UNIFORM IN TIME CONVERGENCE TO BOSE–EINSTEIN CONDENSATION FOR A WEAKLY INTERACTING BOSE GAS WITH AN EXTERNAL POTENTIAL

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ABSTRACT. We consider a gas of weakly interacting bosons in three dimensions subject to an external potential in the mean field regime. Assuming that the initial state of our system is a product state, we show that in the trace topology of one-body density matrices, the dynamics of the system can be described by the solution to the corresponding Hartree type equation. Using a dispersive estimate for the Hartree type equation, we obtain an error term that is uniform in time. Moreover, the dependence of the error term on the particle number is optimal. We also consider a class of intermediate regimes between the mean field regime and the Gross–Pitaevskii regime, where the error term is uniform in time but not optimal in the number of particles.

1. Introduction

Bose-Einstein condensation was proposed by Bose and Einstein [6, 27] in 1924. Since then, the topic has gained great interest in both physics and mathematics [4, 45, 58, 74, 79], in particular after the first experimental observation by Wieman, Cornell and Ketterle [1, 20] in 1995.

We consider the dynamics of a Bose gas of $N$ particles in three dimensions interacting through a symmetric two-body potential in the presence of an external potential. We assume our initial state $\psi_{N,0} \in L^2(\mathbb{R}^3)$ to be fully factorised, i.e.,

$$\psi_{N,0}(x_1, \ldots, x_N) = \prod_{j=1}^{N} u_0(x_j)$$

for some $u_0 \in L^2(\mathbb{R}^3)$ with $\|u_0\|_2 = 1$. The time evolution of our state is governed by the Hamiltonian

$$H_N = \sum_{j=1}^{N} \left( - \Delta_{x_j} + V(x_j) \right) + \frac{\lambda}{N} \sum_{i<j}^{N} w_N(x_i - x_j).$$

The time-evolution $\psi_{N,t}$ is the solution to

$$\begin{cases}
i \partial_t \psi_{N,t} = H_N \psi_{N,t} \\
\psi_{N,t}|_{t=0} = \psi_{N,0}.
\end{cases}$$

Alternatively, one can write

$$\psi_{N,t} = e^{-iH_N t} \psi_{N,0}$$

for every $t \in \mathbb{R}$. Here, $V : \mathbb{R}^3 \to \mathbb{R}$ denotes the external potential and

$$w_N(x) := N^{3\beta} w(N^{\beta} x) \quad \text{for} \quad 0 \leq \beta < 1/3$$

is the interaction potential, where $w \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^2)$ is independent of $N$. We consider weakly interacting bosons, so the coupling constant $\lambda \in \mathbb{R}$ is small in the sense that $|\lambda| \leq \lambda_0$ for some $\lambda_0$ depending on $u_0$, $\|w\|_1$, and $V$. We will give the precise assumptions on the external potential $V$, on $u_0$, and on $\lambda_0$ later.

\footnote{The choice $\beta = 0$ is called mean-field regime, and $\beta = 1$ is called Gross-Pitaevskii regime. Moreover, $\beta \in (0, 1)$ is called intermediate regime between the mean-field regime and the Gross-Pitaevskii regime.}
We prepare a product state that is close to the ‘ground state of an interacting Bose gas in a trap’, which is called the Bose-Einstein condensation state. Then we want to investigate the time evolution of the state after removing the trap and replacing it by a small external potential \( V \).

Since the initial state \( \psi_{N,0} \) is a product state, see (1.1), we expect that, in an appropriate sense, the time-evolved state \( \psi_{N,t} \) is also approximately given by a product state \( \prod_{j=1}^{N} u_{t}(x_{j}) \) for some \( u_{t} \in L^{2}(\mathbb{R}^{3}) \), where \( u_{t} \) is the solution to the corresponding Hartree-type equation (Hartree)

\[
\begin{cases}
i\partial_{t}u_{t} = (-\Delta + V)u_{t} + \lambda(w \ast |u_{t}|^{2})u_{t} \\
u_{t}|_{t=0} = u_{0},
\end{cases}
\]

for \( \beta = 0 \), and to the nonlinear Schrödinger equation (NLS)

\[
\begin{cases}
i\partial_{t}u_{t} = (-\Delta + V)u_{t} + \lambda|u_{t}|^{2}u_{t} \\
u_{t}|_{t=0} = u_{0},
\end{cases}
\]

for \( \beta \in (0, 1/3) \), where \( \lambda = \int_{\mathbb{R}^{3}} dx \, w(x) \).

We cannot expect \( \prod_{j=1}^{N} u_{t}(x_{j}) \) and \( \psi_{N,t} \) to be close in the \( L^{2} \)-norm sense, see [60]. However, we can prove that they are close in the trace topology of one-body density matrices as \( N \to \infty \). More precisely, we define the marginal one-particle density \( \gamma_{N,t}^{(1)} \) by its operator kernel

\[
\gamma_{N,t}^{(1)}(x; y) = \int_{\mathbb{R}^{(N-1)}} dZ \, \overline{\psi_{N,t}(y, Z)} \, \psi_{N,t}(x, Z).
\]

Our main result is the following.

**Theorem 1.1.** Let \( 0 \leq \beta < 1/3 \) and let \( N \in \mathbb{N} \). Suppose that \( w \) is even, real-valued, and \( w \in L^{1}(\mathbb{R}^{3}) \cap L^{2}(\mathbb{R}^{3}) \) with

\[
|w(z)| \leq C_{w}|z|^{-\gamma} \quad \text{for all } |z| \geq 1
\]

for some \( \gamma > 5 \) and \( C_{w} > 0 \). Define

\[
w_{N}(x) := N^{3/2} w(N^{3/2} x).
\]

Let \( V \in W^{2,\infty}(\mathbb{R}^{3}) \) be real-valued and such that there exists a constant \( C^{V} > 0 \) such that

\[
\|e^{i(\Delta + V)f}\|_{\infty} \leq C^{V} \|f\|_{1}
\]

for all \( f \in L^{1}(\mathbb{R}^{3}) \cap L^{2}(\mathbb{R}^{3}) \). Let \( u_{0} \in H^{2}(\mathbb{R}^{3}) \) with \( \|u_{0}\|_{2} = 1 \). We let our system have fully factorized initial data, i.e.,

\[
\psi_{N,0}(x) = \prod_{j=1}^{N} u_{0}(x_{j})
\]

for \( x = (x_{1}, x_{2}, \ldots, x_{N}) \in \mathbb{R}^{3N} \). Let

\[
\psi_{N,t} = e^{-iH_{N}t}\psi_{N,0}
\]

for \( t > 0 \) where \( H_{N} \) is defined in (1.2). Here, we assume that \( |\lambda| \leq \lambda_{0} \) with \( 0 < \lambda_{0} \leq 1 \) depending only on \( \|V\|_{W^{2,\infty}, C^{V}}, \|w\|_{1}, \|u_{0}\|_{H^{2}} \), and \( \|e^{(\Delta + V)u_{0}}\|_{1} \). Then there exists a constant \( C > 0 \) depending only on \( \beta, \|V\|_{W^{2,\infty}, C^{V}}, \|w\|_{1}, \|u_{0}\|_{H^{2}} \), and \( \|e^{(\Delta + V)u_{0}}\|_{1} \) such that

\[
\text{Tr} \left| \gamma_{N,t}^{(1)} - |u_{t}\rangle\langle u_{t}| \right| \leq \begin{cases} CN^{-1} & \text{if } \beta = 0, \\ CN^{-\min(\beta,(1-3\beta)/2)} & \text{if } 0 < \beta < 1/3 \end{cases}
\]

for all \( t > 0 \). In particular, the constant \( C \) does not depend on \( t \) or \( N \).
Remark 1.1.

(1) A special case of Theorem 1.1, namely in the mean field regime and without external potential, i.e. \( \beta = 0 \) and \( V = 0 \), was proved in \([53, \text{Theorem 1.3}]\). Theorem 1.1 is new because we can also treat external potentials and \( 0 < \beta < 1/3 \).

(2) For any \( \lambda \geq 0 \) with \( w \in C^1(\mathbb{R}^3) \) vanishing at infinity, non-negative, spherically symmetric, and decreasing, one can prove the same result without external potential by using \([41, \text{Corollary 3.4}]\). In particular, we can cover large \( \lambda > 0 \).

(3) In many occasions, relevant physical quantities can be obtained from the one-particle density matrix, so Theorem 1.1 tells us that in these occasions we can obtain a good approximation by replacing the one particle density \( \gamma^{(1)}_{N,t} = |\varphi_t\rangle\langle\varphi_t| \).

(4) A similar theorem can be obtained for general \( \lambda \) (without smallness assumption) with time dependent constant \( C \).

(5) The paper covers the mean field regime (\( \beta = 0 \)) and a part of the intermediate regime, namely \( 0 < \beta < 1/3 \). The next goal would be to consider the remaining part of the intermediate regime (\( 1/3 \leq \beta < 1 \)) and the final goal would be the Gross-Pitaevskii (GP) regime (\( \beta = 1 \)).

(6) Note that the scaling in \( N \) conserves the \( L^1 \)-norm of \( w \), i.e. \( \|w\|_{L^1(\mathbb{R}^3)} = \|w_N\|_{L^1(\mathbb{R}^3)} \) but not the \( L^2 \)-norm \( \|w_N\|_{L^2(\mathbb{R}^3)} = N^{3\beta/2} \|w\|_{L^2(\mathbb{R}^3)} \).

Since we use some estimates in terms of \( \|w_N\|_{L^2(\mathbb{R}^3)} \) in our proof, the rate of convergence in Theorem 1.1 for \( \beta > 0 \) is drastically different than for \( \beta = 0 \).

(7) Condition (1.5) will be used in the proof of Lemma 6.1 below.

Our proof is based on \([14, 15, 82]\). We derive the large \( N \) limit with interaction potential \( w \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \) under the existence of external potential \( V \).

We track the dependence of \( \|\varphi_t\|_\infty \), which helps us get a better estimate after using the Grönwall inequality. Applying the time decay estimate obtained in \([23]\), we obtain a time independent rate of convergence. We will explain more details of the proof strategy later in Section 1.2.

1.1. History.

Many-Body Convergence. Now we are going to review the results on the derivation of the effective one-body Schrödinger equation from the many-body Schrödinger equation. To be more precise, a typical result in this direction is that if one has an initial many-body wave function \( \psi_{N,0} \) with

\[
\lim_{N \to \infty} \text{Tr} \left[ \frac{1}{N} \gamma_{N,0} - |u_0\rangle\langle u_0| \right] = 0
\]

for some given \( u_0 \in L^2(\mathbb{R}^3) \), then at later times \( t > 0 \),

\[
\lim_{N \to \infty} \text{Tr} \left[ \frac{1}{N} \gamma_{N,t} - |u_t\rangle\langle u_t| \right] = 0,
\]

where \( u_t \) is a solution of an effective partial differential equation.

For the mean-field regime, i.e., \( \beta = 0 \), with differentiable interaction potential \( w \), Hepp \([48]\) showed \(1.8\). Moreover, Spohn \([83]\) obtained \(1.8\) for a bounded interaction potential \( w \). Ginibre and Velo provided a series of works in this direction \([35, 36, 40]\).

Later, for singular potentials, the derivation of \(1.8\) was obtained by Bardos, Golse, and Mauer \([2]\) for \( L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \)-potentials in one dimension, and by Erdős and Yau \([32]\) for the Coulomb potential in three
dimensions. These works are based on the BBGKY hierarchy method. Therefore, the rate of convergence in these works was not given explicitly. Moreover, the convergence is only established for any time of order one.

For factorized initial states, Rodnianski and Schlein [82] developed a scheme to obtain an explicit rate of convergence. They showed

$$\text{Tr} \left| \frac{1}{N} \gamma_{N,t} - |u_t\rangle \langle u_t| \right| \leq C e^{C t} \sqrt{\frac{1}{N}}$$

(1.9)

for interaction potentials $w$ including the Coulomb interaction potential. Knowles and Pickl [51] provided a scheme to cover more singular potentials than Coulomb with the same rate based on the techniques in [76].

For bounded and integrable interaction potentials, the optimal rate in $N$ was provided in [29]. This work was extended in [13]. In [15], Chen, Lee, and Schlein provided the rate $C e^{C t} \sqrt{N}$, which is optimal in $N$, for the Coulomb interaction potential. This work was generalized in [14] to more singular interaction potentials using Strichartz estimates for the Hartree equation. The time dependence was investigated in [53].

Elgart, Schlein [28] and Michelangeli, Schlein [64] proved convergence for the mean-field Boson star equation, i.e.,

$$H_N = \sum_{j=1}^{N} \left( 1 - \Delta x_j \right)^{1/2} + \frac{\lambda}{N} \sum_{i<j}^{N} \frac{1}{x_i - x_j}.$$
Since there is a correlation among particles, it might not be natural to expect a time independent bound for the norm approximation. In this paper, we provide the convergence in terms of a trace norm approximation of one-particle density matrices instead of a norm approximation of wave functions.

