The adaptive Crouzeix-Raviart element method for convection-diffusion eigenvalue problems

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Abstract: The convection-diffusion eigenvalue problems are hot topics, and computational mathematics community and physics community are concerned about them in recent years. In this paper, we consider the a posteriori error analysis and the adaptive algorithm of the Crouzeix-Raviart nonconforming element method for the convection-diffusion eigenvalue problems. We give the corresponding a posteriori error estimators, and prove their reliability and efficiency. Finally, the numerical results validate the theoretical analysis and show that the algorithm presented in this paper is efficient.

Keywords: convection-diffusion eigenvalue problems, the Crouzeix-Raviart element, a posteriori error analysis, adaptive algorithm

AMS subject classifications. 65N25, 65N30

1 Introduction

The convection-diffusion eigenvalue problems have a strong background in physics, such as the distribution of contaminated material in nuclear waste pollution. Thus, using finite element methods to solve convection-diffusion eigenvalue problems has attracted much attention of scholars. [1, 2, 3] discussed a posteriori error estimates and the adaptive algorithms, [4] an adaptive homotopy approach, [5, 6] extrapolation methods, [7] function value recovery algorithms, [8] spectral element methods, [9, 10] multilevel correction method, and so on. This paper aims at deriving the a posteriori error estimators and the adaptive algorithm of the Crouzeix-Raviart element(C-R element) methods for the convection-diffusion eigenvalue problems.

The adaptive finite element method is a mainstream in scientific computing (see [11, 12, 13, 14]). In past years, the research of the a posteriori error and the adaptive algorithm of convection-diffusion eigenvalue problems used to adopt

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the conforming finite element methods (see [3, 8, 12]). [15] and [16] discussed a posteriori error estimate of the nonconforming methods for Laplace equation and Laplace eigenvalue problem, respectively. Based on the study of [15, 16], this paper first discusses the nonconforming finite element adaptive method for convection-diffusion eigenvalue problems. We give the a posteriori error estimators and prove their reliability and efficiency, and give the adaptive algorithm. Finally we use some numerical examples to verify our theoretical results.

In this paper, $C$ is a positive constant independent of $h$, which may not be the same constant in different places. For simplicity, we use symbol $a \lesssim b$ to replace $a \leq Cb$. The notation $a \approx b$ abbreviates $a \lesssim b \lesssim a$.

2 Preliminaries

Consider the following convection-diffusion eigenvalue problem:

$$- \Delta u + b \cdot \nabla u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,$$

(2.1)

where $\Omega \subset \mathbb{R}^2$ is a polygon bounded domain with boundary $\partial \Omega$.

Let

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v + b \cdot \nabla u v dx, \quad b(u, v) = \int_{\Omega} u v dx.$$

(2.2)

The variational problem associated with (2.1) is given by: Find $(\lambda, u) \in \mathbb{C} \times H^1_0(\Omega), \|u\|_{L^2(\Omega)} = 1$, such that

$$a(u, v) = \lambda b(u, v), \quad \forall v \in H^1_0(\Omega).$$

(2.3)

Let $T_h = \{K\}$ be a regular triangular mesh of $\Omega$.

Let $V_h$ denote the Crouzeix-Raviart nonconforming finite element space over $T_h$. Then, the C-R element approximation of (2.3) is given as follows: Find $(\lambda_h, u_h) \in \mathbb{C} \times V_h, \|u_h\|_{L^2(\Omega)} = 1$, such that

$$a_h(u_h, v) = \lambda_h b(u_h, v), \quad \forall v \in V_h.$$  

(2.4)

where

$$a_h(u_h, v) = \sum_K \int_K \nabla_{h} u_h \nabla v + b \cdot \nabla_{h} u_h v dx.$$  

(2.5)

Since the discrete space $V_h$ is nonconforming, we regard $\nabla_h$ as the gradient operator which is defined elementwise.

The dual problem of (2.1) is as below:

$$- \Delta u^* - \nabla \cdot (b u^*) = \lambda^* u^* \quad \text{in } \Omega, \quad u^* = 0 \quad \text{on } \partial \Omega.$$  

(2.6)

The corresponding variational form of (2.3) is as follows: Find $(\lambda^*, u^*) \in \mathbb{C} \times H^1_0(\Omega), \|u^*\|_{L^2(\Omega)} = 1$, such that

$$a(v, u^*) = \lambda^* b(v, u^*), \quad \forall v \in H^1_0(\Omega),$$  

(2.7)

where

$$a(v, u^*) = \int_{\Omega} \nabla v \cdot \nabla u^* + \nabla v \cdot b u^* dx, \quad b(v, u^*) = \int_{\Omega} v u^* dx.$$  

(2.8)
Then the C-R element approximation of (2.7) is as below: Find \((\lambda^*_h, u^*_h) \in \mathbb{C} \times V_h, \| u^*_h \|_{L^2(\Omega)} = 1\), such that
\[
a_h(v, u^*_h) = \lambda^*_h b(v, u^*_h), \quad \forall v \in V_h,
\]
where
\[
a_h(v, u^*_h) = \sum_T \int_T \nabla v \nabla u^*_h + \nabla v \cdot b u^*_h \, dx, \quad b(v, u^*_h) = \int_\Omega v u^*_h \, dx.
\]

[17] discusses the non-conforming finite element approximation, and proves the error estimates of the discrete eigenvalues obtained by the Adini element, Morley-Zienkiewicz element et al. Due to the reference [17], we can deduce the following Lemma.

