Non-singular rotating metric in ghost-free infinite derivative gravity

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It is well-known that the vacuum solution of Einstein’s theory of general relativity provides a rotating metric with a ring singularity, which is covered by the inner and outer horizons, and an ergo region. In this paper, we will discuss how ghost free, quadratic curvature, Infinite Derivative Gravity (IDG) may resolve the ring-type singularity in nature. It is well-known that a class of IDG actions admit linearized metric solutions which can avoid point-like singularity by a smearing process of the Delta-source distribution induced by non-locality, which makes the metric potential finite everywhere including at \( r = 0 \), and even at the non-linear level may resolve the Schwarzschild singularity. The same action can also resolve the ring singularity in such a way that no horizons are formed in the linear regime, where in this case non-locality plays a crucial role in smearing out a delta-source distribution on a ring. We will also study the full non-linear regime. First, we will argue that the presence of non-local gravitational interaction will not allow the Kerr metric as an exact solution, as there are infinite order derivatives acting on the theta-Heaviside and the delta-Dirac distributions on a ring. Second, we will explicitly show that the Kerr-metric is not a pure vacuum solution when the Weyl squared term, with a non-constant form-factor, is taken into account in the action.

I. INTRODUCTION

Einstein’s theory of general relativity (GR) is indeed a very successful metric theory of gravity which has seen amazing success in the infrared (IR) [1], including the detection of first gravitational wave signal [2]. Inspite of these successes, the classical GR suffers from the ultraviolet (UV) catastrophe at short distances and small time scales, there are blackhole and cosmological singularities [3–5]. It has been recently shown that quadratic curvature infinite derivative gravity (IDG), with infinite covariant derivatives, which is also free from ghosts at the tree-level, can potentially resolve the cosmological [6], and blackhole type singularities [7]. Infinite derivatives acting on a point Delta-Dirac source smears out the singularity by a Gaussian profile [8–10]. At a quantum level the graviton interactions vertex becomes non-local [11–14]; this feature is very similar to string field theory [15–17] and P-adic strings [18].

Most importantly, the gravitational interaction in the UV weakens enough that beyond the effective scale of non-locality, both linear [17] and non-linear equations of motion [25] provide a conformally flat spherically symmetric, static metric solution [8]. A similar scenario also holds in the case of a charged point source [26]. Furthermore, it has been shown that the singularity and the event horizon does not form in a dynamical context at a linear level [27], as a mass gap can be formed which is given by the non-local scale [28]; the non-linear calculation is yet to be shown. In particular, it has been shown that singular solutions such as Schwarzschild metric [30], Kasner metric [31] do not satisfy the field equations in the vacuum when the Weyl squared term, with a non-constant form-factor, is taken into account in the action. Moreover, even in the absence of the Weyl squared term, non-locality is such that the gravitational interaction in the UV weakens enough that beyond the effective scale of non-locality, both linear [17] and non-linear equations of motion [25] provide a conformally flat spherically symmetric, static metric solution [8]. A similar scenario also holds in the case of a charged point source [26].

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1 Non-locality is a feature for any quantum theory of gravity including strings and branes, which are inherently non-local objects, loop quantum gravity, for a review [12], spin foam or dynamical triangulation introduces Wilson loops as a fundamental operator, for a review [20]. The quantum scatterings for such non-local interactions provide a very interesting insight [21, 22], there is a UV-IR connection, and a collective emergent phenomena [2] with a large number of scatterings, the scattering amplitude gets suppressed for external momentum higher than the scale of non-locality, i.e. for \( P^2 > M_s^2 \), the scattering amplitude is exponentially suppressed, and the scale of non-locality gets shifted by \( M_s \rightarrow M_s/\sqrt{N} \) [23]. Furthermore, non-local thermal field theory provides resemblance to a Hagedorn phase as shown in [24].
Schwarzschild metric is not a solution due to the presence of infinite derivatives, as it has been argued in Ref. [8]: the region of non-locality yields a non-vacuum solution as opposed to that in GR. At the cosmological front, such non-locality can potentially replace the cosmological singularity by big bounce [9] or freezing the Universe in the UV [32]. Outside the region of non-locality the gravitational interaction becomes that of GR, thus reproducing all the features of gravity being tested in the IR [33, 34]. In the case of astrophysical Schwarzschild type blackhole, the gravitational radius could be thought of being engulfed by the non-local region, such that \( r_{sch} \leq r_{NL} \), where \( r_{NL} \) signifies the scale of effective non-locality which we will discuss below in more details [8].

The aim of this paper is to understand the rotating metric within IDG, and show how infinite derivatives will smear out the ring type singularity present in the Kerr metric [35]. The gravitational collapse of a rotating system does not generate a point like singularity, but a ring singularity with a rotation on a plane. First, we will show within the linear regime how to resolve a ring-type singularity, by considering a toy-model with a delta-Dirac distribution on a plane. Moreover, we will also explicitly show that order by order in higher covariant derivatives the Kerr-type metric does not solve the field equations in the vacuum if the Weyl squared term, with a non-constant form-factor, is taken into account in the gravitational action.

The most general quadratic curvature action, which is parity invariant and torsion-free, has been derived in Refs. [7, 25, 36] and is given by

\[
S = S_{EH} + S_q = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ \mathcal{R} + \alpha_c (\mathcal{R} F_1(\Box_s) \mathcal{R} + \mathcal{R}^{\mu\nu} F_2(\Box_s) \mathcal{R}_{\mu\nu} + W^{\mu\nu\lambda\sigma} F_3(\Box_s) W_{\mu\nu\lambda\sigma}) \right],
\]

where \( S_{EH} \) corresponds to the Einstein-Hilbert action and \( S_q \) corresponds to the quadratic curvature terms, \( G = 1/M^2_p \) is Newton’s gravitational constant, \( \alpha_c \sim 1/M^2_s \) is a dimensionful coupling, \( \Box_s \equiv \Box/M^2_s \), where \( M_s \) represents the scale of non-locality at which new physics should emerge. In the limit \( M_s \to \infty \), the action reduces to the Einstein-Hilbert term, as expected. The d’Alembertian operator is defined as \( \Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \), where \( \mu, \nu = 0, 1, 2, 3 \), and we work with the mostly positive metric convention, \((-+, +, +, +)\). The three gravitational form factors \( F_i \)'s, are analytic function of \( \Box \) and can be expressed in series representation as follows

\[
F_i(\Box_s) = \sum_{n=0}^{\infty} f_{i,n} \Box^n_s,
\]

and they are reminiscent to any massless theory possessing only derivative interactions. Note that we will always consider operators of \( \Box_s \) which are analytic, and never introduce non-analytic operators such as \( 1/\Box_s \) or \( \ln(\Box_s) \). The complete set of field equations corresponding to the action in Eq. (2) have been derived in Ref. [25],

\[
W^{\mu}_{\alpha\nu\beta} = \mathcal{R}^{\mu}_{\alpha\nu\beta} - \frac{1}{2} \left( \delta_\mu^\beta \mathcal{R}_{\alpha\nu} - \delta_\beta^\alpha \mathcal{R}_{\nu\mu} + \mathcal{R}_\mu^{\nu\alpha\beta} - \mathcal{R}_\nu^{\mu\alpha\beta} g_{\alpha\nu} - \mathcal{R}_\delta^{\mu\alpha\beta} g_{\alpha\nu} \right) + \frac{\mathcal{R}}{6} \left( \delta_\delta^\alpha g_{\alpha\nu} - \delta_\beta^\alpha g_{\alpha\nu} \right)
\]

