OPTIMAL TRANSPORT APPROACH TO MICHAEL-SIMON-SOBOLEV INEQUALITIES IN MANIFOLDS WITH INTERMEDIATE RICCI CURVATURE LOWER BOUNDS

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ABSTRACT. We generalize McCann’s theorem of optimal transport to a submanifold setting and use it to prove Michael-Simon-Sobolev inequalities for submanifolds in manifolds with lower bounds on intermediate Ricci curvatures. The results include a variant of the sharp Michael-Simon-Sobolev inequality in S. Brendle’s [arXiv:2009.13717] when the intermediate Ricci curvatures are nonnegative.

Note: This preprint has not undergone peer review (when applicable) or any post-submission improvements or corrections. The Version of Record of this article is published in Annals of Global Analysis and Geometry, and is available online at https://doi.org/10.1007/s10455-023-09934-9

1. INTRODUCTION

This article is two-fold. In the first part, we generalize McCann’s theorem [28] in optimal transport theory to a submanifold setting and obtain a corresponding change of variable formula. In the second part, we use these results to give optimal transport proofs of Michael-Simon-Sobolev inequalities for compact submanifolds in complete Riemannian manifolds with intermediate Ricci curvatures bounded from below. Most of the arguments are inspired by S. Brendle’s [8], where the Alexandrov-Bakelman-Pucci (ABP) maximum principle was used; a hint to use optimal transport can be found in Brendle’s earlier work [7].

1.1. Generalization of McCann’s Theorem. In optimal transport theory, McCann’s theorem provides the existence and description of optimal transport maps from one Borel probability measure to another on a complete Riemannian manifold when the cost is quadratic (distance squared) and the source measure is absolutely continuous w.r.t. the volume measure. We refer to Section 2 for the terminology and classical results in optimal transport theory.

In the first part of this article, we study an optimal transport problem from a lower dimension to the top dimension on a complete Riemannian manifold with the quadratic cost, in the sense that the two measures are absolutely continuous with respect to the volume measure of a submanifold and that of the ambient manifold, respectively. Our first result is the following:

**Theorem 1.1 (Lemma 3.2, Theorem 3.3, Corollary 3.4).** Let $M$ be a complete Riemannian manifold and $d_M$ the geodesic distance function. Let $\Sigma \subset M$ be a compact submanifold, possibly...
with boundary, and $\Omega \subset M$ a compact regular domain. Let $\mu$ and $\nu$ be Borel probability measures on $\Sigma$ and $\Omega$, respectively, that are absolutely continuous with respect to the corresponding volume measures. Let $\phi$ be a Kantorovich potential function on $\Sigma$ associated to an optimal transport plan of $\mu$ and $\nu$ with the cost function $c := \frac{1}{2} d_M^2$ on $\Sigma \times \Omega$. Then there is a measurable subset $A$ of the normal bundle $T^\perp \Sigma$ and a map $\Phi : A \to \Omega$ defined by
\begin{equation}
\Phi(x, v) = \exp_M x \left( -\nabla_{\Sigma} \phi(x) + v \right)
\end{equation}
where $x \in \Sigma$ and $v \in T^\perp_x \Sigma$, such that

1. $p_{T^\perp \Sigma}(A)$ has full $\mu$-measure in $\Sigma$, where $p_{T^\perp \Sigma} : T^\perp \Sigma \to \Sigma$ is the bundle projection;
2. $\Phi(A)$ has full $\nu$-measure in $\Omega$;
3. For $(x, v) \in A$, the curve $\gamma(t) := \exp_M x \left( -t\nabla_{\Sigma} \phi(x) + tv \right)$ for $t \in [0, 1]$ is a minimizing geodesic;
4. The restricted map at each $x \in p_{T^\perp \Sigma}(A)$ gives a bijection:
\begin{equation}
\Phi(x, \cdot) : A \cap T^\perp_x \Sigma \to \partial^+ \phi(x) \cap \Phi(A).
\end{equation}

Combining Theorem 1.1 above with the transport condition, we deduce the following change of variable formula:

**Theorem 1.2 (Theorem 3.5).** With the same notations as in Theorem 1.1, suppose $f$ and $g$ are the density functions of $\mu$ and $\nu$ w.r.t. the volume measures of $\Sigma$ and $M$, respectively. Then for $\mu$-a.e. $x \in \Sigma$, we have
\[ \int_{A \cap T^\perp_x \Sigma} g \circ \Phi(x, v) \left| \det D\Phi(x, v) \right| dv = f(x). \]

Roughly speaking, Theorem 1.1 identifies the $c$-superdifferential $\partial^+ \phi(x)$ with a specific part of the normal fiber up to null subsets. Compared with the classical McCann’s theorem, where point-wise transportation takes place, our result matches the intuition that in order to compensate the dimensional difference between the source and the target measure, the set-valued map $\partial^+ \phi$ spreads out the point mass at $x$ over the fiber $A \cap T^\perp_x \Sigma$ of dimension equal to the codimension of the submanifold, which is exactly the content of Theorem 1.2. On the other hand, we also observe the same phenomenon, a.k.a. “displacement interpolation,” that mass transportation occurs along minimizing geodesics.

In the literature, optimal transport problems involving submanifolds or dimensional differences have been considered by several authors, e.g.,

- In W. Gangbo and R. J. McCann’s [24], they considered an optimal transport problem between measures on hypersurfaces in the Euclidean space. Their result about the specific shape of $\partial^+ \phi$ (Corollary 1.5, Lemma 1.6 in [24]) can almost be recovered by Theorem 1.1 (more precisely by Lemma 3.2) since the map $\Phi$ becomes linear in the variable $v$ of normal vectors when the ambient space $M$ is Euclidean.
- P. Castillon’s [13] proved a sharp weighted Michael-Simon-Sobolev inequality for submanifolds in the Euclidean space. In the article, Castillon studied an optimal transport problem from a measure on a submanifold to a measure on a linear subspace of the same dimension as the submanifold, and the arguments involved Euclidean orthogonal projections. In contrast, our approach is closer to the classical optimal transport problem by ”raising” the
dimension to the top and considering the transportation between objects of the same top dimension (the set \( A \) and \( \Omega \) in Theorem 1.1).

- B. Pass’s [30] and McCann and Pass’s [29] studied an optimal transport problem from a higher dimension to a lower dimension with general cost functions. This setting is actually considered in the proof of Theorem 1.1 (see Theorem 3.3), and the foliation structure in [30] is realized as the normal bundle structure in our case. In addition, Theorem 2 in [29] is a similar change of variable formula as Theorem 1.2.

- C. Ketterer and A. Mondino’s [27] considered an optimal transport problem between measures of the same dimension which is smaller than that of the ambient space. More precisely, these measures are supported on rectifiable subsets of the same smaller dimension and absolutely continuous with respect to the corresponding Hausdorff measure. We are inspired by [27] to use lower bounds on the intermediate Ricci curvature in the proof of Michael-Simon-Sobolev inequalities in Section 4.

- A recent paper [9] by S. Brendle and M. Eichmair used the Kantorovich dual functions from optimal transport theory to prove a Michael-Simon-Sobolev inequality in the Euclidean space. In comparison, we use the graph-like property of the support of an optimal transport plan (see e.g. Theorem 2.13 in L. Ambrosio and N. Gigli’s [3]; also see Theorem 2.6).

To the best of our knowledge, the optimal transport problem we consider here from a lower dimension to the top dimension is new. We speculate future application of these results and possible generalizations to non-smooth settings, e.g. for subspaces in metric measure spaces.

1.2. Michael-Simon-Sobolev inequalities. In the second part of this article, we use the results from the first part to prove Michael-Simon-Sobolev inequalities for compact submanifolds in complete Riemannian manifolds with intermediate Ricci curvatures bounded from below. The results will be divided into 3 cases where the lower bounds are either zero, positive, or negative.

In the literature, intermediate Ricci curvature lower bounds have been used to prove geometric inequalities for submanifolds, e.g. Brunn-Minkowski type inequalities in [27] and a generalization of Heintze-Karcher Comparison [26] in Y. K. Chahine’s [16] by replacing the sectional curvature condition with integral bounds on the intermediate Ricci curvatures.

1.2.1. Nonnegative Intermediate Ricci curvatures. We begin with the cleanest case where the intermediate Ricci curvatures are nonnegative. The following partially sharp inequality is a variant of Theorem 1.4 in [8]:

**Theorem 1.3.** Let \( M \) be a complete noncompact Riemannian manifold of dimension \( n + m \) with \( \text{Ric}_n, \text{Ric}_m \geq 0 \) and \( m \geq 2 \). Let \( \theta \) be the asymptotic volume ratio of \( M \):

\[
\theta := \lim_{r \to \infty} \frac{\text{vol}_M(B_r(x_0))}{|B^{n+m}| r^{n+m}}
\]

for some (any) point \( x_0 \in M \). Let \( \Sigma \subset M \) be a compact submanifold of dimension \( n \), possibly with boundary \( \partial \Sigma \). Let \( H \) be the mean curvature vector of \( \Sigma \). Let \( f \) be a positive smooth function on \( \Sigma \). Then

\[
(n + m) \frac{1}{m} \left( \int |\nabla f|^{n+1} \right)^{n/(n+1)} \leq \int_{\Sigma} |\nabla \Sigma f| + \int_{\partial \Sigma} f + \int_{\Sigma} f |H|.
\]
Here, $\text{Ric}_n$ and $\text{Ric}_m$ are the intermediate $n$- and $m$-Ricci curvatures of $M$ (see the definitions in Section 4), and $|B^k|$ is the volume of the unit ball in $\mathbb{R}^k$.

Compared with Theorem 1.4 in [8], we here use a weaker assumption that the intermediate Ricci curvatures of the ambient manifold are nonnegative, although our conclusion is also weaker. Nevertheless, our inequality is still sharp when $m = 2$, and in this case the curvature condition is equivalent to $\text{Sec} \geq 0$ (See Remark 4.1). Additionally, as Theorem 1.4 in [8], our inequality also holds (sharply) for hypersurfaces with $m = 1$ since we can raise the codimension by considering $M \times \mathbb{R}$ while preserving the asymptotic volume ratio and the curvature condition $\text{Ric}_1 \geq 0$, which is again equivalent to $\text{Sec} \geq 0$. We remark that the intermediate Ricci curvature assumptions also apply to the original argument in [8].

Our optimal transport proof of Theorem 1.3 is different from the ABP approach in [8], although the structures of the proofs bear a close resemblance. In our proof, the Kantorovich potential $\phi$ from optimal transport (see Theorem 1.1) plays the same role as the solution to a linear elliptic PDE in the ABP approach, and the key computation is the estimate of the Jacobian determinant of the map $\Phi$ along the geodesics $\gamma$ from Theorem 1.1 which is where the curvature condition is used (see Proposition 4.5 and Corollary 4.6 compared with Proposition 4.6 and Corollary 4.7 in [8]). One can also see the connection between these two approaches from the definition of the $c$-superdifferential where a fixed optimal transport plan is supported: for $\tau \in \Sigma$, we have $\zeta \in \partial^c \phi(\tau)$ if and only if the function $\frac{1}{2}d^2_M(\cdot, \zeta) - \phi$ on $\Sigma$ is minimized at $\tau$; this exact statement appeared in the argument of Lemma 4.2 in [8].

In the literature, both optimal transport and the ABP method have been applied to prove many geometric and functional inequalities. For the optimal transport aspect, see e.g. D. Cordero-Erausquin, McCann, and M. Schmuckenschläger’s [19], Cordero-Erausquin, B. Nazaret, and C. Villani’s [18], A. Figalli, F. Maggi, and A. Pratelli’s [21]. For the ABP method, see e.g. N. Trudinger’s [34], X. Cabré’s [11], Y. Wang and X. Zhang’s [36], Cabré, X. Ros-Oton, and J. Serra’s [12], C. Xia and X. Zhang’s [37]. We remark that optimal transport also applies to more general metric measure spaces, e.g. F. Cavalletti and Mondino’s [15], G. Antonelli, E. Pasqualetto, M. Pozzetta, and D. Semola’s [5], Cavalletti and D. Manini’s [14]. Also see [1], [22] for other approaches.

