BLACK HOLES WITH LESS ENTROPY THAN A/4 *

Don N. Page †
CIAR Cosmology Program, Institute for Theoretical Physics
Department of Physics, University of Alberta
Edmonton, Alberta, Canada T6G 2J1

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Abstract

One can increase one-quarter the area of a black hole, A/4, to exceed the total thermodynamic entropy, S, by surrounding the hole with a perfectly reflecting shell and adiabatically squeezing it inward. A/4 can be made to exceed S by a factor of order unity before the shell enters the Planck regime, though practical limitations are much more restrictive. One interpretation is that the black hole entropy resides in its thermal atmosphere, and the shell restricts the atmosphere so that its entropy is less than A/4.
1 Introduction

The Generalized Second Law (GSL) \cite{1} of black hole thermodynamics states that the total thermodynamic entropy $S$ does not decrease, and it further states that for Einsteinian gravity (to which this paper will be restricted, though the generalization to various other theories should be straightforward), $S$ is the GSL entropy

$$S_{GSL} \equiv \frac{1}{4} A + S_m,$$

(1)

where $A$ is the total event horizon area of all black holes and $S_m$ is the entropy of matter outside the black holes. (I am using Planck units in which $\hbar$, $c$, $4\pi\epsilon_0$, Boltzmann’s constant $k$, and the renormalized Newtonian gravitational constant $G$ are all set equal to unity.) Although the Generalized Second Law has only been proved under restricted conditions, such as for quasistationary semiclassical black holes \cite{2}, it is believed to have greater generality, such as to rapidly evolving black holes.

An implicit further assumption that is often made is that the matter entropy $S_m$ cannot be negative. This assumption, plus the GSL, leads to the conclusion that the total entropy is bounded below by one-quarter the total event horizon area:

$$\frac{1}{4} A \leq S.$$

(2)

Here I shall show that the inequality (2) can be violated. This violation can be interpreted as either a violation of the Generalized Second Law (if $S_m$ is assumed to be restricted to nonnegative values) or as an indication that the matter entropy $S_m$ must be allowed to take negative values in order to conform to the GSL.

Briefly, a violation of the inequality (2) can be produced as follows: Take a Schwarzschild black hole of initial mass $M_i$ and radius $2M_i \gg 1$ (in the Planck units used herein) with negligible matter outside the hole and its nearby thermal atmosphere (which is here taken to be part of the hole’s energy and entropy). Assuming the GSL for this initial state, the initial entropy $S_i$ is roughly $A_i/4 = 4\pi M_i^2 \gg 1$, one-quarter the initial area of the hole, since the initial matter entropy $S_m$ is negligible in comparison. (If one considers as matter the thermal atmosphere that forms when the horizon forms in the near-horizon region $r - 2M \ll 2M$, either the entropy of this atmosphere should be considered negligible if it is considered to be part of $S_m$, or it should be considered to be part of the black hole entropy $A/4$; one gets too large a value for the total entropy if one counts both $A/4$ and a large entropy associated with the near-horizon thermal atmosphere. There may indeed be a correction to the entropy of the order of $\ln A$ from the thermal atmosphere and/or from other considerations, and this correction might be a few orders of magnitude larger
than unity for huge black holes, but for the purposes of this paper I shall not count such a correction as “large”—here “large” numbers will mean those many orders of magnitude larger than unity, such as a positive power of the area.)

Now surround the black hole by a spherical perfectly reflecting shell at a radius $r_i$ that is a few times the Schwarzschild radius $2M_i$ of the black hole. This region (outside of the near-horizon atmosphere that is being counted as part of the hole) will soon fill up with thermal Hawking radiation to reach an equilibrium state of fixed energy $M_i$ inside the shell, but for $M_i \gg 1$, all but a negligible fraction ($\sim 1/M_i^2$ in Planck units) of the energy will remain in the hole, which can thus be taken still to have mass $M_i$. Outside the shell, one will have essentially the Boulware vacuum state with zero entropy (plus whatever apparatus that one will use to squeeze the shell in the next step, but this will all be assumed to be in a pure state with zero entropy).

Next, squeeze the shell inward. If this is done sufficiently slowly, this should be an adiabatic process, keeping the total entropy fixed. Also, the outside itself should remain in a zero-entropy pure state, since the perfectly reflecting shell isolates the region outside from the region inside with its black hole and thermal radiation, except for the effects of the gravitational field, which will be assumed to produce negligible quantum correlations between the inside and the outside of the shell (as one would indeed get in a semiclassical approximation in which the geometry is given by a spherically symmetric classical metric). Some of the thermal Hawking radiation will thus be forced into the black hole, increasing its area. So long as the shell is not taken into the near-horizon region $r - 2M \ll 2M$, the radiation forced into the black hole will have negligible energy and so will not increase the black hole area significantly above its initial value $A_i$. (Indeed, some of this tiny increase in the area just compensates for the tiny decrease in the black hole area when it filled the region $r < r_i$ with thermal radiation.)

However, nothing in principle prevents one from squeezing the shell into the near-horizon region, where a significant amount of the near-horizon thermal radiation can be forced into the hole, increasing its mass $M$ and area $A = 4\pi M^2$ significantly. Since the entropy $S$ should not change by this adiabatic process, it remains very nearly at $A_i/4$. Therefore, one ends up with a squeezed black hole configuration with $A > 4S \approx A_i$, or total entropy significantly less than $A/4$. (By significantly less, I mean that $A/4 - S$ is very large in absolute value, not necessarily that it is a significant fraction of $A/4$.)

A simple way to interpret this result is to say that the near-horizon thermal atmosphere contributes a significant fraction (perhaps all) of the black hole entropy. Then when this atmosphere is restricted to a smaller region by a near-horizon shell, its contribution to the total entropy is reduced.
Perhaps the simplest way to incorporate these $S < A/4$ configurations into black hole thermodynamics is modify the Generalized Second Law to state that

$$\tilde{S}_{\text{GSL}} \equiv S_{\text{bh}} + S_m$$

(3)

does not decrease for a suitably coarse-grained nonnegative $S_m$ and for a suitable definition of $S_{\text{bh}}$ that reduces to $A/4 + O(\ln A)$ (in Einstein gravity) when there are no constraints on the near-horizon thermal atmosphere but which is less than $A/4 + O(\ln A)$ when the atmosphere is constrained (and thus has less entropy). One might interpret $S_{\text{bh}}$ as arising entirely from the near-horizon thermal atmosphere, so that if the atmosphere is unconstrained in the vertical direction, its entropy is at least approximately $A/4$. (There is no fundamental difficulty in allowing that in this unconstrained case, $S_{\text{bh}}$ might also have other smaller correction terms, such as a logarithm of the number of fields or a logarithm of $A$ or of some other black hole parameter. It is just that in the unconstrained case, the leading term of $S_{\text{bh}}$ should be proportional to $A$, and the coefficient should be $1/4$, at least in Einstein gravity. I also do assume that in the unconstrained case there is no other term in the black hole entropy going as a positive power of $A$, such as $M = [A/(4\pi)]^{1/2}$.) But if the near-horizon thermal atmosphere is significantly constrained, it has much less entropy.

An alternative (but perhaps less attractive) way to incorporate these $S < A/4$ configurations is to retain the Generalized Second Law in the original form of Eq. (1), which is the special case of Eq. (3) in which $S_{\text{bh}} = A/4$, but now to allow $S_m$ to become negative when one squeezes the black hole. For example, one might use Eq. (1) not to define $S_{\text{GSL}}$ in terms of $A/4$ and $S_m$, but instead to define $S_m$ as the total entropy $S_{\text{GSL}}$ minus the black hole entropy $A/4$. (Of course, this procedure would make the GSL useless for telling what the total entropy is, so then $S_{\text{GSL}}$ would have to be found by some other procedure.)

