Joint distribution of a random sample and an order statistic: A new approach with an application in reliability analysis

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Abstract

This paper considers the joint distribution of elements of a random sample and an order statistic of the same sample. The motivation for this work stems from the important problem in reliability analysis, to estimate the number of inspections we need in order to detect failed components in a coherent system. We consider an \((n - r + 1)\)-out-of-\(n\) system, which is intact until at least \(n - r + 1\) of the components are alive, and it fails if the number of failed components exceeds \(r\). The life time of the system is the \(r\)th order statistic. Assuming that some of the components failed but the system is still functioning, using the results presented in this paper it is possible to find an expected value of the number of inspections we need to do for detecting certain number of failed components.

Keywords: Order statistics, \(k\)-out-of-\(n\) system, joint distributions

1 Introduction

Let \(X_1, X_2, ..., X_n\) be independent and identically distributed (iid) random variables with distribution function (cdf) \(F\) and \(X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}\) be the order statistics. If \(X_1, X_2, ..., X_n\) are corresponding lifetimes of components of a coherent system, then for \(1 \leq r \leq n\) the conditional probability

\[
P\{X_1 \leq x \mid X_{r:n} \leq t\}
\]

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is the distribution of lifetime of any of components given that at the inspection time \( t \) at least \( r \) of the components have failed. The conditional distribution
\[
P\{X_1 \leq x \mid X_{r:n} = t\}
\]
is studied in Nagaraja and Nevzorov (1997) and Nagaraja and Ahmadi (2018) in the context of \((n-r+1)-\)out-of-\(n\) systems whose lifetime \( T(X_1, X_2, ..., X_n) \) is \( X_{r:n} \), i.e. the system that fails if more than \( r \) components fail and the system is intact if at least \((n-r+1)\) of components are alive. The probability \( (2) \) is actually the conditional cdf of any of the components of \((n-r+1)\)-out-of-\(n\) coherent system given that the system failed at time \( t \). Nagaraja and Ahmadi (2018) used \( (2) \) and related joint distributions to find the distribution of number of inspections which is necessary for detecting of all failed components if the system failed at time \( t \).

In this paper we are interested also in conditional distribution
\[
P\{X_1 \leq x \mid t_1 \leq X_{r:n} \leq t_2\},
\]
which can be interpreted as the conditional distribution of any of the components given that the \( r \)th failure has occurred between two inspections at \( t_1 \) and \( t_2 \), i.e. there are \( r \) failed components that we reveal in time interval \([t_1, t_2]\). The conditional distribution \( (3) \) carries information about the life time distribution of any of the components given that the \( r \)th failure has occurred between two inspection times \( t_1 \) and \( t_2 \). The random variables \( X_1, X_2, ..., X_n \) can also be considered as the lifetimes of \( n \) identical items put under life test and then \( (1) \) is the conditional distribution of any of items given that at inspection time \( t \), there are at least \( r \) failed items. In practical applications a system monitoring is important, and it is scheduled at different inspection times. Under double monitoring one may consider the residual and past life functions of the system \((T - t_1 \mid t_1 < T < t_2)\) and \((t_2 - T \mid t_1 < T < t_2)\) which is studied in many research papers including Raqab (2010), Bdair and Raqab (2014), Li and Zhao (2008), Li and Zhang (2008), Parvardeh et al. (2018), Poursaeed (2010), Poursaeed and Nemathollahi (2010a), Poursaeed and Nemathollahi (2010b), Zhang and Meeker (2013), and Zhang and Yang (2010), Eryilmaz (2013), Tavangar and Bairamov (2015), Samadi et al. (2017). In a recent paper Navarro and Cali (2018) consider a system with dependent components assuming that system is exposed to periodical inspections. In the results of these inspections, it may be known that the system was working at time \( t_1 \), but it failed at time \( t_2 \). Under these conditions Navarro and Cali (2018) investigate the system inactivity time \((t_2 - T \mid t_1 < T < t_2)\) for both independent and dependent lifetimes and obtain representations for reliability functions in terms of copula.
The focus of this paper is the joint distribution of \( X_1, X_2, \ldots, X_k \) and \( X_{r:n} \) for \( k < r \). First, we consider the joint distribution of \( X_i \) and \( X_{r:n} \), \( i \in \{1, 2, \ldots, n\} \) as well as the conditional distribution of \( X_1 \) given \( t_1 \leq X_{r:n} \leq t_2 \) for any \( t_1 < t_2 \). The difficulty of finding the joint distribution of random variables \( X_1, X_2, \ldots, X_k \) and \( X_{r:n} \) is concluded in the fact that \( X_{r:n} \) is one of the random variables \( X_1, X_2, \ldots, X_n \). Second, we apply the obtained results to solve an important problem in reliability analysis: the problem of estimating the number of inspections we need in order to detect failed components of a coherent system. Since inspections of the components of the system may sometimes be an expensive action, the optimal planning of periodical inspections is very important. If we interpret \( X_i \)'s as the lifetimes of components of \((n - r + 1)\)-out-of-\( n \) system, then the joint distribution of the random variables \( X_1, X_2, \ldots, X_k \) and \( X_{r:n} \) is necessary to compute the probabilities of the events of type \( X_1 < X_{r:n}, X_2 < X_{r:n}, \ldots, X_k < X_{r:n} \) which are used for computing the probabilities of numbers of inspections we need in order to detect failed components. This paper is organized as follows: in Section 1 we derive the joint distribution of a single observation from the sample and the \( r \)th order statistic of the same sample and consider the conditional distribution of an observation given that the \( r \)th order statistic is between \( t_1 \) and \( t_2 \). Then we consider the joint distributions of several sample observations and an order statistic of the same sample and show that the conditional random variables defined as a set of observations given order statistic are in general dependent, except some special cases. In Section 3 we consider the joint distributions of the set of sample observations and an order statistic. In Section 4 we deal with the distribution of the number of inspections one needs in order to detect failed components in an \((n - r + 1)\)-out-of-\( n \) system and provide a numerical example.

