An Online Multi-unit Auction with Improved Competitive Ratio

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Abstract

We improve the best known competitive ratio (from 1/4 to 1/2), for the online multi-unit allocation problem, where the objective is to maximize the single-price revenue. Moreover, the competitive ratio of our algorithm tends to 1, as the bid-profile tends to “smoothen”. This algorithm is used as a subroutine in designing truthful auctions for the same setting: the allocation has to be done online, while the payments can be decided at the end of the day. Earlier, a reduction from the auction design problem to the allocation problem was known only for the unit-demand case. We give a reduction for the general case when the bidders have decreasing marginal utilities. The problem is inspired by sponsored search auctions.

1 Introduction

We improve the best known competitive ratio (from 1/4 to 1/2), for the online multi-unit allocation problem, which in turn gives a factor of 2 improvement for the online multi-unit auction problem. Moreover, the competitive ratio of our algorithm tends to 1, as the bid-profile tends to “smoothen”. We also give a reduction from the auction problem to the allocation problem when the bidders want multiple copies of the item with decreasing marginal utilities for them. Earlier, such a reduction was known only for the unit-demand case.

Definition 1.1. Online Multi-unit Auction Problem for single demand We have for auction multiple copies of a single item, where the copies are coming online. We have no prior knowledge of how many copies of the item will be produced. Each bidder has a utility $u_i$ for one copy of the good, and bids a value $b_i$ at the beginning of the auction.

The problem is to design an allocation and pricing scheme that has the bids $b_i$ as input and has the following properties:

• (Truthfulness) The auction mechanism should be truthful.

• (Perishable Good) As a new copy comes, either it is allocated to some bidder, who wins the copy, or it is discarded.

• The prices charged to the winning bidders are determined at the end of the auction, when there are no more copies left.

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The goal of the auction is to maximize the revenue of the auctioneer.

The corresponding online multi-unit allocation Problem for single demand is the following:

**Definition 1.2. Online Multi-unit Allocation Problem for single demand** We want to sell multiple copies of a single item, where the copies are coming online. We have no prior knowledge of how many copies of the item will be produced. Each interested bidder sends her bid \( b_i \) at the beginning of the auction.

The problem is to design an allocation scheme that has the bids \( b_i \) as input and as a new copy comes, either it is allocated to some bidder, who wins the copy, or it is discarded. All the winning bidders are charged the same amount at the end of the auction and is less than the bid of all the winning bidders.

The goal of the auction is to maximize the revenue of the auctioneer.

A generalization of the above problem is the multiple demand problem.

**Definition 1.3. Online Multi-unit Auction Problem for Multiple demand** Just like in the single demand problem we have for auction multiple copies of a single item, where the copies are coming online. But in this problem the bidders may bid for multiple copies of the item. The bidders make multiple bids for each copy.

The problem is again to design an allocation and pricing scheme that satisfies the all the conditions of truthfulness, perishableness of the item and charging the price at the end. The goal is to maximize the revenue of the auctioneer.

**Definition 1.4. Competitive Ratio** Given the bids \( \{b_i\} \) and the number of copies of the item that is produced let \( \text{OPT} \) be the revenue of the optimum single price auction. If the expected revenue generated by an online multi-unit auction mechanism is \( \text{REV} \) then the Competitive Ratio of the mechanism is \( \text{REV}/\text{OPT} \).

**Theorem 1.5.** We design an allocation algorithm for the online multi-unit auction problem for unit demand, that achieves a competitive ration of \( 1/2 \).

**Corollary 1.6.** We have a truthful auction mechanism for the online multi-unit auction problem, that achieves a constant competitive ratio.

In the Online Multi-unit Auction problem, we have for auction multiple copies of a single item, where the copies are coming online. We have no prior knowledge of how many copies of the item will be produced. Each bidder has a utility \( u_i \) for one copy of the good, and bids a value \( b_i \) at the beginning of the auction. As a new copy comes, either it is allocated to some bidder, who wins the copy, or it is discarded. The prices charged to the winning bidders are determined at the end of the auction, when there are no more copies left. The goal of the auction is to maximize the revenue of the auctioneer. We use competitive analysis, and compare this revenue to the optimal “single price” revenue, that is the optimal revenue that could have been obtained by charging the same price for all the copies allocated, having known the number of copies produced and the true utilities of the bidders. Further, we want the auction to be strategyproof, that is, the auction is so designed that bidding truthfully (that is \( b_i = u_i \)) is a dominant strategy for every bidder.
Mahdian and Saberi [11] showed that this problem can be reduced to the Online Multi-unit Allocation problem, with a constant factor lost in the competitive ratio. The allocation problem is exactly as before, except that one can assume that the bidders bid their true utility, and the allocation algorithm is also restricted to be single priced, the price being determined at the end of the algorithm. As before, the revenue of the algorithm is compared to the optimum single price revenue, having known the number of copies.