Among these results about inequalities, we point out that a sharp Sobolev inequality for domains in complete manifolds with nonnegative Ricci curvature was proved also in [8] by the ABP method, and by optimal transport in Z. Balogh and A. Kristály’s [6]. One goal of this article is to provide an optimal transport counterpart regarding the proof of the Michael-Simon-Sobolev inequalities for submanifolds, as in [9] when the ambient space is Euclidean.

1.2.2. Positive lower bounds on Intermediate Ricci Curvatures. We apply our optimal transport argument to the case where the intermediate Ricci curvatures are bounded from below by some positive constants. The main result is the following:

**Theorem 1.4.** Let $M$ be a closed Riemannian manifold of dimension $n + m$ with $\text{Ric}_n \geq (n-1)k_1$ and $\text{Ric}_m \geq (m-1)k_2$ for some $k_1, k_2 > 0$ and $m \geq 2$. Let $\Sigma \subset M$ be a compact submanifold of dimension $n$, possibly with boundary $\partial \Sigma$. Let $H$ be the mean curvature vector of $\Sigma$. Let $f$ be
a positive smooth function on $\Sigma$. For $\epsilon > 0$, let $N_\epsilon$ be the $\epsilon$-tubular neighborhood of $\Sigma$. Then we have

$$\left( \frac{\text{vol}_M(M \setminus N_\epsilon)}{|B^m| \text{diam}(M)^m \sin^m \left( \epsilon \sqrt{\frac{k_2(m-1)}{m}} \right)} \right)^{\frac{1}{n}} \left( \int_{\Sigma} f^\frac{n}{n-m} \right)^{\frac{n-m}{n}} \leq \cos \left( \epsilon \sqrt{\frac{k_1(n-1)}{n}} \right) \int_{\Sigma} f + \frac{|\text{diam}(M)|}{n} \left( \int_{\partial \Sigma} f + \int_{\Sigma} |\nabla \Sigma f| + \int_{\Sigma} f|H| \right).$$

The lower bounds here on $\text{Ric}_n$ and $\text{Ric}_m$ are written in this way since $\text{Sec} \geq k > 0$ implies $\text{Ric}_n \geq (n-1)k$ and $\text{Ric}_m \geq (m-1)k$ in general. We also remark that these two curvature bounds are assumed exactly to fit in our argument; they may implicitly imply one another (see Remark 4.1). Additionally, the argument can be modified for the case of hypersurfaces and give a similar inequality.

Theorem 1.6 potentially gives a lower bound on the volume of the tubular neighborhood $N_\epsilon$. On the other hand, by taking $\epsilon \to 0$, we have the following Michael-Simon-Sobolev inequality for the submanifold $\Sigma$:

**Corollary 1.5.** Let $M$ be a closed Riemannian manifold of dimension $n+m$ with $\text{Ric}_n, \text{Ric}_m > 0$. Let $\Sigma \subset M$ be a compact submanifold of dimension $n$, possibly with boundary $\partial \Sigma$. Let $H$ be the mean curvature vector of $\Sigma$. Let $f$ be a positive smooth function on $\Sigma$. Then we have

$$\left( \frac{\text{vol}_M(M)}{|B^m| \text{diam}(M)^m} \right)^{\frac{1}{n}} \left( \int_{\Sigma} f^\frac{n}{n-m} \right)^{\frac{n-m}{n}} \leq \int_{\Sigma} f + \frac{|\text{diam}(M)|}{n} \left( \int_{\partial \Sigma} f + \int_{\Sigma} |\nabla \Sigma f| + \int_{\Sigma} f|H| \right).$$

We remark that Corollary 1.5 also follows from the proof of Theorem 1.3 by assuming $M$ is closed, and hence also holds when $\text{Ric}_n, \text{Ric}_m \geq 0$ (see the discussion in the beginning of Subsection 4.2).

1.2.3. **Negative lower bounds on Intermediate Ricci Curvatures.** In the general case of negative lower bounds, we have the following local result:

**Theorem 1.6.** Let $M$ be a complete Riemannian manifold of dimension $n+m$ with $\text{Ric}_n \geq nk_1$ and $\text{Ric}_m \geq mk_2$ for some $k_1, k_2 < 0$. Let $\Sigma \subset M$ be a compact submanifold of dimension $n$, possibly with boundary $\partial \Sigma$. Let $H$ be the mean curvature vector of $\Sigma$. Let $f$ be a positive smooth function on $\Sigma$. Assume that $\Sigma$ is contained in some geodesic ball $B^M_\frac{1}{2}(x_0)$. Then we have

$$\left( \frac{\text{vol}_M \left( B^M_\frac{1}{2}(x_0) \right)}{|B^m| \sinh^m (r \sqrt{-k_2})} \right)^{\frac{1}{n}} \left( \int_{\Sigma} f^\frac{n}{n-m} \right)^{\frac{n-m}{n}} \leq \cosh (r \sqrt{-k_1}) \int_{\Sigma} f + \frac{\sinh (r \sqrt{-k_1})}{\sqrt{-k_1}} \left( \int_{\partial \Sigma} f + \int_{\Sigma} |\nabla \Sigma f| + \int_{\Sigma} f|H| \right).$$

The lower bounds here on $\text{Ric}_n$ and $\text{Ric}_m$ are written in this way since $\text{Sec} \geq k$ for some $k < 0$ implies $\text{Ric}_n \geq nk$ and $\text{Ric}_m \geq mk$ in general. We also remark that these two curvature bounds
are assumed exactly to fit in our argument; they may implicitly imply one another (see Remark 4.1).

Theorem 1.6 has some local feature as the inequality might become trivial as \( r \to \infty \), e.g. in the \((n + m)\)-hyperbolic space \( \mathbb{H}^{n+m} \) with \( k_1 = -1 = k_2 \). Recently, J. Cui and P. Zhao [20] proved a sharp Michael-Simon-Sobolev inequality for star-shaped hypersurfaces in the hyperbolic space.

1.3. Outline. The outline of this article is as follows. In Section 2 we set up the terminology and summarize some classical results in optimal transport theory. In Section 3 we prove the results about optimal transport in the submanifold setting, including the generalization of McCann’s theorem and the change of variable formula. In Section 4 we prove the Michael-Simon inequalities for compact submanifolds under the assumption that the ambient manifold has lower bounds of various signs on the intermediate Ricci curvatures.

Acknowledgment. The author would like to thank Prof. Aaron Naber for his guidance and inspiring comments. He is also grateful to Prof. Simon Brendle and Prof. Andrea Mondino for their useful suggestions. Additionally, he appreciates Prof. Marco Pozzetta and Prof. Gioacchino Antonelli’s generosity in sharing various relevant references.

2. Preliminaries

In this section, we set up the terminology and summarize some classical results in optimal transport theory, including McCann’s theorem about optimal transport problems on Riemannian manifolds with the distance-squared cost. The contents here can be found in [3], [28] and C. Villani’s [35].

Let \( X \) and \( Y \) be Polish spaces equipped with Borel probability measures \( \mu \) and \( \nu \), respectively. Let \( c : X \times Y \to \mathbb{R} \) be a measurable function. Viewing \( c \) as the cost function, we would like to minimize the total cost to push \( \mu \) forward to \( \nu \). That is, we want to minimize

\[
\int_X c(x, T(x))d\mu(x)
\]

among measurable maps \( T : X \to Y \) with the push-forward condition \( T_\#\mu = \nu \), meaning that \( \mu(T^{-1}(B)) = \nu(B) \) for any Borel set \( B \subset Y \). This problem is the so-called Monge’s formulation of optimal transport, and such minimizing maps are called optimal transport maps.

In general, Monge’s formulation can be ill-posed; for example, there can even be no such a map \( T \) that pushes \( \mu \) forward to \( \nu \). Therefore, we may relax the problem to the following version, known as Kantorovich’s formulation: we minimized

\[
\int_{X \times Y} c(x, y)d\pi(x, y)
\]

among Borel probability measures \( \pi \) on \( X \times Y \) with the transport conditions \( \pi\mid_X = \mu \) and \( \pi\mid_Y = \nu \); such measures are called transport plans. Kantorovich’s formulation is a relaxation since any transport map \( T \) gives rise to a transport plan on \( X \times Y \) by pushing \( \mu \) forward via the map \( \text{id}_X \times T : X \to X \times Y \), and it solves the issue above in Monge’s formulation since transport plans always exist, e.g. the product measure \( \mu \times \nu \). Moreover, the following theorem gives the existence of minimizing measures under mild assumptions:
Remark 2.1 ([3], Theorem 2.5). Assume that the cost function $c$ is lower semicontinuous and bounded from below. Then there exists a minimizer for Kantorovich’s formulation (2.2).

Such minimizing measures are called optimal transport plans. With stronger assumptions, it can be shown that such measures are supported on the graph of set-valued functions with generalized convexity (Theorem 2.6 below); the precise statement requires the following terminologies associated to the cost function $c$. We denote by $\overline{\mathbb{R}}$ the extended real numbers $\mathbb{R} \cup \{\pm \infty\}$.

Definition 2.2 ($c^+$-transform). Let $\psi : Y \to \overline{\mathbb{R}}$ be any function. Its $c^+$-transform $\psi^+ : X \to \overline{\mathbb{R}}$ is defined as

$$\psi^+(x) = \inf_{y \in Y} c(x, y) - \psi(y).$$

Similarly, given $\phi : X \to \overline{\mathbb{R}}$ we can define $\phi^+ : Y \to \overline{\mathbb{R}}$ as

$$\phi^+(y) = \inf_{x \in X} c(x, y) - \phi(x).$$

Definition 2.3 ($c$-concavity). A function $\phi : X \to \overline{\mathbb{R}}$ is $c$-concave if $\phi = \psi^+$ for some $\psi : Y \to \overline{\mathbb{R}}$. Similarly, a function $\psi : Y \to \overline{\mathbb{R}}$ is $c$-concave if $\psi = \phi^+$ for some $\phi : X \to \overline{\mathbb{R}}$.

A simple observation is that $\psi^+ = \psi^{c^+ c^+}$. Thus we have the following lemma:

Lemma 2.4. $\phi$ is $c$-concave if and only if $\phi = \phi^{c^+ c^+}$.

Next we define the following graph-like set:

Definition 2.5 ($c$-superdifferential). Let $\phi : X \to \overline{\mathbb{R}}$ be any function. The $c$-superdifferential $\partial^c \phi \subset X \times Y$ is defined as

$$\partial^c \phi = \{ (x, y) \in X \times Y \mid \phi(x) + \phi^+(y) = c(x, y) \}.$$

The $c$-superdifferential of $\phi$ at $x$, denoted by $\partial^c \phi(x)$, is the set of $y \in Y$ such that $(x, y) \in \partial^c \phi$. Similarly, we can define $c$-superdifferential of $\psi : Y \to \overline{\mathbb{R}}$.

Remark 2.1. Since the cost function $c$ takes value in $\mathbb{R}$, we have that $(x, y) \in \partial^c \phi$ only if both $\phi(x)$ and $\phi^+(y)$ are finite.

Remark 2.2. Since $\phi^+(y) \leq c(z, y) - \phi(z)$ for all $z \in X$, we have that $(x, y) \in \partial^c \phi$ if and only if the function $c(\cdot, y) - \phi$ on $X$ is minimized at $x$. We also remark that if $\phi$ is $c$-concave, then $\partial^c \phi = \partial^c \phi^+$.

