THE ANALYTIC STRUCTURE OF
TRIGONOMETRIC S MATRICES

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ABSTRACT

S-matrices associated to the vector representations of the quantum groups for the classical Lie algebras are constructed. For the $a_{m-1}$ and $c_m$ algebras the complete $S$-matrix is found by an application of the bootstrap equations. It is shown that the simplest form for the $S$-matrix which generalizes that of the Gross-Neveu model is not consistent for the non-simply-laced algebras due to the existence of unexplained singularities on the physical strip. However, a form which generalizes the $S$-matrix of the principal chiral model is shown to be consistent via an argument which uses a novel application of the Coleman-Thun mechanism. The analysis also gives a correct description of the analytic structure of the $S$-matrix of the principle chiral model for $c_m$. 

CERN-TH.6888/93
May 1993
1. Introduction

The complete non-perturbative solution of interacting quantum field theories seems like a hopeless fantasy. In the vast space of two-dimensional quantum field theories, however, there is a small set of theories attracting a large amount of interest due to the fact that they are integrable and hence to a certain degree solvable. At the moment the understanding of such theories is mostly at the level of on-shell physics encoded in the scattering matrix. Even though the $S$-matrix seems to be rather simple, since it factorizes and is specified completely by the two-body $S$-matrix, finding the complete $S$-matrix satisfying all the required properties is a surprisingly difficult task. Also it is worth pointing out that a given $S$-matrix may always be multiplied by CDD factors to give an equivalent $S$-matrix.

Let us summarize the situation so far. First of all, there exist a set of minimal purely elastic $S$-matrices (and hence trivial solutions of the Yang-Baxter equation) which are related to the simply-laced affine Lie algebras. There are particular CDD factors, which depend on a coupling constant which make these $S$-matrices equal to the conjectured $S$-matrices of the corresponding affine Toda field theory (the connection with the field theory being established in perturbation theory) [1]. There are also a set of $S$-matrices which describe the non-simply-laced affine Toda field theories, however, the situation is rather different from the simply-laced theories in that the $S$-matrix does not factor into a ‘minimal’ piece independent of the coupling and furthermore the mass ratios depend on the coupling constant [2].

We now turn to the $S$-matrices that are non-trivial solutions of the Yang-Baxter equation, so that the particle states carry some internal quantum numbers. One has both the rational and trigonometric solutions of the Yang-Baxter equation to hand, the latter depending on a coupling constant. $S$-matrices corresponding to rational solutions have been considered in [3], and describe the principal chiral model for a classical Lie algebra and the Gross-Neveu model for the simply-laced classical Lie algebras. It is worth pointing out that a Gross-Neveu type $S$-matrix for the non-simply-laced algebras can be written down, but the Ansatz fails due to the existence of unexplained singularities on the physical strip. This point highlights the delicate nature of these $S$-matrices; it is simply not good enough to write down an Ansatz for the $S$-matrix elements on some elementary particles, hoping that the bootstrap will close on a conjectured set of particles; in order to claim a consistent $S$-matrix it is necessary to account for all singularities on the physical strip in terms of bound states or anomalous thresholds via the Coleman-Thun mechanism [4].

Trigonometric solutions of the Yang-Baxter equation are more general since they depend on a coupling constant and the rational solutions can be obtained from them in a
certain limit. S matrices constructed from trigonometric solutions have been the subject of much speculation but very little detailed analysis, except for the series associated to $a_{m-1}$ [5,6] and in particular for $a_1$ which gives the soliton $S$-matrix of the sine-Gordon theory [7]. In these cases one can prove that the bootstrap closes on a given set of particles, when the coupling constant satisfies a certain inequality [5]. The problem encountered for other algebras stems from the difficulties in solving the bootstrap equations. However, we shall show how this can be achieved for $c_m$ (in addition to $a_{m-1}$ which has been considered previously).

One of the principle reasons for constructing the trigonometric $S$-matrices is that they are thought to describe the integrable perturbations of certain conformal field theories, typically displaying $W$-algebra type symmetries [6,8].

2. Trigonometric solutions of the Yang-Baxter equation

In this section we describe the trigonometric solutions of the Yang-Baxter Equation (YBE). They are associated to certain deformations of the universal enveloping algebra of a Lie algebra known as a quantum group [9,10]. The solutions can be thought of as intertwiners between tensor products of representations of the algebra:

$$\hat{R}(u) : V_\mu \otimes V_\nu \to V_\nu \otimes V_\mu,$$

(2.1)

where $u$ is the (additive) spectral parameter. Such an ‘$R$-matrix’ has a spectral decomposition

$$\hat{R}(u) = \sum_\lambda \rho_\lambda(u) \mathbb{P}_\lambda,$$

(2.2)

where $\mathbb{P}_\lambda$ is a quantum group invariant homomorphism $V_\mu \otimes V_\nu \to V_\nu \otimes V_\mu$ with the property that $\mathbb{P}_\lambda|_{V_\lambda^\vee} \neq 0$ if and only if $\lambda' = \lambda$. \footnote{We are assuming that every irreducible component $V_\lambda \subset V_\mu \otimes V_\nu$ has multiplicity one.} If $\mu = \nu$ then $\mathbb{P}_\lambda$ is a projection.

In the following we shall use both the language of spectral decompositions and the interaction-round-a-face (IRF) picture when writing down solutions of the YBE. When using the latter we shall denote $\hat{R}(u)$ as $W(u)$. To start with we consider the solutions associated to the vector representation of the algebra. In what follows we shall consider all the classical Lie algebras: $a_{m-1}$, $b_m$, $c_m$ and $d_m$. The set of weights $\Sigma$ of the vector representations are\footnote{In our notation the long roots of $b_m$ have length 2 while for $c_m$ the single long root has length 4.}

$$\Sigma = \{e_1 - (e_1 + \cdots + e_m)/m, \ldots, e_m - (e_1 + \cdots + e_m)/m\}, \text{ for } a_{m-1},$$

$$\Sigma = \{0, \pm e_1, \ldots, \pm e_m\}, \text{ for } b_m, \quad \Sigma = \{\pm e_1, \ldots, \pm e_m\}, \text{ for } c_m, d_m,$$

(2.3)
where the $e_i$’s are a set of orthonormal vectors.

The solution of the YBE is labelled by four weights of the algebra:

$$W\left(\begin{array}{c|c} a & b \\ c & d \end{array}\right| u\right), \ a, b, c, d \in \Lambda^*,$$

with the property that $W$ is only non-zero if $c - a$, $d - c$, $b - a$ and $d - b$ are $\in \Sigma$.

For completeness we now write down the solutions following [11] (see also the review [12]). In the following $\omega$ is a constant which is related to the deformation parameter of the quantum group. For convenience we introduce for $a \in \Lambda^*$

$$a_\mu = \omega(a + \rho) \cdot \mu, \quad \text{for } \mu \in \Sigma \neq 0,$$

$$a_0 = -\omega/2, \quad a_{\mu\nu} = a_\mu - a_\nu, \quad a_{\mu-\nu} = a_\mu + a_\nu,$$

where $\rho$ is the sum of the fundamental weights of the algebra.\(^3\) Also we define

$$[x] = \sin x, \quad \lambda = tg\omega/2,$$

where $g$ is the dual Coxeter number of the algebra and $t$ is the (length)$^2/2$ of the longest root.\(^4\)

For $a_{m-1}$ the solution is

$$W\left(\begin{array}{c|c} a & a + \mu \\ a + \mu & a + 2\mu \end{array}\right| u\right) = [\omega - \lambda u]/[\omega],$$

$$W\left(\begin{array}{c|c} a & a + \mu \\ a + \mu & a + \mu + \nu \end{array}\right| u\right) = [a_{\mu\nu} + \lambda u]/[a_{\mu\nu}],$$

$$W\left(\begin{array}{c|c} a & a + \nu \\ a + \mu & a + \mu + \nu \end{array}\right| u\right) = \frac{[\lambda u]}{[\omega]} \left(\frac{[a_{\mu\nu} + \omega][a_{\mu\nu} - \omega]}{|a_{\mu\nu}|^2}\right)^{1/2},$$

\(^3\) These are the vectors $\omega_i$ with $\omega_i \cdot \alpha_j = (\alpha_j^2/2)\delta_{ij}$ where the $\alpha_j$ are the simple roots.

