Reflection equation for the $N = 3$ Cremmer–Gervais $R$-matrix

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Abstract. We consider the reflection equation of the $N = 3$ Cremmer–Gervais $R$-matrix. The reflection equation is shown to be equivalent to 38 equations which do not depend on the parameter of the $R$-matrix, $q$. Solving those 38 equations, the solution space is found to be the union of two types of spaces, each of which is parameterized by the algebraic variety $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$ and $\mathbb{C} \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$.

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1. Introduction

The Yang–Baxter equation is the sufficient condition for the integrability of the one-dimensional quantum systems (or the two-dimensional classical statistical systems), i.e. it ensures the commutativity of the transfer matrices. Based on the Yang–Baxter equation, the quantum inverse scattering method was developed, which enables us to calculate bulk quantities and correlation functions.

Under the open boundary condition, besides the Yang–Baxter equation, the reflection equation guarantees the existence of the commutative family of transfer matrices [1].

Up to now, there have been many works about the reflection equation [1]–[17]. Taking the XXZ chain for example, the diagonal solution was obtained in [1,2] and the general solution in [3]. For $N \geq 3$-state models, most of the solutions are obtained by imposing initial conditions. There is also an approach from the quantum group [9]–[12]. In [9], some nondiagonal solutions of the reflection equation for the $U_q(\hat{sl}_2)$ $R$-matrix were obtained from the intertwining condition. Moreover, a complete description in terms of current algebra has been accomplished in [10].

In this paper, we consider the reflection equation of the $N = 3$ Cremmer–Gervais $R$-matrix. This $R$-matrix originally appeared in the context of the Toda field theory [18] as a constant $R$-matrix. Recently, the Baxterized $R$-matrix was derived [19] by taking an appropriate trigonometric degeneration of the Shibukawa–Ueno $R$-operator [20]. We determine the full solution space without imposing any conditions such as the initial condition.
The main result is that we found the solution space is the union of two types of spaces, which are parameterized by algebraic varieties. The first type is parameterized by \( \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \), and the solution can be explicitly expressed as

\[
K_1(z, (B_1, B_2) \times (D_1, D_2) \times (E_1, E_2, E_3)) = K_{1,0}(z, D_1, D_2, E_1)
- z^6TK_{1,0}(z^{-1}, D_2, D_1, E_3)T + K_{1,1}(z, B_1, B_2, D_1, E_2)
- z^6TK_{1,1}(z^{-1}, B_2, B_1, D_2, E_2)T,
\]

where \((B_1, B_2) \times (D_1, D_2) \times (E_1, E_2, E_3) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})\) and

\[
K_{1,0}(z, D_1, D_2, E_1) = E_1^2 \begin{pmatrix} D_2^2z^2 & D_1D_2(z^4 - 1) & D_1^2z^2(z^4 - 1) \\ 0 & D_2^2z^2 & D_1D_2(z^4 - 1) \\ 0 & 0 & D_2^2z^2 \end{pmatrix},
\]

\[
K_{1,1}(z, B_1, B_2, D_1, E_2) = -D_1E_2z^2 \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_1 & B_2(1 - z^4) \\ 0 & 0 & B_1z^4 \end{pmatrix},
\]

where \(T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \).

The second type is parameterized by \( \mathbb{C} \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \), and the explicit expression is

\[
K_{II}(z, (b) \times (F_1, F_2) \times (G_1, G_2, G_3)) = b^2\cdot \text{Id} + K_{II,0}(z, F_1, G_1, G_2, G_3)
- z^4TK_{II,0}(z^{-1}, F_2, G_3, -G_2, G_1)T,
\]

where \((b) \times (F_1, F_2) \times (G_1, G_2, G_3) \in \mathbb{C} \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})\) and

\[
K_{II,0}(z, F_1, G_1, G_2, G_3) = -F_1 \begin{pmatrix} G_3 & 0 & G_1(1 - z^4) \\ 0 & G_3 & G_2(1 - z^4) \\ 0 & 0 & G_3z^4 \end{pmatrix}.
\]

In section 2, we define the Cremmer–Gervais \( R \)-matrix and state again the main theorem, which is about the full solution space of the reflection equation. We also derive some properties of the reflection equation coming from the symmetries of the Cremmer–Gervais \( R \)-matrix, which we use for the proof of the main theorem 2.2, given in sections 3 and 4. Section 5 is devoted to the conclusion.

## 2. The \( N = 3 \) Cremmer–Gervais \( R \)-matrix and the reflection equation

### 2.1. The Cremmer–Gervias \( R \)-matrix

We denote the standard orthonormal basis of \( \mathbb{C}^3 \) by \( \{e_0, e_1, e_2\} \). The matrix element \( A_j^i \) of \( A \in \text{End}(\mathbb{C}^3) \) with respect to this basis is defined as

\[
Ae_j = \sum_{i=0}^{2} e_i A_j^i.
\]

We also define \( G \) and \( T \) as

\[
Ge_j = \omega^j e_j, \quad Te_j = e_{2-j}, \tag{2.1}
\]

where \( \omega^3 = 1 \).
The original Cremmer–Gervais $R$-matrix has two parameters besides the spectral parameter [18, 19]. As a solution to the Yang–Baxter equation, it is equivalent to the $R$-matrix with one spectral parameter and one nonspectral parameter which is defined below.

**Definition 2.1.** ([18, 19]) The $N = 3$ Cremmer–Gervais $R$-matrix $R_{CG}(z, q) \in \text{End}(\mathbb{C}^3 \otimes \mathbb{C}^3)$ is defined as

$$
[R_{CG}(z, q)]_{kl}^{ij} = \begin{cases} 
(qz^{-1} - q^{-1}z)/(q - q^{-1})(z - z^{-1}), & \text{for } i = j = k = l, \\
-q^{\text{sgn}(k-l)}/(q - q^{-1}), & \text{for } i = k \neq j = l, \\
z^{\text{sgn}(l-k)}/(z - z^{-1}), & \text{for } l \neq k = j, \\
\text{sgn}(l - k), & \text{for } \min(k, l) < i < \max(k, l), \\
0, & \text{otherwise}.
\end{cases}
$$

(2.2)

**Theorem 2.1.** The Cremmer–Gervais $R$-matrix $R_{CG}(z, q)$ satisfies the Yang–Baxter equation

$$
R_{12}(z_1)R_{13}(z_1 z_2)R_{23}(z_2) = R_{23}(z_2)R_{13}(z_1 z_2)R_{12}(z_1) \in \text{End}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3). \tag{2.3}
$$

We can immediately see from the definition that the Cremmer–Gervais $R$-matrix $R_{CG}(z, q)$ has the following properties.

**Proposition 2.1.** The Cremmer–Gervais $R$-matrix $R_{CG}(z, q)$ has the following properties.

- **Unitarity:** $R_{12}^{CG}(z)R_{21}^{CG}(z^{-1}) = \frac{(q^2 - z^2)(1 - q^2 z^2)}{(q^2 - 1)^2(z^2 - 1)^2}\text{Id}$, \tag{2.4}
- **Conservation law:** $R_{CG}(z)(G \otimes G) = (G \otimes G)R_{CG}(z)$, \tag{2.5}
- **$T$-invariance:** $R_{CG}(z, q) = -(T \otimes T) R_{CG}(z^{-1}, q^{-1})(T \otimes T)$, \tag{2.6}

where $R_{21}(z) = PR_{12}(z)P$, $P(x \otimes y) = y \otimes x$, for any $x, y \in \mathbb{C}^3$.

Equation (2.4) is used to show that $R_{CG}(z, q)$ satisfies the Yang–Baxter equation (2.3). Equation (2.5) means that $[R_{CG}(z)]_{kl}^{ij} = 0$ unless $i + j = k + l$ (mod 3), which the Belavin $R$-matrix also satisfies. On the other hand, (2.6) is the symmetry peculiar to the Cremmer–Gervais $R$-matrix.

**Definition 2.2.** The reflection equation is

$$
R_{12}(z_1/z_2)K_1(z_1)R_{21}(z_1 z_2)K_2(z_2) = K_2(z_2)R_{12}(z_1 z_2)K_1(z_1)R_{21}(z_1/z_2) \in \text{End}(\mathbb{C}^3 \otimes \mathbb{C}^3),
$$

(2.7)

where $R_{12}(z) = R(z)$ is the Cremmer–Gervais $R$-matrix (2.2), and $K_1(z) = K(z) \otimes \text{Id}$, $K_2(z) = \text{Id} \otimes K(z)$.
We regard two solutions to be equivalent if they coincide up to an overall factor. We found two solution spaces \( \mathcal{A}_I \) and \( \mathcal{A}_{II} \) which can be parameterized by the following algebraic varieties, \( \mathcal{U}_I \) and \( \mathcal{U}_{II} \).

**Definition 2.3.** Let \( \mathcal{U}_I, \mathcal{U}_{II} \) be the following algebraic varieties:

\[
\mathcal{U}_I = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}),
\]

\[
\mathcal{U}_{II} = \mathbb{C} \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}).
\]

\( \mathcal{A}_I \) and \( \mathcal{A}_{II} \) is the space of 3 \times 3 matrices which is parameterized by \( \mathcal{U}_I \) and \( \mathcal{U}_{II} \), respectively, as follows.

**Definition 2.4.** We define two spaces of 3 \times 3 matrices \( \mathcal{A}_I \) and \( \mathcal{A}_{II} \) as follows:

\[
\mathcal{A}_I = \{K_I(z, P_I) \mid P_I = (B_1, B_2) \times (D_1, D_2) \times (E_1, E_2, E_3) \in \mathcal{U}_I\},
\]

\[
K_I(z, P_I) = K_{I,0}(z, D_1, D_2, E_1) - z^6 T K_{I,0}(z^{-1}, D_2, D_1, E_3) T
\]

\[
+ K_{I,1}(z, B_1, B_2, D_1, E_2) - z^6 T K_{I,1}(z^{-1}, B_2, B_1, D_2, E_2) T,
\]

where \( K_{I,0}(z, D_1, D_2, E_1) \) and \( K_{I,1}(z, B_1, B_2, D_1, E_2) \) are

\[
K_{I,0}(z, D_1, D_2, E_1) = E_1^T \begin{pmatrix} D_2^2 z^2 & D_1 D_2 (z^4 - 1) & D_2^2 z^2 (z^4 - 1) \\ 0 & D_2^2 z^2 & D_1 D_2 (z^4 - 1) \\ 0 & 0 & D_2^2 z^2 \end{pmatrix},
\]

\[
K_{I,1}(z, B_1, B_2, D_1, E_2) = -D_1 E_2 z^2 \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_1 & B_2 (1 - z^4) \\ 0 & 0 & B_1 z^4 \end{pmatrix}.
\]

\( \mathcal{A}_{II} = \{K_{II}(z, P_{II}) \mid P_{II} = (b) \times (F_1, F_2) \times (G_1, G_2, G_3) \in \mathcal{U}_{II}\},
\]

\[
K_{II}(z, P_{II}) = b z^2 \text{Id} + K_{II,0}(z, F_1, G_1, G_2, G_3)
\]

\[
- z^4 T K_{II,0}(z^{-1}, F_2, G_3, -G_2, G_1) T,
\]

where \( K_{II,0}(z, F_1, G_1, G_2, G_3) \) is

\[
K_{II,0}(z, F_1, G_1, G_2, G_3) = -F_1 \begin{pmatrix} G_3 & 0 & G_1 (1 - z^4) \\ 0 & G_3 & G_2 (1 - z^4) \\ 0 & 0 & G_3 z^4 \end{pmatrix}.
\]
The solutions to the reflection equation of the $N = 3$ Cremmer–Gervais $R$-matrix can be expressed using the spaces of $3 \times 3$ matrices $A_I$ and $A_{II}$ defined in definition 2.4.

**Theorem 2.2 (Main Result).** The solution space $K$ of the $N = 3$ Cremmer–Gervais $R$-matrix (2.2) is the union of $A_I$ and $A_{II}$, and does not depend on the parameters of the $R$-matrix, $q$:

$$K = A_I \cup A_{II}.$$

The following proposition can be checked by a direct calculation.

**Proposition 2.2.** $K_I(z, P_I) \in A_I$ and $K_{II}(z, P_{II}) \in A_{II}$ satisfy unitarity:

$$K_i(z, P_i)K_i(z^{-1}, P_i) = \rho_i(z)\text{Id}, \quad i = I, II,$$

where

$$\rho_I(z) = D_1^2(B_2^2 + D_2^2) + D_2^2(B_1^2 + D_1^2) + (B_1 D_1^3 + B_2 D_2^3 - B_1 B_2 D_1 D_2)(z^2 + z^{-2}) - D_1 D_2(B_1 D_2 + B_2 D_1)(z^4 + z^{-4}) - D_1^2 D_2(z^6 + z^{-6}),$$

$$\rho_{II}(z) = b^2 + F_2^2 G_1^2 + F_1^2 G_2^2 + b(F_2 G_1 - F_1 G_2)(z^2 + z^{-2}) + F_1 F_2 G_1 G_3(z^4 + z^{-4}).$$

We can also see that the solutions have the following transformation properties.