**Time Decay Estimate for One-Body Nonlinear Schrödinger Equations.** Let us first review several results on one-body nonlinear Schrödinger equations without external potentials. Ginibre and Velo [37, Theorem 6.1(1)] considered Hartree type nonlinearities with repulsive interaction potentials \( w \) and they proved the decay estimate

\[
\|u_t\|_q \leq C(1 + |t|)^{-\frac{d}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} \text{ for all } t \neq 0
\]

for all \( q \) with \( \left[ \frac{1}{2} - \frac{1}{d} \right]_+ \leq \frac{1}{q} \leq \frac{1}{2} \), where \( [\cdot]_+ \) denotes the positive part. Note that \( (\frac{1}{2} - \frac{1}{d})^{-1} = (\frac{d-2}{2d})^{-1} = \frac{2d}{d-2} \), so their result corresponds to the energy critical and energy subcritical case. In particular, the case \( q = \infty \) is not covered in dimension \( d = 3 \). Hayashi and Naumkin [46] looked at critical nonlinearities, namely the local nonlinearity

\[
\lambda |u_t|^2 u_t \quad \text{for } \lambda \in \mathbb{R} \text{ and } d \in \{1, 2, 3\}
\]

and the nonlocal nonlinearity

\[
\lambda (|\cdot|^{-1} * |u_t|^2) u_t \quad \text{for } \lambda \in \mathbb{R} \text{ and } d \geq 2
\]

and they can also add non-critical nonlinearities. They showed a dispersive estimate of the form

\[
\|u_t\|_\infty \leq C(1 + |t|)^{-\frac{d}{2}} \text{ for all } t \neq 0.
\]

Later Kato and Pusateri [50] gave an alternative proof of the result in [46], which was based on a careful analysis of the equation in Fourier space.

Grillakis and Machedon [41, Corollary 3.4] considered Hartree type nonlinearities with nonnegative, radial, decreasing interaction potential \( w \in L^1(\mathbb{R}^d) \cap C^1_0(\mathbb{R}^d) \) for sufficiently regular but possibly large initial data. They showed a decay estimate of the form (1.13) using a Grönwall type argument after proving suitable a-priority estimates. Their result was applied in [67], where they showed a norm approximation for the dynamics of many-body quantum systems in the context of Bose-Einstein condensation. Other results without external potentials we would like to mention are [12, 18, 22, 26, 34, 38, 39, 47, 74].

Next, let us turn to results on nonlinear Schrödinger equations with external potentials. In dimension \( d = 1 \), Cuccagna, Georgiev and Visciglia [19] proved a decay estimate of the form (1.13) for subcritical nonlinearities \( \pm |u_t|^{p-1} u_t \) with \( 3 < p < 5 \) and small initial data. The corresponding result for the critical nonlinearities \( \pm |u_t|^{p-1} u_t \) with \( p = 3 \) was proved by Germain, Pusateri and Rousset [33]. Their proof relies on the use of the distorted Fourier transform and the analysis of an oscillatory integral. A similar result was proved by Naumkin [72, 73]. In dimension \( d = 3 \), Pusateri and Soffer [80] considered the nonlinearity \( -u_t^2 \) and small initial data. They showed that

\[
\|u_t\|_\infty \leq C(1 + |t|)^{-(1+\alpha)} \text{ for all } t \neq 0
\]

for some \( \alpha > 0 \). Two other results in dimension \( d = 3 \) with external potentials, which we would like to mention, are scattering in \( H^1 \) for an external potential \( V \) with small negative part by Hong [49], and a classification of the dynamics of solutions to the cubic nonlinear Schrödinger equation with small initial data and a radial external potential \( V \), which is such that the operator \( -\Delta + V \) has exactly one negative eigenvalue, by Nakanishi [65].
1.2. **Strategy of the Proof.** We consider the dynamics of the $N$-dependent Hartree type equation

\[
\begin{align*}
    i\partial_t \varphi_t &= (-\Delta + V) \varphi_t + \lambda (w_N \ast |\varphi_t|^2) \varphi_t \\
    \varphi_t|_{t=0} &= \varphi_0.
\end{align*}
\]  

(1.15)

Note that for $\beta = 0$, $\varphi_t = u_t$ where $u_t$ was defined in (Hartree). For $0 < \beta < 1/3$, we compare $\varphi_t$ to $u_t$ in (NLS) in Section 6 using the Grönwall lemma.

In order to compare $\gamma_{N,t}^{(1)}$ with $|\varphi_t\rangle\langle\varphi_t|$, we will need good control on the time evolution. To this end, we will define a truncated dynamics in Section 4, for which we have good bounds. In Section 5, we then compare the dynamics with its truncated version, for which we deduce good bounds on the dynamics.

Moreover, we divide the dynamics into parity-conserving and non-parity-conserving parts. In order to obtain these bounds, we estimate the time derivative and then use the Grönwall lemma. Since the estimate for the time derivative contains a factor of $\|\varphi_t\|_\infty$ and by Proposition 2.1 below, we know from (2.14) that $t \mapsto \|\varphi_t\|_\infty \in L^1(\mathbb{R})$, the bound we get after applying the Grönwall lemma will be the time independent.

1.3. **Structure of the Paper.** In Section 2, we review the necessary preliminaries. In particular, we review the result on the time decay estimate with an external potential, see 2.1, and the Fock Space formalism for the many-body problem. In Section 3, we prove Theorem 1.1 assuming Propositions 3.1 and 3.2. In Section 4, we introduce the truncated dynamics and prove a bound for it. Section 5 is devoted to proving Propositions 3.1 and 3.2. To this end, we show several results on comparison dynamics. In Section 6, we estimate the difference between the solution of the $N$-dependent Hartree evolution and the limiting nonlinear Schrödinger equation.

1.4. **Notation.**

1. The $L^p$-norm in $\mathbb{R}^3$ can be denoted by $\| \cdot \|_{L^p(\mathbb{R}^3)}$ or $\| \cdot \|_p$.

2. Sometimes, when there can be no confusion, we omit the subscript for norms, for example,

$$
\| \cdot \|_{L^2(\mathbb{R}^3)}, \| \cdot \|_{F}, \text{ or } \langle \cdot , \cdot \rangle_F
$$

are just written as

$$
\| \cdot \|, \| \cdot \|_F, \text{ or } \langle \cdot , \cdot \rangle,
$$

respectively.

3. Our constant $C$ may change from line to line.

4. The indicator function is denoted by $\chi$, for example

$$
\chi(N \leq M) = \begin{cases} 
    1 & \text{if } N \leq M, \\
    0 & \text{if } N > M.
\end{cases}
$$

5. We define the Fourier transform of a function $f \in L^1(\mathbb{R}^3)$ by

$$
\hat{f}(\xi) = \int_{\mathbb{R}^3} dx \, e^{-2\pi i x \cdot \xi} f(x).
$$

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2. Preliminaries

2.1. Time Decay Estimate under the Existence of an External Potential. This subsection is devoted to reviewing the time decay estimate under the existence of external potential.

Proposition 2.1 ([23] Theorem 1.1). Let $d \geq 3$ and let $k \in \mathbb{N}$ be the smallest even number with $k > \frac{d}{2}$. Let $V \in W^{k,\infty}(\mathbb{R}^d)$ be a real-valued function and satisfy

$$\|e^{-it(-\Delta + V)}f\|_{\infty} \leq C^V|t|^{-\frac{d}{2}}\|f\|_1$$

(2.1)

for every $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and some constant $C^V \geq 1$. Let the interaction potential $w \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ be an even, real-valued function. Let $u_0 \in H^k(\mathbb{R}^d)$ and let $u \in C(\mathbb{R}, H^k(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^d))$ be the unique global strong solution to the Hartree type equation

$$\begin{aligned}
0 = & (-\Delta + V)u_x + (w \ast |u|^2)u_x \\
u|_{t=0} = & u_0.
\end{aligned}$$

(2.2)

Assume that the initial data is sufficiently small, that is,

$$\|e^{i(-\Delta+V)}u_0\|_1, \|u_0\|_{H^k} \leq \epsilon_0$$

(2.3)

for some $\epsilon_0 = \epsilon_0(d, \|\|V\|_{W^{k,\infty}}, C^V, \|w\|_1) > 0$.

Then there exists a constant $C_0 = C_0(d, \|\|V\|_{W^{k,\infty}}, C^V, \|w\|_1) \geq 1$ such that

$$\|u_t\|_\infty \leq \frac{C_0}{(1 + |t|)\frac{k}{2}}$$

(2.4)

for all $t \geq 0$. Furthermore, if we assume that

$$\|e^{i(-\Delta+V)}(\partial_x u_t)|_{t=0}\|_1, \|(\partial_x u_t)|_{t=0}\|_{H^k} \leq \tilde{\epsilon}_0$$

(2.5)

for some $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(d, \|\|V\|_{W^{k,\infty}}, C^V, \|w\|_1) > 0$, then

$$\|\partial_x u_t\|_\infty \leq \frac{\tilde{C}_0}{(1 + |t|)\frac{k}{2}}$$

(2.6)

for all $t \geq 0$, where $\tilde{C}_0 = \tilde{C}_0(d, \|\|V\|_{W^{k,\infty}}, C^V, \|w\|_1) > 0$.

Remark 2.1.

(1) Let $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ and assume that

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy \frac{|V(x)||V(y)|}{|x-y|^2} < (4\pi)^2$$

(2.7)

and

$$\sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} dy \frac{|V(y)|}{|x-y|} < 4\pi.$$  

(2.8)

Then [31] Theorem 1.1] there exists a constant $C^V > 0$ depending only on $V$ such that

$$\|e^{-it(-\Delta + V)}f\|_{\infty} \leq C^V|t|^{-\frac{d}{2}}\|f\|_1$$

for all $f \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $t \in \mathbb{R} \setminus \{0\}$.

Hence, the combination of (2.7) and (2.8) is a sufficient condition for (2.1) in dimension three.

(2) Proposition 2.1 remains true for any global solution to the cubic non-linear Schrödinger equation: We may replace the interaction potential by $w = \pm \delta_0$, that is, the nonlinearity is given by $\pm |u_t|^2 u_t$. 


**Idea of the proof.** The proof uses ideas from Grillakis and Machedon [41, Corollary 3.4] and Kato and Pusateri [50]. We only describe the proof strategy for
\[ \|u_t\|_\infty \leq \frac{C_0}{(1 + |t|)^{\frac{d}{2}}} \text{ for all } t \geq 0 \]
as the proof of the corresponding estimate for \( \|\partial_t u_t\|_\infty \) is similar. Define
\[ M(T) := \sup_{0 \leq t \leq T} (1 + |t|)^{\frac{d}{2}} \|u_t\|_\infty + \sup_{0 \leq t \leq T} \|D^k u_t\|_2 + \|u_0\|_2, \]
and note that it suffices to show
\[ M(T) \leq C_0 \text{ for every } T \geq 0 \] (2.10)
for some \( C_0 > 0 \) to be determined. Using Duhamel’s formula and the dispersive estimate
\[ \|e^{-it(-\Delta + V)}f\|_\infty \leq C^V |t|^{-\frac{d}{2}} \|f\|_1 \text{ for all } f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \]
for the operator \(-\Delta + V\), which was part of our assumptions on the external potential \( V\), we obtain after a computation
\[ M(T) \leq \epsilon + CM(T)^3, \]
where \( \epsilon, C > 0 \) do not depend on \( T \) and \( \epsilon \) is small if the initial data is small. If \( \epsilon > 0 \) is small enough, then the graph of the function
\[ f : [0, \infty) \to \mathbb{R}, \quad f(x) := \epsilon + Cx^3 - x \]
has two distinct non-negative zeros, see Figure 2.1.

