ON THE EXISTENCE OF GLOBALLY SOLVABLE VECTOR FIELDS IN SMOOTH MANIFOLDS

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ABSTRACT. Let \((M, \omega_0)\) be a connected paracompact smooth oriented manifold. We establish a necessary and sufficient conditions on an involutive subbundle of \(TM\) such that \(M\) becomes simply connected.

1. INTRODUCTION

It is well known that there exist an obstruction to the existence of \(\dim M - k + 1\) real linearly independent vector fields on a manifold \(M\) in the \(k\)th cohomology group of \(M\), the so called \(k\)th Stiefel-Whitney class. We mean that the \(k\)th Stiefel-Whitney class being nonzero implies that there do not exist everywhere linearly independent vector fields. In particular, the \(\dim M\)th Stiefel-Whitney class is the obstruction to the existence of an everywhere nonzero vector field, and the first Stiefel-Whitney class of a manifold is the obstruction to orientability. Thus if one wishes to assume a hypotheses of existence \(k\) linearly independent vector fields in an orientable manifold it certainly imposes the vanishing of such Stiefel-Whitney classes. But it is not evident that additional hypothesis of integrability and global solvability of \(k\) sections of \(\Gamma(TM)\) the triviality of the all \(l\)th cohomology groups of \(M\) for \(l > \dim M - k + 1\). This fact is encoded in a fundamental theorem of Hormander and Duistermaat (Theorem 6.4.2 [HD], pp 30) which characterizes global solvability of first order real differential operators in a oriented smooth manifold. But its proof is not trivial because it pass troughs a generalization of the theorem above. The generalization lies in a induction over the convexity condition stated in (c) of Theorem 6.4.2 mentioned above. In the reading of proof one realize that the...
condition of convexity is needed only inside the orbits of the vector field as well as compactness is needed only for the intersection of the sublevels \( \{ x \in M : u(x) \leq c \} \) with the orbits in c) of Theorem 6.4.2. Taking this point of view we will extend the concept of convexity above by taking a finitely generated Lie algebra \( a \subseteq \Gamma(TM) \) and for a compact set \( K \subset M \) we define \( \tilde{K} \) to be the set obtained taking all smooth regular paths \( \gamma \) with with endpoints in \( K \) and \( \gamma' \in a \). We will show that in this context the analogous condition c) of Theorem 6.4.2 is equivalent to one of the two condition stated below for the orbits \( O_a(p) \) of \( a \)

(in the sense of Sussmann ([Su])).

\( (C) \) there exists at least \( \dim O_a(p) - 1 \) linearly independent sections of \( \Gamma(T O_a(p)) \) which are globally solvable for every \( p \in M \).

\( (C') \) there exist at least \( \dim O_a(p) - 1 \) linearly independent sections of \( \Gamma(T O_a(p)) \) and \( O_a(p) \) is simply connected for every \( p \in M \).

The condition \( (C') \) is a further restriction on the Stiefel-Whitney classes of \( T O_a(p) \) since we throughout this paper we will assume \( (M, \omega_0) \) will be a para-compact manifold with positive orientation \( \omega_0 \). Then the condition \( (C') \) is entirely determined by the class of homotopy of the CW complex associated to \( M \). Also we will assume the following hypothesis on the a finite set of generators \( \{ X_1, \ldots, X_n \} \) of \( a \supseteq \Gamma(TM) \); set \( d_l \) to be the formal degree of \( X_l \) and assume:

\[
[X_j, X_k] = \sum_{d_l \leq d_j + d_k} c^l_{jk} X_l
\]

in the same sense as defined in the basic work of Nagel et al ([NSW]). We also will denote by \( \sim_a \) the equivalence relation of being in a same orbit \( O_a(p) \) and \( H^1_a O_a(p) \) the first De Rham cohomological group. To avoid introduction of a metric in \( M \) we we will adopt the following definition found in page 111 of the reference [NSW].

