Black hole solutions in 2+1 dimensions

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Abstract

We give circularly symmetric solutions for null fluid collapse in 2+1-dimensional Einstein gravity with a cosmological constant. The fluid pressure $P$ and energy density $\rho$ are related by $P = k\rho$ ($k \leq 1$). The long time limit of the solutions are black holes whose horizon structures depend on the value of $k$. The $k = 1$ solution is the Banados-Teitelboim-Zanelli black hole metric in the long time static limit, while the $k < 1$ solutions give other, 'hairy' black hole metrics in this limit.

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Three dimensional Einstein gravity provides a model for exploring many conceptual questions that arise in four dimensional generally covariant theories. Among these are various issues in quantum gravity and black hole physics.

The study of black holes in three dimensions became possible relatively recently when Banados, Teitelboim and Zanelli (BTZ) discovered a black hole solution [1] in 2+1-dimensional Einstein gravity with a negative cosmological constant. Since then there has been substantial work on black hole physics using this solution [2]. The general BTZ metric is characterized, like the four dimensional Kerr-Newman metric, by its mass, angular momentum and electric charge, but is asymptotically anti-deSitter rather than flat.

One of the questions concerning the BTZ solution is whether it can arise as the long time (static) limit of collapsing distributions of matter. This is indeed the case. There is for example, a three dimensional analog [3] of the Vaidya metric [4] which describes the collapse of a null fluid. The mass of the resulting BTZ black hole depends on the parameters characterizing the collapsing fluid. Other solutions that also describe collapse to the BTZ black hole have been given [5].

This raises the question of whether gravitational collapse of other types of matter also leads, in the asymptotic long time limit, to the BTZ black hole. This question is related to what ‘hairs’ a 2+1 black hole can have.

In this paper we partially address this question. We find exact inhomogeneous and non-static spherically symmetric solutions of the 2+1-dimensional Einstein equations for a collapsing null fluid, with pressure $P$ and energy density $\rho$ related by the equation of state $P = k\rho$. We find that the long time limits of the solutions give black hole metrics more general than the BTZ metric. The BTZ metric arises as the special case $k = 1$, while the metrics for $k < 1$ in general give black hole solutions with multiple apparent horizons.

An inverted approach is used to find the solutions. First the stress-energy tensor is determined from the metric. Then the equation of state and the dominant energy condition are imposed on its eigenvalues. This leads to an equation for the metric function, which is
The precise form of the stress-energy tensor is then displayed. The forms of the apparent horizons in some cases are discussed.

The general circularly symmetric metric in 2+1 dimensions may be written as

\[ ds^2 = -e^{2\psi(r,v)} F(r,v) \, dv^2 + 2e^{\psi(r,v)} \, dvdr + r^2 d\theta^2, \]  

where \( 0 \leq r \leq \infty \) is the proper radial coordinate, \( -\infty \leq v \leq \infty \) is an advanced time coordinate and \( 0 \leq \theta \leq 2\pi \) is the angular coordinate. The mass function \( m(r,v) \) is defined by \( F(r,v) = 1 - 2m(r,v)/r \). The Einstein equations

\[ G_{ab} + \Lambda g_{ab} = 2\pi T_{ab} \]  

give

\[ \psi' = 2\pi r \, T_{rr}, \]

\[ \dot{m} = 2\pi r^2 \, T_{v \, r}, \]

\[ \frac{m}{r} - m' = -\Lambda \, r^2 + 2\pi r^2 \, T_{v \, v}, \]

where the prime and dot denote partial derivatives with respect to \( r \) and \( v \). We will consider the case \( \psi(r,v) = 0 \), which means \( T_{rr} = 0 \).

The stress-energy tensor derived from the above metric may be diagonalized to give the energy density and the principal pressures. The eigenvalue problem is \( T_a^\, b U_b = \lambda U_a \). The \( \theta \) part of the tensor is already diagonal with pressure eigenvalues

\[ P \equiv T_\theta^\, \theta = \frac{G_{\theta\theta} + \Lambda \, r^2}{2\pi r^2} = \frac{1}{2\pi r^3} \left(-r^2 \, m'' + 2 \, r \, m' - 2 \, m + \Lambda \, r^3\right). \]

Since \( \psi = T_{rr} = 0 \) (by choice), \( T_v^\, v = T_r^\, r \) and \( T_r^\, v = 0 \), therefore the \( v-r \) part of the matrix to be diagonalized is

\[ \text{1The same procedure has been used by the author to find null fluid collapse solutions in four-dimensional Einstein gravity.}^{\text{3}} \]
\[ T^b_a = \begin{pmatrix} T^v_v & T^r_v \\ T^r_v & T^v_v \end{pmatrix}, \]  

