PERIODIC SOLUTIONS TO NONLINEAR EULER-BERNOULLI BEAM EQUATIONS

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ABSTRACT. Bending vibrations of thin beams and plates may be described by nonlinear Euler-Bernoulli beam equations with $x$-dependent coefficients. In this paper we investigate existence of families of time-periodic solutions to such a model using Lyapunov-Schmidt reduction and a differentiable Nash-Moser iteration scheme. The results hold for all parameters $(\epsilon, \omega)$ in a Cantor set with asymptotically full measure as $\epsilon \to 0$.

1. INTRODUCTION

Consider one dimensional (1D) nonlinear Euler-Bernoulli beam equations

$$\rho(x)u_{tt} + (p(x)u_{xx})_{xx} = \epsilon f(\omega t, x, u), \quad x \in [0, \pi]$$

with respect to the pinned-pinned boundary conditions:

$$u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0,$$

where $\rho, p$ are positive coefficients, the parameter $\epsilon$ is small, and the nonlinear forcing term $f(\omega t, x, u)$ is $\frac{2\pi}{\omega}$-periodic in time, i.e., $f(\cdot, x, u)$ is $2\pi$-periodic. Obviously, $u = 0$ is not the solution of equation (1.1) if $f(\omega t, x, 0) \neq 0$.

Bending vibrations of thin beams and plates may be described by equation (1.1), which reflects the relationship between the applied load and the beam’s deflection, see [40]. The curve $u(\cdot, x)$ describes the deflection of the beam at some position $x$ in the vertical direction, $p$ is the flexural rigidity and $\rho$ is the density of the beam. Derivatives of the deflection $u$ have physical significance: $u_x$ is the slope of the beam; $-pu_{xx}$ is the bending moment of the beam and $-(pu_{xx})_x$ is the shear force of the beam. Moreover $f$ is distributed load, which may be a function of $x$, $u$ or other variables.

The free vibration of uniform and non-uniform beams attracted many investigators since Bernoulli and Euler derived the governing differential equation in the 18th century. The beams with end springs have been dealt with by many investigators. Many researchers focused on the study of the spectral problems for the following Euler-Bernoulli operators

$$\mathcal{E} u := \frac{1}{\rho}(pu'')'' + Vu,$$

see [1,2,24,35,37] and references therein. Elishakoff et al. considered apparently the first time harmonic form solution (i.e. $u(t, x) = u(x) \sin \omega t$) for linear equation of (1.1) under different boundary conditions, see [20,21]. Under linear boundary feedback control, in [25], Guo was concerned with the Riesz basis property and the stability of such one with boundary conditions

$$\begin{cases}
y(t, 0) = y_x(t, 0) = y_{xx}(t, \pi) = 0, \\
(p(x)y_{xx})_x(t, \pi) = ky(t, \pi),
\end{cases}$$

where $k \geq 0$ is a constant feedback. Despite many studies on the linear model above, the nonlinear problems are less studied due to the challenge of the invertibility of linearized Euler-Bernoulli operators with variable coefficients $\rho, p$. This paper presents the first mathematical analysis for the existence of periodic solutions to...
nonlinear equation (1.1). There are two main challenges in this work: (i) The finite differentiable regularities of the nonlinearity. Clearly, a difficulty when working with functions having only Sobolev regularity is that the Green functions will exhibit only a polynomial decay off the diagonal, and not exponential (or subexponential). A key concept that one must exploit is the interpolation/tame estimates. (ii) The “small divisors problem” caused by resonances. We give the asymptotic formulae of the eigenvalues to the Euler-Bernoulli beam’s problem (2.3). The asymptotic property of the eigenvalues for fourth-order operators on the unit interval are less investigated than for second-order ones, see also [13, 34].

Letting $t \to t/\omega$, equation (1.1) is equivalent to

$$\omega^2 \rho(x) u_{tt} + (p(x) u_{xx})_{xx} = \epsilon f(t, x, u)$$

Hence we look for $2\pi$-periodic solutions in time to (1.4). The existence problem of periodic or quasi-periodic solutions for PDEs has received considerable attention in the last twenty years. The main difficulty in finding periodic solutions of (1.4) is the so-called “small divisors problem” caused by resonances. In fact, the spectrum of

$$\mathcal{M} u := \omega^2 u_{tt} + \frac{1}{\rho} (p u_{xx})_{xx}$$

presents the following form

$$-\omega^2 l^2 + \lambda_j = -\omega^2 l^2 + j^4 + a_j^2 + b + O(1/j), \quad l \in \mathbb{Z}, \quad j \to +\infty.$$

Consequently, above spectrum approaches to zero for almost every $\omega$ under the assumption $b \neq 0$. This causes that the operator $\mathcal{M}$ cannot map, in general, a functional space into itself, but only into a large functional space with less regularity. There are two main approaches to deal with “small divisors problem”. One is the infinite-dimensional KAM (Kolmogorov-Arnold-Moser) theory to Hamiltonian PDEs, refer to Kuksin [32], Wayne [39], and recent results [6, 22, 26]. The other more direct bifurcation approach was established by Craig and Wayne [17] and improved by Bourgain [11, 12] based on Lyapunov-Schmidt reduction and a Nash-Moser iteration procedure, and recent results [8–10].

Up to now, there has been a number of work devoted to the existence of periodic solutions and quasi-periodic solutions for classical beam equation, i.e. equation (1.1) with $\rho(x) = p(x) = 1$. In [27, 33], McKenna et al. investigated the nonlinear beam equation as a model for a suspension bridge and established multiple periodic solutions when a parameter exceeds a certain eigenvalue. By means of the infinite KAM theorem, the existence and stability of small-amplitude quasi-periodic solutions of one dimensional beam equations with boundary conditions (1.2) was obtained in [14, 23, 38]. For high dimensional cases, in [19], Eliasson, Grébert and Kuksin proved that there exist many linearly stable or unstable (for $d \geq 2$) small-amplitude quasi-periodic solutions for nonlinear beam equations

$$u_{tt} + \Delta^2 u + m u + \partial_u f(x, u) = 0, \quad x \in \mathbb{T}^d,$$

where $f(x, u) = u^4 + O(u^5)$. Above proofs are carried out in analytic cases. Recently, based on a differentiable Nash-Moser type implicit function theorem, in [15], Chen et al. gave the existence of quasi-periodic in time solutions of nonlinear beam equations

$$u_{tt} + \Delta^2 u + V(x) u = \epsilon f(\omega t, x, u), \quad x \in M,$$

where $M$ is any compact Lie group or homogenous manifold with respect to a compact Lie group. The nonlinear PDEs with $x$-dependent coefficients have recently attracted considerable attention due to their widely application. In [4, 5], Barbu and Pavel considered the wave equations with $x$-dependent coefficients for the first time. Under the general boundary conditions, periodic or anti-periodic boundary conditions and Dirichle boundary conditions, Ji and Li showed the existence of periodic solutions which periodic $T$ is required to be a rational multiple $\pi$, see [28, 30]. The existence of periodic solutions with $T$ being a irrational multiple $\pi$ for the forced vibrations of a nonhomogeneous string was obtained in [3] and [16] by a Nash-Moser theorem.
1.1. **Main results.** We now state the main results of this paper. To do so, we need to make our notations and assumptions more precise.

Let $\rho, p$ satisfy

$$
\rho(x) = e^{A_{L^2}x^2 + \alpha(x)} > 0, \quad p(x) = p(0)e^{A_{L^2}x^2 + \beta(x)} > 0
$$

with $\alpha(0) + \beta(0) = \alpha(\pi) + \beta(\pi) = 0$. Without loss of generality we assume the following normalization:

$$
\int_0^{\pi} (\rho/p)^{\frac{1}{2}} \, dx = \pi.
$$

In fact, assumption (1.5) on $\rho, p$ is essential for giving the asymptotic forms of the spectrum of Euler-Bernoulli beam operator. Make the Liouville substitution

$$
x = \psi(\xi) \iff \xi = \phi(x) \quad \text{with} \quad \phi(x) := \int_0^x \zeta(s)ds, \quad \zeta(s) = (\rho/p)^{1/4},
$$

together with the unitary Barcilon-Gottlieb transformation $U : L^2((0, \pi), \rho(x)dx) \rightarrow L^2((0, \pi), d\xi)$ by

$$
x(u) \mapsto y(\xi) = (Uu)(\psi(\xi)) = q(\psi(\xi))u(\psi(\xi)) = q(\psi(\xi)), \quad \xi \in [0, \pi],
$$

where $q = p^{1/8} \rho^{3/8} > 0$. Under the conditions that $\rho, p$ satisfy special condition (1.5), in Lemma 5.1 of [1], Badanin and Korotyaev showed that the operators $E$ defined on (1.3) satisfying boundary conditions (1.2) and $\mathcal{H}$ satisfying boundary conditions (1.2) are unitarily equivalent and one has $E = U^{-1}\mathcal{H}U$, where

$$
\mathcal{H}u := u_{xxxx} + 2(p_1u_x)_x + p_2u,
$$

and $p_i, i = 1, 2$ are seen in (5.4)–(5.5) of [1]. Then, They applied the asymptotic formulae of the eigenvalues of the operators $\mathcal{H}$ to study the ones of the operators $E$, see [1].

For all $s \geq 0$, define the following Sobolev spaces $\mathcal{H}^s$ of real-valued functions by

$$
\mathcal{H}^s := \left\{ u : \mathbb{T} \rightarrow \mathcal{H}_p^2((0, \pi); \mathbb{C}), u(t, x) = \sum_{l \in \mathbb{Z}} u_l(x)e^{ilt}, u_l \in \mathcal{H}_p^2((0, \pi); \mathbb{C}), u_{-l} = u_l^*, \|u\|_s < +\infty \right\},
$$

where $u_l^*$ is the complex conjugate of $u_l$, $\|u\|_s^2 := \sum_{l \in \mathbb{Z}} \|u_l\|_{\mathcal{H}_p^2}^2 (1 + l^{2s})$ and

$$
\mathcal{H}_p^2((0, \pi); \mathbb{C}) := \left\{ u \in \mathcal{H}_p^2((0, \pi); \mathbb{C}) : u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0 \right\}.
$$

For all $s > \frac{1}{2}$, one has the classical Sobolev embedding $\mathcal{H}^s \hookrightarrow L^\infty(\mathbb{T}; \mathcal{H}_p^2((0, \pi)); \mathbb{C})$ satisfying

$$
\|u\|_{L^\infty(\mathbb{T}; \mathcal{H}_p^2((0, \pi)))} \leq C(s)\|u\|_s, \quad \forall u \in \mathcal{H}^s,
$$

and algebra property, i.e.,

$$
\|uv\|_s \leq C(s)\|u\|_s\|v\|_s, \quad \forall u, v \in \mathcal{H}^s.
$$

Throughout this paper, our purpose is to look for the solutions in $\mathcal{H}^s$ with respect to $(t, x) \in \mathbb{T} \times [0, \pi]$ and $f \in C_k$ for $k \in \mathbb{N}$ large enough, where

$$
C_k := \left\{ f \in C^1(\mathbb{T} \times [0, \pi] \times \mathbb{R}; \mathbb{R}) : (t, u) \mapsto f(t, \cdot, u) \text{ belongs to } C^k(\mathbb{T} \times \mathbb{R}; \mathcal{H}^2((0, \pi))) \right\}.
$$

**Remark 1.1.** If $f(t, x, u) = \sum_{l \in \mathbb{Z}} f_l(x, u)e^{ilt}$, then $u \mapsto f_l(\cdot, u) \in C^k(\mathbb{R}; \mathcal{H}^2((0, \pi); \mathbb{C}))$ with $f_{-l} = f^*_l$. Moreover it follows from the continuously embedding of $\mathcal{H}^2((0, \pi))$ into $C^1[0, \pi]$ that

$$
\partial_i^2 \partial_j^2 f \in C^1([-l_0, l_0] \times \mathbb{R}; \mathbb{R}), \quad \forall 0 \leq i, j \leq k, \forall f \in C_k.
$$
Denoting by
\[ V := H^2_p(0, \pi), \quad W := \left\{ w = \sum_{l \in \mathbb{Z}, l \neq 0} u_l(x)e^{ilt} \in \mathcal{H}^0 \right\}, \]
we perform the Lyapunov-Schmidt reduction subject to the following decomposition
\[ \mathcal{H}^s = (V \cap \mathcal{H}^s) \oplus (W \cap \mathcal{H}^s) = V \oplus (W \cap \mathcal{H}^s). \]
In fact, for any \( u \in \mathcal{H}^s \), one has \( u(t, x) = u_0(x) + \bar{u}(t, x) \) with \( \bar{u}(t, x) = \sum_{l \neq 0} u_l(x)e^{ilt} \). Then corresponding projectors \( \Pi_V : \mathcal{H}^s \to V \), \( \Pi_W : \mathcal{H}^s \to W \) yields that equation (1.4) is equivalent to
\[
\begin{cases}
(pv'')'' = \epsilon \Pi_V F(v + w) & (Q), \\
L_\omega w = \epsilon \Pi_W F(v + w) & (P),
\end{cases}
\]
where \( u = v + w \) with \( v \in V \), \( w \in W \), and
\[
L_\omega w := \omega^2 \rho(x)w_t + (p(x)w_{xx})_{xx}, \quad F : u \to f(t, x, u).
\]
Note that equations (Q) and (P) are called bifurcation equation and range equation, respectively. Moreover we may write \( f \) as
\[ f(t, x, u) = f_0(x, u) + \bar{f}(t, x, u), \]
where \( \bar{f}(t, x, u) = \sum_{l \neq 0} f_l(x, u)e^{ilt} \). This leads to
\[ \Pi_V F(v) = \Pi_V \bar{f}(t, x, v(x)) = \Pi_V f_0(x, v(x)) + \Pi_V \bar{f}(t, x, v(x)) = f_0(x, v(x)) \text{ for } w = 0. \]
If \( w \) tends to 0, then we simplify the (Q)-equation as
\[ (pv'')'' = \epsilon f_0(x, v), \]
which is also called the infinite-dimensional “zeroth-order bifurcation equation”, see also [3]. We need make the following hypothesis.

**Hypothesis 1.** There exists a constant \( \epsilon_0 \in (0, 1) \) small enough, such that for all \( \epsilon \in [0, \epsilon_0] \), the following system
\[
\begin{cases}
(p(x)v'')(x)'' = \epsilon f_0(x, v(x)), \\
v(0) = v(\pi) = v''(0) = v''(\pi) = 0
\end{cases}
\]
(1.9)
admits a nondegenerate solution \( \hat{v} \in H^2_p(0, \pi) \), i.e., the linearized equation
\[ (ph'')'' = \epsilon f_0'(\hat{v})h \]
(1.10)
possesses only the trivial solution \( h = 0 \) in \( H^2_p(0, \pi) \).