A longer version of most of this paper has already appeared [3], which the reader might like to consult for some details omitted here, but the arguments are sharpened up and expressed more succinctly here. One investigation that was pursued there, but not here, is a closed-form approximation to the static spherically symmetric metric obtained by making a self-consistent nonlinear semiclassical gravitational backreaction calculation with the expectation value of the stress-energy tensor of the vacuum state outside a shell. In contrast, here I shall confine myself to cases in which the gravitational backreaction is sufficiently small that it may be treated as a linear perturbation to the Schwarzschild geometry.
2 Calculation of the Entropy of a Black Hole Inside a Perfectly Reflecting Shell

Let us now try to estimate what the total entropy is of a configuration of an uncharged, nonrotating black hole of mass \( M \) and area \( A = 16\pi M^2 \), in equilibrium with Hawking thermal radiation inside a perfectly reflecting pure-state shell of radius \( R \) and local mass \( \mu \), outside of which one has vacuum. This calculation is somewhat complicated, as it involves specific assumptions about the stress-energy tensor inside and outside the shell and how the adiabatic motion of the shell affects these. In principle, these assumptions could be checked by doing suitable calculations of quantum field theory in nearly-static spacetimes with slowly-moving perfectly reflecting boundaries, but these calculations appear to be so difficult that I have replaced them by what I believe are highly plausible simple physical arguments. Therefore, I do not have a rigorous proof of the validity of my calculations, but I think they are correct, and they do lead to the approximate entropy formula (54) that seems to be eminently reasonable.

As the beginning of the next section indicates, my entropy formula can also easily be derived from an even simpler set of assumptions that are also plausible, though perhaps more open to question than the ones I use in the derivation immediately below. For the reader who accepts the validity of the simpler assumptions of the next section, he or she may wish to skip the present derivation and go immediately to the results of Eqs. (53) and (54) at the end of this section. However, anyone who has doubts about those assumptions may find the present arguments and derivation instructive, as they seem to me stronger than the simpler ones of the next section.

We shall take a semiclassical approximation with a certain set of matter fields, which for simplicity will all be assumed to be massless free conformally coupled fields. Given the field content of the theory, the three parameters \( (M, R, \mu) \) determine the configuration, though the entropy should depend only on \( M \) and \( R \), since the shell and the vacuum outside have zero entropy.

It is convenient to replace the shell radius \( R \) with the classically dimensionless parameter

\[
W = \frac{2M}{R},
\]

which would be 0 if the shell were at infinite radius (though before one reached this limit the black hole inside the shell would become unstable to evaporating away) and 1 if the shell were at the black hole horizon (though in this limit the forces on the shell would have to be infinite). Then we would like to find \( S(M, W) \).

If \( W \) is neither too close to 0 nor to 1, the entropy will be dominated by \( A/4 = 4\pi M^2 \). The dominant relative correction to this will come from effects of the thermal
radiation and vacuum polarization around the hole and so would have a factor of $\hbar$ if I were using gravitational units ($c = G = 1$) instead of Planck units ($\hbar = c = G = 1$). In gravitational units, $\hbar$ is the square of the Planck mass, so to get a dimensionless quantity from that, one must divide by $M^2$ (or by $R^2$, which is just $4M^2/W^2$ with $W$ being of order unity); for the free massless fields under consideration, there are no other mass scales in the problem other than the Planck mass. Therefore, in Planck units, the first relative correction to $A/4$ will have a factor of $1/M^2$ and hence give an additive correction term to $4\pi M^2$ that is of the zeroth power of $M$. Such a term could be a function of $W$, and it could also involve the logarithm of the black hole mass in Planck units, since such a logarithm may be regarded as being of the zeroth power of $M$. However, it will not be proportional to any positive power of $M$.

One might expect that if one proceeded further in this way, one would find that the entropy $S$ is given by $4\pi M^2$ times a whole power series in $1/M^2$, with each term but the zeroth-order one having a coefficient that is a function of $W$ and of the logarithm of the black hole mass. If we had been considering the possibility of massive fields, then these coefficients of the various powers of $1/M^2$ would not be purely functions of $W$ but would also be functions of the masses of the fields. However, for simplicity we shall consider only the free massless field case here. For simplicity I shall also mostly ignore the possible dependence on the logarithm of the black hole mass, so that the coefficients of the powers of $M$ will be assumed to be functions purely of $W$.

In fact, I shall consider only the first two terms in this power series and, for simplicity, drop the possible $\ln M$ dependences:

$$S(M, W) = 4\pi M^2 + f_1(W, \ln M) + f_2(W, \ln M)M^{-2} + \cdots \approx 4\pi M^2 + f_1(W).$$

The function $f_1(W)$ will depend on the massless matter fields present in the theory, most predominantly through the radiation constant

$$a_r = \frac{\pi^2}{30}(n_b + \frac{7}{8}n_f),$$

where $n_b$ is the number of bosonic helicity states and $n_f$ is the number of fermionic helicity states for each momentum. It also proves convenient to define

$$\alpha \equiv \frac{a_r}{384\pi^3} = \frac{n_b + \frac{7}{8}n_f}{11520\pi^3},$$

which makes the entropy density of the thermal Hawking radiation far from the hole (when $R \gg M$ or $W \ll 1$) simply $\alpha/M^3$, and also to set

$$f_1(W) = -32\pi\alpha s(W).$$
where \( s(W) \) depends (weakly) only on the ratios of the numbers of particles of different spins and so stays fixed if one doubles the number of each kind of species. Then my truncated power series expression for the entropy of an uncharged spherical black hole of mass \( M \) at the horizon (and hence horizon radius \( 2M \) and horizon area \( 4\pi M^2 \)) surrounded by a perfectly reflecting shell of radius \( R = 2M/W \) is

\[
S(M, W) \approx 4\pi M^2 - 32\pi \alpha s(W) = \frac{1}{4} A - 32\pi \alpha s(W). \tag{9}
\]

Now I shall evaluate an approximate expression for \( s(W) \) when the perturbation to the Schwarzschild geometry is small from the thermal radiation inside the shell and from the vacuum polarization inside and outside the shell. There will be an additive constant to \( s(W) \) (possibly depending on \( \ln M \)), giving an additive constant to the entropy, that I shall not be able to evaluate, but for simplicity and concreteness I shall assume that \( s(1/2) = 0 \), so that the entropy is \( A/4 \) when the shell is at \( W = 1/2 \) or \( R = 4M \).

First, I shall ignore the Casimir energy and related effects of the shell itself on the fields. I would expect that these effects would give additive corrections to \( s(W) \) that are of order \( W \) or smaller (and so never large compared with unity), whereas the leading term in the perturbative approximation for \( s(W) \) will go as \( 1/W^3 \) (proportional to the volume inside the shell) for \( W \ll 1 \) (shell radius \( R \gg 2M \)) and as \( 1/(1 - W) \) (inversely proportional to the square of the redshift factor to infinity) for \( 1 - W \ll 1 \) (shell radius relatively near the horizon), so one or other of these leading terms will dominate when \( W \) is near 0 or 1. Therefore, I shall take the stress-energy tensor inside the shell to be approximately that of the Hartle-Hawking state in the Schwarzschild geometry, and that outside the shell to be approximately that of the Boulware vacuum.

