One-loop divergences in higher-derivative gravity

Nobuyoshi Ohta

Abstract We give a review of the one-loop divergences in higher derivative gravity theories. We first make the bilinear expansion in the quantum fluctuation on arbitrary backgrounds, introduce a higher-derivative gauge fixing and show that higher-derivative gauge fixing must have ghosts in addition to those naively expected. We give general formulae for the one-loop divergences in such theories, and give explicit results for theories with quadratic curvature terms. In this calculation, we need the heat kernel coefficients for the four-derivative minimal operators and two-derivative nonminimal vector operators, which are summarized. We also discuss the beta functions in the renormalization group, and show that the dimensionless couplings are asymptotically free. The calculation is also extended to the theories with arbitrary functions of $R$ and $R^2_{\mu\nu}$. We show that the result is independent of metric parametrization and gauge on shell.

Keywords
Quantum gravity, Quadratic gravity, Higher-derivative gravity, Perturbation theory, One-loop divergences, Background method, Heat kernel expansion, Effective action, Renormalization group, Asymptotic freedom

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1 Introduction

The study of quantum effects in gravity started with the seminal work of 't Hooft and Veltmann [1], in which one-loop divergences were first studied. It was shown that there exist divergences in the quadratic curvature terms, but these counterterms were shown to be transformed away by a field redefinition. However it was shown later the Einstein gravity is nonrenormalizable at two loops [2]. It was further shown that gravity theory containing quadratic curvature terms is a renormalizable theory [3]. Since this theory contains curvature squares and four derivatives, we refer to such theories as quadratic, four-derivative or higher-derivative gravity. Unfortunately the theory is probably nonunitary in perturbation theory, but see [4] for discussions.

The actual calculation of one-loop divergences in the four-derivative quantum gravity involves some complicated techniques. We use the technique of background field method by separating the quantum fields into backgrounds and fluctuations, and integrate over the fluctuations. To really do this at one loop, we have to know the bilinear forms in the fluctuation fields of all relevant terms, such as Weyl curvature square, scalar curvature square (or other combinations, like Ricci curvature square and Riemann curvature square), and Einstein-Hilbert term.

The next step is to gauge fix the theory. Typically this also involves introduction of the Faddeev-Popov (FP) ghosts which were first discovered by Feynman [5], and introduced in more formal way by DeWitt [6], and later elegantly formulated by Faddeev and Popov [7]. When higher-derivative gauge fixing is chosen, this leads to a complication; in addition to those naively expected [8], we have to include additional ghost contribution [denoted as \(\text{Tr log } \Delta\) in eq. (1)] first noticed in [9, 10], and derived at the one-loop level in [11]. Here instead of examining the one-loop result, we give a more elegant derivation of all the ghosts based on the symmetry principle [12].

The general formula for the divergences is then schematically given as

\[
\Gamma_{\text{div}} = \frac{1}{2} \text{Tr log } \mathcal{H} - \text{Tr log } \Delta \mathcal{H} - \frac{1}{2} \text{Tr log } Y,
\]

where the trace is meant to sum over the spectra of the operators, and the first, second and last terms are the contributions from the graviton, FP ghosts and additional ghosts, respectively. To calculate this, it is most convenient to use heat kernel technique, which is explained here in some details. The calculation involves fourth-order minimal operators in the tensor sector and second-order nonminimal operators for the ghosts. We give the necessary formulae to evaluate these [13, 14]. Then we sum all the contributions to find the one-loop divergences in the quadratic gravity theory. We mention that there is also the technique called Schwinger-DeWitt method, which is basically the same as the heat kernel technique [15]. There were some mistakes in the early calculations of the logarithmic divergences [8, 9, 10], and the correct results were given in [16].

As a simple application of the technique, we also study general higher-derivative theory including an arbitrary function of Ricci tensor squared and Ricci scalar cur-
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We will find the general result for this case, and show that on shell, it depends on neither the parametrizations nor gauge parameters.

2 Bilinear expansion of quadratic terms

We will consider the Euclidean actions of the general form

\[
S = \int d^d x \sqrt{-g} \left[ \frac{1}{\kappa^2} (2\Lambda - R) + \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma R_{\mu\nu\rho\lambda}^2 \right],
\]

where \( \kappa^2 = 16\pi G \) is the \( d \)-dimensional gravitational constant, \( \Lambda \) is the cosmological constant, and \( \alpha, \beta, \gamma \) are the higher-derivative couplings. Though we consider \( d = 4 \) when we give the results for the divergences, here we give the results for general dimension \( d \). It is sometimes more convenient to use a different basis for the higher-derivative terms, namely \( R_{\mu\nu}^2 \), the square of the Weyl tensor

\[
C^2 = R_{\mu\nu\alpha\beta}^2 - \frac{d - 2}{d - 3} R_{\mu\nu}^2 + \frac{2}{(d - 1)(d - 2)} R^2, \tag{3}
\]

and the Gauss-Bonnet combination

\[
E = R_{\mu\nu}^2 - 4 R_{\mu\nu}^2 + R^2, \tag{4}
\]

which is topological for \( d = 4 \) and vanishes identically for \( d = 3 \). Conversely we have

\[
R_{\mu\nu\alpha\beta}^2 = \frac{d - 2}{d - 3} C^2 - \frac{1}{d - 3} E + \frac{1}{d - 1} R^2, \quad R_{\mu\nu}^2 = \frac{d - 2}{4(d - 3)} (C^2 - E) + \frac{d}{4(d - 1)} R^2, \tag{5}
\]

Then the action has the alternative form

\[
S = \int d^d x \sqrt{-g} \left[ \frac{1}{\kappa^2} (2\Lambda - R) + \frac{1}{2\lambda} C^2 - \frac{1}{\rho} E + \frac{1}{\xi} R^2 \right], \tag{6}
\]

where

\[
\lambda = \frac{2(d - 3)}{(d - 2)(\beta + 4\gamma)}, \quad \rho = \frac{4(d - 3)}{(d - 2)\beta + 4\gamma}, \quad \xi = \frac{4(d - 1)}{4(d - 1)\alpha + d\beta + 4\gamma}. \tag{7}
\]

or conversely

\[
\alpha = -\frac{1}{\rho} + \frac{1}{\xi} + \frac{1}{(d - 1)(d - 2)\lambda}, \quad \beta = \frac{4}{\rho} - \frac{2}{(d - 2)\lambda}, \quad \gamma = -\frac{1}{\rho} + \frac{1}{2\lambda}. \tag{8}
\]

Note that in \( d = 3 \), \( C^2 \) and \( E \) both vanish identically. The couplings \( \lambda, \rho \) and \( \xi \) have mass dimension \( 4 - d \). In dimensions higher than three, it is customary to define the
dimensionless combinations
\[ \omega \equiv -\frac{(d-1)\lambda}{\xi}, \quad \theta \equiv \frac{1}{\rho}. \tag{9} \]

We will apply the standard background field method, expanding the metric as
\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{10} \]

In order to derive the effective action at the one-loop level, or to calculate the one-loop beta functions, we need the expansion of the action to second order in \( h_{\mu\nu} \). For this purpose, it is useful to first make the expansion of curvatures in the fluctuations, which are summarized in Appendix A.1.

Using the formulae given in A.1, we find that the terms proportional to \( \alpha \) can be written in the form
\[ ah_{\mu\nu} \left[ \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} - \bar{g}_{\mu\nu} \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} - \bar{g}_{\alpha\beta} \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} + \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} + 2(\bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} - \bar{g}_{\mu\beta} \bar{g}_{\nu\alpha}) \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} - \bar{R} \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} + 2\bar{R} \bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} + \bar{R}_{\mu\nu} \bar{g}_{\alpha\beta} - \frac{1}{4} J_{\mu\nu\alpha\beta} \bar{R}^2 \right] h^{\alpha\beta}. \tag{11} \]

Here and in what follows, \( \Box \equiv \bar{\nabla}_{\mu} \bar{\nabla}^{\mu} \) and a bar indicates that the quantity is evaluated on the background; the indices are raised, lowered and contracted by the background metric \( \bar{g} \), and the covariant derivative \( \bar{\nabla} \) is constructed with the background metric. The tensor \( J \) is defined by
\[ J_{\mu\nu\alpha\beta} = \delta_{\mu\nu,\alpha\beta} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} \tag{12} \]

where
\[ \delta_{\mu\nu,\alpha\beta} = \frac{1}{2}(\bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} + \bar{g}_{\mu\beta} \bar{g}_{\nu\alpha}) \equiv \bar{1}, \tag{13} \]
is the identity in the space of symmetric tensors.

The \( \beta \) terms can be written in the form
\[ \beta h_{\mu\nu} \left[ \frac{1}{2} \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} - \frac{1}{2} \bar{g}_{\alpha\beta} \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} + \frac{1}{4} (\bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} + \bar{g}_{\mu\beta} \bar{g}_{\nu\alpha}) \Box^2 \right] + \frac{1}{2} \bar{R}_{\mu\nu} \bar{g}_{\alpha\beta} \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} + \frac{3}{4} \bar{g}_{\alpha\beta} \bar{R}_{\mu\nu} \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} + \bar{R}_{\mu\nu} \bar{R}_{\alpha\beta} - \frac{3}{2} \bar{R}_{\mu\nu\alpha\beta} + \bar{R}_{\mu\nu\alpha} \bar{R}_{\alpha\beta} + \bar{R}_{\mu\nu\beta} \bar{R}_{\alpha\beta} \tag{14} \]

and the terms proportional to \( \gamma \) are
It can be checked that when arranged in the Gauss-Bonnet combination ($\gamma = \alpha$, $\beta = -4\alpha$) and the background metric is maximally symmetric, one obtains a total derivative. This gives a nontrivial check of the results.

