Tight Lower Bounds for Planted Clique in the Degree-4 SOS Program

Prasad Raghavendra * Tselil Schramm†

Abstract

We give a lower bound of $\tilde{\Omega}(\sqrt{n})$ for the degree-4 Sum-of-Squares SDP relaxation for the planted clique problem. Specifically, we show that on an Erdős-Rényi graph $G(n, \frac{1}{2})$, with high probability there is a feasible point for the degree-4 SOS relaxation of the clique problem with an objective value of $\tilde{\Omega}(\sqrt{n})$, so that the program cannot distinguish between a random graph and a random graph with a planted clique of size $\tilde{O}(\sqrt{n})$. This bound is tight.

We build on the works of Deshpande and Montanari and Meka et al., who give lower bounds of $\tilde{\Omega}(n^{1/3})$ and $\tilde{\Omega}(n^{1/4})$ respectively. We improve on their results by making a perturbation to the SDP solution proposed in their work, then showing that this perturbation remains PSD as the objective value approaches $O(n^{1/2})$.

In an independent work, Hopkins, Kothari and Potechin [HKP15] have obtained a similar lower bound for the degree-4 SOS relaxation.

*UC Berkeley, prasad@cs.berkeley.edu. Supported by NSF Career Award, NSF CCF-1407779 and the Alfred.
P. Sloan Fellowship.
†UC Berkeley, tschramm@cs.berkeley.edu. Supported by an NSF Graduate Research Fellowship (NSF award no 1106400).
1 Introduction

In the Maximum Clique problem, the input consists of a graph $G = (V, E)$ and the goal is to find the largest subset $S$ of vertices all of which are connected to each other. The Maximum Clique problem is NP-hard to approximate within a $n^{1-\epsilon}$-factor for all $\epsilon > 0$ [Has96, Kho01].

Karp [Kar76] suggested an average case version of the Maximum Clique problem on random graphs drawn from the Erdös-Rényi distribution $G(n, \frac{1}{2})$. A heuristic argument shows that an Erdös-Rényi graph $G \sim G(n, \frac{1}{2})$ has a clique of size $(1 - o(1)) \log n$ with high probability: given such a graph, choose a random vertex, then choose one of its neighbors, then choose a vertex adjacent to both, and continue this process until there is no vertex adjacent to the clique. After log $n$ steps, the probability that another vertex can be added is $\frac{1}{n}$, and so after about log $n$ steps this process terminates. This heuristic argument can be made precise, and one can show that this greedy algorithm can find a clique of size $(1 + o(1)) \log n$ in an instance of $G(n, \frac{1}{2})$ in polynomial time.

Indeed, with some work it can be shown that the largest clique in an instance of $G(n, \frac{1}{2})$ actually has size $(2 \pm o(1)) \log n$ with high probability [GM75, Mat76, BE76]. But while some clique of size $(1 \pm o(1)) \log n$ can easily be found in polynomial time (using the heuristic from the previous paragraph), an efficient algorithm for finding the clique of size $2 \log n$ has been much more elusive. In his seminal paper on the probabilistic analysis of combinatorial algorithms, Karp asked whether there exists a polynomial-time algorithm for finding a clique of size $(1 + \epsilon) \log n$ for any fixed constant $\epsilon > 0$ [Kar76]. Despite extensive efforts, there has been no algorithmic progress on this question since.

The planted clique problem is a natural variant of this problem wherein the input is promised to be either a graph drawn from $G \sim G(n, \frac{1}{2})$ or a graph $G \sim G(n, \frac{1}{2})$ with a clique of size $k$ planted within its vertices. The goal of the algorithm is to distinguish between the two distributions.

For $k > (2 + \epsilon) \log n$, there is a simple quasi-polynomial time algorithm that distinguishes the two distributions. The algorithm simply tries all subsets of $(2 + \epsilon) \log n$ vertices, looking for a clique. For a random graph $G(n, \frac{1}{2})$, there are no cliques of size $(2 + \epsilon) \log n$, but there is one in the planted distribution. Clearly, the planted clique problem becomes easier as the planted clique’s size $k$ increases. Yet there are no polynomial-time algorithms known for this problem for any $k < o(\sqrt{n})$. For $k = \Omega(\sqrt{n})$, a result of Alon et al. uses random matrix theory to argue that looking at the spectrum of the adjacency matrix suffices to solve the decision problem [AKS98].

The works of [FK08, BV09] show that, if one were able to efficiently calculate the injective tensor norm of a certain random order-$m$ tensor, then by extending the spectral algorithm of [AKS98] one would have a polynomial-time algorithm for $k > n^{1/m}$. However, there is no known algorithm that efficiently computes the injective tensor norm of an order-$m$ tensor; in fact computing the inective tensor norm is hard to approximate in the general case [HM13].

While algorithmic progress has been slow, there has been success in proving strong lower bounds for the planted clique problem within specific algorithmic frameworks. The first such bound was given by Jerrum, who showed that a class of Markov Chain Monte Carlo algorithms require a super-polynomial number of steps to find a clique of size $(1 + \epsilon) \log n$, for any fixed $\epsilon > 0$, in an instance of $G(n, \frac{1}{2})$ [Jer92]. Feige and Krauthgamer showed that $r$-levels of the Lovász-Schriver SDP hierarchy are needed to find a hidden clique of size $k \geq \tilde{O}(\sqrt{n}/2^r)$ [FK00, FK03]. Feldman et al. show (for the planted bipartite clique problem) that any “statistical algorithm” cannot distinguish in a polynomial number of queries between the random and planted cases for $k < \tilde{O}(\sqrt{n})$ [FGR+12].

More recently, there has been an effort to replicate the results of [FK00, FK03] for the Sum-of-Squares (or SOS) hierarchy, a more powerful SDP hierarchy. The recent work of [MPW15] achieves a $\tilde{O}(n^{1/2r})$-lower bound for $r$-rounds of the SOS hierarchy, by demonstrating a feasible
solution for the level-\(r\) SDP relaxation with a large enough objective value in the random case. The work of [DM15a] achieves a sharper \(\tilde{\Omega}(n^{1/3})\) lower bound for the Meka-Potechin-Wigderson SDP solution, but only for \(r = 2\) rounds; a counterexample of Kelner (which may be found in [Bar14]) demonstrates that the analysis of [DM15a] is tight for the integrality gap instance of [DM15a, MPW15] within logarithmic factors.

This line of work brings to the fore the question: can a \(d = O(1)\)-degree SOS relaxation solve the planted clique problem for some \(k < \sqrt{n}\)? While lower bounds are known for Lovász-Schrijver SDP relaxations for planted clique [FK00, FK03], SOS relaxations can in general be much more powerful than Lovász-Schrijver relaxations. For example, while there are instances of unique games that are hard for poly(log log \(n\))-rounds of the Lovász-Schrijver SDP hierarchy [KS09, RS09], recent work has shown that these instances are solved by degree-8 SOS hierarchy [BBH+12].

Moreover, even the degree-4 SOS relaxation proves to be surprisingly powerful in a few applications:

- First, the work of Barak et al. [BBH+12] shows that a degree 4 SOS relaxation can certify \(2 - to - 4\) hypercontractivity of low degree polynomials over the hypercube. This argument is the reason that hard instances for Lovász-Schriver and other SDP hierarchies constructed via the nois\(y\) hypercube gadgets are easily refuted by the SOS hierarchy.

- Second, a degree-4 SOS relaxation can certify that the 2-to-4 norm of a random subspace of dimension at most \(o(\sqrt{n})\) is bounded by a constant (with high probability over the choice of the subspace) [BBH+12]. This average-case problem has superficial similarities to the planted clique problem.

In this work, we make modest progress towards a lower bound for SOS relaxations of planted clique by obtaining a nearly tight lower bound for the degree-4 SOS relaxation (corresponding to two rounds, \(r = 2\)). More precisely, our main result is the following.

**Theorem 1.1.** Suppose that \(G \sim \mathcal{G}(n, \frac{1}{2})\). Then with probability \(1 - O(n^{-4})\), there exists a feasible solution to the SOS-SDP of degree \(d = 4\) (\(r = 2\)) with objective value \(\sqrt{n} \operatorname{polylog} n\).

Note that by the work of [AKS98], this result is tight up to logarithmic factors. In an independent work, Hopkins, Kothari and Potechin [HKP15] have obtained a similar result.

Our work builds heavily on previous work by Meka, Potechin and Wigderson [MPW15] and Deshpande and Montanari [DM15a]. Since the SDP solution constructed in these works is infeasible for \(k > n^{1/3}\), we introduce a modified SDP solution with objective value \(\tilde{\Omega}(\sqrt{n})\), and prove that for a random graph \(G\) the solution is feasible with high probability. At the parameter setting for which the objective value becomes \(\Omega(n^{1/3})\), the SDP solutions of [DM15a, MPW15] violate the PSDness constraint, or equivalently, there exists a set of test vectors \(X\) such that \(x^T M x < 0\) for all \(x \in X\). Our feasible SDP solution is a perturbation of their solution—we add spectral mass to the solution along the vectors from the set \(X\), then enforce the linear constraints of the SDP program.

### 1.1 Notation

We use the symbol \(\succeq\) to denote the PSD ordering on matrices, saying that \(A \succeq 0\) if \(A\) is PSD and that \(A \succeq B\) if \(A - B \succeq 0\). When we wish to hide constant factors for clarity, we use \(a \lesssim b\) to denote that \(a \leq C \cdot b\) for some constant \(C\).

---

1We have made no effort to optimize logarithmic factors in this work; a more delicate analysis of the required logarithmic factors is certainly possible.
We denote by \( \mathbb{1}_n \in \mathbb{R}^n \) the vector such that \( \mathbb{1}_n(i) \overset{\text{def}}{=} 1 \) \( \forall i \in [n] \), or the all-1’s vector. We denote the normalized version of this vector by \( \tilde{\mathbb{1}}_n \overset{\text{def}}{=} \mathbb{1}_n / \| \mathbb{1}_n \| \). Further, we use \( J_n \overset{\text{def}}{=} \mathbb{1} \mathbb{1}^\top \) and \( Q_n \overset{\text{def}}{=} \tilde{\mathbb{1}} \tilde{\mathbb{1}}^\top \).

We will drop the subscript when \( n \) is clear from context.

In our notation, we at times closely follow the notation of [DM15a], as our paper builds on their results and we recycle many of their bounds. For convenience, we will use the shorthand \( n \overset{\text{def}}{=} n \log n \). We will abuse notation by using \( \binom{n}{2} \) to refer to both the binomial coefficient and to the set \( \binom{n}{2} = \{ (a, b) | a, b \in [n], \ a \neq b \} \). We will also use the notation \( \binom{n}{\leq k} \) to refer to the union of sets \( \bigcup_{i=0}^{k} \binom{n}{i} \).

Throughout the paper, we will (unless otherwise stated) work with some fixed instance \( G \) of \( G(n, \frac{1}{2}) \), and denote by \( A_i \in \mathbb{R}^n \) the “centered” \( i \)th row of the adjacency matrix of \( G \), with \( j \)th entry equal to 1 if the edge \((i, j) \in E \), equal to \(-1\) if the edge \((i, j) \notin E \), and equal to 0 for \( j = i \). We will use \( A_{ij} \) to denote the \( j \)th index of \( A_i \).

1.2 Organization

In Section 2, we give background material on the degree-4 SOS relaxation for the max-clique problem, describe the integrality gap of Deshpande and Montanari for the planted clique problem, and explain the obstacle they face to reach an integrality gap value of \( \tilde{\Omega}(\sqrt{n}) \). We then describe our integrality gap instance, motivating our construction using the obstacle for the Deshpande-Montanari and Meka-Potechin-Wigderson witness, and give an overview of our proof that our integrality gap instance is feasible. In Section 3, we prove that our witness is PSD, completing the proof of feasibility. Section 4 contains our concentration bounds for random matrices that arise within our proofs. In our proof, we reuse several bounds proved by Deshpande and Montanari. As far as possible, we restate the claims from [DM15a] as they are used; for convenience, in Appendix A, we list a few other claims from Deshpande and Montanari that we use in this paper.

2 Preliminaries and Proof Overview

In this section, we describe the degree-4 SOS relaxation for the max-clique problem, and give background on the Deshpande-Montanari witness. We then describe our own modified witness, and give an overview of the proof that our witness is feasible (the difficult part being showing that our witness is PSD). The full proof of feasibility is deferred to Section 3.

2.1 Degree-4 SOS Relaxation for Max Clique

The degree \( d = 4 \) SOS relaxation for the maximum clique problem is a semidefinite program whose variables are \( X \in \mathbb{R}^{\binom{n}{2} \times \binom{n}{2}} \). For a subset \( S \subseteq V \) with \( |S| \leq 2 \), the variable \( X_S \) indicates whether \( S \) is contained in the maximum clique. For a graph \( G \) on \( n \) vertices, the program can be described as follows.

