We compute the SU(2) Casson–Lin invariant for the Hopf link and determine the sign in the formula of Harper and Saveliev relating this invariant to the linking number.

The Casson–Lin invariant $h(K)$ was defined for knots $K$ by X.-S. Lin [1992] as a signed count of conjugacy classes of irreducible SU(2) representations of the knot group $G_K = \pi_1(S^3 \setminus K)$ with traceless meridional image, and Corollary 2.10 of the same paper shows that $h(K)$ is equal to $\frac{1}{2} \text{sign}(K)$, one half the knot signature. E. Harper and N. Saveliev [2010] introduced the Casson–Lin invariant $h_2(L)$ of 2-component links, which they defined as a signed count of certain projective SU(2) representations of the link group $G_L = \pi_1(S^3 \setminus L)$. They showed that $h_2(L)$ equals the linking number of $L = \ell_1 \cup \ell_2$, up to an overall sign: $h_2(L) = \pm \text{lk}(\ell_1, \ell_2)$. Harper and Saveliev [2012] also show that $h_2(L)$ can be regarded as an Euler characteristic associated to a certain SU(2) instanton Floer homology theory, defined by Kronheimer and Mrowka [2011].

The purpose of this note is to determine the sign in the formula of Harper and Saveliev, establishing the following.

**Theorem 1.** If $L = \ell_1 \cup \ell_2$ is an oriented 2-component link in $S^3$, then its Casson–Lin invariant satisfies $h_2(L) = -\text{lk}(\ell_1, \ell_2)$.

We remark that the braid approach in [Harper and Saveliev 2010] is close in spirit to Lin’s original definition, and it shows that $h_2(L)$ is an invariant of oriented links, because the Alexander and Markov theorems hold for oriented links; see Theorems 2.3 and 2.8 of [Kassel and Turaev 2008]. The sign of the invariant $h_2(L)$ depends not only on the choice of orientation on the braid, but also on the more subtle choice of identification of geometric braids with elements in the abstract braid group $B_n$, viewed as a subgroup of $\text{Aut}(F_n)$. Here we follow Conventions 1.13 of [Kassel and Turaev 2008] in making this choice.

Note that extensions of the Casson–Lin invariants to SU($N$) and to oriented links $L$ in $S^3$ with at least two components are presented in [Boden and Harper 2010].

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2016], where, as before, they are defined by counting certain projective SU($N$) representations of the link group $G_L$.

The rest of this paper is devoted to proving Theorem 1.

Proof. The proof of Proposition 5.7 in [Harper and Saveliev 2010] shows that the sign of $\text{lk}(\ell_1, \ell_2)$ in our theorem is independent of $L$. (See also the proof of their Theorem 2 and their general discussion in Section 5.) Thus Theorem 1 will follow from a single computation.

To that end, we will determine the Casson–Lin invariant for the right-handed Hopf link. Since there is just one irreducible projective SU($2$) representation of the link group, up to conjugation, it suffices to determine the sign associated to this one point.

We identify $\text{SU}(2) = \{ x + yi + zj + wk \mid |x|^2 + |y|^2 + |z|^2 + |w|^2 = 1 \}$ with the group of unit quaternions and consider the conjugacy class $C_i = \{ yi + zj + wk \mid |y|^2 + |z|^2 = 1 \} \subset \text{SU}(2)$ of purely imaginary unit quaternions. Notice that $C_i$ is diffeomorphic to $S^2$ and coincides with the set of SU($2$) matrices of trace zero.

Let $L$ be an oriented link in $S^3$, represented as the closure of an $n$-strand braid $\sigma \in B_n$. We follow Conventions 1.13 on page 17 of [Kassel and Turaev 2008] for writing geometric braids $\sigma$ as words in the standard generators $\sigma_1, \ldots, \sigma_{n-1}$. In particular, braids are oriented from top to bottom and $\sigma_i$ denotes a right-handed crossing in which the $(i+1)$-st strand crosses over the $i$-th strand. The braid group $B_n$ gives a faithful right action on the free group $F_n$ on $n$ generators, and here we follow the conventions in [Boden and Harper 2016] for associating an automorphism of $F_n$ to a given braid $\sigma \in B_n$, which we write as $x_i \mapsto x_i^\sigma$ for $i = 1, \ldots, n$. To be precise, to each braid group generator $\sigma_i$ we associate the map $\sigma_i : F_n \to F_n$ given by

$$x_i \mapsto x_{i+1}, \quad x_{i+1} \mapsto (x_{i+1})^{-1} x_i x_{i+1}, \quad x_j \mapsto x_j \ (j \neq i, \ i + 1),$$

and this is a right action, i.e., if $\sigma, \sigma' \in B_n$ are two braids, then $(x_i)^{\sigma \sigma'} = (x_i^\sigma)^{\sigma'}$ for all $1 \leq i \leq n$. Note that each braid $\sigma \in B_n$ fixes the product $x_1 \cdots x_n$.

