Novel Resistive Distance Descriptors on Complex Network

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ABSTRACT Large-scale complex network data poses significant challenges for the analytic ideas and tools to monitor and analyze complex networks. As classical structure descriptors, average path length ($L$), global and local network efficiency ($E_{glob}$ and $E_{loc}$), general and loop clustering coefficient ($C_3$ and $D$), play significant roles in complex network analysis. In this paper, we make use of resistive distance instead of the shortest path distance to suggest resistive distance descriptors, i.e., $L_r$, $\Omega_{glob}$ and $\Omega_{loc}$, $\Omega_3$ and $D_r$. We investigate all the resistive descriptors on classical WS, BA and ER models and find some interesting phenomenons. On one hand, $L_r$ and $\Omega_l$ (resp., $\Omega_{glob}$ and $\Omega_{loc}$) can be used to characterize the features of small-word networks. On the other hand, $L_r$, $\Omega_3$, and $\Omega_{glob}$ can be utilized to measure network invulnerability. To access the effectiveness of the resistive distance descriptors, we conduct extensive numerical simulations on synthetic and real networks. In comparison with the baselines, the proposed metrics show competitive performance on classical network models and are more efficient for networks with small and medium size.

INDEX TERMS resistive distance, clustering coefficient, network efficiency, network invulnerability

I. INTRODUCTION

DurIng the last decades, network science, as a mature interdisciplinary field, has gained considerable attentions. Complex networks describe a wide range of systems in nature and society, such as biological, social, economic, power, traffic systems. These complex systems, which show nontrivial and elaborate characteristics, are neither absolutely random nor perfectly regular. The statistical analysis of graphs, as a field of mathematics, play an important role to understand real-world systems. The three influential statistical graph models, Barabási-Albert model (BA) [1], Watts-Strogatz model (WS) [2] and Erdös-Rényi model (ER) [3], are introduced to generate graphs and to predict some of their topological properties, such as degree distribution, average path length, clustering coefficient. Small-world networks, in terms of WS models, present two significant features of large clustering coefficient and short average path length. Scale-free networks, illustrated by BA models, emphasized that the degrees of nodes follow a power law distribution. An ER random network is also known as a binomial network, whose clustering coefficient is equal to its probability for edge creation.

In recent years, the researches of malicious attacks on complex networks have attracted great attention. Malicious attacks can cause serious failures on various networks, such as traffic network [4], power grid network [5], supply chain network [6], etc. Many network security problems in the real world can be attributed to the invulnerability problem of the networks in terms of cascade failures. Generally, the invulnerability of networks is tightly related to their topological structures. The Scale-free network is robust under the random attack, while vulnerable under the high degree nodes attack [7]. Jalili [8] investigated how the small-worldness of networks changed when nodes are removed randomly or systematically. The network metrics, such as network efficiency, the size of the largest connected component, average path length, clustering coefficient, are commonly used to monitor network invulnerability. The clustering coefficient, introduced by Watts and Strogatz [2] to measure how tightly a node’s neighbors are connected with each other, is an
important statistical tool to describe the structure of complex networks. The concept of efficiency of a network, introduced by Latora and Marchiori [9], is to measure how it exchanges information efficiently. The efficiency \( \epsilon_{ij} \), between nodes \( i \) and \( j \), is inversely proportional to the shortest path distance \( d_{ij} \), denoted by \( \epsilon_{ij} = \frac{1}{d_{ij}} \). By using efficiency measure, the small-world behavior can be interpreted and the small-world network is shown to be both locally and globally efficient.

As an interesting notion of distance on graphs, resistive distance, proposed by Klein and Randić [10] based on an electrical-theoretic approach in 1993, has been investigated by Stephenson and Zelen [11] in 1989 in terms of information-theoretic approach. Let a network be viewed as an electrical circuit, where each edge corresponds to a resistor and each node is a junction between resistors. Consider a unit current enters at source \( i \) and leaves from target \( j \). Effective resistance between \( i \) and \( j \), denoted by \( r_{ij} \), can be denoted by potential difference between them. A high potential difference between nodes \( i \) and \( j \) means that \( i \) and \( j \) are far away. In fact, \( r_{ij} \) satisfies non-negativity, symmetry and triangle inequality and can be viewed as a distance function. Note that \( r_{ij} \leq d_{ij} \), with equality if and only if there is only one path from \( i \) to \( j \), where \( d_{ij} \) represents the shortest path distance between \( i \) and \( j \). Refer to [12]–[15] for different calculation methods of resistive distance. A main topic about resistance distance is its calculation in various graphs. Relevant literatures are available in [16]–[18]. In recent years, many scholars have studied resistive distance in different aspects of complex network, such as community detection [19]–[23], centrality measures [24]–[26], link prediction problem [27] and network robustness [28]–[31], etc. By the way, resistance in the network is intimately connected with the lengths of random walks on the graph. For any \( i \) and \( j \), \( T_{ij} = 2m r_{ij} \), where \( m \) is the number of edges of the \( G \), \( T_{ij} \) indicates the commute time and \( r_{ij} \) represents the resistive distance. Refer to [32] for more details.

In this paper, we aim to design novel descriptors using resistive distance instead of the shortest path distance. To sum up, Our contributions are summarized as follows:

- Novel resistive descriptors, shown in Table 1, are suggested to explore on classical WS, BA and ER models.
- Motivated by \( L \) and \( C_3 \) (resp., \( E_{glob} \) and \( E_{loc} \)), \( L_r \) and \( \Omega_3 \) (resp., \( \Omega_{glob} \) and \( \Omega_{loc} \)) are employed to characterize small-world networks.
- In comparison of \( L \), \( C_3 \) and \( E_{glob} \), \( L_r \), \( \Omega_3 \) and \( \Omega_{glob} \) are utilized to measure network invulnerability and the Kendall’s Tau coefficient is utilized to analyze each resistive descriptor’s correlations with that based on shortest path length.

The rest of the paper proceeds as follows. Section 2 reviews classic structure descriptors and a brief overview of resistive distance. Next, we introduce several novel structure descriptors based on resistive distance in Section 3. Then, Section 4 presents the algorithm descriptions and Section 5 shows experiment results and analysis. Finally, Section 6 concludes this paper with a summary and proposes future directions.

### II. PRELIMINARIES

A complex network can be described as a graph \( G = (V, E) \), where \( V \) denotes the set of nodes and \( E \) represents the set of edges. The Laplacian matrix \( L_G = L_G - A_G \), where \( D_G \) is the degree matrix with \( (D_G)_{ii} = k_i \), and \( A_G \) indicates the adjacency matrix, with \( (A_G)_{ij} = 1 \) when an edge \( (i, j) \) exists and \( (A_G)_{ij} = 0 \) otherwise.