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2 The original action was written in terms of the Riemann tensor (see Ref. [7]), but here we wish to work with the Weyl tensor which is related to the Riemann tensor as follows:

\[
W^{\mu}_{\alpha\nu\beta} = \mathcal{R}^{\mu}_{\alpha\nu\beta} - \frac{1}{2} \left( \delta_\mu^\beta \mathcal{R}_{\alpha\nu} - \delta_\beta^\alpha \mathcal{R}_{\nu\mu} + \mathcal{R}_\mu^{\nu\alpha\beta} - \mathcal{R}_\nu^{\mu\alpha\beta} g_{\alpha\nu} - \mathcal{R}_\delta^{\mu\alpha\beta} g_{\alpha\nu} \right) + \frac{\mathcal{R}}{6} \left( \delta_\delta^\alpha g_{\alpha\nu} - \delta_\beta^\alpha g_{\alpha\nu} \right)
\]
and they are given by

\[
P^{\alpha\beta} = \frac{G^{\alpha\beta}}{8\pi G} + \frac{\alpha_c}{8\pi G} \left( 4G^{\alpha\beta}F_1(\square_\alpha)R + g^{\alpha\beta}RF_1(\square_\alpha)R - 4(\nabla^\alpha \nabla^\beta - g^{\alpha\beta}\square)F_1(\square_\alpha)R \right)
\]

\[
- 2\Omega_1^{\alpha\beta} + g^{\alpha\beta} \left( \Omega_1^{\gamma\delta} + \tilde{\Omega}_1 \right) + 4R^{\alpha}_\mu F_2(\square_\alpha)R^{\mu\beta}
\]

\[
- g^{\alpha\beta} \left( R^{\mu}_{\mu} F_2(\square_\alpha)R^{\nu}_{\mu} - 4\nabla_\mu \nabla_\beta (F_2(\square_\alpha)R^{\mu\alpha}) + 2\square(F_2(\square_\alpha)R^{\alpha\beta}) \right)
\]

\[
+ 2g^{\alpha\beta} \nabla_\mu (F_2(\square_\alpha)R^{\nu}_{\mu}) - 2\Omega_2^{\alpha\beta} + g^{\alpha\beta}(\Omega_2^\gamma + \tilde{\Omega}_2) - 4\Delta_2^{\alpha\beta}
\]

\[
- g^{\alpha\beta} W^{\mu\nu\lambda\sigma} F_3(\square_\alpha)W^{\mu\lambda\sigma\nu} \Omega + 4W^{\alpha}_{\mu\nu\sigma} F_3(\square_\alpha)W^{\mu\nu\sigma}
\]

\[
- 4(\Omega_3^{\alpha\beta} + g^{\alpha\beta}(\Omega_3^\gamma + \tilde{\Omega}_3) - 8\Delta_3^{\alpha\beta})
\]

\[
= - T^{\alpha\beta},
\]

where \( T^{\alpha\beta} \) is the stress-energy tensor of the matter component, and the symmetric tensors are defined as (see Ref. 25):

\[
\Omega_1^{\alpha\beta} = \sum_{n=1}^{\infty} f_{\alpha} \sum_{l=0}^{n-1} \nabla^\alpha R^{(l)}(\nabla^\beta R^{(n-l-1)}), \quad \Omega_1 = \sum_{n=1}^{\infty} f_{\alpha} \sum_{l=0}^{n-1} R^{(l)}(\nabla^\alpha R^{(n-l)}),
\]

\[
\Omega_2^{\alpha\beta} = \sum_{n=1}^{\infty} f_{\alpha} \sum_{l=0}^{n-1} \nabla^\alpha R^{(l)}(\nabla^\beta R^{(n-l-1)}), \quad \Omega_2 = \sum_{n=1}^{\infty} f_{\alpha} \sum_{l=0}^{n-1} R^{(l)}(\nabla^\alpha R^{(n-l)}),
\]

\[
\Delta_2^{\alpha\beta} = \sum_{n=1}^{\infty} f_{\alpha} \sum_{l=0}^{n-1} [\nabla^\alpha R^{(l)}(\nabla^\beta R^{(n-l-1)} - R^{(l)}(\nabla^\alpha R^{(n-l-1)}))],
\]

\[
\Omega_3^{\alpha\beta} = \sum_{n=1}^{\infty} f_{\alpha} \sum_{l=0}^{n-1} W^{\mu\lambda\sigma}(W^{\mu\nu\sigma}(n-l-1)), \quad \Omega_3 = \sum_{n=1}^{\infty} f_{\alpha} \sum_{l=0}^{n-1} W^{\mu\lambda\sigma}(W^{\nu\lambda\sigma}(n-l-1)),
\]

\[
\Delta_3^{\alpha\beta} = \sum_{n=1}^{\infty} f_{\alpha} \sum_{l=0}^{n-1} [\nabla^\alpha R^{(l)}(W^{\beta\sigma\mu}(n-l-1))],
\]

\[
= - T^{\alpha\beta} = - g^{\alpha\beta} T^{\alpha\beta}.
\]

III. STATIC METRIC SOLUTIONS IN IDG

A. Linearized static metric solutions

It has been known that IDG can resolve point-like singularity around the Minkowski background \( \eta \), i.e. \( \eta_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), where \( \eta_{\mu\nu} \) is the Minkowski background and \( h_{\mu\nu} \) is considered as a perturbation. The spherically symmetric spacetime metric in the case of a static source can be written in isotropic coordinates as follows:

\[
ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Psi)d\Omega^2,
\]

(11)
where \( h_{00} = -2\Phi, h_{ij} = -2\Psi \delta_{ij} \). In what follows we will work with the ghost-free condition around an asymptotically Minkowski background, with massless, transverse and traceless graviton, which reduces to GR in the IR \([7, 25]\):

\[
6\mathcal{F}_1(\Box_s) + 3\mathcal{F}_2(\Box_s) + 2\mathcal{F}_3(\Box_s) = 0, \quad a(\Box_s) = 1 + 2\mathcal{F}_2(\Box_s)\Box_s + 4\mathcal{F}_3(\Box_s)\Box_s = e^{-\Box_s}.
\]

In the case of a static source the only non-vanishing component of the energy-momentum tensor is given by \( T_{00} = m\delta^{(3)}(r) \), thus the solutions for the (00) - and (ij) - components of the static metric perturbation \( h_{\mu\nu} \), compatibly with the ghost-free condition in Eq.\((12)\), read \([2]\):

\[
 h_{00}(r) = -2\Phi = \frac{2Gm}{r} \text{Erf} \left( \frac{M_s r}{2} \right), \quad h_{ij}(r) = -2\Psi \delta_{ij} = \frac{2Gm}{r} \text{Erf} \left( \frac{M_s r}{2} \right) \delta_{ij},
\]

which recover GR components in the limit \( M_s r > 2 \). Note that the linearized metric in Eq.\((11)\) with metric potentials given in Eq.\((13)\) is valid from the IR regime (\( M_s r \rightarrow \infty \)) all the way up to the UV regime (\( M_s r \rightarrow 0 \)), provided the inequality

\[
\frac{2GmM_s}{\sqrt{\pi}} < 1 \iff mM_s < M_p^2
\]

holds true. Indeed, at \( r = 0 \) the metric components assume finite constant values \([7]\):

\[
\lim_{r \to 0} h_{00}(r) = \frac{2GmM_s}{\sqrt{\pi}}, \quad \lim_{r \to 0} h_{ij}(r) = \frac{2GmM_s}{\sqrt{\pi}} \delta_{ij}.
\]

Such a static metric turns out to be devoid of curvature singularity and approaches conformal-flatness in the limit \( M_s r \rightarrow 0 \). Moreover, the Ricci scalar and Ricci tensor are non-vanishing and regular, meaning that the non-local gravitational interaction smears the point-source on a spacetime region of size \( 2/M_s \). In particular, the Kretschmann tensor at \( r = 0 \) assumes a finite constant value \([11]\):

\[
\lim_{r \to 0} \mathcal{K} = \lim_{r \to 0} \mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma} = \frac{5G^2m^2M_s^6}{3\pi}.
\]

From the linearized static metric it is very clear that non-locality, through infinite order derivatives, smears out the delta source at \( r = 0 \), avoiding the presence of singular curvature invariants\(^4\).