**Lemma 2.1.** For the C-R nonconforming finite element methods of problem (2.1) and (2.6), the a priori error estimates are given:
\[
\| \nabla_h (u - u_h) \|_{L^2(\Omega)} \lesssim h^r,
\]
\[
\| u - u_h \|_{L^2(\Omega)} \lesssim h^r \| \nabla_h (u - u_h) \|_{L^2(\Omega)},
\]
\[
\| \nabla_h (u^* - u^*_h) \|_{L^2(\Omega)} \lesssim h^r \| \nabla_h (u^* - u^*_h) \|_{L^2(\Omega)}^{1/2},
\]
\[
\| u^* - u^*_h \|_{L^2(\Omega)} \lesssim (h^r \| \nabla_h (u^* - u^*_h) \|_{L^2(\Omega)})^{1/2},
\]
\[
| \lambda - \lambda_h | \lesssim h \| \nabla_h (u - u_h) \|_{L^2(\Omega)} \cdot \| \nabla_h (u^* - u^*_h) \|_{L^2(\Omega)}.
\]

Owing to the above conclusions, we can get the following estimate: there exist some positive constants \(0 < \beta < 1\) and \(h_0 > 0\) (when \(h < h_0\)) with
\[
| \lambda - \lambda_h | \| u \|_{L^2(\Omega)} + | \lambda_h | \| u - u_h \|_{L^2(\Omega)} + \| u - u_h \|_{L^2(\Omega)} \\
\leq \beta \| \nabla_h (u - u_h) \|_{L^2(\Omega)}.
\]

### 3 A posteriori error analysis

Now we introduce some symbols for reading convenience. Suppose \(K\) is one given element of \(T_h\), and \(h_K\) represents the diameter of \(K\). We use \(\varepsilon\) to denote the set of all edges in \(T_h\), \(\varepsilon(\Omega)\) the set of interior edges and \(\varepsilon(K)\) the set of edges of the element \(K\), respectively. For any given edge \(E \in \varepsilon(\Omega)\) with length \(h_E = |E|\), we assign the fixed unit normal \(\nu_E := (\nu_1, \nu_2)\) and tangential vector \(\tau_E := (-\nu_2, \nu_1)\). Once \(\nu_E\) and \(\tau_E\) have been fixed on \(E\), in relation to \(\nu_E\) one defines the elements \(K_- \in T_h\) and \(K_+ \in T_h\), with \(E = K_+ \cap K_-\) and \(\omega_E = K_+ \cup K_-\). Given \(E \in \varepsilon(\Omega)\), we denote by \([v] := (v|_{K_+})|_E - (v|_{K_-})|_E\) the jump of some \(R^d\)-valued function \(v\) defined in \(\Omega\) across \(E\) with \(d = 1, 2\). And throughout this paper, \([\cdot]\) denotes the jump of the piecewise smooth function across the internal edge \(E\), and the trace for the boundary edge \(E\).
Define the a posteriori error estimators on the element $K$ as below:

$$
\eta_{h,K} := (h_K^2 \| \lambda_h u_h + \Delta_h u_h - b \cdot \nabla_h u_h \|_{L^2(K)}^2)^{\frac{1}{2}},
$$

$$
\eta_{h,K}^* := (h_K^2 \| \lambda_h^* u_h^* + \Delta_h u_h^* \nabla_h \cdot u_h^* \|_{L^2(K)}^2)^{\frac{1}{2}},
$$

$$
\eta_{h,K,\nu_E} := \left( \frac{1}{2} \sum_{E \in \partial K} h_E \| \nabla_h u_h \cdot \nu_E \|_{L^2(E)}^2 \right)^{\frac{1}{2}},
$$

$$
\eta_{h,K,\nu_E}^* := \left( \frac{1}{2} \sum_{E \in \partial K} h_E \| \nabla_h u_h^* + b u_h^* \cdot \nu_E \|_{L^2(E)}^2 \right)^{\frac{1}{2}},
$$

$$
\eta_{h,K,\tau_E} := \left( \frac{1}{2} \sum_{E \in \partial K} h_E \| \nabla_h u_h \cdot \tau_E \|_{L^2(E)}^2 \right)^{\frac{1}{2}},
$$

$$
\eta_{h,K,\tau_E}^* := \left( \frac{1}{2} \sum_{E \in \partial K} h_E \| \nabla_h u_h^* + b u_h^* \cdot \tau_E \|_{L^2(E)}^2 \right)^{\frac{1}{2}},
$$

and the residual sum on $K$ are given by

$$
\eta_h(K)^2 := \eta_{h,K}^2 + \sum_{E \in \partial(K), \nu \in \partial \Omega} \eta_{h,K,\nu_E}^2 + \sum_{E \in \partial(K)} \eta_{h,K,\tau_E}^2, \quad (3.1)
$$

$$
\eta_h^*(K)^2 := (\eta_{h,K}^*)^2 + \sum_{E \in \partial(K), \nu \in \partial \Omega} (\eta_{h,K,\nu_E}^*)^2 + \sum_{E \in \partial(K)} (\eta_{h,K,\tau_E}^*)^2. \quad (3.2)
$$

For any $M_h \subset T_h$, define the estimators over $M_h$ by

$$
\eta_h(M_h)^2 := \sum_{K \in M_h} \eta_h(K)^2, \quad \eta_h^*(M_h)^2 := \sum_{K \in M_h} \eta_h^*(K)^2. \quad (3.3)
$$

The left parts of this section aim at proving the reliability and the efficiency of the estimators $\eta_h(T_h)$ and $\eta_h^*(T_h)$. The reliability of the estimators are based on the following lemma (see[14][16]).

**Lemma 3.1.** Under the assumption [2.10] there holds

$$
|a_h(u - u_h, u - u_h)| \leq \min_{v \in H^1_0(\Omega)} \| \nabla_h (u_h - v) \|_{L^2(\Omega)}^2 + \sup_{w \in H^1_0(\Omega)} \frac{|b(\lambda_h u_h, w) - a_h(u_h, w)|}{\| w \|_{L^2(\Omega)}} \| \nabla (u - v) \|_{L^2(\Omega)}, \quad (3.4)
$$

where $(\lambda, u) \in \mathbb{C} \times H^1_0(\Omega)$ and $(\lambda_h, u_h) \in \mathbb{C} \times V_h$ are the solutions to problems [2.3] and [2.4], respectively. For the dual problem, it is similar:

$$
|a_h(u^* - u_h^*, u^* - u_h^*)| \leq \min_{v \in H^1_0(\Omega)} \| \nabla_h (u_h^* - v) \|_{L^2(\Omega)}^2 + \sup_{w \in H^1_0(\Omega)} \frac{|b(\lambda_h^* u_h^*, w) - a_h(u_h^*, w)|}{\| w \|_{L^2(\Omega)}} \| \nabla (u^* - v) \|_{L^2(\Omega)}. \quad (3.5)
$$
Proof. For any $v \in H^1_0(\Omega)$,

