Non-Monotonic Spatial Reasoning
with Answer Set Programming Modulo Theories∗†

Przemysław Andrzej Wałęga, Carl Schultz, Mehul Bhatt

Spatial Reasoning. www.spatial-reasoning.com
The DesignSpace Group, Germany, www.design-space.org

Universities of: Warsaw (Poland), Münster (Germany), Bremen (Germany)

submitted November 2 2015; revised April 11 2016; accepted June 2 2016

Abstract

The systematic modelling of dynamic spatial systems is a key requirement in a wide range of application areas such as commonsense cognitive robotics, computer-aided architecture design, and dynamic geographic information systems. We present ASPMT(QS), a novel approach and fully-implemented prototype for non-monotonic spatial reasoning—a crucial requirement within dynamic spatial systems—based on Answer Set Programming Modulo Theories (ASPMT).

ASPMT(QS) consists of a (qualitative) spatial representation module (QS) and a method for turning tight ASPMT instances into Satisfiability Modulo Theories (SMT) instances in order to compute stable models by means of SMT solvers. We formalise and implement concepts of default spatial reasoning and spatial frame axioms. Spatial reasoning is performed by encoding spatial relations as systems of polynomial constraints, and solving via SMT with the theory of real nonlinear arithmetic. We empirically evaluate ASPMT(QS) in comparison with other contemporary spatial reasoning systems both within and outside the context of logic programming. ASPMT(QS) is currently the only existing system that is capable of reasoning about indirect spatial effects (i.e., addressing the ramification problem), and integrating geometric and qualitative spatial information within a non-monotonic spatial reasoning context.

This paper is under consideration for publication in TPLP.

KEYWORDS: non-monotonic spatial reasoning; answer set programming modulo theories; declarative spatial reasoning; dynamic spatial systems; reasoning about space, actions, and change

∗ This is an extended version of a paper presented at the Logic Programming and Nonmonotonic Reasoning Conference (LPNMR 2015), invited as a rapid communication in TPLP. The authors acknowledge the assistance of the conference program chairs Giovambattista Ianni and Miroslaw Truszczyński.
† This paper comes with an online appendix containing Appendices A-H. The online appendix is available via the supplementary materials link from the TPLP web-site.
1 Introduction

Non-monotonicity is characteristic of commonsense reasoning patterns concerned with, for instance, making default assumptions (e.g., about spatial inertia), counterfactual reasoning with hypotheticals (e.g., what-if scenarios), knowledge interpolation, explanation and diagnosis (e.g., filling the gaps, causal links), and belief revision. Such reasoning patterns, and therefore non-monotonicity, acquire a special significance in the context of spatio-temporal dynamics, or computational commonsense reasoning about space, actions, and change as applicable within areas as disparate as geospatial dynamics, computer-aided design, cognitive vision, and commonsense cognitive robotics (Bhatt 2012). Dynamic spatial systems are characterised by scenarios where spatial configurations of objects undergo a change as the result of interactions within a physical environment (Bhatt and Loke 2008); this requires explicitly identifying and formalising relevant actions and events at both an ontological and (qualitative and geometric) spatial level; for instance:

- within the context of geographical information systems, formalising high-level processes such as desertification and population displacement based on spatial theories about appearance, disappearance, splitting, motion, and growth of regions (Bhatt and Wallgrün 2014);
- within a domain such as (commonsense) cognitive vision from visual imagery (e.g., video, point-clouds etc), the ability to perform spatio-linguistically grounded semantic question-answering about perceived and hypothesised events based on a commonsense qualitative model of space and motion (Suchan and Bhatt 2016a).

Requirements such as in the aforementioned domains call for a deep integration of spatial reasoning within KR-based non-monotonic reasoning frameworks (Bhatt 2012; Bhatt et al. 2011).

In this paper, we select aspects of a theory of dynamic spatial systems — pertaining to spatial inertia, ramifications, and causal explanation — that are inherent to a broad category of dynamic spatio-temporal phenomena, and require non-monotonic reasoning (Bhatt and Loke 2008; Bhatt 2008; Bhatt 2010). For these aspects, we provide an operational semantics and a computational framework for realising fundamental non-monotonic spatial reasoning capabilities based on Answer Set Programming Modulo Theories (Bartholomew and Lee 2013); ASPMT is extended to the qualitative spatial (QS) domain resulting in the non-monotonic spatial reasoning system ASPMT(QS). Spatial reasoning is performed in an analytic manner (e.g., as with reasoners such as CLP(QS) (Bhatt et al. 2011)), where spatial relations are encoded as systems of polynomial constraints; the task of determining whether a set of qualitative spatial constraints are consistent is now equivalent to determining whether the system of polynomial constraints is satisfiable, i.e., Satisfiability Modulo Theories (SMT) with real nonlinear arithmetic, and can be accomplished in a sound and complete manner. Thus, ASPMT(QS) consists of a (qualitative) spatial
representation module and a method for turning tight ASPMT instances into SMT instances in order to compute stable models by means of SMT solvers.

In the following sections we present the relevant foundations of stable model semantics and ASPMT, and then extend this to ASPMT(QS) by defining a relational, qualitative spatial representation module QS, and formalising default spatial reasoning and spatial frame and ramification axioms using choice formulas. We empirically evaluate ASPMT(QS) in comparison with other existing spatial reasoning systems. In the backdrop of our results, and a discussion of related work, we conclude that ASPMT(QS) is the only system, to the best of our knowledge, that operationalises dynamic spatial reasoning — together with a systematic treatment of non-monotonic aspects emanating therefrom, within a KR-based framework.

2 Spatial Representation and Reasoning

Knowledge representation and reasoning about space may be formally interpreted within diverse frameworks such as: (a) geometric reasoning and constructive (solid) geometry (Kapur and Mundy 1988); (b) relational algebraic semantics of ‘qualitative spatial calculi’ (Ligozat 2011); and (c) by axiomatically constructed formal systems of mereotopology and mereogeometry (Aiello et al. 2007). Independent of formal semantics, commonsense spatio-linguistic abstractions offer a human-centred and cognitively adequate mechanism for logic-based automated reasoning about spatio-temporal information (Bhatt et al. 2013).

Research in qualitative spatial representation and reasoning has primarily been driven by the use of relational-algebraic semantics, and development of constraint-based reasoning algorithms to solve consistency problems in the context of qualitative spatial calculi (Ligozat 2011). The key idea has been to partition an infinite quantity space into finite disjoint categories, and utilize the special relational algebraic properties of such a partitioned space for reasoning purposes. Logic-based axiomatisations of topological and mereotopological space, a study of their general computational characteristics from a reasoning viewpoint, and the development of KR-based general reasoning systems have also been thoroughly investigated (Aiello et al. 2007; Bhatt et al. 2011).

