REGULARITY AND INTERSECTIONS OF BRACKET POWERS

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Abstract. Among reduced Noetherian prime characteristic commutative rings, we prove that a regular ring is precisely one where finite intersection of ideals commutes with taking bracket powers. However, reducedness is essential for this equivalence. Connections are made with Ohm-Rush content theory, intersection-flatness of the Frobenius map, and various flatness criteria.

In this paper, all rings are unital, associative, and (unless otherwise specified) commutative.

The condition of being regular is paramount in the theory of commutative Noetherian rings, due in part to its connections to algebraic geometry. Hence an easily tested algebraic condition for it is always welcome. In pursuance of this, we offer:

Main Theorem. Let $R$ be a Noetherian reduced commutative ring of prime characteristic $p > 0$. Then the following are equivalent:

1. $R$ is regular.
2. Finite intersection of ideals commutes with taking bracket powers.
3. For any ideal $I$ and any $x \in R$, we have $I^{[p]} \cap (x^p) = (I \cap (x))^{[p]}$.
4. For any ideal $I$ and any $x \in R$, we have $(I : x)^{[p]} = (I^{[p]} : x^p)$.

The use of the Frobenius to detect important properties of prime characteristic rings, including regularity, the complete intersection property, and Gorensteinness, is not new. We refer the reader to Miller’s survey [Mil03]. Indeed, the equivalence of (1) and (4) of the above theorem was previously proved by Zhang [Zha09], using completely different methods from the ones employed here.

Recall that for a power $p^n$ of $p$, the $p^n$th bracket power of an ideal $I$ is the ideal generated by \{ $x^{p^n} : x \in I$ \}. Kunz [Kun69] showed that for a commutative Noetherian ring $R$ of prime characteristic $p > 0$, $R$ is regular if and only if the Frobenius endomorphism $R \to R$, $r \mapsto r^p$, is flat. Indeed, the proof of the above theorem is really about flatness, so we commence with the Bourbaki flatness criterion, a criterion that this author feels should be better known than it is.

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Proposition 1. [Bou72, Exercise I.2.22] Let $R$ be a ring, not necessarily commutative, and let $M$ be a left $R$-module. Then $M$ is $R$-flat if and only if $(IM :_M x) = (I :_R x)M$ for all finitely generated right ideals $I$ and every $x \in R$, where for a subset $L \subseteq M$, $(L :_M x) := \{z \in M : xz \in L\}$.

Next, we have the Hochster-Jeffries flatness criterion.

Theorem 2. Let $R$ be a ring, not necessarily commutative, and $M$ a left $R$-module. Then $M$ is flat over $R$ if and only if

1. For any finitely generated right ideal $I$ of $R$ and any $x \in R$, we have $(IM \cap xM) = (I \cap xR)M$, and
2. For any $x \in R$, we have $(0 :_M x) = (0 :_R x)M$.

Remark 3. The above is proved by Hochster and Jeffries in [HJ20, Proposition 5.5, iv \Rightarrow i] in case $R$ is commutative, though this is not how they state it. Bourbaki flatness makes the proof simpler though, and it allows the passage to noncommutative rings.

Proof of Theorem 2. First suppose $M$ is flat over $R$. Then (1) is well known (by applying $\cdot \otimes_R M$ to the exact sequence of right $R$-modules $0 \to R/(I \cap xR) \to (R/I) \oplus (R/xR)$), and (2) follows from Bourbaki flatness (Proposition 1).

Conversely, suppose (1) and (2) hold. Let $I$ be a finitely generated right ideal and $x \in R$. Then we have

$$x(IM :_M x) = IM \cap xM = (I \cap xR)M = x(I :_R x)M.$$ 

It follows that

$$IM :_M x = (I :_R x)M + (0 :_M x) = (I :_R x)M + (0 :_R x)M = (I :_R x)M.$$ 

Another appeal to Bourbaki flatness finishes the proof.