The first part of the analysis will be done in a coordinate system \((x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)\) in which the spherically symmetric classical metric has, at each stage of the process, the approximately static form

\[
ds^2 = -e^{2\phi} dt^2 + U^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{10}
\]

with

\[
e^{2\phi} = e^{2\psi} U \tag{11}
\]

and

\[
U = 1 - \frac{2m}{r} = 1 - w \tag{12}
\]

with

\[
w \equiv \frac{2m}{r} = 1 - U = 1 - (\nabla r)^2. \tag{13}
\]
Here $\phi$, $U$, $\psi$, $m$, and $w$ are all functions of the $x^1 = r$ coordinate alone, although they also have a global dependence on the black hole mass $M$ (the value of $r/2$ where $e^{2\phi} = 0$), and (for $r > R$) on the radius $r = R = 2M/W$ of the shell and on the total rest mass energy $\mu$ of the shell. The Einstein equations then give

$$\frac{d\psi}{dr} = 4\pi r (\rho + P) U^{-1} \tag{14}$$

and

$$\frac{dm}{dr} = 4\pi r^2 \rho, \tag{15}$$

where

$$\rho = -\langle T^0_0(r) \rangle \tag{16}$$

is the expectation value of the energy density in the appropriate quantum state, and

$$P = \langle T^1_1(r) \rangle \tag{17}$$

is the corresponding expectation value of the radial pressure, both functions of $r$. The functional form of the expectation value of the tangential pressure $\langle T^2_2(r) \rangle = \langle T^3_3(r) \rangle$ would then follow from the conservation of $\langle T^\mu_\nu \rangle$ but will not be explicitly needed in this paper.

Since we are assuming that the state of the quantum fields inside the shell ($r < R$) is the Hartle-Hawking [4] thermal state, for $r < R$ we have

$$\rho = \rho_H(M, r) \equiv \frac{3\alpha}{32\pi M^4} \varepsilon_H(w) \tag{18}$$

and

$$P = P_H(M, r) \equiv \frac{\alpha}{32\pi M^4} p_H(w), \tag{19}$$

where on the extreme right hand side of each of these two equations I have factored out the dependence on the black hole mass $M$ from that on the radial function $w \equiv 2m/r$ that is classically dimensionless (dimensionless without setting $\hbar = 1$), thereby defining two classically dimensionless functions of $w$, $\varepsilon_H(w)$ and $p_H(w)$.

Similarly, we are assuming that the state of the quantum fields outside the shell ($r > R$) is the Boulware [5] vacuum state, so for $r > R$ we have

$$\rho = \rho_B(M_\infty, r) \equiv \frac{3\alpha}{32\pi M^4_\infty} \varepsilon_B(w) \tag{20}$$

and

$$P = P_B(M_\infty, r) \equiv \frac{\alpha}{32\pi M^4_\infty} p_B(w), \tag{21}$$
thereby defining two new classically dimensionless functions of \( w, \varepsilon_B(w) \) and \( p_B(w) \). Here

\[
M_\infty \equiv m(r = \infty)
\]  

(22)
is the ADM mass at radial infinity.

In some cases one can assume that there is some extra apparatus in the region \( r > R \) holding the shell in. If so, its energy density and radial pressure can simply be included in \( \rho_B \) and \( P_B \). In any case, we shall assume that the shell, and whatever is outside the shell, is in a pure state with zero entropy. Therefore, the only contribution to the entropy will come from the interior to the shell.

Below we shall also need the vacuum polarization part of the stress-energy tensor inside the shell, whose components I shall denote by

\[
\rho_V(M, r) \equiv \rho_H(M, r) - \rho_T(M, r)
\]

\[
\equiv \frac{3\alpha}{32\pi M^4} \varepsilon_V(w) \equiv \frac{3\alpha}{32\pi M^4} (\varepsilon_H(w) - \varepsilon_T(w))
\]

(23)

and

\[
P_V(M, r) \equiv P_H(M, r) - P_T(M, r)
\]

\[
\equiv \frac{\alpha}{32\pi M^4} p_V(w) \equiv \frac{\alpha}{32\pi M^4} (p_H(w) - p_T(w)),
\]

(24)

where \( \rho_T \) and \( P_T \) denote the components of the thermal parts.

I shall assume that the vacuum polarization part is what the Boulware state would give if one had it inside the shell, so that, in my approximation of ignoring Casimir effects, \( \rho_V \) and \( P_V \) have the same dependence on the local mass \( m(r) \) and radius \( r \) as \( \rho_B \) and \( P_B \) do outside the shell (when there is no extra apparatus there).

In the first-order (in \( \alpha/M^2 \)) perturbative calculation being done here, the expectation value of the stress tensor is already first order (except possibly for that of the shell), so its functional dependence on \( m \) can be replaced by its dependence on its zeroth approximation, which is the black hole mass \( M \) for \( r < R \) and the ADM mass \( M_\infty \) for \( r > R \). Therefore, to sufficient accuracy for our purposes, \( \rho_V \) and \( P_V \) can be evaluated by using Eqs. (20) and (21) for \( \rho_B \) and \( P_B \) with the ADM mass \( M_\infty \), which is approximately the value of the local mass \( m(r) \) anywhere outside the massive shell, replaced by the black hole mass \( M \), which is approximately the value of \( m(r) \) anywhere inside the shell. In particular, this implies that we can use

\[
\varepsilon_V(w) = \varepsilon_B(w)
\]

(25)

and

\[
p_V(w) = p_B(w).
\]

(26)
For explicit approximate calculations, it is useful to have explicit approximate formulas for these various components of the stress-energy tensor (though only some of these are necessary for the final result to be given below), given by the equations above from the six functions \( \varepsilon_H(w) \), \( \varepsilon_B(w) \), \( \varepsilon_T(w) = \varepsilon_H(w) - \varepsilon_B(w) \), \( p_H(w) \), \( p_B(w) \), and \( p_T(w) = p_H(w) - p_B(w) \). For simplicity and concreteness, I shall use those obtained for a conformally invariant massless scalar field in the gaussian approximation \( \Box \), which gives

\[
\varepsilon_H(w) = \frac{32\pi M^4}{3\alpha}\rho_H \equiv \frac{(8\pi M)^4}{a_r} \rho_H \approx \frac{1 - (4 - 3w)^2w^6}{(1 - w)^2} - 24w^6 \\
= 1 + 2w + 3w^2 + 4w^3 + 5w^4 + 6w^5 - 33w^6, \tag{27}
\]

\[
\varepsilon_B(w) = \frac{32\pi M^4}{3\alpha}\rho_B \equiv \frac{(8\pi M_{\infty})^4}{a_r} \rho_B \approx \frac{- (4 - 3w)^2w^6}{(1 - w)^2} - 24w^6 \\
= -\frac{1}{(1 - w)^2} + 1 + 2w + 3w^2 + 4w^3 + 5w^4 + 6w^5 - 33w^6, \tag{28}
\]

\[
\varepsilon_T(w) = \varepsilon_H(w) - \varepsilon_B(w) \approx \frac{1}{(1 - w)^2} = \frac{1}{U^2}, \tag{29}
\]

\[
p_H(w) = \frac{32\pi M^4}{\alpha}P_H \equiv \frac{(8\pi M)^4}{3a_r} P_H \approx \frac{1 - (4 - 3w)^2w^6}{(1 - w)^2} + 24w^6 \\
= 1 + 2w + 3w^2 + 4w^3 + 5w^4 + 6w^5 + 15w^6, \tag{30}
\]

\[
p_B(w) = \frac{32\pi M_{\infty}^4}{\alpha}P_B \equiv \frac{(8\pi M_{\infty})^4}{3a_r} P_B \approx \frac{- (4 - 3w)^2w^6}{(1 - w)^2} + 24w^6 \\
= -\frac{1}{(1 - w)^2} + 1 + 2w + 3w^2 + 4w^3 + 5w^4 + 6w^5 + 15w^6, \tag{31}
\]

and

\[
p_T(w) = p_H(w) - p_B(w) \approx \frac{1}{(1 - w)^2} = \frac{1}{U^2}. \tag{32}
\]

Note that this approximation gives

\[
\rho_T \approx 3P_T \approx a_T T_{\text{local}}^4, \tag{33}
\]

just like thermal radiation in flat spacetime, where \( T_{\text{local}} \) is the local value of the Hawking temperature,

\[
T_{\text{local}} \approx \frac{1}{8\pi m} (1 - \frac{2m}{r})^{-1/2}. \tag{34}
\]

The form of \( s(W) \) to be calculated actually depends only on \( \rho_T \) and \( P_T \), so any stress-energy tensor in which they have the massless thermal form given above will
give the same results for the leading correction to the entropy from the position of the shell (when possible \( \ln M \) terms are neglected).

