ON LORENTZIAN TRANS-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study the Trans-Sasakian structure on a manifold with Lorentzian metric. Several interesting results are obtained on the manifold. Also conformally flat Lorentzian Trans-Sasakian manifolds have been studied. Next, in three-dimensional Lorentzian Trans-Sasakian manifolds, explicit formulae for Ricci operator, Ricci tensor and curvature tensor are obtained. Also it is proved that a three-dimensional Lorentzian Trans-Sasakian manifold of type $(\alpha, \beta)$ is locally $\phi$-symmetric if and only if the scalar curvature $r$ is constant provided $\alpha$ and $\beta$ are constants. Finally, we give some examples of three-dimensional Lorentzian Trans-Sasakian manifold.

1. Introduction

Let $M$ be an odd dimensional manifold with Riemannian metric $g$. It is well known that an almost contact metric structure $(\phi, \xi, \eta)$ (with respect to $g$) can be defined on $M$ by a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$. If $M$ has a Sasakian structure (Kenmotsu structure), then $M$ is called a Sasakian manifold (Kenmotsu manifold). Sasakian manifolds and Kenmotsu manifolds have been studied by several authors. In the classification of Gray and Hervella [8] of almost Hermitian manifolds there appears a class, $W_4$, of Hermitian manifolds which are closely related to locally conformally Kaehler manifolds. An almost contact metric structure $(\phi, \xi, \eta, g)$ on $M$ is Trans-Sasakian [17] if $(M \times \mathbb{R}, J, G)$ belongs to the class $W_4$, where $J$ is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(X, f \frac{dt}{dt}) = (\phi X - f\xi, \eta(X) \frac{dt}{dt}),$$

for all vector fields $X$ on $M$, $f$ is a smooth function on $M \times \mathbb{R}$ and $G$ is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [2]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

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for smooth functions $\alpha$ and $\beta$ on $M$. Hence we say that the Trans-Sasakian structure is of type $(\alpha, \beta)$. In particular, it is normal and it generalizes both $\alpha$-Sasakian and $\beta$-Kenmotsu structures. From the formula (1.1) one easily obtains

$$\nabla_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi).$$

(1.2)

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).$$

(1.3)

In 1981, Janssens and Vanhecke introduced the notion of $\alpha$-Sasakian and $\beta$-Kenmotsu manifolds where $\alpha$ and $\beta$ are non zero real numbers. It is known that [6] Trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic ([1], [2]), $\beta$-Kenmotsu ([6]) and $\alpha$-Sasakian ([6]) respectively. The local structure of Trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by Marrero [10]. He proved that a Trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or $\alpha$-Sasakian or $\beta$-Kenmotsu manifold. Trans-Sasakian manifolds have been studied by several authors ([3], [4], [5], [11], [18]).

Let $(x, y, z)$ be cartesian co-ordinates in $\mathbb{R}^3$, then $(\phi, \xi, \eta, g)$ given by

$$\xi = \frac{\partial}{\partial z}, \eta = dz - ydx,$$

$$\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}, \quad g = \begin{pmatrix} e^z + y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix}$$

is a Trans-Sasakian structure of type $(\frac{1}{2e^z}, \frac{1}{2})$ in $\mathbb{R}^3$ [2]. In general, in a three-dimensional $K$-contact manifold with structure tensors $(\phi, \xi, \eta, g)$ for a non-constant function $f$, if we define $\tilde{g} = fg + (1 - f)\eta \otimes \eta$; then $(\phi, \xi, \eta, \tilde{g})$ is a Trans-Sasakian structure of type $(\frac{1}{4}, \frac{1}{2}\xi(ln f))$ [10].

Let $M$ be a differentiable manifold. When $M$ has a Lorentzian metric $g$, that is, a symmetric non degenerate $(0, 2)$ tensor field of index 1, then $M$ is called a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold $M$ has not only spacelike vector fields but also timelike and lightlike vector fields. This difference with the Riemannian case give interesting properties on the Lorentzian manifold. A differentiable manifold $M$ has a Lorentzian metric if and only if $M$ has a 1-dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric. Therefore, it is very natural and interesting idea to define both a Trans-Sasakian structure and a Lorentzian metric on an odd dimensional manifold.