Maximize \[ \sum_{i \in [n]} X_{\{i\},\{i\}} \quad (2.1) \]

subject to \( X_{S_1, S_2} \in [0, 1] \)

\[ X_{S_1, S_2} = X_{S_3, S_4} \quad \forall S_1, S_2 \in \binom{n}{\leq 2} \]

\[ X_{S_1, S_2} = 0 \quad \text{whenever } S_1 \cup S_2 = S_3 \cup S_4 \]

\[ X_{S_1, S_2} = 0 \quad \text{if } S_1 \cup S_2 \text{ is not a clique in } G \]
\[ X_{0,0} = 1 \]
\[ X \geq 0 \]

It is instructive to think of the variable \( X_S \) as a *pseudoexpectation* of the product of indicator variables, or a *pseudomoment*:

\[ X_S = \tilde{E} \left( \prod_{i \in S} \mathbb{I}(i \in \text{clique}) \right). \]

Intuitively, the constraints of the SDP force the solution to behave somewhat like the moments of a probability distribution over integral solutions, although they needn’t correspond to the moments of a true distribution, hence the term *pseudomoment*. For more background, see e.g. [Bar14]. The pseudomoment interpretation of the SDP solution motivates the choice of the witness in the prior work. For example, we may notice that the objective function in this view is simply the pseudoepectation of the size of the planted clique, \( \tilde{E}[\sum_{i \in [n]} \mathbb{I}(i \in \text{clique})] \).

If \( \text{sdpval}(G, 4) \) denotes the optimum value of the SDP relaxation on graph \( G \), then clearly \( \text{sdpval}(G) \) is at least the size of the maximum clique in \( G \). In order to prove a lower bound for degree 4 SOS relaxation on \( G(n, \frac{1}{2}) \), it is sufficient to argue that with overwhelming probability, \( \text{sdpval}(G) \) is significantly larger than the maximum clique on a random graph. This amounts to exhibiting a feasible SDP solution with large objective value, for an overwhelming fraction of graphs sampled from \( G(n, \frac{1}{2}) \). Formally, we will show the following:

**Theorem 2.1** (Formal version of Theorem 1.1). There exists an absolute constant \( c \in \mathbb{N} \) such that

\[ \Pr_{G \sim G(n, \frac{1}{2})} \left\{ \text{sdpval}(G) \geq \sqrt{n} \log^c n \right\} \geq 1 - O(n^{-4}) \]

We obtain Theorem 2.1 by constructing a point, or witness, for each \( G \sim G(n, \frac{1}{2}) \), then proving that the point is feasible with high probability. We defer the description of our witness to Definition 2.8 and Definition 2.9, as we spend Section 2.2 and Section 2.3 motivating our construction; however the curious reader may skip ahead to Definition 2.9 which does not require the knowledge of additional notation.

### 2.2 Deshpande-Montanari Witness

Henceforth, fix a graph \( G \) that is sampled from \( G(n, \frac{1}{2}) \). Both the work of Meka, Potechin and Wigderson [MPW15] and that of Deshpande and Montanari [DM15a] construct essentially the same SDP solution for the degree-4 SOS relaxation.

This SDP solution assigns to each clique of size 1, \ldots, \( d \), a value that depends only on its size (in our case, \( d = 4 \)). In essence, their solution takes advantage of the independence of the \( G(n, p) \) instance. The motivating observation is that the variable \( X_S \) can be thought of as a pseudoepectation of the indicator that \( S \) is a subclique of the planted clique. The idea is then to make this pseudoeexpectation of the indicator consistent with the true expectation under the distribution where a clique of size \( k \) is planted uniformly at random within the instance of \( G(n, p) \). Thus, every vertex is in the clique “with uniform probability:”

\[ \tilde{E}[X_{\{i\}}] \approx \mathbb{E}[\mathbb{I}(i \text{ is in planted clique})] = \frac{k}{n}. \]

Then, the same principle is applied to edges, triangles, and 4-cliques, so that

\[ \tilde{E}[X_S] \approx \mathbb{I}(S \text{ is clique}) \cdot \mathbb{E}[\mathbb{I}(S \text{ is in planted clique})] = \frac{k_{|S|}}{n} \cdot \left( \frac{1}{2} \right)^{|S|^2}. \]
This is the general idea of the SDP solution of [DM15a]. More formally, the SDP solution in [DM15a] is specified by four parameters \( \alpha = \{\alpha_i\}_{i=1}^4 \) as,

\[
M(G, \alpha) = \alpha_{|A\cup B|} \cdot G_{A\cup B},
\]

where for a set of vertices \( A \subseteq V \), \( G_A \) is the indicator that the subgraph induced on \( A \) is a clique. The parameters \( \{\alpha_i\}_{i=1}^4 \) determine the value of the objective function, and the feasibility of the solution. As a convention, we will define \( \alpha_0 = 1 \).

It is easy to check that the solution \( M(G, \alpha) \) satisfies all the linear constraints of the SOS program (2.1), since it assigns non-zero values only to cliques in \( G \). The key difficulty is in showing that the matrix \( M \) is PSD for an appropriate choice of parameters \( \alpha \).

In order to show that \( M(G, \alpha) \succeq 0 \), it is sufficient to show that \( N(G, \alpha) \succeq 0 \) where,

\[
N_{A,B} = \alpha_{|A\cup B|} \cdot \prod_{i \in A \setminus B, j \in B \setminus A} G_{ij},
\]

where \( G_{ij} \) is the indicator for the presence of the edge \((i, j)\). In words, \( N \) is the matrix where the entry \( \{a, b, c, d\} \) is proportional to the indicator of whether \( \{a, c, d\} \) is a clique, but to the indicator of whether \( G \) has as a subgraph the bipartite clique with bipartitions \( \{a, b\} \) and \( \{c, d\} \). It is easy to see that the matrix \( N \) is obtained by dropping from \( N \) the rows and columns corresponding \( \{a, b\} \in \binom{[n]}{2} \) where \((a, b) \notin E(G) \). Hence \( N \succeq 0 \Rightarrow M \succeq 0 \).

Notice that \( N \) is a random matrix whose entries depend on the edges in the random graph \( G \). At the risk of over-simplification, the approach of both the previous works [MPW15] and [DM15a] can be broadly summarized as follows:

1. (Expectation) Show that the expected matrix \( \mathbb{E}[N] \) has sufficiently large positive eigenvalues.

2. (Concentration) Show that with high probability over the choice of \( G \), the noise matrix \( N - \mathbb{E}[N] \) has bounded eigenvalues, so as to ensure that \( N = \mathbb{E}[N] + (N - \mathbb{E}[N]) \succeq 0 \)

Here we will sketch a few key details of the argument in [DM15a]. The matrix \( N \in \mathbb{R}^{\binom{n}{2} \times \binom{n}{2}} \) can be decomposed into blocks \( \{N_{ab}\}_{a,b \in \{0,1,2\}} \) where \( N_{ab} \in \mathbb{R}^{\binom{n}{2} \times \binom{n}{2}} \). Deshpande and Montanari use the Schur complements to reduce the problem of proving that \( N \succeq 0 \) to facts about the blocks \( \{N_{ab}\}_{a,b \in \{0,1,2\}} \). Specifically, they show the following lemma:

**Lemma 2.2.** Let \( A \in \mathbb{R}^{\binom{n}{2} \times \binom{n}{2}} \) be the matrix defined so that \( A_{A,B} = \alpha_{|A|}\alpha_{|B|} \). For \( a, b \in \{0,1,2\} \), let \( H_{a,b} \) be the submatrix of \( N(G, \alpha) - A \) corresponding to monomials \( X_S \) with \(|S| = a + b \). Then \( N(G, \alpha) \) is PSD if and only if

\[
H_{11} \succeq 0, \quad H_{22} - H_{12}^{-1} - H_{11}^{-1} H_{12} \succeq 0
\]

The most significant challenge is to argue that (2.3) holds with high probability. In fact, the inequality only holds for the Deshpande-Montanari SDP solution with high probability for parameters \( \alpha \) for which the objective value is \( o(n^{1/3}) \).

**Expected matrix.** The expected matrix \( \mathbb{E}[H_{22}] \) is symmetric with respect to permutations of the vertices. It forms an association scheme (see [MPW15, DM15a]), by virtue of which its eigenvalues and eigenspaces are well understood. In particular, the following proposition in [DM15a] is an immediate consequence of the theory of association schemes.
Proposition 2.3 (Proposition 4.16 in [DM15a]). $\mathbb{E}[H_{22}]$ has three eigenspaces, $V_0, V_1, V_2$ such that

$$
\mathbb{E}[H_{22}] = \lambda_0 \Pi_0 + \lambda_1 \Pi_1 + \lambda_2 \Pi_2,
$$

where $\Pi_0, \Pi_1, \Pi_2$ are the projections to the spaces $V_0, V_1, V_2$ respectively. The eigenvalues are given by,

$$
\lambda_0(\alpha) \overset{\text{def}}{=} \alpha_2 + (n - 2)\alpha_3 + \frac{(n - 2)(n - 3)}{32} \cdot \alpha_4 - \frac{n(n - 1)}{2} \alpha_2^2 \quad (2.4)
$$

$$
\lambda_1(\alpha) \overset{\text{def}}{=} \alpha_2 + \frac{(n - 4)}{2} \alpha_3 - \frac{(n - 3)}{16} \alpha_4 \quad (2.5)
$$

$$
\lambda_2(\alpha) \overset{\text{def}}{=} \alpha_2 - \alpha_3 + \frac{\alpha_4}{16} \quad (2.6)
$$

Further the eigenspaces are given by,

$$
V_0 = \text{span}\{1\},
$$

$$
V_1 = \text{span}\{u \mid \langle u, 1 \rangle = 0, \ u_{i,j} = x_i + x_j \text{ for } x \in \mathbb{R}^n\},
$$

and

$$
V_2 = \mathbb{R}^{\binom{n}{\leq 2}} \setminus (V_0 \cup V_1),
$$

where we have used $\mathbb{R}^{\binom{n}{\leq 2}}$ to denote the space of vectors of real numbers indexed by subsets of $n$ of size at most 2.

Deviation from Expectation. Given the lower bound on eigenvalues of the expected matrix $\mathbb{E}[H_{22}]$, the next step would be to bound the spectral norm of the noise $H_{22} - \mathbb{E}[H_{22}]$. However, since the eigenspaces of $\mathbb{E}[H_{22}]$ are stratified (for the given $\alpha$), with one large eigenvalue and several much smaller eigenvalues, standard matrix concentration does not suffice to give tight bounds. To overcome this, Deshpande and Montanari split $H_{22}$ and $H^T_{12}H_{11}^{-1}H_{12}$ along the eigenspaces of $\mathbb{E}[H_{22}]$.

More precisely, let us split $H_{22} - \mathbb{E}[H_{22}]$ as

$$
H_{22} - \mathbb{E}[H_{22}] = Q + K
$$

where $Q$ includes all multilinear entries, and $K$ includes all non-multilinear entries, i.e., entries $K(A, B)$ where $A \cap B \neq \emptyset$. Formally,

$$
Q(A, B) = \begin{cases} 
H_{22}(A, B) - \mathbb{E}[H_{22}](A, B) & \text{if } A \cap B = \emptyset \\
0 & \text{otherwise}
\end{cases}
$$

The spectral norm of the matrix $Q$ over the eigenspaces $V_0, V_1, V_2$ is carefully bounded in [DM15a].

Lemma 2.4. (Proposition 4.20, 4.25 in [DM15a]) With probability at least $1 - O(n^{-4})$, all of the following bounds hold:

$$
\|\Pi_a Q \Pi_b\| \lesssim \alpha_4 \mathbb{E}^{3/2} \quad \forall (a, b) \in \{0, 1, 2\}^2 \quad (2.7)
$$

$$
\|\Pi_2 Q \Pi_2\| \lesssim \alpha_4 \mathbb{E} \quad (2.8)
$$

$$
\|K\| \lesssim \alpha_3 \mathbb{E}^{1/2} \quad (2.9)
$$
Proposition 2.3 and Lemma 2.4 are sufficient to conclude that $H_{22} \succeq 0$ for parameter choices of $\alpha$ that correspond to planted clique of size up to $\omega(n^{1/3})$. More precisely, to argue that with high probability $H_{22} \succeq 0$, it is sufficient to argue that, $\mathbb{E}[H_{22}] \succeq \mathbb{E}[H_{22}] - H_{22}$, i.e.,

\[
\begin{bmatrix}
\lambda_0 & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \lambda_2
\end{bmatrix} \succeq \begin{bmatrix}
\|\Pi_0 Q \Pi_0\| & \|\Pi_0 Q \Pi_1\| & \|\Pi_0 Q \Pi_2\| \\
\|\Pi_1 Q \Pi_0\| & \|\Pi_1 Q \Pi_1\| & \|\Pi_1 Q \Pi_2\| \\
\|\Pi_2 Q \Pi_0\| & \|\Pi_2 Q \Pi_1\| & \|\Pi_2 Q \Pi_2\|
\end{bmatrix} + \alpha_3 \pi^{1/2} \cdot \text{Id}. \quad (2.10)
\]

Deshpande and Montanari fix $\alpha_1 = \kappa$, $\alpha_2 = 4\kappa^2$, $\alpha_3 = 8\kappa^3$ and $\alpha_4 = 512\kappa^4$ for a parameter $\kappa$. Using Proposition 2.3 and Lemma 2.4, the above matrix inequality becomes,

\[
\begin{bmatrix}
2\kappa^4 & 0 & 0 \\
0 & 2\kappa^3 & 0 \\
0 & 0 & \kappa^2
\end{bmatrix} \succeq \begin{bmatrix}
\kappa^4 & 0 & 0 \\
0 & \kappa^4 & 0 \\
0 & 0 & \kappa^2
\end{bmatrix},
\]

which can be shown to hold for $\kappa \ll n^{-2/3}$. Eventually, it is necessary to show (2.3), which is stronger than $H_{22} \succeq 0$. This is again achieved by showing bounds on the spectra of $H_{11}^{-1}$ and $H_{12}$. We refer the reader to [DM15a] for more details of the arguments.

