Entanglement thermodynamics

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Abstract: We study entanglement entropy for an excited state by making use of the proposed holographic description of the entanglement entropy. For a sufficiently small entangling region and with reasonable identifications we find an equation between entanglement entropy and energy which is reminiscent of the first law of thermodynamics. We then suggest four statements which might be thought of as four laws of entanglement thermodynamics.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence

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1 Introduction

Thermodynamics provides a useful tool to study a system when it is in thermal equilibrium. In this limit the physics may be described in terms of few macroscopic quantities such as energy, temperature, pressure, entropy and certain chemical potential if the system is charged. There are also laws of thermodynamics which describe how these quantities behave under various conditions. In particular the first law of thermodynamics which is a version of the law of the conservation of energy, tells us how the entropy changes as one changes the energy of the system.

We note, however, that there are several interesting phenomena which occur when the system is far from thermal equilibrium. In fact a rapid change in a system, such as quantum quenches, may bring the system out of equilibrium and indeed it is interesting to study the thermalization process of this quantum system.

Although when the system is far from thermal equilibrium the thermodynamical quantities may not be well defined, it is still possible to compute the entanglement entropy. Therefore entanglement entropy may provide a useful quantity to study excited quantum systems which are far from thermal equilibrium. Of course for a generic quantum system it is difficult to compute the entanglement entropy. Nevertheless, at least, for those quantum systems which have holographic descriptions, one may use the holographic entanglement entropy [1] to explore the behavior of the system.

Another quantity which can be always defined is the energy (or energy density) of the system. It is then natural to pose the question whether there is a relation between the entanglement entropy of an excited state and its energy. Such a question has recently been addressed in a certain situation using holographic entanglement entropy in [2] where it was shown that for a sufficiently small subsystem, the change of the entanglement entropy is proportional to the change of the energy of the subsystem. The proportionality constant is indeed given by the size of the entangling region. To make contact with the first law of thermodynamics the entanglement temperature has been identified with the inverse of the size of the entangling region [2].
The aim of the present article is to further explore a possible generalization of the laws of thermodynamics for quantum entanglement (see also [3]). More precisely using the holographic description of the entanglement entropy at a certain limit for a specific model we suggest several statements which are reminiscent of laws of thermodynamics. This may be thought of as entanglement thermodynamics. We must admit that our results are based on an explicit example and therefore one should be cautious to consider them as a general framework.

The paper is organized as follows. In the next section using the holographic description of the entanglement entropy we will derive an equation which might be considered as the first law of entanglement thermodynamics which make a connection between entanglement entropy, energy and entanglement pressure (to be defined later). In section three we investigate the universal feature of the first law. In section four we suggest several statements which could be considered as other laws of the entanglement thermodynamics. The last section is devoted to discussions.

**Note:** while we were preparing to submit our paper we were aware of the paper [4] where the entanglement pressure has also been studied. Moreover after submitting of our paper to arXiv another paper [5] appeared where the relative entropy has been studied. In this context the contribution of the pressure to the change of the entanglement entropy for an excited state has also been discussed.

## 2 First law of the entanglement thermodynamics

According to the AdS/CFT correspondence [6] gravity on an asymptotically locally AdS provides a holographic description for a strongly coupled quantum field with a UV fixed point. In this context the information of quantum state in the dual field theory is encoded in the bulk geometry. In particular the AdS geometry is dual to the ground state of the dual conformal field theory.

Exciting the dual conformal field theory from the ground state to an excited state holographically corresponds to modifying the bulk geometry from an AdS solution to a general asymptotically local AdS solution. For example if one excites the ground state by heating up the system, the bulk gravity would promote to an AdS black hole.

The aim of this section is to compute the entanglement entropy of an excited state\(^1\) for the case where the entangling region is sufficiently small (below we make it precise what we mean by sufficiency small). Since the entanglement entropy for a small subsystem would probe the UV region of the theory, from an holographic point of view one only needs to know the asymptotic behavior of the bulk geometry.

On the other hand it is known that the most general form of the asymptotically locally AdS may be written in terms of the Fefferman-Graham coordinates as follows

\[
ds^2_{d+1} = \frac{R^2}{r^2} \left( dr^2 + g_{\mu \nu} dx^\mu dx^\nu \right),
\]

\(^1\)Entanglement entropy for excited states in two dimensions has also been studied in [7, 8].
where \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x, r) \) with
\[
h_{\mu\nu}(x, r) = h_{\mu\nu}^{(0)}(x) + h_{\mu\nu}^{(2)}(x) r^2 + \cdots + r^d \left( h_{\mu\nu}^{(d)}(x) + \hat{h}_{\mu\nu}^{(d)}(x) \log r \right) + \cdots \quad (2.2)
\]

The log term is present for even \( d \). The information about the excited state (or the bulk geometry) is encoded in the function \( h_{\mu\nu}(x, r) \). This deformation of pure AdS geometry could be caused by heating up the background or by the back reaction of other fields in the model.\(^2\) Of course in what follows we do not need the explicit form of this function.

To proceed let us fix our notation by reviewing the computations of the holographic entanglement entropy for a strip in an AdS geometry. A \( d+1 \) dimensional AdS solution in the Poincaré coordinates may be written as follows
\[
ds^2 = \frac{R^2}{r^2} (dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu), \quad \mu, \nu = 0, 1, \cdots, d-1. \quad (2.3)
\]

Let us consider an entangling region in the shape of a strip with the width of \( \ell \) given by
\[
-\frac{\ell}{2} \leq x_1 \leq \frac{\ell}{2}, \quad 0 \leq x_i \leq L, \quad i = 2, \cdots, d-1. \quad (2.4)
\]

Following [1] the holographic entanglement entropy may be computed by minimizing a codimension two hypersurface in the bulk geometry which ends on the boundary of the above strip. Then the entanglement entropy is the minimal surface divided by \( 4G_N \) where \( G_N \) is the Newton’s constant of the bulk gravity.

