Maximally Chaotic Dynamical Systems of Anosov—Kolmogorov

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Abstract—The maximally chaotic K-systems are dynamical systems which have nonzero Kolmogorov entropy. On the other hand, the hyperbolic dynamical systems that fulfill the Anosov C-condition have exponential instability of phase trajectories, mixing of all orders and positive Kolmogorov entropy. Therefore the C-condition defines a rich class of maximally chaotic systems which span an open set in the space of all dynamical systems. The interest in Anosov—Kolmogorov C–K systems is associated with the attempts to understand the relaxation phenomena, the foundation of the statistical mechanics, the appearance of turbulence in fluid dynamics, the non-linear dynamics of the Yang—Mills field, the N-body system in Newtonian gravity and the relaxation phenomena in stellar systems and the Black hole thermodynamics. In this respect of special interest are C–K systems that are defined on Reimannian manifolds of negative sectional curvatures and on high-dimensional tori. The classical- and quantum-mechanical properties of maximally chaotic dynamical systems, the application of the C–K theory to the investigation of the Yang—Mills dynamics and gravitational systems as well as their application in the Monte Carlo method will be reviewed.

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1. INTRODUCTION

It seems natural to define the maximally chaotic dynamical K-systems as systems that have nonzero Kolmogorov entropy [1, 2]. A large class of maximally chaotic dynamical systems was constructed by Anosov [3]. These are the systems that fulfill the C-condition. The Anosov C-condition leads to the exponential instability of phase trajectories, to the mixing of all orders and positive Kolmogorov entropy. The examples of maximally chaotic systems were discussed in the earlier investigations [4–15] as well as in [16–25].

In recent years the quantum-mechanical concept of maximally chaotic systems was developed in series of publications [26–33] and references therein. It is based on the analysis of the black holes evaporation thermodynamics and on the investigation of the so called out-of-time-order correlation functions conjectured to diverge exponentially with the exponent linear in temperature and time [26–29]. The interrelation between the classical and quantum mechanical concepts of chaos will be discussed in the next sections.

The review is organised as follows. In the second section the classification of the dynamical systems (DS) by the increase of their statistical-chaotic properties will be presented [14, 15]. These are ergodic, n-fold mixing and finally the K-systems, which have mixing of all orders and nonzero Kolmogorov entropy. The question is: Do the maximally chaotic systems exist?

The Anosov hyperbolic C-systems represent a large class of K-systems defined on the Riemannian manifolds of negative sectional curvatures and on high-dimensional tori. The general properties of the C-systems will be discussed in the third section. From the C-condition follows the strong instability of the phase trajectories and, in fact, the instability is as strong as it can be in principle [3, 9]. The distance between infinitesimally close trajectories increases exponentially and on a closed phase space of the dynamical system this leads to the uniform distribution of almost all trajectories over the whole phase space.

The hyperbolic geodesic flow on closed Riemannian manifolds of negative sectional curvatures will be considered in the fourth section [3, 6–8]. It was proven by Anosov that manifolds with negative sectional curvatures fulfill the C-condition and therefore define a large class of maximally chaotic K-systems. This result provides a powerful tool for the investigation of the Hamiltonian systems.

In the fifth section the classical and quantum dynamics of the Yang—Mills fields is reviewed [27, 28, 30–39, 50, 51]. In the case of space homogeneous gauge fields the Yang—Mills dynamics reduces to the classical-mechanical model, the Yang—Mills classical mechanics (YMCM) [34–39]. The sectional curvatures are negative on the equipotential surface and generate exponential instability of the phase trajectories. The question is to what extent the classical chaos influences the quantum-mechanical properties of the gauge fields. It is an example of fundamental quan-
tum-mechanical matrix system, the so called Yang–Mills quantum mechanics—YMQM [38, 39].