Different in its foundational method involving the use of constraint logic programming (CLP) is the declarative spatial reasoning system CLP(QS) (Bhatt et al. 2011; Schultz and Bhatt 2012; Schultz and Bhatt 2014). CLP(QS) marks a clear departure from other relational algebraically founded methods and reasoning tools by its use of the constraint logic programming framework for formalising the semantics of qualitative spatio-temporal relations and formal spatial calculi. CLP(QS) has demonstrated applications in a range of domains including: architectural design cognition (Bhatt et al. 2014), cognitive vision (Bhatt et al. 2013; Suchan et al. 2014; Suchan and Bhatt 2016a; Suchan and Bhatt 2016b), geospatial dynamics (Bhatt and Wallgrün 2014), and cognitive robotics (Eppe and Bhatt 2013; Spranger et al. 2014; Spranger et al. 2016).
3 Answer Set Programming Modulo Theories

Stable models semantics is the most expanded theory of non-monotonic reasoning with important practical applications. It is capable of expressing a number of non-monotonic and default reasoning types, e.g., causal effects of actions, a lack of information and various completeness assumptions (e.g., absence of ramification yielding state constraints) (Gelfond 2008), which makes it very attractive for reasoning about space, actions, and change in an integrated manner. In what follows, we present a definition of stable models based on syntactic transformations (Bartholomew and Lee 2012) which is a generalization of previous definitions from (Ferraris et al. 2011), (Gelfond and Lifschitz 1988), and (Ferraris 2005). We then present a method for turning tight ASPMT instances into SMT instances.

3.1 Bartholomew – Lee Functional Stable Model Semantics

In what follows we adopt a definition of stable models based on syntactic transformations presented in (Bartholomew and Lee 2012). For predicate symbols (constants or variables) \( u \) and \( c \), expression \( u \leq c \) is defined as shorthand for \( \forall x (u(x) \rightarrow c(x)) \). Expression \( u = c \) is defined as \( \forall x (u(x) \equiv c(x)) \) if \( u \) and \( c \) are predicate symbols, and \( \forall x (u(x) = c(x)) \) if they are function symbols. For lists of symbols \( u = (u_1, \ldots, u_n) \) and \( c = (c_1, \ldots, c_n) \), expression \( u \leq c \) is defined as \( (u_1 \leq c_1) \land \ldots \land (u_n \leq c_n) \), and similarly, expression \( u = c \) is defined as \( (u_1 = c_1) \land \ldots \land (u_n = c_n) \). Let \( \hat{c} \) be a list of distinct predicate and function constants, and let \( \hat{\hat{c}} \) be a list of distinct predicate and function variables corresponding to \( c \). By \( c^\text{pred} \) (\( c^\text{func} \), respectively) we mean the list of all predicate constants (function constants, respectively) in \( c \), and by \( \hat{c}^\text{pred} \) (\( \hat{c}^\text{func} \), respectively) the list of the corresponding predicate variables (function variables, respectively) in \( \hat{c} \). We refer to function constants and predicate constants of arity 0 as object constants and propositional constants, respectively.

Definition 1 (Stable model operator SM)

For any formula \( F \) and any list of predicate and function constants \( c \) (called intensional constants), \( \text{SM}[F; c] \) is defined as

\[
F \land \neg \exists \hat{c} (\hat{c} < c \land F^* (\hat{c})),
\]

where \( \hat{c} < c \) is a shorthand for \( (\hat{c}^{\text{pred}} \leq c^{\text{pred}}) \land \neg (\hat{c} = c) \) and \( F^* (\hat{c}) \) is defined recursively as follows:

- for atomic formula \( F \), \( F^* \equiv F' \land F \), where \( F' \) is obtained from \( F \) by replacing all intensional constants \( c \) with corresponding variables from \( \hat{c} \),
- \( (G \land H)^* = G^* \land H^* \), \( (G \lor H)^* = G^* \lor H^* \),
- \( (G \rightarrow H)^* = (G^* \rightarrow H^*) \land (G \rightarrow H) \),
- \( (\forall x G)^* = \forall x G^* \), \( (\exists x G)^* = \exists x G^* \).

\( \neg F \) is a shorthand for \( F \rightarrow \bot \), \( \top \) for \( \neg \bot \) and \( F \equiv G \) for \( (F \rightarrow G) \land (G \rightarrow F) \).
Definition 2 (Stable model)
For any sentence $F$, a stable model of $F$ on $c$ is an interpretation $I$ of the underlying signature such that $I \models \text{SM}[F; c]$.

3.2 Turning ASPMT into SMT

An SMT instance is a formula in a many-sorted first-order logic, i.e., with fixed meaning (by a background theory) of designated functions and predicates constants. Then, the SMT problem is to check if a given SMT instance has a model that expands the background theory. ASPMT is an generalization of ASP analogous to a generalization of SAT obtained with SMT (the syntax of ASPMT is the same as the syntax of SMT).

It is shown in (Bartholomew and Lee 2013) that a tight part of ASPMT instances can be turned into SMT instances and, as a result, off-the-shelf SMT solvers (e.g., Z3 (De Moura and Bjørner 2008) for arithmetic over reals) may be used to compute stable models of ASP. In order to capture this statement formally in Theorem 1 we firstly introduce notions of Clark normal form, Clark completion and dependency graphs.

Definition 3 (Clark normal form)
Formula $F$ is in Clark normal form (relative to the list $c$ of intensional constants) if it is a conjunction of sentences of the form (1) and (2)
\[
\forall x(G \rightarrow p(x)), \quad (1)
\]
\[
\forall xy(G \rightarrow f(x) = y), \quad (2)
\]
one for each intensional predicate $p$ and each intensional function $f$, where $x$ is a list of distinct object variables, $y$ is an object variable, and $G$ is an arbitrary formula that has no free variables other than those in $x$ and $y$.

Definition 4 (Clark completion)
The completion of a formula $F$ in Clark normal form (relative to $c$), denoted by $\text{Comp}_c[F]$ is obtained from $F$ by replacing each conjunctive term of the form (1) and (2) with (3) and (4) respectively
\[
\forall x(G \equiv p(x)), \quad (3)
\]
\[
\forall xy(G \equiv f(x) = y). \quad (4)
\]

Definition 5 (Dependency graph)
The dependency graph of a formula $F$ (relative to $c$) is a directed graph $DG_c[F] = (V, E)$ such that:

1. $V$ consists of members of $c$,
2. for each \( c, d \in V \), \((c, d) \in E\) whenever there exists a strictly positive occurrence of \( G \rightarrow H \) in \( F \), such that \( c \) has a strictly positive occurrence in \( H \) and \( d \) has a strictly positive occurrence in \( G \),

where an occurrence of a symbol or a subformula in \( F \) is called strictly positive in \( F \) if that occurrence is not in the antecedent of any implication in \( F \).

**Definition 6 (Tight Formula)**
Formula \( F \) is tight (on \( c \)) if \( DG_c[F] \) is acyclic.

The result obtained by Bartholomew and Lee is stated in the following theorem.

**Theorem 1 ((Bartholomew and Lee 2013))**
For a sentence \( F \) in Clark normal form that is tight on \( c \), an interpretation \( I \) that satisfies \( \exists xy(x \neq y) \) is a model of \( SM[F; c] \) iff \( I \) is a model of \( Comp_c[F] \) relative to \( c \).

## 4 ASPMT(QS) – ASPMT with Qualitative Space (QS)

In this section we present our spatial extension of ASPMT, and formalise spatial default rules and spatial frame axioms.

