On generalized fractional kinetic equations

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Abstract
In a recent paper, Saxena et al.[1] developed the solutions of three
generalized fractional kinetic equations in terms of the Mittag-Leffler functions.
The object of the present paper is to further derive the solution of further
generalized fractional kinetic equations. The results are obtained in a com-
 pact form in terms of generalized Mittag-Leffler functions. Their relation to
fundamental laws of physics is briefly discussed.

1 Introduction

The fundamental laws of physics are written as equations for the time evo-
 lution of a quantity $X(t), dX(t)/dt = -AX$, where this could be Maxwell’s
equations or Schrödinger’s equation (if A is limited to linear operators), or it
could be Newton’s law of motion or Einstein’s equations for geodesics (if A
may also be a nonlinear operator [2,3,27]). The mathematical solution (for
linear operators A) is $X(t) = X(0)exp\{-At\}$.

In thermodynamical or statistical applications one is mostly interested in
mean values of the quantity $\langle X(t) \rangle$. In this case, A is a characteristic time
scale $A^{-1} = \tau$ in the evolution equation for $\langle X(t) \rangle$. It then follows that
$\langle X(t) \rangle$ decays exponentially toward equilibrium $\langle X(t) \rangle = \langle X(0) \rangle \exp\{-t/\tau\}$. 

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In 1988, Tsallis [4] generalized the entropic function of Boltzmann-Gibbs statistical mechanics, $s = -\int dx p(x) \ln p(x)$, to nonextensive statistical mechanics with $S_q[p] = \{1 - \int dx [p(x)]^q\} / (q - 1)$ that leads to $q$-exponential distributions $p_q(x) \propto [1 - (1 - q)x^2/kT]^{1/(1-q)}$. Such a distribution reduces to Gaussian distribution for $q = 1$ and for $q = 2$ to a Cauchy-Lorentz distribution, to name two examples. In an attempt to incorporate Lévy distribution into statistical mechanics, Tsallis has also shown that the above distribution becomes a Lévy distribution for $q > 5/3$. Recently, Tsallis [4] used the mathematical simplicity of reaction-type equations, $dX/dY = Y^q$, to emphasize the natural outcome of the above distribution function $p_q(x)$ which corresponds exactly to the solution of the reaction equation of non-linear type. The solution has power-law behavior. In the following we show that the fractional generalization of the linear reaction-type equation also leads to power-law behavior. In both cases, solutions can be expressed in terms of generalized Mittag-Leffler functions.

2 Generalized Mittag-Leffler function

A generalization of the Mittag-Leffler function [5,6]

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an + 1)}$$  \hspace{1cm} (1)

and its generalized form

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an + \beta)} \hspace{1cm} (\alpha, \beta \in C, Re(\alpha) > 0)$$ \hspace{1cm} (2)

was introduced by Prabhakar [7] in terms of the series representation

$$E_{\alpha,\beta}^\gamma(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(an + \beta)(n!)} \hspace{1cm} (\alpha, \beta, \gamma \in C, Re(\alpha) > 0)$$ \hspace{1cm} (3)

where $(\gamma)_n$ is Pochammer’s symbol defined by

$$(\gamma)_n = \begin{cases} 1 & n = 0 \\ \gamma(\gamma+1)\ldots(\gamma+n-1), n \in \mathbb{N} \end{cases} \hspace{1cm} \gamma \neq 0$$ \hspace{1cm} (4)
It is an entire function of order $\rho = \left[ \text{Re}(\alpha) \right]^{-1}$ [7]. This function has been studied by Wiman [8,9], Agarwal [10], Humbert [11] and Humbert and Agarwal [12] and several others. Some special cases of (3) are given below:

(i) $E_\alpha(z) = E_{1,1}^1(z)$,  

(ii) $E_{\alpha,\beta}(z) = E_{1,1}^1(z)$,  

(iii) $\Phi(\beta,\gamma;z) = {}_1F_1(\beta;\gamma;z) = \Gamma(\gamma)E_{1,1}^\beta(z)$,  

where $\Phi(\beta,\gamma;z)$ is Kummer’s confluent hypergeometric function defined in Erdélyi et al. ([13], p. 248, eq.1]). Mellin-Barnes integral representation for the function defined by (3) follows from the integral

$$E^\gamma_{\alpha,\beta}(z) = \frac{1}{2\pi i \Gamma(\gamma)} \int_{\Omega} \frac{\Gamma(-s)\Gamma(\gamma + s)(-z)^s}{\Gamma(\beta + s\alpha)} ds,$$

where $\omega = (-1)^{1/2}$. The contour $\Omega$ is a straight line parallel to the imaginary axis separating the poles of $\Gamma(-s)$ at the points $s = \nu$ ($\nu = 0,1,2,\ldots$) from those of $\Gamma(\gamma + s)$ at the points $s = -\gamma - \nu$ ($\nu = 0,1,2,\ldots$). The poles of the integrand of (8) are assumed to be simple. (8) can be established by calculating the residues at the poles of $\Gamma(-s)$ at the points $s = \nu$ ($\nu = 0,1,2,\ldots$). It follows from (8) that $E^\gamma_{\alpha,\beta}(z)$ can be represented in the form

$$E^\gamma_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} H_{1,2}^{1,1} \left[ -z^{(1-\gamma,1)}; (0,1); (1-\beta,\alpha) \right], \ (\text{Re}(\alpha) > 0; \alpha, \beta, \gamma \in \mathbb{C}),$$

where $H_{1,2}^{1,1}(z)$ is the H-function. A detailed account of the theory and applications of the H-function is available from Mathai and Saxena [14]. This function can also be represented by

$$E^\gamma_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} \Psi_1^{(\gamma,1)}(\beta,\alpha); z,$$

where $\Psi_1(z)$ is a special case of Wright’s generalized hypergeometric function $p\Psi_q(z)$ [15,16] ; also see, Erdélyi et al. ([13], Section 4.1 ), defined by

$$p\Psi_q \left[ \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j n)}{\prod_{j=1}^{q} \Gamma(b_j + B_j n) (n)!} z^n.$$
where $1 + \sum_{j=1}^{n} B_j - \sum_{j=1}^{p} A_j \geq 0$ (equality only holds for appropriately bounded $z$). When $\gamma = 1$, (9) and (10) give rise to (12) and (13) given below:

$$E_{\alpha,\beta}(z) = \Psi_1(1; (1,1); z), \quad (12)$$

$$= H_{1,2}^{1,1}[-z; (0,1); (1,1); (1-\beta, \alpha)], \quad (13)$$

where $Re(\alpha) > 0; \alpha, \beta, \gamma \in C$.

If we further take $\beta = 1$ in (12) and (13) we find that

$$E_{\alpha}(z) = \Psi_1(1; (1,1); z), \quad (14)$$

$$= H_{1,2}^{1,1}[-z; (0,1); (1,1); (1-\beta, \alpha)], \quad (15)$$

for $Re(\alpha) > 0, z \in C$. The following integral gives the Laplace transform of $E_{\alpha,\beta}(z)$.

$$\int_0^{\infty} e^{-pt^{\beta-1}} E_{\alpha,\beta}(at^{\alpha}) dt = p^{-\beta}(1 - ap^{-\alpha})^{-\gamma}, \quad (16)$$

where $Re(p) > |a|^{\frac{1}{Re(\alpha)}}, Re(\beta) > 0, Re(p) > 0$, which can be established by means of the Laplace integral

$$\int_0^{\infty} e^{-pt^{\beta-1}} dt = \frac{\Gamma(\rho)}{p^{\rho}}, Re(p) > 0, Re(\rho) > 0. \quad (17)$$

