DENSE ANALYTIC SUBSPACES IN FRACTAL $L^2$-SPACES

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ABSTRACT. We consider self-similar measures $\mu$ with support in the interval $0 \leq x \leq 1$ which have the analytic functions $\left\{ e^{i2\pi n x} : n = 0, 1, 2, \ldots \right\}$ span a dense subspace in $L^2(\mu)$. Depending on the fractal dimension of $\mu$, we identify subsets $P \subset \mathbb{N}_0 = \{0, 1, 2, \ldots \}$ such that the functions $\{e_n : n \in P\}$ form an orthonormal basis for $L^2(\mu)$. We also give a higher-dimensional affine construction leading to self-similar measures $\mu$ with support in $\mathbb{R}^\nu$. It is obtained from a given expansive $\nu$-by-$\nu$ matrix and a finite set of translation vectors, and we show that the corresponding $L^2(\mu)$ has an orthonormal basis of exponentials $e^{i2\pi \lambda \cdot x}$, indexed by vectors $\lambda$ in $\mathbb{R}^\nu$, provided certain geometric conditions (involving the Ruelle transfer operator) hold for the affine system.

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1. Introduction: Fractal Measures

We use the notation $e_n(x) := e^{i2\pi nx}$, or in complex form $e_n = z^n$ where $z = e^{i2\pi x}$, $x \in \mathbb{R}$. Recall that if $\mu$ is Lebesgue measure on $I = [0,1] \simeq \mathbb{R}/\mathbb{Z} = \mathbb{T}$, then the functions $\{e_n : n \in \mathbb{N}_0\}$ span the Hardy space $H_2$ of analytic functions on $\mathbb{T}$, and $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(I)$ with normalized Lebesgue measure. Functions of the form $F(z) = \sum_{n=0}^{\infty} A_n z^n$ with $\sum_{n=0}^{\infty} |A_n|^2 < \infty$ may be viewed also as defined on $U := \{z \in \mathbb{C} : |z| \leq 1\}$ and $F(z)$ may be identified with its values on the boundary $\partial U = \mathbb{T}$, represented by $z = e^{i2\pi x}$, $x \in \mathbb{R}$. Writing $f(x) := F(e^{i2\pi x})$, and defining $E_z(x) = \sum_{n=0}^{\infty} z^n e^{i2\pi nx}$, we get the familiar reproducing kernel formula

$$ F(z) = \langle E_z | f \rangle = \int_0^1 E_z(x) f(x) \, dx = \sum_{n=0}^{\infty} z^n A_n, \quad |z| < 1. \quad (1.1) $$

Since we show that, for some fractal measures $\mu$ on $I = [0,1]$, there are associated subsets $P \subset \mathbb{N}_0$ with orthogonal and total $\{e_n : n \in P\}$ in $L^2(\mu)$, it follows that this property then carries over to $L^2(\mu)$, i.e., the corresponding Hardy space $H_2(P,\mu)$ is all of $L^2(\mu)$.

In the present paper, we will consider fractal measures $\mu$ supported on compact sets (typically totally disconnected) in $\mathbb{R}^\nu$ which arise from iteration algorithms that are defined from a given affine system of maps in $\mathbb{R}^\nu$. These maps in turn will be defined from an expansive real $\nu$-by-$\nu$ matrix $R$, and a given “sparse” set of translation vectors in $\mathbb{R}^\nu$. The fractal nature of these constructions arises from choosing the number of translation vectors to be strictly smaller than $|\text{det } R|$ where $R$ is the given expansive matrix. Our aim will be to understand the possible orthogonal harmonic function systems in $L^2(\mu)$, and, in particular, to give conditions for $L^2(\mu)$ to have an orthonormal harmonic basis. (This continues earlier work of the co-authors [JoPe97].) Many of the ideas going into the general case are present already in a relatively simple example in a single dimension, and this will be introduced and discussed in Section 3 below.

While there has been a great amount of recent interest in measures which arise from geometric iteration algorithms, see, e.g., [Ban91], [Ban96], [BaGe94], [BoTa87], [Ho93], [LaWa96], [LaWa96d], [KSS95], [Rue88], and [PoSi95], there are relatively few results on the corresponding harmonic analysis of these measures. But recently (see, e.g., [BoTa87] and [Ho93]) diffraction lines in quasicrystals have lent themselves to such a harmonic analysis. In our earlier paper [JoPe97], we noted a connection between certain diffraction spectra and the spectral duality considerations which go into the harmonic analysis of affine self-similarity (for the measures $\mu$ which we sketched above).

While previous harmonic-analysis approaches to self-affine measures (see, e.g., [Str89]) have stressed continuous transforms and asymptotic summation methods, our present approach is discrete and directly based on Fourier series, for those $\mu$ which have the orthogonality property under discussion.

A central tool in our work is a certain double duality: first the usual duality of Fourier analysis, corresponding to the dual variables on either side of the spectral transform; and secondly a duality which derives from our use of matrix scaling. Small scales correspond to compact attractors of fractal Hausdorff dimension, while large scales (“fractals in the large”) correspond to a discrete set of frequencies (in $\nu$ dimensions), $\lambda = (\lambda_1, \ldots, \lambda_\nu) \in \mathbb{R}^\nu$ which label our Fourier basis of orthogonal
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Figure 1. Support of $\mu$

exponentials $e_\lambda(x) := e^{i2\pi\lambda x}$ where $x$ is restricted to the (“small scale”) fractal. In our setup, both scales, small and large, are finitely generated, referring to two given finite subsets $B$ and $L$ in $\mathbb{R}^\nu$ (one on each side of the duality) which are paired in a certain unitary matrix $U(B,L)$, defined from the two sets, and to be described in (3.2) below. The unitary matrix $U(B,L)$ is related to one studied by Hadamard. It turns out that not all configurations of sets $B,L$ allow such a unitary pairing, and there is a further constraint from the dimension $\nu$ of the ambient Euclidean space.

We now turn to the theory, following an illustrative example in a single dimension, i.e., $\nu = 1$.

2. An Example: Scale Four

It is known (see, e.g., [JoPe93], [JoPe96]) that there is a unique probability measure $\mu$ on $\mathbb{R}$ of compact support such that
\[
\int f \, d\mu = \frac{1}{2} \left( \int f \left( \frac{x}{4} \right) \, d\mu(x) + \int f \left( \frac{x}{4} + \frac{1}{2} \right) \, d\mu(x) \right)
\]
for all continuous $f$. In fact, the support $K$ of $\mu$ is the Cantor set obtained by dividing $I = [0,1]$ into four equal subintervals, and retaining only the first and third. (See Figure 1 below.) The Hausdorff dimension of $\mu$ is $d_H = \frac{\ln 2}{\ln 4} = \frac{1}{2}$.

This is a special case of a more general construction in $\nu$ dimensions ($\nu \geq 1$) corresponding to some given real matrix $R$, and a finite subset $B \subset \mathbb{R}^\nu$. It is assumed that
\[
R \text{ has eigenvalues } \xi_i \text{ all satisfying } |\xi_i| > 1.
\]

The subset $B$ is required to satisfy an open-set condition: Introduce
\[
\sigma_b x = R^{-1} x + b, \quad x \in \mathbb{R}^\nu.
\]
It is assumed that there is a nonempty, bounded open set $V$ such that
\[
\bigcup_{b \in B} \sigma_b V \subset V
\]
with the union disjoint corresponding to distinct points in $B$. Our present $\{\sigma_b\}$ systems (see below for the axioms) are special cases of iterated function systems (i.f.s.) considered in [Hut81], see also [Fal80]. There are many interesting more general i.f.s., and that context also leads to measures $\mu$ which satisfy a general
version of the invariance property (2.5), and there is then a corresponding “open-
set assumption”. But for our present affine systems, the splitting property (2.4),
for some open subset $V$ in $\mathbb{R}^\nu$, can be shown in fact to be automatic, see [JoPe96].
In concrete examples, e.g., Figures 1–3 below, the choice of such an open set $V$ is
often apparent from the geometry of the affine maps $\sigma_b$. We do not directly use the
open-set condition (2.4) in our present proofs, except in the application of Theorem
8.3 (Remark 8.5) where the interior of a particular simplex is identified for the open
set $V$, and the property gets used in (8.10). For general i.f.s., the property is needed
for the computation of geometric invariants, such as the Hausdorff dimension, see
[Fal86], for the particular iteration limits; and, for our present systems, it forces
the invariant fractal measure $\mu$ to be locally translation invariant, see [JoPe97,
Appendix]. If $N = \#(B)$, then the corresponding measure $\mu$ on $\mathbb{R}^\nu$ (depending on
$R$ and $B$) has compact support, and satisfies
\[
\int f \, d\mu = \frac{1}{N} \sum_{b \in B} \int f(\sigma_b(x)) \, d\mu(x)
\]
for all continuous $f$. (For more details on the "open-set condition" and affinely
generated fractal measures, we give the following background references: [JoPe94],
[Str94], [Str95], and [Ped97].) Define, for $t \in \mathbb{R}^\nu$, the Fourier transform
\[
\hat{\mu}(t) = \int e^{i2\pi t \cdot x} \, d\mu(x)
\]
with $t \cdot x = \sum_{i=1}^\nu t_i x_i$, we then get
\[
\hat{\mu}(t) = \chi_B(t) \hat{\mu}(R^*-1t)
\]
where
\[
\chi_B(t) := \frac{1}{N} \sum_{b \in B} e^{i2\pi b \cdot t}
\]
and $R^*$ is the transposed matrix. For the above example, this amounts to
\[
\hat{\mu}(t) = \frac{1}{2} (1 + e^{i\pi t}) \hat{\mu} \left( \frac{t}{4} \right), \quad t \in \mathbb{R}.
\]

3. ORTHOGONAL FREQUENCIES AND FRAC TAL HARDY SPACES

Assume that the matrix $R$ in (2.3) has integral entries, and that
\[
RB \subset \mathbb{Z}^\nu, \quad 0 \in B,
\]
but that none of the differences $b - b'$ is in $\mathbb{Z}^\nu$ when $b, b' \in B$ are different.
Furthermore, assume that some subset $L \subset \mathbb{Z}^\nu$ satisfies $0 \in L, \#(L) = N (= \#(B))$,
and the matrix
\[
H_{BL} := N^{-\frac{1}{2}} \left( e^{i2\pi b \cdot l} \right) \text{ is unitary as an } N \times N \text{ complex matrix,}
\]
i.e., $H_{BL}^*H_{BL} = I_N \quad (^* = \text{transposed conjugate}).$
In fact, it can be checked that the assumed non-integrality of the differences $b - b'$
(when $\neq 0$) follows from assuming that $H_{BL}$ is unitary for some $L$ as described.
For our purposes, the assumptions $L \subset \mathbb{Z}^\nu$ and $RB \subset \mathbb{Z}^\nu$ may actually be weakened as follows:
\[
(R^nb) \cdot l \in \mathbb{Z}, \quad \forall n \in \mathbb{N}, \quad b \in B, \quad l \in L.
\]
Details on this more general setup will be given in Section 3 below.

**Lemma 3.1.** With the assumptions, set

\[(3.4) \quad P := \{l_0 + R^* l_1 + \cdots : l_i \in L, \text{ finite sums}\}.\]

Then the functions \(\{e_\lambda : \lambda \in P\}\) are mutually orthogonal in \(L^2(\mu)\) where

\[(3.5) \quad e_\lambda(x) := e^{2\pi \lambda \cdot x}.\]

**Proof.** Let \(\lambda = \sum R^* l_i, \lambda' = \sum R^* l_i'\) be points in \(P\), and assume \(\lambda \neq \lambda'\). Then

\[\langle e_\lambda | e_{\lambda'} \rangle_\mu = \int \overline{e_{\lambda'}} e_{\lambda'} d\mu = \int e^{i2\pi (\lambda' - \lambda) \cdot x} d\mu(x) = \hat{\mu}(\lambda' - \lambda) = \hat{\mu}(l_0' - l_0 + R^* (l_1' - l_1) + \cdots) = \chi_B(l_0' - l_0) \hat{\mu}(l_0' - l_0 + R^* (l_1' - l_1 + R^* (l_2' - l_2) + \cdots)).\]

If \(l_0' \neq l_0\) then \(\chi_B(l_0' - l_0) = 0\) by (2.2). If not, there is a first \(n\) such that \(l_n' \neq l_n\), and then

\[\hat{\mu}(\lambda' - \lambda) = \hat{\mu}(R^* n (l_n' - l_n) + R^* n+1 (l_{n+1}' - l_{n+1}) + \cdots) = \chi_B(l_n' - l_n) \hat{\mu}(l_n' - l_n + l_{n+1} + \cdots) = 0\]

since \(\chi_B(l_n' - l_n) = 0\). \(\square\)

**Corollary 3.2.** Let \(\mu\) be the measure on the line \(\mathbb{R}\) given by \(\{2.1\}\) and with Hausdorff dimension \(d_H = \frac{1}{2}\). (We have \(R = 4, B = \{0, \frac{1}{2}\}\) and \(L = \{0, 1\}\).) Then

\[(3.6) \quad P = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0, 1\}, \text{ finite sums}\},\]

and \(\{e_\lambda : \lambda \in P\}\) is an orthonormal subset of \(L^2(\mu)\).

**Proof.** Immediate from the lemma. \(\square\)

**Lemma 3.3.** Let the subsets \(B, L \subset \mathbb{R}^\nu\), and the matrix \(R\) be as described before Lemma 3.1. Let

\[(3.7) \quad Q_1(t) := \sum_{\lambda \in P} |\hat{\mu}(t - \lambda)|^2, \quad t \in \mathbb{R}^\nu.\]

Then \(\{e_\lambda : \lambda \in P\}\) is an orthonormal basis for \(L^2(\mu)\) if and only if \(Q_1 \equiv 1\) on \(\mathbb{R}^\nu\).

**Proof.** If \(\{e_\lambda : \lambda \in P\}\) is an orthogonal basis for \(L^2(\mu)\), the Bessel inequality is an identity when applied to \(e_t\); that is,

\[1 = \|e_t\|_\mu^2 = \sum_{\lambda} \left|\langle e_\lambda | e_t \rangle_\mu \right|^2 = \sum_{\lambda} |\hat{\mu}(t - \lambda)|^2.\]

Conversely, if this holds, and if \(f \in L^2(\mu) \cap \{e_\lambda : \lambda \in P\}\), then \(\langle e_t | f \rangle_\mu = 0\) for all \(t \in \mathbb{R}^\nu\), or equivalently \(\int e^{-i2\pi t \cdot x} f(x) d\mu(x) = 0\) for all \(t \in \mathbb{R}^\nu\). This implies \(f = 0\) by Stone–Weierstrass applied to the compact support \(\text{supp}(\mu)\). \(\square\)
Remark 3.5. While $L^2(\mu)$ is spanned by $\{z^n : n \in P\}$ ($z = e^{i2\pi x}$), and $P \subset \mathbb{N}_0$, it is of course not the case that
\begin{equation}
\int z^n \, d\mu = 0 \quad \text{for all } n \in \mathbb{N}.
\end{equation}
(The coefficients do vanish for $n \in \{1, 4, 5, 16, 17, \ldots\} = P \setminus \{0\}$, even when $n$ is of the form $n = 4^k (2^l + 1)$, $k \in \mathbb{N}_0$, but not for $n$ in a bigger subset of $\mathbb{N}$.) First of all, we showed in [JoPe96] that $\{z^n : n \in P\}$ is maximally orthogonal; and secondly, if $\mu$ is viewed as a measure on $\mathbb{T}$, then the validity of (3.9) would imply (by the F. and M. Riesz theorem) absolute continuity of $\mu$ with respect to Lebesgue measure $\lambda$, i.e., $\frac{d\mu}{d\lambda} \in L^1(\lambda)$, which is clearly false. It is known (e.g., [JoPe94]) that $\mu$ is purely singular.

Corollary 3.6. There is a canonical isometric embedding $\Phi$ of $L^2(\mu)$ into the subspace $H_2(z^4) + zH_2(z^4)$ of $H_2(z^4)$: $H_2(z^4) := \{f(z^4) : f \in H_2\}$; and it is given by
\begin{equation}
\Phi \left( \sum_{\lambda \in P} \langle c_\lambda \rangle \right) = \sum_{n \in P} c_{4n} z^{4n} + z \sum_{n \in P} c_{4n+1} z^{4n}.
\end{equation}

Proof. Since $\sum_{\lambda \in P} |c_\lambda|^2 < \infty$, and $P = \bigcup_{l \in \{0, 1\}} l + 4P$, with $4P \cap (1 + 4P) = \emptyset$, the representation (3.10) is well defined. Note that $\Phi$ is everywhere defined on $L^2(\mu)$ by Theorem 3.4, and the two functions $f_0(z) = \sum_{n \in P} c_{4n} z^{4n}$ and $f_1(z) = \sum_{n \in P} c_{4n+1} z^{4n}$ are in $H_2(z^4)$.

