Minimal models of compact symplectic semitoric manifolds

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Abstract

A symplectic semitoric manifold is a symplectic 4-manifold endowed with a Hamiltonian \((S^1 \times \mathbb{R})\)-action satisfying certain conditions. The goal of this paper is to construct a new symplectic invariant of symplectic semitoric manifolds, the helix, and give applications. The helix is a symplectic analogue of the fan of a nonsingular complete toric variety in algebraic geometry, that takes into account the effects of the monodromy near focus-focus singularities. We give two applications of the helix: first, we use it to give a classification of the minimal models of symplectic semitoric manifolds, where “minimal” is in the sense of not admitting any blowdowns. The second application is an extension to the compact case of a well known result of Vũ Ngọc about the constraints posed on a symplectic semitoric manifold by the existence of focus-focus singularities. The helix permits to translate a symplectic geometric problem into an algebraic problem, and the paper describes a method to solve this type of algebraic problem.

1 Introduction

The revolution in symplectic toric geometry started in the 1980s with the proof of the convexity of the image of the momentum map \(F = (f_1, \ldots, f_n): (M, \omega) \to \mathbb{R}^n\) associated to a compact symplectic 2n-manifold acted upon by a Hamiltonian \(n\)-dimensional compact connected abelian Lie group \(T\) (i.e. an \(n\)-dimensional torus \(T = (S^1)^n\)), due independently to Guillemin-Sternberg \[8\] and Atiyah \[1\]; such manifolds are called symplectic toric. In fact, \(F(M)\) is the polytope \(\Delta\) equal to the convex hull of the image under \(F\) of the fixed points of the \(T\)-action.

Shortly after, Delzant proved \[2\] that \(\Delta\) encodes all of the information about the manifold \(M\), the form \(\omega\), and the \(\omega\)-preserving \(T\)-action. That is, \(\Delta\) is the only symplectic \(T\)-equivariant invariant of \((M, \omega, F)\). He moreover showed that any simple, rational, smooth polytope \(\Delta\) arises as the image of a momentum map of a symplectic-toric manifold; following Guillemin these polytopes are now called Delzant.

The existence of this action poses restrictions on \((M, \omega)\) and \(F\). For instance, \(F\) only has elliptic singularities; moreover, the fibers are tori of dimension 0 up to \(n\) (in particular, they are submanifolds of \(M\)).
Delzant’s classification was extended in [16, 17] to compact and noncompact symplectic 4-manifolds acted up by the noncompact Lie group $S^1 \times \mathbb{R}$, under certain assumptions (all singularities must be non-degenerate, with none of hyperbolic type, the moment map of the $S^1$-action must be proper, and each fiber contains at most one isolated singularity) these manifolds are called *symplectic semitoric*, and so far are classified when $M$ is 4-dimensional. In this case the momentum map of the $(S^1 \times \mathbb{R})$-action is $F = (f_1, f_2)$, where the Hamiltonian vector field $\mathcal{X}_{f_i}$ is periodic, but not necessarily $\mathcal{X}_{f_2}$. The main novelty with respect to symplectic toric manifolds is that $F$ may have, in addition to elliptic singularities, focus-focus singularities. The fiber containing a focus-focus singularity is not a submanifold, it is homeomorphic to a sphere with its south and north poles identified (i.e. a torus pinched at the focus-focus singularity).

Symplectic semitoric manifolds are characterized by five invariants, one of which is a polygon $P$ constructed from $F(M)$ according to Vu Ngoc [20], by unfolding the singular affine structure induced by $F$ on $F(M)$ as a subset of $\mathbb{R}^2$ (in fact $F(M)$ need not even be convex\(^1\)). The other four invariants account for the effect of the focus-focus singularities and the monodromy around them (a fundamental phenomena studied by Duistermaat [3]), they are: the number of focus-focus singularities; a Taylor series in two variables characterizing the dynamics near the fibers containing focus-focus singularities; a height invariant measuring the volume of certain submanifolds at these singularities; and an index which measures the twist of $F$ viewed as a singular Lagrangian fibration, near focus-focus values of $F$ relative to the global toric momentum map which is used to create the polygon invariant.

**Definition 1.1.** A symplectic toric or symplectic semitoric manifold is *minimal* if it does not admit a blow down.

For a symplectic toric manifold chopping off a corner of $\Delta$ corresponds to $T$-equivariantly blowing up $M$ at a $T$-fixed point, and the inverse operation corresponds to blowing down. To $\Delta$ one can associate a *fan*, the one corresponding to $(M, \omega)$ when viewed as a nonsingular complete toric variety (the explicit relation appears in [4]). Because of this correspondence the search for their minimal model is reduced to an algebraic problem concerning fans associated to Delzant polytopes. If $2n \geq 6$ the problem is still too difficult but when $2n = 4$ Fulton classified the corresponding 2-dimensional fans.

**Theorem 1.2** (W. Fulton 1993). *The inequivalent minimal models of symplectic toric manifolds are $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, and a Hirzebruch surface with parameter $k \neq \pm 1$.*

The Delzant polytopes of the minimal models are: a simplex ($M = \mathbb{CP}^2$ with any multiple of the Fubini-Study form), a rectangle ($M = \mathbb{CP}^1 \times \mathbb{CP}^1$ with any product form), and a trapezoid ($M$ a Hirzebruch surface, with its standard form). The question is whether Fulton’s classification can cover more cases.

**Main Question.** What are the inequivalent minimal models of compact symplectic semitoric manifolds?

\(^1\)and in all important examples it is never a polygon, including the coupled spin-oscillator and the spin-orbit system
Figure 1: The helix is intrinsically constructed by defining a toric momentum map on the preimage of $U$, a neighborhood of the boundary of the image of the momentum map minus a single cut, and collecting the inwards pointing normal vectors of the piecewise linear boundary of the resulting set in $\mathbb{R}^2$.

Even more interesting would be to know whether the question can be answered as an application of the known invariants. However, it is not clear what the effect of blowing up and down is on the known invariants we have just described. The image $F(M)$ is no longer necessarily a polygon, or even a convex set. The polygon $P$ is obtained as the image of a homeomorphism $\varphi: F(M) \to P \subset \mathbb{R}^2$ which unfolds the singular affine structure of $F(M)$ into $P$, taking into account the monodromy (the construction of $\varphi$ is delicate, see [20]). The effect of blowing up or down on $P$ depends on the position of the focus-focus values of $F$, and here is where a new invariant of compact symplectic semitoric manifolds we call the semitoric helix, denote it by $\mathcal{H}$, comes into play. Like in the toric case, $\mathcal{H}$ is given by (an equivalence class of) vectors in $\mathbb{Z}^2$, plus some additional information which we describe later more precisely and which includes the information of focus-focus singularities and monodromy (this does not appear in the toric case).

Analogous to the way in which from a Delzant polygon one constructs a fan, from $P$ one constructs the helix $\mathcal{H}$ (after making some corrections related to the focus-focus singular points), see Figure 4, though the helix can also be constructed directly from $M$, bypassing the polygon, as in Figure 1. The helix $\mathcal{H}$ contains the information encoding blowing up and blowing down, information which appears very difficult to extract from known invariants. And $\mathcal{H}$ generalizes the fan while taking into account for the effects of the monodromy around the focus-focus singularities. Moreover, $\mathcal{H}$ can be studied with algebraic techniques, and can be applied to prove:

**Main Theorem.** There are precisely seven inequivalent families of minimal models of compact symplectic semitoric manifolds. Each model is associated to an explicitly describable helix as given in Theorem 2.4.

One can apply this result to extend a theorem of Vù Ngọc [20] from noncompact to compact symplectic semitoric manifolds: if a compact symplectic semitoric manifold with momentum map $F = (f_1, f_2)$ has more than two focus-focus singular points, then $f_1$ has either a non-unique maximum or a non-unique minimum. The Main Theorem and application can be used to study many integrable systems from classical mechanics such as the spin-orbit system [12, 19] (see Section 9.2). It has precisely one focus-focus singularity with monodromy, and is an example of a compact (nontoric) symplectic semitoric manifold. In a
different direction, and as an application of Fulton’s theorem, in [15] some properties of the
associated moduli spaces of manifolds were studied in detail; it would be interesting to use
the above result to study a semitoric analogue. First steps towards this have been carried out
by the second author in [14], where a natural topology on the space of symplectic semitoric
manifolds is constructed.

We conclude by explaining the idea of the proof of the Main Theorem. The proof of this
theorem operates by translating the problem into algebraic language in which a number of
things are easier to work with. The basic ideas behind this technique were already present
in [10], but they are refined and developed so as to be useful in practical applications. The
algebraic correspondence works as follows. To any semitoric helix, as on the right hand
side of Figure 1, there is a natural way to associate a word of a particular form in $SL_2(\mathbb{Z})$.
The word is $\sigma = ST^a_0 \ldots ST^a_{d-1}$, where the $a_i$ are integers and $S, T \in SL_2(\mathbb{Z})$ represent
the specific matrices given in Equation 4.1. Our attempt to classify helices will operate by
attempting to understand the associated words, which must satisfy two conditions. Firstly,$\sigma$ must be conjugate to $T^c$, where $c \in \mathbb{Z}$ is the number of focus-focus points of the associated
system. Additionally, we need to ensure that our helix wraps only a single time around the
origin before repeating. That is, we can define the number of times a collection of vectors
$v_0, \ldots, v_{d-1}$ winds around the origin by following a path from $v_0$ to $v_0$ which connects $v_i$
to $v_{i+1}$, moves only counterclockwise, and circles the origin the minimal number of times. In
order to detect this winding number from the word $\sigma$ in $SL_2(\mathbb{Z})$, we will need to lift $\sigma$
to the universal cover of $SL_2(\mathbb{R})$, in which we can define a function on words which agrees with the
winding number of the associated helix, which we call the winding number of that word. We
let $G$ denote the preimage of $SL_2(\mathbb{Z})$ in the universal cover of $SL_2(\mathbb{R})$. We are then able to
produce an exact correspondence between minimal semitoric helices and words of particular
form in $G$ that lie in one of a small number of conjugacy classes in $G$.

The key idea in our analysis now follows from the observation that almost all of the
appropriate words conjugate to the correct element of $SL_2(\mathbb{Z})$ have winding numbers that
are too large. In fact, the elements that we are looking for will necessarily have nearly the
smallest possible winding number of any word representing the correct element of $SL_2(\mathbb{Z})$. In
order to properly analyze this, we show that each element of $PSL_2(\mathbb{Z})$ has a unique minimal
word associated to it with the smallest possible winding number. In fact, any representation
of the given element can be reduced to the minimal one by means of a few simple reduction
steps. The thrust of our argument is now to look at the minimal word associated to the
element represented by our helix. Noting that the word corresponding to our helix reduces
to this minimal word in only a few steps, allows us to reduce ourselves to a small number of
possibilities. The main novelty of the paper is precisely this method of proof, which although
it may be natural to algebraists, we have not seen used in symplectic geometry.

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Structure of the article

In Section 2 we state the main results of the paper, Theorem 2.4 and Theorem 2.5. In Section 3 we review some background material regarding symplectic toric manifolds. In Section 4 we define the semitoric helix, outline its construction, and state Theorem 4.15 which is a precise version of Theorem 2.4. In Section 5 we present the construction of a semitoric helix from a symplectic semitoric manifold. In Section 6 we explain the connection between semitoric helices and $\text{SL}_2(\mathbb{Z})$ and in Section 7 we introduce a standard form for elements of $\text{PSL}_2(\mathbb{Z})$. Finally, in Section 8 we use the results of Sections 6 and 7 to prove Theorems 2.4 and 4.15. In Section 9 we follow the argument of the proof of Theorem 4.15 applied to a specific example and also explain the example of the coupled angular momenta system. Section 10 we prove Theorem 2.5.

2 Main results

Definition 2.1. ([16, 17]) A symplectic semitoric manifold is a connected 4-dimensional integrable system $(M, \omega, F = (J, H))$ such that:

1. $J$ is proper, that is, if $K \subset \mathbb{R}$ is compact then $J^{-1}(K)$ is compact;
2. the Hamiltonian vector field $\mathcal{X}_J$ induced by $J$ has periodic flow of period $2\pi$ and the $S^1$-action generated by this flow is effective;
3. all singularities of $F$ are non-degenerate and contain no hyperbolic blocks.

Item (3) refers to the Williamson classification of singularities for integrable systems (see [22]). In [5] Eliasson extends the pointwise classification of singular points implied by Williamson’s classification of Cartan subalgebras of $\mathfrak{sp}(2n)$ [21] to a local normal form for non-degenerate singular points. Since dim$(M) = 4$, item (3) implies that any point $p \in M$ in a symplectic semitoric manifold is one of: completely regular; elliptic-regular; elliptic-elliptic; or focus-focus. In this article we assume all symplectic semitoric manifolds are simple, which means there is at most one focus-focus point in each level set of $J$.

Definition 2.2. Given two symplectic semitoric manifolds $(M, \omega, F)$ and $(M', \omega', F')$ a symplectomorphism $\psi: (M, \omega) \to (M', \omega')$ is a semitoric isomorphism if $\psi^*(J', H') = (J, f(J', H'))$ where $f: \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth map with $\frac{\partial}{\partial y} f \neq 0$ everywhere.

Proposition 2.3. To each compact symplectic semitoric manifold $(M, \omega, F)$ there is an associated semitoric helix denoted $\text{hlx}(M, \omega, F)$ and $\text{hlx}(M, \omega, F) = \text{hlx}(M', \omega', F')$ if $(M, \omega, F)$ and $(M', \omega', F')$ are isomorphic as symplectic semitoric manifolds.

Proposition 2.3 is proven in Section 5.1. There is a natural notion of blowup/down in this context which we describe in detail in Section 4.1, and a symplectic semitoric manifold is said to be minimal if it does not admit a such a blowdown. In [20] it is shown that any symplectic semitoric manifold has only finitely many focus-focus singular points.
Theorem 2.4. Let \((M, \omega, F)\) be a compact symplectic semitoric manifold with \(c > 0\) focus-focus points and \(d > 0\) elliptic-elliptic points. If \(d < 5\) then \((M, \omega, F)\) is minimal if and only if its associated helix is one of:

1. \(c = 1\);
2. \(c = 2\);
3. \(k \neq \pm 2, c = 1\);
4. \(c \neq 2\);
5. \(k \neq \pm 1, 1 - c, c > 0\). If \(d \geq 5\) then \((M, \omega, F)\) is minimal if and only if \(d > 5\) and in this case the associated helix is completely determined by \(c\) and a positively oriented basis \((v_0, v_1)\) of \(\mathbb{Z}^2\).

A complete version of Theorem 2.4 appears as Theorem 4.15 and an example of a helix with \(d > 5\) is shown in Figure 3. The example of the coupled angular momenta system, which is minimal of type (3) with \(k = 1\), is given in Section 9.2. Theorem 4.15 has the following surprising consequence in the study of symplectic semitoric manifolds, which follows from Lemma 10.1 and is proven in Section 10.

Theorem 2.5. If a compact symplectic semitoric manifold has at least two focus-focus singular points and the component of the momentum map with periodic flow achieves its maximum and minimum at a single point each, then the system must have exactly two focus-focus points and be a minimal symplectic semitoric manifold of type (2) from Theorem 4.15.

In [20, Theorem 3] Vũ Ngọc uses an argument related to the Duistermaat-Heckman measure on symplectic semitoric manifolds to prove that there do not exist noncompact symplectic semitoric manifolds for which the component of the momentum map with periodic flow achieves its maximum and minimum at a single point each and which have more than one focus-focus point. Recently, S. Sabatini drew to our attention that she had announced a version of Theorem 2.5 at a conference in 2013 and outlined a different proof from the one given in the present paper.

3 Background: Minimal symplectic toric manifolds

Here we give a pedestrian exposition of toric manifolds and integrable systems from the point of view of symplectic geometry.
An integrable system is a triple \((M, \omega, F)\) where \((M, \omega)\) is a \(2n\)-dimensional symplectic manifold and \(F = (f_1, \ldots, f_n): M \to \mathbb{R}^n\) is a smooth map such that its components \(f_1, \ldots, f_n\) Poisson commute and are independent almost everywhere. That is, \(\omega(X_{f_i}, X_{f_j}) = 0\) for all \(i, j = 1, \ldots, n\) and \((X_{f_1})_p, \ldots, (X_{f_n})_p\) are linearly independent in \(T_p M\) for almost all \(p \in M\), where \(X_{f_i}\) denotes the Hamiltonian vector field of \(f_i\).