Let us explain the rationality of Hypothesis 1. The linearized equation (1.10) possesses only the trivial solution \( h = 0 \) in \( H^2_p(0, \pi) \) for \( \epsilon = 0 \). Hence \( \hat{v} = 0 \) is the nondegenerate solution of (1.9) with \( \epsilon = 0 \). It follows from the implicit function theorem that there exists a constant \( \epsilon_0 \in (0, 1) \) small enough, such that for all \( \epsilon \in [0, \epsilon_0] \), Hypothesis 1 is satisfied. Moreover define
\[ A_\gamma := \left\{ (\epsilon, \omega) \in (\epsilon_1, \epsilon_2) \times (\gamma, +\infty) : \frac{\epsilon}{\omega} \leq \delta_\gamma \gamma^5, |\omega l - \bar{\mu}_j| > \frac{\gamma}{l}, \forall l = 1, \cdots, N_0, \forall j \geq 1 \right\}, \]
where \( \delta_\gamma \) is given in Lemma 2.14, \( N_0 \) is seen in (2.7) and \( \bar{\lambda}_j = \bar{\mu}_j^2, j \geq 1 \) are the eigenvalues of Euler-Bernoulli beam’s problem
\[
\begin{cases}
(p(x)y'')(x)'' = \lambda p(x)y, \\
y(0) = y(\pi) = y''(0) = y''(\pi) = 0.
\end{cases}
\]
(1.11)
Let us state our main theorem as follows.
Theorem 1.2. Assume that Hypotheses 7 holds for some $\hat{\epsilon} \in [0,\epsilon_0]$. Set $\alpha(x), \beta(x) \in H^4(0, \pi), f \in C_k$ for all $k \geq s + \kappa + 3$, where

$$\kappa := 6\tau + 4\sigma + 2$$

with $\sigma = \tau(\tau - 1)/(2 - \tau)$, and fix $\tau \in (1,2), \gamma \in (0, 1)$. Provided that $\rho, p$ satisfy (1.5), if $\frac{K\epsilon}{s} - \hat{\epsilon}$ is small enough, then there exist a constant $K > 0$ depending on $\alpha, \beta, f, \epsilon_0, \hat{\epsilon}, \gamma, \gamma_0, \tau, s, \kappa$, a neighborhood $(\epsilon_1, \epsilon_2)$ of $\hat{\epsilon}$, $0 < r < 1$ and a $C^2$ map $v(\epsilon, w)$ defined on $(\epsilon_1, \epsilon_2) \times \{w \in W \cap \mathcal{H}^s : \|w\|_s < r\}$ with values in $H^2_p(0, \pi)$ satisfying

$$\|v(\epsilon, w) - v(\epsilon, 0)\|_{H^2} \leq K\|w\|_s, \quad \|v(\epsilon, 0) - \hat{v}\|_{H^2} \leq K|\epsilon - \hat{\epsilon}|,$$

a map $\tilde{w} \in C^1(A, W \cap \mathcal{H}^s)$ satisfying

$$\|\tilde{w}\|_s \leq \frac{K\epsilon}{\gamma\omega}, \quad \|\partial_\omega \tilde{w}\|_s \leq \frac{K\epsilon}{\gamma\omega}, \quad \|\partial_\tau \tilde{w}\|_s \leq \frac{K\epsilon}{\gamma\omega},$$

such that for all $(\epsilon, \omega) \in B_\gamma \subset A$,

$$\tilde{w} := v(\epsilon, \tilde{w}(\epsilon, \omega)) + \tilde{w}(\epsilon, \omega) \in H^6(0, \pi) \cap H^2_p(0, \pi), \quad \forall t \in \mathbb{T}$$

is a solution of equation (1.13), where $B_\gamma$ is defined in (2.1). Moreover the Lebesgue measures of the set $B_\gamma \subset A$ and its section $B_\gamma(\epsilon)$ satisfy

$$\text{meas}(B_\gamma(\epsilon) \cap (\omega', \omega'')) \geq (1 - K\gamma)(\omega'' - \omega'), \quad \text{meas}(B_\gamma \cap \Omega) \geq (1 - K\gamma)\text{meas}(\Omega),$$

where $\Omega := (\epsilon', \epsilon'') \times (\omega', \omega'')$ stands for a rectangle contained in $(\epsilon_1, \epsilon_2) \times (2\gamma, +\infty)$.

1.2. Plan of the paper. The rest of the paper is organized as follows. In section 2.1 we solve the $(Q)$-equation by the classical implicit function theorem under Hypothesis 1. The goal of section 2.2 is to solve the $(\bar{P})$-equation under the “first order Melnikov” non-resonance conditions including initialization, iteration and measure estimates. Section 3 is devoted to checking inversion of the linearized operators. Finally, we list the the proof of some related results for the sake of completeness in section 4.

2. PROOF OF THE MAIN RESULTS

The object of this section is to complete the proof of the main results.

2.1. Solution of the $(Q)$-equation. We will first solve the $(Q)$-equation via the classical implicit function theorem.

Lemma 2.1. Let Hypothesis 7 hold for some $\hat{\epsilon} \in [0,\epsilon_0]$. There exists a neighborhood $(\epsilon_1, \epsilon_2)$ of $\hat{\epsilon}$, and a $C^2$ map

$$v : (\epsilon_1, \epsilon_2) \times \{w \in W \cap \mathcal{H}^s : \|w\|_s < r\} \rightarrow H^2_p(0, \pi), \quad (\epsilon, w) \mapsto v(\epsilon, w; \cdot)$$

such that $v(\epsilon, w; \cdot)$ solves the $(Q)$-equation with $v(\hat{\epsilon}, 0; t) = \hat{v}(t)$ and satisfies

$$\|v(\epsilon, w; \cdot) - v(\epsilon, 0; \cdot)\|_{H^2} \leq C\|w\|_s, \quad \|v(\epsilon, 0; \cdot) - \hat{v}(\cdot)\|_{H^2} \leq C|\epsilon - \hat{\epsilon}|$$

for some constant $C > 0$.

Proof. It follows from Hypothesis 1 that the linearized operator

$$h \mapsto (ph'')'' - \hat{\epsilon}f'_0(\hat{v})h$$

is invertible on $V$. Since $f \in C_k$, Lemma 4.5 implies that the following map

$$(\epsilon, w, v) \mapsto (pv'')'' - \hat{\epsilon}f(v + w)$$

belongs to $C^2([\epsilon_1, \epsilon_2] \times (W \cap \mathcal{H}^s) \times V; V)$. Therefore, by the implicit function theorem, there is a $C^2$-path $(\epsilon, w) \mapsto v(\epsilon, w; \cdot)$ such that the conclusions of the lemma hold. □
2.2. **Solution of the (P)-equation.** By virtue of a Nash-Moser iterative theorem, the purpose of this subsection is to solve the (P)-equation, i.e.,

\[ L_\omega w = \epsilon \Pi W F(\epsilon, w), \]

where \( F(\epsilon, w) := F(v(\epsilon, w) + w) \). Let \( W = W_N \oplus W_N^1 \), where

\[
W_N := \left\{ w \in W : w = \sum_{1 \leq |l| \leq N} w_l(x)e^{ilt} \right\}, \quad W_N^1 := \left\{ w \in W : w = \sum_{|l| > N} w_l(x)e^{ilt} \right\}.
\]

Then corresponding projection operators \( P_N : W \rightarrow W_N \), \( P_N^1 : W \rightarrow W_N^1 \) satisfy

(P1) \( \| P_N u \|_{s+\vartheta} \leq N^\vartheta \| u \|_s \), \( \forall u \in H^s, \forall s, \vartheta \geq 0 \).

(P2) \( \| P_N^1 u \|_s \leq N^{-\vartheta} \| u \|_{s+\vartheta} \), \( \forall u \in H^{s+\vartheta}, \forall s, \vartheta \geq 0 \).

Moreover, if \( f \in C_k \) satisfying \( k \geq s' + 3 \) with \( s' \geq s > 1/2 \), then it follows from Lemmata 4.4, 4.5 and Lemma 2.1 that composition operator \( F \) has the following standard properties:

(U1)(Regularity.) \( F \in C^2(\mathcal{H}^s; \mathcal{H}^s) \) and \( F, D_w F, D_w^2 F \) are bounded on \( \{ \|w\|_s \leq 1 \} \), where \( D_w F \) is the Fréchet derivative of \( F \) with respect to \( w \);

(U2)(Tame.) \( \forall u, h, w \in H^s \) with \( \|u\|_s \leq 1 \),

\[
\| F(\epsilon, w) \|_s \leq C(s')(1 + \|w\|_s), \quad \| D_w F(\epsilon, w) h \|_s \leq C(s')(\|w\|_s \|h\|_s + \|h\|_s'),
\]

\[
\| D_w^2 F(\epsilon, w)[h, h] \|_s \leq C(s') \|w\|_s \|h\|_s \|h\|_s + \|h\|_s \|h\|_s'.
\]

(U3)(Taylor Tame.) \( \forall s \leq s' \leq k - 3, \forall w, h \in H^{s'} \) and \( \|w\|_s \leq 1 \), \( \|h\|_s \leq 1 \),

\[
\| F(\epsilon, w + h) - F(\epsilon, w) - D_w F(\epsilon, w)[h] \|_s \leq C \|h\|_s^2,
\]

\[
\| F(\epsilon, w + h) - F(\epsilon, w) - D_w F(\epsilon, w)[h] \|_s' \leq C(s') \|w\|_s \|h\|_s^2 + \|h\|_s \|h\|_s'.
\]

Let us define the linearized operators \( L_N(\epsilon, \omega, w) \) as

\[
L_N(\epsilon, \omega, w)[h] := -L_\omega h + \epsilon P_N \Pi W D_w F(\epsilon, w)[h], \quad \forall h \in W_N,
\]

where \( L_\omega \) is given by (1.8). Denote by \( \lambda_j(\epsilon, w) = \mu_j^2(\epsilon, w), j \in \mathbb{N}^+ \) the eigenvalues of Euler-Bernoulli beam’s problem

\[
\begin{cases}
(p(x)y'')'' - \epsilon \Pi V f'(t, x, v(\epsilon, w(t, x) + w(t, x)y = \lambda \rho(x)y, \\
y(0) = y(\pi) = y''(0) = y''(\pi) = 0,
\end{cases}
\]

where

\[
\mu_j(\epsilon, w) = \begin{cases}
\frac{i\sqrt{-\lambda_j(\epsilon, w)}}, & \text{if } \lambda_j(\epsilon, w) < 0, \\
\sqrt{\lambda_j(\epsilon, w)}, & \text{if } \lambda_j(\epsilon, w) > 0.
\end{cases}
\]

(2.4)

For fixed \( \gamma \in (0, 1), \tau \in (1, 2) \), define \( \Delta_{N, \tau}^\gamma (w) \) by

\[
\Delta_{N, \tau}^\gamma (w) := \left\{ (\epsilon, \omega) \in (\epsilon_1, \epsilon_2) \times (\gamma, +\infty) : |\omega l - \mu_j(\epsilon, w)| > \frac{\gamma}{l^{\tau}}, \right\},
\]

where

\[
|\omega l - j| > \frac{\gamma}{l^{\tau}}, \forall l = 1, 2, \cdots, N, j \geq 1 \right\}.
\]

(2.5)

Note that the non-resonance conditions in (2.5) are trivially satisfied if \( \lambda_j(\epsilon, w) < 0 \).
Lemma 2.2. Let \((\epsilon, \omega) \in \Delta_N^{\gamma, \tau}(w)\) for fixed \(\gamma \in (0, 1), \tau \in (1, 2)\). There exist \(K, K(s') > 0\) such that if
\[
\|w\|_{s+\sigma} \leq 1 \quad \text{with} \quad \sigma := \frac{\tau(\tau - 1)}{2 - \tau},
\] (2.6)
for \(\frac{\epsilon}{\gamma \omega} \leq \delta \leq \frac{\epsilon}{\tau(\sigma)}\) small enough, \(L_N(\epsilon, \omega, w)\) is invertible with
\[
\|L_N^{-1}(\epsilon, \omega, w)h\|_s \leq \frac{K}{\gamma \omega} N^{\tau - 1} |h|_s, \quad \forall s > 1/2,
\]
\[
\|L_N^{-1}(\epsilon, \omega, w)h\|_{s'} \leq \frac{K(s')}{\gamma \omega} N^{\tau - 1} (\|h\|_{s'} + \|w\|_{s'+\sigma} \|h\|_s), \quad \forall s' \geq 1/2.
\]
Note that condition \(\delta \leq \frac{\epsilon}{\tau(\sigma)}\) is to guarantee that Lemma 3.3 holds.

Set
\[
N_n := |e^{\Omega n}| \quad \text{with} \quad \epsilon = \ln N_0,
\] (2.7)
where the symbol \([\cdot]\) denotes the integer part.

Denote by \(A_0\) the open set
\[
A_0 := \left\{ (\epsilon, \omega) \in (\epsilon_1, \epsilon_2) \times (\gamma, +\infty) : |\omega l - \bar{\mu}_j| > \frac{\gamma}{l}, \quad \forall l = 1, \cdots, N_0, j \geq 1 \right\},
\] (2.8)
where \(\bar{\lambda}_j = \bar{\mu}_j; j \geq 1\) are the eigenvalues of Euler-Bernoulli beam’s problem (1.11).

Let us state the inductive theorem.

Theorem 2.3. For \(\frac{\epsilon}{\gamma \omega} \leq \delta_0\) (see Lemma 2.8) small enough, there exists a sequence of subsets \((\epsilon, \omega) \in A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1 \subset A_0\), where
\[
A_n := \left\{ (\epsilon, \omega) \in A_{n-1} : (\epsilon, \omega) \in \Delta_N^{\gamma, \tau}(w_{n-1}) \right\},
\]
and a sequence \(w_n(\epsilon, \omega) \in \mathcal{W}_{N_n}\) satisfying
(S1) \(n \geq 1\) \(\|w_n\|_{s+\sigma} \leq 1\), and \(\|\partial_\omega w_n\|_s \leq \frac{K_1}{\gamma \omega}, \|\partial_k w_n\|_s \leq \frac{K_1}{\gamma \omega}\),
(S2) \(n \geq 1\) \(\|w_k - w_{k-1}\|_s \leq \frac{K_2}{\gamma \omega} N_k^{1-\sigma}, \|\partial_\omega (w_k - w_{k-1})\|_s \leq \frac{K_2}{\gamma \omega} N_k^{1-\sigma}, \|\partial_k (w_k - w_{k-1})\|_s \leq \frac{K_2}{\gamma \omega} N_k^{1-\sigma}\), \(\forall 1 \leq k \leq n\),
(S3) \(n \geq 1\) if \((\epsilon, \omega) \in A_n\), then \(w_n(\epsilon, \omega)\) is a solution of
\[
L_\omega w - eP_{N_n} \mathcal{W}_{N_n} F(\epsilon, w) = 0,
\] (PN_n)
where \(L_\omega\) is defined in (1.3).
(S4) \(n \geq 1\) Setting \(B_n := 1 + \|w_n\|_{s+\kappa}, B_n' := 1 + \|\partial_\omega w_n\|_{s+\kappa}\) and \(B_n'' := 1 + \|\partial_k w_n\|_{s+\kappa}\), there exists some constant \(K = K(N_0)\) such that
\[
B_k \leq C_1 K N_k^{1-\sigma}, \quad B_k' \leq C_2 \frac{K}{\gamma \omega} N_k^{1-\sigma}, \quad B_k'' \leq C_3 \frac{K}{\gamma \omega} N_k^{1-\sigma}, \quad \forall 1 \leq k \leq n,
\]
where \(C_i := C_i(\epsilon, \tau, \sigma); i = 1, 2, 3,\) such that for all \((\epsilon, \omega) \in \cap_{n \geq 0} A_n\), the sequence \(\{w_n = w_n(\epsilon, \omega)\}_{n \geq 0}\) converges uniformly in s-norm to a map
\[
w \in C^1 \left( \cap_{n \geq 0} A_n \cap \{ (\epsilon, \omega) : \epsilon/\omega \leq \delta_4 \gamma^3 \} ; \mathcal{W} \cap \mathcal{H}^s \right).
\]

2.2.1. Initialization. Let us check that (S1)_0, (S3)_0 hold.

Lemma 2.4. For all \((\epsilon, \omega) \in A_0\), the operator \(\frac{1}{\rho} L_\omega\) is invertible with
\[
\| (\frac{1}{\rho} L_\omega)^{-1} h \|_s \leq \frac{K N_0^{1-\pi}}{\gamma \omega} \|h\|_s, \quad \forall s \geq 0, \forall h \in \mathcal{W}_{N_0}
\]
for some constant \(K > 0\).
Lemma 2.6. For all

Proof. It follows from Lemma 2.4 and property (P1) \( C \) for some constants

Thus \( \frac{1}{\rho} L_\omega \) is invertible on \( W_{N_0} \), satisfying the conclusion of the lemma.

Remark 2.5. In the proof of Lemma 2.4 we apply an equivalent scalar product \((\cdot, \cdot)\) on \( H^2_\rho(0, \pi) \) as follows

Thus the map \( U \) is a contraction in

\( \mathcal{B}(0, \rho_0) := \{ w \in W_{N_0} : \| w \|_s \leq \rho_0 \} \) with \( \rho_0 := \frac{\epsilon}{\gamma \omega} K_1 N_0^{\tau-1} \), where \( \epsilon \gamma^{-1} K_1 N_0^{\tau} < r \) (\( r \) is given in Lemma 2.1).

Proof. It follows from Lemma 2.4 and property (U1) that for \( \frac{\epsilon}{\gamma \omega} N_0^{\tau-1} \leq \delta_1 \) small enough

Thus the map \( \mathcal{U}_0 \) is a contraction in \( \mathcal{B}(0, \rho_0) \).

Denote by \( w_0 \) the unique solution of equation \((P_{N_0})\) in \( \mathcal{B}(0, \rho_0) \). Then, for \( \frac{\epsilon}{\gamma \omega} N_0^{\tau-1} \leq \delta_1 \) small enough, applying (P1) and Lemma 2.6 yields

Moreover let us define

It is obvious that \( \mathcal{W}_0(\epsilon, \omega, w_0) = 0 \).