**Definitions.** Let \( a \) a Lie subalgebra of \( \Gamma(TM) \) finitely generated by \( \{ X_1, \ldots, X_n \} \) and assume that Denote by \( C(\delta) \) the class of smooth paths \( \gamma : [0,1] \to M \) such
that
\[ \gamma'(s) = \sum_{i=1}^{n} c_i X_i(\gamma(s)) \]
with \( |c_i| \leq \delta^i \) and all \( c_i \) constants. Define the pseudo-distance
\[ \rho(p, q) = \inf\{ \delta > 0 : \exists \gamma \in C(\delta) \text{ with } \gamma(0) = p, \gamma(1) = q \} \]
and denote by \( \tilde{K}_a \) the set of all paths \( \gamma \in \cup_{\delta>0} C(\delta) \) with \( \gamma(0), \gamma(1) \in K \).

We say that the triplet \( (M, \omega, \omega_0) \) is \( a \)-convex if for every compact subset \( K \subset M \) and \( p \in M \) a pair of conditions holds;

a) \( \tilde{K}_a \) compact and
b) there exists a set \( \{X_1, \ldots, X_{n(p)-1}\} \in \Gamma(TO_a(p)) \) of linearly independent sections such that all its integral orbits are non compact.

Our main result is;

**Theorem A.** Let \( (M, \omega_0) \) be a smooth oriented paracompact connected manifold and \( a \subset \Gamma(TM) \) be a finitely generated Lie algebra. Then three statements are equivalent;

A) \( M \) is \( a \)-convex,
B) \( a \) verifies \( (C) \),
C) \( a \) verifies \( (C') \).

2. **Proof of Theorem A.**

We start the proof with a lemma;

**Lemma.** Let \( a \subset \Gamma(TM) \) be a Lie subalgebra containing by \( k \)-linearly independent globally solvable vector fields. Then there exist a commutative subalgebra \( g \leq a \) generated by \( k \)-linearly independent globally solvable vector fields.

**Proof.** We perform induction on the number \( k \). For \( k = 1 \) we just apply the equivalence between b) and f) in the Theorem 6.4.2 in [DH]. Suppose that the Lemma is true for \( k \) and let \( a \) an Lie algebra containing \( k+1 \)linearly independent set \( \{X_1, \ldots, X_n, X_{n+1}\} \) globally solvable vector fields. Then by induction hypothesis we may assume that \([X_l, X_k] = 0\) if \( 1 \leq l, k \leq n \). Linear independence implies that \( X_{n+1} = a_1 X_1 + \cdots + a_n X_n + Y \) with \( Y \in a \) a non vanishing vector field. We define
the a diffeomorphism $\Phi: M \rightarrow \mathbb{R}^n \times N$ taking as the $n$–first coordinates of $\Phi$ the solutions $X_k t_k = 1$ with its natural ordering and the trivialization comes again from the equivalence between a) and f) in Theorem 6.4.2 in [DH] just by a straightforward inductive argument over the number $k$. It follows that $D \Phi(Y(p)) \in T_p(N_k)$ for every $p \in M$, where $\Pi^{-1}(\Pi(p)) = \mathbb{R}^n \times \Pi(p)$ stands for the $n – k$–last coordinates of $\Phi$. Then by hypothesis for every orbit $\gamma$ of $X_{n+1}$ with endpoints $q_0, q_1 \in K$ there exist another compact set $K' \supset K$ such that $\gamma \subset K'$. Then $\Pi(q_0), \Pi(q_1) \in \Pi(K)$ are endpoints of $\Pi \circ \gamma \subset \Pi(K')$ and $D\Pi(\gamma') = D\Pi(Y(\gamma)) \neq 0$. If an integral orbit $\gamma$ of $X_{n+1}$ is not contained in a compact set $K$ then the same should happens to $\Pi(\gamma)$ with respect to the compact $\Pi(K)$. We now consider the orbits $O_{a_n}(p)$ where $a_n \leq a$ is the Lie subalgebra generated by $\{X_1, ..., X_n\}$. All the orbits $O_{a_n}(p)$ are diffeomorphic to $\mathbb{R}^n$. We can solve $X_{n+1} u = 1$ and consequently if $\gamma$ an integral trajectory of $X_{n+1}$ parameterized by $[0, \infty)$ with $\gamma(0) = p$ then