The paper is organized as follows. In Section 1, we give a brief account of Lorentzian Trans-Sasakian manifolds. After preliminaries, some basic results are given. In Section 4, we study conformally flat Lorentzian Trans-Sasakian manifolds.
In the next section, explicit formulae for Ricci operator, Ricci tensor and curvature tensor are obtained for three-dimensional Trans-Sasakian manifolds. Also it is proved that a three-dimensional Lorentzian Trans-Sasakian manifold of type \((\alpha, \beta)\) is locally \(\phi\)-symmetric if and only if the scalar curvature \(r\) is constant provided \(\alpha\) and \(\beta\) are constants. Finally we construct some examples of three-dimensional Lorentzian Trans-Sasakian manifolds.

2. Lorentzian Trans-Sasakian manifolds

A differentiable manifold \(M\) of dimension \((2n + 1)\) is called a Lorentzian Trans-Sasakian manifold if it admits a \((1,1)\) tensor field \(\phi\), a contravariant vector field \(\xi\), a covariant vector field \(\eta\) and the Lorentzian metric \(g\) which satisfy

\[
\eta(\xi) = -1, \quad \phi^2 = I + \eta \otimes \xi, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad (\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(\xi)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),
\]

for all \(X, Y \in T(M)\).

Also a Lorentzian Trans-Sasakian manifold \(M\) satisfies

\[
\nabla_X \xi = -\alpha(\phi X) - \beta(X + \eta(X))\xi, \quad (\nabla_X \eta)Y = \alpha g(\phi X, Y) + \beta g(\phi X, \phi Y),
\]

where \(\nabla\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\).

If \(\alpha = 0\) and \(\beta \in \mathbb{R}\), the set of real numbers, then the manifold reduces to a Lorentzian \(\beta\)-Kenmotsu manifold studied by Funda Yaliniz, Yildiz, and Turan [20]. If \(\beta = 0\) and \(\alpha \in \mathbb{R}\), then the manifold reduces to a Lorentzian \(\alpha\)-Sasakian manifold studied by Yildiz, Turan and Murathan [21]. If \(\alpha = 0\) and \(\beta = 1\), then the manifold reduces to a Lorentzian Kenmotsu manifold introduced by Mihai, Oiaga and Rosca [15]. Furthermore, if \(\beta = 0\) and \(\alpha = 1\), then the manifold reduces to a Lorentzian Sasakian manifold studied by Ikawa and Erdogan [15]. Also Lorentzian para contact manifolds were introduced by Matsumoto [12] and further studied by the authors ([13],[14],[16]). Trans Lorentzian para Sasakian manifolds have been used by Gill and Dube [7].
3. SOME BASIC RESULTS

In this section, we prove some Lemmas which are needed in the rest of the sections.

**Lemma 3.1.** In a Lorentzian Trans-Sasakian manifold, we have

\[
R(X, Y)\xi = (\alpha^2 + \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) + (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y,
\]

(3.1)

where \( R \) is the curvature tensor.

**Proof.** We have

\[
\nabla_X \nabla_Y \xi = \nabla_X(-\alpha(\phi Y) - \beta(Y + \eta(Y)\xi)) = -(X\alpha)\phi Y - \alpha \nabla_X(\phi Y) - (X\beta)\phi^2 Y - \beta \nabla_XY - \beta(\eta(Y))\xi + \alpha \beta\eta(Y)\phi X + \beta^2 \eta(Y)X + \beta^2 \eta(X)\eta(Y)\xi,
\]

where (2.2) and (2.6) have been used. Hence, in view of the above equation and (2.6), we get

\[
R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]} \xi
\]

\[
= -(X\alpha)\phi Y + (Y\alpha)\phi X - \alpha((\nabla_X \phi Y) - (\nabla_Y \phi X)) - (X\beta)\phi^2 Y + (Y\beta)\phi^2 X - \beta((\nabla_X \eta Y) - (\nabla_Y \eta)X)\xi + \alpha \beta(\eta(Y)\phi X - \eta(X)\phi Y) + \beta^2 \eta(Y)X - \eta(X)Y,
\]

which in view of (2.5) and (2.7) gives (3.1).

