A remark on resolution of terminal singularities

Yu. G. Prokhorov

Abstract. Let \((Z, o)\) be a three-dimensional terminal singularity of type \(cA/r\). We prove that all exceptional divisors over \(o\) with discrepancies \(\leq 1\) are rational.

Let \((Z, o)\) be a three-dimensional terminal singularity of index \(r \geq 1\) and let \(\varphi: \tilde{Z} \to Z\) be a resolution. Let \(S \subset \tilde{Z}\) be an exceptional divisor such that \(\text{Center}(S) = o\). It is clear that the birational type of \(S\) does not depend on \(g\). By [1, 2.14] the surface \(S\) is birationally ruled. We say that the corresponding discrete valuations \(\nu = \nu_S\) of the function field \(K(Z)\) is rational (resp. birationally ruled) if so is the surface \(S\). We are interested in the existence of rational exceptional divisors over \(o \in Z\) with small discrepancies:

**Theorem.** Let \((Z, o)\) be a three-dimensional terminal singularity of type \(cA/r\), \(r \geq 1\) and let \(\nu\) be a divisorial discrete valuation of the function field \(K(Z)\) such that \(a(\nu) \leq 1\) and \(\text{Center}_Z(\nu) = o\). Then \(\nu\) is rational.

Recall that according to the classification [2], [3], \((Z, o)\) belongs to one of the following classes: \(cA/r\), \(cA^4\), \(cA^2\), \(cD/2\), \(cD/3\), \(cE/2\), \(cD\), \(cE\). It was proved in the series of works [4], [5], [6] (see also [7], and [8]) that for any \(i \in \mathbb{N} \setminus r\mathbb{N}\) there exists an exceptional divisor \(S\) with center at \(o\) and discrepancy \(a(S) = i/r\). Exceptional divisors with discrepancies \(< 1\) appear in any resolution. Divisors with discrepancy \(1\) and \(\text{Center} = o\) appear in any *divisorial* resolution, i.e., in a resolution such that the exceptional set is of pure codimension 1.

**Proof of Theorem.** Let \(F \in |-K_Z|\) be a general member. Then \((F, o)\) is a Du Val singularity (of type \(A_n\)) [3, 6.4]. By the Inversion

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of Adjunction [3, 17.6] the pair \((Z, F)\) is plt. Let \(q: Z^q \to Z\) be \(\mathbb{Q}\)-factorialization (see, e.g., [3, 6.7]). Then \(Z^q\) has (terminal) singularities of types \(\text{cA}/r_i\). Indeed, the surface \(F^q := q^{-1}(F)\) contains all singular points of \(Z^q\) and \(F^q \in | - K_{Z^q}|\). Since \(F^q \to F\) is a crepant morphism, \(F^q\) has only singularities of types \(\text{A}_{n_i}\). Thus, replacing \(Z\) with \(Z^q\), we may assume that \(Z\) is \(\mathbb{Q}\)-factorial.

Let \(S\) be an exceptional divisor with center at \(o\) and discrepancy \(a(S) \leq 1\). Then \(a(S, F) < 1\). Since \(K_Z + F\) is linearly trivial, we have \(a(S, F) \leq 0\). Further, there is an 1-complement \(K_F + \Theta\) of \(K_F\) near \(o\) (see [10, 5.2.3]). According to [11, 4.4.1] this complement can be extended to \(Z\), i.e., there is an (integral) Cartier divisor \(F'\) such that \(F'|_F = \Theta\), \(K_Z + F + F' \sim 0\), and \((Z, F + F')\) is lc. Then \(a(S, F + F') = -1\). Now our theorem is a consequence of the following simple fact. 

**Proposition.** Let \((Z, o)\) be a three-dimensional \(\mathbb{Q}\)-factorial terminal singularity and let \(\nu\) be a divisorial discrete valuation of the field \(\mathcal{K}(Z)\). Assume that there is a boundary \(D\) such that the pair \((Z, D)\) is lc and \(a(\nu, D) = -1\). Then

(i) The valuation \(\nu\) is rational or is birationally a ruled surfaces over an elliptic curve;

(ii) if, moreover, \(|D|\) there are at least two components passing through \(o\), then \(\nu\) is rational.

**Proof.** According to [3, 17.10] there is a blowup \(f: X \to Z\) with irreducible exceptional divisor \(S\) representing the valuation \(\nu\) such that the log divisor \(K_X + S + D_X = f^*(K_Z + D)\) is lc. Here \(D_X\) is the proper transform of \(D\). In this situation we have \(\rho(X/Z) = 1\). Therefore, \(D_X \equiv -(K_X + S)\) is \(f\)-ample.