For $\gamma = 1$, (16) reduces to an elegant formula

$$\int_0^{\infty} e^{-pt^{\beta-1}} E_{\alpha,\beta}(at^{\alpha}) dt = p^{-\beta}(1 - ap^{-\alpha})^{-1}, \quad (18)$$

where $Re(\beta) > 0, Re(p) > 0, |p| > |a|^{\frac{1}{Re(\alpha)}}$. In an attempt to investigate the functions which when fractionally differentiated (of any order) reappear, Hartley and Lorenzo [17] came across a special function of the form

$$F_q[-a,t] = t^{q-1} \sum_{n=0}^{\infty} \frac{(-a)^{n+q}}{\Gamma(nq + q)}, Re(q) > 0, \quad (19)$$

$$= t^{q-1} E_{q,q}(-at^q). \quad (20)$$
This function has been studied earlier by Robotnov [18,19] in connection with hereditary integrals for application to continuum mechanics. The Laplace transform of this function is given by

$$L[F_q(a; t)] = \frac{1}{p^q - a}, \quad \text{Re}(q) > 0. \quad (21)$$

A generalization of the F-function is presented by Lorenzo and Hartley [20] by means of the following series representation:

$$R_{\nu,\mu}[a, c, t] = \sum_{n=0}^{\infty} \frac{a^n(t - c)^{(n+1)\nu - \mu - 1}}{\Gamma[(n + 1)\nu - \mu]}, \quad t > c > 0 \quad (22)$$

$$= (t - c)^{\nu - \mu - 1} E_{\nu,\nu - \mu}[a(t - c)^\nu], \quad t > c > 0. \quad (23)$$

The Laplace transform of the R-function is derived by Lorenzo and Hartley [20] in the form

$$L \{R_{\nu,\mu}(a, c, t)\} = e^{-cp} \frac{p^\mu}{p^\nu - a}, \quad c \geq 0, \quad (24)$$

where \(\text{Re}(\nu - \mu) > 0, \text{Re}(p) > 0\). When \(c = 0\), (23) reduces to

$$L \{R_{\nu,\mu}(a, 0, t)\} = \frac{p^\mu}{p^\nu - a}, \quad \text{Re}(\nu - \mu) > 0, \text{Re}(p) > 0. \quad (25)$$

Finally we recall the definition of Riemann-Liouville operator of fractional integration in the form

$$0 D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - u)^{\nu - 1} f(u) du, \quad (26)$$

with \(0 D_t^\alpha f(t) = f(t)\) [21,22,23]. The standard kinetic equation, when integrated, yields

$$N_i(t) - N_0 = c_0 D_t^{-1} N(t), \quad (27)$$

where \(0 D_t^{-1}\) is the standard Riemann-Liouville integral operator. Here it can be mentioned that in the original paper of Haubold and Mathai [24], the number density of the species \(i, N_i = N_i(t)\) is a function of time and \(N_i(t = 0) = N_0\) is the number density of species \(i\) at time \(t = 0\). If we drop the index \(i\) in (27) and replace \(c\) by \(c'\), then the solution of the generalized equation

$$N(t) - N_0 = -c' 0 D_t^{-\nu} N(t), \quad (28)$$
is obtained, Haubold and Mathai [24] as

\[ N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k (ct)^{k\nu}}{\Gamma(k\nu + 1)}. \]  

By virtue of (1), (29) can be written in a compact form as

\[ N(t) = N_0 E_{\nu}(-c^\nu t^\nu). \]  

In the following, we investigate the solution of two more generalized fractional kinetic equations. The results are obtained in a compact form in terms of generalized Mittag-Leffler functions and are suitable for computation. A detailed account of the operators of fractional integration and their applications is available from a recent survey paper of Srivastava and Saxena [25].