The other application of the C–K systems theory was found in the investigation of the relaxation phenomena in stellar systems like globular clusters and galaxies [52–55]. The Maupertuis’s metric is used to reformulate the evolution of N-body system in Newtonian gravity as a geodesic flow on a Riemannian manifold. Investigation of the sectionnal curvature allows to estimate the average value of the exponential divergency of the phase trajectories, the correlation splitting time \( \tau_0 \) and the relaxation time \( \tau = \tau_0 \log(1/\delta v) \) [66, 70] in elliptic galaxies and globular clusters [52]. This time is shorter than the Chandrasekhar’s binary relaxation time [53, 55] and the Hubble time.

Of special interest are continuous C–K systems which are defined on the two-dimensional surfaces embedded into the hyperbolic Lobachevsky plane [43–46]. An example of such system has been defined in a brilliant article published in 1924 by the mathematician Emil Artin [4]. The differential geometry and group-theoretical methods [56] are used to estimate the correlation splitting time \( \tau_0 \) of the classical correlation functions [43].

The quantum-mechanical properties of maximally chaotic systems are of special interest [27–29, 38, 39, 43, 44]. In the eighth section the quantisation of the Artin system, the derivation of the Maass wave function [57], that describes the continuous spectrum, and the behaviour of the quantum-mechanical correlation functions will be discussed [43, 44]. The spectral problem of quantum Artin system has deep number-theoretical origin and was investigated in a series of pioneering articles [57–65]. The that two- and four-point correlation functions decay exponentially in time, reminiscient to the correlation splitting time \( \tau_0 \) of the system in the classical regime [43]. The commutator \( C(\beta, t) \) grows exponentially [44], mimicking the expansion of the initial phase space volume \( \delta v \) over the whole phase space with an exponential rate \( \tau = \tau_0 \log(1/\delta v) \) [66, 70].

In the ninth section we shall demonstrate that the Riemann zeta function zeros [40] define the position and the widths of the resonances of the quantised Artin system [45]. A possible relation of the zeta function zeros and quantum-mechanical spectrum was discussed in the past: David Hilbert seems to have proposed the idea of finding a system whose spectrum contains the zeros of the Riemann \( \zeta(s) \) function [46]. The poles of the S-matrix are located in the complex plane and are expressed in terms of zeros \( u_n \) of the Riemann zeta function \( \zeta \left( \frac{1}{2} - i u_n \right) = 0, \ n = 1, 2, \ldots \)

\[
E = E_n - \frac{\Gamma_n}{2} = \left( \frac{u_n^2}{4} + \frac{3}{16} \right) - \frac{i u_n^2}{2},
\]

where \( E_n \) is the energy and \( \Gamma_n \) is the width of the \( n \)th resonance [45].

In the tenth section the attention will be turn to the investigation of the C–K systems that are defined on high-dimensional tori [3]. It was suggested in 1986 [66] to use the C–K systems to generate high quality pseudorandom numbers for Monte-Carlo simulations [67–71]. The high entropy MIXMAX generator based on C–K system was implemented into the Geant4/CLHEP and ROOT scientific toolkits at CERN [71–74].

2. HIERARCHY OF DYNAMICAL SYSTEMS.

KOLMOGOROV ENTROPY

In ergodic theory the dynamical systems (DS) are classified by the increase of their statistical-chaotic properties [14, 15]. Let \( \mathbf{x} = (q_1, \ldots, q_d, p_1, \ldots, p_d) \) be a point of the phase space \( x \in M \) of the Hamiltonian systems that is equipped with a positive Liouville measure \( d\mu(x) = \rho(q, p) dq_1 \ldots dq_d dp_1 \ldots dp_d \), which is invariant under the Hamiltonian flow. The operator \( T^t \mathbf{x} = x \) defines the time evolution of the trajectories. The n-fold mixing takes place if for any number of sets \( A_1, \ldots, A_n \subset M \) the

\[
\lim_{t_1, \ldots, t_n \to \infty} \mu[A_1 \cap T^{t_1} A_2 \cap \ldots \cap T^{t_n} A_n] = \mu[A_1] \mu[A_2] \ldots \mu[A_n] \mu[B] \text{ or alternatively} [10, 14]
\]

\[
\mathbb{D}(f_n, \ldots, f_1) = \lim_{t_1, \ldots, t_n \to \infty} \left\{ f_n(T^{t_n} x) \ldots f_1(T^{t_1} x) \right\} - \left\{ f_n(x) \ldots f_1(x) \right\} = 0. \tag{2.1}
\]

