Influence of Coulomb interaction on the anisotropic Dirac cone in graphene

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We study the influence of long-range Coulomb interaction on the physical properties of graphene that exhibits an anisotropic Dirac cone by using renormalization group method and \(1/N\) expansion, where \(N\) is the flavor of massless Dirac fermions. Explicit calculations show that the anisotropic fermion velocities flow to an isotropic fixed point in the lowest energy limit in clean graphene. We further incorporate three sorts of disorders, including random chemical potential, random gauge potential and random mass, and find that the interplay of Coulomb interaction and fermion-disorder interaction leads to rich and unusual behaviors, such as breakdown of Fermi liquid and emergence of anisotropic insulator. To make these behaviors more transparent, we calculate several physical quantities after taking into account fermion velocity renormalization. Interestingly, random chemical potential can result in qualitatively different behaviors compared with those generated by random gauge potential and random mass. We also investigate the non-perturbative effects of Coulomb interaction and calculate how dynamical gap generation is affected by the velocity anisotropy. It is found that the dynamical gap is enhanced (suppressed) as the fermion velocities decrease (increase), and that the dynamical gap is suppressed as the velocity anisotropy increases.

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I. INTRODUCTION

Massless Dirac fermions with a relativistic dispersion are known to be the low-energy elementary excitations in a variety of two-dimensional (2D) condensed matter systems, including d-wave superconductors\textsuperscript{[1,2]}, topological insulators\textsuperscript{[3]}, and graphene\textsuperscript{[4–8]}. Different from conventional Schrödinger electron systems with a finite Fermi surface, 2D Dirac fermion systems have discrete Fermi points and a vanishing density of states (DOS) at the lowest energy. Due to this difference, Dirac fermion systems exhibit nontrivial properties that cannot be realized in electron systems with finite Fermi surface. These properties become particularly interesting when massless Dirac fermions couple to some kind of massless bosonic modes. For instance, Dirac fermions may interact strongly with gauge field, long-range Coulomb potential, or critical fluctuation of an order parameter, depending on the concrete materials.

If a Dirac fermion system has an isotropic Dirac cone, there will be an uniform fermion velocity \(v_F\) that can be defined from the kinetic energy \(\varepsilon(k)\) by the relationship, \(v_F \propto \partial \varepsilon(k)/\partial k|_{k_F}\). However, in many cases the Dirac fermion systems may be spatially anisotropic for various reasons. A well-known example is the quasi-2D \(d_{x^2−y^2}\)-wave cuprate superconductors\textsuperscript{[1,2]}, where the massless nodal quasiparticles have a Fermi velocity \(v_F\) and a gap velocity \(v_\Delta\), obtained from the derivatives of Fermi energy and superconducting gap respectively. These two velocities are not equal in magnitude\textsuperscript{[2]}, and their ratio \(v_F/v_\Delta\) can be as large as 10 ~ 20. The velocity ratio is known to strongly affect many observable quantities\textsuperscript{[2]}. Moreover, it is recently discovered that the isotropic Dirac cone of graphene can be made anisotropic once some external force, which might be uniaxial strain\textsuperscript{[9–12]} or external periodic potential\textsuperscript{[13,14]}, is applied to the originally ideal honeycomb lattice. When this happens, Dirac fermions have two different velocities, \(v_1\) and \(v_2\), with their ratio \(\delta = v_2/v_1\) measuring the extent of spatial anisotropy. In addition, it is also possible to realize anisotropic Dirac cone in other Dirac fermion systems.

An interesting and widely studied problem is how the velocity anisotropy in Dirac fermion systems is affected by various interactions. We would like to know whether it is enhanced, weakened, or entirely suppressed. These problems deserve serious and systematic investigations for two reasons. First, the velocity ratio appears in many observable physical quantities, and hence have measurable effects. Second, the interaction-induced nontrivial renormalization of velocity ratio can lead to a number of unusual behaviors. In the current literature, the interactions between Dirac fermions and two sorts of critical bosonic modes have been broadly considered: gauge field and order parameter fluctuation.

Gauge field: It has been proposed by many theorists that many unusual physics of underdoped cuprates can be described by an effective QED\(_3\) theory\textsuperscript{[1]}. Within this effective theory, massless Dirac fermions couple strongly to an emergent U(1) gauge field, which may have different physical origins in different models. Detailed renormalization group (RG) calculations have shown that gauge interaction drives the anisotropic fermion velocities to flow to an isotropic fixed point\textsuperscript{[14,18]}, i.e., \(v_F/v_\Delta \rightarrow 1\), in the low-energy regime. Therefore, the velocity anisotropic is indeed irrelevant, and there might be a restored relativity\textsuperscript{[10]}. Order parameter fluctuation: In the close vicinity of certain quantum phase transitions, massless Dirac fermions may couple to the fluctuation of some order parameters. For instance, the fermions interact with the fluctuation of a nematic order parameter at a nematic quantum critical point\textsuperscript{[19,20]}, which is supposed
to exist in some $d$-wave superconductors. In contrast to the case of gauge interaction, such interaction leads to an extreme anisotropy of fermion velocities $\delta v \rightarrow 0$. Such extreme anisotropy can give rise to a series of intriguing properties, such as non-Fermi liquid behavior $[19,20]$, enhancement of thermal conductivity $[22]$, and suppression of superconductivity $[24]$. Furthermore, Dirac fermions may couple to an incommensurate antiferromagnetic order parameter. It was demonstrated $[20]$ that this coupling is very similar to that between fermions and nematic order parameter, so one could expect an analogous extreme anisotropy in this case.

In this paper, we further investigate the influence of strong interactions in anisotropic Dirac fermion systems. We consider the long-range Coulomb interaction in graphene that exhibits an anisotropic Dirac cone. To address this issue, one can utilize the renormalization group techniques $[27]$. We will pay special attention on how the velocity ratio $\delta$ flows in the low-energy regime and the physical implications of such flow.

Recently, the influence of Coulomb interaction on Dirac fermions with anisotropic dispersion is studied by Sharma et. al. $[28]$, who have performed RG calculations by making perturbative expansion in powers of coupling constant $\alpha_1 = e^2 / v_1 t$, where $v_1$ is supposed to the larger component of the two velocities. It was argued $[28]$ that Coulomb interaction can give rise to unusual behaviors. The RG scheme adopted in Ref. $[28]$ could be improved in two important aspects. First, in the present problem, the Coulomb interaction strength is actually determined by both of the two components of fermion velocities, i.e., $v_1$ and $v_2$, hence the RG calculations performed by making expansion in powers of $\alpha_1$ may not be able to capture all the essential features, especially when the anisotropy becomes strong. Second, the perturbative expansion in powers of coupling constant $\alpha_1$ is valid only in the weak coupling regime, i.e., $\alpha_1 \ll 1$. However, the Coulomb interaction is known to play a much more significant role in the strong coupling regime, which cannot be accessed by the expansion scheme based on small $\alpha_1$.

In order to perform a more general analysis that applies to both weak and strong couplings, here we will make use of the $1/N$-expansion method, with $N$ being the flavor of Dirac fermions. This method proves to be powerful in dealing with field-theoretic models of strongly interacting fermionic systems. Although the physical fermion flavor is taken to be $N = 2$, to be explained in Sec.II, we will consider a general large $N$. An important advantage of this $1/N$-expansion method is that it can be straightforwardly generalized to include the non-perturbative effects of strong Coulomb interaction. After performing explicit RG calculations, we will show that both $v_1$ and $v_2$ increase monotonically with the decreasing energy and that the velocity ratio flowing to unity, i.e., $v_2 / v_1 \rightarrow 1$, in the lowest energy limit. Apparently, the anisotropic Dirac fermion system is driven to approach a stable isotropic fixed point by the Coulomb interaction in the low-energy regime.

It is also interesting to study the effects of quenched disorders, which exist in almost all realistic graphene samples and are known to govern many low-temperature transport properties $[6,8]$. The interactions between Dirac fermions and various disorders have recently stimulated extensive research works $[23,32]$. According to their coupling to Dirac fermions, disorders are usually divided into three classes: random chemical potential, random gauge potential, and random mass. We investigate the interplay of Coulomb interaction and fermion-disorder interaction, and demonstrate that it leads to a series of unusual behaviors, including breakdown of Fermi liquid and emergence of non-Fermi-liquid states. Further, it is shown that random chemical potential exerts very different influence on the system compared with random gauge potential and random mass. To understand these behaviors in more details, we calculate Landau damping rate, DOS and specific heat after taking into account the effects of singular velocity renormalization and then discuss the physical properties of these quantities.

When the Coulomb interaction is sufficiently strong, a finite fermion gap may be dynamically generated through the formation of excitonic particle-hole pairs $[33,44]$. The dynamical gap generation drives an instability of the original semimetal ground state of graphene and leads to semimetal-insulator quantum phase transition. Since the conventional perturbative expansion is unable to study this problem, we will combine $1/N$-expansion with Dyson-Schwinger (DS) gap equation method, and then analyze the non-perturbative effects of strong Coulomb interaction. Our main interest here is the dependence of dynamical gap generation on the fermion velocities and velocity ratio. In the presence of velocity anisotropy, the DS gap equation is formally very complicated. To simplify numerical computations, we introduce a number of approximations and try to extract some common feature from the numerical results. Our results show that, the dynamical gap gets enhanced (suppressed) as the fermion velocities decrease (increase), whereas the dynamical gap is suppressed as the anisotropy increases.

