Cylindrically Symmetric Ground State Solutions for Curl-Curl Equations with Critical Exponent

Xiaoyu Zeng
Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences
P.O. Box 71010, Wuhan 430071, P.R. China

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Abstract

We study the following nonlinear critical curl-curl equation

$$\nabla \times \nabla \times U + V(x)U = |U|^{p-2}U + |U|^4U, \quad x \in \mathbb{R}^3,$$

(0.1)

where $V(x) = V(r, x_3)$ with $r = \sqrt{x_1^2 + x_2^2}$ is 1-periodic in $x_3$ direction and belongs to $L^\infty(\mathbb{R}^3)$. When $0 \not\in \sigma(-\Delta + \frac{1}{r^2} + V)$ and $p \in (4, 6)$, we prove the existence of nontrivial solution for (0.1), which is indeed a ground state solution in a suitable cylindrically symmetric space. Especially, if $\sigma(-\Delta + \frac{1}{r^2} + V) > 0$, ground state solution is obtained for any $p \in (2, 6)$.

Keywords: curl-curl equation; critical exponent; cylindrical symmetry; ground state solutions.

MSC: 35J20, 35J60, 35Q60.

1 Introduction

In this paper, we consider the following nonlinear critical curl-curl equation

$$\nabla \times \nabla \times U + V(x)U = |U|^{p-2}U + |U|^4U, \quad x \in \mathbb{R}^3,$$

(1.1)

where $V(x) \in L^\infty(\mathbb{R}^3)$ is the external potential, $p \in (2, 6)$ is of subcritical growth. Equation (1.1) is related to the standing waves of nonlinear Maxwell’s equation for an inhomogeneous medium. For more physical backgrounds one can refer to [4, 5, 6], etc.

Solution $U \in H(\text{curl}; \mathbb{R}^3)$ of (1.1) is in general a critical point of the energy functional

$$I(U) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times U|^2 + V(x)|U|^2dx - \frac{1}{p} \int_{\mathbb{R}^3} |U|^pdx - \frac{1}{6} \int_{\mathbb{R}^3} |U|^6dx.$$

*Email: zengxy09@126.com
The main difficulty for dealing with equation (1.1) lies in the presence of the curl-curl operator, which causes the energy functional $I$ is strong indefinite in some sense. To overcome this difficulty, there are many researches studied the following equation
\[ \nabla \times \nabla \times U + V(x)U = g(x, U), \quad x \in \mathbb{R}^3, \] (1.2)
where $g(x, U)$ is a subcritical nonlinearity. For the case of $V(x) \equiv 0$, Benci, Fortunato [6] and Azzollini et al. [1] studied equation (1.2) in a cylindrically symmetric vector space which is instructed with divergence free elements. In a different symmetric space D’April and Siciliano also obtained in [9] nontrivial solutions for (1.2). When $V(x) \not\equiv 0$ is cylindrically symmetric, solutions have some kind symmetries were also studied in [4, 10]. Recently, Bartch and Mederski [5] developed the Nehari-Pankov method [21] and considered the ground state and bound state of (1.2) in a bounded domain with the boundary condition $\nu \times U = 0$. Based on this work, Mederski [12] further studied (1.2) where, e.g., $g \sim |U|^{q-1}U$ if $|U| \ll 1$ and $g \sim |U|^{q-1}U$ if $|U| \gg 1$ for $1 < p < 5 < q$ and $V(x) \leq 0$ is periodic and $V \in L^{p-1}(\mathbb{R}^3) \cap L^{q-1}(\mathbb{R}^3)$. The main contribution of [5, 12] is that they can treat equation (1.2) without any assumption on the symmetry of $U$. For the case that $g$ is asymptotically linear growth at infinity, we would like to mention some existence results in the paper [14] by Qin and Tang as well as [16, 17, 18, 19] by Stuart and Zhou for some related topics.

Very recently, Mederski considered the Brezis-Nirenberg problem for equation (1.1) (i.e., removing the term $|U|^{p-2}U$) in a bounded domain, and obtained some existence and nonexistence results of cylindrically symmetric ground state solutions for (1.1) in [13]. In this paper, we focus on problem (1.1) in a suitable cylindrically symmetric space. We always assume that $V(x)$ satisfies
\[ V(x) = V(r, x_3) \in L^\infty(\mathbb{R}^3), \text{ where } r = \sqrt{x_1^2 + x_2^2}. \]

One can search for solutions of the form [1, 4]
\[ U(x) = \frac{u(r, x_3)}{r} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}. \] (1.3)

Taking the fact $\text{div } U = 0$ into account, it follows from (1.1) that $u(r, x)$ satisfies
\[ -\Delta u + \frac{u}{r^2} + V(r, x_3)u = |u|^{p-2}u + |u|^4u, \] (1.4)
where $\Delta = \partial_r^2 + \frac{1}{r} \partial_r + \partial_{x_3}^2$. Correspondingly, we also have
\[ I(U) = J(u), \] (1.5)
where $J \in C^2(E, \mathbb{R})$ is the energy functional of (1.4) and given by
\[ J(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{u^2}{r^2} + V(r, x_3)u^2dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^pdx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6dx, \quad u \in E. \] (1.6)
and its derivative \( J'(u) \in E^* \) is given by
\[
\langle J'(u), \phi \rangle = \int_{\mathbb{R}^3} \nabla u \nabla \phi + \frac{u \phi}{r^2} + V(r, x_3) u \phi \, dx - \int_{\mathbb{R}^3} |u|^{p-2} u \phi \, dx - \int_{\mathbb{R}^3} u^5 \phi \, dx, \quad \forall \phi \in E.
\]