If a bidder wants more than one copy of the item, then he submits multiple bids. The allocation problem is essentially the same. However, the auction problem is more difficult because of truthfulness. The auction has to now consider the possibility that the bidder lies about some subset of his bids, as opposed to lying about the only bid in the unit-demand case.

The problems are inspired by sponsored search auctions, which are a major source of revenue for search engines like Google, Yahoo and MSN. The copies correspond to search queries and the bidders are the advertisers. The auction problem considered here is also a natural extension of the line of work on digital goods auction: from unlimited supply ([6, 7, 8, 9]) to limited supply ([1, 2, 4]), to unknown supply ([11] and this paper). For the sponsored search auction setting, the multiple demand case is more realistic and our reduction for this case is of significant interest.

It can be easily seen that the competitive ratio of any deterministic algorithm for the allocation problem is arbitrarily small. So it is actually surprising that a randomized algorithm can even get a constant competitive ratio. The reason for this difficulty is that the revenue of the algorithm, as a function of the number of copies allocated can have many “peaks” and “valleys”. For any deterministic algorithm, an adversary can make sure that the algorithm either ends up in a valley, or is stuck on a small peak while the optimum is at a larger peak elsewhere. Our allocation algorithm actually solves the following general online problem: let \( f : \mathbb{N} \to \mathbb{R} \) be any “sub-linear” function of natural numbers. Consider an online algorithm that is trying to choose \( n \) to maximize \( f(n) \). The algorithm starts at \( n = 0 \), and at each time step, can choose to stay at \( n \), or go to \( n + 1 \). The competitive ratio of the algorithm is

\[
\frac{f(n)}{\max_{1 \leq i \leq m}\{f(i)\}}
\]

where \( m \) is the total number of time steps. We give an algorithm that achieves a competitive ratio of \( 1/2 \) for this problem.

The simplicity of our algorithm is quite appealing. Whenever the algorithm is at a peak, it has to decide if it has to stay at the peak, or try to get to the next one. What the algorithm does is to simply wait at the current peak for a period of time chosen uniformly at random between \( 1 \) and the maximum distance between peaks seen so far.

The proof of the competitive ratio relies on case analysis since the optimal revenue and the expected revenue of the algorithm vary depending on the total number of copies seen. A good idea of how the analysis goes can be had by considering the following instance: suppose there is one bid of 1 and many bids of \( \epsilon \ll 1 \). In this case the algorithm waits for a time chosen u.a.r between 1 and \( 1/\epsilon \). If the number of copies seen is \( m \leq 1/\epsilon \), then the optimal revenue is 1, while the expected revenue is \( 1 - x + \frac{x^2}{2} \) (where \( x = \epsilon m \)), which is at least \( 1/2 \) when \( x \leq 1 \). If \( m \geq 1/\epsilon \)

\[1\] what we mean is \( f(n)/n \) is decreasing. A function of the form \( f(n) = an^a \) satisfies this condition if and only if \( a \) is at most 1.
then the optimal revenue is $\epsilon m$, while the expected revenue is $\epsilon m - 1/2 \geq \frac{\epsilon m}{2}$.

The reduction in [11] gives a constant competitive ratio for the auction problem. The auction is based on random sampling with computing optimal “price offers”. But when run in an online setting, the prices offered decrease over time, due to which a bidder might regret not getting a copy earlier as the price decreased at a later time. The authors of [11] take care of this situation by a clever implementation that works only when all bidders want only one copy. It is not truthful when the bidders can submit multiple bids. We circumvent this difficulty by combining the random sampling technique with the VCG auction. However, we only get an asymptotic competitive ratio, that is the ratio tends to 1, as a certain bidder dominance parameter tends to 0.

The problems we consider were first studied by Mahdian and Saberi [11]. Other variants have been considered, for instance when the supply is given while the bidders arrive and leave online [10, 3]. Also there is a huge work on digital goods and random sampling auctions. Another interesting case is when the bidders have constant marginal utilities for the copies, but have daily budgets. [4, 1] gave an auction for this case with known supply. Extending it to the online setting is an important open problem.

