Link Floer homology also detects split links

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Abstract

Inspired by work of Lipshitz-Sarkar, we show that the module structure on link Floer homology detects split links. Using results of Ni, Alishahi-Lipshitz, and Lipshitz-Sarkar, we establish an analogous detection result for sutured Floer homology.

1 Introduction

A remarkable feature of modern homology theories in low-dimensional topology is their ability to detect many topological properties of interest. We refer to the introduction of [LS19] for a list of such detection results in Khovanov homology and Heegaard Floer homology. The main theorem of [LS19] is an additional detection result for Khovanov homology: that the module structure on Khovanov homology detects split links. In this short paper, we add one more item to the list: that the analogous module structure on link Floer homology also detects split links.

If \( L \) is a two-component link in \( S^3 \), then its link Floer homology \( \widehat{HFL}(L) \) [OS08a], which takes the form of a finite-dimensional vector space over \( \mathbb{F}_2 = \mathbb{Z}/2 \), is naturally equipped with an endomorphism \( X \) satisfying \( X \circ X = 0 \). Such an endomorphism gives \( \widehat{HFL}(L) \) the structure of a module over \( \mathbb{F}_2[X]/X^2 \). The map \( X \) is defined to be the homological action [Ni14] of a generator of the first relative homology group of the exterior of \( L \). We review the definition of this action in Section 2.

Theorem 1.1. Let \( L \) be a two-component link in \( S^3 \). Then \( L \) is split if and only if \( \widehat{HFL}(L) \) is a free \( \mathbb{F}_2[X]/X^2 \)-module.

Remark 1.2. Before the work of this paper began, Tye Lidman observed and informed the author that link Floer homology should detect split links in this way using results of [Ni13] and arguments similar to those in [LS19]. The proof of Theorem 1.1 appearing in this short paper is due to the author and is independent of the results and arguments in [Ni13] and [LS19]. In particular, it uses sutured manifold hierarchies and does not require citing any deep results in symplectic geometry. However, the proof of Theorem 1.4 appearing here does use such a citation.

Remark 1.3. Link Floer homology detects the Thurston norm of its exterior [OS08b, Theorem 1.1], [Ni09b, Theorem 1.1]. This by itself does not imply that link Floer homology detects split links. For example, the exterior of the Whitehead link has the same Thurston norm as the exterior of a split union of two genus 1 knots.

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More generally, there is a homological action $X_\zeta$ [Ni14] on the sutured Floer homology $\text{SFH}(M,\gamma)$ [Juh06] of a balanced sutured manifold $(M,\gamma)$ satisfying $X_\zeta \circ X_\zeta = 0$ for each $\zeta \in H_1(M,\partial M)$. We prove the following generalization of Theorem 1.1 using a result of Lipshitz-Sarkar [LS19], which builds on results of Alishahi-Lipshitz [AL19] and Ni [Ni13].

\textbf{Theorem 1.4.} Let $(M,\gamma)$ be a balanced sutured manifold, let $\zeta \in H_1(M,\partial M)$, and assume that $\text{SFH}(M,\gamma) \neq 0$. Then $\text{SFH}(M,\gamma)$ is a free $\mathbb{F}_2[X]/X^2$-module with respect to the homological action of $\zeta$ if and only if there is an embedded 2-sphere $S$ in $M$ for which the algebraic intersection number $S \cdot \zeta$ is odd.

\textbf{Remark 1.5.} Let $L$ be a link in $S^3$ with at least two components, and let $C_0, C_1$ be two distinct components of $L$. Let $S^3(L)$ denote the sutured exterior of $L$, and let $\zeta \in H_1(S^3(L),\partial S^3(L))$ be the relative homology class of a path from $C_0$ and $C_1$ in $S^3(L)$. The sutured Floer homology of $S^3(L)$ can be identified with $\text{HFL}(L)$ by [Juh06, Proposition 9.2]. By Theorem 1.4, $\text{HFL}(L)$ is a free $\mathbb{F}_2[X]/X^2$-module with respect to the homological action of $\zeta$ if and only if there exists an embedded 2-sphere in the complement of $L$ which separates $C_0$ from $C_1$.

