Instability of thick brane worlds

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Abstract
We examined 5d thick brane worlds constructed by a real scalar field. The solutions are obtained in terms of a simple form of smooth warp factor. For the case of dS brane, we found a disease of the solutions. For example, it is impossible to construct thick-brane worlds with our simple smoothing. This result is independent of supersymmetry or other symmetries.

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1 Introduction

After the proposal of a thin brane world given in [1], the construction and the use of smooth brane world solutions in higher dimensions have been given by many authors [2, 3]. This direction seems to be natural from the viewpoint of field theory since the thin brane should be considered as a limit of a thick brane which would be obtained as a soliton solution of a higher dimensional field theory coupled with gravity. An interesting and simple case in obtaining a thick brane solution is the 5d gravity with a single real scalar. For supergravity, it is known as a difficult task to find such soliton solutions [4, 5, 6, 7, 8], but many kinds of thick brane solutions are possible in non-supersymmetric case [2, 3, 9, 10, 11, 12, 13, 14]. And various problems, particularly the localization of gravity, have been studied in terms of the latter model.

On the other hand, the fluctuation of the scalar field which constructs the thick brane has not been studied enough. In [2], it was pointed out that, in the thin brane limit, the bulk mass of this field becomes very large and the fluctuation would be frozen leaving only the thin brane as a background configuration. However before taking this limit, we should carefully study the spectrum of this fluctuation. The classical solution for this scalar has generally a form of kink along the fifth coordinate, and the second derivative of its potential is negative around the kink or near the brane. This property would be common to all thick brane models.

In this paper, we consider two types of thick branes, dS and or Poincaré branes. For the thick dS brane case, we show that it is not possible to construct such a brane with a real scalar field. For the thick Poincaré brane case, we show that the scalar fluctuation has a tachyon, when mixing with the scalar fluctuation of the metric is ignored. This tachyonic mode might be removed when the mixing is taken into account [12, 13, 14].

In Section 2, we set our model with a real scalar and show thick brane solutions. In Section 3, the stability of the thick brane worlds is examined. The final section is devoted to conclusion.

2 Thick brane solution

We begin with the action for five-dimensional gravity coupled with a real scalar ($\phi$),

$$S_g = \int d^4x dy \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right\} ,$$

where $\kappa^2$ denotes the five-dimensional gravitational constant, and the form of potential $V$ is not specified at this stage. The equations of motion derived from the action (1) are

$$G_{MN} = \kappa^2 \left\{ \partial_M \phi \partial_N \phi - g_{MN} \left( \frac{1}{2} (\partial \phi)^2 + V \right) \right\} ,$$

$$\frac{1}{\sqrt{-g}} \partial_M \left\{ \sqrt{-g} g^{MN} \partial_N \phi \right\} = \frac{\partial V}{\partial \phi} .$$

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We assume metric of the form
\[ ds^2 = g_{MN} dx^M dx^N = A^2(y) (-dt^2 + a_0^2(t) \gamma_{ij} dx^i dx^j) + dy^2 , \]
where \( \gamma_{ij} = (1 + k \delta_{mn} x^m x^n / 4)^{-2} \delta_{ij} \). The 3d scale factor \( a_0(t) \) is soluble for each \( k \), for example \( a_0 = e^{\sqrt{k} t} \) in the case of \( k = 0 \). We also assume that \( \phi \) depends only on the fifth coordinate \( y \). These assumptions simplify the equations of motion, (2) and (3), as
\[
\frac{A''}{A} + \left( \frac{A'}{A} \right)^2 - \frac{\lambda}{A^2} = -\frac{\kappa^2}{3} \left( \frac{1}{2} (\phi')^2 + V(\phi) \right) ,
\]
\[
\left( \frac{A'}{A} \right)^2 - \frac{\lambda}{A^2} = \frac{\kappa^2}{6} \left( \frac{1}{2} (\phi')^2 - V(\phi) \right) ,
\]
\[
\phi'' + 4 \phi' \frac{A'}{A} = \frac{\partial V}{\partial \phi} ,
\]
where \( ' = d/dy \). From Eqs. (5) and (6), we obtain
\[
(\phi')^2 = -3 \frac{\kappa^2}{\lambda^2} \left( \frac{A''}{A} - \left( \frac{A'}{A} \right)^2 + \frac{\lambda}{A^2} \right) ,
\]
\[
V = -9 \frac{\kappa^2}{2 \lambda^2} \left( \frac{A''}{3A} + \left( \frac{A'}{A} \right)^2 - \frac{\lambda}{A^2} \right) .
\]
The \( \phi' \) and \( V \) defined by Eqs. (8) and (9) automatically satisfy Eq. (7), so Eq. (7) is redundant. If \( A \) is given and smooth everywhere, \( \phi' \) and \( V \) are obtained from the \( A \) and its derivative. The scalar field thus obtained forms a thick brane world.