2 An Algorithm for the Online Multi-unit Allocation Problem

Without loss of generality assuming that the bids are $u_1 \geq u_2 \geq \cdots \geq u_n$, the revenue obtained by allocating $l$ units of the item is $l u_l$. Let $1 = a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < \cdots$ be the critical points of the function $lu_l$, that is, the function $lu_l$ is non-decreasing as $l$ increases from $a_i$ to $b_i$, for all $b_i < l < a_{i+1}$ we have $b_i u_{b_i} > l u_l$ and $b_i u_{b_i} \leq a_{i+1} u_{a_{i+1}}$.

The algorithm is in one of two states, ALLOCATE or WAIT. When it is in ALLOCATE, it allocates the next copy of the item. When it is in WAIT, it discards the next copy. The description of the algorithm is completed by specifying when it transits from one state to the other.

The algorithm is initially in ALLOCATE. It transits from ALLOCATE to WAIT when the number of copies allocated ($X$) is equal to $b_i$ for some $i$. It transits from WAIT to ALLOCATE when the number of copies discarded ($Y$) is equal to a random variable, $T$, for waiting time. $T$ is picked so that it is distributed uniformly between 0 and $D_i$, where

$$D_i = \max_{j \leq i} (a_{j+1} - b_j)$$

(recall that $X = b_i$). We further want to maintain the invariant that $Y$ never exceeds $T$. Equivalently, the value of $T$ can only increase during a run of the algorithm.

We still have to specify how $T$ is picked. Because of the condition that $T$ can only increase, we cannot pick $T$ independently every time we transit to WAIT. Note that if $D_i = D_{i-1}$, then we don’t have to change $T$ at all. If $D_i > D_{i-1}$, then

- w.p. $\frac{D_{i-1}}{D_i}$ don’t change $T$,
- with the remaining probability pick $T$ uniformly at random from the interval $[D_{i-1}, D_i]$. 

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It is easy to see that the resulting $T$ is distributed uniformly in $[0, D_i]$. Note that in case $T$ is not changed, then $Y$ is already equal to $T$, and we transit back to ALLOCATE immediately. Equivalently, we don’t transit to WAIT at all.

**Pseudocode for the Algorithm**

1. initialize STATE = ALLOCATE, i=1, X=Y=T=0;
2. when a new copy is produced
3. If (STATE = ALLOCATE)
4. Allocate the copy to the next bidder;
5. X ++;
6. If (X = $b_i$)
7. If ($D_i > D_{i-1}$)
8. With prob $1 - \frac{D_{i-1}}{D_i}$
9. set $T$ to a random number from the interval $[D_{i-1}, D_i]$;
10. STATE = WAIT;
11. i ++;
12. If (STATE = WAIT)
13. Discard the copy;
14. Y ++;
15. If (Y = $T$)
16. STATE = ALLOCATE
17. GO TO line 2.

3. **Competitive Analysis**

In this section we show that the expected revenue of our algorithm, $ALG$, is at least half of the optimal revenue on hindsight, $OPT$. Let $M$ be the number of copies that is produced at the end of the day. Let $b_i < M \leq b_i+1$.

**Case 1:** If $M \leq a_i+1$, then $OPT$ is $b_iu_{b_i}$.

**Case 2:** If $a_i+1 < M$, then $OPT$ is $Mu_M$.

Recall that $X$ is the number of items sold by the algorithm. Therefore, $ALG = E[Xu_X]$. We approximate $ALG$ in the two cases as follows.

In Case 1, if $X \leq b_i$ then $u_X \geq u_{b_i}$, and if $X > b_i$ then $u_X \geq u_{a_i+1}$.

Hence we have that

$$ALG \geq \Pr[X < b_i]E[X|X < b_i]u_{b_i} + \Pr[X = b_i]b_iu_{b_i} + \Pr[X > b_i]E[X|X > b_i]u_{a_i+1}$$

Also note that by definition $a_i+1u_{a_i+1} \geq b_iu_{b_i} = OPT$. Hence

$$u_{a_i+1} \geq \frac{b_iu_{b_i}}{a_i+1} = OPT \cdot \frac{a_i+1}{a_i+1}.$$
Substituting for the values of \( u_{b_i} \) and \( u_{a_i+1} \), we get

\[
\frac{ALG}{OPT} \geq \Pr[X < b_i]E[X|X < b_i] \frac{1}{b_i} + \Pr[X = b_i] + \Pr[X > b_i]E[X|X > b_i] \frac{1}{a_i+1}.
\]  

(1)

In Case 2, we use the fact that \( u_X \geq u_M \), so \( ALG \geq E[X]u_M \), and since \( OPT = Mu_M \), we need to prove that \( E[X] \) is at least \( M/2 \).

\[
E[X] = \Pr[X < b_i]E[X|X < b_i] + \Pr[X = b_i]b_i + \Pr[X > b_i]E[X|X > b_i].
\]  

(2)

We now give a way to calculate the various probabilities and expectations needed.