A standard application of the Seifert–van Kampen theorem shows that the link complement $S^3 \setminus L$ has fundamental group

$$\pi_1(S^3 \setminus L) = \langle x_1, \ldots, x_n \mid x_i^\sigma = x_i, \ i = 1, \ldots, n \rangle.$$  

We can therefore identify representations in $\text{Hom}(\pi_1(S^3 \setminus L), \text{SU}(2))$ with fixed points in $\text{Hom}(F_n, \text{SU}(2))$ under the induced action of the braid $\sigma$. We further identify $\text{Hom}(F_n, \text{SU}(2))$ with $\text{SU}(2)^n$ by associating to a homomorphism $\varphi$ the
We obtain an orientation on \( f \hat{\gamma} \) where \( \varrho(1) \). Note that \( \sigma : SU(2)^n \to SU(2)^n \) is equivariant with respect to conjugation, so that fixed points come in whole orbits.

Every projective \( SU(2) \) representation can be identified with a fixed point in \( \text{Hom}(F_n, SU(2)) \) under the action of \( \varepsilon \sigma \) for some \( n \)-tuple \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) with \( \varepsilon_i = \pm 1 \) such that \( \varepsilon_1 \cdots \varepsilon_n = 1 \). Notice that the action of \( \varepsilon \sigma \) on \( (X_1, \ldots, X_n) \in SU(2)^n \) preserves the product \( X_1 \cdots X_n \) and is equivariant with respect to conjugation. The Casson–Lin invariant \( h_2(L) \) is then defined as a signed count of orbits of fixed points of \( \varepsilon \sigma \) for a suitably chosen \( n \)-tuple \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \). The choice is made so that the resulting projective representations \( \varrho \) all have \( w_2(\text{Ad} \varrho) \neq 0 \), meaning that the representations \( \text{Ad} \varrho \) do not lift to \( SU(2) \) representations. It has the consequence that for all fixed points \( \varrho \) of \( \varepsilon \sigma \), each \( \varrho(x_i) \) is a traceless \( SU(2) \) element.

We therefore restrict our attention to the subset of traceless representations, which are elements \( \varrho \in \text{Hom}(F_n, SU(2)) \) with \( \varrho(x_j) \in C_i \) for \( j = 1, \ldots, n \). Define \( f : C_i^n \times C_i^n \to SU(2) \) by setting

\[
f(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = (X_1 \cdots X_n)(Y_1 \cdots Y_n)^{-1}.
\]

We obtain an orientation on \( f^{-1}(1) \) by applying the base-fiber rule, using the product orientation on \( C_i^n \times C_i^n \) and the standard orientation on the codomain of \( f \). The quotient \( f^{-1}(1)/\text{conj} \) is then oriented by another application of the base-fiber rule, using the standard orientation on \( SU(2) \). This step uses the fact that, if \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) is chosen so that the associated \( SO(3) \) representation \( \text{Ad} \varrho \) has nontrivial second Stiefel–Whitney class \( w_2 \neq 0 \), then every fixed point of \( \varepsilon \sigma \) in \( \text{Hom}(F_n, SU(2)) \) is necessarily irreducible.

We view conjugacy classes of fixed points of \( \varepsilon \sigma \) as points in the intersection \( \tilde{\Delta} \cap \tilde{\Gamma}_{\varepsilon \sigma} \), where \( \tilde{\Delta} = \Delta/\text{conj} \) is the quotient of the diagonal \( \Delta \subset C_i^n \times C_i^n \), and where \( \tilde{\Gamma}_{\varepsilon \sigma} = \Gamma_{\varepsilon \sigma}/\text{conj} \) is the quotient of the graph \( \Gamma_{\varepsilon \sigma} \) of \( \varepsilon \sigma : C_i^n \to C_i^n \).

