GLOBAL STRONG SOLUTIONS TO THE PLANAR COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH LARGE INITIAL DATA AND VACUUM

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ABSTRACT. This paper considers the initial boundary problem to the planar compressible magnetohydrodynamic equations with large initial data and vacuum. The global existence and uniqueness of large strong solutions are established when the heat conductivity coefficient $\kappa(\theta)$ satisfies

$$C_1(1 + \theta^q) \leq \kappa(\theta) \leq C_2(1 + \theta^q)$$

for some constants $q > 0$, and $C_1, C_2 > 0$.

1. Introduction. Magnetohydrodynamics (MHD) studies the dynamics of conducting fluids in a magnetic field. The MHD finds its way in a very wide range of physical objects, from liquid metals to cosmic plasmas, for example, see [3, 19, 24, 25, 30]. The governing equations of compressible planar magnetohydrodynamic flows, which implies that the flows are uniform in the transverse directions, take the following form:

$$\rho_t + (\rho u)_x = 0,$$

$$\left(\rho u\right)_t + \left(\rho u^2 + P + \frac{1}{2}|b|^2\right)_x = \left(\lambda u_x\right)_x,$$

$$\left(\rho w\right)_t + \left(\rho uw - b\right)_x = \left(\mu w_x\right)_x,$$

$$b_t + (ub - w)_x = \left(\nu b_x\right)_x,$$

$$\left(\rho e\right)_t + \left(\rho ue - (\kappa e_x)x\right)_x = \lambda u_x^2 + \mu |w_x|^2 + \nu |b_x|^2 - Pu_x,$$

where $\rho \geq 0$ denotes the density of the flow, $u \in \mathbb{R}$ the longitudinal velocity, $w \in \mathbb{R}^2$ the transverse velocity, $b \in \mathbb{R}^2$ the transverse magnetic field, and $e$ the internal energy.
energy, respectively. Both the pressure $P$ and the internal energy $e$ are generally related to the density and temperature of the flow according to the equations of state: $P = P(\rho, \theta)$ and $e = e(\rho, \theta)$. The parameters $\lambda = \lambda(\rho, \theta)$ and $\mu = \mu(\rho, \theta)$ denote the bulk and the shear viscosity coefficients, respectively; $\nu = \nu(\rho, \theta)$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, and $\kappa = \kappa(\rho, \theta)$ is the heat conductivity.

The system (1)–(5) are supplemented with the following initial and boundary conditions:

\begin{align}
(u, w, b, \theta_x)|_{\partial \Omega} &= 0, \quad (6) \\
(\rho, u, w, b, \theta)|_{t=0} &= (\rho_0(x), u_0(x), w_0(x), b_0(x), \theta_0(x)), \quad (7)
\end{align}

where $\partial \Omega = \{0, 1\}$ denotes the boundary of the interval $\Omega := (0, 1)$. The conditions (6) mean that the boundary is non-slip and thermally insulated.

There have been a lot of studies on MHD by physicists and mathematicians due to its physical importance, complexity, rich phenomena, and mathematical challenges. Below we mention some mathematical results on existence theory of the compressible MHD equations, the interested readers can refer [3, 19, 24, 25, 30] for complete discussions on physical aspects. We begin with the one-dimensional case. The existence and uniqueness of local smooth solutions were proved firstly in [35], while the existence of global smooth solutions with small smooth initial data was shown in [21]. The exponential stability of small smooth solutions was obtained in [28, 31]. In [33, 13], Hoff and Tsyganov obtained the global existence and uniqueness of weak solutions with small initial energy. Under the technical condition that $\kappa(\rho, \theta)$, depending the temperature $\theta$ only, i.e., $\kappa(\rho, \theta) \equiv \kappa(\theta)$, satisfies

\begin{equation}
C_1(1 + \theta^q) \leq \kappa(\theta) \leq C_2(1 + \theta^q), \quad (8)
\end{equation}

for some constants $q \geq 2$ and $C_1, C_2 > 0$. Chen and Wang [4] proved the existence, uniqueness, and Lipschitz continuous dependence of global strong solutions to the system (1)–(5) with large initial data satisfying

\begin{equation}
0 < \inf_{x \in \Omega} \rho_0(x) \leq \rho_0(x) \leq \sup_{x \in \Omega} \rho_0(x) < \infty; \rho_0, u_0, w_0, b_0, \theta_0 \in H^1(\Omega), \inf_{x \in \Omega} \theta_0 > 0.
\end{equation}

The similar results are obtained in [5, 36] for real gas cases. Fan, Jiang and Nakamura [8] obtained the global weak solutions to the problem (1)–(5) when the initial data satisfying the condition (8) with $q \geq 1$ and

\begin{equation}
\rho_0^{-1}, \rho_0 \in L^\infty(\Omega); \rho_0, u_0, w_0, b_0 \in L^2(\Omega); \theta_0 \in L^1(\Omega), \inf_{x \in \Omega} \theta_0 > 0.
\end{equation}

Later they [9] obtained the existence, the uniqueness and the Lipschitz continuous dependence on the initial data of global weak solutions to the problem (1)–(7) when the initial data lie in the Lebesgue spaces. Recently, Hu and Ju [17] considered the problem (1)–(7) under the assumption that the heat conductivity depends on temperature with

\begin{equation}
\kappa(\theta) = \theta^q, \quad q > 0,
\end{equation}

and obtained the existence and uniqueness of global strong solutions with large initial data. Their methods are different from the one used here. In fact, their arguments were motivated by the ideas developed by Pan and Zhang [29] where the global existence of smooth solutions to the 1-d compressible Navier-Stokes equations with arbitrarily large initial data under the condition (9) was obtained.

For the multi-dimensional compressible MHD equations, there are also many mathematical results on existence of solutions. As mentioned before, Vol’pert and
Hudjaev [35] first obtained the local smooth solutions to the compressible MHD equations. Li, Su and Wang [27] obtained the existence and uniqueness of local in time strong solution with large initial data when the initial density has an positive lower bound. Fan and Yu [11] obtained the strong solution to the compressible MHD equations with vacuum. Kawashima [20] obtained the smooth solutions for two-dimensional compressible MHD equations when the initial data is a small perturbation of given constant state. Umeda, Kawashima and Shizuta [34] obtained the decay of solutions to the linearized MHD equations. Li and Yu [26] obtained the optimal decay rate of small smooth solutions. In [14, 15], Hu and Wang obtained the global existence of weak solutions to the isentropic compressible MHD equations and variational solutions to the full compressible MHD equations, see also [10, 7, 39] for related results. Suen and Hoff [32] obtained the global low-energy weak solutions of the isentropic compressible MHD equations.

It should be pointed out that although there are many progress on compressible MHD equations it is still an open question in obtaining the global strong or smooth solutions to the full compressible MHD equations with large initial data and possible vacuum even in the one dimensional case, see [15].

