On the $\mathcal{N} = 4$, $d = 4$ pure spinor measure factor

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Abstract

In this work, we obtain a simple measure factor for the $\lambda$ and $\theta$ zero-mode integrations in the pure-spinor formalism in the context of an $\mathcal{N} = 4$, $d = 4$ theory. We show that the measure can be defined unambiguously up to BRST-trivial terms and an overall factor, and is much simpler than (although equivalent to) the expression obtained by dimensional reduction from the $\mathcal{N} = 1$, $d = 10$ measure factor. We also give an explicit example of how to obtain the dual to a vertex operator using this measure.

1 Introduction

The prescription for computing tree-level open superstring scattering amplitudes in a manifestly super-Poincaré covariant manner was given by Berkovits some time ago, in the same paper in which the pure spinor superstring was introduced [1]. In this formalism, the unintegrated vertex operators are in the ghost-number 1 cohomology of the BRST operator

$$Q = \frac{1}{2\pi i} \oint dz \lambda^{\hat{\alpha}} d_{\hat{\alpha}}$$  \hspace{1cm} (1.1)

and are functions of the ten-dimensional superspace coordinates $x^\mu$ and $\theta^{\hat{\alpha}}$ for $\mu = 0$ to 9 and $\hat{\alpha} = 1$ to 16. More information on notations and conventions can be found in Appendix A.

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In (1.1), $\lambda^\alpha$ is a pure-spinor ghost variable, i.e. $\lambda^\alpha$ satisfies $\lambda^\gamma \mu \lambda = 0$, and $d_\alpha = p_\alpha - \frac{2}{\alpha'} \left[ \partial x^\mu (\theta^\gamma \mu)\hat{\alpha} + \frac{1}{2} (\theta^\gamma \mu \partial \theta) (\theta^\gamma \mu)\hat{\alpha} \right]$, where $p_\alpha$ is the conjugate momentum to $\theta^\alpha$. The OPE’s

$$p_\alpha(z) \theta^{\hat{\beta}}(w) \sim \frac{\delta^\beta_\alpha}{z - w} \quad \text{and} \quad x^{\mu}(z, \bar{z}) x^{\nu}(w, \bar{w}) \sim -\frac{\alpha'}{2} \eta^{\mu \nu} \log |z - w|^2$$

imply

$$d_\alpha(z) F(x(w), \theta(w)) \sim \frac{1}{z - w} D_\alpha F(x(w), \theta(w)),$$

(1.2)

where $D_\alpha = \frac{\partial}{\partial \theta^\alpha} + (\theta^\gamma \mu)\hat{\alpha} \partial_\mu$ and $F(x, \theta)$ is any superfield. Hence, we write $Q F = \lambda^\alpha D_\alpha F$.

In addition to knowing the vertex operators and OPE’s, one needs to know how to perform the integrations of the $\lambda$ and $\theta$ zero modes. They can be performed by means of the following BRST-invariant measure factor:

$$(\lambda^\gamma \mu \theta^\nu) (\lambda^\gamma \nu \theta^\rho) (\lambda^\gamma \rho \theta^\sigma) (\theta^\mu \nu \rho \sigma \theta).$$

(1.3)

More precisely, the integration is done by keeping only the terms proportional to three $\lambda$’s and five $\theta$’s in this combination.

Thus far, we have written everything in an $N = 1, d = 10$ notation. In order to compute superstring scattering amplitudes in an $N = 4, d = 4$ theory — such as the gauge theory describing the effective world-volume degrees of freedom of a D3-brane, for instance — using the pure-spinor formalism, one needs to know how to perform the integrations of the $\lambda$ and $\theta$ zero modes in that case. In other words, one needs to find a BRST-invariant measure factor analogous to (1.3).

At first, it might seem to be just a matter of dimensional reduction. However, although the particular combination of $\lambda$’s and $\theta$’s of (1.3) is special in ten flat dimensions, since it is the unique (up to an overall factor) $SO(9,1)$ scalar which can be built out of three $\lambda$’s and five $\theta$’s, there is no reason why its dimensional reduction should be preferred over any other BRST-invariant, $SO(3,1) \times SU(4)$ scalar in four dimensions. Therefore, it is important to investigate whether there is any ambiguity in the definition of the $N = 4, d = 4$ measure factor.

In this paper, this issue is studied in detail. In section 2, we write the most general four-dimensional expression with three $\lambda$’s and five $\theta$’s and derive the conditions for it to be BRST invariant. In section 3, we find the independent BRST-trivial combinations of the terms introduced in section 2. In section 4, we present the main results of this paper. We find that the $N = 4, d = 4$ measure factor is unique up to BRST-trivial terms and an overall factor. Moreover, we show that the measure can be written in a much simpler form than the dimensional reduction of (1.3) — the latter has twelve terms, whereas the former has only three. In section 5, we give an example of the use of this measure factor. Finally, section 6 is devoted to our conclusions.
2 BRST equations

In four-dimensional notation, the most general SO(3, 1) \times SU(4)-invariant, real expression one can write with three \( \lambda \)'s and five \( \theta \)'s is

\[
(\lambda^3 \bar{\theta}^\beta) := c_1 \epsilon_{mnj\ell} (\bar{\lambda}_i \bar{\lambda}_k) (\lambda^m \theta^n) (\theta^j \theta^\ell) \\
+ c_2 (\bar{\lambda}_j \bar{\lambda}_k) (\bar{\lambda}_i \bar{\theta}_i) (\theta^i \theta^j) (\theta^k \theta^\ell) \\
+ c_3 \epsilon_{mnj\ell} (\bar{\lambda}_i \bar{\theta}_i) (\lambda^m \theta^n) (\theta^i \theta^j) \\
+ c_4 (\bar{\lambda}_k \bar{\lambda}_j) (\lambda^i \theta^j) (\bar{\theta}_i \bar{\theta}_j) \\
+ c_5 (\bar{\lambda}_i \bar{\theta}_i) (\bar{\lambda}_j \bar{\theta}_j) (\lambda^i \theta^j) (\theta^k \theta^\ell) \\
+ c_6 \epsilon_{mn\ell\ell} (\bar{\lambda}_i \bar{\theta}_i) (\lambda^i \theta^j) (\theta^m \theta^n) (\theta^\ell \theta^\ell) + \text{H.c. ,}
\]  

