Semi-analytic calculation of the monopole order parameter in QCD

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The monopole order parameter of QCD is computed in terms of gauge invariant field strength correlators. Both quantities are partially known from numerical simulations on the lattice. A new insight results on the structure of the confining vacuum.

1. Introduction

The mechanism of confinement by dual superconductivity of the QCD vacuum \cite{123} is confirmed by numerical simulations on a lattice. Chromoelectric flux tubes connecting \(q - \bar{q}\) pairs produced by dual Meissner effect are indeed observed in lattice configurations with the expected form of the electric and magnetic fields\cite{4}. An extensive analysis has been performed by exploring the vacuum by means of an order parameter \((\mu)\) \cite{47}, which is the vacuum expectation value of a magnetically charged operator \(\mu\).

In the confined phase \((\mu) \neq 0\), which implies Higgs breaking of the magnetic U(1) symmetry, in the deconfined phase \((\mu) = 0\) and the magnetic charge is superselected. In SU(N) gauge theory there exist \(N - 1\) magnetic charges and \(N - 1\) independent operators \(\mu^a, (a = 1, \ldots, N - 1)\), which create monopoles of the species \(a\) \cite{10}. They can be written

\[
\mu^a(x, t) = e^{\frac{iq}{2\beta}} \int d^3y \tilde{b}_{\perp}(x-y) \text{Tr}(\Phi^a E)(y,t)
\] (1)

where \(\tilde{b}_{\perp}\) is the field of a Dirac monopole with \(\nabla \tilde{b}_{\perp} = 0\) and \(\nabla \wedge \tilde{b}_{\perp} = \frac{F}{2\pi} + \text{Dirac string}\). \(\Phi^a(x) \equiv U(x,y)\Phi^a_d U^\dagger(x,y)\) (2) with \(U(x,y)\) an arbitrary gauge transformation. We will take for \(U\) a parallel transport to \(x\) from a reference point \(y\) along a path \(C\), \(\langle \mu^a \rangle\) is gauge invariant. In Eq(2)

\[
\langle \mu^a \rangle = \frac{1}{2\pi} \int d\beta \rho^a(\beta)
\] (8)

\(\rho^a\) has been measured by lattice simulation for various gauge theories: compact \(U(1)\) \cite{11}, SU(2) \cite{8} , SU(3) \cite{9} and \(N_f = 2\) QCD \cite{11}. In all these systems \(\rho^a \to \infty\) in the confined phase in the thermodynamical limit \(V \equiv L^3 \to \infty\). By use of Eq(8) this means \(\langle \mu^a \rangle \neq 0\).

In the deconfined phase \(T > T_c\)

\[
\rho^a \approx -|c|L_s + c' \quad \text{or} \quad \langle \mu^a \rangle = 0
\] (9)

In the gauge in which \(\Phi^a = \Phi^a_d\) (Abelian Projection)

\[
\mu^a(x, t) = e^{\frac{iq}{2\beta}} \int d^3y \tilde{b}_{\perp}(x-y) \tilde{E}_{\perp}^a(y,t)
\] (4)

where \(\tilde{E}_{\perp}^a\) is the transverse chromoelectric field along the color direction \(T^a\)

\[
T^a = diag(0, \ldots, 0, 1, -1, 0, \ldots, 0)
\] (5)

In the deconfined phase \(\langle \mu^a \rangle = 0\), \(T^a\) creates a Dirac monopole at \((x, t)\) in the residual gauge symmetry after abelian projection. It proves convenient to use instead of \(\langle \mu^a \rangle\) the susceptibility

\[
\rho^a = \frac{\partial}{\partial \beta} \text{ln} \langle \mu^a \rangle
\] (7)

Here \(\beta\) is the usual variable of the lattice formulation \(\beta \equiv \frac{2N}{g^2}\).

\[
\langle \mu^a \rangle = e^\beta \int_0^1 d\beta \rho^a(\beta)
\] (8)

\[
\rho^a \approx -|c|L_s + c' \quad \text{or} \quad \langle \mu^a \rangle = 0
\] (9)
or, again by Eq(8) $\langle \mu^a \rangle = 0$.

In the critical region $T \approx T_c$ the scaling law holds

$$\rho^a \approx f(\tau L_x^\frac{1}{2})$$

(10)

Here $\tau = 1 - \frac{T}{T_c}$, $\nu$ is the critical index of the correlation length of the order parameter. $\rho^a$ is independent of the choice of the abelian projection $[12]$ $[13]$ $[14]$. Expanding the exponential which defines $\langle \mu^a \rangle$ one has

$$\langle \mu^a \rangle = \Sigma_0 \frac{i q}{2N} \frac{2^n}{n!} \int d^3 y_1 \cdots d^3 y_n b^a_i(\vec{x} - \vec{y}_1) \cdots b^a_n(\vec{x} - \vec{y}_n)$$

$$\equiv \langle \mu^a_i \rangle \equiv \rho^a_i$$

(11)

The notation is $\langle \Phi^a_\tau \rangle$.

