PARTIAL REGULARITY OF SOLUTIONS TO THE FOUR-DIMENSIONAL NAVIER-STOKES EQUATIONS AT THE FIRST BLOW-UP TIME.

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Abstract. The solutions of incompressible Navier-Stokes equations in four spatial dimensions are considered. We prove that the two-dimensional Hausdorff measure of the set of singular points at the first blow-up time is equal to zero.

1. Introduction

In this paper we consider both the Cauchy problem and the initial-boundary value problem for incompressible Navier-Stokes equations in four spatial dimensions with unit viscosity and zero external force:

\[ u_t + u \nabla u - \Delta u + \nabla p = 0, \quad \text{div} \, u = 0 \]  \hspace{1cm} (1.1)

in a smooth domain \( Q_T = \Omega \times (0,T) \subset \mathbb{R}^d \times \mathbb{R} \). Boundary condition \( u|_{\partial \Omega \times [0,T]} = 0 \) is imposed if \( \Omega \neq \mathbb{R}^d \). Here \( d = 4 \) and the initial data \( a \) is in the closure of \( \{ u \in C_0^\infty(\Omega) \ ; \ \text{div} \, u = 0 \} \) in \( L_d(\Omega) \) if \( \Omega \) is bounded, or is in the closure of \( \{ u \in C^\infty(\Omega) \ ; \ \text{div} \, u = 0 \} \) in \( L_d(\Omega) \cap L_2(\Omega) \) if \( \Omega = \mathbb{R}^d \). The local well-posedness of such problems is well-known (see, for example, \[9\] and \[5\]). The solution \( u \) is locally smooth in both spatial and time variables. We are interested in the partial regularity of \( u \) at the first blow-up time \( T \).

Many authors have studied the partial regularity of solutions (in particular, weak solutions) of the Navier-Stokes equations, especially when \( d \) is equal to three. V. Scheffer studied partial regularity in a series of papers \[17,18,20\]. In three space dimensions, he established various partial regularity results for weak solutions satisfying the so-called local energy inequality. For \( d = 3 \), the notion of suitable weak solutions was first introduced in a celebrated paper \[1\] by L. Caffarelli, R. Kohn and...
L. Nirenberg. They called a pair consisting of velocity $u$ and pressure $p$ a suitable weak solution if $u$ has finite energy norm, $p$ belongs to the Lebesgue space $L_{5/4}$, $u$ and $p$ are weak solutions to the Navier-Stokes equations and satisfy a local energy inequality. After proving some criteria for local boundedness of solutions, they established partial regularity of solutions and estimated the Hausdorff dimension of the singular set. They proved that, for any suitable weak solution $u, p$, there is an open subset where the velocity field $u$ is Hölder continuous and they showed that the 1-D Hausdorff measure of the complement of this subset is equal to zero. In [15], with zero external force, F. Lin gave a more direct and sketched proof of Caffarelli, Kohn and Nirenberg’s result. A detailed treatment was later given by O. Ladyzhenskaya and G. A. Seregin in [13]. Very recently, some extended results are obtained in Seregin [16] and Gustafson, Kang and Tsai [6].

For $d = 4$, V. Scheffer proved in [19] that there exists a weak solution $u$ in $\mathbb{R}^4 \times \mathbb{R}^+$ such that $u$ is continuous outside a locally closed set of $\mathbb{R}^4 \times \mathbb{R}^+$ whose 3-D Hausdorff measure is finite. Although Scheffer’s paper is not recent, it appears to us that this is the only published results on the partial regularity of 4-D Navier-Stokes equations.

Remark 1.1. The weak solution considered in [19] doesn’t verify the local energy estimate. The existence of a weak solution satisfying the local energy estimate is still an open problem.

Now let’s state our result. Instead of dealing with weak solutions, we work on classical solutions of 4-D Navier-Stokes equations, which are regular before they blow up. Our first result is the that the singular set at the first blowup time is a compact set with zero 2-D Hausdorff measure. We show this after two partial regularity criterions are obtained. Our proof is conceptually similar to Lin’s in [15], but the problem is technically harder. The main difficulty is the lack of certain compactness. We overcome it by a novelty use of the backward heat kernel (see the proof of Lemma 2.12) and by the use of two appropriate scaled norms of the pressure. It is possible because the nonlinear term is controlled by using the Sobolev embedding theorem, although we don’t have a compact embedding here.

Remark 1.2. In the setting of classical solutions, our result is the 4-D version of Caffarelli, Kohn and Nirenberg’s theorem in [1].

As an application of one of the partial regularity criterions derived in proof of the first result we get our second result: in case $\Omega = \mathbb{R}^4$ if a solution blows up, it must blow up at a finite time.
Remark 1.3. We can prove a similar result in 3-D by using the same argument. Detailed discussions on the long-time behavior of solutions to 3-D Navier-Stokes can be found in J. Heywood [7] and M. Wiegner [24] (and references therein). It seems that we need some new method to deal with the five or higher dimensional case. To the authors’ best knowledge all the existing methods on partial regularity for the Navier-Stokes equations share the following prerequisite condition: in the energy inequality the nonlinear term should be controlled by the energy norm under the Sobolev imbedding theorem. Actually, four is the highest dimension in which we have such condition. In five or higher dimensional, such condition fails. Therefore, we cannot hope the existing methods work in five or higher dimensional case.

The article is organized as follows. Our main theorems (Theorem 2.1-2.5) are given in the following section. Some auxiliary estimates are proved in section 3 and 4 among which Lemma 2.12 plays a crucial role. We give the proof of our main theorems in the last section.

To conclude this Introduction, we explain some notation used in what follows: $\mathbb{R}^d$ is a $d$-dimensional Euclidean space with a fixed orthonormal basis. A typical point in $\mathbb{R}^d$ is denoted by $x = (x_1, x_2, ..., x_d)$. As usual the summation convention over repeated indices is enforced. And $x \cdot y = x_i y_i$ is the inner product for $x, y \in \mathbb{R}^d$. For $t > 0$, we denote $H_t = \mathbb{R}^d \times (0, t)$ and space points are denoted by $z = (x, t)$. Various constants are denoted by $N$ in general and the expression $N = N(\cdot \cdot \cdot)$ means that the given constant $N$ depends only on the contents of the parentheses.