**Lemma 2.1.** We define the action $\text{ad}$ as $\text{ad} X(Y) = XYX^{-1}$ for two $3 \times 3$ matrices $X$ and $Y$. $K_I(z, P_I) \in A_I$ and $K_{II}(z, P_{II}) \in A_{II}$ transform with respect to the action of $G$ and $T$ (2.1) as

$$\text{ad} G(K_i(z, P_i)) = K_i(z, GP_i), \quad i = I, II,$$

$$\text{ad} T(K_i(z, P_i)) = z^6 K_i(z^{-1}, TP_i),$$

$$\text{ad} T(K_{II}(z, P_{II})) = z^4 K_{II}(z^{-1}, TP_{II}),$$

where the action of $G$ and $T$ on $P_I, P_{II}$ is defined by

$$GP_I = (B_1, \omega^2 B_2) \times (\omega^2 D_1, D_2) \times (E_1, \omega E_2, \omega E_3),$$

$$TP_I = -(B_2, B_1) \times (D_2, D_1) \times (E_3, E_2, E_1),$$

$$GP_{II} = (b) \times (\omega F_1, F_2) \times (G_1, \omega G_2, \omega G_3),$$

$$TP_{II} = (b) \times (F_2, F_1) \times (-G_3, G_2, -G_1).$$

$G$ and $T$ act on the algebraic varieties $U_I$ and $U_{II}$, and satisfy $G^3 = \text{Id}, T^2 = \text{Id}$ and $(GT)^2 = \text{Id}$. The common part of $A_I$ and $A_{II}$ is as follows.

**Lemma 2.2.** $A_I \cap A_{II}$, which is the common part of $A_I$ and $A_{II}$, is

$$A_I \cap A_{II} = \mathcal{C} \cup \text{ad} T(\mathcal{C}) = \{\text{Id}\},$$

$$\mathcal{C} = \left\{ \begin{pmatrix} c_3 + c_4 z^2 & 0 & 0 \\ c_2 (z^4 - 1) & c_4 z^2 + c_3 z^4 & 0 \\ c_1 (z^4 - 1) & 0 & c_4 z^2 + c_3 z^4 \end{pmatrix} \right\},$$

where $(c_1, c_2) \times (c_3, c_4) \in \mathbb{C}^2 \times \mathbb{P}^1(\mathbb{C})$.  

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2.2. Properties of the reflection equation of the $N = 3$ Cremmer–Gervais $R$-matrix

Utilizing the symmetries of the Cremmer–Gervais $R$-matrix (2.5) and (2.6), one can show that the solutions to the reflection equation have the following properties.

**Proposition 2.3.** $K(z, q)$ has the following properties:

\begin{align*}
K(z, q) & \in \mathcal{K} \Rightarrow \text{ad} G(K(z, q)) \in \mathcal{K}, \quad (2.23) \\
K(z, q) & \in \mathcal{K} \Rightarrow \text{ad} T(K(z^{-1}, q^{-1})) \in \mathcal{K}. \quad (2.24)
\end{align*}

**Proof of proposition 2.3.** Equations (2.23) and (2.24) follow from the symmetries of $R^{CG}(z, q)$, (2.5) and (2.6), respectively. Since they can be proved similarly, we show (2.24).

Multiplying $T \otimes T$ from the left and right on both sides of the reflection equation

\[
R_{12}(z_1/z_2, q)K_1(z_1, q)R_{21}(z_1z_2, q)K_2(z_2, q) = K_2(z_2, q)R_{12}(z_1z_2, q)K_1(z_1, q)R_{21}(z_1/z_2, q),
\]

and using (2.6), one gets

\[
R_{12}(z_2/z_1, q^{-1})(T_1K_1(z_1, q)T_1)R_{21}(z_1^{-1}z_2^{-1}, q^{-1})(T_2K_2(z_2, q)T_2) = (T_2K_2(z_2, q)T_2)R_{12}(z_1^{-1}z_2^{-1}, q^{-1})(T_1K_1(z_1, q)T_1)R_{21}(z_2/z_1, q^{-1}). \quad (2.26)
\]

Changing $z_i \rightarrow z_i^{-1}, q \rightarrow q^{-1}$ in (2.26), we have

\[
R_{12}(z_1/z_2, q)(T_1K_1(z_1^{-1}, q^{-1})T_1)R_{21}(z_1z_2, q)(T_2K_2(z_2^{-1}, q^{-1})T_2) = (T_2K_2(z_2^{-1}, q^{-1})T_2)R_{12}(z_1^{-1}, q^{-1})(T_1K_1(z_1^{-1}, q^{-1})T_1)R_{21}(z_1/z_2, q), \quad (2.27)
\]

which means that (2.24) holds.

Let us investigate in more detail the general properties of the reflection equation which comes from the symmetries of the Cremmer–Gervais $R$-matrix (2.6). From now on, we prepare some notation.

**Definition 2.5.**

(1) We express the matrix elements of $K(z) \in \text{End}(\mathbb{C}^3)$ using $c^j_i(z, q)$ as

\[
K(z) = \begin{pmatrix}
c^0_0(z, q) & c^1_0(z, q) & c^2_0(z, q) \\
c^0_1(z, q) & c^1_1(z, q) & c^2_1(z, q) \\
c^0_2(z, q) & c^1_2(z, q) & c^2_2(z, q)
\end{pmatrix}. \quad (2.28)
\]

(2) We express the matrix elements of the matrix, which is the left-hand side of the reflection equation subtracted by the right-hand side, as

\[
(i_1i_2 | j_1j_2) := [R_{12}(z_1/z_2, q)K_1(z_1, q)R_{21}(z_1z_2, q)K_2(z_2, q)]^{i_1i_2}_{j_1j_2} - [K_2(z_2, q)R_{12}(z_1z_2, q)K_1(z_1, q)R_{21}(z_1/z_2, q)]^{i_1i_2}_{j_1j_2}, \quad (2.29)
\]

for $i_1, i_2, j_1, j_2 = 0, 1, 2$. 

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(3) $T$ was defined as the matrix which acts on the vector space $\mathbb{C}^3$, i.e. $Te_j = e_{2-j}$ (2.1). We also define the action of $T$ on the index $0, 1, 2$ of the orthonormal basis $\{e_0, e_1, e_2\}$ of $\mathbb{C}^3$ as

$$T(j) := 2 - j,$$

for $j = 0, 1, 2$. □

Using the notations defined in definition 2.5, one has

$$Te_j = e_{T(j)}, \quad (\text{ad } T(K(z)))^i_j = K(z)^{T(i)_j}_i, \quad (\text{ad } T(R(z)))^i_j = R(z)^{T(i)_j}_i.$$

Definition 2.6. We define the space of meromorphic functions of $z, q$ as $\mathcal{M}$:

$$\mathcal{M} := \{f(z, q) \mid f(z, q) \text{ is meromorphic in } z, q\}.$$  

$\mathcal{M}$ is defined as the space of homogeneous polynomials of degree 1 with respect to $c_j^i(z_1, q), i, j = 0, 1, 2$, and the coefficients belong to $\mathcal{M}$.

$$\mathcal{N} = \{g(c(z_1), c(z_2) \mid z_1, z_2, q) \mid g \text{ is a homogeneous polynomial of degree 1 with respect to } c_j^i(z_1, q), i, j = 0, 1, 2 \text{ and the coefficients belong to } \mathcal{M}\}.$$ □

Expressing the matrix elements of $K(z, q)$ as $c_j^i(z, q)$, every element of the reflection equation $(i_1 i_2 \mid j_1 j_2)$ belongs to $\mathcal{N}$ and can be expressed as

$$\sum_{k,l,m,n=0}^2 f_{kl, mn}(z_1, z_2, q)c_k^i(z_1, q)c_m^n(z_2, q),$$

where $f_{kl, mn}(z_1, z_2, q) \in \mathcal{M}$. Comparing the reflection equation (2.25) with (2.26), we find the following holds.

Lemma 2.3. If an element of the reflection equation $(i_1 i_2 \mid j_1 j_2)$ can be expressed as

$$(i_1 i_2 \mid j_1 j_2) = \sum_{k,l,m,n=0}^2 f_{kl, mn}(z_1, z_2, q)c_k^i(z_1, q)c_m^n(z_2, q),$$

where $c_j^i(z, q)$ are the matrix elements of $K(z, q)$ and $f_{kl, mn}(z_1, z_2, q) \in \mathcal{M}$, $(T(i_1)T(i_2) \mid T(j_1)T(j_2))$ can be expressed as

$$(T(i_1)T(i_2) \mid T(j_1)T(j_2)) = \sum_{k,l,m,n=0}^2 f_{kl, mn}(z_1^{-1}, z_2^{-1}, q^{-1})c_{T(k)}^i(z_1, q)c_{T(m)}^j(z_2, q).$$ □

We define the $T$-transformation of an element of $\mathcal{N}$ as follows.

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Definition 2.7. For an element of $\mathcal{N} \subset \mathbb{Z}^2$

$$\sum_{k,l,m,n=0}^{2} f_{kl,cm}(z_1, z_2, q) c_k^n(z_1, q) c_m^n(z_2, q),$$

where $f_{kl,cm}(z_1, z_2, q) \in \mathcal{M}$, we define its $T$-transformation as

$$T \left( \sum_{k,l,m,n=0}^{2} f_{kl,cm}(z_1, z_2, q) c_k^n(z_1, q) c_m^n(z_2, q) \right) := \sum_{k,l,m,n=0}^{2} f_{kl,cm}(z_1^{-1}, z_2^{-1}, q^{-1}) c_{T(k)}^{(k)}(z_1, q) c_{T(m)}^{(m)}(z_2, q).$$

Defining the $T$-transformation as above, lemma 2.3 can be simply expressed as

Corollary 2.1.

$$T(i_1 i_2 | j_1 j_2) = (T(i_1)T(i_2) | T(j_1)T(j_2)). \quad (2.31)$$

We also use the following proposition to prove theorem 2.2, which is about the subgroup of $\mathcal{N}$ invariant under the $T$-transformation.

Proposition 2.4. A subgroup $\tilde{\mathcal{N}}$ of $\mathcal{N}$ is called a $T$-invariant subgroup if it is invariant under the $T$-transformation, i.e. if $T(g) \in \tilde{\mathcal{N}}$ holds for any $g \in \tilde{\mathcal{N}}$. If an element $g \in \mathcal{N}$ can be expressed as the linear combination of elements of $\tilde{\mathcal{N}}$ with $\mathcal{M}$ coefficients, i.e. can be expressed as

$$g = \sum_{\beta} h_{\beta}(z_1, z_2, q) g_{\beta}(c(z_1), c(z_2) | z_1, z_2, q), \quad g_{\beta} \in \tilde{\mathcal{N}}, \quad h_{\beta} \in \mathcal{M}, \quad (2.32)$$

$T(g) \in \mathcal{N}$ can also be expressed as the linear combination of elements of $\tilde{\mathcal{N}}$ with $\mathcal{M}$ coefficients:

$$T(g) = \sum_{\beta} h_{\beta}(z_1^{-1}, z_2^{-1}, q^{-1}) T(g_{\beta}(c(z_1), c(z_2) | z_1, z_2, q)), \quad T(g_{\beta}) \in \tilde{\mathcal{N}}. \quad (2.33)$$

We use (2.24), lemma 2.3, the properties of the reflection equation coming from the symmetry of the Cremmer–Gervais $R$-matrix (2.6), and proposition 2.4, the property which holds for the $T$-invariant subgroup of $\mathcal{N}$, to prove theorem 2.2.
3. Thirty-eight equations equivalent to the reflection equation

Solving the reflection equation is to solve 81 equations \((i_1i_2 | j_1j_2) = 0, i_1, i_2, j_1, j_2 = 0, 1, 2\) for \(c_j^i(z, q)\), \(i, j = 0, 1, 2\). Directly solving the 81 equations is the most straightforward way, which is a tiresome task. Instead, carefully observing these 81 equations, one finds they are equivalent to another set of 38 equations (definition 3.1), which is easier to handle. This equivalence (theorem 3.1) is shown in this section. We briefly outline the procedure. First, we divide the 81 elements of the reflection equation into several subgroups. Such is also done for the set of 38 equations. Then we show three relations (propositions 3.1, 3.2 and 3.3) among the subgroups of the reflection equation and the set of 38 equations. Combining the relations and proposition 2.4, we prove the equivalence between the 81 equations \((i_1i_2 | j_1j_2) = 0, i_1, i_2, j_1, j_2 = 0, 1, 2\) and another set of 38 equations.

