On hypergraph cliques and polynomial programming

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Abstract Motzkin and Straus established a close connection between the maximum clique problem and a solution (namely graph-Lagrangians) to the maximum value of a class of homogeneous quadratic multilinear functions over the standard simplex of the Euclidean space in 1965. This connection provides a new proof of Turán’s theorem. Recently, an extension of Motzkin-Straus theorem was proved for non-uniform hypergraphs whose edges contain 1 or 2 vertices in [13]. It is interesting if similar results hold for other non-uniform hypergraphs. In this paper, we give some connection between polynomial programming and the clique of non-uniform hypergraphs whose edges contain 1, or 2, and more vertices. Specifically, we obtain some Motzkin-Straus type results in terms of the graph-Lagrangian of non-uniform hypergraphs whose edges contain 1, or 2, and more vertices.

Keywords Cliques of hypergraphs · graph-Lagrangians of non-uniform hypergraphs · polynomial programming

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1 Introduction

In 1965, Motzkin and Straus provided a new proof of Turán’s theorem based on a remarkable connection between the maximum clique and the graph-Lagrangian of a graph in [11]. In fact, the connection of graph-Lagrangians and Turán densities can be used to give another proof of the fundamental theorem of Erdős-Stone-Simonovits on

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Turán densities of graphs. This type of connection aroused interests in the study of graph-Lagrangians of $r$-graphs. A generalization of Motzkin-Straus theorem and Erdős-Stone-Simonovits theorem to non-uniform hypergraphs whose edges contain 1 or 2 vertices was given in \cite{13}.

A hypergraph $H = (V, E)$ consists of a vertex set $V$ and an edge set $E$, where every edge in $E$ is a subset of $V$. The set $T(H) = \{ |F| : F \in E \}$ is called the set of edge types of $H$. We also say that $H$ is a $T(H)$-graph. For example, if $T(H) = \{1, 2\}$, then we say that $H$ is a $\{1, 2\}$-graph. If all edges have the same cardinality $r$, then $H$ is called an $r$-uniform hypergraph or $r$-graph. A 2-uniform graph is called a graph. A hypergraph is non-uniform if it has at least two edge types. For any $r \in T(H)$, the level hypergraph $H^r$ is the hypergraph consisting of all edges with $r$ vertices of $H$. We write $H^0$ for a hypergraph $H$ on $n$ vertices with $T(H) = T$. An edge $\{i_1, i_2, \ldots, i_r\}$ in a hypergraph is simply written as $i_1i_2\cdots i_r$ throughout the paper.

For a positive integer $n$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. For a finite set $V$ and a positive integer $i$, let $\binom{V}{i}$ denote the family of all $i$-subsets of $V$. The complete hypergraph $K^r_n$ is a hypergraph on $n$ vertices with edge set $\bigcup_{i \in T} \binom{[n]}{i}$. For example, $K^{(r)}_n$ is the complete non-uniform hypergraph on $n$ vertices. $K^{(r)}_n$ is the non-uniform hypergraph with all possible edges of cardinality at most $r$. The complete graph on $n$ vertices $K^{(2)}_n$ is also called a clique. We also let $[p]^{(r)}$ represent the complete $r$-uniform hypergraph on vertex set $[p]$.

A useful tool in extremal problems of uniform hypergraphs (graphs) is the graph-Lagrangian of a uniform hypergraph (graph).

**Definition 1** For an $r$-uniform graph $H$ with the vertex set $[n]$, edge set $E(H)$, and a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we associate a homogeneous polynomial in $n$ variables, denoted by $\lambda(G, x)$ as follows:

$$\lambda(H, x) := \sum_{i_1i_2\cdots i_r \in E(H)} x_{i_1}x_{i_2}\cdots x_{i_r}.$$  

Let $S := \{x = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}$. Let $\lambda(H)$ represent the maximum of the above homogeneous multilinear polynomial of degree $r$ over the standard simplex $S$. Precisely

$$\lambda(H) := \max\{\lambda(H, x) : x \in S\}.$$

The value $x_i$ is called the weight of the vertex $i$. A vector $x := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is called a feasible weighting for $G$ iff $x \in S$. A vector $y \in S$ is called an optimal weighting for $G$ if $\lambda(G, y) = \lambda(G)$. We call $\lambda(G)$ the graph-Lagrangian of $G$.

**Remark 1** $\lambda(G)$ was called Lagrangian of $H$ in literature \cite{7, 8, 13, 19}. The terminology ‘graph-Lagrangian’ was suggested by Franco Giannessi.

Motzkin and Straus in \cite{11} showed that the graph-Lagrangian of a 2-graph is determined by the order of its maximum clique.

**Theorem 1** (\cite{11}) If $G$ is a 2-graph in which a largest clique has order $t$, then $\lambda(G) = \lambda(K^{(2)}_t) = \lambda([p]^{(2)}) = \frac{1}{2}(1 - \frac{1}{t})$.

Furthermore, the vector $x = (x_1, x_2, \ldots, x_n)$ given by $x_i = \frac{1}{t}$ if $i$ is a vertex in a fixed maximum complete $\{1, 2\}$-subgraph and $x_i = 0$ else is an optimal weighting.

This result provides a solution to the optimization problem for a class of homogeneous quadratic multilinear functions over the standard simplex of an Euclidean plane. The Motzkin-Straus result and its extension were successfully employed in optimization to provide heuristics for the maximum clique problem \cite{1, 3, 9}. It has been also generalized to vertex-weighted graphs \cite{9} and edge-weighted graphs with applications to pattern recognition in image analysis \cite{11, 1, 3, 5, 12, 14}. An attempt to generalize the Motzkin-Straus theorem to hypergraphs is due to Sós and Straus \cite{18}. Recently, in \cite{4, 5} Rota Buló and Pelillo generalized the Motzkin and Straus’ result to
r-graphs in some way using a continuous characterization of maximal cliques other than graph-Lagrangians of
hypergraphs.

