Hodge numbers of Fano threefolds via Landau–Ginzburg models

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Abstract. For each smooth Fano threefold $X$ with Picard number 1 we consider a weak Landau–Ginzburg model, that is a fibration over $\mathbb{C}^1$ given by a certain Laurent polynomial. In the spirit of L. Katzarkov’s program we prove that the number of irreducible components of the central fiber of its compactification is $h^{1,2}(X) + 1$. In particular, it does not depend on the compactification. The question of dependence on the model is open; however we produce examples of different weak Landau–Ginzburg models for the same variety with the same number of components of the central fiber.

Given a smooth Fano variety, Mirror Symmetry predicts the existence of a so called Landau–Ginzburg model — a one-dimensional family of varieties, whose symplectic geometry reflects the algebraic geometry of the Fano variety, and vice versa. There is a number of mirror conjectures (see [CdlOGP91], [Kon94], [KKP06], etc.).

In the paper we study the Hodge structure variations conjecture of Mirror Symmetry. It states that the regularized quantum differential equation of a Fano variety $X$ is of Picard–Fuchs type. Recall that the regularized quantum differential equation is a linear homogenous differential equation polynomially dependent on certain Gromov–Witten invariants — numbers that count rational curves lying on $X$. Picard–Fuchs equation for one-dimensional family of varieties is an equation whose solutions are periods of this family. For a precise definitions see [Prz09] and references therein.

By definition, the relevant Picard–Fuchs equation depends only on relative birational type of a family. In [Prz09] we found that, in the case of smooth Fano threefolds of Picard rank 1 (there are 17 families of them, the so called Iskovskikh list), one can construct dual families of quite specific type. Namely, we defined a (very) weak Landau–Ginzburg model for Fano variety of dimension $N$ to be a pencil of hypersurfaces in $(\mathbb{C}^*)^N$ parameterized by $\mathbb{C}$, or equivalently, a Laurent polynomial $f$ giving such a pencil. We refer the reader to [Prz09] for a list of polynomials for Fano threefolds. The assumption on a type of dual pencils seems to be not restrictive at all: Conjecture 15 in [Prz09] states that weak Landau–Ginzburg models exist for all smooth Fano varieties with Picard rank 1. This conjecture also holds for smooth complete intersections in (weighted) projective spaces and Grassmannians.

Which numerical invariants of Fano variety may be reconstructed from its Landau–Ginzburg model (or, more specific, from a weak one) and how? C. van Enckevort and D. van Straten ([vEvS06]) suggest to extract characteristic numbers of a general anticanonical section of Fano variety writing down a monodromy of the dual family in a specific basis.

L. Katzarkov’s recent idea (see [KKP06] and [KP08]) is to relate the Hodge type of Fano variety to the structure of the central fiber of dual Landau–Ginzburg model and the sheaf of vanishing cycles to this fiber.

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Our Theorem 13 says that the numbers of irreducible components of the central fiber of the standard compactification of weak Landau–Ginzburg model for Fano threefolds found in [Prz09] are the dimensions of their intermediate Jacobians plus 1. Theorem 11 says that this number of components does not in fact depend on the minimal compactification. We end the paper by discussion of the dependence of the number of components over 0 on the particular weak Landau–Ginzburg model. We give some examples demonstrating that this number should not depend on the particular toric weak Landau–Ginzburg model for given Fano threefold.

