Dynamics of the spinning particle in one dimension

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Abstract

The most general $N = 1$ Lagrangian for the spinning particle with local supersymmetry is found and the constraints of the system are analysed. The Dirac quantisation of the model is also investigated.

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1 Introduction

The classical and quantum properties of a particle propagating in a space have found many applications from general relativity to quantum mechanics and quantum field theory. In the past the model that was mostly investigated was a relativistic or non-relativistic particle propagating in a Riemannian manifold with a metric $g$. These results were extended to $N = 1$ supersymmetric particle models in [1]. Later the action of supersymmetric particles with extended supersymmetry was given in [2].

In [3] it was found that rigid supersymmetry in one dimension allows for the construction of more general models than those considered in the past. In particular, the manifold that the particle propagates in can have torsion which is not a closed 3-form. Such models have been found to describe the effective theory of multi-black hole systems [4]. A scalar potential and a coupling to a magnetic field in the action of [3] was added in [5].

In particular it was found in [3] that the $N = 1$ particle with rigid supersymmetry is described by two multiplets $q^\mu$ and $\chi^\alpha$. The superfield $q$ has components $q^\mu$ which are the positions of the particle and a worldline fermion $\lambda^\mu$. The components of $\chi^\alpha$ are a worldline fermion $\chi^\alpha$ and an auxiliary field $y^\alpha$.

In this paper we give the action of an $N = 1$ supersymmetric relativistic particle propagating in a manifold and with couplings which include a metric $g$, torsion $c$, an electromagnetic gauge potential $A$ and a scalar potential $V$. For this we introduce an einbein $e$ and a worldline gravitino $\psi$. Then we give the transformation law of these fields under local supersymmetry and show that our action is invariant up to surface terms. The construction of such an $N = 1$ locally supersymmetric action has been considered before [1]. However in our formalism the geometric interpretation of the various couplings is manifest in the manner of the rigid models in [3]. In addition we consider more general couplings.

Furthermore we use Dirac’s method to analyse the constraints of the system and compute the Poisson bracket algebra of constraints, and then discuss the Dirac quantisation of the model. As expected the constraint associated to supersymmetry is a Dirac operator but now it involves the torsion and it is twisted with respect to a gauge field. The square of the supersymmetry constraint gives the Hamiltonian constraint.

The paper is organised as follows. In section 2, the sigma models with rigid supersymmetry are reviewed. In section 3, the $N = 1$ action and local supersymmetries are given. The constraints of the system are analysed in section 4 and in section 5 its Dirac quantisation is investigated.
The $N = 1$ supersymmetric action

The spinning particle model we use is a one-dimensional sigma model with a worldline superspace $\Xi$ and target manifold $\mathcal{M}$. $\Xi$ is parameterised by one real coordinate $t$, representing time, and one real anticommuting coordinate $\theta$.

The real bosonic superfield $q^\mu(t, \theta)$ describes the position of the spinning particle and its classical spin degrees of freedom. It is a map from the base space into the target manifold in which the particle lives,

$$q : \Xi \rightarrow \mathcal{M} \quad (2.1)$$

There is also a fermionic superfield $\chi^\alpha(t, \theta)$ which can be thought of as a section of the bundle $q^*\epsilon$ over $\Xi$ where $\epsilon$ is a vector bundle over $\mathcal{M}$ [3]. It represents the Yang-Mills sector of the theory and is necessary for the inclusion of a potential term, and generalises the axial spinning coordinate $\psi_5$ in [1].

The most general $N = 1$ spinning particle action with rigid supersymmetry is [4].

$$S = -\int dt d\theta \left[ \frac{1}{2} g_{\mu\nu} Dq^\mu \partial_t q^\nu - \frac{1}{2} h_{\alpha\beta} \chi^\alpha \nabla \chi^\beta + \frac{1}{3!} c_{\mu\nu\rho} Dq^\mu Dq^\nu Dq^\rho ight. \\
+ \frac{1}{2} m_{\mu\alpha\beta} Dq^\mu \chi^\alpha \chi^\beta \\
+ \frac{1}{2} n_{\mu\alpha\beta} Dq^\mu Dq^\nu \chi^\alpha + \frac{1}{3!} l_{\alpha\beta\gamma} \chi^\alpha \chi^\beta \chi^\gamma \\
\left. - f_{\mu\alpha} \partial_t q^\mu \chi^\alpha + A_\mu Dq^\mu + ms_\alpha \chi^\alpha \right] \quad (2.2)$$

The superderivative $D$ is defined by $D = \partial_\theta - \theta \partial_t$, and $\nabla$ is its covariant version with respect to the connection on the fibre bundle,

$$\nabla \chi^\alpha = D\chi^\alpha + Dq^\mu B_{\mu}^{\alpha\beta} \chi^\beta \quad (2.3)$$

where $B_{\mu}^{\alpha\beta}$ is a connection of the bundle $\epsilon$ with fibre metric $h_{\alpha\beta}$. We may assume that the fibre metric $h_{\alpha\beta}$ is compatible with the connection, $\nabla_\mu h_{\alpha\beta} = 0$.