Remark 3.7. (Fractal Hardy spaces) An iteration of the argument from the proof of the corollary yields, for each $n \in \mathbb{N}$, a natural isometric embedding $\Phi_n$ of $L^2(\mu)$ into the subspace of $H_2$ characterized as $n$ increases by:
\begin{align*}
H_2(z^{4n}) + zH_2(z^{4n}) + z^4 H_2(z^{4n}) + z^8 H_2(z^{4n}) + z^{16} H_2(z^{4n}) + z^{17} H_2(z^{4n}) & + \ldots + z^{4^{n+1} - 1} H_2(z^{4n}).
\end{align*}
Specifically, let $n \in \mathbb{N}$ be fixed, and let $P_n = \{l_0 + 4l_1 + \cdots + 4^{n-1}l_{n-1} : l_i \in \{0, 1\}\}$. Then the functions in $\Phi_n(L^2(\mu)) \subset H_2$ have the following characteristic module representation:
\[ \left\{ \sum_{p \in P_n} z^p f_p(z^{4n}) : f_p \in H_2 \right\}. \]
For each $n$, $\Phi_n$ maps into this space, and not onto.

Remark 3.8. (Spectral pairs) Our interest in the problem of finding dense analytic subspaces in $L^2(\mu)$, for probability measures, grew out of our earlier work on spectral pairs, see, e.g., [JoPe91], [JoPe92], [JoPe93a]. Consider subsets $\Omega$ and $\Lambda$ in $\mathbb{R}^d$, with $\Omega$ of finite positive Lebesgue measure, and let $L^2(\Omega)$ be the corresponding
the triadic Cantor set (see Figure 2). It is determined by

\[ \langle \xi, x \rangle \] denotes the pairing between points \( \xi \) in \( \Gamma \) and \( x \) in \( G \). If \( f \mapsto F_\mu f \) extends to an isomorphic isometry (i.e., unitary) of \( L^2(\mu) \) onto \( L^2(\rho) \), then we say that \( (\mu, \rho) \) is a spectral pair. (The case when \( \mu \) is a restriction of Lebesgue measure was studied in [Ped87].) It is clear how the earlier definition of spectral pairs is a special case, even when \( G \) is restricted to the additive group \( \mathbb{R}^n \). But it is not immediate that there are examples \((\mu, \rho)\) of the new spectral pair type which cannot be reduced to the old one.

Theorem 3.4 shows that this is indeed the case (i.e., that there are examples): Let \( G = \mathbb{R} \), and let \( \mu \) be the fractal measure in Theorem 3.4. Let \( \rho = \rho_P \) be the counting measure of \( P \). Then the conclusion in Theorem 3.4 may be restated to the effect that \((\mu, \rho_P)\) is a spectral pair. This is perhaps surprising as earlier work on Fourier analysis of fractal measures, see e.g. [Str90b], [Str93], and [JoPe97], suggested a continuity in the Fourier transform, and also the presence of asymptotic estimates, rather than exact identities.

It can be shown, as a consequence of [JoPe97] Corollary A.5 that if \((\mu, \rho)\) satisfies our spectral-pair property for any measure \( \rho \), then \( \rho = \rho_P \) for some subset \( P \subset \mathbb{R}^n \), i.e., \( L^2(\mu) \) has an orthonormal basis of the form \( \{ e_\lambda : \lambda \in P \} \). The basis for this argument is the finiteness of \( \mu \), when generated from \( \Omega \).

**Remark 3.9.** (The triadic Cantor measure) The significance of the assumptions \((3.1) - (3.2)\) on the pair \( R, B \) lies in the identity \((1.3)\) below, and also in orthogonality. If, for example, we work with the more traditional triadic Cantor set, then the results in Lemmas 3.1 and 3.3 no longer are valid. To see this, take \( R = 3 \) and \( B = \{0, \frac{2}{3}\} \). Let \( \mu_3 \) denote the corresponding measure on \( \mathbb{R} \) with support equal to the triadic Cantor set (see Figure 2). It is determined by

\[
\int f \, d\mu_3 = \frac{1}{2} \left( \int f \left( \frac{x}{3} \right) \, d\mu_3 (x) + \int f \left( \frac{x}{3} + \frac{2}{3} \right) \, d\mu_3 (x) \right), \quad \forall f \in C_c (\mathbb{R}),
\]

has \( d_H = \frac{\ln 2}{\ln 3} \), and satisfies

\[
\hat{\mu}_3 (t) = \frac{1}{2} \left( 1 + e^{i \frac{2}{3} \pi t} \right) \hat{\mu}_3 \left( \frac{t}{3} \right), \quad t \in \mathbb{R}.
\]
Choose \( L = \{0, \frac{1}{3}\} \) so that (3.2) is valid; then the subset \( P_3 \) (which corresponds to \( P = P(L) \) in Lemma 3.3), is
\[
P_3 = \left\{ \frac{3}{4} (l_0 + 3l_1 + 3^2 l_2 + \cdots) : l_i \in \{0, 1\}, \text{ finite sums} \right\},
\]
but the corresponding exponentials \( \{e_\lambda : \lambda \in P_3\} \) are now not mutually orthogonal in \( L^2(\mu_3) \). Take for example the two points \( \lambda = \frac{3}{4} \) and \( \lambda' = \frac{9}{4} \) both in the set \( P_3 \).

The corresponding exponentials \( e_\lambda \) and \( e_{\lambda'} \) are both orthogonal to \( e_0 \), but they are not mutually orthogonal, i.e., \( \langle e_\lambda | e_{\lambda'} \rangle_{\mu_3} \neq 0. \) In fact, for the \( \mu_3 \)-inner product:
\[
\langle e_\lambda | e_{\lambda'} \rangle_{\mu_3} = \widehat{\mu_3} \left( \frac{3}{2} \right) = \frac{1}{4} \widehat{\mu_3} \left( \frac{1}{6} \right) \neq 0.
\]

It can further be shown (see [JoPe97] and Section 6 below) that there is no subset \( P \subset \mathbb{R} \) such that, if \( \rho_P \) denotes the corresponding counting measure on \( \mathbb{R} \), then \( (\mu_3, \rho_P) \) is a spectral pair in the above more general sense (a fortiori, fractions don’t provide a basis either). Similarly, it can be checked that the identity (4.3) in Lemma 4.1 below fails for this pair \( \mu_3, P_3 \), i.e., the triadic one with \( d_H = \frac{\ln 2}{\ln 3} \).

4. Proof of Theorem 3.4: Ruelle’s Transfer Operator

We need only verify the assumption in Lemma 3.3. But when applied to the example in Section 2, the issue becomes showing that \( \sum_{n \in P} |\hat{\mu} (t-n)|^2 = 1, t \in \mathbb{R} \), where the summation in \( n \) is over
\[
P = P(L) = \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0, 1\}, \text{ finite sums} \}.
\]

More generally, let the sets \( B, L \subset \mathbb{R}^\nu \), and the matrix \( R \), be as in Lemma 3.1.

Lemma 4.1. The function
\[
Q_1 (t) := \sum_{\lambda \in P} |\hat{\mu} (t-\lambda)|^2
\]
(where \( P = \{l_0 + R^* l_1 + \cdots : l_i \in L, \text{ finite sums} \} \)) satisfies the functional identity
\[
Q(t) = \sum_{l \in L} |\chi_B (t-l)|^2 Q (R^{-1} (t-l)).
\]
Proof. Let \( t \in \mathbb{R}^\nu \). Then
\[
Q_1 (t) = \sum_{\lambda \in P} |\hat{\mu} (t - \lambda)|^2 \\
= \sum_{\lambda \in P} |\chi_B (t - \lambda)|^2 |\hat{\mu} (R^* t^{-1} (t - \lambda))|^2 \\
= \sum_{\lambda \in P} \sum_{l \in L} |\chi_B (t - l - R^* \lambda)|^2 |\hat{\mu} (R^* t^{-1} (t - l) - \lambda)|^2 \\
= \sum_{l \in L} |\chi_B (t - l)|^2 \sum_{\lambda \in P} |\hat{\mu} (R^* t^{-1} (t - l) - \lambda)|^2 \\
= \sum_{l \in L} |\chi_B (t - l)|^2 Q_1 (R^* t^{-1} (t - l)).
\]
\[\square\]

Remark 4.2. We shall need the following operator \( C \), Ruelle’s transfer operator, defined on functions \( Q \):
\[
C (Q) (t) := \sum_{l \in L} |\chi_B (t - l)|^2 Q (R^* t^{-1} (t - l)).
\]

We also note

Lemma 4.3.

(i) The function \( Q_1 \), defined on \( \mathbb{R}^\nu \) by \( \langle 4.2 \rangle \), has an entire analytic extension to \( \mathbb{C}^\nu \) which is of linear exponential growth in the imaginary direction, i.e., with \( \mathbb{C}^\nu = \mathbb{R}^\nu + i\mathbb{R}^\nu \),
\[
|Q_1 (t + is)| \leq e^{4\pi m \|s\|_2}
\]
where \( m \) is the diameter of the support of \( \mu \) inside \( \mathbb{R}^\nu \).

(ii) The operator \( C \) in \( \langle 4.4 \rangle \) extends naturally to the space \( \mathcal{E}_m \) of entire analytic functions \( f \) on \( \mathbb{C}^\nu \) which satisfy \( \langle 4.5 \rangle \) and \( f (0) = 1 \), and maps this space into itself, i.e., \( C : \mathcal{E}_m \to \mathcal{E}_m \).

Proof. For \( t \in \mathbb{R}^\nu \),
\[
Q_1 (t) = \sum_{\lambda \in P} \hat{\mu} (t - \lambda) \hat{\mu} (\lambda - t) = \sum_{\lambda \in P} \langle e_\lambda | e_t \rangle_\mu \langle e_{-\lambda} | e_{-t} \rangle_\mu.
\]

If the right-hand side is truncated, summing only over the finite subsets in \( P \) given by the restriction \( |\lambda| \leq n \), \( n = 1, 2, \ldots \), then each of the resulting finite sums \( \sum_{\lambda \in P, |\lambda| \leq n} \) defines an entire analytic function in \( t \) (i.e., \( t \) is now allowed to range over \( \mathbb{C}^\nu \), with the convention \( e_t (x) = e^{itx} \), and \( t \cdot x = \sum_{j=1}^\nu t_j x_j \), \( t_j \in \mathbb{C} \), \( x_j \in \mathbb{R} \)). These functions are also in \( L^2 (\mu) \) since the measure \( \mu \) from \( \langle 2.3 \rangle \) has compact support \( \text{supp} (\mu) \) in \( \mathbb{R}^\nu \). Hence
\[
\left| \sum_{\lambda \in P, |\lambda| \leq n} \cdots \right| \leq \sum_{\lambda \in P, |\lambda| \leq n} \left| \sum_{\lambda} \langle e_\lambda | e_t \rangle_\mu \langle e_{-\lambda} | e_{-t} \rangle_\mu \right| \\
\leq \left( \sum_{\lambda} \langle e_\lambda | e_t \rangle_\mu^2 \right)^{\frac{1}{2}} \left( \sum_{\lambda} \langle e_{-\lambda} | e_{-t} \rangle_\mu^2 \right)^{\frac{1}{2}} \\
\leq \|e_t\|_{L^2(\mu)} \|e_{-t}\|_{L^2(\mu)}.
\]
Note this holds for the $P$ summation, for all $n$. Letting $n \to \infty$, we then also get the estimate for the unrestricted summation over $\lambda \in P$. This may be formulated as a pointwise approximation
\[
\lim_{n \to \infty} Q_n(t) = \tilde{Q}(t), \quad t \in \mathbb{C}',
\]
the limit now providing an extension of $Q(\cdot)$, initially given only on $\mathbb{R}'$ by (4.2).

Suppose the support of $\mu$ is contained in $\{x \in \mathbb{R}' : |x| \leq m\}$; then
\[
\|e_t\|_{L^2(\mu)}^2 = \int |e^{ix(t-x)}|^2 \, d\mu(x) = \int e^{-4\pi(\text{Im} t) \cdot x} \, d\mu(x) \leq e^{4\pi m |\text{Im} t|}
\]
where
\[
|\text{Im} t| = \left( \sum_{j=1}^{\nu} |\text{Im} t_j|^2 \right)^{\frac{1}{2}}, \quad t \in \mathbb{C}'.
\]
This means that the functions $Q_n(\cdot \cdot)$ are uniformly bounded in $t \in \mathbb{C}'$, restricted to each strip $\|\text{Im} t\| \leq \text{constant}$. We then conclude by the theorems of Montel and Vitali (see [Neh75, p. 143]) that the limit $\tilde{Q}(t)$ is entire analytic, $t \in \mathbb{C}'$, and serves as an analytic extension of $Q_1$ from (4.2), given only on $\mathbb{R}'$.

The technique which we used above in estimating the extended $Q(t + is)$ in $\mathbb{C}' = \mathbb{R}' + i\mathbb{R}'$ also applies, mutatis mutandis, on the other term, to yield:
\[
\sum_{l \in L} |\chi_B (t + is - l)|^2 \leq N^{-1} \sum_{b \in B} e^{4\pi \|b\|_2 |s|}
\]
by virtue of (4.2). When combined with (4.5), defining $E_m$, and (4.4), the invariance, $C : E_m \to E_m$ follows. We include the details of proof for estimate (4.6) (using unitarity of the matrix in (4.2)):
\[
\sum_{l \in L} |\chi_B (t + is - l)|^2 = N^{-1} \sum_{l \in L} \sum_{b \in B} \left| N^{-\frac{1}{2}} e^{-i2\pi b \cdot l} e^{i2\pi b \cdot (t + is)} \right|^2
\]
\[
= N^{-1} \sum_{b \in B} \left| e^{i2\pi b \cdot (t + is)} \right|^2
\]
\[
= N^{-1} \sum_{b \in B} e^{-4\pi b \cdot s}
\]
\[
\leq N^{-1} \sum_{b \in B} e^{4\pi \|b\|_2 |s|}.
\]
\[\square\]

**Lemma 4.4.** Let $\mu$ be a probability measure with compact support on $\mathbb{R}'$, and let $P \subset \mathbb{R}'$ be such that $\{e_\lambda : \lambda \in P\}$ is orthogonal in $L^2(\mu)$. Set $Q_1(t) = \sum_{\lambda \in P} |\hat{\mu}(t - \lambda)|^2$. Then $Q_1$ is $C^\infty$ on $\mathbb{R}'$; in fact, it has an entire analytic extension. Let $A : L^2(\mu) \to L^2(\mu)$ be the orthogonal projection onto $H_2(P, \mu)$. Then

(i) $Q_1(t) = \|A e_t\|_{L^2(\mu)}^2$;
(ii) $\frac{\partial}{\partial t} Q_1(t)|_{t=0} = 0$,
(iii) $\frac{\partial^2}{\partial t^2} Q_1(t)|_{t=0} = \|Ax_j\|_{\mu}^2 - \|x_j\|_{\mu}^2$, where $\|x_j\|_{\mu}^2 = \int x_j^2 \, d\mu(x)$, and
(iv) $\frac{\partial^n}{\partial t^n} Q_1(t)|_{t=0} = (-1)^n \left( \|x_j^n\|_{\mu}^2 - \|Ax_j^n\|_{\mu}^2 \right)$.

**Proof.** Compute! \[\square\]
Corollary 4.5. Let \( \{ e_\lambda : \lambda \in P \} \) be a set of orthogonal frequencies in \( L^2(\mu) \) and let \( Q_1(t) = \sum_{\lambda \in P} |\hat{\mu}(t - \lambda)|^2 \), \( t \in \mathbb{R}^n \). Assume \( L \) is not contained in a hyperplane in \( \mathbb{R}^n \). Then the set \( P \) is total if and only if

\[
(4.7) \quad \left( \frac{\partial}{\partial t} \right)^{2n} Q_1(t) \big|_{t=0} = 0, \quad j = 1, \ldots, \nu, \quad n = 1, 2, \ldots.
\]

Proof. By Lemmas 3.3 and 4.4 we only need to note that the monomials are dense in \( L^2(\mu) \), by Stone–Weierstrass, and that a vector \( f \in L^2(\mu) \) is in the subspace \( AL^2(\mu) \) if and only if \( \|Af\|_\mu = \|f\|_\mu \). (For more details, see Section 3 below.)