(2.1)

where \( c_1, \ldots, c_6 \) are arbitrary constants and “H.c.” means “Hermitian conjugate”. We use the standard d = 4 two-component spinor notation as described in Appendix A.1. One can convince oneself these are the only non-zero independent terms which can be constructed, keeping in mind that

\[
\lambda^{\alpha i} \bar{\lambda}^\dagger_i = 0 \quad \text{and} \quad (\lambda^i \lambda^j) = \frac{1}{2} \epsilon^{ijkl} (\bar{\lambda}_k \bar{\lambda}_l) ,
\]  

(2.2)

which are the dimensional reduction of \( \lambda^\gamma \lambda^\mu = 0 \). More details on dimensional reduction can be found in Appendix A.2.

The notation we are going to use throughout the paper is such that

\[
(\lambda^3 \bar{\theta}^\beta) =: \sum_{n=1}^{6} c_n T_n + \text{H.c. ,}
\]  

(2.3)

e. i. we define \( T_1, \ldots, T_6 \) to be the independent possible terms as appearing in (2.1). For example, \( T_3 \equiv \epsilon_{mnj\ell} (\bar{\lambda}_i \bar{\theta}_i) (\lambda^i \theta^j) (\theta^k \theta^\ell) (\lambda^m \theta^n) (\theta^i \theta^j) \). The \( T_n \) and their Hermitian conjugates \( T_n^\dagger \) form a basis for four-dimensional expressions made of three \( \lambda \)'s and five \( \theta \)'s. For example, it is not difficult to show the dimensional reduction of (1.3) gives (2.1) with \( c_1 = 1, c_2 = c_4 = 4, c_3 = 3, c_5 = 12 \) and \( c_6 = 2 \), up to an overall factor.

Since we are looking for the pure spinor measure, we are interested in expressions which are annihilated by \( \lambda^\alpha D_\alpha = \lambda^{op} D_{op} + \bar{\lambda}_{\alpha p} \bar{D}^{\alpha p} \). This requirement yields equations for the constants in (2.1). We begin with

\[
\left[ \lambda^\alpha D_\alpha (\lambda^3 \bar{\theta}^\beta) \right]_{\theta^i \theta^j} = \lambda^{op} D_{op} [c_1 T_1] + \bar{\lambda}_{\alpha p} \bar{D}^{\alpha p} [c_2 T_2 + c_3 T_3] ,
\]  

(2.4)

where the subscript \( \theta^i \theta^j \) means “contributions with four \( \theta \)'s and no \( \bar{\theta} \).” The explicit calculation
The gives:

\[
\lambda^{\alpha p} D_{\alpha p} T_1 = \varepsilon_{mnj}(\tilde{\lambda}_i \tilde{\lambda}_k) \left[ (\lambda^m \lambda^n)(\theta^i \theta^j)(\theta^k \theta^\ell) + (\lambda^m \theta^n)(\theta^i \lambda^j)(\theta^k \theta^\ell) \right] \\
= 4 (\tilde{\lambda}_i \tilde{\lambda}_k)(\tilde{\lambda}_j \tilde{\lambda}_\ell)(\theta^j \theta^\ell)(\theta^k \theta^\ell),
\]

(2.5)

where we used (2.2) and \( \lambda^{[i} \lambda^{j]} = -\frac{1}{2} \varepsilon^{\alpha \beta}(\lambda^i \lambda^j) \). Moreover,

\[
\tilde{\lambda}_{\alpha p} \tilde{D}^{\alpha p} T_2 = (\tilde{\lambda}_j \tilde{\lambda}_k)(\tilde{\lambda}_\ell \tilde{\lambda}_i)(\theta^j \theta^\ell)(\theta^k \theta^\ell)
\]

(2.6)

and \( \tilde{\lambda}_{\alpha p} \tilde{D}^{\alpha p} T_3 = 0 \). Therefore

\[
\left[ \lambda^\Delta D_\alpha (\lambda^3 \theta^5) \right]_{\theta^i \theta^j} = 0 \iff c_2 = 4 c_1 .
\]

(2.7)

Proceeding to the next order, we have

\[
\left[ \lambda^\Delta D_\alpha (\lambda^3 \theta^5) \right]_{\theta^i \theta^j} = \lambda^{\alpha p} D_{\alpha p}[c_2 T_2 + c_3 T_3] + \tilde{\lambda}_{\alpha p} \tilde{D}^{\alpha p}[c_4 T_4 + c_5 T_5 + c_6 T_6].
\]

(2.8)

The \( T_2\)-contribution is easy to compute. We get

\[
\lambda^{\alpha p} D_{\alpha p} T_2 = -(\tilde{\lambda}_j \tilde{\lambda}_k)(\tilde{\lambda}_\ell \tilde{\theta}_i)(\lambda^j \theta^\ell)(\theta^k \theta^\ell).
\]

(2.9)

For \( T_3 \), we obtain

\[
\lambda^{\alpha p} D_{\alpha p} T_3 = -\varepsilon_{mnj}(\tilde{\lambda}_i \tilde{\theta}_k) \left[ (\lambda^k \lambda^\ell)(\lambda^m \theta^n)(\theta^i \theta^j) - (\lambda^k \theta^n)(\lambda^m \lambda^j)(\theta^i \theta^j) \right] \\
= 4 (\tilde{\lambda}_j \tilde{\lambda}_k)(\tilde{\lambda}_\ell \tilde{\theta}_i)(\lambda^j \theta^\ell)(\theta^k \theta^\ell).
\]