$$|a_h(u - u_h, u - u_h)| = |a_h(u - u_h, u - v + v - u_h)|$$

$$= |a(u, u - v) - a_h(u_h, u - v) + a_h(u_h, u - v)|$$

$$= |b(u - \lambda_h u + \lambda_h u - u + u_h, u - v) + b(\lambda_h u_h, u - v)|$$

$$+ a_h(u_h, v - u_h) - a_h(u_h, u - v)|$$

$$\leq |b((\lambda - \lambda_h) u - u - u_h) + b(\lambda_h(u - u_h), u - u_h)|$$

$$+ |b((\lambda - \lambda_h) u, v - u_h) + b(\lambda_h(u - u_h), v - u_h)|$$

$$+ |a_h(u - u_h, v - u_h)|$$

$$+ |b(\lambda_h u_h, u - v) - a_h(u_h, u - v)|. \quad (3.6)$$

Due to (2.16), we can get

$$|b((\lambda - \lambda_h) u, u - u_h)| + |b(\lambda_h(u - u_h), u - u_h)|$$

$$\leq \left( |L - \lambda_h| \left\| u \right\|_{L^2(\Omega)} + \left| \lambda_h \right| \left\| u - u_h \right\|_{L^2(\Omega)} \right) \left\| u - u_h \right\|_{L^2(\Omega)}$$

$$\leq \beta^2 \left\| \nabla_h(u - u_h) \right\|_{L^2(\Omega)}^2 . \quad (3.7)$$

Using the Young and Poincaré inequalities, we obtain

$$\beta \left\| \nabla_h(u - u_h) \right\|_{L^2(\Omega)} \left\| v - u_h \right\|_{L^2(\Omega)}$$

$$\leq \frac{1}{\varepsilon^2}(\varepsilon^2 \beta^2 \left\| \nabla_h(u - u_h) \right\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon^2} \left\| v - u_h \right\|_{L^2(\Omega)}^2)$$

$$\leq \frac{1}{\varepsilon^2}(\varepsilon^2 \beta^2 \left\| \nabla_h(u - u_h) \right\|_{L^2(\Omega)}^2 + C_1 \left\| \nabla_h(v - u_h) \right\|_{L^2(\Omega)}^2). \quad (3.9)$$

The inequality (3.6) gives

$$|a_h(u - u_h, v - u_h)|$$

$$= \left| \sum_K \int_K \nabla_h(u - u_h) \cdot \nabla_h(v - u_h) + b \cdot \nabla_h(u - u_h)(v - u_h)dx \right|$$

$$\leq \sum_K \left\{ \left\| \nabla_h(u - u_h) \right\|_{L^2(K)} \left\| \nabla_h(v - u_h) \right\|_{L^2(K)} \right.$$

$$+ \left| b \right| \left\| \nabla_h(u - u_h) \right\|_{L^2(K)} \left\| \nabla_h(v - u_h) \right\|_{L^2(K)} \left\} \leq \left\| \nabla_h(u - u_h) \right\|_{L^2(\Omega)} \left\| \nabla_h(v - u_h) \right\|_{L^2(\Omega)}$$

$$\leq \left| b \right| \varepsilon^2 \left\| \nabla_h(u - u_h) \right\|_{L^2(\Omega)}^2 $$

$$+ \frac{1}{\varepsilon^2} \left| v - u_h \right|_{L^2(\Omega)}^2 \leq \frac{1}{\varepsilon^2}(\varepsilon^2 \beta^2 \left\| \nabla_h(u - u_h) \right\|_{L^2(\Omega)}^2 + C_2 \left\| \nabla_h(v - u_h) \right\|_{L^2(\Omega)}^2). \quad (3.10)$$

Combining (3.7), (3.8), (3.9) with (3.10), we obtain from (3.6)

$$|a_h(u - u_h, u - u_h)| \leq \frac{1}{\varepsilon^2}(\varepsilon^2 \beta^2 + \beta^2 + C_1 \varepsilon^2) \left\| \nabla_h(u - u_h) \right\|_{L^2(\Omega)}^2$$

$$+ \frac{1}{\varepsilon^2} \left\| \nabla_h(u - v) \right\|_{L^2(\Omega)}^2 + |b(\lambda_h u_h, u - v) - a_h(u_h, u - v)|,$$
then, we have
\[
|a_h(u - u_h, u - u_h)| \\
\lesssim \| \nabla h(u_h - v) \|^2_{L^2(\Omega)} + |b(\lambda h u_h, u - v) - a_h(u_h, u - v)| \\
\lesssim \| \nabla h(u_h - v) \|^2_{L^2(\Omega)} \\
+ \frac{|b(\lambda h u_h, u - v) - a_h(u_h, u - v)|}{\| \nabla (u - v) \|^2_{L^2(\Omega)}} \| \nabla (u - v) \|_{L^2(\Omega)} \\
\lesssim \min_{v \in H^1_0(\Omega)} \| \nabla h(u_h - v) \|^2_{L^2(\Omega)} \\
+ \sup_{w \in H^1_0(\Omega)} \frac{|b(\lambda h u_h, w) - a_h(u_h, w)|}{\| w \|_{L^2(\Omega)}} \| \nabla (u - v) \|_{L^2(\Omega)}. \tag{3.11}
\]

Then the proof of (3.4) is finished, and the proof of (3.5) is similar. □

Based on the work of [16, 18], we have the following Lemma:

**Lemma 3.2.** The following estimate is valid:
\[
\| \nabla h(u_h - v) \|^2_{L^2(\Omega)} \lesssim \sum_{K \in \mathcal{E}} h_E \| [\nabla h u_h] \cdot \tau_E \|^2_{L^2(E)}. \tag{3.12}
\]

Let \( S^1_h(\mathcal{T}_h) \) denote the elementwise linear conforming finite element space over \( \mathcal{T}_h \). For the analysis in the rear, we need the Clément-type interpolation operator \( L : H^1_0(\Omega) \rightarrow S^1_h(\mathcal{T}_h) \) with the properties (see [20, 21, 22])
\[
\| \nabla L \varphi \|_{L^2(K)} + \| h^{-1}_K (\varphi - L \varphi) \|_{L^2(K)} \lesssim \| \nabla \varphi \|_{L^2(\omega_K)}, \tag{3.13}
\]
and
\[
\| h^{-1}_E (\varphi - L \varphi) \|_{L^2(E)} \lesssim \| \nabla \varphi \|_{L^2(\omega_K)}, \tag{3.14}
\]
where \( E \in \mathcal{E}(K) \) and \( \varphi \in H^1_0(\Omega) \). In this paper, \( \omega_K \) denotes the element patch defined as
\[
\omega_K := \{ T \in \mathcal{T}_h : \overline{T} \cap K \neq \emptyset \}. \tag{3.15}
\]

Refering to [19], we can prove the following Lemma.