**Definition 3.1.** A symplectic toric manifold is an integrable system \((M, \omega, F)\) such that \((M, \omega)\) is a compact and connected \(2n\)-dimensional symplectic manifold, each \(X_{f_i}\) has \(2\pi\)-periodic flow, and the \(\mathbb{T}^n\)-action produced by these flows is effective.

A convex, compact, rational polygon in \(\mathbb{R}^2\) is a Delzant polygon if the collection of inwards-pointing integer normal vectors to the polygon of minimal length form what is known as a toric fan. For vectors \(v, w \in \mathbb{Z}^2\) let \(\text{det}(v, w)\) denote the determinant of the matrix with columns \(v, w\).

**Definition 3.2.** A toric fan of length \(d \in \mathbb{Z}_{>0}\) is a collection of vectors \((v_0, \ldots, v_{d-1}) \in (\mathbb{Z}^2)^d\) such that
1. \(\text{det}(v_i, v_{i+1}) = 1\) for \(i = 0, \ldots, d - 1\) where \(v_d := v_0\);
2. \(v_0, \ldots, v_{d-1}\) are arranged in counter-clockwise order.

Associated to each toric manifold is a toric fan, formed from the Delzant polygon in this way.

**Definition 3.3.** If \((v_0, \ldots, v_{d-1})\) is a toric fan of length \(d\) such that \(v_i = v_{i-1} + v_{i+1}\) then a new toric fan of length \(d - 1\) can be produced by removing \(v_i\). This operation is known as the blowdown and the inverse operation, inserting the sum of two adjacent vectors, is known as a blowup.

**Definition 3.4.** A toric fan is minimal if \(v_i \neq v_{i-1} + v_{i+1}\) for \(i = 0, \ldots, d - 1\).

Minimal toric fans are those on which a blowdown cannot be performed. A toric fan can be reduced to a minimal toric fan by performing blowdowns until no more are possible. On the other hand, this implies that any toric fan may be obtained from a minimal toric fan by a finite sequence of blowups. Minimal toric manifolds are those that do not admit a symplectic toric blowdown (see Section 4.1).

**Proposition 3.5** ([7]). A blowup/down on a fan corresponds to a blowup/down on the associated toric manifold. In particular, a toric manifold is minimal if and only if its fan is minimal.

Minimal toric fans were classified in [7], and this implies a classification of minimal toric manifolds. The group \(\text{SL}_2(\mathbb{Z})\) acts on a toric fan by acting on each vector in the fan.

**Theorem 3.6** (Fulton [7]). A toric manifold is minimal if and only if its fan is one of the following up to the action of \(\text{SL}_2(\mathbb{Z})\):

1. \(v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}\):
Figure 2: The three possible minimal toric fans (up to $\text{SL}_2(\mathbb{Z})$) listed in Theorem 3.6, where $k \in \mathbb{Z}$ is the parameter for the Hirzebruch trapezoid and in the figure we show the case of $k = -2$.

2. $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$;

3. $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ k \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ for $k \in \mathbb{Z}$, $k \neq 0, \pm 1$.

These fans are shown in Figure 2. Respectively, these are known as the Delzant triangle, the square, and the Hirzebruch trapezoid named for the shapes of their associated Delzant polygons. They correspond, in order, to $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, and a Hirzebruch surface.

4 The semitoric helix

4.1 Toric blowups/downs for symplectic toric and semitoric manifolds

Let $(M, \omega, F)$ be a symplectic toric or symplectic semitoric 4-manifold with $p \in M$ an elliptic-elliptic point. Then there exists complex coordinates $z_1, z_2$ in an open chart $U \subset M$ centered at $p$ such that the symplectic form is given by $\omega = \frac{-i}{2} (dz_1 \wedge dz_1 + dz_2 \wedge d\bar{z}_2)$ and $F(z_1, z_2) = \frac{1}{2} (|z_1|^2 + |z_2|^2) + F(0, 0)$. Let $\phi: U \to \mathbb{C}^2$ denote the map $\phi = (z_1, z_2)$ and let $V = \phi(U)$. Let $B^4(r) \subset \mathbb{C}^2$ denote the standard ball of radius $r > 0$. For any $\lambda > 0$ sufficiently small such that $B^4(\lambda) \subset V$ we can define locally in this chart the toric blowup of weight $\lambda$. Since $p$ is an elliptic-elliptic point this must be possible for some $\lambda > 0$.

Following [13, Section 7.1] define $\tilde{\mathbb{C}}^2 \subset \mathbb{C}^2 \times \mathbb{C}P^1$ to be those pairs $(z, \ell)$ such that $z \in \ell$. That is,

$$\tilde{\mathbb{C}}^2 = \{(z_1, z_2; [w_0, w_1]) \mid w_j z_k = w_{k-1} z_{j+1} \text{ for } j = 0, 1 \text{ and } k = 1, 2\}$$

(the manifold $\tilde{\mathbb{C}}^2$ is the usual (non-symplectic) blowup of $\mathbb{C}^2$ at the origin). There are natural
and for each $r > 0$ define $L(r) = \pi_{c^2}^{-1}(B^4(r))$. For each $\lambda > 0$ define a symplectic form $\rho(\lambda)$ on $\tilde{C}^2$ by $\rho(\lambda) = \pi_{c^2}^{*}\omega_0 + \lambda^2\pi_{\mathbb{C}P^1}^{*}\omega_{FS}$ where $\omega_{FS}$ is the Fubini-Study form on $\mathbb{C}P^1$ and $\omega_0$ is the standard symplectic form on $\mathbb{C}^2$. Finally, with $\lambda$ and $\delta$ chosen small enough so that $B^4(\sqrt{\lambda^2 + \delta^2}) \subset V$, define $\tilde{C}^2_{\lambda} = \left(\mathbb{C}^2 \setminus B^4(\sqrt{\lambda^2 + \delta^2})\right) \cup L(\delta)$. Since $\rho(\lambda) = \omega_0$ outside of $B^4(\sqrt{\lambda^2 + \delta^2})$ there is no problem defining a symplectic structure on $\tilde{M}(p, \lambda) = (M \setminus \phi^{-1}(B^4(\sqrt{\lambda^2 + \delta^2}))) \cap L(\delta)$, which is known as the symplectic toric blowup of $M$ at $p$ of size $\lambda$. This is similar to the standard symplectic blowup except that the choice of chart forces the embedded ball used in this construction to be $\mathbb{R}^2$-equivariantly embedded, where the $\mathbb{R}^2$-action on $M$ comes from the flow of $\mathcal{X}_{F_1}$ and $\mathcal{X}_{F_2}$, which descends to a $\mathbb{T}^2$-action for symplectic toric manifolds and an $(S^1 \times \mathbb{R})$-action for symplectic semitoric manifolds (see [6] for an investigation of symplectic semitoric manifolds as symplectic $(S^1 \times \mathbb{R})$-manifolds). We will not show that this construction is independent of the choices involved because this is a standard fact (again, see [11, 13]).

The inverse of this operation is known as a toric blowdown. Performing a toric blowup or down on a toric manifold corresponds to performing a blowup or down on the associated toric fan. We will see that performing a toric blowup/down on a symplectic semitoric manifold corresponds to performing a combinatorial operation, which we call a blowup/down, on the associated semitoric helix (see Section 4.2). We will often simply call a toric blowup a blowup (and similar for a blowdown).

**Definition 4.1.** A symplectic semitoric manifold $(M, \omega, F)$ is minimal if it does not admit a blowdown.

That is, a symplectic semitoric manifold is minimal if there does not exist any symplectic semitoric manifold $(M', \omega', F')$ such that $(M, \omega, F)$ can be obtained from $(M', \omega', F')$ by a symplectic blowup.

For the present paper we will not be concerned with the size of the blowups since this will not change the associated helix and will not effect whether or not the resulting manifold is minimal. Thus, we will often say "the blowup of $M$ at $p$" to really mean "one of the blowups of $M$ at $p$" or even "the family of all manifolds which can be obtained by performing a blowup of some weight on $M$ at $p.""

**Remark 4.2.** This definition of blowup/down can be extended to be used around any completely elliptic point of any integrable system of any dimension.
4.2 The semitoric helix

Let \((\mathbb{Z}^2)^\infty\) be the space of sequences indexed by \(\mathbb{Z}\) in \(\mathbb{Z}^2\). For \(\{v_i\}_{i \in \mathbb{Z}}, \{w_i\}_{i \in \mathbb{Z}} \in (\mathbb{Z}^2)^\infty\) let \(\sim\) be the equivalence relation on \((\mathbb{Z}^2)^\infty\) given by \(\{v_i\}_{i \in \mathbb{Z}} \sim \{w_i\}_{i \in \mathbb{Z}}\) if and only if there exists \(k, \ell \in \mathbb{Z}\) such that

\[v_i = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k w_{i+\ell}\]

for all \(i \in \mathbb{Z}\). Let \([\{v_i\}_{i \in \mathbb{Z}}] \in (\mathbb{Z}^2)^\infty/\sim\) denote the equivalence class of \([\{v_i\}_{i \in \mathbb{Z}}]\).

**Definition 4.3.** A **semitoric helix** is a triple \(\mathcal{H} = (d, c, [\{v_i\}_{i \in \mathbb{Z}}])\) where \(d \in \mathbb{Z}_{>0}\), \(c \in \mathbb{Z}_{\geq 0}\), and \([\{v_i\}_{i \in \mathbb{Z}}] \in (\mathbb{Z}^2)^\infty/\sim\) such that:

1. \(\det(v_i, v_{i+1}) = 1\) for all \(i \in \mathbb{Z}\);
2. \(v_0, \ldots, v_{d-1}\) are arranged in counter-clockwise order;
3. \(\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} v_i = v_{i+d}\) for all \(i \in \mathbb{Z}\).

We say that a semitoric helix \((d, c, [\{v_i\}_{i \in \mathbb{Z}}])\) has **length** \(d\) and **complexity** \(c\). It is **minimal** if

\[v_i \neq v_{i-1} + v_{i+1}\]

for all \(i \in \mathbb{Z}\).

**Lemma 4.4.** The minimality condition does not depend on the choice of representative of \([\{v_i\}_{i \in \mathbb{Z}}]\).

**Proof.** Let \([w_i]_{i \in \mathbb{Z}} \in [\{v_i\}_{i \in \mathbb{Z}}]\) so there exists \(k, \ell \in \mathbb{Z}\) such that \(v_i = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} w_{i+\ell}\) for all \(i \in \mathbb{Z}\) and thus \(v_j = v_{j-1} + v_{j+1}\) if and only if \(w_{j+\ell} = w_{j+\ell-1} + w_{j+\ell+1}\) by applying \(\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}\).

A minimal semitoric helix is shown in Figure 3. In light of item (3), a semitoric helix of given complexity \(c > 0\) and length \(d\) is determined by any \(d\) consecutive vectors in any representative.

**Definition 4.5.** Let \(\mathcal{H} = (d, c, [\{v_i\}_{i \in \mathbb{Z}}])\) be a semitoric helix. The **blowup of \(\mathcal{H}\) at \(v_i\)** is the helix \((d+1, c, [\{w_i\}_{i \in \mathbb{Z}}])\), where \(\{w_i\}\) is formed from \(\{v_i\}_{i \in \mathbb{Z}}\) by inserting \(v_{i+kd} + v_{i+1+kd}\) between \(v_{i+kd}\) and \(v_{i+1+kd}\) for all \(k \in \mathbb{Z}\). If \(v_j = v_{j-1} + v_{j+1}\) for some \(j \in \mathbb{Z}\) then the **blowdown of \(\mathcal{H}\) at \(v_i\)** is the helix \((d-1, c, [\{u_i\}_{i \in \mathbb{Z}}])\) where \(\{u_i\}\) is produced by removing \(\{v_{j+nd}\}_{n \in \mathbb{Z}}\) from \(\{v_i\}_{i \in \mathbb{Z}}\).
Figure 3: A minimal semitoric helix of length 6 and complexity 2. In the classification from Theorem 4.15 this is a type (7) minimal semitoric helix with $A_0 = ST^2 ST^2$.

4.3 Outline of helix construction

Let $(M, \omega, F = (J, H))$ be a compact symplectic semitoric manifold with $c \in \mathbb{Z}$ focus-focus singular points (symplectic semitoric manifolds always have finitely many focus-focus points [20]). Take $U \subset F(M)$ to be a small open neighborhood of the boundary of $F(M)$ minus a straight line segment $\ell$ which has its endpoints outside of $U$ and exactly one endpoint in $F(M)$, so that $U$ is simply connected. The set $F(M)$ is simply connected by [20, Theorem 3.4]. Since $U$ is simply connected the fibers of $F$ form a trivial torus fibration of $F^{-1}(U)$ so there exists a toric momentum map $F_{\text{toric}}$ on $F^{-1}(U)$ which has the same first component and associated singular Lagrangian fibration as $F$ (as in [20]), and $F_{\text{toric}}(F^{-1}(U))$ minus its interior is a connected union of line segments in $\mathbb{R}^2$. Taking the inwards pointing normal vectors of this segment forms the vectors $v_0, \ldots, v_d$ and these vectors, along with the integer $c$, determine the helix of length $d$ associated to $(M, \omega, F)$, which we denote $\text{hlx}(M, \omega, F)$ and which is known as the semitoric helix associated to $(M, \omega, F)$. The precise construction procedure is in Section 5.

Lemma 4.6. Given a symplectic semitoric manifold $(M, \omega, F)$ there exists exactly one associated semitoric helix.

Lemma 4.6 is restated with more precision later as Lemma 5.2.

Definition 4.7. Let $S_{\text{ST}}$ denote the space of symplectic semitoric manifolds and $S_{\text{H}}$ denote the collection of semitoric helices. The map $\text{hlx}: S_{\text{ST}} \to S_{\text{H}}$ assigns to each symplectic semitoric manifold $(M, \omega, F)$ a semitoric helix $\text{hlx}(M, \omega, F) = \mathcal{H}$, where $\mathcal{H}$ is the semitoric helix associated to $(M, \omega, F)$.

Lemma 4.6 shows that the map $\text{hlx}: S_{\text{ST}} \to S_{\text{H}}$ is well-defined.

Proposition 4.8. Let $(M, \omega, F)$ be a symplectic semitoric manifold with associated helix $\text{hlx}(M, \omega, F)$. The symplectic semitoric manifold $(M', \omega', F')$ can be obtained from $(M, \omega, F)$ by a blowup if and only if the associated helix $\text{hlx}(M, \omega, F)$ can be obtained from $\text{hlx}(M, \omega, F)$ by a blowup of semitoric helices. Moreover, $(M', \omega', F')$ can be obtained from $(M, \omega, F)$ by a
blowdown if and only if the associated helix hlx(M', ω', F') can be obtained from hlx(M, ω, F) by a blowdown of semitoric helices.

Proof. The helix is obtained as the inwards pointing normal vectors on the image of a toric momentum map on a subset of M, and the blowups we have defined are those which produce toric blowups with respect to this momentum map. Thus, the correspondence between toric blowups/downs of toric manifolds and blowups/downs of toric fans implies the result.

4.4 The algebraic technique

Let $S, T \in SL_2(\mathbb{Z})$ be the standard generators given by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

so $SL_2(\mathbb{Z}) = \langle S, T \mid STS = T^{-1}ST^{-1}, S^4 = I \rangle$ and $PSL_2(\mathbb{Z}) = \langle S, T \mid STS = T^{-1}ST^{-1}, S^2 = I \rangle$. We denote by $\mathbb{Z} \ast \mathbb{Z}$ the free group on letters S and T.

**Notation:** Since we consider several groups generated by S and T we use $\equiv_H$ to denote equality in the group H. For instance, $S^4 \equiv_{SL_2(\mathbb{Z})} I$ but $S^4 \not\equiv_{\mathbb{Z} \ast \mathbb{Z}} I$.

Given $v, w \in \mathbb{Z}^2$ we denote by $[v, w]$ the $2 \times 2$ matrix with first column v and second column w and denote by det$(v, w)$ the determinant of $[v, w]$.