A class of dynamical systems which have even stronger chaotic properties was introduced by Kolmogorov in [1, 2]. These are the DS that have a non-zero entropy, so called quasi-regular DS, or K-systems. Let \( \mathcal{A} = \{ A_i \}_{i=1}^n \) (\( n \) is finite or countable) be a measurable partition of the phase space \( M \) into the non-intersecting subsets \( A_i \), which cover the whole phase space \( M \) and then define the entropy of the partition \( \alpha \) as

\[
\alpha = \sum_{i=1}^n \mu(A_i) \ln \mu(A_i).
\]

The entropy of the partition \( \alpha \) with respect to the discrete evolution \( T^n \) is defined as a limit

\[
\alpha(T) = \lim_{n \to \infty} h(\alpha, T^n \alpha) = \lim_{n \to \infty} \frac{h(\alpha \vee T^\alpha \vee \ldots \vee T^{n-1} \alpha)}{n},
\]

where \( n = 1, 2, \ldots \) [1, 2, 12, 16, 18]. This number is equal to the entropy of the partition \( \beta = \alpha \vee T^\alpha \vee \ldots \vee T^{n-1} \alpha \) generated by the iteration of the partition \( \alpha \) by the evolution operator \( T \). The entropy of \( T \) is defined as a supremum:

\[
h(T) = \sup_{\alpha \geq 0} h(\alpha, T^n \alpha),
\]

\( \sup \) mixing \( \alpha \), \( n \)-fold mixing, \( \sup \) mixing \( \sup \) ergodicity [1, 2, 12, 16, 18, 21, 25]. The question is: Do maximally chaotic systems exist? The Anosov hyperbolic C-systems [3] represent a large class of K-systems considered in the next section.
3. HYPERBOLIC ANOSOV C-SYSTEMS

In the fundamental work on geodesic flows on closed Riemannian manifolds \( Q^n \) of negative sectional curvatures [3] Anosov pointed out that the basic property of the geodesic flow on such manifolds is the uniform exponential instability of phase trajectories. The exponential instability of geodesics was studied by Lobachevsky, Hadamard, Artin [4], Hedlund [6] and Hopf [8] and others. The concept of exponential instability appears to be extremely rich, and Anosov suggested to elevate it into a fundamental property of a new class of dynamical systems which he called C-systems\(^1\). The brilliant idea to consider dynamical systems which have local and homogeneous hyperbolic instability of phase trajectories is appealing to the intuition and has very deep physical content. The richness of the concept is expressed by the fact that C-systems occupy a nonzero volume in the space of dynamical systems [3]. Anosov provided an extended list of C–K systems, these are: (i) C-cascades and (ii) the geodesic flow on the Riemannian manifolds of variable negative sectional curvatures [3].

The C-cascade on a d-dimensional compact phase space \( M^d \) is induced by the diffeomorphisms \( T: M^d \rightarrow M^d \) [3]. The iterations are defined by a repeated action of the operator \( T^n, -\infty < n < +\infty \), where \( n \) is an integer number. The tangent space at the point \( x \in M^d \) is denoted by \( R^d_x \). The C-condition requires that the tangent space \( R^d_x \) at each point \( x \) of the d-dimensional phase space \( M^d \) of the dynamical system \( \{ T^n \} \) should be decomposable into a direct sum \( R^d_x = X^k_x \oplus Y^l_x \) of two linear spaces \( X^k_x \) and \( Y^l_x \) with the following properties [3]:

\[
\begin{align*}
\text{a)} & \, \left| \hat{T}^n \xi \right| \leq a \left| \xi \right| e^{-cn} \text{ for } n \geq 0; \\
\text{b)} & \, \left| \hat{T}^n \eta \right| \geq b \left| \eta \right| e^{cn} \text{ for } n \leq 0, \eta \in X^k_x, \\
\text{c)} & \, \left| \hat{T}^n \eta \right| \geq a \left| \eta \right| e^{cn} \text{ for } n \geq 0; \\
\text{d)} & \, \left| \hat{T}^n \eta \right| \leq b \left| \eta \right| e^{cn} \text{ for } n \leq 0, \eta \in Y^l_x,
\end{align*}
\]