(2.10)

The \( T_4\)- and \( T_5\)-contributions are also simple to calculate:

\[
\tilde{\lambda}_{\alpha p} \tilde{D}^{\alpha p} T_4 = -(\tilde{\lambda}_j \tilde{\lambda}_k)(\lambda^j \theta^\ell)(\theta^k \theta^\ell)(\tilde{\lambda}_i \tilde{\theta}_j),
\]

(2.11)

\[
\tilde{\lambda}_{\alpha p} \tilde{D}^{\alpha p} T_5 = -(\tilde{\lambda}_j \tilde{\theta}_k)(\lambda^j \lambda^\ell)(\theta^k \theta^\ell).
\]

(2.12)

Finally, \( \tilde{\lambda}_{\alpha p} \tilde{D}^{\alpha p} T_6 = 0 \). In all, we get our second equation for the coefficients:

\[
\left[ \lambda^\Delta D_\alpha (\lambda^3 \theta^5) \right]_{\theta^i \theta^j} = 0 \iff c_2 + 4 c_3 = c_4 + c_5 .
\]

(2.13)
We now analyze the contributions with equal number of $\theta$'s and $\bar{\theta}$'s:

\[
\lambda^\alpha D_\alpha (\lambda^3 \theta^5) \bigg|_{\theta^2 \bar{\theta}^2} = \lambda^{op} D_{\alpha p} [c_4 T_4 + c_5 T_5 + c_6 T_6] + \bar{\lambda}_{\alpha p} \bar{D}^{\alpha p} [\bar{c}_4 T_4^\dagger + \bar{c}_5 T_5^\dagger + \bar{c}_6 T_6^\dagger].
\] (2.14)

Again, it is straightforward to compute the contributions from $T_4$ and $T_5$:

\[
\lambda^{op} D_{\alpha p} T_4 = - (\bar{\lambda}_i \bar{\lambda}_k) (\lambda^i \theta^k) (\bar{\theta}_i \bar{\theta}_j),
\] (2.15)

\[
\lambda^{op} D_{\alpha p} T_5 = (\bar{\lambda}_i \bar{\lambda}_k) (\bar{\lambda}_j \bar{\theta}_k) (\lambda^k \lambda^\ell) (\theta^i \theta^j). \] (2.16)

The $T_6$-contribution yields

\[
\lambda^{op} D_{\alpha p} T_6 = \varepsilon_{mnkl} (\bar{\theta}_i \bar{\theta}_j) \left[ (\lambda^i \lambda^k) (\lambda^j \theta^k) (\lambda^m \theta^n) - (\lambda^i \theta^k) (\lambda^j \lambda^k) (\lambda^m \theta^n) + (\lambda^i \theta^k) (\lambda^j \lambda^k) (\lambda^m \lambda^n) \right]
= 4 (\bar{\lambda}_i \bar{\lambda}_k) (\lambda^i \theta^k) (\lambda^j \theta^k) (\bar{\theta}_i \bar{\theta}_j).\] (2.17)

These in turn imply, by Hermitian conjugation,

\[
\bar{\lambda}_{\alpha p} \bar{D}^{\alpha p} T_4^\dagger = - (\lambda^i \lambda^k) (\bar{\lambda}_j \bar{\theta}_k) (\bar{\lambda}_i \bar{\theta}_j) (\bar{\theta}^i \bar{\theta}^j),
\] (2.18)

\[
\bar{\lambda}_{\alpha p} \bar{D}^{\alpha p} T_5^\dagger = (\lambda^i \theta^k) (\lambda^j \lambda^k) (\bar{\lambda}_i \bar{\theta}_k) (\bar{\theta}_i \bar{\theta}_j),
\] (2.19)

\[
\bar{\lambda}_{\alpha p} \bar{D}^{\alpha p} T_6^\dagger = 4 (\lambda^i \lambda^k) (\bar{\lambda}_i \bar{\theta}_k) (\bar{\lambda}_j \bar{\theta}_j) (\bar{\theta}^i \bar{\theta}^j).\] (2.20)

Thus we obtain our last equation:

\[
\left[ \lambda^\alpha D_\alpha (\lambda^3 \theta^5) \right]_{\theta^2 \bar{\theta}^2} = 0 \iff \bar{c}_5 = c_4 + 4c_6,
\] (2.21)

as well as its complex conjugate.

Note that the vanishing of the orders $\theta^1 \bar{\theta}^3$ and $\theta^0 \bar{\theta}^4$ implies the complex conjugates of (2.7) and (2.13), since they are just the Hermitian conjugates of the orders $\theta^3 \bar{\theta}^1$ and $\theta^4 \bar{\theta}^0$, respectively.

In summary, we have the following system of equations:

\[
\lambda^\alpha D_\alpha (\lambda^3 \theta^5) = 0 \iff \begin{cases} c_2 = 4c_1 \\ c_2 + 4c_3 = c_4 + c_5 \\ \bar{c}_5 = c_4 + 4c_6 \end{cases},
\] (2.22)

as well as their complex conjugates.
3 BRST-trivial combinations

In the last section, we found the equations which the constants in (2.1) have to satisfy for the expression to be BRST-invariant. Because there are less equations than constants, one might think the $N = 4, d = 4$ pure spinor measure factor is then not unambiguously defined. Fortunately, that is not the case, and the seemingly independent expressions are actually related by BRST-trivial terms, as we show in the following.

