DEGENERATE 0-SCHUR ALGEBRAS AND NIL-TEMPERLEY-LIEB ALGEBRAS

BERNT TORE JENSEN, XIUPING SU AND GUIYU YANG

ABSTRACT. In [11] Jensen and Su constructed 0-Schur algebras on double flag varieties. The construction leads to a presentation of 0-Schur algebras using quivers with relations and the quiver approach naturally gives rise to a new class of algebras. That is, the path algebras defined on the quivers of 0-Schur algebras with relations modified from the defining relations of 0-Schur algebras by a tuple of parameters $\mathbf{t}$. In particular, when all the entries of $\mathbf{t}$ are 1, we have 0-Schur algebras. When all the entries of $\mathbf{t}$ are zero, we obtain a class of degenerate 0-Schur algebras. We prove that the degenerate algebras are associated graded algebras and quotients of 0-Schur algebras. Moreover, we give a geometric interpretation of the degenerate algebras using double flag varieties, in the same spirit as [11], and show how the centralizer algebras are related to nil-Hecke algebras and nil-Temperley-Lieb algebras.

1. Introduction

It is well known that the classical Schur algebras are specialisations of $q$-Schur algebras (see [5] and [6]) at $q = 1$. Analogously, 0-Schur algebras are specialisations of $q$-Schur algebras at $q = 0$. The 0-Schur algebras have been studied by Donkin [6], §2.2] in terms of 0-Hecke algebras of symmetric groups, by Krob and Thibon [14] in connection with noncommutative symmetric functions, by Deng and Yang on their presentations and representation types [3, 4].

A new approach towards 0-Schur algebras was investigated by Jensen and Su [11], by considering the monoid structure of the 0-Schur algebras. Inspired by Beilinson, Lusztig and MacPherson's geometric construction of $q$-Schur algebras [1] and Reineke's work on a monoid structure of Hall algebras [17], Su defined a generic multiplication in the positive part of 0-Schur algebras [20]. The generic multiplication was then generalized by Jensen and Su [11] to give a global geometric construction of 0-Schur algebras. This geometric construction produces a monoid structure, simplifies the multiplication and provides a new approach to studying the structure of 0-Schur algebras. In [12] we gave a construction of indecomposable projective modules and studied homomorphism spaces between projective modules [12]. In an ongoing work [13], we study further irreducible maps, construct idempotents and present 0-Hecke algebras and basic 0-Schur algebras using quivers with relations. Consequently, we obtain an alternative account to the result on extension groups between simple modules by Duchamp, Hivert, and Thibon [8] (see also [9]).

The nature of 0-Schur algebras exposed in [11] leads to several interesting related algebras. First of all, we can modify the generating relations of 0-Schur algebras, relying on multiple parameters $\mathbf{t}$. In particular, when all the parameters are 1, we recover 0-Schur algebras and when all parameters are 0, we obtain a class of basic algebras. We prove that similar to 0-Schur algebras, these newly defined algebras...
have reduced paths as basis (see [12]), which are in one-to-one correspondence with GL(V)-orbits in double flag varieties or certain sets of integral matrices. So they have the same dimension as the corresponding 0-Schur algebras and they are proved to be degenerate 0-Schur algebras. We further show that they are isomorphic to quotients of 0-Schur algebras by naturally defined ideals and to the associated graded algebras of 0-Schur algebras, which notably have a natural structure as a filtered algebra. We also investigate the relation between their centralizer algebras and Nil-Temperley-Lieb algebras.

The remainder of this paper is organised as follows. In Section 2, we give a brief background on $q$-Schur and 0-Schur algebras and discuss how to view the 0-Schur algebra as a filtered algebra. In Section 3, we prove some preliminary results on a family of idempotent ideals. In Section 4, we construct a series of algebras $D_t(n,r)$ using quivers and modified relations of 0-Schur algebras and prove an isomorphism theorem between different $D_t(n,r)$. In Section 5, we first construct the associated graded algebras $DS_0(n,r)$ of 0-Schur algebras and give them a geometric interpretation. We then show that $D_t(n,r)$ for $t = 0$ is a degenerate 0-Schur algebra and prove our main result that the three algebras, $DS_0(n,r)$, $D_0(n,r)$ and the quotient of $S_0(n,n+r)$ modulo a natural idempotent ideal are isomorphic. In Section 6, we discuss relations between centralizer algebras of the degenerate algebras nil-Hecke algebras and nil-Temperley-Lieb algebras.

2. Background on the algebra $S_0(n,r)$

2.1. The algebra $S_q(n,r)$. We first recall the construction of quantised Schur algebras given by Beilinson-Lusztig-MacPherson [1]. Let $k$ be a field and let $V$ be a $k$-vector space of dimension $r$. Let $F$ be the set of $n$-step flags

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V.$$ 

The natural action of $GL(V)$ on the vector space $V$ induces a diagonal action of $GL(V)$ on $F \times F$ defined by

$$g(f, f') = (gf, gf'),$$

where $g \in GL(V)$ and $f, f' \in F$. Denote the orbit of $(f, f')$ by $\mathcal{O}(f, f')$.

Let $\Xi(n,r)$ be the set of $n \times n$-matrices $A = (a_{ij})_{i,j}$ with $a_{ij}$ nonnegative integers and $\sum_{1 \leq i,j \leq n} a_{ij} = r$. Then there is a bijection from $F \times F / GL(V)$ to $\Xi(n,r)$ sending the orbit of $(f, f')$ to $A = (a_{ij})_{i,j}$ with

$$a_{ij} = \dim_k \frac{V_i \cap V_j'}{V_{i-1} \cap V_j' + V_i \cap V_{j-1}'}$$

for $1 \leq i, j \leq n$,

where $f = (V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V)$, $f' = (V_1' \subseteq V_2' \subseteq \cdots \subseteq V_n' = V)$ and $V_0 = V_0' = 0$ by convention. Denote by $e_A$ the orbit in $F \times F$ corresponding to $A$. Consider the diagram

$$\begin{array}{ccc}
F \times F \times F & \xrightarrow{\Delta} & (F \times F) \times (F \times F) \\
\pi \downarrow & & \\
F \times F & & \\
\end{array}$$

where $\pi(f, g, h) = (f, h)$ and $\Delta(f, g, h) = ((f, g), (g, h))$.

Let $\mathbb{Z}[q]$ be the polynomial ring in $q$ over the ring of integers. Following [12], for any given $A, B, C \in \Xi(n,r)$, there is a polynomial $g_{A,B,C} \in \mathbb{Z}[q]$ such
that for all finite fields $k$,
\[ g_{A,B,C}(|k|) = \frac{\pi^{-1}(e_C) \cap \Delta^{-1}(e_A \times e_B)}{|e_C|}, \]
where $|k|$ and $|e_C|$ are the cardinalities of the field $k$ and the orbit $e_C$ over $k$.

Following a remark by Du [7], the $q$-Schur algebra studied by Dipper and James in \[5\] (see also \[6\]) can now be defined as follows.

**Definition 2.1 ([1]).** The quantised Schur algebra $S_q(n,r)$ is the free $\mathbb{Z}[q]$-module with basis \{ $e_A \mid A \in \Xi(n,r)$ \} and with multiplication given by
\[ e_A \cdot e_B = \sum_{C \in \Xi(n,r)} g_{A,B,C}(q) e_C, \quad \text{for all } A, B \in \Xi(n,r). \]

For an $n \times n$-matrix $A = (a_{ij})$, define the row and column vectors of $A$ by
\[ \text{ro}(A) = (\sum_{j=1}^n a_{1j}, \ldots, \sum_{j=1}^n a_{nj}) \quad \text{and} \quad \text{co}(A) = (\sum_{i=1}^n a_{i1}, \ldots, \sum_{i=1}^n a_{in}). \]

Note that if $g_{A,B,C}(q) \neq 0$, then
\[ \text{ro}(A) = \text{ro}(C), \quad \text{co}(A) = \text{ro}(B) \quad \text{and} \quad \text{co}(B) = \text{co}(C). \]

Let $\Lambda(n,r)$ be the set of compositions of $r$ into $n$ parts. For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$, let $\text{diag}(\lambda)$ denote the diagonal matrix $\text{diag}(\lambda_1, \ldots, \lambda_n)$ and write $k_\lambda = e_{\text{diag}(\lambda)}$. By definition, for each $A \in \Xi(n,r)$,
\[ k_\lambda \cdot e_A = \begin{cases} e_A, & \text{if } \lambda = \text{ro}(A); \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad e_A \cdot k_\lambda = \begin{cases} e_A, & \text{if } \lambda = \text{co}(A); \\ 0, & \text{otherwise}. \end{cases} \]

Thus, $\sum_{\lambda \in \Lambda(n,r)} k_\lambda$ is the identity of $S_q(n,r)$.