Here, the space \( E \) is defined by
\[
E := \left\{ u(r, x_3) \in H^1(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} \frac{u^2}{r^2} \, dx < \infty \right\},
\]
and endowed with the norm of
\[
\|u\|_E = \left( \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{|u|^2}{r^2} + u^2 \, dx \right)^{1/2}.
\]

Clearly \( E \hookrightarrow H^1_s(\mathbb{R}^3) := \{ u = u(r, x_3) \in H^1(\mathbb{R}^3) \} \). In general, the integral \( \int_{\mathbb{R}^3} \frac{|u|^2}{r^2} \, dx \) cannot be controlled by \( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \) for all \( u \in H^1_s(\mathbb{R}^3) \), so we have \( E \subsetneq H^1_s(\mathbb{R}^3) \).

We call \( u \in E \setminus \{0\} \) a nontrivial weak solution of (1.4) if \( \langle J'(u), \phi \rangle = 0 \) for any \( \phi \in E \) and denote by
\[
S = \left\{ u : u \text{ is a nontrivial solution of (1.4)} \right\}
\]
the nontrivial solution set of (1.4). Correspondingly, \( u \in E \) is said to be a ground state solution of (1.4) if
\[
u \in \{ u \in S \text{ and } J(u) \leq J(v) \text{ for any } v \in S \}.
\]

**Remark 1.1.** In view of the change (1.3), one can search cylindrically symmetric solution of (1.3) by studying equation (1.4). Moreover, the ground state solution of (1.3) can be defined similarly as that of (1.4). As a consequence of (1.3), any ground state solution of (1.4) corresponds to a ground state solution of (1.3). For these reasons, instead of directly studying equation (1.3), we shall focus ourself on equation (1.4) in what follows.

Define
\[
L^2_{\text{cyl}}(\mathbb{R}^3) := \left\{ u(r, x_3) \in L^2(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} \frac{u^2}{r^4} \, dx < \infty \right\}
\]
and
\[
W^{2,2}_{\text{cyl}}(\mathbb{R}^3) := \left\{ u(r, x_3) \in H^2(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} \frac{u^2}{r^4} \, dx < \infty \right\}.
\]

Let \( L : W^{2,2}_{\text{cyl}}(\mathbb{R}^3) \subset L^2_{\text{cyl}}(\mathbb{R}^3) \to L^2_{\text{cyl}}(\mathbb{R}^3) \) be a self-adjoint operator given by \( L := -\Delta + \frac{1}{r^2} + V(r, x_3) \). Assume that \( V(r, x_3) \) satisfies
\[
(V) \quad V(r, x_3) \in L^\infty(\mathbb{R}^3) \text{ is 1-periodic in } x_3 \text{ direction and } 0 \not\in \sigma(L) \text{ where } \sigma \text{ denotes the spectrum of } L \text{ in } L^2_{\text{cyl}}(\mathbb{R}^3).
\]

Then our main result can be stated as follows.

**Theorem 1.1.** Assume \( V(r, x_3) \) satisfies (V) and \( p \in (4, 6) \) then (1.4) has a ground state solution in \( E \). Especially, if \( \sigma(-\Delta + \frac{1}{r^2} + V) > 0 \) then (1.4) possesses a ground state solution in \( E \) for any \( p \in (2, 6) \).
We intend to prove the above theorem by using Nehari-Pankov method as in [4] where the subcritical case was studied. Let \( (E(\lambda))_{\lambda \in \mathbb{R}} \) be the spectral family of \( L \). Set \( E^- := E(0)L_{cyl}^2 \cap E \) and \( E^+ := (I - E(0))L_{cyl}^2 \cap E \). Then there exists an inner product \((\cdot,\cdot)\) such that the corresponding norm \(\|\cdot\|\) is equivalent to \(\|\cdot\|_E\) and for any \(u = u^+ + u^- \in E^+ + E^-\) there holds that
\[
\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{u^2}{r^2} + V(r,x_3)u^2 dx = \| u^+ \|^2 - \| u^- \|^2.
\]
We note that if \(\sigma(L) \subset (0,\infty)\) then \(\dim E^- = 0\). Upon the above decomposition for \(E\), functional \(J\) defined in (1.6) can be written as
\[
J(u) = \frac{1}{2}\| u^+ \|^2 - \frac{1}{2}\| u^- \|^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx, \ u \in E,
\]
and its derivative \(J'(u) \in E^*\) is given by
\[
\langle J'(u), \phi \rangle = \langle u^+, \phi^+ \rangle - \langle u^-, \phi^- \rangle - \int_{\mathbb{R}^3} |u|^{p-2}u\phi dx - \int_{\mathbb{R}^3} u^5 \phi dx, \ \forall \ \phi \in E. \quad (1.7)
\]
In section [2] we will consider the following minimization problem
\[
c := \inf_{u \in \mathcal{N}} J(u), \quad (1.8)
\]
where \(\mathcal{N}\) is the Nehari-Pankov manifold and defined as
\[
\mathcal{N} := \{ u \in E \setminus \{ 0 \} : \langle J'(u), \phi \rangle = 0 \ \forall \ \phi \in \mathbb{R}u \oplus E^- \}.
\]
From (1.7) one see that \(\langle J'(v), v \rangle < 0\) if \(v \in E^- \setminus \{ 0 \}\), this indicates that
\[
\mathcal{N} \cap E^- = \emptyset.
\]
If we can prove that problem (1.8) is attained by some \(u \in E\), then it is a nontrivial solution, indeed, is a ground state solution of (1.4) since all nontrivial solutions of (1.4) belong to \(\mathcal{N}\). Theorem 1.1 thus can be obtained.