In order to find the independent BRST-trivial combinations of the $T_n$, i.e. the combinations which equal $\lambda^\alpha D_{\bar{\alpha}}$ of something, we start by looking for all independent possible terms with two $\lambda$'s and six $\theta$'s. Keeping (2.2) in mind, we find that there are five:

(i) $\lambda^i \theta^j (\theta^k \lambda^\ell) (\bar{\lambda}_k \bar{\theta}_\ell) (\bar{\theta}_i \bar{\theta}_j)$,

(ii) $\varepsilon^{ijk\ell} (\lambda^i \theta^j) (\lambda^m \theta^k) (\theta^\ell \theta^n) (\bar{\theta}_m \bar{\theta}_n)$,

(iii) $\varepsilon^{ijk\ell} (\bar{\lambda}_i \bar{\theta}_j) (\bar{\lambda}_m \bar{\theta}_k) (\bar{\theta}_\ell \theta^n) (\theta^m \theta^n)$,

(iv) $\varepsilon^{mnj\ell} (\bar{\lambda}_i \bar{\theta}_k) (\lambda^m \theta^n) (\theta^i \theta^j) (\theta^k \theta^\ell)$,

(v) $\varepsilon^{mnj\ell} (\lambda^i \theta^k) (\bar{\lambda}_m \bar{\theta}_n) (\bar{\theta}_i \bar{\theta}_j) (\bar{\theta}_k \bar{\theta}_\ell)$.

Acting with $\lambda^\alpha D_{\bar{\alpha}} = \lambda^{op} D_{op} + \bar{\lambda}_{ap} \bar{D}^{ap}$ on these terms, we obtain BRST-trivial expressions made of three $\lambda$'s and five $\theta$'s. We begin with the first one:

$\lambda^{op} D_{op}(i) = \lambda^i \theta^j (\theta^k \lambda^\ell) (\bar{\lambda}_k \bar{\theta}_\ell) (\bar{\theta}_i \bar{\theta}_j)$

$= \frac{1}{2} \left[ T_4^\dagger - T_5^\dagger \right]$, \hspace{1cm} (3.2)

$\bar{\lambda}_{ap} \bar{D}^{ap}(i) = - (\lambda^i \theta^j) (\theta^k \theta^\ell) (\bar{\lambda}_k \bar{\theta}_\ell) (\bar{\theta}_i \bar{\lambda}_j)$

$= \frac{1}{2} \left[ T_5 - T_4 \right]$. \hspace{1cm} (3.3)

Thus we find the first BRST-trivial expression:

$T_4^\dagger - T_5^\dagger + T_5 - T_4 = \lambda^\alpha D_{\bar{\alpha}} [2 (i)]$. \hspace{1cm} (3.4)

Of course, we could multiply the expression on the left-hand side of this equation by any constant and it would remain BRST-trivial. The same applies to the other boxed expressions we find in the following.
For the second term in (3.1), we have

\[ \lambda^{\alpha p} D^{\alpha p}_{\text{op}}(\text{ii}) = \varepsilon_{i j k \ell} (\bar{\theta}_m \bar{\theta}_n) \left[ (\lambda^i \lambda^j) (\lambda^m \theta^n) (\theta^\ell \theta^\ell) - (\lambda^i \theta^\ell) (\lambda^m \lambda^k) (\theta^\ell \theta^\ell) + (\lambda^i \theta^\ell) (\lambda^m \theta^n) (\lambda^k \theta^\ell) - (\lambda^i \theta^\ell) (\lambda^m \theta^n) (\theta^\ell \theta^k) \right] \]

\[ = 4T_4 - T_6, \quad (3.5) \]

\[ \bar{\lambda}^{\dot{\alpha} p} \bar{D}^{\dot{\alpha} p}(\text{ii}) = -\varepsilon_{i j k \ell} (\lambda^i \theta^\ell) (\lambda^m \theta^n) (\theta^\ell \theta^\ell) (\bar{\theta}_m \bar{\lambda}_n) \]

\[ = T_3. \quad (3.6) \]

Therefore,

\[ T_3 + 4T_4 - T_6 = \lambda^{\dot{\alpha}} D_{\dot{\alpha}}(\text{ii}). \quad (3.7) \]

Since the third term in (3.1) is equal to (ii)\textsuperscript{1},

\[ T_3^{\dagger} + 4T_4^{\dagger} - T_6^{\dagger} = \lambda^{\dot{\alpha}} D_{\dot{\alpha}}(\text{iii}). \quad (3.8) \]

For the fourth term,

\[ \lambda^{\alpha p} D^{\alpha p}_{\text{op}}(\text{iv}) = -\varepsilon_{m n j \ell} (\bar{\lambda}_i \bar{\theta}_k) \left[ (\lambda^m \lambda^n) (\theta^i \theta^j) (\theta^k \theta^\ell) + (\lambda^m \theta^n) (\theta^i \lambda^j) (\theta^k \theta^\ell) - (\lambda^m \theta^n) (\theta^i \theta^j) (\lambda^k \theta^\ell) + (\lambda^m \theta^n) (\theta^i \theta^j) (\theta^k \lambda^\ell) \right] \]

\[ = 4T_2 - T_3, \quad (3.9) \]

\[ \bar{\lambda}^{\dot{\alpha} p} \bar{D}^{\dot{\alpha} p}(\text{iv}) = \varepsilon_{m n j \ell} (\bar{\lambda}_i \bar{\lambda}_k) (\lambda^m \theta^n) (\theta^i \theta^j) (\theta^k \theta^\ell) \]

\[ = T_1. \quad (3.10) \]

Therefore,

\[ T_1 + 4T_2 - T_3 = \lambda^{\dot{\alpha}} D_{\dot{\alpha}}(\text{iv}). \quad (3.11) \]

