Some Relationships and Properties of the Hypergeometric Distribution

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Abstract

The binomial and Poisson distributions have interesting relationships with the beta and gamma distributions, respectively, which involve their cumulative distribution functions and the use of conjugate priors in Bayesian statistics. We briefly discuss these relationships and some properties resulting from them which play an important role in the construction of exact nested two-sided confidence intervals and the computation of two-tailed \( P \)-values. The purpose of this article is to show that such relationships also exist between the hypergeometric distribution and a special case of the Polya (or beta-binomial) distribution, and to derive some properties of the hypergeometric distribution resulting from these relationships.

KEY WORDS: Beta, binomial, gamma, Poisson, and Polya (or beta-binomial) distributions; Conjugate prior distribution; Cumulative distribution function; Posterior distribution.

1. INTRODUCTION

The binomial and Poisson distributions have interesting relationships with the beta and gamma distributions, respectively, which involve their cumulative distribution functions and the use of conjugate priors in Bayesian statistics. We will briefly discuss these relationships and some properties resulting from them in Sections 2 and 3 for the binomial and Poisson distributions, respectively. The resulting properties play an important role in the construction of exact nested two-sided binomial and Poisson confidence intervals, and the computation of exact two-tailed binomial and Poisson \( P \)-values.

The purpose of this article is to show that such relationships also exist between the hypergeometric distribution and a special case of the Polya (or beta-binomial) distribution, and to derive some properties of the hypergeometric distribution resulting from these relationships. We shall do this in Section 4.

2. RELATIONSHIPS AND PROPERTIES OF THE BINOMIAL DISTRIBUTION

Suppose that random variable \( X \) has a binomial distribution with parameters \( n \) and \( p \), denoted by \( X \sim \text{BIN}(n, p) \), where \( n \) is a positive integer and \( 0 \leq p \leq 1 \). Then, for a given \( n \) and for \( 0 < p < 1 \), the probability mass function (pmf) of \( X \), denoted by \( f_X(x \mid p) \), is

\[
f_X(x \mid p) = P(X = x \mid p) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, \ldots, n,
\]

\[
= 0, \quad \text{otherwise},
\]
and \( f_X(0 \mid 0) = f_X(n \mid 1) = 1 \).

Suppose that random variable \( Y \) has a beta distribution with parameters \( \alpha > 0 \) and \( \beta > 0 \), denoted by \( Y \sim \text{BETA}(\alpha, \beta) \). Then the probability density function (pdf) of \( Y \), denoted by \( f_Y(y \mid \alpha, \beta) \), is

\[
f_Y(y \mid \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}, \quad 0 \leq y \leq 1,
\]

where the gamma function \( \Gamma(\kappa) = \int_0^\infty t^{\kappa-1}e^{-t} dt \) for all \( \kappa > 0 \).

Successive integration by parts leads to a relationship between the cumulative distribution functions (cdf's) of the binomial and beta distributions. If \( X \sim \text{BIN}(n, p) \) and \( Y \sim \text{BETA}(i + 1, n - i) \) for integer \( i, 0 \leq i \leq n - 1 \), then

\[
\sum_{x=0}^{i} \binom{n}{x} p^x (1-p)^{n-x} = 1 - \frac{n!}{i!(n-i-1)!} \int_0^p t^i (1-t)^{n-i-1} dt.
\]

That is, \( F_X(i \mid p) = P(X \leq i \mid p) = 1 - P(Y \leq p \mid i+1, n-i) = 1 - F_Y(p \mid i+1, n-i) \). For fixed integer \( i, 0 \leq i \leq n - 1 \), it follows from equation \([1]\) that the function \( P(X \leq i \mid p) \) is continuous and decreasing in \( p \); for fixed integer \( j, 1 \leq j \leq n \), \( P(X \geq j \mid p) = 1 - P(X \leq j-1 \mid p) \) is continuous and increasing in \( p \); and for fixed integers \( i \) and \( j, 1 \leq i \leq j \leq n - 1 \), \( P(i \leq X \leq j \mid p) \) is continuous, and increasing for \( 0 \leq p < p_n(i, j) \) and decreasing for \( p_n(i, j) \leq p \leq 1 \) with maximum at \( p = p_n(i, j) = \{1 + [(n-i) \cdots (n-j)/j \cdots i]^{1/(j-i+1)}\}^{-1} \). Also, \( p_n(0,j) = 0 \) for \( 0 \leq j \leq n - 1 \) and \( p_n(i,n) = 1 \) for \( 1 \leq i \leq n \).