In this paper, unless differently stated, \( G \) represents a connected, undirected and unweighted graph without self-loops and multiple links. Ordinarily, \( N \) denotes network size, \( k_i \), \( \Gamma_i \) and \( G_i \) indicate the degree, the set of neighbors and the induced subgraph of open neighborhood of node \( i \), respectively. \( d_{ij} \) and \( r_{ij} \) represent the shortest path distance and resistive path distance between node \( i \) and \( j \), respectively. Summary of main notations are shown in Table 2.

| A. CLASSIC NETWORK DESCRIPTORS |
|--------------------------------|
| Now, we introduce some commonly used global and local statistical descriptors, which will be used as baselines for comparison in experimental section. |

### Table 2. Summary of main notations.

| notations | descriptions |
|-----------|--------------|
| \( G \) | A simple undirected and connected graph. |
| \( N \) | Network size. |
| \( k_i \) | The degree of node \( i \). |
| \( \Gamma_i \) | The set of neighbors of node \( i \). |
| \( d_{ij} \) | The shortest path distance between node \( i \) and \( j \). |
| \( r_{ij} \) | The resistive path distance between node \( i \) and \( j \). |
| \( d_{ik}/s \) | The shortest path distance between \( j \) and \( k \) not passing through \( i \). |
| \( L_G \) | Laplacian matrix of \( G \). |
| \( D_G \) | Degree matrix of \( G \). |
| \( A_G \) | Adjacency matrix of \( G \). |
| \( R_m(G) \) | Resistive distance matrix of \( G \). |
| \( D_m(G) \) | Shortest path distance matrix of \( G \). |
| \( \tau \) | Kendall’s Tau correlation coefficient. |
| \( G_i \) | The induced subgraph of open neighborhood of \( i \). |
| \( G_i \) | The induced subgraph of closed neighborhood of \( i \). |
| \( D(i) \) | The loop coefficient of an arbitrary node \( i \). |
| \( R(l) \) | The cyclic coefficient of node \( i \). |
| \( C_3(i) \) | The clustering coefficient of node \( i \). |
| \( C_3(G) \) | The average clustering coefficient of \( G \). |
Wiener index ($W_r$) [33]

The Wiener index of a graph is the sum of the shortest-path distances between each pair of reachable nodes, i.e.,

$$W_r = \sum_{i<j} d_{ij},$$

(1)

where $d_{ij}$ is the shortest path distance between $i$ and $j$. The Wiener index is one of the well-known distance-based molecular structure descriptors, which can be used for modeling physical, pharmacologic, biological and other properties of chemical compounds.

Average path length ($L$) [2]

The average path length $L$ denotes the average distance between two generic nodes, i.e.,

$$L = \frac{1}{N(N-1)} \sum_{i\neq j \in G} d_{ij},$$

(2)

where $d_{ij}$ indicates the shortest path distance between $i$ and $j$ and $N$ is the number of nodes of $G$. Many real world networks (graphs) are observed to be small worlds, i.e., the average path length among nodes is small.

Global efficiency ($E_{glob}$) [9]

The global efficiency of $G$, denoted by $E_{glob}(G)$, is defined as

$$E_{glob}(G) = \frac{1}{N(N-1)} \sum_{i\neq j \in G} \frac{1}{d_{ij}},$$

(3)

where $d_{ij}$ indicates the shortest path distance between $i$ and $j$ and $N$ is the number of nodes of $G$. $E_{glob}$ quantifies the exchange of information across the whole network on a global scale, where information is concurrently exchanged.

Local efficiency ($E_{loc}$) [9]

The local efficiency of node $i$, denoted by $E_{loc}(i)$, i.e.,

$$E_{loc}(i) = E_{glob}(G_i),$$

(4)

where $G_i$ represents the induced subgraph of open neighborhood of node $i$. The local efficiency $E_{loc}(i)$ quantifies a network’s resistance to failure on a small scale. The average local efficiency of $G$, denoted by $E_{loc}(G)$, i.e.,

$$E_{loc}(G) = \frac{1}{N} \sum_{i \in G} E_{loc}(i) = \frac{1}{N} \sum_{i \in G} E_{glob}(G_i).$$

(5)

Clustering coefficient ($C_3$) [2]

The clustering coefficient of node $i$, denoted by $C_3(i)$, i.e.,

$$C_3(i) = \frac{2}{k_i(k_i-1)} \sum_{j \neq k \in \Gamma_i} n_{jk},$$

(6)

where $k_i$ and $\Gamma_i$ indicate the degree and the set of neighbors of node $i$, respectively. $n_{jk}=1$ if there is a link between $j$ and $k$, and $n_{jk}=0$ otherwise. $C_3(i)$ gives the ratio of the existing number of links between the neighbors of node $i$ and its maximum possible number $k_i(k_i-1)/2$. $C_3(G)$ denotes the average clustering coefficient of $G$.

Quadrilateral coefficient ($C_4$) [34]

The quadrilateral coefficient of node $i$, denoted by $C_4(i)$, i.e.,

$$C_4(i) = \frac{\sum_{m=1}^{k_i} \sum_{n=m+1}^{k_i} a_i(m,n) + q_i(m,n)}{\sum_{m=1}^{k_i} \sum_{n=m+1}^{k_i} [a_i(m,n) + q_i(m,n)]},$$

(7)

where $m$ and $n$ are neighbors of the node $i$, $q_i(m,n)$ are the number of common neighbors between $m$ and $n$, and $a_i(m,n) = (k_m - \eta_i(m,n))(k_n - \eta_i(m,n))$ with $\eta_i(m,n) = 1 + q_i(m,n) + \theta_{mn}$, where

$$\theta_{mn} = \begin{cases} 1, & (m,n) \in E, \\ 0, & \text{otherwise}. \end{cases}$$

(8)

$C_4$ gives the probability that two neighbors of node $i$ share a common neighbor, which is different from $i$. $C_4(G)$ denotes the average quadrilateral coefficient of $G$.

Cyclic coefficient ($R$) [37]

The cyclic coefficient of node $i$, denoted by $R(i)$, i.e.,

$$R(i) = \frac{2}{k_i(k_i-1)} \sum_{(lm)} \frac{1}{S_{lm}^i},$$

(9)

where $k_i$ indicates the degree of node $i$ and $(lm)$ denotes all the pairs of the neighbors of the node $i$. $S_{lm}^i$ is the size of the smallest loop passing through node $i$ and its pair of neighbors $(l,m)$. $R$ can characterize the cyclic structures of complex networks and it takes values between 0 and $1/3$. If the network is a perfect tree-like structure, $R = 0$. The larger value of $R$ indicates that the network is more cyclic. $R(G)$ denotes the average cyclic coefficient of $G$.

Loop coefficient ($D$) [38]

The loop coefficient of an arbitrary node $i$ is denoted by

$$D(i) = \frac{2}{k_i(k_i-1)} \sum_{j \neq k \in \Gamma_i} \frac{1}{d_{jk/i}},$$

(10)

where $k_i$ represents the degree of $i$, $\Gamma_i$ indicates the set of neighbors of node $i$, and $d_{jk/i}$ is the length of shortest path between $j$ and $k$ not passing through $i$. Loop coefficient varies from 0 to 1. In addition, $D(i)$ coincides with $C_3(i)$, if only the shortest paths $d_{jk/i}=1$ are taken into account. Therefore, Loop coefficient is a generalization of the clustering coefficient. $D(G)$ denotes the average loop coefficient of $G$.

