SOME NOTES CONCERNING THE HOMOGENEITY OF
BOOLEAN ALGEBRAS AND BOOLEAN SPACES

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ABSTRACT. In this article we consider homogeneity properties of Boolean al-
gebras that have nonprincipal ultrafilters which are countably generated.

It is shown that a Boolean algebra $B$ is homogeneous if it is the union of
countably generated nonprincipal ultrafilters and has a dense subset $D$ such
that for every $a \in D$ the relative algebra $B \upharpoonright a := \{ b \in B : b \leq a \}$ is isomorphic
to $B$. In particular, the free product of countably many copies of an atomic
Boolean algebra is homogeneous.

Moreover, a Boolean algebra $B$ is homogeneous if it satisfies the following
conditions:
(i) $B$ has a countably generated ultrafilter,
(ii) $B$ is not c.c.c., and
(iii) for every $a \in B \setminus \{ 0 \}$ there are finitely many automorphisms $h_1, \ldots, h_n$
of $B$ such that $1 = h_1(a) \cup \cdots \cup h_n(a)$.

These results generalize theorems due to Motorov [9] on the homogeneity
of first countable Boolean spaces.

Finally, we provide three constructions of first countable homogeneous
Boolean spaces that are linearly ordered. The first construction gives separ-
able spaces of any prescribed weight in the interval $[\aleph_0, 2^{\aleph_0}]$. The second
construction gives spaces of any prescribed weight in the interval $[\aleph_1, 2^{\aleph_0}]$ that
are not c.c.c. The third construction gives a space of weight $\aleph_1$ which is not
c.c.c. and which is not a continuous image of any of the previously described
eamples.

1. INTRODUCTION

A topological space $X$ is homogeneous if for any two points $x, y \in X$ there is
an autohomeomorphism of $X$ mapping $x$ to $y$. Among the most obvious examples
of homogeneous spaces are topological groups. In the case of topological groups,
translations can be used to show their homogeneity.

If we restrict our attention to zero-dimensional compact spaces, i.e., to Boolean
spaces, topological groups are not interesting from the topological point of view
since infinite compact zero-dimensional groups are all Cantor spaces, that is, they
are homeomorphic to spaces of the form $2^\kappa$ where $\kappa$ is a cardinal (see [2] or [3]).

There is a surprising shortage of examples of homogeneous Boolean spaces other
than Cantor spaces. Interesting examples were provided by Maurice [8], who proved
that for every indecomposable countable ordinal $\gamma$ the lexicographically ordered
space $2^\gamma$ is homogeneous. Here an ordinal $\gamma$ is indecomposable if $\gamma = \alpha + \beta$ with
$\beta > 0$ implies $\beta = \gamma$. If $\gamma > \omega$, then $2^\gamma$ ordered lexicographically does not satisfy the
countable chain condition (c.c.c.) and therefore is not homeomorphic to a Cantor space.

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Cantor spaces and the lexicographically ordered spaces $2^\gamma$, $\gamma$ countable and indecomposable, have the property that not only the spaces themselves, but also their dual Boolean algebras are homogeneous. A Boolean algebra $B$ is homogeneous if for every $a \in B \setminus \{0\}$ the relative algebra $B \upharpoonright a := \{b \in B : b \leq a\}$ is isomorphic to $B$. In general, there is no direct implication between the homogeneity of a Boolean algebra and the homogeneity of its Stone space. Van Douwen [11] constructed a first countable homogeneous Boolean space whose dual Boolean algebra is not homogeneous. And it is well known that the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$ is homogeneous but its Stone space $\beta\omega \setminus \omega$ is not.

However, the homogeneity of first countable Boolean spaces follows from the homogeneity of their dual Boolean algebra. This was noticed independently by Motorov [9] and Koppelberg [5]. Motorov proved that the converse is also true in certain cases. He showed (in topological terms) that the homogeneity of a Boolean algebra follows from the homogeneity of its Stone space if the Boolean algebra is not c.c.c. and every ultrafilter is countably generated. Note that the last condition is equivalent to the first countability of the Stone space.

The main tool in Motorov’s argument is

**Theorem 1.1.** Let $B$ be a Boolean algebra such that every ultrafilter of $B$ is countably generated and $B$ has a dense subset $D$ such that for all $a \in D$, the algebra $B \upharpoonright a$ is isomorphic to $B$. Then $B$ is homogeneous.

Unfortunately, published proofs of Motorov’s results seem to be unavailable. We give the proofs of some generalizations of his theorems. The main observation is that in Theorem 1.1 the assumption “every ultrafilter of $B$ is countably generated” can be weakened to “$B$ is the union of countably generated ultrafilters” (which is equivalent to the Stone space of $B$ having a dense set of points of countable character). This easily implies that the free product of infinitely many copies of an atomic Boolean algebra is homogeneous. Here a Boolean algebra $B$ is atomic if the atoms are dense in $B$, i.e., if the Stone space of $B$ has a dense set of isolated points.

We also show that a Boolean algebra $B$ which is not c.c.c. is homogeneous if it has at least one countably generated ultrafilter and the property that for all $a \in B \setminus \{0\}$ there are finitely many automorphisms $h_1, \ldots, h_n$ of $B$ such that $1 = h_1(a) \cup \cdots \cup h_n(a)$. The latter property is equivalent to the property that every point of the Stone space $X$ of $B$ has a dense orbit with respect to the natural group action of the group $\text{Aut}(X)$ of autohomeomorphisms of $X$.