The rest of paper is organized as follows. In section II, we write down the Hamiltonian and provide the Feynman rules which are used in the following calculations. Three sorts of disorders are introduced explicitly in this section. In section III, we calculate the corrections to the self-energy function of fermions and the fermion-disorder vertex due to the interplay of Coulomb interaction and fermion-disorder interaction. We then derive the RG flow equations for fermion velocities and disorder strength parameters. In section IV, we present numerical solutions of RG equations at four different limits and give a detailed interpretation of the results. In section V, we compute a number of physical quantities after taking into account the velocity renormalization. In Section VI we consider the effects of anisotropy on dynamical gap generation after including the non-perturbative effects of Coulomb interaction. In section VII we summarize our results and discuss their physical implications.
II. MODEL HAMILTONIAN

After monolayer graphene was successfully separated in laboratories \([4, 8]\), a great deal of experimental and theoretical efforts have been devoted to explore its novel and fascinating properties \([4, 8]\). Compared to the conventional metals, the most remarkable new feature of graphene is that its low-energy excitations are massless Dirac fermions having a linear dispersion. Since the fermion DOS vanishes at the neutral Dirac points, the Coulomb interaction between Dirac fermions remains long-ranged after including the dynamical screening due to particle-hole excitations. It is thus widely expected that such long-range Coulomb interaction is responsible for many unusual behaviors of graphene \([4, 8]\).

The physical effects of Coulomb interaction have been extensively investigated, with those on fermion velocity renormalization \([43, 47]\), thermodynamics \([48, 50]\), and electric conductivity \([51, 52, 58]\) being particularly intriguing. Here we are mainly interested in the singular fermion velocity renormalization caused by Coulomb interaction. If the Dirac cone is isotropic, the uniform long-ranged after including the dynamical screening due to Coulomb interaction between Dirac fermions remains long-ranged \([43, 47]\). It is remarkable that the predicted singular renormalization of fermion velocity has already been observed in ultra clean suspended graphene \([59]\), in graphene placed on boron nitride (BN) substrate \([60]\), and in ARPES measurements of quasi-free-standing graphene on silicon carbide (SiC) \([61]\). However, when graphene exhibits an anisotropic Dirac cone, the two components of fermions velocities should be renormalized separately. In this case, the velocity ratio may flow to some nontrivial fixed point.

We now write down the total Hamiltonian of the system. The free Hamiltonian of massless Dirac fermions with anisotropic dispersion is

\[
H_0 = i \sum_{\sigma, 1}^N \int d^2x \bar{\Psi}_\sigma(x)[v_1 \gamma_1 \nabla_1 + v_2 \gamma_2 \nabla_2] \Psi_\sigma(x),
\]

where \(\bar{\Psi} = \Psi^\dagger \gamma_0\). Here we have defined \(4 \times 4\) matrices

\[
\gamma_{0, 1, 2} = (\tau_3, -i \tau_2, i \tau_1) \otimes \tau_3
\]

in terms of Pauli matrices \(\tau_i\) with \(i = 0, 1, 2\), which satisfy the anticommutation relation \(\{\gamma_\mu, \gamma_\nu\} = 2 \text{diag}(1, 1, 1)\). The spin index is \(\sigma\), which takes all integers from 1 to \(N\). The physical value of spin degeneracy is \(N = 2\). However, to perform \(1/N\)-expansion, it is convenient to generalize the flavor to a large, general \(N\). The two spatial components of the anisotropic fermion velocities are \(v_1\) and \(v_2\), respectively. The massless Dirac fermion couple to each other through the long-range Coulomb interaction

\[
H_{\text{ec}} = \frac{1}{4\pi} \sum_{\sigma, \sigma'} \int d^2x d^2x' \rho_\sigma(x) \frac{e^2}{\epsilon |x-x'|} \rho_{\sigma'}(x'),
\]

where \(\rho_\sigma(x) = \Psi_\sigma(x) \gamma_0 \bar{\Psi}_\sigma(x)\) and \(\epsilon\) is dielectric constant whose magnitude is determined by the substrate.

FIG. 1: One-loop Feynman diagram for dynamical screening of Coulomb interaction, where solid line represents the free propagator of Dirac fermions, thin wavy line represents the bare Coulomb interaction function, and thick wavy line represents the dynamically screened Coulomb interaction function.

\[
\text{(a) } \quad \text{(b)}
\]

The disorder scattering process can be described by coupling the Dirac fermions to a random field \(A(x)\) as

\[
H_{\text{dis}} = v_T \sum_{\sigma = 1}^N \int d^2x \bar{\Psi}_\sigma(x) \Gamma \Psi_\sigma(x) A(x). \tag{3}
\]

The random field \(A(x)\) is a quenched, Gaussian variable which satisfies

\[
\langle A(x) \rangle = 0, \quad \langle A(x) A(x') \rangle = \Delta \delta^2(x-x'). \tag{4}
\]

where \(\Delta\) is a dimensionless variance. Here we consider three types of disorders distinguished by the definitions of the \(\Gamma\) matrix \([31]\). More concretely, \(\Gamma = \gamma_0\) for random chemical potential, \(\Gamma = (\gamma_1, \gamma_2)\) and \(v_T = (v_{1T}, v_{2T})\) for random gauge potential, and \(\Gamma = 1_{4 \times 4}\) for random mass.

Starting from \(H_0\), it is easy to obtain the free Dirac fermion propagator

\[
G_0(\omega, k) = \frac{1}{-i \omega \gamma_0 + v_1 k_1 \gamma_1 + v_2 k_2 \gamma_2}. \tag{5}
\]

The bare Coulomb interaction is

\[
D_0(q) = \frac{2\pi e^2}{\epsilon |q|}. \tag{6}
\]

At the one-loop level, the polarization is given by

\[
\Pi(\Omega, q) = -N \int \frac{d\omega}{2\pi} \frac{d^2k}{(2\pi)^2} \text{Tr} [\gamma_0 G_0(\omega, k) \gamma_0 \times G_0(\omega + \Omega, k + q)]
\]

\[
= \frac{N}{8 v_1 v_2} \frac{\epsilon^2 q_1^2}{\sqrt{\Omega^2 + v_1^2 q_1^2 + v_2^2 q_2^2}} + \frac{N}{8 v_1 v_2} \frac{\epsilon^2 q_2^2}{\sqrt{\Omega^2 + v_1^2 q_1^2 + v_2^2 q_2^2}}, \tag{7}
\]

which, according to the diagram shown in Fig. 1 leads to a dressed Coulomb interaction

\[
D(\Omega, q) = \frac{1}{\frac{|q|}{\epsilon} + \frac{N}{8 v_1 v_2} \frac{\epsilon^2 (q_1^2 + q_2^2)}{\sqrt{\Omega^2 + v_1^2 q_1^2 + v_2^2 q_2^2}}}. \tag{8}
\]
III. RENORMALIZATION GROUP ANALYSIS TO THE LEADING ORDER OF $1/N$ EXPANSION

In this section, we first calculate the self-energy corrections of Dirac fermions caused by the interplay of Coulomb interaction and disorder scattering, and then calculate the corrections to the fermion-disorder vertex. On the basis of these results, we will be able to derive the analytical expressions of RG flow equations for fermion velocities and disorder strength parameters. Our calculations are done to the leading order of $1/N$ expansion.

A. Fermion self-energy and vertex correction

The Dirac fermions receive self-energy corrections from both the Coulomb interaction and the fermion-disorder interaction, described by diagrams shown in Fig. 2.

According to Fig. 2(a), the fermion self-energy due to Coulomb interaction is given by

$$\Sigma_c(\omega, k) = -\int \frac{d^2q}{(2\pi)^2} \int \frac{d\Omega}{2\pi} G_0(\Omega + \omega, q + k) \gamma_0 \times D(\Omega, q). \quad (9)$$

After substituting Eq. (5) and Eq. (6) into this expression and performing tedious analytical calculations, which are detailed in Appendix A, we obtain

$$\frac{d\Sigma_c(\omega, k)}{d\ln \Lambda} = -i\omega \gamma_0 C_0 + v_1 k_1 \gamma_1 C_1 + v_2 k_2 \gamma_2 C_2, \quad (10)$$

where

$$C_0 = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty} dx \int_{0}^{2\pi} d\theta \times \frac{-x^2 + \cos^2 \theta + (v_2/v_1)^2 \sin^2 \theta}{(x^2 + \cos^2 \theta + (v_2/v_1)^2 \sin^2 \theta)^2} \mathcal{G}(x, \theta), \quad (11)$$

$$C_1 = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty} dx \int_{0}^{2\pi} d\theta \times \frac{-x^2 + \cos^2 \theta - (v_2/v_1)^2 \sin^2 \theta}{(x^2 + \cos^2 \theta + (v_2/v_1)^2 \sin^2 \theta)^2} \mathcal{G}(x, \theta), \quad (12)$$

$$C_2 = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty} dx \int_{0}^{2\pi} d\theta \times \frac{-x^2 - \cos^2 \theta + (v_2/v_1)^2 \sin^2 \theta}{(x^2 + \cos^2 \theta + (v_2/v_1)^2 \sin^2 \theta)^2} \mathcal{G}(x, \theta), \quad (13)$$

and

$$\mathcal{G}(x, \theta) = \frac{1}{2\pi^2 x} + \frac{N}{8\pi^2 v_1} \frac{\cos^2 \theta + (v_2/v_1)^2 \sin^2 \theta}{\sqrt{x^2 + \cos^2 \theta + (v_2/v_1)^2 \sin^2 \theta}} \quad (14)$$

with $\alpha_1 = \frac{c^2}{v_1^2}$.

