ON THE ACCEPTABLE ELEMENTS

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Abstract. In this paper, we study the set $B(G, \mu)$ of acceptable elements for any $p$-adic group $G$. We show that $B(G, \mu)$ contains a unique maximal element and is represented by an element in the admissible subset of the associated Iwahori-Weyl group.

Introduction

Let $F$ be a finite field extension of $\mathbb{Q}_p$ and $L$ be the completion of the maximal unramified extension of $F$. Let $G$ be a connected reductive algebraic group over $F$ and $\sigma$ be the Frobenius morphism. We denote by $B(G)$ the set of $\sigma$-conjugacy classes of $G(L)$. The set $B(G)$ is classified by Kottwitz in [Ko1] and [Ko2]. This classification generalize the Dieudonné-Manin classification of isocrystals by their Newton polygons.

Let $\tilde{W}$ be the Iwahori-Weyl group of $G$. Let $\{\mu\}$ be a conjugacy class of characters of $G$ defined over $L$. Let $\text{Adm}(\mu) \subset \tilde{W}$ be the admissible subset of $\tilde{W}$ ([KR1]) and $B(G, \mu)$ be the finite subset of $B(G)$ defined by the group-theoretic version of Mazur’s theorem [Ko2, §6].

The main result of this paper is as follows.

Theorem 0.1. The set $B(G, \mu)$ contains a unique maximal element and this element is represented by an element in $\text{Adm}(\mu)$.

For quasi-split groups, this is obvious as the unique maximal element of $B(G, \mu)$ is represented by $t^\mu$. However, it is much more complicated for non quasi-split groups.

This result is an important ingredient in the proof [He3] of Kottwitz-Rapoport conjecture [KR2, Conjecture 3.1] and [Ra, Conjecture 5.2] on the union of affine Deligne-Lusztig varieties. The knowledge of the explicit description of the maximal element of $B(G, \mu)$ is also useful in the study of $\mu$-ordinary locus of Shimura varieties.

Key words and phrases. Newton polygons, $p$-adic groups, affine Weyl groups.
1. Preliminaries

1.1. We first recall the classification of $B(G)$ obtained by Kottwitz in [Ko1] and [Ko2].

For any $b \in G(L)$, we denote by $[b]$ the $\sigma$-conjugacy class of $G(L)$ that contains $b$. Let $\Gamma_F = \text{Gal}(\overline{L}/F)$ be the absolute Galois group of $F$. Let $\kappa_G : B(G) \to \pi_1(G)_{\Gamma_F}$ be the Kottwitz map [Ko2, §7]. This gives one invariant.

Another invariant is given by the Newton map.

Let $S$ be a maximal $L$-split torus that is defined over $F$ and let $T$ be its centralizer. Since $G$ is quasi-split over $L$, $T$ is a maximal torus. We also fix a $\sigma$-invariant alcove $a$ in the apartment of $G_L$ corresponding to $S$.

To an element $b \in G(L)$, we associate its Newton point $\nu_b$. It is a $\sigma$-invariant element in the closed dominant chamber $X^*(T)_Q^+$. By [Ko2, §4.13], the map $B(G) \to X^*(T)_Q^+ \times \pi_1(G)_{\Gamma_F}$, $b \mapsto (\nu_b, \kappa_G(b))$ is injective.

The partial order on $B(G)$ is defined as follows. Let $b, b' \in G(L)$, then $[b] \leq [b']$ if $\kappa_G(b) = \kappa_G(b')$ and $\nu_b \leq \nu_{b'}$, i.e., $\nu_{b'} - \nu_b$ is a non-negative $Q$-linear combination of positive relative coroots.

1.2. We follow [HR]. Let $N$ be the normalizer of $T$. The finite Weyl group associated to $S$ is $W_0 = N(L)/T(L)$. The Iwahori-Weyl group associated to $S$ is $\tilde{W} = N(L)/T(L)_1$, where $T(L)_1$ denotes the unique Iwahori subgroup of $T(L)$. The Frobenius morphism $\sigma$ induces actions on $\tilde{W}$ and $\tilde{W}$, which we still denote by $\sigma$.

Let $\Gamma = \text{Gal}(\overline{L}/L)$. The Iwahori-Weyl group $\tilde{W}$ contains the affine Weyl group $W_a$ as a normal subgroup and

$$\tilde{W} = W_a \rtimes \Omega,$$

where $\Omega \cong \pi_1(G)_{\Gamma}$ is the normalizer of the alcove $a$. The Bruhat order on $W_a$ extend in a natural way to $\tilde{W}$.

Let $G_{sc}$ be the simply connected cover of the derived group of $G$. Denote by $T_{sc}$ the maximal torus of $G_{sc}$ given by the choice of $T$. Then we have a natural injective map $X_*(T_{sc})_{\Gamma} \to X_*(T)_{\Gamma}$. We fix a special vertex of $a$ and represent $\tilde{W}$ and $W_a$ as

$$\tilde{W} = X_*(T)_{\Gamma} \rtimes W_0 = \{t^\lambda w; \lambda \in X_*(T)_{\Gamma}, w \in W_0\},$$
$$W_a = X_*(T_{sc})_{\Gamma} \rtimes W_0 = \{t^\lambda w; \lambda \in X_*(T_{sc})_{\Gamma}, w \in W_0\}. $$
1.3. For any \( w \in \tilde{W} \), we choose a representative in \( N(L) \) and also write it as \( w \). By [He2] §3, any \( \sigma \)-conjugacy class of \( G(L) \) contains an element in \( \tilde{W} \). The restriction of Kottwitz map and Newton map on \( \tilde{W} \subset G(L) \) can be described explicitly as follows.

The map \( N(L) \to G(L) \) induces a map \( \tilde{W} \to B(G) \). Here \( \kappa_G(w) \) is the image of \( w \) under the projection \( \tilde{W} \to \Omega \cong \pi_1(G)_\Gamma \to \pi_1(G)_{\Gamma,F} \).

For any \( w \in \tilde{W} \), we consider the element \( w\sigma \in \tilde{W} \times \langle \sigma \rangle \). There exists \( n \in \mathbb{N} \) such that \( (w\sigma)^n = t^\lambda \) for some \( \lambda \in X_*(T)_\Gamma \). Let \( \nu_{w,\sigma} = \lambda/n \) and \( \tilde{\nu}_{w,\sigma} \) the unique dominant element in the \( W_0 \)-orbit of \( \nu_{w,\sigma} \). It is known that \( \nu_{w,\sigma} \) is independent of the choice of \( n \) and is \( \Gamma \)-invariant. Moreover, \( \tilde{\nu}_{w,\sigma} \) is the Newton point of \( w \) when regarding \( w \) as an element in \( G(L) \).

1.4. Let \( S \) and \( \tilde{S} \) be the set of simple reflections of \( W_0 \) and \( W_a \) respectively. Then \( \sigma(\tilde{S}) = \tilde{S} \). In general \( S \) is not \( \sigma \)-stable since the special vertex of \( a \) may not be \( \sigma \)-invariant. However, we may write \( \sigma \) as \( \sigma = \tau \circ \sigma_0 \), where \( \sigma_0 \) is a diagram automorphism of \( W_0 \) and the induced action of \( \tau \) on the adjoint group \( G_{ad} \) is inner.

