On the parity conjecture in finite-slope families

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Abstract

We generalize to the finite-slope setting several techniques due to Nekovář concerning the parity conjecture for self-dual motives. In particular we show that, for a $p$-adic analytic family, with irreducible base, of symplectic self-dual global Galois representations whose $(\varphi, \Gamma)$-modules at places lying over $p$ satisfy a Panchishkin condition, the validity of the parity conjecture is constant among all specializations that are pure. As an application, we extend some other results of Nekovář for Hilbert modular forms from the ordinary case to the finite-slope case.

Contents

1 Introduction 2

2 Review of $(\varphi, \Gamma)$-modules 3

2.1 Definitions and generalities .............................................. 4

2.2 Numerology of de Rham $(\varphi, \Gamma)$-modules .......................... 5

2.3 The Panchishkin case ......................................................... 9

3 The parity conjecture in families 10

3.1 Local arguments: $\ell = p$ .................................................. 11

3.2 Rational functions on rigid analytic spaces .............................. 14

3.3 Local arguments: $\ell \neq p$ .................................................. 16

3.4 Global data ................................................................. 20

3.5 Global algebraic invariants .............................................. 22

3.6 Global analytic invariants .............................................. 24

3.7 Main result ................................................................. 25

4 Hilbert modular forms 26
1 Introduction

In a series of papers [19, 21, 23, 1, 24, 25], Nekovář introduced a number of useful techniques for treating the parity conjecture for self-dual motives, especially for Hilbert modular forms, at ordinary primes. This conjecture, which is a piece of the Bloch–Kato conjecture, compares the global $\varepsilon$-factor with the dimension modulo 2 of the Selmer group. The purpose of the present paper is to show how recent advances in $p$-adic Hodge theory, namely papers [26, 15] by the present authors and K.S. Kedlaya, can be used to extend these techniques to finite-slope cases.

In [21] in particular, it was shown that the validity of the parity conjecture is constant along certain one-dimensional formal families that are ordinary in the sense of Panchishkin. Our main result extends this conclusion to allow a general rigid analytic space as the base of the family, and weakens the condition at $p$ to require that the motive is only Panchishkin on the level of $(\varphi, \Gamma)$-modules. See Notation 2.3.1 for the definition of the latter condition; in the case of modular forms, it is tantamount to being finite-slope (up to twist), in comparison to ordinary (up to twist). We state our main result more precisely as follows (cf. Theorem 3.7.1).

**Theorem A.** Let $X$ be an irreducible reduced rigid analytic space over $\mathbb{Q}_p$, and $\mathcal{F}$ a locally free coherent $\mathcal{O}_X$-module equipped with a continuous, $\mathcal{O}_X$-linear action of $G_{F,S}$, and a skew-symmetric isomorphism $j: \mathcal{F} \to \mathcal{F}^*(1)$. Assume given, for each $v \in S_p$, a short exact sequence

$$0 \to \mathcal{S}_v^+ \to \mathcal{D}_{\text{rig}}(\mathcal{F}|_{G_{Fv}}) \to \mathcal{D}_v^- \to 0$$

of $(\varphi, \Gamma)$-modules over $\mathcal{R}_{\mathcal{O}_X}(\pi_{Fv})$, with $\mathcal{S}_v^+$ Lagrangian with respect to $j$.

For a (closed) point $P \in X$, we put $V = \mathcal{F} \otimes_{\mathcal{O}_X} \kappa(P)$ and $S_v = \mathcal{S}_v \otimes_{\mathcal{O}_X} \kappa(P)$. Let $X_{\text{alg}}$ be the set of points $P \in X$ such that (1) for all $v \in S_p$, the sequence $S_v$ makes $V|_{G_{Fv}}$ Panchishkin (on the level of $(\varphi, \Gamma)$-modules), and (2) for all $v \in S\backslash S_\infty$ where $\mathcal{F}|_{G_{Fv}}$ is ramified, the associated Weil–Deligne representation $WD(V|_{G_{Fv}})$ is pure.

Then the validity of the parity conjecture for $V$, relating the sign of its global $\varepsilon$-factor to the parity of the dimension of its Bloch–Kato Selmer group, is independent of $P \in X_{\text{alg}}$.

The reader should compare the above statement to [21, (5.3.1)]. We remark that the objects in the statement are well-defined even without the knowledge of conjectures about motivic $L$-functions, such as their analytic continuation and functional equation.

Using this result, some results of Nekovář for Hilbert modular forms generalize readily from the ordinary case to the finite-slope case. The following result gives an example (cf. Theorems 4.2 and 4.3).

**Theorem B.** Let $p \neq 2$ and $F$ a totally real number field. Let $f$ be a Hilbert modular form for $F$ of parallel weight two with trivial central character and finite-slope at $p$. If $[F : \mathbb{Q}]$
is even and $f$ is principal series at all finite places, then additionally assume the condition (A2) of §4 and that $\varepsilon(f) = +1$. Then the parity conjecture holds for $f$.

The following result follows immediately from combining the two theorems above, using the triangulation results of \[15\] to obtain the necessary families $\mathcal{F}_v$ of Panchishkin exact sequences over the relevant eigenvarieties.

**Theorem C.** Let $p \neq 2$, $F$ a totally real number field with $[F : Q]$ odd, and $X$ an irreducible component of the eigenvariety of finite-slope overconvergent $p$-adic Hilbert modular cusp forms for $F$. If $X$ contains a classical, parallel weight two point, then the parity conjecture holds for all classical points of $X$.

At least when $[F : Q]$ is odd, the preceding result should apply to every irreducible component $X$ of the eigenvariety, according to the following folklore conjecture on the global geometry of eigenvarieties, which would generalize results of Buzzard–Kilford \[5\] and Roe \[27\] when $F = Q$ and $p = 2$ and 3, respectively.

**Conjecture.** Let $F$ be a totally real number field, and $X$ an irreducible component of the eigenvariety of finite-slope overconvergent $p$-adic Hilbert modular cusp forms for $F$. For every $\epsilon > 0$ there exists an affinoid subdomain $U_\epsilon$ of weight space $W$ such that the restriction $X \times_W (W \setminus U_\epsilon)$ has an irreducible component $C$ with all slopes strictly less than $\epsilon$.

In particular, for $\epsilon$ sufficiently small depending on $F$, any parallel weight two point of slope strictly less than $\epsilon$ is classical, at least when $X$ is the image under the Jacquet–Langlands correspondence of an irreducible component of the eigenvariety for a definite quaternion algebra in \[4\]. Thus the subspace $C$ of the conjecture forces $X$ to have classical, parallel weight two points.

Here is a roadmap of this paper. In §2 we review the $p$-adic Hodge theory and Galois cohomology of $(\varphi, \Gamma)$-modules. All of the results therein are known to the experts, but appear in no single place in the literature. In §3 we prove Theorem A. While our strategy roughly follows that of Nekovář, our generality forces us to replace certain ingredients of his proof with more technical arguments. In §4 we explain the applications to Hilbert modular forms, and in particular prove Theorem B.

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**2 Review of $(\varphi, \Gamma)$-modules**

In this section, we recall the facts we will need about $(\varphi, \Gamma)$-modules over the Robba ring. The first subsection treats definitions of the objects themselves, the classification of rank
one objects, and the relationship to Galois representations. In the next subsection we collect in one place the fundamental invariants coming from Galois cohomology and $p$-adic Hodge theory, and the interrelations among them. The Panchishkin condition is introduced in the final subsection.

### 2.1 Definitions and generalities

Let $p$ be a prime number, and let $K/Q_p$ be a finite extension with ring of integers $\mathcal{O}_K$ and residue field $k$ of cardinality $q = p^h$. Fix an algebraic closure $K^{alg}$ of $K$ and set $G = \text{Gal}(K^{alg}/K)$ with inertia subgroup $I \subset G$. Write $K_\infty = K(\mu_q^\infty)$, $H = \text{Gal}(K^{alg}/K_\infty)$, and $\Gamma = \text{Gal}(K_\infty/K)$; the $p$-adic cyclotomic character $\chi$ on $G$ has kernel $H$ and maps $G/H = \Gamma$ isomorphically onto an open subgroup of $\mathbb{Z}_p^\times$. (If the dependence on $K$ is relevant, we use a subscript, e.g. $\Gamma_K$.) Let $k'$ be the residue field of $K_\infty$, which is finite, and set $K_0 = W(k)[1/p]$ and $K_0' = W(k')[1/p]$. Write $e'_K = [K_\infty : Q_p^\times]/[k' : k]$ for the ramification degree of $K_\infty/Q_p$. Let $K'_\omega = Q_p^\times$ be the maximal unramified extension of $K_0$.

Let $\pi_K$ be an indeterminate, and for $r > 0$ put

$$\mathcal{R}^r = \mathcal{R}^r(\pi_K) = \{ f(\pi_K) = \sum_{n \in \mathbb{Z}} a_n \pi_K^n \mid a_n \in K_0' \text{ convergent for } p^{-r/(p-1)} \leq |\pi_K| < 1 \}.$$ 

Let $\mathcal{R} = \mathcal{R}(\pi_K) = \bigcup_{r \to 0^+} \mathcal{R}^r(\pi_K)$ be the Robba ring. The theory of the field of norms takes as input a bit of choice involving the meaning of $\pi_K$, and outputs the following variety of structures. One has a constant $C(\pi_K) > 0$ and, for rational numbers $r \in (0, C(\pi_K))$ (which we henceforth tacitly assume when we refer to $r$), a finite flat degree $p$ algebra map $\varphi : \mathcal{R}^r \to \mathcal{R}^r/p$ and a continuous ring-theoretic action of $\Gamma$ on $\mathcal{R}^r$ that commute with one another, compatible with the inclusions for varying $r$. The actions of $\varphi$ and $\Gamma$ on the coefficients $K_0'$ are the arithmetic Frobenius action and the natural one, respectively. For a finite extension $K'/K$ inside $K^{alg}$ and $r < \min\{C(\pi_K), C(\pi_{K'}) e'_K/e'_K \}$, one has canonical inclusions $\mathcal{R}^r(\pi_K) \hookrightarrow \mathcal{R}^r(\pi_{K'})$ that are finite étale of degree $[K'_\omega : K_\infty]$ and compatible with the actions of $\varphi$ and $\Gamma_{K'} \subseteq \Gamma_K$.

In the case $K = Q_p$, each choice of a basis of $\mathbb{Z}_p(1)$ gives rise to a distinguished choice for the meaning of $\pi_Q$, the result of which we call $\pi$. It satisfies $\varphi(\pi) = (1 + \pi)^p - 1$ and $\gamma(\pi) = (1 + \pi)^\chi(\gamma) - 1$. Write $t = \log(1 + \pi) \in \mathcal{R}^r(\pi) \subseteq \mathcal{R}^r(\pi_{K_\infty})$ for Fontaine’s element. One has a family of $\Gamma$-equivariant homomorphisms $t_n = t_n : \mathcal{R}^r \to K_\infty[t]$, indexed by positive integers $n$ for which $1/p^n - 1 \leq r e'_K$, such that $t_{n+1}^{r/p} \circ \varphi = t_n^{r/p}$.

Fix another finite extension $L/Q_p$ as the coefficient field. For $? = \emptyset, r$, write $\mathcal{R}^r_L = \mathcal{R}^r(\pi_K) \otimes_{Q_p} L$, and extend the actions of $\varphi$ and $\Gamma$ to $\mathcal{R}_L$ by $L$-linearity. Note that $\mathcal{R}_L$ is a product of Bézout domains, so a finite projective $\mathcal{R}_L$-module of constant rank is free.

A $\varphi$-module is by definition a finite free $\mathcal{R}_L$-module $D$ equipped with a semilinear action of $\varphi$ such that $\varphi(D) \otimes_{\varphi(\mathcal{R}_L)} \mathcal{R}_L \cong D$. We write $d = d(D) = \text{rank}_{\mathcal{R}_L} D$ (not to be confused with the integer $d_L(D)$ appearing in Section 3 below). Note that there always exists $r$ such that $D$ admits a model $D^r$ over $\mathcal{R}_L^r$, i.e. a finite free $\mathcal{R}_L^r$-submodule $D^r \subset D$ such that,
writing $D^s = D^r \otimes_{\mathbb{R}^s} \mathbb{R}^s \subseteq D$ for $s \in (0, r)$ or $s = \emptyset$, then the inclusion of $\varphi(D^r)$ into $D$ induces an isomorphism $\varphi(D^r) \otimes_{\varphi(\mathbb{R}^r)} \mathbb{R}_{L/p}^{r/p} \cong D^{r/p}$. For $s$ sufficiently small, such $D^s$ is uniquely determined. A $(\varphi, \Gamma)$-module is by definition a $\varphi$-module $D$ equipped with a continuous action of $\Gamma$ commuting with $\varphi$. Since $D^s$ is uniquely determined for small $s$, it is stable by $\Gamma$. If $K'/K$ is a finite extension, we write $D|_{K'} = D \otimes_{\mathbb{R}_L(\pi_{K'})} \mathbb{R}_{L(\pi_{K'})}$ for the resulting $(\varphi, \Gamma_{K'})$-module over $\mathbb{R}_{L(\pi_{K'})}$. Conversely, if $D'$ is a $(\varphi, \Gamma_{K'})$-module over $\mathbb{R}_{L(\pi_{K'})}$, then we define $\text{Ind}_{K'}^K D' = \text{Ind}_{\Gamma_{K'}}^\Gamma D'$, considered as a $(\varphi, \Gamma_K)$-module over $\mathbb{R}_{L(\pi_K)}$.

To each continuous character $\delta: K^\times \to L^\times$, there is associated a rank one $(\varphi, \Gamma)$-module $\mathbb{R}_L(\delta)$. When $K = \mathbb{Q}_p$, one has $\mathbb{R}_L(\delta) = \mathbb{R}_L e_\delta$, where $\varphi(e_\delta) = \delta(p)e_\delta$ and $\gamma(e_\delta) = \delta(\chi(\gamma))e_\delta$ for $\gamma \in \Gamma$; for general $K$ we refer to [15] Construction 6.2.4] for the construction. Every rank one object $D$ isomorphic to $\mathbb{R}_L(\delta_D)$ for a uniquely determined character $\delta_D$. (In this generality, the fact was originally shown in [13] Theorem 1.45] using another language; see [13], Lemma 6.2.13] for a proof in this language.) For an object $D$ of any rank, we write $\det(D)$ for the character of $K^\times$ corresponding to the top exterior power of $D$.

There is a slope theory for $\varphi$-modules [13]. Namely, any $\varphi$-module $D$ is uniquely filtered by a finite list of saturated subobjects such that each graded piece is pure of some $\varphi$-slope $\lambda \in \mathbb{Q}$, and the $\lambda$ are increasing. Although the definition of pureness is somewhat technical, morally it means that $\varphi$ has eigenvalues over a sufficiently large over-ring of $\mathbb{R}_L$ of valuation $\lambda$. We call $D$ étale if is pure of slope 0. When $D$ is a $(\varphi, \Gamma)$-module, the slope filtration of its underlying $\varphi$-module is, by uniqueness, stable under $\Gamma$; we say $D$ is étale if its underlying $\varphi$-module is. Combining work of Fontaine [11], Cherbonnier–Colmez [7], and Kedlaya [14], one obtains a fully faithful, exact $\otimes$-functor $D_{\text{rig}}$ from the category of continuous representations of $G$ over $L$ to the category of $(\varphi, \Gamma)$-modules over $\mathbb{R}_L$, having for essential image the étale objects, and compatible with Galois cohomology and $p$-adic Hodge theoretic invariants to be introduced in the next subsection. In the case of rank one, the functor is simple to describe: a rank one $p$-adic Galois representation is just a character $\delta: G \to L^\times$, and $D_{\text{rig}}(\delta) = \mathbb{R}_L(\delta \circ \text{Art}_K)$, where $\text{Art}_K: K^\times \to G^{\text{ab}}$ is the local Artin map (normalized to take a uniformizer onto a lift of geometric Frobenius).