Finally, the last term in (3.1) is equal to (iv)\textsuperscript{†}, so

\[ T_1^{\dagger} + 4T_2^{\dagger} - T_3^{\dagger} = \lambda^{\dot{\alpha}} D_{\dot{\alpha}}(\text{v}). \quad (3.12) \]
We are now in position to show the $\mathcal{N} = 4$, $d = 4$ measure factor is unique up to BRST-trivial terms and an overall factor. Consider once again the most general real expression with three $\lambda$’s and five $\theta$’s of (2.1). One has

$$ (\lambda^3 \theta^5) = c_1 T_1 + c_2 T_2 + c_3 T_3 + c_4 T_4 + c_5 T_5 + c_6 T_6 + \text{H.c.} \quad (4.1) $$

If this is BRST-invariant, then the constants satisfy the equations (2.22) and their complex conjugates. We are free to add BRST-trivial terms to the above expression. If we add

$$ -c_1 [T_1 + 4T_2 - T_3] + \text{H.c.} $$

to $(\lambda^3 \theta^5)$, we get

$$ (\lambda^3 \theta^5) = (c_2 - 4c_1) T_2 + (c_3 + c_1) T_3 + c_4 T_4 + c_5 T_5 + c_6 T_6 + \text{H.c.} \quad (4.2) $$

where we used (2.22). Furthermore, we can add the BRST-trivial term

$$ -\frac{1}{4} (c_4 + c_5) [T_3 + 4T_4 - T_6] + \text{H.c.} $$

to $(\lambda^3 \theta^5)$, to get

$$ (\lambda^3 \theta^5) = -c_5 T_4 + c_5 T_5 + \frac{1}{4} (c_5 + \bar{c}_5) T_6 + \text{H.c.} \quad (4.3) $$

Finally, if $c_5 = \alpha + i\beta$, with $\alpha, \beta \in \mathbb{R}$, then we can add the BRST-trivial term

$$ -i\beta \left[ T_5 - T_5^\dagger - T_4 + T_4^\dagger \right] $$

to $(\lambda^3 \theta^5)$, thus obtaining

$$ (\lambda^3 \theta^5) = -\alpha \left[ T_4 - T_5 - \frac{1}{2} T_6 + \text{H.c.} \right] . \quad (4.4) $$

This shows that the measure is unique up to BRST-trivial terms and an overall factor.\footnote{Note the equal sign here means “equal up to BRST-trivial terms.”}
The measure can be even further simplified, provided that we relax its reality condition. If we add the BRST-trivial terms
\[\alpha \left[ T_4^\dagger - T_5^\dagger + T_5 - T_4 \right] \quad (4.5)\]
and
\[\frac{1}{2} \alpha [T_3 + 4T_4 - T_6], \quad (4.6)\]
we arrive at
\[ (\lambda^3 \theta^5) = \frac{1}{2} \alpha \left[ T_3 + 4T_5 + T_6^\dagger \right], \quad (4.7)\]
or, more explicitly,
\[
(\lambda^3 \theta^5) = \varepsilon_{m n j l} (\bar{\lambda}_i \bar{\theta}_k)(\lambda^k \theta^j)(\lambda^m \theta^n)(\theta^i \theta^l) + 4 (\bar{\lambda}_i \bar{\theta}_k)(\bar{\lambda}_j \bar{\theta}_l)(\lambda^k \theta^j)(\theta^i \theta^l) \\
+ \varepsilon^{m n l k} (\bar{\lambda}_i \bar{\theta}_k)(\bar{\lambda}_j \bar{\theta}_l)(\bar{\lambda}_m \bar{\theta}_n)(\theta^i \theta^j), \quad (4.8)
\]
where we have dropped the overall factor and recovered the explicit form of each term.

This simple $\mathcal{N} = 4$, $d = 4$ pure spinor measure factor is the main result of this work. Note that, while the dimensional reduction of (1.3) yields twelve independent terms, this expression has only three. In the next section, we give an explicit example of how to obtain the dual to a vertex operator using this measure factor.

5 An example

Consider the following vertex operator from [2]:
\[ V = \frac{1}{4} (\lambda \gamma^\mu \theta)(\lambda \gamma^\nu \theta)(\theta \gamma_{\mu \nu} \xi^s), \quad (5.1)\]
where $\xi^s$ is the zero-momentum gluino antifield. It is easy to show this operator is annihilated by $Q$. The dimensional reduction yields
\[
V = 2 (\bar{\lambda}_j \bar{\lambda}_k)(\theta^i \theta^j)(\theta^k \xi^j) + 3 (\bar{\lambda}_i \bar{\theta}_j)(\theta^i \theta^j)(\bar{\lambda}_k \xi^j) - 2 (\bar{\lambda}_j \bar{\theta}_k)(\theta^i \theta^j)(\bar{\lambda}_i \xi^j) \\
- 6 (\bar{\lambda}_i \bar{\theta}_j)(\lambda^j \theta^k)(\theta^i \theta^j) - \varepsilon_{ijkl}(\lambda^i \theta^j)(\lambda^k \theta^j)(\theta^i \xi^k) + 4 (\bar{\lambda}_j \bar{\theta}_k)(\lambda^k \theta^j)(\theta^i \theta^j) \\
- 2 (\bar{\lambda}_i \bar{\theta}_j)(\lambda^k \theta^j)(\theta^i \xi^k) + (\bar{\lambda}_i \bar{\lambda}_j)(\theta^k \theta^j)(\bar{\theta}_k \xi^j) - 3 \varepsilon_{ijkl}(\lambda^m \theta^i)(\lambda^k \theta^j)(\bar{\theta}_m \xi^j) \\
+ 2 \varepsilon_{ijkl}(\lambda^i \theta^j)(\theta^k \theta^m)(\bar{\lambda}_k \xi^k) \\
+ \text{H.c.}, \quad (5.2)
\]
We can simplify this expression by adding BRST-trivial terms. For example,

\[ Q \left[ \varepsilon_{ijkl}(\lambda^i \theta^j)\theta^k \xi^* \right] = -4 (\bar{\lambda}_i \bar{\lambda}_k)(\theta^j \theta^l)(\theta^k \xi^*) - \varepsilon_{ijkl}(\lambda^i \theta^j)(\theta^l \theta^k \xi^*) \]

\[ = -4t_1 - t_5, \] (5.3a)

where \( t_N \) refers to the \( N \)-th term in \( V \) as appearing in (5.2), without the numerical factor (e.g. \( t_1 \equiv (\bar{\lambda}_j \bar{\lambda}_k)(\theta^j \theta^k)(\theta^l \xi^*) \)). So the combination \( 4t_1 + t_5 \) is BRST-trivial. Of course, this implies \( 4t_1^\dagger + t_5^\dagger \) is also BRST-trivial.