**Theorem 3.1.** \(\{(i_1i_2 | j_1j_2) | i_1, i_2, j_1, j_2 = 0, 1, 2\}\), which are the 81 elements of the reflection equation of the \(N = 3\) Cremmer–Gervais \(R\)-matrix \(R^{CG}(z, q)\), are equivalent to 38 equations which consist of 20 equations \(\{A_j = 0 \mid j = 1, \ldots, 8\} \cup \{B_j = 0 \mid j = 1, \ldots, 7\} \cup \{C_j = 0 \mid j = 1, \ldots, 5\}\) defined in definition 3.1, and their \(T\)-transformed (definition 2.7) equations \(\{TA_j = 0 \mid j = 2, 3, 4, 5, 7, 8\} \cup \{TB_j = 0 \mid j = 1, \ldots, 7\} \cup \{TC_j = 0 \mid j = 1, \ldots, 5\}\).

\(A_1\) and \(A_6\) are essentially self-dual with respect to the \(T\)-transformation, i.e. \(TA_1 = -z_{1}^{-2}z_{2}^{-2}A_1, TA_6 = z_{1}^{-4}z_{2}^{-4}A_6\).

**Definition 3.1.** We define \(\{A_j, j = 1, \ldots, 8\}, \{B_j, j = 1, \ldots, 7\}, \{C_j, j = 1, \ldots, 5\}\) as the polynomials of \(c_j^i(z, q)\), the matrix elements of the \(K(z)\), \(c_j^i(z, q)\), as follows. For simplicity, we denote \(c_j^i(z) = c_j^i(z, q)\).

\[
\begin{align*}
A_1 & := z_{1}^2c_1^0(z_1)c_2^0(z_2) - c_1^2(z_1)z_{1}^2c_1^0(z_2), \\
A_2 & := c_0^0(z_1)z_{2}^2c_1^0(z_2) - z_{1}^2c_1^0(z_1)c_0^0(z_2), \\
A_3 & := c_1^2(z_1)c_2^0(z_2) - c_0^2(z_1)c_1^0(z_2), \\
A_4 & := z_{1}c_1^0(z_1)(c_1^2(z_2) - c_0^2(z_2)) - (c_1^1(z_1) - c_1^0(z_1))z_{1}^2c_1^0(z_2), \\
A_5 & := c_0^2(z_1)c_0^0(z_2) - c_1^2(z_1)c_1^0(z_2) + c_2^0(z_1)c_0^0(z_2) - c_0^2(z_1)c_2^0(z_2), \\
A_6 & := (z_{1}^2 - z_{2}^2)(z_{1}^2z_{2}^2c_0^2(z_1)(c_1^2(z_2) - c_0^2(z_2)) - c_1^2(z_2)(c_0^2(z_1) - c_2^2(z_1))) + (z_{1}^2c_0^0(z_1) - c_2^0(z_1)) \\
& \quad \times z_{2}^2(c_0^0(z_2) - c_1^0(z_2)) - z_{1}^2(c_0^0(z_1) - c_2^0(z_1))(z_{1}^2c_0^0(z_2) - c_2^0(z_2)), \\
A_7 & := c_1^2(z_1)c_0^0(z_2) - c_0^1(z_1)c_2^0(z_2) + c_0^0(z_1)(c_0^0(z_2) - c_2^0(z_2)) - (c_0^0(z_1) - c_2^0(z_1))c_0^0(z_2), \\
A_8 & := z_{1}^2c_0^0(z_1)z_{2}^2c_0^0(z_2) - z_{1}^2c_2^0(z_1)z_{2}^2c_0^0(z_2) \\
& \quad + z_{1}^2c_0^0(z_2)(z_{1}^2c_0^0(z_1) - c_2^0(z_1)) - (z_{1}^2c_0^0(z_2) - c_2^0(z_2))z_{1}^2c_0^0(z_1), \\
B_1 & := c_1^0(z_1)(c_1^2(z_2) - c_0^2(z_2)) - (c_1^1(z_1) - c_1^0(z_1))c_0^0(z_2), \\
B_2 & := c_0^0(z_1)(c_1^2(z_2) - c_0^2(z_2)) - (c_1^1(z_1) - c_2^0(z_2))c_2^0(z_2), \\
B_3 & := c_1^2(z_1)c_2^0(z_2) - (c_1^1(z_1) - c_2^0(z_2))c_2^0(z_2), \\
B_4 & := c_0^0(z_1)(c_1^2(z_2) - c_0^2(z_2)) - (c_1^1(z_1) - c_1^0(z_1))c_0^0(z_2), \\
B_5 & := z_{1}^2c_0^0(z_1)z_{2}^2c_0^0(z_2) - z_{1}^2c_0^0(z_1)z_{2}^2c_0^0(z_2) + z_{1}^4c_1^0(z_1)z_{2}^2c_0^0(z_2) - z_{1}^2c_0^0(z_1)z_{2}^4c_0^0(z_2) \\
& \quad + (z_{1}^4c_1^0(z_1) - c_2^0(z_1))z_{1}^2c_2^0(z_2) - z_{1}^2c_1^0(z_1)z_{2}^4c_0^0(z_2)) \\
& \quad - z_{1}^2(c_1^0(z_1) - c_2^0(z_1))(z_{1}^2c_1^0(z_2) - c_2^0(z_2)),
\end{align*}
\]
Reflection equation for the $N = 3$ Cremmer–Gervais $R$-matrix

Let us make some definitions and prepare propositions to prove theorem 3.1. We first define the groups of polynomials.

**Definition 3.2.** We define $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as the following groups of polynomials:

- $\mathbf{A} := \{A_j \mid j = 1, \ldots, 8\}$
- $\mathbf{B} := \{B_j \mid j = 1, \ldots, 7\}$
- $\mathbf{C} := \{C_j \mid j = 1, \ldots, 5\}$

For a group of polynomials $\mathcal{J}$, let $T\mathcal{J}$ be groups consisting of polynomials which are the $T$-transformed polynomials belonging to $\mathcal{J}$.

$$T\mathcal{J} = \{TX \mid X \in \mathcal{J}\}. \quad (3.1)$$

For a group of polynomials $\mathcal{J}$, we also denote the groups of equations $\{X = 0 \mid X \in \mathcal{J}\}$ by $\mathcal{J}$ as long as it does not cause confusion.

**Definition 3.3.** We define $\mathbf{0}$, $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\mathbf{2}'$, $\mathbf{3}$ and $\mathbf{3}'$ as the following groups of elements of the reflection equation:

- $\mathbf{0} := \{(0|22), (0|11)\}$
- $\mathbf{1} := \{(02|11), (00|21), (02|12), (01|21), (00|00), (02|20), (02|22), (20|22)\}$
- $\mathbf{1}' := \{(00|12), (10|12), (00|20), (00|02)\}$
- $\mathbf{2} := \{(20|21), (01|22), (01|12), (10|10), (12|21), (21|22), (12|22)\}$
- $\mathbf{2}' := \{(10|22), (20|20), (21|12), (11|11), (21|21), (11|22), (10|21)\}$
- $\mathbf{3} := \{(00|10), (20|12), (01|20), (01|02), (11|02)\}$
- $\mathbf{3}' := \{(01|11), (00|01), (10|11), (02|21), (10|02), (10|20), (11|12), (11|21)\}$

Since the coefficients of all the 38 equations $\{A_j = 0 \mid j = 1, \ldots, 8\} \cup \{B_j = 0 \mid j = 1, \ldots, 7\} \cup \{C_j = 0 \mid j = 1, \ldots, 5\}$ do not depend on the parameter of the Cremmer–Gervais $R$-matrix, $q$, $c_j(z, q)$ do not depend on $q$. Thus, we notice the following from theorem 3.1.

**Corollary 3.1.** The solution to the reflection equation of the $N = 3$ Cremmer–Gervais $R$-matrix does not depend on $q$. 

$$\quad \blacksquare$$
Table 1. The elements of the reflection equation \((i_1i_2 \mid j_1j_2)\) and their associated groups. The matrix element \((3i_1 + i_2 + 1, 3j_1 + j_2 + 1)\) is assigned to \((i_1i_2 \mid j_1j_2)\).

\[
\begin{pmatrix}
1 & 3' & 1' & 3 & 0 & 1' & 1' & 1 & 0 \\
T2 & T2' & 3 & T2' & 3' & 2 & 3 & 1 & 2 \\
T1 & T2 & T2' & T3 & 1 & 1 & 1 & 3' & 1 \\
T2 & T2' & 3' & 2 & 3' & 1' & 3' & 2' & 2' \\
T2' & T3' & 3 & T3' & 2' & 3' & T3 & 3' & 2' \\
T2' & T2' & T3' & T1' & T3' & T2' & T2' & T3' & 2 & 2 \\
T1 & T3' & T1 & T1 & T1 & 3 & 2' & 2 & 1 \\
T2 & T1 & T3 & T2 & T3' & 2' & T3 & 2' & 2 \\
0 & T1 & T1' & T1' & 0 & T3 & T1' & T3' & T1 \\
\end{pmatrix}
\]

For \(J = 0, 1, 2, 3, 1', 2', 3'\), let \(TJ\) be groups of the elements of the reflection equation as follows:

\[
TJ = \{(T(i_1)T(i_2)|T(j_1)T(j_2)) \mid (i_1i_2|j_1j_2) \in J\}.
\] (3.2)

For these groups of polynomials, we note the following.

**Lemma 3.1.** (i) \(J \cup TJ\) is invariant under the \(T\)-transformation for each \(J = A, B, C\).

(ii) \(J \cup TJ\) is invariant under the \(T\)-transformation for each \(J = 0, 1, 2, 3, 1', 2', 3'\).

(i) is obvious from the definition. We also note (ii) from the fact that \(T(i_1i_2|j_1j_2)\), which is the \(T\)-transformation of \((i_1i_2|j_1j_2)\), is \((T(i_1)T(i_2)|T(j_1)T(j_2))\) (2.31).

We introduce the following notations for simplicity.

**Definition 3.4.** Let Reflection, Reduced be the following groups of polynomials:

Reflection = \{\((i_1i_2|j_1j_2) \mid i_1, i_2, j_1, j_2 = 0, 1, 2\}\},

Reduced = \(A \cup B \cup C \cup TA \cup TB \cup TC\).

By definition, we obviously have

\[
\text{Reflection} = \bigcup_{J=0,1,2,3,1',2',3'} \{J \cup TJ\}.
\]

For two groups of polynomials \(P\) and \(Q\), let us denote \(P \Rightarrow Q\) if all the polynomials in \(Q\) can be expressed as linear combinations of the polynomials in \(P\) with \(\mathcal{M}\) coefficients. In order to prove theorem 3.1, we prepare three propositions about the relations between the groups of polynomials which have just been defined. The proof of these propositions is given in appendix A.

**Proposition 3.1.** The 4 elements which belong to \(0 \cup T0\) are all 0.

**Proposition 3.2.**

\(1 \cup 2 \cup 3 \cup T2 \Rightarrow 1' \cup 2' \cup 3'\).
Proposition 3.3.

\[ 1 \cup 2 \cup 3 \iff A \cup B \cup C. \]

\[ \blacksquare \]

Proof of theorem 3.1. Theorem 3.1 means

\[ \text{Reflection} \iff \text{Reduced}, \quad (3.3) \]

which can be decomposed into

(i) \( \text{Reflection} \implies \text{Reduced} \),

(ii) \( \text{Reduced} \implies \text{Reflection} \).

We can prove (i) and (ii) by utilizing propositions 3.2, 3.3 and 2.4.

(i) From proposition 3.3, we have

\[ 1 \cup 2 \cup 3 \implies A \cup B \cup C. \]

Combining this with the obvious relation

\[ \text{Reflection} \implies 1 \cup 2 \cup 3, \]

one has

\[ \text{Reflection} \implies A \cup B \cup C. \quad (3.4) \]

Setting \( \tilde{N} = \text{Reflection} \) in proposition 2.4, (3.4) gives

\[ \text{Reflection} \implies TA \cup TB \cup TC. \quad (3.5) \]

Combining (3.4) and (3.5) together, one has (i).

(ii) From proposition 3.3, one has

\[ A \cup B \cup C \implies 1 \cup 2 \cup 3. \]

Combining this with

\[ \text{Reduced} \implies A \cup B \cup C, \]

which is an obvious relation, we have

\[ \text{Reduced} \implies 1 \cup 2 \cup 3. \quad (3.6) \]

Setting \( \tilde{N} = \text{Reduced} \) in proposition 2.4, one gets

\[ \text{Reduced} \implies T1 \cup T2 \cup T3, \quad (3.7) \]

from (3.6). Combining (3.6) and (3.7) gives

\[ \text{Reduced} \implies 1 \cup 2 \cup 3 \cup T1 \cup T2 \cup T3. \quad (3.8) \]

We combine proposition 3.2, (3.8) and \( 1 \cup 2 \cup 3 \cup T1 \cup T2 \cup T3 \implies 1 \cup 2 \cup 3 \cup T2 \) to get

\[ \text{Reduced} \implies 1' \cup 2' \cup 3'. \quad (3.9) \]
Setting $\tilde{\mathcal{N}} = \text{Reduced}$ in proposition 2.4, (3.9) leads to
\[
\text{Reduced} \iff T1' \cup T2' \cup T3'.
\tag{3.10}
\]
From (3.9) and (3.10), we have
\[
\text{Reduced} \iff 1' \cup 2' \cup 3' \cup T1' \cup T2' \cup T3'.
\tag{3.11}
\]
Combining (3.8) and (3.11), one gets (ii).