By virtue of formula (2.9), for \( \frac{\epsilon}{\gamma \omega} N_0^{\tau-1} \leq \delta_1 \) small enough, we obtain

\[ D_w \mathcal{W}_0(\epsilon, \omega, w_0) = \text{Id} - \frac{1}{\rho} L_\omega^{-1} \frac{1}{\rho} P_{N_0} \Pi_W D_w \mathcal{F}(\epsilon, w_0) \]
Since \( \mathcal{U}_0(\epsilon, \omega, w_0) = 0 \) with respect to \( \omega, \epsilon \). Moreover taking the derivative of the identity \( \left( \frac{1}{\rho}L_\omega \right)\left( \frac{1}{\rho}L_\omega \right)^{-1}w = w \) with respect to \( \omega \) yields
\[
\partial_\omega \left( \frac{1}{\rho}L_\omega \right)^{-1}w = -\left( \frac{1}{\rho}L_\omega \right)^{-1} \left( \frac{2}{\rho} \partial_t \right) \left( \frac{1}{\rho}L_\omega \right)^{-1}w.
\]

Then, in view of (2.9), (P1) and Lemma 2.4, we derive
\[
\|\partial_\omega w_0\|_s \leq \frac{K_1\epsilon}{\gamma^2\omega}, \quad \|\partial_\omega w_0\|_s \leq \frac{K_1}{\gamma\omega}.
\]

Above estimates may give that for \( \frac{\rho}{\gamma^2} N_{\theta}^{\tau-1} \leq \delta_1 \) small enough,
\[
\|\partial_\omega w_0\|_{s+\kappa} \leq \tilde{K}^{-1}, \quad \|\partial_\omega w_0\|_{s+\kappa} \leq \tilde{K}(\gamma\omega)^{-1}. \tag{2.11}
\]
Thus we have properties (S1)\(0\), (S3)\(0\).

2.2.2. Iteration. Assume that we have get a solution \( w_n \in W_{N_n} \) of \((P_{N_n})\) satisfying properties (S1)\(_k\)–(S4)\(_k\) for all \( k \leq n \). Next our purpose is to look for a solution \( w_{n+1} \in W_{N_{n+1}} \) of
\[
L_\omega w - \epsilon P_{N_{n+1}} \Pi_W F(\epsilon, w) = 0 \quad (P_{N_{n+1}})
\]
with conditions (S1)\(_{n+1}\)–(S4)\(_{n+1}\).

For \( h \in W_{N_{n+1}} \), denote by
\[
w_{n+1} = w_n + h
\]
a solution of \((P_{N_{n+1}})\). It follows from the fact \( L_\omega w_n = \epsilon P_{N_n} \Pi_W F(\epsilon, w_n) \) that
\[
L_\omega(w_n + h) - \epsilon P_{N_{n+1}} \Pi_W F(\epsilon, w_n + h) = L_\omega h + L_\omega w_n - \epsilon P_{N_{n+1}} \Pi_W F(\epsilon, w_n + h)
\]
\[
= -L_{N_{n+1}}(\epsilon, \omega, w_n)h + R_n(h) + r_n,
\]
where
\[
R_n(h) := -\epsilon P_{N_{n+1}}(\Pi_W F(\epsilon, w_n + h) - \Pi_W F(\epsilon, w_n)) - \Pi_W D_\omega F(\epsilon, w_n)[h]),
\]
\[
r_n := \epsilon P_{N_n} \Pi_W F(\epsilon, w_n) - \epsilon P_{N_{n+1}} \Pi_W F(\epsilon, w_n) = -\epsilon P_{N_n} P_{N_{n+1}} \Pi_W F(\epsilon, w_n).
\]

Since \((\epsilon, \omega) \in A_{n+1} \subseteq A_n\) and \( \frac{\rho}{\gamma\omega} \leq \frac{\rho}{\gamma\omega} \leq \delta_1 N_{\theta}^{\tau-1} \leq \delta \), by means of (S1)\(_n\) and Lemma 2.2, we derive that the linearized operator \( L_{N_{n+1}}(\epsilon, \omega, w_n) \) is invertible with
\[
\|L_{N_{n+1}}^{-1}(\epsilon, \omega, w_n)h\|_s \leq \frac{K}{\gamma\omega} N_{\theta}^{\tau-1} \|h\|_s, \quad \forall s > 1/2, \tag{2.12}
\]
\[
\|L_{N_{n+1}}^{-1}(\epsilon, \omega, w_n)h\|_{s'} \leq \frac{K(s')}{\gamma\omega} N_{\theta}^{\tau-1}(\|h\|_{s'} + \|w\|_{s'+\sigma} \|h\|_s), \quad \forall s' \geq s > 1/2. \tag{2.13}
\]

Define a map
\[
\mathcal{U}_{n+1} : W_{N_{n+1}} \to W_{N_{n+1}}, \quad h \mapsto L_{N_{n+1}}^{-1}(\epsilon, \omega, w_n)(R_n(h) + r_n).
\]
Then solving \((P_{N_{n+1}})\) is reduced to find the fixed point of \( h = \mathcal{U}_{n+1}(h) \).
Lemma 2.7. For \((\epsilon, \omega) \in A_{n+1}\) and \(\frac{\epsilon}{\gamma \omega} \leq \delta_2 \leq \delta_1 N^{-\tau}_0\), there exists \(K_2 > 0\) such that the map \(U_{n+1}\) is a contraction in

\[
B(0, \rho_{n+1}) := \{ h \in W_{N_{n+1}} : \|h\|_s \leq \rho_{n+1} \} \quad \text{with} \quad \rho_{n+1} := \frac{eK_2}{\gamma \omega} N^{-\sigma}_n B_n,
\]

where \(\frac{eK_2}{\gamma \omega} < r\) (is seen in Lemma 2.7). Moreover the unique fixed point \(h_{n+1}(\epsilon, \omega)\) of \(U_{n+1}\) satisfies

\[
\|h_{n+1}\|_s \leq \frac{\epsilon}{\gamma \omega} K_2 N^{\tau-1}_{n+1} N^{-\kappa}_n B_n.
\]

Proof. Using properties (P2), (U2)–(U3) establishes

\[
\|R_n(h)\|_s \leq c \|h\|_s^2, \quad \|r_n\|_s \leq c \gamma \omega N^{-\kappa}_n B_n,
\]

where \(B_n\) is seen in (S4)_n. Based on this together with (2.12), we get

\[
\|U_{n+1}(h)\|_s \leq \frac{eK'}{\gamma \omega} N^{\tau-1}_{n+1} \|h\|_s^2 + \frac{eK'}{\gamma \omega} N^{\tau-1}_{n+1} N^{-\kappa}_n B_n
\leq \frac{eK'}{\gamma \omega} N^{\tau-1}_{n+1} \rho_{n+1}^2 + \frac{eK'}{\gamma \omega} N^{\tau-1}_{n+1} N^{-\kappa}_n B_n.
\]

Obviously, one has \(\sigma > \tau - 1\) according to the fact \(\tau \in (1, 2)\). Then using the definition of \(\rho_{n+1}\), (1.12) and (S4)_n, we derive that for \(\frac{eK'}{\gamma \omega} \leq \delta_2\) small enough,

\[
\frac{eK'}{\gamma \omega} N^{\tau-1}_{n+1} \rho_{n+1} \leq \frac{1}{2}, \quad \frac{eK'}{\gamma \omega} N^{\tau-1}_{n+1} N^{-\kappa}_n B_n \leq \rho_{n+1} \frac{1}{2},
\]

which leads to \(\|U_{n+1}(h)\|_s \leq \rho_{n+1}\). Moreover taking the derivative of \(U_{n+1}\) with respect to \(h\) yields

\[
D_h U_{n+1}(h)[w] = -\epsilon L^{-1}_0(\epsilon, \omega, w_n) P N_{n+1} (\Pi W D_w F(\epsilon, w_n + h) - \Pi W D_w F(\epsilon, w_n)) w.
\]

For \(\frac{e}{\gamma \omega} \leq \delta_3\) small enough, it follows from (U1)–(U2) and (S1)_n that

\[
\|D_h U_{n+1}(h)[w]\|_s \leq \frac{eK'}{\gamma \omega} N^{\tau-1}_{n+1} \|h\|_s \|w\|_s \leq \frac{eK'}{\gamma \omega} N^{\tau-1}_{n+1} \rho_{n+1} \|w\|_s \leq \frac{1}{2} \|w\|_s,
\]

Hence \(U_{n+1}\) is a contraction in \(B(0, \rho_{n+1})\).

Denote by \(h_{n+1}(\epsilon, \omega)\) the unique fixed point of \(U_{n+1}\). With the help of (2.14), (2.16)–(2.17), one has

\[
\|h_{n+1}\|_s \leq \frac{1}{2} \|h_{n+1}\|_s + \frac{eK'}{\gamma \omega} N^{\tau-1}_{n+1} N^{-\kappa}_n B_n,
\]

which arrives at (2.15). \(\square\)

If \(\frac{e}{\gamma \omega} \leq \delta_3\) with \(\delta_3 \leq \delta_2\) is small enough, setting \(h_0 = w_0\), applying Lemmata 2.6–2.7 yields

\[
\|w_{n+1}\|_{s+\sigma} \leq \sum_{i=0}^{n+1} \|h_i\|_{s+\sigma} \overset{(P1)}{\leq} \sum_{i=0}^{n+1} N_i^\sigma \|h_i\|_s \leq \sum_{i=1}^{n+1} N_i^\sigma \frac{eK_2}{\gamma \omega} N^{-\sigma-1}_i + N_0^\sigma \frac{eK_1}{\gamma \omega} N^{\tau-1}_0 \leq 1.
\]

Lemma 2.8. For \((\epsilon, \omega) \in A_{n+1}\) and \(\frac{\epsilon}{\gamma \omega} \leq \delta_4 \leq \delta_3\), the map \(h_{n+1}\) belongs to \(C^1(A_{n+1} \cap \{(\epsilon, \omega) : \epsilon/\omega \leq \delta_4 \gamma^3\}; W_{N_{n+1}})\) and satisfies that for some constant \(K_3 > 0\),

\[
\|\partial_\omega h_{n+1}\|_s \leq \frac{K_3 \epsilon}{\gamma^2 \omega} N^{-1}_{n+1}, \quad \|\partial_\epsilon h_{n+1}\|_s \leq \frac{K_3}{\gamma \omega} N^{-1}_{n+1}.
\]

Proof. Let us define

\[
\mathcal{U}_{n+1}(\epsilon, \omega, h) := -L_\omega (w_n + h) + \epsilon P_{N_{n+1}} \Pi W F(\epsilon, w_n + h).
\]

Lemma 2.7 shows that \(h_{n+1}(\epsilon, \omega)\) is a solution to above equation, i.e.,

\[
\mathcal{U}_{n+1}(\epsilon, \omega, h_{n+1}) = 0,
\]
which carries out
\[ D_h \omega_{n+1}(\epsilon, \omega, h_{n+1}) = \mathcal{L}_{N_{n+1}}(\epsilon, \omega, w_{n+1}) \quad \text{(2.13)} \]

By means of (2.19), the operator \( \mathcal{L}_{N_{n+1}}(\epsilon, \omega, w_{n+1}) \) is invertible with
\[ \| \mathcal{L}_{N_{n+1}}^{-1}(\epsilon, \omega, w_{n+1}) \|_s \leq \| (\mathbb{I} - D_h \mathcal{G}(h_{n+1}))^{-1} \mathcal{L}_{N_{n+1}}^{-1}(\epsilon, \omega, w_{n+1}) \|_s \leq \frac{2K}{\gamma \omega} N_{n+1}^{\tau-1}. \quad \text{(2.21)} \]

Then the implicit function theorem shows \( h_{n+1} \in C^1(A_{n+1}; W_{N_{n+1}}) \), which infers
\[ \partial_{\omega, \epsilon} \omega_{n+1}(\epsilon, \omega, h_{n+1}) + D_h \omega_{n+1}(\epsilon, \omega, h_{n+1}) \partial_{\omega, \epsilon} h_{n+1} = 0. \]

Consequently, using \( w_{n+1} = w_n + h_{n+1} \) and the fact \( L_\omega w_n = eP_{N_n} \Pi W F(\epsilon, w_n) \), we obtain
\[ \partial_{\omega} h_{n+1} = -\mathcal{L}_{N_{n+1}}^{-1}(\epsilon, \omega, w_{n+1}) \partial_{\omega} \omega_{n+1}(\epsilon, \omega, h_{n+1}), \quad \text{(2.22)} \]

where
\[ \partial_{\omega} \omega_{n+1}(\epsilon, \omega, h_{n+1}) = -2\omega \rho(x)(h_{n+1}) + eP_{N_n} \Pi W D_w F(\epsilon, w_n) \partial_{\omega} w_n + eP_{N_n} (\Pi W D_w F(\epsilon, w_{n+1}) - \Pi W D_w F(\epsilon, w_n)) \partial_{\omega} w_n, \quad \text{(2.23)} \]

Furthermore Lemma 2.1 and Lemma 4.5 imply
\[ \| \Pi W d_\epsilon F(\epsilon, w_n) \|_{s+\kappa} \leq C(\kappa) \| w_n \|_{s+\kappa} (1 + \| \partial_\epsilon w_n \|_s) + C(\kappa)(1 + \| \partial_\epsilon w_n \|_{s+\kappa}), \quad \text{(2.25)} \]
\[ \| \Pi W \partial_\epsilon F(\epsilon, w_{n+1}) - \Pi W \partial_\epsilon F(\epsilon, w_n) \|_{s+\kappa} \leq C(1 + \| \partial_\epsilon w_n \|_s) \| h_{n+1} \|_s, \quad \text{(2.26)} \]

Then, it follows from (P1), (U1)–(U2) and (S1)_n that for \( \frac{\epsilon}{\gamma \omega} \leq \delta_4 \) small enough,
\[ \| \partial_{\omega} \omega_{n+1}(\epsilon, \omega, h_{n+1}) \|_s \leq \epsilon K' \gamma^{-1} N_{n+1}^{\tau-1} N_{n+1}^{-\kappa} B_n + \epsilon K' N_{n+1}^{-\kappa} B_n', \quad \text{(2.27)} \]
\[ \| \partial_\epsilon \omega_{n+1}(\epsilon, \omega, h_{n+1}) \|_s \leq \kappa' N_{n+1}^{\tau-1} N_{n+1}^{-\kappa} B_n + \epsilon K' N_{n+1}^{-\kappa} B_n'', \quad \text{(2.28)} \]

where \( B_n, B_n', B_n'' \) are given in (S4)_n. Combining above estimates with (2.21)–(2.22), (S4)_n yields
\[ \| \partial_{\omega} h_{n+1} \|_s \leq \frac{K_3 \epsilon}{\gamma \omega} N_{n+1}^{\tau-1} N_{n+1}^{-1}, \quad \| \partial_\epsilon h_{n+1} \|_s \leq \frac{K_3}{\gamma \omega} N_{n+1}^{\tau-1}, \quad \text{(2.29)} \]

This ends the proof of the lemma. \( \square \)

Thus we complete the proof of properties (S1)_{n+1}–(S3)_{n+1}. Now we are devoted to giving the upper bounds of \( h_{n+1}, \partial_{\omega} h_{n+1} \) in (s + \kappa)-norm, i.e., that (S4)_{n+1} holds.

**Lemma 2.9.** For \( (\epsilon, \omega) \in A_{n+1} \) and \( \frac{\epsilon}{\gamma \omega} \leq \delta_4 \), the first term in (S4)_{n+1} in Theorem 2.3 holds.