Now we use the Einstein equations (14) and (15) with the appropriate \( \rho \) and \( P \) on the right hand side, and with the metric function \( U \) there taking on its approximate Schwarzschild form, \( 1 - 2M/r \) for \( r < R \) and \( 1 - 2M_\infty/r \) for \( r > R \).

We also need to consider the effect of the shell, which has a surface stress-energy tensor with components

\[
S_{0}^{0} = -\frac{\mu}{4\pi R^2}
\]

and

\[
S_{2}^{2} = S_{3}^{3} = -\frac{F}{2\pi R},
\]

where \( \mu \) is the total local mass of the shell, the shell area \( 4\pi R^2 \) multiplied by the local mass-energy per area \( -S_{0}^{0} \) as seen by a local observer fixed on the shell, and \( F \) is the local total tensile force pulling together the two hemispheres of the shell, the circumference \( 2\pi R \) multiplied by the local surface tension (tensile force per length) \( -S_{2}^{2} = -S_{3}^{3} \).

If one integrates the Einstein equations (14) and (15) through the shell and uses the conservation law for the stress-energy tensor, one get the junction conditions [7] in the static case that

\[
\mu = R(U_{-}^{-1/2} - U_{+}^{1/2})
\]

and

\[
8F = \frac{\mu}{R} + (1 + 8\pi R^2 P_{-})U_{-}^{-1/2} - (1 + 8\pi R^2 P_{+})U_{+}^{-1/2},
\]

where

\[
U_{-} = 1 - \frac{2M_{-}}{R}
\]

is the value of \( U \) just inside the shell \( (r = R-) \), where the local mass function \( m \) takes on the value \( M_{-} \), and

\[
U_{+} = 1 - \frac{2M_{+}}{R}
\]

is the value of \( U \) just outside the shell \( (r = R+) \), where the local mass function \( m \) takes on the value \( M_{+} \). Similarly, \( P_{-} \) and \( P_{+} \) are the expectation values of the radial pressure of the respective quantum states just inside and just outside the shell.

Thus we have at least five relevant masses for the configuration: the black hole mass \( M = m(r = 2M) \), the mass \( M_{-} = m(r = R-) \) just inside the shell, the local mass (or local energy) \( \mu \) of the shell itself at radius \( r = R \); the mass \( M_{+} = m(r = R+) \) just outside the shell, and the ADM mass \( M_\infty = m(r = \infty) \) at radial infinity. Since the stress-energy tensor inside the shell is that of the Hartle-Hawking state determined by \( M \) and \( r \), \( M_{-} \) is a function of \( M \) and \( R \). Similarly, since the stress-energy tensor outside the shell is that of the Boulware state determined by \( M_\infty \)
and \( R \) (at least when we do not have an extra apparatus there to hold the shell in place), \( M_+ \) is a function of \( M_\infty \) and \( R \). The junction condition (27) then gives \( \mu \) as a function of \( M_- \), \( M_+ \), and \( R \), and hence as a function of \( M \), \( M_\infty \), and \( R \). One can in principle invert this to get \( M_\infty \) (and hence each of the other masses as well) as a function of \( M \), \( R \), and \( \mu \), or to get \( M \) and each other mass as a function of \( M_\infty \), \( R \), and \( \mu \). The main point is that if we just have a black hole with the Hartle-Hawking thermal state inside a shell, and the Boulware vacuum state outside the shell, the semiclassical configuration (for fixed field content of the quantum field theory) is determined by three parameters, though only two of them (say \( M \) and either \( R \) or \( W = 2M/R \)) are relevant for the entropy which resides purely inside the shell.

To evaluate the function \( s(W) \) in the truncated entropy formula (9), I shall consider an adiabatic process of slowly squeezing the shell, keeping the total entropy constant and thereby getting

\[
\frac{ds}{dW} = \frac{M}{4\alpha} \frac{dM}{dW}
\]

during this process. Since this process is not strictly static, one cannot use precisely the static metric (1) with \( \phi \) and \( U \) (or \( \psi \) and \( m \)) that are purely functions of \( r \) and obey the static Einstein equations (14) and (15). However, one can consider a quasi-static metric in which \( \phi \) and \( U \) (or \( \psi \) and \( m \)) have a very slow dependence on the time coordinate \( t \) and the Einstein equations are only slightly different from Eqs. (14) and (15).

The specific calculation which I shall do will be to have the shell squeeze itself inward by using its own internal energy, so that no apparatus is used outside the shell to push it inward, and so that that outside region has only the Boulware vacuum polarization. The contraction of the shell is assumed to be so slow that it does not excite the vacuum outside it but rather leaves it in the Boulware vacuum state with constant \( M_\infty \). However, as the shell moves in, it is enlarging the Boulware state region, so effectively the shell must be creating a larger volume of vacuum with its vacuum polarization. This means that in the slowly inmoving frame of the shell, there is a flux of energy from the shell into the Boulware region, needed to enlarge the Boulware region while keeping it static where it already exists. [For the stress-energy tensor components of the Boulware vacuum given by Eqs. (28) and (31), this energy influx into the Boulware region is actually negative, so it increases the energy of the shell as it moves inward.]

Similarly, if the inside of the shell were also vacuum that did not get excited by the adiabatic contraction of the shell, there would be a swallowing up of part of the vacuum region by the shell as it moves inward. This would give a flux of (negative) energy from the vacuum inside into the shell, decreasing its energy. Surely this flux into the shell also exists even if the inside is not vacuum, and I assume that it is
given by the vacuum polarization part of the actual stress-tensor there, which I take
to be approximately that of a Boulware state with the same \( m \) and \( r \). The remaining
part of the total stress-energy tensor there, which I am calling the thermal part, and
which is given approximately by Eq. (33), should simply be reflected by the shell
and not give an energy flux into it (in the frame of the slowly contracting shell),
though it will contribute to the force that needs to be counterbalanced very nearly
precisely to obey the static junction equation (38) to high accuracy in order that
the shell not have any significant acceleration relative to a static frame.

In other words, I am assuming that if a shell moves inward through a static
geometry, the vacuum polarization part of the stress-energy tensor will stay static,
with \( T^0_0 = -\rho_V(M, r) \), \( T^1_1 = P_V(M, r) \), and \( T^0_1 = T^1_0 = 0 \) inside the shell, and with
\( T^0_0 = -\rho_B(M_\infty, r) \), \( T^1_1 = P_B(M_\infty, r) \), and \( T^0_1 = T^1_0 = 0 \) outside the shell. Then as
the shell moves through this static stress-energy tensor, in the frame of the shell,
there will be fluxes of energy into or out from the shell on its two sides. In contrast,
I am assuming that the thermal radiation part of the stress-energy tensor will be
perfectly reflected by the shell, so that in the frame of the shell it will give no energy
fluxes into or out from the shell.

