ON A RESULT OF KIYOTA, OKUYAMA AND WADA

JOHN MURRAY

Abstract. M. Kiyota, T. Okuyama and T. Wada recently proved that each 2-block of a symmetric group $\Sigma_n$ contains a unique irreducible Brauer character of height 0. We present a more conceptual proof of this result.

1. Background on bilinear forms

According to the main result in [6], every 2-block of the symmetric group $\Sigma_n$ has a unique irreducible Brauer character of height 0. This is not true for an arbitrary 2-block of a finite group. For example, let $B$ be a real non-principal 2-block which is Morita equivalent to the group algebra of $A_4$ and which has a Klein-four defect group and a dihedral extended defect group (in the sense of [1]). Then one can show that $B$ has three real irreducible Brauer characters of height 0. The non-principal 2-block of $((C_2 \times C_2) : C_9) : C_2$ is of this type.

In this note we place the results of [6] in a more general context using the approach to bilinear forms developed by R. Gow and W. Willems [2]. We use results and notation from [7] for representation theory, from [5] for symmetric groups, and from [8] for bilinear forms in characteristic 2.

Let $G$ be a finite group and let $(K, R, F)$ be a 2-modular system for $G$. So $R$ is a complete discrete valuation ring with field of fractions $K$ of characteristic 0, and residue field $R/J = F$ of characteristic 2. Assume that $K$ contains a primitive $|G|$-th root of unity, and that $F$ is perfect. Then $K$ and $F$ are splitting fields for each subgroup of $G$.

The anti-isomorphism $g \mapsto g^{-1}$ on $G$ extends to an involutory $F$-algebra anti-automorphism $\sigma : FG \to FG$ called the contragredient map. Let $V$ be a right $FG$-module. The linear dual $V^* := \text{Hom}_F(V, F)$ is considered as a right $FG$-module via $(f.x)(v) := f(vx^\sigma)$, for $f \in V^*$, $x \in FG$ and all $v \in V$. The Frobenius automorphism $\lambda \mapsto \lambda^2$ of the field $F$ induces an automorphism $(a_{ij}) \mapsto (a_{ij}^2)$ of the group $\text{GL}_F(V)$. Composing the module map $G \to \text{GL}_F(V)$ with this automorphism
endows $V$ with another $FG$-module structure. This module is called the Frobenius twist of $V$, and is denoted $V^{(2)}$.

Let $V^* \otimes V^*$ be the space of bilinear forms on $V$ and let $\Lambda^2(V^*)$ be the subspace of symplectic bilinear forms on $V$; a bilinear form $b : V \times V \to F$ is symplectic if and only if $b(v, v) = 0$, for all $v \in V$. The quotient space $V^* \otimes V^*/\Lambda^2(V^*)$ is called the symmetric square of $V^*$ and is denoted $S^2(V^*)$.

A quadratic form on $V$ is a map $Q : V \to F$ such that $Q(\lambda v) = \lambda^2 Q(v)$ and $(u, v) \mapsto Q(u + v) - Q(u) - Q(v)$ is a bilinear form on $V$, for all $u, v \in V$ and $\lambda \in F$. Now if $b$ is a bilinear form, its diagonal $\delta(b) : v \mapsto b(v, v)$ is a quadratic form. The assignment $\delta$ is linear with kernel $\Lambda^2(V^*)$. So there is a short exact sequence of vector spaces

\begin{equation}
0 \longrightarrow \Lambda^2(V^*) \longrightarrow V^* \otimes V^* \overset{\delta}{\longrightarrow} S^2(V^*) \longrightarrow 0
\end{equation}

We may identify $S^2(V^*)$ with the space of quadratic forms on $V$. If $Q$ is a quadratic form, its polarization is the associated bilinear form $\rho(Q) : (u, v) \mapsto Q(u + v) - Q(u) - Q(v)$.

