A lower bound for the size of Kakeya sets with respect to hyperplanes in $\mathbb{F}_q^n$

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Abstract

We prove that a subset of $\mathbb{F}_q^n$ that contains a hyperplane in any direction has size at least $q^n - O(q^2)$.

1 Introduction

The classical Euclidean Kakeya problem is an important problem in harmonic analysis: let $E \subset \mathbb{R}^n$ be a compact subset. We say $E$ is Kakeya if it contains a unit line segment in every direction. The problem claims that if $E$ is Kakeya then

$$\dim E = n$$

where $\dim E$ denotes for either the Hausdorff or the Minkowski dimension (see [Tao08]).

To give a better understanding of the problem we explain the Minkowski dimension. Let $E \subset \mathbb{R}^n$ and $\delta > 0$. Then we define $E_\delta = \{ x \in \mathbb{R}^n | \text{dist}(x, E) < \delta \}$ with $\text{dist}(x, E) = \inf \{ \|x - a\| | a \in E \}$. One may think of $E_\delta$ as the union of all open balls with radius $r$ around every point in $E$.

This allows us to define the Minkowski dimension for some $E \subset \mathbb{R}^n$ as

$$\dim E := n - \lim_{\delta \to 0} \frac{\log \text{vol}(E_\delta)}{\log \delta}. \quad (1)$$

A good example to see how the definition works is to compute the dimension of a sphere $S \subset \mathbb{R}^3$ of radius $r$. Then $S_\delta = \{ x \in \mathbb{R}^3 | r - \delta < ||x|| < r + \delta \}$. This denotes for a set that contains the open ball with radius $r + \delta$ where the ball with radius $r - \delta$ has been cut out. Thus

$$\text{vol}(S_\delta) = \frac{4\pi}{3}((r + \delta)^3 - (r - \delta)^3) = \frac{4\pi\delta}{3}(6r^2 + 2\delta^2)$$

which implies by (1) that

$$\dim S = 3 - \lim_{\delta \to 0} \frac{\log \frac{4\pi}{3} + \log \delta + \log(6r^2 + \delta^2)}{\log \delta} = 2.$$
If $B \subset \mathbb{R}^3$ is a ball with radius $r$ instead, then
\[
\text{vol}(B_{\delta}) = \frac{4\pi}{3}(r+\delta)^3
\]
which means by (1) that
\[
\dim B = 3 - \lim_{\delta \to 0} \frac{\log \frac{4\pi}{3} + 3\log(r+\delta)}{\log \delta} = 3.
\]

It is however very hard to solve the Euclidean Kakeya problem which is why the problem is still open for $n > 2$. The case $n = 2$ was solved by Davies in [Dav71].

In 1999, T. Wolff proposed a discrete analogue of the Kakeya problem. Let $\mathbb{F}_q$ be the field with $q$ elements. We define a line to be the translate of a one-dimensional linear subspace. The direction of a given line $w'$ is the unique one-dimensional subspace $w$ such that $w'$ is a translate of $w$. A set $E \subset \mathbb{F}_q^n$ is Kakeya if $E$ contains a line in every direction, i.e. for all $v \in \mathbb{F}_q^n \setminus \{0\}$ there is a $v_0 \in \mathbb{F}_q^n$ such that $\{x_0 + tv | t \in \mathbb{F}_q^n\} \subset E$. The problem then claims that for all $n$ there is a $c > 0$ (only depending on $n$) such that for all Kakeya sets $E \subset \mathbb{F}_q^n$
\[
|E| \geq cq^n.
\]

Z. Dvir was able to solve the problem with $c = \frac{1}{n^5}$ using the so-called polynomial method.

One generalization of the Kakeya problem is the $k$-plane Furstenberg set problem in $\mathbb{F}_q^n$ (see [EE15], Question 1.3). Let $S_k$ be the set of all subspaces $w \subset \mathbb{F}_q^n$ such that $\dim(w) = k$. We define a $k$-plane $w' \subset \mathbb{F}_q^n$ to be a translated $k$-dimensional subspace. Given a $k$-plane $w'$, its direction is defined as the unique subspace $w \in \mathbb{F}_q^n$ such that $w'$ is a translate of $w$. Fix some $c > 0$ and let $E \subset \mathbb{F}_q^n$ be such that for every direction $w \in S_k$ there is at least one $k$-plane $w'$ with $|w' \cap E| \geq q^c$. The problem asks for a lower bound for $|E|$.

