Asymptotic local hypothesis testing between a pure bipartite state and the completely mixed state

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In this paper, we treat an asymptotic hypothesis testing between an arbitrary known bipartite pure state $|\Psi\rangle$ and the completely mixed state under one-way LOCC, two-way LOCC, and separable POVMs. As a result, we derive analytical formulas for the Stein’s lemma type of optimal error exponents under all one-way LOCC, two-way LOCC and separable POVMs, the Chernoff bounds under one-way LOCC POVMs and separable POVMs, and the Hoeffding bounds under one-way LOCC POVMs without any restriction on a parameter and under separable POVMs on a restricted region of a parameter. We also numerically calculate the Chernoff and the Hoeffding bounds under a class of three step LOCC protocols in low dimensional systems, and show that these bounds not only outperform the bounds for one-way LOCC POVMs, but almost approximate the bounds for separable POVMs in the region of parameter where analytical bounds for separable POVMs are derived.

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I. INTRODUCTION

Local discrimination problems, which are the problems to discriminate unknown quantum state from known candidates by means of Local Operation and Classical Communication (LOCC), has been intensively studied in this 10 years [1][26]. This is because this problem is on the intersection of two significantly important topics of quantum information: Quantum State Discrimination [27–30] and Entanglement Theory [31][32]. Quantum state discrimination, which is a protocol to discriminate an unknown states from known candidates, is an essential subroutine for every quantum information processing, since this is the only way to derive classical information from quantum states. On the other hand, entanglement, which is non-local quantum correlation no-increasing under LOCC, is considered to be essential for quantum information processing to outperform its classical counterpart. As a result, by studying local discrimination problems, we can understand basic quantum information processing among spatially separated parties, and also the theory of entanglement itself more deeply.

There are various different problem settings of state discrimination problems except the most conventional problem setting [27][30], like Quantum hypothesis testing [35][36], Quantum State Estimation [37][39], and Classical Capacity of Quantum Channel [40][42]. Thus, there exist various different problem settings on local discrimination problems as well. In this paper, we especially treat local discrimination problems in the form of asymptotic hypothesis testing [13].

In asymptotic hypothesis testing, by measuring many copies of an unknown state, we aim to certify that the unknown state satisfies a given hypothesis $H_1$ (called an “alternative hypothesis”), and in order to do it, we try to reject a hypothesis $H_0$ (called a “null hypothesis”) which is true when $H_1$ is false. When a hypothesis ($H_0$ or $H_1$) consists of a single known state, it is called a simple hypothesis. In this paper, we only treat asymptotic hypothesis testing problems whose both null and alternative hypotheses are simple hypotheses.

In asymptotic hypothesis testing, there exist two different error probabilities, that is, the error probability judging $H_1$ to be true when $H_0$ is true (the type 1 error) and the error probability judging $H_0$ to be true when $H_1$ is true (the type 2 error). There is a trade-off between these error probability, and hence, the way to treat these error probabilities is not unique. At least, the following three different optimal error rates are commonly known, and play important roles in many fields of information theory and statistical inference [44][45]:

1. An asymptotic exponent of the optimal type 2 error under the condition that the type 1 error is upper bounded by a constant.
2. An asymptotic exponent of the optimal type 2 error under the condition that the exponent of the type 1 error is bounded by a constant.
3. An optimum exponent of the average of the type 1 and the type 2 error.

The first optimal error exponent is equal to the relative entropy (or the Kullback-Leibler divergence) [40] between two hypotheses; this fact is called “Stein’s lemma” [47]. The analytical formulas for the second and the third optimal error exponents are called the Hoeffding bound.
and the Chernoff bound \cite{48,49}, respectively. For all these three optimal error exponents, their formulas are recently extended to the case of the quantum hypothesis testing, where both $H_0$ and $H_1$ are single copies of quantum states (say $\rho$ and $\sigma$): the Quantum Stein’s Lemma \cite{50,51}, the Quantum Chernoff Bound \cite{52}, and the Quantum Hoeffding Bound \cite{53,54}.

On the other hand, it is much more difficult to treat an asymptotic quantum hypothesis testing with an additional locality restriction of a POVM (will be called “local asymptotic hypothesis testing”), and little is known for it. As far as we know, only two papers treated very special cases of this problem: the paper by Matthews et al. \cite{55} treats the Chernoff bound under various local POVMs in the case where $\rho$ and $\sigma$ are completely symmetric and anti-symmetric Werner states, respectively, and the paper by Nathanson \cite{56} treats the Chernoff bound under one-way LOCC POVMs in the case where $\rho$ is a pure bipartite state with a maximally Schmidt coefficient $\lambda$, and $\sigma$ satisfies $\text{Tr}\rho\sigma > \lambda$.

In this paper, we treat all three optimal error exponents in the case where $\rho$ is the completely mixed state and $\sigma$ is an arbitrary known bipartite pure state $|\Psi\rangle$. As a class of POVMs, we treat one-way LOCC (Local Operations and one-way Classical Communication) POVMs, two-way LOCC (Local Operations and two-way Classical Communication) POVMs and also separable POVMs. By using the result of the previous paper treating the same hypothesis testing problem in non-asymptotic (single-shot) settings \cite{57}, we derive the following results in a bipartite Hilbert space with local dimensions $d_A$ and $d_B$:

1. The Stein’s lemma type of error exponents are the same for all three classes of local POVMs, and given as $\log d_A + \log d_B - E(|\Psi\rangle)$, where $E(|\Psi\rangle)$ is the entropy of entanglement \cite{33,34}. Moreover, their strong converse bound also coincides the optimal error exponents itself.

2. The Chernoff bound under one-way LOCC POVMs is given as $\log d_A + \log d_B - \log R_n(|\Psi\rangle)$, where $R_n(|\Psi\rangle)$ is the Schmidt rank \cite{33,34}. The Chernoff bound under separable POVMs is given as $\log d_A + \log d_B - LR(|\Psi\rangle)$, where $LR(|\Psi\rangle)$ is the logarithmic robustness of entanglement \cite{51,52}.

3. An analytical formula of the Hoeffding bounds is derived under one-way LOCC POVMs without any restriction on a parameter, and under separable POVMs for a restricted region of a parameter. For other region of the parameter, analytical upper bounds and lower bounds of Hoeffding bounds under Separable POVMs are derived.

4. The Chernoff and the Hoeffding bounds under a class of three step (therefore, two-way) LOCC protocols are numerically calculated for low dimensional systems. As a result, we show that these bounds not only outperform the bounds for one-way LOCC POVMs, but almost approximate the bounds for separable POVMs in the region of parameter where analytical bounds for separable POVMs are derived.

In the above results, the result 2 is remarkable since it gives a new operational meaning for the logarithmic robustness of entanglement in term of this local asymptotic hypothesis testing problem; for another operational meaning of logarithmic robustness, see \cite{58}. The result 4 is also remarkable since, as far as we know, this is first time to find a gap between optimal error exponents under one-way LOCC and under two-way LOCC in asymptotic local discrimination problems; that is, so far, all such gaps are found in optimal error probabilities in non-asymptotic local discrimination problems \cite{36,40}, and it was not known whether such gaps survive in their asymptotic extensions.

The organization of the paper is as follows: In Section II, we give mathematical descriptions of our hypothesis testing problem and known results about optimal error exponents under global POVM. Then, we treat the hypothesis testing problem under one-way LOCC and separable operations in Section III and Section IV, and give analytical expressions of optimal error exponents under these classes of POVM. In Section V, we analyze a special class of three step LOCC (thus, two-way LOCC) protocols for this local hypothesis testing problem. In Section VI, we give and discuss about plots of error exponents corresponding to the Chernoff and Hoeffding bounds in low dimension systems. At last, we summarize the results of our paper in Section VII.

II. PRELIMINARY

In this paper, we always treat a bipartite quantum system and its $n$-copies extension. A single copies of the bipartite Hilbert space is written as $\mathcal{H}_{AB} \overset{\text{def}}{=} \mathcal{H}_A \otimes \mathcal{H}_B$, and its local dimensions are written as $d_A \overset{\text{def}}{=} \dim \mathcal{H}_A$ and $d_B \overset{\text{def}}{=} \dim \mathcal{H}_B$. We use notations like $I_A$, $I_B$, $I_{AB}$, $I'_A$, $I'_B$, and $I'_{AB}$ for identity operations on $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_{AB}$, $\mathcal{H}_A^{\otimes n}$, $\mathcal{H}_B^{\otimes n}$, and $\mathcal{H}_{AB}^{\otimes n}$, respectively. When it is easy to identify the support of an identity operator, we abbreviate them to $I$ in the following part.