With these terminologies introduced, we are now ready to state the following classical theorem:

Theorem 2.6 ([3], Theorem 2.13; also see [32], Theorem 2.2). Assume that the cost function $c : X \times Y \to \mathbb{R}$ is continuous and bounded from below, and there are functions $a \in L^1(\mu)$, $b \in L^1(\nu)$ such that

$$c(x, y) \leq a(x) + b(y).$$

Let $\pi$ be a Borel probability measures on $X \times Y$ with $\pi\big|_X = \mu$ and $\pi\big|_Y = \nu$. Then $\pi$ is optimal if and only if there exists a $c$-concave function $\phi$ on $X$ such that $\max\{\phi, 0\} \in L^1(\mu)$ and $\text{supp } \pi \subset \partial^c \phi$.

Remark 2.3. Such a $c$-concave function $\phi$ is sometimes called a Kantorovich potential.
Now we specify the two spaces $X$ and $Y$ to be compact subsets of a complete Riemannian manifold $M$ and let $c(x, y) = \frac{1}{2}d_M^2(x, y)$ on $X \times Y$ be the cost function, where $d_M$ is the geodesic distance function on $M$. Under this setting, McCann in [28] showed the existence of optimal transport maps, generalizing Brenier’s theorem [10] in the Euclidean setting:

**Theorem 2.7** (McCann’s theorem, [28]). Let $\mu$ and $\nu$ be Borel probability measures on a compact regular domain $X$ and a compact subset $Y$ in a complete Riemannian manifold $M$, respectively. Suppose $\mu$ is absolutely continuous with respect to the volume measure. Then there is an optimal transport map $F$ pushing $\mu$ forward to $\nu$. Furthermore, $F$ is given $\mu$-a.e. by $F(x) = \exp_x(-(\nabla \phi(x)))$ for some $c$-concave function $\phi$ on $X$, and the geodesic $\gamma(t) := \exp_x(-t\nabla \phi(x))$ for $t \in [0, 1]$ is minimizing.

**Sketch proof.** Since the cost function $c$ is continuous and bounded, Theorem 2.1 gives the existence of an optimal transport plan $\pi$, and by Theorem 2.6 we have $\text{supp} \pi \subset \partial^+ \phi$ for some $c$-concave function $\phi$ on $X$. Next, Lemma 2 in [28] showed that $\phi$ is Lipschitz, and hence by Rademacher’s theorem (see e.g. Lemma 4 in [28]) $\phi$ is differentiable a.e. w.r.t. the volume measure $\text{vol}_M$. Since $\mu \ll \text{vol}_M$, $\phi$ is thus differentiable $\mu$-a.e. On the other hand, Lemma 7 in [28] showed that $\partial^+ \phi(x)$ contains only the point $\exp_x(-(\nabla \phi(x)))$ whenever $\phi$ is differentiable at $x$, and in this case the geodesic $\gamma$ is minimizing. It is now direct to check that the map $F(x) = \exp_x(-(\nabla \phi(x)))$ defined for $\mu$-a.e. $x \in X$ is an optimal transport map. \qed

**Remark 2.4.** [28] also proved the uniqueness of the optimal transport map. In addition, the assumptions of Theorem 2.7 can be relaxed; see [23], [25].

3. **Optimal Transport from Measures on Submanifolds**

In this section, we prove a generalization of McCann’s theorem in a submanifold setting, which says that for a.e. $x \in \Sigma$, the $c$-differential $\partial^+ \phi(x)$ can roughly be identified as some specific part of the normal fiber $T^\perp_x \Sigma$. Together with the transport condition, we deduce a change of variable formula. The arguments here are inspired by [8], [28], and the details involve results from [19], [35], F. Santambrogio’s [33], Ambrosio, N. Fusco, and D. Pallara’s [2].

The setting is as follows. Let $M$ be a complete Riemannian manifold of dimension $n + m$. Let $\Omega \subset M$ be a compact regular domain and $\Sigma \subset M$ a compact submanifold of dimension $n$, possibly with boundary. Let $\mu$ and $\nu$ be Borel probability measures on $\Sigma$ and $\Omega$, respectively, and we assume they are absolutely continuous w.r.t. the corresponding volume measures.

**Notation:** The operators and maps associated to $\Sigma$ will be subscripted or superscripted by $\Sigma$, e.g., $\nabla^\Sigma$, $\exp^\Sigma$, $\Delta_\Sigma$. For $x \in \Sigma$ and $v \in T_xM$, we write $v = v^T + v^\perp$ as the orthogonal decomposition of $v$, where $v^T \in T_x\Sigma$ and $v^\perp \in T^\perp_x \Sigma$. We will often write $d$ instead of $d_M$ to denote the geodesic distance function on $M$, and $d_\zeta(\cdot)$ denotes the distance from some fixed point $\zeta \in M$.

Consider the optimal transport problem between $\mu$ and $\nu$ with $c = \frac{1}{2}d^2$ on $\Sigma \times \Omega$ as the cost function. Since $c$ is continuous and bounded, Theorem 2.1 applies to give an optimal transport plan $\pi$ as a measure on $\Sigma \times \Omega$, and Theorem 2.6 shows that $\text{supp} \pi \subset \partial^+ \phi$ for some $c$-concave function $\phi$ on $\Sigma$. By Lemma 2 in [28] and the fact that $d_M$ is dominated by the intrinsic distance on $\Sigma$, $\phi$ is then Lipschitz on $\Sigma$. Thus by Rademacher’s theorem (see e.g. [28], Lemma 4), $\phi$ is differentiable a.e. The following results provide the description we need about $\partial^+ \phi(x)$ whenever $\phi$ is differentiable at $x$; they can be viewed as the submanifold version of Proposition 6 and Lemma 7 in [28].
Proposition 3.1 (Superdifferentiability of distance squared restricted to submanifolds). Let \( x \in \Sigma \setminus \partial \Sigma, \zeta \in M, \) and let \( \sigma : [0,1] \to M \) be a minimizing geodesic in \( M \) from \( \zeta \) to \( x \). Then the function \( \frac{1}{2}d_\Sigma^2 \) on \( \Sigma \) has a supergradient \( \hat{\sigma}(1)^T \) at \( x \), in the sense that

\[
\frac{1}{2}d_\Sigma^2(\exp_\Sigma u) \leq \frac{1}{2}d_\Sigma^2(x) + \langle \hat{\sigma}(1)^T, u \rangle + o(|u|)
\]

for any \( u \in T_x \Sigma \).

Proof. The proof is similar to that of Proposition 6 in \cite{28}. Choose \( \epsilon > 0 \) and a neighborhood \( V \) of \( x \) in \( M \) such that for any \( \eta \in V, \exp_\eta \) maps \( B_\epsilon(0) \subset T_\eta M \) diffeomorphically to some \( U_\eta \supset V \). We first prove the case when \( \zeta \in V \). The proof follows from the following computation: for \( u \in T_x \Sigma \) with small norm,

\[
\frac{1}{2}d_\Sigma^2(\exp_\Sigma u) = \frac{1}{2}d^2(\zeta, \exp_\zeta^M(\exp_\zeta^M)^{-1}\exp_\zeta^V u)
= \frac{1}{2}d^2(\zeta, \exp_\zeta^M(\exp_\zeta^M)^{-1}\exp_\zeta^V u)
\]

where we use Gauss’ lemma in the last step. Thus we have \( \nabla^\Sigma|_x \frac{1}{2}d_\Sigma^2 \) = \( \hat{\sigma}(1)^T \). Applying the chain rule yields \( \nabla^\Sigma|_x \left( \frac{1}{2}d_\Sigma^2 \right) = \hat{\sigma}(1)^T \). (If \( \zeta \neq x \).)

For \( \zeta \notin V \), choose a point \( \eta \) on the geodesic \( \sigma \) such that \( \eta \in V \setminus \{x\} \). From the result above, we have \( \nabla^\Sigma|_x \left( d_\eta \right) = \frac{\hat{\sigma}(1)^T}{|\sigma(1)|} \), and hence

\[
d_\zeta(\exp_\zeta^\Sigma u) \leq d(\zeta, \eta) + d_\eta(\exp_\zeta^\Sigma u)
= d(\zeta, \eta) + \langle \frac{\hat{\sigma}(1)^T}{|\sigma(1)|}, u \rangle + o(|u|)
= d_\zeta(x) + \langle \frac{\hat{\sigma}(1)^T}{|\sigma(1)|}, u \rangle + o(|u|).
\]

Therefore \( d_\Sigma^2 \) has supergradient \( \frac{\hat{\sigma}(1)^T}{|\sigma(1)|} \) at \( x \). Applying the one-sided chain rule \((\cite{28}, Lemma 5)\) yields that \( \frac{1}{2}d_\Sigma^2 \) has supergradient \( d_\zeta(x)\frac{\hat{\sigma}(1)^T}{|\sigma(1)|} = \hat{\sigma}(1)^T \) at \( x \). \( \square \)

Lemma 3.2 (Tangency to submanifolds). Let \( x \in \Sigma \setminus \partial \Sigma \) and \( \zeta \in \partial^c \phi(x) \). If \( \phi \) is differentiable at \( x \), then \( \zeta = \exp_\Sigma^M(\nabla^\Sigma \phi(x) + v) \) for some \( v \in T_x^\Sigma \). In addition, the curve
g(\tau) := \exp_\Sigma^M(\tau \nabla^\Sigma \phi(x) + tv) \) for \( \tau \in [0,1] \) is a minimizing geodesic from \( x \) to \( \zeta \).

Proof. Since \( \zeta \in \partial^c \phi(x) \), for any \( u \in T_x \Sigma \) with small norm, we have

\[
\frac{1}{2}d_\Sigma^2(\exp_\Sigma u) \geq \frac{1}{2}d_\Sigma^2(x) - \phi(x) + \phi(\exp_\Sigma^V u)
\]
\begin{equation}
\frac{1}{2} \frac{d^2}{d\zeta^2}(x) - \phi(x) + \phi(x) + \langle u, \nabla \Sigma \phi(x) \rangle + o(|u|).
\end{equation}

That is to say, the function \( \frac{1}{2} \frac{d^2}{d\zeta^2} \) has a subgradient \( \nabla \Sigma \phi(x) \) at \( x \). On the other hand, let \( \sigma : [0, 1] \to \Sigma \) be a minimizing geodesic from \( \zeta \) to \( x \). By Proposition \[5,1\] above, \( \sigma(1)^T \) is a supergradient of \( \frac{1}{2} \frac{d^2}{d\zeta^2} \) at \( x \). Thus \( \frac{1}{2} \frac{d^2}{d\zeta^2} \) is actually differentiable at \( x \) and \( \sigma(1)^T = \nabla \Sigma \phi(x) \). Therefore \( \sigma(1) = \nabla \Sigma \phi(x) - v \) for some \( v \in T_x \Sigma \). By reversing the direction of \( \sigma \), we conclude
\[ \zeta = \exp^M_{\Sigma}(-\sigma(1)) = \exp^M_{\Sigma}(-\nabla \Sigma \phi(x) + v). \]
\[ \square \]

Lemma \[5,2\] above suggests that we consider the following map:
\[ \Phi(x,v) := \exp^M_{\Sigma}(-\nabla \Sigma \phi(x) + v) \]
for \( x \in \Sigma \) and \( v \in T_x \Sigma \); this is well-defined for a.e. \( x \in \Sigma \) since \( \phi \) is differentiable a.e., and the image of \( \Phi \) then contains \( \partial^c \phi(x) \) for each \( x \). We here remark that since \( \Omega \) is compact, a maximizer argument as in Lemma 7 in \[28\] shows that \( \partial^c \phi(x) \) is never empty.

