Unifying Treatment of Discord via Relative Entropy

Lin Zhang*

Institute of Mathematics, Hangzhou Dianzi University, Hangzhou 310018, P.R. China

Shao-Ming Fei†

School of Mathematical Sciences, Capital Normal University, Beijing 100048, P.R. China

Jun Zhu

Institute of Mathematics, Hangzhou Dianzi University, Hangzhou 310018, P.R. China

Abstract

A new form of zero-discord state via Petz’s monotonicity condition on relative entropy with equality has been derived systematically. A generalization of symmetric zero-discord states is presented and the related physical implications are discussed.

1 Introduction

Relative entropy are powerful tools in quantum information theory [1]. It has a monotonicity property under a certain class of quantum channels and the condition of equality is an interesting and important subject. It is Petz who first studied the equality condition of monotonicity of relative entropy [2, 3]. Later Ruskai obtained similar result in terms of another elegant approach [4]. The most general equality condition along with this line are recently reviewed in [5].

In this note we will make use of the most general equality condition for relative entropy to find the specific form of states which satisfy the zero-discord condition (see details below).

*E-mail: godyalin@163.com
†E-mail: feishm@mail.cnu.edu.cn
Let $H$ denote an $N$-dimensional complex Hilbert space. A state $\rho$ on $H$ is a positive semi-definite operator of trace one. We denote $D(H)$ the set of all the density matrices acting on $H$. If $\rho = \sum_k \lambda_k |u_k\rangle \langle u_k|$ is the spectral decomposition of $\rho$, with $\lambda_k$ and $|u_k\rangle$ the eigenvalues and eigenvectors respectively, then the support of $\rho$ is defined by

$$\text{supp}(\rho) \overset{\text{def}}{=} \text{span}\{|u_k\rangle : \lambda_k > 0\},$$

and the generalized inverse $\rho^{-1}$ of $\rho$ is defined by

$$\rho^{-1} = \sum_{k: \lambda_k > 0} \lambda_k^{-1} |u_k\rangle \langle u_k|.$$

The von Neumann entropy $S(\rho)$ of $\rho$ is defined by

$$S(\rho) \overset{\text{def}}{=} -\text{Tr}(\rho \log \rho),$$

which quantifies information encoded in the quantum state $\rho$. If $\sigma$ is also a quantum state on $H$, then the relative entropy [1] between $\rho$ and $\sigma$ is defined by

$$S(\rho||\sigma) \overset{\text{def}}{=} \begin{cases} \text{Tr}(\rho(\log \rho - \log \sigma)), & \text{if supp}(\rho) \subseteq \text{supp}(\sigma), \\ +\infty, & \text{otherwise}. \end{cases}$$

Let $L(H)$ be the set of all linear operators on $H$. If $X, Y \in L(H)$, then $\langle X, Y \rangle = \text{Tr}(X^\dagger Y)$ defines the Hilbert-Schmidt inner product on $L(H)$. Let $T(H)$ denote the set of all linear super-operators from $L(H)$ to itself. $\Lambda \in T(H)$ is said to be a completely positive super-operator if for each $k \in \mathbb{N}$,

$$\Lambda \otimes 1_{M_k(\mathbb{C})} : L(H) \otimes M_k(\mathbb{C}) \rightarrow L(H) \otimes M_k(\mathbb{C})$$

is positive, where $M_k(\mathbb{C})$ is the set of all $k \times k$ complex matrices. It follows from Choi’s theorem [6] that every completely positive super-operator $\Lambda$ has a Kraus representation

$$\Lambda = \sum_{\mu} \text{Ad}_{M_\mu},$$

that is, for every $X \in L(H)$, $\Lambda(X) = \sum_{\mu} M_\mu X M_\mu^\dagger$, where $\{M_\mu\} \subseteq L(H)$, $M_\mu^\dagger$ is the adjoint operator of $M_\mu$. It is clear that for the super-operator $\Lambda$, there is adjoint super-operator $\Lambda^\dagger \in T(H)$ such that for $A, B \in L(H)$, $\langle \Lambda(A), B \rangle = \langle A, \Lambda^\dagger(B) \rangle$. Moreover, $\Lambda$ is a completely positive super-operator if and only if $\Lambda^\dagger$ is also a completely positive super-operator. A quantum channel is just a trace-preserving completely positive super-operator $\Phi$. If $\Phi$ is also unit-preserving, then it is called unital quantum channel.