To determine the solution space, we use the following 38 equations Reduced' instead of Reduced since they are easier to treat.

**Definition 3.5.** Let $A'_5, A'_6$ and $C'_5$ be the following polynomials.
\[
A'_5 := A_5 + B_4,
\]
\[
A'_6 := z_1^4 c_1^0(z_1) z_2^2 c_0^0(z_2) - z_1^2 c_0^0(z_1) z_2^2 c_1^0(z_2) + c_1^0(z_1) z_2^2 c_1^0(z_2) - z_1^2 c_1^0(z_1) c_1^2(z_2),
\]
\[
+ (z_1^4 c_0^0(z_1) - c_2^0(z_1)) z_2^2 (c_0^0(z_2) - c_2^0(z_2))
\]
\[
- z_1^2 (c_0^0(z_1) - c_2^0(z_1)) (z_2^2 c_0^0(z_2) - c_2^0(z_2)),
\]
\[
C'_5 := c_0^0(z_1) (c_1^0(z_2) - c_2^0(z_2)) - (c_2^0(z_1) - c_2^2(z_1)) c_0^0(z_2)
\]
\[
+ (c_0^0(z_1) - c_1^0(z_1)) (c_2^2(z_2) - c_1^0(z_2)) - (c_2^2(z_1) - c_1^2(z_1)) (c_0^0(z_2) - c_1^0(z_2)).
\]

Let $A', C'$ be the following groups of polynomials:
\[
A' := \{ A_j \mid j = 1, \ldots, 7, 8 \} \cup \{ A'_5, A'_6 \}, \quad C' := \{ C_j \mid j = 1, \ldots, 4 \} \cup \{ C'_5 \}.
\]

Let Reduced' be a group consisting of the following 38 polynomials:
\[
\text{Reduced}' = A' \cup B \cup C' \cup TA' \cup TB \cup TC'.
\]
\[\square\]

$A'_6$ is essentially self-dual with respect to the $T$-transformation, i.e. $TA'_6 = z_1^{-4} z_2^{-4} A'_6$.

**Proposition 3.4.**
\[
\text{Reduced} \iff \text{Reduced}'.
\tag{3.12}
\]
\[\square\]

**Proof of proposition 3.4.** This follows from
\[
A_5, B_4 \iff A'_5, B_4,
\tag{3.13}
\]
\[
TA_5, TB_4 \iff TA'_5, TB_4,
\tag{3.14}
\]
\[
A_1, B_4, TB_4, A_6 \iff A_1, B_4, TB_4, A'_6,
\tag{3.15}
\]
\[
A_6, B_5, TB_5, C_5 \iff A_6, B_5, TB_5, C'_5,
\tag{3.16}
\]
\[
TA_1, B_4, TB_4, TA_6 \iff TA_1, B_4, TB_4, TA'_6,
\tag{3.17}
\]
\[
TA_6, B_5, TB_5, TC_5 \iff TA_6, B_5, TB_5, TC'_5.
\tag{3.18}
\]
For example, (3.15) holds since $A'_6$ can be expressed using $A_1, B_4, TB_4$ and $A_6$ as
\[
A'_6 = A_6 + (z_1^2 z_2^2 - 1) A_1 - z_1^2 z_2^2 B_4 - z_2^2 TB_4.
\]
\[\square\]
4. Solving the reflection equation

In this section, we prove theorem 2.2, i.e. determine the solution space \( \mathcal{K} \) of the reflection equation by solving the 38 equations Reduced'. The outline is as follows. We introduce 9 solution subspaces \( \mathcal{K}_j, i, j = 0, 1, 2 \), where \( \mathcal{K}_j \subset \mathcal{M} \) is an open space whose matrix element \( c_j(z) \) is not identically 0. In order to determine the full solution space \( \mathcal{K} \), we construct the union of 9 open solution subspaces \( \mathcal{K}_j, i, j = 0, 1, 2 \). As a result, the solution space is shown (theorem 4.1) to be the union of two subspaces \( \mathcal{B}_1, \mathcal{B}_2 \) (definition 4.2), which can be parameterized by algebraic varieties \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) (definition 4.1). Analyzing these varieties, we find (theorem 2.2) the solution space is the union of two subspaces \( \mathcal{A}_1, \mathcal{A}_2 \) (definition 2.4), each of which can be parameterized by \( \mathcal{P}(\mathbb{C}) \times \mathcal{P}(\mathbb{C}) \times \mathcal{P}(\mathbb{C}) \) and \( \mathbb{C} \times \mathcal{P}(\mathbb{C}) \times \mathcal{P}(\mathbb{C}) \).

**Definition 4.1.** We define \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) as the following algebraic variety in \( \mathbb{P}^3(\mathbb{C}) \) and \( \mathbb{P}^6(\mathbb{C}) \), respectively:

\[
\mathcal{V}_1 = \left\{ \left( a_0 \frac{a_1}{a_0} a_2 \frac{a_3}{a_0} \frac{a_4}{a_0} \right) \in \mathbb{P}^3(\mathbb{C}) \mid a_j, \bar{a}_j, j = 0 \sim 4 \text{ satisfy 14 relations in (4.2)} \right\},
\]

\[
\mathcal{V}_2 = \left\{ (b, b_0, b_1, b_2, b_3, b_4, b_5) \in \mathbb{P}^6(\mathbb{C}) \mid b, b_j, j = 0 \sim 5 \text{ satisfy 3 relations in (4.4)} \right\}
\]

**Definition 4.2.** Let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) be the following spaces of \( 3 \times 3 \) matrices:

\[
\mathcal{B}_1 = \left\{ K_1(z, \mathcal{Q}_1) \mid \mathcal{Q}_1 = \left( a_0 \frac{a_1}{a_0} a_2 \frac{a_3}{a_0} \frac{a_4}{a_0} \right) \in \mathcal{V}_1 \right\},
\]

**B_2**

\[
K_1(z, \mathcal{Q}_1) = \begin{pmatrix}
\bar{a}_3 + (a_4 - a_3)z^2 - \bar{a}_4z^4 & a_0(z^4 - 1) & a_1z^2(z^4 - 1) \\
(a_2 + \bar{a}_0z^2)(z^4 - 1) & a_3z^2 - (\bar{a}_4 - \bar{a}_3)z^4 & a_4z^2 - (\bar{a}_4 - \bar{a}_3)z^4 - a_3z^6
\end{pmatrix}.
\]

\[
\mathcal{B}_2 = \left\{ K_2(z, \mathcal{Q}_2) \mid \mathcal{Q}_2 = (b, b_0, b_1, b_2, b_3, b_4, b_5) \in \mathcal{V}_2 \right\},
\]

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where

$$K_{II}(z, Q_{II}) = \begin{pmatrix}
  b_3 - b_4 + bz^2 & 0 & b_0(z^4 - 1) \\
  -b_3(z^4 - 1) & -b_4 + bz^4 & b_2(z^4 - 1) \\
  b_1(z^4 - 1) & 0 & bz^2 + (b_3 - b_4)z^4
\end{pmatrix}.$$  \hspace{1cm} (4.8)

In order to prove theorem 2.2, we first prove the following theorem.

**Theorem 4.1.** The solution space $K$ of the $N = 3$ Cremmer–Gervais $R$-matrix (2.2) is the union of $B_I$, $B_{II}$, and does not depend on the parameter of the $R$-matrix, $q$:

$$K = B_I \cup B_{II}.$$  \hspace{1cm} ■

**Proof of theorem 4.1.** From theorem 3.1 and proposition 3.4, it is enough to solve the 38 equations Reduced to determine the solution space of the reflection equation. One notices $K(z, q) = K(z)$ from corollary 3.1. Since we are considering solutions which are not identically 0, at least one of the matrix elements of the reflection equation is not identically 0. Then we have

$$K = \bigcup_{i,j=0}^2 K^i_j,$$  \hspace{1cm} (4.9)

where

$$K^i_j := K \cap U^i_j, \quad U^i_j := \{ K(z) = (c^i_j(z))_{k,l=0,1,2} \in \mathcal{M}^9 \mid c^i_j(z) \neq 0 \} \subset \mathcal{M}^9 \setminus \{0\}.$$  

Thus, if we determine nine subspaces $K^i_j$, $i, j = 0, 1, 2$, one can get $K$ as their union.

For a solution space $K^k_i$, we define its $T$-transformed space $T(K^k_i)$ as follows.

**Definition 4.3.**

$$T(K^k_i) = \{ K'(z) = TK(z^{-1})T \mid K(z) \in K^k_i \}.$$  \hspace{1cm} ■

From (2.24), it is easy to see

$$T(K^k_i) = K^{T(k)}_{T(l)}.$$  \hspace{1cm} (4.10)

From (4.10), the solution space $K^{T(k)}_{T(l)} = K^{2-k}_{2-l}$ can be obtained from $K^k_i$. Thus, in order to obtain $K$, it is enough to determine six spaces

$$K^0_0, K^0_1, K^0_2, K^1_0, K^1_1, K^1_2, K^2_0, K^2_1, K^2_2,$$  \hspace{1cm} (4.11)

out of nine spaces $K^i_j$, $i, j = 0, 1, 2$, and $K^2_1$, $K^2_0$ and $K^1_0$ can be easily obtained from $K^0_1$, $K^0_2$ and $K^1_2$, respectively. Let us further consider solution spaces $K^0_0 \cup K^1_1 \cup K^2_2$ whose diagonal elements are not identically 0. Since several equations in Reduced have terms $c^1_1(z) - c^0_0(z)$, $c^1_1(z) - z^4c^1_0(z)$, $c^1_1(z) - c^2_2(z)$, it is easier to handle the solution spaces in which these terms are not identically 0. Such solution spaces are equivalent to $K^0_0 \cup K^1_1 \cup K^2_2$.

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Lemma 4.1.

\[ K_0^0 \cup K_1^1 \cup K_2^2 = D^0 \cup D^1 \cup D^2, \]

where

\begin{align*}
D^j &:= K \cap V^j, \quad j = 0, 1, 2, \\
V^0 &:= \{ K(z) = (c_l^k(z))_{k,l=0,1,2} \in \mathcal{M}^9 \setminus \{0\} \mid c_1^1(z) - c_0^0(z) \neq 0 \}, \\
V^1 &:= \{ K(z) = (c_l^k(z))_{k,l=0,1,2} \in \mathcal{M}^9 \setminus \{0\} \mid c_1^1(z) - z^4 c_0^0(z) \neq 0 \}, \\
V^2 &:= \{ K(z) = (c_l^k(z))_{k,l=0,1,2} \in \mathcal{M}^9 \setminus \{0\} \mid c_1^1(z) - c_2^2(z) \neq 0 \}.
\end{align*}

Proof of lemma 4.1. This follows from

\[ U_0^0 \cup U_1^1 \cup U_2^2 = V^0 \cup V^1 \cup V^2. \]

From (4.10) and lemma 4.1, one has

Proposition 4.1.

\[ \mathcal{K} = K_0^0 \cup T(K_1^0) \cup K_2^0 \cup T(K_2^0) \cup K_1^1 \cup T(K_1^1) \cup D, \]

where \( D = D^0 \cup D^1 \cup D^2 \).

Thus, one can obtain the solution space \( \mathcal{K} \) by determining

\[ K_1^0, K_2^0, K_1^1, D, \]

instead of (4.11). Among \( K_1^0, K_2^0, K_1^1, D \), we first determine \( K_1^0 \) and \( K_2^0 \). The results, together with those for \( K_2^1 \) and \( K_0^2 \), can be stated as

Proposition 4.2.

(i) \( K_1^0 = B_1|_{a_0 \neq 0} \).
(ii) \( K_2^0 = B_1|_{a_1 \neq 0} \cup B_2|_{a_0 \neq 0} \).
(iii) \( K_1^1 = B_1|_{\bar{a}_0 \neq 0} \).
(iv) \( K_2^1 = B_2|_{\bar{a}_1 \neq 0} \cup B_1|_{b_0 \neq 0} \).

Here, for example, \( B_1|_{a_0 \neq 0} \) is the subspace of \( B_1 \) which satisfies \( a_0 \neq 0 \).

Utilizing (4.10), one can obtain \( K_2^1 \) and \( K_1^0 \) from \( K_1^0 \) and \( K_2^0 \), respectively. Thus, to prove proposition 4.2, it is enough to compute \( K_1^0 \) and \( K_2^0 \). Proposition 4.2 (ii) is proved in appendix B.