Let \( N \) be the order of \( \sigma_0 \). For \( \mu \in X_*(T) \), we set

\[
\mu^\circ = \frac{1}{N} \sum_{i=0}^{N-1} \sigma_i^\circ(\mu) \in X_*(T)_\Q.
\]

Let \( \mu^\sharp \) be the image of \( \mu \) in \( \pi_1(G)_{\Gamma,F} \). Set

\[
B(G, \mu) = \{ [b] \in B(G); \kappa_G(b) = \mu^\sharp, \nu_b \leq \mu^\circ \}.
\]

The elements in \( B(G, \mu) \) are called the (neutral) acceptable elements for \( \mu \).

Let \( \underline{\mu} \) be the image of \( \mu \) in \( X_*(T)_\Gamma \). The \( \mu \)-admissible set is defined as

\[
\text{Adm}(\mu) = \{ w \in \tilde{W}; w \leq t^x(\mu) \text{ for some } x \in W_0 \}.
\]

Now we may reformulate the main theorem 0.1 as follows.

**Theorem 1.1.** We keep the notation as in \( \ref{1.4} \). Set \( B(\tilde{W}, \mu, \sigma) = \{ \tilde{\nu}_{w,\sigma}; w \in t^\mu W_a, \tilde{\nu}_{w,\sigma} \leq \mu^\circ \} \). Then

1. The set \( B(\tilde{W}, \mu, \sigma) \) contains a unique maximal element \( \nu \).
2. There exists an element \( w \in \text{Adm}(\mu) \) with \( \tilde{\nu}_{w,\sigma} = \nu \).

2. The maximal element in \( B(G, \mu) \)

2.1. Let \( G_{ad} \) be the adjoint group of \( G \), i.e., the quotient of \( G \) by its center. Since the buildings of \( G \) and \( G_{ad} \) coincide, the choice of an alcove \( a \) in the building of \( G \) determines an alcove of \( G_{ad} \). Then the Iwahori-Weyl group \( \tilde{W}_{ad} \) of \( G_{ad} \) is \( X_*(T_{ad})_\Gamma \rtimes W_0 \). Let \( \pi : G \to G_{ad} \) be
the projection map. Set \( T_{ad} = \pi(T) \). Then \( \pi \) induces maps \( \tilde{W} \to \tilde{W}_{ad} \) and \( X_*(T^+_{Q}) \to X_*(T^+_{ad}) \), which we still denote by \( \pi \).

It is easy to see that \( \pi(\nu_{w,\sigma}) = \nu_{\pi(w),\sigma} \) for \( w \in \tilde{W} \) and \( \pi \) induces a bijection of posets from \( B(\tilde{W}, \mu, \sigma) \) to \( B(\tilde{W}_{ad}, \pi(\mu), \sigma) \). Thus Theorem 1.1 holds for \( B(\tilde{W}, \mu, \sigma) \) if and only if it holds for \( B(\tilde{W}_{ad}, \pi(\mu), \sigma) \).

2.2. In the rest of this section, we assume that \( G \) is adjoint. We write \( \sigma \) as \( \sigma = \text{Ad}(\tau) \circ \sigma_0 \), where \( \tau \in \tilde{W} \) is a length zero element and \( s_0 \) is a diagram automorphism of \( W_0 \). Then \( \nu_{w,\sigma} = \nu_{w,\sigma_0} \) for all \( w \in \tilde{W} \).

Set \( V = X_*(T)_\Gamma \otimes_\mathbb{Z} \mathbb{R} \). For any \( i \in \mathbb{S} \), let \( \omega_i^\vee \in V \) be the fundamental coweight and \( \alpha_i^\vee \in V \) be the simple coroot. We denote by \( \omega_i, \alpha_i \in V^* \) the fundamental weight and simple root, respectively.

We also fix \( \lambda \in X_*(T)_+ \) such that \( \tau \in t^\lambda W_0 \). For each \( \sigma_0 \)-orbit \( c \) of \( \mathbb{S} \), we set \( \omega_c = \sum_{i \in c} \omega_i \), where \( \omega_i \) is the fundamental weight for \( i \). For any \( \nu \in X_*(T)_Q \), we set \( J(\nu) = \{ s \in S; s(\nu) = \nu \} \) and \( I(\nu) = S \backslash J(\nu) \). If \( \nu = \sigma_0 \), then both \( J(\nu) \) and \( I(\nu) \) are \( \sigma_0 \)-stable.

The follow lemma is essentially contained in [Ch]. Due to its importance, we provide a proof for completeness.

**Lemma 2.1.** Let \( \nu \in X_*(T)_Q^+ \) with \( \sigma_0(\nu) = \nu \). Then \( \nu = \nu_{w,\sigma} \) for some \( w \in t^\mu W_a \) if and only if \( \langle \omega_c, \mu^\circ + \lambda^\circ - \gamma \rangle \in \mathbb{Z} \) for any \( \sigma_0 \)-orbit \( c \) of \( I(\nu) \).

**Proof.** Since \( \nu_{w,\sigma} = \nu \), we have \( w_\tau = t^\gamma x \) for some \( \gamma \in X_*(T)_\Gamma \) and \( x \in W_{I(\nu)} \). Let \( N_0 \) be the order of \( W_0 \times \langle \sigma_0 \rangle \). Then

\[
\nu_{w,\sigma} = \nu_{w_\tau,\sigma_0} = \frac{1}{N_0} \sum_{k=0}^{N_0-1} (x\sigma_0(x) \cdots \sigma_0^{k-1}(x))\sigma_0^k(\gamma)
\]

\[
\in \frac{1}{N_0} \sum_{k=0}^{N_0-1} \sigma_0^k(\gamma) + \sum_{j \in I(\nu)} \mathbb{Q} \alpha_j^\vee.
\]

\[
= \gamma^\circ + \sum_{j \in I(\nu)} \mathbb{Q} \alpha_j^\vee.
\]

If \( w \in t^\mu W_a \), then \( w_\tau \in t^{\mu + \lambda} W_a \) and \( \mu + \lambda - \gamma \in X_*(T_{sc})_\Gamma \). Hence \( \langle \omega_c, \mu^\circ + \lambda^\circ - \gamma \rangle = \langle \omega_c, \mu + \lambda - \gamma \rangle \in \mathbb{Z} \).

On the other hand, let \( a_c = \langle \omega_c, \mu^\circ + \lambda^\circ - \gamma \rangle \in \mathbb{Z} \) for each \( \sigma_0 \)-orbit \( c \) of \( I(\nu) \). We construct an element \( w \in t^\mu W_a \) such that \( \nu_{w,\sigma} = \nu \).

