Quantum conductance problems and the Jacobi ensemble

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In one dimensional transport problems the scattering matrix $S$ is decomposed into a block structure corresponding to reflection and transmission matrices at the two ends. For $S$ a random unitary matrix, the singular value probability distribution function of these blocks is calculated. The same is done when $S$ is constrained to be symmetric, or to be self dual quaternion real, or when $S$ has real elements, or has real quaternion elements. Three methods are used: metric forms; a variant of the Ingham-Seigel matrix integral; and a theorem specifying the Jacobi random matrix ensemble in terms of Wishart distributed matrices.

1 Introduction

In mesoscopic physics, the conductance of certain quasi one-dimensional wires containing scattering impurities exhibit the phenomenon known as universal conductance fluctuations. As first predicted theoretically [11, 17], and soon after observed experimentally [17], for such wires the variance of the conductance is of order $(e^2/h)^2$, independent of sample size or disorder strength. Furthermore the variance decreases by precisely a factor of two if time reversal symmetry is broken by a magnetic field. This conductance problem is fundamental for its relation to the Landauer scattering theory of electronic conduction (see [4] and references therein), and to time reversal symmetry. It is further fundamental for its relation to random matrix theory. Let us revise these points by way of background and motivation for the specific problem of this paper.

In the theoretical description of the conductance problem, basic quantities are the electron fluxes at the left and right hand edges of the wire. These are specified by an $n$-component vector $\vec{I}$ and an $m$-component vector $\vec{I}'$ specifying the complex amplitudes of the available plane wave states travelling into the left and right sides of the wire respectively, as well as an $n$-component vector $\vec{O}$ and an $m$-component vector $\vec{O}'$ for the same states travelling out of the left and right sides of the wire. For definiteness it will be assumed that $n \geq m$.

By definition, the scattering matrix $S$ relates the flux travelling into the conductor to that travelling out,

$$S \begin{bmatrix} \vec{I} \\ \vec{I}' \end{bmatrix} = \begin{bmatrix} \vec{O} \\ \vec{O}' \end{bmatrix}.$$ \hspace{1cm} (1.1)

The scattering matrix is further decomposed in terms of reflection and transmission matrices by

$$S = \begin{bmatrix} r_{n \times n} & r'_{n \times m} \\ t_{m \times n} & t'_{m \times m} \end{bmatrix}.$$ \hspace{1cm} (1.2)
Flux conservation requires
\[ |\vec{I}|^2 + |\vec{J}|^2 = |\vec{\Omega}|^2 + |\vec{\Omega}'|^2, \]
and this implies that \( S \) must be unitary. Furthermore, by relating \( S \) to an evolution operator and thus a Hamiltonian, one can argue (see e.g. [6]) that \( S \) must be symmetric if the system has a time reversal symmetry with \( T^2 = 1 \), and a self dual quaternion matrix when there is a time reversal symmetry with \( T^2 = -1 \).

The immediate relevance of the above formalism is seen by invoking the Landauer scattering theory of electronic conduction. According to this formalism, the conductance \( G \) is given in terms of the transmission matrix \( t_{m \times n} \) (or \( t'_{n \times m} \)) by the so called two probe Landauer formula
\[ G/G_0 = \text{Tr}(t^\dagger t) = \text{Tr}(t'^\dagger t') \tag{1.3} \]
where \( G_0 = 2e^2/h \) is twice the fundamental quantum unit of conductance. Thus to compute \( G \) it suffices to know the distribution of the eigenvalues of \( t^\dagger t \) (or \( t'^\dagger t' \)). In fact the matrix \( S \) can be decomposed in a form which isolates these eigenvalues.

For definiteness suppose there is no time reversal symmetry, so \( S \) is a general unitary matrix. Each block of \( S \) can then be decomposed according to a general singular value decomposition. For example, for the block \( r_{n \times n} \) we have
\[ r_{n \times n} = U_r \Lambda_r V_r^\dagger, \]
where \( U_r, V_r \) are unitary matrices and \( \Lambda_r \) is a rectangular diagonal matrix with entries equal to the positive square roots of the eigenvalues of \( r^\dagger r \) (these eigenvalues are between 0 and 1 since \( r^\dagger r + t^\dagger t = I_{n \times n} \)). The unitarity of \( S \) inter-relates the matrices \( U_r, U_{r'}, \ldots \) (see e.g. [16]) and implies the decomposition
\[ S = \begin{bmatrix} U_r & 0 \\ 0 & U_{r'} \end{bmatrix} L \begin{bmatrix} V_r^\dagger & 0 \\ 0 & V_{r'}^\dagger \end{bmatrix} \tag{1.4} \]
where
\[ L := \begin{bmatrix} \sqrt{1 - \Lambda_r \Lambda_r^T} & i \Lambda_r \Lambda_r^T \\ i \Lambda_r^T/\sqrt{1 - \Lambda_r \Lambda_r^T} & \Lambda_r^T \end{bmatrix}. \tag{1.5} \]