In this paper, we consider an asymptotic hypothesis testing between $n$-copies of an arbitrary known pure-bipartite state $|\Psi\rangle$ having the Schmidt decomposition as

$$|\Psi\rangle \overset{\text{def}}{=} \sum_{i=1}^{d_{\text{min}}} \sqrt{\lambda_i} |i\rangle \otimes |i\rangle,$$

where $d_{\text{min}} \overset{\text{def}}{=} \min(d_A, d_B)$, and $n$-copies of the completely mixed state (or a white noise) $\rho_{\text{mix}}$ under the various restrictions on available POVMs: global POVMs, separable POVMs, one-way LOCC POVMs, two-way
LOCC POVMs. We choose the completely mixed state $ρ_{mix}^⊗n$ as a null hypothesis and the state $|Ψ⟩^⊗n$ as an alternative hypothesis. In the following part of this section, we give definitions of various error exponents and optimal error exponents for a general simple null hypothesis $ρ^⊗n$ and an alternative hypothesis $σ^⊗n$. Thus, $ρ = ρ_{mix}$ and $σ = |Ψ⟩⟨Ψ|$ in our local hypothesis testing problem.

We only treat a two-valued POVM consisting of two POVM elements $T_n$ and $I_n^A − T_n$, where $T_n$ is supported by $H_A ⊗ H_B$. When the measurement result is $T_n$, we judge an unknown state as $σ^⊗n$, and when the measurement result is $I_n^A − T_n$, we judge the unknown state as $ρ^⊗n$.

Thus, the type 1 error is written as

$$α_n(T_n) = \text{Tr}ρ^⊗n T_n,$$

and the type 2 error is written as

$$β_n(T_n) = \text{Tr}σ^⊗n (I_n^A − T_n).$$

As a result, the optimal type 2 error under the condition that the type 1 error is no more than $α$ is written as

$$β_{n, C}(α | σ) = \min_{T_n} \{ β_n(T_n) | α_n(T_n) ≤ α, \{ T_n, I − T_n \} ∈ C \},$$

where $C$ is either $→$, $↔$, $\text{Sep}$, or $g$ corresponding to a classes of one-way LOCC, two-way LOCC, separable and global POVMs, respectively. Similarly, the optimal type 1 error under the condition that the type 2 error is no more than $β$ is written as

$$α_{n, C}(β | σ) = \min_{T_n} \{ α_n(T_n) | β_n(T_n) ≤ β, \{ T_n, I − T_n \} ∈ C \}.$$

It is easily showed that the function $β → α_{n, C}(β | σ)$ is an inverse function of the function $α → β_{n, C}(α | σ)$ in the region where $β_{n, C}(α | σ)$ is strictly decreasing and continuous.

In the case of the Bayesian problem setting, we further assume existence of a prior probability on hypotheses. Suppose there exists a prior probability $(π_0, π_1)$ on the null and alternative hypotheses. Then, the mean error probability is defined as

$$P_n(π_0, π_1; T_n) \overset{def}{=} π_0 α_n(T_n) + π_1 β_n(T_n).$$

Then, for a given class of POVMs $C$, the optimal mean error probability is defined as

$$P_{n, C}(π_0, π_1 | σ) \overset{def}{=} \min \{ P_n(π_0, π_1; T_n) | \{ T_n, I − T_n \} ∈ C \}.$$

For any class of POVMs $C$, the relation between $P_{n, C}(π_0, π_1 | σ)$ and $β_{n, C}(α | σ)$ is given as follows:

$$P_{n, C}(π_0, π_1 | σ) = \inf_{0 ≤ α ≤ 1} π_0 α + π_1 β_{n, C}(α | σ).$$

Similarly, the following formula holds between $P_{n, C}(π_0, π_1 | σ)$ and $α_{n, C}(β | σ)$:

$$P_{n, C}(π_0, π_1 | σ) = \inf_{0 ≤ α ≤ 1} π_0 α + π_1 β_{n, C}(α | σ) + π_1 β.$$
for any class $C$.

It is known that $\beta^{R}_{C,g}$ and $\beta^{R}_{I,g}$ are given for a null $\rho^{x,n}$ and an alternative $\sigma^{x,n}$ hypotheses as

$$\beta^{R}_{C,g}(\rho||\sigma) = \beta^{R}_{I,g}(\rho||\sigma) = \alpha^{R}_{C,g}(\rho||\rho) = \alpha_{I,g}(\rho||\rho) = D(\rho||\sigma), \quad (15)$$

where $D(\rho||\sigma)$ is the relative entropy between $\rho$ and $\sigma$.

This result is called the Stein’s lemma.

For a class of POVMs $C$, the Hoeffding bound $B_{C}(r||\sigma)$ (or $A_{C}(r||\sigma)$) is defined as an optimum type 2 (or 1) error exponent under the restriction where the other error exponent is lower bounded by a constant $r$:

$$B_{C}(r||\sigma) \defeq \sup_{T_{n}} \left\{ \lim_{n \to \infty} \frac{1}{n} \log \alpha_{n}(T_{n}) \right\}$$

$$= \lim_{n \to \infty} \frac{1}{n} \log \alpha_{n}(T_{n}) \geq r, \{T_{n}, I - T_{n}\} \in C, \quad (16)$$

$$A_{C}(r||\sigma) \defeq \sup_{T_{n}} \left\{ \lim_{n \to \infty} \frac{1}{n} \log \beta_{n}(T_{n}) \right\}$$

$$= \lim_{n \to \infty} \frac{1}{n} \log \beta_{n}(T_{n}) \geq r, \{T_{n}, I - T_{n}\} \in C. \quad (17)$$

For global POVMs $C = g$, the Hoeffding bound has the following formula:

$$B_{g}(r||\sigma) = A_{g}(r||\sigma) = \sup_{0 \leq s < 1} \frac{-rs - \log \text{Tr} \rho^{s} \rho^{1-s}}{1-s}. \quad (18)$$

By means of the above analytical formulas for the various optimal error exponents under global POVM, we can calculate these optimal global error exponents for our hypothesis testing problem as follows:

$$\xi_{g}(\rho_{mix}||\Psi) = \alpha^{R}_{C,g}(\rho_{mix}||\Psi) = \alpha^{R}_{C,g}(\rho_{mix}||\Psi) = \log d_{A} + \log d_{B}, \quad (19)$$

$$\beta^{R}_{C,g}(\rho_{mix}||\Psi) = +\infty, \quad (20)$$

$$B_{g}(r||\rho_{mix}||\Psi) = \begin{cases} +\infty & \text{if} \quad 0 \leq r \leq \log d_{A} + \log d_{B}, \\ 0 & \text{otherwise}. \end{cases} \quad (21)$$

From the definition of $\beta^{R}_{C,g}$, Eq. (20) shows the super-exponential convergence of the optimal type 2 error. Moreover, one can easily see that for a fixed $\alpha$, for $n \geq \frac{1}{\log d_{A} + \log d_{B}}$, the optimal type 2 error $\beta_{C}(\rho_{mix}||\Psi)$ is exactly equal to 0.

III. HYPOTHESIS TESTING UNDER ONE-WAY LOCC POVMs

In this section, we consider the case of $C = \rightarrow$, that is, the local hypothesis testing under one-way LOCC POVMs. In the following part of the paper, we mostly treat the case where a null hypothesis is $\rho = \rho_{mix}$ and an alternative hypothesis is $\sigma = \Psi \defeq |\Psi\rangle\langle\Psi|$.

Therefore, when this is the case, we abbreviated these variable in the formula. For example, we write $\beta_{C}(\alpha||\rho_{mix}||\Psi)$ as $\beta_{C}(\alpha)$.

First, from the last paper, we derive the following lemma:

**Lemma 1.**

$$\alpha_{\rightarrow}(\alpha||\rho_{mix}||\Psi) = \alpha_{\rightarrow, g}(\alpha||\rho_{mix}||\sigma_{\Psi}), \quad (22)$$

$$\beta_{\rightarrow}(\alpha||\rho_{mix}||\Psi) = \beta_{\rightarrow, g}(\alpha||\rho_{mix}||\sigma_{\Psi}), \quad (23)$$

where $\sigma_{\Psi}$ is a separable state defined as

$$\sigma_{\Psi} \defeq \sum_{i=1}^{d} \lambda_{i} |i\rangle \otimes |i\rangle \langle i|. \quad (24)$$

(Proof)

From Theorem 1 of the last paper, an optimal one-way LOCC POVM achieving $\beta_{\rightarrow}(\alpha)$ can be chosen to be diagonal in the Schmidt basis of $|\Psi\rangle$ (thus, $\{ |ij\rangle \}_{ij}$ in this case). The same fact is also true for $\beta_{\rightarrow, g}(\alpha)$. Thus, in this case, an optimal $T_{n}$ can be chosen to be diagonal in the Schmidt basis of $|\Psi\rangle^{\otimes n}$; we use the notation $\{ |J(n)\rangle \}_{J_{\infty,n}}$ as an abbreviation of the basis $\{ |i_{1}\rangle \otimes \cdots |i_{n}\rangle \}_{i_{1},\ldots,i_{n}}$.