In the critical region $T \approx T_c$ the scaling law holds $\rho^a \approx f(\tau L_x^\frac{1}{2})$ (10), and are used as an input in stochastic QCD. For those correlators $\langle E^a E^b \rangle = \delta^{ab} \Phi$, so that $\rho^a$ is independent on $a$. This is also the case in lattice determinations of $\rho^a$ $[9]$. The cluster expansion is generically expected to work at large distances, and in the study of confinement we are looking for infrared properties. Anyhow a direct check of it can be obtained by looking at the dependence of $\rho^a$ on $q$. The truncated $\rho^a$ is proportional to $q^2$: higher correlators would introduce terms proportional to higher powers of $q$. Old data $[8]$ $[15]$ seem to agree with $q^2$ but a systematic study of this dependence will be done.

3. The Field Correlators

A general parametrization of field strength correlators dictated by invariance arguments $[20]$ $[21]$ is

$$\Phi_{\mu_1,\nu_1,\mu_2,\nu_2}^a(z_1 - z_2) \equiv \frac{1}{N} (\langle Tr F^a_{\mu_1,\nu_1}(z_1) \rangle) V(z_1, z_2) F^a_{\mu_2,\nu_2}(z_2) V^\dagger(z_1, z_2)$$

(14)

$$\Phi_{\mu_1,\nu_1,\mu_2,\nu_2}^{ab}(z_1 - z_2) = \delta^{ab}$$

$$\langle D(z_1 - z_2) \delta_{\mu_1,\nu_1} \delta_{\mu_2,\nu_2} \rangle + \frac{1}{2} \frac{\partial}{\partial z_{\mu_1}} [D_1(z_1 - z_2) \delta_{\mu_1,\mu_2} \delta_{\nu_1,\nu_2} - z_{\nu_2} \delta_{\mu_1,\nu_2} - z_{\mu_2} \delta_{\mu_1,\nu_1} - z_{\mu_2} \delta_{\nu_1,\nu_2}]$$

(15)

At $T \neq 0$ the electric field correlators do not coincide with the magnetic ones, and there are four form factors, $D_E, D_{1E}, D_H, D_{1H}$.

For correlators of electric fields $E_{i_1}, E_{i_2}$ Eq(15) gives

$$\Phi_{i_1, i_2}^{ab} = \delta^{ab} [\delta_{i_1, i_2} (D_E + \frac{1}{2} D_{1E}) + \frac{\partial}{\partial i_{11}}]$$

(16)

In the convolution with $b_{\perp}$ the derivative terms give 0. For the same reason

$$\delta_{i_1 i_2} \rightarrow \delta_{i_1 i_2} - \frac{k_{i_1} k_{i_2}}{k^2}$$

Going to the Fourier transform we get for $\rho^a$ Eq(13)

$$\rho^a = -\frac{q^2}{16} \frac{\partial}{\partial \beta} [\beta \int \frac{d^3 k}{(2\pi)^3} b^a_1(\vec{k}) b^a_2(-\vec{k})]$$

We shall identify $\Phi^a_{i_1, i_2}$ with the two point correlators defined with a straight line parallel transport.
\[ D_E(k^2) \frac{1}{k^2} [k^2 \delta_{ij}k^2 - k_i k_j] \] (17)

Here \( \bar{D}_E(k^2) \) is the Fourier transform of \((D_E + \frac{1}{2} D_{1E})\).

Since

\[ (k^2 \delta_{ij} - k_i k_j)b^i_\perp(\vec{k})b^j_\perp(\vec{-k}) = |\vec{H}(\vec{k})|^2 \] (18)

we can use the explicit form of \( \vec{H}(\vec{k}) \)

\[ \vec{H}(\vec{k}) = \vec{k} \wedge \vec{b}_\perp(\vec{k}) \] (19)

with \( \vec{n} \) the direction of the Dirac string (we shall call it \( z \)), and get

\[ |\vec{H}(\vec{k})|^2 = -\frac{1}{k^2} + \frac{1}{k_z^2} \] (20)

For \( \rho^a \) we then have

\[ \rho^a = \frac{q^2}{16} \frac{\partial}{\partial \beta} \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{k^2} - \frac{1}{k_z^2} \right) f(k^2) \] (21)

where \( f(k^2) \equiv \frac{k^2}{8 \pi^2} \bar{D}_E(k^2) \).

