CLUSTER POINTS OF JUMPING COEFFICIENTS AND EQUISINGULARITIES OF PLURISUBHARMONIC FUNCTIONS

QUAN GUAN AND ZHENQIAN LI

Abstract. In this article, we will construct a plurisubharmonic function whose jumping coefficients have a cluster point. We also give a class of plurisubharmonic functions which cannot be equisingular to any plurisubharmonic function with generalized analytic singularities.

1. Introduction

1.1. Cluster points of jumping coefficients. Let \( X \) be a complex manifold of dimension \( n \) and \( \varphi \) a quasi-plurisubharmonic (abbr. quasi-psh) function on \( X \). The multiplier ideal sheaf \( \mathcal{I}(\varphi) \) is defined to be the sheaf of germs of holomorphic functions \( f \) such that \(|f|^2 e^{-2\varphi} \) is locally integrable, which is a coherent sheaf of ideals (see [2]).

Let \( x \in X \) be a point. By the solution to Demailly’s strong openness conjecture [7], i.e.,
\[
\mathcal{I}(\varphi) = \mathcal{I}_x(\varphi) := \bigcup_{\varepsilon > 0} \mathcal{I}((1 + \varepsilon)\varphi),
\]
which was posed by Demailly in [1], there is an increasing sequence
\[
0 = \xi_0(\varphi; x) < \xi_1(\varphi; x) < \xi_2(\varphi; x) < \cdots
\]
of real numbers \( \xi_k = \xi_k(\varphi; x) \) such that \( \mathcal{I}(c\cdot\varphi)_x = \mathcal{I}(\xi_k\cdot\varphi)_x \) for \( c \in [\xi_k, \xi_{k+1}) \) and
\[
\mathcal{I}(\xi_{k+1}\cdot\varphi)_x \subsetneq \mathcal{I}(\xi_k\cdot\varphi)_x \quad \text{for every } k.
\]
If the Lelong number \( \nu(\varphi, x) = 0 \), we put \( \xi_1 = +\infty \).

Definition 1.1. ([5], [3]). The real numbers \( \xi_k(\varphi; x) \) is called the jumping coefficients of jumping numbers of \( \varphi \) at \( x \). We say that \( \xi \) is a jumping coefficient of \( \varphi \) on an analytic set \( A \subset X \) if it is a jumping coefficient of \( \varphi \) at some point \( x \in A \). The collection of all jumping coefficients of \( \varphi \) at \( x \) is denoted by \( \text{Jump}(\varphi; x) \).

By the strong openness property of multiplier ideal sheaves, we know that \( \text{Jump}(\varphi; x) \) satisfies the descending chain condition (DCC): any decreasing sequence of jumping coefficients stabilizes. Moreover, it is easy to see that \( \text{Jump}(\varphi; x) \) does not satisfy ACC. By a discussion on Hironaka’s log resolution [3], the sequence \( (\xi_k) \) will go to infinity when \( \varphi \) has analytic singularities, i.e., \( \varphi \) can be written locally as
\[
\varphi = \frac{c}{2} \log \left( \sum_{k=1}^{m} |f_k|^2 \right) + O(1),
\]

Date: January 25, 2018.
2010 Mathematics Subject Classification. 32C35, 32U05, 32U25.
The first author was partially supported by NSFC-11522101 and NSFC-11431013.
Key words. Plurisubharmonic function, Multiplier ideal sheaf, Lelong number, Equisingularity.
where \( c \in \mathbb{R}^+ \) and \( f_k \) are holomorphic functions. However, the sequence \((\xi_k)\) will not go to infinity for general plurisubharmonic functions as we will present, which is very different from the algebraic case. One can refer to [5] and [3] for more properties of jumping coefficients.

In the present subsection, we will construct a plurisubharmonic function \( \varphi \) such that \( \text{Jump}(\varphi; x) \) has a cluster point, i.e., there exists a jumping coefficient \( \xi_{k_0} \) such that for every small enough \( \varepsilon > 0 \) we have a jumping coefficient \( \xi_k \in (\xi_{k_0} - \varepsilon, \xi_{k_0}) \). Indeed, we will prove the following:

**Theorem 1.1.** There exists a plurisubharmonic function \( \varphi \) on \( \mathbb{C}^n \) (\( n \geq 2 \)) such that 1 is a cluster point of \( \text{Jump}(\varphi; o) \).