Subgraph centrality ($SC$) [35]

The subgraph centrality of an arbitrary node $i$ is denoted by

$$SC(i) = \sum_{j=1}^{N} (v_j^i)^2 e^{\lambda_j},$$

(11)
where $v_j$ is an eigenvector of the adjacency matrix $A$ of $G$ corresponding to the eigenvalue $\lambda_j$. Furthermore, the average subgraph centrality of $G$ is denoted by

$$\langle SC \rangle = \frac{1}{N} \sum_{i=1}^{N} SC(i) = \frac{1}{N} \sum_{i=1}^{N} e^{\lambda_i}. \quad (12)$$

By the way, natural connectivity $\tilde{\lambda}$ [36] of $G$, which is highly correlated with subgraph centrality, is denoted by,

$$\tilde{\lambda} = \ln [(\langle SC \rangle)] \approx \ln \left[ \frac{1}{N} \sum_{i=1}^{N} e^{\lambda_i} \right]. \quad (13)$$

Natural connectivity (resp., subgraph centrality), as a measure of structure robustness in complex networks, characterizes the redundancy of alternative routes in a network by quantifying the weighted number of closed walks of all lengths.

**B. EFFECTIVE RESISTANCE**

An electrical network is defined as $N = (G, r) = (V, E, r)$, where $r$ is a function: $E \rightarrow R^+$, and $r_{ij}$ denotes the resistance of the edge $e$. Also, we define an electrical network in the form $N = (G, c)$, where $c$ is a conductance function, and $c_{ij} = \frac{1}{r_{ij}}$ represents the conductance of the edge $e$. For any $e \in E$, if $r_{ij} = 1$, we call the electrical network simple.

Klein and Randić [10] have proved that the effective resistance is indeed a distance function. Previous studies [10], [12], [14], [25] have shown that Laplacian matrix $L_G$ plays a vital role in the theory of the resistive distance. In this work, the resistive matrix of $G$ is defined as $R_m(G)$ and its calculation is shown in algorithm 1 [39]. All the proposed resistive descriptors are calculated based on $R_m(G)$. Here, we will give some related results from [10].

**Definition 2.1:** [10] The total effective resistance, also called Kirchhoff index, is the sum of the effective resistance over all pairs of nodes in $G$:

$$Kf = \sum_{i<j}^2 r_{ij}. \quad (14)$$

It can be expressed in terms of Laplacian eigenvalues as follows:

$$Kf = n \sum_{k=2}^{n} \frac{1}{\mu_k}, \quad (15)$$

where $\mu_1 < \mu_2 \leq \cdots \leq \mu_n$ are the eigenvalues of Laplacian matrix $L$.

**Theorem 2.2:** [10] For all distinct pairs of nodes $i$, $j$ in $G$, $d_{ij} \geq r_{ij}$, with equality if there is only one path between $i$ and $j$.

**Corollary 2.3:** [10] If $G$ is a tree, the conventional distance is the same as the effective resistance distance, i.e., $d_{ij} = r_{ij}$.

### III. NOVEL NETWORK DESCRIPTORS BASED ON RESISTIVE DISTANCE

#### A. AVERAGE PATH LENGTH BASED ON RESISTIVE DISTANCE

**Definition 3.1:** ($L_r$)

The average path length of $G$ is denoted by $L_r$, i.e.,

$$L_r = \frac{1}{N(N-1)} \sum_{i \neq j \in G} r_{ij}. \quad (16)$$

Equivalently,

$$L_r = \frac{2}{N(N-1)} Kf. \quad (17)$$

#### B. EFFICIENCY DESCRIPTORS BASED ON RESISTIVE DISTANCE

**Definition 3.2:** ($\Omega_{\text{glob}}$)

The global efficiency of $G$ is denoted by $\Omega_{\text{glob}}(G)$, i.e.,

$$\Omega_{\text{glob}}(G) = \frac{1}{N(N-1)} \sum_{i \neq j \in G} \frac{1}{r_{ij}}. \quad (18)$$

**Definition 3.3:** ($\Omega_{\text{loc}}$)

The local efficiency of node $i$ is denoted by $\Omega_{\text{loc}}(i)$, i.e.,

$$\Omega_{\text{loc}}(i) = \Omega_{\text{glob}}(G_i), \quad (19)$$

where $G_i$ is the induced subgraph of neighbors of $i$ such that $i \notin G_i$. Usually, $G_i$ is a disconnected graph. For any $j, k \in G_i$, when there is no path between $j$ and $k$, we define $r_{jk} = +\infty$, i.e., $\frac{1}{r_{jk}} = 0$. Besides, the local efficiency of $G$ is denoted by $\Omega_{\text{loc}}(G)$, i.e.,

$$\Omega_{\text{loc}}(G) = \frac{1}{N} \sum_{i \in G} \Omega_{\text{loc}}(i) = \frac{1}{N} \sum_{i \in G} \Omega_{\text{glob}}(G_i). \quad (20)$$

By the way, in this work, another local efficiency $\Omega_{\text{loc}'}$ is defined, i.e.,

$$\Omega_{\text{loc}'}(i) = \Omega_{\text{glob}}(\hat{G}_i), \quad (21)$$

where $\hat{G}_i$ is the induced subgraph of neighbors of $i$ such that $i \in \hat{G}_i$. $\Omega_{\text{loc}'}(G)$ indicates the average local efficiency of $G$. Similarly, the formula based on shortest paths distance is given by

$$E_{\text{loc}'}(G) = E_{\text{glob}}(\hat{G}). \quad (22)$$

Furthermore, $E_{\text{loc}'}(G)$ denotes the average local efficiency of $G$.

#### C. CLUSTERING COEFFICIENT BASED ON RESISTIVE DISTANCE

**Definition 3.4:** ($\Omega_3$)

The resistive clustering coefficient of an arbitrary node $i$ is defined as

$$\Omega_3(i) = \frac{2}{k_i(k_i-1)} \sum_{j \neq k} \sum_{i \in G} \eta_{jk}, \quad (23)$$

where $\eta_{jk} = r_{jk}$ if there is a link between $j$ and $k$, and $\eta_{jk} = 0$ otherwise. $\Omega_3(G)$ denotes the average clustering coefficient of $G$ in terms of resistive distance.
Definition 3.5: \((D_r)\)

The resistive loop coefficient of an arbitrary node \(i\) is defined as

\[
D_r(i) = \frac{2}{k_l(k_l - 1)} \sum_{j \neq k \in \Gamma_i} \frac{1}{r_{jk}}.
\]

(D) EXAMPLE EXPLANATION

Based on the above definitions, some simple examples are presented to explain how the proposed descriptors work in a specific network.

1. The First Example.

As shown in Figure 1 (a), a simple \(G'\) is given. Figure 1 (d)-(h) represent the induced subgraphs \(G'_i\) of open neighborhood of \(i\), while Figure 1 (i)-(m) show the induced subgraphs \(G'_i\) of close neighborhood of \(i\). Figure 1 (b) and (c) describe how to obtain \(G'_2\).