Moreover, we provide three constructions leading to new examples of homogeneous Boolean spaces. In all cases we obtain first countable spaces which are linearly ordered. The first construction yields separable spaces of any prescribed weight in the interval $[\aleph_0, 2^{\aleph_0}]$. These spaces are constructed from nice suborders of $\mathbb{R}$. Note that the space of countable weight is homeomorphic to $2^\omega$ since up to homeomorphism $2^\omega$ is the only Boolean space of countable weight without isolated points.

The second construction uses an easy Löwenheim-Skolem argument and gives homogeneous continuous images of the lexicographically ordered spaces $2^\gamma$, $\gamma < \omega_1$ indecomposable and uncountable. The spaces obtained using this construction can have any prescribed weight in the interval $[\aleph_0, 2^{\aleph_0}]$, and their cellularity equals their weight. The third construction uses a linear order on an Aronszajn tree and yields a space of weight $\aleph_1$ which is not c.c.c. and not a continuous image of any of the previously known examples of first countable homogeneous Boolean spaces.

It should be noted that compact homogenous spaces which are linearly ordered have to be first countable (see [1] or [7]).
2. Generalizing Motorov’s results

As usual, the Stone space of a Boolean algebra $B$ is denoted by $\text{Ult}(B)$ and the Boolean algebra of clopen subsets of a Boolean space $X$ is denoted by $\text{Clop}(X)$. In the following, we will frequently switch between Boolean algebras and their Stone spaces, but our presentation will be mainly in topological terms.

Note that a Boolean algebra $B$ is homogeneous if and only if every nonempty clopen subset of $\text{Ult}(B)$ is homeomorphic to $\text{Ult}(B)$.

**Lemma 2.1.** Let $X$ be a Boolean space such that $\text{Clop}(X)$ is homogeneous. If $x, y \in X$ are points of countable character, then there is an autohomeomorphism of $X$ mapping $x$ to $y$. In particular, $X$ is homogeneous if it is first countable.

**Proof.** Assuming that $X$ is infinite, it follows from the homogeneity of $\text{Clop}(X)$ that $X$ has no isolated points. Let $(A_n)_{n \in \omega}$ and $(B_n)_{n \in \omega}$ be clopen neighborhood bases of $x$ and $y$, respectively. Since $x$ and $y$ are not isolated, we may assume that the sequences $(A_n)_{n \in \omega}$ and $(B_n)_{n \in \omega}$ are strictly decreasing. We may also assume $A_0 = B_0 = X$. For each $n \in \omega$, let $C_n := A_n \setminus A_{n+1}$ and $D_n := B_n \setminus B_{n+1}$ and fix an homeomorphism $h_n : C_n \to D_n$. It is easily checked that $h := \{(x, y)\} \cup \bigcup_{n \in \omega} h_n$ is an autohomeomorphism of $X$ mapping $x$ to $y$. \qed

In order to apply Lemma 2.1 we need a criterion for the homogeneity of Boolean algebras with first countable Stone spaces. A $\pi$-base of a topological space $X$ is a family $\mathcal{F}$ of open subsets of $X$ such that every nonempty open subset of $X$ includes a member of $\mathcal{F}$. A family of clopen subsets of a Boolean space $X$ is a $\pi$-base if and only if it is a dense subset of $\text{Clop}(X)$.

**Lemma 2.2.** Let $X$ be a Boolean space with a dense set of nonisolated points of countable character. Then $\text{Clop}(X)$ is homogeneous if $X$ has a $\pi$-base consisting of clopen sets which are homeomorphic to $X$.

**Proof.** We show that the nonempty clopen subsets of $X$ are pairwise homeomorphic. Let $A$ be a nonempty clopen subset of $X$. Let $x \in A$ be a nonisolated point of countable character. As in the proof of Lemma 2.1, there is a disjoint family $(A_n)_{n \in \omega}$ of nonempty clopen subsets of $A$ such that $A = \{x\} \cup \bigcup_{n \in \omega} A_n$.

Inductively we define sequences $(C_n)_{n \in \{-1\} \cup \omega}$ and $(B_n)_{n \in \omega}$ as follows: Let $C_{-1} := \emptyset$. Let $n \in \omega$ and suppose we have already defined $C_{n-1}$. Since the clopen subsets of $X$ which are homeomorphic to $X$ form a $\pi$-base of $X$, there is a clopen set $B_n \subseteq A_n$ such that $B_n$ is homeomorphic to $X \setminus C_{n-1}$. With this choice, $C_{n-1} \cup B_n \cong X$. Let $C_n := A_n \setminus B_n$. Now

$$A \setminus \{x\} = \bigcup_{n \in \omega} (B_n \cup C_n) = B_0 \cup \bigcup_{n \in \omega} (C_n \cup B_{n+1}).$$

By the choice of the $B_n$, $n \in \omega$, $\bigcup_{n \in \omega} (C_{n-1} \cup B_n)$ is homeomorphic to the disjoint union of $\aleph_0$ copies of $X$. It follows that $A$ is the one-point compactification of the disjoint union of $\aleph_0$ copies of $X$. Since $A$ was arbitrary, it follows that the nonempty clopen subsets of $X$ are pairwise homeomorphic. \qed

Using Lemma 2.2 and Lemma 2.1, we can give an easy proof of the homogeneity of the lexicographically ordered spaces $2^\gamma$, $\gamma$ countable and indecomposable. For every $\alpha < \gamma$ and every $x \in 2^{\alpha+1}$ the set $I_x := \{y \in 2^\gamma : x \subseteq y\}$ is a clopen interval in $2^\gamma$. By the indecomposability of $\gamma$, each $I_x$ is homeomorphic to $2^\gamma$. Clearly,

$$\{I_x : \alpha < \gamma \land x \in 2^{\alpha+1}\}$$

is a $\pi$-base of $2^\gamma$. Thus, $\text{Clop}(2^\gamma)$ is homogeneous by Lemma 2.2. Now the homogeneity of $2^\gamma$ follows from Lemma 2.1.
Another corollary of Lemma 2.2 gives information about free products of atomic Boolean algebras.