There is a modification of this picture that occurs when the inward motion of the
shell squeezes thermal radiation into the black hole so that its mass goes up. While
the hole mass is increasing, the vacuum polarization inside the shell is not quite static
but instead has small \( T^0_1 \) and \( T^1_0 \) terms that, for sufficiently slow adiabatic processes,
are proportional to \( \dot{M} \), the coordinate time derivative of the black hole mass \( M \). In
the present calculation, in which the shell is squeezing itself inward by using its own
internal energy, the ADM mass \( M_\infty \) stays fixed, and so the vacuum stress-energy
tensor outside the shell stays static during the process, under my approximation of
neglecting Casimir-type boundary effects of the shell itself on the quantum field.
For a sufficiently slow inward squeezing of the shell, the \( T^0_1 \) and \( T^1_0 \) terms inside are
small, but over the correspondingly long time of the squeezing they contribute an
effect on the energy balance of the shell that is not completely negligible when one
contemplates squeezing the shell to a final position very near the black hole horizon.

My procedure for calculating the small \( T^0_1 \) and \( T^1_0 \) terms inside the shell is to
assume that the shell squeezing, and all consequent processes, occur so slowly that
\( T^0_0 \) and \( T^1_1 \) are given to high accuracy by the same functions of \( M \) and \( r \) as they are
when the geometry is static, namely \(-\rho_V(M, r)\) and \( P_V(M, r)\). Then I assume that
the vacuum polarization part of the stress-energy tensor is itself conserved away
from the shell, so one can use the conservation of its energy to deduce the radial
derivative of \( e^{\psi} r^2 T^0_1 \).

In particular, if we let the vacuum polarization part of the stress-energy tensor
have the component
\[ T^1_0 = \frac{\alpha \dot{M} w^2}{4\pi M^4} e^{-\psi} f, \tag{42} \]
with the factors chosen so that \( f \) is a function purely of \( w \), then \( T^\mu_{0;\mu} = 0 \) becomes
\[ \frac{\partial f}{\partial r} = \frac{\pi M^2 e^\psi r^2}{\alpha M} \left[ \dot{\rho}_V + \frac{\dot{m}}{rU} (\rho_V + P_V) \right]. \tag{43} \]
For the region inside the shell with \( r \) not too much larger than \( 2M \), one has \( m \approx M \) and \( e^\psi \approx 1 \) (possibly after suitably normalizing the time coordinate \( t \)). Then if one uses Eqs. (23)-(26), one can rewrite Eq. (43) as
\[ \frac{df}{dr} = -\frac{3w}{4} \frac{d}{dw} \left( \frac{\varepsilon_B}{w^4} \right) - \frac{3\varepsilon_B + p_B}{8w^3(1 - w)}. \tag{44} \]
Given the functions \( \varepsilon_B(w) \) and \( p_B(w) \), e.g., as given by Eqs. (28) and (31) from the gaussian approximation for a conformally invariant massless scalar field, one can integrate Eq. (44) to obtain \( f(w) \) up to a constant of integration. Although the constant of integration is not important, it can also be fixed by assuming that an observer that remains at fixed \( w = 2m/r \) as \( m \) changes sees in its frame no energy flux in the limit that \( w \) is taken to unity, which implies that the flux of vacuum polarization energy through the horizon is taken to be zero.

After one calculates the vacuum polarization part of the stress-energy tensor, which gives \( T^1_1 - T^0_0 = \rho_B + P_B \) and \( T^0_0 = 0 \) outside the shell and \( T^1_1 - T^0_0 = \rho_V + P_V \) and \( T^0_0 \) as given by Eq. (12) inside the shell, one can then calculate the fluxes of energy out from and into the shell and insert these into the conservation equations for the surface stress-energy tensor of the shell. For a very slowly expanding or contracting shell, one finds that
\[ d\mu = 4F dR + 4\pi R^2 dR [ (\rho_B + P_B) U^+_{1/2} - (\rho_V + P_V) U^-_{1/2} ] - 4\pi R^2 T^1_0 U^-_{1/2} dt. \tag{45} \]
The first term on the right hand side is the work done by the tensile force within the shell, and the remaining terms are the energy input from the vacuum stress-tensor components \( \rho_B \) and \( P_B \) just outside the shell and the vacuum stress-tensor components \( \rho_V \), \( P_V \), and \( T^0_0 \) just inside the shell.

One now combines this local energy conservation equation for the shell with the static junction equations (37) and (38) that should still apply to high accuracy in this slowly evolving situation to keep the shell radius from accelerating too rapidly. When one also combines this with the integrals of Eq. (15),
\[ M_+ = M + \int_{2M}^R 4\pi r^2 dr \rho_H, \tag{46} \]
\[ M_+ = M_\infty - \int_R^\infty 4\pi r^2d\rho_B, \]  
(47)

one finds
\[
\left(1 - \frac{4\alpha f}{M^2}\right) dM \approx -4\pi R^2dR(\rho_T + P_T) 
\]  
(48)
during the adiabatic contraction of the shell, which, up to the small correction factor involving \( f \), is precisely what one would get in flat spacetime from adiabatically contracting a ball of thermal radiation.

Next, we can use the fact that \( R = 2M/W \) to derive that
\[
\frac{dW}{dM} = \frac{2}{R} \left(1 - \frac{M}{R} \frac{dR}{dM}\right) \approx \frac{2}{R} \left[1 + \frac{M(1 - 4\alpha f/M^2)}{4\pi R^3(\rho_T + P_T)}\right]. 
\]  
(49)

where \( f \) and \( \rho_T + P_T \) are to be evaluated at \( r = R \) or \( w \approx W \). Inserting this back into Eq. (11) then gives
\[
\frac{ds}{dW} \approx \frac{3\varepsilon_B + p_B}{4W^4} \left\{1 + \frac{4\alpha}{M^2} \left[\frac{3\varepsilon_B + p_B}{4W^3} - f\right]\right\}^{-1}. 
\]  
(50)

For massless particles of any spin, it should be a good approximation to take \( \rho_T + P_T \approx (4/3)\varepsilon_T \) in terms of the local temperature \( T_{\text{local}} \), and this implies that \( 3\varepsilon_B + p_B \approx 4/(1 - W)^2 \), so
\[
\frac{ds}{dW} \approx \frac{1}{W^4(1 - W)^2} \left\{1 + \frac{4\alpha}{M^2} \left[\frac{1}{W^3(1 - W)^2} - f\right]\right\}^{-1}. 
\]  
(51)

If we omitted the \( f \) term from the radial flux of vacuum polarization energy when \( M \) changes, then the factor inside the curly brackets above would diverge as one approached the horizon, where \( W = 1 \). This implies that the reciprocal of this factor would cancel the divergence in the factor before it, so \( ds/dW \) would stay finite all the way down to \( W = 1 \), and one would find that the increase of one-quarter the area over the entropy would be limited to an amount of the order of \( \sqrt{\alpha}M \).

For large \( M \) this is large in absolute units, but it is always much smaller than the entropy itself, which is of the order of \( 4\pi M^2 \).

However, one can use the fact that the regularity of the Hartle-Hawking stress-energy tensor at the horizon implies that \( \rho_H + P_H \), and hence \( 3\varepsilon_H + p_H \), must go to zero at least as fast as \( 1 - w \) as one approaches the horizon. (This is easiest to see in the Euclidean section with imaginary time \( t \), on which for fixed coordinates \( \theta \) and \( \varphi \), the horizon is at the center of a regular rotationally symmetric two-surface with angular coordinate \( \kappa t \) with \( \kappa \approx 1/(4M) \) being the black hole surface gravity and with the radial distance being roughly \( 4M\sqrt{1 - w} \) when \( 1 - w \ll 1 \). Then \( P_H = T^1_1 \) is the pressure in the radial direction, and \( -\rho_H = T^0_0 \) is the Euclidean
pressure in the Euclidean angular direction, and regularity at the origin demands that the difference go to zero at least as fast as the square of the radial distance from the origin.) Then one can show that \( f \) cancels the divergence in \( W^{-3}(1-W)^{-2} \) so that \( W^{-3}(1-W)^{-2} - f \) stays finite as one approaches the horizon. In fact, if one chooses the constant of integration of \( f \) so that the flux of vacuum polarization energy through the horizon is zero as \( M \) is slowly changed, then \( W^{-3}(1-W)^{-2} - f \) actually goes to zero linearly with \( 1-W \) as one approaches the horizon. For example, using this constant of integration and the gaussian approximation for \( 3\varepsilon_B \) and \( p_B \) leads to

\[
\frac{ds}{dW} \approx \frac{1}{W^4(1-W)^2} \left\{ 1 + \frac{4\alpha}{M^2W^3}(1-W)(1+3W+6W^2+2W^3+7W^4+13W^5) \right\}^{-1}.
\]