**Proof.** First of all, for \( \frac{\epsilon}{\gamma \omega} \leq \delta_4 \) small enough, we claim
\[ B_{n+1} \leq (1 + N_{n+1}^{-\tau+\kappa}) B_n. \quad \text{(2.29)} \]

Moreover it follows from (2.27) that
\[ N_{n+1}^2 \leq e^{2n+2} < N_{n+2} + 1 \leq 2N_{n+2}. \]
Then (2.29) implies

\[ B_{n+1} \leq B_0 \prod_{k=1}^{n+1} (1 + N_{k+1}^{\tau-1+\sigma}) \leq B_0 \prod_{k=1}^{n+1} (1 + e^{\epsilon^2 \tau-1+\sigma}) \]

\[ \leq \prod_{k=1}^{+\infty} (1 + e^{-\epsilon^2 \tau-1+\sigma}) B_0 e^{\epsilon^2 \tau-1+\sigma} \]

\[ \leq 2^{\tau-1+\sigma} \prod_{k=1}^{+\infty} (1 + e^{-\epsilon^2 \tau-1+\sigma}) B_0 N_{n+2}^{\tau-1+\sigma}, \]

which shows the first term in (S4)_{n+1} by (2.10). Now let us prove above claim (2.29). The definition of \( B_{n+1} \) shows

\[ B_{n+1} \leq 1 + \| w_n \|_{s+\kappa} + \| h_{n+1} \|_{s+\kappa} = B_n + \| h_{n+1} \|_{s+\kappa}. \] (2.30)

This implies that we have to give the upper bound of \( \| h_{n+1} \|_{s+\kappa} \). It follows from Lemma 2.7 and (U2)–(U3) that

\[ \| r_n \| \leq C, \quad \| R_n(h_n+1) \| \leq C \rho_{n+1}^2, \quad \| r_n \|_{s+\kappa} \leq C \rho_{n+1} \| h_{n+1} \|_{s+\kappa}. \]

Hence, using the equality \( h_{n+1} = L_{N_{n+1}}^{-1}(\epsilon, \omega, w_n) (R_n(h_n+1) + r_n) \), we have

\[ \| h_{n+1} \|_{s+\kappa} \leq \frac{eK'_{(\kappa)}}{\gamma \omega} N_{n+1}^{\tau-1+\sigma} B_n + \frac{eK'_{(\kappa)}}{\gamma \omega} N_{n+1}^{\tau-1+\sigma} \rho_{n+1} \| h_{n+1} \|_{s+\kappa}. \]

According to (2.17), one has that for \( \frac{\kappa}{\gamma \omega} \leq \frac{1}{\gamma \omega} \leq \delta_4 \) small enough,

\[ \| h_{n+1} \|_{s+\kappa} \leq \frac{2eK'_{(\kappa)}}{\gamma \omega} N_{n+1}^{\tau-1+\sigma} B_n \leq N_{n+1}^{\tau-1+\sigma} B_n. \] (2.31)

Obviously, (2.29) follows directly from (2.30)–(2.31). □

Let us show the upper bound of \( L_{N_{n+1}}^{-1}(\epsilon, \omega, w_{n+1}) \| w \) (recall (2.20)) in \( (s+\kappa) \)-norm.

Lemma 2.10. For \( (\epsilon, \omega) \in A_{n+1} \) and \( \frac{\kappa}{\gamma \omega} \leq \delta_4 \), one has that for all \( w \in W_{n+1} \),

\[ \| L_{N_{n+1}}^{-1}(\epsilon, \omega, w_{n+1}) \|_{s+\kappa} \leq K_4 \frac{\kappa_1}{\gamma \omega} N_{n+1}^{\tau-1+\sigma} \| w \|_{s+\kappa} \]

\[ + \frac{K_4}{\gamma \omega} N_{n+1}^{2\tau-2} (\| w_n \|_{s+\kappa+\sigma} + \| h_{n+1} \|_{s+\kappa+\sigma}) \| w \|_{s}, \]

for some constant \( K_4 > 0 \).

Proof. Set \( \mathcal{L}(h_{n+1}) := (\text{Id} - D_h U_{n+1}(h_{n+1}))^{-1} \). It is straightforward that

\[ \mathcal{L}(h_{n+1}) = w + D_h U_{n+1}(h_{n+1}) \mathcal{L}(h_{n+1}), \quad \| \mathcal{L}(h_{n+1}) \|_{s} \leq 2 \| w \|_{s}. \] (2.19)

With the help of (2.18), (2.13) and property (U2), we derive

\[ \| D_h U_{n+1}(h_{n+1}) \|_{s+\kappa} \leq \frac{eK'_{(\kappa)}}{\gamma \omega} N_{n+1}^{\tau-1+\sigma} (\| w_n \|_{s+\kappa+\sigma} \| h_{n+1} \|_{s} + \| h_{n+1} \|_{s+\kappa}), \]

which leads to

\[ \| \mathcal{L}(h_{n+1}) \|_{s+\kappa} \leq \| w \|_{s+\kappa} + \frac{eK'_{(\kappa)}}{\gamma \omega} N_{n+1}^{\tau-1+\sigma} (\| w_n \|_{s+\kappa+\sigma} \| h_{n+1} \|_{s} + \| h_{n+1} \|_{s+\kappa}) \| w \|_{s} \]

\[ + \frac{eK'_{(\kappa)}}{\gamma \omega} N_{n+1}^{\tau-1+\sigma} \mathcal{L}(h_{n+1}) \|_{s+\kappa}. \]
For \( \frac{1}{\gamma \omega} \leq \frac{1}{\gamma \omega} \leq \delta_4 \) small enough, by means of (2.17), one has
\[
\|L(h_{n+1})\|_{s+k} \leq 2\|w\|_{s+k} + \frac{2eK''(\gamma)}{\gamma \omega}b_{n+1}^{-\frac{\gamma}{\gamma \omega}}(\|w_n\|_{s+k} + \|h_{n+1}\|_{s+k})\|w\|_s.
\]

Hence we get the conclusion of this lemma according to (2.13). \( \square \)

**Lemma 2.11.** For \((\epsilon, \omega) \in A_{n+1}\) and \(\frac{1}{\gamma \omega} \leq \delta_4\), the last two terms in \((S4)_{n+1}\) in Theorem 2.3 hold.

**Proof.** First of all, for \(\frac{1}{\gamma \omega} \leq \delta_4\) small enough, let us check
\[
B'_{n+1} \leq (1 + N_{n+1}^{\tau-1})B'_n + K'\gamma^{-1}N_{n+1}^{2\tau-\sigma}B_n, \quad B''_{n+1} \leq (1 + N_{n+1}^{\tau-1})B''_n + K'(\gamma \omega)^{-1}N_{n+1}^{2\tau-\sigma}B_n. \quad (2.32)
\]

We only show the upper bound of \(B'_n\), while the upper bound of \(B''_n\) can be proved in a similar manner as employed on the one of \(B'_n\). Denote \(\alpha_1 = \tau - 1, \alpha_2 = 2\tau + \sigma, \alpha_3 = \tau - 1 + \sigma\). The first formula above leads to
\[
B'_n \leq S_1 + S_2, \quad \text{with} \quad S_1 = B'_0 \prod_{k=1}^{n+1} (1 + N_{k}^{\alpha_1}), \quad S_2 = \sum_{k=1}^{n+1} S_{2,k},
\]

where \(S_{2,1} = \frac{K'}{\gamma}N_{n+1}^{\alpha_2}B_n\) and
\[
S_{2,k} = \frac{K'}{\gamma} \left( \prod_{j=2}^{k} \left( 1 + N_{n+1-k}^{\alpha_1} \right) \right) N_{n+1-k}^{\alpha_2}B_{n+1-k}, \quad \forall 2 \leq k \leq n+1.
\]

By proceeding as the proof of Lemma 2.9 on the upper bound on \(B_n\), we obtain
\[
S_1 \leq C(\epsilon, \tau, \sigma)B'_0N_{n+2}^{\alpha_1}.
\]

It follows from the first term in \((S4)_n\) that
\[
S_{2,1} \leq K''\gamma^{-1}B_0e^{\alpha_2e^{2\alpha_1+1}e^{3\alpha_3}} = K''\gamma^{-1}B_0e^{(\alpha_2+\alpha_3)e^{2\alpha_1+1}} \leq C'\gamma^{-1}B_0N_{n+2}^{\alpha_2+\alpha_3}.
\]

On the other hand, one has
\[
\sum_{k=2}^{n+1} S_{2,k} \leq K''\gamma^{-1}B_0 \sum_{k=2}^{n+1} e^{\alpha_2e^{2\alpha_1+1}e^{3\alpha_3}} \leq K''\gamma^{-1}B_0e^{(\alpha_2+\alpha_3)e^{2\alpha_1+1}} \leq C'\gamma^{-1}B_0N_{n+2}^{\alpha_2+\alpha_3}.
\]

Thus formulae (2.10)–(2.11) reads the upper bound of \(B''_{n+1}\).

We now are devoted to verifying (2.32). It follows from the definition of \(B'_n, B''_n\) that
\[
B'_{n+1}, B''_{n+1} \leq 1 + \|\partial_{\omega, \epsilon}w_n\|_{s+k} + \|\partial_{\omega, \epsilon}h_{n+1}\|_{s+k}.
\]

Thus we give the upper bound of \(\partial_{\omega, \epsilon}h_{n+1}\) in \((s + \kappa)\)-norm.

By formula (2.22) and Lemma 2.10, we may obtain
\[
\|\partial_{\omega, \epsilon}h_{n+1}\|_{s+k} \leq \frac{K_4}{\gamma \omega}N_{n+1}^{\tau-1}\|\partial_{\omega, \epsilon}U_{n+1}(\epsilon, \omega, h_{n+1})\|_{s+k}
\]
\[
+ \frac{K_4}{\gamma \omega}N_{n+1}^{2\tau-2}(\|w_n\|_{s+k} + \|h_{n+1}\|_{s+k})\|\partial_{\omega, \epsilon}U_{n+1}(\epsilon, \omega, h_{n+1})\|_s.
\]
Let us show the upper bound of \( \partial_\omega \mathcal{U}_{n+1}(\epsilon, \omega, h_{n+1}) \) in \((s + \kappa)\)-norm. Then, for \( \frac{\gamma}{\gamma_0} \leq \delta_4 \) small enough, applying \((U1)-(U2), (S1)_n\) and \((2.31)\) yields

\[
\| \partial_\omega \mathcal{U}_{n+1}(\epsilon, \omega, h_{n+1}) \|_{s+s+\kappa} \leq C'(\kappa)\omega N_{n+1}^{\tau+1+\sigma} B_n + c C'(\kappa) B'_n,
\]

\[
\| \partial_\omega \mathcal{U}_{n+1}(\epsilon, \omega, h_{n+1}) \|_{s+s+\kappa} \leq C'(\kappa) N_{n+1}^{\tau-1+\sigma} B_n + c C'(\kappa) B'_n.
\]

Moreover, due to \((2.27)-(2.28)\) and \((S4)_n\), we have

\[
\| \partial_\omega \mathcal{U}_n(\epsilon, \omega, h_n) \|_s \leq c C' \gamma^{-1}, \quad \| \partial_\omega \mathcal{U}_{n+1}(\epsilon, \omega, h_{n+1}) \|_{s+s+\kappa} \leq C'.
\]

Hence one has that for \( \frac{\gamma}{\gamma_0} \leq \delta_4 \) small enough,

\[
\| \partial_\omega h_{n+1} \|_{s+s+\kappa} \leq K' \gamma N_{n+1}^{2+\sigma} B_n + \frac{\epsilon K'}{\gamma} \omega N_{n+1}^{\tau-1+\sigma} B'_n, \quad \| \partial_\omega h_{n+1} \|_{s+s+\kappa} \leq K' \gamma N_{n+1}^{2+\sigma} B_n + \frac{\epsilon K'}{\gamma} \omega N_{n+1}^{\tau-1+\sigma} B'_n,
\]

which gives rise to \((2.32)\) because of \((2.33)\).

2.2.3. Whitney extension. Let us define

\[
\hat{A}_n := \left\{ (\epsilon, \omega) \in A_n, \quad \text{dist}((\epsilon, \omega), \partial A_n) > \frac{\gamma_0 \gamma^4}{N_{n+1}^{\tau+1}} \right\}, \quad (2.34)
\]

\[
\tilde{A}_n := \left\{ (\epsilon, \omega) \in A_n, \quad \text{dist}((\epsilon, \omega), \partial A_n) > \frac{2 \gamma_0 \gamma^4}{N_{n+1}^{\tau+1}} \right\} \subset \hat{A}_n, \quad (2.35)
\]

where \( A_n \) is given in Theorem 2.3.

Define a \( C^{\infty} \) cut-off function \( \varphi_n : A_0 \rightarrow [0, 1] \) as

\[
\varphi_n := \begin{cases} 1 & \text{if } (\omega, \epsilon) \in \hat{A}_n, \\
0 & \text{if } (\omega, \epsilon) \in \tilde{A}_n,
\end{cases} \quad \text{with } |\partial_{\omega,\epsilon} \varphi_n| \leq C N_{n+1}^{-1}/(\gamma_0 \gamma^4),
\]

where \( A_0 \) is defined by \((2.8)\), \( \gamma_0 \) will be given in Lemma 2.12. Then,

\[
\hat{h}_n := \varphi_n h_n \in C^1(A_0 \cap \{(\epsilon, \omega) : \epsilon/\omega \leq \delta_4 \gamma^3\}; W_{N_n}).
\]

For \( n \in \mathbb{N}^+ \), it follows from \((2.14)-(2.15), (1.12), (S4)_n\) and Lemma \(2.8\) that

\[
\| \tilde{h}_n \|_s \leq \frac{\tilde{C} \epsilon}{\gamma \omega} N_{n+1}^{-\sigma-1}, \quad \| \tilde{\omega} \tilde{h}_n \|_s \leq \frac{\tilde{C} \gamma_0}{\gamma^3 \omega} N_{n+1}^{-\sigma-1}, \quad \| \tilde{\omega} \tilde{\omega} \tilde{h}_n \|_s \leq \frac{\tilde{C} \gamma_0}{\gamma^5 \omega} N_{n+1}^{-\sigma-1}.
\]

Moreover \( \tilde{w}_n = \sum_{k=0}^{n} \tilde{h}_k \) is an extension of \( w_n \) satisfying \( \tilde{w}_n(\epsilon, \omega) = w_n(\epsilon, \omega) \) for all \( (\epsilon, \omega) \in \hat{A}_n \cap \{(\epsilon, \omega) : \epsilon/\omega \leq \delta_4 \gamma^3\} \). Then \( \tilde{w}(\epsilon, \omega) \) belongs to \( C^1(A_0 \cap \{(\epsilon, \omega) : \epsilon/\omega \leq \delta_5 \gamma^3\}; W \cap \mathcal{H}^s) \) with \( \delta_5 \leq \delta_4 \) and satisfies

\[
\| \tilde{w} \|_s \leq \frac{K \epsilon}{\gamma \omega} r, \quad \| \tilde{\omega} \tilde{w} \|_s \leq \frac{K \gamma_0}{\gamma^3 \omega}, \quad \| \tilde{\omega} \tilde{\omega} \tilde{w} \|_s \leq \frac{K \gamma_0}{\gamma^5 \omega}.
\]

Since \( N_n \leq e^{2n} < N_n + 1 < 2N_n \), formulae \((2.15)\) and \((1.12)\) give that for \( n \geq 1 \),

\[
\| \tilde{w} - \tilde{w}_n \|_s \leq \sum_{k \geq n+1} \frac{\tilde{C} \epsilon}{\gamma \omega} N_k^{-\tau-\sigma-4} \leq \sum_{k \geq n+1} \frac{\tilde{C} \epsilon}{\gamma \omega} e^{-(\tau+\sigma+2)k} \leq \frac{\tilde{C} \epsilon}{\gamma \omega} N_{n+1}^{-\tau-\sigma+2}/2.
\]

(2.36)
Proof of Lemma 2.12.

In fact, here we may take that $|\omega - \bar{\omega}|$ implies which completes the proof of the lemma. □

Lemma 2.12. If $\frac{\bar{\omega}}{\gamma} \leq \frac{\tilde{\omega}}{\gamma} \leq \delta_0 \leq \delta_5$ is small enough, we have that for some $\gamma_0 > 0$,

\[ B_\gamma = \hat{A}_n \cap A_n, \quad n \geq 0. \]

Lemma 2.13. For all $(\varepsilon, w), (\bar{\varepsilon}, \bar{w}) \in (\varepsilon_1, \varepsilon_2) \times \{ W \cap H^5 : \| w \|_s < r \}$, the eigenvalues of $(2.3)$ satisfy that for some constant $\nu > 0$,

\[ |\lambda_j(\varepsilon, w) - \lambda_j(\bar{\varepsilon}, \bar{w})| \leq \nu(|\varepsilon - \bar{\varepsilon}| + \| w - \bar{w} \|_s), \quad \forall j \geq 1. \] (2.39)

Proof. Define

\[ g(t, x) = -e^{\Pi} f'(t, x, v(t, x)) + w(t, x) \in H^2_{0, \pi} \to C^1[0, \pi]. \]

Let $\psi_j(\gamma), j \in \mathbb{N}^+$ denote the eigenfunctions with respect to $\lambda_j(\gamma)$ of problem (2.3). Since the coefficients in problem (2.3) satisfy the assumptions of [31, Theorem 4.4], then it yields

\[ D_\gamma \lambda_j(\gamma)[h] = -\int_{0}^{\pi} (\psi_j(\gamma))^2 h dx. \]

Then, one has

\[ |\lambda_j(\gamma) - \lambda_j(\tilde{\gamma})| = \left| \int_{0}^{1} \int_{0}^{\pi} (\psi_j(g + v(\tilde{g} - g))^2 (\tilde{g} - g) dx dv \right| \]

\[ \leq \max_{v \in [0, 1]} \left| \int_{0}^{\pi} (\psi_j(g + v(\tilde{g} - g))^2 (\tilde{g} - g) dx \right| \]

\[ \leq \| (g - \tilde{g}) / \rho \|_{L^\infty(\mathbb{T}, H^2_{0, \pi})} \max_{v \in [0, 1]} \left| \int_{0}^{\pi} (\psi_j(g + v(\tilde{g} - g))^2 \rho dx \right| \]

\[ \leq \| (g - \tilde{g}) / \rho \|_{H^2(0, \pi)} \leq \nu(|\varepsilon - \bar{\varepsilon}| + \| w - \bar{w} \|_s), \]

which completes the proof of the lemma.