Denote by \( C_0 \) the first zero of \( f \). By a blow-up criterion for \( H^k \)-solutions similar to [11, Theorem 4.10.1], there exists \( T_{\text{max}} \in (0, \infty) \) such that \( M \in C([0, T_{\text{max}}]) \). Furthermore, if \( T_{\text{max}} < \infty \), then
\[ \lim_{T \uparrow T_{\text{max}}} M(T) = \infty. \]
If \( M(0) \leq C_0 \), then by the smallness of the initial data, we deduce that \( T_{\text{max}} = \infty \) and (2.10) holds. \( \square \)
bosonic Fock space is a Hilbert space given by a bigger space, the so-called Fock space. We briefly review the well-known Fock space formalism. The \( \mathcal{F} \)

\[
\mathcal{F} = \bigoplus_{n \geq 0} L^2(\mathbb{R}^3)^{\otimes n},
\]

where \( L^2(\mathbb{R}^3)^{\otimes n} \) is the symmetric subspace of \( L^2(\mathbb{R}^{3n}) \), that is, such that, for

\[
g(x_1, \ldots, x_n) = g(x_{\sigma(1)}, \ldots, x_{\sigma(n)})
\]

for all \( g(x_1, \ldots, x_n) \in L^2(\mathbb{R}^3)^{\otimes n} \) and all \( \sigma \in S_n \), where \( S_n \) denotes the set of all permutations of \( \{1, \ldots, n\} \). We denote an element (or state) \( \psi \in \mathcal{F} \) by

\[
\psi = \psi^{(0)} \oplus \psi^{(1)} \oplus \psi^{(2)} \oplus \cdots = (\psi^{(n)})_{n \geq 0}
\]

where \( L^2(\mathbb{R}^3)^{\otimes n} \) for all \( n \geq 0 \). The inner product on \( \mathcal{F} \) is defined by

\[
\langle \psi_1, \psi_2 \rangle := \sum_{n \geq 0} \left\langle \psi_1^{(n)}, \psi_2^{(n)} \right\rangle_{L^2(\mathbb{R}^{3n})} = \psi_1^{(0)} \psi_2^{(0)} + \sum_{n \geq 1} \int_{\mathbb{R}^{3n}} dx_1 \cdots dx_n \psi_1^{(n)}(x_1, \ldots, x_n) \overline{\psi_2^{(n)}(x_1, \ldots, x_n)}.
\] (2.12)

For \( f \in L^2(\mathbb{R}^3) \), the creation operator \( a^*(f) \) and the annihilation operator \( a(f) \) on \( \mathcal{F} \) are defined by

\[
(a^*(f) \psi)^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)
\] (2.13)

and

\[
(a(f) \psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n + 1} \int_{\mathbb{R}^3} dx f(x) \psi^{(n+1)}(x, x_1, \ldots, x_n).
\] (2.14)

Define

\[
\phi(f) := a^*(f) + a(f).
\] (2.15)

Note that \( \phi(f) \) is self-adjoint, while the creation operator \( a^*(f) \) and the annihilation operator \( a(f) \) are not self-adjoint. We also use operator-valued distributions \( a_x^* \) and \( a_x \) satisfying

\[
a^*(f) = \int_{\mathbb{R}^3} dx f(x) a_x^*, \quad a(f) = \int_{\mathbb{R}^3} dx \overline{f(x)} a_x
\] (2.16)

for any \( f \in L^2(\mathbb{R}^3) \). The annihilation and creation operator satisfy the canonical commutation relation (CCR), which are given by

\[
[a(f), a^*(g)] = (f, g)_{L^2(\mathbb{R}^3)}, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0;
\]
The number operator $\mathcal{N}$ is defined by
\[
\mathcal{N} := \int_{\mathbb{R}^3} dx \, a_x^* a_x
\]
and it satisfies that $(\mathcal{N}\psi)^{(n)} = n\psi^{(n)}$. The domain of the number operator $\mathcal{N}$ is given by
\[
D(\mathcal{N}) = \left\{ \psi \in \mathcal{F} : \sum_{n\geq 1} n^2 \|\psi^{(n)}\|^2 < \infty \right\}
\]

For any bounded operator $J : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, its second quantization $d\Gamma(J) : D(\mathcal{N}) \to \mathcal{F}$ is given by
\[
(d\Gamma(J)\psi)^{(n)} = \sum_{j=1}^n J_j \psi^{(n)}
\]
where $J_j = 1 \otimes \ldots \otimes J \otimes \ldots \otimes 1$ is the operator $J$ acting only on the $j$-th variable. For any compact operator $J : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ with kernel $J(x; y)$, we can write its second quantization $d\Gamma(J)$ in terms of operator-valued distributions as
\[
d\Gamma(J) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy \, J(x; y) a_x^* a_y.
\]

Intuitively speaking, both creation and annihilation operators behave like $\mathcal{N}^{1/2}$ as the following lemma shows:

**Lemma 2.1 (82 Lemma 2.1).** For $\alpha > 0$, let $D(\mathcal{N}^{\alpha}) = \{ \psi \in \mathcal{F} : \sum_{n\geq 1} n^{2\alpha} \|\psi^{(n)}\|^2 < \infty \}$ denote the domain of the operator $\mathcal{N}^{\alpha}$. For any $f \in L^2(\mathbb{R}^3)$ and any $\psi \in D(\mathcal{N}^{1/2})$, we have
\[
\|a(f)\psi\| \leq \|f\| \|\mathcal{N}^{1/2}\psi\|,
\]
\[
\|a^*(f)\psi\| \leq \|f\| \|\mathcal{N} + 1\|^{1/2}\psi\|,
\]
\[
\|\phi(f)\psi\| \leq 2\|f\| \|(\mathcal{N} + 1)^{1/2}\psi\|.\]

Moreover, for any bounded one-particle operator $J$ on $L^2(\mathbb{R}^3)$ and for every $\psi \in D(\mathcal{N})$, we find
\[
\|d\Gamma(J)\psi\| \leq \|J\|\|\mathcal{N}\psi\|.\]

For $f \in L^2(\mathbb{R}^3)$, the Weyl operator $W(f)$ is defined by
\[
W(f) := \exp(a^*(f) - a(f)),
\]
and it satisfies
\[
W(f) = e^{-\|f\|^2/2} \exp(a^*(f)) \exp(-a(f)).
\]

Coherent states $\psi(f)$ can be defined in terms of the Weyl operator as
\[
\psi(f) := W(f)\Omega = e^{-\|f\|^2/2} \exp(a^*(f))\Omega = e^{-\|f\|^2/2} \sum_{n\geq 0} \frac{1}{\sqrt{n!}} f^n^\otimes n.\]

The following lemmata indicate useful properties of the Weyl operator and coherent states:

**Lemma 2.2 (82 Lemma 2.2).** Let $f, g \in L^2(\mathbb{R}^3)$.

1. The commutation relation between the Weyl operators is given by
\[
W(f)W(g) = W(g)W(f)e^{-2\cdot \text{Im}(f,g)} = W(f + g)e^{-\text{Im}(f,g)}.
\]
The Weyl operator is unitary and satisfies
\[ W(f)^* = W(f)^{-1} = W(-f). \]

The coherent states are eigenvectors of annihilation operators, i.e.,
\[ a_x \psi(f) = f(x) \psi(f) \quad \text{and} \quad a(g) \psi(f) = \langle g, f \rangle_{L^2} \psi(f). \]

The commutation relation between the Weyl operator and the annihilation operator (or the creation operator) is
\[ W^*(f) a_x W(f) = a_x + f(x) \quad \text{and} \quad W^*(f) a_x^* W(f) = a_x^* + \overline{f(x)}. \]

The distribution of \( N \) with respect to the coherent state \( \psi(f) \) is Poisson. In particular,
\[ \langle \psi(f), N \psi(f) \rangle = \| f \|^2, \quad \langle \psi(f), N^2 \psi(f) \rangle - \langle \psi(f), N \psi(f) \rangle^2 = \| f \|^2. \]

Let
\[ d_N := \frac{\sqrt{N!}}{N^{N/2} e^{-N/2}}. \] (2.23)

Note that \( C^{-1} N^{1/4} \leq d_N \leq C N^{1/4} \) for some constant \( C > 0 \) independent of \( N \), which can be easily checked by using Stirling’s formula.

Lemma 2.3 ([13 Lemma 6.3]). There exists a constant \( C > 0 \) independent of \( N \) such that, for any \( u_0 \in L^2(\mathbb{R}^3) \) with \( \| u_0 \| = 1 \), we have
\[ \left\| (N + 1)^{-1/2} W^* \left( \sqrt{N} u_0 \right) \frac{(a^*(u_0))^N}{\sqrt{N!}} \right\| \leq \frac{C}{d_N}. \]

Lemma 2.4 ([57 Lemma 7.2]). Let \( P_m \) be the projection onto the \( m \)-particle sector of the Fock space \( \mathcal{F} \) for a non-negative integer \( m \). Then, for any non-negative integer \( k \leq (1/2) N^{1/3} \) and any \( u_0 \in L^2(\mathbb{R}^3) \) with \( \| u_0 \| = 1 \),
\[ \left\| P_{2k} W^* \left( \sqrt{N} u_0 \right) \frac{(a^*(u_0))^N}{\sqrt{N!}} \right\| \leq \frac{2}{d_N} \]
and
\[ \left\| P_{2k+1} W^* \left( \sqrt{N} u_0 \right) \frac{(a^*(u_0))^N}{\sqrt{N!}} \right\| \leq \frac{2(k + 1)^{3/2}}{d_N N^{1/2}}. \]

3. Proof of the Main Result

To consider the problem in the Fock space formalism, we extend the Hamiltonian \( H_N \) in (1.2) to the Fock space by
\[ H_N := \int_{\mathbb{R}^3} dx \, a^*_x \left( -\Delta_x + V(x) \right) a_x + \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy \, w_N (x - y) a^*_y a_x a_y. \] (3.1)

With this definition, we have \( \langle H_N \psi \rangle^{(N)} = H_N \psi^{(N)} \) for any \( \psi \in \mathcal{F} \). The kernel of the one-particle marginal density \( \gamma^{(1)}_\psi \) associated with \( \psi \) is
\[ \gamma^{(1)}_\psi (x; y) = \frac{1}{\langle \psi, N \psi \rangle} \langle \psi, a^*_y a_x \psi \rangle. \] (3.2)

Note that \( \gamma^{(1)}_\psi \) is a trace class operator on \( L^2(\mathbb{R}^3) \) and \( \text{Tr } \gamma^{(1)}_\psi = 1 \). Definition (3.2) is equivalent to (1.4) in the sense that \( \gamma^{(1)}_\psi = \gamma^{(1)}_{N,t} \) for every \( \psi = (1_{n=N} \psi_{N,t})_{n \geq 0} \in \mathcal{F} \) and \( \psi_{N,t} \in L^2(\mathbb{R}^3)^{\otimes N} \).
After a computation, we obtain
\[
L^{(1)}_{N,t}(x,y) = \langle e^{-iH_N t} u_0^N, a_y^* a_x e^{-iH_N t} u_0^N \rangle = \frac{1}{N} \left( u_0^N, e^{-iH_N t} a_y^* a_x e^{-iH_N t} u_0^N \right) = \frac{1}{N} \left( u_0^N, e^{-iH_N t} a_y^* a_x e^{-iH_N t} u_0^N \right) \Omega \left( e^{iH_N t} a_y^* a_x e^{-iH_N t} \right)^N \Omega^*.
\] (3.3)

Let us for a moment discuss the time evolution of a coherent state, taken as \( W^* (\sqrt{N} \varphi_s) \Omega \). We expand \( a_y^* a_x \) around \( \sqrt{N} \varphi_t(y) \varphi_t(x) \), then we are lead to consider the operator
\[
W^* (\sqrt{N} \varphi_s) e^{iH_N (t-s)} (a_x - \sqrt{N} \varphi_t(x)) e^{-iH_N (t-s)} W (\sqrt{N} \varphi_s) = W^* (\sqrt{N} \varphi_s) e^{iH_N (t-s)} W (\sqrt{N} \varphi_t) a_x W^* (\sqrt{N} \varphi_t) e^{-iH_N (t-s)} W (\sqrt{N} \varphi_s).
\] (3.4)

After a computation, we obtain
\[
i \partial_t W^* (\sqrt{N} \varphi_t) e^{-iH_N (t-s)} W (\sqrt{N} \varphi_s) =: \left( \sum_{k=0}^4 L_k(t; s) \right) W^* (\sqrt{N} \varphi_t) e^{-iH_N (t-s)} W (\sqrt{N} \varphi_s),
\] (3.5)

where \( L_k \) is an operator with \( k \) creation and/or annihilation operators. The exact formulas for \( L_k \) are as follows:
\[
L_0(t; s) := \frac{N \lambda}{2} \int_s^t dt \int_{\mathbb{R}^3} dx \left( w_N \ast |\varphi_r|^2 \right)(x) |\varphi_r(x)|^2,
\] (3.6)
\[
L_1(t; s) := L_1 = 0,
\] (3.7)
\[
L_2(t; s) := L_2(t) = \int_{\mathbb{R}^3} dx a_x^* (-\Delta_x + V(x)) a_x + \lambda \int_{\mathbb{R}^3} dx \left( w_N \ast |\varphi_t|^2 \right)(x) a_x^* a_x
\] 
\[+ \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy w_N(x-y) \sqrt{N} \varphi_t(x) \varphi_t(y) a_x^* a_x
\] 
\[+ \frac{\lambda}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy w_N(x-y) \left( \varphi_t(x) \varphi_t(y) a_y^* a_y + \varphi_t(y) \varphi_t(x) a_x a_y \right),
\] (3.8)
\[
L_3(t; s) := L_3(t) = \frac{\lambda}{\sqrt{N}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy w_N(x-y) \left( \varphi_t(x) a_y^* a_y + \varphi_t(y) a_x a_y \right)a_x,
\] (3.9)
\[
L_4(t; s) := L_4 = \frac{\lambda}{2N} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy w_N(x-y) a_x^* a_y a_x a_y.
\] (3.10)

Defining the unitary operator \( \mathcal{U}(t; s) \) by
\[
\mathcal{U}(t; s) := e^{i\omega(t; s)} W^* (\sqrt{N} \varphi_t) e^{-iH_N (t-s)} W (\sqrt{N} \varphi_s),
\] (3.11)
with the phase factor
\[
\omega(t; s) = \int_s^t d\tau L_0(\tau; s),
\]
we get
\[
i \partial_t \mathcal{U}(t; s) = (L_2 + L_3 + L_4) \mathcal{U}(t; s) =: L_N(t) \mathcal{U}(t; s) \quad \text{and} \quad \mathcal{U}(s; s) = I
\] (3.12)
and
\[
W^* (\sqrt{N} \varphi_s) e^{iH_N (t-s)} (a_x - \sqrt{N} \varphi_t(x)) e^{-iH_N (t-s)} W (\sqrt{N} \varphi_s) = \mathcal{U}^*(t; s) a_x \mathcal{U}(t; s),
\] (3.13)
where we used (3.4) and the fact that the operator $e^{i\omega(t; s)}$ is just a multiplication by a complex number.