All the Schmidt-basis-diagonal POVM elements $T_{n}$ are one-way LOCC POVM elements, and can be written down in the form of

$$T_{n} = \sum_{J(n),J'(n)} |J(n)\rangle \langle J'(n)| \otimes |J'(n)\rangle \langle J(n)|, \quad (25)$$

where $T'_{n}$ is a global POVM element. On the other hand for an arbitrary global POVM element $T'_{n}$, the above $T_{n}$ is a one-way LOCC POVM element. For such a POVM element $T_{n}$, $\alpha_{n}(T_{n})$ and $\beta_{n}(T_{n})$ can be written as

$$\alpha_{n}(T_{n}) = \text{Tr} \rho_{mix}^{\otimes n} T'_{n}, \quad (26)$$

$$\beta_{n}(T_{n}) = \text{Tr} \sigma_{\Psi}^{\otimes n} T'_{n}. \quad (27)$$

Substituting the above equations in Eq. (11) and Eq. (15), we derive Eq. (22) and Eq. (23).

By means of the above lemma, we derive analytical formulas for the optimal error exponents:
Theorem 1.

\[
\xi \rightarrow (\rho_{\text{mix}} \Vert \Psi) = \log d_A + \log d_B - \log R_s(\vert \Psi \rangle), \tag{28}
\]

\[
\alpha_{\epsilon, \rightarrow}^R(\rho_{\text{mix}} \Vert \Psi) = \log d_A + \log d_B - E(\vert \Psi \rangle), \tag{29}
\]

\[
\beta_{\epsilon, \rightarrow}(\rho_{\text{mix}} \Vert \Psi) = +\infty, \tag{30}
\]

\[
A_{\rightarrow}(r \vert \rho \vert \sigma) = \log d_A + \log d_B + \sup_{0 \leq s < 1} -rs - \log \sum_{1-s} \lambda_s^s, \tag{31}
\]

\[
B_{\rightarrow}(r \vert \rho \vert \sigma) = \begin{cases} +\infty & \text{if } 0 \leq r \leq \log d_A + \log d_B + \log R_s, \\ \text{otherwise} & \sup_{0 \leq s < 1} -(r-\log d_A - \log d_B)s - \log \sum_{1-s} \lambda_s^{1-s}. \end{cases} \tag{32}
\]

In the above formulas, \(R_s(\vert \Psi \rangle)\) is the Schmidt rank, and \(E(\vert \Psi \rangle)\) is the entropy of entanglement of \(\vert \Psi \rangle\) \(\text{[33, 34, 43, 54]}\).

(Proof)

Lemma \[1\] guarantees

\[
\alpha_{n, \rightarrow}(\rho_{\text{mix}} \Vert \Psi) = \alpha_{n, \sigma}(\rho_{\text{mix}} \Vert \sigma_{\Psi}), \tag{33}
\]

\[
\beta_{n, \rightarrow}(\rho_{\text{mix}} \Vert \Psi) = \beta_{n, \sigma}(\rho_{\text{mix}} \Vert \sigma_{\Psi}). \tag{34}
\]

The second equation and Eq.\[8\] guarantee

\[
P_{n, \rightarrow}(\pi_0, \pi_1 \vert \rho_{\text{mix}} \Vert \Psi) = P_{n, \sigma}(\pi_0, \pi_1 \vert \rho_{\text{mix}} \Vert \sigma_{\Psi}). \tag{35}
\]

The above equation and the definition of \(\xi_C\) guarantee

\[
\xi \rightarrow (\rho_{\text{mix}} \Vert \Psi) = \xi_{\sigma}(\rho_{\text{mix}} \Vert \sigma_{\Psi}). \tag{36}
\]

The similar equations also hold for \(\alpha_{\epsilon, C}^R, \beta_{\epsilon, C}^R, \alpha_C^R\) and \(B_C\). Since analytical formulas are known for these optimal error exponents under global POVMs (Eq.\[11\], Eq.\[15\], and Eq.\[18\]), we can directly calculate these optimal error exponents. As a result, we derive the lemma. \(\Box\)

The equation \(\beta_{\epsilon, \rightarrow} = +\infty\) guarantees that \(\beta_{n, \rightarrow}\) super-exponentially converges to 0 in the limit of \(n \rightarrow \infty\). Indeed, in Section III of the last paper \[53\], it was proved that \(\beta_{n,C}(\alpha) = 0\) for \(C = \rightarrow, \leftrightarrow, \text{sep}\), and \(\alpha \geq 1/d_{\max}\). Since \(\beta_{n, \rightarrow} \geq \beta_{n, \leftrightarrow} \geq \beta_{n, \sigma}\) by the definitions, we derive:

Corollary 1. For \(n \geq -\frac{\log \alpha}{\log d_{\max}}\),

\[
\beta_{n, \rightarrow}(\alpha) = \beta_{n, \leftrightarrow}(\alpha) = \beta_{n, \sigma}(\alpha) = 0. \tag{37}
\]

Thus,

\[
\beta_{\epsilon, \rightarrow} = \beta_{\epsilon, \leftrightarrow} = \beta_{\epsilon, \sigma} = +\infty. \tag{38}
\]

IV. HYPOTHESIS TESTING UNDER SEPARABLE POVMs

In this section, we consider the case of \(C = \text{sep}\), that is, the local hypothesis testing under separable LOCC POVMs.

A. Stein’s Lemma under separable POVMs

In this subsection, we treat the optimal error exponents \(\alpha^R_{\epsilon, \text{sep}}\) and \(\beta^R_{\epsilon, \text{sep}}\), which are the optimal error exponents in the problem setting of the Stein’s Lemma with an additional separability condition on POVMs.

We have derived the equality \(\beta^R_{\sigma} = +\infty\) by means of the result of one-way LOCC POVMs in Corollary \[1\]. Hence, we treat the other error exponent \(\alpha^R_{\epsilon, \text{sep}}\) here. The analytical formula for this error exponent is given as follows:

Theorem 2.

\[
\alpha_{\epsilon, \rightarrow} = \alpha_{\epsilon, \leftrightarrow} = \alpha_{\epsilon, \text{sep}} = \log d_A + \log d_B - E(\vert \Psi \rangle), \tag{39}
\]

where \(E(\vert \Psi \rangle)\) is the entropy of entanglement \(\text{[33, 34, 43, 54]}\).

From Corollary \[1\] and this theorem, we observe the following fact: Although there is a gap on optimal error probabilities among One-way LOCC, Two-way LOCC and separable POVMs in the non-asymptotic local hypothesis testing \[53\], such a difference never appears on the Stein’s lemma type of error exponents in their asymptotic extensions. In the next subsection, we will see that such a difference appears in the Chernoff and Hoeffding types of optimal error exponents. We also note that this theorem gives a new operational interpretation of the entropy of entanglement in terms of the local hypothesis testing.

There is a strong mathematical relation between the optimal error exponent \(\alpha^R_{\epsilon, \text{sep}}\) and the environment-assisted capacities of quantum channel treated in \[53\]. Especially, we can use Lemma 6 of \[54\] to derive an upper bound for \(\alpha^R_{\epsilon, \text{sep}}\). However, we give a direct proof of the theorem without using this lemma in \[54\] to derive a slightly stronger result, that is, the strong converse bound (Corollary \[2\]).