Next, we determine \( K_1^1 \) and \( K_2^1 \). Among \( K_1^1 \) and \( K_2^1 \), solutions which satisfy \( c_1^1(z) \neq 0 \) or \( c_2^2(z) \neq 0 \) or \( c_1^2(z) \neq 0 \) or \( c_2^1(z) \neq 0 \) are included in \( K_1^0, K_2^0, K_1^2 \) or \( K_2^2 \), which have already
been determined in proposition 4.2. Thus, we only need to obtain $\bar{\mathcal{K}}_2 := \mathcal{K}_2 ∩ \{ c_0^0(z) = c_2^0(z) = c_0^1(z) ≡ 0 \}$, $\bar{\mathcal{K}}_0 := \mathcal{K}_0 ∩ \{ c_2^1(z) = c_0^1(z) = c_0^0(z) ≡ 0 \}$ to determine $\bar{\mathcal{K}}_2, \mathcal{K}_0$. It is easy to show

**Proposition 4.3.**

(i) $\bar{\mathcal{K}}_2 = B_1|_{a_0 = \bar{a}_0 = a_1 = \bar{a}_1 = 0, a_2 \neq 0} \cup B_{II}|_{b_0 = b_1 = 0, b_2 \neq 0}$.

(ii) $\bar{\mathcal{K}}_0 = B_1|_{a_0 = \bar{a}_0 = a_1 = \bar{a}_1 = 0, a_2 \neq 0} \cup B_{II}|_{b_0 = b_1 = 0, b_5 \neq 0}$.

It is enough to compute $\bar{\mathcal{K}}_1$ since $\bar{\mathcal{K}}_1$ can be obtained from $\bar{\mathcal{K}}_2$ using (4.10). At last, we consider $\mathcal{D} = \mathcal{D}_0 ∪ \mathcal{D}_1 ∪ \mathcal{D}_2$, solutions whose diagonal elements are not identically 0.

Among $\mathcal{K}(z) ∈ \mathcal{D}$, any solution with some off-diagonal element which is not identically 0 is included in $\mathcal{K}_1^0, \mathcal{K}_2^0, \mathcal{K}_1^1, \mathcal{K}_0^1$ or $\mathcal{K}_2^1$, which have already been determined. To obtain $\mathcal{D}$, what remains to be determined is the solution space $\bar{\mathcal{D}}$, whose off-diagonal elements are all 0:

$$\mathcal{D} = \bar{\mathcal{D}}_0 ∪ \mathcal{D}_1 ∪ \mathcal{D}_2, \quad \bar{\mathcal{D}}^k := \mathcal{D}^k ∪ \{ c_j^k(z) \neq 0 \text{ for } i \neq j \}, \quad k = 0, 1, 2.$$ 

By a direct calculation, it is easy to show

**Proposition 4.4.**

$$\bar{\mathcal{D}} = \left\{ \mathcal{K}(z) = \begin{pmatrix} c_1 + c_2 z^2 & 0 & 0 \\ 0 & c_2 z^2 + c_1 z^4 & 0 \\ 0 & 0 & c_2 z^2 + c_1 z^4 \end{pmatrix} c(z) \mid (c_1, c_2) \in \mathbb{P}^1(\mathbb{C}) \right\} \times \bigcup \left\{ \mathcal{K}(z) = \begin{pmatrix} c_1 + c_2 z^2 & 0 & 0 \\ 0 & c_1 + c_2 z^2 & 0 \\ 0 & 0 & c_2 z^2 + c_1 z^4 \end{pmatrix} c(z) \mid (c_1, c_2) \in \mathbb{P}^1(\mathbb{C}) \right\}.$$

From propositions 4.2, 4.3 and 4.4, $\mathcal{K}$ becomes

$$\mathcal{K} = \mathcal{K}_1^0 \cup \mathcal{K}_2^0 \cup \mathcal{K}_0^1 \cup \mathcal{K}_2^1 \cup \mathcal{K}_1^1 \cup \mathcal{K}_0^0 \cup \mathcal{D} = \mathcal{K}_1^0 \cup \mathcal{K}_2^0 \cup \mathcal{K}_0^1 \cup \mathcal{K}_2^1 \cup \mathcal{K}_1^1 \cup \mathcal{K}_0^0 \cup \bar{\mathcal{D}}$$

$$= B_1|_{a_0 \neq 0} \cup B_1|_{\bar{a}_0 \neq 0} \cup \left\{ B_1|_{a_1 \neq 0} \cup B_{II}|_{b_0 \neq 0} \right\} \cup \left\{ B_1|_{\bar{a}_1 \neq 0} \cup B_{II}|_{b_1 \neq 0} \right\} \times \bigcup \left\{ B_1|_{a_0 = \bar{a}_0 = a_1 = \bar{a}_1 = 0, a_2 \neq 0} \cup B_{II}|_{b_0 = b_1 = 0, b_2 \neq 0} \right\} \times \bigcup \left\{ B_1|_{a_0 = \bar{a}_0 = a_1 = \bar{a}_1 = 0, a_2 \neq 0} \cup B_{II}|_{b_0 = b_1 = 0, b_5 \neq 0} \right\} \cup \bar{\mathcal{D}}$$

$$= \left\{ B_1|_{a_0 \neq 0 \text{ or } \bar{a}_0 \neq 0 \text{ or } a_1 \neq 0 \text{ or } \bar{a}_1 \neq 0} \cup B_1|_{a_0 = \bar{a}_0 = a_1 = \bar{a}_1 = 0, a_2 \neq 0 \text{ or } \bar{a}_2 \neq 0} \right\} \cup \left\{ B_{II}|_{b_0 \neq 0 \text{ or } b_1 \neq 0} \cup B_{II}|_{b_0 = b_1 = 0, b_2 \neq 0} \cup b_5 \neq 0 \right\} \cup \bar{\mathcal{D}} \right\}.$$  

(4.12)

One can easily show the following Lemma about $\bar{\mathcal{D}}$.

doi:10.1088/1742-5468/2010/04/P04005
Lemma 4.2.

\begin{itemize}
    \item[(i)] $B_I|_{a_0=\tilde{a}_0=a_1=\tilde{a}_1=a_2=\tilde{a}_2=0} = \tilde{D}$,
    \item[(ii)] $B_{II}|_{b_0=b_1=b_2=b_5=0} = \tilde{D}$.
\end{itemize}

Thus, from (4.12), lemma 4.2 and

\[
B_I = B_I|_{a_0 \neq 0} \text{ or } \tilde{a}_0 \neq 0 \text{ or } a_1 \neq 0 \text{ or } \tilde{a}_1 \neq 0 \bigcup B_I|_{a_0 = \tilde{a}_0 = a_1 = \tilde{a}_1 = 0, a_2 \neq 0 \text{ or } \tilde{a}_2 \neq 0},
\]

\[
B_{II} = B_{II}|_{b_0 \neq 0} \text{ or } b_1 \neq 0 \bigcup B_{II}|_{b_0 = b_1 = 0, b_2 \neq 0 \text{ or } b_5 \neq 0} \bigcup B_{II}|_{b_0 = b_1 = b_2 = b_5 = 0},
\]

which obviously hold, one gets

\[
\mathcal{K} = B_I \cup B_{II},
\]

which proves theorem 4.1.

Investigating the algebraic varieties $\mathcal{V}_{I}$ and $\mathcal{V}_{II}$ which parameterize $B_I$ and $B_{II}$, one can show that $B_I$ and $B_{II}$ can be described by spaces $\mathcal{A}_{I}$ and $\mathcal{A}_{II}$ (definition 2.4), which are parameterized by simpler algebraic varieties $\mathcal{U}_{I}$ and $\mathcal{U}_{II}$ (definition 2.3).

Proposition 4.5.

\begin{itemize}
    \item[(i)] $B_I = \mathcal{A}_{I}$.
    \item[(ii)] $B_{II} = \mathcal{A}_{II} \cup \{ K(z) = \text{Id} \}$.
\end{itemize}

The proof of proposition 4.5 (i) is given in appendix C. Combining theorem 4.1, proposition 4.5 and

\[
\{ K(z) = \text{Id} \} = A_I|_{D_2 = E_1 = E_2 = 0},
\]

one has

\[
\mathcal{K} = \mathcal{A}_I \cup \mathcal{A}_{II}.
\]

Thus, the proof of theorem 2.2 is completed.

5. Discussion

In this paper, we considered the reflection equation of the $N = 3$ Cremmer–Gervais $R$-matrix. The reflection equation is shown to be equivalent to 38 equations which do not depend on the parameters of the $R$-matrix, $q$. The solution space is determined by solving those 38 equations. We found there are two types, each of which is parameterized by the algebraic variety $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$ and $\mathbb{C} \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$. The Cremmer–Gervais $R$-matrix satisfies the conservation law (2.5) and $T$-invariance (2.6). On the other hand, the critical $\mathbb{Z}_n$ vertex model satisfies the conservation law and the $\mathbb{Z}_n$-invariance. Since there does not exist a simple gauge transformation between these two $R$-matrices, we do not know the relation between the $K$-matrices. However, unexpectedly, the latter variety also appears in the solution of the critical $\mathbb{Z}_3$ vertex model [8]. We mention that one
can also formulate and solve the dual reflection equation [1] for $R$-matrices not satisfying crossing unitarity [17]. It is interesting to extend the analysis to determine the full solution space for the general $N$-state Cremmer–Gervais $R$-matrix. Also interesting is to study the integrable model associated with this $R$-matrix under the periodic or open boundary condition.

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Appendix A. Proof of propositions 3.1, 3.2 and 3.3

A.1. Proof of proposition 3.1

We show proposition 3.1 by showing two lemmas.

Lemma A.1.

\[
(00|11) = 0, \quad (00|22) = 0. \tag{A.1}
\]

Proof of lemma A.1. Directly calculating $(00|ii)$ which is an element of the reflection equation, one has

\[
(00|ii) = \sum_{k_1,k_2,k_3,k_4=0}^2 \left[ R_{12}(z_1/z_2) \right]^{00}_{k_3k_2} \left[ K_1(z_1) \right]^{k_1}_{k_3} \left[ R_{21}(z_1z_2) \right]^{k_1k_2}_{k_3k_4} \left[ K_2(z_2) \right]^{k_4}_{k_4}
\]

\[
= [R_{12}(z_1/z_2)]^{00}_{00} [K_1(z_1)]^{00}_{00} [R_{21}(z_1z_2)]^{00}_{00} [K_2(z_2)]^{00}_{00}
\]

\[
- [K_2(z_2)]^{00}_{00} [R_{12}(z_1z_2)]^{00}_{00} [K_1(z_1)]^{00}_{00} [R_{21}(z_1z_2)]^{00}_{00}
\]

\[
= \{(qz_2/z_1 - q^{-1}z_1/z_2)/(q - q^{-1})(z_1/z_2 - z_2/z_1)\} c_0^0(z_1)
\]

\[
\times \{-q^{-1}/(q - q^{-1})\} c_0^0(z_2) - c_0^0(z_2) \{ -q^{-1}/(q - q^{-1})\} c_0^0(z_2)
\]

\[
\times \{(qz_2/z_1 - q^{-1}z_1/z_2)/(q - q^{-1})(z_1/z_2 - z_2/z_1)\} = 0.
\]

In the second equality, we picked up the terms which involve $[R(z)]^{ij}_{kl} \neq 0$. □

Lemma A.2.

\[
(22|11) = 0, \quad (22|00) = 0. \tag{A.2}
\]

Proof of lemma A.2. Applying lemma 2.3 to $(00|11) = 0$ and $(00|22) = 0$ shown in lemma A.1, one has $T(00|11) = (22|11) = 0$ and $T(00|22) = (22|00) = 0$, respectively. □

Combining lemmas A.1 and A.2, we have proposition 3.1.

\[\text{doi:10.1088/1742-5468/2010/04/P04005}\]
A.2. Proof of proposition 3.2

We prove proposition 3.2 in several steps. First, we show

Lemma A.3.

\[ 1 \implies 1'. \]

\[ \blacksquare \]

Proof of lemma A.3. This follows from

\[ (00|21) \iff (00|12), \quad (A.3) \]

\[ (01|21), (02|22), (20|22) \Rightarrow (10|12), (00|20), (00|02). \quad (A.4) \]

Let us show \( (01|21), (02|22), (20|22) \Rightarrow (10|12) \) for example. Calculating \( (10|12) \), we find that it can be expressed using \( (01|21), (02|22) \) and \( (20|22) \) as

\[ (10|12) = -q^2(01|21) + (q^2 - 1)(q^2z_2^2 - z_1^2)^{-1}(z_1^2(02|22) + z_2^2(20|22)). \quad (A.5) \]

Since every element of \( 1' \) can be expressed as combinations of elements of \( 1 \), we have lemma A.3.