2. Setting and main results

We shall use the notation in [13]. Let $\omega$ be a domain in some finite-dimensional space. Denote $L_p(\omega; \mathbb{R}^n)$ and $W^k_p(\omega; \mathbb{R}^n)$ to be the usual Lebesgue and Sobolev spaces of functions from $\omega$ into $\mathbb{R}^n$. Denote the norm of the spaces $L_p(\omega; \mathbb{R}^n)$ and $W^k_p(\omega; \mathbb{R}^n)$ by $\| \cdot \|_{L_p, \omega}$ and $\| \cdot \|_{W^k_p, \omega}$ respectively. As usual, for any measurable function $u = u(x, t)$ and any $p, q \in [1, +\infty]$, we define

$$\|u(x, t)\|_{L^p_t L^q_x} := \|u(x, t)\|_{L^q_x} \|L^p_t.$$  

For summable functions $p, u = (u_i)$ and $\tau = (\tau_{ij})$, we use the following differential operators

$$\partial_t u = u_t = \frac{\partial u}{\partial t}, \quad u_{i,j} = \frac{\partial u}{\partial x_i \partial x_j}, \quad \nabla p = (p_i), \quad \nabla u = (u_{i,j}),$$

$$\text{div } u = u_{i,i}, \quad \text{div } \tau = (\tau_{ij,j}), \quad \Delta u = \text{div} \nabla u,$$
which are understood in the sense of distributions.

We use the notation of spheres, balls and parabolic cylinders,

\[ S(x_0,r) = \{ x \in \mathbb{R}^4 | |x - x_0| = r \}, \quad S(r) = S(0,r), \quad S = S(1); \]

\[ B(x_0,r) = \{ x \in \mathbb{R}^4 | |x - x_0| < r \}, \quad B(r) = B(0,r), \quad B = B(1); \]

\[ Q(z_0,r) = B(x_0,r) \times (t_0 - r^2, t_0), \quad Q(r) = Q(0,r), \quad Q = Q(1). \]

Also we denote means values of summable functions as follows

\[ [u]_{x_0,r} (t) = \frac{1}{|B(r)|} \int_{B(x_0,r)} u(x,t) \, dx, \]

\[ (u)_{z_0,r} (t) = \frac{1}{|Q(r)|} \int_{Q(x_0,r)} u \, dz. \]

In case \( \Omega = \mathbb{R}^d \), in a well-known paper [9] Kato proved that the problem is locally well-posed. By known local regularity theory for Navier-Stokes equations it can be proved that solutions obtained by Kato’s (also known as mild solutions) is smooth in \( \mathbb{R}^d \times (0, T^*) \) for some \( T^* > 0 \). Meanwhile, for bounded \( \Omega \), it is also known (see [5]) that there exists a unique solution \( u \) of (1.1) satisfying

1. \( u \in C([0, T^*]; L_d) \), \( u(0) = a \) for some \( T^* > 0 \);
2. \( u \in C((0, T^*]; D(A^\alpha)) \) for any \( 0 < \alpha < 1 \);
3. \( \| A^\alpha u(t) \| = o(t^{-\alpha}) \) as \( t \to 0 \).

Here \( A \) is the stokes operator. Moreover, such solution is smooth in \( \Omega \times (0, T^* \). In both cases, let \( T = \sup T^* \) be the first blow-up time. Then \( u \) is a smooth function in \( Q_T \).

Let \( \eta(x) \) be a smooth function on \( \mathbb{R}^4 \) supported in the unit ball \( B(1) \), \( 0 \leq \eta \leq 1 \) and \( \eta \equiv 1 \) on \( B(2/3) \). Let \( z_0 \) be a given point in \( \Omega \times (0, T] \) and \( r > 0 \) a real number such that \( Q(z_0, r) \subset Q_T \). It’s known that for a.e. \( t \in (t_0 - r^2, t_0) \), in the sense of distribution one has

\[ \Delta p = \frac{\partial^2}{\partial x_i \partial x_j} (u_i u_j) \]

\[ = \frac{\partial^2}{\partial x_i \partial x_j} ((u_i - [u_i]_{x_0,r})(u_j - [u_j]_{x_0,r})) \quad \text{in} \ B(x_0,r). \]

For these \( t \), we consider the decomposition \( p = \tilde{p}_{x_0,r} + h_{x_0,r} \) in \( B(x_0,r) \), where \( \tilde{p}_{x_0,r} \) is the Newtonian potential of

\[ (u_i - [u_i]_{x_0,r})(u_j - [u_j]_{x_0,r}) \eta(x/r). \]

Then \( h_{x_0,r} \) is harmonic in \( B(x_0, r/2) \). In the sequel, we omit the indices of \( \tilde{p} \) and \( h \) whenever there is no confusion. The following notation will
be used throughout the article:

\[
A(r) = A(r, z_0) = \text{ess sup}_{t_0 - r^2 \leq t \leq t_0} \frac{1}{r^2} \int_{B(x_0, r)} |u(x, t)|^2 \, dx,
\]

\[
E(r) = E(r, z_0) = \frac{1}{r^2} \int_{Q(z_0, r)} |\nabla u|^2 \, dz,
\]

\[
C(r) = C(r, z_0) = \frac{1}{r^3} \int_{Q(z_0, r)} |u|^3 \, dz,
\]

\[
D(r) = D(r, z_0) = \frac{1}{r^3} \int_{Q(z_0, r)} |p - [h]|_{x_0, r}^{3/2} \, dz.
\]

\[
F(r) = F(r, z_0) = \frac{1}{r^2} \left[ \int_{t_0 - r^2}^{t_0} \left( \int_{B(x_0, r)} |p - [h]|_{x_0, r}^{1 + \alpha} \, dx \right)^{\frac{1 + \alpha}{2 \alpha}} \, dt \right]^{\frac{2\alpha}{1 + \alpha}},
\]

where \( \alpha \in (0, 1) \) is a number to be specified later. Notice that these objects are all invariant under the natural scaling.

Here come our main results of the article.

**Theorem 2.1.** Let \( \Omega \) be a smooth bounded set or the whole space \( \mathbb{R}^4 \) and \((u, p)\) be the solution of (1.1). There is a positive number \( \varepsilon_0 \) satisfying the following property. Assume that for a point \( z_0 \in \Omega \times T \) the inequality

\[
\limsup_{r \downarrow 0} E(r) \leq \varepsilon_0
\]

holds. Then \( z_0 \) is a regular point.

**Theorem 2.2.** Let \( \Omega \) be a smooth bounded set or the whole space \( \mathbb{R}^4 \) and \((u, p)\) be the solution of (1.1). There is a positive number \( \varepsilon_0 \) satisfying the following property. Assume that for a point \( z_0 \in Q_T \) and for some \( \rho_0 > 0 \) such that \( Q(z_0, \rho_0) \subset Q_T \) and

\[
C(\rho_0) + D(\rho_0) + F(\rho_0) \leq \varepsilon_0.
\]

Then \( z_0 \) is a regular point.

