graph and let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence with $n$ sufficiently large satisfying the following:

1. $\pi$ is degree sufficient for $H$, and
2. $d_n \geq k - \alpha(H)$.

There exists a function $f = f(\alpha(H), k)$ such that if $d_1 < n - f(\alpha(H), k)$, then $\pi$ is potentially $H$-graphic.

In Section 3 we present several technical lemmas used in the proofs of Theorems 4 and 5. In Section 4 we prove the Bounded Maximum Degree Theorem, with the proof of Theorems 3 and 4 following in Section 5.
3 Technical Lemmas

We will need the following results from [23] and [31].

**Theorem 6** (Yin and Li). Let \( \pi = (d_1, \ldots, d_n) \) be a nonincreasing graphic sequence and \( k \geq 1 \) be an integer. If \( d_k \geq k - 1 \) and \( d_i \geq 2(k-1) - i \) for \( 1 \leq i \leq k-1 \), then \( \pi \) is potentially \( K_k \)-graphic.

We let \( G \lor H \) denote the standard join of \( G \) and \( H \).

**Lemma 1** (Yin). If \( \pi \) is a potentially \( K_r \lor K_s \)-graphic sequence, then there is a realization of \( \pi \) in which the vertices in the copy of \( K_r \) are the \( r \) vertices of highest degree and the vertices in the copy of \( K_s \) are the next \( s \) vertices of highest degree.

We now present a classical result of Kleitman and Wang [19] that generalizes the Havel-Hakimi Algorithm [16, 18].

**Theorem 7** (Kleitman and Wang). Let \( \pi = (d_1, \ldots, d_n) \) be a nonincreasing sequence of nonnegative integers. If \( \pi_i \) is the sequence defined by

\[
\pi_i = \begin{cases} 
(d_1 - 1, \ldots, d_{d_i} - 1, d_{d_i + 1}, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n) & \text{if } d_i < i \\
(d_1 - 1, \ldots, d_{i-1} - 1, d_{i+1} - 1, \ldots, d_{d_i} - 1, d_{d_i + 2}, \ldots, d_n) & \text{if } d_i \geq i,
\end{cases}
\]

then \( \pi \) is graphic if and only if \( \pi_i \) is graphic.

The process of removing a term from a graphic sequence as described by Theorem 7 is referred to as laying off the term \( d_i \) from \( \pi \). Iteratively applying the Kleitman-Wang algorithm yields the following, which generalizes related lemmas from [3, 13].

**Lemma 2.** Let \( m \) and \( k \) be positive integers and let \( \omega = \omega(n) \) be an increasing function that tends to infinity with \( n \). There exists \( N = N(m, k, \omega) \) such that for all \( n \geq N \), if \( \pi \) is an \( n \)-term graphic sequence such that \( \sigma(\pi) \geq mn + \omega(n) \), then

1. \( \pi \) is potentially \( K_k \)-graphic, or
2. iteratively applying the Kleitman-Wang algorithm by laying off the minimum term in each successive sequence will eventually yield a graphic sequence \( \pi' \) with \( n' \) terms satisfying the following properties: (a) \( \sigma(\pi') \geq mn' + \omega(n') \), (b) the minimum term in \( \pi' \) is greater than \( m/2 \), and (c) \( n' \geq \frac{\omega(n)}{2(k-2)-m} \).

**Proof.** First observe that if \( k \leq 2 \), then the result is trivial, so we assume that \( k \geq 3 \). Let \( \pi = \pi_0 \). If the minimum term of \( \pi_i \) is at least \( m/2 \), let \( \pi' = \pi_i \). Otherwise, obtain the \( n - i - 1 \)-term graphic sequence \( \pi_{i+1} \) by applying Theorem 7 to lay off a minimum term from \( \pi_i \) and then sorting the resulting sequence in nonincreasing order. Note that always \( \sigma(\pi_{i+1}) \geq \sigma(\pi_i) - m \), and by induction we have \( \sigma(\pi_{i+1}) \geq m(n-i-1) + \omega(n) \).

