Wave Turbulence of a Liquid Surface in an External Tangential Electric Field

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Abstract

A direct numerical simulation of the interaction of plane capillary waves on the surface of a liquid dielectric in an external tangential electric field taking into account viscous forces has been performed. It has been shown that the interaction of counter-propagating nonlinear waves can generate a direct energy cascade. In the quasistationary energy dissipation regime, probability density functions for angles of inclination of the boundary tend to a Gaussian distribution and the shape of the boundary becomes complex and chaotic. The spectrum of the surface perturbations in this regime is described by a power law $k^{-5/2}$. The energy spectrum has the form $k^{-3/2}$, which coincides with the Iroshnikov-Kraichnan energy spectrum and indicates that the observed wave turbulence of the liquid surface and the weak magnetohydrodynamic turbulence of interacting Alfvén waves have a related nature.

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It is known that the dynamics of nonlinear capillary waves can be fairly complex. Zakharov and Filonenko [1] showed that the weak turbulence of capillary waves can be developed at the boundary of a liquid. Within a weakly nonlinear model, under the assumption that the amplitudes of capillary waves are small, they demonstrated that the spectrum of surface perturbations $I_k$ in a small-scale region (the gravitational force is negligibly weak) has the power-law form

$$I_k = \langle |\eta_k|^2 \rangle \sim Ck^{-(11/4+d)},$$

where $\eta_k$ is the Fourier transform of the function $\eta$ determining the shape of the boundary of the liquid, $k$ is the length of the wave vector, $C$ is a constant, $d$ is the number of spatial variables of the function $\eta$, and the angle brackets mean angular averaging.

The first numerical experiments definitely confirming the existence of spectrum (1) were reported in [2], and this result was confirmed by physical experiments in [3,4]. Simulating the complete strongly nonlinear hydrodynamic equations, the authors [5] obtained the Zakharov-Filonenko turbulent spectrum. Waves with a large wavelength, whose evolution is determined
primarily by the gravitational force, can exhibit the Phillips \[6\] and Kolmogorov-Zakharov spectra \[7\]. Thus, the effect of capillary and gravitational forces on the development of turbulence of surface waves is well studied (see review \[8\]).

The situation changes significantly in the presence of external electromagnetic fields, which can strongly affect the evolution of the boundary \[9\]. The dynamics of coherent structures on the liquid surface in an electric field has been studied in detail (see, e.g., \[10\]–\[17\]). The chaotic dynamics of the surface of waves in the external field almost has not been studied (except for two experimental works \[18\],\[19\]). For this reason, the study of the wave turbulence of the liquid surface appearing in a high external electric field is very important. This work is devoted to such a study.

A numerical simulation of weakly nonlinear dynamics of plane-symmetric capillary waves on the surface of a dielectric liquid with a high dielectric constant in an external tangential electric field is presented in this work. In a high electric field, when the effect of capillary and gravitational forces is negligibly small, the exact analytical solutions of this problem were first obtained by Zubarev \[20\]. These solutions describe the dispersionless propagation of nonlinear surface waves with an arbitrary shape in the direction of the field or against it \[21\],\[22\]. The interaction of counterpropagating waves results in the appearance of singular points at the boundary, at which the field strength and the velocity of the liquid are discontinuous and the curvature of the surface increases infinitely \[23\]–\[25\]. The calculations performed in this work indicate that the previously observed singularities are smoothed when the surface tension and viscous forces are taken into account. Furthermore, a direct energy cascade and the chaotization of evolution of the system can be observed in the regime of propagation of electrocapillary waves. In the quasistationary energy dissipation regime, the spectrum of surface perturbations tends to a power law, \(k^{-5/2}\).

We consider a potential flow of an ideal incompressible dielectric liquid with infinite depth and a free surface in a uniform horizontal external electric field. Since the problem under consideration is anisotropic because of the existence of the separated direction of the electric field, we consider only plane symmetric waves propagating in the direction parallel to the external field. Let the field strength be directed along the \(x\) axis (correspondingly, the \(y\) axis of the Cartesian coordinate system is perpendicular to it) and have the magnitude \(E\). The shape of the boundary in an unperturbed state corresponds to \(y = 0\).
The dispersion relation for linear waves at the boundary of the liquid has the form [9]

$$\omega^2 = gk + \frac{\varepsilon_0 \varepsilon E^2}{\rho} k^2 + \frac{\sigma}{\rho} k^3,$$

