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Circular Path Following for the Spherical Pendulum on a Cart

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Abstract: This paper investigates the problem of making the cart of a spherical pendulum follow a circular path with constant speed, while guaranteeing that the pendulum does not fall over. A solution methodology is presented which is hierarchical in nature. First, one of the two control inputs is used to stabilize the angle of the pendulum from the vertical axis at a constant value. The remaining control input is used to make the cart converge to the circular path with a desired speed.

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1. INTRODUCTION

The spherical pendulum is a benchmark mechanical system with two degrees-of-freedom and two controls. It may be viewed as a generalization of the simple pendulum, and as a simplified model of human stance. The typical control specifications investigated in the literature are the stabilization of the unstable equilibrium and the swing-up. We refer the reader to Shen et al. (2004); Chaturvedi and McClamroch (2007); Shiriaev et al. (2004) for more details.

The spherical pendulum on a cart has received much less attention in the literature. Unlike the standard spherical pendulum, the spherical pendulum on a cart is an underactuated mechanical system with four degrees-of-freedom (the position of the cart on the plane and the angles of the pendulum) and two controls (the planar force applied to the cart). Being underactuated, the spherical pendulum on a cart constitutes a challenging benchmark system. In Bloch et al. (1998), the authors use the method of controlled Lagrangians to stabilize the inverted equilibrium. In Jankuloski et al. (2012) the authors find a controlled invariant manifold on which the pendulum remains in the upper half-plane and the cart appears to move along bounded orbits.

In this paper, we address a motion control problem which involves making the cart follow a circular path on the horizontal plane, while guaranteeing that the pendulum maintains a constant angle with respect to the vertical axis. We refer to this as the circular path following problem. To the best of our knowledge, this problem as not been investigated before, but it bears some similarity to the problem of making Getz’s bicycle model follow a circular path with constant roll angle, a problem which was solved in Consolini and Maggiore (2013). Indeed, Getz’s bicycle model is mechanically equivalent to a planar pendulum on a cart, with the plane of the pendulum orthogonal to the direction of movement. The problem investigated in this paper is harder because the pendulum is not constrained to lie on a specific plane.

Our strategy to solve the circular path following problem is to use one of the control inputs to stabilize the angle of the pendulum from the vertical axis at a desired constant value. We use the other control input to make the cart converge to a circular path, while making the projection of the pendulum on the horizontal plane point towards the origin of the circle.

Notation. Throughout this paper we denote by $\mathbb{S}^1$ the set of real numbers modulo $2\pi$, diffeomorphic to the unit circle. Whenever convenient, we identify $n$-tuples $(a_1, \ldots, a_n)$ with columns vectors $[a_1 \cdots a_n]$. If $(X, d)$ is a complete metric space and $\Gamma \subset X$ is a closed set, the point-to-set distance of a point $x \in X$ to the set $\Gamma$, denoted by $|x|_\Gamma$, is defined as $|x|_\Gamma := \inf_{y \in \Gamma} d(x, y)$. If $\delta > 0$, we denote $B_\delta(\Gamma) := \{ x \in X : |x|_\Gamma < \delta \}$. When convenient, we use the shorthand notation $c_\alpha$ and $s_\alpha$ for $\cos(\alpha)$ and $\sin(\alpha)$, respectively.

2. PROBLEM FORMULATION

Consider the spherical pendulum depicted in Figure 1. This is a mechanical system comprised of a cart of mass $M$ moving on the horizontal plane, and a pendulum of length $l$ and mass $m$ concentrated at the tip. The pivot point of the pendulum on the cart is a spherical joint. We assume that the cart is fully actuated by two control forces, while the pendulum has no control.
In this paper, we address a motion control problem which is invariant manifold on which the pendulum remains in.

In Jankuloski et al. (2012) the authors find a controlled Lagrangians to stabilize the inverted equilibrium.