Now we show that there is a measurable subset \( A \) of \( T^+ \Sigma \) such that \( \Phi \) becomes an injection when restricted to \( A \); additionally, \( A \) projects to a full \( \mu \)-subset in \( \Sigma \), and \( \Phi(A) \) has full \( \nu \)-measure in \( \Omega \). Our argument here involves the optimal transport problem from \( (\Omega, \nu) \) to \( (\Sigma, \mu) \) as in the classical McCann’s theorem (Theorem \[2,7\]), which gives
\[ F(\zeta) := \exp^M_{\zeta}(-\nabla \Sigma \phi^c(\zeta)) \]
as an optimal transport map from \( \nu \) to \( \mu \) (recall that \( \phi^c \) here is a function on \( \Omega \)). We also need a technical regularity result that \( \phi^c \) functions are semiconcave, in the sense that locally they differ from being geodesically concave by some smooth functions. We refer to Definition 3.8 in \[19\] for the precise definition; the proof of this fact can be found in Proposition 3.14 in \[19\] and Proposition \[5,2\]. Hence by Alexandrov-Bangert theorem (Theorem 3.10 in \[19\]; see Theorem 14.1 in \[35\]), the Alexandrov Hessians \( \text{Hess} \phi \) and \( \text{Hess} \phi^c \) exist a.e., and they define measurable sections of bilinear forms on the tangent bundles.

**Theorem 3.3** (Generalized McCann’s theorem). There is a measurable subset \( A \subset T^+ \Sigma \) such that
\begin{enumerate}
\item \( \Phi \) is well-defined and injective on \( A \);
\item \( p_{T^+ \Sigma}(A) \) is measurable and \( \mu(p_{T^+ \Sigma}(A)) = 1 \);
\item \( \Phi(A) \) is measurable and \( \nu(\Phi(A)) = 1 \).
\end{enumerate}
Here, \( p_{T^+ \Sigma} : T^+ \Sigma \to \Sigma \) is the bundle projection map.

**Proof.** By Alexandrov-Bangert theorem and that \( \mu, \nu \) are absolutely continuous, we can choose Borel subsets \( K \subset \Sigma \) and \( L \subset \Omega \) such that \( \text{Hess} \phi \) exists on \( K \), \( \text{Hess} \phi^c \) exists on \( L \), and \( \mu(K) = 1 = \nu(L) \). Then we have \( \nu(F^{-1}(K)) = \mu(K) = 1 \) from the transport condition. On the other hand, from McCann’s theorem (Theorem \[2,7\]), we can write \( F(L) = p_{\Sigma \times \Omega \to \Sigma} \partial^c \phi^c \cap (\Sigma \times L) \), where \( p_{\Sigma \times \Omega \to \Sigma} \) is the canonical projection map. Hence by the theory of analytic sets (see e.g. Proposition 8.4.4 in \[17\]), \( F(L) \) is measurable. We can now use the transport condition (after extending to Lebesgue measurable subsets) to obtain
\[ \mu(\Sigma \setminus F(L)) = \nu(F^{-1}(\Sigma \setminus F(L))) \leq \nu(\Omega \setminus L) = 0, \]
and thus \( \mu(F(L)) = 1 \). We hereby conclude that \( \mu(K \cap F(L)) = 1 = \nu(F^{-1}(K) \cap L) \).
Fix $\zeta \in F^{-1}(K) \cap L$. Since $\text{Hess} \phi^+(\zeta)$ exists, by Proposition 4.1 in [19] $F(\zeta)$ is not in the cut locus of $\zeta$. Thus there is a unique minimizing geodesic $\sigma : [0, 1] \to M$ from $\zeta$ to $F(\zeta)$, and $\dot{\sigma}(1) = \nabla \left( \frac{1}{2} \mathbf{d}_C^2 \right)_{F(\zeta)}$. This shows that the following map:

$$
\Theta(\zeta) := \left( F(\zeta), -\nabla \left( \frac{1}{2} \mathbf{d}_C^2 \right)_{F(\zeta)} \right)
$$

is well-defined on $F^{-1}(K) \cap L$. On the other hand, by Lemma 7 in [28], we have $F(\zeta) \in \partial^c \phi^+(\zeta)$, which is equivalent to $\zeta \in \partial^c \phi(F(\zeta))$. Since $F(\zeta) \in K$, $\phi$ is differentiable at $F(\zeta)$. Thus by Lemma 3.2 we have $\dot{\sigma}(1)^\perp = \nabla \Sigma \phi(F(\zeta))$. In summary, we have

$$
\Phi \circ \Theta(\zeta) = \exp_{F(\zeta)} \left( -\nabla \Sigma \phi(F(\zeta)) - \nabla \left( \frac{1}{2} \mathbf{d}_C^2 \right)_{F(\zeta)} \right) = \exp_{F(\zeta)}(-\dot{\sigma}(1)) = \zeta.
$$

That is, $\Phi \circ \Theta$ is the identity map on $F^{-1}(K) \cap L$. We now define

$$
A := \Theta(F^{-1}(K) \cap L).
$$

Clearly $\Phi$ is then well-defined and injective on $A$, and both $p_{T^+ \Sigma}(A) = F(F^{-1}(K) \cap L) = K \cap F(L)$ and $\Phi(A) = F^{-1}(K) \cap L$ are measurable with full measures.

To show that $A$ is measurable, we use the following characterization of $A$. Consider the following subsets:

$$
U := \{(x, v) \in T^+ \Sigma | K \mid \gamma(t) := \exp_x^M(-t \nabla \Sigma \phi(x) + tv) \text{ is minimizing for } t \in [0, 1]\};
$$

$$
V := \{(x, v) \in T^+ \Sigma | K \mid \Phi(x, v) \in \partial^c \phi(x)\}.
$$

Clearly $U$ and $V$ are measurable, and it is direct to check that $U \cap V \cap \Phi^{-1}(F^{-1}(K) \cap L) = A$. □

**Remark 3.1.** In the above proof of Theorem 3.3, it suffices to assume that $\phi$ is merely differentiable on $K$ instead of the existence of $\text{Hess}^\Sigma \phi$. However, the existence of the Hessian is needed for the proofs of Michael-Simon-Sobolev inequalities in Section 4.

Theorem 3.3 above has the following corollary:

**Corollary 3.4** ($c$-superdifferential and normal fiber). Define $A_x := A \cap T^+_x \Sigma$ for $x \in p_{T^+ \Sigma}(A)$. Then the restricted map

$$
\Phi(x, \cdot) : A_x \to \partial^c \phi(x) \cap \Phi(A)
$$

is a bijection.

With the results above, we now deduce a change of variable formula from the transport condition:

**Theorem 3.5.** Let $A$ be the subset in Theorem 3.3. Suppose $f$ and $g$ are the density functions of $\mu$ and $\nu$ w.r.t. the volume measures of $\Sigma$ and $M$. Then for $\mu$-a.e. $x \in \Sigma$,

$$
\int_{A_x} g \circ \Phi(x, v) | \det D\Phi(x, v)| dv = f(x).
$$
Proof. First of all, by the disintegration theorem (see e.g. [4], Theorem 5.3.1) we have a family of probability measures \((\pi_x)\) on \(\Omega\) parametrized by \(\mu\)-a.e. \(x \in \Sigma\) such that
\[
\int_{\Sigma \times \Omega} h d\pi = \int_{\Sigma} \int_{\Omega} h(x, \zeta) d\pi_x(\zeta) d\mu(x)
\]
for any nonnegative measurable function \(h\) on \(\Sigma \times \Omega\). Next, we already know that \(\text{supp} \pi \subset \partial^c \phi\). Furthermore by Theorem 3.3, \(\Phi(A)\) has full measure in \(\Omega\), and by Corollary 3.4 we have the bijection via \(\Phi\) between \(A_x\) and \(\partial^c \phi(x) \cap \Phi(A)\) for \(\mu\)-a.e. \(x\). Thus if \(h\) depends only on \(\zeta \in \Omega\), then we can write
\[
\int_{\Omega} h d\nu = \int_{\Sigma} \int_{\partial^c \phi(x) \cap \Phi(A)} h(\zeta) d\pi_x(\zeta) d\mu(x)
\]
(3.5)
\[
= \int_{\Sigma} f(x) \int_{A_x} h \circ \Phi(x, v) d\lambda_x(v) d\nu_{\Sigma}(x),
\]
where \(\lambda_x\) is the push-forward probability measure of \(\pi_x\) by the map \(\Theta\) in Theorem 3.3.

On the other hand, since \(\phi\) is semi-convex, we know that \(\nabla^\Sigma \phi\) is countably Lipschitz (see Theorem 5.34 in [2], or Sec 1.7.6. in [33]), and hence so is \(\Phi\). Thus from the area formula of the map \(\Phi\) and that \(\Phi(A), p_{T \Sigma}(A)\) have full measures, we get
\[
\int_{\Omega} h d\nu = \int_{\Omega} h g d\nu_M
\]
(3.6)
\[
= \int_{\Sigma} \int_{A_x} h \circ \Phi(x, v) g \circ \Phi(x, v) | \det D\Phi(x, v)| d\nu_{\Sigma}(x).
\]

Now let \(a(x)\) be any nonnegative measurable function on \(\Sigma\), and we plug \(h = a \circ F\) into equalities (3.5) and (3.6) above. Then since \(h \circ \Phi(x, v) = a(x)\) from Theorem 3.3, we have
\[
\int_{\Sigma} f(x) a(x) \int_{A_x} d\lambda_x(v) d\nu_{\Sigma}(x) = \int_{\Sigma} a(x) \int_{A_x} g \circ \Phi(x, v) | \det D\Phi(x, v)| d\nu_{\Sigma}(x).
\]
The theorem now follows since \(a\) is arbitrary and \(\lambda_x\) is a probability measure. \(\square\)

4. Proofs of Michael-Simon-Sobolev inequalities

In this section, we apply our generalization of McCann’s theorem to prove Michael-Simon-Sobolev inequalities for compact submanifolds in complete manifolds with intermediate Ricci curvature bounded from below. We here recall the following definitions from [27]:

**Definition 4.1 (p-Ricci Curvature).** For a \(p\)-dimensional plane \(P\) in \(T_x M\) and a vector \(w \in T_x M\), we define the \(p\)-Ricci curvature of \(P\) in the direction of \(w\) as
\[
\text{Ric}_p(P, w) := \text{tr} \left[ \nabla_P \circ (R(w, \cdot)w)|_P \right],
\]
where \(\nabla_P : T_x M \to P\) is the orthogonal projection of \(T_x M\) onto \(P\), and \(R\) is the curvature tensor of \(M\).

Notice that if we choose \(e_1, \ldots, e_p\) as an orthonormal basis of \(P\), then we have
\[
\text{Ric}_p(P, w) = \sum_{i=1}^p \langle R(w, e_i)w, e_i \rangle.
\]
**Definition 4.2** ($p$-Ricci lower bounds). We say that $M$ has $p$-Ricci curvature bounded from below by $K$ if for any $x \in M$, any $w \in T_x M$, and any $p$-dimensional plane $P \subset T_x M$, we have $\text{Ric}_p(P, w) \geq K|w|^2$. In this case we write $\text{Ric}_p \geq K$.

**Remark 4.1.** It is clear that $\text{Sec} \geq 0$ implies $\text{Ric}_p \geq 0$ for all $p$. More generally, we have (see [27], Remark 2.3 for details):

- $\text{Sec} \geq k \geq 0$ implies $\text{Ric}_p \geq (p - 1)k$ for all $1 \leq p \leq \dim M$.
- $\text{Sec} \geq k$ with $k < 0$ implies $\text{Ric}_p \geq pk$ for $1 \leq p \leq \dim M - 1$.
- $\text{Ric}_2 \geq k \geq 0$ is equivalent to $\text{Sec} \geq k \geq 0$.

We also remark that if $\text{Ric}_p \geq K$ for some $2 \leq p \leq \dim M$, then $\text{Ric}_q \geq qK/p$ for any $q$ with $p \leq q \leq \dim M$.

**Remark 4.2.** Some authors (e.g. [16]) use different normalizations for lower bounds on the intermediate Ricci curvatures.