According to Fig. 2(b), the fermion self-energy induced by disorder takes the form

$$\Sigma_{\text{dis}}(\omega) = \Delta v_f^2 \int \frac{d^2k}{(2\pi)^2} \Gamma G_0(k, \omega) \Gamma$$

$$= i\omega v_f^2 \Delta \int \frac{d^2k}{(2\pi)^2} \frac{\Gamma_{\gamma_0} \Gamma}{(\omega^2 + \gamma_0^2 k_x^2 + \gamma_0^2 k_y^2)}. \quad (15)$$

Different from the case of Coulomb interaction, $\Sigma_{\text{dis}}(\omega)$ is independent of momentum, which reflects the fact that disorders are static. We now have

$$\frac{d\Sigma_{\text{dis}}(\omega)}{d\ln \Lambda} = C_g i\omega \gamma_0, \quad (16)$$

where

$$C_g = \frac{v_f^2 \Delta}{2\pi v_1 v_2} \quad (17)$$

for random chemical potential and random mass for random gauge potential, and

$$C_g = \frac{(v_1^2 + v_f^2) \Delta}{2\pi v_1 v_2} \quad (18)$$

for random gauge potential.

We next consider the corrections to the fermion-disorder vertex, which receives contributions from both Coulomb interaction and fermion-disorder interaction, as described by the diagrams shown in Fig. 3. According to Fig. 3(a), the vertex correction due to Coulomb interaction is calculated as follows

$$V_c = -\int \frac{d\Omega}{2\pi} \int \frac{d^2q}{(2\pi)^2} \gamma_0 G_0(\omega, q) v_f \Gamma G_0(\omega, q) \gamma_0 \times D(\Omega, q). \quad (19)$$

After analytical calculations detailed in Appendix B, we have

$$\frac{dV_c}{d\ln \Lambda} = v_f \gamma_0 (-C_0) \quad (20)$$

for random chemical potential;

$$\frac{dV_c}{d\ln \Lambda} = v_f \Gamma (C_0 - C_1 - C_2) \quad (21)$$

for random gauge potential.
for random mass; and
\[ \frac{dV_c}{d\ln \Lambda} = v_T \gamma_1 (-C_1) \]  
(22)
\[ \frac{dV_c}{d\ln \Lambda} = v_T \gamma_2 (-C_2) \]  
(23)
for the \( \gamma_1 \) and \( \gamma_2 \) components of random gauge potential, respectively.

According to Fig. 3(b), the vertex correction due to averaging over disorders are found to be
\[ V_{\text{dis}}(\omega, k) = \Delta v_\gamma^2 \int \frac{d^2 p}{(2\pi)^2} G_0(\omega, p) v_T \Gamma \]
\[ \times G_0(\omega, p + k) \Gamma. \]  
(24)
It is shown that
\[ \frac{dV_{\text{dis}}(\omega, k)}{d\ln \Lambda} = v_T \gamma_0 C_g \]  
(25)
for random chemical potential;
\[ \frac{dV_{\text{dis}}(\omega, k)}{d\ln \Lambda} = 0 \]  
(26)
for random gauge potential; and
\[ \frac{dV_{\text{dis}}(\omega, k)}{d\ln \Lambda} = -v_T \Gamma C_g \]  
(27)
for random mass.

**B. Derivation of the RG equations**

The fermion self-energy corrections and fermion-disorder corrections obtained in the last subsection will be used to derive the relevant RG equations. Including the correction of the interaction, the action for the fermion is
\[ S_\Psi = \sum_{\sigma=1}^{N} \int \frac{d\omega}{2\pi} \frac{d^2 k}{(2\pi)^2} \bar{\Psi}_\sigma(\omega, k) \left[ G^{-1}_0(\omega, k) - \Sigma_C(\omega, k) - \Sigma_{\text{dis}}(\omega, k) \right] \Psi_\sigma(\omega, k) \]
\[ \approx \sum_{\sigma=1}^{N} \int \frac{d\omega}{2\pi} \frac{d^2 k}{(2\pi)^2} \bar{\Psi}_\sigma(\omega, k) \left[ -i\omega \gamma_0 e^l_0 d(-C_0+C_g) \right. \]
\[ + v_1 k_1 e^l_0 d(-C_1) + v_2 k_2 e^l_0 d(-C_2) \]  
(28)
\[ \left. \bar{\Psi}_\sigma(\omega, k) \right] \Psi_\sigma(\omega, k). \]

In the spirit of RG theory \[ 20, 23, 27 \], one can perform the following re-scaling transformation
\[ k_i = k_i' e^{-l}, \]  
(29)
\[ \omega = \omega' e^{-l}, \]  
(30)
\[ \Psi_\sigma(\omega, k) = \Psi_\sigma'(\omega', k') e^l \int d(t+C_0-C_g), \]  
(31)
\[ v_1 = v_1' e^l_0 d(-C_0+C_1+C_g), \]  
(32)
\[ v_2 = v_2' e^l_0 d(-C_0+C_2+C_g), \]  
(33)
which should keep the kinetic term of fermions invariant, namely
\[ S_{\Psi} = \sum_{\sigma=1}^{N} \int \frac{d\omega'}{2\pi} \frac{d^2 k'}{(2\pi)^2} \bar{\Psi}_\sigma(\omega', k') \left[ -i\omega' \gamma_0 + v_1' k_1' \right. \]
\[ + v_2' k_2' \bar{\Psi}_\sigma(\omega', k'). \]  
(34)
After including the influence of interaction, the action for the disorder scattering to the fermion becomes
\[ S_{\text{dis}} = \sum_{\sigma=1}^{N} \int \frac{d\omega}{2\pi} \frac{d^2 k}{(2\pi)^2} \int \frac{d^2 k_1}{(2\pi)^2} \bar{\Psi}_\sigma(\omega, k) \left( \Gamma + V_C \right. \]
\[ + V_{\text{dis}}(\omega, k_1) A(k-k_1). \]  
(35)
Specifically,
\[ S_{\text{dis}} \approx \sum_{\sigma=1}^{N} \int \frac{d\omega}{2\pi} \frac{d^2 k}{(2\pi)^2} \int \frac{d^2 k_1}{(2\pi)^2} \bar{\Psi}_\sigma(\omega, k) v_T \gamma_0 \]
\[ \times e^l_0 d(-C_0+C_g) \Psi_\sigma(\omega, k_1) A(k-k_1); \]  
(36)
for random chemical potential;
\[ S_{\text{dis}} \approx \sum_{\sigma=1}^{N} \int \frac{d\omega}{2\pi} \frac{d^2 k}{(2\pi)^2} \int \frac{d^2 k_1}{(2\pi)^2} \bar{\Psi}_\sigma(\omega, k) v_T \gamma_1 \]
\[ \times e^l_0 d(-C_1) \Psi_\sigma(\omega, k_1) A(k-k_1), \]  
(37)
\[ S_{\text{dis}} \approx \sum_{\sigma=1}^{N} \int \frac{d\omega}{2\pi} \frac{d^2 k}{(2\pi)^2} \int \frac{d^2 k_1}{(2\pi)^2} \bar{\Psi}_\sigma(\omega, k) v_T \gamma_2 \]
\[ \times e^l_0 d(-C_2) \Psi_\sigma(\omega, k_1) A(k-k_1) \]  
(38)
for the \( \gamma_1 \) component and \( \gamma_2 \) component of the random gauge potential respectively; and
\[ S_{\text{dis}} = \sum_{\sigma=1}^{N} \int \frac{d\omega}{2\pi} \frac{d^2 k}{(2\pi)^2} \int \frac{d^2 k_1}{(2\pi)^2} \bar{\Psi}_\sigma(\omega, k) v_T \]
\[ \times e^l_0 d(-C_0+C_1-C_2-C_g) \Psi_\sigma(\omega, k_1) \]
\[ \times A(k-k_1) \]  
(39)
for the random mass. Carry out the scaling \[ 29, 30, 31 \] along with
\[ A(k) = A'(k')e^l, \]  
(40)
and
\[ v_T = v_T', \]  
(41)
for random chemical potential;
\[ v_T_1 = v_T_1' e^l_0 d(-C_0+C_1+C_g), \]  
(42)
\[ v_T_2 = v_T_2' e^l_0 d(-C_0+C_2+C_g) \]  
(43)
for \( \gamma_1 \) and \( \gamma_2 \) component of random gauge potential respectively; and
\[ v_T = v_T' e^l_0 d(-2C_0+C_1+C_2+2C_g) \]  
(44)
FIG. 4: Renormalized fermion velocity for isotropic Dirac cone and without disorder.

for random mass. Then the corresponding action can keep the invariant form as

\[ S_{\text{dis}} = \sum_{\sigma=1}^{N} \int \frac{d\omega'}{2\pi} \frac{d^2k'}{(2\pi)^2} \int \frac{d^2k_1'}{(2\pi)^2} \bar{\Psi}'_\sigma(\omega', k')v'_1 \Gamma 
\times \Psi'_\sigma(\omega', k'_1) A'(k'-k'_1). \]  

(45)

From Eq. (32), (33) and Eq. (41), (42), (43), (44), we can get the renormalization group equations

\[ \frac{dv_1}{dl} = (C_0 - C_1 - C_g) v_1, \]  

(46)

\[ \frac{dv_2}{dl} = (C_0 - C_2 - C_g) v_2, \]  

(47)

\[ \frac{d(v_2/v_1)}{dl} = (C_1 - C_2) \frac{v_2}{v_1}, \]  

(48)

and

\[ \frac{dv_\Gamma}{dl} = 0 \]  

(49)

for random chemical potential;

\[ \frac{dv_{\Gamma 1}}{dl} = (C_0 - C_1 - C_g) v_{\Gamma 1} \]  

(50)

\[ \frac{dv_{\Gamma 2}}{dl} = (C_0 - C_2 - C_g) v_{\Gamma 2} \]  

(51)

for \( \gamma_1 \) and \( \gamma_2 \) component of random gauge potential respectively;

\[ \frac{dv_{\Gamma}}{dl} = (2C_0 - C_1 - C_2 - 2C_g) v_{\Gamma} \]  

(52)

for random mass.

FIG. 5: Renormalized \( v_1, v_2 \) and \( v_2/v_1 \) at fixed coupling \( \alpha_{10} = e^2/\epsilon v_{10} = 1 \) in the absence of disorders.