The spherical pendulum has received much less attention in the literature. Unlike the standard spherical pendulum, the spherical pendulum on a cart is an under-actuated mechanical system with four degrees-of-freedom.

In this section we derive the Lagrangian of the spherical pendulum on a cart depicted in Figure 1. Letting $x \in \mathbb{R}^2$ denote the position of the cart on the horizontal plane measured in the coordinates of frame $\mathbb{I}$, we have

$$x = \rho \begin{bmatrix} \cos \sigma \\ \sin \sigma \end{bmatrix}.$$ 

Letting $\zeta$ denote the unit vector in $\mathbb{R}^3$ representing the direction of the pendulum rod, we have

$$\zeta = \begin{bmatrix} \sin \psi \cos \theta \\ \sin \psi \sin \theta \\ \cos \psi \end{bmatrix}.$$ 

Denoting

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

the tip of the pendulum has coordinates in frame $\mathbb{I}$ given by $P^\top x + l \zeta$. The kinetic energy of the system is

$$K(q, \dot{q}) = \frac{1}{2} M \dot{x}^\top \dot{x} + \frac{1}{2} m \left( P^\top \dot{x} + l \dot{\zeta} \right)^\top \left( P^\top \dot{x} + l \dot{\zeta} \right) = \frac{1}{2} \dot{\theta}^\top D(q) \dot{\theta},$$

neart $\Gamma_1$ for $\Sigma$ if there exists $c > 0$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in B_{\delta}(\Gamma_2) \cap B_{\varepsilon}(\Gamma_1)$, $\phi(t, x_0) \in B_{\varepsilon}(\Gamma_2)$ for all $t \geq 0$. The set $\Gamma_2$ is asymptotically stable near $\Gamma_1$ if it is stable near $\Gamma_1$ and there exists $c > 0$ such that for all $x_0 \in B_{\varepsilon}(\Gamma_1)$, $\| \phi(t, x_0) \|_{\Gamma_2} \to 0$ as $t \to \infty$.

Roughly speaking, $\Gamma_2$ is (asymptotically) stable near $\Gamma_1$ if it is (asymptotically) stable with the restriction that initial conditions be sufficiently close to $\Gamma_1$.

We are ready to state the problem investigated in this paper.

**Circular Path Following Problem.** Given $\tilde{\psi} \in (0, \frac{\pi}{2})$ and $\bar{\rho} > l \sin \tilde{\psi}$, find a smooth feedback $f(q, \dot{q})$ meeting the following control specifications:

(i) There exist a constant $\bar{\omega} > 0$ and a smooth function $\mu : S^1 \to S^1$ such that the compact set

$$\Gamma_1 = \{(q, \dot{q}) \in TQ : (\rho, \psi, \bar{\rho}, \bar{\psi}, 0, \bar{\omega}, 0), \quad (\theta, \dot{\theta}) = (\mu(\sigma), \mu'(\sigma) \dot{\sigma})\},$$

is asymptotically stable for the closed-loop system.

(ii) The set $\Gamma_2 = \{(q, \dot{q}) \in TQ : \psi = \tilde{\psi}, \dot{\psi} = 0\}$ is stable near $\Gamma_1$ for the closed-loop system.

**Remark 3.** Specification (i) roughly translates to requiring the cart to follow a circular path of radius $\bar{\rho}$ with speed $\bar{\omega} > 0$, while keeping the pendulum at a constant angle $\tilde{\psi}$ from the vertical axis. Specification (ii), on the other hand, requires that if the pendulum is initially close to the configuration $(\psi, \dot{\psi}) = (\tilde{\psi}, 0)$, and if the initial condition of the system is not too far from $\Gamma_1$, then during transient the pendulum remains close to the above configuration. Roughly speaking, this guarantees that if during transient a trajectory of the system “overshoots” the set $\Gamma_1$, then it does so without making the pendulum sway too far from the desired inclination $\tilde{\psi} = \dot{\psi}$.