### 3 Generalized fractional kinetic equations

**Theorem 1.** If \( c > 0, \nu > 0, \mu > 0 \), then for the solution of the equation

\[ N(t) - N_0 t^{\mu-1} E_{\nu,\mu}^\gamma [-c^\nu t^\mu] = -c_0^\nu D_t^{-\nu} N(t), \]  

there holds the formula

\[ N(t) = N_0 t^{\mu-1} E_{\nu,\mu}^{\gamma+1} (-c^\nu t^\nu). \]  

**Proof.** By the application of convolution theorem of Laplace transform (Erdélyi et al. [26]) we see that (26) can be written as

\[ L\{ \phi D_t^{-\nu} f(t); p \} = L\left\{ \frac{\mu^{-1}}{\Gamma(\nu)} \right\} L\{ f(t) \}, \]  

\[ = p^{-\nu} F(p), \]  

where \( F(p) = \int_0^\infty e^{-pu} f(u) du, Re(p) > 0. \) Projecting the equation (31) to Laplace transform , we obtain

\[ N(t) = L[N(t); p] = N_0 p^{-\mu} \frac{[1 + (p/c)^{-\nu}]^{-\gamma}}{[1 + (p/c)^{-\nu}]^{1+(\gamma+1)}}. \]  

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On using the formula (16), we find that

\[ L^{-1}\{p^{-\mu} \left[ 1 + (p/c)^{-\nu} \right]^{-(\gamma+1)} \} = t^{\mu-1}E_{\nu,\mu}^\gamma(-c^\nu t^\nu). \] (36)

The result (32) now readily follows from (36).

If we set \( \gamma = 1 \) then \((n)!\) is cancelled. Then in view of the formula

\[ \beta E_{\beta,\gamma}(z) = E_{\beta,\gamma-1}(z) + (1 - \gamma + \beta)E_{\beta,\gamma}(z), \] (37)

which follows from the definition of \( E_{\alpha,\beta}(z) \) given by (3), we arrive at

**Corollary 1.1.** If \( c > 0, \mu > 0, \nu > 0 \), then for the solution of

\[ N(t) - N_0 t^{\mu-1}E_{\nu,\mu}[-c^\nu t^\nu] = -c^\nu \int_0^t D^{-\nu}_t N(t), \] (38)

there holds the formula

\[ N(t) = \frac{N_0 t^{\mu-1}}{\nu} [E_{\nu,\mu-1}(-c^\nu t^\nu) + (1 - \mu + \nu)E_{\nu,\mu}(-c^\nu t^\nu)]. \] (39)

When \( \gamma = 2 \), then by virtue of the following identity

\[
E_{\beta,\gamma}^3(z) = \frac{1}{2\beta^2}[E_{\beta,\gamma-2}(z) + (3\beta + 3 - 2\gamma)E_{\beta,\gamma-1}(z) \\
+ \{2\beta^2 + \gamma^2 + 3\beta - 2\gamma - 3\beta\gamma + 1\}E_{\beta,\gamma}(z)],
\] (40)

which follows as a consequence of the definition (3), we obtain

**Corollary 1.2.** If \( c > 0, \nu > 0, \mu > 0 \), then for the solution of

\[ N(t) - N_0 t^{\mu-1}E_{\nu,\mu}^2[-c^\nu t^\nu] = -c^\nu \int_0^t D^{-\nu}_t N(t), \] (41)

there holds the relation

\[
N(t) = N_0 t^{\mu-1}E_{\nu,\mu}^3(-c^\nu t^\nu),
\]

\[
= N_0 t^{\mu-1} \left[ E_{\nu,\mu-2}(-c^\nu t^\nu) + \{3(\nu + 1) - 2\mu\}E_{\nu,\mu-1}(-c^\nu t^\nu) \\
+ \{2(\nu^2 + \mu^2) + 3\nu - 2\mu - 3\nu\mu + 1\}E_{\nu,\mu}(-c^\nu t^\nu) \right].
\] (43)