Next we show

Lemma A.4.

\[ 1 \cup 2 \cup T2 \Rightarrow 1' \cup 2'. \]

\[ \blacksquare \]

Proof of lemma A.4. We first prove the following relations:

\[ (00|21), (01|22) \Rightarrow (10|22), \quad (A.6) \]

\[ (02|11), (00|00), (12|12) \Rightarrow (21|21), (01|10), \quad (A.7) \]

\[ (02|11), (00|00), (12|12), (10|10) \Rightarrow (11|11), \quad (A.8) \]

\[ (02|11), (00|00), (10|10) \Rightarrow (20|20), \quad (A.9) \]

\[ (01|12), (02|22), (20|22) \Rightarrow (11|22), (10|21). \quad (A.10) \]

For example, \( (02|11), (00|00), (12|12) \Rightarrow (21|21) \) holds since \( (21|21) \) can be expressed using \( (02|11), (00|00) \) and \( (12|12) \) as

\[ (21|21) = q^2(z_1^2 - z_2^2)z_2^{-2}(02|11) + z_1^4z_2^2(q^2 - 1)(z_1^2 - q^2z_2^2)^{-1}(00|00) - z_1z_2^{-2}(12|12). \quad (A.11) \]

From these relations, one has

\[ 1 \cup 2 \cup (12|12) \Rightarrow 2', \quad (A.12) \]

which, combined with lemma A.3, shows lemma A.4.

\[ \square \]
Finally, let us show proposition 3.2. Let us notice the following relations holds:

\[
(00|10), (20|21), (02|12) \Rightarrow (01|11), (00|01), (10|11) \quad (A.13)
\]
\[
(00|10), (20|21), (20|12) \Rightarrow (02|21), \quad (A.14)
\]
\[
(00|10), (20|21), (02|12), (20|12) \Rightarrow (0|02), \quad (A.15)
\]
\[
(01|02), (21|22), (12|22), (02|12) \Rightarrow (10|20), (11|12), (11|21). \quad (A.16)
\]

For example, one can show (A.14) by noting the following equality:

\[
(02|21) = -q^{-2}(20|12) + z_1^{-2}(z_2^2 - z_1^2)(q^2 - 1)^{-1}(20|21) + z_2^{-2}(z_2^2 - z_1^2)(q^2 - 1)^{-1}(02|12). \quad (A.17)
\]

The above relations mean

\[
1 \cup 2 \cup 3 \Rightarrow 3'. \quad (A.18)
\]

Combining this with lemma A.4, one has proposition 3.2.

\[\square\]

A.3. Proof of proposition 3.3

As is the case with proposition 3.2, we show proposition 3.3 by several steps. First, we have

Lemma A.5.

\[1 \iff A.\]

\[\blacksquare\]

Proof of lemma A.5. This follows from the relations below:

\[
(02|11) \iff A_1, \quad (A.19)
\]
\[
(00|21) \iff A_2, \quad (A.20)
\]
\[
(02|12) \iff A_3, \quad (A.21)
\]
\[
(01|21) \iff A_4, \quad (A.22)
\]
\[
(00|00) \iff A_5, \quad (A.23)
\]
\[
(02|20) \iff A_6, \quad (A.24)
\]
\[
(02|22), (20|22) \iff A_7, A_8. \quad (A.25)
\]

For example, calculating (00|00), one has

\[
(00|00) = (q^2 z_2^2 - z_1^2)(q^2 - 1)^{-1}(z_2^2 - z_1^2)^{-1}(1 - z_1^2 z_2^2)^{-1} A_5, \quad (A.26)
\]

which shows (A.23).
Let us next show (A.25). We find by calculation that \((02|22)\) and \((20|22)\) can be expressed by \(A_7\) and \(A_8\) as
\[
(02|22) = (q^2 - 1)^{-1}(z_1^2 - z_2^2)^{-1}(1 - z_1^2 z_2^2)^{-1}(z_1^2 z_2 A_7 + A_8), \tag{A.27}
\]
\[
(20|22) = (q^2 - 1)^{-1}(z_1^2 - z_2^2)^{-1}(z_1^2 z_2^2 - 1)^{-1}(z_1^2 z_2 A_7 + q^2 A_8), \tag{A.28}
\]
from which we find
\[
(02|22), (20|22) \leftarrow A_7, A_8. \tag{A.29}
\]
Solving (A.27) and (A.28) for \(A_7\) and \(A_8\), one gets
\[
(02|22), (20|22) \Rightarrow A_7, A_8, \tag{A.30}
\]
together with (A.29), shows (A.25).
\[
\square
\]
The above relations imply the equivalence between the elements of the reflection equation in 1 and the relations among the matrix elements of \(K(z)\) in \(A\), which is exactly lemma A.5.
\[
\square
\]
Next we prove

Lemma A.6.

\[
1 \cup 2 \iff A \cup B. \tag{A.43}
\]

Proof of lemma A.6. The following relations can be shown:
\[
(20|21), (02|12) \Rightarrow B_1, \tag{A.31}
\]
\[
(01|22), (00|21) \Rightarrow B_2, \tag{A.32}
\]
\[
(01|12), (02|22), (20|22) \Rightarrow B_3, \tag{A.33}
\]
\[
(10|10), (00|00), (02|11) \Rightarrow B_4, \tag{A.34}
\]
\[
(12|21), (00|00), (02|11) \Rightarrow B_5, \tag{A.35}
\]
\[
(21|22), (12|22), (02|12) \Rightarrow B_6, B_7, \tag{A.36}
\]
\[
A_3, B_1 \Rightarrow (20|21), \tag{A.37}
\]
\[
A_2, B_2 \Rightarrow (01|22), \tag{A.38}
\]
\[
A_7, A_8, B_3 \Rightarrow (01|12), \tag{A.39}
\]
\[
A_1, A_5, B_4 \Rightarrow (10|10), \tag{A.40}
\]
\[
A_1, A_5, B_5 \Rightarrow (12|21), \tag{A.41}
\]
\[
A_3, B_6, B_7 \Rightarrow (21|22), (12|22). \tag{A.42}
\]
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For example, (A.39) can be shown by noting that (01|12) can be expressed by $A_7$, $A_8$ and $B_3$ as

$$(01|12) = (q^2 - 1)^{-1}(z_2^2 - z_1^2)^{-1}(z_1^2z_2^2)^{-1}(A_8 + z_1^2z_2^4A_7 + z_2^2(1 - z_1^2z_2^2)B_3). \tag{A.43}$$

From these relations, one has

$$1 \cup 2 \implies B,$$ \tag{A.44}

$$A \cup B \implies 2.$$ \tag{A.45}

Combining these with lemma A.5, we get lemma A.6. \hfill \square

Now let us finally prove proposition 3.3. One can show the following relations among the polynomials:

$$\begin{align*}
(00|10), (20|21), (02|12) & \Rightarrow C_1, \tag{A.46} \\
(20|12), (00|10), (20|21), (02|12) & \Rightarrow C_2, \tag{A.47} \\
(01|20), (02|12), (20|21), (12|22) & \Rightarrow C_3, \tag{A.48} \\
(01|02), (21|22), (12|22), (02|12) & \Rightarrow C_4, \tag{A.49} \\
(11|02), (12|21), (02|11), (00|00) & \Rightarrow C_5, \tag{A.50} \\
A_3, B_1, C_1 & \Rightarrow (00|10), \tag{A.51} \\
A_3, B_1, C_1, C_2 & \Rightarrow (20|12), \tag{A.52} \\
A_3, B_1, B_6, B_7, C_3 & \Rightarrow (01|20), \tag{A.53} \\
B_6, C_2, C_4 & \Rightarrow (01|02), \tag{A.54} \\
A_5, B_5, C_5 & \Rightarrow (11|02). \tag{A.55}
\end{align*}$$

For example, one can show (A.53) since (01|20) can be expressed as combinations of $A_3, B_1, B_6, B_7$ and $C_3$ as

$$(01|20) = (q^2 - 1)^{-1}(z_2^2 - z_1^2)^{-1}(1 - z_1^2z_2^2)^{-1} \\
\times ((z_1^2 + z_2^2)A_3 - z_1^2(1 - z_1^2z_2^2)B_1 + z_2^2B_6 - B_7 + (1 - z_1^2z_2^2)C_3). \tag{A.56}$$

The above relations mean

$$\begin{align*}
1 \cup 2 \cup 3 & \implies C, \tag{A.57} \\
A \cup B \cup C & \implies 3, \tag{A.58}
\end{align*}$$

which, combined with lemma A.6, shows proposition 3.3. \hfill \square

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Appendix B. Proof of proposition 4.2 (ii)

We use the following simple lemma to calculate the solution to the reflection equation.

Lemma B.1. If two meromorphic functions \( X(z), Y(z) \in \mathcal{M} \) satisfy
\[
X(z_1)Y(z_2) = X(z_2)Y(z_1),
\]
there exist constants \( C_1, C_2 \) and a meromorphic function \( f(z) \in \mathcal{M} \) satisfying
\[
X(z) = C_1 f(z), \quad Y(z) = C_2 f(z).
\]
\[
\]

We first use 8 equations \( A_2 = 0, A_3 = 0, A'_3 = 0, A_7 = 0, A_8 = 0, B_2 = 0, B_3 = 0 \) and \( C_4 = 0 \) out of 38 equations Reduced to prove the following.

Proposition B.1. If \( K(z) \in K^0_2, K(z) \) must be expressed in the following form:
\[
K(z) = (((\alpha_4 - \alpha_3)z^2 - (\bar{\alpha}_4 - \bar{\alpha}_3)z^4 + \alpha_7z^6)\text{Id}c(z)
+ \begin{pmatrix}
-\bar{\alpha}_3 - \alpha_7z^2 & \alpha_0 & \alpha_1z^2 \\
\bar{\alpha}_2 + \alpha_5z^2 & 0 & \alpha_0 + \alpha_2z^2 \\
\bar{\alpha}_1 + \alpha_6z^2 & \bar{\alpha}_0z^2 & -\alpha_3z^2
\end{pmatrix}
(z^4 - 1)c(z),
\]
where \( \alpha_i, i = 0, \ldots, 7, \bar{\alpha}_i, i = 0, \ldots, 4 \) are constants, \( \alpha_1 \neq 0, c(z) \neq 0 \) is a meromorphic function and \( \alpha_i, \bar{\alpha}_i \) satisfy four relations:
\[
\alpha_0\bar{\alpha}_0 - \alpha_1\bar{\alpha}_1 = 0, \quad \alpha_0\alpha_3 - \alpha_1\bar{\alpha}_2 = 0, \quad \alpha_0\alpha_2 - \alpha_3\bar{\alpha}_1 = 0, \quad \alpha_0^2 - \alpha_1\alpha_4 = 0.
\]
\[
\]

Proof of proposition B.1. Since we are considering \( K(z) \in K^0_2, c^0_2(z) \) can be expressed as
\[
c^0_2(z) = \alpha_1z^2(z^4 - 1)c(z),
\]
where \( c(z) \neq 0 \) is a meromorphic function of \( z \) and \( \alpha_1 \neq 0 \) is a constant.

From \( A_2 = 0 \) and lemma B.1, one can express \( c^0_1(z) \) as
\[
c^0_1(z) = \alpha_0(z^4 - 1)c(z),
\]
where \( \alpha_0 \) is a constant.

Similarly, utilizing \( A_3 = 0, B_2 = 0, B_3 = 0 \) and lemma B.1, we get
\[
c^2_1(z) = \bar{\alpha}_0z^2(z^4 - 1)c(z),
\]
\[
c^1_1(z) = (\alpha_0 + \alpha_2z^2)(z^4 - 1)c(z),
\]
\[
c^2_2(z) = c^1_1(z) - \alpha_3z^2(z^4 - 1)c(z),
\]
where \( \bar{\alpha}_0, \alpha_2, \alpha_3 \) are constants.

\[\text{doi:10.1088/1742-5468/2010/04/P04005}\]
Next, we substitute (B.3)–(B.7) into $A_7 = 0, A'_3 = 0, C_4 = 0$ and use lemma B.1 to obtain

$$c_0^0(z) = c_1^1(z) - \left(\frac{\alpha_0\alpha_2}{\alpha_1} + \alpha_7z^2\right)(z^4 - 1)c(z), \quad \text{(B.8)}$$

$$c_0^1(z) = \left(\frac{\alpha_0\alpha_3}{\alpha_1} + \alpha_5z^2\right)(z^4 - 1)c(z), \quad \text{(B.9)}$$

$$c_0^2(z) = \left(\frac{\alpha_0\bar{\alpha}_0}{\alpha_1} + \alpha_6z^2\right)(z^4 - 1)c(z), \quad \text{(B.10)}$$

where $\alpha_5, \alpha_6, \alpha_7$ are constants. We set $\bar{\alpha}_1 := \alpha_0\bar{\alpha}_0/\alpha_1, \bar{\alpha}_2 := \alpha_0\alpha_3/\alpha_1, \bar{\alpha}_3 := \alpha_0\alpha_2/\alpha_1$.

