CONCENTRATION AND LIMIT BEHAVIORS OF STATIONARY MEASURES

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Abstract. In this paper, we study limit behaviors of stationary measures of the Fokker-Planck equations associated with a system of ordinary differential equations perturbed by a class of multiplicative including additive white noises. As the noises are vanishing, various results on the invariance and concentration of the limit measures are obtained. In particular, we show that if the noise perturbed systems admit a uniform Lyapunov function, then the stationary measures form a relatively sequentially compact set whose weak*-limits are invariant measures of the unperturbed system concentrated on its global attractor. In the case that the global attractor contains a strong local attractor, we further show that there exists a family of admissible multiplicative noises with respect to which all limit measures are actually concentrated on the local attractor; and on the contrary, in the presence of a strong local repeller in the global attractor, there exists a family of admissible multiplicative noises with respect to which no limit measure can be concentrated on the local repeller. Moreover, we show that if there is a strongly repelling equilibrium in the global attractor, then limit measures with respect to typical families of multiplicative noises are always concentrated away from the equilibrium. As applications of these results, an example of stochastic Hopf bifurcation is provided.

Our study is closely related to the problem of noise stability of compact invariant sets and invariant measures of the unperturbed system.

1. Introduction

Regarded as a physical model, a dynamical system generated from ordinary differential equations is often subject to noise perturbations either from its surrounding environment or from intrinsic uncertainties associated with the system. Analyzing the impact of noise perturbations on the dynamics of the system then becomes a fundamental issue with respect to both modeling and dynamics.

There have been many studies toward this dynamics issue using either a trajectory-based or a distribution-based approach. The trajectory-based approach is often adopted under the framework of random dynamical systems, i.e., skew-product flows with ergodic measure-preserving base flows. By assuming vanishing noise at a reference equilibrium, noise perturbations of essential dynamics of a dynamical system are studied under the random dynamical

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system framework with respect to problems such as noise perturbations of invariant manifolds ([5, 16, 17, 44]), normal forms ([3, 4, 8, 35]), and stochastic bifurcations (see [3] and references therein). For a system of ordinary differential equations subject to white noise perturbations vanishing at a reference equilibrium, we refer the reader to [23] for some study of stochastic stability of the equilibrium (see also [37] for similar studies in infinite dimension).

With respect to general noise perturbations, the distribution-based approach is useful and seemingly necessary to adopt under both frameworks of random dynamical systems and Itô stochastic differential equations. Due to its essential differences from deterministic dynamical systems, much less is known in this direction comparing with cases using the trajectory-based approach. For some important pioneer works on noise perturbations of dynamical systems on a compact manifold from the viewpoint of distributions, we refer the reader to [24, 39] for stochastic stability of flows on a 2-torus or a periodic cycle, to [20] for stochastic stability of equilibria and periodic cycles by introducing large deviation theory, to [15, 31, 30, 46] for stochastic stability of SRB measures, and to [47] for some global stochastic stability characterizations.

In this paper, we adopt the distribution-based approach to study the impact of white noises on basic dynamics of a system of ordinary differential equations in an Euclidean space. More precisely, we consider a system of ordinary differential equations

\[
\dot{x} = V(x), \quad x \in \mathcal{U} \subset \mathbb{R}^n,
\]

where \(\mathcal{U}\) is a connected open set which can be bounded, unbounded, or the entire \(\mathbb{R}^n\), and \(V = (V^i) \in C(\mathcal{U}, \mathbb{R}^n)\). We assume throughout the paper that \(\dot{x} = V(x)\) generates a local flow \(\varphi^t\) on \(\mathcal{U}\). The generality of domain \(\mathcal{U}\) does allow a wide range of applications because many physical models (e.g., those concerning populations and concentrations) are not necessarily defined in the entire \(\mathbb{R}^n\). Adding general multiplicative (i.e., spatially non-homogenous) including additive (i.e., spatially homogenous) white noise perturbations, we obtain the following Itô stochastic differential equations

\[
dx = V(x)dt + G(x)dW, \quad x \in \mathcal{U} \subset \mathbb{R}^n,
\]

where \(W\) is a standard \(m\)-dimensional Brownian motion for some integer \(m \geq n\), and \(G = (g^{ij})_{n \times m}\) is a matrix-valued function on \(\mathcal{U}\), called noise coefficient matrix. For generality, we assume that \(g^{ij} \in W^{1,2p}_{loc}(\mathcal{U})\), \(i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\), for some fixed constant \(p > n\).

The stochastic differential equations (1.2) arise naturally as a non-isolated physical system subject to noise perturbations from its surrounding environments, in which the impact of noises on dynamics is often physically measured in term of distributions. They can also arise naturally from the study of a large scale deterministic but seemingly stochastic system, for instance a so-called mesoscopic system which is partially structured but contains intrinsic uncertainties in a fast time scale due to high complexity, large degree of freedom, lack of full knowledge of mechanisms, the need for organizing a large amount of data, etc. Under some exponential mixing assumptions on the fast dynamics, such a mesoscopic system can have a stochastic reduction of the form (1.2) over any finite time interval in which \(V\) represents the structured field and \(G\) buries all dynamical uncertainties (see e.g., [29, 38]). It has been
argued for a mesoscopic system that in the case of sufficiently high uncertainty, trajectory-based approach using either deterministic or random dynamics modeling would not provide much information to its dynamical description. Instead, a distribution-based approach using stochastic differential equations like (1.2) is necessary to adopt in order to synthesize the typical patterns of dynamics (see [41] and references therein).

An important distribution-based approach for studying diffusion process generated by (1.2) is to use its associated Fokker-Planck equation (also called Kolmogorov forward equation)

\[
(1.3) \quad \begin{cases}
\frac{\partial u(x,t)}{\partial t} = L_A u(x,t), & x \in \mathcal{U}, t > 0, \\
u(x,t) \geq 0, & \int_{\mathcal{U}} u(x,t)dx = 1,
\end{cases}
\]

where \( A = (a_{ij}) = \frac{GG^T}{2} \), called the diffusion matrix, and \( L_A \) is the Fokker-Planck operator defined as

\[
L_A g(x) = \partial_{ij}^2 (a_{ij}(x)g(x)) - \partial_i (V^i(x)g(x)), \quad g \in C^2(\mathcal{U}).
\]

We note that \( a_{ij} \in W^{1,2}_\text{loc}(\mathcal{U}), i, j = 1, 2, \ldots, n \). It is well-known that if the stochastic differential equation (1.2) generates a (local) diffusion process in \( \mathcal{U} \) (e.g., when both \( V \) and \( G \) are locally Lipschitz in \( \mathcal{U} \)), then its transition probability density function, if exists, is actually a (local) fundamental solution of the Fokker-Planck equation (1.3).

In the above and also through the rest of the paper, we use short notions \( \partial_i = \frac{\partial}{\partial x_i} \), \( \partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \), and we also adopt the usual summation convention on \( i, j = 1, 2, \ldots, n \) whenever applicable.

Long time behaviors of solutions of the Fokker-Planck equation (1.3) is governed by the stationary Fokker-Planck equation

\[
(1.4) \quad \begin{cases}
L_A u = \partial_{ij}^2 (a_{ij}u) - \partial_i (V^i u) = 0, \\
u(x) \geq 0, & \int_{\mathcal{U}} u(x)dx = 1,
\end{cases}
\]

which, in the weak form, becomes

\[
(1.5) \quad \begin{cases}
\int_{\mathcal{U}} L_A f(x)u(x)dx = 0, & \text{for all } f \in C^\infty_0(\mathcal{U}), \\
u(x) \geq 0, & \int_{\mathcal{U}} u(x)dx = 1,
\end{cases}
\]

where \( C^\infty_0(\mathcal{U}) \) denotes the space of \( C^\infty \) functions on \( \mathcal{U} \) with compact supports and

\[
L_A = a_{ij} \partial_{ij}^2 + V^i \partial_i
\]

is the adjoint Fokker-Planck operator corresponding to \( A \). Solutions of (1.5) are called weak stationary solutions of (1.3) or stationary solutions corresponding to \( L_A \). More generally, one considers a measure-valued stationary solution \( \mu_A \) of the Fokker-Planck equation (1.3), called a stationary measure of the Fokker-Planck equation (1.3) or a stationary measure corresponding to \( L_A \), which is a Borel probability measure satisfying

\[
(1.6) \quad \int_{\mathcal{U}} L_A f(x)d\mu_A(x) = 0, \quad \text{for all } f \in C^\infty_0(\mathcal{U}).
\]

If a stationary measure \( \mu_A \) is regular, i.e., \( d\mu_A(x) = u_A(x)dx \) for some density function \( u_A \in C(\mathcal{U}) \), then it is clear that \( u_A \) is necessarily a weak stationary solution of (1.3), i.e, it satisfies (1.3). Conversely, according to the regularity theorem in [9], if \( (a_{ij}) \) is everywhere positive
definite in $\mathcal{U}$, then any stationary measure corresponding to $L_A$ must be regular with positive density function lying in $W^{1,p}_{loc}(\mathcal{U})$. In the case that (1.2) generates a diffusion process on $\mathcal{U}$, it is well-known that any invariant measure of the diffusion process is necessarily a stationary measure of the Fokker-Planck equation (1.3), but the converse need not be true. However, under some mild conditions a stationary measure of the Fokker-Planck equation (1.3) is always a sub-invariant measure of some generalized diffusion process (see [10, 13, 14] for discussions in this regard in particular with respect to the uniqueness of stationary measures and their invariance). In this sense, stationary measures of (1.3) may be regarded as generalizations of invariant measures of a classical diffusion process.

The existence of stationary measures of Fokker-Planck equations in $\mathbb{R}^n$ has been extensively investigated (see e.g., [2], [9]-[12], [23], [42] and references therein). In our recent work [26], such existence is investigated for a general domain under certain relaxed Lyapunov conditions. In addition, results concerning non-existence of stationary measures of Fokker-Planck equations in a general domain are also obtained in our work [27] under some anti-Lyapunov conditions, which, together with the existence results in [26], lead to both sufficient and necessary conditions for the existence of stationary measures of Fokker-Planck equations.

While the non-existence of stationary measures of Fokker-Planck equations associated with a family of noise perturbations reflects a strong stochastic instability of the unperturbed deterministic system with respect to these noises, stochastic stability and instability of the deterministic system at a dynamics level can often occur with respect to noise families for which stationary measures do exist for the corresponding Fokker-Planck equations. In order to study the impact of noises on such stochastic stability of the deterministic system when it generates a local flow, a fundamental problem is to classify basic dynamics subjects like compact invariant sets and invariant measures of the local flow that can “survive” from a given family of noise perturbations. This is in fact our main motivation for the present study.

To be more precise, for a fixed drift field $V \in C(\mathcal{U}, \mathbb{R}^n)$, we consider noise coefficients matrices lying in the class

$$\tilde{G} = \{ G = (g^{ij}) : \text{Rank}(G) \equiv n, \ g^{ij} \in W^{1,2p}_{loc}(\mathcal{U}), \ i = 1, 2, \cdots, n, \ j = 1, 2, \cdots, m \}$$

for some fixed $p > n$. The class $\tilde{G}$ gives rise to the following class of diffusion matrices:

$$\tilde{A} = \{ A = (a^{ij}) \in W^{1,p}_{loc}(\mathcal{U}, GL(n, \mathbb{R})) : A = GG^\top \text{ for some } G \in \tilde{G} \}.$$  

(1.7)

To consider small noise perturbations, we will pay particular attention to the so-called null family (resp. bounded null family) $A = \{ A_\alpha \} \subset \tilde{A}$, i.e., a directed net of $\tilde{A}$ which converges to $0$ - the zero matrix, under the topology of $W^{1,p}$-convergence on any compact subsets of $\mathcal{U}$ (resp. $L^\infty$-convergence on $\mathcal{U}$). Then with respect to the system (1.1) and a given null family $A = \{ A_\alpha \} \subset \tilde{A}$, our study amounts to the characterization of behaviors of $A$-limit measures, i.e., sequential limit points, as $A_\alpha \to 0$, of stationary measures $\{ \mu_\alpha \}$ corresponding to $\{ \mathcal{L}_{A_\alpha} \}$ under the weak$^\ast$-topology. Our particular attention will be paid to issues such as invariance and concentration of $A$-limit measures, stochastic stability of compact invariant sets under uniform Lyapunov conditions, and the role played by the multiplicative noises $A$ to the stabilization of a strong local attractor or de-stabilization of a strong local repeller.

Our main results of the paper are as follows.
**Theorem A.** Let $\mathcal{A} = \{A_\alpha\} \subset \hat{\mathcal{A}}$ be a null family. Then the following holds.

a) (Invariance of limit measures) If $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and $\varphi^t$ is a flow on $\mathcal{U}$, then any $\mathcal{A}$-limit measure must be an invariant measure of $\varphi^t$.

b) (Stochastic LaSalle invariance principle) If \( (1.1) \) admits an entire weak Lyapunov (resp. anti-Lyapunov) function $U_0$, then any $\mathcal{A}$-limit measure with compact support is concentrated on the set $S_0 = \{x \in \mathcal{U} : V(x) \cdot \nabla U_0(x) = 0\}$.

c) (Local concentration of limit measures) If $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and $\mathcal{E}_0$ is either a strong local attractor or a strong local repeller of $\varphi^t$, then there is a neighborhood $\mathcal{W}_0$ of $\mathcal{E}_0$ in $\mathcal{U}$ such that $\mu(\mathcal{W}_0 \setminus \mathcal{E}_0) = 0$ for any $\mathcal{A}$-limit measure $\mu$.

Thus, when both $V$ and $A_\alpha$’s are smooth in $\mathcal{U}$, it follows from part a) of Theorem A that the stationary measures $\{\mu_\alpha\}$ corresponding to $\{\mathcal{L}_A\}$, by having smooth density functions, can be regarded as smoothers of their limit invariant measures.

We note that parts b), c) of Theorem A do not require $\varphi^t$ be a flow on $\mathcal{U}$, in which case there is no guarantee that an $\mathcal{A}$-limit measure is an invariant measure of $\varphi^t$. However, in the case that $\varphi^t$ is only a semiflow on $\mathcal{U}$, the invariance of an $\mathcal{A}$-limit measure can be also shown under certain conditions (see Theorem 5.1 c) and Theorem B below).

Given a null family $\mathcal{A} = \{A_\alpha\} \subset \hat{\mathcal{A}}$, if there exists a uniform Lyapunov function in $\mathcal{U}$ with respect to the family $\{\mathcal{L}_A\}$ (see Section 2), then there must exist a stationary measure $\mu_\alpha$ corresponding to each $\mathcal{L}_A$, (Proposition 2.1) and $\varphi^t$ is necessarily a dissipative semiflow on $\mathcal{U}$ (Proposition 2.4) whose global attractor necessarily admits a flow extension (Proposition 6.1).

**Theorem B.** Let $\mathcal{A} = \{A_\alpha\} \subset \hat{\mathcal{A}}$ be a null family. Assume that there is a uniform Lyapunov function in $\mathcal{U}$ with respect to the family $\{\mathcal{L}_A\}$ and denote by $\{\mu_\alpha\}$ the set of all stationary measures corresponding to $\{\mathcal{L}_A\}$. Then the following holds.

a) (Tightness of stationary measures) $\{\mu_\alpha\}$ is $\mathcal{A}$-sequentially null compact in $M(\mathcal{U})$ - the space of Borel probability measures on $\mathcal{U}$, i.e., for any sequence $\{A_k\}_{k=1}^\infty \subset \mathcal{A}$ with $A_k \to 0$, $\{\mu_\alpha\}_{\alpha \in \mathcal{A}}$ is relatively compact in $M(\mathcal{U})$.

b) (Global concentration of limit measures) If $V \in C^1(\mathcal{U}, \mathbb{R}^n)$, then any $\mathcal{A}$-limit measure is an invariant measure of $\varphi^t$ concentrated on the global attractor of $\varphi^t$.

Concentration of $\mathcal{A}$-limit measures on a compact invariant set of $\varphi^t$ is a primary feature of stochastic stability of the invariant set with respect to the noise family $\mathcal{A}$ - the so-called $\mathcal{A}$-stability which we will define in Section 2. For instance, Theorem B actually imply that the global attractor of $\varphi^t$ is $\mathcal{A}$-stable if $\mathcal{A}$ is a so-called invariant null family. Such $\mathcal{A}$-stability is also closely related to the so-called $\mathcal{A}$-stability of an invariant measure of $\varphi^t$. We refer the reader to Sections 2, 3 for more discussions in these regards.

Multiplicative noises actually play important roles in stabilizing a local attractor or destabilizing a local repeller of $\varphi^t$. To analyze such roles played by multiplicative noises, we assume that \( (1.1) \) admits a Lyapunov function in $\mathcal{U}$ whose second derivatives are bounded. This Lyapunov function then becomes a uniform Lyapunov function with respect to a bounded null family in $\hat{\mathcal{A}}$ (Proposition 2.5) for which Theorem B is applicable.
**Theorem C.** Assume $V \in C^1(U, \mathbb{R}^n)$ and that (1.1) admits a Lyapunov function in $U$ whose second derivatives are bounded. Then the following holds.

a) (Noise stabilization of local attractors) If the global attractor of $\varphi^t$ contains a strong local attractor, then there exists a normal null family $\mathcal{A} = \{A_\alpha\} \subset \hat{\mathcal{A}}$ such that stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ exist, form an $\mathcal{A}$-sequentially null compact set in $M(U)$, and any $\mathcal{A}$-limit measure is an invariant measure of $\varphi^t$ concentrated on the local attractor.

b) (Noise de-stabilization of local repellers) If the global attractor of $\varphi^t$ contains a strong local repeller, then there exists a normal null family $\mathcal{A} = \{A_\alpha\} \subset \hat{\mathcal{A}}$ such that stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ exist, form an $\mathcal{A}$-sequentially null compact set in $M(U)$, but all $\mathcal{A}$-limit measures are concentrated away from the local repeller.

c) (Noise instability of repelling equilibria) If the global attractor of $\varphi^t$ contains a strongly repelling equilibrium, then with respect to any normal null family $\mathcal{A} = \{A_\alpha\} \subset \hat{\mathcal{A}}$ on $U$, stationary measures corresponding to $\{\mathcal{L}_{A_\alpha}\}$ exist, form an $\mathcal{A}$-sequentially null compact set in $M(U)$, but all $\mathcal{A}$-limit measures are concentrated away from the equilibrium.

In the above theorem, a normal null family is a more restricted bounded null family defined in Section 2. For definitions of a strong local attractor or repeller and strongly repelling equilibrium of (1.1), we refer the reader to Section 6.