**Corollary 2.3.** Let $X$ be a Boolean space with a dense set of isolated points. Then $\text{Clop}(X^\omega)$ is homogeneous.

**Proof.** Let $D$ be the set of subsets of $X^\omega$ of the form $\{(x_0, \ldots, x_{n-1})\} \times X^{\omega \setminus n}$ where each $x_i \in X$ is isolated. Clearly, $D$ consists of clopen sets that are homeomorphic to $X^\omega$. Since the isolated points are dense in $X$, $D$ is a $\pi$-base of $X^\omega$. Those sequences $(x_i)_{i \in \omega} \in X^\omega$ for which each $x_i$ is isolated in $X$ form a dense subset of $X^\omega$, and each of these sequences is of countable character in $X^\omega$. Now it follows from Lemma 2.2 that $\text{Clop}(X^\omega)$ is homogeneous. \qed

Note that for every cardinal $\kappa$, $\text{Clop}(X^\kappa)$ is isomorphic to the free product of $\kappa$ copies of $\text{Clop}(X)$. It is easily checked that $\text{Clop}(X^\kappa)$ is homogeneous if there is a cardinal $\lambda \leq \kappa$ such that $\text{Clop}(X^\lambda)$ is homogeneous. Therefore Corollary 2.3 implies that for a Boolean space $X$ with a dense set of isolated points, for every infinite cardinal $\kappa$ the Boolean algebra $\text{Clop}(X^\kappa)$ is homogeneous. In other words, if $B$ is an atomic Boolean algebra, then every free product of infinitely many copies of $B$ is homogeneous.

To proceed we need a technical lemma relating the cellularity of a compact space with many autohomeomorphisms to the cellularities of its nonempty open subsets.

For a topological space $X$ let $c(X)$ denote the cellularity of $X$. Recall that $\text{Aut}(X)$ is the group of autohomeomorphisms of $X$. For $x \in X$ the $\text{Aut}(X)$-orbit of $x$ is the set $\{h(x) : h \in \text{Aut}(X)\}$.

**Lemma 2.4.** Let $X$ be compact and infinite. If every $x \in X$ has a dense $\text{Aut}(X)$-orbit, then for every nonempty open subset $O$ of $X$ we have $c(O) = c(X)$.

**Proof.** It is easily checked that all $\text{Aut}(X)$-orbits are dense in $X$ if and only if for every nonempty open set $O \subseteq X$, $\{h(O) : h \in \text{Aut}(X)\}$ covers $X$. Let $O \subseteq X$ be open and nonempty. By the compactness of $X$, there are $n \in \omega$ and $h_1, \ldots, h_n \in \text{Aut}(X)$ such that $X = h_1(O) \cup \cdots \cup h_n(O)$.

Let $A$ be an infinite family of pairwise disjoint subsets of $X$. For some $i \in \{1, \ldots, n\}$, the set $\{A \in \mathcal{A} : A \cap h_i(O) \neq \emptyset\}$ is of size $|\mathcal{A}|$. It follows that $c(O) \geq |\mathcal{A}|$. This implies $c(O) = c(X)$. \qed

Now we have collected the necessary tools to show

**Theorem 2.5.** Let $X$ be a Boolean space which is not c.c.c. and has a point of countable character. Suppose every $x \in X$ has a dense $\text{Aut}(X)$-orbit. Then $\text{Clop}(X)$ is homogeneous.

**Proof.** Since $X$ is not c.c.c., $X$ is infinite. Since every $\text{Aut}(X)$-orbit is dense in $X$, $X$ has no isolated points. Let $x_0 \in X$ be a point of countable character. Since the $\text{Aut}(X)$-orbit of $x_0$ is dense in $X$, $X$ has a dense set of points of countable character. By Lemma 2.2, it remains to show that $X$ has a $\pi$-base consisting of clopen sets which are homeomorphic to $X$.

Let $(U_n)_{n \in \omega}$ be a neighborhood base of $x_0$ consisting of clopen sets. For every $n \in \omega$ there are $m \in \omega$ and $h_1, \ldots, h_m \in \text{Aut}(X)$ such that $X = h_1[U_n] \cup \cdots \cup h_m[U_n]$. It follows that for each $n \in \omega$, $X$ is homeomorphic to a disjoint union of finitely many copies of clopen subsets of $U_n$.

Now let $O$ be a nonempty open subset of $X$. By Lemma 2.4, there is an uncountable family $\mathcal{A}$ of pairwise disjoint nonempty open subsets of $O$. For every $A \in \mathcal{A}$ let $h_A \in \text{Aut}(X)$ be such that $h_A(x_0) \in A$. $h_A$ exists since the $\text{Aut}(X)$-orbit of $x_0$ is dense. For every $A \in \mathcal{A}$ there is $n(A) \in \omega$ such that $h[U_{n(A)}] \subseteq A$. Since $\mathcal{A}$ is uncountable, there is $n_0 \in \omega$ such that for uncountably many $A \in \mathcal{A}$, $n(A) = n_0$. 


Corollary 2.6. Let $X$ be a first countable Boolean space of uncountable cellularity. If every point in $X$ has a dense $\text{Aut}(X)$-orbit, then $\text{Clop}(X)$ and $X$ are both homogeneous. In particular, $X$ is homogeneous if and only if $\text{Clop}(X)$ is.