In the present case where the background is an AdS\(_{d+1} \) geometry, assuming the bulk extension of the surface to be parameterized by \( x_1 = x(r) \), the corresponding area is given by
\[
A_0 = R^{d-1} L^{d-2} \int d\ell \sqrt{1 + x^2}. \quad (2.5)
\]

By making use of the standard procedure one may minimize the area to get [1]
\[
\ell = 2 \int_0^{\tilde{r}_t} dr \frac{(r/\tilde{r}_t)^{d-1}}{\sqrt{1 - (r/\tilde{r}_t)^2(d-1)}} \quad S^{(0)}_{E}(\tilde{r}_t) = 2 \frac{R^{d-1} L^{d-2}}{4G_N} \int_0^{\tilde{r}_t} dr \frac{r^{d-1} \sqrt{1 - (r/\tilde{r}_t)^2(d-1)}}{\tilde{r}_t \sqrt{1 - (r/\tilde{r}_t)^2(d-1)}}, \quad (2.6)
\]

where \( \tilde{r}_t \) is turning point and \( \epsilon \) is a UV cut off. Thus one gets
\[
S^{(0)}_{E} = \frac{L^{d-2} \Gamma(d/2)}{2(d-2)\Gamma(d-2)\Gamma(d/2)} \left[ \frac{1}{\epsilon^{d-2}} - \frac{1}{\ell^{d-2}} \right], \quad (2.7)
\]

Now let us consider a deformation of the AdS geometry which in turn corresponds to dealing with an excited state in the dual field theory. The aim is to compute the entanglement entropy of the strip (2.4) for an excited state when the width of strip is sufficiently small so that only the UV regime of the system will be probed. Holographically this means

\[^2\text{Here we will only consider Einstein gravity. For other gravitational models such as those with higher derivative terms we have other powers in the asymptotic behavior of the metric.}\]
that one needs to compute a codimension two hypersurface on the asymptotically locally AdS geometry which is given by the equation (2.1) in the Fefferman-Graham coordinates.

It is important to note that the deviation from AdS geometry in the bulk does not need to be small. Indeed in what follows, using the notation of the Fefferman-Graham coordinates, we assume that \( h^{(n)}_{\mu} \ell^n \ll 1 \). In fact this is what we mean by “sufficiently small”. Note that in this limit, practically one needs to compute the minimal surface up to order of \( O(h) \).

For the above strip the induced metric in the Fefferman-Graham coordinates is

\[
ds^2 = \frac{R^2}{r^2} \left( 1 + g_{11} x''^2 \right) dr^2 + 2 g_{1i} x' dx^i + g_{ij} dx^i dx^j.
\]

Therefore to find the holographic entanglement entropy one needs to minimize the following area

\[
A = R^{d-1} \int d^{d-2} x dr \frac{\sqrt{g(r)} (1 + G(r) x'^2)}{r^{d-1}}
\]

where \( g(r) = \text{det}(g_{ij}) \) and \( G(r) = g_{11} - g_{1i} g_{ij} g_{j1} \).

To proceed we consider the case where the solution is static. Moreover to find analytic expressions for our results we will assume that the components of the asymptotic metric are independent of \( x_1 \), the direction the width of strip is extended.\(^4\) With these assumptions the equation of motion of \( x \) leads to a constant of motion

\[
\left( \frac{R}{r} \right)^{d-1} \frac{\sqrt{g(r)} G(r) x'}{\sqrt{1 + G(r) x'^2}} = \text{const} = c,
\]

so that

\[
x' = \frac{c}{\sqrt{G(r) \left[ g(r) G(r) \left( \frac{R}{r} \right)^{2(d-1)} - c^2 \right]}}.
\]

The constant \( c \) may be found in terms of the value of the left hand side of the equation (2.10) evaluated at a specific point. Usually the specific point is chosen to be the turning point where \( x' \) diverges. Denoting by \( r_t \) the turning point, one finds

\[
c^2 = g(r_t) G(r_t) \left( \frac{R}{r_t} \right)^{2(d-1)}
\]

It is then straightforward to find the entanglement entropy and the width of the strip as follows

\[
S_E = \frac{1}{2 G_N} \int_0^{r_t} d^{d-2} x dr \left( \frac{R}{r} \right)^{2(d-1)} \frac{\sqrt{g(r)^2 G(r)}}{\sqrt{g(r) G(r) \left( \frac{R}{r} \right)^{2(d-1)} - c^2}}
\]

\[
\ell = 2 \int_0^{r_t} dr \frac{c}{\sqrt{G(r) \left[ g(r) G(r) \left( \frac{R}{r} \right)^{2(d-1)} - c^2 \right]}}
\]

\(^3\)For a thermal geometry it corresponds to the condition of \( T \ell \ll 1 \) where \( T \) is the temperature. See [9] for similar computations.

\(^4\)Actually as far as the leading order behavior is concerned, which is indeed the case for sufficiently small entangling region, both assumptions may be dropped.
To evaluate the above expressions we note that at leading order one has
\[ g(r) = 1 + \text{Tr}(h_{ab}) - h_{11} + \mathcal{O}(h^2), \quad G(r) = 1 + h_{11} + \mathcal{O}(h^2), \quad (2.14) \]
where \(a, b = 1, 2, \cdots, d - 1\). So that \(g(r)G(r) = 1 + \text{Tr}(h_{ab}) + \mathcal{O}(h^2)\).