\(^4\) The values $(g, t)$ are $(m, 1)$, $(2m - 1, 1)$, $(m + 1, 2)$ and $(2m - 2, 1)$ for $a_{m-1}$, $b_m$, $c_m$ and $d_m$, respectively.
where \(\mu, \nu \in \Sigma\) and \(\mu \neq \nu\). For the algebras \(b_m, c_m\) and \(d_m\) the solutions are

\[
W\left(\begin{array}{cc} a & a + \mu \\ a + \mu & a + 2\mu \end{array} \right| u\) = \frac{[\lambda - \lambda u][\omega - \lambda u]}{[\lambda][\omega]}, \quad \text{for } \mu \neq 0,
\]

\[
W\left(\begin{array}{cc} a & a + \mu \\ a + \mu & a + \mu + \nu \end{array} \right| u\) = \frac{[\lambda - \lambda u][a_{\mu\nu} + \lambda u]}{[\lambda][a_{\mu\nu}]}, \quad \text{for } \mu \neq \pm \nu,
\]

\[
W\left(\begin{array}{cc} a & a + \nu \\ a + \mu & a + \mu + \nu \end{array} \right| u\) = \frac{[\lambda - \lambda u][\lambda u]}{[\lambda][\omega]} \times \frac{[a_{\mu\nu} + \omega][a_{\mu\nu} - \omega]}{[a_{\mu\nu}]^2}^{1/2}, \quad \text{for } \mu \neq \pm \nu,
\]

\[W\left(\begin{array}{cc} a & a + \nu \\ a + \mu & a \end{array} \right| u\) = \frac{[\lambda u][a_{\mu\nu} + \omega - \lambda + \lambda u]}{[\lambda][a_{\mu\nu} + \omega]} (G_{\alpha\mu}G_{\alpha\nu})^{1/2} + \delta_{\mu\nu} \frac{[\lambda - \lambda u][a_{\mu\nu} + \omega + \lambda u]}{[\lambda][a_{\mu\nu} + \omega]}, \quad \text{for } \mu \neq 0,
\]

\[
W\left(\begin{array}{cc} a & a \\ a & a \end{array} \right| u\) = \frac{[\lambda + \lambda u][2\lambda - \lambda u]}{[\lambda][2\lambda]} - \frac{[\lambda u][\lambda - \lambda u]}{[\lambda][2\lambda]} J_\alpha,
\]

where \(\mu, \nu \in \Sigma\) and

\[
G_{\alpha\mu} = \sigma \frac{s(a_{\mu\nu} + \omega)}{s(a_{\mu\nu})} \prod_{\kappa \neq \pm \mu, 0} \frac{[a_{\mu\kappa} + \omega]}{[a_{\mu\kappa}]}, \quad \text{for } \mu \neq 0, \quad G_{\alpha\alpha} = 1,
\]

\[J_\alpha = \sum_{\kappa \neq 0} \frac{[\alpha_{\kappa} + \omega/2 - 2\lambda]}{[\alpha_{\kappa} + \omega/2]} G_{\alpha\kappa}.
\]

In the above \(\sigma\) is \(-1\) for \(c_m\) otherwise being \(1\). The function \(s(x) = [tx]\) for \(b_m\) and \(c_m\), and \(s(x) = 1\) for \(d_m\).

The solutions satisfy a set of conditions, in addition to the YBE, which will be important for the construction of an \(S\)-matrix (our terminology follows that of [12]).

(i) The standard initial condition.

\[W\left(\begin{array}{cc} a & b \\ c & d \end{array} \right| 0\) = \delta_{bc}, \tag{2.10}\]

(ii) The unitarity condition.

\[
\sum_{\epsilon} W\left(\begin{array}{cc} a & e \\ c & d \end{array} \right| u\) W\left(\begin{array}{cc} a & b \\ e & d \end{array} \right| -u\) = \varrho(u) \delta_{bc}, \tag{2.11}
\]

where

\[
\varrho(u) = \frac{[\omega - \lambda u][\omega + \lambda u]}{[\omega]^2}, \quad \text{for } a_{m-1},
\]

\[
= \frac{[\lambda - \lambda u][\omega - \lambda u][\lambda + \lambda u][\omega + \lambda u]}{[\lambda]^2[\omega]^2}, \quad \text{for } b_m, c_m \text{ and } d_m. \tag{2.12}
\]
(iii) Crossing symmetry \((b_m, c_m \text{ and } d_m \text{ only})\).

\[
W \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigg| u \bigg) = W \begin{pmatrix} c & a \\ d & b \end{pmatrix} \bigg| 1 - u \bigg) \left( \frac{G_b G_c}{G_a G_d} \right)^{1/2},
\]

(2.13)

where

\[
G_a = \varepsilon(a) \prod_{k=1}^{m} s(a_i) \prod_{1 \leq i < j \leq m} [a_i - a_j][a_i + a_j],
\]

(2.14)

where \(a = \sum_{i=1}^{m} a_i e_i\) defines the \(a_i\) and \(\varepsilon(a)\) is a sign factor chosen so that \(\varepsilon(a+\mu)/\varepsilon(a) = \sigma\). \(G_a\) is related to \(G_{a,\mu}\) in (2.9) by \(G_{a,\mu} = G_{a+\mu}/G_a\).

For \(a_{m-1}\) the representation is complex and so the solution of the YBE does not satisfy a crossing symmetry relation involving only the vector representation.

The above solutions correspond to the unrestricted solutions of the YBE where the variables \(\{a, b, c, d\}\) are any weights of the algebra. We shall be primarily interested in the restricted models which are obtained for the particular values

\[
\omega = \frac{\pi}{t(g + k)}, \quad k = 1, 2, \ldots
\]

(2.15)

and the weights are restricted to lie in the set of integrable weights of the affine algebra at level \(k\) projected onto the weight lattice of the finite algebra. In practice, this means that we restrict the allowed weights to the dominant weights satisfying

\[
a \cdot \theta \leq k,
\]

(2.16)

where \(\theta\) is the highest root of the algebra. We denote the set of weights satisfying this condition as \(\Lambda^*(k)\).

3. The vector-vector scattering matrices

The solutions of the YBE equation that we wrote down in the last section naturally lead to \(S\)-matrices for a set of kinks. We denote a kink state by \(K_{ab}(\theta)\), where \(a\) and \(b\) are two vacua of the theory and \(\theta\) is the rapidity of the kink. In an integrable field theory we need only consider the \(S\)-matrix for the process

\[
K_{ac}(\theta_1) + K_{cd}(\theta_2) \rightarrow K_{ab}(\theta_2) + K_{bd}(\theta_1)
\]

(3.1)

since all the other \(S\)-matrix elements are determined in terms of these. The idea is to find an \(S\)-matrix for a theory associated to a classical Lie algebra with vacua in one-to-one correspondence with \(\Lambda^*(k)\) (or \(\Lambda^*\) in the unrestricted model) and hence kinks with a
The topological charge being a weight of the vector representation of the algebra. The $S$-matrix of the process (3.1) is related to the solution of the YBE as

$$\tilde{S} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) |u\rangle = Y(u)W \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) f(u) \left( \frac{G_a G_d}{G_b G_c} \right)^{f(u)/2},$$

(3.2)

for some scalar function $Y(u)$, where $u = (\theta_1 - \theta_2)/i\pi$ is the rapidity difference of the incoming kinks. The other scalar factor is included in order to simplify the implementation of crossing symmetry. The scalar function $Y(u)$ must be chosen so that the $S$-matrix axioms are satisfied, subject to minimality, meaning that we search for $Y(u)$ which ensures the axioms are fulfilled with the minimum number of poles and zeros on the physical strip (the region $0 \leq \text{Re}(u) \leq 1$).