Suppose that the binomial parameter \( p \) is unknown and we wish to estimate it. In Bayesian statistics, information obtained from the data \( x \), a realization of \( X \sim \text{BIN}(n, p) \), is combined with prior information about \( p \) that is specified in a “prior distribution” with pdf \( g(p) \) and summarized in a “posterior distribution” with pdf \( h(p \mid x) \) which is derived from the joint distribution \( f_X(x \mid p)g(p) \), and according to Bayes formula is

\[
h(p \mid x) = \frac{f_X(x \mid p)g(p)}{\int_0^1 f_X(x \mid p)g(p) \, dp}.
\]

Because \( h(p \mid x) \) is generally not available in closed form, the favoured types of priors until the introduction of Markov chain Monte Carlo methods have been those allowing explicit computations, namely “conjugate priors.” These are prior distributions for which the corresponding posterior distributions are themselves members of the original prior family, the Bayesian updating being accomplished through updating of parameters. For a realization \( x \) of \( X \sim \text{BIN}(n, p) \), a family of conjugate priors is the family of beta distributions \( \text{BETA}(\alpha, \beta) \) where we note from equation \([2]\) that for \( x = 0, 1, \ldots, n \),

\[
h(p \mid x) = \frac{\binom{n}{x} p^x (1-p)^{n-x} \Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{p^{\alpha-1}(1-p)^{\beta-1}}{\Gamma(\alpha + \beta + n)}
\]

\[
\times \frac{\Gamma(\alpha + \beta + n)}{\Gamma(\alpha + x)\Gamma(\beta + n - x)} p^{\alpha+x-1}(1-p)^{\beta+n-x-1}, \quad 0 \leq p \leq 1,
\]

\[
= 0, \quad \text{otherwise},
\]

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That is, the posterior distribution is also beta with updated parameters \( \alpha + x \) and \( \beta + n - x \).

3. RELATIONSHIPS AND PROPERTIES OF THE POISSON DISTRIBUTION

Suppose that random variable \( X \) has a Poisson distribution with parameter \( \lambda \geq 0 \), denoted by \( X \sim \text{POI}(\lambda) \). Then, for \( \lambda > 0 \), the pmf of \( X \), denoted by \( f_X(x \mid \lambda) \), is

\[
    f_X(x \mid \lambda) = P(X = x \mid \lambda) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \ldots,
\]

and \( f_X(0 \mid 0) = 1 \).

Suppose random variable \( Y \) has a gamma distribution with parameters \( \alpha > 0 \) and \( \beta > 0 \), denoted by \( Y \sim \text{GAM}(\alpha, \beta) \). Then the pdf of \( Y \), denoted by \( f_Y(y \mid \alpha, \beta) \), is

\[
    f_Y(y \mid \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} e^{-y/\beta}, \quad y > 0,
\]

\[
    = 0, \quad \text{otherwise}.
\]

Successive integration by parts leads to a relationship between the cdf’s of the Poisson and gamma distributions. If \( X \sim \text{POI}(\lambda) \) and \( Y \sim \text{GAM}(i + 1, 2) \) for nonnegative integer \( i \), then

\[
    \sum_{x=0}^{i} \frac{e^{-\lambda} \lambda^x}{x!} = 1 - \frac{1}{2^{i+1}!} \int_0^{2\lambda} t^i e^{-t/2} dt. \tag{3}
\]

That is, \( F_X(i \mid \lambda) = P(X \leq i \mid \lambda) = 1 - P(Y \leq 2\lambda \mid i + 1, 2) = 1 - F_Y(2\lambda \mid i + 1, 2) \). For fixed nonnegative integer \( i \), it follows from equation (3) that the function \( P(X \leq i \mid \lambda) \) is continuous and decreasing in \( \lambda \); for positive integer \( j \), \( P(X \geq j \mid \lambda) = 1 - P(X \leq j - 1 \mid \lambda) \) is continuous and increasing in \( \lambda \); and for \( 1 \leq i \leq j \), \( P(i \leq X \leq j \mid \lambda) \) is continuous, and increasing for \( 0 \leq \lambda < \lambda(i, j) \) and decreasing for \( \lambda \geq \lambda(i, j) \) with maximum at \( \lambda = \lambda(i, j) = (i \cdots j)^{1/(j-i+1)} \). Also, \( \lambda(0, j) = 0 \) for \( j \geq 0 \).