(3.2)

where the constants \( a, b \) and \( c \) are positive and are the same for all \( x \in M^d \) and all \( \xi \in X^k_x, \eta \in Y^l_x \). The linear spaces \( X^k_x \) and \( Y^l_x \) are invariant with respect to the derivative mapping \( \hat{T}^n X^k_x = X^k_{T^n x}, \hat{T}^n Y^l_x = Y^l_{T^n x} \) and represent the exponentially contracting and expanding linear spaces (see Fig. 1). The \( X^k_x \) and \( Y^l_x \) are the target vector spaces to the contracting and expanding foliations \( \Sigma^k_x \) and \( \Sigma^l_x \) [3]. The C-condition represents a powerful criterion for the identification of maximally chaotic DS and the main task reduces to the examination of the C-condition in each particular case.

4. THE HYPERBOLIC C-SYSTEMS ON RIEMANNIAN MANIFOLDS

Let us consider a Riemannian manifold \( Q \) with the local coordinates \( q^\alpha(s) \in Q, \alpha = 1, 2, \ldots, 3N \) and the velocity vector \( u^\alpha = dq^\alpha ds, \alpha = 1, 2, \ldots, 3N \). The proper time \( s \) along the \( q^\alpha(s) \) is equal to the length, while the Riemannian metric on \( Q \) is defined as \( ds^2 = g_{ab} dq^a dq^b \), and therefore \( g_{ab} u^a u^b = 1 \). The infinitesimal deformation of geodesics congruence \( q^\alpha(s, v) \) is defined as a vector \( \delta q^\alpha = \frac{d\delta q^\alpha}{dv} dv \). The resulting phase space manifold \((q(s), u(s)) \in M \) has a bundle structure with the base \( q \in Q \) and the spheres \( S^{3N-1} \) of unit tangent vectors \( u^\alpha \) as fibers. The geodesic equation has the form \( \frac{d^2 q^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dq^\beta}{ds} \frac{dq^\gamma}{ds} = 0 \), and the Jacobi equation for the relative acceleration depends on the Riemann curvature: \( \frac{D^2 \delta q^\alpha}{ds^2} = -R^\alpha_{\beta\gamma\delta} \delta q^\beta \delta q^\gamma \delta q^\delta \). Following Anosov it is convenient to define the deviation norm \( |\delta q|^2 \equiv g_{ab} \delta q^a \delta q^b \) and represent the Jacobi equation in the form

\[
\frac{d^2}{ds^2} |\delta q|^2 = -2K(q, u, \delta q) |u \wedge \delta q|^2 + 2|\delta u|^2,
\]

(4.3)

where \( K(q, u, \delta q) \) is the sectional curvature in the two-dimensional directions defined by the velocity vector \( u^\alpha \) and the deviation vector \( \delta q^\beta : K(q, u, \delta q) = \sum_{\alpha, \beta} g_{ab} \delta q^a \delta q^b \).
where the index \( a = 1, ..., N^2 - 1 \) for \( SU(N) \) group and the constraint has the form \( \hat{n} = [\bar{A}_a, A_i] = 0 \). It is natural to call this system the Yang–Mills Classical Mechanics (YMC) \([37–39]\). It is a mechanical system with \( d \cdot (N^2 - 1) \) degrees of freedom. The YMC has a number of conserved integrals: the space and isospin angular momenta and the energy integral (5.7). The question is if there exist additional conservation integrals. Performing the substitution \( A = O_iEO_i^T \), when \( d = 3, N = 2 \) and \( E = (x(t), y(t), z(t)) \) is a diagonal matrix and \( O_1, O_2 \) are orthogonal matrices the Hamiltonian (5.7) will take the form \([35–38]\)