The warp factor of the thin brane world is written as \( A = A(|y|) \) and singular at \( y = 0 \), the position of the thin brane. Smoothing out the singularity corresponds to introducing the thickness to the brane. Here we introduce a smooth warp factor of the form \( A = A(f(b, y)) \) with
\[
f(b, y) = \frac{1}{b} \log(\cosh(by)) ,
\]
replacing \( |y| \) by \( f \) without changing the functional form of \( A(|y|) \). This replacement was adopted also in [3], and the original thin brane is recovered in the large limit of parameter \( b \), since \( f \to |y| \) as \( b \to \infty \). Although there are many other smoothed forms and parametrizations [2, 9, 10, 11, 14], we take (10) for convenience.

3 Instability of thick dS brane

The stability of the thick brane solutions is investigated in two manners. First we check the reality of \( \phi \) through (5).
For $\lambda > 0$ and $\Lambda < 0$, we have a warp factor $A$ of the form

$$A(y) = \sqrt{\frac{\lambda}{\mu}} \sinh [\mu(y_H - f(y))],$$

leading to

$$(\phi')^2 = \frac{3\mu^2}{\kappa^2} \frac{b}{\cosh^2(by) \sinh^2[\mu(y_H - f)]} - 1,$$

where $\mu = \sqrt{-\Lambda/6}$ and $\sinh[\mu y_H] = \mu/\sqrt{\lambda}$. Obviously, there exists a region of negative $(\phi')^2$ for $y$ satisfying $\sinh[2\mu(y_H - f)] < 2\mu/b$, near the extended horizon $y = \hat{y}_H > y_H$ defined by $A(f(\hat{y}_H)) = 0$. Thus, the thick-brane solution is unstable in the region. A similar analysis is applicable for the case of $\lambda > 0$ and $\Lambda > 0$, and the same result is obtained also for this case. The unstable region disappears in the limit of either $b \to \infty$ or $\lambda \to 0$. The former corresponds to the thin-brane limit, and the latter discussed below does to the limit of $y_H \to \infty$.

As for $\lambda = 0$ and $\Lambda < 0$, we have

$$A(y) = e^{-\mu f(y)}$$

leading to

$$(\phi')^2 = \frac{3\mu^2}{\kappa^2} \frac{b}{\cosh^2(by)} > 0.$$  

Thus, the reality condition for $\phi$ is satisfied. Solving (13), we get

$$\phi = v \arctan [\tanh \frac{by}{2}], \quad v = \sqrt{12\mu / \kappa^2 b},$$

and eventually reach to the so-called sine-Gordon potential

$$V = \frac{b^2 v^2}{16} \left(1 + \cos \frac{4\phi}{v} - \frac{\kappa^2 v^2}{3} (1 - \cos \frac{4\phi}{v}) \right).$$

Note that the potential minimum at $\phi = \pm \pi v/4$ is negative for finite $\kappa^2$. This implies that the AdS$_5$ bulk configuration is recovered in the thin brane limit. Further, we notice that the part proportional to $\kappa^2$ is negative for any $\phi$ and it provides an attractive force for the fluctuation of $\phi$ from its classical configuration, the kink.

Now, we consider the second point to be examined. In general, the stability of a field configuration is assured by the positivity of the second derivative of the potential around the configuration. For the case of $\lambda = 0$, it turns out to be

$$\frac{d^2V}{d\phi^2} = b^2 (1 + 4\frac{\mu}{b} \frac{\sinh^2(by) - 1}{\cosh^2(by)}) \equiv -M^2(y),$$

indicating that $M^2(y)$ is negative near the center of the thick brane, $y \sim 0$. It is a non-trivial problem to see whether this region causes the instability of the thick brane.
solution or not. On the other hand, we should notice another important point. The scalar fluctuation mixes in general with the scalar components of the metric \[12, 13, 14\], when the scalar takes a nontrivial background configuration just as the case considered here.