5. THE EXAMPLE REVISITED

Let \( \nu = 1, B = \{0, \frac{1}{3}\} \), \( L = \{0,1\} \), and \( R = 4 \). Let \( \mu \) be the corresponding measure \( (dH(\mu) = \frac{1}{2}) \), and let

\[
(5.1) \quad P = P(L) = \{l_0 + 4l_1 + 4^2l_2 + \cdots : l_i \in \{0,1\}, \text{finite sums}\}.
\]

Lemma 5.1. The function

\[
Q_1(t) = \sum_{\lambda \in P} |\hat{\mu}(t - \lambda)|^2
\]

satisfies the following identity:

\[
(5.2) \quad Q_1(t) = \cos^2 \left( \frac{\pi t}{2} \right) Q_1 \left( \frac{t}{4} \right) + \sin^2 \left( \frac{\pi t}{2} \right) Q_1 \left( \frac{t-1}{4} \right).
\]

Let \( C \) be the operator defined by the right-hand side in \( (5.2) \). On the convex space \( W_0 := \{Q \in C^1 \left( \left[ -\frac{1}{4}, 0 \right] \right) : Q(0) = 1 \} \), introduce the metric

\[
(5.3) \quad d(Q_1, Q_2) := \sup \{|Q_1'(t) - Q_2'(t)| : t \in \left[ -\frac{1}{4}, 0 \right]\}.
\]

Then \( (W_0, d) \) is a complete metric space, and

\[
(5.4) \quad d(C(Q_1), C(Q_2)) \leq \left( \frac{1}{4} + \frac{\pi \sqrt{3}}{16} \right) d(Q_1, Q_2), \quad \forall Q_1, Q_2 \in W_0.
\]

Proof. First note that the functional identity \( (5.2) \) in the lemma is a special case of \( (4.3) \) in Lemma 4.1 above. It is clear that \( C1 = 1 \) where \( 1 \) denotes the constant function. In the general case of Lemma 4.1 that follows from \( (3.2) \).

For \( Q_1, Q_2 \in W_0 \), we have \( Q_1(t) - Q_2(t) = \int_0^t (Q_1'(s) - Q_2'(s)) \, ds \), and therefore \( |Q_1(t) - Q_2(t)| \leq \frac{1}{3} d(Q_1, Q_2) \). In particular, \( (W_0, d) \) is a complete metric space. We have \( C \) inducing an operator in the convex set \( W_0 \), and

\[
C(Q')(t) = \frac{1}{4} \left( \cos^2 \left( \frac{\pi t}{2} \right) Q' \left( \frac{t}{4} \right) + \sin^2 \left( \frac{\pi t}{2} \right) Q' \left( \frac{t-1}{4} \right) \right)
\]

\[
+ \frac{\pi}{2} (-\sin \pi t) \, Q \left( \frac{t}{4} \right) + \frac{\pi}{2} (\sin \pi t) \, Q \left( \frac{t-1}{4} \right)
\]

\[
= \frac{1}{4} \left( \cos^2 \left( \frac{\pi t}{2} \right) Q' \left( \frac{t}{4} \right) + \sin^2 \left( \frac{\pi t}{2} \right) Q' \left( \frac{t-1}{4} \right) \right)
\]

\[
+ \frac{\pi}{2} (\sin \pi t) \int_{\frac{t}{4}}^{\frac{t-1}{4}} Q'(s) \, ds.
\]
For \( Q_1, Q_2 \in W_0 \), the first term is estimated above by \( \frac{1}{4} \sup_{t \in [-\frac{4}{3}, 0]} |Q_1'(t) - Q_2'(t)| \) sine the two affine transformations \( t \mapsto \frac{t}{3} \) and \( t \mapsto \frac{t}{3} - \frac{1}{3} \) leave the interval \([-\frac{1}{3}, 0] \) invariant. The second term is bounded by
\[
\frac{\pi}{2} \cdot \frac{\sqrt{3}}{2} \int_{-\frac{4}{3}}^{0} |Q_1'(s) - Q_2'(s)| \, ds
\]
where we estimate \(|\sin \pi t|\) by \( \frac{\sqrt{3}}{2} \) in the interval \( t \in [-\frac{1}{3}, 0] \). Since the integral is bounded by \( \frac{1}{4} \sup_{s \in [-\frac{4}{3}, 0]} |Q_1'(s) - Q_2'(s)| \), the result follows.

**Corollary 5.2.** Since the constant function \( Q \equiv 1 \) in \([-\frac{1}{3}, 0]\) satisfies \( C(Q) = Q \) by (5.3), we conclude that \( Q_1 \) from (6.2) must be constant in \([-\frac{1}{3}, 0]\) by the Banach fixed-point principle. Theorem 8.4 now follows by Lemma 5.1.

### 6. One Dimension: Even and Odd Scales

If a subset \( P \subset \mathbb{R} \) is orthogonal in the sense that \( \{e_\lambda : \lambda \in P\} \) is orthogonal in \( L^2(\mu) \), for some fixed probability measure \( \mu \) on \( \mathbb{R} \), then so is the reflected set \(-P\), and the translates \( t + P \) when \( t \in \mathbb{R} \) is fixed. In reviewing orthogonal subsets \( P \) we shall therefore impose the condition \( 0 \in P \) for simplicity.

We now examine the following three cases for \( \mu \) as they illustrate the nature of the assumptions made above, as well as the variety of the possible cases: For the subset \( B = \{0, \frac{1}{2}\} \), we have considered the two scales \( R = 2 \) and \( R = 4 \), and also \( R = 3 \) separately in (6.10) below. The resulting measures \( \mu_2, \mu_4, \) and \( \mu_3 \) are determined by their Fourier transforms, as follows, (6.1), (6.2), and (6.10):

\[
\hat{\mu}_2(t) = \prod_{n=0}^{\infty} \frac{1}{2} \left(1 + e^{i\frac{\pi}{2n}}\right) = \prod_{n=1}^{\infty} e^{i\frac{\pi}{2n}} \cos \left(\frac{\pi t}{2n}\right)
\]
\[
= e^{i\pi t} \sum_{n=1}^{\infty} \frac{i}{2n} \prod_{n=1}^{\infty} \cos \left(\frac{\pi t}{2n}\right) = e^{i\pi t} \frac{\sin(\pi t)}{\pi t},
\]
where in the last step we used a familiar infinite product formula for \( \cos \left(\frac{\pi x}{2}\right) \).

The infinite product coincides with the integral \( \int_{0}^{1} e^{i2\pi xt} \, dx = \frac{e^{i\pi t} - 1}{i2\pi t} \) since \( \mu_2 \) is Lebesgue measure on \( I \). Similarly,

\[
\hat{\mu}_4(t) = \prod_{n=0}^{\infty} \frac{1}{2} \left(1 + e^{i\frac{4\pi}{2n}}\right) = e^{i\pi \frac{4\pi}{2n}} \prod_{n=0}^{\infty} \cos \left(\frac{\pi t}{2\cdot 4^n}\right).
\]

(The \( \hat{\mu}_3 \) product will be discussed separately.) We picked the set \( L = \{0, 1\} \subset \mathbb{Z} \) to satisfy condition (3.2) above. Then from (6.1)–(6.4) we get:

\[
P_2 = \{l_0 + 2l_1 + 2^2l_2 + \cdots : l_i \in \{0, 1\}, \text{ finite sums}\} = N_0 = \{0, 1, 2, \ldots \}
\]
and

\[
P_4 = \{l_0 + 4l_1 + 4^2l_2 + \cdots : l_i \in \{0, 1\}, \text{ finite sums}\} = \{0, 1, 4, 5, 16, 17, \ldots \}.
\]

We get explicit formulas for the zero sets of the transforms,

(6.3) \( \mathcal{Z}(\hat{\mu}_2) = \mathbb{Z} \setminus \{0\} \) (= the nonzero integers),
and

\[(6.4) \quad \mathbb{Z} (\hat{\mu}_4) = \{4^n \cdot (1 + 2\mathbb{Z}) : n = 0, 1, 2, \ldots \}\]

(= powers of 4 times odd integers)

where \(\mu_2\) is Lebesgue measure, supported on \(I = [0, 1]\), while \(\mu_4\) is the fractal measure with \(d_H = \frac{1}{2}\) and support on the Cantor set from Figure 1. In the first case, it is classical that \(H_2 (P_2, \mu_2)\) is the familiar Hardy space in \(L^2 (\mu_2)\). That is because \(L^2 (\mu_2)\) may be identified with \(L^2\) of the circle, and we are back to classical Fourier series, i.e., \(H_2\) spanned by \(1, z, z^2, \ldots\) in \(L^2 (\mathbb{T})\). The contrast between the two cases is made clear from a comparison of the respective functions \(Q_2\) and \(Q_4\) (derived from \((6.7)\)) as follows:

\[Q_2 (t) = \sum_{n \in P_2} |\hat{\mu}_2 (t - n)|^2 = \sum_{n=0}^{\infty} \left| \frac{\sin \pi (t - n)}{\pi (t - n)} \right|^2 = \sum_{n=0}^{\infty} \frac{1}{(t - n)^2},\]

where we provide the following interpretation of the last expression (see [Art64] and [Car95]): Let \(\Gamma (t)\) be the gamma function, with its meromorphic extension resulting from the functional equation

\[\Gamma (t) \Gamma (1 - t) = \frac{\pi}{\sin (\pi t)} ,\]

recalling

\[\Gamma (t) = \int_0^{\infty} e^{-x^t} x^{t-1} dx ,\]

for positive \(t\). When \(t > 0\),

\[\frac{d}{dt} \left( \frac{\Gamma' (t)}{\Gamma (t)} \right) = \sum_{n=0}^{\infty} \frac{1}{(t + n)^2} =: \psi (t) ,\]

is well defined, and negative \(t\) is from the meromorphic extension. It follows that

\[(6.5) \quad Q_2 (t) = \| A_2 e_t \|_{L^2 (\mu_2)}^2 = \frac{\sin^2 (\pi t)}{\pi^2} \psi (-t) ,\]

where \(A_2\) denotes the projection in \(L^2\) onto the Hardy space \(H_2\). (Note that it is not immediate from this formula \((6.5)\) that \(Q_2 (t)\) is entire analytic.) In contrast, for \(Q_4 (t)\) we have

\[Q_4 (t) = \| A_4 e_t \|_{L^2 (\mu_4)}^2 \equiv 1;\]

i.e., as noted in Theorem \(3.4\), \(A_4 = I\), or, equivalently, the \(\mu_4\)-Hardy space is all of \(L^2 (\mu_4)\), and embedded as a subspace of \(H_2\) via

\[H_2 (z^4) \oplus zH_2 (z^4) .\]

(See Corollary \(3.4\) for details.)

In the \(\mu_2\) case, the operator \(C\), which is analogous to \(C\) for \(\mu_4\) in \((5.2)\), is

\[(6.6) \quad C (Q) (t) = \cos^2 \left( \frac{\pi t}{2} \right) Q \left( \frac{t}{2} \right) + \sin^2 \left( \frac{\pi t}{2} \right) Q \left( \frac{t - 1}{2} \right) .\]

In the \(\mu_2\) case, the convex \(W_0\) is

\[(6.7) \quad W_0 = \{ Q \in C^1 ([0, 1]) : Q (0) = 1 \} .\]
It can be checked that $C$ is then not strictly contractive in $W_0$ relative to the metric
\begin{equation}
    d(Q_1, Q_2) = \sup_{t \in [-1,0]} |Q'_1(t) - Q'_2(t)|,
\end{equation}
and, as noted, $C$ has fixed points in $W_0$ other than the constant function $Q \equiv 1$. The contractivity constant can be checked in this case (i.e., \(6.6\), \(6.7\), \(6.8\) corresponding to $R = 2$) to be $1/2 + \pi/4$ which is $> 1$ in contrast to the $R = 4$ case in \(\text{(5.3)}\) where the contractivity constant is $1/4 + \pi\sqrt{3}/16$ ($< 1$). This illustrates the assumption
\begin{equation}
    \#(B) < |\det R|
\end{equation}
which will be placed on the general systems from Section \(7\) below.

Set $z = e^{i2\pi t}$, and $g(z) = \cos^2(\pi t) = \frac{1}{2} + \frac{1}{2}z + \frac{1}{4}z^2$. Then \(6.6\) may be rewritten in complex form as
\begin{equation}
    C(Q)(z) = \sum_{w \in z^\mu} g(w) Q(w),
\end{equation}
which is a special case of the familiar Perron–Frobenius–Ruelle operator \[\text{Rue94}\].

The presence of the above mentioned two independent solutions (in $W_0$) to $C(Q) = Q$ is not predicted by Ruelle’s theory. The Perron–Frobenius eigenvalue problem $C(Q) = Q$ is actually studied in wavelet theory, see, e.g., \[\text{Dau92}, \text{Chapter 6}\]. Since \(\{z \in \mathbb{T}: g(z) = 1\}\) is just a singleton there is only one periodic solution $Q$ in $W_0$. This is consistent with the second solution from \(\text{(5.5)}\) above being non-periodic.

We also considered $B = \{0, \frac{2}{3}\}$ and $R = 3$, where the measure $\mu_3$ has $d_H = \ln 2/\ln 3$ (see Figure \[\text{3}\]), and is given by
\begin{equation}
    \hat{\mu}_3(t) = \prod_{n=1}^\infty \frac{1}{2} \left(1 + e^{i\frac{2\pi nt}{3}}\right) = e^{\pi t} \prod_{n=1}^\infty \cos \left(\frac{2\pi t}{3^n}\right)
\end{equation}
with
\begin{equation}
    Z(\hat{\mu}_3) = \left\{\frac{3^n}{4} (1 + 2\mathbb{Z}) : n = 1, 2, 3, \ldots \right\}.
\end{equation}

The contrast to the earlier case is that the sets $L$ which make \(\text{(3.2)}\) valid, e.g., $L = \{0, \frac{1}{2}\}$, are not contained in $\mathbb{Z}$. It can be checked (see Theorem \[\text{3.1}\] below) that $L^2(\mu_3)$ does not contain orthogonal families \(\{e_\lambda : \lambda \in P\}\) when $P \subset \mathbb{R}$ has three, or more, distinct elements. In particular, there are no infinite orthogonal sets of exponentials in $L^2(\mu_3)$. It can be checked, from our formula for $Z(\hat{\mu}_3)$, that $P_4$ \((= \{0, 1, 4, 5, 16, 17, \ldots \})\) is a unique maximal, orthogonal subset for $\mu_4$, containing $0$, and contained in $\mathbb{N}_0$.

We finally mention that $\mu_{3,4} := \mu_3 * \mu_4$ (convolution) has $P_4$ as a maximal orthogonal set for $L^2(\mu_{3,4})$. (Recall $Z(\hat{\mu}_3) \cap Z(\hat{\mu}_4) = \emptyset$, from \(\text{(6.4)}\) and \(\text{(6.11)}\) above, and $\hat{\mu}_{3,4} = \hat{\mu}_3 \cdot \hat{\mu}_4$.) But \(\{e_\lambda : \lambda \in P_4\}\) is not total in $L^2(\mu_{3,4})$; or, equivalently,
\begin{equation}
    Q(t) := \sum_{n \in P_4} |\mu_{3,4}(t-n)|^2
\end{equation}
is not constant on $\mathbb{R}$.

The possibilities described above may be summarized in the following two theorems \((\nu = 1)\) for measures on $\mathbb{R}$ with affine self-similarity. For both results, we have $R, b$ determined as follows: $R \in \mathbb{Z} \setminus \{0\}$, $b \in \mathbb{R} \setminus \{0\}$; defining $\sigma_b(x) = R^{-1}x$. 

σ_b(x) = R^{-1}x + b, recall that the corresponding probability measure µ on \(R\) depends on both \(R\) and \(b\), being determined uniquely by

\[
\mu = \frac{1}{2} (\mu \circ \sigma_0^{-1} + \mu \circ \sigma_b^{-1})
\]

when \(|R| > 1\).

**Theorem 6.1.** Let \(R\) be an odd integer, \(R \neq \pm 1\), and let \(µ\) be the corresponding measure. Then any set of \(µ\)-orthogonal exponentials contains at most two elements.

**Proof.** Recall that

\[
\hat{µ}(t) = \prod_{n=0}^{\infty} \frac{1}{2} \left(1 + e^{it\frac{\pi}{R}n}\right) = e^{i\frac{\pi}{R}b} \prod_{n=0}^{\infty} \cos \left(\frac{\pi bt}{R^n}\right)
\]

and therefore

\[
Z(\hat{µ}) = \left\{ \frac{R^n}{2b} (2Z + 1) : n = 0, 1, \ldots \right\}.
\]

If \(\gamma_j, j = 1, 2, 3\), are such that the \(e^{\gamma_j}\)'s are mutually orthogonal in \(L^2(µ)\), then the differences \(\gamma_i - \gamma_j (i \neq j)\) are in \(Z(\hat{µ})\). Let \(\lambda_1 = \gamma_1 - \gamma_2, \lambda_2 = \gamma_2 - \gamma_3, \lambda_0 = \gamma_1 - \gamma_3, \) and \(\lambda_j = \frac{R^n}{2b} (2z_j + 1), z_j \in \mathbb{Z}\). Since

\[
\lambda_1 + \lambda_2 = \lambda_0,
\]

we get

\[
R^{\lambda_1} (2z_1 + 1) + R^{\lambda_2} (2z_2 + 1) = R^{\lambda_0} (2z_0 + 1),
\]

which is a contradiction: the left-hand side is even while the right-hand side is odd.

**Theorem 6.2.** Let \(R\) be an even \((\neq 0)\) integer, \(|R| \geq 4\), and let \(b \in \frac{1}{2} \mathbb{Z} \setminus \{0\}\). Let \(z_0 = \frac{1}{R}\) and set \(P := z_0 \left\{ l_0 + Rl_1 + R^2l_2 + \cdots : l_i \in \{0, 1\}, \text{finite sums} \right\}\). Then \(\{e_\lambda : \lambda \in P\}\) is an orthonormal basis for \(L^2(µ)\).