(7)

This has one eigenvalue \( \lambda \) which gives the energy density \( \rho \), namely

\[ \rho \equiv -\lambda = -T^v_v = \frac{1}{2\pi \tau^3} \left(r \ m' - m - \Lambda \ r^3 \right). \]  

(8)

The corresponding eigenvector is \( v_a = (1, 0, 0) \), (in the coordinates \((v, r, \theta)\)), and is lightlike. Therefore the stress-energy tensor, (which follows from \( \psi = 0 \)), is of Type II \[ 7 \] Its non-vanishing components are

\[ T_{vv} = \rho \left(1 - \frac{2m}{r}\right) + \frac{\dot{m}}{2\pi \tau^2}, \]
\[ T_{vr} = -\rho, \]
\[ T_{\theta\theta} = P \ g_{\theta\theta}, \]  

(9)

which may be succinctly written using the lightlike vectors \( v_a \) and \( w_a = \left(\frac{F}{2}, 0, 0\right) \) as

\[ T_{ab} = \frac{\dot{m}}{2\pi \tau^2} \ v_a v_b + \rho \left( v_a w_b + v_b w_a \right) + P \left( g_{ab} + v_a w_b + v_b w_a \right). \]  

(10)

The last two terms on the right hand side make up a perfect fluid-like contribution to the stress-energy tensor, with the difference that the velocities are lightlike. A stationary observer with 3-velocity \( S^a = (1/\sqrt{F}, 0, 0) \) and a rotating observer with 3-velocity \( R^a = (\sqrt{2}/F, 0, 1/r) \) see respectively the energy densities

\[ T_{ab} S^a S^b = \frac{\dot{m}}{2\pi \tau^2 F} + \rho, \]  

(11)
\[ T_{ab} R^a R^b = 2 \left( \frac{\dot{m}}{2\pi \tau^2 F} + \rho \right) + P. \]  

(12)

\[ ^2\text{Stress-energy tensors are classified by their eigenvectors. In four dimensions, Type I stress-energy tensors have one timelike and three spacelike eigenvectors, and Type II tensors have one lightlike and two spacelike eigenvectors. Most physical fields are of Type I, except for certain null fluids, which are of type Type II.} \]
The stress-energy tensor (10) satisfies the dominant energy condition if the following three conditions are met [7]:

\[ P \geq 0, \quad \rho \geq P, \quad \text{and} \quad T_{ab} w^a w^b > 0, \tag{13} \]

To satisfy the first two of these conditions (13), we impose the equation of state \( P = k\rho \), with \( k \leq 1 \). This gives the equation

\[ r^2 m'' - r (2 - k) m' + (2 - k) m = \Lambda (1 + k) r^3 \tag{14} \]

for the mass function, which has the general solution

\[ m(r, v) = g(v) r + h(v) r^{2-k} + \frac{\Lambda r^3}{2}, \tag{15} \]

where \( g(v) \) and \( h(v) \) are arbitrary functions. Therefore from (6) and (8) we have explicitly that

\[ P = k \frac{(1 - k) h(v)}{2\pi r^{k+1}} = k\rho. \tag{16} \]

Hence we must have \( h(v) \geq 0 \) for positive pressure and energy density. Note that when \( k = 1 \), \( P = \rho = 0 \), and as we will see below, this gives the BTZ metric in the long time limit.

The last requirement in (13) for the dominant energy condition leads to

\[ \dot{m} = \dot{g}(v) r + \dot{h}(v) r^{2-k} > 0. \tag{17} \]

Physically this means that the matter within a radius \( r \) increases with time, which corresponds to an implosion. This is easily satisfied by choosing \( \dot{g}(v) \) and \( \dot{h}(v) \) to be positive everywhere.

In summary, we have shown that the metric

\[ ds^2 = -(1 - 2 g(v) - 2 h(v) r^{1-k} - \Lambda r^3) dv^2 + 2 dvdr + r^2 d\theta^2 \tag{18} \]

is a solution of the 2+1-dimensional Einstein equations for the null fluid stress-energy tensor (10) with \( P = k\rho \). Since \( k \leq 1 \), the cosmological constant must be negative for this metric.
to be asymptotically Lorentzian, in which case it is asymptotically anti-deSitter. Therefore in the following we set $\Lambda = -1/l^2$.