### 4.1 The Qualitative Spatial Domain QS

Qualitative spatial calculi can be classified into two groups: topological and positional calculi. With topological calculi such as the Region Connection Calculus (RCC) (Randell et al. 1992), the primitive entities are spatially extended regions of space, and could possibly even be 4D spatio-temporal histories, e.g., for motion-pattern analyses. Alternatively, within a dynamic domain involving translational motion, point-based abstractions with orientation calculi could suffice (e.g., using the Oriented-Point Relation Algebra (OPRA\(_m\)) (Moratz 2006)). The qualitative spatial domain (QS) that we consider in the formal framework of this paper encompasses the following ontology.

**QS1. Domain Entities in QS** Domain entities in QS include circles, triangles, points, segments, convex polygons, and egg-yolk regions. While our method is applicable to a wide range of 2D and 3D spatial objects and qualitative relations for example, as defined in (Pesant and Boyer 1994; Bouhineau 1996; Pesant and Boyer 1999; Bouhineau et al. 1999; Bhatt et al. 2011; Schultz and Bhatt 2012; Schultz and Bhatt 2014), for brevity and clarity we primarily focus on a 2D spatial domain:

- a point is a pair of reals \( x, y \),
- a line segment is a pair of end points \( p_1, p_2 \) \((p_1 \neq p_2)\),
- a circle is a centre point \( p \) and a real radius \( r \) \((0 < r)\),
• a triangle is a triple of vertices (points) \( p_1, p_2, p_3 \) such that \( p_3 \) is placed to the \textit{left} of the directed segment \( p_1, p_2 \), i.e. \textit{left} with respect to the direction of the segment from \( p_1 \) to \( p_2 \),

• a convex polygon is defined by a list of \( n \) vertices (points) \( p_1, \ldots, p_n \) (spatially ordered counter-clockwise) such that \( p_k \) is \textit{left of} the directed segment \( p_i, p_j \) for all \( 1 \leq i < j < k \leq n \),

• an egg yolk region\(^1\) is defined by a circular upper and lower approximation \( c^+, c^- \) such that \( c^- \) is a \textit{proper part} of \( c^+ \).

QS2. Spatial Relations in QS We define a range of spatial relations with the corresponding polynomial encodings. Examples of spatial relations in QS include:

Relative Orientation. Left, right, collinear orientation relations between points and segments, and parallel, perpendicular relations between segments (Lee 2014).

Mereotopology. Part-whole and contact relations between regions (Varzi 1996; Randell et al. 1992).

4.2 Spatial representations in ASPMT(QS)

Spatial representations in ASPMT(QS) are based on parametric functions and qualitative relations, defined as follows.

\textit{Definition 7 (Parametric function)}

A parametric function is an \( n \)-ary function

\[ f_n : D_1 \times D_2 \times \ldots \times D_n \to \mathbb{R}, \]

such that for any \( i \in \{1 \ldots n\}, D_i \) is a type of spatial object, e.g., Points, Circles, Polygons, etc.

As an example consider the following parametric functions

\[ x : \text{Circles} \to \mathbb{R}, \]

\[ y : \text{Circles} \to \mathbb{R}, \]

\[ r : \text{Circles} \to \mathbb{R}, \]

which return the position values \( x, y \) of a circle’s centre and its radius \( r \), respectively. Then, circle \( c \in \text{Circles} \) may be described by means of parametric functions as follows:

\[ x(c) = 1.23 \land y(c) = -0.13 \land r(c) = 2. \]

\(^1\) The egg-yolk method of modelling regions with indeterminante boundaries (Cohn and Gotts 1996) can be employed to characterise a class of regions (including polygons) that satisfies topological and relative orientation relations (Schultz and Bhatt 2015a). Each egg-yolk region is an equivalence class for all regions that are contained within the upper approximation (the egg white), and completely contain the lower approximation (the egg yolk).
Definition 8 (Qualitative spatial relation)

A qualitative spatial relation is an n-ary predicate

\[ Q_n \subseteq D_1 \times D_2 \times \ldots \times D_n, \]

such that for any \( i \in \{1 \ldots n\} \), \( D_i \) is a type of spatial object. For each \( Q_n \) there is a corresponding formula of the form

\[ \forall d_1 \in D_1 \ldots \forall d_n \in D_n \left( p_1(d_1, \ldots, d_n) \land \ldots \land p_m(d_1, \ldots, d_n) \rightarrow Q_n(d_1, \ldots, d_n) \right), \]

where \( m \in \mathbb{N} \) and for any \( i \in \{1, \ldots, m\} \), \( p_i \) is a polynomial equation or inequality over parametric functions involving \( d_1, \ldots, d_n \).

As an example consider a qualitative spatial relation \( EC \subseteq \text{Circles} \times \text{Circles} \), informally interpreted as “two circles are externally connected”. The corresponding formula of the relation states that two circles are in the \( EC \) relation whenever the distance between their centers is equal to the sum of their radii and is as follows:

\[ \forall c_1, c_2 \in \text{Circles} \left( ((x(c_1) - x(c_2))^2 + (y(c_1) - y(c_2))^2 = (r(c_1) + r(c_2))^2 \rightarrow EC(c_1, c_2) \right), \]

Proposition 1

Each qualitative spatial relation according to Definition 8 may be represented as a tight formula in Clark normal form.

Proof

Follows directly from Definitions 3 and 8.

Thus, qualitative spatial relations belong to a part of ASPMT that may be turned into SMT instances by transforming the implications in the corresponding formulas into equivalences (Clark completion). The obtained equivalence between polynomial expressions and predicates enables us to compute relations whenever parametric information is given, and vice versa, i.e., computing possible parametric values when only the qualitative spatial relations are known.

Many relations from existing qualitative calculi may be represented in ASPMT(QS) according to Definition 8; our system can express the polynomial encodings presented in, e.g., (Pesant and Boyer 1994; Bouhineau 1996; Pesant and Boyer 1999; Bouhineau et al. 1999; Bhatt et al. 2011). In what follows we give some illustrative results (see Appendix B for proofs).

Proposition 2

Each relation of Interval Algebra (IA) (Allen 1983) and Rectangle Algebra (RA) (Guesgen 1989) may be defined in ASPMT(QS).

Proposition 3

Each relation of the Left-Right Algebra (LR) (Scivos and Nebel 2004) may be defined in ASPMT(QS).
Proposition 4
Each relation of RCC–5 (Randell et al. 1992) in the domain of convex polygons with a finite number of vertices may be defined in ASPMT(QS).

Proposition 5
Each relation of Cardinal Direction Calculus (CDC) (Frank 1991) may be defined in ASPMT(QS).