### 3. Modeling

In this section we derive the Lagrangian of the spherical pendulum on a cart depicted in Figure 1. Letting $x \in \mathbb{R}^2$ denote the position of the cart on the horizontal plane measured in the coordinates of frame $\mathbb{I}$, we have

$$x = \rho \begin{bmatrix} \cos \sigma \\ \sin \sigma \end{bmatrix}.$$
The equations of motion have the standard form in three steps.

1. Find \( \mu : S^1 \to S^1 \) and \( \bar{\omega} > 0 \) such that the set \( \Gamma_1 \) in (1) is controlled invariant for (3) (i.e., there exists a smooth feedback \( f \) making \( \Gamma_1 \) invariant).

2. Use the radial control input \( f_\rho \) to make the set \( \Gamma_2 \) in (2) asymptotically stable near \( \Gamma_1 \). In particular, the feedback will asymptotically enforce the relations \( \dot{\psi}, \dot{\bar{\psi}} = 0 \) near \( \Gamma_1 \).

3. We investigate a reduced-order control system representing the dynamics on \( \Gamma_2 \), and design the control input \( f_\sigma \) to asymptotically stabilize \( \Gamma_1 \). This is achieved by noticing that \( \Gamma_1 \) is a closed orbit, linearizing the dynamics along it, and designing a linear feedback stabilizing the portion of the dynamics transverse to the orbit.

5. SOLUTION OF THE CIRCULAR PATH FOLLOWING PROBLEM.

In this section we develop a solution to the circular path following problem unfolding in three steps.

1. Find \( \mu : S^1 \to S^1 \) and \( \bar{\omega} > 0 \) such that the set \( \Gamma_1 \) in (1) is controlled invariant for (3) (i.e., there exists a smooth feedback \( f \) making \( \Gamma_1 \) invariant).

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3. We investigate a reduced-order control system representing the dynamics on \( \Gamma_2 \), and design the control input \( f_\sigma \) to asymptotically stabilize \( \Gamma_1 \). This is achieved by noticing that \( \Gamma_1 \) is a closed orbit, linearizing the dynamics along it, and designing a linear feedback stabilizing the portion of the dynamics transverse to the orbit.

5. SOLUTION OF THE CIRCULAR PATH FOLLOWING PROBLEM.

In this section we develop a solution to the circular path following problem unfolding the steps outlined in the previous section.

5.1 Step 1: Determination of \( \mu \) and \( \bar{\omega} \)

Specification (i) requires the set \( \Gamma_1 \) in (3) to be asymptotically stable. A necessary condition for a set to be stable for the closed-loop system is that the set be positively invariant for the closed-loop system or, equivalently, that there exists a smooth feedback making \( \Gamma_1 \) invariant. The set \( \Gamma_1 \) is a smooth closed curve in \( TQ \) described by the smooth map \( X : S^1 \to TQ \), \( X(\sigma) := (\bar{q}(\sigma), \dot{\bar{q}}(\sigma)) \), where

\[
\bar{q}(\sigma) := (\sigma, \rho, \mu(\sigma), \bar{\omega})
\]

\[
\dot{\bar{q}}(\sigma) := (\bar{\omega}, 0, \mu^\prime(\sigma)\bar{\omega}, 0).
\]

The set \( \Gamma_1 \) is controlled invariant if and only if

\[
[0_{2 \times 2} \ I_2] \begin{bmatrix} (M + m)\rho^2 + \rho m \rho c \sigma - m \rho g \sigma c \psi \\ 0 & M + m \rho s - m g \sigma c \psi \\ 0 & M + m \rho s - m g \sigma c \psi \\ \bar{\omega} - m \rho s c \psi & \bar{\omega} - m g \sigma c \psi \end{bmatrix} \frac{\partial}{\partial \sigma} \bar{q} = 0,
\]

where \( \bar{q} = (\partial \bar{q}/\partial \sigma) \bar{\omega} \).