The action includes the torsion term $c_{\mu\nu\rho}$ and an electromagnetic potential $A_\mu$. The potential of the theory is described in terms of $s_\alpha$ [4], $V(q) = ms_\alpha s^\alpha/2$. Also present are the Yukawa couplings $m_{\mu\alpha\beta}, n_{\mu\alpha\beta}, l_{\alpha\beta\gamma}$ and $f_{\mu\alpha}$.

The action is invariant under worldline translations generated by $H = \partial_t$,

$$\delta^H_t q^\mu = \epsilon \dot{q}^\mu \quad \delta^H_t \chi^\mu = \epsilon \dot{\chi}^\mu \quad (2.4)$$
and supersymmetry transformations generated by \( Q = \partial_\theta + \theta \partial_t \),

\[
\delta_\zeta q^\mu = \zeta Q q^\mu \quad \delta_\zeta \chi^\alpha = \zeta Q \chi^\alpha \tag{2.5}
\]

where \( \zeta \) is anticommuting. The algebra of these two transformations is

\[
[\delta_\zeta, \delta_\eta] q^\mu = 2 \zeta \eta \dot{q}^\mu = 2 \delta_\zeta^{(H)} q^\mu \tag{2.6}
\]

which realises the commutator \( \{Q, Q\} = 2H \) of the \( N = 1 \) supersymmetry algebra.

We expand the action in components defined by

\[
q^\mu = q^\mu| \quad \chi^\alpha = \chi^\alpha| \tag{2.7}
\]

\[
\lambda^\mu = Dq^\mu| \quad y^\alpha = \nabla \chi^\alpha| \tag{2.8}
\]

where the line means evaluation at \( \theta = 0 \). We are following the standard convention of using the same letter for a superfield and its lowest component. The field \( y^\alpha \) is auxiliary and we will eliminate it later. Expanding the action (2.2) into component form, we find that

\[
S = \int dt \left[ \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu + \frac{1}{2} g_{\mu\nu} \lambda^\mu \nabla_t^{(+)\lambda^\nu} - \frac{1}{2} h_{\alpha\beta} y^\alpha y^\beta - \frac{1}{2} h_{\alpha\beta} \chi^\alpha \nabla_t \chi^\beta + \frac{1}{4} G_{\mu\nu\alpha\beta} \lambda^\mu \lambda^\nu \lambda^\alpha \lambda^\beta - \frac{1}{3!} \nabla_{[\mu} e_{\nu\rho\sigma]} \lambda^\mu \lambda^\nu \lambda^\rho \lambda^\sigma + \frac{1}{2} m_{\mu\alpha|\beta} \dot{q}^\mu \chi^\alpha + m_{\mu\alpha\beta} \lambda^\mu y^\alpha \chi^\beta - \frac{1}{2} \nabla_{[\mu} m_{\nu\alpha\beta]} \lambda^\mu \lambda^\nu \chi^\alpha \chi^\beta + \frac{1}{2} n_{\mu\nu\alpha} \dot{q}^\mu \chi^\nu \chi^\alpha - \frac{1}{2} n_{\mu\nu\alpha|\beta} \lambda^\mu \lambda^\nu \chi^\alpha \chi^\beta + \frac{1}{2} l_{\alpha\beta\gamma} \lambda^\alpha \chi^\beta \chi^\gamma - \frac{1}{3!} \nabla_{[\mu} l_{\nu\alpha\beta\gamma]} \lambda^\mu \lambda^\nu \chi^\alpha \chi^\beta \chi^\gamma + f_{\mu\alpha} \dot{q}^\mu y^\alpha + f_{\mu\alpha} \dot{\lambda}^\mu \chi^\alpha + \nabla_{[\mu} f_{\nu\alpha\beta]} \lambda^\mu \dot{q}^\nu \chi^\alpha + A_{\mu\lambda^\mu} - \nabla_{[\mu} A_{\nu]} \lambda^\mu \lambda^\nu - m s_{\alpha} y^\alpha - m \nabla_{[\mu} s_{\alpha} \lambda^\mu \chi^\alpha \right] \tag{2.9}
\]