**Remark 2.3.** It is worth noting that the object under estimation in condition (2.1) involves the gradient of \( u \) while the objects in condition (2.2) involve only \( u \) and \( p \) themselves. However, by using condition (2.1) one can obtain a better estimate of the Hausdorff dimension of the set of all singular points.

**Theorem 2.4.** Let \( \Omega \) be a smooth bounded set or the whole space \( \mathbb{R}^4 \) and \((u, p)\) be the solution of (1.1). Then the 2-D Hausdorff measure of the set of singular points in \( \Omega \times T \) is equal to zero.
Theorem 2.5. Assume $\Omega$ is the whole space $\mathbb{R}^4$. Let $(u,p)$ be the solution of (1.1). If the solution does not blow up in finite time, then $u$ is bounded and smooth in $\mathbb{R}^4 \times (0, +\infty)$.

In the sequel, we shall make use of the following well-known interpolation inequality.

Lemma 2.6. For any functions $u \in W_2^1(\mathbb{R}^4)$ and real numbers $q \in [2, 4]$ and $r > 0$,
\[
\int_{B_r} |u|^q \, dx \leq N(q) \left[ \left( \int_{B_r} |\nabla u|^2 \, dx \right)^{q-2} \left( \int_{B_r} |u|^2 \, dx \right)^{2-q/2} + r^{-(q-2)} \left( \int_{B_r} |u|^2 \, dx \right)^{q/2} \right].
\]

Let $(u, p)$ be the solution of the Navier-Stokes equation (1.1).

Lemma 2.7. (i) We have
\[
u \in L_\infty(0, T; L_2(\Omega; \mathbb{R}^4)) \cap L_2(0, T; W_2^1(\Omega; \mathbb{R}^4)) \cap L_3(Q_T),
\]
and
\[
p \in L_{3/2}(Q_T).
\]
(ii) For $0 < t \leq T$ and for all non-negative function $\psi \in C_0^\infty(\Omega \times (0, \infty)),
\]
the following generalized energy inequality is satisfied
\[
\text{ess sup}_{0 < s \leq t} \int_{\Omega} |u(x, s)|^2 \psi(x, s) \, dx + 2 \int_{Q_t} |\nabla u|^2 \psi \, dxds
\leq \int_{Q_t} \{|u|^2 (\psi_t + \Delta \psi) + (|u|^2 + 2p)u \cdot \nabla \psi\} \, dxds.
\]

Sketch of the proof. To prove $u \in L_\infty(0, T; L_2(\Omega; \mathbb{R}^4)) \cap L_2(0, T; W_2^1(\Omega; \mathbb{R}^4)) \cap L_3(Q_T)$, it is sufficient to multiply the first equation of (1.1) by $u$ and integrate by parts. By using Lemma 2.6 with $q = 3$ and integrating in $t$, we obtain $u \in L_3(Q_T)$. Since in $Q_T$ it holds that
\[
\Delta p = \frac{\partial^2}{\partial x_i \partial x_j} u_i u_j,
\]
(2.5) follows from the Calderón-Zygmund’s estimate. Next, let’s prove part (ii). First, notice that for $0 < t < T$, (2.6) can be obtained by multiplying the first equation of (1.1) by $u \psi$, integrating by parts and integrating with respect to $t$. For the case when $t = T$, due to part (i) and Hölder’s inequality both sides of (2.6) are finite. Then it remains to let $t \to T$ and take the limit on both sides.
We shall prove Theorem 2.4 in three steps. First, we want to control $A, C, D, F$ in a smaller ball by the their values in a larger ball under the assumption that $E$ is sufficiently small. Similar results can be found in [13] or [15] in case when the space dimension is three.

**Lemma 2.8.** Suppose $\gamma \in (0, 1)$, $\rho > 0$ are constants and $Q(z_0, \rho) \subset Q_T$. Then we have

$$C(\gamma \rho) \leq N\left[\gamma^{-3} A^{1/2}(\rho) E(\rho) + \gamma^{-9/2} A^{3/4}(\rho) E^{3/4}(\rho) + \gamma C(\rho)\right],$$

where $N$ is a constant independent of $\gamma, \rho$ and $z_0$.

**Lemma 2.9.** Suppose $\alpha \in (0, 1/2]$, $\gamma \in (0, 1/3]$, $\rho > 0$ are constants and $Q(z_0, \rho) \subset Q_T$. Then we have

$$F(\gamma \rho) \leq N(\alpha) \left[\gamma^{-2} A^{\frac{1+\alpha}{1+\alpha}}(\rho) E^{\frac{2+\alpha}{1+\alpha}}(\rho) + \gamma^{\frac{4+\alpha}{1+\alpha}} F(\rho)\right].$$

where $N(\alpha)$ is a constant independent of $\gamma, \rho$ and $z_0$. In particular, for $\alpha = 1/2$ we have,

$$D(\gamma \rho) \leq N\left[\gamma^{-3} A^{1/2}(\rho) E(\rho) + \gamma^{5/2} D(\rho)\right].$$

Moreover, it holds that

$$D(\gamma \rho) \leq N(\alpha) \left[\gamma^{-3} (A(\rho) + E(\rho))^{3/2} + \gamma^{(9-3\alpha)/(2+2\alpha)} F^{3/2}(\rho)\right].$$

**Lemma 2.10.** Suppose $\theta \in (0, 1/2]$, $\rho > 0$ are constants and $Q(z_0, \rho) \subset Q_T$. Then we have

$$A(\theta \rho) + E(\theta \rho) \leq N\theta^{-2} \left[C^{2/3}(\rho) + C(\rho) + C^{1/3}(\rho) D^{2/3}(\rho)\right].$$

In particular, when $\theta = 1/2$ we have

$$A(\rho/2) + E(\rho/2) \leq N[C^{2/3}(\rho) + C(\rho) + C^{1/3}(\rho) D^{2/3}(\rho)].$$

As a conclusion, we obtain

**Proposition 2.11.** For any $\varepsilon_0 > 0$, there exists $\varepsilon_1 > 0$ small such that for any $z_0 \in Q_T \cup (\mathbb{R}^4 \times \{T\})$ satisfying

$$\limsup_{r \to 0} E(r) \leq \varepsilon_1,$$

we can find $\rho_0$ sufficiently small such that

$$A(\rho_0) + E(\rho_0) + C(\rho_0) + D(\rho_0) + F(\rho_0) \leq \varepsilon_0.$$

In the second step, our goal is to estimate the values of $A, E, C$ and $F$ in a smaller ball by the values of themselves in a larger ball.