A consequence of the Kleitman-Wang algorithm is that if \( \pi_i \) is potentially \( K_k \)-graphic, then \( \pi_0 \) is potentially \( K_k \)-graphic. By the results of [10], [13] and [20], [21], and [22] we know that \( \sigma(K_k, n) = 2(k-2)n - (k-1)(k-2) + 2 \) for \( n \geq \binom{k}{2} + 3 \). Thus, if \( \sigma(\pi_i) \geq 2(k-2)(n-i)+2 \), then \( \pi_i \) is potentially \( K_k \)-graphic. Therefore, if \( \pi_0 \) is potentially \( K_k \)-graphic, then \( \pi_i \) is potentially \( K_k \)-graphic for all \( i \).

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for some $i \geq 0$, then $\pi$ is potentially $K_k$-graphic. Considering the case when $i = 0$, we may assume that $m < 2(k - 2)$. If $i \geq n - \frac{\omega(n)}{2(k - 2) - m}$, then
\[
\sigma(\pi_i) - 2(k - 2)(n - i) \geq (m - 2(k - 2))(n - i) + \omega(n) \\
\geq (m - 2(k - 2)) \left( \frac{\omega(n)}{2(k - 2) - m} \right) + \omega(n) \\
\geq 0.
\]
Therefore, if $i \geq n - \frac{\omega(n)}{2(k - 2) - m}$ and $\frac{\omega(n)}{2(k - 2) - m} \geq \left( \frac{k}{2} \right) + 3$, then $\pi_i$ is potentially $K_k$-graphic and consequently $\pi$ is potentially $K_k$-graphic.

It now follows that if $\pi$ is not potentially $K_k$-graphic, then the process of applying the Kleitman-Wang algorithm must output $\pi' = \pi_i$ for some $i < n - \frac{\omega(n)}{2(k - 2) - m}$. Since $\omega$ is increasing and $n' \leq n$, it follows that $\sigma(\pi') \geq mn' + \omega(n')$.

\section{Proof of the Bounded Maximum Degree Theorem}

We are now ready to prove the Bounded Maximum Degree Theorem.

\textit{Proof.} Let $V(H) = \{u_1, \ldots, u_k\}$, indexed such that $d_H(u_i) \geq d_H(u_j)$ when $i \leq j$. Let us assume that $\pi$ satisfies the hypothesis of the theorem, but is not potentially $H$-graphic. In a realization $G$ of $\pi$, let $S = \{v_1, \ldots, v_k\}$ be a set of vertices such that $d_G(v_i) = d_i$ and let $H_S$ be the graph with vertex set $S$ in which two vertices $v_i, v_j$ are adjacent if and only if $u_iu_j \in E(H)$. If all edges of $H_S$ are edges of $G$, then $H_S$ is a subgraph of $G$ isomorphic to $H$ and $\pi$ is potentially $H$-graphic. Hence, let us assume that $G$ is a realization which maximizes $|E(G) \cap E(H_S)|$, but that the edge $v_iv_j \in E(H_S)$ while $v_iv_j \not\in E(G)$. Now, because $d_G(v_i) \geq d_H(u_i) = d_{H_S}(v_i)$ and $d_G(v_j) \geq d_H(u_j) = d_{H_S}(v_j)$, there must exist (not necessarily distinct) vertices $a_i$ and $a_j$ such that $v_ia_i, v_ja_j \in E(G)$, while $v_ia_i, v_ja_j \not\in E(H_S)$.

We begin by showing that many vertices of $\pi' = \pi_i$ have neighbors in $V(G) - S$.

\textbf{Claim 1.} There are at most
\[
g(\alpha(H), k) = \binom{k}{\alpha(H)} \left( 2 \binom{k - \alpha(H)}{2} + \alpha(H) - 1 \right)
\]
vertices $w$ in $V(G) - S$ such that $N(w) \subseteq S$.

\textit{Proof.} Let us assume that there are at least $g(\alpha(H), k) + 1$ vertices $w$ in $V(G) - S$ such that $N(w) \subseteq S$. By the hypothesis of the theorem, each vertex $w \in V(G)$ satisfies $d_G(w) \geq k - \alpha(H)$. For each vertex $w$ such that $N(w) \subseteq S$, let $S_w$ be a set of $k - \alpha(H)$ vertices such that $S_w \subseteq N(w)$. Let $\ell = 2 \binom{k - \alpha(H)}{2} + \alpha(H)$. By the pigeonhole principle, there exists an independent set $\{w_1, \ldots, w_\ell\}$ of vertices and a $k - \alpha(H)$ element subset $\hat{S}$ of $S$ such that $S_{w_i} = \hat{S}$ for each $i$ between 1 and $\ell$.

