On a polynomial transformation of hypergeometric equations, Heun’s differential equation and exceptional Jacobi polynomials

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Abstract. This paper addresses a general method of polynomial transformation of hypergeometric equations. Examples of some classical special equations of mathematical physics are generated. Heun’s equation and exceptional Jacobi polynomials are also treated.

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1. Introduction

The singularities of linear ordinary differential equation (ODE), Fuchsian or not Fuchsian, can move, change, coalesce, etc. by a transformation acting on the independent variable $x$, or on the dependent variable $y$ in the equation $L\{y(x)\} = 0$. For instance, in mathematical physics, one uses to transform a given equation, like the Schroedinger equation of the harmonic oscillator, into a well known equation, as the Hermite differential equation, changing both the independent variable (rescaling) and the dependent variable (wave function). In this exemple, the alone singularity at the infinity stays at infinity.

In 1971, Kimura [1] investigated in detail all Fuchsian differential equations $F\{y(x)\} = 0$ reducible to the hypergeometric equation $H\{z(x)\} = 0$ by a linear transformation $y(x) = P_0(x)z(x) + P_1(x)z′(x)$, with $P_0(x)$ and $P_1(x)$ rational functions of $x$. In order to compute $P_0(x)$ and $P_1(x)$, he assumed the same set of singularities and the same monodromy group for $H\{y(x)\} = 0$ and $F\{z(x)\} = 0$, giving information
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about the structures and properties of $P_0(x)$ and $P_1(x)$. The equation $F\{y(x)\} = y'' + A_1(x)y' + A_2(x)y = 0$, ($y = y(x)$), being Fuchsian and written as a generalized Heun’s equation contains regular singular points with many parameters, but not all free, in order to eliminate logarithmic situations and irreducibility (equation not factorizable). Many theorems and properties allow to compute heavily $P_0(x)$ and $P_1(x)$ for each peculiar situations, but without using the coupled differential equations satisfied by $P_0(x)$ and $P_1(x)$ given by the author. (See [2]-[13] for more details on Heun’s equations and orthogonal polynomials).

The aim of this paper is to reverse this approach, allowing to eliminate the Fuchsian constraints: inject an arbitrary linear transformation $y = A(x)z + B(x)z'$ in the hypergeometric equation and build a second order differential equation for $F(z(x))$, not always Fuchsian, but with 2 families of arbitrary differentiable functions, polynomial or not. This approach allows also to use any solution $z(x)$ of the hypergeometric equation, polynomial or not, including possibly solutions of second kind. It is surprising that recently introduced new orthogonal polynomials, also called exceptional polynomials (Jacobi, Laguerre, etc.) [6, 9], can be also built from the same linear transformation long time ago given in the above mentioned seminal paper by Kimura. Recall that the concept of exceptional orthogonal polynomials was introduced by Gómez-Ullate et al [14, 15]. Within the Sturm-Liouville theory they constructed $X_1$ Laguerre and $X_1$ Jacobi polynomials, which turned out to be the first members of the infinite families. Then Quesne and collaborators [16, 17] reformulated their results in the framework of quantum mechanics and shape-invariant potentials. The merit of quantum mechanical reformulation resides in the fact that the orthogonality and completeness of the obtained eigenfunctions are guaranteed. Besides, the well established solution mechanism of shape invariance combined with the Crum’s method [18], or the so-called factorization method [19], or the susy quantum mechanics [20] is available. A nice discussion on these aspects is presented in [21] and references therein.

2. General setting

Consider the hypergeometric equation:

$$
\sigma(x)z''(x) + \tau(x)z'(x) + \lambda z(x) = 0,
$$

(1)

where $\sigma \equiv \sigma(x)$ is a polynomial of degree less or equal to 2, and $\tau \equiv \tau(x)$ a polynomial of degree exactly equal to 1; $\lambda$ is a constant. Let the following transformation:

$$
y(x) = A(x)z(x) + B(x)z'(x)
$$

(2)

with $A = A(x)$ and $B = B(x)$ polynomials of degrees $r$ and $s$, respectively. This transformation appears for instance in the recent development of exceptional $X_1$ Laguerre and Jacobi polynomials [6, 9] and also in the problem of reducing Fuchsian ordinary differential equations into hypergeometric equations (ODEs) like Heun’s equations [11, 12, 13].
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In this work, we build a general linear second order ordinary differential equation $\mathcal{L}_2[y(x)] = 0$ satisfied by the function $y(x)$, polynomial or not, and we investigate a set of situations.