**Lemma 2.12.** Suppose $\rho > 0$, $\theta \in (0, 1/3]$ are constants and $Q(z_1, \rho) \subset Q_T$. Then we have

$$A(\theta \rho) + E(\theta \rho) \leq N\theta^2 A(\rho) + N\theta^{-3}[A(\rho) + E(\rho) + F(\rho)]^{3/2},$$

where $N$ is a constant independent of $\rho, \theta$ and $z_1$. 
Lemma 2.13. Suppose $\rho > 0$ is constants and $Q(z_1, \rho) \subset Q_T$. Then we can find $\theta_1 > 0$ small such that
\[
A(\theta_1 \rho) + E(\theta_1 \rho) + F(\theta_1 \rho) \leq \frac{1}{2} [A(\rho) + E(\rho) + F(\rho)] + N(\theta_1)[A(\rho) + E(\rho) + F(\rho)]^{3/2},
\] (2.15)
where $N$ is a constant independent of $\rho$ and $z_1$.

Proposition 2.14. For any $\varepsilon_2 > 0$, there exists $\varepsilon_0 > 0$ small such that: if for some $z_0 \in Q_T \cup (\mathbb{R}^4 \times \{T\})$ and $\rho_0 > 0$ satisfying $Q(z_0, \rho_0) \subset Q_T$ and
\[
C(\rho_0) + D(\rho_0) + F(\rho_0) \leq \varepsilon_0,
\] (2.16)
then for any $\rho \in (0, \rho_0/4)$ and $z_1 \in Q(z_0, \rho/4)$ we have
\[
A(\rho, z_1) + C(\rho, z_1) + E(\rho, z_1) + F(\rho, z_1) \leq \varepsilon_2.
\] (2.17)

Finally, we apply Schoen’s trick to prove the main theorems.

3. Proof of Proposition 2.11

We will prove these lemma briefly. For more detail, we refer the reader to [13].

Proof of Lemma 2.8. Denote $r = \gamma \rho$. We have, by using Poincaré’s inequality and Cauchy’s inequality,
\[
\int_{B(x_0, r)} |u|^2 \, dx = \int_{B(x_0, r)} (|u|^2 - [|u|^2]_{x_0, \rho}) \, dx + \int_{B(x_0, r)} [|u|^2]_{x_0, \rho} \, dx
\]
\[
\leq N\rho \int_{B(x_0, \rho)} |\nabla u||u| \, dx + \left(\frac{r}{\rho}\right)^4 \int_{B(x_0, \rho)} |u|^2 \, dx
\]
\[
\leq N\rho \left(\int_{B(x_0, \rho)} |\nabla u|^2 \, dx\right)^{1/2} \left(\int_{B(x_0, \rho)} |u|^2 \, dx\right)^{1/2} + \left(\frac{r}{\rho}\right)^4 \int_{B(x_0, \rho)} |u|^2 \, dx
\]
\[
\leq N\rho^2 A^{1/2}(\rho) \left(\int_{B(x_0, \rho)} |\nabla u|^2 \, dx\right)^{1/2} + \left(\frac{r}{\rho}\right)^4 \left(\int_{B(x_0, \rho)} |u|^3 \, dx\right)^{2/3} \rho^{1/3}.
\]
Owing to Lemma [2.6] with $q = 3$ and using the inequality above, one gets
\[
\int_{B(x_0, r)} |u|^3 \, dx \leq N \left[ \int_{B(x_0, r)} |\nabla u|^2 \, dx \right] \rho A^{1/2}(\rho) \\
+ \rho^{3r^{-2}} A^{3/4}(\rho) \left( \int_{B(x_0, \rho)} |\nabla u|^2 \, dx \right)^{3/4} + \frac{(\frac{T}{\rho})^4}{\rho} \int_{B(x_0, \rho)} |u|^3 \, dx
\]

By integrating with respect to $t$ on $(t_0 - r^2, t_0)$ and applying Hölder’s inequality, we get
\[
\int_{Q(x_0, r)} |u|^3 \, dz \leq N \left[ \int_{Q(x_0, \rho)} |\nabla u|^2 \, dz \right] \rho A^{1/2}(\rho) \\
+ \rho^{3r^{-3/2}} A^{3/4}(\rho) \left( \int_{Q(x_0, \rho)} |\nabla u|^2 \, dz \right)^{3/4} + \frac{(\frac{T}{\rho})^4}{\rho} \int_{Q(x_0, \rho)} |u|^3 \, dz.
\]

The conclusion of Lemma [2.8] follows immediately.

**Proof of Lemma [2.9].** Denote $r = \gamma \rho$. Recall the decomposition of $p$ introduced in Section [2] and by using Calderón-Zygmund’s estimate, Lemma [2.6] with $q = 2(1 + \alpha)$ and Poincaré inequality, one has
\[
\int_{B(x_0, r)} |\tilde{p}_{x_0, r}(x, t)|^{1+\alpha} \, dx \\
\leq N \int_{B(x_0, r)} |u - [u]_{x_0, r}|^{2(1+\alpha)} \, dx \\
\leq N \left( \int_{B(x_0, r)} |\nabla u|^2 \, dx \right)^{2\alpha} \left( \int_{B(x_0, r)} |u - [u]_{x_0, r}|^2 \, dx \right)^{1-\alpha} \\
+ Nr^{-4\alpha} \left( \int_{B(x_0, r)} |u - [u]_{x_0, r}|^2 \, dx \right)^{1+\alpha} \\
\leq N \left( \int_{B(x_0, r)} |\nabla u|^2 \, dx \right)^{2\alpha} \left( \int_{B(x_0, r)} |u|^2 \, dx \right)^{1-\alpha}. \quad (3.1)
\]

Here we also use the inequality
\[
\int_{B(x_0, r)} |u - [u]_{x_0, r}|^2 \, dx \leq \int_{B(x_0, r)} |u|^2 \, dx.
\]

Similarly,
\[
\int_{B(x_0, \rho)} |\tilde{p}_{x_0, \rho}|^{1+\alpha} \, dx \leq N \left( \int_{B(x_0, \rho)} |\nabla u|^2 \, dx \right)^{2\alpha} \left( \int_{B(x_0, \rho)} |u|^2 \, dx \right)^{1-\alpha}. \quad (3.2)
\]

Since $h_{x_0, \rho}$ is harmonic in $B(x_0, \rho/2)$, any Sobolev norm of $h_{x_0, \rho}$ in a smaller ball can be estimated by any of its $L_p$ norm in $B(x_0, \rho/2)$. 