**Definition 3.1.** For all \( i \geq 1 \) let \( T_i \) be the value of the random variable \( T \) chosen at phase \( i \).

\( T_i \) is the number of number of items we plan to discard before allocating the \((b_i + 1)\)-th element. Also note that \( T_i \) is distributed uniformly between 0 and \( D_i \), and \( T_{i-1} \leq T_i \) for all \( i \). Also, it is easy to see from the description of the algorithm that \( X \geq M - T_i \) when \( X < b_i \), and \( X = M - T_i \) when \( X > b_i \). This gives us the following lemmas. Let \( M' := M - b_i \).

**Lemma 3.2.** For all \( i \geq 1 \) the following statements hold, (let \( T_0 = 0 \)),

1. \( X < b_i \Leftrightarrow T_{i-1} > M' \).
2. \( X = b_i \Leftrightarrow T_{i-1} \leq M' \) and \( T_i \geq M' \).
3. \( X > b_i \Leftrightarrow (T_{i-1} < M' \) and \( T_i < M') \Leftrightarrow T_i < M' \).

**Lemma 3.3.**

\[
E[X|X < b_i] \geq M - E[T_{i-1}|T_{i-1} > M'] = M - \frac{M' + D_{i-1}}{2}.
\]

\[
E[X|X > b_i] = M - E[T_i|T_i < M'] = M - \frac{\min\{M', D_i\}}{2}.
\]

However, the probability of the events \( X < b_i \), \( X = b_i \) and \( X > b_i \) depend upon the order of \( J_i := b_i + D_{i-1}, a_{i+1} \) and \( M \). So we consider all possible orders of these 3 quantities separately. Table 1 shows the probabilities for all the cases.

### 3.1 Analyzing all the Cases

A few observations first: from Lemma 3.3

\[
E[X|X < b_i] = M - \frac{M' + D_{i-1}}{2} \geq \frac{M}{2}
\]

since \( D_{i-1} \leq b_i = M - M' \) implies \( M' + D_{i-1} \leq M \). And when \( M' \leq D_{i} \),

\[
E[X|X > b_i] = M - \frac{M}{2} = \frac{M + b_i}{2}.
\]
| Case 1a | $J_i < M \leq a_{i+1}$ | $D_{i-1} \leq M' \leq D_i$ | 0 | $1 - \frac{M'}{D_{i-1}}$ | $\frac{M'}{D_i}$ |
|---------|---------------------|---------------------|---------------|---------------------|---------------------|
| Case 1b | $M \leq J_i < a_{i+1}$ | $M' \leq D_{i-1} \leq D_i$ | $1 - \frac{M'}{D_{i-1}}$ | $\frac{M'}{D_{i-1}} - \frac{M'}{D_i}$ | $\frac{M'}{D_i}$ |
| Case 1c | $M \leq a_{i+1} \leq J_i$ | $M' \leq D_{i-1} = D_i$ | $1 - \frac{M'}{D_{i-1}}$ | 0 | $\frac{M'}{D_{i-1}}$ |
| Case 2a | $J_i < a_{i+1} \leq M$ | $D_{i-1} < D_i \leq M'$ | 0 | 0 | 1 |
| Case 2b | $a_{i+1} \leq M \leq J_i$ | $M' \leq D_{i-1} = D_i$ | $1 - \frac{M'}{D_{i-1}}$ | 0 | $\frac{M'}{D_{i-1}}$ |
| Case 2c | $a_{i+1} \leq J_i < M$ | $D_{i-1} = D_i \leq M'$ | 0 | 0 | 1 |

Table 1: Probability of the event $X < b_i$, $X = b_i$ and $X > b_i$ for the six different cases

**Case 1a:** $[J_i \leq M \leq a_{i+1}]$ Set $x = \frac{M'}{D_i}$ and $y = \frac{D_i}{a_{i+1}}$. Then $\frac{M'}{a_{i+1}} = xy$ and $\frac{b}{a_{i+1}} = 1 - y$. Substituting for the probabilities and expectations in (1),

$$\frac{ALG}{OPT} \geq 1 - x + x \left(1 + \frac{xy}{2} - y\right) = 1 + \frac{x^2y}{2} - xy =: \alpha.$$ 

Therefore $\alpha$ is minimized when $x = 1$, and at this point, $\alpha = 1 - y/2 \geq 1/2$ since $y \leq 1$.