**Lemma 3.2.** For a Lorentzian Trans-Sasakian manifold, we have

\[
\eta(R(X, Y)Z) = (\alpha^2 + \beta^2)(g(X, Z)\eta(Y) - g(Y, Z)\eta(X)).
\]

(3.2)

**Proof.** We have from (3.1),

\[
g(R(X, Y)\xi, Z) = (\alpha^2 + \beta^2)(\eta(Y)g(X, Z) - \eta(X)g(Y, Z)) + 2\alpha\beta(\eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z)) + (Y\alpha)g(\phi X, Z) - (X\alpha)g(\phi Y, Z) + (Y\beta)g(\phi^2 X, Z) - (X\beta)g(\phi^2 Y, Z),
\]

Now interchanging \( \xi \) and \( Z \) in the above equation, we get

\[
-g(R(X, Y)Z, \xi) = (\alpha^2 + \beta^2)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)) + 2\alpha\beta(\eta(Y)g(\phi X, \xi) - \eta(X)g(\phi Y, \xi)) + (Y\alpha)g(\phi X, \xi) - (X\alpha)g(\phi Y, \xi) + (Y\beta)g(\phi^2 X, \xi) - (X\beta)g(\phi^2 Y, \xi).
\]

After simplification, we find,

\[
g(R(X, Y)Z, \xi) = (\alpha^2 + \beta^2)(g(X, Z)\eta(Y) - g(Y, Z)\eta(X)),
\]
which gives (3.2).

**Lemma 3.3.** For a Lorentzian Trans-Sasakian manifold, we have

\[ R(X;Y)\xi = (\alpha^2 + \beta^2 - \xi \beta)\phi Y + (2\alpha \beta - \xi \alpha)\phi Y. \]  

**(3.3)**

**Proof.** Replacing \( X \) by \( \xi \) in (3.1), we get (3.3).

**Lemma 3.4.** In a \((2n+1)\)-dimensional Lorentzian Trans-Sasakian manifold, we have

\[
S(X;Y) = (2n\alpha^2 + \beta^2 - \xi \beta)\eta(X) + (2n-1)(X\beta)
\]

\[
- (\phi X)\alpha + \psi(2\alpha \beta \eta(X) + X\alpha),
\]

**(3.4)**

\[
Q\xi = (2n\alpha^2 + \beta^2 - \xi \beta)\xi + (2n-1)grad\beta
\]

\[
- \phi(grad\alpha) + \psi(2\alpha \beta \xi + grada),
\]

**(3.5)**

where \( S \) is the Ricci curvature and \( Q \) is the Ricci operator given by

\[
S(X,Y) = g(QX,Y) \quad and \quad \psi = \sum_{i=1}^{2n+1} \epsilon_i g(\phi e_i, e_i).
\]

**Proof.** Let \( M \) be an \((2n+1)\)-dimensional Lorentzian Trans-Sasakian manifold. Then the Ricci tensor \( S \) of the manifold \( M \) is defined by

\[
S(X;Y) = \sum_{i=1}^{2n+1} \epsilon_i g(R(e_i, X)Y, e_i),
\]

where \( \epsilon_i = g(e_i, e_i) \), \( \epsilon_i = \pm 1 \). From (3.1), we have

\[
S(X;\xi) = (\alpha^2 + \beta^2)\eta(X) \sum_{i=1}^{2n+1} g(e_i, e_i)g(e_i, e_i) - \sum_{i=1}^{2n+1} \eta(e_i)g(e_i, e_i)g(X, e_i)
\]

\[
+ 2\alpha \beta \eta(X) \sum_{i=1}^{2n+1} g(e_i, e_i)g(\phi e_i, e_i) - \sum_{i=1}^{2n+1} \eta(e_i)g(e_i, e_i)g(\phi X, e_i)
\]

\[
- \sum_{i=1}^{2n+1} (\epsilon_i \alpha)g(e_i, e_i)g(\phi X, e_i) + \sum_{i=1}^{2n+1} (X\alpha)g(e_i, e_i)g(\phi e_i, e_i)
\]

\[
- \sum_{i=1}^{2n+1} (\epsilon_i \beta)g(e_i, e_i)g(\phi^2 X, e_i) + \sum_{i=1}^{2n+1} (X\beta)g(e_i, e_i)g(\phi e_i, e_i)
\]

\[
= (2n\alpha^2 + \beta^2 - \xi \beta)\eta(X) + (2n-1)(X\beta)
\]

\[
- (\phi X)\alpha + \psi(2\alpha \beta \eta(X) + X\alpha)
\]

