Non-equilibrium thermodynamics as gauge fixing

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Abstract

By considering a thermodynamic force as gauge field, we extend constitutive equations of Onsager’s non-equilibrium thermodynamics to non-linear equations. In Onsager’s non-equilibrium thermodynamics, the thermodynamic force corresponds to a pure gauge, for which the constitutive equations are obtained by gauge fixing. If we extend the thermodynamic force from pure gauge to physical one, we obtain the non-linear constitutive equations of non-equilibrium thermodynamics.

1 Introduction

Onsager’s theory is the most important one in non-equilibrium thermodynamics with linear constitutive equations \[1, 2\], in which constitutive equations for currents are derived from the minimum energy dissipation principle. Later on, this argument was supported by the path integral representation of the probability distribution \[3, 4, 5, 6, 7\]. Onsager’s theory holds in the case of linear constitutive equations, but it is not well understood in the non-linear case (for the latest research see \[8, 9\]). Recently, Sugamoto pointed out with his collaborators including the present author that thermodynamic force can be viewed as a gauge field \[10\].

In this paper we discuss this statement more definitely by means of gauge fixing, and derive the non-linear constitutive equation by adding the free action of the usual electromagnetism.

The paper contains the following sections. In the next section, we review the electromagnetism in a pure gauge as an useful analogy in the later discussions. In section 3, non-equilibrium thermodynamics is introduced using a differential form. In section 4, we examine gauge properties of the thermodynamic force.

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We extend thermodynamic force to thermodynamical gauge field in section 5. In section 6, we discuss this gauge theory in the path integral method. The final section is devoted to discussion.

In addition, the paper contains the following appendices. In Appendix A, the meaning of the time dependence of $S(a, t)$ is discussed. In Appendix B, we examine restriction from the second law of thermodynamics. In Appendix C, we discuss how non-linear effects work in dimensional analysis. In Appendix D, a simple example is derived in our model.

2 Electromagnetism

First, we consider electromagnetism as a typical gauge theory. The notion of a pure gauge and its gauge transformation (local phase shift) is important in later sections in order to consider thermodynamics as a gauge theory.

As is well known, a pure gauge $A_\mu$ is described as

$$
d\theta(x) = A_\mu(x)dx^\mu, \quad (1)
$$

where

$$
A_\mu(x) \equiv \frac{\partial \theta(x)}{\partial x^\mu}, \quad (2)
$$

using phase $\theta(x)$. The gauge transformation

$$
\delta A_\mu(x) = \partial_\mu \epsilon(x) \quad (3)
$$

causes a phase shift

$$
\delta \theta(x) = \epsilon(x). \quad (4)
$$

At the pure gauge, electromagnetic field $F$ is zero,

$$
F = dA = dd\theta = 0. \quad (5)
$$

For a general gauge potential $A_\mu$, $F$ satisfies Maxwell’s equations,

$$
\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0, \quad (6)
$$

$$
\partial^\mu F_{\mu\nu} = 0. \quad (7)
$$

3 Differential form of non-equilibrium thermodynamics

A linear constitutive equation in non-equilibrium thermodynamics is expressed as

$$
a^i = L^{ij}X_j, \quad (8)
$$
where $a^i$ is extensive quantity for entropy, $X_j$ is thermodynamic force for $a^j$, and $L^{ij}$ is Onsager’s phenomenological matrix. According to Onsager’s theory [1, 2], (8) is obtained by taking variation
\[
\delta \left( -\frac{1}{2} R_{ij} \dot{a}^i \dot{a}^j + X_i \dot{a}^i \right) = 0
\]
(9)
with respect to $\dot{a}^i$, where $R_{ij}$ is the inverse matrix of $L^{ij}$,
\[
R_{ij} L^{jk} = \delta^k_i.
\]
(10)
As its dual version, (8) is also obtained by taking variation
\[
\delta \left( -\frac{1}{2} L^{ij} X_i X_j + X_i \dot{a}^i \right) = 0
\]
(11)
with respect to $X_i$.

At a first glance, thermodynamics and electromagnetism are different theories. In thermodynamics, an important quantity, entropy $S(a, t)$, is described in terms of extensive quantities $a$’s and time $t$. If we describe, however, entropy $S(a, t)$ in terms of differential form, then quantities analogous to Lagrangian $L$ and gauge field $A$ appear.

