From Quantum Affine Kac-Moody Algebras to Drinfeldians and Yangians

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Abstract. A general scheme of construction of Drinfeldians and Yangians from quantum non-twisted affine Kac-Moody algebras is presented. Explicit description of Drinfeldians and Yangians for all Lie algebras of the classical series $A$, $B$, $C$, $D$ is given in terms of a Chevalley basis.

1. Introduction

Rational and trigonometric deformations of a universal enveloping algebra $U(g[u])$ ($g$ is a finite-dimensional complex simple Lie algebra) - Yangian $Y_\eta(g)$ and a quantum ($q$-deformed) affine algebra $U_q(g[u])$ - are playing increasingly a role in the theory of integrable systems and the quantum field theory. For applications it is useful to know various realizations (or bases) of these deformations. Among various realizations the Chevalley basis plays a special role due to its compactness and simplicity. In the case of the quantum affine algebras the Chevalley basis was originally introduced in the defining relations while the Yangians were initially defined in terms of a finite Cartesian basis (the first Drinfeld realization $^2$) as well as in terms of a special infinite basis (the second Drinfeld realization $^3$). It was not clear whether does exist a Chevalley basis for the Yangians.

Recently in $^12$, $^13$ it was shown that the Yangian can be realized in terms of Chevalley basis and, moreover, there was found a simple connection of this realization with the corresponding quantum affine algebra (also see $^14$, $^15$). This connection allows to introduce a new Hopf algebra called Drinfeldian which depends on two deformation parameters $q$ and $\eta$. The Drinfeldian $D_{q\eta}(g)$ can be considered as quantization of $U(q[u])$ in the direction of a classical $r$-matrix which is a sum of simplest rational and trigonometric $r$-matrices. When the parameter $q$ goes to $1$ we obtain the Yangian $Y_\eta(g)$ in terms of the Chevalley basis.

In this paper we remind once more the procedure of construction of the Drinfeldian $D_{q\eta}(g)$ and the Yangian $Y_\eta(g)$ in terms of the Chevalley basis starting from

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the defining relations of the quantum affine algebra \( U_q(\mathfrak{g}[u]) \). General defining relations for \( D_{q\eta}(\mathfrak{g}) \) depend on some vector \( \hat{e}_{-\theta} \in U_q(\mathfrak{g}) \), where \( \theta \) is a maximal root of the Lie algebra \( \mathfrak{g} \). We can choose such vector \( \hat{e}_{-\theta} \) in order to obtain more simple explicit formulas for the defining relations of \( D_{q\eta}(\mathfrak{g}) \) and \( Y_q(\mathfrak{g}) \). Here we give these explicit defining relations of Drinfeldians and Yangians for all Lie algebras of the classical series \( A, B, C, D \).

The plan of this paper is as follows. In Section 2 a fundamental result of the paper by A. Belavin and V. Drinfeld \( \cite{1} \) concerning classification of the classical Yang-Baxter equation (CYBE) is briefly presented. In Section 3 we demonstrate that a sum of the simplest rational and trigonometric \( r \)-matrices is also a solution of CYBE. In order to compare our realization of the Yangian with Drinfeld’s realizations we present two Drinfeld’s realizations in explicit form in Section 4. Section 5 deals with the quantum algebra \( U_q(\mathfrak{g}[u]) \) for an arbitrary simple finite-dimensional complex Lie algebra \( \mathfrak{g} \). In Section 6 we show how to construct a two-parameter deformation or the Drinfeldian \( D_{q\eta}(\mathfrak{g}) \) starting from \( U_q(\mathfrak{g}[u]) \). At \( q = 1 \) the Drinfeldian \( D_{q\eta}(\mathfrak{g}) \) coincides with the Yangian \( Y_q(\mathfrak{g}) \). In Sections 7–10 we give the explicit description of Drinfeldians and Yangians in the terms of the Chevalley basis for all Lie algebras of the classical series \( A, B, C \) and \( D \).

2. Classical Yang-Baxter equation (CYBE).

Belavin-Drinfeld classification of CYBE solutions

The classical Yang-Baxter equation (CYBE) is an equation of the form

\[
[r_{12}(u_1, u_2), r_{13}(u_1, u_3) + r_{23}(u_2, u_3)] + [r_{13}(u_1, u_3), r_{23}(u_2, u_3)] = 0 ,
\]

where \( r(u, v) \) is a function of two variables \( u, v \) with values in \( \mathfrak{g} \otimes \mathfrak{g} \) (\( \mathfrak{g} \) is a finite-dimensional complex simple Lie algebra). Let \( \{e_i\} \) be a basis of \( \mathfrak{g} \) then \( r(u, v) \) has the form

\[
r(u, v) = \sum_{i,j} f_{ij}(u, v) e_i \otimes e_j ,
\]

where \( f_{ij}(u, v) \) are usual functions of the variables \( u \) and \( v \). The notations \( r_{12}(u_1, u_2) \), \( r_{13}(u_1, u_3) \) and \( r_{23}(u_2, u_3) \) are defined as follows:

\[
r_{12}(u_1, u_2) = \sum_{i,j} f_{ij}(u_1, u_2) e_i \otimes e_j \otimes 1
\]

etc. One usually assumes that the function \( r(u, v) \) satisfies the following additional properties:

(i) the unitary condition: \( r_{12}(u, v) = -r_{21}(v, u) \),

(ii) the function \( r(u, v) \) depends only on the difference \( z := u - v \).

For the expression (2.2) these properties are equivalent to

\[
r(z) = \sum_{i,j} f_{ij}(z) e_i \otimes e_j ,
\]

where the functions \( f_{ij}(z) \) satisfy the relation

\[
f_{ij}(z) = -f_{ji}(-z) .
\]

In 1982 A. Belavin and V. Drinfeld \( \cite{1} \) classified such \( z \)-dependent solutions of CYBE \( \cite{1} \) provided that \( r(z) \) is a “non-degenerate” meromorphic function. They showed that there exist three type of the solutions: rational, trigonometric...
and elliptic. Namely, if \( f_{ij}(z) \) are rational functions of the variable \( z \) then the corresponding solution \( R_{ij} \) is called rational, if \( f_{ij}(z) = \phi_{ij}(\exp(kz)) \), where \( \phi_{ij}(z) \) are rational functions, then the corresponding solution \( R_{ij} \) is called trigonometric and, finally, if the functions \( f_{ij}(z) \) are elliptic the solution is elliptic. The elliptic solution exists only for the case \( g = sl_{n} \).

The classical Yang-Baxter equation \( (2.4) \) has a quantum analog, the quantum Yang-Baxter equation (QYBE), \( (r(z) \to R(z)) \):

\[
R^{12}(z_{1} - z_{2}) R^{13}(z_{1} - z_{3}) R^{23}(z_{2} - z_{3}) = R^{23}(z_{2} - z_{3}) R^{13}(z_{1} - z_{3}) R^{12}(z_{1} - z_{2}),
\]

(2.6)

\[
R(z) = 1 + hr(z) + O(h^{2}).
\]

(2.7)

Where \( h \) is a deformation parameter. Every solution \( r(z) \) corresponds to a quantum deformation of \( U(g[u]) \) - a quantum group. The simplest rational \( r \)-matrix corresponds to a quantum group named Yangian (introduced by Drinfeld in 1985, [2]). The simplest trigonometric solution of QYBE corresponds to a \( q \)-deformation of \( U(g[u]) \), the elliptic \( r \)-matrix corresponds to an elliptic quantum group.

3. A rational-trigonometric solution of CYBE

It turns out that a sum of the simplest rational and trigonometric \( r \)-matrices is also a solution of CYBE. For the sake of simplicity we take the case \( g = sl_{2} := sl(2, \mathbb{C}) \). The simplest rational \( r_{rt}(u, v) \) and trigonometric \( r_{tr}(u, v) \) solutions of CYBE have the form\(^1\)

\[
r_{rt}(u - v) = \frac{\eta}{u - v} \Omega_{2},
\]

(3.1)

\[
r_{tr}(u - v) = h \left( \frac{\exp(u - v) + 1}{\exp(u - v) - 1} \Omega_{2} + e_{-\alpha} \otimes e_{\alpha} - e_{\alpha} \otimes e_{-\alpha} \right).
\]

Here \( \{h_{\alpha}, e_{\pm \alpha}\} \) is a standard Cartan-Weyl basis of \( sl_{2} \) with the defining relations:

\[
[e_{\alpha}, e_{-\alpha}] = h_{\alpha}, \quad [h_{\alpha}, e_{\pm \alpha}] = \pm(\alpha, \alpha)e_{\pm \alpha} \quad ((\alpha, \alpha) = 2).
\]

(3.2)

The element \( \Omega_{2} \) is called the Casimir two-tensor:

\[
\Omega_{2} = \frac{1}{2} \left( \Delta_{0}(C_{2}) - C_{2} \otimes 1 - 1 \otimes C_{2} \right),
\]

where

\[
C_{2} = \frac{1}{2} h_{\alpha}^{2} + e_{-\alpha}e_{\alpha} + e_{\alpha}e_{-\alpha}
\]

(3.3)

is a second order Casimir element, and the symbol \( \Delta_{0} \) is the primitive co-multiplication

\[
\Delta_{0}(x) = x \otimes 1 + 1 \otimes x \quad (\forall x \in sl_{2}).
\]

(3.4)

In the trigonometric \( r \)-matrix we make change of the variables \( \exp u \to u, \exp v \to v \) then the trigonometric \( r \)-matrix takes the form

\[
r_{tr}(u, v) = \left( \frac{u + v}{u - v} \Omega_{2} + e_{\alpha} \otimes e_{-\alpha} - e_{\alpha} \otimes e_{-\alpha} \right).
\]

(3.5)

\[^{1}\text{In contrast to (2.4), here in (3.1) the deformation parameters } \eta \text{ and } h \text{ are introduced explicitly in the } r \text{-matrices.}\]
This function already does not depend on the difference $u - v$, but it also satisfies CYBE. Thus, we have to abandon the condition (ii) for $r(u, v)$. However, right now it is easy to see that the sum of the rational and trigonometric solutions of CYBE is also a solution of CYBE. Indeed, we have

$$r_{\text{tr}}(u + \frac{a}{2}, v + \frac{a}{2}) = r_{\text{tr}}(u, v) + \frac{a}{\eta} r_{\text{rt}}(u, v)$$ \quad \text{for} \ a \in \mathbb{C}.$$  \hspace{1cm} (3.7)  

Since $r_{\text{tr}}(u, v)$ for any $u$ and $v$ satisfies CYBE

$$[r_{\text{tr}}^{12}(u_1, u_2), r_{\text{tr}}^{13}(u_1, u_3) + r_{\text{tr}}^{23}(u_2, u_3)] + [r_{\text{tr}}^{13}(u_1, u_3), r_{\text{tr}}^{23}(u_2, u_3)] = 0$$ \hspace{1cm} (3.8)  

the sum $r_{\text{tr}, r}(u, v) := r_{\text{tr}}(u, v) + r_{\text{rt}}(u, v)$ also satisfies CYBE.