2.3 Problematic Subspace

The SDP solution described above ceases to be PSD at $\kappa \simeq n^{-2/3}$ which corresponds to an objective value of $O(n^{1/3})$. The specific obstruction to $H_{22} \succeq 0$ arises out of (2.10). More precisely, the bottom $2 \times 2$ principal minor which yields the constraint,

\[
\begin{bmatrix}
\lambda_1 & \Pi_1 Q \Pi_2 \\
\Pi_2 Q \Pi_1 & \lambda_2
\end{bmatrix} \simeq \begin{bmatrix}
n\kappa^3 & -2n^{3/2}\kappa^4 \\
-2n^{3/2}\kappa^4 & \kappa^2
\end{bmatrix} \succeq 0
\]

forcing $\kappa \ll n^{-2/3}$. It is clear that the problematic vectors $x \in \mathbb{R}^{\binom{n}{2}}$ for which $x^T H_{22} x < 0$ are precisely those for which $x^T \Pi_2 Q \Pi_1 x < 0$ and $|x^T \Pi_2 Q \Pi_2 x|$ is large, i.e., $\Pi_2 x$ aligns with the subspace $Q(V_1 \oplus V_0)$.

In fact, we identify a specific subspace $W$ that is problematic for the [DM15a] solution. To describe the subspace, let us fix some notation. Define the random variable $A_{ij}$ to be $-1$ if $(i, j) \notin E$, and $+1$ otherwise. We follow the convention that $A_{ii} = 0$.

Lemma 2.5. Let the vectors $a_1, \ldots, a_n \in \mathbb{R}^{\binom{n}{2}}$ be defined so that $a_i(k, \ell) \overset{\text{def}}{=} A_{ik} A_{i\ell}$, and let $W \overset{\text{def}}{=} \text{span}\{a_1, \ldots, a_n\}$. Then with probability at least $1 - O(n^{-4})$, \[
\|\Pi_2 Q - \Pi_2 \Pi W Q\| \lesssim \alpha_4 \pi
\]

Proof. This is an immediate observation from the various matrix norm bounds in [DM15a] (specifically Lemma A.2, Lemma A.3 and Observation A.5). We defer the detailed proof to Appendix A.1.  

Since $\|\Pi_2 Q \Pi_1\| \gg \alpha_4 \pi$, the above lemma implies that all the vectors with large singular values for $Q$ are within the subspace $W$. Furthermore, we will show the following lemma which clearly articulates that $W$ is the sole obstruction to $H_{22} \succeq 0$.

\footnote{Here, we have identified the matrices $\mathbb{E}[H_{22}]$ and $\mathbb{E}[H_{22}] - H_{22}$, which are matrices in $\mathbb{R}^{\binom{n}{2} \times \binom{n}{2}}$, with the $3 \times 3$ matrices corresponding to diagonalizing $\mathbb{E}[H_{22}]$ according to the three eigenspaces $V_0, V_1, V_2$ of the expectation $\mathbb{E}[H_{22}]$. This is analogous to decomposing any quadratic form $v^T H_{22} v$ into $v^T (\Pi_0 + \Pi_1 + \Pi_2) H_{22} (\Pi_0 + \Pi_1 + \Pi_2) v$.}
Lemma 2.6. Suppose $\alpha \in \mathbb{R}_+^4$ satisfies
\[
\begin{align*}
\min(\lambda_0(\alpha), \lambda_1(\alpha), \lambda_2(\alpha)) &\gg \alpha_3 m^{1/2}, \\
\lambda_0(\alpha) &> \lambda_1(\alpha) \gg \alpha_4 m^{3/2}, \\
\lambda_2(\alpha) &\gg \alpha_4 m
\end{align*}
\] (2.11)
(2.12)
(2.13)
then with probability $1 - O(n^{-d})$,
\[
H_{22} \geq \frac{1}{4} \cdot E[H_{22}] - \frac{16||Q||^2}{\lambda_1} \cdot \Pi_2 \Pi_W \Pi_2.
\]

Proof. Fix $\theta = \frac{16||Q||^2}{\lambda_1}$. Recall that $H_{22} - E[H_{22}] = Q + K$. We can write the matrix
\[
H_{22} + \theta \cdot \Pi_2 \Pi_W \Pi_2 = B_{W^\perp} + B_W + B_K + \frac{1}{4} E[H_{22}],
\]
where
\[
B_{W^\perp} = \frac{1}{4} E[H_{22}] + \left[ \begin{array}{ccc} 
\Pi_0 \Pi_0 & \Pi_0 \Pi_1 & \Pi_0 Q(\Pi - \Pi_W) \Pi_2 \\
\Pi_1 \Pi_0 & \Pi_1 \Pi_1 & \Pi_1 Q(\Pi - \Pi_W) \Pi_2 \\
\Pi_2 (\Pi - \Pi_W) \Pi_0 & \Pi_2 (\Pi - \Pi_W) \Pi_1 & \Pi_2 Q \Pi_2 
\end{array} \right]
\]
and
\[
B_W = \frac{1}{4} E[H_{22}] + \left[ \begin{array}{ccc} 
0 & 0 & \Pi_0 Q \Pi_W \Pi_2 \\
0 & 0 & \Pi_1 Q \Pi_W \Pi_2 \\
\Pi_2 \Pi_W \Pi_0 & \Pi_2 \Pi_W \Pi_1 & \theta \cdot \Pi_2 \Pi_W \Pi_2 
\end{array} \right]
\]
and $B_K = K + \frac{1}{4} E[H_{22}]$.

It is sufficient to show that $B_{W^\perp}, B_W$ and $B_K \succeq 0$. Using Proposition 2.3 and (2.9), $B_K \succeq (\lambda_0 - \alpha_3 m^{1/2}) \Pi_0 + (\lambda_1 - \alpha_3 m^{1/2}) \Pi_1 + (\lambda_2 - \alpha_3 m^{1/2}) \Pi_2 \succeq 0$ when condition (2.11) holds. Using Proposition 2.3, Lemma 2.4 and Lemma 2.5 we can write,
\[
B_{W^\perp} \succeq \frac{1}{4} \left[ \begin{array}{ccc} 
\lambda_0 & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \lambda_2 
\end{array} \right] - \alpha_4 \cdot \left[ \begin{array}{ccc} 
\pi^{3/2} & \pi^{3/2} & \pi \\
\pi^{3/2} & \pi^{3/2} & \pi \\
\pi & \pi & \pi 
\end{array} \right]
\]
which is PSD given the bounds on $\lambda_1, \lambda_2, \lambda_3$ in conditions (2.12) and (2.13). To see this, one shows that all the $2 \times 2$ principal minors are PSD.

On the other hand, for any $x \in \mathbb{R}^{(2)}$, we can write
\[
x^T B_W x \geq \lambda_0 ||\Pi_0 x||^2 + \frac{\theta}{2} ||\Pi_W \Pi_2 x||^2 - 2 ||Q|| ||\Pi_W \Pi_2 x|| ||\Pi_0 x|| + \lambda_1 ||\Pi_1 x||^2 + \frac{\theta}{2} ||\Pi_W \Pi_2 x||^2 - 2 ||Q|| ||\Pi_W \Pi_2 x|| ||\Pi_1 x||
\]
Now we will appeal to the fact that a quadratic $r(p, q) = ap^2 + 2bpq + cq^2 \geq 0$ for all $p, q \in \mathbb{R}$ if $b^2 < 4ac$ and $a > 0$. Since $\theta \lambda_1, \theta \lambda_0 \geq 16||Q||^2$ by condition (2.12), it is easily seen that the above quadratic form is always non-negative, implying that $B_W \succeq 0$. □

\footnote{Here again we diagonalize according to the subspaces $V_0, V_1, V_2$, as in (2.10)}
An immediate corollary of the proof of the above lemma is the following.

**Corollary 2.7.** Under the hypothesis of Lemma 2.6, with probability $1 - O(n^{-4})$,

$$H_{22} - K \succeq \frac{1}{2} \cdot \mathbb{E}[H_{22}] - \frac{16\|Q\|^2}{\lambda_1} \cdot \Pi_2 \Pi W \Pi_2.$$

The above corollary is a consequence of the fact that $H_{22} - K = B_W + B_{W \perp} + \frac{1}{2} \mathbb{E}[H_{22}]$.

### 2.4 The Corrected Witness

Suppose we have an unconstrained matrix $M$ that we wish to modify as little as possible so as to ensure $M \succeq 0$. Given a test vector $w$ so that $w^T M w < 0$, the natural update to make is to take $M' = M + \beta \cdot w w^T$ for a suitably chosen $\beta$. This would suggest creating a new SDP solution by setting $H_{22}' = H_{22} + \beta \sum_{i \in [n]} a_i a_i^T$.

Unfortunately, the SOS SDP relaxation has certain hard constraints, namely that the non-clique entries are fixed at zero. Moreover, the entry $X_{S_1, S_2}$ must depend only on $S_1 \cup S_2$. Setting the SDP solution matrix to $H_{22} + \beta \sum_{i \in [n]} a_i a_i^T$ would almost certainly violate both these constraints. It is thus natural to consider multiplicative updates to the entries of the matrix which clearly preserve the zero entries of the matrix.

Specifically, the idea would be to consider an update of the form $M' = M + \beta D_w M D_w$ where $D_w$ is the diagonal matrix with entries given by the vector $w$. If the matrix $M$ has a significantly large eigenvalue along $1$, i.e., $M \succeq \lambda_0 \cdot \mathbb{1} \mathbb{1}^T + \Delta$, for some matrix $\Delta$ with $\|\Delta\| \ll \lambda_1$, then this multiplicative update has a similar effect as an additive update,

$$M' \succeq M + \beta \cdot \lambda_0 \cdot w w^T + \beta D_w \Delta D_w,$$

where the norm of the final “error” term $\beta D_w \Delta D_w$ is relatively small. Recall that, in our setting, the Deshpande Montanari SDP solution matrix $N$ does have a large eigenvalue along $\mathbb{1}$. We now formally describe our SDP solution, first as a matrix according to the intuition given above, and then as a set of pseudomoments.

**Definition 2.8** (Corrected SDP Witness, matrix view). Let $\hat{a}_1, \ldots, \hat{a}_n \in \mathbb{R}^{(\leq 2)}$ be defined so that

$$\hat{a}_i(A) = \begin{cases} 0 & |A| < 2 \\ a_i(A) & |A| = 2. \end{cases}$$

Define $\hat{D}_i \in \mathbb{R}^{(\leq 2)}$ to be the diagonal matrix with $\hat{a}_i$ on the diagonal. Define $\hat{K}$ to be the restriction of $N(G, \alpha)$ to the non-multilinear entries. Also let

$$N'(G, \alpha) = N(G, \alpha) + \beta \cdot \sum_{i \in [n]} \hat{D}_i \left( N(G, \alpha) - \hat{K} \right) \hat{D}_i,$$

where $\beta = \frac{1}{100 \sqrt{n \log n}}$. Then our **SDP witness** is the matrix $M'$, defined so that

$$M'(G, \alpha) = \mathcal{P} \left( N'(G, \alpha) \right),$$

where $\mathcal{P}$ is the projection that zeros out rows and columns corresponding to pairs $(i, j) \notin E$. 