The dual $S^2(V)^*$ of the symmetric square $S^2(V)$ of $V$ is the space of symmetric bilinear forms on $V$. As $\text{char}(F) = 2$, each symplectic form is symmetric. If $b$ is a symmetric bilinear form, $\delta(b)$ is additive and hence can be identified with a linear map $V^{(2)} \to F$. Thus there is a short exact sequence:

\begin{equation}
0 \longrightarrow \Lambda^2(V^*) \longrightarrow S^2(V^*) \overset{\delta}{\longrightarrow} V^{(2)*} \longrightarrow 0
\end{equation}

All of these $F$-spaces are $FG$-modules, and the maps are $FG$-module homomorphisms. It is a singular feature of the characteristic 2-theory that $S^2(V^*)$ and $S^2(V)^*$ need not be isomorphic as $FG$-modules.

Now let $b$ be a bilinear form on $V$. We say that $b$ is $G$-invariant if the associated map $v \mapsto b(v, )$ for $v \in V$, is an $FG$-module map $V \to V^*$. We say that $b$ is nondegenerate if this map is an $F$-isomorphism. Taking $G$-fixed points in (2) we get a long exact sequence of the form

\[ 0 \longrightarrow \Lambda^2(V^*)^G \longrightarrow S^2(V)^{G} \overset{\delta}{\longrightarrow} V^{(2)*G} \longrightarrow H^1(G, \Lambda^2(V^*)) \longrightarrow \ldots \]

In particular, if $V^{(2)*G} = 0$, then each $G$-invariant symmetric bilinear form on $V$ is symplectic. Now the trivial $FG$-module equals its Frobenius twist. A simple argument then shows:

**Lemma 1.** If $V \cong V^*$, and $V$ has no trivial $G$-submodules, then each $G$-invariant symmetric bilinear form on $V$ is symplectic.

We will make use of Fong's Lemma:
**Lemma 2.** Let $V$ be an absolutely irreducible non-trivial $FG$-module. Then $V \cong V^*$ if and only if $V$ affords a nondegenerate $G$-invariant symplectic bilinear form. In particular $\dim(V)$ is even.

Let $t, h \in G$, with $t$ an involution and $h$ not an involution. Define quadratic forms $Q_t$ and $Q_h$ on $FG$ by setting, for $u = \sum_{g \in G} u_g g \in FG$

\begin{equation}
(3) \quad Q_t(u) = \sum_{\{g, tg\} \subseteq G} u_g u_{tg},
Q_h(u) = \sum_{g \in G} u_g u_{hg}.
\end{equation}

Then each $G$-invariant quadratic form on $FG$ is a linear combination of $Q_t$’s and $Q_h$’s.

2. **Real 2-blocks of defect zero**

Assume that $G$ has even order, and that $B$ is a real 2-block of $G$ which has a trivial defect group. Equivalently $B$ is a simple $F$-algebra which is a $\sigma$-invariant $FG \times G$-direct summand of $FG$. Moreover, $B$ has a unique irreducible $K$-character $\chi$ and a unique simple module $S$.

Let $e_B$ be the identity element (or block idempotent) of $B$. Then

$$e_B = e_1 + e_2 + \cdots + e_d,$$

where $d = \dim_F(S)$ and the $e_i$ are pairwise orthogonal primitive idempotents in $FG$. Each $e_iFG$ is isomorphic to $S$. In particular $S$ is a projective $FG$-module.

Let $M$ be an $RG$-lattice whose character is $\chi$. Then $M/J(R)M \cong S$, as $FG$-modules. Now $M$ has a quadratic geometry because $\chi$ has Frobenius-Schur indicator $+1$. Thus $S$ has a quadratic geometry.

By [2] there exists an involution $t$ in $G$ such that the restriction of the form $Q_t$ of (3) to $e_1FG$ is non-degenerate. It follows that $e_1$ can be chosen so that $e_1 = e_1^\sigma = te_1^\sigma t$. We note that it can be shown that $\langle t \rangle$ is an extended defect group of $B$ and $S$ is a direct summand of $F\bar{C}_G(t)^\uparrow G$.

As $e_B = e_B^\sigma$, we have $e_B = e_1 + e_2^\sigma + \cdots + e_d^\sigma$, and each $e_i^\sigma$ is primitive in $FG$ and $e_i e_i^\sigma = 0 = e_i^\sigma e_1$, for $i > 1$.