**Proof.** The only part of the proof which is not included in the previous discussion is the strict contractivity of the operator \(C : Q \to C(Q)\), from (4.4). It specializes to

\[
C(Q)(t) = \cos^2(\pi bt) Q \left(\frac{t}{R}\right) + \sin^2(\pi bt) Q \left(\frac{t - \frac{z_0}{R}}{R}\right).
\]

Let \(J\) denote the closed interval between 0 and \(\frac{z_0}{R}\). Depending on the signs, we can have 0 as an endpoint to the left or the right. Set \(W_0 = \{Q \in C^1(J) : Q(0) = 1\}\) and

\[
d(Q_1, Q_2) := \sup_{t \in J} |Q'_1(t) - Q'_2(t)|.
\]

Then the argument from the proof of Theorem 3.4 leads to the following strict contractivity for \(Q \to C(Q)\) with an explicit formula for the contractivity constant \((\gamma < 1)\):

\[
d(C(Q_1), C(Q_2)) \leq \gamma d(Q_1, Q_2) \quad \forall Q_1, Q_2 \in W_0,
\]
where
\[ \gamma = \gamma (R) := \frac{\pi}{2R} \sin \left( \frac{\pi}{|R| - 1} \right) + \frac{1}{R}. \]

(The details of the proof of this contractivity property are the same, \textit{mutatis mutandis}, as the corresponding ones in Section 4 for the special case \( R = 4, b = \frac{1}{2} \).) Specifically, one of the terms in the estimation of \( d (C (Q_1), C (Q_2)) \) will be
\[ \pi b |\sin (2\pi bt)| \int_{\frac{t}{R} - z_0}^{\frac{t}{R}} |Q_1' (s) - Q_2' (s)| \, ds \]
for the case when \( \frac{t - z_0}{R} \leq \frac{t}{R} \), and \( t \in J \). The sin-term may be estimated by
\[ |\sin \left( \frac{2\pi b z_0}{|R|} \right)| = \sin \left( \frac{\pi}{|R| - 1} \right) \]
and the integral by \( \left| z_0 \right| d (Q_1, Q_2) \), and the result follows.

Note that \( \gamma = \gamma (R) \) is a function only of \( R \), i.e., independent of the second constant \( b \). Also note that \( R \mapsto \left| \gamma (R) \right| \) depends only on \( |R| \). In the range of \( |R| \), i.e., \( |R| \geq 4 \), \( |\gamma (R)| \) is decreasing, starting at \( \frac{\pi}{16} \sqrt{3 + \frac{1}{4}} (< 1) \) when \( |R| = 4 \). Finally, note that the interval \( J \) is chosen to be invariant under the two transformations, \( t \mapsto \frac{t}{R} \) and \( t \mapsto \frac{t - z_0}{R} \). This completes the proof. \( \square \)

7. Three Dimensions

Let \( B \) and \( L \) be two finite subsets in \( \mathbb{R}^\nu \), both containing 0, having the same number of elements \( N \), i.e., \( \# (B) = \# (L) = N \); and assume further that the matrix \( H_{BL} \) in (3.2) is unitary. Also suppose \( L \subset \mathbb{Z}^\nu \). A \( \nu \)-by-\( \nu \) matrix \( R \) will be given satisfying the following four conditions:

(i) \( R = (a_{ij})_{i,j=1}^\nu, a_{ij} \in \mathbb{Z}, \)
(ii) \( R (B) \subset \mathbb{Z}^\nu, \)
(iii) the eigenvalues \( \xi_i \) of \( R \) satisfy \( |\xi_i| > 1, \)
(iv) \( N < |\det R|. \)

Define
\[ P = P (L) = \{ l_0 + R^* l_1 + R^* 2 l_2 + \cdots : l_i \in L, \text{finite sums} \}, \]
\[ \sigma_b (x) = R^{-1} x + b, \]
\[ \tau_l (x) = R^* x + l, \]
\[ \rho_l = \tau_l^{-1} = \sigma_l^*, \quad \text{and} \]
\[ \chi_B (t) = N^{-1} \sum_{b \in B} e^{i 2\pi b \cdot t}. \]

If \( \mu \) is the measure (depending on \( B \) and \( R \)) satisfying (2.5), then
\[ \hat{\mu} (t) = \chi_B (t) \hat{\mu} (R^{-1} t), \quad t \in \mathbb{R}^\nu. \]
Clearly then
\[ P = \bigcup_{l \in L} (l + R^* P), \]
and it follows from the unitarity of \( H_{BL} \) and (2.5) that this union is disjoint. This is equivalent to the assertion that the representation
\[ \lambda = l_0 + R^* l_1 + R^* 2 l_2 + \cdots \quad (l_i \in L) \]
of points in $P(L)$ is unique. Then we proved in Lemmas 4.1 and 4.4 that

$$Q(t) := \sum_{\lambda \in P} |\hat{\mu}(t - \lambda)|^2$$

is entire analytic, and that it satisfies (4.3).

The attractor $X(L)$ of $L$ is the (unique) compact subset $X$ of $\mathbb{R}^\nu$ satisfying

$$R^* X = \bigcup_{l \in L} (X - l).$$

To prove that $P(L)$, the “fractal in the large”, which is constructed from some given scaling system $(R, B, L)$ in $\mathbb{R}^\nu$, provides an orthogonal basis of exponentials $\{e_\lambda : \lambda \in P(L)\}$ for $L^2(\mu)$, using the Ruelle transfer operator, we must exclude the special case when $L$ is contained in a hyperplane in $\mathbb{R}^\nu$. But this condition is in fact necessary as we show in Section 9 below. Moreover, this restriction on $L$, together with the Hadamard matrix property (3.2) for the two sets $B$ and $L$, is stringent, and we showed in $\textit{IoPe96}$ that it can hold only in dimensions $\nu = 1, 3,$ and higher. While we can get examples for $\nu = 2$ (the plane) where the axioms, including (3.2), are satisfied, we showed in $\textit{IoPe96}$ that those planar examples will “collapse” down into a line: The possible examples in the plane will either be essentially one-dimensional, or they will not have $H_2(P(L))$ equal to $L^2(\mu)$, but only a proper subspace; see Section 9 for more details.

The following scholium for $\mathbb{R}^\nu$ will serve as a guiding principle for the results in Sections 8 and 9 below. We will also there discuss two metrics on the testing functions $Q$, both involving the gradient $\nabla Q$, but one giving a better contractivity condition than the other. The proofs, and the evaluation of contractivity constants, will be postponed to Sections 8 and 9.

**Scholium 7.1.** The operator

$$C(Q)(t) := \sum_{l \in L} |\chi_B(t - l)|^2 Q(\rho_l(t))$$

acts boundedly on functions defined on the convex hull of $X(L)$.

**Proof.** Details to follow: see the proof of Theorem 8.3 below, and Section 9. \qed

The contractivity property of the scholium refers to the $C^1$ metric on the convex hull of $X(L)$ applied to the convex set of $C^1$ functions $Q$ satisfying $Q(0) = 1$, and the proof requires all the listed assumptions, including (4.1)–(4.3) for the matrix $R$.

But with weaker assumptions, we can still get orthogonality of $\{e_\lambda : \lambda \in P(L)\}$ and uniqueness for the $L$-expansion:

**Lemma 7.2.** With the following assumptions:

(7.2) $L \subset \mathbb{Z}^\nu$,
(7.3) $R(B) \subset \mathbb{Z}^\nu$,
(7.4) $H_{BL}H_{BL} = I_N$,

we have that the representation $\lambda = l_0 + R^* l_1 + R^* 2 l_2 + \cdots$ for points in $P(L)$ is unique, and the exponentials $\{e_\lambda : \lambda \in P(L)\}$ are mutually orthogonal in $L^2(\mu)$ where $\mu$ is the probability measure on $\mathbb{R}^\nu$ subject to

$$\mu = N^{-1} \sum_{b \in B} \mu \circ \sigma_b^{-1}.$$
Lemmas 3.1 and 3.3 will apply provided we can get contractive in the convex set $V\in\sigma$. Assume that indexed by $b$ by induction.) Suppose not, i.e., $l\neq l'$; then we saw that
\[\chi_B(l-l'+R^*(p-p')) = 0\]
for some $l,l'\in L$ and $p,p'\in P(L)$, then $l=l'$. (The uniqueness will then follow by induction.) Suppose not, i.e., $l\neq l'$; then we saw that
\[\chi_B(l-l'+R^*(p-p')) = 0.\]
Since $\chi_B(0) = 1$, we have a contradiction.

The orthogonality of $e_\lambda$ and $e_{\lambda'}$ for distinct $\lambda$ and $\lambda'$ in $P(L)$ follows from the representation
\[\tilde{\mu}(t) = \prod_{n=0}^{\infty} \chi_B(R^{-n}t)\]
applied to $t = \lambda - \lambda'$. The first nonzero term in the $L$-expansion of $\lambda - \lambda'$ yields a factor in the product of the form
\[\chi_B(l-l'+R^*(p-p'))\]
and we noted that this vanishes whenever $l\neq l'$ in $L$. It follows that $\langle e_{\lambda'}|e_\lambda \rangle_{\tilde{\mu}} = \tilde{\mu}(\lambda - \lambda') = 0$ which is the desired orthogonality.

Consider subsets $B,L\subset\mathbb{R}^\nu$ and an integer matrix $R$ with the properties mentioned in Lemma 3.4. We then have dual transforms
\[\sigma_l^*(t) := R^{-1}(t-l), \quad t\in\mathbb{R}^\nu\]
indexed by $l\in L$. (Compare to the associated transformations $\{\sigma_b : b\in B\}$ in (2.3.) Assume that $\{\sigma^*_l : l\in L\}$ satisfies the “open-set condition” (2.3) for some open set $V^*\subset\mathbb{R}^\nu$ which contains 0 in its closure. We now turn to basis properties of the exponentials $\{e_\lambda : \lambda\in P\}$ when $P = \{l_0 + R^*l_1 + \cdots : l_i\in L\}$. Note that Lemmas 3.4 and 3.5 will apply provided we can get
\[Q\mapsto C(Q)(t) := \sum_{l\in L} |\chi_B(t-l)|^2 Q(\sigma_l^*(t))\]
contractive in the convex set
\[(7.5) \quad W_0 := \left\{Q\in C^1(\overline{V^*}) : Q(0) = 1\right\}.\]
Note that some $\overline{V^*}$ may possibly be scaled down to get $C : V_0 \to V_0$ strictly contractive. This will be studied in detail in Sections 8 and 9.
Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertrieben können. — D. Hilbert [Hil26, p. 170]

Figure 3. The Eiffel Tower (Example 7.3)
Example 7.3. The conditions are satisfied in the following example ([JoPe96, Example 7.4], the Eiffel Tower, see Figure 3):

\[ \nu = 3, \]
\[ R = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \]
\[ B = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{cases}, \]
\[ L = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{cases}. \]

(7.6)

The natural candidate for a subset \( P \subset \mathbb{R}^3 \) such that \( \{ e_\lambda : \lambda \in P \} \) is an orthonormal basis in \( L^2(\mu) \) is \( P = \{ l_0 \pm 2t_1 \pm 2^2t_2 + \cdots : t_i \in L, \text{ finite sums} \} \). If \( \lambda \in P \), the three coordinates \( \lambda = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) are all in \( \mathbb{N}_0 \). One of the metrics on the corresponding space \( W_0 \) will be

\[ d(Q_1, Q_2) := \sum_{j=1}^\nu \sup_{\mathbb{V}^{-}} \left| \frac{\partial}{\partial t_j} (Q_1 - Q_2) \right|. \]

(7.7)

The contractivity proof generalizes that of \( \nu = 1 \) in Section 5. Note (in the general case \( \nu > 1 \))

\[ \sum_{l \in L} \frac{\partial}{\partial t_j} |\chi_B(t - l)|^2 = 0, \]

since

\[ \sum_{l \in L} |\chi_B(t - l)|^2 = 1, \quad t \in \mathbb{R}^\nu. \]

(7.8)

Pick a term in (7.7) which is positive, e.g.,

\[ \frac{\partial}{\partial t_1} |\chi_B(t)|^2 \geq 0; \]

then

\[ \sum_{l \in L \setminus \{0\}} \frac{\partial}{\partial t_1} |\chi_B(t - l)|^2 = - \frac{\partial}{\partial t_1} |\chi_B(t)|^2. \]

(7.9)

(The significance of (7.7) and (7.8) will become clear in Lemma 7.5 below.) Make line integrals connecting 0 to \( R^{\nu-1}(t - l) \) for each \( l \in L \setminus \{0\} \), or even curve integrals (if \( \mathbb{V}^{-} \) isn’t convex). Hence, replacing \( Q_1 - Q_2 \) by \( Q \), we find that

\[ \sum_{l \in L} \frac{\partial}{\partial t_1} |\chi_B(t - l)|^2 Q (\sigma_l^*(t)) \]

is estimated by a constant times

\[ \sum_{j=1}^\nu \left\| \frac{\partial}{\partial t_j} Q \right\|_{\infty, \mathbb{V}^{-}} = \| \nabla Q \|_{\infty, \mathbb{V}^{-}}. \]
The property (7.8) is important for the argument, and we include the (easy) proof. It is based on the assumption (3.2) placed on the two given finite subsets $B$ and $L$ in $\mathbb{R}^\nu$. We now calculate from the left-hand side of (7.8):

$$
\sum_{l \in L} |\chi_B(t-l)|^2 = N^{-2} \sum_{l \in L} \left| \sum_{b \in B} e_{t-l}(b) \right|^2
$$

$$
= N^{-2} \sum_{l \in L} \sum_{b \in B} \sum_{b' \in B} e_{t-l}(b'b')
$$

$$
= N^{-2} \sum_{b \in B} \sum_{b' \in B} e_{t-bb'}
$$

$$
= N^{-1} \sum_{b \in B} e_{t-bb'}
$$

$$
= N^{-1} \sum_{b \in B} e_{t}(0) = N^{-1} N = 1,
$$

which is the claim in (7.8).

Remark 7.4. For the Eiffel Tower example we may take

$$(7.10)$$

$$\mathcal{P} = \{(t_1, t_2, t_3) : -a \leq t_i \leq 0\}$$

where $a \in \mathbb{R}_+$ is determined to get the contractive property on

$$Q : \mathbb{R}^3 \to C(\mathbb{R}^3)\left(t = \sum_{i \in L} |\chi_B(t-l)|^2 Q(\sigma_{t_l}(t))\right).$$

Set

$$d(Q_1, Q_2) = \frac{3}{2} \sum_{i=1}^3 \left| \frac{\partial}{\partial t_i} (Q_1 - Q_2) \right|_{\infty}_{\mathcal{P}},$$

and

$$W_0(a) = \{Q \in C^1(-a \leq t_i \leq 0) : Q(0) = 1\}.$$

Then

$$\frac{\partial}{\partial t_j} C(Q)(t) = \frac{1}{2} \sum_{l \in L} |\chi_B(t-l)|^2 \frac{\partial}{\partial t_j} Q(\sigma_{t_l}(t)) + \text{second term},$$

and therefore

$$d(C(Q_1), C(Q_2)) \leq \frac{1}{2} d(Q_1, Q_2) + \text{second term}.$$  

The “second term” comes from evaluating $\frac{\partial}{\partial t_l} |\chi_B(t-l)|^2$ when

$$(7.11)$$

$$\chi_B(t-l) = \frac{1}{4} \left(1 \pm e^{i\pi t_1} \pm e^{i\pi t_2} \pm e^{i\pi t_3}\right).$$

We use the estimate

$$\left| \frac{\pi}{4} \sum_{j=1}^3 \int_{-a}^0 \frac{\partial}{\partial t_j} Q \right| \leq \frac{\pi}{4} \sum_{j=1}^3 \left| \frac{\partial}{\partial t_j} Q \right|_{\infty}$$

and complete the proof of the contractive property of the operator $Q : C(Q)$:

We get a strict contraction if $a \in \mathbb{R}_+$ is chosen (so small that) $\frac{1}{2} + \frac{2\pi}{\pi} < 1$, i.e., $a < \frac{\pi}{\pi}$. Note however that the set (7.10) will not be invariant under the affine dual maps $t \mapsto R^{-1}(t-l)$, $l \in L$, unless $R$ is expanded by a certain scale, to be specified in the following section. We show there that strict contractivity of
Proof. In the formula for $Q \mapsto C(Q)$ may always be obtained by such a “rescaling”. (See the details below in Proposition 7.4.) We also give a different metric on the functions $Q$ which yields sharper results. Proposition 7.4 implies that the contractivity is indeed strict for the matrix $R = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{pmatrix}$, $r \geq 3$, of Example 7.3. We then get $Q \equiv 1$ by the Banach fixed point principle. Therefore $\frac{\partial^m}{\partial t_1^m \partial t_2^m \partial t_3^m} Q(t) |_{t=0} = 0$ if $m = m_1 + m_2 + m_3 \geq 1$. Hence $x_1^{m_1} x_2^{m_2} x_3^{m_3} \in H_2(P(L), \mu)$ and $H_2(P(L), \mu) = L^2(\mu)$ as claimed.