The group $G = \langle S, T \mid STS = T^{-1}ST^{-1} \rangle$ is isomorphic to the preimage of $SL_2(\mathbb{Z})$ in the universal cover of $SL_2(\mathbb{R})$ [10, Proposition 3.7], as in the following diagram:

$$\begin{array}{ccc}
G & \xrightarrow{\rho} & \tilde{SL}_2(\mathbb{R}) \\
\downarrow & & \downarrow \\
SL_2(\mathbb{Z}) & \xleftarrow{i} & SL_2(\mathbb{R})
\end{array}$$

where $\tilde{SL}_2(\mathbb{R})$ denotes the universal cover of $SL_2(\mathbb{R})$, which has fundamental group $\mathbb{Z}$. Above $\rho: G \rightarrow SL_2(\mathbb{R})$ denotes the map that takes G isomorphically to the preimage of $SL_2(\mathbb{Z})$ in $SL_2(\mathbb{R})$ given by

$$\rho(T) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}_{0 \leq t \leq 1} \quad \text{and} \quad \rho(S) = \begin{pmatrix} \cos \left( \frac{\pi t}{2} \right) & -\sin \left( \frac{\pi t}{2} \right) \\ \sin \left( \frac{\pi t}{2} \right) & \cos \left( \frac{\pi t}{2} \right) \end{pmatrix}_{0 \leq t \leq 1}$$

(as in [10, Proposition 3.7]), $i: SL_2(\mathbb{Z}) \hookrightarrow SL_2(\mathbb{R})$ is the inclusion map and the other two maps are the natural projections. Each element of the kernel of the natural projection from G to $SL_2(\mathbb{Z})$, denoted ker$(G \rightarrow SL_2(\mathbb{Z}))$, represents a closed loop in $SL_2(\mathbb{R})$. Let $(\mathbb{R}^2)^* := \mathbb{R}^2 \setminus \{(0, 0)\}$.

**Definition 4.9.** Given any closed loop $\tilde{\gamma}: [0, 1] \rightarrow (\mathbb{R}^2)^*$, $\tilde{\gamma}(0) = \tilde{\gamma}(1)$, we denote by wind($\tilde{\gamma}$) $\in \mathbb{Z}$ the usual winding number of $\tilde{\gamma}$ in $(\mathbb{R}^2)^*$.
Define \( pr: \text{SL}_2(\mathbb{R}) \to (\mathbb{R}^2)^* \) by

\[
pr: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix}.
\]

Since \( \pi_1(\text{SL}_2(\mathbb{R})) \cong \pi_1((\mathbb{R}^2)^*) \cong \mathbb{Z} \) and \( pr \) sends a generator of \( \pi_1(\text{SL}_2(\mathbb{R})) \) to a generator of \( \pi_1((\mathbb{R}^2)^*) \), \( pr \) induces an isomorphism at the level of fundamental groups.

**Definition 4.10.** Given any loop \( \gamma: [0,1] \to \text{SL}_2(\mathbb{R}) \), \( \gamma(0) = \gamma(1) \), we define the **winding number** of \( \gamma \), denoted \( \text{wind}(\gamma) \), by

\[
\text{wind}(\gamma) := \text{wind} \left( pr(\gamma) \right).
\]

In the following section we extend the map \( \text{wind} \circ \rho: \ker(G \to \text{SL}_2(\mathbb{Z})) \to \mathbb{Z} \) to all of \( G \).

### 4.4.1 The winding number

Let \( W: \mathbb{Z} \ast \mathbb{Z} \to \mathbb{Z} \) be the homomorphism generated by \( W(S) = \frac{1}{4} \) and \( W(T) = -\frac{1}{12} \). Since \( W(STS) = W(T^{-1}ST^{-1}) \), \( W \) descends to a map on \( G \) which we also denote \( W \). The map is known as the **winding number** \([10]\) because if \( \sigma \in \ker(G \to \text{SL}_2(\mathbb{Z})) \) then \( W(\sigma) \) agrees with \( \text{wind}(\rho(\sigma)) \) as in Definition 4.10, where \( \rho \) is as in Equation 4.2.

**Lemma 4.11** ([10]). Given \( \sigma \in \ker(G \to \text{SL}_2(\mathbb{Z})) \), \( W(\sigma) = \text{wind}(pr \circ \rho(\sigma)) \).

**Proof.** The map \( W \) is a homomorphism and \( W(S^4) = \text{wind}(\rho(S^4)) \). Since \( S^4 \) is a generator of \( \ker(G \to \text{SL}_2(\mathbb{Z})) \cong \mathbb{Z} \) this uniquely defines it. \( \square \)

For a semitoric helix \( \mathcal{H} = (d, c, \{v_i\}_{i \in \mathbb{Z}}) \) let \( -\mathcal{H} = (d, c, \{-v_i\}_{i \in \mathbb{Z}}) \). If \( \mathcal{H} = \mathcal{H}' \) or \( \mathcal{H} = -\mathcal{H}' \) we write \( \mathcal{H} = \pm \mathcal{H}' \). A cyclic permutation of a list \( (a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d \) of integers is given by

\[
(a_{k \mod d}, a_{k+1 \mod d}, \ldots, a_{k+d-1 \mod d})
\]

for some \( k \in \mathbb{Z} \).

**Proposition 4.12.** Associated to any semitoric helix of length \( d \) and complexity \( c > 0 \) there is a list of integers \( (a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d \) which satisfy

\[
ST^{a_0} \cdots ST^{a_{d-1}} =_G S^4 X^{-1} T^c X
\]

for some \( X \in G \). This list of integers is unique up to cyclic permutation, and any such list of integers is associated to some semitoric helix. Semitoric helices \( \mathcal{H} \) and \( \mathcal{H}' \) have the same length, complexity, and associated integers if and only if \( \mathcal{H} = \pm \mathcal{H}' \).

Proposition 4.12 is proven in Section 6.

**Definition 4.13.** A word \( \eta \in \mathbb{Z} \ast \mathbb{Z} \) is **S-positive** if it can be written using only non-negative powers of \( S, T, \) and \( T^{-1} \).

To classify the minimal models of symplectic semitoric manifolds we will show that the associated word of a minimal helix (as in Proposition 4.12) is very close to the following standard form in \( \text{PSL}_2(\mathbb{Z}) \).
**Theorem 4.14** (Standard form in $\text{PSL}_2(\mathbb{Z})$). If $X \in \text{SL}_2(\mathbb{Z})$ there exists a unique string $\overline{X} \in \mathbb{Z}^* \mathbb{Z}$ such that $X = {\text{PSL}_2(\mathbb{Z})} \overline{X}$ and

$$\overline{X} = \mathbb{Z}^* \mathbb{Z} T^b S T^{a_0} \ldots S T^{a_{d-1}}$$

where $a_i > 1$ for $i = 0, \ldots, d - 2$. Moreover, $W(\overline{X}) \leq W(\eta)$ for all $S$-positive $\eta \in \mathbb{Z}^* \mathbb{Z}$ satisfying $\eta = {\text{PSL}_2(\mathbb{Z})} X$.

We call $\overline{X}$ the standard form of $X$. Theorem 4.14 is proven in Section 7.

### 4.5 Main result: minimal models of symplectic semitoric manifolds

Let

$$S = \left\{ A \in \text{SL}_2(\mathbb{Z}) \mid \overline{A} = S T^{a_0} \ldots S T^{a_{d-1}}, \text{ for } d > 5, a_{d-1} \notin \{0, 1\} \right\}. \quad (4.3)$$

Recall a semitoric helix of length $d$ is determined by specifying the complexity and any $d$ consecutive vectors in any representative of the helix.

**Theorem 4.15.** Suppose that $(M, \omega, F)$ is a minimal compact symplectic semitoric manifold with $c > 0$ focus-focus points and associated semitoric helix $(d, c, [(v_i)_{i \in \mathbb{Z}}]) = \text{hlx}(M, \omega, F)$. If $d < 5$ then the representative $\{v_i\}_{i \in \mathbb{Z}}$ can be chosen to be exactly one of the following:

| type | length | $v_0, \ldots, v_{d-1}$ | complexity |
|------|--------|--------------------------|------------|
| (1)  | $d = 2$| $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ | $c = 1$    |
| (2)  | $d = 2$| $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ | $c = 2$    |
| (3)  | $d = 3$| $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ k \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ | $k \neq \pm 2$ |
| (4)  | $d = 3$| $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ | $c \neq 2$ |
| (5)  | $d = 4$| $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ k \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ | $k \neq \pm 1, 0$ |
| (6)  | $d = 4$| $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} k \\ -1 \end{pmatrix}$ | $k \neq -1$ for $k \neq 1 - c$ |

Otherwise, $d \geq 5$, in which case $d > 5$, and we say that the symplectic semitoric manifold and helix are minimal of type (7). There is a one-to-one correspondence between minimal helices of type (7) and the set $S \times \mathbb{Z}_{>0}$. Given $c \in \mathbb{Z}_{>0}$ and a basis $v_0, v_1$ of $\mathbb{Z}^2$ satisfying $[v_0, v_1] \in S$ then the corresponding minimal helix of type (7) is determined by the following procedure: Let $a_0, \ldots, a_{d-1}, d \in \mathbb{Z}$ be the unique integers which satisfy

$$S^2[v_0, v_1] - T^c[v_0, v_1] = \mathbb{Z}^* \mathbb{Z} S T^{a_0} \ldots S T^{a_{d-1}}. \quad (4.4)$$
Then the recurrence relation
\[ v_j = a_{j-2}v_{j-1} + v_{j-2} \]
for \( j = 0, \ldots, d - 1 \) and given \( v_0, v_1 \) determines the vectors \( \{v_i\}_{0 \leq i < d} \) which, along with the complexity \( c \), determine the helix, \( \mathcal{H} \).

Types (1)-(6) are shown in Theorem 2.4 and a representative example of type (7) is shown in Figure 3. Theorem 4.15 is a direct consequence of Lemma 8.3.

**Corollary 4.16.** Suppose that \( \mathcal{H} \) is a minimal helix of length \( d > 4 \). Then \( \mathcal{H} \) is of type (7) from Theorem 4.15 and there exists a representative \( \{v_i\}_{i \in \mathbb{Z}} \) such that \( \mathcal{H} = (d, c, [\{v_i\}_{i \in \mathbb{Z}}]) \) and the following hold:
1. \( v_0 = -v_2 \);
2. there exists \( k \in \mathbb{Z} \) with \( 2 < k < d \) such that \( v_k \) is \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) or its negative.

Corollary 4.16 is proven in Section 8.

**4.5.1 Idea of proof of Theorem 4.15**

In the proof of Theorem 4.14, the standard form in \( \text{PSL}_2(\mathbb{Z}) \), we use a reduction algorithm with four steps. Three of these steps reduce the winding number by \( 1/2 \) and the remaining step, which corresponds to a blowdown, does not change the winding number. We will see, by Lemmas 7.5 and 7.7, that if \( a_0, \ldots, a_{d-1} \) is associated to a semitoric helix then
\[
W(ST^{a_0} \cdots ST^{a_{d-1}}) - W(ST^{a_0} \cdots ST^{a_{d-1}}) = \begin{cases} 1, & X =_{\text{PSL}_2(\mathbb{Z})} T^k \\ \frac{1}{2}, & \text{otherwise} \end{cases}
\]
and thus we know that \( ST^{a_0} \cdots ST^{a_{d-1}} \) can be reduced to the standard form from Theorem 4.14 by using only one or two of the moves which reduce \( W \) along with any number of blowdowns.

This observation allows us to prove Lemma 8.3, which classifies all minimal words satisfying Equation (6.2). This implies Theorem 4.15, which is proven in Section 8. The method of the proof of Theorem 4.15 is carried out on a specific example in Section 9.1.

**5 From symplectic semitoric manifolds to helices**

In this section we give the details of the construction of the semitoric helix outlined in Section 4.3. To do this, we need the following result of Vũ Ngọc, adapted slightly to fit the present situation.

**Theorem 5.1** (Follows from [20, Theorem 3.8]). If \((M, \omega, F)\) is a symplectic semitoric manifold and \( U \subset F(M) \) is simply connected, open as a subset of \( F(M) \), and contains no values of focus-focus points of \( F \) then there exists a smooth function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that \( f \circ F \) is a momentum map for a Hamiltonian \( \mathbb{T}^2 \)-action on \( F^{-1}(U) \) and \( f \) fixes the first component, i.e. there exists a function \( f^{(2)} : \mathbb{R}^2 \rightarrow \mathbb{R} \) such that \( f(x, y) = (x, f^{(2)}(x, y)) \).
Such a function \( f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is known as a \textit{straightening map} for the symplectic semitoric manifold \((M, \omega, F)\).

### 5.1 Intrinsic construction of the helix

Let \((M, \omega, F = (J, H))\) be a compact symplectic semitoric manifold and we will construct the associated semitoric helix, \(\text{hxl}(M, \omega, F)\). The images under \(F\) of the elliptic-regular and elliptic-elliptic singular points all lie in the boundary \(\partial F(M)\) and there are finitely many focus-focus points, whose images lie in the interior \(\text{int}(F(M))\) (see [20]). Choose a set \(U' \subset F(M)\) such that

1. \(U'\) is open as a subset of \(F(M)\);
2. \(U'\) contains \(\partial F(M)\);
3. \(U'\) does not contain the image of any focus-focus point;
4. \(U'\) has fundamental group \(\mathbb{Z}\).

This is possible because \(F(M)\) is simply connected [20, Theorem 3.4] and compact (by assumption). For instance, \(U'\) could be chosen to be the set of all points in \(F(M)\) less than a distance of \(\varepsilon\) from the boundary for a sufficiently small \(\varepsilon > 0\). Let \(\ell \subset F(M)\) be any line segment starting from a point in \(F(M) \setminus U'\) and ending outside \(F(M)\) which intersects \(\partial F(M)\) in exactly one connected component and does not include any singular points of maximal rank of \(F(M)\). Let \(U = U' \setminus \ell\). We call such a subset a \textit{helix neighborhood} for \((M, \omega, F)\), see the first step of Figure 1.

By Theorem 5.1 there exists a straightening map \(f: \mathbb{R}^2 \rightarrow \mathbb{R}^2\) so that \(\mu = f \circ F\) is the momentum map for a Hamiltonian \(T^2\)-action on \(F^{-1}(U)\). Thus, \(f(\partial F(M) \cap U)\) is piecewise linear of finitely many segments each with rational slope, because it is the image of the elliptic-regular and elliptic-elliptic singular points of \((F^{-1}(U), \omega, \mu)\) and this system has only finitely many elliptic-elliptic fixed points. Let \(d \in \mathbb{Z}\) be one less than the number of segments so that there are \(d + 1\) segments in this piecewise linear curve and let \(v_0, \ldots, v_d \in \mathbb{Z}^2\) be the consecutive primitive vectors normal to these segments facing towards the interior of \(f(U)\), numbered so that \(v_0, \ldots, v_{d-1}\) are arranged in counter-clockwise order, as shown in the last step of Figure 1.

The relationship between \(v_0\) and \(v_d\) is determined by the monodromy from the focus-focus points of the system. In [20] Vű Ngoc studies the monodromy effect of focus-focus points on toric momentum maps defined on the preimage of the momentum map image minus a few "cuts" that remove the focus-focus points and keep the set simply connected. The proof holds for other simply connected sets, such as the set \(U\), and in this case implies that \(v_d = Tcv_0\) because the set \(U\) loops around all \(c\) focus-focus points of the system.

Finally, by Definition 4.3 part (3) \(v_0, \ldots, v_d\) extend to a unique semitoric helix \(\mathcal{H} = (d, c, \{v_i\}_{i \in \mathbb{Z}})\). We say that \(\mathcal{H}\) is associated to the given symplectic semitoric manifold \((M, \omega, F)\).
Now we must show that the semitoric helix constructed in this was is the unique one associated to $M$. That is, we show the helix does not depend on the choices of open set $U'$, line segment $\ell$, and straightening map $f$ made during the construction.

**Lemma 5.2.** There is precisely one semitoric helix associated to each symplectic semitoric manifold

**Proof.** Let $(M, \omega, F)$ be a symplectic semitoric manifold with $d$ elliptic-elliptic points and $c$ focus-focus points. Any semitoric helix produced from the above construction must have length $d$ and complexity $c$. Now let $\mathcal{H}_j = (c, d, \{v_i^j\}_{i \in \mathbb{Z}})$ be a semitoric helix constructed from $(M, \omega, F)$ as above using a set $U'_j$, line segment $\ell_j$, and straightening map $f_j$ for $j = 1, 2$. We will show $\mathcal{H}_1 = \mathcal{H}_2$.