\section{Gauge fixing and ghosts}

The first attempt at calculating one-loop divergences was made in \cite{8} with higher-order gauge fixing, but it was pointed out that it did not correctly incorporate the FP ghosts. The problem is that an additional ghost contribution was missing, which was first considered in \cite{9,10}. But it was not clear why we need such additional ghosts, and later clarified in \cite{11}; this showed it must be there by carefully examining the one-loop amplitude in the path integral formulation. Here we explain more elegant way of introducing the correct FP ghosts including the additional ghosts for higher-derivative gauge fixing \cite{12}, which is valid not only for one loop but also for all loops because it is based on the exact BRST (Becchi-Rouet-Stora-Tyutin) symmetry of the system.

The BRST transformation for the fields is found to be

\[
\delta B g_{\mu\nu} = -\delta \lambda \left[ g_{\mu\nu} \nabla^2 \chi + g_{\rho\mu} \nabla^2 \chi \right] \equiv -\delta \lambda \mathcal{D}_{\mu\nu} \chi \nabla^2 \chi, \quad \delta B e^\mu = -\delta \lambda e^\mu \nabla^2 \chi, \quad \delta B \bar{\epsilon}_\mu = i\delta \lambda B_\mu, \quad \delta B B_\mu = 0, \tag{16}
\]

which is nilpotent. Here $e^\mu$, $\bar{\epsilon}_\mu$ and $B_\mu$ are the FP ghost, anti-ghost and an auxiliary field, respectively, and $\delta \lambda$ is an anticommuting parameter. The gauge fixing term and the FP ghost terms are concisely written as

\[
\mathcal{L}_{GF+FP}/\sqrt{-g} = -i\delta B \left[ \bar{\epsilon}_\mu Y^{\mu\nu} (\chi - \frac{a}{2} B_\nu) \right]/\delta \lambda \]

\[
= B_\mu Y^{\mu\nu} \chi - i\bar{\epsilon}_\mu Y^{\mu\nu} \left( \nabla^2 \mathcal{D}_{\delta\nu,\rho} + b \nabla^2 \mathcal{D}_{\delta\nu,\rho} \right) e^\rho - \frac{a}{2} B_\mu Y^{\mu\nu} B_\nu \]

\[
= \frac{1}{2d} \chi_\mu Y^{\mu\nu} \chi - \frac{a}{2} B_\mu Y^{\mu\nu} B_\nu + i\bar{\epsilon}_\mu Y^{\mu\nu} \Delta_{\delta h,\nu} e^\rho, \tag{17}
\]

where

\[
\chi_\mu \equiv \nabla^2 h_{\lambda\mu} + b \nabla^2 h, \quad \bar{B}_\mu \equiv B_\mu - \frac{1}{a} \chi_\mu, \tag{18}
\]
are the gauge fixing function and a field imposing the gauge condition, respectively. \( Y^{\mu\nu} \) is a derivative operator for higher-derivative gauge fixing

\[
Y_{\mu\nu} \equiv -\bar{g}_{\mu\nu} \Box - c \nabla_{\mu} \nabla_{\nu} + d \nabla_{\nu} \nabla_{\mu}, \quad \Delta_{g_{h,\mu\nu}} \equiv -\bar{g}_{\mu\nu} \Box - (1 + 2b) \nabla_{\mu} \nabla_{\nu} - R_{\mu\nu},
\]

(19)

\( \Delta_{g_{h,\mu\nu}} \) is the ghost kinetic term, and \( a, b, c \) and \( d \) are gauge parameters. For one-loop calculation, we can replace \( \nabla \) by \( \hat{\nabla} \). We see from (17) that we get the determinant factor \( (\text{Det} \ Y^{\mu\nu})^{1/2} \) after we perform the path integral over \( \hat{B}_\mu \) and FP ghosts \( \hat{c}_\mu \) and \( \hat{c}' \). (The factor \( Y^{\mu\nu} \) in the first term combines into the graviton contribution.) In the standard way of Faddeev and Popov, the factor of \( Y \) can be easily missed [8], but in this formulation we see why there must be this factor. Note also that the field \( B_\mu \) is an auxiliary field for low-derivative gauge fixing, but here it becomes dynamical due to the higher-derivative gauge fixing with the factor \( Y^{\mu\nu} \).

We choose the gauge parameters such that the nonminimal four derivative terms \( \hat{v}_\mu \hat{v}_\nu \hat{v}_\alpha \hat{v}_\beta, \bar{g}_{\mu\nu} \hat{v}_\alpha \hat{v}_\beta \) and \( \bar{g}_{\nu\beta} \hat{v}_\mu \hat{v}_\alpha \) cancel. This leads to the choice \([18, 19]\)

\[
a = \frac{1}{\beta + 4\gamma}, \quad b = \frac{4\alpha + \beta}{4(\gamma - \alpha)}, \quad c - d = \frac{2(\gamma - \alpha)}{\beta + 4\gamma} - 1.
\]

(20)

In order to simplify the gauge-fixing term, we will further choose \( d = 1 \). Then, the quadratic terms in the action can be written in the form \( h^{\mu\nu} K_{\alpha\beta} h_{\alpha\beta} \), where

\[
K = K \Box^2 + D_{\rho\lambda} \hat{\nabla}^\rho \hat{\nabla}^\lambda + W.
\]

(21)

The explicit forms of the coefficients are

\[
(K)_{\mu\nu, ab} = \frac{\beta + 4\gamma}{4} \left( \bar{g}_{\mu a} \bar{g}_{\nu b} + \frac{4\alpha + \beta}{4(\gamma - \alpha)} \bar{g}_{\mu} \bar{g}_{ab} \right),
\]

(22)

\[
(D_{\rho\lambda})_{\mu\nu, ab} = -2\gamma \bar{g}_{\rho\beta} \bar{R}_{\alpha\mu\lambda} + 4\gamma \bar{g}_{\nu\beta} \bar{R}_{\alpha\lambda\rho} + (\beta + 3\gamma) \bar{g}_{\rho\lambda} \bar{R}_{\mu\alpha\nu} - (2\beta + 4\gamma) \bar{g}_{\alpha\rho} \bar{g}_{\nu\beta} \bar{R}_{\mu\lambda}
\]

\[
-2\gamma \bar{g}_{\nu\beta} \bar{g}_{\rho\mu} \bar{g}_{\lambda} + (\beta + 3\gamma) \bar{g}_{\nu\rho} \bar{R}_{\beta\mu\alpha} - 2\alpha \bar{g}_{\rho\lambda} \bar{g}_{\nu\beta} \bar{R}_{\mu\lambda} + 2\alpha \bar{g}_{\mu\nu} \bar{g}_{\rho\lambda} \bar{R}_{\beta\alpha} + 2\gamma \bar{g}_{\mu\nu} \bar{R}_{\rho\lambda}
\]

\[
+ \left( \frac{\alpha}{2} \bar{R} - \frac{1}{4k^2} \right) (\bar{g}_{\mu a} \bar{g}_{\nu b} \bar{g}_{\rho\lambda} - \bar{g}_{\mu} \bar{g}_{\nu} \bar{g}_{ab} \bar{R}_{\rho\lambda} - 2\gamma \bar{g}_{\nu\beta} \bar{g}_{\rho a} \bar{g}_{\alpha\lambda} + 2\bar{g}_{\mu\nu} \bar{g}_{ab} \bar{g}_{\rho\lambda},
\]

(23)
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\[(W)_{\mu\nu,\alpha\beta} = \frac{3}{2} \gamma \bar{g}_{\nu\beta} \bar{R}_\rho^{\mu\lambda} \bar{R}_{\alpha\mu\lambda} + 4 \gamma \bar{R}_{\rho\nu\mu\lambda} \bar{R}_\rho^{\mu\lambda} - \gamma \bar{R}_{\rho\nu\mu\lambda} \bar{R}_\rho^{\mu\lambda} + (\beta + 5 \gamma) \bar{R}_{\rho\mu\lambda\nu} \bar{R}_\rho^{\mu\lambda} \]

+6\gamma \bar{R}_\rho^{\mu\lambda} \bar{R}_{\rho\mu\lambda\nu} + \left( \frac{\beta}{2} + \gamma \right) \bar{R}_{\mu\nu} \bar{R}_\nu + \left( \alpha \bar{R} - \frac{1}{2k^2} \right) \left( \frac{1}{2} \bar{R}_{\mu\nu} \bar{R}_\nu + \frac{\beta}{2} \bar{R}_{\mu\nu} \bar{R}_\mu - \bar{g}_{\mu\nu} \bar{R}_{\alpha\beta} \right)

+\alpha \bar{R}_{\mu\nu} \bar{R}_{\alpha\beta} + \frac{1}{8} \left( \alpha \bar{R}^2 + \beta \bar{R}_{\rho\lambda} + \gamma \bar{R}_{\rho\lambda\sigma} \bar{R}_\sigma - \frac{1}{k^2} \left( \bar{R} - 2 \Lambda \right) \right) \left( \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} - 2 \bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} \right)

+\left( \frac{5}{2} \beta + 4 \gamma \right) \bar{g}_{\nu\beta} \bar{R}_{\rho\mu} \bar{R}_\rho^{\alpha\lambda} - \bar{g}_{\mu\nu} \bar{R}_\alpha^{\rho\lambda} \bar{R}_{\rho\lambda\sigma} - \beta \bar{g}_{\alpha\beta} \bar{R}_{\mu\nu} \bar{R}_\rho^{\rho\lambda} - \left( \beta + 4 \gamma \right) \bar{g}_{\nu\beta} \bar{R}_{\rho\mu} \bar{R}_\rho^{\rho\lambda} \bar{R}_{\rho\mu\alpha\lambda}.