The differential form of entropy $S(a, t)$ is expressed as
\[
dS(a, t) = \Phi(a, t) dt + X_i(a, t) da^i,
\]
(12)
with
\[
\Phi(a, t) \equiv \frac{\partial S(a, t)}{\partial t},
\]
(13)
\[
X_i(a, t) \equiv \frac{\partial S(a, t)}{\partial a^i},
\]
(14)
where $\Phi(a, t)$ is called the dissipation function.

We will take the following notations:
\[
X_0 \equiv \Phi,
\]
(15)
\[
a^0 \equiv t,
\]
(16)
and $X_\mu$, $a^\mu$ by
\[
\{X_\mu\} = X_0, X_1, \ldots X_N,
\]
(17)
\[
\{a^\mu\} = a^0, a^1, \ldots, a^N.
\]
(18)
Then, (60) is rewritten as
\[
dS(a) = X_\mu(a) da^\mu,
\]
(19)
\footnote{In non-equilibrium thermodynamics, the time dependence of $S(a, t)$ is natural when we consider path integral formalism and WKB approximation. See Appendix A.}
where

\[ X_\mu(a) \equiv \frac{\partial S(a)}{\partial a^\mu}. \]  

(20)

If we describe \( a^\mu \) using parameter \( \tau \), we get non-equilibrium thermodynamical Lagrangian \( L \) as follows:\textsuperscript{3}

\[ L(a, \frac{da}{d\tau})d\tau = X_\mu(a) \frac{da^\mu}{d\tau} d\tau. \]  

(21)

### 4 Gauge transform property of \( X_\mu \)

In order to know the properties of the above non-equilibrium thermodynamical Lagrangian \( L \), we have to choose an “appropriate gauge fixing condition” so that we can get the Onsager’s phenomenological transfer matrix \( L^{ij} \) and its constitutive equations for \( X_i \).

The Lagrangian \( L \) in (21) has gauge symmetry described by the transformation

\[ \delta X_\mu(a) = \partial_\mu \epsilon(a). \]  

(22)

The transformation (22) causes entropy shift,

\[ \delta S(a) = \epsilon(a). \]  

(23)

Since the gauge transform (22) introduces entropy’s shift (23), we want to fix this gauge. To fix gauge, we take a gauge fixing condition,

\[ X_0(a) = \frac{1}{2} L^{ij} X_i(a) X_j(a), \]  

(24)

Here, we assume \( \tau = t \) to fix the parametrization invariance.

Finally, the thermodynamical Lagrangian \( L \) becomes

\[ L(a, \dot{a}) dt = \frac{1}{2} L^{ij} X_i(a) X_j(a) dt + X_i(a) \dot{a}^i dt. \]  

(25)

Since \( X_i \) is a nondynamical field, the equation of motion for \( X_i \) produces a constitutive equation \textsuperscript{[1,2]}

\[ \dot{a}^i = L^{ij} X_j(a). \]  

(26)

Substituting (26) into (25), we obtain

\[ L(a, \dot{a}) dt = \frac{1}{2} R_{ij} \dot{a}^i \dot{a}^j dt. \]  

(27)

\textsuperscript{2}The parameter \( \tau \) is an arbitrary parameter, and it can always be introduced into a dynamical system.

\textsuperscript{3}This corresponds to the Rayleigh dissipation function.
5 Thermodynamical gauge theory

According to gauge theory, if we consider the thermodynamic force as a gauge potential, then the equation (25), with Onsager’s phenomenological matrix, corresponds to a pure gauge theory. On the other hand, if we introduce the physical gauge theory instead of pure one into this thermodynamic system, then we will obtain a different equation of motion for $X_\mu$, reflecting gauge-fixing condition for the physical gauge theory. This equation of motion is similar to Maxwell’s equations in electromagnetism. Furthermore, we can obtain the non-linear constitutive equation of non-equilibrium thermodynamics.

So far we define thermodynamic gauge field $F$ by

$$F(a) = dX(a), \quad (28)$$

but because $dS(a) = X(a)$, $F(a)$ is always zero,

$$F(a) = ddS(a) = 0. \quad (29)$$

Now, we will generalize this, and assume that $X_\mu$ is not a pure gauge field, but a gauge field in a physical gauge theory. The action

$$S_F[X] = \int d^{N+1} a F_{\mu\nu} F^{\mu\nu} \quad (30)$$

is added to the original action with gauge fixing condition,

$$S[a, X, \lambda] = \int d\tau \left( X_\mu(a) \frac{da^\mu}{d\tau} - \lambda \left( X_0(a) + \frac{1}{2} L^{ij} X_i(a) X_j(a) \right) \right), \quad (31)$$

where $\lambda$ is a Lagrange multiplier.