where $\omega$ is the frequency; $g$ is the gravitational acceleration; $\varepsilon_0$ is the electric constant; $\varepsilon$ ($\varepsilon \gg 1$) and $\rho$ and $\sigma$ are the dielectric constant and density of the liquid, respectively; and $\sigma$ is the surface tension coefficient. For further analysis, it is convenient to introduce the dimensionless variables

$$y \rightarrow y/k_0, \quad x \rightarrow x/k_0, \quad t \rightarrow t \cdot t_0, \quad E \rightarrow \beta \cdot E_0,$$

where $k_0 = (g \rho / \sigma)^{1/2}$, $t_0 = (\sigma / g^3 \rho)^{1/4}$, and $E_0^2 = (\sigma g \rho)^{1/2} / \varepsilon_0 \varepsilon$ are the characteristic values of the wavenumber, time, and electric field strength, respectively, and $\beta = E / E_0$ is the dimensionless electric field strength. In particular, for water ($\varepsilon \approx 81$), $\lambda = 2\pi / k_0 \approx 1.7 \text{ cm}$, $t_0 \approx 0.01 \text{ s}$, and $E_0 \approx 1.9 \text{ kV/cm}$.

Below, we consider the region of electrocapillary waves $\beta^2 + k \gg 1/k$, i.e., wavelengths for which the effect of the gravitational force can be neglected. The dispersion relation (2) can be represented in the dimensionless form

$$\omega^2 = (\beta^2 + k) k^2.$$

The equations of motion of the boundary of the liquid up to the second order terms can be represented in the form [20]

$$\psi_t = \eta_{xx} + \frac{1}{2} \left[ \beta^2 (\hat{k} \eta^2 - \eta_x^2) + \hat{k} \psi^2 - \psi_x^2 \right] +$$

$$+ \beta^2 \left[ -\hat{k} \eta + \hat{k} (\eta \hat{k} \eta) + \partial_x (\eta \psi_x) \right] + \hat{D}_k \psi,$$

$$\eta_t = \hat{k} \psi - \hat{k} (\eta \hat{k} \psi) - \partial_x (\eta \psi_x) + \hat{D}_k \eta,$$

where $\psi$ is the function determining the potential of the velocity of the liquid at the boundary, $\hat{k}$ is the integral operator having the form: $\hat{k} f_k = |k| f_k$ in the Fourier representation, and the operator $\hat{D}_k$ describes viscosity and is defined in the $k$ space as

$$\hat{D}_k = -\gamma (|k| - |k_d|)^2, \quad |k| \geq |k_d|; \quad \hat{D}_k = 0, \quad |k| < |k_d|.$$

Here, $\gamma$ is a constant, and $k_d$ is the wavenumber determining the spatial scale at which the energy dissipation occurs. This definition of viscosity is standard at the simulation of the wave turbulence of the liquid surface in the absence of an electric field [20].
Equations (4) and (5) are Hamiltonian and can be derived as the variational derivatives

\[
\frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \quad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}.
\]

Here,

\[
H = H_0 + H_1 = \frac{1}{2} \int \left( \psi \dot{k} \psi + \beta^2 \eta \dot{k} \eta + \eta_x^2 \right) dx - \frac{1}{2} \int \eta \left( [\dot{k} \psi^2 - \psi_x^2] + \beta^2 [\dot{k} \eta^2 - \eta_x^2] \right) dx,
\]

is the Hamiltonian of the system specifying the total energy. Here, \( H_0 \) and \( H_1 \) are the linear and quadratic terms, respectively. For the weakly nonlinear model, the condition \( H_1/H_0 \ll 1 \) should be satisfied.

For high fields (\( \beta \gg 1 \)), i.e., in the absence of capillary and viscous forces, equations of motion of the boundary (4) and (5) allow a pair of exact solutions in the form

\[
\psi = \pm \beta \dot{H} \eta,
\]

where \( \dot{H} \) is the Hilbert transform operator: \( \dot{H} f_k = i \text{sign}(k) f_k \). These solutions are obtained by Zubarev [20]. As mentioned above, they describe the propagation of waves without distortions in the direction of the field or against it (depending on the sign in the above formula).

To minimize the effect of coherent structures (collapses or solitons), initial conditions for Eqs. (4) and (5) are taken in the form of two counter-propagating interacting wavepackets:

\[
\eta_1(x) = \sum_{i=1}^{3} a_i \cos(k_i x), \quad \eta_2(x) = \sum_{i=1}^{3} b_i \cos(p_i x), \quad (6)
\]

\[
\eta(x, 0) = \eta_1 + \eta_2, \quad \psi(x, 0) = \beta (\dot{H} \eta_1 - \dot{H} \eta_2).
\]

The numerical experiments were performed for three field strengths \( \beta_1^2 = 35, \beta_2^2 = 30, \beta_3^2 = 25 \). The spatial derivatives and integral operators were calculated using pseudo-spectral methods with the total number \( N \) of harmonics, and the time integration was performed by the fourth-order explicit RungeKutta method with the step \( dt \). To stabilize the numerical scheme, the amplitudes of higher harmonics with a wavenumber above \( k_s \) were equated to zero at each step in time; i.e., a low-frequency filter was used (similar to [2]).