(Proof of Theorem \[2\])

By the definitions, we have

\[
\underline{\alpha}^R_{\epsilon, \rightarrow} \leq \underline{\alpha}^R_{\epsilon, \leftrightarrow} \leq \underline{\alpha}^R_{\epsilon, \text{sep}}. \tag{40}
\]

Thus, from Eq.\[29\], we immediately derive

\[
\underline{\alpha}_{\epsilon, \rightarrow} \geq \log d_A + \log d_B - E(\vert \Psi \rangle). \tag{41}
\]

Hence, what we need to prove is the inequality:

\[
\overline{\alpha}^R_{\epsilon, \text{sep}} \leq \log d_A + \log d_B - E(\vert \Psi \rangle). \tag{42}
\]

In order to show the inequality, we define \(\overline{\alpha}^R_{1, \text{sep}}\) as

\[
\overline{\alpha}^R_{1, \text{sep}} = \sup_{\{T_n\}_{n=1}} \left\{ \lim_{n \rightarrow \infty} -\frac{1}{n} \log \alpha_{n, \text{sep}}(T_n) \mid \lim_{n \rightarrow \infty} \beta_{n, \text{sep}}(T_n) < 1, \{T_n, I - T_n\} \in \text{Sep} \right\}. \tag{43}
\]
where $\limsup$ is the limit superior. By the definition of $\tau^R_{\text{sep}}$, there exists a sequence of separable POVMs $\mathcal{T}_n \uparrow_{n \to 1}$ such that $\limsup -\frac{1}{n}\alpha_n(T_n) = \alpha^R_{\text{sep}}$ and $\beta_n(T_n) \leq \epsilon$ for all $n$. This fact and the definition of $\alpha^R_{n,\text{sep}}$ guarantees

$$\alpha^R_{1,\text{sep}} \geq \tau^R_{\text{sep}}$$

(44)

for all $0 < \epsilon < 1$.

In the following, we show that when a sequence of tests $\{T_n, I - T_n\} \in \mathcal{S}_{\text{sep}}$ satisfies $\lim_{n \to \infty}\beta_n(T_n) < 1$, the following inequality holds:

$$\lim n_{\to \infty} - \frac{1}{n}\log \alpha_n,\text{sep}(T_n) \leq \log d_A + \log d_B - E(\langle \Psi \rangle).$$

(45)

There exist a small real number $\delta > 0$ and an integer $N_1$ such that

$$\text{Tr}\langle \Psi | I - T_n | \Psi \rangle < 1 - \delta$$

(46)

for any number $n \geq N_1$. For an arbitrary small real number $\delta' > 0$ and the above given $\delta > 0$, there exists an integer $N_2$ such that $\text{Tr} L_{\delta',n}^A \geq 1 - \delta'^2/32$ where $L_{\delta',n}^A := \{\rho_{\text{se}}^n - e^{-nE(\langle \Psi \rangle - \delta')} \leq 0\}$, and $X \leq 0$ is the projection to the subspace spanned by eigenvector with a non-positive eigenvalue of $X$. $L_{\delta',n}^B$ is also defined similarly.

Thus, $\text{Tr}\langle \Psi | L_{\delta',n}^A \otimes L_{\delta',n}^B \geq (1-\delta'^2/32) \geq 1 - \delta'^2/16$. We choose a unitary $U$ such that

$$\|L_{\delta',n}^A \otimes L_{\delta',n}^B\|_1 \geq \text{Tr}(L_{\delta',n}^A \otimes L_{\delta',n}^B)\langle \Psi |(I - L_{\delta',n}^A \otimes L_{\delta',n}^B)U, $$

where $\| \cdot \|_1$ is the trace norm. Using the Schwartz inequality concerning $\text{Tr} A^\dagger B$ between two matrices $A$ and $B$, we obtain

$$\|L_{\delta',n}^A \otimes L_{\delta',n}^B\|_1 \leq \text{Tr}(L_{\delta',n}^A \otimes L_{\delta',n}^B)\langle \Psi |(I - L_{\delta',n}^A \otimes L_{\delta',n}^B)\Psi.$$ 

Thus,

$$\|L_{\delta',n}^A \otimes L_{\delta',n}^B\|_1 \leq \sqrt{(\langle \Psi |(I - L_{\delta',n}^A \otimes L_{\delta',n}^B)\Psi)}.$$ 

Similarly,

$$\|\Psi |(I - L_{\delta',n}^A \otimes L_{\delta',n}^B)\|_1 \leq \sqrt{(\langle \Psi |(I - L_{\delta',n}^A \otimes L_{\delta',n}^B)\Psi}).$$

Hence, for $n \geq \max\{N_1, N_2\}$,

$$\|\Psi |(I - L_{\delta',n}^A \otimes L_{\delta',n}^B)\|_1 \leq \sqrt{(\langle \Psi |(I - L_{\delta',n}^A \otimes L_{\delta',n}^B)\Psi}).$$

Thus, the non-normalized vector $|\Psi\rangle := (L_{\delta',n}^A \otimes L_{\delta',n}^B)|\Psi\rangle$ satisfies

$$\text{Tr} |\Psi\rangle \langle \Psi | T_{n_k} \geq \text{Tr} |\Psi\rangle \langle \Psi | T_{n_k} - \| \langle \Psi | - |\Psi\rangle \langle \Psi | \| \leq \delta - \delta/2 = \delta/2.$$

Since the largest Schmidt coefficient of $|\Psi\rangle$ is no more than $e^{-nE(\langle \Psi \rangle - \delta')}$, any separable state $\sigma$ satisfies

$$\langle \Psi | \sigma \rangle \langle \Psi | \leq e^{-nE(\langle \Psi \rangle - \delta')}.$$

Because $T_n$ has a separable form,

$$e^{-nE(\langle \Psi \rangle - \delta')} \text{Tr} T_n \geq \text{Tr} |\Psi\rangle \langle \Psi | T_{n_k}.$$

Thus,

$$\text{Tr} T_n \geq \delta e^{-nE(\langle \Psi \rangle - \delta')}/2.$$

Therefore,

$$\alpha_{n,\text{sep}}(T_n) = \frac{\text{Tr} T_n}{d_A d_B} \geq \frac{\delta e^{-nE(\langle \Psi \rangle - \delta')}}{2d_A d_B},$$

(47)

which implies that

$$\lim_{n \to \infty} - \frac{1}{n}\log \alpha_{n,\text{sep}}(T_n) \leq \lim_{n \to \infty} - \frac{1}{n}\log \alpha_{n,\text{sep}}(T_n) \leq \log d_A + \log d_B - E(\langle \Psi \rangle) + \delta'.$$

Since $\delta' > 0$ is arbitrary, we obtain $[15]$ At the last part of this section, we consider the strong converse bound $\alpha^R_{C} (\rho | \sigma)$ defined by Eq.(14). As we have seen in Section 11 for global POVMs $C = g$, it is known that $\alpha_{C}^R = \alpha_{g}^R$ [36]. Actually, the similar equality holds for $C = \rightarrow, \leftrightarrow, \text{sep}$ in the case when $\rho = \rho_{\text{mix}}$ and $\sigma = \Psi$:

**Corollary 2.** For $C = \rightarrow, \leftrightarrow, \text{sep},$

$$\alpha_{C}^R(\rho_{\text{mix}} | \Psi) = \alpha_{C}^R(\rho_{\text{mix}} | \Psi) = \log d_A + \log d_B - E(\langle \Psi \rangle).$$

(48)

**(Proof)**

Since we have proved Theorem 2 we only need to prove

$$\alpha_{C}^R \leq \log d_A + \log d_B - E(\langle \Psi \rangle).$$

(49)

The proof of this inequality is almost the same as the last half of the proof of Theorem 2.
The proof of this inequality is almost the same as the last half of the proof of Theorem 2.

The difference comes from Eq. (50). Because of the definition of $\alpha_{n,sep}(T_{n})$, this equation does not hold for all large $n$ in general. What we can say is: there exist a small real number $\delta > 0$, a subsequence $\{n_k\}$ such that

$$\text{Tr}(\Psi |I-T_{n_k}|\Psi) < 1 - \delta$$

for any number $k$. As a result, we derive Eq. (47) only for the subsequence. That is, we derive

$$\alpha_{n,sep}(T_{n_k}) \geq \frac{\delta_0 n_k}{2d_A^k d_B^k},$$

for large $k$. This inequality implies

$$\lim_{n \to \infty} \frac{1}{n} \log \alpha_{n,sep}(T_n) \leq \log d_A + \log d_B - E(\|\Psi\|) + \delta'.$$

B. Chernoff bound for separable POVM

In this subsection, we treat the optimal error exponent $\xi_{sep}$, which is the Chernoff bound with an additional separability condition on POVMs.