Denote by $E_{ij}$ the elementary $n \times n$ matrix with a single nonzero entry 1 at $(i, j)$. We denote by $e_{i,\lambda}$ (resp. $f_{j,\lambda}$) the basis element of $S_q(n,r)$ corresponding to the matrix that has column vector $\lambda$ and the only nonzero off diagonal entry is 1 at $(i, i+1)$ (resp. $(j+1, j)$).

### 2.2. Definition of $S_0(n,r)$. From now on, let $k$ be algebraically closed. In [11], Jensen and Su defines a generic multiplication of orbits in $\mathcal{F} \times \mathcal{F}$ given by
\[ e_A \cdot e_B = \begin{cases} e_C, & \text{if } \Delta^{-1}(e_A \times e_B) \neq \emptyset; \\ 0, & \text{otherwise}, \end{cases} \]
where $e_C$ is the unique open orbit in $\pi \Delta^{-1}(e_A \times e_B)$. This defines an associative $\mathbb{Z}$-algebra $G(n,r)$ with basis $\Xi(n,r)$ and in fact $G(n,r)$ is isomorphic to the 0-Schur algebra (Theorem 7.2.1 in [11]).

\[ S_0(n,r) = S_q(n,r) \otimes_{\mathbb{Z}[q]} \mathbb{Z}, \]
where $\mathbb{Z}$ is viewed as the $\mathbb{Z}[q]$-module $\mathbb{Z}[q]/(q)$. As the multiplication in $G(n,r)$ is much simplified (e.g. the multiplication of two orbits is either 0 or again an orbit), in the rest part of this paper, we will take $G(n,r)$ as the 0-Schur algebra $S_0(n,r)$. 

2.3. The fundamental multiplication rules. Note that $S_0(n,r)$ is generated by $e_{i,\lambda}$, $f_{i,\lambda}$ and $k_{\lambda}$, where $1 \leq i \leq n-1$ and $\lambda \in \Lambda(n,r)$ (see Lemma 6.9 in [11]). Let

$$e_i = \sum_{\lambda \in \Lambda(n,r)} e_{i,\lambda} \quad \text{and} \quad f_i = \sum_{\lambda \in \Lambda(n,r)} f_{i,\lambda}.$$ 

Note that for any given orbit $e_A$, by the definition of the multiplication,

$$e_i e_A = e_{i,co(A)} e_A.$$ 

This says that only one term remains in the product and the same for $e_A e_i$, $e_A f_i$ and $f_i e_A$. The following are the fundamental multiplication rules in $S_0(n,r)$, which describe the action of generators on basis elements.

**Lemma 2.2** (Lemma 6.11, [11]). Let $e_A \in S_0(n,r)$ with $ro(A) = \lambda$.

1. If $\lambda_{i+1} > 0$, then $e_i e_A = e_X$, where $X = A + E_{i,p} - E_{i+1,p}$ with $p = \max \{j \mid a_{i+1,j} > 0\}$.
2. If $\lambda_i > 0$, then $f_i e_A = e_Y$, where $Y = A - E_{i,p} + E_{i+1,p}$ with $p = \min \{j \mid a_{i,j} > 0\}$.

Symmetrically, there are the following formulas.

**Lemma 2.3** (Lemma 2.2, [12]). Let $e_A \in S_0(n,r)$ with $co(A) = \mu$.

1. If $\mu_{i+1} > 0$, then $e_A f_i = e_X$, where $X = A + E_{p,i} - E_{p,i+1}$ with $p = \max \{j \mid a_{i,j+1} > 0\}$.
2. If $\mu_i > 0$, then $e_A i = e_Y$, where $Y = A - E_{p,i} + E_{p,i+1}$ with $p = \min \{j \mid a_{i,i} > 0\}$.

2.4. Presenting $S_0(n,r)$ by quiver with relations. Let

$$\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0) \in \mathbb{Z}^n,$$

where the only nonzero entries 1 and −1 are at the $i$th- and $(i+1)$th-positions, respectively. We define a quiver $\Sigma(n,r)$ with vertices corresponding to $\lambda \in \Lambda(n,r)$ and arrows

and let $J \subseteq \mathbb{Z} \Sigma(n,r)$ be the ideal generated by the binomial relations

$$P_{ij,\lambda} = k_\mu P_{ij} k_\lambda,$$

$$N_{ij,\lambda} = k_\mu N_{ij} k_\lambda,$$

$$C_{ij,\lambda} = k_{\lambda+\alpha_i-\alpha_j} C_{ij} k_\lambda,$$

where

$$P_{ij} = \begin{cases} e_i^2 e_j - e_i e_j e_i & \text{for } i = j - 1, \\ -e_i e_j e_i + e_j e_i^2 & \text{for } i = j + 1 \quad \text{and} \quad \mu = \begin{cases} \lambda + 2\alpha_i + \alpha_j & \text{if } i = j \pm 1 \\ \lambda + \alpha_i + \alpha_j & \text{otherwise}; \end{cases} \\ e_i e_j - e_j e_i & \text{otherwise}, \end{cases}$$

$$N_{ij} = \begin{cases} -f_i f_j f_i + f_j f_i^2 & \text{for } i = j - 1, \\ f_i^2 f_j - f_i f_j f_i & \text{for } i = j + 1 \quad \text{and} \quad \mu = \begin{cases} \lambda - 2\alpha_i - \alpha_j & \text{if } i = j \pm 1 \\ \lambda - \alpha_i - \alpha_j & \text{otherwise}; \end{cases} \\ f_i f_j - f_j f_i & \text{otherwise}, \end{cases}$$
and \[
C_{ij} = e_i f_j - f_j e_i - \delta_{ij} \left( \sum_{\lambda_{i+1}=0} k_{\lambda} - \sum_{\lambda_i=0} k_{\lambda} \right).
\]

The quotient algebra \( \mathbb{Z} \Sigma(n, r)/J \) is isomorphic to \( S_0(n, r) \), by an isomorphism that maps the arrows \( e_{i, \lambda}, f_{i, \lambda} \) and vertices \( k_{\lambda} \) to the corresponding elements in \( S_0(n, r) \) (Theorem 7.1.2 in [11]). The relations \( P_{ij, \lambda}, N_{ij, \lambda} \) are usually called the Serre relations of \( S_0(n, r) \). The relations \( C_{ij, \lambda} \) is a commutative relation when \( i \neq j \) or \( \lambda_i \lambda_{i+1} \neq 0 \), an idempotent relation when exactly one of \( \lambda_i, \lambda_{i+1} \) is zero, and an empty relation otherwise.

### 2.5. Bases of reduced paths and \( S_0(n, r) \) as a filtered algebra.

We say that a path in \( \Sigma(n, r) \) is reduced if it is not equal to a path of shorter length in \( \Sigma(n, r) \) modulo \( J \). Equivalence classes of reduced paths form a multiplicative basis for \( S_0(n, r) \), which coincides with the basis \( e_A, A \in \Sigma(n, r) \). There are in general many paths equal to \( e_A \), modulo \( J \), but by the fundamental multiplication rules and the presentation of \( S_0(n, r) \), the numbers of occurrences of \( e_i \) and \( f_i \) in any reduced path are

\[
E(e_A)_i = \sum_{t \leq i < m} a_{t, m} \quad \text{and} \quad F(e_A)_i = \sum_{m \leq i < t} a_{t, m},
\]

respectively.

**Lemma 2.4.** For any \( 1 \leq i \leq n - 1 \), we have

\[
E(e_A)_i + E(e_B)_i - E(e_A \cdot e_B)_i = F(e_A)_i + F(e_B)_i - F(e_A \cdot e_B)_i \geq 0.
\]

**Proof.** The generating relations \( P_{ij} \) and \( N_{ij} \) do not change the length of a path. The length of a path decreases, when we apply the relations \( C_{ii} \), that is, we replace an \( e_i f_i \) or \( f_i e_i \) by an idempotent. Note that in this case, where the length of a path does decrease, the numbers of the occurrences of \( e_i \) and \( f_i \) decrease at the same pace. Therefore

\[
E(e_A)_i + E(e_B)_i - E(e_A \cdot e_B)_i = F(e_A)_i + F(e_B)_i - F(e_A \cdot e_B)_i \geq 0,
\]

as stated. \( \square \)

We define vectors \( E(e_A) \) and \( F(e_A) \) by

\[
E(e_A) = (E(e_A)_i)_i \quad \text{and} \quad F(e_A) = (F(e_A)_i)_i.
\]

If we compare tuples \((E(e_A), F(e_A))\) componentwise, we have the following.