It has been reviewed in [5] that,
Lemma 1.1. Let \( \rho, \sigma \in D(\mathcal{H}) \), \( \Phi \in T(\mathcal{H}) \) be a quantum channel. If \( \text{supp}(\rho) \subseteq \text{supp}(\sigma) \), then \( S(\Phi(\rho)||\Phi(\sigma)) \leq S(\rho||\sigma) \); moreover

\[
S(\Phi(\rho)||\Phi(\sigma)) = S(\rho||\sigma) \quad \text{if and only if} \quad \Phi^\dagger \sigma \circ \Phi(\rho) = \rho,
\]

where \( \Phi^\dagger = \text{Ad}_{\sigma^{1/2}} \circ \Phi^\dagger \circ \text{Ad}_{\sigma^{-1/2}} \).

Moreover, for a tripartite state, one has \([7, 8]\).

Lemma 1.2. Let \( \rho_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C) \) for which strong subadditivity is saturated for both triples \( ABC \), \( BAC \). Then \( \rho_{ABC} \) must have the following form:

\[
\rho_{ABC} = \bigoplus_{i,j} p_{ij} \rho_{a_i}^{(i)} \otimes \rho_{b_j}^{(i)} \otimes \rho_{(k)}^c,
\]

where \( k \) is a function only of \( i, j \) in the sense that

\[
k = k(i,j) = k_1(i) = k_2(j) \quad \text{whenever} \quad p_{ij} > 0.
\]

In particular, \( k \) need only be defined where \( p_{ij} > 0 \) so that it is not necessarily constant. By collecting the terms of equivalent \( k \) we can write

\[
\rho_{ABC} = \bigoplus_k p_k \rho_{AB}^{(k)} \otimes \rho_{C}^{(k)},
\]

where

\[
p_k \rho_{AB}^{(k)} = \sum_{i,j,k(i,j) = k} p_{ij} \rho_{a_i}^{(i)} \otimes \rho_{b_j}^{(i)} \otimes \rho_{b_j}^{(j)}.
\]

2 Quantum discord

Consider a bipartite system \( AB \) composed of subsystems \( A \) and \( B \). Let \( \rho_{AB} \) be the density operator of \( AB \), and \( \rho_A \) and \( \rho_B \) the reduced density operators. The total correlation between the systems \( A \) and \( B \) is measured by the quantum mutual information

\[
I(\rho_{AB}) = S(\rho_A) - S(\rho_A|\rho_B),
\]

where

\[
S(\rho_A|\rho_B) = S(\rho_{AB}) - S(\rho_B)
\]

is the entropy of \( A \) conditional on \( B \). The conditional entropy can also be introduced by a measurement-based approach. Consider a measurement locally performed on \( B \),
which can be described by a set of projectors \( \Pi_B = \{ \Pi_{B,\mu} \} = \{|b_\mu\rangle\langle b_\mu|\} \). The state of the quantum system, conditioned on the measurement of the outcome labeled by \( \mu \), is

\[
\rho_{AB,\mu} = \frac{1}{p_{B,\mu}} (\mathbb{1}_A \otimes \Pi_{B,\mu}) \rho_{AB} (\mathbb{1}_A \otimes \Pi_{B,\mu}) ,
\]

where

\[
p_{B,\mu} = \text{Tr} \left( (\mathbb{1}_A \otimes \Pi_{B,\mu}) \rho_{AB} (\mathbb{1}_A \otimes \Pi_{B,\mu}) \right) = \langle b_\mu | \rho_{AB} | b_\mu \rangle > 0
\]
denotes the probability of obtaining the outcome \( \mu \), and \( \mathbb{1}_A \) denotes the identity operator for \( A \). The conditional density operator \( \rho_{AB,\mu} \) allows for the following alternative definition of the conditional entropy:

\[
S(\rho_{AB} | \{ \Pi_{B,\mu} \}) = \sum_\mu p_{B,\mu} S(\rho_{AB,\mu}) = \sum_\mu p_{B,\mu} S(\rho_{A,\mu}) ,
\]

where

\[
\rho_{A,\mu} = \text{Tr}_B (\rho_{AB,\mu}) = (1/p_{B,\mu}) \langle b_\mu | \rho_{AB} | b_\mu \rangle .
\]

Therefore, the quantum mutual information can also be defined by

\[
I(\rho_{AB} | \{ \Pi_{B,\mu} \}) = S(\rho_A) - S(\rho_{AB} | \{ \Pi_{B,\mu} \}) .
\]

The quantities \( I(\rho_{AB}) \) and \( I(\rho_{AB} | \{ \Pi_{B,\mu} \}) \) are classically equivalent but distinct in the quantum case.