By Theorem 1.1 one can obtain that the plurisubharmonic function \( \varphi \) will not be equisingular to any plurisubharmonic function with analytic singularities (here, “equisingular” for \( \varphi_1 \) and \( \varphi_2 \) means that \( e^{-2\varphi_1} - e^{-2\varphi_2} \) is locally integrable). We will consider general case in the next subsection.

1.2. Equisingular problem on plurisubharmonic functions. In [1] (see also [2]), Demailly proved that, for any given quasi-psh function \( \varphi \) on compact Hermitian manifold \( X \), one can choose equisingular quasi-psh functions \( \varphi_j \) (\( j = 1, 2, \ldots \)) on \( X \), which are smooth outside the poles and decreasingly convergent to \( \varphi \). In [6], Guan showed that the above result does not hold if \( \varphi_j \) are supposed to have analytic singularities.

A plurisubharmonic function \( \varphi \) will be said to have **generalized analytic singularities** if for any point \( x_0 \in X \) there exists a ball \( B_r(x_0) \) such that for every complex line \( L \subset B_r(x_0) \), \( \varphi|_L \) has analytic singularities. The set of plurisubharmonic functions on \( X \) with generalized analytic singularities is denoted by \( \mathcal{E}(X) \). In particular, \( \mathcal{E}(X) \) contains any \( \varphi \in \operatorname{Psh}(X) \), which can be written locally as

\[
\varphi = \log \left( \sum_{j=1}^N \prod_{k=1}^{m_j} |f_{jk}|^{\lambda_{jk}} \right) + O(1),
\]

where \( \lambda_{jk} \in \mathbb{R}^+ \) and \( f_{jk} \) are holomorphic functions.

Now, it is natural to put forward such a question:

**Question.** For any given \( \varphi \in \operatorname{Psh}(X) \), can one choose \( \varphi_A \in \mathcal{E}(X) \) such that \( \varphi_A \) is equisingular to \( \varphi \)?

In this subsection, we establish the following result on equisingularities of plurisubharmonic functions.

**Theorem 1.2.** Let \( \chi \) be a convex increasing function on \( \mathbb{R} \) with \( \lim_{t \to -\infty} \chi'(t) = C_0 > 0 \). Let \( \varphi \) be a plurisubharmonic function near \( o \in \mathbb{C}^n \) with \( C_0 \cdot \nu(\varphi, o) \geq n \) and \( \varphi_A \in \mathcal{E}(\mathbb{C}^n) \) has generalized analytic singularities. If \( e^{-2\varphi} - e^{-2\varphi_A} \) is locally integrable near \( o \), then \( \lim_{t \to -\infty} |\chi(t) - C_0 t| < \infty \).

**Remark 1.1.** (1) By Theorem 1.2, for any plurisubharmonic function \( \varphi \) near \( o \in \mathbb{C}^n \) with \( \nu(\varphi, o) > 0 \), there exists a convex increasing function \( \chi \) on \( \mathbb{R} \) such that \( e^{-2\chi} - e^{-2\varphi} \) is not locally integrable near \( o \).

(2) By a similar discussion in [6], we have that, for any complex manifold \( X \) (compact or noncompact) with \( \dim X \geq 2 \) and \( z_0 \in X \), there exists a quasi-psh
function \( \varphi \) on \( X \) such that for any plurisubharmonic function \( \varphi_A \) with generalized analytic singularities near \( z_0 \), \( e^{-2\varphi} - e^{-2\varphi_A} \) is not locally integrable near \( z_0 \).

Note that in Theorem 1.2, the Lelong number \( \nu(\chi \circ \varphi, o) \geq n \) is necessary. For Lelong number near 1 case, following from the plurisubharmonic function analytic singularities near \( z \) (Theorem 2.1).

