STATISTICAL STUDY ON THE NUMBER OF INJECTIVE LINEAR FINITE TRANSUDCERS

Ivone Amorim\(^{(A)}\)  António Machiavelo\(^{(B)}\)  Rogério Reis\(^{(A)}\)

CMUP, Faculdade de Ciências da Universidade do Porto, Portugal

\(^{(A)}\) \{ivone.amorim,rvr\}@dcc.fc.up.pt
\(^{(B)}\) ajmachia@fc.up.pt

Abstract
The notion of linear finite transducer (LFT) plays a crucial role in some cryptographic systems. In this paper we present a way to get an approximate value, by random sampling, for the number of non-equivalent injective LFTs. By introducing a recurrence relation to count canonical LFTs, we show how to estimate the percentage of \(\tau\)-injective LFTs. Several experimental results are presented, which by themselves constitute an important step towards the evaluation of the key space of those systems.

1. Introduction

In this work we present a statistical study on the number of non-equivalent linear finite transducers that are injective with some delay. This study is motivated by the application of these transducers in Cryptography. A transducer, in this context, is a finite state sequential machine given by a quintuple \(\langle X, Y, S, \delta, \lambda \rangle\), where: \(X, Y\) are the nonempty input and output alphabets, respectively; \(S\) is the nonempty finite set of states; \(\delta : S \times X \to S\), \(\lambda : S \times X \to Y\), are the state transition and output functions, respectively. These transducers are deterministic and can be seen as having all the states as final. Every state in \(S\) can be used as initial state, and this gives rise to a transducer in the usual sense, i.e., one that realises a rational function. Therefore, in what follows, a transducer is a family of classical transducers that share the same underlying digraph.

A finite transducer is called linear if its transition and output functions are linear maps. Linear finite transducers play a core role in a family of cryptosystems, named FAPKCs, introduced in a series of papers by Tao \cite{11, 12, 13}. Those schemes seem to be a good alternative to the classical ones, being computationally attractive and thus suitable for application on devices with very limited computational resources, such as satellites, cellular phones, sensor networks, and smart cards \cite{12}.
Roughly speaking, in these systems, the private key consists of two injective transducers, denoted by $M$ and $N$ in Figure 1, where $M$ is a linear finite transducer (LFT), and $N$ is a non-linear finite transducer (non-LFT) of a special kind, whose left inverses can be easily computed. The public key is the result of applying a special product, $C$, for transducers to the original pair, obtaining a non-LFT, denoted by $C(M, N)$ in Figure 1. The crucial point is that it is easy to obtain an inverse of $C(M, N)$ from the inverses of its factors, $M^{-1}$ and $N^{-1}$, while it is believed to be hard to find that inverse without knowing those factors. On the other hand, the factorization of a transducer seems to be a hard problem by itself [16].

LFTs are fundamental in the FAPKC systems because their invertibility theory is of core importance in the security of these systems. They also play a crucial role in the key generation process, since in these systems a pair (public key, private key) is formed using one injective LFT and two injective non-LFTs, as explained above. Consequently, for these cryptosystems to be feasible, injective LFTs have to be easy to generate, and the set of non-equivalent injective LFTs has to be large enough to make an exhaustive search intractable.

Several studies were made on the invertibility of LFTs [8, 9, 15, 16, 6, 2], and some attacks to the FAPKC systems were presented [3, 15, 10]. However, as far as we know, no study was conducted to determine the size of the key space of these systems.

Amorim et al [1] introduced a notion of canonical LFT and proved that each equivalence class has exactly one canonical LFT. Using this and a way to test if two LFTs are equivalent, they proved a result that allows to compute the size of the equivalence class of a given LFT. Two necessary and sufficient conditions for a LFT to be injective with some delay $\tau$ are also well known [2, Theorem 3.4]. In this paper we use these results to estimate the number of non-equivalent LFTs that are injective with some delay. The obtained estimate can be used to compute the size of the key spaces of the mentioned cryptographic systems. We also give a recurrence relation to count the number of canonical LFTs, and get an approximated value for the percentage of equivalence classes formed by injective LFTs. Knowing this percentage is crucial to conclude if random generation of LFTs is a feasible option to generate keys. Several algorithms and experimental results are also presented. All the algorithms were implemented in Python using some Sage [7] modules, to deal with matrices.

The paper is organized as follows. In Section 2 we introduce the basic definitions and some preliminary results. In Section 3 we start by presenting two algorithms, one to test if a LFT is injective with some delay $\tau$, and the other to determine the equivalence class size of a given
Let $M$ be a nonempty finite set called the set of states and $\lambda$ the output function. The formal definition of a finite transducer (FT) is the following.

As usual, for a finite set $A$, we let $|A|$ denote the cardinality of $A$, $A^n$ be the set of words of $A$ with length $n$, where $n \in \mathbb{N}$, and $A^0 = \{\varepsilon\}$, where $\varepsilon$ denotes the empty word. We put $A^* = \bigcup_{n \geq 0} A^n$, the set of all finite words, and $A^\omega = \{a_0a_1 \cdots a_n \cdots | a_i \in A\}$ is the set of infinite words. Finally, $|\alpha|$ denotes the length of $\alpha \in A^*$.

The formal definition of a finite transducer (FT) is the following.

**Definition 2.1** A finite transducer is a quintuple $(X, Y, S, \delta, \lambda)$, where: $X$ is a nonempty finite set, called the input alphabet; $Y$ is a nonempty finite set, called the output alphabet; $S$ is a nonempty finite set called the set of states; $\delta: S \times X \rightarrow S$, called the state transition function; and $\lambda: S \times X \rightarrow Y$, called the output function.

Let $M = \langle X, Y, S, \delta, \lambda \rangle$ be a finite transducer. The state transition function $\delta$ and the output function $\lambda$ can be extended to finite words, i.e. elements of $X^*$, recursively, as follows:

\[
\begin{align*}
\delta(s, \varepsilon) &= s \\
\delta(s, x\alpha) &= \delta(\delta(s, x), \alpha) \\
\lambda(s, \varepsilon) &= \varepsilon \\
\lambda(s, x\alpha) &= \lambda(s, x) \lambda(\delta(s, x), \alpha),
\end{align*}
\]

where $s \in S$, $x \in X$, and $\alpha \in X^*$. In an analogous way, $\lambda$ may be extended to $X^\omega$.