The graph-Lagrangian of a hypergraph has been a useful tool in hypergraph extremal problems. For example,
Sidorenko [13] and Frankl-Furedi [7] applied graph-Lagrangians of hypergraphs in finding Turán densities of
hypergraphs. Frankl and Rödl [8] applied it in disproving Erdös long standing jumping constant conjecture. In
most applications, we need an upper bound for the graph-Lagrangian of a hypergraph.

Note that the graph-Lagrangian of an r-uniform graph can be viewed as the supremum of densities of its
blow-ups multiplying a constant \(\frac{1}{r!}\). The graph-Lagrangian of a non-uniform hypergraph defined in [13]
is the supremum of densities of its blow-ups.

**Definition 2** For a hypergraph \(H_n^T\) with \(T(H) = T\) and a vector \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\), define
\[
\lambda'(H_n^T, x) := \sum_{r \in T} (r! \sum_{i_1i_2\ldots i_r \in E(H)} x_{i_1}x_{i_2}\ldots x_{i_r}).
\]

Let \(S = \{x = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}\). The Graph-Lagrangian of \(H_n^T\), denoted by
\(\lambda'(H_n^T)\), is defined as
\[
\lambda'(H_n^T) := \max\{\lambda'(H_n^T, x) : x \in S\}.
\]
The value \(x_i\) is called the weight of the vertex \(i\). A vector \(y \in S\) is called an optimal weighting for \(H\) if
\(\lambda'(H, y) = \lambda'(H)\).

In [13], Peng et al. gave a generalization of Motzkin-Straus result to \(\{1, 2\}\)-graphs.

**Theorem 2** ([13]) If \(H\) is a \(\{1, 2\}\)-graph and the order of its maximum complete \(\{1, 2\}\)-subgraph is \(t\), where \(t \geq 2\), then
\(\lambda'(H) = \lambda'(K_{t}^{\{1,2\}}) = 2 - \frac{1}{t}\).

Furthermore, the vector \(x = (x_1, x_2, \ldots, x_n)\) given by \(x_i = \frac{1}{t}\) if \(i\) is a vertex in a fixed maximum complete
\(\{1, 2\}\)-subgraph and \(x_i = 0\) else is an optimal weighting.

Some related Motzkin-Straus type results in terms of graph-Lagrangians for non-uniform hypergraphs can be
found in [10].

In [14], a more general question is proposed.

**Problem 1** Let \(H\) be an \(\{r_0, r_1, r_2, \ldots, r_m\}\)-graph, \(r_0 < r_1 < r_2 < \ldots < r_m\), with vertex set \(V(H) = [n]\) and edge
set \(E(H)\). Let \(S = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}\). Let \(\alpha_i, 1 \leq i \leq m\) be positive constants. For \(x \in S\), let
\[
L_{\{\alpha_1, \alpha_2, \ldots, \alpha_m\}}(H, x) := \sum_{i_1i_2\ldots i_{r_0} \in E(H^0)} x_{i_1}x_{i_2}\ldots x_{i_{r_0}} + \alpha_{r_1} \sum_{i_1i_2\ldots i_{r_1} \in E(H^{r_1})} x_{i_1}x_{i_2}\ldots x_{i_{r_1}} + \ldots + \alpha_{r_m} \sum_{i_1i_2\ldots i_{r_m} \in E(H^{r_m})} x_{i_1}x_{i_2}\ldots x_{i_{r_m}}.
\]

The polynomial optimization problem of \(H\) is
\[
L_{\{\alpha_1, \alpha_2, \ldots, \alpha_m\}}(H) := \max\{L(H, x) : x \in S\}.
\]

We sometimes simply write \(L_{\{\alpha_1, \alpha_2, \ldots, \alpha_m\}}(H, x)\) and \(L_{\{\alpha_1, \alpha_2, \ldots, \alpha_m\}}(H)\) as \(L(H, x)\) and \(L(H)\) if there is no con-
fusion. The value \(x_i\) is called the weight of the vertex \(i\). A vector \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) is called a feasible
solution to (1) if and only if \(x \in S\). A vector \(y \in S\) is called a solution to optimization problem (1) if and only if
\(L(H, y) = L(H)\).
Remark 2 Let $H$ be an $\{r_0, r_1, r_2, \ldots, r_m\}$-graph, $r_0 < r_1 < r_2 < \ldots < r_m$, with vertex set $V(H) = [n]$ and edge set $E(H)$. Clearly, $\lambda'(H, x) = r_0! L_{\lambda_{r_0}}(r_1, \ldots, r_m)(H, x)$. Hence we can view $L(H)$ as subgraph weighted graph-Lagrangian of $H$.

Peng etc. in [14] gave some Motzkin-Straus type results to $\{1, r\}$-graphs and $\{1, 2, 3\}$-graphs for the polynomial programming (1).