IV. NUMERICAL RESULTS

In this section, we present numerical solutions of RG equations obtained in Sec III and discuss their physical implications. In order to examine the effects of various physical mechanisms and parameters, it is helpful to analyze the results at different limits. First, we consider the case of isotropic Dirac cone in the absence of disorders. Second, we consider the case of anisotropic Dirac cone in the absence of disorders. Third, we consider isotropic Dirac cone in the presence of disorders. Finally, we turn to anisotropic Dirac cone in the presence of disorders.
A. Clean isotropic case

We first consider graphene with isotropic Dirac cone and uniform velocity, \( v_1 = v_2 = v \), and assume the sample is clean. In this case, the velocity flows as

\[
\frac{dv}{dl} = C v, \tag{53}
\]

where

\[
C = \frac{4}{N\pi^2} \left[ 1 - \frac{1}{\lambda} \arccos(\lambda) \right] \begin{cases} \frac{1}{\sqrt{1 - \lambda}} & \lambda < 1 \\ \frac{1}{\sqrt{\lambda^2 - 1}} \arccosh(\lambda) & \lambda > 1 \\ 1 & \lambda = 1 \end{cases}
\]

with \( \lambda = \frac{\delta}{N\pi} \). This result has previously been obtained by Son \[47\]. The renormalized fermion velocity, shown in Fig. 4, increases monotonically in the low-energy regime. It is interesting that this behavior is recently observed in experiments \[59–61\].

B. Clean anisotropic case

We then consider clean graphene with an anisotropic Dirac cone. We obtain the following flow equations of fermion velocities \( v_{1,2} \) and their ratio,

\[
\frac{dv_1}{dl} = (C_0 - C_1) v_1, \tag{54}
\]

\[
\frac{dv_2}{dl} = (C_0 - C_2) v_2, \tag{55}
\]

\[
\frac{d(v_2/v_1)}{dl} = (C_1 - C_2) \frac{v_2}{v_1}, \tag{56}
\]

where \( C_{0,1,2} \) are given in Sec.III. From Fig. 5, it is easy to see that both \( v_1 \) and \( v_2 \) increase monotonically with the decreasing energy, and that the velocities flow to the isotropic limit at the lowest energy, i.e., \( v_2/v_1 \to 1 \) as \( l \to \infty \). Apparently, the velocity anisotropy is irrelevant, analogous to the case of QED\(_3\). Notice this conclusion is different from the non-monotonic flow of velocity ratio claimed in Ref. \[28\].

It is now necessary to make a comparison between our results with those of Ref. \[28\]. Sharma et al. investigated the influence of Coulomb interaction on Dirac fermion systems with an anisotropic dispersion by performing perturbative expansion in powers of \( \alpha_1 = e^2/v_1 \epsilon \) \[28\]. They found that the system will flow to three different fixed points, depending on the initial values of \( \alpha_1 \) and \( \delta = v_2/v_1 - 1 \) (notice that the meaning of \( \delta \) in our paper is different from Ref. \[28\]), where it is assumed that \( v_2 < v_1 \) with \( \delta = -1 \) representing an infinite anisotropic limit. When \( \alpha_1 \) is small and \( \delta \) is smaller than certain critical value, the flow of \( \delta \) is not monotonic. In particular, the anisotropy of fermion dispersion initially increases with decreasing energy in a range of energy scale, but finally goes to an isotropic limit at the lowest energy. If \( \alpha_1 \) is strong enough, the system can become an anisotropic insulator. If \( \alpha_1 \) is small and \( |\delta| \) is larger than certain critical value, the system will become a quasi-one-dimensional non-Fermi liquid. It is obvious that these results differ significantly from ours.

We would point out that the perturbative expansion presented in Ref. \[28\] is valid if the Coulomb interaction is weak. This expansion scheme is no longer valid when the Coulomb interaction becomes strong. In addition, the Coulomb interaction strength actually depend on both \( \alpha_1 \) and \( \delta \). Since \( \delta \) itself also flows strongly with the varying energy, it seems questionable to make perturbative expansion in powers of \( \alpha_1 \) alone. In order to avoid these problems and make RG analysis reliable for both weak and strong couplings, we have chosen to study the influence of Coulomb interaction on velocity anisotropy by means of \( 1/N \) expansion. As shown in our results, the anisotropic system flows to an isotropic fixed point.

When the Coulomb interaction becomes sufficiently strong, the perturbative \( 1/N \) expansion is also invalid since the interaction may lead to an excitonic instability. We do agree with Ref. \[28\] on the opinion that the excitonic insulating transition should be investigated by means of a non-perturbative method. This issue will be addressed in Sec.\[VI\] by constructing and solving the self-consistent DS gap equation.

C. Disordered and isotropic case

Now we come to the case of isotropic Dirac cone in the presence of disorders. After assuming \( v_1 = v_2 = v \) and introducing disordered potentials, we find that

\[
\frac{dv}{dl} = (C - C_g) v, \tag{57}
\]

and that

\[
\frac{dv_T}{dl} = 0 \tag{58}
\]

for random chemical potential;

\[
\frac{dv_T}{dl} = (C - C_g) v_T \tag{59}
\]

for random gauge potential;

\[
\frac{dv_T}{dl} = 2(C - C_g) v_T \tag{60}
\]

for random mass. Apparently, the flows of \( v_T \) in random gauge potential and random mass are very similar, but are quite different from random chemical potential.

The fixed point for random gauge potential and random mass can be obtained by setting

\[
C - C_g = 0. \tag{61}
\]
This problem has been studied earlier by Stauber et al., who have made expansion in powers of interaction strength \[30\]. They have shown that \( \alpha^* \propto \Delta^{-1} \) for random chemical potential, \( \alpha^* \propto \Delta \) for random gauge potential, and \( \alpha^* \propto \Delta^3 \) for random mass. Our calculations are performed by using \( \frac{1}{N} \) expansion method and have reached quantitatively different results, depicted in Fig. 6. However, in agreement with the qualitative results of Stauber et al. \[30\], we find that, the fixed points for both random gauge potential and random mass are stable, whereas there are no stable fixed points for random chemical potential. We now present our results for random chemical potential, random gauge potential and random mass respectively in the following.

For random chemical potential, Eq. (58) implies that \( v_{\Gamma} = v_{\Gamma 0} \).

\[ v_{\Gamma} = v_{\Gamma 0}. \]  

The lines of fixed points are shown in Fig. 6(a). Apparently, the fixed points are unstable in this case. Fig. 7(a) shows the velocity flow at different values of \( \alpha_0 \). If \( \alpha_0 \)
is smaller than some critical value $\alpha^*(\Delta)$, the velocity increases continuously as the energy scale is decreasing, and the effective strength of Coulomb interaction flows to the infinitely weak coupling limit. In this case, the weak Coulomb interaction is obviously irrelevant. However, when $\alpha_0 > \alpha^*(\Delta)$, the fermion velocity decreases with decreasing energy scale, and finally vanishes at certain finite energy scale, which means the effective strength of Coulomb interaction is greatly enhanced and flows to an infinitely strong coupling limit before $l$ approaches infinity. Such unusual behaviors may be interpreted as a signature for the emergence of an interaction-driven insulating phase \[30\]. We emphasize that such kind of insulating phase (if exists in reality) is signalled by the absence of any stable fixed point and the divergence of interaction strength. It is completely different from the excitonic insulating phase \[33–44\] which is characterized by the generation of a finite fermion gap and accompanied by the happening of spontaneous symmetry breaking.

For random gauge potential, Eq. (57) and Eq. (60) combine to yield

$$\frac{v_T}{v} = \frac{v_{T0}}{v_0}. \quad (63)$$

The lines of fixed points are shown in Fig. 6(b). The fixed points are stable in this case. We see from Fig. 7(b) that the fermion velocity either increases or decreases with the decreasing energy scale, depending on the concrete value of $\alpha_0$, but finally are saturated to certain finite values. This behavior is consistent with previous results obtained in Ref. \[30\] and Ref. \[31\]. It is argued in Ref. \[31\] that such disorder dependent fixed point can give rise to a number of interesting properties, such as nonuniversal minimum dc conductivity.

For random mass, Eq. (57) and Eq. (60) lead to

$$\frac{v_T}{v^2} = \frac{v_{T0}}{v_0^2}. \quad (64)$$

As shown in Fig. 6(c), the fixed points are stable. According to Fig. 7(c), the fermion velocity are also saturated to finite values at low-energy limit, similar to the case of random gauge potential. An apparent conclusion is that random chemical potential leads to distinct behaviors compared with random gauge potential and random mass.

**D. Disordered and anisotropic case**

We finally come to the general and most interesting case in which both anisotropy and disorder are present. We will show that Coulomb interaction and fermion-disorder coupling can result in rich behaviors. The physical properties are very complicated and determined by several parameters, including Coulomb coupling $\alpha_0$ and bare velocity ratio $\delta_{10} = v_{20}/v_{10}$. To simplify our analysis, we fix the coupling at $\alpha_{10} = 1$ and examine how the two velocities and their ratio flow as $\delta_0$ is varying.

For random chemical potential, we already know that $v_T$ remains a constant, i.e., $v_T = v_{T0}$. At a fixed Coulomb coupling $\alpha_{10} = 1$, the renormalized velocities $v_{1,2}$ and their ratio $v_2/v_1$ for different bare values of ratio $\delta_0$ are presented in Fig. 8. From Fig. 7, we already know that the isotropic velocity $v$ increases monotonically with growing $l$ at fixed coupling $\alpha_{10} = 1$. In the anisotropic case, there is a critical value $\delta_0$, in the range $0.5 < \delta_0 < 0.9$. If $\delta_{0c} < \delta_0 \leq 0.9$, both $v_1$ and $v_2$ increase monotonically as the energy scale is decreasing, whereas the velocity ratio $v_2/v_1$ approaches unity at the lowest energy, which corresponds to an isotropic fixed point. On
the other hand, if \(0.5 \leq \delta_0 < \delta_{0c}\), both \(v_1\) and \(v_2\) decrease with the decreasing energy, and are driven to vanish at certain finite energy. The latter behavior suggests the disappearance of well-defined quasiparticles, and may indicate the appearance of an anisotropic insulating phase.