The proof of Theorem C actually contains explicit conditions on the noise family $\mathcal{A}$ under which a) or b) above holds. For instance, in the case of a), these conditions actually quantify how strong the noises should be away from the local attractor in order to achieve the desired stabilization. We refer the reader to Section 4 for details.

We note that Theorem C can actually be re-stated as such that the strong local attractor in a) is $\mathcal{A}$-stable with respect to some invariant, bounded null family $\mathcal{A}$, the strong local repeller in b) is strongly $\mathcal{A}$-unstable with respect to some invariant, bounded null family $\mathcal{A}$, and the repelling equilibrium in c) is strongly $\mathcal{A}$-unstable with respect to any invariant, normal null family $\mathcal{A}$.

This paper is organized as follows. Section 2 is a preliminary section in which we mainly review some fundamental properties of stationary measures of Fokker-Planck equations associated with (1.2) and introduce various notions of stochastic stability for compact invariant sets and invariant measures of $\varphi^t$. In Section 3, we study behaviors and concentration of limit measures. Possible local concentration of a limit measure on a strong local attractor or repeller will be studied and invariance or semi-invariance of limit measures will be discussed based on some new characterizations on invariant and semi-invariant measures of $\varphi^t$. Theorems A, B will be proved in this section. In Section 4, we prove Theorem C by characterizing noise families that stabilize a strong local attractor or de-stabilize a strong local repeller. In Section 5, we demonstrate applications of our main results by considering an example of stochastic Hopf bifurcation in which the existence of a stochastically stable cycle is observed. Section 6 is an Appendix in which we summarize some basic notions and dynamical properties for a system of dissipative ordinary differential equations.
We remark that if the system (1.1) is defined on $U \times M$, where $M$ is a smooth, compact manifold without boundary (e.g. $M = \mathbb{T}^k$, the $k$-torus), then one can modify the definitions of uniform Lyapunov and anti-Lyapunov functions by replacing the domain $U \subset \mathbb{R}^n$ with $U \times M$ so that all results in Sections 3, 4 hold with respect to such a generalized domain.

Through the rest of the paper, for simplicity, we will use the same symbol $| \cdot |$ to denote the absolute value of a number, the norm of a vector, a matrix, and the Lebesgue measure of a set. For any connected open set $\Omega \subset U$ and any integer $0 \leq M < \infty$ or $M = \infty$, we denote by $C^M_0(\Omega)$ the set of functions in $C^M(\Omega)$ with compact supports in $\Omega$ and by $C^M_b(\Omega)$ the set of functions with bounded derivatives up to order $M$. When $M = 0$, we denote $C^0_0(\Omega), C^0_b(\Omega)$ simply by $C^0(\Omega), C^b(\Omega)$ respectively.

2. Preliminary

In this section, we will review some fundamental properties of stationary measures of the Fokker-Planck equation associated with (1.2) from [25, 26] and define various notions of stochastic stability for compact invariant sets and invariant measures. We will also recall a Harnack inequality to be used in later sections.

2.1. Lyapunov function and stationary measures. Let $A = (a^{ij})$ be a given everywhere positive definite, $n \times n$ matrix-valued function on $U$ such that $a^{ij} \in W^1_\text{loc}(U)$, $i,j = 1, 2, \cdots, n$. The matrix-valued function corresponds to an adjoint Fokker-Planck operator:

$$\mathcal{L}_A = a^{ij} \partial^2_{ij} + V^i \partial_i$$

which defines a weak form of stationary Fokker-Planck equation (1.5).

Let $\Omega$ be a connected open subset of $U$. We recall from [25, 26] that a non-negative function $U \in C(\Omega)$ is a compact function if i) $U(x) < \rho_M$, $x \in \Omega$; and ii) $\lim_{x \to \partial\Omega} U(x) = \rho_M$, where $\rho_M = \sup_{x \in \Omega} U(x)$ is called the essential upper bound of $U$. For each $\rho \in [0, \rho_M)$, we denote $\Omega_\rho = \{x \in \Omega: U(x) < \rho\}$ as the $\rho$-sublevel set of $U$ and $U^{-1}(\rho) = \{x \in \Omega: U(x) = \rho\}$ as the $\rho$-level set of $U$. In the above, the notion $\partial \Omega$ and limit $x \to \partial \Omega$ are defined through a unified topology which identifies the extended Euclidean space $\mathbb{E}^n = \mathbb{R}^n \cup \partial \mathbb{R}^n$ with the closed unit ball $\mathbb{B}^n = \mathbb{B}^n \cup \partial \mathbb{B}^n$ in $\mathbb{R}^n$ so that $\partial \mathbb{R}^n$, consisting of infinity elements of all rays, is identified with $\partial \mathbb{E}^n = \mathbb{S}^{n-1}$ (see [25, 26] for details). Therefore, when $\Omega = \mathbb{R}^n$, the limit $x \to \partial \mathbb{R}^n$ is simply equivalent to $x \to \infty$.

We also recall from [25, 27] the following notions of Lyapunov-like and anti-Lyapunov-like functions.

Definition 2.1. A $C^2$ compact function $U$ is called a Lyapunov function (resp. anti-Lyapunov function) in $\Omega$ with respect to $\mathcal{L}_A$, if there is a $\rho_m \in (0, \rho_M)$, called essential lower bound of $U$, and a constant $\gamma > 0$, called Lyapunov constant (resp. anti-Lyapunov constant) of $U$, such that

$$\mathcal{L}_A U(x) \leq -\gamma, \quad (\text{resp.} \geq \gamma), \quad x \in \tilde{\Omega} = \Omega \setminus \Omega_{\rho_m},$$

where $\tilde{\Omega}$ is called essential domain of $U$. If $\gamma = 0$ in the above, then $U$ is referred to as a weak Lyapunov function (resp. weak anti-Lyapunov function) in $\Omega$ with respect to $\mathcal{L}_A$. 
Proposition 2.1. ([26, Theorem A]) If there is a Lyapunov function in $U$ with respect to $\mathcal{L}_A$, then a stationary measure $\mu_A$ corresponding to $\mathcal{L}_A$ exists and is regular in the sense that $\partial_\mu_A(x) = u_A(x) dx$ for some weak stationary solution $u_A \in W^{1,p}_\text{loc}(U)$ corresponding to $\mathcal{L}_A$. Moreover, if the Lyapunov function is unbounded, then the stationary measure is unique.

Below, we recall from [25, 26] some measure estimates which are derived based on the level set method introduced in these works. In fact, these estimates only require the weaker regularity condition on the drift field $V$ that $V^i \in L^p_{\text{loc}}(U)$, $i = 1, \cdots, n$.

Proposition 2.2. Assume that there is a Lyapunov function $U$ in $\Omega$ with respect to $\mathcal{L}_A$, with Lyapunov constant $\gamma$, essential lower bound $\rho_m$ and upper bound $\rho_M$. Let $\Omega_\rho$ denote the $\rho$-sublevel set of $U$ for each $\rho \in [\rho_m, \rho_M]$. Then the following holds for any stationary measure $\mu$ corresponding to $\mathcal{L}_A$:

a) ([26, Lemma 4.1]) For any $\rho_0 \in (\rho_m, \rho_M)$, there exists a constant $C_{\rho_0, \rho} > 0$ depending only on $\rho, \rho_0$ such that

$$
\mu(U \setminus \Omega_{\rho_0}) \leq \gamma^{-1} C_{\rho_0, \rho} \lambda_{\rho_0, \rho} |A|_{C(\Omega_{\rho_0} \setminus \Omega_{\rho_m})} |\nabla U|^2_{C(\Omega_{\rho_0} \setminus \Omega_{\rho_m})} \mu(\Omega_{\rho_0} \setminus \Omega_{\rho_m}).
$$

b) ([25, Theorem A b)]) If, in addition,

$$
\nabla U(x) \neq 0, \quad \forall x \in U^{-1}(\rho) \text{ for a.e. } \rho \in [\rho_m, \rho_M),
$$

$$
a^{ij}(x) \partial_i U(x) \partial_j U(x) \leq H(\rho), \quad x \in \partial \Omega_\rho, \quad \rho \in [\rho_m, \rho_M)
$$

for some non-negative measurable function $H$ defined on $[\rho_m, \rho_M)$, then

$$
\mu(U \setminus \Omega_\rho) \leq e^{\gamma \int_{\rho_0}^{\rho} \frac{1}{H(t)} dt}, \quad \rho \in [\rho_m, \rho_M).
$$

Proposition 2.3. ([25, Theorem B a)]) Assume that there is an anti-Lyapunov function $U$ in $\Omega$ with respect to $\mathcal{L}_A$, with anti-Lyapunov constant $\gamma$, essential lower bound $\rho_m$ and upper bound $\rho_M$. Let $\Omega^*_\rho$ denote the $\rho$-sublevel set of $U$ for each $\rho \in [\rho_m, \rho_M)$. If $U$ satisfies (2.2), (2.3) with respect to a non-negative measurable function $H$ defined on $[\rho_m, \rho_M)$, then the following holds for any stationary measure $\mu$ corresponding to $\mathcal{L}_A$:

$$
\mu(\Omega_\rho \setminus \Omega^*_\rho) \geq \mu(\Omega_\rho \setminus \Omega^*_\rho) e^{\gamma \int_{\rho_0}^{\rho} \frac{1}{H(t)} dt}, \quad \rho \in (\rho_0, \rho_M),
$$

where $\Omega^*_\rho = \Omega_{\rho_m} \cup U^{-1}(\rho_m) = \{x \in U : U(x) \leq \rho_m\}$.

2.2. Null family. Consider the class $\tilde{A}$ of diffusion matrices defined in (1.7).

Definition 2.2. 1) A null family (resp. bounded null family) $A = \{A_\alpha\} \subset \tilde{A}$ is a directed net of $\tilde{A}$ which converges to 0 - the zero matrix, under the topology of $W^{1,p}$-convergence on any compact subsets of $U$ (resp. $L^\infty$-convergence on $U$).

2) A bounded null family $\{A_\alpha = (a^{ij}_\alpha)\} \subset \tilde{A}$ is said to be a normal null family if for any pre-compact open subset $\Omega$ of $U$,

$$
\sup_{\alpha} \frac{\Lambda_\alpha(\Omega)}{\lambda_\alpha(\Omega)} < \infty,
$$

where, for each $\alpha$,

$$
\Lambda_\alpha(\Omega) = \sup_{x \in \Omega} \Lambda_\alpha(x), \quad \lambda_\alpha(\Omega) = \inf_{x \in \Omega} \lambda_\alpha(x)
$$
with \( \Lambda_\alpha(x) = \sqrt{\sum_{i,j} |a_{ij}(x)|^2} \) and \( \lambda_\alpha(x) = \inf_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\xi^T A_\alpha(x) \xi}{|\xi|^2} \) for each \( x \in \Omega \).

We note that in the above \( \lambda_\alpha(x) \) is just the smallest eigenvalue of \( A_\alpha(x) \) and \( \Lambda_\alpha(x) \) is strictly bigger than the largest eigenvalue of \( A_\alpha(x) \).

On \( \tilde{A} \), the topology of \( W^{1,\bar{p}} \)-convergence on any compact subsets of \( \mathcal{U} \) is metrizable. Through the rest of the paper, we denote an equivalent metric generating this topology by \( d \) and denote the \( L^\infty \) norm on the subspace of bounded elements of \( \tilde{A} \) by \( | \cdot | \) for short.

**Remark 2.1.** By Sobolev embedding \( W^{1,\bar{p}}(\Omega) \hookrightarrow C(\overline{\Omega}) \) for any pre-compact open set \( \Omega \subset \mathcal{U} \) with \( C^1 \) boundary \( \partial \Omega \), if \( A = \{ A_\alpha \} \subset \tilde{A} \) is a null family, then \( A_\alpha \to 0 \) uniformly on any compact subsets of \( \mathcal{U} \). This is because \( \mathcal{U} \) can be approximated from inside by pre-compact open subsets with smooth boundary.

**Definition 2.3.** Let \( A \subset \tilde{A} \) be a null family.

1) \( A \) is said to be **invariant** if for each \( A = \frac{GG^T}{2} \in A, \mathcal{U} \) is invariant with respect to the diffusion process corresponding to \( G \) or \( A \), i.e., with probability one, any solution of (1.2) starting in \( \mathcal{U} \) remains in \( \mathcal{U} \) for all positive time of existence.

2) \( A \) is said to be **admissible** if for each \( A = \frac{GG^T}{2} \in A \), there exists a stationary measure corresponding to \( \mathcal{L}_A \).

**Remark 2.2.** Let \( A \) be a null family.

1) With our assumptions on \( V \) and \( G \in \tilde{G} \), local existence of solutions of (1.2) is guaranteed. The invariance of a null family \( A \) only requires that such solutions corresponding to each \( A = \frac{GG^T}{2} \in A \) do not escape \( \mathcal{U} \) when time evolves, and local uniqueness of them is not even required.

2) When \( \mathcal{U} = \mathbb{R}^n \), it is obvious that any null family \( A \) is automatically invariant. When \( \mathcal{U} \) admits a boundary in \( \mathbb{R}^n \), the invariance property of a null family \( A \) naturally leads to suitable boundary conditions (on \( V \) or \( A \in A \) or both) of reflection, stopping, or general Wentzell types (see [18, 19, 45]) for (1.2) or (1.3). For instance, if (1.1) admits a Lyapunov function in \( \mathcal{U} \) and each \( A \in A \) satisfies \( A(x) \to 0 \) as \( x \to \partial \mathcal{U} \), then it is not hard to see that \( A \) is invariant.

3) For a null family \( A \) to be both invariant and admissible, certain uniform Lyapunov-like conditions (see below) are usually needed. When \( \mathcal{U} \) is bounded, it is true by the theory of elliptic equations that if each \( A \in A \) is positive definite on \( \mathcal{U} \) then there exists a stationary measure in \( \mathcal{U} \) corresponding to each \( \mathcal{L}_A \) (11, 26). But such a null family will fail to be invariant. To ensure the admissibility of a null family \( A \) in the case of degeneracy of \( A \in A \) on \( \partial \mathcal{U} \), uniform Lyapunov-like conditions are still necessary (see [26]).

**Definition 2.4.** Let \( A = \{ A_\alpha \} \subset \tilde{A} \) be a null family and \( \Omega \subset \mathcal{U} \) be a connected open set. A \( C^2 \) compact function \( U \) is a **uniform Lyapunov function** (resp. **uniform anti-Lyapunov function** in \( \Omega \) with respect to \( A \) or \( \{ \mathcal{L}_{A_\alpha} \} =: \{ \mathcal{L}_A \}_{A \in A} \) if it is a Lyapunov function (resp. anti-Lyapunov function) in \( \Omega \) with respect to each \( \mathcal{L}_A, A \in A \), and the essential lower bound \( \rho_m \) and Lyapunov (resp. anti-Lyapunov) constant \( \gamma \), of \( U \), are independent of \( A \in A \).
Remark 2.3. 1) It follows immediately from Proposition 2.1 that if there is a uniform Lyapunov function in $\mathcal{U}$ with respect to a null family $A$, then $A$ is admissible.

2) Consider $A_{\epsilon} = \epsilon A$, $0 < \epsilon \leq 1$, for a fixed $A = (a^{ij}) \in \hat{A}$. Then it is clear that $\{A_{\epsilon}\}$ is a null family, and in fact a normal null family if $A$ is uniformly positive definite on $U$. We note that a uniform Lyapunov function can be easily obtained with respect to such a null family. Assume that $U$ is a Lyapunov function in $\Omega \subset U$ with respect to the operator $L_A$ with Lyapunov constant $\gamma$. If $U$ is quasi-convex (i.e., the Hessian matrix $D^2 U$ is positive semi-definite) near $\partial \Omega$, then

$$L_{A_{\epsilon}} = \epsilon a^{ij}(x) \partial_{ij}^2 U(x) + V^i(x) \partial_i U(x) \leq a^{ij}(x) \partial_{ij}^2 U(x) + V^i(x) \partial_i U(x) \leq -\gamma$$

for all $x \in \Omega$ sufficiently close to $\partial \Omega$, i.e., $U$ becomes a uniform Lyapunov function in $\Omega$ with respect to the family $\{L_{A_{\epsilon}}\}$.

A uniform Lyapunov (resp. anti-Lyapunov) function with respect to a null family is actually a Lyapunov (resp. anti-Lyapunov) function of the system (1.1).

Proposition 2.4. Suppose that $U$ is a uniform Lyapunov (resp. anti-Lyapunov) function in a connected open set $\Omega \subset U$ with respect to a null family $A = \{A_{\alpha}\} \subset \hat{A}$. Then $U$ must be a Lyapunov (resp. anti-Lyapunov) function of (1.1) in $\Omega$. Consequently, (1.1) generates a positive (resp. negative) semiflow which is dissipative (resp. anti-dissipative) in $\Omega$.

Proof. We only prove the case when $U$ is a uniform Lyapunov function in $\Omega$ with respect to $\{L_{A_{\alpha}}\}$.

Denote $\rho_n$, respectively $\rho_M$, as the essential lower, respectively upper, bound of $U$, $\gamma$ as a uniform Lyapunov constant of $U$, and $\Omega_{\rho}$ as the $\rho$-sublevel set of $U$ for each $\rho \in [\rho_n, \rho_M]$. Then

$$L_{A_{\alpha}} U(x) = \epsilon a^{ij}(x) \partial_{ij}^2 U(x) + V^i(x) \partial_i U(x) \leq -\gamma, \quad x \in \Omega \setminus \bar{\Omega}_{\rho_m}.$$ 

By taking limit $A_{\alpha} \to 0$ in the above, we have

$$V(x) \cdot \nabla U(x) \leq -\gamma, \quad x \in \Omega \setminus \bar{\Omega}_{\rho_m},$$

i.e., $U$ is a Lyapunov function of (1.1) in $\Omega$. It follows from Proposition 6.2 that (1.1) generates a positive semiflow which is dissipative in $\Omega$.

Conversely, with respect to a bounded null family, a uniform Lyapunov function can be naturally obtained from a Lyapunov function of (1.1).

Proposition 2.5. Let $\Omega \subset U$ be a connected open set and suppose that (1.1) admits a $C^2$ Lyapunov (resp. anti-Lyapunov) function $U$ whose second derivatives are bounded in $\Omega$. Then $U$ is a uniform Lyapunov (resp. anti-Lyapunov) function in $\Omega$ with respect to any bounded null family $A = \{A_{\alpha}\} \subset \hat{A}$ with $|A_{\alpha}| \ll 1$.