Furthermore, we substitute (B.3)–(B.10) in $A_8 = 0$. The result is

$$(z_1^4 - 1)(z_2^4 - 1)(z_2^4c(z_2) - (\alpha_1c_1^1(z_1) + ((\alpha_1\alpha_3 - \alpha_0^2)z_1^2 - \alpha_1\alpha_7z_1^6)c(z_1))) - (\alpha_1c_1^1(z_2) + ((\alpha_1\alpha_3 - \alpha_0^2)z_2^2 - \alpha_1\alpha_7z_2^6)c(z_2))z_1^2c(z_1)) = 0.$$  

Using lemma B.1, one has

$$c_1^1(z) = \left(\frac{\alpha_0^2}{\alpha_1} - \alpha_3\right)z^2 + \bar{\alpha}_4z^4 + \alpha_7z^6)c(z), \quad \text{(B.11)}$$

where $\bar{\alpha}_4$ is a constant. Setting $\alpha_4 := \alpha_0^2/\alpha_1, \bar{\alpha}_4 := -\bar{\alpha}_4 + \bar{\alpha}_3$, (B.11) becomes

$$c_1^1(z) = ((\alpha_4 - \alpha_3)z^2 - (\bar{\alpha}_4 - \bar{\alpha}_3)z^4 + \alpha_7z^6)c(z). \quad \text{(B.12)}$$

Substituting (B.12) into (B.7), (B.8), one sees $c_2^2(z)$ and $c_0^0(z)$ are expressed as

$$c_2^2(z) = (\alpha_4z^2 - (\bar{\alpha}_4 - \bar{\alpha}_3)z^4 + (\alpha_7 - \alpha_3)z^6)c(z), \quad \text{(B.13)}$$

$$c_0^0(z) = (\bar{\alpha}_3 + (\alpha_4 + \alpha_7 - \alpha_3)z^2 - \alpha_4z^4)c(z). \quad \text{(B.14)}$$

From (B.3), (B.4), (B.5), (B.6), (B.9), (B.10), (B.12), (B.13), (B.14) and $\bar{\alpha}_1 := \alpha_0\bar{\alpha}_0/\alpha_1, \bar{\alpha}_2 := \alpha_0\alpha_3/\alpha_1, \bar{\alpha}_3 := \alpha_0\alpha_2/\alpha_1, \alpha_4 := \alpha_0^2/\alpha_1$, one sees that if $K(z) \in \mathcal{K}_3^0$, the matrix elements $c_i^j(z), i, j = 0, 1, 2$ of $K(z)$ must be expressed as (B.1), and $\alpha_i, \bar{\alpha}_i$ should satisfy (B.2). Thus, proposition B.1 is proved. \qed

Proposition B.1 is just a necessary condition to be a solution since we only used 8 equations of the 38 equations Reduced’. For (B.1) to be a solution, the rest of the 30 equations, into which (B.1) have been substituted, must hold for any $z$. These conditions are relations between the coefficients. For example, $TA_2 = 0$ becomes

$$\alpha_6\bar{\alpha}_0z_1^2z_2^2(z_1^2 - z_2^2)(z_1^4 - 1)(z_2^4 - 1)c(z_1)c(z_2) = 0. \quad \text{(B.15)}$$

One must have

$$\alpha_6\bar{\alpha}_0 = 0, \quad \text{(B.16)}$$

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for (B.15) to hold for any \( z \). Similarly, the following relations must be fulfilled. We denote the equations which yield the relations to the right of them:

\[
\begin{align*}
\alpha_6\alpha_0 &= 0, & TA_3 &= 0, & TC_1 &= 0, & (B.17) \\
(\bar{\alpha}_0 - \alpha_5)\bar{\alpha}_0 &= 0, & TA_4 &= 0, & (B.18) \\
\bar{\alpha}_2\alpha_0 - \alpha_3\alpha_4 - (\alpha_2\bar{\alpha}_0 - \bar{\alpha}_3\bar{\alpha}_4) + \alpha_7\alpha_4 &= 0, & A'_6 &= 0, & (B.19) \\
\bar{\alpha}_0\bar{\alpha}_2 - \alpha_3\bar{\alpha}_1 + \alpha_7\bar{\alpha}_1 - \alpha_6\bar{\alpha}_3 &= 0, & TA_\tau &= 0, & (B.20) \\
\bar{\alpha}_0\alpha_5 - \bar{\alpha}_1\bar{\alpha}_4 - \alpha_6\alpha_4 &= 0, & TA_8 &= 0, & (B.21) \\
\alpha_7\alpha_0 &= 0, & B_1 &= 0, & TB_7 &= 0, & (B.22) \\
(\bar{\alpha}_0 - \alpha_5)\bar{\alpha}_1 + \alpha_6\bar{\alpha}_2 &= 0, & TB_2 &= 0, & (B.23) \\
\alpha_7\bar{\alpha}_1 - \alpha_6\bar{\alpha}_3 &= 0, & TB_3 &= 0, & (B.24) \\
(\bar{\alpha}_0 - \alpha_5)\alpha_0 &= 0, & B_4 &= 0, & (B.25) \\
\bar{\alpha}_2\alpha_0 - \alpha_3\alpha_4 &= 0, & B_5 &= 0, & (B.26) \\
\alpha_7\alpha_3 - \alpha_6\alpha_1 &= 0, & B_5 &= 0, & (B.27) \\
\alpha_2\bar{\alpha}_0 - \bar{\alpha}_3\bar{\alpha}_4 + \alpha_7(\alpha_3 - \alpha_4) - \alpha_6\alpha_1 &= 0, & TB_5 &= 0, & (B.28) \\
\alpha_5\bar{\alpha}_3 - \bar{\alpha}_1\alpha_2 + \alpha_6\alpha_0 - \alpha_7\bar{\alpha}_2 &= 0, & TB_6 &= 0, & (B.29) \\
\alpha_2\alpha_4 - \alpha_0\bar{\alpha}_3 - (\alpha_1\bar{\alpha}_0 - \alpha_0\bar{\alpha}_4) &= 0, & B_7 &= 0, & (B.30) \\
(\bar{\alpha}_0 - \alpha_5)\alpha_1 - \alpha_7\alpha_2 &= 0, & B_7 &= 0, & C'_3 &= 0, & (B.31) \\
\bar{\alpha}_2\alpha_4 - \bar{\alpha}_0\alpha_3 - (\bar{\alpha}_1\alpha_0 - \alpha_3\alpha_4) + \alpha_6\alpha_2 &= 0, & TB_7 &= 0, & (B.32) \\
\alpha_1\bar{\alpha}_0 - \alpha_0\bar{\alpha}_4 &= 0, & C_1 &= 0, & (B.33) \\
\bar{\alpha}_1\alpha_0 - \bar{\alpha}_0\alpha_4 &= 0, & TC_1 &= 0, & (B.34) \\
\alpha_7\bar{\alpha}_0 &= 0, & TC_1 &= 0, & TC_2 &= 0, & (B.35) \\
(\bar{\alpha}_0 - \alpha_5)\alpha_3 - \alpha_6\alpha_2 &= 0, & TC_3 &= 0, & (B.36) \\
\alpha_0\bar{\alpha}_3 - \alpha_2\bar{\alpha}_1 + \alpha_6\alpha_0 &= 0, & TC_4 &= 0, & (B.37) \\
\alpha_2\bar{\alpha}_2 - \alpha_3\bar{\alpha}_3 &= 0, & C'_5 &= 0, & (B.38) \\
(\bar{\alpha}_0 - \bar{\alpha}_3)\alpha_0 + \alpha_2\bar{\alpha}_2 - \alpha_3\bar{\alpha}_3 &= 0, & TC'_5 &= 0. & (B.39) \\
\end{align*}
\]

From proposition B.1, we also have

\[
\begin{align*}
\alpha_0\bar{\alpha}_0 - \alpha_1\bar{\alpha}_1 &= 0, & (B.40) \\
\alpha_0\alpha_3 - \alpha_1\bar{\alpha}_2 &= 0, & (B.41) \\
\alpha_0\alpha_2 - \bar{\alpha}_1\bar{\alpha}_4 &= 0, & (B.42) \\
\alpha^2_0 - \alpha_1\alpha_4 &= 0. & (B.43) \\
\end{align*}
\]
Among (B.16)–(B.43), we first consider the following six relations:

\[
\left\{ \begin{array}{l}
\alpha_6\bar{\alpha}_0 = 0 \ (B.16), \\
\alpha_5\alpha_0 = 0 \ (B.17), \\
(\bar{\alpha}_0 - \alpha_5)\bar{\alpha}_0 = 0 \ (B.18), \\
(\bar{\alpha}_0 - \bar{\alpha}_5)\alpha_0 = 0 \ (B.25), \\
\alpha_7\bar{\alpha}_0 = 0 \ (B.35), \\
\alpha_7\alpha_0 = 0 \ (B.22) \\
\end{array} \right.
\]

These six relations are equivalent to

\[
(A)\{\alpha_6 = \alpha_7 = 0, \alpha_5 = \bar{\alpha}_0\} \quad \text{or} \quad (B)\{\alpha_0 = \bar{\alpha}_0 = 0\}.
\]

Let us investigate the cases (A) and (B), separately.

\( (A) \) \( \alpha_6 = \alpha_7 = 0, \alpha_5 = \bar{\alpha}_0 \)

Substituting \( \alpha_6 = \alpha_7 = 0, \alpha_5 = \bar{\alpha}_0 \), the matrix elements of \( K(z) \) (B.1) become

\[
K(z) = \left( \begin{array}{ccc}
\bar{\alpha}_3 + (\alpha_4 - \alpha_3)z^2 - \bar{\alpha}_4z^4 & \alpha_0(z^4 - 1) & \alpha_1z^2(z^4 - 1) \\
(\bar{\alpha}_2 + \bar{\alpha}_0z^2)(z^4 - 1) & (\alpha_4 - \alpha_3)z^2 - (\bar{\alpha}_4 - \bar{\alpha}_3)z^4 & (\alpha_0 + \alpha_2z^2)(z^4 - 1) \\
\bar{\alpha}_1(z^4 - 1) & \bar{\alpha}_0z^2(z^4 - 1) & \alpha_4z^2 - (\bar{\alpha}_4 - \bar{\alpha}_3)z^4 - \alpha_3z^6 \\
\end{array} \right) \times c(z),
\]

(B.44)

and the relations (B.16)–(B.43) are reduced to

\[
\begin{align*}
\alpha_9\bar{\alpha}_0 - \alpha_1\bar{\alpha}_1 &= 0, \quad \text{(B.45)} \\
\bar{\alpha}_2\alpha_0 - \alpha_3\alpha_4 - (\alpha_2\bar{\alpha}_0 - \bar{\alpha}_3\bar{\alpha}_4) &= 0, \quad \text{(B.46)} \\
\alpha_9\alpha_2 - \alpha_3\alpha_1 &= 0, \quad \text{(B.47)} \\
\bar{\alpha}_0\alpha_2 - \alpha_3\bar{\alpha}_1 &= 0, \quad \text{(B.48)} \\
\alpha_0^2 - \alpha_1\alpha_4 &= 0, \quad \text{(B.49)} \\
\bar{\alpha}_0^2 - \bar{\alpha}_1\bar{\alpha}_4 &= 0, \quad \text{(B.50)} \\
\bar{\alpha}_2\alpha_0 - \alpha_3\alpha_4 &= 0, \quad \text{(B.51)} \\
\alpha_2\bar{\alpha}_0 - \bar{\alpha}_3\bar{\alpha}_4 &= 0, \quad \text{(B.52)} \\
\alpha_0\alpha_3 - \alpha_1\bar{\alpha}_2 &= 0, \quad \text{(B.53)} \\
\bar{\alpha}_0\alpha_3 - \bar{\alpha}_1\alpha_2 &= 0, \quad \text{(B.54)} \\
\alpha_1\alpha_0 - \alpha_0\bar{\alpha}_4 - (\alpha_2\alpha_4 - \alpha_0\bar{\alpha}_3) &= 0, \quad \text{(B.55)} \\
\bar{\alpha}_1\alpha_0 - \bar{\alpha}_0\alpha_4 - (\bar{\alpha}_2\alpha_4 - \bar{\alpha}_0\alpha_3) &= 0, \quad \text{(B.56)} \\
\bar{\alpha}_1\alpha_0 - \bar{\alpha}_0\alpha_4 &= 0, \quad \text{(B.57)} \\
\alpha_1\alpha_0 - \alpha_0\alpha_4 &= 0, \quad \text{(B.58)} \\
\alpha_2\alpha_0 - \alpha_3\alpha_3 &= 0. \quad \text{(B.59)}
\end{align*}
\]

Noting that (B.46) can be obtained by subtracting both the left-hand and right-hand sides of (B.51) by those of (B.52), the 15 relations (B.45)–(B.58) are equivalent to 14 relations (4.2) \((I_j, j = 1, \ldots, 8), (Tl_j, j = 3, \ldots, 8)\). Thus, in the case (A), the matrix

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elements of the solution $K(z)$ is given by (B.44) and the coefficients must satisfy the relations $(I_{ij}, j = 1, \ldots, 8), (T I_{ij}, j = 3, \ldots, 8)$. This corresponds to $B_1$ in definition 4.2 with $a_1 \neq 0$.