The unitarity constraint which follows from the hypothesis that the space of states of the theory is complete can be satisfied by virtue of the completeness relation (2.11) if $f(u) = cu$, for some constant $c$, and

$$Y(u)Y(-u) = 1/\varrho (f(u)).$$

(3.3)

Crossing symmetry of the $S$-matrix relates the process (3.1) with the process $K_{cd} + K_{db} \rightarrow K_{ca} + K_{ab}$, where $K$ is the charge conjugate kink. Notice, that for the $b_m$, $c_m$ and $d_m$ algebras if $a - b \in \Sigma$ then $b - a \in \Sigma$, due to the fact that the representations are real, therefore we may take $K_{ab} \equiv K_{ab}$. In these cases crossing symmetry requires

$$\tilde{S} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) |u\rangle = \tilde{S} \left( \begin{array}{cc} c & a \\ d & b \end{array} \right) |1 - u\rangle,$$

(3.4)

which may be satisfied by virtue of (2.13) if $f(u) = u$ and

$$Y(1 - u) = Y(u).$$

(3.5)

The situation for $a_{m-1}$ is somewhat different since the vector representation is not conjugate to itself. However, this case has been dealt with elsewhere [5,6]. One finds that a crossing symmetry relation can be satisfied when kinks in the conjugate vector representation are included in the spectrum and the minimal solution for $Y(u)$ is

$$Y_{m,k}(u) = \exp \left\{ \int_0^\infty dx \frac{2 \sinh(mux/2)}{x \sinh[(k + m)x] \sinh(mx)} \right\} \times (\cosh(kx) \cosh(mux/2) - \cosh[(m + k - 2)x] \cosh(mx(u/2 - 1))] \right\},$$

(3.6)

where $k$ is a parameter related to $\omega$ as in (2.15). $Y_{m,k}(u)$ can also be expressed in terms of products of Gamma functions [6]. The function $Y_{m,k}(u)$ satisfies two important identities:

$$Y_{m,k}(u)Y_{m,k}(-u) = \frac{\sin^2 \omega}{\sin(\omega + \lambda u) \sin(\omega - \lambda u)},$$

(3.7)
and
\[
Y_{m,k}(1-u)Y_{m,k}(1+u) = \frac{\sin^2 \omega}{\sin(\lambda - \lambda u) \sin(\lambda + \lambda u)},
\]
(3.8)
where as before \( \omega = \pi/(m+k) \) and \( \lambda = m\omega/2 \).

Using these two relations we can now write down \( S \)-matrices for the other algebras
by noting that a solution of (3.3) and (3.5) is
\[
Y(u) = Y_{tg,tk}(u) Y_{tg,tk}(1-u) \frac{\sin \lambda}{\sin \omega}, \quad \text{for } b_m, c_m, d_m.
\]
(3.9)

Summarizing we have
\[
\tilde{S}\left( \begin{array}{cc} a & b \\ c & d \end{array} \bigg| u \right) = W\left( \begin{array}{cc} a & b \\ c & d \end{array} \bigg| u \right) \left( \frac{G_aG_d}{G_bG_c} \right)^{u/2} \\
\times \begin{cases} 
Y_{m,k}(u), & \text{for } a_{m-1}, \\
Y_{tg,tk}(u)Y_{tg,tk}(1-u) \frac{\sin \lambda}{\sin \omega}, & \text{for } b_m, c_m, d_m.
\end{cases}
\]
(3.10)

It is straightforward to show using (2.10) and the fact that \( Y_{tg,tk}(0) = 1 \) and \( Y_{tg,tk}(1) = \sin \omega/\sin \lambda \) that
\[
\tilde{S}\left( \begin{array}{cc} a & b \\ c & d \end{array} \bigg| 0 \right) = \delta_{bc}.
\]
(3.11)

Next we turn to an investigation of the analytic structure structure of the above \( S \)-matrices. The factor \( Y_{tg,tk}(u) \) has no poles or zeros on the physical strip which means that
there are no bound state resonances. So the \( S \)-matrix defines a complete theory for the
cases \( b_m, c_m \) and \( d_m \) with kinks associated to the vector representation of the algebra. For
the case \( a_{m-1} \) the \( S \)-matrix does not make sense on its own because the requirement of
crossing symmetry means that kinks in the conjugate representation should be included in
the spectrum.

We now consider the case of the \( a_{m-1} \) theory in more detail. In order to satisfy
crossing symmetry the conjugate vector representation must be included in the spectrum. This is achieved in a dynamical way by a generalization of the original Ansatz (3.10):
\[
S\left( \begin{array}{cc} a & b \\ c & d \end{array} \bigg| u \right) = X(u)\tilde{S}\left( \begin{array}{cc} a & b \\ c & d \end{array} \bigg| u \right),
\]
(3.12)
where the additional CDD factor \( X(u) \) does not upset the unitarity or crossing symmetry
relations but provides an additional pole on the physical strip. In order to motivate the
form of the scalar factor consider the spectral decomposition of the \( R \)-matrix (see appendix A):
\[
\hat{R}(u) = \frac{\sin(\omega - \lambda u)}{\sin \omega} P_{2\omega_1} + \frac{\sin(\omega + \lambda u)}{\sin \omega} P_{\omega_2},
\]
(3.13)
where $P_{2\omega_1}$ and $P_{\omega_2}$ are the quantum group invariant projectors onto the representations with highest weights $2\omega_1$ and $\omega_2$ which appear in the tensor product of two vector representations. The idea is that $X(u)$ should have a simple pole at $u = \omega/\lambda = 2/m$ at the place where the $R$-matrix projects onto the second fundamental representation: $\tilde{R}(u) \propto P_{\omega_2}$. This determines the form of the scalar factor:

$$X(u) = \frac{\sin\left(\frac{\pi u}{2} + \frac{\pi}{m}\right)}{\sin\left(\frac{\pi u}{2} - \frac{\pi}{m}\right)}.$$

(3.14)

As a consequence a particle transforming in the second fundamental representation in included in the spectrum and the $S$-matrix elements of this new state can then be found using the bootstrap equations. The process is then repeated: the new $S$-matrix has simple poles on the physical strip which are interpreted in terms of states propagating in the direct or crossed channels. For the unrestricted models it was shown in [5] that the procedure terminates on a finite set of particles corresponding to all the fundamental representations of $a_{m-1}$ (in particular the conjugate vector representation) if the coupling constant satisfies the inequality $\omega < 2\pi/m$. Notice that the restricted models, for which $\omega = \pi/(m + k)$, all lie in this region. The most important comment to make about this procedure is that the positions of the poles are dictated by the fusion structure of the solutions of the YBE, which in turn determines the ratios of the masses to be

$$m_a \propto \sin\left(\frac{\pi a}{m}\right), \quad a = 1, 2, \ldots, m - 1.$$

(3.15)

Here the particle with mass $m_a$ transforms in the $a^{th}$ fundamental representation. This is the mass spectrum of the $S$-matrix of the $a_{m-1}$ affine Toda field theory. This is not surprising because the extra CDD factor in (3.12) is the minimal\footnote{By minimal we mean the part of the Toda $S$-matrix which is independent of the coupling constant.} $S$-matrix of the aforementioned theory for the particle associated to the vector representation. In fact the poles on the physical strip of the non-diagonal theory are completely determined by the minimal Toda $S$-matrix since the rest of the $S$-matrix has no poles or zeros on the physical strip. It is important to point out that the minimal Toda scalar $S$-matrix has double poles on the physical strip, but these are explained in terms of the anomalous thresholds produced by ‘on-shell’ diagrams [1]. We will discuss the full solution of the bootstrap equations in the next section.

We can now try to construct $S$-matrices for the other classical Lie algebras along the same lines as for the $a_{m-1}$ algebra. The first thing to consider is the spectral decomposition of the solution of the Yang-Baxter equation for vector-vector scattering. It is known that
\[
\tilde{R}(u) = \frac{\sin(\lambda - \lambda u) \sin(\omega - \lambda u) + \sin(\lambda - \lambda u) \sin(\omega + \lambda u) + \sin(\lambda + \lambda u) \sin(\omega + \sigma \lambda u)}{\sin \omega \sin \lambda} \]

where \( \sigma = -1 \) for \( c_m \) and 1 otherwise. So if we wish to include the second fundamental representation in the spectrum we would require a simple pole at \( u = \omega/\lambda = 2/tg \); however, notice that for \( c_m \) the residue at this point projects onto the irreducible representation, whereas for the other algebras the residue projects onto a reducible representation consisting of the second fundamental representation plus a trivial representation: we are forced to conclude that this second particle transforms in this reducible representation in these cases. A simple pole at this point can be introduced by including a CDD factor

\[
X(u) = \frac{\sin \left( \frac{\omega u}{2} + \frac{\pi}{tg} \right) \sin \left( \frac{\pi}{tg} + \frac{\pi}{2} - \frac{\omega u}{2} \right)}{\sin \left( \frac{\omega u}{2} - \frac{\pi}{tg} \right) \sin \left( \frac{\pi}{tg} - \frac{\pi}{2} + \frac{\omega u}{2} \right)}
\]

which also has a simple pole corresponding to the process where the second particle is exchanged in the crossed-channel as demanded by crossing symmetry. The form of \( X(u) \) ensures that it satisfies unitarity and crossing independently of the non-diagonal part of the \( S \)-matrix.

