GENERATORS OF SPLIT EXTENSIONS OF ABELIAN GROUPS BY CYCLIC GROUPS

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ABSTRACT. Let $G \simeq M \rtimes C$ be an $n$-generator group which is a split extension of an Abelian group $M$ by a cyclic group $C$. We study the Nielsen equivalence classes and T-systems of generating $n$-tuples of $G$. The subgroup $M$ can be turned into a finitely generated faithful module over a suitable quotient $R$ of the integral group ring of $C$. When $C$ is infinite, we show that the Nielsen equivalence classes of the generating $n$-tuples of $G$ correspond bijectively to the orbits of unimodular rows in $M^{n-1}$ under the action of a subgroup of $GL_{n-1}(R)$. Making no assumption on $C$, we exhibit a complete invariant of Nielsen equivalence in the case $M \simeq R$. As an application, we classify Nielsen equivalence classes and T-systems of soluble Baumslag-Solitar groups, split metacyclic groups and lamplighter groups.

1. Introduction

Given a finitely generated group $G$, denote by $\text{rk}(G)$ the minimal number of its generators. For $n \geq \text{rk}(G)$, let $V_n(G)$ be the set of generating $n$-vectors of $G$, i.e., the set of elements in $G^n$ whose components generate $G$. In order to classify generating vectors, we can rely on a well-studied equivalence relation on $V_n(G)$, namely the Nielsen equivalence relation: two generating $n$-vectors are said to be Nielsen equivalent if they can be related by a finite sequence of transformations taken in the set $\{L_{ij}, I_i; 1 \leq i \neq j \leq n\}$ where $L_{ij}$ and $I_i$ replace the component $g_i$ of $g = (g_1, \ldots, g_n) \in V_n(G)$ by $g_j g_i$ and $g_i^{-1}$ respectively and leave the other components unchanged.

We recommend [Lub11, Eva07, Pak01, LM93] to the reader interested in Nielsen equivalence and its applications. Let $F_n$ be the free group with basis $x = (x_1, \ldots, x_n)$. The Nielsen equivalence relation turns out to be generated
by an \( \text{Aut}(F_n) \)-action. Indeed, the set \( V_n(G) \) identifies with the set \( \text{Epi}(F_n, G) \) of epimorphisms from \( F_n \) onto \( G \) via the bijection \( g \mapsto \pi_g \) with \( \pi_g \) defined by \( \pi_g(x) = g \). Therefore defining \( g\psi \) for \( \psi \in \text{Aut}(F_n) \) through \( \pi_g\psi = \pi_g \circ \psi \) yields a right group action of \( \text{Aut}(F_n) \) on \( V_n(G) \). Because \( \text{Aut}(F_n) \) has a set of generators which induce the elementary Nielsen transformations \( L_{ij} \) and \( I_i \) [LS77, Proposition 4.1], this action generates the Nielsen equivalence relation.

In this article we are concerned with finitely generated groups \( G \) containing an Abelian normal subgroup \( M \) and a cyclic subgroup \( C \) such that \( G = MC \) and \( M \cap C = 1 \). Denoting by \( \sigma \) the natural map \( G \to G/M \simeq C \), such a group \( G \) fits into the split exact sequence

\[
0 \longrightarrow M \longrightarrow G \overset{\sigma}{\longrightarrow} C \longrightarrow 1
\]

where the arrow from \( M \) to \( G \) is the inclusion \( M \subset G \). The cyclic group \( C = \langle a \rangle \) is finite or infinite and is given together with a generator \( a \). The action of \( \text{Aut}(F_n) \) on \( V_n(G) \) is known to be transitive if \( n > \text{rk}(G) + 2 \) [Eva93, Theorem 4.9]. Our goal is to describe the \( \text{Aut}(F_n) \)-orbits for the three exceptional values of \( n \), namely \( \text{rk}(G) \), \( \text{rk}(G) + 1 \) and \( \text{rk}(G) + 2 \). To this end, we relate the problem of classifying Nielsen equivalence classes to a pure module-theoretic problem involving \( M \). The conjugacy action of \( C \) on \( M \) defined by \( c \circ m = cmc^{-1} \), with \( m \in M \) and \( c \in C \) extends linearly to \( \mathbb{Z}[C] \), turning \( M \) into a module over \( \mathbb{Z}[C] \). Let \( \text{ann}(M) \) be the annihilator of \( M \). Then \( M \) is a faithful module over

\[
R = \mathbb{Z}[C]/\text{ann}(M).
\]

Let \( \text{rk}_R(M) \) be the minimal number of generators of \( M \) considered as an \( R \)-module. Denote by \( \text{Um}_n(M) \) the set of elements in \( M^n \) whose components generate \( M \) as an \( R \)-module and set \( \text{Um}_n(M) = \emptyset \) if \( n < \text{rk}_R(M) \). The group \( \text{GL}_n(R) \) acts on \( \text{Um}_n(M) \) by matrix right-multiplication. There are two subgroups of \( \text{GL}_n(R) \) which are relevant to us. The first is \( \text{E}_n(R) \), the subgroup generated by the elementary matrices, i.e., the matrices that differ from the identity by a single off-diagonal element (agreeing that \( E_1(R) = \{1\} \)). The second is \( \text{D}_n(T) \), the subgroup of diagonal matrices with coefficients in \( T \), the subgroup of \( R^n \) made of trivial units. We call a unit in \( R^n \) a trivial unit, if it lies in the image of \( \pm C \) by the natural map \( \mathbb{Z}[C] \to R \). Our first result establishes a connection between the \( \text{Aut}(F_n) \)-orbits of generating \( n \)-vectors and the orbits of unimodular rows in \( M \) with size \( n - 1 \) under the action of

\[
\Gamma_{n-1}(R) = \text{D}_{n-1}(T)\text{E}_{n-1}(R).
\]

Additional definitions are needed to state this result. Denoting by \( |C| \) the cardinal of \( C \), we define the norm element of \( Z[C] \) by 0 if \( C \) is infinite, and by \( 1 + a + \cdots + a^{|C|} \) otherwise. Let \( \nu(G) \) be the image in \( R \) of the norm element of \( Z[C] \) via the natural map. Let \( \pi_{ab} : G \to G_{ab} \) be the abelianization
homomorphism of $G$ and let $M_C$ be the largest quotient of $M$ with a trivial $C$-action. We assume throughout this paper that $n \geq 2$ whenever the integer $n$ refers to the size of generating vectors of $G$, in which case $n \geq \text{rk}(G)$ necessarily holds. Let $\varphi_a : \text{Um}_{n-1}(M) \to V_n(G)$ be defined by $\varphi_a(m) = (m, a)$. This is elementary to check that $\varphi_a$ induces a map

$$\Phi_a : \text{Um}_{n-1}(M)/\Gamma_{n-1}(R) \to V_n(G)/\text{Aut}(F_n)$$

**Theorem A** (Theorems 3 and 4). The map $\Phi_a$ is surjective if and only if at least one of the following holds:

(i) $n > \text{rk}(G_{ab})$.

(ii) $C$ is infinite.

(iii) $\text{rk}(M_C) < \text{rk}(G)$ and $M_C$ is not isomorphic to $\mathbb{Z}^{\text{rk}(G)-1}$.

In particular, $\Phi_a$ is surjective if $\nu(G)$ is nilpotent. It is bijective if $C$ is infinite.

Observe that if $\Phi_a$ is surjective for $n = \text{rk}(G)$, then $\text{rk}_R(M) = \text{rk}(G) - 1$. Consequently Theorem A doesn’t address the situation for which $n = \text{rk}(G)$ and $\text{rk}_R(M) = \text{rk}(G)$ hold simultaneously (actually, none of our results does, except Proposition 4 below). Instead, it characterizes conditions under which the equality $\text{rk}_R(M) = \text{rk}(G) - 1$ prevails.

Combined with various assumptions on $C$, $M$ or $R$ (e.g., $C$ is infinite and $R$ is Euclidean), Theorem A provides a complete description of Nielsen equivalence classes as stated in Corollary 4 below. For the next results, we suppose in particular that $M \simeq R$. Therefore $G \simeq R \rtimes C$ is generated by $a$ and the identity $b$ of the ring $R$. At this stage, a few examples may help understand the kind of groups we want to address. Assume $C$ is the cyclic subgroup of $\text{GL}_2(\mathbb{Z})$ generated by some invertible matrix $a$. Let $b$ be the 2-by-2 identity matrix and let $G$ be the semi-direct product $\mathbb{Z}^2 \rtimes_a C$ where the action of $a$ on $\mathbb{Z}^2$ is the obvious one. It is readily checked that $\text{rk}(G) = 2$ if and only if $M \simeq \mathbb{Z}^2$ is a cyclic $\mathbb{Z}[C]$-module. If this holds, then $M$ naturally identifies with the subring $R = Za + Zb$ of the ring of 2-by-2 matrices over $\mathbb{Z}$ and we can certainly write $G \simeq R \rtimes_a C$. If the minimal polynomial of $a$ is moreover irreducible and if $\alpha \in \mathbb{C}$ is one of its roots, then $G$ identifies in turn with the semi-direct product $G(\alpha) = \mathbb{Z}[\alpha^\pm 1] \rtimes_\alpha \langle \alpha \rangle$ where $\alpha$ acts on $\mathbb{Z}[\alpha^\pm 1] \subset \mathbb{C}$ via complex multiplication. For arbitrary choices of $\alpha \in \mathbb{C}$, the family $G(\alpha)$ provides us with countably many interesting non-isomorphic examples. For instance, if $\alpha$ is transcendent over $\mathbb{Q}$, then $\mathbb{Z}[\alpha^\pm 1]$ is isomorphic to the ring $\mathbb{Z}[X^\pm 1]$ of univariate Laurent polynomials over $\mathbb{Z}$. In this case, the group $G(\alpha)$ is isomorphic to the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$ which is the subject of our last application.

Let us return to the presentation of our results, assuming that $M \simeq R$ so that $G \simeq R \rtimes C$ is a two-generated group. Under this assumption, we exhibit
a complete invariant of Nielsen equivalence for generating pairs. In addition, if \( n = 3 \) and \( C \) is finite, or if \( n = 4 \), we prove that \( \text{Aut}(F_n) \) acts transitively on \( V_n(G) \). Note that if \( n = 3 \) and \( C \) is infinite, then Theorem \( \text{A} \) reduces the study of the Nielsen equivalent triples to a pure ring-theoretic problem.

Our invariant is based on a map \( D \) defined as follows. If \( \nu(G) = 0 \) there is a unique derivation \( d \in \text{Der}(C, R) \) satisfying \( d(a) = 1 \) (see Section \ref{sec:3.4}). For \( g = (rc, r'c') \in G^2 \) with \( (r, r') \in R^2 \) and \( (c, c') \in C^2 \), we set then \( D(g) = r'\delta(c) - r\delta(c') \in R \). If \( \nu(G) \neq 0 \), we set furthermore

\[
(2) \quad D(g) = D(\pi_{\nu(G)}(g)) \in R/\nu(G)R
\]

where \( \pi_{\nu(G)} \) stands for the natural map \( R \times C \rightarrow R/\nu(G)R \times C \) and the right-hand side of \( (2) \) is defined as above. The first of the following observations will enable us to construct the actual invariant.

**Proposition B** (Lemma \ref{lem:13} and Proposition \ref{prop:6}). Let \( G \simeq R \times C \) and let \( g \in G^2 \).

(i) If \( g \in V_2(G) \), then \( D(g) \in (R/\nu(G)R)^\times \).

(ii) Assume \( \nu(G) \) is nilpotent. Then \( g \) generates \( C \) if and only if \( \sigma(g) \) generates \( C \) and \( D(g) \in (R/\nu(G)R)^\times \).

Let \( \Lambda = R/\nu(G)R \), \( T_\Lambda = \pi_{\nu(G)}R(T) \) and let \( \Delta : V_2(G) \rightarrow \Lambda^\times/T_\Lambda \) be defined by \( \Delta(g) = T_\Lambda D(g) \). The map \( \Delta \) is the invariant we needed and we are now in position to describe the Nielsen equivalence classes of generating \( n \)-vectors of \( G \simeq R \times C \) for \( n = 2, 3 \) and \( 4 \). For \( n \geq 2 \), let us denote by \( n_n(G) \) the cardinality of the set of Nielsen equivalence classes of generating \( n \)-vectors of \( G \).

**Theorem C** (Theorems \ref{thm:7} \& \ref{thm:8} and \ref{thm:9}). Let \( G \simeq R \times C \). Then the following hold:

(i) Two generating pairs \( g, g' \) of \( G \) are Nielsen equivalent if and only if \( \pi_{ab}(g) \) and \( \pi_{ab}(g') \) are Nielsen equivalent and \( \Delta(g) = \Delta(g') \).

(ii) If \( C \) is infinite or \( G_{ab} \) is finite, then \( \Delta \) induces a bijection \( V_2(G)/\text{Aut}(F_2) \simeq \Lambda^\times/T_\Lambda \). In particular \( n_2(G) = |\Lambda^\times/T_\Lambda| \).

(iii) If \( \text{SL}_2(R) = E_2(R) \), e.g., \( C \) is finite, then \( n_3(G) = 1 \).

(iv) \( n_4(G) = 1 \).

Assertion (i) of Theorem \( \text{C} \) provides us with an algorithm which decides whether or not two generating pairs of \( G \) are Nielsen equivalent. Indeed, the first condition in (i) can be determined by means of the Diaconis-Graham determinant \[DG99\] while the second condition can be reduced to the ideal membership problem in \( \mathbb{Z}[X^\pm] \) which is solvable \[PU99, Asc04\].

Consider now the left group action of \( \text{Aut}(G) \) on \( V_n(G) \) where we define \( \phi g \) for \( \phi \in \text{Aut}(G) \) by \( \pi_{\phi g} = \phi \circ \pi_{g} \), using the identification of \( V_n(G) \) with \( \text{Epi}(F_n, G) \). This action clearly commutes with the right \( \text{Aut}(F_n) \)-action introduced earlier so that \( (\phi, \psi)g \doteq \phi g \psi^{-1} \) is an action of \( \text{Aut}(G) \times \text{Aut}(F_n) \).
on $V_n(G)$. Following B.H. Neumann and H. Neumann [NN51], we call the orbits of this action the $T$-systems of generating $n$-vectors of $G$, or concisely, the $T_n$-systems of $G$. We denote by $t_n(G)$ the cardinality of the $T_n$-systems of $G$. In order to present estimates of $t_2(G)$ and $t_3(G)$, we introduce two more definitions. Let $A(C)$ be set of the automorphisms of $C$ which are induced by automorphisms of $G$ preserving $M$. Let $A'(C)$ be the subgroup of $\text{Aut}(C)$ generated by $A(C)$ and the involution $c \mapsto c^{-1}$.