One can also show

\[ Q \left[ (\bar{\lambda}_i \bar{\theta}_j)(\theta^j \theta^k)(\theta^l \xi^*) \right] = -t_1 - t_4 - t_6, \] (5.3b)

\[ Q \left[ \bar{\theta}_i \bar{\theta}_j)(\lambda^j \theta^k)(\theta^m \xi^*) \right] = t_4 + t_1^\dagger - t_8^\dagger, \] (5.3c)

\[ Q \left[ \bar{\theta}_i \bar{\theta}_j)(\lambda^j \theta^k)(\theta^l \xi^*) \right] = t_6 + t_7 + t_3^\dagger + t_8^\dagger, \] (5.3d)

\[ Q \left[ \varepsilon_{ijkl}(\lambda^j \theta^k)(\theta^m \xi^*) \right] = -3t_8 - t_9 - t_{10}. \] (5.3e)

as well as their Hermitian conjugates. Then we can add the BRST-trivial amount

\[-2t_1 - 3t_2 + 2t_3 - 9t_4 + t_5 - 4t_6 + 2t_7 - t_8 - 2t_9 - 2t_{10} + \text{H.c.} \]

(5.4)

to \( V \) to obtain a BRST-equivalent vertex operator given by

\[ V' = -15 (\bar{\lambda}_i \bar{\theta}_j)(\theta^j \theta^k)(\theta^l \xi^*) \]

\[ + 5 \varepsilon_{ijkl}(\lambda^j \theta^k)(\theta^m \theta^l)(\theta^i \xi^*) + \text{H.c.} \] (5.5)

Finally, multiplying by an overall factor and dropping the prime, we arrive at the simplest form

\[ V = 3 (\bar{\lambda}_i \bar{\theta}_j)(\theta^j \theta^k)(\theta^i \xi^*) + \varepsilon_{ijkl}(\lambda^j \theta^k)(\theta^m \theta^l)(\theta^i \xi^*) + \text{H.c.} \] (5.6)

Now we may look for the dual to this vertex operator, meaning the BRST-closed expression with one \( \lambda \) and two \( \theta \)'s whose product with \( V \) gives something proportional to the measure (4.8). The dual is certainly going to contain the gluino field \( \xi^\alpha \), and the substitution

\[ \xi^\alpha \xi_{\dot{\alpha}} \rightarrow \delta^\alpha_{\dot{\alpha}} \] (5.7)

can be used to determine it.

Comparing (5.6) with (4.8), we expect the dual to contain a term with \( \varepsilon_{ijkl} \). It is easy to see
there is only one such term:

$$\tilde{V}_1 = \kappa_1 \varepsilon_{ijkl}(\lambda^i \theta^j)(\theta^k \xi^l),$$

where $\kappa_1$ is a constant to be determined. Then, using (5.7), we obtain

$$V \tilde{V}_1 = \kappa_1 [3T_3 + 3T_5].$$

To obtain a term proportional to $T_6^\dagger$, we need a term with $\bar{\lambda}$. There are not many, and it is not difficult to show the following one works:

$$\tilde{V}_2 = \kappa_2 (\bar{\lambda}_i \bar{\theta}_j)(\theta^i \xi^j),$$

for a constant $\kappa_2$ to be determined shortly. Making use of (5.7) once again, we get

$$V \tilde{V}_2 = -\kappa_2 \left[3T_5 + T_6^\dagger\right].$$

Hence, if $\kappa_1 = \frac{1}{3}$ and $\kappa_2 = -1$, we have

$$V \tilde{V} := V(\tilde{V}_1 + \tilde{V}_2) = T_3 + 4T_5 + T_6^\dagger,$$

which means the dual to $V$ is given by

$$\tilde{V} = \frac{1}{3} \varepsilon_{ijkl}(\lambda^i \theta^j)(\theta^k \xi^l) - (\bar{\lambda}_i \bar{\theta}_j)(\theta^i \xi^j).$$

One can show this expression is BRST-invariant, as it should be.

6 Conclusion

In this work, we have obtained a simple measure factor for the $\lambda$ and $\theta$ zero-mode integrations in the pure-spinor formalism in the context of an $\mathcal{N} = 4$, $d = 4$ theory. We have shown that the measure can be defined unambiguously up to BRST-trivial terms and an overall factor, and is much simpler than (although equivalent to) the expression obtained by dimensional reduction from the $\mathcal{N} = 1$, $d = 10$ measure factor. We have also given an explicit example of how to obtain the dual to a vertex operator using this measure.

We expect these results to be useful for the computation of disk scattering amplitudes of states propagating in the world-volume of a D3-brane or open-closed superstring amplitudes of states close to the AdS$_5$ boundary [6].
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A Appendix

A.1 Two-component spinor notation

The four-dimensional Lorentz group SO(3, 1) is locally isomorphic to SL(2, C), which has two distinct fundamental representations. One of them is described by a pair of complex numbers

\[ \psi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{A.1} \]

with transformation law

\[ \psi'_\alpha = \Lambda^\beta_\alpha \psi_\beta, \quad \Lambda \in \text{SL}(2, \mathbb{C}), \tag{A.2} \]

and is called \((\frac{1}{2}, 0)\) or left-handed chiral representation.