where \( G_{\mu\nu\alpha\beta} \) is the curvature of the vector bundle connection \( B^\alpha_{\mu \beta} \),

\[
G_{\mu\nu}^\alpha = \partial_\mu B^\alpha_{\nu \beta} - \partial_\nu B^\alpha_{\mu \beta} + B^\alpha_{\mu \gamma} B^\gamma_{\nu \beta} - B^\alpha_{\nu \gamma} B^\gamma_{\mu \beta} \tag{2.10}
\]

and \( \nabla^{(+)\mu} \) is the covariant derivative including the torsion,

\[
\nabla_t^{(+)\mu} \lambda^\mu = \nabla_t \lambda^\mu - \frac{1}{2} \tau^\mu_{\nu\rho} \dot{q}^\nu \lambda^\rho \tag{2.11}
\]
The $N = 1$ supersymmetry transformations of the component fields are

$$
\begin{align*}
\delta \zeta q^\mu &= \zeta \lambda^\mu \\
\delta \zeta \lambda^\mu &= -\zeta \dot{q}^\mu \\
\delta \zeta \chi^\alpha &= \zeta (\dot{y}^\alpha - \lambda^\mu B^\alpha_{\mu \beta} \chi^\beta) \\
\delta \zeta y^\alpha &= -\zeta (\nabla_t \chi^\alpha + \lambda^\mu B^\alpha_{\mu \beta} y^\beta) + \frac{1}{2} \zeta \lambda^\mu \lambda^\nu \chi^\beta G^{\alpha \beta}_{\mu \nu}
\end{align*}
$$

(2.12)  (2.13)  (2.14)  (2.15)

### 3 Supergravity in one dimension

To construct the $N = 1$ supergravity action, we gauge the rigid supersymmetry by promoting the supersymmetry parameter $\zeta$ to a local parameter, $\zeta = \zeta(t)$. In general this will destroy invariance of (2.9) under supersymmetry, because we will get terms proportional to $\dot{\zeta}$. Therefore it is necessary to introduce gauge fields whose transformations will cancel with the $\dot{\zeta}$ terms arising from varying (2.9).

The method to find how these fields appear in the action, and their transformations, is the Noether technique. We illustrate this technique on a simple Lagrangian in flat space,

$$
\mathcal{L}_0 = \frac{1}{2} \eta_{\mu \nu} \dot{q}^\mu \dot{q}^\nu + \frac{1}{2} \eta_{\mu \nu} \lambda^\mu \dot{\lambda}^\nu
$$

(3.1)

Taking the supersymmetry transformation, with $\zeta = \zeta(t)$,

$$
\delta \zeta \mathcal{L}_0 = \dot{\zeta} \eta_{\mu \nu} \lambda^\mu \dot{q}^\nu
$$

(3.2)

up to surface terms, which vanish in the action. To cancel (3.2), consider the Lagrangian to first order in a parameter $g$,

$$
\mathcal{L}_1 = \mathcal{L}_0 + g \psi \eta_{\mu \nu} \lambda^\mu \dot{q}^\nu
$$

(3.3)

where $\psi$ is a gauge field with $\delta \zeta \psi = -g^{-1} \dot{\zeta}$. Then the variation of the new Lagrangian vanishes to zeroth order in $g$.

Continuing this process for all orders of $g$, we find

$$
\mathcal{L} = \frac{1}{2} e^{-1} \eta_{\mu \nu} \dot{q}^\mu \dot{q}^\nu + \frac{1}{2} \eta_{\mu \nu} \lambda^\mu \dot{\lambda}^\nu + g e^{-1} \psi \eta_{\mu \nu} \dot{q}^\mu \lambda^\nu
$$

(3.4)