If the link \( L \) is the closure of a 2-strand braid, as it is for the Hopf link, then \( \varepsilon = (-1, -1) \) is the only choice whose associated \( SO(3) \) bundle has \( w_2 \neq 0 \). Furthermore, in this case the intersection \( \tilde{\Delta} \cap \tilde{\Gamma}_{\varepsilon \sigma} \) takes place in the pillowcase \( f^{-1}(1)/\text{conj} \), which is defined as the quotient

\[
P = \{(a, b, c, d) \in C_i^4 \mid ab = cd \}/\text{conj}.
\]

It is well known that \( P \) is homeomorphic to \( S^2 \). To see this, first conjugate so that \( a = i \), then conjugate by elements of the form \( e^{i\theta} \) to arrange that \( b \) lies in the \((i, j)\)-circle. A straightforward calculation using the equation \( ab = cd \) shows that \( d \) must also lie on the \((i, j)\)-circle. Clearly \( c \) is determined by \( a, b, d \). We thus obtain an embedded 2-torus of elements of \( C_i^4 \) satisfying \( ab = cd \), parametrized by

\[
g(\theta_1, \theta_2) = (i, e^{k\theta_1}i, e^{k(\theta_2-\theta_1)}i, e^{k\theta_2}i)
\]
for \( \theta_1, \theta_2 \in [0, 2\pi) \), which maps onto \( P \). It is easy to verify that this is a two-to-one submersion, except when \( \theta_1, \theta_2 \in \{0, \pi\} \). This realizes \( P \) as a quotient of the torus by the hyperelliptic involution. In particular, this involution is orientation-preserving, and away from the four singular points of \( P \), we can lift all orientation questions up to the torus.

Let \( L \) be the right-handed Hopf link, which we view as the closure of the braid \( \sigma = \sigma_1^2 \in B_2 \), and suppose \( \varepsilon = (-1, -1) \). The intersection \( \hat{\Delta} \cap \hat{\Gamma}_{\varepsilon} \sigma \) consists of only one point, the conjugacy class of \( g(\frac{\pi}{2}, \frac{\pi}{2}) \), that is, the point \([i, j, i, j]\) \( \in P \). (This is easily verified using the action of \( \sigma_2^2 \) on \( F_2 = \langle x, y \rangle \); see Figure 1.) Thus, in order to pin down the sign of the Casson–Lin invariant \( h_2(L) \), we must determine the orientations of \( \hat{\Delta}, \hat{\Gamma}_{\varepsilon}, \) and \( P \) at this point.

Notice that
\[
\begin{align*}
\frac{\partial}{\partial \theta_1} g(\theta_1, \theta_2) &= (0, e^{k\theta_1}j, -e^{k(\theta_2-\theta_1)}j, 0), \\
\frac{\partial}{\partial \theta_2} g(\theta_1, \theta_2) &= (0, 0, e^{k(\theta_2-\theta_1)}j, e^{k\theta_2}j).
\end{align*}
\]

Evaluating at \( \theta_1 = \theta_2 = \frac{\pi}{2} \) gives two tangent vectors \( u_1 := (0, -i, -j, 0) \) and \( u_2 := (0, 0, j, -i) \) to \( C_i^4 \) which span a complementary subspace in \( \ker df \) to the orbit tangent space. Therefore, an ordering of these vectors determines an orientation on \( P = f^{-1}(1) / \text{conj} \).

The orbit tangent space is spanned by the three tangent vectors
\[
\begin{align*}
v_1 &:= \frac{\partial}{\partial t} \bigg|_{t=0} e^{it}(i, j, i, j)e^{-it} = (2k, 0, 0), \\
v_2 &:= \frac{\partial}{\partial t} \bigg|_{t=0} e^{jt}(i, j, i, j)e^{-jt} = (-2k, 0, -2k), \\
v_3 &:= \frac{\partial}{\partial t} \bigg|_{t=0} e^{kt}(i, j, i, j)e^{-kt} = (2j, -2i, 2j, -2i).
\end{align*}
\]

Then \( \{u_1, u_2, v_1, v_2, v_3\} \) is a basis for \( \ker(df|_{(i, i, j)}) = T(i, i, i, j)f^{-1}(1) \). We choose vectors \( w_1 = (k, 0, 0, 0), \ w_2 = (0, k, 0, 0), \ w_3 = (j, 0, 0, 0) \) to extend this to a basis for \( T(i, i, i, j)C_i^4 \).

The orientation conventions in the definition of \( h_2(L) \) (see Section 5d of [Harper and Saveliev 2010]) involve pulling back the orientation from \( \mathfrak{su}(2) = T_1 \mathfrak{SU}(2) \) by \( df \) to obtain a coorientation for \( \ker(df|_{(i, i, j)}) \). With that in mind, we compute the action of \( df \) on \( \{w_1, w_2, w_3\} \), namely, \( df(w_1) = -j, \ df(w_2) = i, \ df(w_3) = k \).