We may assume that $U'_1 = U'_2$ by replacing each with $U' = U'_1 \cap U'_2$ and using the restricted straightening maps. Now $U_1 = U' \setminus \ell_1$ and $U_2 = U' \setminus \ell_2$ and, assuming $U \cap \ell_1 \neq U \cap \ell_2$, the set $U_1 \cap U_2$ has two connected components (if $U \cap \ell_1 = U \cap \ell_2$ the remainder of the proof simplifies). Denote these two components by $A$ and $B$ ordered so that $v_0^1, \ldots, v_k^1$ are the inwards pointing normal vectors of the boundary of $f_1(A)$ and $v_{k+1}^1, \ldots, v_d^1$ are the inwards pointing normal vectors of $f_1(B)$.

Since $A \subset U_j$ for $j = 1, 2$ we see $f_j|_A \circ F : F^{-1}(A) \to \mathbb{R}^2$ is a toric momentum map for $j = 1, 2$. Thus, by [20, Theorem 3.8] there exists $k_A \in \mathbb{Z}$ and $x_A \in \mathbb{R}^2$ such that

$$f_1|_A = T^{k_A} \circ f_2|_A + x_A \quad (5.1)$$

and similarly there exists $k_B \in \mathbb{Z}$ and $x_B \in \mathbb{R}^2$ such that

$$f_1|_B = T^{k_B} \circ f_2|_B + x_B. \quad (5.2)$$

Thus, $v_i^1 = T^{k_A} v_{i+d-k}^2$ for $i = 0, \ldots, k$ and $v_k^1 = T^{k_B} v_{i-k+1}^2$ for $i = k+1, \ldots, d$. Now, $\{v_i^2\}_{i \in \mathbb{Z}}$ is equivalent in $\mathbb{Z}^2/\sim$ to $\{\tilde{v}_i^2\}_{i \in \mathbb{Z}}$ defined by $\tilde{v}_i^2 = T^{k_A} v_{i+d-k}^2$ and thus $v_i^1 = \tilde{v}_i^2$ for $i = 0, \ldots, k$ and

$$v_k^1 = T^{k_B} v_{i-k}^2 = T^{k_B} T^{-c} v_{i-k+1}^2 = T^{k_B} T^{-c} T^{-k_A} v_{(i-k+1)+d-k} = T^{k_B-k_A-c} \tilde{v}_i^2$$

for $i = k, \ldots, d$ because $\mathcal{H}_2$ has complexity $c$. Thus, $v_i^1 = \tilde{v}_i^2$ for all $i \in \mathbb{Z}$ if $k_B - k_A = c$, in which case the proof is complete.

By Equation (5.1) $f_1$ and $T^{k_A}$ differ by a translation on $A$ so $f_1|_B = T^{c}(T^{k_A} \circ f_2)|_B + x'_B$ for some $x'_B \in \mathbb{R}^2$ because this is precisely the effect of the monodromy of the set $U_1 \cap U_2$ encircling all of the $c$ focus-focus points of the system (see [20, Theorem 3.8]). Combining this with Equation (5.2) we see that $T^{c} \circ T^{k_A} \circ f_2|_B = T^{k_B} \circ f_2|_B + x''_B$ for some $x''_B \in \mathbb{R}^2$ and thus $k_B - k_A = c$ as desired. \hfill $\Box$

Recall Proposition 2.3, that the helix is an invariant of semitoric isomorphism type.

**Proof of Proposition 2.3.** Let $(M, \omega, F)$ and $(M', \omega', F')$ be symplectic semitoric manifolds and let $\phi : M \to M'$ be a semitoric isomorphism, so there exists a smooth map $f : \mathbb{R}^2 \to \mathbb{R}$ with $\partial f / \partial y \neq 0$ such that $\phi^*F' = (J, f(J, H))$. This implies they must each have the same
number of focus-focus points and elliptic-elliptic points. Let \( d \in \mathbb{Z} \) be the number of elliptice-elliptic points and let \( c \in \mathbb{Z} \) be the number of focus-focus points. Let \( U \subset F'(M') \) be a helix neighborhood for \((M', \omega', F')\), which is to say it is an open subset that can be used to construct the helix associated to \((M', \omega', F')\) as is done above, and let \( g: \mathbb{R}^2 \to \mathbb{R} \) be a straightening map for \( U \). This means there exists some \( g^{(2)}: \mathbb{R}^2 \to \mathbb{R} \) such that \( g(x, y) = (x, g^{(2)}(x, y)) \) and \( g \circ F \) is a toric momentum map on \( F^{-1}(U) \). The semitoric helix associated to \((M', \omega', F')\) is \( \mathcal{H} = (d, c, \{v_i\}_{i \in \mathbb{Z}}) \) where \( v_0, \ldots, v_{d-1} \) are the inwards pointing normal vectors of the piecewise linear boundary of \( g(U) \) and \( v_i \) for \( i < 0 \) and \( i \geq d \) is determined by Definition 4.3 part (3) from the other vectors and the complexity.

The map \( \phi \) descends to the map \( \tilde{\phi}: F(M) \to F'(M') \) given by \( \tilde{\phi}(x, y) = (x, f(x, y)) \) so the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{\phi} & M' \\
\downarrow{F} & & \downarrow{F'} \\
F(M) & \xrightarrow{\tilde{\phi}} & F'(M')
\end{array}
\]

The set \( \tilde{\phi}^{-1}(U) \) is a helix neighborhood for \((M, \omega, F)\). Let \( \tilde{g} = g \circ \tilde{\phi} \) and notice that \( \tilde{g} \) is a straightening map for \( \tilde{\phi}^{-1}(U) \subset F(M) \). Indeed, \( \tilde{g} \) clearly preserves the first component (it is the composition of maps which preserve the first component) and the second component of \( \tilde{g} \circ F: M \to \mathbb{R} \), which is \( g^{(2)}(J, f(J, H)) \), has \( 2\pi \)-periodic Hamiltonian flow because \( g^{(2)}(J, f(J, H)) = \phi^*(g^{(2)}(J', H')) \), \( \phi \) is a symplectomorphism, and \( g \) is a straightening map for \((M', \omega', F')\) so \( g^{(2)}(J', H') \) has \( 2\pi \)-periodic flow. The inwards pointing normal vectors of the piecewise linear portion of the boundary of \( g(\tilde{\phi}^{-1}(U)) \) generate the helix for \((M, \omega, F)\), which we denote \( \mathcal{H} \). Thus, \( \mathcal{H} = \mathcal{H}' \) because \( g(\tilde{\phi}^{-1}(U)) = g(U) \), and since the helix constructed in this way is unique by Lemma 5.2 the helix for \((M, \omega, F)\) agrees with the helix for \((M', \omega', F')\). \( \square \)

To prove that each possible semitoric helix is associated to some symplectic semitoric manifold we need to invoke the classification of symplectic semitoric manifolds, particularly the semitoric polygon invariant.

### 5.2 Delzant semitoric polygons

Here we quickly review the definition of a Delzant semitoric polygon from [17] so we can explain the relationship between semitoric polygons and the semitoric helix.

Let \( \pi: \mathbb{R}^2 \to \mathbb{R} \) denote projection onto the first component and for any \( \lambda \in \mathbb{R} \) let \( \ell_\lambda = \pi^{-1}(\lambda) \). A weighted polygon of complexity \( c \in \mathbb{Z}_{\geq 0} \) is a triple \( \Delta_w = (\Delta, (\ell_\lambda)_{j=1}^c, (\epsilon_j)_{j=1}^c) \) where

1. \( \Delta \subset \mathbb{R}^2 \) is a convex, closed (possibly non-compact), rational polygon;
2. \( \epsilon_j \in \{\pm 1\} \) for \( j = 1, \ldots, c \);
3. \( \lambda_j \in \text{int}(\pi(\Delta)) \) for \( j = 1, \ldots, c \);
4. \( \lambda_1 < \lambda_2 < \ldots < \lambda_c \).
Let $G_c = \{\pm 1\}^c$ and $\mathcal{G} = \{(T^t)^k : k \in \mathbb{Z}\}$ where $T^t$ is the transpose of the matrix $T$ given in Equation (4.1). Given $k \in \mathbb{Z}$ and any vertical line $\ell = \ell_\lambda$, $\lambda \in \mathbb{R}$, define $t^\ell_k : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$t^\ell_k(x, y) = \begin{cases} (x, y), & x \leq \lambda \\ (x, k(x - \lambda) + y), & x > \lambda \end{cases}$$

and for $\vec{u} = (u_j)_{j=1}^c \in \mathbb{Z}^c$ and $\vec{\lambda} = (\lambda_j)_{j=1}^c \in \mathbb{R}$ let $t_{\vec{u},\vec{\lambda}} = t^u_{\lambda_1} \circ \cdots \circ t^u_{\lambda_c}$. The group $G_c \times \mathcal{G}$ acts on a weighted polygon by

$$((\epsilon'_j)_{j=1}^c, (T^t)^k) \cdot \left(\Delta, (\ell_\lambda)_{j=1}^c, (\epsilon_j)_{j=1}^c\right) = \left(t_{\vec{u},\vec{\lambda}} \circ (T^t)^k(\Delta), (\ell_\lambda)_{j=1}^c, (\epsilon'_j \epsilon_j)_{j=1}^c\right),$$

where $\vec{u} = ((\epsilon_j - \epsilon'_j)/2)_{j=1}^c$. A weighted polygon is called admissible if this action of $G_c \times \mathcal{G}$ preserves its convexity. Let $\text{WPoly}_c(\mathbb{R}^2)$ denote the space of all admissible weighted polygons of complexity $c \in \mathbb{Z}_{\geq 0}$. An element of $\text{WPoly}_c(\mathbb{R}^2) / G_c \times \mathcal{G}$ is known as a semitoric polygon.

Let $\Delta_w = (\Delta, (\ell_\lambda)_{j=1}^c, (\epsilon_j)_{j=1}^c)$ be a weighted polygon of complexity $c \in \mathbb{Z}_{\geq 0}$. Let $p \in \Delta$ be a vertex and let $v, w \in \mathbb{Z}^2$ be the inwards pointing normal vectors to the edges adjacent to $p$ of minimal length ordered so that $\det(v, w) > 0$. Such vectors exist because $\Delta$ is rational. The vertex $p$ satisfies:

1. the Delzant condition if $\det(v, w) = 1$;
2. the hidden Delzant condition if $\det(T v, w) = 1$;
3. the fake condition if $\det(T v, w) = 0$.

Let

$$\partial^\text{top} \Delta = \{(x, y) : x \in \pi(\Delta), y = \sup\{y_0 \in \mathbb{R} : (x, y_0) \in \Delta\}\}$$

denote the top boundary of $\Delta$.

**Definition 5.3.** ([17]) Let $[\Delta_w] \in \text{WPoly}_c(\mathbb{R}^2) / G_c \times \mathcal{G}$ be a semitoric polygon and suppose that $\Delta_w$ is a representative of the form $\Delta_w = (\Delta, (\ell_\lambda)_{j=1}^c, (+1)_{j=1}^c)$. Then $[\Delta_w]$ is a Delzant semitoric polygon if

1. $\Delta \cap \ell_\lambda$ is either compact or empty for all $\lambda \in \mathbb{R}$;
2. each point in $\partial^\text{top} \Delta \cap \ell_\lambda$ satisfies either the hidden Delzant or fake condition, and is hence known as a hidden or fake corner, respectively, for $j = 1, \ldots, c$;
3. all other vertices of $\Delta$ satisfy the Delzant condition and are known as Delzant corners.

We say $[\Delta_w]$ is compact whenever $\Delta$ is compact for one, and hence all, representatives.

Every symplectic semitoric manifold determines a Delzant semitoric polygon [17, 20], which is compact if the manifold is compact.

### 5.3 Construction of the helix from the polygon invariant

The helix can also be constructed from the V\~{u} Ngoc polygon associated to the symplectic semitoric manifold [20]. Here we give a brief outline of that construction, shown in Figure 4. Let $(M, \omega, F)$ be a compact symplectic semitoric manifold.
Figure 4: The helix can be recovered from the semitoric polygon invariant by "unwinding" the polygon to correct for the effect of the focus-focus points and then removing the resulting repeated vectors.

Step 1: **Construct polygon:** Associated to \((M, \omega, F)\) is a semitoric polygon

\[ [\Delta_w] = [(\Delta, (\ell_{\lambda_j})^c_{j=1}, (\epsilon_j^c)_{j=1}^c)] \]

as in [20, Theorem 3.9] and described above in Section 5.2.

Step 2: **Construct semitoric fan:** \(\Delta\) is rational, so take the collection of inwards pointing integer normal vectors \(w_0, \ldots, w_{m-1}\) of minimal length to its edges (this is known as a semitoric fan, see [10]). These can be chosen so that the corner between \(w_{m-1}\) and \(w_0\) is not on any of the vertical lines \(\ell_{\lambda_j}\);

Step 3: **Correct for monodromy effect:** Each consecutive pair of vectors \((w_j, w_{j+1})\) is labeled as either fake, hidden, or Delzant depending on the type of corner of \(\Delta\) it corresponds to. For each \(j\) such that \((w_j, w_{j+1})\) is either hidden or fake replace \(w_{j+1}, \ldots, w_{m-1}\) by \(T w_{j+1}, \ldots, T w_{m-1}\).

Label the new list of vectors \(w'_0, \ldots, w'_{m-1}\);

Step 4: **Remove repeated vectors:** Now each pair \((w'_i, w'_{i+1})\) either satisfies \(\det(w'_i, w'_{i+1}) = 1\) or \(w'_i = w'_{i+1}\). For each \(j\) such that \(w'_j = w'_{j+1}\) remove \(w'_{j+1}\) from the list and when all repeated vectors are removed denote the remaining vectors by \(v_0, \ldots, v_{d-1}\). Notice \(\det(v_i, v_{i+1}) = 1\) for all \(i = 0, \ldots, d - 2\);

Step 5: **Extend to helix:** By condition (3) of Definition 4.3 there exists a unique helix of length \(d\) and complexity \(c\) (the number of focus-focus points of the original symplectic semitoric manifold) with the given \(v_0, \ldots, v_{d-1}\) from the previous step.

**Remark 5.4.** Let \([(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^c)]\) be a compact semitoric polygon which has no hidden corners and such that all of the fake corners are consecutive (while traversing the boundary of \(\Delta\)) and let \(v_0, \ldots, v_{d-1}\) be the primitive integral inwards pointing normal vectors to every edge of \(\Delta\) which is adjacent to at least one Delzant corner. The associated semitoric helix is the unique helix of length \(d\) and complexity \(c\) with the given \(v_0, \ldots, v_{d-1}\).
Remark 5.5. Proposition 2.3 also follows from the fact that the semitoric polygon invariant is an invariant of the semitoric isomorphism type and the above construction of the helix from the semitoric polygon, but we have chosen to prove Proposition 2.3 in a way which is independent of the existence of the semitoric polygon invariant.

5.4 Surjectivity of the helix map

Lemma 5.6. Given any semitoric helix $\mathcal{H}$ there exists a symplectic semitoric manifold $(M, \omega, F)$ such that $\text{hlx}(M, \omega, F) = \mathcal{H}$.