and hence from (3.4), we get (3.5).
Remark 3.5. If in a \((2n + 1)\)-dimensional Lorentzian Trans-Sasakian manifold of type \((\alpha, \beta)\) we consider \(\phi(\text{grad}\alpha) = (2n - 1)\text{grad}\beta\), then
\[
\xi\beta = g(\xi, \text{grad}\beta) = \frac{1}{2n - 1} g(\xi, \phi(\text{grad}\alpha)) = \frac{1}{2n - 1} \eta(\phi(\text{grad}\alpha)) = 0
\]
and
\[
X\beta = g(X, \text{grad}\beta) = \frac{1}{2n - 1} g(X, \phi(\text{grad}\alpha)) = \frac{1}{2n - 1} g(\phi X, \text{grad}\alpha) = \frac{1}{2n - 1} (\phi X)\alpha
\]
and hence (3.4) and (3.5) are reduced to
\[
S(X, \xi) = 2n(\alpha^2 + \beta^2)\eta(X) + \psi(2\alpha\beta\eta(X) + X\alpha)
\]
and
\[
Q\xi = (2n(\alpha^2 + \beta^2) - \xi\beta)\xi + \psi(2\alpha\beta\xi + \text{grad}\alpha),
\]
respectively.

4. CONFORMALLY FLAT LORENTZIAN TRANS-SASAKIAN MANIFOLDS

In this section we consider conformally flat Lorentzian Trans-Sasakian manifold \(M^{2n+1}(\phi, \xi, \eta, g)\) \((n > 1)\). The conformal curvature tensor \(C\) is given by
\[
C(X, Y)Z = R(X, Y)Z - \frac{1}{2n - 1}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{(2n)(2n - 1)}[g(Y, Z)X - g(X, Z)Y], \tag{4.1}
\]
where \(r\) is the scalar curvature of \(M\).

For conformally flat manifold, we have \(C(X, Y)Z = 0\) for \(n > 1\) and hence from (4.1) we have
\[
\bar{R}(X, Y, Z, W) = \frac{1}{2n - 1}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) + g(Y, Z)S(X, W) - g(X, Z)S(Y, W)]
- \frac{r}{(2n)(2n - 1)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)], \tag{4.2}
\]
where \(g(R(X, Y)Z, U) = \bar{R}(X, Y, Z, U)\). Setting \(W = \xi\) in (4.2) we get
\[ \eta(R(X,Y)Z) = \frac{1}{2n-1}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y) \\
+ g(Y,Z)S(X,\xi) - g(X,Z)S(Y,\xi)] \\
- \frac{r}{(2n)(2n-1)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{4.3} \]

Replacing \( Y \) by \( \xi \) in (4.3) and using (3.2) and (3.6), we get

\[ S(X,Z) = \left( (\alpha^2 + \beta^2)(1 - 6n) + \frac{r}{2n} - 4\alpha\beta\psi \right) \eta(X)\eta(Z) \\
- \left[ \eta(Z)(X\alpha) + \eta(X)(Z\alpha) \right] \psi. \tag{4.4} \]

This leads to the following:

**Theorem 4.1.** A conformally flat Lorentzian Trans Sasakian manifold \( M^{2n+1} \) \((\phi, \xi, \eta, g) \) \((n > 1)\) is an \( \eta \)– Einstein manifold provided \( \psi = \text{trace} \phi = 0 \) and \( \phi(\text{grad} \alpha) = (2n-1)\text{grad} \beta \).

**Corollary 1.** A conformally flat Lorentzian \( \beta \)– Kenmotsu manifold \( M^{2n+1} \) \((\phi, \xi, \eta, g) \) \((n > 1)\) is an \( \eta \)– Einstein manifold.

5. **Three- dimensional Lorentzian Trans- Sasakian manifolds**

Since the conformal curvature tensor vanishes in a three-dimensional Riemannian manifold, therefore we get

\[ R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\
- \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \tag{5.1} \]

where \( Q \) is the Ricci operator, that is, \( g(QX,Y) = S(X,Y) \) and \( r \) is the scalar curvature of the manifold.

From Lemma 2.4, in a three- dimensional Lorentzian Trans-Sasakian manifold we have

\[ S(X,\xi) = (2(\alpha^2 + \beta^2) - \xi \beta)\eta(X) + (X\beta) \\
- (\phi X)\alpha + \psi(2\alpha\beta\eta(X) + X\alpha), \tag{5.2} \]

\[ Q\xi = (2(\alpha^2 + \beta^2) - \xi \beta)\xi + \text{grad} \beta \\
- \phi(\text{grad} \alpha) + \psi(2\alpha\beta \xi + \text{grad} \alpha). \tag{5.3} \]
Now, in the following theorem, we obtain an expression for Ricci operator in a three-dimensional Lorentzian Trans-Sasakian manifold.