In the present paper we study the global existence and uniqueness of large strong solutions to the planar compressible magnetohydrodynamic equations (1)-(5) with large initial data and vacuum. We focus on the perfect gas case:

\[ P(\rho, \theta) := R\rho\theta, \quad e := C_V\theta, \]

where \( R > 0 \) is the gas constant and \( C_V > 0 \) is the heat capacity of the gas at constant volume. We will consider the case that the coefficients \( \lambda, \mu, \) and \( \nu \) are positive constants and the heat conductivity coefficient depends on the temperature \( \theta \) only, i.e., \( \kappa(\rho, \theta) \equiv \kappa(\theta) \).

The main result in this paper reads as follows.

**Theorem 1.1.** Let \( \kappa \in C^2[0, \infty) \) satisfies the condition (8) for some \( q > 0 \). Suppose that the initial data \( (\rho_0, u_0, w_0, b_0, \theta_0) \) satisfy \( \rho_0 \geq 0, \theta_0 \geq 0, \rho_0 \in H^3(\Omega), u_0 \in H^4(\Omega) \cap H^2(\Omega), w_0, b_0 \in H^2(\Omega) \cap H^2(\Omega), \theta_0 \in H^2(\Omega), (\theta_0)|_{\partial \Omega} = 0 \) and the following compatibility condition:

\[
\begin{align*}
\lambda(u_0)_{xx} - \left(\rho_0\theta_0 + \frac{3}{2}|b_0|^2\right)_x &= \sqrt{\rho_0} g_1, \\
\mu(w_0)_{xx} - (b_0)_x &= \sqrt{\rho_0} g_2, \\
(\kappa(\theta_0)(\theta_0)_x)_x + |(u_0)_x|^2 + |(w_0)_x|^2 + |(b_0)_x|^2 &= \sqrt{\rho_0} g_3,
\end{align*}
\]

for some \( g_1, g_2, g_3 \in L^2(\Omega) \). Then, for any \( T > 0 \), there exists a unique global solution \( (\rho, u, w, b, \theta) \) to the problem (1)-(7) such that

\[
\begin{align*}
\rho &\in L^\infty([0, T]; H^2(\Omega)), \quad (u, w, b, \theta) \in L^\infty([0, T]; H^1(\Omega)), \\
(u, w, b, \theta) \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)), \\
(\sqrt{\rho} u_t, \sqrt{\rho} w_t, b_t, \sqrt{\rho} \theta_t) &\in L^\infty(0, T; L^2(\Omega)), \\
(u_t, w_t, b_t, \theta_t) &\in L^2(0, T; H^1(\Omega)).
\end{align*}
\]

There are two ingredients in our result comparing with the previous results on one-dimensional MHD equations mentioned above. First, in our result the initial density may contain the vacuum provided that it satisfies the compatible conditions (10). Next, a relaxed condition on the heat conductivity coefficient is permitted. In fact, in (8), we only need \( q > 0 \) while in [4, 5, 36] required \( q \geq 2 \) and in [8] with
\( q \geq 1 \). We point out that the assumption \( q > 0 \) paly a crucial role in our arguments, see Lemmas 2.3, 2.5, 2.7, and 2.9 below.

**Remark 1.1.** As has been observed in [6], the lack of a positive lower bound of \( \rho_0 \) should be compensated with some conditions on the initial data \((\rho_0, w_0, b_0, \theta_0)\). Roughly speaking, the compatibility condition (10) is equivalent to the \( L^2 \)-integrability of \( \sqrt{\rho} u_t, \sqrt{\rho} w_t, \) and \( \sqrt{\rho} \theta_t \) at \( t = 0 \), as can be shown formally by letting \( t \to 0 \) in (2), (3), and (5). Hence the condition (10) plays a key role in the estimates of \((\sqrt{\rho} u_t, \sqrt{\rho} w_t, b_t, \sqrt{\rho} \theta_t)\). This was observed and justified rigorously in [6] for viscous polytropic fluids. Note that the condition (10) is satisfied automatically for all initial data \((\rho_0, u_0, w_0, b_0, \theta_0)\) with the regularity presented in Theorem 1.1 whenever \( \rho_0 \) is bounded away from zero.

**Remark 1.2.** It is possible to extend our results to the one-dimensional compressible MHD system with more general state of equations:

\[
P = \rho^2 \frac{\partial e}{\partial \rho} + \theta \frac{\partial P}{\partial \theta}
\]

with some additional assumptions.

**Remark 1.3.** When there is no vacuum initially, we can improve the results in [8] to the case \( q > 0 \) by applying the arguments developed here.

We remark that when taking \( w = b = 0 \) the system (1)–(7) reduces to the well-known one-dimensional full Navier-Stokes equations and there are a lot of studies on this system. In the case of that the initial density is bounded away from zero, Kazhikhov and Shelukhin [23] first obtained the global smooth solutions for large initial data three decades ago, see also [1, 40, 41, 22] for different extensions. Recently, Huang and Li [18] obtain the global smooth solutions to the full Navier-Stokes system with possible vacuum and large oscillations provided that the total initial energy is sufficient small. Wen and Zhu obtained the global smooth solutions to the one-dimensional full Navier-Stokes system [37] and symmetric higher dimensional full Navier-Stokes system [38] with large initial data. For the variational or weak solutions to the full Navier-Stokes system, see [2, 12].

We give a few words on the strategy of the proof on our main result. Since the initial data may contains vacuum we first construct the regularized initial density \( \rho_{\delta}(x) = \rho_0 + \delta \) for any \( \delta > 0 \). Next, for each fixed \( \delta \), we can obtain the local and uniqueness existence of strong solutions. Third, we establish sufficient \textit{a priori} estimates uniformly with \( \delta \). Combining the local existence result, and the uniformly a priori estimates, we obtain the desired global existence result by taking the limiting as \( \delta \to 0^+ \) and applying standard continuity argument. We remark that the key point in the whole proof is to obtain uniformly a priori estimates where some ideas developed in [37, 38] are adapted. Comparing with [37], the main additional difficulties are due to the presence of the magnetic field and its interaction with the hydrodynamic motion of the flow of large oscillation. We shall deal with the terms involving the magnetic field very carefully, see especially Lemmas 2.5-2.7 below.

Before ending this introduction we recall the following auxiliary inequalities.

