Low Frequency Characteristics of a Two-Degree-of-Freedom Vibration System with Complex Rigid Constraints

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Abstract. The dynamic model of a two-degree-of-freedom impact vibration system with complex rigid constraints under the action of periodic excitation force is established. Through the multi-objective and multi-parameter numerical simulation, the types and distribution areas of the periodic impact motions of the system in the $(\omega, \delta)$-parameter plane are obtained. The transition law of the fundamental impact motions of the system in the low frequency area is discussed, the generation mechanism of the tongue-shaped regions appearing in the transition process and the types of subharmonic motions in the tongue-shaped regions are analyzed.

1. Introduction

Due to the requirements of production and installation and the limitation of processing accuracy, there are unavoidable clearances and constraints between various parts of the mechanical system and between parts and fixed boundaries, as a result, impact motions occur when the mechanical system is working. In the actual mechanical vibration system, there are often many constraints and clearances, which makes the dynamic characteristics of the mechanical system more and more complicated. How to reveal its dynamic characteristics under complex constraints is particularly important. This provides theoretical guidance for effectively using the impact characteristics of the mechanical system and avoiding the harm caused by impact motions. Scholars at home and abroad have conducted in-depth research from many aspects such as qualitative analysis, numerical simulation, and experimental verification. Shaw [1] studied a class of single-degree-of-freedom vibration systems and found that there are complex dynamic phenomena such as basic periodic impact motions, subharmonic impact motions and chaos, and discussed the possible singularity caused by the grazing bifurcation, which makes smooth dynamic research methods cannot be directly applied to the bifurcation study of non-smooth systems. The singularity of the grazing bifurcation will produce new dynamic behaviors, and the relationship between the grazing bifurcation and the saddle-node bifurcation has also been studied [2-4]. Shock and vibration systems sometimes exhibit the chattering-impact motions with sticking property in the low frequency regions [5-7]. Luo [8-9] studied the impact vibration system with one-sided rigid constraint and double-sided rigid constraint respectively, and defined the basic periodic motion group $p/1$. Wiercigroch [10-12] studied nonlinear dynamic behavior in cutting systems.

In this paper, for a two-degree-of-freedom mechanical vibration system with complex rigid constraints, a multi-parameter co-simulation research method is used to analyze the existence areas and transition law of fundamental impact motions and subharmonic motions within the the $(\omega, \delta)$-parameter plane.
2. Mechanical Model

Fig. 1 shows a mechanical model of a two-degree-of-freedom vibration system with complex rigid constraints. The masses \( M_i \) \((i=1,2)\) are connected to the supporting foundation by linear springs \( K_i \) and linear dampers \( C_i \). The harmonic excitation forces \( P_i \sin(\Omega t + \tau) \) act on the masses \( M_i \), where \( P_i, \Omega \) and \( \tau \) represent the excitation forces amplitudes, excitation frequency, and phase angle respectively. The displacement of the masses \( M_i \) is \( x_i \), when the displacement of the mass \( M_i \) is \( |x_i| = B_i \), the mass \( M_i \) hits the rigid constraint \( A1 \); Similarly, when the displacement of the mass \( M_2 \) is \( x_2 = -B_2 \), the mass \( M_2 \) hits the rigid constraint \( A2 \); when the relative displacement of \( M_1 \) and \( M_2 \) is \( x_1 - x_2 = B_2 \), a rigid collision occurs between the two masses. The recovery coefficient of rigid collision is \( R \), and the transient velocities before and after collision of masses \( M_i \) are expressed by \( \dot{x}_i \) and \( \ddot{x}_i \) respectively.

\[
\delta_i = \frac{B_i K_1}{P_1 + P_2}, \quad f = \frac{P_2}{P_2 + R}, \quad i = 1, 2
\]  
(1)

The system’s non-dimensional motion differential equation under the action of periodic excitation force can be described by following equations:

\[
x_i + 2\zeta \dot{x}_i + x_i = (1-f) \sin(\omega t + \tau),
\]

\[
\frac{\mu_m}{1-\mu_m} \dddot{x}_1 + \frac{\mu_m}{1-\mu_m} \dddot{x}_2 + x_2 = f \sin(\omega t + \tau), \quad |x_1| < \delta_1, \quad x_2 > -\delta_1, \quad |x_1 - x_2| < \delta_2
\]  
(2)

\[
\dot{x}_{i+} = -R \dot{x}_{i-} \quad (|x_i| = \delta_i), \quad \ddot{x}_{i+} = -R \ddot{x}_{i-} \quad (x_2 > -\delta_i)
\]  
(3)

\[
\frac{\mu_m}{1-\mu_m} \dddot{x}_{i+} + \dddot{x}_{i+} = \frac{\mu_m}{1-\mu_m} \dddot{x}_{i-} + \dddot{x}_{i-}, \quad R = (\dot{x}_{i+} - \dot{x}_{i-}) / (\ddot{x}_{i+} - \ddot{x}_{i-}), \quad (x_2 - x_1) = \delta_i
\]  
(4)