**Theorem 5.1.** In a three-dimensional Lorentzian Trans Sasakian manifold, the Ricci operator is given by

\[ QX = \left( \frac{r}{2} + \xi \beta - (\alpha^2 + \beta^2) + \psi(\xi \alpha - 2\alpha \beta) \right)X + \left( \frac{r}{2} + \xi \beta - 3(\alpha^2 + \beta^2) - 4\alpha \beta \psi \right)\eta(X)\xi - \eta(X)(\text{grad} \beta - \phi(\text{grad} \alpha) + \psi(\text{grad} \alpha)) - (X \beta - (\phi X) \alpha + \psi(X \alpha))\xi + (2\alpha \beta - \xi \alpha) \phi X. \] (5.4)

**Proof.** For a three-dimensional Lorentzian Trans Sasakian manifold, from (5.1) and (5.2), we have

\[ R(X,Y)\xi = \eta(Y)QX - \eta(X)QY - \left( \frac{r}{2} + \xi \beta - 2(\alpha^2 + \beta^2) - 2\alpha \beta \psi \right)[X\eta(Y) - Y\eta(X)] + (Y\beta - (\phi Y)\alpha + (Y\alpha)\psi)X - (X\beta - (\phi X)\alpha + (X\alpha)\psi)Y. \] (5.5)

In view of (3.1) and (5.5), we obtain

\[ 2\alpha \beta(\eta(Y)\phi X - \eta(X)\phi Y) + (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y = \eta(Y)QX - \eta(X)QY - \left( \frac{r}{2} + \xi \beta - (\alpha^2 + \beta^2) - 2\alpha \beta \psi \right) \]
\[ [\eta(Y)X - \eta(X)Y] + (Y\beta - (\phi Y)\alpha + (Y\alpha)\psi)X - (X\beta - (\phi X)\alpha + (X\alpha)\psi)Y. \]

Putting \( Y = \xi \) in the above equation, we get (5.4). \( \square \)

**Corollary 2.** In a three-dimensional Lorentzian Trans Sasakian manifold, Ricci tensor and curvature tensor are given respectively by

\[ S(X,Y) = \left( \frac{r}{2} + \xi \beta - (\alpha^2 + \beta^2) + \psi(\xi \alpha - 2\alpha \beta) \right)g(X,Y) + \left( \frac{r}{2} + \xi \beta - 3(\alpha^2 + \beta^2) - 4\alpha \beta \psi \right)\eta(X)\eta(Y) + \eta(X)[-Y\beta + (\phi Y)\alpha - \psi(Y \alpha)] - \eta(Y)(X\beta - (\phi X)\alpha + \psi(X \alpha)) + (2\alpha \beta - \xi \alpha)g(\phi X, Y). \] (5.6)
and

\[
R(X, Y)Z = \left(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 + \beta^2) + 2\psi(\xi\alpha - 2\alpha\beta))\right)\left[g(Y, Z)X - g(X, Z)Y\right] \\
+ g(Y, Z)\left[\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi\right]\eta(X)\xi \\
+ \eta(X)(\phi(\text{grad}\alpha) - \psi(\text{grad}\alpha) - \text{grad}\beta) - (X\beta - (\phi X)\alpha + \psi(X\alpha))\xi] \\
+ g(X, Z)\left[\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi\right]\eta(Y)\xi \\
+ \eta(Y)(\phi(\text{grad}\alpha) - \psi(\text{grad}\alpha) - \text{grad}\beta) - (Y\beta - (\phi Y)\alpha + \psi(Y\alpha))\xi] \\
+ \left[\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi\right]\eta(Y)\eta(Z) \\
+ \eta(Y)(-Z\beta + (\phi Z)\alpha - \psi(Z\alpha)) - \eta(Z)(Y\beta - (\phi Y)\alpha + \psi(Y\alpha))\eta(X) \\
- \left[\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi\right]\eta(X)\eta(Z) \\
+ \eta(X)(-Z\beta + (\phi Z)\alpha - \psi(Z\alpha)) - \eta(Z)(X\beta - (\phi X)\alpha + \psi(X\alpha))\eta(Y) \\
+(2\alpha\beta - \xi\alpha)\left[g(\phi Y, Z)X - g(\phi X, Z)Y\right].
\] (5.7)

Equation (5.6) follows from (5.4). Using (5.4) and (5.6) in (5.1), the curvature tensor in a three-dimensional Lorentzian Trans-Sasakian manifold is given by (5.7).