Let $\mathcal{P}$ be a family of $\binom{k - \alpha(H)}{2}$ disjoint pairs of vertices from $\{w_1, \ldots, w_\ell\}$, and to each pair $\{v_i, v_j\}$ of vertices in $\hat{S}$ associate a distinct pair $P(i, j) \in \mathcal{P}$. If $v_iv_j$ is not an edge
in $G$ and $P(i, j) = \{w_r, w_s\}$, then we perform the 2-switch that replaces the edges $v_iw_r$ and $v_jw_s$ with the edges $v_iv_j$ and $w_rw_s$. In this way arrive at a realization of $\pi$ in which the graph induced by $\tilde{S}$ is complete, while maintaining the property that the vertices of $\tilde{S}$ are joined to $\{w_{\ell-\alpha(H)+1}, \ldots, w_{\ell}\}$. This produces a realization of $\pi$ that contains $K_{k-\alpha(H)} \lor \tilde{K}_{\alpha(H)}$ as a subgraph, contradicting the assumption that $\pi$ is not potentially $H$-graphic and completing the proof of the claim.

We will now use the fact that many vertices of $V(G) - S$ have neighbors in $V(G) - S$ to exhibit a 2-switch that inserts the edge $v_iv_j$ into $G$ at the expense of the edges $v_ia_i$ and $v_ja_j$ while preserving each edge in $H_S$. Let

$$\begin{align*}
f(\alpha(H), k) &= |g(\alpha(H), k) + 4k^2| + k + 1,
\end{align*}$$

and let $d_i = n - 1 - f(\alpha(H), k)$. If we let

$$X_i = \{v \in (V(G) - S) - N_{G-S}(a_i) : d_{G-S}(v) > 0\}$$

and

$$X_j = \{v \in (V(G) - S) - N_{G-S}(a_j) : d_{G-S}(v) > 0\},$$

then by Claim $\Pi$ we have that $|X_i|, |X_j| \geq 4k^2$.

Let $Y_i = N_{G-S}(X_i)$ and $Y_j = N_{G-S}(X_j)$.

By assumption, $\pi$ is not potentially $H$-graphic, and is therefore not potentially $K_k$-graphic. As such, Theorem $\mathbf{3}$ implies that each vertex in $G - S$ has degree at most $2k - 4 < 2k$. Let $y_i$ be a vertex in $Y_i$, and let $x_i$ be a neighbor of $y_i$ in $X_i$. There are at least $4k^2 - 1$ vertices in $X_j$ that are not $x_i$, and at least $4k^2 - 1 - (2k - 5)$ of these vertices have a neighbor in $Y_j$ that is not $y_i$. Since vertices in $V(G) - S$ have degree at most $2k - 4$, there are more than $(4k^2 - 2k + 4)/(2k) = 2k - 1 + 2/k$ vertices in $Y_j$ that are not $y_i$ that furthermore have a neighbor in $X_j$ that is not $x_i$. Since $d(y_i) < 2k$, there exists a vertex $y_j$ such that $y_j \neq y_i$ and $y_j$ has a neighbor $x_j \in X_j$ such that $x_j \neq x_i$. In this case, exchanging the edges $v_ia_i, v_ja_j, x_iy_i$ and $x_jy_j$ for the nonedges $v_iv_j, a_ix_i, a_jx_j$ and $y_iy_j$ yields a realization of $\pi$ that contradicts the maximality of $G$.

\section{5 Proofs of Theorems $\mathbf{3}$ and $\mathbf{4}$}

We begin by proving Theorem $\mathbf{3}$.

\textbf{Proof.} Every realization $G$ of $\tilde{\pi}_i(H, n)$ is a complete graph on $k - i$ vertices joined to an $n - k + i$-vertex graph $G_i$ with maximum degree $\nabla_i - 1$. Any $k$-vertex subgraph of $G$ contains at least $i$ vertices in $G_i$. Thus $H$ is not a subgraph of $G$ since every $i$-vertex induced subgraph of $H$ has maximum degree at least $\nabla_i$.