The first derivative of (2) gives, using (1):
\[
\sigma y' = \tilde{A}z + \tilde{B}z' 
\]
with $\tilde{A} = \sigma A' - B\lambda, \tilde{B} = \sigma A + \sigma B' - \tau B$.

The derivative of (3) can be written as
\[
\sigma(\sigma y')' = \tilde{A}z + \tilde{B}z' 
\]
with $\tilde{A} = \sigma A' - B\lambda, \tilde{B} = \sigma A + \sigma B' - \tau B$ which can be expanded in terms of $\sigma, \tau, \lambda$ as follows:
\[
\tilde{A} = A''\sigma^2 + \sigma A' \sigma' - 2\lambda \sigma B' - \lambda A \sigma + \lambda B \tau 
\]
\[
\tilde{B} = 2A'\sigma^2 - \lambda \sigma B + \sigma A \sigma' + \sigma^2 B'' + \sigma B' \sigma' 
- 2\sigma B' \tau - \sigma B \tau' - \tau A \sigma + B \tau^2.
\]

We then arrive at the following determinantal equation giving the ordinary differential equation satisfied by $y(x)$ :
\[
\mathcal{L}_2[y(x)] = \begin{vmatrix}
    y & A & B \\
    \sigma y' & \tilde{A} & \tilde{B} \\
    \sigma(\sigma y')' & \tilde{A} & \tilde{B}
\end{vmatrix} = 0. \tag{7}
\]

The following presentation is also useful:
\[
\hat{\mathcal{L}}_2[y(x)] = \begin{vmatrix}
    y & A & B \\
    \sigma y' & \hat{A} & \hat{B} \\
    \sigma^2 y'' & \hat{A} & \hat{B}
\end{vmatrix} = 0 \tag{8}
\]
with
\[
\hat{A} = \sigma \tilde{A}' - B \lambda - \sigma' \tilde{A} \tag{9}
\]
\[
\hat{B} = \sigma \tilde{A} + \sigma \tilde{B}' - \tilde{B}(\tau + \sigma'). \tag{10}
\]

**Remark 2.1** The following observations are in order:

(i) The coefficient of $y''$ is $\sigma^2(AB - B\tilde{A})$ and introduces new singular points depending on the degrees $r$ and $s$ of the polynomials $A$ and $B$. In general the obtained ODE is no more Fuchsian.

(ii) This general formulation does not take into account the possible logarithmic solutions and apparent singularities for $\mathcal{L}_2[y(x)] = 0$ which strongly restrict the choice of the polynomials $A$ and $B$. Such a freedom allows to cover a larger class of equations with $r + s + 2$ parameters such as Heun, generalized Heun equations and their various confluenes and to propose some solutions.
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(iii) An obvious way to remain in the Fuchsian’s class is to choose $B(x) = \sigma(x)$ or more generally $B(x) = S(x)\sigma(x)$. Such a simplified choice has been already considered in the work by Kimura [1] and in recent literature on exceptional polynomials (see [3] and references therein).

2.1. Basic choice: $A(x)$ is arbitrary and $B(x) = \sigma(x)$

The choice $B(x) = \sigma(x)$ generates a simpler equation, eliminating the factor $\sigma^2$ in the coefficient of $y''$:

$$\mathcal{L}_2[y(x)] = \begin{pmatrix} y & A & \sigma \\ y' & A' - \lambda & A + \sigma' - \tau \\ (\sigma y')' & C & D \end{pmatrix} = 0,$$  \hspace{1cm} (11)

where

$$C = A''\sigma + A'\sigma' + \lambda(\tau - 2\sigma' - A)$$  \hspace{1cm} (12)

$$D = \sigma(\sigma'' - \lambda - \tau' + 2A') + \sigma'(\sigma' - 2\tau + A) + \tau(\tau - A)$$  \hspace{1cm} (13)

yielding the second order differential equation:

$$\mathcal{L}_2[y(x)] = \sigma Py'' + Qy' + Ry = 0$$  \hspace{1cm} (14)

with

$$P \equiv P(x) = A^2 + A(\sigma' - \tau) + \sigma(\lambda - A')$$  \hspace{1cm} (15)

$$Q \equiv Q(x) = A''\sigma^2 + \sigma(\lambda\sigma' + \tau - 2A') + \lambda(\tau - \sigma' - \sigma'')$$  \hspace{1cm} (16)

$$R \equiv R(x) = \sigma[A''(\tau - A - \sigma') + A'(2\lambda + 3\lambda - \tau') + \lambda(\tau' - 2\lambda\lambda - \sigma'') + 2\lambda A + \sigma' - (A - \tau)(\lambda A - A'\tau)]$$  \hspace{1cm} (17)