Symmetries relate \( V_r, V_{r'} \) to \( U_r, U_{r'} \). As already remarked, in the case of a time reversal symmetry \( T^2 = 1 \), \( S \) must be symmetric, while for \( T^2 = -1 \), \( S \) must be self dual quaternion real. These symmetries require that
\[ V_r^\dagger = U_r^T, \quad V_{r'}^\dagger = U_{r'}^T, \quad \text{and} \quad V_r^\dagger = U_r^D, \quad V_{r'}^\dagger = U_{r'}^D \tag{1.6} \]
respectively. In (1.4) the operation \( D \), for an \( n \times n \) matrix \( A \) with real quaternion elements regarded as a \( 2n \times 2n \) matrix with complex elements, is specified by
\[ A^D = Z_{2n} A^T Z_{2n}^{-1}, \quad Z_{2n} := I_n \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \]

The relevance of random matrix theory comes from hypothesizing that apart from the constraint imposed by a time reversal symmetry, in the regime relevant to universal conductance fluctuations, the scattering matrix \( S \) is effectively a random matrix chosen with Haar (uniform) measure. With this assumption the non-zero elements of \( \Lambda_r \), which in turn are equal to the square root of the non-zero eigenvalues of \( t^\dagger t \) and are thus in the interval \((0,1)\), has a joint probability density function (p.d.f.) proportional to
\[ \prod_{j=1}^{m} \lambda_j^{\beta_n} \prod_{1 \leq j < k \leq m} |\lambda_j^2 - \lambda_k^2|^\beta, \quad \alpha := n - m + 1 - 1/\beta, \tag{1.7} \]
where $\beta = 1, 2, 4$ according to $S$ being constrained to be symmetric, no constraints, or constrained to be self dual quaternion. In its full generality this result is stated without proof in Beenakker \[4\] eq. (2.9)], where it is attributed to Brouwer. In the case $n = m$ it was derived in \[2, 14\]. With knowledge of (1.7), it can be deduced from \[18\] that with $\alpha$ fixed
\[
\lim_{m, n \to \infty} \text{Var}(G/G_0) = \frac{1}{8 \beta},
\] (1.8)
which is in quantitative agreement with experiment.

However the issue of fluctuation formulas is not our concern here. Rather we seek to specify three distinct derivations of (1.7) in the case $\beta = 2$, each of which we believe to have particular features of interest. The first derivation to be given uses the method of metric forms. This method applies to all three cases, and in fact details will be given in the case $\beta = 1$. The second derivation uses versions of the Ingham-Seigel integral; while the third relies on knowledge of the Jacobi random matrix ensemble as derived from Wishart distributed matrices. These latter two derivations generalize (1.7) in the case $\beta = 2$, and also allow $\beta$ generalizations relating to the block decomposition of unitary matrices, however now with $\beta = 1$ referring to the elements of $S$ being real, and $\beta = 4$ to the elements being real quaternion. Some aspects of statistical properties of the block decomposition of real orthogonal matrices (i.e. unitary matrices with real elements) have been previously given in \[18\].

### 2 Metric forms

Let $X = [x_{jk}]_{j,k=1,...,N}$ be an $N \times N$ matrix. Let $\{x_\mu\}$, where $\mu$ labels both rows and columns, be the set of independent real and imaginary parts in $X$. The metric form of the line element $ds$ is defined by
\[
(ds)^2 := \text{Tr}(dXdX^\dagger) = \sum_{\mu \text{ diag.}} (dx_\mu)^2 + 2 \sum_{\mu \text{ upper triangular}} (dx_\mu)^2.
\]

The corresponding volume measure is
\[
(dX) = \bigwedge_{\mu \text{ diag.}} dx_\mu \bigwedge_{\mu \text{ upper triangular}} dx_\mu.
\] (2.1)