Let
\[
\hat{L}_2(t) := \int_{\mathbb{R}^3} dx \ a_x^* \left( -\Delta_x + V(x) \right) a_x + \lambda \int_{\mathbb{R}^3} dx \ (w_N * |\varphi_t|^2) a_x^* a_x \\
+ \lambda \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy \ w_N(x-y) \varphi_t(x) \varphi_t(y) a_x^* a_x \\
+ \frac{\lambda}{2N^{3\beta/2}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy \ w_N(x-y) (\varphi_t(x) \varphi_t(y) a_x^* a_y + \varphi_t(x) \varphi_t(y) a_x a_y).
\]

(3.14)

Also we let $\hat{L} := \hat{L}_2 + L_4$ and define the unitary operator $\hat{U}(t; s)$ by
\[
i\partial_t \hat{U}(t; s) = \hat{L}(t) \hat{U}(t; s) \quad \text{and} \quad \hat{U}(s; s) = 1.
\]

(3.15)

Since $\hat{L}$ does not change the parity of the number of particles,
\[
\langle \Omega, \hat{U}^*(t; 0) a_y \hat{U}(t; 0) \Omega \rangle \leq \langle \Omega, \hat{U}^*(t; 0) a_x^* \hat{U}(t; 0) \Omega \rangle = 0.
\]

(3.16)

**Remark 3.1.** Note that in (3.15), $\hat{L}_2 = L_2$ for $\beta = 0$. However, for $0 < \beta < 1/3$, formally the last term of $\hat{L}_2$ converges to 0 as $N \to \infty$. So we can think of $\hat{L}_2$ as $L_2$ without its last term. This explains the jump of the rate of convergence in (1.6).

Now, we have the following bounds for $E_t^{(1)}(J)$ and $E_t^{(2)}(J)$, which will be defined and Proposition 3.1 and Proposition 3.2. They will be employed in the proof of Theorem 1.1.

**Proposition 3.1.** Suppose that the assumptions in Theorem 1.1 hold. For any compact Hermitian operator $J$ on $L^2(\mathbb{R}^3)$, let
\[
E_t^{(1)}(J) := \frac{d}{dN} \left( \sum_{\mathbb{R}^3} W^*(\sqrt{N} u_0) \frac{(a_x^* (u_0))^N}{\sqrt{N}!} \Omega, \hat{U}_t^*(t; 0) d\Gamma(J) \hat{U}(t; 0) \Omega \right).
\]

Then, there exist constants $C$ depending only on $\|w\|_{L^1(\mathbb{R}^3)}, \|w\|_{L^2(\mathbb{R}^3)}, \|V\|_{L^\infty} \text{ and } C^V$ such that
\[
|E_t^{(1)}(J)| \leq \frac{C\|J\|}{N}.
\]

**Proposition 3.2.** Suppose that the assumptions in Theorem 1.1 hold. For any compact Hermitian operator $J$ on $L^2(\mathbb{R}^3)$, let
\[
E_t^{(2)}(J) := \frac{d}{dN} \left( \sum_{\mathbb{R}^3} W^*(\sqrt{N} u_0) \frac{(a_x^* (u_0))^N}{\sqrt{N}!} \Omega, \hat{U}_t^*(t; 0) \phi(J \varphi_t) \hat{U}(t; 0) \Omega \right).
\]

Then, there exist constants $C$ depending only on $\|w\|_{L^1(\mathbb{R}^3)}, \|w\|_{L^2(\mathbb{R}^3)}, \|V\|_{L^\infty} \text{ and } C^V$ such that
\[
|E_t^{(2)}(J)| \leq \begin{cases} 
C\|J\|N^{-1} & \text{if } \beta = 0 \\
C\|J\|N^{-\frac{1+3\beta}{2}} & \text{if } 0 < \beta < 1/3
\end{cases}
\]

The proofs of these propositions will be given in Section 5.

*Proof of Theorem 1.1.* For any $0 \leq \beta < 1/3$ let $\varphi_t$ be the solution to (recall: $w_N(x) := N^{3\beta} w(N^\beta x)$)
\[
\begin{aligned}
i\partial_t \varphi_t &= (-\Delta + V) \varphi_t + \lambda (w_N * |\varphi_t|^2) \varphi_t \\
|\varphi_t|_{t=0} &= u_0.
\end{aligned}
\]

(3.17)
We will first show that
\[
\text{Tr} \left| \gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| \right| \leq \begin{cases} CN^{-1} & \text{if } \beta = 0 \\ CN^{-1+3\beta} & \text{if } 0 < \beta < 1/3. \end{cases}
\]
Later we combine this estimate with the estimate
\[
\text{Tr} \left| |\varphi_t\rangle\langle\varphi_t| - |u_t\rangle\langle u_t| \right| \leq 2\|\varphi_t - u_t\|_2 \leq CN^{-\beta},
\]
where \(u_t\) is a solution of (Hartree) or (NLS).
Recall that
\[
\gamma_{N,t}^{(1)}(x; y) = \frac{1}{N} \langle \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, e^{iH_N t} a_y^* a_x e^{-i\hat{H}_N t} \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega \rangle.
\]
From the definition of the creation operator in (2.13) and the definition of \(d_N\) in (2.22), we can easily find that
\[
\{0, 0, \ldots, 0, a_0^{\odot N}, 0, \ldots\} = \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega,
\]
where the \(a_0^{\odot N}\) on the left-hand side is in the \(N\)-particle sector of the Fock space. Recall that \(P_N\) is the projection onto the \(N\)-particle sector of the Fock space. From (2.22), we find that
\[
\frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega = \frac{\sqrt{N!}}{N^{N/2}e^{-N/2}} P_N W(\sqrt{N} u_0) \Omega = d_N P_N W(\sqrt{N} u_0) \Omega.
\]
Since \(\hat{H}_N\) does not change the number of particles, we also have that
\[
\gamma_{N,t}^{(1)}(x; y) = \frac{d_N}{N} \langle \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, e^{iH_N t} a_y^* a_x e^{-i\hat{H}_N t} \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega \rangle
\]
\[
= \frac{d_N}{N} \langle \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, e^{iH_N t} a_y^* a_x e^{-i\hat{H}_N t} P_N W(\sqrt{N} u_0) \Omega \rangle
\]
\[
= \frac{d_N}{N} \langle \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, P_N e^{iH_N t} a_y^* a_x e^{-i\hat{H}_N t} W(\sqrt{N} u_0) \Omega \rangle
\]
\[
= \frac{d_N}{N} \langle \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, e^{iH_N t} a_y^* a_x e^{-i\hat{H}_N t} W(\sqrt{N} u_0) \Omega \rangle,
\]
where we used that \(P_N \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega = \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega\) in the last step. To simplify it further, we use the relation
\[
e^{iH_N t} a_x e^{-i\hat{H}_N t} = W(\sqrt{N} u_0) \mathcal{U}^*(t; 0)(a_x + \sqrt{N} \varphi(t)) \mathcal{U}(t; 0) W^*(\sqrt{N} u_0),
\]
which follows from the first equality in Lemma (2.24), the definition of \(\mathcal{U}\) in (3.11) and the unitary of the Weyl operator, see Lemma (2.22). By (3.20) and an analogous result for the creation operator, we obtain that
\[
\gamma_{N,t}^{(1)}(x; y) = \frac{d_N}{N} \langle \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, e^{iH_N t} a_y^* a_x e^{-i\hat{H}_N t} W(\sqrt{N} u_0) \Omega \rangle
\]
\[
= \frac{d_N}{N} \langle \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, W(\sqrt{N} u_0) \mathcal{U}^*(t; 0) \left( a_y^* + \sqrt{N} \varphi(t) \right) \left( a_x + \sqrt{N} \varphi(t) \right) \mathcal{U}(t; 0) \Omega \rangle.
\]
Since the space of compact operators is the dual to the space of the trace class operators, and since
\[
\phi_t(y)\varphi_t(x) = \frac{d_N}{N} \left( \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, W(\sqrt{N}u_0)\mathcal{U}^*(t;0)a_0\mathcal{U}(t;0)\Omega \right)
\]
\[
+ \frac{d_N}{\sqrt{N!}} \left( \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, W(\sqrt{N}u_0)\mathcal{U}^*(t;0)a_0\mathcal{U}(t;0)\Omega \right)
\]
\[
+ \frac{d_N}{\sqrt{N!}} \left( \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, W(\sqrt{N}u_0)\mathcal{U}^*(t;0)a_0\mathcal{U}(t;0)\Omega \right).
\]
(3.21)

Recall the definition of \(E_t^{(1)}(J)\) and \(E_t^{(2)}(J)\) in Propositions 3.1 and 3.2. For any compact one-particle
Hermitian operator \(J\) on \(L^2(\mathbb{R}^3)\), we have
\[
\text{Tr} \left( J \left( \gamma_{N,t}^{(1)} - \varphi_t \langle \varphi_t \rangle \right) \right) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy \ J(x,y) \left( \gamma_{N,t}^{(1)}(y; x) - \varphi_t(y)\varphi_t(x) \right)
\]
\[
= \frac{d_N}{N} \left( \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, W(\sqrt{N}u_0)\mathcal{U}^*(t;0)d\Gamma(J)\mathcal{U}(t;0)\Omega \right)
\]
\[
+ \frac{d_N}{\sqrt{N!}} \left( \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, W(\sqrt{N}u_0)\mathcal{U}^*(t;0)\phi(J)\mathcal{U}(t;0)\Omega \right)
\]
\[
= E_t^{(1)}(J) + E_t^{(2)}(J).
\]

The second step can be seen by using the expression for \(d\Gamma(J)\) in terms of operator-valued distributions in
(2.19), (3.21), the definition of annihilation and creation operators in terms of operator-valued distributions in
(2.16) and the definition of \(\phi\) in (2.15). Thus, from Propositions 3.1 and 3.2 we find that
\[
\left| \text{Tr} \left( J \left( \gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle \varphi_t| \right) \right) \right| \leq \begin{cases} C||J||N^{-1} & \text{if } \beta = 0 \\ C||J||N^{-1+\beta} & \text{if } 0 < \beta < 1/3. \end{cases}
\]
(3.22)

Since the space of compact operators is the dual to the space of the trace class operators, and since \(\gamma_{N,t}^{(1)}\) and
\(|\varphi_t\rangle\langle \varphi_t|\) are Hermitian and trace class, there exists an orthonormal basis \((f_k)_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^3)\) and a sequence
of real numbers \((s_k)_{k \in \mathbb{N}} \subset \mathbb{R}\) with \(\sum_{k \in \mathbb{N}} s_k < \infty\) such that
\[
\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle \varphi_t| = \sum_{k \in \mathbb{N}} s_k |f_k\rangle\langle f_k|.
\]
(3.23)

Now for any \(K \in \mathbb{N}\) choose the Hermitian compact operator
\[
J_K := \sum_{k=1}^K \text{sign}(s_k) |f_k\rangle\langle f_k|,
\]
(3.24)
and note that \(|J_K|| < 1\). We have
\[
\text{Tr} \left| \gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle \varphi_t| \right| = \sum_{k \in \mathbb{N}} s_k = \lim_{K \to \infty} \sum_{k=1}^K s_k = \lim_{K \to \infty} \left| \text{Tr} \left( J_K \left( \gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle \varphi_t| \right) \right) \right|.
\]
(3.25)

Combining this with (3.22), we obtain
\[
\text{Tr} \left| \gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle \varphi_t| \right| \leq \begin{cases} CN^{-1} & \text{if } \beta = 0 \\ CN^{(3\beta-1)/2} & \text{if } 0 < \beta < 1/3, \end{cases}
\]
which shows the theorem for $\beta = 0$ because $\varphi_t = u_t$ for $\beta = 0$.