Theorem 3 [14] Let $\alpha_r > 0$ be a constant. Let $H$ be a $\{1, r\}$-graph. If both the order of its maximum complete $\{1, r\}$-subgraph and the order of its maximum complete $\{1\}$-subgraph are $t$, where $t \geq \left\lceil \frac{(\alpha_r - (r - 2))^{r - 1}}{(r - 2)\alpha_r^{r - 3}} \right\rceil$, then

$$L_{\{1, r\}}(H) = L_{\{1\}}(K_r^{\{1, r\}}) = 1 + \alpha_r \frac{\prod_{i=1}^{t-1} (t - i)}{t!}.$$

Furthermore, the vector $x = (x_1, x_2, \ldots, x_n)$ given by $x_i = \frac{1}{t}$ if $i$ is a vertex in a fixed maximum complete $\{1, r\}$-subgraph and $x_i = 0$ else is a solution to the optimization problem (1) with $m = 1$ and $r_0 = 1$.

Theorem 4 [14] Let $\alpha_2, \alpha_3 > 0$ be constants. Let $H$ be a $\{1, 2, 3\}$-graph. If both the order of its maximum complete $\{1, 2, 3\}$-subgraph and the order of its maximum complete $\{1\}$-subgraph are $t$, where $t \geq \left\lceil \frac{(\alpha_2 + \alpha_3)^2 - \alpha_3}{\alpha_2 - 2} \right\rceil$, then

$$L_{\{1, 2, 3\}}(H) = L_{\{1\}}(K_r^{\{1\}, 2}) = 1 + \alpha_2 \frac{t - 1}{2t} + \alpha_3 \frac{(t - 1)(t - 2)}{6t^2}.$$

Furthermore, the vector $x = (x_1, x_2, \ldots, x_n)$ given by $x_i = \frac{1}{t}$ if $i$ is a vertex in a fixed maximum complete $\{1, 2, 3\}$-subgraph and $x_i = 0$ else is a solution to the corresponding optimization problem.

In this paper, we will prove other Motzkin-Straus type results to non-uniform hypergraphs whose edges contain 1, 2, and more vertices for (1). Here are our main results.

Theorem 5 (a) Let $\alpha_r > 0$ be a constant. Let $H$ be a $\{2, r\}$-graph. If both the order of its maximum complete $\{2, r\}$-subgraphs and the vertex order of $H^2$ are $t$, where $t \geq \left\lceil \frac{\alpha_r}{r\alpha_r - 2} \right\rceil + 1$, then

$$L_{\{2, r\}}(H) = L_{\{2\}}(K_r^{\{2\}}) = 1 + \frac{t - 1}{2t} + \frac{\prod_{i=1}^{t-1} (t - i)}{t!}.$$

(b) Let $\alpha_2, \alpha_r > 0$ be constants. Let $H$ be a $\{1, 2, r\}$-graph. If both the order of its maximum complete $\{1, 2, r\}$-subgraphs and the vertex order of $H^2$ are $t$, where $t \geq \left\lceil \frac{\alpha_2}{\alpha_2 - 2} \right\rceil + 1$, then

$$L_{\{1, 2, r\}}(H) = L_{\{1, 2\}}(K_r^{\{1, 2\}}) = 1 + \alpha_2 \frac{t - 1}{2t} + \alpha_r \frac{\prod_{i=1}^{t-1} (t - i)}{t!}.$$

Furthermore, the vector $x = (x_1, x_2, \ldots, x_n)$ given by $x_i = \frac{1}{t}$ if $i$ is a vertex in a fixed maximum complete $\{1, 2, 3\}$-subgraph and $x_i = 0$ else is a solution to the corresponding optimization problem in both (a) and (b).

Theorem 6 (a) Let $\alpha_r > 0$ be a constant. Let $H$ be a $\{2, r\}$-graph. If the order of its maximum complete $\{2, r\}$-subgraphs is $t$, and the number of edges in $H^2$, say $m$, satisfies $\left\lceil \frac{t}{2} \right\rceil \leq m \leq \left\lceil \frac{t}{2} \right\rceil + t - 2$, where $t \geq \left\lceil \frac{\alpha_r}{r\alpha_r - 2} \right\rceil + 1$, then

$$L_{\{2, r\}}(H) = L_{\{2\}}(K_r^{\{2\}}) = 1 + \alpha_r \frac{t - 1}{2t} + \frac{\prod_{i=1}^{t-1} (t - i)}{t!}.$$
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(b) Let \( c_2, \alpha > 0 \) be constants satisfying \( \alpha \geq \frac{2r}{r - 2} \). Let \( H \) be a \( \{1, 2, r\}\)-graph. If the order of its maximum complete \( \{1, 2, r\}\)-subgraphs is \( t \), and the number of edges in \( H^2 \), say \( m \), satisfies \( \binom{t}{2} \leq m \leq \binom{t}{2} + t - 2 \), where \( t \geq \frac{\alpha}{\alpha(r - 2)} + 1 \) and \( \alpha \geq \frac{2r}{r - 2} \), then

\[
L(c_2, \alpha) \left( H \right) = L(c_2, \alpha) \left( K_t(1, 2, r) \right) = 1 + c_2 \frac{t - 1}{2r} + \alpha \prod_{i=1}^{r-1} \frac{(t - i)}{r^{i-1}}.
\]

Furthermore, the vector \( x = (x_1, x_2, \ldots, x_n) \) given by \( x_i = \frac{1}{r} \) if \( i \) is a vertex in a fixed maximum complete \( \{1, 2, 3\}\)-subgraph and \( x_i = 0 \) else is a solution to the corresponding optimization problem in both (a) and (b).

Applying Theorems 5, 6, Remark 2, and by choosing appropriate coefficients in the polynomial programming \( (1) \), it is easy to see that the following results hold.

**Corollary 1** (a) Let \( H \) be a \( \{2, r\}\)-graph. If both the order of its maximum complete \( \{2, r\}\)-subgraphs and the vertex order of \( H^2 \) are \( t \), where \( t \geq \frac{r(r-1)}{2} + 1 \), then \( \lambda'(H) = \lambda'(K_t(2, r)) \).