\[
H_{FS} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{g^2}{2}(x^2 y^2 + y^2 z^2 + z^2 x^2) + T_{YM},
\]

where \( T_{YM} \) is the energy of the Yang–Mills “top” spinning in space and isospace. The YMCM is equivalent to the geodesic flow on a Riemannian manifold with the Maupertuis’s metric and the sectional curvatures on the equipotential surface \( x^2 y^2 + y^2 z^2 + z^2 x^2 = 1 \) are negative and generate the exponential instability of the trajectories. The question is: What are the quantum-mechanical properties of the classically chaotic system (5.7), (5.8) and what is the structure of its spectrum and of the wave functions? The Schrödinger equation of the Yang–Mills quantum-mechanical system (YMQM) represents a special class of quantum-mechanical matrix models \([38, 39]\). The ground state wave function \( i(x) = \Phi(x)\sqrt{D(x)} \) fulfills the equation \([39]\)

\[
\Phi = E\Phi,
\]

where \( D(x) = \left| (y-x_1)^2 (y-x_2)^2 (y-x_3)^2 \right| \). The analytical investigation of Eq. (5.9) is a challenging problem because it cannot be solved by separation of variables as far as all canonical symmetries are already extracted and the residual system possesses no continuous symmetries. The energy spectrum of YMQM system is discrete \([38, 39]\), the energy levels “repulse” similar to the eigenvalues of the matrices with randomly distributed elements \([47–49]\). In the next section the gravitational \( N \)-body problem will be presented \([52]\).
6. COLLECTIVE RELAXATION OF STELLAR SYSTEMS

The $N$-body problem in Newtonian gravity can be formulated as a geodesic flow on Riemannian manifold with the conformal Maupertuis metric $ds^2 = (E - U)d\rho^2 = W \sum_{\alpha=1}^{3N} (dq^\alpha)^2$, where $W = E - U$, the $\{q^\alpha\}$ are the coordinates of the stars and $U = G \sum_{a \neq b} \frac{M_a M_b}{|r_a - r_b|^2}$. The sectional curvature has the form

$$R_{ab\mu\nu} = \left[ - \frac{1}{3N} \frac{\partial W}{\partial q^a} \delta q^a_{\mu} \delta q^b_{\nu} - \theta \right].$$

(6.10)

The number of stars in a galaxy is very large, $N \gg 1$ and the dominant term in sectional curvature (6.10) is negative: $K(q, u, \delta q) = -\frac{1}{4} \left( \frac{V W}{W^3} \right)^2 < 0$. The deviation Eq. (4.4) will take the form $\frac{d^2}{dt^2} |\delta q_1|^2 \geq \left( \frac{V W}{W^3} \right)^2 |\delta q_1|^2$ and the correlation splitting time $\tau_0$ can be defined as

$$\tau_0 = \frac{W}{\sqrt{V W}} [66, 70, 52].$$

For the elliptic galaxies and globular clusters it is

$$\tau_0 \approx 10^8 \text{yr} \left( \frac{V}{10 \text{ km/s}} \right) \left( \frac{n}{1 \text{ pc}^{-3}} \right)^{-2/3} \left( \frac{M}{M_\odot} \right)^{-1}.$$  

(6.11)

The relaxation time $\tau = \tau_0 \log(1/\delta v)$ is by few orders of magnitude shorter than the Chandrasekhar’s binary relaxation time [53–55] and is less than the Hubble time.