Suppose next that $V$ is a $B$-module, equipped with a (possibly degenerate) $G$-invariant symmetric bilinear form $\langle , \rangle$. The $G$-invariance is equivalent to $\langle ux, v \rangle = \langle u, vx^\sigma \rangle$, for all $u, v \in V$ and $x \in FG$. Now $e_1 e_i = 0$, for $i > 1$. So

$$\langle V e_1, V e_i^\sigma \rangle = 0, \quad \text{for } i > 1.$$

Following [6], we define a bilinear form $b$ on the $F$-space $Ve_1$ by

$$b(ue_1, ve_1) := \langle ue_1, ve_1 t \rangle, \quad \text{for all } ue_1, ve_1 \in Ve_1.$$
Then $b$ is symmetric, as

$$b(ue_1, ve_1) = \langle ue_1 t, ve_1 \rangle = \langle ve_1, u e_1 t \rangle = b(ve_1, u e_1).$$

Now consider the radicals of the forms

$$\text{rad}(V) := \{ u \in V \mid \langle u, v \rangle = 0, \forall v \in V \},$$

$$\text{rad}(Ve_1) := \{ u e_1 \in Ve_1 \mid b(ue_1, ve_1) = 0, \forall ve_1 \in Ve_1 \}.$$

We include a proof of Lemma 4.5 of [6] for the benefit of the reader:

**Lemma 3.** $\text{rad}(Ve_1) = \text{rad}(V)e_1$ and $Ve_1/\text{rad}(Ve_1) \cong (V/\text{rad}(V))e_1$.

**Proof.** Let $u \in \text{rad}(V)$ and $ve_1 \in V e_1$. Then

$$b(ue_1, ve_1) = \langle u e_1, ve_1 t \rangle = \langle u, ve_1 t e_1^\sigma \rangle = 0.$$

So $\text{rad}(Ve_1) \supseteq \text{rad}(V)e_1$. Now let $ue_1 \in \text{rad}(Ve_1)$ and $v \in V$. Writing $v = \sum_{i=1}^d ve_i^\sigma$, we have

$$\langle u e_1, v \rangle = \sum_{i=1}^d \langle u e_1, ve_i^\sigma \rangle = \langle u e_1, v e_1^\sigma \rangle = b(ue_1, v e_1) = 0.$$

So $\text{rad}(Ve_1) \subseteq \text{rad}(V)e_1$. The stated equality follows.

We have an $F$-vector space map $\phi : Ve_1 \to (V/\text{rad}(V))e_1$ such that $\phi(ve_1) = ve_1 + \text{rad}(V)$. Now $(v + \text{rad}(V))e_1 = ve_1 + \text{rad}(V)e_1 \subseteq \text{rad}(V)e_1 \subseteq \text{rad}(V)$. So $\phi$ is onto. Moreover, $\ker(\phi) = \text{rad}(V)e_1$. The stated isomorphism follows from this.

\[\square\]

3. Brauer characters of symmetric groups

Let $n$ be a positive integer. Corresponding to each partition $\lambda$ of $n$, there is a Young subgroup $\Sigma_\lambda$ of $\Sigma_n$ and a permutation $R \Sigma_n$-module $M^\lambda := \text{Ind}_{\Sigma_\lambda}^{\Sigma_n}(R \Sigma_\lambda)$. This module has a $\Sigma_n$-invariant symmetric bilinear form with respect to which the permutation basis is orthonormal. The *Specht lattice* $S^\lambda$ is a uniquely determined $R$-free $R \Sigma_n$-submodule of $M^\lambda$ c.f. [5] 4.3. Then $S^\lambda \otimes_R K$ is an irreducible $K \Sigma_n$-module and all irreducible $K \Sigma_n$-modules arise in this way.

Now $S^\lambda$ is usually not a self-dual $R \Sigma_n$-module; the dual module $S^\lambda := S^{\lambda^\vee}$ is naturally isomorphic to $S^{[\lambda^\vee]} \otimes_R S^\lambda$ where $\lambda^\vee$ is the transpose partition to $\lambda$. Note that $S^{[\lambda^\vee]}$ is the 1-dimensional *sign module*.

Set $\overline{S^\lambda} := S^\lambda/JS^\lambda$. Then $\overline{S^\lambda}$ is a Specht module for $F \Sigma_n$. It inherits an $\Sigma_n$-invariant symmetric bilinear form $\langle , \rangle$ from $S^\lambda$. This form is nonzero if and only if $\lambda$ is 2-regular (i.e. if $\lambda$ has different parts).