From these definitions it follows that, for all $s \in S, \alpha \in X^*$, and for all $\beta \in X^* \cup X^\omega$,

\[
\lambda(s, \alpha\beta) = \lambda(s, \alpha) \lambda(\delta(s, \alpha), \beta).
\]

A crucial concept to recall here is the concept of injective FT. In fact, there are two notions of injectivity that are behind the invertibility property of FTs used for cryptographic purposes: the concept of $\omega$-injectivity and the concept of injectivity with some delay $\tau$, with $\tau \in \mathbb{N}$.

**Definition 2.2** A finite transducer $M = \langle X, Y, S, \delta, \lambda \rangle$ is said to be $\omega$-injective, if

\[
\forall s \in S, \forall \alpha, \alpha' \in X^\omega, \quad \lambda(s, \alpha) = \lambda(s, \alpha') \implies \alpha = \alpha'.
\]

That is, for any $s \in S$, and any $\alpha \in X^\omega$, $\alpha$ is uniquely determined by $s$ and $\lambda(s, \alpha)$.

**Definition 2.3** A finite transducer $M = \langle X, Y, S, \delta, \lambda \rangle$ is said to be injective with delay $\tau$ or $\tau$-injective, with $\tau \in \mathbb{N}$, if

\[
\forall s \in S, \forall x, x' \in X, \forall \alpha, \alpha' \in X^\tau, \quad \lambda(s, x\alpha) = \lambda(s, x'\alpha') \implies x = x'.
\]

That is, for any $s \in S$, $x \in X$, and $\alpha \in X^\tau$, $x$ is uniquely determined by $s$ and $\lambda(s, x\alpha)$.
It is quite obvious that if an FT is injective with some delay $\tau \in \mathbb{N}$, then it is injective with delay $\tau'$, for $\tau' \geq \tau$, which implies that is also $\omega$-injective. The reverse is also true. Tao [10, Corollary 1.4.3] showed that if $M = \langle X, Y, S, \delta, \lambda \rangle$ is a $\omega$-injective FT, then there exists a non-negative integer $\tau \leq \frac{|S|(|S|-1)}{2}$ such that $M$ is $\tau$-injective.

The notions of equivalent states and minimal transducer considered here are the classical ones.

**Definition 2.4** Let $M_1 = \langle X, Y_1, S_1, \delta_1, \lambda_1 \rangle$ and $M_2 = \langle X, Y_2, S_2, \delta_2, \lambda_2 \rangle$ be two FTs. Let $s_1 \in S_1$, and $s_2 \in S_2$. One says that $s_1$ and $s_2$ are equivalent, and denotes this relation by $s_1 \sim s_2$, if $\forall \alpha \in X^*$, $\lambda_1(s_1, \alpha) = \lambda_2(s_2, \alpha)$.

**Definition 2.5** A finite transducer $M = \langle X, Y, S, \delta, \lambda \rangle$ is called minimal if it has no pair of equivalent states.

We now introduce the notion of equivalent transducers used in this context.

**Definition 2.6** Let $M_1 = \langle X, Y_1, S_1, \delta_1, \lambda_1 \rangle$ and $M_2 = \langle X, Y_2, S_2, \delta_2, \lambda_2 \rangle$ be two FTs. $M_1$ and $M_2$ are said to be equivalent, and we denote this by $M_1 \sim M_2$, if the following two conditions are satisfied: $\forall s_1 \in S_1$, $\exists s_2 \in S_2$ : $s_1 \sim s_2$ and $\forall s_2 \in S_2$, $\exists s_1 \in S_1$ : $s_1 \sim s_2$.

This relation $\sim$ is an equivalence relation on the set of FTs. To simplify, an equivalence class formed by $\omega$-injective FTs is said to be $\omega$-injective. Analogously, an equivalence class of $\tau$-injective FTs, for some $\tau \in \mathbb{N}$, is said to be $\tau$-injective.

Finally, we give the definition of what is called a linear finite transducer (LFT).

**Definition 2.7** If $X, Y$ and $S$ are vector spaces over a field $F$, and both $\delta : S \times X \to S$ and $\lambda : S \times X \to Y$ are linear maps, then the finite transducer $M = \langle X, Y, S, \delta, \lambda \rangle$ is called linear over $F$, and we say that $\dim(S)$ is the size of $M$.

If $X, Y$, and $S$ have dimensions $\ell, m$ and $n$, respectively, then there exist matrices $A \in \mathcal{M}_{n,n}(F)$, $B \in \mathcal{M}_{n,\ell}(F)$, $C \in \mathcal{M}_{m,n}(F)$, and $D \in \mathcal{M}_{m,\ell}(F)$, such that

$$\delta(s, x) = As + Bx \quad \text{and} \quad \lambda(s, x) = Cs + Dx,$$

for all $s \in S, x \in X$. The matrices $A, B, C, D$ are called the LFT structural matrices, and $\ell, m, n$ are called the LFT structural parameters. An LFT such that $C$ is the null matrix (with the adequate dimensions) is called trivial.

Let $L$ be the set of LFTs over a field $F$, and let $L_n$ denote the set of LFTs of size $n$. The restriction of the equivalence relation $\sim$ to $L$ is also represented by $\sim$. Its restriction to $L_n$ is denoted by $\sim_n$.

**Definition 2.8** Let $M \in L_n$ with structural matrices $A, B, C, D$. The matrix

$$\Delta_M = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$
is called the diagnostic matrix of $M$.

Amorim et al. [1] introduced a notion of canonical LFT and proved that each equivalence class of LFTs has exactly one canonical LFT, which is minimal. Here, we recall the notions of canonical LFT and of standard basis, used to define it.

**Definition 2.9** Let $V$ be a $k$-dimensional vector subspace of $\mathbb{F}^n$, where $\mathbb{F}$ is a field. The unique basis $\{b_1, b_2, \ldots, b_k\}$ of $V$ such that the matrix $[b_1 \ b_2 \ \cdots \ b_k]^T$ is in row echelon form will be here referred to as the standard basis of $V$.

**Definition 2.10** Let $M = \langle X, Y, S, \delta, \lambda \rangle$ be a linear finite transducer. One says that $M$ is a canonical LFT if $\{\Delta_M e_1, \Delta_M e_2, \ldots, \Delta_M e_n\}$ is the standard basis of $\{\Delta_M s \mid s \in S\}$, where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of $S$.

In the same work it is also proved a fundamental result about the size of LFTs equivalence classes. It gives a way to compute the number of LFTs in $L_{n_2}$ that are equivalent to minimal LFTs in $L_{n_1}$, for $n_2 \geq n_1$. This result will be essential in Section 4. to deduce the recurrence relation that gives the number of canonical LFTs.