Proof. Let $\mathcal{H} = (d, c, [\{v_i\}_{i \in \mathbb{Z}}])$ be a semitoric helix. Define a collection of vectors $w_0, \ldots, w_{d+c}$ by $w_i = v_i$ for $i = 0, \ldots, d-1$ and $w_i = T^{i-d+1}v_{d-1}$ for $i = d, \ldots, d+c$. Then $\det(w_i, w_{i+1}) = 1$ for $i = 0, \ldots, d-1$, $\det(Tw_i, w_{i+1}) = \det(Tw_i, Tw_{i}) = 0$ for $i = d, \ldots, d+c-1$, and $w_0 = w_{d+c}$ by the periodicity requirement on the helix $\mathcal{H}$. The vectors $w_0, \ldots, w_{d+c-1}$ are arranged counter-clockwise so there exists a polygon $\Delta \subset \mathbb{R}^2$ with $d+c$ edges which has these as inwards pointing normal vectors. The polygon $\Delta$ has $d$ Delzant corners $c$ fake corners, and since $T$ does not change the $y$-value of a vector we see that either all of the fake corners are on the top boundary of $\Delta$ or all of the fake corners are on the bottom boundary of $\Delta$. Let $\lambda_i$ be the horizontal position of the $i$th fake corner and we may number these so that $\lambda_1 < \lambda_2 < \ldots < \lambda_c$ since each vertical line intersects the top and bottom boundaries at most once each. If the fake corners are on the top boundary let $\epsilon_j = +1$ for $j = 1, \ldots, c$ and otherwise let $\epsilon_j = -1$ for $j = 1, \ldots, c$. Then, $\Delta_w = [(\Delta, (\ell_{\lambda_j})_{j=1}^c, (\epsilon_j)_{j=1}^c)]$ is a Delzant semitoric polygon with associated semitoric helix $\mathcal{H}$. By [17, Theorem 4.6] there exists a symplectic semitoric manifold with $[\Delta_w]$ as its semitoric polygon.

Remark 5.7. Lemma 5.6 shows that the map $\text{hlx}: S_{\text{ST}} \rightarrow S_{\text{H}}$ is surjective by producing a right inverse, but this map is not injective. In terms of the Pelayo-Vũ Ngọc invariants this is because the helix does not encode any information about the Taylor series invariant, the volume invariant, the twisting index, the horizontal position of the focus-focus points, or the lengths of the edges of the semitoric polygon.

6 Semitoric helices and $\text{SL}_2(\mathbb{Z})$

In this Section we prove Proposition 4.12, which is the tool we use to translate questions about semitoric helices into questions about words on letters $S$ and $T$.

Lemma 6.1. Given any semitoric helix $\mathcal{H} = (d, c, [\{v_i\}_{i \in \mathbb{Z}}])$ there exists a list of integers $(a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d$ such that

$$a_i \mod d v_{i+1} = v_i + v_{i+2}$$

(6.1)

for all $i \in \mathbb{Z}$. Furthermore, given $v_0$, $v_1$, and $(a_0, \ldots, a_{d-1})$ the helix can be recovered.
Proof. Let $\mathcal{H} = (d, c, \{v_i\}_{i \in \mathbb{Z}})$. Let $A_i = [v_i, v_{i+1}]$ and write $v_{i+2}$ in the $(v_i, v_{i+1})$ basis as $v_{i+2} = b_i v_i + a_i v_{i+1}$, for $a_i, b_i \in \mathbb{Z}$. Thus,

$$A_i \begin{pmatrix} 0 & b_i \\ 1 & a_i \end{pmatrix} = A_{i+1}$$

and since $A_i, A_{i+1} \in \text{SL}_2(\mathbb{Z})$ we see the determinant of each side is 1 so $b_i = -1$ and $v_{i+2} + v_i = a_i v_{i+1}$ as desired. Conversely, given $v_0, v_1$ and $(a_0, \ldots, a_{d-1})$ the helix can be recovered by using the recurrence relation Equation (6.1).

**Definition 6.2.** The $(a_0, \ldots, a_{d-1}) \in \mathbb{Z}^2$ in Lemma 6.1 are the associated integers to $\mathcal{H}$.

Recall $\widetilde{\text{SL}_2(\mathbb{R})}$ denotes the universal cover of $\text{SL}_2(\mathbb{R})$ with base point at the identity, so $\alpha \in \widetilde{\text{SL}_2(\mathbb{R})}$ is a continuous map $\alpha: [0, 1] \to \text{SL}_2(\mathbb{R})$ satisfying $\alpha(0) = I$. The group $G$ is isomorphic to the preimage of $\text{SL}_2(\mathbb{Z})$ in $\widetilde{\text{SL}_2(\mathbb{R})}$ [10, Proposition 3.7] via the homomorphism $\rho: G \to \widetilde{\text{SL}_2(\mathbb{Z})}$ generated by its action on $S$ and $T$ given in Equation 4.2. The operation in $G$ is concatenation of paths. If $\alpha, \beta \in \widetilde{\text{SL}_2(\mathbb{R})}$ then $\alpha, \beta: [0, 1] \to \text{SL}_2(\mathbb{R})$ and we define $\alpha \beta \in \widetilde{\text{SL}_2(\mathbb{R})}$ by

$$\alpha \beta(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2} \\ \alpha(1) \beta(2t - 1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

That is, the path $\alpha \beta$ is obtained by traveling first along the path $\alpha$ and then along the path produced by multiplying each element of the path $\beta$ on the left by $\alpha(1)$. It turns out that the path produced by traveling first along $\beta$ and then along $\alpha$ multiplied on the right by $\beta(1)$ is homotopic to $\alpha \beta$. The next result follows from the fact that the fundamental group of a topological group is abelian (see [9, Section 3.C, Exercise 5]), but we prove it here for completeness.

**Lemma 6.3.** If $\alpha, \beta \in \widetilde{\text{SL}_2(\mathbb{R})}$ then the paths in $\text{SL}_2(\mathbb{R})$ from $I$ to $\alpha(1) \beta(1)$ given by

$$\gamma_0(t) = \begin{cases} \beta(2t), & 0 \leq t \leq \frac{1}{2} \\ \alpha(2t - 1) \beta(1), & \frac{1}{2} < t \leq 1 \end{cases}$$

and

$$\gamma_1(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2} \\ \alpha(1) \beta(2t - 1), & \frac{1}{2} < t \leq 1 \end{cases}$$

are homotopic.

**Proof.** A continuous homotopy between them is given by

$$\gamma_s(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{s}{2} \\ \alpha(s) \beta(2t - s), & \frac{s}{2} \leq t \leq \frac{1+s}{2} \\ \alpha(2t-1) \beta(1), & \frac{1+s}{2} \leq t \leq 1 \end{cases}$$

for $0 \leq s \leq 1$, which is shown in Figure 5. Indeed, $\gamma_s$ is continuous because $\gamma_s(s/2) = \alpha(s)$ since $\beta(0) = I$ and $\gamma_s((1+s)/2) = \alpha(s) \beta(1)$. It is left to the reader to check that it is a homotopy from $\gamma_0$ to $\gamma_1$. \qed
Recall the map \( \text{pr}: \text{SL}_2(\mathbb{R}) \to (\mathbb{R}^2)^* \), where \((\mathbb{R}^2)^* = \mathbb{R}^2 \setminus \{(0,0)\}\), given by \(\text{pr}([v_1, v_2]) = v_1\). Since \(\pi_1(\text{SL}_2(\mathbb{R})) \cong \pi_1((\mathbb{R}^2)^*) \cong \mathbb{Z}\) and

\[
\text{pr}\begin{pmatrix}
\cos(2\pi t) & -\sin(2\pi t) \\
\sin(2\pi t) & \cos(2\pi t)
\end{pmatrix} = \begin{pmatrix}
\cos(2\pi t) \\
\sin(2\pi t)
\end{pmatrix}
\]

for \(0 \leq t \leq 1\) we see \(\text{pr}\) sends a generator of \(\pi_1(\text{SL}_2(\mathbb{R}))\) to a generator of \(\pi_1((\mathbb{R}^2)^*)\), so \(\text{pr}^*: \pi_1(\text{SL}_2(\mathbb{R})) \to \pi_1((\mathbb{R}^2)^*)\) is an isomorphism.

**Definition 6.4.** Let \(\theta: (\mathbb{R}^2)^* \to [0, 2\pi)\) be the usual angle coordinate from polar coordinates on \(\mathbb{R}^2\) and let \(R_\phi: \mathbb{R}^2 \to \mathbb{R}^2\) be rotation by the angle \(\phi \in [0, 2\pi)\). We say a path \(\gamma: [0, 1] \to (\mathbb{R}^2)^*\) **travels counter-clockwise at most one full rotation** if there exists some \(\phi \in [0, 2\pi)\) such that \(t \mapsto \theta(R_\phi(\gamma(t)))\) is an increasing function for \(t \in (0, 1)\).

**Lemma 6.5.** Let \(\mathcal{H} = (d, c, \{v_i\}_{i \in \mathbb{Z}})\) be a semitoric helix and let \(A_0 = [v_0, v_1]\). If \(\sigma \in G\) is given by

\[
\sigma = G \cdot ST^{a_0} \cdots ST^{a_{d-1}}
\]

then \(\text{pr}(A_0 \rho(\sigma))\) is homotopic to a path from \(v_0\) to \(v_{d-1}\) which travels counter-clockwise at most one full rotation.

**Proof.** Let \(A_i = [v_i, v_{i+1}]\) for \(1 \leq i \leq d - 1\) and recall \(A_i = A_{i-1}ST^{a_{i-1}}\). Thus,

\[
\text{pr}(A_{i-1} \rho(ST^{a_{i-1}}))
\]

is a path from \(v_i\) to \(v_{i+1}\) which is homotopic to

\[
\gamma_i(t) = \text{pr}\left(A_{i-1} \begin{pmatrix}
\cos \left( \frac{\pi t}{2} \right) & -\sin \left( \frac{\pi t}{2} \right) + ta_{i-1} \cos \left( \frac{\pi t}{2} \right) \\
\sin \left( \frac{\pi t}{2} \right) & \cos \left( \frac{\pi t}{2} \right) + ta_{i-1} \sin \left( \frac{\pi t}{2} \right)
\end{pmatrix} \right) = \cos \left( \frac{\pi t}{2} \right) v_{i-1} + \sin \left( \frac{\pi t}{2} \right) v_i
\]

for \(0 \leq t \leq 1\). The path \(\gamma_i\) travels only counter-clockwise at most one full rotation from \(v_{i-1}\) to \(v_i\) so the composition of paths \(\gamma_1, \ldots, \gamma_{d-1}\) travels counter-clockwise from \(v_0\) to \(v_{d-1}\). The result follows because \(v_0, \ldots, v_{d-1}\) are arranged in counter-clockwise order.

\[\square\]
Lemma 6.6. The integers \((a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d\) are associated to a semitoric helix of complexity \(c \geq 0\) if and only if

\[
ST^{a_0} \ldots ST^{a_{d-1}} =_G S^4 X^{-1} T^c X
\]  

(6.2)

for some \(X \in G\). If \(\mathcal{H} = (d, c, [\{v_i\}_{i \in \mathbb{Z}}])\) is a semitoric helix with associated integers \((a_0, \ldots, a_{d-1})\) then \(A_0 = [v_0, v_1]\) satisfies \(X =_G A_0\).

Proof. Let \(A_i = [v_i, v_{i+1}]\). By Lemma 6.1 and the fact that

\[
\begin{pmatrix} 0 & -1 \\ 1 & a_i \end{pmatrix} = ST^{a_i}
\]

we find that \(A_{i+1} =_{\text{SL}_2(\mathbb{Z})} A_i ST^{a_i}\) for all \(i \in \mathbb{Z}\). We conclude that

\[
A_d =_{\text{SL}_2(\mathbb{Z})} A_0 ST^{a_0} \ldots ST^{a_{d-1}}
\]

and since \(T^c A_0 =_{\text{SL}_2(\mathbb{Z})} A_d\) this implies that

\[
ST^{a_0} \ldots ST^{a_{d-1}} =_{\text{SL}_2(\mathbb{Z})} A_0^{-1} T^c A_0.
\]

Since \(S^4\) generates the kernel of the projection \(G \to \text{SL}_2(\mathbb{Z})\) we have that

\[
ST^{a_0} \ldots ST^{a_{d-1}} =_G S^{4k} A_0^{-1} T^c A_0
\]

for some \(k \in \mathbb{Z}\). This is because \(S^2\) is in the center of \(\text{SL}_2(\mathbb{Z})\) and when reducing an element of \(\text{SL}_2(\mathbb{Z})\) we can assume that the relation \(S^4 =_{\text{SL}_2(\mathbb{Z})} I\) is not used until the last step. Rearranging we have

\[
A_0 ST^{a_0} \ldots ST^{a_{d-1}} A_0^{-1} T^{-c} =_G S^{4k}
\]

(6.3)

To complete the proof we must only show that \(k = 1\) in Equation (6.3).

Let \(\sigma, \eta \in G\) be given by

\[
\sigma =_G ST^{a_0} \ldots ST^{a_{d-1}} \quad \text{and} \quad \eta =_G A_0 \sigma A_0^{-1} T^{-c}
\]

so Equation (6.3) becomes \(\eta =_G S^{4k}\). Since \(W(S^{4k}) = k\), it is sufficient to show that \(W(\eta) = 1\). Recall \(\pi_1(\text{SL}_2(\mathbb{R}))\) is abelian so the class of a loop is well-defined without fixing the basepoint. By Lemma 4.11, \(\eta =_{\text{SL}_2(\mathbb{Z})} I\) implies that \(W(\eta) = \text{wind}(\rho(\eta))\). By Lemma 6.3 \(\rho(\eta)\) is homotopic to

\[
\left( \rho'(\eta) \right) (t) = \begin{cases} 
\rho(A_0^{-1})(4t), & 0 \leq t \leq \frac{1}{4} \\
(\rho(\sigma))(4t - 1) A_0^{-1}, & \frac{1}{4} \leq t \leq \frac{1}{2} \\
\sigma A_0^{-1} \left( (\rho(T^{-c}))(4t - 2) \right), & \frac{1}{2} \leq t \leq \frac{3}{4} \\
(\rho(A_0))(4t - 3) \sigma A_0^{-1} T^{-c}, & \frac{3}{4} \leq t \leq 1.
\end{cases}
\]

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Let \( \gamma_0 : [0, 1] \to \text{SL}_2(\mathbb{R}) \) be the path from \( A_0^{-1} \) to itself given by

\[
\gamma_0(t) = \begin{cases} 
(\rho(\sigma))(2t) A_0^{-1}, & 0 \leq t \leq 1/2 \\
\sigma A_0^{-1} \left( (\rho(T^{-c}))(2t - 1) \right), & 1/2 \leq t \leq 1. 
\end{cases}
\tag{6.4}
\]

The paths \( \gamma_0 \) and \( \rho'(\eta) \) are homotopic via the homotopy

\[
(\rho'(\eta))(t) = \begin{cases} 
\rho(A_0^{-1})\left(\frac{4t}{s}\right), & 0 \leq t \leq \frac{s}{4} \\
\left( (\rho(\sigma))\left(\frac{4t-s}{2-s}\right) \right) A_0^{-1}, & 2/4 \leq t \leq 1/2 \\
\sigma A_0^{-1} \left( (\rho(T^{-c}))(\frac{4t-2}{2-s}) \right), & 1/2 \leq t \leq \frac{4-s}{4} \\
\left( (\rho(A_0))\left(\frac{4t+s-4}{s}\right) \right) \sigma A_0^{-1} T^{-c}, & \frac{4-s}{4} \leq t \leq 1
\end{cases}
\]

for \( 0 < s \leq 1 \) where \( \gamma_0 \) is defined as above. Thus, to complete the proof we only must show \( \text{wind}(\gamma_0) = 1 \) where \( \gamma_0 \) is as in Equation (6.4). By Lemma 6.5, the path

\[
\text{pr}\left( (\rho(\sigma))(2\cdot) \right) : [0, 1/2] \to (\mathbb{R}^2)^* 
\]

is homotopic to a path which travels counter-clockwise at most one full rotation. The path

\[
\text{pr}\left( \sigma A_0^{-1} \rho(T^{-c})(2\cdot - 1) \right) : [1/2, 1] \to (\mathbb{R}^2)^*,
\]

travels only counter-clockwise and cannot cross the line \( \{ y = 0 \} \), so it completes at most one half-rotation. Since \( \sigma A_0^{-1} T^{-c} =_{\text{SL}_2(\mathbb{Z})} A_0^{-1} \), the path \( \gamma_0 \) thus circles the origin an integer number of times, so we conclude that \( \text{wind}(\text{pr}(\gamma_0)) = \text{wind}(\gamma_0) = 1 \). This completes the proof.