Suppose now a change of variables $\{x_\mu\} \mapsto \{y_\mu\}$ is made such that $(ds)^2$ is a symmetric quadratic form in $\{dy_\mu\}$,
\[
(ds)^2 = \sum_{\mu, \nu} g_{\mu, \nu} dy_\mu dy_\nu, \quad g_{\mu, \nu} = g_{\nu, \mu}.
\]

The corresponding volume measure is then (see e.g. \[12\])
\[
(\det[g_{\mu, \nu}])^{1/2} \bigwedge_{\mu} dy_\mu.
\] (2.2)

In \[6\] this formalism has been used to derive the result (1.7) in the case $\beta = 2$. Here the details will be given for the case $\beta = 1$, when (1.7) reads
\[
S = \begin{bmatrix} U_r & 0 \\ 0 & U_{r'} \end{bmatrix} L \begin{bmatrix} U_r^T & 0 \\ 0 & U_{r'}^T \end{bmatrix},
\] (2.3)
will be given.
For a general matrix $A$, let $\delta A := A^\dagger dA$, where $dA$ denotes the matrix of differentials of the elements of $A$. Using the fact that for $A$ unitary $(\delta A)^\dagger = -\delta A$, it follows from (2.3) that

$$
\begin{bmatrix}
U_1^\dagger & 0 \\
0 & U_r^\dagger
\end{bmatrix} dS
\begin{bmatrix}
\delta U_r & 0 \\
0 & \delta U_r
\end{bmatrix} L + dL - L \begin{bmatrix}
\delta U_r & 0 \\
0 & \delta U_r
\end{bmatrix}.
$$

(4.2)

In $\text{Tr}(dSd^\dagger)$, the right hand side of (2.3) can effectively be substituted for $dS$. Doing this and simplifying using

$$
\text{Tr} \left( \begin{bmatrix}
\delta U_r & 0 \\
0 & \delta U_r
\end{bmatrix} L^\dagger dL \right) = 0
$$

gives

$$
\text{Tr}(dSd^\dagger) = \text{Tr} A_1 \tilde{A}_1 + \text{Tr}(A_2 + \tilde{A}_2) + \text{Tr}(A_3 \tilde{A}_3) + \text{Tr}(dLd^\dagger)
$$

(2.5)

where, with $I$ denoting the identity matrix,

$$
A_1 = \sqrt{I - \Lambda_1^T} \delta U_r - \delta U_r \sqrt{I - \Lambda_1^T}
$$

$$
A_2 = (\Lambda_1 \delta U_r - \delta U_r \Lambda_1)(\Lambda_1^T \delta U_r - \delta U_r \Lambda_1^T)
$$

$$
A_3 = \sqrt{I - \Lambda_1^T} \delta U_r - \delta U_r \sqrt{I - \Lambda_1^T}.
$$

We recall that the diagonal elements of $\Lambda_t$ are the positive square roots of the eigenvalues of $tt^\dagger$. Because $t = t_{m \times n}$ and thus has rank $m$, $tt^\dagger$ must have $n - m$ zero eigenvalues, so with the diagonal elements of $\Lambda_1 \Lambda_1^T$ denoted $\lambda_j$ ($j = 1, \ldots, n$), as is consistent with the notation of (1.7), we have $\lambda_{m+1} = \cdots = \lambda_n = 0$. Using this fact we see that in component form the four terms in (2.5), $T_1, \ldots, T_4$ say, can be expanded to read

$$
T_1 = \sum_{k=1}^n (1 - \lambda_k^2) |(\delta U_r)_{kk} - (\delta U_r)_{kk}|^2 + \sum_{1 \leq k < l \leq n} \frac{1}{2} \left( \sqrt{1 - \lambda_k^2} + \sqrt{1 - \lambda_l^2} \right)^2 |(\delta U_r)_{lk} - (\delta U_r)_{lk}|^2
$$

$$
+ \left( \sum_{1 \leq k < l \leq m} + \sum_{k=1}^m \sum_{l=m+1}^n \right) \frac{1}{2} \left( \sqrt{1 - \lambda_k^2} - \sqrt{1 - \lambda_l^2} \right)^2 |(\delta U_r)_{lk} + (\delta U_r)_{lk}|^2
$$