At first, we calculate \(W_r, L, E_{glob}(G'), K_f, L_r,\) and \(\Omega_{glob}(G')\). The matrix of the shortest path distance of \(G'\) is

\[
D_m[G'] = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 2 \\
2 & 1 & 1 & 0 & 1 \\
3 & 1 & 2 & 1 & 0 \\
4 & 2 & 2 & 1 & 2 \\
\end{pmatrix}
\]

According to Eq. (1) and \(D_m[G']\),

\[
W_r = \sum_{i<j} d_{ij} = d_{01} + d_{02} + d_{03} + d_{04} + d_{12} + d_{13} + d_{14} + d_{23} + d_{24} + d_{34} = 1 + 1 + 1 + 2 + 1 + 2 + 1 + 1 + 2 = 14.
\]

According to Eq. (2) and \(D_m[G']\),

\[
L = \frac{2}{N(N-1)} W_r = 1.4.
\]

According to Eq. (3) and \(D_m[G']\),

\[
E_{glob}(G') = \frac{1}{N(N-1)} \sum_{i \neq j \in G'} \frac{1}{d_{ij}} = \frac{2}{[5 * (5 - 1)]} \left( \frac{1}{d_{01}} + \frac{1}{d_{02}} + \frac{1}{d_{03}} + \frac{1}{d_{04}} + \frac{1}{d_{12}} + \frac{1}{d_{13}} + \frac{1}{d_{14}} + \frac{1}{d_{23}} + \frac{1}{d_{24}} + \frac{1}{d_{34}} \right) = 0.1 \left( 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{2} \right) = 0.8.
\]

In addition, according to algorithm 1, the resistive distance matrix of \(G'\) is

\[
R_m[G'] = \begin{pmatrix}
0 & 0 & 0.62 & 0.62 & 1.5 \\
1 & 0.62 & 0 & 0.62 & 1.63 \\
2 & 0.5 & 0.62 & 0 & 0.62 & 1 \\
3 & 0.62 & 1 & 0.62 & 0 & 1.63 \\
4 & 1.5 & 1.63 & 1 & 1.63 & 0 \\
\end{pmatrix}
\]

According to Eq. (14) and \(R_m[G']\),

\[
K_f = \sum_{i<j} r_{ij} = r_{01} + r_{02} + r_{03} + r_{04} + r_{12} + r_{13} + r_{14} + r_{23} + r_{24} + r_{34} = 0.62 + 0.5 + 0.62 + 1.5 + 0.62 + 1 + 1.63 + 0.62 + 1 + 1.63 = 9.74.
\]

According to Eq. (17) and \(R_m[G']\),

\[
L_r = \frac{2}{N(N-1)} K_f = 0.974.
\]
According to Eq. (18) and $R_m[G']$,
\[
\Omega_{glob}(G') = \frac{1}{N(N-1)} \sum_{i \neq j \in G'} \frac{1}{r_{ij}} = \frac{2}{[4 \times (4-1)]} \left( \frac{1}{r_{01}} + \frac{1}{r_{02}} + \frac{1}{r_{03}} + \frac{1}{r_{04}} + \frac{1}{r_{12}} + \frac{1}{r_{13}} + \frac{1}{r_{14}} + \frac{1}{r_{23}} + \frac{1}{r_{24}} + \frac{1}{r_{34}} \right) \approx 0.1\left(0.62 + \frac{1}{1.63} + 0.62 + \frac{1}{1 + 1.63} + \frac{1}{1.63} + \frac{1}{1 + 0.62} + \frac{1}{0.62} + \frac{1}{1.5} + \frac{1}{0.62} + \frac{1}{1.63} + \frac{1}{1 + 1.63} + \frac{1}{1 + 1.63} + \frac{1}{1 + 0.62} + \frac{1}{0.62} + \frac{1}{1.5} \right) = 1.23.
\]

Next, in terms of node 2, $E_{loc}(2)$, $\Omega_{loc}(2)$, $\Omega_3(2)$, $D_r(2)$, $E_{loc}(2)$ and $\Omega_{loc}(2)$ are computed below.

As shown in Figure 1 (b) and (c), remove node 2 from $G'$ and obtain $G_2^r$, which is the induced subgraph of neighbors of node 2. $G_2^r$ has two components $C_1$ and $C_2$. As is well known, a graph is a tree if only if it is connected and has no loop. More than one tree can constitute a forest. Clearly, both $C_1$ and $C_2$ are trees, and $C_2$ is a trivial tree. $G_2^r$ is a forest. According to Corollary 2.3, for any $i$ and $j$ of a tree, $r_{ij} = d_{ij}$. Thus, the distance matrix of $G_2^r$ can be denoted as
\[
R_m[G_2^r] = D_m[G_2^r] = \begin{pmatrix}
0 & 1 & 3 & 4 \\
0 & 0 & 1 & 1 & +\infty \\
0 & 1 & 0 & 2 & +\infty \\
3 & 1 & 2 & 0 & +\infty \\
4 & +\infty & +\infty & +\infty & +\infty \\
\end{pmatrix}.
\]

According to Eqs. (3)-(4) and $D_m[G_2^r]$,
\[
E_{loc}(2) = E_{glob}(G_2^r) = \frac{1}{N(N-1)} \sum_{i \neq j \in G_2^r} \frac{1}{d_{ij}} = \frac{2}{[4 \times (4-1)]} \left( \frac{1}{d_{01}} + \frac{1}{d_{03}} + \frac{1}{d_{04}} + \frac{1}{d_{13}} + \frac{1}{d_{14}} + \frac{1}{d_{34}} \right) = 2\left(\frac{1}{1} + 0 + \frac{1}{2} + 0 + 0\right) = 0.42.
\]

Similarly, according to Eqs. (18)-(19) and $R_m[G_2^r]$,
\[
\Omega_{loc}(2) = \Omega_{glob}(G_2^r) = \frac{1}{N(N-1)} \sum_{i \neq j \in G_2^r} \frac{1}{r_{ij}} = \frac{2}{[4 \times (4-1)]} \left( \frac{1}{r_{01}} + \frac{1}{r_{02}} + \frac{1}{r_{03}} + \frac{1}{r_{04}} + \frac{1}{r_{12}} + \frac{1}{r_{13}} + \frac{1}{r_{14}} + \frac{1}{r_{23}} + \frac{1}{r_{24}} + \frac{1}{r_{34}} \right) = 2\left(\frac{1}{1} + 0 + \frac{1}{2} + 0 + 0\right) = 0.42.
\]

In this example, for any $i \in G'$, $G'_i$ is either a tree or a forest (see Figure 1 (d)-(g)). That is, only one shortest path exists between any pair of connected nodes in $G_i'$. In other words, the resistive distance equals to the shortest path distance between them. Thus, for any $i \in G'$, $\Omega_{loc}(i) = E_{loc}(i)$, which can be seen in Table 3.