(24)

where we have dropped terms with two derivatives acting on a background curvature, and performed the symmetrizations \( \mu \leftrightarrow \nu, \alpha \leftrightarrow \beta \) and \( (\mu, \nu) \leftrightarrow (\alpha, \beta) \).

In order to use the heat kernel formula, we have to put this operator into the form

\[\mathcal{H} = K^{-1} \mathcal{K} = \square^2 + V_{\rho\lambda} \nabla^\rho \nabla^\lambda + U,\]

where

\[\left(K^{-1}\right)_{\mu\nu}^{\alpha\beta} = \frac{4}{\beta + 4 \gamma} \left( \delta_{\mu\nu}^{\alpha\beta} - \Omega \bar{g}_{\mu\nu} \bar{g}^{\alpha\beta} \right),\]

with

\[\Omega = \frac{4\alpha + \beta}{\Sigma}, \quad \Sigma \equiv 4(\gamma - \alpha) + d(4 \alpha + \beta).\]

The form of the coefficients \( V_{\rho\lambda} \) and \( U \) is complicated. First \( V \) is given by

\[V_{\rho\lambda} = \frac{4}{\beta + 4 \gamma} \sum_{i=1}^{20} b_i k_i,\]

where

\[k_1 = \bar{g}_{\nu\beta} \bar{g}^{\rho\lambda} \bar{R}_{\mu\lambda}, \quad k_2 = \delta_{\mu\nu,\alpha\beta} \bar{g}^{\rho\lambda}, \quad k_3 = \bar{g}^{\rho\lambda} \bar{R}_{\mu\nu} \bar{R}_\rho, \quad k_4 = \delta_{\nu\rho} \bar{R}_{\mu\lambda} \bar{R}_\rho \bar{R}_{\lambda\mu}, \quad k_5 = \delta_{\rho\nu} \bar{R}_{\mu\lambda} \bar{R}_\rho \bar{R}_{\lambda\mu}, \quad k_6 = \delta_{\nu\rho} \bar{R}_{\mu\lambda} \bar{R}_\rho \bar{R}_{\lambda\mu}, \quad k_7 = \frac{1}{2} (\delta_{\nu\rho} \bar{R}_{\mu\lambda} + \delta_{\nu\lambda} \bar{R}_{\mu\rho}), \quad k_8 = \bar{g}_{\nu\beta} \delta_{\mu\rho} \bar{R}_{\alpha\lambda \rho \lambda}, \quad k_9 = \bar{g}_{\nu\beta} \bar{R}_{\alpha\rho \lambda \rho \lambda}, \quad k_{10} = \frac{1}{2} (\delta_{\alpha\rho} \bar{R}_{\mu\nu} \bar{R}_{\lambda\lambda} + \delta_{\mu\nu} \bar{R}_{\lambda\lambda} \bar{R}_{\rho\lambda}), \quad k_{11} = \bar{g}_{\nu\beta} \bar{R}_{\mu\lambda} \bar{R}_\rho, \quad k_{12} = \bar{g}_{\nu\beta} \bar{R}_{\mu\rho}, \quad k_{13} = \bar{g}_{\nu\beta} \bar{R}_{\rho\lambda \rho \lambda}, \quad k_{14} = \bar{g}_{\mu\rho} \bar{g}^{\rho\lambda} \bar{R}_{\mu\lambda}, \quad k_{15} = \bar{g}_{\mu\rho} \delta_{\rho\lambda} \bar{R}_{\mu\lambda}, \quad k_{16} = \bar{g}_{\nu\beta} \delta_{\rho\lambda} \bar{R}_{\mu\lambda}, \quad k_{17} = \bar{g}_{\mu\rho} \delta_{\rho\lambda} \bar{R}_{\mu\lambda}, \quad k_{18} = \bar{g}_{\mu\rho} \delta_{\rho\lambda} \bar{R}_{\mu\lambda}, \quad k_{19} = \bar{g}_{\mu\rho} \bar{g}_{\nu\beta} \bar{R}_{\rho\lambda}, \quad k_{20} = \bar{g}_{\nu\beta} \bar{g}_{\rho\lambda} \bar{R}_{\rho\lambda}.\]

(28)

and
These results are given in Refs. [18, 19], but they still contained errors, and this was corrected in [16]. Here we give the correct results using heat kernel expansion.

The partition function for the one-loop is obtained from the quadratic terms of the action as

\[ S^2 = \alpha \tilde{R}^2 + \beta \tilde{R}^{2 \mu \nu} + \gamma \tilde{R}^{3 \mu \nu \lambda} - \frac{1}{k^2}(\tilde{R} - 2\Lambda), \]  

These results are given in Refs. [18, 19].

### 4 General formula for one-loop divergences

Correcting the errors in [8], the results for one-loop divergences were given in [9, 10], but they still contained errors, and this was corrected in [16]. Here we give the correct results using heat kernel expansion.

The partition function for the one-loop is obtained from the quadratic terms of the action as
for a real scalar field, and the effective action is given by
\[ \Gamma = -\log Z = \frac{1}{2} \text{Tr} \log \Delta. \] (35)

If the fluctuations are anticommuting fields like the FP ghosts, the sign should be opposite. If the fluctuations are complex fields (or independent two hermitian fields), the front factor should be 1.

Suppose we know the eigenvalues and eigenfunctions of the operator \( \Delta \):
\[ \Delta \phi_n = \lambda_n \phi_n. \] (36)

Then we can evaluate (35) as
\[ \frac{1}{2} \text{Tr} \log \Delta = \frac{1}{2} \sum_n \log \lambda_n. \] (37)

We define the zeta function for \( \Delta \):
\[ \zeta_\Delta(s) = \sum_{n=1}^\infty \lambda_n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr}(e^{-t\Delta}), \] (38)

and obtain
\[ \frac{1}{2} \text{Tr} \log \Delta = \frac{1}{2} \frac{d}{ds} \zeta_\Delta(s) \bigg|_{s=0} = \frac{1}{2} \frac{d}{ds} \left[ \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr}(e^{-t\Delta}) \right] \bigg|_{s=0}. \] (39)

For a differential operator of order \( p \) in \( d \) dimensions, the heat kernel \( e^{-t\Delta} \) has the expansion
\[ \text{Tr}(e^{-t\Delta}) = \int \frac{d^d x}{(4\pi)^{d/2}} \sqrt{g} \sum_{n=0}^\infty b_{2n}(\Delta) t^{(2n-d)/p}. \] (40)

Typically we have the operators of \( p = 4 \) and \( p = 2 \). The formulae are given separately. The divergent part of the effective action for \( p = 4 \) operator is evaluated as
\[ \Gamma^{(4)} = -\frac{1}{2} \int \frac{d^d x}{(4\pi)^{d/2}} \sqrt{g} \int_{1/\Lambda_{UV}^4}^{1/\mu^4} dt \left[ t^{-\frac{d}{4}-1} b_0 + t^{-\frac{d-2}{4}-1} b_2 + \cdots + t^{-1} b_d + \cdots \right] \]
\[ = -\frac{1}{2} \int \frac{d^d x}{(4\pi)^{d/2}} \sqrt{g} \left[ \Lambda_{UV}^d b_0 + \Lambda_{UV}^{d-2} (d-2)/4 b_2 + \cdots + \log \frac{\Lambda_{UV}^d}{\mu^d} b_d + \text{finite} \right], \] (41)

where the heat kernel coefficients are given in the subsection 4.1. Here \( \Lambda_{UV} \) is an ultraviolet cutoff which should be distinguished from the cosmological constant \( \Lambda \).
The divergent part of the effective action for $p = 2$ operator is given by

$$
\Gamma^{(2)} = -\frac{1}{2} \int \frac{d^d x}{(4\pi)^{d/2}} \sqrt{g} \int_{1/\Lambda_{UV}^2}^{1/\mu^2} dt \left[ t^{-d/2} b_0 + t^{-\frac{d}{2} - 1} b_0 + \cdots + t^{-1} b_d + \cdots \right]
$$

$$
= -\frac{1}{2} \int \frac{d^d x}{(4\pi)^{d/2}} \sqrt{g} \left[ \Lambda_{UV}^d b_0 + \Lambda_{UV}^{d-2} b_0 + \cdots + \log \frac{\Lambda_{UV}^2}{\mu^2} b_d + \text{finite} \right]. \quad (42)
$$

These are relevant for the contributions from the ghost and the operator $Y$.