Suppose that $\lambda$ is 2-regular. Then $D^\lambda := S^\lambda/\text{rad}(\overline{S^\lambda})$ is a simple $F \Sigma_n$-module, and all simple $F \Sigma_n$-modules arise uniquely in this way. The $D^\lambda$ are evidently self-dual. Indeed, $\langle , \rangle$ induces a nondegenerate form on $D^\lambda$, which by Fong’s Lemma is symplectic if $D^\lambda$ is non-trivial.
Note that \( S^{[w]} \) is the trivial \( F\Sigma_n \)-module, as \( \text{char}(F) = 2 \). It follows that the dual of a Specht module in characteristic 2 is a Specht module:

\[
\overline{S}_\lambda \cong S^\lambda^t.
\]

Let \( B \) be a 2-block of \( \Sigma_n \). Then \( B \) is determined by an integer weight \( w \) such that \( n - 2w \) is a nonnegative triangular number \( k(k+1)/2 \). The partition \( \delta := [k, k-1, \ldots, 2, 1] \) is called the 2-core of \( B \). Each defect group of \( B \) is \( \Sigma_n \)-conjugate to a Sylow 2-subgroup of \( \Sigma_{2w} \).

Recall that the 2-core of a partition \( \lambda \) is obtained by successively stripping removable domino shapes from \( \lambda \). We attach to \( B \) all partitions of \( n \) which have 2-core \( \delta \).

Set \( m := n - 2w \) and identify \( \Sigma_{2w} \times \Sigma_m \) with a Young subgroup of \( \Sigma_n \). Now \( \Sigma_m \) has a 2-block \( B_\delta \) of weight 0 and 2-core \( \delta \). This block is real and has a trivial defect group. Moreover, \( S^K_\delta \) is the unique irreducible \( K\Sigma_m \)-module in \( B_\delta \) and \( D^\delta = \overline{S}\delta \) is the unique simple \( B_\delta \)-module. It is important to note that \( D^\delta \) is a projective \( F\Sigma_m \)-module and every \( F\Sigma_m \)-module in \( B_\delta \) is semi-simple.

Let \( e_\delta \) be the block idempotent of \( B_\delta \). Following Section 2 choose an involution \( t \in \Sigma_m \) and a primitive idempotent \( e_1 \) in \( F\Sigma_m \) such that \( e_1 = e_1 e_\delta \) and \( e_1^t = e_1 \). Note that \( \dim_F(D^\delta e_1) = 1 \).

Let \( \mu \) be a 2-regular partition in \( B \). Regard \( V := \overline{S}\mu e_\delta \) as an \( F\Sigma_{2w} \times \Sigma_m \)-module by restriction. Then \( V e_1 \) is an \( F\Sigma_{2w} \)-module, as the elements of \( \Sigma_{2w} \) commute with \( e_1 \). Indeed

\[
V \cong V e_1 \otimes_F D^\delta \quad \text{as } F\Sigma_{2w} \times \Sigma_m \text{-modules}.
\]

Now \( \overline{S}\mu \) and hence \( V \) affords a \( \Sigma_{2w} \times \Sigma_m \)-invariant symmetric bilinear form \( \langle , \rangle \) such that \( V/\text{rad}(V) = D^\mu e_\delta \). It then follows from Lemma 3 that we may use the identity \( e_1^t = e_1 \) to construct a symmetric bilinear form \( b \) on \( V e_1 \). Moreover, \( V e_1/\text{rad}(V e_1) \cong D^\mu e_1 \). So the \( F\Sigma_{2w} \)-module \( D^\mu e_1 \) inherits a nondegenerate symmetric bilinear form \( b \). Reviewing the construction of \( b \) from \( \langle , \rangle \), we see that \( b \) is \( \Sigma_{2w} \)-invariant (as \( t \in \Sigma_m \) commutes with all elements of \( \Sigma_{2w} \), and \( \langle , \rangle \) is \( \Sigma_n \)-invariant).