For use both in the detailed estimates in Example 7.3 and also in Sections 8 and 9 below, we need the following lemma. Its statement involves some notation: $\| \cdot \|_2$ for the $l^2$-norm in $\nu$ dimension, $\| \cdot \|_{op}$ for the corresponding operator norm on $\nu \times \nu$ matrices, i.e.,

$$\|R\|_{op} := \max \{\|Rx\|_2 : x \in \mathbb{R}^\nu, \|x\|_2 = 1\}.$$ 

Finally, let $| \cdot |_{\infty,Y}$ be the supremum-norm applied to functions on $Y$, i.e., taking supremum over some simplex $Y \subset \mathbb{R}^\nu$.

**Lemma 7.5.** Let the setting be as in Scholium 7.1, corresponding to a given expansive (see (2.3)) scaling matrix $R$, and a finite set $L$ of translation vectors in $\mathbb{R}^\nu$, such that $0 \in L$. Define $L^* = L \setminus \{0\}$.

(i) Then, for every set of coefficients $\{c_l\}_{l \in L}, c_l \in \mathbb{C}$, such that $\sum_{l \in L} c_l = 0$, we have

$$S_c(t) = \sum_{l \in L} c_l Q (\rho_l(t)) = \sum_{l \in L^*} c_l (Q (R^{-1}(t-l)) - Q (R^{-1}t))$$

where $\rho_l(t) := R^{-1} (t-l)$.

(ii) Since

$$Q (R^{-1}(t-l)) - Q (R^{-1}t) = - \int_0^1 (R^{-1}(\nabla l) Q) (R^{-1} (t - sl)) \ ds,$$

we get the estimate

$$|S_c(t)| \leq \sum_{l \in L^*} |c_l| \int_0^1 |R^{-1}(\nabla l) Q (R^{-1} (t - sl))| \ ds,$$

and, if $Y$ is the convex hull of the attractor $X$ (i.e., $X$ is the solution to $X = L + RX$), then, for all $t \in Y$,

$$|S_c(t)| \leq \|R^{-1}\|_{op} \sum_{l \in L^*} |c_l| \|l\|_2 \|\nabla Q\|_2|_{\infty,Y}.$$ 

(iii) In case $R$ is diagonalizable with a single positive eigenvalue, then $Y$ (= conv $X$, i.e., the smallest convex subset in $\mathbb{R}^\nu$ invariant under all the $\rho_l$’s) is equal to the simplex $Y'$ in $\mathbb{R}^\nu$ generated by the points $y_l := -(R-I)^{-1} l$, $l \in L^*$.

**Proof.** In the formula for $S_c(t)$, substitute $c_0 = - \sum_{l \in L} c_l$. Then

$$S_c(t) = - \sum_{l \in L^*} c_l Q (R^{-1}t) + \sum_{l \in L^*} c_l Q (R^{-1}(t-l))$$

$$= \sum_{l \in L^*} c_l (Q (R^{-1}(t-l)) - Q (R^{-1}t))$$
as claimed. The difference terms inside the sum may then be estimated using the auxiliary functions $\varphi_t (s) := Q \left( R^{-1} (t - sl) \right)$, $0 \leq s \leq 1$, where

$$\frac{d}{ds} \varphi_t (s) = -R^{-1} (\nabla_I) Q \left( R^{-1} (t - sl) \right),$$

and noting that, for each $s \in [0, 1]$, $l \in L$, and for $t \in Y$, the points $R^{-1} (t - sl)$ are all in $Y$.

To prove the asserted invariance of the simplex $Y'$, let $Y$ denote the convex hull (in $\mathbb{R}^n$) of the attractor $X$ of the affine system $\{\rho_l : l \in L\}$ (i.e., the unique $X$ ($\subset \mathbb{R}^n$) satisfying $X = L + RX$). Note that, for each $l \in L$, $y_l = -(R - I)^{-1} l$ is the unique fixed point of $\rho_l$, including the case $l = 0$, where $\rho_0 (0) = 0$.

For $n \in \mathbb{N}$, and $(l_1, l_2, \ldots, l_n) \in L \times L \times \cdots \times L$, let $y (l_1, \ldots, l_n)$ be the fixed point of $\rho_{l_1} \circ \rho_{l_2} \circ \cdots \circ \rho_{l_n}$, and let $X_n$ be the set of all the fixed points $y (l_1, \ldots, l_n)$ as $(l_1, \ldots, l_n)$ varies over $X_1^n L$. Then it is known (see [Hut81, Theorem 1]) that $X = \bigcup_{n \in \mathbb{N}} X_n$, and therefore

$$(7.12) \quad Y = \text{conv } X = \bigcup_{n \in \mathbb{N}} \text{conv } X_n$$

where conv denotes the convex hull. Since $\rho_0 (0) = 0$, conv $X_1$ is the simplex $Y'$ generated by $\{y_l : l \in L^*\}$ where $y_l = -(R - I)^{-1} l$, i.e.,

$$Y' = \text{conv } X_1 = \left\{ \sum_{l \in L^*} s_l y_l : s_l \geq 0, \sum_{l \in L^*} s_l \leq 1 \right\}.$$

Using now the contractivity property of the affine maps $\{\rho_l : l \in L\}$ implied by (2.2), we infer by induction that, if $R$ is diagonalizable with a single positive eigenvalue, then

$$(7.13) \quad X_n \subset Y' \quad \text{for all } n \in \mathbb{N},$$

and, therefore, by (7.13), that $Y \subset Y'$. A further volume consideration shows that in fact $Y = Y'$. The invariance of the simplex follows, as claimed.

To prove (7.13), proceed by induction, checking first that $X_2 \subset \text{conv } X_1$ when $R$ has the form $R = \left( \begin{array}{cccc} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{array} \right)$ for some $r \in \mathbb{N}$. Every $y \in X_2$ has the representation

$$y = (I - R^2)^{-1} (l_0 + Rl_1)$$

for some $l_0, l_1 \in L$. It follows that $y = (R + I)^{-1} x_0 + R (R + I)^{-1} x_1$ where $x_i := -(R - I)^{-1} l_i \in X_1$ for $i = 0, 1$. So when the scaling matrix $R$ has the stated form, it follows that $y = x_0 + x_1 \in \text{conv } X_1$.

Hence $X_2 \subset \text{conv } X_1$. Now suppose (7.13) has been proved for $n$, and let $y \in X_{n+1}$. Then

$$y = (I - R^{n+1})^{-1} (l_0 + Rl_1 + \cdots + R^n l_n)$$

for some $l_i \in L$, and it follows that there are $z \in X_n$ ($z = (I - R^n)^{-1} (l_0 + Rl_1 + \cdots + R^{n-1} l_{n-1})$), and $x \in X_1$, such that

$$y = \frac{1}{1 - R^n} z + \frac{1}{1 - R^n} x.$$

Clearly $\frac{1}{1 - R^n} + \frac{1}{1 - R^n} = 1$, so that, when $R$ has the stated form, it follows from the inductive hypothesis that $y \in \text{conv } (X_n \cup X_1) \subset Y'$, concluding the proof.

Note that $Y$ is not in general contained in $\text{conv } (X_1)$, even if $R$ is in diagonal form. For examples when $\text{conv } (X)$ is of a more complex nature, see [BrJo96b, Sections 9–10 and the graphics].
The first application is to the contractivity of the $C$-operator in Example 7.3. Let the $R$ in (7.4) be variable; i.e., $R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & r \end{pmatrix}$, rather than the $r = 2$ version (7.6), and assume $r \in \mathbb{N}$, $r \geq 2$. Then, for each $r$, we have a corresponding simplex (as in the lemma). It is given by parameters $s_i \geq 0$, $i = 1, 2, 3$, such that $s_1 + s_2 + s_3 \leq 1$. Specifically,

\[(7.14)\]

\[Y(r) = \left\{ t = \left( \frac{t_1}{t_2}, \frac{t_3}{t_4} \right) \in \mathbb{R}^3 : t_1 = \frac{s_2 + s_3}{1-r}, \ t_2 = \frac{s_1 + s_3}{1-r}, \ t_3 = \frac{s_1 + s_2}{1-r}, \ s_1 \geq 0 \ \forall i, \ \sum_{i=1}^{3} s_i \leq 1 \right\}. \]

(See Figure 4.) Recall that $Y := Y(r)$ is (by the lemma) then invariant under the four affine transformations,

\[t \mapsto \frac{t - l}{r} \quad (l \in L),\]

and also $\frac{t - s_3}{s_1} \in Y$, whenever $t \in Y$ and $s \in [0,1]$.

Let $B$ and $L \subset \mathbb{R}^3$ be as in (7.4) of Example 7.3. For $r \in \mathbb{N}$, let

\[P_r(L) = \{ l_0 + rl_1 + r^2l_2 + \cdots : l_i \in L, \ \text{finite sums} \}, \]

\[C_r(f)(t) = \sum_{l \in L} |\chi_B(t - l)|^2 \left( \frac{t - l}{r} \right), \quad f \in C^\infty(\mathbb{R}^3), \ t \in \mathbb{R}^3, \]

and

\[Q_r(t) := \sum_{\lambda \in P_r(L)} |\hat{\mu}_r(t - \lambda)|^2, \quad t \in \mathbb{R}^3. \]

We then have the following conclusions concerning this example.

(i) If $r$ is even, then (3.3) holds, and we then have the eigenvalue equation $C_r(Q_r) = Q_r$, satisfied (by Lemma 4.1).

(ii) If $r \geq 3$, then $C_r$ is contractive in the supremum-gradient norm (Proposition 7.6 below). Hence, if $r \geq 4$ and even, then the exponentials

\[E_r(L) := \{ e_\lambda : \lambda \in P_r(L) \} \]

form an orthonormal basis in $L^2(\mu_r)$.

(iii) But (i)–(iii) do not carry over to the case when $r$ in $\mathbb{N}$ is odd. For example, it can be checked directly that $C_3(Q_3) \neq Q_3$, and that $E_3(L)$ is not orthogonal in $L^2(\mu_3)$. (The points $(0,0,0)$ and $(4,4,0)$ are both in $P_3(L)$, but not orthogonal in $L^2(\mu_3)$. Specifically, $\hat{\mu}_3(4,4,0) = \hat{\mu}_3(\frac{4}{3},\frac{4}{3},0) \neq 0$ by a direct check.)

Proposition 7.6. In the Eiffel Tower example (Example 7.3), the operator $Q \mapsto C_r(Q)$ is contractive when $r \geq 3$ on the space of functions $Q \in C^\infty(\mathbb{R}^3)$ such that $Q(0) = 0$, and relative to the norm

\[\|Q\|_{Y(3)} := \sup_{t \in Y(3)} \|\nabla Q(t)\|\]

where $\|\nabla Q\| := \sum_{j=1}^{3} \left| \frac{\partial}{\partial y_j} Q \right|$. 

\[\]
Proof. The first term in $C_r (Q)$ of Example 7.3 is (7.9), and we will restrict, for simplicity, to functions $Q \in C^\infty (\mathbb{R}^3)$ such that $Q (0) = 0$. For the derivative terms $\frac{\partial}{\partial t_1} |\chi_B (t - l)|^2$ of (7.9), we have

$$\sum_{l \in L^*} \left| \frac{\partial}{\partial t_1} |\chi_B (t - l)|^2 \right| = \frac{\pi}{2} \sum_{l \in L^*} |\operatorname{Im} \chi_B (t - l)|$$

$$= \frac{\pi}{8} \sum_{l \in L^*} \left| \sum_{k=1}^3 (\pm) \sin (\pi t_k) \right|$$

with the sign-convention (and vanishing of one of the three terms in the sum) from (7.11) above.
Hence

\[(7.15) \quad \sum_{l \in L^*} \left| \frac{\partial}{\partial t_1} \varphi_l(t - l) \right|^2 \leq \frac{3\pi}{8} \sum_{k=1}^{3} |\sin(\pi t_k)| \leq \frac{3\pi}{4} \sin \left( \frac{\pi}{r - 1} \right)\]

does not depend on \(r\). (If \(r = 2\), we bound \(|\sin(\pi t_k)|\) by 1.) The last estimate is from evaluation of the maximum of the \(\sum_{k=1}^{3}\)-term when \(t\) is restricted to the simplex \(Y\) (see also (7.14)). (The maximum is attained at one of the vertices.)

Returning to the estimate on the sum (7.15), we have (using Lemma 7.5):

\[(7.16) \quad \sum_{l \in L^*} \left| \frac{\partial}{\partial t_1} \varphi_l(t - l) \right|^2 \leq \frac{3\pi}{4} \sin \left( \frac{\pi}{r - 1} \right) \frac{1}{r} \sum_{l \in L^*} \int_{t = 0}^{1} \left| \nabla Q \left( \frac{t - sl}{r} \right) \right| ds\]

where estimate (7.15) was used. But when \(t\) is restricted to \(Y\) (\(r\), then \(\frac{t - sl}{r} \in Y\), and the term under the integral may in that case be estimated by a supremum over \(Y\). The corresponding supremum of \(\sum_{j=1}^{3} \left| \frac{\partial}{\partial t_j} Q \right|\) will be denoted simply \(\|\nabla Q\|\), but a more specific terminology would be \(\|\nabla Q\|_{\infty, Y(r)}\). Continuing estimate (7.16), we get, for \(t \in Y\),

\[\|\nabla Q\| \leq \frac{3\pi}{2} \sin \left( \frac{\pi}{r - 1} \right) \frac{1}{r} \operatorname{vol}_3 (Y) \|\nabla Q\|.\]

Since \(\operatorname{vol}_3 (Y) = \frac{1}{3(r - 1)^3}\), then we get

\[\|\nabla Q\| \leq \frac{\pi}{2r (r - 1)^3} \sin \left( \frac{\pi}{r - 1} \right) \|\nabla Q\|.\]

The other term in the estimate of \(\left| \frac{\partial}{\partial t_1} C_r(Q)(t) \right|\) is \(\frac{1}{r} \left| C_r \left( \frac{\partial}{\partial t_1} Q \right) \right|\), which, for \(t \in Y\) (\(r\)), is bounded above by \(\frac{1}{r} \left| \frac{\partial}{\partial t_1} Q \right|_{\infty, Y(r)}\). The identity (7.8) was used a second time for that. The same estimates work also for the other partial derivatives \(\left| \frac{\partial}{\partial t_j} C_r(Q)(t) \right|\), and summing the estimates on \(\sum_{j=1}^{3} \left| \frac{\partial}{\partial t_j} C_r(Q) \right|\), we get

\[\|\nabla C_r(Q)\| \leq \frac{1}{r} \|\nabla Q\| \cdot \left( 1 + \frac{3\pi}{2 (r - 1)^3} \sin \left( \frac{\pi}{r - 1} \right) \right) \|\nabla Q\|\]  

**Summary.** With respect to this gradient-norm, \(C_r\) has \(\frac{1}{r} \cdot \left( 1 + \frac{3\pi}{2 (r - 1)^3} \sin \left( \frac{\pi}{r - 1} \right) \right)\) as operator bound.

This estimate does not get us contractivity of \(Q \mapsto C_r(Q)\) (on functions \(Q \in C^\infty\), restricted to \(Y\) (\(r\)), and satisfying \(Q(0) = 0\), when \(r = 2\); but, when \(r = 3\), we get the number \(\frac{1}{3} \left( 1 + \frac{3\pi}{2 (3 - 1)^3} \right) \approx .53 < 1\) as an upper bound on the contractivity constant.

**Remark 7.7.** Note that, rather than using the simplex \(Y\) in the estimates for the operator \(C_r\) of Example 7.3, we could alternatively have used simply the enveloping unit cube \(B\) positioned as follows: \(-1 \leq t_j \leq 0, j = 1, 2, 3, \) in \(\mathbb{R}^3\). (See Figure 3.) But its volume is one, as opposed to \(\operatorname{vol}_3 (Y) = \frac{1}{3(r - 1)^3}\), which gives the present much better contractivity constant for \(C_r\).
8. A Scaling Property for Matrix \(L^2\)-Spaces

Let the system \((R, B, L)\) be as described, i.e., \(R\) is a \(\nu\)-by-\(\nu\) integral matrix which is strictly expansive. The subsets \(B\) and \(L\) in \(\mathbb{R}^\nu\) will be assumed to satisfy (3.1)-(3.2), specifically \(R(B) \subset \mathbb{Z}^\nu\) and \(L \subset \mathbb{Z}^\nu\). If \(R\) is scaled by some \(r \in \mathbb{N}\), then the new system \((rR, B, L)\) satisfies the same conditions. We introduce the transformations
\[
\sigma_{r, b}(x) := (rR)^{-1} x + b, \quad x \in \mathbb{R}^\nu,
\]
and the corresponding measure \(\mu_r\) (of course also depending on \(B\)). The fixed-point property of \(\mu_r\) is
\[
\mu_r = \frac{1}{N} \sum_{b \in B} \mu_r \circ \sigma_{r, b}^{-1},
\]
where \(N = \#(B) (= \#(L))\), and we recall that \(\mu_r = \mu\) for \(r = 1\). The corresponding formula for the Fourier transform \(\widehat{\mu_r}\) is
\[
\widehat{\mu_r}(t) = \chi_B(t) \widehat{\mu_r}\left((rR)^{-1} t\right),
\]
where, as before,
\[
\chi_B(t) = \frac{1}{N} \sum_{b \in B} e^{2\pi i b \cdot t}, \quad t \in \mathbb{R}^\nu.
\]
The transformation corresponding to (4.3) is
\[
C_r(Q)(t) = \sum_{l \in L} |\chi_B(t - l)|^2 Q\left(r^{-1}R^*(t - l)\right).
\]

Example 8.1. We have considered the following two measures, \(\mu_1\) and \(\mu_2\), in one dimension, arising from \(B = \{0, \frac{1}{2}\}, L = \{0, 1\}\), and \(R = 2\). Then \(rR = 2, 4\) for \(r = 1, 2\), and we showed that \(\mu_1\) was Lebesgue measure on \([0, 1]\) while \(\mu_2\) was the measure with \(d_H(\mu_2) = \frac{1}{2}\) from Theorem 3.4. We considered the eigenvalue problem \(C_r(Q) = Q\), and showed that \(Q \equiv 1\) is the only solution normalized to \(Q(0) = 1\) when \(r = 2\), but not when \(r = 1\).