2 The joint distributions of the random variables and their order statistics

Throughout this paper we assume that \( X_1, X_2, \ldots, X_n \) be iid random variables with cdf \( F \) and \( X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n} \) be the order statistics. Where it is needed we will assume that \( F \) is an absolutely continuous cdf with pdf \( F'(x) = f(x) \) supported in \([0, \infty)\) and \( X_i, i \in \{1, 2, \ldots, n\} \) are lifetimes of the components of coherent system of \( n \) components.
Theorem 1 The joint distribution of $X_1$ and $X_{r:n}$ is

$$P(X_1 \leq x, X_{r:n} \leq t) = \begin{cases} 
F(x) \sum_{i=1}^{n-1} \binom{n-1}{i} F^i(t)(1 - F(t))^{n-1-i}, & x \leq t \\
\left[ F(x) \sum_{i=1}^{n-1} \binom{n-1}{i} F^i(t)(1 - F(t))^{n-1-i} - \binom{n-1}{r-1}(F(x) - F(t))F^{r-1}(t)(1 - F(t))^{n-r} \right], & x > t
\end{cases}$$

(4)

Proof. a) Let $x \leq t$. We have

$$P(X_1 \leq x, X_{r:n} \leq t) = P(X_1 \leq x, \text{at least } r - 1 \text{ of } X_2, X_3, ..., X_n \text{ are less or equal than } t)$$

$$= F(x)P(\text{exactly } r - 1 \text{ of } X_2, X_3, ..., X_n \text{ are less or equal than } t)$$

$$= F(x) \sum_{i=1}^{n-1} \binom{n-1}{i} F^i(t)(1 - F(t))^{n-1-i}$$

b) Let $x > t$. Using the total probability formula one can write

$$P(X_1 \leq x, X_{r:n} \leq t)$$

$$= P(X_1 \leq x, X_1 > t, X_{r:n} \leq t) + P(X_1 \leq x, X_1 \leq t, X_{r:n} \leq t)$$

$$= P(X_1 \leq x, X_1 > t, \text{at least } r \text{ of } X_2, X_3, ..., X_n \text{ are less than or equal to } t)$$

$$+ P(X_1 \leq x, X_1 \leq t, \text{at least } r - 1 \text{ of } X_2, X_3, ..., X_n \text{ are less than or equal to } t)$$

$$= (F(x) - F(t)) \sum_{i=r}^{n-1} \binom{n-1}{i} F^i(t)(1 - F(t))^{n-1-i}$$

$$+ F(t) \sum_{i=r}^{n-1} \binom{n-1}{i} F^i(t)(1 - F(t))^{n-1-i}$$

$$= (F(x) - F(t)) \sum_{i=r-1}^{n-1} \binom{n-1}{i} F^i(t)(1 - F(t))^{n-1-i}$$

$$- (F(x) - F(t)) \binom{n-1}{r-1} F^{r-1}(t)(1 - F(t))^{n-r}$$

$$+ F(t) \sum_{i=r-1}^{n-1} \binom{n-1}{i} F^i(t)(1 - F(t))^{n-1-i}$$

$$= \sum_{i=r-1}^{n-1} \binom{n-1}{i} F^i(t)(1 - F(t))^{n-1-i}(F(x) - F(t) - F(t))$$

$$- (F(x) - F(t)) \binom{n-1}{r-1} F^{r-1}(t)(1 - F(t))^{n-r}.$$
The theorem is thus proved. ■

**Corollary 1** The conditional distribution of **X** \(_1\) given \(X_{r:n} \leq t\) is

\[
P\{X_1 \leq x \mid X_{r:n} \leq t\} = \begin{cases} 
F(x) \sum_{i=r-1}^{n-1} \binom{n-1}{i} F^i(t)(1 - F(t))^{n-1-i} \\
\times \left( \sum_{i=r-1}^{n-1} \binom{n}{i} F^i(t)(1 - F(t))^{n-1-i} \right)^{-1} & \text{if } x \leq t \\
F(x) \sum_{i=r-1}^{n-1} \binom{n-1}{i} F^i(t)(1 - F(t))^{n-1-i} \\
- \binom{n-1}{r-1}(F(x) - F(t))F^{r-1}(t)(1 - F(t))^{n-r} & \text{if } x > t \\
\times \left( \sum_{i=r}^{n} \binom{n}{i} F^i(t)(1 - F(t))^{n-i} \right)^{-1} & \end{cases}
\]

(5)

**Proof.** Follows from Theorem 1. ■

Below in Figure 1 we provide for illustration the graph of the joint distribution \(P\{X_1 \leq x, X_{r:n} \leq t\}\) for \(F(x) = 1 - \exp(-x), x \geq 0, n = 15\) and \(r = 7\).

![Graph of joint distribution](image)

Figure 1. The graph of \(P\{X_1 \leq x, X_{r:n} \leq t, n = 15, r = 7, F(x) = 1 - \exp(-x), x \geq 0\}.

**Remark 1** Special cases. Because of the importance of formula (5) (or (4)) for our research, we can verify it with the special cases \(r = n\) and \(r = 1\), which can be computed by using the properties of extreme order statistics.
a) Let \( r = n \). Then if \( x < t \), we have

\[
P\{X_1 \leq x, X_{n:n} \leq t\} = P\{X_1 \leq x, X_{n:n} \leq t\} = P\{X_1 \leq x, X_1 \leq t, ..., X_n \leq t\} = F(x)F^{n-1}(t)
\]

and if \( x \geq t \) we have

\[
P\{X_1 \leq x, X_{n:n} \leq t\} = P\{X_1 \leq x, X_{n:n} < t\} = P\{X_1 \leq x, X_1 \leq t, ..., X_n \leq t\} = F^n(t).
\]

Therefore,

\[
P\{X_1 \leq x, X_{n:n} \leq t\} = \begin{cases} F(x)F^{n-1}(t) & \text{if } x \leq t \\ F^n(t) & \text{if } x > t \end{cases}
\]

(6)

It is clear that

\[
P\{X_1 \leq x \mid X_{n:n} \leq t\} = \begin{cases} F(x) & \text{if } x \leq t \\ \frac{F(x) - (F(x) - F(t))(1 - F(t))^{n-1}}{1 - (1 - F(t))^n} & \text{if } x > t \end{cases}
\]

(7)

Now, let \( r = n \) in (4) or (5) and we clearly obtain (7) or (6).

b) Let \( r = 1 \). Then one can write

\[
P\{X_1 \leq x, X_{1:n} \leq t\} = P\{X_1 \leq x, X_{1:n} \leq t\}
\]

\[
= P\{X_1 \leq x\} - P\{X_1 \leq x, X_{1:n} > t\}
\]

\[
= F(x) - P\{X_1 \leq x, X_1 > t, ..., X_n > t\}
\]

\[
= \begin{cases} F(x) & \text{if } x \leq t \\ F(x) - (F(x) - F(t))(1 - F(t))^{n-1} & \text{if } x > t \end{cases}.
\]