Definition 2.9 (Corrected SDP Witness, pseudomoments view). Let $\beta = \frac{1}{100\sqrt{n \log n}}$, and let $\alpha \in \mathbb{R}^4_+$ be a set of parameters, to be fixed later. For a subset $S \subseteq [n]$, let $G[S]$ be the graph induced on $G$ by $S$. For any subset of at most 4 vertices $S \subseteq [n]$, $|S| \leq 4$, we define

$$
\tilde{E}[X_S](G, \alpha) = \begin{cases} 
\frac{c_4(\alpha)}{\binom{n}{4}} + \beta \sum_{v \in [n]} (-1)^{\# \{\text{edges from } v \text{ to } S \text{ in } G\}} & |S| = 4 \text{ and } G[S] = \text{clique}, \\
\frac{c_{|S|}(\alpha)}{\binom{n}{|S|}} & |S| \leq 3 \text{ and } G[S] = \text{clique}, \\
0 & \text{otherwise,}
\end{cases}
$$

where $c_{|S|}(\alpha)$ is some factor chosen for each $|S| \in \{0, \ldots, 4\}$ depending on the choice of $\alpha$, which we will set later to ensure that the final moments matrix is PSD.

Proposition 2.10. For $\beta = \frac{1}{100\sqrt{n \log n}}$, and $\alpha_4 \leq \frac{1}{2}$, with probability at least $1 - O(n^{-5})$, the solution $N'(G, \alpha)$ does not violate any of the linear constraints of the planted clique SDP.

Proof. First, $M'(S_1, S_2) = M(S_1, S_2)$ whenever $|S_1 \cup S_2| < 4$ so these entries satisfy the constraints of the SDP. If $|S_1 \cup S_2| = 4$ then $M'(S_1, S_2)$ is given by,

$$
M'(S_1, S_2) = \alpha_4 \cdot \mathbb{I}[S_1 \cup S_2 \text{ is a clique}] \cdot \left( 1 + \beta \sum_{i \in [n]} \prod_{j \in S_1 \cup S_2} A_{ij} \right).
$$

Notice that $M'(S_1, S_2)$ is non-zero only if $S_1 \cup S_2$ is a clique, and it depends only on $S_1 \cup S_2$. Moreover, $\sum_{i \in [n]} \prod_{j \in S_1 \cup S_2} A_{ij}$ is a sum over iid mean 0 random variables and therefore satisfies,

$$
\mathbb{P} \left( \left| \sum_{i \in [n]} \prod_{j \in S_1 \cup S_2} A_{ij} \right| \leq 100\sqrt{n \log n} \right) \geq 1 - O(n^{-10}).
$$

A simple union bound over all subsets $S_1 \cup S_2$ shows that $M'(S_1, S_2) \in [0, 1]$ for all of them with probability at least $1 - O(n^{-5})$. \qed

It now remains to verify that $N'(G, \alpha) \succeq 0$. We will do this by verifying the Schur complement conditions, as in [DM15a]. Analogous to the submatrix $H_{22}$, one can consider the corresponding submatrix $H'_{22}$ of $N'$. The expression for $H'$ is as follows:

$$
H'_{22} \overset{\text{def}}{=} H_{22} + \sum_{i \in [n]} D_i (H_{22} + \frac{1}{10} \alpha_2 \binom{n}{2} I_{\binom{n}{2}} - K) D_i,
$$

Here $D_i$ is the matrix with $a_i$ on the diagonal, and $K$ is the matrix corresponding to the non-multilinear entries (entries corresponding to monomials like $x_a^2 x_b x_c$), and $I_{\binom{n}{2}}$ is the all-1s matrix. The matrices $H_{12}$ and $H_{11}$ are unchanged, and so we must simply verify that $H'_{22} \succeq H_{12}^\top H_{11}^{-1} H_{12}$ and that $H'_{22} \succeq 0$.

This concludes our proof overview. In Section 3, we verify the Schur complement conditions and prove our main result, and in Section 4 we give the random matrix concentration results upon which we rely throughout the proof.

3 Proof of the Main Result

In this section, we will demonstrate that $H'_{22} \succeq 0$, and that $H'_{22} \succeq H_{12}^\top H_{11}^{-1} H_{12}$. This will allow us to conclude that our solution matrix is PSD, and therefore is a feasible point for the degree-4 SOS relaxation.
Proposition 2.3
Lemma 4.5
Lemma 2.2
Lemma 4.3
by
Corollary 2.7
When convenient, we will also use the shorthand $c$ to the
Theorem 3.1.
Fix $\theta$.
Proof. We define $\lambda_0, \lambda_1, \lambda_2$ from Proposition 2.3 and bounded by,
$$
\lambda_0 \geq \frac{\alpha_4 \alpha_2^2}{64} = \frac{\gamma^2 \rho^2}{64} \\
\lambda_1 \geq \frac{\alpha_3 \alpha_2}{4} = \frac{\gamma^3 \rho^3}{4n^{1/2}} \\
\lambda_2 \geq \frac{\alpha_3}{2} = \frac{\gamma \rho^2}{2n}
$$
When convenient, we will also use the shorthand $c_1 \equiv n^{1/2} \lambda_1, c_2 \equiv n \lambda_2, c_3 \equiv n^{3/2} \lambda_3$, and $c_4 \equiv n^2 \lambda_4$.

3.1 Proving that $H'_{22} \succeq 0$
Here we will make a first step towards verifying the Schur complement conditions of Lemma 2.2 by showing that $H'_{22} \succeq 0$. Specifically, we will show the following stronger claim.

Theorem 3.1. For $\beta = \frac{1}{100n \log n}$ and $\gamma = \log^4 n, \rho < \log^{-20} n$, the following holds with probability at least $1 - O(n^{-4})$,
$$
H'_{22} \succeq \frac{1}{8} \mathbb{E}[H_{22}] + \frac{\beta \lambda_0}{16} \cdot \Pi W
$$
Proof. Fix $\theta = \frac{\|Q\|^2}{n \lambda_1}$. By definition of $H'_{22}$, we have
$$
H'_{22} = H_{22} + \beta \cdot \sum_{i \in [n]} D_i(H_{22} + \frac{1}{16} \alpha_2^2 J - K) D_i.
$$
Define $P_W = \sum_{i \in [n]} a_i a_i^T$. We can apply Lemma 2.6 to the $H_{22}$ term and Corollary 2.7 for $H_{22} - K$,
$$
H'_{22} \succeq \frac{1}{4} \mathbb{E}[H_{22}] - \theta \cdot \Pi_2 \Pi W \Pi_2 + \beta \cdot \sum_{i \in [n]} D_i \left( \frac{1}{2} \mathbb{E}[H_{22}] - \theta \Pi_2 \Pi W \Pi_2 + \frac{1}{16} \alpha_2^2 J \right) D_i.
$$
Now we will appeal to a few matrix concentration bounds that we show in Section 4. First, with probability $1 - O(n^{-5})$, the vectors $\{a_i^{\otimes 2}\}$ are nearly orthogonal, and therefore form a well-conditioned basis for the subspace $W$.
$$
P_W \succeq \frac{n^2}{2} \cdot \Pi W. \quad \text{(see Lemma 4.3)}
$$
Also, the vectors $\{a_i^{\otimes 2}\}$ have negligible projection on to the eigenspaces $V_0, V_1$ which implies that with overwhelming probability,
$$
\|\Pi_2 \Pi W \Pi_2 - \Pi W\| \leq \frac{\log^2 n}{n} \cdot \text{Id}, \quad \text{(see Lemma 4.5)}
$$
Finally, $W$ is an $n$ dimensional space. Each $D_{i} \Pi_{2} \Pi_{1} W \Pi_{2} D_{i}$ has only $n$ non-zero singular values each of which is $O(1)$. Moreover, multiplying on the left and right by $D_{i}$ acts as a random linear transformation/ random change of basis. Intuitively, this suggests that $\sum_{i} D_{i} \Pi_{2} \Pi_{1} W \Pi_{2} D_{i}$ has $n^2$ eigenvalues all of which are roughly $O(1)$. In fact, with probability $1 - O(n^{-5})$,

$$\sum_{i} D_{i} \Pi_{2} \Pi_{1} W \Pi_{2} D_{i} \preceq O(n) \cdot \Pi_{0} + \log^{2} n \cdot \operatorname{Id} \quad \text{(see Lemma 4.6)}$$

Substituting these bounds we get,

$$H' \succeq \frac{1}{4} \mathbb{E}[H_{22}] + \left( \frac{\beta \lambda_{0}}{8} - \theta \right) \cdot \Pi_{W} - \left( \frac{\theta \log^{2} n}{n} + \beta \theta \log^{2} n \right) \cdot \operatorname{Id} - \beta \theta \cdot O(n) \cdot \Pi_{0}$$

By Lemma 2.4, with probability at least $1 - O(n^{-4})$, $\|Q\| \lesssim \frac{\alpha_{4} \sqrt{n}}{\lambda_{1}}$. Substituting this bound for $\theta = \frac{\|Q\|^{2}}{\lambda_{1}}$ along with (3.2), finishes the proof for our choice of parameters. The details are presented below for completeness.

$$\frac{\theta}{\beta \lambda_{0}} \lesssim \frac{\alpha_{4}^{2} n^{3}}{\alpha_{3} n} \cdot \frac{\sqrt{n} \log n}{\alpha_{4} n^{2}} = \log^{4} n \cdot \gamma^{3} \rho \ll 1,$$

$$\frac{\beta \theta n}{\lambda_{0}} \lesssim \frac{1}{\sqrt{n} \log n} \cdot \frac{\alpha_{4}^{2} n^{3}}{\alpha_{3} n} \cdot \frac{1}{\alpha_{4} n^{2}} \lesssim \log^{2} n \cdot \gamma^{3} \rho \ll 1$$

Clearly $\lambda_{0} > \lambda_{1} > \lambda_{2}$ and

$$\lambda_{2} \gg 100 \theta \left( \beta \log^{2} n + \frac{\log^{2} n}{n} \right),$$

because

$$\frac{\theta \beta \log^{2} n}{\lambda_{2}} \lesssim \frac{\alpha_{4}^{2} n^{3}}{\alpha_{3} n} \cdot \frac{1}{\sqrt{n} \log n} \cdot \frac{\log^{2} n}{\alpha_{2}} = \log^{4} n \cdot \gamma^{8} \rho^{3} \ll 1.$$

\[\square\]

### 3.2 Bounding singular values of $H_{12}$

Towards bounding the eigenvalues of $H_{22}^{\perp} H_{11}^{-1} H_{12}$, Deshpande and Montanari [DM15a] observe the following properties of $H_{21}$ with regards to the spaces $V_{0}, V_{1}$ and $V_{2}$.

**Lemma 3.2** (Consequence of Propositions 4.18 and 4.27 in [DM15a]). Let $Q_{n} \in \mathbb{R}^{n \times n}$ be the orthogonal projector to the space spanned by $1_{n}$. Let $p = \frac{1}{2}$. For the matrix $H_{21}$, we have that for sufficiently large $n$, with probability $1 - O(n^{-5})$,

$$\| \mathbb{E}[H_{21}] - H_{21} \| \leq \alpha_{3} \sqrt{n},$$

and

$$\| \Pi_{0} \mathbb{E}[H_{21}] Q_{n} \| \leq \frac{1}{4} n^{3/2} \alpha_{3} + 2 \alpha_{2} n^{1/2} \quad \Pi_{0} \mathbb{E}[H_{21}] Q_{n}^{\perp} = 0$$

$$\Pi_{1} \mathbb{E}[H_{21}] Q_{n} = 0 \quad \| \Pi_{1} \mathbb{E}[H_{21}] Q_{n}^{\perp} \| \leq \alpha_{2} n^{1/2}$$

$$\Pi_{2} \mathbb{E}[H_{21}] Q_{n} = 0 \quad \Pi_{2} \mathbb{E}[H_{21}] Q_{n}^{\perp} = 0.$$

Unfortunately, the bound of [DM15a] on $\| \mathbb{E}[H_{21}] - H_{21} \|$ is insufficient for our purposes, and we require a more fine-grained bound on the deviation from expectation. In fact, outside the problematic subspace $W$, we show that $H_{21} - \mathbb{E}[H_{21}]$ is much better behaved.
Proposition 3.3. Let $W = \text{span}_{i \in [n]}\{a_i\}$, and let $\Pi_W$ be the projector to that space. With probability at least $1 - O(n^{-4})$, the following holds for every $x \in \mathbb{R}^n$

$$\|x^T \Pi_2 (H_{21} - \mathbb{E}[H_{21}])\|^2 \lesssim \alpha_3^2 \frac{\|x\|^2}{n} \left( \|\Pi_W x\|^2 + \frac{\log^2 n}{n} \|x\|^2 \right)$$

Proof. From Lemma 3.2, we have that

$$\Pi_2 \mathbb{E}[H_{21}] = 0.$$ 
Thus we may work exclusively with the difference from the expectation; for convenience, we let $U = H_{21} - \mathbb{E}[H_{21}]$. Fix $A = \{a, b\}$ and $c$, so that $|\{a, b\}| = 2$. By inspection, the entry $(A, c)$ of $U$ is given by the polynomial

$$U_{A, c} = \frac{\alpha_3}{4}(A_{ac}A_{bc} + (A_{ac} + A_{bc})).$$
Thus, the columns of $U$ are in $W \cup V_1$. So we have that

$$\Pi_2 U = \Pi_2(\Pi_W + \Pi_{V_1 \setminus W} + \Pi_{W \cup V_1})U = \Pi_2 \Pi_W U + \Pi_2 \Pi_{V_1 \setminus W} U + \Pi_{W \cup V_1} U = \Pi_2 \Pi_W U,$$
where the latter two terms were eliminated because $V_1 \perp V_2$ and $W \cup V_1 \perp \text{span}\{\text{col}(U)\}$.