For each \( \sigma_0 \)-orbit of \( J(\nu) \), we choose a representative. Let \( x \) be the product of these representatives (in some order). Then \( x \) is a \( \sigma_0 \)-twisted Coxeter element of \( W_{I(\nu)} \) in the sense of [Sp, 7.3]. For each \( \sigma_0 \)-orbit \( c \) of \( I(\nu) \), we choose a representative \( \iota_c \). Let \( \alpha^\vee_c \) be the corresponding simple coroot in \( X_*(T_{sc})_\Gamma \). Set \( \beta = \mu + \lambda - \sum_c a_c \alpha^\vee_c \) and \( w = t^\beta x \iota^{-1} \in t^\mu W_a \).
Write \( \beta = h + r \) with \( r \in \sum_{j \in J(v)} \mathbb{Q} \alpha_j^\vee \) and \( h \in \sum_{i \in I(v)} \mathbb{Q} \omega_i^\vee \). Then
\[
\nu_{w, \sigma} = \nu_{w, \sigma_0} = \frac{1}{N_0} \sum_{i=0}^{N_0-1} (x\sigma_0)^i(\beta)
\]
\[
= h^\circ + \frac{1}{N_0} \sum_{k=0}^{N_0-1} (x\sigma_0)^k(r)
\]
\[
= h^\circ + h^\circ + \lambda^\circ - \sum_c a_c(\alpha_i^\vee)_c^\circ - r^\circ,
\]
where the fourth equality is due to the fact that \( x \) is \( \sigma_0 \)-elliptic in \( W_{J(v)} \).
Hence for any \( \sigma_0 \)-orbit \( c \) of \( I(v) \) and any \( j \in J(v) \), we have
\[
\langle \omega, \mu^\circ + \lambda^\circ - \nu_{w, \sigma} \rangle = \langle \omega, \sum_{c'} a_{c'}(\alpha_{i_{c'}}^\vee)^{c'} \rangle = a_c = \langle \omega, \mu^\circ + \lambda^\circ - v \rangle
\]
and
\[
\langle \alpha_j, \mu^\circ + \lambda^\circ - \nu_{w, \sigma} \rangle = \langle \alpha_j, \mu^\circ + \lambda^\circ \rangle = \langle \alpha_j, \mu^\circ + \lambda^\circ - v \rangle,
\]
which means \( \nu_{w, \sigma} = v \) as desired. \( \square \)

**Corollary 2.2.** \( \mu^\circ \in B(G, \mu) \) if and only if \( \langle \omega, \lambda^\circ \rangle \in \mathbb{Z} \) for any \( \sigma_0 \)-orbit \( c \) of \( I(\mu^\circ) \). In this case, \( \mu^\circ \) is a priori the maximal Newton polygon of \( B(G, \mu) \).

2.3. We follow [Ch, §6]. For any \( \sigma_0 \)-stable subset \( B \) of \( X_\ast(T)_\mathbb{Q}^\dagger \), we define
\[
C_{\geq B} = \{ v \in X_\ast(T)_\mathbb{Q}^\dagger; \sigma_0(v) = v \text{ and } v \geq b, \forall b \in B \}.
\]
We say \( B \) is reduced if \( C_{\geq B} \subseteq C_{\geq B'} \) for any \( \sigma_0 \)-stable proper subset \( B' \subseteq B \).
For any \( i \in S \), let
\[
pr(i) : V = \mathbb{R} \omega_i^\vee \otimes \sum_{j \neq i} \mathbb{R} \alpha_j^\vee \to \mathbb{R} \omega_i^\vee
\]
be the projection map.
Now we prove part (1) of Theorem 1.1

2.4. **Proof of Theorem 1.1 (1).** For any \( \sigma_0 \)-orbit \( c \) of \( S \) and \( i \in c \), we define \( e_i \in \mathbb{Q} \omega_i^\vee \) by
\[
\langle \omega_i, e_i \rangle = \frac{1}{\#C} \max\{ \{ t \in \langle \omega_c, \mu^\circ + \lambda^\circ \rangle + \mathbb{Z}; t \leq \langle \omega_c, \mu^\circ \rangle \} \cup \{ 0 \} \}.
\]
Let \( E_0 = \{ e_i; i \in S \} \) and \( E \subseteq E_0 \) be a \( \sigma_0 \)-stable subset which is reduced and satisfies \( C_{\geq E} = C_{\geq E_0} \). Let \( I(E) = \{ i \in S; e_i \in E \} \). By [Ch, Theorem 6.5], there exists an element \( \nu \in C_{\geq E} \) defined by
\[ \mu^\circ + \lambda^\circ - \nu \in X_*(T_{sc})_T \otimes \mathbb{Z} \mathbb{R}, \quad I(\nu) = I(E) \text{ and } \langle \omega_j, \nu \rangle = \langle \omega_j, e_j \rangle \text{ for } j \in I(E), \] which satisfies \( C_{\geq \nu} = C_{\geq E} = C_{\geq E_0} \). Since \( \mu^\circ \in C_{\geq E} \), we have \( \nu \leq \mu^\circ \). By Lemma 2.1, \( \nu \in B(\tilde{W}, \mu, \sigma) \).

Let \( \nu' \in B(\tilde{W}, \mu, \sigma) \). Set \( E(\nu') = \{ \text{pr}_{(j)}(\nu'): j \in I(\nu') \} \). By Lemma 2.1 and the inequality \( \nu' \leq \mu^\circ \), we have, for any \( \sigma_0 \)-orbit \( c \) of \( I(\nu') \) and \( j \in c \), that
\[
\| c \cdot \langle \omega_j, \text{pr}_{(j)}(\nu') \rangle \| \leq \| c \cdot \langle \omega_j, \mu^\circ \rangle \| = \langle \omega_c, \mu^\circ \rangle + \mathbb{Z}
\]
and
\[
\| c \cdot \langle \omega_j, \text{pr}_{(j)}(\nu') \rangle \| \leq \| c \cdot \langle \omega_j, \mu^\circ \rangle \| = \langle \omega_c, \mu^\circ \rangle.
\]
So \( \langle \omega_j, \text{pr}_{(j)}(\nu') \rangle \leq \langle \omega_j, e_j \rangle \), that is, \( \text{pr}_{(j)}(\nu') \leq e_j \leq \nu \) for \( j \in I(\nu') \). By \( \text{[Ch, Lemma 6.2 (i)]} \), we deduce that \( \nu' \leq \nu \). Therefore \( \nu \) is the unique maximal element of \( B(\tilde{W}, \mu, \sigma) \).

3. Reduction to the irreducible case

Lemma 3.1. Let \( \tau \in \Omega \). Then Theorem 1.1 holds for \( (\tilde{W}, \mu, \sigma) \) if and only if it holds for \( (\tilde{W}, \mu, \tau \sigma \tau^{-1}) \).

Proof. For any \( v \in V, \tilde{\tau}(v) = \tilde{v} \). Thus \( \tau \text{Adm}(\mu) \tau^{-1} = \text{Adm}(\mu) \) and \( \tilde{\nu}_{w, \tau \sigma \tau^{-1}} = \tilde{\tau}(\nu_{w^{-1} \tau \sigma^{-1} \sigma^{-1}}) = \nu_{w^{-1} \tau \sigma^{-1} \sigma^{-1}} \). Therefore \( B(\tilde{W}, \mu, \tau \sigma \tau^{-1}) = B(\tilde{W}, \mu, \sigma) \). Since conjugation by \( \tau \) preserves the Bruhat order, Theorem 1.1 (2) holds for \( (\tilde{W}, \mu, \sigma) \) if and only if it holds for \( (\tilde{W}, \mu, \tau \sigma \tau^{-1}) \).

3.1. In the rest of this section, we assume that \( G \) is adjoint and \( \sigma \) acts transitively on the set of connected components of the affine Dynkin diagram of \( W_a \). In other words, \( \tilde{W} = \tilde{W}_1 \times \cdots \tilde{W}_m \), where \( \tilde{W}_1 \cong \cdots \cong \tilde{W}_m \) are extended affine Weyl groups of adjoint type with connected affine Dynkin diagram and \( \sigma(\tilde{W}_1) = \tilde{W}_2, \cdots, \sigma(\tilde{W}_m) = \tilde{W}_1 \). After conjugating a suitable element in \( \Omega \), we may assume that \( \sigma = \text{Ad}(\tau) \circ \sigma_0 \) with \( \tau \in \tilde{W}_m \). We may write \( \mu \) as \( \mu = (\mu_1, \cdots, \mu_m) \), where \( \mu_i \) is a dominant coweight for \( \tilde{W}_i \). Set \( \gamma = \sum_{i=1}^{m} \sigma_{m-i}(\mu_i) \). Then the natural projection induces a bijection from \( B(\tilde{W}, \mu, \sigma) \) to \( B(\tilde{W}, \gamma, \sigma^m) \).