(52)

Therefore, we see that the correction term that is first order in \( \alpha/M^2 \) inside the curly brackets of Eqs. (50)-(52) does not diverge as one takes \( 1-W \) to zero but instead always remains small. Therefore, we can drop it (as we have also neglected other finite corrections that are linear in \( \alpha/M^2 \)) and integrate the zeroth-order part of Eq. (52) to get an explicit formula for \( s(W) \):

\[
s(W) \approx \int_{W=0}^{1} \frac{dw}{W^4(1-w)^2} = \frac{1}{1-W} + 4\ln \frac{W}{1-W} - \frac{1}{3W^3} - \frac{1}{W^2} - \frac{3}{W} + \frac{32}{3}.
\]

(53)

As discussed above, I arbitrarily chose the constant of integration of this integral to make \( s(W) = 0 \) at \( W = 1/2 \) or \( R = 4M \), but this is not likely to be valid, and there are also Casimir-energy effects from the shell and corrections to Eqs. (29) and (32) that would give correction terms at least of order \( W \) and likely also of the order of a constant and of order \( 1/W \). From the logarithmic terms I am also ignoring, I would also expect there to be corrections of the order of \( \ln M \).

Finally, we can insert this form for \( s(W) \) into Eq. (34) to get

\[
S(M,W) = \frac{1}{4} A \left[ 1 - \frac{8\alpha}{M^2} s(W) \right] = 4\pi M^2 - 32\pi \alpha s(W)
\]

\[
\approx 4\pi M^2 - 32\pi \alpha \left[ \frac{1}{1-W} + 4\ln \frac{W}{1-W} - \frac{1}{3W^3} - \frac{1}{W^2} \right] + O \left( \frac{1}{W} \right)
\]

\[
\approx 4\pi M^2 - 32\pi \alpha \left[ \frac{R}{R-2M} + 4\ln \frac{2M}{R-2M} - \frac{R^3}{24M^3} - \frac{R^2}{4M^2} \right]
\]

\[
= 4\pi M^2 - \frac{m_b + \frac{7}{2}n_f}{360} \left[ \frac{1}{1-2M/R} - 4\ln \left( \frac{R}{2M} - 1 \right) - \frac{R^3}{24M^3} - \frac{R^2}{4M^2} \right],
\]

(54)

where after the second approximate equality I have dropped the \( O(1/W) \) terms in Eq. (34) that I suspect are always dominated by corrections to my approximations that I have not included. Although I have retained four terms from \( s(W) \) inside the square
brackets, only the first two terms should be kept when $1 - W = 1 - 2M/R \ll 1$ (shell very near the horizon), and only the next two terms should be retained when $W = 2M/R \ll 1$ (shell very large compared with the black hole).

One might question the validity of getting a large energy influx into the black hole horizon from squeezing the reflecting shell deep into the near-horizon region. In my calculation it came from assuming that the vacuum-polarization part of the stress-energy tensor inside the contracting shell is absorbed by the shell and hence has no effect on changing the mass of the hole, whereas the thermal-radiation part is perfectly reflected inward by the shell and hence increases the black hole mass. But someone might object that the total stress-energy tensor inside the shell is small, so that manipulating it would not seem to be able to increase the black hole mass significantly.

However, another way of seeing that a significant increase of the black hole mass is reasonable is to consider the fact that the ADM mass at infinity, $M_\infty$, is fixed, and that as the shell is moved inward, it opens up a larger and larger region of Boulware vacuum outside it, which has negative energy density. Therefore, for fixed $M_\infty$, the mass just outside the shell, $M_+$, increases as the shell is moved inward. Because the inward-moving shell is converting part of its local energy $\mu$ into doing work against the pressure difference across it (greater pressure on the inside from the Hawking radiation inside, or one could equivalently say greater tension on the outside from the fact that one has the Boulware state outside), the mass just inside the shell, $M_-$, increases even more than $M_+$ does as the shell is moved inward. The small total value of the energy density inside the shell means that the black hole mass, $M$, is very nearly the same as $M_-$ and hence increases significantly as the reflecting shell is moved deep into the near-horizon atmosphere of the black hole.

3 Alternative Justifications of the Black Hole Entropy Formula

The result indicated by Eqs. (53) and (54) is precisely the same that one would obtain by taking the geometry to be Schwarzschild with a thermal bath of radiation with local Hawking temperature

$$T_{\text{local}} = \frac{1}{8\pi M} \left(1 - \frac{2M}{r}\right)^{-1/2}$$

and entropy density $(4/3)\alpha_4 T_{\text{local}}^3$, and then taking the total entropy to be $4\pi M^2$ plus the entropy difference between that inside the shell at radius $R$ and that inside the radius $4M$. If one naively integrates this assumed entropy density all the way down to the horizon, one would get a divergence, but one can take the attitude that
this divergence is regulated so that the entropy in this thermal atmosphere below some radius like $4M$ (the precise value of which doesn’t matter much, since the assumed entropy density is this region is of the order of $\alpha/M^3$) is the black hole entropy $S_{bh} \approx A/4 = 4\pi M^2$. Then one can say that if the shell is put at a much larger radius, the entropy of the thermal Hawking radiation outside $4M$ or so would be matter entropy $S_m$ that would add to $S_{bh}$, which is certainly an uncontroversial assumption.

What I have found from my consideration of having the shell squeezed in adiabatically is that if the shell is put much nearer the horizon than a radius of $4M$ or so is, then the entropy is correspondingly less than the usual black hole entropy $S_{bh} \approx A/4 = 4\pi M^2$. Because the thermal atmosphere is restricted from filling up the region to $4M$ or so, it does not have the entropy needed to make the total entropy as large as $A/4$.

Another way to justify this result is to start with a zero-entropy perfectly reflecting shell at $R = 4M$ ($W = 1/2$) and vacuum outside, so that the initial entropy is roughly $A/4$ (i.e., up to an additive correction of the order of unity, plus a possible correction of the order of $\ln A$). Then, without changing the total entropy, construct a new zero-entropy perfectly reflecting shell at a value of $W = 2M/R$ much nearer unity. Next, adiabatically pump out the thermal radiation between the two shells, so that the region in between becomes a vacuum region with zero entropy. Finally, adiabatically discard the outer zero-entropy shell. If the thermal radiation pumped out is discarded (e.g., sent to infinity) and is no longer counted in the entropy of the configuration, and if discarding the outer shell does not change the entropy, this whole process should reduce the entropy being counted by that of the original thermal radiation in the region between the shells. This entropy is the difference between that of the thermal radiation and that of the vacuum in that region, so there should be no inherent ambiguities from any supposed renormalization of the entropy. If one ignores the backreaction of this radiation on the metric, one can calculate the entropy from the Hawking temperature and from the radiation eigenmodes and their frequencies in the region between the two shells. When the inner shell has $1 - W \ll 1$, this entropy is given, to a good approximation, by the difference between the values of $S(M,W)$ for the two values of $W$ corresponding to the two shells.
4 Fundamental Limitations on the Range of Validity of the Black Hole Entropy Formula