(8)

It is clear that

\[
P\{X_1 \leq x \mid X_{1:n} \leq t\} = \begin{cases} \frac{F(x)}{1 - (1 - F(t))^n} & \text{if } x \leq t \\ \frac{F(x) - (F(x) - F(t))(1 - F(t))^{n-1}}{1 - (1 - F(t))^n} & \text{if } x > t \end{cases}
\]

(9)

Now, let \( r = 1 \) in (4) and (5) and one obtains (8) and (9).
Remark 2 Let $I_p(a,b) = \frac{1}{B(a,b)} \int_0^p t^{a-1}(1-t)^{b-1} dt$ be an incomplete beta function and $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$ be a beta function. Since, $\sum_{i=r}^{n-1} \binom{n-1}{i} F_i(t) (1-F(t))^{n-1-i} = I_F(r, n-r)$ and $\sum_{i=r}^{n-1} \binom{n-1}{i} F_i(t) (1-F(t))^{n-1-i} = I_F(r-1, n-r+1)$, for $1 < r < n$, (5) can also be written as

$$P\{X_1 \leq x \mid X_{r:n} \leq t\} =
\begin{cases}
F(x)I_F(r-1, n-r+1) \\
\times (I_F(r, n-r+1))^{-1}
\end{cases}
\quad \text{if } x \leq t
$$

$$= \begin{cases}
[F(x)I_F(r-1, n-r+1) \\
-\binom{n-1}{r-1} (F(x) - F(t)) F^{r-1}(t) (1-F(t))^{n-r}] \\
\times (I_F(r, n-r+1))^{-1}
\end{cases}
\quad \text{if } x > t
$$

(10)

where $I_p(a,b) = \frac{1}{B(a,b)} \int_0^p t^{a-1}(1-t)^{b-1} dt$ is an incomplete beta function and $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$ is a beta function.
Theorem 2 Let $0 < t_1 < t_2$. Then

$$P\{X_1 \leq x \mid t_1 \leq X_{r:n} \leq t_2\} = \begin{cases} \left[ F(x) \sum_{i=r-1}^{n-1} \binom{n-1}{i} \left[ F^i(t_2)(1 - F(t_2))^{n-1-i} - F^i(t_1)(1 - F(t_1))^{n-1-i} \right] \right] \times \left( \sum_{i=r}^{n} \binom{n}{i} F^i(t)(1 - F(t))^{n-i} \right)^{-1}, & x \leq t_1 \\ \left[ F(x) \sum_{i=r-1}^{n-1} \binom{n-1}{i} \left[ F^i(t_2)(1 - F(t_2))^{n-1-i} - F^i(t_1)(1 - F(t_1))^{n-1-i} \right] - \binom{n-1}{r-1}(F(x) - F(t_1)) F^{r-1}(t_1)(1 - F(t_1))^{n-r} \right] \times \left( \sum_{i=r}^{n} \binom{n}{i} F^i(t)(1 - F(t))^{n-i} \right)^{-1}, & t_1 \leq x \leq t_2 \\ \left[ F(x) \sum_{i=r-1}^{n-1} \binom{n-1}{i} \left[ F^i(t_2)(1 - F(t_2))^{n-1-i} - F^i(t_1)(1 - F(t_1))^{n-1-i} \right] - \binom{n-1}{r-1}(F(x) - F(t_2)) F^{r-1}(t_2)(1 - F(t_2))^{n-r} \right] \times \left( \sum_{i=r}^{n} \binom{n}{i} F^i(t)(1 - F(t))^{n-i} \right)^{-1}, & x > t_2 \end{cases}$$

(12)
It is clear that for $1 < r < n$ \[12\] can be written as

$$
P(X_1 \leq x \mid t_1 \leq X_{r:n} \leq t_2) = \begin{cases} 
F(x) \left[F_{F(t_2)}(r-1, n-r+1) - F_{F(t_1)}(r-1, n-r+1)\right] 
& \times \left[(I_{F(t_2)}(r, n-r+1) - I_{F(t_1)}(r, n-r+1))^{-1}\right] 
& \text{if } x < t_1 

\{F(x)I_{F(t_2)}(r-1, n-r+1) - I_{F(t_1)}(r-1, n-r+1)\} 
& \times \left[(I_{F(t_2)}(r, n-r+1) - I_{F(t_1)}(r, n-r+1))^{-1}\right] 
& \text{if } t_1 \leq x \leq t_2 

\{F(x)I_{F(t_2)}(r-1, n-r+1) - I_{F(t_1)}(r-1, n-r+1)\} 
& \times \left[(I_{F(t_2)}(r, n-r+1) - I_{F(t_1)}(r, n-r+1))^{-1}\right] 
& \text{if } x > t_2 
\end{cases} \tag{13}$$

and also as

$$
P(X_1 \leq x \mid t_1 \leq X_{r:n} \leq t_2) = \begin{cases} 
F(x) \left[F_{F(t_2)}(r-1, n-r+1) - F_{F(t_1)}(r-1, n-r+1)\right] 
& \times \left[(I_{F(t_2)}(r, n-r+1) - I_{F(t_1)}(r, n-r+1))^{-1}\right] , \quad x < t_1 

\{F(x)I_{F(t_2)}(r-1, n-r+1) - (F(x) - F(t_1))I_{F(t_1)}(r, n-r) 
& -F(t_1)I_{F(t_1)}(r-1, n-r+1)\} 
& \times \left[(I_{F(t_2)}(r, n-r+1) - I_{F(t_1)}(r, n-r+1))^{-1}\right] , \quad t_1 \leq x \leq t_2 \tag{14} 

\{F(x) - F(t_2)\}I_{F(t_2)}(r, n-r) + F(t_2)I_{F(t_2)}(r-1, n-r+1) 
& -F(x) - F(t_1))I_{F(t_1)}(r, n-r) - F(t_1)I_{F(t_1)}(r-1, n-r+1)\} , \quad x > t_2.