**Corollary 2.5.** \( S_0(n, r) \) is a filtered algebra with degree function \( e_A \mapsto (E(e_A), F(e_A)) \).

For \( j \geq i \), let

\[
e(i, j) = e_i \cdot e_{i+1} \cdot \ldots \cdot e_j \quad \text{and} \quad f(j, i) = f_j \cdot \ldots \cdot f_{i+1} \cdot f_i.
\]

We give two explicit descriptions of reduced paths equal to a basis element \( e_A \), analogous to the monomial and PBW-basis of \( S_q(n, r) \), respectively.

**Lemma 2.6.** Let \( e_A \in S_0(n, r) \) with \( A = (a_{i, j})_{i, j} \). Then we can write \( e_A \) as reduced paths as follows.

1. \( e_A = (\prod_{j=0}^{n} \prod_{i=0}^{n} e_i^{\sum_{1 \leq p \leq l} a_{p, i+1}}) \cdot (\prod_{s=1}^{n-1} \prod_{l=0}^{n} f_l^{\sum_{l \leq p \leq s} a_{p, s}}) \cdot k_{\text{co}(A)};\)
2. \( e_A = (\prod_{j=n}^{n} \prod_{i=1}^{n} e(i, j-1)^{a_{ij}}) \cdot (\prod_{s=0}^{n-1} \prod_{l=0}^{n} f(i, l)^{a_{ij}}) \cdot k_{\text{co}(A)}.\)

**Proof.** The formula follows by repeatedly applying the fundamental multiplication rules in Lemma 2.2. In both (1) and (2), the numbers of occurrences of \( e_i \) and \( f_i \) are \( E(e_A)_i \) and \( F(e_A)_i \). So the paths are reduced. \( \square \)
We explain by an example on how to achieve the above formulae.

**Example 2.7.** Let $A = (a_{ij})$, where

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$ 

Note that $e_B = k_{\text{col}(A)}$. Using the fundamental multiplication rules, we have the following.

$$\begin{align*}
(1) \quad & B = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow{f_2^e} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 6 \end{pmatrix} \xrightarrow{f_1^e} \begin{pmatrix} 0 & 0 & 0 \\ 8 & 1 & 0 \\ 0 & 6 & 6 \end{pmatrix} \xrightarrow{f_2^e} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 6 & 6 \end{pmatrix} e_1 \xrightarrow{(e_1 e_2)^2} \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 5 & 6 & 6 \end{pmatrix} e_2^e \xrightarrow{(e_1 e_2)^2} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix} = A.
\end{align*}$$

That is,

$$e_A = e_1^2 \cdot e_2^6 \cdot e_1 \cdot f_2^5 \cdot f_1^8 \cdot f_2^6 \cdot e_B = e_1^{a_{13}} \cdot e_2^{(a_{13}+a_{23})} \cdot e_1^{a_{12}} \cdot f_2^{a_{31}} \cdot f_1^{(a_{21}+a_{31})} \cdot f_2^{a_{32}} \cdot e_B.$$

$$\begin{align*}
(2) \quad & B = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow{f_2^e} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 6 \end{pmatrix} \xrightarrow{(f_2 f_1)^3} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 6 & 6 \end{pmatrix} \xrightarrow{f_1^3} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 6 & 6 \end{pmatrix} e_1 \xrightarrow{(e_1 e_2)^2} \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 5 & 6 & 6 \end{pmatrix} e_2e^e \xrightarrow{(e_1 e_2)^2} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix} = A.
\end{align*}$$

That is,

$$e_A = e_2^4 \cdot (e_1 e_2)^2 \cdot e_1 \cdot f_1^3 \cdot (f_2 f_1)^5 \cdot f_2^6 \cdot e_B = e(2,2)^4 \cdot e(1,2)^2 \cdot f(1,1)^3 \cdot f(2,1)^5 \cdot f(2,2)^6 \cdot e_B.$$

In both processes, the the lower triangular parts/upper triangular parts are created column-wise. The main differences are, for instance in the lower triangular parts, in (2) each step creates an entry $a_{ij}$ at a time, starting from the lowest entry and then moving upwards, while in (1) it goes downwards and in each step we apply the maximal times of $f_i$ so that afterwards we have exactly right entry in row $i$.

### 3. Idempotents and idempotent ideals of $S_0(n,r)$

The set of compositions $\Lambda(n,r)$ (i.e. the vertices in the quiver $\Sigma(n,r)$) can be drawn on an $(n-1)$-simplex, where compositions with zero entries lie on the boundary. We call an idempotent $k_3$ boundary if $\lambda$ lies on the boundary of the simplex, and interior otherwise. In this section, we are interested in the ideal generated by all boundary idempotents. We obtain a dimension formula for the quotient algebra, which turns out to be useful when we consider degenerate 0-Schur algebras Section 5.
Let \( I_i(n, r) \) be the ideal generated by the idempotents \( k_\lambda \) with the number of nonzero entries in \( \lambda \) less than or equal to \( i \). That is, \( I_i(n, r) \) is generated by idempotents corresponding to \((i - 1)\)-faces of the simplex. There are in general several \((i - 1)\)-faces in \( \Sigma(n, r) \), but as the following lemma shows, any one of them contains enough idempotents to generate the whole of \( I_i(n, r) \).

**Lemma 3.1.** The ideal \( I_i(n, r) \) is generated by all idempotents \( k_\lambda \) lying in any chosen \((i - 1)\)-face in \( \Sigma(n, r) \).

**Proof.** For any \( \lambda = (\ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots) \in \Lambda(n, r) \),

\[
k_\lambda = f^\lambda_{i+1} k_\mu e^\lambda_{i+1},
\]

where \( \mu = (\ldots, \lambda_{i-1}, \lambda_{i+1}, 0, \ldots) \) with the other entries equal to those of \( \lambda \). Therefore \( k_\lambda \) is contained in the ideal generated by \( k_\mu \). Similarly, \( k_\mu \) is contained in the ideal generated by \( k_\lambda \), and so these two ideals are equal. This shows that \( I_i(n, r) \) is generated by all idempotents \( k_\lambda \) lying in any chosen \((i - 1)\)-face in \( \Sigma(n, r) \). \( \square \)

Denote by \( I(n, r) \) the ideal of \( S_0(n, r) \) generated by the boundary idempotents. If \( r \geq n \), then \( I(n, r) = I_{n-1}(n, r) \). If \( r < n \), then \( I(n, r) = S_0(n, r) \).

**Lemma 3.2.** The ideal \( I(n, r) \) has a basis consisting of \( e_A \), where \( A \in \Xi(n, r) \) is a matrix with at least one diagonal entry equal to 0.

**Proof.** Denote by \( S \) the subspace of \( S_0(n, r) \) spanned by \( e_A \) with at least one diagonal entry of \( A \) equal to 0. We first show that \( S \) is an ideal. It is enough to prove that for any generator \( x \) and any \( e_D \in S \), both \( xe_D \) and \( e_DX \) are contained in \( S \). We only prove that \( xe_D \in S \) for \( x = e_j \) and \( D = (d_{ij})_{ij} \) with \( d_{ii} = 0 \), as the other cases can be done similarly. Let \( e_X = e_j e_D \) and \( X = (x_{ij})_{ij} \). By Lemma 2.2, multiplying \( e_j \) with \( e_D \) from the left only affects two entries in the \((j + 1)\)th-row and \( j \)th-row in \( D \), respectively. We have either \( x_{ii} = d_{ii} = 0 \) or \( x_{ii} = 1 \), which occurs only if \( d_{i+1,i+1} = 0 \), but then

\[
x_{i+1,i+1} = 0
\]
as well. In either case \( xe_D = e_X \in S \), as required, and so \( S \) is an ideal in \( S_0(n, r) \).

As \( I(n, r) \) is generated by idempotents \( k_\lambda \) with \( \lambda_i = 0 \) for some \( i \), we have \( k_\lambda \in S \) and therefore \( I(n, r) \subseteq S \), since \( S \) is an ideal.