The one-sided quantum discord is defined by:

\[
D_B(\rho_{AB}) = \inf_{\Pi_B} \left\{ I(\rho_{AB}) - I(\rho_{AB} | \Pi_B) \right\} .
\]

If we denote the nonselective von Neumann measurement performed on \( B \) by

\[
\Pi_B(\rho_{AB}) = \sum_\mu (\mathbb{1}_A \otimes \Pi_{B,\mu}) \rho_{AB} (\mathbb{1}_A \otimes \Pi_{B,\mu}) = \sum_\mu p_{B,\mu} \rho_{A,\mu} \otimes |b_\mu\rangle\langle b_\mu| ,
\]

then the quantum discord can be written alternatively as

\[
D_B(\rho_{AB}) = \inf_{\Pi_B} \left\{ S(\rho_{AB} | |\rho_A \otimes \rho_B\rangle\langle \rho_A \otimes \rho_B|) - S(\Pi_B(\rho_{AB}) | |\rho_A \otimes \rho_B\rangle\langle \rho_A \otimes \rho_B|) \right\}
\]

\[
= \inf_{\Pi_B} \left\{ S(\rho_{AB} | \Pi_B(\rho_{AB})) - S(\rho_B | |\Pi_B(\rho_B)\rangle\langle \rho_B|) \right\} .
\]

Apparently, \( D_B(\rho_{AB}) \geq 0 \) from Lemma [11].

The symmetric quantum discord of \( \rho_{AB} \) is defined by [9],

\[
D(\rho_{AB}) = \inf_{\Pi_A \otimes \Pi_B} \left\{ S(\rho_{AB} | |\Pi_A \otimes \Pi_B(\rho_{AB})\rangle\langle \rho_A \Pi_B(\rho_B)|) - S(\rho_A | |\Pi_A(\rho_A)\rangle\langle \rho_A|) - S(\rho_B | |\Pi_B(\rho_B)\rangle\langle \rho_B|) \right\} .
\]
For the symmetric quantum discord of $\rho_{AB}$, one still has that

$$D(\rho_{AB}) = \inf_{\Pi_A \otimes \Pi_B} \{ S(\rho_{AB}||\rho_A \otimes \rho_B) - S(\Pi_A \otimes \Pi_B(\rho_{AB}))||\Pi_A \otimes \Pi_B(\rho_A \otimes \rho_B)) \}. \tag{2.1}$$

The symmetric quantum discord of $\rho_{A_1...A_N}$ for $N$-partite systems are defined by:

$$D(\rho_{A_1...A_N})$$

$$= \inf_{\Pi_{A_1} \otimes \cdots \otimes \Pi_{A_N}} \left\{ S(\rho_{A_1...A_N}||\Pi_{A_1} \otimes \cdots \otimes \Pi_{A_N}(\rho_{A_1...A_N})) - \sum_{i=1}^{N} S(\rho_{A_i}||\Pi_{A_i}(\rho_{A_i})) \right\}$$

$$= \inf_{\Pi_{A_1} \otimes \cdots \otimes \Pi_{A_N}} \left\{ S(\rho_{A_1...A_N}||\rho_{A_1} \otimes \cdots \otimes \rho_{A_N}) - S(\Pi_{A_1} \otimes \cdots \otimes \Pi_{A_N}(\rho_{A_1...A_N}||\Pi_{A_1} \otimes \cdots \otimes \Pi_{A_N}(\rho_{A_1} \otimes \cdots \otimes \rho_{A_N}))) \right\},$$

which is non-negative, $D(\rho_{A_1...A_N}) \geq 0$.