We also note that in view of the existence of Sobolev critical exponent in equation (1.4), to study problem (1.8), formally we need to prove the energy of \(J\) is strictly less than some critical value, which is related to the following minimization problem.
\[
\hat{S} := \inf_{0 \neq u \in E} \left( \frac{\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{|u|^2}{r^2} dx}{(\int_{\mathbb{R}^3} |u|^6 dx)^{\frac{1}{3}}} \right), \quad (1.9)
\]
Repeating the argument of [8, Theorem 3], one can prove that (1.9) can be achieved by a nonnegative minimizer \(\Phi \in E\) and \(\Phi\) satisfies the following equation
\[
- \Delta \Phi + \frac{\Phi}{r^2} = \Phi^5 \text{ in } \mathbb{R}^3. \quad (1.10)
\]
From (1.9) and (1.10) one can easily check that
\[
\int_{\mathbb{R}^3} |\nabla \Phi|^2 + \frac{\Phi^2}{r^2} dx = \int_{\mathbb{R}^3} |\Phi|^6 dx = \hat{S}^\frac{2}{3}. \quad (1.11)
\]
Moreover, the results of [2] propositions 5 and 6] indicate that $\Phi \in L^\infty(\mathbb{R}^3)$ and

$$\limsup_{|x| \to \infty} |x|^v \Phi(x) < \infty \text{ for any } v < \frac{1+\sqrt{5}}{2}. \tag{1.12}$$

This especially yields that $\Phi \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$.

In what follows we denote by $C$ the universal positive constant unless specified. The open ball centered at $x_0$ with radius $R$ is denoted by $B_R(x_0)$ and $|\cdot|_{L^q}$ means the norm in $L^q(\mathbb{R}^3)$.

## 2 Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1 by using the Nehari-Pankov method. For this purpose, we recall two key lemmas which have been proved in previous papers, so we omit the proof here. Firstly, similar to the argument of [7, proposition 2.2], we have the following lemma.

**Lemma 2.1.** If $V(x)$ satisfies (V), then for any $\mu \in \mathbb{R}$ there exists $C_0, C_1 > 0$ such that $|u|_{L^\infty} \leq C_0 \|u\|_E \leq C_1 |u|_{L^2}$ for all $u \in E(\mu)L^2_c(\mathbb{R}^3) \cap E$.

Repeating the argument of [4] lemmas 19-21 one can prove that for any $u \in \mathcal{N}$ the norm of $\nabla J(u)$ can be controlled by its tangential component. Here $\nabla J(u)$ denotes the unique correspondence of $J'(u)$ in $E$. Precisely, we have

**Lemma 2.2.** Let $N_0$ be a bounded subset of $\mathcal{N}$. There exists $C_0 > 0$ such that the following holds: for any $u \in N_0$, $\nabla J(u) = \tau + \sigma$ where $\tau \in T_u \mathcal{N}$ is the tangential component of $\nabla J(u)$ and $\sigma \perp \tau$ is the transversal component of $\nabla J(u)$, then,

$$\|\nabla J(u)\| \leq C_0 \|\tau\|.$$  

The following lemma tells that the norm $\|u\|$ and $J(u)$ are both bounded below on the manifold $\mathcal{N}$.

**Lemma 2.3.** Assume $V(x)$ satisfies (V), then there exist $C_1, C_2 > 0$ such that

$$\|u\| \geq C_1 \text{ and } \|u\| \leq C_2 (J(u))^{\frac{2}{5}} \text{ for all } u \in \mathcal{N}. \tag{2.1}$$

**Proof.** For any $u \in \mathcal{N}$, then

$$J(u) = (\frac{1}{2} - \frac{1}{p}) \int_{\mathbb{R}^3} |u|^p dx + \frac{1}{3} \int_{\mathbb{R}^3} |u|^6 dx \geq 0. \tag{2.2}$$

Therefore, we have $\|u^-\| \leq \|u^+\|$. Moreover, it follows from $\langle J'(u), u^+ \rangle = 0$ that for any $\delta > 0$,

$$\|u^+\|^2 = \int_{\mathbb{R}^3} |u|^p u^+ dx + \int_{\mathbb{R}^3} u^5 u^+ dx \leq \delta \int_{\mathbb{R}^3} |u|u^+ dx + C|u|^5 u^+ dx \leq \delta \|u\|_{L^2} |u^+|_{L^2} + C(\delta) |u|^5 |u^+|_{L^6} \leq \delta \|u\|^2 + C \|u\|^6 \tag{2.3}$$
Taking $\delta < \frac{1}{3}$, we then deduce that there exists $C_1 > 0$ such that
\[ \|u\| \geq C_1 > 0. \]
Furthermore, it follows from (2.2) and (2.3) that
\[ \frac{\|u\|^2}{2} \leq \delta \|u\|^2 + C|u|_{L^6}^5|u_+^2|_{L^6} \leq \delta \|u\|^2 + C|u|_{L^6}^5\|u\| \]
\[ \leq \delta \|u\|^2 + C(J(u))_5^{\frac{5}{6}}\|u\| \]
This indicates that $\|u\| \leq C_2(J(u))_5^{\frac{5}{6}}$. \hfill \square

Stimulated by the arguments of [2, 3, 7, 20], we can apply Lemmas 2.1 - 2.2 to obtain the following theorem. Which yields Theorem 1.1 if condition $c < \frac{1}{3}S_\delta^\frac{3}{2}$ is satisfied.