\times \left[(I_{F(t_2)}(r, n-r+1) - I_{F(t_1)}(r, n-r+1))^{-1}\right] 
\end{cases}
$$

2.1 Dependency

Consider now the conditional random variables $Y_{1}^{(n,t)} = (X_1 \mid X_{n:n} \leq t)$, $Y_{2}^{(n,t)} = (X_2 \mid X_{n:n} \leq t)$, ..., $Y_{n}^{(n,t)} = (X_n \mid X_{n:n} \leq t)$.

**Proposition 1** $Y_{1}^{(n,t)}, Y_{2}^{(n,t)}, ..., Y_{n}^{(n,t)}$ are iid and $Y_{1}^{(n,t)} \overset{d}{=} (X_1 \mid X_1 \leq t)$.
Proof. The joint distributions of random variables $Y_1^{(n,t)}$ and $Y_2^{(n,t)}$ can be easily found as follows:

\[
P\{Y_1(n,t) \leq x_1, Y_2(n,t) \leq x_2\} = \frac{P\{X_1 \leq x_1, X_2 \leq x_2, X_{n:n} \leq t\}}{F^n(t)}
\]

\[
= \left\{ \begin{array}{ll}
\frac{F(x_1)F(x_2)}{F^n(t)} & \text{if } x_1 < x_2 < t \text{ or } x_2 < x_1 < t \\
\frac{F(x_1)}{F(t)} & \text{if } x_1 < t < x_2 \\
\frac{F(x_2)}{F(t)} & \text{if } x_2 < t < x_1 \\
1 & \text{if } x_1 > t \text{ and } x_2 > t
\end{array} \right.
\]

\[
= P\{Y_1(n,t) \leq x_1\}P\{Y_2(n,t) \leq x_2\}
\]

Consider the random variables $Z_1^{(n,t)} = (X_1 \mid X_{1:n} \leq t)$, $Z_2^{(n,t)} = (X_2 \mid X_{1:n} \leq t)$, ..., $Z_n^{(n,t)} = (X_n \mid X_{1:n} \leq t)$.

**Proposition 2** The random variables $Z_1^{(n,t)}, Z_2^{(n,t)}, ..., Z_n^{(n,t)}$ are dependent.

**Proof.** Applying the total probability formula one can write

\[
P\{Z_1(n,t) \leq x_1, Z_2(n,t) \leq x_2\} \leq x_1, Z_2(n,t) \leq x_2\} = \frac{P\{X_1 \leq x_1, X_2 \leq x_2, X_{1:n} \leq t\}}{P\{X_{1:n} \leq t\}}
\]

\[
= \frac{P\{X_1 \leq x_1, X_2 \leq x_2\} - P\{X_1 \leq x_1, X_2 \leq x_2, X_{1:n} > t\}}{1 - (1 - F(t))^n}
\]

\[
= \left\{ \begin{array}{ll}
(1 - (1 - F(t))^{-n}F(x_1, x_2), & x_1 < t \text{ or } x_2 < t \\
(1 - (1 - F(t))^{-n}[F(x_1, x_2) - (F(x_1) - F(t))(1 - F(t))^{n-2}], & x_1 > t \text{ and } x_2 > t
\end{array} \right.
\]
Comparing (15) with (8), it can be observed that \( Z_1^{(n,t)} \) and \( Z_2^{(n,t)} \) are dependent. ■

From the Proposition 2 we see that the random variables \( Z_1^{(n,t)}, Z_2^{(n,t)}, \ldots, Z_n^{(n,t)} \) are dependent. However, if we consider the random variables \( T_1^{(n,t)} = (X_1 | X_{1:n} > t), T_2^{(n,t)} = (X_2 | X_{1:n} > t), \ldots, T_n^{(n,t)} = (X_n | X_{1:n} > t) \), it is interesting to observe that they are iid.

**Proposition 3** The random variables \( T_1^{(n,t)}, T_2^{(n,t)}, \ldots, T_n^{(n,t)} \) are iid and \( T_1^{(n,t)} = (X_1 | X_{1:n} > t) \).

**Proof.** Consider

\[
P\{T_1(n,t) \leq x_1, T_2(n,t) \leq x_2\} = \frac{P\{X_1 \leq x_1, X_2 \leq x_2, X_{1:n} > t\}}{P\{X_{1:n} > t\}}
\]

\[
= \frac{P\{X_1 \leq x_1, X_2 \leq x_2, X_1 > t, \ldots, X_n > t\}}{(1 - F^n(t))}
\]

\[
= \begin{cases} 
(1 - F(t))^{-2}(F(x_1) - F(t))(F(x_2) - F(t)) & \text{if } x_1 > t, x_2 > t \\
0 & \text{if } x_1 \leq t \text{ or } x_2 \leq t 
\end{cases}
\]

(16)

It is clear that

\[
P\{T_1^{(n,t)} \leq x\} = P\{X_1 \leq x | X_{1:n} > t\} = \begin{cases} 
(1 - F(t))^{-1}(F(x) - F(t)) & \text{if } x > t \\
0 & \text{if } x \leq t 
\end{cases}
\]

(17)

Comparing (16) and (17) we see that \( T_1^{(n,t)}, T_2^{(n,t)}, \ldots, T_n^{(n,t)} \) are iid random variables.

**Proposition 4** Let \( 1 \leq r < n \). The random variables \( Y_1^{(r,t)} = (X_1 | X_{r:n} \leq t), Y_2^{(r,t)} = (X_2 | X_{r:n} \leq t), \ldots, Y_n^{(r,t)} = (X_n | X_{r:n} \leq t) \) are dependent.

**2.2 The absolutely continuous case and the conditional distribution of a sample observation given an order statistic**

The conditional distribution of a sample in case where \( F \) is absolutely continuous underlying distribution was first considered by Nagaraja and Nevzorov (1997) for a single observation and Ahmadi (2018) for multiple observations with many interesting characterization results and applications in reliability.
It follows from Theorem 2 that if the distribution function \( F(t) \) is absolutely continuous with \( F'(t) = f(t) \), then
\[
\lim_{h \to 0} P \{ X_1 \leq x \mid t \leq X_{r,n} \leq t + h \} = \begin{cases} 
\frac{(I_F(t+h)(r, n-r+1) - I_F(t)(r, n-r+1))^{-1}}{F(x)} \left[ I_F(t+h)(r-1, n-r+1) - I_F(t)(r-1, n-r+1) \right] & \text{if } x < t \\
\frac{(I_F(t+h)(r, n-r+1) - I_F(t)(r, n-r+1))^{-1}}{F(x)} \left\{ (I_F(t+h)(r-1, n-r+1) - I_F(t)(r-1, n-r+1) \right \} & \text{if } t \leq x \leq t + h \\
\frac{(I_F(t+h)(r, n-r+1) - I_F(t)(r, n-r+1))^{-1}}{F(x)} \left\{ (I_F(t+h)(r-1, n-r+1) - I_F(t)(r-1, n-r+1) \right \} & \text{if } x > t + h \\
\frac{(n-1)(F(x) - F(t))F^{r-1}(t)(1 - F(t))^{n-r}}{I_F(t+h)(r, n-r+1) - I_F(t)(r, n-r+1)} \right\} & 
\end{cases}
\]