| No. | Index | Degree | $h^{12}$ | $\#$ comp. over 0 | No. | Index | Degree | $h^{12}$ | $\#$ comp over 0 |
|-----|-------|--------|---------|------------------|-----|-------|--------|---------|------------------|
| 1   | 1     | 2      | 52      | 53               | 10  | 1     | 22     | 0       | 1               |
| 2   | 1     | 4      | 30      | 31               | 11  | 2     | 8      | 1       | 21              | 22              |
| 3   | 1     | 6      | 20      | 21               | 12  | 2     | 8      | 2       | 10              | 11              |
| 4   | 1     | 8      | 14      | 15               | 13  | 2     | 8      | 3       | 5               | 6               |
| 5   | 1     | 10     | 10      | 11               | 14  | 2     | 8      | 4       | 2               | 3               |
| 6   | 1     | 12     | 7       | 8                | 15  | 2     | 8      | 5       | 0               | 1               |
| 7   | 1     | 14     | 5       | 6                | 16  | 3     | 27     | 2       | 0               | 1               |
| 8   | 1     | 16     | 3       | 4                | 17  | 4     | 64     | 0       | 1               |                 |
| 9   | 1     | 18     | 2       | 3                |     |       |        |         |                 |                 |

Table 1. Fano threefolds.

In the paper Fano variety means smooth Fano variety over $\mathbb{C}$ with Picard number 1.

1. Mirror symmetry conjecture for threefolds

We state a version of Mirror Symmetry conjecture of Hodge structure variations adopted to our goals. More precise see in [Prz09] and references therein.

Let $X$ be a Fano threefold. Let

$$a_{ij} = \langle (-K_X)^i, (-K_X)^{3-j}, -K_X \rangle_{j-i+1}, \quad 0 \leq i \leq 3, \; j > 0,$$

be a Gromov–Witten invariant whose meaning is an expected number of rational curves of anticanonical degree $j-i+1$ that intersect general representatives of homological classes dual to $(-K_X)^i, (-K_X)^{3-j}, -K_X$.

It turns out that such numbers determine Gromov–Witten theory of $X$. Moreover, regularized quantum $\mathcal{D}$-module (or, equivalently, Dubrovin’s second structural connection) for $X$ may be represented by a differential equation of type $D3$. Coefficients of this equation are polynomials in $a_{ij}$’s as follows.

**Definition 1.** Consider a ring $\mathcal{D} = \mathbb{C}[t, \frac{\partial}{\partial t}]$ and a differential operator $D = t \frac{\partial}{\partial t} \in \mathcal{D}$. The regularized quantum differential operator or operator of type $D3$ associated with Fano threefold $X$ is an operator

$$L_X = D^3 - t(2D + 1)(\lambda D^2 + (a_{11} + \lambda)D + \lambda a_{11} + \lambda) + t^2(2D + 1)((a_{11} + \lambda)^2D^2 + \lambda^2D^2 + 4(a_{11} + \lambda)\lambda D^2 - a_{12}D^2 - 2a_{01}D^2 + 8(a_{11} + \lambda)\lambda D - 2a_{12}D + 2\lambda^2D + 4a_{01}D + 2(a_{11} + \lambda)\lambda^2D + 6(a_{11} + \lambda)\lambda$$

$$+ \lambda^2 - 4a_{01}) - t^3(2D + 3)(D + 1)(\lambda^2(a_{11} + \lambda) + (a_{11} + \lambda)\lambda a_{12} + a_{02}) - (a_{11} + \lambda)a_{01} - a_{01}\lambda + t^4(D + 3)(D + 2)(D + 1)(-\lambda^2a_{12} + 2a_{02}\lambda + \lambda^2(a_{11} + \lambda)^2 - a_{03} + a_{01}^2 - 2a_{01}(a_{11} + \lambda)\lambda),$$

defined up to a shift $\lambda \in \mathbb{C}$. 

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Equations of this type (of type $DN$) were studied in [GS07].

**Definition 2.** (A unique) analytic solution of the equation $L_X I = 0$ of type

$$I_{I_0}^{X} = 1 + a_1 t + a_2 t^2 + \ldots \in \mathbb{C}[t], \quad a_i \in \mathbb{C},$$

is called the fundamental term of the regularized $I$-series of $X$.

This series is the constant term (with respect to cohomology) of regularized $I$-series for $X$, i.e. of generating series for 1-pointed Gromov–Witten invariants.