The next question is the range of \( W \) over which one would expect that Eq. (54) is approximately valid. For very small \( W \) or very large \( R \), one essentially has a black hole of mass \( M \) surrounded by a much bigger volume, \( V \approx 4\pi R^3/3 \), of radiation in nearly flat spacetime with Hawking temperature \( 1/(8\pi M)^{-1} \), energy density roughly \( 3\alpha/(32\pi M^4) \), and entropy density roughly \( \alpha/M^3 \). The dominant term for the total energy of the radiation is \( E_r \approx \alpha R^3/(8M^4) \), and from Eq. (54), the dominant term for the total entropy of the radiation is \( S_r \approx 4\pi \alpha R^3/(3M^3) \). This agrees with the standard expression for the entropy of thermal radiation of energy \( E_r \) in a volume \( V \),

\[
S_r = \frac{4}{3}(a_r V)^{1/4}E_r^{3/4} \approx \frac{4\pi}{3} \alpha^{1/4}(8RE_r)^{3/4}.
\]  

(56)

For fixed total energy \( M_\infty = M + E_r \ll R \), the total entropy

\[
S \approx 4\pi M^2 + S_r \approx 4\pi(M_\infty - E_r)^2 + \frac{4\pi}{3} \alpha^{1/4}(8RE_r)^{3/4}
\]  

(57)

is indeed extremized for

\[
E_r \approx \frac{\alpha R^3}{8M^4} = \frac{\alpha R^3}{8(M_\infty - E_r)^4},
\]  

(58)

but the extremum is a local entropy maximum if and only if \( 5E_r \leq M_\infty \) or \( 4E_r \leq M \) \[8\], which implies that one needs \( R \leq (2M^5/\alpha)^{1/3} \) or

\[
W \geq \left( \frac{4\alpha}{M^2} \right)^{1/3}
\]  

(59)

for thermodynamic stability.

For smaller values of \( W \) (larger values of \( R \)), the radiation energy \( E_r \) is more than 20% of the total available energy \( M_\infty \) (assumed to be held fixed), and then if the black hole emits some extra radiation and shrinks, it heats up more than the radiation does, leading to an instability in which the black hole radiates away completely. On the other hand, if the black hole absorbs some extra radiation, it will grow and cool down more than the surrounding radiation, therefore cooling down more and absorbing more radiation, until the radiation energy \( E_r \) drops to the lower positive root of Eq. (58), which is less than \( 0.2M_\infty \) and hence is at least locally stable. However, for larger values of \( W \), obeying the inequality \( \[8\] \), the net feedback to extra emission or absorption by the black hole is negative, so that the corresponding configuration is locally stable with fixed total energy \( M_\infty \).
At the opposite extreme, the question is how small $1 - W$ can be. Here the fundamental limit is the Planck regime, which is the boundary of the semiclassical approximation being used in this paper. The Boulware vacuum energy density $\rho_B$ just outside a massless shell (so that the mass just outside, $M_+$, is very nearly the same as the black hole mass $M$; for positive shell mass $\mu$, $\rho_B$ would have an even greater magnitude) is, for very small $U = 1 - W$, $\rho_B \sim -3\alpha/(32\pi M^4 U^2)$. Suppose the semiclassical theory is valid until the orthonormal Einstein tensor component $G_0^0 = -8\pi \rho_B \sim 3\alpha/(4M^4 U^2)$ reaches a maximum value of, say, $C_M$, which would be expected to be of order unity (orthonormal curvature component of the order of the Planck value). This gives the restriction

$$U = 1 - W \geq \left( \frac{3\alpha}{4C_M M^4} \right)^{1/2}. \quad (60)$$

For $U = 1 - W \ll 1$, the spatial distance from the shell to the horizon is $D \sim 4MU^{1/2}$, so this restriction on $U$ gives a minimum distance the shell can be from the horizon:

$$D \geq \left( \frac{192\alpha}{C_M} \right)^{1/4}, \quad (61)$$
in Planck units, as all quantities are in this paper unless otherwise specified.

If we combine the restriction (60) with the lower bound on $W$ from Eq. (59) and re-express the combined restriction as a restriction on the radius $R$ of the shell, we get

$$2M + \frac{1}{M} \sqrt{\frac{3\alpha}{C_M}} \leq R \leq \left( \frac{2M^5}{\alpha} \right)^{1/3}. \quad (62)$$

Alternatively, in terms of the distance $D$ of the shell to the horizon (which is $D \sim R$ for $R \gg 2M$), we get

$$\left( \frac{192\alpha}{C_M} \right)^{1/4} \leq D \leq \left( \frac{2M^5}{\alpha} \right)^{1/3}. \quad (63)$$

If we now insert the restriction (60) or (61) into the asymptotic form of the total entropy (54) for $U = 1 - W \ll 1$, which is

$$S(M,W) \sim 4\pi M^2 - \frac{32\pi \alpha}{U} \sim 4\pi M^2 \left( 1 - \frac{128\alpha}{D^2} \right) \sim 4\pi M^2 - \frac{8\pi \alpha}{U} \sim 4\pi M^2 \left( 1 - \frac{32\alpha}{D^2} \right), \quad (64)$$

we get the limitation

$$S(M,W) \geq 4\pi M^2 \left( 1 - 16\sqrt{\frac{\alpha C_M}{3}} \right) = 4\pi M^2 \left( 1 - \sqrt{\frac{C_M}{135\pi}} (n_b + \frac{7}{8} n_f) \right). \quad (65)$$
This can be re-expressed as a limitation on how much the area $A$ of a black hole can exceed four time the entropy, $4S$:

$$A - 4S \leq A \sqrt{\frac{C_M}{135\pi}} (nb + \frac{7}{8}nf).$$

(66)

Therefore, unless we have $N \equiv nb + 7nf/8$, the effective number of one-helicity particles, comparable to or greater than $\sqrt{135\pi / C_M} \approx 20.6/\sqrt{C_M}$, the fractional increase of the black hole area $A$ above $4S$ is restricted to be rather small, though even just $N = 4$ from two-helicity gravitons and photons would give a fractional increase of about 19% if the curvature limitation $C_M$ is one in Planck units.

### 5 Limitations from Imperfectly Reflecting Shells

Another limitation on the reduction of entropy below $A/4$ by a reflecting shell is the fact that no shell can be a perfect reflector. For example, Smolin has argued [3] that no realistic shell can be a good reflector of gravitational radiation. Therefore, the entropy of the gravitational radiation part of the black hole thermal atmosphere cannot be significantly reduced by surrounding the hole with a realistic shell.

Strictly speaking, no shell is a perfect reflector of any radiation, so if one waits for a sufficiently long time that true thermal equilibrium of the radiation sets in, no shell can stop the region outside from also being thermal. In fact, if one waits long enough for the shell itself to come into complete thermal equilibrium with the radiation, the shell will evaporate and become part of the radiation, with, for example, most of its baryons either decaying, falling into the hole, or getting expelled. However, one can consider squeezing the black hole atmosphere with a shell over a shorter timescale than the timescale for the shell to disintegrate. If the shell is a very good (but not perfect) reflector of some kinds of radiation (apparently never possible for gravitational radiation), then one can imagine squeezing the shell sufficiently slowly that it is nearly adiabatic but sufficiently rapidly that the squeezing is over a timescale short in comparison with the timescale for a significant amount of radiation to leak through the shell. Then the entropy of that kind of radiation can be greatly reduced from its thermal values outside the shell, and hence the total entropy can be reduced by the reduction of the entropy in the thermal atmosphere of the kind of radiation that is practically completely confined to lie within the shell for that intermediate timescale.
6 Further Practical Limits on Entropy Reduction Below $A/4$

We found in Eq. (54) that for a neutral spherical black hole of area $A = 4\pi M^2$ surrounded by a perfectly reflecting shell at $R - 2M \ll M$, the entropy is roughly

$$S \approx \frac{1}{4} A - \frac{32\pi \alpha R}{R - 2M} = \frac{1}{4} A - \frac{n_b + \frac{7}{8} n_f}{360(1 - 2M/R)}.$$  \hspace{3cm} (67)$$

The last term represents the leading term for the reduction of the entropy below one-quarter the area. Let us ask how large this term can be for various assumptions about the shell.