Now we estimate

$$\text{Tr} \left| \varphi_t \langle \varphi_t | - | u_t \rangle \langle u_t | \right|.$$ 

First by the triangle inequality and the Cauchy-Schwarz inequality we have

$$\text{Tr} \left| \varphi_t \langle \varphi_t | - | u_t \rangle \langle u_t | \right| = \text{Tr} \left| \varphi_t \langle \varphi_t | - | u_t \rangle \langle \varphi_t | + | u_t \rangle \langle \varphi_t | - | u_t \rangle \langle u_t | \right|$$

$$= \text{Tr} \left| \varphi_t \langle \varphi_t | - u_t \rangle | + \varphi_t - u_t \rangle \langle u_t | \right|$$

$$\leq 2 \| \varphi_t - u_t \|_2.$$ 

In order to conclude, we need to show

$$\| \varphi_t - u_t \|_2 \leq CN^{-\beta}.$$ 

This follows from Lemma [6.1] in Section sec:comparison-of-one-body-dynamics below. 

\[\square\]

4. Truncation Dynamics

First, we introduce a truncated time-dependent generator with fixed $M > 0$ as follows:

$$L_N^{(M)} (t) = \int_{\mathbb{R}^3} dx a_x^* (-\Delta_x + V(x)) a_x$$

$$+ \lambda \int_{\mathbb{R}^3} dx \left( w_N * | \varphi_t |^2 \right) (x) a_x^* a_x + \lambda \int_{\mathbb{R}^3} dx dy w_N(x-y) \overline{\varphi_t(x)} \varphi_t(y) a_x^* a_x$$

$$+ \frac{\lambda}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy w_N(x-y) \left( \varphi_t(x) \varphi_t(y) a_x^* a_x + \overline{\varphi_t(x)} \varphi_t(y) a_x a_x \right)$$

$$+ \frac{\lambda}{\sqrt{N}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy w_N(x-y) a_x^* \left( \varphi_t(y) a_y \chi(N \leq M) + \varphi_t(y) \chi(N \leq M) a_y^* \right)$$

$$+ \frac{\lambda}{2N} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy w_N(x-y) a_x^* a_y a_x.$$ 

We remark that $M$ will be chosen to be $M = N^{1-3\beta}$ later in the proof of Lemma [5.1]. Define a unitary operator $U_N^{(M)}$ by

$$i \frac{d}{dt} U_N^{(M)} (t; s) = L_N^{(M)} (t) U_N^{(M)} (t; s) \quad \text{and} \quad U_N^{(M)} (s; s) = 1.$$ (4.1)

**Lemma 4.1.** Suppose that the assumptions in Theorem [7.1] hold and let $U_N^{(M)}$ be the unitary operator defined in (4.1). Then, for any $j \in \mathbb{N}$ there exists a constant $K = K(C_0, \| V \|_{W^{2,\infty}}, C_V, \| w \|_1, \| w \|_2, j) > 0$ such that for all $N \in \mathbb{N}$, $M > 0$, $\psi \in \mathcal{F}$, and $t, s \in \mathbb{R}$,

$$\langle U_N^{(M)} (t; s) \psi, (N + 1)^j U_N^{(M)} (t; s) \psi \rangle \leq C \langle \psi, (N + 1)^j \psi \rangle \exp \left( K \left( 1 + \sqrt{M/N^{1-3\beta}} \right) \right).$$

**Proof.** Following the proof of [32] Lemma 3.5, see [32] (3.15), we have

$$\frac{d}{dt} \langle U_N^{(M)} (t; 0) \psi, (N + 1)^j U_N^{(M)} (t; 0) \psi \rangle = \lambda (A + B),$$ (4.2)
where

\[ A := 2 \sum_{k=0}^{j-1} \binom{j}{k} (-1)^k \text{Im} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy w_N(x - y)\varphi_t(x)\varphi_t(y) \]

\[ \times \left( \mathcal{U}_N^{(M)}(t; 0)\psi, \left( N^{k/2}a_x^*a_y^* (N + 2)^{k/2} + (N + 1)^{k/2}a_x^*a_y^* (N + 3)^{k/2} \right) \mathcal{U}^{(M)}_N(t; 0)\psi \right) \]

\[ B := \frac{2}{\sqrt{N}} \sum_{k=0}^{j-1} \binom{j}{k} \text{Im} \int_{\mathbb{R}^3} dx \left( \mathcal{U}_N^{(M)}(t; 0)\psi, a_x^*a(w_N(x - \cdot)\varphi_t) \chi(\mathcal{N} \leq M) (N + 1)^{k/2}a_x N^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \right). \]

To control the contribution from the first term on the right-hand side of (4.2), we use bounds of the form

\[ \left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy w_N(x - y)\varphi_t(x)\varphi_t(y) \left( \mathcal{U}_N^{(M)}(t; 0)\psi, (N + 1)^{k/2}a_x^*a_y^* (N + 3)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \right) \right| \]

\[ \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy \left| w_N(x - y)\varphi_t(x)\varphi_t(y) \right| \left| a_x (N + 1)^{k/2}a_y^* \mathcal{U}^{(M)}_N(t; 0)\psi \right| \left| a_y (N + 3)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \right| \]

\[ \leq \| w_N \|_1 \| \varphi_t \|_\infty^2 \left( \int_{\mathbb{R}^3} dx \left| a_x (N + 1)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \right|^2 \right)^{1/2} \left( \int_{\mathbb{R}^3} dy \left| a_y (N + 3)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \right|^2 \right)^{1/2} \]

\[ \leq \| w_N \|_1 \| \varphi_t \|_\infty^2 \| (N + 3)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \|^2. \]

Here we used Young’s inequality in the second step and the definition of the number operator in (2.17) and \( \| w_N \|_1 = \| w \|_1 \) in the last step.

On the other hand, to control the second integral on the right-hand side of (4.2), we use that

\[ \left| \int_{\mathbb{R}^3} dx \left( \mathcal{U}_N^{(M)}(t; 0)\psi, a_x^*a(w_N(x - \cdot)\varphi_t) \chi(\mathcal{N} \leq M)(N + 1)^{k/2}a_x N^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \right) \right| \]

\[ \leq \int_{\mathbb{R}^3} dx \left| a_x (N + 1)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \right| \left| a_x^*a(w_N(x - \cdot)\varphi_t) \chi(\mathcal{N} \leq M) \right| \left| a_x N^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \right| \]

\[ \leq M^{1/2} \sup_x \| w_N(x - \cdot)\varphi_t \| \| (N + 1)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \|^2 \]

\[ \leq M^{1/2} \| w_N \|_2 \| \varphi_t \|_\infty \| (N + 1)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \|^2 \]

\[ \leq M^{1/2} N^{3/2} \| w_N \|_2 \| \varphi_t \|_\infty \| (N + 1)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \|^2. \]

Using |\( \lambda \)| \( \leq 1 \), this gives us that

\[ \left| \frac{d}{dt} \mathcal{U}_N^{(M)}(t; 0)\psi, (N + 1)^{k/2} \mathcal{U}_N^{(M)}(t; 0)\psi \right| \]

\[ \leq 2 \sum_{k=0}^{j-1} \binom{j}{k} 2 \| w \|_1 \| \varphi_t \|_\infty^2 \| (N + 3)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \|^2 \]

\[ + \frac{2}{\sqrt{N}} \sum_{k=0}^{j-1} \binom{j}{k} M^{1/2} N^{3/2} \| w \|_2 \| \varphi_t \|_\infty \| (N + 1)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \|^2 \]

\[ \leq 2 \sum_{k=0}^{j-1} \binom{j}{k} \left( 2 \| w \|_1 \| \varphi_t \|_\infty^2 + \frac{1}{\sqrt{N}} M^{1/2} N^{3/2} \| w \|_2 \| \varphi_t \|_\infty \right) \| (N + 3)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \|^2 \]

\[ \leq 2 \cdot 2^j \left( 2 \| w \|_1 \| \varphi_t \|_\infty^2 + \left( \frac{M}{N^{1/3}} \right)^{1/2} \| w \|_2 \| \varphi_t \|_\infty \right) \right) \| (N + 1)^{k/2} \mathcal{U}^{(M)}_N(t; 0)\psi \|^2 \]
Applying the Grönwall lemma together with Proposition 2.1, we get

\[
\langle U_N^{(M)}(t; 0)\psi, (N + 1)^2U_N^{(M)}(t; 0)\psi \rangle.
\]

Lemma 5.2 \([82, \text{Lemma } 3.6]\) such that

\[
0 \leq 2 \cdot 6^j \left( 2\|w\|_1 \|\varphi_s\|_\infty^2 + \left( \frac{M}{N^{1-3\beta}} \right)^{1/2} \|w\|_2 \|\varphi_s\|_\infty \right) \langle U_N^{(M)}(t; 0)\psi, (N + 1)^2U_N^{(M)}(t; 0)\psi \rangle.
\]

Applying the Grönwall lemma together with Proposition 2.1 we get

\[
\langle U_N^{(M)}(t; 0)\psi, (N + 1)^2U_N^{(M)}(t; 0)\psi \rangle
\]

\[
\leq \langle \psi, (N + 1)^j\psi \rangle \exp \left( \int_0^t \text{d}s \cdot 6^j \left( 2\|w\|_1 \|\varphi_s\|_\infty^2 + \left( \frac{M}{N^{1-3\beta}} \right)^{1/2} \|w\|_2 \|\varphi_s\|_\infty \right) \right)
\]

\[
\leq \langle \psi, (N + 1)^j\psi \rangle \exp \left( \int_0^t \text{d}s \cdot 6^j \left( 2\|w\|_1 \|C_0(1 + |s|)^{-3} + \left( \frac{M}{N^{1-3\beta}} \right)^{1/2} \|w\|_2 \|C_0(1 + |s|)^{-3/2} \right) \right)
\]

\[
\leq \langle \psi, (N + 1)^j\psi \rangle \exp \left( K \left( 1 + \left( \frac{M}{N^{1-3\beta}} \right)^{1/2} \right) \right)
\]

This gives us the desired lemma. \(\square\)

5. Comparison of Dynamics

The main goal of this section is to provide important lemmata to prove Proposition 3.1 and 3.2.

**Lemma 5.1.** Suppose that the assumptions in Theorem 1.1 hold. Let \(U(t; s)\) be the unitary evolution defined in (3.12). Then for any \(\psi \in F\) and \(j \in N\), there exists a constant \(C \equiv C(j, C_0, \|V\|_{L^\infty}, C_V, \|w\|_1, \|w\|_2) > 0\) such that

\[
\langle U(t; s)\psi, N^jU(t; s)\psi \rangle \leq C \langle \psi, (N + 1)^{2L_{j, \beta} + 2}\psi \rangle,
\]

where

\[
L_{j, \beta} := \left\lceil \frac{j + (3\beta/2)}{1 - 3\beta} \right\rceil .
\]

In this section we modify the proof given in previous articles, for example, [82] or [14] and the references therein. In [14, 82], the Hardy inequality was used to cover singular interaction potentials. However, here we use Hölder’s inequality and Proposition 2.1. Moreover, we need to take into account the scaling of the interaction potential for \(\beta > 0\).

We now begin the proof of Lemma 5.1. To prove the lemma, we compare the dynamics of \(U\) and \(U^{(M)}\) in Lemma 5.3. To do so, we recall weak bounds on the \(U\) dynamics.

**Lemma 5.2 ([82 Lemma 3.6]).** For arbitrary \(t, s \in \mathbb{R}\) and \(\psi \in F\), we have

\[
\langle \psi, U(t; s)NU(t; s)\psi \rangle \leq 6 \langle \psi, (N + N + 1)\psi \rangle.
\]

Moreover, for every \(\ell \in N\), there exists a constant \(C(\ell)\) such that

\[
\langle \psi, U(t; s)NU(t; s)\psi \rangle \leq C(\ell) \langle \psi, (N + N)^{2\ell}\psi \rangle,
\]

\[
\langle \psi, U(t; s)NU(t; s)\psi \rangle \leq C(\ell) \langle \psi, (N + N)^{2\ell+1}(N + 1)\psi \rangle
\]

for all \(t, s \in \mathbb{R}\) and \(\psi \in F\).

