Abstract

We consider a set of gauge invariant terms in higher order effective Lagrangians of the strongly interacting scalar of the electroweak theory. The terms are introduced in the framework of the hidden gauge symmetry formalism. The usual gauge term is known to stabilize the skyrmion but only in the large vector mass limit. We find that adding higher-order gauge terms is insufficient to insure stability. We then proceed to analyze other gauge invariant interaction terms. Some of the conclusions also apply to QCD skyrmions.

PACS number(s): 11.30.Na, 11.30.Rd, 12.15.Cc
I. INTRODUCTION

Although there has been an enormous amount of theoretical work on the scalar sector of the electroweak theory \cite{1}, it remains the part of the Standard Model which is the worst known and this situation will most probably prevail until we reach accelerator energies that cover scales well over the electroweak symmetry breaking scale. The minimal description proposed in the Standard Model relies on spontaneous symmetry breaking but even in this scheme two possibilities arise: either there is a scalar particle with mass below 1 TeV (otherwise unitarity is violated in $2 \rightarrow 2$ scattering) or the Higgs sector becomes strongly interacting. In the limiting case where the mass of the Higgs tends to infinity, the scalar sector (Goldstone particles) becomes equivalent to the gauged non-linear $\sigma$-model \cite{2}. The analogy with low-energy hadron physics suggests the addition of Skyrme-like \cite{3} terms to an effective Lagrangian which in turn could induce new states, the so-called weak skyrmions. Weak skyrmions are not exclusive to the spontaneous symmetry breaking mechanism. Indeed, the analogy between QCD and Technicolor is even more intuitive; in this case the weak skyrmions could be identified to technibaryons \cite{4}.

The stabilizing term suggested by Skyrme is added in the context of an effective theory of scalar fields. (The scalar fields describe pions in QCD and Goldstone bosons in the Weinberg-Salam theory). Several approaches have been proposed to stabilize the soliton. The most obvious approach, taken by Skyrme, is to add just the higher order term in the derivatives of the fields which is needed for stability. On the other hand, one could consider a very general effective Lagrangian similar to that proposed by Gipson and Tze \cite{5} some time ago and look at the stability properties of the static solution. Most of the work in that direction is limited to order $O(\partial^4)$ terms since it becomes increasingly difficult to keep track of all possible terms as the number of derivatives increases. Both of these approaches involve a certain degree of arbitrariness and lack an underlying principle that would justify the choice of terms. The hidden gauge symmetry (HGS) formulation proposed by Bando et al. \cite{6} is more elegant in that respect. A new gauge symmetry is introduced. The symmetry is spontaneously broken giving a mass to the new vector field and it turns out that in the limit where this mass goes to infinity, the gauge field contribution is equivalent to a stabilizing Skyrme term \cite{7}. The stabilization is effective however only in that limit \cite{8}.

Our purpose in this paper is to examine the stability properties of generalized Lagrangian based on the HGS formalism. We will first consider higher order gauge terms. We will require that these terms be gauge invariant in order to retain the spontaneous nature of the gauge symmetry breaking. They will be generally constructed out of covariant derivatives. The motivations behind these new objects are the following: (a) The terms can be viewed as new higher-order gauge terms of some sort and/or as counterterms when they are dynamically generated. \cite{8} (b) As mentioned above, the terms are not responsible for symmetry breaking. (c) Some combinations of gauge terms (e.g. polynomials of $F_{\mu\nu}$’s) are especially interesting to many respect. (d) The large vector mass limit leads to an effective scalar Lagrangian with a Skyrme term. These new terms will contribute to higher orders in the derivative of the scalar fields. Previous works \cite{6,9,10} have shown that skyrmions exist for certain classes of all-orders effective Lagrangian. (e) The HGS does not stabilize the soliton in general but that there is a distinct possibility that the new gauge terms will help. The second part of our analysis will consider higher-order gauge-invariant interaction terms.
We begin by a description of a strongly-interacting scalar sector of the Weinberg-Salam theory and of the hidden gauge symmetry formulation. In section 3, the stability of the weak skyrmion is analyzed with respect to the HGS. We proceed to prove that the higher-order gauge terms are not sufficient to stabilize the soliton using arguments similar to that of Ezawa and Yanagida [8]. Section 4 is devoted to other higher-order gauge-invariant terms, the interaction terms, and contains final remarks.