It remains to show that \( S \subseteq I(n, r) \). Suppose that \( e_A \in S \) has the diagonal entry \( a_{ii} = 0 \), for some \( 1 \leq i \leq n \). We will show that \( e_A \in I(n, r) \). Let \( B \) be the matrix with

\[
b_{ss} = \left\{ \begin{array}{ll}
\sum_{s \leq t \leq i} a_{ts} & \text{if } s < i, \\
0 & \text{if } s = i, \\
\sum_{i \leq t \leq s} a_{ts} & \text{if } s > i.
\end{array} \right.
\]
on the diagonal and

\[
b_{st} = \left\{ \begin{array}{ll}
0 & \text{if } i \geq s > t \\
0 & \text{if } i \leq s < t \\
a_{st} & \text{otherwise}
\end{array} \right.
\]
off the diagonal. The \( i' \)th row in \( B \) is zero, and so we have \( e_B = k_\lambda e_B \in I(n, r) \), where \( \lambda = \text{ro}(B) \). To complete the proof we will construct elements \( x \) and \( y \) such that

\[xye_B = e_A,\]

which implies that \( e_A \in I(n, r) \). The element \( y \) is a product of generators \( e_j \) for \( j > i \) and \( x \) is a product of \( f_j \) for \( j \leq i \). Multiplying \( e_B \) with \( x \) and \( y \) produces
the right entries at \((s, t)\) in the two zero region \(i \geq s > t\) and \(i \leq s < t\) from the diagonal entries in \(B \). Explicitly, similar to Lemma \(2.6\) we have

\[
y = \prod_{s=n-1}^{i} \prod_{t=i}^{s} e_{t}^{i} = \left( e_{1}^{a_{i}n_{1} + a_{i+1}n} \ldots e_{n-1}^{\sum_{i \leq p \leq i} a_{p}} \right) \ldots e_{i}^{a_{i,i+1}}.
\]

and similarly for \(x\). Now by the fundamental multiplication rules in Lemma \(2.2\)

\[
xye_{B} = e_{A},
\]
as required. \(\square\)

The lemma shows that \(I(n, r)\) is a summand of \(S_{0}(n, r)\) as \(\mathbb{Z}\)-modules, and so \(S_{0}(n, r)/I(n, r)\) is a free \(\mathbb{Z}\)-module. We have the following formulae which will be used Section 5.

**Corollary 3.3.**

1. \(\text{rank } I(n, r) = \sum_{s=1}^{n} \binom{n}{s} \frac{n^2 + r - n - 1}{r + s - n} \).
2. \(\text{rank } S_{0}(n, r)/I(n, r) = \frac{n^2 + r - n - 1}{r - n} \).

Note that in part (1), each term in the sum counts the matrices with exactly \(s\) zero diagonal entries.

4. **Modified algebras of \(S_{0}(n, r)\)**

In the remaining of this paper, we will work with algebras defined over a field \(\mathbb{F}\). By modifying generating relations in Section \(2.4\), we have a series of modified algebras \(D_{\ell}(n, r)\) of \(S_{0}(n, r)\). We will show that for a particular \(\ell\), \(D_{\ell}(n, r)\) is a degeneration of \(S_{0}(n, r)\) in next section.

Let \(B(n, r) \subseteq \Lambda(n, r)\) be the set of elements corresponding to boundary idempotents. i.e., this is the set of \(\lambda\) such that there is some \(\lambda_{i} = 0\). Let

\[
\ell = (t_{i, \lambda})_{1 \leq i \leq n-1, \lambda \in B(n, r)}
\]

with each entry of \(\ell\) in \(\mathbb{F}\). Denote by \(\mathcal{J}(\ell)\) the ideal of \(\mathbb{F}\Sigma(n, r)\) generated by \(P_{ij, \lambda}, N_{ij, \lambda}\) and \(C_{ij, \lambda}(\ell)\), where \(P_{ij, \lambda}\) and \(N_{ij, \lambda}\) are defined as in Section \(2.4\) and

\[
(5) \quad C_{ij, \lambda}(\ell) = k_{\lambda + \alpha_{i} - \alpha_{j}} C_{ij}(\ell) k_{\lambda},
\]

where

\[
C_{ij}(\ell) = e_{i}f_{j} - f_{j}e_{i} - \delta_{ij} \cdot t_{i, \lambda} \cdot \left( \sum_{\lambda_{i+1} = 0} k_{\lambda} - \sum_{\lambda_{i} = 0} k_{\lambda} \right).
\]

Note that when \(\lambda_{i} = \lambda_{i+1} = 0\), then \(C_{ij, \lambda}(\ell)\) is an empty relation.

**Definition 4.1.** \(D_{\ell}(n, r) := \mathbb{F}\Sigma(n, r)/\mathcal{J}(\ell)\).

**Remark 4.2.** It is enough to define \(\ell\) only on vertices corresponding to \(B(n, r)\). Note that for \(i \neq j\), \(C_{ij}(\ell) = e_{i}f_{j} - f_{j}e_{i}\). By \([5]\) there is a commutative diagram as follows.

\[
\begin{array}{ccc}
k_{\lambda} & \xrightarrow{e_{i}} & k_{\lambda + \alpha_{i}} \\
\downarrow f_{i} & & \downarrow f_{j} \\
k_{\lambda - \alpha_{j}} & \xrightarrow{e_{i}} & k_{\lambda + \alpha_{i} - \alpha_{j}}
\end{array}
\]

When \(i = j\) and either \(\lambda_{i} \neq 0\) or \(\lambda_{i+1} \neq 0\), we have the following three cases.
is section that this is a degenerate 0-Schur algebra.

For the first case, we have the commutative relation at $k_\lambda$ as follows.

$\lambda$ is boundary and the relations are not admissible if $t_{i,\lambda} \neq 0$. When all the $t_{i,\lambda} = 1$, we recover 0-Schur algebra $S_0(n, r)$. When all the $t_{i,\lambda} = 0$, we have a basic algebra and we will prove in the following section that this is a degenerate 0-Schur algebra.

Note that "bad" parameter $t$ may produce "bad" relations, in the sense that it may force some idempotents to be zero, which might eventually lead to the collapse of the whole algebra $D_2(n, r)$, i.e., $D_2(n, r) = 0$. The following lemma shows what $t$ is bad. For $\mu \in \Lambda(n, r)$, let $\mu s_i$ be the composition given by permuting the $i$-th and the $(i + 1)$-th entries of $\mu$.

**Lemma 4.3.** Suppose that $\lambda = \mu s_i$ in $B(n, r)$ with $\lambda_i = 0$. If $t_{i,\lambda}^{\lambda_i+1} \neq t_{i,\mu}^{\lambda_i+1}$ then $k_\lambda = 0$ or $k_\mu = 0$ in $D_2(n, r)$.

**Proof.** By definition we get

$$k_\lambda f_i e_i k_\lambda = t_{i,\lambda} k_\lambda, \quad k_\mu e_i f_i k_\mu = t_{i,\mu} k_\mu$$

in $D_2(n, r)$.

Consequently, by the first equation in (6) we get

$$k_\lambda f_i^{\lambda_i+1} e_i^{\lambda_i+1} k_\lambda = t_{i,\lambda}^{\lambda_i+1} k_\lambda, \quad k_\mu e_i^{\lambda_i+1} f_i^{\lambda_i+1} k_\mu = t_{i,\mu}^{\lambda_i+1} k_\mu.$$

Hence

$$(k_\lambda f_i^{\lambda_i+1} e_i^{\lambda_i+1} k_\lambda)^2 = t_{i,\lambda}^{2\lambda_i+1} k_\lambda$$

$$= k_\lambda f_i^{\lambda_i+1} e_i^{\lambda_i+1} f_i^{\lambda_i+1} e_i^{\lambda_i+1} k_\lambda$$

$$= k_\lambda f_i^{\lambda_i+1} (k_\mu f_i^{\lambda_i+1} e_i^{\lambda_i+1} k_\mu) e_i^{\lambda_i+1} k_\lambda$$

$$= t_{i,\mu}^{\lambda_i+1} k_\lambda f_i^{\lambda_i+1} e_i^{\lambda_i+1} k_\lambda$$

$$= t_{i,\mu}^{\lambda_i+1} t_{i,\lambda}^{\lambda_i+1} k_\lambda,$$

i.e.,

$$(7) \quad t_{i,\lambda}^{\lambda_i+1} (t_{i,\mu}^{\lambda_i+1} - t_{i,\mu}^{\lambda_i+1}) k_\lambda = 0.$$