**Lemma 3.3.** The following estimates are valid:
\[
\sup_{w \in H^1_0(\Omega)} \frac{|b(\lambda h u_h, w) - a_h(u_h, w)|}{\| \nabla w \|_{L^2(\Omega)}} \lesssim \left( \sum_{K \in \mathcal{T}_h} \eta^2_{h,K} + \sum_{E \in \mathcal{E}(\Omega)} \eta^2_{h,K,\nu E} \right)^{\frac{1}{2}}, \tag{3.16}
\]
\[
\sup_{w \in H^1_0(\Omega)} \frac{|b(w, \lambda^*_h u^*_h) - a_h(u, w)|}{\| \nabla w \|_{L^2(\Omega)}} \lesssim \left( \sum_{K \in \mathcal{T}_h} (\eta^2_{h,K})^2 + \sum_{E \in \mathcal{E}(\Omega)} (\eta^2_{h,K,\nu E})^2 \right)^{\frac{1}{2}}. \tag{3.17}
\]

**Proof.** Using the estimates (3.13) and (3.14) and integrating by parts,
we can deduce that
\[
| b(\lambda_h u_h, w) - a_h(u_h, w) | = | (\lambda_h u_h, w - L w)_{L^2(\Omega)} - a_h(u_h, w - L w)_{L^2(\Omega)} |
\]
\[
= \sum_K \int_K \lambda_h u_h (w - L w) dx - \sum_K \int_K -\Delta h u_h (w - L w)
\]
\[
+ b \cdot \nabla h u_h (w - L w) ds - \int_{\partial K} \frac{\partial u_h}{\partial \nu} (w - L w) ds |
\]
\[
= \sum_K \int_K (\lambda_h u_h + \Delta_h u_h - b \cdot \nabla h u_h) (w - L w) dx - \sum_K \int_{\partial K} \frac{\partial u_h}{\partial \nu} (w - L w) ds |
\]
\[
\leq \sum_K \| \lambda_h u_h + \Delta_h u_h - b \cdot \nabla h u_h \|_{L^2(K)} \| h_K \| \| \nabla w \|_{L^2(\omega_K)}
\]
\[
+ \sum_{E \in \epsilon(\Omega)} \| \nabla h u_h \cdot \nu_E \|_{L^2(E)} \cdot h_E^{\frac{1}{2}} \| \nabla w \|_{L^2(\omega_E)}
\]
\[
\leq \left( \sum_K h_K^2 \| \lambda_h u_h + \Delta_h u_h - b \cdot \nabla h u_h \|_{L^2(K)}^2 \right)^{\frac{1}{2}} \left( \sum_K \| \nabla w \|_{L^2(\omega_K)}^2 \right)^{\frac{1}{2}}
\]
\[
+ \left( \sum_{E \in \epsilon(\Omega)} h_E \| \nabla h u_h \cdot \nu_E \|_{L^2(E)}^2 \right)^{\frac{1}{2}} \| \nabla w \|_{L^2(\Omega)}
\]
\[
\leq \left( \sum_{K \in T_h} \eta_{h,K}^2 + \sum_{E \in \epsilon(\Omega)} \eta_{h,E}^2 \right)^{\frac{1}{2}} \| \nabla w \|_{L^2(\Omega)}. \tag{3.18}
\]

This ends the proof. The proof of (3.17) is similar. □

Combining Lemma 3.2 with Lemma 3.3, we can get the reliability of the a posteriori error estimators.

**Theorem 3.1.** Let \((\lambda, u) \in C \times H^1_0(\Omega)\) and \((\lambda_h, u_h) \in C \times V_h\) be the solutions to problems (2.3) and (2.4), and let \((\lambda^*, u^*) \in C \times H^1_0(\Omega)\) and \((\lambda_h^*, u_h^*) \in C \times V_h\) be the solutions to problems (2.7) and (2.9), respectively. Under the assumption (2.10) there holds
\[
\| u - u_h \|_h^2 \lesssim \eta_h(T_h)^2, \tag{3.19}
\]
\[
\| u^* - u_h^* \|_h^2 \lesssim \eta_h^*(T_h)^2, \tag{3.20}
\]
\[
| \lambda - \lambda_h | \lesssim \eta_h(T_h)^2 + \eta_h^*(T_h)^2. \tag{3.21}
\]

**Proof.** Combining Lemmas 3.1-3.3 we get (3.19) and (3.20). Substituting (3.19) and (3.20) into (2.15) yields (3.21). □

Next, we shall prove the efficiency of the a posteriori error estimators.

