A rigidity theorem for Moor-bialgebras

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Abstract: We introduce the operad Moor, dual of the operad NAP and the notion of Moor-bialgebras. We warn the reader that the compatibility relation linking the Moor-operation with the Moor-cooperation is not distributive in the sense of Loday. Nevertheless, a rigidity theorem (à la Hopf-Borel) for the category of connected Moor-bialgebras is given. We show also that free permutative algebras can be equipped with a Moor-cooperation whose compatibility with the permutative product looks like the infinitesimal relation.

Notation: In the sequel $K$ is a characteristic zero field and $\Sigma_n$ is the group of permutations over $n$ elements. If $A$ is an operad, then the $K$-vector space of $n$-ary operations is denoted as usual by $A(n)$. We adopt Sweedler notation for the binary cooperation $\Delta$ on a $K$-vector space $V$ and set $\Delta(x) = x_{(1)} \otimes x_{(2)}$.

1 Introduction

The well-known Hopf-Borel theorem states that any connected cocommutative commutative bialgebra (Hopf algebra) $\mathcal{H}$ is free and cofree over its primitive part $\text{Prim } \mathcal{H}$. Otherwise stated;

Theorem 1.1 (Hopf-Borel) For any cocommutative commutative bialgebra $\mathcal{H}$ the following is equivalent.

1. $\mathcal{H}$ is connected;
2. $\mathcal{H}$ is isomorphic to $\text{Com}(\text{Prim } \mathcal{H})$ as a bialgebra;
3. $\mathcal{H}$ is isomorphic to $\text{Com}^c(\text{Prim } \mathcal{H})$ as a coalgebra.
In the theory developed by J.-L. Loday, this result is rephrased by saying that the triple of operads \((Com, Com, Vect)\) is good. Some good triples of operads of type \((A, A, Vect)\) or \((C, A, Vect)\) have been found since and a summary can be found in [6]. The aim of this paper is to produce another good triple of operads of this form but without using the powerful theorems of J.-L. Loday [6], simply because his first Hypothesis \((H0)\) is not fulfilled by our objects.

From our coalgebra framework on weighted directed graphs [4, 3], we describe a directed graph by two cooperations \(\Delta_M\) and \(\tilde{\Delta}_M\) verifying:

\[
(\Delta_M \otimes id)\Delta_M = (id \otimes \Delta_M)\tilde{\Delta}_M.
\]

To code a bidirected graph, we have to add the extra condition \(\tau \Delta_M = \tilde{\Delta}\), where \(\tau\) is the usual flip map. The previous equation becomes,

\[
(id \otimes \tau)(\Delta_M \otimes id)\Delta_M = (\Delta_M \otimes id)\Delta_M.
\]

Such coalgebras were called \(L\)-cocommutative in [4]. On the algebra side, this yield \(K\)-vector spaces equipped with a binary operation \(<\) verifying,

\[
(x < y) < z = (x < z) < y.
\]

Such algebras came out in the work of M. Livernet [5] under the name nonassociative permutative algebras, \(NAP\)-algebras for short. The operad \(NAP\) of \(NAP\)-algebras is important because it is related to the operad \(preLie\) of \(preLie\)-algebras. Indeed, the triple of operads \((NAP, preLie, Vect)\) has been shown to be good by M. Livernet [5]. Requiring the operation \(<\) to be associative leads to permutative algebras, or \(Perm\)-algebras for short [1]. In this paper, we introduce the dual, in the sense of Ginzburg and Kapranov [6], of \(NAP\)-algebras, called \(Moor\)-algebras in Sections 2-3 and give a rigidity theorem for the category of connected \(Moor\)-bialgebras in Section 4, that is the triple of operads \((Moor, Moor, Vect)\) is good. This category is interesting, as we said, for we cannot apply the powerful results of J.-L. Loday [6] since the compatibility relation linking the cooperation and the operation of a \(Moor\)-bialgebra is not distributive as required in [6], Hypothesis \((H0)\). We end with Section 5, where we show that the free permutative algebra over a \(K\)-vector space \(V\) can be equipped with a \(Moor\)-cooperation whose compatibility relation with the permutative product looks like the nonunital infinitesimal relation.