Meanwhile, For node 2, $\Gamma_2 = \{0,1,3,4\}$ and $k_2 = 4$. According to Eq. (23) and $R_m[G']$,
\[
\Omega_3(2) = \frac{2}{k_2(k_2 - 1)} \sum_{j \neq k \in \Gamma_2} \eta_{jk} = \frac{2}{4 \times (4-1)} (r_{01} + r_{03}) = \frac{2}{4 \times (4-1)} (0.62 + 0.62) = 0.21.
\]

According to Eq. (24) and $R_m[G']$,
\[
D_r(2) = \frac{2}{k_2(k_2 - 1)} \sum_{j \neq k \in \Gamma_2} \frac{1}{r_{jk}} = \frac{2}{4 \times (4-1)} \left( \frac{1}{r_{01}} + \frac{1}{r_{03}} + \frac{1}{r_{04}} + \frac{1}{r_{13}} + \frac{1}{r_{14}} + \frac{1}{r_{34}} \right) = \frac{2}{4 \times (4-1)} \left( 0.62 + 0.62 + 1 + 0 + 1.5 + 1.63 \right) = 1.02.
\]

See Figure 1 (a) and (k), $G'$ and $G'_2$ represent the same graph. Thus, according to Eqs. (21) and (22),
\[
\Omega_{loc'}(2) = \Omega_{glob}(G'_2) = \Omega_{glob}(G') = 1.23,
\]
and,
\[
E_{loc'}(2) = E_{glob}(G'_2) = E_{glob}(G') = 0.8.
\]

We turn to calculate $\Omega_{loc'}(1)$ and $E_{loc'}(1)$. See Figure 1 (j) for $G'_1$. The shortest path distance matrix and resistive distance matrix is
\[
D_m[G'_1] = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}.
\]

Then, in terms of $D_m[G'_1]$ and $R_m[G'_1]$,
\[
E_{loc'}(1) = E_{glob}(G'_1) = \frac{2}{3(3-1)} (1 + 1 + 1) = 1.
\]
In contrast, the resistive matrix of $G_2''$ is

$$R_m[G_2'] = \begin{pmatrix}
0 & 3 & 5 & 6 & 7 & 8 & 1 \\
0 & 1 & 2 & 3 & +∞ & +∞ & +∞ & +∞ \\
3 & 1 & 0 & 1 & +∞ & +∞ & +∞ & +∞ \\
4 & 2 & 1 & 0 & 1 & +∞ & +∞ & -∞ & +∞ \\
5 & 3 & 2 & 1 & 0 & -∞ & +∞ & +∞ & +∞ \\
6 & +∞ & +∞ & +∞ & +∞ & 0 & 1 & 1.67 & 1.67 \\
7 & +∞ & +∞ & +∞ & +∞ & 1 & 0.67 & 0.67 & 0 \\
8 & +∞ & +∞ & +∞ & +∞ & 1.67 & 0.67 & 0.67 & 0 \\
1 & +∞ & +∞ & +∞ & +∞ & 1.67 & 0.67 & 0.67 & 0
\end{pmatrix}.$$ 

Similarly, according to Eqs. (18)-(19) and $R_m[G_2']$,

$$Ω_{loc}(2) = Ω_{glob}(G_2')$$

$$\Omega_{loc}(2) = Ω_{glob}(G_2') = \frac{2}{[8 * (8 - 1)]} \left( \frac{1}{r_{03}} + \frac{1}{r_{04}} + \frac{1}{r_{05}} + \frac{1}{r_{34}} + \frac{1}{r_{35}} - \frac{1}{r_{45}} + \frac{1}{r_{56}} + \frac{1}{r_{67}} \right)$$

$$\Omega_{loc}(2) = Ω_{glob}(G_2') = \frac{2}{[8 * (8 - 1)]} \left( \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} \right)$$

For the sake of simplicity, other descriptors’s details are omitted in this example, whose results are also shown in Table 3. Moreover, the proposed descriptors are labelled in bold.

IV. ALGORITHM DESCRIPTION

Algorithm 1 Calculate $R$

**Input:** $G(V, E)$

**Output:** $R_m(G)$

1: $n = |G(V, E)|$
2: $A_G = \text{adjacency\_matrix}(G)$
3: $D_G = \text{degree\_matrix}(A_G)$
4: $L_G = D_G - A_G$
5: $J = ||1||$
6: $X = ||x_i|| = (L + \frac{1}{n} * J)^{-1}$
7: $\bar{X} = \text{diag}(x_1, ..., x_n)$
8: $R_m(G) = ||r_i|| = \bar{X}J + J\bar{X} - 2X$

Algorithm 2 Calculate $Kf$

**Input:** $G(V, E)$

**Output:** $Kf$

1: $\mu\text{List} = \text{laplacian\_spectrum}(G)$
2: $Kf = n \sum_{k=2}^{n} \frac{1}{\mu_k}$

All the resistive quantities in this work are calculated in algorithms 1-6, respectively.

The calculation method of resistive matrix is shown in algorithm 1. The input is a graph $G$ and the output $R_G$ represents resistive matrix of $G$. Note that at line 2, line 3 and line 4, $A_G$, $D_G$ and $L_G$ indicate adjacency matrix, degree matrix and Laplacian matrix, respectively. At line 4, $J$ is a
square matrix and all of its elements are 1. Lines 6-8 present the formulations of $R_G$. Moreover, the entry of $R_G$, denoted by $r_{ij}$, can also be expressed as $r_{ij} = x_{ii} + x_{jj} - 2x_{ij}$, where $X = \|x_{ij}\| = (L + \frac{1}{2} J)^{-1}$. Refer to [39] for more details.

Algorithm 2 presents the calculation method of Kirchhoff index according to Eq. (15). $K_f$ is closely related to the spectrum of the Laplacian matrix $L_G$ (for more details see [10]). At line 2, $\mu\text{List}$ denotes a list of all the eigenvalues of Laplacian matrix $L_G$.

Algorithm 3 displays how to calculate average resistive path length $L_r$ according to Eq. (16). At first, we compute resistive matrix $R_G$ according to algorithm 1 (see line 2). Subsequently, $L_r$ is obtained by $R_G$, which is given at lines 3-10.

Algorithm 4 shows how to calculate resistive global efficiency $\Omega_{\text{glob}}$ according to Eq. (18). The execution of the method is similar to algorithm 3. At line 2, $R_G$ is obtained. From line 3 to line 12, $\Omega_{\text{glob}}$ is computed by $R_G$. Note that $R_G[i, i] = 0$ for any $i$.

Algorithm 5 presents the method of local resistive efficiency $\Omega_{\text{loc}}$ according to Eq. (19) and (20). Input is also a graph $G$. Two outputs, $\Omega_{\text{loc}}(v)$ and $\Omega_{\text{loc}}(G)$, denote local resistive efficiency of node $v$ and that of graph $G$, respectively. At lines 5 and 6, $v_{\text{neighbors}}$ indicates neighbors of $v$ and $v_{\text{n_len}}$ is its number. $G_v$ is the spanning graph