In order to show the importance of the mixing, we first switch it off. The resultant eigenvalue equation for the fluctuation (\(\delta \phi\)) of \(\phi\) around its classical solution is obtained, by representing it as \(\delta \phi = \int dm \varphi(m, x) \Phi_m(y)\), as

\[
\Phi''_m + \frac{4A'}{A} \Phi'_m + \frac{m^2}{b^2 A^2} \Phi_m = M^2 \Phi_m.
\]  

Before performing a numerical analysis, we make a small speculation on this equation. By introducing the variable \(q = by\) and the parameter \(\xi = \mu/b = v^2 \kappa^2/12\), (17) is written as

\[
\ddot{\Phi}_m + \frac{4A'}{A} \dot{\Phi}_m + \frac{m^2}{b^2 A^2} \Phi_m = \tilde{M}^2 \Phi_m,
\]  

where

\[
A = \cosh^{-\xi}(q), \quad \tilde{M}^2 = (1 + 4\xi) \frac{\sinh^2(q) - 1}{\cosh^2(q)}.
\]

where the dot \(\cdot\) denotes \(\partial_q\). Equation (18) shows that its eigenvalue \(m^2/b^2\) depends only on \(\xi\): \(m^2/b^2 = P(\xi)\). In the limit of small \(\kappa^2\) with \(v\) and \(b\) fixed, \(A\) tends to 1. In such a flat space-time, as a well-known fact, \(\delta \phi\) has a zero mode leading to \(P(0) = 0\) at \(\xi = 0\). At small \(\xi\), \(m^2\) is then represented as

\[
m^2 = c\xi b^2 = c\mu b
\]

with a constant \(c\). Figure 1 shows that numerical solution yields \(c = -2.73\). The fact of \(c < 0\) is understandable, as shown below, through the the Shroedinger-type eigenvalue equation derived from (18), since in the eigenvalue equation the potential becomes more attractive near the brane when \(\xi\) is put on.

Here we take the boundary condition for \(\Phi_m\) as \(\Phi'_m(0) = 0\), which is corresponding to the Neuman condition used for the Planck brane. While \(\Phi_m(0)\) is arbitrary due to the linearity of the equation. The ground state wave function is shown in the Fig.2. It is localized on the brane.

The results shown above could be understood more qualitatively according to the usual formulation. Rewrite (17), with new quantities \(z\) and \(u(z)\) defined by \(\partial z/\partial y = \pm A^{-1}\) and \(\Phi_m = A^{-3/2} u(z)\), as

\[
[-\partial_z^2 + V_g(z)] u(z) = m^2 u(z),
\]  

where the potential \(V_g(z)\) is given as

\[
V_g(z) = \frac{9}{4} (A')^2 + \frac{3}{2} A A'' + A^2 M^2.
\]
The first two terms of the potential $V_{g}(z)$ are the pure gravitational part and they would be important in obtaining the tachyonic state given above. Further the finiteness of $\kappa$ in the third term is also important. To see the total gravitational effects, we compare the potential $V_{g}$ with the case of $\kappa = 0$ (denoted by $V_{0}$). In Fig.2, we show them for $\mu = b = 1$. For the case of $V_{0}$, the lowest eigenvalue is seen at zero. While the potential becomes deep and its shape is the famous volcano type when the gravitational effect works. As a result, the ground state energy is pull down to the tachyonic point which is shown in the Fig.2 by $E_{g}$. These results are naturally understood.

Now we consider the mixing of the scalar fluctuation with the metric ones. In this case, the potential $V_{g}$ in the corresponding Schrödinger-like equation (20) is modified and it changes to a positive-definite one [12, 13, 14], so that the scalar fluctuation has no tachyon.

4 Conclusion

We have studied, through two conditions, whether the thick-brane world constructed from the real scalar field is stable or not. For the thick dS brane case, with a positive 4d cosmological constant ($\lambda > 0$), the thick-brane world breaks, near the horizon, the reality condition of the scalar field ($\phi$). This difficulty is cured by taking the limit of $\lambda \to 0$ or the thin brane limit. When we consider the Poincaré brane with no $\lambda$, on the other hand, the thick-brane world recovers the reality condition, and the fluctuation of $\phi$ does not generate any tachyonic mode by an effect of the coupling between the

Fig. 1: Tachyonic eigenemass as a function of $b$. Here $\mu$ is fixed at 1. The solid line shows the exact solution calculated numerically, while the dashed one does a line $m^{2} = -2.73b$. 
Fig. 2: Potential $V_g$ are shown for $\mu = b = 1$. The case of $\kappa = 0$ is shown by the dotted curve denoted by $V_0$, and the point denoted by $E_g = -2.963$ is the lowest eigenvalue with the potential $V_g$. By the dotted curve $W_f$, the ground state wave function is expressed.

The configuration of the thick brane constructed above, in general, depends on how to smooth the warp factor $A(y)$ of the thin-brane world. A different type of smoothing is proposed in [14]. In this type of smoothing, the stability of thick dS brane may depend on parameters taken; the thick brane seems to be unstable for large thickness of the brane, that is, for $n < 1$ in the notation of [14]. Further analysis is thus highly expected.

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