Consider a minimal log terminal modification \(g: Y \to X\) of the pair \((X, S + D_X)\) (see, e.g., [11, 3.1.3]), i.e., a blowup such that \(Y\) is \(\mathbb{Q}\)-factorial and

\[K_Y + S_Y + D_Y + E = g^*(K_X + S + D_X),\]

is dlt. Here \(S_Y, D_Y\) are proper transforms of \(S, D_X\), respectively, and \(E = \sum E_i\) is a (reduced) exceptional divisor with \(a(E_i, S + D_X) = -1\). Denote \(\Delta := \text{Diff}_{S_Y}(D_Y + \sum E_i)\) and \(\Omega := f^*D_X|_{S_Y}\). By [3, 17.7] the surface \(S_Y\) is normal and the pair \((S_Y, \Delta)\) is lc. The assertion of (i) easily follows by the lemma below.

To prove (ii) we assume that \(S_Y\) is not rational and \(|D|\) is not irreducible. Then \(S \cap |D_X|\) has at least two irreducible components. So is \(S_Y \cap |D_Y + E|\). By the lemma below \(|\Delta|\) has exactly two components (contained in \(|D_Y + E|\)) and the pair \((S_Y, \Delta)\) is plt. Further, \((S_Y, \Delta - \varepsilon\Omega)\) is klt whenever \(0 < \varepsilon\). For \(0 < \varepsilon \ll 1\), the pair \((S_Y, \Delta - \varepsilon\Omega)\)


is a klt log del Pezzo (because \( \Omega \) is nef and big). In this situation, \( S_Y \) is rational (see, e.g., [11], 5.4.1).

**LEMMA.** Let \((S, \Delta)\) be a projective log surface with \( \kappa(S) = -\infty \). Assume that \( K_X + \Delta \) is lc and numerically trivial and the surface \( S \) is nonrational. Then \( S \) is birationally ruled over an elliptic curve and there exists at most two divisors with discrepancy \( a(\Delta) = -1 \).

**Proof** (cf. [10], 6.9). Replace \( S \) with its minimal resolution and \( \Delta \) with its crepant pull-back. There is a contraction \( \phi: S \to C \) (with general fiber \( \mathbb{P}^1 \)) onto a curve \( C \) of genus \( p_a(C) \geq 1 \). In this situation pair \((S, \Delta)\) has only canonical singularities and all components of \( \Delta \) are horizontal [11, 8.2.2-8.2.3]. Hence, divisors with discrepancy \( a(\Delta) = -1 \) are exactly components of \( \lfloor \Delta \rfloor \). As an immediate consequence we have that the number of divisors with discrepancy \(-1\) is at most two. If \( \lfloor \Delta \rfloor \neq 0 \), then for any component \( \Delta_i \subset \lfloor \Delta \rfloor \) we have \( 2p_a(\Delta_i) - 2 \leq (K_S + \Delta) \cdot \Delta_i = 0 \). Therefore, \( p_a(C) \leq p_a(\Delta_i) \leq 1 \). It remains to consider the case, when the pair \((S, \Delta)\) is klt. Again for any component \( \Delta_i \subset \text{Supp}(\Delta) \) we have \( \Delta_i^2 \leq 0 \) (otherwise \((S, \Delta + \varepsilon \Delta_i)\) is a klt log del Pezzo). As above, \( p_a(C) \leq p_a(\Delta_i) = \frac{1}{2}(K_S + \Delta_i) \cdot \Delta_i + 1 \leq \frac{1}{2}(K_S + \Delta) \cdot \Delta_i + 1 = 1 \).

Note that the assertion (ii) of our theorem is not true for other types of terminal singularities:

**Example** ([4]). Let \((Z, o)\) be a terminal \(cAx/2\)-singularity \( \{x^2 + y^2 + z^{4m} + t^{4m} = 0\}/\mathbb{Z}_2(0, 1, 1, 1) \). Consider the weighted blowup with weight \( \frac{1}{2}(2m, 2m + 1, 1, 1) \). Then the exceptional divisor \( S \) is given in \( \mathbb{P}(2m, 2m + 1, 1, 1) \) by the equation \( x^2 + z^{4m} + t^{4m} = 0 \). It is reduced, irreducible, and \( a(S) = 1/2 \). It is easy to see that \( S \) is a birationally ruled surface over a hyperelliptic curve of genus \( 2m - 1 \).

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**Department of Algebra, Faculty of Mathematics, Moscow State Lomonosov University, Moscow 117234, Russia**

*E-mail address: prokhor@mech.math.msu.su*