We use a superscript $^*$ to denote internal Hom into the unit object. For any $(\varphi, \Gamma)$-module $D$, we write $D(n) = D \otimes_{\mathbb{R}^n} D_{\text{rig}}(\mathbb{Q}_p(1))$ for $n \in \mathbb{Z}$.

### 2.2 Numerology of de Rham $(\varphi, \Gamma)$-modules

Writing $\Delta \subseteq \Gamma$ for the $p$-torsion subgroup (which is trivial if $p \neq 2$) and fixing a choice of $\gamma \in \Gamma$ mapping to a topological generator in $\Gamma/\Delta$, for any object $X$ with commuting actions of $\varphi$ and $\Gamma$ (resp. an action of $\Gamma$), one defines $H^*_{\varphi, \gamma}(X)$ (resp. $H^*_{\gamma}(X)$) to be the cohomology of the complex concentrated in degrees $[0, 2]$ given by

$$
C^*_{\varphi, \gamma}(X) = [X^\Delta \xrightarrow{\varphi^{-1} \gamma^{-1}} X^\Delta \oplus X^\Delta \xrightarrow{\gamma^{-1} - \varphi} X^\Delta]
$$

(resp. in degrees $[0, 1]$ given by $C^*_{\gamma}(X) = [X^\Delta \xrightarrow{\gamma^{-1}} X^\Delta]$). One has a canonical identification $H^0_{\gamma}(X) \cong X^\Gamma$, and an identification $H^1_{\gamma}(X) \approx X^\Gamma$ determined up to a constant $\mathbb{Z}_p$-multiple (which we fix). When $X = D$ or $D[1/t]$, where $D$ is a $(\varphi, \Gamma)$-module, one calls $H^*_{\varphi, \gamma}(X)$
Lemma 2.2.1. Let $0 \to D_1 \to D_2 \to D_3 \to 0$ be a short exact sequence of de Rham $(\varphi, \Gamma)$-modules. Then for $? = \text{st}, \text{pst}$ the sequence $0 \to \mathbf{D}_{\varphi}(D_1) \to \mathbf{D}_{\varphi}(D_2) \to \mathbf{D}_{\varphi}(D_3) \to 0$ is exact.
Proof. The claim for \( ? = \text{pst} \) follows from the \( p \)-adic monodromy theorem, and one deduces from this the claim for \( ? = \text{st} \) because Hilbert’s Theorem 90 shows that taking \( G \)-invariants is exact in this case. \( \square \)

We will have need for the following \( \Gamma \)-equivariant version of elementary divisors over \( K_\infty [t] \), whose proof we leave as an exercise.

**Lemma 2.2.2.** Suppose \( V \) is a (nonzero) finite-dimensional \( K \)-vector space, considered with trivial \( \Gamma \)-action, and let \( \mathcal{L} = V \otimes_K K_\infty [t] \) and \( \mathcal{W} = \mathcal{L}[1/t] \) have the obvious semilinear actions of \( \Gamma \). Let \( \mathcal{L}' \) be another \( \Gamma \)-stable \( K_\infty [t] \)-lattice in \( \mathcal{W} \). Then there exist a \( K \)-basis \( \{ v_i \} \) of \( V \) (hence also a \( K_\infty [t] \)-basis of \( \mathcal{L} \)) and integers \( a_i \) such that \( \mathcal{L}' \) is the \( K_\infty [t] \)-span of the \( t^{a_i} v_i \).

For the rest of this section, assume that \( D \) is de Rham, so that by definition

\[
h^0_\gamma (D^\text{diff}_+ (D)[1/t]) = [K : Q_p] d = d_- + d_+,
\]

where \( d_- = d_- (D) \) (resp. \( d_+ = d_+ (D) \)) are the number of negative (resp. nonnegative) Hodge–Tate weights of \( D \). (The multiplicity \( d_- \), which is only used in this Section [2] should not be confused with the integer \( d^- \) appearing in Section [3].) The \( \Gamma \)-equivariant elementary divisor theory, applied to \( V = D^\text{diff}_0 (D) \) and \( \mathcal{L}' = D^\text{diff}_+ (D) \) (forgetting the structure of the coefficient field \( L \)), shows that \( D^\text{diff}_+ (D) \) is isomorphic, as a \( K_\infty [t] \)-module with \( \Gamma \)-action, to \( \bigoplus_{i=1}^{[L : Q_p]} t^{a_i} K_\infty [t] \) for integers \( a_i \). Each factor \( t^{a_i} K_\infty [t] \) for \( a_i \) positive (resp. \( a_i \) nonpositive) contributes 1/\( [L : Q_p] \) to the multiplicity of the negative (resp. nonnegative) Hodge–Tate weight \( -a_i \) on the one hand, and contributes 0 (resp. 1/\( [L : Q_p] \)) to \( h^0_\gamma (D^\text{diff}_+ (D)) \) for \( i = 0, 1 \) on the other hand. From this, one deduces easily that, for \( i = 0, 1 \), one has \( h^0_\gamma (D^\text{diff}_+ (D)[1/t]) = [K : Q_p] d, \ h^1_\gamma (D^\text{diff}_+ (D)) = d_+ \), and the natural map \( H^i_\gamma (D^\text{diff}_+ (D)) \to H^i_\gamma (D^\text{diff}_+ (D)[1/t]) \) is injective. Connecting homomorphisms \( \delta_i^D \) for \( i = 0, 1 \), are constructed in [17, Theorem 2.8] so that the sequence

\[
0 \longrightarrow H^0_{\varphi, \gamma} (D) \longrightarrow H^0_{\varphi, \gamma} (D[1/t]) \oplus H^0_{\gamma} (D^\text{diff}_+ (D)) \longrightarrow H^0_{\gamma} (D^\text{diff}_+ (D)[1/t]) \\
\delta^D_0 \longrightarrow H^1_{\varphi, \gamma} (D) \longrightarrow H^1_{\varphi, \gamma} (D[1/t]) \oplus H^1_{\gamma} (D^\text{diff}_+ (D)) \longrightarrow H^1_{\gamma} (D^\text{diff}_+ (D)[1/t]) \\
\delta^D_0 \longrightarrow H^2_{\varphi, \gamma} (D) \longrightarrow 0
\]

is exact, where “\( \longrightarrow \)” denotes the difference of the natural maps. One should view (2.2.1) as a generalization of the fundamental exact sequence in \( p \)-adic Hodge theory.

One defines subspaces \( H^i_\gamma (D) \) of \( H^1 (D) \), for \( ? = e, f, \text{st}, g+, g \), by the formulas

\[
H^i_e (D) = \ker (H^1_{\varphi, \gamma} (D) \to H^1_{\varphi, \gamma} (D[1/t])), \\
H^i_f (D) = \ker (H^1_{\varphi, \gamma} (D) \to H^1_{\gamma} (D[1/t])), \\
H^i_{\text{st}} (D) = \ker (H^1_{\varphi, \gamma} (D) \to H^1_{\gamma} (D[1/t, \log \pi])), \\
H^i_{g+} (D) = \ker (H^1_{\varphi, \gamma} (D) \to H^1_{\gamma} (D^\text{diff}_+ (D))), \\
H^i_g (D) = \ker (H^1_{\varphi, \gamma} (D) \to H^1_{\gamma} (D^\text{diff}_+ (D)[1/t])).
\]
One may rephrase these definitions as follows. Define $K_{e} = \mathbb{Q}_{p}$, $K_{f} = K_{st} = K_{0}$, and $K_{g+} = K_{g} = K$, and define $D_{e}(D) = H^{0}(D[1/t])$, $D_{f} = D_{\text{crys}}$, $D_{st}$ as given above, $D_{g+} = D_{\text{dR}}$, and $D_{g} = D_{\text{dR}}$, so that $D_{f}(D)$ is a $(K_{f} \otimes \mathbb{Q}_{p}, \mathbb{L})$-module for each of $e, f, st, g+, g$. If the class $c \in H^{1}(D)$ corresponds to the extension of $(\varphi, \Gamma)$-modu-les $0 \rightarrow D \rightarrow E_{c} \rightarrow \mathbb{R} \rightarrow 0$, then one has $c \in H^{1}_{st}(D)$ if and only if $\text{rank}_{K_{f} \otimes \mathbb{Q}_{p}, \mathbb{L}} D_{f}(D) + 1$ (the ranks considered as locally constant functions on $\text{Spec}(K_{f} \otimes \mathbb{Q}_{p}, L)$).

There are obvious inclusions $H^{1}_{e}(D) \subseteq H^{1}_{f}(D) \subseteq H^{1}_{st}(D) \subseteq H^{1}_{g}(D)$ and $H^{1}_{g+}(D) \subseteq H^{1}_{g}(D)$. Two of these inclusions are in fact equalities: applying Lemma 2.2.1 to the reformulated definitions shows that $H^{1}_{st}(D) = H^{1}_{g}(D)$, and because $H^{1}_{f}(D_{\text{dif}}(D)) \subseteq H^{1}_{f}(D_{\text{dif}}(D)[1/t])$ one has $H^{1}_{g+}(D) = H^{1}_{g}(D)$. The injectivity of $H^{1}_{\gamma}(D_{\text{dif}}(D))[1/t]) \rightarrow H^{1}_{\gamma}(D_{\text{dif}}(D)[1/t])$ implies that $\text{img} \delta^{0}_{D} = H^{1}_{\gamma}(D)$, whence $h^{1}_{\gamma}(D) = d_{-} + h^{0}(D) - h^{0}(D[1/t])$, and it is shown in [2, Corollary 1.4.5] that $h^{1}_{\gamma}(D) = d_{-} + h^{0}(D)$. It is also shown in [2, Corollary 1.4.10] that $H^{1}_{f}(D)$ and $H^{1}_{f}(D^{*}(1))$ are orthogonal complements under Tate duality, and similarly the commutative diagram of perfect pairings in [17, Lemma 2.15(1)], namely

\[
H^{0}_{\gamma}(D_{\text{dif}}(D)[1/t]) \times H^{1}_{\gamma}(D_{\text{dif}}(D^{*}(1))[1/t]) \xrightarrow{\delta^{0}_{D}} K \otimes_{\mathbb{Q}_{p}} \mathbb{L}
\]

implies that $\text{img} \delta^{0}_{D} = H^{1}_{\gamma}(D)$ is the orthogonal complement of $\text{ker}(i) = H^{1}_{g}(D^{*}(1))$. It follows that $h^{1}_{\gamma}(D) = h^{1}_{f}(D^{*}(1))$ and $h^{1}_{g}(D) = h^{1}_{e}(D^{*}(1))$, where we write $h^{1}_{\gamma}(D) = h^{1}(D) - h^{1}_{e}(D) = \dim_{L} H^{1}(D)/H^{1}_{g}(D)$ for $? = e, f, g$. We summarize the dimensions as follows.

**Formulary 2.2.3.**

\[
\begin{align*}
    h^{0}(D) &= h^{2}(D^{*}(1)) = \dim_{L} \text{Fil}^{0} D_{\text{crys}}(D)_{\varphi = 1}, \\
    h^{1}(D) &= h^{1}(D^{*}(1)) = [K : \mathbb{Q}_{p}]d + h^{0}(D) + h^{0}(D^{*}(1)), \\
    h^{0}(D[1/t]) &= \dim_{L} D_{\text{crys}}(D)_{\varphi = 1}, \\
    h^{1}(D[1/t]) &= [K : \mathbb{Q}_{p}]d + h^{0}(D[1/t]), \\
    h^{2}(D[1/t]) &= 0, \\
    h^{1}_{e}(D) &= h^{1}_{g}(D^{*}(1)) = d_{-} + h^{0}(D) - h^{0}(D[1/t]), \\
    h^{1}_{f}(D) &= h^{1}_{f}(D^{*}(1)) = d_{-} + h^{0}(D), \\
    h^{1}_{g}(D) &= h^{1}_{g+}(D) = h^{1}_{st}(D) = d_{-} + h^{0}(D) + h^{0}(D^{*}(1)[1/t]).
\end{align*}
\]

For completeness, we describe the graded pieces of the filtration

\[
0 \subseteq H^{1}_{e}(D) \subseteq H^{1}_{f}(D) \subseteq H^{1}_{g}(D) \subseteq H^{1}(D).
\]

From the exact sequence (2.2.1) one sees immediately that

\[
H^{1}_{e}(D) = \text{img} \delta^{0}_{D} \cong D_{\text{dR}}(D)/(D_{\text{dR}}(D) + D_{\text{crys}}(D)_{\varphi = 1}).
\]
Hence, also, $H^1(D)/H^1_{\text{crys}}(D)$ is dual to $D_{\text{dif}}(D^*/(1)) / (D_{\text{dif}}^+(D^*/(1))+D_{\text{crys}}(D^*/(1))_{\varphi=1})$. Similarly, it follows from the short exact sequence

$$0 \to H^0_\gamma(D[1/t]/(\varphi - 1) \to H^1_{\varphi,\gamma}(D[1/t]) \to H^1_{\gamma}(D[1/t])_{\varphi=1} \to 0$$

that $H^1_{\gamma}(D)/H^1_{\text{crys}}(D) \hookrightarrow H^0_{\gamma}(D[1/t]/(\varphi-1)) = D_{\text{crys}}(D)/(\varphi-1)$, and comparing the dimension on the right with Formulary [2.2.3] we see that this inclusion is an isomorphism. Hence, also, $H^1_{\gamma}(D)/H^1_{\text{crys}}(D)_{\varphi=1}$ is dual to $D_{\text{crys}}(D^*/(1))/(\varphi-1)$.

**Remark 2.2.4.** All numerical invariants of $D$ defined in this section have been arranged to be invariant under replacing $L$ by a finite extension $L'$, and $D$ by the $(\varphi, \Gamma)$-module over $\mathcal{R}_{L'}$ given by $D \otimes_L L' = D \otimes_{\mathcal{R}_L} \mathcal{R}_{L'}$. Therefore, in arguments involving these constants we may enlarge $L$ in this manner without any harm.

### 2.3 The Panchishkin case

**Notation 2.3.1.** We say a $(\varphi, \Gamma)$-module $D$ is *Panchishkin* if it sits in a short exact sequence $S : 0 \to D^+ \to D \to D^- \to 0$ of $(\varphi, \Gamma)$-modules with each $D^\pm$ de Rham with $d_\pm(D^\pm) = [K : \mathbb{Q}_p] d(D^\pm)$, i.e. $D^+$ (resp. $D^-$) only has negative (resp. nonnegative) Hodge–Tate weights.