Proof of Proposition 4.12. Let \( \mathcal{H} = (d, c, \{v_i\}_{i \in \mathbb{Z}}) \) be a semitoric helix. Then there exists associated integers \( (a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d \) as in Definition 6.2 by Lemma 6.1. If \( \{w_i\}_{i \in \mathbb{Z}} \in \mathcal{H} \) then by Definition 4.3 there exists some \( k, \ell \in \mathbb{Z} \) such that \( v_i = T^k w_{i+\ell} \) for all \( i \in \mathbb{Z} \). In this case \( a_i v_{i+1} = v_i + v_{i+2} \) implies that \( a_i w_{i+1+\ell} = w_{i+\ell} + w_{i+2+\ell} \) and denoting \( a_j := a_{j \mod d} \) this implies that \( a_{i-\ell} w_{i+1} = w_i + w_{i+2} \). Thus, the associated integers for \( \{w_i\}_{i \in \mathbb{Z}} \) are given by \( (a_{-\ell}, a_{1-\ell}, \ldots, a_{d-1-\ell}) \) which agrees with those integers for \( \{v_i\}_{i \in \mathbb{Z}} \) up to cyclic permutation, as desired.

Suppose \( (a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d \) is a list of integers satisfying

\[
ST^{a_0} \cdots ST^{a_{d-1}} = G \ 4 \ X^{-1} T^{-c} X
\]

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Figure 6: The loop $\rho(\eta)$ from the proof of Lemma 6.6, which has winding number 1.

for some $c \in \mathbb{Z}_{>0}$. Let $A_0 \in \text{SL}_2(\mathbb{Z})$ be any matrix satisfying $X =_{\text{SL}_2(\mathbb{Z})} A_0$ and define $v_0, v_1 \in \mathbb{Z}^2$ so that $A_0 = [v_0, v_1]$. Then define $v_2, \ldots, v_{d-1}$ by $v_i = a_{i-2}v_{i-1} - v_{i-2}$ for $i = 2, \ldots, d - 1$. Use the relationship $v_{i+d} = T^c v_i$ to extend $v_0, \ldots, v_{d-1}$ to $\{v_i\}_{i \in \mathbb{Z}}$. Since $W(ST^{a_0} \ldots ST^{a_{d-1}}) = 1$, the vectors $v_0, \ldots, v_{d-1}$ are in counter-clockwise order and by construction $\det(v_i, v_{i+1}) = 1$ for all $i \in \mathbb{Z}$, so $\{v_i\}_{i \in \mathbb{Z}}$ is a semitoric helix with the prescribed associated integers.

If $\mathcal{H}$ and $\mathcal{H}'$ satisfy $\mathcal{H} = \pm \mathcal{H}'$ then they have the same associated integers since those integers are defined by a linear equation, Equation (6.1), which is invariant under the action of $-I$.

Conversely, suppose that $\mathcal{H} = (d, c, \{v_i\}_{i \in \mathbb{Z}})$ and $\mathcal{H}' = (d, c, \{v'_i\}_{i \in \mathbb{Z}})$ are semitoric helices of the same length, complexity, and associated integers. Let $A_0 = [v_0, v_1]$ and $A'_0 = [v'_0, v'_1]$ and let $a_0, \ldots, a_{d-1}$ be the common associated integers. Then

$$ST^{a_0} \ldots ST^{a_{d-1}} =_{\text{SL}_2(\mathbb{Z})} A_0^{-1} T^c A_0 \quad \text{and} \quad ST^{a_0} \ldots ST^{a_{d-1}} =_{\text{SL}_2(\mathbb{Z})} (A'_0)^{-1} T^c (A'_0)$$

so $A_0^{-1} T^c A_0 =_{\text{SL}_2(\mathbb{Z})} (A'_0)^{-1} T^c (A'_0)$. Thus $A'_0 A_0^{-1}$ commutes with $T^c$, so $A'_0 A_0^{-1} =_{\text{SL}_2(\mathbb{Z})} \pm T^k$ for some $k \in \mathbb{Z}$, and so we may assume $A_0 =_{\text{SL}_2(\mathbb{Z})} A'_0$ because $T^k$ is already included in the equivalence relation on helices. Finally, $v_0 = \pm v'_0$ and $v_1 = \pm v'_1$ implies $v_i = \pm v'_i$ by Equation (6.1).

\section{Standard form in $\text{PSL}_2(\mathbb{Z})$ and the winding number}

First we prove several lemmas which will be needed in the proof of Theorem 4.14.

\begin{lemma}
If $\sigma \in \mathbb{Z} * \mathbb{Z}$ is $S$-positive and $\sigma =_{\text{PSL}_2(\mathbb{Z})} I$ then $W(\sigma) \geq 0$ where $W(\sigma) = 0$ if and only if $\sigma$ is the empty word.
\end{lemma}

\begin{proof}
If $\sigma$ is the empty word then $W(\sigma) = 0$ and the claim holds. Assume $\sigma$ is not the empty word. Since $\sigma$ is $S$-positive up to conjugation by $T$, which does not change $W(\sigma)$,
we may write it as \( \sigma = \sum_{i} S^{a_i} \) for some \( a_0, \ldots, a_{d-1} \in \mathbb{Z} \). We define a sequence of vectors \( v_0, \ldots, v_{d-1} \in \mathbb{Z}^2 \) by choosing any \( v_0, v_1 \in \mathbb{Z}^2 \) with \( \det(v_0, v_1) = 1 \) and defining \( v_2, \ldots, v_{d-1} \) by \( v_{i+2} = -v_i + a_i v_{i+1} \) for \( i = 0, \ldots, d-3 \). Let \( \gamma : [0, 1] \to (\mathbb{R}^2)^* \) be a path which connects \( v_0, \ldots, v_{d-1} \) in order and travels only counter-clockwise. Then, \( W(\sigma) = \text{wind}(\gamma) \) and \( \text{wind}(\gamma) > 0 \) because \( \gamma \) must travel at least once around the origin to move only counter-clockwise and return to \( \gamma(0) \).

\[ \square \]

**Lemma 7.2.** If \( X \in \text{PSL}_2(\mathbb{Z}) \) then there exists some \( q \in \frac{1}{12} \mathbb{Z} \) such that \( w(\sigma) \geq q \) for all \( \sigma \in \mathbb{Z} \ast \mathbb{Z} \) which are \( S \)-positive and satisfy \( \sigma = \text{PSL}_2(\mathbb{Z}) X \).

**Proof.** Since \( S = \text{PSL}_2(\mathbb{Z}) S^{-1} \) every element of \( \text{PSL}_2(\mathbb{Z}) \) has a \( S \)-positive representation. Fix some \( S \)-positive \( \eta \in \mathbb{Z} \ast \mathbb{Z} \) such that \( \eta = \text{PSL}_2(\mathbb{Z}) X^{-1} \) and let \( q = -W(\eta) \). Let \( \sigma \) be any \( S \)-positive element of \( \mathbb{Z} \ast \mathbb{Z} \) such that \( \sigma = \text{PSL}_2(\mathbb{Z}) X \). Now \( \sigma \eta = \text{SL}_2(\mathbb{Z}) I \), so \( W(\sigma \eta) \geq 0 \) by Lemma 7.1. This means \( W(\sigma) + W(\eta) \geq 0 \) so \( W(\sigma) \geq q \) and the result follows because \( q \) does not depend on the choice of \( \sigma \).

The following is a special case of [10, Lemma 3.8], but for the sake of being self-contained we include the proof here.

**Lemma 7.3.** Suppose \( d > 0 \) and \( b, a_0, \ldots, a_{d-1} \in \mathbb{Z} \) are such that

\[
T^b S^{a_0} \ldots S^{a_{d-1}} =_{\text{PSL}_2(\mathbb{Z})} I. \tag{7.1}
\]

Then \( d > 1 \). If \( d = 2 \) then \( a_0 = 0 \) and \( a_1 = -b \). If \( d > 2 \) then \( a_i \in \{0, \pm 1\} \) for some \( 0 \leq i < d-2 \).

**Proof.** The group \( \text{PSL}_2(\mathbb{Z}) \) acts faithfully on the extended real line \( \mathbb{R} \cup \{\infty\} \) by \( T(x) = x + 1, S(x) = \frac{1}{x} \) for \( x \in \mathbb{R} \setminus \{0\} \), \( T(0) = 0 \), \( S(0) = \infty \), \( T(\infty) = \infty \), and \( S(\infty) = 0 \).

If \( d = 1 \), then Equation (7.1) states that \( T^b S^{a_0} =_{\text{PSL}_2(\mathbb{Z})} I \) which is impossible because \( T^b S^{a_0}(\infty) = b \neq \infty \).

If \( d = 2 \), then, after conjugating by \( S^{a_1} \), Equation (7.1) states that \( S^{a_1+b} S^{a_0} =_{\text{PSL}_2(\mathbb{Z})} I \) and evaluation of both sides at infinity gives \( S(a_1+b) = \infty \) which implies \( a_1 = -b \). Thus, \( S^2 S^{a_0} =_{\text{PSL}_2(\mathbb{Z})} I \) so \( a_0 = 0 \).

Finally, suppose \( d > 2 \) and \( a_i \notin \{0, \pm 1\} \) for all \( i = 0, \ldots, d-2 \). Conjugate Equation (7.1) by \( S^{a_{d-1}} \) to produce

\[
S^{a_{d-1}+b} S^{a_0} \ldots S^{a_{d-2}} =_{\text{PSL}_2(\mathbb{Z})} I. \tag{7.2}
\]

Let \( y = S^{a_0} \ldots S^{a_{d-2}}(\infty) \) so Equation (7.2) implies \( S^{a_{d-1}+b}(y) = \infty \). On the other hand, \( S^{a_{d-2}}(\infty) = 0 \) and if \( x \in \mathbb{R} \) with \( |x| < 1 \) then \( 0 < |S^k(x)| < 1 \) for any integer \( k \notin \{\pm 1, 0\} \), so

\[
|y| = |S^{a_0} \ldots S^{a_{d-3}}(0)| \in (0, 1),
\]

and thus

\[
S^{a_{d-1}+b}(y) = \frac{1}{y + a_{d-1} + b} \neq \infty,
\]

forming a contradiction.

\[ \square \]
Lemma 7.4. \( ST^{-n}S =_{\text{PSL}_2(\mathbb{Z})} (TST)^n \) for \( n \geq 0 \).

Proof. First \( STS =_{\text{PSL}_2(\mathbb{Z})} T^{-1}ST^{-1} \) implies \( S =_{\text{PSL}_2(\mathbb{Z})} TSTST \) so \( ST^{-1}S =_{\text{PSL}_2(\mathbb{Z})} TST \) since \( S =_{\text{PSL}_2(\mathbb{Z})} S^{-1} \). Now,

\[
ST^{-n}S =_{\text{PSL}_2(\mathbb{Z})} (ST^{-1}S)^n =_{\text{PSL}_2(\mathbb{Z})} (TST)^n
\]

for \( n > 0 \), and if \( n = 0 \) the claim reduces to \( S^2 =_{\text{PSL}_2(\mathbb{Z})} I \). \( \square \)

7.1 Standard form for elements of \( \text{PSL}_2(\mathbb{Z}) \)

In this section we prove Theorem 4.14.

Proof of Theorem 4.14. Let \( \sigma \in \mathbb{Z} \ast \mathbb{Z} \) any \( S \)-positive word with \( \sigma =_{\text{PSL}_2(\mathbb{Z})} X \). There are three steps to the reduction algorithm we will use on \( \sigma \), where Reduction 2 holds by Lemma 7.4. The reductions are:

- **Reduction 1** replace \( S^2 \) with \( I \);
- **Reduction 2** replace \( ST^{-n}S \) with \( (TST)^n \), for some \( n > 0 \);
- **Reduction 3** replace \( STS \) with \( T^{-1}ST^{-1} \);

To reduce the word we iteratively apply Reduction 1, Reduction 2, and Reduction 3 until no more are possible. Each of these reductions preserves the value of \( \sigma \) in \( \text{PSL}_2(\mathbb{Z}) \) and recall that the winding number cannot decrease indefinitely by Lemma 7.2. Reduction 1 and Reduction 2 reduce the winding number while Reduction 3 preserves the winding number but reduces the number of times \( S \) appears in the word, which is bounded below by zero. Thus, this process must terminate and after the reduction the word will be of the required form.

Now we will show uniqueness. Suppose that \( \sigma, \eta \in \mathbb{Z} \ast \mathbb{Z} \) with \( \sigma =_{\text{PSL}_2(\mathbb{Z})} \eta \) and \( \sigma =_{\mathbb{Z} \ast \mathbb{Z}} T^bST^{a_0}\ldots ST^{a_{d-1}}, \ \eta =_{\mathbb{Z} \ast \mathbb{Z}} T^{b'}ST^{a'_0}\ldots ST^{a'_{d'-1}} \), where \( a_i, a'_j > 1 \) for \( i = 0, \ldots, d-2 \) and \( j = 0, \ldots, d'-2 \). First assume \( \min(d, d') \leq 1 \), and in this case assume \( d \geq d' \).

If \( d' = 0 \) then \( T^{b-b'}ST^{a_0}\ldots ST^{a_{d-1}} =_{\text{PSL}_2(\mathbb{Z})} I \) which contradicts Lemma 7.3 unless \( d = 0 \), in which case \( T^{b-b'} =_{\text{PSL}_2(\mathbb{Z})} I \) so \( b = b' \). If \( d' = 1 \) then

\[
T^{b-b'}ST^{a_0}\ldots ST^{a_{d-1}}ST^0 =_{\text{PSL}_2(\mathbb{Z})} I
\]

so \( a_{d-1} - a'_0 \in \{0, \pm 1\} \) by Lemma 7.3. Consider the cases if \( d > 1 \). If \( a_{d-1} - a'_0 = 0 \) then

\[
T^{b-b'}ST^{a_0}\ldots ST^{a_{d-2}}S^2 =_{\text{PSL}_2(\mathbb{Z})} I
\]

which contradicts Lemma 7.3 after replacing \( S^2 \) by \( I \). If \( a_{d-1} - a'_0 = -1 \) then

\[
T^{b-b'}ST^{a_0}\ldots ST^{a_{d-2}}ST^{-1}S =_{\text{PSL}_2(\mathbb{Z})} I
\]

which contradicts Lemma 7.3 after replacing \( ST^{-1}S \) by \( TST \). Finally, if \( a_{d-1} - a'_0 = 1 \), then

\[
T^{b-b'}ST^{a_0}\ldots ST^{a_{d-2}-1}ST^{-1} =_{\text{PSL}_2(\mathbb{Z})} I
\]
which contradicts Lemma 7.3 unless \( a_{d-2} = 2 \). This process is repeated to conclude that 
\[ a_0 = \ldots = a_{d-2} = 2 \] 
so \( T^{b-b'}(ST^2)^{d-1} STS =_{\text{PSL}_2(\mathbb{Z})} I \) which implies \( T^{b-b'-1}ST^{-d} =_{\text{PSL}_2(\mathbb{Z})} I \).

By Lemma 7.3 this cannot hold. Thus, \( d = 1 \), in which case \( T^{b-b'}ST^{a_0-a_0'}S =_{\text{PSL}_2(\mathbb{Z})} I \), so \( b - b' = 0 \) and \( a_0 - a_0' = 0 \) by Lemma 7.3.