Proof. The homogeneity of $\text{Clop}(X)$ follows immediately from Theorem 2.4. The homogeneity of $X$ now follows from Lemma 2.1. □

3. Examples of homogeneous Boolean spaces

The homogeneous Boolean spaces we are going to construct will be Stone spaces of interval algebras of certain linear orders. As usual, if $(L, \leq)$ is a linear order, we use $<$ to denote $\leq \setminus \{\neq\}$. Similarly, if $<$ is transitive and irreflexive, we use $\leq$ to denote $\leq \cup \{\neq\}$.

Definition 3.1. Let $(L, \leq)$ be a linear order. The interval algebra $B(L)$ of $L$ is the subalgebra of $\mathcal{P}(L)$ generated by the intervals $[x, y)$, $x, y \in L$, $x < y$.

Every element of $B(L)$ is a finite union of intervals of the form $[x, y)$, $x, y \in L \cup \{\infty\}$, $x < y$, and of the form $(\infty, x)$, $x \in L \cup \{\infty\}$ (see [4]).

The Stone space of an interval algebra $B(L)$ is homeomorphic to the linear order of initial segments of $L$ (see [1]). Using this fact, we can characterize those linear orders whose interval algebras have a first countable Stone space.

Call a subset $S$ of a linear order $L$ coinitial if for all $a \in L$ there is $b \in S$ such that $b \leq a$. The coinitiality of $L$ is the least size of a coinitial subset of $L$, which is the same as the cofinality of the reversed order.

Lemma 3.2. The Stone space of an interval algebra $B(L)$ is first countable if and only if every initial segment of $L$ has a countable cofinality and every final segment has a countable coinitiality.

Proof. Let $X$ be the set of initial segments of the linear order $L$. $X$ itself is linearly ordered by $\subseteq$. Suppose $X$ is first countable. We show that every initial segment of $L$ is of countable cofinality. The proof that every final segment of $L$ is of countable coinitiality is symmetric.

By the first countability of $X$, for every $x \in X$, the set $\{y \in X : y \subsetneq x\}$ is of countable cofinality. Let $x \in X$ be nonempty and assume that $x$ has no last element. Then the set $\{(-\infty, a) : a \in x\}$ is cofinal in $\{y \in X : y \subsetneq x\}$. Therefore, $\{(-\infty, a) : a \in x\}$ is of countable cofinality. This implies that $x$ is of countable cofinality.

Now suppose that every initial segment of $L$ is of countable cofinality and that every final segment of $L$ is of countable coinitiality. To show the first countability of $X$, it suffices to prove that for all $x \in X$ the following two conditions hold:

1. If in $X$ there is no largest element below $x$, then $x$ is the first element of $X$ or there is a countable sequence in $X$ converging to $x$ from the left.
2. If in $X$ there is no smallest element above $x$, then $x$ is the last element of $X$ or there is a countable sequence in $X$ converging to $x$ from the right.

We show only the first condition since the proof of the second condition is symmetric. Suppose that there is no largest element in $X$ which is below $x \in X$. Assume further that $x$ is not the first element of $X$. Then $x$ as a subset of $L$ is nonempty and does not have a last element. By our assumption on $L$, there is a sequence $(a_n)_{n \in \omega}$ which is cofinal in $x$. We may assume that $(a_n)_{n \in \omega}$ is strictly increasing. The sequence $\langle (-\infty, a_n) \rangle_{n \in \omega}$ of initial segments of $L$ converges to $x$ from the left. □
Lemma 3.2 easily implies

**Corollary 3.3.** If the linear order \( L \) has no uncountable sequences (indexed by ordinals) which are strictly increasing or strictly decreasing, then the Stone space of \( B(L) \) is first countable. In particular, the Stone space of \( B(L) \) is first countable if \( L \) has a countable dense subset.

**Proof.** If \( L \) has an initial segment of uncountable cofinality, then it has a strictly increasing sequence of length \( \omega_1 \). Similarly, if \( L \) has a final segment of uncountable cofinality, then it has a strictly decreasing sequence of length \( \omega_1 \).

If \( L \) has a strictly increasing or strictly decreasing sequence of length \( \omega_1 \), then it is not c.c.c. and therefore cannot have a countable dense subset. \( \square \)

The following lemma provides an easy criterion for the homogeneity of an interval algebra.

**Lemma 3.4.** Let \( L \) be a linear order with the property that every two nonempty open intervals of \( L \) are isomorphic. Then \( B(L) \) is homogeneous.

**Proof.** Since every two nonempty open intervals of \( L \) are isomorphic, also the intervals of the form \([x, y)\) with \( x \in L, y \in L \cup \{\infty\}, x < y, \) and \((x, y) \neq \emptyset\) are pairwise isomorphic. It follows that for all \( n \in \omega \) and all \( x_0, \ldots, x_{2n+1} \in L \cup \{-\infty, \infty\}\) with \( x_0 < \cdots < x_{2n+1} \) and \((x_0, x_{2n+1}) \neq \emptyset, (x_0, x_1) \cup \bigcup_{i=1}^{n} [x_{2i}, x_{2i+1}] \) is isomorphic to \( L \).