| Metrics                          | $G'$          | $G''$         |
|---------------------------------|---------------|---------------|
| $E_{\text{loc}}$                | 0.83 1.0 0.42 | 1.0 0.33 0.83 |
| $\Omega_{\text{loc}}$           | 0.83 1.0 0.42 | 1.0 0.33 0.83 |
| $E_{\text{loc}}'$               | 0.92 1.0 0.8  | 1.0 0.71 0.92 |
| $\Omega_{\text{loc}}'$          | 1.57 1.5 1.23 | 1.5 1.5 1.85  |
| $SC$                            | 4.36 3.10 5.21| 3.79 6.63 16.92|
| $C_4$                           | 1.0 1.0 1.0   | 1.0 1.2 1.0   |
| $C_3$                           | 0.67 1.0 0.33 | 1.0 0.25 0.67|
| $\Omega_3$                      | 0.41 0.5 0.21 | 0.48 0.45 0.14|
| $D$                             | 0.83 1.0 0.42 | 1.0 0.83 1.0  |
| $D_r$                           | 1.41 2.0 1.02 | 2.08 2.22 1.21|
| $\Omega_{\text{loc}}(v)$        |              |               |
| $\Omega_{\text{loc}}(G)$        |              |               |
| $cc_{\text{list}}$              |              |               |
| $v_{\text{sum}}$                |              |               |
| $\Omega_{\text{loc}}(v) = v_{\text{sum}}/[v_{\text{n_len}} * (v_{\text{n_len}} - 1)]$ |               |
| $\Omega_{\text{loc}}(G) = \Omega_{\text{loc}}(v)$ |               |
| $\Omega_{\text{glob}} = \sum_{i,j \in \text{cc}} \frac{1}{r_{ij}}$ |               |
| $v_{\text{sum}} = v_{\text{sum}} + cc_{\text{sum}}$ |               |
| $\Omega_{\text{glob}} = \Omega_{\text{loc}}(G) / n$ |               |
Algorithm 6 Calculate $D_r$

**Input:** $G(V,E)$

**Output:** $D_r(v), D_r(G)$

1. $n = |G(V,E)|$;
2. $R_G = \text{calculate } R(G)$;
3. for $v$ in $G$ do
4. \hspace{1em} $v\_neighbors = \text{Neighbors}(v)$;
5. \hspace{1em} $\text{deg} = \text{Degree}(v,G)$;
6. \hspace{1em} if $\text{deg} == 1$ then
7. \hspace{2em} $D_r(v) = 0$;
8. \hspace{2em} continue;
9. \hspace{1em} end if
10. \hspace{1em} $\text{sum} = 0$;
11. \hspace{1em} $\text{potential} = [\text{deg} \ast (\text{deg} - 1)] / 2$;
12. \hspace{1em} for $u, w$ in $v\_neighbors$ do
13. \hspace{2em} $\text{sum} = \text{sum} + 1 / R_G[u,w]$;
14. \hspace{1em} end for
15. \hspace{1em} $D_r(G) = D_r(G) + D_r(v)$;
16. end for
17. $D_r(G) = D_r(G) / n$;

of node $v$ at line 7. If $v\_n\_len$ equals to 1, $G_v$ is a trivial graph and $\Omega_{loc}(v)=0$ (See lines 8-11). Otherwise, the set of connected components of $G_v$, denoted by $cc\_list$ at line 12, is computed. Usually, $G_v$ is an unconnected graph. Therefore, at lines 14-17, $\Omega_{loc}(v)$ and $\Omega_{loc}(G)$ are obtained by enumerating each connected component of $G_v$.

Algorithm 6 computes loop coefficient $D_r$ according to Eq. (24). Similarly, resistive matrix $R_G$ of $G$ is obtained at first at line 2. $v\_neighbors$ also represents the set of neighbors of $v$ at line 4. If $v$ has only a neighbor, $D_r(v)=0$, which is displayed at lines 5-9. Otherwise, $D_r(v)$ is calculated through lines 10-16, where $R_G[u,w]$ indicates the resistive distance between node $u$ and $w$.

V. EXPERIMENTAL RESULTS AND ANALYSIS

In this section, we assess the effectiveness of our descriptors on computer-generated and real-world networks, and make comparisons with baseline metrics. All the experiments are run on a PC equipped with a 2.7 GHz Intel Core i5-7200U CPU and 8.00G memory operating in the Microsoft windows 10 environment. Our codes are conducted relying upon python programming language. Especially, python packages, such as networkx, numpy and matplotlib, are utilized for modeling the networks, computation of data and visualization, respectively.

A. COMPUTER-GENERATED NETWORKS

In this section, we considered our proposed descriptors for three representative models of complex networks, the WS model [2], the ER model [3] and the BA model [1]. We calculate every descriptor’s value for each classical network model. This experiment is conducted with network size $N=1000$. As Tables 4-6 show, the values of all the descriptors in this work are given, for Small-world networks, ER random networks and Scale-free networks, respectively. Moreover, the values in terms of resistive distance are emphasized in bold.

1) Small-world networks

In this work, we construct small-world networks by Watts-Strogatz models. The initial graph is taken to be a one-dimensional lattice of $N=1000$ nodes. It meets the periodic boundary conditions, i.e., each node is connected to its first $k=4$ neighbors. The small-world network is created by rewiring each edge of the lattice at random with probability $p$. The transition from a regular lattice ($p=0$) to random network ($p=1$) can be obtained by varying $p$ in a continuous way. Table 4 shows the values of structural quantities with $p$ ranging progressively from 0 to 0.4.

As shown in Table 4, Kirchhoff index ($K_f$) and average resistive path length ($L_r$) decrease as $p$ increases, which is consistent with Wiener index ($W_f$) and average path length ($L$). Overall, all the clustering coefficient descriptors, i.e., $C_3$, $\Omega_3$, $C_4$, $R$, $D$, $D_r$, show similar actions with $p$. $C_3$ and $C_4$ also decrease with increased $p$, so do $R$ and $D$. Compared with $C_3$, $\Omega_3$ displays a consistent performance, i.e., 0.241, 0.2013, 0.1715, 0.1416, 0.1187, 0.0959, 0.0913, 0.0677, 0.0478 with increased $p$. $D$ doesn’t have a perfect performance, but it overall decreases when $p$ increases significantly.

To compare $L$ and $L_r$ for the WS network model, Figure 3 (a) illustrates the data normalized by the $L(0)$ and $L_r(0)$, which are 125.3754 and 3385, respectively. Figure 3 shows the plot of the normalized average shortest path length $L(p)/L(0)$, the normalized the average resistive path length $L_r(p)/L_r(0)$, the normalized ordinary clustering coefficient $C_3(p)/C_3(0)$ and the normalized resistive clustering coefficient $\Omega_3(p)/\Omega_3(0)$ as functions of the rewiring probability $p$. The seminal work of Watts and Strogatz [2] uses the combination of high clustering coefficient and short average path length to define the characteristics of small-world networks. They found that, when rewiring probability $p < 0.01$, average path length $L$ (black line) has a rapid drop, whereas the clustering coefficient $C_3$ (blue line) stays almost constant. Interestingly, the resistive clustering coefficient $\Omega_3$ (purple line) also stays nearly unchanged when $p < 0.01$ (see Figure 3 (a)) and drops to 0 when $p=1$ (see Figure 3 (b)). Likewise, rewiring just a few edges ($p < 0.01$) also can lead to a dramatic reduction in the average resistive path length $L_r$ (red line).

