Tracker fields from nonminimally coupled theory

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Abstract

We extend the concept of quintessence to flat nonminimally coupled scalar–
tensor theories of gravity. By means of Noether’s symmetries for the cosmological pointlike Lagrangian $\mathcal{L}$, it is possible to exhibit exact solutions for a class of models depending on a free parameter $s$. This parameter comes out in the relationship existing between the coupling $F(\varphi)$ and the potential $V(\varphi)$ because of such a symmetry for $\mathcal{L}$. When inverse power–law potentials are taken into account, a whole family of exact solutions parametrized by such an $s$ is proposed as a class of tracker fields, and some considerations are made about them.

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I. INTRODUCTION

Recently, astronomical observations have indicated a strong evidence of an accelerated universe \[1\]–\[3\]. Together with measurements of the cosmic microwave background and the mass power spectrum (see \[1\], for example), they suggest that a large amount of the energy density of the universe should have a negative pressure. A way of describing the \textit{missing} component of the energy needed to reach, for instance, the critical energy density is \textit{quintessence} \[5\]–\[7\], \[4\]. Essentially, this is a spatially inhomogeneous and slowly evolving fraction of the total energy. We may consider it as given by a scalar field $\varphi$ slowly rolling down its potential $V(\varphi)$ and such that $-1 < w_\varphi < 0$, being $w_\varphi \equiv p_\varphi/\rho_\varphi$, where $p_\varphi$ and $\rho_\varphi$ are, respectively, the pressure and the energy density of the scalar field. Actually, recent considerations \[8\] fix the interval $-1 < w_\varphi \lesssim -0.6$ as the more suitable one for such a scalar field to effectively represent quintessence. (A cosmological constant $\Lambda$, which mimics vacuum energy density, also produces a negative pressure, but this is such that $p_\Lambda/\rho_\Lambda = -1$.)

Within the scenario created by quintessence, there is a twofold problem. One of its two aspects is the so-called \textit{fine-tuning problem}, based on the question why $\rho_\varphi$ appears to be so small with respect to typical particle physics scales. The other aspect, called \textit{cosmic coincidence} \[9\], requires that the initial conditions have to be set precisely in order to explain why $\rho_\varphi$ and the matter energy density $\rho_m$ should appear of the same order today. This poses problems on the theoretical choice for the energy fraction which seems to be missing.

More recently, a special form of quintessence has been introduced to avoid the coincidence problem. It is called \textit{tracker field} \[7\], \[10\] and works like an attractor solution to the equations of motion, even if it is not really a fixed point, since it is time dependent and $\rho_\varphi/\rho_m$ changes as $\varphi$ evolves, leaving later cosmology independent of the early conditions. Of the kinds of potentials proposed for quintessence, two have been more studied for tracker solutions, namely, $V(\varphi) = M^4 + \alpha \varphi^{-\alpha}$ and $V(\varphi) = M^4[exp(M_P/\varphi) - 1]$, where $M$ and $\alpha > 0$ are free parameters, and $M_P$ is the Planck mass. The family of tracker solutions is then parametrized by $M$, whose value can be fixed by the measured value of $\Omega_m$ today. Such
specific forms of potentials have been chosen because of their importance in particle physics models [11–14]. Anyway, to our knowledge, Ratra and Peebles [15,16] were the first ones to study the influence of an inverse power–law potential in cosmology with a scalar field.

A general study of specific features of tracker solutions has been made [7,10], also leading to the introduction of the important function $\Gamma$, in order to fulfill the so–called tracker equation. Essentially, it is shown that “tracking behavior with $w_Q < w_B$ occurs for any potential in which $\Gamma \equiv V''V/(V')^2 > 1$ and is nearly constant” for any possible initial value of the scalar field $Q$. (Here, prime denotes derivative with respect to $Q$, and $B$ indicates background.) As a consequence, once a certain potential has been assigned, the existence of tracking solutions can be tested without solving the equations of motion.