(b) Let \( H \) be a \( \{1, 2, r\}\)-graph. If both the order of its maximum complete \( \{1, 2, r\}\)-subgraphs and the vertex order of \( H^2 \) are \( t \), where \( t \geq \frac{r(r-1)}{2} + 1 \), then \( \lambda'(H) = \lambda'(K_t(1, 2, r)) \).

Furthermore, the vector \( x = (x_1, x_2, \ldots, x_n) \) given by \( x_i = \frac{1}{r} \) if \( i \) is a vertex in a fixed maximum complete \( \{1, 2, 3\}\)-subgraph and \( x_i = 0 \) else is a solution to the corresponding optimization problem in both (a) and (b).

**Corollary 2** (a) Let \( 3 \leq r \leq 4 \). Let \( H \) be a \( \{2, r\}\)-graph. If the order of its maximum complete \( \{2, r\}\)-subgraphs is \( t \), and the number of edges in \( H^2 \), say \( m \), satisfies \( \binom{t}{2} \leq m \leq \binom{t}{2} + t - 2 \), where \( t \geq \frac{r(r-1)}{2} + 1 \), then

\[
\lambda'(H) = \lambda'(K_t(2, r)) = \frac{t - 1}{t} + \frac{\prod_{i=1}^{r-1} (t - i)}{t^{r-1}}.
\]

(b) Let \( 3 \leq r \leq 4 \). Let \( H \) be a \( \{1, 2, r\}\)-graph. If the order of its maximum complete \( \{1, 2, r\}\)-subgraphs is \( t \), and the number of edges in \( H^2 \), say \( m \), satisfies \( \binom{t}{2} \leq m \leq \binom{t}{2} + t - 2 \), where \( t \geq \frac{r(r-1)}{2} + 1 \), then

\[
\lambda'(H) = \lambda'(K_t(1, 2, r)) = 1 + \frac{t - 1}{t} + \frac{\prod_{i=1}^{r-1} (t - i)}{t^{r-1}}.
\]

Furthermore, the vector \( x = (x_1, x_2, \ldots, x_n) \) given by \( x_i = \frac{1}{r} \) if \( i \) is a vertex in a fixed maximum complete \( \{1, 2, 3\}\)-subgraph and \( x_i = 0 \) else is a solution to the corresponding optimization problem in both (a) and (b).

The rest of the paper is organized as follows. Some useful results are summarized in Section 2. The proofs of Theorems 5, 6 are given in Section 3. Further Motzkin-Straus type results for \( \{2, r_3, \ldots, r_m\}\)-graphs and \( \{1, 2, r_3, \ldots, r_m\}\)-graphs are given in Section 3 as well.

2 Some Preliminary Results

We will impose an additional condition on any solution \( x = (x_1, x_2, \ldots, x_n) \) to the polynomial programming \( (1) \):

(i) \( x_1 \geq x_2 \geq \ldots \geq x_\theta \geq 0 \).

(ii) \( \{i : x_i > 0\} \) is minimal, i.e., if \( y \) is a feasible solution to the polynomial programming \( (1) \) satisfying \( \{|i : y_i > 0|\} < \{|i : x_i > 0|\} \), then \( L(H, y) < L(H) \).

For a hypergraph \( H = (V, E) \), \( i \in V \), and \( r \in T(H) \), let \( E'_i = \{A \in V^{(r-1)}, A \cup \{i\} \in E'\} \). For a pair of vertices \( i, j \in V \), let \( E''_{i,j} = \{B \in V^{(r-2)}B \cup \{i, j\} \in E'\} \). Let \( (E'_i)' = \{A \in V^{(r-1)}, A \cup \{i\} \in V^{(r)} \setminus E'\} \), \( (E''_{ij})' = \{B \in V^{(r-2)}B \cup \{i, j\} \in V^{(r)} \setminus E'\} \), and \( E''_{i,j} = E'_i \cap (E''_{ij})' \). Let \( L(E'_i, x) = \alpha_i \lambda(E'_i, x) \), where \( \alpha_0 = 1 \). And \( L(E''_{i,j}, x) \) and \( L(E''_{i,j}, x) \) are defined similarly.
Let $E_i = \bigcup_{r \in T(H)} E_{ij}^r$, $E_{ij} = \bigcup_{r \in T(H)} E_{ij}^r$, and $E_{ij} = \bigcup_{r \in T(H)} L(E_{ij}^r, x)$. Let $L(E_i, x) = \bigcup_{r \in T(H)} L(E_{ij}^r, x)$. And $L(E_{ij}, x)$ and $L(E_{i\setminus j}, x)$ are defined similarly. Note that $L(E_i, x) = \frac{\partial L(H, x)}{\partial x_i}$ and $L(E_{ij}, x) = \frac{\partial L(H, x)}{\partial x_j}$.

Let $H = ([n], E)$. For $e \in E$, and $i, j \in [n]$ with $i < j$, define
\[
C_{i \setminus j}(e) = \begin{cases} (e \setminus \{j\}) \cup \{i\} & \text{if } i \notin e \text{ and } j \in e, \\ e & \text{otherwise.} \end{cases}
\]

and $C_{i \setminus j}(e) = \{C_{i \setminus j}(e) : e \in E\} \cup \{e, C_{i \setminus j}(e) \in E\}$.

We say that $H$ is left-compressed if $C_{i \setminus j}(E) = E$ for every $1 \leq i \leq j$.