Lemma 3.2. We keep the notations in 3.1. If Theorem 1.1 (2) holds for \( (\tilde{W}_m, \gamma, \sigma^m) \), then it holds for \( (\tilde{W}, \mu, \sigma) \).

Proof. Let \( \nu \) be the maximal element in \( B(\tilde{W}_m, \gamma, \sigma^m) \). By assumption, there exists \( w \in \text{Adm}(\gamma) \) such that \( \tilde{\nu}_{w, \sigma^m} = \nu \). By definition, there exists \( x \) in the finite Weyl group associated to \( \tilde{W}_m \) such that \( w < t^x(\Omega) \). Since \( \ell(t^x(\Omega)) = \sum_{i=1}^{m} \ell(t^x(\sigma_{m-i}(\mu_i))) \), there exists \( w_i \in \tilde{W}_m \) for each \( i \) such that \( w = w_1 \cdots w_m \) and \( w_i < t^x(\sigma_{m-i}(\mu_i)) \) for all \( i \). Hence
\[ \sigma_0^{i-m}(w_i) \leq t^{x(\mu)}. \] Set \( y = (\sigma_0^{i-m}(w_i), \ldots, w_m) \in \bar{W}. \) Then \( y \in \text{Adm}(\mu) \) and \( \nu_{y,\sigma} = (\sigma_{\nu_{w,s}}^{i-m}, \ldots, \nu_{w,s}^{i-m}). \) Hence \( \bar{\nu}_{y,\sigma} = (\sigma_0\nu, \ldots, \nu) \) is the maximal element in \( B(\bar{W}, \mu, \sigma). \) \( \square \)

4. Reduction to the superbasic case

4.1. Let \( \epsilon \in \bar{W} \rtimes \langle \sigma \rangle. \) We say that \( \epsilon \) is superbasic if \( \ell(\epsilon) = 0 \) and each \( \epsilon \)-orbit on \( \bar{S} \) is a union of the connected components of the affine Dynkin diagram of \( \bar{W}. \) By [HN1, 3.5], \( \epsilon \) is superbasic if and only if \( W_0 = W_0^{m_1} \times \cdots \times W_0^{m_n}, \) where \( W_i \) is an affine Weyl group of type \( A_{n_i-1} \) and \( \epsilon \) gives an order \( n_i m_i \) permutation on the set of simple reflections of \( W_i^{m_i}. \)

4.2. The main purpose of this section is to reduce to the case where \( \sigma \) is superbasic. We keep the assumption in §2.2.

For any \( J \subset S, \) let \( W_J \) be the subgroup of \( W_0 \) generated by \( s_j \) for \( j \in J \) and \( J W_0 \) be the set of minimal coset representatives in \( W_J \backslash W_0. \) Let \( \bar{W}_J = X_s(T)_T \rtimes W_J. \)

We regard \( \sigma \) as an element in \( \bar{W} \rtimes \langle \sigma_0 \rangle. \) We will construct a superbasic element in \( \bar{W}_J \rtimes \langle \sigma_0 \rangle \) for a suitable subset \( J \subset S \) with \( \sigma_0(J) = J. \) We follow the approach in [HN2 §5].

Let \( V^\sigma \) be the fixed point set of \( \sigma. \) Since \( \sigma \) is an affine transformation on \( V \) of finite order, \( V^\sigma \) is a nonempty affine subspace. Set \( V' = \{ v-c; v \in V^\sigma \}, \) where \( e \) is an arbitrary point of \( V^\sigma. \) Then \( V' \) is the (linear) subspace of \( V \) parallel to \( V^\sigma. \) We choose a generic point \( v_0 \) of \( V' \), i.e., for any root \( \alpha, \langle \alpha, v \rangle = 0 \) implies that \( \langle \alpha, v' \rangle = 0 \) for all \( v' \in V'. \) Let \( \bar{v}_0 \) be the unique dominant element of the \( W_0 \)-orbit of \( v_0. \) We set \( I = I(\bar{v}_0) \) and \( J = J(\bar{v}_0). \) Let \( z \in J W_0 \) be the unique element with \( \bar{v}_0 = z(\bar{v}_0). \) Set \( \sigma^J = z\sigma z^{-1}. \)

Lemma 4.1. The element \( \sigma^J \) is a superbasic element in \( \bar{W}_J \rtimes \langle \sigma_0 \rangle. \)

Proof. Since \( \sigma(0) = \lambda, \sigma(v_0) = v_0 + \lambda \) and \( \sigma^J(\bar{v}_0) = \bar{v}_0 + z(\lambda). \) Write \( \sigma^J \) as \( \sigma^J = t^{\xi(\lambda)}u \sigma_0 \) for some \( u \in W_0. \) Then \( u \sigma_0(\bar{v}_0) = \bar{v}_0. \) Therefore \( \sigma^J(\bar{v}_0) = u^{-1}\bar{v}_0 \) is the unique dominant element in the \( W_0 \)-orbit of \( v_0. \) Hence \( \bar{v}_0 = \sigma(\bar{v}_0) = u^{-1}\bar{v}_0. \) Therefore \( u \in W_J \) and \( \sigma_0(J) = J. \)

Let \( \ell_J \) be the length function on \( \bar{W}_J \rtimes \langle \sigma_0 \rangle. \) By [HN2, Proposition 3.2], \( \ell_J(\sigma^J) = 0. \) Since \( v_0 \) is generic, by [HN2 §5.5], \( V^{\sigma_0} \subset V^{W_J}. \) Therefore there is no nonempty subset of \( J \) that is stable under \( \sigma^J. \) Hence each orbit of \( \sigma^J \) on the set of simple reflections of \( \bar{W}_J \) is a union of connected components of the affine Dynkin diagram of \( \bar{W}_J. \) Hence \( \sigma^J \) is superbasic. \( \square \)
Lemma 4.2. We keep the notations in §4.2. Then
\( (1) \) \( z(\lambda)^0 \in \sum_{j \in J} Q\alpha_j^0. \)
\( (2) \) Let \( c \) be a \( \sigma_0 \)-orbit of \( S \). Then \( \langle \omega_c, \lambda^0 \rangle \in \mathbb{Z} \) if and only if \( c \in I. \)

Proof. Assume \( z(\lambda) \in r + h \) with \( r \in \sum_{j \in J} Q\alpha_j^0 \) and \( h \in \sum_{i \in I} Q\alpha_i^0. \) Since \( \sigma \) is of length zero, we have \( \nu_{t(z(\lambda))u, \sigma_0} = 0 \), which implies \( h^\circ = 0. \) Hence \( z(\lambda)^0 \in \sum_{j \in J} Q\alpha_j^0 \) and (1) is proved.