4.3 Polynomial Semantics for QS

Analytic geometry gives us a general, sound, and complete way of working with spatial relations. Analytic geometry can be applied to encode the semantics of high-level qualitative spatial relations using systems of real polynomial constraints; determining spatial consistency is then equivalent to determining satisfiability of these polynomial constraints. Given polynomial constraints over a set of real variables X, the constraints are satisfiable if there exists some real value for each variable in X such that all the polynomial constraints are simultaneously satisfied. As an example, consider the definition of a relative qualitative distance relation such as nearer than: point a is nearer than b with respect to a reference point c if the distance between a, c is less than the distance between b, c, denoted nearer_than(a, b, c) (Figure 1):

\[(a_x - c_x)^2 + (a_y - c_y)^2 < (b_x - c_x)^2 + (b_y - c_y)^2.\]

Similarly, the relation of point p3 being left of segment s_{p1,p2} is encoded as the following polynomial constraint (Bhatt et al. 2011):

\[p_3 \text{ left of } s_{p_1,p_2} \equiv_{df} (x_2 - x_1)(y_3 - y_1) > (y_2 - y_1)(x_3 - x_1)\]

If there exists an assignment of real values to the variables (e.g., \(a_x = 3, c_x = 10\cdot 5\) in the nearness relation) that satisfies the polynomial equations and inequalities, then the qualitative spatial relations are consistent. Continuing with the example of relative nearness relations, consider a qualitative spatial description with the relations: nearer_than(a, b, c), nearer_than(b, a, c). This is encoded in the following polynomial constraints:

\[(a_x - c_x)^2 + (a_y - c_y)^2 < (b_x - c_x)^2 + (b_y - c_y)^2,\]

Tarski famously proved that the theory of real-closed fields is decidable via quantifier elimination (see (Collins 1975; Arnon et al. 1984; Collins and Hong 1991) for an overview and algorithms); i.e., in a finite amount of time we can determine the consistency (or inconsistency) of any formula consisting of quantifiers (\(\forall, \exists\)) over the reals, and polynomial equations and inequalities combined using logical connectors (\(\land, \lor, \neg\)). Thus, by encoding spatial relations as systems of polynomial constraints (i.e., analytic geometry) we can employ polynomial constraint solving methods that are guaranteed to determine (in)consistency, giving us sound and complete spatial reasoning.
(a_x - c_x)^2 + (a_y - c_y)^2 > (b_x - c_x)^2 + (b_y - c_y)^2,

which can be reformulated as

\[ d_{ac} < d_{bc}, \quad d_{ac} > d_{bc}. \]

As the real value \( d_{ac} \) cannot be both greater and smaller than \( d_{bc} \), this system of polynomial constraints is inconsistent, and no configuration of points (within a Euclidean space) exists that can satisfy this set of qualitative spatial constraints.

A range of qualitative spatial relations can be similarly encoded in the form of polynomial equations and inequalities, a number of spatial relations known from qualitative calculi may be expressed (Propositions 2-5, and Appendix B). Examples of specific polynomial encodings for topological and orientation relations can be found in (Pesant and Boyer 1994; Bouhineau 1996; Pesant and Boyer 1999; Bhatt et al. 2011; Schultz and Bhatt 2014; Schultz and Bhatt 2015b) (also, Appendix H presents optimisations for encodings).

Since ASPMT(QS) enables the use of polynomial equations and inequalities, a number of spatial relations known from qualitative calculi may be expressed (Propositions 2-5, and Appendix B).

Table 1: Polynomial encodings of RCC relations between circles \( c_1 \), \( c_2 \) (omitting inverses), where \( x_i = x(c_i), y_i = y(c_i), r_i = r(c_i) \), and \( \Delta(c_1, c_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 \).

| RCC Relation                  | Polynomial Encoding                                      |
|-------------------------------|----------------------------------------------------------|
| contact (C)                   | \( \Delta(c_1, c_2) \leq (r_1 + r_2)^2 \)               |
| discrete from (DR)            | \( \Delta(c_1, c_2) \geq (r_1 + r_2)^2 \)               |
| disconnects (DC)              | \( \Delta(c_1, c_2) > (r_1 + r_2)^2 \)                 |
| externally connects (EC)      | \( \Delta(c_1, c_2) = (r_1 + r_2)^2 \)                 |
| overlaps (O)                  | \( \Delta(c_1, c_2) < (r_1 + r_2)^2 \)                 |
| partially overlaps (PO)       | \( (r_1 - r_2)^2 < \Delta(c_1, c_2) < (r_1 + r_2)^2 \) |
| part of (P)                   | \( \Delta(c_1, c_2) \leq (r_1 - r_2)^2 \land (r_1 \leq r_2) \) |
| proper part of (PP)           | \( \Delta(c_1, c_2) \leq (r_1 - r_2)^2 \land (r_1 \leq r_2) \) |
| tangential proper part (TPP)  | \( \Delta(c_1, c_2) = (r_1 - r_2)^2 \land (r_1 < r_2) \) |
| nontangential proper part (NTPP)| \( \Delta(c_1, c_2) < (r_1 - r_2)^2 \land (r_1 < r_2) \) |
| equal (EQ)                    | \( x_1 = x_2 \land y_1 = y_2 \land r_1 = r_2 \)      |

Fig. 2: RCC relations between two circular regions.

5 ASPMT(QS) – System Overview and Implementation

The implementation of ASPMT(QS) builds on ASPMT2SMT (Bartholomew and Lee 2014) – a compiler translating a tight fragment of ASPMT into SMT instances. Our

\footnote{The region connection calculus (RCC) is a spatial logic of topological relations between regions (Randell et al. 1992). The theory is based on a single connects predicate \( C(x, y) \) interpreted as the topological closures of regions \( x \) and \( y \) having at least one point in common (i.e., the regions touch at their boundaries or their interiors overlap).}
system consists of an additional module for spatial reasoning and Z3 as the SMT solver. As our system operates on a tight fragment of ASPMT, input programs need to fulfil certain requirements, described in the following section. As output, our system either produces the stable models of the input programs, or states that no such model exists.

5.1 Syntax of Input Programs

The input program to our system needs to be \( f\text{-plain} \) to use Theorem 1 from (Bartholomew and Lee 2012).

\textbf{Definition 9 (f-plain formula)}

Let \( f \) be a function constant. A first–order formula is called \( f\text{-plain} \) if each atomic formula:

- does not contain \( f \), or
- is of the form \( f(t) = u \), where \( t \) is a tuple of terms not containing \( f \), and \( u \) is a term not containing \( f \).

Additionally, the input program needs to be \textit{av-separated}, i.e. no variable occurring in an argument of an uninterpreted function is related to the value variable of another uninterpreted function via equality (Bartholomew and Lee 2014). The input program is divided into declarations of:

- \textit{sorts} (data types),
- \textit{objects} (particular elements of given types),
- \textit{constants} (functions),
- \textit{variables} (variables associated with declared types).

The second part of the program consists of clauses. ASPMT(QS) supports:

- \textit{connectives}: \&, |, not, \text{-}, <, <=, >=, >, !=, +, =, *

with their usual meaning. Additionally, ASPMT(QS) supports the following as native / first-class entities:

- \textit{basic spatial objects types}, e.g., \texttt{point}, \texttt{segment}, \texttt{circle}, \texttt{triangle};
- \textit{parametric functions describing objects parameters}, e.g., \texttt{x(point)}, \texttt{r(circle)};
- \textit{qualitative relations}, e.g., \texttt{rccEC(circle,circle)}, \texttt{coincident(point,circle)}.