### 2 On \(Moor\)-algebras

Define the operad \(Moor\), \((Moor\) because a typical element of a \(Moor\)-algebra looks like \((\ldots ((x_1 x_2) x_3) \ldots) x_n\) whose parentheses are concentrating at the beginning, reminding boats being moored one behind the other) to be the free operad on one binary operation \(<\) divided out by the following set of relations:

\[
R := \{ (x < y) < z = (x < z) < y; \ x < (y < z) \}.
\]
If $V$ stands for a $K$-vector space, then $S(V)$ stands for the symmetric module over $V$, that is:

$$S(V) := K ⊕ \bigoplus_{n>0} S^n(V),$$

where $S^n(V)$ is the quotient of $V^\otimes n$ by the usual action of the symmetric group $\Sigma_n$. A typical element of $S^n(V)$ will be written $v_1 \vee v_2 \vee \ldots \vee v_n$, where the $v_i \in V$.

**Theorem 2.1** The following hold.

1. The dual of the operad NAP is the operad Moor.
2. The free Moor-algebra over a $K$-vector space $V$ is $V \otimes S(V)$ as a $K$-vector space equipped with the operation $\prec$ defined by:

$$v \otimes \omega \prec v' \otimes \omega' = v \otimes \omega \vee v',$$

if $\omega' \in K$ and vanishes otherwise.
3. The generating series of the operad Moor is,

$$f_{\text{Moor}}(x) := xe^x = \sum_{n>0} \frac{x^n}{n!}.$$

**Proof:** Observe that the free binary operad $F$ with one binary operation obey the relation $\dim F(3) = 12$. We get $\dim NAP(3) = 9$ and $\dim Moor(3) = 3$. As in $F(3)$, quadratic relations defining $NAP(3)$ are orthogonal (see [2]) to those defining $Moor(3)$, the dual of $NAP$ is Moor. Let $V$ be a $K$-vector space. The $K$-vector space $V \otimes S(V)$ equipped with the operation $\prec$:

$$\prec : V \otimes S(V) \otimes V \otimes S(V) \rightarrow V \otimes S(V), \quad v \otimes \omega \prec v' \otimes \omega' = v \otimes \omega \vee v',$$

if $\omega' \in K$ and vanishes otherwise is a Moor-algebra. Observe that $i : V \rightarrow V \otimes K \rightarrow V \otimes S(V)$ defined by $i(v) := v \otimes 1_K$ realises the expected embedding. Let $(A, \prec_A)$ be a Moor-algebra and $f : V \rightarrow A$ be a map. Define $\tilde{f} : V \otimes S(V) \rightarrow A$ by,

$$\tilde{f}(v \otimes 1_K) := f(v),$$

$$\tilde{f}(v \otimes v_1 \cdots v_p) := (\cdots((f(v) \prec_A f(v_1)) \prec_A f(v_2)) \cdots \prec_A f(v_{p-1})) \prec_A f(v_p)).$$

Then, $\tilde{f}$ is a Moor-morphism and the only one such that $\tilde{f} \circ i = f$. For the last item, observe that in a Moor-algebra only these monomials,

$$(\text{left combs : (lc)}) \quad (\cdots(v_1 \prec v_2) \prec v_3) \cdots \prec v_{n-1} \prec v_n),$$

do not vanish. Indeed, one can model n-ary operations of the Moor-operad with planar rooted binary trees whose nodes are decorated by $\prec$. For instance, $x \prec (y \prec z)$ is represented by $\Uparrow \Uparrow \Uparrow \Uparrow$, so $\Uparrow \Uparrow \Uparrow \Uparrow = 0$ and only left combs survive. Therefore, we get $n(n-1)!$ such left combs but because of the relation $(x \prec y) \prec z = (x \prec z) \prec y$, the relation $(lc)$ is invariant under the action of the symmetric group $\Sigma_{n-1}$. Hence, $\dim Moor(n) = n$. □
3 The cofree Moor-coalgebra

Let \( i \) be an integer. By \( v^i \), we mean \( v \vee \ldots \vee v \), times \( i \). In the sequel, we set by induction, for all \( n > 0 \), \( \Delta^{(1)}_n = \Delta_n \) and \( \Delta^{(n)}_n := (\Delta_n \otimes \text{id}_{(n-1)}) \Delta^{(n-1)}_n \) for any cooperation \( \Delta_n \) of a Moor-coalgebra \( \mathcal{H} \). We get the following two propositions by dualising the corresponding results in the proof of Theorem 2.1.