We will now show that this is general, also for the examples in higher dimension. We show that the eigenvalue problem \(C_r(Q) = Q\) always will have a unique (normalized) solution provided only that \(r\) is taken large enough.

The proof is based on two technical points:

Lemma 8.2. For the directional derivative \(\nabla_a, a \in \mathbb{R}^\nu\), of \(|\chi_B(t)|^2\), we have
\[
\nabla_a |\chi_B(t)|^2 = -2\pi N^{-2} \sum_{b, b' \in B} a \cdot (b - b') \sin [2\pi (b - b') \cdot t], \quad t \in \mathbb{R}^\nu.
\]

Proof. An elementary computation. Recall the notation \(\nabla_a := a \cdot \nabla = \sum_{j=1}^\nu a_j \frac{\partial}{\partial x_j}\) relative to the canonical coordinate system in \(\mathbb{R}^\nu\).

With the assumptions from above on the system \((R, B, L)\) in \(\mathbb{R}^\nu\), we have the following estimate for the contractivity constant \(\gamma\) of
\[
C(Q)(t) = \sum_{l \in L} |\chi_B(t - l)|^2 Q\left(R^*(t - l)\right).
\]
Let $\mathbf{Y}$ denote the convex hull of the attractor $\mathbf{X}(L)$ of the maps
\begin{equation}
(8.4) \quad \rho_l(t) := R^{-1}(t - l), \quad t \in \mathbb{R}^\nu,
\end{equation}
indexed by $l \in L$.

We replace the norm $\| \cdot \|_\infty, \mathbf{Y}$ from Section 8.4 by the following alternative norm:
\begin{equation}
(8.5) \quad \| Q \|_{1, \mathbf{Y}} := \| \nabla Q \|_2 = \int_\mathbf{Y} |\nabla Q|_2 \, dm,
\end{equation}
where $m$ denotes Lebesgue measure.

**Theorem 8.3** ($L^1$-Theorem). Let $(R, B, L)$ be a system in $\mathbb{R}^\nu$ satisfying (8.4)–(8.3), $0 \in L$. Let $C$ be the operator given by (8.5), and let $\mathbf{Y}$ denote the convex hull of the attractor $\mathbf{X}_\nu$. Let $\| Q \|_{1, \mathbf{Y}}$ be given by (8.5), and let
\[ \beta := 2\pi \text{diam}(B) \max_{b, b' \in B, l \in L} \| \sin(2\pi(b - b')(\cdot - l)) \|_\infty. \]

Let $\| T \|_{op}$ be the operator norm, and $\| T \|_{hs} := \left( \sum_{j,k=1}^{\nu} |t_{j,k}|^2 \right)^{1/2}$ the Hilbert-Schmidt norm for a $\nu \times \nu$ matrix $T$. Then we have
\begin{equation}
(8.6) \quad \| CQ \|_{1, \mathbf{Y}} \leq |\det R| \left( 1 - \frac{1}{N} \right) \beta \| R^{-1} \|_{op} \max_{l \in L} \| T_l \|_2 + N \| R^{-1} \|_{hs} \| Q \|_{1, \mathbf{Y}},
\end{equation}
for any $C^1$-function $Q$ such that $Q(0) = 0$. If further $\mathbf{Y} \cap (\mathbf{Y} - l)$ is a set of Lebesgue measure zero for any $l \in L$ with $l \neq 0$, then we have the sharper estimate
\begin{equation}
(8.7) \quad \| CQ \|_{1, \mathbf{Y}} \leq |\det R| \left( 1 - \frac{1}{N} \right) \beta \| R^{-1} \|_{op} \max_{l \in L} \| T_l \|_2 + \| R^{-1} \|_{hs} \| Q \|_{1, \mathbf{Y}},
\end{equation}
for any $C^1$-function $Q$ such that $Q(0) = 0$.

**Remark 8.4.** Note that this result is pointless in one dimension ($\nu = 1$), for then
\[ |\det R| \| R^{-1} \|_{hs} = 1, \]
while this product gets small in higher dimensions. In our setup (JoPe96), the $L^1$-Theorem can only be used if $|\det R| \| R^{-1} \|_{hs} < 1$ is possible for some invertible integer matrices $R$. Actually this may work better if we take the generalized setup (8.3) seriously, and, e.g., take $R = \frac{1}{2} L_\nu$, since then $|\det R| \| R^{-1} \|_{hs} \to 0$ as $\nu \to \infty$.

However, it is not yet clear if there are any nontrivial examples allowing us to use only the $L^1$-Theorem.

**Proof.** We have
\begin{align*}
\| CQ \|_{1, \mathbf{Y}} &= \left\| \sum_{l \in L} |\chi_B(t - l)|^2 Q \circ \rho_l \right\|_1 \\
&= \left\| \sum_{l \in L} \left( \nabla |\chi_B(t - l)|^2 \right) Q \circ \rho_l + \sum_{l \in L} |\chi_B(t - l)|^2 \nabla (Q \circ \rho_l) \right\|_1 \\
&\leq \left\| \sum_{l \in L} \left( \nabla |\chi_B(t - l)|^2 \right) Q \circ \rho_l \right\|_1 + \left\| \sum_{l \in L} |\chi_B(t - l)|^2 \nabla (Q \circ \rho_l) \right\|_1.
\end{align*}
As before we estimate the two terms in this sum, starting with the first term. Note that Lemma 7.3 implies
\[ \sum_{l \in L} \left( \nabla |\chi_B(t - l)|^2 \right) = 0 \]
for all \( t \). Setting \( L^* := L \setminus \{0\} \), it follows that

\[
\sum_{l \in L} \left( \nabla |x_B(t - l)|^2 \right) Q \circ \rho_l(t) = \sum_{l \in L^*} \left( \nabla |x_B(t - l)|^2 \right) (Q \circ \rho_l(t) - Q \circ \rho_0(t))
\]

\[
= - \sum_{l \in L^*} \left( \nabla |x_B(t - l)|^2 \right) \int_0^1 (R^* - t) \cdot (\nabla Q (R^* - t - sR^* - l)) \, ds.
\]

for all \( t \). So we have

\[
\left\| \sum_{l \in L} \left( \nabla |x_B(\cdot - l)|^2 \right) Q \circ \rho_l \right\|_1
\]

\[
\leq \left\| \sum_{l \in L^*} \left( \nabla |x_B(\cdot - l)|^2 \right) \int_0^1 (R^* - l) \cdot (\nabla Q (R^* - l - sR^* - l)) \, ds \right\|_1
\]

\[
\leq \left\| \sum_{l \in L^*} \left( \nabla |x_B(\cdot - l)|^2 \right) \int_0^1 |\nabla Q (R^* - l - sR^* - l)| \, ds \right\|_1
\]

\[
\leq \max_{l \in L^*} \left\| \nabla |x_B(\cdot - l)|^2 \right\|_2 \max_{l \in L^*} |R^* - l|_2 \int_0^1 |\nabla Q (R^* - l - sR^* - l)| \, ds \right\|_1.
\]

We continue the estimate by estimating the last factor in this product.

\[
\left\| \sum_{l \in L^*} \int_0^1 |\nabla Q (R^* - l - sR^* - l)|_2 \, ds \right\|_1
\]

\[
= \int \sum_{l \in L^*} \int_0^1 |\nabla Q (R^* - l - sR^* - l)|_2 \, ds \, dt
\]

\[
= \int_0^1 \sum_{l \in L^*} \int |\nabla Q (R^* - l - sR^* - l)|_2 \, dt \, ds.
\]

Making the substitution \( u = R^* - l - sR^* - l = \rho_s(t) \) we have \( dt = |\det R| \, du \), so that

\[
\left\| \sum_{l \in L^*} \int_0^1 |\nabla Q (R^* - l - sR^* - l)|_2 \, ds \right\|_1
\]

\[
= \int_0^1 \sum_{l \in L^*} \int |\nabla Q (u)|_2 |\det R| \, du \, ds
\]

\[
\leq \int_0^1 \sum_{l \in L^*} \int |\nabla Q (u)|_2 |\det R| \, du \, ds
\]

\[
\leq (N - 1) |\det R| \left\| |\nabla Q|_2 \right\|_1.
\]

The second to last estimate used the fact that \( \rho_s(t) \subset Y \), which in turn is a consequence of the fact that \( (\rho_s(t))_{0 \leq s \leq 1} \) is the line connecting \( \rho(t) \) and \( \rho_0(t) \), the invariance \( \rho_Y \subset Y \), and the convexity of \( Y \). So we have shown

\[
\left\| \sum_{l \in L} \left( \nabla |x_B(\cdot - l)|^2 \right) Q \circ \rho_l \right\|_1
\]

\[
\leq \max_{l \in L^*} \left\| \nabla |x_B(\cdot - l)|^2 \right\|_2 \max_{l \in L^*} |R^* - l|_2 (N - 1) |\det R| \left\| |\nabla Q|_2 \right\|_1.
\]
Estimating the $\| R^{* - 1} l \|_2$ term,

$$\left\| \sum_{l \in L} \left( \nabla |\chi_B(\cdot - l)|^2 \right) Q \circ \rho_l \right\|_1 \leq (1 - N^{-1}) \beta \| R^{-1} \|_{op}^{\max} \| l \|_2 \| \det R \|_1 \| Y \|.$$  

The other estimate is

$$\left\| \sum_{l} |\chi_B(\cdot - l)|^2 \left| \nabla (Q \circ \rho_l) \right|_2 \right\|_1 = \int_Y \sum_{l} |\chi_B(t - l)|^2 \left| \nabla (Q \circ \rho_l) (t) \right|_2 dt$$

$$\leq \sum_{l} \int_Y \left| \nabla (Q \circ \rho_l) (t) \right|_2 dt$$

$$\leq \sum_{l} \int_Y \left| \nabla (Q \circ \rho_l) (t) \right|_2 \| R^{-1} \|_{hs} dt$$

$$= |\det R| \sum_{l} \int_{\rho_l Y} \left| \nabla (Q \circ \rho_l) (u) \right|_2 \| R^{-1} \|_{hs} du$$

$$\leq |\det R| N \| \nabla Q \|_2 \| R^{-1} \|_{hs}.$$  

The last inequality used $\rho_l Y \subset Y$; and it can be improved if $Y \cap (Y - l)$ is a set of Lebesgue measure zero for every $l \in L^*$. In that case, we have

$$\left\| \sum_{l} |\chi_B(\cdot - l)|^2 \left| \nabla (Q \circ \rho_l) \right|_2 \right\|_1 \leq |\det R| \| \nabla Q \|_2 \| R^{-1} \|_{hs}.$$  

It is now easy to complete the proof. □

Remark 8.5. The following additional arguments may be used in a sequence of steps (to be described below) leading to the elimination of the factor $|\det (R)|$ in the two estimates (8.6) and (8.7) in the conclusion of Theorem 8.3. After estimating $\| \nabla Q \|_{L^1(Y)}$ by a constant times

$$\sum_{l \in L} \int_Y \left| \nabla (Q \circ \rho_l) \right|_2 dt$$

for $Q \in C^\infty(\mathbb{R}^\nu)$ such that $Q(0) = 0$, we may change variables in the $\int & \cdots \int \cdots$ terms in the summation (8.8). Let $Y$ have the non-overlap property (2.4), i.e., suppose that the Lebesgue measure of the possible overlap in the union

$$T (Y) := \bigcup_{l \in L} \rho_l (Y)$$

is zero. For the sum in (8.8), we then have

$$\sum_{l \in L} \int_Y \left| \nabla Q \circ \rho_l \right| dt = |\det (R)| \int_{T(Y)} \left| \nabla Q \right| dt$$

and

$$\int_{T(Y)} \left| \nabla Q \right| dt \leq \sup_{T(Y)} \left| \nabla Q \right| \operatorname{vol}_\nu (T (Y))$$

$$= \sup_{T(Y)} \left| \nabla Q \right| \frac{N \operatorname{vol}_\nu (Y)}{|\det (R)|}.$$
Then combining estimates (8.8) and (8.10), we get
\[ \sum_{t \in L} \int_Y |(\nabla Q) \circ \rho_t| \, dt \leq N \text{vol}_v(Y) \sup_{T(Y)} |\nabla Q|, \]
When combined in turn with the estimate from Theorem 8.3, we get
\[ \|\nabla C(Q)\|_{L^1(Y)} \leq N \sup_{T(Y)} |\nabla Q| F_v(R) \text{vol}_v(Y), \]
where \( F_v(R) \) is an expression (see Theorem 8.3 and Section 8) which depends on the operator norm, and the Hilbert-Schmidt norm, applied to \( R \) for \( r \)
and also that we are finished after showing \( C^n(Q) = 0 \) when \( Q(0) = 0 \). Since
\[ (8.11) \]
and \( X \) has Lebesgue measure zero when \( N < |\det R| \), the limits can be estimated further.
Recall that the \( T^n \)-terms in the intersection (8.11) are defined from iteration of
\[ (8.9), \]
i.e.,
\[ T^n(Y) = \bigcup_{(l_1, \ldots, l_n) \in X^n_L} \rho_{l_1} \circ \cdots \circ \rho_{l_n}(Y), \]
and also that we are finished after showing \( C^n(Q_0) \to 0 \) for all \( Q_0 \in C^\infty(\mathbb{R}^n) \) such that \( Q_0(0) = 0 \), because the fixed point \( C(Q_1) = Q_1 \) \( (Q_1 = 1 - Q_0) \) will then be unique subject to \( Q_1(0) = 1 \).

Remark 8.6. The constant \( \beta \) in (8.4) may be easily calculated for the three-dimensional example in Figure 3, i.e., Example 7.3. We have \( \beta = 2\pi \text{diam}(B) = 2\pi \frac{1}{\sqrt{2}} = \pi \sqrt{2} \). Calculating \( \det (R) \) in this example, it follows from the formula for \( F_3(R) \) that the number \( (\pi \sqrt{2} + 1) F_3(R) < 1 \) is a contractive upper bound for the constant \( \gamma \) when \( R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 0 \end{pmatrix} \), and \( r \in \mathbb{N} \) is taken sufficiently large. See formulas (0.13) and (0.16) in Section 4 for details. We expect \( r = 2 \) will suffice. (Hence we get strict contractivity in this new metric as claimed.) An application of Lemma 3.3 then shows that \( L^2(\mu_r) \) of Figure 3 has an orthonormal basis \( \{e_\lambda : \lambda \in P_r \} \) when \( P_r \) is given in (7.4).

Corollary 8.7. When the contractivity constant \( \gamma_r \) is calculated for the system \((rR, B, L)\) as described above, then \( \lim_{r \to \infty} \gamma_r = 0 \), and it follows that \( C_r \) is strictly contractive when \( r \) increases. As a consequence, if \( L \) is not contained in a hyperplane in \( \mathbb{R}^n \), then the exponentials \( \{e_\lambda : \lambda \in P_r(L)\} \) will form an orthonormal basis for \( L^2(\mu_r) \) where \( P_r(L) = \{l_0 + rR^l l_1 + \cdots : l_i \in L, \ \text{finite sums}\} \).

Remark 8.8. The formula for the contractivity constant \( \gamma_r \) contains a factor \( r^{-1} \), but we also note that the convex hull \( Y_r \) of \( X_r(L) \) (= the attractor of \( \{\rho_{r,l} : l \in L\} \)) decreases as a function of \( r \). (See the next section for details.)
9. Estimates and Duality of Scales: Fractals in the Large

In this section, we provide a context in \( \mathbb{R}^\nu \) (the dimension \( \nu \) arbitrary) which utilizes two types of duality: first the traditional time/frequency duality of harmonic analysis, and secondly the duality between the iteration limits when the scale (given by a relative integral \( \nu \)-by-\( \nu \) matrix \( R \)) gets arbitrarily small, versus when it gets arbitrarily large.