**Theorem 2.1.** If $c < \frac{1}{3}S_\delta^\frac{3}{2}$ and $p \in (2, 6)$, then problem (1.5) can be achieved by some $u \in E$. Moreover, $u$ is a ground state solution of (1.4).

**Proof.** Using Ekeland’s variational principle [15], there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}$ of $c$ such that $J(u_n) \to c$ and $(J|_{\mathcal{N}})'(u_n) \to 0$. Furthermore, from Lemma 2.3 we see that $\{u_n\}$ is bounded in $E$. Thus, $J'(u_n) \to 0$ by applying Lemma 2.2. In summary, we obtained a Palais-Smale sequence
\[ J(u_n) \rightharpoonup c \text{ and } J'(u_n) \to 0 \text{ in } E^*. \] (2.4)

We first rule out the case of vanishing. For otherwise, if
\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} |u_n|^2 dx = 0 \text{ for any } R > 0. \] (2.5)
It follows from [11, Lemma 1.1] that
\[ u_n \rightharpoonup 0 \text{ in } L^q(\mathbb{R}^3) \text{ for any } q \in (2, 6). \] (2.6)

Thus,
\[ J(u_n) - \frac{1}{2}\langle J'(u_n), u_n \rangle = \frac{1}{3} \int_{\mathbb{R}^3} |u_n|^6 dx + o(1) \rightharpoonup c. \] (2.7)
From (2.6) and Lemma 2.1 we deduce from the fact $\langle J'(u_n), u_n^- \rangle = 0$ that
\[ \|u_n^+\|^2 = - \int_{\mathbb{R}^3} |u_n|^4 u_n^- w_n dx - \int_{\mathbb{R}^3} |u_n|^{p-2} u_n^- w_n dx \]
\[ \leq |u_n|_{L^5}^5 |u_n^-|_{L^\infty} + |u_n|_{L^p}^{p-1} |u_n^-|_{L^p} + o(1) \to 0. \] (2.8)
Rewrite $u_n^+ = w_n + z_n$, where $w_n \in E(\mu)L_{cyl}^2 \cap E$ and $z_n \in (I - E(\mu))L_{cyl}^2 \cap E$, $\mu > 0$ is larger enough (which will be determined later). As a consequence of (2.2) we have $\langle J'(u_n), w_n \rangle = o(1)$. Similar to the argument of (2.8), one can obtain that
\[ \|w_n\|^2 = \int_{\mathbb{R}^3} |u_n|^4 w_n dx + \int_{\mathbb{R}^3} |u_n|^{p-2} w_n dx + o(1) \rightharpoonup 0. \]
Therefore,
\[ |u_n - z_n|_{L^6} = |u_n^- + z_n|_{L^6} \leq C|u_n^- + z_n| \rightarrow 0, \tag{2.9} \]
and
\[ \|z_n\|^2 = \int_{\mathbb{R}^3} |\nabla z_n|^2 + \frac{|z_n|^2}{r^2} + V(r, x_3)|z_n|^2 dx \]
\[ = \langle J'(u_n), u_n \rangle - \|w_n\|^2 + \|u_n^-\|^2 + \int_{\mathbb{R}^3} |u_n|^6 dx + \int_{\mathbb{R}^3} |u_n|^p dx \tag{2.10} \]
\[ = \int_{\mathbb{R}^3} |u_n|^6 dx + o(1). \]

Since \( \mu \int_{\mathbb{R}^3} |z_n|^2 dx \leq \int_{\mathbb{R}^3} |\nabla z_n|^2 + \frac{|z_n|^2}{r^2} + V|z_n|^2 dx \), we have \( (\mu - |V|_{L^\infty}) \int_{\mathbb{R}^3} |z_n|^2 dx \leq \int_{\mathbb{R}^3} |\nabla z_n|^2 + \frac{|z_n|^2}{r^2} dx \). For any \( \delta > 0 \), we take \( \mu > 0 \) large enough such that \( |V|_{L^\infty} \leq \delta(\mu - |V|_{L^\infty}) \). Then
\[ \delta \int_{\mathbb{R}^3} |\nabla z_n|^2 + \frac{|z_n|^2}{r^2} dx \geq \delta(\mu - |V|_{L^\infty}) \int_{\mathbb{R}^3} |z_n|^2 dx \geq |V|_{L^\infty} \int_{\mathbb{R}^3} |z_n|^2 dx. \]
This implies that
\[ (1 - \delta) \int_{\mathbb{R}^3} |\nabla z_n|^2 + \frac{|z_n|^2}{r^2} dx \leq \int_{\mathbb{R}^3} |\nabla z_n|^2 + \frac{|z_n|^2}{r^2} + V|z_n|^2 dx = \|z_n\|^2. \tag{2.11} \]