**Theorem D** (Theorem 9). Let $G \simeq R \rtimes C$.

(i) The cardinality $t_2(G)$ is finite and we have $t_2(G) \leq |\text{Aut}(C)/A'(C)|$, with equality if $R$ is a characteristic subgroup of $G$. If $C$ is infinite or $G_{ab}$ is finite, then $t_2(G) = 1$.

(ii) If $C$ is infinite and $R$ is characteristic in $G$, then we have $|A(C)| \leq 2$, $n_3(G) \leq |A(C)|t_2(G)t_3(G)$ and $|A(C)|t_3(G) \geq |\text{SK}_1(R)|$, where $\text{SK}_1(R)$ denotes the special Whitehead group of $R$.

Note that assertion (i) of Theorem D generalizes Brunner’s theorem [Bru74, Theorem 2.4] according to which 2-generated Abelian-by-(infinite cyclic) groups have only one $T_2$-system; another kind of generalization is given in Theorem 5 below. The next theorem shows that there is no upper bound for $t_2(G)$ and $t_3(G)$ when $G$ is a 2-generated split extension of an Abelian group by a cyclic group.

**Theorem E** (Corollary 5, Theorem 10 and Example 1). For every integer $N \geq 1$, there exist

(i) a group $G_N$ of the form $R \rtimes C$ with $C$ finite such that $t_2(G_N) \geq N$,

(ii) a two-generated Abelian-by-(infinite cyclic) group $H_N$ such that $n_3(H_N) = t_3(H_N) = N$.

For comparison, Dunwoody constructed for every $N \geq 1$ a two-step nilpotent 2-group on two generators with at least $N$ $T_2$-systems [Dun63]. We were not able to settle the question as to whether $t_3(G)$ is finite for $G$ a general two-generated Abelian-by-(infinite cyclic) group. In contrast to our examples, the free metabelian group on two generators has infinitely many $T_3$-systems but only one Nielsen class of generating pairs; the latter facts can be deduced from [Eva99, Eva94].

We now turn to applications of Theorems C and D for three different classes of two-generated groups, namely the soluble Baumslag-Solitar groups, the split metacyclic groups and the lamplighter groups. A *Baumslag-Solitar group* is a group with a presentation of the form

$$BS(k, l) = \langle a, b \mid ab^k a^{-1} = b^l \rangle$$
for \( k,l \in \mathbb{Z} \setminus \{0\} \). Brunner proved that \( BS(2,3) \) has infinitely many \( T_2 \)-systems whereas its largest metabelian quotient has only one \( T_2 \)-system [Bru74, Theorem 3.2]. The group \( BS(k,l) \) is soluble if and only if \(|k| = 1 \) or \(|l| = 1 \). As a result, a soluble Baumslag-Solitar group is isomorphic to \( BS(1,l) \) for some \( l \in \mathbb{Z} \setminus \{0\} \) and hence admits a semi-direct decomposition \( \mathbb{Z}[1/l] \rtimes_l \mathbb{Z} \) where the canonical generator of \( \mathbb{Z} \) acts as the multiplication by \( l \) on \( \mathbb{Z}[1/l] = \{ \frac{z}{l^i}; z \in \mathbb{Z}, i \in \mathbb{N} \} \).

**Corollary F.** Let \( G = BS(1,l) \) with \( l \in \mathbb{Z} \setminus \{0\} \) and let \( n_n, t_n \ (n = 2,3) \) be defined as in Theorems C and D. Then the following hold:

(i) \( n_2(G) \) is finite if and only if \( l = \pm p^d \) for some prime number \( p \in \mathbb{N} \) and some non-negative integer \( d \). In this case, \( n_2(G) = \max(d,1) \).

(ii) \( t_2(G) = n_3(G) = 1 \).

Define \( \mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z} \) for \( d \geq 0 \) (thus \( \mathbb{Z}_0 = \mathbb{Z} \)) and let \( \varphi(d) \) be the cardinal of \( \mathbb{Z}_d^* \). A split *metacyclic group* is a semi-direct product of the form \( \mathbb{Z}_k \rtimes_\alpha \mathbb{Z}_l \) with \( k,l \geq 0 \). Here the canonical generator of \( \mathbb{Z}_d \) is denoted by \( a \) and acts on \( \mathbb{Z}_k \) as the multiplication by \( \alpha \in \mathbb{Z}_k^* \).

**Corollary G.** Let \( G = \mathbb{Z}_k \rtimes_\alpha \mathbb{Z}_l \ (k,l \geq 0) \) and let \( n_n, t_n \ (n = 2,3) \) be defined as in Theorems C and D. Then the following hold.

(i) \( n_2(G) = \frac{\varphi(\lambda)}{\omega} \), where \( \lambda \geq 0 \) is such that \( \mathbb{Z}_\lambda \simeq \mathbb{Z}_k/\nu(G)\mathbb{Z}_k \) and \( \omega \) is the order of the subgroup of \( \mathbb{Z}_\lambda^* \) generated by \(-1\) and the image of \( \alpha \).

(ii) \( t_2(G) = n_3(G) = 1 \).

(iii) If \( G \) is finite, then \( |V_2(G)| = \frac{k\varphi(k)}{e\varphi(e)} |V_2(\mathbb{Z}_l)| \), where \( e \geq 1 \) is such that \( \mathbb{Z}_e \simeq \mathbb{Z}_k/(1-\alpha)\mathbb{Z}_k \). In addition every Nielsen equivalence class of generating pairs has \( \frac{|V_2(G)|}{n_2(G)} \) elements.

Corollary G encompasses in particular [MW03, Theorem 4.5] which classifies generating pairs of dihedral groups modulo Nielsen equivalence.

A two-generated *lamplighter group* is a restricted wreath product of the form \( \mathbb{Z}_k \wr \mathbb{Z}_l \) with \( k,l \geq 0 \). Such a group reads also as \( R \rtimes C \) with \( C = \mathbb{Z}_d \) and \( R = \mathbb{Z}_k[C] \), the integral group ring of \( C \) over \( \mathbb{Z}_k \). We are able to determine the number of Nielsen classes and \( T \)-systems of any two-generated lamplighter groups with the exception of \( \mathbb{Z}_l \rtimes \mathbb{Z}_2 \).

**Corollary H** (Corollaries C and D). Let \( G = \mathbb{Z}_k \wr \mathbb{Z}_l \ (k,l \geq 0 \text{ and } k,l \neq 1) \) and let \( n_n \) and \( t_n \ (n = 2,3) \) be defined as in Theorems C and D. Then the following hold.

(i) \( t_2(G) = 1 \).

(ii) If \( \mathbb{Z}_k \) or \( \mathbb{Z}_l \) is finite, then \( n_3(G) = 1 \).
(iii) Assume that $\mathbb{Z}_k$ is finite and $\mathbb{Z}_l$ is infinite. Then $n_2(G)$ is finite if and only if $k$ is prime; in this case $n_2(G) = \max(\frac{k-1}{2}, 1)$.
(iv) Assume that $\mathbb{Z}_k$ is infinite and $\mathbb{Z}_l$ is finite. Then $n_2(G)$ is finite if and only if $l \in \{2, 3, 4, 6\}$; in this case $n_2(G) = 1$.

The case of a finite two-generated lamplighter group is addressed by Corollary 7 below and a formula for $|V_2(\mathbb{Z}_k \wr \mathbb{Z}_l)|$ is derived. For the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$, we show that classifying Nielsen equivalence classes and T-systems of generating triples tightly relates to an open problem in ring theory.

Corollary I (Corollary 10). Let $G = \mathbb{Z} \wr \mathbb{Z}$ and let $n_2, n_3$ and $t_3$ be defined as in Theorems C and D. Then we have $n_2(G) = 1$ and $n_3(G) \leq 2t_3(G)$. In addition, the following are equivalent:

(i) $n_3(G) = 1$.
(ii) $t_3(G) = 1$.
(iii) The ring $R$ of univariate Laurent polynomials over $\mathbb{Z}$ satisfies $SL_2(R) = E_2(R)$ (cf. [Abr08, Conjecture 5.3], [BMS02, Open Problem MA1], [BM82, Open problem]).

The paper is organized as follows. Section 2 deals with notation and gathers known facts on generalized Euclidean rings. It concludes on some classical results about units in group rings which we shall revisit. These results are collected in Lemma 6 and are used to handle lamplighter groups in Section 5. Section 3 is dedicated to the proof of Theorem A. In Section 3.1, we determine the conditions under which the map $\Phi_a$ of Theorem A is surjective. Section 4 is dedicated to the proofs of Theorems C, D and E. Section 5 presents applications to Baumslag-Solitar groups, split metacyclic groups and lamplighter groups.

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2. Preliminary results

2.1. Notation and definitions. We set in this section the notation and the definitions used throughout the article. A parallel is drawn between generating vectors of a group and unimodular rows of a module.

Rings. All considered rings are commutative rings with identity. Given a ring $R$, we denote by $\mathcal{J}(R)$ its Jacobson radical, i.e., the intersection of all its maximal ideals. We denote by $\text{nil}(R)$ the nilradical of $R$, i.e., the intersection of all its prime ideals. The nilradical coincides with the set of nilpotent elements
Let $M$ be a finitely generated $R$-module. Then an $R$-epimorphism of $M$ is an $R$-automorphism \cite[Theorem 2.4]{Mat89}, a fact that we will use without further notice. Let $N \subseteq M$ be finitely generated $R$-modules and $I \subseteq J(R)$ an ideal of $R$. Then the identity $N + IM = M$ implies $N = M$. We refer to the latter fact as Nakayama’s lemma \cite[Theorem 2.2’s corollary]{Mat89}. Apart from Section 2.2, the ring $R$ will always be a quotient of $\mathbb{Z}[X^{\pm 1}]$, the ring of univariate Laurent polynomials over $\mathbb{Z}$. In this case, we have $\text{nil}(R) = J(R)$ \cite[Theorem 4.19]{Eis95} and we shall consistently denote $\varphi$ the Euler totient function, so that $\varphi(d) = |\mathbb{Z}_d^\times|$ if $d > 0$. We set $\mathbb{Z}_d \doteq \mathbb{Z}/dz$ for $d \geq 0$. Thus the additive group of $\mathbb{Z}_d$ is the cyclic group with $d$ elements if $d > 0$ whereas $\mathbb{Z}_0 = \mathbb{Z}$. We denote by $\varphi$ the Euler totient function, so that $\varphi(d) = |\mathbb{Z}_d^\times|$ if $d > 0$. We set furthermore $\varphi(0) \doteq 2$.

**Orbits of generating vectors.** Let $G$, $H$ be groups and let $f \in \text{Hom}(G, H)$. We denote by $1_G$ the trivial element of $G$. For $g = (g_i) \in G^n$, we set $f(g) = (f(g_i))$. Thus the component-wise left action of $\text{Aut}(G)$ on $V_n(G)$ reads as $\phi g = \phi(g)$ for $(\phi, g) \in \text{Aut}(G) \times V_n(G)$. This action clearly coincides with the $\text{Aut}(G)$-action introduced earlier. Let us examine the component-wise counterpart of the $\text{Aut}(F_n)$-action we previously defined via the identification of $V_n(G)$ with $\text{Epi}(F_n, G)$. For $\psi \in \text{Aut}(F_n)$, set $w_i(x) \doteq \psi(x_i) \in F_n$, with $i = 1, \ldots, n$. Then the $\text{Aut}(F_n)$-action also reads as $g \psi = (w_i(g))$.

**Orbits of unimodular rows.** For $r \in R$ and $i \neq j$, we denote by $E_{ij}(r) \in \text{GL}_n(R)$ the elementary matrix with ones on the diagonal and whose $(i, j)$-entry is $r$, all other entries being 0. For $u \in R^\times$, we denote by $D_i(u) \in \text{GL}_n(R)$ the diagonal matrix with ones on the diagonal except at the $(i, i)$-entry, which is set to $u$. Recall that $E_n(R)$ is the subgroup of $\text{GL}_n(R)$ generated by elementary matrices. Given a subgroup $U$ of $R^\times$, we define $D_n(U)$ as the subgroup of $\text{GL}_n(R)$ generated by the matrices $D_i(u)$ with $u \in U$. Following P.M. Cohn \cite{Coh66}, we denote by $\text{GE}_n(R)$ the subgroup generated by $E_n(R)$ and $D_n(R^\times)$. Let $\Gamma$ be a subgroup of $\text{GL}_n(R)$ and let $M$ be an $R$-module. Two rows $r, r' \in M^n$ are said to be $\Gamma$-equivalent if there is $\gamma \in \Gamma$ such that $r' = r \gamma$.

**Elementary rank and stable rank.** We say that $n \geq 1$ belongs to the elementary range of $R$ if $E_{n+1}(R)$ acts transitively on $\text{Um}_{n+1}(R)$. The elementary rank of $R$ is the least integer $\text{er}(R)$ such that $n$ lies in the elementary range of $R$ for every $n \geq \text{er}(R)$. We say that $k$ lies in the stable range of $R$ if for every $n \geq k$ and every $(r_i) \in \text{Um}_{n+1}(R)$ there is $(s_i) \in R^n$ such that $(r_1 + s_1 r_{n+1}, r_2 + s_2 r_{n+1}, \ldots, r_n + s_n r_{n+1}) \in \text{Um}_{n}(M)$. The stable rank of $R$ is the least integer $\text{sr}(M)$ lying in the stable range of $M$. By \cite[Proposition 11.3.11]{MR87} we have:

\begin{equation}
1 \leq \text{er}(R) \leq \text{sr}(R).
\end{equation}
If $R$ is moreover Noetherian, the Bass Cancellation Theorem asserts [MR87, Corollary 6.7.4] that
\begin{equation}
\text{sr}(R) \leq \dim \text{K-null}(R) + 1.
\end{equation}

Let $R$ be a quotient of $\mathbb{Z}[X^\pm 1]$ and let $\alpha$ by the image of $X$ in $R$ via the quotient map. Recall that $T$ denotes the subgroup of $R^\times$ generated by $-1$ and $\alpha$. Let now $\Gamma_n(R)$ stand for the subgroup generated by $E_n(R)$ and $D_n(T)$. Since $D_n(R^\times)$ normalizes $E_n(R)$, we have $\Gamma_n(R) = D_n(T)E_n(R)$.

**Lemma 1.** For every $n \geq 2$, the group $\Gamma_n(R)$ is generated by $D_n(T)$ together with the elementary matrices $E_{ij}(1)$ with $1 \leq i \neq j \leq n$.