Homological actions on Heegaard Floer homology for closed oriented 3-manifolds were originally defined in [OS04, Section 4.2.5]. These homological actions on Heegaard Floer homology and the closely-related construction of twisted Heegaard Floer homology are well-studied. See [Ni09a, HN10, Ni13, HN13, AL19, LS19, HL20] for further connections to non-separating spheres and to the module structure on Khovanov homology.

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2 Preliminaries

See [Gab83, Gab87, Juh06, Juh08, Juh10] for the definitions of balanced sutured manifolds, nice surface decompositions, and sutured Floer homology. In this paper, we use sutured Floer homology with coefficients in the field $\mathbb{F}_2 = \mathbb{Z}/2$. We first provide some examples of sutured manifolds which also serve to set notation.

\textbf{Examples.} A product sutured manifold is a sutured manifold of the form $([-1, 1] \times \Sigma, [-1, 1] \times \partial \Sigma)$ where $\Sigma$ is a compact oriented surface. It is balanced if $\Sigma$ has no closed components.

Let $L$ be a link in a closed oriented connected 3-manifold $Y$. The sutured exterior $Y(L)$ of the link is the balanced sutured manifold obtained from $Y$ by deleting a regular neighborhood of $L$ and adding two oppositely oriented meridional sutures on each boundary component.

If $Y$ is a closed oriented connected 3-manifold, then let $Y(n)$ be the balanced sutured manifold obtained by deleting $n$ disjoint open balls from $Y$ and adding a suture to each boundary component. We similarly define $(M,\gamma)(n)$ when $(M,\gamma)$ is a connected balanced sutured manifold.

\textbf{Definition} [Gab83, Definition 2.10]. A sutured manifold $(M,\gamma)$ is \textit{taut} if $M$ is irreducible and $R(\gamma)$ is norm-minimizing in $H_2(M,\gamma)$.
Remark 2.1. Balanced product sutured manifolds are taut. If \( L \) is a two-component link in \( S^3 \), then \( S^3(L) \) is taut if and only if \( L \) is not split. Except for \( S^3(1) \), any balanced sutured manifold of the form \( Y(n) \) or \( (M, \gamma)(n) \) for \( n \geq 1 \) is not irreducible and therefore is not taut.

**Theorem 2.2** [Gab83, Theorem 4.2], [Juh08, Theorem 8.2]. If \( (M, \gamma) \) is a taut balanced sutured manifold, then there is a sequence of nice surface decompositions

\[
(M, \gamma) \xrightarrow{\mathcal{S}_1} (M_1, \gamma_1) \xrightarrow{\mathcal{S}_2} \cdots \xrightarrow{\mathcal{S}_n} (M_n, \gamma_n)
\]

where \( (M_n, \gamma_n) \) is a balanced product sutured manifold.

Let \( (M, \gamma) \) be a balanced sutured manifold. We review Ni’s definition of the homological action of a relative homology class \( \zeta \in H_1(M, \partial M) \) on SFH\((M, \gamma)\) [Ni14, Section 2.1]. These actions are an extension of the homological actions on Heegaard Floer homology defined in [OS04, Section 4.2.5]. We then recall Ni’s result that these homological actions are compatible with nice surface decompositions [Ni14, Theorem 1.1].

**Definition** (Homological actions on sutured Floer homology). Let \( (\Sigma, \alpha, \beta) \) be an admissible balanced diagram for \( (M, \gamma) \), and let \( \omega = \sum k_i \alpha_i \) be a formal finite sum of properly embedded oriented curves \( \alpha_i \) on \( \Sigma \) with integer coefficients \( k_i \). Each \( \alpha_i \) is required to intersect the \( \alpha \)- and \( \beta \)-curves transversely and to be disjoint from every intersection point of the \( \alpha \)- and \( \beta \)-curves.