The following theorem describes the structure of symmetric zero-discord states:

**Theorem 2.1.** $D(\rho_{AB}) = 0$ if and only if

$$\rho_{AB} = \sum_{\mu,\nu} \frac{p_{AB,\mu\nu}}{p_{A,\mu}p_{B,\nu}} \sqrt{\rho_A \Pi_{A,\mu}} \sqrt{\rho_B \Pi_{B,\nu}} \sqrt{\rho_B}$$

for both von Neumann measurements $\Pi_A = \{\Pi_{A,\mu}\}$ and $\Pi_B = \{\Pi_{B,\nu}\}$, where

$$p_{A,\mu} = \text{Tr}(\Pi_{A,\mu} \rho_A), \quad p_{B,\nu} = \text{Tr}(\Pi_{B,\nu} \rho_B), \quad p_{AB,\mu\nu} = \text{Tr}(\Pi_{A,\mu} \otimes \Pi_{B,\nu} \rho_{AB}).$$

**Proof.** Clearly, $\text{supp} (\rho_{AB}) \subseteq \text{supp} (\rho_A) \otimes \text{supp} (\rho_B) = \text{supp} (\rho_A \otimes \rho_B)$ [10]. Since $D(\rho_{AB}) = 0$, from Eq. (2.1), it follows that there exist two von Neumann measurement $\Pi_A = \{\Pi_{A,\mu}\}$ and $\Pi_B = \{\Pi_{B,\nu}\}$ such that

$$S(\Pi_A \otimes \Pi_B(\rho_{AB})||\Pi_A \otimes \Pi_B(\rho_A \otimes \rho_B)) = S(\rho_{AB}||\rho_A \otimes \rho_B).$$

Assume that $\sigma = \rho_A \otimes \rho_B, \Phi = \Pi_A \otimes \Pi_B$ in Lemma 1.1 Therefore $D(\rho_{AB}) = 0$ if and only if

$$S(\Pi_A \otimes \Pi_B(\rho_{AB})||\Pi_A \otimes \Pi_B(\rho_A \otimes \rho_B)) = S(\rho_{AB}||\rho_A \otimes \rho_B).$$

Namely

$$\rho_{AB} = \Phi_{\sigma}^\dagger \circ \Phi(\rho_{AB}) = ((\Pi_{A,\rho_A}^\dagger \circ \Pi_A) \otimes (\Pi_{B,\rho_B}^\dagger \circ \Pi_B))(\rho_{AB})$$

Therefore

$$\rho_{AB} = \sum_{\mu,\nu} \frac{p_{AB,\mu\nu}}{p_{A,\mu}p_{B,\nu}} \sqrt{\rho_A \Pi_{A,\mu}} \sqrt{\rho_B \Pi_{B,\nu}} \sqrt{\rho_B}.$$
Accordingly we have

**Corollary 2.2.** \(D_B(\rho_{AB}) = 0\) if and only if

\[
\rho_{AB} = \sum_{\mu} \rho_{A,\mu} \otimes \sqrt{\rho_B} \Pi_{B,\mu} \sqrt{\rho_B}
\]  

(2.2)

for some von Neumann measurement \(\Pi_B = \{\Pi_{B,\mu}\}\), where

\[
\rho_{A,\mu} = \frac{1}{p_{B,\mu}} \text{Tr}_B (\mathbb{1}_A \otimes \Pi_{B,\mu} \rho_{AB}), \quad p_{B,\mu} = \text{Tr} (\Pi_{B,\mu} \rho_B).
\]

**Remark 2.3.** Suppose that the von Neumann measurement in Eq. (2.2) is \(\Pi_B = \{\Pi_{B,\mu}\} = \{|b_\mu\rangle\langle b_\mu|\}\). Then we can assert that \(|b_\mu\rangle\) is the eigenvectors of \(\rho_B\). This can be seen as follows. From Eq. (2.2) it follows that

\[
\Pi_B(\rho_{AB}) = \sum_{\mu} \rho_{A,\mu} \otimes \Pi_B (\sqrt{\rho_B} \Pi_{B,\mu} \sqrt{\rho_B}).
\]

(2.3)

Actually,

\[
\Pi_B(\rho_{AB}) = \sum_{\mu} (\mathbb{1}_A \otimes \Pi_{B,\mu}) \rho_{AB} (\mathbb{1}_A \otimes \Pi_{B,\mu}) = \sum_{\mu} p_{B,\mu} \rho_{A,\mu} \otimes \Pi_{B,\mu}.
\]

(2.4)