**Theorem 2.11** Let $M_1$ be a minimal LFT over $\mathbb{F}_q$ with structural parameters $\ell, m, n_1$, and let $n_2 \geq n_1$. Then, the number of finite transducers $M \in L_{n_2}$ which are equivalent to $M_1$ is

$$(q^{n_2} - 1)(q^{n_2} - q) \cdots (q^{n_2} - q^{-r})q^{(n_2 + \delta)(n_2 - r)},$$

where $r = \text{rank}(\Delta_{M_1})$.

We now recall the notion of Smith normal form (SNF) of a matrix and the well known result (see [4] or [5, Theorem II.9]) that ensures its existence.

**Theorem 2.12** Let $R$ be a principal ideal domain. Every matrix $A \in \mathcal{M}_{m,n}(R)$ is equivalent to a matrix of the form

$$D = \text{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0) = \begin{bmatrix}
    d_1 & & & \\
    & \ddots & & \\
    & & d_r & \\
    & & & 0 \\
    0 & & & \\
    & & & \ddots \\
    & & & & 0
\end{bmatrix}$$

where $r = \text{rank}(A)$, $d_i \neq 0$ and $d_i \mid d_{i+1}$, i.e. $d_i$ divides $d_{i+1}$, for $1 \leq i \leq r - 1$. The matrix $D$ is called the Smith normal form of $A$, and the elements $d_i$ are called the invariant factors of $A$.

### 3. Estimation of the number of $\tau$-injective equivalence classes

In this section we show how to estimate the number of non-equivalent LFTs that are $\tau$-injective, for some $\tau \in \mathbb{N}$, by generating LFTs at random. Subsection 3.1 is devoted to explain how to
implement an algorithm in Python to test if a given LFT is injective with some delay $\tau$ using the Sage system. In Subsection 3.2 we present an algorithm that, given a LFT, computes the size of its equivalence class. Finally, in Subsection 3.3 we explain how these algorithms can be used to get an approximated value for the number of $\tau$-injective equivalence classes, i.e., the number of non-equivalent LFTs.

### 3.1. Checking if a LFT is injective with delay $\tau \in \mathbb{N}$

Let $M = (\mathcal{X}, \mathcal{Y}, S, \delta, \lambda)$ be a LFT over a field $\mathbb{F}$ defined by the structural matrices $A, B, C, D$ and with structural parameters $\ell, m, n$. Starting at a state $s_0$ and reading an input sequence $x_0x_1x_2 \ldots$, one gets a sequence of states $s_0s_1s_2 \ldots$ and a sequence of outputs $y_0y_1y_2 \ldots$ satisfying the relations $s_{t+1} = \delta(s_t, x_t) = As_t + Bx_t$ and $y_t = \lambda(s_t, x_t) = Cs_t + Dx_t$, for all $t \geq 0$. Now, let

$$X(z) = \sum_{t \geq 0} x_t z^t, \quad Y(z) = \sum_{t \geq 0} y_t z^t, \quad Q(z) = \sum_{t \geq 0} s_t z^t,$$

regarded as elements of the $\mathbb{F}[[z]]$-modules $\mathbb{F}[[z]]^\ell, \mathbb{F}[[z]]^m, \mathbb{F}[[z]]^n$, respectively, where $\mathbb{F}[[z]]$ is the ring of formal power series over $\mathbb{F}$. Amorim et al [2] showed that

$$Y(z) = G(z)s_0 + H(z)X(z)$$

where $G(z) = C(I - Az)^{-1}$ and $H(z) = C(I - Az)^{-1}Bz + D$. The matrices $G \in \mathcal{M}_{m,n}(\mathbb{F})[[z]]$ and $H \in \mathcal{M}_{m,\ell}(\mathbb{F})[[z]]$ are called, respectively, the free response matrix and the transfer function matrix of the transducer. In the same paper the authors also proved that

$$H(z) = \frac{1}{f(z)} \left( C(I - Az)^*Bz + f(z)D \right), \quad (1)$$

where $f(z) = \det(I - Az)$, and $P^*$ denotes the adjoint matrix of $P$. Consider the multiplicatively closed set $\mathcal{S} = \{ 1 + zb(z) \mid b(z) \in \mathbb{F}[z] \}$, and let $\mathbb{F}[z]_\mathcal{S} = \left\{ \frac{f}{g} \mid f \in \mathbb{F}[z], g \in \mathcal{S} \right\}$ be the ring of fractions of $\mathbb{F}[z]$ relative to $\mathcal{S}$. Then, the transfer function matrix of a LFT is in $\mathcal{M}(\mathbb{F}[z]_\mathcal{S})$. Since $\mathbb{F}[z]_\mathcal{S}$ is a principal ideal domain, and $z$ is its unique irreducible element, up to units, the SNF of every transfer function matrix $H(z)$, with rank $r$, is of the form

$$\mathcal{D}_{n_0,n_1,\ldots,n_u} = \text{diag}(I_{n_0}, zI_{n_1}, \ldots, z^nI_{n_u}, 0, \ldots, 0) = \begin{bmatrix} I_{n_0} & \cdots & 0 \\ zI_{n_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z^nI_{n_u} \end{bmatrix},$$

where $n_i \geq 0$, for $0 \leq i \leq u$; $n_u \neq 0$ unless $H(z) = 0$, and $\sum_{i=0}^un_i = r$. We now restate the result [2] that gives two necessary and sufficient conditions for a transducer to be injective with some delay $\tau$. We put $n_i = 0, \forall i > u$. 
Theorem 3.1 Let $\mathcal{X}, \mathcal{Y}$ and $S$ be vector spaces over a field $\mathbb{F}$, with dimensions $\ell, m$ and $n$, respectively. Let $M = (\mathcal{X}, \mathcal{Y}, S, \delta, \lambda)$ be a LFT, and let $H \in \mathcal{M}_{m, \ell}(\mathbb{F}[z]_S)$ be its transfer function matrix. Let $D = D_{n_0, n_1, \ldots, n_u}$ be the Smith normal form of $H$, and assume $n_u \neq 0$. Then, the following conditions are equivalent:

(i) $M$ is injective with delay $\tau$;

(ii) $\sum_{i=0}^r n_i = \ell$;

(iii) there is $H' \in \mathcal{M}_{\ell, m}(\mathbb{F}[z]_S)$ such that $H' H = z^\tau I$.

In Algorithm 1 one can read the definition of the function $\text{IsInjective}$, which tests if a LFT over $\mathbb{F}_2$, defined by its structural matrices, $A, B, C, D$, is $\tau$-injective by checking condition 3 of the previous theorem.