6. Locally $\phi$-symmetric three-dimensional Lorentzian Trans-Sasakian manifolds with trace $\phi = \psi = 0$

The notion of locally $\phi$-symmetry was first introduced by T. Takahashi [19] on a Sasakian manifold. In this paper we study locally $\phi$-symmetric three-dimensional Lorentzian Trans-Sasakian manifolds.

**Definition 6.1.** A three-dimensional Lorentzian Trans-Sasakian manifold is said to be locally $\phi$-symmetric if

\[
\phi^2(\nabla_\mathcal{W} R)(X, Y)Z = 0,
\] (6.1)

where $W, X, Y, Z$ are horizontal vector fields, that is $W, X, Y, Z$ are orthogonal to $\xi$. 

Let $M$ be a three-dimensional Lorentzian Trans-Sasakian manifold with $\text{trace} \phi = \psi = 0$. Then its curvature tensor is given by

\[
R(X, Y)Z = (r^2 + 2\xi^2 - 2(\alpha^2 + \beta^2))[g(Y, Z)X - g(X, Z)Y]
+ g(Y, Z)[r^2 + \xi^2 - 3(\alpha^2 + \beta^2)]\eta(X)\xi
+ \eta(X)(\phi(\text{grad} \alpha) - \text{grad} \beta) - (X\beta - (\phi X)\alpha)\xi
+ g(X, Z)[r^2 + \xi^2 - 3(\alpha^2 + \beta^2)]\eta(Y)\xi
+ \eta(Y)(\phi(\text{grad} \alpha) - \text{grad} \beta) - (Y\beta - (\phi Y)\alpha)\xi
+ [r^2 + \xi^2 - 3(\alpha^2 + \beta^2)]\eta(Y)\eta(Z)
+ \eta(Y)(-Z\beta + (\phi Z)\alpha) - \eta(Z)(Y\beta - (\phi Y)\alpha)\eta(X)
- [r^2 + \xi^2 - 3(\alpha^2 + \beta^2)]\eta(X)\eta(Z)
+ \eta(X)(-Z\beta + (\phi Z)\alpha) - \eta(Z)(X\beta - (\phi X)\alpha)\eta(Y)
+ (2\alpha\beta - \xi \alpha)[g(\phi Y, Z)X - g(\phi X, Z)Y].
\]

(6.2)

Differentiating (6.2) we get

\[
(\nabla_w R)(X, Y)Z = \frac{dr(W)}{2} + 2(\nabla_w (\xi\beta)) - 4(\alpha dW + \beta dW)]
+ [g(Y, Z)X - g(X, Z)Y] + g(Y, Z)[\frac{dr(W)}{2} + (\nabla_w (\xi\beta))]
- 6(\alpha dW + \beta dW)\eta(X)\xi + (r^2 + \xi^2 - 3(\alpha^2 + \beta^2))[(\nabla_w \eta)(\eta(X)\xi + \eta(X)\nabla_w \xi) + (\nabla_w \eta)(\phi(\text{grad} \alpha) - \text{grad} \beta) + \eta(X)(\nabla_w (\phi(\text{grad} \alpha) - \text{grad} \beta))
+ (\nabla_w (X\beta - (\phi X)\alpha))\eta + (X\beta - (\phi X)\alpha)\nabla_w \xi]
- g(X, Z)[\frac{dr(W)}{2} + (\nabla_w (\xi\beta)) - 6(\alpha dW + \beta dW)]\eta(Y)\xi
+ [r^2 + \xi^2 - 3(\alpha^2 + \beta^2))\eta(Y)\eta(X)\nabla_w \xi]
+ (\nabla_w (Y\beta - (\phi Y)\alpha))\xi + (Y\beta - (\phi Y)\alpha)\nabla_w \xi\xi
- Y[(\nabla_w (Y\beta - (\phi Y)\alpha))\eta(Z) + (Y\beta - (\phi Y)\alpha)\nabla_w \eta)Z
+ \nabla_w (Z\beta - (\phi Z)\alpha)\eta(Y) + (Z\beta - (\phi Z)\alpha)\nabla_w \eta)Y
- (\frac{dr(W)}{2} + (\nabla_w (\xi\beta)) - 6(\alpha dW + \beta dW))\eta(Y)\eta(Z)
\]
ON LORENTZIAN TRANS-SASAKIAN MANIFOLDS