**Case 1b:** $[b_i \leq M \leq J_i \leq a_{i+1}]$ As observed earlier, we have that $E[X|X < b_i] \geq \frac{M}{2} \geq \frac{b}{2}$ and $E[X|X > b_i] = \frac{M+b}{2} \geq b_i$. Setting $x = \frac{M'}{D_{i-1}}$ and using (1) again,

$$\frac{ALG}{OPT} \geq (1 - x) \frac{1}{2} + x \left(1 - \frac{D_{i-1}}{D_i} + \frac{D_{i-1}}{D_i} \frac{b_i}{a_{i+1}}\right).$$
It is enough to prove that \( \frac{D_{i+1}}{D_i} \left( 1 - \frac{b_i}{a_{i+1}} \right) \leq \frac{1}{2} \). This follows from the fact that \( a_{i+1} = b_i + D_i \geq 2D_{i-1} \).

**Case 1c:** \( b_i \leq M \leq a_{i+1} \leq J_i \) As before we have that \( E[X|X < b_i] \geq \frac{M}{2} \geq \frac{b_i}{2} \) and \( E[X|X > b_i] = \frac{M+b_i}{2} \geq a_{i+1} \). The last inequality follows because \( a_{i+1} \leq J_i = b_i + D_{i-1} \leq b_i + M \). Plugging these back in (1) gives \( \frac{\text{ALG}}{\text{OPT}} \geq 1/2 \).

**Case 2a:** \( J_i \leq a_{i+1} \leq M \) From (2) and Lemma 3.3 \( E[X] = M - \frac{D_i}{2} \geq M/2 \) since \( M \geq D_i \).

**Case 2b:** \( a_{i+1} \leq M \leq J_i \) In this case, it is enough to show that both \( E[X|X < b_i] \) and \( E[X|X > b_i] \) are bigger than \( M/2 \). From Lemma 3.3 \( E[X|X < b_i] \geq \frac{M}{2} \). \( E[X|X > b_i] = M - \frac{D_i}{2} \geq \frac{M}{2} \).

**Case 2c:** \( a_{i+1} \leq J_i \leq M \) The analysis is identical to Case 2a.

In fact, the competitive ratio of our algorithm is \( 1 - \epsilon \) if \( \epsilon \geq \max \left\{ \frac{D_{i-1}}{b_i}, \frac{D_i}{a_{i+1}} \right\} \). The proof is essentially the same as above.

### 4 Designing Truthful Mechanism

Let \( B = \{1, 2, \ldots, n\} \) be the set of bidders. Each bidder can make multiple bids. We will design a truthful mechanism which has good competitive ratio. Our mechanism will use an online multi-unit allocation algorithm as a sub-routine. Under a bidder-dominance assumption, the competitive ratio of our mechanism will be \( (1-\epsilon)\alpha \) where \( \alpha \) is the competitive ratio of the allocation algorithm we use as our subroutine.

**The Mechanism:** We divide the set of bidders into two groups \( S \) and \( T \) by placing each bidder randomly into either of the groups. On each set of bidders \( S \) and \( T \) we will have fictitious runs of the allocation algorithm. Let the fictitious run of the allocation algorithm on the set \( S \) (respectively \( T \)) allocates \( x(S,k) \) (respectively \( x(T,k) \)) copies when \( k \) copies are produced.

Now when the \( j \)-th copy is produced, if \( j \) is even we compute \( x(S,j/2) \). If at that time the number of copies allocated to bidders in \( T \) is less than \( x(S,j/2)(1-6\gamma) \) then we allocate the \( j \)-th copy to \( T \) otherwise discard the copy. Similarly, if \( j \) is odd we compute \( x(T,(j+1)/2) \) and if the number of copies allocated to bidders in \( S \) is less than \( x(T,(j+1)/2)(1-6\gamma) \) then we allocate the \( j \)-th copy to \( S \) otherwise discard the copy.

Finally let \( x_{\text{final}}(S) \) and \( x_{\text{final}}(T) \) copies are allocated to bidders in \( S \) and \( T \) respectively. The prices charged are the VCG payments, that is, as if we ran a VCG auction to sell \( x_{\text{final}}(S) \) copies to bidders in \( S \).