**Theorem 3.2.** Assume the conditions of Theorem 3.1 hold, then
\[
\eta_h(T_h)^2 \lesssim \| u - u_h \|^2_h, \tag{3.22}
\]
\[
\eta_h^*(T_h)^2 \lesssim \| u^* - u_h^* \|^2_h. \tag{3.23}
\]

**Proof.** 1. **Proof of** \( \sum_{K \in T_h} \eta_{h,K}^2 \lesssim \| \nabla (u - u_h) \|^2_{L^2(\Omega)} \)

Given \(K \in T_h\), let \( b_K = 27 \lambda_1 \lambda_2 \lambda_3 \) with \( \lambda_i \), \( i = 1, 2, 3 \). Define
\[
v_K = b_K (\lambda_h u_h + \Delta h u_h - b \cdot \nabla h u_h) \tag{3.24}
\]
Then, we have
\[
\| \lambda_h u_h + \Delta_h u_h - b \cdot \nabla_h u_h \|_{L^2(K)}^2 \lesssim (\lambda_h u_h + \Delta_h u_h - b \cdot \nabla_h u_h, v_K)_{L^2(K)}
\]
\[
= (\lambda_h u_h - \lambda u + \Delta_h u_h - b \cdot \nabla_h u_h, v_K)_{L^2(K)}
\]
\[
= (\lambda_h u_h - \lambda u, v_K)_{L^2(K)} + (-\Delta u + b \cdot \nabla u + \Delta_h u_h - b \cdot \nabla_h u_h, v_K)_{L^2(K)}
\]
\[
= (\lambda_h u_h - \lambda u, v_K)_{L^2(K)} + (-\Delta u + \lambda_h u_h, v_K)_{L^2(K)}
\]
\[
= (\lambda_h u_h - \lambda u, v_K)_{L^2(K)} + (\nabla_h (u - u_h), \nabla v_K)_{L^2(K)}
\]
\[
+ (b \cdot \nabla u - b \cdot \nabla_h u_h, v_K)_{L^2(K)}.
\] (3.25)

Using the Young inequalities in (3.25) to obtain
\[
|\lambda_h u_h - \lambda u, v_K)_{L^2(K)}| \leq \|\lambda_h u_h - \lambda u\|_{L^2(K)} \|v_K\|_{L^2(K)}
\]
\[
\lesssim \|\lambda_h u_h - \lambda u\|_{L^2(K)} \|\lambda_h u_h + \Delta_h u_h - b \cdot \nabla_h u_h\|_{L^2(K)}
\]
\[
\leq \frac{1}{2} \frac{1}{\varepsilon^2} \|\lambda_h u_h - \lambda u\|_{L^2(K)}^2 + \varepsilon^2 \|\lambda_h u_h + \Delta_h u_h - b \cdot \nabla_h u_h\|_{L^2(K)}^2.
\] (3.26)

Thanks to the assumption (3.13) and using the Young inequalities we can have
\[
|\nabla_h (u - u_h), \nabla v_K)_{L^2(K)}| \leq \|\nabla_h (u - u_h)\|_{L^2(K)} \|\nabla v_K\|_{L^2(K)}
\]
\[
\lesssim h_K^{-1} \|\nabla_h (u - u_h)\|_{L^2(K)} \|v_K\|_{L^2(K)}
\]
\[
\leq \frac{1}{2} \frac{1}{\varepsilon^2} \cdot h_K^{-2} \|\nabla_h (u - u_h)\|_{L^2(K)}^2 + \varepsilon^2 \|\lambda_h u_h + \Delta_h u_h - b \cdot \nabla_h u_h\|_{L^2(K)}^2.
\] (3.27)

and
\[
|(b \cdot \nabla u - b \cdot \nabla_h u_h, v_K)_{L^2(K)}| \leq \|b \cdot \nabla u - b \cdot \nabla_h u_h\|_{L^2(K)} \|v_K\|_{L^2(K)}
\]
\[
\leq \frac{1}{2} \frac{1}{\varepsilon^2} \|b \cdot \nabla u - b \cdot \nabla_h u_h\|_{L^2(K)}^2 + \varepsilon^2 \|\lambda_h u_h + \Delta_h u_h - b \cdot \nabla_h u_h\|_{L^2(K)}^2.
\] (3.28)

then combining (3.26) - (3.28) can yield:
\[
\eta_h^2 = h_K^2 \|\lambda_h u_h + \Delta_h u_h - b \cdot \nabla_h u_h\|_{L^2(K)}^2
\]
\[
\lesssim \|\nabla_h (u - u_h)\|_{L^2(K)}^2 + h_K^2 \|\lambda_h u_h - \lambda u\|_{L^2(K)}^2
\]
\[
+ h_K^2 \|b \cdot \nabla u - b \cdot \nabla_h u_h\|_{L^2(K)}^2.
\] (3.29)

Then, we have
\[
\sum_{K \in T_h} \eta_h^2 \lesssim \|\nabla_h (u - u_h)\|_{L^2(K)}^2 + \sum_{K \in T_h} h_K^2 \|\lambda_h u_h - \lambda u\|_{L^2(K)}^2
\]
\[
+ \sum_{K \in T_h} h_K^2 \|b \cdot \nabla u - b \cdot \nabla_h u_h\|_{L^2(K)}^2
\]
\[
\lesssim \|\nabla_h (u - u_h)\|_{L^2(K)}^2.
\] (3.30)
2. Proof of \[ \sum_{E \in \varepsilon(\Omega)} n_E^2 h_E \lesssim \| \nabla h(u - u_h) \|_{L^2(\Omega)}^2 \]

Given any edge \( E \in \varepsilon(\Omega) \), let \( b_E \in H^1_0(\omega_E) \) denote the piecewise polynomial function vanishing at the midside point of \( E \). Define

\[ v_E = b_E[\nabla h u_h] \cdot \nu_E. \quad (3.31) \]

Then we have

\[ \| [\nabla h u_h] \cdot \nu_E \|_{L^2(E)}^2 \approx ([\nabla h u_h] \cdot v_E)_{L^2(E)} = \int_{\omega_E} \Delta h u_h \cdot \mathbf{v}_E dx + \int_{\omega_E} \nabla h u_h \cdot \nabla v_E dx. \]