In order to derive an analytical formula $\xi_{sep}$, we need one of the main results of Section VI of the previous paper 53, where we proved the equivalence between the hypothesis testing under separable POVMs and a different type global hypothesis with a composite alternative hypothesis. In the paper 53, we only treated a single copy case. However, it can easily extend to the $n$-copies case. In the $n$-copies case, the corresponding global hypothesis testing on $(\mathbb{C}^{d_{min}})^{\otimes n}$ is a hypothesis testing between an arbitrary pure state $|\psi\rangle^\otimes n$ and a set of states

$$\{|\phi_{\vec{K}}^n\rangle\}_{\vec{K} \in \mathbb{Z}_{2}^{d_{min}}},$$

where $\{|J\rangle\}_{J=1}^{d_{max}}$ is equivalent to the standard basis $\{|i_1\rangle \otimes \cdots \otimes |i_n\rangle\}_{i_1,\ldots,i_n}$ of the space $(\mathbb{C}^{d_{min}})^{\otimes n}$ under the relabeling of the basis vectors. Then, for a two-valued POVM $\{S_n, I - S_n\}$ on $(\mathbb{C}^{d_{min}})^{\otimes n}$, the type 1 error $a_n(S_n)$ and the type 2 error $b_n(S_n)$ are defined as

$$a_n(S_n) \equiv \max _{\vec{K} \in \mathbb{Z}_{2}^{d_{min}}} a_{n,\vec{K}}(S_n),$$

$$b_n(S_n) \equiv \langle \psi | \otimes I_{d_{min}} - S_n |\psi\rangle^\otimes n,$$

where $I_{d_{min}}$ is an identity operator on $(\mathbb{C}^{d_{min}})^{\otimes n}$, and $a_{n,\vec{K}}(S_n)$ is an error probability defined as

$$a_{n,\vec{K}}(S_n) \equiv \langle \phi_{\vec{K}}^n | S_n |\phi_{\vec{K}}^n\rangle$$

The optimal type 2 error under the restriction of the condition that the type 1 error is no more than $a$ can be written as

$$b_n \left( a \left[ |\phi_{\vec{K}}^n\rangle \right] \left|\psi\rangle \right| \right) \equiv \min _{0 \leq S_n \leq I_{d_{min}}} \{ b_n(S_n) |a_n(S_n) \leq a \}$$

The optimal type 2 error under the restriction of the condition that the type 1 error is no more than $a$ can be written as

$$b_n \left( a \left[ |\phi_{\vec{K}}^n\rangle \right] \left|\psi\rangle \right| \right) \equiv \min _{0 \leq S_n \leq I_{d_{min}}} \{ b_n(S_n) |a_n(S_n) \leq a \}$$

In the following part of this subsection, we often abbreviate $b_n \left( a \left[ |\phi_{\vec{K}}^n\rangle \right] \left|\psi\rangle \right| \right)$ as $b_n(a)$. Then, in the present notation, the statement of the theorem can be written as follows:

**Theorem 3** (Theorem 5 of 53).

$$\beta_{n,sep}(\alpha|\rho_{mix}\rangle |\Psi\rangle) = b_n \left( \alpha a_{n,\rho_{mix}} \left[ |\phi_{\vec{K}}^n\rangle \right] \left|\psi\rangle \right| \right),$$

where $|\psi\rangle$ is a pure state on $\mathbb{C}^{d_{min}}$ defined as $|\psi\rangle \equiv \sum_i \sqrt{\nu_i} |i\rangle$ by using the Schmidt coefficients $\{\nu_i\}_{i=1}^{d_{min}}$ of $|\Psi\rangle$.

We define $Q_n(\kappa_0, \kappa_1; S_n)$ as the mean error probability of the above global hypothesis testing under a given POVM $\{S_n, I - S_n\}$ and a given prior $(\kappa_0, \kappa_1)$:

$$Q_n(\kappa_0, \kappa_1; S_n) \equiv \min _{0 \leq S_n \leq I_{d_{min}}} Q_n(\kappa_0, \kappa_1; S_n).$$

Then, an optimal mean probability is defined as

$$Q_n(\kappa_0, \kappa_1) \equiv \min _{0 \leq S_n \leq I_{d_{min}}} Q_n(\kappa_0, \kappa_1; S_n).$$

**Theorem 3** immediately leads the following lemma:

**Lemma 2.**

$$P_{n,sep}(\pi_0, \pi_1|\rho_{mix}\rangle |\Psi\rangle) = \left( \pi_0 a_{n,\rho_{mix}} + \pi_1 \right) \cdot Q_n(\kappa_0(n), \kappa_1(n)),$$

where $\kappa_0(n)$ and $\kappa_1(n)$ are defined as

$$\kappa_0(n) \equiv \left( \pi_0 a_{n,\rho_{mix}} + \pi_1 \right)^{-1} \cdot \pi_0 \cdot d_{max},$$

$$\kappa_1(n) \equiv \left( \pi_0 a_{n,\rho_{mix}} + \pi_1 \right)^{-1} \cdot \pi_1.$$

In the above lemma, Eq. (62) means that the Chernoff bound under separable POVMs are equal to the exponent of the optimal error probability of the above-mentioned global hypothesis testing problem with the prior $(\kappa_0(n), \kappa_1(n))$ which converges to (0, 1) in the limit $n \to \infty$.

**Proof of Lemma 2.**

Eq. (62) is just a direct consequence of Eq. (61).
Eq. (61) can be derived as follows:
\[
P_{n,se}(\pi_0, \pi_1) = \inf_{0 \leq \alpha, \beta \leq 1} \pi_0 \alpha + \pi_1 \beta_{n,se}(\alpha)
\]
\[
= \inf_{0 \leq \alpha, \beta \leq 1} \pi_0 \alpha + \pi_1 b_n(\alpha d_{max})
\]
\[
= \inf_{0 \leq \alpha, \beta \leq 1} \pi_0 \alpha + \pi_1 b_n(\alpha')
\]
\[
= \left(\frac{\pi_0}{d_{max}} + \pi_1\right) Q_n(\kappa_0^{(n)}, \kappa_1^{(n)})
\]  
(65)

In the above equations, we used Eq. (58) in the first and fifth line, Theorem 3 in the second line. We also used the fact \( b_n(\alpha') = b_n(1) \) for all \( \alpha' \geq 1 \) in the fourth line.

In order to evaluate \( \xi_{\text{se}} \), we start from the following lemma:

**Lemma 3.**
\[
\xi_{\text{se}}(\rho_{\text{mix}} \| \Psi)
\leq \lim_{n \to \infty} -\frac{1}{n} \log P_{n,g}(\kappa_0^{(n)}, \kappa_1^{(n)} \mid |\phi_0\rangle \| |\psi\rangle),
\]  
(66)

where \( |\phi_0\rangle \) is a pure state on \( \mathbb{C}^{d_{\text{min}}} \) defined as
\[
|\phi_0\rangle = \frac{1}{\sqrt{d_{\text{min}}}} \sum_{i=1}^{d_{\text{min}}} |i\rangle.
\]  
(67)

**(Proof)** From the definition of \( Q_n(\kappa_0^{(n)}, \kappa_1^{(n)}) \) and \( P_{n,g}(\kappa_0^{(n)}, \kappa_1^{(n)} \mid |\phi_0\rangle \| |\psi\rangle) \), we derive
\[
Q_n(\kappa_0^{(n)}, \kappa_1^{(n)})
\leq \min_{0 \leq S_n \leq I} \max_{\kappa \in \mathbb{S}_{d_{max}}^{n}} \kappa_0^{(n)} a_{n,\kappa}(S_n) + \kappa_1^{(n)} b_n(S_n)
\geq \min_{0 \leq S_n \leq I} \kappa_0^{(n)} a_{n,\vec{0}}(S_n) + \kappa_1^{(n)} b_n(S_n)
\]
\[
= P_{n,g}(\kappa_0^{(n)}, \kappa_1^{(n)} \mid |\phi_0\rangle \| |\psi\rangle),
\]  
(68)

where \( \vec{0} \) is the zero vector in \( \mathbb{Z}_2^{d_{max}} \). In the first line of the above inequalities, we used Eq. (63) and Eq. (64). The last equality can be derived by using the fact \( |\phi_0^{(n)}\rangle = |\phi_0\rangle^{\otimes n} \).

Eq. (68) and Eq. (62) guarantee Eq. (66). The following lemma gives an analytical formula for the right-hand-side of Eq. (66):

**Lemma 4.**
\[
\lim_{n \to \infty} -\frac{1}{n} \log P_{n,g}(\kappa_0^{(n)}, \kappa_1^{(n)} \mid |\phi_0\rangle \| |\psi\rangle)
\]
\[
= -2 \log \langle \psi | \phi_0 \rangle + \log d_{\text{max}}.
\]  
(69)

**(Proof)** Since we only need to consider the case when \( n \) is large, we only consider the case \( \langle \psi | \phi_0 \rangle^{2n} \leq \frac{1}{2} \).
Finally, we can derive the inequality \( \xi_{\text{sep}} \) as follows:

\[
\xi_{\text{sep}} = \lim_{n \to \infty} \max_{0 \leq S_n \leq t} -\frac{1}{n} \log \left( \kappa_0^{(n)} a_n(S_n) + \kappa_1^{(n)} b_n(S_n) \right) \\
\geq \lim_{n \to \infty} -\frac{1}{n} \log \left( \kappa_0^{(n)} a_n(S_{n,0}) + \kappa_1^{(n)} b_n(S_{n,0}) \right) \\
= -2 \log \langle \psi | \phi_0 \rangle + \log d_{\text{max}},
\]

where we use Eq. (76) and Eq. (77) in the second equality.