It proves necessary to introduce a second gauge field $e$ which transforms under local supersymmetry as $\delta \zeta e = 2 \zeta \psi$. It is an einbein which is the gauge field associated with the diffeomorphisms of the worldline.
It is also necessary to modify the transformation for $\lambda^\mu$ to

$$\delta_\zeta \lambda^\mu = -e^{-1}\zeta(\dot{q}^\mu + g\psi^\mu) \quad (3.5)$$

Applying the Noether method to (2.9), the $N = 1$ supergravity action is

$$S = \int dt \left[ \frac{1}{2} e^{-1} g_{\mu\nu}(\dot{q}^\mu + \psi^\mu)(\dot{q}^\nu + \psi^\nu) + \frac{1}{2} g_{\mu\nu} \lambda^\mu \nabla_t \lambda^\nu ight.$$

$$\left. - \frac{1}{2} e h_{\alpha\beta} y^\alpha y^\beta - \frac{1}{2} h_{\alpha\beta} \chi^\alpha \nabla_t \chi^\beta + \frac{1}{4} e G_{\mu\nu\alpha\beta} \lambda^\mu \chi^\alpha \chi^\beta \right.$$

$$\left. + \frac{1}{2} e G_{\mu\nu\alpha\beta}(\dot{q}^\mu + \frac{2}{3}\psi^\mu) \lambda^\nu \chi^\alpha - \frac{1}{3!} e \nabla_{[\mu} c_{\nu\rho\delta]} \lambda^\mu \lambda^\nu \lambda^\rho \lambda^\sigma \right.$$

$$\left. + \frac{1}{2} m_{\mu\nu\alpha\beta} \dot{q}^\mu \chi^\nu \chi^\alpha + e m_{\mu\nu\alpha\beta} \lambda^\mu y^\alpha y^\beta - \frac{1}{2} e \nabla_{[\mu} m_{\nu]\alpha\beta} \lambda^\mu \lambda^\nu \lambda^\alpha \chi^\beta \right.$$

$$\left. + n_{\mu\nu\alpha}(\dot{q}^\mu + \frac{1}{2}\psi^\mu) \lambda^\nu \chi^\alpha - \frac{1}{2} e n_{\mu\nu\alpha} \lambda^\mu \lambda^\nu y^\alpha - \frac{1}{2} e \nabla_{[\mu} n_{\nu]\alpha\beta} \lambda^\mu \lambda^\nu \chi^\alpha \chi^\beta \right.$$

$$\left. - \frac{1}{2} e l_{\alpha\beta\gamma} \chi^\alpha \chi^\beta \chi^\gamma - \frac{1}{3!} e l_{\alpha\beta\gamma} \chi^\alpha \chi^\beta \chi^\gamma - \frac{1}{3!} e \nabla_{[\mu} l_{\nu]\alpha\beta\gamma} \lambda^\mu \lambda^\nu \lambda^\alpha \chi^\beta \chi^\gamma \right.$$

$$\left. + f_{\mu\alpha}(\dot{q}^\mu + \psi^\mu) y^\alpha + f_{\mu\alpha} \dot{\lambda}^\mu \chi^\alpha + \nabla_\mu f_{\nu\alpha} \lambda^\mu (\dot{q}^\nu + \psi^\nu) \chi^\alpha \right.$$

$$\left. + A_\mu \dot{q}^\mu + e \nabla_{[\mu} A_{\nu]} \lambda^\mu \lambda^\nu - m_{\alpha}(e y^\alpha + \psi^\alpha) - e m \nabla_{[\mu} s_{\nu]\alpha} \lambda^\mu \chi^\alpha \right]$$

$$\quad (3.6)$$

We observe that the Lagrangian is not obtained by the simple minimal coupling rule $\dot{q}^\mu \rightarrow \dot{q}^\mu + \psi^\mu$.

The geometric interpretation of the couplings with the above action is manifest. The above action is similar to that of models with rigid supersymmetry in [3]. After various redefinitions of the couplings and the fields, one can recover the action constructed in [3] in the special case where $f_{\mu\alpha} = 0$. However, if $f_{\mu\alpha} \neq 0$ then (3.4) is more general.

The $N = 1$ supersymmetry transformations for the components fields become

$$\delta_\zeta q^\mu = \zeta \lambda^\mu \quad (3.7)$$

$$\delta_\zeta \lambda^\mu = -\zeta e^{-1}(\dot{q}^\mu + \psi^\mu) \quad (3.8)$$

$$\delta_\zeta \chi^\alpha = \zeta (y^\alpha - \lambda^\alpha B_{\mu\beta}\chi^\beta) \quad (3.9)$$

$$\delta_\zeta y^\alpha = -e^{-1}\zeta(\nabla_t \chi^\alpha + \psi y^\alpha) - \zeta \lambda^\mu B_{\mu\beta} \chi^\beta + \frac{1}{2} \zeta \lambda^\mu \lambda^\nu \chi^\beta G_{\mu\nu\beta}^\alpha \quad (3.10)$$
The einbein $e$ and gravitino $\psi$ transform as

$$\delta_\zeta e = 2\zeta \psi, \quad \delta_\zeta \psi = -\dot{\zeta}$$  \hspace{1cm} (3.11)$$