Similarly,

$$(k_\mu e_i^{\lambda_i+1} f_i^{\lambda_i+1} k_\mu)^2 = t_{i,\mu}^{2\lambda_i+1} k_\mu$$

$$= t_{i,\mu}^{\lambda_i+1} t_{i,\lambda}^{\lambda_i+1} k_\mu,$$

i.e.,

$$(8) \quad t_{i,\mu}^{\lambda_i+1} (t_{i,\mu}^{\lambda_i+1} - t_{i,\lambda}^{\lambda_i+1}) k_\mu = 0.$$

If $t_{i,\lambda}^{\lambda_i+1} \neq t_{i,\mu}^{\lambda_i+1}$, then $k_\lambda = 0$ or $k_\mu = 0$ by (7) and (8). This proves the lemma. □
Remark 4.4. We want to avoid the case where $D_{\underline{t}}(n,r) = 0$. So from now on we only consider $\underline{t}$ with

$$t_{\lambda,i} = t_{\mu,i}, \text{ if } \lambda = \mu s_i \text{ and } \lambda_i = 0. \tag{9}$$

Next we consider when the two algebras $D_{\underline{t}}(n,r)$ and $D_{\underline{s}}(n,r)$ are isomorphic. Suppose that $\underline{t}$ and $\underline{s}$ satisfy condition (9) and $t_{i,\lambda}s_{i,\lambda} \neq 0$ for all $1 \leq i \leq n-1$ and $\lambda \in B(n,r)$. Define a map $\Phi$:

$$D_{\underline{t}}(n,r) \rightarrow D_{\underline{s}}(n,r)$$

given by

$$k_\lambda \mapsto k_\lambda, f_{i,\lambda} \mapsto f_{i,\lambda} \text{ and } e_{i,\lambda} \mapsto \frac{t_{i,\lambda}}{s_{i,\lambda}} e_{i,\lambda},$$

and define a map $\Psi$:

$$D_{\underline{s}}(n,r) \rightarrow D_{\underline{t}}(n,r)$$

given by

$$k_\lambda \mapsto k_\lambda, f_{i,\lambda} \mapsto f_{i,\lambda} \text{ and } e_{i,\lambda} \mapsto \frac{s_{i,\lambda}}{t_{i,\lambda}} e_{i,\lambda}.$$

For $\underline{a} = (a_1, a_2, \ldots, a_{n-1}) \in \mathbb{F}^{n-1}$, we write $\underline{t} = \underline{a} \underline{s}$ if $t_{i,\lambda} = a_i s_{i,\lambda}$ for all $1 \leq i \leq n-1$ and $\lambda \in B(n,r)$. We have the following proposition.

**Proposition 4.5.** The two algebras $D_{\underline{t}}(n,r) \cong D_{\underline{s}}(n,r)$ are isomorphic via the maps $\Phi$ and $\Psi$ defined above if and only if $\underline{t} = \underline{a} \underline{s}$, where $\underline{a} \in \mathbb{F}^{n-1}$ and $a_i \neq 0$ for all $1 \leq i \leq n-1$.

**Proof.** Suppose that the two algebras are isomorphic via the maps $\Phi$ and $\Psi$. As $\Phi$ and $\Psi$ are algebra homomorphisms, the images of generators should satisfy the generating relations of $D_{\underline{s}}(n,r)$ and $D_{\underline{t}}(n,r)$, respectively. By definition, when $\lambda_i \lambda_{j+1} \neq 0$ and $i \neq j$, we have the following equation.

$$\Phi(e_i f_j k_\lambda - f_j e_i k_\lambda) = \frac{t_{i,\lambda-\alpha_j}}{s_{i,\lambda-\alpha_j}} e_i f_j k_\lambda - \frac{t_{i,\lambda}}{s_{i,\lambda}} f_j e_i k_\lambda$$

$$= \left(\frac{t_{i,\lambda-\alpha_j}}{s_{i,\lambda-\alpha_j}} - \frac{t_{i,\lambda}}{s_{i,\lambda}}\right) e_i f_j k_\lambda$$

$$= 0. \tag{10}$$

This implies that

$$\frac{t_{i,\lambda-\alpha_j}}{s_{i,\lambda-\alpha_j}} = \frac{t_{i,\lambda}}{s_{i,\lambda}}$$

for $\lambda \in B(n,r)$ with $\lambda_i \lambda_{j+1} \neq 0$ and $i \neq j$.

Similarly, when $\lambda_i \lambda_{i+1} \neq 0$, we have the following equation.

$$\Phi(e_i f_i k_\lambda - f_i e_i k_\lambda) = \frac{t_{i,\lambda-\alpha_i}}{s_{i,\lambda-\alpha_i}} e_i f_i k_\lambda - \frac{t_{i,\lambda}}{s_{i,\lambda}} f_i e_i k_\lambda$$

$$= \left(\frac{t_{i,\lambda-\alpha_i}}{s_{i,\lambda-\alpha_i}} - \frac{t_{i,\lambda}}{s_{i,\lambda}}\right) e_i f_i k_\lambda$$

$$= 0.$$

This implies that

$$\frac{t_{i,\lambda-\alpha_i}}{s_{i,\lambda-\alpha_i}} = \frac{t_{i,\lambda}}{s_{i,\lambda}} \tag{11}$$
for $\lambda \in B(n, r)$ with $\lambda, \lambda_{i+1} \neq 0$. So

$$t = as.$$ as stated.

On the other hand, supposed that $t = as$ for some $a \in \mathbb{F}^{n-1}$ with all $a_i \neq 0$. By (10) and (11), we know that $\Phi$ is compatible with the commutative relations. Further, when $i = j - 1$, $\lambda_{i+1} \geq 2$ and $\lambda_{j+1} \geq 1$, the following Serre relation can be deduced from (10) and (11).

$$\Phi(e_i^2 e_j k_{\lambda} - e_i e_j e_i k_{\lambda}) = \frac{t_{i,\lambda} + a_i + a_j}{s_i,\lambda + a_i + a_j} \frac{t_{j,\lambda} + a_i}{s_j,\lambda + a_i} e_i e_j e_i k_{\lambda} - \frac{t_{i,\lambda} + a_i + a_j}{s_i,\lambda + a_i + a_j} \frac{t_{j,\lambda} + a_i}{s_j,\lambda + a_i} e_i e_j k_{\lambda}$$

$$= \frac{t_{i,\lambda} + a_i + a_j}{s_i,\lambda + a_i + a_j} \frac{t_{j,\lambda} + a_i}{s_j,\lambda + a_i} e_i e_j k_{\lambda} - \frac{t_{i,\lambda} + a_i + a_j}{s_i,\lambda + a_i + a_j} e_i^2 e_j k_{\lambda}$$

$$= 0.$$

Similarly we can prove that the Serre relations on $f_i$ can be deduced from (10) and (11). And the relations $C_{ij,\lambda}(t)$ can be proved directly without requirements on $t_i,\lambda$ and $s_{ij,\lambda}$. So $\Phi$ is a homomorphism of algebras with $\Psi$ the inverse, and so an isomorphism. This proves the proposition. \[\square\]

**Corollary 4.6.** Suppose that $t_{i,\lambda} = t_i \neq 0$ for all $\lambda \in B(n, r)$ and $1 \leq i \leq n - 1$. Then $D_2(n, r) \cong S_0(n, r)$.

5. The degeneration $DS_0(n, r)$ of $S_0(n, r)$

5.1. A new algebra defined on double flag varieties. Let $DS_0(n, r)$ be the $\mathbb{F}$-space with basis $\{e_A \mid A \text{ is in } \Xi(n, r)\}$. Define a multiplication in $DS_0(n, r)$ as follows,

$$e_A \ast e_B = \begin{cases} e_A \cdot e_B & \text{if } E(e_A) + E(e_B) = E(e_A \cdot e_B); \\ 0 & \text{otherwise.} \end{cases}$$

In other words, by Lemma 2.1 and Corollary 2.3 $DS_0(n, r)$ is the associated graded algebra of the filtered algebra $S_0(n, r)$. We often skip the multiplication signs in case of no confusion.

We give a geometric interpretation of the condition in the definition of $\ast$. Suppose that

$$e_A = [f, g], e_B = [g, h], \text{ and } e_C = [f', h']$$

with

$$e_A \cdot e_B = e_C$$

in $S_0(n, r)$. Recall that $V$ is an $r$-dimensional vector space defined over a field $k$. We view an $n$-step flag in $V$ as a representation of a linear quiver $A_n$, where vertex 1 is a source and vertex $n$ is a sink. Denote the indecomposable projective representation of $A_n$ by $P_i$.