In the presence of random gauge potential, we know from Eqs. (46), (47), (50), and (51) that

\[
\frac{v_{1f}}{v_1} = \frac{v_{10}}{v_{10}} \quad \text{and} \quad \frac{v_{2f}}{v_2} = \frac{v_{20}}{v_{20}}, \tag{65}
\]

The RG flows of velocities \(v_{1,2}\) and ratio \(v_2/v_1\) are presented in Fig. 9. It is easy to observe that, both \(v_1\) and \(v_2\) increase initially but cease to increase after reaching some finite values. In addition, the ratio \(v_2/v_1\) eventually flows to an isotropic limit, i.e., \(v_2/v_1 \rightarrow 1\), at the lowest energy. Comparing to the clean and anisotropic case, \(v_2/v_1\) flows to unity more rapidly.

For random mass, the flows of \(v_1\), \(v_2\) and \(v_2/v_1\) are qualitatively very similar to those in the case of random gauge potential, and hence will not be presented. In particular, \(v_2/v_1 \rightarrow 1\) at the lowest energy, recovering the isotropy. However, the flow of \(v_1\) is different. In the presence of random mass, \(v_1\) are saturated to finite values (not shown here).

V. INFLUENCE OF FERMION VELOCITY RENORMALIZATION

In the last section, we have already shown that the long-range Coulomb interaction, sometimes in collaboration with disorders, can have remarkable effects on the low-energy properties of fermion velocities and velocity ratio. These effects should manifest themselves in various physical quantities. In order to make these effects more transparent, in this section we calculate several physical quantities, including quasiparticle damping rate, density of states and specific heat, and discuss the physical implications of the results.

A. Landau damping rate

Landau damping rate is an important quantity that is frequently used to characterize the effects of inter-particle interactions and to judge whether an interacting many-body system is a Fermi liquid or not. This quantity is intimately related to the wave renormalization function, which can be calculated as follows,

\[
Z_f(\omega) = \frac{1}{\left| 1 - \frac{\omega}{\omega_0} \text{Re} \Sigma^R(\omega) \right|}, \tag{66}
\]

where \(\text{Re} \Sigma^R(\omega)\) is the real part of the retarded fermion self-energy function. However, taking advantage of the RG scheme used in this paper, it is more convenient to write it in the following form

\[
Z_f = e^{\int_0^{l_0} (C_0 - C_g) dl}. \tag{67}
\]

Using the results obtained in Sec.III, it is easy to get

\[
\frac{dZ_f}{dl} = (C_0 - C_g) Z_f, \tag{68}
\]

where \(C_0\) and \(C_g\), given in Sec.IIIA, represent effects of Coulomb interaction and disorders, respectively.
In the clean limit, \( C_g = 0 \), we have

\[
\frac{dZ_f}{dl} = C_0 Z_f. \tag{69}
\]

As shown in Fig. 10, \( Z_f \) initially decreases with growing \( l \), but is saturated to a finite value as \( l \to \infty \), independent of the values of bare velocity ratio. The finiteness of \( Z_f \) indicates that the Dirac quasiparticles are well-defined and the system is a Fermi liquid. These results are well consistent with previous RG analysis presented in Ref. 10.

In the presence of disorders, the initial value of \( C_g \) becomes finite. We show the flowing behaviors of \( Z_f \) with growing \( l \) in the presence of random chemical potential in Fig. 11. At fixed coupling \( \alpha_{10} = 1 \), there is a critical value \( \delta_0 = 0.9 \). If \( \delta_0 < \delta_0 \leq 0.9 \), \( Z \) approaches certain finite value as \( l \to \infty \) and the system is a stable Fermi liquid. In this case, the fermion velocities and \( Z_f \) decrease rapidly to zero at certain finite energy scale. These unusual behaviors indicate the breakdown of Fermi liquid and may correspond to the emergence of an insulating phase, where observable quantities (including DOS and specific heat) should all vanish in the low-energy regime.

The \( l \)-dependence of \( Z_f \) in the presence of random gauge potential and random mass are shown in Fig. 12 and Fig. 13 respectively. The most noticeable common feature of these figures is that \( Z_f \) vanishes as \( l \to \infty \), which is independent of the concrete values of bare velocity ratio \( \delta_0 \). This property obviously indicates the breakdown of Fermi liquid and the emergence of non-Fermi liquid behaviors. More concretely, \( Z \) behaves as

\[
\lim_{l \to \infty} \ln(1/Z_f)/l = \eta \quad (0 < \eta < 1) \tag{70}
\]

in the low energy limit. Here, the magnitude of constant \( \eta \) is determined by the parameters \( \alpha_{10}, \nu_{20}/\nu_{10} \) and \( \nu_{10} \Delta/\nu_{10} \). We can further write

\[
Z_f \propto (e^{-l})^\eta. \tag{71}
\]

Rewriting the energy as \( \omega = \omega_0 e^{-l} \), it is then easy to obtain the real part of retarded self-energy

\[
\text{Re}\Sigma^R(\omega) \propto \omega^\eta. \tag{72}
\]

Using Kramers-Kronig relation, we obtain the imaginary part of retarded self-energy

\[
\text{Im}\Sigma^R(\omega) \propto \omega^\eta, \tag{73}
\]

which is typical non-Fermi liquid behavior since \( 0 < \eta < 1 \). Therefore, both random gauge potential and random mass can lead to breakdown of Fermi liquid and emergence of non-Fermi liquid ground state.

We have seen in this subsection that the interplay of Coulomb interaction and fermion-disorder interaction can lead to a series of interesting behaviors in graphene samples with an anisotropic Dirac cone. The system may be a normal Fermi liquid, a non-Fermi liquid, or an exotic insulator, depending on the concrete values of bare velocity ratio and the sorts of disorders. In particular, random chemical potential behaves quite differently from random gauge potential and random mass.

### B. Density of States

We now study the influence of Coulomb interaction on density of states (DOS) using the method employed by Xu et. al. 21. The DOS \( \rho(\omega) \) is defined as

\[
\rho(\omega) = N \int \frac{dk_1dk_2}{(2\pi)^2} \text{Tr} \{ \text{Im} [G_{\text{ret}}(\omega, v_1k_1, v_2k_2)] \}
\]

\[
= \frac{N}{v_1v_2} \int \frac{dk_1'dk_2'}{(2\pi)^2} \text{Tr} \{ \text{Im} [G_{\text{ret}}(\omega, k_1', k_2')] \}. \tag{74}
\]
where $G_{\text{ret}}(\omega, k_1, k_2)$ is the retarded two-point Green’s function (propagator) of Dirac fermions. In the absence of Coulomb interaction, $G_{\text{ret}}(\omega, k_1, k_2)$ is simply the retarded free fermion propagator and it is well-known that DOS exhibits a linear energy dependence, i.e., $\rho(\omega) \propto \omega$. We now study the effects of Coulomb interaction on DOS by explicit calculations.

For certain given energy $\omega$, the corresponding momentum scale is

$$\tilde{p} = \frac{\omega}{\max (v_1, v_2)},$$

which leads to

$$\frac{d \ln \omega}{d \ln \tilde{p}} = 1 + \frac{d \ln \max (v_1, v_2)}{d \ln \tilde{p}}.$$

Now the scaling equation for $\rho(\omega)$ reads

$$\frac{d \ln \rho}{d \ln \omega} = \frac{d \ln \rho}{d \ln \tilde{p} \frac{d \ln \omega}{d \ln \tilde{p}}}$$

$$= - \frac{d \ln \rho}{d \ln \Lambda \left(1 - \frac{d \ln \max (v_1, v_2)}{d \ln \Lambda}\right)}$$

$$= \frac{1 + C_0 - C_g + \frac{d \ln (v_1, v_2)}{d \ln \Lambda}}{1 - \frac{d \ln \max (v_1, v_2)}{d \ln \Lambda}}.$$

Using the RG solutions of fermion velocities obtained in Sec.III, we have

$$\frac{d \ln \rho}{d \ln \omega} = \frac{1 + 3C_0 - C_1 - C_2 - 3C_g}{1 - C_0 + C_1 + C_g}$$

for $v_1 > v_2$, and

$$\frac{d \ln \rho}{d \ln \omega} = \frac{1 + 3C_0 - C_1 - C_2 - 3C_g}{1 - C_0 + C_2 + C_g}$$

for $v_2 > v_1$.

In the clean limit with $C_g = 0$, the behaviors of $\rho(\omega)$ for different bare ratios are shown in Fig. 14(a). In the low-energy regime, $\omega \to 0$, we find that

$$\frac{\rho(\omega)}{\omega} \sim \frac{1}{\ln(\omega)}.$$

When there is random gauge potential or random mass, the corresponding $\rho(\omega)$ for different bare ratios are shown in Fig. 14(b) and (c). It can be found that $\rho(\omega)$ behaves as

$$\rho(\omega) \sim \omega^\eta \quad (0 < \eta < 1)$$

in the limit $\omega \to 0$. Comparing this expression to the linear $\omega$-dependence of DOS obtained in the non-interacting

---

FIG. 12: Wave renormalization factor for different $v_{20}/v_{10}$ at fixed coupling $\alpha_{10} = 1$ in the presence of random gauge potential with $v_{10}^2\Delta/v_{10}^2 = 0.05$.