Consider a torus $\mathbb{C} \times ^3_m = \text{Spec} \mathbb{C}[x^\pm 1, y^\pm 1, z^\pm 1]$ and a function $f$ on it. This function may be represented by Laurent polynomial: $f = f(x, x^{-1}, y, y^{-1}, z, z^{-1})$. Let $\phi_f(i)$ be the constant term (i.e. the coefficient at $x^0y^0z^0$) of $f^i$. Put

$$\Phi_f = \sum_{i=0}^\infty \phi_f(i) \cdot t^i \in \mathbb{C}[t].$$

**Definition 3.** The series $\Phi_f = \sum_{i=0}^\infty \phi_f(i) \cdot t^i$ is called the constant terms series of $f$.

The following theorem is a sort of mathematical folklore (see, for instance, [Prz08, Proposition 2.3]). It states that the constant terms series of Laurent polynomial is the main period of a pencil given by this polynomial.

**Theorem 4.** Consider a pencil $(\mathbb{C}^*)^n \to \mathbb{B} = \mathbb{P} [u : v] \setminus (0 : 1)$ given by Laurent polynomial $f \in \mathbb{C}[\mathbb{Z}^n]$ with fibers $Y_\alpha = \{1 - \alpha f = 0\}, \alpha \in \mathbb{C} \setminus \{0\} \cup \{\infty\}$. Let the Newton polytope of $f$ contain 0 in the interior. Let $t \in \mathbb{B}$ be the local coordinate around $(0 : 1)$. Then there is a fiberwise $(n - 1)$-form $\omega_t \in \Omega_{(\mathbb{C}^*)^n/\mathbb{B}}^{n-1}$ and (locally defined) fiberwise $(n - 1)$-cycle $\Delta_t$ such that

$$\Phi_f(t) = \int_{\Delta_t} \omega_t.$$

**Definition 5.** Let $X$ be a smooth Fano threefold and $I_{I_0}^{X} \in \mathbb{C}[t]$ be its fundamental term of regularized $I$-series. The Laurent polynomial $f \in \mathbb{C}[\mathbb{Z}^3]$ is called a very weak Landau–Ginzburg model for $X$ if (up to the shift $f \mapsto f + \alpha, \alpha \in \mathbb{C}$)

$$\Phi_f(t) = I_{I_0}^{X}(t).$$

The Laurent polynomial $f \in \mathbb{C}[\mathbb{Z}^3]$ is called a weak Landau–Ginzburg model for $X$ if it is a very weak Landau–Ginzburg model for $X$ and for almost all $\lambda \in \mathbb{C}$ the hypersurface $\{1 - \lambda f = 0\}$ is birational to a K3 surface.

Let $\Delta \subset N_\mathbb{R} = \mathbb{Z}^3 \otimes _\mathbb{Z} \mathbb{R}$ be a Newton polytope of $f$. Let $\Delta^* \subset M = N^*$ be its dual polytope, that is $\Delta^* = \{p \in M \mid \forall x \in \Delta (p, x) \geq -1\}$. Let $m\Delta^* = \{p \in M \mid \frac{p}{m} \in \Delta^*\}$. Let $E(m)$ be the number of lattice points in $m\Delta^*$. Then $f$ is called toric weak Landau–Ginzburg model for $X$ if it is a weak Landau–Ginzburg model for $X$ and

$$E(m) = \frac{m(m+1)(2m+1)}{12} (-K_X)^3 + 2m + 1.$$

**Remark 6.** In particular, if $f$ is very weak Landau–Ginzburg model, then $L_X = PF_f$, where $PF_f$ is a Picard–Fuchs operator for a pencil given by $f$.

**Remark 7.** Let $T$ be the toric variety with the fan whose rays are generated by vertices of $\Delta$. Then $E(m)$ is its Hilbert polynomial. Thus, the weak Landau–Ginzburg model is toric if Hilbert polynomials for $X$ and $T$ coincide.