Note that the even indexed copies will be allocated only to bidders in \( T \) and the odd-indexed copies will be allocated only to bidders in \( S \). But the bids of bidders in \( S \) decides how many (odd-indexed) copies will be allocated to bidders in \( T \) and vice versa. This mechanism is similar to that in [8] on digital good auction with unlimited supplies except that in [8] the bids of bidders in \( S \) decides the cut off price for bidders in \( T \) and vice-versa.
If $M$ is the number of copies of the item that are finally produced we denote by $OPT = OPT(B, M)$ the revenue obtained by the optimal single price allocation algorithm.

**Definition 4.1.** For any price $p$ and any bidder $i$ we denote by $n(i, p)$ the number of bids of bidder $i$ that are more than $p$.

We define the bidder dominance parameter $\eta$ as

$$\eta = \max_{i, p} n(i, p)p.$$  

**Theorem 4.2.** The above mechanism is a truthful mechanism. If all the bids are from a finite set of prices (say $Q$) and if

$$\frac{1}{\eta} = \Omega \left( \log \left( \frac{|Q|}{\delta} \right) \left( \frac{1}{\epsilon^2} \right) \right)$$

and $\gamma = \frac{\epsilon}{8}$ then with probability more than $(1 - \delta)$ our mechanism guarantees a revenue of at least $\alpha OPT(1 - \epsilon)$ on expectation, where $\alpha$ is the competitive ratio of the allocation algorithm that we use as the subroutine.

In the rest of this section we will give a sketch of the proof of the theorem. The detailed proof of the theorem is in the Appendix. The proof is similar to that in [8].

The proof that the mechanism is truthful follows from the facts that the number of copies allocated to each half is independent of the number of bids of the bidders in that half and the fact that pricing is determined by the VCG auction.

The proof of the competitive ratio is in two stages. The first thing to notice is that since the bidders are split randomly into two sets the optimal revenue we can obtain from either of the sets is on expectation nearly half of what we can obtain from the whole set.

The second thing is that the discounting factor of $(1 - 6\gamma)$ ensures that w.h.p the eventual winners in $S$ (respectively $T$) are charged at least as much as our allocation algorithm charges during its fictitious run on the set $T$ (respectively $S$).

Note that the bound on the bidder dominance gives us an upper bound on $n(i, p)$ that is the number of bids on any bidders that is more than $p$. This is essential for our analysis.

Since the bidders are split randomly into two sets, the optimal revenue we can obtain from either of the sets is on expectation nearly half that we can obtain from the whole set. Let $OPT(S, j)$ denote the revenue generated after $j$ copies are produced by a fictitious run of the optimal single price allocation algorithm on $S$. By McDiarmid’s Inequality and the bound on the bidder dominance, with probability at least $(1 - O(\delta))$ we have $OPT(S, \lceil M/2 \rceil) > (1/2 - \gamma)OPT$, where $M$ is the final number of copies produced. Similarly we have $OPT(T, \lfloor M/2 \rfloor) > (1/2 - \gamma)OPT$.

For the second stage we again notice that since the set of bidders was partitioned randomly the number of bids more than $p$ is w.h.p divided evenly among the two sets $S$ and $T$. Again from the McDiarmid’s Inequality and from the bound on the bidder dominance ratio we have that the number of bid in $S$ that are more than $p$ is w.h.p much more than $(1 - 6\gamma)$ times the number of bids in $T$ that are more than $p$ (and vice versa).
Let $ALG(S, \lceil M/2 \rceil)$ and $ALG(T, \lfloor M/2 \rfloor)$ be the revenue generated by the fictitious run of our allocation algorithm on $S$ and $T$ respectively. Now since the allocation algorithm is $\alpha$ competitive we have that on expectation $ALG(S,j) > \alpha OPT(S,j)$. From this it follows that with probability at least $1 - O(\delta)$ the revenue we earned on expectation is more than

$$ALG(S, \lceil M/2 \rceil)(1 - 6\gamma) + ALG(T, \lfloor M/2 \rfloor)(1 - 6\gamma) > \alpha(1 - 6\gamma)(1 - 2\gamma)OPT > \alpha(1 - 8\gamma)OPT$$

5 Conclusion and Open Problems

The optimal competitive ratio for the allocation problem is open. [11] showed an upper bound of $e/(e+1)$ for any randomized algorithm. The instance for which they show this upper bound is when there is one bid of 1 and many bids of $\varepsilon$. For this particular instance, the following algorithm gets a competitive ratio of $2/3$: with probability $1/3$, allocate just one copy and get a revenue of 1, and with probability $2/3$, run our algorithm. We conjecture that this algorithm can be generalized to get a $2/3$ competitive ratio. Also, a better upper bound proof will probably have to consider instances with multiple peaks, where the ratio of the $D_i$’s to the $a_{i+1}$’s is large.