FIG. 13: Wave renormalization factor for different $v_{20}/v_{10}$ at fixed coupling $\alpha_{10} = 1$ in the presence of random mass with $v_{10}^2\Delta/v_{10}^2 = 0.05$.
case, we know that $\eta$ reflects the corrections arising from Coulomb interaction.

Our result of $\rho(\omega)$ is different from that obtained in Ref. \[31\], where it is shown that the fermion velocity is saturated to a finite value in the presence of random gauge potential and that $\rho(\omega) \sim \omega$ due to the fact that the dynamic exponent $z = 1$ in the low-energy regime. However, we notice that the non-trivial corrections to DOS actually come from both the change of dynamic exponent and the non-trivial wave function renormalization. Although the dynamic exponent $z = 1$, the wave renormalization function receives a non-trivial correction given in Eq. (71) and lead to non-Fermi liquid like damping rate of Dirac fermions. Consequently, the behaviors of low-energy DOS are disorder dependent, as can be seen from Eq. (81).

C. Specific heat

To calculate specific heat, we also follow the method used in Ref. \[21\]. The free energy $\mathcal{F} = T \ln Z/V$ has the following singular part

$$\mathcal{F} = (\xi_x \xi_y)^{-1},$$

where $\xi_x \sim 1/T$, $\xi_y \sim v_1 \xi_x$, and $\xi_y \sim v_2 \xi_x$. In an anisotropic graphene, the free energy receives a contribution,

$$\mathcal{F} \sim \frac{1}{v_1 v_2} T^3,$$

so the specific heat can be given by

$$C_v = -T \frac{\partial^2 \mathcal{F}}{\partial T^2} \sim \frac{1}{v_1 v_2} T^2.$$

At energy $T$, the corresponding momentum scale is

$$\tilde{p} = \frac{T}{\max(v_1, v_2)},$$

which then leads to

$$\frac{d \ln T}{d \ln \tilde{p}} = 1 + \frac{d \ln \max(v_1, v_2)}{d \ln \tilde{p}}.$$

Now the scaling equation for $C_{\Psi}$ is

$$\frac{d \ln C_{\Psi}}{d \ln T} \sim 2 + \frac{d \ln \left(\frac{1}{v_1 v_2}\right)}{d \ln \tilde{p}} \frac{d \ln \max(v_1, v_2)}{d \ln \tilde{p}}$$

$$\sim 2 + \frac{d \ln \left(\frac{1}{v_1 v_2}\right)}{d \ln \tilde{p}} \left(1 + \frac{d \ln \max(v_1, v_2)}{d \ln \tilde{p}}\right)$$

$$\sim 2 - \frac{d \ln \left(\frac{1}{v_1 v_2}\right)}{d \ln \Lambda} \left(1 - \frac{d \ln \max(v_1, v_2)}{d \ln \Lambda}\right).$$

It is easy to find that

$$\frac{d \ln C_{\Psi}}{d \ln T} \sim 2 + \frac{2C_0 - C_1 - C_2 - 2C_g}{1 - C_0 + C_1 + C_g}$$

for $v_1 > v_2$, and

$$\frac{d \ln C_{\Psi}}{d \ln T} \sim 2 + \frac{2C_0 - C_1 - C_2 - 2C_g}{1 - C_0 + C_2 + C_g}$$

for $v_2 > v_1$. 

FIG. 14: (a) DOS for different $v_{10}/v_{10}$ at fixed coupling $\alpha_{10} = 1$ in the absence of disorders; (b) DOS for different $v_{20}/v_{10}$ at fixed coupling $\alpha_{10} = 1$ in the presence of random gauge potential with $v_{10}^2 \Delta/v_{10}^2 = 0.05$; (c) Wave renormalization factor for different $v_{20}/v_{10}$ at fixed coupling $\alpha_{10} = 1$ in the presence of random mass with $v_{10}^2 \Delta/v_{10}^2 = 0.05$. 


In clean graphene, the specific heat $C_v(T)$ for different parameters is shown in Fig. 15 (a). We can see that $C_v(T)/T^2 \to 0$ in the limit of $T \to 0$. More specifically, we have

$$\frac{C_v(T)}{T^2} \sim \frac{1}{\ln(T)} \quad (90)$$

which is consistent with the results obtained previously in Ref. [48].

In the presence of random gauge potential or random mass, the corresponding specific heat $C_v(T)$ for different parameters are shown in Fig. 15 (b) and (c). $C_v(T)$ behaves as

$$C_v(T) \sim T^2 \quad (91)$$

in the limit of $T \to 0$. This result is due to the fact that fermion velocities are saturated to finite values and that dynamic exponent $z \to 1$ in the low energy regime.

VI. DYNAMICAL GAP GENERATION IN ANISOTROPIC GRAPHENE

Recently, it has been predicted in both theoretical and numerical works [33, 44] that the long-range Coulomb interaction between massless Dirac fermions in graphene may generate a dynamical gap by forming excitonic pairs and consequently lead to semimetal-insulator transition. This gap-generating mechanism is of great interest to theorists because it can be considered as a concrete condensed-matter realization of the non-perturbative phenomenon of dynamical chiral symmetry breaking, which was originally proposed by Nambu and Jona-Lasinio [62] and has played a significant role in the development of modern particle physics [63-64]. From a technological point of view, a gapped graphene is more promising than gapless one as a candidate material for manipulating novel electronic devices [41-65]. For these reasons, dynamical gap generation and the resultant semimetal-insulator transition have stimulated considerable effort in recent years.

Earlier analytical calculations based on DS equation approach have reached an optimistic conclusion that a dynamical gap can be generated by Coulomb interaction in suspended clean graphene [33-44]. However, a number of approximations are adopted in these works, which more or less weakens the reliability of the above conclusion (for a brief review of these approximations, see Ref. [41]). More recently, we have revisited this problem by improving most of these approximations and found that the Coulomb interaction in suspended graphene is indeed not strong enough to open a dynamical gap [44]. At the same time, experimentalists have measured the energy spectrum of suspended graphene at ultra-low temperatures and observed no evidence of insulating behavior [59, 66]. A key factor that weakens the effective Coulomb interaction is the unusual renormalization of fermion velocity. It is known that the Coulomb interaction coupling $\alpha \propto e^2/v$ with $v$ being the universal fermion velocity in isotropic graphene. As $v$ diverges in the lowest energy limit, $\alpha$ tends to vanish, which means the effective interaction strength is significantly reduced. Apparently, fermion velocity renormalization plays a crucial role in this problem, and needs to be carefully treated.

As shown in the last sections, the renormalization of fermion velocities in anisotropic graphene can be quite different from that in the case of isotropic graphene. Such
an important difference may lead to remarkable change of the effective strength of Coulomb interaction. It is therefore very interesting to investigate how dynamical gap generation is affected by the velocity anisotropy. In this section, we study whether the velocity anisotropy in clean graphene enhances or suppresses dynamical gap generation. The non-perturbative DS equation approach \cite{33,38,42,43,62,64,67} will be used to address this issue since the conventional perturbative expansion is unable to tackle the non-perturbative phenomenon of dynamical gap generation. Moreover, in this section we will not consider the effects of disorders, which are very difficult to be incorporated in the DS equation \cite{36,43}.

After including the interaction induced self-energy corrections, the free fermion propagator, given in Eq. (5), will be renormalized to the following full propagator

\[
G(\omega, k) = \frac{1}{-i\omega A_0 \gamma_0 + v_1 k_1 A_1 \gamma_1 + v_2 k_2 A_2 \gamma_2 + m}.
\]

where \( m \) represents a finite dynamical gap and \( A_{0,1,2} \) are the three components of wave function renormalization. According to the Feynman diagram shown in Fig. 16, the dressed fermion propagator is related to the free one via the following DS equation,

\[
G^{-1}(\varepsilon, p) = G^{-1}_0(\varepsilon, p) + \int \frac{d\omega}{2\pi} \frac{d^2 k}{(2\pi)^2} \gamma_0 G(\omega, k) \gamma_0 \times D(\varepsilon - \omega, p - k).
\]

To the lowest order of \( 1/N \) expansion, we take \( A_0 = A_1 = A_2 = 1 \) for simplicity and substitute Eq. (92) into Eq. (93). After straightforward calculations, we obtain an integral equation for the dynamical gap \( m \),

\[
m(\varepsilon, p_1, p_2) = \int \frac{d\omega}{2\pi} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \frac{m(\omega, k_1, k_2)}{\omega^2 + v_1^2 k_1^2 + v_2^2 k_2^2 + m^2(\omega, k_1, k_2)} \times \frac{1}{\frac{|q|}{2\pi} + N \frac{\pi v_q q_1^2 + v_2 q_2^2}{\sqrt{12 + v_1^2 q_1^2 + v_2^2 q_2^2}}},
\]

where \( \Omega = \varepsilon - \omega \), \( q_1 = p_1 - k_1 \) and \( q_2 = p_2 - k_2 \). This nonlinear equation is complicated and needs to be numerically solved. A finite fermion gap is generated by the Coulomb interaction once this equation develops a non-trivial solution. In the anisotropic case, the equation of \( m \) depends on energy and two components of momentum separately, which makes it difficult to solve the integral equation numerically. In order to simplify numerical computations, we adopt several frequently used approximations, and then compare the results obtained under these approximations. If some common features can be extracted from all the results, then we can qualitatively judge whether spatial anisotropy is in favor of dynamical gap generation or not.

We will consider six different approximations. First, we consider the instantaneous approximation which drops the energy dependence of polarization function \cite{33–38,42,43,62–64,67} as follows

\[
m(p_1, p_2) = \frac{1}{2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \frac{m(k_1, k_2)}{\sqrt{k_1^2 + \delta^2 k_2^2 + m^2(k_1, k_2)}} \times \frac{1}{\frac{|q|}{2\pi} + N \frac{\pi v_q q_1^2 + v_2 q_2^2}{\sqrt{12 + v_1^2 q_1^2 + v_2^2 q_2^2}}},
\]

where \( \delta = v_2/v_1 \). In the derivation of this gap equation, we have performed the re-scaling transformations

\[
v_1 p_{1,2} \rightarrow p_{1,2}, \quad v_1 k_{1,2} \rightarrow k_{1,2}.
\]

Such transformation will also be used in the calculations to be performed below.