Proof. We only consider the case that $U$ is a Lyapunov function of $\varphi^t$ in $\Omega$, i.e.,

$$V(x) \cdot \nabla U(x) \leq -\gamma, \quad x \in \tilde{\Omega},$$

where $\tilde{\Omega}$ is the closure of $\Omega$. Then

$$L_{A_{\alpha}} = \epsilon a^{ij}(x) \partial_{ij}^2 U(x) + V^i(x) \partial_i U(x) \leq a^{ij}(x) \partial_{ij}^2 U(x) + V^i(x) \partial_i U(x) \leq -\gamma$$

for all $x \in \Omega$ sufficiently close to $\partial \Omega$, i.e., $U$ becomes a uniform Lyapunov function in $\Omega$ with respect to the family $\{L_{A_{\alpha}}\}$.
where \( \tilde{\Omega}, \gamma \) are essential domain, Lyapunov constant, of \( U \), respectively. Let \( \mathcal{A} = \{ A_\alpha \} = \{(a_i^j)\} \subset \tilde{\mathcal{A}} \) be a bounded null family. Then with \(|A_\alpha| \ll 1\),
\[
\mathcal{L}_{A_\alpha} U(x) = a_i^j \partial_i^j U(x) + V(x) \cdot \nabla U(x) \leq -\frac{\gamma}{2}, \quad x \in \tilde{\Omega},
\]
whence holds with \( \epsilon > 0 \) for every \( f \in \mathcal{C}_b(\Omega) \). It is well-known that \( M(\Omega) \) with the weak\(^*\)-topology is metrizable.

The following result is well-known (see e.g., [4, Chapter II, Theorem 6.1]).

**Proposition 2.6.** Let \( \{ \mu_\ell \}_{\ell=1}^\infty \) be a sequence in \( M(\Omega) \) and \( \mu \in M(\Omega) \). Then the following statements are equivalent.

1. \( \lim_{\ell \to \infty} \mu_\ell = \mu \) under the weak\(^*\)-topology.
2. \( \limsup_{\ell \to \infty} \mu_\ell(C) \leq \mu(C) \) for any closed subset \( C \) of \( \Omega \).
3. \( \liminf_{\ell \to \infty} \mu_\ell(W) \geq \mu(W) \) for any open subset \( W \) of \( \Omega \).
4. \( \lim_{\ell \to \infty} \mu_\ell(B) = \mu(B) \) for any Borel subset \( B \) of \( \Omega \) whose boundary has zero \( \mu \)-measure.

By Prokhorov's Theorem (see e.g., [8, Theorem 5.1]), a subset \( \mathcal{M} \subset M(\Omega) \) is relatively sequentially compact in \( M(\Omega) \) if it is tight, i.e., for any \( \epsilon > 0 \) there exists a compact subset \( K_\epsilon \subset \Omega \) such that \( \mu(\Omega \setminus K_\epsilon) < \epsilon \) for all \( \mu \in \mathcal{M} \). We note that if \( \Omega \) is compact, then any subset of \( M(\Omega) \) is tight.

**Definition 2.5.** Let \( \mathcal{A} = \{ A_\alpha \} \subset \tilde{\mathcal{A}} \) be an admissible null family and \( \Omega \subset \mathcal{U} \) be a Borel set. Consider the set \( \{ \mu_\alpha \} \) of all stationary measures corresponding to \( \{ \mathcal{L}_{A_\alpha} \} \). For each \( \mu_\alpha \), we denote \( \mu_\alpha^\Omega = \frac{\mu_\alpha}{\mu_\alpha(\Omega)} \) when \( \mu_\alpha(\Omega) > 0 \), called a *normalized stationary measure in \( \Omega \).*

1. An \( \mathcal{A} \)-limit measure \( \mu_\alpha^\Omega \) in \( \Omega \) is a sequential limit point, as \( A_\alpha \to 0 \), of the set \( \{ \mu_\alpha^\Omega \} \) in \( M(\Omega) \). The set of all \( \mathcal{A} \)-limit measures in \( \Omega \) is denoted by \( \mathcal{M}_A^\Omega \). If \( \Omega = \mathcal{U} \), then we simply denote \( \mathcal{M}_A^\Omega \) by \( \mathcal{M}_A \) and call each measure \( \mu \in \mathcal{M}_A \) an \( \mathcal{A} \)-limit measure.
2. The set \( \{ \mu_\alpha^\Omega \} \) is said to be \( \mathcal{A} \)-sequentially null compact in \( M(\Omega) \) if any sequence \( \{ \mu_\ell^\Omega \} \) in the set with \( A_\ell \to 0 \) is relatively compact in \( M(\Omega) \).

**Remark 2.4.** Let \( \mathcal{A} = \{ A_\alpha \} \subset \tilde{\mathcal{A}} \) be an admissible null family on \( \mathcal{U} \) and \( \Omega \subset \mathcal{U} \) be a connected open set.

1. By the aforementioned regularity theorem in [3], \( \mu_\alpha(\Omega) > 0 \) in this situation.
2. We note that when \( \Omega = \mathcal{U} \), the normalized stationary measure \( \mu_\ell^\mathcal{U} \) coincides with the original stationary measure \( \mu_\alpha \) corresponding to \( \mathcal{L}_{A_\alpha} \), for each \( \alpha \).
3. If the set \( \{ \mu_\alpha^\Omega \} \) is \( \mathcal{A} \)-sequentially null compact in \( M(\Omega) \), then \( \mathcal{M}_A^\Omega \) is clearly non-empty. In general, \( \mathcal{M}_A^\Omega \) can be an empty set. However, when \( \Omega \) is a pre-compact open subset of \( \mathcal{U} \) (i.e., its closure in \( \mathbb{R}^n \) is a compact subset of \( \mathcal{U} \)), it follows from the regularity theorem...
again that each stationary measure corresponding to $L_A$ is regular, hence each $\mu_\alpha^{\Omega}$ is also a probability measure on $\bar{\Omega}$. This enables us to consider the set $M_\Omega^{\bar{\Omega}}$ of all $A$-limit measures of $\{\mu_\alpha^{\Omega}\}$ in $M(\bar{\Omega})$, which is always non-empty because $\{\mu_\alpha^{\Omega}\}$ is tight, hence relatively sequentially compact in $M(\bar{\Omega})$.

4) If a sub-domain $\Omega \subset \mathcal{U}$ is considered in (1.6) instead of $\mathcal{U}$, then one can speak of a stationary measure corresponding to $L_A$ in $\Omega$. A normalized stationary measure $\mu_\alpha^{\Omega}$ in $\Omega$ is necessarily a stationary measure corresponding to $\{L_A^\alpha\}$ in $\Omega$. When $\Omega$ is a pre-compact open subset of $\mathcal{U}$, stationary measures corresponding to $\{L_A^\alpha\}$ in $\Omega$ other than the normalized ones $\{\mu_\alpha^{\Omega}\}$ also exist. In fact, depending on the boundary conditions imposed, there are infinitely many such stationary measures in $\Omega$ (see [11, 26] for details). Though most of our results in this paper hold for all stationary measures corresponding to $\{L_A^\alpha\}$ in $\Omega$, we only work with the normalized ones $\{\mu_\alpha^{\Omega}\}$ because they are noise or diffusion relevant.

We now define the following notion of stochastic stability with respect to a given admissible null family of noise perturbations.

**Definition 2.6.** Let $\Omega \subset \mathcal{U}$ be a connected open set and $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an invariant and admissible null family. Let $\{\mu_\alpha\}$ be the set of all stationary measures corresponding to $\{L_{A_\alpha}\}$ and $\{\mu_\alpha^{\Omega}\}$ be the set of normalized stationary measures on $\Omega$.

1) A compact invariant set $J \subset \Omega$ of $\varphi^t$ is said to be relatively $\mathcal{A}$-stable in $\Omega$ if for any $\epsilon > 0$ and any open neighborhood $W$ of $J$ in $\mathcal{U}$ there exists a $\delta > 0$ such that $\mu_\alpha^{\Omega}(\Omega \setminus W) < \epsilon$ whenever $d(A_\alpha, 0) < \delta$. An invariant measure $\mu$ of $\varphi^t$ in $\Omega$ is said to be relatively $\mathcal{A}$-stable in $\Omega$ if $\{\mu_\alpha^{\Omega}\}$ converges to $\mu$ in $M(\Omega)$ as $A_\alpha \to 0$.

2) A compact invariant set or invariant measure is said to be relatively $\mathcal{A}$-unstable in $\Omega$ if it is not relatively $\mathcal{A}$-stable in $\Omega$.

3) A relatively $\mathcal{A}$-stable (resp. relatively $\mathcal{A}$-unstable) compact invariant set or invariant measure in $\mathcal{U}$ is simply said to be $\mathcal{A}$-stable (resp. $\mathcal{A}$-unstable). A compact invariant set $J$ is said to be strongly $\mathcal{A}$-unstable if $\text{supp}(\mu) \cap J = \emptyset$ for all $\mu \in M_A$.

**Remark 2.5.** 1) Clearly the admissibility of a null family $\mathcal{A}$ is necessary in defining (relative) $\mathcal{A}$-stability, otherwise the (relative) $\mathcal{A}$-stability is void.

2) The consideration of stationary measures instead of invariant measures allows a general characterization of stochastic stability because they often exist regardless whether (1.2) generates a diffusion process or (1.3) generates a generalized diffusion process, not talking about the fact that even when a diffusion or a generalized diffusion process can be defined its invariant measures need not exist. Such a consideration is natural in the sense that compact invariant sets, respectively invariant measures, of (1.1) are indeed stationary with respect to its induced flow on compact sets, respectively on the space of Borel probability measures.

3) The invariance of $\mathcal{A}$ is also necessary in defining (relative) $\mathcal{A}$-stability because for such a stability one would like to only restrict the consideration to “noise relevant” stationary measures. This is particularly so when $\mathcal{U}$ is bounded. In this case, stationary measures corresponding to $L_A$ for any $A \in \tilde{\mathcal{A}}$ always exist, but depending on the boundary conditions
imposed, many of which need not be associated with true stationary processes if \( \mathcal{A} \) fails to be invariant (see also Remark 2.2 and Remark 2.4).

4) Relative \( \mathcal{A} \)-stability of a compact invariant set or an invariant measure of \( \varphi^t \) in \( \Omega \) only says that the normalized stationary measures \( \{ \mu^\Omega_\alpha \} \) rather than the restricted stationary measures \( \{ \mu_\alpha |_{\Omega} \} \) are “attracted” to the set or the measure as \( A_\alpha \to 0 \), i.e., it can happen that eventually all or part of the original stationary measures \( \{ \mu_\alpha \} \) corresponding to \( \{ L_{A_\alpha} \} \) still “escape” from \( \Omega \) and concentrate elsewhere. To ensure the eventual concentration of all restricted stationary measures \( \{ \mu_\alpha |_{\Omega} \} \) in \( \Omega \), an additional condition \( \text{supp}(\mu) \cap \Omega \neq \emptyset, \mu \in \mathcal{M}_\mathcal{A} \) need to be imposed.

2.4. Harnack inequality. Consider the differential operator

\[
Lu := \partial_i (a^{ij}(x) \partial_j u + b^i(x) u) + c^i(x) \partial_i u + d(x) u,
\]

where \( a^{ij}, b^i, c^i, d, i, j = 1, \cdots, n \), are measurable and bounded functions on a connected open set \( \Omega \subset \mathbb{R}^n \). Assume that

\[
a^{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2, \quad x \in \Omega, \ \xi \in \mathbb{R}^n
\]

for some constant \( \lambda > 0 \), and there exist constants \( \Lambda > 0, \nu \geq 0 \) such that

\[
\sum_{i,j} |a^{ij}(x)|^2 \leq \Lambda^2, \quad \lambda^{-2} \sum_i (|b^i(x)|^2 + |c^i(x)|^2) + \lambda^{-1} |d(x)| \leq \nu^2
\]

for all \( x \in \Omega \).

The following Harnack inequality is well-known (see Theorem 8.20 of [21]).

**Proposition 2.7.** (Harnack inequality) Assume that the operator \( L \) satisfies (2.7) and (2.8). Let \( u \in W^{1,2}(\Omega) \) be any non-negative solution of

\[
Lu(x) = 0, \quad x \in \Omega.
\]

Then for any ball \( B_{4R}(y) \subset \Omega \), we have

\[
\sup_{B_R(y)} u \leq C \inf_{B_R(y)} u,
\]

where the constant \( C \) can be estimated by

\[
C \leq C_0^{(\lambda/\lambda + \nu R)}
\]

for some constant \( C_0 = C_0(n) \) depending only on \( n \).

3. General properties of limit measures

In this section, we consider some general limit properties of stationary measures of a family of Fokker-Planck equations as diffusion coefficients tend to zero. Among these properties, the concentration of limit measures will play an important role in characterizing stochastic stability of compact invariant sets of the corresponding unperturbed deterministic system at both local and global levels.

For a fixed vector field \( V \in C(\mathcal{U}, \mathbb{R}^n) \) in (1.1), we consider the class \( \tilde{\mathcal{A}} \) of diffusion matrices defined in (1.7). When \( V \in C^1(\mathcal{U}, \mathbb{R}^n) \), (1.1) generates a \( C^1 \) local flow on \( \mathcal{U} \) which will be denoted by \( \varphi^t \) through this and the next section.
This section (as well as next section) uses some basic dynamics notions of a local flow such as dissipation and anti-dissipation, (strong) attractors and repellers, (entire, weak) Lyapunov and anti-Lyapunov functions, and (positive, negative) invariant measures. The definition of these notions and some of their fundamental properties are reviewed in the Appendix (Section 6).

3.1. Invariance of limit measures. We first give some new characterizations of invariant and semi-invariant measures of $\varphi^t$ in a subset which are useful in the study of limit behaviors of stationary measures of Fokker-Planck equations.

**Proposition 3.1.** Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and let $\Omega \subset \mathcal{U}$ be an open invariant set of $\varphi^t$. Then $\mu \in M(\Omega)$ is an invariant measure of $\varphi^t$ on $\Omega$ if and only if

$$
\int_{\Omega} V(x) \cdot \nabla h(x) \, d\mu(x) = 0, \quad h \in C^1_0(\Omega).
$$

**Proof.** It follows from Riesz Representation Theorem that $\nu_1 = \nu_2 \in M(\Omega)$ if and only if

$$
\int_{\Omega} h(x) \, d\nu_1(x) = \int_{\Omega} h(x) \, d\nu_2(x), \quad h \in C_b(\Omega).
$$

As any $\nu \in M(\Omega)$ is a Borel regular measure, (3.2) is equivalent to

$$
\int_{\Omega} h(x) \, d\nu_1(x) = \int_{\Omega} h(x) \, d\nu_2(x), \quad h \in C_b(\Omega).
$$

For a given $t \in \mathbb{R}$, define $\mu^t \in M(\Omega)$: $\mu^t(B) =: \mu(\varphi^{-t}(B))$ for any Borel set $B \subset \Omega$. Then the invariance of $\mu$ is equivalent to $\mu = \mu^t$ for any $t \in \mathbb{R}$, which, by (3.2) and (3.3), is equivalent to

$$
\int_{\Omega} h(x) \, d\mu(x) = \int_{\Omega} h(x) \, d\mu^t(x) = \int_{\Omega} h(\varphi^t(x)) \, d\mu(x), \quad t \in \mathbb{R},
$$

for all $h \in C_b(\Omega)$ or all $h \in C^1_b(\Omega)$.

For any $h \in C^1_b(\Omega)$, consider the function

$$
f_h(t) := \int_{\Omega} h(\varphi^t(x)) \, d\mu(x), \quad t \in \mathbb{R}.
$$

Then, for any $t, s \in \mathbb{R}$, we have by the flow property that

$$
f_h(t + s) = \int_{\Omega} h(\varphi^{t+s}(x)) \, d\mu(x) = \int_{\Omega} h \circ \varphi^t(\varphi^s(x)) \, d\mu(x) = f_{h \circ \varphi^s}(s).
$$

It follows that

$$
f_h'(t) = f'_{h \circ \varphi^t}(0), \quad h \in C^1_b(\Omega), \quad t \in \mathbb{R}.
$$

If $\mu$ is an invariant measure of $\varphi^t$ in $\Omega$, then for any $h \in C^1_b(\Omega)$, (3.4) implies that $f_h(t) = f_h(0)$ for all $t \in \mathbb{R}$. Taking derivatives yields that $f'_h(t) \equiv 0$. In particular, $f'_h(0) = 0$, i.e., (3.4) holds.

Conversely, suppose that (3.4) holds. For any $h \in C^1_b(\Omega)$ and any $t \in \mathbb{R}$, using the fact that $\varphi^t : \Omega \to \Omega$ is a $C^1$ diffeomorphism, it is easy to see that $h \circ \varphi^t \in C^1_b(\Omega)$. Applications of (3.5) and (3.4) with $h \circ \varphi^t$ in place $h$ yield that

$$
f_h'(t) = f'_{h \circ \varphi^t}(0) = \int_{\Omega} V(x) \cdot \nabla (h \circ \varphi^t)(x) \, d\mu(x) = 0,
$$

as desired.
i.e.,
\[ \int_{\Omega} h(x) \, d\mu(x) = \int_{\Omega} h(\varphi^t(x)) \, d\mu(x), \quad h \in C_0^1(\Omega), \ t \in \mathbb{R}. \]
Since \( C_0^1(\Omega) \) is dense in \( C_0(\Omega) \), (3.4) holds. Hence \( \mu \) is an invariant measure of \( \varphi^t \) in \( \Omega \).

**Proposition 3.2.** Assume \( V \in C^1(\mathcal{U}, \mathbb{R}^n) \) and let \( \Omega \subset \mathcal{U} \) be an open, pre-compact, positively (resp. negatively) invariant set of \( \varphi^t \). Then \( \mu \in M(\Omega) \) is a positively (resp. negatively) invariant measure of \( \varphi^t \) on \( \Omega \) if and only if
\[ \int_{\Omega} V(x) \cdot \nabla h(x) \, d\mu(x) = 0, \quad h \in C_0^1(\Omega). \]

**Proof.** The proof follows from the same argument as that of Proposition 3.1 with \( C_0(\Omega), C_0^1(\Omega), \mathbb{R}_+ \) (resp. \( \mathbb{R}_- \)) in places of \( C_0(\Omega), C_0^1(\Omega), \mathbb{R} \) respectively.

Part a) of Theorem A follows from part b) of the following result when taking \( \Omega = \mathcal{U} \).