**Lemma 3.1** Let \((\mathcal{H}, \Delta_n)\) be a coalgebra whose cooperation verifies \( \Delta^{(2)}_n = (\text{id} \otimes \tau)\Delta^{(2)}_n \). For all \( n > 0 \), the map:

\[
\Delta^{(n)}_n : \mathcal{H} \to \mathcal{H}^{\otimes (n+1)}, \quad x \mapsto \Delta^{(n)}(x) := x_{n+1} \otimes x_n \otimes \ldots \otimes x_1 \otimes x_2 \otimes x_1,
\]

has its last \( n \) components invariant by \( \Sigma_n \).

**Proof:** Fix \( i = 1, \ldots, n - 1 \). The following,

\[
\Delta^{(n)}_n = (\Delta^{(n-i-1)}_n \otimes \text{id}_{(i+1)}) \circ (\Delta^{(2)}_n \otimes \text{id}_{(i-1)}) \circ \Delta^{(i-1)}_n,
\]

\[
= (\Delta^{(n-i-1)}_n \otimes \text{id}_{(i+1)}) \circ ((\text{id} \otimes \tau)\Delta^{(2)}_n \otimes \text{id}_{(i-1)}) \circ \Delta^{(i-1)}_n,
\]

\[
= (\text{id}_{(n-i)} \otimes \tau \otimes \text{id}_{(i-1)}) \circ (\Delta^{(n-i-1)}_n \otimes \text{id}_{(i+1)}) \circ (\Delta^{(2)}_n \otimes \text{id}_{(i-1)}) \circ \Delta^{(i-1)}_n,
\]

shows that the last \( n \) components of \( \Delta^{(n)}_n \) are invariant by the transpositions \((i, i+1)\) for all \( i = 1, \ldots, n - 1 \), hence the claim. \( \square \)

**Proposition 3.2** The cofree Moor-coalgebra over a \( K \)-vector space \( V \) is:

\[
\text{Moor}^c(V) := V \otimes S(V),
\]

as a \( K \)-vector space equipped with the following co-operation \( \delta \) defined as follows:

\[
\delta(v \otimes 1_K) = 0,
\]

\[
\delta(v_1 \otimes v_2^{i_2} \vee \ldots \vee v_n^{i_n}) = \sum_{k=2}^{n} (v_1 \otimes v_2^{i_2} \vee \ldots \vee v_k^{i_k-1} \vee \ldots \vee v_n^{i_n}) \otimes (v_k \otimes 1_K).
\]

Let \( \Gamma V^{\otimes n} \) be the \( K \)-vector space of tensors invariant through the usual action of \( \Sigma_n \). For all \( n > 1 \), define \( \mathcal{j}_n : V \otimes \Gamma V^{\otimes n} \to V \otimes S^n(V) \) by \( \mathcal{j}_n(\sum_{\sigma \in \Sigma_n} v \otimes v_1 \otimes \ldots \otimes v_n) = v \otimes v_1 \vee \ldots \vee v_n \). They are bijective maps since \( K \) is a characteristic zero field.

**Proposition 3.3** If \((\mathcal{H}, \Delta_n)\) is a Moor\(^c\)-coalgebra and \( f : \mathcal{H} \to V \) a linear map, set by induction \( f^{(1)} = f \) and \( f^{(n)} = f^{(n-1)} \otimes f \). Then, the map \( \tilde{f} : \mathcal{H} \to \text{Moor}^c(V) \) defined by:

\[
\tilde{f} := \sum_{n=1}^{\infty} \mathcal{j}_n \circ f^{(n+1)} \circ \Delta^{(n)}_n,
\]

is the unique coalgebra morphism verifying \( \pi \circ \tilde{f} = f \), where \( \pi : \text{Moor}^c(V) \to V \) is the canonical projection.
4 On Moor-bialgebras

4.1 Definition

In the sequel, we set for any $v_1, \ldots, v_n \in V$:

$$(\ldots(v_1 \prec v_2) \prec \ldots) \prec v_n) := [v_1|v_2, \ldots, v_n].$$

By definition, a Moor-bialgebra $H$ is the data of:

1. A graduated Moor-algebra $H := \bigoplus_{p \geq 0} H_p$,
2. A Moor-cooperation $\Delta_H : H \rightarrow H \otimes 2$, i.e., verifying:
   $$(id \otimes \Delta_H)\Delta_H = 0,$$
   $$((\Delta_H \otimes id)\Delta_H = (id \otimes \tau)(\Delta_H \otimes id)\Delta_H,$$
3. The Moor-operation and cooperation have to be related by the following compatibility condition:
   $$\Delta_H(x \prec y) = x \otimes e(y) + (x^{(1)} \prec y) \otimes x^{(2)},$$
   for any $x, y \in H$, where $e : H \rightarrow H_1$ is the canonical projection.