Usually, the scalar field representing quintessence has been considered as minimally coupled to gravity and only more recently nonminimal coupling has been introduced in such a context [17–19]. In this connection, we consider interesting to refer again to scalar–tensor theories of gravity, in which the scalar field $\varphi$ is nonminimally coupled to gravity and also inverse power–law potentials for $\varphi$ have been studied, leading to exact solutions for $\varphi(t)$ and for the scale factor $a(t)$ of the universe (see [20] and references therein for a review on this topic). Our main purpose is to outline how a family of exact tracker solutions can be derived from a flat nonminimally coupled theory with inverse power–law potential, expliciting also how the tracker equation is fulfilled in such a way.

In what follows, section 2 is devoted to a short review of some basic notions of flat nonminimally coupled theories, and section 3 selects a special class of solutions. In section 4 we identify such a class as a family of tracker solutions, and in section 5 we draw conclusions.

II. NONMINIMALLY COUPLED THEORIES

As it is well known, one of the main reasons why nonminimally coupled scalar–tensor theories of gravity have received a special degree of attention is that they seem to play an important role in inflationary cosmology (see [20], for instance). In the following, however,
we will limit ourselves to concentrate on what can be useful here, trying to draw a sort of narrow and straightforward path to our specific goal, namely deriving tracker solutions. First of all, we deal with flat (i.e., with the curvature scalar \( k = 0 \)) models described by the action

\[
A = \int d^4x \sqrt{-g} \left[ F(\varphi)R + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) + L_m \right],
\]

(1)

where \( g \) is the determinant of the metric \( g_{\mu\nu} \), \( R \) the curvature scalar, semicolon indicates covariant derivative, and the functions \( F(\varphi) \) and \( V(\varphi) \) are not specified; \( L_m \) is the Lagrangian of an ordinary perfect fluid noncoupled to the scalar field \( \varphi \). \( F(\varphi) \) expresses the nonminimal coupling of \( \varphi \) with gravity and is such that, when \( F \equiv F_0 \equiv -\frac{1}{2} \) (using units such that \( 8\pi G = c = \hbar = 1 \)), action in Eq. (1) reduces to the usual one in the flat minimal coupling case.

The field equations can be derived by varying Eq. (1) with respect to \( g_{\mu\nu} \) and they can be written as

\[
G_{\mu\nu} = \tilde{T}_{\mu\nu} \equiv -\frac{1}{2F(\varphi)} T^{(\text{tot})}_{\mu\nu},
\]

(2)

where \( G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor, and \( \tilde{T}_{\mu\nu} \) is a quantity related to the total stress–energy tensor

\[
T^{(\text{tot})}_{\mu\nu} \equiv T^{(\varphi)}_{\mu\nu} + T^{(m)}_{\mu\nu}.
\]

(3)

Here, the tensor

\[
T^{(\varphi)}_{\mu\nu} \equiv \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \varphi \partial^\alpha \varphi + g_{\mu\nu} V(\varphi)
\]

\[
+ 2g_{\mu\nu} \Box F(\varphi) - 2F(\varphi)_{,\mu\nu}
\]

(4)

represents the scalar field source, while \( T^{(m)}_{\mu\nu} \) is the standard perfect fluid matter source (and \( \Box \) is the usual d’Alembert operator). Varying with respect to \( \varphi \), we get the Klein–Gordon equation ruling the dynamics of the scalar field \( \varphi \)

\[
\Box \varphi - RF'(\varphi) + V'(\varphi) = 0,
\]

(5)
denoting the prime the derivative with respect to $\varphi$. It is possible to show that Eq. (5) is nothing else but the contracted Bianchi identity \cite{24,20}, which means that the effective stress–energy tensor $\tilde{T}_{\mu\nu}$ introduced in Eq. (2) is a zero–divergence tensor, coherently with Einstein’s theory of gravity \cite{22}.