**B. Towards non-linear static metric solution**

In IDG, since there is a new scale, \( M_s \), which introduces non-locality at the level of interactions, and therefore within the length scale, \( 2/M_s \), the gravitationally bound system behaves like a non-local system, where the Ricci scalar and the Ricci tensor are non-zero, see Refs. \([8, 11]\). In Ref.\([8]\) it has been argued that the inequality in Eq.\((14)\) is a non-perturbative statement which has to hold in both linear and non-linear regime. This is due to the fact that the modifications are warranted at the level of the full action, see Eq.\((2)\), when \( S_g \gg S_{EH} \), the non-local scale \( M_s \) determines the physics in the UV regime. Furthermore, the scale of non-locality itself acts as a collective and an emergent phenomena \([33]\), when there are more and more gravitons which are held within a gravitational potential; the scale of non-locality shifts as \( M_s \rightarrow M_s/\sqrt{N} \), where \( N \) is the number of gravitons held within a bound state of a self-gravitating potential. The decrease in the value of \( M_s \) can now be compensated by the increase in the mass of the compact object, such that the inequality in Eq.\((14)\) turns out to be always satisfied for any mass, \( m \), and the gravitational potential remains bounded by one, i.e. \( mM_s \ll M_p^2 \), and therefore no event horizon will appear. In this respect, the system has some similar features as that of the fuzz-ball \([14]\).

Another important fact, in GR the Schwarzschild metric, whose components contain the mass \( m \), is derived by imposing the boundary condition at the origin, i.e. by putting a delta-Dirac distribution at \( r = 0 \) \([46, 47]\). It is worth emphasizing that without putting a delta-Dirac at the origin the Einstein equations and the Schwarzschild

\(^3\) See also Ref. \([10]\) for a pedagogical review on tree-level unitarity in local and non-local theories of gravity.

\(^4\) In Ref. \([12]\), it has been shown that in higher curvature gravity with more than 4 derivatives, the delta source gets smeared out, as for example in sixth order theory of gravity, and the linearized metric turns out to be singularity-free. However, such local theories still suffer from the presence of ghosts at the tree-level.
metric would not be valid at \( r = 0 \), as it was rigorously studied in the theory of distributions by the authors in Refs. [40, 17]. In IDG, we have non-local form-factors acting on the delta-Dirac distribution, that would generate a more complicated expression on the left-hand side of the equations of motion involving infinite order derivatives of the delta, which generically would not correspond to a point-like source, but to a source with a non-point-like support, as can be seen by this example:

\[
e^{\alpha \partial^2} \delta(x) = e^{\alpha \partial^2} \int \frac{dk}{2\pi} e^{ikx} = \int \frac{dk}{2\pi} e^{-\alpha k^2} e^{ikx} = \frac{1}{\sqrt{4\pi \alpha}} e^{-\frac{x^2}{4\alpha}}.
\]

Furthermore, in Ref. [30], the authors studied the complete field equations including the Weyl part of the action, see Eq.(2). It was shown that the Schwarzschild-like metric, seen as a Ricci flat solution, \( R = 0, \ R_{\mu\nu} = 0 \), does not pass through the field equations at each order in \( \square \), indeed the contribution of the Weyl term, \( W_{\mu\nu\rho\sigma} \square^{\rho} W_{\mu\nu\rho\sigma} \), does not vanish for \( n \geq 1 \). At each order in \( \square \), the field equations the Bianchi identity holds, and at each and every order in \( \square \), the metric potential as \( 1/r \) must pass through the equations of motion, but it does not. Instead, at order \( \square^n \) with \( n \geq 1 \), the Weyl part of the equations of motion would provide [30]

\[
P^{\alpha\beta} \sim f_n \Omega \left( \frac{1}{r^{6+2n}} \right),
\]

going from \( n \) to \( n + 1 \), and this series does not get any accidental cancellation, because \( f_n \) are just numbers. Similar conclusions were drawn for spacetimes with metric potentials given by \( \Phi(r) \sim 1/r^n \). Furthermore, in Ref. [8] an interesting mathematical non-linear solution was found by studying the equations of motion, Eq.(1), in the UV regime for \( r \ll 2/M_s \), the metric solution was non-singular and approached conformal-flatness at \( r = 0 \).

It was postulated that in the IDG, the entire spacetime metric is regular in the static case, inside the non-local region, i.e. \( r \ll 2/M_s \), without any singularity. Therefore, perturbation theory can be trusted all the way from \( r = 0 \) to \( r \to \infty \) as long as \( mM_s < M^2 \); in the linear regime such a spacetime metric can also be expressed in spherically coordinates and reads [8]:

\[
ds^2 = - \left( 1 - \frac{2Gm}{r} \right) dt^2 + \left( 1 + \frac{2Gm}{r} \right) dr^2 + r^2 d\Omega^2.
\]

We wish to apply similar arguments in the case of a rotating Kerr-metric, for which the essential point is to show that there cannot be any ring type singularity within IDG. \(^5\)

### IV. RING SINGULARITY

Let us briefly recall the Kerr metric in rational polynomial coordinates, which is given by [49]:

\[
ds^2 = \left( 1 - \frac{2mr}{r^2 + a^2 \chi^2} \right) dt^2 - \frac{4ma \left( 1 - \chi \right)}{a^2 + r^2} dt d\varphi + \frac{r^2 + a^2 \chi^2}{r^2 - 2mr + a^2} dr^2 + \left( r^2 + a^2 \chi^2 \right) \frac{d\chi^2}{1 - \chi^2} + \left( 1 - \chi^2 \right) \left( \frac{r^2 + a^2}{r^2 + a^2 \chi^2} \right) d\varphi^2,
\]

where \( \chi = \cos \theta \) is the transformation used to bring the standard Boyer-Lindquist coordinates, while \( m \) is the mass and \( J = am \) is the angular momentum, with \( a \) being the rotation parameter. One of the key observation is that the Kerr metric has a ring-singularity which is described by the equation (see Ref. [50] for a nice discussion)

\[
r^2 + a^2 \cos^2 \theta = 0,
\]

where it is clear that \( a \) corresponds to the radius of the ring, while \( r \) is the radial coordinate in Boyer-Lindquist coordinates, which are defined in terms of the Cartesian ones as follows:

\[
x = \sqrt{r^2 + a^2 \sin \theta \cos \varphi},
\]

\[
y = \sqrt{r^2 + a^2 \sin \theta \sin \varphi},
\]

\[
z = r \cos \theta.
\]

\(^5\) There were attempts to understand the Kerr metric in IDG, see [48]. However, we have found an error in our analysis, which we have rectified here. Unfortunately, the rotation was not taken into account correctly in the paper.
The Kretschmann scalar blows up when Eq. (21) is satisfied, i.e. when \( r = 0 \) and \( \theta = \pi / 2 \), which in Cartesian coordinates means
\[
x^2 + y^2 = a^2, \quad z = 0,
\] (23)

namely the ring singularity lies on a plane, which is perpendicular to the rotation axis.

Let us first discuss the physics in the linear regime, in analogy with the static case. We can consider a framework in which the source is a delta-Dirac distribution on a ring of radius \( a \) which is rotating with a constant angular velocity \( \omega \) in the plane \( x-y \) \( (z = 0) \). Thus, the (00)-component of the energy momentum tensor of the source is given by\(^6\)
\[
T_{00} = m\delta(z) \frac{\delta(x^2 + y^2 - a^2)}{\pi}.
\] (24)

Since the ring is also rotating, we also have the following non-vanishing components of the stress-energy tensor:
\[
T_{0i} = T_{00}v_i,
\] (25)

where \( v_i \) is the tangential velocity whose magnitude can be expressed as \( v = \omega a \), and assuming that the rotation happens around the \( z \)-axis, we have
\[
v_x = -y\omega, \quad v_y = x\omega, \quad v_z = 0.
\] (26)

Note that this choice of the source, in analogy with the static case, is compatible with the fact that in order for the Einstein equations and the Kerr metric to be defined in the entire spacetime we need a stress-energy tensor which is not totally zero. In fact, by using the theory of distribution, in Ref. \([47]\) it was rigorously shown that the stress-energy tensor for a Kerr metric has a structure similar to the one we have written in Eq. (24). For example, for the (00)-component of the Einstein tensor in the case of the Kerr metric one has \( G_{00} \sim m\delta(z)\delta(x^2 + y^2 - a^2) \) \([47]\). A general linearized metric, which can describe the spacetime in presence of a rotating source can be written, in isotropic coordinates, as
\[
ds^2 = -(1 + 2\Phi)dt^2 + 2\hat{h} \cdot d\vec{r}dt + (1 - 2\Psi)dx^2,
\] (27)

where \( h_{00} = -2\Phi < 1 \), \( h_{ij} = -2\Psi \delta_{ij} < 1 \) and \( h_{0i} = h_i < 1 \) signify the weak-field and the slow rotation regime, and now the metric components depend on the isotropic radius, \( r \), which should not be confused with the Boyer-Lindsquit radial coordinate used above. To find the form of the metric components, we would need to solve the following differential equations:
\[
e^{-\nabla^2 / M^2} \nabla^2 \Phi(\vec{r}) = e^{-\nabla^2 / M^2} \nabla^2 \Psi(\vec{r}) = 4Gm\delta(z)\delta(x^2 + y^2 - a^2),
e^{-\nabla^2 / M^2} \nabla^2 h_{0x}(\vec{r}) = -16Gmw\delta(z)\delta(x^2 + y^2 - a^2),
e^{-\nabla^2 / M^2} \nabla^2 h_{0y}(\vec{r}) = 16Gm\omega \delta(z)\delta(x^2 + y^2 - a^2);
\] (28)

where we are assuming the ghost-free condition in Eq. (12). To solve the differential equations in Eq. (28) we can go to the Fourier space and then anti-transform back to coordinate space; thus, first of all, we need to compute the Fourier transforms of the stress-energy tensor components, i.e. of \( T_{00} \) and \( T_{0i} \).