In Lemma 4.5, we bound $\|\Pi_0 \Pi_W \Pi_{01}\| \leq O\left(\frac{\log^2 n}{n}\right)$. So we have that

$$\|x^T \Pi_2 \Pi_W\|^2 \leq x^T (\text{Id} - \Pi_0) \Pi_W (\text{Id} - \Pi_0) x = \|\Pi_W x\|^2 - 2x^T \Pi_0 \Pi_W x + x^T \Pi_0 \Pi_W \Pi_0 x \leq \|\Pi_W x\|^2 + 2\|\Pi_W x\| \|\Pi_0 \Pi_W x\| + \|x\|^2 \|\Pi_0 \Pi_W \Pi_0\| \leq 2\|\Pi_W x\|^2 + 2\frac{\log^2 n}{n} \|x\|^2,$$
where we have applied the inequality $a^2 + b^2 \geq 2ab$. The conclusion follows by noting that $\|x^T \Pi_2 H_{21}\| = \|x^T \Pi_2 \Pi_W U\| \leq \|x^T \Pi_2 \Pi_W\| \cdot \|U\|$, and that by Lemma 3.2 $\|U\| \lesssim \alpha_3^2 n$.

3.3 Bounding singular values of $H_{12}^T H_{11}^{-1} H_{12}$

We will bound $H_{21}^T H_{11}^{-1} H_{12}$, as in our bounds on $H_{22}'$, by splitting the matrix up according to the eigenspaces $\Pi_0, \Pi_1, \Pi_2$.

Theorem 3.4. Let $c_1, \ldots, c_4$ be as defined in (3.1). For the choice of $\alpha$ in (3.1), we have that with probability $1 - O(n^{-5})$,

$$H_{12}^T H_{11}^{-1} H_{12} \preceq \frac{c_3}{c_2} \cdot \Pi_0 + \frac{c_3^2 + c_3 \log^2 n}{c_1 n^{1/2}} \cdot \Pi_1 + \frac{c_3 \log^4 n}{c_1 n^{3/2}} \cdot \Pi_2 + \frac{c_3^2 \log^2 n}{c_1 n^{1/2}} \cdot \Pi_W$$

Proof. For each $a, b \in \{0, 1\}$, define the matrix $U_{ab} = \Pi_a (\mathbb{E}[H_{21}] - H_{21}) \Pi_b$. We can verify that for our choice of $\alpha$ the conditions of Lemma A.7 are met, and so we conclude that with probability $1 - O(n^{-5})$,

$$H_{11}^{-1} \preceq \frac{1}{n(\alpha_2 - 2\alpha_1^2)} \cdot Q_n + \frac{1}{\alpha_1} \cdot Q_n^\perp$$

For $x \in \mathbb{R}^n$, let $y_U = (\mathbb{E}[H_{12}] - H_{12}) \Pi_2 x$, $z_U = (\mathbb{E}[H_{12}] - H_{12})(\Pi_0 + \Pi_1)x$, $x_A = \mathbb{E}[H_{12}] \Pi_0 x$, $x_B = \mathbb{E}[H_{12}] \Pi_1 x$ and $x_C = \mathbb{E}[H_{12}] \Pi_2 x$. In order to simplify our computations, we will use the following observation.

13
**Observation 3.5.** Given $A \in \mathbb{R}^{n \times m}$ with $A \succeq 0$ and vectors $x_1, \ldots, x_t \in \mathbb{R}^m$,

$$\left( \sum_{i \in [t]} x_i \right)^T A \left( \sum_{i \in [t]} x_i \right) \leq t \cdot \sum_{i \in [t]} x_i^T Ax_i$$

**Proof.** If we set $y_i = A^{1/2} x_i$ then the inequality reduces to,

$$\| \sum_{i \in [t]} y_i \|^2 \leq t \sum_{i \in [t]} \| y_i \|^2,$$

which is an immediate consequence of triangle inequality for $\| \cdot \|$ and Cauchy-Schwartz inequality. \hfill \square

Using **Observation 3.5,**

$$x^T H_{11}^{-1} H_{11} x_{12} \leq 5 \left( x_A^T H_{11}^{-1} x_A + x_B^T H_{11}^{-1} x_B + x_C^T H_{11}^{-1} x_C + y_U^T H_{11}^{-1} y_U + z_U^T H_{11}^{-1} z_U \right).$$

To simplify the calculations and make the dominant terms apparent, let us fix $\alpha_1 = c_1/\sqrt{n}$, $\alpha_2 = c_2/n$, $\alpha_3 = c_3/n^{3/2}$ and $\alpha_4 = c_4/n$ wherein each $c_i \in [1/\log^{200} n, 1]$. First, observe that $1/\alpha_1 \gg 1/(n\alpha_2 - 2\alpha_1^2)$ for this setting of parameters.

For the terms $x_A^T H_{11}^{-1} x_A$, $x_B^T H_{11}^{-1} x_B$ and $x_C^T H_{11}^{-1} x_C$ we can write,

$$x_A^T H_{11}^{-1} x_A \lesssim \| \Pi_0 x \|^2 \left( \frac{1}{n(\alpha_2 - 2\alpha_1^2)} \cdot \| \Pi_0 E[H_{21}] Q_n \| \right)^2 \lesssim \alpha_2 + \alpha_3 n^2 \| \Pi_0 E[H_{21}] Q_n \|^2$$

$$x_B^T H_{11}^{-1} x_B \lesssim \| \Pi_1 x \|^2 \left( \frac{1}{n(\alpha_2 - 2\alpha_1^2)} \cdot \| \Pi_1 E[H_{21}] Q_n \| \right)^2 \lesssim \frac{\alpha_2^2}{\alpha_1} \| \Pi_1 x \|^2$$

$$x_C^T H_{11}^{-1} x_C \lesssim \| \Pi_2 x \|^2 \left( \frac{1}{n(\alpha_2 - 2\alpha_1^2)} \cdot \| \Pi_2 E[H_{21}] Q_n \| \right)^2 \lesssim \frac{\alpha_2^2}{\alpha_1} \| \Pi_2 E[H_{21}] Q_n \|^2$$

$$z_U^T H_{11}^{-1} z_U \lesssim \| (\Pi_0 + \Pi_1) x \|^2 \left( \frac{1}{n(\alpha_2 - 2\alpha_1^2)} \cdot \| H_{21} - E[H_{21}] \| \right)^2 \lesssim \frac{\alpha_3^2}{\alpha_1} \| \Pi_0 E[H_{21}] Q_n \|^2$$

Finally, we have

$$y_U^T H_{11}^{-1} y_U \lesssim \| (H_{12} - E[H_{12}]) \Pi_2 x \|^2 \left( \frac{1}{n(\alpha_2 - 2\alpha_1^2)} \right) \lesssim \frac{\alpha_3^2}{\alpha_1} \| \Pi_2 E[H_{21}] Q_n \|^2$$

The conclusion follows by grouping the projections, and taking the dominating terms as $n$ grows. \hfill \square
3.4 Proof of $H'_{22} \succeq H_{12}^T H_{11}^{-1} H_{12}$

Theorem 3.6. For the choice of $\alpha$ given in (3.1), we have that $H'_{22} \succeq H_{12}^T H_{11}^{-1} H_{12}$ with probability $1 - O(n^{-4})$.

Proof. Recall that by Theorem 3.1 with probability at least $1 - O(n^{-4})$,

$$H'_{22} \succeq \frac{\lambda_0}{4} \cdot \Pi_0 + \frac{\lambda_1}{4} \cdot \Pi_1 + \frac{\lambda_2}{4} \cdot \Pi_2 + \frac{\beta \lambda_0}{16} \Pi_W.$$ 

By our choice of the parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, the conclusion of Theorem 3.4 implies that

$$H'_{22} - H_{12}^T H_{11}^{-1} H_{12} \succeq \left( \frac{1}{4} \lambda_0 - \frac{c_3^2}{c_2} \right) \Pi_0 + \left( \frac{1}{4} \lambda_1 - \frac{c_2^2 + c_3^2 \log^2 n}{c_1 n^{1/2}} \right) \Pi_1$$

$$+ \left( \frac{1}{4} \lambda_2 - \frac{c_3 \log^4 n}{c_1 n^{3/2}} \right) \Pi_2 + \left( \frac{1}{16} \beta \lambda_0 - \frac{c_3^2 \log^2 n}{c_1 n^{1/2}} \right) \Pi_W$$

$\succeq 0,$

as desired. The details of the calculation are spelled out below for the sake of completeness; we verify that the coefficient of each projector is non-negative.

For the space $\Pi_0$,

$$\frac{c_3^2}{c_2} \cdot \frac{1}{\lambda_0} = \frac{c_3^2}{c_2 c_4} = \gamma^{-1} \ll 1.$$ 

For $\Pi_1$,

$$\frac{c_2^2 + c_3^2 \log^2 n}{c_1 n^{1/2}} \cdot \frac{1}{\lambda_1} \lesssim \frac{c_2^2 + c_3^2 \log^2 n}{c_3 c_1} = \frac{1}{\gamma} + \gamma^3 \rho^2 \log^2 n \ll 1.$$ 

For $\Pi_2$,

$$\frac{c_3 \log^4 n}{c_1 n^{3/2}} \cdot \frac{1}{\lambda_2} \lesssim \frac{1}{n^{1/2}} \cdot \frac{c_3 \log^4 n}{c_1 c_2} \ll 1.$$ 

Finally for $\Pi_W$,

$$\frac{c_3 \log^2 n}{c_1 n^{1/2}} \cdot \frac{1}{\beta \lambda_0} = \frac{c_3 \log^3 n}{c_1 c_4} = \rho \log^3 n \ll 1.$$ 

This concludes the proof.

\[\square\]

3.5 Proof of Main Theorem

We finally have the components needed to prove Theorem 2.1.

Proof of Theorem 2.1. First, we recall that independent of our choice of $\alpha$, the SDP solution defined in Definition 2.8 does not violate any of the linear constraints of (2.1), as shown in Proposition 2.10. To meet the program constraints, it remains to show that for the choice of $\alpha$ given in (3.1), our solution is PSD.

The solution matrix $M'(G, \alpha)$ is a principal submatrix of $N'(G, \alpha)$, and so $N' \succeq 0$ implies $M' \succeq 0$. We prove that $N'$ satisfies the Schur complement conditions from Lemma 2.2 with high probability. Observing that $H'_{11} = H_{11}$ and $H'_{12} = H_{12}$, we apply Theorem 3.6, which states that
for our choice of $\alpha$, $H_{22}^2 \geq H_{12}^TH_{11}^{-1}H_{21}$. For the $H_{11}$ term, we apply the lower bound on the eigenvalues of $H_{11}$ given by [DM15a] (Lemma A.7), which state that so long as $\alpha_1 - \alpha_2 \gg \alpha_2 n^{-1/2}$ and $\alpha_2 - 2\alpha_1^2 \geq 0$, we have $H_{11} \succeq 0$ with probability $1 - O(n^{-5})$. For our choice of $\alpha_1, \alpha_2$, we have
\[
\frac{\alpha_2}{n^{1/2}} = \frac{\gamma \rho^2}{n^{3/2}} < \frac{\rho}{n^{1/2}} - \frac{\gamma \rho^2}{n} = \alpha_1 - \alpha_2,
\]
and
\[
\alpha_2 - 2\alpha_1^2 = \frac{\gamma \rho^2 - 2\rho^2}{n} \gg 0,
\]
and so we may conclude that $H_{11} \succeq 0$. Therefore by a union bound, the conditions of Lemma 2.2 are satisfied with probability $1 - O(n^{-4})$, and $N'(G, \alpha) \succeq 0$ and so our solution satisfies the PSDness constraint.

It remains only to prove that the objective value is $\tilde{\Omega}(\sqrt{n})$. The objective value is simply $\sum_{i \in [n]} \alpha_1 = n\alpha_1 = \rho n^{1/2}$, concluding our proof.

4 Concentration of Projected Matrices

In this section, we give bounds on the spectra of random matrices that are part of the correction term. Though we are able to recycle many of the spectral bounds of Deshpande and Montanari [DM15a], in our modification to their witness, we introduce new matrices which also require description and norm bounds.