Fixing a homogeneous and isotropic (FRW) metric reduces the relevant variables to $a$ and $\varphi$, i.e. the scale factor and the scalar field, each one a function of $t$ only. As a matter of fact, field equations (2) can be reduced to two ordinary differential equations ($k = 0$)

$$H^2 + \frac{\dot{F}}{F} H + \frac{\rho_\varphi}{6F} + \frac{p_m}{6F} = 0, \quad (6)$$

$$\dot{H} = \frac{\dot{\varphi}^2}{4F} - \frac{1}{2} \left( H^2 + \frac{\rho_\varphi}{6F} \right) - \frac{\dot{F}}{2F} + \frac{p_m}{4F} + \frac{p_m}{6F}, \quad (7)$$

where dot indicates the time derivative, $H \equiv \dot{a}/a$, $p_m = w_m \rho_m$ is the equation of state for ordinary fluid matter, and $p_\varphi = w_\varphi \rho_\varphi$ is the equation of state for the scalar field $\varphi$, having defined its pressure and energy density, respectively, as

$$p_\varphi = \frac{1}{2} \dot{\varphi}^2 - V(\varphi), \quad (8)$$

$$\rho_\varphi = \frac{1}{2} \dot{\varphi}^2 + V(\varphi). \quad (9)$$

This implies, thus, that

$$w_\varphi = \frac{p_\varphi}{\rho_\varphi} = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)}. \quad (10)$$

Now, it is very interesting, and for us very important, to notice that Eq. (5) (rewritten in the FRW flat case) and Eq. (7) can be seen as the Euler–Lagrange equations of the point Lagrangian

$$\mathcal{L} = 6aa^2 F(\varphi) + 6\dot{a}\dot{\varphi}a^2 F'(\varphi) + a^3(p_\varphi + p_m), \quad (11)$$

Eq. (3) being equivalent to $E_\mathcal{L} = 0$, where $E_\mathcal{L}$ is the energy. The configuration space is then given by $Q \equiv \{a, \varphi\}$ (the minisuperspace) and the tangent space by $TQ \equiv \{a, \dot{a}, \varphi, \dot{\varphi}\}$,
being the *coordinates* associated with the pointlike Lagrangian $\mathcal{L}$ just the scale factor $a$ and the scalar field $\varphi$, with *velocities* $\dot{a}$, $\dot{\varphi}$. It is important to stress that from Eq. (11) we get that $E_{\mathcal{L}} = \text{const.}$; making the homogeneous and isotropic limit of Einstein field equations (2) implies to choose such a constant equal to zero. According to Noether’s theorem, the existence of a symmetry for the dynamics derived from $\mathcal{L}$ involves a constant of motion. As a consequence, Noether symmetries in cosmology give the possibility to infer some transformations of variables which often lead to deduce exact cosmological solutions [23,24,20]. Such solutions, though obtained by means of a procedure suggested by the existence of this kind of symmetries, are actually independent of it and could also be got by chance, suitably choosing the *right* transformation of variables. That is, a Noether symmetry simply *suggests* that such a transformation should exist and gives an easy way to find it: once we have the *right* way to write down equations, we can easily solve them and get a solution. We have to stress, also, that, in order to verify the existence of a Noether symmetry, we find a way of assigning the functions $F(\varphi)$ and $V(\varphi)$ for which a Noether symmetry exists, leading to remarkable results in many cases (see again [20] and references therein for several examples). In the following, we will introduce a class of exact solutions which deserves a special attention in the context we are working in.

**III. A SPECIAL CLASS OF SOLUTIONS**

First of all, let us notice that we start from the pointlike Lagrangian (11). It represents a whole class of theories, since a particular model is assigned by specifying $F(\varphi)$ and $V(\varphi)$. Examining the existence of Noether symmetries when the matter content is dust ($p_{m} = 0$), as a first result [23,20] one deduces the relevant relation

$$V(\varphi) = V_{0}F(\varphi)^{p(s)},$$  \hspace{1cm} (12)

where $V_{0} > 0$ is an arbitrary constant assumed always positive, and

$$p(s) = \frac{3(s + 1)}{2s + 3}. \hspace{1cm} (13)$$
Therefore, the potential, through its exponent \( p(s) \), depends on the free parameter \( s \). (The case \( s = -3/2 \) is degenerate and has been studied separately \[26,20\]. When \( s = -1 \), \( p(s) \) is zero and \( V(\varphi) \) becomes a constant; this situation has also been treated apart \[20\].)