**Proof.** For $1 \leq i \neq j \leq n$ and $(r, r') \in R^2$, $\beta \in \{\alpha^\pm 1\},$ we have the following identities: $D_i(\beta)E_{ij}(r)D_i(\beta)^{-1} = E_{ij}(\beta r)$ and $E_{ij}(r)E_{ij}(r') = E_{ij}(r+r')$. Since $R$ is generated as a ring by $\alpha$ and $\alpha^{-1}$, the result follows. \[\square\]

2.2. GE-rings. Our study of Nielsen equivalence is significantly simplified when restricting to a class of rings $R$ which are similar to Euclidean rings in a specific sense. This is why we introduce the following definitions and elementary results. A ring $R$ is termed a GE$_n$-ring if $\text{GE}_n(R) = \text{GL}_n(R)$, which is equivalent to say that $\text{SL}_n(R) = E_n(R)$. Indeed, we have $\text{GE}_n(R) = D_n(R^\times)E_n(R)$ and a matrix $D \in D_n(R^\times)$ lies in $\text{SL}_n(R)$ if and only if it lies in $E_n(R)$ by Whitehead’s lemma [Wei13, Lemma 1.3.3]. Thus the latter equality implies the former, the converse being obvious. A ring $R$ is called a generalized Euclidean ring in the sense of P. M. Cohn [Coh66], or a GE-ring for brevity, if it is a GE$_n$-ring for every $n \geq 1$. Euclidean rings are known to satisfy this property [HO89, Theorem 4.3.9]. The reader is invited to check the two following elementary lemmas.

**Lemma 2.** The following assertions hold:

(i) $R$ is a GE$_2$-ring if and only if $1$ lies in the elementary range of $R$, i.e., $E_2(R)$ acts transitively on $\text{Um}_2(R)$.

(ii) If $\text{er}(R) = 1$ then $R$ is a GE-ring. In particular, $R$ is a GE-ring if its stable rank is $1$.

A semilocal ring, i.e., a ring with only finitely many maximal ideals, has stable rank $1$ [Bas64, Corollary 6.5]. As a result, semilocal rings, and Artinian rings in particular, are GE-rings.

**Lemma 3.** The following assertions hold:

(i) Let $J$ be an ideal contained in $J(R)$. Then $R$ is a GE-ring if and only if $R/J$ is a GE-ring [Gel77, Proposition 5].

(ii) Assume $R$ is a direct sum of rings. Then $R$ is a GE-ring if and only if each of its direct factors is a GE-ring. [Coh66, Theorem 3.1]
Lemma 4. Let $R$ be an Artinian ring. Then every homomorphic image of $R[X^\pm 1]$ is a GE-ring.

Proof of Lemma 4. Since $J(R) = \text{nil}(R)$, we have $J(R)[X^\pm 1] \subset J(R[X^\pm 1])$. As the factor ring $P = R[X^\pm 1]/J(R)[X^\pm 1]$ is isomorphic to a direct sum of Euclidean rings we deduce from Lemma 4 that $R[X^\pm 1]$ is a GE-ring. Let us consider a quotient $Q$ of $R[X^\pm 1]$. Then $Q/J(Q)$ is a quotient of $P$ and is therefore a direct sum whose factors are Euclidean or Artinian. As a result $Q$ is a GE-ring.

Remark 1. If $sr(R) = r < \infty$, then it easy to prove that any matrix in $\text{GL}_n(R)$ for $n > r$ can be reduced to a matrix of the form $\begin{pmatrix} A & 0 \\ 0 & I_{n-r} \end{pmatrix}$ with $A \in \text{GL}_r(R)$ by elementary row transformations. Thus $R$ is a GE-ring if it is a $\text{GE}_n$-ring for every $n \leq r$.

2.3. The univariate Laurent polynomials ring and its quotients. Because our focus is on the cyclic action of $C$ on $M$ in the exact sequence (1), the ring $\mathbb{Z}[X^\pm 1]$ of univariate Laurent polynomials over $\mathbb{Z}$ plays a prominent role in the article. We collect in this section preliminary facts about $\mathbb{Z}[X^\pm 1]$ and its quotients. These facts relate to row reduction and unit group description; they have an important bearing on quantitative statements in our applications.

A ring $R$ is said to be completable if every unimodular row over $R$ can be completed into an invertible square matrix over $R$, or equivalently, if $\text{GL}_n(R)$ acts transitively on $\text{Um}_n(R)$ for every $n \geq 1$.

Lemma 5. Every homomorphic image of $\mathbb{Z}[X^\pm 1]$ is completable.

Proof. If $R$ is isomorphic to $\mathbb{Z}[X^\pm 1]$, then $R$ is completable by [Sus77, Theorem 7.2]. So we can assume that $\dim_{\text{Krull}}(R) \leq 1$. Let $n \geq 2$ and $r \in \text{Um}_n(R)$. Since $sr(R) \leq 2$ we can find $E \in \text{E}_n(R)$ such that $rE = (r_1, r_2, 0, \ldots, 0)$. Let $A \in \text{SL}_2(R)$ be such that $(r_1, r_2)A = (1, 0)$ and set $B = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Then we have $rEB = (1, 0, \ldots, 0)$. 

The following is at the core of Theorem C.

Theorem 1. [Guy16b, Theorem A] Let $C$ be a finite cyclic group. Then every homomorphic image of $\mathbb{Z}[C]$ is a GE-ring.

The last lemma of this section will come in handy in Section 3 when scrutinizing lamplighter groups. Before we can state this lemma, we need to introduce some notation. Given a rational integer $d > 0$ we set $\zeta_d = e^{2\pi i / d}$ and let 

$$\lambda_d : \mathbb{Z}[X] \to \mathbb{Z}[\zeta_d]$$
be the ring homomorphism induced by the map $X \mapsto \zeta_d$. Given a set $\mathcal{D}$ of positive rational integers, we define

$$
\lambda_\mathcal{D} : \mathbb{Z}[X] \to \prod_{d \in \mathcal{D}} \mathbb{Z}[\zeta_d]
$$

by $\lambda_\mathcal{D} = \prod_{d \in \mathcal{D}} \lambda_d$ and set $\mathcal{O}(\mathcal{D}) \doteq \lambda_\mathcal{D}(\mathbb{Z}[X^\pm 1])$. Let $\alpha \doteq \lambda_\mathcal{D}(X) \in \mathcal{O}(\mathcal{D})$. Recall that a unit in $\mathcal{O}(\mathcal{D})^\times$ is said to be trivial if it lies in $T$, the subgroup generated by $-1$ and $\alpha$.

**Lemma 6.** Let $\mathcal{D}$ be a non-empty set of positive rational integers.

1. The torsion-free rank of $\mathcal{O}(\mathcal{D})^\times$ is $\sum_{d \in \mathcal{D}, n > 2} \frac{\varphi(d)}{2} - 1$.
2. Assume $\mathcal{D}$ is the set of divisors of $l$, with $l \geq 2$.
   
   (i) Any non-trivial unit of finite order in $\mathcal{O}(\mathcal{D} \setminus \{1\})$ is of the form $u \left(1 + \sum_{i \in E} \alpha^i\right)$ for some trivial unit $u$ and some non-empty subset $E \subseteq \{1, 2, \ldots, l-1\}$.
   
   (ii) If $l \in \{2, 3, 4, 6\}$, then units of $\mathcal{O}(\mathcal{D} \setminus \{1\})$ are trivial.

**Proof.** The proofs of 1 and 2.i essentially adapts [AA69, Theorems 3 and 4] to the rings $\mathcal{O}(\mathcal{D})$ under consideration; we provide them for completeness.

(1) Since the additive groups of $R = \prod_{d \in \mathcal{D}} \mathbb{Z}[\zeta_d]$ and $\mathcal{O}(\mathcal{D})$ are free Abelian groups of the same rank $r = \sum_{d \in \mathcal{O}(\mathcal{D})} \varphi(d)$, the latter group is of finite index $k$ in the former. By Dirichlet’s Unit Theorem the group $R^\times$ is finitely generated and its torsion-free rank is $\sum_{d \in \mathcal{D}, n > 2} \frac{\varphi(d)}{2} - 1$. Therefore it is sufficient to prove that $\mathcal{O}(\mathcal{D})^\times$ is of finite index in $R^\times$. This certainly holds if every unit $u \in \mathbb{Z}[\zeta_d]$ for $d \in \mathcal{D}$ is of finite order modulo $\mathcal{O}(\mathcal{D})$. To see this, consider the principal ideal $I$ of $\mathbb{Z}[\zeta_d]$ generated by $k$. Since $\mathbb{Z}[\zeta_d]/I$ is finite, there is $k' \geq 1$ such that $u^{k'} \equiv 1 \mod I$. Therefore $u^{k'} = 1 + k\zeta$ for some $\zeta \in \mathbb{Z}[\zeta_d]$. As $k\zeta \in \mathcal{O}(\mathcal{D})$, we deduce that $u^{k'} \in \mathcal{O}(\mathcal{D})$, which completes the proof.

(2). Let $g \in \mathcal{O}(\mathcal{D})$ such that $pr_1 : \mathcal{O}(\mathcal{D}) \to \mathcal{O}(\mathcal{D} \setminus \{1\})$ maps $g$ to a unit of finite order. Identifying $\mathcal{O}(\mathcal{D})$ with $\mathbb{Z}[C]$ for $C = \mathbb{Z}/l\mathbb{Z}$, we can write $g = \sum_{c \in C} a_c c$ with $a_c \in \mathbb{Z}$. For $d$ dividing $n$, let $\pi_d$ be the natural projection of $\mathcal{O}(\mathcal{D})$ onto $\mathbb{Z}[\zeta_d]$, let $\chi_d = \pi_d|C$ and $\rho_d = \pi_d(g) = \sum_{c \in C} a_c \chi_d(c)$. Since $pr_1(g)$ is of finite order, $\rho_d$ is a root of unity for every divisor $d > 1$. The characters $\chi_d$ form a complete set of inequivalent characters of $C$ by [AA69, Lemma 2]. Therefore we have

$$
\sum_{c \in C} a_c \chi(c) = \rho_x
$$

for every $\chi \in \hat{C}$, the character group of $C$, where $\rho_x$ is a root of unity if $\chi \neq 1$. Using the orthogonality relation of characters we obtain

$$
n a_c = \sum_{\chi \in \hat{G}} \rho_x \overline{\chi(c)}$$
for every $c \in C$. Hence $|a_c - a_{c'}| \leq \frac{2(l-1)}{n} < 2$ for every $c, c' \in C$. Replacing $g$ by $\varepsilon(g - k \sum_{c \in C} c)$ for a suitable choice of $\varepsilon \in \{\pm 1\}$ and $k \in \mathbb{Z}$, we can assume that $a_c \in \{0, 1\}$ for every $c$. Replacing $g$ by $cg$ for some suitable choice of $c \in C$, we can assume that $a_{1_c} = 1$. This ensures eventually that the image of $g$ in $O(D \setminus \{1\})$ has the desired form.

(2).ii. If $l \in \{2, 3\}$, then $O(D \setminus \{1\}) = \mathbb{Z}[\zeta_l]$ and this ring has only trivial units. Assume now $l = 4$. Since $R = O(\{2, 4\})$ embeds into $\mathbb{Z} \times \mathbb{Z}[\iota]$, it has at most 8 units. It is easily checked that there are exactly 8 trivial units in $R$. Therefore all units are trivial. Assume eventually that $l = 6$. Since $R = O(\{2, 3, 6\})$ embeds into $\mathbb{Z} \times \mathbb{Z}[\zeta_3] \times \mathbb{Z}[\zeta_3]$, it has only units of finite orders. Considering projections on each of the three factors, it is routine to check that no element of the form $1 + \sum_{i \in E} \alpha^i$ with $\emptyset \neq E \subset \{1, 2, 3, 4, 5\}$ is a unit in $R$. This proves that $R$ has only trivial units by 2.ii. $\square$

2.4. Nielsen equivalence in finitely generated Abelian groups. We present in this section the classification of generating tuples modulo Nielsen equivalence in finitely generated Abelian groups. This result is instrumental in Section 3.1 when reducing generating vectors to a standard form. Different parts of the aforementioned classification were obtained by different authors: [NN51, Satz 7.5], [Dun63, Section 2’s lemmas], [Eva93, Lemma 4.2] [LM93, Example 1.6] [DG99]. The classification reaches its most complete form with the following:

**Theorem 2.** [Oan11, Theorem 1.1] Let $G$ be a finitely generated Abelian group whose invariant factor decomposition is

$$\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_k}, \quad 1 \neq d_1 | d_2 | \cdots | d_k, \quad d_i \geq 0$$

Then every generating $n$-vector $g$ with $n \geq k = \text{rk}(G)$ is Nielsen equivalent to $(\delta e_1, e_2, \ldots, e_k, 0, \ldots, 0)$ for some $\delta \in \mathbb{Z}_{d_1}^\times$ and where $e = (e_i) \in G^k$ is defined by $(e_i)_1 = 1 \in \mathbb{Z}_{d_1}$ and $(e_i)_j = 0$ for $j \neq i$.

- If $n > k$, then $\delta$ above can be replaced by the identity.
- If $n = k$ then $\delta$ is unique, up to multiplication by $-1$.

In particular $G$ has only one Nielsen equivalence class of generating $n$-vectors for $n > k$ and only one $T_k$-system while it has $\max(\varphi(d_1)/2, 1)$ Nielsen equivalence classes of generating $k$-vectors where $\varphi$ denotes the Euler totient function extended by $\varphi(0) = 2$.

In the remainder of this section, we consider decompositions with cyclic factors which might differ from the invariant factor decomposition of $G$.

**Corollary 1.** [Guy16a, Corollary C] Let $G = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_k}$ with $d_i \geq 0$ for $i = 1, \ldots, k$. Let $d$ be the greatest common divisors of the integers $d_i$. For
Let $G \simeq \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_k}$ be a decomposition of $G$ in cyclic factors such that $k = \text{rk}(G)$. The identity elements of each factor ring in such decomposition form a generating vector of $G$. We refer to this vector as the generating vector naturally associated to the given decomposition. If $e$ is such a vector we define $\text{det}_e$ as in Corollary [1]

**Remark 2.** In the case $k = \text{rk}(G)$, Corollary [1] shows in particular that $g \in V_k(G)$ is Nielsen equivalent to $(\text{det}_e(m)e_1, e_2, \ldots, e_k)$ where $e = (e_i)$ is the generating vector naturally associated to the given cyclic decomposition of $G$.

### 3. Nielsen equivalence classes and T-systems of $M \rtimes C$

Throughout this section we assume that $G$ is a group which fits in the exact sequence [1I] and $n \geq \max(\text{rk}(G), 2)$. We denote by $\alpha$ the image of a favored generator $a$ of $C$ via the natural map $\mathbb{Z}[C] \to R$. Computing with powers of group elements in $G$ shall be facilitated by the following notation. For $u \in R^\times$ and $l \in \mathbb{Z} \cup \{\infty\}$, let

$$\partial_u(l) = \begin{cases} 
1 + u + \cdots + u^{l-1} & \text{if } l > 0, \\
0 & \text{if } l = 0, \infty, \\
-u^{-1}\partial_{u^{-1}}(-l) & \text{if } l < 0.
\end{cases}$$

For every $l \in \mathbb{Z}$ we have then $(1-u) \partial_u(l) = 1-u^l$. In particular $(1-\alpha)\nu(G) = 0$ for $\nu(G) = \partial_\alpha(\langle C \rangle)$. If $C$ is infinite then $\partial_\alpha$ is the composition of the Fox derivative over $C$ [Fox53] with the natural embedding of $C$ into $R = \mathbb{Z}[C]$. For $k, l \in \mathbb{Z}$ and $m \in M$, we have the identity $(ma^k)^l = (\partial_\alpha(l)m)a^k$.