We assume that $D$ is Panchishkin for the rest of this section. We warn the reader that unlike in the case of Galois representations, the sequence $S$ is not necessarily uniquely determined by $D$. Using $\Gamma$-equivariant elementary divisors, Lemmas [2.2.2, 2.2.1] the Panchishkin condition implies that $D$ is de Rham, and $d_\pm = d_\pm(D) = [K : \mathbb{Q}_p] d(D^\pm)$. (In fact $D$ is semistable if both $D^\pm$ are, by Lemma [2.2.1].) One also deduces from $\Gamma$-equivariant elementary divisors that the extension of $K_\infty[[t]]$-modules with $\Gamma$-actions

$$0 \to D_{\text{dif}}^+(D^+) \to D_{\text{dif}}^+(D) \to D_{\text{dif}}^+(D^-) \to 0$$

is split, that for $i = 0, 1$ one has $H^i_{\gamma}(D_{\text{dif}}^+(D^+)) = 0$ so $H^i_{\gamma}(D_{\text{dif}}^+(D)) \cong H^i_{\gamma}(D_{\text{dif}}^+(D^-))$, and that $h^i_{\gamma}(D_{\text{dif}}^-(D^-)) = d_\gamma$. In particular, $H^1_{\gamma}(D^+) = H^1(D^+)$. Letting $\delta^g_S : H^i(D^-) \to H^{i+1}(D^+)$ denote the usual connecting map for the long exact cohomology sequence for $S$, a simple diagram chase shows that the sequence

$$0 \to H^0(D^+) \to H^0(D) \to H^0(D^-) \xrightarrow{\delta^g_S} H^1_{\gamma}(D^+) \to H^1_{\gamma}(D) \to H^1_{\gamma}(D^-) \to X \to 0, \quad (2.3.1)$$

is exact, where $X$ satisfies

$$\dim_L X = h^0((D^+)^*/(1)[1/t]) + h^0((D^-)^*/(1)[1/t]) - h^0(D^*/(1)[1/t]).$$

**Lemma 2.3.2.** Let $D$ be Panchishkin, and assume that $D_{\text{crys}}(D)_{\varphi=1} = D_{\text{crys}}(D^*/(1))_{\varphi=1} = 0$. Then $\delta^g_S$ and the functorial map $H^1(D^+) \to H^1(D)$ fit into a short exact sequence

$$0 \to H^0(D^-) \xrightarrow{\delta^g_S} H^1(D^+) \to H^1_{\gamma}(D) \to 0.$$
Proof. In fact, we already have $H^1_1(D^+) = H^1(D^+)$, and Formulary \[2.2.3\] gives $H^0(D) = H^0(D^+)$ = 0 and $H^1_1(D) = H^1_1(D^+)$, so it suffices to check the surjectivity on the right. In light of (2.3.1), the desired cokernel has dimension $\dim H^1_1(D^+) - \dim_L X$. Noting that $d_-(D^-) = 0$ and our hypothesis implies $h^0(D^*[1/t]) = 0$, Formulary \[2.2.3\] shows that

$$\dim H^1_1(D^-) - \dim_L X = h^0(D^-) - h^0((D^+)^*[1/t]).$$

Since this dimension is a nonnegative, it gives the second inequality in

$$h^0(D^-[1/t]) \geq h^0(D^-) \geq h^0((D^+)^*[1/t]).$$

Since our hypotheses are symmetric under replacing $D$ (resp. $D^\pm$, $S$) with $D^*[1]$ (resp. $(D^\pm)^*[1]$), the reverse inequality also holds, and in fact there is equality throughout. The claim follows.

3 The parity conjecture in families

In this section we show how to generalize \[21\] (always tacitly taking into account \[22\]) to our setting. Our exposition will try to be self-contained and at the same time include a complete discussion of the compatibility with that paper.

In the first subsection we treat those purely local facts taking place at primes over $p$, essentially generalizing \[21, \S3\]. The second subsection, which is preparatory, gives definition and basic facts concerning rational functions on rigid analytic spaces. In the third subsection, we treat those purely local facts taking place at primes not over $p$. In the final subsection we treat the purely global facts, essentially generalizing \[21, \S\S4–5\].

In the first and third subsections, we assume the reader is familiar with Weil–Deligne representations and local $\varepsilon$-factors as in \[21, \S\S1–2\], whose notations and claims we use without modification. Namely, $K/\mathbb{Q}_\ell$ denotes a finite extension with residue field $\mathbb{k}$ of cardinality $q = \ell^h$. We write $G_K$ (resp. $I_K$, $W_K$) for its absolute Galois group (resp. inertia group, Weil group), and $\nu : W_K/I_K \xrightarrow{\sim} \mathbb{Z}$ for the isomorphism which sends a lift $f \in G_K$ of the geometric Frobenius (i.e. inducing $x \mapsto x^{q-1}$ on $k$) to 1. Let $\text{unr}(\alpha)$ be the one-dimensional Weil–Deligne representation over $L$ with $N = 0$, $I$ acting trivially, and $f$ acting through multiplication by $\alpha$. For example, $WD(\mathbb{Q}_p(1)) = \text{unr}(q^{-1})$. One has $\text{unr}(\alpha)^* \cong \text{unr}(\alpha^{-1})$. Recall the notation $\text{sp}(m)$ from \[21\] Example 1.2.3, and that $\text{sp}(m)^* \cong \text{sp}(m)/(1 - m)$. In particular, $(\text{unr}(\alpha) \otimes \text{sp}(m))^*(1) \cong \text{unr}(q^{m-2}\alpha^{-1}) \otimes \text{sp}(m)$. The object $\text{unr}(\alpha) \otimes \text{sp}(m)$ is pure of weight $n$ if and only if $\alpha$ is a $q$-Weil number of weight $n + m - 1$.

Note that rank one Weil–Deligne representations are identified with certain characters of the Weil group, so composing with the local Artin map allows us to view them as characters of $K^\times$, just as is done with rank one representations of the Galois group. For example, $\text{unr}(\alpha)$ is identified with the character that is trivial on $\mathcal{O}_K^\times$ and sends a uniformizer to $\alpha$. For any Weil–Deligne representation $\Delta$, we let $\det(\Delta)$ be the character of $K^\times$ corresponding to the top exterior power of $\Delta$. 

10
3.1 Local arguments: $\ell = p$

This subsection is concerned with [21 §3]. Here we let $K$ be as in the start of this section, and we assume that the characteristic $\ell$ of $k$ is equal to $p$. We also fix a finite extension $L/Q_p$.

The basic invariants of $p$-adic Galois representations of $p$-adic fields in [21 §3.1] (namely, Galois cohomology, Fontaine’s functors, and Bloch–Kato’s subspaces) are replaced by their generalizations to $(\varphi, \Gamma)$-modules in the preceding section, but no other modifications are necessary to that subsection.

Fix a field extension $E/K_0^{ur}$ admitting an embedding $\tau : L \hookrightarrow E$. Fontaine’s recipe for attaching a Weil–Deligne representation $WD_\tau(V)$ to a semistable representation representation $V$ of $G_K$ over $L$, described in [21 §3.2], passes first to $D_{pst}(V)$, and from $D_{pst}(V)$ to $WD_\tau(V)$. Since the second step does not require the weak admissibility of $D_{pst}(V)$, it may applied verbatim to $D_{pst}(D)$ for a potentially semistable $(\varphi, \Gamma)$-module to obtain $WD_\tau(D)$. Namely, if $\rho_D$ denotes the usual $G_K$-action on $D_{pst}(D)$, the space

$$WD_\tau(D) = D_{pst}(D) \otimes_{K_0^{ur} \otimes Q_p, L, \text{id} \otimes \text{E}} E$$

with $w \in W_K$ acting $E$-linearly by $\rho_D(w) \circ \varphi^{h(w)} \otimes \text{id}$ and $N$ acting by $N \otimes \text{id}$ is a Weil-Deligne representation over $K$. The isomorphism class of $WD_\tau(D)$ does not depend on $\tau$, so we abbreviate it to $WD(D)$.

We shall need the following invariants for a potentially semistable $(\varphi, \Gamma)$-module $D$ (which should not be confused with the $R_L$-rank $d(D)$ and the multiplicity $d_- (D)$ appearing in Section 2):

$$d_L(D) = \sum_{i \in \mathbb{Z}} i \dim_L \text{Gr}^i D_{dR}(D), \quad d_-(D) = \sum_{i < 0} i \dim_L \text{Gr}^i D_{dR}(D).$$

The only further modification we require to [21 §3.2] is to Proposition 3.2.6, which is replaced by the following analog.

**Lemma 3.1.1.** For each potentially semistable $(\varphi, \Gamma)$-module $D$, we have

$$\det_E(WD(D))) \cdot (-1) = (-1)^{d_L(D)} \det(D) \cdot (-1). \quad (3.1.1)$$

**Proof.** As in the proof of [21 Proposition 3.2.6] (with $V$ replaced by $D$), it suffices to treat the case when $D$ is of rank one, and hence is associated to a continuous character $\chi_D : K^\times \rightarrow L^\times$. The potential semistability of $D$ forces $\chi_D$ to take the form of $\chi \prod_{\sigma : K \rightarrow L} (\sigma \circ \chi_\pi)^{-n_\sigma}$ with $\chi$ a character for which $\chi(O_K^\times)$ is finite, and $\chi_\pi$ the Lubin-Tate character defined as in [21 Example 3.2.2(4)]. The equality (3.1.1) for the character $\sigma \circ \chi_\pi$ is treated in the proof of [21 Proposition 3.2.6] and for the character $\chi$ is an easy direct computation. \(\square\)

Some statements in [21 §3.3] become false for $(\varphi, \Gamma)$-modules. In Definition 3.3.1 therein (and replaced by Notation 2.3.1), the $D^\pm$ are no longer uniquely determined. In Proposition 3.3.2(1), the first equality holds (for a Panchishkin $D$), but the second equality may now
fail (although it is never used in that paper or this one). Proposition 3.3.2(2) is replaced by Lemma 2.3.2 above; note that not requiring any restriction to $K' / K$ over which $D$ becomes semistable gives a slightly stronger result.

We conclude this subsection with a discussion of [21 Proposition 3.3.3] (by replacing it with Proposition 3.1.2 below). Since uniqueness of the Panchishkin filtration fails, in our generality we will have to assume the conclusion of part (1), that $j$ induces isomorphisms $D^\pm \sim (D^\mp)^*(1)$. Part (3) holds verbatim which we include as (3.1.2) below. We ignore part (2) in the present paper, which is only used in [21] to prove part (4), and we now prove (4) directly.

**Proposition 3.1.2.** Let $D$ be a Panchishkin $(\varphi, \Gamma)$-module, pure (of weight $-1$), with a symplectic pairing $j: D \sim D^*(1)$ that induces isomorphisms $D^\pm \sim (D^\mp)^*(1)$. Put $\Delta^j = WD(D^j)$ for $? = \emptyset, \pm$. Then

$$
(det_E(\Delta^+))(1)/\det(D^+)(1) = (-1)^{d_e(D^+)} = (-1)^{d_e(D)}, \quad \text{and} \quad \varepsilon(\Delta) = (-1)^{h^0(D^-)}(-1)^{d_e(D)} \det(D^+)(1).
$$

(3.1.2)

(3.1.3)

**Proof.** The equality (3.1.2) follows from applying Lemma 3.1.1 below; note that not requiring any restriction to $K/K$ over which $D$ becomes semistable gives a slightly stronger result. Proposition 3.3.2 above; note that not requiring any restriction to $K' / K$ over which $D$ becomes semistable gives a slightly stronger result. Proposition 3.3.2 above; note that not requiring any restriction to $K' / K$ over which $D$ becomes semistable gives a slightly stronger result. Proposition 3.3.2 above; note that not requiring any restriction to $K' / K$ over which $D$ becomes semistable gives a slightly stronger result. Proposition 3.3.2 above; note that not requiring any restriction to $K' / K$ over which $D$ becomes semistable gives a slightly stronger result. Proposition 3.3.2 above; note that not requiring any restriction to $K' / K$ over which $D$ becomes semistable gives a slightly stronger result.

Let $V^{A,\lambda}$ denote the generalized $\lambda$-eigenspace of a linear operator $A$ on a vector space $V$, we claim that

$$
\dim_E \Delta^{-,j,N=0,f=1} \equiv \dim_E \Delta^{-,j,N=0,\lambda \sim 1} \pmod 2.
$$

(3.1.4)

This follows from [21 Lemma 2.2.2] as soon as we show that $f$ is orthogonal for a nondegenerate, symmetric pairing on $\Delta^{-,j,N=0,f=1}$. To exhibit this pairing, consider the diagram

$$
0 \to \Delta^{+,f,\sim 1} \to \Delta^{j,\sim 1} \to \Delta^{-,f,\sim 1} \to 0
$$

$$
0 \to \Delta^{+,f,\sim 1} \to \Delta^{j,\sim 1} \to \Delta^{-,f,\sim 1} \to 0,
$$

where the vertical arrows are $N$. Our purity hypothesis on $V$ shows that the middle vertical arrow is an isomorphism, so the snake lemma gives an isomorphism $a: \Delta^{-,N=0,f=1} \cong \Delta^{+,f,\sim 1}/N$. On the other hand, $j$ induces $b: \Delta^{+,f,\sim 1}/N \cong (\Delta^{-,N=0,\lambda \sim 1})^*(1)$. Write $\langle \cdot, \cdot \rangle_{\text{b.o.a.}}: \Delta^{-,N=0,f=1} \times \Delta^{-,N=0,\lambda \sim 1} \to E(1)$ for the nondegenerate pairing adjoint to $b \circ a$, and $\langle \cdot, \cdot \rangle_{j}: \Delta \times \Delta \to E(1)$ for the nondegenerate pairing adjoint to $j$. Given $\delta, \delta' \in \Delta^{-,N=0,\lambda \sim 1}$, if $\delta, \delta' \in \Delta^{j,\sim 1}$ denote arbitrary lifts (respectively), one computes from the definitions of $a$ and $b$ that $\langle \delta, \delta' \rangle_{\text{b.o.a.}} = \langle N(\delta), \delta' \rangle_j$. Because $N$ is adjoint to $-N$ and $j$ is symplectic, it follows that $\langle \cdot, \cdot \rangle_{\text{b.o.a.}}$ is symmetric. To see that $f$ is orthogonal, we note that $f \delta, f \delta'$ are lifts of $f \delta, f \delta'$ (respectively), and compute that

$$
\langle N(f \delta), f \delta' \rangle_j = p\langle f N(\delta), f \delta' \rangle_j = pf\langle N(\delta), \delta' \rangle_j = \langle N(\delta), \delta' \rangle_j.
$$

12
Since $(\langle \cdot, \cdot \rangle_{boa})$ is $I$-invariant by construction, the desired pairing on $\Delta^{-, I, N=0,f \sim 1}$ is obtained by restriction.

We return to considering the desired formula (3.1.3). Each term appearing in it is left unchanged upon replacing $\Delta$ by $\Delta^f_{ss}$. For the left hand side, this is [21] (2.1.2.1)]. For the term $(-1)^{h_0(D^-)}$, this was just shown. For the remaining terms, this is obvious. So, we assume always that $\Delta$ is Frobenius-semisimple.