We obtain our bounds by employing the trace power method. The trace power method uses the fact that if $X \in \mathbb{R}^{n \times n}$ is a symmetric matrix, then for even powers $k$, $\text{Tr}(X^k) = \sum_{i \in [n]} \lambda_i(X)^k \geq \lambda_{\text{max}}(X)^k$. By bounding $\mathbb{E}[\text{Tr}(X^k)]^{1/k}$ for a sufficiently large $k$, we essentially obtain bounds on the infinity norm of the vector of eigenvalues, i.e., a bound on the spectral norm of the matrix $X$. A formal statement follows, and the proof is given in Appendix A.1 for completeness.

Lemma 4.1. Suppose an $n \times n$ random matrix $M$ satisfies $\mathbb{E}[\text{Tr}(M^k)] \leq n^{\alpha k + \beta} \cdot (\gamma k)!$ for any even integer $k$, where $\alpha, \beta, \gamma$ are constants. Then
\[
\mathbb{P} \left( \|M\| \leq \eta^{-1/\log n} \cdot n^\alpha \cdot \log n \right) \geq 1 - \eta.
\]

Our concentration proofs consist of, for each matrix in question, obtaining a bound on $\mathbb{E}[\text{Tr}(X^k)]$. The expression $\mathbb{E}[\text{Tr}(X^k)]$ is a sum over products along closed paths of length $k$ in the entries of $X$. In our case, the entries of the random matrix $X$ are themselves low degree polynomials in random variables $\{A_{ij}\}_{i,j \in [n]}$ where $A_{ij}$ is the centered random variable that indicates whether the edge $(i, j)$ is part of the random graph $G$. Thus $\text{Tr}(X^k)$ can be written out as a polynomial in the random variables $\{A_{ij}\}_{i,j \in [n]}$. Since the random variables $\{A_{ij}\}_{i,j \in [n]}$ are centered (i.e., $\mathbb{E}[A_{ij}] = 0$), almost all of the terms in $\mathbb{E}[\text{Tr}(X^k)]$ vanish to zero. The nonzero terms are precisely those monomials in which every variable appears with even multiplicity.

For the purpose of moment calculations, we borrow much of our terminology from the work of Deshpande and Montanari [DM15a]. Every monomial in random variables $\{A_{ij}\}_{i,j \in [n]}$ corresponds to a labelled graph $(F = (V, E), \ell)$ that consists of a graph $F = (V, E)$ and a labelling $\ell : V \to [n]$ that maps its vertices to $[n]$. A labelling of $F$ contributes (is nonzero in expectation), if and only if every pair $\{i, j\}$ appears an even number of times as a label of an edge in $F$. The problem of bounding $\mathbb{E}[\text{Tr}(X^k)]$ reduces to counting the number of the number of contributing labeled graphs.

For example, for a given matrix $X$, we may have a bound on the number of “vertices” and “edges” in a term of $\mathbb{E}[\text{Tr}(X^k)]$ as a function of $k$. In that case, we may use the following proposition, which
allows us to bound the number of such graphs in which every variable \( A_{ij} \), corresponding to an edge between vertices \( i \) and \( j \), appears at least twice.

**Proposition 4.2.** Let \( F = (V, E) \) be a multigraph and let \( \ell : V \to [n] \) be a labelling such that each pair \((i, j)\) appears an even number of times as the label of an edge in \( E \). Then,

\[
|\{\ell(v) | v \in V \}| \leq \frac{|E|}{2} + (\# \text{ connected components of } F)
\]

**Proof.** From \( F \), we form a new graph \( F' \) by identifying all the nodes with the same label; thus, the number of nodes in \( F' \) is the number of labels in \( F \). We then collapse the parallel edges in \( F' \) to form the graph \( H \); since each labelled edge appears at least twice, the number of edges in \( H \) is at most half that in \( F \). The number of nodes in \( H \) (and thus labels in \( F \)) is at most the number of edges in \( H \) plus the number of connected components; this is tight when \( H \) is a forest. Thus the number of distinct labels in \( F \) is at most \(|E|/2 + c\), where \( c \) is the number of components in \( F \). \( \square \)

We apply this lemma, as well as simple inductive arguments, to bound the number of contributing terms in \( \mathbb{E}[X^k] \) for the matrices \( X \) in question, and this allows us to bound their norms. We give the concentration proofs the following subsection.

### 4.1 Proofs of Concentration

Let \( G \) be an instance of \( G(n, \frac{1}{2}) \). As in the preceding sections, define the vector \( A_i \in \mathbb{R}^n \) so that \( A_i(j) = 1 \) if \((i, j) \in E(G)\), \( A_i(j) = -1 \) if \((i, j) \notin E(G)\), and \( A_i(i) = 0 \). Again as in the preceding sections, define \( a_1, \ldots, a_n \in \mathbb{R}^2 \) by setting \( a_i \) to be the restriction of \( A_i^\otimes 2 \) to coordinates corresponding to unordered pairs, i.e., \( a_i\{c, d\} = A_{ic}A_{id} \) for all \( \{c, d\} \in \binom{[n]}{2} \). We will continue to use the notation \( W = \text{span}_{i \in [n]}(a_i) \), and the notation \( D_i = \text{diag}(a_i) \).

We begin with a lemma that shows that the \( a_i \) are close to an orthogonal basis for \( W \).

**Lemma 4.3.** If \( P_W = \sum_i a_i a_i^T \) then with probability at least \( 1 - O(n^{-5}) \),

\[
\frac{1}{n^2}(1 + o(1)) \cdot P_W \geq \Pi_W \geq (1 - o(1))\frac{1}{n^2} \cdot P_W
\]

**Proof.** By definition, the vectors \( a_1, \ldots, a_n \) form a basis for the subspace \( W \).

Let \( \mathcal{R} \) be the matrix whose \( ith \) row is \( a_i \). We will use matrix concentration to analyze the eigenvalues of \( \mathcal{R} \mathcal{R}^T \), which are identical to the nonzero eigenvalues of \( P_W = \mathcal{R}^T \mathcal{R} \).

The \((i, j)\)th entry of \( \mathcal{R} \mathcal{R}^T \) is \( \langle a_i, a_j \rangle = \frac{1}{2}(A_i^\otimes 2, A_j^\otimes 2) = \frac{1}{2}(A_i, A_j)^2 \). When \( i = j \), this is precisely \( \frac{1}{2}(n - 1)^2 \), and so \( 2 \mathcal{R} \mathcal{R}^T = (n - 1)^2 \cdot \text{Id}_n + B \), where \( B \) is a matrix that is 0 on the diagonal and equal to \( \langle A_i^\otimes 2, A_j^\otimes 2 \rangle \) in the \((i, j)\)th entry for \( i \neq j \).

Let \( M = B - \mathbb{E}[B] = B - (n - 2)(J_n - \text{Id}_n) \). We will use the trace power method to prove that \( \|M\| = O(n^{3/2}) \). The \((i, j)\)th entry of \( M \) is given by 0 for \( i = j \), and when \( i \neq j \)

\[
M(i, j) = \langle A_i, A_j \rangle^2 - (n - 2) = \left( \sum_{p, q} A_{ip}A_{iq}A_{jp}A_{jq} \right) - (n - 2) = \sum_{p \neq q} A_{ip}A_{iq}A_{jp}A_{jq}.
\]

The expression \( \text{Tr}(M^k) \) is a sum over monomial products over variables \( \{A_{ip}\}_{i, p \in [n]} \), where each monomial product corresponds to a labelling \( \mathcal{L} : F \to [n] \) of a graph \( F \). Each entry in \( M_{ij} \) corresponds to a sum over links, where each link is a cycle of length 4, with the vertices \( i, j \) on opposite ends of the cycle, and the necessarily distinct vertices \( p, q \) are on the other opposite ends of a cycle. We will refer to \( i, j \) as the center vertices and \( p, q \) as the peripheral vertices of the link.
Each edge \((u, v)\) of the link is weighted by \(A_{uv}\). Since \(A_{ii} = 0\) for all \(i \in [n]\), for every contributing labelling, it can never be the case that one of \(p, q = i\). Each monomial product in the summation \(\text{Tr}(M^k)\) corresponds to a labelling \((F, \mathcal{L})\) of the graph \(F\), where \(F\) is a cycle with \(k\) links. \(F\) has \(4k\) edges, and in total it has \(3k\) vertices.

The quantity \(\text{Tr}(M^k)\) is equal to the sum over all labellings of \(F\). Taking the expectation, terms in \(\mathbb{E}[\text{Tr}(M^k)]\) which contain a variable \(A_{uv}\) with multiplicity 1 have expectation 0. Thus, \(\mathbb{E}[\text{Tr}(M^k)]\) is equal to the number of labellings of \(F\) in which every edge appears an even number of times.

We prove that any such contributing labelling \(\mathcal{L} : F \to [n]\) has at most \(3k/2 + 1\) unique vertex labels. We proceed by induction on \(k\), the length of the cycle. In the base case, we have a cycle on two links; by inspection no such cycle can have more than 5 labels, and the base case holds.

Now, consider a cycle of length \(k\). If every label appears twice, then we are done, since there are \(3k\) vertices in \(F\). Thus there must be a vertex that appears only once.

There can be no peripheral vertex whose label does not repeat, since the two center vertices neighboring a single peripheral vertex cannot have the same label in a contributing term, as \(M(i, i) = 0\). Now, if there exists a center vertex \(i\) whose label does not repeat, it must be that there is a matching between its \(p, q\) neighbors so that every vertex is matched to a vertex of the same label; we identify these same-label vertices and remove \(i\) and two of its neighbors from the graph, leaving us with a cycle of length \(k - 1\), having removed at most one label from the graph. The induction hypothesis now applies, and we have a total of at most \(3(k - 1)/2 + 2 \leq 3k/2 + 1\) labels, as desired.

Thus, there are at most \(3k/2 + 1\) unique labels in any contributing term of \(\mathbb{E}[\text{Tr}(M^k)]\). We may thus conclude that \(\mathbb{E}[\text{Tr}(M^k)] \leq n^{3k/2 + 1} \cdot (3k/2 + 1)^3k\), and applying Lemma 4.1, we have that \(\|M\| \lesssim n^{3k/2}\) with probability at least \(1 - O(n^{-5})\).

Therefore, \(2\mathcal{R}\mathcal{R}^T = ((n - 1)^2 - n + 2)\text{Id}_n + (n - 2)J_n + M\), and we may conclude that all eigenvalues of \(\mathcal{R}\mathcal{R}^T\) are \((1 \pm o(1)) \cdot n^2\), which implies the same of \(P_W = \mathcal{R}^T\mathcal{R}\). Since the range of \(P_W\) and \(\Pi_W\) is the same, we finally have that with probability \(1 - o(1)\)
\[
(1 + o(1))/n^2 \cdot P_W \succeq \Pi_W \succeq (1 - o(1))/n^2 \cdot P_W,
\]
as desired. 

The following lemma allows us to approximate the projector to \(V_0 \cup V_1\) by a matrix that is easy to describe; we will use this matrix as an approximation to the projector in later proofs.

**Lemma 4.4.** Let \(\Pi_{01}\) be the projection to the vector space \(V_0 \cup V_1\). Let \(P_{01} \in \mathbb{R}^{(\binom{n}{2})^2}\) be a matrix defined as follows:
\[
P_{01}(ab, cd) = \begin{cases} 
\frac{2}{n-1} & |\{a, b, c, d\}| = 2 \\
\frac{1}{n-1} & |\{a, b, c, d\}| = 3 \\
0 & |\{a, b, c, d\}| = 4.
\end{cases}
\]

Then
\[
\Pi_{01} \succeq P_{01} \succeq \left(\frac{n-2}{n-1}\right) \cdot \Pi_{01},
\]
Proof. We will write down a basis for $V_0 \cup V_1$, take a summation over its outer products, and then argue that this summation approximates $\Pi_{01}$. The vectors $v_1, \ldots, v_n \in \mathbb{R}^{(2)}$ are a basis for $V_1 \cup V_0$:

$$v_i(a, b) = \begin{cases} \frac{1}{\sqrt{n-1}} & \{a, b\} = \{i, \cdot\} \\ 0 & \text{otherwise.} \end{cases}$$

For any two $v_i, v_j$, we have $\langle v_i, v_j \rangle = \frac{1}{n-1}$. Let $U \in \mathbb{R}^{n \times n}$ be the matrix whose $i$th column is given by $v_i$. Notice that the eigenvalues of $\sum_i v_i v_i^T = UU^T$ are equal to the eigenvalues of $U^T U$, and that $U^T U = \frac{1}{n-1} J_n + \frac{n-2}{n-1} I_n$. Therefore, as both matrices have the same column and row spaces,

$$\Pi_{01} \succeq \sum_i v_i v_i^T \succeq \frac{n-2}{n-1} \Pi_{01},$$

Now, let $P_{01} = \sum_i v_i v_i^T$; we can explicitly calculate the entries of $P_{01}$,

$$P_{01}(ab, cd) = \begin{cases} \frac{2}{n-1} & |\{a, b, c, d\}| = 2 \\ \frac{1}{n-1} & |\{a, b, c, d\}| = 3 \\ 0 & |\{a, b, c, d\}| = 4. \end{cases}$$

The conclusion follows. \qed

We will require the fact that $W$ lies mostly outside of $V_0 \cup V_1$, which we prove in the following lemma.