Each of the classical $r$-matrices $r_{\text{tr}}(u, v)$, $r_{\text{tr}}(u, v)$ and $r_{\text{rt}}(u, v)$ defines its own Lie bialgebra $([\mathfrak{sl}_2|u], \delta_1)$ \cite{2}, where $\delta_1$ is the cocommutator given by $\delta_1(x) = [x \otimes 1 + 1 \otimes x, r_1]$, $(x \in \mathfrak{sl}_2|u)$, $r_1$ is one of our classical $r$-matrices). Each of these Lie bialgebras corresponds to a quantum deformation of $U(\mathfrak{g}|u)$.

Using the comultiplication formulas of the quantum algebras $U_q(\mathfrak{sl}_2|u)$ and $Y_q(\mathfrak{sl}_2)$ one can show that the bialgebras for $r_{\text{tr}}(u, v)$ and $r_{\text{tr}}(u, v)$ coincide with semiclassical limits of $U_q(\mathfrak{sl}_2|u)$ and $Y_q(\mathfrak{sl}_2)$, i.e., for example, $\delta_{\text{tr}} = \Delta_{\text{op}}^\eta - \Delta_{\text{q}}$ (mod $\hbar^2$) where $\Delta_{\text{op}}^\eta$ is a comultiplication opposite to $\Delta_q$, and analogously for $\delta_{\text{rt}}$.

The bialgebra with the rational-trigonometric $r$-matrix $r_{\text{rt}}(u, v) := r_{\text{rt}}(u, v) + r_{\text{tr}}(u, v)$ corresponds to a two-parameter deformation of $U(\mathfrak{g}|u)$, which we call Drinfeldian. Thus, the Drinfeldian $D_q(\mathfrak{sl}_2)$ is a quantization of $U(\mathfrak{sl}_2|u)$ in direction of the classical $r$-matrix $r_{\text{rt}}(u, v)$ and moreover the quantum algebra $U_q(\mathfrak{sl}_2|u)$ and the Yangian $Y_q(\mathfrak{sl}_2)$ are limit quantum algebras of $D_q\mathfrak{sl}_2\mathfrak{sl}_2$ when the parameters of deformation $\eta$ goes to 0 and $q$ goes to 1, correspondingly.

Below we present a very simple scheme for construction of defining relations of the Drinfeldian $D_q\mathfrak{sl}_2\mathfrak{sl}_2$ for any finite-dimensional complex simple Lie algebra $\mathfrak{g}$ using the defining relations of the quantum ($q$-deformed) algebra $U_q(\mathfrak{g}|u)$, and a limited pass $q \to 1$ gives us very simple defining relations for the Yangian $Y_q(\mathfrak{g})$. Thus, we find very simple connection between the quantum affine algebra $U_q(\mathfrak{g}|u)$ and the Yangian $Y_q(\mathfrak{g})$.

In order to compare our realization of the Yangian in terms of a Chevalley basis with Drinfeld’s realizations at first we present the Drinfeld realizations in an explicit form. There are three realizations of $Y_q(\mathfrak{g})$, given by Drinfeld \cite{2,4}. We give two of them.

### 4. Drinfeld’s realizations of Yangians

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. Fix a non-zero invariant bilinear form $(\ ,\ )$ on $\mathfrak{g}$, and let $\{I_\alpha\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to $(\ ,\ )$.

1). The first Drinfeld realization of Yangians.

**Definition 4.1.** \cite{4} The Yangian $Y_q(\mathfrak{g})$ is generated as an associative algebra over $\mathbb{C}[[\eta]]$ by the Lie algebra $\mathfrak{g}$ and elements $J(x)$, $x \in \mathfrak{g}$, with the defining relations:

$$J(\lambda x + \mu y) = \lambda J(x) + \mu J(y),$$

$$J([x, y]) = [x, J(y)]$$ \quad \text{for} \ x, y \in \mathfrak{g}, \ \lambda, \mu \in \mathbb{C}.$$ \hspace{1cm} (4.1)


\[ \text{if } \sum_i [x_i, y_i] = 0 \text{ for } x_i, y_i \in \mathfrak{g} \Rightarrow \]

\[ \sum_i [J(x_i), J(y_i)] = \frac{\eta^2}{12} \sum_{\alpha, \beta, \gamma} \{ [[x_i, I_\alpha], [y_i, I_\beta]], I_\gamma \} \{ I_\alpha, I_\beta, I_\gamma \}, \]

\[ \text{if } \sum_i [[x_i, y_i], z_i] = 0 \text{ for } x_i, y_i, z_i \in \mathfrak{g} \Rightarrow \]

\[ \sum_i [J(x_i), J(y_i)], J(z_i)] = \frac{\eta^2}{4} \sum_{\alpha, \beta, \gamma} f(x_i, y_i, z_i, I_\alpha, I_\beta, I_\gamma) \times \{ I_\alpha, I_\beta, J(I_\gamma) \}, \]

where\(^2\) the notations are used \( \{a_1, a_2, a_3\} := (1/6) \sum_{i \neq j \neq k} a_i a_j a_k \) and \( f(x, y, z, a, b, c) := \) 

\[ \text{Alt Sym} ([x, [y, a]], [[z, b], c]). \] 

A comultiplication map \( (\Delta_\eta : Y_\eta(\mathfrak{g}) \to Y_\eta(\mathfrak{g}) \otimes Y_\eta(\mathfrak{g})) \), an antipode \( (S_\eta : Y_\eta(\mathfrak{g}) \to Y_\eta(\mathfrak{g})) \) and a counite \( (\varepsilon_\eta : Y_\eta(\mathfrak{g}) \to \mathbb{C}) \) are given by the formulas \( (x \in \mathfrak{g}) \)

\[ \Delta_\eta(x) = x \otimes 1 + 1 \otimes x, \]

\[ \Delta_\eta(J(x)) = J(x) \otimes 1 + 1 \otimes J(x) + \frac{\eta}{2} [x \otimes 1, \Omega_2], \]

\[ S_\eta(x) = -x, \quad S_\eta(J(x)) = -J(x) + \frac{\eta}{4} \lambda x, \]

\[ \varepsilon_\eta(x) = \varepsilon_\eta(J(x)) = 0, \quad \varepsilon_\eta(1) = 1, \]

where \( \Omega_2 \) is the Casimir two-tensor \( (\Omega_2 = \sum I_\alpha \otimes I_\alpha) \), \( \lambda \) is the eigenvalue of the Casimir operator \( C_2 = \sum I_\alpha I_\alpha \) in the adjoint representation of \( \mathfrak{g} \) in \( \mathfrak{g} \).

We may specialize the formal parameter \( \eta \) to any complex number \( \nu \in \mathbb{C} \), however, the resulting Hopf algebra \( Y_\nu(\mathfrak{g}) \) (over \( \mathbb{C} \)) is essentially independent of \( \nu \), provided that \( \nu \neq 0 \). It means that any two Hopf algebras \( Y_\nu(\mathfrak{g}) \) and \( Y_{\nu, \nu} \) with \( \nu \neq \nu'; \nu, \nu' \neq 0 \) are isomorphic. Thus, we can as well take \( \nu = 1 \) and therefore the parameter \( \eta \) is dropped. However, for convenience of passage to the limit \( \eta \to 0 \) we shall remain the formal parameter \( \eta \).

2). The second Drinfeld realization of Yangians. Let \( \mathcal{A} = (a_{ij})^l_{i,j=1} \) be a standard Cartan matrix of \( \mathfrak{g} \), \( \Pi := \{ \alpha_1, \ldots, \alpha_l \} \) be a system of simple roots \( (l \text{ is rank of } \mathfrak{g}) \), and let \( B_{ij} := \frac{1}{2}(\alpha_i, \alpha_j) \).

**Theorem 4.2.** [3] The Yangian \( Y_\eta(\mathfrak{g}) \) is isomorphic to the associative algebra over \( \mathbb{C}[[\eta]] \) with the generators:

\[ \xi_+^i, \quad \xi_-^i, \quad \varphi_n \quad \text{for } i = 1, 2, \ldots, l; \quad n = 0, 1, 2, \ldots, \]

\[ \text{in the case } \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \text{ there is a more complicated relation instead } \xi_+ \text{ (see [2]).} \]
and the following defining relations:

\begin{align}
(4.8) & \quad [\varphi_{in}, \varphi_{jm}] = 0 , \\
(4.9) & \quad [\varphi_{i0}, \xi^\pm_{jm}] = \pm 2B_{ij} \xi^\pm_{jm} , \\
(4.10) & \quad [\xi^+_m, \xi^-_m] = \delta_{ij} \varphi_{j n+m} , \\
(4.11) & \quad [\varphi_{i n+1}, \xi^\pm_{jm}] - [\varphi_{i n}, \xi^\pm_{j m+1}] = \pm \eta B_{ij} (\varphi_{i n} \xi^\pm_{jm} + \xi^\pm_{jm} \varphi_{i n}) , \\
(4.12) & \quad [\xi^\pm_{i n+1}, \xi^\pm_{j m}] - [\xi^\pm_{i n}, \xi^\pm_{j m+1}] = \pm \eta B_{ij} (\xi^\pm_{i n} \xi^\pm_{jm} + \xi^\pm_{jm} \xi^\pm_{im}) , \\
(4.13) & \quad \text{Sym}_{n_1, n_2, \ldots, n_k} \left[ \xi^\pm_{i n_1}, \ldots, \xi^\pm_{i n_k}, \xi^\pm_{j m_1}, \ldots, \xi^\pm_{j m_k} \right] = 0 \quad \text{for } i \neq j, \quad k = 1 - A_{ij} .
\end{align}

Explicit formulas for the action of the comultiplication \( \Delta_q \) on the generators \( \xi^\pm_{in}, \varphi_{in} \) are rather cumbersome (see [5]) and they are not given here.

For the Yangian \( Y_q(\text{sl}(n, \mathbb{C})) \) Drinfeld has also given a third realization. It is presented in terms of \( RLL \)-relations (see details in [3], [9], [10]).

All these realizations of Yangians are not minimal, i.e., they are not given in terms of a Chevalley basis. However, the minimal realization exists and we present it here using connection of Yangians with quantum non-twisted affine algebras.