The axioms below are such that the scaling matrix \( R \) is only required to be integral with respect to the two given (finite) translation sets \( B \) and \( L \); see (3.3), or (12.2) below. In this general setup, we give explicit estimates on the contractivity constant, making it clear how it depends on the given data \((R, B, L)\) and the dimension \( \nu \). The estimates are gradient-supremum-norm ones.

Consider a triple \((R, B, L)\) such that \( R \) is an expansive \( \nu \times \nu \) matrix with real entries and \( B \) and \( L \) are subsets of \( \mathbb{R}^\nu \) satisfying:

\[
N := \#B = \#L; \tag{9.1}
\]

\[
R^{n}b \cdot l \in \mathbb{Z}, \text{ for any } n \in \mathbb{N}, b \in B, l \in L; \tag{9.2}
\]

\[
H_{B,L} := N^{-1/2} (e^{i2\pi b \cdot l})_{b \in B, l \in L} \text{ is a unitary } N \times N \text{ matrix.} \tag{9.3}
\]

We introduce two dynamical systems

\[
\sigma_b(x) := R^{-1}x + b; \\
\tau_l(x) := R^*x + l
\]

and the corresponding inverse functions

\[
\omega_b(x) := R(x - b); \\
\rho_l(x) := R^{-1}(x - l)
\]

where \( x \in \mathbb{R}^\nu, b \in B, \) and \( l \in L. \) We showed in \[JoPe94\] that one may think of \((\sigma_b)\) and \((\tau_l)\) as dual dynamical systems. It is known that there are unique non-empty compact sets \( X_{\sigma} \) and \( X_{\rho} \) such that

\[
X_{\sigma} = \bigcup_{b \in B} \sigma_b(X_{\sigma}) = B + R^{-1}X_{\sigma} \tag{9.4}
\]

and

\[
X_{\rho} = \bigcup_{l \in L} \rho_l(X_{\rho}) = R^{-1}(X_{\rho} - L); \tag{9.5}
\]

and it is easy to see that

\[
X_{\sigma} = \left\{ \sum_{k=0}^{\infty} R^{-k}b_k : b_k \in B \right\} \quad \text{and} \quad X_{\rho} = \left\{ -\sum_{k=1}^{\infty} R^{-k}l_k : l_k \in L \right\}.
\]

By uniqueness, these sets are the closures of the respective orbits of 0 under the maps \((\sigma_b)\), respectively \((\rho_l)\). In \[JoPe96\], we argued that it is natural to think of \( X_{\rho} \) as a dual of the attractor \( X_{\sigma} \). Let \( \mu = \mu_{\sigma} \) be the unique probability measure on \( \mathbb{R}^\nu \) such that

\[
\mu = N^{-1} \sum_{b \in B} \mu \circ \sigma_b^{-1}. \tag{9.6}
\]
Then this measure has \( X_\sigma \) as its support. Similarly, \( X_\rho \) is the support of the unique probability measure \( \mu' = \mu_\rho \) satisfying
\[
\mu' = N^{-1} \sum_{l \in L} \mu' \circ \rho_l^{-1}.
\]

The corresponding expansive orbits
\[
\mathcal{L} = X_\tau := \left\{ \sum_{k=0}^{n} R^k l_k : n \in \mathbb{N}, l_k \in L \right\}
\]
\[
\mathcal{B} = X_\omega := \left\{ -\sum_{k=1}^{n} R^k b_k : n \in \mathbb{N}, b_k \in B \right\}
\]
of 0 under \( \tau \) and \( \omega \) are also important. It should now be clear that the pairs \((\sigma_b, \tau_l)\) and \((\rho_l, \omega_b)\) share many properties, which however we do not always make explicit in the following. Notice that the set \( \mathcal{L} \) has a recursive property similar to (9.3):
\[
\mathcal{L} = L + RL.
\]

We discussed uniqueness of the corresponding decompositions of elements in \( \mathcal{L} \) above. We also investigated when the functions \( \{ e_\lambda : \lambda \in \mathcal{L} \} \) form an orthonormal basis for \( L^2(\mu) \). One may equally well ask when \( \{ e_\lambda : \lambda \in \mathcal{B} \} \) is an orthonormal basis for \( L^2(\mu') \), but that is an essentially equivalent question. It turns out that the measure \( \mu' \) is important for our investigation. We will see that the set \( \mathcal{B} \) has properties that imply that translates of \( X_\sigma \) tile \( \mathbb{R}^d \), when \( N = \| \det R \| \). The assumptions on \((R, B, L)\) insure that the functions \( \{ e_\lambda : \lambda \in \mathcal{L} \} \) are orthogonal, so we really only have to investigate totality.

The following result, combined with [LaWa96a, Theorems 1.1 and 1.2], shows that, if \( N = \| \det R \| \), then \( X_\sigma \) and \( X_\rho \) are translation tiles. We must also check that \( \mathcal{L} \) is uniformly discrete, i.e., that there is an \( \varepsilon > 0 \) such that \( \| \lambda - \lambda' \|_2 \geq \varepsilon \) for all \( \lambda, \lambda' \in \mathcal{L} \). That follows from our assumptions on \((R, B, L)\), as can be verified by an induction argument.

**Lemma 9.1.** Each point \( \lambda \in \mathcal{L} \) has a unique representation \( \lambda = \sum R^k l_k \) with \( l_k \in L \). Similarly each point in \( \mathcal{B} \) has a unique representation of the form \( \sum R^k b_k \) with \( b_k \in B \).

**Proof.** Each point \( \lambda \in \mathcal{L} \) has a representation of the form \( \lambda = l + R^p \) with \( l \in L \) and \( p \in \mathcal{L} \). Suppose \( l - l' + R^p (p - p') = 0 \) where \( l, l' \in L \) and \( p, p' \in \mathcal{L} \). We will show that \( l = l' \), the desired result, then follows by induction. If \( b \in B \), then
\[
1 = e_b (l - l' + R^p (p - p'))
\]
\[
= e_b (l - l') e_b (R^p (p - p'))
\]
\[
= e_b (l - l') e_{Rb} (p - p')
\]
\[
= e_b (l - l'),
\]
where the last equality used (9.2). If \( l \neq l' \) then we have \( \sum_{b \in B} e_b (l - l') = 0 \) by (9.3). But this contradicts (9.11) because this sum would be \( N \) if (9.11) holds, hence \( l = l' \) as we had to show.

We will show that \( \mathcal{L} \) has the basis property by showing that \( C \) from (4.4) acts contractively on a set of functions including \( Q_1 \) of (8.7) and \( 1 \) (where \( 1 \) denotes the constant function) and thereby showing \( \sum_{\lambda \in \mathcal{L}} |\mu (\cdot - \lambda)|^2 = 1 \). That is, we
want to use the uniqueness part of the Banach fixed-point theorem on a suitably chosen metric space. It is natural to consider $C$ as an operator on a set of smooth functions $Q$ satisfying $Q(0) = 1$. However, the appropriate choice of a metric is not so obvious. The present discussion is based on gradient-supremum-norm estimates.

Let $H_2(L)$ denote the subspace of $L^2(\mu)$ spanned by the orthonormal set $\{e_\lambda : \lambda \in \mathcal{L}\}$. Any $e_t$, $t \in \mathbb{C}$, is in $L^2(\mu)$, so $H_2(L)$ is a subspace of $L^2(\mu)$. We will show that $H_2(L) = L^2(\mu)$ for certain systems $(R, B, L)$ satisfying (9.1)−(9.3). Let $A$ denote the orthogonal projection of $L^2(\mu)$ onto $H_2(L)$, that is

$$Af := \sum_{\lambda \in \mathcal{L}} \langle e_\lambda | f \rangle \mu e_\lambda$$

for any $f \in L^2(\mu)$. Our next result is a useful criterion for when $H_2(L) = L^2(\mu)$.

**Lemma 9.2.** Let $\partial_j := \frac{\partial}{\partial z_j}$. If $0 \in L$, then $H_2(L) = L^2(\mu)$ if and only if $\partial_1^{n_1} \cdots \partial_{\nu}^{n_\nu} Q_1(0) = 0$ for all $n_j \geq 0$ with $n_1 + \cdots + n_\nu \geq 1$, where $Q_1(t) = \sum_{\lambda \in \mathcal{P}} |\hat{\mu}(t - \lambda)|^2$.

**Proof.** Restating the definition of $Q_1$, we have $Q_1(t) = \|Ae_t\|_\mu^2$ for all $t$. If $H_2(L) = L^2(\mu)$, then $Ae_t = e_t$ for all $t$, hence $Q_1(t) = 1$ for all $t \in \mathbb{R}$, in particular, $\partial_1^{n_1} \cdots \partial_{\nu}^{n_\nu} Q_1(0) = 0$ for all $n_j \geq 0$ with $n_1 + \cdots + n_\nu \geq 1$. Conversely, suppose $\partial_1^{n_1} \cdots \partial_{\nu}^{n_\nu} Q_1(0) = 0$ for all $n_j \geq 0$ with $n_1 + \cdots + n_\nu \geq 1$. By Lemma 1.3 we have a power series expansion $Q_1(z) = \sum_{\eta \in \mathcal{P}} a_\eta z^n$; and, by our assumption, $a_n = 0$ unless $n = 0$. Hence $Q_1$ is a constant function. Since $0 \in L$, we have $Q_1(t) = 1$ for all $t \in \mathbb{R}$. It follows that $e_t \in H_2(L)$ for all $t \in \mathbb{R}$, so an application of the Stone–Weierstrass theorem implies that $H_2(L)$ contains all continuous functions on $X$, hence $H_2(L)$, being a dense subspace of $L^2(\mu)$, must be equal to $L^2(\mu)$ as we wanted to show. \hfill $\square$

Let $Y$ denote the convex hull of the attractor $X_\rho$ given by (9.3), let $\|Q\|_\infty := \sup_{y \in Y} |Q(y)|$, and let

$$\|Q\|_Y := \|\nabla Q\|_2$$

(9.12)

where $|z|_2 := \left(\sum_{j=1}^{\nu} |z_j|^2\right)^{1/2}$ is the usual Hilbert norm on $\mathbb{C}$. We begin by finding an explicit formula for the operator norm of $C$ acting on a suitable set of smooth functions.

**Theorem 9.3.** Let $(R, B, L)$ be a system in $\mathbb{R}$ satisfying (9.1)−(9.3), $0 \in L$. Let $C$ be the operator given by (1.4), let $Y$ denote the convex hull of the attractor $X_\rho$ given by (9.3), and let $\|Q\|_Y$ be given by (9.12). If $L$ spans $\mathbb{R}$, and if there exists $\gamma < 1$ so that $\|CQ\|_Y \leq \gamma \|Q\|_Y$ for all $Q$ in a set of $C^1$-functions containing $1 - Q_1$, where $Q_1(t) = \sum_{\lambda \in \mathcal{P}} |\hat{\mu}(t - \lambda)|^2$, then $H_2(L) = L^2(\mu)$.

**Proof.** Since $C1 = 1$ and $CQ_1 = Q_1$ we have

$$\|1 - Q_1\|_Y = \|C(1 - Q_1)\|_Y \leq \gamma \|1 - Q_1\|_Y$$

so $Q_1(y) = 1$ for all $y \in Y$. Since $\mathcal{Y}$ is a convex set containing 0, and $\mathcal{Y}$ is not contained in any proper subspace of $\mathbb{R}$, it follows that $\partial_1^{n_1} \cdots \partial_{\nu}^{n_\nu} Q_1(0) = 0$ for all $n_j \geq 0$ with $n_1 + \cdots + n_\nu \geq 1$. Hence an application of Lemma 9.2 completes the proof. \hfill $\square$
Remark 9.4. In the statement and proof of Theorem 9.3, Y could be replaced by any set containing \( X_\gamma \). But Y will be chosen so as to produce optimal estimates.

The restricting assumption in Theorem 9.3 (and also in the corresponding results in Sections 6 and 9) that the set \( L \subset \mathbb{R}^3 \) not be contained in a hyperplane is necessary. In [JoPe96, Example 7.3], we considered a triple \((R, B, L)\) in \( \mathbb{R}^2 \) where \( \nu = 2, \ N = 3 \), and where the conditions in Theorem 9.3 all hold, except that \( L \), in this [JoPe96] example, is contained in a line in the plane. Specifically, take \( R = (\frac{0}{6}, 0) \), \( B = \{(\frac{0}{6}, \frac{2}{3}) \}, \ L = \{(\frac{0}{6}, \pm t) \}. \) Properties \( (9.1) - (9.3) \) are clearly satisfied for the example. Let \( \mu \) be the unique probability measure on \( \mathbb{R}^2 \) satisfying

\[
\hat{\mu} (t) = \chi_B (t) \hat{\mu} \left( \frac{t}{6} \right), \quad t \in \mathbb{R}^2,
\]

where

\[
\chi_B (t) = \frac{1}{3} \left( 1 + e^{i \pi t_1} + e^{i \pi t_2} \right).
\]

Let \( P (L), Q_1 \), and \( C \) be defined (as usual):

\[
P (L) = \{ l_0 + 6l_1 + 6^2 l_2 + \cdots : l_i \in L, \text{ finite sums} \},
\]

\[
Q_1 (t) := \sum_{\lambda \in P(L)} |\hat{\mu} (t - \lambda)|^2,
\]

and

\[
C (f) (t) = \sum_{l \in L} |\chi_B (t - l)|^2 f \left( \frac{t - l}{6} \right), \quad t \in \mathbb{R}^2.
\]

From Lemma 4.1, we get the eigenvalue identity \( C (Q_1) = Q_1 \) satisfied, and conclude that the exponentials \( E (P (L)) = \{ e_\lambda : \lambda \in P (L) \} \) are orthogonal in \( L^2 (\mu) \). Moreover, the argument from the proof of Proposition 7.4 gets us contractivity for \( C \), i.e.,

\[
(9.13) \quad \| \nabla C (f) \|_Y \leq \gamma \| \nabla f \|_Y
\]

for some constant \( \gamma < 1 \), and for all \( f \in C^\infty (\mathbb{R}^2) \) such that \( f (0) = 0 \); where

\[
\| \nabla f \|_Y := \sup_{t \in Y} \left( \sum_{j=1}^{2} | \partial_j f (t) | \right),
\]

and \( Y \) is the L-attractor. But \( Y \) is a finite line in \( \mathbb{R}^2 \) by Lemma 7.5; in fact, \( Y = \{ (t_1, t_2) : t_2 = -t_1, \ |t_1| \leq \frac{2}{15} \}. \) From the uniqueness of the Banach fixed point, we get that \( Q_1 \equiv 1 \) on \( Y \), but not on \( \mathbb{R}^2 \). Indeed, if we had \( Q_1 \equiv 1 \) on \( \mathbb{R}^2 \), then it would follow from Lemma 9.2 that \( H_2 (P (L)) = L^2 (\mu) \), i.e., that \( E (P (L)) \) is an orthonormal basis for \( L^2 (\mu) \). But it is not, as the function \( f_\perp (x) = e^{i (x_1 + x_2)} \) is in \( L^2 (\mu) \), but not in \( H_2 (P (L)) \), since \( \{ \frac{1}{t} \} \perp P (L) \). That would make \( f_\perp \) constant on every line of slope 45 degrees and therefore constant on \( X (B) \). But we can see that it is not from the formula for \( \mu \) and its support, the attractor \( X (B) \). This attractor has Hausdorff dimension \( \frac{\ln 3}{\ln 6} \); see Figure 3, and the formula for \( X (B) \),

\[
(9.14) \quad X (B) = \left\{ \sum_{i=0}^{\infty} b_i \frac{1}{6^i} : b_i \in B \right\}.
\]
From (9.13), and Lemma 4.3, we conclude that $Q_1$ has the form $Q_1(t_1, t_2) = F(t_1 + t_2)$ for some entire analytic function $F$ of one variable. If $E(P(L))$ were an orthonormal basis for $L^2(\mu)$, it would follow similarly that every $f \in L^2(\mu)$ had the representation $f(x_1, x_2) = g_f(x_1 - x_2)$ for some function $g_f$ of one variable, which we noted is not the case: that would make $f(x_1, x_2) = e^{i(x_1 + x_2)}$ constant on $X(B)$ which it is not, as can be checked by inspection of (9.14).

The argument shows that functions $f \in H_2(P(L))$ are characterized by the following representation: let

$$\Gamma_{l_1} := \{ \lambda \cdot l_1 : \lambda \in P(L) \}.$$ 

Then $f(x_1, x_2) = \sum_{\xi \in \Gamma_{l_1}} c(\xi) e_{\xi}(x_1 - x_2)$, where $\sum_{\xi \in \Gamma_{l_1}} |c(\xi)|^2 < \infty$.