Now $\Delta^{N}_{ss}$ is semisimple, so it splits into $\Delta^{+, N}_{ss} \oplus \Delta^{-, N}_{ss} \cong (\Delta^{+, N}_{ss} \oplus (\Delta^{+, N}_{ss})^*)^1$, and the argument of [21] Proposition 2.2.1(4) shows that $\varepsilon(\Delta^{N}_{ss}) = \det(\Delta^{-, N}_{ss})(-1)$. We compute that the right hand side of (3.1.3) is

$$(-1)^{h_0(D^-)}(-1)^{d^-|D|} \det(D^+)(-1)$$

$$= (-1)^{h_0(D^-)} \det(\Delta^+)(-1)$$

$$= (-1)^{h_0(D^-)} \det(\Delta^{+, N}_{ss})(-1)$$

$$= (-1)^{h_0(D^-)} \varepsilon(\Delta^{N}_{ss})$$

$$= (-1)^{h_0(D^-)} \varepsilon(\Delta) / \det(-f|\Delta^I/\Delta^{I, N=0})$$

by [21] Proposition 2.2.1(4)

Combining this with (3.1.4), our desired identity (3.1.3) is thus equivalent to the claim that

$$(3.1.5)$$

Recall that $I$ acts semisimply, and moreover any Weil–Deligne representation $\Delta_0$ is functorially a direct sum $\Delta_0' \oplus \Delta_0''$, where $\Delta_0'$ is the component where $I$ acts nontrivially. After applying this decomposition to $\Delta_0 = \Delta^\gamma$, $\gamma = 0, \pm$, in showing (3.1.5) we may assume that inertia acts trivially. Multiplying both sides of (3.1.5) through by $\det(-f|\Delta^{N=0})$, and using the evenness of $d = \dim_E \Delta$ and [21] (1.2.2)(2)] to compute the resulting right hand side, we are left to show that

$$(3.1.6)$$

If necessary, enlarge $E$ to include the eigenvalues of $f$ on $\Delta$. One can decompose $\Delta$ into a sum of subobjects $\Delta_i$ of dimension $d_i$, pairwise orthogonal for the symplectic pairing, each pair $(\Delta_i, j|\Delta)$ isomorphic to one of

(1) $\text{unr}(\alpha_i) \otimes \text{sp}(d_i/2) \oplus \text{unr}(q^{d_i/2-2}\alpha_i^{-1}) \otimes \text{sp}(d_i/2)$ with $\alpha_i$ a $q$-Weil number of weight $d_i/2 - 2$, equipped with the symplectic form $\langle (x, x'), (y, y') \rangle = x'(y) - y'(x)$,

(2) $\text{unr}(\alpha_i) \otimes \text{sp}(d_i)$ with $\alpha_i = \pm q^{d_i/2-1}$, equipped with its unique symplectic form (a twist of the one in [21] Example 1.2.3)).

This follows from [21] (1.2.4): Note that $\Delta$ has trivial inertia action and is Frobenius-semisimple. In the case (1) we may assume that the term $X$ appearing in loc. cit. is of the form $\text{unr}(\alpha) \otimes \text{sp}(m)$, with the condition on $\alpha_i$ coming from purity. In the case (2) the term $\rho$ appearing in loc. cit. is of the form $\text{unr}(\alpha) \otimes \text{sp}(m)$, with the fact that $\alpha = \pm q^{m/2-1}$ being forced by the self-dual condition.
In the case where $\Delta_i$ is of the form (1), one immediately computes that $\Delta_i^{N=0} \cong \text{unr}(q^{1-d_i/2} \alpha_i) \oplus \text{unr}(q^{1-d_i/2} q^{-d_i/2} \alpha_i^{-1})$, so $\det(-f|\Delta_i^{N=0}) = q^{-d_i/2}$. In the case where $\Delta_i$ is of the form (2), one has $\Delta_i^{N=0} \cong \text{unr}(q^{1-d_i} \alpha_i)$, so $\det(-f|\Delta_i^{N=0}) = \mp q^{-d_i/2}$. Thus, (3.1.6) is equivalent to the claim that $\dim_E \Delta^{-,N=0,f=1}$ has the same parity as the number of factors of the form (2) with $\pm_i = +$.

Functorially splitting $\Delta'$, where $? = \emptyset, \pm$, in the form $\Delta' \oplus \Delta''$, where $f$ has eigenvalues on $\Delta'$ (resp. $\Delta''$) belonging (resp. not belonging) to $q^Z$, we replace $\Delta'$ with its first summand because the second makes no contribution to this congruence. Since each $\Delta_i$ of the form (1) has two indecomposable summands, and each $\Delta_i$ of the form (2) has one indecomposable summand and now has $\pm_i = +$, the claim (3.1.6) is equivalent to the claim that $\dim_E \Delta^{-,N=0,f=1}$ has the same parity as the number of indecomposable summands of $\Delta$. To count the latter, purity implies that each indecomposable summand $\Delta_0$ of $\Delta$ as Weil–Deligne representation has $\Delta_0^{f=1}$ one-dimensional, so that the number of indecomposable summands is equal to $\dim_E \Delta^{f=1}$. Thus we are to show that

$$\dim_E \Delta^{-,N=0,f=1} \equiv \dim_E \Delta^{f=1} \pmod{2}. \quad (3.1.7)$$

Write $r_\pm = \dim_E \Delta^{\pm,f=1}$ so that the right hand side of (3.1.7) is $r_- + r_+$, and note that duality gives $r_+ = \dim_E \Delta^{-,f=q^{-1}}$. Purity implies that $N: \Delta^{f=1} \to \Delta^{f=q^{-1}}$ is an isomorphism, so $N: \Delta^{-,f=1} \to \Delta^{-,f=q^{-1}}$ is surjective. It follows that the left hand side of (3.1.7) is $r_- - r_+$. As $r_+ + r_- \equiv r_- - r_+ \pmod{2}$, this completes the proof of the proposition.

\[\square\]

### 3.2 Rational functions on rigid analytic spaces

In this preparatory subsection, we let $L$ be a complete nonarchimedean field, and we let $X$ be a reduced rigid analytic space over $L$. We briefly present here the definition and basic properties of the $L$-algebra $\kappa(X)$ of rational functions on $X$, for lack of a suitable reference in the literature.

Throughout this paper, if $R$ is a commutative ring, $\text{Frac}(R) = T^{-1}R$ denotes the total fraction ring of $R$, where $T \subset R$ is the set of non-zero divisors of $R$.

**Definition 3.2.1.** A rational function on $X$ is a rule $f$ that associates to each affinoid subdomain $U = \text{Max}(A)$ of $X$ an element $f_U \in \text{Frac}(A)$, in a manner compatible with restriction morphisms for inclusions of affinoid subdomains. We write $\kappa(X)$ for the $L$-algebra of such $f$.

One could equivalently define $\kappa(X)$ to be the categorical limit of the $L$-algebras $\text{Frac}(A)$ as $\text{Max}(A)$ ranges over all affinoid subdomains inside $X$. Note that if $X = \text{Max}(A)$ is affinoid then $\kappa(X) = \text{Frac}(A)$, because $X$ is the final affinoid subdomain of $X$.

The following first results suffice for our purposes.

**Proposition 3.2.2.** The $L$-algebra of rational functions has the following properties.

1. The formation of $\kappa(X)$ is functorial for pullback under dominant morphisms.
2. If $U$ is a dense Zariski open in $X$, then the map $\kappa(X) \to \kappa(U)$ is injective.
(3) Assigning to each affinoid subdomain $U = \text{Max}(A)$ the $L$-algebra $\text{Frac}(A)$ defines a sheaf $\mathcal{M}_X$ on $X$ for the weak topology on $X$ (generated by affinoid subdomains). In particular, $\kappa(X) = \Gamma(X, \mathcal{M}_X)$.

(4) If $\pi : \tilde{X} \to X$ is a normalization, then $\pi_* (\mathcal{M}_{\tilde{X}}) = \mathcal{M}_X$ and $\kappa(X) = \kappa(\tilde{X})$.

(5) If $\{X_i\}_{i \in I}$ is the set of irreducible components of $X$, then $\kappa(X) = \prod_{i \in I} \kappa(X_i)$.

(6) If $X$ is irreducible, then $\kappa(X)$ is a field.

Proof. The claims (1) and (2) are obvious.

The claim (3) is essentially [12, Theorem 4.6.6]; the theorem therein was stated for the covers by rational subdomains because the Gerritzen–Grauert theorem only appeared later, but the proof works verbatim for the weak topology.

When $X$ is affinoid it is obvious that $\kappa(X) = \kappa(\tilde{X})$. Passing to the construction of (3), this gives $\pi_* (\mathcal{M}_{\tilde{X}}) = \mathcal{M}_X$ in general, whence taking global sections of the latter identity yields $\kappa(X) = \kappa(\tilde{X})$ in general. This shows (4).

We treat (5). The irreducible components $X_i$ of $X$ are defined to be the images of the connected components $\tilde{X}_i$ of $\tilde{X}$. Moreover, each $\tilde{X}_i \to X_i$ is a normalization. Thus (4) reduces us to the case where $X$ is normal, and $\{X_i\}_{i \in I}$ is none other than the set of connected components of $X$, in which case the claim is obvious.

In the setting of (6), we immediately reduce to the case where $X$ is normal. Then each affinoid subdomain $U = \text{Max}(A)$ of $X$ is a finite disjoint union of connected affinoid subdomains $U_i = \text{Max}(A_i)$, and each $\text{Frac}(A_i)$ is a field. Thus we are reduced to showing that if $f \in \kappa(X)$ is not the zero element then, for each connected affinoid subdomain $U$ of $X$, the element $f_U \in \text{Frac}(U)$ is nonzero. By definition, the fact that $X$ is connected means that for any two points $P, Q \in X$, there exists a chain of connected affinoid subdomains $U_1, \ldots, U_N$ of $X$ with $P \in U_1, Q \in U_N$, and $U_i \cap U_{i+1}$ nonempty. Given nonzero $f$, choose $V$ such that $f_V$ is nonzero, and arbitrarily choose $P \in V$. Given any $U$, choose arbitrary $Q \in U$, and connect $P, Q$ with a chain $U_1, \ldots, U_N$ as in the definition of connectedness. Insert $V$ at the beginning of the chain, and $U$ at the end of the chain, so that now $U_1 = V$ and $U_N = U$. It suffices to see that if $f_{U_1}$ is nonzero then so is $f_{U_{i+1}}$. Indeed, choose any connected affinoid subdomain $W_i$ contained in $U_i \cap U_{i+1}$, write $U_i = \text{Max}(A_i)$ and $W_i = \text{Max}(B_i)$, and note that the maps

$$p_i : \text{Frac}(A_i) \to \text{Frac}(B_i), \quad p'_i : \text{Frac}(A_{i+1}) \to \text{Frac}(B_i)$$

are homomorphisms of fields, hence injective, and $p_i(f_{U_i}) = f_{W_i} = p'_i(f_{U_{i+1}})$. \hfill \Box

Caution 3.2.3. If $U$ is a dense Zariski open in $X$, the natural map $\kappa(x) \to \kappa(U)$ is not in general surjective. For example, when $X = \text{Max}(\mathcal{Q}_p(z))$ and $U = X \setminus \{0\}$, the function $f(z) = \prod_{n \geq 1} \left(1 + \frac{p^n}{z - p^n}\right)$ belongs to $\kappa(U)$ but not to $\kappa(X)$.

Assume the characteristic of $L$ is not two, and write $\{X_i\}_{i \in I}$ for the set of irreducible components of $X$. We say that $f \in \kappa(X)$ has values in $\{\pm 1\}$ if it belongs to the subgroup $\prod_{i \in I} \{\pm 1\}$ of $\prod_{i \in I} \kappa(X_i)^\times = \kappa(X)^\times$. 

15
3.3 Local arguments: $\ell \not= p$

Here we let $K$ be as in the start of this section, and we assume that the characteristic $\ell$ of $k$ is not equal to $p$.

**Proposition 3.3.1.** Let $X$ be a reduced rigid analytic space over $\mathbb{Q}_p$, and let $\mathcal{T}$ be a locally free coherent $\mathcal{O}_X$-module equipped with a continuous, $\mathcal{O}_X$-linear action of $G_K$. Then the usual recipe (see, for example, [21, (5.1.4)]) associates to $\mathcal{T}$ a Weil–Deligne representation structure $WD(\mathcal{T})$ on $\mathcal{T}$, in a manner that is functorial in $\mathcal{T}$ and commutes with arbitrary base change in $X$.

**Proof.** By gluing it suffices to treat the case where $X = \text{Max}(A)$ is affinoid, and in this case it suffices to know the claim that the wild inertia subgroup $I'_K \subset G_K$ acts on $\mathcal{T}$ through a finite quotient. To see this, [6, Lemme 3.18] shows that there exists a unit ball subalgebra $A_0 \subset A$ and a finite, flat, $G_K$-stable $A_0$-lattice $\mathcal{T}_0 \subset \mathcal{T}$. By replacing $\mathcal{T}_0$ by its direct sum with another finite flat $A_0$-module with trivial $G_K$-action, we may assume it is free, say of rank $d$. A cofinal system of open subgroups of $\text{Aut}_{A_0}(\mathcal{T}_0)$ is given by $1 + p^n \text{End}_{A_0}(\mathcal{T}_0) \approx 1 + p^d M_d(A_0)$, where $d$ is the rank of $\mathcal{T}_0$. Note that $(1 + p^n M_d(A_0))/(1 + p^{n+1} M_d(A_0)) \cong M_d(A_0/p)$ for $n \gg 0$, where the right hand side is annihilated by $p$. Since $I'_K$ is a pro-$\ell$ group, this implies that the order of its image in $\text{Aut}_{A_0}(\mathcal{T}_0/p^n)$ stabilizes for $n \gg 0$, and therefore its image in $\text{Aut}_{A_0}(\mathcal{T}_0)$ is finite, as was desired. \qed

**Notation 3.3.2.** Define a local datum $\mathcal{X} = (X, \mathcal{T}, j)$ (for $K$) to consist of:

- a reduced rigid analytic space $X$ over $\mathbb{Q}_p$,
- a locally free coherent $\mathcal{O}_X$-module of rank $d$ equipped with a continuous, linear $\mathcal{O}_X$-action of $G_K$, and
- a skew-symmetric, $\mathcal{O}_X[G_K]$-linear morphism $\langle \cdot, \cdot \rangle_j : \mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{T} \to \mathcal{O}_X(1)$ whose adjoint map $j : \mathcal{T} \to \mathcal{T}^*(1)$ is generically on $X$ an isomorphism (so $d$ is even).

If $\mathcal{X}$ is a local datum as above and $f : Y \to X$ is a morphism of reduced rigid analytic spaces, then the tuple $f^*\mathcal{X} = (Y, f^*\mathcal{T}, f^*j)$ is again a local datum if and only if $f^*j$ is generically on $Y$ an isomorphism.