Notice that the ordered triple \( \{df(w_1), df(w_2), df(w_3)\} = \{-j, i, k\} \) gives the same orientation as the standard basis for \( \mathfrak{su}(2) \). Thus, the base-fiber rule gives the coorientation \( \{w_1, w_2, w_3\} \) on \( \ker df \), so we choose the orientation \( O_{\ker df} \) on \( \ker df \) such that \( O_{\{w_1, w_2, w_3\}} \oplus O_{\ker df} \) agrees with the product orientation on \( C_i^2 \times C_i^2 \).

The orientation on the pillowcase \( P \) is then obtained by applying the base-fiber rule a second time to the quotient (1), using \( O_{\ker df} \) to orient \( f^{-1}(1) \) and giving the
orbit tangent space the orientation induced from that on SU(2) as well. We claim that
the basis \( \{u_1, u_2\} \) for the tangent space to the pillowcase has the opposite orientation.
To see this, we note that \( \{v_1, v_2, v_3\} \) is the fiber orientation for \( \text{SO}(3) \to f^{-1}(1) \to P \)
and compare \( S = \{w_1, w_2, w_3, u_1, u_2, v_1, v_2, v_3\} \) to the product orientation on \( C_i^2 \times C_i^2 \). Using the basis \( \{(j, 0), (k, 0), (0, k), (0, i)\} \) for \( T(i, j)(C_i^2) \), we see that
\[
\beta = \{(j, 0, 0, 0), (k, 0, 0, 0), (0, k, 0, 0), (0, i, 0, 0), (0, 0, j, 0), (0, 0, k, 0), (0, 0, 0, k), (0, 0, 0, i)\}
\]
is an oriented basis for \( T(i, j, i, j)C_i^4 = T(i, j)C_i^2 \times T(i, j)C_i^2 \) with the product orientation.\(^1\)

Let \( M \) be the matrix expressing the vectors in \( S \) in terms of the basis \( \beta \). Since
\[
M = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\
1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 
\end{bmatrix},
\]
one easily computes that \( \det M = -8 \), confirming our claim that \( \{u_2, u_1\} \) is a positively oriented basis for the pillowcase tangent space.

Recall that \( L \) is the right-handed Hopf link, which we represent as the closure of the braid \( \sigma = \sigma_1^2 \in B_2 \). For \( \varepsilon = (-1, -1) \), as in Figure 1, one can verify that
\[
\varepsilon \sigma(X, Y) = (-Y^{-1}XY, -Y^{-1}X^{-1}YXY).
\]
Consider the curve \( \alpha(\theta) = (i, e^{k\theta}i) \), passing through the point \( (i, j) \in C_i^2 \) when \( \theta = \frac{\pi}{2} \), which is transverse to the orbit \([[(i, j)]]) \). Then \( (\alpha(\theta), \alpha(\theta)) \) and \( (\alpha(\theta), \varepsilon \sigma \circ \alpha(\theta)) \) are curves in \( \Delta \) and \( \Gamma_{\varepsilon \sigma} \), respectively, and both are necessarily transverse to the orbit in

\(^1\)As explained in Section 5d of [Harper and Saveliev 2010], the invariant \( h_2(L) \) is independent of the choice of orientation on \( C_i \). In fact, \( C_i^2 \) can be oriented arbitrarily provided one uses the product orientation on \( C_i^2 \times C_i^2 \).
Thus, we can compare the orientations induced by the parametrizations $[(\alpha(\theta), \alpha(\theta)) \text{ and } [(\alpha(\theta), \varepsilon \circ \alpha(\theta))])$ of $\hat{\Delta}$ and $\hat{\Gamma}_{e^2}$ to the pillowcase orientation determined above, namely $\{u_2, u_1\}$. The velocity vectors for the paths $(\alpha(\theta), \alpha(\theta)) = (i, e^{k\theta}i, i, e^{k\theta}i)$ and $(\alpha(\theta), \varepsilon \circ \alpha(\theta)) = (i, e^{k\theta}i, -e^{2k\theta}i, -e^{3k\theta}i)$ at $\theta = \frac{\pi}{2}$ are given by $(0, -i, 0, -i) = u_1 + u_2$ and $(0, -i, 2j, -3i) = u_1 + 3u_2$, respectively.

The Casson–Lin invariant is defined as the intersection number $h_2(L) = \langle \hat{\Delta}, \hat{\Gamma}_{e^2} \rangle$, and in our case the sign of the unique intersection point in $\hat{\Delta} \cap \hat{\Gamma}_{e^2}$ is determined by comparing the orientation of $\{u_1 + u_2, u_1 + 3u_2\}$ with $\{u_2, u_1\}$. Since the change of basis matrix $\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ has negative determinant, it follows that $h_2(L) = -1$.  

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