Write \( \lambda = z(\lambda) + \theta \) for some \( \theta \in X_*(T_{sc})_\Gamma. \) We have
\[
\langle \omega_c, \lambda^0 \rangle = \langle \omega_c, z(\lambda)^0 \rangle + \langle \omega_c, \theta \rangle = \langle \omega_c, r^0 \rangle + \langle \omega_c, \theta \rangle = \langle \omega_c, r \rangle \quad \text{mod } \mathbb{Z}.
\]
Hence \( \langle \omega_c, \lambda^0 \rangle \in \mathbb{Z} \) if \( c \subset I. \) On the other hand, since \( \sigma^J \) is a superbasic element of \( \tilde{W}_J, \tilde{W}_J \) has only type \( A \) factors. One may check directly that \( \langle \omega_c, r \rangle = \langle \omega_c^J, r \rangle \notin \mathbb{Z} \) for any \( \sigma_0 \)-orbit \( c \) of \( I \) and (2) is proved. \( \square \)

Proposition 4.3. The maximal Newton point of \( B(\tilde{W}, \mu, \sigma) \) is contained in the natural inclusion \( B(\tilde{W}_J, \mu, \sigma^J) \hookrightarrow B(\tilde{W}, \mu, \sigma). \)

Proof. We denote by \( \omega_c^J \in \sum_{j \in J} Q\alpha_j^J \) the corresponding fundamental coweight of \( \Phi_J \) and set \( \omega_c^J = \sum_{j \in c} \omega_c^J \) for any \( \sigma_0 \)-orbit \( c \) of \( J. \) Let \( \nu \) be the maximal Newton point of \( B(\tilde{W}, \mu, \sigma). \) By the proof of Theorem 4.2.1 (1), for each \( \sigma_0 \)-orbit \( c \) of \( S, \)
\[
\langle \omega_c, \mu^0 \rangle \geq \langle \omega_c, \nu \rangle \geq \langle \omega_c, \mu^0 + \lambda^0 \rangle - \lfloor \langle \omega_c, \lambda^0 \rangle \rfloor.
\]

Let \( c \) be a \( \sigma_0 \)-orbit of \( I. \) By Lemma 4.2.2 (2), \( \langle \omega_c, \lambda^0 \rangle \in \mathbb{Z}. \) Hence \( \langle \omega_c, \nu \rangle \) and \( \mu^0 - \nu \in \sum_{j \in J} Q\alpha_j^0. \)

By Lemma 4.2.1 (1), \( z(\lambda)^0 \in \sum_{j \in J} Q\alpha_j^0. \) Thus \( \mu^0 + z(\lambda)^0 - \nu \in \sum_{j \in J} Q\alpha_j^0. \) Now \( \nu^J \) is a \( \sigma_0 \)-orbit in \( I(\nu) \cap J. \) Then
\[
\langle \omega_c^J, \mu^0 + z(\lambda)^0 - \nu \rangle = \langle \omega_c, \mu^0 + z(\lambda)^0 - \nu \rangle = \langle \omega_c, \mu^0 + \lambda^0 - \nu \rangle - \langle \omega_c, \theta \rangle,
\]
where \( \theta = \lambda - z(\lambda) \in X_*(T_{sc})_\Gamma. \) By Lemma 2.1 \( \langle \omega_c^J, \mu^0 + \lambda^0 - \nu \rangle \in \mathbb{Z}. \) Hence \( \langle \omega_c^J, \mu^0 + z(\lambda)^0 - \nu \rangle \in \mathbb{Z}. \) Again by Lemma 2.1, \( \nu \in B(\tilde{W}_J, \mu, \sigma^J). \) \( \square \)

Lemma 4.4. Let \( J \subset S \) and \( x \in \mathcal{J}W_0. \) If \( w, w' \in \tilde{W}_J \) with \( w \leq_J w' \) for the Bruhat order of \( \tilde{W}_J, \) then \( z^{-1}wz \leq z^{-1}w'z \) for the Bruhat order of \( \tilde{W}. \)

Proof. It suffices to consider the case where \( w' = ws_\alpha \) for some positive affine root \( \alpha \) of \( \tilde{W}_J. \) Since \( w' \geq w, \lambda(w) \) is again a positive affine root of \( \tilde{W}_J. \) Since \( x^{-1} \) sends positive affine roots of \( \tilde{W}_J \) to positive affine roots of \( \tilde{W}, \) \( z^{-1}wz = z^{-1}wzs_{\lambda^{-1}(\alpha)} \) and \( z^{-1}w(\alpha) \) is a positive affine root of \( \tilde{W}. \) Hence \( z^{-1}w'z \geq z^{-1}wz. \) \( \square \)
Corollary 4.5. If Theorem 1.1 (2) holds for \( B(\tilde{W}, \mu, \sigma) \), then it holds for \( B(\tilde{W}, \mu, \sigma) \).

Proof. Let \( \nu \) be the maximal Newton point of \( B(\tilde{W}, \mu, \sigma) \), which is also the maximal Newton point of \( B(\tilde{W}_J, \mu, \sigma) \) by Proposition 4.3. By assumption, there exist \( w_1 \in t^\mu(W_a \cap \tilde{W}_J) \) and \( x_1 \in \tilde{W}_J \) such that \( \tilde{\nu}_{w_1, \sigma} = \nu \) and \( w_1 \preceq_J t^{x_1(\mu)} \). Here \( \preceq_J \) is the Bruhat order on \( \tilde{W}_J \) defined with respect to \( J \). Let \( w = z^{-1}w_1z \) and \( x = z^{-1}x_1 \). Then we have \( \tilde{\nu}_{w, \sigma} = \nu, w \in t^\mu W_a \) and \( w \preceq t^x(\mu) \) as desired. \( \square \)

5. The superbasic case

5.1. In this section, we consider the extended affine Weyl group \( \tilde{W} \) of \( G = GL_n \). Then \( \tilde{W} \cong \mathbb{Z}^n \rtimes \mathfrak{S}_n \), where \( \mathfrak{S}_n \) is the permutation group of \( \{1, 2, \ldots, n\} \) which acts on \( \mathbb{Z}^n \cong \bigoplus_{i=1}^n \mathbb{Z}e_i^\vee \) by \( w(e_i^\vee) = e_{w(i)}^\vee \) for \( w \in \mathfrak{S}_n \). Let \( \{e_i\}_{i=1}^n \) be the dual basis. Set \( d = \sum_{i=1}^n e_i \) and \( d^\vee = \sum_{i=1}^n e_i^\vee \). The simple roots and fundamental weights are given by \( \alpha_i = e_i - e_{i+1} \) and \( \omega_i = -\frac{1}{n} d + \sum_{j=1}^i e_j \) respectively for \( i \in [1, n-1] \).

Set \( \sigma_{m,n} = \sum_{j=1}^n e_j^\vee \).

For any positive integer \( m < n \), let \( \sigma_{m,n} = t^{\sigma_{m,n}} u_{m,n} \in t^{\sigma_{m,n}} W_0 \) be the unique length zero element with \( u_m \in W_0 \). Then any superbasic element in \( \tilde{W} \) is of the form \( t^{\sum_{j=1}^n e_j^\vee} \sigma_{m,n} \) for some \( m \) coprime to \( n \).

The main purpose of this section is to prove the following result.

Proposition 5.1. Let \( m < n \) be a positive integer coprime to \( n \). Let \( \mu \in \mathbb{Z}^n \) be a dominant coweight of \( G = GL_n \). Then there exists \( \bar{w} \in \tilde{W} \) and \( x \in W_0 \) such that \( \bar{w} < t^{x(\mu)} \) and \( \bar{\nu}_{\bar{w}, \sigma_{m,n}} - \frac{m}{n} d^\vee \) equals the unique maximal Newton point \( \nu \) of \( B(\tilde{W}, \mu, t^{-\frac{m}{n} d^\vee} \sigma_{m,n}) \).

The proof will be given in §5.6.