In what follows in order to simplify the expressions, it is found useful to define the following parameters
\[ \gamma(r) = \text{Tr}(h_{ab}), \quad \beta(r) = h_{11}, \quad f(r, r_t) = \sqrt{1 - \left(\frac{r}{r_t}\right)^{2(d-1)}}. \quad (2.15) \]

In this notation at the first order in \(h\) one arrives at
\[ \ell = \int_0^{r_t} \frac{(r/r_t)^{d-1}}{f(r, r_t)} \left[ 2 + \frac{\gamma(r_t) - \gamma(r)}{f^2(r, r_t)} - \beta(r) \right] dr \quad (2.16) \]
It is worth noting that the aim was to compute the entanglement entropy for an excited and compare it with the ground state which is represented by an AdS solution. Actually we are interested in the change of the entanglement entropy caused by the change of the state. Therefore we keep the entangling surface fixed. Since \(\ell\) is kept fixed while the geometry is deformed the turning point should also be changed. Indeed assuming \(r_t = \tilde{r}_t + \delta r_t\) with \(\tilde{r}_t\) being the turning point for the pure AdS case, one finds
\[ \delta r_t = -\frac{1}{2a_d} \int_0^{\tilde{r}_t} \frac{f(r, \tilde{r}_t)}{(r/\tilde{r}_t)^{d-1}} \left[ \Gamma(0) + \Gamma(2) r^2 + \cdots + \Gamma(d) r^d + \hat{\Gamma}(d) r^d \ln r \right] dr \quad (2.17) \]
where
\[ a_d = \int_0^1 \frac{\xi^{d-1}}{\sqrt{1 - \xi^{2(d-1)}}} d\xi, \quad (2.18) \]
Moreover the width of the strip \(\ell\) is the same as that in pure AdS geometry that is given by the equation (2.6) which is \(\ell = 2\tilde{r} a_d\). \(\tilde{r}_t\) is the turning point for pure AdS geometry.

It is now straightforward to compute the entanglement entropy up to order of \(\mathcal{O}(h)\). In fact expanding the expression of the entanglement entropy one finds\(^5\)
\[ S_E = S_E^{(0)}(\tilde{r}_t) + \frac{R^{d-1}}{4G_N} \int_0^{\tilde{r}_t} dr \left( \Gamma^{(0)} + \Gamma^{(2)} r^2 + \cdots + \Gamma^{(d)} r^d + \hat{\Gamma}^{(d)} r^d \ln r \right), \quad (2.19) \]
where \(S_E^{(0)}(\tilde{r}_t)\) is the holographic entanglement entropy for the strip in a pure AdS\(_{d+1}\) geometry given in the equation (2.6).

By making use of the Fefferman-Graham expansion for the asymptotic form of the metric one arrives at
\[ \Delta S_E = \frac{R^{d-1}}{4G_N} \int_0^{\tilde{r}_t} dr \left( \Gamma^{(0)} + \Gamma^{(2)} r^2 + \cdots + \Gamma^{(d)} r^d + \hat{\Gamma}^{(d)} r^d \ln r \right), \quad (2.20) \]

\(^5\)Note that we could have found this result by replacing \(r_t\) with \(\tilde{r}_t\) in the first equation of (2.13), then expanding at the first order in \(h\) and requiring \(\delta r_t = 0\). We would like to thank the referee for his/her comment on this point.
where the change of the entanglement entropy is defined by $\Delta S_E = S_E - S_E^{(0)}(\tilde{r}_t)$, and also

$$\Gamma^{(n)} = \frac{1}{r^{d-1}} \int d^{d-2}x \frac{\text{Tr}(h_{ab}^{(n)})}{f(r, \tilde{r}_t)} - \frac{f(r, \tilde{r}_t)}{r^{d-1}} \int d^{d-2}x h_{11}^{(n)},$$

$$\tilde{\Gamma}^{(d)} = \frac{1}{r^{d-1}} \int d^{d-2}x \frac{\text{Tr}(h_{ab}^{(d)})}{f(r, \tilde{r}_t)} - \frac{f(r, \tilde{r}_t)}{r^{d-1}} \int d^{d-2}x h_{11}^{(d)}. \tag{2.21}$$

Using this expansion it is straightforward to perform the integration over $r$. Indeed for $d > 2$ one finds

$$\int_{\epsilon}^{\tilde{r}_t} dr \; \Gamma^{(n)} r^n = \frac{1}{(d-2-n)e^{d-2-n}} \int d^{d-2}x \left( \text{Tr}(h_{ab}^{(n)}) - h_{11}^{(n)} \right)$$

$$- \frac{F(d-1, d-1-n)}{r_t^{d-2-n}} \int d^{d-2}x \left( \text{Tr}(h_{ab}^{(n)}) - \frac{d-1}{n+1} h_{11}^{(n)} \right)$$

$$= \frac{1}{(d-2-n)e^{d-2-n}} \frac{1}{r_t^{d-2-n}} M^{(n)}, \tag{2.22}$$

where $\epsilon$ is a UV cut off, and

$$F(m, n) = \frac{2F_1 \left( \frac{1}{2}, \frac{1-n}{2m}, \frac{2n+1-n}{2m}, 1 \right)}{n-1}, \tag{2.23}$$

with $2F_1$ being the hypergeometric function. Note that for even $d$ and $n = d-2$ one finds just a logarithmic divergence as $N^{(d-2)} \ln \frac{\epsilon}{r_t}$ while for odd $d$ and $n = d-1$ the result is finite and is given by $M^{(d-1)} \tilde{r}_t$. On the other hand for arbitrary $d$ for $n = d$ it leads to a finite term given by $\tilde{r}_t^2 M^{(d)}$. More precisely, using the fact that in general at leading order $\text{Tr}(h_{\mu\nu}^{(d)}) = \mathcal{A}$ with $\mathcal{A}$ being the trace anomaly one finds

$$\int_{0}^{\tilde{r}_t} dr \; \Gamma^{(d)} r^d = -F(d-1, -1) \tilde{r}_t^2 \int d^{d-2}x \left( h_{tt}^{(d)} + \mathcal{A} - \frac{d-1}{d+1} h_{11}^{(d)} \right). \tag{2.24}$$

Note that for odd $d$ the anomaly term is zero. One should add that when $d$ is an even number we have another term coming from $\Gamma^{(d)}$ which can similarly be calculated leading to an $\ln \tilde{r}_t$ contribution to the entanglement entropy.