The potential energy is \( V(q) = mg \cos \psi \). We now define the Lagrangian \( L : TQ \to \mathbb{R} \) as \( L(q, \dot{q}) = K(q, \dot{q}) - V(q) \).

Using the definition of the control vector \( f = (f_\rho, f_\sigma) \) in Figure 2, the Euler-Lagrange equation gives

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Bf,
\]

with

\[
B := \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}.
\]

The equations of motion have the standard form

\[
D(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla V(q) = Bf,
\]

where \( C \) is a matrix formed using the Christoffel coefficients associated with the mass matrix \( D(q) \) (see Spong et al. (2006)).

4. SOLUTION STRATEGY

5. SOLUTION OF THE CIRCULAR PATH FOLLOWING PROBLEM.

In this section we develop a solution to the circular path following problem unfolding following the steps outlined in the previous section.

5.1 Step 1: Determination of \( \mu \) and \( \bar{\omega} \)

Specification (i) requires the set \( \Gamma_1 \) in (3) to be asymptotically stable. A necessary condition for a set to be stable for the closed-loop system is that the set be positively invariant for the closed-loop system or, equivalently, that there exists a smooth feedback making \( \Gamma_1 \) invariant. The set \( \Gamma_1 \) is a smooth closed curve in \( TQ \) described by the smooth map \( X : S^1 \to TQ \), \( X(\sigma) := (\bar{q}(\sigma), \dot{\bar{q}}(\sigma)) \), where

\[
\bar{q}(\sigma) := (\sigma, \rho, \mu(\sigma), \bar{\omega})
\]

\[
\dot{\bar{q}}(\sigma) := (\bar{\omega}, 0, \mu^\prime(\sigma)\bar{\omega}, 0).
\]

The set \( \Gamma_1 \) is controlled invariant if and only if

\[
[0_{2 \times 2} \ I_2] \begin{bmatrix} (M + m)\rho^2 + \rho m \rho c \sigma - m \rho g \sigma c \psi \\ 0 & M + m \rho s - m g \sigma c \psi \\ 0 & M + m \rho s - m g \sigma c \psi \\ \bar{\omega} - m \rho s c \psi & \bar{\omega} - m g \sigma c \psi \end{bmatrix} \frac{\partial}{\partial \sigma} \bar{q} = 0,
\]

where \( \bar{q} = (\partial \bar{q}/\partial \sigma) \bar{\omega} \).

The above two equations are parametrized by \( \mu(\sigma) \) and \( \bar{\omega} \). One can show they hold if and only if

\[
\bar{\omega} = \sqrt{\frac{g \tan \psi}{\bar{\rho} - \bar{\rho} \sin \psi}}
\]

(6)

\[
\mu(\sigma) = \sigma + \pi.
\]

Remark 4. The constant \( \bar{\omega} \) is well-defined because in the statement of the circular path following problem we assumed that \( \bar{\rho} > \bar{\rho} \sin \psi \). With the definition of \( \mu(\sigma) \) in (6), we may now give a precise physical interpretation of the target set \( \Gamma_1 \). Namely, \( \Gamma_1 \) is a closed orbit along which the cart moves around a circle of radius \( \bar{\rho} \) counterclockwise with constant speed \( \bar{\omega} \). The pendulum has a constant inclination angle \( \psi \) from the vertical, and its projection on the horizontal plane is a vector pointing towards the centre of the circle. This configuration is depicted in Figure 3.
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The equations of motion have the standard form

\[ \dot{x} = f(x, t) \]

Our solution to the circular path following problem unfolds

\[ \Gamma \]

5.1 Step 1: Determination of

for the closed-loop system is that the set be positively

\[ \Gamma \]

As pointed out in Section 5.1, the set \( \Gamma \) is a smooth
closed curve with parametrization (using (4) and (6))

\[ (\sigma, \dot{\sigma}, \rho, \dot{\rho}) \]

with \( \omega \) as in (6). By the construction in Section 5.1, \( \Gamma \) is an orbit of (8), and one can show that \( \Gamma_1 \) is invariant when \( f_\sigma = 0 \).