II. THE SCALAR SECTOR AND HIDDEN GAUGE SYMMETRY

The scalar fields of the Standard Model can be written in the form a $2 \times 2$ matrix $\Phi(x)$:

$$\Phi = \sqrt{2} \begin{pmatrix} \phi^0 & -\phi^{-} \\ \phi^{-} & \phi^0 \end{pmatrix}.$$  \hspace{1cm} (1)

In this notation, the Lagrangian of the scalar sector becomes

$$L_\Phi = \frac{1}{4} \text{Tr} \, D_\mu \Phi D^{\mu} \Phi^\dagger - \lambda \left[ \frac{1}{2} \text{Tr} \, \Phi \Phi^\dagger - v^2 \right]^2$$ \hspace{1cm} (2)

where $v \sim 250 \text{ GeV}$ is the vacuum expectation value (VEV) of the scalar field, $\lambda$ is related to $v$ through $\lambda = \frac{M_H^2}{8 v^2}$ and $D_\mu$ is the covariant derivative corresponding to the $SU(2)_L \otimes U(1)_Y$ gauge interactions. When the gauge interactions in (2) are switched off (i.e. $D_\mu \rightarrow \partial_\mu$), the remaining Lagrangian possess a global symmetry under the group $G = SU(2)_L \otimes SU(2)_R$. This symmetry is in turn spontaneously broken to a diagonal subgroup $H = SU(2)_V$ since the scalar field acquire a non-vanishing VEV, $\langle \Phi(x) \rangle = v$. In this case, it is easy to separate the Higgs degree of freedom from the remaining scalar fields (Goldstone bosons) by rewriting them in terms of a real scalar field $\eta(x)$ and a matrix $U(x) \in SU(2)$ respectively such that $\Phi(x) = \eta(x)U(x)$. In the absence of gauge interactions, $L_\Phi$ becomes:

$$L_\Phi = \frac{\eta^2}{4} \text{Tr} \, \partial_\mu \eta \partial^\mu \eta^\dagger + \frac{1}{4} \text{Tr} \, \partial_\mu \eta \partial^\mu \eta - \lambda \left( \eta^2 - v^2 \right)^2.$$ \hspace{1cm} (3)

A large Higgs mass corresponds to a strong coupling $\lambda$ and in the limit $\lambda \rightarrow \infty$, $\eta$ must be set to the value of the VEV, $\langle \eta \rangle = v$, for consistency. The Higgs field decouples leaving only Goldstone bosons which take values in the quotient space $G/H$ and the Lagrangian reduces to that of the non-linear $\sigma$-model

$$L_\sigma = \frac{f^2}{4} \text{Tr} \, \partial_\mu U \partial^\mu U^\dagger$$ \hspace{1cm} (4)

with $f = v$.

Both the global symmetry group $SU(2)_L \otimes SU(2)_R$ (chiral symmetry) and the relation of the scalar sector of the theory to the non-linear $\sigma$-model suggest that the low-energy effective Lagrangian approach used in the pion sector of strong interactions can be applied here as well. The Skyrme Lagrangian is particularly interesting in that respect since it would induce weak skyrmions constructed as soliton configurations of Goldstone bosons fields.

In order to introduce a Skyrme-like effective Lagrangian for the scalar sector of the electroweak theory, we use the formulation of local hidden gauge symmetry of the non-linear $\sigma$-model [3][4]. The procedure introduce a Yang-Mills field $V_\mu$ and the scalar sector
is represented by the group $G \otimes SU(2)_V$ where $SU(2)_V$ is the gauge group. The scalar sector can then be conveniently described by two $2 \times 2$ unitary matrices $L(x) \in SU(2)_L$ and $R(x) \in SU(2)_R$ with the transformation properties

$$L(x) \rightarrow h_L L(x) h_V(x), \quad R(x) \rightarrow h_R R(x) h_V(x)$$

where $h_i \in SU(2)_i$. The most general Lagrangian for $L(x)$ and $R(x)$ up to order two in derivative of the fields is

$$\mathcal{L}_S = -\frac{f^2}{4} \text{Tr} \left[ (L^\dagger \partial_\mu L - R^\dagger \partial_\mu R)^2 - a(L^\dagger \partial_\mu L + R^\dagger \partial_\mu R)^2 \right]$$

(5)

When the $SU(2)_L \otimes U(1)_Y$ electroweak interactions and the new $SU(2)_V$ gauge interactions are included, the most general scalar Lagrangian containing only two derivatives reads:

$$\mathcal{L}_S = -\frac{f^2}{4} \text{Tr} \left[ (L^\dagger D_\mu L - R^\dagger D_\mu R)^2 - a(L^\dagger D_\mu L + R^\dagger D_\mu R)^2 \right]$$

(6)

with

$$D_\mu L = (\partial_\mu + W_\mu) L - LV_\mu, \quad D_\mu R = (\partial_\mu + Y_\mu) R - RV_\mu$$

(7)

and

$$W_\mu \equiv i g W_\mu^k \tau^k, \quad Y_\mu \equiv i g' Y_\mu^3, \quad V_\mu \equiv i g_V V_\mu^k \tau^k$$

(8)

where $g, g'$ and $g_V$ are the respective electroweak and hidden symmetry couplings, $f \sim 250 \text{ GeV}$ and $a$ is a positive real number. In $\mathcal{L}_S$ the gauge fields $V_\mu$ are not dynamical. Using the equation of motion for $V_\mu$

$$V_\mu = \frac{1}{2} \left[ L^\dagger (\partial_\mu + W_\mu) L + R^\dagger (\partial_\mu + Y_\mu) R \right]$$

leads to the gauged non-linear $\sigma$-model:

$$\mathcal{L}_{G\sigma} = \frac{f^2}{4} \text{Tr} (D_\mu U^\dagger) (D^\mu U)$$

(9)

when $U = LR^\dagger$. Note that for the $U$ fields

$$D_\mu U = (\partial_\mu + W_\mu) U - U Y_\mu$$

which means that $U$ has no $SU(2)_V$ interactions even though both $L(x)$ and $R(x)$ are interacting with the gauge fields $V_\mu$, i.e. the gauge symmetry is hidden.