We deduce from (2.6)-(2.11) that
\[ c + o(1) = J(u_n) = \frac{1}{2}\|z_n\|^2 + \frac{1}{2}\|w_n\|^2 - \frac{1}{2}\|u_n^-\|^2 - \frac{1}{6}\int_{\mathbb{R}^3} |u_n|^6 dx + \frac{1}{p}\int_{\mathbb{R}^3} |u_n|^p dx \]
\[ = \frac{1}{2}\|z_n\|^2 - \frac{1}{6}\int_{\mathbb{R}^3} |u_n|^6 dx + o(1) \geq \frac{1 - \delta}{2}\int_{\mathbb{R}^3} |\nabla z_n|^2 + \frac{|z_n|^2}{r^2} dx - \frac{c}{2} + o(1) \]
\[ \geq \frac{(1 - \delta)\hat{S}}{2}|z_n|_{L^6}^2 - \frac{c}{2} + o(1) = \frac{(1 - \delta)\hat{S}}{2}|u_n|_{L^6}^2 - \frac{c}{2} + o(1) \]
\[ \geq \frac{(1 - \delta)(3\hat{c})\frac{3}{2}\hat{S}}{2} - \frac{c}{2} + o(1). \]

From (2.1) we see that \( c > \beta > 0 \) for some \( \beta > 0 \). Letting \( n \rightarrow \infty \), the above inequality thus implies that \( c \geq \frac{1}{3}(1 - \delta)\frac{3}{2}\hat{S}\hat{S} \) for all \( \delta > 0 \). This leads to a contradiction for the condition \( c < \frac{1}{3}\hat{S}\hat{S} \) is assumed.

Now, since we have proved that vanishing (2.3) cannot occur. Thus, there exist \( R, \eta > 0 \) and a sequence \( \{x_n = (y_n, x_{3n})\} \subset \mathbb{R}^3 \) (without loss of generality, we assume that \( x_n \in \mathbb{Z}^3 \)) such that
\[ \limsup_{n \rightarrow \infty} \int_{B_R(x_n)} |u_n|^2 dx \geq \eta > 0. \tag{2.12} \]
We claim that
\[ \limsup_{n \rightarrow \infty} |y_n| < \infty. \tag{2.13} \]
Hence, by passing to subsequence, there exists $u$. Furthermore, in view of proposition 17 that for any $l \in \mathbb{N} \setminus \{0, 1\}$, there exist $n_l > 0$ and $g_1, g_2, \ldots, g_l \in O(2)$ such that when $n > n_l$, there holds that

$$B_R(g_k y_n, x_{3n}) \cap B_R(g_j y_n, x_{3n}) = \emptyset \quad \text{for any} \quad k \neq j.$$ 

Together with (2.12) and (2.14) this yields that

$$\lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n|^2 dx \geq \lim_{l \to \infty} \sum_{k=1}^l \int_{B_R(g_k y_n, x_{3n})} |u_n|^2 dx \geq \lim_{l \to \infty} \eta = +\infty.$$ 

This is impossible and thus claim (2.13) is proved.

Let $\bar{x}_n = (0, x_{3n})$, in view of (2.13) we may assume that (2.12) still holds by replacing $x_n$ with $\bar{x}_n$. Set $\bar{u}_n(x) = u_n(x + \bar{x}_n)$, then

$$\limsup_{n \to \infty} \int_{B_R(0)} |ar{u}_n|^2 dx \geq \eta > 0.$$ 

Moreover, since $V(r, x_3)$ is 1-periodic in $x_3$ direction, we see that $\{\bar{u}_n\} \subset \mathcal{N}$ satisfies

$$J(u_n) = J(\bar{u}_n) \xrightarrow{n \to \infty} c \quad \text{and} \quad J'(u_n) = J'(\bar{u}_n) \xrightarrow{n \to \infty} 0 \quad \text{in} \quad E^*.$$ 

Hence, by passing to subsequence, there exists $u \in E$ such that $\bar{u}_n \xrightarrow{n \to \infty} u$ in $E$ and $J'(u) = 0$. From (2.15) we see that $u \neq 0$, so $u$ is a nontrivial solution of (1.4) by (2.16).

Furthermore, in view of $u, \bar{u}_n \in \mathcal{N}$, it follows from Fatou’s lemma that

$$c = \lim_{n \to \infty} J(\bar{u}_n) = \lim_{n \to \infty} \left[ \frac{1}{2} - \frac{1}{p} \right] \int_{\mathbb{R}^3} |\bar{u}_n|^p dx + \frac{1}{3} \int_{\mathbb{R}^3} |\bar{u}_n|^6 dx \geq \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^3} |u|^p dx + \frac{1}{3} \int_{\mathbb{R}^3} |u|^6 dx = J(u) \geq c.$$ 

This implies that $u$ is a minimizer of (1.8), so it is a ground state solution of (1.4). 

From Theorem 2.1 we see that, to finish the proof of Theorem 1.1 it remains to show that the energy value $c < \frac{1}{3} S^\frac{2}{3}$. Our following three lemmas tell that this condition can be verified under the assumptions of Theorem 1.1.