Figure 4 shows the plot of the ordinary global efficiency $E_{glob}$, local efficiency $E_{loc}$ and their corresponding resistive efficiency $\Omega_{glob}$ and $\Omega_{loc}$ as func-
### TABLE 4. The values of global structural quantities for Small-world model networks with $N=1000$, $k=4$

| $p$ | $p = 0$ | $p = 0.05$ | $p = 0.1$ | $p = 0.15$ | $p = 0.2$ | $p = 0.25$ | $p = 0.3$ | $p = 0.35$ | $p = 0.4$ |
|-----|---------|------------|-----------|------------|-----------|-----------|-----------|------------|-----------|
| $W_r$ | 62625000.0 | 5749503.0 | 4356787.0 | 3722952.0 | 3414384.0 | 3198683.0 | 3107978.0 | 2995138.0 | 2894453.0 |
| $K_f$ | 16756053.0 | 980659.0 | 692470.0 | 629470.0 | 513145.0 | 485477.0 | 470996.0 | 453559.0 | 430107.0 |
| $L$ | 1253754.0 | 115105.0 | 87223.0 | 74534.0 | 68356.0 | 64038.0 | 63483.0 | 59693.0 | 57947.0 |
| $L_r$ | 335547.0 | 19633.0 | 13863.0 | 11511.0 | 10273.0 | 9719.0 | 9429.0 | 908.0 | 8611.0 |

| $E_{glob}$ | 0.0244 | 0.1018 | 0.1293 | 0.1484 | 0.1602 | 0.17 | 0.1712 | 0.1805 | 0.186 |
| $Q_{glob}$ | 0.0648 | 0.5362 | 0.754 | 0.9039 | 1.0084 | 1.0706 | 1.0996 | 1.1534 | 1.2095 |
| $E_{loc}$ | 0.7222 | 0.5934 | 0.4896 | 0.3879 | 0.3184 | 0.252 | 0.2334 | 0.1687 | 0.1192 |
| $Q_{loc}$ | 0.7222 | 0.5934 | 0.4896 | 0.3879 | 0.3184 | 0.252 | 0.2334 | 0.1687 | 0.1192 |
| $K_f'$ | 0.85 | 0.8292 | 0.8121 | 0.7936 | 0.7814 | 0.7682 | 0.7644 | 0.751 | 0.7401 |
| $r_{loc}$ | 1.5497 | 1.4105 | 1.2955 | 1.1782 | 1.0993 | 1.0209 | 0.9967 | 0.9185 | 0.8596 |

### TABLE 5. The values of global structural quantities for E-R model networks with 1000 nodes

| $p$ | $p = 0.025$ | $p = 0.05$ | $p = 0.075$ | $p = 0.1$ | $p = 0.125$ | $p = 0.15$ | $p = 0.175$ | $p = 0.2$ |
|-----|-------------|-------------|-------------|-----------|------------|-----------|-----------|-----------|
| $W_r$ | 1246066.0 | 1012174.0 | 962992.0 | 949349.0 | 936499.0 | 923723.0 | 911855.0 | 896685.0 |
| $K_f$ | 43356.0 | 20702.0 | 13609.0 | 10237.0 | 8099.0 | 6706.0 | 5783.0 | 5015.0 |
| $L$ | 2.4946 | 2.0264 | 1.9279 | 1.9066 | 1.8749 | 1.8493 | 1.8255 | 1.7992 |
| $L_r$ | 0.0868 | 0.0414 | 0.0272 | 0.0205 | 0.0162 | 0.0134 | 0.0116 | 0.01 |
| $E_{glob}$ | 0.4259 | 0.5123 | 0.5371 | 0.5497 | 0.5626 | 0.5754 | 0.5872 | 0.6004 |
| $Q_{glob}$ | 117.819 | 24.3779 | 36.9353 | 49.0375 | 61.908 | 74.7083 | 86.5957 | 99.8116 |
| $E_{loc}$ | 0.0361 | 0.2539 | 0.4222 | 0.4926 | 0.5414 | 0.5705 | 0.5866 | 0.6003 |
| $Q_{loc}$ | 0.0366 | 0.5284 | 2.1129 | 4.265 | 7.1549 | 10.6958 | 14.5922 | 19.5381 |
| $K_f'$ | 0.5514 | 0.5439 | 0.55 | 0.559 | 0.5696 | 0.5809 | 0.592 | 0.6044 |
| $r_{loc}$ | 0.7149 | 1.373 | 2.8325 | 4.9288 | 7.8016 | 11.3396 | 15.2387 | 20.1892 |

### TABLE 6. The values of global structural quantities for Scale-free model networks with 1000 nodes

| $n$ | $m = 1$ | $m = 2$ | $m = 3$ | $m = 4$ | $m = 5$ | $m = 6$ |
|-----|---------|---------|---------|---------|---------|---------|
| $W_r$ | 3191651.0 | 2054321.0 | 173680.0 | 1567484.0 | 1481918.0 | 1412634.0 |
| $K_f$ | 3191651.0 | 488722.0 | 283127.0 | 200405.0 | 153874.0 | 124875.0 |
| $L$ | 6.3897 | 4.1128 | 3.4761 | 3.138 | 2.9668 | 2.8281 |
| $L_r$ | 6.3897 | 0.9784 | 0.5668 | 0.4012 | 0.3081 | 0.25 |
| $E_{glob}$ | 0.1782 | 0.259 | 0.3049 | 0.3361 | 0.3544 | 0.3714 |
| $Q_{glob}$ | 0.1782 | 1.1216 | 1.9467 | 2.7526 | 3.597 | 4.443 |
| $E_{loc}$ | 0.0 | 0.0272 | 0.0535 | 0.0483 | 0.0508 | 0.0684 |
| $Q_{loc}$ | 0.0 | 0.0272 | 0.0535 | 0.0507 | 0.0558 | 0.0793 |
| $K_f'$ | 0.92 | 0.7659 | 0.6957 | 0.6605 | 0.6346 | 0.6201 |
| $r_{loc}$ | 0.92 | 0.7659 | 0.7413 | 0.7393 | 0.7319 | 0.7631 |

| $SC$ | 26.5633 | 46.7249 | 216.3778 | 72471.6463 | 625879.296 | 17777606.290 |
| $C_3$ | 0.0 | 0.0268 | 0.0316 | 0.0399 | 0.0921 | 0.0473 |
| $O_3$ | 0.0 | 0.006 | 0.0045 | 0.0048 | 0.0041 | 0.0004 |
| $C_4$ | 0.0 | 0.0037 | 0.0026 | 0.0022 | 0.0021 | 0.002 |
| $R$ | 0.0 | 0.1835 | 0.2019 | 0.2141 | 0.2191 | 0.2249 |
| $D$ | 0.0 | 0.3087 | 0.3574 | 0.3935 | 0.408 | 0.4268 |
| $D_r$ | 0.1685 | 2.5137 | 3.8584 | 5.6262 | 6.4824 | 7.855 |
3) Scale-free networks

Scale-free networks are constructed by BA model, which refers to Barabási-Albert preferential attachment model. A graph of N nodes is expanded by attaching new nodes each with m edges that are preferentially attached to existing nodes with high degree. Table 6 shows the values of structural quantities for the BA networks with N = 1000 and m ranging from 1 to 6. As m increases, W_r (resp., K_f), L (resp., L_r) decrease, E_{glob} (resp., Ω_{glob}), E_{loc} (resp., Ω_{loc}) increase, R and D (resp., D_r) increase, respectively. That is, resistive descriptors, K_f, L_r, Ω_{glob}, Ω_{loc}, D_r, have similar characteristics in BA model networks. However, Ω_3 displays different results from C_3 in BA model networks.