The abovementioned qualitative spatial relations and parametric functions do not need to be defined in the program (they are predefined in our spatial reasoning module). Similarly, object types do not need to be defined. The user only needs to define names of objects and the type they belong to, e.g., “\( a :: circle \)”, stands for an object “\( a \)” that is a circle.
Example 1: combining topology and size

Consider a program describing three circles \(a\), \(b\), and \(c\) such that \(a\) is discrete from \(b\), \(b\) is discrete from \(c\), and \(a\) is a proper part of \(c\), declared as follows:

\[
\begin{align*}
\text{:- constants} \\
a &::\text{circle}; \\
b &::\text{circle}; \\
c &::\text{circle}.
\end{align*}
\]

\[
\text{:- not rccDR} (a, b), \\
\text{:- not rccDR} (b, c), \\
\text{:- not rccPP} (a, c).
\]

ASPMT\((\text{QS})\) checks if the spatial relations are satisfiable. In the case of a positive answer, a parametric model and computation time are presented. The above mentioned program generates the following output:

\[
\begin{align*}
r(a) & = 1.0 & r(b) & = 1.0 & r(c) & = 2.0 \\
x(a) & = 1.0 & x(b) & = 1.0 & x(c) & = 1.0 \\
y(a) & = 1.0 & y(b) & = 4.0 & y(c) & = 0.5
\end{align*}
\]

This example demonstrates that ASPMT\((\text{QS})\) is capable of computing whether a given set of qualitative relations is consistent. In this case we show that ASPMT\((\text{QS})\) may compute RCC–5 relations between circles (for details see (Randell et al. 1992)).

Now, consider the addition of a further constraint to the program stating that circles \(a\), \(b\), and \(c\) have the same radius:

\[
\text{:- ar=} R1 \& br=} R2 \& cr=} R3 \& (R1!=R2 \mid R2!=R3 \mid R1!=R3).
\]

The new program is an example of combining different types of qualitative information, namely topology and size, which is a non-trivial research topic within the relation algebraic spatial reasoning community; relation algebraic-based solvers such as GQR (Ganter et al. 2008; Wölfl and Westphal 2009) will not correctly determine inconsistencies in general for arbitrary combinations of different types of relations (orientation, shape, distance, etc.). In this case, ASPMT\((\text{QS})\) correctly determines that the spatial constraints are inconsistent:

\[
\text{UNSATISFIABLE; Z3 time in milliseconds: 0; Total time in milliseconds: 32}
\]

Example 2: combining topology and relative orientation

Given three circles \(a\), \(b\), \(c\) let \(a\) be proper part of \(b\), \(b\) discrete from \(c\), and \(a\) in contact with \(c\), declared as follows:

\[
\begin{align*}
\text{:- constants} \\
a &::\text{circle}; \\
b &::\text{circle}; \\
c &::\text{circle}.
\end{align*}
\]

\[
\text{:- not rccPP} (a, b), \\
\text{:- not rccDR} (b, c), \\
\text{:- not rccCP} (a, c).
\]

Given this basic qualitative information, ASPMT\((\text{QS})\) is able to refine the topological relations to infer that (Figure 3a):

- \(a\) is a *tangential proper part of* \(b\)
- both \(a\) and \(b\) are *externally connected to* \(c\)

as stated in the following output:
We then add an additional constraint that the centre of \( a \) is \textit{left of} the directed segment between the centres \( b \) to \( c \).

ASPMT(QS) determines that this is inconsistent:

\[
\text{UNSATISFIABLE;}
\]

In other words the centres must be \textit{collinear} (therefore the configuration presented in Figure 3b is not possible).

![Figure 3: Reasoning about consistency and refinement by combining topology and relative orientation.](image)

### 6 Empirical Evaluation and Examples

In this section we present an empirical evaluation of ASPMT(QS) in comparison with other existing spatial reasoning systems. The range of problems demonstrates the unique, non-monotonic spatial reasoning features that ASPMT(QS) provides beyond what is possible using other currently available systems. Table 2 presents run times obtained by Clingo – an ASP grounder and solver (Gebser et al. 2014), GQR – a binary constraint calculi reasoner (Gantner et al. 2008), CLP(QS) – a Constraint Logic Programming system over qualitative spatial domains (Bhatt et al. 2011) and our ASPMT(QS) implementation. Tests were performed on an Intel Core 2 Duo 2.00 GHZ CPU with 4 GB RAM running Ubuntu 14.04. The polynomial encodings of topological relations are presented in Table 1.

#### 6.1 Ramification Problem

The following two problems, \textit{Growth} and \textit{Motion}, were introduced in (Bhatt 2008). Consider the initial situation \( S_0 \) presented in Figure 4, consisting of three cells: \( a, b, c \), such that \( a \) is a non-tangential proper part of \( b: \text{rccTPP}(a, b, 0) \), and \( b \) is externally connected to \( c: \text{rccEC}(b, c, 0) \).

---

4 The fundamental distinction between CLP(QS) and ASPMT(QS) is the declarative framework that has been extended, in particular, Constraint Logic Programming does not facilitate non-monotonic reasoning and thus cannot be used to formalise default rules, spatial inertia, etc.
Table 2: Total time results (i.e., preprocessing and solving) of performed tests. “—” indicates that the problem can not be formalised, “I” indicates that indirect effects can not be formalised, “D” indicates that default rules can not be formalised.

| Problem      | Clingo  | GQR    | CLP(QS) | ASPMT(QS) |
|--------------|---------|--------|---------|-----------|
| Growth       | 0.004s  | 0.014s | 1.623s  | 0.169s    |
| Motion       | 0.004s  | 0.013s | 0.449s  | 0.167s    |
| Attach I     | 0.008s  | —      | 3.139s  | 0.625s    |
| Attach II    | —       | —      | 2.789s  | 0.268s    |

Figure 4: Indirect effects of $growth(a, 0)$ and $motion(a, 0)$ events.

> Growth. Let $a$ grow in step $S_0$: the event $growth(a, 0)$ occurs and leads to a successor situation $S_1$. The direct effect of $growth(a, 0)$ is a change of a relation between $a$ and $b$ from $rccNTPP(a, b, 0)$ to $rccEQ(a, b, 1)$ (i.e., $a$ is equal to $b$). No change of the relation between $a$ and $c$ is directly stated, and thus we must derive the relation $rccEC(a, c, 1)$ as an indirect effect.

> Motion. Let $a$ move in step $S_0$: the event $motion(a, 0)$ leads to a successor situation $S_1$. The direct effect is a change of the relation $rccTPP(a, b, 0)$ to $rccTPP(a, b, 1)$ ($a$ is a tangential proper part of $b$). In the successor situation $S_1$ we must determine that the relation between $a$ and $c$ can only be either $rccEC(a, c, 1)$ or $rccEC(a, c, 1)$.

GQR provides no support for domain-specific reasoning, and thus we encoded the problem as two distinct qualitative constraint networks (one for each simulation step) and solved them independently, i.e., with no definition of $growth$ and $motion$. Thus, GQR is not able to produce any additional information about indirect effects. As Clingo lacks any mechanism for analytic geometry, we implemented the RCC8 composition table and thus it inherits the incompleteness of relation algebraic reasoning. While CLP(QS) facilitates the modelling of domain rules such as $growth$, there is no native support for default reasoning and thus we forced $b$ and $c$ to remain unchanged between simulation steps, otherwise all combinations of spatially consistent actions on $b$ and $c$ are produced without any preference (i.e., leading to the frame problem).

In contrast, ASPMT(QS) can express spatial inertia, and derives indirect effects
directly from spatial reasoning: in the Growth problem we can use ASPMT(QS) to abduce that \( a \) has to be concentric with \( b \) in \( S_0 \) (otherwise a move event would also need to occur).\(^5\) Checking global consistency of scenarios that contain inter-dependent spatial relations is a crucial feature that is enabled by the method of polynomial encodings and is provided only by CLP(QS) and ASPMT(QS).