So the proposal for the \( S \)-matrix for vector-vector scattering has the form

\[
S(u) = X(u)\tilde{S}(u),
\]

where \( \tilde{S}(u) \) is the \( S \)-matrix in (3.10) and \( X(u) \) is the CDD factor (3.14) or (3.17) which is the minimal \( S \)-matrix for the vector-vector scattering of an associated affine Toda theory. The associated Toda theories are the following:

\[
a_{m-1} \rightarrow a^{(1)}_{m-1}, \quad b_m \rightarrow a^{(2)}_{2m-1}, \quad c_m \rightarrow d^{(2)}_{m+1}, \quad d_m \rightarrow d^{(1)}_m.
\]

The associated algebra is in fact the one whose Cartan matrix is equal to the transpose of the Cartan matrix of the untwisted affinization of the original algebra. We emphasize that \( X(u) \) is not the Toda \( S \)-matrix itself but rather the part which is independent of the coupling constant (for the non-simply laced algebras we refer to the naïve \( S \)-matrix elements that one would write down for the particles with the classical mass ratios [1] as opposed to the actual \( S \)-matrix of the Toda theory [2]).

On the basis of this there are serious objections to the \( S \)-matrices that we have proposed. Firstly, for \( b_m \) and \( d_m \) the vector representation cannot be considered as the elementary particle from which all the others follow as bound states because the spinor
representations could not be produced in this way. One might think that one could define a consistent \(S\)-matrix which does not include the spinor representations and for which the vector representation is elementary, however this cannot be so because the spinor makes an appearance in the correct interpretation of the higher order poles. In other words such an \(S\)-matrix would have some unexplained higher order poles and must consequently be rejected.\(^6\)

The second thought is that if one starts with the spinor representations in a similar way then the \(S\)-matrices that we have written down would follow by fusion, in other words we have only written down part of the \(S\)-matrix.

There is, however, a more serious objection which applies to the non-simply-laced theories. The objection arises from the fact that for the non-simply-laced algebras the minimal Toda \(S\)-matrix part of the conjecture does not by itself define a consistent \(S\)-matrix. The reason is that as written down the scalar \(S\)-matrices for the \(a^{(2)}_{2m-1}\) and \(d^{(2)}_{m+1}\) Toda theories have multiple poles on the physical strip which cannot be explained by the spectrum of particles which follows from the classical Toda Lagrangian [1]:

\[
\begin{align*}
    a^{(2)}_{2m-1} : m_j &\propto \begin{cases} 
    1 & j = m \\
    2 \sin \left[ j \frac{\pi}{2m-1} \right] & j = 1, 2, \ldots, m - 1,
    \end{cases} \\
    d^{(2)}_{m+1} : m_j &\propto \sin \left[ j \frac{\pi}{2(m+1)} \right], \quad j = 1, 2, \ldots, m.
\end{align*}
\]

The best way to see this is to consider the \(a^{(2)}_{2m-1}\) and \(d^{(2)}_{m+1}\) \(S\)-matrices as a subset of the minimal \(S\)-matrix of a larger Toda theory. The \(a^{(2)}_{2m-1}\) \(S\)-matrix is the \(d^{(1)}_{2m}\) \(S\)-matrix for the subset of particles \(\{2, 4, \ldots, 2m\}\), whilst the \(d^{(2)}_{m+1}\) \(S\)-matrix is the \(d^{(1)}_{m+2}\) for the subset of particles \(\{1, 2, \ldots, m\}\) (i.e. without the spinor and anti-spinor particles). The point now is that the sub-\(S\)-matrix has multiple poles on the physical strip which require the full spectrum of the larger theory in order to be explained.

In fact the resolution of this problem for the non-simply-laced Toda theories is surprisingly subtle. What happens is that the ratios of the particle masses get renormalized in a non-trivial way such that the mass ratios depend on the coupling constant with the consequence that the analytic structure becomes modified. The new \(S\)-matrix no longer suffers from the pathologies of the naïve one. The non-diagonal \(S\)-matrices under discussion would, on the face of it, suffer the same fate as their naïve non-simply-laced Toda counterparts, since it is the naïve Toda factor that provides the poles on the physical strip; however, in this case we cannot hypothesize that the particle masses get renormalized in some non-trivial way because the positions of the poles must match the fusing of the \(\tilde{S}\) factor and this fusing structure is completely rigid: the positions of the poles are determined completely.

\(^6\) It is just possible for the matrix factor of the \(S\)-matrix to provide some judicious zeros, but this is not the case.
If the $S$-matrices conjectured in (3.12) are to be completely consistent then the part arising from the fusion of $\widetilde{S}$ must provide some appropriate zeros in order to cancel the unwanted poles of the Toda theory factor. In the next section we show how this can occur by solving the bootstrap equation for the $a_{m-1}$ and $c_m$ cases. The obstruction for dealing with the $b_m$ and $d_m$ algebras in the same way, apart from the problem that the vector particle is not elementary, is that the relevant spectral decompositions of the $R$-matrices are not known.

$S$-matrices associated to the vector representations of the classical Lie algebras have recently been proposed in [13]. However, this reference uses an incorrect form for the spectral decomposition (3.16) and misses the crucial factor of $t$ in the above equations which render the result invalid for $c_m$. It also asserts that the masses of the theories are those of the corresponding classical Toda theory; whereas our results show that they are actually those of the related classical Toda theories (3.20).

4. The solution of the bootstrap for $a_{m-1}$ and $c_m$

In this section we will explicitly solve the bootstrap for the $a_{m-1}$ and $c_m$ theories. This involves the resolution of two interlocking problems. Firstly we must find the spectral decompositions of the $R$-matrix on the fundamental representations and then we must show how the scalar factor provides the correct analytic structure on the physical strip.

Fortunately, for the representations that are necessary to construct the $S$-matrices, there is well-established technology for finding the spectral decompositions of the associated $R$-matrices. We explain how the spectral decompositions are found in appendix A. However, knowing the spectral decompositions is not enough for our purposes, they only tell us the form of the $R$-matrix up to a scalar function of the spectral parameter. In fact, we must solve the bootstrap equations for which the overall scalar factors are determined and indeed crucial for the correct form of the $S$-matrix. We derive these scalar factors in appendix B.

Our results are the following. We denote by $\hat{R}^{ab}(u)$ the $R$-matrix between the $a^{th}$ and $b^{th}$ fundamental representations (where the highest weight of the $a^{th}$ fundamental representation is $\omega_a = e_1 + e_2 + \cdots + e_a$), and we choose without loss of generality $b \geq a$. For $a_{m-1}$ we have

$$\hat{R}^{ab}(u) = Z_{1}^{ab}(u) \sum_{k=0}^{\min(m-b,a)} (-)^{k+1} \rho_{k}^{ab}(u) \prod_{i=k+\omega_{b+k}}}^{m} \omega_{a-k}, \quad (4.1)$$
with
\[
\rho_{k}^{ab}(u) = \prod_{p=1}^{k} \{2p + b - a\} \prod_{p=k+1}^{\min(m-b,a)} \{-2p - b + a\},
\]
(4.2)
\[
Z_{1}^{ab}(u) = \prod_{j=1}^{a} \prod_{k=1}^{b-1} \{2j + 2k - a - b\} \prod_{p=\min(m-b,a)+1}^{a} \{-2p - b + a\}.
\]
where we take \(\omega_m = \omega_0 = 0\) and
\[
\{x\} = \frac{\sin (\omega x/2 + \lambda u)}{\sin \omega}.
\]
(4.3)