Under complex gap conditions, it is very difficult to study the impact vibration mode of the system according to the masses \( M_i \), especially for the mass \( M_1 \) has a two-sided collision at constraint \( A1 \), and at the same time there is a one-sided collision with \( M_2 \) at constraint \( A12 \). For convenience, the entire system is described in terms of the number of impacts of the masses \( M_i \) on the constrains. For the one-sided collision at the constraints \( A12 \) and \( A2 \), the symbol \( p/n \) is used to describe, for the two-sided collision at the constraints \( A1 \), the symbol \( n-p-q \) is used to describe, where \( n \) represents the number of periods of the excitation force of the system within one vibration period \((n = 1, 2, 3, \ldots); p \) represents the number of impacts at one-sided constraint \((p = 0, 1, 2, 3, \ldots); p \) and \( q \) represent the
number of impacts at the left and right sides of two-sided constraint \((p, q = 0, 1, 2, 3, \ldots)\). To determine the \(n, p,\) and \(q\) of each constraint, the following Poincaré sections are established:

\[
\sigma_n = \{(x_1, x_1, x_2, x_2, t) \in R^4 \times T | x_i = x_{i\text{min}}, \text{mod}(t = 2\pi/\omega)\},
\]

\[
\sigma^L_{AI} = \{(x_1, x_1, x_2, x_2, t) \in R^4 \times T | x_i = \delta_x, x_i > 0\},
\]

\[
\sigma^R_{AI} = \{(x_1, x_1, x_2, x_2, t) \in R^4 \times T | x_i = -\delta_x, x_i < 0\},
\]

\[
\sigma_{AI2} = \{(x_1, x_1, x_2, x_2, t) \in R^4 \times T | x_1 - x_2 = \delta_x, x_1 - x_2 > 0\},
\]

\[
\sigma_{A2} = \{(x_1, x_1, x_2, x_2, t) \in R^4 \times T | x_1 = -\delta_x, x_1 < 0\}
\]  \hspace{1cm} (5)

Combined with the number of fixed points at the established Poincaré section, the number of cycles and the number of collisions can be judged. The Poincaré section \(\sigma_n\) is the minimum displacement of the mass \(M_1\) in the period of the excitation force, the number of fixed points on the Poincaré section represents the period of the excitation force; The number of fixed points on the Poincaré sections \(\sigma^L_{AI}\) and \(\sigma^R_{AI}\) represents the number of impacts of the mass \(M_1\) on the left and right sides of the constraint A1, respectively. The number of fixed points on the Poincaré sections \(\sigma_{A2}\) represents the number of impacts of the mass \(M_2\) on the constraint A2. The number of fixed points on the Poincaré sections \(\sigma_{AI2}\) represents the number of mutual impacts between the mass \(M_1\) and the mass \(M_2\).

The impact Poincaré map of the vibratory systems can be simply expressed as

\[
X^{(i+1)} = f(X^{(i)}, \mu)
\]

where: \(X^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_1^{(i+1)}, x_2^{(i+1)})^T, X^{(i+1)} = (x_1^{(i+1)}, x_2^{(i+1)}, x_1^{(i+1)}, x_2^{(i+1)})^T, X \in R^4, \mu\) are parameters, \(\mu \in R^m, m=8\).

3. Types and Distribution Areas of the Periodic Impact Motions of the Vibration System under Complex Constraints

According to equation (1), there are five non-dimensionalized parameter values that can be easily determined as \(\mu_m \in (0, 1), \mu_k \in (0), \mu_\zeta \in (0, 1), f \in [0, 1], R \in [0, 1].\) In this model, the clearances can be divided into two categories, that is, the clearance \(\delta_1\) between the two masses and the fixed constraints, the clearance \(\delta_2\) between the two masses. Consider the situation when \(\delta_1\) takes a fixed value and \(\delta_2 = \delta\) changes continuously. Select the parameters \(\mu_m = 0.5, \mu_k = 0.5, \mu_\zeta = 0.5, f = 0\) as the reference parameters. The excitation frequency \(\omega\) and the clearance \(\delta\) between the two masses are used as bifurcation parameters. Through multi-parameter co-simulation, the distribution areas of the system's periodic impact vibration on the \((\omega, \delta)\)-parameter plane, the transition law between the adjacent periodic impact vibrations and the bifurcation characteristics can be obtained. On the \((\omega, \delta)\)-parameter plane, different colors are used to distinguish different types of motions, and \(p\) is used to identify one-sided collisions, \(n-p-q\) is used to identify two-sided collisions, and unmarked gray areas are mainly quasi-period and chaotic motions, or periodic motions in which the number of cycles or the number of collisions are too large and the area are too small.