Consider \( x < t \). \( \frac{1}{h} (I_F(t+h)(a, b) - I_F(t)(a, b)) \to \frac{1}{B(a,b)} F^{n-1}(t)(1 - F(t))^{b-1} f(t) \), as \( h \to 0 \). Therefore,
\[
\lim_{h \to 0} P \{ X_1 \leq x \mid t \leq X_{r,n} \leq t + h \} = \lim_{h \to 0} \frac{F(x) [I_F(t+h)(r-1, n-r+1) - I_F(t)(r-1, n-r+1)]}{I_F(t+h)(r, n-r+1) - I_F(t)(r, n-r+1)} \left[ \frac{1}{B(r,n+r+1)} F^{r-2}(t)(1 - F(t))^{n-r} f(t) \right] = \frac{r-1}{n} F(x) \frac{F(x) - F(t))F^{r-1}(t)(1 - F(t))^{n-r}}{F(t)}.
\]

Consider \( t \leq x < t + h \). Then \( \frac{1}{h} (F(x) - F(t)) \to f(t) \), as \( h \to 0 \) and
\[
\lim_{h \to 0} P \{ X_1 \leq x \mid t \leq X_{r,n} \leq t + h \}
= \lim_{h \to 0} \left\{ \frac{F(x) [I_F(t+h)(r-1, n-r+1) - I_F(t)(r-1, n-r+1)]}{I_F(t+h)(r, n-r+1) - I_F(t)(r, n-r+1)} + \frac{(n-1)(F(x) - F(t))F^{r-1}(t)(1 - F(t))^{n-r}}{I_F(t+h)(r, n-r+1) - I_F(t)(r, n-r+1)} \right\}
= \frac{r-1}{n} F(x) + \frac{1}{n} = \frac{r}{n}.
\]
Consider \( x > t + h \). Then \( \frac{(n-1)}{B(r,n-r+1)} = \frac{1}{n} \) and

\[
\lim_{h \to 0} P\{X_1 \leq x \mid t \leq X_{r,n} \leq t + h\} = \lim_{h \to 0} \frac{F(x)[I_{F(t+h)}(r-1,n-r+1) - I_{F(t)}(r-1,n-r+1)]}{I_{F(t+h)}(r,n-r+1) - I_{F(t)}(r,n-r+1)} \frac{1}{n} - \frac{1}{n} \frac{d}{dh} \left[F(x) - F(t)\right] \frac{F^{r-1}(t+h)(1 - F(t))^n - (F(x) - F(t))F^{r-1}(t)(1-F(t))^{n-r}}{n(1-F(t))} \frac{F^{r-1}(1-F(t))^{n-r}}{n} + \frac{r}{n}.
\]

Therefore

\[
P\{X_1 \leq x \mid X_{r,n} = t\} = \lim_{h \to 0} P\{X_1 \leq x \mid t \leq X_{r,n} \leq t + h\} = \begin{cases} \frac{r-1}{n} \frac{F(x)}{F(t)} & \text{if } x < t \\ \frac{(n-r)(F(x)-F(t))}{n(1-F(t))} + \frac{r}{n} & \text{if } x \geq t \end{cases}.
\]

(18)

This cdf has a jump at the point \( x = t \) and \( F(t) - F(t - 0) = \frac{1}{n} \). Formula (15) was first presented in Nagaraja and Nevzorov (1997). For more results on conditional distributions of two or multiple observations given \( X_{r,n} = t \) see the recent paper of Ahmadi and Nagaraja (2018). In this comprehensive paper the joint pdf of \( X_1, X_2, ..., X_k \) given \( X_{r,n} \) has been derived under the condition that \( X_1, X_2, ..., X_k \) are distinct than \( X_{r,n} \). The result is applied for calculating the number of inspections that one needs to detect all failed components after the system failure.

Below in Figure 2 we provide the graphs of cdfs and pdfs of \( P\{X_1 \leq x \mid X_{r,n} \leq t\} \) and \( P\{X_1 \leq x \mid X_{r,n} = t\} \) in the case of exponential underlying distributions.
Figure 2. The graphs of cdfs and pdfs of $P\{X_1 \leq x \mid X_{r,n} \leq t\}$ (red, above) and $P\{X_1 \leq x \mid X_{r,n} = t\}$ (blue, below) for $F(x) = 1 - \exp(-x)$, $x \geq 0$, $n = 10$, $r = 4$, and $t = 2$

Remark 3 It is interesting to point out that the cdf (18) has a jump at the point $x = t$ and $F(t) - F(t - 0) = \frac{1}{n}$, while (5) is continuous at the point $x = t$.

3 The joint distributions of a set of observations and an order statistic

Now we are interested in joint distributions of $X_1, X_2, \ldots, X_k$ and $X_{r,n}$, $1 \leq r \leq n$ and $1 < k < r$. Consider first $k = 2$. If $x_1 \leq t$ and $x_2 \leq t$, then we have

$$F_{1,2,r}(x_1, x_2, t) = P\{X_1 \leq x_1, X_2 \leq x_2, X_{r,n} \leq t\} = P\{X_1 \leq x_1, X_2 \leq x_2, \text{at least} \quad r - 2 \text{ of } X_3, X_4, \ldots, X_n \text{ are less than or equal to } t\}$$

$$= F(x_1)F_2(x_2) \sum_{i=r-2}^{n-2} \binom{n-2}{i} F(t)(1 - F(t))^{n-i}$$

$$= F(x_1)F_2(x_2)I_{F(t)}(r-2, n-r+1).$$

(20)