5. Quantum algebra \( U_q(\widehat{\mathfrak{g}[u]}) \)

Again, let \( \mathfrak{g} \) be a finite-dimensional complex simple Lie algebra of rank \( l \) with the standard Cartan matrix \( A = (a_{ij})_{i,j=1}^l \), with the system of simple roots \( \Pi := \{ \alpha_1, \ldots, \alpha_l \} \), and with the Chevalley basis \( h_{\alpha_i}, e'_{\pm \alpha_i}, (i = 1, 2, \ldots, l) \). Let \( \theta \) be the maximal positive root of \( \mathfrak{g} \). The corresponding non-twisted affine algebra \( \widehat{\mathfrak{g}} \) is generated by \( \mathfrak{g} \) and the additional affine elements \( e'_{\pm (\delta - \theta)} \) and \( h_{\delta - \theta} \). It is well known [5] that \( \widehat{\mathfrak{g}} \cong \mathfrak{g}[u, u^{-1}] \).

Let \( \mathfrak{g}[u] \) be a Lie algebra generated by \( \mathfrak{g} \), the positive root vector \( e'_{\delta - \theta} = ue'_{-\theta} \) and the Cartan element \( h_{\delta - \theta} \), i.e. \( \mathfrak{g}[u] \cong \mathfrak{g}[u] \oplus Ch_\delta \), where \( h_\delta \) is a central element.

The standard defining relations of \( \mathfrak{g}[u] \) and its universal enveloping algebra \( U(\mathfrak{g}[u]) \) are given by the formulas:

\begin{align}
(5.1) & \quad [h_\delta, \text{everything}] = 0 , \\
(5.2) & \quad [h_{\alpha_i}, h_{\alpha_j}] = 0 , \\
(5.3) & \quad [h_{\alpha_i}, e'_{\pm \alpha_j}] = \pm (\alpha_i, \alpha_j) e'_{\pm \alpha_j} , \\
(5.4) & \quad [e'_{\alpha_i}, e'_{-\alpha_j}] = \delta_{ij} h_{\alpha_i} , \\
(5.5) & \quad (\text{ad} e'_{\pm \alpha_j})^{n_{ij}} e'_{\pm \alpha_j} = 0 \quad \text{for } i \neq j, \quad n_{ij} := 1 - a_{ij} , \\
(5.6) & \quad [h_{\alpha_i}, e'_{\delta - \theta}] = - (\alpha_i, \theta) e'_{\delta - \theta} , \\
(5.7) & \quad [e'_{-\alpha_i}, e'_{\delta - \theta}] = 0 , \\
(5.8) & \quad (\text{ad} e'_{\alpha_i})^{n_{i0}} e'_{\delta - \theta} = 0 \quad \text{for } n_{i0} = 1 + 2(\alpha_i, \theta)/(\alpha_i, \alpha_i) , \\
(5.9) & \quad [e'_{\alpha_i}, e'_{\delta - \theta}, e'_{\delta - \theta}] = 0 \quad \text{for } \mathfrak{g} \neq sl_2 \text{ and } (\alpha_i, \theta) \neq 0 , \\
(5.10) & \quad [[[e'_{\alpha_i}, e'_{\delta - \alpha}], e'_{\delta - \alpha}], e'_{\delta - \alpha}] = 0 \quad \text{for } \mathfrak{g} = sl_2 .
\end{align}
Here "ad" is the adjoint action of $\hat{g}[u]$ in $\hat{g}[u]$, i.e. $(ad x)y = [x, y]$ for $x, y \in \hat{g}[u]$.

The relations (5.9) relate to the case $g \neq \mathfrak{sl}_2$, and the relation (5.10) belongs to the case $g = \mathfrak{sl}_2$ (in this case $\theta = \alpha$).

The defining relations of the quantum algebra $U_q(\hat{g})$, which is a $q$-deformation of $U(\hat{g}[u])$, can be obtained from the defining relations of the quantum algebra $U_q(\hat{g})$ by removing the relations with the negative affine root vector $e_{-\alpha + \theta}$.

If $k_\delta^\pm := q^\pm h_\delta$, $k_{\alpha_i}^\pm := q^\pm h_{\alpha_i}$, $e_{\pm \alpha_i}$ ($i = 1, 2, \ldots, l$), and $e_{\pm (\delta - \theta)}$ is the Chevalley basis of $U_q(\hat{g})$ then the standard defining relations of $U_q(\hat{g}[u])$ are given by the formulas:

\begin{align*}
(5.11) & \quad [k_\delta^\pm, \text{everything}] = 0, \\
(5.12) & \quad k_{\alpha_i}^\pm k_{\alpha_i}^\pm = k_{\alpha_i}^\pm k_{\alpha_i}^\pm, \\
(5.13) & \quad k_\delta k_\delta^{-1} = k_\delta^{-1} k_\delta = k_\alpha^\pm k_\alpha^{-1} = k_\alpha^{-1} k_\alpha = 1, \\
(5.14) & \quad k_\alpha e_{\pm \alpha_j} k_\alpha^{-1} = q^{(\alpha_i, \alpha_j)} e_{\pm \alpha_j}, \\
(5.15) & \quad [e_{\alpha_i}, e_{-\alpha_i}] = \frac{k_\alpha - k_\alpha^{-1}}{q - q^{-1}}, \\
(5.16) & \quad (ad_q e_{\pm \alpha})^{n_{ij}} e_{\pm \alpha_j} = 0 \quad \text{for} \ i \neq j, \ n_{ij} := 1 - a_{ij}, \\
(5.17) & \quad k_\alpha e_{\delta - \theta} k_\alpha^{-1} = q^{-(\alpha_i, \theta)} e_{\delta - \theta}, \\
(5.18) & \quad [e_{-\alpha_i}, e_{\delta - \theta}] = 0, \\
(5.19) & \quad (ad_q e_{\alpha})^{n_{i0}} e_{\delta - \theta} = 0 \quad \text{for} \ n_{i0} = 1 + 2(\alpha_i, \theta)/(\alpha_i, \alpha_i), \\
(5.20) & \quad [[e_{\alpha_i}, e_{\delta - \theta}]_q, e_{\delta - \theta}]_q = 0 \quad \text{for} \ g \neq \mathfrak{sl}_2 \ \text{and} \ (\alpha_i, \theta) \neq 0, \\
(5.21) & \quad [[[e_{\alpha_i}, e_{\delta - \theta}]_q, e_{\delta - \theta}]_q, e_{\delta - \theta}]_q = 0 \quad \text{for} \ g = \mathfrak{sl}_2,
\end{align*}

where $(ad_q e_{\beta}) e_{\gamma}$ is the $q$-commutator:

\begin{equation}
(5.22) \quad (ad_q e_{\beta}) e_{\gamma} := [e_{\beta}, e_{\gamma}]_q := e_{\beta} e_{\gamma} - q^{(\beta, \gamma)} e_{\gamma} e_{\beta}.
\end{equation}

The comultiplication $\Delta_q$, the antipode $S_q$, and the counit $\varepsilon_q$ of $U_q(\hat{g}[u])$ are given by

\begin{align*}
(5.23) & \quad \Delta_q(k_\gamma^\pm) = k_\gamma^\pm \otimes k_\gamma^\pm \quad (\gamma = \delta, \alpha_i), \\
& \quad \Delta_q(e_{-\alpha_i}) = e_{-\alpha_i} \otimes k_{\alpha_i} + 1 \otimes e_{-\alpha_i}, \\
& \quad \Delta_q(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + k_{\alpha_i}^{-1} \otimes e_{\alpha_i}, \\
& \quad \Delta_q(e_{\delta - \theta}) = e_{\delta - \theta} \otimes 1 + k_{\delta - \theta}^{-1} \otimes e_{\delta - \theta}, \\
& \quad S_q(k_\gamma^\pm) = k_\gamma^\pm \quad (\gamma = \delta, \alpha_i), \\
& \quad S_q(e_{-\alpha_i}) = -e_{-\alpha_i} k_{\alpha_i}^{-1}, \\
& \quad S_q(e_{\alpha_i}) = -k_{\alpha_i} e_{\alpha_i}, \\
& \quad S_q(e_{\delta - \theta}) = -k_{\delta - \theta} e_{\delta - \theta},
\end{align*}
We set
\[ k_{\delta-\theta} = k_\delta k_{a_1}^{-n_1} k_{a_2}^{-n_2} \ldots k_{a_l}^{-n_l} \]
if \( \theta = n_1\alpha_1 + n_2\alpha_2 + \ldots + n_l\alpha_l \).

It is easy to check that the following proposition is valid.

**Proposition 5.1.** There is a one-parameter group of Hopf algebra automorphisms \( \mathfrak{T}_a \) of \( U_q(\mathfrak{g}[u]) \), \( a \in \mathbb{C} \), given by
\[ \mathfrak{T}_a(k) = k, \quad \mathfrak{T}_a(e_{\pm \alpha}) = e_{\pm \alpha}, \quad \mathfrak{T}_a(e_{\delta-\theta}) = a e_{\delta-\theta}. \]

Now starting from the quantum algebra \( U_q(\mathfrak{g}[u]) \) we construct the two-parameter Hopf algebra which coincides with the Yangian \( Y_q(\mathfrak{g}) \) at \( q = 1 \).

### 6. Drinfeldian \( D_{q_0}(\mathfrak{g}) \)

The quantum algebra \( U_q(\mathfrak{g}[u]) \) does not contain any singular elements when \( q \) goes to 1 (\( q = \exp(h) \)). This means that \( \lim_{q \to 1} x \in U(\mathfrak{g}[u]) \) for any \( x \in U_q(\mathfrak{g}[u]) \). Now we extend \( U_q(\mathfrak{g}[u]) \) by singular elements, i.e. we consider the algebra
\[ \tilde{U}_q(\mathfrak{g}[u]) := U_q(\mathfrak{g}[u]) \otimes \mathbb{C}[u] \mathbb{C}(h) . \]

This algebra contains singular elements, i.e. elements which have not any limit when \( q \to 1 \). For example, if \( x \in U_q(\mathfrak{g}[u]) \) and \( \lim_{q \to 1} x \neq 0 \) then the element \( x/(q - q^{-1}) \) is singular.