The gauge bosons become dynamical when we add their kinetic term to the Lagrangian. They can be interpreted either as new fundamental gauge bosons or as dynamically generated ones in which case the kinetic term is introduced as a counterterm with respect to radiative corrections (e.g. the $\rho$-meson in QCD, techni-$\rho$ in Technicolor,...). It has the usual form

$$\mathcal{L}_X = -\frac{1}{2g_X} \text{Tr} F^X_\mu F^{X\mu\nu}$$

(10)
where $g_X$ and $F^X_{\mu\nu}$ stand for the coupling and field strength of each gauge field $X = W, Y, V$. The total Lagrangian is

$$\mathcal{L}_{tot} = \mathcal{L}_{G\sigma} + \mathcal{L}_W + \mathcal{L}_Y + \mathcal{L}_V. \quad (11)$$

Focusing on the effect of the hidden symmetry contribution, we take the limit $g_V \gg g, g'$ ($g, g' \to 0$) and neglect contributions of the gauge bosons $W$ and $Y$ from hereon. The vector bosons $V$ then receives its mass from the same mechanism as the standard gauge bosons; in this case we get $m^2_V = f^2 g^2_V a$.

### III. STABILITY OF THE SOLITON AND HIGHER ORDER GAUGE TERMS

There are no stable solitonic solution for the gauged non-linear $\sigma$-model (eq. (9)). Furthermore, Ezawa and Yanagida have shown that the hidden gauge symmetry does not stabilize the soliton in general. On the other hand, it is interesting to note that the total Lagrangian $\mathcal{L}_{tot}$ becomes equivalent to the Skyrme Lagrangian in the limit $m_V \to \infty$, $g, g' \to 0$ where it induces stable skyrmions. In the more general case where $m_V$ is finite, the solutions correspond at best to saddle points which are unstable configurations. Other types of solitonic solutions we may look for are local minima (classically stable soliton) or global minima (absolutely stable soliton). Let us first examine how these conclusions are obtained.

The equation of motion for the hidden gauge field $V_\mu$ in (11) is ($g, g' \to 0$)

$$V_\mu = \frac{1}{2}(L^\dagger \partial_\mu L + R^\dagger \partial_\mu R) + \frac{1}{m^2_V} [D^{V\nu}, [D_{V\mu}, D_{V\nu}]] \quad (12)$$

with $D^{V\alpha} = \partial_\alpha + V_\alpha$. We can use this equation to perform an expansion in terms of derivatives of the fields, then

$$\mathcal{L}_{tot} = -\frac{f^2}{4} \text{Tr} L_\mu L^\mu + \frac{1}{32g^2_V} \text{Tr} [L_\mu, L_\nu]^2 + \frac{af^2}{m^4_V} \text{Tr} [D^{V\nu}, [D_{V\mu}, D_{V\nu}]]^2 \quad (13)$$

where $L_\mu = U^\dagger \partial_\mu U$. The first and second terms in $\mathcal{L}_S$ are the non-linear $\sigma$-model Lagrangian and the Skyrme term respectively. Therefore, up to four derivatives the hidden symmetry approach leads to the Skyrme Lagrangian

$$\mathcal{L}_{Skyrme} = -\frac{f^2}{4} \text{Tr} L_\mu L^\mu + \frac{1}{32g^2_V} \text{Tr} [L_\mu, L_\nu]^2 \quad (14)$$

This result which also corresponds to the large $m_V$ limit gives a stable soliton because of the right sign of the Skyrme term. Another interesting fact, which is characteristic in this approach, is that the strength of $g_V$ determines the magnitude of the physical parameters of the weak skyrmion (size, mass, ...) for $m_V \to \infty$. It also corresponds to a vector field solution of

$$V_\mu = \frac{1}{2}(L^\dagger \partial_\mu L + R^\dagger \partial_\mu R). \quad (15)$$
However, it is easy to see that higher-order contributions in the derivative expansion are necessary and determinative for the stability of the skyrmion since they regulate the small distance behavior. The mass (or static energy) of the soliton scales as

\[ M_{\text{tot}} \equiv - \int d^3r \mathcal{L}_{\text{tot}} = c_2 R + c_4 R^{-1} + c_6 R^{-3} + c_8 R^{-5} + \ldots \]  