From Eq. (2.3) and Eq. (2.4), we have

\[
\Pi_B (\sqrt{\rho_B} \Pi_{B,\mu} \sqrt{\rho_B}) = p_{B,\mu} \Pi_{B,\mu},
\]

which implies that

\[
\begin{cases}
\Pi_{B,\mu} \sqrt{\rho_B} \Pi_{B,v} \sqrt{\rho_B} \Pi_{B,\mu} = 0, & \text{if } \mu \neq v, \\
\Pi_{B,\mu} \sqrt{\rho_B} \Pi_{B,\mu} \sqrt{\rho_B} \Pi_{B,\mu} = p_{B,\mu} \Pi_{B,\mu}, & \text{otherwise.}
\end{cases}
\]

(2.5)

That is,

\[
\begin{cases}
\left| \langle b_\mu | \sqrt{\rho_B} | b_v \rangle \right|^2 = 0 \quad \text{if } \mu \neq v, \\
\langle b_\mu | \sqrt{\rho_B} | b_\mu \rangle = \sqrt{p_{B,\mu}} = \sqrt{\langle b_\mu | \rho_B | b_\mu \rangle} \quad \text{otherwise.}
\end{cases}
\]

Thus we conclude that \(|b_\mu\rangle\) is the eigenvectors of \(\rho_B\).

For general multipartite case we have
Corollary 2.4. \( D(\rho_{A_1\ldots A_N}) = 0 \) if and only if

\[
\rho_{A_1\ldots A_N} = \sum_{\mu_1,\ldots,\mu_N} \frac{p_{A_1\ldots A_N,\mu_1\ldots \mu_N}}{p_{A_1,\mu_1} \cdots p_{A_N,\mu_N}} \sqrt{\rho_{A_1}} \Pi_{A_1,\mu_1} \sqrt{\rho_{A_1}} \otimes \cdots \otimes \sqrt{\rho_{A_N}} \Pi_{A_N,\mu_N} \sqrt{\rho_{A_N}}
\]

for \( N \) von Neumann measurements \( \Pi_{A_i} = \{ \Pi_{A_i,\mu_i} \} \), where

\[
p_{A_i,\mu_i} = \text{Tr} \left( \Pi_{A_i,\mu_i} \rho_{A_i} \right) (i = 1, \ldots, N), \quad p_{A_1\ldots A_N,\mu_1\ldots \mu_N} = \text{Tr} \left( \Pi_{A_1,\mu_1} \otimes \cdots \otimes \Pi_{A_N,\mu_N} \rho_{A_1\ldots A_N} \right).
\]

In order to obtain a connection with strong subadditivity of quantum entropy \([11]\), we associate each von Neumann measurement \( \Pi_B = \{ \Pi_{B,\mu} \} \) with a system \( C \) as follows:

\[
\sigma_{ABC} = V \rho_{AB} V^+ = \sum_{\mu,\nu} (1_A \otimes \Pi_{B,\mu}) \rho_{AB} (1_A \otimes \Pi_{B,\nu}) \otimes |\mu\rangle \langle \nu|_C,
\]

where

\[
V|\psi_C\rangle \overset{\text{def}}{=} \sum_{\mu} \Pi_{B,\mu} |\psi_B\rangle \otimes |\mu\rangle_C
\]

is an isometry from \( B \) to \( BC \). From Eq. (2.6) we have

\[
\sigma_{AB} = \text{Tr}_C \left( V \rho_{AB} V^+ \right) = \Pi_B(\rho_{AB}) = \sum_{\mu} p_{B,\mu} \rho_{A,\mu} \otimes \Pi_{B,\mu},
\]

\[
\sigma_{BC} = \text{Tr}_A \left( V \rho_{AB} V^+ \right) = \sum_{\mu,\nu} \Pi_{B,\mu} \Pi_{B,\nu} \otimes |\mu\rangle \langle \nu|_C,
\]

\[
\sigma_B = \sum_{\mu} p_{B,\mu} \Pi_{B,\mu},
\]

where \( p_{B,\mu} = \text{Tr} \left( \rho_B \Pi_{B,\mu} \right) \). This implies that the conditional mutual information between \( A \) and \( C \) conditioned on \( B \) is

\[
I(A; C | B)_\sigma \overset{\text{def}}{=} S(\sigma_{AB}) + S(\sigma_{BC}) - S(\sigma_{ABC}) - S(\sigma_B)
\]

\[
= \sum_{\mu} p_{B,\mu} S(\rho_{A,\mu}) + S(\rho_B) - S(\rho_{AB})
\]

\[
= S(\rho_{AB} || \rho_A \otimes \rho_B) - S(\Pi_B(\rho_{AB})) || \rho_A \otimes \Pi_B(\rho_B)).
\]