Checking the algebra of the new transformations (3.7),

$$\left[\delta_\zeta, \delta_\eta\right] q^\mu = 2e^{-1} \zeta \eta (\dot{q}^\mu + \psi \lambda^\mu) \quad \left[\delta_\zeta, \delta_\eta\right] \lambda^\mu = 2e^{-1} \zeta \eta (\dot{\lambda}^\mu - e^{-1} \psi \dot{q})$$  \hspace{1cm} (3.12)$$

$$= 2e^{-1} (\delta_{\zeta \eta}^{(H)} + \delta_{\zeta \eta} \psi) q^\mu = 2e^{-1} (\delta_{\zeta \eta}^{(H)} + \delta_{\zeta \eta} \psi) \lambda^\mu$$  \hspace{1cm} (3.13)$$

from which we obtain

$$\left[\delta_\zeta, \delta_\eta\right] = 2e^{-1} (\delta_{\zeta \eta}^{(H)} + \delta_{\zeta \eta} \psi)$$  \hspace{1cm} (3.14)$$

and the same is true on the components $\chi^\alpha$, $y^\alpha$.

For invariance of (3.10) under worldline diffeomorphisms, we need to specify the action of $\delta_\epsilon^{(H)}$ on the einbein and the gravitino,

$$\delta_\epsilon^{(H)} e = \partial_t (\epsilon e) \quad \delta_\epsilon^{(H)} \psi = \partial_t (\epsilon \psi)$$  \hspace{1cm} (3.15)$$

and those for $q^\mu$, $\lambda^\mu$, $\chi^\alpha$ and $y^\alpha$ are unchanged.

4 Hamiltonian Analysis

To investigate the Hamiltonian dynamics of the system described by (4.2), we follow the Dirac-Bergman procedure [8] to analyse the constraints. This will be important when we come to quantise the system in the next section.

At this stage we introduce vielbeins $e_{\mu}^i$, $f_{\alpha}^a$ so that $g_{\mu \nu} = \eta_{ij} e_{\mu}^i e_{\nu}^j$ and $h_{\alpha \beta} = \eta_{ab} f_{\alpha}^a f_{\beta}^b$ where $\eta_{ij}$ and $\eta_{ab}$ are the flat metrics on the manifold and vector bundle respectively. We will use latin letters for vielbein indices and greek letters otherwise. We take

$$\lambda^i = e_{\mu}^i \lambda^\mu, \quad \chi^a = f_{\alpha}^a \chi^\alpha$$  \hspace{1cm} (4.1)$$

as our new fermion fields. This ensures that in the next section, the Dirac brackets, hence commutation relations, between $p$ and the fermions are zero. It is also necessary to set the Yukawa coupling $f_{\mu \alpha} = 0$.

Adopting this notation, and eliminating the auxiliary field $y^\alpha$ from (2.3)
using its equation of motion, gives the action

\[
S = \int dt \left[ \frac{1}{2} g_{\mu\nu} e^{-1}(\dot{q}^\mu + \psi e^{i\nu} \lambda^i)(\dot{q}^\nu + \psi e^{i\nu} \lambda^j) + \frac{1}{2} \eta_{ij} \dot{\lambda}^i \nabla_t \lambda^j - \frac{1}{2} \eta_{ab} \chi^a \nabla_t \chi^b 
\right.
\]

\[
+ \frac{1}{2} e h_{ab} Y^a Y^b + \frac{1}{4} e G_{ijab} \dot{\lambda}^i \dot{\lambda}^j \chi^a \chi^b 
\]

\[
+ \frac{1}{2} c_{ijk}(\dot{e}^i + \frac{2}{3} \psi e^i \lambda^k) - \frac{1}{3!} e e^i_{[i} \nabla_{[j} c_{jkl]} \lambda^i \dot{\lambda}^j \lambda^k \lambda^l 
\]

\[
+ \frac{1}{2} m_{\muab} \dot{\chi}^a \chi^b - \frac{1}{2} e e^a_{[i} \nabla_{[j} m_{ijab} \lambda^i \chi^a \lambda^b 
\]

\[
+ n_{ija} (\dot{\lambda}^a + \frac{1}{2} \psi \lambda^i \lambda^a) - \frac{1}{2} e e^a_{[i} \nabla_{[j} n_{ijab} \lambda^i \lambda^j \lambda^k \chi^a 
\]

\[
- \frac{1}{3!} e l_{abc} \chi^a \chi^b \chi^c - \frac{1}{3!} e e^a_{[i} \nabla_{[j} l_{abc} \chi^a \lambda^b \chi^c 
\]

\[
+ A_{\mu} \dot{\chi}^a - e e^a_{[i} \nabla_{[\mu} A_{i]} \lambda^j \lambda^j - \psi m s a \chi^a - e e^a_{[i} \nabla_{[\mu} s a \lambda^i \chi^a 
\]