Lemma 4. Let $R$ be a (commutative) integral domain and $M$ a torsion-free $R$-module such that for any finitely generated ideal $I$ of $R$ and any $x \in R$, we have $(I \cap xR)M = IM \cap xM$. Then $M$ is flat over $R$.

Proof. By Hochster-Jeffries flatness (Theorem 2), it is enough to show that $\text{ann}_R(a)M = (\text{ann}_M a)M$ for all $a \in R$. But if $a = 0$, then $\text{ann}_R(a) = R$ and $\text{ann}_M(a) = M$, whence $\text{ann}_R(a)M = RM = M = \text{ann}_M a$. On the other hand, if $a \neq 0$, then $\text{ann}_R(a) = 0$, and by torsion-freeness we also have $\text{ann}_M(a) = 0$, so $\text{ann}_R(a)M = 0M = 0 = \text{ann}_M(a)$.

Next, note the following connection between zero-divisors and intersection of bracket powers of principal ideals.

Lemma 5. Let $R$ be a commutative Noetherian ring of prime characteristic $p > 0$. Suppose that for all $x, y \in R$, we have $((x^p) \cap (y^p)) = ((x) \cap (y))^{[p]}$. Then every nonzero zero-divisor in $R$ has a nonzero nilpotent multiple.
Proof. First, note that since bracket powers and finite intersection commute with localization, as does nilpotency, we may assume \((R, m)\) is local.

Let \(0 \neq a\) be a zero-divisor of \(R\). Assume it has no nonzero nilpotent multiple. Let \(P\) be an associated prime of \(R\) that contains \(\text{ann}(a)\). Without loss of generality (by passing to a multiple), we may assume \(P = \text{ann}(a)\).

Now let \(Z := \{\text{ann } x : 0 \neq x \in P\}\). Then \(Z\) is a nonempty set of proper ideals of \(R\), so the fact that \(R\) is Noetherian implies that \(Z\) has a maximal element \(Q\), which is then prime by the usual arguments. Let \(x \in P\) with \(Q = \text{ann } x\). Then we have \(P = \text{ann}(a^t)\) (since \(a\) is non-nilpotent) and \(Q = \text{ann}(x^t)\) (since \(Q\) is maximal in \(Z\)) for all positive integers \(t\).

Fix a positive integer \(t\), and let \(b = a^t\) and \(y = x^t\). We have \(g = cb = d \cdot (b + y)\) for some \(c, d \in R\). Thus, \((c - d)b^2 = dby = 0\), whence \((c - d) = \text{ann}(b^2) = P\). Say \(c = d + \pi, \pi \in P\).

Then we have \(0 = \pi b = (c - d)b = dy\), so \(d \in \text{ann}(y) = Q\). We have shown that \((b) \cap (b + y) \subseteq Qb\).

Conversely, if \(q \in Q\), then \(q b = q(b + y) \in (b) \cap (b + y)\) for any ideals \(M, N\) of \(R\) with localization at maximal ideals, and since any localization of a reduced ring is reduced, we have that \((3)\) holds when \(R\) is a local ring.

Thus, \(Qa^t = \mathfrak{m}Qa^t\), so the fact that \(\mathfrak{m}\) is non-nilpotent) and \(Q = \text{ann}(x^t)\) (since \(Q\) is maximal in \(Z\)).

Hence, \(Q\) is a torsion-free \((\mathfrak{m}^\infty)\)-module, the Frobenius is flat by Lemma 4, and we are done by the Kunz regularity criterion.

Proof of Main Theorem. By the Kunz regularity criterion [Kun69], \(R\) is regular iff the maps \(f_q : R \to R\) given by \(a \mapsto a^q, q\) a power of \(p\), are flat, iff \(f_p\) is flat. Moreover, we have \(f_q(I)R = I^{[q]}\). Hence, the equivalence of (1) and (4) follows directly from Bourbaki flatness. Moreover, the implication \((1) \implies (2)\) follows from the well-known fact that for any flat \(R\)-module \(M\), we have \(IM \cap JM = (I \cap J)M\) for any ideals \(I, J\) of \(R\) (cf. the proof of Theorem 2), plus a trivial induction step. The implication \((2) \implies (3)\) is obvious. Hence, we need only prove \((3) \implies (1)\).