In our present case, the graviton contribution (25) is $p = 4$ and the ghost contributions are $p = 2$. So the one-loop part of our effective action is

$$
\Gamma^{1-\text{loop}} = \frac{1}{2} \text{Tr} \log \mathcal{H} - \text{Tr} \log \Delta_{gh} - \frac{1}{2} \text{Tr} \log Y
$$

$$
= -\int \frac{d^d x}{2(4\pi)^{d/2}} \sqrt{g} \left[ \Lambda_{UV}^d \left( b_0(\mathcal{H}) - b_0(\Delta_{gh}) - \frac{1}{2} b_0(Y) \right) 
+ \Lambda_{UV}^{d-2} \left( 2b_2(\mathcal{H}) - 2b_2(\Delta_{gh}) - b_2(Y) \right)
+ \log \frac{\Lambda_{UV}^2}{\mu^2} \left( 2b_4(\mathcal{H}) - 2b_4(\Delta_{gh}) - b_4(Y) \right) \right]. \quad (43)
$$

Now we start the evaluation of these contributions.

### 4.1 Contributions from $p = 4$ minimal operator

First we have to evaluate the contribution from (25) which is $p = 4$ minimal operator. The coefficients in the heat kernel expansion for spin 2 are [13]:

$$
b_0(\mathcal{H}) = \frac{\Gamma(d/4)}{2\Gamma(d/2)} \frac{d(d+1)}{2}, \quad (44)
$$

$$
b_2(\mathcal{H}) = \frac{\Gamma((d-2)/4)}{2\Gamma((d-2)/2)} \frac{d+1}{2} \left[ \frac{\tilde{R}}{6} + \frac{1}{2d} \tilde{V}_\mu^\mu \right], \quad (45)
$$

$$
b_4(\mathcal{H}) = \frac{\Gamma(d/4)}{2\Gamma((d-2)/2)} \left[ \frac{1}{90} \tilde{R}_{\mu\nu\rho\tau}^2 - \frac{1}{90} \tilde{R}_{\mu\nu}^2 + \frac{1}{36} \tilde{R}^2 + \frac{1}{6} \Omega_{\rho\lambda} \Omega^{\rho\lambda} - \frac{2}{d-2} U 
- \frac{1}{6(d-2)} (2\tilde{R}_{\rho\lambda} V^{\rho\lambda} - \tilde{R} V^{\rho\lambda}) + \frac{1}{4(d^2 - 4)} (V^{\rho\lambda} V^{\rho\lambda} + 2V_{\rho\lambda} V^{\rho\lambda}) 
+ \frac{1}{15} \tilde{\Box} \tilde{\Box} + \frac{d+4}{6(d^2 - 4)} \tilde{\Box} V^{\rho\lambda} + \frac{2(d+1)}{3(d^2 - 4)} \tilde{\Box} \tilde{\Box} + \frac{2(d+1)}{3(d^2 - 4)} \tilde{\Box} \tilde{\Box} \right]. \quad (46)
$$

where $\hat{1}$ is the identity defined in (13) and $\Omega_{\rho\lambda}$ is the commutator of the covariant derivatives acting on the tensor $h^{\mu\rho}$: $\Omega_{\rho\lambda} = [\tilde{\nabla}_\rho, \tilde{\nabla}_\lambda]$. The traces should
be taken over the space of symmetric tensors with the identity (13). These give
\[
\text{tr}(\hat{1}) = \frac{d(d+1)}{2}, \quad \text{tr}(\Omega_{\mu\nu} \Omega^{\mu\nu}) = -(d+2)\bar{R}^2_{\mu\nu\alpha\beta},
\]
We need more explicit formulae for the traces. Because these are very complicated for general dimensions, we give the results for \(d = 4\) and omit total derivative terms. We find
\[
\text{tr} U = \delta^{\mu\nu,\alpha\beta} U_{\mu\nu,\alpha\beta} = A_1 R^2_{\mu\nu\rho\lambda} + A_2 R^2_{\mu\nu} + A_3 \bar{R}^2 - A_4 \frac{\bar{R}}{k^2} - A_5 \frac{\Lambda}{k^2},
\]
where
\[
A_1 = 3, \quad A_2 = \frac{8}{3} + \frac{4\lambda}{\xi}, \quad A_3 = \frac{1}{3} + \frac{2\lambda}{\xi}, \quad A_4 = 3\lambda, \quad A_5 = \frac{2}{9}(84\lambda - \xi).
\]
and
\[
\text{tr} (V^{\mu\rho} \bar{R}) = B_1 \bar{R}^2 - B_2 \frac{\bar{R}}{k^2},
\]
where
\[
B_1 = -\frac{68}{3} + \frac{32\lambda}{\xi}, \quad B_2 = -20\lambda + \frac{2\xi}{3}.
\]
Next
\[
\text{tr} (V^{\rho\lambda} \bar{R}_{\mu\nu}) = C_1 \bar{R}^2_{\mu\nu} + C_2 \bar{R}^2 - C_3 \frac{\bar{R}}{k^2},
\]
where
\[
C_1 = \frac{8}{3} - \frac{8\lambda}{\xi}, \quad C_2 = -\frac{19}{3} + \frac{10\lambda}{\xi}, \quad C_3 = -5\lambda + \frac{\xi}{6}.
\]
Finally
\[
\frac{1}{48} \text{tr}(V^{\mu\rho} V^{\rho\lambda}) + \frac{1}{24} \text{tr}(V_{\mu\rho} V^{\rho\lambda}) = D_1 \bar{R}^2_{\mu\nu\rho\lambda} + D_2 \bar{R}^2_{\mu\nu} + D_3 \bar{R}^2 - D_4 \frac{\bar{R}}{k^2} + D_5 \frac{1}{k^2},
\]
where
\[
D_1 = 6, \quad D_2 = \frac{2(18\lambda^2 + 6\lambda \xi + 113\xi^2)}{27\xi^2}, \quad D_3 = \frac{576\lambda^2 - 240\lambda \xi - 47\xi^2}{54\xi^2},
\]
\[
D_4 = \frac{-180\lambda^2 + 30\lambda \xi + \xi^2}{18\xi}, \quad D_5 = \frac{180\lambda^2 + \xi^2}{72}.
\]
4.2 Contribution from the $p = 2$ nonminimal vector operator

To find the contribution from the vector operator, we need the contribution from $p = 2$ nonminimal vector operator. The general form of the $p = 2$ nonminimal operator is

$$\Delta = -\tilde{g}^{\mu\nu}\square + a\tilde{\phi}^{\mu}\tilde{\phi}^{\nu} + X^{\mu\nu},$$  \hfill (55)

The coefficients in the heat kernel expansion (40) have been calculated in [14]. For the general case, we have

$$b_0 = (1 - a)^{-d/2} + d - 1,$$ \hfill (56)

$$b_2 = \left(\frac{d^2 - d - 6}{6d} + (1 - a)^{-d/2}\frac{6 + (1 - a)d}{6d}\right)\bar{R} + \frac{1 - d - (1 - a)^{-d/2}}{d}X,$$ \hfill (57)

$$b_4 = \frac{d - 16 + (1 - a)^{2-d/2}}{180}\bar{R}_{\mu
u}^{\alpha\beta}
+ \frac{1}{180ad(d^2 - 4)}\left[ -360d - (d^4 - d^3 + 116d^2 - 296d - 360)a + (1 - a)^{-d/2}[-360a
+ 4(90 - 74a + 28a^2 + a^3)d - 60a(1 - a)d^2 - (1 - a)^2a d^3]\bar{R}_{\mu
u}^2
+ \frac{1}{72ad(d^2 - 4)}\left[72(2 - a) + (16d - 16d^2 - d^3 + d^4)a
+ (1 - a)^{-d/2}[-72(2 - a) - 4a(a^2 + 4a - 14)d + 12a(1 - a)d^2 + a(1 - a)^2d^3]\bar{R}^2
+ \frac{1}{6ad(d^2 - 4)}\left[1 - 24a + 12a + ad^2 - ad^3 + (1 - a)^{-d/2}[24 - 12a
+ 2a(a - 4)d - a(1 - a)d^2]\bar{R}X + 2\left[-12a + 4(3 - 2a)d + 5a d^2
+ (1 - a)^{-d/2}[12a - 2(6 - 4a + a^2)d + a(1 - a)d^2]\bar{R}_{\mu
u}X^{\mu\nu}
+ 3\left[-4a + ad + (1 - a)^{-d/2}(-4 + 2a + ad)\right]X^2 + 3\left[-4a + 2(2 - a)d - 2ad^2
+ ad^3 + (1 - a)^{-d/2}[-4a + 2(2 - a)d]\right]X_{\mu\nu}X^{\mu\nu}\right] + \text{(total derivative terms)},$$ \hfill (58)

where $X = X^{\mu}$.