(16)

when \( r \) is scaled according to \( r \to rR^{-1} \) where, \( c_n \) is the coefficient corresponding to the term with \( n \) derivatives. The existence of a global minimum at a finite \( r \) (\( r \neq 0, \infty \)) in the mass indicates the presence of a stable classical soliton. The large \( R \) behavior of the mass is dominated by the first term and since \( c_2 \) is positive the size of the soliton must remain finite. Clearly, in the Skyrme Lagrangian (up to order four in derivatives), there exists a stable weak skyrmion because \( c_4 \) turns out to be positive and the series stops at this order. However for \( \mathcal{L}_{\text{tot}} \), the small distance behavior (large momentum) is dominated by the remaining terms of the series which are generated by the third term in (13). There is absolute stability if there exist a global minimum for \( M_{\text{tot}}(R) \) and if it occurs at a nonvanishing skyrmion size \( R \).

For the purpose of showing that the Lagrangian \( \mathcal{L}_{\text{tot}} \) does not induce stable solitons, we introduce the usual hedgehog ansatz for the chiral field \( U \):

\[ U = \exp (i \tau \cdot \hat{r} F(r)) \]  

(17)

and the most general spherically symmetric parameterization for the vector field:

\[ V_0 = 0, \quad V_m = i \tau^n (\delta_{mn} \frac{H(r)}{2r} + \hat{r}_m \hat{r}_n \frac{K(r)}{2r} + \epsilon_{mnp} \hat{r}_p G(r)) \]  

(18)

where \( \tau^n \) are the Pauli matrices, \( F(r) \), \( G(r) \), \( H(r) \) and \( K(r) \) are functions of \( r \), \( \delta_{mn} = \delta_m r_n \) and \( \hat{r} \) is the unit vector. Both expressions (17) and (18) are invariant under the combined rotations of spin and isospin. Substituting the last expression in (16) leads to the static energy of the system

\[ M_{\text{tot}} = M_- + M_+ + M_V, \quad \text{with} \]

\[ M_- = \frac{2 \pi f^2}{m_V} \int d\rho \frac{2 \sin^2 F}{\rho^2} \left[ \frac{2 \sin^2 F}{\rho^2} + F'^2 \right] \]

\[ M_+ = \frac{2 \pi a f^2}{m_V} \int d\rho \left[ 2(1 - \cos F - G)^2 + 2H^2 + 2K^2 \right] \]  

(19)

\[ M_V = \frac{2 \pi a f^2}{m_V} \int d\rho \left[ \frac{(G^2 - 2G + H^2)^2}{\rho^2} + 2 \left( G' - \frac{HK}{\rho} \right)^2 + 2 \left( H' + \frac{GK - K}{\rho} \right)^2 \right] \]

where we use the dimensionless variable \( \rho = m_V r \) (recall that \( m_V^2 = f^2 g^2 a \)). \( M_V \) is the contribution of the gauge field kinetic term \( \mathcal{L}_V \) and \( M_- \) and \( M_+ \) correspond to the first and second term of the expression (3) for \( \mathcal{L}_S \). By comparison, the Skyrme Lagrangian involves no gauge field and the static energy is given by

\[ M_{\text{Skyrme}} = \frac{2 \pi f^2}{m_V} \int d\rho \frac{2 \sin^2 F}{\rho^2} \left[ \frac{2 \sin^2 F}{\rho^2} + F'^2 \right] + \frac{2 \pi f^2}{m_V} \frac{\sin^2 F}{\rho^2} \left( \frac{\sin^2 F}{\rho^2} + 2F'^2 \right) \]
The static energy in (19) shows a discrete symmetry under the transformation $(F, G, H, K) \rightarrow (F, G, -H, -K)$ and clearly the static solution will be parity conserving only for $H(r) = K(r) = 0$ otherwise it will break parity spontaneously. The $N = 1$ topological soliton that we are interested in corresponds to the boundary conditions $F(0) = \pi$, $G(0) = 2$, $H(0) = K(0) = 0$, and $F(\infty) = G(\infty) = H(\infty) = K(\infty) = 0$.

However, if we consider a more specific solution

$$G(r) = g(r) \sin^2 \theta, \quad H(r) = g(r) \sin \theta \cos \theta, \quad K(r) = 0$$

where $\theta$ is a constant parameter, the static energies reduce to

$$M_+ = \frac{4\pi a f^2}{m_V} \int d\rho \left[ (1 - \cos F)^2 + \left( g^2 - 2g(1 - \cos F) \right) \sin^2 \theta \right]$$

$$M_V = \frac{2\pi a f^2}{m_V} \int d\rho \left[ \frac{g^2(g - 2)^2}{\rho^2} + 2g^2 \right] \sin^2 \theta.$$