Inspection of (9.14) yields the following representation of points $x = (x_1, x_2)$ in $X(B)$:

$$x = \frac{1}{2} \sum_{i=0}^{\infty} \eta_i \left( \frac{\varepsilon_i}{1 - \varepsilon_i} \right) \frac{1}{6^i},$$
where \( \varepsilon_i, \eta_i \in \{0, 1\} \). Let \( \pi_\pm (x) := x_1 \pm x_2 \), and consider the lines \( l_\pm (h) \) given by \( x_1 \pm x_2 = h \), respectively, for \( h \in \mathbb{R}, |h| \leq 1 \). Expanding \( h \) in base 6,

\[
h = \frac{1}{2} \sum_{i=0}^{\infty} h_i \frac{1}{6^i}, \quad h_i \in \{0, \pm 1, \pm 2\},
\]

we have, for the intersections \( X_\pm (h) := X(B) \cap l_\pm (h) \), that \( X_+ (h) \neq \emptyset \) if and only if \( h_i \in \{0, 1\} \). In that case

\[
X_+ (h) = \left\{ x \in X(B) : \pi_+ (x) = h \right\} = \left\{ \frac{1}{2} \sum_{i=0}^{\infty} h_i \left( \varepsilon_i - \frac{1}{6^i} \right) : \varepsilon_i \in \{0, 1\} \right\}.
\]

Hence both \( H := \{ h \in \mathbb{R} : X_+ (h) \neq \emptyset \} \), and the \( X_+ (h)'s \), are fractals (embedded in one dimension). The respective Hausdorff dimensions are \( H \) of the intersections

\[
\|\gamma\|_{\mathcal{H}} := \left( \sum_{i=1}^{\infty} \|t_{j,k}\|_1 \right)^{1/2}
\]

is the Hilbert-Schmidt norm for a \( \nu \times \nu \) matrix \( T \).

**Theorem 9.5.** Let \((R,B,L)\) be a system in \( \mathbb{R}^2 \) satisfying (9.1)–(9.3). \( 0 \in L \). Let \( C \) be the operator given by (1.4), and let \( Y \) denote the convex hull of the attractor \( X_\rho \) given by (9.5). If \( \|Q\|_Y \) is given by (9.12) and

\[
\gamma := (N - 1)^2 N^{-1} \beta \|R^{-1}\|_{op} \max_{l \in L} \|l\|_2 \quad \left( \sum_{j,k=1}^{\nu} |t_{j,k}| \right)^{1/2}
\]

where \( \beta := 2\pi \text{diam}(B) \max_{b,b' \in B} \|\sin(2\pi(b-b')(\cdot - l))\|_\infty \),

\[
\|R\|_{op} \quad \text{the operator norm of } R, \quad \|T\|_{hs} := \left( \sum_{j,k=1}^{\nu} |t_{j,k}| \right)^{1/2}
\]

for a \( \nu \times \nu \) matrix \( T \), then we have

\[
\|CQ\|_Y \leq \gamma \|Q\|_Y,
\]

for any \( C^1 \)-function \( Q \) such that \( Q(0) = 0 \).
Proof. Begin by observing that
\[
|\chi_B(t - l)|^2 = N^{-2} \sum_{b, b'} e^{i2\pi(b - b')(t - l)} = N^{-2} \sum_{b, b'} \cos(2\pi(b - b')(t - l))
\]
so
\[
\left\| \nabla \left( |\chi_B(t - l)|^2 \right) \right\|_2 = N^{-2} \sum_{b, b'} 2\pi |b - b'|_2 |\sin(2\pi(b - b')(t - l))|.
\]
Since the $N$ terms with $b = b'$ are 0, it follows that
\[
\text{(9.17)} \quad \left\| \nabla \left( |\chi_B(\cdot - l)|^2 \right) \right\|_2 \leq (1 - N^{-1})\beta
\]
for each $l \in L$. Since $\rho_l X_\rho \subset X_\rho$ and $\rho_l$ is affine we have $\rho_l Y \subset Y$. If $0 \leq Q(t) \leq 1$, then clearly we have the following estimate:
\[
\text{(9.18)} \quad \left\| CQ \right\|_Y = \left\| \nabla CQ \right\|_2 \leq \left\| \nabla \sum_l |\chi_B(\cdot - l)|^2 Q(\rho_l t) \right\|_2 \leq \left\| \nabla \sum_l |\chi_B(\cdot - l)|^2 Q \circ \rho_l \right\|_2 + \left\| \nabla \sum_l |\chi_B(\cdot - l)|^2 \nabla(Q \circ \rho_l) \right\|_2.
\]
Let us begin by estimating the first term in this last sum. Using Lemma 7.5 we have
\[
\sum_l \nabla \left( |\chi_B(t - l)|^2 \right) Q(\rho_l(t)) = -\sum_{l \in L} \nabla \left( |\chi_B(t - l)|^2 \right) \int_0^1 (R^{s-1} l) \cdot \nabla Q \left( R^{s-1} (t - s l) \right) ds.
\]
Since $(R^{s-1} (t - s l))_{0 \leq s \leq 1}$ is the line connecting $\rho_0(t)$ and $\rho_1(t)$ it follows from (9.17) that the right hand side is dominated by
\[
\text{(9.19)} \quad \left\| \sum_{l \neq 0} \nabla \left( |\chi_B(\cdot - l)|^2 \right) Q \circ \rho_l \right\| \leq (N - 1) (1 - N^{-1}) \beta \left\| R^{-1} \right\|_{op} \max_{l \in L} |||Q|||_Y.
\]
The $(N - 1)$ factor is from an application of Lemma 7.5. In the estimates above we used the invariance $\rho_l Y \subset Y$ and the convexity of $Y$.
To bound the second term in (9.18), we begin by noticing that Lemma 4.1 and (7.3) lead to
\[
\text{(9.20)} \quad \left\| \sum_l |\chi_B(\cdot - l)|^2 |\nabla(Q \circ \rho_l)|_2 \right\|_\infty \leq \max_{l \in L} \left\| \nabla(Q \circ \rho_l) \right\|_2 \infty.
\]
If \( c_j \) denotes the \( j \)th column in \( R^{-1} \), then
\[
\frac{\partial}{\partial t_j} Q \circ \rho_t = ((\nabla Q) \circ \rho_t) \cdot c_j
\]
so that
\[
|\nabla (Q \circ \rho_t)|_2 = \left( \sum_{j=1}^\nu |((\nabla Q) \circ \rho_t) \cdot c_j|^2 \right)^{1/2}
\]
\[
\leq |((\nabla Q) \circ \rho_t)|_2 \left( \sum_{j=1}^\nu |c_j|^2 \right)^{1/2}
\]
\[
\leq \|Q\|_Y \|R^{-1}\|_h_a,
\]
and it follows that
\[
\left\| \sum_l |\chi_B(\cdot - l)|^2 |\nabla (Q \circ \rho_t)|_2 \right\|_\infty \leq \|R^{-1}\|_h_a \|Q\|_Y,
\]
hence the proof is complete. \( \square \)

**Corollary 9.6.** Let \((R, B, L)\) satisfy (9.1)–(9.3), and, for \( r \in \mathbb{N} \), let
\[
\mathcal{L}_r := \left\{ \sum_{k=0}^n (rR^*)^k l_k : n \in \mathbb{N}, l_k \in L \right\}
\]
and let \( \mu_r \) be the probability measure solving
\[
\mu_r = N^{-1} \sum_{b \in B} \mu_r \circ \sigma_{r,b}^{-1}
\]
where \( \sigma_{r,b}(x) := (rR)^{-1} x + b \). Then \( \{e_\lambda : \lambda \in \mathcal{L}_r\} \) is an orthonormal basis for \( L^2(\mu_r) \) for \( r \) sufficiently large.

**Proof:** This is an immediate consequence of Lemma 3.3 and Theorem 9.5 since \( \|rR\|_{op}^{-1} \to 0 \) and \( \left\| (rR)^{-1} \right\|_h_a \to 0 \) as \( r \to \infty \). \( \square \)

In one dimension, with \( R \) an integer, and \( N = 2 \), the general situation is summarized in the following strengthening of Theorem 6.2 above.

**Corollary 9.7.** Suppose \( \nu = 1, N = 2, B = \{0, b\} \), with \( b \in \mathbb{R}\setminus\{0\} \), \( R \) is an integer with \( |R| \geq 2 \), and \( \mu \) is given by (6.12). If \( R \) is odd, then \( L^2(\mu) \) does not have a basis of exponentials for any \( b \in \mathbb{R}\setminus\{0\} \). If \( R \) is even, and \( |R| \geq 4 \), then \( L^2(\mu) \) has a basis of exponentials for all \( b \in \mathbb{R}\setminus\{0\} \). If \( R \geq 4 \) and even, then \( L \) may be chosen in such a way that \( L^2(\mu) \) is identified as the boundary values of analytic functions in the unit disc.

**Proof:** The argument for odd \( R \) is in the proof of Theorem 6.1.

Suppose \( R \) is even, and let \( L := \{0, \frac{1}{2b}\} \). Then (9.1), (9.3) are satisfied, and we can easily find the convex hull, \( Y \), of the attractor \( X_\rho \). (See Lemma 7.5(i).) In fact, if \( R > 0 \), then \( Y \) is the closed interval with endpoints \( \frac{1}{2b(R-1)} \) and 0; while if \( R < 0 \), then \( Y \) is the closed interval with endpoints \( \frac{-1}{2b(R-1)} \) and \( \frac{-1}{2bR(R-1)} \). Using the notation from Theorem 9.3, we have \( \|R\|_{op} = |R|, \|R^{-1}\|_h_a = |R^{-1}| \), and
\[
\beta = 2\pi |b| \|\sin 2\pi b \|_\infty.
\]
From the characterizations of \( Y \), it follows that
\[
\beta = 2\pi |b| \left| \sin \frac{\pi}{R - 1} \right|, \quad \text{if } R > 0,
\]
and
\[
\beta = 2\pi |b| \sin \frac{\pi}{R^2 - 1}, \quad \text{if } R < 0.
\]
Using \( N = 2 \), and \( \max_{l \in L} b|l|_2 = \frac{1}{129} \), we conclude that
\[
\gamma = \frac{\pi}{2R} \left| \sin \frac{\pi}{R - 1} \right| + \frac{1}{R}, \quad \text{if } R > 0,
\]
and
\[
\gamma = \frac{\pi}{2|R|} \left| \sin \frac{\pi}{R^2 - 1} \right| + \frac{1}{|R|}, \quad \text{if } R < 0.
\]
Thus \( \gamma < 1 \) if \( |R| \geq 4 \), but not if \( R = -2, 2 \).

If \( R = 2 \), then \( \{ e_\lambda : \lambda \in \mathbb{Z} \} \) is an orthonormal basis for \( L^2(\mu) \), and clearly \( \{ e_\lambda : \lambda \in L \} \) cannot be an orthonormal basis for \( L^2(\mu) \) for any choice of \( L = \{0, l\} \) with \( l \neq 0 \); in fact, depending on the sign of \( l \), the set \( L \) will consist solely of non-negative real numbers, or solely of non-positive real numbers. We do not know what happens for general \( b \neq 0 \).

If \( R = -2 \), then we do not know if \( L^2(\mu) \) has a basis of exponentials for some \( b \in \mathbb{R} \setminus \{0\} \).

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References

[Art64] E. Artin, The Gamma Function, Holt, Rinehart and Winston, New York, 1964, translated by M. Butler from the German original, Einführung in die Theorie der Gammafunktion, Hamburger Mathematische Einzelschriften, vol. 1, Verlag B.G. Teubner, Leipzig, 1931.

[BaGe94] Christoph Bandt and G. Gelbrich, Classification of self-affine lattice tilings, J. London Math. Soc. (2) 50 (1994), 581–593.

[Ban91] Christoph Bandt, Self-similar sets 5: Integer matrices and fractal tilings of \( \mathbb{R}^n \), Proc. Amer. Math. Soc. 112 (1991), 549–562.

[Ban96] Christoph Bandt, Self-similar tilings and patterns described by mappings, Proceedings of NATO Advanced Study Institute: Mathematics of aperiodic order (R. Moody and J. Patera, eds.), Kluwer, to appear.

[BoTa87] Enrico Bombieri and Jean E. Taylor, Quasicrystals, tilings, and algebraic number theory: Some preliminary connections, The legacy of Sonya Kovalevskaya (Cambridge, Mass., and Amherst, Mass., 1985) (Linda Keen, ed.), Contemp. Math., vol. 64, American Mathematical Society, Providence, RI, 1987, pp. 241–264.

[BrJo96b] Ola Bratteli and Palle E.T. Jorgensen, Iterated function systems and permutation representations of the Cuntz algebra, Mem. Amer. Math. Soc., to appear; funct-an/9612002.

[Car95] H.P. Cartan, Elementary Theory of Analytic Functions of One or Several Complex Variables, Dover, New York, 1995.

[Dau92] Ingrid Daubechies, Ten lectures on wavelets, CBMS-NSF Regional Conf. Ser. in Appl. Math., vol. 61, Society for Industrial and Applied Mathematics, Philadelphia, 1992.
DENSE ANALYTIC SUBSPACES IN FRACTAL $L^2$-SPACES

[Fal86] K.J. Falconer, The geometry of fractal sets, Cambridge Tracts in Mathematics, vol. 85, Cambridge University Press, Cambridge, 1986.

[Fug74] Bent Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, J. Funct. Anal. 16 (1974), 101–121.

[Hil26] D. Hilbert, Über das Unendliche, Math. Ann. 95 (1926), 161–190.

[Hof95] A. Hof, On diffraction by aperiodic structures, Comm. Math. Phys. 169 (1995), 25–43.

[Hut81] J.E. Hutchinson, Fractals and self similarity, Indiana Univ. Math. J. 30 (1981), 713–747.

[JoPe91] P.E.T. Jorgensen and S. Pedersen, An algebraic spectral problem for $L^2(\Omega), \Omega \subset \mathbb{R}^n$, C. R. Acad. Sci. Paris Ser. I Math. 312 (1991), 495–498.

[JoPe92] P.E.T. Jorgensen and S. Pedersen, Spectral theory for Borel sets in $\mathbb{R}^n$ of finite measure, J. Funct. Anal. 107 (1992), 72–104.

[JoPe93a] P.E.T. Jorgensen and S. Pedersen, Group-theoretic and geometric properties of multivariable Fourier series, Exposition. Math. 11 (1993), 309–329.

[JoPe93b] P.E.T. Jorgensen and Steen Pedersen, Harmonic analysis of fractal measures induced by representations of a certain $C^*$-algebra, Bull. Amer. Math. Soc. (N.S.) 29 (1993), 228–234.

[JoPe94] Palle E.T. Jorgensen and Steen Pedersen, Harmonic analysis and fractal limit-measures induced by representations of a certain $C^*$-algebra, J. Funct. Anal. 125 (1994), 90–110.

[JoPe95] Palle E.T. Jorgensen and Steen Pedersen, Estimates on the spectrum of fractals arising from affine iterations, Fractal geometry and stochastics (Christoph Bandt, Siegfried Graf, and Martina Zähle, eds.), Progress in Probability, vol. 37, Birkhäuser, Basel, 1995, pp. 191–219.

[JoPe96] Palle E.T. Jorgensen and Steen Pedersen, Harmonic analysis of fractal measures, Constr. Approx. 12 (1996), 1–30.

[JoPe97] Palle E.T. Jorgensen and Steen Pedersen, Spectral pairs in Cartesian coordinates, preprint, University of Iowa, 1997.

[KSS95] Mike Keane, Meir Smorodinsky, and Boris Solomyak, On the morphology of $\lambda$-expansions with deleted digits, Trans. Amer. Math. Soc. 347 (1995), 967–983.

[LaWa96c] Jeffrey C. Lagarias and Yang Wang, Tiling the line with translates of one tile, Invent. Math. 124 (1996), 341–365.

[LaWa96d] Jeffrey C. Lagarias and Yang Wang, Self-affine tiles in $\mathbb{R}^n$, Adv. Math. 121 (1996), 21–49.

[Neh75] Z. Nehari, Conformal Mapping, Dover, New York, 1975.

[Ped87] S. Pedersen, Spectral theory of commuting self-adjoint partial differential operators, J. Funct. Anal. 73 (1987), 122–134.

[Ped97] S. Pedersen, Fourier Series and Geometry, preprint, Wright State University, 1997.

[PoSi95] Mark Pollicott and Koral Simon, The Hausdorff dimension of $\lambda$-expansions with deleted digits, Trans. Amer. Math. Soc. 347 (1995), 967–983.

[Rue88] D. Ruelle, Noncommutative algebras for hyperbolic diffeomorphisms, Invent. Math. 93 (1988), 1–13.

[Rue94] D. Ruelle, Dynamical zeta functions for piecewise monotone maps of the interval, CRM Monograph Series, vol. 4, American Mathematical Society, Providence, 1994.

[Str89] Robert S. Strichartz, Fourier asymptotics of fractal measures, J. Funct. Anal. 89 (1990), 154–187.

[Str90c] Robert S. Strichartz, Self-similar measures and their Fourier transforms, I, Indiana Univ. Math. J. 39 (1990), 797–817.

[Str93] Robert S. Strichartz, Self-similar measures and their Fourier transforms, II, Trans. Amer. Math. Soc. 336 (1993), 335–361.

[Str94] Robert S. Strichartz, Self-similarity in harmonic analysis, J. Fourier Anal. Appl. 1 (1994), 1–37.

[Str95] Robert S. Strichartz, Fractals in the large, preprint, Cornell University, 1995.