$$\lim_{t \to \infty} u(\gamma(t)) = \infty$$

and consequently $u$ will not be smooth in $M$ if $\Pi(\gamma)$ has compact closure. (Another argument: Denote by $\omega_n$ the positive orientation induced from $(M, \omega_0)$. Since $\omega_n(X_1, ..., X_n) > 0$ we can solve $X_{n+1} u = \omega_n(X_1, ..., X_n)^{-1}$ entailing that $du \wedge \omega_n(X_{n+1}, X_1, ..., X_n) = 1$. Let $K \subset M$ such that $K \cap O_{a_n}(\gamma(t))$ is compact and

$$1 \, dt = \int_{\Pi^{-1}(\Pi(\gamma(t)))} \chi_K \omega_n$$

for every $p \in M$. If $\gamma$ an integral trajectory of $X_{n+1}$ parameterized by $[0, \infty)$ with $\gamma(0) = p$ then

$$\int_{\Pi(\gamma)} \left( \int_{\Pi^{-1}(\Pi(p))} \chi_K \omega_n \right) du = \int_{\Pi^{-1}(\Pi(\gamma))} du \wedge \chi_K \omega_n = \infty$$

consequently if $\Pi(\gamma)$ if compact closure because then the quantity on the left is finite. Thus any integral trajectory of $Y$ is unbounded and $Y$ verify the condition d) in Theorem 6.4.2 in [DH] is globally solvable in $N_n$ indeed and the equivalence between f) and j) in Theorem 6.4.2 in [DH] applies to conclude the induction. □

Now if B) is true we apply the Lemma to $\Gamma(T O^C(p))$ to find out that is diffeomorphic to $\mathbb{R}^{n(p)-1} \times \gamma$ for some smooth regular connected curve $\gamma$ and with
$u(p) = \dim \mathcal{O}_\mathcal{E}(p)$. We consider a non vanishing section of $\gamma' \in \Gamma(T\gamma)$ which verifies

$$\omega_{\mathcal{O}_\mathcal{E}(p)}(\gamma'(p), X_1(\gamma(p)), ..., X_{n(p)-1}(p)) > 0$$

and define

$$u(t) = \int_0^t \omega_{\mathcal{O}_\mathcal{E}(\gamma(s))}(\gamma'(s), X_1(\gamma(s)), ..., X_{n(p)-1}(\gamma(s))) \, ds$$

is a strictly monotonous function of $t \in \mathbb{R}$ and consequently proper showing that $\gamma$ is diffeomorphic to an open interval and the vector field $X_{\dim \mathcal{O}_\mathcal{E}(p)}(p) = \gamma'(p)$ is indeed globally solvable in $M$ by $d$) in Theorem 6.4.2 in [DH]. Then we apply again the Lemma to find out that $\mathcal{O}_\mathcal{E}(p) \simeq R^{\dim \mathcal{O}_\mathcal{E}(p)}$. Observe that $\mathcal{O}_\mathcal{E}(p) \cap \tilde{K}_a = \mathcal{O}_\mathcal{E}(p) \cap K_a$ comes from a result of Sussmann ([Su]) and consequently compactness of $\tilde{K}_a$ is equivalent to compactness of its intersection with an arbitrary orbit $\mathcal{O}_\mathcal{E}(p)$.

On the other hand the condition (C) implies that $\mathcal{O}_\mathcal{E}(p) \simeq R^{\dim \mathcal{O}_\mathcal{E}(p)}$ with $\tilde{K}_a \cap \mathcal{O}_\mathcal{E}(p)$ corresponding to the convex envelope of $K \cap \mathcal{O}_\mathcal{E}(p)$ in $R^{\dim \mathcal{O}_\mathcal{E}(p)}$. Since the Lie algebra $a$ is finitely generated and $\mathcal{O}_\mathcal{E}(p) \simeq R^{\dim \mathcal{O}_\mathcal{E}(p)}$ the metric $\rho$ is well defined by $\rho$ in $\mathcal{O}_\mathcal{E}(p)$. Denote by $\text{diam}$ the set function induced in $\mathcal{O}_\mathcal{E}(p)$ by the Euclidean metric in $R^{\dim \mathcal{O}_\mathcal{E}(p)}$ with the inherited orientation from $M$ (see Theorem 1. & 3, page 110 in [NSW]). It follows that if $p, q \in K \cap \mathcal{O}_\mathcal{E}(p)$ then