$$
T_2 = 2 \sum_{k=1}^m \lambda_k^2 |(\delta U_r)_{kk} - (\delta U_r)_{kk}|^2 + \sum_{1 \leq k < l \leq m} \left\{ \frac{1}{2} (\lambda_k + \lambda_k) |(\delta U_r)_{lk} - (\delta U_r)_{lk}|^2
$$

$$
+ \frac{1}{2} (\lambda_k - \lambda_l) |(\delta U_r)_{lk} + (\delta U_r)_{lk}|^2 \right\} + \sum_{k=1}^m \sum_{l=m+1}^n \lambda_k^2 |(\delta U_r)_{lk}|^2
$$

$$
T_3 = \sum_{k=1}^m (1 - \lambda_k^2) |(\delta U_r)_{kk} - (\delta U_r)_{kk}|^2 + \sum_{1 \leq k < l \leq m} \frac{1}{2} \left( \sqrt{1 - \lambda_k^2} + \sqrt{1 - \lambda_l^2} \right)^2 |(\delta U_r)_{lk} - (\delta U_r)_{lk}|^2
$$

$$
+ \frac{1}{2} \left( \sqrt{1 - \lambda_k^2} - \sqrt{1 - \lambda_l^2} \right)^2 |(\delta U_r)_{lk} + (\delta U_r)_{lk}|^2
$$

$$
T_4 = \sum_{j=1}^m (\delta \lambda_j)^2
$$

In general a symmetric unitary matrix has the same number of independent elements as a real symmetric matrix of the same rank, so $S$ has $\frac{1}{2}(m + n)(m + n + 1)$ independent elements. We thus seek this same number of independent differentials in $T_1, \ldots, T_4$. These are

$$
(\delta U_r)_{ij} |_{1 \leq k \leq n}
$$

and

$$
(\delta U_r)_{kk} |_{1 \leq k \leq m}
$$


and
\[(\delta U)^{(i)}_{lk}, \ (\delta U)^{(i)}_{ik}, \ (\delta U)^{(r)}_{lk}, \ (\delta U)^{(r)}_{ik}, \ 1 \leq k < l \leq m,\]
as well as
\[(\delta U)^{(i)}_{lk}, \ (\delta U)^{(i)}_{ik}, \ 1 \leq l \leq m \text{ & } m+1 \leq l \leq n\]
and
\[(\delta U)^{(i)}_{ik}, \ m+1 \leq l < k \leq n \quad d\lambda_j \ 1 \leq j \leq m,\]
which indeed tally to \(\frac{1}{2}(m+n)(m+n+1)\).

We see from the expressions for \(T_1, T_2, T_3\) that the contribution to the metric form from the differentials with subscripts \(lk\) such that \(1 \leq k < l \leq m\) is
\[
4 \sum_{1 \leq k < l \leq m} \left( 1 + \sqrt{1 - \lambda^2_k} \sqrt{1 - \lambda^2_l} \right) \left( (\delta U^{(i)}_{lk})^2 + (\delta U^{(r)}_{ik})^2 \right) \\
+ \left( 1 - \sqrt{1 - \lambda^2_k} \sqrt{1 - \lambda^2_l} \right) \left( (\delta U^{(i)}_{lk})^2 + (\delta U^{(r)}_{ik})^2 \right) \\
- 2\lambda_k \lambda_l (\delta U^{(i)}_{lk}) (\delta U^{(r)}_{ik}) + 2\lambda_k \lambda_l (\delta U^{(i)}_{ik}) (\delta U^{(r)}_{lk}). \tag{2.6}
\]
Setting \(\sqrt{1 - \lambda^2_k} \sqrt{1 - \lambda^2_l} =: a\) for notational convenience, this portion of the metric form contributes to \((\det[g_{jk}])^{1/2}\) in 2.2 \(2 \times 2\) block factors which is proportional to
\[
\prod_{k<l} \begin{vmatrix} 1+a & -\lambda_k \lambda_l \\ -\lambda_k \lambda_l & 1+a \end{vmatrix}^{1/2} \begin{vmatrix} 1-a & \lambda_k \lambda_l \\ \lambda_k \lambda_l & 1-a \end{vmatrix}^{1/2} = \prod_{k<l} |\lambda^2_k - \lambda^2_l|. \tag{2.7}
\]