**Lemma 1.2** ([12, 37]). Let \( \Omega = (0,1) \) be an interval in \( \mathbb{R}^1 \).

(i). Assume that \( \rho \) is a non-negative function satisfying

\[
0 < M \leq \int_{\Omega} \rho \, dx \leq K,
\]

\[
\rho_0 \delta(x) = \rho_0 + \delta
\]
for two constants $M$ and $K$. Then, for any $v \in H^1(\Omega)$, it holds
\[
\|v\|_{L^\infty(\Omega)} \leq \frac{K}{M}\|v_x\|_{L^2(\Omega)} + \frac{1}{M}\int_\Omega \rho v dx,
\]

(ii) Assume further that $v$ satisfies
\[
\|[\rho v]\|_{L^1(\Omega)} \leq C.
\]
Then for any $r > 0$, there exists a positive constant $C = C(M, K, r, C)$ such that
\[
\|v^r\|_{L^\infty(\Omega)} \leq C\|(v^r)_x\|_{L^2(\Omega)} + C.
\]

2. Proof of Theorem 1.1. In this section we will prove Theorem 1.1 by considering the initial density $\rho_0 = \rho_0 + \delta$, as mentioned before, to get a sequence of approximate solutions to (1)–(7), then taking $\delta \to 0^+$ after making some a priori estimates uniformly for $\delta$. Since the proof of the local existence and uniqueness of strong solutions to the approximate problem is now standard (see, for example [6, 11]), thus we only need to establish the uniform estimates.

Below we still use $(\rho, u, w, b, \theta)$ to denote the smooth solutions of approximate problem to (1)–(7). We shall denote $Q_T := \Omega \times [0, T]$ with $T > 0$ and omit the spatial domain $\Omega$ in integrals for convenience. We use $C$ to denote the constants which are independent of $\delta$, but possible depending on $T$, and may change from line to line.

To begin with the proof, we notice that the total mass and energy in the system (1)–(5) are conserved. In fact, by rewriting (1)–(5), one has
\[
\begin{align*}
\mathcal{E}_t + \int \left[ u \left( \mathcal{E} + P + \frac{1}{2} |b|^2 \right) - w \cdot b \right]_x = & \left( \lambda u u_x + \mu w \cdot w_x + \nu b \cdot b_x + \kappa(\theta)\theta_x \right)_x, \\
(\rho \mathcal{S})_t + (\rho \mathcal{S})_x = & \left( \lambda u u_x + \mu w \cdot w_x + \nu b \cdot b_x + \kappa(\theta)\theta_x \right)_x
\end{align*}
\]
(11)
where $\mathcal{E}$ and $\mathcal{S}$ are the total energy and the entropy, respectively,
\[
\mathcal{E} := \rho \left( \theta + \frac{1}{2} |u|^2 + |w|^2 \right) + \frac{1}{2} |b|^2, \quad \mathcal{S} := \ln \theta - \ln \rho.
\]
Integrating (1), (11) and (12) over $Q_T$, we have

Lemma 2.1.
\[
\int \rho(x,t)dx = \int \rho_0(x)dx, \quad \int \mathcal{E}(x,t)dx = \int \mathcal{E}(x,t = 0)dx,
\]
\[
\int (\rho \ln \rho + \rho \ln \theta)(x,t)dx + \int_Q \left( \frac{u_x^2}{\theta} + \frac{|w_x|^2 + |b_x|^2}{\theta} + \frac{\kappa(\theta)\theta_x^2}{\theta^2} \right) dxdt \leq C.
\]

Lemma 2.2.
\[
0 \leq \rho(x,t) \leq C, \quad (x, t) \in \overline{Q}_T.
\]

Proof. We need only to estimate the upper bound. The proof is similar to that in [8], here we present it for completeness. From (1) and (2), we have
\[
(\rho u)_t = \tilde{P}_x, \quad \tilde{P}_x := \lambda u x - \rho u^2 - P - \frac{1}{2} |b|^2.
\]
Denote
\[
\phi := \int_0^t \tilde{P}(x, \tau)d\tau + \int_0^{x} \rho_0(\xi)u_0(\xi)d\xi,
\]
we have
\[ \phi_x = \rho u, \quad \phi_t = \tilde{P}, \quad \phi_x|_{\partial \Omega} = 0, \quad \phi|_{t=0} = \int_0^x \rho_0(\xi) u_0(\xi) d\xi. \] (14)

By virtue of Lemma 2.1 and Cauchy inequality, it holds
\[ \|\phi_x\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \quad \left| \int \phi dx \right| \leq C. \]
Hence
\[ \|\phi\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C. \] (15)

Now, denoting \( F := e^{\phi} \) and using (14), we have after a straightforward calculation that
\[ D_t(\rho F) := \partial_t(\rho F) + u \partial_x(\rho F) = -\left( P + \frac{1}{2} |b|^2 \right) \rho F \leq 0, \]
which together with (15) implies (13) immediately.

**Lemma 2.3.** Let \( 0 < \alpha < \min \{1, q\} \) be any given constant, then it holds
\[ \int \int_{Q_T} \left( \frac{\lambda u_x^2}{\theta^\alpha} + \mu |w_x|^2 + \nu |b_x|^2 \right) dxdt \leq C, \]
(16)
\[ \int_0^T \|\theta\|_{L^\infty}^{q-\alpha+1} dt \leq C. \] (17)

**Proof.** The proof is similar to that given in [38] for symmetric Navier-Stokes equations, we present it here for completeness. Multiplying (5) by \( \theta^{-\alpha} \) and integrating the result over \( Q_T \), we have
\[ \int \int_{Q_T} \left( \frac{\lambda u_x^2}{\theta^\alpha} + \mu |w_x|^2 + \nu |b_x|^2 \right) dxdt + \alpha \int \int_{Q_T} \frac{\nu(\theta)^2}{\theta^{1+\alpha}} dxdt \]
\[ = \int \int_{Q_T} \{ (\rho \theta)_t + (\rho u \theta)_x + P u_x \} \theta^{-\alpha} dxdt. \] (18)

Using (1), Lemma 2.1, and Young’s inequality, the first two terms on the right-hand side of (18) can be bounded as
\[ \int \int_{Q_T} \{ (\rho \theta)_t + (\rho u \theta)_x \} \theta^{-\alpha} dxdt = \int \int_{Q_T} (\rho \theta \theta^{-\alpha} + \theta_x \rho u \theta^{-\alpha}) dxdt \]
\[ = \frac{1}{1-\alpha} \int \int_{Q_T} [ (\rho \theta^{1-\alpha})_t + (\rho u \theta^{1-\alpha})_x ] dxdt \]
\[ = \frac{1}{1-\alpha} \left[ \int \rho \theta^{1-\alpha} dx - \int \rho_0 \theta_0^{1-\alpha} dx \right] \]
\[ \leq \frac{1}{1-\alpha} \int \rho \theta^{1-\alpha} dx \]
\[ \leq \frac{C}{1-\alpha} \left[ \int \theta dx + \int \rho dx \right] \]
\[ \leq C. \] (19)