Having found these expressions and taking into account that $\ell = 2\tilde{r}_t a_d$, one can find the variation of the entanglement entropy, $\Delta S_E$, as a function of $\ell$. More precisely for odd $d$ one has

$$\Delta S_E = \frac{R^{d-1}}{4G_N} \sum_{n<d-2} \left( \frac{1}{(d-2-n)e^{d-2-n}} N^{(n)} + \frac{(2a_d)(d-2-n)}{\ell^{d-2-n}} M^{(n)} \right) + \frac{R^{d-1}M^{(d-1)}}{8G_N a_d} \ell$$

$$- \frac{R^{d-1}F(d-1, -1)}{16a_d^2 G_N} \ell^2 \int d^{d-2}x \left( h_{tt}^{(d)} - \frac{d-1}{d+1} h_{11}^{(d)} \right) + \cdots, \tag{2.25}$$
while for even $d$ one gets

$$
\Delta S_E = \frac{R^{d-1}}{4G_N} \sum_{n<d-2} \left( \frac{1}{(d-2-n)^{d-2-n}} N^{(n)} + \frac{(2a_d)^{d-2-n}}{\ell^{d-2-n}} M^{(n)} \right)
+ \frac{R^{d-1}N^{(d-2)}}{4G_N} \ln \frac{2\epsilon a_d}{\ell} 
- \frac{R^{d-1}F(d-1,-1)}{16a_d^2 G_N} \ell^2 \int d^{d-2}x \left( h^{(d)}_{tt} + A - \frac{d-1}{d+1} \hat{h}^{(d)}_{11} \right)
- \frac{R^{d-1}F(d-1,-1)}{16a_d^2 G_N} \ell^2 \ln \frac{\ell}{2a_d} \int d^{d-2}x \left( \hat{h}^{(d)}_{tt} - d \frac{d-1}{d+1} \hat{h}^{(d)}_{11} \right) + \cdots .
$$

(2.26)

Here we have used the fact that Tr$(\hat{h}^{(d)}_{\mu\nu}) = 0$.

Entanglement entropy is divergent due to short range interactions near the boundary of the entangling surface and thus a UV cut off is needed. The coefficient of the most divergent term is proportional to the area of the entangling surface. Although in the expressions of the change of the entanglement entropy, $\Delta S_E$, the divergent term coming from the AdS geometry has already been subtracted, it still has divergent terms whose coefficients are given by $N^{(n)}$ for $n \leq d-2$. It is worth to note that all of these terms are given in terms of $h^{(0)}_{\mu\nu}$ and its derivatives whose precise form may be found from the holographic renormalization procedure [10]. Therefore as soon as the boundary becomes curved we have extra divergent terms due to the extrinsic curvature of the boundary. Note that for the flat boundary all $N^{(n)}$’s vanish.

On the other hand the most relevant terms of the finite parts of $\Delta S_E$ are given in terms of $M^{(n)}$ which is again given in terms of $h^{(0)}_{\mu\nu}$ and its derivatives which vanish for the flat boundary. There is also one extra important term which comes from the $d$ th order of the Fefferman-Graham expansion. Note that this term does not depend on $h^{(0)}_{\mu\nu}$ and therefore its contribution remains non-zero even for the flat boundary. Indeed it has a very interesting feature as we explore below.

When one excites the ground state to an excited state, the energy of the system is increased and generally one gets non-zero expectation value for the energy momentum tensor. By making use of the holographic renormalization one can compute this expectation value. Indeed one has [10]

$$
\langle T_{\mu\nu} \rangle = \frac{dR^{d-1}}{16\pi G_N} h^{(d)}_{\mu\nu}.
$$

(2.27)

In other words from the dual gravity point of view the expectation value of the energy momentum tensor is given by $h^{(d)}_{\mu\nu}$, which is exactly the extra contribution to the holographic entanglement entropy as we just mentioned. Therefore the extra non-trivial contribution to the entanglement entropy is coming from expectation value of the energy-momentum tensor which does depend on the excited state we are considering. More precisely one finds

$$
\Delta S_E^{\text{finite}} = \sum_n (\cdots) \frac{M^{(n)}}{\ell^{d-2-n}} - \frac{\pi F(d-1,-1)\ell}{a_d^2} \times
\times \left( \Delta E - \frac{d-1}{d+1} \int \Delta P_{d-1} dV_{d-1} + \frac{dR^{d-1}}{16\pi G_N} \int \mathcal{A} dV_{d-1} \right) + \cdots ,
$$

(2.28)
where \((\cdots)\) stands for a numerical factor and \(dV_{d-1} = \ell d^{d-2}x\). Moreover the energy and entanglement pressure are defined by

\[
\Delta E = \int dV_{d-1}\langle T_{tt}\rangle, \quad \Delta P_x = \langle T_{11}\rangle.
\]

(2.29)

It is worth mentioning that since in general the system is not in the thermal equilibrium, the pressure \(P_x\) should not be identified with that in the thermodynamics and indeed it was the reason we called it entanglement pressure. Note also that only entanglement pressure normal to the entangling surface appears in the finite part of the change of the entanglement entropy.