**Remark 3** (Equivalent definition of left-compressed) A $T(H)$-hypergraph $H = ([n], E)$ is left-compressed if and only if for any $r \in T(H)$, $j_1 j_2 \cdots j_r \in E$ implies $i_1 i_2 \cdots i_r \in E$ provided $i_p \leq j_p$ for every $1 \leq p \leq r$. Equivalently, a $T(H)$-hypergraph $H = ([n], E)$ is left-compressed if and only if for any $r \in T(H)$, $E_{j_1}^r = \emptyset$ for any $1 \leq i < j \leq n$.

**Lemma 1** ([15]) Let $H = ([n], E)$ be a $T(H)$-graph, $i, j \in [n]$ with $i < j$ and $x = (x_1, \ldots, x_n)$ be a solution to the polynomial programming (1). Write $H_{i \setminus j} = ([n], C_{i \setminus j}(E))$. Then,
\[
L(H, x) \leq L(H_{i \setminus j}, x).
\]

**Lemma 2** ([14]) If $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = x_{k+2} = \ldots = x_n = 0$ and $x = (x_1, x_2, \ldots, x_n)$ is a solution to the polynomial programming (1), then (a) $\frac{\partial L(H, x)}{\partial x_1} = \frac{\partial L(H, x)}{\partial x_2} = \ldots = \frac{\partial L(H, x)}{\partial x_k}$. This is equivalent to $L(E_i, x) = L(E_{ij}, x)$ for $1 \leq i < j \leq k$. (b) $\forall 1 \leq i < j \leq k$, there exists an edge $e \in E(H)$ such that $\{i, j\} \subseteq e$.

**Remark 4** (a) Lemma 2 part (a) implies that
\[
x_j L(E_{ij}, x) + L(E_{i\setminus j}, x) = x_i L(E_{ij}, x) + L(E_{ij}, x).
\]
In particular, if $H$ is left-compressed, then
\[
(x_i - x_j)L(E_{ij}, x) = L(E_{i\setminus j}, x)
\]
for any $i, j$ satisfying $1 \leq i < j \leq k$ since $E_{\setminus i} = \emptyset$.

(b) If $G$ is left-compressed, then for any $i, j$ satisfying $1 \leq i < j \leq k$,
\[
x_i - x_j = \frac{L(E_{ij}, x)}{L(E_{ij}, x)}
\]
holds. If $G$ is left-compressed and $E_{\setminus i} = \emptyset$ for $i, j$ satisfying $1 \leq i < j \leq k$, then $x_i = x_j$.

(c) By (2), if $H$ is left-compressed, then a solution $x = (x_1, x_2, \ldots, x_n)$ to the optimization problem (1) must satisfy
\[
x_1 \geq x_2 \geq \ldots \geq x_n \geq 0.
\]

In [15], [20], and [16], the following theorems for 3-graphs and r-graphs were proved, respectively.

**Theorem 7** ([15]) Let $m$ and $t$ be positive integers satisfying
\[
\left(\frac{t}{3}\right) \leq m \leq \left(\frac{t}{3}\right) + \left(\frac{t - 1}{2}\right).
\]
Let $H$ be a 3-graph with $m$ edges and containing a clique of order $t$. Then $\lambda(H) = \lambda([t]^{(3)})$.

**Theorem 8** ([20]) Let $m$ and $t$ be integers satisfying $\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t - 1}{2}$. Let $G$ be a 3-graph with $m$ edges, if $G$ does not contain a complete subgraph of order $t$, then $\lambda(G) < \lambda([t]^{(3)})$.

**Theorem 9** ([16]) Let $m$ and $t$ be positive integers satisfying
\[
\left(\frac{t}{r}\right) \leq m \leq \left(\frac{t}{r}\right) + \left(\frac{t - 1}{r - 1}\right) - (2^{r-3} - 1)(\frac{t - 1}{r - 2} - 1).
\]
Let $H$ be an $r$-graph on $t + 1$ vertices with $m$ edges and containing a clique of order $t$. Then $\lambda(G) = \lambda([t]^{(r)})$. 
3 Proofs of main results

In order to prove Theorems 5 and 6, we begin with two lemmas. In the rest of the paper an optimal (feasible) weighting for \( H \) refers to a solution (feasible) to the polynomial programming (1) unless specifically stated.

**Lemma 3**  (a) Let \( H \) be a \([2, r]\)-graph. If both the order of its maximum complete \([2, r]\)-subgraphs and the vertex order of \( H^2 \) are \( t \), and \( H' \) is \([v]^{(r)}\), where \( s \geq t \geq \frac{\alpha}{(r-2)!} + 1 \), then \( L(H) = L(K^{(2, r)}_t) \).

(b) Let \( H \) be a \([1, 2, r]\)-graph. If both the order of its maximum complete \([1, 2, r]\)-subgraphs and the vertex order of \( H^2 \) are \( t \), \( H' \) is \([u]^{(1)}\), and \( H'' \) is \([v]^{(r)}\), where \( u \geq 4 \) and \( v \geq 4 \), then \( L(H) = L(K^{(1, 2, r)}_t) \).

**Proof of Lemma 3** We only give the proof of (b). The proof of (a) is similar to (b).

Applying Lemma 2a) and a direct calculation, we get a solution \( y \) to the polynomial programming (1) when \( H = K^{(1, r)}_t \) which is given by \( y_i = 1/t \) for each \( i \) and \( y_i = 0 \) else. So \( L(K^{(1, 2, r)}_t) = 1 + \alpha_2 \frac{1}{t^2} + \alpha_0 \frac{1}{t!} \).