Next, we prove Theorem $\mathbf{4}$. 

\begin{flushright}$\Box$\end{flushright}
Proof. Let $\pi$ be an $n$-term graphic sequence with $n > N(\omega, H)$ such that $\sigma(\pi) \geq \overline{\sigma}(H)n + \omega(n)$. It suffices to show that $\pi$ is potentially $H$-graphic. If $\pi$ is potentially $K_k$-graphic, then the result is trivial. Therefore, by Theorem 4, we may assume that $d_{k+1} \leq 2k - 4$. Furthermore, throughout the proof we may assume that when applying Lemma 2 to $\pi$ or a residual sequence obtained from $\pi$, the resulting sequence satisfies conclusion (2) of that lemma.

The proof has two stages. The first is a process that reduces $\pi$ to a residual sequence called $\pi_\ell$. The second entails constructing a realization of $\pi$ containing $G$ from an appropriate realization of $\pi_\ell$ and then reversing the steps of the reduction of $\pi$.

To initialize Stage 1, apply Lemma 2 to $\pi$ to obtain an $n_0$-term graphic sequence $\pi_0 = (d_1^{(0)}, \ldots, d_{n_0}^{(0)})$ with minimum entry at least $\overline{\sigma}(H)/2$ and $\sigma(\pi_0) \geq \overline{\sigma}(H)n_0 + \omega(n_0)$.

The process terminates after generating the graphic sequence $\pi_i$ of length $n_i$ after stage $i$ such that $\pi_i = (d_1^{(i)}, \ldots, d_{n_i}^{(i)})$ satisfies $d_1^{(i)} \leq n_i - f(\alpha(H), k - i)$ (here $f$ is defined in the proof of Theorem 5). If the process terminates after step $i$, then set $\pi_\ell = \pi_i$.

Otherwise, let $\hat{\pi}_i$ be a realization of $\pi_i$ in which the vertex of degree $d_1^{(i)}$, which we call $v_1^{(i)}$, is adjacent to the next $d_1^{(i)}$ vertices of highest degree; such a realization exists as a consequence of Theorem 7. From $\hat{\pi}_i$, obtain the graph $\hat{\pi}_i'$ by deleting the nonneighbors of $v_1^{(i)}$, and let $\pi_i' = \pi(\hat{\pi}_i')$. Apply Lemma 2 to $\pi_i'$ to obtain the graphic sequence $\pi_i''$. Finally, lay off the largest term from $\pi_i''$ to obtain $\pi_{i+1}$ and let $n_{i+1}$ denote the length of $\pi_{i+1}$.

Let $G_i$ be a realization of $\pi_i$. We claim that there is a realization $G$ of $\pi$ such that $K_i \vee G_i$ is an induced subgraph of $G$. The proof is by induction on $i$, with the base case $i = 0$ being immediate. Thus, let $i > 0$.

We wish to obtain a realization $G_{i-1}$ of $\pi_{i-1}$ that contains $G_i$ as an induced subgraph such that $v_1^{(i-1)}$ is joined to all the vertices in $G_i$. Adding a vertex adjacent to each vertex of $G_i$ gives a realization $G_{i-1}''$ of $\pi_{i-1}''$. A realization $G_{i-1}'$ of $\pi_{i-1}'$ is obtained from $G_{i-1}''$ by iteratively adding vertices to $G_{i-1}''$ and joining them to those vertices whose degrees were reduced by the Kleitman-Wang algorithm. This process leaves $G_{i-1}'''$ as an induced subgraph of $G_{i-1}'$, which furthermore contains $G_i$ dominated by a single vertex.

Finally, note that deleting the nonneighbors of $v_1^{(i-1)}$ from a realization $G_{i-1}$ of $\pi_{i-1}$ does not change the graph induced by $N_{G_{i-1}}[v_1^{(i-1)}]$. Therefore, we may assume that $G_{i-1}$ is a realization of $\pi_{i-1}$ with $G_{i-1}'$ as the subgraph induced by the closed neighborhood of a vertex of maximum degree. By induction it follows that there is a realization $G$ of $\pi$ such that $G_i$ is an induced subgraph of $G$ and there is a clique of size $i$ that dominates the copy of $G_i$.