For the case of \(h_{\mu\nu}^{(0)} = 0\) where the geometry is asymptotically AdS solution one has

\[
h_{\mu\nu}(x,r) = h_{\mu\nu}^{(d)}(x) r^d.
\]

(2.30)

Note that in this case since the boundary is flat the anomaly term is zero and therefore the equation (2.28) reads

\[
\Delta S_E = \frac{\pi \ell}{2d C_0^2} \left( \Delta E - \frac{d-1}{d+1} \int dV_{d-1} \Delta P_x \right),
\]

(2.31)

where

\[
C_0 = \sqrt{\pi} \frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)}, \quad C_1 = \sqrt{\pi} \frac{\Gamma\left(\frac{d}{d-1}\right)}{\Gamma\left(\frac{d+1}{2(d-1)}\right)}.
\]

(2.32)

Following [2] one may define entanglement temperature in terms of the width of the strip. More generally the entanglement temperature is proportional to the inverse of typical size of the entangling region and the proportionality constant depends on the shape of the entangling region. In particular in the present case the corresponding temperature may be given by \(T_E = \frac{2dC_0^2}{\pi \ell^2} \frac{1}{d}\). Assuming \(h_{\mu\nu}^{(d)}\) to be constant the above equation may be recast to the following form\(^6\)

\[
\Delta E = T_E \Delta S_E + \frac{d-1}{d+1} V_{d-1} \Delta P_x
\]

(2.33)

where \(V_{d-1}\) is the volume of the entangling region. Due to its similarity with the first law of thermodynamics we would like to consider this expression as the first law of entanglement thermodynamics.

The way the energy and the entanglement pressure were defined suggests that the holographic equation of state should be given by \(\text{Tr}(h_{\mu\nu}^{(d)}) = A\). In particular for the flat boundary where \(A = 0\) and when the solution is isotropic the equation of state becomes \(h_{tt}^{(d)} = (d-1)h_{11}^{(d)}\). Using the holographic renormalization and the definition of the entanglement pressure the equation of state in the dual field theory is \(E = (d-1)P_x\), where \(E\) is energy density. In this case the first law of entanglement thermodynamics reads\(^7\)

\[
T_E \Delta S_E = \frac{d}{d+1} \Delta E.
\]

(2.34)

\(^6\)It is important to note that although for \(d = 2\) we find logarithmic divergences, the final result is the same.

\(^7\)In comparison with the result of [2] one has an extra \(\frac{d}{d-1}\) which is due to our definition of entanglement temperature.
An explicit example of such a situation is the AdS Schwarzschild background whose metric is given by

\[ ds^2 = \frac{R^2}{\rho^2} \left( -f(\rho)dt^2 + \frac{d\rho^2}{f(\rho)} + \sum_{i=1}^{d-1} dx_i^2 \right), \quad f(\rho) = 1 - \left( \frac{\rho}{\rho_H} \right)^d. \]  

(2.35)

where \( \rho_H \) is the radius of horizon. By making use of the coordinate transformation

\[ \frac{dr}{r} = \frac{d\rho}{\rho f^{1/2}}, \]  

(2.36)

one may recast the metric to the Fefferman-Graham coordinates as follows

\[ ds^2 = \frac{R^2}{r^2} (dr^2 + g_{\mu\nu} dx^\mu dx^\nu), \]  

(2.37)

whose asymptotic behavior of the metric components are

\[ g_{tt} = -1 + h_{tt}^{(d)} r^d = -1 + \frac{4(d-1)}{d} \rho_H r^d, \quad g_{aa} = 1 + h_{aa}^{(d)} r^d = 1 + \frac{4}{d} \rho_H r^d. \]  

(2.38)

Note that in this case one observes \( h_{tt}^{(d)} = (d-1) h_{aa}^{(d)} = \frac{4(d-1)}{d} \rho_H^d \). For an entangling region given by the strip (2.4) the energy and the entanglement pressure are given by \( \Delta E = \frac{4(d-1)}{d} \rho_H^d V_{d-1} \) and \( \Delta P_x = \frac{4}{d} \rho_H^d \), respectively. Plugging these expressions in the first law of the entanglement thermodynamics one can easily find the change of the entanglement entropy for the AdS Schwarzschild black hole as follows

\[ T_E \Delta S_E = \frac{4(d-1)}{d+1} \rho_H^d V_{d-1}. \]  

(2.39)

3 Universal features of the first law

In the previous section in order to introduce the first law of the entanglement thermodynamics we have considered the entanglement entropy for a strip. It is then natural to see to what extent the resultant first law is universal. In this section we will consider the holographic entanglement entropy for a system in the form of a sphere to address this question.

To proceed let us first write down the boundary metric at a fixed time in the spherical coordinates

\[ ds^2 = \frac{R^2}{r^2} (dr^2 + g_{ij} dx^i dx^j) = \frac{R^2}{r^2} (dr^2 + g_{\rho\rho} d\rho^2 + 2g_{\rho\alpha} d\rho d\theta^\alpha + \rho^2 g_{\alpha\beta} d\theta^\alpha d\theta^\beta), \]  

(3.1)

where

\[ g_{\rho\rho} = \Omega^i g_{ij} \Omega^j, \quad g_{\rho\alpha} = \Omega^i g_{ij} \frac{\partial \Omega^j}{\partial \theta^\alpha}, \quad g_{\alpha\beta} = \frac{\partial \Omega^i}{\partial \theta^\alpha} g_{ij} \frac{\partial \Omega^j}{\partial \theta^\beta}. \]  

(3.2)

Here \( \Omega^i \)'s are the angular elements with the condition \( \sum_i \Omega^i \Omega^i = 1 \).
Now the aim is to study the entanglement entropy for a sphere with a radius \( \ell \) in the boundary. Setting \( \rho = \rho(r) \) the induced metric on the codimension two hypersurface in the bulk is given by

\[
ds^2 = \frac{R^2}{r^2} \left[ (1 + g_{\rho\rho} \rho^2) \, dr^2 + 2 \rho \rho' g_{\rho\alpha} \, dr \, d\theta^\alpha + \rho^2 g_{\alpha\beta} \, d\theta^\alpha d\theta^\beta \right]
\]

(3.3)

Therefore to compute the holographic entanglement entropy one needs to minimize the following area

\[
A = R^{d-1} \int dr d\Omega_{d-2} \rho^{d-2} \frac{\sqrt{g(1 + G \rho^2)} \rho'}{r^{d-1}}
\]

(3.4)

where \( g = \det(g_{\alpha\beta}) \) and \( G = g_{\rho\rho} - g_{\rho\alpha} g_{\rho\beta}^{-1} g_{\beta\rho} \).