**Contribution from the ghost operator $\Delta_{gh}$**

For the ghost operator $\Delta_{gh}$ in (19), we see that we have $a = -1 - \frac{\Delta_{gh}}{2(\gamma - a)} \equiv \sigma_g$ and $X_{\mu\nu} = -\bar{R}_{\mu\nu}$ in (55). The above formulae give, for $d = 4$, 

...
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\[ b_0(\Delta_{gh}) = (1 - \sigma_g)^{-2} + 3, \tag{59} \]

\[ b_2(\Delta_{gh}) = \frac{6\sigma_g^2 - 13\sigma_g + 10}{6(1 - \sigma_g)^2} \bar{R}, \tag{60} \]

\[ b_4(\Delta_{gh}) = \frac{-11 R^2}{180} \bar{R}_{\mu \nu \rho \lambda} - \frac{2\sigma_g^2 + 26\sigma_g - 43}{90(1 - \sigma_g)^2} \bar{R}_{\mu \nu} + \frac{5\sigma_g^2 - 10\sigma_g + 8}{36(1 - \sigma_g)^2} \bar{R}^2. \tag{61} \]

**Contribution from the the additional ghost operator**

For the operator \( \bar{\nabla}_\mu \bar{\nabla}_\nu \chi^\nu \) in (19), we first note that \([ \bar{\nabla}_\mu, \bar{\nabla}_\nu ] \chi^\nu = -\bar{R}_{\mu \nu} \chi^\nu\). Using this relation, we find that we have \( a = 1 + 2 \frac{\sigma - Y}{\beta + \gamma} \equiv \sigma_Y \) and \( X_{\mu \nu} = \bar{R}_{\mu \nu} \) in (55), and the above formulae give, for \( d = 4 \),

\[ b_0(Y) = (1 - \sigma_Y)^{-2} + 3, \tag{62} \]

\[ b_2(Y) = \frac{3\sigma_Y - 2}{6(1 - \sigma_Y)} \bar{R}, \tag{63} \]

\[ b_4(Y) = \frac{-11 R^2}{180} \bar{R}_{\mu \nu \rho \lambda} + \frac{43}{90} \bar{R}_{\mu \nu} - \frac{1}{9} \bar{R}^2. \tag{64} \]

### 5 One-loop divergences and asymptotic freedom

We are now ready to calculate the divergences in the quadratic curvature theory on the general backgrounds. For this purpose, we need to know the heat kernel coefficients. In this section, we restrict the dimension of our spacetime to four.

Putting the results for \( \mathcal{H} \) into eq. (46), we get

\[ b_0(\mathcal{H}) = \frac{\sqrt{\pi}}{2} \left[ \frac{5\bar{R}}{3} + \frac{1}{8} \left( B_1 \bar{R} - B_2 \frac{1}{\kappa} \right) \right], \tag{65} \]

\[ b_2(\mathcal{H}) = \frac{\sqrt{\pi}}{2} \left[ \frac{5\bar{R}}{3} + \frac{1}{8} \left( B_1 \bar{R} - B_2 \frac{1}{\kappa} \right) \right], \tag{66} \]

\[ b_4(\mathcal{H}) = \frac{1}{2} \bar{R}^2 \bar{R}_{\mu \nu \rho \lambda} \left( -\frac{8}{9} - A_1 + D_1 \right) - \bar{R}_{\mu \nu} \left( \frac{1}{9} + A_2 + \frac{1}{6} C_1 - D_2 \right) \]

\[ + \bar{R}^2 \left( \frac{5}{18} - A_3 + \frac{1}{12} B_1 - \frac{1}{6} C_2 + D_3 \right) + \frac{1}{\kappa^2} \bar{R} \left( A_4 - \frac{1}{12} B_2 + \frac{1}{6} C_3 - D_4 \right) \]

\[ + \frac{1}{\kappa^2} \left( \Lambda A_5 + \frac{1}{\kappa^2} D_5 \right), \tag{67} \]

where the constants \( A_i, B_i, C_i \) and \( D_i \) are given in (48) – (54).

Collecting other contributions from ghosts and \( Y \), we finally get
\[\Gamma^{1\text{-loop}} = -\int \frac{d^4x}{(4\pi)^2} \left[ \frac{133}{20} C^2 + \left( 10 \frac{\lambda^2}{\xi^2} - 5 \frac{\lambda^4}{\xi^4} + \frac{5}{36} \right) \bar{R}^2 - \frac{196}{45} E \right] + \frac{(30 \lambda - \xi)(4 \lambda + \xi)}{12 \xi} \left[ \frac{84 \lambda - \xi}{9 \xi} \frac{2}{3} \Lambda + \frac{180 \lambda^2 + \xi^2}{72 \xi} \right] \log \frac{\Lambda^2_{\text{UV}}}{\mu^2} \]

Note that the coefficient of the Euler term \(E\) is independent of any coupling, in particular of its coupling \(\rho\). This is to be expected, because the Euler term is a topological term and is a total derivative itself, and as such it does not contribute to the Hessian and therefore to quantum effects [21]. Thus it is a universal result that it is independent of the coupling \(\rho\) whatever the approximation is (beyond one loop).

The correspondence between the cutoff and the dimensional regularization is

\[\log \frac{\Lambda^2_{\text{UV}}}{\mu^2} \leftrightarrow \frac{2}{(4\pi)^2} \text{log} \frac{\Lambda^2_{\text{UV}}}{\mu^2}\]

and one can try to compare the results with the existing literature (see for example, [16, 22]).

Note also that there are strange terms with coefficients \(\sqrt{\pi}\) in the quadratic divergences. Indeed, if we consider the minimal operators \(F_1 = \Box + P_1\) and \(F_2 = \Box + P_2\) and consider the quadratic divergences of \(\text{Tr} \log(F_1 F_2)\), it appears that we get such factor of \(\sqrt{\pi}\) from the formula \(b_2\) in (66), whereas if we write it as \(\text{Tr} \log F_1 + \text{Tr} \log F_2\), we do not get \(\sqrt{\pi}\) as is clear from (57) for the same quantity. This clearly indicates that the coefficients of the quadratic divergences depend on how we calculate. Thus the power divergences are not universal and do not lead to physical effects like renormalization group scaling. The above results are just those obtained by the naive application of the formulae but should not be taken seriously.

On the other hand, the logarithmic divergences are universal. From the coefficients, we can determine the beta functions for the dimensionless couplings: recall that together with the bare terms, the coefficients of \(C^2\) should give the renormalized coupling

\[\frac{1}{2 \lambda_R} = \frac{1}{2 \lambda_B} - \frac{133}{(4\pi)^2} \log \frac{\Lambda^2_{\text{UV}}}{\mu^2}. \quad (69)\]

Since the bare coupling \(\lambda_B\) does not depend on the renormalization scale \(\mu\), differentiation of the expression with respect to \(\log \mu\) gives

\[-\frac{1}{2 \lambda_R^2} \frac{d \lambda_R}{d \mu} = \frac{133}{(4\pi)^2} \frac{20}{20} \quad (70)\]

Omitting the subscript \(R\), this gives the beta function of the coupling \(\lambda\):
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\[
\frac{d\lambda}{d\mu} = \beta_\lambda = -\frac{1}{(4\pi)^2} \frac{133}{10} \lambda^2, \tag{71}
\]

Similarly we find the beta functions for other dimensionless couplings:

\[
\beta_\xi = -\frac{1}{(4\pi)^2} \left( 10\lambda^2 - 5\lambda\xi + \frac{5}{36}\xi^2 \right),
\]

\[
\beta_\nu = -\frac{1}{(4\pi)^2} \frac{196}{45} \nu^2. \tag{72}
\]

The first equation (71) tells us that the coupling \(\lambda\) goes to zero from positive \(\lambda\). Similarly the other couplings also go to zero. This is known as asymptotic freedom [9, 10, 16, 17, 18, 19, 20]. However it has recently been discovered that there are fixed point at finite values for these couplings [21] in addition to these Gaussian fixed points. See, however, [23].

6 Divergences for \(f(R, R_{\mu\nu}^2)\) gravity

As an interesting case of higher-derivative gravity, here we present the one-loop divergences for \(f(R, R_{\mu\nu})\) gravity on the Einstein space [24]:

\[
\bar{R}_{\mu\nu} = \frac{\bar{R}}{d\bar{g}_{\mu\nu}}. \tag{73}
\]

Moreover we consider a general parametrization of the fluctuations

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \tag{74}
\]

where the fluctuation is expanded:

\[
\delta g_{\mu\nu} = \delta g_{\mu\nu}^{(1)} + \delta g_{\mu\nu}^{(2)} + \delta g_{\mu\nu}^{(3)} + \ldots, \tag{75}
\]

where \(\delta g_{\mu\nu}^{(n)}\) contains \(n\) powers of \(h_{\mu\nu}\). We will parametrize the first two terms of the expansion as follows:

\[
\delta g_{\mu\nu}^{(1)} = h_{\mu\nu} + m\bar{g}_{\mu\nu}h, \]

\[
\delta g_{\mu\nu}^{(2)} = \omega h_{\mu\nu} h^\rho + m h_{\mu\nu} + m \left( \omega - \frac{1}{2} \right) \bar{g}_{\mu\nu} h^{\alpha\beta} h_{\alpha\beta} + \frac{1}{2} m^2 \bar{g}_{\mu\nu} h^2. \tag{76}
\]