First, consider the case that the shell is held up entirely by its own stresses, with no external forces (other than gravity) on it. In particular, we shall consider the static shell junction conditions (37) and (38), applying the strong energy condition to the shell so that its surface stress obeys the inequality $S_2^2 = S_3^2 \leq -S_0^0$. As we shall soon see, it then turns out that $U_- = 1 - 2M/R \approx 1 - W = 1 - 2M/R$ cannot be very small, so the terms involving the pressures inside and outside the shell are then negligible. Then the strong energy condition applied to the junction conditions (37) and (38) imply that $U_+ U_- \geq 1/25$, and since Eq. (37) implies that a shell with positive local mass has $U_+ < U_-$, we see that $1 - W \approx U_- > 1/5$, or $R > 2.5M$. If Eq. (67) applied for such a large value of $1 - W$, it would then give

$$\frac{1}{4} A - S \approx \frac{n_b + \frac{7}{8} n_f}{360(1 - 2M/R)} < \frac{n_b + \frac{7}{8} n_f}{72},$$  \hspace{3cm} (68)$$

a quite negligible decrease in the entropy, unless somehow $n_b + (7/8)n_f$ is very large. This decrease could indeed be dominated by effects that I have ignored, such as Casimir-energy effects, and, to an even greater degree, by possible terms proportional to $\ln M$.

Next, consider the case that the shell has charge $Q$, so that its electrostatic repulsion holds it up. Since we found above that the stresses within the surface of the shell are quite ineffectual in holding up the shell at $R - 2M \ll M$, let us drop them from the junctions equations but add the tension of the electromagnetic field outside the shell and assume that that tension is much greater than the radial pressures (or tensions) of the quantum fields. Then the junction conditions (37) and (38) become

$$\frac{\mu}{R} = U_-^{1/2} - U_+^{1/2}$$  \hspace{3cm} (69)$$

and

$$0 = 8F = \frac{\mu}{R} + U_-^{-1/2} - (1 - \frac{Q^2}{R^2})U_+^{-1/2},$$  \hspace{3cm} (70)$$
Now for fixed charge-to-mass ratio $Q/\mu$, if we let $\gamma = (\mu/R)/U_{-}^{1/2} < 1$, Eq. (69) implies that $U_{+}^{1/2} = (1 - \gamma)(\mu/R)$, which when inserted into Eq. (70) gives

$$\frac{1}{1 - 2M/R} \approx \frac{1}{U_{-}} = 1 - \gamma + \gamma(Q/\mu)^{2} < (Q/\mu)^{2}. \quad (71)$$

If we take the charge-to-mass ratio of an electron, we get $(Q/\mu)^{2} \approx 4.17 \times 10^{42}$. If we then suppose that somehow a shell of electrons reflects electromagnetic (but not other) radiation and thereby manages to keep the electromagnetic field in its Boulware vacuum state outside the shell (rather than in the Hartle-Hawking thermal state that exists within the shell), then $n_{b} = 2$ (from the two helicities of photons) and $n_{f} = 0$, so one gets

$$\frac{1}{4} A - S \approx \frac{1}{180(1 - 2M/R)} < 2.31 \times 10^{40}. \quad (72)$$

Of course, there are severe problems in attaining anything near this limit. First, electrons in a shell around a black hole, even if in static equilibrium as I have calculated they can be, will not be in stable equilibrium, and some unknown mechanism would have to be invoked to keep the shell in place. Second, without specifying how the electrons are to be kept in place, it is hard to say how they will respond to the black hole thermal radiation impinging upon them from below. However, it is interesting that the upper limit given by Eq. (72) for the reduction in the entropy below one-quarter the (neutral) black-hole area, from a shell held up by electrostatic forces, is so large (because the charge-to-mass ratio of an electron is so large).

For a somewhat more nearly realistic example of a shell around a black hole, consider a thin aluminum foil that is charged so that, like the shell of pure electrons, the electrostatic forces balance the gravitational forces. In this case there will be limitations from the mass density $\rho$ of the foil, the minimum practical thickness $\tau$ of the foil and the maximum charge per surface area, $\sigma$, that it can hold.

In the earlier and longer version of this paper, I took the aluminum foil to have the parameters (in conventional and Planck units respectively)

$$\rho \approx 2.70 \text{ g/cm}^{2} \approx 5.23 \times 10^{-94}, \quad (73)$$

$$\tau = 0.0005 \text{ cm} \approx 3.09 \times 10^{29} \quad (74)$$

(about 100 times the Meissner magnetic penetration depth of about 50 nm, so that the shell should be a very nearly perfect reflector of the electromagnetic part of the Hawking radiation inside it), and

$$\sigma \approx 3.34 \times 10^{12} \text{ e/cm}^{2} \approx 7.46 \times 10^{-55} \quad (75)$$
(so that if the electric surface charge density \( \sigma \) were an excess of electrons, the probability for one to tunnel off would be a very small number, chosen to be \( \exp(-100) \), in some suitable atomic time unit). I then calculated that the charge-to-mass ratio of the aluminum foil is
\[
\frac{Q}{\mu} = \frac{\sigma}{\rho \tau} \approx 4.61 \times 10^9. \tag{76}
\]

Then, by the same analysis used above for the pure electron shell, one finds that if one takes \( \mu/R = U_{1/2} = \mu/Q \), one gets
\[
\frac{1}{1 - 2M/R} = \left( \frac{Q}{\mu} \right)^2 \approx 2.12 \times 10^{19}, \tag{77}
\]
and hence the reduction of the entropy from excluding the thermal photons from above the shell is
\[
\Delta S \equiv \frac{1}{4} A - S \approx \frac{1}{180(1 - 2M/R)} \approx 1.18 \times 10^{17}. \tag{78}
\]

This is extremely tiny in comparison with the total entropy of the black hole,
\[
S \approx 4\pi M^2 \approx \frac{1}{4\pi \sigma^2} \approx 3.58 \times 10^{106}, \tag{79}
\]
but it is very large in absolute value. Indeed, it is much larger than any correction to the entropy that I have ignored if, as I would expect, such corrections are no larger than some number of the order of \( \ln M \), which is in this case about 121.4.

This reduction in the entropy from the naive value of \( A/4 \) (or of this plus or minus a correction of the order of \( \ln M \)) means that the number of states is fewer by a factor of about
\[
e^{\Delta S} \sim 10^{51,000,000,000,000,000}, \tag{80}
\]
which is quite a large factor. Therefore, even though the reduction in the entropy is a tiny fraction of the total entropy, it is large in absolute value, and hence, when exponentiated, it gives an enormous factor in the reduction of the total number of states. It is this sense in which the entropy of a black hole can be significantly reduced below \( A/4 \) by restricting its thermal atmosphere to lie below that of a reflecting shell at very small values of \( 1 - 2M/R \).
7 Conclusions and Acknowledgments

Thus we have seen that by placing a reflecting shell around a black hole, we can make the entropy have a value that is below one-quarter its area (or one-quarter its area plus a positive or negative correction of the order of the logarithm of the area in Planck units). If we are allowed an idealized perfectly reflecting shell that can be placed within roughly one Planck length of the horizon, then this entropy reduction can be of the same order as the area of the hole. For a more realistic shell, such as a superconducting aluminum foil, the entropy reduction can only be a tiny fraction of the area, but it still can be huge in absolute units (much larger than other corrections arising from, say, logarithms of the black hole mass or area), markedly reducing the number of black hole states from what would be erroneously estimated by exponentiating one quarter the horizon area.

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