**Theorem 3.1.** Let \( \mathcal{A} = \{ A_{\alpha} \} \subset \hat{\mathcal{A}} \) be an admissible null family and \( \Omega \subset \mathcal{U} \) be a connected open set. Then the following holds for any \( \mu \in M^\Omega_{\mathcal{A}} \).

a) \[ \int_{\Omega} V(x) \cdot \nabla h(x) \, d\mu(x) = 0 \] for all \( h \in C_0^1(\Omega) \).

b) Suppose that \( V \in C^1(\mathcal{U}, \mathbb{R}^n) \) and that \( \Omega \) is an invariant set of \( \varphi^t \). Then \( \mu \) is an invariant measure of \( \varphi^t \) on \( \Omega \).

c) Suppose that \( V \in C^1(\mathcal{U}, \mathbb{R}^n) \), and that there exists an open, pre-compact, positively (resp. negatively) invariant set of \( \varphi^t \) in \( \Omega \) containing \( \text{supp}(\mu) \). Then \( \mu \) is an invariant measure of \( \varphi^t \) on \( \Omega \).

**Proof.** Denote \( \{ \mu^\Omega_{\alpha} \} \) as the set of all normalized stationary measures corresponding to \( \{ \mathcal{L}_{A_{\alpha}} \} \) in \( \Omega \). Let \( \{ A_l = (a_{i}^{ij}) \} \subset \mathcal{A} \) and \( \{ \mu_l =: \mu^\Omega_l \} \subset \{ \mu^\Omega_{\alpha} \} \) be sequences such that \( A_l \to 0 \) and \( \mu_l \to \mu \) in \( M(\Omega) \), as \( l \to \infty \).

We note by the definition of stationary measures that
\[ \int_{\Omega} (a_{i}^{ij} \partial_j h(x) + V^i \partial_i h(x)) \, d\mu_l(x) = 0, \quad h \in C_0^\infty(\Omega), \]
for all \( l \). Using the uniform convergence of \( (a_{i}^{ij}) \) to 0 on any compact subsets of \( \mathcal{U} \) and the weak*-convergence of \( \mu_l \) to \( \mu \), we have by simply taking limit \( l \to \infty \) in (3.6) that
\[ \int_{\Omega} V(x) \cdot \nabla h(x) \, d\mu(x) = 0, \quad h \in C_0^\infty(\Omega). \]
It follows that a) holds because \( C_0^\infty(\Omega) \) is dense in \( C_0^1(\Omega) \).

When \( \Omega \) is an invariant set of \( \varphi^t \), the invariance of \( \mu \) follows from Proposition 3.1. This proves b).

To prove c), we let \( \hat{\Omega} \) be an open, pre-compact, positively (resp. negatively) invariant set of \( \varphi^t \) in \( \Omega \) containing \( \text{supp}(\mu) \). For any \( \hat{h} \in C_0^1(\hat{\Omega}) \), we let \( h \in C_0^1(\Omega) \) be an extension of \( \hat{h} \). Then by a), we have
\[ \int_{\Omega} V(x) \cdot \nabla \hat{h}(x) \, d\mu(x) = \int_{\Omega} V(x) \cdot \nabla h(x) \, d\mu(x) = 0. \]
It follows from Proposition 3.2 that $\mu$ is a positively (resp. negatively) invariant measure of $\varphi^t$ in $\tilde{\Omega}$. It follows from Proposition 6.5 that $\mu$ is an invariant measure of $\varphi^t$ in $\tilde{\Omega}$ supported on the $\omega$-limit set $\omega(\tilde{\Omega})$ (resp. $\alpha$-limit set $\alpha(\tilde{\Omega})$) of $\tilde{\Omega}$. It follows that $\mu$ is an invariant measure of $\varphi^t$ on $\Omega$ because $\mu(\Omega \setminus \tilde{\Omega}) = 0$. □

3.2. Stochastic counterpart of the LaSalle invariance principle. For a deterministic dynamical system, the classical LaSalle invariance principle (see Proposition 6.3) is an important tool in locating $\omega$-limit sets using a Lyapunov-like function. For the stochastic system (1.2), we show below that a similar principle can be obtained for locating the support of a limit measure.

For a Borel set $\Omega \subset \mathcal{U}$ and an admissible null family $A = \{A_\alpha\} \subset \tilde{\mathcal{A}}$, we denote

$$J^\Omega_A := \bigcup \{\text{supp}(\mu) : \mu \in M^\Omega_{\tilde{\mathcal{A}}} \}.$$  \hspace{1cm} (3.7)

In the case $\Omega = \mathcal{U}$, we simply denote $J^\Omega_A$ by $J_A$.

**Proposition 3.3.** Let $A = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be an admissible null family and $\Omega \subset \mathcal{U}$ be a connected open set. Then the following holds.

a) If (1.1) admits an entire weak Lyapunov (resp. anti-Lyapunov) function $U$ in $\Omega$, then any $\mu \in M^\Omega_{\tilde{\mathcal{A}}}$ with compact support satisfies

$$\text{supp}(\mu) \subset S = \{x \in \Omega : V(x) \cdot \nabla U(x) = 0\}.$$  

b) If the set $J^\Omega_A$ is contained in a compact set $J \subset \Omega$ and there is an entire weak Lyapunov (resp. anti-Lyapunov) function $U_0$ of (1.1) defined in a neighborhood of $J$, then any $\mu \in M^\Omega_{\tilde{\mathcal{A}}}$ satisfies

$$\text{supp}(\mu) \subset S_0 = \{x \in J : V(x) \cdot \nabla U_0(x) = 0\}.$$  

**Proof.** To prove a), we let $\rho_0 > 0$ be such that the $\rho_0$-sublevel set $\Omega_{\rho_0}$ of $U$ contains the support of $\mu$. Applying Theorem 3.1a) to a function $h \in C^1_0(\Omega)$ satisfying $h \equiv U$ on $\Omega_{\rho_0}$ yields that

$$0 = \int_{\Omega} V(x) \cdot \nabla h(x) \, d\mu(x) = \int_{\Omega_{\rho_0}} V(x) \cdot \nabla U(x) \, d\mu(x) = \int_{\Omega_{\rho_0} \cap (\Omega \setminus S)} V(x) \cdot \nabla U(x) \, d\mu(x).$$

Since $V(x) \cdot \nabla U(x)$ is of constant sign on $\Omega_{\rho_0} \cap (\Omega \setminus S)$, $\mu(\Omega_{\rho_0} \cap (\Omega \setminus S)) = 0$, i.e., supp$(\mu) \subset \Omega_{\rho_0} \cap S \subset \mathcal{S}$.

To prove b), we note by a) that supp$(\mu) \subset S_0 = \{x \in \Omega_0 : V(x) \cdot \nabla U_0(x) = 0\}$, where $\Omega_0$ is a neighborhood of $J$ in which $U_0$ is defined. By the definition of $J^\Omega_A$ in (3.7), we also have supp$(\mu) \subset J$. It follows that supp$(\mu) \subset J \cap S_0$, from which b) follows. □

We note that part a) of Proposition 3.3 with $\Omega = \mathcal{U}$ is precisely the part b) of Theorem A.
3.3. Some general properties on \( \mathcal{A} \)-stability. We first give an equivalent condition for a compact invariant set of \( \varphi^t \) to be relatively \( \mathcal{A} \)-stable or \( \mathcal{A} \)-stable.

**Proposition 3.4.** Let \( \Omega \subset \mathcal{U} \) be a connected open set, \( \mathcal{J} \subset \Omega \) (resp. \( \mathcal{J} \subset \mathcal{U} \)) be a compact invariant set of \( \varphi^t \), and \( \mathcal{A} = \{A_0\} \subset \hat{\mathcal{A}} \) be an invariant and admissible null family. Then \( \mathcal{J} \) is relatively \( \mathcal{A} \)-stable in \( \Omega \) (resp. \( \mathcal{A} \)-stable) if and only if the set \( \{\mu_{\alpha}^\Omega\} \) of all normalized (resp. the set \( \{\mu_{\alpha}\} \) of all) stationary measures corresponding to \( \{\mathcal{L}_{\mathcal{A}_\alpha}\} \) in \( \Omega \) (resp. in \( \mathcal{U} \)) is \( \mathcal{A} \)-sequentially null compact in \( M(\Omega) \) (resp. in \( M(\mathcal{U}) \))

**Proof.** To show the sufficiency, we suppose for contradiction that \( \mathcal{J} \) is not relatively \( \mathcal{A} \)-stable in \( \Omega \). Then there exists an \( \epsilon_0 > 0 \), an open neighborhood \( W \) of \( \mathcal{J} \) in \( \Omega \) and a sequence \( \{A_l\} \subset \mathcal{A} \) such that \( A_l \rightarrow 0 \) but \( \mu_{A_l}^\Omega(\Omega \setminus W) \geq \epsilon_0 \) for all \( l \). Since \( \{\mu_{\alpha}^\Omega\} \) is relatively sequentially compact, we may assume without loss of generality that \( \mu_{A_l}^\Omega \) converges, say, to some \( \mu \in M_{\mathcal{A}}^\Omega \). Since \( \Omega \setminus W \) is a closed subset of \( \Omega \) under the restricted topology, Proposition 2.6 implies that \( \mu(\Omega \setminus W) \geq \limsup_{l \rightarrow \infty} \mu_{A_l}^\Omega(\Omega \setminus W) \geq \epsilon_0 \). Thus, \( \text{supp}(\mu) \nsubseteq \mathcal{J} \), a contradiction to the fact that \( \mathcal{J}_{\mathcal{A}_l} \subset \mathcal{J} \).

To show the necessity, we let \( \{A_l\} \subset \mathcal{A} \) be a sequence such that \( A_l \rightarrow 0 \). Let \( \epsilon > 0 \) be given. We also take any compact neighborhood \( W \) of \( \mathcal{J} \) in \( \Omega \). Since \( \mathcal{J} \) is relatively \( \mathcal{A} \)-stable, there exists a positive integer \( N \) such that

\[
\mu_{A_l}^\Omega(\Omega \setminus W) < \epsilon, \quad \ell \geq N,
\]

i.e., \( \mu_{A_l}^\Omega(W) \geq 1 - \epsilon \) for all \( \ell \geq N \). Since each \( \mu_{A_l}^\Omega \) is a Borel regular measure, there is a compact subset \( B \) of \( \Omega \) such that \( \mu_{A_l}^\Omega(B) \geq 1 - \epsilon \) for all \( 1 \leq \ell \leq N \). Let \( K = B \cup W \). Then \( K \) is a compact subset of \( \Omega \) and \( \mu_{A_l}^\Omega(K) \geq 1 - \epsilon \) for all \( \ell \). This shows that \( \{\mu_{A_l}^\Omega\}_{\ell=1}^\infty \) is tight, hence relatively compact in \( M(\Omega) \).

Let \( \mu \in M_{\mathcal{A}}^\Omega \). Then there exists a sequence \( \{A_l\} \subset \mathcal{A} \) with \( A_l \rightarrow 0 \) such that \( \lim_{l \rightarrow \infty} \mu_{A_l}^\Omega = \mu \) under the weak*-topology. Again, using the relative \( \mathcal{A} \)-stability of \( \mathcal{J} \) in \( \Omega \), we see that for any \( \epsilon > 0 \) and any compact neighborhood \( W \) of \( \mathcal{J} \) in \( \Omega \), \( 3.3 \) holds with the present \( \epsilon, W \), and \( \{\mu_{A_l}^\Omega\} \). Since \( \Omega \setminus W \) is an open subset of \( \Omega \), Proposition 2.6 implies that

\[
\mu(\Omega \setminus W) \leq \liminf_{l \rightarrow \infty} \mu_{A_l}^\Omega(\Omega \setminus W) \leq \epsilon.
\]

Since \( \epsilon \) and \( W \) are arbitrary, we have \( \mu(\Omega \setminus \mathcal{J}) = 0 \) by the continuity of probability measures, i.e., \( \text{supp}(\mu) \subset \mathcal{J} \).

**Corollary 3.1.** Assume \( V \in C^1(\mathcal{U}, \mathbb{R}^n) \). Let \( \mathcal{A} = \{A_0\} \subset \hat{\mathcal{A}} \) be an invariant and admissible null family and \( \Omega \subset \mathcal{U} \) be a connected open set. If \( \mathcal{J}_{\mathcal{A}}^\Omega \) (resp. \( \mathcal{J}_{\mathcal{A}} \)) is a relatively \( \mathcal{A} \)-stable set in \( \Omega \) (resp. \( \mathcal{A} \)-stable set), then it is the smallest relatively \( \mathcal{A} \)-stable set in \( \Omega \) (resp. \( \mathcal{A} \)-stable set), i.e., it contains no proper relatively \( \mathcal{A} \)-stable subset in \( \Omega \) (resp. \( \mathcal{A} \)-stable set).

**Proof.** It follows from Proposition 3.4 immediately.

The following result establishes some connections between (relatively) \( \mathcal{A} \)-stable sets and measures.
Proposition 3.5. Let $\mathcal{A} = \{A_\alpha\} \subset \bar{\mathcal{A}}$ be an invariant and admissible null family and $\Omega \subset U$ be a connected open set. Then the following holds.

a) Let $\mu \in M(\Omega)$ (resp. $\mu \in M(U)$) be an invariant measure of $\varphi^t$ with compact support. If $\mu$ is relatively $\mathcal{A}$-stable in $\Omega$ (resp. $\mathcal{A}$-stable), then so is $\text{supp}(\mu)$ as a compact invariant set.

b) Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$. Let $\mathcal{J} \subset \Omega$ (resp. $\mathcal{J} \subset \mathcal{U}$) be a compact invariant set of $\varphi^t$ which is uniquely ergodic and contained in an open, pre-compact, positively or negatively invariant subset of $\varphi^t$ in $\Omega$ (resp. in $\mathcal{U}$). If $\mathcal{J}$ is relatively $\mathcal{A}$-stable in $\Omega$ (resp. $\mathcal{A}$-stable), then so is the unique ergodic measure on $\mathcal{J}$.

Proof. Denote $\{\mu^\Omega_\alpha\}$ as the set of all normalized stationary measures in $\Omega$ corresponding to the family $\{L_{A_\alpha}\}$.

a) We note that $\text{supp}(\mu)$ is a compact invariant set of $\varphi^t$ in $\Omega$. Since $\mu$ is relatively $\mathcal{A}$-stable in $\Omega$, $\{\mu^\Omega_\alpha\}$ converges to $\mu$ in $M(\Omega)$ as $A_\alpha \to 0$. Let $W$ be an open neighborhood of $\text{supp}(\mu)$ in $\Omega$. Then it follows from Proposition 2.6 and the fact $\mu(\Omega \setminus \text{supp}(\mu)) = 0$ that

$$\limsup_{A_\alpha \to 0} \mu^\Omega_\alpha(\Omega \setminus W) \leq \mu(\Omega \setminus W) = 0.$$ 

Thus for any $\epsilon > 0$, there exists $\delta > 0$ such that $\mu^\Omega_\alpha(\Omega \setminus W) < \epsilon$ whenever $d(A_\alpha, 0) < \delta$, i.e., $\mu$ is relatively $\mathcal{A}$-stable in $\Omega$.

b) Let $\nu$ be the unique invariant measure of $\varphi^t$ on $\mathcal{J}$. Suppose for contradiction that $\nu$ is not relatively $\mathcal{A}$-stable in $\Omega$. Then there are sequences $\{A_t = (a_{ij}^t)\}_{t=1}^\infty \subset \mathcal{A}$ and $\{\mu^\Omega_t\} \subset \{\mu^\Omega_\alpha\}$ such that $A_t \to 0$ but $\mu^\Omega_t$ is not convergent to $\nu$ in $M(\Omega)$, as $t \to \infty$. Since $\mathcal{J}$ is relatively $\mathcal{A}$-stable in $\Omega$, Proposition 3.4 implies that $\{\mu^\Omega_{A_t}\}$ is relatively sequentially compact. Without loss of generality, we assume $\lim_{t \to \infty} \mu^\Omega_{A_t} = \nu^* \neq \nu$. By Theorem 3.1 (c) and Proposition 3.4, $\nu^*$ is an invariant measure of $\varphi^t$ on $\mathcal{J}$. But since $\mathcal{J}$ is uniquely ergodic, we must have $\nu^* = \nu$, a contradiction.

We now consider the case when there exists a uniform Lyapunov function corresponding to $\{L_{A_\alpha}\}$ in $\Omega$. In this case, we recall from Proposition 2.4 that $\varphi^t$ must be dissipative in $\Omega$, in particular, $\Omega$ is a positively invariant set and contains a maximal attractor of $\varphi^t$ (Proposition 6.1).

Lemma 3.1. Let $\mathcal{A} = \{A_\alpha\} \subset \bar{\mathcal{A}}$ be an admissible null family and $\Omega \subset U$ be a connected open set. Suppose that there is a uniform Lyapunov function in $\Omega$ with respect to $\{L_{A_\alpha}\}$ with essential lower bound $\rho_m$ and upper bound $\rho_M$. Then the following holds.

a) The family $\{\mu^\Omega_\alpha\}$ of all normalized stationary measures in $\Omega$ corresponding to $\{L_{A_\alpha}\}$ is $\mathcal{A}$-sequentially null compact in $M(\Omega)$; 

b) $\text{supp}(\mu) \subset \bar{\Omega}_{\rho_m}$ for any $\mu \in M^\Omega_{A_\alpha}$, where $\bar{\Omega}_{\rho_m}$ denotes the $\rho_m$-sublevel set of $U$.

Proof. Denote $\gamma$ as the uniform Lyapunov constant of $U$ and $\Omega_\rho$ as the $\rho$-sublevel set of $U$ in $\Omega$ for each $\rho \in [\rho_m, \rho_M]$.

To prove a), we let $\{A_t = (a_{ij}^t)\} \subset \mathcal{A}$ be a sequence with $A_t \to 0$ as $t \to \infty$, and denote $\mu_t := \mu_t^\Omega$ for all $t$. For a fixed $\rho_0 \in (\rho_m, \rho_M)$, we have by Proposition 2.2 a) that

$$\mu_t(\Omega \setminus \bar{\Omega}_{\rho_0}) \leq \mu_t(\Omega \setminus \Omega_{\rho_0}) \leq \gamma^{-1} C_{\rho_m, \rho_M} |A_t|_{C(\Omega_{\rho_0})} \mu_t \left( \Omega_{\rho_0} \setminus \bar{\Omega}_{\rho_m} \right).$$ 


for all \( l = 1, 2, \ldots \), where \( C_{\rho_m, \rho_0} \) is a constant depending only on \( \Omega, \rho_0, \rho_m \). It follows that for any \( \epsilon > 0 \), there is a natural number \( l_0 \) such that
\[
\mu_l(\Omega \setminus \bar{\Omega}_{\rho_0}) < \epsilon, \quad l = l_0 + 1, \ldots .
\]
For each \( l = 1, \ldots , l_0 \), clearly we can find a compact set \( S_l \subset \Omega \) such that \( \mu_l(\Omega \setminus S_l) < \epsilon \). Let \( K_\epsilon := \bar{\Omega}_{\rho_0} \cup \left( \bigcup_{l=1}^{l_0} S_l \right) \). Then \( \mu_l(\Omega \setminus K_\epsilon) < \epsilon \) for all \( l \). This shows that the sequence \( \{\mu_l\} \) is tight, hence relatively sequentially compact in \( M(\Omega) \).