The equation of motion for $X_\mu$ is

$$\frac{da^0}{d\tau} - \lambda = \partial_\mu F^{\mu 0}(X) = \partial_i E^i(X), \quad (32)$$

$$\frac{da^i}{d\tau} - \lambda L^{ij} X_j(a) = \partial_\mu F^{\mu i}(X) = \dot{E}^i(X) + \partial_j B^{ji}(X), \quad (33)$$

where $E_i$ and $B_{ij}$ are

$$E_i(X) \equiv F_{i0}(X) = \partial_i \Phi(X) - \dot{X}_i \equiv L^{ik} X_j \partial_i X_k - \dot{X}_i,$$

$$B_{ij}(X) \equiv F_{ij}(X) = \partial_i X_j - \partial_j X_i. \quad (34)$$

We call $E_i(X)$ and $B_{ij}(X)$ thermodynamic electric field and thermodynamic magnetic field, respectively.

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4It’s dimension is $N + 1$ which is the number of components of $X$. 5
(32) is solved for $\lambda$,

$$\lambda = \frac{da^0}{d\tau} - \partial_i E^i,$$

and we take static gauge $\tau = t$, then (33) yields

$$\dot{a}^i = (1 + \partial_k E^k) L^{ij} X_j + \dot{E}^i + \partial_j B^{ji}.$$

From

$$\partial_i E^i = L^{il} L^{jk} \partial_i X_j \partial_l X_k + L^{il} L^{jk} X_j \partial_i X_k - L^{ij} \partial_i \dot{X}_j,$$

we get a non-linear constitutive equation of non-equilibrium thermodynamics,

$$\dot{a}^i = L^{ij} X_j + \xi^i$$

where $\xi^i$ is a term from the thermodynamic gauge fields:

$$\xi^i = L^{ml} L^{jk} L^{in} \partial_i X_j \partial_m X_k X_n + L^{ml} L^{jk} L^{in} \partial_i X_j \partial_m \partial_n X_k X_n - L^{kj} L^{il} \partial_k \dot{X}_j X_l$$

$$+ L^{jk} L^{il} \dot{X}_j \partial_l X_k + L^{jk} L^{il} X_j \partial_l \dot{X}_k - L^{ij} \dot{X}_j$$

$$+ L^{jk} L^{il} \partial_j \partial_l X_k - L^{jk} L^{il} \partial_j \partial_l X_k.$$

What we have done is, by introducing the usual kinetic action of electromagnetism into the thermodynamics, the gauge field $X_i(a)$ can have the additional transverse components, in addition to the longitudinal (or the pure gauge) component existing in the Onsager’s non-equilibrium thermodynamics. Then, the non-linear constitutive equation is obtained.

The second law of thermodynamics restricts thermodynamic gauge fields to

$$\dot{E}^i + \partial_j B^{ji} \geq - \left( \frac{1}{2} + \partial_k E^k \right) L^{ij} X_j.$$

From the dimensional analysis in Appendix C, the non-linear term dominates and our model with a non-linear constitutive equation becomes effective at long relaxation time.

Thermodynamic gauge field naturally leads to an oscillatory phenomena.

6 Path Integral

Here, let us discuss this gauge theory in the path integral representation. For the thermodynamic variable $a(t)$, let $P[a]$ be the probability that a thermodynamic path will be realized with. We will call entropy for this probability as path entropy, given by

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5 See Appendix B for details.

6 As a simple example, see Appendix D.
\[ S[a] \equiv -k_B \log P[a]. \] (42)

The probability of the transition from state \((a_i, t_i)\) to \((a_f, t_f)\) can be written by path integration,

\[ P(a_f, t_f | a_i, t_i) = \int D a P[a], \] (43)

where the path is fixed at times \(t_i\) and \(t_f\). The path entropy can generally be described by the entropic Lagrangian,

\[ S[a] = \int_{t_i}^{t_f} L(a, \dot{a}) dt, \] (44)

\[ L = \dot{S} = X(a) \dot{a} + \Phi(X(a), a). \] (45)

Here, in order to be able to regard \(X\) as a free variable, we insert

\[ \int D X \delta \left( X(t) - \frac{\partial S}{\partial a} \right) = \int D X D \eta e^{i \int dt \eta (X - \frac{\partial S}{\partial a})} = 1 \] (46)

into the path integral. Then, the entropic Lagrangian is rewritten as

\[ L = X \dot{a} + \Phi(X, a) + i \eta \left( X - \frac{\partial S}{\partial a} \right). \] (47)