7. CORRELATION FUNCTIONS OF CLASSICAL ARTIN SYSTEM

The Artin system [4] is defined on the fundamental region $\mathcal{F}$ of the Lobachevsky plane obtained by the identification of points congruent with respect to the modular group $SL(2, Z)$, a discrete subgroup of the Lobachevsky plane isometries $SL(2, R)$. The fundamental region $\mathcal{F}$ in this case is a hyperbolic triangle [43–45] shown on Fig. 2. The time evolution of the physical observables $\{f(x, y, \theta)\}$ is defined on the phase space $(x, y, \theta) \in M$, where $\tau = x + iy \in F$ and $\theta \in S^1$ is a direction of a unit velocity vector. The Liouville measure is $d\mu = \frac{dx dy}{y^2} d\theta$ and the invariant product of functions is

$$(f_1, f_2) = \int_0^{2\pi} d\theta \int_{F} f_1(x, y, \theta) f_2(x, y, \theta) \frac{dx dy}{y^2} [41, 56].$$

A two-point correlation function (2.1) has the form [43]:

$$\langle \mathcal{D}_{ij}(f_1, f_2) \rangle \equiv \int_0^{2\pi} \int_{F} f_1(x, y, \theta) \times f_2(x', y', \theta, \tau, \theta', \theta') \frac{dx dy}{y^2} d\theta.$$  

(7.12)

We found that the upper bound on the correlations functions is [43]

$$|\mathcal{D}_{ij}(f_1, f_2)| \leq C_{ij} e^{-2K|l|},$$  

(7.13)

where the Poincaré metric is $ds^2 = \frac{dx^2 + dy^2}{Ky^2}$ and $N$ is the weight of the automorphic functions. The correla-
tion splitting time is therefore \( \tau_0 = \frac{2}{NK} \). Considering a set of initial trajectories occupying a small volume \( \delta V \) in the phase space of a C—K system, one can ask how fast this small phase volume will be uniformly distributed over the whole phase space \([66, 70]\). This characteristic time interval \( \tau \) defines the relaxation time at which the system reaches a stationary distribution \([66, 70]\). Because the entropy defines the expansion rate of the phase space volume it follows that \( \tau = \tau_0 \log(1/\delta V) \) with large hierarchy between \( \tau_0 \) and \( \tau \) \([66, 70]\).

8. CORRELATION FUNCTIONS OF QUANTUM ARTHIN SYSTEM

As we have seen, the classical correlation functions decay exponentially \([43, 70]\), and our aim is to investigate the behaviour of the quantum-mechanical correlation functions \([44, 45]\). It was conjectured \([27]\) that the behaviour of the out-of-time-order correlation functions play a crucial role in diagnosing the classical chaos. To investigate the correlation functions it is necessary to know the spectrum and the corresponding wave functions. The spectral problem has deep number-theoretical origin \([57–65]\). The continuous spectrum is described by the Maass wave function \([57]\) and has the form \([44, 45]\)

\[
\psi_p(x, y) = e^{-ip\varphi} + \frac{\theta(1 + ip)}{\theta(2 - ip)} e^{ip\varphi} + \frac{4}{\theta(1 + ip)} \sum_{l=1}^{\infty} \tau_{ip}(l) K_{ip}(2\pi l^2 \xi) \cos(2\pi lx),
\]

(8.14)

where \( \int dy/\gamma = \ln \gamma = \tilde{y} \) is a physical distance on the fundamental triangle and the corresponding momentum is \( p_\gamma \). The \( e^{-ip\varphi} \) describes the incoming and \( e^{ip\varphi} \) the outgoing plane waves, the reflection amplitude is a pure phase \( S(p) = \frac{\theta(1 + ip)}{\theta(2 - ip)} = \exp[i\varphi(p)] \). where

\[
\theta(1 - ip) = \frac{\zeta(1 - 2ip)\Gamma(1 - ip)}{\pi^2}.
\]