The existence of such symmetries for Lagrangian \((11)\) implies that we can find a differential equation for \( F(\varphi) \) which has a general solution expressed by an elliptical integral of second kind, but we will limit our attention here to the particular solution

\[
F(\varphi) = k_0 \varphi^2 ,
\]  

where \( k_0 < 0 \) is an arbitrary constant. (Negative values of \( F(\varphi) \) are necessary to disregard repulsive gravity.) Then, Eq. \((14)\) inserted into Eq. \((12)\) gives

\[
V(\varphi) = [V_0 k_0 p(s)] \varphi^{2p(s)} .
\]

It is possible to see that there is a whole family of exact solutions for the time evolutions of \( a \) and \( \varphi \), and that they can be expressed as \[20\]

\[
a(\tau) = \xi(s) \tau^r ,
\]

\[
\varphi(\tau) = \zeta(s) \tau^{6/\chi(s)} ,
\]

where

\[
\chi(s) \equiv -\frac{6s}{2s+3} , \quad \zeta(s) \equiv \left[ \frac{\chi(s)}{3} \right]^{3/\chi(s)} ,
\]

\[
\xi(s) \equiv \zeta(s)^{-2/[3p(s)]} , \quad r \equiv \frac{2s^2 + 9s + 6}{s(s+3)}
\]

are parameters depending on \( s \). Time \( \tau \) is, actually, a rescaled time.

Apart from the cases showing pathologies in the solutions \( (s = 0 \) and \( s = -3) \), which have to be discussed separately \[25\], it is important to notice that the right sign of the coupling, i.e. \( F(\varphi) < 0 \), implies

\[
-2 < s < -1 .
\]
When $s$ varies in such an interval, we have an infinite number of exact solutions of the forms given in Eqs. (16) and (17). Asymptotically, we have

$$a(\tau) \approx \tau^r, \varphi(\tau) \approx \tau^{-(2s+3)/s}.$$  

(21)

That is, depending on the values of $s$, the scale factor $a(\tau)$ can have asymptotic Friedmann, power–law and pole-like behaviors. For instance, when $|s| \gg 0$, it is $a(\tau) \approx \tau^2$. But, in the range of values in Eq. (20), it can only be Friedmannian or power–law.

As to the scalar field $\varphi$, it diverges for $s < -3/2$ and converges for $s > -3/2$. In what follows, the range of values anyway chosen for $s$ will always be in the latter interval.

IV. A FAMILY OF EXACT TRACKER SOLUTIONS

Let us pose

$$\alpha \equiv -2p(s) > 0.$$  

(22)

From the definition of $p(s)$ in Eq. (13), this implies

$$-\frac{3}{2} < s < -1,$$  

(23)

and Eq. (15) gives

$$V(\varphi) = \left(V_0 k_0^{-\alpha/2}\right) \varphi^{-\alpha},$$  

(24)

with $\alpha$ always both positive and even. Substituting $\alpha$ into Eqs. (16), (17), (18), and (19) yields

$$a(\tau) = \left(\frac{2 + \alpha}{3}\right)^{4/[\alpha(2+\alpha)]} \tau^{2[2\alpha^2+9\alpha+6]/3(\alpha^2+6\alpha+8)},$$  

(25)

$$\varphi(\tau) = \left(\frac{2 + \alpha}{3}\right)^{3/(2+\alpha)} \tau^{2/(2+\alpha)}.$$  

(26)

On the other hand, from Eq. (10) we see that, being
\[ x \equiv \frac{\dot{\varphi}^2}{2V} > 0 \] (27)