5.2. We first show that Proposition 5.1 implies Theorem 1.1 (ii).

By §2.1, we may assume that \( \tilde{W} \) is the Iwahori-Weyl group of an adjoint \( p \)-adic group \( G \). Then by Lemma 3.2 and Lemma 4.3, it suffices to prove the case where \( G = PGL_n \) and \( \sigma \) is superbasic, which follows from Proposition 5.1 and §2.1.

5.3. Now we give an algorithm to construct the maximal element in \( B(\tilde{W}, \mu, \sigma_{m,n}) \).

We recall the definition of \( \mathfrak{a} \)-sequence and \( \chi_{m,n} \) in [Hel §3 & §5].

Let \( r \in \mathbb{N} \) and \( \chi \in \mathbb{Z}^r \). For each \( j \in [1, r] \) we define \( \mathfrak{a}_{\chi}^j : \mathbb{Z}_{\geq 0} \to \mathbb{Z} \) by \( \mathfrak{a}_{\chi}^j(k) = \chi(j - k) \). Here we identify \( l \) with \( l + r \) for \( l \in \mathbb{Z} \). We say \( i \gg_{\chi} j \) if \( \mathfrak{a}_{\chi}^l \geq \mathfrak{a}_{\chi}^j \) in the sense of lexicographic order. If \( \gg_{\chi} \) is a linear order, we define \( \epsilon_{\chi} \in \mathfrak{S}_r \) such that \( \epsilon_{\chi}(i) < \epsilon_{\chi}(j) \) if and only if \( i >_{\chi} j \).
Define $\chi_{m,n} \in \mathbb{Z}^n$ by $\chi_{m,n}(i) = [im] - [(i - 1)m]$ for $i \in [1, n]$. Set $\epsilon_{m,n} = \epsilon_{\chi_{m,n}} \in \mathcal{G}_n$. Since $m$ and $n$ are co-prime, it is well defined. Note that $\epsilon_{m,n}(\chi_{m,n}) = \varpi_{m,n}$.

5.4. Let $S = \cup_{1 \leq i \leq j \leq n} \mathbb{Z}^{[i,j]}$, whose elements are called segments. Let $\eta \subset S$ be a segment. Assume $\eta \in \mathbb{Z}^{[i,j]}$. We call $h(\eta) = i$ and $t(\eta) = j$ the head and the tail of $\eta$ respectively. We call the nonnegative integer $j - i + 1$ the size of $\eta$. We set

$$|\eta| = \sum_{k = h(\eta)}^{t(\eta)} \eta(k), \quad \text{av}(\eta) = \frac{1}{t(\eta) - h(\eta) + 1}|\eta|.$$ 

Let $[i', j'] \subset [i, j]$ be a sub-interval, we call the restriction $\eta|_{[i', j']}$, defined by $\eta$, to $[i', j']$ a subsegment of $\eta$ and write $\eta|_i = \eta|_{[i,i]}$. Let $\theta$ be another segment such that $h(\theta) = t(\eta) + 1$. We denote by $\eta \vee \theta \in \mathbb{Z}^{[h(\eta), t(\theta)]}$ the natural union of $\eta$ and $\theta$. For $k \in \mathbb{Z}$, we denote by $\eta[k]$ the $k$-shift of $\eta$ defined by $\eta[k](i) = \eta(i + k)$. We say two segments are of the same type if they can be identified with each other up to some shift.

For $\eta \in \mathbb{Q}^n \cong \mathbb{Q}^{1,n}$ we denote by $\text{Con}(\eta) \in \mathbb{Q}^2$ the convex hull of the points $(0, 0)$ and $(k, |\eta|_{[1,k]})$ for $k \in [1, n]$. We say a subsegment $\gamma$ of $\eta$ is sharp if $\text{av}(\gamma)$ is maximal/minimal among all subsegments of $\eta$ with the same head/tail. If $\eta = \gamma^1 \vee \gamma^2 \vee \cdots \vee \gamma^s$ with each $\gamma^k$ a sharp subsegments, then the points $(0, 0)$ and $(t(\gamma^i), |\gamma^1 \vee \cdots \vee \gamma^s|)$ in $\mathbb{R}^2$ for $i \in [1, s]$ lie on the boundary of $\text{Con}(\eta)$ and their convex hull is just $\text{Con}(\eta)$. We call the dominant vector

$$\text{sl}(\text{Con}(\eta)) = (\text{av}(\gamma^1) \vee \cdots \vee \text{av}(\gamma^s)) \in \mathbb{Q}^s$$

the slope sequence of $\text{Con}(\eta)$. Here for any $\gamma \in S$, we define $\text{av}(\gamma) \in \mathbb{Z}^{[h(\gamma), t(\gamma)]}$ by $\text{av}(\gamma)(i) = \text{av}(\gamma)$ for $i \in [h(\gamma), t(\gamma)]$.

Let $\mu \in \mathbb{Z}^n$ be a dominant coweight. Define $\mu_{m,n} = \mu + \chi_{m,n}$. Then

$$\langle \omega_i, \mu_{m,n} \rangle = \langle \omega_i, \mu \rangle - \left(\frac{mi}{n} - \left\lfloor \frac{mi}{n} \right\rfloor \right) = \langle \omega_i, \mu + \varpi_{m,n} \rangle - \left[\langle \omega_i, \varpi_{m,n} \rangle \right].$$

According to the proof of Theorem 1.1 (1), the slope sequence $\nu = \text{sl}(\text{Con}(\mu_{m,n}))$ is the unique maximal Newton point of $B(G, \mu, \sigma_{m,n})$.

**Example 5.2.** Now we provide an example.

For a sequence of (distinct) elements $i_1, i_2, \ldots, i_r$ in $[1, n]$, we denote by $\text{cyc}(i_1, i_2, \ldots, i_r) \in \mathcal{G}_n$ the cyclic permutation $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_r \mapsto i_1$, which acts trivially on the remaining elements of $[1, n]$. 
Let $n = 8$, $m = 5$ and $\mu = (1, 1, 1, 0, 0, 0, 0, 0)$. Then $\chi_{m,n} = (0, 1, 0, 1, 1, 0, 1, 1) \in \mathbb{Z}^8$, $\epsilon_{m,n} = \text{cyc}(1, 6, 7, 4, 5, 2, 3, 8)$ and $u_{m,n} = \text{cyc}(6, 3, 8, 5, 2, 7, 4, 1)$.

We have the following sharp decomposition:

$$
\mu_{m,n} = (1, 2, 1, 1, 0, 1, 1, 1) = (1, 2) \vee (1) \vee (1, 1) \vee (0, 1, 1).
$$

Hence $\nu = (\frac{3}{2}, \frac{3}{2}, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. Moreover, one checks that

$$
t_{\epsilon_{m,n}(\mu)} \sigma_{m,n} > t_{\epsilon_{m,n}(\mu)} \sigma_{m,n} \text{cyc}(8, 3)
> t_{\epsilon_{m,n}(\mu)} \sigma_{m,n} \text{cyc}(8, 3) \text{cyc}(1, 3)
> t_{\epsilon_{m,n}(\mu)} \sigma_{m,n} \text{cyc}(8, 3) \text{cyc}(1, 3) \text{cyc}(1, 2).
$$

Set $\bar{w} = t_{\epsilon_{m,n}(\mu)} \sigma_{m,n} \text{cyc}(8, 3) \text{cyc}(1, 3) \text{cyc}(1, 2) \sigma_{m,n}^{-1}$. Then $\bar{w} < t_{\epsilon_{m,n}(\mu)}$ and $\nu_{\bar{w}, \sigma_{m,n}} = \nu$. This verifies Proposition 5.1 in this case.