Moreover the non-degeneracy of $\hat{v} = v(\bar{v}, 0)$ means that $\lambda_j(\bar{v}, 0) \neq 0$ for all $j \geq 1$. Then, formula (2.39) implies

\[ \nu_0 := \inf \{ |\lambda_j(\varepsilon, \omega) : j \geq 1, \varepsilon \in (\varepsilon_1, \varepsilon_2), \| w \|_s < r \} > 0. \]

In fact, here we may take that $|\varepsilon_2 - \varepsilon_1|, r$ are smaller than ones in Lemma 2.1 and use such a number $\nu_0$ for the proof of Lemma 2.12.

Proof of Lemma 2.12. Obviously, we have $\hat{A}_n \subseteq A_n, \forall n \in \mathbb{N}$. Moreover we claim that

(F1): If $\frac{\bar{\omega}}{\gamma} \leq \delta_6$ small enough, then there exists $\gamma_0 > 0$ such that for all $(\varepsilon, \omega) \in B_\gamma$,

\[ B \left( (\varepsilon, \omega), \frac{2\gamma_0^4}{N_{\gamma}^4} \right) \subseteq A_n. \]

This implies that $(\varepsilon, \omega)$ may belong to $\hat{A}_n$ for all $n \in \mathbb{N}$. 

Denote by $\lambda_j(\varepsilon, \bar{w}) = \mu_j^2(\varepsilon, \bar{w}), j \in \mathbb{N}^+$ the eigenvalues of Euler-Bernoulli beam’s problem

\[ \begin{cases} (py''')'' - e^{\Pi} f'(v(\varepsilon, \bar{w}) + \bar{w})y = \lambda_\gamma y, \\ y(0) = y(\pi) = y''(0) = y''(\pi) = 0. \end{cases} \]

Moreover let us define

\[ B_\gamma := \left\{ (\varepsilon, \omega) \in (\varepsilon_1, \varepsilon_2) \times (2\gamma, +\infty) : |\omega - \bar{\omega}| > \frac{2\gamma}{\varepsilon}, \forall l = 1, \ldots, N_0, \forall j \geq 1, \right. \]

\[ \left. \frac{\bar{\omega}}{\gamma} \leq \delta_7 \gamma^2, |\omega - j| > \frac{2\gamma}{\varepsilon}, |\omega - \mu_j(\varepsilon, \bar{w})| > \frac{2\gamma}{\varepsilon}, \forall l \geq 1, \forall j \geq 1 \right\}. \] (2.38)
Let us check claim (F1) by induction. If \( \gamma_0 \leq \frac{1}{2} \), for all \((\bar{\epsilon}, \bar{\omega}) \in B \left( (\epsilon, \omega), \frac{2\gamma_0 \gamma^4}{N_0^{\tau+1}} \right)\), one has

\[
|\bar{\omega}l - \bar{\mu}_j| \geq |\omega_l - \bar{\mu}_j| - |\omega - \bar{\omega}| l > \frac{2\gamma}{l^\tau} - \frac{2\gamma_0 \gamma^4}{N_0^{\tau+1}} l \geq \frac{\gamma}{l^\tau} + \frac{2\gamma_0 \gamma^4}{N_0^{\tau+1}} \geq \frac{\gamma}{l^\tau}, \quad \forall 1 \leq l \leq N_0,
\]

which gives rise to \((\bar{\epsilon}, \bar{\omega}) \in A_n\).

Suppose that

\[
B \left( (\epsilon, \omega), \frac{2\gamma_0 \gamma^4}{N_n^{\tau+1}} \right) \subseteq A_n.
\]

It is clear that \((\epsilon, \omega) \in \bar{A}_n\), which leads to \(\bar{w}_n(\epsilon, \omega) = w_n(\epsilon, \omega)\).

Finally, we show that claim (F1) holds at \((n+1)\)-th step. For \(\gamma_0 \leq \frac{1}{2}\), a similar argument yields that for all \((\bar{\epsilon}, \bar{\omega}) \in B \left( (\epsilon, \omega), \frac{2\gamma_0 \gamma^4}{N_{n+1}^{\tau+1}} \right)\),

\[
|\omega_l - j| \geq |\omega_l - j| - |\omega - \bar{\omega}| l > \frac{2\gamma}{l^\tau} - \frac{2\gamma_0 \gamma^4}{N_{n+1}^{\tau+1}} l \geq \frac{\gamma}{l^\tau} + \frac{2\gamma_0 \gamma^4}{N_{n+1}^{\tau+1}} \geq \frac{\gamma}{l^\tau}, \quad \forall 1 \leq l \leq N_{n+1}.
\]

For brevity, denote \(\mu_{j,n}(\bar{\epsilon}, \bar{\omega}) = \lambda_{j,n}(\bar{\epsilon}, \bar{\omega}) := \lambda_j(\epsilon, w_n(\bar{\epsilon}, \bar{\omega})), \mu_{\bar{\mu}}(\epsilon, \omega) = \tilde{\lambda}_j(\epsilon, \omega) := \lambda_j(\epsilon, \bar{w}(\epsilon, \omega))\). Then, it follows from (2.38), (S1) and (2.37) that

\[
|\lambda_{j,n}(\bar{\epsilon}, \bar{\omega}) - \tilde{\lambda}_j(\epsilon, \omega)| = \frac{|\lambda_{j,n}(\bar{\epsilon}, \bar{\omega}) - \tilde{\lambda}_j(\epsilon, \omega)|}{|\lambda_{j,n}(\bar{\epsilon}, \bar{\omega}) + |\tilde{\lambda}_j(\epsilon, \omega)|} \leq \frac{1}{\sqrt{N_0}} |\lambda_{j,n}(\bar{\epsilon}, \bar{\omega}) - \tilde{\lambda}_j(\epsilon, \omega)|
\]

\[
\leq \frac{\nu}{\sqrt{N_0}} \left( |\bar{\epsilon} - \epsilon| + \|w_n(\bar{\epsilon}, \bar{\omega}) - \bar{w}(\epsilon, \omega)\|_s \right)
\]

\[
\leq \frac{\nu}{\sqrt{N_0}} |\bar{\epsilon} - \epsilon| + \frac{\nu}{\sqrt{N_0}} \|w_n(\bar{\epsilon}, \bar{\omega}) - w_n(\bar{\epsilon}, \omega)\|_s + \frac{\nu}{\sqrt{N_0}} \|\bar{w}_n(\epsilon, \omega) - \bar{w}(\epsilon, \omega)\|_s
\]

\[
\leq \frac{\nu}{\sqrt{N_0}} \left( \frac{2\gamma_0 \gamma^4}{N_{n+1}^{\tau+1}} + \frac{2K_1 \gamma_0 \gamma^4}{N_{n+1}^{\tau+1}} + \frac{C_\tau}{\gamma_0} N_{n+1}^{-(\tau+\sigma+2)/2} \right).
\]

Since \(-(\tau + \sigma + 2)/2 \leq -\tau\), we infer that for \(\gamma_0, \frac{\nu}{\sqrt{N_0}} \) small enough,

\[
|\mu_{j,n}(\bar{\epsilon}, \bar{\omega}) - \tilde{\mu}_j(\epsilon, \omega)| \leq \frac{\gamma}{2l^\tau}.
\]

Consequently, for all \((\epsilon_1, \omega_1) \in B \left( (\epsilon, \omega), \frac{2\gamma_0 \gamma^4}{N_{n+1}^{\tau+1}} \right)\), we can obtain that for \(\gamma_0, \frac{\nu}{\sqrt{N_0}} \) small enough,

\[
|\omega_1 l - \mu_{j,n}(\bar{\epsilon}, \bar{\omega})| \geq |\omega_l - \tilde{\mu}_j(\epsilon, \omega)| - |\omega - \bar{\omega}| l - |\mu_{j,n}(\bar{\epsilon}, \bar{\omega}) - \tilde{\mu}_j(\epsilon, \omega)|
\]

\[
> \frac{2\gamma}{l^\tau} - \frac{2\gamma_0 \gamma^4}{N_{n+1}^{\tau+1}} l - \frac{\gamma}{l^\tau}, \quad \forall l = 1, \ldots, N_{n+1}.
\]

The proof is completed.

\[\square\]

Let \(\Omega := (\epsilon', \epsilon'') \times (\omega', \omega'')\) stand for a rectangle contained in \((\epsilon_1, \epsilon_2) \times (2\gamma, +\infty)\) and set

\[
\nu_1 := \inf \{ |\mu_{j+1}(\epsilon, \omega) - \mu_j(\epsilon, \omega)| : j \geq 1, \epsilon \in (\epsilon_1, \epsilon_2), \|w\|_s < r \} > 0,
\]

\[
\nu_2 := \inf \{ |\mu_{j+1}(\epsilon, \omega) - \mu_j(\epsilon, \omega)| : j \geq 1, \epsilon(\omega) \in B_\gamma \},
\]

where \(\mu_{j}(\epsilon, \omega) = \lambda_j(\epsilon, \omega) := \lambda_j(\epsilon, w(\epsilon, \omega))\). The proof of formula (2.40) will be given in the appendix. Moreover, we assume \(\omega'' - \omega' \geq 1\).
Lemma 2.14. For fixed $\epsilon \in (\epsilon', \epsilon'')$, the measure estimate on $B_\gamma(\epsilon)$ satisfies
\[
\text{meas}(B_\gamma(\epsilon) \cap (\omega', \omega'')) \geq (1 - Q_\gamma)(\omega'' - \omega')
\]
for some constant $Q > 0$, where $B_\gamma(\epsilon) := \{\omega : (\epsilon, \omega) \in B_\gamma\}$. Furthermore
\[
\text{meas}(B_\gamma \cap \Omega) \geq (1 - Q_\gamma)\text{meas}(\Omega) = (1 - Q_\gamma)(\omega'' - \omega')(\epsilon'' - \epsilon').
\]

Proof. Let $(B_\gamma(\epsilon))^c$ denote the complementary set of $B_\gamma(\epsilon)$. Using the definition of $B_\gamma$ yields
\[
(B_\gamma(\epsilon))^c \subset \Omega^1(\epsilon) \cup \Omega^2 \cup \Omega^3,
\]
where $\Omega^1(\epsilon) = \bigcup_{l \geq 1, j \geq 1} \Omega^1_{l,j}(\epsilon)$, $\Omega^2 = \bigcup_{l \geq 1, j \geq 1} \Omega^2_{l,j}$, $\Omega^3 = \bigcup_{l \geq 1, j \geq 1} \Omega^3_{l,j}$, and
\[
\Omega^1_{l,j}(\epsilon) := \left\{ \omega \in (\omega', \omega'') : |\omega l - \tilde{\mu}_j(\epsilon, \omega)| \leq \frac{2\gamma}{l^r} \right\},
\]
\[
\Omega^2_{l,j} := \left\{ \omega \in (\omega', \omega'') : |\omega l - \tilde{\mu}_j| \leq \frac{2\gamma}{l^r} \right\},
\]
\[
\Omega^3_{l,j} := \left\{ \omega \in (\omega', \omega'') : |\omega - j| \leq \frac{2\gamma}{l^r} \right\}.
\]
We now show the upper bound of $\text{meas}(\Omega^1(\epsilon))$. It follows from (2.39) and the definition of $\nu_0$ that
\[
|\tilde{\lambda}_j(\epsilon, \omega_1) - \tilde{\lambda}_j(\epsilon, \omega)| = \frac{|\tilde{\lambda}_j(\epsilon, \omega_1) - \tilde{\lambda}_j(\epsilon, \omega)|}{|\tilde{\lambda}_j(\epsilon, \omega_1)| + |\tilde{\lambda}_j(\epsilon, \omega)|} \leq \frac{1}{\nu_0} |\tilde{\lambda}_j(\epsilon, \omega_1) - \tilde{\lambda}_j(\epsilon, \omega)|
\]
\[
\leq \frac{\nu}{\sqrt{\nu_0}} ||\tilde{w}(\epsilon, \omega_1) - \tilde{w}(\epsilon, \omega)||_s \leq \frac{e\nu K(\gamma_0)}{\sqrt{\nu_0} \gamma^8 \omega} |\omega_1 - \omega|,
\]
which leads to
\[
|\partial_\omega \tilde{\mu}_j(\epsilon, \omega)| \leq \frac{e\nu K(\gamma_0)}{\sqrt{\nu_0} \gamma^8 \omega}.
\]
Let $g(\omega) := \omega l - \tilde{\mu}_j(\epsilon, \omega)$. Hence it is clear that for $\frac{\epsilon}{\gamma \omega} \leq \delta_7$ small enough,
\[
\partial_\omega g(\omega) = l - \partial_\omega \tilde{\mu}_j(\epsilon, \omega) \geq l/2, \quad \forall l \geq 1,
\]
which carries out
\[
\text{meas}(\Omega^1_{l,j}(\epsilon)) \leq \frac{|g(\omega_1) - g(\omega_2)|}{\min |\partial_\omega g(\omega)|} \leq \frac{8\gamma}{l^{r+1}}.
\]
For fixed $l$, we get
\[
\omega' l - \frac{2\gamma}{l^r} \leq \tilde{\mu}_j(\epsilon, \omega) \leq \omega'' l + \frac{2\gamma}{l^r} \quad \text{if} \quad \Omega^1_{l,j}(\epsilon) \neq \emptyset.
\]
Since $B_\gamma \subseteq \tilde{A}_n$ (recall Lemma 2.13), one has $w(\epsilon, \omega) = \tilde{w}(\epsilon, \omega)$, which leads to $\lambda_j(\epsilon, \omega) = \tilde{\lambda}_j(\epsilon, \omega)$ on $B_\gamma \cap \{(\epsilon, \omega) : \epsilon/\omega \leq \delta_7 \gamma^5\}$. Then, formula (2.36) implies $\nu_2 \geq \nu_1 > 0$ for $\frac{\epsilon}{\gamma \omega} \leq \delta_7$ small enough. Thus one has
\[
\tilde{z}_j \leq \frac{1}{\nu_1} (l(\omega'' - \omega') + \frac{4\gamma}{l^r}) + 1,
\]
where $\tilde{z}_j$ denotes the number of $j$. Consequently, we obtain
\[
\text{meas}(\Omega^1(\epsilon)) \leq \sum_{l=1}^{+\infty} \frac{8\gamma}{l^{r+1}} \left( \frac{1}{\nu_1} (l(\omega'' - \omega') + \frac{4\gamma}{l^r}) + 1 \right) \leq \sum_{l=1}^{+\infty} \frac{8\gamma}{l^{r+1}} Q'' l(\omega'' - \omega') \leq Q' \gamma(\omega'' - \omega').
\]

The argument to prove the upper bounds of $\text{meas}(\Omega^2)$ and $\text{meas}(\Omega^3)$ is analogous to the one used as above. We will sketch the proof for simplicity. This shows formula (2.41).
Moreover we get
\[
\text{meas}(B_\gamma \cap \Omega) = \int_{t'} e^{-\gamma} \text{meas}(B_\gamma(\varepsilon) \cap (\omega', \omega'')) \, dt' \geq (1 - Q_\gamma) \text{meas}(\Omega).
\]
This ends the proof of the lemma. \[\square\]

Theorem 1.2 follows from Lemma 2.1, Lemma 2.14 and Theorem 2.3.