For $\theta = \pi/2$, the vector field becomes identical to the monopole ansatz

$$V_0 = 0, \quad V_m = i \tau \epsilon_{mnp} r^2 \frac{G(r)}{2r}, \quad (21)$$

but decreasing $\theta$ from $\pi/2$ to 0 continuously eliminates smoothly the higher order terms in $M_{tot}$ with the final result

$$M_{tot}(\theta = 0) = \frac{2\pi f^2}{m_V} \int d\rho \rho^2 \left[ \left( \frac{2\sin^2 F}{\rho^2} + F^2 \right) + \frac{2a}{\rho^2} (1 - \cos F)^2 \right]$$

The global minimum in the static energy (mass of the soliton) is found at soliton size $R = 0$. In other words, a static solution for a gauge field of the form (18) will decay (e.g. sphaleron) and in the absence of any restraining term, the soliton will shrink to zero size and disappear. One may wonder how does this relates to the result obtained for large $m_V$ limit (i.e. stable skyrmions)? When $m_V \rightarrow \infty$, the vector field hardly propagates and it is forced into the static configuration (17) which has the monopole form (21).

Note that the configuration in (20) does minimize $M_{tot}$ only for $\theta = 0$ or $\pi/2$. Indeed, minimizing $M_{tot}$ with respect to $K$ leads to the equation of motion

$$K = \rho \frac{H(G - 1)' - H'(G - 1)}{H^2 + (G - 1)^2 + \rho^2/2} = \rho \frac{g' \sin \theta \cos \theta}{g(g - 2) \sin^2 \theta + 1 + \rho^2/2}.$$  

instead of $K = 0$ in (20). Otherwise the conclusion remains intact.

Consider now a higher-order gauge Lagrangian $\tilde{L}_V$. We will be interested in contributions to $\tilde{L}_V$ which can be expressed as polynomials of $F^{\mu\nu}$’s.

$$\text{Tr} \left( F^{V\mu} F^{V\mu} \right), \quad \text{Tr} \left( F^{V\mu} F^{V\nu} F^{V\lambda} F^{V\mu} \right), \quad \text{Tr} \left( F^{V\mu} F^{V\nu} \right)^2, \quad \text{Tr} \left( F^{V\mu} F^{V\nu} F^{V\lambda} F^{V\rho} F^{V\mu} \right), \quad \text{etc...}$$

(22)

These terms are a natural generalization of the gauge field kinetic term which can be used to build an effective Lagrangian. They are, of course, manifestly gauge invariant. Our
interest here lies mainly in the fact that in the limit \( m_V \to \infty \), the field strength tends to 
\[
F_{\mu\nu}^V = [D_{\nu}, D_{\nu\nu}] \to -\frac{1}{4} L[L_{\mu}, L_{\nu}]L = -\frac{1}{4} L f_{\mu\nu}L.
\] Consequently, they also have the same \( SU(2) \) and Lorentz structure as the objects analyzed in refs. [9] and [12].

Let us recall some of these results. For the hedgehog static solution one can write
\[
L_0 = 0, \quad L_m = i\tau^n(\alpha(r)\delta_{mn}^T + \beta(r)\bar{r}_m\bar{r}_n + \gamma(r)\epsilon_{nmp}\bar{r}^p)
\]
where \( \alpha, \beta \) and \( \gamma \) are
\[
\alpha(r) = \frac{\sin F \cos F}{r}, \quad \beta(r) = F', \quad \gamma(r) = \frac{\sin^2 F}{r}.
\]
The commutator takes the general form
\[
f_{ij} = -2i\tau^n \left[ \alpha^2\epsilon_{ijk}\delta_k^m + (\alpha^2 + \gamma^2)\epsilon_{ijk}\bar{r}_k\bar{r}_n + \beta\gamma(\delta_{jn}^T\bar{r}_i - \delta_{in}^T\bar{r}_j) + \alpha(\beta - \alpha)(\epsilon_{ikn}\bar{r}_j - \epsilon_{jkn}\bar{r}_i) \right].
\] (24)
The calculations in ref. [12] led to a number of properties of trace such as those in (22). First, all traces are found to be polynomials of the following combinations, \( a_- = (\alpha^2 + \gamma^2) \) and \( b_- = \beta^2 \). They can be written as
\[
\text{Tr} (f_{\mu\nu})^n = \text{const} \cdot \sum_{m=0}^{[n]} \kappa_m a_-^{n-m}(b_- - a_-)^m
\] (25)
where \((f_{\mu\nu})^n\) is a generic form for any combination of \( n \) \( f_{\mu\nu} \)'s, \([k]\) is the integer part of \( k \) and \( \kappa_0/\kappa_1 \) is found to be 3/\( n \). For example,
\[
\text{Tr} f_{\mu\nu}f^{\mu\nu} = 16a_-[a_- + 2b_-], \quad \text{Tr} f_{\mu\nu}f^{\nu\lambda}f_\lambda^\mu = 96a_-^2b_-,...
\]
Furthermore, one can construct a special class [9] of such combinations which is at most linear in \( b_- \) (or of degree two in derivatives of \( F \)). These Lagrangians give a very simple form
\[
\text{Tr} (f_{\mu\nu})^n = \text{const} \cdot a_-^{n-1}[3a_- + n(b_- - a_-)]
\] (26)
and lead to a chiral angle equation which is tractable since it is of degree two, despite the fact that the former involve \( 2n \) derivatives terms. It also turns out that for this class of Lagrangians \( \text{const} = 0 \) for \( n \) odd \( \geq 5 \).