For \(m+1 \leq k < l \leq n\), the coefficient of \((\delta U^{(i)}_{lk})\) in \(T_1\) is independent of the \(\lambda\)'s and so for the present purposes can be ignored. For \(1 \leq k \leq m\), \(m+1 \leq l \leq n\), we see from \(T_1\) and \(T_2\) that the corresponding contribution to the metric form is
\[
4 \sum_{k=1}^{m} \sum_{l=m+1}^{n} \left( 1 + \sqrt{1 - \lambda^2_k} \sqrt{1 - \lambda^2_l} \right) \left( (\delta U^{(i)}_{lk})^2 + (\delta U^{(r)}_{ik})^2 \right) \\
+ \left( 1 - \sqrt{1 - \lambda^2_k} \sqrt{1 - \lambda^2_l} \right) \left( (\delta U^{(i)}_{lk})^2 + (\delta U^{(r)}_{ik})^2 \right) \\
- 2\lambda_k \lambda_l (\delta U^{(i)}_{lk}) (\delta U^{(r)}_{ik}) + 2\lambda_k \lambda_l (\delta U^{(i)}_{ik}) (\delta U^{(r)}_{lk}).
\]
This contributes to the volume form a factor proportional to
\[
\prod_{k=1}^{m} \prod_{l=m+1}^{n} \lambda_k = \prod_{k=1}^{m} \lambda_k^{n-m}. \tag{2.8}
\]

We read off that the contribution to \((\det[g_{jk}])^{1/2}\) from the coefficients of the terms \((d\lambda_j)^2\) in \(T_4\) is proportional to
\[
\prod_{k=1}^{m} \frac{1}{(1 - \lambda^2_k)^{1/2}}. \tag{2.9}
\]

It remains to calculate the contribution from the differentials on the diagonal. For \(m+1 \leq k \leq n\), the coefficient of \((\delta U^{(i)}_{kk})\) is a constant so these differentials can be ignored. We read off from \(T_1, T_2, T_3\) that the contribution to the metric form from the remaining differentials is
\[
2 \sum_{k=1}^{m} \left( 2 - \lambda^2_k \right) \left( (\delta U^{(i)}_{kk})^2 + (\delta U^{(r)}_{kk})^2 \right) + 2\lambda^2_k (\delta U^{(i)}_{kk}) (\delta U^{(r)}_{kk}).
\]
The contribution to the volume form is thus proportional to
\[
\prod_{k=1}^{m} \begin{vmatrix} 2 - \lambda^2_k & \lambda^2_k \lambda^2_k \\ \lambda^2_k & 2 - \lambda^2_k \end{vmatrix}^{1/2} \propto \prod_{k=1}^{m} (1 - \lambda^2_k)^{1/2}. \tag{2.10}
\]
Multiplying together (2.7)–(2.10) gives (1.7) in the case \(\beta = 1\).
3 Matrix integrals

Let

\[ I_{m,n}^{(2)}(Q_m) := \int e^{\frac{i}{2} Tr(H - Q_m)} \left( \det(H_m - \mu I_m) \right)^{-n} (dH_m) \]  

(3.1)

where \( H_m, Q_m \) are \( m \times m \) Hermitian matrices, and suppose \( n \geq m, \Im \mu > 0 \). For \( Q_m \) positive definite, it has been proved by Fyodorov [7] that

\[ I_{m,n}^{(2)}(Q_m) = \frac{2^{m} \pi^{m(m+1)/2} \beta^{m(m+1)/2}}{\prod_{j=0}^{m-1} \Gamma(n - j)} \left( \det(\frac{i}{2} Q_m) \right)^{(n-m)/2} e^{\frac{1}{2} \mu \Tr Q_m}. \]  

(3.2)

(Here we have taken \( (dH_m) := \prod_{j=0}^{m} dh_{jj}^{(i)} \prod_{j<k} dh_{jk}^{(i)} dh_{jk}^{(j)} \) which differs by a factor of 2 in the product over \( j < k \) to the convention adopted in [7].) This matrix integral may be regarded as being of the type first evaluated by Ingham and Siegel (see [7] and references therein). In a subsequent work [8], (3.1) was used to derive (3.2), where now \( \beta = 1 \) and \( \beta = 4 \), using suitable variants of (3.1), where now \( \beta = 1 \) and \( \beta = 4 \) refers to the decomposition \( \mu \) with \( S \) having real and real quaternion elements respectively.