Similarly we have

\[
I(A; B | C)_\sigma = S(\rho_{AB} || \rho_A \otimes \rho_B) - S(\Pi_B(\rho_{AB})) || \rho_A \otimes \Pi_B(\rho_B)).
\]

That is,

\[
I(A; C | B)_\sigma = I(A; B | C)_\sigma = S(\rho_{AB} || \rho_A \otimes \rho_B) - S(\Pi_B(\rho_{AB})) || \rho_A \otimes \Pi_B(\rho_B)). \tag{2.7}
\]
If Eq. (2.7) vanishes for some von Neumann measurement $\Pi_B = \{\Pi_{B,\mu}\}$, $I(A;C|B)_\sigma = I(A;B|C)_\sigma = 0$, then from Lemma 1.2(i),
\[
\sigma_{ABC} = \bigoplus_k p_k \sigma^{(k)}_A \otimes \sigma^{(k)}_{BC}.
\]
If $D_B(\rho_{AB}) = S(\rho_{AB}||\rho_A \otimes \rho_B) - S(\Pi_B(\rho_{AB})||\rho_A \otimes \Pi_B(\rho_B))$ for some von Neumann measurement $\Pi_B$, then
\[
D_B(\rho_{AB}) = I(A;B|C)_\sigma.
\]
There exists a famous protocol—state redistribution—which gives an operational interpretation of conditional mutual information $I(A;B|C)_\sigma$ [12, 13]. This amounts to give implicitly an operational interpretation of quantum discord [14, 15].

3 A generalization of zero-discord states

Denote
\[
\Omega_0 A \overset{\text{def}}{=} \{\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B) : D_A(\rho_{AB}) = 0\},
\]
\[
\Omega_0 \overset{\text{def}}{=} \{\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B) : D(\rho_{AB}) = 0\}.
\]
Suppose $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, with two marginal density matrices are $\rho_A = \text{Tr}_B(\rho_{AB})$ and $\rho_B = \text{Tr}_A(\rho_{AB})$, respectively. A sufficient condition for zero-discord states has been derived in [16]: if $\rho_{AB} \in \Omega_0 A$, then $[\rho_{AB}, \rho_A \otimes \mathbb{1}_B] = 0$.

A characterization of $[\rho_{AB}, \rho_A \otimes \mathbb{1}_B] = 0$ is obtained in [17], $[\rho_{AB}, \rho_A \otimes \mathbb{1}_B] = 0$ if and only if $\rho_{AB} = \Pi_A(\rho_{AB})$, where $\Pi_A = \{\Pi_{A,\mu}\}$ is some positive valued measurement for which each projector $\Pi_{A,\mu}$ is of any rank. That is,
\[
\rho_{AB} = \sum_{\mu} (\Pi_{A,\mu} \otimes \mathbb{1}_B) \rho_{AB} (\Pi_{A,\mu} \otimes \mathbb{1}_B).
\]
States $\rho_{AB}$ such that $[\rho_{AB}, \rho_A \otimes \mathbb{1}_B] = 0$ are called lazy ones with particular physical interpretations [17]. Consider general evolution of the state in a finite-dimensional composite system $AB$:
\[
\frac{d}{dt}\rho_{AB,t}\bigg|_{t=\tau} = -i[H_{AB}, \rho_{AB,\tau}],
\]
where the total Hamiltonian is $H_{AB} \equiv H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B + H_{\text{int}}$, which consists of the system, the environment and the interaction Hamiltonians. Clearly, it is required that
$$\text{Tr}_A (H_{\text{int}}) = \text{Tr}_B (H_{\text{int}}) = 0.$$ For the system $A$, the change rate of the system entropy at a time $\tau$ is given by [16]:

$$\left[ \frac{d}{dt} S(\rho_{A,\tau}) \right]_{t=\tau} = -i \text{Tr} \left( H_{\text{int}} [\rho_{AB,\tau}, \log_2 (\rho_{A,\tau} \otimes 1_B)] \right).$$  \hspace{1cm} (3.1)