Since in the present case the above expression treated as a one dimensional action does not have a constant of motion in order to find \( \rho \) one needs to solve its equation of motion

\[
\left[ \frac{1}{r^{d-1}} \frac{g G \rho^2 \rho'}{\sqrt{g(1 + G \rho^2)}} \right]' = (d - 2) \rho^{d-3} \frac{1}{r^{d-1}} \sqrt{g(1 + G \rho^2)}
\]

(3.5)

It is easy to check that for the ground state where the dual gravity is given by an AdS_{d+1} geometry a solution of the above equation with the boundary condition \( \rho_0(r = 0) = \ell \) is \( \tilde{\rho}_0 = \sqrt{\ell^2 - r^2} \). It is then evident that the turning point is also given by \( \tilde{r}_t = \ell \). Note also that in this case \( G = 1 \) and

\[
g^{(0)}_{\alpha\beta} = \frac{\partial \Omega^i}{\partial \theta^\alpha} \delta_{ij} \frac{\partial \Omega^j}{\partial \theta^\beta}.
\]

(3.6)

Following our previous study the aim is to find the entanglement entropy for an excited state for a sufficiently small entangling region. To do so, one needs to expand the expression for the area which at leading order it yields

\[
A(\rho, \ell) = A(\rho_0, \ell) + \delta_g A(\rho_0, \ell).
\]

(3.7)

Here

\[
A(\rho_0, \ell) = R^{d-1} \int_0^\ell dr d\Omega_{d-2} \rho_0^{d-2} \frac{g^{(0)}(1 + \rho_0^2)}{r^{d-1}},
\]

(3.8)

is the minimal area for the case where the dual theory is an AdS_{d+1} geometry and

\[
\delta_g A(\rho_0, \ell) = \frac{R^{d-1}}{2} \int_0^\ell dr d\Omega_{d-2} \rho_0^{d-2} \frac{g^{(0)}(1 + \rho_0^2)}{r^{d-1}} \left[ g_0^{\alpha\beta} h_{\alpha\beta} + \frac{\rho_0^2}{1 + \rho_0^2} h_{\rho\rho} \right],
\]

(3.9)

is the the first order correction to the minimal area due to the deviation from the AdS geometry. With these expressions it is easy to find the change of the entanglement entropy as follows

\[
\Delta S_E = \frac{\delta_g A(\rho_0, \ell)}{4G_N} = \frac{R^{d-1}}{8G_N} \int_0^\ell \frac{(\ell^2 - r^2)^{\frac{d-3}{2}}}{r^{d-1}} \ell \left[ \text{Tr}(h_{ab}) - \frac{\ell^2 - r^2}{\ell^2} h_{\rho\rho} \right] dr d\Omega_{d-2}.
\]

(3.10)
Now we need to use the Fefferman-Graham expansion for the metric to find an expansion for the change of the entanglement entropy. Actually the result has the same structure as that in the strip case. Namely there are divergent terms which must be regulated by introducing a UV cut off and they all vanish when the boundary is flat.

Let us consider the case where $h_{\mu\nu} = h_{(d)}^{(d)} r^d$ then one arrives at

$$T_E \Delta S_E = \Delta E - \frac{d-1}{d+1} \int \Delta P_\rho \, dV_{d-1} \quad (3.11)$$

where $T_E = \frac{d}{2\pi}$ and $dV_{d-1} = \rho^{d-2} d\rho \, d\Omega_{d-2}$. As we have already mentioned in the case of the strip, it is important to note that when the system is isotropic then there is a relation between pressure and energy. In the present case where the entangling region is a sphere this condition is automatically satisfied leading to the relation of $\Delta E = (d-1) \int \Delta P_\rho dV_{d-1}$.

Therefore in this case the above equation reads

$$\tilde{T}_E \Delta S_E = \Delta E, \quad (3.12)$$

in agreement with [5, 15].

To conclude we note that although the numerical factor in the definition of the entanglement temperature is different from that in the strip case, the final form of the first law is the same. In particular the numerical factor in front of the pressure term is universal and only the entanglement pressure normal to the entangling surface appears in the first law. Therefore the first law could be universal and the only shape dependent parameter is the numerical factor in the definition of entanglement temperature.

### 4 Laws of entanglement thermodynamics

In the previous sections based on the holographic description of the entanglement entropy and for explicit examples we have found a relation between entanglement entropy, energy and entanglement pressure which using the similarity with the thermodynamics could be thought of as the first law of entanglement thermodynamics. It is then natural to pose the question whether there are other laws similar to what we have in the thermodynamics. The aim of this section is to introduce other statements that could be identified as other laws of entanglement thermodynamics.

There is a natural statement for the second law of entanglement thermodynamics. It is indeed the strong subadditivity that must be satisfied by the entanglement entropy $[11, 12]$. According to the strong subadditivity for any given three subsystems $A$, $B$ and $C$ that do not intersect one has

$$S_{E(A \cup B)} + S_{E(C \cup B)} \geq S_{E(A \cup B \cup C)} + S_{E(B)}. \quad (4.1)$$

Note that by setting $B$ empty in the above expression, one arrives at

$$S_{E(A)} + S_{E(C)} \geq S_{E(A \cup C)}. \quad (4.2)$$

---

We would like to thank the referee for a comment on this point.
It is worth noting that although the entanglement entropy is divergent due to UV effects, the divergent parts of the entanglement entropy drop from both sides. In fact this inequality is also satisfied by the finite part of the entanglement entropy. It is known that holographic entanglement entropy defined as a minimal surface in the bulk does satisfy this inequality too [13]. Therefore one may suggest this relation as the second law of the entanglement thermodynamics.