**Lemma 2.4.** Let $\Phi$ be the nonnegative solution given by equation (1.10) and set

$$\varphi_\varepsilon(x) = \varepsilon^{-\frac{3}{2}} \Phi(\frac{x}{\varepsilon}) = \varphi^+_{\varepsilon}(x) + \varphi^-_{\varepsilon}(x) \in E^+ \oplus E^-,$$

then,

$$\int_{\mathbb{R}^3} |\nabla \varphi_\varepsilon|^2 dx = \int_{\mathbb{R}^3} |\varphi_\varepsilon|^2 dx = \int_{\mathbb{R}^3} |\varphi_\varepsilon|^2 dx = O(\varepsilon^{-\frac{2}{3} + 3}) \quad \forall \quad q \in [2, 6), \quad (2.17)$$
\[
|\varphi^-_\varepsilon|_{L^\infty}, \|\varphi^-_\varepsilon\|, |\varphi^+\varepsilon|_{L^2} \leq O(\varepsilon), \quad (2.18)
\]
\[
|\varphi^+\varepsilon|_{L^5} \leq C\varepsilon^{\frac{2}{5}}, \quad \|\varphi_\varepsilon^+\|^2 = S^\frac{2}{5} + O(\varepsilon^2), \quad (2.19)
\]
and
\[
\left|\int_{\mathbb{R}^3} |\varphi_\varepsilon|_6^6 dx - \int_{\mathbb{R}^3} |\varphi^+\varepsilon|_6^6 dx\right| \leq C\varepsilon^{\frac{2}{5}}. \quad (2.20)
\]

**Proof.** From (1.11) and (1.12) we see that \( \Phi(x) \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \), then direct calculations show that (2.17) holds. Using (2.17) we see that \( |\varphi^-_\varepsilon|_{L^2} \leq C|\varphi_\varepsilon^+|_{L^2} \leq C\varepsilon \), this yields (2.18) by applying Lemma 2.1 (2.17) and (2.18) indicate that
\[
|\varphi^+_\varepsilon|_{L^5} = |\varphi^+_\varphi^-_\varepsilon|_{L^5} \leq C(|\varphi^+_\varepsilon|_{L^5} + |\varphi^-_\varepsilon|_{L^5}) \leq C\varepsilon^{\frac{2}{5}}.
\]

Moreover, in view of \( V \in L^\infty(\mathbb{R}^3) \) there holds that
\[
\|\varphi^+\varepsilon\|^2 = \int_{\mathbb{R}^3} |\nabla \varphi_\varepsilon|^2 + \frac{\varphi^2}_\varepsilon}{r^2} + V|\varphi_\varepsilon|^2 dx + \|\varphi^-_\varepsilon\|^2 = S^\frac{2}{5} + O(\varepsilon^2).
\]
Thus (2.19) is obtained.

To prove (2.20), we first deduce from the convexity of \(| \cdot |_{L^6}\) and Lemma 2.1 that
\[
\int_{\mathbb{R}^3} |\varphi_\varepsilon|_6^6 dx = \int_{\mathbb{R}^3} |\varphi^+_\varepsilon + \varphi^-_\varepsilon|_6^6 dx \geq \int_{\mathbb{R}^3} |\varphi^+_\varepsilon|_6^6 dx + 6 \int_{\mathbb{R}^3} (\varphi^+_\varepsilon)^5 \varphi^-_\varepsilon dx
\]
\[
\geq \int_{\mathbb{R}^3} |\varphi^+_\varepsilon|_6^6 dx - 6 \int_{\mathbb{R}^3} |\varphi^+_\varepsilon|_6^5 dx |\varphi^-_\varepsilon|_{L^\infty} \geq \int_{\mathbb{R}^3} |\varphi^+_\varepsilon|_6^6 dx - C\varepsilon^{\frac{2}{5}}, \quad (2.21)
\]
where (2.18) and (2.19) is used in the last inequality. Similarly, it follows from (2.17) and (2.18) that
\[
\int_{\mathbb{R}^3} |\varphi^+_\varepsilon|_6^6 dx = \int_{\mathbb{R}^3} |\varphi\varepsilon - \varphi^-_\varepsilon|_6^6 dx \geq \int_{\mathbb{R}^3} |\varphi_\varepsilon|_6^6 dx - C\varepsilon^{\frac{2}{5}}.
\]
This together with (2.21) yields (2.20). \( \square \)

Let
\[
M_\varepsilon := \{ u = u^- + t\varphi^+_\varepsilon : u^- \in E^- \text{ and } ||u|| < R \},
\]
Then we have

**Lemma 2.5.** (i) If \( \varepsilon > 0 \) is small enough, there exists \( R > 0 \) independent of \( \varepsilon \) such that
\[
\sup_{u \in \partial M_\varepsilon} J(u) \leq 0 \text{ and } \sup_{u \in M_\varepsilon} J(u) < \infty. \quad (2.22)
\]

(ii) There exists \( 0 < \rho < R \) such that \( J(u) \geq \alpha \) for all \( u \in E^+ \cap \partial B_\rho(0) \).