The description of Bachmuth’s IA automorphisms will considerably ease off the study of Nielsen equivalence in $G \simeq M \rtimes C$. Recall that $F_n$ denotes the free group on $x = (x_1, \ldots, x_n)$. For $\psi \in \text{Aut}(F_n)$, let $\overline{\psi}$ be the automorphism of $\mathbb{Z}^n$ induced by $\psi$. We denote by $e = (e_i)$ the image of $x$ by the abelianization homomorphism $F_n \to (F_n)_{ab} = \mathbb{Z}^n$. The map $\psi \mapsto \overline{\psi}$ is an epimorphism from $\text{Aut}(F_n)$ onto $\text{GL}_n(\mathbb{Z})$ [LS77] Proposition I.4.4] whose kernel is denoted by $\text{IA}(F_n)$. This group clearly contains the isomorphisms $\varepsilon_{ij}$ defined by $\varepsilon_{ij}(x_i) = x_j^{-1}x_i x_j$ and $\varepsilon_{ij}(x_k) = x_k$ if $k \neq i$. In turn, $\text{IA}(F_n)$ is generated by the automorphisms $\varepsilon_{ijk}$ defined by $\varepsilon_{ijk}(x_i) = x_i [x_j, x_k]$ and $\varepsilon_{ijk}(x_l) = x_l$ for $l \neq i$ [LS77] Chapter I.4] where $[x, y] = xyx^{-1}y^{-1}$.

**3.1. Reduction to an a-row.** We discuss here circumstances under which a generating $n$-vector $g$ of $G$ can be Nielsen reduced to an a-row, i.e., a generating $n$-vector of the form $(m, a)$ with $m \in \text{Um}_{n-1}(R)$ and $a$ a favored generator of
We first observe that any generating $n$-vector $g$ is Nielsen equivalent to $(m, ma)$ for some $m \in M^{n-1}$ and some $m \in M$. Indeed, we can find $\psi \in \text{Aut}(F_n)$ such that $\sigma(g)\psi = (1,\alpha)$ using Theorem 2. This proves the claim. We shall establish conditions under which the element $m$ can be cancelled by a subsequent Nielsen transformation.

**Lemma 7.** Let $g = (m, ma) \in G^n$ with $m \in M^{n-1}$ and $m \in M$. Then $g$ generates $G$ if and only if $(m, \nu(G)m)$ generates $M$ as an $R$-module.

**Proof.** Assume first that $g \in V_n(G)$. Given $m' \in M$ there exists $w \in F_n$ such that $m' = w(m, ma)$. We can write $w = vx_s^a$ with $v$ lying in the normal closure of $\{x_1, \ldots, x_{n-1}\}$ in $F_n$ and $s \in Z$. Since conjugation by $ma$ induces multiplication by $\alpha$ on $M$, $\nu(g)$ lies in the $R$-submodule of $M$ generated by $m$.

When every generating $n$-vector of $G$ can be Nielsen reduced to a $\alpha$-row, we say that $G$ enjoys property $N_n(a)$. If $N_n(a)$ holds for $n = \text{rk}(G)$ then the equality $\text{rk}_R(M) = \text{rk}(G) - 1$ must be satisfied. The converse does not hold, as the latter is equivalent to the weaker property $N_n(C)$ according to which every generating $n$-vector can be Nielsen reduced to a $\alpha$-row with $\alpha$ ranging among generators of $C$, see Theorem 3 below.

**Lemma 8.** Let $g = (m, ma) \in V_n(G)$ with $m \in M^{n-1}$ and $m \in M$. Then the following hold:

(i) If $m \in (1-\alpha)M$ then $m$ generates $M$ as an $R$-module.

(ii) If $\nu(G)$ is nilpotent then $m$ generates $M$ as an $R$-module.

(iii) If $m$ generates $M$ as an $R$-module then $g$ is Nielsen equivalent to $(m, a)$.

**Proof.** We know that $(m, \nu(G)m)$ generates $M$ as an $R$-module by Lemma 7. If $m \in (1-\alpha)M$ then $\nu(G)m = 0$. Hence $m$ generates $M$ as an $R$-module, which proves (i). If $\nu(G) \in J(R)$, the same conclusion follows from Nakayama’s
lemma, which proves \((ii)\). Let us prove \((iii)\). If \(m = (m_i)\) generates \(M\) as an \(R\)-module then \(m\) is a sum of elements of the form \(k\alpha'm'\) with \(k, l \in \mathbb{Z}\) and \(m' \in \{m_1, \ldots, m_{n-1}\}\). We can subtract each of these terms from \(m\) in the last entry of \(g\) by applying transformations of the form \(e_{i,n}'\) and \(L_{n,i}^{-1}\).

Combining assertions \((ii)\) and \((iii)\) of Lemma \(8\) yields:

**Lemma 9.** If \(\nu(G)\) is nilpotent then \(N_n(a)\) holds for every \(n \geq \text{rk}(G)\).

Let \(\pi_{ab} : G \to G_{ab}\) be the abelianization homomorphism of \(G\). Let \(\pi_C\) be the natural map \(M \to M_C = M/(1 - \alpha)M\). Then we have \(G_{ab} = M_C \times C\) and \(\pi_{ab} = \pi_C \times \sigma\).

**Proposition 1.** Let \(g \in V_n(G)\) and assume that at least one of the following holds:

\[(i) \quad n > \text{rk}(G_{ab}),\]
\[(ii) \quad n > \text{rk}(M_C) \text{ and } M_C \text{ is not free over } \mathbb{Z}.

Then \(g\) is Nielsen equivalent to a vector \((m, a)\) with \(m \in \text{Um}_{n-1}(M)\).

**Proof.** Let \(k = \text{rk}(M_C)\). Observe first that both assumptions imply \(n > k\).

Let \(Z_{d_1} \times \cdots \times Z_{d_k}\) be the invariant factor decomposition of \(M_C\). Let then \(Z_{d_1} \times \cdots \times Z_{d_{n-1}} \times C\) be the decomposition of \(G_{ab}\) where \(d_i = 1\) if \(i > k\). Define \(e = (e_i) \in G_{ab}^{n-1}\) by \(e_i = 1 \in \mathbb{Z}_{d_i}\) if \(i \leq k\), \(e_i = 0\) otherwise. Set \(\bar{g} = \pi_{ab}(g)\).

Suppose now that \((ii)\) holds so that \(Z_{d_1}\) must be finite. Let \(\delta \in \mathbb{Z}_{d_1}^{\times}\) be a lift of \(\delta \equiv \text{det}_e(\bar{g})\) and let \(\bar{g}' = (\bar{g}, e_2, \ldots, e_{n-1}, a)\). Since \(\text{det}_e(\bar{g}) = \delta = \text{det}_e(\bar{g})\), the vectors \(\bar{g}\) and \(\bar{g}'\) are Nielsen equivalent by Corollary \(1\). By Theorem \(2\) this is also true if we assume \((i)\) and set \(\delta = 1 \in \mathbb{Z}_{d_1}\). Hence under assumption \((i)\) or \((ii)\) there is \(\psi \in \text{Aut}(F_n)\) such that \(\bar{g}' = \bar{g}\psi\). Then \(\bar{g}' \equiv \bar{g}\psi\) is of the form \((m, ma)\) with \(m = (m_1, \ldots, m_{n-1}) \in M^{n-1}\) and \(m \in M\) such that \(\pi_C(m) = 0\).

Applying Lemma \(8\) gives the result for any of the two hypotheses.

**Proposition 2.** Assume \(C\) is finite and let \(n = \text{rk}(G)\). Suppose moreover that \(M_C\) is isomorphic to \(\mathbb{Z}^{n-1}\). Let \(g \in V_n(G)\). Then \(g\) is Nielsen equivalent to a vector \((m, a^k)\) with \(m \in \text{Um}_{n-1}(M)\) and \(k = \pm \text{det}_e(\bar{g})\) where \(\bar{g} = \pi_{ab}(g)\) and \(e\) is any basis of \(M_C\).

**Proof.** Let \(e\) be a basis of \(M_C\) over \(\mathbb{Z}\). By Theorem \(2\) we can assume that \(\bar{g} = (e, a^k)\) for some \(k\) such that \(k = \pm \text{det}_e(\bar{g})\). Hence \(g = (m, ma^k)\) for some \(m \in M^{n-1}\) and \(m \in (1 - \alpha)M\). Then \(g\) is Nielsen equivalent to \((m, a^k)\) by Lemma \(8\).

Eventually, we present a characterization of property \(N_n(a)\) which establishes the first part of Theorem \(3\).
Theorem 3. Property $\mathcal{N}_n(a)$ holds if and only if at least one of the following does:

(i) $n > \text{rk}(G_{ab})$.
(ii) $C$ is infinite.
(iii) $\text{rk}(G) > \text{rk}(M_C)$ and $M_C$ is not isomorphic to $\mathbb{Z}^{\text{rk}(G)-1}$.

Property $\mathcal{N}_n(C)$ holds if and only if $n > \text{rk}(R)$.

Proof. The result follows from Lemma 8 and Propositions 1 and 2. \qed

Corollary 2. Let $n \geq \text{rk}(G)$. Then property $\mathcal{N}_n(c)$ holds for some generator $c$ of $C$ if and only if it holds for all generators $c$ of $C$.

□

3.2. Nielsen equivalence related to $\Gamma_{n-1}(R)$-equivalence. In this section we scrutinize the relation between Nielsen equivalence of generating $n$-vectors and $\Gamma_{n-1}(R)$-equivalence of unimodular rows. We prove there another part of Theorem 3. We establish now a partial converse to Lemma 10.

Lemma 10. If $m, m' \in \text{Um}_{n-1}(M)$ are $\Gamma_{n-1}(R)$-equivalent, then $(m, a), (m', a) \in V_n(G)$ are Nielsen equivalent.

Proof. Since $(mE_{ij}(1), a) = (m, a)L_{ij}$ for $1 \leq i \neq j \leq n-1$ and $(mD_i(\alpha), a) = (m, a)e_{i, n}$ for $1 \leq i \neq j \leq n-1$, we deduce from Lemma 4 that $(m, a)$ and $(m', a)$ are Nielsen equivalent. \qed

We establish now a partial converse to Lemma 10.

Proposition 3. Assume $C$ is infinite. If $(m, a), (m', a) \in V_n(G)$ are Nielsen equivalent then $m, m' \in \text{Um}_{n-1}(M)$ are $\Gamma_{n-1}(R)$-equivalent.

Proof. Suppose that $(m', a) = (m, a)\psi$ for some $\psi \in \text{Aut}(F_n)$. We claim that $\psi$ is of the form $\psi_0\psi_1L$ where

- $\psi_0 \in \text{IA}(F_n)$,
- $\psi_1 \in \text{Aut}(F_{n-1})$, i.e., $\psi_1(x_n) = x_n$ and $\psi_1$ leaves $F_{n-1} = F(x_1, \ldots, x_{n-1})$ invariant,
- $L$ belongs to the group generated by the automorphisms $L_{n,j}$.

To see this, consider the automorphism $\overline{\psi} \in \text{GL}_n(\mathbb{Z})$ induced by $\psi$. Since $\overline{\psi}(e_n) = e_n$, we can find a product of lower elementary matrices

$$\overline{L} \triangleq E_{n,1}(\mu_1) \cdots E_{n,n-1}(\mu_{n-1})$$

with $\mu_i \in \mathbb{Z}$ such that $\overline{\psi} \cdot \overline{L} \in \text{GL}_{n-1}(\mathbb{Z})$. Let $\psi_1 \in \text{Aut}(F_{n-1})$ be an automorphism inducing $\overline{\psi} \cdot \overline{L}$ on $\mathbb{Z}^{n-1}$. Let $L$ be the product of automorphisms $L_{n,j}^{\mu_j}$ with $\mu_j \in \mathbb{Z}$. Then $L$ induces $\overline{L}$ and by construction we have $L\psi_1^{-1} \in \text{IA}(F_n)$, which proves the claim.
The action of every IA-automorphism \( \varepsilon_{ijk} \) on \((m, a)\) leaves \(\sigma(m, a)\) invariant and induces a transformation on \(m\) which lies in \(\Gamma_{n-1}(R)\). The same holds for every automorphism in \(\text{Aut}(F_{n-1})\) and every automorphism \(L_{ij}\) with \(i > j\). Let \(g = (m, a)\psi_0\psi_1\). Then we can write \(g = (n, ma)\) with \(m \in M\) and where \(n\) is \(\Gamma_{n-1}(R)\)-equivalent to \(m\). As \(\sigma(gL) = \sigma(m', a) = (1_{C^{n-1}}, a)\) we deduce that \(\mu_j = 0\) for every \(j\), i.e., \(L = 1\). Hence \(m' = n\) which yields the result. \(\Box\)

### 3.3. Nielsen equivalence classes.

In this section we complete the proof of Theorem A by means of Theorem 3 below. We also discuss assumptions under which Theorem 3 allows to enumerate efficiently Nielsen equivalence classes.

Recall that the map \(\varphi_a : U_{m-1}(M) \to V_n(G)\) is defined by \(\varphi_a(m) = (m, a)\).

**Theorem 4.** The map \(\varphi_a\) induces a map \(\Phi_a : U_{m-1}(M)/\Gamma_{n-1}(R) \to V_n(G)/\text{Aut}(F_n)\)

(i) Property \(N_n(a)\) holds if and only if \(\Phi_a\) is surjective.
(ii) If \(C\) is infinite then \(\Phi_a\) is bijective.

**Proof.** It follows from Lemma 10 that \(\Phi_a\) is well-defined. Assertion (i) is trivial while assertion (ii) results from Lemma 8 and Proposition 3. \(\Box\)

**Corollary 3.** Assume that \(M \simeq R\). If \(n > sr(R) + 1\), then \(G = M \times C\) has only one Nielsen equivalence class of generating \(n\)-vectors.