\[ -(\frac{r}{2} + \xi \beta - 3(\alpha^2 + \beta^2)) \]
\[ ((\nabla_W \eta) Y \eta(Z) + \eta(Y)(\nabla_W \eta)Z) \]
\[ + X[(\nabla_W (X \beta - (\phi X) \alpha)) \eta(Z) + (X \beta - (\phi X) \alpha)(\nabla_W \eta)Z \]
\[ + \nabla_W (Z \beta - (\phi Z) \alpha)) \eta(X) + (Z \beta - (\phi Z) \alpha)(\nabla_W \eta)X \]
\[ - (\frac{dr(W)}{2} + (\nabla_W (\xi \beta)) - 6(\alpha(W) + d\beta(W))) \eta(X) \eta(Z) \]
\[ - (\frac{r}{2} + \xi \beta - 3(\alpha^2 + \beta^2)) \]
\[ ((\nabla_W \eta) X \eta(Z) + \eta(X)(\nabla_W \eta)Z) \]
\[ +(2(\nabla_W (\alpha \beta)) - (\nabla_W (\xi \alpha)))[g(\phi Y, Z)X - g(\phi X, Z)Y]. \quad (6.3) \]

Suppose that \( \alpha \) and \( \beta \) are constants and \( X, Y, Z, W \) are orthogonal to \( \xi \). Then using \( \phi \xi = 0 \) and (6.1), we get
\[ \phi^2(\nabla_W R)(X, Y)Z = (\frac{dr(W)}{2})[g(Y, Z)X - g(X, Z)Y]. \quad (6.4) \]

Thus we can state the following:

**Theorem 6.2.** A three-dimensional Lorentzian Trans-Sasakian manifold of type \((\alpha, \beta)\) is locally \(\phi\)-symmetric if and only if the scalar curvature \( r \) is constant provided \( \alpha \) and \( \beta \) are constants.

### 7. Examples

**Example 7.1:** We consider the three-dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\} \), where \((x, y, z)\) are standard co-ordinate of \( \mathbb{R}^3 \).

The vector fields
\[ e_1 = z(\frac{\partial}{\partial x} + y \frac{\partial}{\partial z}), \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z} \]
are linearly independent at each point of \( M \).

Let \( g \) be the Riemannian metric defined by
\[ g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0, \]
\[ g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1. \]

Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in \chi(M) \).

Let \( \phi \) be the \((1, 1)\) tensor field defined by
\[ \phi(e_1) = -e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0. \]

Then using the linearity of \( \phi \) and \( g \), we have
\[ \eta(e_3) = -1, \]
\[ \phi^2Z = Z + \eta(Z)e_3, \]
\[ g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W), \]
for any \( Z, W \in \chi(M) \).
Then for \( e_3 = \xi \), the structure \((\phi, \xi, \eta, g)\) defines an Lorentzian structure on \( M \).
Let \( \nabla \) be the Levi-Civita connection with respect to metric \( g \) and \( R \) be the curvature tensor of \( g \). Then we have
\[ [e_1, e_2] = ye_2 - z^2 e_3, \quad [e_1, e_3] = -\frac{1}{z} e_1 \quad \text{and} \quad [e_2, e_3] = -\frac{1}{z} e_2. \]
Taking \( e_3 = \xi \) and using Koszul formula for the Riemannian metric \( g \), we can easily calculate
\[ \nabla_{e_1} e_3 = -\frac{1}{z} e_1 + \frac{1}{z^2} e_2, \quad \nabla_{e_1} e_2 = -\frac{1}{2} z^2 e_3, \]
\[ \nabla_{e_1} e_1 = -\frac{1}{z} e_3, \quad \nabla_{e_2} e_3 = -\frac{1}{z} e_2 + \frac{1}{2} z^2 e_1, \]
\[ \nabla_{e_2} e_2 = ye_1 - \frac{1}{z} e_3, \quad \nabla_{e_2} e_1 = \frac{1}{2} z^2 e_3 - ye_2, \]
\[ \nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = \frac{1}{2} z^2 e_1, \quad \nabla_{e_3} e_1 = -\frac{1}{2} z^2 e_2. \]
From the above it can be easily seen that \((\phi, \xi, \eta, g)\) is an Lorentzian Trans-Sasakian structure on \( M \). Consequently \( M^3(\phi, \xi, \eta, g) \) is an Lorentzian Trans-Sasakian manifold with \( \alpha = \frac{1}{2} z^2 \neq 0 \) and \( \beta = \frac{1}{z} \neq 0 \).