To show b), we note that \( \Omega \setminus \bar{\Omega}_{\rho_0} \) is open. Applying Proposition 2.6, we have by taking limit \( l \to \infty \) in (3.9) that
\[
0 = \lim_{l \to \infty} \inf \mu_l(\Omega \setminus \bar{\Omega}_{\rho_0}) \geq \mu(\Omega \setminus \bar{\Omega}_{\rho_0}).
\]
This implies \( \text{supp}(\mu) \subset \bar{\Omega}_{\rho_0} \) by the continuity of probability measures and the arbitrariness of \( \rho_0 \in (\rho_m, \rho_M) \).

Note that Theorem B a) is just Lemma 3.1 a) when taking \( \Omega = \mathcal{U} \).

**Theorem 3.2.** Assume \( V \in C^1(\mathcal{U}, \mathbb{R}^n) \). Let \( \mathcal{A} = \{A_\alpha\} \subset \bar{\mathcal{A}} \) be an admissible null family and \( \Omega \subset \mathcal{U} \) be a connected open set. If there is a uniform Lyapunov function in \( \Omega \) with respect to \( \{\mathcal{L}_{A_\alpha}\} \), then the following holds:

a) \( \mathcal{M}_\mathcal{A}^\Omega \neq \emptyset \) and each \( \mu \in \mathcal{M}_\mathcal{A}^\Omega \) is an invariant measure of \( \varphi^t \) in \( \Omega \) supported on \( J \) - the maximal attractor of \( \varphi^t \) in \( \Omega \).

b) If \( \mathcal{A} \) is invariant, then \( \mathcal{J}_\mathcal{A}^\Omega \) and \( J \) are the smallest and largest relatively \( \mathcal{A} \)-stable sets in \( \Omega \), respectively.

**Proof.** a) Denote by \( \{\mu_\alpha^\Omega\} \) the set of all normalized stationary measures in \( \Omega \) corresponding to \( \{\mathcal{L}_{A_\alpha}\} \). By Lemma 3.1 a), \( \mathcal{M}_\mathcal{A}^\Omega \neq \emptyset \). Let \( \mu \in \mathcal{M}_\mathcal{A}^\Omega \). Then there is a sequence \( \{\mu_l := \mu_\alpha^\Omega\} \) such that \( \mu = \lim_{l \to \infty} \mu_l \), where \( \{A_l = (a_{ij}^l)\} \subset \mathcal{A} \) is a sequence such that \( A_l \to 0 \) as \( l \to \infty \). Denote \( U \) as a uniform Lyapunov function in \( \Omega \) with respect to \( \{\mathcal{L}_{A_\alpha}\} \), \( \rho_m \) as an essential lower bound of \( U \), \( \rho_M \) as the essential upper bound of \( U \), and \( \Omega_\rho \) as the \( \rho \)-sublevel set of \( U \) for each \( \rho > 0 \). By Lemma 3.1 b), we have \( \text{supp}(\mu) \subset \bar{\Omega}_{\rho_m} \).

For fixed \( \rho \in (\rho_m, \rho_M) \), by Proposition 2.4, \( \Omega_\rho \) is an open, pre-compact, positively invariant set of \( \varphi^t \) in \( \Omega \) containing \( \text{supp}(\mu) \), so it follows from Theorem 3.1 c) and Proposition 6.5 that \( \mu \) is an invariant measure of \( \varphi^t \) supported on the maximal attractor \( J = \omega(\Omega_{\rho_m}) \) of \( \varphi^t \) in \( \Omega_{\rho_m} \), which is also the maximal attractor of \( \varphi^t \) in \( \Omega \).

b) We note by the invariance of any \( \mu \in \mathcal{M}_\mathcal{A}^\Omega \) that \( \mathcal{J}_\mathcal{A}^\Omega \) is a compact invariant set of \( \varphi^t \) in \( \Omega \) and by a) that \( \mathcal{J}_\mathcal{A}^\Omega \subset J \). It follows from Proposition 3.4, Lemma 3.1 a) and Corollary 3.1 that both \( \mathcal{J}_\mathcal{A}^\Omega \) and \( J \) are relatively \( \mathcal{A} \)-stable sets in \( \Omega \) with \( \mathcal{J}_\mathcal{A}^\Omega \) being the smallest relatively \( \mathcal{A} \)-stable set in \( \Omega \). Since the maximal attractor \( J \) is the largest compact invariant set of \( \varphi^t \) in \( \Omega \), the proof is complete.

Note that Theorem 3.2 a) immediately implies Theorem B b) when \( \Omega = \mathcal{U} \).

**Corollary 3.2.** Assume \( V \in C^1(\mathcal{U}, \mathbb{R}^n) \). If \( \mathcal{A} \) admits a \( C^2 \) Lyapunov function whose second derivatives are bounded on \( \mathcal{U} \), then the global attractor \( J \) in \( \mathcal{U} \) is \( \mathcal{A} \)-stable with respect to any invariant, bounded null family \( \mathcal{A} \).
Proof. It follows from Proposition 2.5, Remark 2.3 1), and Theorem 3.2 with $\Omega = \mathcal{U}$. ☐

3.4. Local concentration of limit measures. For a given admissible null family $\mathcal{A} = \{A_\alpha\} \subset \hat{\mathcal{A}}$, if there exists a uniform Lyapunov function with respect to $\{\mathcal{L}_{A_\alpha}\}$, then when $\Omega = \mathcal{U}$ Theorem 3.2 asserts that any limit measure $\mu \in \mathcal{M}_\mathcal{A}$ is an invariant measure of $\varphi^t$ supported on the global attractor $\mathcal{J}$ of $\varphi^t$. In this case, possible local concentration of $\mu$ is clear because any invariant measure of a flow defined on a compact metric space is supported on the closure of the recurrent set of the flow (13).

In case that a uniform Lyapunov function is not known on $\mathcal{U}$, information may still be obtained about possible local concentration of a limit measure $\mu \in \mathcal{M}_\mathcal{A}$ on a compact invariant set $\mathcal{E} \subset \mathcal{U}$ of $\varphi^t$, i.e., there is a neighborhood $\mathcal{W}$ of $\mathcal{E}$ in $\mathcal{U}$ such that $\mu(\mathcal{W} \setminus \mathcal{E}) = 0$, provided that a uniform Lyapunov or anti-Lyapunov function is known on $\mathcal{U}$.

We first consider a pre-compact set $\Omega \subset \mathcal{U}$ in which a uniform anti-Lyapunov function with respect to $\{\mathcal{L}_{A_\alpha}\}$ exists. We note that $\mathcal{M}_\mathcal{A}^\Theta$ is always non-empty (see Remark 2.4) even if $\mathcal{M}_\mathcal{A}^\Theta$ is empty. We also note by Proposition 3.4 that, with the existence of a uniform anti-Lyapunov function in $\Omega$ with respect to $\{\mathcal{L}_{A_\alpha}\}$, $\varphi^t$ must be anti-dissipative in $\Omega$. In particular, $\Omega$ contains a maximal repeller of $\varphi^t$.

**Lemma 3.2.** Let $\mathcal{A} = \{A_\alpha\} \subset \hat{\mathcal{A}}$ be a null family. Suppose that $\Omega$ is a connected, open, and pre-compact subset of $\mathcal{U}$ and that there is a uniform anti-Lyapunov function $U$ with respect to $\{\mathcal{L}_{A_\alpha}\}$ in $\Omega$. Then the following holds.

a) The family $\{\mu^\alpha_{\Theta}\}$ of normalized stationary measures in $\Omega$ corresponding to $\{\mathcal{L}_{A_\alpha}\}$ is relatively sequentially compact in $M(\hat{\Omega})$.

b) For any $\mu \in \mathcal{M}_\mathcal{A}^\Theta$, $\text{supp}(\mu) \subset \bar{\Omega}_{\rho_m} \cup \partial \Omega$, where $\rho_m$ is the essential lower bound of $U$ and $\Omega_{\rho_m}$ is the $\rho_m$-sublevel set of $U$.

c) If $V \in C^1(\mathcal{U}, \mathbb{R}^n)$, then for any $\mu \in \mathcal{M}_\mathcal{A}^\Theta$, $\text{supp}(\mu) \subset \mathcal{R} \cup \partial \Omega$, where $\mathcal{R}$ is the maximal repeller of $\varphi^t$ in $\Omega$. Moreover, if $\mu(\mathcal{R}) \neq 0$, then $\mu |_{\mathcal{R}}$ is an invariant measure of $\varphi^t$ on $\mathcal{R}$, and consequently, $\mu$ is an invariant measure of $\varphi^t$ in $\Omega$ if $\mu(\partial \Omega) = 0$.

**Proof.** a) holds because $\{\mu^\alpha_{\Theta}\}$, as a family of Borel probability measures on the compact set $\hat{\Omega}$, is tight.

To prove b), we let $\{A_l = (a_l^{ij})\} \subset \mathcal{A}$ and $\{\mu_l := \mu^\Theta_l\}$ be sequences such that $A_l \to 0$ and $\mu_l \to \mu \in M(\hat{\Omega})$ as $l \to \infty$. Denote $\rho_M$ as the essential upper bound of $U$, $\gamma$ as the uniform anti-Lyapunov constant of $U$, and $\Omega_\rho$ as the $\rho$-sublevel set of $U$ in $\Omega$ for each $\rho \in [\rho_m, \rho_M)$.

For each $\rho \in [\rho_m, \rho_M)$, we have

$$a_l^{ij}(x) \partial_i U(x) \partial_j U(x) \leq |A_l|_{C(\hat{\Omega})} |\nabla U|_{C(\Omega_\rho)}^2, \quad x \in \partial \Omega_\rho.$$  

Thus, for a given $\rho_0 \in (\rho_m, \rho_M)$, it follows from Proposition 2.5 that

$$(3.11) \quad \mu_l(\Omega_{\rho_0} \setminus \Omega_{\rho}^*) \leq \mu_l(\Omega_{\rho} \setminus \Omega_{\rho_m}^*) e^{-\frac{\gamma^2 c_2}{c_1}},$$

where $\Omega_{\rho_m}^* = \Omega_{\rho_m} \cup U^{-1}(\rho_m)$, $a_l = |A_l|_{C(\hat{\Omega})}$, and $C_\rho = \frac{\rho - \rho_0}{|\nabla U|_{C(\Omega_\rho)}^2}$, $\rho \in (\rho_0, \rho_M)$.

Since $A_l \to 0$ and $\mathcal{L}_{A_l} U(x) \geq \gamma$ for all $l$ and $x \in \hat{\Omega} = \Omega \setminus \Omega_{\rho_m}$, taking limit $l \to \infty$ yields that $V(x) \cdot \nabla U(x) \geq \gamma$ in $\hat{\Omega}$. Hence $V(x) \cdot \nabla U(x) \geq \gamma$ on the the closure of $\hat{\Omega}$ in $\Omega$, i.e.,
on the set \( \{ x \in \Omega : U(x) \geq \rho_m \} \). For any \( \rho \in [\rho_m, \rho_M] \) and \( x \in \partial \Omega_\rho \), the Implicit Function Theorem implies that in the vicinity of \( x \), \( \partial \Omega_\rho \) is a local \( C^2 \) hypersurface which coincides with a portion of the level surface \( U^{-1}(\rho) \). Since \( \partial \Omega_\rho \) is a compact set, it consists of finitely many \( C^2 \) components, each of which is oriented. Thus, \( \Omega_\rho \) is a \( C^2 \) domain and

\[
(3.12) \quad \partial \Omega_\rho = U^{-1}(\rho), \quad \text{for any } \rho \in [\rho_m, \rho_M].
\]

In particular, \( \Omega^*_{\rho_m} = \bar{\Omega}_{\rho_m} \). Since \( a_t \to 0 \), an application of Proposition \( \mathbf{2.6} \) to \( \mathbf{(3.11)} \) yields that \( \mu(\Omega_{\rho_0} \setminus \bar{\Omega}_{\rho_m}) = 0 \) for any \( \rho_0 \in (\rho_m, \rho_M) \). It follows that \( \mu(\Omega \setminus \bar{\Omega}_{\rho_m}) = 0 \). Hence \( \text{supp}(\mu) \subseteq \Omega_{\rho_m} \cup \partial \Omega \).

To prove c), we observe that if \( \mu(\bar{\Omega}_{\rho_m}) = 0 \), then by b), \( \text{supp}(\mu) \subseteq \partial \Omega \). Now suppose \( \mu(\bar{\Omega}_{\rho_m}) \neq 0 \) and let \( \rho_*, \rho_0 \in (\rho_m, \rho_M) \) be fixed such that \( \rho_* < \rho_0 \). We have by b) that \( \mu(\partial \Omega_{\rho_0}) = 0 \). It follows from Proposition \( \mathbf{2.6} \) that \( \lim_{t \to \infty} \mu_t(\Omega_{\rho_0}) = \mu(\Omega_{\rho_0}) \), and therefore

\[
\tilde{\mu}_l = \frac{\mu_l|_{\Omega_{\rho_0}}}{\mu_l(\Omega_{\rho_0})}
\]

is a sequence of stationary measures corresponding to \( \{ \mathcal{L}_A \} \) in \( \Omega_{\rho_0} \) which converges to

\[
\tilde{\mu} = \frac{\mu|_{\Omega_{\rho_0}}}{\mu(\Omega_{\rho_0})}
\]

as \( l \to \infty \). We clearly have \( \text{supp}(\tilde{\mu}) \subseteq \Omega_{\rho_*} \) because \( \rho_m < \rho_* \). Since \( V(x) \cdot \nabla U(x) \geq \gamma \) in \( \{ x \in \Omega : U(x) \geq \rho_m \} \), \( \Omega_{\rho_*} \) is a negatively invariant, pre-compact open subset of \( \Omega_{\rho_0} \). It follows from Theorem \( \mathbf{3.1} \) with \( \bar{\Omega} \) in place of \( \mathcal{U} \) and \( \Omega_{\rho_0} \) in place of \( \Omega \) that \( \tilde{\mu} \) is an invariant measure of \( \varphi^t \) on \( \Omega_{\rho_0} \). Moreover, by Proposition \( \mathbf{6.5} \) \( \tilde{\mu} \) is supported on \( \alpha(\Omega_{\rho_0}) = \mathcal{R} \) - the maximal attractor of \( \varphi^t \) in \( \Omega \). Therefore, \( \text{supp}(\tilde{\mu}) \subseteq \mathcal{R} \). By b), \( \text{supp}(\mu) \subseteq \mathcal{R} \cup \partial \Omega \). Since \( \frac{\mu|_{\mathcal{R}}}{\mu(\mathcal{R})} = \tilde{\mu}|_{\mathcal{R}} \), the proof is complete.

\( \square \)

**Theorem 3.3.** Assume \( V \in C^1(\mathcal{U}, \mathbb{R}^n) \). Let \( \mathcal{A} = \{ A_\alpha \} \subset \hat{\mathcal{A}} \) be an admissible null family, \( \mu \in \mathcal{M}_A \), and \( \Omega \subset \mathcal{U} \) be a connected open set. Then the following holds.

a) If there exists a uniform Lyapunov function in \( \Omega \) with respect to \( \{ \mathcal{L}_A \} \), then \( \mu(\Omega \setminus \mathcal{J}) = 0 \), where \( \mathcal{J} \) is the maximal attractor of \( \varphi^t \) in \( \Omega \).

b) If \( \Omega \) is pre-compact in \( \mathcal{U} \) and there exists a uniform anti-Lyapunov function in \( \Omega \) with respect to \( \{ \mathcal{L}_A \} \), then \( \mu(\Omega \setminus \mathcal{R}) = 0 \), where \( \mathcal{R} \) is the maximal repeller of \( \varphi^t \) in \( \Omega \).

**Proof.** Let \( \{ A_l = (a^{ij}_l) \} \subset \mathcal{A} \) be a sequence such that \( A_l \to 0 \) and \( \{ \mu_l \} \) be a sequence of stationary measures corresponding to \( \{ \mathcal{L}_{A_l} \} \) such that \( \mu_l \to \mu \in \mathcal{M}(\mathcal{U}) \) as \( l \to \infty \). Also let \( U \) be a uniform Lyapunov function in \( \Omega \) in the case a) and a uniform anti-Lyapunov function in \( \Omega \) in the case b), with essential lower bound \( \rho_m \) and upper bound \( \rho_M \). For each \( \rho \in [\rho_m, \rho_M) \), denote \( \Omega_\rho \) as the \( \rho \)-sublevel set of \( U \).

If \( \mu(\Omega) = 0 \), then the theorem holds automatically. We now suppose that \( \mu(\Omega) \neq 0 \). For any \( \rho_0 \in (\rho_m, \rho_M) \), applications of \( \mathbf{(3.9), (3.11), (3.12)} \) to

\[
\tilde{\mu}_l = \frac{\mu_l|_{\Omega}}{\mu_l(\Omega)}
\]
in place of $\mu_l$ for the case a), b) respectively yield that
\begin{align}
\mu(\Omega \setminus \Omega_{\rho_0}) = 0, \text{ in the case a),} \\
\mu(\Omega_{\rho_0} \setminus \Omega_{\rho_0}) = 0, \text{ in the case b),}
\end{align}
for any $\rho_0 \in (\rho_m, \rho_M)$. Now let $\rho_0 \in (\rho_m, \rho_M)$ be fixed. Then (3.13) and (3.14) imply that
$$
\mu(\partial \Omega_{\rho_0}) = 0.
$$
Hence it follows from Proposition 2.6 that
$$
\rho
$$
is a sequence of stationary measures corresponding to \{L_\mu\} in $\Omega_{\rho_0}$ which converges to \(\mu^*\).

We note that $U$ is also a uniform Lyapunov function in $\Omega_{\rho_0}$ in the case a) and a uniform anti-Lyapunov function in $\Omega_{\rho_0}$ in the case b), with respect to $\{L_\nu\}$.