### 4.1. Manifolds with nonnegative intermediate Ricci curvatures.

The goal of this section is to prove Theorem 1.3, and we here use the notations from Section 3. In addition, we denote the second fundamental form of the submanifold $\Sigma$ by $\Pi$, and $H = \text{tr} \Pi$ is the mean curvature vector.

For parameters $0 < \sigma < 1$ and $r > 0$ large enough, define the following annulus-like subset

$$\Omega = \{ p \in M : \sigma r \leq d(x, p) \leq r, \forall x \in \Sigma \}.$$

By approximation, we may assume that $\Omega$ is a regular domain. Consider the optimal transport problem of the following two measures

$$\mu = \frac{1}{\text{vol}(\Omega)} \int_{\Sigma} f \text{vol}_\Sigma, \quad \nu = \frac{1}{\text{vol}(\Omega)} \text{vol}_{M[\Omega]},$$

with the cost $c(x, \zeta) = \frac{1}{2}d(x, \zeta)^2$ on $\Sigma \times \Omega$. By the generalized McCann’s theorem (Theorem 3.3), we have a $c$-concave function $\phi$ on $\Sigma$, and the map

$$\Phi(x, v) := \exp_x (-\nabla_\Sigma^* \phi(x) + v)$$

defined on the subset $A \subset T^\perp \Sigma$ with the specific properties. Note that these objects may depend implicitly on the parameters $\sigma, r$.

The first step of the proof is the following inequality from integration by parts; this is also the last step in the arguments of [9]:

**Lemma 4.3.**

$$-\int_{\Sigma} f \Delta_{\Sigma^*}^\phi \leq r \int_{\omega^\Sigma} f + r \int_{\Sigma} |\nabla_\Sigma^* f|,$$

where $\Delta_{\Sigma^*}^\phi$ is the trace of the Alexandrov Hessian of $\phi$.

**Proof.** Recall that since $\phi$ is $c$-concave, Proposition A.2 yields that $\phi$ is semiconcave. Then by Alexandrov-Bangert theorem, the Alexandrov Hessian $\text{Hess}_\Sigma^* \phi$ exists a.e.; additionally, the Lebesgue decomposition of the distributional Laplacian $[\Delta_{\Sigma^*} \phi]$ (as a measure) can be written as

$$[\Delta_{\Sigma^*} \phi] = \Delta_{\Sigma^*}^\phi \text{vol}_\Sigma + [\Delta_{\Sigma^*} \phi]^s,$$

and the singular part $[\Delta_{\Sigma^*} \phi]^s$ is nonpositive.
For $\epsilon > 0$ small enough, let $\lambda_\epsilon$ be a smooth cut-off of $\Sigma$ w.r.t. $\partial \Sigma$, i.e. $\lambda_\epsilon \geq 0$, $\lambda_\epsilon \equiv 1$ on $\Sigma_\epsilon := \{ x \in \Sigma \mid d_\Sigma(x, \partial \Sigma) > \epsilon \}$ and $\lambda_\epsilon \in C_c^\infty(\Sigma \setminus \partial \Sigma)$. Using the decomposition (Line 4.1) above, we then have

$$\int_\Sigma \Delta_\Sigma(\lambda_\epsilon f) \phi \, d\text{vol}_\Sigma = \int_\Sigma \lambda_\epsilon f \, d[\Delta_\Sigma \phi] \leq \int_\Sigma \lambda_\epsilon f \Delta_{\Sigma_\epsilon}^\alpha \phi \, d\text{vol}_\Sigma.$$  

On the other hand, integrating by parts with $|\nabla^\Sigma \phi(x)| \leq r$ since $\Phi(x, v) \in \Omega$, we have

$$\int_\Sigma \Delta_\Sigma(\lambda_\epsilon f) \phi \, d\text{vol}_\Sigma = -\int_\Sigma \langle \nabla^\Sigma(\lambda_\epsilon f), \nabla^\Sigma \phi \rangle \, d\text{vol}_\Sigma \geq -r \int_\Sigma \lambda_\epsilon |\nabla^\Sigma f| \, d\text{vol}_\Sigma - r \int_\Sigma f |\nabla^\Sigma \lambda_\epsilon| \, d\text{vol}_\Sigma.$$  

The lemma now follows by combining these two inequalities and taking the limit as $\epsilon \to 0$.  

The second step is to use the change of variable formula (Theorem 3.5), which gives

$$\int_A |\det D\Phi(x, v)| \, dv = \frac{\text{vol}(\Omega)}{\int_\Sigma f} \frac{n}{n-1} f_{n-1}(x)$$  

for $\mu$-a.e. $x \in \Sigma$.

The third step is to estimate $\det D\Phi(\overline{x}, \overline{v})$ for fixed $(\overline{x}, \overline{v}) \in A$ by $\Delta_{\Sigma_\epsilon}^\alpha \phi(\overline{x})$. As in Proposition 4.6 of [8] and in Chapter 14 of [35], we use the Jacobi fields from the geodesic variation

$$\Phi_t(x, v) = \exp_x(-t\nabla^\Sigma \phi(x) + tv)$$  

for $t \in [0, 1]$ and denote $\overline{\gamma}(t) := \Phi_t(\overline{x}, \overline{v})$. The following lemma is similar to Lemma 4.3 in [8] and will be used in the comparison argument later:

**Lemma 4.4.** $n - \Delta_{\Sigma}^\alpha \phi(\overline{x}) - \langle H(\overline{x}), \overline{v} \rangle \geq 0$.

**Proof.** From Corollary 3.4, we have $\overline{\gamma}(1) = \Phi(\overline{x}, \overline{v}) \in \partial^+ \phi(\overline{x})$. Thus the function $\frac{1}{2}d_{\overline{\gamma}(1)}^2 - \phi$ on $\Sigma$ is minimized at $\overline{x}$. For any curve smooth $\gamma : [0, 1] \to M$ satisfying $\gamma(0) \in \Sigma$ and $\gamma(1) = \overline{\gamma}(1)$, we then have

$$\frac{1}{2} \int_0^1 |\gamma'(t)|^2 \, dt - \phi(\gamma(0)) \geq \frac{1}{2} \left( \int_0^1 |\gamma'(t)| \, dt \right)^2 - \phi(\gamma(0)) \geq \frac{1}{2} \phi(\gamma(0), \gamma(1)) - \phi(\gamma(0)) \geq \frac{1}{2} \langle \overline{\gamma}(0), \overline{\gamma}(1) \rangle - \phi(\overline{\gamma}(0)) = \frac{1}{2} \int_0^1 |\overline{\gamma}'(t)|^2 \, dt - \phi(\overline{\gamma}(0)).$$

That is, $\overline{\gamma}$ minimizes the functional $\frac{1}{2} \int_0^1 |\gamma'(t)|^2 \, dt - \phi(\gamma(0))$ among such curves $\gamma$. Hence by the second variation formula, we get

$$-\langle \Pi(Z(0), Z(0)), \overline{\gamma}'(0) \rangle - \text{Hess}_{\overline{\gamma}}^\Sigma \phi(Z(0), Z(0)) + \int_0^1 (|Z'(t)|^2 - R(\overline{\gamma}(t), Z(t), \overline{\gamma}(t), Z(t))) \, dt \geq 0$$
for any smooth vector field $Z$ along $\overline{\gamma}$ satisfying $Z(0) \in T_{x} \Sigma$ and $Z(1) = 0$.

Now let $e_1, \ldots, e_n$ be an orthonormal basis of $T_{x} \Sigma$, and let $E_i$ be the parallel transport of $e_i$ along the geodesic $\overline{\gamma}$. Plugging in $\overline{\gamma}(0) = -\nabla_{\Sigma}^{\Sigma} \phi(\overline{x}) + \overline{x}$, $Z(t) = (1-t)E_i(t)$ into the formula above and summing over $i = 1, \ldots, n$, we get

$$-\langle H(\overline{x}), \overline{\gamma} \rangle - \Delta_{\Sigma}^{\Sigma} \phi(\overline{x}) + n \geq \int_{0}^{1} (1-t)^2 \text{Ric}_n(P_t, \overline{\gamma}(t)) \, dt \geq 0,$$

where $P_t$ is the $n$-plane in $T_{\overline{\gamma}(t)} M$ spanned by $\{E_i\}$, and we use the condition $\text{Ric}_n \geq 0$. \hfill $\square$

The following computation is the key of the whole estimation on $\det D\Phi(\overline{x}, \overline{\gamma})$. It can be viewed as a slight improvement of Proposition 4.6 in [8], and the idea is to consider the traces of two diagonal blocks instead of the individual diagonal entries.

**Proposition 4.5.** The function

$$t^{-m} \left( 1 - \frac{t}{n} \left( \Delta_{\Sigma}^{\Sigma} \phi(\overline{x}) + \langle H(\overline{x}), \overline{\gamma} \rangle \right) \right)^{-n} | \det D\Phi_t(\overline{x}, \overline{\gamma}) |$$

is decreasing in $t \in (0, 1)$, where $\Phi_t$ is defined in Line 4.3.

**Proof.** Let $e_1, \ldots, e_n$ be an orthonormal basis of $T_{x} \Sigma$. Let $(x_1, \ldots, x_n)$ be a system of geodesic normal coordinates on $\Sigma$ around the point $\overline{x}$ with $\frac{\partial}{\partial x_i} = e_i$ at $\overline{x}$. Let $\{\nu_{n+1}, \ldots, \nu_{n+m}\}$ be a local orthonormal frame of $T_{x} \Sigma$, chosen so that $\langle \nabla_{e_i} \nu_{\alpha}, \nu_{\beta} \rangle = 0$ at $\overline{x}$. We write a normal vector $v = \sum_{\alpha=n+1}^{n+m} v_{\alpha} \nu_{\alpha}$. With this understood, $(x_1, \ldots, x_n, v_{n+1}, \ldots, v_{n+m})$ is a local coordinate system on the total space of $T_{x} \Sigma$.

For each $1 \leq i \leq n$, we denote by $E_i(t)$ the parallel transport of $e_i$ along $\overline{\gamma}$. Moreover, for each $1 \leq i \leq n$, we denote by $X_i(t) = (D\Phi_t)e_i$ the unique Jacobi field along $\overline{\gamma}$ satisfying $X_i(0) = e_i$ and

$$\begin{align*}
\langle X_i'(0), e_j \rangle &= -(\text{Hess}_{\overline{x}} \phi)(e_i, e_j) - \langle \text{II}(e_i, e_j), \overline{\gamma} \rangle, \\
\langle X_i'(0), \nu_\beta \rangle &= -(\text{II}(e_i, \nabla_{\overline{x}} \phi(\overline{x})), \nu_\beta)
\end{align*}$$

for all $1 \leq j \leq n$ and all $n+1 \leq \beta \leq n+m$. For each $n+1 \leq \alpha \leq n+m$, we denote by $N_{\alpha}(t)$ the parallel transport of $\nu_{\alpha}$ along $\overline{\gamma}$. Moreover, for each $n+1 \leq \alpha \leq n+m$, we denote by $Y_{\alpha}(t) = (D\Phi_t)\nu_{\alpha}$ the unique Jacobi field along $\overline{\gamma}$ satisfying $Y_{\alpha}(0) = 0$ and $Y'_{\alpha}(0) = \nu_{\alpha}$. We have that $X_1(t), \ldots, X_n(t), Y_{n+1}(t), \ldots, Y_{n+m}(t)$ are linearly independent for each $t \in (0, 1)$ since $\overline{\gamma}$ is minimizing from Lemma 3.2.