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Algorithm 1 Testing if a LFT over $\mathbb{F}_2$ is injective with some delay $\tau$.

```python
def IsInjective(A, B, C, D, tau):
    Ring = GF(Integer(2))['z']
    (z, ) = Ring._first_gens(1)
    poly = identity_matrix(A.nrows()) - A * z
    fH = C * poly.adjoint() * B * z + poly.det() * D
    DH = D.H.elementary_divisors()
    for i in D.H if i != 0
    return B.ncols() == len([j for j in D.H if j <= z * tau])
```

In lines 4–7 the SNF of $H(z)$ is computed as follows. It starts by using the Sage function $\text{elementary_divisors}$ to determine the invariant factors of $f(z) H(z) \in \mathcal{M}_{m, \ell}(\mathbb{F}_2[z])$. Since units are irrelevant in the SNF computation, and $f(z)$ is a unit in $\mathbb{F}_2[z]_S$, one can find the invariant factors of $H(z)$ from the invariant factors of the matrix $f(z) H(z)$ using the following straightforward result.

Proposition 3.2 Let $D_{fH} = \text{diag}(d_1', d_2', \ldots, d_r', 0, \ldots, 0)$ be the SNF of $f(z) H(z)$ and $D_H = \text{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0)$ the SNF of $H$. Then,

$$\forall i \in \{1, \ldots, r\}, \ d_i = \gcd(d_i', z^u),$$

where $z^u$ is the biggest power of $z$ that divides $d_i'$.

Having this, and since the entries of the matrix $f(z) H(z)$ belong to $\mathbb{F}_2[z]$, the algorithm starts by defining the ring $\mathbb{F}_2[z]$ (line 2), and $z$ as a variable in that ring (line 3). The expression $\text{identity_matrix}(A.nrows())$, as the name suggests, returns the identity matrix whose size is the number of rows of $A$, i.e., $n$. The matrix $f(z) H(z)$ is then computed using the expression (1), and the algorithm uses functions $\text{adjoint}$ and $\text{det}$, to compute the adjoint and the determinant of a matrix, respectively (line 5). The invariant factors of $f(z) H(z)$ are computed using the function $\text{elementary_divisors}$ (line 6). Since to check if condition 3 of Theorem 3.1 is verified one just needs to count the invariant factors of $H(z)$ that are less or equal to
We apply Proposition 3.2 in the algorithm by replacing $z^u$ with $z^{r+1}$ in expression (2) (line 7). The algorithm then returns True if the number of invariant factors of $H(z)$ which divide $z^r$ is equal to $\ell$, i.e., is equal to the number of columns of the matrix $B$. It returns False otherwise.

### 3.2. Determining the size of equivalence classes

In a previous work \[1\], it was proved that, given a LFT over $\mathbb{F}_q$, $M$, with structural matrices $A, B, C, D$ and structural parameters $\ell, m, n \in \mathbb{N}\setminus\{0\}$, the size of $[M]_\sim^n$ is given by the following expression:

$$|[M]_\sim^n| = \prod_{i=0}^{r-1} (q^n - q^i) \cdot q^{(n+\ell)(n-r)}, \quad \text{where } r = \text{rank}(\Delta_M).$$  

Algorithm 2 shows the definition of `EquivClassSize` that computes the size of an equivalence class using expression (3) for $q = 2$. This takes as input the structural matrices $A, B, C, D$, and the structural parameters $\ell, m, n$ are determined using functions `nrows` and `ncols` (lines 2–4). To determine the value of $r$, it calls functions `stack` and `rank`. The first is used to create the LFT diagnostic matrix (lines 5–7), and the second is used to determine the rank of that matrix (line 8). The size of the equivalence class is then easily obtained through a loop (lines 9–12).

### Algorithm 2 Determining the size of equivalence classes.

```python
def EquivClassSize(A, B, C, D):
    l = B.ncols()
    m = C.nrows()
    n = A.nrows()
    K = copy.deepcopy(C)
    for j in {1, \ldots, n-1}:
        K = K.stack(K*A)
    r = K.rank()
    size = 1
    for j in {0, \ldots, r-1}:
        size = size * (2**n - 2**j)
    size = size * 2**(n + l) * (n - r)
    return size
```

### 3.3. Computing an approximated value for the number of $\tau$-injective equivalence classes

Let $\mathcal{E}$ be the set of equivalence classes of LFTs over $\mathbb{F}_q$ with structural parameters $\ell, m, n$. Let $\mathcal{I}_\tau \subseteq \mathcal{E}$ be the set of the $\tau$-injective equivalence classes, i.e., $\mathcal{I}_\tau = \{[M]_\sim^n \in \mathcal{E} \mid M \text{ is } \tau\text{-injective}\}$. One wants to estimate $|\mathcal{I}_\tau|$. 
It is easy to generate random LFTs because, given a triple of structural parameters, one just needs to generate structural matrices $A, B, C, D$ with the appropriate sizes. Using the method described in Subsection 3.1, it is also possible to test if a LFT is injective with some delay $\tau$. Hence, one can get an approximated value for $|I_\tau|$ with simple random sampling, as we will see in the remaining of this subsection.

Let $L_{\ell,m,n}$ be the set of LFTs with structural parameters $\ell, m, n$. Let $\mathcal{R}$ be a multiset of randomly generated LFTs in $L_{\ell,m,n}$, and $\eta_E$ the number of occurrences in $\mathcal{R}$ of transducers that belong to a class $E \in \mathcal{E}$. Let $p_E$ be the probability that a LFT in $L_{\ell,m,n}$ is in the class $E \in \mathcal{E}$, that is, $p_E = \frac{|E|}{|L_{\ell,m,n}|}$. One knows that $\frac{\eta_E}{|\mathcal{R}|}$ is an approximated value for $p_E$, and that the larger the sample size $|\mathcal{R}|$, the better will the approximation be.

Take $E \in \mathcal{E}$, and let:

$$\mu_E = \begin{cases} \frac{1}{p_E} & \text{if } E \in I_\tau \\ 0 & \text{otherwise} \end{cases}.$$  \hfill (4)

Trivially,

$$|I_\tau| = \sum_{E \in \mathcal{I}_\tau} 1 = \sum_{E \in \mathcal{I}_\tau} p_E = \sum_{E \in \mathcal{E}} p_E \mu_E.$$  

Consequently,

$$|I_\tau| \approx \sum_{E \in \mathcal{E}} \frac{\eta_E}{|\mathcal{R}|} \mu_E = \frac{1}{|\mathcal{R}|} \sum_{E \in \mathcal{E}} \eta_E \mu_E.$$  

Since, obviously, $M \in E$ if and only if $E = [M]_{\sim_n}$, one has

$$\sum_{E \in \mathcal{E}} \eta_E \mu_E = \sum_{M \in \mathcal{R}} \mu_{[M]_{\sim_n}}, \; \text{and } |I_\tau| \approx \frac{1}{|\mathcal{R}|} \sum_{M \in \mathcal{R}} \mu_{[M]_{\sim_n}}.$$  

Therefore we can get an approximated value for $|I_\tau|$ using a simple function as the one presented in Algorithm 3.