5.5. Similar to [He1, §5], we use the Euclidean algorithm to give a recursive construction of $\chi_{m,n}$, which plays a crucial role in the proof of Proposition 5.1.

Let $D = \{(i, j) \in \mathbb{Z}^2; i < j \text{ are co-prime}\}$. We define $f : D \rightarrow D \sqcup \{(1, 1)\}$ by

$$
f(m, n) = \begin{cases} 
(m(\lfloor \frac{m}{n} \rfloor + 1) - n, m), & \text{if } \frac{n}{m} \geq 2; \\
(n - (n - m) \lfloor \frac{n}{n-m} \rfloor, n - m), & \text{otherwise}.
\end{cases}
$$

Define two types of segments $1_{m,n}$ and $0_{m,n}$ in $S$ by

$$(1_{m,n}, 0_{m,n}) = \begin{cases} 
((0(\lfloor \frac{n}{n-m} \rfloor - 1), 1), (0(\lfloor \frac{n}{n-m} \rfloor), 1)), & \text{if } \frac{n}{m} \geq 2; \\
((0, 1(\lfloor \frac{n}{n-m} \rfloor)), (0, 1(\lfloor \frac{n}{n-m} \rfloor - 1))), & \text{otherwise}.
\end{cases}
$$

For $\eta \in S$ and $k \in [h(\eta), t(\eta)]$, set

$$
\eta(k)_{m,n} = \begin{cases} 
1_{m,n}, & \text{if } \eta(k) = 1; \\
0_{m,n}, & \text{otherwise}.
\end{cases}
$$

For $k \in [h(\eta), t(\eta)]$, let $\eta_{m,n,k}$ be the shift of $\eta(k)_{m,n}$ whose head is determined recursively as follows:

$$
h(\eta_{m,n,k}) = \begin{cases} 
h(\eta), & \text{if } k = h(\eta); \\
t(\eta_{m,n,k-1}) + 1, & \text{if } k > h(\eta).
\end{cases}
$$

Now we define $\phi_{m,n} : S \rightarrow S$ by $\phi_{m,n}(\eta) = \eta_{m,n,h(\eta)} \vee \cdots \vee \eta_{m,n,t(\eta)}$ for $\eta \in S$.

If $f^{h-1}(m, n)$ is defined, we set $\phi_{m,n,h} = \phi_{f^{h-1}(m, n)} \circ \cdots \circ \phi_{m,n}$. Using the Euclidean algorithm, one checks that

$$
\phi_{m,n,h}(\chi_{f^{h}(m, n)}) = \chi_{m,n}.
$$
We say a subsegment $\gamma$ of $\chi_{m,n}$ is of level $h$ if it is the image of some subsegment $\gamma^h$ of $\chi_{f^h(m,n)}$ under the map $\phi_{m,n,h}$. When $h = 1$ and $\gamma^h$ is of size one, we say $\gamma$ is an elementary subsegment of $\chi_{m,n}$.

Let $\beta^1$ and $\gamma^1$ be two segments of $\chi^1 = \chi_{f(m,n)}$ and let $\gamma$ be a level one subsegment of $\chi = \chi_{m,n}$. Using the Euclidean algorithm, we have the following basic facts:

(a) $\text{av}(\beta^1) \geq \text{av}(\gamma^1)$ if and only if $\text{av}(\phi_{m,n}(\beta^1)) \geq \text{av}(\phi_{m,n}(\gamma^1))$.

(b) Each sharp subsegment of $\gamma$ with the same head is of level one.

(c) If moreover $\gamma$ is an elementary subsegment of $\chi$, then $\mathbf{a}_\chi^j < \mathbf{a}_\chi^{h(\gamma)-1}$ and $\mathbf{a}_\chi^j < \mathbf{a}_\chi^{h(\gamma)}$ for $j \in [h(\gamma), t(\gamma) - 1]$.

(d) $\mathbf{a}_\chi^j < \mathbf{a}_\chi^k$ if and only if $\mathbf{a}_{\chi,(\phi_{m,n}(\chi^1[i,j]))} < \mathbf{a}_{\chi,(\phi_{m,n}(\chi^1[i,j]))}$.

(e) $\epsilon_{m,n}(n) = 1$.

5.6. Proof of Proposition [5.1]. For $\eta \in \mathcal{S}$ we set $x_\eta = \text{cyc}(h(\eta), h(\eta) + 1, \ldots, t(\eta)) \in \mathcal{S} = \mathcal{S}_i \cup \mathcal{S}_j \cup \mathcal{S}_l$. Similarly, for a sequence $c = (c^1, \ldots, c^\mu)$ of segments, we set $x_c = x_{c^1, \ldots, c^\mu} = x_{c^1} \cdots x_{c^\mu}$. If $\eta = c^1 \lor \cdots \lor c^\mu$, we say $c$ is decomposition of $\eta$. Now we are ready to prove Proposition [5.1].

Write $\chi = \chi_{m,n}, \theta = \mu_{m,n}$ and $\epsilon = \epsilon_{m,n}$. For $h \in Z_{\geq 0}$ we set $\phi_h = \phi_{m,n,h}$ and $\chi^h = \chi_{f^h(m,n)}$. By [5.4] we have $\nu = \text{sl}(\text{Con}(\theta))$. The proof will proceed as follows. First we construct a suitable sharp decomposition $c$ of $\theta$. One checks directly $\nu_{w,c,\text{id}} = \text{sl}(\text{Con}(\theta)) = \nu$, where $w_c = t^\theta x_c \in \mathcal{W}$. Then we show that

$$\epsilon w_c \epsilon^{-1} < t^{\epsilon(\mu)} \sigma_{m,n} = \epsilon t^\theta x_\theta \epsilon^{-1}.$$ 

Set $\tilde{w} = \epsilon w_c \epsilon^{-1} \sigma_{m,n}^{-1}$. Then $\tilde{w} < t^{\epsilon(\mu)}$ and $\nu_{\tilde{w},\sigma_{m,n}} = \nu_{w_c,\epsilon^{-1},\text{id}} = \epsilon(\nu)$. This completes our proof.

Assume $I(\mu) = \{j \in [1, n - 1]; \langle \alpha_j, \mu \rangle \neq 0\} = \{b_1, b_2, \ldots, b_{r-1}\}$ with $b_1 < b_2 < \cdots < b_{r-1}$. We set $b_0 = 0$ and $b_r = n$. Set $\theta^i = \theta^{[b_{i-1}+1, b_i]}$ for $i \in [1, r]$. Then $\theta = \theta^1 \lor \cdots \lor \theta^r$. Suppose we have a sharp decomposition $c_i$ of $\theta^i$ for each $i \in [1, r]$. Since $\chi \in \{0, 1\}^{[1, n]}$ and $\theta = \mu + \epsilon$, for any subsegment $\eta^i$ (resp. $\eta^j$) of $\theta^i$ (resp. $\theta^j$) we have $\text{av}(\eta^i) \geq \text{av}(\eta^j)$ if $i < j$. Therefore the natural union $c = c_1 \lor \cdots \lor c_r$ forms a sharp decomposition of $\theta$.