The continuous energy spectrum is given by the formula \( E = p^2 + \frac{1}{4} \). The wave functions of the discrete spectrum were known only numerically \([59, 60, 65]\). The correlation functions defined in \([27]\) are:

\[
\mathcal{D}_4(\beta, t) = \langle A(t)B(0)e^{-\beta H} \rangle,
\]

\[
\mathcal{D}_4(\beta, t) = \langle A(t)B(0)A(t)B(0)e^{-\beta H} \rangle
\]

and \( C(\beta, t) = -\langle [A(t), B(0)]^2 e^{-\beta H} \rangle \). We were considering the Louiville type operators \( A \) and \( B \) of the form:

\[ A(N) = e^{-2\pi N}, N = 1, 2, \ldots \] \([44]\). We found that two- and four-point correlation functions decay exponentially in time, reminiscent to the correlation splitting time \( \tau_0 \) of the system in the classical regime \((7.13)\) \([43]\). The commutator \( C(\beta, t) \) grows exponentially \([44]\), mimicking the expansion of the initial phase space volume \( \delta V \) growing at an exponential rate \( \tau = \tau_0 \log(1/\delta V) \) \([66, 70]\).

9. RESONANCES AND RIEMANN ZETA FUNCTION ZEROS

Here we shall demonstrate that the Riemann zeta function zeros define the position and the widths of the resonances of the quantised Artin system \([45]\). In physical terms it is narrow infinitely long waveguide stretched out to infinity and a cavity resonator attached to it at the bottom (Fig. 2). As the energy of the incoming wave comes close to the eigenmodes of the cavity a pronounced resonance behaviour shows up in the scattering amplitude \([45]\). The position of the S-matrix poles is defined by the absence of the incoming waves in \((8.14)\)

\[
\theta(1 - ip_\gamma) = 0, \ [45]\)

and is determined by zeros of the Riemann zeta function \( \zeta(1 - ip_n) = 0, n = 1, 2, \ldots \) \([40]\). The location of poles is at complex momenta \( p_n = \frac{u_n}{2} - i \times \frac{1}{4} \), \( n = 1, 2, \ldots \), thus:

\[
E = p_n^2 + \frac{1}{4} = \frac{u_n^2}{4} + \frac{3}{16} - i \frac{u_n}{4}.
\]

(9.15)

These are the resonances \( E = E_n - i \frac{\Gamma_n}{2} \), where \( E_n = \frac{u_n^2}{4} + \frac{3}{16} \) and \( \Gamma_n = \frac{u_n}{2} \).

10. C-CASCADES AND MIXMAX RANDOM NUMBER GENERATOR

Let us consider the second class of the C—K systems defined on high-dimensional tori \([3]\) and their application in the Monte Carlo method \([66–71]\). The automorphisms of a torus are generated by the transformations \( x_i \to \sum_{j=1}^{n} T_{ij} x_j, (\text{mod} 1) \), where \( \text{Det} T = 1 \) and matrix \( T \) has no eigenvalues on the unit circle. The entropy of the Anosov automorphisms on a torus is equal to the sum \([3, 19, 20, 22–24]\):

\[
h(T) = \sum_{|\lambda| = 1} \ln |\lambda| \]

and depends on the spectrum of the operator \( T \). The strong instability of phase trajectories leads to the statistical behaviour of the Anosov C-systems \([17]\). The time average \( f_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \) behaves as a superposition of
quantities which are statistically weakly dependent and fluctuations around the space average \( \langle f \rangle = \int f(x) dx \) have at large \( N \to \infty \) a Gaussian distribution \([17, 70]\):

\[
\lim_{N \to \infty} \mu(x : \sqrt{N} (\bar{f}_N(x) - \langle f \rangle) < z) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma^2}} dy.
\]  

(10.16)

We were able to express the standard deviation in terms of entropy \( \sigma_f^2 = \sum_{n=-\infty}^{\infty} \frac{M_n^2}{128\pi^2} e^{-Anh(T)} \). It was suggested in 1986 in [66] to use the hyperbolic C–K systems to generate high quality pseudorandom numbers for Monte-Carlo simulations used in high energy experiments at CERN for the design of the efficient particle detectors and for the statistical analysis of the experimental data [73]. The entropies of the C–K systems suggested in [66–71] are linearly increasing with the dimension of the operator \( T \). The MIXMAX random number generators are currently made available in a portable implementation in the C++ language at hepforge.org [71] and were implemented into the Geant4/CLHEP and ROOT toolkits at CERN [72–74].

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ADDITIONAL INFORMATION

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