A morphism of Moor-bialgebras is a morphism of graduated Moor-algebras and a morphism of Moor-coalgebras. Observe that this compatibility relation is not distributive in the sense of J.-L. Loday [6]. By $\text{Prim } H := \ker \Delta_H$ we mean the $K$-vector space of primitive elements.

**Proposition 4.1** Let $H$ be a Moor-bialgebra. Then, $\ker \Delta_H = \hat{H}_1 \oplus \hat{H}$, where $\hat{H} := \bigoplus_{j \in J} H_j$, with $J$ a suitable subset of $\mathbb{N} \setminus \{0, 1\}$. If it exists, $H$ is a Moor-algebra equipped with the following action:

$$\hat{H}_1 \otimes \hat{H} \rightarrow \hat{H}, \quad h_1 \otimes h \mapsto h_1 \prec h.$$ 

Moreover, $H_* := \bigoplus_{p>1} H_p$ acts on $\ker \Delta_H$ on the right via:

$$\ker \Delta_H \otimes H_* \rightarrow \ker \Delta_H, \quad a \otimes h \mapsto a \prec h.$$

**Proof:** For $j > 0$, set $\hat{H}_j$ the $K$-vector space of primitive elements of degree $j$. For $h_j$ and $h_{j'}$ two primitive elements of degrees resp. $j \geq 1$ and $j' > 1$, one has:

$$\Delta_H(h_j \prec h_{j'}) = h_j \otimes e(h_{j'}) = 0,$$
hence $h_j \prec h_{j'}$ of degree $j + j'$ is primitive. If $h \in \mathcal{H}_*$, then $\Delta(h_j \prec h) = h_j \otimes e(h) = 0$ hence the last claim.

Set $\Delta^{(n)} := (\Delta_n \otimes id_{(n-1)}) \Delta^{(n-1)}_{\mathcal{H}}$ with $id_{(n-1)} = \underbrace{id \otimes \ldots \otimes id}_{\text{times } n-1}$ for all $n \geq 1$. By definition, a Moor-bialgebra is is said to be connected if $\mathcal{H} = \bigcup_{r>0} \mathcal{F}_r \mathcal{H}$, where the filtration $(\mathcal{F}_r \mathcal{H})_{r>0}$ is defined as follows:

\[(\text{The primitive elements :}) \text{ Prim } \mathcal{H} := F_1 \mathcal{H} := \ker \Delta_n \subset \mathcal{H}_1.\]

Set $\Delta^{(n)} := (\Delta_n \otimes id_{(n-1)}) \Delta^{(n-1)}_{\mathcal{H}}$ with $id_{(n-1)} = \underbrace{id \otimes \ldots \otimes id}_{\text{times } n-1}$ for all $n \geq 1$. Then,

$$F_r \mathcal{H} := \ker \Delta^{(r)}_{\mathcal{H}}.$$

Here is an example of connected Moor-bialgebras.

**Theorem 4.2** Let $V$ be a $K$-vector space. The free Moor-algebra over $V$ is a connected Moor-bialgebra.

**Proof:** Let $V$ be a $K$-vector space. Define the co-operation $\Delta$ by induction as follows:

$$\Delta(v \otimes 1_K) := 0,$$

$$\Delta(x \prec y) = x \otimes \pi(y) + (x_{(1)} \prec y) \otimes x_{(2)},$$

for any $v \in V$, $x, y \in Moor(V)$, where $\pi : Moor(V) \rightarrow i(V)$ is the canonical projection map. As $x \prec y = 0$ for all $x, y \in Moor(V)$ and $y$ of degree at least 2, we have to check that $\Delta(x \prec y)$ vanishes. But,