Let us now transpose these results to an arbitrary operator \( \zeta_\mu \) and the commutator
\( \zeta_{\mu\nu} \equiv [\zeta_\mu, \zeta_\nu] \) and consider a Lagrangian from the commutators \( \zeta_{\mu\nu} \equiv [\zeta_\mu, \zeta_\nu] \)
\[
\tilde{L} = a_2 \text{Tr} (\zeta_{\mu\nu}\zeta^{\mu\nu}) + a_3 \text{Tr} (\zeta_\mu^\nu\zeta_\nu^\lambda\zeta_\lambda^\mu) + a_4 \text{Tr} (\zeta_{\mu\nu}\zeta^{\mu\nu})^2 + \cdots
\] (27)
It is easy to see that any \( SU(2) \) commutator with the same Lorentz structure as (24) may be written as:
\[
\zeta_{ij} \equiv [\zeta_i, \zeta_j] = -2i\tau^n \left[ \alpha_1\epsilon_{ijk}\delta_k^m + \alpha_2\epsilon_{ijk}\bar{r}_k\bar{r}_n + \alpha_3(\delta_{jn}^T\bar{r}_i - \delta_{in}^T\bar{r}_j) + \alpha_4(\epsilon_{ikn}\bar{r}_j - \epsilon_{jkn}\bar{r}_i) \right].
\]
(28)
with $\alpha_i \equiv \alpha_i(r)$ and for practical purposes here we set $\zeta_0 = 0$. It is then easy to show that (25) and (26) apply to the $\zeta_{\mu\nu}$ commutators by substituting $f_{\mu\nu} \to \zeta_{\mu\nu}$ and

$$a_- \to a_\zeta = \alpha_2 \quad \text{and} \quad a_- b_- \to a_\zeta b_\zeta = (\alpha_1 + \alpha_4)^2 + \alpha_3^2.$$  

In general, a Lagrangian constructed from all orders of $\zeta_{\mu\nu}$ leads to a static energy which can be written as:

$$\tilde{M} \equiv - \int d^3 r \tilde{\mathcal{L}} = 4\pi \int r^2 dr \sum_{n=1}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \kappa_{n,m} a_\zeta^n b_\zeta^m (b_\zeta - a_\zeta)^m$$  

(29)

where according to (26), $\kappa_{n,0}/\kappa_{n,1} = 3/n$. The special class of combinations which is at most linear in $b_\zeta$ gives an even simpler form since $\kappa_{n,m} = 0$ for $m > 1$:

$$\tilde{M} = 4\pi \int r^2 dr \left[3\chi(a_\zeta) + (b_\zeta - a_\zeta)\chi'(a_\zeta)\right]$$  

(30)

where $\chi(x) = \sum_{n=1}^{\infty} \kappa_n x^n$ and $\chi'(x) = d\chi(x)/dx$. Note that for $\tilde{M}$ to be positive, it is sufficient to impose the conditions $\kappa_{n,m} \geq 0$ and $(b_\zeta - a_\zeta) \geq 0$.

Let us now consider the higher-order gauge terms of (22). The field strength in those expressions have the same commutator structure as in (28) and indeed one finds that

$$F_{ij}^V = -2i\tau^n \left[\alpha_1 \epsilon_{ijk} \delta_r^{\kappa_n} + \alpha_2 \epsilon_{ijk} \tilde{r}_k \tilde{r}_n + \alpha_3 (\delta^{T}_{jmn} \tilde{r}_i - \delta^{T}_{jnm} \tilde{r}_i) + \alpha_4 (\epsilon_{ikm} \tilde{r}_k \tilde{r}_j - \epsilon_{ijkm} \tilde{r}_k \tilde{r}_j)\right].$$  

(31)

with

$$\alpha_1 = \frac{2G + H^2}{2r^2}, \quad \alpha_2 = \frac{G(G - 2) + H^2}{2r^2}, \quad \alpha_3 = \frac{rH' + (G - 1)K}{2r^2}, \quad \alpha_4 = \frac{2G - rG' - H(H - K)}{2r^2}.$$  

We can then use the above results for the gauge field case by substituting $\zeta_{ij} \to F_{ij}^V$ and $a_\zeta$, $b_\zeta \to a_V$, $b_V$ with

$$a_V = \frac{G^2 - 2G + H^2}{2r^2} \quad \text{and}$$

$$a_V b_V = \frac{1}{4r^4} \left[(rG' - HK)^2 + (rH' + (G - 1)K)^2\right].$$  

(32)