\right]

(4.2)

where for convenience we define

\[
Y_a = m_{ijab} \lambda^i \lambda^b - \frac{1}{2} n_{ija} \lambda^i \lambda^j - \frac{1}{2} l_{abc} \chi^a \chi^b - m s a
\]

(4.3)

and the two covariant derivatives are with respect to the spin connections \(\omega_{\mu i}\) on the manifold and \(\Omega_{\mu a b}\) on the vector bundle,

\[
\nabla_t \lambda^i = \partial_t \lambda^i + \dot{\lambda}^i_{\mu} \omega_{\mu k} \lambda^k 
\]

(4.4)

\[
\nabla_t \chi^a = \partial_t \chi^a + \dot{\chi}^a_{\mu} \Omega_{\mu a b} \chi^b 
\]

(4.5)

The canonical momenta for \(\lambda^i\) and \(\dot{q}^i\) are

\[
\pi_i = -\frac{1}{2} \eta_{ij} \dot{\lambda}^j 
\]

(4.6)

\[
p_{\mu} = e^{-1} g_{\mu\nu} (\dot{q}^\nu + \psi e^{i\nu} \lambda^i) + \frac{1}{2} \omega_{\mu jk} \dot{\lambda}^j \lambda^k - \frac{1}{2} \Omega_{\mu ab} \chi^a \chi^b 
\]

\[
+ \frac{1}{2} e^{i}_{\mu} c_{ijk} \lambda^j \lambda^k + \frac{1}{2} m_{\muab} \chi^a \chi^b + e^{i}_{\mu} n_{ija} \lambda^i \lambda^a + A_{\mu} 
\]

(4.7)

respectively. Similarly the canonical momenta for \(\chi^a\), \(e\) and \(\psi\) are

\[
\pi_{\chi a} = \frac{1}{2} \eta_{ab} \chi^b, \quad \pi_e = 0, \quad \pi_{\psi} = 0
\]

(4.8)

Clearly the system is constrained, as would be expected. The explicit constraints are

\[
\phi_i = \pi_i + \frac{1}{2} \eta_{ij} \dot{\lambda}^j \approx 0, \quad \phi_{\chi a} = \pi_{\chi a} - \frac{1}{2} \eta_{ab} \chi^b \approx 0 
\]

(4.9)

\[
\phi_e = \pi_e \approx 0, \quad \phi_{\psi} = \pi_{\psi} \approx 0
\]

(4.10)
where the $\approx$ denotes weak equality, in other words equality up to linear combinations of the other constraints.

The constrained Hamiltonian can then be found to be

$$H_c = \frac{1}{2} e \eta_{ij} P^i P^j - \psi \lambda^i (P_i + \frac{1}{3} c_{ijk} \lambda^j \lambda^k + \frac{1}{2} n_{ija} \lambda^j \chi^a)$$

$$- \frac{1}{2} e \eta_{ab} Y^a Y^b - \frac{1}{4} e G_{ijab} \lambda^i \lambda^j \chi^a \chi^b$$

$$+ \frac{1}{3!} e e^{\mu}_{[i} \nabla_{\mu} c_{ijkl]} \lambda^i \lambda^j \lambda^k \lambda^l + \frac{1}{2} e e^{\mu}_{[i} \nabla_{\mu} m_{ijab} \lambda^i \lambda^j \lambda^a \chi^b$$

$$+ \frac{1}{3!} e e^{\mu}_{[i} \nabla_{\mu} l_{abc} \lambda^i \lambda^a \chi^b \chi^c$$

$$+ \frac{1}{3!} \psi l_{abc} \lambda^a \chi^b \chi^c + e e^{\mu}_{[i} \nabla_{\mu} A_{ij]} \lambda^i \lambda^j$$

$$+ m \psi s_{a} \lambda^a + e e^{\mu}_{[i} \nabla_{\mu} m s_{a} \lambda^i \chi^a$$