$$\Delta(x \prec y) = x \otimes \pi(y) + (x_{(1)} \prec y) \otimes x_{(2)} = 0,$$

because $\pi(y) = 0$ since the degree of $y$ is at least 2 and $x_{(1)} \prec y = 0$ for the same reason. By induction one proves:

$$\Delta([v_1|v_2, \ldots, v_n]) = \sum_{k=2}^{n} [v_1|v_2, \ldots, \hat{v}_k, \ldots, v_n] \otimes (v_k \otimes 1_K),$$

for all $v_1, \ldots, v_n \in V$ and where the hat notation means as usual that the involved element vanishes. From that formula, it is straightforward to check that the co-operation $\Delta$ verifies:

$$(id \otimes \Delta)\Delta = 0,$$

$$(\Delta \otimes id)\Delta = (id \otimes \tau)(\Delta \otimes id)\Delta.$$

Hence, the free Moor-algebra over $V$, which is graduated by construction, is a Moor-bialgebra.

6
The map $\phi(V) : Moor(V) \to Moor^c(V)$, defined as follows:

$$
\phi(V)(v \otimes 1_K) = v \otimes 1_K,
$$

$$
\phi(V)([v_1|v_2^2, \ldots, v_n^{i_n}]) = i_2! \ldots i_n! v_1 \otimes v_2^{i_2} \vee \ldots \vee v_n^{i_n},
$$

is an isomorphism of $Moor$-coalgebras. It suffices to observe that:

$$
\delta(v_1 \otimes v_2^{i_2} \vee \ldots \vee v_n^{i_n}) = \sum_{k=2}^{n} \left( (v_1 \otimes v_2^{i_2} \vee \ldots \vee v_k^{i_k-1} \vee \ldots \vee v_n^{i_n}) \otimes (v_k \otimes 1_K) \right),
$$

and:

$$
\Delta([v_1|v_2^2, \ldots, v_n^{i_n}]) = \sum_{k=2}^{n} i_k [v_1|v_2^{i_2} \vee \ldots \vee v_k^{i_k-1} \vee \ldots \vee v_n^{i_n}] \otimes (v_k \otimes 1_K),
$$

Thus:

$$
\delta(\phi(V)([v_1|v_2^2, \ldots, v_n^{i_n}])) = i_2! \ldots i_n! \sum_{k=2}^{n} (v_1 \otimes v_2^{i_2} \vee \ldots \vee v_k^{i_k-1} \vee \ldots \vee v_n^{i_n}) \otimes (v_k \otimes 1_K),
$$

and:

$$
(\phi(V) \otimes \phi(V))\Delta([v_1|v_2^2, \ldots, v_n^{i_n}]) = (\phi(V) \otimes \phi(V))\left( \sum_{k=2}^{n} i_k [v_1|v_2^{i_2} \vee \ldots \vee v_k^{i_k-1}, \ldots, v_n^{i_n}] \otimes (v_k \otimes 1_K) \right) = \sum_{k=2}^{n} i_2! \ldots i_k(i_k - 1)! \ldots i_n! (v_1 \otimes v_2^{i_2} \vee \ldots \vee v_k^{i_k-1} \vee \ldots \vee v_n^{i_n}) \otimes (v_k \otimes 1_K).
$$

Hence, $\phi(V)$ is a coalgebra morphism and is bijective since $K$ is a characteristic zero field. Therefore, $\ker \Delta = (Moor(V))_1$ and the filtration being given by the $((Moor(V))_n)_{n>0}$, the free $Moor$-algebra over $V$ is a connected $Moor$-bialgebra. 

\hfill \Box

**Lemma 4.3** A connected $Moor$-bialgebra $\mathcal{H}$ is generated by its primitive elements. Moreover, $\ker \Delta_\mathcal{H} = \mathcal{H}_1$.