The problem now is to design \( f_\sigma \) so as to asymptotically stabilize the closed orbit \( \Gamma_1 \). For this, we appeal to the classical theory of stability of closed orbits found in (Hale, 1980, Chapter VI). In general terms, the idea is as follows.

Consider a smooth dynamical system \( \Sigma : \dot{x} = f(x) \), with \( x \in \mathbb{R}^n \), and let \( \bar{x}(t) \) be a twice-differentiable closed orbit of \( \Sigma \). Let \( T > 0 \) be the minimal period of \( \bar{x}(t) \). The linearization of \( \Sigma \) along \( \bar{x} \) is

\[ \dot{\tilde{x}} = (df_{\bar{x}(t)}) \tilde{x} = A(t) \tilde{x} \]

This is a linear T-periodic differential equation. The eigenvalues of the monodromy matrix of (10) are called the characteristic multipliers of (10). We have the following result.

**Theorem 5.** (Hale (1980)). The characteristic multipliers of the linear T-periodic system (10) are \( \{ \mu_1, \ldots, \mu_n \} \subset \mathbb{C} \). If \( |\mu_i| = 1 \), \( i = 1, \ldots, n-1 \), then the closed orbit \( \text{Range}(\tilde{x}(\cdot)) \) is asymptotically stable.

In the special case when \( A \) is independent of \( t \), one has the following.

**Corollary 6.** If the matrix \( A \) in (10) is constant, then \( A \) has eigenvalues \( \{0, \lambda_1, \ldots, \lambda_{n-1} \} \), and the closed orbit \( \text{Range}(\tilde{x}(\cdot)) \) is asymptotically stable provided that \( \text{Re}(\lambda_i) < 0 \), \( i = 1, \ldots, n-1 \).

The eigenspace associated with the eigenvalue at 0 is the tangent space to the closed orbit.

We now apply these ideas in our context. Referring to system (8), define the state

\[ x = (\sigma, \dot{\sigma}, \rho, \dot{\rho}, \theta, \dot{\theta}, \psi, \dot{\psi}) \]

and the error from the parametrization of \( \Gamma_1 \) in (9),

\[ \tilde{x} = x - (\sigma, \omega, \rho, 0, \sigma + \pi, \omega, 0, 0) \]

Since Theorem 5 concerns dynamical systems without control, we define a linear feedback \( f_\sigma (\tilde{x}) = K \tilde{x} \).

We linearize (8) along \( \Gamma_1 \). The result has the form

\[ \dot{\tilde{x}} = A \tilde{x} + B K \tilde{x} \]

where \( H \) is the Hurwitz matrix

\[ H = \begin{pmatrix} 0 & 1 \\ -K_\rho & -K_\theta \end{pmatrix} \]

and the pair \((A, B)\) can be shown to be controllable for all \( \psi \in (0, \pi/2) \) and all \( \rho > l \sin \psi \). In light of the block-upper triangular structure of the system matrix and the fact that \( H \) is already Hurwitz, appealing to Corollary 6 we need to design \( K \) such that the matrix \( A + BK \) has \( n-1 \) eigenvalues with negative real part, and an eigenvalue at 0 with eigenvector \( v = (1, 0, 0, 0, 1, 0) \). This is a subspace stabilization problem for a linear time-invariant system.