By Cauchy inequality, Lemmas 2.1 and 2.2, we have
\[ \int \int_{Q_T} P u_x \theta^{-\alpha} dxdt \leq \frac{\lambda}{2} \int \int_{Q_T} \frac{u_x^2}{\theta^\alpha} dxdt + C \int \int_{Q_T} \rho^2 \theta^{2-\alpha} dxdt \]
≤ \frac{\lambda}{2} \int_{Q_T} \frac{u^2}{\theta^\alpha} dx dt + C \int_0^T \|\theta\|_{L^\infty}^{1-\alpha} dt. \quad (20)

Noticing $0 < \alpha < \min\{1, q\}$, the Hölder inequality and Lemma 1.2 imply that
\[
C \int_0^T \|\theta\|_{L^\infty}^{1-\alpha} dt \leq C + \int_0^T \|	heta^-\theta_x\|_{L^2} dt \\
\leq C + C \int_0^T \left( \int \frac{\theta^2 \theta^{1-\alpha}}{\theta^{1+\alpha}} dx \right)^{1/2} dt \\
\leq C + \frac{\alpha}{2} \int_{Q_T} \frac{\kappa(\theta) \theta_x^2}{\theta^{1+\alpha}} dx dt \quad (21)
\]
for $q \geq 1 - \alpha$, while
\[
C \int_0^T \|\theta\|_{L^\infty}^{1-\alpha} dt \leq C + \int_0^T \|	heta^-\theta_x\|_{L^2} dt \\
\leq C + C \int_0^T \left( \int \frac{\theta^2 \theta^{1-\alpha}}{\theta^{1+\alpha}} dx \right)^{1/2} dt \\
\leq C + \frac{\alpha}{2} \int_{Q_T} \frac{\kappa(\theta) \theta_x^2}{\theta^{1+\alpha}} dx dt + C \int_0^T \|\theta\|_{L^\infty}^{1-\alpha} dt \quad (22)
\]
for $0 < q < 1 - \alpha$.

Putting (19)-(22) into (18) implies (16). By (21), (22) and (16), we can easily get (17). \qed

**Lemma 2.4.**
\[
\int_{Q_T} (\lambda u^2_x + \mu|w_x|^2 + \nu|b_x|^2) dx dt \leq C. \quad (23)
\]

**Proof.** Multiplying (2) by $u$, using (1), and integrating the result over $\Omega$, we see that
\[
\frac{1}{2} \frac{d}{dt} \int \rho u^2 dx + \int \lambda u^2_x dx = \int Pu_x dx - \int u b \cdot b_x dx. \quad (24)
\]

Similar, multiplying (3) by $w$, using (1), and integrating the result over $\Omega$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int \rho|w|^2 dx + \int \mu|w_x|^2 dx = \int b_x \cdot w dx = - \int b \cdot w_x dx. \quad (25)
\]

Multiplying (4) by $b$ and then integrating them over $\Omega$, we infer that
\[
\frac{1}{2} \frac{d}{dt} \int |b|^2 dx + \int \nu|b|^2 dx = \int b \cdot b_x dx + \int b \cdot b_x dx. \quad (26)
\]

Summing up (24), (25) and (26), using (13) and (17), we get
\[
\frac{1}{2} \frac{d}{dt} \int (\rho u^2 + \rho|w|^2 + |b|^2) dx + \int (\lambda u^2_x + \mu|w_x|^2 + \nu|b_x|^2) dx = \int Pu_x dx. \quad (27)
\]

By Young inequality, Lemmas 2.1 and 2.2, the estimate (17), and the condition that $0 < \alpha < \min\{1, q\}$, we have
\[
\int_{Q_T} Pu_x dx dt \leq C \int_{Q_T} \rho^2 \theta^{2} dx dt + \frac{1}{2} \int_0^T \|u_x\|_{L^2}^2 dt
\]
\[
\begin{align*}
\leq C & \int_0^T \|\theta\|_{L^\infty} dt + \frac{1}{2} \int_0^T \|u_x\|^2_{L^2} dt \\
= C + C & \int_0^T \|\theta\|_{L^\infty}^{q-\alpha+1} dt + \frac{1}{2} \int_0^T \|u_x\|^2_{L^2} dt \\
\end{align*}
\]

Integrating (27) over \([0, T]\) and applying the above inequality give (23).

**Lemma 2.5.**

\[
\int (\lambda u_x^2 + \mu |w_x|^2 + \nu |b_x|^2 + P^2 + \rho \theta^{q+2}) dx + \int_Q (\rho u_t^2 + \rho |w_x|^2 + |b_x|^2 + \nu |b_{xx}|^2 + \kappa^2 (\theta \theta_x^2)) dx dt \leq C.
\]

**Proof.** Multiplying (2) by \(u_t\), and then integrating them over \(\Omega\), we infer that

\[
\frac{1}{2} \frac{d}{dt} \int u_t^2 dx + \int \rho u_t^2 dx + \int \rho u u_x u_t dx = -\int \left( \left( P + \frac{1}{2} |b|^2 \right) \right) u_t dx \\
= \int \left( P + \frac{1}{2} |b|^2 \right) u_t dx - \int \left( P + \frac{1}{2} |b|^2 \right) t u_x dx \\
= \frac{d}{dt} \int \left( P + \frac{1}{2} |b|^2 \right) u_x dx - \int \left( P + \frac{1}{2} |b|^2 \right) t \left( u_x - P - \frac{1}{2} |b|^2 \right) dx \\
- \frac{d}{dt} \int \left( P + \frac{1}{2} |b|^2 \right) \frac{d}{dx} dx.
\]

From (2) and (13), and Lemma 2.2, we obtain

\[
\begin{align*}
\int \rho u u_x u_t dx & \leq \frac{1}{16} \int \rho u_t^2 dx + C \int \rho u_x^2 dx \\
& \leq \frac{1}{16} \int \rho u_t^2 dx + C \|\rho\|_{L^\infty} \|u\|^2_{L^\infty} \int u_x^2 dx \\
& \leq C + \frac{1}{16} \int \rho u_t^2 dx + C \left( \int u_x^2 dx \right)^2.
\end{align*}
\]

Using (31), (32), (2), (13), and Lemma 2.2, we obtain

\[
- \int \left( P + \frac{1}{2} |b|^2 \right) \left( \lambda u_x - P - \frac{1}{2} |b|^2 \right) dx \\
= - \int [\lambda u_x^2 + \mu |w_x|^2 - Pu_x - b \cdot (u b - w)] \left( \lambda u_x - P - \frac{1}{2} |b|^2 \right) dx
\]
Multiplying (5) by $\theta_x - uP + b \cdot b_x$, using (13), and integrating the result over $\Omega$, we derive
\[ \frac{\mu}{2} \frac{d}{dt} \int \left| w_x \right|^2 dx + \int \rho |w_t|^2 dx \]
\[ = \int b_x \cdot w_t dx - \int \rho uw_x \cdot w_t dx \]
\[ = -\int b \cdot w_{xt} dt - \int \rho uw_x \cdot w_t dx \]
\[ = -\frac{d}{dt} \int b \cdot w_x dx + \int b_t \cdot w_x dx - \int \rho uw_x \cdot w_t dx \]
\[ \leq -\frac{d}{dt} \int b \cdot w_x dx + \frac{1}{16} \int |b_t|^2 dx + C \int |w_x|^2 dx \]
\[ + \frac{1}{16} \int \rho |w_t|^2 dx + C \|u_x\|^4_{L^2} + C \|w_x\|^4_{L^2}. \]  
(34)