Since the von Neumann entropy $S(\rho_X)$ of $\rho_X$ quantifies the degree of decoherence of the system $X = A, B$, it follows that the system entropy rates are independent of the $AB$ coupling if and only if

$$\left[ \frac{d}{dt} S(\rho_{A,\tau}) \right]_{t=\tau} = 0,$$

which is equivalent to the following expression:

$$[\rho_{AB,\tau}, \log_2 (\rho_{A,\tau} \otimes 1_B)] = 0 \iff [\rho_{AB,\tau}, \rho_{A,\tau} \otimes 1_B] = 0.$$

In view of this point, the entropy of quantum systems can be preserved from decoherence under any coupling between $A$ and $B$ if and only if the composite system states are lazy ones.

From the symmetry with respect to $A$ and $B$, one has

$$\left[ \frac{d}{dt} S(\rho_{B,\tau}) \right]_{t=\tau} = -i \text{Tr} \left( H_{\text{int}} [\rho_{AB,\tau}, 1_A \otimes \log(\rho_{B,\tau})] \right).$$  \hspace{1cm} (3.2)

Due to that

$$\left[ \frac{d}{dt} I(\rho_{AB,\tau}) \right]_{t=\tau} = \left[ \frac{d}{dt} S(\rho_{A,\tau}) \right]_{t=\tau} + \left[ \frac{d}{dt} S(\rho_{B,\tau}) \right]_{t=\tau} - \left[ \frac{d}{dt} S(\rho_{AB,\tau}) \right]_{t=\tau},$$

and

$$\left[ \frac{d}{dt} S(\rho_{AB,\tau}) \right]_{t=\tau} = 0,$$

we have further

$$\left[ \frac{d}{dt} I(\rho_{AB,\tau}) \right]_{t=\tau} = -i \text{Tr} \left( H_{\text{int}} [\rho_{AB,\tau}, \log(\rho_{A,\tau} \otimes \rho_{B,\tau})] \right).$$  \hspace{1cm} (3.3)

We can see from Eq. (3.3) that the total correlation is preserved under any coupling between $A$ and $B$ if and only if the mutual entropy rate of composite system $AB$ is zero:

$$\left[ \frac{d}{dt} I(\rho_{AB,\tau}) \right]_{t=\tau} = 0,$$

which is equivalent to the following expression:

$$[\rho_{AB,\tau}, \log(\rho_{A,\tau} \otimes \rho_{B,\tau})] = 0 \iff [\rho_{AB,\tau}, \rho_{A,\tau} \otimes \rho_{B,\tau}] = 0.$$

Similarly, we have:
Proposition 3.1. If $\rho_{AB} \in \Omega^0$, then $[\rho_{AB}, \rho_A \otimes \rho_B] = 0$.

Moreover,

Proposition 3.2. $[\rho_{AB}, \rho_A \otimes \rho_B] = 0$ if and only if $\rho_{AB} = \Pi_A \otimes \Pi_B(\rho_{AB})$, where $\Pi_X = \{\Pi_{X,\alpha}\}$ are some PVM for which each projector $\Pi_{X,\alpha}$, where $(X,\alpha) = (A,\mu), (B,\nu)$, are of any rank. That is,

$$\rho_{AB} = \sum_{\mu,\nu} (\Pi_{A,\mu} \otimes \Pi_{B,\nu}) \rho_{AB} (\Pi_{A,\mu} \otimes \Pi_{B,\nu}).$$

Proof. Let the spectral decompositions of $\rho_{A,T}$ and $\rho_{B,T}$ be

$$\rho_{A,T} = \sum_{\mu} p_{\mu} \Pi_{A,T}, \quad \rho_{B,T} = \sum_{\nu} q_{\nu} \Pi_{B,T},$$

respectively, where $\{\Pi_{A,\mu}\}$ and $\{\Pi_{B,\nu}\}$ are the orthogonal projectors of any rank, such that $\{p_{\mu}\}$ and $\{q_{\nu}\}$ are non-degenerate, respectively. Then $\{\Pi_{A,\mu} \otimes \Pi_{B,\nu}\}$ are orthogonal eigen-projectors of $\rho_{A,T} \otimes \rho_{B,T}$. Since $[\rho_{AB}, \rho_A \otimes \rho_B] = 0$ is equivalent to $[\rho_{AB}, \Pi_{A,\mu} \otimes \Pi_{B,\nu}] = 0$ for all $\mu, \nu$, it follows from $\sum_{\mu,\nu} \Pi_{A,\mu} \otimes \Pi_{B,\nu} = \mathbb{1}_A \otimes \mathbb{1}_B$ that

$$\rho_{AB} = \sum_{\mu,\nu} (\Pi_{A,\mu} \otimes \Pi_{B,\nu}) \rho_{AB} (\Pi_{A,\mu} \otimes \Pi_{B,\nu}).$$

The converse follows from direct computation.