**6.2 Geometric Reasoning and the Frame Problem**

In problems Attachment I and Attachment II the initial situation \( S_0 \) consists of three objects (circles), namely car, trailer and garage as presented in Figure 5. Initially, the trailer is attached to the car: \( \text{rccEC(car, trailer, 0)} \), \( \text{attached(car, trailer, 0)} \). The successor situation \( S_1 \) is described by \( \text{rccTPP(car, garage, 1)} \). The task is to infer the possible relations between the trailer and the garage, and the necessary actions that would need to occur in each scenario.

There are two domain-specific actions: the car can move, \( \text{move(car, X)} \), and the trailer can be detached, \( \text{detach(car, trailer, X)} \) in simulation step \( X \). Whenever the trailer is attached to the car, they remain \( \text{rccEC} \). The car and the trailer may be either completely outside or completely inside the garage.

![Figure 5: Non-monotonic reasoning with additional geometric information.](image_url)

\( S_0 \)

\( S_1 \)

\( ^5 \) That is, we can define a binary predicate \text{concentric} between two circles that is \text{true} when the centre points of the circles are equal, and \text{false} when they are not equal. ASPMT(QS) abduces that \text{not concentric} is unsatisfiable, i.e. that the circles are necessarily concentric.
and then moved into the garage: (b) the car, together with the trailer attached to it, moved into the garage:

\[ r(\text{car}) = 2, r(\text{trailer}) = 2 \text{ and } r(\text{garage}) = 3. \]

Case (b) is now inconsistent, and we must determine that the only possible solution is (a).

These domain-specific rules require default reasoning: “typically the trailer remains in the same position” and “typically the trailer remains attached to the car”. The later default rule is formalised in ASPMT(QS) by means of the spatial default. The formalisation of such rules addresses the frame problem. GQR is not capable of expressing the domain-specific rules for detachment and attachment in Attachment I and Attachment II. Neither GQR nor Clingo is capable of reasoning with a combination of topological and numerical information, as required in Attachment II. As CLP(QS) cannot express default rules, we can not capture the notion that, for example, the trailer should typically remain in the same position unless we have some explicit reason for determining that it moved; once again this leads to an exhaustive enumeration of all possible scenarios without being able to specify preferences, i.e. the frame problem, and thus CLP(QS) will not scale in larger scenarios.

The results of the empirical evaluation show that ASPMT(QS) is the only system that is capable of (a) non-monotonic spatial reasoning, (b) expressing domain-specific rules that also have spatial aspects, and (c) integrating both qualitative and numerical information. Regarding the greater execution times in comparison to CLP(QS), we have not yet fully integrated available optimisations (e.g., based on spatial symmetry-driven pruning strategies (Schultz and Bhatt 2015b)) with respect to spatial reasoning; this is one of the directions of future work as the known performance gains are truly significant (Schultz and Bhatt 2015b).

### 6.3 Abductive Reasoning

We show how abductive reasoning may be achieved in ASPMT(QS). Consider an application where the spatial configuration of objects is recorded in discrete time points (e.g., geospatial information collected about cities or a dynamic environment observed by a mobile robot). We consider the situation presented in Figure 6, where the following data about spatial relations between \(a, b, c\) in time points \(t_1, t_2, t_3\) is available:

\[
\begin{align*}
\text{rccPP}(c,a,t1). & \quad \text{rccPP}(b,a,t1). \\
\text{rccPP}(c,a,t2). & \quad \text{rccEC}(b,a,t2). \\
\text{rccPP}(c,a,t3). & \quad \text{rccEC}(b,a,t3). \\
\text{l}(b,t1)=R1 \land r(b,t2)=R2 \land R1 \Rightarrow R2. \\
\end{align*}
\]

Having a knowledge base about available actions that may be performed and a number of rules describing the spatial behaviour of objects, ASPMT(QS) infers what actions had to be performed between time points, namely:

\[
\text{move}(b,t1)=\text{true}. \quad \text{move}(b,t2)=\text{true}. \quad \text{grow}(b,t2)=\text{true}.
\]

Abductive reasoning requires default rules such as inertia which states that if no
action is performed, then object locations and parameters remain the same by default. This rule enables the inference that the position of \( c \) with respect to \( a \) is the same in all three time steps. The example illustrated in Figure 6 shows that ASPMT(QS) checks the global consistency of various spatial relations. Therefore, it infers relations occurring as a result of performed actions (indirect effects) even if there is more than one possible solution, e.g., the relation between \( c \) and \( b \) in \( t_3 \) may be either \( rccDC(b, c, t_3) \) or \( rccEC(b, c, t_3) \).

7 Discussion and Related Work

Our research investigates the formal modelling and computational aspects of space and spatio-temporal dynamics (Bhatt and Loke 2008; Bhatt 2012) from the viewpoint of the artificial intelligence (AI) sub-disciplines of knowledge representation and reasoning (KR), commonsense reasoning, geometric and qualitative spatial reasoning, and more broadly, spatial information theory and spatial cognition and computation.

Beyond the state of the art  Research in qualitative spatial representation and reasoning (QSTR) has focussed on the construction of relation-algebraically founded qualitative spatial and temporal calculi, complexity results thereof, and formalisations of (topological) spatial logics. Furthermore, specialized relation-algebraically founded methods and prototypical spatial reasoning tools – working as black-box systems outside of any systematic KR method – have been a niche too. What is still missing in the field of spatial representation and reasoning is a modular, unifying KR framework of space, action, and change (Bhatt 2012) that would seamlessly integrate with or be accessible via general KR languages and frameworks in AI, and be applicable in a wide-range of application domains. Primarily, our research is motivated by addressing this gap.
KR based methods and tools addressing spatio-temporal dynamics

With a focus on commonsense reasoning about space, action, and change, the principal focus and long-term agenda of our research is to develop methods and generally usable tools for visuo-spatial problem solving with spatio-temporal configurations and dynamic phenomena at the scale of everyday human perception, abstraction, interpretation, and interaction. This agenda is being driven by the application domains that are being pursued independently in the areas of spatio-temporal narrative interpretation and synthesis, spatial computing for architecture design, cognitive vision and robotics, and geospatial dynamics (Bhatt et al. 2013). This level of generality, and our emphasis on mixed qualitative-quantitative spatial reasoning in the context of state of the art KR methods, namely constraint logic and answer set programming paradigms, will inherently offer a robust and scalable representational and computational foundation for the class of application areas that inspire and guide the basic research questions reported in this paper.

Related work on space and motion (Shanahan 1995) describes a default reasoning problem, analogous to the classic frame problem, which arises when an attempt is made to construct a logic-based calculus for reasoning about the movement of objects in a real-valued co-ordinate system. (Shanahan 1995)’s all-encompassing theory alludes to a unification of spatial, temporal and causal aspects at representational and computational levels. The use of commonsense reasoning about the physical properties of objects within a first-order logical framework has been investigated by (Davis 2008; Davis 2011). The aim here is to combine commonsense qualitative reasoning about “continuous time, Euclidean space, commonsense dynamics of solid objects, and semantics of partially specified plans” (Davis 2011). (Cabalar and Santos 2010) investigate the formalization of the commonsense representation that is necessary to solve spatial puzzles involving non-trivial objects such as holes and strings. (Bhatt and Loke 2008) explicitly formalize a dynamic spatial systems approach for the modelling of changing spatial domains using the Situation Calculus (McCarthy and Hayes 1969). A dynamic spatial system here is regarded as a specialization of the generic dynamic systems approach (Sandewall 1994; Reiter 2001) for the case where sets of qualitative spatial relationships (grounded in formal spatial calculi) undergo change as a result of actions and events in the system.