**Proof of Theorem 1.2** With the help of Theorem 2.3 and the step of Whitney extension (recall (2.34)–(2.36)), \(\tilde{w}(\varepsilon, \omega)\) solves \((P)\)-equation in (1.7). Moreover, \(\tilde{w}(\varepsilon, \omega)\) is in \(C^1(A_\gamma; W \cap H^s)\). For \(\varepsilon, \omega \leq \delta_\gamma\) small enough, by virtue of the fact \(\|\tilde{w}\|_s < r\) (recall (2.36)), Lemma 2.1 presents that \(v(\varepsilon, w)\) solves \((Q)\)-equation in (1.7). Then, one has that
\[
\tilde{u}(\varepsilon, \omega) := v(\varepsilon, \tilde{w}(\varepsilon, \omega)) + \tilde{w}(\varepsilon, \omega) \in H^2_p(0, \pi) \oplus (W \cap H^s)
\]
is a solution of equation (1.4). Meanwhile, estimates (1.13)–(1.14) can be obtained by (2.1) and (2.36).

Since \(\tilde{u}\) solves \(-(p(x)u_{xx})_{xx} = \varepsilon f(t, x, u) - \omega^2 p(x)u_{tt},\) we obtain
\[
-(p(x)\tilde{u}_{xx})_{xx} \in H^2(0, \pi), \quad \forall t \in \mathbb{T}.
\]
Moreover, \(\alpha, \beta\) are in \(H^4(0, \pi),\) which gives \(\rho, p \in H^5(0, \pi)\) according to (1.5), then, one has \(\tilde{u}(t, x) \in H^6(0, \pi) \cap H^2_p(0, \pi) \rightarrow C^5[0, \pi]\) for all \(t \in \mathbb{T}. \quad \square\)

3. **INVERTIBILITY OF LINEARIZED OPERATORS**

Let us complete the proof of Lemma 2.1. More precisely, we have to give the invertibility of operators \(L_N(\varepsilon, \omega, w)\) (recall (2.2)), which is the core of any Nash-Moser iteration.

We rewrite \(L_N(\varepsilon, \omega, w)\) as
\[
L_N(\varepsilon, \omega, w)[h] := L_1(\varepsilon, \omega, w)[h] + L_2(\varepsilon, w)[h], \quad \forall h \in W_N,
\]
where
\[
L_1(\varepsilon, \omega, w)[h] := -L_\omega h + \epsilon P_N \Pi_W f'(t, x, v(\varepsilon, \omega, w) + w) h,
\]
\[
L_2(\varepsilon, w)[h] := \epsilon P_N \Pi_W f'(t, x, v(\varepsilon, w) + w) D_w v(\varepsilon, w)[h].
\]
Let \(b(t, x) := f'(t, x, v(\varepsilon, w, w(t, x))) + w(t, x)\). Using (4.11), \(\|w\|_{s+\sigma} \leq 1\) and Lemma 2.1 we derive
\[
\|b\|_s \leq \|b\|_{s+\sigma} \leq C, \quad \forall s > 1/2, \quad (3.1)
\]
\[
\|b\|_{s'} \leq C(s')(1 + \|w\|_{s'}), \quad \forall s' \geq s > 1/2. \quad (3.2)
\]
With the help of decomposing
\[
b(t, x) = \sum_{k \in \mathbb{Z}} b_k(x) e^{ikt}, \quad h(t, x) = \sum_{1 \leq |l| \leq N} h_l(x) e^{ilt},
\]
the operator \(L_1(\varepsilon, \omega, w)\) can be written as
\[
L_1(\varepsilon, \omega, w)[h] = \sum_{1 \leq |l| \leq N} \left( \omega^2 t^2 \rho h_l - \frac{1}{\rho} (p h_l')'' \right) e^{ilt} + \epsilon P_N \Pi_W \left( \sum_{k \in \mathbb{Z}, 1 \leq |l| \leq N} b_{k-l} h_l e^{ikt} \right)
\]
\[
= \rho L_{1,D}[h] - \rho L_{1,ND}[h],
\]
where \(b_0 = \Pi_V f'(t, x, v(\varepsilon, w) + w)\) and
\[
L_{1,D}[h] = \sum_{1 \leq |l| \leq N} \left( \omega^2 t^2 h_l - \frac{1}{\rho} (p h_l')'' + \frac{b_0}{\rho} h_l \right) e^{ilt},
\]
\[
L_{1,ND}[h] = -\frac{\epsilon}{\rho} \sum_{1 \leq |l|, |k| \leq N, l \neq k} b_{k-l} h_l e^{ikt}.
\]
Let us apply the results of [11] cf. Theorem 1.2, Proposition 6.2] to give the asymptotic formulae of the eigenvalues to problem \((2.3)\) for \(\rho, p\) satisfying \((1.5)\).

**Lemma 3.1.** Denote by \(\lambda_j(\epsilon, w)\) and \(\psi_j(\epsilon, w)\), \(j \in \mathbb{N}^+\) the eigenvalues and the eigenfunctions of problem \((2.3)\) respectively. One has

\[
\lambda_1(\epsilon, w) < \lambda_2(\epsilon, w) < \cdots < \lambda_j(\epsilon, w) < \cdots
\]

with \(\lambda_j(\epsilon, w) \to +\infty\) as \(j \to +\infty\), and for all \(\epsilon \in (\epsilon_1, \epsilon_2)\), \(w \in \{W \cap H^s : \|w\|_s < r\}\),

\[
\lambda_j(\epsilon, w) = j^4 + 2j^2v_0 + v_1(\epsilon, w) - \rho_j(\epsilon, w) + \frac{o(1)}{j} \quad \text{as} \quad j \to +\infty.
\]

Here,

\[
v_0 = \vartheta(\pi) - \vartheta(0) + \frac{1}{\pi} \int_0^\pi \tilde{f}(x) \, dx,
\]

\[
v_1(\epsilon, w) = \epsilon(\pi) - \epsilon(0) + \frac{1}{\pi} \int_0^\pi \tilde{g}(x) \zeta(\pi) \, dx + \frac{\epsilon_0^2}{2} - \frac{1}{\pi} \int_0^\pi \Pi V f'(v(\epsilon, w) + w)(x) \zeta(x) \, dx,
\]

\[
\rho_j(\epsilon, w) = \frac{1}{\pi} \int_0^\pi \left( -\epsilon \Pi V f'(v(\epsilon, w) + w)(x) \zeta(x) + \frac{\alpha''''(x) - \beta''''(x)}{4\zeta^3(x)} \right) \cos \left( 2j \int_0^x \zeta(z) \, dz \right) \, dx,
\]

where

\[
\vartheta = \frac{3\alpha + 5\beta}{2\xi}, \quad \zeta = (\rho/p)^{1/4}, \quad \tilde{f} = \frac{5\alpha^2 + 5\beta^2 + 6\alpha\beta}{4} \geq \frac{\alpha^2 + \beta^2}{2} \geq 0,
\]

\[
\epsilon = \frac{1}{\zeta^3} \left( \frac{2\beta^3}{3} - \frac{\eta_+^3}{2} - 2\eta_+ - (3 - \eta_-)\eta_+ - (3 - \eta_-)\eta_+ + (3 - \eta_-)^\prime\eta_+ - (\eta_-)^\prime - \frac{(\eta_-)^\prime}{4} \right),
\]

\[
\tilde{g} = \frac{1}{8\zeta^4} \left( ((\eta_-)^\prime - \eta_-^2 - 2\xi)^2 - 8((\eta_+ - 2\xi)^2 - 8((\eta_+)^\prime - 2\eta_+)^2) \right),
\]

\[
\zeta = \frac{\alpha + 3\beta}{2}, \quad \eta = \eta_+ \eta_-, \quad \eta_\pm = \beta \pm \alpha.
\]

And \(\psi_j(\epsilon, w)\) form an orthogonal basis of \(L^2(0, \pi)\) with the scalar product

\[(y, z)_{L^2} := \int_0^\pi pyz \, dx.\]

Moreover, define an equivalent scalar product \((\cdot, \cdot)_{\epsilon, w}\) on \(H^2_\rho(0, \pi)\) by

\[(y, z)_{\epsilon, w} := \int_0^\pi \rho y''z'' - \epsilon \Pi V f'(v(\epsilon, w) + w)yz + M \rho yz \, dt\]

with

\[
L_1 \|y\|_{H^2} \leq \|y\|_{\epsilon, w} \leq L_2 \|y\|_{H^2}, \quad \forall y \in H^2_\rho(0, \pi)
\]

for some constants \(L_1, L_2 \geq 0\). The eigenfunctions \(\psi_j(\epsilon, w)\) are also an orthogonal basis of \(H^2_\rho(0, \pi)\) with respect to the scalar product \((\cdot, \cdot)_{\epsilon, w}\) and one has that for all \(y = \sum_{j \geq 1} \hat{y}_j \psi_j(\epsilon, w)\),

\[
\|y\|^2_{L^2_\rho} = \sum_{j \geq 1} (\hat{y}_j)^2, \quad \|y\|^2_{\epsilon, w} = \sum_{j \geq 1} (\lambda_j(\epsilon, w) + M)(\hat{y}_j)^2,
\]

where \(\lambda_j(\epsilon, w) + M > 0\) for \(M > 0\) large enough.

**Proof.** Since the asymptotic formulae of the eigenvalues can be get similar by Theorem 1.2 and Proposition 6.2 of [11], we just need to check \((3.7)-(3.8)\).
Using Poincaré inequality yields \( \| y' \|_{L^2(0, \pi)} \leq C \| y'' \|_{L^2(0, \pi)} \). A simple calculation gives \((3.7)\). Moreover, one has
\[-(\psi''(e, w))'' - e \pi V f'(t, x, v(e, w) + w) \psi_j(e, w) = \lambda_j(e, w) \psi_j(e, w).\]
Multiplying above equality by \( \psi_j(e, w) \) and integrating by parts yields
\[(\psi_j, \psi_j')_{e, w} = \delta_{j, j'} \lambda_j(e, w),\]
which implies \((3.8)\) for all \( y = \sum_{j \geq 1} \hat{y}_j \psi_j(e, w) \).

Formula \((3.1)\) shows the equivalent norm of the s-norm restricted to \( W \cap H^s \), i.e.,
\[L^2_1 \| w \|^2_s \leq \sum_{|l| \geq 1} (\lambda_j(e, w) + M)(\hat{w}_{l,j})^2 (1 + l^2)^s \leq L^2_2 \| w \|^2_s \]
for all \( w = \sum_{|l| \geq 1} \hat{w}_{l,j} \psi_j(e, w) e^{ilt} \), where \( L_i, i = 1, 2 \) are given in \((3.7)\). Moreover, applying Lemma \((3.1)\) yields that for \( h_t = \sum_{j = 1}^{+\infty} \hat{h}_{l,j} \psi_j(e, w) \),
\[\omega^2 l^2 h_t - \frac{1}{\rho} (p h''_t)'' + \frac{b_n}{\rho} h_t = \sum_{j = 1}^{+\infty} (\omega^2 l^2 - \lambda_j(e, w)) \hat{h}_{l,j} \psi_j(e, w).\]
Then, \( L_{1,D} \) is a diagonal operator on \( W_N \).

Define \( L_{1,D}^{\frac{1}{2}} \) by
\[|L_{1,D}|^{\frac{1}{2}} h = \sum_{1 \leq |l| \leq N, j \geq 1} \sqrt{\omega^2 l^2 - \lambda_j(e, w)} \hat{h}_{l,j} \psi_j(e, w) e^{ilt}, \quad \forall h \in W_N.\]
If \( \omega^2 l^2 - \lambda_j(e, w) \neq 0, \forall 1 \leq |l| \leq N, \forall j \geq 1 \), its invertibility is
\[|L_{1,D}|^{-\frac{1}{2}} h := \sum_{1 \leq |l| \leq N, j \geq 1} \frac{1}{\sqrt{\omega^2 l^2 - \lambda_j(e, w)}} \hat{h}_{l,j} \psi_j(e, w) e^{ilt}.\]

Hence we can rewrite \( L_N(e, \omega, w) \) as
\[L_N(e, \omega, w) = \rho |L_{1,D}|^{\frac{1}{2}} (|L_{1,D}|^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}} - R_1 - R_2) |L_{1,D}|^{\frac{1}{2}},\]
where
\[R_1 = |L_{1,D}|^{-\frac{1}{2}} L_{1,D}^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}}, \quad R_2 = -|L_{1,D}|^{-\frac{1}{2}} (1/\rho L_2) |L_{1,D}|^{-\frac{1}{2}}.\]
Consequently, it follows from the definitions of \( L_{1,D} \), \( L_{1,D}^{\frac{1}{2}} \) that
\[(|L_{1,D}|^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}}) h = \sum_{1 \leq |l| \leq N, j \geq 1} \text{sign}(\omega^2 l^2 - \lambda_j(e, w)) \hat{h}_{l,j} \psi_j(e, w) e^{ilt}, \quad \forall h \in W_N.\]
Then, it is invertible with
\[\|(|L_{1,D}|^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}})^{-1} h\|_s \leq \frac{L_2}{L_1} h_s, \quad \forall h \in W_N, \forall s \geq 0.\]
Therefore, \( L_N(e, \omega, w) \) may be reduced to
\[L_N(e, \omega, w) = \rho |L_{1,D}|^{\frac{1}{2}} (|L_{1,D}|^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}})(\text{Id} - R) |L_{1,D}|^{\frac{1}{2}},\]
where \( R = R_1 + R_2 \) with
\[R_1 = (|L_{1,D}|^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}})^{-1} R_1, \quad R_2 = (|L_{1,D}|^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}} L_{1,D}^{\frac{1}{2}})^{-1} R_2.\]
To verify the invertibility of the operator \( \text{Id} - R \) in \((3.12)\), we have to suppose some non-resonance conditions.
For $\tau \in (1, 2)$, assume the following “Melnikov’s” non-resonance conditions:

$$|\omega l - \mu_j(\epsilon, w)| > \frac{\gamma}{l^\tau}, \quad \forall 1 \leq l \leq N, \forall j \geq 1. \quad (3.13)$$

It follows from the definition of $\mu_j(\epsilon, w)$ (recall (2.4)) that

$$|\omega^2 l^2 - \lambda_j(\epsilon, w)| = |\omega l - \mu_j(\epsilon, w)||\omega l + \mu_j(\epsilon, w)| > \frac{\gamma \omega}{l^\tau - 1}, \quad \forall 1 \leq l \leq N, \forall j \geq 1. \quad (3.14)$$

Furthermore denote

$$\omega_l := \min_{j \geq 1} |\omega^2 l^2 - \lambda_j(\epsilon, w)| = |\omega^2 l^2 - \lambda_j^*(\epsilon, w)|, \quad \forall 1 \leq |l| \leq N. \quad (3.15)$$

It is clear that $\omega_l = \omega_{-l}$ for all $1 \leq l \leq N$.

**Lemma 3.2.** Given assumption (3.13), for all $s \geq 0, h \in W_N$, the operator $|\Sigma_{1, D}|^{-\frac{1}{2}}$ is invertible with

$$|||\Sigma_{1, D}|^{-\frac{1}{2}} h||_s \leq \frac{\sqrt{2} L_2}{\sqrt{\gamma \omega L_1}} \|h\|_{s+\frac{\tau}{2}}; \quad (3.16)$$

$$|||\Sigma_{1, D}|^{-\frac{1}{2}} h||_s \leq \frac{\sqrt{2} L_2}{\sqrt{\gamma \omega L_1}} N^\frac{\tau}{2} \|h\|_s. \quad (3.17)$$

**Proof.** Since $|l|^{-1}(1 + l^{2s}) < 2(1 + |l|^{2s+\tau-1})$ for all $1 \leq |l| \leq N$, by (3.9), (3.14)–(3.15), one has

$$|||\Sigma_{1, D}|^{-\frac{1}{2}} h||_s^2 \leq \frac{1}{L_1^2} \sum_{1 \leq |l| \leq N, j \geq 1} \frac{\lambda_j(\epsilon, w) + M(h_{l,j})^2}{\omega_l}(1 + l^{2s})$$

$$\leq \frac{2}{\gamma \omega L_1} \sum_{1 \leq |l| \leq N, j \geq 1} (\lambda_j(\epsilon, w) + M(h_{l,j})^2 (1 + |l|^{2s+\tau-1})$$

$$\leq \frac{2L_2^2}{\gamma \omega L_1^2} \|h\|_{s+\frac{\tau}{2}}^2 \leq \frac{2L_2^2 N^{\tau-1}}{\gamma \omega L_1^2} \|h\|_s^2.$$

This shows the conclusion of the lemma. $\Box$

The next step is to verify the upper bounds of $\|R_i h\|_{s'}, i = 1, 2$ for all $s' \geq s > 1/2$. Moreover, for $\tau \in (1, 2)$, we also assume “Melnikov’s” non-resonance conditions:

$$|\omega l - j| > \frac{\gamma}{l^\tau}, \quad \forall 1 \leq l \leq N, \forall j \geq 1. \quad (3.18)$$

(In fact, condition (3.18) will be used for the proof of (F2).)