In the case a), it follows from Theorem 3.2 with $\Omega$ in place of $\mathcal{U}$ and $\Omega_{\rho_0}$ in place of $\Omega$ that $\mu^*(\Omega_{\rho_0} \setminus \mathcal{J}) = 0$, implying that $\mu(\Omega_{\rho_0} \setminus \mathcal{J}) = 0$. In the case b), since $\mu^*(\partial \Omega_{\rho_0}) = 0$, we have by Lemma 3.2 c) with $\Omega$ in place of $\mathcal{U}$ and $\Omega_{\rho_0}$ in place of $\Omega$ that $\mu^*(\Omega_{\rho_0} \setminus \mathcal{R}) = 0$, implying that $\mu(\Omega_{\rho_0} \setminus \mathcal{R}) = 0$. This completes the proof because $\rho_0 \in (\rho_m, \rho_M)$ is arbitrary.

Part a) of the following result is precisely part c) of Theorem A.

**Corollary 3.3.** Assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and $\varphi^t$ admits a strong local attractor $\mathcal{J}_0$ (resp. strong local repeller $\mathcal{R}_0$) with an isolating neighborhood $\mathcal{W}_{\mathcal{J}_0}$ (resp. $\mathcal{W}_{\mathcal{R}_0}$). Then the following holds for any admissible null family $\mathcal{A} = \{A_\alpha\} \subset \mathcal{A}$.

a) For any $\mu \in \mathcal{M}_A$, $\mu(\mathcal{W}_{\mathcal{J}_0} \setminus \mathcal{J}_0) = 0$ (resp. $\mu(\mathcal{W}_{\mathcal{R}_0} \setminus \mathcal{R}_0) = 0$).

b) $\mathcal{J}_0$ is relatively $\mathcal{A}$-stable in $\mathcal{W}_{\mathcal{J}_0}$ if $\mathcal{A}$ is invariant.

**Proof.** We note by the proof of Proposition 6.4 that (1.1) admits a $C^2$ Lyapunov (resp. anti-Lyapunov) function $U$ in $\mathcal{W} =: \mathcal{W}_{\mathcal{J}_0}, \mathcal{W}_{\mathcal{R}_0}$. Since $\mathcal{W}$ is a pre-compact subset of $\mathcal{U}$, all second derivatives of $U$ are bounded in $\mathcal{W}$. Let $\mathcal{A} = \{A_\alpha\} \subset \mathcal{A}$ be any null family. Without loss of generality, we assume $d(A_\alpha, 0) \ll 1$. It follows from Proposition 2.5 that $U$ is a uniform Lyapunov (resp. anti-Lyapunov) function in $\mathcal{W}$ with respect to $\{L_{A_\alpha}\}$. By Theorem 3.3, $\mu(\mathcal{W} \setminus \mathcal{J}_0) = 0$ (resp. $\mu(\mathcal{W} \setminus \mathcal{R}_0) = 0$). This proves a).

Using the uniform Lyapunov function $U$ in $\mathcal{W}_{\mathcal{J}_0}$ with respect to $\{L_{A_\alpha}\}$ and the fact that $\mathcal{J}_0$ is the maximal attractor of $\varphi^t$ in $\mathcal{W}_{\mathcal{J}_0}$, b) follows from Theorem 3.2 with $\Omega = \mathcal{W}_{\mathcal{J}_0}$. 

4. Stabilization and de-stabilization via multiplicative noises

Throughout the section, we assume $V \in C^1(\mathcal{U}, \mathbb{R}^n)$ and that (1.1) admits a $C^2$ Lyapunov function with bounded second derivatives on $\mathcal{U}$. For any bounded null family $\mathcal{A} = \{A_\alpha\} \subset \mathcal{A}$, it follows from Proposition 2.5, Remark 2.3, 1), and Theorem 3.2 that $\mathcal{A}$ is admissible, $\mathcal{M}_A \neq \emptyset$, and any limit measure $\mu \in \mathcal{M}_A$ is an invariant measure of $\varphi^t$ supported on the global attractor $\mathcal{J}$ of $\varphi^t$, i.e., $\mathcal{J}$ is $\mathcal{A}$-stable if $\mathcal{A}$ is also invariant. Moreover, by Corollary 3.3 if $\mathcal{J}$ contains a strong local attractor $\mathcal{J}_0$ (resp. repeller $\mathcal{R}_0$), then $\mu$ may be locally concentrated.
on $\mathcal{J}_0$ (resp. $\mathcal{R}_0$). In fact, we believe that local concentration on a strong local repeller can be expected only when it further contains a sub-local attractor, as suggested by Theorem 4.2 below.

In this section, we will demonstrate an important role played by multiplicative noise perturbations. We will show that if $\mathcal{J}$ contains a strong local attractor $\mathcal{J}_0$ of $\varphi^t$, then one can design a particular invariant, bounded null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ by adding stronger noise away from the local attractor such that any $\mu \in \mathcal{M}_\mathcal{A}$ is (globally) concentrated on $\mathcal{J}_0$, i.e., $\mathcal{J}_0$ is $\mathcal{A}$-stable. On the contrary, if $\mathcal{J}$ contains a strong local repeller $\mathcal{R}_0$ of $\varphi^t$, then one can design a particular invariant, bounded null family $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ by adding stronger noise on and near the repeller such that any $\mu \in \mathcal{M}_\mathcal{A}$ is (globally) concentrated away from $\mathcal{R}_0$, i.e., $\mathcal{R}_0$ is strongly $\mathcal{A}$-unstable.

In the case that $\mathcal{R}_0$ is a strongly repelling equilibrium, we will show that it is strongly $\mathcal{A}$-unstable with respect to any invariant, normal null family $\mathcal{A} \subset \tilde{\mathcal{A}}$.

4.1. Stabilizing a local attractor and de-stabilizing a local repeller. Let $\mathcal{E}_0$ be either a strong local attractor or a strong local repeller of $\varphi^t$ in $\mathcal{U}$. We denote the isolating neighborhood of $\mathcal{E}_0$ by $\mathcal{W}_0$. Then by the proof of Proposition 6.4 there is a positive function $U_0 \in C^2(\mathcal{U})$ and constants $\gamma_0, \hat{\rho} > 0$ such that $\partial \mathcal{W}_0 = \{U_0(x) = \hat{\rho}\}$ and

$$|V(x) - \nabla U_0(x)| > \gamma_0 |\nabla U_0(x)|, \quad x \in \partial \mathcal{W}_0 = \partial \Omega^0_\rho,$$

where $\Omega^0_\rho$ denotes the $\rho$-sublevel set of $U_0$ for each $\rho > 0$. By modifying $U_0$ away from $\partial \mathcal{W}_0$, we can assume without loss of generality that $U_0(x) < \hat{\rho}$ for all $x \in \mathcal{W}_0$, so that $U_0$ becomes a compact function in $\mathcal{W}_0$. We note that $\mathcal{W}_0 = \tilde{\Omega}^0_\rho = \partial \Omega^0_\rho \cup U_0^{-1}(\hat{\rho})$.

Lemma 4.1. Let $\mathcal{E}_0$, $\mathcal{W}_0$, $U_0$, $\hat{\rho}$, $\gamma_0$, $\Omega^0_\rho$ be as in the above, and $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$ be a null family on $\mathcal{U}$. Then there are positive constants $\rho_*, \rho^*$, depending only on $\mathcal{W}_0, U_0$, with $\rho_* < \hat{\rho} < \rho^*$, such that as $d(A_\alpha, 0) \ll 1$,

$$\mu_\alpha(\Omega^0_{\rho^*} \setminus \mathcal{W}_0) = \mu_\alpha(\Omega^0_{\rho_*} \setminus \mathcal{W}_0) \leq e^{-\frac{C_1}{a_1(\alpha)}}, \quad \text{when } \mathcal{E}_0 \text{ is a local attractor},$$

$$\mu_\alpha(\mathcal{W}_0 \setminus \tilde{\Omega}^0_{\rho_*}) = \mu_\alpha(\mathcal{W}_0 \setminus \Omega^0_{\rho_*}) \leq e^{-\frac{C_2}{a_2(\alpha)}}, \quad \text{when } \mathcal{E}_0 \text{ is a local repeller}$$

for all stationary measures $\{\mu_\alpha\}$ corresponding to $\{\mathcal{L}_{A_\alpha}\}$, where

$$C_1 = \frac{\gamma_0(\hat{\rho} - \rho_*) \min_{x \in \partial \Omega^0_\rho} |\nabla U_0(x)|}{2 \max_{\rho_* \leq U_0(x) \leq \hat{\rho}} |\nabla U_0(x)|^2},$$

$$a_1(\alpha) = \max_{\rho_* \leq U_0(x) \leq \hat{\rho}} |A_\alpha(x)|,$$

$$C_2 = \frac{\gamma_0(\rho^* - \hat{\rho}) \min_{x \in \partial \Omega^0_\rho} |\nabla U_0(x)|}{2 \max_{\rho_* \leq U_0(x) \leq \rho^*} |\nabla U_0(x)|^2},$$

$$a_2(\alpha) = \max_{\rho_* \leq U_0(x) \leq \rho^*} |A_\alpha(x)|.$$

Proof. By (4.1), there are $\rho_*, \rho^*$ with $\rho_* < \hat{\rho} < \rho^*$ such that, for all $x \in \tilde{\Omega}^0_{\rho_*} \setminus \Omega^0_{\rho_*}$, $\nabla U_0(x) \neq 0$ and

$$|\mathcal{L}_{A_\alpha} U_0(x)| = \left| a_2 \partial^2_{ij} U_0(x) + V(x) \cdot \nabla U_0(x) \right| \geq \frac{\gamma_0 \min_{x \in \partial \Omega^0_\rho} |\nabla U_0(x)|}{2},$$

where $\gamma_0 > 0$ is the constant in Proposition 6.4.
as \(d(A,0) \ll 1\). Since \(W_0\) is connected, by making \(\rho_*, \rho^*\) sufficiently close to \(\hat{\rho}\) if necessary, we may assume without loss of generality that \(\Omega^0_\rho\) is a connected open set for each \(\rho \in (\rho_*, \rho^*)\).

As in the proof of Lemma 3.2, the Implicit Function Theorem implies that \(\Omega^0_\rho\) is a \(C^2\) domain and \(\Omega^0_\rho = \Omega^0_\rho \cup U_0^{-1}(\rho)\) for each \(\rho \in (\rho_*, \rho^*)\).

In the case that \(\mathcal{E}_0\) is a strong local attractor, the remaining part of the lemma follows from Proposition 2.2 b) with \(\Omega = \Omega^0_\rho\) in place of \(\Omega\) and

\[
\rho_M =: \rho^*, \quad \rho_m = \rho_*, \quad \rho =: \hat{\rho}, \quad \gamma =: \frac{\gamma_0 \min_{x \in \partial \Omega^0_\rho} |\nabla U_0(x)|}{2}, \quad H = a_1(\alpha) \max_{\rho \leq U_0(x) \leq \hat{\rho}} |\nabla U_0(x)|^2.
\]

In the case that \(\mathcal{E}_0\) is a strong local repeller, the remaining part of the lemma follows from Proposition 2.3 with \(W_0\) in place of \(\Omega\) and

\[
\rho =: \rho^*, \quad \rho_m = \rho_*, \quad \rho_0 =: \tilde{\rho}, \quad \gamma =: \frac{\gamma_0 \min_{x \in \partial \Omega^0_\rho} |\nabla U_0(x)|}{2}, \quad H = a_2(\alpha) \max_{\rho \leq U_0(x) \leq \tilde{\rho}} |\nabla U_0(x)|^2.
\]

Theorem C a), b) follow from the following result.

**Theorem 4.1.** Assume \(V \in C^1(\mathcal{U}, \mathbb{R}^n)\) and that \((1.1)\) admits a \(C^2\) Lyapunov function with bounded second derivatives on \(\mathcal{U}\). Then the following holds.

a) For any strong local attractor \(\mathcal{J}_0\) lying in \(\mathcal{U}\), there is a normal null family \(\mathcal{A} \subset \hat{\mathcal{A}}\) on \(\mathcal{U}\) which is both invariant and admissible such that \(\mathcal{M}_\mathcal{A} \neq \emptyset\) and each limit measure \(\mu \in \mathcal{M}_\mathcal{A}\) is an invariant measure of \(\varphi^t\) supported on \(\mathcal{J}_0\). Consequently, \(\mathcal{J}_0\) is \(\mathcal{A}\)-stable.

b) For any strong local repeller \(\mathcal{R}_0\) lying in \(\mathcal{U}\), there is a normal null family \(\mathcal{A} \subset \hat{\mathcal{A}}\) which is both invariant and admissible such that \(\mathcal{M}_\mathcal{A} \neq \emptyset\) and each limit measure \(\mu \in \mathcal{M}_\mathcal{A}\) is an invariant measure of \(\varphi^t\) supported on the maximal attractor \(\mathcal{J}\) of \(\varphi^t\) in \(\mathcal{J} \setminus W_0\), where \(\mathcal{J}\) is the global attractor of \(\varphi^t\) and \(W_0\) is an isolating neighborhood of \(\mathcal{R}_0\). Consequently, \(\mathcal{J}_*\) is \(\mathcal{A}\)-stable and \(\mathcal{R}_0\) is strongly \(\mathcal{A}\)-unstable.

**Proof.** Let \(\mathcal{E}_0\) be either \(\mathcal{J}_0\) or \(\mathcal{R}_0\), and \(W_0\) be an isolating neighborhood of \(\mathcal{E}_0\). We note that since \(\mathcal{U}\) is dissipative, \(\mathcal{E}_0\) must lie in the global attractor \(\mathcal{J}\) of \(\varphi^t\).

First of all, we fix a pre-compact, connected open neighborhood \(\Omega_0\) of \(\mathcal{J}\) in \(\mathcal{U}\). Let \(\mathcal{A} = \{A_\alpha\} = \{(a^\alpha_i)\} \subset \hat{\mathcal{A}}\) be any bounded null family. We have by Proposition 2.5 that there is a uniform Lyapunov function with respect to \(\{L_{A_\alpha}\}\) in \(\mathcal{U}\). Hence by Remark 2.1 1) \(\mathcal{A}\) is admissible. Denote \(\{\mu_{A_\alpha}\}\) as the set of all stationary measures corresponding to \(\{L_{A_\alpha}\}\).

It follows from Lemma 3.1 a) that \(\{\mu_{A_\alpha}\}\) is \(\mathcal{A}\)-sequentially null compact in \(M(\mathcal{U})\) and hence \(\mathcal{M}_\mathcal{A} \neq \emptyset\). By Theorem 3.2 any \(\mu \in \mathcal{M}_\mathcal{A}\) is an invariant measure of \(\varphi^t\) supported on the global attractor \(\mathcal{J}\).

If \(\mathcal{U} = \mathbb{R}^n\), then \(\mathcal{A}\) is clearly invariant. Otherwise, one can modify each \(A \in \mathcal{A}\) near \(\partial \mathcal{U}\) such that \(A(x) \to 0\) as \(x \to \partial \mathcal{U}\). Since \((1.1)\) admits a Lyapunov function in \(\mathcal{U}\), it is not hard to see that the modified normal null family \(\mathcal{A}\) becomes invariant.

Let \(\rho_*, \rho^*, \tilde{\rho}, \Omega_\rho^0, \rho \in (\rho_*, \rho^*)\), be as in Lemma 4.1. Recall that \(\rho_* < \tilde{\rho} < \rho^*\) and \(\bar{W}_0 = \bar{\Omega}^0_{\tilde{\rho}}\).

By taking \(\rho^*\) further small if necessary, we can assume \(\Omega^0_{\rho^*} \subset \Omega_0\).
For fixed \( \tilde{\rho}^* \in (\tilde{\rho}, \rho^*) \), \( \hat{\rho}_* \in (\rho_*, \hat{\rho}) \), we consider sets

\[
D = \begin{cases} 
\bar{\Omega}_0 \setminus \Omega^0_{\hat{\rho}^*}, & \text{when } \mathcal{E}_0 \text{ is a local attractor}, \\
\bar{\Omega}^0_{\tilde{\rho}^*}, & \text{when } \mathcal{E}_0 \text{ is a local repeller}, 
\end{cases}
\]

\[
D^* = \begin{cases} 
\Omega^0_{\rho^*} \setminus \Omega^0_{\hat{\rho}^*}, & \text{when } \mathcal{E}_0 \text{ is a local attractor}, \\
\bar{\Omega}^0_{\tilde{\rho}^*} \setminus \Omega^0_{\hat{\rho}^*}, & \text{when } \mathcal{E}_0 \text{ is a local repeller}, 
\end{cases}
\]

\[
D_* = \begin{cases} 
\bar{\Omega}^0_{\rho^*} \setminus \Omega^0_{\hat{\rho}^*}, & \text{when } \mathcal{E}_0 \text{ is a local attractor}, \\
\bar{\Omega}^0_{\tilde{\rho}^*} \setminus \Omega^0_{\hat{\rho}^*}, & \text{when } \mathcal{E}_0 \text{ is a local repeller}. 
\end{cases}
\]

Let \( \Omega \) be a fixed small neighborhood of \( D \) in \( \mathcal{U} \) with \( \bar{\Omega} \cap D_* = \emptyset \). Then there are small balls \( \{B_R(y_k)\}_{k=1}^N \) of radius \( R \) centered at \( y_k \in D \), \( k = 1, \cdots, N \), whose union covers \( D \), and

\[
\bigcup_{k=1}^N B_R^k(y_k) \subset \Omega.
\]

Using the facts that \( \bar{\Omega}^0_{\hat{\rho}^*} \subset \bar{\Omega}_0, \bar{\Omega}_0 \) is connected, and \( \Omega^0_{\rho^*} \) is a connected open set with oriented \( C^2 \) boundary for each \( \rho \in (\rho_*, \rho^*) \), we see that \( D \) has finitely many components, say, \( D_1, D_2, \cdots, D_I \). Since \( \partial \bar{\Omega}^0_{\hat{\rho}^*} \subset \text{int}(\bar{\Omega}^0_{\rho^*} \setminus \Omega^0_{\hat{\rho}^*}) \) and \( \partial \bar{\Omega}^0_{\tilde{\rho}^*} \subset \text{int}(\bar{\Omega}^0_{\rho^*} \setminus \Omega^0_{\hat{\rho}^*}) \), it is not hard to see that \( |D_i \cap D^*| > 0 \) for all \( i = 1, 2, \cdots, I \).