For the auction problem, the competitive ratio for the unit-demand case is quite small, and that for the multiple demand case holds only asymptotically. Getting it to a reasonably large constant (or proving that it is impossible) is an important open problem.

The most common scenario in sponsored search auctions is that the bidders have a constant utility for multiple copies of the item, but with a daily budget. Our allocation algorithm works for this case as well, but the reduction from the auction problem is not truthful. Borgs et al [4] give a truthful auction for the offline case with budgets, using the standard random sampling techniques with price offers. However, it is not clear how to extend their auction to the online case. The difficulty is the same as that for the multiple-demand case, that the price offers are decreasing over time. But unlike the multiple-demand case, there is no VCG auction for the budgets case, so our reduction does not work.

6 Acknowledgements

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References

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Appendix

A  Proof of Theorem 4.2

We will now give the detailed proof of Theorem 4.2. Let Allocate be the allocation algorithm that we use as our subroutine. The algorithm decides to allocate \( x(S, j) \) copies at price \( p(S, j) \) when \( j \) copies are produced. So the Allocate\((S, j)\) generate revenue \( ALG(S, j) = x(S, j)p(S, j) \).

Let the optimal single price allocation algorithm decides to allocate \( x^*(S, j) \) copies at price \( p^*(S, j) \) and the optimal revenue is \( OPT(S, i) = x^*(p, i)p^*(S, i) \). Let the Allocate\((S, j)\) have a competitive ratio of \( \alpha \). That is for all \( S \) and \( j \),

\[
ALG(S, j) \geq \alpha OPT(S, j)
\]

If \( M \) is the number of copies of the item that are finally produced we denote the optimal revenue as \( OPT = OPT(B, M) \).

Definition A.1. For any price \( p \) let \( n(S, p) \), \( n(T, p) \) and \( n(B, p) \) be the set of bids more that \( p \) that are made by bidders in \( S \), \( T \) and \( B \) respectively.
Let $Y_i$ be the indicator variable indicating whether the bidder $i$ is in $S$ or not. Let $f_p(Y_1, \ldots, Y_n)$ calculate the number of bids more than or equal to $p$ that are in $S$, that is $f_p(Y_1, \ldots, Y_n) = n(S, p)$. Note that since the bidders are randomly placed in $S$ or $T$ we have

$$E[f_p(Y_1, Y_2, \ldots, Y_n)] = \frac{n(B, p)}{2}$$

Let $c_i$ is the maximum change in the value of $f_p$ if we change the value of $Y_i$. Note that $c_i$ is equal to the number of bids of bidder $i$ that are more than $p$, that is $c_i = n(i, p)$. But from our assumption we have

$$1 > \frac{\eta}{\eta} < \frac{OPT}{n(i, p)p}$$

So $n(i, p) < \eta OPT/ p$. Hence

$$\sum c_i^2 < \eta OPT \frac{\sum c_i}{p} = \eta OPT \frac{n(B, p)}{p}$$

By McDiarmid’s Inequality we have for a fixed $p$

$$\Pr \left[ \left| \frac{n(B, p)}{2} - n(S, p) \right| > \gamma n(B, p) \right] < \exp \left( -\frac{2\gamma^2 n(B, p)^2}{\sum c_i^2} \right) < \exp \left( -\frac{2p\gamma^2 n(B, p)}{\eta OPT} \right) \quad (3)$$

**Lemma A.2.** With probability at least $(1 - 2|Q| \exp(-2\gamma^2/\eta))$

$$OPT(S, \lceil M/2 \rceil) + OPT(T, \lfloor M/2 \rfloor) > (1 - 2\gamma) OPT$$

**Proof.** Note that $n^*(B, p)p > OPT$. From Equation 3 we have that for a fixed $p$ if $p = p^*(B, M)$ then

$$\Pr \left[ \left| \frac{n(B, p^*(B, M))}{2} - n(S, p^*(B, M)) \right| > \gamma n(B, p^*(B, M)) \right] < \exp \left( -\frac{2\gamma^2}{\eta} \right)$$

Now since $p$ takes values from the set $Q$ so by union bound we have for any $p = p^*(B, M)$ with probability at least $(1 - |Q| \exp(-2\gamma^2/\eta))$ we have

$$\left| \frac{n(B, p^*(B, M))}{2} - n(S, p^*(B, M)) \right| > \gamma n(B, p^*(B, M))$$

That is, with probability at least $(1 - |Q| \exp(-2\gamma^2/\eta))$ there are more than $(1/2 - \gamma)n(B, p^*(B, M))$ bids in $S$ are more than $p^*(B, M)$, for any $p^*(B, M)$.