Let \( a \in B(L) \setminus \{\emptyset\} \). Then for some \( n \in \omega \) there are \( x_0, \ldots, x_{2n+1} \in L \cup \{-\infty, \infty\}\) with \( x_0 < \cdots < x_{2n+1} \) such that either \( x_0 = -\infty \) and \( a = (x_0, x_1) \cup \bigcup_{i=1}^{n} [x_{2i}, x_{2i+1}] \) or \( x_0 \in L \) and \( a = [x_0, x_1) \cup \cdots \cup [x_{2n}, x_{2n+1}) \). In either case, it is easily checked that \( B(L) \setminus x \) is isomorphic to the interval algebra of \((x_0, x_1) \cup \bigcup_{i=1}^{n} [x_{2i}, x_{2i+1}] \). Since \((x_0, x_1) \cup \bigcup_{i=1}^{n} [x_{2i}, x_{2i+1}] \) is isomorphic to \( L, B(L) \setminus a \) is isomorphic to \( B(L) \). \( \square \)

Combining the information we have gathered so far we obtain

**Lemma 3.5.** Let \( L \) be a linear order such that every nonempty open interval of \( L \) isomorphic to \( L \). Then \( \text{Ult}(B(L)) \) is homogeneous if and only if \( L \) has no strictly increasing or strictly decreasing sequences of length \( \omega_1 \). In particular, \( \text{Ult}(B(L)) \) is homogeneous if \( L \) is separable.

**Proof.** Since \( L \) is isomorphic to every one of its nonempty open intervals, \( B(L) \) is homogeneous by Lemma 3.4. If \( L \) has no strictly increasing or strictly decreasing sequences of uncountable length, then \( \text{Ult}(B(L)) \) is first countable by Corollary 3.3 and homogeneous by Lemma 2.3. Now suppose that \( L \) has a strictly increasing or strictly decreasing sequence of uncountable length. Then the Stone space of \( B(L) \), being homeomorphic to the linear order of initial segments of \( L \), is not first countable and therefore cannot be homogeneous, as mentioned in the introduction. \( \square \)

It remains to construct linear orders with the properties required in Lemma 3.5. We first construct some separable linear orders.

**Lemma 3.6.** For every cardinal \( \kappa \in [\aleph_0, 2^{\aleph_0}] \) there is a separable dense linear order \( L \) of size \( \kappa \) without endpoints which is isomorphic to every one of its nonempty open intervals.

**Proof.** We define two operations \( f, f^- : \mathbb{R}^3 \to \mathbb{R} \) as follows: For \( x, y \in \mathbb{R} \) with \( x < y \) let \( g_{x,y} \) be an order isomorphism from \( \mathbb{R} \) onto \((x, y)\). If \( x, y, z \in \mathbb{R} \) are such that \( x < y, \) we let \( f(x, y, z) := g_{x,y}(z) \). Otherwise let \( f(x, y, z) := 0. \) If \( x, y, z \in \mathbb{R} \) are such that \( x < z < y \), we put \( f^-(x, y, z) := g_{x,y}^{-1}(z) \). Otherwise let \( f^-(x, y, z) := 0 \).

Now let \( \kappa \in [\aleph_0, 2^{\aleph_0}] \) be a cardinal. Let \( L \) be a subset of \( \mathbb{R} \) of size \( \kappa \) such that \( \mathbb{Q} \subseteq L \) and \( L \) is closed under \( f \) and \( f^- \). Then \( L \) is separable since \( \mathbb{Q} \subseteq L. \) For
Lemma 2.1, $X := \text{Ult}(\lambda)$. There is a dense linear order $L, \kappa, \gamma$ such that $L, \kappa, \gamma$ is a first countable homogeneous Boolean space. Since $2^{\omega_1}$ is a linear order with $\gamma > \omega$, it is the Stone space of an interval algebra (see [4]). Let $L$ be a first countable homogeneous Boolean space. Since $2^{\omega_1}$ is homogeneous, too.

Corollary 2.6, $B(\lambda) = \text{Ult}(\lambda)$ is homogeneous by Lemma 3.5. □

In order to construct first countable homogeneous Boolean spaces of a given weight in the interval $[\aleph_1, 2^{\aleph_0}]$ that are not c.c.c., one can start with an indecomposable countable ordinal $\gamma > \omega$ and use the downward Löwenheim-Skolem theorem to get a continuous image of the lexicographically ordered space $2^{\gamma}$ with the right properties.

Theorem 3.8. For every cardinal $\kappa \in [\aleph_0, 2^{\aleph_0}]$ there is a first countable homogeneous Boolean space $X$ of weight and cellularity $\kappa$.

Proof. Let $\gamma > \omega$ be a countable indecomposable ordinal. Then the lexicographically ordered space $2^{\gamma}$ is a first countable homogeneous Boolean space. Since $2^{\gamma}$ is linearly ordered, it is the Stone space of an interval algebra (see [4]). Let $L$ be a first countable homogeneous Boolean space.

Since $\text{Ult}(B(L))$ is first countable, in $L$ there is no strictly increasing or strictly decreasing sequence of length $\omega_1$. The cellularity of $2^{\gamma}$ is $2^{\aleph_0}$, as is the weight. By Corollary 2.6, $B(L) = \text{Ult}(B(L))$.

Let $\lambda$ be a sufficiently large cardinal and consider the structure $(H_\lambda, \in)$ where $H_\lambda$ is the family of sets whose transitive closure is of size $< \lambda$. Fix an antichain $A \subseteq B(L)$ of size $2^{\aleph_0}$ and let $M$ be an elementary submodel of $(H_\lambda, \in)$ of size $\kappa$ such that $L, A \in M$ and $\kappa \subseteq M$.