A qualitatively similar behavior was observed in all three experiments. The calculations did not involve the mechanical pumping of the energy of the system. Hence, the average steepness of the boundary \( \mu = < |\eta_x| > \) was determined only by initial conditions (6). The calculation was performed with the parameters \( N = 2048, dt = 10^{-5}, \gamma = 10, k_d = 500, k_s = 624 \). All numerical experiments demonstrate the formation of an inertial interval in the wavenumber
range $30 < k < 150$. A decrease in the field strength is accompanied by an increase in the nonlinearity threshold necessary for the formation of a direct energy cascade. In particular, three experiments gave the average steepness of the boundary $\mu_1 \approx 0.116$, $\mu_2 \approx 0.121$, $\mu_3 \approx 0.127$, respectively.

We present in more detail the results of calculation for $\beta_1^2 = 35$ (at high field strengths, experiments demonstrated strong singularities, which can correspond to the formation of vertical liquid jets [27]). The amplitudes and wavenumbers in initial conditions (6) were chosen empirically to minimize the deviation of the model from weakly nonlinear. The calculations reported below were performed with the parameters

$$a_1 = 0.01125, a_2 = 0.0165, a_3 = 0.01,$$
$$b_1 = 0.0125, b_2 = 0.0145, b_3 = 0.01,$$
$$k_1 = 4, k_2 = 6, k_3 = 8, \quad p_1 = 5, p_2 = 7, p_3 = 9.$$

Figure 1a shows the time dependence of the relative change $\Delta H$ in the energy of the system. It is seen that, beginning with a certain time, energy dissipation occurs with almost a constant rate. This behavior correlates well with the time dependence of the space-period-averaged magnitude $< |K| >$ of the curvature of the boundary shown in Fig. 1b. The magnitude $< |K| >$ reaches a maximum at $t \approx 3.2 \cdot 10^3$ and, then, decreases almost monotonically. Figure 1c shows the time dependence of the ratio $H_1/H_0$, characterizing the nonlinearity level of the system. The plot does not demonstrate such a transition regime, and the ratio $H_1/H_0$ remains small during the entire integration time interval. The average nonlinearity level in the process of evolution of the system was small, about $1.5 \cdot 10^{-2}$.

Figure 2 shows the shape of the boundary of the liquid at the initial time and at $t = 3.16 \cdot 10^3$. It is seen that this boundary in the quasistationary energy dissipation regime has a complex irregular shape. In this case, the probability density functions for angles of inclination of the boundary become very close to a Gaussian distribution (see Fig. 3). This behavior indicates the absence of strong spacetime correlations and, consequently, the possible appearance of the Kolmogorov spectrum of wave turbulence.

Figure 4 shows the time-averaged spectrum of surface perturbations of the boundary of the liquid:

$$I_\beta = (T)^{-1} \int_{t_1}^{t_2} |\eta_k|^2 dt,$$
where $T = t_2 - t_1$ is the time averaging period (averaging was performed in a stationary regime).

As seen in Fig. 4, the resulting spectrum of surface perturbations is far from the classical Zakharov-Filonenko spectrum and can be approximated with a high accuracy by a power law $k^{-2.5}$. It is remarkable that the slope of the spectrum in the wave turbulence regime is the same.
for all performed calculations. Figure 5 shows the compensated spectra for three numerical experiments with different values. It is seen that the empirical dependence obtained describes the numerical experiments very well. The results of calculations can generally be interpreted as the detection of a new spectrum of wave turbulence of the liquid surface different from classical spectra for capillary and gravitational waves.