Let \( \zeta \in H_1(M, \partial M) \) denote the relative homology class that \( \omega \) represents. Any relative first homology class of \( M \) is represented by such a relative 1-cycle on \( \Sigma \).

Let \( \text{SFC}(\Sigma, \alpha, \beta) \) be the sutured Floer chain complex whose differential \( \partial \) is defined with respect to a suitable family of almost complex structures. In particular

\[
\partial \mathbf{x} = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y)} \# \hat{\mathcal{U}}(\phi) \cdot \mathbf{y}
\]

for each \( x \in T_\alpha \cap T_\beta \). Any Whitney disc \( \phi \in \pi_2(x, y) \) has an associated 2-chain on \( \Sigma \) called its domain \( D(\phi) \). Let \( \partial_{\alpha,\beta} D(\phi) \) be the part of \( \partial D(\phi) \) lying in the \( \alpha \)-circles, thought of as an oriented multi-arc from \( x \) to \( y \). Define \( X_\zeta : \text{SFC}(\Sigma, \alpha, \beta) \to \text{SFC}(\Sigma, \alpha, \beta) \) by

\[
X_\zeta \cdot \mathbf{x} = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y)} (\omega \cdot \partial_{\alpha,\beta} D(\phi)) \# \hat{\mathcal{U}}(\phi) \cdot \mathbf{y}
\]

where \( \omega \cdot \partial_{\alpha,\beta} D(\phi) \) is the algebraic intersection number mod 2.

It is shown in [Ni14, OS04] that \( X_\zeta \) is a chain map, and its induced map on homology, also denoted \( X_\zeta \), squares to zero. Furthermore, the map on homology is independent of the choice of \( \omega \) representing \( \zeta \in H_1(M, \partial M) \) and the choice of admissible Heegaard diagram. This induced map on SFH\((M, \gamma)\) is called the homological action of \( \zeta \). Unless otherwise stated, \( X_\zeta \) refers to the map on homology. Since \( X_\zeta \circ X_\zeta = 0 \), we may view SFH\((M, \gamma)\) as a module over \( \mathbb{F}_2[X]/X^2 \) where the action of \( X \) is \( X_\zeta \). Note that if \( \zeta = 2\zeta' \) is an even homology class, then \( X_\zeta = 0 \). More generally, \( X_\zeta \) is additive in \( \zeta \), which is to say that \( X_{\zeta + \zeta'} = X_\zeta + X_{\zeta'} \).

**Example.** We will make use of the following direct computation. Let \( \Sigma \) denote an annulus, and let \( \alpha, \beta \) be embedded essential curves which intersect transversely in two points. Then
$(\Sigma, \alpha, \beta)$ is an admissible diagram for the balanced sutured manifold $S^3(2)$. Then the two points in the intersection $T_\alpha \cap T_\beta$ can be labeled $x, y$ so that there are two Whitney discs from $x$ to $y$. Each has a unique holomorphic representative, and they cancel in the differential of $SFC(\Sigma, \alpha, \beta)$ so $\dim_{\mathbb{F}_2} SFH(S^3(2)) = 2$. Let $\omega$ be an embedded oriented arc in $\Sigma$ whose endpoints lie on different boundary components and which intersects $\alpha \cup \beta$ transversely in exactly two points. Note that $\omega$ represents a generator $\zeta$ of $H_1(S^3(2), \partial S^3(2))$. Then $X_\zeta \cdot x = y$ so $SFH(S^3(2)) \cong F_2[X]/X^2$ as a module with respect to the action of $\zeta$.

Essentially the same computation shows that $\widehat{HF}(S^3\times S^3) \cong F_2[X]/X^2$ as a module with respect to the action of a generator of $H_1(S^3 \times S^3)$.