Solving (47) for \(X\), we obtain the constitutive equation,

\[ \dot{a}^i = -\frac{\partial \Phi}{\partial X_i} - i \eta^i. \] (48)

Here, if we take \(\Phi\) as

\[ \Phi = -\frac{1}{2} L_{ij} X_i X_j, \] (49)

the constitutive equation yields

\[ \dot{a}^i = L_{ij} X_j - i \eta^i. \] (50)

Substituting this expression into \(\dot{S}\), we get

\[ \dot{S} = \frac{1}{2} \dot{a}^2 + i \eta \left( \dot{a} - \frac{\partial S}{\partial a} \right) - \frac{1}{2} \eta^2, \] (51)

and solving this equation for \(\eta\), we get

\[ \dot{a}^i = L_{ij} \frac{\partial S}{\partial a^j} - i \eta^i. \] (52)

Substituting this expression into \(\dot{S}\), we obtain

\[ \dot{S} = \frac{1}{2} \left( \frac{\partial S}{\partial a} \right)^2 - \frac{\partial S}{\partial a} i \eta^i. \] (53)
Then, the first term of $\dot{S}$ gives the the entropy increasing law of macroscopic irreversible process, and the second term is an effect of fluctuation.

Next, let us assume that $\Phi$ is gauge-fixed like

$$dS = X(a)da + \Phi(a)dt.$$

Then, we understand that before gauge fixing, $dS$ is

$$dS = X(a)da + \Phi(a)dt.$$  \hspace{1cm} (54)

This action before gauge fixing is invariant under gauge transformation,

$$X(a) \rightarrow X(a) + \frac{\partial \epsilon}{\partial a},$$  \hspace{1cm} (56)

$$\Phi(a) \rightarrow \Phi(a) + \frac{\partial \epsilon}{\partial t}.$$  \hspace{1cm} (57)

Then, this theory is originally a gauge theory of $A_\mu = (\Phi, X).$  \hspace{1cm} (58)

From the above discussion, the differential form representation of non-equilibrium thermodynamics is natural from the viewpoint of the path integral representation, and the gauge fixing is naturally introduced to obtain a specific thermodynamical physics.

### 7 Discussion

We have assumed that the thermodynamic force is a gauge potential and generalized this potential from a pure gauge to a physical gauge potential. By gauge fixing (24) and adding a thermodynamical gauge field action (30), we have obtained a non-linear constitutive equation for non-equilibrium thermodynamics (41). If we choose another gauge fixing condition, we may get another thermodynamical physics. This will be discussed elsewhere.

The viewpoint that the existing theory is some kind of gauge theory with gauge fixing may be useful to generalize the theory. One of the most interesting applications of this method is application to classical mechanics, where a momentum may be regarded as a kind of statistic [11].

The entropy of a thermodynamic path, which is essential for the differential form representation of non-equilibrium thermodynamics, is also a key to the detailed fluctuation theorem [12]. Then, the gauge fixing method and the gauge principle may help understand the second law of thermodynamics.

All arguments so far are in the realm of classical theory. Reconsideration of our discussion in the context of quantum theory is important. In the latter case, the quantum fluctuation may be added to (53).

For other applications, application to phase transition phenomena can be considered. In time-dependent Ginzburg-Landau theory [13], the free energy...
is expressed in terms of a complex order parameter field and have the gauge symmetry of this parameter’s phase shift. By applying our gauge fixing method to time-dependent Ginzburg-Landau theory, we may obtain understanding between these two gauge symmetries.

Finally, optics and non-equilibrium thermodynamics may have a closer relationship than we have thought of, such as the invariance of the line element of light under the gauge transformation. We will investigate such relationship in future work.

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A. The time dependence of $S(a, t)$

In this Appendix, the meaning of the time dependence of $S(a, t)$ is discussed. As is well-known in classical mechanics \[\text{[14]}\], for the action

\[
S[x] = \int dt L(x, \dot{x}), \tag{59}
\]

the differential of the action is equal to

\[
dS(x, t) = p\, dx - H\, dt. \tag{60}
\]

Then, we obtain

\[
p = \frac{\partial S(x, t)}{\partial x}, \tag{61}
\]

\[
-H = \frac{\partial S(x, t)}{\partial t}. \tag{62}
\]

The last equation is Hamilton-Jacobi equation. The application of the Hamilton-Jacobi-Equation to fluctuation phenomena is studied by Kitahara \[\text{[15]}\].