**Algorithm 3** Estimating the number of non-equivalent LFTs.

```python
    def EstCountInjective(nr, l, m, n, tau):
        count = 0
        for i in {1, ..., nr}:
            A, B, C, D = RandomLFT(l, m, n)
            if IsInjective(A, B, C, D, tau):
                count = count + 1/Probability(A, B, C, D)
        return count/nr
```

The function `EstCountInjective` takes as input the sample size, represented by the variable $nr$, the structural parameters $\ell, m, n$, and the delay $\tau$. It calls the following three functions:

- **RandomLFT**: a function such that, given the parameters $\ell, m,$ and $n$, returns matrices $A \in \mathcal{M}_{n,n}(\mathbb{F}_2)$, $B \in \mathcal{M}_{n,\ell}(\mathbb{F}_2)$, $C \in \mathcal{M}_{m,n}(\mathbb{F}_2)$, and $D \in \mathcal{M}_{m,\ell}(\mathbb{F}_2)$, whose entries were uniformly randomly generated;
• **IsInjective**: the function defined in Subsection 3.1.
• **Probability**: a function such that, given the structural matrices of a LFT, \( M \), it returns \( p[M] \sim \) using the function **EquivClassSize** presented in Subsection 3.2.

Given an input, the algorithm starts by initializing the variable **count** with the value 0 (line 2). Then, at each iteration of the loop, generates a LFT, let us say \( M \), and if \( M \) is injective with delay \( \tau \) it adds, to the variable **count**, the value of \( \mu[M] \sim \) (lines 3–6). This way, when the loop is finished, one has \( \text{count} = \sum_{M \in R} \mu[M] \sim \), where \( R \) is the set of the \( nr \) random generated LFTs. It returns \( \text{count} / nr \), that is, an estimate for \( |I_\tau| \).

### 4. Estimating the percentage of \( \tau \)-injective equivalent classes

In this section, we first deduce a recurrence relation that, given \( \ell, m, n \in \mathbb{N} \setminus \{0\} \), counts the number of canonical LFTs over \( F_q \) with structural parameters \( \ell, m, n \). Then we show how to estimate the percentage of \( \tau \)-injective equivalence classes.

Let \( \ell, m, n \in \mathbb{N} \setminus \{0\} \), and consider the following notation:

- \( L_{\ell,m,n} \) denotes the total number of LFTs over \( F_q \) in \( L_{\ell,m,n} \);
- \( T_{\ell,m,n} \) denotes the number of trivial LFTs over \( F_q \) in \( L_{\ell,m,n} \);
- \( mL_{\ell,m,n} \) denotes the number of non-trivial LFTs over \( F_q \) in \( L_{\ell,m,n} \) that are minimal;
- \( mL_{\ell,m,n} \) denotes the number of non-trivial LFTs over \( F_q \) in \( L_{\ell,m,n} \) that are not minimal;
- \( C_{\ell,m,n} \) denotes the number of canonical LFTs over \( F_q \) in \( L_{\ell,m,n} \).

It is obvious that \( L(\ell, m, n) = T(\ell, m, n) + mL(\ell, m, n) + mL(\ell, m, n). \)

The number of trivial transducers is easy to find: since a LFT is trivial when \( C = 0 \), the entries of the other matrices \( (A, B, \text{ and } D) \) can take any value and, therefore,

\[
T(\ell, m, n) = q^{2+\ell(m+n)}.
\]

The set of non-trivial LFTs in \( L_{\ell,m,n} \) that are minimal is formed by the equivalence classes that have a canonical LFT. Notice that, from Theorem 2.11 those classes all have the same cardinality. Let \( EC(n) \) be the size of the equivalence class \([M] \sim \), where \( M \) is a canonical transducer in \( L_{\ell,m,n} \). Then, also from Theorem 2.11 \( EC(n) = \prod_{i=0}^{n-1}(q^n - q^i). \)

Therefore,

\[
mL(\ell, m, n) = EC(n) \cdot C(\ell, m, n) = \prod_{i=0}^{n-1}(q^n - q^i) \cdot C(\ell, m, n).
\]

Now, let us see how to determine \( mL(\ell, m, n) \) for all \( \ell, m, n \in \mathbb{N} \setminus \{0\} \).

For \( n = 1 \), all the non-trivial LFTs are canonical. Therefore \( mL(\ell, m, 1) = 0 \), and

\[
C(\ell, m, 1) = L(\ell, m, 1) - T(\ell, m, 1)
\]
For \( n = 2 \), \( \overline{mL}(\ell, m, n) \) is the number of transducers in \( \mathcal{L}_{\ell,m,2} \) that are equivalent to transducers in \( \mathcal{L}_{\ell,m,1} \). Theorem \([2.11]\) tells us a way to compute the number of LFTs in \( \mathcal{L}_{\ell,m,n_2} \) that are equivalent to minimal transducers in \( \mathcal{L}_{\ell,m,n_1} \), for \( n_2 \geq n_1 \). Let \( \text{NM}(\ell, n_1, n_2) \) be that value, that is, \( \text{NM}(\ell, n_1, n_2) = \prod_{i=0}^{n_1-1}(q^{n_2} - q^i) \cdot q^{(n_2 + \ell)(n_2 - n_1)} \). Then,

\[
\overline{mL}(\ell, m, 2) = C(\ell, m, 1) \cdot \text{NM}(\ell, 1, 2) = C(\ell, m, 1) \cdot (q^2 - 1) \cdot q^{\ell+2}
\]

For \( n = 3 \), the set of non-minimal LFTs if formed by the LFTs that are equivalent to minimal transducers in \( \mathcal{L}_{\ell,m,1} \), and the ones that are equivalent to minimal transducers in \( \mathcal{L}_{\ell,m,2} \). Therefore,

\[
\overline{mL}(\ell, m, 3) = C(\ell, m, 1) \cdot \text{NM}(\ell, 1, 3) + C(\ell, m, 2) \cdot \text{NM}(\ell, 2, 3)
\]

\[
= \sum_{i=1}^{2} C(\ell, m, i) \cdot \text{NM}(\ell, i, 3) = \sum_{i=1}^{2} C(\ell, m, i) \cdot \prod_{j=0}^{i-1}(q^3 - q^j) \cdot q^{(\ell+3)(3-i)}
\]