In general, an all-order Lagrangian $\tilde{\mathcal{L}}_V$ constructed from all powers of $F_{\mu\nu}^V$ leads to a static energy which can be written:

$$\tilde{M}_V \equiv - \int d^3 r \tilde{\mathcal{L}}_V = 4\pi \int r^2 dr \sum_{n=1}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \kappa_{n,m} a_V^n b_V^m (b_V - a_V)^m$$  

(33)

or, according to (30), the special class of Lagrangians which leaves the degree of the chiral angle equation equal to two:

$$\tilde{M}_V = 4\pi \int r^2 dr \left[3\chi_V(a_V) + (b_V - a_V)\chi'_V(a_V)\right].$$  

(34)
It is interesting to note that, even away from the limit \( m_V \to \infty \), \( \tilde{M}_V \) is only quadratic in the derivatives and this leads to equations with an overall degree of two. Some examples of stable solitons were given in ref. [10]; they correspond to the limit \( m_V \to \infty \) of (34).

We would like to address here the question of absolute stability and the presence of a non trivial global minimum in the static energy. In other words, are the gauge field terms introduced in (22) sufficient to stabilize the soliton? We use the same argument as above to prove that this is not the case. Consider the gauge field configuration in (20). The functions \( a_V \) and \( b_V \) become

\[
a_V = \frac{\sin^2 \theta}{2r^2} g(g - 2), \quad a_V b_V = \frac{g^2}{4r^2} \sin^2 \theta.
\]

The static energy

\[
\tilde{M}_V = 4\pi \int r^2 dr \sum_{n=1}^{\infty} \sum_{m=0}^{n-2} \kappa_{n,m} \left[ \frac{\sin^2 \theta}{2r^2} \right]^{n-m} [g(g - 2)]^{n-2m} \left[ \frac{g^2}{2} - \frac{\sin^2 \theta}{2r^2} g^2(g - 2)^2 \right]^m
\]

coming from the gauge terms, falls monotonously as \( \theta \) goes from \( \pi/2 \) to 0. Therefore, all gauge terms contributions vanish for \( \theta = 0 \) leaving no stabilizing terms in \( M_{tot} \). Given a soliton configuration, it will tend to shrink to zero size and vanish. For this kind of generalized Lagrangian, there remains a possibility to find other solutions which lead to the same contribution to the static energy \( \tilde{M}_V = 0 \). This, of course, depends on the specific Lagrangian (more precisely the set of parameters \( \kappa_{n,m} \)) but in general, it would lead to non-trivial contributions from \( G, H \) and \( K \) in \( M_+ \) in (19). Unfortunately, it is easy to see that the same conclusions regarding the stability apply for this case as well.

We note that even for arbitrary functions \( G, H \) and \( K \), the quantities \( a_V \) and \( b_V \) are independent of the derivatives of \( K \). Indeed, recalling for example the results of (32) and (34), we find that when we consider only the static energy \( \tilde{M}_V \), it is minimized for

\[
K = \frac{H(G - 1)' - H'(G - 1)}{H^2 + (G - 1)^2}.
\]

This solution coincides with the field configuration in (20) only in the limits \( \theta = 0, \pi/2 \) but these are just the relevant limits we are interested in. In any case, rewriting \( a_V \) and \( b_V \) with the above substitution leads to

\[
a_V = \frac{J^2 - 1}{2r^2} \quad \text{and} \quad a_V b_V = \frac{1}{4r^2} J^2,
\]

where \( J^2 \equiv H^2 + (G - 1)^2 \). The consequence of this last relation is that it is possible to minimize the static energy for the gauge term with respect to the function \( J \) alone independently of the specific choice for \( H \) or \( G \). One could chose \( H = 0 \) (or \( G = 1 \)) without loss of generality in which case \( K = 0 \) and \( J = G - 1 \) (or \( J = H \)). This symmetry is however explicitly broken by the Lagrangian \( L_S \) as can be seen from \( M_+ \) in (19).

**IV. OTHER GAUGE INVARIANT TERMS**

Since the gauge terms of the HGS Lagrangian is insufficient to guarantee stability, it is also customary to add some sort of stabilizing Skyrme-like term to \( L_{tot} \), preferably a “gauge invariant” Skyrme term of the form [8][13]:

...
The first term is the usual Skyrme term. At this order (four derivatives), these are the only contributions that lead to a positive Hamiltonian which is second order in time derivatives.