If $H$ takes a constant value $E$, then we have
\[ S(x,t) = -Et + \bar{S}(x), \quad (63) \]

and (60) becomes

\[ dS(x,t) = \bar{p}dx - Edt \quad (64) \]

\[ \bar{p} \equiv \frac{\partial \bar{S}(x)}{\partial x} \quad (65) \]

The entropy that does not depend on time is this \( \bar{S}(x) \).

**B. Restriction from the second law of thermodynamics**

We want to discuss the restriction from the second law of thermodynamics on our model with a non-linear constitutive equation.

The entropy production is

\[ \dot{S} = \frac{\partial S}{\partial a^i} \dot{a}^i + \Phi = \frac{1}{2} L^{ij} X_i X_j + X_i \xi^i \quad (66) \]

\[ X_i \xi^i = X^2 (\partial_i X_k)^2 + X^2 X^k \partial^2 X_k - X^2 \partial \cdot \dot{X} + X^i \dot{X}^j \partial_i X_j + X^i X^j \partial_i \dot{X}_j - X^i \ddot{X}_i \quad (67) \]

Since (60) can be written as

\[ S(a,t) = \Phi t + \bar{S}(a), \quad (68) \]

we get

\[ dS(a,t) = \bar{X}_i(a) da^i + \Phi dt, \quad (69) \]

\[ \bar{X}_i(a) \equiv \frac{\partial \bar{S}(a)}{\partial a^i}. \quad (70) \]

Then, under (68), (67) becomes

\[ \bar{X}_i \xi^i = \bar{X}^2 (\partial_i \bar{X}_k)^2 + \bar{X}^2 \dot{X}^i \partial^2 \bar{X}_i \quad (71) \]

Here we note, \( \dot{E}_i(\bar{X}) = B_{ij}(\bar{X}) = 0 \).

If it is assumed that the second law of thermodynamics can not be broken, this requires the condition of

\[ \frac{1}{2} + (\partial_i \bar{X}_k)^2 + \bar{X}_i \partial^2 \bar{X}_i \geq 0. \quad (72) \]
If we use Ruppiner metric [16]:

\[ g^R_{ij} = \frac{\partial^2 \bar{S}}{\partial a_i \partial a_j} = \frac{\partial \bar{X}_i}{\partial a_j}, \]  

(73)

this condition gives

\[ 2 \partial^k (\bar{X}^i g^R_{ik}) \geq -1 \]  

(74)

if \( g^R_{ij} \) is constant, then

\[ 2 g^{Rik} g^R_{ik} \geq -1. \]  

(75)

Because \( L^{ij} \) is positive definite,

\[ g^{Rik} g^R_{ik} \geq 0 \]  

(76)

always holds.

More generally, in term of thermodynamic electric field and thermodynamic magnetic field, the non-linear constitutive equation is

\[ \dot{a}^i = (1 + \partial_k E^k) L^{ij} X_j + \dot{E}^i + \partial_j B^{ji}. \]  

(77)

Then, the entropy production can be given by

\[ \dot{S} = \left( \frac{1}{2} + \partial_k E^k \right) L^{ij} X_i X_j + X_i (\dot{E}^i + \partial_j B^{ji}). \]  

(78)

Therefore, the restriction from the second law of thermodynamics on our model with a non-linear constitutive equation yields

\[ \dot{E}^i + \partial_j B^{ji} \geq - \left( \frac{1}{2} + \partial_k E^k \right) L^{ij} X_j. \]  

(79)

C. Dimensional analysis

To know physics, the dimensional analysis is important. From this analysis, we obtain relations,

\[ [S] = [k_B], \]  

(80)

\[ [\Phi] = \frac{[S]}{[t]} = \frac{k_B}{[t]}, \]  

(81)

\[ \left[ \frac{\partial}{\partial a^i} \right] = \frac{1}{[a^i]}, \]  

(82)

\[ [X_i] = \frac{[S]}{[a^i]} = \frac{k_B}{[a^i]}, \]  

(83)

\[ \frac{[a^i]}{[t]} = [L^{ij}] [X_j] = [L^{ij}] \frac{k_B}{[a^i]}. \]  

(84)
Under these relations, we insert a basic constant of time, \( \hat{t} \) and we get

\[
\lambda = \frac{d\hat{a}^0}{d\tau} - k_B \hat{t} \partial_i E^i
\] (93)

\[
\dot{a}^i = (1 + k_B \hat{t} \partial_k E^k) L^{ij} X_j + k_B \hat{t} \left( \dot{E}^i + \partial_j B^{ji} \right)
\] (94)

Then, if \( \hat{t} \) is long, then the non-linear term dominates and our model with a non-linear constitutive equation becomes effective.