(4.11)

where $Y^a$ was defined in (4.3) and

$$P_i = e^{\mu}_{i} p_{\mu} - \frac{1}{2} \omega_{ijk} \lambda^j \lambda^k + \frac{1}{2} \Omega_{iab} \chi^a \chi^b$$

$$- \frac{1}{2} c_{ijk} \lambda^j \lambda^k - \frac{1}{2} m_{iab} \chi^a \chi^b - n_{ija} \lambda^j \chi^a - A_i$$

(4.12)

The primary Hamiltonian is defined to be

$$H_p = H_c + \phi_i u^i + \phi_{\chi a} u^a_{\chi} + \phi_{e} \psi + \phi_{\psi} u_{\psi}$$

(4.13)

where the $u$ are all Lagrange multipliers (and $u^i$, $u^a_{\chi}$, $u_{\psi}$ are anticommuting).

We require that the constraints (4.9) hold for all time,

$$\dot{\phi}_i = \{ \phi_i, H_p \} \approx 0 \quad \dot{\phi}_{\chi a} = \{ \phi_{\chi a}, H_p \} \approx 0$$

$$\dot{\phi}_e = \{ \phi_e, H_p \} \approx 0 \quad \dot{\phi}_{\psi} = \{ \phi_{\psi}, H_p \} \approx 0$$

(4.14) (4.15)

Each condition either determines a multiplier or leads to a new constraint. We assume the canonical Poisson brackets

$$\{ q^\mu, P_\nu \} = \delta^\mu_\nu$$

$$\{ \chi^a, \pi_{\chi b} \} = -\delta^a_b$$

$$\{ \lambda^i, \pi_{\lambda j} \} = -\delta^i_j$$

$$\{ e, \pi_e \} = 1$$

$$\{ \psi, \pi_\psi \} = -1$$

(4.16) (4.17) (4.18) (4.19) (4.20)
Imposing (4.15) requires the secondary constraints

\[
\varphi_e = \frac{1}{2} \eta_{ij} P_i P_j - \frac{1}{2} \eta_{ab} Y^a Y^b - \frac{1}{4} G_{ijab} \chi^i \chi^a \chi^b \\
+ \frac{1}{3!} e^\mu_{[i} \nabla_{\mu} c_{jkl]} \chi^i \chi^j \chi^k + \frac{1}{2} e^\mu_{[i} \nabla_{\mu} m_{j]ab} \chi^i \chi^j \chi^b \\
+ \frac{1}{2} e^\mu_{[i} \nabla_{\mu} n_{]ija} \chi^i \chi^j \chi^a + \frac{1}{3!} e^\mu_{[i} \nabla_{\mu} l_{abc} \chi^i \chi^b \chi^c
\]

(4.21)

\[
\varphi_\psi = \lambda^i (P_i + \frac{1}{3} c_{ijk} \lambda^j \lambda^k + \frac{1}{2} m_{ija} \lambda^j \chi^a) - \frac{1}{3!} l_{abc} \chi^a \chi^b \chi^c - m s a \chi^a
\]

(4.22)

In fact these are the charges of \( H, Q \) respectively. It can be checked that both of these are conserved over time,

\[
\{ \varphi_e, H_p \} \approx 0 \quad \{ \varphi_\psi, H_p \} \approx 0
\]

(4.23)

so they give rise to no new constraints.

The remaining conditions (4.14) determine

\[
u^i = e \eta^{ij} \{ \varphi_e, \phi_j \} + \psi \eta^{ij} \{ \varphi_\psi, \phi_j \}
\]

(4.24)

\[
u^a = -e \eta^{ab} \{ \varphi_e, \phi_\chi^b \} - \psi \eta^{ab} \{ \varphi_\psi, \phi_\chi^b \}
\]

(4.25)

Observe that the constrained Hamiltonian \( H_c \) can be written in terms of the secondary constraints

\[H_c = \varphi_e e + \varphi_\psi \psi\]

(4.26)

Essentially this is because \( \varphi_e = \{ \pi_e, H_c \} \) and \( \varphi_\psi = \{ \pi_\psi, H_c \} \) and the Hamiltonian \( H_c \) is linear in the gauge fields \( e \) and \( \psi \).