**Remark 2.3.** Let $Y$ be a closed oriented connected 3-manifold. Then there is an identification $\widehat{HF}(Y) = SFH(Y(1))$ [Juh06, Proposition 9.1]. The homological actions on $\widehat{HF}(Y)$ defined in [OS04] correspond to the homological actions on $SFH(Y(1))$ using the natural identification $H_1(Y) = H_1(Y(1), \partial Y(1))$.

**Remark 2.4.** Let $(M, \gamma)$ and $(N, \beta)$ be balanced sutured manifolds, and let $\zeta \in H_1(M, \partial M)$ and $\xi \in H_1(N, \partial N)$. Consider the disjoint union $(M \sqcup N, \gamma \cup \beta)$ and the relative homology class $\zeta \oplus \xi \in H_1(M, \partial M) \oplus H_1(N, \partial N) = H_1(M \sqcup N, \partial(M \sqcup N))$.

The homological action $X_{\zeta \oplus \xi}$ on $SFH(M \sqcup N, \gamma \cup \beta) = SFH(M, \gamma) \otimes_{F_2} SFH(N, \beta)$ is given by

$$X_{\zeta \oplus \xi}(a \otimes b) = (X_\zeta a) \otimes b + a \otimes (X_\zeta b).$$

**Theorem 2.5 [Ni14, Theorem 1.1].** Let $(M, \gamma) \xrightarrow{\Sigma} (M', \gamma')$ be a nice surface decomposition of balanced sutured manifolds. Let $i_\# : H_1(M, \partial M) \to H_1(M', \partial M')$ be the map induced by the inclusion map $i : (M, \partial M) \to (M', \partial M')$. Then $SFH(M', \gamma')$ is isomorphic as an $F_2[X]/X^2$-module to a direct summand of $SFH(M, \gamma)$ as an $F_2[X]/X^2$-module.

The following lemma addresses the special case where the decomposing surface $S$ is a product disc. Recall that a decomposing surface $D$ in a sutured manifold $(M, \gamma)$ is called a product disc [Gab87, Definitions 0.1] if $D$ is a disc and $|D \cap s(\gamma)| = 2$.

**Lemma 2.6 [Juh06, Lemma 9.13].** Let $(M, \gamma)$ be a balanced sutured manifold, let $(M, \gamma) \xrightarrow{\Sigma} (M', \gamma')$ be a product disc decomposition, and let $\zeta \in H_1(M, \partial M)$. Then there is an isomorphism $SFH(M, \gamma) \cong SFH(M', \gamma')$ of $F_2[X]/X^2$-modules where $X$ acts on $SFH(M, \gamma)$ and $SFH(M', \gamma')$ by the homological actions of $\zeta$ and $i_\#(\zeta)$, respectively.

**Proof.** The lemma follows directly from the definition of the homological action and the proof of [Juh06, Lemma 9.13].

We will use the following two connected-sum formulas for the homological action on sutured Floer homology.
Lemma 2.7. Let \((M, \gamma)\) be a balanced sutured manifold, and let \(Y\) be a closed oriented 3-manifold. Fix \(\zeta \in H_1(M#Y, \partial(M#Y))\) and write \(\zeta = \zeta' + \zeta''\) according to the decomposition \(H_1(M\#Y, \partial(M\#Y)) = H_1(M, \partial M) \oplus H_1(Y)\). Then there is an isomorphism of \(F_2[X]/X^2\)-modules

\[
\text{SFH}(M\#Y, \gamma) \cong \text{SFH}(M, \gamma) \otimes_{F_2} \widehat{\text{HF}}(Y)
\]

where \(X\) acts on \(\text{SFH}(M\#Y, \gamma)\) by \(X_\zeta\) while \(X\) acts on the right-hand side by

\[
X(a \otimes b) = (X_\zeta a) \otimes b + a \otimes (X_\zeta b).
\]

Proof. As observed in [Juh06, Proposition 9.15], there is a product disc decomposition

\[
(M\#Y, \gamma) \xrightarrow{D} (M, \gamma) \sqcup Y(1).
\]