This process can be generalized to get:

\[
\overline{mL}(\ell, m, n) = \sum_{i=1}^{n-1} C(\ell, m, i) \cdot \text{NM}(\ell, i, n).
\]

Therefore, given \( \ell, m, n \in \mathbb{N} \setminus \{0\} \), the number of canonical LFTs with structural parameters \( \ell, m, n \) satisfies the following recurrence relation:

\[
\begin{cases}
C(\ell, m, 1) = (q^m - 1)q^{\ell(m+1)+1} \\
C(\ell, m, n) = \frac{1}{\text{EC}(n)} \cdot (L(\ell, m, n) - T(\ell, m, n) - \overline{mL}(\ell, m, n)), \text{ for } n \geq 2
\end{cases}
\]

where

\[
L(\ell, m, n) = q^{m\ell + n(\ell + m + n)}, \quad \text{EC}(n) = \prod_{i=0}^{n-1}(q^n - q^i), \quad T(\ell, m, n) = q^{n^2 + \ell(m + n)},
\]

\[
\overline{mL}(\ell, m, n) = \sum_{i=1}^{n-1} C(\ell, m, i) \cdot \text{NM}(\ell, i, n), \quad \text{and } \text{NM}(\ell, i, n) = \prod_{j=0}^{i-1}(q^n - q^j) \cdot q^{(n+\ell)(n-i)}.
\]

We define \textbf{CountCT} (Algorithm 4) taking as input a triple \( \ell, m, n \in \mathbb{N} \setminus \{0\} \), and using the previous recurrence relation to compute the number of canonical LFTs with structural parameters \( \ell, m, n \). It starts by checking if \( n = 1 \) and, if that is true, it computes \( C(\ell, m, 1) \) using expression \([3]\) (lines 2–3). If \( n \geq 2 \), it computes \( \text{EC}(n), L(\ell, m, n), T(\ell, m, n) \) and \( \overline{mL}(\ell, m, n) \) using the expressions given above.

**Algorithm 4** Counting the number of canonical LFTs.

1. \textbf{def} CountCT(\( l, m, n \)):
2. 1. \textbf{if} \( n = 1 \):
3. 2. \textbf{return} \((2 \ast m - 1) \ast 2 \ast (l \ast (m + 1) + 1)\)
4. 3. \textbf{else} :
The function `CountCT` computes the exact number of canonical LFTs which have structural parameters $\ell, m, n \in \mathbb{N} \setminus \{0\}$. Thus, it can be used to count the exact number of equivalence classes that contain at least one LFT with structural parameters $\ell, m, n$. Given a triple $\ell, m, n \in \mathbb{N} \setminus \{0\}$, one just needs to sum up the number of canonical LFTs that have structural parameters $\ell, m, n'$, for $n' \leq n$. Since the function `EstCountInjective` defined in Algorithm 3 gives an approximate value for the number of equivalence classes of LFTs with structural parameters $\ell, m, n \in \mathbb{N} \setminus \{0\}$ that are $\tau$-injective, we can obtain, using these two functions, an estimated value for the percentage of $\tau$-injective equivalence classes. The function `EstPercInjective` (Algorithm 5) implements this process.

### Algorithm 5 Estimating the percentage of $\tau$-injective equivalence classes.

```python
def EstPercInjective(nr, l, m, n, tau):
    EC = 0
    for i in {1, \ldots, n}:
        EC = EC + CountCT(l, m, i)
    return EstCountInjective(nr, l, m, n, tau)/EC
```

5. Experimental results

In this Section we present some experimental results on the number of $\omega$-injective and $\tau$-injective equivalent classes of LFTs over $\mathbb{F}_2$, for some values of $\tau \in \mathbb{N}$. Recall that if a LFT is $\tau$-injective for some $\tau \in \mathbb{N}$, then it is $\omega$-injective.

For each triple of structural parameters $\ell, m, n$, with $\ell \in \{1, \ldots, 5\}$, $m = 5$ and $n \in \{1, \ldots, 10\}$, we uniformly randomly generated a sample of 20 000 LFTs. With these samples we estimate the number of $\tau$-injective equivalence classes, for $\tau \in \{0, 1, \ldots, 10\}$, using `EstCountInjective` defined above (Algorithm 3). The total number of equivalence classes was obtained using the recurrence relation to count canonical LFTs. Then, using the previous results, we computed an approximated value for the percentage of $\tau$-injective equivalence classes of LFTs. The size of each sample is sufficient to ensure the statistical significance with a 99% confidence level.
within a 1% error margin. The sample size is calculated with the formula \( N = \left( \frac{z}{\epsilon} \right)^2 \), where \( z \) is obtained from the normal distribution table such that \( P(-z < Z < z) = \gamma \), \( \epsilon \) is the error margin, and \( \gamma \) is the desired confidence level.

In Table 1, we present the approximated values for the number of 10-injective equivalence classes when \( m = 5 \), and \( n \), \( l \) range in \( \{1, \ldots, 10\} \) and \( \{1, \ldots, 5\} \), respectively. We chose to show the results for \( \tau = 10 \) because this value is large enough to draw conclusions about the number of \( \omega \)-injective equivalence classes. From the results obtained, one can observe an exponential growth on the number of 10-injective equivalence classes, as \( n \) and \( \ell \) increase. Consequently, the number of \( \omega \)-injective equivalence classes also grows exponentially.