Thus, by using Poincaré inequality one can obtain
\[
\int_{B(x_0,r)} |h_{x_0,\rho} - [h_{x_0,\rho}]_{x_0,r}|^{1+\alpha} \, dx \\
\leq N r^{1+\alpha} \int_{B(x_0,r)} |\nabla h_{x_0,\rho}|^{1+\alpha} \, dx \\
\leq N r^{5+\alpha} \sup_{B(x_0,r)} |\nabla h_{x_0,\rho}|^{1+\alpha}
\]
\[
\leq N \left( \frac{r}{\rho} \right)^{5+\alpha} \int_{B(x_0,\rho/2)} |h_{x_0,\rho}(x,t) - [h_{x_0,\rho}]_{x_0,\rho}|^{1+\alpha} \, dx \\
\leq N \left( \frac{r}{\rho} \right)^{5+\alpha} \left[ \int_{B(x_0,\rho)} |p(x,t) - [h_{x_0,\rho}]_{x_0,\rho}|^{1+\alpha} \, dx \right].
\]
(3.3)

Combining (3.2) and (3.3) together yields,
\[
\int_{B(x_0,r)} |p(x,t) - [h_{x_0,\rho}]_{x_0,r}|^{1+\alpha} \, dx \\
\leq N \left( \int_{B(x_0,\rho)} |\nabla u(x,t)|^2 \, dx \right)^{2\alpha} \left( \int_{B(x_0,\rho)} |u(x,t)|^2 \, dx \right)^{1-\alpha} \\
+ N \left( \frac{r}{\rho} \right)^{5+\alpha} \int_{B(x_0,\rho)} |p(x,t) - [h_{x_0,\rho}]_{x_0,\rho}|^{1+\alpha} \, dx.
\]
(3.4)

Since \( \tilde{p}_{x_0,r} + h_{x_0,r} = p = \tilde{p}_{x_0,\rho} + h_{x_0,\rho} \) in \( B(x_0, r) \), by Hölder’s inequality
\[
\int_{B(x_0,r)} |[h_{x_0,\rho}]_{x_0,r} - [h_{x_0,\rho}]_{x_0,r}|^{1+\alpha} \, dx \\
= N r^4 |[h_{x_0,\rho}]_{x_0,r} - [h_{x_0,\rho}]_{x_0,r}|^{1+\alpha} \\
= N r^4 |[\tilde{p}_{x_0,\rho}]_{x_0,r} - [\tilde{p}_{x_0,\rho}]_{x_0,r}|^{1+\alpha} \\
\leq N \int_{B(x_0,r)} |\tilde{p}_{x_0,\rho}|^{1+\alpha} + |\tilde{p}_{x_0,\rho}|^{1+\alpha} \, dx.
\]
(3.5)

From (3.1), (5.2), (3.4) and (3.5) we get
\[
\int_{B(x_0,r)} |p(x,t) - [h_{x_0,\rho}]_{x_0,r}|^{1+\alpha} \, dx \\
\leq N \left( \int_{B(x_0,\rho)} |\nabla u(x,t)|^2 \, dx \right)^{2\alpha} \left( \int_{B(x_0,\rho)} |u(x,t)|^2 \, dx \right)^{1-\alpha} \\
+ N \left( \frac{r}{\rho} \right)^{5+\alpha} \int_{B(x_0,\rho)} |p(x,t) - [h_{x_0,\rho}]_{x_0,\rho}|^{1+\alpha} \, dx.
\]
(3.6)
Raising to the power $1/(2\alpha)$ and integrating with respect to $t$ in $(t_0 - r^2, t_0)$ complete the proof of (2.8) and also (2.9).

To prove (2.10), we use a slightly different estimate from (3.3). Again, since $h$ is harmonic in $B(x_0, \rho/2)$, we have

\[
\int_{B(x_0,r)} |h_{x_0,\rho} - [h_{x_0,\rho}]_{x_0,r}|^{3/2} dx \\
\leq N r^{3/2} \int_{B(x_0,r)} |\nabla h_{x_0,\rho}|^{3/2} dx \\
\leq N r^{11/2} \sup_{B(x_0,r)} |\nabla h_{x_0,\rho}|^{3/2} \\
\leq N \rho^{3/2 + 6/(1 + \alpha)} \left[ \int_{B(x_0,\rho)} |h_{x_0,\rho}|^{2/3} (x,t) - [h_{x_0,\rho}]_{x_0,r}^{1+\alpha} dx \right]^{2(1+\alpha)} \\
\leq N \rho^{3/2 + 6/(1 + \alpha)} \left[ \int_{B(x_0,\rho)} |p(x,t) - [h_{x_0,\rho}]_{x_0,r}^{1+\alpha} dx \right]^{2(1+\alpha)} \\
+ \left[ \int_{B(x_0,\rho)} |\bar{p}_{x_0,\rho}(x,t)|^{1+\alpha} dx \right]^{3(1+\alpha)}.
\]  

(3.7)

Similar to (3.6), we obtain

\[
\int_{B(x_0,r)} |p(x,t) - [h_{x_0,\rho}]_{x_0,r}|^{3/2} dx \\
\leq N \left( \int_{B(x_0,\rho)} |\nabla u(x,t)|^2 dx \right)^{1/2} \left( \int_{B(x_0,\rho)} |u(x,t)|^2 dx \right)^{1/2} \\
+ N \rho^{-1/2} \left[ \int_{B(x_0,\rho)} |p(x,t) - [h]_{x_0,\rho}|^{1+\alpha} dx \right]^{3/2(1+\alpha)} \\
+ \left( \int_{B(x_0,\rho)} |\nabla u(x,t)|^2 dx \right)^{3/(1+\alpha)} \left( \int_{B(x_0,\rho)} |u(x,t)|^2 dx \right)^{3(1+\alpha)/(2(1+\alpha))} \right].
\]  

(3.8)

Integrating with respect to $t$ in $(t_0 - r^2, t_0)$ and applying Hölder’s inequality complete the proof of (2.10).