(B) $a_0 = \bar{a}_0 = 0$

From (B.40), (B.41), (B.42), (B.43) and $a_1 \neq 0$, one must have $\bar{a}_1 = \bar{a}_2 = \bar{a}_3 = \alpha_4 = 0$. Thus the matrix elements of $K(z)$ (B.1) become

$$K(z) = \begin{pmatrix}
\alpha_7 - \alpha_3 - \bar{a}_4 z^2 & 0 & \alpha_1 (z^4 - 1) \\
\alpha_5 (z^4 - 1) & -\alpha_3 - \bar{a}_4 z^2 + \alpha_7 z^4 & \alpha_2 (z^4 - 1) \\
\alpha_6 (z^4 - 1) & 0 & -\bar{a}_4 z^2 + (\alpha_7 - \alpha_3) z^4
\end{pmatrix} z^2 c(z), \quad (B.60)$$

and the relations (B.16)–(B.43) are reduced to three relations

$$\begin{align*}
\alpha_1 \alpha_6 - \alpha_7 \alpha_3 &= 0, \quad (B.61) \\
\alpha_6 \alpha_2 + \alpha_3 \alpha_5 &= 0, \quad (B.62) \\
\alpha_2 \alpha_7 + \alpha_5 \alpha_1 &= 0. \quad (B.63)
\end{align*}$$

In case (B), the matrix elements of the solution $K(z)$ is given by (B.60), and the coefficients must satisfy the relations (B.61), (B.62) and (B.63). This corresponds to $B_{II}$ in definition 4.2 with $b_0 \neq 0$.

Investigating the two cases (A) and (B), one has $K^0_2 = B_1|_{a_1 \neq 0} \cup B_{II}|_{b_0 \neq 0}$. Thus, the proposition 4.2 (ii) is proved.

**Appendix C. Proof of proposition 4.5 (i)**

Observing the algebraic variety $V_1$ which parameterizes $B_1$, one notices the following Lemma holds.

**Lemma C.1.** The points in $V_1$ fulfill the equation

$$I_0: a_0 \tilde{a}_0 - a_4 \bar{a}_4 = 0,$$

if any one of $a_0, \tilde{a}_0, a_1, \tilde{a}_1, a_2, \tilde{a}_2, a_3$ or $\bar{a}_3$ is nonzero.

The 15 equations $(I_{ij}, j = 1, \ldots, 8), (T I_{ij}, j = 3, \ldots, 8)$ and $I_0$ can be conveniently expressed as

$$\text{rank} \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \tilde{a}_0 & \bar{a}_4 \\ a_4 & a_0 & \tilde{a}_3 & \tilde{a}_2 & a_1 & \tilde{a}_0 \end{vmatrix} = 1.$$  

By utilizing lemma C.1, the following holds.

**Proposition C.1.**

$$V_1 = V^0_1 \cup V^1_1,$$

where

$$V^0_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 0 & \bar{a}_4 \end{pmatrix} \in \mathbb{P}^9(\mathbb{C}) \middle| (a_4, \bar{a}_4) \in \mathbb{P}^1(\mathbb{C}), a_4 \bar{a}_4 \neq 0 \right\},$$

$$V^1_1 = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 & \bar{a}_4 \end{pmatrix} \in \mathbb{P}^9(\mathbb{C}) \middle| \text{rank} \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \tilde{a}_0 & \bar{a}_4 \\ a_4 & a_0 & \tilde{a}_3 & \tilde{a}_2 & a_1 & \tilde{a}_0 \end{vmatrix} = 1 \right\}.$$

Studying $V^1_1$ which is the main part of $V_1$, we find that the following variety is included.

\[\text{doi:10.1088/1742-5468/2010/04/P04005}\]
Definition C.1.

\[ S = \{(c_0, c_1, c_2, c_3, c_4, c_5) \in \mathbb{P}^5(\mathbb{C}) \mid c_j, j = 0 \sim 5 \text{ satisfy 3 relations in (C.3) } S_1, S_2, S_3\}, \tag{C.2} \]

\[ S_1: c_0 c_1 - c_3 c_4 = 0, \quad S_2: c_1 c_2 - c_4 c_5 = 0, \quad S_3: c_2 c_3 - c_5 c_0 = 0. \tag{C.3} \]

The coordinates of points of \( V_1 \) can be parameterized using the projective varieties \( \mathbb{P}^1(\mathbb{C}) \) and \( S \).

Proposition C.2.

\[ V_1 = \mathcal{W}, \]

where

\[ \mathcal{W} = \left\{ \left( \begin{array}{cccc} A_1 A_2 & A_1^2 & B_2 B_2 & B_1 B_2 & A_2^2 \\ A_1 A_2 & A_1^2 & B_2 B_1 & B_1 B_2 & A_2^2 \end{array} \right) \in \mathbb{P}^9(\mathbb{C}) \mid (B_1, B_2) \right\}, \]

\[ \cong \mathbb{P}^1(\mathbb{C}), (A_1, \bar{A}_1, B_2, A_2, \bar{A}_2, \bar{B}_1) \in S \right\}. \]

Proof of proposition C.2. Let us first take a look at 3 equations \( I_2, I_3, TI_3 \) out of the 15 ones \((I_j, j = 0, \ldots, 8), (TI_j, j = 3, \ldots, 8)\). From \( I_2, I_3 \) and \( TI_3 \), we see that \( a_j, \bar{a}_j, j = 0, \ldots, 4 \) can be parameterized as

\[ a_2 = B_2 \bar{B}_2, \quad \bar{a}_2 = B_1 \bar{B}_1, \quad a_3 = B_1 \bar{B}_2, \quad \bar{a}_3 = \bar{B}_1 B_2, \]

\[ a_0 = A_1 A_2, \quad a_1 = A_1^2, \quad a_4 = A_2^2, \]

and

\[ a_0 = A_1 A_2, \quad a_1 = A_1^2, \quad a_4 = A_2^2, \]

where \( A_1, A_2, B_1, B_2, \bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{B}_2 \in \mathbb{C} \).

Substituting these parameterizations into the remaining 12 equations among \( (I_j, j = 0, \ldots, 8), (TI_j, j = 3, \ldots, 8) \) and calculating \( V_1|_{a_1 \neq 0}, \mathcal{V}_1|_{a_1 \neq 0}, \mathcal{V}_1|_{a_1 = \tilde{a}_1 = 0, a_4 \neq 0}, \mathcal{V}_1|_{a_1 = \tilde{a}_1 = 0, a_4 = 0} \), one finds

\[ V_1|_{a_1 \neq 0} = \left\{ \left( \begin{array}{cccc} A_1 A_2 & A_1^2 & B_2 B_2 & B_1 B_2 & A_2^2 \\ A_1 A_2 & A_1^2 & B_2 B_1 & B_1 B_2 & A_2^2 \end{array} \right) \right. \]

\[ \in \mathbb{P}^9(\mathbb{C}) \mid A_1 \in \mathbb{C}^\times, (B_1, B_2) \in \mathbb{P}^1(\mathbb{C}), (A_1, \bar{A}_1, B_2, A_2, \bar{A}_2, \bar{B}_1) \in S \right\}, \tag{C.4} \]

\[ = \mathcal{W}|_{A_1 \neq 0}, \]

\[ V_1|_{a_1 \neq 0} = \mathcal{W}|_{A_1 \neq 0}, \tag{C.5} \]

\[ V_1|_{a_1 = \tilde{a}_1 = 0, a_4 \neq 0} = \left\{ \left( \begin{array}{cccc} 0 & 0 & 0 & A_2^2 \\ 0 & 0 & B_1 \bar{B}_1 & B_1 B_2 \end{array} \right) \right. \]

\[ \in \mathbb{P}^9(\mathbb{C}) \mid A_2 \in \mathbb{C}^\times, B_1 \in \mathbb{C}, (B_1, B_2) \in \mathbb{P}^1(\mathbb{C}) \right\}, \tag{C.6} \]

\[ = \mathcal{W}|_{A_1 = \tilde{A}_1 = 0, A_2 \neq 0}. \]

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\[
\mathcal{V}_1^0 |_{a_1 = \bar{a}_1 = 0, a_2 \neq 0} = \mathcal{W}|_{A_1 = \bar{A}_1 = 0, A_2 \neq 0},
\]

\[
\mathcal{V}_1^1 |_{a_1 = \bar{a}_1 = a_4 = \bar{a}_4 = 0} = \begin{cases} 
0 & 0 \\
B_2 \bar{B}_2 & B_1 \bar{B}_2 & 0
\end{cases} 
\in \mathbb{P}^9(\mathbb{C}) | (B_1, B_2) \in \mathbb{P}^1(\mathbb{C}), (\bar{B}_1, \bar{B}_2) \in \mathbb{P}^1(\mathbb{C}) 
= \mathcal{W}|_{A_1 = \bar{A}_1 = A_2 = \bar{A}_2 = 0}. 
\]

From (C.4), (C.5), (C.6), (C.7) and (C.8) and the following obvious relations
\[
\mathcal{V}_1^1 = \mathcal{V}_1^1 |_{a_1 \neq 0} \cup \mathcal{V}_1^1 |_{\bar{a}_1 \neq 0} \cup \mathcal{V}_1^1 |_{a_1 = \bar{a}_1 = 0, a_4 \neq 0} \cup \mathcal{V}_1^1 |_{a_1 = \bar{a}_1 = a_4 = \bar{a}_4 = 0}, 
\]
\[
\mathcal{W} = \mathcal{W}|_{A_1 \neq 0} \cup \mathcal{W}|_{\bar{A}_1 \neq 0} \cup \mathcal{W}|_{A_1 = \bar{A}_1 = 0, A_2 \neq 0} \cup \mathcal{W}|_{A_1 = \bar{A}_1 = A_2 = \bar{A}_2 = 0};
\]
one has
\[
\mathcal{V}_1^1 = \mathcal{W},
\]
which proves proposition C.2.

The algebraic variety $\mathcal{S}$ is isomorphic with the Segre threefold $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$.

**Proposition C.3.**

\[
\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \cong \mathcal{S}.
\]

The map $\psi$
\[
\psi : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \rightarrow \mathcal{S},
\]
\[
\psi((D_1, D_2) \times (E_1, E_2, E_3)) = (D_1 E_1, D_2 E_3, D_1 E_2, D_2 E_1, D_1 E_3, D_2 E_2),
\]
parameterizes the points in $\mathcal{S}$ by $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$. Combining this map with proposition C.2, one sees that the points in $\mathcal{V}_1^1$ can be parameterized by the projective variety $\mathcal{U}_1 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$ (definition 2.3).

**Proposition C.4.** Any point in $\mathcal{V}_1^1$ can be parameterized by $\mathcal{U}_1$ as
\[
\mathcal{V}_1^1 = \left\{ \begin{pmatrix} D_1 D_2 E_2^2 & D_2^2 E_1^2 & D_1 E_2 B_2 & D_1 E_2 B_1 & D_2^2 E_1^2 \\ D_1 D_2 E_3^2 & D_2^2 E_3^2 & D_2 E_2 B_1 & D_2 E_2 B_2 & D_2^2 E_3^2 \end{pmatrix} \right| 
(B_1, B_2) \times (D_1, D_2) \times (E_1, E_2, E_3) \in \mathcal{U}_1 \right\}.
\]

Let us denote the solution space corresponding to $\mathcal{V}_1^0$ and $\mathcal{V}_1^1$ as $\mathcal{A}_1^0$ and $\mathcal{A}_1^1$, respectively.

**Definition C.2.**

$$
\mathcal{A}_1^0 = \{ K_1(z, \mathcal{Q}_1) \mid \mathcal{Q}_1 \in \mathcal{V}_1^0 \}, \quad \mathcal{A}_1^1 = \{ K_1(z, \mathcal{Q}_1) \mid \mathcal{Q}_1 \in \mathcal{V}_1^1 \}. 
$$
Noting that $\mathcal{A}_1^0$ is included in $\mathcal{A}_1^1$ as
\[ \mathcal{A}_1^0 = \{ K(z) = \text{Id} \} = \mathcal{A}_1^1 |_{D_2 = E_1 = E_2 = 0}, \]
and utilizing proposition C.4, one has
\[ \mathcal{A}_1 = \mathcal{A}_1^0 \cup \mathcal{A}_1^1 = \mathcal{A}_1^1 = \{ K_1(z, \mathcal{P}_1) \mid \mathcal{P}_1 \in \mathcal{U}_1 \} = \mathcal{B}_1, \]
which proves proposition 4.5 (i).

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