Let $B := B(L) \cap M$. By elementarity, $B = B(L \cap M)$. $L \cap M$ is a linear order without strictly increasing or strictly decreasing sequences of length $\omega_1$. Thus, $X := \text{Ult}(B)$ is first countable. Again by elementarity, $B$ is homogeneous. By Lemma 2.1, $X$ is homogeneous, too.

Since $\kappa \subseteq M$, $B$ is of size $\kappa$, and so is $A \cap M$. It follows that the cellularity of $X$ is $\kappa$. □

There is another interesting example of a first countable homogeneous Boolean space. This one is constructed from a linear order on an Aronszajn tree and is not the continuous image of any of the first countable homogeneous Boolean spaces mentioned so far.

Recall that a tree is Aronszajn if it is of height $\omega_1$, has only countable levels, and does not include an uncountable chain. If $T$ is a tree ordered by $\supseteq$, we will always assume that incomparable elements of $T$ are disjoint.

Lemma 3.9. There is a dense linear order $L$ of size $\aleph_1$ without endpoints and with the following properties:

(i) $L$ has no strictly increasing or strictly decreasing sequences of length $\omega_1$.
(ii) $L$ is isomorphic to every one of its nonempty open intervals.
(iii) $L$ is not c.c.c.
(iv) $B(L)$ has a subset which is an Aronszajn tree (ordered by $\supseteq$).
Proof. For two functions $f$ and $g$ with the same domain we write $f =^* g$ if $f$ and $g$ agree on all but finitely many points of their common domain. Using the construction of an Aronszajn tree given in [6], we obtain a sequence $(f_\alpha)_{\alpha < \omega_1}$ such that each $f_\alpha$ is a 1-1 function from $\alpha$ into $\mathbb{Q} \cap (0,1)$ and for all $\alpha, \beta < \omega_1$ with $\alpha < \beta$, $f_\alpha =^* f_\beta \upharpoonright \alpha$.

Now for each $\alpha < \omega_1$ let

$$S_\alpha := \{ f \in {}^\alpha \mathbb{Q} : f \text{ is 1-1 and } f =^* f_\alpha \}$$

and $T_\alpha := \bigcup_{\beta < \alpha} S_\beta$. $T := \bigcup_{\alpha < \omega_1} T_\alpha$ ordered by inclusion is an Aronszajn tree.

We define a linear order on $T$. Let $x, y \in T$ be such that $x \neq y$. If $x$ and $y$ are incomparable with respect to $\subseteq$, put $\Delta(x, y) := \min \{ \nu \in \text{dom}(x) : x(\nu) \neq y(\nu) \}$ and let $x < y$ if $x(\Delta(x, y)) < y(\Delta(x, y))$. If $x \subseteq y$ and $\text{dom}(x) = \alpha$, let $x < y$ if $y(\alpha) > \pi$ and $y < x$ if $y(\alpha) < \pi$.

In other words, $T$ is ordered lexicographically after identifying each $x \in T$ with the function $x \sim \pi$ where $\sim$ denotes the concatenation of sequences and $\pi$ is identified with the sequence of length one with value $\pi \in \mathbb{R}$.

Claim 3.10. $(T, \subseteq)$ is not c.c.c.

It is wellknown that $T$ ordered by reverse inclusion is not c.c.c. For example, for every $\alpha < \omega_1$ let $x \in S_{\alpha+1}$ be such that $x(\alpha) = 0$. Since the $x_\alpha$, $\alpha < \omega_1$, are 1-1, $(x_{\alpha})_{\alpha < \omega_1}$ is an antichain in $(T, \supseteq)$. Note that for all $x \in T$, $\mathbb{Q} \setminus \text{ran}(x)$ is infinite since otherwise the construction of the $f_\alpha$, $\alpha < \omega_1$, would break down at some point. Thus, for every $\alpha < \omega_1$ there are $p_\alpha, q_\alpha \in \mathbb{Q} \setminus \text{ran}(x_\alpha)$ with $p_\alpha < q_\alpha$. Now for each $\alpha < \omega_1$, $x_{\alpha} \sim q_\alpha$ and $x_{\alpha} \sim p_\alpha$ are elements of $T$. Clearly, $(\{x_{\alpha} \sim p_\alpha, x_{\alpha} \sim q_\alpha : \alpha < \omega_1\})$ is an uncountable family of pairwise disjoint nonempty open intervals of $(T, \subseteq)$.

Claim 3.11. In $(T, \subseteq)$ there is no strictly increasing or strictly decreasing sequence of length $\omega_1$.

We only show that there is no strictly increasing sequence of length $\omega_1$ since the proof for decreasing sequences is the same. Suppose $(x_\alpha)_{\alpha < \omega}$ is a sequence in $T$ that is strictly decreasing (with respect to $\subseteq$). We construct a strictly increasing subsequence $(y_\alpha)_{\alpha < \omega}$ such that

1. for all $\alpha, \beta < \omega_1$ with $\alpha < \beta$, $\text{dom}(y_\alpha) < \text{dom}(y_\beta)$ and
2. for all $\alpha, \beta, \gamma < \omega_1$, if $\alpha < \beta < \gamma$ and $\delta = \text{dom}(y_\alpha)$, then $y_\beta \upharpoonright (\delta + 1) = y_\gamma \upharpoonright (\delta + 1)$.

In order to construct $(y_\alpha)_{\alpha < \omega}$ first note that by the countability of every $T_\alpha$, $\alpha < \omega_1$, we can thin out $(x_\alpha)_{\alpha < \omega_1}$ and assume $\text{dom}(x_\alpha) < \text{dom}(x_\beta)$ for all $\alpha, \beta < \omega_1$ with $\alpha < \beta$.