| \( n \) | \( \ell \) | 1 | 2 | 3 | 4 | 5 |
|-----|-----|-----|-----|-----|-----|-----|
| 1   |     | 3.91 \times 10^{13} | 2.42 \times 10^{16} | 1.44 \times 10^{17} | 7.66 \times 10^{16} | 2.97 \times 10^{10} |
| 2   |     | 3.34 \times 10^{19} | 4.17 \times 10^{19} | 5.13 \times 10^{19} | 5.92 \times 10^{14} | 5.29 \times 10^{10} |
| 3   |     | 2.45 \times 10^{27} | 6.15 \times 10^{29} | 1.54 \times 10^{12} | 3.70 \times 10^{14} | 7.39 \times 10^{16} |
| 4   |     | 1.66 \times 10^{99} | 8.45 \times 10^{11} | 4.26 \times 10^{14} | 2.10 \times 10^{17} | 9.24 \times 10^{19} |
| 5   |     | 1.10 \times 10^{14} | 1.12 \times 10^{14} | 1.13 \times 10^{17} | 1.14 \times 10^{14} | 1.05 \times 10^{21} |
| 6   |     | 7.17 \times 10^{12} | 1.45 \times 10^{16} | 2.96 \times 10^{19} | 5.97 \times 10^{22} | 1.15 \times 10^{25} |
| 7   |     | 4.61 \times 10^{14} | 1.87 \times 10^{18} | 7.64 \times 10^{21} | 3.10 \times 10^{23} | 1.22 \times 10^{25} |
| 8   |     | 2.96 \times 10^{18} | 2.40 \times 10^{20} | 1.96 \times 10^{24} | 1.60 \times 10^{26} | 1.28 \times 10^{28} |
| 9   |     | 1.90 \times 10^{22} | 3.08 \times 10^{22} | 5.04 \times 10^{26} | 8.24 \times 10^{29} | 1.33 \times 10^{31} |
| 10  |     | 1.22 \times 10^{26} | 3.95 \times 10^{24} | 1.29 \times 10^{29} | 4.23 \times 10^{33} | 1.37 \times 10^{35} |

Table 1: Approximated values for the number of injective classes when \( m = 5 \) and \( \tau = 10 \).

The results on the percentage of \( \tau \)-injective equivalence classes are exhibited in Tables 2 to 5. Each of these tables presents the approximated percentage for a given value of \( \ell \in \{2, \ldots, 5\} \), while \( n \) and \( \tau \) range in \( \{1, \ldots, 10\} \) and \( \{0, 1, \ldots, 10\} \), respectively.

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 99.88 | 95.21 | 95.21 | 95.21 | 95.21 | 95.21 | 95.21 | 95.21 | 95.21 | 95.21 | 95.21 |
| 2   | 99.5 | 97.06 | 97.2 | 97.2 | 97.2 | 97.2 | 97.2 | 97.2 | 97.2 | 97.2 | 97.2 |
| 3   | 99.82 | 98.27 | 98.58 | 98.62 | 98.62 | 98.62 | 98.62 | 98.62 | 98.62 | 98.62 | 98.62 |
| 4   | 91.1 | 99.07 | 99.53 | 99.57 | 99.57 | 99.57 | 99.57 | 99.57 | 99.57 | 99.57 | 99.57 |
| 5   | 91.01 | 99.18 | 99.72 | 99.74 | 99.74 | 99.74 | 99.74 | 99.74 | 99.74 | 99.74 | 99.74 |
| 6   | 91.04 | 99.37 | 99.92 | 99.95 | 99.96 | 99.96 | 99.96 | 99.96 | 99.96 | 99.96 | 99.96 |
| 7   | 91.75 | 99.12 | 99.69 | 99.73 | 99.74 | 99.74 | 99.74 | 99.74 | 99.74 | 99.74 | 99.74 |
| 8   | 96.64 | 99.31 | 99.76 | 99.81 | 99.81 | 99.81 | 99.81 | 99.81 | 99.81 | 99.81 | 99.81 |
| 9   | 96.6 | 99.18 | 99.74 | 99.75 | 99.75 | 99.75 | 99.75 | 99.75 | 99.75 | 99.75 | 99.75 |
| 10  | 96.85 | 99.39 | 99.85 | 99.89 | 99.89 | 99.89 | 99.89 | 99.89 | 99.89 | 99.89 | 99.89 |

Table 2: Approximated percentage value for \( \ell = 2 \) and \( m = 5 \).

In Table 2 we present the results for \( \ell = 2 \). The results show that, in this case, when \( n \) increases, there is a significant increase in the percentage of \( \tau \)-injective LFTs, for \( \tau \geq 1 \). Nonetheless, when \( n = 1 \) the percentage of 1-injective (and consequently \( \omega \)-injective) LFTs is already very high (above 95%). This suggests that, in this case, there is also a very high probability of a uniform random generated LFT be \( \omega \)-injective.

The results for \( \ell = 3 \), presented in Table 3, also show a significant growing of the values with \( n \). A more careful observation of the column \( \tau = 10 \), allow us to conclude that when \( n \geq 3 = \ell \),
the percentage of $\omega$-injective LFTs is above 95%.

| $n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 79.42 | 88.48 | 88.48 | 88.48 | 88.48 | 88.48 | 88.48 | 88.48 | 88.48 | 88.48 |
| 2   | 79.08 | 92.77 | 93.61 | 93.61 | 93.61 | 93.61 | 93.61 | 93.61 | 93.61 | 93.61 |
| 3   | 79.19 | 94.98 | 96.54 | 96.68 | 96.68 | 96.68 | 96.68 | 96.68 | 96.68 | 96.68 |
| 4   | 79.22 | 96.31 | 98.27 | 98.37 | 98.38 | 98.38 | 98.38 | 98.38 | 98.38 | 98.38 |
| 5   | 79.69 | 96.89 | 99.04 | 99.28 | 99.29 | 99.29 | 99.29 | 99.29 | 99.29 | 99.29 |
| 6   | 79.68 | 97.14 | 99.39 | 99.66 | 99.70 | 99.71 | 99.71 | 99.71 | 99.71 | 99.71 |
| 7   | 79.21 | 97.37 | 99.58 | 99.79 | 99.84 | 99.85 | 99.85 | 99.85 | 99.85 | 99.85 |
| 8   | 79.72 | 97.22 | 99.52 | 99.79 | 99.82 | 99.82 | 99.82 | 99.82 | 99.82 | 99.82 |
| 9   | 79.50 | 97.32 | 99.56 | 99.85 | 99.90 | 99.91 | 99.91 | 99.91 | 99.91 | 99.91 |
| 10  | 80.07 | 97.64 | 99.83 | 100   | 100   | 100   | 100   | 100   | 100   | 100   |

Table 3: Approximated percentage value for $\ell = 3$ and $m = 5$.