8 Conclusion and Outlook

We have presented ASPMT(QS), a novel approach for reasoning about space and spatial change within a KR paradigm. By integrating dynamic spatial reasoning within a KR framework, namely answer set programming (modulo theories), our system can be used to model behaviour patterns that characterise high-level processes, events, and activities as identifiable with respect to a general characterisation of commonsense reasoning about space, actions, and change (Bhatt 2012; Bhatt and Loke 2008).

ASPM(T(QS)) is capable of sound and complete spatial reasoning, and combining qualitative and quantitative spatial information when reasoning non-monotonically;
this is due to the approach of encoding spatial relations as polynomial constraints, and solving using SMT solvers with the theory of real nonlinear arithmetic. We have demonstrated that no other existing spatial reasoning system is capable of supporting the non-monotonic spatial reasoning features (e.g., spatial inertia, ramification, causal explanation) in the context of any systematic knowledge representation and reasoning method, be it a mainstream method such as answer set programming, or otherwise.

This work opens up several opportunities: the spatio-temporal ontology can be extended in many interesting ways with support for richer spatial relational as well as object domains. The polynomial encodings may be further optimised, and this is a topic that we have ourselves devoted considerable attention to (Schultz and Bhatt 2015b) (Appendix H). Most interestingly, there exist many possibilities to build additional modules directly on top of ASPMT(QS) concerning aspects such as visibility, spatio-temporal motion patterns (based on space-time histories) etc from the viewpoint of applications in cognitive vision, cognitive robotics, eye-tracking or visual perception (Suchan et al. 2014; Suchan and Bhatt 2016a; Suchan and Bhatt 2016b), and to provide support for specialized, domain-specific spatial-linguistic characterisations identifiable in a range of spatial assistance systems and assistive technologies encompassing (spatial) logic, (spatial) language, and (spatial) cognition (Bhatt et al. 2013).

Acknowledgments.

We acknowledge the support provided by the Polish National Science Centre grant 2011/02/A/HS1/0039, and the DesignSpace Group (www.designspace.org).

References

Aiello, M., Pratt-Hartmann, I. E., and Benthem, J. F. v. 2007. Handbook of Spatial Logics. Springer-Verlag New York, Inc., Secaucus, NJ, USA.

Allen, J. F. 1983. Maintaining knowledge about temporal intervals. Communications of the ACM 26, 11, 832–843.

Arnon, D. S., Collins, G. E., and McCallum, S. 1984. Cylindrical Algebraic Decomposition I: The basic algorithm. SIAM Journal on Computing 13, 4, 865–877.

Bartholomew, M. and Lee, J. 2012. Stable models of formulas with intensional functions. In KR.

Bartholomew, M. and Lee, J. 2013. Functional stable model semantics and answer set programming modulo theories. In Proceedings of the Twenty-Third international joint conference on Artificial Intelligence. AAAI Press, 718–724.

Bartholomew, M. and Lee, J. 2014. System aspmt2smt: Computing ASPMT Theories by SMT Solvers. In Logics in Artificial Intelligence. Springer, 529–542.

Bhatt, M. 2008. (Some) Default and Non-Monotonic Aspects of Qualitative Spatial Reasoning. In AAAI-08 Technical Reports, Workshop on Spatial and Temporal Reasoning. 1–6.
Bhatt, M. 2010. Commonsense inference in dynamic spatial systems: Epistemological requirements. In Proceedings of the Twenty-Third International Florida Artificial Intelligence Research Society Conference, May 19-21, 2010, Daytona Beach, Florida. AAAI Press.

Bhatt, M. 2012. Reasoning about space, actions and change: A paradigm for applications of spatial reasoning. In Qualitative Spatial Representation and Reasoning: Trends and Future Directions. IGI Global, USA.

Bhatt, M., Guesgen, H., Wölfl, S., and Hazarika, S. 2011. Qualitative spatial and temporal reasoning: Emerging applications, trends, and directions. Spatial Cognition & Computation 11, 1, 1–14.

Bhatt, M., Lee, J. H., and Schultz, C. 2011. CLP(QS): A Declarative Spatial Reasoning Framework. In COSIT 2011 - Spatial Information Theory. Springer-Verlag, Berlin, Heidelberg, 210–230.

Bhatt, M. and Loke, S. 2008. Modelling dynamic spatial systems in the situation calculus. Spatial Cognition and Computation 8, 1, 86–130.

Bhatt, M., Schultz, C., and Freksa, C. 2013. The ‘Space’ in Spatial Assistance Systems: Conception, Formalisation and Computation. In Representing space in cognition: Interrelations of behavior, language, and formal models. Series: Explorations in Language and Space, T. Tenbrink, J. Wiener, and C. Claramunt, Eds. 978-0-19-967991-1, Oxford University Press.

Bhatt, M., Schultz, C. P. L., and Thosar, M. 2014. Computing narratives of cognitive user experience for building design analysis: KR for industry scale computer-aided architecture design. In KR 2014, C. Baral, G. D. Giacomo, and T. Eiter, Eds. AAAI Press.

Bhatt, M., Suchan, J., and Schultz, C. 2013. Cognitive Interpretation of Everyday Activities – Toward Perceptual Narrative Based Visuo-Spatial Scene Interpretation. In Computational Models of Narrative (CMN) 2013., a satellite workshop of CogSci 2013: The 35th meeting of the Cognitive Science Society., M. Finlayson, B. Fisseni, B. Loewe, and J. C. Meister, Eds. OpenAccess Series in Informatics (OASIcs), Dagstuhl, Germany.

Bhatt, M. and Wallgrün, J. O. 2014. Geospatial narratives and their spatio-temporal dynamics: Commonsense reasoning for high-level analyses in geographic information systems. ISPRS Int. J. Geo-Information 3, 1, 166–205.

Bouhineau, D. 1996. Solving geometrical constraint systems using CLP based on linear constraint solver. In Artificial Intelligence and Symbolic Mathematical Computation. Springer, 274–288.

Bouhineau, D., Trilling, L., and Cohen, J. 1999. An application of CLP: Checking the correctness of theorems in geometry. Constraints 4, 4, 383–405.

Cabalar, P. and Santos, P. E. 2010. Formalising the fisherman’s folly puzzle. Artificial Intelligence In Press, Corrected Proof, –.

Cohn, A. G. and Gotts, N. M. 1996. The ‘egg-yolk’representation of regions with indeterminate boundaries. Geographic objects with indeterminate boundaries 2, 171–187.

Collins, G. E. 1975. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. In Automata Theory and Formal Languages 2nd GI Conference Kaiserslautern, May 20–23, 1975. Springer, 134–183.

Collins, G. E. and Hong, H. 1991. Partial cylindrical algebraic decomposition for quantifier elimination. Journal of Symbolic Computation 12, 3, 299 – 328.