The result now follows from Remark 2.4 and Lemma 2.6. \(\square\)

Lemma 2.8. Let \((M, \gamma)\) be a connected sum of balanced sutured manifolds \((N_1, \beta_1)\) and \((N_2, \beta_2)\), and let \(\zeta \in H_1(M, \partial M)\). Then there is an isomorphism of \(F_2[X]/X^2\)-modules

\[
\text{SFH}(M, \gamma) \cong \text{SFH}(N_1, \beta_1) \otimes_{F_2} \text{SFH}(N_2, \beta_2) \otimes_{F_2} \text{SFH}(S^3(2))
\]

where \(X\) acts on \(\text{SFH}(M, \gamma)\) by the homological action of \(\zeta\), and \(X\) acts on the right-hand side by

\[
X(a \otimes b \otimes c) = (X_{\zeta_1} a) \otimes b \otimes c + a \otimes (X_{\zeta_2} b) \otimes c + a \otimes b \otimes (X_{\zeta})
\]

for certain classes \(\zeta_1 \in H_1(N_1, \partial N_1), \zeta_2 \in H_1(N_2, \partial N_2),\) and \(\xi \in H_1(S^3(2), \partial S^3(2))\).

Let \(S\) be the 2-sphere in \(M\) along which the connected sum is formed. If \(S \cdot \zeta\) is odd, then \(\text{SFH}(S^3(2)) \cong F_2[X]/X^2\) with respect to the homological action of \(\xi\). If \(S \cdot \zeta\) is even, then \(X_\zeta = 0\).

Proof. Again as observed in [Juh06, Proposition 9.15], there are product disc decompositions

\[
(M, \gamma) \xrightarrow{D} (N_1, \beta_1) \sqcup (N_2, \beta_2)(1) \xrightarrow{D'} (N_1, \beta_1) \sqcup (N_2, \beta_2) \sqcup S^3(2)
\]

Let \(\zeta_1 \oplus \zeta_2 \oplus \xi\) be the image of \(\zeta\) in

\[
H_1(M_1, \partial M_1) \oplus H_1(M_2, \partial M_2) \oplus H_1(S^3(2), \partial(S^3(2))).
\]

Note that \(\xi\) is an even class if and only if \(S \cdot \zeta\) is even. Furthermore, a direct computation gives an isomorphism \(\text{SFH}(S^3(2)) \cong F_2[X]/X^2\) as modules when \(X\) is the homological action of a generator. The result now follows from Remark 2.4 and Lemma 2.6. \(\square\)

Before turning to the main results, we record here an algebraic lemma.

Lemma 2.9. Let \(M_1, \ldots, M_k\) be a collection of finitely-generated \(F_2[X]/X^2\)-modules, and view \(M = M_1 \otimes_{F_2} \cdots \otimes_{F_2} M_k\) as an \(F_2[X]/X^2\)-module where the action of \(X\) is defined by

\[
X(m_1 \otimes \cdots \otimes m_k) = \sum_{i=1}^k m_1 \otimes \cdots \otimes m_{i-1} \otimes (Xm_i) \otimes m_{i+1} \otimes \cdots \otimes m_k.
\]

Then \(M\) is free if and only if at least one of the \(M_i\) is free.
We provide a quick proof of Theorem 1.1 before turning to its generalization. This result of Alishahi-Lipshitz builds on work of Ni [Ni13]. See also [HN13, Theorem 4 and Corollary 5.2].

3 Main results

The following lemma contains the main argument of this short paper. Using this lemma, we provide a quick proof of Theorem 1.1 before turning to its generalization.

**Lemma 3.1.** Let $(M, \gamma)$ be a taut balanced sutured manifold with $\zeta \in H_1(M, \partial M)$. Then $\text{SFH}(M, \gamma)$ is not a free $F_2[X]/X^2$-module with respect to the homological action of $\zeta$.