It is worth noticing that appropriate choices of coefficients in the polynomial $A(x)$ reducing equation (11) to hypergeometric equations generate known or unknown contiguous relations between the solutions $y(x)$. To cite a few, for instance with $\sigma(x) = B(x) = 1 - x^2$, and $\tau(x) = \beta - \alpha - (\alpha + \beta + 1)x$ and $\lambda = (n - 1)(n + \alpha + \beta)$, the functions $y(x)$ are reduced to the Jacobi polynomials $P^{(\alpha,\beta)}_{n-1}(x)$ as polynomial solution. The equation (14) coincides, after simplification, with the equation

$$\mathcal{L}_2[y(x)] = (1 - x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]y'(x) + (n - 1)(n + \alpha + \beta)y(x) = 0$$  \hspace{1cm} (18)

with $y(x) = P^{(\alpha,\beta)}_{n-1}(x)$ as solution. This proves therefore the well known contiguous relation:

$$(2n + \alpha + \beta)(1 - x^2)\frac{d}{dx}P_n^{(\alpha,\beta)}(x) - n[\alpha - \beta - (2n + \alpha + \beta)x]P_n^{(\alpha,\beta)}(x) = 2(n + \alpha)(n + \beta)P_n^{(\alpha,\beta)}_{n-1}(x)$$  \hspace{1cm} (19)
On a polynomial transformation of hypergeometric equations, Heun’s differential equation and exceptional Jac providing the polynomial coefficient $A(x)$ of $P_n^{(\alpha, \beta)}(x)$ after division of the equation (19) by the factor $2(n + \alpha)(n + \beta)$. Of course if $\lambda$ is arbitrary, non polynomial solutions as the second kind solutions of (11) generate similar contiguous relations for the $z(x)$ as for the polynomial solutions $y(x)$.

2.2. Second choice: $A$ and $B$ are constants

This trivial case leads to another simpler equation:

$$L_2[y(x)] = \begin{vmatrix} y & A & B \\ \sigma y' & -\lambda B & \sigma A - \tau B \\ \sigma(\sigma y')' & -\lambda(\sigma A - \tau B) & \sigma(A\sigma' - B\tau' - A\tau - \lambda B) + B\tau^2 \end{vmatrix} = 0 \quad (20)$$

providing the second order differential equation:

$$L_2[y(x)] = \sigma P y'' + Q y' + R y = 0 \quad (21)$$

with

$$P \equiv P(x) = \sigma(x)[A(\sigma A - \tau B) + \lambda B^2] \quad (22)$$

$$Q \equiv Q(x) = \sigma^2(\tau A^2 + AB\tau') + \sigma(\lambda B^2\tau + \lambda\sigma' B^2 - AB\tau^2 - \sigma' AB\tau) \quad (23)$$

$$R \equiv R(x) = \lambda A^2\sigma^2 + \sigma(\lambda^2 B^2 - \lambda A\sigma' + \lambda B^2\tau' - \lambda B\tau). \quad (24)$$

This equation is also not hypergeometric in general, but using suitable choices we can deduce again contiguous relations.

- As illustration, if $\sigma(x) = x$, $\tau(x) = 1 + \alpha - x$ and $\lambda = n$, the $z(x)$ are Laguerre polynomials $L_n^{(\alpha)}(x)$. The choice $A = 1 = -B$ gives again a Laguerre polynomial for $y(x)$ confirming the well known relation

$$L_n^{(\alpha)}(x) - [L_n^{(\alpha)}(x)]' = [L_{n+1}^{(\alpha)}(x)]' = L_n^{(\alpha+1)}(x). \quad (25)$$