The problem is solvable because \((A, B)\) is controllable. Define an invertible matrix \( P \in \mathbb{R}^{6 \times 6} \) whose first column is the vector \( v \) just defined. The isomorphism \( z = P^{-1} \tilde{x} \) transforms \((A, B)\) to \((\hat{A}, \hat{B})\) with the form

\[ \hat{A} = \begin{pmatrix} 0 & A_{12} \\ 0_{5 \times 1} & A_{22} \end{pmatrix}, \hat{B} = B_1 [B_2] \]

Since \((A, B)\) is controllable, so is \((\hat{A}, \hat{B})\), which in turn implies that the same holds for \((A_{22}, B_2)\). Let \( K \) be such that \( A_{22} + B_2 K \) has all eigenvalues with negative real part, and set

\[ K = [0 \hat{K}] P^{-1} \begin{pmatrix} 0 & 0 \end{pmatrix} \]

By this construction, \( A + BK \) has one eigenvalue at zero with eigenvector \( v \) and all other eigenvalues with negative real part. Returning to the linear time-invariant system (11) representing the linearization of (8) along the closed orbit \( \Gamma_1 \), this system has the same properties (the eigenvector associated with the eigenvalue at 0 is now \( (v, 0, 0) \)). By Corollary 6, \( \Gamma_1 \) is asymptotically stable for the closed-loop system formed by (8) with feedback

\[ f_\sigma = K \begin{pmatrix} 0 & \sigma - \tilde{\omega} & -\tilde{\rho} & \dot{\rho} & -\sigma & -\pi \theta & -\tilde{\omega} & 0 & 0 \end{pmatrix} \]

5.4 Summary

Since \( \Gamma_1 \) is asymptotically stable, by the discussion in Section 5.2 we also have that \( \Gamma_2 \) is asymptotically stable near \( \Gamma_1 \). In conclusion, for any \( \psi \in (0, \pi/2) \) and any \( \rho > l \sin \psi \), the feedback \((f_\rho, f_\sigma)\), with \( f_\rho \) defined in (7) and \( f_\sigma \) defined in (12), with constant \( \tilde{\omega} \) defined in (6), solves the circular path following problem.

6. SIMULATION RESULTS

We consider a spherical pendulum with geometrical and mechanical parameters \( l = 1 \text{ m}, M = 1 \text{ kg}, m = 0.1 \text{ kg} \) and the circular path following problem with \( \tilde{\rho} = 10 \text{ m}, \psi = \frac{\pi}{2} \text{ rad} \). The corresponding value of the angular velocity is \( \tilde{\omega} = 1.0274 \text{ rad/s} \). We show numerical simulations of the system with feedback \( f_\sigma \) given in (7) with \( K_\rho = 20, K_\theta = 100 \) and \( f_\rho \) given in (12) with \( K \) designed according to the LQR method. In particular, we take

\[ K = 110.3 5.5078 -4.92 -275.421 -68.0462 0 0 \]

The system is initialised at \( \sigma = 0 \text{ rad}, \rho = 12 \text{ m}, \theta = \pi \text{ rad}, \psi = 10^{-3} \text{ rad}, \dot{\sigma} = 0.2 \text{ rad/s}, \dot{\rho} = 0 \text{ m/s}, \dot{\theta} = 0.5 \text{ rad/s} \) and \( \dot{\psi} = 0 \text{ rad/s} \). Fig. 4 and 5 show the asymptotic convergence of \( \rho \) to \( \tilde{\rho} \) and of \( \theta - \sigma \) to \( \pi \) respectively. The convergence of the polar angle \( \psi \) to its desired value \( \tilde{\psi} \) is much faster than the other state variables (see Fig. 6). In Fig. 7 the trajectory of the cart is depicted. It is apparent that the cart approaches a circle centered in the origin and of radius \( \tilde{\rho} \).
7. CONCLUSION

In this paper we considered a simple model of a spherical pendulum on a cart and we dealt with the problem of making the cart follow a circular path with constant speed, while guaranteeing that the pendulum does not fall over. The two-dimensional control input achieving this objective was constructed by means of a hierarchical approach. More precisely, one of the two components was designed as a function of the state and of the other component so that to stabilize the angle of the pendulum from the vertical axis at a constant value. The remaining control input was designed to make the cart converge to the circular path with a desired speed. Future work will focus partly on the stabilization of our model to closed paths other than the one considered here and partly on the adaptation of the method to a more realistic, possibly nonholonomic, model of the spherical pendulum on the cart.

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