C. Hoeffding bound for separable POVMs

In this subsection, we analyze the Hoeffding bound of our hypothesis testing under separable POVMs.

Similar to the case of the Chernoff bound, we utilize Theorem 3 to derive a relationship between the hypothesis testing under separable POVMs and the global hypothesis testing with a composite alternative hypothesis. We define the Hoeffding bound of the corresponding global hypothesis testing as:

\[
b^R(r) = \sup_{\{S_n\}_{n=1}^\infty} \{ \lim_{n \to \infty} -\frac{1}{n} b_n(S_n) \} \\
\leq \frac{1}{n} a_n(S_n) \geq r, \ 0 \leq S_n \leq I^*_{d_{\text{min}}} \}.
\]

Then, Theorem 3 is immediately rewritten as:

**Lemma 5.** For \( r \geq \log d_{\text{max}} \),

\[ B_{\text{sep}}(r) = b^R(r - \log d_{\text{max}}). \]  

On the other hand, we can evaluate \( b^R(r) \) as follows:

**Lemma 6.**

\[
b^R(r) = +\infty, \quad \text{if} \ 0 \leq r \leq -\log |\langle \psi | \phi_0 \rangle|^2, \]  

\[
b^R(r) \leq -\log |\langle \psi | \phi_0 \rangle|^2, \quad \text{if} \ r > -\log |\langle \psi | \phi_0 \rangle|^2. \]  

(Proof)

First, by choosing \( S'_n = |\psi\rangle \otimes |\psi\rangle \), we observe \( \lim_{n \to \infty} -\frac{1}{n} a_n(S'_n) = -\log |\langle \psi | \phi_0 \rangle|^2 \) and \( \lim_{n \to \infty} -\frac{1}{n} \log b_n(S'_n) = +\infty. \) These equations guarantee Eq. (81).

Second, we will show Eq. (82). Suppose \( a_{0,n}(S_n) \) is defined as \( a_{0,n}(S_n) \equiv |\langle \phi_0 | \otimes |S_n| \phi_0 \rangle|^2 \), and \( b^R_0(r) \) is defined as

\[
b^R_0(r) = \sup_{\{S_n\}_{n=1}^\infty} \{ \lim_{n \to \infty} -\frac{1}{n} b_n(S_n) \} \\
\leq \frac{1}{n} a_{0,n}(S_n) \geq r, \ 0 \leq S_n \leq I^*_{d_{\text{min}}} \}. \]

Then, the inequality \( a_{0,n}(S_n) \leq a_n(S_n) \) guarantees

\[
b^R(r) \leq b^R_0(r). \]  

On the other hand, by the definition, we observe \( b^R_0(r) = B_g(r | \phi_0 \rangle | \psi \rangle). \) This fact and Eq. (19) guarantee

\[
b^R_0(r) = \begin{cases} 
+\infty, & \text{if} \ 0 \leq r \leq -\log |\langle \psi | \phi_0 \rangle|^2, \\
-\log |\langle \psi | \phi_0 \rangle|^2, & \text{if} \ r > -\log |\langle \psi | \phi_0 \rangle|^2.
\end{cases} \]  

This equation together with Eq. (82) guarantees Eq. (83).

\[ \square \]

The Stein-type’s strong converse bound \( \alpha_{\text{sep}}^{R1} \) also gives information about the value of \( B_{\text{sep}}(r) \) as follows:

**Lemma 7.** For \( r > \alpha_{\text{sep}}^{R1}, \ B_{\text{sep}}(r) = 0. \)

(Proof)

Suppose a sequence of separable POVMs \( \{T_n\}_{n=1}^\infty \) satisfies \( \lim_{n \to \infty} -\frac{1}{n} \log a(T_n) > \alpha_{\text{sep}}^{R1}. \) Then, from the definition of \( \alpha_{\text{sep}}^{R1}, \ \lim_{n \to \infty} -\frac{1}{n} \log \beta_n(T_n) = 0. \) This fact guarantees the statement of the lemma.

Finally, we evaluate \( B_{\text{sep}}(r) \) as follows:

**Theorem 5.**

\[
B_{\text{sep}}(r) = \begin{cases} 
+\infty, & \text{if} \ 0 \leq r \leq \log d_A + \log d_B - LR(|\Psi\rangle), \\
0, & \text{if} \ r > \log d_A + \log d_B - E(|\Psi\rangle).
\end{cases} \]  

(86)  

\[
B_{\text{sep}}(r) \leq \log d_{\text{min}} - LR(|\Psi\rangle), \\
\text{if} \ r > \log d_A + \log d_B - LR(|\Psi\rangle). \]  

(87)  

(88)

(Proof)

Lemma 5, 6, 7, and Corollary 2 guarantee this theorem.

\[ \square \]

From this theorem, we can also evaluate \( A_{\text{sep}}(r) \):

**Corollary 3.** For \( r \geq \log d_{\text{min}} - LR(|\Psi\rangle), \ A_{\text{sep}}(r) \) is given as

\[
A_{\text{sep}}(r) = \log d_A + \log d_B - LR(|\Psi\rangle), \]  

and for \( 0 \leq r \leq \log d_{\text{min}} - LR(|\Psi\rangle), \) it is evaluated as

\[
\log d_A + \log d_B - E(|\psi\rangle) \leq A_{\text{sep}}(r) \leq \log d_A + \log d_B - E(|\psi\rangle). \]  

(90)

(Proof)

First, the second inequality of Eq. (81) comes from the inequality \( A_{\text{sep}}(r) \leq \alpha_{\text{sep}}^{R1}. \) Second, Eq. (83) guarantees

\[
A_{\text{sep}}(r) \geq \log d_{\text{max}} - \log |\langle \psi | \phi_0 \rangle|^2, \ \forall r > 0. \]  

(91)  

This guarantees the first inequality of Eq. (89). Finally, Eq. (85) guarantees that there is no sequence of separable
POVMs \( \{ T_n \}_{n=1}^{\infty} \) satisfying \( \lim -\frac{1}{n} \alpha_n(T_n) > \log d_{\max} - \log \langle \psi | \phi_0 \rangle^2 \) and \( \lim -\frac{1}{n} \beta_n(T_n) > -\log \langle \psi | \phi_0 \rangle^2 \). This fact and the inequality (91) guarantee the equality (93). \( \square \)

Here, we note that when \( |\Psi\rangle \) is a maximally entangled state, we have \( \log d_A + \log d_B - LR(|\Psi\rangle) = \log d_A + \log d_B - E(|\Psi\rangle) - \log d_{\max} \) and \( \log d_{\min} - LR(|\Psi\rangle) = 0 \). Hence, Theorem 5 and Corollary 8 completely determine the behavior of \( B_{sep}(r) \) and \( A_{sep}(r) \) in this case, respectively.

V. HYPOTHESIS TESTING UNDER TWO-WAY LOCC POVM

In this section, we consider the case of \( C \leftrightarrow \), that is, the local hypothesis testing under separable LOCC POVMs.

In the previous paper [59], we showed that there is a gap between optimal error probabilities under one-way and two-way LOCC POVMs in non-asymptotic problem settings of this local hypothesis testing. On the other hand, in asymptotic problem settings, we have already proved that there is no difference among one-way, two-way LOCC, and separable POVMs in terms of the Stein’s lemma type of the optimal error exponents \( \alpha_C \) and \( \beta_C \) in Theorem 12. Hence, in this section, we concentrate ourselves on the Chernoff and Hoeffding bounds under two-way LOCC POVMs, and ask a question whether there is a gap between one-way and two-way LOCC POVMs.

Since the definition of the two-way LOCC is mathematically complicated in comparison to that of one-way LOCC and separable operations, it seems impossible to derive analytical formula for the Chernoff and Hoeffding bounds under two-way LOCC POVM. Hence, we try to derive a lower bound of the Chernoff and Hoeffding bounds under two-way LOCC POVM by numerical calculations.

Since infinite number of parameters are necessary to describe general two-way LOCC operations, it is impossible to optimize the error exponent numerically for general two-way LOCC protocols. Thus, here, we consider only a special class of tree step LOCC protocols which can outperform any one-way LOCC protocols in non-asymptotic problem settings [22, 58]. However, even if we restrict ourselves on the special class of three step LOCC protocols, we need to treat infinitely many parameters in an asymptotic situation. Thus, in order to simplify the analysis furthermore, we only consider the situation where Alice and Bob apply the three step LOCC protocols for each single copy of an unknown state, and then, they apply an optimal classical hypothesis testing protocol on the resulted unknown classical probability distribution.