Let us define an $(n+m) \times (n+m)$-matrix $P(t)$ by

$$P_{ij}(t) = \langle X_i(t), E_j(t) \rangle, \quad P_{i\beta}(t) = \langle X_i(t), N_{\beta}(t) \rangle, \quad P_{\alpha j}(t) = \langle Y_{\alpha}(t), E_j(t) \rangle, \quad P_{\alpha \beta}(t) = \langle Y_{\alpha}(t), N_{\beta}(t) \rangle,$$

for $1 \leq i, j \leq n$ and $n+1 \leq \alpha, \beta \leq n+m$. Moreover, we define an $(n+m) \times (n+m)$-matrix $S(t)$ by

$$S_{ij}(t) = R(\overline{\gamma}(t), E_i(t), \overline{\gamma}(t), E_j(t)), \quad S_{i\beta}(t) = R(\overline{\gamma}(t), E_i(t), \overline{\gamma}(t), N_{\beta}(t)), \quad S_{\alpha j}(t) = R(\overline{\gamma}(t), N_{\alpha}(t), \overline{\gamma}(t), E_j(t)), \quad S_{\alpha \beta}(t) = R(\overline{\gamma}(t), N_{\alpha}(t), \overline{\gamma}(t), N_{\beta}(t)).$$
for $1 \leq i, j \leq n$ and $n + 1 \leq \alpha, \beta \leq n + m$. Clearly $S(t)$ is symmetric. Since the vector fields $X_1(t), \ldots, X_n(t), Y_{n+1}(t), \ldots, Y_{n+m}(t)$ are Jacobi fields, we obtain
\[
P''(t) = -P(t)S(t).
\]
The initial conditions of $P$ are
\[
P(0) = \begin{bmatrix} \delta_{ij} & 0 \\ 0 & 0 \end{bmatrix}, \quad P'(0) = \begin{bmatrix} (-\text{Hess}_{\Sigma} \phi)(e_i, e_j) - \langle \Sigma(e_i, e_j), \nu \rangle & \langle -\Sigma(e_i, \nabla^\Sigma \phi), \nu \rangle \\ 0 & 0 \end{bmatrix}.
\]
In particular, the matrix $P'(0)P(0)^T$ is symmetric. Moreover, the matrix
\[
\frac{d}{dt}(P'(t)P(t)^T) = P''(t)P(t)^T + P'(t)P'(t)^T = -P(t)S(t)P(t)^T + P'(t)P'(t)^T
\]
is symmetric for each $t$. Thus, we conclude that the matrix $P'(t)P(t)^T$ is symmetric for all $t$.

Since $X_1(t), \ldots, X_n(t), Y_{n+1}, \ldots, Y_{n+m}(t)$ are linearly independent for each $t \in (0, 1)$, the matrix $P(t)$ is invertible for $t \in (0, 1)$. With $P'(t)P(t)^T$ being symmetric, it follows that the matrix $Q(t) := P(t)^{-1}P'(t)$ is symmetric. Furthermore, $Q$ satisfies the Riccati equation
\[
Q'(t) = P(t)^{-1}P''(t) - P(t)^{-1}P'(t)P(t)^{-1}P'(t) = -S(t) - Q(t)^2
\]
for all $t \in (0, 1)$. We write
\[
Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}
\]
where $Q_1$ is $n \times n$, $Q_2$ is $n \times m$, and $Q_3$ is $m \times m$, and $Q_1, Q_3$ are symmetric.

We now use the curvature conditions to derive differential inequalities for $\text{tr} Q_1$ and $\text{tr} Q_3$. Let $\text{tr}_n$ and $\text{tr}_m$ denote the trace of the upper-left $n \times n$ block and the lower-right $m \times m$ block in an $(n + m) \times (n + m)$ matrix, respectively. First of all, since $\text{Ric}_n, \text{Ric}_m \geq 0$, we have
\[
-\text{tr}_n S(t) = -\text{Ric}_n(\mathcal{E}_t, \mathcal{T}(t)) \leq 0;
\]
\[
-\text{tr}_m S(t) = -\text{Ric}_m(\mathcal{N}_t, \mathcal{V}(t)) \leq 0,
\]
where $\mathcal{E}_t, \mathcal{N}_t \subset T_{\mathcal{V}(t)}M$ are the the spaces spanned by $\{E_i\}_{i=1}^n$ and $\{N_\alpha\}_{\alpha=n+1}^{n+m}$, respectively. Next, by Cauchy inequality we have
\[
\text{tr}_n(Q^2) = \text{tr}(Q_1^2 + Q_2Q_2^T) \geq \text{tr}(Q_1^2) \geq \frac{1}{n} (\text{tr} Q_1)^2;
\]
\[
\text{tr}_m(Q^2) = \text{tr}(Q_2^2 + Q_3Q_3^T) \geq \text{tr}(Q_3^2) \geq \frac{1}{m} (\text{tr} Q_3)^2.
\]
Combined with the Riccati equation\ref{eq:Riccati} above, we have the differential inequalities:
\[
\text{tr} Q'_1 \leq -\frac{1}{n} (\text{tr} Q_1)^2, \quad \text{tr} Q'_3 \leq -\frac{1}{m} (\text{tr} Q_3)^2.
\]
To derive the initial conditions of $\text{tr} Q_1$ and $\text{tr} Q_3$, consider the asymptotic expansion
\[
P(t) = \begin{bmatrix} \delta_{ij} + O(t) & O(t) \\ O(t) & t \delta_{\alpha\beta} + O(t^2) \end{bmatrix},
\]
implying
\[
P(t)^{-1} = \begin{bmatrix} \delta_{ij} + O(t) & O(1) \\ O(1) & t^{-1} \delta_{\alpha\beta} + O(1) \end{bmatrix}
\]
as $t \to 0$. Moreover, using the Jacobi field equations we have
\[
P'(t) = \begin{bmatrix}
-\text{Hess}_\Sigma \phi(e_i, e_j) - \langle \Pi(e_i, e_j), \nu \rangle + O(t) & O(1) \\
O(t) & \delta_{\alpha \beta} + O(t)
\end{bmatrix}
\]
as $t \to 0$. Consequently, the matrix $Q(t) = P(t)^{-1}P'(t)$ satisfies the asymptotic expansion
\[
Q(t) = \begin{bmatrix}
-\text{Hess}_\Sigma \phi(e_i, e_j) - \langle \Pi(e_i, e_j), \nu \rangle + O(t) & O(1) \\
O(1) & t^{-1}\delta_{\alpha \beta} + O(1)
\end{bmatrix}
\]
as $t \to 0$, and we thus have the initial conditions
\[
(4.6) \quad \text{tr } Q_1 \sim -\Delta^\Sigma_{\alpha \beta} \phi(\nu) - \langle H(\nu), \nu \rangle + O(t), \quad \text{tr } Q_3 \sim \frac{m}{t} + O(1)
\]
as $t \to 0$. Hence we can find a small number $\tau_0 \in (0, 1)$ such that
\[
\text{tr } Q_1(\tau) < -\Delta^\Sigma_{\alpha \beta} \phi(\nu) - \langle H(\nu), \nu \rangle + \sqrt{\tau} =: G(\tau), \quad 0 < \text{tr } Q_3(\tau) < \frac{2m}{\tau}
\]
for all $\tau \in (0, \tau_0)$. For such a fixed $\tau$, a standard ODE comparison principle together with Lemma 4.4 implies
\[
\text{tr } Q_1(t) \leq \frac{G(\tau)}{1 + \frac{(t-\tau)G(\tau)}{t}}, \quad \text{tr } Q_3(t) \leq \frac{m}{t - \frac{\tau}{2}}
\]
for all $t \in (\tau, 1)$. Passing to the limit as $\tau \to 0$, we conclude that
\[
\text{tr } Q_1(t) \leq \frac{-\Delta^\Sigma_{\alpha \beta} \phi(\nu) - \langle H(\nu), \nu \rangle}{1 - \frac{t}{n}(-\Delta^\Sigma_{\alpha \beta} \phi(\nu) - \langle H(\nu), \nu \rangle)}, \quad \text{tr } Q_3(t) \leq \frac{m}{t}
\]
for $t \in (0, 1)$, and therefore
\[
(4.7) \quad \text{tr } Q(t) = \text{tr } Q_1(t) + \text{tr } Q_3(t) \leq \frac{m}{t} + \frac{-\Delta^\Sigma_{\alpha \beta} \phi(\nu) - \langle H(\nu), \nu \rangle}{1 + \frac{t}{n}(-\Delta^\Sigma_{\alpha \beta} \phi(\nu) - \langle H(\nu), \nu \rangle)}
\]
for all $t \in (0, 1)$.

We can now estimate $\det P$. From the asymptotic expansion of $P$ in Line (4.5), we have $\lim_{t \to 0} t^{-m} \det P(t) = 1$. Since $P(t)$ is invertible for each $t \in (0, 1)$, it follows that $\det P(t) > 0$ for such $t$, and thus we can consider the function $\log \det P(t)$. Using the estimate of $\text{tr } Q$ in Line 4.7 above, we obtain
\[
\frac{d}{dt} \log \det P(t) = \text{tr}(Q(t)) \leq \frac{m}{t} + \frac{-\Delta^\Sigma_{\alpha \beta} \phi(\nu) - \langle H(\nu), \nu \rangle}{1 + \frac{t}{n}(-\Delta^\Sigma_{\alpha \beta} \phi(\nu) - \langle H(\nu), \nu \rangle)}
\]
\[
= m \frac{d}{dt} \log(t) + n \frac{d}{dt} \log \left(1 + \frac{t}{n}(-\Delta^\Sigma_{\alpha \beta} \phi(\nu) - \langle H(\nu), \nu \rangle)\right)
\]
for all $t \in (0, 1)$. Consequently, the function
\[
t \mapsto t^{-m} \left(1 + \frac{t}{n}(-\Delta^\Sigma_{\alpha \beta} \phi(\nu) - \langle H(\nu), \nu \rangle)\right)^{-n} \det P(t)
\]
is decreasing for $t \in (0, 1)$. Finally observe that $| \det D\Phi_\nu(\nu, \nu) | = \det P(t)$ for all $t \in (0, 1)$ since $\{E_i\}_{i=1}^n \cup \{N_\alpha\}_{\alpha=n+1}^{n+m}$ forms an orthonormal basis. This finishes the proof of the proposition. \qed

Since $\lim_{t \to 0} t^{-m} | \det D\Phi_\nu(\nu, \nu) | = 1$ and $\Phi_1 = \Phi$, we have the estimate:
Corollary 4.6 ([8], Corollary 4.7). The Jacobian determinant of $\Phi$ satisfies

$$|\det D\Phi(x,v)| \leq \left(1 - \frac{1}{n}(\Delta^{ac}_{\Sigma}\phi(x) + \langle H(x), v \rangle)\right)^n$$

for $(x, v) \in A$.