**Lemma 4.5.** With probability at least $1 - O(n^{-\omega(\log n)})$,

$$\|\Pi_{01} W \Pi_{01}\| \leq O\left(\frac{\log^2 n}{n}\right).$$

**Proof.** Call $M = \Pi_{01} W \Pi_{01}$. We will apply the trace power method to $M$. By Lemma 4.4 and Lemma 4.3, we may exchange $\Pi_W$ for $\frac{1+o(1)}{n^2} \sum_i a_i a_i^T$ and $\Pi_{01}$ for $P_{01}$. Letting $M^{(k)} = \left(\frac{1+o(1)}{n^2} P_{01} P_W\right)^k$, we have by the cyclic property of the trace that $\mathbb{E} [\text{Tr}(M^{(k)})] \leq \text{Tr}(M^{(k)})$.

We consider the expression for $\mathbb{E} [\text{Tr}(M^{(k)})]$. Let a *chain* consists of a set of quadruples $\{a_\ell, b_\ell, c_\ell, d_\ell\} \in [n]^4$ such that for each $\ell \in [k]$, we have $|\{a_\ell, b_\ell\} \cap \{c_\ell - 1, d_\ell - 1\}| \geq 1$ (where we identify $a_\ell$ with $a_{\ell \mod k}$). Let $C_k$ denote the set of all chains of size $k$. We have that,

$$\text{Tr}(M^{(k)}) \leq \text{Tr}(M^{(k)}) = \sum_{i_1, \ldots, i_k} \sum_{\{a_\ell, b_\ell, c_\ell, d_\ell\} \in [n]^4} \prod_{\ell=1}^k \left(1 + \frac{o(1)}{n^2}\right) \cdot r_\ell \cdot A_{i_\ell} A_{i_\ell+1} A_{b_\ell} A_{b_{\ell+1}} A_{c_\ell} A_{c_{\ell+1}},$$

where $r_\ell = \frac{1}{n-1}$ or $\frac{2}{n-1}$ depending on whether one or both of $a_\ell, b_\ell$ are common with the following link in the chain. The quantity $\text{Tr}(M^{(k)})$ consists of cycles of $k$ links, each link is a star on 4 outer vertices $a_\ell, b_\ell, c_\ell, d_\ell$ with center vertex $i_\ell$, and the non-central vertices of the link must have at least one vertex in common with the next link, so each link has 4 edges and the cycle is a connected graph. See the figure below for an illustration (dashed lines indicate vertex equality, and are not edges).

![Diagram of a connected graph with cycles of links](image_url)
Each term in the product has a factor of at most \( \frac{2(1+o(1))}{n^3} \), due to the scaling of the entries of \( P_0 \) and \( P_W \). Thus we have

\[
\mathbb{E}[\text{Tr}(M^k)] \leq \left( \frac{3}{n^3} \right)^k \sum_{i_1, \ldots, i_k} \sum_{\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} \in \mathcal{C}_n} \mathbb{E} \left[ \prod_{\ell=1}^k A_{i_\ell, a_{i_\ell}} A_{i_\ell, b_{i_\ell}} A_{i_\ell, c_{i_\ell}} A_{i_\ell, d_{i_\ell}} \right].
\]

The only contributing terms correspond to those for which every edge variable in the product has even multiplicity. Each contributing term is a connected graph and has 4k edges and at most 5k vertices where every labeled edge appears twice, so we may apply Proposition 4.2 to conclude that there are at most 2k + 1 labels in any such cycle. We thus have that

\[
\mathbb{E}[\text{Tr}(M^k)] \leq \left( \frac{3}{n^3} \right)^k \cdot n^{2k+1} \cdot (5k)!,
\]

and applying Lemma 4.1, we conclude that \( \|M\| \lesssim \frac{\log^2 n}{n} \) with probability \( 1 - O(n^{-\omega(\log n)}) \), as desired.

We combine the above lemmas to bound the norm of one final matrix that arises in the computations in Theorem 3.1.

**Lemma 4.6.** With probability \( 1 - O(n^{-5}) \),

\[
\sum_w D_w \Pi_2 \Pi_W \Pi_2 D_w \preceq O(n) \cdot \Pi_0 + O(\log^2 n) \cdot \text{Id}_n.
\]

**Proof.** We begin by replacing \( \Pi_2 \) with \( (1 - \Pi_0) \), as by Lemma 4.5, \( \Pi_0 \) can be replaced by \( P_0 \) which has a convenient form. For any vector \( x \in \mathbb{R}^{(n)} \),

\[
x^T \left( \sum_i D_i \Pi_2 \Pi_W \Pi_2 D_i \right) x = x^T \left( \sum_i D_i \Pi_W D_i \right) x - 2x^T \left( \sum_i D_i \Pi_0 \Pi_W D_i \right) x + x^T \left( \sum_i D_i \Pi_0 \Pi_W \Pi_0 \Pi_W D_i \right) x
\]

\[
\leq \sum_i \left( \|\Pi_W D_i x\|^2 + 2\|\Pi_W \Pi_0 \Pi_W D_i x\| \cdot \|\Pi_W D_i x\| + \|\Pi_W \Pi_0 \Pi_W D_i x\|^2 \right)
\]

\[
\leq 2x^T \left( \sum_i D_i \Pi_W D_i \right) x + 2 \left( \sum_i (D_i x)^T \Pi_0 \Pi_W \Pi_0 D_i x \right)
\]

\[
\leq 2x^T \left( \sum_i D_i \Pi_W D_i \right) x + 2n \|\Pi_0 \Pi_W \Pi_0 \| \cdot \|x\|^2,
\]

where to obtain the second line we have applied Cauchy-Schwarz, to obtain the third line we have used the fact that \( a^2 + b^2 \geq 2ab \), and to obtain the final line we have used the fact that \( \|D_i x\| = \|x\| \).

Now, the second term is \( O(\log^2 n) \cdot \|x\|^2 \) with overwhelming probability by Lemma 4.5. It remains to bound the first term. To this end, we apply Lemma 4.3 to replace \( \Pi_W \) with \( \frac{1+o(1)}{\sqrt{n}^2} \cdot P_W = \frac{1+o(1)}{\sqrt{n}^2} \cdot \sum_i a_i a_i^T \). Let \( M = \frac{1}{n^2} \cdot \sum_i D_i P_W D_i \). An entry of \( M \) has the form

\[
M(ab, cd) = \frac{1}{n^2} \left( n + \sum_{i \neq j} A_{ia} A_{ib} A_{ia} A_{id} A_{ja} A_{jb} A_{ja} A_{jb} \right).
\]
Thus we can see that $M = \frac{1}{n} J_{\frac{n}{2}} + \frac{1+o(1)}{n^2} BB^T$, where $J_{\frac{n}{2}}$ is the all-ones matrix in $\mathbb{R}^{\frac{n}{2} \times \frac{n}{2}}$ and $B$ is the matrix whose entries have the form

$$B(ab, ij) = A_{ia} A_{ib} A_{ja} A_{jb}.$$

The matrix $B$ is actually equal to the matrix $J_{\frac{n}{4}}$ from [DM15a], and by Lemma A.3 has $\|B\| \leq \tilde{O}(1)$ with probability $1 - O(n^{-5})$. We can thus conclude that with probability $1 - O(n^{-5})$, $\|M - \frac{1}{n} J_{\frac{n}{2}}\| \leq \frac{1+o(1)}{n^2} \|B\|^2 \leq \tilde{O}(1)$, and so $x^T M x \leq \frac{1+o(1)}{n^2} \langle x, 1_{\frac{n}{2}} \rangle^2 + x^T (M - n^{-1} J) x \leq O(n) \cdot \|\Pi_0 x\|^2 + \tilde{O}(1) \cdot \|x\|^2$, which gives the desired result. \hfill $\Box$

Acknowledgements

We thank Satish Rao for many helpful conversations.

We also greatly acknowledge the comments of anonymous reviewers in helping us improve the manuscript.

References

[AKS98] Noga Alon, Michael Krivelevich, and Benny Sudakov, Finding a large hidden clique in a random graph, Random Struct. Algorithms 13 (1998), no. 3-4, 457–466.

[Bar14] Boaz Barak, Sum of squares: upper bounds, lower bounds, and open questions (lecture notes, fall 2014), 2014.

[BBH+12] Boaz Barak, Fernando G. S. L. Brandão, Aram Wettroth Harrow, Jonathan A. Kelner, David Steurer, and Yuan Zhou, Hypercontractivity, sum-of-squares proofs, and their applications, Proceedings of the 44th Symposium on Theory of Computing Conference, STOC 2012, New York, NY, USA, May 19 - 22, 2012, 2012, pp. 307–326.

[BE76] Bella Bollobas and Paul Erdős, Cliques in random graphs, Mathematical Proceedings of the Cambridge Philosophical Society 80 (1976), 419–427.

[BV09] S. Charles Brubaker and Santosh Vempala, Random tensors and planted cliques, Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, 12th International Workshop, APPROX 2009, and 13th International Workshop, RANDOM 2009, Berkeley, CA, USA, August 21-23, 2009. Proceedings (Irit Dinur, Klaus Jansen, Joseph Naor, and José D. P. Rolim, eds.), Lecture Notes in Computer Science, vol. 5687, Springer, 2009, pp. 406–419.

[DM15a] Yash Deshpande and Andrea Montanari, Improved sum-of-squares lower bounds for hidden clique and hidden submatrix problems, Proceedings of The 28th Conference on Learning Theory, COLT 2015, Paris, France, July 3-6, 2015 (Peter Grünwald, Elad Hazan, and Satyen Kale, eds.), JMLR Proceedings, vol. 40, JMLR.org, 2015, pp. 523–562.

[DM15b] Yash Deshpande and Andrea Montanari, Improved sum-of-squares lower bounds for hidden clique and hidden submatrix problems, CoRR abs/1502.06590 (2015).
Vitaly Feldman, Elena Grigorescu, Lev Reyzin, Santosh Vempala, and Ying Xiao, *Statistical algorithms and a lower bound for planted clique*, Electronic Colloquium on Computational Complexity (ECCC) 19 (2012), 64.

Uriel Feige and Robert Krauthgamer, *Finding and certifying a large hidden clique in a semirandom graph*, Random Struct. Algorithms 16 (2000), no. 2, 195–208.

Uriel Feige and Robert Krauthgamer, *The probable value of the lovász-schrijver relaxations for maximum independent set*, SIAM Journal on Computing 32 (2003), 2003.

Alan M. Frieze and Ravi Kannan, *A new approach to the planted clique problem*, IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2008, December 9-11, 2008, Bangalore, India, 2008, pp. 187–198.

Geoffrey R. Grimmett and Colin J. H. McDiarmid, *On colouring random graphs*, Mathematical Proceedings of the Cambridge Philosophical Society 77 (1975), 313–324.

Johan Håstad, *Clique is hard to approximate within n^{1-epsilon}*, 37th Annual Symposium on Foundations of Computer Science, FOCS ’96, Burlington, Vermont, USA, 14-16 October, 1996, 1996, pp. 627–636.

Samuel B. Hopkins, Pravesh Kothari, and Aaron Potechin, *SoS and planted clique: Tight analysis of MPW moments at all degrees and an optimal lower bound at degree four*.

Aram Wettroth Harrow and Ashley Montanaro, *Testing product states, quantum merlin-arthur games and tensor optimization*, J. ACM 60 (2013), no. 1, 3.

Mark Jerrum, *Large cliques elude the metropolis process.*, Random Struct. Algorithms 3 (1992), no. 4, 347–360.

Richard Karp, *The probabilistic analysis of some combinatorial search algorithms*, Algorithms and Complexity: New Directions and Recent Results (1976), 1–19.

Subhash Khot, *Improved inaproximability results for maxclique, chromatic number and approximate graph coloring*, 42nd Annual Symposium on Foundations of Computer Science, FOCS 2001, 14-17 October 2001, Las Vegas, Nevada, USA, 2001, pp. 600–609.

Subhash Khot and Rishi Saket, *SDP integrality gaps with local ell_1-embeddability*, 50th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2009, October 25-27, 2009, Atlanta, Georgia, USA, 2009, pp. 565–574.

David Matula, *The largest clique size in a random graph*, Tech. report, Southern Methodist University, Dallas, 1976.

Raghu Meka, Aaron Potechin, and Avi Wigderson, *Sum-of-squares lower bounds for planted clique*, Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing, STOC 2015, Portland, OR, USA, June 14-17, 2015, 2015, pp. 87–96.