This problem implies the finite field Kakeya problem namely when $c = k = 1$. Note that $c = k$ means $w' \subset E$. We therefore consider this case as the generalization of the Kakeya problem.

**Definition 1** (Kakeya set with respect to $k$-planes). A set $E \subset \mathbb{F}_q^n$ is Kakeya with respect to $k$-planes if for every $w \in S_k$ there is a $k$-plane $w' \subset \mathbb{F}_q^n$ in direction $w$ such that $w' \subset E$.

We will only discuss the size of Kakeya sets w.r.t. to $(n-1)$-planes also known as hyperplanes and will therefore call a set Kakeya if it is Kakeya w.r.t. hyperplanes. From the result from Dvir one would expect that for every $n$ there is a constant $c > 0$ such that for every Kakeya set $E \subset \mathbb{F}_q^n$ the inequality $|E| \geq cq^n$ from the original problem holds. In fact, we are able to show the following theorem.

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**Theorem 1.** Every set $E \subset \mathbb{F}_q^n$ that is Kakeya fulfills

$$|E| \geq \frac{q^{2n} - q^n}{q^n + q^2 - 2q}.$$ 

A known proof, independent of Dvirs method, solves the problem when $n = 2$ and gives

$$|E| \geq \frac{q(q + 1)}{2}$$

as a bound for every set $E \subset \mathbb{F}_q^2$ that is Kakeya. This proof is a discrete version of the previously mentioned proof in [Dav71]. We were not able to trace it back in the literature, but to read an exposition of this proof see [Sla14].

We modify and generalize the proof so that it gives Theorem 1. What is surprising is that while for $n = 2$ the asymptotic bound is $|E| \geq \frac{q^2}{2} - O(q)$, for $n > 2$ it is $|E| \geq q^n - O(q^2)$ rather than the expected $|E| \geq cq^n - O(q^{n-1})$ with some $c \in (0, 1)$.

## 2 The proof

Before being able to prove the theorem we need to prove two lemmas. Let $S = S_{n-1}$.

**Lemma 1.** The set $S$ fulfills

$$|S| = \frac{q^n - 1}{q - 1}.$$ 

**Proof.** Let

$$K = \{(k_1, \ldots, k_{n-1}) \in \mathbb{F}_q^n \times \cdots \times \mathbb{F}_q^n | \dim \text{span}(k_1, \ldots, k_{n-1}) = n - 1\}.$$ 

We may choose any element of $\mathbb{F}_q^n \setminus \{0\}$ for $k_1$. If $k_2 \in \text{span}(k_1)$ then $\text{span}(k_1, k_2) = \text{span}(k_1)$. If however $k_2 \notin \text{span}(k_1)$ then $\text{span}(k_1) \subset \text{span}(k_1, k_2)$ and hence $\dim \text{span}(k_1, k_2) = 2$. Thus $k_2 \in \mathbb{F}_q^n \setminus \text{span}(k_1)$. One can easily generalize this argument and conclude that $k_h \in \mathbb{F}_q^n \setminus \text{span}(k_1, \ldots, k_{h-1})$. Therefore

$$|K| = \prod_{h=0}^{n-2} (q^n - q^h).$$

Define $\phi : K \mapsto S$ to be $\phi(k_1, \ldots, k_{n-1}) = \text{span}(k_1, \ldots, k_{n-1})$. Thus

$$|K| = \sum_{w \in S} |\text{fiber}_\phi(w)|.$$
We count the number of \((k_1, \ldots, k_{n-1}) \in \text{fiber}_\phi(w)\). We may choose any element from \(w\setminus\{0\}\) for \(k_1\). Using a similar argument as above we see that \(k_h \in w\setminus\text{span}(k_1, \ldots, k_{h-1})\) which shows that 

\[
|\text{fiber}_\phi(w)| = \prod_{h=0}^{n-2} (q^{n-1} - q^h).
\]