Let \( \tilde{e}_{-\theta} \in U_q(\mathfrak{g}[u]) \) be any element of the weight \(-\theta\), i.e.
\[ k_{\alpha_i} \tilde{e}_{-\theta} k_{\alpha_i}^{-1} = q^{-\langle \alpha_i, \theta \rangle} \tilde{e}_{-\theta}, \]
and, moreover,
\[ \lim_{q \to 1} \tilde{e}_{-\theta} = e'_{-\theta}, \]
where \( e'_{-\theta} \) is a root vector of \( \mathfrak{g} \) with the minimal weight \(-\theta\). Let us introduce the new affine generator
\[ \xi_{\delta-\theta} = e_{\delta-\theta} + \frac{\eta}{q - q^{-1}} \tilde{e}_{-\theta}. \]
Then the defining relations (5.17)-(5.21) take the form
\[ k_{\alpha_i} \xi_{\delta-\theta} k_{\alpha_i}^{-1} = q^{-\langle \alpha_i, \theta \rangle} \xi_{\delta-\theta}, \]
\[ [e_{-\alpha_i}, \xi_{\delta-\theta}] = \tau [e_{-\alpha_i}, \tilde{e}_{-\theta}], \]
\[ (\text{ad}_q e_{\alpha_i})^{n_0} \xi_{\delta-\theta} = \tau (\text{ad}_q e_{\alpha_i})^{n_0} \tilde{e}_{-\theta} \]
for \( n_0 = 1 + 2\langle \alpha_i, \theta \rangle / \langle \alpha_i, \alpha_i \rangle \), and
\[ [e_{\alpha_i}, \xi_{\delta-\theta} q \xi_{\delta-\theta}] = -\tau^2 [e_{\alpha_i}, \tilde{e}_{-\theta}] q [\tilde{e}_{-\theta}] q \]
\[ + \tau ([e_{\alpha_i}, \tilde{e}_{-\theta}] q [\xi_{\delta-\theta}] q + [e_{\alpha_i}, \xi_{\delta-\theta}] q [\tilde{e}_{-\theta}] q). \]
for \( g \neq \mathfrak{sl}_2 \) and \((\alpha, \theta) \neq 0\),

\[
[[[e_\alpha, \xi_{\delta-\alpha}]_q, \xi_{\delta-\alpha}]_q, \xi_{\delta-\alpha}]_q = \tau \left( [[[e_\alpha, \tilde{e}_{-\alpha}]_q, \xi_{\delta-\alpha}]_q, \xi_{\delta-\alpha}]_q + [[[e_\alpha, \xi_{\delta-\alpha}]_q, \xi_{\delta-\alpha}]_q, \xi_{\delta-\alpha}]_q + [[[e_\alpha, \xi_{\delta-\alpha}]_q, \xi_{\delta-\alpha}]_q, \xi_{\delta-\alpha}]_q \right) \\
- \tau^2 \left( [[[e_\alpha, \tilde{e}_{-\alpha}]_q, \xi_{\delta-\alpha}]_q + [[[e_\alpha, \xi_{\delta-\alpha}]_q, \xi_{\delta-\alpha}]_q, \xi_{\delta-\alpha}]_q + [[[e_\alpha, \xi_{\delta-\alpha}]_q, \xi_{\delta-\alpha}]_q, \xi_{\delta-\alpha}]_q \right)
\]

(6.9)

for \( g = \mathfrak{sl}_2 \). Here and elsewhere \( \tau := \eta/(q - q^{-1}) \).

The formulas (6.28) for the comultiplication \( \Delta_{q\eta} \) and the antipode \( S_{q\eta} \) are rewritten as follows

\[
\Delta_{q\eta}(x) = \Delta_q(x) \quad (x \in U_q(g) \otimes \mathbb{C}k_\delta),
\]

(6.10)

\[
\Delta_{q\eta}(\xi_{\delta-\theta}) = \xi_{\delta-\theta} \otimes 1 + k_{\delta-\theta}^{-1} \otimes \xi_{\delta-\theta}
+ \tau \left( \Delta_q(\tilde{e}_{-\theta}) - \tilde{e}_{-\theta} \otimes 1 - k_{\delta-\theta}^{-1} \otimes \tilde{e}_{-\theta} \right),
\]

(6.11)

\[
S_{q\eta}(x) = S_q(x) \quad (x \in U_q(g) \otimes \mathbb{C}k_\delta),
\]

(6.12)

\[
S_{q\eta}(\xi_{\delta-\theta}) = -k_{\delta-\theta} \xi_{\delta-\theta} + \tau \left( S_q(\tilde{e}_{-\theta}) + k_{\delta-\theta} \tilde{e}_{-\theta} \right).
\]

(6.13)

The following theorem is valid.

**Theorem 6.1.** If the element \( \tilde{e}_{-\theta} \in U_q([g][u]) \) satisfies the conditions (6.2) and (6.3) then the right-hand sides of the relations (6.4)–(6.9) and (6.11), (6.13) are nonsingular at \( q = 1 \).

Thus, the associative algebra \( D_{q\eta}(g) \) over \( \mathbb{C}[[h, \eta]] \) generated by the quantum algebra \( U_q(g) \), the central element \( k_{\delta-\theta}^{-1} \) and by the affine element \( \xi_{\delta-\theta} \) with the relations (6.5)–(6.9), where \( \tilde{e}_{-\theta} \in U_q([g][u]) \) satisfies the conditions (6.2) and (6.3), and with the Hopf structure given by the formulas (6.10)–(6.13) is nonsingular at \( q = 1 \).

**Theorem 6.2.** (i) The Hopf algebra \( D_{q\eta}(g) \) is a two-parameter quantization of \( U([g][u]), ([g][u] := g[[u] \oplus h_\delta) \), in the direction of a classical \( r \)-matrix which is a sum of the simplest rational and trigonometric \( r \)-matrices.

(ii) The Hopf algebra \( D_{q=1,\eta}(g) := D_{q\eta}(g)/\hbar D_{q\eta}(g) \) is isomorphic to the Yangian \( Y_q(g) \) (with the additional central element \( h_\delta \)). Moreover, \( D_{q\eta=0}(g) = U_q([g][u]) \).

By virtue of this theorem, the Hopf algebra \( D_{q\eta}(g) \) can be called as the rational-trigonometric quantum algebra or the \( q \)-deformed Yangian. We also call this algebra as the Drinfeldian (in honor of V.G. Drinfeld, who first observed existence of such subalgebra in the extension (6.3) of the \( q \)-deformed affine algebra \( U_q(g) \) (p.1)).

**Remark 6.3.** Since the defining relations for \( D_{q\eta}(g) \) and \( U([g][u]) \) in terms of the Chevalley basis differ only in the right-hand sides of the relations (6.4)–(6.9) and (6.11)–(6.13), therefore the Dynkin diagram of \( [g][u] \) can be also used for classification of the Drinfeldian \( D_{q\eta}(g) \) and the Yangian \( Y_q(g) \).

As an immediate consequence of Proposition 6.1 we have the following result.
Proposition 6.4. There is a one-parameter group of Hopf algebra automorphisms $\mathfrak{T}_a$ of $D_{q\eta}(g)$, $a \in \mathbb{C}$, given by

$$\mathfrak{T}_a(x) = x \ (x \in U_q(g) \otimes \mathbb{k}_\delta),$$

(6.14)

$$\mathfrak{T}_a(\xi_{\delta-\theta}) = \left(1 - (q - q^{-1})a\right)\xi_{\delta-\theta} + \eta a \xi_{\delta-\theta}.$$ 

The relations between the Drinfeldian $D_{q\eta}(g)$ and the algebras $U_q(g[u]), \ Y_q(g), \ U(g[u])$ (and also their subalgebras) are shown in the picture:

$$U_q(g) \subset D_{q\eta}(g) \xrightarrow{\eta \to 0} U_q(g[u]) \supset U_q(g) \quad q \to 1$$
$$U(g) \subset Y_q(g) \xrightarrow{\eta \to 0} U(g[u]) \supset U(g).$$

Fig. 1. Diagram of the limit Hopf algebras of the Drinfeldian $D_{q\eta}(g)$ and their subalgebras. The arrows show passages to the limits.

7. Drinfeldians and Yangians for Lie algebras $A_{l-1}$ ($l \geq 3$)

An explicit description of the Drinfeldian $D_{q\eta}(g)$ and Yangian $Y_q(g)$ for the simple Lie algebra $g = A_1 \simeq sl_2 := sl(2, \mathbb{C})$ was given in [12] [13] and, therefore, we consider here the case $g = A_{l-1}$, where $l \geq 3$.

We first recall the defining relations of the $q$-deformed universal enveloping algebra $U_q(sl_l)$ ($sl_l := sl(l, \mathbb{C}) \simeq A_{l-1}$) and structure of its Cartan-Weyl basis.

Let $\Pi := \{\alpha_1, \ldots, \alpha_{l-1}\}$ be a system of simple roots of $sl_l$ endowed with the following scalar product: $\langle \alpha_i, \alpha_j \rangle = (\alpha_j, \alpha_i), \ (\alpha_i, \alpha_i) = 2, \ (\alpha_i, \alpha_{i+1}) = -1, \ (\alpha_i, \alpha_j) = 0$ ($|i - j| > 1$). The corresponding Dynkin diagram is presented on the picture:

![Dynkin diagram of sl_l](image)

Fig. 2. The Dynkin diagram of the Lie algebra $sl_l$.

Usually, it is convenient to realize a positive root system $\Delta_+$ (with respect to $\Pi$) and a total root system $\Delta = \Delta_+ \cup (-\Delta_+)$ of the classical Lie algebras $A_{l-1}, \ B_l, \ C_l$ and $D_l$ in terms of an orthonormalized basis of a $l$-dimensional Euclidian space. Namely, let $e_i$ ($i = 1, 2, \ldots, l$) be an orthonormalized basis of a $l$-dimensional Euclidian space $\mathbb{R}^l$: $\langle e_i, e_j \rangle = \delta_{ij}$. In the terms of $e_i$ the systems $\Pi$, $\Delta_+$ and $\Delta$ of $sl_l$ are presented as follows:

(7.1) \[ \Pi = \{ \alpha_i = e_i - e_{i+1} \ | \ i = 1, 2, \ldots, l-1 \}, \]

(7.2) \[ \Delta_+ = \{ e_i - e_j \ | \ 1 \leq i < j \leq l \}; \]

(7.3) \[ \Delta = \Delta_+ \cup (-\Delta_+) = \{ e_i - e_j \ | \ i \neq j; \ i, j = 1, 2, \ldots, l \}. \]

The root $\theta := \alpha_1 + \alpha_2 + \ldots + \alpha_l = e_1 - e_l$ is maximal. We shall use instead of the simple Lie algebra $sl_l$ its central extension $gl_l$. 