So far our suggestions and statements about the laws of entanglement thermodynamics were based on rigorous computations. To proceed for other possible laws we note that although we will use an explicit example to explore them, we should admit that our suggestions are based on speculation.

The most important part of our study is the definition of the entanglement temperature. Although from dimensional analysis and also from our experiences in thermodynamics and hydrodynamics it is natural to consider the inverse of the typical size of the entangling region as the temperature, having a non-universal numerical factor in its definition makes us wonder to what extent it is a well defined quantity.

Of course as long as we are considering entangling regions with a fixed shape the numerical factor is universal [2]. On the other hand having different shapes may lead to a puzzle as to define the temperature. Apart from this ambiguity, in what follows for a fixed shape we suggest a statement which could be considered as the zeroth law of entanglement thermodynamics.

Consider two entangling regions given by two strips with widths \( \ell_1 \) and \( \ell_2 \), respectively. If we bring these two regions close together we get another strip whose width at most could be \( \ell_3 = \ell_1 + \ell_2 \). Therefore we have the following inequality for the entanglement temperatures before and after joining the systems.

\[
\frac{1}{T_{1E}} + \frac{1}{T_{2E}} \geq \frac{1}{T_{3E}}. \tag{4.3}
\]

It is easy to argue that such a relation could also be satisfied when the entangling regions are spheres.

Let us now proceed to introduce the third law of entanglement thermodynamics. To do so, consider the finite part of the entanglement entropy of a strip for an excited state. Using the definition of the entanglement temperature, up to order of \( \mathcal{O}(T_E^{-2}) \), one gets

\[
S_{E}^{\text{finite}} = \frac{R^{d-1}}{8G_N} \left[ (\tilde{B}_0 L^{d-2} + B_0 M^{(0)}) T_E^{d-2} + \sum_{n \geq 1} B_n M^{(n)} T_E^{d-2-n} \right] + \frac{1}{T_E} \left( \Delta E - \frac{d-1}{d+1} \Delta P \Delta V d^{-1} \right) \tag{4.4}
\]

where \( \tilde{B}_0, B_n \) are numerical factors. For sufficiently large entanglement temperature (small size) the finite part of the entanglement entropy diverges as\(^9\)

\[
S_{E}^{\text{finite}} \sim T_E^{d-2}. \tag{4.5}
\]

\(^9\)For \( d = 2 \) it diverges logarithmically.
Therefore in principle the finite part of entanglement entropy goes to infinity for sufficiently higher entanglement temperature. We note, however, that due to a natural UV cut off in the theory there is a natural cut off for temperature preventing to get infinite entanglement entropy.

Note that as we increase the temperature, the dominant divergent parts comes from the ground state which corresponds to the AdS geometry. It is then possible to argue that the above statement is also valid for other shapes of the entangling region. We would like to suggest the above statement as the third law of entanglement thermodynamics.

5 Discussions

In this paper based on the holographic description of the entanglement entropy and within an explicit example we have suggested four laws for the quantum entanglement entropy which are reminiscent of the laws of thermodynamics. The corresponding laws of entanglement thermodynamics may be summarized as follows.

- **Zeroth law**: the entanglement temperature is proportional to the inverse of the typical size of the entangling region and for two subsystems $A$ and $B$ one has
  \[
  \frac{1}{T_{(A)E}} + \frac{1}{T_{(B)E}} \geq \frac{1}{T_{(A\cup B)E}}. \tag{5.1}
  \]

- **First law**: there is a relation between the energy of the system and the entanglement entropy as follows
  \[
  \Delta E = T_E \Delta S_E + \frac{d-1}{d+1} V_{d-1} \Delta P_\perp,
  \tag{5.2}
  \]
  where $\Delta P_\perp$ is the entanglement pressure normal to the entangling surface.

- **Second law**: entanglement entropy enjoys strong subadditivity
  \[
  S_{E(A\cup B)} + S_{E(C\cup B)} \geq S_{E(A\cup B\cup C)} + S_{E(B)}. \tag{5.3}
  \]

- **Third law**: there is an upper bound on the entanglement temperature preventing to have an infinite entanglement entropy.

In this paper we have considered entanglement entropy for a static case where the corresponding background geometry was time independent. It is, however, possible to show that the final results also hold for time dependent cases. Of course when we are dealing with a time dependent geometry, in general, one should use the covariant holographic entanglement entropy in which the entanglement entropy is given by a codimension two hypersurface which is extremal [14].

We note, however, that as long as we are interested in a sufficiently small subsystem we could still use the static solution leading to the same result for the first law. The reason is as follows. Consider a time dependent excited state above a vacuum solution. From the bulk point of view it corresponds to a time dependent deviation from an AdS solution.
There are several sources which contribute to the change of the holographic entanglement entropy. The change may be caused by the change of the turning point, the change of the solution and the change of the metric. The interesting point is that at leading order which is what we are interested in, the change of entanglement entropy is completely given by the change of metric (see the equation (3.8)). In other words one has \[ \Delta S_E = \frac{1}{4G_N} \int d^{d-1}x \sqrt{\det(g^{(0)}_{\text{in}})(g^{(0)}_{\text{in}})_{ab}^{-1}g^{(1)}_{\text{in} ab}}, \] (5.4)

where \(g^{(0)}_{\text{in}}\) and \(g^{(1)}_{\text{in}}\) are the induced metrics on the codimension two hypersurface in the bulk for the cases of AdS geometry and the perturbation above it, respectively. It is, now, clear that from the AdS case which is static one can read the first law. Indeed the result is the same as that we considered in the previous section. Therefore the first law we have introduced in this paper may also be applied for the time dependent case (see for example [16]). Of course to have the second law one needs to further assume the null energy condition for the excited state [17].