We will associate to a local datum two subsets $X_{\text{alg}} \subseteq X_{\text{pair}} \subseteq X$. First, $X_{\text{pair}}$ is the set of points $P \in X$ for which there exists $u \in \text{Frac}(\mathcal{O}^\wedge_{X,P})^\times$ such that $u j_P^* : \mathcal{T}_P^* \to \mathcal{T}^*(1)_P$ is an isomorphism of $\mathcal{O}_{X,P}^\wedge$-modules. Equivalently, by Nakayama’s lemma, $u j_P^*$ sends $\mathcal{T}_P^\wedge$ into $\mathcal{T}^*(1)_P$ and $\overline{u j} = u j_P^* \otimes_{\mathcal{O}_{X,P}} \kappa(P) : \mathcal{T} \otimes_{\mathcal{O}_X} \kappa(P) \to \mathcal{T}^*(1) \otimes_{\mathcal{O}_X} \kappa(P)$ is an isomorphism. We say that such $u$ guarantees $P \in X_{\text{pair}}$. For $u \in \kappa(X)^\times$, we write $X_{\text{pair}}^u$ for the set of $P \in X$ such that the image of $u$ in $\text{Frac}(\mathcal{O}^\wedge_{X,P})^\times$ guarantees $P \in X_{\text{pair}}$. If $f : Y \to X$ is a morphism such that $f^*\mathcal{X}$ is again a local datum then one has $f^{-1}(X_{\text{pair}}) \subseteq Y_{\text{pair}}$. If moreover $f^*(u)$ is well-defined then $f^{-1}(X_{\text{pair}}^u) \subseteq Y_{\text{pair}}^f$. 

16
Proposition 3.3.3. Let $\mathcal{X}$ be a local datum.

1. If $\iota: Y \subseteq X$ is an admissible open subset, then $\iota^* \mathcal{X}$ is a local datum and satisfies $Y_{\text{pair}} = Y \cap X_{\text{pair}}$.

2. If $X = \text{Max}(A)$ is affinoid, then for each $P \in X_{\text{pair}}$ there exists $u \in \text{Frac}(A)^{\times}$ such that $P \in X_u$.

3. For each $u \in \kappa(X)^{\times}$ the subset $X^u_{\text{pair}} \subseteq X$ is a dense Zariski open subspace. In particular, $X_{\text{pair}}$ is also a dense Zariski open subspace.

Proof. Claim (1) is clear from the definitions of a local datum and $X_{\text{pair}}$.

For (2), since $\text{Frac}(A)$ depends only on the normalization of $A$, it suffices to assume $A$ is normal, and therefore also irreducible. Choose $u_1 \in \text{Frac}(\mathcal{O}_{X,P}^\wedge)^{\times}$ guaranteeing $P \in X^u_{\text{perf}}$, and write $u_1 = s_1^{-1}f_1$ with $f_1, s_1 \in \mathcal{O}_{X,P}^\wedge$ and $s_1$ a non-zerodivisor. By the definition of $\mathcal{O}_{X,P}$, we may find an affinoid subdomain $\text{Max}(A') \subseteq \text{Max}(A)$ containing $P$ and $f_2, s_2 \in A$ such that $f_2 \equiv f_1$ and $s_2 \equiv s_1$ modulo a high power of the maximal ideal at $P$. We may take $\text{Max}(A')$ to be connected, and therefore also irreducible because $A$ is normal. It follows that $s_2$ is a non-zerodivisor. But the morphism $A \rightarrow A'$ is injective with dense image for the spectral norm $|\cdot|_{A'}$ on $A'$, so we may choose $f_3, s_3 \in A$ with $|f_3 - f_2|_{A'} < |f_2|_{A'}$ and $|s_3|_{A'} - |s_2|_{A'} < |s_2|_{A'}$. It is easy to check that $u = s_3^{-1}f_3 \in \text{Frac}(A)$ satisfies $P \in X^u_{\text{perf}}$.

Consider (3). We may assume $X = \text{Max}(A)$ is affinoid. By Nakayama’s lemma, the conditions defining membership $P \in X^u_{\text{pair}}$ in terms of $uj$ are Zariski open. On the other hand, writing $u = s^{-1}f$ with $f, s \in A$ and $s$ a non-zerodivisor, away from the union of the vanishing loci of $f, s$ one has $X^u_{\text{perf}} = X^1_{\text{perf}}$, and by definition of a local datum the right hand side is dense in $X$. The final claim also follows, because part (2) shows that $X_{\text{pair}} = \bigcup_{u \in \text{Frac}(A)^{\times}} X^u_{\text{pair}}$.

Given a datum $\mathcal{X}$ as above with $X$ irreducible, it follows from the preceding proposition that the Weil–Deligne representation $WD(\mathcal{X}) \otimes_{\mathcal{O}_X} \kappa(X)$ is symplectic self-dual, and then [21 Proposition 2.2.1(2)] gives $\varepsilon(WD(\mathcal{X}) \otimes_{\mathcal{O}_X} \kappa(X)) \in \{\pm 1\}$. If we do not assume $X$ to be irreducible, working one irreducible component at a time gives an element of $\kappa(X)^{\times}$ with values in $\{\pm 1\}$. Similarly, if $P \in X^u_{\text{pair}}$ then $uj$ makes $WD(\mathcal{X}) \otimes_{\mathcal{O}_X} \kappa(P)$ symplectic self-dual, so that $\varepsilon(WD(\mathcal{X}) \otimes_{\mathcal{O}_X} \kappa(P)) \in \{\pm 1\}$.

Finally, we define $X_{\text{alg}}$ to be $X_{\text{pair}}$ if $\mathcal{X}$ is unramified, and otherwise to be the set of points $P \in X_{\text{pair}}$ such that $\mathcal{X} \otimes_{\mathcal{O}_X} \kappa(P)$ is pure of weight $-1$.

Proposition 3.3.4. Suppose $\mathcal{X}$ is a local datum with $X$ irreducible and $P \in X_{\text{alg}}$. Then one has $\varepsilon(\mathcal{X} \otimes_{\mathcal{O}_X} \kappa(P)) = \varepsilon(\mathcal{X} \otimes_{\mathcal{O}_X} \kappa(X))$. Moreover, $\mathcal{X} \otimes_{\mathcal{O}_X} \kappa(P)$ is ramified if and only if $\mathcal{X}$ is.

Proof. The first claim is trivial if $\mathcal{X}$ is unramified, because then both sides of the desired identity are 1, so we assume that $\mathcal{X}$ is ramified.

Write, for brevity, $\mathcal{Y} = \mathcal{X} \otimes_{\mathcal{O}_X} \kappa(X)$ and $V = \mathcal{X} \otimes_{\mathcal{O}_X} \kappa(P)$. In this proof we will have no need for the underlying $G_K$-representation structures on $\mathcal{X}, \mathcal{Y}, V$, so we use the latter symbols to denote these objects only equipped with their Weil–Deligne representation structures. I.e., we identify $WD(\mathcal{X}) = \mathcal{X}, WD(\mathcal{Y}) = \mathcal{Y}$, and $WD(V) = V$. 17
We begin with some reductions using implicitly, and repeatedly, the fact that formation of $\varepsilon$-factors commutes with extension of coefficient fields. First, we may assume that $X$ is normal, and it suffices to replace $X$ by an arbitrary small connected affinoid subdomain $\text{Max}(A)$ containing $P$. By Proposition 3.3.3(2) we may choose $u$ such that $P \in X^u_\text{pair}$, and by Proposition 3.3.3(3) by replacing $j$ by $uj$ and shrinking $X$ we may assume $j$ is an isomorphism. Further shrinking $X$, we may assume $\mathcal{T}$ is free. Then blowing up $X$ along the $(\text{rank}_A \text{coker}(N^r \mid \mathcal{T}))$th Fitting ideals of each of the $\text{coker}(N^r \mid \mathcal{T})$ (see, for example, [15 §6.3]), and again replacing $X$ by a small neighborhood of a preimage of $P$, we may assume that each of the $\text{img}(N^r \mid \mathcal{T})$ are flat and moreover free.

Letting $S$ be a free polynomial variable, we may replace $A$ by a finite integral extension over which the characteristic polynomial $Q(S) = \det(S - f \mid \mathcal{T})$ splits, and $P$ with any maximal ideal of this extension lying over it. In particular, the reduction $\bar{Q}(S) = \det(S - f \mid V)$ splits over $\kappa(P)$. If we write $V_i$ for the maximal $f$-stable subspace whose eigenvalues are strictly pure of weight $i - 1$, then the $\bar{Q}_i(S) = \det(S - f \mid V_i)$ for different $i \in \mathbb{Z}$ are coprime in $\kappa(P)[S]$. Because $V$ is pure of weight $-1$ we have $V = \bigoplus_{i \in \mathbb{Z}} V_i$, and the factorization $\bar{Q} = \prod_i \bar{Q}_i$. There is a unique lifting of this factorization to $Q = \prod Q_i$ in $A[S]$ with the $Q_i$ monic: $Q_i$ is characterized as the largest monic divisor of $Q$ all of whose roots have image in $\kappa(P)$ strictly pure of weight $i - 1$. Since $Q_i, Q_j$ generate the unit ideal in $\kappa(P)[S]$ if $i \neq j$, after perhaps shrinking $X$ around $P$ one has that $Q_i, Q_j$ generate the unit ideal in $A[S]$ if $i \neq j$. It follows that the decomposition $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is obtained by specializing at $P$ the decomposition $\mathcal{F} = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}_i$, where $\mathcal{F}_i = \ker(Q_i(f) \mid \mathcal{F})$. We set $\mathcal{V}_i = \mathcal{V} \otimes_A \kappa(X)$ to obtain a decomposition $\mathcal{V} = \bigoplus_{i \in \mathbb{Z}} \mathcal{V}_i$. The $\mathcal{F}_i$ are projective by construction, and one has $\text{det}(S - f \mid \mathcal{F}_i) = \det(S - f \mid V_i) = Q_i(S)$.

Since $\mathcal{V}_i$ is the subspace of $\mathcal{V}$ spanned by those generalized $f$-eigenvectors whose eigenvalues are roots of $Q_i(S)$, it follows from the characterization of the $Q_i$ above that $N(\mathcal{V}_i) \subseteq \mathcal{V}_{i-2}$. Because $\mathcal{F}_i = \mathcal{V}_i \cap \mathcal{F}$, this implies that also $N(\mathcal{F}_i) \subseteq \mathcal{F}_{i-2}$. For $i \geq 0$, the purity of $V$ (because $P \in X_{\text{alg}}$) implies that the map $N^i: V_i \to V_{i-1}$ is an isomorphism, so $N^i: \mathcal{F}_i \to \mathcal{F}_{i-1}$ becomes isomorphism after perhaps shrinking $X$ around $P$, and therefore $N^i: \mathcal{V}_i \to \mathcal{V}_{i-1}$ is also an isomorphism. It follows from the characterization of the monodromy filtration given in [21 (1.3.1)] that $\{ \mathcal{V}_i \}_i$ is a splitting of the monodromy filtration on $\mathcal{V}$. We conclude that, in the notations of [21 (1.3.6)], one has

$$d_i(\mathcal{V}) = \dim_{\kappa(X)} \mathcal{V}_i = \text{rank}_A \mathcal{F}_i = \dim_{\kappa(P)} V_i = d_i(V),$$

and therefore also $m_i(\mathcal{V}) = m_i(\mathcal{F})$ and $\text{rank}(N^r \mid \mathcal{V}) = \text{rank}(N^r \mid V)$ for $r \geq 0$. It is easy to see that the natural map

$$\text{img}(N^r \mid \mathcal{T}) \otimes_A \kappa(P) \to \text{img}(N^r \mid V) \quad (3.3.1)$$

is surjective. However, we have arranged that $\text{img}(N^r \mid \mathcal{T})$ is a projective $A$-module, so the preceding numerics show that the source and target of the above map have the same $\kappa(P)$-dimension. It follows that $(3.3.1)$ is an isomorphism.

We return to $\varepsilon$-factors. One has factorizations

$$\varepsilon(\mathcal{V}) = \varepsilon(\mathcal{V}^{N_{ss}}) \det(-f \mid \mathcal{V}_g/\mathcal{V}_f), \quad \varepsilon(V) = \varepsilon(V^{N_{ss}}) \det(-f \mid V_g/V_f)$$

18
using [21] (2.1.2.3); in each equation the first two terms belong to \(\{\pm 1\}\) and therefore the third does too. It suffices to compare the factors on the right hand sides. An argument is given to treat each factor in the proof of [21, Proposition 2.2.4], and these arguments apply directly to our setting, granted one knows the following two facts. The first fact is that the natural maps

\[ T^\text{N-ss} \otimes_A \kappa(P) \xrightarrow{\sim} V^\text{N-ss}, \quad T_g / T_f \otimes_A \kappa(P) \xrightarrow{\sim} V_g / V_f \]

are isomorphisms. The first is obvious, and the second follows from the isomorphism (3.3.1) and the fact that \(N\) induces an isomorphism

\[ (T_g / T_f) \xrightarrow{\sim} \text{img}(N | T). \]

The second fact is that the module \(T\) (resp. \(T_g / T_f\)) is flat, so that one can form the trace (resp. determinant) of an endomorphism in a manner compatible with base change. But for \(T\) (resp. for \(T_g / T_f\)), this is obvious (resp. follows from the isomorphism (3.3.2) and passing to a direct summand).

We now treat the second claim, continuing with the notations just introduced, and in particular considering \(T\) and \(V\) as Weil–Deligne representations. It suffices to show that inertia (resp. monodromy) acts trivially on \(T\) if and only if it does on \(V\). For inertia, this follows from the fact that the unique \(I_K\)-stable decomposition \(T = T_I K \oplus T'\) is compatible with arbitrary base change. For monodromy, this becomes clear after we may make the same blowups and shrinkings of \(X\) around \(P\) as in the proof of the first claim, to obtain the isomorphism (3.3.1) with \(\text{img}(N | T)\) a projective \(A\)-module.

We associate to the local datum \(X\) the complex \(C^\bullet(G_K, T)\) of continuous cochains of \(G_K\) on \(T\), and also the complex \(C^\bullet_{\text{unr}}(G_K, T) = [T_{I_K} \xrightarrow{f-1} T_{I_K}']\) in degrees 0, 1.

**Proposition 3.3.5.** For each \(P \in X_{\text{pair}}\) for which WD(V) is pure, after perhaps shrinking \(X\) around \(P\), the complexes \(C^\bullet(G_K, T)\) and \(C^\bullet_{\text{unr}}(G_K, T)\) are acyclic.

**Proof.** It suffices to work in a sufficiently small affinoid neighborhood \(\text{Max}(A)\) of \(P\). For the first complex, the argument of [15, Lemma 4.1.5] shows that it suffices to check that \(C^\bullet(G_K, T) \otimes_A \kappa(P)\) is acyclic; this claim follows from the base change result of [26, Theorem 1.4(1)] together with [21, 4.2.2(1)]. For the second complex, we take mapping fibers of \(f - 1\) on the distinguished triangle

\[ T_{I_K} \rightarrow C^\bullet(I_K, T) \rightarrow T_{I_K}[-1] \]

to obtain the distinguished triangle

\[ C^\bullet_{\text{unr}}(G_K, T) \rightarrow C^\bullet(G_K, T) \rightarrow [T_{I_K} \xrightarrow{f-1} T_{I_K}'], \]

with the last complex concentrated in degrees 1, 2. We compute

\[ T_{I_K} / (f - 1) \otimes_A \kappa(P) \cong (T \otimes_A \kappa(P))_{G_K} = 0 \]

by the purity of \(T \otimes_A \kappa(P)\). In particular, after shrinking \(X\) around \(P\) we have that \(f - 1\) is surjective, hence bijective, on \(T_{I_K}\). This forces \([T_{I_K} \xrightarrow{f-1} T_{I_K}']\) to be acyclic, and so is \(C^\bullet_{\text{unr}}(G_K, T)\) by (3.3.3). \(\square\)
3.4 Global data

The rest of this section is concerned with adapting [21 §§4–5] to the setting of rigid analytic spaces. In order to state our main theorem, we will first need to invoke a good deal of notation, consisting mostly of straightforward modifications of [21 §§4–5]. These notations will be in force through the remainder of this section.