Theorem 2.1. (\([4]\), Theorem 2.1). Let \( \Omega \subset \mathbb{C}^n \) be a bounded pseudoconvex domain, and let \( H \) be an affine linear subspace of \( \mathbb{C}^n \) of codimension \( p \geq 1 \) given by an orthogonal system \( s \) of affine linear equations \( s_1 = \cdots = s_p = 0 \). For every \( \beta < p \), there exists a constant \( C_{\beta,n,\Omega} \) depending only on \( \beta, n \) and the diameter of \( \Omega \), satisfying the following property. For every plurisubharmonic function \( \varphi \in Psh(\Omega) \) and \( f \in \mathcal{O}(\Omega \cap H) \) with \( \int_{\Omega \cap H} |f|^2 e^{-\varphi} d\lambda_H < +\infty \), there exists an extension \( F \in \mathcal{O}(\Omega) \) of \( f \) such that

\[
\int_{\Omega} |F|^2 |s|^{-\beta} e^{-\varphi} d\lambda_n \leq C_{\beta,n,\Omega} \int_{\Omega \cap H} |f|^2 e^{-\varphi} d\lambda_H,
\]

where \( d\lambda_n \) and \( d\lambda_H \) are the Lebesgue volume elements in \( \mathbb{C}^n \) and \( H \) respectively.

Theorem 2.2. (see \([2]\), Theorem 2.18). Let \( T \) be a closed positive current of bidimension \((p,p)\). Then \( T \) can be written as a convergent series of closed positive currents

\[
T = \sum_{k=1}^{+\infty} \lambda_k [A_k] + R,
\]

where \([A_k]\) is a current of integration over an irreducible analytic set of dimension \( p \), and \( R \) is a residual current with the property that \( \dim E_c(R) < p \) for every \( c > 0 \). This decomposition is locally and globally unique: the sets \( A_k \) are precisely the \( p \)-dimensional components occurring in the upperlevel sets \( E_c(T) \), and \( \lambda_k = \min_{z \in A_k} \nu(T,x) \) is the generic Lelong number of \( T \) along \( A_k \).

The proof of Lemma 4.7 in \([8]\) implies

Lemma 2.3. Let \( p : \Delta^2 \to \Delta, z = (z_1, z_2) \mapsto z_1 \) and \( \varphi \in Psh(\Delta^2) \). For almost all \( z_1 \in \Delta \) (in the sense of the Lebesgue measure on \( \Delta \)), the level set of Lelong numbers of \( \varphi|_{p^{-1}(z_1)} \) satisfies

\[
\{z|\nu(\varphi|_{p^{-1}(z_1)}, z) = 0\} = (\{z|\nu(\varphi, z) = 0\} \cap p^{-1}(z_1)).
\]

For the proof of Theorem 2.2 we also need the following two Lemmas.

Lemma 2.4. Let \( \varphi_1, \varphi_2 \) be two subharmonic functions near \( o \in \mathbb{C} \). If \( e^{-2\varphi_1} - e^{-2\varphi_2} \) is locally integrable near \( o \) and the Lelong number \( \nu(\varphi_1, o) \geq 1 \), then \( \nu(\varphi_1, o) = \nu(\varphi_2, o) \).
Proof. As $e^{-2\varphi_1} - e^{-2\varphi_2}$ is locally integrable near $o$, it follows from $\nu(\varphi_1, o) \geq 1$ that $\nu(\varphi_2, o) \geq 1$.

Suppose that $\nu(\varphi_1, o) = \nu(\varphi_2, o)$ does not hold. Without loss of generality, we put $\nu(\varphi_1, o) > \nu(\varphi_2, o)$. Let $a \in (\nu(\varphi_2, o), \min\{\nu(\varphi_1, o), \nu(\varphi_2, o) + 1\})$. Then, $\varphi_1 - (a - 1) \log |z|$ are subharmonic near $o$ and

$$
\nu(\varphi_1 - (a - 1) \log |z|, o) > 1, \quad \nu(\varphi_2 - (a - 1) \log |z|, o) < 1.
$$

That is to say $e^{-2(\varphi_1 - (a - 1) \log |z|)} - e^{-2(\varphi_2 - (a - 1) \log |z|)}$ is not locally integrable near $o$. However, the local integrability of $e^{-2\varphi_1} - e^{-2\varphi_2}$ near $o$ implies that

$$
|z|^{2(a-1)}(e^{-2\varphi_1} - e^{-2\varphi_2}) = e^{-2(\varphi_1 - (a - 1) \log |z|)} - e^{-2(\varphi_2 - (a - 1) \log |z|)}
$$

is locally integrable near $o$, which is a contradiction.

\[\square\]

Lemma 2.5. Let $\varphi_A = C_1 \log |z| + O(1)$ near $o \in \mathbb{C}$ and $\varphi$ a subharmonic function near $o \in \mathbb{C}$ with $\nu(\varphi, o) = C_2$, where $C_1, C_2$ are positive constants. Let $\chi$ be a convex increasing function on $\mathbb{R}$ such that $\lim_{t \to -\infty} \chi(t) = C_0 > 0$ with $C_0 C_2 \geq k$ for some $k \in \mathbb{N}^+$. If $|z|^{2(k-1)}(e^{-2\chi \varphi} - e^{-2\varphi_A})$ is locally integrable near $o$, then $\lim_{t \to -\infty} |\chi(t) - C_0 t| < \infty$.