In Tables 4 and 5 we present the results for $\ell = 4$ and $\ell = 5$, respectively. Again, the percentage of $\omega$-injective LFTs is quite high for $n \geq \ell$.

| $n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 59.09 | 73.64 | 73.64 | 73.64 | 73.64 | 73.64 | 73.64 | 73.64 | 73.64 | 73.64 |
| 2   | 59.70 | 81.83 | 84.60 | 84.60 | 84.60 | 84.60 | 84.60 | 84.60 | 84.60 | 84.60 |
| 3   | 59.50 | 85.33 | 90.49 | 91.07 | 91.07 | 91.07 | 91.07 | 91.07 | 91.07 | 91.07 |
| 4   | 59.76 | 87.84 | 93.85 | 95.01 | 95.13 | 95.13 | 95.13 | 95.13 | 95.13 | 95.13 |
| 5   | 59.01 | 88.77 | 95.79 | 97.35 | 97.60 | 97.64 | 97.64 | 97.64 | 97.64 | 97.64 |
| 6   | 59.38 | 89.29 | 96.39 | 98.14 | 98.48 | 98.52 | 98.53 | 98.53 | 98.53 | 98.53 |
| 7   | 59.93 | 89.49 | 96.97 | 98.76 | 99.14 | 99.19 | 99.22 | 99.22 | 99.22 | 99.22 |
| 8   | 59.43 | 89.30 | 97.14 | 98.87 | 99.35 | 99.49 | 99.51 | 99.51 | 99.51 | 99.51 |
| 9   | 59.93 | 89.91 | 97.40 | 99.34 | 99.81 | 99.95 | 99.97 | 99.98 | 99.98 | 99.98 |
| 10  | 59.81 | 89.46 | 97.64 | 99.51 | 99.99 | 100   | 100   | 100   | 100   | 100   |

Table 4: Approximated percentage value for $\ell = 4$ and $m = 5$.

| $n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 29.29 | 44.63 | 44.63 | 44.63 | 44.63 | 44.63 | 44.63 | 44.63 | 44.63 | 44.63 |
| 2   | 30.26 | 53.48 | 59.11 | 59.11 | 59.11 | 59.11 | 59.11 | 59.11 | 59.11 | 59.11 |
| 3   | 29.75 | 51.69 | 68.60 | 71.09 | 71.09 | 71.09 | 71.09 | 71.09 | 71.09 | 71.09 |
| 4   | 30.13 | 61.15 | 75.19 | 80.37 | 81.63 | 81.63 | 81.63 | 81.63 | 81.63 | 81.63 |
| 5   | 29.98 | 62.05 | 85.95 | 88.83 | 89.21 | 89.24 | 89.24 | 89.24 | 89.24 | 89.24 |
| 6   | 29.21 | 62.69 | 79.92 | 88.01 | 91.37 | 92.52 | 92.79 | 92.79 | 92.79 | 92.79 |
| 7   | 29.35 | 62.83 | 80.43 | 88.92 | 92.98 | 94.87 | 95.30 | 95.30 | 95.30 | 95.30 |
| 8   | 29.78 | 63.60 | 81.02 | 89.20 | 94.50 | 96.43 | 97.33 | 97.62 | 97.67 | 97.67 |
| 9   | 30.07 | 63.39 | 81.08 | 90.05 | 94.57 | 96.71 | 97.85 | 98.35 | 98.48 | 98.50 |
| 10  | 28.97 | 62.58 | 80.92 | 90.70 | 95.22 | 97.24 | 98.34 | 98.87 | 99.14 | 99.25 |

Table 5: Approximated percentage value for $\ell = 5$ and $m = 5$.

Observing all the tables, it can be noticed that the approximated percentage value, specially for low values of $n$, suffers a big reduction when $\ell$ increases from 1 to 5. However, the growth, as a function of $n$, is much steeper for higher values of $\ell$. This ensures that, for a not so large value of $n$, the percentage of $\omega$-injective LFTs is very high. Therefore, if one uniformly random generates LFTs, it is highly probable to get $\omega$-injective ones.

We give here the results of an additional experiment, taking $\ell = m = 8$, $n \in \{1, \ldots, 10\}$ and $\tau \in \{0, 1, \ldots, 10\}$. The percentages of $\tau$-injective LFTs obtained are presented in Table 6.
Again, for values of $n$ slightly larger than $\ell$ and $m$, one can see that the percentage of $\omega$-injective LFTs is very high.

| $n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 29.01 | 43.59 | 43.59 | 43.59 | 43.59 | 43.59 | 43.59 | 43.59 | 43.59 | 43.59 |
| 2   | 29.11 | 52.44 | 57.91 | 57.91 | 57.91 | 57.91 | 57.91 | 57.91 | 57.91 | 57.91 |
| 3   | 29.71 | 58.98 | 69.04 | 71.44 | 71.44 | 71.44 | 71.44 | 71.44 | 71.44 | 71.44 |
| 4   | 29.11 | 59.60 | 73.92 | 80.16 | 80.16 | 80.16 | 80.16 | 80.16 | 80.16 | 80.16 |
| 5   | 28.76 | 60.80 | 77.23 | 86.94 | 87.51 | 87.51 | 87.51 | 87.51 | 87.51 | 87.51 |
| 6   | 28.52 | 62.01 | 79.32 | 87.49 | 90.88 | 92.30 | 92.55 | 92.55 | 92.55 | 92.55 |
| 7   | 28.33 | 61.79 | 80.11 | 88.77 | 92.99 | 94.61 | 95.16 | 95.29 | 95.29 | 95.29 |
| 8   | 28.98 | 62.25 | 80.95 | 89.98 | 94.20 | 96.11 | 97.09 | 97.47 | 97.55 | 97.55 |
| 9   | 29.09 | 62.39 | 80.84 | 89.94 | 94.51 | 96.96 | 97.94 | 98.36 | 98.56 | 98.59 |
| 10  | 29.01 | 62.86 | 81.34 | 90.75 | 95.36 | 97.63 | 98.56 | 99.00 | 99.28 | 99.34 |

Table 6: Approximated percentage value for $\ell = 8$ and $m = 8$.

### 6. Conclusion

We presented a way to get an approximated value for the number of non-equivalent LFTs that are injective with some delay $\tau$. We also give a recurrence relation to determine the number of canonical LFTs, and show how to get an approximated value for the percentage of equivalence classes formed by injective LFTs.

From the experimental results presented in the previous section we may draw two very important conclusions. First, that the number of injective equivalence classes is very high and seems to grow exponentially as the structural parameters $\ell$ and $n$ increase. This implies that a brute force attack to the linear part of the key space may not be feasible. Second, that the percentage of equivalence classes of $\omega$-injective LFTs, with structural parameters $\ell, m, n$, is very high, for values of $n$ slightly larger than $\ell$ and $m$. The LFTs used in Cryptography satisfy the condition $n = h\ell + km$, where $h, k \in \mathbb{N} \setminus \{0\}$, which guarantees that $n$ is large enough so that there is a very high percentage of $\omega$-injective LFTs of that size. Therefore, random generation of LFTs is a feasible option to generate keys.

These results constitute an important step towards the evaluation of the key space. A similar study is required for the non-LFTs used in the FAPKCs.

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