We refer to [15] for definitions and background on (“arithmetic”) families of $(ϕ, Γ)$-modules over rigid analytic spaces.

For the rest of this section, $F$ is a number field, $S$ is a fixed finite set of places of $F$ containing the places $S_p$ above $p$ and the archimedean places $S_∞$, $F_S$ is a maximal algebraic extension of $F$ unramified outside $S$, and $G_{F,S} = \text{Gal}(F_S/S)$. For a place $v$ of $F$ we fix an algebraic closure $F_v^{\text{alg}}$ of $F_v$ and an embedding $F_S \hookrightarrow F_v^{\text{alg}}$, giving maps $G_{F,v} \to G_{F,S}$. If $X$ has an action of $G_{F,S}$, we write $X_v$ for $X$ with the action restricted to $G_{F,v}$.

**Definition 3.4.1.** Define a global point datum $(L, V, j, \{S_v\}_{v \in S_p})$ to consist of:

- a finite extension $L/\mathbb{Q}_p$,
- a finite-dimensional $L$-vector space $V$ of dimension $d$ equipped with a continuous, $L$-linear action of $G_{F,S}$,
- a skew-symmetric, $L[G_{F,S}]$-linear morphism $\langle \cdot, \cdot \rangle : V \otimes_L V \to L(1)$ whose adjoint map $j : V \to V^*(1)$ is an isomorphism,
- for each $v \in S_p$, a short exact sequence
  $$S_v : 0 \to D^+_v \to D_v \to D^-_v \to 0$$
  of $(ϕ, Γ)$-modules over $\mathcal{R}_L(\pi_{F_v})$, where $D_v = D_{\text{rig}}(V_v)$, such that $\langle D^+_v, D^+_v \rangle_{D_{\text{rig}}(j)} = 0$, rank$_{\mathcal{R}_L(\pi_{F_v})} D^\pm_v = d/2$, and $S_v$ makes $D_v$ Panchishkin, and
- for every finite place $v$ where $V_v$ is ramified, $V_v$ is pure (necessarily of weight $−1$).

Usually we will write simply $V$ for the tuple.

Define a global datum $\mathcal{X} = (X, \mathcal{T}, j, \{\mathcal{T}_v\}_{v \in S_p})$ to consist of:

- a reduced rigid analytic space $X$ over $\mathbb{Q}_p$,
- a locally free coherent $\mathcal{O}_X$-module $\mathcal{T}$ of rank $d$ equipped with a continuous, $\mathcal{O}_X$-linear action of $G_{F,S}$,
- a skew-symmetric, $\mathcal{O}_X[G_{F,S}]$-linear morphism $\langle \cdot, \cdot \rangle : \mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{T} \to \mathcal{O}_X(1)$ whose adjoint map $j : \mathcal{T} \to \mathcal{T}^*(1)$ is generically on $X$ an isomorphism (so $d$ is even), and
- for each $v \in S_p$, a short exact sequence
  $$\mathcal{T}_v : 0 \to \mathcal{D}^+_v \to \mathcal{T}_v \to \mathcal{D}^-_v \to 0$$
  of $(ϕ, Γ)$-modules over $\mathcal{R}_{\mathcal{O}_X}(\pi_{F_v})$, where $\mathcal{D}_v = D_{\text{rig}}(\mathcal{T}_v)$, such that $\langle \mathcal{D}^+_v, \mathcal{D}^+_v \rangle_{D_{\text{rig}}(j)} = 0$ and rank$_{\mathcal{R}_{\mathcal{O}_X}(\pi_{F_v})} \mathcal{D}^\pm_v = d/2$. 

20
Note that for any place \( v \notin S_p \cup S_{\infty} \), the tuple \( \mathcal{P}_v = (X, \mathcal{I}_v, j) \) is a local datum for \( F_v \).

A global point datum \( (L, V, j, \{S_v\}_{v \in S_p}) \) determines a global datum upon taking \( X = \{P\} \) to be the spectrum of \( L \). Conversely, when \( P \in X \) is fixed we write \( L = \kappa(P), V = \mathcal{I} \otimes_{\mathcal{O}_L} \kappa(P), j = \mathcal{I} \otimes_{\mathcal{O}_L} \kappa(P), \) and \( S_v = \mathcal{I} \otimes_{\mathcal{O}_L} \kappa(P), \) for \( ? = 0, \pm \). In the sequel we will specify some points \( P \in X \) for which \( (L, V, j, \{S_v\}_{v \in S_p}) \) is a global point datum.

Global data admit the same functorialities as local data, and the notations \( X_{\text{pair}} \) and \( X_{\text{pair}} \) carry over changing only “local” to “global” throughout. In particular, when \( X = \{P\} \) arises from a global point datum, one has \( X = X_{\text{pair}} \).

We have placed no conditions on points \( P \) of \( X_{\text{pair}} \) concerning the duality of \( D_v^\pm \), because the next proposition shows the duality to be automatic.

**Proposition 3.4.2.** If \( u \) guarantees \( P \in X_{\text{pair}} \), then \( D_{\text{rig}}(uj) \) induces, for each \( v \in S_p \), isomorphisms \( D_v^\pm \to (D_v^\pm)^*(1) \).

**Proof.** Since the map \( D_v^\pm \to (D_v^\pm)^*(1) \) has full rank by the nondegeneracy of \( uj \), we just need to check that it has saturated image, or equivalently that the top exterior power \( \bigwedge^{d/2} D_v^\pm \to \bigwedge^{d/2}(D_v^\pm)^*(1) \) has saturated image. Indeed, one has an identification

\[
\bigwedge^{d/2} D_v^\pm \otimes_{\mathcal{O}_v} (\bigwedge^{d/2} D_v^\pm) \cong \bigwedge^d D_v = \mathcal{O}_{\kappa(P)}(\pi_{F_v})(d/2)
\]

arising from the sequence \( S_v \) and the identification \( \bigwedge^d V \cong \kappa(P)(d/2) \) of [21 (1.2.2)(2)], hence there exists an isomorphism \( \bigwedge^{d/2} D_v^\pm \cong (\bigwedge^{d/2} D_v^\pm)^*(d/2) = \bigwedge^{d/2}(D_v^\pm)^*(1) \), and any nonzero endomorphism of a rank one object is an isomorphism and in particular has saturated image. (One can check that the two maps agree up to sign, but we do not need this fact.) \( \square \)

**Notation 3.4.3.** We define \( X_{\text{alg}} \) to be the set of points \( P \in X_{\text{pair}} \) satisfying

- for each \( v \in S_p \), the sequence \( S_v \) makes \( D_v \) Panchishkin, and
- for each \( v \notin S_{\infty} \) such that \( \mathcal{I}|_{G_{F_v}} \) is ramified, \( V_v \) is pure (of weight \(-1\)).

We make some remarks on this notation. First, the purity hypothesis always applies in particular to \( v \in S_p \), but only applies to \( v \) belonging to \( S \). Second, for \( v \notin S_p \cup S_{\infty} \), Proposition 3.3.4 shows that \( V_v \) is ramified if and only if \( \mathcal{I}_v \) is ramified. Thus, if \( P \in X_{\text{alg}} \cap X_{\text{pair}}^{\text{an}} \), then \( (\kappa(P), V, uj, \{S_v\}_{v \in S_p}) \) is a global point datum. The choice of \( u \) will not matter, so we abusively use \( V \) to denote this tuple. Third, if \( f: Y \to X \) is a morphism such that \( f^* \mathcal{I} \) is a global datum, then one has \( f^{-1}(X_{\text{alg}}) \subseteq Y_{\text{alg}} \). Finally, for any place \( v \notin S_p \cup S_{\infty} \), the subset \( X_{\text{alg}} \) for \( \mathcal{I} \) is contained in the subset \( X_{\text{alg}} \) for \( \mathcal{I}_v \).

**Example 3.4.4.** The data \( (R, \mathcal{I}, \{\mathcal{I}_v^\pm\}_{v \in S_p}, P, u) \), assumed given in [21 (5.1.2)] give rise to a global datum \( \mathcal{I} = (\text{Spf}(R), \mathcal{I}^\pm, \{\mathcal{I}_v^\pm\}_{v \in S_p}) \) and a point \( P \in X_{\text{alg}} \cap X_{\text{pair}}^{\text{an}} \).
3.5 Global algebraic invariants

In this subsection we treat the variation of Selmer groups in families. We continue with the notations and hypotheses of Subsection 3.4.

Let $T$ be either $\mathcal{T}$ where $\mathcal{X} = (X, \mathcal{T}, j, \{\mathcal{S}_v\}_{v \in S_p})$ is a global datum, or $V$ where $(L, V, j, \{S_v\}_{v \in S_p})$ is a global point datum. For $v \in S_p$ write $\mathcal{D}_{\text{rig}}(T_v)^+ = \mathcal{D}^+_v, D^+_v$ (respectively). Choose a set $\Sigma$ with $S_p \subseteq \Sigma \subseteq S \setminus S_\infty$ of primes at which $T$ is ramified. Then the Selmer complex (cf. [21 §5.2]) is defined by

$$\tilde{C}^*_f(G_{F,S}, T; \Sigma) = \text{Fib} \left( C^*(G_{F,S}, T) \oplus \bigoplus_{v \in S - S_\infty} U_v^+(T) \rightarrow \bigoplus_{v \in S - S_\infty} C^*(G_{F_v}, T) \right),$$

where $\text{Fib}(f) = \text{Cone}(f)[-1]$ denotes the mapping fiber of a morphism of complexes, and

$$U_v^+(T) = \begin{cases} C^*_{\psi, \gamma}(\mathcal{D}_{\text{rig}}(T_v)^+), & v \in S_p, \\ 0, & v \in \Sigma - S_p, \\ C^*_{\text{unr}}(G_{F_v}, T) = [T^1 \xrightarrow{f_v - 1} T^1], & v \in S - \Sigma. \end{cases}$$

In the case $T = \mathcal{T}$, it is easy to see that the formation of these complexes commutes (up to canonical quasi-isomorphism) with base change to open subsets of $X$. In particular, their cohomologies form coherent sheaves on $X$.

When the choice of $S$ and $\Sigma$ is not important we will write $\tilde{C}^*_f(F, T)$ for the Selmer complex, $\tilde{H}^*_f(F, T)$ for its cohomology, and $\tilde{h}^1_f(F, T)$ for its generic rank when $T = \mathcal{T}$ (resp. dimension when $T = V$). By the following proposition, which is a straightforward adaptation of [21 Proposition 4.2.2] and [21 Proposition 5.2.2(2)] to our setup, the rational function (resp. integer) $\tilde{h}^1_f(F, T)$ is independent of choice of $S$ and $\Sigma$.

For $V$ a global point datum, we use $H^1_f(F, V)$ to denote the usual Bloch–Kato Selmer group (see, for example, [21 (4.1.1)]), and set $h^1_f(F, V) = \dim_L H^1_f(F, V)$.

**Proposition 3.5.1.** (1) Suppose $(X, \mathcal{T}, j, \{\mathcal{S}_v\}_{v \in S_p})$ is a global datum. For each $P \in X_{\text{alg}}$, after perhaps shrinking $X$ around $P$, the complex $\tilde{C}^*_f(G_{F,S}, \mathcal{T}; \Sigma)$ is canonically independent (up to quasi-isomorphism) of $S$ and $\Sigma$.

(2) Suppose $(L, V, j, \{S_v\}_{v \in S_p})$ is a global point datum. The complex $\tilde{C}^*_f(G_{F,S}, V; \Sigma)$ is canonically independent of (up to quasi-isomorphism) of $S$ and $\Sigma$. Moreover, we have $\tilde{h}^1_f(F, V) = h^1_f(F, V) + \sum_{v \in S_p} h^0(D_v^-)$.

**Proof.** In both (1) and (2), the independence with respect to $\Sigma$ follows from Proposition[3.3.5] and the independence with respect to $S$ is a general fact [20 Proposition 7.8.8]. We take $\Sigma = S_p$ from now on. By the definition of the Selmer complex as a mapping fiber and Lemma[2.3.2] we deduce an exact sequence

$$H^0(G_{F,S}, V) \rightarrow \oplus_{v \in S_p} H^0(D_v^-) \rightarrow \tilde{H}^1_f(F, V) \rightarrow H^1_f(F, V) \rightarrow 0.$$ 

The purity of $V$ forces $H^0(G_{F,S}, V) = 0$, and the desired identity follows. \(\square\)
We have seen that the ability to use the choice $\Sigma = S_p$ allows one to compare $\tilde{H}^1_j(F,V)$ to $H^1_j(F,V)$. On the other hand, taking $\Sigma$ to be the set of all ramified primes for $\mathcal{T}$ allows one to use general base-change results without complications, which is an ingredient in the proof of Proposition 3.5.2 below.

**Proposition 3.5.2.** If $\mathcal{X}$ is a global datum with $X$ irreducible and $P \in X_{\text{alg}}$, then one has $\tilde{h}^1_j(F,V) \equiv \tilde{h}^1_j(F,\mathcal{T}) \pmod{2}$.

*Proof.* We use implicitly, and repeatedly, the fact that the formation of $\tilde{h}^1_j(F,\mathcal{T})$ commutes replacing $X$ by an admissible open subset, or by any space $Y$ with a morphism $f : Y \to X$ that is generically an isomorphism (although we make no claim about whether $\tilde{h}^1_j(F,\mathcal{T})$ itself commutes with the latter base change). In particular, if $u$ guarantees $P \in X_{\text{pair}}$ then we may replace $j$ by $uj$ and shrink $X$ around $P$ so that $j$ is an isomorphism, and moreover $\mathcal{T}$ is free. We may also replace $X$ with a resolution of singularities (and $P$ by a point in its preimage) and, after a constant scalar extension, shrink $X$ to have the form $X = \operatorname{Max}(A)$ with $A = \kappa(P)(T_1, \ldots, T_r)$, with $P$ corresponding to $T_1 = \ldots = T_r = 0$.

Define the increasing system of closed subspaces $\{P\} = X_0 \subset X_1 \subset \cdots \subset X_r = X$ by letting $X_j$ correspond to $T_{j+1} = \cdots = T_r = 0$ with inclusion $\iota_j : X_j \hookrightarrow X$, so that $X_j = \operatorname{Max}(A_j)$ with $A_j = \kappa(P)(T_1, \ldots, T_j)$. Write $\mathcal{T}_j = \iota^*_j \mathcal{T}$ for brevity. Note that each $\mathcal{X}_j = \iota^*_j \mathcal{X}$ is a global datum with $X_{j,\text{pair}} = X_j$ and $P \in X_{j,\text{alg}}$.