(where now dot indicates derivative with respect to rescaled time \( \tau \)), we can write

\[ w_\varphi = \frac{x - 1}{x + 1}. \] (28)

Thus, it is clear that it is always \( w_\varphi > -1 \). For a constant \( \varphi \), i.e. a constant potential (which mimics a cosmological constant term), we should get \( w_\varphi = -1 \). We have in general

\[ x \equiv \frac{\dot{\varphi}^2}{2V} = \frac{k_0^{\alpha/2}}{2V_0} \dot{\varphi}^2 \varphi^\alpha > 0, \] (29)

implying the constraint

\[ k_0^{\alpha/2} > 0, \] (30)

which is always true.

One of the main requests that the scalar field has to satisfy, in order for it to be seen as a good tracker field, is that \( \dot{\varphi}^2 < V(\varphi) \), i.e. that \( x < 1 \). Requiring that \( 0 < x < 1 \) then poses the condition

\[ 0 < k_0^{\alpha/2} < \frac{27V_0}{2(2 + \alpha)}. \] (31)

Of course, \( 0 < x < 1 \) also implies that \( w_\varphi < 0 \), so that it is \(-1 < w_\varphi < 0\), namely what is needed for the scalar field to be interpreted as quintessence. Recently, it has been claimed that constraints from large–scale structure together with SNIa data imply \( w_\varphi < 0.6 \) with 95\% of confidence level \( ^8 \), which forces the field of variation for \( x \) to be

\[ 0 < x < 0.25. \] (32)

This, in turn, yields

\[ 0 < k_0^{\alpha/2} < \frac{27V_0}{8(2 + \alpha)}. \] (33)

Constraints given by Eq. (31) are also consistent with a direct calculation of \( w_\varphi \) from Eq. (10). As a matter of fact, inserting into it the expression of Eq. (26) for \( \varphi(\tau) \), we get

10
\[ w_\varphi = \frac{2k_0 \alpha/2(2 + \alpha) - 27V_0}{2k_0 \alpha/2(2 + \alpha) + 27V_0}. \]  

(34)

That is, \( w_\varphi < 0 \) implies Eq. (31).

A further restriction can be found on the values of \( s \), already such that Eq. (23) holds, if one looks at equipartition at the end of inflation for inverse power–law potentials. This, being \( \alpha \) even, constrains to \( \alpha > 5 \) \[10\], so that

\[ -\frac{3}{2} < s < -\frac{21}{16}. \]  

(35)

In \[10\], in a minimal coupling regime, there were also introduced two important equations, the equation of motion and the tracker equation. As a matter of fact, the first one can be easily generalized to the nonminimal coupling situation, giving

\[ \pm \frac{V'}{V} = 3 \sqrt{\frac{\kappa(1 + w_\varphi)}{\Omega_\varphi}} \left[ 1 + \frac{1}{6} \frac{d \ln x}{d \ln a} + \frac{2F'}{\dot{\varphi}H} \left( 2H^2 + \dot{H} \right) \right], \]  

(36)

where \( \kappa \equiv 8\pi G/3 \) (we are changing now our units following the current literature), \( \Omega_\varphi \equiv \kappa \rho_\varphi/H^2 \), and \( F' = 2k_0 \dot{\varphi} \). (The \( \pm \) signs, respectively, depend on whether \( V' > 0 \) or \( V' < 0 \).) \( F'' = 0 \) gives the minimal coupling case, and all cosmological solutions converge to the tracking solution, which is such that \( w_\varphi \) is nearly constant and less than \( w_B \) (where \( B \) indicates background), implying that \( 1 + w_\varphi = O(1) \) and therefore \( \dot{\varphi}^2 \approx \Omega_\varphi H^2 = \kappa \rho_\varphi \), so that

\[ \frac{V'}{V} \approx \frac{1}{\sqrt{\Omega_\varphi}} \approx \frac{H}{\dot{\varphi}}. \]  

(37)

This is referred to as the tracker equation \[10\].