**Lemma 5.1.** $\dim f' \cap h' \geq \dim f \cap g + \dim g \cap h - \dim g$. Consequently,

$$\dim f' \cap h' = \dim f \cap g + \dim g \cap h - \dim g$$

if and only if

$$E(e_A) + E(e_B) = E(e_A \cdot e_B).$$

**Proof.** Note that

$$E(e_A) = \dim f - \dim f \cap g, E(e_B) = \dim g - \dim h \cap g, E(e_A e_B) = \dim f' - \dim f' \cap h'$$

and $f \cong f'$ as representations. Now the lemma follows from Lemma 2.1. \[\square\]
Therefore the product $\ast$ can be defined as below.

**Lemma 5.2.**

$$[f,g] \ast [g,h] = \begin{cases} [f',h'] & \text{if } \dim f' \cap h' = \dim f \cap g + \dim g \cap h - \dim g, \\ 0 & \text{otherwise.} \end{cases}$$

**Example 5.3.** Let $A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$e_A e_B = e_C,$$

but

$$e_A \ast e_B = 0.$$  

By Lemma 3.2, the $e_C$ in Example 5.3 is contained in the ideal $I(2,5)$. This indicate that there should be a link between $DS_0(n,r)$ and the quotient algebra $S_0(n,r)/I(n,r)$. We will explore the relation and their relation to $D_0(n,r)$ in the remaining part of this section.

**Lemma 5.4.** $DS_0(n,r)$ is generated by $e_{i,\lambda}$, $f_{i,\lambda}$ and $k_{\lambda}$ for $\lambda \in \Lambda(n,r)$ and $1 \leq i \leq n-1$.

**Proof.** The lemma follows from the definition of multiplication in $DS_0(n,r)$ and the construction of the basis elements in Lemma 2.6.

Recall the algebra $\mathbb{F}\Sigma(n,r)/J(\mathbb{F})$ in Section 4. In particular, we have

$$D_0(n,r) = \mathbb{F}\Sigma(n,r)/J(\mathbb{F}).$$

The difference between $D_0(n,r)$ and $S_0(n,r)$ is that the idempotent relations in $S_0(n,r)$ are replaced by zero relations. Recall also for $j \geq i$,

$$e(i,j) = e_i \cdot e_{i+1} \cdot \ldots \cdot e_j.$$

**Lemma 5.5.** Let $i < j$ and $a, b \geq 0$ such that $j - i > a + b$. Then

$$e(i,j) \cdot e(i + a, j - b) = e(i + a, j - b) \cdot e(i,j)$$

in $D_0(n,r)$.

**Proof.** It suffices to prove that $e(i,j) \cdot e_l = e_l \cdot e(i,j)$ for all $l = i, \ldots, j$. In fact, due to the commutativity relation $e_me_{m'} = e_{m'}e_m$ when $|m - m'| > 1$, we need only prove $e(i,j) \cdot e_l = e_l \cdot e(i,j)$ for $(i,j) = (l - 1, l), (l - 1, l + 1), (l, l + 1)$. All the three cases follow from the relations $P_{mm'}$.

**Proposition 5.6.** $\dim_\mathbb{F} D_0(n,r) = \dim_\mathbb{F} S_0(n,r)$

**Proof.** Observe that in both algebras, a complete list of representatives of nonzero paths is a basis. In particular, the paths of the form in Lemma 2.6 (2), which are all reduced paths, form a basis for $S_0(n,r)$.

As the relations $P_{ij}$ and $N_{ij}$ are the same in both algebra, if two nonzero paths are the same in $D_0(n,r)$, they are then the same in $S_0(n,r)$. Further, any reduced path that is nonzero in $S_0(n,r)$ is also nonzero in $D_0(n,r)$. Indeed, take a nonzero reduced path $\rho = \ldots e_{i,\lambda} \ldots f_{j,\mu} \ldots$ in $\Sigma(n,r)$. Now suppose that $0 = \rho \in D_0(n,r)$. This implies that using relations $P_{ij,\lambda}$, $N_{ij,\lambda}$,

$$\rho = \ldots e_{i,\lambda - \alpha_i} f_{i,\lambda} f_{i,\lambda} k_{\lambda} \ldots \text{ or } \ldots k_{\lambda} f_{i,\lambda} \alpha_i e_{i,\lambda} \ldots \quad (\dagger)$$

with $\lambda$ at the boundary. So the new expression (\dagger) also holds in $S_0(n,r)$. By the relations $C_{ij,\lambda}$,

$$e_{i,\lambda - \alpha_i + \alpha_{i+1}} f_{i,\lambda} k_{\lambda} = k_{\lambda} f_{i,\lambda} \alpha_i e_{i,\lambda} = k_{\lambda},$$
which contradicts the minimality of the number of arrows in \( \rho \). So \( \rho \) is a non-zero path in \( D_\mathcal{Q}(n, r) \). Therefore

\( \dim_\mathbb{F} D_\mathcal{Q}(n, r) \geq \dim_\mathbb{F} S_0(n, r) \).

Next we claim that any reduced path \( \rho \) in \( D_\mathcal{Q}(n, r) \) is equal to a path of the form (2) in Lemma 2.6 and thus by the observation and the inequality (12), the dimensions of the two algebras are the same. We proceed the proof of the claim by induction on \( \rho \). When the length of \( \rho \) is at most 1, then it already has the required form. Assume that the length is larger than 1, and that \( \rho = \rho' e_i k_{\lambda} \). We have

\( \rho = \rho' e_i k_{\lambda} \) or \( \rho = f_i k_{\lambda} \)

where \( \rho' \) is a path of length one less than \( \rho \). By the induction hypothesis, we may write \( \rho' \) of the form (2) in Lemma 2.6

\[ \rho' = EFk_{\lambda}, \]

where \( E = e(i_s, j_s) \cdot \ldots \cdot e(i_1, j_1) \) and \( F = f(i'_1, j'_1) \cdot \ldots \cdot f(i'_t, j'_t) \). We first consider the case \( \rho = \rho' e_i k_{\lambda} \). Since \( \rho \) is reduced, we have

\[ \rho = EFe_i k_{\lambda} = Ee_i Fk_{\lambda}. \]

If \( j_m \neq i - 1 \) for all \( m \), the relations \( P_{ij} \) give us the required form

\[ Ee_i Fk_{\lambda} = e(i_s, j_s) \cdot \ldots \cdot e(i_i, j_i) \cdot e(i, i) \cdot \ldots \cdot e(i_1, j_1) Fk_{\lambda}, \]

where \( l \) is the smallest integer such that \( j_l > i \).

Let \( m \) be the smallest integer with \( j_m = i - 1 \). We have

\[ Ee_i = e(i_s, j_s) \cdot \ldots \cdot e(i_m, j_m + 1) \cdot e(i_1, j_1). \]

By repeatedly applying Lemma 5.3 and relations \( P_{ij} \), we can move \( e(i_m, j_m + 1) \) to the appropriate position.

The case \( \rho = \rho f_i k_{\lambda} \) is similar, and so we skip the details. In either case, the path \( \rho \) is equal to a path of the form (2) in Lemma 2.6, as claimed. \( \square \)

**Remark 5.7.** By Corollary 4.6 and Proposition 5.6, the algebra \( D_\mathcal{Q}(n, r) \) is a degeneration of the 0-Schur algebra \( S_0(n, r) \).

The following is the main result of this paper.

**Theorem 5.8.** We have isomorphisms

\[ S_0(n, r + n)/I(n, r + n) \cong D_\mathcal{Q}(n, r) \cong DS_0(n, r), \]

as \( \mathbb{F} \)-algebras.

**Proof.** We first prove the first isomorphism. Embed \( \Sigma(n, r) \) into the interior of \( \Sigma(n, r + n) \) via \( \phi : k_{\lambda} \mapsto k_\mu \) and embed the arrows correspondingly, where \( \mu = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_n + 1) \in \Lambda(n, r + n) \). Then observe that the relations \( \phi(P_{ij, \lambda}), \phi(N_{ij, \lambda}) \) and \( \phi(C_{ij, \lambda}(\mathcal{Q})) \) hold in \( S_0(n, r + n)/I(n, r + n) \). So we have a surjective map

\[ D_\mathcal{Q}(n, r) \twoheadrightarrow S_0(n, r + n)/I(n, r + n). \]

By Corollary 3.3

\[ \dim_\mathbb{F} S_0(n, r + n)/I(n, r + n) = \binom{n^2 + (n + r) - n - 1}{n + r - n} = \binom{n^2 + r - 1}{r}, \]

which is the dimension of \( S_0(n, r) \). So by Proposition 5.6

\[ \dim_\mathbb{F} S_0(n, r + n)/I(n, r + n) = \dim_\mathbb{F} D_\mathcal{Q}(n, r). \]
Thus the two algebras are isomorphic.