After shrinking $X$ (and therefore the $X_j$) around $P$ by rescaling the parameters $T_j$, we may assume (Proposition 3.3.5) that all the complexes $C^\bullet(G_{F_v}, \mathcal{T}_j)$ and $C^\bullet_{\text{unr}}(G_{F_v}, \mathcal{T}_j)$, $v \in S \setminus (S_p \cup S_{\infty})$, are acyclic and (Proposition 3.5.1(2)) the $\tilde{C}^\bullet_j(F,\mathcal{T}_j)$ are independent of $S$ and $\Sigma$. Because we may take $\Sigma$ to be the set of ramified places for $\mathcal{T}$, it then follows immediately from general base-change results that the natural map is a quasi-isomorphism for $j > 0$,

$$\tilde{C}^\bullet_j(F,\mathcal{T}_j) \otimes_{A_{j-1}} \xrightarrow{L} C^\bullet_j(F,\mathcal{T}_{j-1}).$$

In particular, one has short exact sequences

$$0 \to \tilde{H}^1_j(F,\mathcal{T}_j)[T_j] \to \tilde{H}^1_j(F,\mathcal{T}_{j-1}) \to \tilde{H}^{1+1}_j(F,\mathcal{T}_j)[T_j] \to 0,$$

where $(-)[T_j]$ denotes the subset of elements killed by $T_j$. If $j > 0$ then the localization $A_{j,(T_j)}$ is a discrete valuation ring with uniformizer $T_j$, fraction field $\operatorname{Frac}(A_{j})$, and residue field $\operatorname{Frac}(A_{j-1})$. We thus obtain short exact sequences

$$0 \to \tilde{H}^1_j(F,\mathcal{T}_j)(T_j) \to \tilde{H}^1_j(F,\mathcal{T}_{j-1}) \otimes_{A_{j-1}} \operatorname{Frac}(A_{j-1}) \to \tilde{H}^{1+1}_j(F,\mathcal{T}_j)(T_j) \to 0. \quad (3.5.1)$$

One may now apply the arguments of [24] Proposition 5.2.2 parts (4) through (7), using the above short exact sequence in place of part (3) there, over each of the discrete valuation rings $A_{j,(T_j)}$, $j = 1, \ldots, r$, to show the following in turn:

(a) Applying [20] Theorem 10.2.3 and Proposition 10.2.5] to (3.5.1) for $i = 1$, there exists a nondegenerate skew-symmetric pairing

$$\tilde{H}^2_j(F,\mathcal{T}_j)(T_j),A_{j,(T_j)}\text{-tors} \times \tilde{H}^2_j(F,\mathcal{T}_j)(T_j),A_{j,(T_j)}\text{-tors} \to (\operatorname{Frac} A_j)/A_{j,(T_j)}.$$
(b) By (a) and the structure of symplectic modules over a DVR, there exists a finite-length
\( A_{j,(T_j)} \)-module \( Z_j \) such that \( \tilde{H}^2_f(F, \mathcal{F}_j)(T_j) \cdot A_{j,(T_j)} \)-tors \( \cong Z_j \otimes Z_j \).

(c) By looking at (3.5.1) for \( i = 0 \), the \( A_{j,(T_j)} \)-module \( \tilde{H}^1_f(F, \mathcal{F}_j)(T_j) \) is torsion-free and hence free of rank \( \tilde{h}^1_f(F, \mathcal{F}_j) \).

(d) By (b) and (3.5.1) for \( i = 1 \), one has \( \tilde{h}^1_f(F, \mathcal{F}_j) \equiv \tilde{h}^1_f(F, \mathcal{F}_{j-1}) \pmod{2} \).

One then assembles the congruences
\[
\tilde{h}^1_f(F,V) = \tilde{h}^1_f(F,\mathcal{F}_0) \equiv \tilde{h}^1_f(F,\mathcal{F}_1) \equiv \cdots \equiv \tilde{h}^1_f(F,\mathcal{F}_r) = \tilde{h}^1_f(F,\mathcal{F}) \pmod{2}
\]
to complete the proof of the proposition. \( \square \)

### 3.6 Global analytic invariants

In this subsection we treat the variation of global \( \varepsilon \)-factors in families. We continue with the notations and hypotheses of Subsection 3.4.

Let \( (L, V, j, \{S_v\}_{v \in S_p}) \) be a global point datum. To every finite place \( v \) one can attach a local \( \varepsilon \)-factor \( \varepsilon_v(V) = \varepsilon(WD(V_v)) \in \{\pm 1\} \) (when \( v \in S_p \), recall that \( V_v \) is Panchishkin and hence de Rham). All \( v \) where \( V \) is unramified, and in particular all \( v \notin S \), satisfy \( \varepsilon_v(V) = 1 \). We do not know how to compute the individual factors \( \varepsilon_v(V) \) for archimedean places \( v \in S_\infty \), but we can compute their product, namely
\[
\varepsilon_\infty(V) = (-1)^{r_2(F) \dim_L V/2} (-1)^{d^-(V)},
\]
where \( r_2(F) \) is the number of pairs of complex embeddings of \( F \) and, as in Subsection 3.1,
\[
d^-(V_v) := \sum_{i < 0} i \dim_L \text{Gr}^i \text{D}_{\text{dR}}(V_v) \text{ for } v \in S_p \quad \text{and} \quad d^-(V) := \sum_{v \in S_p} d^-(V_v).
\]

**Remark 3.6.1.** Let \( M \) be a motive over \( F \) with coefficients in a number field \( E \), symplectic self-dual and pure (of weight \(-1\)), let \( \mathfrak{p} \) be a prime of \( E \) lying over \( p \), and suppose \( L = \bar{E}_\mathfrak{p} \) and \( V \) is the \( \mathfrak{p} \)-adic étale realization of \( M \). By classifying the possible Hodge structures attached to \( M \) and applying [10, §§5.2–5.3], one can compute that for each embedding \( \iota : E \hookrightarrow \mathbb{C} \), the product of archimedean \( \varepsilon \)-factors of \( M \) (with respect to \( \iota \)) is given by
\[
\varepsilon_\infty(\iota M) = \prod_{u \in S_\infty} \varepsilon_u(\iota M) = (-1)^{r_2(F) \dim_L V/2} (-1)^{\sum_{i < 0} [F_u : \mathbb{R}] i d^i_u(\iota M)},
\]
where \( d^i_u(\iota M) = \dim_{\mathbb{C}} \text{Gr}^i M_{\text{dR}} \otimes_{E \otimes_{\mathbb{Q}} F} \mathbb{C} \) for any embedding \( \bar{u} : F \hookrightarrow \mathbb{C} \) giving rise to \( u \). We claim that \( \varepsilon_\infty(\iota M) = \varepsilon_\infty(V) \). In fact, by inspection one has
\[
\varepsilon_\infty(\iota M) = (-1)^{r_2(F) \dim_L V/2} \varepsilon_\infty(\iota \text{Ind}^F_Q M) \quad \text{and} \quad \varepsilon_\infty(V) = (-1)^{r_2(F) \dim_L V/2} \varepsilon_\infty(\text{Ind}^F_Q V),
\]
24
so it suffices to treat the case where \( F = \mathbb{Q} \). Then \( S_\infty = \{ \infty \} \) and \( S_p = \{ p \} \), and one has

\[
d_{\infty}(\ell M) = \dim_C \text{Gr}^i M_{\text{dR}} \otimes_{E, c} C = \dim_E \text{Gr}^i M_{\text{dR}} = \dim_L \text{Gr}^i M_{\text{dR}} \otimes E L,
\]

\[
\text{Gr}^i M_{\text{dR}} \otimes E L \cong \text{Gr}^i D_{\text{dR}}(V_p),
\]

which imply the desired claim. (Another argument to show that \( \varepsilon_{\infty}(\ell M) = \varepsilon_{\infty}(V) \) is given in \cite{22}, and this argument gives more information than ours, but we warn that some of its steps are invalid when \( F \) is not totally real.)

We define the global \( \varepsilon \)-factor as the product of the local \( \varepsilon \)-factors:

\[
\varepsilon(V) = \varepsilon_\infty(V) \cdot \prod_{v \notin S_\infty} \varepsilon_v(V) \in \{ \pm 1 \}.
\]

Applying Proposition \ref{prop:3.1.2} for each \( v \in S_p \) gives

\[
\varepsilon(V) = (-1)^{\sum v \in S_p h^0(D_v)} (-1)^{r_2(F) \dim_L V/2} \prod_{v \in S_p} (\det(D_v^+))(-1) \cdot \prod_{v \notin S_p \cup S_\infty} \varepsilon_v(V).
\]

We define the modified \( \varepsilon \)-factor to be

\[
\tilde{\varepsilon}(V) = (-1)^{\sum v \in S_p h^0(D_v)} \varepsilon(V)
\]

\[
= (-1)^{r_2(F) \dim_L V/2} \prod_{v \in S_p} (\det(D_v^+))(-1) \cdot \prod_{v \notin S_p \cup S_\infty} \varepsilon_v(V).
\]

Let \( \mathcal{X} = (X, \mathcal{I}, j, \{ \mathcal{I}_v \}_{v \in S_p}) \) be a global datum. For each place \( v \notin S_p \cup S_\infty \) we form \( \varepsilon_v(\mathcal{I}) = \varepsilon(WD(\mathcal{I}_v) \otimes_{\mathcal{O}_X} \kappa(X)) \), a rational function on \( X \) with values in \( \{ \pm 1 \} \). Then we set

\[
\tilde{\varepsilon}(\mathcal{I}) = (-1)^{r_2(F) \text{rank}_{\mathcal{O}_X} \mathcal{I}/2} \prod_{v \in S_p} (\det(\mathcal{I}_v^+))(-1) \cdot \prod_{v \notin S_p \cup S_\infty} \varepsilon_v(\mathcal{I}),
\]

again a rational function on \( X \) with values in \( \{ \pm 1 \} \).

**Proposition 3.6.2.** If \( \mathcal{X} \) is a global datum with \( X \) irreducible and \( P \in X_{\text{alg}} \), then \( \tilde{\varepsilon}(V) = \tilde{\varepsilon}(\mathcal{I}) \).

**Proof.** It suffices to compare \( \tilde{\varepsilon}(V) \) and \( \tilde{\varepsilon}(\mathcal{I}) \) factor-by-factor. The first two factors are obviously unchanged by specialization from \( X \) to \( P \), and for the final factor one invokes Proposition \ref{prop:3.3.4}.

\[\square\]

### 3.7 Main result

We continue with the notations and hypotheses of Subsection \ref{subsec:3.4} as well as the two intervening subsections.

Suppose \((L, V, j, \{ S_v \}_{v \in S_p})\) is a global point datum. Then the by parity conjecture we mean the claim that the sign \( \varepsilon(V)/(−1)^{h_{\text{dR}}(F,V)} \in \{ \pm 1 \} \) is equal to 1. When \( V \) is semisimple,
the Fontaine–Mazur conjecture predicts that $V$ is the $p$-adic étale realization of a motive, and when $V$ is motivic in this sense the Bloch–Kato conjecture would imply the claim of the parity conjecture. Although we make no predictions on the validity of the claim if $V$ is not motivic, the results of this paper apply to a general global point datum regardless of whether $V$ is motivic.

Our main result is the following. It generalizes [21, Theorem 5.3.1], as per Example 3.4.4.

**Theorem 3.7.1.** Let $\mathcal{X}$ be a global datum with $X$ irreducible. Then one has
\[
\varepsilon(V)/(-1)^{\tilde{h}_1^j(F,V)} = \tilde{\varepsilon}(\mathcal{T})/(-1)^{\tilde{\tilde{h}}_1^j(F,\mathcal{T})},
\]
and in particular the sign $\varepsilon(V)/(-1)^{h_1^j(F,V)}$ is independent of $P \in X_{alg}$.

**Proof.** One assembles the equalities
\[
\varepsilon(V)/(-1)^{h_1^j(F,V)} = \tilde{\varepsilon}(\mathcal{T})/(-1)^{\tilde{h}_1^j(F,\mathcal{T})} = \tilde{\tilde{h}}_1^j(F,\mathcal{T}),
\]
the first by (3.6.2) and Proposition 3.5.1(2), and the second by Proposition 3.6.2 and Proposition 3.5.2.

### 4 Hilbert modular forms

The previous section is concerned with relating the parity conjecture of two different objects, when they are members of the same family. By contrast, the present section gives a supply of objects where the parity conjecture is actually known.

In numerous works, Nekovář has presented techniques allowing one to prove the parity conjecture for a wide class of Hilbert modular forms, subject most significantly to the condition that they be ordinary (up to twist) at places above $p$; see for example [24], where many of the techniques are employed. We gather these techniques into one place, and use the main result of the preceding section to weaken the condition to being finite-slope (up to twist) at places above $p$.

Fix a prime $p \neq 2$.

We fix $F$ a totally real field and an algebraic closure $F_{alg}/F$. In contrast to the previous sections, here the letter $K$ will always denote a finite extension of $F$ inside $F_{alg}$. When $S$ is a finite set of primes of $F$, we let $K_S$ be the maximal extension of $K$ inside $F_{alg}$ that is unramified outside places lying over those in $S \cup \{\infty\}$, and we set $G_{K,S} = \text{Gal}(K_S/K)$. For brevity, if $I \subseteq \mathcal{O}_F$ is a nonzero ideal, we write $G_{K,I}$ for $G_{K,S}$ with $S$ taken to be the set of primes dividing $I$.

We consider cuspidal Hilbert newforms $f$ over $F$ of parallel weight 2, trivial central character $\omega_f = 1$, level $N = N_f$, and coefficients embedded in a finite extension of $Q_p$. If $f$ has CM, we let its field of CM be $K'/F$. There is associated to $f$ a two-dimensional continuous $G_{F,N}$-representation $V = V_f$. We assume $f$ is finite-slope (up to twist) at $p$, which means that for every prime $\nu$ of $F$ lying over $p$, the local constituent at $\nu$ of its associated
automorphic representation is not supercuspidal; this is equivalent to \( \mathbf{D}_{\text{rig}}(V|_{G_{Fv}}) \) being Panchishkin.

We consider quadratic CM extensions \( K/F \) of discriminant \( D = D_K \) and quadratic character \( \eta = \eta_K \) over \( F \), the latter considered interchangeably as either a homomorphism \( G_{F,D} \to \{ \pm 1 \} \) or as a homomorphism \( \mathbf{A}_F^\times/F^\times \to \{ \pm 1 \} \). Given also \( f \), we write \( f_\eta \) for the newform associated to the twist \( f \otimes \eta \), so that \( V_{f_\eta} = V_f \otimes \eta \).

Given \( K \) and a prime \( P \) of \( F \) lying over \( p \), we let \( \tilde{G}_n \) denote the ring class group of conductor \( P^n \), defined by \( \mathbf{A}_F^\times/(K^\times K_\infty O_{F_p}^\times) \) where \( K_\infty = K \otimes \mathbb{Q} \mathbf{R} \) and \( O_{P^n} = O_F + P^n O_K \). Let \( \tilde{G}_\infty = \varprojlim_n \tilde{G}_n \), and let \( \Delta \) be its (finite) torsion subgroup. Let \( G_\infty = \tilde{G}_\infty/\Delta \), so that \( G_\infty \) is noncanonically isomorphic to \( \mathbb{Z}[F_p^\times \mathbb{Q}_p] \), and let \( G_n \) denote the quotient of \( \tilde{G}_n \) by the image of \( \Delta \). When \( K \) and \( P \) are understood, we consider finite-order continuous characters \( \chi \) of \( G_\infty \). The conductor of \( \chi \) is by definition \( P^n \), for the smallest integer \( n \geq 0 \) such that \( \chi \) factors through \( G_n \). Note that the image of \( \mathbf{A}_F^\times \) in \( \tilde{G}_\infty \) is contained in \( \Delta \), so that \( \chi|_{\mathbf{A}_F^\times} = 1 \).

