Theoretical hydrodynamics has progressed through the invention of a series of abstract fluids (perfect, inviscid, incompressible, and so on) that enable the tractable description of certain physical aspects of three-dimensional (3D) fluid systems \[1\]. Among the most useful of these idealizations has been that of the incompressible Newtonian fluid, which models the low-Reynolds number flow of simple and weakly-associated liquids, for example. While there are many physical realizations of such fluids in 3D, there have been none which satisfy the basic requirements in 2D, i.e., which is homogeneous in density and viscosity, and obeys the laws of conservation of mass, energy, and momentum. Currently studied 2D fluids include soap films \[2\], which are highly compressible in-plane due their facile response to stress (resulting in thickness changes); and few-nanometer thick, freely suspended, fluid smectic liquid crystal films \[3\] which, by virtue of their lamellar structure, are quantized in thickness to an integral multiple of several nanometers. In this Letter, we describe the Brownian diffusion of micron-scale inclusions in freely suspended smectic A liquid crystal films a few nanometers thick and several millimeters in diameter depends strongly on the air surrounding the film. Near atmospheric pressure, the three-dimensionally coupled film/gas system is well described by Hughes/Pailthorpe/White hydrodynamic theory but at lower pressure \((p \lesssim 70 \text{ torr})\), the diffusion coefficient increases substantially, tending in high vacuum toward the two-dimensional limit where it is determined by film size. In the absence of air, the films are found to be a nearly ideal physical realization of a two-dimensional, incompressible Newtonian fluid.

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Experimental Realization of an Incompressible Newtonian Fluid in Two Dimensions

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The Brownian diffusion of micron-scale inclusions in freely suspended smectic A liquid crystal films a few nanometers thick and several millimeters in diameter depends strongly on the air surrounding the film. Near atmospheric pressure, the three-dimensionally coupled film/gas system is well described by Hughes/Pailthorpe/White hydrodynamic theory but at lower pressure \((p \lesssim 70 \text{ torr})\), the diffusion coefficient increases substantially, tending in high vacuum toward the two-dimensional limit where it is determined by film size. In the absence of air, the films are found to be a nearly ideal physical realization of a two-dimensional, incompressible Newtonian fluid.

Since friction from the air plays such an important role in determining the hydrodynamic behavior of inclusions in smectic films, understanding how the inclusion-air interactions can be tuned by varying the ambient pressure...
is of fundamental interest. As the air pressure is reduced, the mean free path $\lambda$ of the air molecules is expected to increase, as indicated in Fig. 2(d). At sufficiently low pressure, the surrounding air can not be regarded simply as an incompressible, continuous fluid and the well-established SD/HPW model based on low Reynolds number hydrodynamics can no longer be used to predict the mobilities of inclusions. Here we explore the effects of varying the ambient air pressure on the Brownian motion of inclusions in FSLC films, showing that as the air is removed, the system evolves from a pseudo-3D regime where coupling to the air is dominant to a regime in which the hydrodynamics are determined by confinement at the boundaries, as predicted for an ideal 2D fluid.

Homogeneous FSLC films a few molecular layers thick are robust preparations [3] provide an ideal platform for studying hydrodynamics in reduced dimensions [21]. In previous experiments, we described the Brownian motion of silicone oil droplets embedded in such films with the ambient air at atmospheric pressure [13]. These droplets form lens-shaped inclusions are insoluble in liquid crystal and whose size remains constant over long time intervals, typically for more than half an hour, which far exceeds the time required to perform a typical measurement.

The liquid crystal material used in our experiment is 8CB (4′-n-octyl-4′-cyanobiphenyl), which is in the fluid smectic A phase at room temperature. The saturation vapor pressure of 8CB is very low (around $10^{-3}$ torr), enabling us to study the microhydrodynamics of inclusions in the film over a wide range of Knudsen number ($\lambda/2R$, the reduced mean free path). The density and viscosity of 8CB are $\rho \approx 0.96$ g/cm$^3$ [23] and $\eta \approx 0.052$ Pa·s [24] respectively. Each smectic layer of thickness $3.17$ nm was formed by spreading a small amount of the liquid crystal across a 4 mm-diameter hole in a glass cover slip and were then observed using reflected light video microscopy. The film thickness $h$, an integral number $N$ of smectic layers (typically $2 \leq N \leq 6$), is determined precisely by comparing the reflectivity of the film with black glass [20]. A resistive filament coated in silicone oil is then electrically heated in order to generate an oil vapor, some of which makes its way to the film where it eventually condenses and forms visible droplets such as those shown in Fig. 2a, with radii between 2 and 50 $\mu$m. A double-sealed rotary pump is then used to reduce the pressure to $3 \times 10^{-3}$ torr, after which a turbo pump is used to get the film chamber down to $10^{-4}$ torr. The shape and thickness of the oil droplets is measured by analyzing the interference fringes (see Fig. 2b) formed in monochromatic light [27].