Recall that $x^*(S, \lceil M/2 \rceil)$ is the optimal allocation to bidders in $S$ when $\lceil M/2 \rceil$. So we have

$$OPT(S, \lceil M/2 \rceil) > \left( \frac{1}{2} - \gamma \right) OPT$$

Similarly with probability at least $(1 - |Q| \exp(-2\gamma^2/\eta))$ we have

$$OPT(T, \lfloor M/2 \rfloor) > \left( \frac{1}{2} - \gamma \right) OPT$$
**Corollary A.3.** With probability at least \((1 - 2|Q|\exp(-2\gamma^2 / \eta))\) we have the following two inequalities

\[
x(S, [M/2]) > \alpha \left( \frac{1}{2} - \gamma \right) \frac{OPT}{p(S, [M/2])}
\]

\[
x(T, [M/2]) > \alpha \left( \frac{1}{2} - \gamma \right) \frac{OPT}{p(S, [M/2])}
\]

From Equation 3 and Corollary A.3 we see that for any fixed \(p\) if \(p = p(S, [M/2])\) then

\[
\Pr \left[ \left| \frac{n(B, p)}{2} - n(S, p) \right| > \gamma n(B, p) \right] < \exp \left( \frac{-2\gamma^2 p(n(B, p))}{\eta OPT} \right) < \exp \left( \frac{-2\gamma^2 (1/2 - \gamma)}{\eta} \right)
\]

Using union bound we obtain that with probability \((1 - 2|Q|\exp(-2\gamma^2 (1/2 - \gamma) / \eta) - 2|Q|\exp(-2\gamma^2 / \eta))\) we have for all \(p = p(S, [M/2])\)

\[
\left| \frac{n(B, p)}{2} - n(S, p) \right| > \gamma n(B, p)
\]

and for all \(p = p(T, [M/2])\) we have

\[
\left| \frac{n(B, p)}{2} - n(T, p) \right| > \gamma n(B, p)
\]

For \(p = p(S, [M/2])\) we have

\[
n(i, p) < \eta OPT / p < \eta x(S, [M/2]) \frac{x(S, [M/2])}{(1/2 - \gamma)\alpha} < \gamma x(S, [M/2])
\]

(4)

The algorithm decides to allocate \(x(S, [M/2])\) copies to bidders in \(S\) at price \(p(S, [M/2])\). So by our mechanism we allocate \((1 - 6\gamma)x(S, [M/2])\) copies to bidders in \(T\). By the above inequalities we know that with probability at least \((1 - 4|Q|\exp(-2\gamma^2 (1/2 - \gamma) / \eta))\)

\[
n(T, p(S, [M/2])) > (1 - 6\gamma)n(S, p(S, [M/2])) + 2\gamma n(S, p(S, [M/2]))
\]

Thus there are at least \(2\gamma n(S, p(S, [M/2]))\) losing bid in \(T\) that bids more than \(p(S, [M/2])\). From Equation 4 we see that no bidder has more than \(\gamma n(S, p(S, [M/2]))\) bids above \(p(S, [M/2])\). So by the VCG auction pricing system each winner in \(T\) pays at least \(p(S, [M/2])\) per copy. So the revenue we get from \(T\) is at least

\[
p(S, i)(1 - 6\gamma)x(S, [M/2]) = ALG(S, [M/2])(1 - 6\gamma)
\]

Similarly the revenue we get from \(S\) is at least

\[
ALG(T, [M/2])(1 - 6\gamma)
\]

So with probability \((1 - 4|Q|\exp(-2\gamma^2 (1/2 - \gamma) / \eta))\) our revenue earned is at least

\[
ALG(S, [M/2])(1 - 6\gamma) + ALG(T, [M/2])(1 - 6\gamma) > \alpha(1 - 6\gamma)(1 - 2\gamma)OPT > \alpha(1 - 8\gamma)OPT
\]

So if \(\gamma = \epsilon / 8\) and \((2\gamma^2 (1/2 - \gamma) / \eta)) > \log(4|Q| / \delta)\) then with probability \((1 - \delta)\) the total revenue earned on expectation is at least \(\alpha(1 - \epsilon)OPT\).