Due to

\[ \int_{\omega_E} \lambda u v dx = \int_{\omega_E} \nabla u \cdot \nabla v dx + \int_{\omega_E} b \cdot \nabla v dx. \]

and (3.13), (3.32) can be estimated as

\[ \int_{\omega_E} \nabla h(u_h - u) \cdot \nabla \mathbf{v}_E dx - \int_{\omega_E} \mathbf{b} \cdot \nabla \mathbf{v}_E dx + \int_{\omega_E} (\lambda u + \Delta h u_h) \mathbf{v}_E dx \]

\[ = \int_{\omega_E} \nabla h(u_h - u) \cdot \nabla \mathbf{v}_E dx - \int_{\omega_E} \mathbf{b} \cdot \nabla (u - u_h) \mathbf{v}_E dx \]

\[ + \int_{\omega_E} (- \mathbf{b} \cdot \nabla h u_h + \Delta h u_h + \lambda h u_h) \mathbf{v}_E dx + \int_{\omega_E} (\lambda u - \lambda h u_h) \mathbf{v}_E dx \]

\[ \lesssim \| \nabla h(u_h - u) \|_{L^2(\omega_E)} \| \nabla \mathbf{v}_E \|_{L^2(\omega_E)} \| \nabla h(u_h - u) \|_{L^2(\omega_E)} \| \mathbf{v}_E \|_{L^2(\omega_E)} \]

\[ + \| - \mathbf{b} \cdot \nabla h u_h + \Delta h u_h + \lambda h u_h \|_{L^2(\omega_E)} \| \mathbf{v}_E \|_{L^2(\omega_E)} \]

\[ \lesssim h_E^{-1} \| \nabla h(u_h - u) \|_{L^2(\omega_E)} \| \mathbf{v}_E \|_{L^2(\omega_E)} \]

\[ \lesssim h_E^{-1} \| \nabla h(u_h - u) \|_{L^2(\omega_E)}^2. \quad (3.33) \]

Then, we obtain

\[ \sum_{E \in \varepsilon(\Omega)} n_E^2 h_E = \sum_{E \in \varepsilon(\Omega)} h_E \| [\nabla h u_h] \cdot \nu_E \|_{L^2(E)}^2 \]

\[ \lesssim \| \nabla h(u_h - u) \|_{L^2(\Omega)}^2. \quad (3.34) \]

3. Proof of \[ \sum_{E \in \varepsilon(\Omega)} n_E^2 h_E \lesssim \| \nabla h(u_h - u) \|_{L^2(\Omega)}^2 \]

With the edge bubble function \( b_E \) as in (3.31), we define

\[ v_E = b_E[\nabla h u_h] \cdot \tau_E. \quad (3.35) \]

Then, we have

\[ \| [\nabla h u_h] \cdot \tau_E \|_{L^2(E)}^2 \approx ([\nabla h u_h] \cdot \tau_E, v_E)_{L^2(E)} \]

\[ = \int_E [\nabla h u_h] \cdot \tau_E \cdot \mathbf{v}_E ds. \quad (3.36) \]
Noting that \( \nu_E = (n_x, n_y) \) and \( \tau_E = (-n_y, n_x) \), (3.30) can be estimated as

\[
\int_{E} \left[ -(u_h)_x n_y + (u_h)_y n_x \right] \cdot \nabla u_E \, ds \\
= \int_{\partial E} -(u_h)_x n_E - (u_h)_y (\nabla u_E)_y + (u_h)_y n_E + (u_h)_y (\nabla u_E)_x \, dx \\
= \int_{\partial E} \nabla_h(u_h) \cdot \text{curl} u_E \, dx \\
= \int_{\partial E} \nabla_h(u_h - u) \cdot \text{curl} u_E \, dx.
\]

(3.37)

where \( \text{curl} u_E = (-\nabla u_E)_y, (\nabla u_E)_x \) and \( \int_{\partial E} \nabla u \cdot \text{curl} u_E \, dx = 0 \).

An application of the inverse estimate leads to

\[
\sum_{E \in \mathcal{E}} \eta_{h,E}^2 = \sum_{E \in \mathcal{E}} h_E \left\| \nabla_h u_h \right\|_{L^2(E)}^2 \\
\lesssim \left\| \nabla_h (u_h - u) \right\|_{L^2(\Omega)}^2.
\]

(3.38)

Thanks to the following conclusion

\[
\left\| \nabla_h (u_h - u) \right\|_{L^2(\Omega)} \lesssim a_h(u^* - u_h^*, u^* - u_h^*),
\]

(3.39)

combining (3.30), (3.31) with (3.38), we obtain (3.28). The proof of (3.28) is similar. \( \square \)

Combining Lemmas 3.1, 3.2, 3.3 and Theorem 3.2, we derive the following theorem:

**Theorem 3.3.** Let \( (\lambda, u) \in \mathbb{C} \times H^1_0(\Omega) \) and \( (\lambda_h, u_h) \in \mathbb{C} \times V_h \) be the solution to problems (2.3) and (2.9), respectively. Then

\[
a_h(u - u_h, u - u_h) \approx \eta_h^2.
\]

(3.40)

Let \( (\lambda^*, u^*) \) and \( (\lambda_h^*, u_h^*) \) be the eigenpairs of the adjoint problems (2.7) and (2.10), respectively. Then

\[
a_h(u^* - u_h^*, u^* - u_h^*) \approx (\eta_h^*)^2.
\]

(3.41)

### 4 The adaptive algorithm and numerical results

Using the a posteriori error estimates and consulting the existing standard algorithm (see, e.g., [1 2 3]), we obtain the following adaptive algorithm of the C-R element for the convection-diffusion eigenvalue problem (2.1):

**Algorithm 1.**

1. Choose parameter \( 0 < \theta < 1 \).
2. Pick any initial mesh \( T_{h_0} \) with mesh size \( h_0 \).
3. Solve (2.4) and (2.9) on \( T_{h_0} \) for discrete solution \((\lambda_{h_0}, u_{h_0}, u_{h_0}^*)\).
4. Let \( l = 0 \).
5. Compute the local indicators \( \eta_{h_l}(K)^2 + \eta_{h_l}^*(K)^2 \).
6. Construct \( T_{h_l} \subset T_{h_l} \) by **Marking Strategy E** and parameter \( \theta \).
7. Refine \( T_{h_l} \) to get a new mesh \( T_{h_{l+1}} \) by **Refine**.
8. Solve (2.4) and (2.9) on \( T_{h_{l+1}} \) for discrete solution \((\lambda_{h_{l+1}}, u_{h_{l+1}}, u_{h_{l+1}}^*)\).
**Step 8.** Let $l = l + 1$ and go to Step 4.