Next, if we set \( \gamma = 0 \), then by virtue of the identity \( E_{\nu,\mu}^0(z) = \frac{1}{\Gamma(\mu)} \), we arrive at another result given by Saxena et al. [1].
Theorem 2. If $c > 0, b \geq 0, \text{Re}(p) > 0, \nu > \mu + 1$, then for the solution of

$$N(t) - N_0 R_{\nu,\mu}(-c', b, t) = -c'\, _0D_t^{-\nu} N(t), \quad (44)$$

there holds the formula

$$N(t) = \frac{N_0}{\nu} (t - b)^{\nu-\mu-1} [E_{\nu,\nu-\mu-1}(-c'(t - b)^\nu) + (\mu + 1) E_{\nu,\nu-\mu}(-c'(t - b)^\nu)]. \quad (45)$$

Proof. Taking Laplace transform of both sides of (45), it gives

$$N(t) = L\{N(t); p\} = N_0 L^{-1} \left[ \frac{e^{-bp^{\mu-\nu}}}{\{1 + (p^{\nu}/p^\nu)\}^2} \right]$$

$$= N_0 L^{-1} \left[ \frac{e^{-bp^{\mu-\nu}}}{\{1 + (p^{\nu}/p^\nu)\}^2} \right]$$

$$= N_0 \sum_{n=0}^{\infty} \frac{(2)_n(-c)^{\nu_n}(t - b)^{\nu_n}}{(n)! (\nu + \nu - \mu)} L^{-1}[e^{-bp^{\mu-\nu-n\nu}}]$$

$$= N_0 \sum_{n=0}^{\infty} \frac{(2)_n(-c)^{\nu_n}(t - b)^{\nu_n}}{(n)! (\nu + \nu - \mu)} L^{-1}[e^{-bp^{\mu-\nu-n\nu}}]$$

$$= N_0 \left[ \frac{e^{-bp^{\mu-\nu}}}{\{1 + (p^{\nu}/p^\nu)\}^2} \right]$$

$$= N_0 \left[ \frac{e^{-bp^{\mu-\nu}}}{\{1 + (p^{\nu}/p^\nu)\}^2} \right]$$

$$= N_0 \left[ \frac{e^{-bp^{\mu-\nu}}}{\{1 + (p^{\nu}/p^\nu)\}^2} \right]$$

which is same as (45). This completes the proof of theorem 2. If we set $\mu = 0$, theorem 2 reduces to

Corollary 2.1. If $c > 0, b \geq 0, \nu > 1$, then for the solution of

$$N(t) - N_0 R_{\nu,0}(-c', b, t) = -c'\, _0D_t^{-\nu} N(t), \quad (46)$$

there holds the formula

$$N(t) = \frac{N_0}{\nu} (t - b)^{\nu-\mu-1} [E_{\nu,\nu-\mu-1}(-c'(t - b)^\nu) + (\mu + 1) E_{\nu,\nu-\mu}(-c'(t - b)^\nu)]. \quad (47)$$

For $b = 0$, theorem 2 yields...
Corollary 2.2. If $c > 0, \nu > \mu + 1$, then for the solution of

$$N(t) - N_0 R_{\nu,\mu}(-c^\nu, 0, t) = -c^\nu \partial_t D_t^{-\nu} N(t), \tag{48}$$

there holds the formula

$$N(t) = \frac{N_0}{\nu} t^{\nu - \mu - 1} [E_{\nu,\nu-\mu-1}(-c^\nu t^\nu) + (\mu + 1) E_{\nu,\nu-\mu}(-c^\nu t^\nu)]. \tag{49}$$

If we further take $\mu = 0$ then the above corollary reduces to the following result:

If $c > 0, \nu > 1$, then the solution of

$$N(t) - N_0 F_{\nu}[-c^\nu, t] = -c^\nu D_t^{-\nu} N(t), \tag{50}$$

is given by

$$N(t) = \frac{N_0 t^{\nu-1}}{\nu} [E_{\nu-1}(-c^\nu t^\nu) + E_{\nu,\nu}(-c^\nu t^\nu)]. \tag{51}$$

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