**Proof.** By Theorem 2.2, we may find a sequence of nice surface decompositions from $(M, \gamma)$ to a product sutured manifold $(N, \beta)$. Then $\text{SFH}(N, \beta)$ is isomorphic to a direct summand of $\text{SFH}(M, \gamma)$ as an $F_2[X]/X^2$-module by Theorem 2.5 where $X$ acts on $\text{SFH}(N, \beta)$ by some homological action. Since $\dim F_2 \text{SFH}(N, \beta) = 1$ by [Juh06, Proposition 9.4], it cannot be a free $F_2[X]/X^2$-module. Thus $\text{SFH}(M, \gamma)$ is not a free $F_2[X]/X^2$-module.

**Proof of Theorem 1.1.** Recall that $\text{HFL}(L) = \text{SFH}(S^3(L))$ [Juh06, Proposition 9.2] where $S^3(L)$ is the sutured exterior of $L$. If $L$ is split, then $\text{SFH}(S^3(L))$ is a free $F_2[X]/X^2$-module by a computation in a carefully chosen Heegaard diagram. This computation can be formalized in the following way. If $L$ is the split union of the knots $K$ and $J$, then $S^3(L) = S^3(K) \# S^3(J)$. By Lemma 2.8, there is an isomorphism of $F_2[X]/X^2$-modules

$$\text{SFH}(S^3(L)) \cong \text{SFH}(S^3(K)) \otimes_{F_2} \text{SFH}(S^3(J)) \otimes_{F_2} \text{SFH}(S^3(2))$$

where the action of $X$ on the right-hand side is given by $\text{Id} \otimes \text{Id} \otimes X_\zeta$ where $\zeta$ is a generator of $H_1(S^3(2), \partial S^3(2))$. Since $\text{SFH}(S^3(2)) \cong F_2[X]/X^2$ as modules with respect to the action of $\zeta$, it follows from Lemma 2.9 that $\text{SFH}(S^3(L))$ is a free $F_2[X]/X^2$-module.

If $L$ is not split, then $S^3(L)$ is taut, so $\text{SFH}(S^3(L))$ is not free by Lemma 3.1.

The next lemma is a direct consequence of [LS19, Lemma 5.6], which Lipshitz-Sarkar prove using [AL19, Theorem 1.1]. This result of Alishahi-Lipshitz builds on work of Ni [Ni13]. See also [HN13, Theorem 4 and Corollary 5.2].

**Lemma 3.2.** Let $Y$ be an irreducible closed oriented 3-manifold with $\zeta \in H_1(Y)$. Then $\widehat{\text{HF}}(Y)$ is not a free $F_2[X]/X^2$-module with respect to the homological action of $\zeta$.

**Proof.** We use the notation in [LS19] without reintroducing it. Since $S \cdot \zeta = 0$ for all embedded 2-spheres $S$ in $Y$, the unrolled homology of $\widehat{\text{CF}}(Y)$ with respect to $\zeta$ is nontrivial by [LS19, Lemma 5.6]. The $E^1$-page of the spectral sequence associated to the horizontal filtration on the unrolled complex of $\widehat{\text{CF}}(Y)$ is the unrolled complex of $\widehat{\text{HF}}(Y)$ viewed as an $F_2[X]/X^2$-module with respect to $\zeta$. Since the $E^\infty$-page is nonzero, the $E^2$-page is also nonzero so $\text{HF}(Y)$ is not a free $F_2[X]/X^2$-module.
Proof of Theorem 1.4. We first prove the result under the assumption that \((M, \gamma)\) is strongly-balanced. We then prove the general statement by reducing to this case. A balanced sutured manifold \((M, \gamma)\) is strongly-balanced [Juh08, Definition 3.5] if for each component \(F\) of \(\partial M\), we have the equality \(\chi(F \cap R_+(\gamma)) = \chi(F \cap R_-(\gamma))\).

Under the assumption that \((M, \gamma)\) is strongly-balanced, suppose there exists a 2-sphere \(S\) in \(M\) for which \(S \cdot \zeta\) is odd. There are two cases.