Let $1 \leq i \leq r$. We will construct inductively the subsegments $\zeta^i_1, \gamma^i_1, \xi^i_1$ for $j \in [1, l_i]$ (some of them may be empty) such that

(a) $\gamma^i_0 = \theta^i$ and $\gamma^i_{j-1} = \zeta^i_1 \lor \gamma^i_1 \lor \xi^i_1$ for $j \in [1, l_i]$;

(b) $\zeta^i_1$ and $\xi^i_1$ are sharp subsegments of $\gamma^i_{j-1}$; any sharp subsegment of $\gamma^i_1$ is also a sharp subsegment of $\gamma^i_{j-1}$; $\gamma^i_1$ is a sharp subsegments of itself (self-sharp).
(c) For any $j$, $\epsilon z_{i,j-1} \epsilon^{-1} > \epsilon z_{i,j} \epsilon^{-1}$.

Here
\[ z_{i,j} = t^\theta y_{i-1} x_{i}^j v_{i,j}; \]
\[ y_{i} = x_{c_1} \cdots x_{c_{i-1}}; \]
\[ x_{i}^j = x_{c_1 + \cdots + c_j}, \ldots, x_{i}^j \cdots x_{i}; \]
\[ v_{i,j} = x_{c_1 + \cdots + \gamma_i} \cyc(t(\gamma_i^j), n)) = \cyc(h(\gamma_i^j), \ldots, t(\gamma_i^j), b_i + 1, \ldots, n). \]

Once we have (a), (b) and (c) for all $i$ and $j$, then
\[ c_i = (\xi_1^i, \xi_2^i, \ldots, \xi_t_i, \eta_i^i, \zeta_i^i) \]
forms a sharp decomposition of $\theta^i$, and
\[ \epsilon t^\theta x_\theta \epsilon^{-1} = \epsilon z_{1,0} \epsilon^{-1} > \cdots > \epsilon z_{1,t_i+1} \epsilon^{-1} = \epsilon z_{2,0} \epsilon^{-1} > \cdots > \epsilon z_{r,t_i+1} \epsilon^{-1} = \epsilon w_e \epsilon^{-1} \]
as desired.

The construction is as follows. Suppose for $1 \leq k < i$ and $0 \leq l \leq j$, $c_k, z_l, \xi_l, \eta_l$ are already constructed, and moreover $\epsilon z_{i,j-1} \epsilon^{-1} > \epsilon z_{i,j} \epsilon^{-1}$.
We construct $\xi_i^{j+1}, \eta_i^{j+1}, \zeta_i^{j+1}$ and show that $\epsilon z_{i,j} \epsilon^{-1} > \epsilon z_{i,j+1} \epsilon^{-1}$.

If $\gamma_i^j$ is empty, there is nothing to do. Otherwise, we assume $\gamma_i^j$ is of level $h$ but not of level $h+1$. Then $\gamma_i^j = \phi_h(\iota)$ for some subsegment $\iota$ of $\chi^h$.

Case (I): $\iota$ is not a subsegment of any elementary subsegment of $\chi^h$.
Then there exist unique subsegments $\zeta, \gamma$ and $\xi$ of $\chi^h$ such that $\gamma$ is of level one, $\zeta$ (resp. $\xi$) is a proper subsegment of some elementary segment of $\chi^h$ with the same tail (resp. head), and $\iota = \zeta \cup \gamma \cup \xi$. Without loss of generality, we may assume that none of $\gamma$, $\xi$ and $\zeta$ is empty.

Define $\xi_i^{j+1} = \phi_h(\zeta)$, $\gamma_i^{j+1} = \phi_h(\gamma)$ and $\zeta_i^{j+1} = \phi_h(\xi)$. Note that $\av(\chi^h|_b(\zeta), t(\zeta))$ (resp. $\av(\chi^h|_b(\xi), t(\xi))$) is maximal (resp. minimal) among all subsegments of $\chi^h$ with the same head (resp. tail). Therefore, (b) follows from §5.5 (a) & (b). For (c), it suffices to show that
\[ \epsilon z_{i,j+1} \epsilon^{-1} < \epsilon z_{i,j} \cyc(n, \cyc(h(\xi_i^{j+1}), n)) \epsilon^{-1}; \]
\[ \epsilon z_{i,j} \cyc(n, \cyc(h(\xi_i^{j+1}), n)) \epsilon^{-1} < \epsilon z_{i,j} \epsilon^{-1}. \]

Note that $\epsilon z_{i,j+1} \epsilon^{-1} = \epsilon z_{i,j} \cyc(n, \cyc(h(\xi_i^{j+1}), n)) \cyc(h(\xi_i^{j+1})-1, t(\xi_i^{j+1})) \epsilon^{-1}$. By §5.5 (c), we have $a_h^{(\xi_i^{j+1})^{-1}} > a_h^{t(\xi_i^{j+1})}$. Hence by §5.5 (d), $\epsilon h(\xi_i^{j+1})-1 < \epsilon(t(\xi_i^{j+1})))$. If $\theta(h(\xi_i^{j+1})) > \theta(b_i + 1)$, then (d) holds. If $\theta(h(\xi_i^{j+1})) =$
\( \theta(b_i + 1) \), then \( \chi(b_i + 1) = 1 > 0 = \chi(h(\xi_i^{j+1})) \). Hence \( \epsilon(b_i + 1) < \epsilon(h(\xi_i^{j+1})) \). Then (d) still holds.

Since \( \gamma \neq \emptyset \), \( t(\xi_i^{j+1}) \neq n \). By §5.5(e), \( 1 = \epsilon(n) < \epsilon(t(\xi_i^{j+1})) \).

If \( \theta(t(\xi_i^{j+1}) + 1) < \theta(h(\xi_i^{j+1})) \), then (e) holds. If \( \theta(t(\xi_i^{j+1}) + 1) = \theta(h(\xi_i^{j+1})) \), then \( \chi(h(\xi_i^{j+1})) = \chi(t(\xi_i^{j+1}) + 1) = 0 \). Since \( \zeta \) is a proper subsegment of some elementary segment of \( \chi^h \) and shares the same tail with it, by §5.5(c) we have that \( A_{\chi^h}^{\xi_i^{j+1}} > A_{\chi^h}^{h(\xi_i^{j+1})} \), and by §5.5(d) we have that \( A_{\chi^h}^{\xi_i^{j+1}} > A_{\chi^h}^{h(\xi_i^{j+1})} \). So \( \epsilon(h(\xi_i^{j+1})) > \epsilon(t(\xi_i^{j+1}) + 1) \). Then (e) still holds.

Case (II): \( \iota \) is a subsegment of some elementary subsegment of \( \chi^h \).

We define \( l_i = j \) and the construction of \( c_i \) is finished. One checks directly that \( \iota \) is self-sharp, hence so is \( \gamma_i^l = \phi_h(\iota) \) by §5.5(a) & (b). If \( t(\gamma_i) = n \), the induction step is finished. Otherwise, it remains to show

\[
(\text{f}) \quad \epsilon z_i \epsilon^{-1} > \epsilon z_i \epsilon^{-1} \cyc(t(\gamma_i^l), n) \epsilon^{-1} = \epsilon z_i \epsilon^{-1}.
\]

Note that \( 1 = \epsilon(n) < \epsilon(t(\gamma_i)) \). If \( \theta(h(\gamma_i^l)) > \theta(b_i + 1) \), (f) holds. Otherwise, we have \( \theta(h(\gamma_i^l)) = \theta(b_i + 1) \), \( \chi(h(\gamma_i^l)) = 0 \) and \( \chi(b_i + 1) = 1 \) since \( b_i \in I(\mu) \). Hence \( \epsilon(h(\gamma_i^l)) > \epsilon(b_i + 1) \) and (f) still holds.

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