Lemma 4. Suppose that \( \mu \neq [k + 2w, k - 1, \ldots, 2, 1] \). Then \( D^\mu e_1 \) affords a non-degenerate \( \Sigma_{2w} \)-invariant symplectic bilinear form.

Proof. In view of Lemma 1 and the discussion above, it is enough to show that \( D^\mu e_1 \) has no trivial \( F\Sigma_{2w} \)-submodules. Suppose otherwise. Then \( F\Sigma_{2w} \otimes_F D^\delta \) is a submodule of the restriction of \( D^\mu \) to \( \Sigma_{2w} \times \Sigma_m \). But \( D^\mu \) is a submodule of \( \overline{S}\mu \). So \( D^\delta \) is a submodule of \( \text{Hom}_{F\Sigma_{2w}}(F\Sigma_{2w}, \overline{S}\mu) \) as \( F\Sigma_m \)-modules.
We have $F$-isomorphisms
\[
\text{Hom}_{F \Sigma_2} (F_{\Sigma_2}, S_\mu) \cong \text{Hom}_{F \Sigma_n} (M^{[2,1^m]}_{2w}, S_\mu),
\]
by Eckmann-Shapiro
\[
\cong \text{Hom}_{F \Sigma_n} (S_\mu, M^{[2,1^m]}_{2w}),
\]
as $M^{[2,1^m]}_{2w}$ is self-dual.

As $\mu$ is 2-regular, it follows from [3, 13.13] that $\text{Hom}_{F \Sigma_n} (S_\mu, M^{[2,1^m]}_{2w})$ has a basis of semistandard homomorphisms. The argument of Theorem 4.5 of [1] now applies, and shows that
\[
\text{Hom}_{F \Sigma_2} (F_{\Sigma_2}, S_\mu) \cong S_{\mu\setminus[1^{2w}]}^\delta
\]
as $F \Sigma_m$-modules.

Here $\mu\setminus[1^{2w}]$ is a skew-partition of $m$; it is empty if $\mu_1 < 2w$ (in which case $\text{Hom}_{F \Sigma_2} (F_{\Sigma_2}, S_\mu) = 0$). Otherwise its diagram is the set of nodes in the Young diagram of $\mu$ not in the top $2w$ rows of the first column.

Now $S_{\mu\setminus[1^{2w}]}^\delta$ has an $F \Sigma_m$-submodule isomorphic to $D^\delta$ if and only if $S_{\Sigma_m}^\delta\setminus[1^{2w}]$ has an $K \Sigma_m$-submodule isomorphic to $S_{\Sigma_m}^\delta$, as $D^\delta = S_K^\delta$, and using the projectivity of $D^\delta$.

The multiplicity of $S_{\Sigma_m}^\delta$ in $S_{\mu\setminus[1^{2w}]}^\delta$ is the number of $\mu\setminus[2w]$-tableau of type $\delta' = \delta$ which are strictly increasing along rows and nondecreasing down columns. Suppose for the sake of contradiction that such a tableau $T$ exists.

We claim that $\mu_i \leq k - i + 2$ for $i = 2, \ldots, k$, and $\mu_i = 0$ for $i > k + 1$. This is true for $i = 2$, as the entries in the second row of $T$ are different. Suppose that $i \geq 2$ and $\mu_{i-1} \leq k - i + 3$. But $\mu_i < \mu_{i-1}$, as $\mu$ is 2-regular. So $\mu_i \leq k - i + 2$, proving our claim.

On the other hand, $\mu_i \geq \delta_i = k - i + 1$, for $i = 1, \ldots, k$, as $\mu$ has 2-core $\delta$. It follows that $\mu\setminus\delta$ consists of the last $\mu_1 - k$ nodes in the first row of $\mu$, and a subset of the nodes $(2, k), (3, k-1), \ldots, (k, 2), (k+1, 1)$. On the other hand, $\mu$ has 2-core $\delta$. So $\mu\setminus\delta$ is a union of domino shapes. It follows that $T$ does not exist if $\mu \neq [k + 2w, k - 1, \ldots, 2, 1]$. This contradiction completes the proof of the Lemma. \hfill $\square$

Suppose that $G$ is a finite group and that $B$ is a 2-block of $G$ with defect group $P \leq G$. Then it is known that $[G : P]_2$ divides the degree of every irreducible Brauer character in $B$. Recall that a Brauer character in $B$ has height zero if the 2-part of its degree is $[G : P]_2$. We now prove the main result of [6].