The final step is to combine the results in the previous steps. From the change of variable formula (Line 4.2) and the corollary above, for $\mu$-a.e. $x \in \Sigma$ we have

$$\frac{\text{vol}(\Omega)}{\int_{\Sigma} f_{\Sigma}^{\frac{n}{n-1}}(x)} \leq \int_{A_{x}} |\det D\Phi(x,v)|dv \leq \int_{A_{x}} \left(1 - \frac{1}{n}(\Delta^{ac}_{\Sigma}\phi(x) + \langle H(x), v \rangle)\right)^n dv.$$  

Since $\Phi(x, v) \in \Omega$, we have $|v| \leq d(x, \Phi(x, v)) < r$. Thus by Cauchy inequality, we have

$$\int_{A_{x}} \left(1 - \frac{1}{n}(\Delta^{ac}_{\Sigma}\phi(x) + \langle H(x), v \rangle)\right)^n dv \leq \int_{A_{x}} \left(1 - \frac{1}{n}\Delta^{ac}_{\Sigma}\phi(x) + \frac{1}{n}|H(x)|r\right)^n dv,$$

and the integrand now no longer depends on $v$. Again since $\Phi(x, v) \in \Omega$, we have

$$\int_{A_{x}} dv \leq \int_{\{y \in T_{x}^\perp \Sigma : \sigma r^2 < |v|^2 + |\nabla \phi(x)|^2 < r^2\}} dv$$

(4.8)

$$= |B^m| \left((r^2 - |\nabla \phi(x)|^2)^{\frac{m}{2}} - (\sigma^2 r^2 - |\nabla \phi(x)|^2)^{\frac{m}{2}}\right)$$

$$\leq \frac{m}{2}|B^m|r^m(1 - \sigma^2),$$

where we use the mean value theorem with $m \geq 2$ in the last step. Consequently, we have the following inequality:

$$\frac{\text{vol}(\Omega)}{\int_{\Sigma} f_{\Sigma}^{\frac{n}{n-1}}(x)} \leq \left(1 - \frac{1}{n}\Delta^{ac}_{\Sigma}\phi(x) + \frac{1}{n}|H(x)|r\right)^n \frac{m}{2}|B^m|r^m(1 - \sigma^2).$$

Taking the $n$-th root, multiplying by $f$, and integrating over $\Sigma$ on both sides, we get

$$n \left(\frac{2\text{vol}(\Omega)}{m|B^m|(1 - \sigma^2)}\right)^\frac{1}{n} \left(\int_{\Sigma} f_{\Sigma}^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq r^m \int_{\Sigma} (nf - f \Delta^{ac}_{\Sigma}\phi + rf|H|).$$

Using the inequality from integration by parts (Lemma 4.3), we get

(4.9)

$$n \left(\frac{2\text{vol}(\Omega)}{m|B^m|(1 - \sigma^2)}\right)^\frac{1}{n} \left(\int_{\Sigma} f_{\Sigma}^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq r^m \int_{\Sigma} f + r \int_{\partial \Sigma} f + r \int_{\Sigma} |\nabla \Sigma f| + r \int_{\Sigma} f|H|.$$

Recall the definition of the set $\Omega = \{ y \in M : \sigma r \leq d(y,x) \leq r, \forall x \in \Sigma \}$. Since $\sigma < 1$, we have $\sigma(2 - \delta) < \delta$ for any $\delta < 1$ close enough to 1. Choose any $x_0 \in \Sigma$. By triangle inequality, we then have $B_{\sigma(2 - \delta)r}(x_0) \subset \Omega$ for $r$ large enough, where $B_{\sigma,b}(x_0)$ denotes the annulus in $M$ centered at $x_0$ with radii $a < b$. We divide both sides of Line 4.9 by $r^{\frac{n+2}{n}}$ and let $r \rightarrow \infty$ while keeping $\sigma, \delta$ fixed. This gives

$$n\theta^\frac{1}{n} \left(\frac{2|B^{n+m}|(\delta^n + m - (\sigma(2 - \delta))(n+1))}{m|B^m|(1 - \sigma^2)}\right)^\frac{1}{n} \left(\int_{\Sigma} f_{\Sigma}^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq \int_{\partial \Sigma} f + \int_{\Sigma} |\nabla \Sigma f| + \int_{\Sigma} f|H|. $$
Finally we take the limit $\delta \to 1^-$ and then let $\sigma \to 1^-$ to conclude
\[ n \theta \frac{1}{2} \left( \frac{(n + m) \mid B^{n+m} \mid}{m \mid B^m \mid} \right) \frac{1}{n} \left( \int_{\Sigma} f \frac{\partial f}{\partial t} \right)^\frac{n-1}{n} \leq \int_{\partial \Sigma} f + \int_{\Sigma} |\nabla f| + \int_{\Sigma} |H| \].
This finishes the proof of Theorem 1.3.

4.2. Manifolds with positive lower bounds on intermediate Ricci curvatures. In this subsection, we assume $\text{Ric}_n \geq (n-1)k_1, \text{Ric}_m \geq (m-1)k_2$ for some $k_1, k_2 > 0$ on the ambient complete manifold $M$. Then since $\text{Ric} \geq (n-1)k_1 + (m-1)k_2 > 0$, Myers’ theorem (see e.g. [31], Theorem 6.3.3) yields that $M$ is compact (without boundary). Hence there is no guarantee that the torus-like subset $\Omega = \{ p \in M : \sigma r \leq d(x, p) \leq r, \forall x \in \Sigma \}$ considered in the previous subsection will have positive volume since $r$ cannot be arbitrarily large. Nevertheless, taking $\Omega = M, r = \text{diam}(M)$ in the proof of Theorem 1.3 (see Subsection 4.1 above) with little adjustment in Line 4.8 yields Corollary 1.5.

On the other hand, it is possible to derive a more general result, namely Theorem 1.4, by modifying the proof of Theorem 1.3. The idea is to use the positive bounds on the intermediate Ricci curvatures in the Riccati equation (Line 4.4) to get another type of differential inequalities. In that case, the speed of the transporting geodesic $|\gamma'(t)| = |\nabla \phi(x) + v|$ comes into play, which can be controlled by our choice of the target domain $\Omega$ since $\gamma$ is a minimizing. We here choose $\Omega$ as the complement of a tubular neighborhood of $\Sigma$ to obtain a lower bound on the speed.

Proof of Theorem 1.4. Most of the notations and arguments are from the previous Subsection 4.1. Recall that $N_{\epsilon}$ is the $\epsilon$-tubular neighborhood of $\Sigma$ for some $\epsilon > 0$, and we may assume that $\Omega = M \setminus N_{\epsilon}$ has positive volume. Consider the optimal transport problem between the following two probability measures:

\[ \mu = \frac{1}{\int_{\Sigma} f \frac{\partial f}{\partial t} \text{vol}_{\Sigma}}, \quad \nu = \frac{1}{\text{vol}(\Omega)} \text{vol}_{M \setminus \Omega} \]

on $\Sigma$ and $\Omega$ respectively, with the same cost function $c(x, \zeta) = \frac{1}{2} d_M^2 (x, \zeta)$ on $\Sigma \times \Omega$. By the generalized McCann’s theorem (Theorem 5.5), we have a $c$-concave function $\phi$ on $\Sigma$, and the map

\[ \Phi(x, v) := \exp_x (-\nabla \Sigma \phi(x) + v) \]

defined on the set $A \subset T^\perp \Sigma$ such that $\Phi(A)$ has full measure in $\Omega$. Since $\Phi(x, v) \in \Omega$, we have

\[ \epsilon^2 \leq d^2(x, \Phi(x, v)) = |\nabla \Sigma (x)|^2 + |v|^2 \leq \text{diam}(M)^2. \]

Now we estimate $|\det D \Phi(x, v)|$ for fixed $(x, v) \in A$ by modifying Proposition 4.5. Recall the Riccati equation (Line 4.4):

\[ Q' = -S - Q^2. \]

By $\text{Ric}_n \geq (n-1)k_1, \text{Ric}_m \geq (m-1)k_2$ and Cauchy inequality, we get

\[ \text{tr} \ Q_1' \leq -(n-1)k_1 \epsilon^2 - \frac{1}{n} (\text{tr} \ Q_1)^2, \quad \text{tr} \ Q_3' \leq -(m-1)k_2 \epsilon^2 - \frac{1}{m} (\text{tr} \ Q_3)^2. \]

We have the same initial conditions for $\text{tr} \ Q_1$ and $\text{tr} \ Q_3$ as in Line 4.6. Since $k_1, k_2 > 0$ and $n, m \geq 2$, standard ODE comparison arguments as those from Line 4.6 to Line 4.7 then imply

\[ \frac{d}{dt} \log \det P = \text{tr} \ Q = \text{tr} \ Q_1(t) + \text{tr} \ Q_3(t) \leq \frac{d}{dt} (n \log \cos G_1 + m \log \cos G_2) \]
for \( t \in (0, 1) \), where
\[
G_1(t) = -t\epsilon \sqrt{\frac{k_1(n-1)}{n}} + \arctan\left(\frac{-\Delta_{\Sigma}^{ac} \phi(x) - \langle H(x), v \rangle}{\epsilon \sqrt{k_1 n (n-1)}}\right); \quad G_2(t) = -t\epsilon \sqrt{\frac{k_2 (m-1)}{m}} + \frac{\pi}{2}.
\]

**Remark 4.3.** To see that there is no blow-up for \( t \in (0, 1) \), assume the contrary that there is some \( 0 < t_0 < 1 \) such that \( G_1(t) \) tends to \( -\frac{\pi}{2} \) as \( t \to t_0^- \). Then \( \text{tr} \ Q \) will tend to \( -\infty \) as \( t \to t_0^- \). However this is impossible since \( \text{tr} \ Q \) is continuous on \( [0, 1) \). Thus the inequality will hold for \( t \in (0, 1) \) and in fact \( G_1(t) \in (\frac{-\pi}{2}, \frac{\pi}{2}) \). For the same reason we have \( G_2(t) \in (\frac{-\pi}{2}, \frac{\pi}{2}) \).

Consequently, the following function in \( t \) is decreasing:
\[
\frac{\det P(t)}{\cos^n G_1(t) \cos^m G_2(t)}.
\]

Since \( \lim_{t \to 0} t^{-m} \det P(t) = 1 \) and \( \det P(1) = |\det D\Phi(x, v)| \), we have
\[
|\det D\Phi(x, v)| \leq \lim_{t \to 0} \frac{\cos^n G_1(t) \cos^m G_2(t) \det P(t)}{\cos^n G_1(t) \cos^m G_2(t)} = \left[ \cos \left(\epsilon \sqrt{\frac{k_1(n-1)}{n}}\right) - \frac{1}{n} \text{sinc} \left(\epsilon \sqrt{\frac{k_1(n-1)}{n}}\right) (\Delta_{\Sigma}^{ac} \phi(x) + \langle H(x), v \rangle) \right]^n \\
\cdot \text{sinc}^m \left(\epsilon \sqrt{\frac{k_2(m-1)}{m}}\right)
\leq \left[ \cos \left(\epsilon \sqrt{\frac{k_1(n-1)}{n}}\right) - \frac{1}{n} \text{sinc} \left(\epsilon \sqrt{\frac{k_1(n-1)}{n}}\right) (\Delta_{\Sigma}^{ac} \phi(x) - |H(x)| \text{diam}(M)) \right]^n \\
\cdot \text{sinc}^m \left(\epsilon \sqrt{\frac{k_2(m-1)}{m}}\right),
\]

where we use that \( |v| \leq \text{diam}(M) \). Combined with the change of variable formula of \( D\Phi \) (Line 4.2) and using \( A_x \subset \{ v \in T_{x, \Sigma}^+ \mid |v| \leq \text{diam}(M) \} \), we have
\[
\frac{\text{vol}(\Omega)}{\int_{\Sigma} f^\frac{n-1}{n} \, dx} f^\frac{n-1}{n}(x) = \int_{A_x} |\det D\Phi(x, v)| dv \\
\leq \left[ \cos \left(\epsilon \sqrt{\frac{k_1(n-1)}{n}}\right) - \frac{1}{n} \text{sinc} \left(\epsilon \sqrt{\frac{k_1(n-1)}{n}}\right) (\Delta_{\Sigma}^{ac} \phi(x) - |H(x)| \text{diam}(M)) \right]^n \\
\cdot \text{sinc}^m \left(\epsilon \sqrt{\frac{k_2(m-1)}{m}}\right) |B^m| \text{diam}(M)^m.
\]

Now we take the \( n \)-th root on both sides, multiply by \( f \), integrate over \( \Sigma \) and use the integration by parts inequality (Lemma 4.3) with \( r = \text{diam}(M) \) to finish the proof.
4.3. Manifolds with negative lower bounds on intermediate Ricci curvatures. We prove Theorem 1.6 in this subsection, where we assume $\text{Ric}_n \geq nk_1, \text{Ric}_m \geq mk_2$ for some $k_1, k_2 < 0$ on the ambient complete manifold $M$. As in Subsection 4.2, these lower bounds will be used in the Riccati equation (Line 4.4) to give another type of differential inequalities involving the speed of transporting geodesics. Here we choose the target domain $\Omega$ as a geodesic ball to get an upper bound on the speed.