For one copy of the unknown state \( |\Psi\rangle \) or \( \rho_{mix} \), the tree-step LOCC protocol is described as follows:

1. Alice measures her state by a POVM \( \{ M_\omega \}_{\omega \in \mathcal{P}(d_A)} \).
   Here, \( \mathcal{P}(d_A) \) is a power set (a set of all subsets) of a finite set \( \{1, \ldots, d_A\} \) including an empty set \( \emptyset \). For a non empty set \( \omega \), \( M_\omega \) is defined as \( M_\omega \overset{\text{def}}{=} \sum_{h \in \omega} m_{hi}^A |h\rangle \langle h| \) in terms of positive coefficients \( m_{hi}^A \), where \( \{ |h\rangle \}_{i=1}^{d_A} \) is Alice’s part of the Schmidt basis of \( |\Psi\rangle \). Then, \( M_\emptyset \) is defined as \( M_\emptyset \overset{\text{def}}{=} I_A - \sum_{\omega \in \mathcal{P}(d_A) \setminus \emptyset} M_\omega \). When Alice’s result is \( \emptyset \), Alice and Bob stop the protocol and conclude the unknown state to be \( \rho_{mix} \). Otherwise, they continue the protocol.

2. At the second step, Bob measures his state by a POVM \( \{ N_\omega \}_{j=0}^{\omega} \) depending on Alice’s measurement result \( \omega \), where \( |\omega| \) is a size of set \( \omega \). For \( j \neq 0 \), \( N_\omega \) is defined as \( N_\omega \overset{\text{def}}{=} |\xi_j^\omega\rangle \langle \xi_j^\omega| \), where \( \{ |\xi_j^\omega\rangle \}_{j=1}^{|\omega|} \) is a mutually unbiased basis of the subspace \( \text{span} \{ |h\rangle \}_{h \in \omega} \). Then, \( N_\emptyset \) is defined as \( N_\emptyset \overset{\text{def}}{=} I_B - \sum_{j=1}^{|\omega|} N_\omega \). When Bob derive the result 0, Alice and Bob stop the protocol and conclude the unknown state to be \( \rho_{mix} \). Otherwise, they continue the protocol.

3. At the third step, Alice measures her states by a POVM \( \{ O_\omega \}_{k=0}^{\omega} \), where \( O_0 \overset{\text{def}}{=} I_A - O_1 \). Here, \( O_0 \) is chosen as Alice’s state after the Bob’s measurement in the case when the given state is \( |\Psi\rangle \), and is defined as \( O_0 \overset{\text{def}}{=} \sqrt{M_{\rho_A}} (|\zeta_j^0\rangle \langle \zeta_j^0|) \sqrt{M_{\rho_A}} \), where \( \rho_A \overset{\text{def}}{=} \text{Tr}_B |\Psi\rangle \langle \Psi| \), and \( T \) is the transposition in the Schmidt basis of \( |\Psi\rangle \). When Alice’s measurement result \( k \) is 0, Alice and Bob conclude the unknown state to be \( |\Psi\rangle \), and otherwise they conclude the unknown state to be \( \rho_{mix} \).

Here, we note that the diagonal elements of the first Alice’s POVM \( \{ m^A_{hi} \}_{\omega,h} \) are the only free parameters on this three step LOCC protocol.

After applying the above protocol for each single copy of \( n \)-copies of an unknown state, Alice and Bob derive \( n \)-copies of unknown classical probability distributions. We call the unknown distribution \( P_{|\Psi\rangle}(\omega, j, k) \) or \( P_{\rho_{mix}}(\omega, j, k) \) depending on the initial state \( |\Psi\rangle \) or \( \rho_{mix} \), where \( \omega, j, k \) are the results of Alice’s first, Bob’s, and Alice’s second POVMs, respectively. Thus, by applying an asymptotically-optimal classical hypothesis testing, the Chernoff and Hoeffding types of error rates achieve the classical Chernoff and Hoeffding bounds between \( P_{|\Psi\rangle}(\omega, j, k) \) and \( P_{\rho_{mix}}(\omega, j, k) \), respectively. By optimizing them over \( \{ m^A_{hi} \}_{\omega,h} \), we derive the best error exponent which can be achieved by the above three step LOCC protocols. We write such bounds as \( \xi_{\leftrightarrow} \) and \( \xi_{\leftrightarrow} \), where \( \xi_{\leftrightarrow} \) is the Chernoff type, and \( \xi_{\leftrightarrow} \) is the Hoeffding
type. Hence, those are defined as:

\[
\tilde{\xi}_{\omega} \overset{\text{def}}{=} \sup_{\{m_{h}^{h}\}_{\omega,h}} \left\{ \xi_{\omega} \left( P_{\rho_{mix}}^{h}(\omega,j,k)\| P_{\omega}(\omega,j,k) \right) \mid 0 \leq m_{h}^{h} \leq 1, \sum_{\omega \in O_{h}} m_{h}^{h} \leq 1 \right\} \tag{92}
\]

\[
\tilde{A}_{\omega}(r) \overset{\text{def}}{=} \sup_{\{m_{h}^{h}\}_{\omega,h}} \left\{ A_{\omega}(r) \left| P_{\rho_{mix}}^{h}(\omega,j,k)\| P_{\omega}(\omega,j,k) \right| \mid 0 \leq m_{h}^{h} \leq 1, \sum_{\omega \in O_{h}} m_{h}^{h} \leq 1 \right\} \tag{93}
\]

In the above formula, \( O_{h} \in \mathcal{P} (\mathcal{P}(d_{A})) \) is defined by the relation \( \omega \in O_{h} \Leftrightarrow h \in \omega \) for \( \omega \in \mathcal{P}(d_{A}) \). \( \xi_{\omega}(p\|q) \) and \( A_{\omega}(r\|q) \) are the classical Chernoff and Hoeffding bounds, respectively. By their definitions, these error exponents are upper and lower bounded by the optimal error exponents of one-way and two-way LOCC POVM, respectively:

**Corollary 4.**

\[
\xi_{\omega} \leq \tilde{\xi}_{\omega} \leq \hat{\xi}_{\omega}, \tag{94}
\]

\[
A_{\omega}(r) \leq \tilde{A}_{\omega}(r) \leq \hat{A}_{\omega}(r). \tag{95}
\]

**Proof.**

The proofs of Eq.(92) and Eq.(93) are essentially the same. Their second inequalities comes directly from their definitions, since \( \xi_{\omega} \) and \( A_{\omega}(r) \) are optimal Chernoff and Hoeffding error exponents for the particular class of two-way LOCC POVMs. On the other hand, Lemma 1 guarantees that an optimal one-way LOCC protocol is just the projection onto the Schmidt basis of \( |\Psi\rangle \) followed by classical information processing. Actually, the projection onto the Schmidt basis is included in the above three step LOCC protocols when \( m_{h}^{h} = 1 \) for \( \omega = \{1\}, \ldots, \{d_{A}\} \), and \( m_{h}^{h} = 0 \) for the other choices of \( \omega \). This fact guarantees the first inequalities of Eq.(92) and Eq.(93).

By means of these analytical formulas Eq.(11) and Eq.(12), which are also applicable to the classical states \( P_{\rho_{mix}}(i,j,k) \) and \( P_{\omega}(i,j,k) \), we derive the following formulas for \( \xi_{\omega} \) and \( A_{\omega}(r) \) after straightforward calculations:

\[
\tilde{\xi}_{\omega} = \sup_{0 \leq s \leq 1, \{m_{h}^{h}\}_{\omega,h}} \left\{ f(s, \{m_{h}^{h}\}_{\omega,h}) \mid 0 \leq m_{h}^{h} \leq 1, \sum_{\omega \in O_{h}} m_{h}^{h} \leq 1 \right\}, \tag{96}
\]

\[
\tilde{A}_{\omega}(r) = \sup_{0 \leq s \leq 1, \{m_{h}^{h}\}_{\omega,h}} \left\{ \frac{f(s, \{m_{h}^{h}\}_{\omega,h}) - rs}{1 - s} \mid 0 \leq m_{h}^{h} \leq 1, \sum_{\omega \in O_{h}} m_{h}^{h} \leq 1 \right\}, \tag{97}
\]

where an objective function \( f(s, \{m_{h}^{h}\}_{\omega,h}) \) is given as

\[
f(s, \{m_{h}^{h}\}_{\omega,h}) = -\log \text{Tr} P_{\rho_{mix}}(i,j,k)^{1-s} P_{\omega}(i,j,k)^{s} = -\log \left[ \left( 1 - \frac{1}{d_{A}} \sum_{\omega \in \mathcal{P}(d_{A})} \sum_{h \in \omega} m_{h}^{h} \right)^{1-s} \cdot \left( 1 - \sum_{\omega \in \mathcal{P}(d_{A})} \sum_{h \in \omega} m_{h}^{h} \lambda_{h} \right)^{s} \right] \cdot \left( \sum_{h \in \omega} m_{h}^{h} \lambda_{h} \right)^{2s-1}. \tag{98}
\]

In the next section, we give plots of numerical calculation.