Note also that since \( p \) is odd, another quadratic extension \( K'/F \) has \( K' \neq K \) if and only if \( K' \nsubseteq (F^{\text{alg}})_\ker \chi \).

With \( f, K, \) and \( \chi \) as above, there are the Bloch–Kato Selmer groups

\[
H^1_f(F, V), \quad H^1_j(F, V_{f_\eta}), \quad H^1_f(K, V_f \otimes \chi), \quad H^1_f(K, V_f) \cong H^1_f(F, V_f) \oplus H^1_f(F, V_{f_\eta}),
\]

of respective dimensions

\[
h^1_f(f), \quad h^1_j(f_\eta), \quad h^1_j(f, \chi), \quad h^1_j(f/K) = h^1_j(f) + h^1_j(f_\eta).
\]

On the other hand, we write

\[
\varepsilon(f), \quad \varepsilon(f_\eta), \quad \varepsilon(f, \chi), \quad \varepsilon(f/K) = \varepsilon(f)\varepsilon(f_\eta),
\]

respectively, for the sign of the functional equation of

\[
L(f, s), \quad L(f_\eta, s), \quad L(f, \chi, s), \quad L(f, 1, s) = L(f, s)L(f_\eta, s),
\]

where the latter two are Rankin–Selberg \( L \)-functions. There are also local signs \( \varepsilon_v(\cdot) \), and the product formulas \( \varepsilon(\cdot) = \prod_v \varepsilon_v(\cdot) \). For each \( \varepsilon \in \{ f, f_\eta, (f, \chi), f/K \} \) the parity conjecture for \( ? \) is the claim that \((-1)^{h^1_j(\varepsilon)} = \varepsilon(\cdot) \). If the parity conjecture holds for \( f/K \), then the parity conjecture for \( f \) is equivalent to the parity conjecture for \( f_\eta \). These definitions all agree with the setup in the previous section; in particular, the product of these epsilon factors at infinity is given by \eqref{eqn:product epsilon factors}.

The following result is our main contribution to the circle of ideas of this section.

**Theorem 4.1.** Given any \( f, K, P \) as above, the parity conjecture for \( f/K \) is equivalent to the parity conjecture for \( (f, \chi) \) for any \( \chi \) as above.

**Proof.** This follows from the finite-slope hypothesis and Theorem \[3.7.1\] using the family of twists of \( V|_{G_{K,N,D}} \) by continuous characters of \( G_\infty \). We only need to verify that \( V|_{G_{K,N,D}} \) and
$V|G_{K,ND} \otimes \chi$ are pure. Since $\chi$ has finite order, it suffices to check for $V|G_{K,ND}$, and for the latter it suffices to know purity for $V$. This fact follows from combining the Ramanujan conjecture and local-global compatibility for Hilbert modular forms, which were shown in increasing generality by many people, the final cases being treated in [31 Theorem 1] and [28 Theorem 1], respectively.

To treat the parity conjecture for $f$, we will momentarily divide into two cases. The precise inputs needed depend on which case we are in, but the proof generally proceeds in both cases by the following two steps, which first appeared in [19]. Step 1 is to verify the parity conjecture for $f/K$ for some class of $K$. First, one invokes nonvanishing results for Rankin–Selberg $L$-values or CM points on Jacobians of Shimura curves, which involve a possibly high-conductor twist $\chi$. Second, one uses Euler system machinery to convert these nonvanishing results into bounds on Selmer groups, to obtain the parity conjecture for $(f, \chi)$. Finally, one uses the above theorem to get rid of $\chi$.

Step 2 is to use nonvanishing theorems for quadratic-twisted $L$-values to choose a $K$ as in Step 1 where the Euler system method is powerful enough to give the parity conjecture for $f$ itself.

We say that $f$ admits an indefinite case if it satisfies the equivalent conditions of [28 Proposition 2.10.2], one of which being that $[F : Q]$ is odd or $f$ is not principal series at some finite place. Note that this condition is invariant under replacing $f$ by a quadratic twist.

Theorem 4.2. Recall that we have assumed $p$ is odd, and $f$ has trivial central character and is finite-slope (up to twist) at $p$. If $f$ admits an indefinite case, then the parity conjecture holds for $f$.

Proof. Step 1. We show the following claim: Assume that $K$ satisfies $\varepsilon(f/K) = -1$, and that if $f$ has CM then $K' \neq K$. Then the parity conjecture holds for $f/K$.

Fix an archimedean place $\tau$ of $F$. Let $\Sigma$ be the set of places $v$ of $F$ not equal to $\tau$ such that $\eta_v(-1)\varepsilon_v(f/K) = -1$. All infinite places $v$ of $F$ not equal to $\tau$ belong to $\Sigma$, because $\eta_v(-1) = -1$ and $\varepsilon_v(f/K) = 1$. At each finite $v \in \Sigma$, $K/F$ is not split and $f$ is special or supercuspidal. There exists a unique quaternion algebra $B/F$ that has nonsplit set $\Sigma$. The Jacquet–Langlands correspondence associates $f$ to a cuspidal automorphic representation $\pi'$ of $B^\times$ whose vectors are differentials on a system of $B$-Shimura curves. We fix an $F$-algebra embedding $t: K \hookrightarrow B$, allowing us to form $K$-CM points on these Shimura curves.

We apply [11 Theorem 2.5.1]; let us specify how our situation fits into the notations and hypotheses there. We use the quaternion algebra $B$ above. In the data (2.4.1) of loc. cit., we use (D1) $s = 1$, $P_1 = P$, (D2) $c = 1$, (D3) the class $C$ is nontrivial but otherwise arbitrary, (D4) the quotient $A_0$ is to be specified momentarily, and (D5) $\chi_0 = 1$ (because our $\chi = 1$).

In (2.4.3) of loc. cit. the set $S$ consists of those primes distinct from $P$ that ramify in $K/F$, and $\chi_1 = 1$ because $\chi_0 = 1$. It remains to choose $A_0$ among those in the Shimura curve Jacobians corresponding to $\pi'$ so that the equivalent conditions of Proposition 2.4.10(1) of loc. cit. are satisfied. But $c = 1$ implies all primes in $S$ ramify in $K/F$, and in particular do not split in $K/F$, so the discussion of (2.4.11) of loc. cit. produces a nonzero $\sigma\ell$: if the vector $v$ satisfying $\sigma\ell(v) \neq 0$ is fixed by the level $H$, then some quotient $A_0$ of the Shimura
curve Jacobian of level $H$ will do. In summary, there exist $K$-CM points on $A_0$ of arbitrarily high conductor that are nontorsion.

Then we apply [23] Theorem 3.2, to $A_0$, $S_B = \Sigma$, a nontorsion $K$-CM point guaranteed by the preceding paragraph, and any character $\alpha$ whose component of the $K$-CM point is nontorsion (note that $\alpha$ is necessarily a finite-order character of $G_\infty$). Therefore one has $(-1)^{h_f^1(f, \alpha)} = -1$, which, combined with [1] Proposition 2.6.2(2)] with $\alpha$ in place of $\chi$, establishes the parity conjecture for $(f, \alpha)$. We now appeal to Theorem 4.1 to obtain the claim.

Step 2. [13] Theorem B] gives infinitely many $K_1, K_2$ (distinct from $K'$ if $f$ has CM) with

$$
\varepsilon(f/K_1) = \varepsilon(f_{\eta_{K_1}}/K_2) = -1, \quad \{\text{ord}_{s=1} L(f_{\eta_{K_1}}, s), \text{ord}_{s=1} L((f_{\eta_{K_1}})_{\eta_{K_2}}, s)\} = \{0, 1\}.
$$

Applying Step 1 twice shows that the parity conjectures for $f, f_{\eta_{K_1}}, (f_{\eta_{K_1}})_{\eta_{K_2}}$ are all equivalent. But letting $g \in \{f_{\eta_{K_1}}, (f_{\eta_{K_1}})_{\eta_{K_2}}\}$ be such that $\text{ord}_{s=1} L(g, s) = 0$, it follows from [25, Corollary 2.10.4] that $h_f^1(g) = 0$, so that the parity conjecture holds for $g$. 

If $f$ does not admit an indefinite case, so that $\varepsilon(f/K) = +1$ for all $K$, more hypotheses are necessary. The following condition (A2) for $f$ arises in [25].

(A2) there exists $g \in G_{F,N}$ such that $\det(1-gX|V) = (1-\lambda_1 X)(1-\lambda_2 X)$ with $\lambda_1^2 = 1 \neq \lambda_2^2,$

and if $f$ has CM then moreover $\lambda_n^2 \neq 1$ for all $n \geq 1$.

Note that condition (A2) is invariant under replacing $f$ by a quadratic twist.

**Theorem 4.3.** Recall we have assumed $p$ is odd, and $f$ has trivial central character and is finite-slope (up to twist) at $p$. If $f$ does not admit an indefinite case, condition (A2) holds, and $\varepsilon(f) = +1$, then the parity conjecture holds for $f$.

**Proof.** Step 1. We show the following claim: Assume that $D$ is coprime to $N$, that $P$ is split in $K/F$, and that if $f$ has CM then $K \neq K'$. Then the parity conjecture holds for $f/K$.

Because $P$ is split in $K/F$, for all $\chi$ the sign $\varepsilon(f, \chi) = \varepsilon(f/K) = 1$ is constant, and by [9, Theorem 1.4], one can find $\chi$ of sufficiently large $P$-power conductor such that $L(f, \chi, 1) \neq 0$. Then [25, Theorem A] shows that $h_f^1(f, \chi) = 0$. Thus the parity conjecture holds for $(f, \chi)$, hence also for $f/K$ by Theorem 4.1.

Step 2. Fix two primes $Q_1, Q_2$ of $F$ not dividing $NP$. [13] Theorem B] gives infinitely many $K$ (distinct from $K'$ if $f$ has CM) such that every prime dividing $NP$ is split in $K/F$, the primes $Q_1, Q_2$ ramify in $K/F$, and $L(f, 1) \neq 0$. The proof of [25, Theorem B] shows that $h_f^1(f_n) = 0$, so that the parity conjecture holds for $f_n$. But Step 1 shows that the parity conjecture for $f_n$ is equivalent to the parity conjecture for $f$.  

**References**

[1] E. Aflalo and J. Nekovář, Non-triviality of CM points in ring class field towers. With an appendix by Cristophe Cornut. *Israel J. Math.* **175** (2010), 225–284.
[2] D. Benois, A generalization of Greenberg’s $L$-invariant. *Amer. J. Math.* **133** (2011), no. 6, 1573–1632.

[3] D. Blasius, Hilbert modular forms and the Ramanujan conjecture. *Aspects Math.* **E37** (2006), 35–56.

[4] K. Buzzard, Eigenvarieties, in *L-functions and Galois representations*, 59–120, *London Math. Soc. Lecture Note Ser.*, **320**, Cambridge Univ. Press, Cambridge, 2007.

[5] K. Buzzard and L. J. P. Kilford, The 2-adic eigencurve at the boundary of weight space. *Compos. Math.* **141** (2005), no. 3, 605–619.

[6] G. Chenevier, Un application des variétés de Hecke des groupes unitaires. Preprint, [http://www.math.polytechnique.fr/~chenevier/pub.html](http://www.math.polytechnique.fr/~chenevier/pub.html).

[7] F. Cherbonnier and P. Colmez, Représentations $p$-adiques surconvergentes. *Invent. Math.* **133** (1998), no. 3, 581–611.

[8] B. Conrad, Irreducible components of rigid spaces. *Ann. Inst. Fourier (Grenoble)* **49** (1999), no. 2, 473–541.

[9] C. Cornut and V. Vatsal, Nontriviality of Rankin–Selberg $L$-functions and CM points. *London Math. Soc. Lecture Note Ser.* **320** (2007), 121–186.

[10] P. Deligne, Valeurs de Foncitons $L$ et périodes d’intégrales. With an appendix by N. Koblitz and A. Ogus. *Proc. Sympos. Pure Math.* **XXXIII** (1979), 313–346.

[11] J.-M. Fontaine, Représentations $p$-adiques des corps locaux. I. The Grothendieck Festschrift, vol. II, 249–309.

[12] J. Fresnel and M. van der Put, *Rigid analytic geometry and its applications* Progress in Mathematics **218** Boston, MA: Birkhäuser. xii, 296 p. (2004).

[13] S. Friedberg and J. Hoffstein, Nonvanishing theorems for automorphic $L$-functions on $GL(2)$. *Ann. of Math. (2)* **142** (1995), no. 2, 385–423.

[14] K. S. Kedlaya, Slope filtrations for relative Frobenius. *Astérisque* **319** (2008), 259–301.

[15] K. S. Kedlaya, J. Pottharst, and L. Xiao, Cohomology of arithmetic families of $(\varphi, \Gamma)$-modules. *J. Amer. Math. Soc.* **27** (2014), no. 4, 1043–1115.

[16] R. Liu, Cohomology and duality for $(\varphi, \Gamma)$-modules over the Robba ring. *Int. Math. Res. Not.* 2008, no. 3, Art. ID rnm150, 32 pp.

[17] K. Nakamura, Iwasawa theory of de Rham $(\varphi, \Gamma)$-modules over the Robba ring. *J. Inst. Math. Jussieu* **13** (2014), no. 1, 65–118.
[18] K. Nakamura, Classification of two-dimensional split trianguline representations of \( p \)-adic fields. *Compos. Math.* **145** (2009), no. 4, 865–914.

[19] J. Nekovář, On the parity of ranks of Selmer groups. II. *C. R. Acad. Sci. Paris Sér. I Math.* **332** (2001), no. 2, 99–104.

[20] J. Nekovář, *Selmer complexes*, *Astérisque* **310** (2006), viii+559 pp.

[21] J. Nekovář, On the parity of ranks of Selmer groups. III. *Doc. Math.* **12** (2007), 243–274.

[22] J. Nekovář, Erratum for “On the parity of ranks of Selmer groups. III”. *Doc. Math.* **14** (2009), 191–194.

[23] J. Nekovář, The Euler system method for CM points on Shimura curves. *London Math. Soc. Lecture note Ser.* **320** (2007), 471–547.

[24] J. Nekovář, Growth of Selmer groups of Hilbert modular forms over ring class fields. *Ann. Sci. Éc. Norm. Supér. (4)* **41** (2008), no. 6, 1003–1022.

[25] J. Nekovář, Level raising and anticyclotomic Selmer groups for Hilbert modular forms of weight two. *Canad. J. Math.* **64** (2012), no. 3, 588–668.

[26] J. Pottharst, Analytic families of finite-slope Selmer groups. *Algebra & Number Theory* **7** (2013), no. 7, 1571–1612.

[27] D. Roe, The 3-adic eigencurve at the boundary of weight space. Preprint, [http://people.ucalgary.ca/~roed/writings/research/eigencurve.pdf](http://people.ucalgary.ca/~roed/writings/research/eigencurve.pdf).

[28] C. Skinner, A note on the \( p \)-adic Galois representations attached to Hilbert modular forms, *Doc. Math.* **14** (2009), 241–258.