Second, we utilize the following approximation \cite{37}

\[
m(p_1, p_2) = \frac{1}{2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \frac{m(k_1, k_2)}{\sqrt{k_1^2 + \delta^2 k_2^2 + m^2(k_1, k_2)}} \times \frac{1}{\frac{|q|}{2\pi} + N \frac{\pi v_q q_1^2 + v_2 q_2^2}{\sqrt{12 + v_1^2 q_1^2 + v_2^2 q_2^2}}},
\]

The corresponding gap equation has the form

\[
m(p_1, p_2) = \frac{1}{2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \frac{m(k_1, k_2)}{\sqrt{k_1^2 + \delta^2 k_2^2 + m^2(k_1, k_2)}} \times \frac{1}{\frac{|q|}{2\pi} + N \frac{\pi v_q q_1^2 + v_2 q_2^2}{\sqrt{12 + v_1^2 q_1^2 + v_2^2 q_2^2}}},
\]

Third, we consider the approximation used in Ref. \cite{38}, which assumes that \( m(p_0, k) \) is energy-independent, i.e.,

\[
m(p_0, p_1, p_2) \rightarrow m(p_1, p_2).
\]

Applying this approximation leads to

\[
m(p_1, p_2) = \alpha_1 \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \frac{1}{\sqrt{k_1^2 + \delta^2 k_2^2 + m^2(k_1, k_2)}} 	imes \frac{m(k_1, k_2)}{\sqrt{k_1^2 + \delta^2 k_2^2 + m^2(k_1, k_2)}}.
\]

FIG. 16: Diagrams for fermion self-energy in the DS equation approach. The thick solid line represents the dressed fermion propagator.
where

\[ J(d, g) = \frac{(d^2 - 1) \left[ \pi - gc(d) \right] + dg^2 c(g)}{d^2 + g^2 - 1}, \quad (102) \]

with

\[ c(x) = \begin{cases} \frac{2}{\sqrt{1-x}} \cos^{-1}(x) & x < 1 \\ \frac{2}{\sqrt{x-1}} \cosh^{-1}(x) & x > 1 \\ 2 & x = 1 \end{cases}, \quad (103) \]

\[ d = \sqrt{\frac{k_1^2 + \delta^2 k_2^2 + m^2(k_1, k_2)}{q_1^2 + \delta^2 q_2^2}}, \quad (104) \]

\[ g = \frac{N \pi \alpha_1 \sqrt{q_1^2 + \delta^2 q_2^2}}{4 \delta \sqrt{q_1^2 + q_2^2}}, \quad (105) \]

In these approximations, the fermion velocities \( v_1 \) and \( v_2 \) are assumed to take bare values. However, both \( v_1 \) and \( v_2 \) are indeed strongly renormalized by the Coulomb interaction. To incorporate the feedback effects of strong velocity renormalization on the DS generation, we can replace the bare fermion velocities by the renormalized, momentum-dependent velocities \[ v_{1,2} \rightarrow v_{1,2}(k), \]

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which are determined by the solutions of Eq.(54) and Eq.(55), and then solve the new gap equations.

We present numerical results of the \( \delta \) and \( \alpha_1 \) dependence of \( m(0) \) obtained by applying the above three approximations in (a), (b) and (c) of Fig. 17 respectively. We then replace the bare velocities appearing in Eq.(96), Eq.(99) and Eq.(101) by the corresponding renormalized velocities, and show the results in (d), (e) and (f) of Fig. 17 respectively. From these six figures, we see that the quantitative results of dynamical gap is very sensitive to the concrete approximations, and significantly differ from each other.

Strictly speaking, all the results of \( m(0) \) presented in Fig. 17 may not correspond to the precise values of the dynamical gap. Nevertheless, one can extract a common feature from the results obtained in all these six cases: at a fixed \( \alpha_1 = \frac{e^2}{v_1 \epsilon} \), the dependence of \( m(0) \) on bare velocity ratio \( \delta \) is not monotonic. As \( \delta \) is growing from zero, the dynamical gap \( m(0) \) first increases, reaches its maximal value at certain critical ratio \( \delta_c \), and then decreases rapidly. This common feature is independent of the concrete magnitudes of coupling constant \( \alpha_1 \), provided that \( \alpha_1 \) is sufficiently large. Certainly, the precise positions of the peaks of \( m(0) \) are strongly case dependent.

We first look at the results presented in Fig. 17 (a-c). At certain fixed ratio \( \delta \), we see that the dynamical gap is always enhanced as the coupling \( \alpha_1 \) increases, which corresponds to decrease of both \( v_1 \) and \( v_2 \) if \( \epsilon \) does not change. If we fix the value of coupling \( \alpha_1 \) and increase the ratio \( \delta \), the dynamical gap is initially enhanced but then gets depressed once \( \delta \) exceeds some critical value. For fixed \( \alpha_1 \), the deceasing of \( \delta \) from 1 represents two effects:

\[ \frac{m(0)}{v_1 \Delta}, \quad (a) \]

\[ \frac{m(0)}{v_1 \Delta}, \quad (b) \]

\[ \frac{m(0)}{v_1 \Delta}, \quad (c) \]

\[ \frac{m(0)}{v_1 \Delta}, \quad (d) \]

\[ \frac{m(0)}{v_1 \Delta}, \quad (e) \]

\[ \frac{m(0)}{v_1 \Delta}, \quad (f) \]

FIG. 17: Dependence of dynamical gap \( m(0) \) on bare velocity ratio \( \delta \) obtained under a series of approximations: (a) Approximation (95); (b) Approximation (98); (c) Approximation (100). In (d), (e) and (f), the bare fermion velocities used to obtain the results of (a), (b) and (c) are replaced by the renormalized velocities obtained from the solutions of (54) and (55).
reduction of $v_1$ and enhancement of velocity anisotropy. The non-monotonic dependence of the dynamic gap on $\delta$ implies that these two effects are competing with each other. Since the first effect always enhances dynamical gap, the second effect should always suppress dynamical gap. At fixed coupling $\alpha_1$, increasing $\delta$ from 1 also leads to two effects: growth of $v_2$ and reduction of velocity anisotropy. Both of these two effects tend to suppress the dynamical gap. Indeed, Fig. 17(a-c) clearly shows that the dynamical gap is always suppressed as $\delta$ increase from $\delta = 1$.

It is also interesting to make a comparison between Fig. 17(a) and Fig. 17(a)(d). For smaller values of bare ratio $\delta_0$, velocity renormalization tends to promote the happening of dynamical gap generation. However, for relatively larger values of $\delta_0$, velocity renormalization tends to suppress the dynamical gap. The same conclusion can be drawn if we compare Fig. 17(b) with Fig. 17(c), and compare Fig. 17(c) with Fig. 17(f).

In this section, we have acquired only the $\delta$-dependence of dynamical gap for several fixed values of coupling $\alpha$. Unfortunately, it is hard to obtain a quantitatively reliable $\delta$-dependence of critical coupling $\alpha_c$ which separates the semimetal and insulating phases, primarily because of the complexity of anisotropic DS equation. However, the unusual $\delta$-dependence of dynamical gap presented in Fig. 17 suggests that it is both interesting and necessary to solve the anisotropic DS equation more precisely. We expect large scale Monte Carlo simulation \cite{33, 10} can be performed to investigate this issue and clarify some crucial problems.

VII. SUMMARY AND DISCUSSIONS

In this paper, we have investigated the influence of long-range Coulomb interaction on various properties of Dirac fermions in the context of graphene with a spatial anisotropy by performing detail RG calculations based on 1/N expansion. We find the renormalized fermion velocities increase monotonically as the energy scale decreases and the system approaches a stable isotropic fixed point in the low-energy regime.

The effects of three types of static disorders, including random chemical potential, random gauge potential and random mass are considered. The interplay of Coulomb interaction and fermion-disorder interaction leads to a series of highly unusual behaviors, such as breakdown of Fermi liquid and emergence of anisotropic insulating phase. In order to see these unusual behaviors more explicitly, we calculate several physical quantities, which include Landau damping rate, DOS and specific heat, after taking into account the singular renormalization of fermion velocities. Among the three types of disorders, random chemical potential results in qualitatively different behaviors compared to the cases of random gauge potential and random mass.

We have further studied the non-perturbative effects of Coulomb interaction and included the velocity anisotropy into the DS gap equation. We have acquired the dependence of dynamical gap on coupling $\alpha_1$ and velocity ratio $\delta$. Our results indicate that decreasing (increasing) velocities enhance (suppress) dynamic gap. In addition, increasing the velocity anisotropy weakens the effective strength of Coulomb interaction and therefore leads to suppression of dynamical gap generation.