**Proof.** The result follows from Theorem 3 and Theorem 4i together with inequality (3). \(\Box\)

We consider now several hypotheses under which the problem of counting Nielsen equivalence classes is particularly tractable. One of these hypotheses is that \(R\) be quasi-Euclidean, i.e., \(R\) enjoys the following row reduction property shared by Euclidean rings: for every \(n \geq 2\) and every \(r = (r_1, \ldots, r_n) \in R^n\), there exist \(E \in E_n(R)\) and \(d \in R\) such that \((d, 0, \ldots, 0) = rE\) (see [AJLL14] for a comprehensive survey on quasi-Euclidean rings). If \(R\) is a Noetherian quasi-Euclidean ring, then \(M\) admits an invariant factor decomposition, i.e., a decomposition of the form \(R/a_1 \times R/a_2 \times \cdots \times R/a_n\) with \(R \neq a_1 \supset a_2 \supset \cdots \supset a_n\) where the ideals \(a_i\) are referred to as the invariant factors of \(M\) (see [Guy16a, Lemma 1]).

**Corollary 4.** Let \(G = M \times C\) and \(n = \text{rk}(G)\). Then the following hold:

(i) If \(M\) is free over \(R\), \(C\) is infinite and \(R\) is a \(GE_{n-1}\)-ring, then \(n_n(G) = |R^x/T|\).
(ii) If \(C\) is infinite and \(R\) is quasi-Euclidean, then \(n_n(G) = |\Lambda^x/T_\Lambda|\) where \(\Lambda = R/a_1\), \(a_1\) is the first invariant factor of \(M\) and \(T_\Lambda\) is the image of \(T\) in \(\Lambda\) by the natural map.
(iii) If \( R \) is quasi-Euclidean then \( n_{n+1}(G) = 1 \).

Proof. (i). As \( C \) is infinite, it follows from Theorem 3 that \( M \cong R^{n-1} \). For \( m \in \text{Um}_{n-1}(M) \), let \( \text{Mat}(m) \in \text{GL}_{n-1}(R) \) be the matrix whose columns are the components of \( m \). For every \( E \in \Gamma_{n-1}(R) \) the identity \( \text{Mat}(mE) = \text{Mat}(m)E \) holds. As \( R \) is a GE\(_{n-1}\)-ring, we have \( \Gamma_{n-1}(R) = \text{D}_{n-1}(T)\text{SL}_{n-1}(R) \). We deduce from Whitehead’s lemma that \( \text{Mat}(m) \) can be reduced to \( D_{n-1}(u) \) via right multiplication by \( E \), where \( u \) is a member of a transversal of \( R^\times/T \). Therefore \( n_n(G) \leq |R^\times/T| \). Considering that \( uT = \det(\text{Mat}(m)E)T \) holds for every \( E \in \Gamma_{n-1}(R) \), we conclude that \( n_n(G) = |R^\times/T| \).

(ii). By [Guy16a, Theorem A and Corollary C], we have \( \text{Um}_{n}(M)/\Gamma_{n}(R) \cong \Lambda^\times/T\Lambda \). Theorem 4 implies \( n_n(G) = |\Lambda^\times/T\Lambda| \).

(iii). By [Guy16a, Corollary B], \( \text{Um}_{n}(M)/\Gamma_{n}(R) \) is reduced to one element. Theorems 3 and 4 imply \( n_{n+1}(G) = 1 \). \( \qed \)

We examine in the next proposition the structural implication of \( M \) being free over \( R \) with \( R \)-rank equal to \( \text{rk}(G) \).

**Proposition 4.** Assume that \( \text{rk}_R(M) = n = \text{rk}(G) \) and \( M \) is the direct sum of \( n \) cyclic factors, i.e.,

\[
M = R/a_1 \times \cdots \times R/a_n
\]

where the \( a_i \) are ideals of \( R \). Let \( a = a_1 + \cdots + a_n \). Then \( \nu(G) \) is invertible modulo \( a \). In addition, \( C \) is finite and \( G/aM = \mathbb{Z}_d^\times \times C \) where \( d = |R/a| < \infty \) is prime to \( |C| \).

Proof. We can assume without loss of generality that \( a = \{0\} \). Let \( e = (e_i) \) be a basis of \( M \) over \( R \) and let \( g \in V_n(G) \) for \( n = \text{rk}(G) \). We can also suppose that \( g = (m, ma) \) with \( m \in M^{n-1} \) and \( m \in M \). By Lemma 7 the vector \( (m, \nu(G)m) \) generates \( M \) as an \( R \)-module. Therefore the map \( e \mapsto (m, \nu(G)m) \) induces an \( R \)-automorphism of \( M \). This shows that \( e_n = \text{rk}(G)m' \) for some \( m' \in M \). Hence a relation of the form \( \sum_{i=1}^{n-1} r_i e_i + (\nu(G)r_n - 1)e_n = 0 \), with \( r_i \in R \) holds in \( M \). It follows that \( \nu(G) \) is invertible. Thus \( M = \nu(G)M \) is \( C \) invariant so that \( G = M_C \times C \). As a result \( M = M_C \) is a free \( \mathbb{Z}_d \)-module with \( d = |R| \) or \( d = 0 \). Since \( \text{rk}(M) = \text{rk}(G) \), the group \( C \) must be finite, \( d \) must be non-zero and prime to \( |C| \). \( \qed \)

3.4. **T-systems.** In this section we prove results on the T-systems of \( G \cong M \times C \) under the assumption that \( M \) is free over \( R \). These results will be specialized in Section 4.3 so as to prove Theorem 10. We present first all the definitions needed for describing \( \text{Aut}(G) \). Let \( c \mapsto \overline{c} \) be the restriction to \( C \) of the natural map \( \mathbb{Z}[C] \twoheadrightarrow R \). We call \( d : C \twoheadrightarrow M \) a derivation if
\[ d(cc') = d(c) + \mathbf{c}d(c') \] holds for every \((c, c') \in C^2\). For \(d \in \text{Der}(C, M)\), denote by \(X_d\) the automorphism of \(G\) defined by
\[ mc \mapsto md(c)c \]
For \(t \in \text{Aut}_R(M)\), denote by \(Y_t\) the automorphism of \(G\) defined by
\[ mc \mapsto t(m)c \]

The following lemma underlines the link between \(\text{Der}(M, C)\) and the automorphisms of \(G\) which leave \(M\) point-wise invariant.

**Lemma 11.** Let \(m \in M\). Then the following are equivalent:

1. \(\nu(G)m = 0\).
2. There exists \(d \in \text{Der}(C, M)\) such that \(d(a) = m\).
3. There exists \(X \in \text{Aut}(G)\) such that \(X(a) = ma\).

If one of the above holds, then the derivation \(d\) in (ii) is uniquely defined by \(d(a^k) = \partial_a(k)m\) for every \(k \in \mathbb{Z}\). If in addition the restriction to \(M\) of the automorphism \(X\) in (iii) is the identity then \(X(mc) = md(c)c\) for every \(m \in M, c \in C\).

\[ \square \]

We denote by \(A(C)\) the subgroup of the automorphisms of \(C\) induced by automorphisms of \(G\) preserving \(M\). The following result is referred to in [Guy12, Proposition 4] where it is a key preliminary to the study of 2-generated \(G\)-limits in the space of marked groups.

**Proposition 5.** Assume \(M\) is a free \(R\)-module and \(\nu(G) = 0\). Let \(n = \text{rk}(G)\) and let \(g, g' \in V_n(G)\). Then the following are equivalent:

1. \(g\) and \(g'\) are related by an automorphism of \(G\) preserving \(M\).
2. \(\sigma(g)\) and \(\sigma(g')\) are related by an automorphism in \(A(C)\).

*Proof.* Clearly, assertion (i) implies (ii). Let us prove the converse. Replacing \(g\) by \(\phi g\) for some automorphism \(\phi\) of \(G\) that preserves \(M\), we can assume without loss of generality that \(\sigma(g) = \sigma(g')\). Replacing \(g\) and \(g'\) by \(g\psi\) and \(g'\psi\) respectively for some \(\psi \in \text{Aut}(F_n)\), we can also assume that \(\sigma(g) = \sigma(g') = (1_{C^{n-1}}, a)\) and hence write \(g = (m, ma)\) and \(g' = (m', m'a)\) with \(m, m' \in M^{n-1}\) and \(m, m' \in M\). By Lemma \(\square\) the rows \(m\) and \(m'\) are bases of \(M\). By Lemma \(\square\) there is \(d' \in \text{Der}(C, M)\) such that \(d'(a) = m - m'\). Let \(t \in \text{Aut}_R(M)\) be defined by \(t(m) = m'\). Then we have \(g = Y_t X_{d'} g'\), which proves the result. \(\square\)

Recall that \(G\) is said to have property \(\mathcal{N}_n(a)\) if every of its generating \(n\)-vectors can be Nielsen reduced to an \(a\)-row.

**Theorem 5.** Let \(n = \text{rk}(G)\). Assume that one of the following holds:

1. Property \(\mathcal{N}_n(a)\) holds and \(M\) is free over \(R\).
(ii) $\text{rk}_R(M) = n$ and $M$ is free over $R.$

Then $G$ has only one $T_n$-system.

Proof. Assume (i) holds. Given an $R$-basis $e$ of $M,$ we shall prove that any $g \in V_n(G)$ is in the same $T_n$-system as $(e, a).$ By Theorem 3 we can assume without loss of generality that $g$ is of the form $(m, a)$ with $m \in \text{Um}_{n-1}(M).$ The $R$-endomorphism $t$ mapping $e$ to $m$ is an $R$-isomorphism. Thus $(m, a) = Y_t(e, a).$ Assume that (ii) holds. The group $G$ is then a finite Abelian group by Proposition 4. Thus the result follows from Theorem 2. \hfill $\Box$

We denote by $A'(C)$ the subgroup of $\text{Aut}(C)$ generated by $A(C)$ and the automorphism of $C$ which maps $a$ to $a^{-1}.$ Here is the most general result we obtain regarding the count of $T$-systems.

Theorem 6. Let $n = \text{rk}(G).$ Assume that $\text{rk}_R(M) = n - 1$ and $M$ is free over $R.$ Then $\tau_n(G) \leq |\text{Aut}(C)/A'(C)|,$ with equality if $M$ is a characteristic subgroup of $G$ and $M_C$ is isomorphic to $\mathbb{Z}^n.$

In order to prove the above theorem, we will use the following lemma, which is a simple variation on results found in [Sze04].

Lemma 12. Let $\text{Aut}_Z(M)$ be the group of $\mathbb{Z}$-automorphisms of $M.$

(i) Let $(\tau, \theta) \in \text{Aut}_Z(M) \times \text{Aut}(C)$ such that $\theta(a) = a^k$ and

$$\tau(mc^{-1}) = \theta(c)\tau(m)\theta(c)^{-1}$$

for all $m \in M,$ $c \in C.$ Then $Y_{\tau, \theta} : mc \mapsto \tau(m)\theta(c)$ is an automorphism of $G$ and $X \mapsto X^k$ induces a ring automorphism $\overline{\theta}$ of $R$ which satisfies

$$\tau(rm) = \overline{\theta}(r)\tau(m)$$

for all $m \in M,$ $r \in R.$

(ii) Let $d \in \text{Der}(C, M),$ $t \in \text{Aut}_R(M)$ and $(\tau, \theta)$ as in (i). Then we have

$$\tau^{-1} \circ d \circ \theta \in \text{Der}(C, M), \quad \tau^{-1} \circ t \circ \tau \in \text{Aut}_R(M)$$

and

$$Y_{\tau, \theta} X_d Y_{\tau, \theta}^{-1} = X_{\tau^{-1} \circ d \circ \theta}, \quad Y_{\tau, \theta} Y Y_{\tau, \theta}^{-1} = X_{\tau^{-1} \circ t \circ \tau}.$$

(iii) Let $\phi \in \text{Aut}(G)$ such that $\phi(M) = M.$ Then there is $d \in \text{Der}(C, M)$ and $(\tau, \theta)$ as in (i), such that $\phi = X_d Y_{\tau, \theta}.$ In particular, every automorphism in $A(C)$ is induced by some $Y_{\tau, \theta} \in \text{Aut}(G).$

Proof. The proofs of assertions (i) and (ii) are straightforward verifications.

(iii). Let $\tau$ be the restriction of $\phi$ to $M$ and let $\theta$ the automorphism of $C$ induced by $\phi.$ It is easy to check that $(\tau, \theta)$ satisfy the conditions of (i). Let $X = \phi Y_{\tau, \theta}^{-1}.$ Then the restriction of $X$ to $M$ is the identity and there is $m \in M$ such that $X(a) = ma.$ By Lemma 11 there is $d \in \text{Der}(C, M)$ such that $X = X_d.$ \hfill $\Box$
Proof of Theorem 6. By Theorems 3 and 5, we can assume, without loss of generality that $C$ is finite and $M_C$ is isomorphic to $\mathbb{Z}^n$. Using Proposition 2 and reasoning with a basis $e$ of $M$ as in the proof of Theorem 5, we see that every generating $n$-vector falls into the $T_n$-system of $(e, a^k)$ for some $k$ coprime with $|C|$. It follows from Lemma 12 that $(e, \theta(a^k))$ lies the $T_n$-system of $(e, a^k)$ for every $\theta \in A'(C)$. Therefore $t_n(G) \leq |\text{Aut}(C)/A'(C)|$.

Assume now that $M$ is a characteristic subgroup of $G$. If $(e, a)$ lies in the $T_n$-system of $(e, a^k)$ for some $k$ coprime with $|C|$, then we can find $\phi \in \text{Aut}(G)$ such that $\phi(e, a)$ is Nielsen equivalent to $(e, a^k)$. By Lemma 12, we have $\phi(e, a) = (e', m\theta(a))$ for some basis $e'$ of $M$, some $m \in M$ and $\theta \in A(C)$. By Lemma 8, the vector $(e', \theta(a))$ is Nielsen equivalent to $(e, a^k)$. Proposition 2 implies that $\theta(a) = a^{\pm k}$, hence there is $\theta' \in A'(C)$ such that $\theta'(a) = a^k$. □

4. Nielsen equivalence classes and $T$-systems of $R \times C$

In this section we assume that $M \simeq R$, i.e., $G = \langle a, b \rangle$ is a split extension of the form $R \times_C C$ with $C = \langle a \rangle$, while $b$ is the identity of the ring $R$ and $a$ acts on $R$ as the multiplication by $\alpha \in R^\times$.

4.1. Nielsen equivalence of generating pairs. We prove here the first two assertions of Theorem 6. Our first step consists in defining an invariant of Nielsen equivalence named $\Delta_a$. If $\nu(G) = 0$ there is a unique derivation $d_a \in \text{Der}(C, R)$ satisfying $d_a(a) = 1$. For $g = (g, g') = (r, r') G^2$ with $(r, r') \in R^2$ and $(c, d') \in C^2$, we set

$$D_a(g) = r'd_a(c) - rd_a(d') \in R$$

It is easily checked that $[g, g'] = (1 - \alpha)D_a(g)$. If $\nu(G) \neq 0$, we set further

$$D_a(g) = D_a(\pi_{G}(R)(g))) \in R/\nu(G)R$$

where the right-hand side is defined as in (5).