Recently Lewkowycz and Maldacena [18] have introduced a generalized gravitational entropy for classical Euclidean gravity solutions. More precisely consider metrics that end on a boundary which has a direction with the topology of a circle. Note that the solution is not necessarily symmetric under the U(1) rotation of the circle. Moreover the boundary need not to be a true asymptotic boundary of the metric and indeed it is just a place where the boundary conditions are imposed. One may associate an entropy to this solution. If the circle never shrinks in the interior of the bulk geometry the corresponding entropy is zero. If it does, the entropy is given by the area of a codimension two hypersurface in the bulk of the solution which, for the Einstein gravity, satisfies the minimal area condition. In fact at this hypersurface the circle shrinks to zero size.

Using the results of the present paper and the fact that when solutions are symmetric under the U(1) rotation the above construction reduces to the Gibbons-Hawking computation of the black hole entropy, one may wonder that there could be a generalized laws of thermodynamics for the generalized gravitational entropy. It would be interesting to explore this possibility.

It is worth noting that besides the holographic entanglement entropy there are other interesting quantities which have been studied in the literature. These quantities include the geometric entropy [19, 20] and its generalization when one has fractionalized charges [21]. It would also be interesting to see if the first law can also be obtained for these quantities.

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References

[1] S. Ryu and T. Takayanagi, Holographic derivation of entanglement entropy from AdS/CFT, Phys. Rev. Lett. 96 (2006) 181602 [hep-th/0603001] [insPIRE].
[2] J. Bhattacharya, M. Nozaki, T. Takayanagi and T. Ugajin, Thermodynamical property of entanglement entropy for excited states, Phys. Rev. Lett. 110 (2013) 091602 [arXiv:1212.1164] [inSPIRE].

[3] D.V. Fursaev, 'Thermodynamics' of minimal surfaces and entropic origin of gravity, Phys. Rev. D 82 (2010) 064013 [Erratum ibid. D 86 (2012) 049903] [arXiv:1006.2623] [inSPIRE].

[4] W.-Z. Guo, S. He and J. Tao, Note on entanglement temperature for low thermal excited states in higher derivative gravity, JHEP 08 (2013) 050 [arXiv:1305.3182] [inSPIRE].

[5] D.D. Blanco, H. Casini, L.-Y. Hung and R.C. Myers, Relative entropy and holography, JHEP 08 (2013) 060 [arXiv:1305.2682] [inSPIRE].

[6] J.M. Maldacena, The large-N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113] [hep-th/9711200] [inSPIRE].

[7] A.E. Mosaffa, Symmetric orbifolds and entanglement entropy for primary excitations in two dimensional CFT, arXiv:1208.3204 [inSPIRE].

[8] A.F. Astaneh and A.E. Mosaffa, Holographic entanglement entropy for excited states in two dimensional CFT, JHEP 03 (2013) 135 [arXiv:1301.1495] [inSPIRE].

[9] W. Fischler and S. Kundu, Strongly coupled gauge theories: high and low temperature behavior of non-local observables, JHEP 05 (2013) 098 [arXiv:1212.2643] [inSPIRE].

[10] S. de Haro, S.N. Solodukhin and K. Skenderis, Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence, Commun. Math. Phys. 217 (2001) 595 [hep-th/0002230] [inSPIRE].

[11] E.H. Lieb and M.B. Ruskai, A fundamental property of quantum-mechanical entropy, Phys. Rev. Lett. 30 (1973) 434 [inSPIRE].

[12] E. Lieb and M. Ruskai, Proof of the strong subadditivity of quantum-mechanical entropy, J. Math. Phys. 14 (1973) 1938 [inSPIRE].

[13] M. Headrick and T. Takayanagi, A holographic proof of the strong subadditivity of entanglement entropy, Phys. Rev. D 76 (2007) 106013 [arXiv:0704.3719] [inSPIRE].

[14] V.E. Hubeny, M. Rangamani and T. Takayanagi, A covariant holographic entanglement entropy proposal, JHEP 07 (2007) 062 [arXiv:0705.0016] [inSPIRE].

[15] M. Nozaki, T. Numasawa, A. Prudenziati and T. Takayanagi, Dynamics of entanglement entropy from Einstein equation, Phys. Rev. D 88 (2013) 026012 [arXiv:1304.7100] [inSPIRE].

[16] M. Nozaki, T. Numasawa and T. Takayanagi, Holographic local quenches and entanglement density, JHEP 05 (2013) 080 [arXiv:1302.5703] [inSPIRE].

[17] R. Callan, J.-Y. He and M. Headrick, Strong subadditivity and the covariant holographic entanglement entropy formula, JHEP 06 (2012) 081 [arXiv:1204.2309] [inSPIRE].

[18] A. Lewkowycz and J. Maldacena, Generalized gravitational entropy, arXiv:1304.4926 [inSPIRE].

[19] M. Fujita, T. Nishioka and T. Takayanagi, Geometric entropy and hagedorn/deconfinement transition, JHEP 09 (2008) 016 [arXiv:0805.3118] [inSPIRE].
[20] I. Bah, L.A. Pando Zayas and C.A. Terrero-Escalante, Holographic geometric entropy at finite temperature from black holes in global Anti de Sitter spaces, *Int. J. Mod. Phys. A* 27 (2012) 1250048 [arXiv:0809.2912] [rsSPIRE].

[21] D. Allahbakhshi and M. Alishahiha, Probing fractionalized charges, arXiv:1301.4815 [rsSPIRE].