It is clear that $F_{1,2,r}(x_1, x_2, t)$ has the following form

$$F_{1,2,r}(x_1, x_2, t) = \begin{cases} 
F(x_1)F(x_2)I_{F(t)}(r-2, n-r+1), & x_1 \leq t, x_2 \leq t \\
\psi_1(x_1, x_2, t) & x_1 \leq t, x_2 > t \\
\psi_2(x_1, x_2, t) & x_1 > t, x_2 \leq t \\
\psi_3(x_1, x_2, t) & x_1 > t, x_2 > t 
\end{cases}$$
For the cases \( x_1 \leq t, x_2 > t \) or \( x_1 > t, x_2 \leq t \) \( \circ \) \( x_1 > t, x_2 > t \) the functions \( \psi_i(x_1, x_2, t), i = 1, 2, 3 \) can easily be calculated by using the total probability formula considering different cases, i.e.

\[
P\{X_1 \leq x_1, X_2 \leq x_2, X_{r:n} \leq t\} = P\{X_1 \leq x_1, X_2 \leq x_2, X_1 \leq t, X_2 \leq t, X_{r:n} \leq t\} \\
+ P\{X_1 \leq x_1, X_2 \leq x_2, X_1 > t, X_2 \leq t, X_{r:n} \leq t\} \\
+ P\{X_1 \leq x_1, X_2 \leq x_2, X_1 \leq t, X_2 > t, X_{r:n} \leq t\} \\
+ P\{X_1 \leq x_1, X_2 \leq x_2, X_1 > t, X_2 > t, X_{r:n} \leq t\}.
\]

It is also clear that \( F_{1,2;r}(x_1, x_2, t) \) is continuous for all values of \( x_1, x_2 \) and \( t \). The pdf is

\[
f_{1,2;r}(x_1, x_2, t) = \begin{cases} 
\frac{1}{B(r-k,n-r+1)} F(t)^{r-3}(1 - F(t))^{n-r} f(t), & x_1 \leq t, x_2 \leq t \\
\frac{\partial^2 \psi_1(x_1, x_2, t)}{\partial x_1 \partial t}, & x_1 \leq t, x_2 > t \\
\frac{\partial^2 \psi_2(x_1, x_2, t)}{\partial x_1 \partial t}, & x_1 > t, x_2 \leq t \\
\frac{\partial^2 \psi_3(x_1, x_2, t)}{\partial x_1 \partial t}, & x_1 > t, x_2 > t
\end{cases}
\]

Actually, our interest is concentrated on the part of the joint distribution \( P\{X_1 \leq x_1, X_2 \leq x_2, ..., X_k \leq x_k, X_{r:n} \leq t\} \) for \( x_1 < t, ..., x_k < t \).

**Lemma 3** For any \( x_1, x_2, ..., x_k \leq t \) it is true that

\[
F_{1,2,..,k:r}(x_1, x_2, ..., x_k, t) = \begin{cases} 
F(x_1)F(x_2)\cdots F(x_k)I_{F(t)}(r-k,n-r+1), & 1 \leq k < r \\
F(x_1)F(x_2)\cdots F(x_k), & k \geq r
\end{cases}
\]  

(21)

**Proof.** Indeed, if \( k < r \) then \( P\{X_1 \leq x_1, X_2 \leq x_2, ..., X_k \leq x_k, X_{r:n} \leq t\} = P\{X_1 \leq x_1, X_2 \leq x_2, ..., X_k \leq x_k, X_{r:n} \leq t\} = P\{X_1 \leq x_1, X_2 \leq x_2, ..., X_k \leq x_k, \text{at least } r-k \text{ of } X_{k+1}, ..., X_n \text{ are less than or equal to } t\} = F(x_1)F(x_2)\cdots F(x_k) \sum_{i=r-k}^{n-k} P\{\text{exactly } i \text{ of } X_{k+1}, ..., X_n \text{ are less than or equal to } t\} \sum_{i=r-k}^{n-k} (n-k) F^i(t)(1 - F(t))^{n-k-i}.

If \( r > k \), then it is clear that \( P\{X_1 \leq x_1, X_2 \leq x_2, ..., X_k \leq x_k, X_{r:n} \leq t\} = P\{X_1 \leq x_1, X_2 \leq x_2, ..., X_k \leq x_k, X_{r:n} \leq t\} = P\{X_1 \leq x_1, X_2 \leq x_2, ..., X_k \leq x_k\}, \text{since } x_1, x_2, ..., x_k \leq t \). \( \blacksquare \)

From the Lemma 1 it follows that for \( x_1, x_2, ..., x_k \leq t \) the joint pdf of \( X_1, X_2, ..., X_k, X_{r:n} \) is

\[
f_{1,2,..,k:r}(x_1, x_2, ..., x_k, t) = \begin{cases} 
\frac{1}{B(r-k,n-r+1)} F^{r-k-1}(t)(1 - F(t))^{n-r} f(t) \prod_{i=1}^{k} f(x_i), & 1 \leq k < r \\
f(x_1)f(x_2)\cdots f(x_k), & k \geq r
\end{cases}
\]  

(22)
Theorem 4 It is true that for \( k < r \),

\[
P\{X_1 \leq X_{r:n}, X_2 \leq X_{r:n}, \ldots, X_k \leq X_{r:n}\} = \frac{(n-k)!(r-1)!}{n!(r-k-1)!}.
\]

Proof. Let \( k < r \). Then using (22) one can write

\[
P\{X_1 < X_{r:n}, X_2 < X_{r:n}, \ldots, X_k < X_{r:n}\} = \frac{1}{B(r-2, n-r+1)} \int_0^t \cdots \int_0^t F^{r-k-1}(t) (1 - F(t))^{n-r} f(t)
\]

\[
\times \prod_{i=1}^k f(x_i) dx_1 dx_2 \cdots dx_k dt
\]

\[
= \frac{1}{B(r-2, n-r+1)} \int_0^1 (1-t)^{n-r} dt = \frac{B(r, n-r+1)}{B(r-2, n-r+1)} = \frac{(n-k)!(r-1)!}{n!(r-k-1)!}.
\]