Once oil droplets appear on the film, the chamber is carefully tilted in order to maneuver a droplet of desired radius $a$ into the center of the film (of radius $R$), after which the film is leveled to minimize gravitational drift, enabling us to capture several thousand images at a video frame rate of 24 fps while the droplet is in the field of view and far from the film boundaries, as shown schematically in Fig. 2c).

After reducing the pressure of the film chamber to $10^{-4}$ torr, closing the valve between the pump and the vacuum buffer chamber allows us to maintain quasi-constant pressure around the film for dozens of minutes, during which we are able to make video recordings of droplet motion. The pressure is then gradually increased by injecting small amounts of air into the system, allowing us to obtain a series of movies showing the Brownian motion of the droplet as the chamber pressure increases in steps from $3 \times 10^{-4}$ torr to 633 torr. These videos are decomposed into sequential images and the size and position of the droplet are determined using algorithms based on Canny’s method for edge detection [28] and Taubin’s method for object identification [29]. The diffusion coefficient is obtained after analytically removing any drift [13].

The diffusion coefficient of a typical droplet near the center of the film is plotted as a function of ambient pressure in Fig. 2(d). The green curve shows the corresponding SD/HPW theory with the air viscosity corrected for pressure [30]. The observed variation of diffusion coefficient is well described by this model at pressures close to atmospheric ($p \gtrsim 70$ torr but the experimental data deviate significantly from the theory at lower pressure, increasing monotonically as the pressure is reduced before saturating at very high vacuum, at the limit corresponding to 2D boundary confinement (horizontal red dashed line in Fig. 2(d)). The observed behavior falls in three distinct hydrodynamic regimes: (1) Near atmospheric pressure ($p \gtrsim 70$ torr), the mean free path of the air molecules ($\lambda \sim 7$ $\mu$m) is much less than the diameter of the film. In this regime, the air may be regarded as
a large oil droplet. (c) Cartoon of an oil droplet of radius \( R \) near the center of a film of radius \( R \). (d) Diffusion coefficient of a single droplet (\( a = 8 \mu m \)) near the center of a film (\( R = 2 \text{ mm}, N = 3 \) layers) as a function of surrounding air pressure (symbols). The green curve corresponds to SD/HPW theory. The black curve shows the diffusion predicted by a model assuming free air molecules impinging on the film. The horizontal dashed red line shows the 2D confinement limit predicted by Saffman for vanishingly small air viscosity and no-slip boundaries. The dashed black line shows the mean free path of the air molecules scaled by the film diameter (\( \lambda_{\text{reduced}} = \lambda/(2R) \)) as a function of pressure. The background shading indicates three distinct behavioral regimes corresponding to different air pressure ranges: free molecules, film and slip, and continuity. (e) Reduced mobility of single oil droplets in a film in high vacuum as a function of reduced film radius. The model curve is Saffman’s prediction for a 2D fluid with no-slip boundaries.

In both 3D and 2D fluids, inclusions near a rigid boundary experience a “wall effect” which reduces their mobility. To study the 2D “wall effect”, in the 1940s, White measured the drag on metal wires falling sideways through viscous liquids confined between two vertical bounding walls and found that at low Reynolds numbers, the mean free path is several mm long, a distance comparable to the diameter of the film. In this regime, the ambient air can be regarded as an ensemble of collisionless molecules that obey a Maxwell-Boltzmann velocity distribution.

In order to model the behavior of droplets at the lowest pressures, we may approximate the total drag force \( F \) as the sum of two terms, one arising from confinement forces acting from the impingement of air molecules on the inclusion, and on indirect frictional forces resulting from changes of streamlines in the film caused by collisions with air molecules. Calculations based on kinetic theory indicate that the unit frictional force as a function of droplet speed \( U \) and surrounding air pressure \( p \) may be written

\[
F_{\text{air}} = p\sqrt{\pi \nu_m}/(2kT)U,
\]

where \( m \) is the mass of an air molecule, \( k \) the Boltzmann constant, and \( T \) the temperature. The net inverse droplet mobility may be written as \( 1/\mu = 1/\mu_b + 1/\mu_{\text{air}} \). Since \( \mu_b \) is independent of pressure, the mobility can be expressed in the form

\[
\frac{1}{\mu} = \frac{1}{\mu_b} + \text{const} \times p,
\]

where the constant can be found by fitting the experimental data at low pressure. This model is plotted as the black curve in Fig. 2(d).