Define by \( I_{m,n}^{(1)}(Q_m) \) the matrix integral (3.1) with \( H_m \) and \( Q_m \) now real symmetric. Also, define by \( I_{m,n}^{(4)}(Q_m) \) the same matrix integral but with \( H_m \) and \( Q_m \) now self dual quaternion Hermitian matrices (such matrices regarded as \( 2m \times 2m \) Hermitian matrices are doubly degenerate; adopt the convention that the operations \( \Tr \) and \( \det \) include only distinct eigenvalues). In [7] it is remarked that the method of derivation given therein to deduce that the evaluation of \( I_{m,n}^{(2)}(Q_m) \) can also be used to deduce the evaluation of \( I_{m,n}^{(1)}(Q_m) \), which reads

\[ I_{m,n/2}^{(1)}(Q_m) = \frac{2^{m} \pi^{m(m+1)/2} \beta^{m(m+1)/2}}{\prod_{j=0}^{m-1} \Gamma(n - j)/2} \left( \det(\frac{i}{2} Q_m) \right)^{(n-m)/2} e^{\frac{1}{2} \mu \Tr Q_m}. \]  

(3.3)

Applying the same method to \( I_{m,n}^{(4)}(Q_m) \) gives

\[ I_{m,2n}^{(4)}(Q_m) = \frac{2^{m} \pi^{m(m+1)/2}}{\prod_{j=0}^{m-1} \Gamma(2(n - j))} \left( \det(\frac{i}{2} Q_m) \right)^{2n-m-1} e^{\frac{1}{2} \mu \Tr Q_m}. \]  

(3.4)

Hence, all three cases we have

\[ I_{m,\beta n/2}^{(\beta)}(Q_m) = C_{m,n}^{(\beta)} \left( \det(Q_m) \right)^{(\beta/2)(n-m-1-2/\beta)} e^{\frac{1}{2} \mu \Tr Q_m}, \]  

(3.5)

where \( C_{m,n}^{(\beta)} \) is independent of \( Q_m \).

Let \( U \) be an \( N \times N \) random unitary matrix, with real \( (\beta = 1) \), complex \( (\beta = 2) \) and real quaternion \( (\beta = 4) \), chosen with Haar measure. Generalizing (1.2), decompose \( U \) into blocks

\[ U = \begin{bmatrix} A_{n_1 \times n_2} & C_{n_1 \times (N-n_2)} \\ B_{(N-n_1) \times n_2} & D_{(N-n_1) \times (N-n_2)} \end{bmatrix}. \]  

(3.6)

Since \( U \) is unitary, we require

\[ AA^\dagger + CC^\dagger = I_{n_2}, \]  

(3.7)

together with three similar equations involving \( B \) and \( D \) which given \( A \) and \( C \) can always be satisfied. The idea of [18] is to regard (3.7) as a constraint in the space of general rectangular matrices \( A, C \) with entries of the type required by the index \( \beta \). Thus in this viewpoint the distribution of \( A \) is given by

\[ \int \delta(AA^\dagger + CC^\dagger - I_{n_2})(dC). \]  

(3.8)
In (3.8) the delta function is a product of scalar delta functions, one for each independent real and imaginary component of $A^\dagger A + C^\dagger C - I_{n_2}$. It is proportional to the matrix integral

$$\int e^{-i\text{Tr}(H(AA^\dagger + CC^\dagger - I) + C^\dagger C - I_{n_2})}(dH)$$

where $H$ is an Hermitian matrix with elements of the type $\beta$. Following [8] we would like to substitute (3.9) for the delta function in (3.8), and change the order of integration. The integrations over $(dC)$ are simply Gaussian integrals. For the resulting function of $H$ to be integrable around $H = 0$, the replacement $H \mapsto H - i\mu I_{n_1}$ in the exponent of (3.9) must made. Doing this and computing the Gaussian integrals gives that (3.8) is proportional to

$$\lim_{\mu \to 0^+} \int \left( \det(H - i\mu I_{n_1}) \right)^{-\beta(N-n_2)/2} e^{i\text{Tr}(H(I_{n_1} - AA^\dagger))}(dH)$$