The quantum algebra $U_q(\mathfrak{g}_l)$ is generated by the Chevalley elements $e_{i,-1} := e_{i,-1}$, $e_{i+1,-i} := e_{i+1,-i}$ $(i = 1, 2, \ldots, l-1)$, and $q^{\pm e_{i,-1}}$ $(i = 1, 2, \ldots, l)$ with the defining relations:

$$q^{\epsilon_{i,-1}}q^{-\epsilon_{i,-1}} = q^{-\epsilon_{i,-1}}q^{\epsilon_{i,-1}} = 1,$$
$$q^{\epsilon_{i,-1}}q^{\epsilon_{j,-1}} = q^{\epsilon_{j,-1}}q^{\epsilon_{i,-1}},$$
$$q^{\epsilon_{i,-1}}e_{j,-k}q^{-\epsilon_{i,-1}} = q^{\delta_{i,j}-\delta_{i,k}}e_{j,-k} (|j-k| = 1),$$
$$[e_{i,-1}, e_{j,-j+1}] = \delta_{ij} q^{\epsilon_{i,-1}+\epsilon_{j,-1}-\epsilon_{i,-1}} - q^{-\epsilon_{i,-1}-\epsilon_{j,-1}-\epsilon_{i,-1}} q^{-1},$$
$$[e_{i,-1}, e_{j,-1}] = 0 \text{ for } |i-j| \geq 2,$$
$$[e_{i+1,-i}, e_{j+1,-j}] = 0 \text{ for } |i-j| \geq 2,$$
$$|[e_{i,-1}, e_{j,-1}]_{q}, e_{j,-1}]_{q} = 0 \text{ for } |i-j| = 1,$$
$$|[e_{i+1,-i}, e_{j+1,-j}]_{q}, e_{j+1,-j}]_{q} = 0 \text{ for } |i-j| = 1.$$

The Hopf structure on $U_q(\mathfrak{g}_l)$ is given by the following formulas for a comultiplication $\Delta_q$, an antipode $S_q$, and a counit $\epsilon_q$:

$$\Delta_q(q^{\pm e_{i,-1}}) = q^{\pm e_{i,-1}} \otimes q^{\pm e_{i,-1}},$$
$$\Delta_q(e_{i,-1}) = e_{i,-1} \otimes 1 + q^{\epsilon_{i+1,-i}-\epsilon_{i,-1}} \otimes e_{i+1,-i},$$
$$\Delta_q(e_{i+1,-i}) = e_{i+1,-i} \otimes q^{\epsilon_{i,-1}+\epsilon_{i+1,-i}+1} + 1 \otimes e_{i+1,-i},$$
$$S_q(q^{\pm e_{i,-1}}) = q^{\mp e_{i,-1}},$$
$$S_q(e_{i,-1}) = -q^{\epsilon_{i,-1}+\epsilon_{i+1,-1}} e_{i+1,-1},$$
$$S_q(e_{i+1,-i}) = -e_{i+1,-i} q^{\epsilon_{i+1,-i}+\epsilon_{i,-1}},$$
$$\epsilon_q(q^{\pm e_{i,-1}}) = 1, \quad \epsilon_q(e_{i,-j}) = 0 \text{ (if } |i-j| = 1 \text{).}$$

For construction of the composite root vectors $e_{i,-j}$ for $|i-j| \geq 2$ we fix the following normal ordering of the positive root system $\Delta_+$ (see [11] [6] [7])

$$(\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3, \ldots, \epsilon_1 - \epsilon_l, \epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4, \ldots, \epsilon_2 - \epsilon_l, \ldots),$$
$$(\epsilon_{l-3} - \epsilon_{l-2}, \epsilon_{l-3} - \epsilon_{l-1}, \epsilon_{l-3} - \epsilon_l, \epsilon_{l-2} - \epsilon_{l-1}, \epsilon_{l-2} - \epsilon_l, \epsilon_{l-1} - \epsilon_l).$$

According to this ordering we set

$$e_{i,-j} := [e_{i,-k}, e_{k,-j}]_{q^{-1}}, \quad e_{j,-i} := [e_{j,-k}, e_{k,-i}]_q,$$

where $1 \leq i < k < j \leq l$. It should be stressed that the structure of the composite root vectors is independent of choice of the index $k$ in the r.h.s. of the definition (7.9). In particular, we have

$$e_{i,-j} := [e_{i,-1}, e_{i+1,-j}]_{q^{-1}} = [e_{i,-1}, e_{i+1,-j}]_{q^{-1}},$$
$$e_{j,-i} := [e_{j,-1}, e_{i+1,-i}]_q = [e_{j,-1}, e_{i+1,-i}]_q,$$

where $2 \leq i+1 < j \leq l$. General properties of the Cartan-Weyl basis $\{e_{i,-j}\}$ can be found in [7].

---

3If $q^{\epsilon_{i,-1}} e_{i,-1} = 1$ then we obtain the quantum algebra $U_q(\mathfrak{g}_l)$. 
As it was noted above the Dynkin diagrams of the non-twisted affine algebras can be also used for classification of the Drinfeldians and the Yangians. In the case of \( \hat{\mathfrak{sl}}_l \) (\( \mathfrak{gl}_l \)), the Dynkin diagram of the corresponding affine Lie algebra \( A^{(1)}_{l-1} \simeq \hat{\mathfrak{sl}}_l \) (\( \hat{\mathfrak{gl}}_l \)) is presented by the picture \[ \hat{\delta} - \hat{\theta} \]

![Dynkin Diagram](attachment:diagram.png)

Fig. 3. The Dynkin diagram of the affine Lie algebra \( \hat{\mathfrak{sl}}_l \).

The defining relations for the generators of \( D_{q\eta}(\mathfrak{g}) \) presented in the previous section depend explicitly on the choice of the element \( \tilde{e}_{-\theta} \in U_q(\mathfrak{g}) \) of the weight \( -\theta \), such that \( \lim_{q \to 1} \tilde{e}_{-\theta} = e'_{-\theta} \in \mathfrak{g} \). Here we present specification to the case \( \mathfrak{g} = \mathfrak{gl}_l \) and we set

\[
(7.11) \quad \tilde{e}_{-\theta} = q^{e_{1,-i+1}+e_{i,-1}}. 
\]

After some calculations we obtain the following result.

**Theorem 7.1.** The Drinfeldian \( D_{q\eta}(\mathfrak{gl}_l) \) \((l \geq 3)\) is generated (as a unital associative algebra over \( \mathbb{C}[[\log q, \eta]] \)) by the algebra \( U_q(\mathfrak{gl}_l) \) and the elements \( \xi_{\delta - \theta} \), \( q^{\pm \xi} := q^{\pm h} \) with the relations:

\[

(7.12) \quad [q^{\pm \xi}, \text{everything}] = 0, \\

(7.13) \quad q^{\pm e_{1,-i}} \xi_{\delta - \theta} = q^{\pm 1} \xi_{\delta - \theta} q^{\pm e_{1,-i}}, \\

(7.14) \quad q^{\pm e_{1,-i}} \xi_{\delta - \theta} = \xi_{\delta - \theta} q^{\pm e_{1,-i}} \quad \text{for} \; i = 2, 3, \ldots, l - 1, \\

(7.15) \quad q^{\pm e_{1,-i}} \xi_{\delta - \theta} = q^{\pm 1} \xi_{\delta - \theta} q^{\pm e_{1,-i}}, \\

(7.16) \quad [\xi_{\delta - \theta}, e_{1,-i+1}] = 0 \quad \text{for} \; i = 1, 2, \ldots, l - 1, \\

(7.17) \quad [e_{1,-i+1}, \xi_{\delta - \theta}] = 0 \quad \text{for} \; i = 2, 3, \ldots, l - 2, \\

(7.18) \quad [e_{1,-2}, [e_{1,-2}, \xi_{\delta - \theta}]]_q = 0, \\

(7.19) \quad [[[\xi_{\delta - \theta}, e_{1,-i}], 1], e_{1,-i}]_q = 0, \\

(7.20) \quad [[[e_{1,-2}, \xi_{\delta - \theta}], q_{\xi_{\delta - \theta}}]]_q = \eta q^{e_{1,-i+1}+e_{i,-1}} \times \\

\times (q^{-2}[e_{1,-2}, e_{1,-i}]\xi_{\delta - \theta} - e_{1,-i}[e_{1,-2}, \xi_{\delta - \theta}]_q), \\

(7.21) \quad [[[\xi_{\delta - \theta}, [\xi_{\delta - \theta}, e_{1,-1}], q_{\xi_{\delta - \theta}}]]_q = \eta q^{e_{1,-i+1}+e_{i,-1}+1} \times \\

\times (q[e_{1,-1}, e_{1,-i}]\xi_{\delta - \theta} - e_{1,-i}[\xi_{\delta - \theta}, e_{1,-1}]_q). 

\]

The Hopf structure of \( D_{q\eta}(\mathfrak{gl}_l) \) is defined by the formulas \( (7.9) - (7.22) \) for \( U_q(\mathfrak{gl}_l) \) (i.e. \( \Delta_{q\eta}(x) = \Delta_q(x) \), \( S_{q\eta}(x) = S_q(x) \) for \( x \in U_q(\mathfrak{gl}_l) \)), and \( \Delta_q(q^{\pm \xi}) = q^{\pm \xi} \otimes q^{\pm \xi} \).
where we use the notation \([x] := (q^x - q^{-x})/(q - q^{-1})\).

It is not difficult to check that the substitution \(\xi_{\delta-\theta} = q^{e_{\ell-1} + e_{l-1}}\) satisfies the relations \((7.20)\) and \((7.21)\), i.e. there is a simple homomorphism \(D_{q\eta}(\mathfrak{g}_l) \to U_q(\mathfrak{g}_l)\). Moreover, the both sides of the relations \((7.20)\) and \((7.21)\) are equal to zero independently. Therefore, we can construct a "evaluation representation" \(\rho_{ev}\) of \(D_{q\eta}(\mathfrak{g}_l)\) in \(U_q(\mathfrak{g}_l) \otimes \mathbb{C}[u]\) as follows

\[
\rho_{ev}(q^x) = \begin{cases} 1 & \text{if } x = 0 \leq l-1 \\ q^x & \text{if } x = e_{\ell-1} + e_{l-1} \end{cases}
\]

(7.24)

It is obvious that

\[
D_{q=0}(\mathfrak{g}_l) \simeq U_q(\mathfrak{g}_l) \simeq \mathbb{C}[u]
\]

as Hopf algebras. If \(q \to 1\) then the limit Hopf algebra \(D_{q=1}(\mathfrak{g}_l)\) is isomorphic to the Yangian \(Y_q(\mathfrak{g}_l)\) with \(\check{e} \neq 0\) [12]:

\[
D_{q=1}(\mathfrak{g}_l) \simeq Y_q(\mathfrak{g}_l).
\]

(7.26)

By setting \(q = 1\) in \((7.20)\) and \((7.21)\), we obtain the defining relations of the Yangian \(Y_q(\mathfrak{g}_l)\) and its Hopf structure in the Chevalley basis. This result can be formulated as the theorem.