There are two special cases of this solution which are already known. One is the 2+1 dimensional analog \[3\] of the Vaidya metric \[4\], which arises for $k = 1$. This means vanishing $\rho$ and $P$. Then the only non-vanishing component of the stress-energy tensor \[10\] is $T_{vv} = \dot{m}/2\pi r^2 = \dot{g}(v)/2\pi r$. The other is the BTZ black hole \[1\], which arises when $g(v)=$constant, in addition to $k = 1$. (We note that when $k = 1$, $h(v)$ disappears from the stress-energy tensor \[10\], therefore setting $k = 1$ is the same as setting $h(v) = 0$).

We now consider the evolution of the apparent horizons for some cases of the metric \[18\]. The radius of the apparent horizon $r_{AH}$ is given by the equation $g^{ab}\partial_a r \partial_b r = 0$, which gives the equation $m(r,v; c_i) = r/2$ for $r_{AH}(v; c_i)$. The $c_i$ are any constants that appear in the mass function. The long time $v \to \infty$ limit of $r_{AH}$ gives the asymptotic radius of the apparent horizon as a function of the constants $c_i$. This is a measure of the black hole mass for asymptotically flat or (anti)-deSitter spacetimes, namely $M_{BH} := \lim_{v \to \infty} r_{AH}(v; c)/2$.

We focus on the cases where the $v \to -\infty$ limit of $g(v)$ and $h(v)$ are zero, so that there are no initial horizons, and their $v \to +\infty$ limit are finite non-zero constants, so that there are final horizons. To be specific, we assume in the following that $\lim_{v \to \infty} g(v) = A (> 0)$, and $\lim_{v \to \infty} h(v) = B (> 0)$. These conditions may be met easily by appropriate choices of the arbitrary functions $g(v)$ and $h(v)$ in the solutions. The radii of the apparent horizons in the $v \to \infty$ limit of the metric \[18\] are then given by

$$r^2 - 2B l^2 r^{1-k} + l^2 (1 - 2A) = 0.$$  

(19)

This equation leads to a range of polynomial equations for the asymptotic apparent horizons depending on the values of $k \leq 1$. We will consider a few special cases.

For $k = 1$ the radius of the apparent horizon is

$$r_{AH} = l \sqrt{2A - 1},$$

(20)

which is the single horizon of the circularly symmetric uncharged BTZ black hole. (As noted above, $k = 1$ also implies $B = 0$).
For $k = 0$, the pressure vanishes but the energy density $\rho$ does not. The apparent horizon equation (19) is quadratic with solutions

$$r_{AH} = l^2 B \pm \sqrt{l^4 B^2 - l^2 (1 - 2A)}.$$  \hspace{1cm} (21)

This is like the Reissner-Nordstrom black hole in four dimensions with charge $l \sqrt{1 - 2A}$ and mass $l^2 B$, with the extreme case occurring when $l B = \sqrt{1 - 2A}$. An important difference however is that there is no curvature singularity at $r = 0$ in the metrics (18). We note that the charged BTZ black hole does not give Eqn. (21) for the horizons [3] because, among other things, the electric potential in two spatial dimensions is proportional to $\ln r$.

The next simplest case is $k = 1/2$ for which Eqn. (19) is quartic. All other values of $k$ give higher order polynomials which may be solved numerically. The main observation is that in general there will be multiple horizons, and the long time limits will give black hole solutions more general then the BTZ black hole. Therefore 2+1-dimensional black holes can have ‘null fluid hair’ in the static limit.

An explicit example of a hairy black hole is provided by taking the arbitrary functions $g(v)$ and $h(v)$ in the mass function (15) to be $g(v) = A \tanh v$ and $h(v) = B (1 + \tanh v)$, for positive constants $A$ and $B$. (The dominant energy condition is satisfied for these choices). Then in the $v \to \infty$ static limit, the mass function becomes $m(r) = A r + 2B r^{2-k} + \Lambda r^3/2$. Therefore the constant $A$ determines the black hole mass, and $B$ represents the ‘null fluid hair’ of the resulting static black hole. There are of course many possible choices for $g(v)$ and $h(v)$ that lead to such black holes in the static long time limit.

In conclusion, we have given a new class of null fluid collapse solutions (18) of the 2+1 dimensional Einstein equations with negative cosmological constant for the stress-energy tensor (10). A special case of these solutions is the BTZ black hole. It would be of interest to study the global structures and thermodynamics of these black hole metrics.

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