**Lemma 3.3.** Supposed that (3.13) and (3.18) hold, if $\|w\|_{s+\sigma} \leq 1$ with $\sigma$ being seen in (2.6), then there exists some constant $L(s') > 0$ such that for all $s' \geq s > 1/2$,

$$\|R_i h\|_{s'} \leq \frac{c L(s')}{2\gamma^3 \omega} (\|h\|_{s'} + \|w\|_{s'+\sigma} \|h\|_{s}), \quad \forall h \in W_N. \quad (3.19)$$

**Proof.** Let us first claim the following:

**(F2):** Fix $\tau \in (1, 2), \gamma \in (0, 1)$ and $\omega > \gamma$. Provided (3.13) and (3.18) hold for all $|l|, |k| \in \{1, \cdots , N\}$ with $l \neq k$, there is some constant $\tilde{L} > 0$ such that

$$\omega_{l} \omega_{k} \geq \frac{\tilde{L}^2 \gamma^6 \omega^2}{|l - k|^{2\sigma}}.$$
Using formula (3.10) and the definitions of $\mathcal{L}_{1,ND}$, $|\mathcal{L}_{1,D}|^{-\frac{1}{2}}$ yields

$$R_1h = |\mathcal{L}_{1,D}|^{-\frac{1}{2}} \mathcal{L}_{1,ND} \left( \sum_{1 \leq |l| \leq N, j \geq 1} \frac{\hat{h}_{l,j}}{\sqrt{|\omega^2l^2 - \lambda_j(\epsilon, w)|}} \psi_j(\epsilon, w)e^{i\lambda t} \right)$$

$$= -\epsilon |\mathcal{L}_{1,D}|^{-\frac{1}{2}} \left( \sum_{1 \leq |l| \leq N, l \neq k, j \geq 1} \frac{\hat{h}_{l,j}}{\sqrt{|\omega^2l^2 - \lambda_j(\epsilon, w)|}} \frac{b_{k-l}}{\rho} \psi_j(\epsilon, w)e^{i\lambda t} \right)$$

$$= -\epsilon \sum_{1 \leq |l| \leq N, l \neq k, j \geq 1} \frac{\hat{h}_{l,j}}{\sqrt{|\omega^2l^2 - \lambda_j(\epsilon, w)|}} \frac{b_{k-l}}{\rho} \psi_j(\epsilon, w).$$

This leads to

$$(R_1h)_k = -\epsilon \sum_{1 \leq |l| \leq N, l \neq k, j \geq 1} \frac{\hat{h}_{l,j}}{\sqrt{|\omega^2l^2 - \lambda_j(\epsilon, w)|}} \frac{b_{k-l}}{\rho} \psi_j(\epsilon, w).$$

Combining this with (3.7)–(3.8) and (F2), we can obtain

$$\| (R_1h)_k \|_{H^2} \leq \frac{\epsilon L_2}{L_1} \sum_{1 \leq |l| \leq N, l \neq k} \frac{1}{\sqrt{\omega l}} \| b_{k-l}/\rho \|_{H^2} \| h_l \|_{H^2}$$

$$\leq \frac{\epsilon L_2}{\gamma^3 \omega LL_1} \sum_{1 \leq |l| \leq N, l \neq k} \| b_{k-l}/\rho \|_{H^2} |k-l|^\sigma \| h_l \|_{H^2}. \quad (3.20)$$

Let us define

$$\tau(x) := \sum_{1 \leq |l| \leq N} \| b_{l}/\rho \|_{H^2} |k-l|^\sigma \| h_l \|_{H^2} e^{i\lambda t} \quad \text{with } b_0 = 0,$$

$$p(x) := \sum_{l \in \mathbb{Z}} \| b_{l}/\rho \|_{H^2} |l|^\sigma e^{i\lambda t}, \quad q(x) := \sum_{1 \leq |l| \leq N} \| h_l \|_{H^2} e^{i\lambda t}.$$ 

Clearly, $\tau = P_N(pq)$ and for all $s' \geq s > \frac{1}{2}$,

$$\|p\|_{s'} \leq \sqrt{2} \|1/\rho\|_{H^2} \|b\|_{s'+\sigma} \leq C'(s')(1 + \|w\|_{s'+\sigma}), \quad \|q\|_{s'} = \|h\|_{s'}.$$

Hence we can deduce that for $\|w\|_{s'+\sigma} \leq 1$,

$$\|R_1h\|_{s'} \leq \frac{\epsilon L_2}{\gamma^3 \omega LL_1} \|\tau\|_{s'} \leq \frac{\epsilon L_2}{\gamma^3 \omega LL_1} C(s')(\|p\|_{s'}\|q\|_s + \|p\|_s\|q\|_{s'})$$

$$\leq \frac{\epsilon L_2 C''(s')}{\gamma^3 \omega LL_1} (\|w\|_{s'+\sigma} \|h\|_s + \|h\|_{s'}).$$

Combining above inequality with (3.11) completes the proof of the lemma if we take $L(s')/2 = \frac{L_2 C''(s')}{\gamma^3 \omega LL_1}$.

**Lemma 3.4.** Under the non-resonance conditions (5.13), for $\|w\|_{s'+\sigma} \leq 1$ with $\sigma$ being seen in (2.6), we have

$$\|R_2h\|_{s'} \leq \frac{\epsilon L(s')}{2\gamma^3 \omega} (\|h\|_{s'} + \|w\|_{s'+\sigma} \|h\|_s), \quad \forall h \in W_N, \forall s' \geq s > \frac{1}{2}. \quad (3.21)$$

**Proof.** It is straightforward that $\sigma > \tau - 1$ due to the fact $\tau \in (1, 2)$. Moreover, it follows from that Lemma 2.1 that

$$D_{\omega} \psi(\epsilon, w) [\mathcal{L}_{1,D}^{-\frac{1}{2}} h] \in H^2_p(0, \pi).$$
Consequently, by virtue of (3.11) and (3.12), we have
\[
\| R_2 h \|_{s'} \leq \frac{\epsilon \sqrt{L_2}}{\sqrt{\gamma \omega L_1}} \| 1/\rho \|_{H^2} \| b(t, x) D(w, v) \| \mathcal{L}_{1, D}^{-\frac{1}{2}} h \|_{s'+\frac{1}{2}}.
\]
(3.13)
\[
\leq \frac{\epsilon \sqrt{L_2}}{\sqrt{\gamma \omega L_1}} \| 1/\rho \|_{H^2} |C'(s') (\| b \|_{s'+\sigma} \mathcal{L}_{1, D}^{-\frac{1}{2}} h \|_{s'} + \| b \|_{s'+\sigma} \| \mathcal{L}_{1, D}^{-\frac{1}{2}} h \|_{s'+\frac{1}{2}}) \|_{s'}
\]
(3.14)
\[
\leq \frac{2 \epsilon L_2 C''(s')}{\gamma \omega L_1^2} (\| w \|_{s'+\sigma} \| h \|_s + \| h \|_{s'}). \tag{3.15}
\]

Hence, using (3.11) yields the conclusion of the lemma if \( L(s')/2 = \frac{2L_2 C''(s')}{L_1^2} \).

\[\square\]

**Lemma 3.5.** Given (3.13) and (3.18), if \( \| w \|_{s+\sigma} \leq 1 \) and \( \epsilon L(s')/(\gamma^3 \omega) \leq c \) is small enough, one has that the operator \((\text{Id} - \mathcal{R})\) is invertible with
\[
\|(\text{Id} - \mathcal{R})^{-1} h\|_s \leq 2(\| h \|_s + \| w \|_{s'+\sigma} \| h \|_s), \quad \forall h \in W_N, \forall s' \geq s > 1/2. \tag{3.22}
\]

**Proof.** If \( \epsilon L(s')/(\gamma^3 \omega) \leq c \) small enough, it follows from Lemmata 3.3.4 that for \( \| w \|_{s+\sigma} \leq 1 \), one has
\[
\| \mathcal{R} h \|_s \leq \epsilon L(s')(\gamma^3 \omega)^{-1} \| h \|_s \leq 1/2 \| h \|_s \quad \text{with } L \leq L(s').
\]

Then, by Neumann series, the operator \((\text{Id} - \mathcal{R})\) is invertible in \((W_N, \| \cdot \|_s)\).

Next, let us claim the following:

(F1): If \( \| w \|_{s+\sigma} \leq 1 \), then
\[
\| \mathcal{R}^n h \|_{s'} \leq (\epsilon L(s')(\gamma^3 \omega)^{-1})^n (\| h \|_{s'} + \| w \|_{s'+\sigma} \| h \|_s), \quad \forall h \in W_N, \forall n \in \mathbb{N}^+.
\]
(3.23)

Hence, for \( \epsilon L(s')(\gamma^3 \omega)^{-1} \leq c(s') \leq c \) small enough, above inequality reads
\[
\|(\text{Id} - \mathcal{R})^{-1} h\|_{s'} = \|(\text{Id} + \sum_{n \in \mathbb{N}^+} \mathcal{R}^n) h\|_{s'} \leq \| h \|_{s'} + \sum_{n \in \mathbb{N}^+} \| \mathcal{R}^n h \|_{s'}
\]
\[
\leq \| h \|_{s'} + \sum_{n \in \mathbb{N}^+} (\epsilon L(s')(\gamma^3 \omega)^{-1})^n (\| h \|_{s'} + \| w \|_{s'+\sigma} \| h \|_s)
\]
\[
\leq 2\| h \|_{s'} + 2\| w \|_{s'+\sigma} \| h \|_s
\]
for all \( h \in W_N \) and \( s' \geq s > 1/2 \).

Let us prove (F1) by induction. Formulae (3.19) and (3.21) show that for \( n = 1 \),
\[
\| \mathcal{R} h \|_{s'} \leq \epsilon L(s')(\gamma^3 \omega)^{-1} (\| h \|_{s'} + \| w \|_{s'+\sigma} \| h \|_s).
\]
Assume that (3.23) holds for \( n = l \) with \( l \in \{1 \in \mathbb{N}^+ : l \geq 2\} \). Let us check that (3.22) holds for \( n = l + 1 \). Based on the assumption for \( n = l \), we can obtain
\[
\| \mathcal{R}^{l+1} h \|_{s'} = \| \mathcal{R}^l(\mathcal{R} h) \|_{s'} \leq (\epsilon L(s')(\gamma^3 \omega)^{-1})^l (\| \mathcal{R} h \|_{s'} + \| w \|_{s'+\sigma} \| \mathcal{R} h \|_s)
\]
\[
\leq (\epsilon L(s')(\gamma^3 \omega)^{-1})^l (\epsilon L(s')(\gamma^3 \omega)^{-1} \| h \|_{s'} + (\epsilon L(s')(\gamma^3 \omega)^{-1} \| h \|_{s'} + (\epsilon L(s')(\gamma^3 \omega)^{-1}) \| w \|_{s'+\sigma} \| h \|_s)
\]
\[
\leq (\epsilon L(s')(\gamma^3 \omega)^{-1})^{l+1} (\| h \|_{s'} + (l+1)\| w \|_{s'+\sigma} \| h \|_s),
\]
which completes the proof of (F1).

\[\square\]

Let us complete the proof of Lemma 2.1

**Proof of Lemma 2.1** It follows from formulæ (3.17), (3.22) and (3.11) that
\[
\| \mathcal{L}^{-1}_N(\epsilon, \omega, w) h \|_{s'} \leq \frac{K(s')}{\gamma \omega} N \| h \|_{s'} + \| w \|_{s'+\sigma} \| h \|_s,
\]
(3.24)
where \( K(s') := \frac{8L_3^3}{L_1} \|1/\rho\|_{H^2} \). In particular, we get that for \( \|w\|_{s+\sigma} \leq 1 \),

\[
\|L_N^{-1}(\epsilon, \omega, w)h\|_s \leq \frac{K}{\gamma \omega} N^{\tau-1} \|h\|_s.
\]

Now we are devoted to checking (F2). Since \( \alpha, \beta \) belong to \( H^4(0, \pi) \), which shows \( \rho, p \in H^5(0, \pi) \) according to (1.5). If \( \Pi V f'(t, x, v(\epsilon, w) + w) \in H^5_0(0, \pi) \), one has

\[
m(\epsilon, w) \in H^1(0, \pi),
\]

where \( m(\epsilon, w) := -\epsilon \Pi V f'(v(\epsilon, w) + w)\zeta + \frac{\alpha'''' - \beta''}{4\zeta^3} \), and \( \zeta \) is defined in (3.6). Then integrating by parts implies

\[
|\rho_j(\epsilon, w)| = \left| \frac{1}{\pi} \int_0^\pi m(\epsilon, w)(x) \cos \left( 2j \int_0^x \zeta(z)dz \right) dx \right| \leq \frac{\|m/\zeta\|_{L^1}}{j},
\]

where \( \rho_j(\epsilon, w) \) is defined in (3.5). Thus it follows from (3.4) that for \( M > 0 \) large enough,

\[
\lambda_j(\epsilon, w) = j^4 + 2j^2 \upsilon_0 + \upsilon_1(\epsilon, w) + r(\epsilon, w) \quad \text{with } |r(\epsilon, w)| \leq M \quad \text{as } j \to +\infty.
\]

Using (2.4), (3.24) and Taylor expansion yields that there exists \( J_0 > \max\{2|\upsilon_0|, 1\} > 0 \) large enough such that for all \( j > J_0 \),

\[
|\mu_j(\epsilon, w) - (j^2 + \upsilon_0)| \leq \frac{M}{j^2}
\]

for \( M > 0 \) large enough. Moreover if \( \omega_2^j l^2 - \lambda_{J_0 + 1}(\epsilon, w) > 0 \), then \( j^* \geq J_0 + 1 \), where \( j^* \) is seen in (3.15). Hence there exists \( J_1 := J_1(J_0) > 0 \) such that for every \( l > J_1/\omega \),

\[
j^* \geq M\sqrt{\omega l}.
\]

Proof of (F2). Let \( l, k \geq 1 \) with \( l \neq k \) and set \( \omega_l = \min_{j \geq 1} |\omega_2^j l^2 - \lambda_j(\epsilon, w)| = |\omega_2^2 l^2 - \lambda_j^*(\epsilon, w)| \) and \( \omega_k = |\omega_2^k l^2 - \lambda_j^*(\epsilon, w)| \).

Case 1: \( 2|k - l| > (\max \{k, l\})^\varsigma \), where \( \varsigma = (2 - \tau)/\tau \in (0, 1) \) by \( \tau \in (1, 2) \). It follows form (3.14) that

\[
\omega_l \omega_k > \frac{(\gamma_\omega)^2}{(kl)^{\tau-1}} \geq \frac{(\gamma_\omega)^2}{\left(\max \{k, l\}\right)^{2(\tau-1)}} > \frac{(\gamma_\omega)^2}{2^{2(\tau-1)/\varsigma}|k - l|^{2(\tau-1)/\varsigma}}.
\]

Case 2: \( 0 < 2|k - l| \leq (\max \{k, l\})^\varsigma \). Then either \( k > l \) or \( l > k \) holds. Using the fact \( \varsigma \in (0, 1) \), in the first case \( 2l > k \) and in the latter \( 2k > l \), i.e.,

\[
k/2 < l < 2k.
\]

(i) If \( \lambda_j^*(\epsilon, w), \lambda_l^*(\epsilon, w) < 0 \), then \( \omega_l \geq \omega_2^2 l^2, \omega_k \geq \omega_2^2 k^2 \), which leads to

\[
\omega_l \omega_k \geq \omega^2 > \gamma^2 \omega^2.
\]

(ii) We consider either \( \lambda_j^*(\epsilon, w) < 0 \) or \( \lambda_l^*(\epsilon, w) < 0 \). Then in the first case

\[
\omega_l \omega_k \geq \omega_2^2 l^2 \frac{\gamma_\omega}{k^{\tau-1}} > 2^{1-\tau} \gamma_\omega^3 > 2^{1-\tau} \gamma^2 \omega^2,
\]

and in the latter

\[
\omega_l \omega_k \geq \omega_2^2 k^2 \frac{\gamma_\omega}{l^{\tau-1}} > 2^{1-\tau} \gamma_\omega^3 > 2^{1-\tau} \gamma^2 \omega^2.
\]
(iii) We restrict our attention to the case \( \lambda_j^* (\epsilon, w) > 0, \lambda_i^* (\epsilon, w) > 0 \). Suppose \( \max \{ k, l \} = k \geq k_* \) with \( k_* := \max \left\{ \frac{2J_0}{\omega}, \left( \frac{6M}{M^2\omega} \right)^{\frac{1}{2}} \right\} \). It follows from (3.18), (3.25)-(3.26) and \( \tau \in (1, 2) \) that

\[
\left| \omega (l - k) - \left( (\tau^*)^2 - (j^*)^2 \right) \right| \geq \frac{\gamma}{|l - k|^2} - \frac{M}{M^2\omega l} - \frac{M}{M^2\omega k} > \frac{2\tau^*}{k^*} - \frac{3M}{M^2\omega k} > \frac{\gamma}{2k^*} + \frac{\gamma}{k^*} - \frac{3M}{M^2\omega k} > \frac{1}{2} \left( \frac{\gamma}{k^*} + \frac{\gamma}{l^*} \right),
\]

which implies

\[
\text{either } |\omega k - \mu_j^* (\epsilon, w)| > \frac{\gamma}{2k^*} \quad \text{or } |\omega l - \mu_i^* (\epsilon, w)| > \frac{\gamma}{2k^*}\]

holds. The same conclusion is reached if \( \max \{ k, l \} = l \geq k_* \). Without loss of generality, we suppose \( |\omega k - \mu_i^* (\epsilon, w)| > \frac{\gamma}{2k^*} \). Then

\[
\omega_k = |\omega^2 k^2 - \lambda_i^* (\epsilon, w)| = |\omega k - \mu_i^* (\epsilon, w)||\omega k + \mu_i^* (\epsilon, w)| > \frac{\gamma^2 \omega}{2} k^{1-\varsigma},
\]

which shows

\[
\omega_k \geq \frac{\gamma \omega}{l^*} k^{1-\varsigma} > \frac{(\gamma \omega)^2}{2\tau} k^{2-\tau-\varsigma} = \frac{(\gamma \omega)^2}{2\tau} \quad \text{for } l < 2k;
\]

where \( \varsigma \) is taken as \( (2 - \tau) / \tau \) to guarantee \( 2 - \tau - \varsigma \tau = 0 \).