**Proof.** The proof can be found in [82]. \(\square\)

Now we are ready to compare the dynamics of \(U\) and \(U^{(M)}\).
Lemma 5.3. Suppose that the assumptions in Theorem \[\text{[1.1]}\] hold. Then, for every \(j \in \mathbb{N}\) and \(\psi \in \mathcal{F}\), there exists a constant \(C \equiv C(j, C_0, |V|_{W^{2,\infty}}, C_1^v, \|w\|_1, \|w\|_2) > 0\) such that for all \(t, s \in \mathbb{R}\)

\[
\left| \langle \mathcal{U}(t; s)\psi, \mathcal{N}^j \mathcal{U}(t; s) - \mathcal{U}_N^{(M)}(t; s)\psi \rangle \right| \\
\leq C(N^{1-3\beta}/M)^{L_{i,\beta}} \|(N + 1)^{L_{i,\beta}+1}\psi\|^2 \exp \left( K(1 + \sqrt{M/N^{1-3\beta}}) \right) \quad (5.1)
\]

and

\[
\left| \langle \mathcal{U}_N^{(M)}(t; 0)\psi, \mathcal{N}^j \mathcal{U}(t; 0) - \mathcal{U}_N^{(M)}(t; 0)\psi \rangle \right| \\
\leq C(N^{1-3\beta}/M)^{L_{i,\beta}} \|(N + 1)^{L_{i,\beta}+1}\psi\|^2 \exp \left( K(1 + \sqrt{M/N^{1-3\beta}}) \right) \quad (5.2)
\]

Proof. To simplify the notation, we consider the case \(s = 0\) and \(t > 0\) only; other cases can be treated in a similar manner. To prove the first inequality of the lemma, we expand the difference of the two evolutions as follows:

\[
\langle \mathcal{U}(t; 0)\psi, \mathcal{N}^j \mathcal{U}(t; 0) - \mathcal{U}_N^{(M)}(t; 0)\psi \rangle = \langle \mathcal{U}(t; 0)\psi, \mathcal{N}^j \mathcal{U}(t; 0)(1 - \mathcal{U}^*(t; 0)\mathcal{U}_N^{(M)}(t; 0))\psi \rangle
\]

\[
= -i \int_0^t ds \langle \mathcal{U}(t; 0)\psi, \mathcal{N}^j \mathcal{U}(t; 0)(\mathcal{L}_N(s) - \mathcal{L}_N^{(M)}(s))\mathcal{U}_N^{(M)}(s; 0)\psi \rangle
\]

\[
= -\frac{i\lambda}{\sqrt{N}} \int_0^t ds \int_{\mathbb{R}^3} dx dy w_N(x - y)
\times \langle \mathcal{U}(t; 0)\psi, \mathcal{N}^j \mathcal{U}(t; s)a^*_x(\chi N > M + \varphi_s(y)\chi(N > M))a_x\mathcal{U}_N^{(M)}(s; 0)\psi \rangle
\]

\[
= -\frac{i\lambda}{\sqrt{N}} \int_0^t ds \int_{\mathbb{R}^3} dx a_x\mathcal{N}^j \mathcal{U}(t; 0)\psi, a(w_N(x - \cdot)\varphi_s)\chi(N > M)a_x\mathcal{U}_N^{(M)}(s; 0)\psi \rangle
\]

\[
- \frac{i\lambda}{\sqrt{N}} \int_0^t ds \int_{\mathbb{R}^3} dx a_x\mathcal{N}^j \mathcal{U}(t; 0)\psi, \chi(N > M)a^*(w_N(x - \cdot)\varphi_s)a_x\mathcal{U}_N^{(M)}(s; 0)\psi \rangle.
\]

Hence, using \(|\lambda| \leq 1\),

\[
\left| \langle \mathcal{U}(t; 0)\psi, \mathcal{N}^j \mathcal{U}(t; 0) - \mathcal{U}_N^{(M)}(t; 0)\psi \rangle \right|
\leq \frac{1}{\sqrt{N}} \int_0^t ds \int_{\mathbb{R}^3} dx \|a_x\mathcal{N}^j \mathcal{U}(t; s)\mathcal{N}^j \mathcal{U}(t; 0)\psi\|
\times \|a(w_N(x - \cdot)\varphi_s)a_x\chi(N > M + 1)\mathcal{U}_N^{(M)}(s; 0)\psi\|
\]

\[
+ \frac{1}{\sqrt{N}} \int_0^t ds \int_{\mathbb{R}^3} dx \|a_x\mathcal{N}^j \mathcal{U}(t; s)\mathcal{N}^j \mathcal{U}(t; 0)\psi\|
\times \|a^*(w_N(x - \cdot)\varphi_s)a_x\chi(N > M)\mathcal{U}_N^{(M)}(s; 0)\psi\|
\]

\[
\leq \frac{2}{\sqrt{N}} \int_0^t ds \sup_x \|w_N(x - \cdot)\varphi_s\|
\times \int_{\mathbb{R}^3} dx \|a_x\mathcal{N}^j \mathcal{U}(t; s)\mathcal{N}^j \mathcal{U}(t; 0)\psi\| \|a_x(N + 1)^{1/2}\chi(N > M)\mathcal{U}_N^{(M)}(s; 0)\psi\|
\]

\[
\leq \frac{2}{\sqrt{N}} \int_0^t ds \sup_x \|w_N(x - \cdot)\varphi_s\|
\times \|\mathcal{N}^{1/2}\mathcal{U}^*(t; s)\mathcal{N}^j \mathcal{U}(t; 0)\psi\| \|(N + 1)\chi(N > M)\mathcal{U}_N^{(M)}(s; 0)\psi\|. \quad (5.3)
\]
Since $\chi(\mathcal{N} > M) \leq (\mathcal{N}/M)^L$ for any $L > 1$, we find that, from [82] (3.30),

\[
\left| \mathcal{U}(t; 0)\psi, \mathcal{N}^2(\mathcal{U}(t; 0) - \mathcal{U}^{(M)}(t; 0))\psi \right|
\leq CN^j \| (\mathcal{N} + 1)^{j+1}\psi \|
\times \int_0^t ds \sup_x \| w_N(x - \cdot)\varphi_s \|_2 (\mathcal{U}^{(M)}_N(s; 0)\psi, (\mathcal{N} + 1)^2\chi(\mathcal{N} > M)\mathcal{U}^{(M)}_N(s; 0)\psi)^{1/2}
\leq CN^j \| (\mathcal{N} + 1)^{j+1}\psi \|
\times \int_0^t ds N^{3\beta/2} \| \varphi_s \|_\infty (\mathcal{U}^{(M)}_N(s; 0)\psi, (\mathcal{N} + 1)^2\chi(\mathcal{N} > M)\mathcal{U}^{(M)}_N(s; 0)\psi)^{1/2}
\leq CN^{j+(3\beta/2)} \| (\mathcal{N} + 1)^{j+1}\psi \| (\mathcal{U}^{(M)}_N(s; 0)\psi, (\mathcal{N} + 1)^2\chi(\mathcal{N} > M)\mathcal{U}^{(M)}_N(s; 0)\psi)^{1/2} \int_0^t ds \| \varphi_s \|_\infty
\]

where

\[
C = C(j, \| V \|_{W^{2, \infty}}, C^V, \| w \|_1, \| w \|_2)
\]

and

\[
L_{j, \beta} := \left\lfloor \frac{j + (3\beta/2)}{1 - 3\beta} \right\rfloor.
\]

By Lemma 5.1 $L_{j, \beta} := j + (3\beta/2)$ and $\| \varphi_s \|_\infty \leq C(1 + s)^{3/2}$, see Proposition 2.1, we conclude that

\[
\left| \mathcal{U}(t; 0)\psi, \mathcal{N}^2(\mathcal{U}(t; 0) - \mathcal{U}^{(M)}(t; 0))\psi \right|
\leq C(N/M)^{j+(3\beta/2)} \| (\mathcal{N} + 1)^{L_{j, \beta}+1}\psi \|^2 \exp \left( K(1 + \sqrt{M/N^{1-3\beta}}) \right) \int_0^t ds (1 + s)^{3/2}
\leq C(N/M)^{j+(3\beta/2)} \| (\mathcal{N} + 1)^{L_{j, \beta}+1}\psi \|^2 \exp \left( K(1 + \sqrt{M/N^{1-3\beta}}) \right)
\leq C(N^{1-3\beta}/M)^{L_{j, \beta}} \| (\mathcal{N} + 1)^{L_{j, \beta}+1}\psi \|^2 \exp \left( K(1 + \sqrt{M/N^{1-3\beta}}) \right).
\]

To prove (5.2), we proceed similarly; analogously to (5.3) we find

\[
\mathcal{U}^{(M)}_N(t; 0)\psi, \mathcal{N}^2 \left( \mathcal{U}(t; 0) - \mathcal{U}^{(M)}_N(t; 0) \right)\psi
\]

\[
= -\frac{i\lambda}{\sqrt{N}} \int_0^t ds \int_{\mathbb{R}^3} dx \langle a_s \mathcal{U}(t; s)^* N^2 \mathcal{U}^{(M)}_N(t; 0)\psi, a(w_N(x - \cdot)\varphi_s)\chi(\mathcal{N} > M)a_s \mathcal{U}^{(M)}_N(s; 0)\psi \rangle
\]

\[
-\frac{i\lambda}{\sqrt{N}} \int_0^t ds \int_{\mathbb{R}^3} dx \langle a_s \mathcal{U}(t; s)^* N^2 \mathcal{U}^{(M)}_N(t; 0)\psi, \chi(\mathcal{N} > M)a^*(w_N(x - \cdot)\varphi_s)a_s \mathcal{U}^{(M)}_N(s; 0)\psi \rangle
\]

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and thus, by $|\lambda| \leq 1$, $\chi(N > M) \leq (\frac{\lambda}{M})^L$ for any $L > 1$, $\|w_N\|_2 = N^{3\beta/2} \|w\|_2$ and Proposition 2.1

$$\left\| U_N^{(M)}(t; 0)\psi_N^j \left( U(t; 0) - U_N^{(M)}(t; 0) \right) \psi \right\|
\leq \frac{C}{\sqrt{N}} \int_0^t ds \sup_x \|w_N(x - \cdot)\|_2 \|N^{1/2}U(t; x)N^jU_N^{(M)}(t; 0)\psi\| \|N\chi(N > M)U_N^{(M)}(s; 0)\psi\|
\leq \frac{C}{\sqrt{N}} \int_0^t ds \|w_N\|_2 \|\|N\|\| \|N^{1/2}U(t; s)N^jU_N^{(M)}(t; 0)\psi\| \|N\chi(N > M)U_N^{(M)}(s; 0)\psi\|
\leq \frac{C}{M^{L_j,\beta} \sqrt{N^{1-3\beta}}} \|w\|_2 \int_0^t ds C_0(1 + |s|)^{-3/2} \|N^{1/2}U(t; s)N^jU_N^{(M)}(t; 0)\psi\| \|N^{L_j,\beta+1}U_N^{(M)}(s; 0)\psi\|
. \hspace{1cm} (5.4)$$

By Lemma 5.2, Lemma 4.1 and $j + 1/2 \leq L_j,\beta + 1$, we have

$$\|N^{1/2}U(t; s)N^jU_N^{(M)}(t; 0)\psi\| \leq 6\|(N + N + 1)^{1/2}N^{j+1/2}U_N^{(M)}(t; 0)\psi\|
\leq 12N^{j+1/2}\|(N + 1)N^{j+1/2}U_N^{(M)}(t; 0)\psi\|
\leq CN^{j+1/2}\|(N + 1)N^{j+1/2}\psi\| \exp \left(\frac{K}{1 + \sqrt{M/N^{1-3\beta}}}\right)
\leq CN^{j+1/2}\|(N + 1)^{L_j,\beta+1}\psi\| \exp \left(\frac{K}{1 + \sqrt{M/N^{1-3\beta}}}\right). \hspace{1cm} (5.5)$$

By Lemma 4.1, we have

$$\|N^{L_j,\beta+1}U_N^{(M)}(s; 0)\psi\| \leq C\|(N + 1)^{L_j,\beta+1}\psi\| \exp \left(\frac{K}{1 + \sqrt{M/N^{1-3\beta}}}\right). \hspace{1cm} (5.6)$$

Combining (5.4), (5.5) and (5.6), we obtain

$$\left\| U_N^{(M)}(t; 0)\psi_N^j \left( U(t; 0) - U_N^{(M)}(t; 0) \right) \psi \right\|
\leq \frac{C}{M^{L_j,\beta} \sqrt{N^{1-3\beta}}} \|N^{j+1/2}\|(N + 1)^{L_j,\beta+1}\psi\|^2 \exp \left(\frac{2K}{1 + \sqrt{M/N^{1-3\beta}}}\right)
\leq \frac{C}{M^{L_j,\beta} \sqrt{N^{1-3\beta}}} \|N^{j+3\beta/2}\|(N + 1)^{L_j,\beta+1}\psi\|^2 \exp \left(\frac{K}{1 + \sqrt{M/N^{1-3\beta}}}\right)
\leq \frac{C}{M^{L_j,\beta} \sqrt{N^{1-3\beta}}} \|(N + 1)^{L_j,\beta+1}\psi\|^2 \exp \left(\frac{K}{1 + \sqrt{M/N^{1-3\beta}}}\right), \hspace{1cm} (5.7)$$

where we used $j + 3\beta/2 \leq (1 - 3\beta)L_j,\beta$ in the last step. Again, we have

$$C = C(j, C_0, \|V\|_{W^{2,\infty}}, C^V, \|w\|_1, \|w\|_2).$$

This gives us the desired lemma.

Let us now prove Lemma 5.1.