Multiplying (3) by $b_t - \nu b_{xx}$, integrating the result over $\Omega$, and using Cauchy inequality, we have
\[ \frac{d}{dt} \int \left| b_x \right|^2 dx + \int (|b_t|^2 + \nu |b_{xx}|^2) dx \]
\[ = \int (w - ub) \cdot (b_t - \nu b_{xx}) dx \]
\[ \leq C \int \left| w_x \right|^2 dx + C \|u_x\|^4_{L^2} + C \|b_x\|^4_{L^2} + \frac{1}{16} \int (|b_t|^2 + \nu |b_{xx}|^2) dx. \]  
(35)

Multiplying (5) by $\theta^{q+1}$, using (8), and integrating the result over $\Omega$, we find that
\[ \frac{d}{dt} \int \rho \theta^{q+2} dx + C \int \kappa^2 \theta_x^2 dx \]
Multiplying the above equation by $2\rho \theta^{q+1}$, where we have used the estimate:

$$\int x^2 \rho \theta^{q+1} dx \leq C \left(1 + \|\kappa \theta_x\|_{L^2}\right)$$

which can be derived from Lemma 1.2 and (8).

Combining (29), (30), (33), (34) and (35) with (36), we arrive at (28).

**Lemma 2.6.**

$$\int (\rho_x^2 + \rho_t^2) dx + \int Q \theta (u_{xx}^2 + |w_{x}|^2) dx dt \leq C.$$

**Proof.** Applying the operator $\partial_x$ to (1) gives

$$\rho_{xt} + \rho_{xx} u + 2\rho_x u_x + \rho u_{xx} = 0.$$

Multiplying the above equation by $2\rho_x$, integrating the result over $\Omega$, and using (13) and (28), we find that

$$\frac{d}{dt} \int \rho_x^2 dx = -3 \int \rho_x^2 u_x dx - 2 \int \rho \rho_x u_{xx} dx$$

$$= -3 \int \rho_x^2 \left(\lambda u_x - P - \frac{1}{2} |b|^2\right) dx - 3 \int \rho_x^2 \left(p + \frac{1}{2} |b|^2\right) dx$$

$$- 2 \int \rho \rho_x u_{xx} dx$$

$$\leq -3 \int \rho_x^2 \left(\lambda u_x - P - \frac{1}{2} |b|^2\right) dx$$

$$- 2 \int \rho \rho_x u_{xx} dx$$

$$\leq 3 \left\|\lambda u_x - P - \frac{1}{2} |b|^2\right\|_{L^\infty} \int \rho_x^2 dx + C \|\rho\|_{L^\infty} \|\rho_x\|_{L^2} \|u_{xx}\|_{L^2}$$

$$\leq C \left(\left\|\lambda u_x - P - \frac{1}{2} |b|^2\right\|_{L^2} + \left\|\lambda u_x - P - \frac{1}{2} |b|^2\right\|_{L^2}\right) \int \rho_x^2 dx$$

$$+ C \|\rho_x\|_{L^2} \|u_{xx}\|_{L^2}$$

$$\leq C (1 + \|\sqrt{\rho} u_t\|_{L^2}) \|\rho_x\|_{L^2}^2 + C \|u_{xx}\|_{L^2}^2.$$  

(39)

On the other hand, it follows from (2) that

$$\|u_{xx}\|_{L^2} \leq C(\|\sqrt{\rho} u_t\|_{L^2} + \|\rho u_x\|_{L^2} + \|P_x\|_{L^2} + \|b \cdot b_x\|_{L^2})$$

$$\leq C (1 + \|\sqrt{\rho} u_t\|_{L^2} + \|u_x\|_{L^2}^2 + \|\theta_x\|_{L^2} + \|\rho_x\|_{L^2} + \|\theta\|_{L^\infty})$$

$$\leq C (1 + \|\sqrt{\rho} u_t\|_{L^2} + \|u_x\|_{L^2}^2 + \|\kappa(\theta)\theta_x\|_{L^2} + \|\rho_x\|_{L^2} + \|\theta\|_{L^\infty}).$$  

(40)
Inserting (40) into (39), using Lemmas 13 and 2.5, (37), and the Gronwall inequality, we have
\[
\|\rho_x\|_{L^\infty(0,T;L^2(\Omega))} \leq C. \quad (41)
\]
It follows from (1), (13), (28) and (41) that
\[
\|\rho_t\|_{L^\infty(0,T;L^2(\Omega))} \leq C.
\]
Thus (40) yields
\[
\|u_{xx}\|_{L^2(0,T;L^2(\Omega))} \leq C.
\]
It follows from (3), (13), and (28) that
\[
\|w_{xx}\|_{L^2(0,T;L^2(\Omega))} \leq C.
\]

Lemma 2.7.
\[
\int (\rho^2_t + \rho|w_t|^2 + |b_t|^2 + u_{xx}^2 + |w_{xx}|^2 + |b_{xx}|^2 + \kappa^2(\theta)\theta_x^2)dx
\]
\[
+ \iint_{Q_T} \left( \lambda u_{xt}^2 + \mu|w_{xt}|^2 + \nu|b_{xt}|^2 + \rho\theta_t^2 + \theta_{xx}^2 \right)dxdt \leq C. \quad (42)
\]

Proof. Applying \(\partial_t\) to (2), we see that
\[
p\mu_{tt} + p\mu_{xt} - \lambda u_{tt} = - \left( P + \frac{1}{2} |b|^2 \right)_{xt} - \rho_t u_t - \rho_t u_{tx} - \rho_t u_x.
\]
Multiplying the above equation by \(u_t\), integrating the result over \(\Omega\), and using (1), Lemmas 2.5 and 2.6, and Cauchy inequality, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int \nabla u_t^2 dx + \int \lambda u_{xt}^2 dx
\]
\[
= \int \left( P + \frac{1}{2} |b|^2 \right) u_{xt} dx - 2 \int \rho u_t u_{xt} dx - \int \rho_t u_t u_x dx - \int \rho u_{xx}^2 dx
\]
\[
\leq \int (\rho \theta_t + \theta_{tt} + b \cdot b_t) u_{xt} dx + 2\sqrt{\rho} \|u_t\|_{L^2} \sqrt{\rho} \|u_{tt}\|_{L^\infty} \|u_t\|_{L^\infty} \|u_{xx}\|_{L^2}
\]
\[
+ \|\rho_t\|_{L^2} \|u_t\|_{L^\infty} \|u_x\|_{L^2} \|u_{xx}\|_{L^\infty} + \|u_x\|_{L^\infty} \int \rho u_{tt}^2 dx
\]
\[
\leq \epsilon_1 \int u_{tt}^2 dx + C \int \rho \theta_t^2 dx + \|\theta_t\|_{L^\infty}^2 + \|b_t\|_{L^2}^2
\]
\[
+ C \int \rho u_{tt}^2 dx + \|u_{xx}\|_{L^2} \int \rho u_t^2 dx + C \quad (43)
\]
for any \(0 < \epsilon_1 < 1\).