Mapping the vertices $k_\lambda$ and the arrows $e_{i,\lambda}$ and $f_{i,\lambda}$ to the corresponding generators of $DS_0(n, r)$ defines an algebra homomorphism $\psi$ from $D_0(n, r)$ to $DS_0(n, r)$. By Lemma 5.4, $\psi$ is an epimorphism. Note that by definition,

$$\dim_F DS_0(n, r) = \dim_F S_0(n, r)$$

and thus by Proposition 5.6,

$$\dim_F DS_0(n, r) = \dim_F D_0(n, r).$$

Therefore, $\psi$ is an isomorphism and thus the three algebras are isomorphic as claimed.

Remark 5.9.  
(1) By the isomorphisms in Theorem 5.8, we can view any of the three algebras as a degeneration of $S_0(n, r)$.

(2) The multiplication $\star$ in Lemma 5.7 gives a geometric interpretation of the multiplications in $DS_0(n, r)$. Thus the space of $GL(V)$-orbits in the double flag varieties $F \times F$ associated with the multiplication $\star$ gives a geometric construction of the three algebras in Theorem 5.8.

6. CENTRALIZER ALGEBRAS OF $D_0(n, r)$

In this section, we discuss relations between the 0-Hecke algebra $H_0(r)$, the nil-Hecke algebra $NH_0(r)$, the nil-Temperley-Lieb algebra $NTL(r)$ and degenerate 0-Schur algebras. Throughout this section, we let $\alpha = (1, \ldots, 1)$ be the composition in $\Lambda(r, r)$ with all the entries equal to 1.

6.1. $H_0(r)$, $NH_0(r)$ and $DS_0(r, r)$. We first recall some definitions and key facts. We then show that the associated graded algebras of $H_0(r)$ and $NH_0(r)$, with the filtration induced by the degree function discussed in Section 2.5, are isomorphic to a subalgebra of $k_\alpha DS_0(r, r)k_\alpha$.

Definition 6.1 ([2], [8]). The 0-Hecke algebra, denoted by $H_0(r)$, is the $F$-algebra generated by $T_1, T_2, \ldots, T_r$ with defining relations

$$\begin{cases} T_i^2 = T_i, & \text{for } 1 \leq i \leq r - 1; \\ T_i T_j = T_j T_i, & \text{for } 1 \leq i, j \leq r - 1 \text{ with } |i - j| > 1; \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & \text{for } 1 \leq i \leq r - 2. \end{cases}$$

Theorem 6.2 (Theorem 10.4, [11]). The algebras $H_0(r)$ and $k_\alpha S_0(r, r)k_\alpha$ are isomorphic via the the map $T_i \mapsto k_\alpha f_i e_i k_\alpha$.

Definition 6.3 ([18]). The nil-Hecke algebra, denoted by $NH_0(r)$, is the unital $F$-algebra generated by $T_1, T_2, \ldots, T_r$ with defining relations

$$\begin{cases} T_i^2 = 0, & \text{for } 1 \leq i \leq r - 1; \\ T_i T_j = T_j T_i, & \text{for } 1 \leq i, j \leq r - 1 \text{ with } |i - j| > 1; \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & \text{for } 1 \leq i \leq r - 2. \end{cases}$$

Assume that $w = s_{i_1} \cdots s_{i_t} = s_{j_1} \cdots s_{j_t}$ are reduced expressions in the symmetric group $S_r$ on $r$ letters, where $s_i$ is the transposition $(i, i+1)$. As reduced expressions of $w$ can be obtained from each other by using the braid relations only (see [15] [21]), we have

$$T_{i_1} \cdots T_{i_t} = T_{j_1} \cdots T_{j_t}.$$
in both $H_0(r)$ and $NH_0(r)$, and thus the element

$$T_w = T_{i_1} \cdots T_{i_t}$$

is well-defined in both algebras. One can also deduce that $\{T_w \mid w \in S_r\}$ is a basis for both algebras. Further,

$$(15) \quad \text{in } H_0(r), \quad T_i T_w = \begin{cases} T_{s_i w}, & \ell(s_i w) = \ell(w) + 1; \\ T_w, & \text{otherwise,} \end{cases}$$

and

$$(16) \quad \text{in } NH_0(r), \quad T_{w_1} T_{w_2} = \begin{cases} T_{w_1 w_2}, & \ell(w_1 w_2) = \ell(w_1) + \ell(w_2); \\ 0, & \text{otherwise}, \end{cases}$$

where $\ell : W \to \mathbb{N} \cup \{0\}$ is the length function of elements in $S_r$.

By Theorem 6.2, the filtration of the 0-Schur algebra $S_0(r, r)$ discussed in Section 2.5 induces a filtration on $H_0(r)$. When the length equation in (15) or (16) holds, the multiplications of $T_w$ and $T_{w'}$ in $H_0(r)$ and $NH_0(r)$ are the same, so we also have a filtration on $NH_0(r)$. Further, together with the fact that $DS_0(r, r)$ is the associated graded algebra of $S_0(r, r)$, this implies the following.

**Proposition 6.4.** The associated graded algebras of $H_0(r)$ and $NH_0(r)$ are isomorphic to the algebra $k \alpha DS_0(r, r) k_{\alpha}$.

### 6.2. Nil-Temperley-Lieb algebras

The nil-Temperley-Lieb algebra $NTL(r)$ is the quotient algebra of $NH_0(r)$ modulo the ideal generated by $T_i T_{i+1} T_i$ for $1 \leq i \leq r-2$ (see e.g. [10]).

**Lemma 6.5.** The algebra $NTL(r)$ has a basis consisting of $T_w$, where $w$ does not contain a subword of the form $s_i s_j s_i$ in any of its reduced expressions, for any $i$ and $j$ with $|i-j| = 1$.

**Proof.** It follows from the fact that $\{T_w \mid w \in S_r\}$ is a basis of $NH_0(r)$ and the multiplication in (16). \qed

Note that $\dim NTL(r)$ is the Catalan number $\frac{1}{r+1} \binom{2r}{r}$ and there is a useful combinatorial parametrization of the elements in a basis of $NTL(r)$, using Dyck words. Pictorially, Dyck words can be described using peak pictures in a triangle with $r$ dots on each edge (cf [19]). For instance, when $r = 3$, the five peak pictures are as follows.
Let 

\[ x_i = k_\alpha f_i e_i k_\alpha \in k_\alpha DS_0(n, r) k_\alpha. \]

Denote by \( DS(r, \alpha) \) the subalgebra of \( k_\alpha DS_0(n, r) k_\alpha \) generated by \( x_i \) for \( 1 \leq i \leq r - 1 \). When \( r = 3 \), the algebra \( DS(r, \alpha) \) is five dimensional, the orbit basis elements are determined by the following matrices.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

Similar to \( T_w \),

\[ x_w = x_{i_1} \ldots x_{i_t} \]

is well-defined, where \( s_{i_1} \ldots s_{i_t} \) is a reduced expression of \( w \). By direct computation following the fundamental multiplication rules and the definition of \( DS_0(n, r) \), we have the following lemma.

**Lemma 6.6.** The elements \( x_i \) for \( 1 \leq i \leq r - 1 \) satisfy the generating relations of \( NTL(r) \). Consequently, there is an epimorphism

\[ NTL(r) \rightarrow DS(r, \alpha), \ T_i \mapsto x_i, \ \forall i. \]

Any matrix that determines a nonzero orbit in \( DS(r, \alpha) \) is a permutation matrix. We call an nonzero entry that is below the diagonal a peak entry. For a peak entry \((i, j)\), which implies that \( j < i \), we call the entries \((j, j)\) and \((i, i)\) the feet of the peak. For the five matrices above, which determine the orbit basis elements in \( DS(r, \alpha) \), we connect the peaks, feet and diagonal 1-entry, using zig-zag lines as follows.

\[
\begin{array}{ccccccccccc}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

In this way, we obtain a well-defined one-to-one correspondence between the basis elements of \( NTL(3) \) and \( DS(3, \alpha) \). In fact, such a one-to-one correspondence exists for all \( r \).