Here the states $\rho_{AB}$ satisfying the condition $[\rho_{AB}, \rho_A \otimes \rho_B] = 0$ are just the generalization of zero-symmetric discord states and lazy states are the generalization of zero discord states.

4 Conclusion

We have studied the well-known monotonicity inequality of relative entropy under completely positive linear maps, by deriving some properties of symmetric discord. A new form of zero-discord state via Petz’s monotonicity condition on relative entropy with equality has been derived systematically. The results are generalized for the zero-discord states.

There is a more interesting and challenging problem which can be considered in the future study: What is a sufficient and necessary condition for the vanishing conditional mutual entropy rates at a time $\tau$:

$$\left[\frac{d}{dt} I(A : B|E)_{\rho}\right]_{t=\tau} = 0,$$

where $I(A : B|E)_{\rho} = S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E)$. 

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References

[1] M. Ohya and D. Petz, Quantum Entropy and Its Use. Springer-Verlag, Heidelberg, 
1993. Second edition 2004.

[2] D. Petz, Sufficiency of Channels over von Neumann algebras. Quart. J. Math. 
Oxford(2), 39(1988), 97–108.

[3] D. Petz, Monotonicity of quantum relative entropy revisited. Rev. Math. Phys. 
15(1):79–91(2003).

[4] M. Ruskai, Inequalities for quantum entropy: A review with conditions for equality. 
J. Math. Phys. 43, 4358 (2002).

[5] F. Hiai, M. Mosonyi, D. Petz and C. Bény, Quantum $f$-divergence and error correc-
tion. Rev. Math. Phys. 23:691-747(2011).

[6] M. Choi, Completely positive linear maps on complex matrices. Lin. Alg. Appl. 10, 
285 (1975).

[7] N. Linden and A. Winter, A New Inequality for the von Neumann Entropy. Commn. 
Math. Phys. 259, 129–138 (2005).

[8] J. Cadney, N. Linden, and A. Winter, Infinitely Many Constrained Inequalities for 
the von Neumann Entropy. IEEE Trans. Inf. Theor. 58(6), 3657-3663 (2012).

[9] C. Rulli and M. Sarandy, Global quantum discord in multipartite systems. Phys. 
Rev. A 84, 042109 (2011).

[10] R. Renner, Security of Quantum Key Distribution, PhD thesis. arXiv:quant-
ph/0512258.
[11] A. Datta, A Condition for the Nullity of Quantum Discord. arXiv:quant-ph/1003.5256.

[12] I. Devetak and J. Yard, Exact Cost of Redistributing Multipartite Quantum States. Phys. Rev. Lett. 100, 230501 (2008).

[13] J. Yard and I. Devetak, Optimal Quantum Source Coding With Quantum Side Information at the Encoder and Decoder. IEEE Trans. Info. Theor. 55, 5339–5351 (2009).

[14] V. Madhok and A. Datta, Interpreting quantum discord through quantum state merging. Phys. Rev. A. 83, 032323 (2011).

[15] D. Cavalcanti, L. Aolita, S. Boixo, K. Modi, M. Piani, and A. Winter1, Operational interpretations of quantum discord. Phys. Rev. A. 83, 032324 (2011).

[16] A. Ferraro, L. Aolita, D. Cavalcanti, F. M. Cucchietti and A. Acín, Allmost all quantum states have nonclassical correlations. Phys. Rev. A 81, 052318 (2010).

[17] C. A. Rodríguez-Rosario, G. Kimura, H. Imai and A. Aspuru-Guzik, Sufficient and Necessary Condition for Zero Quantum Entropy Rates under any Coupling to the Environment. Phys. Rev. Lett. 106, 050403 (2011).