**Proof:** Let $x \in \mathcal{F}_r\mathcal{H}$ with $r$ minimal and belongs to $\mathcal{H}_p$, $p > 0$ which is not primitive. Write $\Delta_\mathcal{H}(x) = x_{(1)} \otimes x_{(2)}$ as a sum of independents vectors. We get $0 = (id \otimes \Delta_\mathcal{H})(\Delta_\mathcal{H}(x)) = x_{(1)} \otimes \Delta_\mathcal{H}(x_{(2)})$. Hence $\Delta_\mathcal{H}(x_{(2)}) = 0$ and the $x_{(2)}$ are primitive elements and belongs to $\mathcal{H}_1$. Moreover, $0 = \Delta^{(r)}_\mathcal{H}(x) = \Delta^{(r-1)}_\mathcal{H}(x_{(1)}) \otimes x_{(2)}$ which leads to $\Delta^{(r-1)}_\mathcal{H}(x_{(1)}) = 0$ and the $x_{(1)} \in \mathcal{F}_{r-1}\mathcal{H}$. Therefore,

$$
\Delta^{(r-1)}_\mathcal{H}(x) = x_{(1)} \otimes \ldots \otimes x_{(r)},
$$

where the $x_{(i)}$ for $1 \leq i \leq r$ are primitive. However,

$$
\Delta^{(r-1)}_\mathcal{H}(x - [x_{(1)}|x_{(2)}, \ldots, x_{(r)}]) = 0,
$$

hence $x - [x_{(1)}|x_{(2)}, \ldots, x_{(r)}]$ is a primitive element and belongs to $\mathcal{H}_1$. As $x \in \mathcal{H}_p$, $[x_{(1)}|x_{(2)}, \ldots, x_{(r)}] \in \mathcal{H}_r$, we get $p = r$ and $x = [x_{(1)}|x_{(2)}, \ldots, x_{(r)}]$. 

\hfill \Box
4.2 A rigidity theorem for connected Moor-bialgebras

**Theorem 4.4** A connected Moor-bialgebra $H$ is free and cofree over its primitive part $\text{Prim } H$, that is the following is equivalent for any Moor-bialgebra $H$:

1. $H$ is connected;
2. $H$ is isomorphic to $\text{Moor}(\text{Prim } H)$ as a Moor-bialgebra;
3. $H$ is isomorphic to $\text{Moor}^c(\text{Prim } H)$ as a Moor-coalgebra.

**Proof:** Let $H$ be a connected Moor-bialgebra. Since, $\text{Moor}(\text{Prim } H)$ is free, we get:

$$
\begin{array}{ccc}
\text{Prim } H & \xrightarrow{i} & \text{Moor}(\text{Prim } H) \\
& \Downarrow{j} & \Downarrow{i} \\
& & H
\end{array}
$$

where $\tilde{i}$ is the unique Moor-morphism verifying $\tilde{i} \circ i = j$, where $i$ and $j$ are the canonical injections. Via Lemma 4.3, $\tilde{i}$ is surjective. Since $\text{Moor}^c(\text{Prim } H)$ is cofree, we get:

$$
\begin{array}{ccc}
H & \xrightarrow{\tilde{e}} & \text{Moor}^c(\text{Prim } H) \\
& \Downarrow{\pi} & \Downarrow{\pi} \\
& & \text{Prim } H
\end{array}
$$

with $\tilde{e}$ the unique morphism of coalgebra extending the canonical projection $e$. Still set by induction $\Delta^{(1)}_n = \Delta_n$ and $\Delta^{(n)}_n := (\Delta_n \otimes \text{id}_{(n-1)})\Delta^{(n-1)}_n$. Set $e^{\otimes 1} = e$ and $e^{\otimes n} = e^{\otimes (n-1)} \otimes e$. Recall the coalgebraic morphism $\tilde{e}$ is given as follows:

$$
\tilde{e}(x) = \sum_{n=1}^{\infty} j_n \circ e^{\otimes (n+1)} \circ \Delta^{(n)}_n (x).
$$

As a connected Moor-bialgebra is generated by its primitive elements, we focus on elements $x := [x_1|x_2, \ldots, x_n]$, with the $x_i$ primitive. As expected,

$$
\tilde{e}([x_1|x_2^{i_2}, \ldots, x_n^{i_n}]) = i_2! \ldots i_n! \ x_1 \otimes x_2^{i_2} \lor \ldots \lor x_n^{i_n}.
$$

Hence, we get on the whole $H$, $\phi(\text{Prim } H) = \tilde{e} \circ \tilde{i}$ where $\phi(\text{Prim } H)$ is defined in the proof of Theorem 4.2. Since $\tilde{i}$ is surjective (Lemma 4.3) and $\phi(\text{Prim } H)$ is bijective, $\tilde{i}$ is injective and is an isomorphism. Hence $\tilde{e}$ is also an isomorphism of Moor-coalgebras since $\tilde{e} = \phi(\text{Prim } H) \circ \tilde{i}^{-1}$.

\[\square\]
5 A Moor-cooperation over free Perm-algebras

Permutative algebras have been introduced in \([1]\). In fact the following holds.