Assume \(R\) satisfies \((3)\), and suppose the implication \((3) \implies (1)\) holds when \(R\) is local. Since both finite intersections and bracket powers commute with localization at maximal ideals, and since any localization of a reduced ring is reduced, we have that \((3)\) holds for the reduced local ring \(R_m\) for any maximal ideal \(m\). Thus by assumption, \(R_m\) is a regular local ring, so \(R\) is regular since \(m\) was arbitrary. Thus, we are reduced to the local case.

If \(R\) is an integral domain and \((3)\) holds, then since \(R^{1/p}\) is a torsion-free \(R\)-module, the Frobenius is flat by Lemma 4 and we are done by the Kunz regularity criterion.

It remains to show that when \((R, m)\) is local and \((3)\) holds, \(R\) is an integral domain. But this follows from Lemma 5.

The obvious question now is: Do the equivalences in the Main Theorem still hold when \(R\) is nonreduced? The answer is no.
Example 6. Let $R = k[x]/(x^2)$, where $k$ is a field and $x$ an indeterminate over $k$. Clearly $R$ is local and Noetherian but not regular, since $\dim R = 0$ but the maximal ideal requires a nonzero generator. However, let $I$, $J$ be nonzero ideals of $R$. Since the 3 ideals of $R$ are linearly ordered, without loss of generality we have $J \subseteq I$. Thus, $I^{[q]} \cap J^{[q]} = J^{[q]} = (I \cap J)^{[q]}$.

Remark 7 (Ohm-Rush content and Frobenius roots). Recall that for a commutative ring $R$, a module $M$ is Ohm-Rush OR72, ES16, or weakly intersection flat for ideals LI20, if for all collections $\{ I_\alpha \}$ of ideals of $R$, we have $\bigcap_\alpha I_\alpha M = (\bigcap_\alpha I_\alpha)M$. The condition on a Noetherian local ring of positive prime characteristic that the Frobenius endomorphism is a flat Ohm-Rush module is an important condition in tight closure theory (cf. HI94, Kat08, Sha12). In particular, it allows, given an ideal $J$ and a power $q$ of $p$, to find a unique smallest ideal $I$ such that $J \subseteq I^{[q]}$, which is known as $J^{[1/q]}$ (e.g. see Sha12, Definition 9.5). In particular, we intersect all the ideals that have this property. But this intersection is just the Ohm-Rush content of the ideal OR72, ES16 via the iterated Frobenius endomorphism, so the Ohm-Rush property itself is enough to guarantee such a minimal member. Hence it is natural to consider whether the weaker condition of being an Ohm-Rush module is enough to force flatness of the Frobenius, and hence regularity. We see above that the answer is yes for reduced rings, but no otherwise. It might be interesting to characterize the class of (nonreduced) rings for which the Ohm-Rush property holds for the Frobenius endomorphism, of which the above example is a member.

When $R$ is complete, finite intersection of ideals commuting with bracket powers is enough to ensure the property for arbitrary intersections of ideals:

Proposition 8. Let $R$ be a complete Noetherian local ring of prime characteristic $p > 0$. Suppose that for any ideals $I, J$ of $R$, we have $I^{[p]} \cap J^{[p]} = (I \cap J)^{[p]}$. Then the Frobenius endomorphism $f : R \to R$ is Ohm-Rush.

Proof. We need to show that for any $x \in R$, there is a unique smallest ideal $I$ with $x \in I^{[p]}$.