Since \( K^{(1, 2, r)}_t \subset H \), clearly \( L(H) \geq L(K^{(1, 2, r)}_t) \). Thus to prove Theorem 3 we only need to prove that \( L(H) \leq L(K^{(1, 2, r)}_t) \).

Denote \( M_{\alpha,\beta,\gamma} = \max \{ L(H) : H \text{ is a } [1, 2, r]-\text{graph with } H' = [u]^{(1)}, H'' = [v]^{(r)} \text{ and both the order of its maximum complete } [1, 2, r]-\text{subgraph and the vertex order of } H^2 \text{ are } t \} \). We can assume that \( L(H) = M(t + 1, t, [1, 2, r]) \), i.e. \( H \) is an extremal graph. We can assume that \( H \) is left-compressed. If \( H \) is not left-compressed, performing a sequence of left-compressing operations (i.e. replace \( E \) by \( e_{i}/-j(E) \) if \( e_{i}/-j(E) \neq E \), we get a left-compressed \([1, 2, r]\)-graph with \( H' \) with the same number of edges, \( H'^1 = [u]^{(1)}, H'^r = [v]^{(r)} \), and both the order of its maximum complete \([1, 2, r]\)-subgraph and the vertex order of \( H'^2 \) are still \( t \). By Lemma 3, \( H' \) is also an extremal graph. We give the proof of the case \( u \leq v \) below. The proof for the case \( v \leq u \) is similar and the details will not be given.

Let \( x = (x_1, x_2, \ldots, x_r) \) be an optimal weighting of \( H \). By Remark 4(c), \( x_1 \geq x_2 \geq \ldots \geq x_r \geq 0 \). By Remark 4(b) we may assume that \( x_1 = \cdots = x_t = x_{t+1} = \cdots = x_{t+1} = \cdots = x_r \).

First we show that \( x_r = 0 \). Assume that \( x_r > 0 \) for a contradiction. By Lemma 2(a), \( L(E_1, x) = L(E_r, x) \). Assume \( u < v \) since otherwise we only need to prove that \( x_u = 0 \). Hence

\[
1 + \alpha_2 (1 - x_1 - x_{t+1} - \cdots - x_r) = L(E_1, x) = x_r L(E_r, x)。
\]

Since \( 0 < L(E_r, x) \leq \frac{\alpha(1-x_r)}{(r-2)!} < \frac{\alpha}{(r-2)!} \), then \( (t-1)x_1 < \frac{\alpha}{\alpha_2 (r-2)!} (1-x_r) \), i.e. \( t \leq \frac{\alpha}{\alpha_2 (r-2)!} + 1 \), which contradicts to \( t \geq \frac{\alpha}{(r-2)!} + 1 \).

Since \( x_1 = \cdots = x_r = 0 \), we can assume that the hypergraph is on \([u] \). Next we show that \( x_u = 0 \). Assume that \( x_u > 0 \) for a contradiction. By Lemma 2(b), \( L(E_1, x) = L(E_u, x) \). Hence

\[
\alpha_2 (1 - x_1 - x_{t+1} - \cdots - x_u) = L(E_1, x) = x_u L(E_u, x) = x_u L(E_r, x) = x_r L(E_r, x)。
\]

Since \( 0 < L(E_r, x) \leq \frac{\alpha(1-x_r)}{(r-2)!} < \frac{\alpha}{(r-2)!} \), then \( (t-1)x_1 < \frac{\alpha}{\alpha_2 (r-2)!} (1-x_r) \), i.e. \( t \leq \frac{\alpha}{\alpha_2 (r-2)!} + 1 \), which contradicts to \( t \geq \frac{\alpha}{(r-2)!} + 1 \).

This completes the proof of (b).

**Proof of Theorem 5** We only give the proof of (b). The proof of (a) is similar to (b). Since \( K^{(1, 2, r)}_t \subset H \), clearly \( L(H) \geq L(K^{(1, 2, r)}_t) \). Thus we only need to prove that \( L(H) \leq L(K^{(1, 2, r)}_t) \). Assume \( H' \) is on vertex set \([n] \). Let \( \overline{E} = [n]^{(1)} \), \( \overline{E} = [n]^{(r)} \) and \( \overline{T} = E^2 \cup \overline{E} \). Then \( L(H) \leq L(\overline{T}) \). By Lemma 3(b), \( L(H) \leq L(K^{(1, 2, r)}_t) \). This completes the proof of (b).
Now we are ready to prove Theorem 6 by applying Theorem 5.

Proof of Theorem 6: We only give the proof of (b). The proof of (a) is similar to (b). Since $K_1^{[1,2,r]} \subset H$, clearly $L(H) \geq L(K_1^{[1,2,r]})$. Thus to prove Theorem 5 we only need to prove that $L(H) \leq L(K_2^{[1,2,r]})$.

Denote $M_{(m,t,\{1,2,r\})} = \max \{ L(H) : H \text{ is a } \{1,2,r\}-\text{graph with } m \text{ edges in } H^* \}$. The order of its maximum complete $\{1,2,r\}$-subgraphs is $t$. We can assume that $L(H) = L(m,t,\{1,2,r\})$, i.e., $H$ is an extremal graph. We can assume that $H$ is left-compressed. If $H$ is not left-compressed, performing a sequence of left-compressing operations (i.e., replacing $E$ by $E_i$ if $E_i(E) \neq E$), we will get a left-compressed $\{1,2,r\}$-graph $H'$ with the same number of edges. And the order of the maximum complete $\{1,2,r\}$-subgraphs is still $t$. By Lemma 1, $H'$ is also an extremal graph.