Lemma 13. We have $D_a(g) \in (R/\nu(G)R)^\times$ for every $g \in V_2(G)$.

Proof. We can assume without loss of generality that $\nu(G) = 0$.

(i) Let $g = (rak', r'a^k) \in V_2(G)$ with $(r, r') \in R$ and $k, k' \in \mathbb{Z}$. We first observe that

$$D_a(gL_{12}) = \alpha^{k'}D_a(g)$$
$$D_a(gL_{21}) = \alpha^kD_a(g)$$
$$D_a(gI_1) = -\alpha^{-1}D_a(g)$$

Thus $D_a(g \text{Aut}(F_2)) = TD_a(g)$. We know from Lemma 8 that $g$ is Nielsen equivalent to $(r, a)$ for some $r \in R^\times$. Therefore $D_a(g) \in rT$ which shows that $D_a(g)$ is invertible. □
Remark 3. Assume $\nu(G) = 0$. Let $c$ be a generator of $C$ and let $d_c \in \text{Der}(C, R)$ such that $d_c(c) = 1$. It is easily checked that $d_c = d_c(a)d_a$ and the identity $d_c(c) = 1$ implies that $d_c(a) \in R^\times$. For such elements $c$ there is thus only one map $D_c$ up to multiplication by a unit of $R$.

Let $\Lambda \cong R/\nu(G)R$, $T_\Lambda \cong \pi_{\nu(G)R}(T)$ and define $\Delta_a : V_2(G) \rightarrow \Lambda^\times / T_\Lambda$ by

$$\Delta_a(g) \equiv T_\Lambda D_a(g)$$

In the course of Lemma 13’s proof we actually showed

**Lemma 14.** The map $\Delta_a$ is $\text{Aut}(F_2)$-invariant.

The last stepping stone to the theorem of this section is the crucial

**Lemma 15.** Let $g = (r, a)$, $g' = (r', a)$ with $r, r' \in R^\times$. Then the following are equivalent:

(i) $g$ and $g'$ are Nielsen equivalent.
(ii) $\pi_{\nu(G)R}(g)$ and $\pi_{\nu(G)R}(g')$ are Nielsen equivalent.
(iii) $\Delta_a(g) = \Delta_a(g')$.

**Proof.** (i) $\Rightarrow$ (ii). This follows from the $\text{Aut}(F_2)$-equivariance of $\pi_{\nu(G)R}$.

(ii) $\Rightarrow$ (iii). This follows from Lemma 14.

(iii) $\Rightarrow$ (i). The result is trivial if $\nu(G) = 0$, thus we can assume that $C$ is finite. By hypothesis, there exist $k \in \mathbb{Z}$, $r_\nu \in R$ and $\epsilon \in \{\pm 1\}$ such that $r' = \epsilon \alpha^k r + r_\nu \nu(G)$. Replacing $g'$ by a conjugate if needed, we can assume that $k = 0$. Taking the inverse of the first component of $g$ if needed, we can moreover assume that $\epsilon = 1$, so that $r' = r + \nu(G)r_\nu$. Since $r'$ is a unit, we can argue as in the proof of Lemma 13(iii) to get $\psi \in \text{Aut}(F_2)$ such that $(r', a)\psi = (r', r_\nu a)$. We have then $(r', a)\psi L_{1,2}^{[C]} = (r, r_\nu a)$. Since $r$ is a unit, we can cancel $r_\nu$ using another automorphism of $F_2$. \hfill \square

**Lemma 16 (Z[1]).** Let $I \subset R$ be an ideal which is contained in all but finitely many maximal ideals of $R$. Then the natural map $R^\times \rightarrow (R/I)^\times$ is surjective.

**Proof.** Let $m_1, \ldots, m_k$ be the maximal ideals of $R$ not containing $I$ and let $J = (\bigcap_i m_i) \cap I$. By the Chinese Remainder Theorem, the map

$$\rho : r + J \mapsto (r + I, r + m_1, \ldots, r + m_k)$$

is a ring isomorphism from $R/J$ onto $R/I \times R/m_1 \times \cdots \times R/m_k$. Given $u \in (R/I)^\times$ we can find $v = \tilde{u} + J \in (R/J)^\times$ such that $\rho(v) = (u, 1 + m_1, \ldots, 1 + m_k)$. Hence we have $u = \tilde{u} + I$. As $J \subset \mathcal{J}(R)$, we also have $\tilde{u} \in R^\times$. \hfill \square

Before we can state the main result of this section, we need to introduce some notation. Given an ideal $I$ of $R$, we denote by $\pi_I$ the natural group epimorphism $R \rtimes_c C \rightarrow R/I \rtimes_{c+1} C$. 
Theorem 7. Let $g, g' \in V_2(G)$, $e = \pi_{ab}(b, a)$ and $R_C = R/(1 - \alpha)R$.

1. The following are equivalent:
   
   (i) The pairs $g$ and $g'$ are Nielsen equivalent.
   
   (ii) The pairs $\pi_I(g)$ and $\pi_I(g')$ are Nielsen equivalent for every
   
   $I \in \{(1 - \alpha)R, \nu(G)R\}$.
   
   (iii) $\det_e \circ \pi_{ab}(g) = \pm \det_e \circ \pi_{ab}(g')$ and

   $\Delta_a(g) = \Delta_a(g')$.

2. If $C$ is infinite or $R_C$ is finite then $\Delta_a$ is surjective and the above conditions are equivalent to $\Delta_a(g) = \Delta_a(g')$. In this case $\Delta_a$ is a complete invariant of Nielsen equivalence for generating pairs.

Proof. 1.(i) $\Rightarrow$ (ii). This follows from the Aut($F_2$)-equivariance of $\pi_I$.

1.(ii) $\Rightarrow$ (iii). We deduce the identity $\det_e \circ \pi_{ab}(g) = \pm \det_e \circ \pi_{ab}(g')$ from

Theorem 2 and the identity $\Delta_a(g) = \Delta_a(g')$ from Lemma 13.

1.(iii) $\Rightarrow$ (i). Suppose first that $C$ is infinite or $R/(1 - \alpha)R$ is finite. By

Theorem 3 we know that $g$ and $g'$ can be Nielsen reduced to $(r, a)$ and $(r', a)$ for some $r, r' \in R^\times$. By Lemma 15 the pairs $g$ and $g'$ are Nielsen equivalent. Suppose now that $C$ is finite and $R/(1 - \alpha)R$ is infinite. By Theorem 2 we know that $g$ and $g'$ can be Nielsen reduced to $(r, a^k)$ and $(r', a^{k'})$ for some $r, r' \in R^\times$ and $k, k' \in \mathbb{Z}$ such that $k \equiv \pm \det_e \circ \pi_{ab}(g)$ and $k' \equiv \pm \det_e \circ \pi_{ab}(g')$ modulo $|C|$. We deduce from Theorem 2 that $k' \equiv \pm k \mod |C|$. Replacing $g$

by $gI_2$ if needed, we can assume that $a^k = a^{k'}$. Thanks to Remark 3, we can argue as in the first part of the proof where $a$ is replaced by $a^k$, which proves the Nielsen equivalence of $g$ and $g'$.

2. We are left with the proof of $\Delta_a$’s surjectivity. Clearly, it suffices to show that the natural map $R^\times \to (R/\nu(G)R)^\times$ is surjective. This is trivial if $C$ is infinite since $\nu(G) = 0$ in this case. So let us assume that $R/(1 - \alpha)R$ is finite. Since we have $(1 - \alpha)\nu(G) = 0$, the ring element $\nu(G)$ belongs to every maximal ideal of $R$ which doesn’t contain $1 - \alpha$. Hence it belongs to all but finitely many maximal ideals of $R$. The result follows then from Lemma 16.

We conclude this section with an algorithmic characterization of generating pairs.

Proposition 6. Assume $\nu(G)$ is nilpotent and let $g \in G^2$. Then the following are equivalent:

(i) $g$ generates $G$.

(ii) $\sigma(g)$ generates $C$ and $D_a(g) \in (R/\nu(G)R)^\times$.

Proof. (i) $\Rightarrow$ (ii). This follows from $\sigma$’s surjectivity and Lemma 13.

(ii) $\Rightarrow$ (i). Since $\sigma(g)$ generates $C$ there is $\psi \in \text{Aut}(F_2)$ such that $\sigma(g)\psi = (1_C, a)$. Replacing $g$ by $g\psi$, we can then assume that $g = (r, r'a)$ for some
r, r' ∈ R. Since Δₐ(g) = r + ν(G)R ∈ (R/ν(G)R)ₓ and ν(G) ∈ J(R) we deduce that r ∈ Rₓ. Therefore g generates G. □

4.2. Nielsen equivalence of generating triples and quadruples. In this section we prove the last two assertions of Theorem C. Since rk(G) = 2 and dimₐₗ(K(R)) ≤ 2, Corollary 3 ensures that G has only one Nielsen class of generating n-vectors for n > 4. Using Theorem 4 in combination with a theorem of Suslin [Sus77, Theorem 7.2], we show in Theorem 8 below that this remains true if n ≥ 3. Recall that the map

\[ Φₐ : rΓₙ₋₁(R) ↪ (r, a) \text{Aut}(Fₙ) \]

defined in Theorem 4 is a bijection from Umₙ₋₁(R)/Γₙ₋₁(R) onto Vₙ(G)/Aut(Fₙ) provided that C is infinite.

**Lemma 17.** Let n ≥ 3. If R is a GEₙ₋₁-ring, then G has only one Nielsen equivalence class of generating n-vectors.

**Proof.** Let g ∈ Vₙ(G). We shall show that g is Nielsen equivalent to g₁ = (r₁, a) with r₁ = (1, 0, ..., 0) ∈ Umₙ₋₁(R). As n > rk(G), the property Νₙ(a) holds by Theorem 3. Therefore g can be Nielsen reduced to a vector of the form (r, a) with r ∈ Umₙ₋₁(R). Since R is a GEₙ₋₁-ring and R is completable by Lemma 5, the group Eₙ₋₁(R) acts transitively on Umₙ₋₁(R). Hence r can be transitioned to r₁ under the action of Eₙ₋₁(R). Lemma 10 implies that g is Nielsen equivalent to g₁. □

Our forthcoming result on generating triples involves the following definitions from algebraic K-theory. For every n ≥ 1, the map A ↦ \( \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \)
defines an embedding from SLₙ(R) into SLₙ₊₁(R), respectively from Eₙ(R) into Eₙ₊₁(R). Denote by SL(R) and E(R) the respective ascending unions. Then E(R) is normal in SL(R) and the group SK₁(R), the *special Whitehead group of R*, is the quotient SL(R)/E(R). The next lemma shows in particular that the image in SK₁(R) of a matrix in SL₂(R) depends only on its first row.

**Lemma 18.** Let R be any commutative ring with identity. Denote by E₂(R) the normal closure of E₂(R) in SL₂(R). Let ρ : SL₂(R) ↪ Um₂(R) be defined by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b) \). Then the map ρ induces a bijection from SL₂(R)/E₂(R) onto Um₂(R)/E₂(R).

**Proof.** For every A, B ∈ SL₂(R) the identity ρ(AB) = ρ(A)B holds. Therefore the map \( \hat{ρ} : A\hat{E}_₂(R) ↪ \rho(A)\hat{E}_₂(R) \) is well defined. Let (a, b) ∈ Um₂(R) and let a', b' ∈ R be such that aa' + bb' = 1. Then A ∈ Um₂(R)
and \((a, b) = \rho(A)\), so that \(\rho\), and hence \(\hat{\rho}\) is surjective. Let us prove that \(\hat{\rho}\) is injective. Consider for this \(A, B \in \text{SL}_2(R)\) such that \(\rho(A) = \rho(B)\). Multiplying \(A\) on the right by a matrix in \(\hat{E}_2(R)\) if needed, we can assume that \(\rho(A) = \rho(B)\). Thus \(\rho(AB^{-1}) = \rho(A)B^{-1} = \rho(B)B^{-1} = (1, 0)\), which shows that \(AB^{-1} \in E_2(R)\). The result follows. \(\Box\)

The Mennicke symbol \([r]\) of \(r \in \text{Um}_2(R)\) is the image in \(\text{SK}_1(R)\) of any matrix of \(\text{SL}_2(R)\) whose first row is \(r\). We are now in position to prove

**Theorem 8.** The following hold:

(i) If \(C\) is infinite then \(V_3(G) / \text{Aut}(F_3)\) surjects onto \(\text{SK}_1(R)\).

(ii) If \(R\) is a \(\text{GE}_2\)-ring, e.g., \(C\) is finite, then \(n_3(G) = 1\).

(iii) \(n_4(G) = 1\).

**Proof.**

(i) By Theorem 1 we can identify \(V_3(G) / \text{Aut}(F_3)\) with \(\text{Um}_2(R) / \Gamma_2(R)\). The classical properties of the Mennicke symbol imply that the map \([\cdot] : \text{Um}_2(R) \to \text{SK}_1(R)\) is \(\Gamma_2(R)\)-invariant [Lam06, Proposition 3.4]. This yields a map \(\text{Um}_2(R) / \Gamma_2(R) \to \text{SK}_1(R)\). By Remark 1, the latter map is surjective.

(ii) This is Lemma 17 for \(n = 3\).

(iii) We can assume that \(C\) is infinite since Lemma 17 applies otherwise. If \(\text{dim}_{K_{	ext{null}}}(R) \leq 1\) then \(G\) has only one Nielsen equivalence class of generating quadruples by Corollary 3. Thus can we also suppose that \(R = \mathbb{Z}[X^{\pm 1}]\). Since \(V_4(G) / \text{Aut}(F_4)\) identifies with \(\text{Um}_3(R) / \Gamma_3(R)\) by Theorem 4 and since \(E_3(R)\) acts transitively on \(\text{Um}_3(R)\) by [Sus77, Theorem 7.2], the group \(G\) has only one Nielsen equivalence class of generating quadruples. \(\Box\)

### 4.3. T-systems of generating pairs.

This section is dedicated to the proofs of Theorem 12 and 13. Recall that we denote by \(n_n(G)\) the number of Nielsen equivalence classes of generating \(n\)-vectors of \(G\) and by \(t_n\) the number its \(T_n\)-systems of \(G\), both numbers may be infinite. Combining Theorems 3, 5 and 6 we obtain:

**Theorem 9.** The following hold:

(i) If \(C\) is infinite or \(G_{ab}\) is finite, then \(t_2(G) = 1\).

(ii) If \(C\) is finite and \(G_{ab}\) is infinite then \(t_2(G) \leq |\text{Aut}(C)/A'(C)|\), with equality if \(R\) is a characteristic subgroup of \(G\).