The theorem is thus proved. ■

4 Number of inspections we need in order to detect failed components in an \((n-r+1)\)-out-of-\(n\) system

Consider a coherent system with \((n-r+1)\)-out-of-\(n\) structure and assume that the cdf \( F(x) = P\{X_1 \leq x\} \) is absolutely continuous and \( F'(x) = f(x) \). The \((n-r+1)\)-out-of-\(n\) system is intact until at least \( n-r+1 \) of the components are alive, and it fails if the number of failed components exceed \( r \), and the lifetime of this system is \( T(X_1, X_2, \ldots, X_n) = X_{r:n} \). Assume that under periodical inspections we get information about the state of the system and replace the failed components with functioning ones. We are interested in the following problem: in \((n-r+1)\)-out-of-\(n\) system some \( k < r \) components may fail, the system, however, will still be working (because of \((n-r+1)\)-out-of-\(n\) structure). In the planning of periodical inspections to detect failed components and replace them with the working ones, an important question is: what is the probability that we need \( m \) inspections to detect \( k \) failed components? Define a random variable \( N_{r:n}(k) \) to be a number of periodical inspections we need to detect \( k \) failed components. The expected value of \( N_{r:n}(k) \) will be a required average number. In engineering designs of many technical systems, the cost of inspections is high, and information about the expected number of inspections may
reduce expenses. To understand the random variable \( N_{r,n}(k) \) we assume that the components of the system are shown as \( A_1, A_2, ..., A_n \) and the corresponding lifetimes are \( X_1, X_2, ..., X_n \). For example, let \( n = 6, k = 2, r = 4 \), then \( N_{4:6}(2) = 2 \) means that in two inspections we detect 2 failed items. This can be done as follows: in the first inspection we see that \( A_1 \) is failed, (therefore, we must have \( X_1 < X_{4:6} \)) and in the second inspection we see that \( A_2 \) is failed (then, we must have \( X_2 < X_{4:6} \)). If \( N_{4:6}(2) = 3 \) this means that we detect 2 failed components in 3 inspections, and this can be done as follows: in the first inspection, we have \( A_1 \) failed, in the second inspection we have \( A_2 \) is alive, and in the third inspection we have \( A_3 \) failed; or in the first inspection we have \( A_1 \) is alive, in the second inspection we have \( A_2 \) failed and in the third inspection we have \( A_3 \) failed. These events can be represented in strings of zeros and ones as follows: \( N_{4:6}(2) = 2 \leftrightarrow \{11\}; \quad N_{4:6}(2) = 3 \leftrightarrow \{101, 011\}; \quad N_{4:6}(2) = 4 \leftrightarrow \{1001, 0101, 0011\}, \quad N_{4:6}(2) = 5 \leftrightarrow \{10001, 01001, 00101, 00011\} \) etc. The following theorem allows to calculate the distribution of random variable \( N_{r,n}(k) \).

**Theorem 5** For \( 1 \leq k < r \) it is true that

\[
P\{N_{r,n}(k) = m\} = \left(\frac{m-1}{k-1}\right) \sum_{j=0}^{m-k} (-1)^j \binom{m-k}{j} \frac{(n-k-j)!(r-1)!}{n!(r-k-j-1)!} m = k, ..., n - r + k + 1
\]  

(23)

**Proof.** Consider the random variables

\[
\xi_i = \begin{cases} 
1 & \text{if } X_i \leq X_{r:n}, \\
0 & \text{otherwise}, \end{cases} i = 1, 2, ..., n.
\]

It is clear that \( \xi_1, \xi_2, ..., \xi_n \) are exchangeable. This can be easily understood by considering, for example, the following two probabilities:

\[
P\{\xi_1 = 1, \xi_2 = 0\} = P\{X_1 \leq X_{r:n}, X_2 > X_{r:n}\} = P\{X_1 \leq X_{r:n}\} - P\{X_1 \leq X_{r:n}, X_2 \leq X_{r:n}\}
\]

\[
P\{\xi_1 = 0, \xi_2 = 1\} = P\{X_1 > X_{r:n}, X_2 \leq X_{r:n}\} = P\{X_2 \leq X_{r:n}\} - P\{X_1 \leq X_{r:n}, X_2 \leq X_{r:n}\}.
\]

We will use the following formula for exchangeable binary variables (see George and Bowman (1995)):

\[
P\{\xi_1 = 1, \xi_2 = 1, ..., \xi_k = 1, \xi_{k+1} = 0, ..., \xi_m = 0\} = \sum_{j=0}^{m-k} (-1)^j \binom{m-k}{j} \lambda_{k+j},
\]

(24)
where
\[ \lambda_k = P\{\xi_1 = 1, \xi_2 = 1, \ldots, \xi_k = 1\}. \]

Using exchangeability and (24) we have

\[
P\{N_{r,n}(k) = m\} = \sum_{i_1, i_2, \ldots, i_m} P\{X_{i_1} \leq X_{r,n}, \ldots, X_{i_k} \leq X_{r,n}, X_{i_{k+1}} > X_{r,n}, \ldots, X_{i_m} > X_{r,n}\}
\]

\[
= \binom{m-1}{k-1} P\{\xi_1 = 1, \xi_2 = 1, \ldots, \xi_k = 1, \xi_{k+1} = 0, \ldots, \xi_m = 0\}
\]

\[
= \binom{m-1}{k-1} \sum_{j=0}^{m-k} (-1)^j \binom{m-k}{j} \lambda_{k+j}
\tag{25}
\]

From the Theorem 3, one can write

\[
\lambda_{k+j} = P\{\xi_1 = 1, \xi_2 = 1, \ldots, \xi_{k+j} = 1\}
\]

\[
= \frac{(n-k-j)!(r-1)!}{n!(r-k-j-1)!}
\tag{26}
\]

Taking (26) into account in (25), one obtains (23). The theorem is thus proved. ■

**Numerical example**

**Example 1** Below in Table 1 we present numerical values of \(P\{N_{r,n}(k) = m\}\) for particular values of
\textstyle n = 12, r = 5, k = 3 \text{ and } n = 12, r = 7, k = 2, m = k, k + 1, \ldots, n - r + k + 1.

\begin{center}
\begin{tabular}{|c|c|}
\hline
$m$ & $P\{N_{5:12}(3) = m\}$ \\
\hline
3 & \frac{1}{55} = 0.01818 \\
4 & \frac{8}{165} = 0.04849 \\
5 & \frac{14}{165} = 0.08485 \\
6 & \frac{4}{33} = 0.12121 \\
7 & \frac{5}{33} = 0.15152 \\
8 & \frac{28}{165} = 0.16970 \\
9 & \frac{28}{165} = 0.16970 \\
10 & \frac{8}{55} = 0.14545 \\
11 & \frac{1}{11} = 0.09091 \\
\hline
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{|c|c|}
\hline
$m$ & $P\{N_{7:12}(2) = m\}$ \\
\hline
- & - \\
2 & \frac{5}{22} = 0.22727 \\
3 & \frac{3}{11} = 0.27273 \\
4 & \frac{5}{22} = 0.22727 \\
5 & \frac{5}{33} = 0.15152 \\
6 & \frac{25}{308} = 0.08117 \\
7 & \frac{5}{154} = 0.03247 \\
8 & \frac{1}{132} = 0.00756 \\
- & - \\
\hline
\end{tabular}
\end{center}