A. Smearing out the ring singularity at the linearized level

Let us start to compute the corresponding gravitational potential, \( \Phi = \Psi \); thus, we need the Fourier transform of the ring-distribution in Eq. (24):
\[
\mathcal{F}[\delta(z)\delta(x^2 + y^2 - a^2)] = \int dx dy dz \delta(z)\delta(x^2 + y^2 - a^2) e^{ik_x x} e^{ik_y y} e^{ik_z z}.
\] (29)

It can be computed by performing the integral in cylindrical coordinates:
\[
x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad z = z,
\] (30)

\(^6\) Note that the factor \( \pi \) in the denominator of Eq. (24) comes from the fact that \( \delta(x, y) \equiv \delta(x)\delta(y) = \pi\delta(x^2 + y^2) \).
so that,
\[
\mathcal{F}[\delta(z)\delta(x^2 + y^2 - a^2)] = \int_{-\infty}^{\infty} dz \delta(z) e^{ik_z z} \int_{0}^{2\pi} d\varphi e^{ik_x \rho \cos\varphi} e^{ik_y \rho \sin\varphi} = \int_{0}^{\infty} d\rho \delta(\rho^2 - a^2) \left(2\pi I_0 \left(i\rho \sqrt{k_x^2 + k_y^2}\right)\right)
\]
\[= \pi \int_{0}^{\infty} d(\rho^2 - a^2) I_0 \left(i\rho \sqrt{k_x^2 + k_y^2}\right) = \pi I_0 \left(ia \sqrt{k_x^2 + k_y^2}\right),
\]
where \(I_0\) is a Modified Bessel function, which is also defined in terms of the Bessel function as \(I_0(x) = J_0(ix)\). By anti-transforming, we obtain the gravitational potential in coordinate space:
\[
\Phi(\vec{r}) = -4\pi Gm \int \frac{d^3k}{(2\pi)^3} e^{-k^2/M_s^2} I_0 \left(ia \sqrt{k_x^2 + k_y^2}\right) e^{ik_x x} e^{ik_y y} e^{ik_z z},
\]
where \(d^3k = dk_x dk_y dk_z\) and \(k^2 = k_x^2 + k_y^2 + k_z^2\). In order to study whether the ring singularity is still present or not in IDG, we can simplify the integral in Eq.(32), by considering ourselves on the \(x\)-\(y\) \((z = 0)\) plane, where the ring singularity lies in GR. Thus, by setting \(z = 0\), and going to cylindrical coordinates, \(k_x = \zeta \cos\varphi\), \(k_y = \zeta \sin\varphi\), \(k_z = k_z\), we can rewrite the integral in Eq.(32) as follows:
\[
\Phi(\rho) = -Gm \int_{0}^{\infty} d\zeta I_0 \left(ia\zeta\right) I_0 \left(i\zeta \rho\right) \text{Erfc} \left(\frac{\zeta}{M_s}\right),
\]
which in the limit \(M_s \to \infty\) gives the metric potential in the case of GR:
\[
\Phi_{GR}(\rho) = -Gm \int_{0}^{\infty} d\zeta I_0 \left(ia\zeta\right) I_0 \left(i\zeta \rho\right).
\]
The two integrals in Eqs. (33) and (34) cannot be solved analytically, but we can compute them numerically and check whether any singularity is present. Very interestingly, from the numerical computation one can explicitly see that for \( x^2 + y^2 = a^2 \) the gravitational potential in GR diverges, while in IDG it turns out to be singularity-free; see Fig. 1. This is what we have expected to be physically; non-local gravitational interaction smears out a ring distribution very similarly to the case of a point source [6–8, 41].

Furthermore, we can trust the linear regime all the way up to \( \rho = 0 \), as long as \( 2\Phi(0) < 1 \). The integral in Eq. (34) can be evaluated analytically at \( \rho = 0 \):

\[
\Phi(0) = -\frac{Gm}{a} \text{Erf} \left( \frac{M_a}{2} \right),
\]

so that the linearized regime requires:

\[
2\frac{Gm}{a} \text{Erf} \left( \frac{M_a}{2} \right) < 1. \tag{36}
\]

Note that, since \( \text{Erf} \left( \frac{M_a}{2} \right) < 1 \) for any value of the argument, the case \( a > 2Gm \) always satisfies the inequality; while in the opposite case \( a < 2Gm \) the weak-field inequality is satisfied as long as

\[
a < \frac{2}{M_a}, \tag{37}
\]

which means that the radius of the ring is engulfed by the scale non-locality. This results suggest that non-locality in the gravitational interaction can indeed help us to avoid ring type singularity, which is impossible to resolve within GR.

B. Computing \( h_{0i} \) components for a rotating ring

So far we have only computed the static gravitational potential generated by a delta-Dirac distribution on the ring. We now wish to study the components \( h_{0i} \), which are related to the fact that the ring is also rotating with a constant angular velocity \( \omega \). We would need to compute the following Fourier transforms:

\[
\mathcal{F}[x\delta(z)\delta(x^2 + y^2 - a^2)] = \int dx dy dz x\delta(z)\delta(x^2 + y^2 - a^2)e^{ikx}e^{iky}e^{ikz}, \tag{38}
\]

\[
\mathcal{F}[y\delta(z)\delta(x^2 + y^2 - a^2)] = \int dx dy dz y\delta(z)\delta(x^2 + y^2 - a^2)e^{ikx}e^{iky}e^{ikz}. \tag{39}
\]

The computation can be performed by using cylindrical coordinates as done in Eq. (31):

\[
\mathcal{F}[x\delta(z)\delta(x^2 + y^2 - a^2)] = \int_{-\infty}^{\infty} dz \delta(z)e^{ikz} \int_0^{\infty} d\rho \rho^2 \delta(\rho^2 - a^2) \int_0^{2\pi} d\varphi e^{ikx}e^{iky}e^{ikz} \cos \varphi ,
\]

\[
= \int_0^{\infty} d\rho \rho^2 \delta(\rho^2 - a^2) \left( \frac{2\pi k_x}{\sqrt{k_x^2 + k_y^2}} I_1 \left( i\rho \sqrt{k_x^2 + k_y^2} \right) \right) \tag{40}
\]

\[
= \pi a \frac{k_x}{\sqrt{k_x^2 + k_y^2}} I_1 \left( i\rho \sqrt{k_x^2 + k_y^2} \right) ,
\]

and by following similar steps we also obtain:

\[
\mathcal{F}[y\delta(z)\delta(x^2 + y^2 - a^2)] = \pi a \frac{k_y}{\sqrt{k_x^2 + k_y^2}} I_1 \left( i\rho \sqrt{k_x^2 + k_y^2} \right) , \tag{41}
\]
where $I_1$ is another Modified Bessel function. We can now express the components $h_{0i}$ in coordinate space as anti-transforms:

$$h_{0x}(\vec{r}) = 16Gm\omega a \int \frac{d^3k}{(2\pi)^3} \frac{e^{-k^2/M_s^2}}{k^2} \frac{k_x}{\sqrt{k_x^2 + k_y^2}} I_1 \left( i\alpha \sqrt{k_x^2 + k_y^2} \right) e^{ik_xx} e^{ik_yy} e^{ik_zz},$$

$$h_{0y}(\vec{r}) = -16Gm\omega a \int \frac{d^3k}{(2\pi)^3} \frac{e^{-k^2/M_s^2}}{k^2} \frac{k_y}{\sqrt{k_x^2 + k_y^2}} I_1 \left( i\alpha \sqrt{k_x^2 + k_y^2} \right) e^{ik_xx} e^{ik_yy} e^{ik_zz}. $$