We next claim that $\sigma(\pi_i) \geq (\overline{\sigma}(H) - 2i)n_i + \omega(n_i)/2^i$ and the minimum term in $\pi_i$ is at least $\overline{\sigma}(H)/2 - i$ for all $i \in \{0, \ldots, \ell\}$. Again we use induction on $i$. First observe that $\sigma(\pi_0) \geq \overline{\sigma}(H)n_0 + \omega(n_0)$ and (as we assume that condition (2) holds in all applications of Lemma 2) the minimum term of $\pi_0$ is at least $\overline{\sigma}(H)/2$. Let

$$M = \max_{0 \leq i \leq k - \alpha(H) - 1} \{2f(\alpha(H), k - i)(2k - 4)\}.$$

Creating $\pi_i'$ entails deleting at most $f(\alpha(H), k - i)$ vertices each with degree at most $2k - 4$, which completes Stage 1.
and consequently, by induction,
\[ \sigma(\pi_i^t) \geq \sigma(\pi_i) - M \geq (\bar{\sigma}(H) - 2i)n_i + \omega(n_i)/2^i - M. \]

By assumption, \( n_i \) is sufficiently large, so we have \( \omega(n_i)/2^i \geq 2M \), and therefore \( \sigma(\pi_i^t) \geq (\bar{\sigma}(H) - 2i)n_i + \omega(n_i)/2^{i+1} \). If \( n_i'' \) is the number of terms in \( \pi_i'' \), then applying Lemma 2 to obtain \( \pi_i'' \) yields \( \sigma(\pi_i'') \geq (\bar{\sigma}(H) - 2i)n_i'' + \omega(n_i'')/2^{i+1} \). Since the maximum term in \( \pi_i'' \) is \( n_i'' - 1 \), removing this term and applying Theorem 4 yields
\[
\sigma(\pi_{i+1}) \geq (\bar{\sigma}(H) - 2i)n_i'' + \omega(n_i'')/2^{i+1} - 2(n_i'' - 1) \\
\geq (\bar{\sigma}(H) - 2(i + 1))n_{i+1} + \omega(n_{i+1})/2^{i+1}.
\]

It remains to show that \( \pi_t \) has a realization containing a \( k - \ell \)-vertex subgraph of \( H \), or some supergraph thereof. In \( \pi_t \), let \( t = \max\{i : d_i^{(\ell)} \geq k - \ell - 1\} \). First suppose that \( t \geq k - \ell - \alpha(H) \). In this case, \( \pi_t \) is degree sufficient for \( K_{k - \alpha(H) - \ell} \cup K_{\alpha(H)} \). Thus, since the minimum term in \( \pi_t \) is at least \( \bar{\sigma}(H)/2 - \ell \), the Bounded Maximum Degree Theorem (Theorem 4) implies that \( \pi_t \) is potentially \( K_{k - \alpha(H) - \ell} \cup K_{\alpha(H)} \)-graphic. Therefore there is a realization of \( \pi_t \) containing \( K_{k - \alpha(H) - \ell} \cup K_{\alpha(H)} \) which in turn contains a \( k - \ell \)-vertex subgraph of \( H \).

Now suppose that \( t < k - \ell - \alpha(H) \). Let \( F_t \) denote an \( i \)-vertex induced subgraph of \( H \) that achieves \( \Delta(F_t) = \nabla_i(H) \). We show that \( \pi_t \) is degree sufficient for \( K_t \cup F_{k-\ell-t} \). First observe that since \( t \leq k - \ell - \alpha(H) - 1 \) we have \( \bar{\sigma}(H) - 2\ell \geq 2t - \nabla_{k-\ell-t}(H) - 1 \). Thus
\[
\sigma(\pi_t) \geq (2t - \nabla_{k-\ell-t}(H) - 1)n\ell + \omega(n_t)/2^t.
\]
However, if \( \pi_t \) is not degree sufficient for \( K_t \cup F_{k-\ell-t} \), then
\[
\sigma(\pi_t) \leq t(n\ell - 1) + (k - \ell - t - 1)(k - \ell - 2) + (n\ell - k + l + 1)(t + \nabla_{k-\ell-t}(H) - 1) \\
< (2t + \nabla_{k-\ell-t}(H) - 1)n\ell + \omega(n_t)/2^t
\]
provided that \( n\ell \) is sufficiently large. Thus \( \pi_t \) is degree sufficient for \( K_t \cup F_{k-\ell-t} \). Note that \( \alpha(F_{k-\ell-t}) \) may be less than \( \alpha(H) \). Therefore we are not able to immediately apply the Bounded Maximum Degree Theorem to obtain a realization of \( \pi_t \) containing \( K_t \cup F_{k-\ell-t} \).