For \(c_m\) we have
\[
\tilde{R}^{ab}(u) = Z_{1}^{ab}(u) \sum_{j=0}^{a} \sum_{k=0}^{a-j} (-)^{j+k} \rho_{jk}^{ab}(u) \prod_{p=1}^{\omega_{b+j-k+\omega_{a-j-k}+1}} \prod_{p=\min(m-b,a)+1}^{a} \{-2p - b + a\},
\]
(4.4)
where
\[
\rho_{jk}^{ab}(u) = \prod_{p=1}^{j} \{2p + b - a\} \prod_{q=1}^{k} \{2(m+1) + 2q - a - b\}
\]
\times \prod_{p=j+1}^{\min(m-b,a)} \{-2p - b + a\} \prod_{q=k+1}^{a} \{-2(m+1) - 2q + a + b\}.
\]
(4.5)
\[
Z_{2}^{ab}(u) = \left(\frac{\sin \lambda}{\sin \omega}\right)^{ab} \prod_{j=1}^{a} \prod_{k=1}^{b-1} \{2j + 2k - a - b\} \{a + b - 2(m+1) - 2j - 2k\}
\]
\times \prod_{p=\min(m-b,a)+1}^{a} \{-2p - b + a\}.
\]

The full \(S\)-matrix is then equal to
\[
\tilde{S}_{(k)}^{ab}(u) = Y^{ab}(u) \tilde{R}^{ab}(u),
\]
(4.6)
with
\[
Y^{ab}(u) = \prod_{j=1}^{a} \prod_{k=1}^{b} Y \left(u + \frac{1}{tg} (2j + 2k - a - b - 2) \right).
\]
(4.7)
where \(Y(u)\) is given by (3.6) for \(a_{m-1}\) and (3.9) for \(c_m\). In (4.6) we have explicitly indicated the dependence on the parameter \(k\) via \(\omega\) in (2.15). The normalization factors \(Z_{1}^{ab}(u)\) and \(Z_{2}^{ab}(u)\) arise from solving the bootstrap equations starting with \(\tilde{R}(u)\) and are crucial for
the correct form of the $S$-matrix since they can provide zeros on the physical strip and ensure that the unitarity condition is satisfied. To illustrate the latter point we notice that

$$
\tilde{R}^{ab}(u)\tilde{R}^{ba}(-u) = \prod_{j=1}^{a} \prod_{k=1}^{b} g \left( u + \frac{1}{tg} (2j + 2k - a - b - 2) \right), \quad (4.8)
$$

where $g(u)$ is defined in (2.12). So using (4.7) along with (3.7) and (3.8) we deduce that

$$
\tilde{S}^{ab}(u)\tilde{S}^{ba}(-u) = I^b \otimes I^a. \quad (4.9)
$$

We are now in a position to investigate the analytic structure of $\tilde{S}^{ab}(u)$ on the physical strip. For $a_{m-1}$ there are no poles or zeros on the physical strip, whereas for $c_m$ there are no poles but if $a + b > m$ there is a series of simple zeros at

$$
u = \frac{a + b - 2j + 2}{2(m + 1)}, \quad j = 1, 2, \ldots, a + b - m. \quad (4.10)
$$

A consistent $S$-matrix is made by appending a suitable minimal Toda factor which provides the necessary pole structure. Generalizing the case for $a_{m-1}$ one is tempted to try the Gross-Neveu (GN) type Ansatz

$$
S^{ab}_{(k)}(u) = X^{ab}(u)\tilde{S}^{ab}_{(k)}(u), \quad (4.11)
$$

where $X^{ab}(u)$ is the minimal Toda factor of the associated algebra (3.19). However, we shall argue below that this is inconsistent for the non-simply-laced algebras due to the appearance of spurious poles on the physical strip which cannot be explained. Rather we shall find that a consistent $S$-matrix is given by the tensor product form which generalizes the $S$-matrix of the Principal Chiral Model (PCM):

$$
S^{ab}_{(k,l)}(u) = X^{ab}(u)\tilde{S}^{ab}_{(k)}(u) \otimes \tilde{S}^{ab}_{(l)}(u). \quad (4.12)
$$

For this $S$-matrix the particles transform in the reducible representations $V_{\omega_a} \otimes V_{\omega_a}$ where $V_{\omega_a}$ is the representation of the algebra with highest weight $\omega_a$.

Let us verify the above statements for algebras $a_{m-1}$ and $c_m$. The former case is easy to discuss. The minimal Toda factor is associated to the algebra $a_{m-1}^{(1)}$:

$$
X^{ab}(u) = \prod_{j=|a-b|+1}^{a+b-1} (j + 1)(j - 1), \quad (4.13)
$$

with the notation

$$
(j) = \frac{\sin \left( \frac{\pi u}{2} + \frac{\pi j}{2tg} \right)}{\sin \left( \frac{\pi u}{2} - \frac{\pi j}{2tg} \right)}, \quad (4.14)
$$

13
where $tg = m$ in this case. The pole in $X^{ab}(u)$ at $(a+b)/m$ (if $a+b < m$) or $2-(a+b)/m$ (if $a+b > m$) corresponds to particle $a+b$ or $a+b-m$, respectively, in the direct channel. One can verify directly using (4.1) that the residues at these poles are $P_{\omega_{a+b}}$ and $P_{\omega_{a+b-m}}$, respectively, as required for consistency. The pole at $|a-b|/m$ corresponds to the particle $|a-b|$ in the crossed channel. The element $S^{ab}(u)$ also exhibits double poles at

$$u = \frac{a+b-2k}{m}, \quad k = 1, 2, \ldots, \min(a, b) - 1. \quad (4.15)$$

These double poles are understood in terms of the Coleman-Thun mechanism [4] which was originally posited in order to explain the double poles in the sine-Gordon $S$-matrix. Essentially, the poles are Landau singularities of the ordinary sort, which appear as poles in two-dimensions and branch-points in four-dimensions, and are associated to ‘on-shell’ diagrams. In the present example, figure 1 gives the on-shell diagram causing the $k^{th}$ double pole in (4.15) for the case $a, b \leq m/2$ (other cases are found by crossing). We therefore conclude that both the generalized GN- and PCM-type $S$-matrices are consistent for $a_{m-1}$.

1. On-shell diagram giving double pole.

The scalar factor for $c_m$ is the $d^{(2)}_{m+1}$ minimal Toda $S$-matrix. This $S$-matrix can be understood as the $S$-matrix of the $d^{(1)}_{m+2}$ excluding the spinor and anti-spinor particles. So

$$X^{ab}(u) = \prod_{j=|a-b|+1}^{a+b-1} (j+1)(j-1)(2g-j+1)(2g-j-1), \quad (4.16)$$

where $g = m + 1$ is the dual Coxeter number of $c_m$ and the bracket notation is defined in (4.14). So $X^{ab}(u)$ in general exhibits poles of order 1,2,3 and 4 on the physical strip. Consider a GN-type $S$-matrix. The analytic structure of $S^{ab}_{(k)}(u)$ is deduced by combining that of $X^{ab}(u)$ with the zeros (4.10) of the $\tilde{S}^{ab}(u)$ part. Rather than analyzing the $S$-matrix in full detail we shall exhibit a singularity which has no explanation in terms of the particle spectrum. To this end one notices that $X^{a_{m+1-a}}(u)$ has a double pole at $u = 1/2$ which is explained in terms of the on-shell diagrams in figure 2 involving the spinor and anti-spinor
particles of the $d_{m+2}^{(1)}$ theory.\(^7\) \(\tilde{S}_{ab}^{(1)}(u)\) has a simple zero at $u = 1/2$; hence $S_{ab}^{(1)}(u)$ has a simple pole at $u = 1/2$ which cannot be explained in terms of the particle spectrum. At this stage we are forced to conclude that there cannot be a consistent $S$-matrix of the GN-type.\(^8\)

2. On-shell diagram giving double pole in $X_{a,m+1-a}^{a,m+1-a}(u)$ for $d_{m+2}^{(1)}$.

The PCM-type of $S$-matrix, however, does not have this spurious pole since there is an extra zero to cancel it. The analytic structure on the physical strip is illustrated in figure 3 where the crosses represent poles.

3. Poles in the PCM-type $S$-matrix for $c_m$.

Notice that there are only simple poles and double poles. Some of the simple poles can be understood in terms of direct or crossed channel resonances of the particles and we consider these ones first. It is useful to distinguish three cases.

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\(^7\) I am grateful to Patrick Dorey for explaining this point.

\(^8\) It might be possible to enlarge the set of particles and then close the bootstrap, however, the new particle would transform in a reducible representation involving non-fundamental representations.
(i) $a + b < m + 1$. In this case there are four simple poles. The simple poles at $(a+b)/2(m+1)$ and $1 - |a-b|/2(m+1)$ correspond to the exchange of particles $a + b$ and $|a - b|$ in the direct channel, respectively, with residues proportional to $P_{\omega_{a+b}}$ and $P_{\omega_{|a-b|}}$. The simple poles at $|a - b|/2(m+1)$ and $1 - (a + b)/2(m+1)$ correspond to the exchange of $|a - b|$ and $a + b$ in the crossed channel, respectively.