It is most likely that a complete effective Lagrangian will involve all-order terms. In that respect, the generalization of the previous calculations to other gauge-invariant all-order terms constructed out of powers of $d_{\mu\nu}$ is straightforward. The same constructions with $f_{\mu\nu}$ were considered in refs. [9]. So we consider an all-order Lagrangian $\tilde{L}_+$ of the form (27) with $\zeta_{\mu\nu} \to d_{\mu\nu}$. We may attempt to construct a general Lagrangian or some special class. Since the commutators $d_{\mu\nu}$ can be written as in (28), the result will inevitably be cast in a form like (29) or even (30) with $a_+^+$ and $b_+^+$ defined as

$$a_+^+ = \frac{2}{r^2} \left[ (G - 1 + \cos F)^2 + H^2 \right] \quad \text{and} \quad b_+^+ = \frac{2}{r^2} K^2$$

(37)

We note that no derivatives of $F, G, H$ and $K$ are involved here at any orders. Furthermore, for a gauge field configuration of the form (20), the functions $a_+^+$ and $b_+^+$ become

$$a_+^+ = \frac{2}{r^2} \left[ g^2 \sin^2 \theta - 2g \sin^2 \theta (1 - \cos F) + (1 - \cos F)^2 \right] \quad \text{and} \quad b_+^+ = 0.$$

Then $\tilde{L}_+$ contribution to the static energy is

$$\tilde{M}_+ \equiv - \int d^3r \, \tilde{L}_+ = 4\pi \int r^2 dr \sum_{n=1}^{\infty} \kappa_n a_+^n.$$

(38)

Applying Derrick’s test for stability under scale transformation could lead to a stable soliton since there are terms of order $O(r^{-2n})$ that are potentially enough to stabilize the soliton. There is, for example, absolute stability when all $\kappa_n > 0$.

We conclude with a few remarks. The results presented here are only a limited set of all-orders terms that could contribute to an effective Lagrangian describing the hidden gauge symmetry and the scalar sector of the electroweak theory. The analysis is limited to objects which have the structure of powers of a commutator (any type of commutator, not only $f_{\mu\nu}, d_{\mu\nu}$ and $F^V_{\mu\nu}$). Mixed terms (e.g. $\text{Tr} (f_{\mu\nu}d^{\mu\lambda} \ldots)$) were not considered here. The generalization of our calculations to such terms is straightforward but in absence of a specific model their interest remains academic. Despite these limitations, the Lagrangians considered here are attractive since they involve manifestly gauge-invariant all-orders contributions in a very general way. This led to a number of results regarding the energy of a static solution (mass of the soliton). In general, higher-order terms can induce weak skyrmions but gauge terms alone are not sufficient to absolutely stabilize the soliton. Most of the conclusions reported here apply to QCD skyrmions where the vector meson is generally interpreted as the $\rho$-meson. Finally, it is clear that whether or not weak skyrmions exist, the presence of higher-order terms in an effective Lagrangian for the strongly interacting scalar sector of
the electroweak theory are interesting since they do affect a number of observables. The low-energy theorems however remain unaffected by the addition of higher order terms since they generally rely on the lowest order terms [14].

This research was supported by the Natural Science and Engineering Research Council of Canada and by the Fonds pour la Formation de Chercheurs et l’Aide à la Recherche du Québec.
REFERENCES

[1] M.S. Chanowitz, Ann. Rev. Nucl. Part. Sci. 38 323 (1988).
[2] M. Veltman, Acta Phys. Pol. B8 475 (1977); T. Appelquist and C. Bernard, Phys. Rev. D22 200 (1980).
[3] T.H.R. Skyrme, Proc. R. Soc. London. A2603 127 (1961).
[4] E. Farhi and L. Susskind, Phys. Rep. 74 277 (1981).
[5] J.M. Gipson and H.C. Tze, Nucl. Phys. B183 524 (1981).
[6] M. Bando, T. Kugo, S. Uehara, K. Yamawaki and T. Yanagida, Phys. Rev. Lett. 54 1215 (1985); M. Bando, T. Kugo and K. Yamawaki, Phys. Rep. 164 217 (1988).
[7] R. Casalbuoni, S. de Curtis, D. Dominici and R. Gatto, Phys. Lett. B155 95 1985 ; Nucl. Phys. B282 235 (1987); A. Dobado and M.J. Herrero, Nucl. Phys. B319 491 (1989).
[8] Z.F. Ezawa and T. Yanagida, Phys. Rev. D33 247 (1986).
[9] L. Marleau, Phys. Lett. B235 141 (1990).
[10] S. Dubé and L. Marleau, Phys. Rev. D41 1606 (1990); L. Marleau, Phys. Rev. D43 885 (1991); A.D. Jackson, C. Weiss and A. Wirzba, Nucl. Phys. A529 741 (1991); K. Gustafsson and D.O. Riska, Two-phase structure of infinite order skyrmion, University of Helsinki preprint HU-TFT-93-11, January 1993.
[11] L. Marleau, Hidden gauge symmetry and the weak skyrmions, Proc. of Beyond the Standard Model, Ottawa, June 22-24 1992, World Scientific, ed. S. Godfrey.
[12] L. Marleau, Phys. Rev. D45 1776 (1991).
[13] F.R. Klinkhamer, Z. Phys. C31 623 (1986).
[14] M. Harada, T. Kugo and K. Yamawaki, Proving the low energy theorem of hidden local symmetry, Kyoto University preprint DPNU-93-01 and KUNS-1178, March 1993.