**Proof of Lemma 2.10** Let $r = \theta \rho$. In the energy inequality (2.6), we put $t = t_0$ and choose a suitable smooth cut-off function $\phi$ such that

\[
\psi \equiv 0 \text{ in } Q_{t_0} \setminus Q(z_0, \rho), \quad 0 \leq \psi \leq 1 \text{ in } Q_T,
\]

\[
\psi \equiv 1 \text{ in } Q(z_0, r), \quad |\nabla \psi| < N \rho^{-1}, \quad |\partial_t \psi| + |\nabla^2 \psi| < N \rho^{-2} \text{ in } Q_{t_0}.
\]
By using (2.6) and because \( u \) is divergence free, we get
\[
A(r) + 2E(r) \leq \frac{N}{r^2} \left[ \int_{Q(z_0, \rho)} |u|^2 \, dz \right]^{1/2} \left[ \int_{Q(z_0, \rho)} \left( |u|^2 + 2|p - [h]_{x_0, \rho}| |u| \right) \, dz \right].
\]
Due to H"older's inequality, one can obtain
\[
\int_{Q(z_0, \rho)} |u|^2 \, dz \leq \left( \int_{Q(z_0, \rho)} |u|^3 \, dz \right)^{2/3} \left( \int_{Q(z_0, \rho)} \, dz \right)^{1/3} \leq \rho^3 C^{2/3}(\rho),
\]
\[
\int_{Q(z_0, \rho)} |p - [h]_{x_0, \rho}| |u| \, dz \leq \left( \int_{Q(z_0, \rho)} |p - [h]_{x_0, \rho}|^{3/2} \, dz \right)^{2/3} \left( \int_{Q(z_0, \rho)} |u|^3 \, dz \right)^{1/3} \leq N \rho^3 D^{2/3}(\rho) C^{1/3}(\rho).
\]
Then the conclusion of Lemma 2.10 follows immediately.

**Proof of Proposition 2.11.** Let's prove first (2.13) without the presence of \( F \) on the left-hand side. For a given point \( z_0 = (x_0, t_0) \in Q_T \cup (\mathbb{R}^4 \times \{T\}) \) satisfying (2.12), choose \( \rho_0 > 0 \) such that \( Q(z_0, \rho_0) \subset Q_T \). Then for any \( \rho \in (0, \rho_0] \) and \( \gamma \in (0, 1/6) \), by using (2.11),
\[
A(\gamma \rho) + E(\gamma \rho) + C(\gamma \rho) + D(\gamma \rho) \leq N[C^{2/3}(2\gamma \rho) + C(2\gamma \rho) + D(2\gamma \rho)].
\]
This estimate, (2.7) and (2.9) together with Young’s inequality imply
\[
A(\gamma \rho) + E(\gamma \rho) + C(\gamma \rho) + D(\gamma \rho) \leq N[\gamma^{2/3} C^{2/3}(\rho) + \gamma^{5/2} D(\rho) + \gamma C(\rho) + \gamma A(\rho)] + N \gamma^{-100}(E(\rho) + E^3(\rho)) \leq N \gamma^{2/3}[A(\rho) + E(\rho) + C(\rho) + D(\rho)] + N \gamma^{-100}(E(\rho) + E^3(\rho)).
\]
It is easy to see that for any \( \epsilon_3 > 0 \), there’re sufficiently small real numbers \( \gamma \leq 1/(2N)^{3/2} \) and \( \epsilon_1 \) such that if (2.12) holds then for all small \( \rho \) we have
\[
N \gamma^{2/3} + N \gamma^{-100}(E(\rho) + E^3(\rho)) < \epsilon_3/2
\]
By using (3.9) we reach
\[
A(\rho_1) + C(\rho_1) + D(\rho_1) \leq \epsilon_3
\]
for some \( \rho_1 > 0 \) small enough. To include \( F \) in the estimate, it suffices to use (2.8).
4. Proof of Proposition 2.14

Proof of Lemma 2.12. Let \( r = \theta \rho \). Define the backward heat kernel as

\[
\Gamma(t, x) = \frac{1}{4\pi^2(r^2 + t_1 - t)^2} e^{-\frac{|x - x_1|^2}{2(r^2 + t_1 - t)}}.
\]

In the energy inequality (2.6) we put \( t = t_1 \) and choose \( \psi = \Gamma \phi := \Gamma \phi_1(t) \), where \( \phi_1, \phi_2 \) are suitable smooth cut-off functions satisfying

- \( \phi_1 \equiv 0 \) in \( \mathbb{R}^4 \setminus B(x_1, \rho) \), \( 0 \leq \phi_1 \leq 1 \) in \( \mathbb{R}^4 \), \( \phi_1 \equiv 1 \) in \( B(x_1, \rho/2) \)
- \( \phi_2 \equiv 0 \) in \( (-\infty, t_1 - \rho^2) \cup (t_1 + \rho^2, +\infty) \), \( 0 \leq \phi_2 \leq 1 \) in \( \mathbb{R} \), \( \phi_2 \equiv 1 \) in \( (t_1 - \rho^2/4, t_1 + \rho^2/4) \), \( |\phi_2'| \leq N\rho^{-2} \) in \( \mathbb{R} \), \( |\nabla \phi_1| < N\rho^{-1} \), \( |\nabla^2 \phi_1| < N\rho^{-2} \) in \( \mathbb{R}^4 \). (4.1)

By using the equality

\[
\Delta \Gamma + \Gamma_t = 0,
\]

we have

\[
\int_{B(x_0, \rho)} |u(x, t)|^2 \Gamma(t, x) \phi(x, t) \, dx + 2 \int_{Q(x_0, \rho)} |\nabla u|^2 \Gamma \phi \, dz 
\]

\[
\leq \int_{Q(x_0, \rho)} \left| u \right|^2 \left( \Gamma \phi_t + \Gamma \Delta \phi + 2\nabla \phi \nabla \Gamma \right) + (|u|^2 + 2p)u \cdot (\Gamma \nabla \phi + \phi \nabla \Gamma) \, dz. \tag{4.2}
\]

After some straightforward computations, it is easy to see the following three properties:

(i) For some constant \( c > 0 \), on \( \bar{Q}(z_1, r) \) it holds that

\[
\Gamma \phi = \Gamma \geq cr^{-4}.
\]

(ii) For any \( z \in Q(z_1, \rho) \), we have

\[
|\phi(z) \nabla \Gamma(z)| + |\nabla \phi(z) \Gamma(z)| \leq N r^{-5}.
\]

(iii) For any \( z \in Q(z_1, \rho) \setminus Q(z_1, r) \), we have

\[
|\Gamma(z) \phi_t(z)| + |\Gamma(z) \Delta \phi(z)| + |\nabla \phi \nabla \Gamma| \leq N \rho^{-6},
\]

These properties together with (4.2) and (4.1) yield

\[
A(r) + E(r) \leq N[\theta^2 A(\rho) + \theta^{-3}(C(\rho) + D(\rho))]. \tag{4.3}
\]

Owing to Lemma 2.6 with \( q = 3 \), one easily gets

\[
C(\rho/3) \leq NC(\rho) \leq N[ A(\rho) + E(\rho)]^{3/2}. \tag{4.4}
\]

By using (2.10) with \( \gamma = 1/3 \), we have

\[
D(\rho/3) \leq N[ A(\rho) + E(\rho) + F(\rho)]^{3/2}. \tag{4.5}
\]
Upon combining (4.3) (with $\rho/3$ in place of $\rho$), (4.4) and (4.5) together, the lemma is proved.