Under the conditions of the selected reference parameters, when \(f = 0\), the entire excitation force is applied to the mass \(M_1\). Under this condition, the mass \(M_1\) moves under the action of the excitation force within the displacement range limited by the constraint A1; The mass \(M_1\) and the mass \(M_2\) generate impact vibration between each other at the constraint A12, and drive the mass \(M_2\) to move
through the impact vibration. Due to the limited displacement of mass $M_1$ and no exciting force on mass $M_2$, there is no impact vibration on constraint A2. The impact vibration at constraint A1 and constraint A2 are mainly studied under this condition.

![Figure 2](image1.png)

**Figure 2.** Existence regions of different types of periodic-impact motions in the $(\omega, \delta)$-parameter plane

![Figure 3](image2.png)

**Figure 3.** Bifurcation diagrams and time series of complete chattering-impact motions: (a) $\delta = 0.25$ bifurcation diagram $\dot{x}_1 - \dot{x}_2(w)$; (b) $\delta = 0.25$ bifurcation diagram $x_{1,\text{imp}}(w)$; (c) $\delta = 0.25, \omega = 0.3$ time series; (c) $\delta = 0.25, \omega = 0.1$ time series; (c1) and (d1) detail of (c) and (d) respectively.

In the case of large clearance range and high excitation force frequency, the system is the impactless 1-0-0 at constraint A1 and the impactless 0/1 at constraint A12, as shown in Fig.2. The system presents the fundamental group of impact motion in the range of low exciting frequency. The fundamental group of impact motions means that they have the same number of excitation force cycles ($n = 1$) and have different number of impacts $p$ (or $q$) = 0, 1, 2, ..., $m$ ... i.e., $p$ (or $q$)/1 impact motions. At constraint A1, when the clearance $\delta$ is small, it is mainly shows as the impact of mass $M_1$ on the left side of constraint A1, as the clearance increases, it changes into a two-sided impact state; In the low-frequency region, fundamental 1-0-0 is mainly caused by one-sided impact. With the $\omega$ decrease, along with the Grazing bifurcation, the number of impacts $p$ increases one by one, and gradually shifts to incomplete chatting-impact vibration. The basic periodic motion 1-$p$-0 in the $\omega$-parameter plane band is narrow and not easy to observe. When the $\omega$ further decreases and passes the sliding bifurcation boundary of incomplete chatting-impact vibration, the incomplete
chatting-impact vibration shifts to the complete chatting-impact vibration with sticking occurs on the left side of constraint A1.

At constraint A12, the impact vibration between the mass $M_1$ and $M_2$ each other shows more typical impact dynamics in the low-frequency region. Complex and regular subharmonic vibration groups may exist on the boundary lines of fundamental motions $p/1$ and $(p+1)/1$, which are generated by the interaction between fundamental motions $p/1$ and subharmonic motions $p/n$. In the $(\omega, \delta)$-parameter plane of constraint A12, As shown in Fig.2(b), with the decrease of excitation frequency $\omega$ in the low-frequency region accompanied by the occurrence of Grazing bifurcation, the number of impacts $p$ of the fundamental $p/1$ motions increases by one and transition to the $(p+1)/1$ motions. When the number of impact vibrations $p$ increases to a sufficiently large value and the minimum impact velocity is not yet zero, the system will enter incomplete chatting-impact vibration $p/1$. When the number of impacts $p$ is large enough and the minimum impact velocity decays to zero, the system crosses the sliding bifurcation line and enters complete chatting-impact vibrations. It is represented by CIS/1 in the parameter plane. In Fig.3 we can see that when the clearance value is $\delta = 0.25$, the bifurcation diagram of the relative impact velocity $\dot{x}_1 - \dot{x}_3$ of the system at the constraint A12 with respect to the excitation force frequency $\omega$, where the red dots indicate the complete chattering-impact motions with sticking property. The complete chattering-impact motions with sticking property shows the characteristic of “bulge” on the bifurcation diagram. In the sticking region of the bifurcation diagram, select the time series when takes $\omega = 0.3$ and $\omega = 0.1$ respectively. In Fig.3(c),(d), and its enlarged view Fig.3(c1),(d1), it can be clearly seen that as the excitation force frequency $\omega$ decreases the time series of the system within a single excitation force sticking time gradually expands.