(iii) Assume that \(C\) is infinite and \(R\) is a characteristic subgroup of \(G\). Then we have \(n_3(G) \leq |A(C)|n_2(G)t_3(G)\) and \(|A(C)|t_3(G) \geq |\text{SK}_1(R)|\).

**Proof.**

(i) Combine Theorems 3 and 5.

(ii) This is a specialization of Theorem 6 to \(G = R \rtimes C\).

(iii) Consider the action of \(\text{Aut}(G)\) on \(V_3(G) / \text{Aut}(F_3)\) defined by \(\phi \cdot (g \text{Aut}(F_3)) = (\phi g) \text{Aut}(F_3), \ g \in V_3(G), \phi \in \text{Aut}(G)\)
Regarding the first inequality, it suffices to show that the stabilizer $S_g$ of $g \Aut(F_3)$ has index at most $|A(C)|n_2(G)$ in $\Aut(G)$ for every $g \in V_3(G)$. By Theorem 3, such a triple $g$ is Nielsen equivalent to $(r, s, a)$ for some $r, s \in R$. Since $R = sR + tR = \mathbb{Z}[\alpha^\pm 1]$, we easily see that every automorphism $X_\alpha$ stabilizes $g \Aut(F_3)$. For an automorphism $Y_{r,1}$ of $G$, we observe that $\tau \in \Aut_{\mathbb{Z}}(R)$ is actually an $R$-automorphism, so that $\tau$ is the multiplication some unit of $R$. If $\tau$ is a trivial unit, we see that $Y_\tau = Y_{r,1}$ stabilizes $g \Aut(F_3)$ considering conjugates of the first two components of $g$. If $A(C)$ contains an automorphisms $\theta$ which maps $a$ to $a^{-1}$, we let $\phi_{-1}$ be an automorphism of the form $Y_{r,\theta}$ whose image is $\theta$ through the natural map $\Aut(G) \to A(C)$. Otherwise we set $\phi_{-1} = 1$. Let $V$ be a transversal of $R^\times/T$. It follows from Lemma 12 that $\{Y_\tau \phi; \tau \in \{0, 1\}, \phi(b) \in V\}$ is a transversal of $\Aut(G)/S_g$. Since $n_2(G) = |R^\times/T|$ by Theorem 4, we deduce that $|\Aut(G)/S_g| \leq |A(C)|n_2(G)$, which completes the proof of the first inequality.

In order to prove the second inequality, consider the action of $\Aut(G)$ on $SK_1(R)$ defined by $\phi \cdot [r] = [\phi(r)]$ for $(\phi, r) \in \Aut(G) \times \Um_2(R)$ and where $[r]$ denotes the Mennicke symbol of $r$. The map $(r, a) \Aut(F_3) \to [r]$ induces an $\Aut(G)$-equivariant map $\mu$ from $V_3(G)/\Aut(F_3)$ onto $SK_1(R)$. Every automorphism $\phi \in \{X_d, Y_\tau; d \in \Der(C, R), \tau \in \Aut_R(R)\}$ fixes every symbol $[r]$. It follows from Lemma 12 that the $\Aut(G)$-action on $SK_1(R)$ factors through an $A(C)$-action and we have $\Aut(G) \backslash SK_1(R) \simeq A(C) \backslash SK_1(R)$. Therefore $\mu$ induces a surjective map from $\Aut(G) \backslash V_3(G)/\Aut(F_3)$ onto $A(C) \backslash SK_1(R)$, which yields the result.

With Corollary 5 below, we show that $G$ can have arbitrarily many $T_2$-systems when $C$ is finite but $G_{ab}$ isn’t.

**Corollary 5.** Let $q = p^d$ and $N = q - 1$, with $p$ a prime integer and $d \geq 2$ an even integer. Let $R = \mathbb{Z}[X^\pm]/(X - 1)I$ with $I = p\mathbb{Z}[X^\pm] + P(X)\mathbb{Z}[X^\pm]$, $P \in \mathbb{Z}[X]$ a polynomial of degree $d$ whose reduction modulo $p$ is an irreducible factor of $\Phi_n(X)$. Then the subgroup $C$ of $R^\times$ generated by the image of $X$ has $N$ elements and the number of $T_2$-systems of $G = R \rtimes C$ is $t_2(G) = \varphi(N)/d$.

We will use the following straightforward consequence of Lemma 12.

**Lemma 19.** Let $k \in \mathbb{Z}$. The following are equivalent:

(i) There is $\theta \in A(C)$ such that $\theta(a) = a^k$.

(ii) The map $X \mapsto X^k$ induces a ring automorphism of $R$.

□

**Lemma 20.** The two following hold:
(i) Let $g = ra^k \in G$ with $r \in R$, $k \in \mathbb{Z}$. Then $g$ centralizes its conjugacy class if and only if $(1 - \alpha^k)^2 = 0 = (1 - \alpha)(1 - \alpha^k)r$.

(ii) Let $\omega$ be the order of $\alpha$ in $R^\times$. Assume that $\omega = |C|$ and for every $k \in \mathbb{Z}$, we have $(1 - \alpha^k)^2 \neq 0$ whenever $\alpha^k \neq 1$. Then $R$ is a characteristic subgroup of $G$.

Proof. Assertion (i) is a direct consequence of the identity

$$[g, hgh^{-1}] = (1 - \alpha^k) \left( (1 - \alpha^k)r - (1 - \alpha^k)r' \right),$$

where $h = r'a^k$.

In order to prove (ii), consider $\phi \in \text{Aut}(G)$ and write $\phi(b) = ra^k$ where $b$ is the identity of the ring $R$. Since $b$ centralizes its conjugacy class, so does $\phi(b)$. By (i), we have $(1 - \alpha^k)^2 = 0$, which yields $\alpha^k = 1$. As $\omega = |C|$, we deduce that $\phi(b) = r$ and hence $\phi(R) = R$. □

Proof of Corollary 2. By the Chinese Remainder Theorem, the ring $R$ identifies with $\mathbb{Z} \times \mathbb{F}_q$ where $\mathbb{F}_q = \mathbb{Z}_p[X]/P(X)\mathbb{Z}_p[X]$ is the field with $q$ elements. As a result, $C = \mathbb{F}_q^\times$ has $N$ elements and the ring automorphisms of $R$ induced by maps of the form $X \mapsto X^k$ correspond bijectively to powers of the Frobenius endomorphism of $\mathbb{F}_q$. Lemma 20’s hypotheses are easily checked so that $R$ is a characteristic subgroup of $G$. By Lemma 19 we have then $|A(C)| = d$ and hence $|A'(C)| = d$ for $d$ is even. By Theorem 9 we obtain $t_2(G) = |\text{Aut}(C)/A'(C)| = \varphi(N)/d$. □

Regarding $T$-systems of generating triples, we obtain the following sharp statement thanks to the $K$-theory of orders of arithmetic type.

**Theorem 10.** Let $N \geq 3$, $\zeta_N = e^{2\pi i/N}$ and $\alpha = 1 + N^2\zeta_N$. Define the ring $S$ as the localization $\mathbb{Z}[\zeta_N]_{\alpha}$ of $\mathbb{Z}[\zeta_N]$ at $\alpha$ and let $R = \mathbb{Z}[\alpha^{\pm 1}] \subset S$. Let $G = R \rtimes_{\alpha} \mathbb{Z}$ where the canonical generator of $\mathbb{Z}$ acts on $R$ as the multiplication by $\alpha$. Then we have $n_3(G) = t_3(G) = N$.

We first need to establish

**Proposition 7.** Let $R$ be as in Theorem 10. Then $E_2(R)$ is normal in $\text{SL}_2(R)$ and we have $\text{SL}_2(R)/E_2(R) \simeq \text{SK}_1(R) \simeq \mathbb{Z}_N$.

Proof. Since $R$ contains $N^2S$ and $S$ is a Dedekind ring of arithmetic type, the ring $R$ is an order of arithmetic type in the sense of B. Lieh [Lie81]. It has moreover infinitely many units. By [Lie81] Formulas (5) and (19), the group $E_2(R)$ is normal in $\text{SL}_2(R)$ and $\text{SL}_2(R)/E_2(R)$ is a quotient of $\text{SK}_1(S, \mathbb{F}_2^2)$ where $\mathbb{F}_2^2 = \{x \in R \mid xS \subset R\}$ is the conductor of $R$ in $S$. Let us show that $\mathbb{F}_2^2 = N^2S$, i.e., the exponent of the Abelian group $S/R$ is $N^2$. As $\{1, \zeta_N, \ldots, \zeta_N^{(N-1)}\}$ is a basis of $\mathbb{Z}[\zeta_N]$ over $\mathbb{Z}$, the exponent of $\mathbb{Z}[\zeta_N]/(\mathbb{Z} + N^2\mathbb{Z}[\zeta_N])$ is $N^2$. Setting $\overline{\alpha} = N^2\mathbb{Z}[\zeta_N]$ over $\mathbb{Z}$, we observe that $S/R = \mathbb{Z}[\zeta_N]_{\alpha}/(\mathbb{Z} + N^2\mathbb{Z}[\zeta_N])_{\alpha} = \mathbb{Z}$.
\[(\mathbb{Z}[\zeta_N]/(\mathbb{Z} + N^2\mathbb{Z}[\zeta_N]))_\pi \simeq \mathbb{Z}[\zeta_N]/(\mathbb{Z} + N^2\mathbb{Z}[\zeta_N])\]. Therefore the exponent of \(S/R\) is also \(N^2\), which proves the claim.

We shall establish that

\[(6) \quad SK_1(R) \simeq SK_1(R, f)\]

The inclusion map from \(R\) into \(S\) induces an epimorphism \(SK_1(R, f) \to SK_1(S, f)\). If \((6)\) holds, then \(SK_1(R)\) maps homomorphically onto \(SK_1(S, f)\), which is in turn isomorphic to both \(SK_1(S, f^2)\) and \(\mathbb{Z}_N\) by the Bass-Milnor-Serre theorem \([Mag02\text{, Theorem 11.33}]. Using \([Lie81\text{, Formula (19)\]}, we see that there is an epimorphism from \(SK_1(R)\) onto \(SL_2(R)/E_2(R)\). By Remark \([1]\) there is also an epimorphism from \(SL_2(R)/E_2(R)\) onto \(SK_1(R)\). Since both groups are finite, there are isomorphic.

It only remains to prove \((6)\). To do so, we first observe that \(R/f\simeq \mathbb{Z}_{N^2}\). Indeed, setting \(Q = (\mathbb{Z} + N^2\mathbb{Z}[\zeta_N])/N^2\mathbb{Z}[\zeta_N]\), it is straightforward to show that \(Q \simeq \mathbb{Z}_{N^2}\) and subsequently \(R/f = (\mathbb{Z} + N^2\mathbb{Z}[\zeta_N])_\alpha/(N^2\mathbb{Z}[\zeta_N])_\alpha \simeq Q_\pi \simeq \mathbb{Z}_{N^2}\). In the exact sequence (see, e.g., \([Mag02\text{, Theorem 13.20 and Example 13.22}\])

\[K_2(R/f) \to SK_1(R, f) \to SK_1(R) \to SK_1(R/f)\]

the last term, namely \(SK_1(R/f)\), is trivial since \(R/f\) is finite. In addition, the image of \(K_2(R/f)\) in \(SK_1(R, f)\) is also trivial. Indeed \(K_2(R/f)\) is generated by the Steinberg symbol \(\{-1 + f, -1 + f\}_{R/f}\) because \(R/f \simeq \mathbb{Z}_{N^2}\) (see \([Mag02\text{, Exercices 13A.10 and 15C.10}\]). As \(\{-1, -1\}_R\) is a lift of the previous symbol in \(K_2(R)\) our claim follows and the proof is now complete. \(\square\)

**Proof of Theorem \([10]\)** By Theorem \([4]\), the set \(V_3(G)/\text{Aut}(F_3)\) identifies with \(Um_2(R)/G_2(R)\) via \((r, a)\text{Aut}(F_3) \mapsto rG_2(R)\). Since \(E_2(R)\) is normal in \(SL_2(R)\) by Proposition \([7]\) the set \(Um_2(R)/E_2(R)\) identifies in turn with \(SL_2(R)/E_2(R) \simeq SK_1(R)\) by Lemma \([18]\) and the identification is given by \(r \mapsto [r]\). Consequently \(Um_2(R)/E_2(R) \simeq Um_2(R)/G_2(R)\) and it follows that \(\mathfrak{e}_3(G) = |SK_1(R)|\). Since \(R\) is a domain, Lemma \([20]\) \(ii\) applies so that \(R\) is a characteristic subgroup of \(G\). As the complex modulus of \(\alpha \in \mathbb{Z}[\zeta_N]\) is greater than 1, the integer \(\alpha\) is not a unit of \(\mathbb{Z}[\zeta_N]\). Therefore \(\alpha\) and \(\alpha^{-1}\) have distinct minimal polynomials over \(\mathbb{Q}\) and hence \(\alpha \mapsto \alpha^{-1}\) doesn’t induce a ring endomorphism of \(R\). By Lemma \([19]\) the group \(A(C)\) is trivial and reasoning as in the proof of Theorem \([9]\) \(iii\), we obtain a bijective map \(\text{Aut}(G) \setminus V_3(G)/\text{Aut}(F_3) \to SK_1(R)\). In particular, \(\mathfrak{e}_3(G) = |SK_1(R)|\). We eventually observe that \(|SK_1(R)| = N\) by Proposition \([7]\) which completes the proof. \(\square\)

**Example 1.** Let \(\alpha = 1 + 3\zeta_3\) and \(R = \mathbb{Z}[\alpha^{\pm 1}]\). Reasoning as in the proof of Theorem \([10]\), we can show that \(E_2(R)\) is normal in \(SL_2(R)\) and \(SL_2(R)/E_2(R) \simeq \)
SK_1(R) \simeq \mathbb{Z}_2. As the quadratic residue
\left(1 + \frac{3 + 9\sqrt{-3}}{2}\right)_2
is -1, the Mennicke symbol \([3, 1 + \frac{3 + 9\sqrt{-3}}{2}]\) generates SK_1(R). Therefore G = 
R \rtimes_{\alpha} \mathbb{Z} has exactly two Nielsen equivalence classes of generating triples which
coincide with the distinct T_3-systems of (r, a) for r \in \{(1, 0), (3, 1 + \frac{3 + 9\sqrt{-3}}{2})\}.
In contrast G has only one Nielsen class of generating pairs since the units of
R are easily seen to be generated by -1 and \alpha.

5. Baumslag-Solitar groups, split metacyclic groups and
lamplighter groups

This section is dedicated to the proofs of the Corollaries F, G, H and I. We
consider first the soluble Baumslag-Solitar group BS(1, l) = \mathbb{Z}[1/l] \rtimes l \mathbb{Z}
with l \in \mathbb{Z} \setminus \{0\}.