Let $x = (x_1, x_2, \ldots, x_n)$ be an optimal weighting for $H$. Then $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = \ldots = x_n = 0$. First we show that the vertex order of $H^*$ is at most $t+1$. Assume that there is $ij \in H^*$ with $k \geq j \geq t+2$. We define a new feasible weighting $y$ for $H$ as follows. Let $y_j = x_i$ for $l \neq j-1, j$, $y_j = 0$ and $y_{j-1} = x_{j-1} + x_j$. By Lemma 2, we have $L(E_{j-1}, x) = L(E_j, x)$. Note that $(j - 1) \notin E^2$ for $j \geq t+1$. Hence

$$L(H, y) - L(H, x) = \sum_{u \in \{1,2,r\}} x_j [L(E_u^{\alpha} - E_j^r, x)] - \sum_{u \in \{1,2,r\}} x_j^2 [L(E_u^{\alpha} - E_{j-1}, x)]$$

$$= -x_j^2 L(E_j^r - E_{j-1}, x).$$

Since $y_j = 0$, we may remove all the edges containing $j$ from $E$ to form a new 3-graph $\overline{H} = ([n], \overline{E})$ with $|\overline{E}| = |E| - |E_j|$ and $L(\overline{H}, y) = L(H, y)$. Since $n \leq (\lambda)^t + t - 2$, we have $(t-1)(j - 1) \notin E^2$. Let $\overline{H} = \overline{H} \cup \{(t-1)(j - 1)\}$. Then $\overline{H}$ is a $\{1,2,r\}$-graph. The order of its maximum complete $\{1,2,r\}$-subgraph is still $t$. The number of edges in $\overline{H}$ satisfies $(\lambda)^t \leq m \leq (\lambda)^t + t - 2$. Recalling that $\alpha_2 \geq \frac{\alpha_0}{2(t-2)!}$ and $x_1 \geq x_2 \geq \ldots \geq x_{t-1} \geq x_{t-1} + x_{t} > 0$, we have

$$L(\overline{H}, y) - L(H, x) = x_j \alpha (x_j - x_j) - x_j^2 L(E_j^r - E_{j-1}, x)$$

$$\geq x_j \alpha (x_j - x_j) - \frac{\alpha}{r-2} x_j^2$$

$$\geq 0$$

since $0 < L(E_j^r - E_{j-1}, x) \leq \frac{\alpha (1-x_j-x_j)^{r-2}}{(r-2)!} < \frac{\alpha}{r-2}$. This contradicts to that $H$ is an extremal graph. Hence the order of the 2-graph is at most $t + 1$.

Next we prove $L(H) \leq L(K_1^{[1,2,r]})$.

Let $x = (x_1, x_2, \ldots, x_n)$ be an optimal weighting for $H$. Then $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = \ldots = x_n = 0$. We define a new feasible weighting $z$ as follows. Let $z_l = x_l$ for $l \neq t, t + 1, z_t = 0$ and $z_{t+1} = x_t + x_{t+1}$. By Lemma 2, we have $L(E_t^r, x) = L(E_{t+1}, x)$. Note that $(t+1) \notin E^2$. Hence

$$L(H, x) - L(H, z) = \sum_{u \in \{1,2,r\}} x_j [L(E_u^{\alpha} - E_j^r, x)] - \sum_{u \in \{1,2,r\}} x_j^2 [L(E_u^{\alpha} - E_{j+1}, x)]$$

$$= -x_j^2 L(E_j^r - E_{j+1}, x).$$

Since $z_t = 0$ we may remove all the edges containing $t$ from $E$ to form a new 3-graph $H^* = ([k], E^*)$ with $|E^*| = |E| - |E_t|$ and $L(H^*, y) = L(H, y)$. Since $n \leq (\lambda)^t + t - 2$, we have $(t-1)(t+1) \notin E^2$. Let $H^{**} := H^* \cup \{(t-1)(t+1)\}$. Then $H^{**}$ is a $\{1,2,r\}$-graph. Recalling that $\alpha_2 \geq \frac{\alpha_0}{2(t-2)!}$ and $x_1 \geq x_2 \geq \ldots \geq x_{t-1} \geq x_t \geq 0$, we have

$$L(H^{**}, y) - L(H^*, x) = x_j \alpha (x_j + x_{j+1}) - x_j^2 \lambda (E_j^r, x)$$

$$\geq x_j \alpha (x_j + x_{j+1}) - \frac{\alpha}{r-2} x_j^2$$

$$\geq 0.$$
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Since \(0 < L(E_{i+1}^{r_i}) \leq \frac{\alpha_2(1-x_i-x_{i+1}+\cdots-x_r)}{(r-2)!} < \frac{\alpha_2}{(r-2)!}\). Since the vertex order of \((H^*)^2\) is \(t\), we have \(L(H^*) \leq L(K_i^{(1,2,r)})\) by Theorem 5. Hence \(L(H) = L(H, x) \leq L(H^*, y) \leq L(H^*) = L(K_i^{(1,2,r)})\). This completes the proof of part (b).

Using the method given in the proof of Theorem 5 we may generalize Theorem 5 for \(\{2, r_3, \cdots, r_l\}\)-graphs (\(\{1,2, r_3, \cdots, r_l\}\)-graphs, respectively) in the following way, where \(r_3 < \cdots < r_m\), with vertex set \(V(H) = [n]\) and edge set \(E(H)\).