Prasad Raghavendra and David Steurer, *Integralty gaps for strong SDP relaxations of UNIQUE GAMES*, 50th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2009, October 25-27, 2009, Atlanta, Georgia, USA, 2009, pp. 575–585.
A Matrix Norm Bounds from Deshpande and Montanari

In this appendix, we give for completeness a list of the bounds proven by Deshpande and Montanari [DM15a] that were not included in the body above out of space or expository considerations.

Definition A.1. Let $A = \{a, b\} \subset [n]$ be disjoint from $B = \{c, d\} \subset [n]$. For $\eta \in \{1, \ldots, 4\}$ and for $\nu(\eta) \in \binom{\eta}{2}$ we define the matrices $J^{\eta}_{\nu(\eta)}$ as follows:

\[
\begin{align*}
J^{1}_{1,1}(A, B) &= A_{ac} \\
J^{1}_{1,2}(A, B) &= A_{ad} \\
J^{1}_{1,3}(A, B) &= A_{bc} \\
J^{1}_{1,4}(A, B) &= A_{bd}
\end{align*}
\]

\[
\begin{align*}
J^{2}_{1,1}(A, B) &= A_{ac}A_{bd} \\
J^{2}_{1,2}(A, B) &= A_{ac}A_{bc} \\
J^{2}_{1,3}(A, B) &= A_{ac}A_{ad} \\
J^{2}_{1,4}(A, B) &= A_{ac}A_{bd}
\end{align*}
\]

\[
\begin{align*}
J^{3}_{1,1}(A, B) &= A_{ac}A_{bd}A_{bc} \\
J^{3}_{1,2}(A, B) &= A_{ac}A_{bc}A_{bd} \\
J^{3}_{1,3}(A, B) &= A_{ac}A_{ad}A_{bd} \\
J^{3}_{1,4}(A, B) &= A_{ac}A_{ad}A_{bc}
\end{align*}
\]

\[
\begin{align*}
J^{4}_{1,1}(A, B) &= A_{ac}A_{bd}A_{bc}A_{bd} \\
J^{4}_{1,2}(A, B) &= A_{ac}A_{bc}A_{bd}A_{bd} \\
J^{4}_{1,3}(A, B) &= A_{ac}A_{ad}A_{bd}A_{bd} \\
J^{4}_{1,4}(A, B) &= A_{ac}A_{ad}A_{bc}A_{bd}
\end{align*}
\]

And further, letting $\mathcal{P} : \mathbb{R}^{\binom{\eta}{2} \times \binom{\eta}{2}}$ to be the matrix projector such that

\[
(M_{\mathcal{P}})_{A,B} = \begin{cases} 
M_{A,B} & |A \cup B| = 4 \\
0 & \text{otherwise}
\end{cases}
\]

we define $J^{\eta}_{\nu,\eta} = \mathcal{P}J^{\eta}_{\nu}$, and finally we define $J^{\eta}_{\nu} \overset{\text{def}}{=} \frac{\eta}{8\eta} \alpha_4 \cdot J^{\eta}_{\nu}$ (as in Deshpande and Montanari), so that $Q = \sum_{\eta=1}^{4} \sum_{\nu=1}^{\binom{\eta}{2}} J^{\eta}_{\nu}$.

Notice that since we have defined $A_{ii} = 0$ and since $|\{a, b\}| = |\{c, d\}| = 2$, we have $J^{1}_{4,1} = J^{2}_{4,1}$.

For some of the terms, the $J$ is never considered; however for some terms it is cleaner to bound the spectral norm of $J$ in the subspace $V_2$, and so Deshpande and Montanari provide trace power method bounds on the difference in norm:

Lemma A.2 (Lemma 4.26 in [DM15b]). With probability at least $1 - 6n^{-5}$, for each $\eta \leq 2$ and for each $\nu \leq \binom{\eta}{2}$,

\[
\|J^{\eta}_{\nu} - J^{\eta}_{\nu}\| \lesssim \alpha_4 \bar{n}.
\]

Deshpande and Montanari use the trace power method to bound the norm of $Q$ by bounding the norms of the $J^{\eta}_{\nu}$ individually. Some of the $J^{\eta}_{\nu}$ matrices have Wigner-like behavior.

Lemma A.3 (Lemmas 4.21, 4.22 in [DM15b]). With probability $1 - O(n^{-5})$, we have that for each $(\eta, \nu) \in \{(2, 1), (2, 6), (3, \cdot), (4, 1)\}$,

\[
\|J^{\eta}_{\nu}\| \lesssim \alpha_4 \cdot \bar{n}.
\]

A select few of the $J^{\eta}_{\nu}$ have larger eigenvalues.

Lemma A.4 (Lemmas 4.23, 4.24 in [DM15b]). With probability $1 - O(n^{-4})$, we have that for each $(\eta, \nu) \in \{(1, \cdot), (2, 2), (2, 3), (2, 4), (2, 5)\}$,

\[
\|J^{\eta}_{\nu}\| \lesssim \alpha_4 \cdot \bar{n}^{3/2}.
\]

We also give a short proof of an observation of Deshpande and Montanari, which states that some of the $J^{\eta}_{\nu}$ vanish when projected to $V_2$:
Observation A.5 (Lemmas 4.23, 4.24 in [DM15a]). Let $\Pi_2$ be the projector to $V_2$. Then always,

$$\left\| \Pi_2 \left( \sum_{\nu=1}^4 \tilde{J}_{1,\nu} \right) \right\| = 0,$$
and similarly,

$$\left\| \Pi_2 (J_{2,3} + \tilde{J}_{2,5}) \right\| = 0.$$

Proof. The proof follows from noting that the range of both of these sums of $J_{\eta,\nu}$ is in $V_1$. Consider some vector $v \in \mathbb{R}^{(t_2)}$; let $v' \overset{\text{def}}{=} \left( \sum_{\nu=1}^4 J_{1,\nu} \right) v$. We will look at the entry of $v'$ indexed by the disjoint pair $A = \{a, b\}$. By definition of the $J_{1,\nu}$, we have that

$$v'_A = \sum_{c,d \in [n]} \left( (A_{a,c} + A_{a,d}) + (A_{b,c} + A_{b,d}) \right) v_{c,d}$$

$$= \left( \sum_{c,d} (A_{a,c} + A_{a,d}) v_{c,d} \right) + \left( \sum_{c,d} (A_{b,c} + A_{b,d}) v_{c,d} \right),$$

and so by the characterization of $V_1$ from Proposition 2.3 the vector $v' \in V_1$. The conclusion follows.

Finally, we use a bound on the norm of the matrix $K$, which is the difference of $H_{2,2}$ and the non-multilinear entries.

Lemma A.6 (Lemma 4.25 in [DM15b]). Let $K$ be the restriction of $H_{2,2} - \mathbb{E}[H_{2,2}]$ to entries indexed by sets of size at most 3. With probability at least $1 - n^{-5}$,

$$\|K\| \leq \tilde{O}(\alpha_3 n^{1/2}).$$

We also require bounds on the matrices used in the Schur complement steps. The bounds of Deshpande and Montanari suffice for us, since we do not modify moments of order less than 4.

Lemma A.7 (Consequence of Proposition 4.19 in [DM15b]). Define $Q_n \in \mathbb{R}^{n, n}$ to be the orthogonal projection to the space spanned by $\tilde{1}$. Suppose that $\alpha$ satisfies $\alpha_1 - \alpha_2 \geq \Omega(\alpha_2 n^{-1/2})$ and $\alpha_2 - 2\alpha_1^2 \geq 0$, $\alpha_1 \geq 0$. Then with probability at least $1 - n^{-5}$,

$$H_{1,1} \geq 0,$$

$$H_{1,1}^{-1} \leq \frac{1}{n(\alpha_2 p - \alpha_1^2 p)} Q_n + \frac{2}{\alpha_1} Q_n^\perp.$$

A.1 Additional Proofs

We prove Lemma 2.5, which follows almost immediately from the bounds of [DM15a].

Proof of Lemma 2.5. Using the matrices from Definition A.1 and Observation A.5, we have that

$$\Pi_2 Q = \Pi_2 (\tilde{J}_{2,4} + \tilde{J}_{2,2}) + \Pi_2 \left( J_{2,4} - \tilde{J}_{2,4} + J_{2,2} - \tilde{J}_{2,2} \right) + \Pi_2 \left( \sum_{\nu=1}^4 J_{3,\nu} + J_{4,1} \right),$$

$$\Pi_2 \Pi_W Q = \Pi_2 \Pi_W (\tilde{J}_{2,4} + \tilde{J}_{2,2}) + \Pi_2 \Pi_W \left( J_{2,4} - \tilde{J}_{2,4} + J_{2,2} - \tilde{J}_{2,2} \right) + \Pi_2 \Pi_W \left( \sum_{\nu=1}^4 J_{3,\nu} + J_{4,1} \right),$$

The rest of the proof follows from noting that the range of both of these sums of $J_{\eta,\nu}$ is in $V_1$. Consider some vector $v \in \mathbb{R}^{(t_2)}$; let $v' \overset{\text{def}}{=} \left( \sum_{\nu=1}^4 J_{1,\nu} \right) v$. We will look at the entry of $v'$ indexed by the disjoint pair $A = \{a, b\}$. By definition of the $J_{1,\nu}$, we have that

$$v'_A = \sum_{c,d \in [n]} \left( (A_{a,c} + A_{a,d}) + (A_{b,c} + A_{b,d}) \right) v_{c,d}$$

$$= \left( \sum_{c,d} (A_{a,c} + A_{a,d}) v_{c,d} \right) + \left( \sum_{c,d} (A_{b,c} + A_{b,d}) v_{c,d} \right),$$

and so by the characterization of $V_1$ from Proposition 2.3 the vector $v' \in V_1$. The conclusion follows.

Finally, we use a bound on the norm of the matrix $K$, which is the difference of $H_{2,2}$ and the non-multilinear entries.

Lemma A.6 (Lemma 4.25 in [DM15b]). Let $K$ be the restriction of $H_{2,2} - \mathbb{E}[H_{2,2}]$ to entries indexed by sets of size at most 3. With probability at least $1 - n^{-5}$,

$$\|K\| \leq \tilde{O}(\alpha_3 n^{1/2}).$$

We also require bounds on the matrices used in the Schur complement steps. The bounds of Deshpande and Montanari suffice for us, since we do not modify moments of order less than 4.

Lemma A.7 (Consequence of Proposition 4.19 in [DM15b]). Define $Q_n \in \mathbb{R}^{n, n}$ to be the orthogonal projection to the space spanned by $\tilde{1}$. Suppose that $\alpha$ satisfies $\alpha_1 - \alpha_2 \geq \Omega(\alpha_2 n^{-1/2})$ and $\alpha_2 - 2\alpha_1^2 \geq 0$, $\alpha_1 \geq 0$. Then with probability at least $1 - n^{-5}$,

$$H_{1,1} \geq 0,$$

$$H_{1,1}^{-1} \leq \frac{1}{n(\alpha_2 p - \alpha_1^2 p)} Q_n + \frac{2}{\alpha_1} Q_n^\perp.$$
where we have used the fact that the columns of $J_{2,4}$ and $J_{2,2}$ lie in $W$. We thus have

$$
\Pi_2 Q - \Pi_2 \Pi_W Q = \Pi_2 (I - \Pi_W) \left( J_{2,2} - \tilde{J}_{2,2} + J_{2,4} - \tilde{J}_{2,4} + \sum_{\nu=1}^{4} J_{3,\nu} + J_{4,1} \right),
$$

And by the bounds $\|J_{3,\cdot}\| \leq \alpha_4 \cdot \pi$ and $\|J_{4,1}\| \leq \alpha_4 \cdot \pi$ from Lemma A.3 and the bounds $\|J_{2,4} - \tilde{J}_{2,4}\| \leq \alpha_4 \pi$ and $\|J_{2,2} - \tilde{J}_{2,2}\| \leq \alpha_4 \pi$ from Lemma A.2, and because $\Pi_2 (I - \Pi_W)$ is a projection, the conclusion follows.

Now, we prove that the trace power method works, for completeness.

**Proof of Lemma 4.1.** The proof follows from an application of Markov’s inequality. We have that for even $k$,

$$
P[\|M\| \geq t] = P[\|M^k\| \geq t^k] 
\leq P[\text{Tr}(M^k) \geq t^k] 
\leq \frac{1}{t^k} \mathbb{E}[\text{Tr}(M^k)] 
\leq \frac{1}{t^k} \sqrt{\pi \gamma k} \left( \frac{\gamma k}{e} \right)^k n^{\alpha k + \beta},
$$

where we have applied Stirling’s approximation in the last step. Choosing $k = O(\log n)$ and $t = O\left( \eta^{-1/k} \cdot \gamma \cdot \log n \cdot n^\alpha \right)$ completes the proof.