Finally, assume \( d, d' > 1 \) and assume that \( a_{d-1} \neq a_{d'-1}' \), otherwise cancel \( ST^{a_{d-1}} \) from both sides. In this case we see that \( \sigma \eta^{-1} =_{\text{PSL}_2(\mathbb{Z})} I \) implies
\[
T^b ST^{a_0} \ldots ST^{a_{d-1}-a_{d'-1}} ST^{-a_{d'-2}} \ldots ST^{-a_0} S =_{\text{PSL}_2(\mathbb{Z})} I
\]
and since some power of \( T \) must be in \( \{0, \pm 1\} \) by Lemma 7.3, but \( a_i, a_j' > 1 \) for \( i = 0, \ldots, d-2, j = 0, \ldots, d' - 2 \) since \( \sigma \) and \( \eta \) are in standard form, we conclude \( a_{d-1} - a_{d'-1}' \in \{0, \pm 1\} \). We have assumed \( a_{d-1} \neq a_{d'-1}' \) so \( a_{d-1} - a_{d'-1}' = \pm 1 \), and furthermore we can assume \( a_{d-1} - a_{d'-1}' = 1 \), otherwise exchange \( \sigma \) and \( \eta \). Then choose maximal \( k \in \mathbb{Z}_{\geq 0} \) such that \( a_{d-2} = a_{d-3} = \ldots = a_{d-2-(k-1)} = 2 \) where \( k = 0 \) if \( a_{d-2} \neq 2 \). If \( k < d - 1 \) then
\[
\sigma \eta^{-1} =_{\text{PSL}_2(\mathbb{Z})} T^b ST^{a_0} \ldots ST^{a_{d-2-k}} (ST^2)^k (STS) T^{-a_{d'-2}} \ldots ST^{-a_0} ST^{-b'}
\]
\[
=_{\text{PSL}_2(\mathbb{Z})} T^b ST^{a_0} \ldots ST^{a_{d-2-k}}(TST)^k ST^{-a_{d'-2-1}} \ldots ST^{-a_0} ST^{-b'}
\]
\[
=_{\text{PSL}_2(\mathbb{Z})} T^b ST^{a_0} \ldots ST^{a_{d-2-k}}(ST^{-1}ST^{-d}) \ldots ST^{-a_0} ST^{-b'}.
\]

Since \( a_{d-2-k} - 1 > 1 \) and \( -a_{d'-2-k} - 1 < -1 \) this expression cannot evaluate to the identity in \( \text{PSL}_2(\mathbb{Z}) \) by Lemma 7.3. Otherwise, \( k = d - 1 \), in which case
\[
\sigma \eta^{-1} =_{\text{PSL}_2(\mathbb{Z})} T^b(ST^2)^{d-1}(STS) T^{-a_{d'-2}} \ldots ST^{-a_0} ST^{-b'}
\]
\[
=_{\text{PSL}_2(\mathbb{Z})} T^{b-1}(TST)^{d-1} ST^{-a_{d'-2-1}} \ldots ST^{-a_0} ST^{-b'}
\]
\[
=_{\text{PSL}_2(\mathbb{Z})} T^{b-1} ST^{-a_{d'-2-d}} \ldots ST^{-a_0} ST^{-b'}.
\]

which again cannot evaluate to the identity in \( \text{PSL}_2(\mathbb{Z}) \) by Lemma 7.3. This completes the proof of uniqueness.

Lastly, we will show the standard form has minimal winding number. Let \( X \in \text{PSL}_2(\mathbb{Z}) \) and suppose \( \eta \in \mathbb{Z} \ast \mathbb{Z} \) is \( S \)-positive with \( \eta =_{\text{PSL}_2(\mathbb{Z})} X \). Then \( \eta \) can be reduced to the standard form of \( X \), denoted \( \overline{X} \in \mathbb{Z} \ast \mathbb{Z} \), by following the reduction algorithm at the beginning of the proof. Since each of \textbf{Reduction 1-Reduction 3} in the algorithm either preserves or reduces the winding number, \( W(\overline{X}) \leq W(\eta) \).

\[ \square \]

### 7.2 Standard forms and the winding number

Recall that given any \( X \in \text{PSL}_2(\mathbb{Z}) \) we denote by \( \overline{X} \in \mathbb{Z} \ast \mathbb{Z} \) the standard form of \( X \), as given in Theorem 4.14.

**Lemma 7.5.** If \( X \in \text{PSL}_2(\mathbb{Z}) \setminus \{T^k\}_{k \in \mathbb{Z}} \) then
\[
W(\overline{X}) + W(\overline{X}^{-1}) = \frac{1}{2}.
\]
Proof. Write \( X = T^b ST^{a_0} \ldots ST^{a_{d-1}} \) and since \( X \neq T^k \) for any \( k \in \mathbb{Z}, d > 0 \). Now, \( W\left( (X)^{-1} \right) = -W(X) \) where

\[
(X)^{-1} = S^{-1}T^{-a_{d-1}} \ldots S^{-1}T^{-a_0}S^{-1}T^{-b}.
\]

We will reduce \((X)^{-1}\) to standard form using the reduction steps in the proof of Theorem 4.14 and keep track of the winding number. Replacing each \( S^{-1} \) by \( S \) increases the winding number by \( d/2 \). Now replace each \( ST^{-a_i}S \) with \((TST)^{a_i}\) for each even index \( i \) which at most increases the odd indexed powers of \( T \) by 2. Since each \( a_i \geq 2 \) for \( i = 0, \ldots, d - 2 \) we do the replacement \( ST^{-a_i+2}S = (TST)^{a_i-2} \) for odd \( 0 < i < d - 3 \) and the replacement \( ST^{-a_i+1}S = (TST)^{a_i-1} \) for \( i = 1 \) and the highest odd \( i \leq d - 2 \). Thus we have now used \( ST^{-n}S = (TST)^n \), for varying values of \( n > 0 \), a total of \( d - 1 \) times decreasing \( W \) by \( 1/2 \) each time. The word produced in this way is now in standard form so it is equal to \( X^{-1} \) and

\[
W\left( (X)^{-1} \right) = -W(X) + \frac{d}{2} - \frac{d-1}{2} = -W(X) + \frac{1}{2}
\]
as desired. \( \Box \)

We can now prove that in many cases the first power of \( T \) in \( X \) and the last power of \( T \) in \( X^{-1} \) must sum to 1.

Lemma 7.6. For \( X \in \text{PSL}_2(\mathbb{Z}) \) write

\[
X =_{\text{zsl}} T^b ST^{a_0} \ldots ST^{a_{d-1}} \quad \text{and} \quad X^{-1} =_{\text{zsl}} T^{b'} ST^{a'_0} \ldots ST^{a'_{d'-1}}.
\]

Then

\[ a_{d-1} + b' = a'_{d'-1} + b = 0 \]

if \( X =_{\text{PSL}_2(\mathbb{Z})} T^k ST^a \) or \( X =_{\text{PSL}_2(\mathbb{Z})} T^k \) for some \( k, a \in \mathbb{Z} \), and

\[ a_{d-1} + b' = a'_{d'-1} + b = 1 \]

otherwise.

Proof. The cases of \( X =_{\text{PSL}_2(\mathbb{Z})} T^k ST^a \) and \( X =_{\text{PSL}_2(\mathbb{Z})} T^k \) are easily checked. Suppose \( X \) is not of that form. Since \( \overline{X}^{-1}X =_{\text{sl}} I \) by Lemma 7.3 some power of \( T \) that is not at the front or end of the word must be \(-1, 1, \) or \( 0 \). Since \( \overline{X} \) and \( \overline{X^{-1}} \) are in standard form, \( X \neq_{\text{PSL}_2(\mathbb{Z})} T^k ST^a \), and \( X \neq_{\text{PSL}_2(\mathbb{Z})} T^k \) this means that \( a'_{d'-1} + b \in \{ \pm 1, 0 \} \).

If \( a'_{d'-1} + b = 0 \) then \( S^2 \) is a subword of \( \overline{X^{-1}}X \) which can be replaced by \( I \) and if \( a'_{d'-1} + b = -1 \) then \( ST^{-1}S \) is a subword of \( \overline{X^{-1}}X \) which can be replaced by \( TST \). In either case this means that \( W\left( (X)^{-1}X \right) \leq W\left( (X)^{-1} \right) + W(X) - \frac{1}{2} = 0 \) where the last equality is by Lemma 7.5. By Lemma 7.1 \( W(\overline{X}X^{-1}) \geq 0 \) with equality only when \( X = I \). Since \( X \neq I \) we must have \( a'_{d'-1} + b = 1 \). The same analysis on \( \overline{X} \overline{X^{-1}} \) implies that \( a_{d-1} + b' = 1 \). \( \Box \)
Lemma 7.7. Let $X \in \text{PSL}_2(\mathbb{Z})$ and $c \in \mathbb{Z}_{>0}$. Then $X^{-1}T^cX = G X^{-1}T^cX$ and in particular $W(X^{-1}T^cX) = W(X^{-1}T^cX)$.

Proof. If $X = \text{PSL}_2(\mathbb{Z}) T^k$ for some $k \in \mathbb{Z}$ then $X^{-1}T^cX = _{\mathbb{Z}_{>0}} X^{-1}T^cX = _{\mathbb{Z}_{>0}} T^c$ so the result holds. If $X = _{\mathbb{Z}_{>0}} T^k ST^a$ for some $k, a \in \mathbb{Z}$ then there are two cases. If $c > 1$ then $X^{-1}T^cX = _{\mathbb{Z}_{>0}} X^{-1}T^cX = _{\mathbb{Z}_{>0}} T^c ST^{-k}$ so the result holds. If $c = 1$ then $X^{-1}T^cX = _{\mathbb{Z}_{>0}} T^c ST^{-k}$ and the result still holds.

If $X \neq _{\mathbb{Z}_{>0}} T^k$ and $X \neq _{\mathbb{Z}_{>0}} T^k ST^a$ for all $k, a \in \mathbb{Z}$, then $X^{-1}T^cX$ is already in standard form for any $c > 0$ by Lemma 7.6, so $X^{-1}T^cX = _{\mathbb{Z}_{>0}} X^{-1}T^cX$. \hfill \square

8 Minimal models for semitoric helices

Definition 8.1. An $S$-positive word with no leading $T$, $ST^{a_0} \ldots ST^{a_{d-1}} \in \mathbb{Z} * \mathbb{Z}$, is minimal if and only if $a_0, \ldots, a_{d-1} \neq 1$.

Minimal words are those associated to minimal helices.

Lemma 8.2. Suppose $\sigma = ST^{a_0} \ldots ST^{a_{d-1}} \in \mathbb{Z} * \mathbb{Z}$ is minimal and there exists $X \in G \setminus \{S^2T^k\}_{t,k \in \mathbb{Z}}$ such that

$$\sigma = G S^4X^{-1}T^cX.$$ 

Then, after cyclically reordering $a_0, \ldots, a_{d-1}$ if necessary, $a_0 \leq 0$ and one of the following hold:

(i) $a_0 = 0$ and $\sigma = _{\mathbb{Z}_{>0}} T^{a_1} ST^{a_2} \ldots ST^{a_{d-1}}$;

(ii) $a_0 < 0$ and $\sigma = _{\mathbb{Z}_{>0}} (TST)^{-a_0} T^{a_1} ST^{a_2} \ldots ST^{a_{d-1}}$.

Proof. Notice $W(\sigma) = W(S^4X^{-1}T^cX) = 1 - \frac{c}{12}$ while

$$W(\sigma) = W(X^{-1}T^cX) = W(X^{-1}T^cX)$$

$$= W(X^{-1}) + W(X) - \frac{c}{12} = \frac{1}{2} - \frac{c}{12}$$

by Lemmas 7.7 and 7.5 since $X \neq _{\mathbb{Z}_{>0}} T^k$ for any $k \in \mathbb{Z}$. Thus, $W(\sigma) \neq W(\sigma)$ so $\sigma$ is not in standard form. This means that $a_j \leq 1$ for some fixed $j \in \{0, \ldots, d-2\}$ and since $\sigma$ is minimal this implies $a_j \leq 0$.

If $a_j = 0$ for some $j \in \{0, \ldots, d-2\}$ then reorder so $a_0 = 0$ and $\sigma = S^2T^{a_1} ST^{a_2} \ldots ST^{a_{d-1}}$. Notice that $\eta = T^{a_1} ST^{a_2} \ldots ST^{a_{d-1}}$ satisfies $\eta \neq _{\mathbb{Z}_{>0}} T^{a_1} ST^{a_2} \ldots ST^{a_{d-1}}$ and also notice $W(\eta) = W(\sigma) - \frac{1}{2} = W(\sigma)$. All steps in the reduction algorithm in the proof of Theorem 4.14 reduce the winding number, except for the blowdown $STS \rightarrow T^{-1}ST^{-1}$, so the only possible step to reduce $\eta$ into standard form is a blowdown. For a blowdown to be possible we must have $a_j = 1$ for some $j \in \{1, \ldots, d-1\}$, contradicting the minimality of $\sigma$. Thus, $\eta = _{\mathbb{Z}_{>0}} \eta$ so $\eta \neq _{\mathbb{Z}_{>0}} \eta$. 

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Otherwise, \(a_j \neq 0\) for all \(j \in \{0, \ldots, d-1\}\) so, after cyclically reordering, we may assume \(a_0 < 0\). In this case let
\[
\eta' = (TST)^{-a_0}T^{a_1}ST^{a_2} \ldots ST^{a_1}
\]
and notice \(\eta' =_{\text{PSL}_2(\mathbb{Z})} \sigma\) so \(\overline{\eta'} =_{\mathbb{Z} \ast \mathbb{Z}} \sigma\). Again, \(W(\eta') = W(\sigma)\) so the only possible reduction move would be a blowdown, but if a blowdown could be performed on \(\eta'\) that would contradict the minimality of \(\sigma\), except in the case that \(a_1 = 0\), which we have assumed does not occur. Thus, \(\sigma = \eta'\).

Here we classify all words associated to minimal semitoric helices. Recall \(S\) from Equation (4.3).

**Lemma 8.3** (Classification of minimal words). Let \(\mathcal{H}\) be a helix with complexity \(c > 0\). If \(\mathcal{H}\) is minimal then there is an associated word \(\sigma \in \mathbb{Z} \ast \mathbb{Z}\) which is exactly one of the following, where \(A_0 = [v_0, v_1]\) for some \(\{v_i\}_{i \in \mathbb{Z}}\) such that \(\mathcal{H} = (d, c, [\{v_i\}_{i \in \mathbb{Z}}])\).

| type | \(\sigma \in \mathbb{Z} \ast \mathbb{Z}\) | \(c\) | \(A_0\) |
|------|---------------------------------|------|--------|
| (1)  | \(\sigma = ST^{-1}ST^{-4}\) | \(c = 1\) | \(ST^{-2} = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}\) |
| (2)  | \(\sigma = ST^{-2}ST^{-2}\) | \(c = 2\) | \(ST^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\) |
| (3)  | \(\sigma = S2T^aST^{-a-2}, a \neq 1, -3\) | \(c = 1\) | \(ST^{-a-1} = \begin{pmatrix} 0 & -1 \\ 1 & -a - 1 \end{pmatrix}\) |
| (4)  | \(\sigma = ST^{-1}ST^{-1}ST^{c-1}\) | \(c \neq 2\) | \(I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) |
| (5)  | \(\sigma = S2T^aST^cST^{-a}, a \neq \pm 1\) | \(c \neq 1\) | \(ST^{-a} = \begin{pmatrix} 0 & -1 \\ 1 & -a \end{pmatrix}\) |
| (6)  | \(\sigma = S2T^aS2T^{c-a}, a \neq 1, c - 1\) | \(c > 0\) | \(I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) |
| (7)  | \(\sigma = S2A_0^{-1}T^cA_0\) | \(c > 0\) | \(A_0 \in S\) |

where \(a \in \mathbb{Z}\) is a parameter.

**Proof.** Suppose that \(\sigma =_{\mathbb{Z} \ast \mathbb{Z}} ST^{a_0} \ldots ST^{a_{d-1}}\) is minimal and associated to a semitoric helix \(\mathcal{H}\) of length \(d\) and complexity \(c > 0\). By Lemma 6.6 there exists some \(X \in G\) such that
\[
\sigma =_G S^4X^{-1}T^cX.
\]  
(8.1)

We will proceed by cases on \(X\), and show that in each case that \(\sigma\) is one of type (1)-(7) in the statement of the Lemma.
Case I: $X = \text{PSL}_2(\mathbb{Z}) T^k$ for some $k \in \mathbb{Z}$. This implies that
\[
ST^{a_0} \ldots ST^{a_{d-1}} =_G S^4,
\]
and so $(a_0, \ldots, a_{d-2}, a_{d-1} - c) \in \mathbb{Z}^d$ are associated to a minimal toric fan. Such words are completely classified in [10, Lemma 4.8] and we conclude either $d = 3$ and $a_0 = a_1 = a_2 - c = -1$, which is minimal only when $c \neq 2$, or $d = 4$ and $a_0 = a_2 = 0$, $a_3 = c - a_1$, which is minimal only when $a \neq 1, c - 1$. Thus $\sigma$ is either of type (4) or (6).