We can also introduce the function \( \Gamma \equiv V''V/(V')^2 \), used in \[10\] for a test on the tracker behavior. Let us notice, then, that we have at all times

\[ \Gamma \equiv \frac{V''V}{(V')^2} = \frac{2p(s) - 1}{2p(s)} = 1 + \frac{1}{\alpha} = \text{const.}, \]  

(38)

which is the major condition for a tracker behavior. It is \( \Gamma > 1 \) for \( s \) in the interval in Eq. (23), being

11
\[ \Gamma = \frac{4s + 3}{6(s + 1)} = 1 - \frac{2s + 3}{6(s + 1)}. \]  

(39)

Of course, the range of values in Eq. (35) for \( \alpha > 5 \) is contained in the one in Eq. (23), still implying therefore \( \Gamma > 1 \).

The tracker equation is then

\[
\Gamma = 1 + \frac{1}{(1 + w_\varphi) \left[ 6 + \dot{x} + \frac{12F'}{\dot{x}H} \left( 2H^2 + \dot{H} \right) \right]}
\]

\[
\times \left\{ \frac{2\dot{x}}{\dot{\varphi}(1 + x)^2} - \frac{\Omega_\varphi'}{\Omega_\varphi} \sqrt{\frac{\Omega_\varphi}{\kappa(1 + w_\varphi)}} \right\}
\]

\[
\times \left\{ \dot{x} + \frac{\dot{\varphi}}{H} d_{\varphi} \left[ \frac{12F'}{\dot{\varphi}H} \left( 2H^2 + \dot{H} \right) \right] \right\},
\]

(40)

giving back what can be written for the minimal coupling case (when \( F' = 0 \)). Here, \( \dot{x} \equiv d \ln x / d \ln a \) and \( \ddot{x} \equiv d^2 \ln x / d \ln a^2 \).

When an inverse power–law potential is considered, as shown in [27] for the minimal coupling case and in [17] for a nonminimal coupling case (with \( F(\varphi) \equiv \xi \varphi^2 / 2 \), being \( \xi \) a constant, but with a slightly different Lagrangian for \( \varphi \)), if the perfect fluid with \( \rho_B \propto a^{-3(1+w_B)} \) dominates, so that \( a \propto t^{2/[3(1+w_B)]} \), then the following relation holds

\[ w_\varphi \approx \frac{w_B \alpha - 2}{\alpha + 2}. \]

(41)

On the other hand, if \( \Gamma \) is nearly constant, Eq. (11) in the minimal coupling regime implies [10] that a solution exists in which \( w_\varphi \) is also nearly constant and \( x, \dot{x}, \ddot{x} \) become nearly zero. This also leads to Eq. (41).

\[ w_B - w_\varphi = \frac{2(\Gamma - 1)(w_B - 1)}{1 + 2(\Gamma - 1)}, \]

(42)
evidenting the fact that \( \Gamma > 1 \) is equivalent to \( w_B > w_\varphi \) in a matter dominated situation.

In [17] it is found a solution for \( \varphi \) of the same type as in Eq. (28). Thus, the situation described therein is practically similar to ours, and we can import some of its considerations. Assuming that the universe is matter dominated, \( a \propto t^{2/[3(1+w_B)]} \) implies that \( H = [2/3(1+w_B)]t^{-1} \), and the Klein–Gordon equation (5) can be written as
\[
\dot{\phi} + \frac{2}{1 + w_B} \frac{1}{\tau} \dot{\phi} - \frac{8}{(1 + w_B)} \left[ \frac{4}{3(1 + w_B)} - 1 \right] k_0 \frac{1}{\tau^2} \phi - \alpha V_0 k_0^{-\alpha/2} \phi^{-\alpha - 1} = 0 .
\] (43)

As demonstrated in [17], this involves, for example, that we can take Eq. (41) as valid also in our context, even if we did not write Eq. (40) in such a way to clearly evidentiate that behavior when \(x, \dot{x}, \ddot{x}\) are negligible and \(w_\phi\) is constant.