**Marking Strategy E**

Given parameter $0 < \theta < 1$:

**Step 1.** Construct a minimal subset $\hat{T}_{h_i}$ of $T_{h_i}$ by selecting some elements in $T_{h_i}$ such that

$$\sum_{K \in \hat{T}_{h_i}} (\eta_{h_i}(K)^2 + \eta^*_{h_i}(K)^2) \geq \theta (\eta_{h_i}(T_{h_i})^2 + \eta^*_{h_i}(T_{h_i})^2).$$

**Step 2.** Mark all the elements in $\hat{T}_{h_i}$.

Next, we will present some numerical experiments by using the triangular C-R element. We use MATLAB 2012 together with the package of IFEM [23] to solve the (2.4) and (2.9) as below. For simplicity of the presentation, we use the following notations:

- $\lambda_{k,h}$: the $k$-th finite element eigenvalue.
- $\lambda_k$: the $k$-th exact eigenvalue.
- $\Phi(\lambda_{k,h})$: the a posteriori error indicator for $\lambda_{k,h}$.
- $N_{k,i}(b_i)$: number of degrees of freedom for $\lambda_{k,h}$ after the $i$-th iteration when $b = (b_1, 0)^T$.

**Example 1.** Let $\Omega = (0, 1)^2$ and $b = (b_1, b_2)^T$. Consider the convection-diffusion eigenvalue problem (2.1) whose eigenvalues are

$$\frac{b_1^2 + b_2^2}{4} + \pi^2 (j^2 + i^2),$$

where $j, i \in N_+$. We know that $\lambda_1 = \frac{b_1^2 + b_2^2}{4} + 2\pi^2$, $\lambda_2 = \lambda_3 = \frac{b_1^2 + b_2^2}{4} + 5\pi^2$. We restrict our attention to the case of $b = (1, 0)^T$, $b = (3, 0)^T$, and $b = (10, 0)^T$. Some adaptive refined meshes are shown in Figures 1 and 2 and the numerical results are shown in table 1. From the results we can see that the a posteriori error indicators presented in this paper are efficient and reliable, which is consistent with our theoretical analysis. But we have to note that the numerical eigenvalues do not perform that well when $b = (10, 0)^T$. This is probably the consequence of the performance of linear algebra routine on a convection dominated problem.

**Example 2.** Consider the convection-diffusion eigenvalue problem (2.1) on $\Omega = (0, 2)^2 \setminus [1, 2]^2$. Since the exact eigenvalues of (2.1) are unknown, we choose the approximate eigenvalues with high accuracy to replace them. For $b = (1, 0)^T$, $b = (3, 0)^T$, and $b = (10, 0)^T$, respectively, some adaptive refined meshes are shown in Figures 4 and 5 and the numerical results are shown in table 2. From the results we can see that for the convection parameters $b = (1, 0)^T$, $b = (3, 0)^T$, and $b = (10, 0)^T$, the a posteriori error indicators can reflect the general trend of the error of discrete eigenvalues but similar to Example 1 the numerical eigenvalues do not perform that well when $b = (10, 0)^T$.

**References**

[1] J. Gedicke and C. Carstensen, “A posteriori error estimators for
Convection-diffusion eigenvalue problems,” Computer Methods in Applied Mechanics and Engineering, vol. 268, pp. 160-177, 2014.

[2] R. Rannacher, “Adaptive FE eigenvalue computation with applications to hydrodynamic stability,” in Advances in Mathematical Fluid Mechanics, pp. 425–450, Springer, Berlin, Germany, 2010.

[3] Y. Yang, L. Sun, H. Bi and H. Li, “A note on the residual type a posteriori error estimates for finite element eigenpairs of nonsymmetric elliptic eigenvalue problems,” Applied Numerical Mathematics, vol. 82, pp. 51-67, 2014.

[4] C. Carstensen, J. Gedicke, V. Mehrmann and A. Miedlar, “An adaptive homotopy approach for non-selfadjoint eigenvalue problems,” Numerische Mathematik, vol. 119, no. 3, pp. 557-583, 2011.

[5] T.Lü and Y. Feng, “Splitting extrapolation based on domain decomposition for finite element approximations,” Science in China Series E:Technological Sciences, vol. 40, no. 2, pp. 144-155, 1997.

[6] Y. Yang, H. Bi and S. Li, “The extrapolation of numerical eigenvalues by finite elements for differential operators,” Applied Numerical Mathematics, vol. 69, pp. 59-72, 2013.

[7] A. Naga and Z. Zhang, “Function value recovery and its application in eigenvalue problems,” SIAM Journal on Numerical analysis, vol. 50, no. 1, pp. 272-286, 2012.

[8] J. Han and Y. Yang, “A class of spectral element methods and its a priori/a posteriori error estimates for 2nd-order elliptic eigenvalue problems,” Abstract and Applied Analysis, vol. 2013, 14 pages, 2013.

[9] Y. Yang and J. Han, “Multilevel finite element discretizations based on local defect correction for nonsymmetric eigenvalue problems,” Computers and Mathematics with Applications, vol. 70, pp. 1799-1816, 2015.

[10] Z. Peng, H. Bi, H. Li and Y. Yang, “A Multilevel correction method for Convection-diffusion eigenvalue problems,” Mathematical Problems in Engineering, Vol. 2015, pp. 1-10, 2015.

[11] M. Ainsworth and J.T. Oden,“A posterior error estimation in Finite element Analysis,” Wiley-Inter science, New York, 2011.

[12] I. Babuska and W.C. Rheinboldt, “Error estimates for adaptive finite element computations,” SIAM J.Numer.Anal., Vol. 15, pp. 736-754, 1978.

[13] R. Verfürth, “A Posteriori Error Estimation Techniques,” Oxford University Press, USA, 2013.

[14] Z. Shi and M. Wang, “Finite Element Methods,” Beijing, Scientific Publishers, 2013.

[15] C. Carstensen, J. Hu and A. Orlando, “Framework for the a posteriori error analysis of nonconforming finite elements,” SIAM. J. Numer. Anal, vol. 45, pp. 68-82, 2007.
Figure 1: the adaptively refined meshes of 1st eigenvalue after 6th iteration when $b = (1, 0)^T$, $b = (3, 0)^T$, and $b = (10, 0)^T$, respectively.