In an ideal 2D fluid of finite size, therefore, the only drag experienced by a disk-like inclusion should come from confinement forces arising from long-range hydrodynamic interactions with the fluid boundaries. Our experiments confirm that in high vacuum (\( p \lesssim 0.003 \text{ torr} \)), the frictional drag from the remaining air molecules is much smaller than the hydrodynamic confinement force and can be neglected. In this regime, the freely suspended SmA liquid crystal film approaches a true 2D fluid and exhibits purely 2D hydrodynamics. To verify that we were really in the 2D limit, we analyzed the Brownian motion of droplets of different sizes in films of different radii under high vacuum. The reduced mobility \( m = 4\pi \nu_m \mu \) of these inclusions as a function of reduced film radius \( R/a \) is plotted in Fig. 2(e). The observed mobility follows the predictions of SD theory quite well, increasing logarithmically with inclusion size as expected for a system with 2D hydrodynamic behavior. The observed mobilities are slightly larger than predicted by the model, an effect which might be due to deviations from ideal, no-slip boundary conditions resulting from the presence of a meniscus.

Over the surfaces of the film and oil droplet. (3) At very low pressure (\( p \lesssim 0.02 \text{ torr} \)), the mean free path of free molecules is comparable to the film diameter. In this regime, the ambient air can be regarded as an ensemble of collisionless molecules that obey a Maxwell-Boltzmann velocity distribution.

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number, the presence of the walls affected the mobility of the wires even when they were many hundreds of wire diameters away, with the mobility depending logarithmically on the ratio of wall separation to wire radius $\frac{d}{a}$. Recent measurements of inclusion mobility in very thick smectic films, in which the Saffman length is greater than the film size and the influence of the air is relatively small, also showed the effects of the boundary [37].

Eliminating the environmental drag on a thin smectic film by removing the ambient air seemed a promising way of studying wall effects in a true 2D fluid. We therefore measured the mobilities of included oil droplets both parallel and perpendicular to a straight boundary in high vacuum. The experimental observations, shown in Fig. 3 were compared with the model of Jeffrey and Onishi [38], who solved the Navier-Stokes equations for 2D flow around a translating cylinder near a plane wall, assuming small Reynolds number flow. For translation respectively parallel and perpendicular to a straight film boundary, the predicted reduced mobilities are:

$$m_\parallel = 4\pi h \mu_\parallel = \ln \left(\frac{d + \sqrt{d^2 - a^2}}{a}\right), \quad (1a)$$

$$m_\perp = 4\pi h \mu_\perp = \ln \left(\frac{d + \sqrt{d^2 - a^2}}{a}\right) - \frac{\sqrt{d^2 - a^2}}{d}, \quad (1b)$$

For large values of $d/a$, these expressions simplify to $m_\parallel \approx \ln[2d/a]$ and $m_\perp \approx \ln[2d/a] - 1$. In contrast to 3D fluids, where the wall effect on mobility decays within a few dozen inclusion radii [39], the influence of the boundary extends a long distance into a 2D fluid and the hydrodynamic behavior of inclusions is predicted to remain anisotropic at large distances from the wall. Our experimental results confirm this behavior, as seen in Fig. 3. The measured mobilities are on average slightly higher than the theory but are generally in good agreement, except very close to the wall. This may be due to deviations from true no-slip boundary conditions [40] at the meniscus, as mentioned previously.

In summary, we have described the Brownian motion of single inclusions in freely suspended smectic A liquid crystal films as the pressure of the surrounding air is reduced from one atmosphere to a high vacuum. The inclusion mobility was characterized in three hydrodynamic regimes: near atmospheric pressure, where diffusion follows HPW theory, in partial vacuum (the slip regime), and in high vacuum, where we observe motion limited by 2D confinement effects. The parallel and perpendicular mobilities of an inclusion in high vacuum near the edge of the film increase logarithmically with distance from the boundary as predicted by theory, with an anisotropic character that persists far into the film. The observations suggest that thin, freely suspended smectic films in high vacuum are a nearly ideal experimental realization of a two-dimensional fluid. This work opens the way for more general hydrodynamic studies in the 2DIN limit, of such phenomena as driven micro rheological flow, high Reynolds number turbulence, energy cascades, and jets.

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