Multiplying (5) by \(\kappa(\theta)\theta_t = \left( \int_0^\theta k(\xi) d\xi \right) \), integrating them over \(\Omega\), and using (1) and Lemmas 2.5 and 2.6, we deduce that
\[
\frac{1}{2} \frac{d}{dt} \int \kappa^2(\theta)\theta_t^2 dx + \int \rho \kappa(\theta)\theta_t^2 dx
\]
\[
= - \int \rho u_t \kappa(\theta)\theta_t dx - \int \rho \theta u_{xt} \kappa(\theta)\theta_t dx
\]
\[
+ \int (\lambda u_{xx}^2 + \mu |w_{xx}|^2 + \lambda |b_{xx}|^2) \left( \int_0^\theta k(\xi) d\xi \right) dx
\]
Multiplying (45) by \(\theta\) and integrating the result over \(\Omega\), and using (1) and Lemmas 2.5 and 2.6, we have

\[
\begin{align*}
\leq & \epsilon_2 \int \rho\kappa(\theta)\theta^2 dx + C \int \rho u^2\kappa(\theta)\theta^2 dx + C \int \rho^2\kappa(\theta)u^2 dx \\
& + \frac{d}{dt} \left( \int (\lambda u_x^2 + \mu |w_x|^2 + \nu |b|^2) \int_0^\theta \kappa(\xi) d\xi dx \right) \\
& - 2 \int (u_x u_{xt} + w_x \cdot w_{xt} + b_x \cdot b_{xt}) \int_0^\theta \kappa(\xi) d\xi dx \\
\leq & \epsilon_2 \int \rho\kappa(\theta)\theta^2 dx + C \int \kappa\theta^2 u_x^2 + C\|u_x\|_{L^2}^2 \int \rho^2 \kappa(\theta) dx \\
& + \frac{d}{dt} \left( \int (\lambda u_x^2 + \mu |w_x|^2 + \nu |b|^2) \int_0^\theta \kappa(\xi) d\xi dx \right) \\
& + C\|u_x\|_{L^2}^2 \|w_x\|_{L^2}^2 + \|b_x\|_{L^2}^2 \\
\leq & \epsilon_2 \int \rho\kappa(\theta)\theta^2 dx + \epsilon_3(\|u_x\|_{L^2}^2 + \|w_x\|_{L^2}^2 + \|b_x\|_{L^2}^2) \\
& + \frac{d}{dt} \left( \int (\lambda u_x^2 + \mu |w_x|^2 + \nu |b|^2) \int_0^\theta \kappa(\xi) d\xi dx + C\|\theta(1 + \theta^3)\|_{L^2}^2 \right)
\end{align*}
\]

for any \(0 < \epsilon_2, \epsilon_3 < 1\).

Applying the operator \(\partial_t\) to (3) gives

\[
\rho w_{tt} + \rho u w_{xt} - \mu w_{xt} = -\rho_t w_1 - \rho u_t w_x - \rho u_t w_x + b_{xt}.
\]

Multiplying (45) by \(w_t\), integrating the result over \(\Omega\), and using (1) and Lemmas 2.5 and 2.6, we have

\[
\frac{1}{2} \frac{1}{2} \frac{d}{dt} \int \rho|w_t|^2 dx + \int \mu|w_{xt}|^2 dx \\
= -2 \int \rho u w_t \cdot w_{xt} dx - \int \rho u w_x \cdot w_t dx \\
- \int \rho u_t w_x \cdot w_t dx - \int b_x \cdot w_{xt} dx \\
\leq 2\|\sqrt{\rho} w_t\|_{L^2} \|\sqrt{\rho} u\|_{L^\infty} \|w_{xt}\|_{L^2} + \|\rho_t\|_{L^2} \|u\|_{L^\infty} \|w_x\|_{L^2} \|w_t\|_{L^\infty} \\
+ \|\sqrt{\rho} w_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|w_{xt}\|_{L^2} + \|b_x\|_{L^2} \|w_{xt}\|_{L^2} \\
\leq C\|\sqrt{\rho} w_t\|_{L^2} \|w_{xt}\|_{L^2} + C\|w_t\|_{L^\infty} \\
+ C\|\sqrt{\rho} w_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|w_{xt}\|_{L^2} + C\|w_{xt}\|_{L^2} \\
\leq \epsilon_4\|w_{xt}\|_{L^2}^2 + C + C\|\sqrt{\rho} w_t\|_{L^2}^2 + C\|w_{xt}\|_{L^2} \\
\leq \epsilon_4\|w_{xt}\|_{L^2}^2 + C + C\|\sqrt{\rho} w_t\|_{L^2}^2 + C\|w_{xt}\|_{L^2} \
\]

for any \(0 < \epsilon_4 < 1\).
Applying the operator $\partial_t$ to (4) gives

$$b_{tt} - \nu b_{xt} = -(ub - w)_{xt}.$$  

Multiplying the above equation by $b_t$, integrating the result over $\Omega$, and using Lemma 2.5, we find that

$$\frac{1}{2} \frac{d}{dt} \int |b_t|^2 dx + \int \nu |b_{xt}|^2 dx = \int (ub - w)_t \cdot b_{xt} dx$$

$$\leq \int (u_t b + b_t u - w_t) \cdot b_{xt} dx$$

$$\leq \left( \|b\|_{L^\infty} \|u_t\|_{L^2} + \|u\|_{L^\infty} \|b_t\|_{L^2} + \|w_t\|_{L^2} + \|b\|_{L^2} \right) \|b_{xt}\|_{L^2}$$

$$\leq C \left( \|u_t\|_{L^2} + \|b_t\|_{L^2} + \|w_t\|_{L^2} \right) \|b_{xt}\|_{L^2}$$

$$\leq C \left( \|u_{xt}\|_{L^2} + \|b_t\|_{L^2} + \|w_{xt}\|_{L^2} \right) \|b_{xt}\|_{L^2}$$

$$\leq \frac{\nu}{2} \|b_{xt}\|_{L^2}^2 + C \left( \lambda \|u_{xt}\|_{L^2}^2 + \mu \|w_{xt}\|_{L^2} + \|b_t\|_{L^2}^2 \right). \quad (47)$$