If we, only in a speculative way, compare Eq. (34), “exact” and always valid at any time, and Eq. (41), “approximated” and valid only when \(\rho_B \gg \rho_\phi\), we find

\[
k_0^{\alpha/2} = \frac{\alpha(1 + w_B)}{4(2 + \alpha)} , \quad V_0 = \frac{4 + \alpha(1 - w_B)}{54} .
\] (44)

This forces the constant \(k_0^{\alpha/2}\) to be

\[
k_0^{\alpha/2} = \frac{1}{2} - \frac{27V_0}{2(2 + \alpha)}
\] (45)

and, according to Eq. (31), leads to

\[
\frac{2 + \alpha}{54} < V_0 < \frac{2 + \alpha}{27}
\] (46)

or, from Eq. (33), to

\[
\frac{4(2 + \alpha)}{135} < V_0 < \frac{2 + \alpha}{27} .
\] (47)

Also, substituting Eq. (44) directly into Eq. (31) and Eq. (33), respectively, gives

\[
w_B < \frac{2}{\alpha} , \quad w_B < \frac{4 - 3\alpha}{5\alpha}
\] (48)

or, equivalently,

\[
\alpha < \frac{2}{w_B} , \quad \alpha < \frac{4}{5w_B + 3} .
\] (49)

Let us notice, then, that the second relation in Eq. (48) always implies \(w_B < 0\) (which is not good for ordinary matter), while the first relation in Eq. (48) gives the right constraint \(w_B < 1\). Also, the second relation in Eq. (49) yields \(\alpha < 4/3\) for dust \((w_B = 0)\) and \(\alpha < 1/2\).
for stiff matter \((w_B = 1)\), while the first relation in Eq. (13) lets us accept any value of \(\alpha\) for dust, and gives \(\alpha < 6\) for radiation \((w_B = 1/3)\) or \(\alpha < 2\) for stiff matter. Finally, we can say that these considerations seem to imply that Eq. (32) limits too much the variability of \(x\).

Let us also notice that Eq. (31) could be read as
\[
 w_B \approx \frac{2 - (2 + \alpha)|w_\phi|}{\alpha}. \tag{50}
\]
Now, for ordinary matter (i.e., when \(0 \leq w_B \leq 1\)), it comes out (always disregarding the approximated equality)
\[
\frac{2 - \alpha}{2 + \alpha} \leq |w_\phi| \leq \frac{2}{2 + \alpha}, \tag{51}
\]
implying (for \(\alpha = 2\)) \(0 \leq w_\phi \leq 1/2\), or (for \(\alpha = 4\)) \(0 < |w_\phi| \leq 1/3\), for instance. This means, then, that values \(w_\phi \lesssim 0.6\) are however possible, even if considerations made above let us understand that there may also be values of \(w_\phi\) such that \(0.6 < w_\phi < 0\).

Of course, all these kinds of considerations have to be taken just as indicative, since Eq. (11) is not always valid and is usually read in one way: once we assign a specific value of \(w_B\), then it gives an approximated value of \(w_\phi\). For example, \(w_B = 0\) gives \(w_\phi \approx -2/(2 + \alpha) < 0\) for any \(\alpha > 0\), and from \(w_B = 1/3\) it is found \(w_\phi \approx (\alpha - 6)/[3(2 + \alpha)]\) (giving the critical value \(\alpha = 6\), such that \(\alpha > 6\) implies positive \(w_\phi\) and \(\alpha > 6\) implies negative \(w_\phi\)).