Suppose that the Poisson parameter \( \lambda \) is unknown and we wish to estimate it using Bayesian methods. For a realization \( x \) of \( X \sim \text{POI}(\lambda) \), a family of conjugate priors is the family of gamma distributions \( \text{GAM}(\alpha, \beta) \) where for \( x = 0, 1, 2, \cdots \), the pdf \( h(\lambda \mid x) \) of the posterior distribution is given by

\[
    h(\lambda \mid x) = \frac{e^{-\lambda} \lambda^x}{x!} \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta} \int_0^{\lambda_x} \frac{e^{-\lambda} \lambda^x}{x!} \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta} d\lambda
\]

\[
    = \frac{1}{\beta/(1 + \beta)^{\alpha+x+1} \Gamma(\alpha + x)} \lambda^{\alpha+x-1} e^{-\lambda/[\beta/(1+\beta)]}, \quad \lambda > 0,
\]

\[
    = 0, \quad \text{otherwise.
\]

That is, the posterior distribution is also gamma with updated parameters \( \alpha + x \) and \( \beta/(1 + \beta) \).
Suppose that integer-valued random variable $X$ has a hypergeometric distribution with parameters $n$, $M$, and $N$, denoted by $X \sim \text{HYP}(n, M, N)$, where $n$, $M$, and $N$ are integers with $1 \leq n \leq N$ and $0 \leq M \leq N$. Then, for given $n$ and $N$, and for $0 < M < N$, the pmf of $X$, denoted by $f_X(x \mid M)$, is

$$ f_X(x \mid M) = P(X = x \mid M) = \binom{M}{x} \binom{N-M}{n-x} \frac{(N-M)}{(n-M)}, \quad \max(0, n - N + M) \leq x \leq \min(n, M), $$

$$ = 0, \quad \text{otherwise}, \quad (4) $$

and $f_X(0 \mid 0) = f_X(n \mid N) = 1$.

Suppose that random variable $Y$ has a specially defined discrete distribution with parameters $a$, $b$, and $c$, denoted by $Y \sim \text{ABC}(a, b, c)$, where $a$, $b$, and $c$ are nonnegative integers. Then, for $c > 0$, the pmf of $Y$, denoted by $f_Y(y \mid a, b, c)$, is

$$ f_Y(y \mid a, b, c) = P(Y = y \mid a, b, c) = \binom{a+y}{a} \binom{b+c-y}{b} \frac{(a+b+c+1)}{(a+b+1)}, \quad y = 0, 1, \ldots, c, $$

$$ = 0, \quad \text{otherwise}, $$

and $f_Y(0 \mid a, b, 0) = 1$. We note that formula (12.16) of Feller (1968, p.65) can be used to prove that

$$ \sum_{y=0}^{c} \binom{a+y}{a} \binom{b+c-y}{b} = \binom{a+b+c+1}{a+b+1}. $$

We also note that the ABC distribution is just a special case of the Polya (or beta-binomial) distribution (Dyer and Pierce, 1993, p.2130). From equation (4), it easily follows that $P(X \leq n \mid M) = 1$ for $0 \leq M \leq N$. For $0 \leq i < n \leq N$ and $0 \leq M \leq N$, we have from equation (4) that

$$ \binom{N}{n} P(X \leq i \mid M) = \sum_{x=0}^{i} \binom{M}{x} \binom{N-M}{n-x} $$

$$ = \sum_{x=0}^{i} \binom{M}{x} \left[ \binom{N-M-1}{n-x-1} + \binom{N-M-1}{n-x} \right] $$

$$ = \sum_{x=0}^{i} \binom{M}{x} \binom{N-M-1}{n-x-1} + \sum_{x=0}^{i} \binom{M}{x} \binom{N-M-1}{n-x} $$

$$ = \sum_{x=1}^{i+1} \binom{M}{x-1} \binom{N-M-1}{n-x} + \sum_{x=0}^{i} \binom{M}{x} \binom{N-M-1}{n-x} $$
where by definition \( \binom{M}{i} = 0 \), \( \binom{M}{i} = 0 \) if \( M < i \), and \( \binom{N-M-1}{n-i-1} = 0 \) if \( M > N-n+i \). Furthermore, from the recursion relationship in equation (5), it follows that

\[
P(X \leq i \mid M) = \sum_{k=M}^{N-n+i} \binom{k}{i} \frac{\binom{N-k}{n-i-1}}{\binom{N}{n}}
\]

\[
= \sum_{k=M-i}^{N-n} \binom{i+k}{i} \frac{\binom{n-i-1+N-n-k}{n-i-1}}{\binom{N}{n}}
\]

\[
= 1 - \sum_{k=0}^{M-1} \binom{i+k}{i} \frac{\binom{n-i-1+N-n-k}{n-i-1}}{\binom{N}{n}}. \tag{6}
\]