Combining (43), (44), (37), and (46) with (47), taking $\epsilon_i$ ($i = 1, \ldots, 4$) small enough, and integrating the resulting inequality over $(0, t)$, we conclude that

$$\int \rho u_x^2 + \rho |w|_t^2 + |b_t|^2 + \kappa^2 (\theta_x^2) dx + \int_{Q_T} \left( \lambda u_{xt}^2 + \mu |w_{xt}|^2 + \nu |b_{xt}|^2 + \rho \theta_t^2 \right) dx dt \leq C. \quad (48)$$

where we have used the following estimate:

$$\int_0^t \left( \frac{d}{dt} \int (\lambda u_x^2 + \mu |w_x|^2 + \nu |b_x|^2) \int_0^\theta \kappa(\xi) d\xi dx \right) d\tau$$

$$\leq \int (\lambda u_x^2 + \mu |w_x|^2 + \nu |b_x|^2) dx \int_0^\theta \kappa(\xi) d\xi \bigg|_{L^\infty} + C$$

$$\leq C \int_0^\theta \kappa(\xi) d\xi \bigg|_{L^\infty} + C$$

$$\leq C \left( \int_0^\theta \kappa(\xi) d\xi \right)_{L^2} + C$$

$$\leq C \kappa(\theta) \theta_x \|_{L^2} + C \leq \epsilon^5 \kappa(\theta) \theta_x \|_{L^2} + C$$

for any $0 < \epsilon_5 < 1$.

It follows from (2), (3), (4), (48), and Lemmas 2.5 and 2.6 that

$$\int (\lambda u_{xt}^2 + \mu |w_{xt}|^2 + \nu |b_{xt}|^2) dx \leq C.$$

Noting the above estimate, (5), and (48), it follows that

$$\|\theta_{xx}\|_{L^2} \leq C \int (\theta_x^4 + u_x^4 + |w_x|^4 + |b_x|^4 + \rho \theta_t^2 + u^2 \theta_x^2 + \theta^2 u_x^2) dx$$

$$\leq C + C \int \theta_x^2 dx + C \int \rho \theta_t dx$$
we infer that

\[
\int \theta_x^2 \, dx + C \int \rho \theta_t^2 \, dx
\]

which yields

\[
\| \theta_{xx} \|_{L^2}^2 \leq C + C \int \rho \theta_t^2 \, dx.
\]  \hspace{1cm} (49)

**Lemma 2.8.**

\[
\int (\rho_{xx}^2 + \rho_{xt}^2) \, dx + \int_{Q_T} (\rho_{tt}^2 + u_{xxt}^2) \, dxdt \leq C.
\]  \hspace{1cm} (50)

**Proof.** Applying the operator \( \partial_x^2 \) to (1) gives

\[
\rho_{xx} = -\rho_{xx}u_x - 3\rho_{xx}u_x - 3\rho_{xx}u_x - \rho_{xxx}.
\]

Multiplying the above equation by \( 2\rho_{xx} \), integrating the result over \( \Omega \), and using Lemmas 2.7 and 2.6, we find that

\[
\frac{d}{dt} \int \rho_{xx}^2 \, dx
\]

\[
= -5 \int \rho_{xx}^2 u_x \, dx - 6 \int \rho_x \rho_{xx} u_{xx} \, dx - 2 \int \rho \rho_{xx} u_{xxx} \, dx
\]

\[
\leq 5\| u_x \|_{L^\infty} \int \rho_{xx}^2 \, dx + 6\| \rho_x \|_{L^\infty} \| \rho_{xx} \|_{L^2} \| u_{xx} \|_{L^2} + 2\| \rho \|_{L^\infty} \| \rho_{xx} \|_{L^2} \| u_{xxx} \|_{L^2}
\]

\[
\leq C \int \rho_{xx}^2 \, dx + C \int u_{xxx}^2 \, dx + C.
\]  \hspace{1cm} (51)

Applying \( \partial_x \) to (2), integrating the result over \( \Omega \), and using Lemmas 2.6 and 2.7, we infer that

\[
\| u_{xxx} \|_{L^2}
\]

\[
\leq \left\| (\rho u)_{xt} + \rho u_{xxt} + \rho u_{uxx} + \rho u_{xx} + \rho u_{x} + 2\rho_x \theta_x + \rho \theta_{xx} + b \cdot b_{xx} + |b_x|^2 \right\|_{L^2}
\]

\[
\leq \| \rho_u \|_{L^\infty} \| u_t \|_{L^\infty} + \| \rho \|_{L^\infty} \| u_{xt} \|_{L^2} + \| \rho_x \|_{L^2} \| u \|_{L^\infty} \| u_x \|_{L^2} + \| \theta \|_{L^\infty} \| \rho_{xx} \|_{L^2}
\]

\[
+ 2\| \rho_x \|_{L^\infty} \| \theta_x \|_{L^2} + \| \rho \|_{L^\infty} \| \theta_{xx} \|_{L^2} + \| b \|_{L^\infty} \| b_{xx} \|_{L^2} + \| b_x \|_{L^2}
\]

\[
\leq C \| u_t \|_{L^\infty} + C \| u_{xt} \|_{L^2} + C + C \| \rho_{xx} \|_{L^2} + C \| \rho_x \|_{L^\infty} + C \| \theta_{xx} \|_{L^2}
\]

\[
\leq C \| u_{xx} \|_{L^2} + C + C \| \rho_{xx} \|_{L^2} + C \| \theta_{xx} \|_{L^2}.
\]

Inserting the above estimates into (51), and using Lemma 2.7 and the Gronwall inequality, we get

\[
\int \rho_{xx}^2 \, dx + \int_{Q_T} u_{xxt}^2 \, dxdt \leq C.
\]
Since
\[ \rho_{xt} = -(\rho u)_{xx}, \]
it is easy to show that
\[ \int \rho_{xx}^2 dx \leq C \int (\rho^2 u_{xx}^2 + \rho_{xx}^2 u_{x}^2 + \rho_{xx}^2 u_{xx}^2) dx \]
\[ \leq C \int (u_{xx}^2 + \rho_{xx}^2) dx + C \|u_x\|_\infty^2 \int \rho_x^2 dx \leq C. \]

Finally, noting
\[ \rho_{u} = -(\rho u)_{xt} = -(\rho_{xt}u + \rho_{u}x + \rho_{u}u + \rho_{u}tx), \]
it holds that
\[ \iint_{Q_T} \rho_{xt}^2 dx dt \leq C \|u\|_{L^\infty(\Omega)}^2 \iint_{Q_T} \rho_{xt}^2 dx dt + C \|\rho_t\|_{L^\infty(\Omega)}^2 \iint_{Q_T} u_{xt}^2 dx dt \]
\[ + C \|\rho_x\|_{L^\infty(\Omega)}^2 \iint_{Q_T} u_{x}^2 dx dt + C \|\rho\|_{L^\infty}^2 \iint_{Q_T} u_{xt}^2 dx dt \]
\[ \leq C \iint_{Q_T} (\rho_{xt}^2 + u_{x}^2 + u_{t}^2 + u_{xt}^2) dx dt \]
\[ \leq C + C \iint_{Q_T} u_{xt}^2 dx dt \leq C. \]

\[ \square \]