**Remark 8.** All polynomials in [Prz09, Table 1, are toric weak Landau–Ginzburg models. 
Conjecture 9 (Mirror Symmetry of Hodge structure variations). For any Fano variety $X$ there exists a one-dimensional family $Y \to \mathbb{C}$ whose Picard–Fuchs $\mathcal{D}$-module is isomorphic to a regularized quantum $\mathcal{D}$-module for $X$.

Assume that dim $X = 3$ and $Y = (\mathbb{C}^*)^3$. Then this conjecture reduces to the following one.

Conjecture 10. For any smooth Fano threefold $X$ with Picard number 1 there exists a (toric) (weak) Landau–Ginzburg model.

This conjecture is proven in [Prz09]. There is also a list of weak Landau–Ginzburg models for all 17 threefolds.

2. Main results

We call weak Landau–Ginzburg models for Fano threefolds listed in [Prz09] or the corresponding pencil standard.

The relative compactification of a pencil is called minimal if a general fiber and a total space of compactified pencil are smooth and there is no relative birational map (isomorphic on the torus) to a family with the same properties.

We call the following relative compactification of each standard pencil standard.

If $f$ corresponds to a complete intersection, compactify fibers of the pencil in a product of projective spaces as in Proposition 9 in [Prz09]. Otherwise compactify fibers in a projective space using natural embedding $\mathbb{A}[x, y, z] \hookrightarrow \mathbb{P}[x : y : z : t]$. Normalize threefold we get if it is not normal (two cases: sextic double solid and double Veronese cone, 1-st and 11-th in Table 1 correspondingly) by blowing up divisors. We get a singular variety whose singular set is the union of curves. There are two types of such curves: “horizontal”, intersecting all fibers, and “vertical”, lying over 0. Blow them up one by one. We typically get new singular curves upon blowup and (maybe) ordinary double points. These curves are du Val along the line as well. Repeat until we arrive at a threefold with ordinary double points. Any small resolution of these points (it turns out algebraic) is called the standard compactification.

Theorem 11. Let $Y \to \mathbb{C}$ and $\tilde{Y} \to \mathbb{C}$ be two minimal compactifications of a standard weak Landau–Ginzburg model. Then $Y$ and $\tilde{Y}$ are birationally isomorphic in codimension 2.

Proof. Compactify the standard pencil in a projective space or in a product of projective spaces as for a standard compactification. We get no “new” divisors over 0. A general fiber of the family we get is isomorphic to a K3 surface. So general fibers are the same for all minimal compactifications. This in particular means that all “horizontal” divisors we get appear in each compactification.

Resolving singularities for this compactification we normalize a variety (the normalization is unique). Resolving singular curves as in the standard compactification we see that on each step of resolution a singular locus is a union of curves which are du Val along a line and (probably) ordinary double points. One-dimensional singularities have a unique minimal resolution for a general transversal section. This means that each exceptional divisor we get appears in each compactification. Ordinary double points have two minimal resolutions, which are isomorphic in codimension 2. Thus minimal compactifications are biregular in codimension 2. □
Remark 12. The proof of this theorem involves that the standard compactification is minimal.

**Theorem 13.** Let $X$ be a smooth Fano threefold with Picard rank 1. Let $f : (\mathbb{C}^*)^3 \to \mathbb{C}$ be its standard weak Landau–Ginzburg model. Let $k_X$ be a number of irreducible components of the central fiber of minimal compactification of $f$. Then $k_X = h^{12}(X) + 1$.

**Proof.** By Theorem 11 it is enough to consider a standard compactification of $f$. The proof is given by direct calculations in all 17 cases. □

**Example 14.** Consider a variety $V_{16}$ (8-th in Table 1). Its standard weak Landau–Ginzburg model is

$$\frac{(x + 1)(y + 1)(z + 1)(x + y + 1)}{xyz}.$$ 

The compactification in a projective space is a family of quartics

$$\{(x + t)(y + t)(z + t)(x + y + z + t) = \lambda xyzt\} \subset \mathbb{A}[\lambda] \times \mathbb{P}[x : y : z : t].$$

There are 4 components of the central fiber $\lambda = 0$. Singularities are the disjoint union of 9 “horizontal” lines. All of them are products of du Val singularities of type $A_1$ in the neighborhood of a general point. After blowing them up we get three ordinary double points in the central fiber. So finally we get no new components of the central fiber and $k_{V_{16}} = 4 = h^{12}(V_{16}) + 1$.