On the other hand, anyway, taking the tracker condition in Eq. (37) into account, we can immediately control its validity, being
\[
\frac{V'}{V} = -\frac{\alpha}{\phi} \approx \tau^{\frac{1 + w_\phi}{1 + w_B} - 1}, \tag{52}
\]
\[
\frac{H}{\phi} \approx \tau^{\frac{1 + w_\phi}{1 + w_B} - 1}. \tag{53}
\]
As a global consequence, we can assert that (21) is a good family of exact tracker solutions, parametrized by the constants \(V_0\) and \(k_0\), being
\[
M^{4+\alpha} \equiv V_0k_0^{-\alpha/2}. \tag{54}
\]
If we consider relations in Eq. (44), apart from the contribution of the ordinary matter content, the parameter \( M \) depends only on \( s \)

\[
M = \left\{ \frac{4s}{8(1+w_B)(2s+3)} \left[ s \left( \frac{1+3w_B}{1+w_B} \right) + 3 \right] (2s+3)/[2(s+3)] \right\}.
\]

(55)

The observational constraint \( \Omega_\varphi \approx 0.7 \) today, when \( \varphi \approx O(M_P) \), implies that \( V(\varphi \approx M_P) \approx \rho_{m_0} \), being \( \rho_{m_0} \approx 10^{-47}\text{GeV}^4 \) the current matter density. This gives

\[
V(\varphi) = M^{4+\alpha} \varphi^{-\alpha} \approx M^{4+\alpha} M_P^{-\alpha} \approx \rho_{m_0},
\]

(56)

so that \( M \approx (\rho_{m_0} M_P^{-\alpha})^{1/(4+\alpha)} > 1\text{GeV} \), in good comparison to particle physics scale, when \( \alpha \gtrsim 2 \), that is \( s \lesssim -1.2 \). (This shows again that, since such a value does not respect Eq. (35), constraining to \( \alpha > 5 \) should be reconsidered.)

V. CONCLUSIONS

In the picture offered by observations in recent times, a good cosmological scenario needs an energy component with negative pressure. Of course, the simplest and most extreme candidate is a cosmological constant, but other softer proposals exist. Among these, quintessence has been advanced resuming, generalizing, and suitably readapting older ideas on cosmology with a scalar field. Initially thought of as a field acting like an attractor on other solutions, a more refined version of it has been proposed more recently, namely the tracker field. This kind of field tends to isotropize the universe at late times, nicely solving the coincidence problem. In that, it finds an ideal convergence with what is claimed by Wald’s theorem on isotropization in homogeneous cosmologies with a positive cosmological constant.

In this paper, we have shown that the requested features of such a field when \( V(\varphi) \) has an inverse power–law behavior can be obtained looking at a kind of exact solutions already present in the past literature on nonminimally coupled scalar–tensor theories of gravity (see [20], for example). Even if it was deduced in a very peculiar way there, nevertheless it has
been reintroduced as, say, an *ad hoc* tracker solution only more recently. For these reasons, it has seemed interesting to connect the solution for $\varphi$ to the solution for $a$, placing all the discussion in the context of a well developed theory. To be precise, then, a family of exact tracker solutions has been found here, depending on the values of a parameter $s$ which is crucial for the relationship existing between the coupling $F(\varphi)$ and the potential $V(\varphi)$ in our context.

Such a relationship is a condition for a Noether symmetry to exist in the cosmological scenario involved by the model proposed through action in Eq. (1). This condition appears to be very important to us, because it seems to imply tracker fields in a very natural way, based on the apparent naturality of Noether symmetries in cosmology [20].

Also, in the context of nonminimally coupled scalar–tensor theories of gravity, there is no dramatic difference between quintessence and cosmological constant proposals, introduced to solve the puzzle offered by recent observations. It can be shown that there is, in fact, an evident possibility to generalize Wald’s theorem [28] in order to get an asymptotic cosmological constant in many nonminimal theories, without introducing it *a priori* (see [29] and references therein). As a matter of fact, for many choices of $F(\varphi)$, a time dependent $\Lambda$–term can be defined, asymptotically approaching a constant (that is, a *cosmological* constant). (In [29] several examples are considered.) In this connection, it is noteworthy that some interesting comments have also been made [30] on the fact that, with respect to the coincidence problem, an inverse power–law potential is not really different from a cosmological constant.

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