However, in this case, \( \pi_t \) is degree sufficient for \( K_t \cup \overline{K}_{k-\ell-t} \), and the minimum term of \( \pi_t \) is at least \( k - \ell - t \) (since \( \sigma(\pi_t) \geq (2k - t + \nabla_{t}(H) - 1 - 2\ell)n\ell \)). Thus by the Bounded Maximum Degree Theorem, there is a realization \( G_t \) of \( \pi_t \) containing \( K_t \cup \overline{K}_{k-\ell-t} \). If \( v_1, \ldots, v_{k-\ell} \) are the \( k - \ell \) vertices of highest degree in \( G_t \), then by Lemma 1 we may assume that \( \{v_1, \ldots, v_t\} \) is a clique that is completely joined to \( \{v_{i+1}, \ldots, v_{k-\ell}\} \). Delete \( v_1, \ldots, v_t \) from \( G_t \) to obtain \( G_t' \), and let \( \pi_t' = \pi(G_t') \), with the order of \( \pi_t' \) coming from the ordering of \( \pi_t \). Thus the first \( k - \ell - t \) terms in \( \pi_t' \) correspond to the vertices \( \{v_{t+1}, \ldots, v_{k-\ell}\} \) in \( G_t \). Since the minimum degree of the vertices in \( G_t \) is at least \( k - \ell - \alpha(H) \) and \( t < k - \ell - \alpha(H) \), the minimum term in \( \pi_t' \) is at least 1. It remains to show that there is a realization of \( \pi_t' \) that contains a copy of \( F_{k-\ell-t} \) on the vertices \( \{v_{t+1}, \ldots, v_{k-\ell}\} \).
To construct such a realization, place a copy of $F_{k-\ell-t}$ on the vertices $v_{t+1}, \ldots, v_{k-\ell}$. Join any remaining edges incident to $\{v_{t+1}, \ldots, v_{k-\ell}\}$ to distinct vertices among the remaining $n_\ell - (k-\ell)$ vertices. It remains to show that there is a graph on the remaining $n_\ell - (k-\ell)$ vertices that realizes the residual sequence. This sequence has at least $n_\ell - (k-\ell) - (k-\ell-t)(k-\ell-3)$ positive terms and with the maximum term being at most $k-\ell-2$. By the Erdős-Gallai criteria \cite{9} such a sequence is graphic provided that $n_\ell'$ is sufficiently large.

As noted above, Theorem 2 follows immediately from Theorems 3 and 4.

6 Conclusion

Having determined the asymptotic value of the potential function for general $H$, it may be of interest to study the structure of those $n$-term graphic sequences that are not potentially $H$-graphic, but whose sum is close to $\sigma(H, n)$. This line of inquiry would be related to recent work of Chudnovsky and Seymour \cite{6}, which for an arbitrary graphic sequence $\pi$ gives a partial structural characterization of those graphic sequences $\pi'$ for which no realization of $\pi'$ contains any realization of $\pi$ as an induced subgraph.

References

[1] A. Busch, M. Ferrara, M. Jacobson, H. Kaul, S. Hartke and D. West, Packing of Graphic $n$-tuples, to appear in J. Graph Theory.

[2] N. Bushaw and N. Kettle. Turán numbers of Multiple Paths and Equibipartite Trees, Combin. Probab. Comput. 20 (2011), 837–853

[3] G. Chen, M. Ferrara, R. Gould and J. Schmitt, Graphic Sequences with a Realization Containing a Complete Multigraph Subgraph, Discrete Math. 308 (2008), 5712–5721.

[4] G. Chen, R. Gould, F. Pfender and B. Wei, Extremal Graphs for Intersecting Cliques, J. Combin. Theory Ser. B 89 (2003), 159–181.

[5] G. Chen, J. Li and J. Yin, A variation of a classical Turán-type extremal problem, European J. Comb. 25 (2004), 989–1002.

[6] M. Chudnovsky and P. Seymour, Rao’s Degree Sequence Conjecture, submitted.

[7] Z. Dvořák and B. Mohar, Chromatic number and complete graph substructures for degree sequences, submitted.

[8] P. Erdős, Z. Füredi, R. Gould and D. Gunderson, Extremal Graphs for Intersecting Triangles, J. Combin. Theory Ser. B 64 (1995), 89–100.