**Example 7.2:** We consider the three-dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\} \), where \((x, y, z)\) are standard coordinate of \( \mathbb{R}^3 \).
The vector fields
\[ e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z} \]
are linearly independent at each point of \( M \).
Let \( g \) be the Riemannian metric defined by
\[ g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0, \]
\[ g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1 \]
that is, the form of the metric becomes
\[ g = \frac{dx^2 + dy^2 - dz^2}{z^2}. \]
Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in \chi(M) \).
Let \( \phi \) be the \((1, 1)\) tensor field defined by
\[ \phi(e_1) = -e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0. \]
Then using the linearity of \( \phi \) and \( g \), we have
\[ \eta(e_3) = -1, \]
\[ \phi^2 Z = Z + \eta(Z)e_3, \]
\[ g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W), \]
for any \( Z, W \in \chi(M) \).

Then for \( e_3 = \xi \), the structure \((\phi, \xi, \eta, g)\) defines an Lorentzian structure on \( M \).

Let \( \nabla \) be the Levi-Civita connection with respect to metric \( g \). Then we have
\[
[e_1, e_3] = e_1 e_3 - e_3 e_1
= z \frac{\partial}{\partial x} \left( z \frac{\partial}{\partial z} \right) - z \frac{\partial}{\partial z} \left( z \frac{\partial}{\partial x} \right)
= z^2 \frac{\partial^2}{\partial x \partial z} - z^2 \frac{\partial^2}{\partial z \partial x} - z \frac{\partial}{\partial x}
= -e_1.
\]

Similarly
\[
[e_1, e_2] = 0 \quad \text{and} \quad [e_2, e_3] = -e_2.
\]

The Riemannian connection \( \nabla \) of the metric \( g \) is given by
\[
2g(\nabla_X Y, Z) = X g(Y, Z) + Y g(Z, X) - Z g(X, Y)
- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),
\]
which known as Koszul’s formula.

Using (7.1) we have
\[
2g(\nabla_{e_1} e_3, e_1) = -2g(e_1, e_1)
= 2g(-e_1, e_1).
\]

Again by (7.1)
\[
2g(\nabla_{e_1} e_3, e_2) = 0 = 2g(-e_1, e_2)
\]
and
\[
2g(\nabla_{e_1} e_3, e_3) = 0 = 2g(-e_1, e_3).
\]

From (7.2), (7.3) and (7.4) we obtain
\[
2g(\nabla_{e_1} e_3, X) = 2g(-e_1, X),
\]
for all \( X \in \chi(M) \).

Thus
\[
\nabla_{e_1} e_3 = -e_1.
\]

Therefore, (7.1) further yields
\[
\nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = -e_3,
\]
\[
\nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_1 = 0,
\]
\[
\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0.
\]

(7.5) tells us that the manifold satisfies (1.3) for \( \alpha = 0, \beta = 1 \) and \( \xi = e_3 \). Hence the manifold is a Lorentzian Trans-Sasakian manifold of type \((0, 1)\). It is known that
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.
\]

(7.6)
With the help of the above results and using (7.6) it can be easily verified that
\[ R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1, \]
\[ R(e_1, e_2)e_2 = -e_1, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_1, e_3)e_2 = 0, \]
\[ R(e_1, e_2)e_1 = e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = -e_3. \]
From the expression of the curvature tensor it follows that the manifold is of constant curvature \(-1\). Hence the manifold is locally \(\phi\)-symmetric. Also from the above expressions of the curvature tensor, we obtain
\[ S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = -2. \]
Similarly, we have
\[ S(e_2, e_2) = -2, \quad S(e_3, e_3) = 2. \]
Therefore,
\[ r = S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) = -6. \]
Thus the scalar curvature \(r\) is constant. Hence Theorem 6.1 is verified.

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