We now restrict \( \{A_\alpha\} \) to be a normal null family satisfying \( A_\alpha \in C^1(\mathcal{U}, GL(n, \mathbb{R})) \) for each \( \alpha \) and

\[
\sup_\alpha \tilde{\lambda}_\alpha(\Omega) < 1,
\]

where, for each \( \alpha \),

\[
\tilde{\lambda}_\alpha(\Omega) = \sup_{x \in \Omega} |\partial A_\alpha(x)|.
\]

For each \( \alpha \), denote \( u_\alpha \) as the weak stationary solution corresponding to \( \mathcal{L}_{A_\alpha} \) that is associated with the measure \( \mu_\alpha \). We also let \( \lambda_\alpha, \gamma_\alpha \) be the quantities defined as in Definition 2.2 on each pre-compact open subset of \( \mathcal{U} \) for the restricted family \( \{A_\alpha\} \).

Note that the above neighborhood \( \Omega \) of \( D \) can be chosen such that the number of components of \( \Omega \) is at most \( I \). Therefore, applying Proposition 2.7 in \( \Omega \) with \( b^i = \partial_j a^{i,j} - V^i \), \( c^i = d \equiv 0, i = 1, \cdots, n \), we have

\[
\sup_{D_i} u_\alpha(x) \leq C_0 (C_1 + \frac{C_2}{\lambda_\alpha(\Omega)}) \inf_{D_i(y_i)} u_\alpha(x), \quad i = 1, \cdots, N,
\]

where \( C_0 \geq 1 \) is a positive constant depending only on \( n \) and \( \Omega \), \( C_1 = \sup_\alpha \frac{\lambda_\alpha(\Omega)}{\lambda_\alpha(\Omega)} < \infty \), and \( C_2 = 2n(\sup_\alpha \tilde{\lambda}_\alpha(\Omega) + \sup_{x \in \Omega} |V(x)|) \leq 2n(1 + \sup_{x \in \Omega} |V(x)|) < \infty \).

Let \( C_3 = C_0 NC_1 \) and \( C_* = NC_2 R \ln C_0 \). Then for each \( i = 1, 2, \cdots, I \),

\[
\sup_{D_i} u_\alpha(x) \leq C_3 e^{\frac{C_2}{\lambda_\alpha(\Omega)}} \inf_{D_i} u_\alpha(x) \leq C_3 e^{\frac{C_2}{\lambda_\alpha(\Omega)}} \inf_{D_i \cap D^*} u_\alpha(x)
\]

\[
\leq \frac{C_3}{|D_i \cap D^*|} e^{\frac{C_2}{\lambda_\alpha(\Omega)}} \mu_\alpha(D_i \cap D^*) \leq \frac{C_3}{|D_i \cap D^*|} e^{\frac{C_2}{\lambda_\alpha(\Omega)}} \mu_\alpha(D^*).
\]

By Lemma 2.1 there is a positive constant \( C^* \) depending only on \( U_0 \) and \( D_* \) such that

\[
\mu_\alpha(D^*) \leq e^{-\frac{C^*}{\lambda_\alpha(D^*)}},
\]
as \(|A_\alpha| \ll 1\).

It now follows from (4.3) and (4.4) that
\[
\mu_\alpha(D) \leq \sum_{i=1}^{I} |D_i| \sup_{D_i} u_\alpha(x) \leq \sum_{i=1}^{I} \frac{|D_i|C_3}{|D_i \cap D^*|} e^{-\frac{\gamma_\alpha(D_i \cap D^*)}{\lambda_\alpha(D_i \cap D^*)}}.
\]

Therefore, if we further restrict \(\{A_\alpha\}\) to be such that
\[(4.5)\]
\[
\frac{\lambda_\alpha(\Omega)}{\Lambda_\alpha(D_*)} > \frac{C_s + 1}{C^*}
\]
for all \(\alpha\), then \(\mu_\alpha(D) \to 0\) as \(A_\alpha \to 0\). Since \(D = \overline{\Omega}_0 \setminus \Omega^0_{\tilde{\rho}^*}\) when \(E_0\) is a local attractor and \(D = \overline{\Omega}^0_{\tilde{\rho}^*}\) when \(E_0\) is a local repeller, it follows that
\[(4.6)\]
\[
\begin{cases}
\mu_\alpha(\Omega_0 \setminus \overline{\Omega}^0_{\tilde{\rho}^*}) \to 0, & \text{when } E_0 \text{ is a local attractor,} \\
\mu_\alpha(\Omega^0_{\tilde{\rho}^*}) \to 0, & \text{when } E_0 \text{ is a local repeller,}
\end{cases}
\]
as \(A_\alpha \to 0\). As \(\Omega\) and \(D_*\) are separated by a positive distance, it is easy to construct a null families, still denoted by \(A\), such that both (4.2) and (4.5) hold simultaneously.

Let \(\mu \in \mathcal{M}_A\). Then it follows from Proposition 2.6 and (4.6) that \(\mu\) is supported on \(\overline{\Omega}_0 \setminus \overline{\Omega}^0_{\tilde{\rho}^*}\) in the case a) and on \(\Omega_0 \setminus \overline{\Omega}^0_{\tilde{\rho}^*}\) in the case b), both of which are positively invariant set of \(\varphi^t\). It follows from Proposition 6.5 that \(\mu\) is actually supported on \(\omega(\overline{\Omega}_0 \setminus \overline{\Omega}^0_{\tilde{\rho}^*})\) which equals \(J_0\) in the case a) and on \(\omega(\Omega_0 \setminus \overline{\Omega}^0_{\tilde{\rho}^*})\) which equals \(J_*\) in the case b). By Proposition 3.4, both \(J_0\) and \(J_*\) are \(A\)-stable. \(\square\)

The above theorem naturally applies to families of the form \(\{\epsilon A\} \subset A\), where \(\epsilon > 0\) is a small parameter and \(A\) is a bounded, everywhere positive definite, \(n \times n\) matrix-valued \(C^1\) function on \(U\).

**Corollary 4.1.** Assume \(V \in C^1(U, \mathbb{R}^n)\) and that (1.1) admits a \(C^2\) Lyapunov function with bounded second derivatives on \(U\). Then there are bounded, everywhere positive definite, \(n \times n\) matrix-valued \(C^1\) functions \(A_i, i = 1, 2\), on \(U\), such that a), b) of Theorem 4.1 hold respectively for the family \(\{\epsilon A_i\}, i = 1, 2\), as \(\epsilon \to 0\).

**Proof.** For any bounded, everywhere positive definite, \(n \times n\) matrix-valued \(C^1\) function \(A\) on \(U\), \(\{\epsilon A\}\) is clearly a normal null family. In the case that \(U \neq \mathbb{R}^n\), the invariance of \(\{\epsilon A\}\) is guaranteed if \(A(x) \to 0\) as \(x \to \partial U\). The corollary now follows from Theorem 4.1.

In fact, by the proof of Theorem 4.1, we just need to choose \(A = (a^{ij})\) such that
\[
\frac{\inf_{\Omega} \lambda_A(x)}{\max_{D_*} \Lambda_A(x)} > \frac{C_s + 1}{C^*},
\]
where \(\Omega, D_* , C_s, C^*\) are as in the proof of Theorem 4.1, \(\Lambda_A(x) = \sqrt{\sum_{i,j} |a^{ij}(x)|^2}\), and \(\lambda_A(x)\) is the smallest eigenvalue of \(A(x)\) for each \(x \in U\). \(\square\)
4.2. Uniform de-stabilization of a locally repelling equilibrium. We note that if \( R_0 \) is a strong local repeller, then in general not every normal null family \( A \subset \hat{A} \) can de-stabilize \( R_0 \). This is because \( R_0 \) may be further decomposed to contain a strong local attractor for which Theorem 4.1a is applicable.

We now show that if \( R_0 \) is a strongly repelling equilibrium (see Definition 6.4), then it is de-stabilized by any normal null family \( A \subset \hat{A} \). This particularly implies part c) of Theorem C.

**Theorem 4.2.** Assume \( V \in C^1(U, \mathbb{R}^n) \) and that (1.1) admits a \( C^2 \) Lyapunov function with bounded second derivatives on \( U \). Let \( x_0 \) be a strongly repelling equilibrium of \( \varphi^t \) contained in the global attractor \( J \) of \( \varphi^t \) and \( W \) be an isolating neighborhood of \( x_0 \). Then with respect to any normal null family \( \mathcal{A} = \{ A_\alpha \} \subset \hat{A}, \mathcal{M}_A \neq \emptyset \), and moreover, each \( \mu \in \mathcal{M}_A \) is supported on the maximal attractor \( J_\alpha \) of \( \varphi^t \) in \( J \setminus W \). Consequently, with respect to any invariant, normal null family \( \mathcal{A}, J_\alpha \) is \( A \)-stable and \( \{ x_0 \} \) is strongly \( A \)-unstable.

**Proof.** Let \( \mathcal{A} = \{ A_\alpha \} = \{(a_{ij}^\alpha)\} \subset \hat{A} \) be any bounded null family such that \( |A_\alpha| \ll 1 \). By Proposition 2.5, \( \{ \mathcal{L}_{A_\alpha} \} \) admits a uniform Lyapunov function in \( U \). By Proposition 2.1 for each \( \alpha \), there exists a stationary measure \( \mu_\alpha \) corresponding to \( \{ \mathcal{L}_{A_\alpha} \} \) which is also regular, i.e., \( d\mu_\alpha(x) = u_\alpha(x)dx \) for some stationary solution \( u_\alpha \in W^{1, P}(U) \) corresponding to \( \mathcal{L}_{A_\alpha} \) in \( U \).

Without loss of generality, we assume \( x_0 = 0 \). In virtue of Definition 6.1, let \( U \) be an entire weak anti-Lyapunov function associated with \( W \) that satisfies (6.3)–(6.5). We note that since \( W \subset J \), it is a pre-compact set in \( U \). For each \( \rho \in [0, \rho_M] \), where \( \rho_M \) is the essential upper bound of \( U \), we denote \( \Omega_\rho \) as the \( \rho \)-sublevel set of \( U \). We note by (6.5) that \( \partial \Omega_\rho = U^{-1}(\rho) \) and hence \( \Omega_\rho = \{ x \in W : U(x) \leq \rho \} \) for any \( \rho \in (0, \rho_M) \).

Fix \( \bar{\rho} \in (0, \rho_M) \). Let \( \lambda_0^\rho := \inf_{x \in \Omega_\rho} \lambda_0(x), \lambda_0^\rho := \inf_{x \in \Omega_\rho} \lambda_0(x), \lambda_0(x), \lambda_0(x) \) denote the smallest eigenvalues of \( A_\alpha(x), D^2U(x) \), respectively. Then for each parameter \( \alpha \),

\[
\mathcal{L}_{A_\alpha}U(x) = a_{ij}^\alpha(x)\partial_{ij}U(x) + V(x) \cdot \nabla U(x)
\]

\[
\geq a_{ij}^\alpha(x)\partial_{ij}U(x) = \text{tr} \left( A_\alpha(x)D^2U(x) \right) \geq \lambda_0^\rho \lambda_0 =: \gamma_0^\rho, \quad x \in \Omega_\rho,
\]

i.e., \( U \) is an anti-Lyapunov function with respect to \( \{ \mathcal{L}_{A_\alpha} \} \) on \( \Omega_{\hat{\rho}} \) with the essential lower bound \( \rho_m = 0 \).

By (6.3), (6.4) and the Taylor’s expansion of \( U \) at the point 0, it is easy to see that there is a constant \( C_2 > 0 \) and \( \rho_* < \hat{\rho} \) such that

\[
|\nabla U(x)|^2 \leq C_2 U(x), \quad x \in \Omega_{\hat{\rho}}.
\]

Since \( U \) is of the class \( C^1 \) and \( U(x) > 0 \) on the compact set \( \Omega_{\bar{\rho}} \setminus \Omega_{\rho_*} \), we can make \( C_2 \) larger if necessary so that

\[
|\nabla U(x)|^2 \leq C_2 U(x), \quad x \in \Omega_{\bar{\rho}} \setminus \Omega_{\rho_*},
\]

i.e., there exists a constant \( C_2 = C_2(\bar{\rho}) > 0 \) such that

\[
|\nabla U(x)|^2 \leq C_2 U(x), \quad x \in \Omega_{\bar{\rho}}.
\]

It follows that

\[
a_{ij}^\alpha(x)\partial_iU(x)\partial_jU(x) \leq C_2 A_\alpha \rho =: H^\alpha(\rho), \quad x \in \Omega_{\rho}, \rho \in [0, \bar{\rho}],
\]
where \( \Lambda_\alpha = \Lambda_\alpha(\tilde{\rho}) := \sup_{x \in \Omega_\rho} |A_\alpha(x)| \). We note that, since each \( \mu_\alpha \) is a regular stationary measure and \( \Omega_{\rho_0} = \{0\} \), \( \mu_\alpha(\Omega_{\rho_0}) = 0 \). Applying Proposition 2.3 with respect to the stationary measure

\[
\bar{\mu}_\alpha = \frac{\mu_\alpha(\Omega_{\rho})}{\mu_\alpha(\Omega_{\tilde{\rho}})}
\]

corresponding to \( \mathcal{L}_{A_\alpha} \) in \( \mathcal{W} \), we have

\[
\mu_\alpha(\Omega_{\rho_0}) = \mu_\alpha(\Omega_{\rho_0} \setminus \Omega_{\rho_0}) \leq e^{-C_f \int_{\rho_0}^{\tilde{\rho}} \overline{J} \, d\rho}, \quad \rho_0 \in (0, \tilde{\rho}),
\]

where \( C = C(\tilde{\rho}) = \frac{\lambda_{\rho_0}^{\overline{J}}}{C_2(\rho)} \inf_{0} \frac{\lambda_0^{\overline{J}}}{\Lambda_\alpha(\rho)} \), which is a positive constant since \( \mathcal{A} = \{A_\alpha\} \) is a normal null family.

It follows from Theorem 3.2 that \( \mathcal{M}_A \neq \emptyset \), and that any \( \mathcal{A} \)-limit measure is an invariant measure of \( \varphi^t \) supported on the global attractor \( J \). Let \( \mu \in \mathcal{M}_A \). Since \( \Omega_{\rho_0} \) is open, we have by applying Proposition 2.6 to (4.7) and taking limit \( A_\alpha \to 0 \) that

\[
\mu(\Omega_{\rho_0}) \leq e^{-C_f \int_{\rho_0}^{\tilde{\rho}} \overline{J} \, d\rho}, \quad \rho_0 \in (0, \tilde{\rho}).
\]

By letting \( \rho_0 \to 0 \) in the above, we have

\[
\mu(\{0\}) = 0.
\]

For any given \( \rho \in (0, \tilde{\rho}) \), we have by (6.5) and Proposition 2.5 that \( \{L_{A_\alpha}\} \) admits a uniform anti-Lyapunov function in \( \Omega_\rho \). It also follows from (6.5) that \( \{0\} \) is the maximal repeller of \( \varphi^t \) in \( \Omega_\rho \). Applying Theorem 3.3 b) and (4.8), we have

\[
\mu(\Omega_{\rho}) = \mu(\Omega_{\rho} \setminus \{0\}) = 0.
\]

Since \( \rho \) is arbitrary, \( \mu(\Omega_{\rho}) = 0 \). Since \( \tilde{\rho} \in (0, \rho_{\mathcal{U}}) \) is also arbitrary, \( \mu(\mathcal{W}) = 0 \).

Since \( \text{supp}(\mu) \subset J \) and \( J \setminus \mathcal{W} \) is a positively invariant set of \( \varphi^t \), it follows from Proposition 6.5 that \( \mu \) is actually supported on the \( \omega \)-limit set \( J_\omega = \omega(J \setminus \mathcal{W}) \). The \( \mathcal{A} \)-stability of \( J_\omega \) follows from Proposition 3.4 Remark 2.3 1), and Lemma 3.1 a).

**Corollary 4.2.** Assume \( V \in C^1(\mathcal{U}, \mathbb{R}^n) \) and that (1.1) admits a \( C^2 \) Lyapunov function with bounded second derivatives on \( \mathcal{U} \). Let \( x_0 \in \mathcal{U} \) be an equilibrium of \( \varphi^t \) such that all eigenvalues of \( DV(x_0) \) have positive real parts. Then \( x_0 \) is a strongly repelling equilibrium, and consequently the conclusion of Theorem 4.2 holds for \( x_0 \).

**Proof.** By standard theory of dissipative dynamical systems [3,1] Sections 17 and 18], \( x_0 \) is a strongly repelling equilibrium. In fact, there is a positive definite \( n \times n \) matrix \( B \) such that

\[
(DV(x_0))^\top B + B(DV(x_0)) = I.
\]

Let \( U(x) = (x - x_0)^\top B(x - x_0) \). Then \( U \) is a desired entire weak anti-Lyapunov function satisfying (6.3) in the vicinity of \( x_0 \). \( \square \)

**Corollary 4.3.** Assume \( V \in C^1(\mathcal{U}, \mathbb{R}^n) \) and that (1.1) admits a \( C^2 \) Lyapunov function with bounded second derivatives on \( \mathcal{U} \). Then Theorem 4.2 and Corollary 4.2 hold for any family of the form \( \{\epsilon A\} \) when \( \epsilon \to 0 \), where \( A \in \mathcal{A} \) is such that \( \sup_{x \in \mathcal{U}} |A(x)| < \infty \) and \( A(x) \to 0 \) as \( x \to \partial \mathcal{U} \) when \( \mathcal{U} \neq \mathbb{R}^n \).

**Proof.** Theorem 4.2 and Corollary 4.2 clearly hold because \( \{\epsilon A\} \) is an invariant normal null family. \( \square \)
5. An Example of Stochastically Stable Limit Cycles

We demonstrate by an example the application of our results in obtaining stochastically stable limit cycles, in connection with stochastic Hopf bifurcations.