That is, if \( X \sim \text{HYP}(n, M, N) \) and \( Y \sim \text{ABC}(i, n-i-1, N-n) \) for integer \( i, 0 \leq i < n \leq N \), then

\[
F_X(i \mid M) = P(X \leq i \mid M) = 1 - P(Y \leq M - i - 1 \mid i, n-i-1, N-n) = 1 - F_Y(M-i-1 \mid i, n-i-1, N-n)
\]

where, in particular,

\[
P(X \leq i \mid M) = 1, \quad \text{if} \quad 0 \leq M \leq i,
\]

\[
= 0, \quad \text{if} \quad N-n+i < M \leq N. \tag{7}
\]

For \( 0 < i \leq j < n \leq N \) and \( 0 \leq M \leq N \), we have from equation (5) that

\[
\binom{N}{n} P(i \leq X \leq j \mid M) = \binom{N}{n} P(X \leq j \mid M) - \binom{N}{n} P(X \leq i-1 \mid M)
\]

\[
= \binom{M}{j} \binom{N-M-1}{n-j-1} + \binom{N}{n} P(X \leq j \mid M+1)
\]

\[
- \binom{M}{i-1} \binom{N-M-1}{n-i} - \binom{N}{n} P(X \leq i-1 \mid M+1)
\]

\[
= \binom{M}{j} \binom{N-M-1}{n-j-1} - \binom{M}{i-1} \binom{N-M-1}{n-i}
\]

\[
+ \binom{N}{n} P(i \leq X \leq j \mid M+1). \tag{8}
\]
Similar to the determination of equation (8), it follows from the recursion relationship in equation (8) that

\[ P(i \leq X \leq j | M) = \sum_{k=M}^{N-n+j} \binom{k}{j} \binom{N-k-1}{n-j-1} / \binom{N}{n} - \sum_{l=M}^{N-n+i-1} \binom{l}{i-1} \binom{N-l-1}{n-i} / \binom{N}{n} \]

\[ = \sum_{k=M-j}^{N-n-j} \binom{j+k}{j} \binom{n-j-1+N-n-k}{n-j-1} / \binom{N}{n} \]

\[ - \sum_{l=M-i+1}^{N-n} \binom{i-1+l}{i-1} \binom{n-i+N-n-l}{n-i} / \binom{N}{n} \]

\[ = \sum_{l=0}^{M-i} \binom{i-1+l}{i-1} \binom{n-i+N-n-l}{n-i} / \binom{N}{n} \]

\[ - \sum_{k=0}^{M-j-1} \binom{j+k}{j} \binom{n-j-1+N-n-k}{n-j-1} / \binom{N}{n} \]  \hspace{1cm} (9)

where, in particular,

\[ P(i \leq X \leq j | M) = 0, \quad \text{if either} \quad 0 \leq M < i \quad \text{or} \quad N-n+j < M \leq N. \]  \hspace{1cm} (10)

We note in equation (8) that the difference

\[ \binom{M}{j} \binom{N-M-1}{n-j-1} - \binom{M}{i-1} \binom{N-M-1}{n-i} = -\binom{N-i}{n-i} < 0, \quad \text{if} \quad M = i-1, \]

\[ = \binom{N-n+j}{j} > 0, \quad \text{if} \quad M = N-n+j. \]  \hspace{1cm} (11)

and for \( i \leq M < N-n+j \), the same difference

\[ \binom{M}{j} \binom{N-M-1}{n-j-1} - \binom{M}{i-1} \binom{N-M-1}{n-i} = \frac{M!(N-M-1)!}{j!(M-j)!(n-j-1)!(N-M-n+j)!} - \frac{M!(N-M-1)!}{(i-1)!(M-i+1)!(n-i)!(N-M-n+i-1)!} \]

\[ = \frac{1}{M!(N-M-1)!} \left[ \frac{1}{(j-1)!(N-M-n+j)\cdots(N-M-n+i)} - \frac{1}{(M-i+1)\cdots(M-j+1)!(n-i)\cdots(n-j)} \right] \]  \hspace{1cm} (12)