B. THE KENDALL’S TAU FOR WS, ER AND BA MODELS

To evaluate the performance of the proposed structural descriptors, we investigate the Kendall’s Tau (τ) between the ranking scores of different methods. Higher τ suggests better performance. Given two ranking scores of X and Y, if τ is more closer to 1, it means that X and Y are more highly correlated. Figure 5 shows the Kendall’s Tau correlation coefficient matrices between the ranking scores given by eleven local structural properties, i.e., C_4, C_5, Ω_3, SC, D, R, D_r, E_{loc}, E_{loc'}, Ω_{loc}, Ω_{loc'}. Three classic networks models, i.e., WS, ER and BA, are considered with network size N=500. For WS networks, k=5, and p ranging from 0.05 to 0.25, i.e., 0.05, 0.1, 0.15, 0.2, 0.25. For ER networks, p, probability for edge creation, varies from 0.05 to 0.4, i.e., 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4. For BA networks, the values of m,
number of edges attached from a new node to existing nodes, are 1, 2, 3, 4, 5. We classify all the structural properties into three comparative groups, i.e., $C_3$, $\Omega_3$, and $\Omega_{loc}$ for the first group, $D$, $R$, and $D_r$ for the second group, $E_{loc}$, $E_{loc'}$, $\Omega_{loc}$, and $\Omega_{loc'}$ for the last group. As shown in Figure 5, the values, bordered in red, green, and yellow, corresponds to the matrix of the first, second and third group, respectively. The following discussions suggest more details for each group.

First, we make an assumption that the Kendall’s Tau between $C_3$ and $\Omega_3$ is inversely related to the standard average clustering coefficient ($C_3$) of the target network. A network with a high average clustering coefficient indicates that the network is more clustered. That is to say, there are more paths for any pair of nodes. However, in general, for any two nodes, less paths can lead to a observation that the resistive distance between them is more close to 1. In terms of a node, resistive clustering coefficient $\Omega_3$ is more close to typical clustering coefficient $C_3$. As is well known, average clustering coefficient of small-world networks decreases with the increasing rewiring probability, while that of ER random networks increases with the probability for edge creation increases progressively.

Second, we turn to a discussion of $D$, $R$ and $D_r$. $D$ is close to $R$. As $D$ considers the length of shortest path, while $R$ applies that of the smallest cycle. However, regardless of each model, the correlation values, between $D_r$ and $D$ ($R$), has no functional relation to its parameter.

Third, the last group, i.e., $E_{loc}$, $E_{loc'}$, $\Omega_{loc}$, and $\Omega_{loc'}$, displays some interesting results. Comparatively, we have more concerns about the values between $E_{loc}$ and $\Omega_{loc}$. Excitedly, in WS and BA models, the Kendall’s $\tau$, in terms of any network’s parameters, are close to 1 or equal to 1. In addition, in WS and BA models, under the condition of network’s parameters that we set, for each node $v$ of the network, it is possible that $E_{loc}(v)=\Omega_{loc}(v)$. Refer to Table 4 and 6 for the comparisons between $E_{loc}(v)$ and $\Omega_{loc}(v)$. As described in Section III, if $G_v$ is a tree, for any node $i, j$ in $G_v$, $r_{ij}=d_{ij}$, where $r_{ij}$ and $d_{ij}$ are resistive distance and shortest path distance, respectively. Therefore, if $G_v$ is a tree, $E_{loc}(v)=\Omega_{loc}(v)$.

Besides, in ER model, $\tau=1$, 0.97, 0.87, 0.77, 0.73, when $p=0.02, 0.04, 0.06, 0.08, 0.1$, respectively. It indicates an inverse association between $\tau$ and $p$. Moreover, $p$ equals to $C_3$ in ER networks. In other words, $\tau$ between $E_{loc}$ and $\Omega_{loc}$, in terms of ER random networks, is inverse to the ordinary clustering coefficient ($C_3$). For more details for this group, see the matrices bordered in yellow, shown in Figure 5.

The experiments are carried on eight real networks from disparate fields, including one transportation network (US-
Air), one biological network (bio-CE-GN), one communication network (Email), two collaboration networks (NS and Jazz), three social networks (Karate, Dolphin and Football).

In brief, USAir [45] is the US air transportation network. CE-GN [47] is a biological network and its links are inferred by gene neighbourhoods of bacterial and archaeal orthologs. Email [44] is a network of E-mail interchanges between members of the Rovira i Virgili University. NS [46] is a co-authorship network of scientists working on network science. Jazz [43] is a collaboration network between jazz musicians consisting of 198 nodes and 2,742 edges. Karate [40] is an undirected social network of friendships between 34 club members of a karate club at a US university in the 1970s; Dolphin [41] is an undirected social network of frequent associations between 62 dolphins during a community living off Doubtful Sound, New Zealand; Football [42] is an undirected social network of American football games between Division IA colleges during regular season Fall 2000. The values of structural quantities for eight real networks are shown in Table 7.

### C. INTENTIONAL ATTACKS BASED ON DC AND BC

In this section, experiments are conducted for the BA ($N=1000, n=4$), WS ($N=1000, m=4, p=0.3$), ER ($N=1000, p=0.01$) and real networks under Degree Centrality attack and Betweenness Centrality attack. Email network is selected as an example for real networks. Different invulnerability metrics, including average path length ($L$ and $L_r$), network efficiency ($E_{glob}$ and $\Omega_{glob}$), clustering coefficient ($C_3$ and $\Omega_3$) are considered. The results under DC attack and BC attack are shown in Figure 6 and Figure 7, respectively. Clearly, the curve of each novel vulnerability metric is similar with that of the baseline method. Moreover, we compare the running time for calculating four global structural metrics ($L$, $L_r$, $E_{glob}$, $\Omega_{glob}$) in different network, as shown in Table 8. Obviously, the resistive descriptors have better performance. Resistive metrics are based on matrix calculation, while the shortest path metrics are based on graph searching algorithms. From this section’s results, matrix calculation are more efficient than graph searching algorithms. In other words, global metrics based on resistive distance are more efficient in the networks with small and medium size.

### VI. CONCLUSION

In this work, analogous to classical statistical descriptors, novel resistive descriptors, including Kirchhoff index ($K_f$), average path length ($L_r$), clustering coefficient ($\Omega_3$), loop coefficient ($D_r$), global efficiency ($\Omega_{glob}$) and local efficiency...
FIGURE 6. Comparisons of different invulnerability metrics under Degree Centrality attack.

FIGURE 7. Comparisons of different invulnerability metrics under Betweenness Centrality attack.
(Ωloc) are suggested. We investigate all the resistive descriptors on classical WS, BA and ER models. Interestingly, Ω3 and Lr (resp., Ωglob and Ωloc) could be used to characterize the features of small-world networks. Besides, Lr, Ω3 and Ωglob are considered to measure network invulnerability based on DC and BC attack. Experiments are conducted on networks with small and medium size. The results show that, in terms of running time, resistive distance depending on matrix calculation is more efficient than the shortest-path distance mainly based on the searching algorithms. In the future, we will explore the potential use of the proposed structure descriptors in other issues of complex networks including community detection, link prediction, privacy preservation, etc.

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