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Appendix A: Fermion self-energy due to Coulomb interaction

In this appendix, we provide the details for the calculations of fermion self-energy due to Coulomb interaction. The self-energy is given by

$$\Sigma_C(\omega, \mathbf{k}) = -\int \frac{d^2q}{(2\pi)^2} \int \frac{d\Omega}{2\pi} \gamma_0 G_0(\Omega + \omega, \mathbf{q} + \mathbf{k}) \gamma_0 \times V(\Omega, \mathbf{q}) \times K \left( \frac{(\mathbf{q} + \mathbf{k})^2}{\Lambda^2} \right) K \left( \frac{\mathbf{q}^2}{\Lambda^2} \right). \quad (A1)$$

An ultraviolet cutoff is introduced by multiplying both fermion propagator and boson propagator by a smooth cutoff function $K(k^2/\Lambda^2)$. Here $K(y)$ is an arbitrary function with $K(0) = 1$. It falls off rapidly with $y \approx 1$, e.g., $K(y) = e^{-y}$. However, the results will be independent of the particular choices of $K(y)$. Now the self-energy becomes

$$\Sigma_C(\omega, \mathbf{k}) = -\int \frac{d^2q}{(2\pi)^2} \int \frac{d\Omega}{2\pi} \gamma_0 G_0(\Omega + \omega, \mathbf{q} + \mathbf{k}) \gamma_0 \times V(\Omega, \mathbf{q}) K \left( \frac{(\mathbf{q} + \mathbf{k})^2}{\Lambda^2} \right) K \left( \frac{\mathbf{q}^2}{\Lambda^2} \right). \quad (A2)$$

Namely,

$$\Sigma_C(K) = -\int \frac{d^3Q}{(2\pi)^3} F(Q + K) V(Q) K \left( \frac{(\mathbf{q} + \mathbf{k})^2}{\Lambda^2} \right) \times K \left( \frac{\mathbf{q}^2}{\Lambda^2} \right), \quad (A3)$$

where

$$F(\Omega + \omega, \mathbf{q} + \mathbf{k}) = i (\Omega + \omega) \gamma_0 + v_1 (q_1 + k_1) \gamma_1 + v_2 (q_2 + k_2) \gamma_2 \frac{1}{(\Omega + \omega)^2 + v_1^2 (q_1 + k_1)^2 + v_2^2 (q_2 + k_2)^2}. \quad (A4)$$
and $K \equiv (\omega, k)$ and $Q \equiv (q, \varepsilon)$ are 3-momenta. One can make the following expansion to the first order of $K_\mu$,

$$
F(Q + K)K \left( \frac{(q + k)^2}{\Lambda^2} \right)
$$

$$
\approx K_\mu \left[ \frac{\partial F(Q)}{\partial Q_\mu} K \left( \frac{q^2}{\Lambda^2} \right) + F(Q) \frac{2q_k}{\Lambda^2} K' \left( \frac{q^2}{\Lambda^2} \right) \right].
$$

(A5)

where $K_\mu = (\omega, k)$, $k_\mu = (0, k)$. Therefore, the self-energy is rewritten as

$$
\Sigma_C(K) = -K_\mu \int \frac{d^3Q}{(2\pi)^3} \left[ \frac{\partial F(Q)}{\partial Q_\mu} V(Q) K^2 \left( \frac{q^2}{\Lambda^2} \right) + F(Q) V(Q) \frac{2q_k}{\Lambda^2} K' \left( \frac{q^2}{\Lambda^2} \right) K' \left( \frac{q^2}{\Lambda^2} \right) \right].
$$

(A6)

which yields

$$
d\Sigma(K) = \frac{d\Sigma(K)}{d\ln \Lambda} = K_\mu \int \frac{d^3Q}{(2\pi)^3} \left[ \frac{\partial F(Q)}{\partial Q_\mu} V(Q) \left( \frac{q^2}{\Lambda^2} \right) + F(Q) V(Q) \frac{2q_k}{\Lambda^2} K' \left( \frac{q^2}{\Lambda^2} \right) \right] \times K'' \left( \frac{q^2}{\Lambda^2} \right)\right].
$$

(A7)

Converting to cylindrical co-ordinates by defining

$$
Q_\rho = y\Lambda (v_1 x, \cos \theta, \sin \theta),
$$

(A8)

$$
Q_\mu = (v_1 x, \cos \theta, \sin \theta),
$$

(A9)

$$
q_\mu = y\Lambda (0, \cos \theta, \sin \theta),
$$

(A10)

$$
\hat{q}_\mu = (0, \cos \theta, \sin \theta),
$$

(A11)

$$
d^3Q = y^2 \Lambda^2 v_1 dx dy d\theta,
$$

(A12)

we have

$$
d\Sigma(K) = \frac{d\Sigma(K)}{d\ln \Lambda} = K_\mu \frac{v_1}{2\pi^3} \int_{-\infty}^{+\infty} dx \int_0^{2\pi} d\theta \left[ \frac{\partial F(Q)}{\partial Q_\mu} + F(Q) \frac{2q_k}{\Lambda^2} \right]
$$

$$
\times V(Q) \int_0^{+\infty} dyy K \left( \frac{y^2}{\Lambda^2} \right) K' \left( \frac{y^2}{\Lambda^2} \right) + F(Q) V(Q) \frac{2q_k}{\Lambda^2} K' \left( \frac{y^2}{\Lambda^2} \right) K'' \left( \frac{y^2}{\Lambda^2} \right)\right].
$$

(A13)

Since

$$
\int_{0}^{+\infty} dyy K \left( \frac{y^2}{\Lambda^2} \right) K' \left( \frac{y^2}{\Lambda^2} \right) = \frac{1}{4},
$$

(A14)

$$
\int_{0}^{+\infty} dyy^3 K \left( \frac{y^2}{\Lambda^2} \right) K' \left( \frac{y^2}{\Lambda^2} \right) = \frac{1}{4},
$$

(A15)

one can further obtain

$$
d\Sigma(K) = \frac{d\Sigma(K)}{d\ln \Lambda} = -K_\mu \frac{v_1}{8\pi^3} \int_{-\infty}^{+\infty} dx \int_0^{2\pi} d\theta \frac{\partial F(Q)}{\partial Q_\mu} V(Q).
$$

(A16)

where

$$
F(Q) = \frac{1}{v_1} \frac{i}{x^2} + \cos^2 \theta + (v_2/v_1)^2 \sin^2 \theta,
$$

(A17)

$$
V(Q) = v_1 G(x, \theta),
$$

(A18)

with

$$
G(x, \theta) = \frac{1}{\sqrt{2\pi v}} + \frac{C}{\sqrt{2\pi v}} (x^2 + \cos^2 \theta + (v_2/v_1)^2 \sin^2 \theta)
$$

(A19)

Finally, the self-energy function is determined by

$$
d\Sigma(K) = \frac{d\Sigma(K)}{d\ln \Lambda} = -K_\mu \frac{v_1}{8\pi^3} \int_{-\infty}^{+\infty} dx \int_0^{2\pi} d\theta \frac{\partial F(Q)}{\partial Q_\mu} V(Q)
$$

$$
= -i \omega \gamma_0 C_0 + v_1 k_1 \gamma_1 C_1 + v_2 k_2 \gamma_2 C_2
$$

(A20)

where

$$
C_0 = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty} dx \int_0^{2\pi} d\theta \frac{\partial F(Q)}{\partial Q_\mu} V(Q)
$$

$$
\times (x^2 + \cos^2 \theta + (v_2/v_1)^2 \sin^2 \theta) G(x, \theta),
$$

(A21)

$$
C_1 = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty} dx \int_0^{2\pi} d\theta \frac{\partial F(Q)}{\partial Q_\mu} V(Q)
$$

$$
\times (x^2 + \cos^2 \theta - (v_2/v_1)^2 \sin^2 \theta) G(x, \theta),
$$

(A22)

$$
C_2 = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty} dx \int_0^{2\pi} d\theta \frac{\partial F(Q)}{\partial Q_\mu} V(Q)
$$

$$
\times (x^2 - \cos^2 \theta + (v_2/v_1)^2 \sin^2 \theta) G(x, \theta),
$$

(A23)

Appendix B: Fermion-Disorder vertex correction due to Coulomb interaction

The correction to fermion-disorder vertex due to Coulomb interaction is given by

$$
V_C = -i \frac{\partial Q}{\partial \Omega} \int \frac{d^2 Q}{(2\pi)^2} \gamma_0 G_0(Q) v_1 \Gamma G_0(Q) \gamma_0 V(Q)
$$

$$
\times V(\Omega, q).
$$

(B1)

One can impose a momentum cutoff by multiplying both fermion and boson propagators by a smooth function $K(k^2/\Lambda^2)$, and then obtain

$$
V_C = -v_1 \int \frac{d^2 Q}{(2\pi)^2} \gamma_0 G_0(Q) \Gamma G_0(Q) \gamma_0 V(Q)
$$

$$
\times K^3 \left( \frac{Q^2}{\Lambda^2} \right).
$$

(B2)

Therefore, we have

$$
\frac{dV_C}{d\ln \Lambda} = 6v_1 \int \frac{d^2 Q}{(2\pi)^2} \gamma_0 G_0(Q) \Gamma G_0(Q) \gamma_0 V(Q)
$$

$$
\times K^3 \left( \frac{Q^2}{\Lambda^2} \right) K' \left( \frac{Q^2}{\Lambda^2} \right) \left( \frac{Q^2}{\Lambda^2} \right).
$$

(B3)
After converting to cylindrical coordinates defined by Eq. (A.8)-(A.12), it is easy to get

\[
\frac{dV_C}{d\Lambda} = \frac{v_F^3}{4\pi^2} \int_{-\infty}^{0} \frac{d\theta}{\gamma_0 G(\hat{Q}) \Gamma G(\hat{Q})} \frac{d\gamma_0 V(\hat{Q})}{\gamma_0}
\]

\[
\times \int_{0}^{+\infty} dyyK(2)(y^2)K'(y^2),
\]

where

\[
\int_{0}^{+\infty} dyyK(2)(y^2)K'(y^2) = -\frac{1}{6}
\]

Finally, we obtain

\[
\frac{dV_C}{d\ln \Lambda} = -\frac{v_F^3}{8\pi^2} \int_{-\infty}^{0} \frac{d\theta}{\gamma_0 H(\hat{Q})}.
\]

where

\[
H(\hat{Q}) = \gamma_0 G(\hat{Q}) \Gamma G(\hat{Q}) \gamma_0 V(\hat{Q}).
\]

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