$$C_1 \text{diam}(K \cap \mathcal{O}_\mathcal{E}(p)) \leq \rho(q, p) \leq C_2 \text{diam}^{1/\max d_j}(K \cap \mathcal{O}_\mathcal{E}(p))$$

from Proposition 1.1 page 107 in [NSW]. Consequently

$$\tilde{K}_a \cap \mathcal{O}_\mathcal{E}(p) \subset \{ q \in \mathcal{O}_\mathcal{E}(p) : \rho(q, p) \leq C_3 \text{diam}^{1/\max d_j}(K \cap \mathcal{O}_\mathcal{E}(p)) \}$$

and $\{ \rho(\cdot, p) \leq c \} \cap \mathcal{O}_\mathcal{E}(p)$ is always compact for every $c \in \mathbb{R}$. Moreover if $(x_1, ..., x_{n(p)})$ are the coordinates of $R^{\dim \mathcal{O}_\mathcal{E}(p)}$ then the inverse image of $(\partial/\partial x_1, ..., \partial/\partial x_{n(p)})$ in $\Gamma(T\mathcal{O}_\mathcal{E}(p))$ will verify the property b) for $a-$convexity, completing the first part of the proof.

Now if A) is true, since $\mathcal{O}_\mathcal{E}(p)$ is a manifold and locally we may assume that $V \simeq R^{\dim \mathcal{O}_\mathcal{E}(p)}$ for an open convex neighborhood $V \subset M$ of $p \in M$. Then $(V, a, \omega_0)$ verifies (C) and $\{ \rho(\cdot, p) \leq c \} \cap \mathcal{O}_\mathcal{E}(p)$ remains compact for small $c \in \mathbb{R}$. On the other hand by besides the existence of global linearly independent set $\{ X_1, ..., X_{n(p)-1} \}$ in $\Gamma(T\mathcal{O}_\mathcal{E}(p))$ by the property b) of $a-$convexity, the pseudo-distances defined by
but taking only smooth paths tangent to the linear span of the set \( \{X_1, \ldots, X_n\} \) generating \( \mathfrak{a} \) is equivalent to the former one by Theorem 3. &4. Equivalent pseudo-distances, page 111 in [NSW]. As a consequence the set \( \{X_1, \ldots, X_{n(p)-1}\} \) will be a set of globally solvable vector fields of \( \Gamma(T\mathcal{O}_a(p)) \) by the property \( a \)-convexity and the condition \( (C) \) is verified and \( A) \) implies \( B) \). That \( B) \) implies \( C) \) is just application of the Lemma together the equivalence \( f) \) in in Theorem 6.4.2 in [DH]. Then it is left to prove that \( C) \) implies \( B) \). Since \( \mathfrak{a} \) is finitely generated we may select a set \( \{X_1, \ldots, X_n\} \subset \Gamma(TM) \) generating \( \mathfrak{a} \) and define the second order operator \( H = X_1^2 + \cdots + X_n^2 \) which is well defined in every orbit \( \mathcal{O}_a(p) \) where it is hypoelliptic by a result of Hormander [Ho]). It is a consequence of the Bony’s maximum principle ([Bo]) that a twice differentiable function \( u \) verifying \( Hu \geq 0 \) have upperlevel sets \( \{u \geq c\} \cap \mathcal{O}_a(p) \) is relatively non compact in \( \mathcal{O}_a \). On the other hand if we assume also that the sublevel sets \( \{u \leq c\} \cap \mathcal{O}_a(p) \) are compact and the 2th Stiefel-Whitney class of \( T\mathcal{O}_a(p) \) is trivial there exist \( n(p) - 1 \) linearly independent sections \( \{X_1, \ldots, X_{n(p)-1}\} \subset \Gamma(T\mathcal{O}_a(p)) \). Since the orbits \( \mathcal{O}_a(p) \) are positively oriented with the inherited \( n(p)-1 \) differential form \( \omega_{\mathcal{O}_a(p)}(\cdot, X_1, \ldots, X_{n(p)-1}) \) is nonvanishing. But it vanishes in the tangent space of the orbit generated by the the linearly independent set of vector fields \( \{X_1, \ldots, X_{n(p)-1}\} \) which dimension is \( n(p) - 1 \) or \( n(p) \), the latter incompatible with the positivity of the orientation in \( \mathcal{O}_a(p) \). If the first De Rham cohomology group of \( \mathcal{O}_a(p) \) is trivial then locally we can find smooth function \( u \) with \( du = \omega_{\mathcal{O}_a(p)}(\cdot, X_1, \ldots, X_{n(p)-1}) \) which is constant in the components of orbits generated by \( \{X_1, \ldots, X_{n(p)-1}\} \). Then for any choice of \( X_{n(p)} \in \Gamma(T\mathcal{O}_a(p)) \) linearly independent from \( \{X_1, \ldots, X_{n(p)-1}\} \) such that \( \omega_{\mathcal{O}_a(p)}(X_{n(p)}, X_1, \ldots, X_{n(p)-1}) > 0 \) will verify \( X_{n(p)}^2 e^{-\kappa u^2}(p) > 0 \) for large positive \( \kappa(K) \) and all \( p \in K \), a compact set. The paracompactness of \( M \) allows one to find sequence of compacts \( K_n \) such that \( \text{int} K_n \subset K_{n+1} \) and smooth \( \chi_{K_n} \) with \( \chi_{K_n}(q) = 1 \) if \( q \in K_n \), \( \chi_{K_n}(q) = 0 \) if \( q \in M \setminus K_{n+1} \) and \( 0 \leq \chi_{K_n}(q) \leq 1 \) for all
q ∈ M. Denote by \( \kappa(K_n) \) a constant verifying