By using cylindrical coordinates, as already done to get the expression in Eq. (33), and setting $z = 0$, we can obtain similar expressions for the cross-terms:

$$h_{0x}(x,y) = 4Gm\omega a \frac{y}{\rho} \int_0^\infty d\zeta I_1 (i\alpha\zeta) I_1 (i\zeta\rho) \text{Erfc} \left( \frac{\zeta}{M_s} \right),$$

$$h_{0y}(x,y) = -4Gm\omega a \frac{x}{\rho} \int_0^\infty d\zeta I_1 (i\alpha\zeta) I_1 (i\zeta\rho) \text{Erfc} \left( \frac{\zeta}{M_s} \right), $$

where remember that $\rho = \sqrt{x^2 + y^2}$ is the radial cylindrical coordinate in the plane $z = 0$. Note that since $\theta = \pi/2$, we have

$$\frac{x}{\rho} = \cos \varphi, \quad \frac{y}{\rho} = \sin \varphi,$$

thus all the radial dependence and the singularity structure are taken into account by the following integral:

$$H(\rho) := \int_0^\infty d\zeta I_1 (i\alpha\zeta) I_1 (i\zeta\rho) \text{Erfc} \left( \frac{\zeta}{M_s} \right), $$

FIG. 2: In this plot we have shown the results of the numerical computation for the integrals in Eqs. (17) and (18), and the behavior of the same function in the case of the multipole expansion in Eq. (55). The blue line corresponds to the behavior of the function $H_{GR}$, and so of the cross-term in GR; the orange line to the behavior of the function $H_{IDG}$, and so of the cross-term in IDG; while the dashed red line represents the cross-term in the case of the multipole expansion. We have chosen $a = 1$ and $M_s = 1.5$. We can notice that the metric components $h_{0i}$ blow up in GR for $\rho = a = 1$, while they are finite in IDG; moreover, the metric coming from the multipole expansion is a very good approximation outside the source, i.e. for $\rho > a$. 

where $\iota$ is another Modified Bessel function. We can now express the components $h_{0i}$ in coordinate space as antitransforms: 

$$h_{0x}(\vec{r}) = 16Gm\omega a \int \frac{d^3k}{(2\pi)^3} \frac{e^{-k^2/M_s^2}}{k^2} \frac{k_x}{\sqrt{k_x^2 + k_y^2}} I_1 \left( i\alpha \sqrt{k_x^2 + k_y^2} \right) e^{ik_xx} e^{ik_yy} e^{ik_zz},$$

$$h_{0y}(\vec{r}) = -16Gm\omega a \int \frac{d^3k}{(2\pi)^3} \frac{e^{-k^2/M_s^2}}{k^2} \frac{k_y}{\sqrt{k_x^2 + k_y^2}} I_1 \left( i\alpha \sqrt{k_x^2 + k_y^2} \right) e^{ik_xx} e^{ik_yy} e^{ik_zz}.$$ 

By using cylindrical coordinates, as already done to get the expression in Eq. (33), and setting $z = 0$, we can obtain similar expressions for the cross-terms:

$$h_{0x}(x,y) = 4Gm\omega a \frac{y}{\rho} \int_0^\infty d\zeta I_1 (i\alpha\zeta) I_1 (i\zeta\rho) \text{Erfc} \left( \frac{\zeta}{M_s} \right),$$

$$h_{0y}(x,y) = -4Gm\omega a \frac{x}{\rho} \int_0^\infty d\zeta I_1 (i\alpha\zeta) I_1 (i\zeta\rho) \text{Erfc} \left( \frac{\zeta}{M_s} \right), $$

where remember that $\rho = \sqrt{x^2 + y^2}$ is the radial cylindrical coordinate in the plane $z = 0$. Note that since $\theta = \pi/2$, we have

$$\frac{x}{\rho} = \cos \varphi, \quad \frac{y}{\rho} = \sin \varphi,$$

thus all the radial dependence and the singularity structure are taken into account by the following integral:

$$H(\rho) := \int_0^\infty d\zeta I_1 (i\alpha\zeta) I_1 (i\zeta\rho) \text{Erfc} \left( \frac{\zeta}{M_s} \right).$$
which in the limit $M_s \to \infty$ gives the GR case:

$$H_{GR}(\rho) := \int_0^{\infty} d\zeta I_1(\zeta \rho). \tag{48}$$

The two integrals in Eqs. (47) and (48) cannot be solved analytically but we can compute them numerically and check the absence of any singularities. As it also happens for the potentials $h_{00}$ and $h_{0i}$, the cross-term $h_{0i}$ show the presence of a ring singularity in GR; indeed, from the numerical analysis one can explicitly see that for $x^2 + y^2 = a^2$ the function $H_{GR}$ diverges in GR. While in IDG the cross-term turns out to be singularity-free; indeed, the function $H$ is finite everywhere.

We have shown that the presence of non-local gravitational interaction, through infinite covariant derivatives, can avoid the ring singularity in a toy-model where the source is assumed to be a delta-Dirac distribution on a ring. In analogy with the static scenario, also in this case we have infinite derivatives which are responsible for the smearing out of the delta-Dirac ring distribution. Note also that at the origin, $\rho = 0$, the cross-term vanishes and this implies that in IDG the spacetime metric approaches conformal-flatness; indeed, at $r = 0$ the rotating metric becomes the static one, which has already been shown to be conformally-flat at the origin [11].

In the IR regime, for $\rho \gg a$, the metric components found above match extremely well with the case of GR. Indeed, for distances larger than the radius of the ring and the scale of non-locality, i.e. $\rho \gg 2/M_s > a$, we recover the Lense-Thirring metric (see Appendix A). Indeed, the leading order of the diagonal components are given by

$$h_{00}(\rho) \sim \frac{2GM}{\rho}, \quad h_{ij}(\rho) \sim \frac{2GM}{\rho} \delta_{ij}, \tag{49}$$

while for the cross-term, the function $H(\rho)$ becomes:

$$H(\rho) \sim \frac{a}{2\rho^2}, \tag{50}$$

which implies

$$h_{0x} \sim \frac{2Gma^2 \omega}{\rho^3} y, \quad h_{0y} \sim -\frac{2Gma^2 \omega}{\rho^3} x. \tag{51}$$

To exactly recover the Lense-Thirring metric in Eq. (A9) at large distances, we need to identify $J = ma^2 \omega$, which is nothing but the relation $J = I \omega$, where $I = ma^2$ is the moment of inertia of the delta-Dirac ring distribution [7], thus we obtain:

$$ds^2 \rightarrow - \left(1 - \frac{2GM}{\rho}\right) dt^2 + \frac{4GJ}{\rho^3} (y dx dt - xd y dt) + \left(1 + \frac{2GM}{\rho}\right) (d\rho^2 + \rho^2 d\Omega^2). \tag{52}$$

Note that the GR limits in the last equations have been expressed in the case $z = 0$, in terms of the coordinate $\rho = \sqrt{x^2 + y^2}$, but it is clear that they also hold for the radial coordinate $r = \sqrt{x^2 + y^2 + z^2}$.

### C. Rotating metric outside the source: multipole expansion in IDG

Thus, we have consistently shown that the spacetime metric generated by a delta-Dirac distribution on a ring approaches the GR limit at large distances, which means at distances much larger than the scale of non-locality, so that the metric becomes Lense-Thirring. We now wish to determine the generic form of the metric in IDG outside the rotating source, without assuming any large distance limit.