Evaluating the matrix integral using (3.5) shows that the distribution of $A$ is proportional to

$$\left( \det(I_{n_2} - A^\dagger A) \right)^{(\beta/2)(N-n_1-n_2+1-2/\beta)}.$$

Note that for this to be normalizable, we must have

$$N - n_1 - n_2 \geq 0.$$

Suppose that in addition to (3.11) we have $n_1 \geq n_2$. Then $A$ has rank $n_1$ and so $AA^\dagger$ has $n_1 - n_2$ zero eigenvalues and (3.10) can be written

$$\left( \det(I_{n_2} - A^\dagger A) \right)^{(\beta/2)(N-n_1-n_2+1-2/\beta)}.$$

Setting $Y = A^\dagger A$ we know (see e.g. [6]) that

$$dA \propto (\det Y)^{(\beta/2)(n_1-n_2+1-2/\beta)}(dY),$$

so we have from (3.12) that the distribution of $Y$ is proportional to

$$\left( \det(Y)^{(\beta/2)(n_1-n_2+1-2/\beta)} \right) \left( \det(I_{n_2} - Y) \right)^{(\beta/2)(N-n_1-n_2+1-2/\beta)}. $$

Denote the eigenvalues of $Y$ by $y_1, \ldots, y_{n_2}$. Using the fact that the eigenvalue dependent portion of the Jacobian for an Hermitian matrix, with elements of the type $\beta$, when changing variables to its eigenvalues and eigenvectors is $\prod_{j<k} |y_k - y_j|^\beta$, we read off from (3.14) that the eigenvalue p.d.f. of $Y$ is proportional to

$$\prod_{j=1}^{n_2} y_j^{(\beta/2)(n_1-n_2+1-2/\beta)} (1 - y_j)^{(\beta/2)(N-n_1-n_2+1-2/\beta)} \prod_{j<k} |y_k - y_j|^\beta. $$

In the case

$$n_1 = n_2 = m, \quad N = n + m,$$

the matrix $A$ in (3.6) coincides with $r$ in (1.2), and the $y_j$ in (3.15) are then the eigenvalues of $r^\dagger r$. The non-zero values of the singular values of the submatrix $t$ in (1.2) are the $\lambda_j$’s in (1.7). The decomposition (1.4) tells us that $y_j = 1 - \lambda_j^2$. Indeed, making this change of variable in (3.15), and making the substitutions (3.16), reclaims (1.7) in the case $\beta = 2$. 

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For $\beta = 2$ and general $n_1 \geq n_2$, $N \geq n_1 + n_2$, a result equivalent to (3.15) is derived in a recent work of Simon and Moustakas [15]. Motivated by a quantum dot problem with three leads, they decomposed the $N \times N$ scattering matrix $S$ into a $3 \times 3$ block structure:

$$S = \begin{bmatrix} r_{11} & t_{12} & t_{13} \\ t_{21} & r_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$

where each $r_{ii}$ is $N_i \times N_i$ and $t_{ij}$ is $N_i \times N_j$, with $N = N_1 + N_2 + N_3$. For $S$ a random unitary matrix with Haar measure, it is shown that the eigenvalues of $t_{12}t_{12}^\dagger$ have p.d.f. given by (3.15) with $\beta = 2$, $n_1 = N$, $n_2 = N_2$. This is consistent with our result because the distribution of $S$ is unchanged by interchanging rows and columns. The first and second block columns in (3.17) can be interchanged, effectively giving the decomposition with $t_{12} = A_{N_1 \times N_2}$.

We remark that for $y_j \ll 1$ the term involving $(1 - y_j)$ in (3.15) can be ignored. The resulting p.d.f. is then the $x_j \ll 1$ limit of the singular values p.d.f. of the matrix product $X^\dagger X$, where $X$ is an $n_1 \times n_2$ random Gaussian matrix with real ($\beta = 1$), complex ($\beta = 2$) or real quaternion elements ($\beta = 4$) (see e.g. [6]). This is consistent with a recent result of Jiang [10], who quantifies the degree to which the entries of orthogonal or unitary random matrices can be approximated by entries of Gaussian random matrices with real or complex entries respectively.

## 4 Random projections

In a recent work Collins [5] computed the distribution of certain products of random orthogonal projections. Here this approach, suitably modified, will be used to derive (3.15). Knowledge is required of the following result [13, 6].