It is convenient to use York decomposition

\[
h_{\mu\nu} = h_{\mu\nu}^{TT} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \nabla_\mu \nabla_\nu \sigma - \frac{1}{d} \bar{g}_{\mu\nu} \nabla^2 \sigma + \frac{1}{d} \bar{g}_{\mu\nu} h, \tag{77}
\]

where

\[
\frac{d\lambda}{d\mu} = \beta_\lambda = -\frac{1}{(4\pi)^2} \frac{133}{10} \lambda^2, \tag{71}
\]
\[ \nabla \mu h_{\mu \nu}^{TT} = 0; \quad \tilde{g}^{\mu \nu} h_{\mu \nu}^{TT} = 0; \quad \nabla \nu \xi^\mu = 0. \]

We then find that the Hessian is
\[ S^{(2)} = \int d^{d}x \sqrt{\tilde{g}} \left[ h_{\mu \nu}^{TT} H^{TT} h_{\mu \nu}^{\prime \prime} + \xi_{\mu} H^{\mu} \xi_{\nu} + \sigma H^{\sigma \sigma} \sigma + \sigma H^{\sigma} h + h H^{\sigma} \sigma \right. \]
\[ + \left. h H^{\mu} \right], \] (78)

where
\[ H^{TT} = \frac{1}{2} \left[ \tilde{f}_{\mu} \left( \Delta_{L2} - 4 \tilde{R} \frac{d}{d} \right) - \tilde{f}_{R} \left( \Delta_{L2} - \frac{2 \tilde{R}}{d} \right) \right] - (1 - 2 \omega)(1 + md) \tilde{E}, \] (79)
\[ H^{\mu} = \frac{1}{2} \left( \frac{d-1}{d} \right)^{2} \left[ P \Delta_{L0} \left( \Delta_{L0} - \tilde{R} \frac{d}{d-1} \right) + Q \Delta_{L0} \left( \Delta_{L0} - \tilde{R} \frac{d}{d-1} \right) \right] - \frac{d(1 - 2 \omega)(1 + md)}{2(d - 1)} \tilde{E} \Delta_{L0} \left( \Delta_{L0} - \tilde{R} \frac{d}{d-1} \right), \] (80)
\[ H^{\sigma \sigma} = \frac{1}{2} \left( \frac{d-1}{d} \right)^{2} \left[ P \Delta_{L0} \left( \Delta_{L0} - \tilde{R} \frac{d}{d-1} \right) + Q \Delta_{L0} \left( \Delta_{L0} - \tilde{R} \frac{d}{d-1} \right) \right] \Delta_{L0} \left( \Delta_{L0} - \tilde{R} \frac{d}{d-1} \right), \] (81)
\[ H^{\sigma h} = \frac{1}{2} \left( \frac{d-1}{d} \right)^{2} \left[ P \Delta_{L0} \left( \Delta_{L0} - \tilde{R} \frac{d}{d-1} \right) + Q \Delta_{L0} \left( \Delta_{L0} - \tilde{R} \frac{d}{d-1} \right) \right] \Delta_{L0} \left( \Delta_{L0} - \tilde{R} \frac{d}{d-1} \right) \Delta_{L0} \left( \Delta_{L0} - \tilde{R} \frac{d}{d-1} \right), \] (82)
\[ H^{hh} = \frac{1}{4(d-1)^{2}(1+md)} \tilde{E}, \] (83)

where \( \Delta_{L2}, \Delta_{L1} \) and \( \Delta_{L0} \) are the Lichnerowicz Laplacians defined as
\[ \Delta_{L2} T_{\mu \nu} = -\tilde{\nabla}^{2} T_{\mu \nu} + \tilde{R}_{\mu} T_{\nu \rho} + \tilde{R}_{\mu \nu} T^{\rho \sigma} - \tilde{R}_{\mu \nu \rho} T^{\sigma \rho}, \]
\[ \Delta_{L1} V_{\mu} = -\tilde{\nabla}^{2} V_{\mu} + \tilde{R}_{\mu} V_{\rho}, \]
\[ \Delta_{L0} S = -\tilde{\nabla}^{2} S, \] (84)

and the subscripts on \( f \) denote derivatives with respect to its arguments:
\[ f_{R} = \frac{\partial f}{\partial \tilde{R}}, \quad f_{X} = \frac{\partial f}{\partial X}, \quad f_{RR} = \frac{\partial^{2} f}{\partial \tilde{R}^{2}}, \quad f_{RX} = \frac{\partial^{2} f}{\partial \tilde{R} \partial X}, \quad f_{XX} = \frac{\partial^{2} f}{\partial X^{2}}, \] (85)

with
\[ X \equiv \tilde{R}_{\mu \nu}^{2}. \] (86)

We have also used the shorthands.
We can rewrite this as

\[ P = \tilde{f}_{RR} + \frac{4}{d^2} \tilde{f}_{XX} + 4 \tilde{R} \tilde{f}_{RX} + \frac{d}{2(d-1)} \tilde{f}_X \]  

(87)

\[ Q = \frac{d-2}{2(d-1)} \tilde{f}_R + \frac{3d^2 - 10d + 8}{2d(d-1)^2} \tilde{R} \tilde{f}_X \]  

(88)

and

\[ \tilde{E} \equiv \frac{2}{d} \tilde{R} \tilde{f}_R - \frac{4 \tilde{R}^2}{d^2} \tilde{f}_X = 0, \]  

(89)

is the field equation evaluated on the Einstein space (73).

Our gauge fixing is

\[ S_{GF} = \frac{1}{2a} \int d^d x \sqrt{\tilde{g}} \tilde{g}^\mu \nu F_\mu F_\nu, \]  

(90)

with

\[ F_\mu = \tilde{\nabla}_a h^\alpha_{\mu} - \frac{\tilde{b} + 1}{d} \tilde{\nabla}_\mu h, \]  

(91)

and \( a \) and \( \tilde{b} \) are gauge parameters. This can be rewritten as

\[ S_{GF} = -\frac{1}{2a} \int d^d x \sqrt{\tilde{g}} \left[ \xi_\mu \left( \Delta_{L,1} - \frac{2 \tilde{R}}{d} \right) \tilde{g}^\mu + \frac{(d-1-b)^2}{d^2} \Delta \Delta_{L,0} \left( \Delta_{L,0} - \frac{\tilde{R}}{d-1-b} \right)^2 \chi \right], \]  

(92)

in terms of the new field

\[ \chi = \frac{(d-1)\Delta_{L,0} - \tilde{R}}{(d-1-b)\Delta_{L,0} - \tilde{R}} \tilde{\nabla}_\sigma - \frac{b(1+md)}{(d-1-b)\Delta_{L,0} - \tilde{R}} h, \]  

(93)

where \( b = \tilde{b}/(1+md) \).

The ghost action contains a nonminimal operator

\[ S_{gh} = i \int d^d x \sqrt{\tilde{g}} \tilde{C}_\mu \left( \delta_\nu \tilde{\nabla}^2 + \left( 1 - \frac{2b+1}{d} \right) \tilde{\nabla}_\mu \tilde{\nabla}^\nu + \tilde{R}_\mu^\nu \right) C_\nu, \]  

(94)

We can rewrite this as

\[ S_{gh} = i \int d^d x \sqrt{\tilde{g}} \left[ \tilde{C}_\mu \left( \Delta_{L,1} - \frac{2 \tilde{R}}{d} \right) C_\mu + \frac{2d-1-b}{d} \tilde{C}_\nu L \left( \Delta_{L,0} - \frac{\tilde{R}}{d-1-b} \right) C_\nu \right], \]  

(95)

in terms of the transverse and longitudinal parts of the ghost field:

\[ C_\nu = C_\nu^T + \tilde{\nabla}_\nu C^L = C_\nu^T + \tilde{\nabla}_\nu \frac{1}{\sqrt{\Delta_{L,0}}} C^L, \]  

(96)
and the same for \( \mathcal{C} \). This change of variables has unit Jacobian.

\[
S_{gh} = i \int d^d x \sqrt{\bar{g}} \left[ \bar{C}^T \bar{C}_\mu \left( \Delta_{L1} - \frac{2 \bar{R}}{d} \right) C^T \right] + 2 \frac{d-1-b}{d} \bar{C}^T \left( \Delta_{L0} - \frac{\bar{R}}{d-1-b} \right) C^T ] .
\]

(97)

Unless we (1) set \( \omega = \frac{1}{2} \), or (2) put \( m = -\frac{1}{d} \) or (3) go on shell, the effective action is gauge dependent. If we impose the on-shell condition (89), the effective action is gauge independent and is independent of \( \omega \) and \( m \). In this case, we find

\[
\Gamma = \frac{1}{2} \text{Tr} \log \left( \Delta_{L2} - \frac{4 \bar{R}}{d} - \frac{\bar{f}_R}{f_X} \right) + \frac{1}{2} \text{Tr} \log \left( \Delta_{L2} - \frac{2 \bar{R}}{d} \right)
+ \frac{1}{2} \text{Tr} \log \left( \Delta_{L0} - \frac{R}{d-1} + \frac{Q}{P} \right) - \frac{1}{2} \text{Tr} \log \left( \Delta_{L1} - \frac{2 \bar{R}}{d} \right) .
\]

(98)

If \( \bar{f}_X = 0 \), the first contribution is absent.