Figure 2: the adaptively refined meshes of 2nd eigenvalue after 6th iteration when $b = (1, 0)^T$, $b = (3, 0)^T$, and $b = (10, 0)^T$, respectively.

[16] Y. Li, “A posteriori error analysis of nonconforming methods for the eigenvalue problem,” Jrl Syst Sci & Complexity, vol. 22, pp. 495-502, 2009.

[17] Y. Yang, J. Han and H. Bi, “Non-conforming finite element methods for transmission eigenvalue problem,” Computer Methods in Applied Mechanics and Engineering, vol. 307, pp. 144-163, 2016.

[18] C. Carstensen and J. Hu, “A unifying theory of a posteriori error control for nonconforming finite element methods,” Numer. Math, vol. 107, pp. 473-502, 2007.

[19] R. Verfürth, “A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques” Wiley-Teubner, 2007.

[20] P. Clément, “Approximation by finite functions using local regularization,” RAIRO Ana Numer, vol. 9, pp. 77-84, 1975.

[21] C. Carstensen, “Quasi-interpolation and a poeteriori error analsis in finite element methods,” M²NA, vol. 33, pp. 1187-1202, 1999.

[22] C. Bernardi and V. Girault, “A local regularisation operator for triangular and quadrilateral finite elements,” SIAM J. Numer. Anal, vol. 35, pp. 1893-1916, 1998.

[23] L. chen, “IFEM: an innovative finite element methods package in MATLAB,” Technical Report, University of California at Irvine, 2009.
Figure 3: $\Omega = (0, 1)^2$, the first eigenvalue and the second eigenvalue

Figure 4: the adaptively refined meshes of 1st eigenvalue after 6th iteration when $b = (1, 0)^T$, $b = (3, 0)^T$, and $b = (10, 0)^T$, respectively.

Figure 5: the adaptively refined meshes of 8th eigenvalue after 6th iteration when $b = (1, 0)^T$, $b = (3, 0)^T$, and $b = (10, 0)^T$, respectively.

Figure 6: $\Omega = (0, 2)^2 \setminus [1, 2]^2$, the first eigenvalue and the 8th eigenvalue
Table 1: The 1st and 2nd eigenvalues on $\Omega = (0, 1)^2$ with $H = \sqrt{\frac{2}{16}}$.

| $k$ | $l$ | $N_{k,l}(1)$ | $\lambda_{k}(1)$ | $N_{k,l}(3)$ | $\lambda_{k}(3)$ | $N_{k,l}(10)$ | $\lambda_{k}(10)$ |
|-----|-----|--------------|-----------------|-------------|----------------|--------------|----------------|
| 1   | 6   | 1 046        | 2.977997        | 6113        | 21.976516      | 4171         | 44.755867      |
| 1   | 18  | 5 026        | 19.986582       | 35943       | 21.987155      | 20781        | 44.743331      |
| 1   | 30  | 17 171       | 19.988992       | 191642      | 21.989001      | 122039       | 44.740570      |
| 1   | 38  | 5 13308      | 19.989039       | 577955      | 21.989681      | 374065       | 44.739512      |
| 1   | 39  | 5 90647      | 19.989101       | 648352      | 21.989119      | 428658       | 44.739520      |
| 1   | 40  | 675033       | 19.989153       | 751651      | 21.989156      | 493913       | 44.739547      |
| 2   | 6   | 4757         | 49.54975734     | 5630        | 51.53186078    | 4174         | 74.18697239    |
| 2   | 18  | 17 413       | 49.57934179     | 24916       | 51.58036988    | 24714        | 74.28997551    |
| 2   | 30  | 58 20202     | 49.59209659     | 96381       | 51.59363069    | 144114       | 74.33904514    |
| 2   | 38  | 130 004      | 49.5950385      | 244514      | 51.5966467     | 436679       | 74.3446992     |
| 2   | 39  | 140 739      | 49.59507564     | 266852      | 51.59682511    | 503989       | 74.34528594    |
| 2   | 40  | 155 888      | 49.59561150     | 292042      | 51.59684387    | 578175       | 74.34579087    |

Table 2: The 1st and 8th eigenvalues on $\Omega = (0, 2)^2 \setminus [1, 2]^2$ with $H = \sqrt{\frac{2}{16}}$.

| $k$ | $l$ | $N_{k,l}(1)$ | $\lambda_{k}(1)$ | $N_{k,l}(3)$ | $\lambda_{k}(3)$ | $N_{k,l}(10)$ | $\lambda_{k}(10)$ |
|-----|-----|--------------|-----------------|-------------|----------------|--------------|----------------|
| 1   | 6   | 3278        | 9.868593        | 3876        | 11.863678      | 2686         | 34.725519      |
| 1   | 18  | 19789       | 9.885896        | 23820       | 11.885939      | 13141        | 34.655247      |
| 1   | 30  | 111 849     | 9.889023        | 135412      | 11.889674      | 84307        | 34.648104      |
| 1   | 38  | 345699      | 9.889490        | 412412      | 11.889511      | 266844       | 34.641987      |
| 1   | 39  | 394 793     | 9.889514        | 474643      | 11.889540      | 307313       | 34.641930      |
| 1   | 40  | 452 105     | 9.889551        | 545221      | 11.889562      | 354041       | 34.641676      |
| 8   | 6   | 4103        | 49.336511       | 3671        | 51.372524      | 2902         | 74.010829      |
| 8   | 12  | 87 911      | 49.476412       | 7272        | 51.428181      | 6220         | 74.123357      |
| 8   | 18  | 20 394      | 49.536447       | 14189       | 51.510658      | 15446        | 74.193993      |
| 8   | 23  | 34 097      | 49.564431       | 481420      | 51.595113      | 33382        | 74.229178      |
| 8   | 24  | 37 940      | 49.567477       | 738241      | 51.596140      | 38817        | 74.231469      |
| 8   | 25  | 41 081      | 49.573008       | 1433565     | 51.596812      | 45354        | 74.242124      |