Let $\{ I_\alpha \}_{\alpha \in \Lambda}$ be the set of ideals $J$ with $x \in J^{[p]}$, indexed by the index set $\Lambda$. Set $c(x) := \bigcap \{ I_\alpha : \alpha \in \Lambda \}$. By the argument in the proof of Kat08 Proposition 5.3], there is a countable subset $\{ \alpha_i : i \in \mathbb{N} \}$ of $\Lambda$ such that $c(x) = \bigcap_{i=1}^{\infty} I_{\alpha_i}$. For each $i \in \mathbb{N}$, set $J_i := \bigcap_{h=1}^{i} I_{\alpha_h}$. Then this is a decreasing sequence of ideals whose intersection is $c(x)$. Moreover, for each $i$, we have $x \in \bigcap_{h=1}^{i} I_{\alpha_h}^{[p]} = (\bigcap_{h=1}^{i} I_{\alpha_h})^{[p]} = J_i^{[p]}$, where the first equality follows from the assumption on pairs of ideals. Chevalley’s lemma Che43, Lemma 7 then guarantees that for each $n \in \mathbb{N}$, there is some $i(n) \in \mathbb{N}$ such that $J_{i(n)} \subseteq m^n + c(x)$. Thus, for each $n$, we have

$$x \in \bigcap_n (J_{i(n)}^{[p]}) \subseteq \bigcap_n (m^n + c(x))^{[p]} = \bigcap_n (m^{[p]})^n + c(x)^{[p]} = c(x)^{[p]}.$$
with the last equality by the Krull intersection theorem. Since \( x \in c(x)[p] \)
and \( c(x) \) is contained in all ideals \( J \) with \( x \in J[p] \), it follows that \( c(x) \) is the
unique smallest such ideal.

Recall that the Frobenius closure \( I^F \) of an ideal \( I \) in a prime characteristic
ring consists of all those elements \( x \in R \) such that there exists some \( n \in \mathbb{N} \)
with \( x^{p^n} \in I[p^n] \). \( F \)-purity is a property of rings that implies that every ideal
coincides with its Frobenius closure. It is considered to be a much weaker
property than regularity.

Example 6 indicates that there might be a connection between the proper-
ties in the Main Theorem for non-reduced rings and the regula rity property
for the reduced structure of the ring. Indeed this is so, in the presence of
\( F \)-purity.

**Proposition 9.** Let \( R \) be a commutative Noetherian ring of prime charac-
teristic \( p > 0 \) such that
\[
\begin{align*}
(1) & \quad (I \cap xR)[p] = I[p] \cap x^pR \quad \text{for all ideals } I \text{ and } x \in R, \text{ and} \\
(2) & \quad R_{\text{red}} \text{ is } F\text{-pure.}
\end{align*}
\]

Then \( R_{\text{red}} \) is regular.

**Proof.** Let \( a \) be an ideal of \( R_{\text{red}} \) and \( \alpha \in R_{\text{red}} \). Let \( I \) be an ideal of \( R \)
and \( x \in R \) whose residues in \( R_{\text{red}} \) are \( a \) and \( \alpha \) respectively. Let \( N \) be the
nilradical of \( R \). Then there is some power \( q \) of \( p \) such that \( N^q = 0 \).

Now let \( \beta \in a[p] \cap (\alpha^p) \). Let \( y \in R \) with residue class equal to \( \beta \). Then
\[
y^q \in (I[p] + N)[q] \cap ((x^p) + N)[q] = I[pq] \cap (x^{pq}) = (I \cap (x))[pq],
\]
where the last equality follows from (1) and induction on \( \log_p q \).

It follows that \( \beta^q \in (a \cap (\alpha))[pq] = ((a \cap (\alpha))[p])[q] \), so that \( \beta \in ((a \cap (\alpha))[p])^F \).
But every ideal in \( R_{\text{red}} \) is Frobenius closed, so \( \beta \in (a \cap (\alpha))[p] \). Thus,
\( a[p] \cap (\alpha^p) \subseteq (a \cap (\alpha))[p] \), whence equality holds. Then by the Main Theorem,
\( R_{\text{red}} \) is regular. \( \square \)

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