Consider stochastic differential equations

\begin{align}
 dx &= (bx - y - x(x^2 + y^2))dt + \sqrt{\epsilon}g^{11}(x, y)dW_1 + \sqrt{\epsilon}g^{12}(x, y)dW_2, \\
 dy &= (x + by - y(x^2 + y^2))dt + \sqrt{\epsilon}g^{21}(x, y)dW_1 + \sqrt{\epsilon}g^{22}(x, y)dW_2,
\end{align}

(5.1)

where \( b, \epsilon \) are parameters, \( g^{ij} \in W^{1,2}_\text{loc}(\mathbb{R}^2), \ i, j = 1, 2, \) for some \( p > 2, \) and \( G(x, y) = (g^{ij}(x, y)) \) is everywhere non-singular and bounded on \( \mathbb{R}^2. \) Denote \( V_b(x, y) = (bx - y - x(x^2 + y^2), x + by - y(x^2 + y^2))^\top \) and \( A(x, y) = \frac{G(x, y)G^\top(x, y)}{2}. \) Also denote the adjoint Fokker-Planck operators associated with (5.1) by \( \mathcal{L}_{V_b,\epsilon A}. \)

Let \( U(x, y) = x^2 + y^2. \) Then it is clear that

\begin{align}
 V_b \cdot \nabla U &= 2U(b - U)
\end{align}

(5.2)

which approaches to \(-\infty\) as \( x^2 + y^2 \to \infty. \) It follows that, for each \( b, \) the unperturbed system of (5.1) is dissipative; in particular, it generates a (positive) semiflow \( \varphi_b^\epsilon \) on \( \mathbb{R}^2 \) which admits a global attractor \( J_b. \) Moreover, since \( A(x, y) \) is bounded, Proposition 2.5 implies that \( U \) is in fact a uniform Lyapunov function in \( \mathbb{R}^2 \) with respect to the family \( \{\mathcal{L}_{V_b,\epsilon A}\} \) when \( 0 < \epsilon \ll 1. \) We have by Proposition 2.1 that each \( \mathcal{L}_{V_b,\epsilon A} \) admits a unique stationary measure \( \mu_{\epsilon x A}, \) by Lemma 3.1 a) that the set of limit measures of \( \{\mu_{\epsilon x A}\} \) as \( \epsilon \to 0 \) is non-empty, and by Theorem 3.2 that each limit measure is an invariant measure of \( \varphi_b^\epsilon \) supported on \( J_b. \)

When \( b = 0, J_b = \{0\}. \) Hence \( \mu_{\epsilon x A} \to \delta_0 \) as \( \epsilon \to 0. \) Consequently, \( \{0\} \) and \( \delta_0 \) are \( \{\epsilon\}-\)stable.

When \( b > 0, J_b = \Omega_b - \) the closed disk of radius \( \sqrt{b} \) centered at the origin. We note that \( U \) is also an entire weak anti-Lyapunov function in \( \Omega_b \) satisfying (6.3)-(6.5) for \( x_0 = 0. \) An application of Corollary 4.11 in the open disk \( \Omega_b \) yields that each limit measure of \( \{\mu_{\epsilon x A}\} \) is supported on \( \partial\Omega_b =: C_b \) - the circle with radius \( \sqrt{b} \) as \( \epsilon \to 0. \) Since \( C_b \) is a periodic invariant cycle of \( \varphi_b^\epsilon, \) it is uniquely ergodic with the Haar measure \( \mu_b \) as the only invariant measure. Therefore, \( \mu_{\epsilon x A} \to \mu_b \) as \( \epsilon \to 0. \) Consequently, \( C_b \) and \( \mu_b \) become \( \{\epsilon\}-\)stable in this case.

This example thus demonstrates an interesting phenomenon of stochastic Hopf bifurcation in which the repelling equilibrium \( \{0\} \) is invisible when \( b > 0 \) under noise perturbations.

6. Appendix

In this section, we will summarize some general notions and fundamental properties of dynamics of ordinary differential equations including dissipation and invariant measures. Some materials concerning dissipation are taken from the Appendix of [27] for a general domain \( U \subset \mathbb{R}^n. \)

Consider (1.1) for a continuous \( V. \) Throughout the section, we assume that solutions of (1.1) are locally unique (e.g., \( V \) is locally Lipschitz continuous). Then (1.1) generates a (continuous) local flow \( \varphi^t : U \to U, \) where for each \( \xi \in U, \) \( \varphi^t(\xi) \) is the solution of (1.1) for \( t \) lying in the maximal interval \( \mathcal{I}_\xi \) of existence in \( U. \) A set \( B \subset U \) is called an invariant set (resp. positively invariant set, resp. negatively invariant set) of \( \varphi^t \) if \( \mathcal{I}_\xi = \mathbb{R} \) (resp. \( \mathcal{I}_\xi \supset \mathbb{R}^+, \) resp. \( \mathcal{I}_\xi \supset \mathbb{R}^- \) for all \( \xi \in B, \) and \( \varphi^t(B) = B \) for all \( t \in \mathbb{R} \) (resp. \( \varphi^t(B) \subset B \) for all \( t \in \mathbb{R}^+, \) resp. \( \varphi^t(B) \supset B \) for all \( t \in \mathbb{R}^- \)).
resp. $\varphi^t(B) \subset B$ for all $t \in \mathbb{R}_-$. If $\mathcal{U}$ itself is an invariant set (resp. positively invariant set, resp. negatively invariant set) of $\varphi^t$, $\varphi^t$ is called a flow (resp. positive semiflow, resp. negative semiflow). We note that if $\varphi^t$ is a flow, then for each $t \in \mathbb{R}$, $\varphi^t : \mathcal{U} \to \mathcal{U}$ is a homeomorphism.

6.1. **Dissipation v.s. anti-dissipation.** For any set $B \subset \mathcal{U}$ such that $\cup_{t \in \mathbb{R}_+} \varphi^t(B)$ (resp. $\cup_{t \in \mathbb{R}_-} \varphi^t(B)$) exists and is pre-compact in $\mathcal{U}$, we recall that the $\omega$-limit set of $B$ (resp. $\alpha$-limit set of $B$) is defined as

$$
\omega(B) = \cap_{t \geq 0} \cup_{t \geq t} \varphi^t(B) \quad \text{(resp. } \alpha(B) = \cap_{t \leq 0} \cup_{t \leq t} \varphi^t(B)).
$$

**Definition 6.1.** Let $\Omega \subset \mathcal{U}$ be a connected open set.

1) $\varphi^t$ is said to be dissipative (resp. anti-dissipative) in $\Omega$ if $\Omega$ is a positively (resp. negatively) invariant set of $\varphi^t$ and there exists a compact subset $B$ of $\Omega$ with the property that for any $\xi \in \Omega$ there exists a $t_0(\xi) > 0$ such that $\varphi^t(\xi) \in B$ as $t \geq t_0(\xi)$ (resp. $t \leq -t_0(\xi)$).

2) When $\Omega$ is a positively (resp. negatively) invariant set of $\varphi^t$, the maximal attractor $J$ (resp. maximal repeller $R$) of $\varphi^t$ in $\Omega$ is a compact invariant set of $\varphi^t$ which attracts (resp. repels) any pre-compact subset $K$ of $\Omega$, i.e., $\omega(K) \subset J$ (resp. $\alpha(K) \subset R$), or equivalently, $\lim_{t \to +\infty} \text{dist}(\varphi^t(K), J) = 0$ (resp. $\lim_{t \to -\infty} \text{dist}(\varphi^t(K), R) = 0$), where for any two subsets $A, B$ of $\mathbb{R}^n$, $\text{dist}(A, B)$ denotes the Hausdorff semi-distance from $A$ to $B$.

3) Suppose that the maximal attractor $J$ (resp. maximal repeller $R$) of $\varphi^t$ exists in $\Omega$. If $\Omega = \mathcal{U}$, then we call $J$ (resp. $R$) the global attractor (resp. global repeller) of $\varphi^t$. Otherwise, it is called a local attractor (resp. local repeller) of $\varphi^t$.

It is clear that maximal attractor or repeller of $\varphi^t$ in $\Omega$, if exists, must be unique. Consequently, global attractor or repeller of $\varphi^t$ is unique.

**Proposition 6.1.** Let $\Omega \subset \mathcal{U}$ be a connected open set.

1) $\varphi^t$ is dissipative (resp. anti-dissipative) in $\Omega$ if and only if $\Omega$ is a positively (resp. negatively) invariant set and admits a maximal attractor (resp. repeller) of $\varphi^t$.

2) If $\varphi^t$ is dissipative (resp. anti-dissipative) in $\Omega$, then the maximal attractor $J$ (resp. maximal repeller $R$) of $\varphi^t$ in $\Omega$ equals

$$
J = \bigcup_{B \subset \Omega \text{ pre-compact}} \omega(B) \quad \text{(resp. } R = \bigcup_{B \subset \Omega \text{ pre-compact}} \alpha(B)).
$$

**Proof.** See Proposition 6.2 in [27].

**Definition 6.2.** Let $\Omega \subset \mathcal{U}$ be a connected open set and $U \subset C^1(\Omega)$ be a compact function with essential upper bound $\rho_M$. For each $\rho \in [0, \rho_M)$, denote $\Omega_{\rho}$ as the $\rho$-sublevel set of $U$.

1) $U$ is a weak Lyapunov function (resp. weak anti-Lyapunov function) of (1.1) in $\Omega$ if

$$
(6.1) \quad V(x) \cdot \nabla U(x) \leq 0 \quad \text{(resp. } \geq 0), \quad x \in \Omega \setminus \tilde{\Omega}_{\rho_m},
$$

where $\rho_m \in (0, \rho_M)$ is a constant, called essential lower bound of $U$, and $\tilde{\Omega}$ is called essential domain of $U$ in $\Omega$.

If (6.1) holds for all $x \in \Omega$ instead of $\tilde{\Omega}$, then $U$ is called an entire weak Lyapunov function (resp. entire weak anti-Lyapunov function) of (1.1) in $\Omega$. 


2) $U$ is a Lyapunov function (resp. anti-Lyapunov function) of (1.1) in $\Omega$ if there exists a constant $\gamma > 0$, called a Lyapunov constant of $U$, such that

$$V(x) \cdot \nabla U(x) \leq -\gamma \quad (\text{resp.} \geq \gamma), \quad x \in \bar{\Omega} = \Omega \setminus \bar{\Omega}_{\rho_m},$$

where $\rho_m \in (0, \rho_M)$ is a constant, called essential lower bound of $U$, and $\bar{\Omega}$ is called essential domain of $U$ in $\Omega$.

**Proposition 6.2.** Let $\Omega \subset U$ be a connected open set.

a) If (1.1) admits a weak Lyapunov (resp. anti-Lyapunov) function in $\Omega$, then $\Omega$ must be a positively (resp. negatively) invariant set of $\varphi^t$.

b) If (1.1) admits a Lyapunov (resp. anti-Lyapunov) function in $\Omega$, then it must be dissipative (resp. anti-dissipative) in $\Omega$, with the maximal attractor (resp. repeller) in $\Omega$ being $\omega(\Omega_{\rho_m})$ (resp. $\alpha(\Omega_{\rho_m})$), where $\rho_m$ is the essential lower bound and $\Omega_{\rho_m}$ is the $\rho_m$-sublevel set of the Lyapunov (resp. anti-Lyapunov) function.

**Proof.** See Proposition 6.3 in [27].

The following is a strong version of the well-known LaSalle invariance principle ([33, Chapter 2, Theorem 6.4]) for locating $\omega$-limit sets in a domain which admits an entire weak Lyapunov function.

**Proposition 6.3.** Let $\Omega \subset U$ be a connected open set. If (1.1) admits an entire weak Lyapunov function $U$ in $\Omega$, then for any $x_0 \in \Omega$,

$$\omega(x_0) \subset S,$

where $\omega(x_0)$ is the $\omega$-limit set of $x_0$.

We note that the original LaSalle invariance principle does not require $U$ to be a compact function (in this case, the conclusion of Proposition 6.3 only holds for points which have bounded positive orbits).

**Definition 6.3.** A compact invariant set $J \subset U$ (resp. $R \subset U$) of $\varphi^t$ is called a strong local attractor (resp. strong local repeller) of $\varphi^t$ if there is a connected, positively (resp. negatively) invariant neighborhood $W$ of $J$ (resp. $R$) in $U$, called an isolating neighborhood, with oriented $C^2$ boundary such that

1) $J$ attracts $W$ (resp. $R$ repels $W$);

2) $V(x) \cdot \nu(x) < 0$ (resp. $V(x) \cdot \nu(x) > 0$), $x \in \partial W$, where $\nu(x)$ denotes the outward unit normal vector of $\partial W$ at $x \in \partial W$.

**Remark 6.1.** The neighborhood $W$ in Definition 6.3 is indeed an isolating neighborhood in the usual sense, i.e., any entire orbits of $\varphi^t$ lying in $W$ must lie in $J$ (resp. $R$).

**Proposition 6.4.** A local attractor (resp. repeller) of $\varphi^t$ is a strong local attractor (resp. repeller $R$) if and only if it has an isolating neighborhood on which (1.1) admits a Lyapunov (resp. anti-Lyapunov) function.
Proof. We only treat the case of a local attractor $\mathcal{J}$ of $\varphi^t$.

To show the necessity, we parametrize the boundary $\partial \mathcal{W}$ of an isolating neighborhood $\mathcal{W}$ in Definition 6.3 by a $C^2$ function $U$, i.e., $\partial \mathcal{W} = \{ x \in \mathcal{U} : U(x) = \text{constant} \}$. Let $\gamma = \min_{x \in \partial \mathcal{W}} |V(x) \cdot \nu(x)|$. Then $\gamma > 0$. Since $\nu(x) = \nabla U(x)/|\nabla U(x)|$,

$$V(x) \cdot \nabla U(x) \leq -\gamma |\nabla U(x)| \leq -\gamma \min_{x \in \partial \mathcal{W}} |\nabla U(x)| =: -\tilde{\gamma}$$

for all $x \in \partial \mathcal{W}$. It follows that there is a neighborhood $\tilde{\mathcal{W}}$ of $\partial \mathcal{W}$ in $\mathcal{W}$ such that

$$V(x) \cdot \nabla U(x) < -\frac{\tilde{\gamma}}{2}, \quad x \in \tilde{\mathcal{W}},$$

i.e., $U$ becomes a Lyapunov function of (1.1) in $\mathcal{W}$.

To prove the sufficiency, we let $\mathcal{W}$ be an isolating neighborhood of $\mathcal{J}$ on which a Lyapunov function $U$ of (1.1) exists. Using $C^2$ approximation, we may assume without loss of generality that $U$ is of the class $C^2$ in a connected open set $\tilde{\mathcal{W}} \supset \mathcal{W}$ on which $U$ remains as a Lyapunov function of $\varphi^t$. Then it is clear that Definition 6.3 (2) is satisfied on $\partial \mathcal{W}$. $\square$

Definition 6.4. An equilibrium $x_0$ of (1.1) is called a (local) strongly repelling equilibrium if there is a connected open set $\mathcal{W}$ containing $x_0$, called an isolating neighborhood of $x_0$, and a compact function $U \in C^2(\mathcal{W})$ such that

(6.3) the Hessian matrix $D^2U$ is everywhere positive definite in $\mathcal{W},$

(6.4) $U(x_0) = 0, \quad \nabla U(x_0) = 0; \quad U(x) > 0, \quad x \in \mathcal{W} \setminus \{x_0\},$ and

(6.5) $V(x) \cdot \nabla U(x) > 0, \quad x \in \mathcal{W} \setminus \{x_0\}.$

Remark 6.2. 1) We note that a strongly repelling equilibrium $x_0$ of (1.1) is necessarily a strong local repeller.

2) By a time reversing application of Converse Lyapunov Theorem (see e.g., [36, Theorem 4.2.1], [7, Theorem V.2.12]), if $x_0$ is a (local) repelling equilibrium in the usual sense, then there is a Lipschitz continuous function $U$ in a neighborhood of $x_0$ which is an almost everywhere entire weak anti-Lyapunov function and satisfies (6.4), (6.5) almost everywhere.

6.2. Invariant measures. For a Borel set $\Omega \subset \mathbb{R}^n$, we denote by $M(\Omega)$ the set of Borel probability measures on $\Omega$ furnished with the weak$^*$-topology.

For an invariant (resp. positively invariant, negatively invariant) Borel set $\Omega \subset \mathcal{U}$ of $\varphi^t$, $\mu \in M(\Omega)$ is said to be an invariant measure (resp. positively invariant measure, negatively invariant measure) of $\varphi^t$ on $\Omega$ if for any Borel set $B \subset \Omega$,

$$\mu(\varphi^{-t}(B)) = \mu(B), \quad t \in \mathbb{R} \quad (\text{resp. } t \in \mathbb{R}^+, \quad t \in \mathbb{R}^-),$$

where $\varphi^{-t}(B) = \{ x \in \Omega : \varphi^t(x) \in B \}$.

We note that if $\Omega$ is an invariant Borel set, then it follows from the flow property that any positively or negatively invariant measure on $\Omega$ must be invariant.

An invariant measure $\mu$ is an ergodic measure if any invariant Borel set has either full $\mu$-measure or zero $\mu$-measure. A compact invariant set admits at least one ergodic measure. It is said to be uniquely ergodic if it admits only one invariant measure.
Proposition 6.5. Let $\Omega \subset U$ be a positively (resp. negatively) invariant Borel set in $U$ which admits a positively (resp. negatively) invariant measure $\mu$ of $\varphi^t$. Then the following holds.

1) If $\Omega$ is pre-compact in $U$, then $\mu$ is an invariant measure of $\varphi^t$ supported on $\omega(\Omega)$ (resp. $\alpha(\Omega)$).

2) If $\Omega$ is a connected open subset of $U$ and $\varphi^t$ is dissipative (resp. anti-dissipative) in $\Omega$, then $\mu$ is supported on the maximal attractor (resp. repeller) of $\varphi^t$ in $\Omega$.

Proof. 1) can be easily seen from the definitions of invariant measures, positively (resp. negatively) invariant measures, $\omega$-limit (resp. $\alpha$-limit) sets.

To prove 2), we first consider the case that $\varphi^t$ is dissipative in $\Omega$. Let $J$ denote the maximal attractor of $\varphi^t$ in $\Omega$. Since $J$ is a compact invariant set, it is sufficient to show that $\mu(J) = 1$.

For any given $\epsilon > 0$, we have by the Borel regularity of $\mu$ that there exist a compact subset $B$ of $\Omega$ and an open neighborhood $W$ of $J$ lying in $\Omega$ such that $\mu(B) > 1 - \epsilon$ and $\mu(W \setminus J) < \epsilon$. Since $J$ attracts $B$, there exists $t > 0$ such that $\varphi^t(B) \subseteq W$. Thus,

$$\mu(J) \geq \mu(W) - \epsilon \geq \mu(\varphi^t(B)) - \epsilon = \mu((\varphi^t)^{-1}(\varphi^t(B))) - \epsilon \geq \mu(B) - \epsilon \geq 1 - 2\epsilon.$$ 

Since $\epsilon$ is arbitrary, $\mu(J) = 1$.

The case when $\varphi^t$ is anti-dissipative in $\Omega$ is similar. \hfill $\square$

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