(iv) Let \( \max \{ j, k \} \leq j_* \). If \( j_* = \frac{2J_0}{\omega} \),

\[
\omega_j \omega_k \geq \frac{(\gamma \omega)^2}{(j^*)^2 (\tau - 1)} > \frac{(\gamma \omega)^2}{(2J_1/\omega)^2 (\tau - 1)} \quad \text{for } j_*>\gamma, \gamma \in (0, 1), \tau \in (1, 2), \quad \gamma^2 \omega^2 > \frac{(2J_1)^2 (\tau - 1)}{(2J_1/\omega)^2 (\tau - 1)}.
\]

On the other hand, for \( j_* = \left( \frac{6M}{M^2\omega} \right)^{\frac{1}{2}} \), one has

\[
\omega_j \omega_k \geq \frac{(\gamma \omega)^2}{(k^*)^2 (\tau - 1)} > \frac{(\gamma \omega)^2}{(6M)^2 (\tau - 1)} = \gamma^2 \omega^2 \left( \frac{\gamma \omega M^2}{6M} \right)^{\frac{1}{2}} > \frac{\gamma^6 \omega^2}{(6M)^2 M^4}.
\]

Since \( \omega_l = \omega_{-l} \), the remainder of the lemma may be proved in the similar way as above with \( l \geq 1, k \leq -1 \), or \( l \leq -1, k \geq 1 \), or \( l, k \leq -1 \). Thus this ends the proof of (F2).

4. APPENDIX

Similar to the proof of Lemmata 2.1–2.3 in [7], we may get the following Lemmata 4.1–4.3.

**Lemma 4.1** (Moser-Nirenberg). \( \forall u_1, u_2 \in \mathcal{H}^s \cap \mathcal{H}^s \) with \( s' \geq 0 \) and \( s > \frac{1}{2} \), one has

\[
\|u_1 u_2\|_{s'} \leq C(s') \left( \|u_1\|_{L^\infty(\mathcal{T}, H^2(0, \pi))} \|u_2\|_{s'} + \|u_1\|_{s'} \|u_2\|_{L^\infty(\mathcal{T}, H^2(0, \pi))} \right)
\]

\[
\leq C(s') \left( \|u_1\|_{s'} \|u_2\|_{s'} + \|u_1\|_{s'} \|u_2\|_{s} \right).
\]

**Lemma 4.2** (Logarithmic convexity). Setting \( 0 \leq a' \leq a \leq b \leq b' \) with \( a + b = a' + b' \), one has

\[
\|u_1\|_{a} \|u_2\|_{b} \leq \Theta \|u_1\|_{a'} \|u_2\|_{b'} + (1 - \Theta) \|u_2\|_{a'} \|u_1\|_{b'}, \quad \forall u_1, u_2 \in \mathcal{H}^{b'},
\]

where \( \Theta = \frac{b' - a}{b' - a} \). Particularly, this holds:

\[
\|u\|_{a} \|u\|_{b} \leq \|u\|_{a'} \|u\|_{b'}, \quad \forall u \in \mathcal{H}^{b'}.
\]

(4.3)
Denote by $\mathcal{C}_k$ the following space composed by the time-independent functions:

$$
\mathcal{C}_k := \left\{ f \in C^1([0, \pi] \times \mathbb{R}; \mathbb{R}) : u \mapsto f(\cdot, u) \text{ is in } C^k(\mathbb{R}; H^2(0, \pi)) \right\}.
$$

**Lemma 4.3.** If $f \in \mathcal{C}_1$, then the composition operator $u(x) \mapsto f(x, u(x))$ belongs to $C(H^2(0, \pi); H^2(0, \pi))$ with

$$
\|f(x, u(x))\|_{H^2} \leq C \left( \max_{u \in [-1, 1]} \|f(\cdot, u)\|_{H^2} + \max_{u \in [-1, 1]} \|\partial_u f(\cdot, u)\|_{H^2} \|u\|_{H^2} \right),
$$

where $\mathcal{U} := \|u\|_{L^\infty(0, \pi)}$. In particular, one has

$$
\|f(x, 0)\|_{H^2} \leq C.
$$

With the help of Lemmata 4.1–4.3, we can obtain the following lemma.

**Lemma 4.4.** Let $f \in \mathcal{C}_k$ with $k \geq 1$. Then, for all $s > \frac{1}{2}$, $0 \leq s' \leq k - 1$, the composition operator $u(t, x) \mapsto f(t, x, u(t, x))$ is in $C(\mathcal{H}^s \cap \mathcal{H}^{s'}; \mathcal{H}^{s'})$ with

$$
\|f(t, x, u)\|_{s'} \leq C(s', \|u\|_s)(1 + \|u\|_{s'}).
$$

**Proof.** For all $s' = q \in \mathbb{N}$ with $q \leq k - 1$, we show that

$$
\|f(t, x, u)\|_q \leq C(q, \|u\|_s)(1 + \|u\|_q), \quad \forall u \in \mathcal{H}^s \cap \mathcal{H}^q,
$$

and that

$$
f(t, x, u_n) \to f(t, x, u) \quad \text{as } u_n \to u \text{ in } \mathcal{H}^s \cap \mathcal{H}^q.
$$

Let us check $\mathcal{L}5(4.7)$ by a recursive argument. For $q = 0$ ($k = 1$), using $\mathcal{L}6$ yields

$$
\|f(t, x, u)\|_0 \leq C \max_{t \in \mathbb{T}} \|f(t, \cdot, u(t, \cdot))\|_{H^2(0, \pi)} \leq C' \left( 1 + \max_{t \in \mathbb{T}} \|u(t, \cdot)\|_{H^2(0, \pi)} \right)
$$

and that

$$
\|\partial_t f(t, x, u)\| \leq C(\|u\|_s), \quad \max_{t \in \mathbb{T}} \|\partial_u f(t, \cdot, u(t, \cdot))\|_{H^2(0, \pi)} \leq C(\|u\|_s).
$$

Moreover, a similar argument as $\mathcal{L}7$ can yield that for $k \geq 2$,

$$
\|\partial_t f(t, x, u)\|_0 \leq C(\|u\|_s), \quad \max_{t \in \mathbb{T}} \|\partial_u f(t, \cdot, u(t, \cdot))\|_{H^2(0, \pi)} \leq C(\|u\|_s).
$$

Hence, according to the continuity property in Lemma 4.3 and the compactness of $\mathbb{T}$, we derive

$$
\|f(t, x, u_n) - f(t, x, u)\|_0 \leq C \max_{t \in \mathbb{T}} \|f(t, \cdot, u_n(t, \cdot)) - f(t, \cdot, u(t, \cdot))\|_{H^2(0, \pi)} \to 0
$$

as $u_n \to u$ in $\mathcal{H}^s \cap \mathcal{H}^0$.

Suppose that $\mathcal{L}5(4.7)$ holds for $q = \ell$ with $\ell \in \mathbb{N}^+$. Let us show that it holds for $q = \ell + 1$ with $\ell + 1 \leq k - 1$.

Since $\partial_t f, \partial_u f \in \mathcal{C}_{k-1}$, above assumption yields that for all $u \in \mathcal{H}^s \cap \mathcal{H}^{s+1}$,

$$
\|\partial_t f(t, x, u)\| \leq C(\|u\|_s)(1 + \|u\|_\ell), \quad \|\partial_u f(t, x, u)\| \leq C(\|u\|_s)(1 + \|u\|_\ell).
$$

Setting $\mathcal{L}8(4.9)$, we write $h$ as $h(t, x) = \sum_{t \in \mathbb{Z}} h_t(x)e^{i\ell t}$. Clearly, one has $h(t, x) = \sum_{t \in \mathbb{Z}} \frac{i}{\ell} q_t(x)e^{i\ell t}$. Thus, we have

$$
\|h(t, x)\|_{\ell+1}^2 = \sum_{t \in \mathbb{Z}} \|h_t\|_{H^2}^2 = \sum_{t \in \mathbb{Z}} \|h_t\|_{H^2}^2 + \|\partial_t f(t, x, u)\|_\ell^2
$$

This gives rise to

$$
\|f(t, x, u)\|_{\ell+1} \leq \|f(t, x, u)\|_0 + \|\partial_t f(t, x, u)\|_\ell + \|\partial_u f(t, x, u)\|_\ell,
$$

(4.10)
which leads to that for $q = 1$ ($\xi = 0$),
\[
\|f(t, x, u)\|_1 \leq \|f(t, x, u)\|_0 + \|\partial_t f(t, x, u)\|_0 + C \max_{t \in [0, T]} \|\partial_u f(t, \cdot, u(t, \cdot))\|_{H^2(0, \pi)} \|\partial_x u\|_0
\]
\[
\leq 2C(\|u\|_s) + C'(\|u\|_s)\|u\|_1 \leq C(1, \|u\|_s)(1 + \|u\|_1)
\]
because of (4.8), where $C(1, \|u\|_s) := \max\{2C(\|u\|_s), C'(\|u\|_s)\}$. Obviously, one has
\[
\begin{cases}
   s_1 < \xi < s_1 + 1 < \xi + 1, & \xi = 1, \\
   s_1 < s_1 + 1 < \xi < \xi + 1, & \forall \xi \geq 2,
\end{cases}
\]
where $s_1 \in (1/2, \min(1, s))$. Combining this with (4.3), we can obtain
\[
\|u\|_t \|u\|_{s_1 + 1} \leq \|u\|_{t+1} \|u\|_{s_1} \leq \|u\|_{t+1} \|u\|_s.
\]
Thus it follows from (4.7)-(4.10), (4.1) that
\[
\|f(t, x, u)\|_{t+1} \leq C(\|u\|_s) + C(\xi, \|u\|_s)(1 + \|u\|_\xi) + C(\xi)\|\partial_u f(t, x, u)\|_\xi \|\partial_t u\|_{L^\infty(T, H^2(0, \pi))}
\]
\[
\quad + C(\xi)\|\partial_u f(t, x, u)\|_{L^\infty(T, H^2(0, \pi))} \|u\|_{t+1}
\]
\[
\leq C(\|u\|_s) + C(\xi, \|u\|_s)(1 + \|u\|_\xi) + C(\xi)C(\|u\|_s)(1 + \|u\|_\xi)\|u\|_{s_1+1}
\]
\[
\leq C(\xi + 1, \|u\|_s)(1 + \|u\|_{t+1}),
\]
where $C(\xi + 1, \|u\|_s) = 4 \max\{C(\|u\|_s), C(\xi, \|u\|_s), C(\xi)C(\|u\|_s)(1 + \|u\|_s), C(\xi)C(\|u\|_s)\}$.

Finally, we assume that (4.6) holds for $q = \xi$. Using the inequality (4.10) yields that (4.6) also holds for $p = \xi + 1$ with $\xi + 1 \leq k - 1$.

When $s'$ is not an integer, we can obtain the result by the Fourier dyadic decomposition. The argument is similar to the proof of the Lemma A.1 in [18].

**Lemma 4.5.** Let us define a map $F$ as
\[
F : \mathcal{H}^s \cap \mathcal{H}^{s'} \to \mathcal{H}^s, \quad u \mapsto f(t, x, u).
\]
If $f \in \mathcal{C}_k$ with $k \geq 3$, for all $0 \leq s' \leq k - 3$, $F$ is $C^2$ with respect to $u$ and
\[
D_u F(u)[h] = \partial_u f(t, x, u) h, \quad D_{u}^2 G(u)[h, h] = \partial_{uu}^2 f(t, x, u) h^2, \quad \forall h \in \mathcal{H}^s \cap \mathcal{H}^{s'}
\]
with
\[
\|\partial_u f(t, x, u)\|_{s'} \leq C(s', \|u\|_s)(1 + \|u\|_s'), \quad \|\partial_{uu}^2 f(t, x, u)\|_{s'} \leq C(s', \|u\|_s)(1 + \|u\|_s'). \quad (4.11)
\]
**Proof.** Since $\partial_u f, \partial_{uu}^2 f$ are in $\mathcal{C}_{k-1}, \mathcal{C}_{k-2}$, Lemma [4.3] shows that the maps $u \mapsto \partial_u f(t, x, u), u \mapsto \partial_{uu}^2 f(t, x, u)$ are continuous and that formula (4.11) holds. We now verify that $F$ is $C^2$ respect to $u$. Applying the continuity property of $u \mapsto \partial_u f(t, x, u)$, we deduce
\[
\|f(t, x, u + h) - f(t, x, u) - \partial_u f(t, x, u) h\|_{s'} \leq \|h\| \int_0^1 (\partial_u f(t, x, u + vh) - \partial_u f(t, x, u)) dv\|_{s'}
\]
\[
\leq C(s')\|h\|_{\max\{s, s'\}} \max_{v \in [0, 1]} \left(\|\partial_u f(t, x, u + vh) - \partial_u f(t, x, u)\|_{\max\{s, s'\}}\right)
\]
\[
= o(\|h\|_{\max\{s, s'\}}),
\]
which leads to
\[
D_u F(u)[h] = \partial_u f(t, x, u) h, \quad \forall h \in \mathcal{H}^s \cap \mathcal{H}^{s'}
\]
with $u \mapsto D_u F(u)$ being continuous. In addition,
\[
\partial_u f(t, x, u + vh) - \partial_u f(t, x, u) h - \partial_{uu}^2 f(t, x, u) h^2 = h^2 \int_0^1 (\partial_{uu}^2 f(t, x, u + vh) - \partial_{uu}^2 f(t, x, u)) dv.
\]
The same discussion as above yields that $F$ is twice differentiable with respect to $u$ and that $u \mapsto D_u^2 F(u)$ is continuous. \[\square\]
Proof of formula (2.40). If \( j > \max\{J_0, 2\sqrt{M}\} \), it follows from formula (3.25) that
\[
\inf_{\substack{\epsilon \in (\epsilon_1, \epsilon_2) \\ w \in \{W \cap H^s: \|w\|_s < \rho\}}} |\mu_{j+1}(\epsilon, w) - \mu_j(\epsilon, w)| \geq 1 - |\mu_{j+1}(\epsilon, w) - ((j + 1)^2 + v_0)|
- |\mu_j(\epsilon, w) - (j^2 + v_0)|
\geq 1 - \frac{2M}{j^2} > \frac{1}{2}
\]

If \( 0 \leq j \leq \max\{J_0, 2\sqrt{M}\} \), it is clear that
\[
\delta_j := \inf_{\substack{\epsilon \in (\epsilon_1, \epsilon_2) \\ w \in \{W \cap H^s: \|w\|_s < \rho\}}} |\mu_{j+1}(\epsilon, w) - \mu_j(\epsilon, w)|.
\]

Thus we complete the proof. □

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