**Proof.** (i). To prove (2.22) it is sufficient to prove that there exists \( R = R(\varepsilon) > 0 \) large enough such that
\[
J(u^- + t\varphi^+_\varepsilon) \leq 0 \text{ for any } ||u^- + t\varphi^+_\varepsilon|| \geq R. \quad (2.23)
\]
We argue in two different cases. Firstly, if \( \| \tau \varphi^+ \| \leq \frac{1}{2} \| u^- \| \),

\[
J(u^- + \tau \varphi^+) \leq \frac{1}{2} \| \tau \varphi^+ \|^2 - \frac{1}{2} \| u^- \|^2 \leq \frac{1}{4} \| u^- \|^2 < 0.
\]

(2.24)

On the other hand, if \( \| \tau \varphi^+ \| > \frac{1}{2} \| u^- \| \), then

\[
R^2 \leq \| u^- + \tau \varphi^+ \|^2 = \| u^- \|^2 + \| \tau \varphi^+ \|^2 \leq 5t^2 \| \varphi^+ \|^2,
\]

together with (2.19) we see that if \( \varepsilon > 0 \) is small enough,

\[
t^2 \geq \frac{R^2}{5 \| \varphi^+ \|^2} \geq \frac{R^2}{10 \hat{S}^2}.
\]

(2.25)

Moreover, it follows from Lemma 2.1, (2.17), (2.19) and (2.20) that

\[
\int_{\mathbb{R}^3} |u^- + \tau \varphi^+|^6 dx \geq t^6 \int_{\mathbb{R}^3} |\varphi^+|^6 dx + 6 \int_{\mathbb{R}^3} (\tau \varphi^+)^5 u^- dx \\
\geq t^6 \int_{\mathbb{R}^3} |\varphi^+|^6 dx - 6 |t|^5 \| u^- \| \int_{\mathbb{R}^3} |\varphi^+|^5 dx \geq t^6 \int_{\mathbb{R}^3} |\varphi^+|^6 dx - C \varepsilon^\frac{7}{2} |t|^5 \| u^- \| \\
\geq t^6 \int_{\mathbb{R}^3} |\varphi^+|^6 dx - C \varepsilon^\frac{7}{2} t^6 \| \varphi^+ \| \geq t^6 (\hat{S}^\frac{3}{2} + O(\varepsilon^3)) - C \hat{S}^\frac{3}{2} t^\frac{7}{2} \geq \frac{\hat{S}^\frac{3}{2}}{2} t^6.
\]

(2.26)

Combine with (2.25), we see that if \( R > 0 \) is suitable large,

\[
J(u^- + \tau \varphi^+) \leq \frac{1}{2} \| \tau \varphi^+ \|^2 - \frac{1}{6} \int_{\mathbb{R}^3} |u^- + \tau \varphi^+|^6 dx \leq \hat{S}^\frac{3}{2} t^2 - \frac{\hat{S}^\frac{3}{2}}{12} t^6 < 0.
\]

This together with (2.24) gives (2.23). From (2.23) we see that \( \sup_{u \in \partial M_\varepsilon} J(u) \leq 0 \).

Moreover, since \( J \) maps bounded sets into bounded sets, therefore \( \sup_{u \in M_\varepsilon} J(u) < \infty \).

(ii) Since \( |u|_{L^q} \leq C \| u \| \) for any \( q \in [2, 6] \). For any \( u \in E^+ \cap \partial B_\rho(0) \) we have

\[
J(u) \geq \frac{1}{2} \| u \|^2 - C_1 \| u \|^p - C_2 \| u \|^6 \geq C_3 \| u \|^2 - C_4 \| u \|^6 > C_\rho > 0.
\]

\[\Box\]

Lemma 2.6. Let \( c \) be defined as in (1.8), then

\[
c < \frac{1}{3} \hat{S}^\frac{3}{2}
\]

(2.27)

if one of the following conditions holds:

(i) \( V(r, x_3) \) satisfies (V) and \( p \in (4, 6) \);

(ii) \( V(r, x_3) \in L^\infty(\mathbb{R}^3) \) satisfies \( 0 < \sigma(-\Delta + \frac{1}{r^2} + V) \) and \( p \in (2, 6) \).
Thus, \( \beta \) is small enough, then there exists subsequence, we can assume that there exists \( u_n \) such that

\[
\lim_{n \to \infty} J(u_n) = \gamma.
\]

It then follows from (2.19) that

\[
\sup_{n \in \mathbb{N}} J = \sup \{ J \} \geq \alpha > 0.
\]

Therefore, there exists \( \{u_n = t_n \varphi_\varepsilon^+ + u_n^-\} \subset M_\varepsilon \) such that \( \lim_{n \to \infty} J(u_n) = \gamma \). Up to subsequence, we can assume that there exists \( u_0 = t_0 \varphi_\varepsilon^+ + u_0^- \in M_\varepsilon \) such that

\[
u_n = t_n \varphi_\varepsilon^+ + u_n^- \xrightarrow{n} u_0 = t_0 \varphi_\varepsilon^+ + u_0^- \text{ in } E.
\]

This implies that \( t_n \xrightarrow{n} t_0 \). Moreover, since \( \psi(u) := \frac{1}{2} ||u^-||^2 + \frac{1}{6} \int \varphi_\varepsilon^+ dx + \frac{1}{2} \int |u|^p dx \)

is weakly sequentially lower semicontinuous in \( E \), we thus deduce that

\[
\gamma = \lim_{n \to \infty} J(u_n) \leq J(u_0) \leq \gamma.
\]

Thus, \( u_0 \in M_\varepsilon \) is a maximum point of \( J \) on \( \mathbb{R} \varphi_\varepsilon \oplus E^- \). This yields that \( u_0 \in \mathcal{N} \) and \( c \leq \sup_{n \in M_\varepsilon} J \). Hence, to obtain (2.27), it is sufficient to prove that

\[
\sup_{M_\varepsilon} J < \frac{1}{3} \hat{S}^2.
\]

\[ (i). \] For any \( u = u^- + t \varphi_\varepsilon^+ \in M_\varepsilon \), since \( ||u^-|| \leq R \), we deduce from (2.19) that if \( \varepsilon > 0 \) is small enough, then there exists \( \beta = \beta(R) > 0 \) such that

\[
||t|| \leq \beta \text{ and } ||u^-|| \leq \beta.
\]