\[
\kappa(K_n) \min_{p \in K_n} \omega_{O_a(p)}(X_{n(p)}, X_1, \ldots, X_{n(p)-1}) \geq \sup_{p \in K_{n-1}} |X^2_{n(p)} \chi_{K_n} u|.
\]

Then by a suitable choice of constants \( \kappa(K_n) \) we find out that the function

\[
u_0 = \sum_{n=1}^{\infty} \chi_{K_n} e^{-\kappa(K_n)u^2}
\]

is smooth and \( X^2_{n(p)} u_0 > 0 \) for all \( p \in M \). If the integral trajectory \( \gamma \) of \( X_{n(p)} \) starting at \( p \in O_a(p) \) remains in a compact set \( K \) then at some point \( q \) of this compact

\[
\omega_{O_a(p)}(X_{n(p)}(q), X_1(q), \ldots, X_{n(p)-1}(q)) = 0
\]

which is a contradicts the positivity of the orientation \( \omega_{O_a(p)}(q) \). It follows that the condition \( c \) in in Theorem 6.4.2 in [DH] is verified for \( X_{n(p)} \) and it is globally integrable and we apply \( f \) in the same theorem to write \( O_a(p) = N_{n(p)-1} \times \mathbb{R} \).

Since the submanifolds \( N_{n(p)-1} \times t \) inherits same properties of \( O_a(p) \) we apply induction to conclude that \( O_a(p) \simeq \mathbb{R}^{n(p)} \), finishing the proof. □

Before finish this paper we must remark the the extraordinary semblance between this form of presenting global solvability for a real vector field in a manifold with the parallelization Theorem A in a work of Greene & Shiohama ([GS]), but in the absence of a Riemannian structure or the necessity of emptiness of the critical set of \( u \). The Theorem B in [GS] says that when the manifold is not simply connected the obstruction to global solvability will be located in the singular set of the convex function pointing to further investigation of this phenomena. Also one also may apply the generalized Tietze-Nakajima theorem in [KB] together the Lemma above to show that the manifold \( M \) has as projection of the first \( n \)–coordinates of \( \Phi \) a convex subset of \( \mathbb{R}^n \) when \( \Phi \) is proper.

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