In the whole transition process, the fundamental group of impact motions $p/1(p \geq 1)$ appears as a gradually narrowed band in the low frequency region, and the band from $p=1$ to $p=10$ is depicted in the $(\omega, \delta)$-parameter plane. It can be found that with the gradual decrease of the excitation force frequency $\omega$, the transition law of the fundamental group of impact motions and the incomplete and complete chatting-impact vibration can be summarized as follows:

$$
\omega \downarrow p/1 \leftarrow \ldots \leftarrow p/1 \leftarrow \ldots \leftarrow (p+1)/1 \leftarrow p/1 \leftarrow \ldots \leftarrow 4/1 \leftarrow 3/1 \leftarrow 2/1 \leftarrow 1/1
$$

**Figure 4.** Partial description of Fig.2(b)- tongue-shaped regions during the transition from the fundamental group of impact motions $p/1$ to $(p+1)/1$

Fig. 4 is a partial description of Fig. 2(b), it can be seen that under the interaction of complex clearances, at the constraint A12, there are many complex and regular tongue-shaped regions on the boundary line of fundamental $p/1$ transfer to $(p+1)/1$. The tongue-shaped region is surrounded by the bare-grazing bifurcation line of fundamental $p/1$ motion on the upper boundary and the period doubling line of fundamental $(p+1)/1$ motion on the lower boundary, the main subharmonic shock
vibration contained in the form is $(np + 1)/n$. Taking the tongue-shaped region during the transition of the fundamental vibration from 4/1 to 5/1 as an example, it can be observed from a partially enlarged view of the tongue-shaped domain in Fig.4(b) that the upper boundary is the bare-grazing bifurcation line $G_{4/1}$ of the fundamental vibration 4/1 and the lower boundary is the period doubling bifurcation line $PD_{5/1}$ of fundamental 5/1. The tongue-shaped region contains subharmonic vibrations 10/2, 9/2, 13/3, 17/4, 18/4 where the 10/2 motion is generated by the period doubling bifurcation of the 5/1 motion, the 9/2 motion and the 10/2 motion are shifted from each other by the grazing bifurcation and the saddle-node bifurcation. Fig.5 is a bifurcation diagram when longitudinally traversing the tongue-shaped domain with $\delta$ as the bifurcation parameter when selecting $\omega = 0.228$. Fig. 5(a) is a bifurcation diagram of the relative impact velocities $\dot{x}_1 - \dot{x}_2$ of the mass $M_1$ and mass $M_2$ with respect to the clearance $\delta$ at the constraint A12, where the number of branches can determine the number of impacts $p$; Fig. 5(b) is the bifurcation diagram of the minimum displacement $x_{imp}$ of the mass $M_1$ in one excitation force period with respect to the clearance $\delta$, where the number of branches can determine the number of excitation force periods of the system.

**Figure 5.** Bifurcation diagram of the system when it longitudinally traverses the tongue-shaped region between 4/1 and 5/1 motions, $\omega = 0.228$: (a) bifurcation diagram $\dot{x}_1 - \dot{x}_2(\delta)$; (b) bifurcation diagram $x_{imp}(\delta)$.

### 4. Conclusion

A two-degree-of-freedom impact vibration system with complex rigid constraints under the action of periodic excitation force will impact with multiple collision surfaces, which will produce complex impact dynamics. When the clearance $\delta$ and the periodic excitation force frequency $\omega$ are used as variables to study the motion mode and distribution areas of the system, the $(\omega, \delta)$-parameter plane at constraint A12 exhibits more typical transition laws and bifurcation characteristics. In the low-frequency region, the fundamental impact motion $p/1$ at the constraint A12 crosses the real-grazing bifurcation line as the excitation force frequency $\omega$ decreases, and the number of impact motion $p$ increases once to become $(p+1)$. When the number of impacts $p$ grows large enough and the impact velocity is still not zero, the system enters incomplete chatting-impact vibration. When the excitation force frequency $\omega$ continues to decrease and the system crosses the sliding bifurcation boundary of incomplete chatter-impact vibration, the complete chattering-impact motions with sticking property appear, the impact masses finally become constrained to the stop in the excitation period. As the excitation force frequency $\omega$ decreases, the time series of the system within a single excitation force sticking time gradually expands. During the transition of the basic periodic shock vibration from $p/1$ to $(p+1)/1$, tongue-shaped regions containing subharmonic motions are interspersed. The tongue-shaped regions’ upper boundary is bare-grazing bifurcation line of
fundamental $p/l$ motion and lower boundary is the period doubling line of fundamental $(p+1)/l$ motion. Among tongue-shaped regions the main form of subharmonic motions is $(np+1)/n$.

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