[9] P. Erdős and T. Gallai, Graphs with prescribed degrees, Matematiki Lapor 11 (1960), 264–274 (in Hungarian).
[10] P. Erdős, M. Jacobson and J. Lehel, Graphs Realizing the Same Degree Sequence and their Respective Clique Numbers, Graph Theory, Combinatorics and Applications (eds. Alavi, Chartrand, Oellerman and Schwenk), Vol. I, 1991, 439–449.

[11] P. Erdős and M. Simonovits, A Limit Theorem in Graph Theory, Studia Sci. Math. Hungar. 1 (1966), 51–57.

[12] P. Erdős and A. Stone, On the Structure of Linear Graphs, Bull. Amer. Math. Soc. 52 (1946), 1087–1091.

[13] M. Ferrara, Graphic Sequences with a Realization Containing a Union of Cliques, Graphs Comb. 23 (2007), 263–269.

[14] M. Ferrara and J. Schmitt, A General Lower Bound for Potentially $H$-Graphic Sequences, SIAM J. Discrete Math. 23 (2009), 517–526.

[15] R. Gould, M. Jacobson, and J. Lehel, Potentially $G$-graphic degree sequences, Combinatorics, Graph Theory, and Algorithms (eds. Alavi, Lick and Schwenk), Vol. I, New York: Wiley & Sons, Inc., 1999, 387–400.

[16] S.L. Hakimi, On the realizability of a set of integers as degrees of vertices of a graph, J. SIAM Appl. Math, 10 (1962), 496–506.

[17] S. Hakimi and E. Schmeichel, Graphs and their degree sequences: A survey. In Theory and Applications of Graphs, volume 642 of Lecture Notes in Mathematics, Springer Berlin / Heidelberg, 1978, 225–235.

[18] V. Havel, A remark on the existence of finite graphs (Czech.), Časopis Pěst. Mat. 80 (1955), 477–480.

[19] D. Kleitman and D. Wang, Algorithms for constructing graphs and digraphs with given valences and factors, Discrete Math. 6 (1973) 79–88.

[20] J. Li and Z. Song, An extremal problem on the potentially $P_k$-graphic sequences, The International Symposium on Combinatorics and Applications (W.Y.C. Chen et. al., eds.), Tanjin, Nankai University 1996, 269–276.

[21] J. Li and Z. Song, The smallest degree sum that yields potentially $P_k$-graphical sequences, J. Graph Theory 29 (1998), 63–72.

[22] J. Li, Z. Song, and R. Luo, The Erdős-Jacobson-Lehel conjecture on potentially $P_k$-graphic sequences is true, Science in China, Ser. A, 41 (1998), 510–520.

[23] J. Li and J. Yin, Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size, Discrete Math. 301 (2005), 218–227.

[24] J. Li and J. Yin, An extremal problem on potentially $K_{r,s}$-graphic sequences, Discrete Math. 260 (2003), 295–305.
[25] J. Li and J. Yin, A variation of a conjecture due to Erdős and Sós, *Acta Math. Sin.* **25** (2009), 795–802.

[26] W. Mantel, Problem 28, soln by H. Gouwentak, W. Mantel, J Teixeira de Mattes, F. Schuh and W.A. Wythoff, *Wiskundige Opgaven* **10** (1907), 60–61.

[27] A. R. Rao, The clique number of a graph with a given degree sequence. In *Proceedings of the Symposium on Graph Theory*, volume 4 of *ISI Lecture Notes*, Macmillan of India, New Delhi, 1979, 251–267.

[28] S. B. Rao, A survey of the theory of potentially $P$-graphic and forcibly $P$-graphic degree sequences. In *Combinatorics and graph theory*, volume 885 of *Lecture Notes in Math.*, Springer, Berlin, 1981, 417–440.

[29] N. Robertson and Z. Song, Hadwiger number and chromatic number for near regular degree sequences, *J. Graph Theory* **64** (2010), 175–183.

[30] P. Turán, Eine Extremalaufgabe aus der Graphentheorie, *Mat. Fiz. Lapook* **48** (1941), 436–452.

[31] J. Yin, A Rao-type characterization for a sequence to have a realization containing a split graph, *Discrete Math.* **311** (2011), 2485–2489.