**Proposition 5.1** Let \(V\) be a \(K\)-vector space. Then, the \(K\)-vector space \(V \otimes S(V)\) equipped with the operation \(\boxdot\) defined by,
\[
v_1 \otimes v_2 \boxdot \ldots \boxdot v_n \boxdot w_1 \otimes w_2 \boxdot \ldots \boxdot w_m = v_1 \otimes v_2 \boxdot \ldots \boxdot v_n \boxdot w_1 \boxdot w_2 \boxdot \ldots \boxdot w_m,
\]
for all \(v_i, w_j \in V\), is the free Perm-algebra over \(V\).

A Perm-algebra \(P\) is said to be unital if it exists an element denoted by 1 such that \(x \boxdot 1 = x\), for all \(x \in P\), the symbols \(1 \boxdot x\), \(1 \boxdot 1\) being not defined. The augmented free Perm-algebra over a \(K\)-vector space \(V\), \(K \oplus Perm(V)\), is a unital Perm-algebra.

Let \(V\) be a \(K\)-vector space. On \(V \otimes S(V)\), one can define the left and right maps as follows:
\[
l(1_K) = 0, \quad l(v_1 \otimes v_2 \boxdot \ldots \boxdot v_n) = v_1 \otimes 1_K, \quad r(v_1 \otimes v_2 \boxdot \ldots \boxdot v_n) = \frac{1}{n-1} \sum_{i=2}^{n} v_i \otimes v_2 \boxdot \ldots \boxdot \hat{v}_i \boxdot \ldots \boxdot v_n,
\]
\[
r(v_1 \otimes 1_K) = 1_K, \quad r(1_K) = 0,
\]
for all \(v_i \in V\).

**Proposition 5.2** Let \(V\) be a \(K\)-vector space. Then, the augmented free Perm-algebra over a \(K\)-vector space \(V\), \(K \oplus Perm(V)\), can be equipped with a Moor-cooperation \(\Delta\) verifying the following compatibility relation:
\[
\Delta(1_K) = 0.
\]
\[
\Delta(x \boxdot y) = (x_1 \boxdot y) \otimes x(2) + (x \boxdot y_1) \otimes y(2) + (x \boxdot r(y)) \otimes l(y),
\]
for all \(x, y \in K \oplus Perm(V)\).

**Proof**: Recall in \(V \otimes S(V)\) the existence of the following Moor-cooperation \(\Delta\):
\[
\Delta(v_1 \otimes v_2 \boxdot \ldots \boxdot v_n) = \sum_{k=2}^{n} (v_1 \otimes v_2 \boxdot \ldots \boxdot \hat{v}_k \boxdot \ldots \boxdot v_n) \otimes (v_k \otimes 1_K),
\]
defined for all \(v_i \in V\). Add \(\Delta(1_K) := 0\). Hence, set \(x = v_1 \otimes v_2 \boxdot \ldots \boxdot v_n\) and \(y = w_1 \otimes w_2 \boxdot \ldots \boxdot w_m\) and observe that
\[
\Delta(x \boxdot y) = (x_1 \boxdot y) \otimes x(2) + (x \boxdot y_1) \otimes y(2) + (x \boxdot r(y)) \otimes l(y),
\]
holds. \(\square\)

**Acknowledgments**: Many thanks to M. Livernet and J.-L. Loday for usefull discussions.
References

[1] F. Chapoton. Un endofoncteur de la théorie des opérades.

[2] V. Ginzburg and M. Kapranov. Koszul duality for operads. Duke Math. J. 76 (1994) 203–272.

[3] Ph. Leroux. An equivalence of categories motivated by weighted directed graphs. arXiv:0709.3453.

[4] Ph. Leroux. An algebraic framework of weighted directed graphs. Int. J. Math. Math. Sci., 58, 2003.

[5] M. Livernet. A rigidity theorem for Pre-Lie algebras. J.P.A.A., 207:1–18, 2006.

[6] J.-L. Loday. Generalized bialgebras and triples of operads. arXiv:math.QA/0611885.