The other fundamental representation, called \((0, \frac{1}{2})\) or right-handed chiral, is obtained by complex conjugation:

\[ \bar{\psi}^\dot{\alpha} = \bar{\Lambda}^\dot{\beta}_{\dot{\alpha}} \bar{\psi}_{\dot{\beta}}, \quad \bar{\Lambda}^\dot{\beta}_{\dot{\alpha}} = (\Lambda^\beta_\alpha)^*, \tag{A.3} \]

The dot over the indices indicates the representation to which we refer.

The indices with and without dot are raised and lowered in the following way:

\[ \psi^\alpha = \varepsilon^{\alpha\beta} \psi_\beta, \quad \bar{\chi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\chi}_{\dot{\beta}}; \tag{A.4a} \]

\[ \psi_\alpha = \varepsilon_{\alpha\beta} \psi^\beta, \quad \bar{\chi}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\chi}^{\dot{\beta}}, \tag{A.4b} \]

where \(\varepsilon\) is antisymmetric and has the properties

\[ \varepsilon^{12} = -\varepsilon^{12} = -\varepsilon_{12} = 1 \implies \varepsilon_{\alpha\beta} \varepsilon^{\beta\gamma} = \delta^\gamma_\alpha, \quad \varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon^{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\gamma}}_{\dot{\alpha}}. \tag{A.5} \]

For spinorial derivatives, raising or lowering the indices involve an extra sign. For example, \(D^\alpha_i = -\varepsilon^{\alpha\beta} D_{\beta i}\).

The convention for contraction of spinorial indices is

\[ \psi^\alpha \lambda_\alpha =: (\psi \lambda), \quad \bar{\chi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} =: (\bar{\chi} \bar{\xi}) \tag{A.6} \]
In SL(2, \mathbb{C}) notation, a four-component Dirac spinor is represented by a pair of chiral spinors:

\[ \Psi_D = \left( \begin{array}{c} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{array} \right). \]  
(A.7)

For a Majorana spinor, \( \bar{\chi}^{\dot{\alpha}} = (\psi_\alpha) \). The Dirac matrices are

\[ \Sigma^a = \left( \begin{array}{cc} 0 & (\sigma^a)_{\alpha\dot{\alpha}} \\ (\bar{\sigma}^a)^{\dot{\alpha}\alpha} & 0 \end{array} \right), \]  
(A.8)

where the matrices \( \sigma^a \) \( (a = 0, \ldots, 3) \) are defined as

\[ (\sigma^a)_{\alpha\dot{\alpha}} = (-\mathbb{I}_2, \bar{\sigma})_{\alpha\dot{\alpha}}, \quad (\bar{\sigma}^a)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon_{\alpha\beta} (\sigma^a)_{\beta\dot{\beta}} = (-\mathbb{I}_2, -\bar{\sigma})^{\dot{\alpha}\alpha}, \]  
(A.9)

with \( \mathbb{I}_2 \) the \( 2 \times 2 \) identity matrix and \( \bar{\sigma} \) the Pauli matrices

\[ \sigma^1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma^2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma^3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \]  
(A.10)

and have the following properties:

\[ (\sigma^a)_{\alpha\dot{\alpha}}(\bar{\sigma}_b)^{\dot{\beta}\beta} = -2\delta^\beta_\beta \delta^\alpha_{\dot{\alpha}}, \quad (\sigma_a)_{\alpha\dot{\alpha}}(\bar{\sigma}^b)^{\dot{\alpha}\alpha} = -2\delta^b_b, \]
\[ \sigma^a\bar{\sigma}^b = -\eta^{ab} + \sigma^{ab}, \quad \bar{\sigma}^a\sigma^b = -\eta^{ab} + \bar{\sigma}^{ab}, \quad \sigma^{ab} = -\sigma^{ba}, \quad \bar{\sigma}^{ab} = -\bar{\sigma}^{ba}, \quad (\sigma^{ab})_{\alpha\dot{\alpha}} = (\bar{\sigma}^{ab})^{\dot{\alpha}\alpha} = 0, \]
(A.11)

with \( \eta^{ab} = \text{diag}(-1, 1, 1, 1) \). These properties imply \( \{\Sigma^a, \Sigma^b\} = -2\eta^{ab} \mathbb{I}_4 \).

**A.2 Dimensional reduction**

Since in the text we write expressions both in ten- and four-dimensional notation, it is important to clarify our notation and conventions. Breaking the SO(9,1) Lorentz symmetry to SO(3,1) \( \times \) SO(6) \( \simeq \) SO(3,1) \( \times \) SU(4), an SO(9,1) vector \( v^\mu \) \( (\mu = 0, \ldots, 9) \) decomposes as

\[ v^\mu \rightarrow (v^a, v^{[ij]}), \]
(A.12)

where \( v^a \) \( (a = 0, \ldots, 3) \) transforms under the representation 4 of SO(3,1) and \( v^{[ij]} = -v^{[ji]} \) \( (i, j = 1, \ldots, 4) \) transforms under the 6 of SU(4). The relation between the 6 of SU(4) and the 6 of SO(6) is given by the SO(6) Pauli matrices \( (\rho_I)^{ij} = -(\rho_I)^{ji} \) \( (I = 1, \ldots, 6) \) in the following
way:
\[ v^{ij} = \frac{1}{2i} (\rho^I)^{ij} v^{I+3}. \] (A.13)