**Proposition 4.1.** Let $c$ and $d$ be $n_1 \times m$ and $n_2 \times m$ Gaussian random matrices, all elements identically and independently distributed (i.i.d.) where $n_2, n_2 \geq m$, indexed by the parameter $\beta = 1, 2$ or 4. The parameter $\beta$ specifies that the elements are real ($\beta = 1$), complex ($\beta = 2$) or real quaternion ($\beta = 4$). Let $C = c^\dagger c$, $D = d^\dagger d$. The distribution of $J := (C + D)^{-1/2}C(C + D)^{-1/2}$ is proportional to

$$(\det J)^{a\beta/2}(\det(I - J))^{b\beta/2}$$

with

$$a = n_1 - m + 1 - 2/\beta, \quad b = n_2 - m + 1 - 2/\beta.$$  

The p.d.f. (4.1) is referred to as the Jacobi ensemble (see e.g. [6]). To make use of Proposition 4.1 let $X$ be a $N \times m$, $N \geq m$ i.i.d. Gaussian random matrix indexed by $\beta$. Define $Q_1$ to be the top $n \times N$ submatrix of an $N \times N$ unitary matrix indexed by $\beta$. Because $Q_1Q_1^\dagger = I_n$, we have that $Q_1X$ is distributed as an $n \times m$ i.i.d. Gaussian random matrix indexed by $\beta$ (see [6, Thm. 2.3.1]). Let $Q_2$ be the bottom $(N - n) \times N$ submatrix of $U$ so that

$$Q_1^\dagger Q_1 + Q_2^\dagger Q_2 = I_N.$$  

(4.2)

Because $Q_2Q_2^\dagger = I_{N-n}$, $Q_2X$ is distributed as an $(N - n) \times m$ i.i.d. Gaussian matrix indexed by $\beta$.

Now set

$$Y = X^\dagger Q_1^\dagger Q_1 X, \quad Z = X^\dagger Q_2^\dagger Q_2 X.$$  

(4.3)
Denoting by $W_p^{(\beta)}(n)$ the (Wishart) distribution of the matrix product $A^\dagger A$, where $A$ is an $n \times p$, $(n \geq p)$ i.i.d. Gaussian random matrix indexed by $\beta$, we see that the distribution of $Y$ is $W_m^{(\beta)}(n)$ while the distribution of $Z$ is $W_m^{(\beta)}(N-n)$. The result of Proposition 4.1 tells us that the eigenvalue p.d.f. of

$$(Y + Z)^{-1/2}Y(Y + Z)^{-1/2} = (X^\dagger X)^{-1/2}Y(X^\dagger X)^{-1/2}$$

is given by the Jacobi ensemble (4.1) with $n_1 = n$, $n_2 = N-n$.

According to the singular value decomposition, we can write

$$X = U_1 \Lambda U_2$$

for $U_1$ an $N \times N$ unitary matrix, $U_2$ an $m \times m$ unitary matrix, and $\Lambda$ an $N \times m$ diagonal matrix, with diagonal entries equal to the positive square root of the eigenvalues of $X^\dagger X$. It follows from (4.5) that

$$(X^\dagger X)^{1/2} = U_2 [\Lambda]_{m,m} U_2$$

(4.6)

where $[\Lambda]_{m,m}$ refers to the top $m \times m$ sub-block of $\Lambda$, while (4.6) and (4.5) together imply

$$X(X^\dagger X)^{-1/2} = [U_1]_{N,m} U_2^\dagger.$$  

(4.7)

Substituting for $Y$ in (4.4) according to (4.3), then using (4.7) to substitute for $X(X^\dagger X)^{-1/2}$ tells us that the distribution of

$$U_2[U_1]_{N,m}^\dagger Q_1 Q_1[U_1]_{N,m} U_2^\dagger$$

(4.8)

is given by (4.1). We note

$$U_1^\dagger[U_1]_{N,m} = \text{diag}(1, \ldots, 1, 0, \ldots, 0)$$

where there are $m$ 1’s. Since the distribution of $Q_1$ is unchanged upon multiplication by a unitary matrix, the distribution of (4.8) is the same as the distribution of

$$U_2[Q_1 Q_1]_{m,m} U_2^\dagger.$$  

(4.9)

In terms of the decomposition (3.6) this in turn has the same distribution as

$$A^\dagger A \Big|_{n_1 \mapsto n, n_2 \mapsto m}.$$  

Hence we have the distribution of (4.10) is given by (4.11) with

$$a = n - m + 1 - 2/\beta, \quad b = N - n - m + 1 - 2/\beta,$$

which is in precise agreement with (3.11) upon the replacements $n_1 \mapsto n$, $n_2 \mapsto m$ in the latter.

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