**Theorem 7.2.** The Yangian \(Y_q(\mathfrak{g}_l)\) \((l \geq 3)\) is generated (as a unital associative algebra over \(\mathbb{C}[\eta]\)) by the algebra \(U(\mathfrak{g}_l)\) and the elements \(\xi_{\delta-\theta}, \check{e}\) with the relations:

\[
[e_{1-1}, \xi_{\delta-\theta}] = 0, \quad [\xi_{\delta-\theta}, \check{e}] = 0 \quad \text{for } i = 2, 3, \ldots, l-1, \quad [\xi_{\delta-\theta}, e_{l-1}+1] = 0, \quad [e_{l-1}+1, e_{l-1}] = 0 \quad \text{for } i = 2, 3, \ldots, l-2.
\]

(7.27)

\[
\frac{[e_{1-1}, \xi_{\delta-\theta}]}{[e_{1-2}, [e_{1-2}, \xi_{\delta-\theta}]]} = 0, \quad \frac{[\xi_{\delta-\theta}, e_{l-1}+1]}{[\xi_{\delta-\theta}, e_{l-1-1}]} = 0.
\]

(7.28)
The Hopf structure of the Yangian is trivial for the element \( \xi_{\delta - \theta} \):

\[
\Delta_{\eta}(\xi_{\delta - \theta}) = \xi_{\delta - \theta} \otimes 1 + 1 \otimes \xi_{\delta - \theta} + \eta \left( \frac{1}{2} \hat{c} \otimes e_{l, -1} + \sum_{i=1}^{l} e_{l, -i} \otimes e_{i, -1} \right),
\]

\[
S_{\eta}(\xi_{\delta - \theta}) = -\xi_{\delta - \theta} + \eta \left( \frac{1}{2} \hat{c} e_{l, -1} + \sum_{i=1}^{l} e_{l, -i} e_{i, -1} \right).
\]

8. Drinfeldians and Yangians for Lie algebras \( B_l \) (\( l \geq 3 \))

The Dynkin diagram of the simple Lie algebra \( B_l \simeq so_{2l+1} \) is presented on the picture:

![Dynkin diagram](image)

Fig. 4. The Dynkin diagram of the Lie algebra \( so_{2l+1} \).

In the terms of the orthonormalized basis \( \epsilon_i \) (\( i = 1, 2, \ldots, l \)) the root systems \( \Pi, \Delta_+ \) and \( \Delta \) of \( so_{2l+1} \) are given as follows:

\[
\Pi = \{ \alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{l-1} = \epsilon_{l-1} - \epsilon_l, \alpha_l = \epsilon_l \},
\]

\[
\Delta_+ = \{ \epsilon_i \pm \epsilon_j, \epsilon_k \mid 1 \leq i < j \leq l; k = 1, 2, \ldots, l \},
\]

\[
\Delta = \Delta_+ \cup ( -\Delta_+) = \{ \pm \epsilon_i \pm \epsilon_j, \pm \epsilon_k \mid i \neq j; i, j, k = 1, 2, \ldots, l \}.
\]

The root \( \theta := \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_l = \epsilon_1 + \epsilon_2 \) is maximal.

The quantum algebra \( U_q(so_{2l+1}) \) is generated by the Chevalley \( U_q(gl_l) \)-elements \( e_{i, -1} \) with the relations (7.35), and the additional elements \( e_{\pm l} := e_{\pm l, 1} \) with the relations:

\[
q^{\epsilon_{i, -1} e_{\pm l}} q^{-\epsilon_{i, -1}} = q^{\delta_{i, l}} e_{\pm l},
\]

\[
[e_{l, -1}] = q^{-\delta_{i, l}} e_{l, -1},
\]

\[
[e_{l, -1} e_{j}, e_{l}] = 0 \quad \text{for} \quad j \neq l, \quad |i-j| = 1,
\]

\[
[e_{i, -1} e_{j}, e_{l}] = 0 \quad \text{for} \quad i \neq l, \quad |i-j| = 1,
\]

\[
[[e_{l, -1}, e_{l}]] = 0,
\]

\[
[[[e_{l, -1}, e_{l}]] q, e_{l}] = 0.
\]
The Hopf structure on $U_q(\mathfrak{so}_{2l+1})$ is given by the formulas (8.5) and also
\begin{align}
\Delta_q(e_l) &= e_l \otimes 1 + q^{\epsilon_{l-1}} \otimes e_l, \\
\Delta_q(e_{-1}) &= e_{-1} \otimes q^{\epsilon_{l-1}} + 1 \otimes e_{-1},
\end{align}

(8.6) $S_q(e_l) = -q^{\epsilon_{l-1}} e_l$, $S_q(e_{-1}) = -e_{-1} q^{-\epsilon_{l-1}}$.

(8.7) $\varepsilon_q(e_{\pm 1}) = 0$.

For construction of the composite root vectors $e_{i-j}$ ($|i-j| > 1$), $e_{i,j}$, $e_{-i-j}$ ($i < j$), and $e_{\pm i}$ ($1 \leq i \leq l-1$) we fix the following normal ordering of the positive root system: $\Delta_+$
\begin{align}
(\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3, \ldots, \epsilon_1 - \epsilon_l, \epsilon_1, \epsilon_1 + \epsilon_l, \ldots, \epsilon_1 + \epsilon_l, \epsilon_1 + \epsilon_l, \epsilon_1 + \epsilon_l),
(\epsilon_2 - \epsilon_3, \\
\epsilon_2 - \epsilon_4, \ldots, \epsilon_2 - \epsilon_l, \epsilon_2, \epsilon_2 + \epsilon_l, \ldots, \epsilon_2 + \epsilon_l, \epsilon_2 + \epsilon_l), \\
\epsilon_{l-1} - \epsilon_l, \epsilon_{l-1} - \epsilon_{l-2}, \epsilon_{l-1} + \epsilon_l, \epsilon_{l-1} + \epsilon_{l-2}, \epsilon_{l-1} + \epsilon_l),
(\epsilon_{l-1} - \epsilon_l, \epsilon_{l-1} - \epsilon_{l-1}, \epsilon_{l-1} - \epsilon_l, \epsilon_{l-1} - \epsilon_l),
\epsilon_l.
\end{align}

According to this ordering we set
\begin{align}
e_{i-j} &:= [e_{i-j-1}, e_{i+1-j}]q^{-1}, \\
e_{j-i} &:= [e_{j-i-1}, e_{i+1-j}q]
\end{align}
(8.9) for $2 \leq i + 1 < j \leq l$,
\begin{align}
e_i &:= [e_{i-1}, e_i]q^{-1}, \\
e_{-i} &:= [e_{-1}, e_{-i}]q
\end{align}
(8.10) for $1 \leq i < l$,
\begin{align}
e_{i,l} &:= [e_i, e_l]q^{-1}, \\
e_{-l,-i} &:= [e_{-l}, e_{-i}]q
\end{align}
(8.11) for $1 \leq i < l$,
\begin{align}
e_{i,j} &:= [e_{i,j+1}, e_{j,-j-1}]q^{-1}, \\
e_{-j,-i} &:= [e_{j+1-j}, e_{j-1}-i]q
\end{align}
(8.12) for $1 \leq i < j < l$.

The extended Dynkin diagram of $B_l$ (or the Dynkin diagram of the corresponding affine Lie algebra $\mathfrak{so}_{2l+1}$) is presented by the picture:

Fig. 5. The Dynkin diagram of the Lie algebra $\mathfrak{so}_{2l+1}$.

Specializing the general formulas (6.3) to the case $\mathfrak{g} = \mathfrak{so}_{2l+1}$ with the vector
\begin{align}
\tilde{e}_{-\theta} &= q^{\epsilon_{1,-1} + \epsilon_{2,-2} e_{-2,-1}}.
\end{align}
(8.13)

after some calculations we obtain the following result.
Theorem 8.1. The Drinfeldian $D_{q\eta}(\mathfrak{so}_{2l+1})$ ($l \geq 3$) is generated (as a unital associative algebra over $\mathbb{C}[[\log q, \eta]]$) by the algebra $U_q(\mathfrak{so}_{2l+1})$ and the elements $\xi_{\delta-\theta}$, $q^{\pm \hat{c}} := q^{\pm h}$ with the relations:

\begin{align*}
(8.14) & \quad [q^{\pm \hat{c}}, \text{everything}] = 0 , \\
(8.15) & \quad q^{\pm e_{i,-1}} \xi_{\delta-\theta} = q^{\mp 1} \xi_{\delta-\theta} q^{\pm e_{i,-1}} \quad \text{for } i = 1, 2 , \\
(8.16) & \quad q^{\pm e_{i,-1}} \xi_{\delta-\theta} = \xi_{\delta-\theta} q^{\pm e_{i,-1}} \quad \text{for } i = 3, 4, \ldots , l ,
\end{align*}

\begin{align*}
(8.17) & \quad [e_{2,-1}, \xi_{\delta-\theta}] = \eta q^{e_{2,-1} + e_{2,-2}} \times \\
& \quad \times \left( \sum_{k=3}^{l} (-1)^k q^{k-3} e_{k,-1} e_{k,-1} + \frac{(-1)^{l-1} q^{l-1}}{q+1} e^{-1}_{-2} \right) , \\
(8.18) & \quad [e_{1,-2}, \xi_{\delta-\theta}] = \eta q^{2e_{2,-2}} \times \\
& \quad \times \left( \sum_{k=3}^{l} (-1)^k q^{k-3} e_{k,-2} e_{k,-2} + \frac{(-1)^{l-1} q^{l-1}}{q+1} e^{-2}_{-2} \right) , \\
(8.19) & \quad [e_{3,-2}, \xi_{\delta-\theta}] = 0 , \\
(8.20) & \quad [e_{2,-3}, [e_{2,-3}, \xi_{\delta-\theta}]]_q = 0 , \\
(8.21) & \quad [[e_{2,-3}, \xi_{\delta-\theta}]_q, \xi_{\delta-\theta}]_q = 0 , \\
(8.22) & \quad [e_{i,-j}, \xi_{\delta-\theta}] = 0 \quad (3 \leq i, j \leq l, \ |i-j| = 1) , \\
(8.23) & \quad [e_{\pm l}, \xi_{\delta-\theta}] = 0 .
\end{align*}

Since the explicit formulas of $\Delta_{q\eta}(\xi_{\delta-\theta})$ and $S_{q\eta}(\xi_{\delta-\theta})$ for the cases $B_l$, $C_l$ and $D_l$ are cumbersome they are not written here. Analogous formulas are also not given for the corresponding Yangians.