**Example 15.** Consider a variety $V_{18}$ (9-th in Table 1). Its standard weak Landau–Ginzburg model is

$$\frac{(x + y + z)(x + xz + xy + xyz + z + y + yz)}{xyz}.$$ 

The compactification in a projective space is a family of quartics

$$\{(x + y + z)(xt^2 + xzt + xyt + xyz + zt^2 + yzt) = \lambda xyzt\} \subset \mathbb{A}[\lambda] \times \mathbb{P}[x : y : z : t].$$

There are 2 components of the central fiber $\lambda = 0$. Singularities are 3 “horizontal” lines globally of type $A_1$ along a line, 3 “horizontal” lines globally of type $A_2$ along a line and one “horizontal” line $\ell$ of type $A_1$ along a line away from the central fiber. The intersection of two components of the central fiber is a plane cubic with one node; this node lies on $\ell$. Blowing up $\ell$ we get one more “vertical” line globally of type $A_1$ along a line. Two components of fiber over 0 intersect now by the union of two lines (one of them is a singularity of our threefold); these lines intersect by 2 points. Blowing the rest singularity up we get 3 surfaces over 0; each 2 of them intersect by a rational curve; 3 such lines intersect by 2 points. So finally we get $k_{V_{18}} = 3 = h^{12}(V_{18}) + 1$.

Remark 16. The most interesting cases of compactifications are described in details in [KP08].

**Problem 17.** Generalize Theorem 13 for higher dimensions.

**Question 18.** May Theorem 11 be generalized? That is, let $f$ be (toric) weak Landau–Ginzburg model for Fano variety. Is it true that all minimal compactifications of $f$ are birational in codimension 2?
3. Discussion

In [Prz09] we prove that any Fano threefold has a weak Landau–Ginzburg model. Theorem 13 shows that they “know” the dimension of intermediate Jacobian of corresponding Fano variety. Moreover, Theorem 11 shows that compactified Landau–Ginzburg model is determined (up to flops) by the weak one. However given Fano variety may have several weak Landau–Ginzburg models. So the natural question is the following. Given Fano threefold consider classes of its weak Landau–Ginzburg models up to fiberwise birational isomorphism (i.e. ones that have “different compactifications”). Is it true that numbers of components over 0 of the compactification of (toric) weak Landau–Ginzburg models from all classes are the same? If not, what requirements on weak Landau–Ginzburg models guarantee that such numbers for admissible polynomials are the same?

By several reasons for Fano threefolds of large degree it is natural to consider toric weak Landau–Ginzburg models with the following properties. Firstly their Newton polytopes should contain only one strictly internal integral point. In the other words, consider toric varieties given by fans whose rays are generated by vertices of Newton polytopes we discuss. Then these toric varieties should have canonical singularities. Secondly these toric varieties should be of Picard rank 1. We call such weak Landau–Ginzburg models and their Newton polytopes canonical.

Remark 19. Unfortunately, it is not enough to consider canonical toric weak Landau–Ginzburg models for all Fano threefolds. That is, sextic double solid has no canonical weak Landau–Ginzburg model as there is no integral polytope of volume \( \frac{2}{3} \) containing only one strictly internal integral point. By the similar reasons there is no canonical weak Landau–Ginzburg model for double Veronese cone. All the rest Fano threefolds have canonical weak Landau–Ginzburg models.

Proposition 20. Let \( f \) be canonical weak Landau–Ginzburg model for \( \mathbb{P}^3 \). Then all fibers of its minimal compactification are irreducible and smooth except for 4 fibers over roots of degree 4 from 256. All of them are K3 surfaces with one ordinary double point.