Suppose we have found $y_\delta \in \{x_\nu : \nu < \omega_1\}$ with $\text{dom}(y_\delta) = \delta$. The set $C := \{x_\nu : \delta + 1 : y_\delta < x_\nu \land \nu < \omega_1\}$ is a countable linear order (ordered by $\subseteq$). For every $x \in C$ let

$$\text{ext}(x) := \{x_\nu : x_\nu \upharpoonright \delta + 1 = x \land \nu < \omega_1\}.$$ 

Clearly, $(\text{ext}(x))_{x \in C}$ is a partition of $\{x_\nu : y_\delta < x_\nu \land \nu < \omega_1\}$ into countably many convex sets. Since for all $x, y \in C$ with $x < y$ all elements of $\text{ext}(x)$ are below all elements of $\text{ext}(y)$, for cofinality reasons $C$ has a last element $x$. Now $\text{ext}(x)$ is a final segment of $\{x_\nu : y_\delta < x_\nu \land \nu < \omega_1\}$. Choose $y_{\delta + 1} \in \text{ext}(x)$. This finishes the successor step of the construction.

Now suppose that $\alpha < \omega_1$ is a limit ordinal and $y_\beta$ has been chosen for all $\beta < \alpha$. Since $\{x_\nu : \nu < \omega_1\}$ is of cofinality $\aleph_1$, the set $\{y_\beta : \beta < \alpha\}$ is bounded in $\{x_\nu : \nu < \omega_1\}$. Pick $y_\alpha \in \{x_\nu : \nu < \omega_1\}$ such that $y_\beta < y_\alpha$ for all $\beta < \alpha$. It is easily checked that this construction yields a sequence with the desired properties.

To finish the proof of Claim 3.11, for each $\alpha < \omega_1$ let $\delta_\alpha := \text{dom}(y_\alpha)$ and consider the sequence $(y_{\alpha+1} \upharpoonright \delta_\alpha)_{\alpha < \omega_1}$. By the choice of the $y_\alpha$, $\alpha < \omega_1$, for all $\alpha, \beta < \omega_1$
with \( \alpha < \beta \) we have \( y_{\alpha+1} \upharpoonright \delta_\alpha \nsubseteq y_{\beta+1} \upharpoonright \delta_\beta \), contradicting the fact that \( T \) has no uncountable chains with respect to \( \subseteq \).

In order to prove that \( (T, \leq) \) is isomorphic to each of its nonempty open intervals it suffices to show

**Claim 3.12.** For every \( x \in T \), \( (T, \leq) \) is isomorphic to \( (x, \infty) \) and to \( (-\infty, x) \).

We only show \( T \cong (x, \infty) \) since the proof of \( T \cong (-\infty, x) \) is symmetric. Let \( \delta := \text{dom}(x) + 1 \).

For each \( y \in S_\delta \) let \( \text{ext}(y) := \{ z \in T : \delta \subseteq \text{dom}(z) \land \delta \upharpoonright z = y \} \). As before, \( \text{ext}(y) \) is a convex subset of \( T \). For \( y, z \in S_\delta \), \( (\text{ext}(y), \leq | \text{ext}(y)) \) and \( (\text{ext}(z), \leq | \text{ext}(z)) \) are isomorphic by the isomorphism mapping every \( y' \in \text{ext}(y) \) to \( z \uparrow y' \upharpoonright (\text{dom}(y') \setminus \delta) \).

As suborders of \( (T, \leq) \), \( T_{\delta+1} \) and \( T_{\delta+1} \cap (x, \infty) \) both are countable dense linear orders without endpoints. \( S_\delta \) is a dense and co-dense subset of \( T_{\delta+1} \) and \( S_\delta \cap (x, \infty) \) is a dense and co-dense subset of \( T_{\delta+1} \cap (x, \infty) \). By the usual back-and-forth argument, there is an isomorphism \( \varphi \) between \( T_{\delta+1} \) and \( T_{\delta+1} \cap (x, \infty) \) mapping \( S_\delta \) onto \( S_\delta \cap (x, \infty) \).

For every \( y \in S_\delta \) let \( \varphi_y \) be an isomorphism between \( \text{ext}(y) \) and \( \text{ext}(\varphi(y)) \). Now we can construct an isomorphism \( \psi \) between \( T \) and \( (x, \infty) \) by letting \( \psi(y) := \varphi(y) \) for every \( y \in T_{\delta} \) and \( \psi(y) := \varphi_y(b(y)) \) for every \( y \in T \setminus T_{\delta} \).

Finally, we have to find an Aronszajn tree inside \( B(T, \leq) \). Let \( T' \) be the subtree of \( T \) consisting of those elements of \( T \) whose ranges are subsets of \( (0, 1) \). As \( (T, \subseteq) \), \( (T', \subseteq) \) is an Aronszajn tree. This is the place where we take advantage of the fact that the ranges of the \( f_\alpha \), \( \alpha < \omega_1 \), are subsets of \( (0, 1) \). Mapping every \( x \in T' \) to the interval \( [x^-0, x^-1) \) of \( (T, \leq) \), we obtain an embedding of \( (T', \subseteq) \) into \( (B(T, \leq), \supseteq) \).

Note that this embedding maps incomparable elements of \( T' \) to disjoint members of \( B(T, \leq) \).

**Theorem 3.13.** There is a homogeneous Boolean space of weight \( \aleph_1 \) which is first countable, not c.c.c., and not a continuous image of any of the lexicographically ordered spaces \( 2^\gamma \), \( \gamma < \omega_1 \).