**Lemma 2.9.**
\[ \int \rho_{t}^2 dx + \iint_{Q_T} |(\kappa(\theta)\theta_x)_t|^2 dx dt \leq C. \]  (52)

**Proof.** Applying \( \partial_t \) to (5) gives
\[ \rho \theta_{tt} + \rho u \theta_{xt} - (\kappa(\theta) \theta_{x})_{xt} \]
\[ = 2(u_x u_{xt} + w_x w_{xt} + b_x b_{xt}) - \rho u_{xt} - \rho_t u_x - \rho_u \theta_x - \rho_{u}\theta_x - \rho_{u\theta_x}. \]

Multiplying the above equation by \( \kappa(\theta) \theta_t = \left( \int_{0}^{\theta} \kappa(\xi) d\xi \right)_t \), integrating them over \( \Omega \), and using (1), \( (\kappa_{\theta}x) = (\kappa_{\theta}x), \) Lemmas 2.7 and 2.8, we infer that
\[ \frac{1}{2} \frac{d}{dt} \int \rho \kappa(\theta) \theta_t^2 dx + \int \left| (\kappa(\theta))_{x} \right|^2 dx \]
\[ = \frac{1}{2} \int \rho \theta_t^2 (\kappa(\theta))_t dx + \frac{1}{2} \int \rho u \theta_t (\kappa(\theta))_{x} dx \]
\[ + 2 \int (u_x u_{xt} + w_x w_{xt} + b_x b_{xt}) \kappa(\theta) \theta_t dx \]
\[ - \int (P u_{xt} + P_t u_x + \rho_t \theta_t + \rho_{u} \theta_x + \rho_{u} u_x + \rho_{u} \theta_x) \kappa(\theta) \theta_t dx \]
\[ \leq \frac{1}{2} \| \kappa(\theta) \|_{L^\infty} \int \rho \theta_t^2 dx + \frac{1}{2} \int \rho \theta_t^2 dx \|u\|_{L^\infty} \| (\kappa(\theta))_{x} \|_{L^\infty} \]
\[ + 2 (\|u_x\|_{L^2} \|u_{xt}\|_{L^2} + \|w_x\|_{L^2} \|w_{xt}\|_{L^2} + \|b_x\|_{L^2} \|b_{xt}\|_{L^2}) \| \kappa(\theta) \theta_t \|_{L^\infty} \]
\[ + (\|P\|_{L^\infty} \|u\|_{L^2} + \|P_t\|_{L^2} \|\theta\|_{L^\infty} \|u\|_{L^\infty} \]
\[ + \|\sqrt{\rho} \theta_t\|_{L^2} \|\sqrt{\rho}\|_{L^\infty} \|u_x\|_{L^\infty} \| \kappa(\theta) \theta_t \|_{L^\infty} \]
\[ + \int (\rho u)_{x} \kappa \theta_t^2 dx + \|\rho_{t}\|_{L^2} \|u\|_{L^\infty} \| \theta_x \|_{L^2} \| \kappa(\theta) \theta_t \|_{L^\infty} \]
Inserting them into (53) and using the Gronwall inequality, we arrive at (52).

Lemma 2.10.

It is easy to verify that

$$\int \theta^2_{xx} dx + \int \int_{Q_T} \theta_{xx}^2 dxdt \leq C. \tag{56}$$

Proof. It follows from (49) and (52) that

$$\int \theta_{xx}^2 dx \leq C. \tag{57}$$

It is easy to verify that

$$\int_0^T \theta_t^2 dx dt \leq \int_0^T \| \theta (\theta_x) \|^2_{L_\infty} dx dt \leq C \int_0^T \| (\theta_x) \|^2_{L_2} dx dt \leq C. \tag{58}$$

Since

$$\theta_{xt} = (\theta (\theta_x))_x - \theta' (\theta_x) \theta_x,$$

by applying (58) and Cauchy inequality, we have

$$\int \int_{Q_T} \theta_{xx}^2 dxdt \leq C \int \int_{Q_T} \theta_t^2 \theta_x^2 dxdt \leq C \int \int_{Q_T} \theta_t^2 \theta_x^2 dxdt \leq C + \int \int_0^T \| \theta_t \|^2_{L_\infty} \| \theta_x \|^2_{L_2} dx dt \leq C + \int \int_0^T \| \theta_x \|^2_{L_\infty} dx dt \leq C. \tag{59}$$
Applying the operator $\partial_x$ to (5) gives
\[
\kappa(\theta) \theta_{xx} = -3\kappa'(\theta) \theta_x \theta_{xx} - \kappa''(\theta) \theta_x^3 - 2(\lambda u_x u_{xx} + \mu w_x \cdot w_{xx} + \nu b_x \cdot b_{xx}) \\
- \rho_x \theta_t - \rho \theta_{xt} - (\rho \theta u_x)_{x} - (\rho \theta u_x)_{x},
\]
whence
\[
\int \theta_x^2 dx \leq \int \kappa(\theta) \theta_x^2 dx \leq C \int \theta_x^2 dx + C \int \theta_x^2 dx \\
+ C \int (u_x^2 u_{xx} + |w_x|^2 |w_{xx}|^2 + |b_x|^2 |b_{xx}|^2) dx \\
+ C \int \rho_x^2 dx + C \|\theta_x\|_{L^\infty}^2 \int \theta_x^2 dx + C \|\rho u\|_{L^\infty}^2 \int \theta_x^2 dx \\
+ C \|\rho\|_{L^\infty}^2 \|\theta\|_{L^\infty}^2 \|u_x\|_{L^2}^2 + C \|\rho\|_{L^\infty}^2 \|\theta_x\|_{L^2}^2 \|u_x\|_{L^2}^2 \\
+ C \|\rho\|_{L^\infty}^2 \|\theta\|_{L^2}^2 \|u_{xx}\|_{L^2}^2 \\
\leq C + C \int \theta_x^2 dx + C \int \theta_x^2 dx,
\]
by Lemma 2.7, Lemma 2.8 and Lemma 2.9.

The estimates (58), (59) and (60) imply
\[
\int_0^T \int_{Q_T} \theta_x^2 dx dt \leq C.
\]

By combining all the estimates obtained above, we get sufficient a priori estimates uniformly with $\delta$ to take the limit $\delta \to 0^+$ and then extend the local strong solutions to be global one. Since the process is standard (see, for example [6, 11]), we omit them here for brevity. Hence the proof of Theorem 1.1 is completed.

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