(ii) $a + b = m + 1$. In this case there are two simple poles. The simple poles at $|a - b|/2(m+1)$ and $1 - |a-b|/2(m+1)$ correspond to the exchange of particle $|a - b|$ in the crossed and direct channel, respectively. In the latter case the residue is proportional to $P_{\omega_{|a-b|}}$.

(iii) $a + b > m + 1$. In this case there are four simple poles, however, only two of them correspond to direct or cross channel resonances. The simple poles at $|a - b|/2(m+1)$ and $1 - |a-b|/2(m+1)$ correspond to the exchange of particle $|a - b|$ in the crossed and direct channel, respectively. In the latter case the residue is proportional to $P_{\omega_{|a-b|}}$.

We now turn to the remaining singularities. Firstly if $a + b > m + 1$ there are simple poles at $(a+b)/2(m+1)$ and $1 - (a + b)/2(m+1)$. Secondly there are double poles are in the union of the set

$$u = \frac{a + b - 2k}{2(m+1)}, \quad k = 1, 2, \ldots, \min(a, b) - 1,$$

and its crossed version, with the exceptions that for $a + b > m + 1$ $u = (a + b)/2(m+1)$ and $u = 1 - (a + b)/2(m+1)$ are simple rather than double poles and when $a + b = m + 1$ $u = 1/2$ is regular.

There exists at least one on-shell diagram at the positions of the double poles, which involve particles in the spectrum and which lead to double poles on kinematical grounds. These on-shell diagrams are the same as those in figure 1 (plus the crossed versions). If $a + b > m + 1$ one would expect to see triple poles at $(a+b)/2(m+1)$ and $1 - (a + b)/2(m+1)$ coming from the factor $X^{ab}(u)$, whereas in fact the $S$-matrix only has simple poles. It is rather unconventional to have an $S$-matrix, some of whose simple poles do not correspond to direct or cross channel resonances. However, [14] discusses a recent example of such an eventuality in non-simply-laced Toda theories. These strange occurrences are explained by a generalization of the Coleman-Thun mechanism [4], in the sense that there are on-shell diagrams which at first sight seem to lead to higher order poles but which on closer inspection actually only lead to simple poles because some sub-$S$-matrix element in the diagram has a zero. We shall find that a similar mechanism is at work in the example under discussion, except that here the softening of the singularity is not due to some sub-$S$-matrix element having a zero but rather is intimately bound up with the fact that the particles carry internal quantum numbers and one must sum all the quantum numbers of particles on internal lines.
The diagram which looks like it would lead to a triple pole at \((a + b)/2(m + 1)\) (and \(1 - (a + b)/2(m + 1)\) by crossing) is illustrated in figure 4 (with the incoming particles coming from the left), where the grey blob is the S-matrix element \(S^{m+1-a-b,m+1-a-b}(u)\) evaluated at \(u \equiv u_0 = 1 - (a + b)/2(m + 1)\) and the black blobs represent projection operators.

4. On-shell diagram for \(a + b > m + 1\).

As it stands, standard kinematical arguments would indicate that the diagram yields a double pole. However the S-matrix element \(S^{m+1-a-b,m+1-a-b}(u)\) has a simple pole at \(u_0\) corresponding to the exchange of \(2(m + 1) - a - b\) in the direct channel; hence the expected singularity is a triple pole. However, this argument is too naïve and we must investigate the residue more closely. The behaviour of the internal S-matrix element in the vicinity of the pole to \(O(u - u_0)\) is

\[
S^{m+1-a-b,m+1-a-b}(u) = \frac{1}{u - u_0} \left( a\mathbb{P}' + (u - u_0) \sum_\mu b_\mu \mathbb{P}_\mu \right) \otimes \left( a\mathbb{P}' + (u - u_0) \sum_\mu b_\mu \mathbb{P}_\mu \right),
\]

where \(\mathbb{P}' = \mathbb{P}_{\omega_2(m+1)-a-b}\), \(a\) and \(b_\mu\) are constants and the sums are over the highest weights of the representations that appear in the tensor product \(V_{\omega_{m+1-a-b}} \otimes V_{\omega_{m+1-a-b}}\). The relevant observation is that the representation with highest weight \(\omega_{2(m+1)-a-b}\) does not actually appear in the tensor product \(V_{\omega_a} \otimes V_{\omega_b}\) (remember that \(a + b > m + 1\)). So if we isolate the group-theoretic contributions from one of the factors in the tensor product to the S-matrix we conclude that the terms of the form \(\mathbb{P}_0 \mathbb{P}' \mathbb{P}_0\), where \(\mathbb{P}_0\) represents the product of the two projection operators on the external legs and hence is a homomorphism from \(V_{\omega_a} \otimes V_{\omega_b}\) to \(V_{\omega_{m+1-a-b}} \otimes V_{\omega_{m+1-a-b}}\), must vanish. Hence when the diagram is evaluated one actually picks up the contribution from the central S-matrix element at \(O(u - u_0)\), rather than at \(O(u - u_0)^{-1}\). In other words the contributions from the more singular diagrams are zero after one has summed over all the intermediate states. So the actual singularity
is a simple pole rather than a triple pole since each triangle contributes a simple pole from its kinematical factors.

It is important to be aware of the fact that the $S$-matrix $X^{ab}(u)$ also has fourth order poles which can be explained solely in terms of the set of particles in the $d_{m+1}^{(2)}$ spectrum. These singularities are reduced to poles of second order in the PCM-type $S$-matrix. In order for this to occur the residues of the diagrams corresponding to the fourth order poles must vanish. Such an eventuality is possible because the particles carry internal quantum numbers — as we have seen from the preceding arguments. Unfortunately, an analysis of the residues is a rather formidable problem and is not within the scope of the present article.

Before we move on to discuss the rational limits of the $S$-matrices we first pause to consider the $d_m$ and $b_m$ cases. As we have pointed out for these algebras the vector is not the elementary particle, rather it is the spinor (and anti-spinor) particle. Nevertheless our $S$-matrices should generate a subset of the full $S$-matrix. One finds that in these cases the bound states are not associated to the fundamental representations, as in the $a_{m-1}$ and $c_m$ cases. In fact the $a^\text{th}$ particle transforms in the reducible representation [15]

$$W_a = \bigoplus_{j=0}^{a-2j \geq 0} V_{\omega_a-2j}.$$

(4.19)

Unfortunately the spectral decompositions have not been found for these representations.

5. The rational limit

The rational limit of the $S$-matrices are obtained by taking $k \to \infty$ and the resulting $S$-matrices are those of some well-known quantum field theories. Furthermore one can show by explicit computation that the ‘important’ analytic structure of the $S$-matrix is not affected by the limit, in the sense that no poles or zeros from the $\tilde{S}$ factor can wander onto or off the physical strip. The rational $S$-matrices are actually invariant under the group associated to the Lie algebra in question (on the contrary the trigonometric $S$-matrices of the last section are invariant under the quantum group).