where as \( M \) increases, the term \( 1/(N-M-n+j)\cdots(N-M-n+i) \) increases and the term \( 1/(M-i+1)\cdots(M-j+1) \) decreases so that as \( M \) increases between \( i-1 \) and \( N-n+j \), the difference \( \binom{M}{j} \binom{N-M-1}{n-j-1} - \binom{M}{i-1} \binom{N-M-1}{n-i} \) goes from being negative to being positive and staying positive.
In summary, \( P(X \leq n \mid M) \) equals 1 for \( 0 \leq M \leq N \), and for fixed integer \( i \), \( 0 \leq i < n \leq N \), we see from equations (8) and (7) that \( P(X \leq i \mid M) \) equals 1 for \( 0 \leq M \leq i \), is decreasing for \( i < M \leq N - n + i \), and equals 0 for \( N - n + i < M \leq N \); \( P(X \geq n + 1 \mid M) \) equals 0 for \( 0 \leq M \leq N \), and for fixed integer \( j \), \( 1 \leq j \leq n \leq N \), \( P(X \geq j \mid M) = 1 - P(X \leq j - 1 \mid M) \) equals 0 for \( 0 \leq M \leq j - 1 \), is increasing for \( j - 1 < M \leq N - n + j - 1 \), and equals 1 for \( N - n + j - 1 < M \leq N \); and we see from equations (8) to (12) that for fixed integers \( i \) and \( j \), \( 0 < i \leq j < n \leq N \) where we define

\[
M_{n,N}(i, j) = \min\{M \mid i \leq M \leq N - n + j \} \quad \text{and} \quad \binom{M}{j} \binom{N-M}{n-j-1} \geq \binom{M-i}{j-1} \binom{N-M-i}{n-j-1},
\]

\( P(i \leq X \leq j \mid M) \) equals 0 for \( 0 \leq M < i \), is increasing for \( i \leq M < M_{n,N}(i, j) \), is decreasing for \( M_{n,N}(i, j) + 1 < M \leq N - n + j \), and equals 0 for \( N - n + j < M \leq N \) with maximum at either \( M_{n,N}(i, j) \) if \( \binom{M}{j} \binom{N-M}{n-j-1} > \binom{M-i}{j-1} \binom{N-M-i}{n-j-1} \) for \( M = M_{n,N}(i, j) \) so that \( P(i \leq X \leq j \mid M_{n,N}(i, j)) > P(i \leq X \leq j \mid M_{n,N}(i, j) + 1) \) or maximum at both \( M_{n,N}(i, j) \) and \( M_{n,N}(i, j) + 1 \) if \( \binom{M}{j} \binom{N-M}{n-j-1} = \binom{M-i}{j-1} \binom{N-M-i}{n-j-1} \) for \( M = M_{n,N}(i, j) \) so that \( P(i \leq X \leq j \mid M_{n,N}(i, j)) = P(i \leq X \leq j \mid M_{n,N}(i, j) + 1) \).

Suppose that the hypergeometric parameters \( n \) and \( N \) are known but \( M \) is not and we wish to estimate it using Bayesian methods. For a realization \( x \) of \( X \sim \text{HYP}(n, M, N) \), a family of conjugate priors for \( M - x \) is the family of discrete distributions \( \text{ABC}(a, b, N) \) where for \( x = 0, 1, \ldots, n \), the pmf \( h(M \mid x) \) of the posterior distribution for \( M \) is given by

\[
h(M \mid x) = \frac{\binom{M}{j} \binom{n-j}{a} \binom{N-M}{b} \binom{a+x+n-M}{a+b+n+1}}{\sum_{M=x}^{N-n+x} \binom{M}{j} \binom{n-j}{a} \binom{N-M}{b} \binom{a+x+n-M}{a+b+n+1}}, \quad x \leq M \leq N - n + x, \]

\[= 0, \quad \text{otherwise}, \quad (13)\]

from which it easily follows that the pmf \( h(M - x \mid x) \) of the posterior distribution for \( M - x \) is given by

\[
h(M - x \mid x) = \frac{\binom{a+x+x-M-x}{a+x} \binom{b+n-x+M+n-M-x}{b+n-x} \binom{a+x+b+n-x+M-n-M-x}{a+x+b+n-x+1}}{\binom{a+x+b+n-x+M-n-M-x}{a+x+b+n-x+1}}}, \quad 0 \leq M - x \leq N - n, \]

\[= 0, \quad \text{otherwise}. \quad (14)\]

That is, the posterior distribution for \( M - x \) is also \( \text{ABC} \) with updated parameters \( a + x \), \( b + n - x \), and \( N - n \).

Finally, we note that as a family of conjugate priors for the hypergeometric distribution \( \text{HYP}(n, M, N) \), the family of discrete distributions \( \text{ABC}(a, b, N) \) has, in addition to unimodal members, strictly increasing members \( \text{ABC}(a, 0, N) \), strictly decreasing members \( \text{ABC}(0, b, N) \), and the discrete uniform distribution \( \text{ABC}(0, 0, N) \).
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