Proof. As $\chi'' \geq 0$ and $\lim_{t \to -\infty} \chi'(t) = C_0 > 0$, we have $\chi'(t) \geq C_0$ for any $t \in \mathbb{R}$. Then, $\chi(t) - C_0 t$ is a convex increasing function on $\mathbb{R}$. Suppose that $\lim_{t \to -\infty} |\chi(t) - C_0 t| < \infty$ does not hold. Then $\lim_{t \to -\infty} \chi(t) - C_0 t = -\infty$.

Since $\nu(\chi \circ \varphi, o) = C_0 C_2 \geq k$, then $\nu(\chi \circ \varphi - (k - 1) \log |z|, o) \geq 1$. It follows from the integrability of $|z|^{2(k-1)}(e^{-2\chi \varphi} - e^{-2\varphi_A})$ near $o$ that $|z|^{2(k-1)}e^{-2\varphi_A}$ is not locally integrable near $o$, which implies that $\varphi_A - (k - 1) \log |z|$ is subharmonic near $o \in \mathbb{C}$ and $\nu(\varphi_A - (k - 1) \log |z|, o) \geq 1$. By Lemma 2.5, we have

$$
\nu(\chi \circ \varphi - (k - 1) \log |z|, o) = \nu(\varphi_A - (k - 1) \log |z|, o),
$$

which implies $\nu(\chi \circ \varphi, o) = \nu(\varphi_A, o) = C_1 \geq k$.

By the assumption, we have $\lim_{z \to 0} (\chi \circ \varphi - C_0 \varphi) = -\infty$. Then, we infer from $\varphi \leq C_2 \log |z| + O(1)$ that $\lim_{z \to 0} (\chi \circ \varphi - C_0 C_2 \log |z|) = -\infty$. Hence,

$$
\lim_{z \to 0} (e^{-2(\chi \circ \varphi - C_1 \log |z|)} - e^{-2(\varphi_A - C_1 \log |z|)}) = +\infty.
$$

It follows from $\nu(\chi \circ \varphi, o) = \nu(\varphi_A, o) = C_1 \geq k$ that $|z|^{2(k-1)}(e^{-2\chi \varphi} - e^{-2\varphi_A})$ is not locally integrable near $o$, which is a contradiction. \[\square\]

3. Proof of main results

We are now in a position to prove our main results.

Proof of Theorem 1.1. Let

$$
\varphi(z) = \log |z_1| + \sum_{k=1}^{N_{\varepsilon}} \alpha_k \log (|z_1| + \frac{|z_2|^2}{k^2})^{\beta_k},
$$

where $\alpha_k = M^{-k}, \beta_k = M^{2k}$ and $M \geq 2$. Then $\varphi \in Psh(\mathbb{C}^n)$, $\mathcal{J}(\varphi)_o = (z_1) \cdot \mathcal{O}_n$ and $\nu(\varphi, (0, z_2)) = 1, \forall z_2 \neq 0$.

Given $\varepsilon \in (0, 1]$. As $(z_1, z_2) \cdot \mathcal{O}_n \subseteq \mathcal{J}((1-\varepsilon)\varphi)_o$, there exists an integer $N > 0$ such that $z_2^N \in \mathcal{J}((1-\varepsilon)\varphi)_o$. Note that $\mathcal{J}((1-\varepsilon)\varphi)_o \subset \mathcal{J}((1-\varepsilon)\varphi_k)_o$ for every $k$, where $\varphi_k = \log |z_1| + \alpha_k \log (|z_1| + \frac{|z_2|^2}{k^2})^{\beta_k}$. To prove the desired result, it is
sufficient to show that for any integer $m > 0$, there exists $k_0$ and $\varepsilon_0 > 0$ such that $z^n_m \notin \mathcal{F}((1 - \varepsilon)\varphi_{k_0}), \forall \varepsilon \in (0, \varepsilon_0].$