**VI. PLOTS OF THE ERROR EXPONENTS**

In this section, we give plots of various error exponents for our local hypothesis testing problem. Since there is no difference among one-way LOCC, two-way LOCC and separable POVMs in terms of the error exponents corresponding to Stein’s Lemma, here, we only plot error exponents corresponding to the Chernoff and the Hoeffding bounds.

[FIG. 1: The Chernoff bounds for \( |\Psi\rangle = \sqrt{\lambda}(11) + \sqrt{1-\lambda}(22) \). The lines labeled “Global”, “Separable”, “Two-way LOCC”, and “One-way LOCC” are plots of \( \xi_{\omega}, \xi_{sep}, \xi_{\omega}, \) as functions of \( \lambda \), respectively.

FIG.1 and FIG.2 are plots of the Chernoff bounds \( \xi_{\omega}, \xi_{sep}, \xi_{\omega}, \) as functions of \( \lambda \), respectively. \( \xi_{\omega} \) and \( \xi_{sep} \) were calculated via analytical expressions (25) and (72), respectively. On the other hand, \( \xi_{\omega} \) is numerically calculated by means of Eq.(46) and Eq.(49). We chose \( |\Psi\rangle \) as \( \sqrt{\lambda}(11) + \sqrt{1-\lambda}(22) \) in FIG.1, and \( \sqrt{\lambda}(11) + \sqrt{1-\lambda}(22) \) in FIG.2. Here, we observe that \( \xi_{\omega} \) is always strictly larger than \( \xi_{\omega} \) except the case when \( |\Psi\rangle \) is a product state or a maximally entangled state, and, moreover, it well approximates \( \xi_{sep} \).]
LOCC" are plots of "Global", "Sep. upper bound", "Sep. lower bound", "Two-way LOCC", and "One-way LOCC" are plots of $\lambda$, $\xi_{sep}$, $\xi_{\alpha}$ as functions of $\lambda$, respectively.

FIG. 2: The Chernoff bounds for $|\Psi\rangle = \sqrt{\lambda}(|11\rangle + |22\rangle + |33\rangle) + \sqrt{1-\lambda}|44\rangle$. The lines labeled “Global”, “Sep. upper bound”, “Sep. lower bound”, “Two-way LOCC”, and “One-way LOCC” are plots of $\xi_\alpha$, $\xi_{sep}$, $\xi_{\alpha}$, $\xi_{\alpha}$ as functions of $\lambda$, respectively.

This means that the above three step LOCC protocols not only outperform the best one-way LOCC protocols, but they are even near to optimum over all two-way LOCC protocols in the viewpoint of the Chernoff bounds.

Next we give plot of the Hoeffding bounds $A_C(r)$ against a parameter $r$. Before show the plots, we gives two remarks. First, from the definition, the Hoeffding bound is equal to the Stein’s Lemma type of the error exponent in the limit of $r \to +0$. Thus, the y-intercepts of the plots for one-way LOCC $A_\rightarrow(r)$, two-way LOCC $A_\leftrightarrow(r)$, and separable POVM $A_{sep}(r)$ are all equal to $a_{\alpha,\leftrightarrow} = a_{\alpha,\rightarrow} = a_{\alpha,sep} = \log d_A + \log d_B - E(|\Psi\rangle)$, although, as we can observe from FIG. 3 and 4, the convergence of $A_\leftrightarrow(r)$ in the limit $r \to +0$ is very slow. Second, when a given $r_0$ satisfies $A_C(r_0) = r_0$, the Hoeffding bound for $r_0$ is equal to the Chernoff bound: $A_C(r_0) = \xi_C$. We can actually find such a $r_0$ as an intersection of the graphs of $y = A_C(r)$ and $y = x$. Thus, we can derive the value of $\xi_C$ from the plots of $A_C(r)$.

For one-way LOCC POVMs, we have an analytical expression of $A_\rightarrow(r)$ via Theorem I. On the other hand, for separable POVMs, since we do not know an analytical formula for $A_{sep}(r)$ available for all range of $r > 0$, we use analytical upper $\overline{A}_{sep}(r)$ and lower bounds $\underline{A}_{sep}(r)$ of $A_{sep}(r)$ derived from Corollary III instead of $A_{sep}(r)$:

$$
A_{\leftrightarrow}(r) \overset{\text{def}}{=} \begin{cases} 
\log d_A + \log d_B - E(|\Psi\rangle), & \text{if } 0 \leq r \leq \log d_{min} - LR(|\Psi\rangle) \\
\log d_A + \log d_B - LR(|\Psi\rangle), & \text{if } r \geq \log d_{min} - LR(|\Psi\rangle)
\end{cases}
$$

(99)

$$
\underline{A}_{sep}(r) \overset{\text{def}}{=} \log d_A + \log d_B - LR(|\Psi\rangle)
$$

(100)

FIG.3 and FIG.4 are plots of the Hoeffding bounds $A_\rightarrow(r)$, $\overline{A}_{sep}(r)$, $\underline{A}_{sep}(r)$, $A_\rightarrow(r)$ as functions of the parameter $r$, respectively. Here, $\overline{A}_{\leftrightarrow}(r)$ is numerically calculated by means of Eq. (97) and Eq. (98). We chose $|\Psi\rangle$ as $\frac{1}{\sqrt{3}}|11\rangle + \frac{1}{\sqrt{2}}|22\rangle$ in FIG.3, and $\left(\frac{1}{\sqrt{5}}|11\rangle + \frac{2}{\sqrt{2}}|22\rangle\right)^{\otimes 2}$ in FIG.4.

From FIG. 3 and FIG. 4, we observe that $\overline{A}_{\leftrightarrow}(r)$ well approximates $\overline{A}_{sep}(r)$ if $A_\rightarrow(r) < \overline{A}_{sep}(r)$, and $\overline{A}_{\leftrightarrow}(r)$ is $A_\rightarrow(r)$ otherwise. As a result, we well approximates $A_{sep}(r) = \overline{A}_{sep}(r) = \overline{A}_{sep}(r)$ in the region $r \geq \log d_{min} - LR(|\Psi\rangle)$. Thus, we observe that, at least in low dimensional systems and in this region of the parameter $r$, the above three step LOCC protocols are near to optimum over all two-way LOCC protocols in the viewpoint of the Hoeffding bounds, too. Here, we note: this is the first result showing the existence of a gap between first error exponents under an optimal one-way LOCC and under a two-way LOCC protocols in asymptotic settings of local discrimination problems.
VII. SUMMARY

In this paper, we treated an local asymptotic hypothesis testing between an arbitrary known bipartite pure state $|\Psi\rangle$ and the completely mixed state. As a result, we showed that: the Stein’s lemma type of optimal error exponents are given as $\log d_A + \log d_B - E(|\Psi\rangle)$ for all one-way LOCC, two-way LOCC and separable POVMs. The Chernoff bounds are given as $\log d_A + \log d_B - \log R_s(|\Psi\rangle)$ for one-way LOCC POVMs, and as $\log d_A + \log d_B - LR(|\Psi\rangle)$ for separable POVMs. In these formulas, $E(|\Psi\rangle)$, $R_s(|\Psi\rangle)$, and $LR(|\Psi\rangle)$ are the entropy of entanglement, the Schmidt rank, the logarithmic robustness of entanglement of $|\Psi\rangle$, respectively. In the view point of the entanglement theory, these formulas give new operational interpretations for these entanglement measures. Moreover, we derive analytical formulas of the Hoeffding bounds under one-way LOCC POVM without any restriction on a parameter, and under separable POVM on a restricted region of a parameter. Finally, we numerically calculated the Chernoff and the Hoeffding bounds under a particular class of three step LOCC protocols in low dimensional systems, and showed that these bounds not only outperform the bounds for One-way LOCC, but almost approximate the bounds for Separable POVMs in the region of parameter where analytical bounds for Separable POVMs are derived. As far as we know, this is a first time to show existence of a gap between optimal error exponent under one-way LOCC POVM and two-way LOCC POVM in “asymptotic local discrimination problems”.

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