- If $\sigma = 1$ and $\tau = -2x$, the equation (14) takes the form:

$$L_2[y(x)] = y''[(A^2 + B^2\lambda + 2ABx) - y'[2AB + 2x(A^2 + B^2\lambda) + 4x^2AB] + y\lambda[B^2(\lambda - 2) + A(A + 2xB)] = 0 \quad (26)$$

which can be further simplified to give, for the particular case of $B = 0$ with arbitrary $A$, the well known Hermite equation:

$$L_2[y(x)] = y'' - 2xy' + \lambda y = 0. \quad (27)$$

In the opposite, when $A = 0$ and $B$ is an arbitrary function, the equation (21) is transformed into a modified Hermite equation

$$L_2[y(x)] = y'' - 2xy' + \lambda y = 0 \text{ with } \lambda = \lambda - 2. \quad (28)$$
3. Link with the Heun equations and exceptional Jacobi polynomials

With \( A = \alpha x + \beta \) and \( \sigma = x^2 - x \), the degrees of \( P, Q, R \) in equation (14) are 2, 3 and 2, respectively. In order to generate a HEUN equation

\[
y'' + \left[ \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-\mu} \right] y' + \frac{\alpha \beta x + \rho}{x(x-1)(x-\mu)} y = 0, \tag{29}
\]

the coefficients \( \alpha, \beta \) and \( \tau \) must be chosen such that \( P(x) \) reduces to \( x - \mu \), and \( Q(x) = (x - \mu)Q_1(x) \) and \( R(x) = (x - \mu)R_1(x) \), where \( Q_1(x) \) and \( R_1(x) \) are polynomials. Other appropriate choices also produce confluent Heun equations [11]. All these situations generate Heun equations with explicit solutions \( y(x) = A(x)z(x) + B(x)z'(x) \). Kimura gives in [1] solutions in two relevant cases: (i) for \( \epsilon = -1 \) with \( A(x) \) of degree 1 and \( B(x) = x(x-1) \), and (ii) for \( \epsilon = -2 \) with \( A(x) \) of degree 2 and \( B(x) \) of degree 3. The Kimura method is very nice, but complicated and restrictive, the confluent equations being excluded. Its intrinsic complexity resides in the step increasing the degree of polynomials \( A(x) \) and \( B(x) \). In our approach, the problem is entirely algebraic, even not excluding, of course, also some difficulty.

Finally, let us mention that the exceptional \( X_1-Jacobi \) polynomials investigated in [6] can be also easily retrieved from our method. Indeed, set, for \( g, h \neq \{-1/2, -3/2, -5/2, \ldots\} : \)

\[
\zeta(\eta) = \frac{g - h}{2} \eta + \frac{g + h + 1}{2}, \quad \zeta(\eta) = \frac{g - h}{2} \eta + \frac{g + h + 3}{2}.
\]

and consider the polynomials

\[
A(\eta) = \frac{1}{k + h + \frac{1}{2}} \left( h + \frac{1}{2} \right) \zeta(\eta)
\]

and

\[
B(\eta) = \frac{1}{k + h + \frac{1}{2}} \left( 1 + \eta \right) \zeta(\eta)
\]

of degrees 1 and 2, respectively. Let \( z(\eta) \) be the Jacobi polynomial parametrized as [6]:

\[
P_k(\eta) = \frac{(g + \frac{1}{2})_k}{k!} \sum_{j=0}^{k} \frac{(-k)_j (k + g + h + 2)_j}{j! (g + \frac{1}{2})_j} \left( \frac{1 - \eta}{2} \right)^j. \tag{30}
\]

Then the transformation (2) gives the \( X_1-Jacobi \) polynomials \( y(\eta) \equiv \tilde{P}_k(\eta) \) satisfying the following differential equation:

\[
(1 - \eta^2)y''(\eta) + \left( h - g - (g + h + 3)\eta - 2 \frac{(1 - \eta^2)\zeta'(\eta)}{\zeta(\eta)} \right)y'(\eta)
\]

\[
+ \left( - \frac{2(h + \frac{1}{2})(1 - \eta)\zeta'(\eta)}{\zeta(\eta)} + k(k + g + h + 2) + g - h \right)y(\eta) = 0. \tag{31}
\]

Transforming \( \eta \) into a new variable and assigning adequate relations between parameters lead to specific Heun’s differential equations as shown, for instance in [6], with the particular variable change \( \eta = 1 - 2x \). For more details about Heun’s equation and differential equation describing the exceptional Jacobi polynomials, see [6] and references therein.
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