The starting point is again the linearized metric in Eq. (27). The components $h_{00}$ and $h_{ij}$ will be the same already obtained in the static case in Eq. (13), while to compute the $(0i)$-components we can consider a multipole expansion for $\text{Erf} \left( M_s |\vec{r} - \vec{r}'| / 2 / |\vec{r} - \vec{r}'| \right)$, given by

$$\frac{1}{|\vec{r} - \vec{r}'|} \text{Erf} \left( \frac{M_s |\vec{r} - \vec{r}'|}{2} \right) = \frac{1}{r} \text{Erf} \left( \frac{M_s r}{2} \right) + \left( \frac{1}{r^3} \text{Erf} \left( \frac{M_s r}{2} \right) - \frac{M_s}{\sqrt{\pi} r^2} e^{-\frac{M_s^2 r^2}{4}} \right) \sum_{j=1}^3 x^j x'^j + \cdots. \tag{53}$$

\[7\] Note that in the model we are considering, the relation $J = am$ does not hold, but the angular momentum is related to the parameter $a$ through the momentum of inertia of the source.
which recovers the GR case in Eq. (A2) in the regime $M/sr \gg 2$, as expected. Such a multipole expansion holds true for $r > r' \sim a$, which means outside the source. By using Eq. (53) and by following the same steps as in the case of GR in Appendix A, we can now compute the $h_{0i}$ components

$$h_{0i}(r') = 4G \int d^3r' T_{0i}(r') \frac{M_s |r - r'|}{|r - r'|} \text{Erf} \left( \frac{M_s |r - r'|}{2} \right)$$

$$= 2G \left( \frac{1}{r^3} \text{Erf} \left( \frac{M_s r}{2r} \right) - \frac{M_s}{\sqrt{\pi} r^2} e^{-\frac{M^2 r^2}{4}} \right) \sum_{j=1}^{3} x^j \varepsilon_{ijk} J^k$$

$$= 2G \left( \frac{1}{r^3} \text{Erf} \left( \frac{M_s r}{2r} \right) - \frac{M_s}{\sqrt{\pi} r^2} e^{-\frac{M^2 r^2}{4}} \right) (\vec{r} \times \vec{J})_i,$$  

(54)

We can move from Cartesian to isotropic coordinates, so that the $d\phi dt$ component of the metric will be given by:

$$2\vec{h} \cdot \vec{d}dt = 4GJ \left( \frac{1}{r} \text{Erf} \left( \frac{M_s r}{2} \right) - \frac{M_s}{\sqrt{\pi} r^2} e^{-\frac{M^2 r^2}{4}} \right) (ydxdt - xdydt)$$

$$= -4GJ \left( \frac{1}{r} \text{Erf} \left( \frac{M_s r}{2r} \right) - \frac{M_s}{\sqrt{\pi} r^2} e^{-\frac{M^2 r^2}{4}} \right) \sin^2 \theta d\phi dt,$$  

(55)

which in the regime $M/sr \gg 2$ recovers the GR case in Eq. (A8), as expected. Moreover, by expressing $J = I\omega = ma^2 \omega$ and imposing $|h_{0i}| \sim GmM^2 \omega a^2 < 1$, we can notice that the slow rotation regime means $\omega < 1/a$.

Thus, the linearized spacetime metric in Eq. (27) outside the source, $r > a$, in the case of IDG reads:

$$ds^2 = - \left( 1 - \frac{2Gm}{r} \text{Erf} \left( \frac{M_s r}{2} \right) \right) dt^2 + \left( 1 + \frac{2Gm}{r} \text{Erf} \left( \frac{M_s r}{2} \right) \right) (dr^2 + r^2 d\Omega^2)$$

$$- 4GJ \left( \frac{1}{r} \text{Erf} \left( \frac{M_s r}{2r} \right) - \frac{M_s}{\sqrt{\pi} r^2} e^{-\frac{M^2 r^2}{4}} \right) \sin^2 \theta d\phi dt.$$  

(56)

From Fig. 1 and 2, it is very clear that the metric constructed by using the multipole expansion is a very good approximation to describe the spacetime outside the source, $r > a$; while in the regime $M/sr \gg 2$, we recover the GR predictions, indeed Eq. (56) reduces to the Lense-Thirring metric in Eq. (A9), as expected.

Summarizing, we have now computed the linearized metric for a delta-Dirac distribution on a rotating ring, and shown that no ring singularity is present. Outside the ring, the spacetime metric in IDG can be also well described by using a multipole expansion and is given by the metric in Eq. (56). Finally, far from the region of non-locality $r \gg 2/M_s$, which also means far from the source, we consistently recover the GR predictions, indeed the metric becomes Eq. (A9).

Thus, in the case of a rotating source we have a hierarchy of scales: the radius of the source $a$, the Schwarzschild radius $r_{sch} = 2Gm$ and the scale of non-locality $r_{NL} \sim 2/M_s$, which have to satisfy the following set of inequalities to preserve the linearity

$$r_{NL} \sim \frac{2}{M_s} > r_{sch} = \frac{2m}{M_p^2} > a.$$  

(57)

In the IDG case, as long as the inequality in Eq. (57) holds, the spacetime metric generated by a slowly rotating source is valid all the way from $r = \infty$ up to $r = 0$, and it turns out to be free from any curvature singularity, and also devoid of any horizons, see Eq. (56). Furthermore, since in our case, the $h_{00}$ component is always bounded below unity, there is no ergo-region, as first pointed out in [19].

V. NON-KERR TYPE METRIC IN THE FULL NON-LINEAR THEORY

So far we have only worked in the linear regime and proposed strong arguments in favor of the absence of any ring-singularity in IDG. We now wish to move towards the full non-linear regime, and show that the Kerr metric does not solve the full non-linear field equations in Eq. (4). In particular, we will show that the Kerr-like metric is not a vacuum solution, when we consider the Weyl squared term in the action, with a non-constant form factor, $F_3(\Box_s) \neq \text{const}$. 


A. Infinite order derivatives acting on a delta distribution on a ring

As it was argued in Ref. [8], strictly speaking the Schwarzschild metric in GR is not a vacuum solution, indeed there is a delta-Dirac distribution at the origin, so that the stress-energy tensor is non-vanishing at \( r = 0 \) in absence of the Weyl squared term \( W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} \), the full non-linear IDG field equations will not allow the Schwarzschild metric as a solution, due to the presence of infinite order covariant derivatives acting on a delta-Dirac source.

We can argue the same also in the case of the Kerr metric. As it was rigorously shown in Ref. [47] by using the theory of distribution, we can argue that neither the Kerr metric is completely a vacuum solution; for example the Schwarzschild metric as a solution, due to the presence of infinite order covariant derivatives acting on a delta-Dirac source.

In order to obtain some insight into this problem, let us first consider the right hand side of Eq.(20), is a vacuum solution for the full non-linear equations (4), i.e. let us impose

\[
\mathcal{R} = 0, \quad \mathcal{R}_{\mu\nu} = 0; \quad (59)
\]

while the Weyl-tensor is non-vanishing, but coincide with the Riemann tensor. Let us now check whether the Kerr metric is allowed as a vacuum solution with \( P_{\alpha\beta} = 0 \), and \( \mathcal{R} = 0 \) and \( \mathcal{R}_{\mu\nu} = 0 \) (which also means \( G^{\alpha\beta} = 0 \)). Thus, by imposing Eq.(59), the full field equations in Eq.(4) become

\[
P^{\alpha\beta} = 0 = P_3^{\alpha\beta} = \frac{\alpha c}{8\pi G} \left( -g^{\alpha\beta} W^{\mu\nu\lambda\sigma} F_3(\square_s) W_{\mu\nu\lambda\sigma} + 4 W^{\alpha}_{\mu\nu\sigma} F_3(\square_s) W^{\beta\mu\nu\sigma} \right.
\]

\[
- 8 \nabla_\mu \nabla_\nu (F_3(\square_s) W^{\beta\mu\nu\alpha}) - 2 \Omega_3^{\alpha\beta} + g^{\alpha\beta} (\Omega_3^{3\gamma} + \bar{\Omega}_3) - 8 \Delta_3^{\alpha\beta} \right).
\]

(60)

In order to obtain some insight into this problem, let us first consider the right hand side of \( P_3^{\alpha\beta} \) up to second order in \( \square_s \), namely

\[
F_3(\square_s) = (f_{30} + f_{31} \square_s + f_{32} \square_s^2),
\]

(61)

and study the field equations order by order, as we had done for the static case in Ref. [30]. After some computations (see also Appendix B), we have obtained the following interesting results:

- **At the zeroth order in \( \square_s \):** This is the case of local fourth order gravity of Stelle [51]. The terms are proportional to \( f_{30} \), all the terms cancel each other, namely

\[
P_3^{(0)\alpha\beta} (\square_s) = 0.
\]

(62)

In this case, the action corresponds to a theory with fourth order derivatives, very similar to the case of local quadratic curvature gravity:

\[
S = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left( \mathcal{R} + \alpha_c \left[ f_{10} \mathcal{R}^2 + f_{20} \mathcal{R}^{\mu\nu} \mathcal{R}_{\mu\nu} + f_{30} W^{\mu\nu\lambda\sigma} W_{\mu\nu\lambda\sigma} \right] \right).
\]

(63)

Since we are requiring the condition in Eq.(59), the full field equations in Eq.(4) are explicitly reduced to Eq.(60), where the only relevant terms that remains to be analyzed is the one corresponding to the form-factor coefficient.
\[ f_{30}^{8}: \]

\[
 f_{30} \left( -g^{\alpha\beta} W^{\mu\nu\lambda\sigma} W_{\mu\nu\lambda\sigma} + 4W^{\alpha}_{\mu\nu\sigma} W^{\beta\mu\nu\sigma} - 8\nabla_{\mu} \nabla_{\nu} W^{\beta\mu\nu\alpha} \right). \tag{64}
\]

From the direct evaluation of this last expression, for the Kerr-like metric, we obtain

\[
 -g^{\alpha\beta} W^{\mu\nu\lambda\sigma} W_{\mu\nu\lambda\sigma} + 4W^{\alpha}_{\mu\nu\sigma} W^{\beta\mu\nu\sigma} = 0 \tag{65}
\]

where, for example, the (00)-component is given by

\[
 -g^{00} W^{\mu\nu\lambda\sigma} W_{\mu\nu\lambda\sigma} = \frac{(a^2\chi^2 + r^2)^6 \left(-a^6\chi^6 + 15a^4r^2\chi^4 - 15a^2r^4\chi^2 + r^6\right)}{(a^2\chi^2 + r^2)^{14} (a^2 + r(r - 2Gm))} \times 48(Gm)^2 (a^2\chi^2 + r^2 - 2rGm) (a^4\chi^2 + a^2r (r\chi^2 + r - 2\chi^2Gm + 2Gm) + r^4)
\]

\[
 = -4W^{0}_{\mu\nu\sigma} W^{0\mu\nu\sigma} ,
\]

and,

\[
 \nabla_{\mu} \nabla_{\nu} W^{\beta\mu\nu\alpha} = 0 , \tag{66}
\]

i.e. the local contribution from the Weyl squared term with a constant form factor \( f_{30} \), vanishes in 4 dimensions, which was already expected as it is a reminiscence of the Gauss-Bonnet topological invariant, which means that the Kerr metric is still an exact solution for the local quadratic curvature gravity. Similar conclusion were drawn in Ref. [51].

- **At the first order in \( \Box \):** Even though the Weyl contribution vanishes at zeroth order, this is not the case for the higher powers of box, i.e. \( \Box^n \), with \( n > 0 \). Indeed, at the first order in box we obtain

\[
 P^{(1)\alpha\beta}_{3} (\Box) = \frac{\alpha c}{8\pi G f_{31}} \begin{pmatrix} a_{00} & 0 & 0 & a_{03} \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ a_{30} & 0 & 0 & a_{33} \end{pmatrix} , \tag{68}
\]

with the dimensionless matrix elements given by

\[
 a_{00} = \frac{144G^2m^2 \left(-8a^4Gm + a^2r \left(100G^2m^2 - 8Gmr + 5r^2\right) + 5r^4(r - 2Gm)\right)}{r^{11}M_s^2 (a^2 + r(r - 2Gm))} , \\
 a_{03} = -\frac{288a^3G^3m^3 \left(4a^2 + 25r(r - 2Gm)\right)}{r^{11}M_s^2 (a^2 + r(r - 2Gm))} , \\
 a_{11} = -\frac{1008G^2m^2 \left(4a^4 + 5a^2r(r - 2Gm) + r^2(r - 2Gm)^2\right)}{r^{12}M_s^2} , \\
 a_{22} = \frac{144G^2m^2 \left(28a^2 + r(21r - 50Gm)\right)}{r^{12}M_s^2} , \\
 a_{30} = -\frac{288a^3G^3m^3 \left(4a^2 + 25r(r - 2Gm)\right)}{r^{11}M_s^2 (a^2 + r(r - 2Gm))} , \\
 a_{33} = \frac{144G^2m^2 \left(r \left(100G^2m^2 - 92Gmr + 21r^2\right) - 8a^2Gm\right)}{r^{11}M_s^2 (a^2 + r(r - 2Gm))} ;
\]

where we have fixed the equatorial plane, \( \chi = \cos(\pi/2) = 0 \), without any loss of generality.

We can also compute the two-rank symmetric tensor \( P^{(2)\alpha\beta}_{3} (\Box) \) at higher order in box, see for example Appendix B for the computations of the second order in box and for the explicit expression of \( P^{(2)\alpha\beta}_{3} (\Box) \).

---

\(^8\) Notice that, by definitions [3], \( \Omega_3^{\alpha\beta}, \Omega_3^{\alpha\gamma}, \Omega_3^\alpha \) and \( \Delta_3^{\alpha\beta} \) have no dependence on \( f_{30} \).
Generic orders in $\Box_s$: We can now ask what would happen for generic higher orders in $\Box_s$. Note that for the Kerr metric one has $\Box_s \sim \frac{2r}{M^2(r^2 + x^2 + y^2)} \partial^9$, and by dimensional analysis we can find the behavior of the lowest order in power of $1/r$ at each order in box. We have already seen that the lowest order in $1/r$ at one box goes like $1/r^{10}$, and at two boxes we have $1/r^{12}$; see Appendix B. By proceeding in the same way, we can notice that at third order in box, the lowest contribution in powers of $1/r$ is

$$f_{33} \frac{G^2 m^2}{r^{14} M^6},$$

and at fourth order in box

$$f_{34} \frac{G^2 m^2}{r^{16} M^6}.$$

Finally, we can hint that at $n$-th order in box, the lowest contribution in powers of $1/r$ will be always proportional to

$$f_{3n} \frac{G^2 m^2}{r^{8+2n} M^{2n}}.$$

By just looking at the lowest order contributions at each order in box, we can notice that the tensor $P^{\alpha \beta}_3$ satisfies the following relation:

$$P^{\alpha \beta}_3 \sim f_{31} \mathcal{O} \left( \frac{1}{r^{10}} \right) + f_{32} \mathcal{O} \left( \frac{1}{r^{12}} \right) + \cdots + f_{3n} \mathcal{O} \left( \frac{1}{r^{8+2n}} \right) + \cdots,$$

from which it is clear that in order to vanish we would require an unlikely fine-tuning among all coefficients $f_{3n}$. In this respect, Kerr-like metric as in Eq. (20) cannot be a vacuum solution of the full non-linear field equations in Eq. (4), indeed it does not pass through at any order in box, $W_{\mu \nu \rho \sigma} \Box^n W^{\mu \nu \rho \sigma}$ with $n \geq 1$.

VI. CONCLUSIONS

Let us briefly conclude our study. In this paper we have studied rotating metric in the case of ghost free IDG [7]. First, we have worked in the linear regime and found the spacetime metric in the case of stress-energy tensor given by a delta-Dirac distribution on a rotating ring. In GR, this kind of source generates a metric solution which suffers from the presence of a ring singularity, where the Kretschmann scalar blows up, and indeed the metric components diverge on the ring, i.e. for $x^2 + y^2 = a^2$ and $z = 0$, which mimics the ring singularity appearing in the Kerr metric [35].

Instead, we have found that in the IDG the spacetime metric turns out to be singularity-free, and very interestingly for $r \to 0$ the metric becomes conformally-flat, i.e. the cross-term vanishes at the origin, where the metric coincides with the static one [7, 8]. Moreover, the linear approximation can be trusted all the way from the IR to the UV regime, provided we require slow rotation, $mM_s < M_p^2$ and $a < 2/M_s$. The last inequality means that the region of non-locality has to engulf the ring-source of radius $a$.