Case II: $X \neq \text{PSL}_2(\mathbb{Z}) T^k$ for all $k \in \mathbb{Z}$. In light of Equation (8.1) apply Lemma 8.2 to $\sigma$ and conclude that, after passing to an equivalent helix by cyclically permuting,
\[
\sigma =_{Z \times Z} ST^{a_0} \ldots ST^{a_{d-1}}
\]
satisfies either
1. $a_0 = 0$; or
2. $a_j \neq 0$ for all $j = 0, \ldots, d - 1$ and $a_0 < 0$.
If $a_0 = 0$ then
\[
\bar{\sigma} = T^{a_1} ST^{a_2} \ldots ST^{a_{d-1}}
\]
and otherwise
\[
\bar{\sigma} = (TST)^{-a_0} T^{a_1} ST^{a_2} \ldots ST^{a_{d-1}}.
\]
By cyclically permuting the $a_i$, which corresponds to conjugating Equation (8.1) by $ST^{a_i}$ for various $i$, we change $X$, but $X \neq T^k$ still holds. We now have three further cases on $X$.

Case IIa: $X = \text{PSL}_2(\mathbb{Z}) T^k ST^a$ for $k, a \in \mathbb{Z}$. First assume $a_0 = 0$. If $c = 1$ then
\[
X^{-1} T^c X =_{Z \times Z} T^{-a} ST^{a-1}
\]
so $\sigma = S^2 T^{-a} ST^{a-1}$ which is minimal for $a \neq \pm 2$ and is of type (3). If $c \neq 1$ then
\[
X^{-1} T^c X = T^{-a} ST^c ST^a
\]
so $\sigma = S^2 T^{-a} ST^c ST^a$ which is minimal if $a \neq \pm 1$ and is of type (5).

Now suppose $a_0 \neq 0$. Then
\[
\bar{\sigma} = (TST)^{-a_0} T^{a_1} ST^{a_2} \ldots ST^{a_{d-1}}
\]
and $a_0 < 0$. If $c = 1$ then $\bar{\sigma} = T^{-a} ST^{a-1}$ so $a = -2$ and thus $\bar{\sigma} = (TST) T^{-4}$ so $\sigma = ST^{-1} ST^{-4}$, which is of type (1). If $c = 2$, then $\bar{\sigma} = T^{-a} ST^2 ST^{a}$ which means $a = -1$ and $a_0 = -2$ so $\sigma = ST^{-2} ST^{-2}$, which is of type (2). If $c > 2$, then $\bar{\sigma} = T^{-a} ST^c ST^{a}$ which means $a = -1$ and $a_0 = -1$ so $\sigma = ST^{-1} ST^{-1} ST^{-1}$, which is of type (4).

Case IIb: $X \neq \text{PSL}_2(\mathbb{Z}) T^k T^k ST^a$ for all $k, a \in \mathbb{Z}$ and $a_i \neq 0$ for all $i = 0, \ldots, d - 1$. In this case $a_0 < 0$. If $d = 2$, then $\sigma = ST^{a_0} ST^{a_1}$ and $\bar{\sigma} = (TST)^{-a_0} T^{a_1}$ which means $(TST)^{-a_0} T^{a_1} =_{Z \times Z} X^{-1} T^c X$. Since $X^{-1} T^c X$ starts with $TS$ it must end with $S$ by Lemma 7.6 so $a_1 = -1$. Now $W(\sigma) = 1 - \epsilon/12$ from Equation (8.1) and
\[
W(\sigma) = W(ST^{-n} ST^{-1}) = \frac{1}{12} (7 + n)
\]

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and we have
\[
(TST)^{5-c}T^{-1} =_{\mathbb{Z}^{*}} \overline{X}^{-1}T^{c}X.
\] (8.2)

The right side of Equation (8.2) contains $T^{c+1}$ while the highest power of $T$ on the left side is $T^2$, so $c > 0$ implies $c = 1$. Thus we obtain $\sigma = ST^{-4}ST^{-1}$ and have type (3).

If $d > 2$ then
\[
(TST)^{-a_0T^{a_1}ST^{a_2} \ldots ST^{a_{d-1}}} =_{\mathbb{Z}^{*}} \overline{X}^{-1}T^{c}X
\]
implies that $\overline{X}^{-1}T^{c}X$ must end with $S$ by Lemma 7.6, so $a_{d-1} = 0$ which contradicts our assumption in this case.

Case IIc: $X \neq_{\text{PSL}(Z)} T^kT^kST^a$ for all $k, a \in \mathbb{Z}$ and $a_i = 0$ for some $i \in \{0, \ldots, d - 1\}$.
We have already used Lemma 8.2 to establish that either $a_0 = 0$ or $a_j \neq 0$ for all $j = 0, \ldots, d - 1$. Thus, in this case, $a_0 = 0$ so
\[
\sigma = S^{2}\overline{\sigma} = S^{2}\overline{X}^{-1}T^{c}X
\]
which is minimal if $X$ does not end with $ST$ and $\overline{X}^{-1}$ does not begin with $TS$, and is of type (7).

Proof of Theorem 4.15. Let $H = (d, c, \{v_i\}_{i \in \mathbb{Z}})$ be a minimal semitoric helix with associated integers $(a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d$. Then $\sigma = ST^{a_0} \ldots ST^{a_{d-1}}$ is a minimal word and, passing to an equivalent helix if necessary, we conclude $\sigma$ must be of some type (1)-(7) in Lemma 8.3. Types (1)-(6) for $\sigma$ in Lemma 8.3 correspond to types (1)-(6) for $H$ in Theorem 4.15. Notice these each have length $d < 5$. 

Otherwise, $\sigma$ must be of type (7), which means there exists some $X =_{G} A_0$, where $A_0 = [v_0, v_1] \in S$ and
\[
\sigma =_{\mathbb{Z}^{*}} S^{2}\overline{X}^{-1}T^{c}X.
\]
Since $A_0 \in S$ notice that $A_0 = ST^{a_0} \ldots ST^{a_{d-1}}$ with $\ell \geq 2$, which implies that $\sigma$ has at least six occurrences of $S$, so the length $d$ of $H$ satisfies $d \geq 6$.

Proof of Corollary 4.16. From Equation (4.4) in Theorem 4.15 we see that $a_0 = 0$ and so the given recurrence relation $v_j = a_{j-2}v_{j-1} + v_{j-2}$ with $j = 2$ gives $v_2 = -v_0$.

Suppose $H$ has associated integers $a_0, \ldots, a_{d-1}$ and let $A_0 = [v_0, v_1]$. Then
\[
S^{2}\overline{A}_0^{-1}T^{c}\overline{A}_0 =_{\mathbb{Z}^{*}} ST^{a_0} \ldots ST^{a_{d-1}},
\] (8.3)
implies
\[
S^{2}\overline{A}_0^{-1}T^{c} =_{\mathbb{Z}^{*}} ST^{a_0} \ldots ST^{a_{k+1}}
\] (8.4)
for some $k \in \mathbb{Z}$, since $\overline{A}_0$ starts with $S$. By the recurrence relation $a_i v_{i+1} = v_i + v_{i+2}$ we see
\[
[v_{k+2}, v_{k+3}] =_{\text{SL}(Z)} A_0 ST^{a_0} \ldots ST^{a_{k+1}}.
\] (8.5)
Combining Equations (8.4) and (8.5) yields $[v_{k+2}, v_{k+3}] =_{\text{PSL}(Z)} A_0 S^2A_0^{-1}T^{c}$ which implies
\[
[v_{k+2}, v_{k+3}] =_{\text{PSL}(Z)} T^{c} =_{\text{PSL}(Z)} \left( \begin{array}{cc} 1 & c \\ 0 & 1 \end{array} \right)
\]
so $v_{k+2}$ is the required vector. 

9 Examples

9.1 A representative example of Theorem 4.15

Suppose that $\mathcal{H} = (d, c, \{ v_i \}_{i \in \mathbb{Z}})$ is a minimal semitoric helix of length $d > 4$. By Corollary 4.16, $\mathcal{H}$ is of type (T) and the representative $\{ v_i \}_{i \in \mathbb{Z}}$ can be chosen to satisfy $v_0 = -v_2$. Then $\mathcal{H}$ is determined by its complexity $c$ and the basis $(v_0, v_1)$ of $\mathbb{Z}^2$.

Let $\mathcal{H}$ have complexity $c = 2$ and

$$v_0 = \left( \frac{-1}{2} \right) \text{ and } v_1 = \left( \frac{-2}{3} \right).$$

Next we compute $\mathcal{H}$. Define $A_0 := [v_0, v_1]$, which means that $A_0 = ST^2ST^2$ in terms of the generators $S, T$. Define $A_i = [v_i, v_{i+1}]$ for $i \in \mathbb{Z}$ and notice that $\det(v_i, v_{i+1}) = \det(v_{i+1}, v_{i+2}) = 1$ implies that there exists some $a_i \in \mathbb{Z}$ so that

$$[v_{i+1}, v_{i+2}] = [v_i, v_{i+1}] \begin{pmatrix} 0 & -1 \\ 1 & a_i \end{pmatrix}$$

which means that $A_{i+1} = A_i ST^{a_i}$ for all $i \in \mathbb{Z}$. Then, $A_d = \text{SL}_2(\mathbb{Z}) A_{d-1} ST^{a_{d-1}} = \text{SL}_2(\mathbb{Z}) A_0 ST^{a_0} ST^{a_{d-2}} \ldots ST^{a_{d-1}}$ and since $\mathcal{H}$ is of length $d$ and complexity 2, $A_d = T^2 A_0$. Thus $T^2 A_0 = \text{SL}_2(\mathbb{Z}) A_0 ST^{a_0} \ldots ST^{a_{d-1}}$ which implies

$$ST^{a_0} \ldots ST^{a_{d-1}} = \text{SL}_2(\mathbb{Z}) A_0^{-1} T^2 A_0.$$  \hspace{1cm} (9.1)

Our goal is now to recover $a_0, \ldots, a_{d-1}$. Let $\sigma =_{\mathbb{Z} \ast \mathbb{Z}} ST^{a_0} \ldots ST^{a_{d-1}}$. Lifting Equation (9.1) to the group $G$ yields

$$\sigma =_G S^{4k} A_0^{-1} T^2 A_0,$$ \hspace{1cm} (9.2)

for some choice of $k \in \mathbb{Z}$. The fact that the vectors $v_0, \ldots, v_{d-1}$ in the semitoric helix are arranged in counter-clockwise order forces $k = 1$ by Proposition 4.12.

By substituting for $A_0$ and using the relations of $G$ on Equation (9.2) we see that

$$\sigma =_G S^4 (ST^2 ST^2)^{-1} T^2 (ST^2 ST^2) =_G S^2 T^{-1} ST^2 ST^3 ST^2 ST^2$$

and by taking the standard form of both sides of this equality we have

$$\sigma =_{\mathbb{Z} \ast \mathbb{Z}} T^{-1} ST^2 ST^3 ST^2 ST^2.$$  \hspace{1cm} (9.3)

From this we deduce $W(\sigma) - W(\sigma) = \frac{10}{12} - \frac{4}{12} = \frac{1}{2}$ which means that $\sigma$ can be reduced to $\tilde{\sigma}$ by a substitution of either

$$S^2 =_{\text{PSL}_2(\mathbb{Z})} I \text{ or } ST^{-n} S =_{\text{PSL}_2(\mathbb{Z})} (TST)^n$$

combined with several applications of the blowdown operation, $STS =_{\text{PSL}_2(\mathbb{Z})} T^{-1} ST^{-1}$. This comes from observation that $S^2 =_{\text{PSL}_2(\mathbb{Z})} I$ and $ST^{-n} S =_{\text{PSL}_2(\mathbb{Z})} (TST)^n$ each reduce
the winding number by $\frac{1}{2}$ and the proof of Theorem 4.14, which uses an algorithm consisting of these three reductions to put an element of $\text{PSL}_2(\mathbb{Z})$ in standard form.

To finish, we recall that $\mathcal{H}$ is minimal, so it does not admit a blowdown. This means $\sigma$ may be obtained from $\overline{\sigma}$ by a sequence of blowups followed by either the addition of a $S^2$ term or the replacement of $(TST)^n$ by $ST^{-n}S$ for some $n > 0$. By examining the form of $\sigma$ in Equation (9.3) we see the requirement that $a_0 = 0$ forces the addition of $S^2$ at the front of the word and the requirement that $a_i \neq 1$ for all $i$ implies that no blowups may be performed before adding $S^2$ to the front which forces $\sigma = 2s_s T^{-1}SST^2ST^3ST^2ST^2$. Thus $d = 7$ and $a_0 = 0, a_1 = -1, a_2 = 2, a_3 = 2, a_4 = 3, a_5 = 2, a_6 = 2$. Since we were given $v_0$ and $v_1$ these values determine $\mathcal{H}$ by the recurrence relation Equation (6.1). An image of the first seven vectors in this helix is given in Figure 3.

### 9.2 Coupled angular momenta

Consider $S^2 \times S^2$ with coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$, where $S^2$ is the 2-sphere. The Delzant polygon of $S^2 \times S^2$ endowed with the toric integrable system given by $F = (z_1, z_2)$ with any product symplectic form is a rectangle, where the length of the sides are determined by the symplectic area of each copy of $S^2$. Its associated fan is formed by the normal vectors to the faces of the polygon, given by $(1, 0), (0, 1), (-1, 0), (0, -1)$. If instead we consider the symplectic semitoric manifold with $F = (J, H)$ on $S^2 \times S^2$ with the standard product symplectic form where

\[
J = z_1 + \frac{5}{2}z_2 \\
H = \frac{1}{2}z_1 + \frac{1}{2}(x_1x_2 + y_1y_2 + z_1z_2)
\]

then we obtain the coupled spin system. In [12] it is shown that this is indeed a symplectic semitoric manifold and the authors find the two representatives of the polygons associated to the system obtained by Vû Ngôc’s cutting procedure. One of these polygons has vertices $(-3.5, 0), (-1.5, 2), (3.5, 2), (1.5, 0)$. The semitoric helix associated to this manifold, which can be recovered from the polygons as described in Section 5.3, is represented by the three vectors

\[
v_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}
\]

and is minimal of type (3) from Theorem 4.15 with $k = 1$. This is shown in Figure 7. Since the system has only one focus-focus point, the helix can be obtained as the inwards pointing normal vectors of the semitoric polygon computed in [12], where $v_0$ is chosen to be the inwards pointing normal vector of an edge adjacent to the fake corner produced by the focus-focus point. This is because in systems with one focus-focus point the same toric momentum map that produces the semitoric polygon also works for the construction of the semitoric helix, as described in Section 5.1 (see Remark 5.4).
10 Proof of Theorem 2.5

Since any helix may be obtained from a minimal one by a finite sequence of blowups, Theorem 4.15 implies results about the non-minimal helices as well.

Lemma 10.1. Any semitoric helix of complexity $c > 2$ includes the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or its negative.

Proof. We will show that this vector is in every minimal semitoric helix of complexity $c > 2$, and since every semitoric helix can be produced by a sequence of blowups on a minimal semitoric helix and blowups do not remove vectors from the helix or change the complexity, the result will follow.

Since $c > 2$ the only possible types for minimal models are types (4)-(7). By Theorem 4.15 we see that (4), (5), and (6) include the required vector. Helices of type (7) include the required vector by Corollary 4.16.

Proof of Theorem 2.5. Suppose that $(M, \omega, F = (J, H))$ is a compact symplectic semitoric manifold with $c \geq 2$ focus-focus points and that $J$ achieves its maximum and minimum at a single point each. This means that $F(M)$ does not include as its boundary a vertical line segment, which in turn, since a straightening map cannot produce a vertical wall, implies that $\text{hlx}(M, \omega, F)$ does not include a vector on the $x$-axis.

Lemma 10.1 states that if $c > 2$ then $\text{hlx}(M, \omega, F)$ includes a vector on the $x$-axis, so this case cannot occur. Otherwise, $c = 2$. Since blowups do not change the complexity, Theorem 4.15 implies that $\text{hlx}(M, \omega, F)$ can be obtained from a minimal semitoric helix of either type (2) or type (7) by a sequence of blowups. By Corollary 4.16 any helix of type (7) includes a vector on the $x$-axis and thus any blowup of a helix of type (7) must also include such a vector. Thus, this case cannot occur.

Suppose that $\text{hlx}(M, \omega, F)$ can be obtained from a minimal helix of type (2) via a nonzero number of blowups. There are two distinct blowups which can be performed on a minimal
helix of type (2), adding either the vector
\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\]
or the vector
\[
\begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
In either case a vector on the x-axis is including in the resulting helix, so this case cannot occur either. The only remaining case is that \((M, \omega, F)\) is minimal of type (2). \qed

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