Table 1. Values of $P\{N_{r,n}(k) = m\}$

$n = 12, r = 5, k = 3, m = 3, 4, \ldots, 11$ (left)

$n = 12, r = 7, k = 2, m = 2, 3, \ldots, 8$ (right)

The expected value of $N_{5:12}(3)$ is

\[ EN_{5:12}(3) = \sum_{m=3}^{11} mP\{N_{5:12}(3) = m\} = 7.8, \]

and the expected value of $N_{7:12}(2)$ is

\[ EN_{5:12}(3) = \sum_{m=3}^{11} mP\{N_{7:12}(2) = m\} = 3.7143. \]

Therefore, in 8-out-of-12 system for detecting three failed components, we need an average of 8 inspections and for a 6-out-of-12 system to detect 2 failed components, we need an average of 4 inspections.

4.1 A discussion on mean residual and mean past functions

Consider a coherent system with $(n - r + 1)$–out–of–$n$ structure with life times of the components having cdf $F$ and pdf $f$. Assume that under periodical inspections, we get information about the state of the system. For example, we may know that at inspection time $t_1$ system was functioning, but at the next inspection time $t_2 > t_1$ it appeared to have failed. The exact failure time, however, is not known, i.e. it is censured in time interval $(t_1, t_2)$. One may be interested in residual life of any of the components.
having this information, i.e. the conditional mean residual life function of the components given that the failure of the system has occurred at time interval \((t_1, t_2)\). More precisely, we consider a function
\[
\varphi_n(t_1, t_2) = E\{X_1 - t_2 \mid t_1 < X_{r:n} < t_2\}
\]
and call it the mean residual life (MRL) function of the component of system failed in \((t_1, t_2)\). Another important function is
\[
\psi_n(t_1, t_2) = E\{t_2 - X_2 \mid t_1 < X_{r:n} < t_2\},
\]
the mean past (MP) function of the components given that the system has failed in \((t_1, t_2)\). These two functions may be important for reliability engineers, because the knowledge of \(\varphi_n(t_1, t_2)\) will help to determine expected residual life of the components having information about the censured failure time of the system under periodical inspections. The function \(\psi_n(t_1, t_2)\) provide information about the inactivity time of the components under the conditions described above. The pdf of conditional distribution \(P\{X_1 \leq x \mid t_1 \leq X_{r:n} \leq t_2\}\) is
\[
f_{x|t_1 \leq X_{r:n} \leq t_2}(x \mid t_1, t_2) = \begin{cases} 
(I_F(t_2) r, n - r + 1) - I_F(t_1) r, n - r + 1) & \text{if } x < t_1 \\
\times f(x) [I_F(t_2) (r - 1, n - r + 1) - I_F(t_1) (r - 1, n - r + 1)] & \\
(I_F(t_2) (r, n - r + 1) - I_F(t_1) (r, n - r + 1)) & \text{if } t_1 \leq x \leq t_2 . \\
\times f(x) [I_F(t_2) (r, n - r) - I_F(t_1) (r, n - r)] & \text{if } x > t_2.
\end{cases}
\]
We have
\[
\varphi_n(t_1, t_2) = E\{X_1 - t_2 \mid t_1 < X_{r:n} < t_2\} = \frac{1}{I_F(t_2) (r, n - r + 1) - I_F(t_1) (r, n - r + 1)} \int_0^{t_1} xf(x)dx \\
+ \frac{I_F(t_2) (r - 1, n - r + 1) - I_F(t_1) (r, n - r)}{I_F(t_2) (r, n - r + 1) - I_F(t_1) (r, n - r + 1)} \int_{t_1}^{t_2} xf(x)dx \\
+ \frac{I_F(t_2) (r, n - r) - I_F(t_1) (r, n - r)}{I_F(t_2) (r, n - r + 1) - I_F(t_1) (r, n - r + 1)} \int_{t_2}^{\infty} xf(x)dx - t_2
\]
The mean past function $\psi_n(t_1,t_2)$ can be written as follows:

$$
\psi_n(t_1,t_2) = E\{t_2 - X_1 \mid t_1 < X_{r:n} < t_2\} = 
\frac{1}{I_{F(t_2)}(r,n-r+1) - I_{F(t_1)}(r,n-r+1)} \int_0^{t_1} xf(x)dx - 
\frac{I_{F(t_2)}(r-1,n-r+1) - I_{F(t_1)}(r,n-r)}{I_{F(t_2)}(r,n-r+1) - I_{F(t_1)}(r,n-r+1)} \int_{t_1}^{t_2} xf(x)dx - 
\frac{I_{F(t_2)}(r,n-r) - I_{F(t_1)}(r,n-r)}{I_{F(t_2)}(r,n-r+1) - I_{F(t_1)}(r,n-r+1)} \int_{t_2}^{\infty} xf(x)dx.
$$

(29)

According to Remark 3, the conditional distribution $P\{X_1 \leq x \mid t_1 < X_{r:n} < t_2\}$ is continuous and the MRL and MP functions of the components given that system fails in $[t_1,t_2]$ can be easily calculated from (28) and (29).

**Conclusion 1** In this paper we consider the joint distribution of elements of random sample and the order statistic of the same sample. The joint distributions expressed in terms of binomial sums and incomplete beta functions are presented, and the dependence between related conditional random variables is discussed. The distribution results are used to solve an important problem in reliability analysis, to detect the failed components of coherent system. In particular, we consider the $(n-r+1)$-out-of-$n$ system which can function even though $k$ of the components ($k < r$) have failed. The number of inspections we need to detect certain number of failed components is important information which can help to control costs. In $(n-r+1)$-out-of-$n$ system we define a random variable $N_{r:n}(k)$ which is the number of inspections we need to detect $k$ components and find the distribution of this random variable. The expected value of $N_{r:n}(k)$ provide important information which can be used in the planning of periodical inspections of coherent systems.

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