The rational limit of the $R$-matrices is easily obtained by taking the $k \to \infty$ limits of (4.1) and (4.4), which means replacing (4.3) with

$$\{x\} = (x + tgu)/2,$$

(5.1)

and taking the $p$’s to be the Lie algebra, rather than quantum group, invariant homomorphisms. This latter limit is taken because as $k \to \infty$ the deformation parameter of the quantum group $q \to -1$ and so the quantum group reduces to the Lie algebra.
The factor $Y_{tg,tk}(u)$ also has a good limit:

$$Y_{tg,\infty}(u) = \frac{1}{tg} \frac{\Gamma \left( 1 - \frac{u}{2} \right) \Gamma \left( \frac{1}{tg} + \frac{u}{2} \right)}{\Gamma \left( 1 + \frac{u}{2} \right) \Gamma \left( 1 + \frac{1}{tg} - \frac{u}{2} \right)}.$$  \hspace{1cm} (5.2)

Using these results we find for vector-vector scattering

$$\tilde{S}_{(\infty)}(u) = \frac{\Gamma \left( 1 - \frac{u}{2} \right) \Gamma \left( \frac{1}{m} + \frac{u}{2} \right)}{\Gamma \left( 1 + \frac{u}{2} \right) \Gamma \left( \frac{1}{m} - \frac{u}{2} \right)} \left[ \mathbb{P}_{2\omega_1} + \left( \frac{\frac{2}{m} + u}{\frac{2}{m} - u} \right) \mathbb{P}_{\omega_2} \right],$$  \hspace{1cm} (5.3)

for $a_{m-1}$ and

$$\tilde{S}_{(\infty)}(u) = \frac{\Gamma \left( 1 - \frac{u}{2} \right) \Gamma \left( \frac{1}{tg} + \frac{u}{2} \right) \Gamma \left( \frac{1}{t^2} + \frac{1}{2} - \frac{u}{2} \right)}{\Gamma \left( 1 + \frac{u}{2} \right) \Gamma \left( \frac{1}{tg} - \frac{u}{2} \right) \Gamma \left( \frac{1}{t^2} - \frac{u}{2} \right) \Gamma \left( \frac{1}{tg} + \frac{1}{2} + \frac{u}{2} \right)} \times \left[ \mathbb{P}_{2\omega_1} + \left( \frac{\frac{2}{t^2} + u}{\frac{2}{t^2} - u} \right) \mathbb{P}_{\omega_2} + \left( \frac{1 + u}{1 - u} \right) \left( \frac{\frac{2}{t^2} + \sigma u}{\frac{2}{t^2} - u} \right) \mathbb{P}_0 \right],$$  \hspace{1cm} (5.4)

for the other algebras.

A careful comparison of the rational limits of our $S$-matrices (5.3) and (5.4) with those of the principal chiral model [3],9 in which the particles transform in a tensor product of fundamental representations of the algebra, confirms that

$$S_{PCM}^{ab}(u) = S_{(\infty,\infty)}^{ab}(u) = \tilde{S}_{(\infty)}^{ab}(u) \otimes \tilde{S}_{(\infty)}^{ab}(u) X^{ab}(u),$$  \hspace{1cm} (5.5)

for $a_{m-1}$ and $c_m$ (and $b_m$ and $d_m$ for $a = b = 1$). This relation explains our use the nomenclature ‘PCM-type’ for the $S$-matrix of (4.12).

For $a_{m-1}$ we can also consider the $S$-matrix

$$S_{GN}^{ab}(u) = S_{(\infty)}^{ab}(u) = \tilde{S}_{(\infty)}^{ab}(u) X^{ab}(u),$$  \hspace{1cm} (5.6)

which is the $S$-matrix of the $SU(m)$ Gross-Neveu model [3,16].

The fact that the Gross-Neveu $S$-matrix for the non-simply-laced algebras violates the bootstrap was noted in [3]. It has been claimed [3] that the $S$-matrix of the principal chiral model for $c_m$ violates the bootstrap due to the appearance of simple poles with no explanation in terms of direct or cross channel poles. However, the analysis of the last section shows how this problem is resolved and the poles can be properly understood via a generalization of the Coleman-Thun mechanism.

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9 In comparing our expressions with this reference it is important to notice that the $S$-matrices defined there are equal to ours up to a permutation of the outgoing particles.
6. **The situation at \( k = 1 \)**

In this section we consider what happens to the \( S \)-matrices when the level \( k = 1 \). It is known from the study of quantum groups that when \( q \) is a root of unity, so \( k \) is an integer, that the representation theory has to modified. This modification is implemented automatically by moving to the IRF picture and means that the usual rule for decomposing tensor products is truncated on the set of representations which correspond to highest weights of level \( k \). This would entail appropriate modifications of the spectral decompositions. When \( k = 1 \) only level one representations survive. It is significant that in the cases where the particles are associated to higher level representations, i.e. for the \( b_m \) and \( d_m \) theories where there are representations at level 2, then the representations (4.19) are reducible and contain a level 1 component, so that all the particles remain in the spectrum.\(^{10}\)

As we mentioned, the appropriate way to write down the \( S \)-matrix for \( k = 1 \) is via the IRF formalism. At \( k = 1 \) the allowed weights are simply \( \Lambda^*(1) \), that is

\[
\begin{align*}
\{0, \omega_1, \omega_2, \ldots, \omega_{m-1}\}, & \quad \text{for } a_{m-1}, \\
\{0, \omega_1, \omega_m\}, & \quad \text{for } b_m, \\
\{0, \omega_1, \omega_2, \ldots, \omega_m\}, & \quad \text{for } c_m, \\
\{0, \omega_1, \omega_{m-1}, \omega_m\}, & \quad \text{for } d_m.
\end{align*}
\]

(6.1)

For each of the representations \( W_a \) (recall from (4.19) that they are in general reducible for \( b \) and \( d \) series) one can draw an admissibility diagram consisting of a set of nodes for each element of \( \Lambda^*(1) \) with \( a \) joined to \( b \) by an oriented link if \( a - b \) is a weight of the representation. For example, in figures 5 and 6 we give the admissibility diagrams for the vector spinor and anti-spinor representations of \( d_m \) and for the vector and spinor representations of \( c_m \).

One notices immediately a qualitative difference between the admissibility diagrams of the simply-laced and non-simply-laced algebras. In the former case the admissibility diagrams are trivial in the sense that the space of paths on the diagram from a given starting point and given length is just one-dimensional. This means that the resulting solution of the YBE is trivial:

\[
\tilde{S}_{ab}^{(1)}(u) = 1.
\]

(6.2)

This means that the GN-type \( S \)-matrix for the simply-laced algebras is related to the PCM-type \( S \)-matrix:

\[
S_{ab}^{(k)}(u) = S_{ab}^{(k,1)}(u),
\]

(6.3)

\(^{10}\) This observation seems also to be true for the known solutions of the YBE for the exceptional algebras as well.
and furthermore
\[
S^{ab}_{(1,1)}(u) = S^{ab}_{(1)}(u) = X^{ab}(u),
\]  \hspace{1cm} (6.4)

the minimal Toda $S$-matrix.

On the contrary, for the non-simply-laced algebras the admissibility diagrams are not trivial when $k = 1$. In other words at $k = 1$ the $S$-matrix is still non-diagonal and a relations like (6.3) and (6.4) do not hold.

7. Discussion

We have constructed factorizable $S$-matrices for trigonometric solutions of the YBE for the vector representations of all the classical Lie algebras. By appending a suitable CDD factor the $S$-matrices have singularities corresponding to new states. For the $a_{m-1}$ and $c_m$ algebras, where the relevant spectral decompositions are known, one can solve the bootstrap equations to find that the spectrum consists of particles transforming in the fundamental representations of the algebra. For $b_m$ and $d_m$ the situation is less clear and in any case the vector particle is not the fundamental particle; rather one would expect
this to be the spinor (and anti-spinor for $d_m$).

The simplest Ansatz for the $S$-matrix, generalizing that of the Gross-Neveu model, fails for $c_m$ due to the existence of singularities on the physical strip which cannot be explained in terms of the spectrum of particles. A generalized principal chiral model Ansatz, for which the particles transform in a tensor product of representations, is shown to be consistent and all singularities on the physical strip can be accounted for.

It has been argued that the $a_{m-1}$ trigonometric $S$-matrices describe certain integrable deformations of some coset conformal field theories [6,8]. Consider the coset conformal field theory $x_k \times x_l / x_{k+l}$, for some Lie algebra $x$. There is a particular relevant operator in the theory of holomorphic dimension

$$\Delta = 1 - \frac{g}{k+l+g}, \quad (7.1)$$

where $g$ is as before the dual Coxeter number of $x$, which leads to a massive theory with higher spin integrals of motion. The natural conjecture is that the $S$-matrix of this integrable theory is precisely the generalized PCM-type $S$-matrix $S_{(k,l)}^{ab}(u)$ [17]. Notice that the $x_1 \times x_k / x_{k+1}$ cases are described by the GN-type $S$-matrix for the simply-laced algebras only (see section 6). The conjecture could perhaps be placed on a better footing by employing the technology of the Thermodynamic Bethe Ansatz to investigate the ultra-violet limit of the theories. The relation of the trigonometric solutions to the YBE and Bethe Ansatz systems has been considered in [3,18,19].