(a) The sphere \(S\) is non-separating.

Then \((M, \gamma)\) is the connected sum of a strongly-balanced sutured manifold \((N, \beta)\) with \(S^1 \times S^2\), where \(S\) is a copy of \(pt \times S^2\) in the \(S^1 \times S^2\) summand. If \(\zeta = \zeta' \oplus \zeta''\) under the natural identification \(H_1(M, \partial M) = H_1(N, \partial N) \oplus H_1(S^1 \times S^2)\), then \(\zeta''\) is an odd multiple of a generator \(\xi\) of \(H_1(S^1 \times S^2)\) by the assumption that \(S \cdot \zeta\) is odd. By Lemma 2.7, there is an isomorphism of \(\mathbb{F}_2[X]/X^2\)-modules

\[
\text{SFH}(M, \gamma) \cong \text{SFH}(N, \beta) \otimes_{\mathbb{F}_2} \text{SFH}(S^1 \times S^2),
\]

where the actions of \(X\) on the right-hand side is \(X_{\zeta'} \otimes \text{Id} + \text{Id} \otimes X_{\zeta''}\). Since \(\zeta'' - \xi\) is even, we know that \(X_{\zeta''} = X_\xi\). By a direct computation, \(\text{SFH}(S^1 \times S^2) = \mathbb{F}_2[X]/X^2\) as an \(\mathbb{F}_2[X]/X^2\)-module with respect to the action of \(\xi\). It follows that \(\text{SFH}(M, \gamma)\) is free by Lemma 2.9.

(b) The sphere \(S\) is separating.

Then \((M, \gamma)\) is the connected sum of sutured manifolds \((N_1, \beta_1)\) and \((N_2, \beta_2)\) along the sphere \(S\). Since \((M, \gamma)\) is strongly-balanced, both \((N_1, \beta_1)\), \((N_2, \beta_2)\) are as well. By Lemma 2.8, there is an isomorphism of \(\mathbb{F}_2[X]/X^2\)-modules

\[
\text{SFH}(M, \gamma) \cong \text{SFH}(N_1, \beta_1) \otimes_{\mathbb{F}_2} \text{SFH}(N_2, \beta_2) \otimes_{\mathbb{F}_2} \text{SFH}(S^3(2))
\]

where the action of \(X\) on the right-hand side is given by

\[
X_{\zeta_1} \otimes \text{Id} \otimes \text{Id} + \text{Id} \otimes X_{\zeta_2} \otimes \text{Id} + \text{Id} \otimes \text{Id} \otimes X_\xi
\]

for classes \(\zeta_i \in H_1(N_i, \partial N_i)\) and \(\xi \in H_1(S^3(2), \partial S^3(2))\). Furthermore, by the assumption that \(S \cdot \zeta\) is odd, Lemma 2.8 implies that \(\text{SFH}(S^3(2)) \cong \mathbb{F}_2[X]/X^2\) as modules with respect to the action of \(\xi\). Thus \(\text{SFH}(M, \gamma)\) is a free \(\mathbb{F}_2[X]/X^2\)-module by Lemma 2.9.

Now assume that \(S \cdot \zeta\) is even for every embedded 2-sphere \(S\) in \(M\), where \((M, \gamma)\) is strongly-balanced. To show that \(\text{SFH}(M, \gamma)\) is not a free \(\mathbb{F}_2[X]/X^2\)-module with respect to the action of \(\zeta\), we use Lemmas 2.7, 2.8, and 2.9 to reduce to the irreducible case, which is then handled by Lemmas 3.1 and 3.2. Write \(M\) as a connected sum

\[
M = N_1 \# \cdots \# N_k \# Y_1 \# \cdots \# Y_\ell \# (S^1 \times S^2)^m
\]