By setting $q = 1$ in (8.13–8.23) we obtain the defining relations of the Yangian $Y_\eta(\mathfrak{so}_{2l+1})$ in the Chevalley basis. This result is formulated as the theorem.

Theorem 8.2. The Yangian $Y_\eta(\mathfrak{so}_{2l+1})$ ($l \geq 3$) is generated (as a unital associative algebra over $\mathbb{C}[\eta]$) by the algebra $U(\mathfrak{so}_{2l+1})$ and the elements $\xi_{\delta-\theta}$, $\hat{c}$ with
the relations:

\begin{align*}
(8.24) & \quad [\hat{c}, \text{everything}] = 0, \\
(8.25) & \quad [e_{i,-i}, \xi_{\delta-\theta}] = \xi_{\delta-\theta} \quad \text{for } i = 1, 2, \\
(8.26) & \quad [e_{i,-i} \xi_{\delta-\theta}] = 0 \quad \text{for } i = 3, 4, \ldots, l, \\
(8.27) & \quad [e_{2,-1}, \xi_{\delta-\theta}] = \eta \left( \sum_{k=3}^{l} (-1)^k e_{-k,-1} e_{k,-1} + \frac{(1)^{l-1}}{2} \right) e_{-1}, \\
(8.28) & \quad [e_{1,-2}, \xi_{\delta-\theta}] = \eta \left( \sum_{k=3}^{l} (-1)^k e_{-k,-2} e_{k,-2} + \frac{(1)^{l-1}}{2} \right) e_{-2}, \\
(8.29) & \quad [e_{3,-2}, \xi_{\delta-\theta}] = 0, \\
(8.30) & \quad [e_{2,-3}, [e_{2,-3}, \xi_{\delta-\theta}]] = 0, \\
(8.31) & \quad [[e_{2,-3}, \xi_{\delta-\theta}], \xi_{\delta-\theta}] = 0, \\
(8.32) & \quad [e_{i,-j}, \xi_{\delta-\theta}] = 0 \quad (3 \leq i, j \leq l, |i - j| = 1), \\
(8.33) & \quad [e_{\pm l}, \xi_{\delta-\theta}] = 0.
\end{align*}

9. Drinfeldians and Yangians for Lie algebras $C_l$ ($l \geq 2$)

The Dynkin diagram of the simple Lie algebra $C_l \simeq \mathfrak{sp}_{2l} := \mathfrak{sp}(2l, \mathbb{C})$ is presented on the picture:

\begin{center}
\begin{tikzpicture}
  \node[draw, circle] (a1) at (0,0) {$\alpha_1$};
  \node[draw, circle] (a2) at (1,0) {$\alpha_2$};
  \node[draw, circle] (al1) at (4,0) {$\alpha_{l-1}$};
  \node[draw, circle] (al) at (5,0) {$\alpha_l$};
  \draw (a1) -- (a2);
  \draw (a2) -- (al1);
  \draw (al1) -- (al);
  \draw (al) node[right] {$\cdots$};
\end{tikzpicture}
\end{center}

Fig. 6. The Dynkin diagram of the Lie algebra $\mathfrak{sp}_{2l}$.

In the terms of the orthonormalized basis $e_i$ ($i = 1, 2, \ldots, l$) the root systems $\Pi$, $\Delta_+$ and $\Delta$ of $\mathfrak{sp}_{2l}$ are presented as follows:

\begin{align*}
(9.1) & \quad \Pi = \{ \alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{l-1} = e_{l-1} - e_l, \alpha_l = 2e_l \}, \\
(9.2) & \quad \Delta_+ = \{ \epsilon_i \pm \epsilon_j, 2\epsilon_k | 1 \leq i < j \leq l; k = 1, 2, \ldots, l \}, \\
(9.3) & \quad \Delta = \Delta_+ \cup (-\Delta_+) = \{ \pm \epsilon_i \pm \epsilon_j, \pm 2\epsilon_k | i \neq j; i, j, k = 1, 2, \ldots, l \}.
\end{align*}

The root $\theta := 2\alpha_1 + \ldots + 2\alpha_{l-1} + \alpha_l = 2\epsilon_1$ is maximal.

The quantum algebra $U_q(\mathfrak{sp}_{2l})$ is generated by the Chevalley $U_q(\mathfrak{gl}_l)$-elements $e_{i,-i-1} := e_{\epsilon_i-\epsilon_{i+1}}, e_{i+1,-i} := e_{\epsilon_i+1-\epsilon_i} (i = 1, 2, \ldots, l - 1), q^{\pm \epsilon_i} - 1 (i = 1, 2, \ldots, l)$ satisfying the relations (C4) and the additional elements $e_{l,l} := e_{2\epsilon_l}, e_{-l,-l} := e_{-2\epsilon_l}$ satisfying the relations (A4).
with the relations:
\[ q^{e_{i,-1}} e_{\pm l, \pm l} q^{-e_{i,-1}} = q^{\pm 2\delta_{i,l}} e_{\pm l, \pm l}, \]
\[ [e_{l,l}, e_{-l,-l}] = q^{2\delta_{l,l}} \frac{q^{-2\delta_{l,-l}}}{q - q^{-2}}, \]
\[ [e_{i,-j}, e_{l,l}] = 0 \quad \text{for} \quad j \neq l, \quad |i - j| = 1, \]
\[ [e_{i,-j}, e_{-l,-l}] = 0 \quad \text{for} \quad i \neq l, \quad |i - j| = 1, \]
\[ [[e_{i-1,-1}, e_{l,l}]_{q^2}, e_{l,l}]_q = 0, \]
\[ [[e_{l-1,-1}, e_{l,l}]_{q^2}, e_{l,l}]_q = 0, \]
\[ [e_{i-1,-1}, [e_{i-1,-1}, e_{l,l}]_{q^2}] = 0, \]
\[ [e_{l-1,-1}, [e_{l-1,-1}, e_{l,l}]_{q^2}] = 0. \]

The Hopf structure on \( U_q(\mathfrak{sp}_{2l}) \) is given by the formulas \([\ref{eq:hopf}] - [\ref{eq:hopf}]\) and also

\[ \Delta_q(e_{l,l}) = e_{l,l} \otimes 1 + q^{-2e_{l,-1}} \otimes e_{l,l}, \]
\[ \Delta_q(e_{-l,-l}) = e_{-l,-l} \otimes q^{2e_{l,-1}} + 1 \otimes e_{-l,-l}, \]
\[ S_q(e_{l,l}) = -q^{2e_{l,-1}} e_{l,l}, \quad S_q(e_{-l,-l}) = -e_{-l,-l} q^{-2e_{l,-1}}, \]
\[ e_q(e_{\pm l, \pm l}) = 0. \]

For construction of the composite root vectors \( e_{i,-j} \) (\( |i - j| > 1 \)), \( e_{i,j}, e_{-j,-i} \) \( (i \leq j) \) we fix the following normal ordering of the positive root system \( \Delta_+ \):

\[ (\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3, \ldots, \epsilon_1 - \epsilon_j, 2\epsilon_1, \epsilon_1 + \epsilon_j, \ldots, \epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_2), (\epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4, \ldots, 2\epsilon_2, \epsilon_2 + \epsilon_3, \ldots, \epsilon_2 + \epsilon_4, \ldots, \epsilon_2 + \epsilon_j), (\epsilon_i - \epsilon_1, \epsilon_i - \epsilon_2, \ldots, \epsilon_i - \epsilon_j), (\epsilon_j - \epsilon_1, 2\epsilon_j, \epsilon_j + \epsilon_1, \epsilon_j + \epsilon_2), 2\epsilon_j. \]

According to this ordering we set
\[ e_{i,-j} := [e_{i-1,-1}, e_{i+1,-j}]_{q^{-1}}, \quad e_{j,-i} := [e_{j-1,-1}, e_{j+1,-i}]_{q^{-1}} \]
for \( 2 \leq i + 1 < j \leq l \),
\[ e_{i,l} := [e_{i-1,-1}, e_{l,l}]_{q^{-1}}, \quad e_{-i,-l} := [e_{l-1,-1}, e_{l,l}]_{q^{-1}} \]
for \( 1 \leq i < l \),
\[ e_{i,i} := [e_{i-1,-1}, e_{i,l}]_{q^{-1}}, \quad e_{-i,-i} := [e_{l-1,-1}, e_{l,i}]_{q^{-1}} \]
for \( 1 \leq i < l \),
\[ e_{i,j} := [e_{i,j+1}, e_{j,-j-1}]_{q^{-1}}, \quad e_{j,-i} := [e_{j+1,-j}, e_{-j-1,-i}]_{q^{-1}} \]
for \( 1 \leq i < j < l \).

The extended Dynkin diagram of \( C_l \) (or the Dynkin diagram of the corresponding affine Lie algebra \( \tilde{\mathfrak{sp}}_{2l} \)) is presented by the picture:

\[ \begin{array}{c}
\delta - \theta \\
\alpha_1 \\
\alpha_2 \\
\cdots \\
\alpha_{l-1} \\
\alpha_l 
\end{array} \]

Fig. 7. The Dynkin diagram of the Lie algebra \( \tilde{\mathfrak{sp}}_{2l} \).
Specializing the general formulas and (6.5)–(6.8) to the case $g = \mathfrak{sp}_{2l}$ with the element
\begin{equation}
\hat{e}_{-\theta} = q^{2e_{\bar{i},-1}} e_{-1,-1},
\end{equation}
after some calculations we obtain the following result.

**Theorem 9.1.** The Drinfeldian $D_{q\theta}(\mathfrak{sp}_{2l})$ ($l \geq 2$) is generated (as a unital associative algebra over $\mathbb{C}[[\log q, \eta]]$) by the algebra $U_q(\mathfrak{sp}_{2l})$ and the elements $\xi_{\delta-\theta}$, $q^{\pm \hat{c}} := q^{\pm h_\delta}$ with the relations:
\begin{equation}
[q^{\pm \hat{c}}, \text{everything}] = 0,
\end{equation}
\begin{equation}
q^{\pm e_{\bar{i},-1}} \xi_{\delta-\theta} = q^{2} \xi_{\delta-\theta} q^{\pm e_{\bar{i},-1}},
\end{equation}
\begin{equation}
q^{\pm e_{\bar{i},-1}} \xi_{\delta-\theta} = \xi_{\delta-\theta} q^{\pm e_{\bar{i},-1}} (i = 2, 3, \ldots, l),
\end{equation}
\begin{equation}
[e_{\bar{2},-1}, \xi_{\delta-\theta}] = 0,
\end{equation}
\begin{equation}
[e_{\bar{i},-j}, \xi_{\delta-\theta}] = 0 \quad (2 \leq i, j \leq l, |i - j| = 1),
\end{equation}
\begin{equation}
[e_{-l,-1}-l, \xi_{\delta-\theta}] = \eta q^{2e_{\bar{i},-1}} e_{-l,-1},
\end{equation}
\begin{equation}
[e_{l,l}, \xi_{\delta-\theta}] = \eta [2] q^{2(e_{\bar{i},-1}+e_{\bar{i},-1})} e_{l,l}.
\end{equation}

By setting $q = 1$ in (9.15)–(9.26), we obtain the defining relations of the Yangian $Y_q(\mathfrak{sp}_{2l})$ in the Chevalley basis. This result can be formulated as the theorem.