It then follows from (2.19) that

\[
\int |\nabla u|^2 + \frac{|u|}{r^2} + V|u|^2 dx = ||t \varphi_\varepsilon^+||^2 - ||u^-||^2 = \hat{S}^2 t^2 + O(\varepsilon^2) - ||u^-||^2.
\]

Similar to (2.20), we deduce from Lemmas 2.1, 2.4 and (2.29) that

\[
\int |u|^6 dx \geq t^6 \int |\varphi_\varepsilon^+|^6 dx - C_1 \varepsilon^3 t^5 ||u^-|| \geq t^6 \int |\varphi_\varepsilon^+|^6 dx - C_1 \varepsilon^3 - C_1 \varepsilon^2 ||u^-||
\]

\[
= \hat{S}^3 t^6 - C_1 \varepsilon^3 - C_1 \varepsilon^2 ||u^-||,
\]

and

\[
\int |u|^p dx = \int |t \varphi_\varepsilon - \varphi_\varepsilon^+ - u^-|^p dx \geq |t|^p \int |\varphi_\varepsilon|^p dx - p \int |t \varphi_\varepsilon|^{p-1} |t \varphi_\varepsilon^+ + u^-| dx
\]

\[
\geq |t|^p \int |\varphi_\varepsilon|^p dx - C_2 (||\varphi_\varepsilon||_{L^\infty} + ||u^-||_{L^\infty}) \int |t \varphi_\varepsilon|^{p-1} dx
\]

\[
\geq |t|^p \int |\varphi_\varepsilon|^p dx - C_2 (||\varphi_\varepsilon||_{L^\infty} + ||u^-||) \int |t \varphi_\varepsilon|^{p-1} dx
\]

\[
\geq |t|^p \int |\varphi_\varepsilon|^p dx - C_2 \int |t \varphi_\varepsilon|^{p-1} dx \geq |t|^p C_2 \varepsilon^{-\frac{p}{2}+3} - C_2 \varepsilon^{-\frac{p-1}{2}+3}.
\]
Using (2.30)-(2.32) and the fact that $-\|u^--\|^2 + C_1 \epsilon \|u^-\| \leq C_3 \epsilon$, we see that

$$J(u) \leq \hat{S}\frac{t^2}{2} - \frac{6^6}{6} - |t|^p C_2 \epsilon^{-\frac{p}{2}+3} + C_2 \epsilon^{-\frac{p}{2}+3} + C_3 \epsilon. \tag{2.33}$$

If $|t| \leq \frac{1}{2}$, then $\left(\frac{t^2}{2} - \frac{6^6}{6}\right) < \frac{1}{8}$, and it is obvious to see that

$$J(u) \leq \frac{1}{8} \hat{S}\frac{t^2}{2} + o(1) < \frac{1}{4} \hat{S}\frac{t^2}{2}. \tag{2.34}$$

On the other hand, if $\frac{1}{2} < |t| \leq \eta$, we then deduce from (2.33) and the fact $\sup_{t \in \mathbb{R}} \left(\frac{t^2}{2} - \frac{6^6}{6}\right) = \frac{1}{3}$ that

$$J(u) \leq \frac{1}{3} \hat{S}\frac{t^2}{2} - \frac{1}{2p} C_2 \epsilon^{-\frac{p}{2}+3} + C_2 \epsilon^{-\frac{p}{2}+3} + C_3 \epsilon \leq \frac{1}{3} \hat{S}\frac{t^2}{2} - \frac{1}{2p} C_4 \epsilon^{-\frac{p}{2}+3}, \tag{2.35}$$

where $p \in (4,6)$ is used in the last inequality. In brief, we deduce from (2.34) and (2.35) that

$$J(u) \leq \max \left\{ \frac{1}{4} \hat{S}\frac{t^2}{2}, \frac{1}{3} \hat{S}\frac{t^2}{2} - \frac{1}{2p} O(\epsilon^{-\frac{p}{2}+3}) \right\} \text{ for all } u \in M_\epsilon.$$

Thus (2.28) holds if $\epsilon > 0$ is small.

(ii). If $0 < \sigma(-\Delta + \frac{1}{t^2} + V)$, then $E^- = \{0\}$ and thus

$$M_\epsilon = \{t \varphi_\epsilon : t \geq 0 \text{ and } \|t \varphi_\epsilon\| < R\}.$$

It then follows from Lemma 2.4 and $p \in (2,6)$ that

$$J(t \varphi_\epsilon) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla \varphi_\epsilon|^2 + \frac{1}{r^2} + V|\varphi_\epsilon|^2 \ dx - \frac{|t|^p}{p} \int_{\mathbb{R}^3} |\varphi_\epsilon|^p \ dx - \frac{1}{6} \int_{\mathbb{R}^3} |\varphi_\epsilon|^6 \ dx$$

$$= \hat{S}\frac{t^2}{2} - \frac{6^6}{6} + C_5 \epsilon^2 - \frac{|t|^p}{p} C_6 \epsilon^{-\frac{p}{2}+3} \leq \frac{1}{3} \hat{S}\frac{t^2}{2} - C_7 \epsilon^{-\frac{p}{2}+3}.$$

Then (2.28) follows by taking $\epsilon > 0$ small enough. \qed

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