Finally, the $a_{m-1}$ trigonometric $S$-matrices have been proposed to describe the scattering of solitons in complex $a_m^{(1)}$ Toda field theory [5,20]. This has been established via semi-classical techniques. It is now known that all complex affine Toda theories admit soliton solutions [21], and it is natural to ask whether the trigonometric $S$-matrices for the other algebras describe the scattering of these solitons. This may be true for the simply-laced algebras, for which the soliton $S$-matrix would be the generalized GN-type $S$-matrix. For the non-simply-laced algebras the situation is much less clear. Experience with the real Toda theories in these cases shows that the resolution of the problems may be surprisingly subtle and perhaps will require new types of solution to the YBE appropriate to a set of particles whose mass ratios depend on a coupling constant.

I would like to thank Patrick Dorey for many conversations on $S$-matrices and also Gustav Delius for some useful discussions. Also I would like to thank Niall Mackay for pointing out an error in an earlier version of the manuscript which allowed for a significant improvement.
Appendix A: The spectral decompositions for $a_{m-1}$ and $c_m$

In this appendix we derive the spectral decompositions of the $R$-matrices on the fundamental representations of the $a_{m-1}$ and $c_m$ algebras. We follow the approach of [22] although our method is also a direct translation into the quantum group of [23] which found the spectral decompositions of the rational $R$-matrices for $c_m$.

The first point to make is an $R$-matrix cannot be associated with any two representations of the quantum group. It is necessary that the representations are \textit{affinizable} in the language of [22]. It can be shown that all the fundamental representations of $a_{m-1}$ and $c_m$ have this property. However, it is not true for the fundamental representations of the other algebras, where the affinizable representations are reducible in general [10,15].

Now we apply the technology of [22,23] to find the spectral decompositions of the $R$-matrices on the fundamental representations. One first constructs the Tensor Product Graph (TPG). This is a graph is constructed by letting the irreducible components of the tensor product $V_\mu \otimes V_\nu$ be the nodes joined by a link if $V_\lambda$ and $V_\sigma$ have opposite parity and $V_\sigma \subset \text{adjoint} \otimes V_\lambda$. The parity of an irreducible component $V_\lambda$ is defined to be $\pm 1$ according to whether $V_\lambda$ appears symmetrically or anti-symmetrically in the tensor product (in the limit $q \to 1$).

For $V_{\omega_a} \otimes V_{\omega_b}$ (with $b \geq a$ without loss of generality) of $a_{m-1}$, using the notation $j, k \equiv V_{\omega_{j+k}}$ and $j \equiv V_{\omega_j}$ we have the TPG

$$
\begin{align*}
  a, b & \leftrightarrow a - 1, b + 1 & \cdots & \leftrightarrow & a - \min(m - b, a), b + \min(m - b, a). \\
  a, b & \leftrightarrow a - 1, b + 1 & \cdots & \leftrightarrow & a - \min(m - b, a), b + \min(m - b, a).
\end{align*}
$$

(A.1)

For $c_m$ we have for $a + b \leq m$

$$
\begin{align*}
  a, b & \leftrightarrow a - 1, b + 1 & \cdots & \leftrightarrow & 1, a + b - 1 & \leftrightarrow & a + b \\
  a - 1, b - 1 & \leftrightarrow a - 2, b & \cdots & \leftrightarrow & a + b - 2 \\
  \vdots & \vdots & & & \vdots \\
  1, b - a + 1 & \leftrightarrow b - a + 2 \\
  b - a
\end{align*}
$$

(A.2)

whilst if $a + b > m$ then the graph truncates at the $(m - b + 1)$th column.

The spectral decomposition of the $R$-matrix has the form

$$
\tilde{R}^{ab}(x) = \sum_{\mu} \rho_\mu(x) P_\mu, 
$$

(A.3)

23
where the sum is over the representations that appear in the tensor product $V_{\omega_a} \otimes V_{\omega_b}$ and hence is a sum over nodes of the TPG. $x$ is the multiplicative spectral parameter. If there is an arrow from $\nu$ to $\mu$ on the TPG then the coefficients $\rho_\mu(x)$ and $\rho_\nu(x)$ satisfy
\[
\frac{\rho_\mu(x)}{\rho_\nu(x)} = \frac{xq^{I(\mu)/2} - x^{-1}q^{I(\nu)/2}}{x^{-1}q^{I(\mu)/2} - xq^{I(\nu)/2}}, \tag{A.4}
\]
where $q$ is the deformation parameter of the quantum group and $I(\mu) = (\mu + 2\rho) \cdot \mu$ (where $\rho$ is the sum of the fundamental weights) is the eigenvalue of the quadratic Casimir on the representation with highest-weight $\mu$.

The spectral decompositions follow from applying the rule (A.4) recursively from $\omega_a + \omega_b$ (for the case of $c_m$ it is important to notice that the result is independent of the path). With
\[
x = \exp(i\lambda u), \quad q = -\exp(-i\omega), \tag{A.5}
\]
one finds (4.1) and (4.4) up to an overall multiplicative factor.

**Appendix B: Zeros and the bootstrap**

In this appendix we explain how the solution of the bootstrap equations leads to the zeros in (4.1) and (4.4). From the spectral decompositions of the $R$-matrices we know that
\[
\tilde{S}^{ab}(u_{ab}) \propto \mathbb{P}_{\omega_c}, \tag{B.1}
\]
for some appropriate value of $u_{ab}^c$, in which case we say that there is a fusion $ab \rightarrow c$. This also implies that there are fusions $a\bar{c} \rightarrow \bar{b}$ and $b\bar{c} \rightarrow \bar{a}$ where $\bar{a}$ is the charge conjugate particle (for $a_{m-1} \bar{a} = m - a$ whilst for the other algebras $\bar{a} = a$). The identity
\[
u_{ab}^c + \nu_{a\bar{c}}^{\bar{b}} + \nu_{b\bar{c}}^{\bar{a}} = 2, \tag{B.2}
\]
holds between the fusing parameters. The bootstrap equations give the $S$-matrix elements of $c$ in terms of $a$ and $b$ which follows from the fusion relation between the $R$-matrices:
\[
\tilde{R}^{dc}(u) = \left( I^b \otimes \tilde{R}^{da}(u + \nu_{a\bar{c}}^{\bar{b}}) \right) \left( \tilde{R}^{db}(u - \nu_{b\bar{c}}^{\bar{a}}) \otimes I^a \right), \tag{B.3}
\]
restricted on the left and right to the subspace $V_{\omega_c} \subset V_{\omega_b} \otimes V_{\omega_a}$. In the above $\nu = 1 - u$ and $I^a$ is the identity on $V_{\omega_a}$. Consider the case $d = a$ evaluated at $u = -\nu_{a\bar{c}}^{\bar{b}}$ then
\[
\tilde{R}^{ac}(-\nu_{a\bar{c}}^{\bar{b}}) = \left( I^b \otimes \tilde{R}^{aa}(0) \right) \left( \tilde{R}^{ab}(-\nu_{ab}^c) \otimes I^a \right) \tag{B.4}
\]
using (B.2). But from the spectral decompositions (4.1) and (4.4) one finds $\tilde{R}^{aa}(0) \propto I^a \otimes I^a$ and that $\tilde{R}^{ab}(-u_{ab}^c)$ has no component proportional to $P_{\omega_c}$, hence

$$\tilde{R}^{ac}(-u_{ac}^b) = 0.$$  

(B.5)

We now apply (B.5) and (B.3) recursively to find zeros of $\tilde{R}^{ab}(u)$. Starting from $\tilde{R}^{11}(u)$ ((3.13) and (3.16)) (B.5) implies that $\tilde{R}^{12}(-1/tg) = 0$. For all the algebras except $a_{m-1}$ crossing symmetry would imply $\tilde{R}^{12}(1 + 1/tg) = 0$ in addition. By the recursive use of (B.3) we find that $\tilde{R}^{ab}(u)$ for $b \geq a$ has a set of zeros at

$$u = \frac{1}{tg}(a + b - 2j - 2k), \quad j = 1, 2, \ldots, a, \quad k = 1, 2, \ldots, b - 1,$$

(B.6)

and for all the algebras except $a_{m-1}$ their crossed values as well.

The remaining zeros in (4.1) and (4.4) are accounted for in a different way. When $a + b > m$ then we saw in appendix A that the TPG gets truncated which implies an additional set of zeros at

$$u = \frac{1}{tg}(2k + b - a), \quad k = m - b + 1, m - b + 2, \ldots, a.$$  

(B.7)

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