The reported calculations were performed for quite high external electric fields. A decrease in the field strength is accompanied by an increase in the nonlinearity level necessary for the
implementation of the direct cascade energy. The Zakharov-Filonenko spectrum of turbulence (1) was not obtained in the case of zero external electric field. This can be due to the absence of conditions for three-wave resonance of capillary waves in one-dimensional geometry. Indeed, it is easy to show that the conditions for three-wave resonance of capillary waves ($\omega^2 = k^3$),

\[ \omega = \omega_1 + \omega_2, \quad k = k_1 + k_2, \]

cannot be satisfied in one-dimensional geometry. In the presence of a high electric field, i.e., at $\beta \gg 1$, the dispersion relation (3) for low wavenumbers $k < \beta^2$ can be written in the form

\[ \omega^2 = \beta^2 k^2. \]

For this relation, conditions (7) are satisfied at any $k$ values. Thus, the probability of satisfaction of the conditions for three-wave resonance of electrocapillary waves (3) increases with the field strength.

It is noteworthy that wave turbulence studied in this work is very similar to magnetohydrodynamic (MHD) wave turbulence of interacting Alfvén waves. It is known that waves in an ideally conducting liquid can propagate without distortions along the direction of the external magnetic field. Interaction is possible only between counterpropagating waves, and this interaction is elastic [28]. Surface waves in the high electric field regime studied in this work have the same properties [23]. The classical result of the study of MHD wave turbulence is the Iroshnikov-Kraichnan spectrum [29,30]. We briefly illustrate the derivation of this spectrum.

Figure 5: Compensated spectra of surface perturbations for various electric field strengths:
(1) $k^{2.5}I_\beta$ ($\beta^2 = 35$), (2) $0.1k^{2.5}I_\beta$ ($\beta^2 = 30$), (3) $0.01k^{2.5}I_\beta$ ($\beta^2 = 25$), and (red dashed straight line) $k^{-2.5}$. 
and show that the resulting empirical dependence \( I_\beta \sim k^{-5/2} \) can be closely related to MHD wave turbulence.

In the presence of a high field, the time of interaction \( \tau \) between counter-propagating quasi-particles (harmonics) of the magnetic field (in our case, the electric field) with the wavelength \( \lambda \sim k^{-1} \) and velocity \( v_k \) can be estimated as

\[
\tau \sim \frac{1}{k V_A},
\]

where \( V_A = \sqrt{\varepsilon_0 \varepsilon E^2 / \rho} \) is an analog of the Alfvén velocity for our problem. Velocity perturbations are small, i.e., \( v_k / V_A \ll 1 \). The velocity \( v_k \) is related to the spectral density of the kinetic energy \( E_T(k) \) in the one-dimensional case as \( v_k \sim [E_T(k)]^{1/2} \). For the elastic interaction of waves, the spectral densities of the kinetic \( E_T \) and potential (electrostatic) \( E_U \) energies are proportional to each other \( E_T \sim E_U \). At an elementary interaction event, the wavepacket is distorted \( \Delta v_k \sim v_k^2 / V_A \) (this expression follows from the assumption of the dominant effect of quadratic terms). The number of “collisions” \( N_{int} \) necessary for a significant distortion of the wavepacket is estimated as \[31\]

\[
N_{int} = \frac{V_A}{\Delta v_k} \sim \left( \frac{V_A}{v_k} \right)^2 \gg 1.
\]

Then, the energy redistribution time can be introduced as \( \tau_E \sim N_{int} / k V_A \). The energy dissipation rate \( s \) should be independent of the wavenumber (condition of the local energy transfer)

\[
v_k^2 / \tau_E = s.
\]

Substituting the expression for \( \tau_E \) into this equality, one can obtain the spectral distributions for the velocity \( v_k \) and, therefore, for the kinetic \( E_T \) and potential \( E_U \) energies in the turbulent regime,

\[
v_k^2 \sim (s V_A)^{1/2} k^{-1/2}, \quad E_U \sim E_T \sim (s V_A)^{1/2} k^{-3/2}.
\] \( (8) \)

Expressions \( (8) \) are classical Iroshnikov-Kraichnan spectra. In the case of a high field, the velocity fluctuations \( v_k \) are related to surface perturbations \( \eta_k \) as \( v_k \sim V_A k \eta_k \). Consequently, the desired spectrum of surface perturbations is obtained from Eq. \( (8) \) in the form

\[
< |\eta_k|^2 > \sim I_\beta \sim s^{1/2} V_A^{-3/2} k^{-5/2}.
\] \( (9) \)

Thus, the spectrum \( I_\beta \) obtained in numerical experiments is in qualitative agreement with Eq. \( (9) \), which directly follows from the Iroshnikov-Kraichnan energy spectrum \( (8) \) for MHD...
wave turbulence. To conclude, the results obtained in this work are applicable for a ferromagnetic liquid with a high dielectric constant in an external magnetic field. These results, together with the data obtained in [18-19], indicate the existence of a new spectrum of the wave turbulence of the liquid surface in an external magnetic (electric) field.

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