Proof of Corollary F (i). Let G = BS(1, l). By Theorem 7, we have
n_2(G) = |R^\times/\langle \pm \alpha \rangle| with R = \mathbb{Z}[1/l] and \alpha = l. The prime divisors of n
form a basis of a free Abelian subgroup of R^\times of index 2. Thus n_2(G) is finite
if only if l = \pm p^d for some prime p and some d \geq 0. If d = 0, then R = \mathbb{Z} and
clearly n_2(G) = 1. Otherwise n_2(G) = |\langle \pm p^d \rangle/\langle \pm p^d \rangle| = d.

(ii). By Theorem 9 (equivalently Brunner’s theorem [Bru74]) we have
t_2(G) = 1. Since \mathbb{Z}[1/l] is Euclidean, it follows from Theorem C (iii) that
n_3(G) = 1.

We consider now split metacyclic groups, i.e., semi-direct products of the
form G = \mathbb{Z}_k \rtimes_{\alpha} \mathbb{Z}_l with k, l \geq 0 and \alpha \in \mathbb{Z}_k^\times.

Proof of Corollary G (i). By Theorem 7, we have
n_2(G) = |(R/\nu(G)R)^\times/\langle \pm \alpha \rangle|
with R = \mathbb{Z}_k. Thus n_2(G) = \frac{2\lambda}{\omega} follows from the definitions of \lambda and \omega.

(ii). By Theorem 9 we have t_2(G) = 1. Since \mathbb{Z}_k is a GE-ring, it follows from Theorem C (iii) that n_3(G) = 1.

(iii). Let \mathbf{g} \in V_2(\mathbb{Z}_l). Since Aut(F_2) acts transitively on \mathbb{Z}_l by Theorem
2 the number of preimages of \mathbf{g} in V_2(G) with respect to the abelianization homomorphism
\pi_{ab} does not depend on \mathbf{g}. Hence it suffices to compute this
number for \mathbf{g} = (\overline{b}, a) where \overline{b} denotes the image of b, the canonical generator
of \mathbb{Z}_k, in \mathbb{Z}_e \simeq \mathbb{Z}_k/(1 - \alpha)\mathbb{Z}_k. A generating pair \mathbf{g} \in V_2(G) which maps to
(\overline{b}, a) via \pi_{ab} is of the form (r, sa) with r \in 1 + (1 - \alpha)\mathbb{Z}_k and s \in (1 - \alpha)\mathbb{Z}_k. It
follows from Lemma 8 (i) that a pair of this form generates G if and only if r is,
in addition, a unit. By Lemma 16 the natural map \mathbb{Z}_k \rightarrow \mathbb{Z}_k/(1 - \alpha)\mathbb{Z}_k \simeq \mathbb{Z}_e
induces an epimorphism $\mathbb{Z}_k^\times \twoheadrightarrow \mathbb{Z}_l^\times$. Therefore the number of preimages of $\mathbb{G}$ is $\frac{k\nu(k)}{e\nu(e)}$. As a result, we obtain $|V_2(G)| = \frac{k\nu(k)}{e\nu(e)} |V_2(\mathbb{Z}_l)|$. Since $t_2(G) = 1$, the automorphisms of $G$ acts transitively on the set $V_2(G)/\text{Aut}(F_2)$. Therefore the Nielsen equivalence classes of generating pairs have the same number of elements given by the ratio $\frac{|V_2(G)|}{n_2(G)}$. \hfill \Box

We consider eventually the two-generated lamplighter groups, i.e., the restricted wreath products of the form $G = \mathbb{Z}_k \wr \mathbb{Z}_l$ with $k, l \geq 0$ and $k, l \neq 1$. Such a group $G$ reads also as $G = R \times_a C$ with $C = \mathbb{Z}_k = \langle a \rangle$ and $R = \mathbb{Z}_k[C] \cong \mathbb{Z}_k[X^\pm 1]/(X^l - 1)\mathbb{Z}_k[X^\pm 1]$. As before, we denote by $T$ the subgroup of $R^\times$ generated by $-1$ and $a$. We also set $\Lambda = R/\nu(G)R$ and $T_\Lambda = \pi_\nu(G)R(T)$, like in Section 4.1.

**Corollary 6.** Let $k, l \geq 0$ with $k, l \neq 1$ and let $G = \mathbb{Z}_k \wr \mathbb{Z}_l$. Then the following hold.

1. $t_2(G) = 1$.
2. If $\mathbb{Z}_k$ is finite or $\mathbb{Z}_l$ is infinite, then $n_2(G) = |\Lambda^\times /T_\Lambda|$.
3. If $\mathbb{Z}_k$ or $\mathbb{Z}_l$ is finite, then $n_3(G) = 1$.

**Proof.** (i). If $\mathbb{Z}_k$ is finite, or $\mathbb{Z}_l$ is infinite, then $t_2(G) = 1$ by Theorem 7.ii. Otherwise, Theorem 7.ii applies and $t_2(G) \leq |\text{Aut}(C)/A'(C)|$. It is easy to see that $X \mapsto X^i$ induces a ring automorphism of $R$ for every $i$ coprime with $l$. Thus $A'(C) = \text{Aut}(C)$ by Lemma 19 which implies $t_2(G) = 1$.

(ii). This is an immediate consequence of Theorem 7.ii.

(iii). If $\mathbb{Z}_k$ is finite then $R$ is a GE-ring by Lemma 1. If $\mathbb{Z}_l$ is finite then $R$ is GE-ring by Theorem 1. Therefore $n_3(G) = 1$ by Theorem 8.ii. \hfill \Box

**Corollary 7.** Assume that both $\mathbb{Z}_k$ and $\mathbb{Z}_l$ are finite and non-trivial. Given a prime divisor $p$ of $k$, we denote by $\nu_1(p, d)$ the number of distinct irreducible factors of

$$1 + X + \cdots + X^{l-1}$$

in $\mathbb{Z}_p[X]$ which are monic of degree $d$. Let $l' = 2l$ if $k \neq 2$, $l' = l$ otherwise. Then we have

$$n_2(\mathbb{Z}_k \wr \mathbb{Z}_l) = \frac{k^{l-1}}{l'} \prod_{p,d} (1 - \frac{1}{p^d})^{\nu_1(p, d)}$$

where $p$ ranges over the prime divisors of $k$ and $d$ over the positive integers.

**Proof.** Since $\nu(G)R = \mathbb{Z}_k \nu(G)$, the ring $\Lambda$ has $k^{l-1}$ elements. A finite ring is in particular Artinian, therefore $\text{nil}(\Lambda) = \mathcal{J}(\Lambda)$ is the product of all maximal ideals. Each maximal ideal $\mathfrak{m}$ is generated by a prime divisor $p$ of $k$ and the image in $\Lambda$ of a polynomial $P \in \mathbb{Z}_k[X]$ whose reduction modulo $p$ is an
irreducible monic factor of $1 + X + \cdots + X^{l-1}$. Hence $\Lambda/\mathfrak{m} = \mathbb{F}_{p^d}$ where $d$ is the degree of $P$. We deduce from the Chinese Remainder Theorem that

$$| (\Lambda / \text{nil}(\Lambda))^\times | = \prod_{p,d} (p^d - 1)^{\nu_{\mathfrak{m}}(p,d)}.$$ 

Thus

$$| \Lambda^\times | = | (\Lambda / \text{nil}(\Lambda))^\times | | 1 + \text{nil}(\Lambda) | = k^{l-1} \prod_{p,d} (1 - \frac{1}{p^d})^{\nu_{\mathfrak{m}}(p,d)}.$$ 

We conclude the proof in observing that $l' = |T_\Lambda|$. \hfill \Box

Given a prime divisor $p$ of $k$, we denote by $\mu_{l'}(p,d)$ the number of distinct irreducible factors of $1 - X^l$ in $\mathbb{Z}_p[X]$ which are monic of degree $d$. Reasoning as above, it is straightforward to establish the formula

$$| (\mathbb{Z}_k[\mathbb{Z}_l])^\times | = k^l \prod_{p,d} (1 - \frac{1}{p^d})^{\mu_{l'}(p,d)}.$$ 

where $p$ ranges over the prime divisors of $k$.

For $p$ a prime, we denote by $\lambda_p(l)$ the $p$-adic valuation of $l$. By [DG99, Remark 3.1], we have

$$| V_2(\mathbb{Z}_l) | = \prod_p p^{2(\lambda_p(l) - 1)}(p^2 - 1).$$ 

where $p$ ranges over the prime divisors of $l$.

**Corollary 8.** Assume that both $\mathbb{Z}_k$ and $\mathbb{Z}_l$ are finite and non-trivial. Then we have

$$| V_2(\mathbb{Z}_k \wr \mathbb{Z}_l) | = \frac{k^{l-1}}{\varphi(k)} | (\mathbb{Z}_k[\mathbb{Z}_l])^\times | | V_2(\mathbb{Z}_l) |$$ 

and the number of elements in a Nielsen equivalence class of generating pairs is

$$k^{l-1} l' | V_2(\mathbb{Z}_l) |.$$ 

where $l'$ is as in Corollary 7.

**Proof.** Let $G = \mathbb{Z}_k \wr \mathbb{Z}_l$ and let $\mathfrak{g} \in V_2(\mathbb{Z}_l)$. Reasoning as in the proof of Corollary 7 we see that the number of preimages of $\mathfrak{g}$ in $V_2(G)$ with respect to the abelianization homomorphism $\pi_{ab}$ does not depend on $\mathfrak{g}$. Let us compute this number for $\mathfrak{g} = (\overline{b}, a)$ where $\overline{b}$ denotes the image of $b$ in $\mathbb{Z}_k$. Using Lemma 8, we see that a lift $\mathfrak{g} \in G^2$ of $(\overline{b}, a)$ generates $G$ if and only if it is of the form $(r, sa)$ with $r \in (1+(1-\alpha)\mathbb{Z}_k[\mathbb{Z}_l]) \cap (\mathbb{Z}_k[\mathbb{Z}_l])^\times$ and $s \in (1-\alpha)\mathbb{Z}_k[\mathbb{Z}_l]$. By Lemma 16, the natural map $\mathbb{Z}_k[\mathbb{Z}_l] \to \mathbb{Z}_k$ induces an epimorphism $(\mathbb{Z}_k[\mathbb{Z}_l])^\times \to \mathbb{Z}_k^\times$. 


Therefore the number of preimages of $\mathfrak{g}$ is $\frac{\left([\mathbb{Z}_k[X]]^\times\right)^{k^{l-1}}}{\mathbb{Z}_k^\times}$, which proves the first formula.

Since $t_2(G) = 1$, the automorphism group of $G$ acts transitively on the set $V_2(G)/\text{Aut}(F_2)$. Therefore the Nielsen equivalence classes of generating pairs have the same number of elements and this number is the ratio
\[ \frac{|V_2(G)|}{n_2(G)} = \frac{kl'}{\varphi(k)} \prod_{p,d}(1 - \frac{1}{p^d})^\mu(p,d) - \nu(p,d) |V_2(\mathbb{Z}_t)| \]
where $p$ ranges over the prime divisors of $k$. The integer $\mu_l(p, d) - \nu_l(p, d)$ is the number of monic irreducible polynomials in $\mathbb{Z}_p[X]$ of degree $d$ which divides $1 - X^l$ but not $1 + X + \cdots + X^{l-1}$. Therefore $\mu_l(p, d) - \nu_l(p, d) = 1$ if $d = 1$ and it cancels otherwise. Thus we have $\prod_{p,d}(1 - \frac{1}{p^d})^\mu(p,d) - \nu(p,d) = \prod_p(1 - \frac{1}{p}) = \frac{\varphi(k)}{k}$, which gives the result.

**Corollary 9.** Let $k, l \geq 0$ and $k, l \neq 1$.

(i) Assume that $\mathbb{Z}_k$ is finite and $\mathbb{Z}_l$ is infinite. Then $n_2(G)$ is finite if and only if $k$ is prime; in this case $n_2(G) = \max(\frac{k-1}{2}, 1)$.

(ii) Assume that $\mathbb{Z}_k$ is infinite and $\mathbb{Z}_l$ is finite. Then $n_2(G)$ is finite if and only if $l \in \{2, 3, 4, 6\}$; in this case $n_2(G) = 1$.

**Proof.** (i). The result follows from Corollary 6 and the isomorphisms
\[ (\mathbb{Z}_k[X^\pm 1])^\times \cong \mathbb{Z}_k^\times \times U_X \times U_{X-1} \times \mathbb{Z}^\rho. \]

\[ T \cong \{\pm 1\} \times \{1\} \times \{1\} \times \mathbb{Z}. \]

where $\rho$ is the number of prime divisors of $k$ and $U_Y = 1 + Y \text{nil}(\mathbb{Z}_k)[Y]$ with $Y \in \{X^\pm 1\}$ (see e.g., [Lam06, Exercise 3.17] where the units in the ring of Laurent polynomials are determined).

(ii). As $\Lambda = \mathbb{Z}[X]/(1 + X + \cdots + X^{l-1})\mathbb{Z}[X]$, the ring $\Lambda$ identifies with $\mathcal{O}(\mathcal{D} \setminus \{1\})$ where $\mathcal{D}$ is the set of divisors of $l$. By Lemma 6 the group $\Lambda^\times$ is finite if and only if $l \in \{2, 3, 4, 6\}$; in this case the equality $\Lambda^\times = T_\Lambda$ holds.

Since $n_2(G) = \max(\varphi(l)/2, 1) |\Lambda^\times/T_\Lambda|$ by Theorem 7, the result follows.

**Corollary 10.** Let $G = \mathbb{Z} \wr \mathbb{Z}$. Then we have $n_3(G) \leq 2t_3(G)$ and the following are equivalent:

(i) $n_3(G) = 1$.

(ii) $t_3(G) = 1$.

(iii) The ring $\mathbb{Z}[X^\pm 1]$ is a $\text{GE}_2$-ring.

**Proof.** It follows from Lemma 20 that $R = \mathbb{Z}[X^\pm 1]$ is characteristic in $G$. The inequality $n_3(G) \leq 2t_3(G)$ is then a consequence of Theorem 9.

The implication (i) $\Rightarrow$ (ii) is obvious while the equivalence (i) $\Leftrightarrow$ (iii) results from Theorem 3 and Lemma 4.
In order to prove \((ii) \Rightarrow (i)\), fix \(g_0 \in V_2(G)\) and let \(g\) be an arbitrary generating triples of \(G\). As \(t_3(G) = 1\), we deduce that \(g\) is Nielsen equivalent to a triple of the form \((1_G, g_1)\) with \(g_1 \in V_2(G)\). By Corollary \(\text{C} ii\), we have \(n_2(G) = 1\) so that \((1_G, g_1)\) is Nielsen equivalent \((1_G, g_0)\). Therefore \(n_3(G) = 1\).

\[\square\]

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