**Theorem 9.2.** The Yangian $Y_q(\mathfrak{sp}_{2l})$ ($l \geq 2$) is generated (as a unital associative algebra over $\mathbb{C}[\eta]$) by the algebra $U(\mathfrak{sp}_{2l})$ and the elements $\xi_{\delta-\theta}$, $\hat{c}$ with the relations:
\begin{equation}
[\hat{c}, \text{everything}] = 0,
\end{equation}
\begin{equation}
[e_{\bar{i},-1}, \xi_{\delta-\theta}] = -2\xi_{\delta-\theta},
\end{equation}
\begin{equation}
[e_{\bar{i},-i}, \xi_{\delta-\theta}] = 0 \quad (i = 2, 3, \ldots, l),
\end{equation}
\begin{equation}
[e_{\bar{2},-1}, \xi_{\delta-\theta}] = 0,
\end{equation}
\begin{equation}
[e_{\bar{i},-2}, [e_{\bar{i},-2}, [e_{\bar{i},-2}, \xi_{\delta-\theta}]]] = 0,
\end{equation}
\begin{equation}
[[e_{\bar{i},-1}, \xi_{\delta-\theta}], \xi_{\delta-\theta}] = 0,
\end{equation}
\begin{equation}
[e_{\bar{i},-j}, \xi_{\delta-\theta}] = 0 \quad (2 \leq i, j \leq l, |i - j| = 1),
\end{equation}
\begin{equation}
[e_{-l,-1}-l, \xi_{\delta-\theta}] = \eta e_{-l,-1},
\end{equation}
\begin{equation}
[e_{l,l}, \xi_{\delta-\theta}] = 2\eta e_{l,l}.
\end{equation}

10. Drinfeldians and Yangians for Lie algebras $D_l$ ($l \geq 4$)

The Dynkin diagram of the simple Lie algebra $D_l \simeq \mathfrak{so}_{2l}$ ($l \geq 4$) is presented on the picture:
In the terms of the orthonormalized basis $\epsilon_i$ ($i = 1, 2, \ldots, l$) the root systems $\Pi$, $\Delta_+$ and $\Delta$ of $\mathfrak{so}_{2l}$ are presented as follows:

$$
(10.1) \quad \Pi = \{\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \epsilon_{l-2} - \epsilon_{l-1}, \alpha_{l-1} = \epsilon_{l-1} - \epsilon_l, \alpha_l = \epsilon_{l-1} + \epsilon_l\},
$$

$$
(10.2) \quad \Delta_+ = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq l\},
$$

$$
(10.3) \quad \Delta = \{\pm \epsilon_i \pm \epsilon_j \mid i \neq j; i, j = 1, 2, \ldots, l\}.
$$

The root $\theta := \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l = \epsilon_1 + \epsilon_2$ is maximal.

The quantum algebra $U_q(\mathfrak{so}_{2l})$ is generated by the Chevalley $U_q(\mathfrak{gl}_l)$-elements $e_i, i = 1, 2, \ldots, l$ with the relations (7.4) and the additional elements $e_{l-1,l} := \epsilon_{l-1} + \epsilon_l$, $e_{-l-1,-l} := \epsilon_{l-1} - \epsilon_l$ with the relations:

$$
(10.4) \quad q^{e_{i-1,i}}e_{l-1,l}q^{-e_{i-1,i}} = q^{\delta_{i-1,i-1}}e_{l-1,l},
$$

$$
q^{e_{i-1,i}}e_{-l-1,-l+1}q^{-e_{i-1,i}} = q^{\delta_{i-1,i-1}}e_{-l-1,-l+1},
$$

$$
[e_{l-1,l}, e_{-l-1,-l+1}] = \frac{q^{e_{l-1,-l+1}+e_{l-1,l}} - q^{-e_{l-1,-l+1}-e_{l-1,l}}}{q-q^{-1}},
$$

$$
[e_{l-1,l}, e_{l-1,l+1}] = 0 \quad (j \neq l-1, |i-j| = 1),
$$

$$
[e_{l-1,l}, e_{l-1,l+1}] = 0 \quad (i \neq l-1, |i-j| = 1),
$$

$$
[e_{l-1,l}, e_{l-1,l+1}] = 0.
$$

The Hopf structure on $U_q(\mathfrak{so}_{2l})$ is given by the formulas (10.4)–(10.7) and also

$$
(10.6) \quad \Delta_q(e_{l-1,l}) = e_{l-1,l} \otimes 1 + q^{e_{l-1,-l+1}+e_{l-1,l}} \otimes e_{l-1,l},
$$

$$
\Delta_q(e_{-l-1,-l+1}) = e_{-l-1,-l+1} \otimes 1 + q^{-e_{l-1,-l+1}-e_{l-1,l}} \otimes e_{-l-1,-l+1},
$$

$$
S_q(e_{l-1,l}) = -q^{e_{l-1,-l+1}+e_{l-1,l}} e_{l-1,l},
$$

$$
S_q(e_{-l-1,-l+1}) = -e_{-l-1,-l+1} q^{-e_{l-1,-l+1}-e_{l-1,l}},
$$

$$
(10.8) \quad e_q(e_{l-1,l}) = e_q(e_{l+1,-l}) = 0.
$$

For construction of the composite root vectors $e_{i,j}$ ($|i-j| = 1$), $e_{i,j}$, $e_{-j,-i}$ ($i < j$, $i \neq l-1$) we fix the following normal ordering of the positive root system.
According to this ordering we set

\[
\Delta_+ = (\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3, \ldots, \epsilon_1 - \epsilon_l),
\]

(10.9)

\[
\Delta_- = (\epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4, \ldots, \epsilon_2 - \epsilon_l, \epsilon_1 + \epsilon_l, \ldots, \epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_2),
\]

(10.10)

\[
(\epsilon_{l-1} - \epsilon_{l-2}, \epsilon_{l-2} - \epsilon_{l-3} + \epsilon_{l-3}, \epsilon_{l-3} + \epsilon_{l-4} + \epsilon_{l-4}).
\]

(10.11)

for 1 \leq l < j \leq l,

(10.12)

for 1 \leq i < j < l-2,

(10.13)

for 1 \leq i < j < l-2.

The Drinfeldian diagram of \(D_1\) (or the Dynkin diagram of the corresponding affine Lie algebra \(\hat{\mathfrak{so}}_{2l}\) \((l \geq 4)\)) is presented by the picture:

![Dynkin diagram of \(\hat{\mathfrak{so}}_{2l}\)](image)

Specializing the general formulas (6.5)–(6.8) to the case \(g = \mathfrak{so}_{2l}\) with the vector

\[
\tilde{c}_{\delta-\theta} = q^{\epsilon_1, -1 + \epsilon_2, -2} e_{-2, -1},
\]

(10.13)

after some calculations we obtain the following result.

**Theorem 10.1.** The Drinfeldian \(D_q(\mathfrak{so}_{2l})\) \((l \geq 4)\) is generated (as a unital associative algebra over \(\mathbb{C}[[\log q, \eta]]\)) by the algebra \(U_q(\mathfrak{so}_{2l})\) and the elements \(\xi_{\delta-\theta}\), \(q^{\pm \tilde{c}}\) with the relations:

\[
[q^{\pm \tilde{c}}, \text{everything}] = 0,
\]

(10.14)

\[
q^{\pm \epsilon_{i,-1}} \xi_{\delta-\theta} = q^{\mp 1} \xi_{\delta-\theta} q^{\pm \epsilon_{i,-1}} \quad \text{for } i = 1, 2,
\]

(10.15)

\[
q^{\pm \epsilon_{i,-1}} \xi_{\delta-\theta} = \xi_{\delta-\theta} q^{\pm \epsilon_{i,-1}} \quad \text{for } i = 3, 4, \ldots, l,
\]

(10.16)

\[
[e_{-2, -1}, \xi_{\delta-\theta}] = \eta q^{\epsilon_{1,-1} + \epsilon_{2,-2}} \sum_{k=3}^{l} (-1)^k q^{k-3} e_{-k,-1} e_{k,-1},
\]

(10.17)

\[
[e_{1,-2}, \xi_{\delta-\theta}] = \eta q^{2 \epsilon_{2,-2}} \sum_{k=3}^{l} (-1)^k q^{k-3} e_{-k,-2} e_{k,-2},
\]

(10.18)
By setting $q = 1$ in (10.14)–(10.24), we obtain the defining relations of the Yangian $Y_q(\mathfrak{so}_{2l})$ in the Chevalley basis. This result is formulated as the theorem.

**Theorem 10.2.** The Yangian $Y_q(\mathfrak{so}_{2l})$ ($l \geq 4$) is generated (as a unital associative algebra over $\mathbb{C}[\eta]$) by the algebra $U(\mathfrak{so}_{2l})$ and the elements $\xi_{\delta - \theta}, \hat{c}$ with the relations:

1. $[\hat{c}, \text{everything}] = 0$,
2. $[e_{i, -i}, \xi_{\delta - \theta}] = \xi_{\delta - \theta}$ for $i = 1, 2$,
3. $[e_{i, -i}, \xi_{\delta - \theta}] = 0$ for $i = 3, 4, \ldots, l$,
4. $[e_{2, -1}, \xi_{\delta - \theta}] = \eta \sum_{k=3}^{l} (-1)^k e_{-k, -1} e_{k, -1}$,
5. $[e_{1, -2}, \xi_{\delta - \theta}] = \eta \sum_{k=3}^{l} (-1)^k e_{-k, -2} e_{k, -2}$,
6. $[e_{3, -2}, \xi_{\delta - \theta}] = 0$,
7. $[e_{2, -3}, \xi_{\delta - \theta}] = 0$,
8. $[e_{2, -3}, \xi_{\delta - \theta}] = 0$,
9. $[e_{i, -j}, \xi_{\delta - \theta}] = 0$ (3 \leq i, j \leq l, |i - j| = 1),
10. $[e_{l-1, l}, \xi_{\delta - \theta}] = 0$,
11. $[e_{l-1, l+1}, \xi_{\delta - \theta}] = 0$.

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