**Theorem 5.** Let $B$ be a 2-block of $\Sigma_n$. Then $B$ contains a unique irreducible Brauer character of height 0.

**Proof.** Suppose as above that $B$ has weight $w$ and 2-core $\delta$, and let $\theta$ be a height zero irreducible Brauer character in $B$. Then $\theta$ is the Brauer character of $D^\mu$ for some 2-regular partition $\mu$ of $n$ belonging to $B$. 


Let \( P \) be a vertex of \( D^\mu \). Then \( P \) is a defect group of \( B \). We may assume that \( P \) is a Sylow 2-subgroup of \( \Sigma_{2w} \). It is easy to show that \( N_{\Sigma_n}(P) = P \times \Sigma_m \), a subgroup of \( \Sigma_{2w} \times \Sigma_m \).

Let \( B_0 \) denote the principal 2-block of \( \Sigma_{2w} \). Then \( B_0 \otimes B_\delta \) is the Brauer correspondent of \( B \) with respect to \( (\Sigma_n, P, \Sigma_{2w} \times \Sigma_m) \). So the Green correspondent of \( D^\mu \) with respect to \( (\Sigma_n, P, \Sigma_{2w} \times \Sigma_m) \) has the form \( U^\mu \otimes D^\delta \), where \( U^\mu \) is an indecomposable \( \Sigma_{2w} \)-direct summand of \( D^\mu e_1 \) which belongs to \( B_0 \). Moreover, \( U^\mu \) is the unique component of \( D^\mu e_1 \) that has vertex \( P \).

If \( \mu = [k + 2w, k - 1, \ldots , 2, 1] \) it can be shown that \( U^\mu \) is the trivial \( F\Sigma_{2w} \)-module. Suppose that \( \mu \neq [k + 2w, k - 1, \ldots , 2, 1] \). Lemma 4 implies that \( D^\mu e_1 \) has a symplectic geometry. It then follows from the first proposition in [3] that \( U^\mu \) has a symplectic geometry. In particular \( \dim(U^\mu) \) is even.

Now the 2-part of \( \dim(U^\mu \otimes D^\delta) \) divides \( 2|\Sigma_m|_2 = 2[\Sigma_n : P]_2 \). A standard result on the Green correspondence implies that the 2-part of \( \dim(D^\delta) \) divides \( 2|\Sigma_n : P|_2 \). This contradicts the assumption that \( \theta \) has height zero, and completes the proof.

4. Acknowledgement

B. Külshammer drew my attention to the preprint [6] during a visit to Jena in April 2011. S. Kleshchev suggested I look at the restrictions of dual Specht modules, and D. Hemmer clarified the ‘fixed-point functors’ used to prove Lemma 4. G. Navarro gave me the example of the group of order 72 which has a 2-block with 3 real irreducible Brauer characters of height 0.

References

[1] R. Gow, Real 2-blocks of characters of finite groups, Osaka J. Math. 25 (1988), 135–147.
[2] R. Gow, W. Willems, Quadratic geometries, projective modules and idempotents, J. Algebra 160 (1993), 257–272.
[3] R. Gow, W. Willems, A note on Green correspondence and forms, Comm. Algebra 23 (4) (1995), 1239–1248.
[4] D. J. Hemmer, Fixed-point functors for symmetric groups and Schur algebras, J. Algebra 280 (2004), 295–312.
[5] G. D. James, The representation theory of the symmetric groups, Lecture Notes in Math. 682, Springer-Verlag, 1978.
[6] M. Kiyota, T. Okuyama, T. Wada, The heights of irreducible Brauer characters in 2-blocks of the symmetric groups, to appear in J. Algebra.
[7] H. Nagao, Y. Tsushima, Representations of Finite Groups, Academic Press, Inc., 1989.
[8] W. Willems, *Duality and forms in representation theory*, Progress in Math. 95 (1991), 509–520.

Department of Mathematics, National University of Ireland, Maynooth, Co. Kildare, Ireland

E-mail address: John.Murray@maths.nuim.ie