Davis, E. 2008. Pouring liquids: A study in commonsense physical reasoning. Artif. Intell. 172, 12-13, 1540–1578.

Davis, E. 2011. How does a box work? A study in the qualitative dynamics of solid objects. Artificial Intelligence 175, 1, 299–345.
De Moura, L. and Bjørner, N. 2008. Z3: An efficient smt solver. In Tools and Algorithms for the Construction and Analysis of Systems. Springer, 337–340.

Eppe, M. and Bhatt, M. 2013. Narrative based postdictive reasoning for cognitive robotics. In COMMONSENSE 2013: 11th International Symposium on Logical Formalizations of Commonsense Reasoning.

Ferraris, P. 2005. Answer sets for propositional theories. In Logic Programming and Nonmonotonic Reasoning. Springer, 119–131.

Ferraris, P., Lee, J., and Lifschitz, V. 2011. Stable models and circumscription. Artificial Intelligence 175, 1, 236–263.

Frank, A. U. 1991. Qualitative spatial reasoning with cardinal directions. In 7. Österreichische Artificial-Intelligence-Tagung/Seventh Austrian Conference on Artificial Intelligence. Springer, 157–167.

Gantner, Z., Westphal, M., and Wölfl, S. 2008. GQR-A fast reasoner for binary qualitative constraint calculi. In Proc. of AAAI. Vol. 8.

Gebser, M., Kaminski, R., Kaufmann, B., and Schaub, T. 2014. Clingo = ASP+ control: Preliminary report. arXiv preprint arXiv:1405.3694.

Gelfond, M. 2008. Answer sets. Handbook of knowledge representation 1, 285.

Gelfond, M. and Lifschitz, V. 1988. The stable model semantics for logic programming. In ICLP/SLP, Vol. 88. 1070–1080.

Guesgen, H. W. 1989. Spatial reasoning based on Allen’s temporal logic. Technical Report TR-89-049. International Computer Science Institute Berkeley.

Kapur, D. and Mundy, J. L., Eds. 1988. Geometric Reasoning. MIT Press, Cambridge, MA, USA.

Lee, J. H. 2014. The complexity of reasoning with relative directions. In 21st European Conference on Artificial Intelligence (ECAI 2014).

Ligozat, G. 2011. Qualitative Spatial and Temporal Reasoning. Wiley-ISTE.

McCarthy, J. and Hayes, P. J. 1969. Some philosophical problems from the standpoint of artificial intelligence. In Machine Intelligence 4, B. Meltzer and D. Michie, Eds. Edinburgh University Press, 463–502.

Moratz, R. 2006. Representing relative direction as a binary relation of oriented points. In ECAI, G. Brewka, S. Coradeschi, A. Perini, and P. Traverso, Eds. Frontiers in Artificial Intelligence and Applications, vol. 141. IOS Press, 407–411.

Pesant, G. and Boyer, M. 1994. QUAD-CLP (R): Adding the power of quadratic constraints. In Principles and Practice of Constraint Programming. Springer, 95–108.

Pesant, G. and Boyer, M. 1999. Reasoning about solids using constraint logic programming. Journal of Automated Reasoning 22, 3, 241–262.

Randell, D. A., Cui, Z., and Cohn, A. G. 1992. A spatial logic based on regions and connection. KR 92, 165–176.

Reiter, R. 2001. Knowledge in action: Logical foundations for describing and implementing Dynamical systems. MIT Press.

Sandewall, E. 1994. Features and Fluents (Vol. 1): The Representation of Knowledge about Dynamical Systems. Oxford University Press, Inc., New York, NY, USA.

Schultz, C. and Bhatt, M. 2012. Towards a Declarative Spatial Reasoning System. In 20th European Conference on Artificial Intelligence (ECAI 2012).

Schultz, C. and Bhatt, M. 2014. Declarative spatial reasoning with boolean combinations of axis-aligned rectangular polytopes. In ECAI 2014 - 21st European Conference on Artificial Intelligence. 795–800.

Schultz, C. and Bhatt, M. 2015a. Encoding Relative Orientation and Mereotopology Relations with Geometric Constraints in CLP(QS). In 1st Workshop on Logics for Qualitative Modelling and Reasoning (LQMR’15). Lodz, Poland.
Walęga, Schultz, Bhatt

Schultz, C. and Bhatt, M. 2015b. Spatial symmetry driven pruning strategies for efficient declarative spatial reasoning. In COSIT 2015 - Spatial Information Theory. 331–353.

Scivos, A. and Nebel, B. 2004. The Finest of its Class: The Natural, Point-Based Ternary Calculus LR for Qualitative Spatial Reasoning. In C. Freksa et al. (2005), Spatial Cognition IV. Reasoning, Action, Interaction: International Conference Spatial Cognition. Lecture Notes in Computer Science Vol. 3343, Springer, Berlin Heidelberg. Vol. 3343. 283–303.

Shanahan, M. 1995. Default reasoning about spatial occupancy. Artif. Intell. 74, 1, 147–163.

Spranger, M., Suchan, J., and Bhatt, M. 2016. Robust natural language processing - combining reasoning, cognitive semantics and construction grammar for spatial language. In IJCAI 2016: International Joint Conference on Artificial Intelligence., New York, United States. AAAI Press. to appear.

Spranger, M., Suchan, J., Bhatt, M., and Eppe, M. 2014. Grounding dynamic spatial relations for embodied (robot) interaction. In PRICAI 2014: Pacific Rim Int. Conf. on Artificial Intelligence, D. N. Pham and S. Park, Eds. Lecture Notes in Computer Science, vol. 8862. Springer, 958–971.

Suchan, J. and Bhatt, M. 2016a. Semantic Question-Answering with Video and Eye-Tracking Data – AI Foundations for Human Visual Perception Driven Cognitive Film Studies. In IJCAI 2016: International Joint Conference on Artificial Intelligence., New York, United States. AAAI Press.

Suchan, J. and Bhatt, M. 2016b. The Geometry of a Scene: On Deep Semantics for Visual Perception Driven Cognitive Film Studies. In WACV 2016: IEEE Winter Conference on Applications of Computer Vision (WACV 2016)., Lake Placid, NY, USA, IEEE. IEEE.

Suchan, J., Bhatt, M., and Santos, P. E. 2014. Perceptual narratives of space and motion for semantic interpretation of visual data. In Computer Vision - ECCV 2014 Workshops - Zurich, Switzerland, September 6-7 and 12, 2014, Proceedings, Part II. 339–354.

Varzi, A. C. 1996. Parts, wholes, and part-whole relations: The prospects of mereotopology. Data & Knowledge Engineering 20, 3, 259–286.

Walęga, P. A., Bhatt, M., and Schultz, C. P. L. 2015. ASPMT(QS): Non-Monotonic Spatial Reasoning with Answer Set Programming Modulo Theories. In Logic Programming and Nonmonotonic Reasoning - 13th International Conference, LPNMR 2015, Lexington, KY, USA, September 27-30, 2015. Proceedings. Lecture Notes in Computer Science, vol. 9345. Springer, 488–501.

Wölfl, S. and Westphal, M. 2009. On combinations of binary qualitative constraint calculi. In IJCAI 2009. 967–973.