Identifying our correlators with those of the stochastic model simply explains that \( \rho^a \) is independent both on \( a \) and on the abelian projection. At large \( \beta \) (deconfined phase) \( f(k^2) \) can be approximated by first order perturbation theory

\[ f(k^2) = \frac{1}{2Nk} \] (22)

The only dependence on \( \beta \) is the explicit factor in Eq(21) so that

\[ \rho^a = \frac{q^2}{16N} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} - \frac{1}{k_z^2} \] (23)

The integral is easily computed with UV cut-off \( \frac{1}{a} \) (\( a \) the lattice spacing) and IR cut-off \( \frac{1}{L_{\text{iso}}} \) (\( L_s \) the spacial size of the lattice). The result is

\[ \rho^a = \frac{q^2}{16N} \frac{1}{(2\pi)^2} (-\sqrt{2}L_s + 2\ln(L_s) + \text{const}) \] (24)

By comparison with Eq(8) this means that \( \langle \mu^a \rangle = 0 \) in the thermodynamical limit \( L_s \to \infty \). The correlators have been measured on the lattice both at \( T = 0 \) and at \( T \neq 0 \). In the range of distances \( 1 fm \leq x \leq 1 fm \) they are well parametrized by a form \[ \text{(17)} \]

\[ D_E = Ae^{-\frac{L_s}{x}} + \frac{b}{x^4}e^{-\frac{L_s}{x}} \] (25)

with \( \lambda_a \approx 2\lambda_b \) and \( \lambda_b \approx .3 fm \) \( A \) and \( b \) are independent on \( \beta \) within statistical errors in the range from \( T = 0 \) up to \( T \approx .95T_c \). Approaching further \( T_c \) \( A \) rapidly decreases to zero.\[19\]

In fact the parametrization Eq(25) cannot be valid at shorter distances, where the operator product expansion and the non-existence of condensates of dimension less than 4 require that

\[ D_E \approx \frac{b}{2} \left( \frac{1}{(x+ie)^4} + \frac{1}{(x-ie)^4} \right) + c + dx^2 \] (26)

The prescription on the singularity is the same as in perturbation theory. At larger distances a stronger infrared cut-off at some distance \( \Lambda \), must exist, since colored particles cannot propagate at infinite distance. This feature needs a further numerical investigation of correlators at large distance on the lattice.

Up to \( T \approx .95T_c \) the only dependence on \( \beta \) in Eq(21) is again the explicit factor so that the result coincides with Eq(24) except that the lattice size \( L_s \) is replaced by the infrared cut-off \( \Lambda \)

\[ \rho^a = \frac{q^2}{16} \frac{1}{(2\pi)^2} (-\sqrt{2}\Lambda + 2\ln(\Lambda) + \text{const}) \] (27)

The integral of the exponential term of Eq(25) is included in the constant. This expression gives a finite value of \( \rho^a \) for \( T < T_c \) independent of the volume and hence \( \langle \mu^a \rangle = 0 \). This means dual superconductivity for any finite value of the UV cut-off \( a \). However in the continuum limit \( a \to 0 \) \( \rho^a \) diverges so that \( \langle \mu^a \rangle \) needs a renormalization. This is similar to what happens for the Polyakov line \[23\]. Existing Lattice data support this statement [See Fig(2) of ref.\[3\]] , but we plan to do a more systematic investigation of this issue. By approaching the critical temperature both the IR cut-off \( \Lambda \) and the coefficient \( A \) in Eq(25) strongly depend on \( \beta : \Lambda \) diverges and \( A \) tends to zero. A more detailed calculation, which will be reported elsewhere, gives

\[ \rho^a = \frac{q^2}{16N} \frac{\partial}{\partial \beta} \left( \frac{1}{(2\pi)^2} (-\sqrt{2} \Lambda + 2\ln(\Lambda) \right) + \text{const}) \] (28)

Numerical determinations of the dependence of \( A_E \) on of the temperature around \( T_c \) exist in the
literature[19]. Not much is known about the behavior of $\Lambda$. Further study is needed to understand how the scaling law of $\rho^a$ Eq(10) described in Sect 1 comes out of Eq(28).

4. Conclusions

We close with a few remarks.

Checking the dependence $\rho^a \propto q^2$ is a test of the Stochastic Vacuum model.

The independence of $\rho^a$ on $a$ and on the abelian projection are also an important test of it.

The existence of confinement depends on a strong infrared cut-off of the field correlators. This can be directly checked on lattice.

$\rho^a$ diverges in the continuum limit, but provides a good description of confinement at any fixed value of the UV cut-off.

An interesting interplay emerges of confinement with infrared properties of gauge invariant field strength correlators.

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