Proof of Theorem 1.6. Again, most of the notations and arguments are from Subsection 4.1. Consider the optimal transport problem between the following two probability measures:

$$
\mu = \frac{1}{\int_{\Sigma} f \frac{\nu}{\nu M} \text{vol}_\Sigma}, \quad \nu = \frac{1}{\text{vol}(\Omega)} \text{vol}_{M \setminus \Omega}
$$

on $\Sigma$ and $\Omega = B^M(x_0)$, respectively, with the cost function $c(x, \zeta) = \frac{1}{2}d^2_M(x, \zeta)$ on $\Sigma \times \Omega$. By the generalized McCann’s theorem (Theorem 3.3), we have a $c$-concave function $\phi$ on $\Sigma$, and the map

$$
\Phi(x, v) := \exp_x(-\nabla^\Sigma \phi(x) + v)
$$

defined on $A \subset T^1 \Sigma$ such that $\Phi(A)$ has full measure in $\Omega$. Since $\Phi(x, v) \in \Omega$, we have

$$
|\nabla^\Sigma(x)|^2 + |v|^2 = d^2(x, \Phi(x, v)) \leq r^2.
$$

Now we estimate $|\det D\Phi(x, v)|$ for fixed $(x, v) \in A$ by modifying Proposition 4.5. Recall the Riccati equation (Line 4.4):

$$Q' = -S - Q^2.$$

By $\text{Ric}_n \geq nk_1, \text{Ric}_m \geq mk_2$ and Cauchy inequality, we get

$$
\text{tr} Q'_1 \leq -nk_1 r^2 - \frac{1}{n} (\text{tr} Q_1)^2, \quad \text{tr} Q'_3 \leq -mk_2 r^2 - \frac{1}{m} (\text{tr} Q_3)^2.
$$

Again, we have the same initial conditions for $\text{tr} Q_1$ and $\text{tr} Q_3$ as in Line 4.6 and hence we can find a small number $\tau_0 \in (0, 1)$ such that

$$
\text{tr} Q_1(\tau) < -\Delta^\Sigma_{\phi}(\overline{x}) - \langle H(\overline{x}), \overline{v} \rangle + \sqrt{r}, \quad \text{tr} Q_3(\tau) < \frac{2m}{\tau}, -mk_2 r^2 - \frac{1}{m} (\text{tr} Q_3(\tau))^2 < 0
$$

for all $\tau \in (0, \tau_0)$. By a standard ODE comparison principle and taking the limit as $\tau \to 0^+$, we conclude

$$
\frac{d}{dt} \log \det P = \text{tr} Q_1(t) + \text{tr} Q_3(t) \leq \frac{d}{dt} (n \log \cosh G_1 + m \log \sinh G_2),
$$

for $t \in (0, 1)$, where

$$
G_1(t) = \text{tr} \sqrt{-k_1 + \tanh^{-1} \left( \frac{-\Delta^\Sigma_{\phi}(x) - \langle H(x), v \rangle}{rn \sqrt{-k_1}} \right)}, \quad G_2(t) = tr \sqrt{-k_2}.
$$

Therefore the following function in $t$ is decreasing:

$$
\frac{\det P(t)}{\cosh^n G_1(t) \sinh^m G_2(t)}.
$$

Since $\lim_{t \to 0} t^{-m} \det P(t) = 1$ and $\det P(1) = |\det D\Phi(x, v)|$, we have

$$
|\det D\Phi(x, v)| \leq \lim_{t \to 0} \frac{\cosh^n G_1(1) \sinh^m G_2(1) \det P(t)}{\cosh^n G_1(t) \sinh^m G_2(t)}.
$$
\[
\begin{align*}
&= \left[ \cosh(r \sqrt{-k_1}) - \frac{\sinh(r \sqrt{-k_1})}{r n \sqrt{-k_1}} (\Delta^\infty_{\Sigma} \phi(x) + \langle H(x), v \rangle) \right]^n \left( \frac{\sinh(r \sqrt{-k_2})}{r \sqrt{-k_2}} \right)^m \\
&\leq \left[ \cosh(r \sqrt{-k_1}) - \frac{\sinh(r \sqrt{-k_1})}{r n \sqrt{-k_1}} (\Delta^\infty_{\Sigma} \phi(x) - |H(x)| r) \right]^n \left( \frac{\sinh(r \sqrt{-k_2})}{r \sqrt{-k_2}} \right)^m,
\end{align*}
\]
where we use that \(|v| \leq r\). Combined with the change of variable formula of \(D\Phi\) (Line 4.2) and using \(A_x = \{ v \in T^1_x \Sigma \mid |v| \leq r \}\), we have
\[
\frac{\text{vol}(B^r_\Sigma(x_0))}{\int_{\Sigma} f^{\frac{n}{n-r}}(x)} = \int_{A_x} |\det D\Phi(x, v)| dv \\
\leq \left[ \cosh(r \sqrt{-k_1}) - \frac{\sinh(r \sqrt{-k_1})}{r n \sqrt{-k_1}} (\Delta^\infty_{\Sigma} \phi(x) - |H(x)| r) \right]^n \left( \frac{\sinh(r \sqrt{-k_2})}{r \sqrt{-k_2}} \right)^m |B^m|^r.
\]
Now we take the \(n\)-th root on both sides, multiply by \(f\), integrate over \(\Sigma\) and use the integration by parts inequality (Lemma 4.3) to finish the proof. \(\square\)

APPENDIX A. C-CONCAVE FUNCTIONS ON SUBMANIFOLDS

In this appendix, we show that \(c\)-concave functions on submanifolds are semiconcave. This follows essentially from the Hessian upper bound (see [19], Lemma 3.11) of such functions. Recall the notations that \(c = \frac{1}{2} d^2_M\) is the cost function defined on \(\Sigma \times \Omega\), where \(\Sigma \subset M\) is a compact submanifold, and \(\Omega \subset \bar{M}\) is a compact (regular) domain.

First we prove the following adaption of Lemma 3.12 in [19] under the submanifold setting:

**Lemma A.1 (Hessian upper bound for distance squared restricted to \(\Sigma\)).** Fix \(x \in \Sigma\) and \(\zeta \in M\). Let \(\gamma\) be a minimal geodesic in \(M\) from \(x\) to \(\zeta\). Suppose \(k < 0\) is a lower bound for the sectional curvature along \(\gamma\). Define \(b(s) = \coth s\). Then for each \(u \in T_x \Sigma\), we have
\[
\limsup_{s \to 0} \frac{d^2(\exp^\Sigma_x su) + d^2(\exp^\Sigma_x -su) - 2d^2_\zeta(x)}{s^2} \leq 4b \left( \frac{d_\zeta(x)}{2} \sqrt{-k} \right) + 2d_\zeta(x) |H(u, u)|.
\]
Here, \(H\) is the second fundamental form of \(\Sigma\).

**Proof.** Let \(l = d_\zeta(x)\). Assume that \(\gamma : [0, l] \to M\) is parametrized by arc length. Let \(\tilde{\zeta} = \gamma(\frac{1}{2})\) be the middle point of \(\gamma\). By triangle inequality, we have
\[
2d^2_\zeta(\exp^\Sigma_x su) \leq (d(\zeta, \tilde{\zeta}) + d(\tilde{\zeta}, \exp^\Sigma_x su))^2 \leq 2 \left( \frac{l}{2} \right)^2 + 2d^2_\zeta(\exp^\Sigma_x su),
\]
and a similar bound holds for \(d^2_\zeta(\exp^\Sigma_x -su)\). Hence we have
\[
\frac{d^2_\zeta(\exp^\Sigma_x su) + d^2_\zeta(\exp^\Sigma_x -su) - 2d^2_\zeta(x)}{s^2} \leq \frac{d^2_\zeta(\exp^\Sigma_x su) + d^2_\zeta(\exp^\Sigma_x -su) - 2d^2_\zeta(x)}{s^2}.
\]
Since \(\gamma\) is minimizing, \(x\) is not in the cut locus of the middle point \(\tilde{\zeta}\). Thus as \(s \to 0\) the limit exists on the right hand side, and we have
\[
\limsup_{s \to 0} \frac{d^2_\zeta(\exp^\Sigma_x su) + d^2_\zeta(\exp^\Sigma_x -su) - 2d^2_\zeta(x)}{s^2} \leq 2 \Hess^\Sigma_x (d^2_\zeta_\Sigma)(u, u)
\]
\[ = 2 \left( \text{Hess}^M_x \left( d_\zeta^2 \right)(u, u) + \langle \nabla (d_\zeta^2)(x), \text{II}(u, u) \rangle \right). \]

Now we apply Lemma 3.12 in [19] to \( \tilde{\zeta} \) and obtain
\[ \text{Hess}^M_x \left( d_\zeta^2 \right)(u, u) \leq 2 \left( \frac{1}{2} \sqrt{-k} \right). \]

The lemma then follows from Cauchy inequality and that \(| \nabla (d_\zeta^2)(x) | = 2d_\zeta(x) = l. \)

We now prove the main result in this appendix, which can be viewed as the submanifold version of Proposition 3.14 in [19]:

**Proposition A.2** (c-concave functions on \( \Sigma \) are semiconcave). Let \( \Sigma \subset M \) be a compact submanifold, and let \( \Omega \subset M \) be a compact domain. Let \( \phi \) be a c-concave function on \( \Sigma \) (w.r.t the pair \( (\Sigma, \Omega) \)). Then \( \phi \) is semiconcave on \( \Sigma \).

**Proof.** For \( x \in \Sigma \), since the function \( \frac{1}{2}d_x^2 - \phi^c+ \) is continuous on the compact set \( \Omega \), there is some \( \zeta \in \Omega \) such that
\[
\frac{1}{2}d_\zeta^2(\zeta) - \phi^c+(\zeta) = \inf_{\eta \in \Omega} \frac{1}{2}d_\eta^2(x, \eta) - \phi^c+(\eta) = \phi^c+c+\phi(x) = \phi(x).
\]

Hence \( \zeta \in \partial^c \phi(x) \). Then for any \( u \in T_x \Sigma \) of small norm, we have
\[
\phi (\exp^\Sigma_x u) \leq \phi(x) + \frac{1}{2} d_\zeta^2(\exp^\Sigma_x u) - \frac{1}{2} d_\zeta^2(x).
\]
Therefore by Lemma A.1, we have for \( u \in T_x \Sigma \) of unit length,
\[
\limsup_{s \to 0} \frac{\phi(\exp^\Sigma_x su) + \phi(\exp^\Sigma_x -su) - 2\phi(x)}{s^2} \leq \frac{1}{2} \limsup_{s \to 0} \frac{d_\zeta^2(\exp^\Sigma_x su) + d_\zeta^2(\exp^\Sigma_x -su) - 2d_\zeta^2(x)}{s^2} \leq 2b \left( \frac{1}{2} \sqrt{-k} \right) + l | \text{II}(u, u) |,
\]
where the function \( b \) is defined in Lemma A.1 and \( l = d(x, \zeta) \). Now since both \( \Sigma \) and \( \Omega \) are compact, \( l \) admits a universal upper bound, and we can choose a universal lower bound \( k \) of the sectional curvature along any minimal geodesic starting in \( \Sigma \) and ending in \( \Omega \). In addition, the second fundamental form \( \text{II} \) admits a universal upper bound since \( \Sigma \) is compact. Hence we have
\[
\limsup_{s \to 0} \frac{\phi(\exp^\Sigma_x su) + \phi(\exp^\Sigma_x -su) - 2\phi(x)}{s^2} \leq C
\]
for some constant \( C \) independent of \( x \in \Sigma \) and \( u \in T_x \Sigma \). By Lemma 3.11 in [19], we conclude that \( \phi \) is semiconcave on \( \Sigma \). \( \square \)
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