Proof of Lemma 5.1. Let $M = N^{1-3\beta}$. Then by Lemmata 4.1 and 5.3 we get

$$\left\langle U(t; s)\psi, N^j(U - U_N^{(M)}) (t; s) \psi \right\rangle
= \left\langle U(t; s)\psi, N^j(U - U_N^{(M)}) (t; s) \psi \right\rangle + \left\langle (U - U_N^{(M)}) (t; s)\psi, N^jU_N^{(M)} (t; s) \psi \right\rangle
+ \left\langle U_N^{(M)} (t; s)\psi, N^jU_N^{(M)} (t; s) \psi \right\rangle.$$
Proof. This leads us the desired result. □

Recall the definition of \( \hat{U}(t; s) \) in (3.15). In the next lemma, we prove an estimate for the evolution with respect to \( \hat{U} \).

**Lemma 5.4.** Suppose that the assumptions in Theorem 1.1 hold. Then, for any \( \psi \in \mathcal{F} \) and \( j \in \mathbb{N} \), there exists a constant \( C = C(C_0, \|V\|_{W^{2, \infty}}, C^V, \|w\|_1) > 0 \) such that

\[
\langle \hat{U}(t; s) \psi, N^j \hat{U}(t; s) \psi \rangle \leq C(\psi, (N + 1)^j \psi).
\]

**Proof.** Let \( \hat{\psi} = \hat{U}(t; s) \psi \) and assume without loss of generality that \( t \geq s \). We have

\[
\frac{d}{dt} \langle \hat{\psi}, (N + 1)^j \hat{\psi} \rangle = \langle \hat{\psi}, [(\hat{L}_2 + L_4), (N + 1)^j] \hat{\psi} \rangle
\]

\[
= -\frac{1}{N^{3/2}} \text{Im} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy \, w_N(x - y) \varphi_1(x) \varphi_1(y) \langle \hat{\psi}, [a_x^* a_y^*, (N + 1)^j] \hat{\psi} \rangle
\]

\[
= -\frac{1}{N^{3/2}} \text{Im} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy \, w_N(x - y) \varphi_1(x) \varphi_1(y) \langle \hat{\psi}, a_x^* a_y^* ((N + 3)^j - (N + 1)^j) \hat{\psi} \rangle
\]

\[
= \frac{1}{N^{3/2}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy \, w_N(x - y) \varphi_1(x) \varphi_1(y) \langle \hat{\psi}, a_x^* a_y^* ((N + 3)^j - (N + 1)^j) \hat{\psi} \rangle
\]

Then, using Young’s inequality, one gets

\[
\frac{d}{dt} \langle \hat{\psi}, (N + 1)^j \hat{\psi} \rangle \leq \frac{1}{N^{3/2}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy \, |w_N(x - y)| |\varphi_1(x)| |\varphi_1(y)|\langle (N + 3)^{(j-1)/2} a_x^* \hat{\psi} \rangle
\]

\[
\times \langle (N + 3)^{(j-1)/2} a_x^* \hat{\psi} \rangle
\]

\[
\leq \frac{1}{N^{3/2}} \langle \varphi_1 \rangle^2_\infty \left( \int_{\mathbb{R}^3} dx \, (N + 3)^{(j-1)/2} a_x^* \hat{\psi} \langle (N + 3)^{(j-1)/2} a_x^* \hat{\psi} \rangle \right)^{1/2}
\]

Since, for all \( j \in \mathbb{N} \), \( |(N + 3)^j - (N + 1)^j| \leq C_j (N + 1)^{-1} \) for some \( C_j > 0 \), we have that

\[
\frac{d}{dt} \langle \hat{U}(t; s) \psi, (N + 1)^j \hat{U}(t; s) \psi \rangle \leq \frac{C_j}{N^{3/2}} \langle \varphi_1 \rangle^2_\infty \langle (N + 1)^j \hat{U}(t; s) \psi \rangle^2
\]

\[
= \frac{C_j}{N^{3/2}} \langle \varphi_1 \rangle^2_\infty \langle \hat{U}(t; s) \psi, (N + 1)^j \hat{U}(t; s) \psi \rangle.
\]

Here, note that \( C_j \) can change from line to line. Applying Grönwall’s lemma with Proposition 2.1, we conclude that

\[
\langle \hat{U}(t; s) \psi, (N + 1)^j \hat{U}(t; s) \psi \rangle \leq \langle \psi, (N + 1)^j \psi \rangle \exp \left( \int_s^t \frac{C_j}{N^{3/2}} C_0^2 \frac{1}{(1 + |r|)^3} \right)
\]

\[
\leq C(\psi, (N + 1)^j \psi).
\]

Hence, we get the result. □
Lemma 5.5. For all $\psi \in \mathcal{F}$ and $f \in L^2(\mathbb{R}^3)$, we have the following inequalities with a constant $C = C(C_0, \|V\|_{W^{2,\infty}}, C^0, \|w\|_1, \|w\|_2) > 0$.

If $\beta = 0$,
\[
\|(N + 1)^{3/2} \mathcal{L}_3(t)\psi\| \leq \frac{C}{N} \|\varphi_t\|_{L^\infty} \|N + 1\|^{(j + 3)/2} \psi, \quad \tag{5.8}
\]
\[
\|(N + 1)^{3/2} (\mathcal{L}_2(t) - \mathcal{L}_2(t))\psi\| = 0, \quad \tag{5.9}
\]
\[
\|(N + 1)^{3/2} (\mathcal{U}^*(t; 0)\phi(f)\mathcal{U}(t; 0) - \mathcal{U}^*(t; 0)\phi(f)\mathcal{U}(t; 0))\Omega\| \leq \frac{C\|f\|}{N}. \quad \tag{5.10}
\]

If $0 < \beta < 1/3$,
\[
\|(N + 1)^{3/2} \mathcal{L}_3(t)\psi\| \leq \frac{C}{\sqrt{N^{1-3\beta}}} \|\varphi_t\|_{L^\infty} \|N + 1\|^{(j + 3)/2} \psi, \quad \tag{5.11}
\]
\[
\|(N + 1)^{3/2} (\mathcal{L}_2(t) - \mathcal{L}_2(t))\psi\| \leq C\|\varphi_t\|_{L^\infty} \|N + 1\|^{(j + 3)/2} \psi, \quad \tag{5.12}
\]
\[
\|(N + 1)^{3/2} (\mathcal{U}^*(t; 0)\phi(f)\mathcal{U}(t; 0) - \mathcal{U}^*(t; 0)\phi(f)\mathcal{U}(t; 0))\Omega\| \leq \frac{C\|f\|}{\sqrt{N^{1-3\beta}}}. \quad \tag{5.13}
\]

Proof. For (5.8) and (5.11), we copy the proof from [57] Lemma 5.3 with $\|w_N\|_2 = N^{3/2}/\|w\|_2$.

For (5.9) and (5.12), we follow the proof from [57] Lemma 5.3 with $\|w_N\|_1 = \|w\|_1$, and we replace $\mathcal{L}_3$ by
\[
\mathcal{L}_2 - \tilde{\mathcal{L}}_2 = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy w_N(x - y) (\varphi_t(x)\varphi_t(y) a_x^* a_y + \varphi_t(x)\varphi_t(y) a_x a_y).
\]

For (5.10) and (5.13), we follow the proof from [57] Lemma 5.4 with Lemma 5.4, (5.8), (5.11), (5.9), and (5.12).

Proof of Proposition 3.1. Recall that
\[
E_t^{(1)}(J) = \frac{d_N}{N} \left\langle W^*(\sqrt{N} u_0) \frac{(a^*(u_0))^N}{\sqrt{N}!} \Omega, \mathcal{U}^*(t; 0) d\Gamma(J) \mathcal{U}(t; 0)\Omega \right\rangle.
\]

We begin by noting that
\[
|E_t^{(1)}(J)| = \left| \frac{d_N}{N} \left\langle W^*(\sqrt{N} u_0) \frac{(a^*(u_0))^N}{\sqrt{N}!} \Omega, \mathcal{U}^*(t; 0) d\Gamma(J) \mathcal{U}(t; 0)\Omega \right\rangle \right| 
\]
\[
\leq \frac{d_N}{N} \left\| (N + 1)^{-\beta} W^*(\sqrt{N} u_0) \frac{(a^*(u_0))^N}{\sqrt{N}!} \Omega \right\| \left\| (N + 1)^{\beta} \mathcal{U}^*(t; 0) d\Gamma(J) \mathcal{U}(t; 0)\Omega \right\|.
\]

Recall that, by Lemma 2.3,
\[
\|(N + 1)^{-\beta} W^*(\sqrt{N} u_0) \frac{(a^*(u_0))^N}{\sqrt{N}!} \Omega \leq \frac{C}{d_N}
\]
and, by applying Lemma 5.1 twice,
\[
\|(N + 1)^{\beta} \mathcal{U}^*(t; 0) d\Gamma(J) \mathcal{U}(t; 0)\Omega \| \leq C \|(N + 1)^{J_{\beta, a+1}} d\Gamma(J) \mathcal{U}(t; 0)\Omega \|
\]
\[
\leq C \|J\| \|(N + 1)^{J_{\beta, a+2}} \mathcal{U}(t; 0)\Omega \| \leq C \|J\| \|(N + 1)^{J_{\beta, a+2, a+1}} \| = C \|J\|,
\]
we obtain
\[
|E_t^{(1)}(J)| \leq \frac{C\|J\|}{N},
\]
which is the desired result. □
Proof of Proposition 5.2 Let

\[ \mathcal{R}(f) = \mathcal{U}^*(t; 0) \phi(f) \mathcal{U}(t; 0) - \mathcal{U}^*(t; 0) \phi(f) \mathcal{U}(t; 0). \]

Then

\[
|E^{(2)}(J)| = \frac{dN}{\sqrt{N}} \left( W^* (\sqrt{N} u_0) \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, \mathcal{U}^*(t; 0) \phi(J \varphi_t) \mathcal{U}(t; 0) \Omega \right) \\
+ \frac{dN}{\sqrt{N}} \left( W^* (\sqrt{N} u_0) \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega, \mathcal{R}(J \varphi_t) \Omega \right) \\
\leq \frac{dN}{\sqrt{N}} \left\| \sum_{k=0}^{\infty} (N+1)^{-\frac{2}{5}} P_{2k+1} W^* (\sqrt{N} u_0) \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega \right\| \left\| (N+1)^{\frac{1}{4}} \mathcal{U}^*(t; 0) \phi(J \varphi_t) \mathcal{U}(t; 0) \Omega \right\| \\
+ \frac{dN}{\sqrt{N}} \left\| (N+1)^{-\frac{3}{4}} W^* (\sqrt{N} u_0) \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega \right\| \left\| (N+1)^{\frac{1}{2}} \mathcal{R}(J \varphi_t) \Omega \right\|.
\]

(5.14)

Let \( K = \frac{1}{2} N^{1/3} \). By Lemmata 2.3 and 2.4, we have

\[
\sum_{k=0}^{K} \left\| (N+1)^{-\frac{2}{5}} P_{2k+1} W^* (\sqrt{N} u_0) \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega \right\|^2 \\
\leq \sum_{k=0}^{K} \left\| (N+1)^{-\frac{2}{5}} P_{2k+1} W^* (\sqrt{N} u_0) \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega \right\|^2 \\
+ \frac{1}{K^2} \sum_{k=K}^{\infty} \left\| (N+1)^{-1/2} P_{2k+1} W^* (\sqrt{N} u_0) \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega \right\|^2 \\
\leq \left( \frac{K}{(k+1)^2 d_N^2 N} \right) + \frac{C}{N^{4/3}} \left\| (N+1)^{-1/2} W^* (\sqrt{N} u_0) \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega \right\|^2 \leq \frac{C}{d_N^2 N^{4/3}}.
\]

Using Lemma 5.4

\[
\left\| (N+1)^{\frac{1}{4}} \mathcal{U}^*(t; 0) \phi(J \varphi_t) \mathcal{U}(t; 0) \Omega \right\| \leq C \left\| (N+1)^{\frac{1}{4}} \phi(J \varphi_t) \mathcal{U}(t; 0) \Omega \right\| \\
\leq C \left\| (N+1)^{3/4} \phi(J \varphi_t) \mathcal{U}(t; 0) \Omega \right\| = C \left\| \phi(J \varphi_t) (N+2)^{3/4} \mathcal{U}(t; 0) \Omega \right\| \\
\leq C \left\| J \varphi_t \right\| \left\| (N+2)^{3/4} \mathcal{U}(t; 0) \Omega \right\| \leq C \left\| J \right\| \left\| (N+2)^{7/4} \Omega \right\| = C \left\| J \right\|.
\]

For the second term on the right-hand side of (5.14), we use Lemma 2.3 (5.10), and 5.13, for \( f = J \varphi_t \).