Theorem 10 (a) Let \(H\) be a \(\{2, r_3, \cdots, r_m\}\)-graph, where \(r_3 < \cdots < r_m\). If both the order of its maximum complete \(\{2, r_3, \cdots, r_m\}\)-subgraph and the order of \(\{2\}\)-graph are \(t\), where \(t \geq (m-2)\frac{\alpha_2}{(r_2-2)!} + 1\) then

\[
L(H) = L(K_i^{(2, r_3, \cdots, r_m)}).
\]

(b) Let \(H\) be a \(\{1,2, r_3, \cdots, r_m\}\)-graph, where \(r_3 < \cdots < r_m\). If both the order of its maximum complete \(\{1,2, r_3, \cdots, r_m\}\)-subgraph and the order of \(\{2\}\)-graph are \(t\), where \(t \geq (m-2)\frac{\alpha_2}{(r_2-2)!} + 1\) then

\[
L(H) = L(K_i^{(1,2, r_3, \cdots, r_m)}).
\]

We remark that the proof of Theorem 10 is similar to the proof of Theorem 5. For instance, to prove Theorem 10(b), we change (4) to

\[
1 + \alpha_2(1-x_1-x_{i+1}+\cdots-x_r) + \sum_{i \in \{3, \ldots, m\}} L(E_{i+1}^{r_i}, x) + \sum_{v \in \{3, \ldots, m\}} x_v L(E_{i+1}^{r_i}, x) - \sum_{i \in \{3, \ldots, m\}} x_i L(E_{i+1}^{r_i}, x) = 0.
\]

And we also change (5) to

\[
\alpha_2(1-x_1-x_{i+1}+\cdots-x_r) + \sum_{i \in \{3, \ldots, m\}} L(E_{i+1}^{r_i}, x) + \sum_{v \in \{3, \ldots, m\}} L(E_{i+1}^{r_i}, x) - \sum_{i \in \{3, \ldots, m\}} x_i L(E_{i+1}^{r_i}, x) = 0.
\]

We can make other responding changes easily. We omit the detail of the proof here.

By using Theorems 9, 7, and 8 we have

Theorem 11 (a) Let integers \(m\) and \(t\) satisfy \((\frac{1}{2}) \leq m \leq (\frac{2r-3}{3}) - (2r-3 - 1)\left((\frac{r_2-2}{2}) - 1\right)\). Let \(H\) be a \(\{2, r\}\)-graph (\(\{1,2, r\}\)-graph, respectively) with \(m\) edges in \(H^\prime\) and \(t+1\) vertices. If both the vertex order of its maximum complete \(\{2, r\}\)-subgraphs (\(\{1,2, r\}\)-subgraphs, respectively) and the vertex order of its maximum complete \(\{2\}\)-subgraphs (\(\{1,2\}\)-subgraphs, respectively) are \(t\). Then \(\lambda'(H) = \lambda'(K_i^{(2, r)})\) (\(\lambda'(H) = \lambda'(K_i^{(1,2,r)})\), respectively).

(b) Let integers \(m\) and \(t\) satisfy \((\frac{1}{2}) \leq m \leq (\frac{2r-3}{3}) - (2r-3 - 1)\left((\frac{r_2-2}{2}) - 1\right)\). Let \(H\) be a \(\{1,3\}\)-graph (\(\{1,2,3\}\)-graph, respectively) with \(m\) edges in \(H^\prime\). If both the order of its maximum complete \(\{2, 3\}\)-subgraphs (\(\{1,2, 3\}\)-subgraphs, respectively) and the order of its maximum complete \(\{2\}\)-subgraphs (\(\{1,2\}\)-subgraphs, respectively) are \(t\). Then \(\lambda'(H) = \lambda'(K_i^{(2,3)})\) (\(\lambda'(H) = \lambda'(K_i^{(1,2,3)})\), respectively).

(c) Let integers \(m\) and \(t\) satisfy \((\frac{1}{2}) \leq m \leq (\frac{2r-3}{3}) - (2r-3 - 1)\left((\frac{r_2-2}{2}) - 1\right)\). Let \(H\) be a \(\{1,3\}\) with \(m\) edges in \(H^\prime\). Then, if its maximum complete 3-graph is \(K_i^{(3)}\), we have \(\lambda'(H) = \lambda(K_i^{(1,3)})\); otherwise \(\lambda'(H) < \lambda(K_i^{(1,3)})\).
Proof (a) Let $H$ be a $\{2, r\}$-graph with $m$ edges in $H'$. Let $x = (x_1, x_2, \ldots, x_n)$ be an optimal weighting for $H$ for graph-Lagrangian function. Then, use Theorem 1,

$$
\lambda'(H) = \lambda'(H, x) = 2! \sum_{ij \in E} x_i x_j + r! \sum_{\{i_1 i_2 \ldots i_r\} \in E'} x_{i_1} x_{i_2} \cdots x_{i_r} \\
\leq (1 - \frac{1}{t}) + r! \sum_{\{i_1 i_2 \ldots i_r\} \in E'} x_{i_1} x_{i_2} \cdots x_{i_r}.
$$

By Theorem 9, we have \( \sum_{\{i_1 i_2 \ldots i_r\} \in E'} x_{i_1} x_{i_2} \cdots x_{i_r} \leq \lambda'(t|^{(r)}) \). Hence \( \lambda'(H) \leq (1 - \frac{1}{t}) + \lambda'(t|^{(r)}) = \lambda'(K_t^{(2,r)}) \).

On the other side, let $x_1 = x_2 = \ldots = x_t = \frac{1}{t}$. We have \( \lambda'(H, x) = \lambda'(K_t^{(2,r)}) \). Therefore \( \lambda'(H) = \lambda'(K_t^{(2,r)}) \).

The proof of the other results are similar. Note that we use Theorem 7 in part (b) and Theorem 8 in part (c). We omit the details.

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