These matrices have the properties \[4\]
\[
(\rho^I)^{ij} (\rho^I)^{jk} = 2 \eta^{IJ} \delta^I_k,
(\rho^I)^{ij} = \frac{1}{2} \varepsilon_{ijk} (\rho^I)^{k\ell},
(\rho^I)^{ij} (\rho^J)^{ij} = -2 \varepsilon_{ijk},
\]
where \(\eta^{IJ} = \text{diag}(1,1,1,1,1)\) and \(\varepsilon_{ijk}\) is the SU(4)-invariant, totally antisymmetric tensor such that \(\varepsilon^{1234} = 1\). Analogously, one can define the tensor \(\varepsilon^{ijk}\) such that \(\varepsilon^{1234} = 1\). These satisfy the relation
\[
\varepsilon_{ijk} \varepsilon^{k\ell m} = 4 \delta^m_i \delta^n_j.
\] (A.15)

A left-handed Majorana-Weyl spinor \(\xi^\alpha (\hat{\alpha} = 1, \ldots, 16)\) transforming under the 16 of SO(9, 1) decomposes as
\[
\xi^{\hat{\alpha}} \rightarrow (\xi^{\alpha}, \bar{\xi}^{\dot{\alpha}}),
\] (A.16)
where we use the standard two-component notation for chiral spinors \((\alpha = 1, 2 ; \hat{\alpha} = \hat{1}, \hat{2})\) and \(\xi^{\alpha}\) (resp. \(\xi^{\hat{\alpha}}\)) transforms under the representation 4 (resp. 4) of SU(4). Analogous conventions apply to right-handed Majorana-Weyl spinors of SO(9, 1).

We also need to know how to translate the SO(9, 1) Pauli matrices \((\gamma^\mu)^{\hat{\alpha}\hat{\beta}}\) and \((\gamma^\mu)^{\hat{\alpha}\hat{\beta}}\) to the language of SO(3, 1) × SU(4). Based on [5], we propose the following ansatz for the non-vanishing components:
\[
(\gamma^a)_{(\alpha i)}^{(\alpha)} = \delta^i_\alpha (\sigma^a)_{\alpha\hat{\alpha}} = (\gamma^a)_{(\alpha i)}^{(\hat{\alpha})}
(\gamma^{[k\ell]})_{(\alpha i)}^{(\beta j)} = 2 \varepsilon_{\alpha\beta} \delta_k^{i\ell} \delta^\ell_j
(\gamma^{[k\ell]})_{(\alpha i)}^{(\hat{\beta} \dot{\beta})} = \varepsilon_{\alpha\beta} \delta_k^{i\ell} \delta^\ell_j
\] (A.17)

for \((\gamma^\mu)^{\hat{\alpha}\hat{\beta}}\) and
\[
(\gamma^a)_{(\alpha i)}^{(\alpha)} = \delta^i_\alpha (\bar{\sigma}^a)^{\hat{\alpha}\hat{\beta}} = (\gamma^a)_{(\alpha i)}^{(\hat{\alpha})}
(\gamma^{[k\ell]})_{(\alpha i)}^{(\beta j)} = \varepsilon^{\alpha\beta} \varepsilon^{i\ell j}
(\gamma^{[k\ell]})_{(\alpha i)}^{(\hat{\beta} \dot{\beta})} = 2 \varepsilon^{\hat{\alpha}\hat{\beta}} \delta_k^{i\ell} \delta^\ell_j
\] (A.18)
for \((\gamma^\mu)^{\dot{a}\dot{b}}\). It is straightforward to show that the above matrices satisfy the usual relation

\[
(\gamma^\mu)^{\dot{a}\dot{b}}(\gamma^\nu)^{\dot{\alpha}\dot{\beta}} + (\gamma^\nu)^{\dot{a}\dot{b}}(\gamma^\mu)^{\dot{\alpha}\dot{\beta}} = -2\eta^{\mu\nu}\delta_{\dot{a}}^\dot{\alpha} ,
\]

(A.19)

with \(\eta^{ij|k\ell} := \frac{1}{2} \varepsilon^{ijk\ell}\).

As an example, we show how to obtain the dimensional reduction of the pure spinor constraints \(\lambda \gamma^\mu \lambda = 0\) using (A.17). For \(\lambda \gamma^a \lambda = 0\), we have

\[
\lambda^{\dot{a}} (\gamma^a)^{\dot{a}\dot{b}} \lambda^{\dot{b}} = 0 \iff \lambda^{\dot{a}} (\gamma^a)^{(\dot{a})(\dot{a})} \tilde{\lambda}_j^{\dot{a}} + \tilde{\lambda}_j^{\dot{a}} (\gamma^a)^{\dot{a}(\dot{a})} \lambda^{\dot{a}} = 2\lambda^{\dot{a}} (\sigma^a)_{\dot{a}a} \tilde{\lambda}_j^{\dot{a}} = 0 ,
\]

whence

\[
\lambda^{\dot{a}} \tilde{\lambda}_j^{\dot{a}} = 0 .
\]

(A.20)

For \(\lambda \gamma^{ij} \lambda = 0\), we have

\[
\lambda^{\dot{a}} (\gamma^{ij})^{\dot{a}\dot{a}} \lambda^{\dot{b}} = 0 \iff \lambda^{\dot{a}} (\gamma^{ij})_{(\dot{a})(\dot{a})} \lambda^{\dot{b}} + \tilde{\lambda}_k^{\dot{a}} (\gamma^{ij})^{\dot{a}(\dot{a})} \lambda^{\dot{b}} = 2(\lambda^i \lambda^j) - \varepsilon^{ijkl}(\tilde{\lambda}_k \tilde{\lambda}_l) = 0 ,
\]

whence

\[
(\lambda^i \lambda^j) = \frac{1}{2} \varepsilon^{ijkl}(\tilde{\lambda}_k \tilde{\lambda}_l) .
\]

(A.21)

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