**Theorem 6.7.** The two algebras \( NTL(r) \) and \( DS(r, \alpha) \) are isomorphic.

**Proof.** First note the nonzero orbits \( e_A \) in \( DS(r, \alpha) \) form a basis and each orbit \( e_A \) is uniquely determined by the matrix \( A \). By Lemma 6.6 there is a surjective morphism from \( NTL(r) \) to \( DS(r, \alpha) \). So it suffices to show that there exists a nonzero element generated by the \( x_i \)'s such that the corresponding matrix produces the peak picture.

First identify the triangle, in which we draw the peak pictures, with the lower triangular part of an \( r \times r \) matrix. The peaks give us peak entries, \((i_1, j_1), \ldots, (i_s, j_s)\) where for any \( l \),

\[ j_l < i_l < i_{l+1} \text{ and } j_l < j_{l+1}. \]

Let

\[ x^{(l)} = x_{i_{l-1}} \ldots x_{j_l} x_{j_l}, \text{ and } x = x^{(1)} \ldots x^{(s)}. \]
By the fundamental multiplication rules, the multiplication of \(x_{i+1}\) with \(x_i \ldots x_j k_\alpha\) moves a 1 on or above the diagonal further up in the same column and a 1 on or below the diagonal further down in the same column and thus
\[
E(x_{i+1}) + E(x_i \ldots x_j k_\alpha) = E(x_{i+1}x_i \ldots x_j k_\alpha).
\]
That is, the equality in the definition of \(\star\) holds. Therefore,
\[
x^{(s)} k_\alpha \neq 0.
\]
Similarly,
\[
x^{(l)}(x^{(l+1)} \ldots x^{(s)} k_\alpha) \neq 0
\]
and when compared with \(x^{(l+1)} \ldots x^{(s)} k_\alpha\), it has a new peak at \((i_l, j_l)\). Thus we have found a nonzero element \(x = xk_\alpha\) such that the corresponding matrix gives us the required peak picture. □

**Example 6.8.** In this example we demonstrate the process of obtaining the matrix with two peaks above Theorem 6.7 using the construction in the proof of the theorem. The peak picture has two peak entries \((2, 1)\) and \((3, 2)\). By definition,
\[
x^{(2)} = x_2 \text{ and } x^{(1)} = x_1.
\]
Then \(x^{(2)} k_\alpha\) and \(x^{(1)}x^{(2)} k_\alpha\) are the orbit basis elements corresponding to the matrices, respectively,
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]

We have the following special version of Lemma 5.2.

**Lemma 6.9.** Use the same notation Lemma 5.2. Further, assume that all the flags are isomorphic to \(\bigoplus_{i=1}^n P_i\) and \([f, g] = x_i\) for some \(i\). Then the following is true in \(DS(r, \alpha)\).
\[
[f, g] \star [g, h] = \begin{cases} [f', h'] & \text{if } \dim f' \cap h' = \dim g \cap h - 1; \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** Note that as a representation, \(g/(f \cap g)\) is isomorphic to the simple representation of the linear quiver \(A_n\) at vertex \(i\), since \([f, g] = x_i\). Now the lemma follows from Lemma 5.1 and 5.2. □

**Remark 6.10.** In the light of Lemma 6.9, Theorem 6.7 gives a geometric realisation of nil-Temperley-Lieb algebras, via double flag varieties.

### 6.3. An observation.
Using the presentation of \(S_0(r, r)\) in Section 2.4, we define a new algebra \(\hat{DS}_0(r, r)\) as a quotient algebra of \(\mathbb{F} \Sigma(r, r)\). This new algebra has the same relations as \(S_0(r, r)\) except that the idempotent relations
\[
e_{i, \alpha} e_{i, \lambda} = k_\lambda, \quad f_{i, \alpha} e_{i, \mu} = k_\mu
\]
are replaced by
\[
e_{i, \alpha} f_{i, \lambda} = 0 \quad \text{and} \quad f_{i, \alpha} e_{i, \mu} = 0,
\]
for \(1 \leq i \leq r - 1\), where \(\lambda = \alpha + \alpha_i\), \(\mu = \alpha - \alpha_i\), \(\alpha\) is the composition in \(\Lambda(r, r)\) with all the entries equal to 1 and \(\alpha_i\) is defined in Section 2.4. The new algebra \(\hat{DS}_0(r, r)\) has fewer zero relations coming from idempotent relations in \(S_0(r, r)\) than \(D_2(r, r)\). We
only force those idempotent relations that go via the centre $k_\alpha$ of the simplex $\Sigma(n, r)$ to be 0. Let
\[ y_i = e_{i,a} - f_{i,a}. \]

Note that by the multiplication rules,
\[ y_i = e_i f_{i,a} = e_{i,a} f_i, \]
because once the starting or ending idempotent is given, then the product of a sequence of $e_i$s and $f_j$s is determined, we don’t need to indicate the composition indices $\lambda$ for the other factors.

Lemma 6.11. The elements $y_i$, where $1 \leq i \leq r - 1$, satisfy the generating relations of $NTL(r)$.

Proof. It can be checked directly that when $|i - j| > 1$, $y_i y_j = y_j y_i$ for $1 \leq i, j \leq r - 1$. So it suffices to prove that $y_i y_{i+1} y_i = y_{i+1} y_i y_{i+1} = 0$. Since the proof is similar, we only prove that $y_i y_{i+1} y_i = 0$.

For $1 \leq i \leq r - 1$, $\alpha - \alpha_i + \alpha_{i+1}$ is of the form $(1, \ldots, 1, 0, 3, 0, 1, \ldots, 1)$, which has 3 at the $(i + 1)$th entry. We first prove that $k_{\alpha_i - \alpha_i + \alpha_{i+1}} = 0$. We have
\[
k_{\alpha_i - \alpha_i + \alpha_{i+1}} = f_i e_i f_{i+1} e_{i+1} f_{i+1} k_{\alpha_i - \alpha_i + \alpha_{i+1}}
= f_i e_i f_{i+1} e_{i+1} f_{i+1} k_{\alpha_i - \alpha_i + \alpha_{i+1}}
= f_i e_i f_{i+1} e_{i+1} f_{i+1} k_{\alpha_i - \alpha_i + \alpha_{i+1}}
= 0,
\]
by (17). Consequently,
\[
y_i y_{i+1} y_i = e_i f_{i+1} e_{i+1} f_{i+1} e_{i+1} f_{i+1} e_{i+1} f_{i+1} e_{i+1} f_{i+1} = 0.
\]
This proves the lemma. \qed

Example 6.12 (An observation).

The composition of two paths is written in the form such as $\beta_3 : 5 \rightarrow 6$, $\beta_5 : 6 \rightarrow 7$, then $\beta_5 \beta_3 : 5 \rightarrow 7$. The algebra $DS_0(3, 3)$ is the path algebra of $\Sigma(3, 5)$ with zero
relations
\[ \beta_1 \beta_2 = 0, \beta_4 \beta_3 = 0, \beta_5 \beta_6 = 0, \beta_8 \beta_7 = 0 \]
and the other generating relations of \( S_0(3, 3) \).

By computation we have that \( k_\alpha \hat{\mathcal{D}}S_0(3, 3)k_\alpha \) is in fact generated by
\[ y_1 = \beta_2 \beta_1 = \beta_7 \beta_8 \quad \text{and} \quad y_2 = \beta_6 \beta_5 = \beta_3 \beta_4. \]

The vertices 1, 4, 10 correspond to compositions \( \lambda = (0, 0, 3), \mu = (0, 3, 0), \nu = (3, 0, 0) \), respectively. By Lemma [6.11] and similar proof or the generating relations in \( \hat{\mathcal{D}}S_0(3, 3) \), we have
\[ k_\lambda = 0, \quad k_\mu = 0, \quad k_\nu = 0 \]
and
\[ y_1 y_2 y_1 = y_2 y_1 y_2 = 0. \]

So the algebra \( k_\alpha \hat{\mathcal{D}}S_0(3, 3)k_\alpha \) is 5-dimensional and \( 1, y_1, y_2, y_1 y_2, y_2 y_1 \) form a basis. So
\[ k_\alpha \hat{\mathcal{D}}S_0(3, 3)k_\alpha \cong NTL(3). \]

It would be interesting to find out whether \( k_\alpha \hat{\mathcal{D}}S_0(r, r)k_\alpha \cong NTL(r) \) for any \( r \).

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BTJ: Department of Mathematical Sciences, NTNU in Gjøvik, Norwegian University of Science and Technology, 2802 Gjøvik, Norway.
Email: bernt.jensen@ntnu.no

XS: Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom.
Email: xs214@bath.ac.uk

GY: School of Science, Shandong University of Technology, Zibo 255000 , China
Email: yanggy@mail.bnu.edu.cn