**Proof of Lemma 2.13.** Due to (2.8) and (2.14), for any $\gamma, \theta \in (0, 1/3]$, we have

\[
F(\gamma \theta \rho) \leq N \left[ \gamma^{-2} (A(\theta \rho) + E(\theta \rho)) + \gamma^{(3-\alpha)/(1+\alpha)} F(\theta \rho) \right] \\
\leq N \gamma^{-2} \theta^2 A(\rho) + \gamma^{(3-\alpha)/(1+\alpha)} \theta^{-2} F(\rho) \\
+ N \gamma^{-2} \theta^{-3} [A(\rho) + E(\rho) + F(\rho)]^{3/2} \tag{4.6}
\]

\[
A(\gamma \theta \rho) + E(\gamma \theta \rho) \leq (\gamma \theta)^2 A(\rho) + (\gamma \theta)^{-3} [A(\rho) + E(\rho) + F(\rho)]^{3/2}. \tag{4.7}
\]

Now we put $\alpha = 1/27$ such that

\[(3 - \alpha)/(1 + \alpha) = 20/7 > 2.\]

In Section 5 we will give more explanation why we choose $\alpha = 1/27$. Now one can choose and fix $\gamma$ and $\theta$ sufficiently small such that

\[N[\gamma^{-2} \theta^2 + \gamma^{20/7} \theta^{-2} + (\gamma \theta)^2] \leq 1/2.\]

Upon adding (4.6) and (4.7), we obtain

\[
A(\gamma \theta \rho) + E(\gamma \theta \rho) + F(\gamma \theta \rho) \leq \frac{1}{2} A(\rho) + N[A(\rho) + E(\rho) + F(\rho)]^{3/2},
\]

where $N$ depends only on $\theta$ and $\gamma$. After putting $\theta_1 = \gamma \theta$, the lemma is proved.

**Proof of Proposition 2.14.** Take the constant $\theta_1$ from Lemma 2.13. Due to Lemma 2.10 we may choose $\varepsilon_0, \varepsilon' > 0$ small enough such that

\[
A(\rho_0/2) + E(\rho_0/2) + C(\rho_0/2) + D(\rho_0/2) + F(\rho_0/2) \leq \varepsilon'.
\]

\[
2\varepsilon' + 8N(\theta_1) \varepsilon_0^{3/2} \leq \min(4\varepsilon', \theta_1^2 \varepsilon_2), \tag{4.8}
\]

where the constant $N(\theta_1)$ is the same one as in (2.15). Since $z_1 \in Q(z_0, \rho/4)$, we have

\[
Q(z_1, \rho_0/4) \subset Q(z_0, \rho_0/2) \subset Q_\varepsilon,
\]

\[
A(\rho_0/4, z_1) + E(\rho_0/4, z_1) + F(\rho_0/4, z_1) \leq 4\varepsilon'.
\]

By using (4.8) and (2.15), one obtains inductively for $k = 1, 2, \ldots$,

\[
A(\theta_1^k \rho_0/4, z_1) + E(\theta_1^k \rho_0/4, z_1) + F(\theta_1^k \rho_0/4, z_1) \leq \min\{\theta_1^k \varepsilon_2, 4\varepsilon'\},
\]

Thus, for any $\rho \in (0, \rho_0/4]$, it holds that

\[
A(\rho, z_1) + E(\rho, z_1) + F(\rho, z_1) \leq \varepsilon_2.
\]

To include the term $C(\rho, z_1)$ in the estimate, it suffices to use (4.4). The proposition is proved.
5. Proof of Theorem 2.1-2.5

Proof of Theorem 2.1 and 2.2. Let $z_0 \in Q_T \cup (\mathbb{R}^4 \times \{T\})$ be a given point. Proposition 2.11 and 2.14 imply that for any $\varepsilon_2 > 0$ there exist small numbers $\varepsilon_1, \varepsilon_0, \rho_0 > 0$ such that either

$$\limsup_{r \to 0} E(r, z_0) \leq \varepsilon_1$$

or

$$C(\rho_0) + D(\rho_0) + F(\rho_0) \leq \varepsilon_0$$

holds true, we can find $\rho_1 > 0$ so that $Q(z_0, \rho_1) \subset Q_T$ and for any $z_1 \in Q(z_0, \rho_1/2), \rho \in (0, \rho_1/2)$ we have

$$C(\rho, z_1) + F(\rho, z_1) \leq \varepsilon_2.$$  

Let $\delta \in (0, \rho_1^2/4)$ be a number and denote

$$M_\delta = \max_{Q(z_0, \rho_1/2) \cap \bar{Q}_{T-\delta}} d(z)|u(z)|,$$

where

$$d(z) = \min[\text{dist}(x, \partial \Omega), (t + \rho_1^2/4 - T)^{1/2}].$$

Lemma 5.1. If (5.3) holds true for a sufficiently small $\varepsilon_2$, then

$$\sup_{Q(z_0, \rho_1/4)} |u(z)| < +\infty.$$  

Proof. If for all $\delta \in (0, \rho_1^2/4)$ we have $M_\delta \leq 2$, then there’s nothing to prove. Otherwise, suppose for some $\delta$ and $z_1 \in \hat{Q}(z_0, \rho_1/2) \cap \bar{Q}_{T-\delta},$

$$M := M_\delta = |u(z_1)|d(z_1) > 2.$$  

Let $r_1 = d(z_1)/M < d(z_1)/2$. We make the scaling as follows:

$$\bar{u}(y, s) = r_1 u(r_1 y + x_1, r_1^2 s + t_1),$$

$$\bar{p}(y, s) = r_1 p(r_1 y + x_1, r_1^2 s + t_1).$$

It’s known that the pair $(\bar{u}, \bar{p})$ satisfies the Navier-stokes equations (1.1) in $Q(0, 1)$. Obviously,

$$\sup_{Q(0,1)} |\bar{u}| \leq 2, \quad |\bar{u}(0,0)| = 1.$$  

Due to the scaling-invariant property of our objects $A, E, C, D$ and $F$, in what follows we look them as objects associated to $(\bar{u}, \bar{p})$ at the origin. For any $\rho \in (0, 1]$, we have

$$C(\rho) + F(\rho) \leq \varepsilon_2.$$  

Recall what we did before in the proof of Lemma 2.9. Since $\bar{u}$ is bounded in $Q(0, 1)$, we have