**Proof.** Let \( L \) be a linear order as in Lemma 3.3. Then \( \text{Ult}(B(L)) \) is first countable by Corollary 3.3. \( \text{Ult}(B(L)) \) is homogeneous by Lemma 3.3 and \( B(L) \) is of size \( \aleph_1 \) since \( L \) is. \( B(L) \) is not c.c.c. since \( L \) is not c.c.c.

Now suppose that \( \text{Ult}(B(L)) \) is a continuous image of the lexicographically ordered space \( 2^\gamma \) for some \( \gamma < \omega_1 \). Then \( B(L) \) embeds into \( \text{Clop}(2^\gamma) \). By condition (iv) of Lemma 3.9, this implies that \( \text{Clop}(2^\gamma) \) has a subset \( T \) such that \( (T, \supseteq) \) is an Aronszajn tree. This contradicts

**Claim 3.14.** Let \( T \subseteq \text{Clop}(2^\gamma) \) be such that \( (T, \supseteq) \) is a tree whose levels are all countable. Then \( T \) is countable.

Every element \( a \) of \( \text{Clop}(2^\gamma) \) can be uniquely written as a finite union of clopen intervals that are maximal convex subsets of \( a \). We may assume that \( \gamma \) is infinite. If \( a \in \text{Clop}(2^\gamma) \) is nonempty, let \( \text{depth}(a) \) be the least ordinal \( \alpha < \gamma \) such that there are \( x, y \in a \) with \( x < y \) and \( \Delta(x, y) = \alpha \) such that the closed interval \( [x, y] \) is a maximal convex subset of \( a \).

For every \( a \in T \) let \( \text{height}(a) \) be the ordertype of \( (\{ b \in T : b \supseteq a \}, \supseteq) \). We show that for every \( a \in T \) and every \( \alpha < \omega_1 \) the following statement holds:

\((*)_{a, \alpha}\) The set \( \{ b \in T : b \subseteq a \land \text{depth}(b) \leq \text{depth}(a) + \alpha \} \) is countable.

The claim follows from this since there is no \( b \in T \) with \( \text{depth}(b) > \gamma \).

We show \((*)_{a, \alpha}\) by induction on \( \alpha < \omega_1 \) simultaneously for all \( a \in T \). Let \( a \in T \). We start with proving \((*)_{a, 1}\) since \((*)_{a, 0}\) is trivial, i.e., there is no \( b \in T \) with \( b \subseteq a \) and \( \text{depth}(b) < \text{depth}(a) \).
If \( b \in T \) is such that \( b \subseteq a \) and depth\( (b) = \text{depth}(a) \), then there are \( x, y \in b \) such that \( x < y, \ [x, y] \) is a maximal convex subset of \( b \), and \( \Delta(x, y) = \text{depth}(a) \). It is easily checked that there is \( z \in a \) such that either

a) \( z < x, \ [z, y] \) is a maximal convex subset of \( a \), and depth\( (a) = \Delta(z, y) \) or

b) \( y < z, \ [x, z] \) is a maximal convex subset of \( a \), and depth\( (a) = \Delta(x, z) \).

Suppose that \( \{ b \in T : b \subseteq a \land \text{depth}(b) = \text{depth}(a) \} \) is uncountable. Then there is \( p \in a \) such that for uncountably many \( b \in T \) with \( b \subseteq a, p \) occurs as \( y \) in \( a \) or as \( x \) in \( b \). This implies that \( \{ b \in T : b \subseteq a \land p \in b \} \) is uncountable. By our assumption on trees ordered by \( \supseteq \), any two elements of \( T \) are either disjoint or comparable, and thus \( \{ b \in T : b \subseteq a \land p \in b \} \) is a chain in \( T \). But this contradicts the fact that Clop\( (2^\omega) \) does not include any uncountable wellordered chain. This finishes the proof of \((*)_{a,1}\).

Now let \( \alpha = \beta + 1 \) for some \( \beta < \omega_1 \) and suppose we have \((*)_{b,\beta}\) for all \( b \in T \). Let \( a \in T \). By \((*)_{a,\beta}\), there are only countably many \( b \in T \) with \( a \supseteq b \) and depth\( (b) < \text{depth}(a) + \beta \). Let \( \delta < \omega_1 \) be a bound for the heights of such \( b \). If \( b \in T \) is minimal (with respect to the order \( \supseteq \) on \( T \)) with \( a \supseteq b \) and depth\( (b) = \text{depth}(a) + \beta \), then height\( (b) \leq \delta + 1 \). It follows that there are only countably many \( b \in T \) that are minimal with \( a \supseteq b \) and depth\( (b) = \text{depth}(a) + \beta \). Since \( (T, \supseteq) \) is a tree, every \( b \in T \) with \( a \supseteq b \) and depth\( (b) = \text{depth}(a) + \beta \) is above a minimal element of \( T \) with these properties. Applying \((*)_{b,1}\) for every minimal \( b \in T \) with \( a \supseteq b \) and depth\( (b) = \text{depth}(a) + \beta \), we obtain \((*)_{a,\beta+1}\).

Finally suppose that \( \alpha < \omega_1 \) is a limit ordinal and for all \( a \in T \) and all \( \beta < \alpha \) we have \((*)_{a,\beta}\). Then for all \( a \in T \) the set

\[
\{ b \in T : b \subseteq a \land \text{depth}(b) < \text{depth}(a) + \beta \} = \\
\bigcup_{\beta<\alpha} \{ b \in T : b \subseteq a \land \text{depth}(b) < \text{depth}(a) + \beta \}
\]

is countable, which shows \((*)_{a,\alpha}\). \( \square \)

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