Furthermore, we have shown that outside the source, $r > 2/M_s$, the spacetime metric can be well described in terms of a multipole expansion; see Eq. (56). We have also consistently checked that at large distances, i.e. $r \gg 2/M_s$, we recover the GR predictions, namely the spacetime metric becomes that of Lense-Thirring.

We then studied the full non-linear IDG field equations. First, we have argued that a Kerr-like metric cannot be a vacuum solution in IDG, even for a simpler action without the Weyl squared term. Indeed, strictly speaking the Kerr-metric is not completely Ricci-flat, but there is a non-zero stress-energy tensor, see Eq. (59), thus the presence of infinite order covariant derivatives acting on a theta-Heaviside and a delta-Dirac distribution on a ring, generally, will not generate a point-like source, which implies that the Kerr-metric will not be a vacuum solution of the IDG field equations, Eq. (4).

Finally, we have analyzed the full field equations including the Weyl squared term, and shown that the Kerr metric, seen as Ricci-flat, will not pass as a vacuum solution if the form-factor $\mathcal{F}_3(\Box_s)$ is not constant; indeed, the Weyl

\[ \mathcal{F}_3(\Box_s) = \frac{1}{M^2} g^{\nu \mu} \nabla_\nu \nabla_\mu = \frac{1}{M^2 (r^2 + a^2 + r^2)} \left( a^2 + r (r - 2Gm) \partial^2 + 2(r - Gm) \partial_r \right). \]
contribution does not vanish at each order in box. These results are very good hints that allow us to argue that in
the ghost free IDG, the ring singularity will be solved in both linear and non-linear regime. In this way the notion
of rotating blackhole that we have in GR, would be different in IDG, i.e. without singularity, devoid of any event
horizons and ergo region. Indeed, our study will have an interesting impact in astrophysical blackholes, which should
be discussed elsewhere in some details. Hopefully, our study will shed some light in presence of LIGO/VIRGO data,
and understanding the spacetime near a rotating non-singular compact object.

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Appendix A: Lense-Thirring metric in general relativity

In GR it is well-known that the spacetime far from a rotating source can be well described by the Lense-Thirring
metric, which can be also be derived with a top-down approach by starting from the Kerr metric and imposing
weak-field and slow rotation.

In this Appendix we will briefly review the derivation of the Lense-Thirring metric from a bottom-up approach,
by working in the linear regime. The starting point is the linearized metric in Eq.(27), where the (00)- and
(ij)-components can be easily found by solving the standard Poisson equation for the Newtonian potential and we obtain:

$$\Phi(r) = \Psi(r) = -\frac{Gm}{r}. \quad (A1)$$

As for the cross-term we can compute it by making use of the following multipole expansion:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} + \frac{1}{r^3} \sum_{j=1}^{3} x^j x'^j + \cdots. \quad (A2)$$

Thus, the $h_{0i}$ components read:

$$h_{0i}(\vec{r}) = 4G \int d^3r' \frac{T_{0i}(\vec{r}')}{|\vec{r} - \vec{r}'|} = \frac{4G}{r} \int d^3r' T_{0i}(\vec{r}') + \frac{4G}{r^3} \sum_{j=1}^{3} \int d^3r' x'^j T_{0j}(\vec{r}')$$

$$= \frac{2G}{r^3} \sum_{j=1}^{3} x^j \varepsilon_{ijk} J^k$$

$$= \frac{2G}{r^3} (\vec{r} \times \vec{J})_i, \quad (A3)$$

where from the third line to the fourth we have used the continuity equation which implies $\int d^3r' T_{0i}(\vec{r}') = 0^{10}$, and

\[10\] The vanishing of the integral $\int d^3r' T_{0i}(\vec{r}')$ can be proved in the following way:

$$\int d^3r' T^{ii}(\vec{r}') = \int d^3r' T^{ik}(\vec{r}') \delta^i_k = \int d^3r' T^{ik}(\vec{r}') \frac{\partial x'^i}{\partial x^k} = -\int d^3r' \frac{\partial T^{ik}(\vec{r}')}{\partial x^k} x'^i = 0, \quad (A4)$$

which is ensured by the continuity equation:

$$\frac{\partial T^{ik}}{\partial x^k} = 0. \quad (A5)$$
we have introduced the angular momentum of the source (the Levi-Civita tensor is defined such that $\varepsilon_{123} = +1$):

$$\vec{J} = \int d^3r' \vec{r}' \times \vec{P} \iff J^i = \int d^3r' \varepsilon_{ijk} T_{0j} x'^i,$$

(A6)

where $P_j = T_{0j}$ is $j$-th component of the density momentum of the source; in particular we have used the formula:

$$\int d^3r' \varepsilon_{ijk} T_{0j} x'^i = \frac{1}{2} \varepsilon_{ijk} J^k. \quad (A7)$$

Note that we can define the rotation of the source along the $z$-axis, such that $\vec{J} = J \hat{z}$; thus, the $h_{0i}$ components in the spacetime metric in Eq. (27) will be given by:

$$2 \vec{h} \cdot d\vec{x} dt = \frac{4G}{r^3} (r \times \vec{J})_1 dx dt = -\frac{4GJ}{r^3} \sin^2 \theta d\varphi dt,$$

where from the second to the third line we have used the relations $x = r \sin \theta \cos \varphi$ and $y = r \sin \theta \sin \varphi$. Thus, the spacetime metric far outside a rotating source in GR is given by

$$ds^2 = -\left(1 - \frac{2Gm}{r}\right) dt^2 - \frac{4GJ}{r^3} \sin^2 \theta d\varphi dt + \left(1 + \frac{2Gm}{r}\right) \left(dr^2 + r^2 d\Omega^2\right), \quad (A9)$$

which is called Lense-Thirring metric \[45\].

Appendix B: Second order contributions from the Weyl term

We now wish to present the explicit expression of the two-rank symmetric tensor $P_{3}^{\alpha \beta}$ at second order in $\Box_s$. It is given by:

$$P_{3}^{(2)\alpha \beta}(\Box_s) = \frac{\alpha_c}{8\pi G} f_{32} \left( \begin{array}{ccc} a_{00} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{33} \end{array} \right), \quad (B1)$$

with the dimensionless matrix elements, defined as

$$a_{00} = \frac{576G^2 m^2}{r^{15} M_s^4 (a^2 + r(r - 2Gm))} \left[ 4a^4 G mr (89Gm - 66r) - 72a^6 Gm ight. \right.$$  
$$\left. + a^2 r^2 (-1578G^3 m^3 + 927G^2 m^2 r - 656G m^2 + 140r^3) + r^5 (939G^2 m^2 - 744G m r + 140r^2) \right],$$

$$a_{03} = - \frac{1152a G^3 m^3}{r^{15} M_s^4 (a^2 + r(r - 2Gm))} \left[ 36a^4 - 2a^2 r (89Gm - 52r) + r^2 (789G^2 m^2 - 696G mr + 148r^2) \right],$$

$$a_{11} = \frac{576G^2 m^2}{r^{15} M_s^4} \left[ 2a^3 (193Gm - 50r) + a^2 r (-967G^2 m^2 + 718G mr - 120r^2) \right. \right.$$  
$$\left. + r^2 (390G^3 m^3 - 459G^2 m^2 r + 172G mr - 20r^3) \right],$$
Moreover, we can also see, for example, how the component $P^{22}$ looks like up to $\Box^2$:

\[
P^{22} = \frac{\alpha_s}{8\pi G} \left[ f_{31} \left( \frac{3024 a^2 G^2 m^2}{r_{10} M_s^2} - \frac{7200 G^3 m^3}{r_{11} M_s^2} + \frac{4032 a^2 G^2 m^2}{r_{12} M_s^2} \right) 
+ f_{32} \left( \frac{46080 G^2 m^2}{r_{12} M_s^2} - \frac{307584 G^3 m^3}{r_{13} M_s^2} + \frac{57600 a^2 G^2 m^2}{r_{14} M_s^2} + 454464 G^4 m^4}{r_{15} M_s^2} - \frac{245376 a^2 G^3 m^3}{r_{15} M_s^2} \right) \right] + \cdots \tag{B2}
\]

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