$$\int_{Q(0, 1/3)} |\bar{p}_{0, 1}|^{14} dz \leq \int_{Q(0, 1/2)} |\bar{p}_{0, 1}|^{14} dz \leq N, \quad (5.7)$$

$$\int_{B(0, 1/3)} |\bar{h}_{0, 1}(z) - [\bar{h}_{0, 1}]_{0, 1/3}|^{14} dx$$

$$\leq N \sup_{B(0, 1/3)} |\nabla \bar{h}_{0, 1}(x, t)|^{14}$$

$$\leq N \left( \int_{B(0, 1/2)} |\bar{h}_{0, 1} - [\bar{h}_{0, 1}]_{0, 1/2}|^{28/27} dx \right)^{27/2},$$

and

$$\int_{Q(0, 1/3)} |\bar{h}_{0, 1}(z) - [\bar{h}_{0, 1}]_{0, 1/3}|^{14} dz$$

$$\leq N \int_{-1/9}^{0} \left( \int_{B(0, 1/2)} |\bar{h}_{0, 1} - [\bar{h}_{0, 1}]_{0, 1/2}|^{28/27} dx \right)^{27/2} dt$$

$$\leq N(1 + F^{14}(1)). \quad (5.8)$$

Estimates (5.7) and (5.8) yield

$$\int_{Q(0, 1/3)} |\bar{p}(z) - [\bar{h}_{0, 1}]_{0, 1/3}|^{14} dz \leq N. \quad (5.9)$$

Because $(\bar{u}, \bar{p})$ satisfies the equation

$$\bar{u}_t - \Delta \bar{u} = \text{div}(\bar{u} \otimes \bar{u}) - \nabla (\bar{p} - [\bar{h}_{0, 1}]_{0, 1/3})$$

in $Q(0, 1)$. Owing to (5.5), (5.9) and the classical Sobolev space theory of parabolic equation, we have

$$\bar{u} \in W^{1, 1/2}_{14}(Q(0, 1/4)), \quad \|\bar{u}\|_{W^{1, 1/2}_{14}(Q(0, 1/4))} \leq N. \quad (5.10)$$

Since $1/2 - 6/14 = 1/14 > 0$, owing to the Sobolev embedding theorem (see [11]), we obtain

$$\bar{u} \in C^{1/14}(Q(0, 1/5)), \quad \|\bar{u}\|_{C^{1/14}(Q(0, 1/5))} \leq N,$$

where $N$ is a universal constant independent of $\varepsilon_1$ and $\varepsilon_2$. Therefore, we can find $\delta_1 < 1/5$ independent of $\varepsilon_1, \varepsilon_2$ such that

$$|\bar{u}(x, t)| \geq 1/2 \quad \text{in } Q(0, \delta_1). \quad (5.11)$$

Now we choose $\varepsilon_2$ small enough which makes (5.11) and (5.6) a contradiction. The lemma is proved. □
Theorem 2.1 and 2.2 follow immediately from Lemma 5.1.

**Proof of Theorem 2.4.** Take the number \( \varepsilon_1 \) in Lemma 5.1. Denote

\[ \Omega^* := \{ z \in \Omega \times \{ T \} \mid \limsup_{r \downarrow 0} E(r, z) \leq \varepsilon_1 \} \]

It is well known that 2-D Hausdorff measure of \( \Omega \setminus \Omega^* \) is zero. By using Lemma 5.1, for any \( z \in \Omega^* \) we can find \( \rho > 0 \) such that \( u \) is bounded in \( Q(z, \rho) \). Then there’s no blow-up at \( z \) and \( z \) is a regular point. The theorem is proved.

**Proof of Theorem 2.5.** For any \( \alpha_0 \in [0, 1] \), due to Lemma 2.7 (i), the interpolation inequality (2.3) with \( r = +\infty, q = 2(1 + \alpha_0) \) and Hölder’s inequality, one can easily get

\[ \| u \|_{L^1_t(L_{(1+\alpha_0)/(2\alpha_0)}^{\infty} \times \mathbb{R}^+)} < +\infty. \]  

(5.12)

Since \((u, p)\) satisfies

\[ \Delta p = \frac{\partial^2}{\partial x_i \partial x_j} (u_i u_j) \quad \text{in} \quad \mathbb{R}^4 \times \mathbb{R}^+, \]

Due to Calderón-Zygmund’s estimate, we have

\[ \| p \|_{L^1_t(L_{(1+\alpha_0)/(2\alpha_0)}^{\infty} \times \mathbb{R}^+)} < +\infty. \]  

(5.13)

Because of (5.12) with \( \alpha_0 = 1/2 \) and (5.13) with \( \alpha_0 = 1/2, \alpha \), and again by Calderón-Zygmund’s estimate, for any \( \varepsilon_4 \in (0, 1) \) we can find \( R \geq 1 \) sufficiently large such that for any \( z_0 \in \mathbb{R}^4 \times (R, +\infty) \) it holds that

\[ C(1, z_0) + \| p \|_{L^1_t(L_{3/2}^{\infty} \times \mathbb{R}^+)} + \| p \|_{L^1_t(L_{1+\alpha_0}/(2\alpha_0)}^{\infty} \times \mathbb{R}^+)} \leq \varepsilon_4, \]  

(5.14)

\[ \| \tilde{p}_{z_0, 1} \|_{L^1_t(L_{3/2}^{\infty} \times \mathbb{R}^+)} + \| \tilde{p}_{z_0, 1} \|_{L^1_t(L_{1+\alpha_0}/(2\alpha_0)}^{\infty} \times \mathbb{R}^+)} \leq \varepsilon_4. \]  

(5.15)

Thus,

\[ \| h_{z_0, 1} \|_{L^1_t(L_{3/2}^{\infty} \times \mathbb{R}^+)} + \| h_{z_0, 1} \|_{L^1_t(L_{1+\alpha_0}/(2\alpha_0)}^{\infty} \times \mathbb{R}^+)} \leq 2\varepsilon_4. \]  

(5.16)

After combining (5.14) and (5.16) together, it is clear by using Hölder’s inequality that

\[ C(1, z_0) + D(1, z_0) + F(1, z_0) \leq N \varepsilon_4, \]

where \( N \) is independent of \( \varepsilon_4 \).

Then owing to Proposition 2.14 and Lemma 5.1 for sufficiently small \( \varepsilon_4 \) we can find a uniform upper bound \( M_0 > 0 \) such that for any \( z_0 \in \mathbb{R}^4 \times (R, +\infty) \)

\[ \sup_{z \in Q(z_0, 1/4)} |u(z)| \leq M_0. \]

Therefore, \( u \) will not blow up as \( t \) goes to infinity, and Theorem 2.5 is proved.
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