The divergent part of the effective action can be computed by the heat kernel methods. On an Einstein background in four dimensions, with the help of the heat kernel coefficients for the Lichnerowicz operator summarized in Appendix A.2, the logarithmically divergent part is found to be [24]

\[
\Gamma_{\log}(\bar{g}) = \frac{1}{720(4\pi)^2} \int d^4 x \sqrt{\bar{g}} \log \left( \frac{\Lambda^2_{UV}}{\mu^2} \right) \left[ -826 \bar{R}_{\mu
u\rho\sigma}^2 + 509 \bar{R}^2 - \frac{300 \bar{f}_R}{f_X} - \frac{900 \bar{f}_R^2}{f_X} + 320 \bar{f}_R \left( 3 \bar{f}_R + 2 \bar{f}_X \right) \right] \frac{240 \bar{R} (3 \bar{f}_R + 2 \bar{f}_X)}{8 \bar{f}_X + 12 \bar{f}_{RR} + 48 \bar{f}_{RX} + 3 \bar{R}^2 \bar{f}_{XX}} - \frac{320 (3 \bar{f}_R + 2 \bar{f}_X)^2}{\left( 8 \bar{f}_X + 12 \bar{f}_{RR} + 48 \bar{f}_{RX} + 3 \bar{R}^2 \bar{f}_{XX} \right)^2} .
\]

(99)

where \( \Lambda_{UV} \) stands for a cutoff and we introduced a reference mass scale \( \mu \).

For the choice

\[
f(R, X) = \alpha \bar{R}^2 + \beta X ,
\]

(100)

it reduces to

\[
\Gamma_{\log}(\bar{g}) = \frac{1}{(4\pi)^2} \int d^4 x \sqrt{\bar{g}} \log \left( \frac{\Lambda^2_{UV}}{\mu^2} \right) \left[ \frac{413}{360} \bar{R}_{\mu
u\rho\sigma}^2 - \frac{1200 \alpha^2 + 200 \alpha \beta - 183 \beta^2}{240 \beta^2} \bar{R}^2 \right] ,
\]

(101)

which is the standard universal result in higher-derivative gravity.

On the other hand if we put

\[
f(R, X) = f(R) ,
\]

(102)

we obtain
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\[ \Gamma_{\log}(\bar{g}) = \frac{1}{(4\pi)^2} \int d^4x \sqrt{\bar{g}} \log \left( \frac{\Lambda_{UV}^4}{\mu^2} \right) \left[ \frac{71}{120} \bar{R}_{\mu
u\rho\sigma}^2 + \frac{433}{1440} \bar{R}^2 + \frac{f_{R\bar{R}}}{12f_{RR}} - \frac{f_R^2}{36f_{RR}^2} \right], \] (103)

which agrees with the results of [25, 26].

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Appendix

A.1 Expansion of curvatures up to second order

Here we summarize our conventions and formulae necessary in the text. We give these such that they are valid for any dimension \( d \).

Our signature of the metric is \((- , + , \cdots +)\) and the curvature tensors are given as

\[ R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\beta\nu} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\beta\mu}, \]
\[ R_{\mu\nu} = \bar{R}^\alpha_{\mu\alpha\nu}. \] (104)

The backgrounds are denoted with overbar. Expansion around the background gives

\[ \Gamma^\alpha_{\mu\nu} = \bar{\Gamma}^\alpha_{\mu\nu} + \Gamma^{(1)}_{\mu\nu} + \Gamma^{(2)}_{\mu\nu}, \] (105)

where

\[ \Gamma^{(1)}_{\mu\nu} = \frac{1}{2} (\bar{\nabla}_\nu h^\alpha_{\mu} + \bar{\nabla}_\mu h^\alpha_{\nu} - \bar{\nabla}^\alpha h_{\mu\nu}), \] (106)
\[ \Gamma^{(2)}_{\mu\nu} = -\frac{1}{2} h^{\alpha\beta} (\bar{\nabla}_\nu h_{\mu\beta} + \bar{\nabla}_\mu h_{\nu\beta} - \bar{\nabla}_\beta h_{\mu\nu}). \] (107)

Note that

\[ \sqrt{-g} = \sqrt{\bar{g}} \left[ 1 + \frac{1}{2} h + \frac{1}{8} (h^2 - 2h^2_{\mu\nu}) + O(h^3) \right], \] (108)

where \( h \equiv h_{\mu\nu} \). We find, to the second order,
\[ R^\mu_{\nu\alpha\beta} = R^\mu_{\nu\alpha\beta} + R^\mu_{(1)}_{\nu\alpha\beta} + R^\mu_{(2)}_{\nu\alpha\beta}, \]

\[ R^\mu_{(1)}_{\nu\alpha\beta} = \frac{1}{2} (\bar{\nabla}_\nu \bar{\nabla}_\alpha h^\mu_{\beta} - \bar{\nabla}_\nu \bar{\nabla}_\beta h^\mu_{\alpha} - \nabla_\beta \nabla_\alpha h^\mu_{\nu} + \nabla_\beta \nabla_\nu h^\mu_{\alpha}) \]

\[ + \frac{1}{2} \bar{R}_{\gamma\alpha\beta} h^\mu_{\nu} + \frac{1}{2} \bar{R}^\mu_{\gamma\alpha\beta} h^\nu_{\gamma}, \] \hspace{1cm} (109)

\[ R^\mu_{(2)}_{\nu\alpha\beta} = -\frac{1}{2} h^\mu_{\gamma} \nabla_\gamma (\bar{\nabla}_\nu \bar{\nabla}_\alpha h^\gamma_{\beta} - \bar{\nabla}_\nu \bar{\nabla}_\beta h^\gamma_{\alpha}) - \frac{1}{4} \nabla_\alpha h^\mu_{\nu} (\bar{\nabla}_\beta h^\gamma_{\nu} + \bar{\nabla}_\nu h^\gamma_{\beta} - \bar{\nabla}_\gamma h^\nu_{\beta}) \]

\[ + \frac{1}{4} \bar{\nabla}_\nu h^\mu_{\alpha} (\bar{\nabla}_\beta h^\nu_{\alpha} + \bar{\nabla}_\alpha h^\nu_{\beta} - \bar{\nabla}_\gamma h^\nu_{\beta}) - \frac{1}{4} \bar{\nabla}_\nu h^\mu_{\gamma} (\bar{\nabla}_\beta h^\nu_{\gamma} + \bar{\nabla}_\gamma h^\nu_{\beta} - \bar{\nabla}_\gamma h^\nu_{\beta}) \]

\[ - (\alpha \leftrightarrow \beta). \] \hspace{1cm} (110)

Similarly

\[ R^\mu_{(1)}_{\nu\mu} = -\frac{1}{2} (\bar{\nabla}_\nu \bar{\nabla}_\alpha h - \bar{\nabla}_\alpha \bar{\nabla}_\nu h + \nabla_\nu \nabla_\alpha h - \bar{R}_{\alpha\mu\nu} h^\alpha_{\beta} + \nabla_\mu h^\alpha_{\beta} + \frac{1}{2} \bar{R}_{\nu\alpha\beta} h^\nu_{\alpha} + \frac{1}{2} \bar{R}_{\nu\alpha\beta} h^\nu_{\alpha}, \]

\[ R^\mu_{(2)}_{\nu\mu} = \frac{1}{2} \bar{R}_\mu (h^\alpha_{\beta} \bar{\nabla}_\nu h^\alpha_{\beta}) - \frac{1}{2} \bar{\nabla}_\nu (h^\alpha_{\beta} (\bar{\nabla}_\mu h^\beta_{\nu} + \bar{\nabla}_\nu h^\beta_{\mu} - \bar{\nabla}_\mu h^\beta_{\nu})) \]

\[ - \frac{1}{4} \bar{\nabla}_\nu h^\alpha_{\beta} + \nabla_\nu h^\alpha_{\beta} - \nabla_\beta h^\alpha_{\nu} (\bar{\nabla}_\mu h^\alpha_{\nu} + \bar{\nabla}_\nu h^\alpha_{\mu} - \bar{\nabla}_\gamma h^\nu_{\beta}) + \nabla_\mu h^\alpha_{\nu} (\bar{\nabla}_\gamma h^\alpha_{\nu} + \bar{\nabla}_\nu h^\alpha_{\gamma} - \bar{\nabla}_\gamma h^\nu_{\gamma}), \]

\[ R^\nu = \bar{\nabla}_\nu h^\mu - \bar{R}_{\nu\mu} h^\nu. \]

\[ R^\nu = \frac{1}{2} \bar{R}_\mu (h^\alpha_{\beta} \bar{\nabla}_\nu h^\alpha_{\beta}) - \frac{1}{2} \bar{\nabla}_\nu (h^\alpha_{\beta} (2h^\beta_{\gamma} - \bar{\nabla}_\beta h)) - \frac{1}{4} \bar{\nabla}_\nu h^\alpha_{\beta} + \nabla_\nu h^\alpha_{\beta} - \nabla_\beta h^\alpha_{\nu} (\bar{\nabla}_\mu h^\alpha_{\nu} + \bar{\nabla}_\nu h^\alpha_{\mu} - \bar{\nabla}_\gamma h^\nu_{\beta}) + \nabla_\mu h^\alpha_{\nu} (\bar{\nabla}_\gamma h^\alpha_{\nu} + \bar{\nabla}_\nu h^\alpha_{\gamma} - \bar{\nabla}_\gamma h^\nu_{\gamma}), \]