Proof. There are 5 canonical polytopes of volume \( \frac{64}{3} \). Three of them give the following weak Landau–Ginzburg models.

\[
\begin{align*}
    x + y + z + \frac{1}{xyz}, \\
    \frac{(x + 1)^2}{xyz} + \frac{y}{z} + z, \\
    x + \frac{y}{x} + \frac{z}{x} + \frac{1}{xy} + \frac{1}{xz}.
\end{align*}
\]

One may compactify them in a way similar to the standard compactification and prove the statement of the proposition. \( \square \)

Proposition 21. Let \( f \) be canonical Landau–Ginzburg model for smooth quadric in \( \mathbb{P}^4 \). Then all fibers of its minimal compactification are irreducible and smooth except for 3 fibers over roots of degree 3 from 108 and a fiber over 0. All of them are K3 surfaces with one ordinary double point.

\(^1\)Canonicity means that polytope contains exactly one strictly internal integral point.
Proof. There are 5 canonical polytopes of volume $\frac{54}{3!}$. Four of them give the following weak Landau–Ginzburg models.

\[
\frac{(x+1)^2}{xyz} + y + z,
\]
\[
x + y + z + \frac{1}{xy} + \frac{1}{xz},
\]
\[
\frac{(x+y)^2}{x} + \frac{1}{xy} + \frac{z}{x} + \frac{y}{xz},
\]
\[
\frac{(x+1)^3}{xyz} + \frac{y}{z} + \frac{2}{z} + \frac{2x}{z} + \frac{z^2}{y}.
\]

One may compactify them in a way similar to the standard compactification and prove the statement of the proposition. □

The similar picture holds for Fano threefolds with non-trivial intermediate Jacobians.

Example 22. Consider a complete intersection of 2 quadrics in $\mathbb{P}^5$. Consider two following toric weak Landau–Ginzburg models for it.

\[
\frac{(x+1)^2(y+1)^2}{xyz} + z,
\]
\[
x + y + z + \frac{1}{xyz} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xyz.
\]

Compactifying them in a way similar to the standard compactification (it is more convenient to change variables $a = \frac{1}{x}$ in the second case) one may get families of $K3$ surfaces. Both of them have 2 fibers with ordinary double points (over $\pm 8$) and the central fiber consisting of 3 components. Singularities of the central fiber (the intersection of its components) are 3 rational curves. All these curves intersect in 2 points.

Remark 23. Singularities of toric weak Landau–Ginzburg models we get agree with expectations of Homological Mirror Symmetry. That is, a derived category of $\mathbb{P}^3$ is generated by $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2),$ and $\mathcal{O}(3)$, a derived category of quadric is generated by $\mathcal{B}, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$, where $\mathcal{B}$ is a category generated by one element, and a derived category of complete intersection of 2 quadrics is generated by $\mathcal{D}^b(C), \mathcal{O}, \mathcal{O}(1)$, where $C$ is a curve of genus 2.

Remark 24. Coordinates of singular fibers of weak Landau–Ginzburg model are determined by its Picard–Fuchs equation. However the number of components of fiber over 0 does not. The example is the following. Let $X$ be a complete intersection of 2 quadrics in $\mathbb{P}^5$. Consider the following weak (but not toric!) Landau–Ginzburg model for $X$:

\[
\left( x + \frac{1}{x} \right) \left( y + \frac{1}{y} \right) \left( z + \frac{1}{z} \right).
\]

The number of components over 0 of its minimal compactification is 30, but $h^12(X) = 2$.

Problem 25. One may check that toric weak Landau–Ginzburg models from Proposition 20 (resp. Proposition 21, Example 22) are relatively birational. This means that smooth fibers of their minimal compactifications are isomorphic (two birationally isomorphic smooth $K3$ surfaces are isomorphic). Moreover, singular fibers with non-zero coordinates are also isomorphic as they are isomorphic in the neighborhood of singularities. This means that these minimal compactifications differ by birational transforms concentrated over zero. Describe these birational transforms.
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