Altogether, we have

\[
\left\| (N+1)^{j/2} \mathcal{R}(f) \Omega \right\| \leq \begin{cases} 
C \left\| J \right\|^{N-1} & \text{if } \beta = 0 \\
C \left\| J \right\| N^{-1/3} & \text{if } 0 < \beta < 1/3
\end{cases}
\]

which is the desired conclusion.

\[ \square \]

6. Comparison of the One-Body Dynamics

In this section we prove an estimate for \( \| u_t - \varphi_t \| \) using Grönwall’s lemma and Proposition 2.1.
Lemma 6.1. Let \( u_t \) be the solution to
\[
\begin{align*}
    i\partial_t u_t &= (-\Delta + V)u_t + \lambda a|u_t|^2u_t \\
    u_t |_{t=0} &= u_0
\end{align*}
\] (6.1)
and let \( \varphi_t \) be the solution to
\[
\begin{align*}
    i\partial_t \varphi_t &= (-\Delta + V)\varphi_t + \lambda (w_N*|\varphi_t|^2)\varphi_t \\
    \varphi_t |_{t=0} &= u_0.
\end{align*}
\] (6.2)
Then there exists a constant \( C = C(C_0, \|V\|_{\infty}, CV, C_w, \|w\|_1, \gamma) > 0 \) such that
\[
\|u_t - \varphi_t\|_2 \leq CN^{-\beta}.
\] (6.3)
Remark 6.1. A similar result
\[
\|u_t - \varphi_t\|_2 \leq CN^{-\alpha/2}
\] (6.4)
for
\[\alpha := \beta \frac{\gamma - 3}{\gamma - 2}\]
holds if we assume \( \gamma > 3 \). In this case, we use a different proof technique which will be explained in Remark 6.2.

Proof. We have
\[
\frac{d}{dt} \|u - \varphi\|_2^2 = 2 \text{Re}
\left(-i\langle u - \varphi, i(u - \varphi)\rangle\right) = 2\lambda \text{Im}\langle u - \varphi, a|u|^2u - (w_N*|\varphi|^2)\varphi\rangle,
\] (6.5)
where we used that \( \langle u - \varphi, (-\Delta + V)(u - \varphi)\rangle \) is real-valued. Now
\[
\text{Im}\langle u - \varphi, a|u|^2u - (w_N*|\varphi|^2)\varphi\rangle
= \text{Im}\langle u - \varphi, (a|u|^2 - w_N*|\varphi|^2)u\rangle + \text{Im}\langle u - \varphi, (w_N*|\varphi|^2)(u - \varphi)\rangle
+ \text{Im}\langle u - \varphi, (w_N*|\varphi|^2)\varphi\rangle
= \text{Im}\langle -\varphi, (a|u|^2 - w_N*|\varphi|^2)u\rangle + \text{Im}\langle -\varphi, (w_N*|u|^2)\varphi\rangle
=: (I) + (II).
\]
Here we used that the second term vanishes since \( w_N*|u|^2 \) is real-valued. We estimate \( (I) \) and \( (II) \) separately. For \( (I) \), note that the Cauchy-Schwarz inequality, the conservation of the \( L^2 \)-norm and \( \|u_0\|_2 = 1 \) give us that
\[
| (I) | \leq \|\varphi\|_2\|(a|u|^2 - w_N*|u|^2)u\|_2
\leq \|u_0\|_2\|(a|u|^2 - w_N*|u|^2)\|_{\infty}\|u\|_2
= \|u_0\|_2\|(a|u|^2 - w_N*|u|^2)\|_{\infty}
= \|(a|u|^2 - w_N*|u|^2)\|_{\infty}.
\]
Hence,
\[
| (I) | \leq \|\varphi\|_2\|(a|u|^2 - w_N*|u|^2)u\|_2
\leq \|u_0\|_2\|(a|u|^2 - w_N*|u|^2)\|_{2}\|u\|_{\infty}
= \|u_0\|_2\|(a|u|^2 - w_N*|u|^2)\|_{2}\frac{C_0}{(1 + |t|)^{3/2}}.
\]
Using Plancherel’s identity, Hölder’s inequality, and $L^p$ interpolation, we have
\[
\left\| \alpha |u|^2 - w_N * |u|^2 \right\|_2 = \left\| \hat{w}_N(0) - \hat{w}_N(\cdot) |u|^2 \right\|_2 \\
= \left\| \hat{w}(0) - \hat{w}(\frac{\cdot}{N^\beta}) |u|^2 \right\|_2 \leq \left\| \hat{w}(0) - \hat{w}(\frac{\cdot}{N^\beta}) \right\|_\infty \left\| |u|^2 \right\|_2 \\
\leq \left\| \hat{w}(0) - \hat{w}(\frac{\cdot}{N^\beta}) \right\|_\infty \left\| |u|^2 \right\|_2 = \left\| \hat{w}(0) - \hat{w}(\frac{\cdot}{N^\beta}) \right\| \left\| u \right\|^2_4.
\]
Note that $w$ is even and so is $\hat{w}$. This implies $\nabla \hat{w}(0) = 0$. Thus, we have
\[
\hat{w} \left( \frac{\xi}{N^\beta} \right) - \hat{w}(0) = \frac{\xi}{N^\beta} (\nabla \hat{w})(0) + \left( \frac{\xi}{N^\beta} \right)^2 \langle \xi, D^2 \hat{w}(s) \xi \rangle = \langle \xi, D^2 \hat{w}(s) \xi \rangle
\]
for some $s \in \mathbb{R}^3$. Moreover,
\[
\| D^2 \hat{w} \|_\infty \leq (2\pi)^2 \| | \cdot |^2 w \|_1 \leq C < \infty
\]
since $|w(x)| \leq C|x|^{-\gamma}$ with $\gamma > 5$. Hence,
\[
\| (I) \| \leq \| u_N \|_2 \| (\alpha |u|^2 - w_N * |u|^2) \|_2 \| C_0 \| (1 + |t|)^{3/2} \leq \frac{C}{(1 + |t|)^{3/2}} N^{2\beta}.
\]

Next, let us estimate $(II)$. We have by the Cauchy-Schwarz inequality and Young’s convolutional inequality,
\[
\| (II) \| = \left| \text{Im} \langle u - \varphi, w_N * (|u|^2 - |\varphi|^2) \varphi \rangle \right| \\
\leq \| u - \varphi \|_2 \| w_N * (|u|^2 - |\varphi|^2) \varphi \|_2 \\
\leq \| u - \varphi \|_2 \| w_N * (|u|^2 - |\varphi|^2) \|_2 \| \varphi \|_\infty \\
\leq \| u - \varphi \|_2 \| w_N \|_1 \| |u|^2 - |\varphi|^2| \|_2 \| \varphi \|_\infty \\
\leq \| u - \varphi \|_2 \| w \|_1 \| (|u| - |\varphi|)(|u| + |\varphi|) \|_2 \| \varphi \|_\infty \\
\leq \| u - \varphi \|_2 \| w \|_1 \| |u| - |\varphi| \|_2 \| |u| + |\varphi| \|_\infty \| \varphi \|_\infty \\
\leq \| u - \varphi \|_2 \| w \|_1 (\|u\|_\infty + \| \varphi \|_\infty) \| \varphi \|_\infty \\
\leq \frac{1}{(1 + |t|)^{3/2}} \| u - \varphi \|_2.
\]

It follows that
\[
\frac{d}{dt} \| u - \varphi \|_2^2 \leq 2|\lambda| (|I| + (II)) \leq 2|\lambda| |(I) + (II)| \\
\leq CN^{-2\beta} \frac{1}{(1 + |t|)^3} + C \frac{1}{(1 + |t|)^3} \| u - \varphi \|_2 =: \varepsilon_N(t) + \alpha(t) \| u - \varphi \|_2.
\]

Using the Grönwall lemma and $u_{|t=0} = \varphi_0 = \varphi_{t=0}$, we obtain
\[
\| u - \varphi \|_2^2 \leq \int_0^t ds \, e^{\int_s^t dr \, \alpha(r)} \varepsilon_N(s) \leq C \int_0^t ds \, N^{-2\beta} \frac{1}{(1 + |s|)^{3/2}} \leq CN^{-2\beta}.
\]
Thus we conclude that
\[
\| u - \varphi \|_2 \leq CN^{-\beta},
\]
which finishes the proof.
Remark 6.2 (Another proof: worse decay in $N$ but $w(z) \lesssim |z|^{-\gamma}$ for $|z| \geq 1$ with $\gamma > 3$). We follow the strategy of the proof of Lemma 6.1. The only change is that we estimate $(I)$ differently: We have

$$\left|\int (I)\right| \leq \|\varphi\|_2 \|a|u|^2 - w_N \ast |u|^2\|_2$$

and so by the fundamental theorem of calculus, and

$$\left|\int (I)\right| \leq \|\varphi\|_2 \|a|u|^2 - w_N \ast |u|^2\|_2.$$

Hence, using Proposition 2.1 and $\|w_N\|_1 = \|w\|_1$, we obtain

$$\left|\int_{B(x, N^{-\alpha})} dy \ w_N(x-y) \ (|u(x)|^2 - |u(y)|^2)\right|$$

$$\leq \int_{B(x, N^{-\alpha})} dy \ |w_N(x-y)| \left( |u(x)|^2 - |u(y)|^2 \right)$$

$$\leq \int_{B(x, N^{-\alpha})} dy \ |w_N(x-y)| C\|u\|_\infty \|u\|_{H^4 N^{-\alpha}}$$

$$\leq C\|w\|_1 \|u\|_\infty \|u\|_{H^4 N^{-\alpha}}$$

$$\leq C\|w\|_1 \frac{C_0}{(1 + |t|)^{3/2}} C_0 N^{-\alpha}$$

$$\leq C N^{-\alpha} \frac{1}{(1 + |t|)^{3/2}}.$$

Let us now estimate the part of the integral for $|x-y| < N^{-\alpha}$. By assumption, we know that for all $z \in \mathbb{R}^3$ with $|z| \geq 1$, we have

$$|w(z)| \leq \frac{C}{|z|^\gamma}.$$

We get, using the change of variables $\tilde{z} = N^3 z$

$$\left|\int_{B(x, N^{-\alpha})^c} dy \ w_N(x-y) \ (|u(x)|^2 - |u(y)|^2)\right|$$

$$\leq \int_{B(x, N^{-\alpha})^c} dy \ |w_N(x-y)| 2\|u\|_\infty^2.$$
Then it follows from chain rule and (1.3.1) that
\[
\int_{B(0,N^{-\alpha})} dz \, \left| w_N(z) \right|
\leq \frac{2C_0}{(1 + |t|)^3} \int_{B(0,N^{-\alpha})} dz \, \left| N^\beta w_N(z) \right|
\]
where we used that
\[
(\beta - \alpha)(3 - \gamma) = \left( \beta - \frac{\beta + 3 - \gamma}{\gamma - 2} \right) (3 - \gamma) = \beta \frac{3 - \gamma}{\gamma - 2} = -\alpha
\]
in the last step.

To sum up, we get for any \( x \in \mathbb{R}^3 \)
\[
\left| a|u(x)|^2 - (w_N * |u|^2)(x) \right|
\leq \left| \int_{B(x,N^{-\alpha})} dy \, w_N(x-y) \left( |u(x)|^2 - |u(y)|^2 \right) \right|
\leq CN^{-\alpha} \frac{1}{(1 + |t|)^{3/2}} + C\frac{1}{(1 + |t|)^3}
\leq CN^{-\alpha} \frac{1}{(1 + |t|)^{3/2}}.
\]
Thus,
\[
|\langle I \rangle| \leq \| a|u|^2 - w_N * |u|^2 \|_\infty \leq C N^{-\alpha} \frac{1}{(1 + |t|)^{3/2}}.
\]

**Lemma 6.2** (Grönwall lemma). Let \( T > 0 \) and \( \phi, \phi', \varepsilon, \alpha \in L^1(0,T) \). Suppose that \( \phi \) satisfies the following inequality
\[
\phi'(t) \leq \alpha(t)\phi(t) + \varepsilon(t)
\]
for all \( t \in [0,T] \). Then we have
\[
\phi(t) \leq e^{A(t)}\phi(0) + \int_0^t ds \, e^{A(t)-A(s)}\varepsilon(s)
\]
where \( A(t) := \int_0^t \alpha(s)ds \).

**Proof.** Since \( \phi, \phi' \in L^1(0,t) \hookrightarrow C([0,t]) \) for all \( t \in (0,T) \) and \( \varepsilon, \alpha \in L^1(0,T) \), all terms are well-defined. Let
\[
\psi(t) := e^{-A(t)}\phi(t).
\]
Then it follows from chain rule and (1.3.1) that
\[
\psi'(t) = e^{-A(t)}\left( \phi'(t) - \alpha(t)\phi(t) \right) \leq e^{-A(t)}\varepsilon(t).
\]
By integrating both sides from 0 to $t$, we obtain

$$\psi(t) - \psi(0) \leq \int_0^t ds \, e^{-A(s)} \varepsilon(s).$$

Using the definition of $\psi$, we get

$$e^{-A(t)} \phi(t) - \phi(0) \leq \int_0^t ds \, e^{-A(s)} \varepsilon(s).$$

Thus, inequality (1.3.2) holds in $[0, T]$. □

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