\[ = \frac{3}{4} \bar{\nabla}_\nu h^\alpha_{\mu} \bar{\nabla}^\alpha h^\mu_{\nu} + h^\mu_{\nu} \bar{\nabla}^\nu h^\mu_{\nu} - h^\nu_{\mu} h^\mu_{\nu} - h^\nu_{\mu} \bar{\nabla}^\nu h^\mu_{\nu} - h^\nu_{\mu} \bar{\nabla}^\nu h^\mu_{\nu} - h^\nu_{\mu} \bar{\nabla}^\nu h^\mu_{\nu} - h^\nu_{\mu} \bar{\nabla}^\nu h^\mu_{\nu}, \]

\[ - \frac{1}{2} \bar{R}_{\nu\alpha\beta} h^\mu_{\alpha\nu} - \frac{1}{4} \bar{R}_{\nu\alpha\beta} h^\mu_{\alpha\nu} - \bar{R}_{\alpha\beta\gamma} h^\mu_{\alpha\beta} h^\nu_{\gamma}, \] \hspace{1cm} (111)

where \( h^\mu_{\nu} = \bar{\nabla}^\nu h^\mu_{\nu}. \) Note that \( \bar{R}^\mu_{\nu\mu} R^\nu_{\mu\nu} \neq R^\mu_{(1)}_{\nu\mu}, \) because the latter has additional contribution from \( h^\mu_{\nu} \bar{R}^\nu_{\mu\nu}. \) When total derivative terms are dropped, \( R^\mu_{(2)} \) makes the contribution to the action

\[ R^\mu_{(2)} = \frac{1}{4} (h^\mu_{\nu\mu} \bar{\nabla}^\nu h^\mu_{\nu} + h^\nu_{\mu} - 2h^\mu_{\nu^2} + 2 \bar{R}_{\alpha\beta\gamma} h^\mu_{\alpha\beta} h^\nu_{\gamma} + 2 \bar{R}_{\alpha\beta\gamma} h^\mu_{\alpha\beta} h^\nu_{\gamma} + 2 \bar{R}_{\alpha\beta\gamma} h^\mu_{\alpha\beta} h^\nu_{\gamma} h^\rho_{\delta}). \] \hspace{1cm} (112)

We use the notation \( \approx \) to denote equality up to total derivatives.
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A.2 Heat kernel coefficients for $p = 2$ minimal operator

For minimal operator $\Delta = -\nabla^2 + E$, the general formulae for any spin are

\[ b_0 = tr \hat{1}, \quad b_2 = \frac{1}{6} \hat{R} tr \hat{1} - tr E, \]

\[ b_4 = \frac{1}{180} \left( \hat{R}^2_{\mu \nu \alpha \beta} - \hat{R}^2_{\mu \nu} + \frac{5}{2} \hat{R}^2 + 6 \nabla^2 \hat{R} \right) tr \hat{1} + \frac{1}{3} tr E^2 \]

\[- \frac{1}{6} \hat{R} tr E + \frac{1}{12} tr \Omega^2_{\mu \nu} - \frac{1}{6} \nabla^2 tr E, \]  

(113)

where $\Omega_{\mu \nu} = [\nabla_{\mu}, \nabla_{\nu}]$ is the curvature from the covariant derivatives for each spin, and the traces should be taken using the identity $\eta_{\mu \nu}$ for the vector and (13) for the symmetric tensor. The formulae in subsection 4.2 are for spin 1, but with an additional nonminimal term.

Using these formulae (113) to the Lichnerowicz Laplacians (84), we find, for spin 0

\[ b_0(\Delta_{L0}) = 1, \quad b_2(\Delta_{L0}) = \frac{1}{6} \hat{R}, \]

\[ b_4(\Delta_{L0}) = \frac{1}{180} \left( \hat{R}^2_{\mu \nu \alpha \beta} - \hat{R}^2_{\mu \nu} + \frac{5}{2} \hat{R}^2 + 6 \nabla^2 \hat{R} \right), \]

(114)

and for spin 1

\[ b_0(\Delta_{L1}) = d, \quad b_2(\Delta_{L1}) = \frac{d - 6}{6} \hat{R}, \]

\[ b_4 = \frac{d - 15}{180} \hat{R}^2_{\mu \nu \alpha \beta} - \frac{d - 90}{180} \hat{R}^2_{\mu \nu} + \frac{d - 12}{72} \hat{R}^2 + \frac{d - 5}{30} \nabla^2 \hat{R}, \]

(115)

since $\Omega_{\mu \nu} A_\alpha = \hat{R}_{\mu \nu \alpha \beta} A^\beta \equiv (\Omega_{\mu \nu})_{a \beta} A^\beta$ and

\[ \tr \Omega^2_{\mu \nu} = \hat{R}_{\mu \nu \alpha \beta} \hat{R}^{\alpha \beta} = -\hat{R}^2_{\mu \nu \alpha \beta} \]

(116)

For spin 2, we find

\[ b_0(\Delta_{L2}) = \frac{d(d + 1)}{2}, \quad b_2(\Delta_{L2}) = \frac{d^2 - 11d - 24}{12} \hat{R}, \]

\[ b_4(\Delta_{L2}) = \frac{d^2 - 29d + 480}{360} \hat{R}^2_{\mu \nu \alpha \beta} - \frac{d^2 - 359d - 1080}{360} \hat{R}^2_{\mu \nu} \]

\[ + \frac{d^2 - 23d - 48}{144} \hat{R}^2 + \frac{d^2 - 9d - 20}{60} \nabla^2 \hat{R}, \]

(117)

since $(\Omega^2_{\mu \nu})_{a \beta, \rho \sigma} = \hat{R}_{\mu \nu \gamma \rho \delta \beta \sigma} + \hat{R}_{\mu \nu \alpha \rho} \hat{R}^{\alpha \beta \sigma} + \hat{R}_{\mu \nu \beta \sigma} \hat{R}^{\mu \nu \alpha \rho} + \hat{R}_{\mu \nu \gamma \sigma} \hat{R}^{\mu \nu \alpha \rho}$ and
\[ \text{tr}(\Omega^2_{\mu
u}) = \frac{1}{2} \left( \text{sum over } \alpha = \rho, \beta = \sigma \right) + \text{sum over } \alpha = \sigma, \beta = \rho \]

\[ = -(d+2)\tilde{R}^2_{\mu\nu\rho\sigma}. \]  

These are the results when the fields do not have any constraints. If the fields have constraint such as transverse, suitable subtraction is needed for spins 1 and 2.

Spin 0 does not have any constraint, so their heat kernel coefficients are the same as above. For the transverse spin 1, we have \( b_n(\Delta_{L,1}^{T}) = b_n(\Delta_{L,1}) - b_n(\Delta_{L,0}) \). Hence

\[ b_0(\Delta_{L,1}^{T}) = d - 1, \quad b_2(\Delta_{L,1}^{T}) = \frac{d-7}{6}\tilde{R}, \]

\[ b_4(\Delta_{L,1}^{T}) = \frac{d-16}{180}R_{\mu\nu\rho\sigma}^2 - \frac{d-91}{180}R_{\mu\nu}^2 + \frac{d-13}{72}\tilde{R}^2 + \frac{d-6}{30}\nabla^2\tilde{R}. \]  

(119)

For the Einstein space, \( b_4(\Delta_{L,1}^{T}) \) reduces to

\[ b_4(\Delta_{L,1}^{T}) = \frac{d-16}{180}R_{\mu\nu\rho\sigma}^2 + \frac{5d^2-67d+182}{360d}\tilde{R}^2 + \frac{d-6}{30}\nabla^2\tilde{R}. \]  

(120)

Similarly those for transverse and traceless spin 2, we have \( b_n(\Delta_{L,2}^{TT}) = b_n(\Delta_{L,2}) - b_n(\Delta_{L,1}^{T}) - 2b_n(\Delta_{L,0}) \), so

\[ b_0(\Delta_{L,2}^{TT}) = \frac{(d+1)(d-2)}{2}, \quad b_2(\Delta_{L,2}^{TT}) = \frac{(d+1)(d-14)}{12}\tilde{R}, \]

\[ b_4(\Delta_{L,2}^{TT}) = \frac{d^2-31d+508}{360}R_{\mu\nu\rho\sigma}^2 - \frac{d^2-361d-902}{360}R_{\mu\nu}^2 + \frac{(d+1)(d-26)}{144}\tilde{R}^2 + \frac{(d+1)(d-12)}{60}\nabla^2\tilde{R}. \]  

(121)

For the Einstein space, \( b_4(\Delta_{L,2}^{TT}) \) reduces to

\[ b_4(\Delta_{L,2}^{TT}) = \frac{d^2-31d+508}{360}R_{\mu\nu\rho\sigma}^2 + \frac{5d^3-127d^2+592d+1804}{720d}\tilde{R}^2 + \frac{(d+1)(d-12)}{60}\nabla^2\tilde{R}. \]  

(122)

Finally we also need the following formulae which also follow from (113):

\[ b_0(\Delta+aR) = b_0(\Delta), \]

\[ b_2(\Delta+aR) = b_2(\Delta) - aRb_0(\Delta), \]

\[ b_4(\Delta+aR) = b_4(\Delta) - aRb_2(\Delta) + \frac{1}{2}a^2R^2b_0(\Delta). \]  

(123)

We can evaluate the one-loop divergent part (42) of the effective action (98) by using eqs. (114), (119) – (123).
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