**D. Simple example**

As a simple example, we assume,

\[
X_i = g^R_{ij} a^j.
\] (95)

\[
\partial_j X_i = g^R_{ij}
\] (96)

\[
\xi^i = k_B \hat{t} \left( (g^R_{nk})^2 L^{in} g^R_{np} a^p + L^{jk} L^{li} g^R_{jp} \dot{a}^p + g^R_{lk} - L^{ij} g^R_{jp} \ddot{a}^p \right)
\] (97)

In this case, we can derive the following we get second order linear differential equation,
\[ A_p^i \ddot{a}^p + B_p^i \dot{a}^p + C_p^i a^p = 0, \quad (98) \]
\[ A_p^i \equiv k_B \tilde{t} \dot{L}^i_{\ j} g_{jp}, \quad (99) \]
\[ B_p^i \equiv (1 - k_B \tilde{t} L^j L^k \dot{g}_{jkp} g_{kl}), \quad (100) \]
\[ C_p^i \equiv -(1 + k_B \tilde{t} L^R_i L^g_{g_p g_p} g_{R k}). \quad (101) \]

If we modify the coefficient of the first term in (98) to be 1, we get
\[ \ddot{a}^i + D_p^i \dot{a}^p + E_p^i a^p = 0, \quad (102) \]
\[ D_p^i \equiv (A^{-1})^i_j B_j^p = \left( \frac{1}{k_B \tilde{t}} g_{R ik} R_{kp} - L^i_{\ j} g_{jkp} \right), \quad (103) \]
\[ E_p^i \equiv (A^{-1})^i_j C_j^p = \epsilon \delta^i_p, \quad (104) \]
\[ \epsilon \equiv - \left( \frac{1}{k_B \tilde{t}} + (g_{ml}^R)^2 \right). \quad (105) \]

Here we note
\[ [D, E] = 0. \quad (106) \]

If \( \tilde{t} \to 0 \), we get well known linear constitutive equation.

We take a linear transformation of \( b \),
\[ a^p = P_p^o b^o, \quad (107) \]
\[ \ddot{b}^i + (P^{-1} D P)^i_p \dot{b}^p + \epsilon b^p = 0. \quad (108) \]

If we assume it can be diagonalized,
\[ (P^{-1} D P)^i_p = \lambda_{(i)} \delta^i_p, \quad (109) \]
we get
\[ \ddot{b}^i + \lambda_{(i)} \dot{b}^i + \epsilon b^i = 0, \quad (110) \]
\[ \omega^2 + \lambda_{(i)} \omega + \epsilon = 0, \quad (111) \]
\[ \omega = \frac{-\lambda_{(i)} \pm \sqrt{\lambda_{(i)}^2 - 4 \epsilon}}{2}. \quad (112) \]

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Under $\tilde{t} \to \infty$,

$$D^i_p = -L^{ik} g^R_{kp}, \quad (113)$$

$$\epsilon = -\left(g^R_{km}\right)^2. \quad (114)$$

If we assume for simplicity,

$$L^{ik} g^R_{kp} = \alpha \delta^i_p, \quad (115)$$

then

$$\alpha = \frac{L^{ik} g^R_{ki}}{N}, \quad (116)$$

$$\epsilon = -L^{ik} L^{lm} g^R_{km} g^R_{ks} = -N \alpha^2, \quad (117)$$

$$P^i_j = \delta^i_j. \quad (118)$$

we get

$$\ddot{a}^i - \alpha \dot{a}^i + N \alpha^2 a^i = 0, \quad (119)$$

$$\omega = \frac{\alpha \pm \sqrt{\alpha^2 - 4N \alpha^2}}{2} = \frac{1 \pm i \sqrt{(4N - 1)} \alpha}{2}. \quad (120)$$

Then, we get solution that decays while oscillating,

$$a(t) = a(0) e^{-\frac{\sqrt{(4N - 1)} \alpha}{2} t} \cos \frac{1}{2} \left(\frac{L^{ik} g^R_{ki}}{N}\right) t \quad (121)$$

If we take limit $N \to \infty$, the oscillating of (121) disappears,

$$a(t) = a(0) e^{-L^{ik} g^R_{ki} t}. \quad (122)$$

Non-equilibrium macroscopic oscillatory phenomena is discussed in the context of chemical reaction [17].

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