Considering the integration of $|z_2|^m e^{-(1-\varepsilon)\varphi_k}$ over unit bidisk $\Delta \times \Delta$, we have

$$\int_{\Delta \times \Delta} \left( |z_2|^m e^{-(1-\varepsilon)\varphi_k} \right)^2 d\lambda_2$$

$$= \int_{\Delta^*} \left( \frac{1}{|z_1|^{(1-\varepsilon)(1+\alpha_k)}} \right)^2 \frac{\sqrt{-1}}{2} dz_1 \wedge d\bar{z}_1 \int_{\Delta} \left( \frac{|z_2|^m}{1 + |z_2|^k \varphi_k} \right)^2 \frac{\sqrt{-1}}{2} dz_2 \wedge d\bar{z}_2$$

$$= \int_{\Delta^*} \left( \frac{1}{|z_1|^{(1-\varepsilon)(1+\alpha_k)}} \right)^2 \frac{\sqrt{-1}}{2} dz_1 \wedge d\bar{z}_1 \int_{\Delta} \left( \frac{|z_2|^m}{1 + |z_2|^k \varphi_k} \right)^2 \frac{\sqrt{-1}}{2} dw \wedge d\bar{w}$$

$$\geq C \int_{\Delta^*} \left( \frac{1}{|z_1|^{(1-\varepsilon)(1+\alpha_k) - m/k}} \right)^2 \frac{\sqrt{-1}}{2} dz_1 \wedge d\bar{z}_1.$$ 

where $C > 0$ is some constant and $\Delta$ is the domain of integration in the new variable. Then, for any $m$, it follows from $\alpha_k = M^{-k}$ and $\beta_k = M^{2k}$ that there exists $k_0$ and $\varepsilon_0 > 0$ such that $(1 - \varepsilon_0)(1 + \alpha_{k_0}) - m/k_{k_0} \geq 1$, which implies $z^n_m \notin \mathcal{F}((1 - \varepsilon)\varphi_{k_0}), \forall \varepsilon \in (0, \varepsilon_0].$

\textbf{Proof of Theorem 1.2} Let

$$\xi : \mathbb{C}^n \to \mathbb{C}^n, \ z \mapsto (z_1, z_1 z_2, \ldots, z_1 z_n)$$

be a holomorphic mapping. Then $\xi$ is an isomorphism outside $\{z_1 = 0\}$. Hence, for sufficiently small polydisc $\Delta^n_\varepsilon$, we have

$$\int_{\Delta^n_\varepsilon} |z_1|^{2(n-1)} (e^{-2x(\varphi \circ \xi)} - e^{-2(\varphi_A \circ \xi)}) d\lambda_n = \int_{\xi(\Delta^n_\varepsilon) \setminus \{z_1 = 0\}} (e^{-2x(\varphi_A - \varphi_A A)} d\lambda_n < \infty.$$ 

That is to say, for almost every $(z_2, \ldots, z_n) \in \Delta^n_\varepsilon$, we have

$$\int_{\Delta^n_\varepsilon} |z_1|^{2(n-1)} (e^{-2x(\varphi \circ \xi)} - e^{-2(\varphi_A \circ \xi)}) d\lambda_n < \infty.$$ 

It follows from Lemma 2.3 that $\lim_{t \to -\infty} |\chi(t) - C_0 t| < \infty.$

\textbf{Proof of Theorem 1.3} It is enough to prove the dimension two case. In fact, if $e^{-2\varphi} - e^{-2\varphi_A}$ is locally integrable near the origin $o = (\alpha', \alpha'')$ for general $n (n \geq 3)$ case, then for almost all $(a_3, \ldots, a_n) \in \mathbb{C}^{n-2}$ near $\alpha''$, $(e^{-2\varphi} - e^{-2\varphi_A})_{\{z_3 = a_3, \ldots, z_n = a_n\}}$ is locally integrable near $\alpha'$, which is a contradiction.