where the \(N_i\) are irreducible compact 3-manifolds with nonempty boundary and the \(Y_i\) are irreducible closed 3-manifolds. A quick way to see that such a decomposition exists uses the Grushko-Neumann theorem on the ranks of free products of finitely-generated groups and the Poincaré conjecture (for example, see [Mil62]). The assumption that \((M, \gamma)\) is strongly-balanced implies \((N_i, \beta_i)\) is also strongly-balanced where \(\beta_i\) are the sutures inherited from
Because $\text{SFH}(M, \gamma) \neq 0$, it follows that $\text{SFH}(N_i, \beta_i) \neq 0$ so $(N_i, \beta_i)$ is taut by [Juh06, Proposition 9.18]. Note that $S \cdot \zeta$ is even for each sphere along which the connected sums are formed and for each sphere of the form $pt \times S^2$ in each $S^1 \times S^2$ summand. The result now follows from Lemmas 2.7, 2.8, 2.9, 3.1, and 3.2.

We now reduce to the case that $(M, \gamma)$ is strongly-balanced. As explained in [Juh08, Remark 3.6], we may construct a strongly-balanced sutured manifold $(M', \gamma')$ from a given balanced sutured manifold $(M, \gamma)$ so that there is a sequence of product disc decompositions from $(M', \gamma')$ to $(M, \gamma)$. To construct $(M', \gamma')$, we repeat the following procedure, which is sometimes referred to as a contact 1-handle attachment: fix two components $F_1, F_2$ of $\partial M$ for which $\chi(F_i \cap R_+(\gamma)) \neq \chi(F_i \cap R_-(\gamma))$, choose discs $D_i$ centered at points on $s(\gamma) \cap F_i$, and identify $D_1$ with $D_2$ by an orientation-reversing map so that the sutures $s(\gamma) \cap D_1$ and $s(\gamma) \cap D_2$ are identified. Do this identification in such a way that the orientations of the sutures are reversed. The resulting manifold naturally inherits an orientation and sutures for which there is at least one fewer boundary component $F$ with $\chi(F \cap R_+(\gamma)) \neq \chi(F \cap R_-(\gamma))$. This reverse of this procedure is a product disc decomposition along $D_1 = D_2$.

Let $(M, \gamma)$ be a balanced sutured manifold, and let $\zeta \in H_1(M, \partial M)$. Construct a strongly-balanced sutured manifold $(M', \gamma')$ from $(M, \gamma)$ as explained. Let $z \subset M$ be a properly-embedded oriented 1-manifold representing $\zeta$ which is disjoint from the boundary used in the construction of $(M', \gamma')$. Then $z$ represents a class $\zeta' \in H_1(M', \partial M')$ for which $i_* (\zeta') = \zeta$ under the sequence of product disc decompositions from $(M', \gamma')$ to $(M, \gamma)$. Note that $\text{SFH}(M, \gamma)$ and $\text{SFH}(M', \gamma')$ are isomorphic as $F_2[X]/X^2$-modules with respect to the actions of $\zeta$ and $\zeta'$, respectively, by Lemma 2.6.

If there is an embedded 2-sphere $S$ in $M$ with $S \cdot \zeta$ odd, then the same 2-sphere viewed in $M'$ also has $S \cdot \zeta'$ odd. Thus $\text{SFH}(M', \gamma')$ and $\text{SFH}(M, \gamma)$ are free. Conversely, if $\text{SFH}(M, \gamma)$ is free, then $\text{SFH}(M', \gamma')$ is free as well, so there is an embedded 2-sphere $S'$ in $(M', \gamma')$ which intersects $z$ transversely in an odd number of points. By using an innermost argument, we may compress $S'$ along discs in the product discs of the sequence of decompositions from $(M', \gamma')$ to $(M, \gamma)$ to obtain a collection of embedded 2-spheres $S_i$ in $M$ for which $\sum_i S_i \cdot \zeta = S' \cdot \zeta'$ is odd. Thus there is at least one $S_i$ for which $S_i \cdot \zeta$ is odd. □

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