Therefore 

\[
|S| = \prod_{h=0}^{n-2} \frac{q^n - q^h}{q^{n-1} - q^h} = \prod_{h=1}^{n-1} \frac{q^{h+1} - 1}{q^h - 1} = \frac{q^n - 1}{q - 1}.
\]

**Lemma 2.** Let \(A\) and \(B\) be finite sets and let \(\phi : A \to B\) be a map. If 
\(C = \{(a_1, a_2) \in A \times A|\phi(a_1) = \phi(a_2)\}\) then 
\[|B| \geq \frac{|A|^2}{|C|}.\]

**Proof.** Let \(b \in B\) and define \(C_b = \{(a_1, a_2) \in C|\phi(a_1) = b\}\). Note that 
\[|C| = \sum_{b \in B} |C_b|.
\]

For a fixed \(b\) one may choose any \(a_1 \in \text{fiber}_\phi(b)\) and any \(a_2 \in \text{fiber}_\phi(b)\). Thus 
\[|C_b| = |\text{fiber}_\phi(b)|^2.\]

By applying the Cauchy-Schwarz inequality we obtain 
\[\left( \sum_{b \in B} |\text{fiber}_\phi(b)| \cdot 1 \right)^2 \leq \sum_{b \in B} |\text{fiber}_\phi(b)|^2 \sum_{b \in B} 1\]
which is by the previous observations equivalent to the lemma.

**Proof of Theorem 1.** For every \(w \in S\) take a \(v_w \in \mathbb{F}_q^n\) such that \(w' = v_w + w\) is contained in \(E\). Let 
\[I = \{(w, v) \in S \times E|v \in w'\}.
\]

There are \(|S|\) possibilities for \(w\) while every \(w\) contributes \(q^{n-1}\) points since 
\[|w'| = q^{n-1}.\]

Thus \(|I| = |S|q^{n-1}\).

Let \(\gamma : I \to E\) be \(\gamma(w, v) = v\) and define 
\[W = \{(w_1, w_2, v) \in S \times S \times E|v \in w'_1 \cap w'_2\}.
\]
It is easy to see that there is a bijection between $W$ and $K = \{((w_1, v_1), (w_2, v_2)) \in I \times I | v_1 = v_2\}$ given by $(w_1, w_2, v) \mapsto ((w_1, v), (w_2, v))$. This implies by Lemma 2 that

$$|E| \geq \frac{|I|^2}{|W|}. \tag{2}$$

The last step before being able to obtain the final inequality is to compute $|W|$. To do so, we differ two cases:

1. $w_1 = w_2$: This case contributes $|I|$ elements since for every $(w_1, v) \in I$ there is a $(w_1, w_1, v) \in W$ and vice versa.

2. $w_1 \neq w_2$: Note first that by assumption $w_1 \parallel w_2$. By the dimension formula for subspaces $w_1' \cap w_2'$ has cardinality $q^{n-2}$. There are $|S|(|S| - 1)$ ordered pairs of not parallel hyperplanes. Hence this case contributes $|S|(|S| - 1)q^{n-2}$ elements to $W$.

This leaves us with

$$|W| = |I| + |S|(|S| - 1)q^{n-2}.$$  

By using (2) and Lemma 1 we deduce that

$$|E| \geq \frac{|S|^2q^{2n-2}}{|S|^2 - |S| + |S(q)|q^{n-2}} = \frac{q^{2n} - q^n}{q^n + q^2 - 2q}$$

Note how our final bound for $|E|$ has the form $|E| \geq q^n - O(q^2)$ for $n > 2$ and $|E| \geq \frac{q^n}{2q} - O(q)$ for $n = 2$. Namely, when $n > 2$ the leading term in the denominator is $q^n$ while it is $2q^2$ when $n = 2$.

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