Assume that $e^{-2\varphi} - e^{-2\varphi_A}$ is integrable on a relatively compact neighborhood $U_0$ of $o$. Then, $e^{-2\varphi} - e^{-2\varphi_A}$ is integrable on $U_0$ for any $0 \leq c \leq 1$. As $\mathcal{F}(\varphi) = (z_1) : \mathcal{O}_{C_2}$ on $U_0$, it follows that $e^{-2\varphi_A}$ is not locally integrable near $\{z_1 = 0\}|_{U_0}$. By Theorem 2.2 on $U_0$, we have $\varphi_A = \lambda \log |z_1| + \psi$, where $\lambda \geq 1$ and $\psi$ is a plurisubharmonic function near $o$ such that $(\{z_1 = 0\}, o) \not\subseteq \{\nu(\psi, z) \geq c, o\}, \forall c \in \mathbb{Q}^+$. It follows from Lemma 2.3 that $\psi$ is locally integrable near $0, z_2$, which is a contradiction to the local integrability of $e^{-2\varphi_A}$ near $0, z_2$. Hence, we have $\varphi_A = \log |z_1| + \psi$ on $U_0.$
Since \( \psi|_{z_1=0} \not\equiv -\infty \), then the Lelong number \( \nu(\psi|_{z_1=0}, o') < \infty \). Hence, we have an integer number \( N > 0 \) such that
\[
z_N^2 \in \mathcal{I}(\psi|_{z_1=0}) \subset \mathcal{I}(c\psi|_{z_1=0})_o, \quad \forall c \in (0,1).
\]
By Theorem 2.1 there exists a holomorphic function \( F_c \) on a Stein neighborhood \( V_0 \subset \subset U_0 \) of \( o \) such that
\[
F_c(z) \mid_{V_0 \cap \{z_1=0\}} = z_N^2 \mid_{V_0 \cap \{z_1=0\}} \quad \text{and} \quad F_c(z)_o \in \mathcal{I}(c\psi + c\log |z_1|)_o = \mathcal{I}(c\varphi_A)_o.
\]
Let \( F_c(z) = z_N^2 + z_1 \cdot G_c(z) \) for some holomorphic function \( G_c(z) \) near \( o \). By the assumption, we have
\[
(z_1) \cdot O_2 = \mathcal{I}(\varphi)_o = \mathcal{I}(\varphi_A)_o \subset \mathcal{I}(c\varphi_A)_o.
\]
It follows that \( z_N^2 = F_c(z) - z_1 \cdot G_c(z) \in \mathcal{I}(c\varphi_A)_o \) for any \( c \in (0,1) \). However, by the proof of Theorem 1.1 we know that there exists \( \varepsilon_0 > 0 \) such that
\[
z_N^2 \not\in \mathcal{I}((1-\varepsilon)\varphi)_o, \quad \forall \varepsilon \in (0,\varepsilon_0].
\]
which is a contradiction to the assumption, i.e., \( e^{-2\varphi} - e^{-2\varphi_A} \) is not locally integrable near \( o \). \qed

Acknowledgements. Both authors would like to sincerely thank our supervisor, Professor Xiangyu Zhou for valuable discussions and useful comments. The second author is also grateful to Beijing International Center for Mathematical Research for its good working conditions.

References

[1] J.-P. Demailly, *Multiplier ideal sheaves and analytic methods in algebraic geometry*, School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), 1–148, ICTP Lect. Notes, 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.

[2] J.-P. Demailly, *Analytic Methods in Algebraic Geometry*, Higher Education Press, Beijing, 2010.

[3] J.-P. Demailly, *Extension of holomorphic functions defined on non reduced analytic subvarieties*, preprint, arXiv: 1510.05230.

[4] J.-P. Demailly, J. Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 4, 525–556.

[5] L. Ein, R. Lazarsfeld, K. E. Smith, D. Varolin, *Jumping coefficients of multiplier ideals*, Duke Math. J. 123 (2004), no. 3, 469–506.

[6] Q. A. Guan, *Nonexistence of decreasing equisingular approximations with logarithmic poles*, J. Geom. Anal., published online on March 16, 2016; DOI: 10.1007/s12220-016-9702-2.

[7] Q. A. Guan, X. Y. Zhou, *A proof of Demailly’s strong openness conjecture*, Ann. of Math. 182 (2015), 605–616. See also arXiv: 1311.3781.

[8] Q. A. Guan, X. Y. Zhou, *Characterization of multiplier ideal sheaves with weights of Lelong number one*, Adv. Math. 285 (2015), 1688–1705.

Qi'an Guan: School of Mathematical Sciences, and Beijing International Center for Mathematical Research, Peking University, Beijing, 100871, China.

E-mail address: guanqian@amss.ac.cn

Zhenqian Li: School of Mathematical Sciences, Peking University, Beijing, 100871, China.

E-mail address: lizhenqian@amss.ac.cn