5 Quantisation

We observe that the constraints \( \phi_i \) and \( \phi_\chi^a \) are both second class, whereas

\[
\phi_e
\]

(5.1)

\[
\phi_\psi
\]

(5.2)

\[
\varphi_e = \varphi_e + \eta^{ij} \{ \varphi_e, \phi_i \} \phi_j - \eta^{ab} \{ \varphi_e, \phi_\chi^a \} \phi_\chi^b
\]

(5.3)

\[
\varphi_\psi = \varphi_\psi + \eta^{ij} \{ \varphi_\psi, \phi_i \} \phi_j - \eta^{ab} \{ \varphi_\psi, \phi_\chi^a \} \phi_\chi^b
\]

(5.4)

are all first class. Defining the Dirac bracket as

\[
\{ A, B \}_D = \{ A, B \} + \eta^{ij} \{ A, \phi_i \} \{ \phi_j, B \} - \eta^{ab} \{ A, \phi_\chi^a \} \{ \phi_\chi^b, B \}
\]

(5.5)
then Poisson brackets between the primed constraints are weakly equal to Dirac brackets between the original unprimed constraints.

The extra terms on the right of (5.5) give rise to new relations

\[ \{ \lambda^i, \lambda^j \}_D = \eta^{ij} \]
\[ \{ \chi^a, \chi^b \}_D = -\eta^{ab} \]

We can check that the \( N = 1 \) supersymmetry algebra is still obeyed,

\[ \{ \varphi'_\psi, \varphi'_\psi \} \approx \{ \varphi_\psi, \varphi_\psi \} \]
\[ H_p = \varphi_\psi e + \varphi_\psi \psi + \phi_e u_e + \phi_\psi u_\psi \]

so it is vanishing weakly, as is expected for a gravitational system.

In Dirac’s process of quantisation, fields become operators acting on some Hilbert space and second class constraints are imposed as operator conditions on the states. First class constraints generate unphysical degrees of freedom, so it is necessary to fix a gauge,

\[ e = 1 \quad \quad \psi = 0 \]

Moving over to the quantised system, Dirac brackets become (anti) commutation relations. One realisation of this algebra is using the standard Clifford algebra generators

\[ \{ \gamma^i, \gamma^j \} = 2\eta^{ij} \quad \quad \{ \gamma^a, \gamma^b \} = 2\eta^{ab} \]

If the dimension of the manifold is even, for example \( d = 4 \), then we have an element of the algebra, \( \gamma^{d+1} \), satisfying \( (\gamma^{d+1})^2 = -1 \) and \( \{ \gamma^{d+1}, \gamma^i \} = 0 \) so the following realisation exists,

\[ \hat{\lambda}^i = \frac{1}{\sqrt{2}} \gamma^i \otimes 1 \quad \quad \hat{\chi}^a = \frac{1}{\sqrt{2}} \gamma^{d+1} \otimes \gamma^a \]

The \( \gamma^{d+1} \) ensures that \( \hat{\lambda}^i \) and \( \hat{\chi}^a \) anticommute.
If the manifold $\mathcal{M}$ is not even dimensional then the following realisation may be used,

$$\hat{\chi} = \frac{1}{2} \gamma^i \otimes 1 \otimes \sigma_1 \quad \hat{\chi} = \frac{1}{2} 1 \otimes \gamma^a \otimes \sigma_2$$  \hspace{1cm} (5.15)

where $\sigma_i$ are the Pauli spin matrices which satisfy $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$. For the $N$-extended case, see for example [9].

In the following we will assume that the dimension of $\mathcal{M}$ is even and use the first realisation given. Then the second class constraints are imposed as conditions on physical states,

$$\hat{Q} |_{\text{phys}} = 0 \quad \hat{H} |_{\text{phys}} = 0$$  \hspace{1cm} (5.16)

where

$$\hat{Q} = - \frac{1}{\sqrt{2}} (\gamma^i \otimes 1) \epsilon^\mu_i \frac{\partial}{\partial q^\mu} - \frac{1}{4\sqrt{2}} \omega_{ijk}(\gamma^i \gamma^j \otimes 1) - \frac{1}{4\sqrt{2}} \Omega_{iab}(\gamma^i \otimes \gamma^{ab})$$

$$+ \frac{1}{12\sqrt{2}} c_{ijk}(\gamma^j \otimes 1) - \frac{1}{12\sqrt{2}} m_{iab}(\gamma^i \otimes \gamma^{ab}) - \frac{1}{4\sqrt{2}} n_{ija}(\gamma^i \gamma^{d+1} \otimes \gamma^a)$$

$$- \frac{1}{\sqrt{2}} A_i(\gamma^i \otimes 1) + \frac{1}{12\sqrt{2}} l_{abc}(\gamma^{d+1} \otimes \gamma^{abc}) - \frac{1}{\sqrt{2}} m_s(\gamma^{d+1} \otimes \gamma^a)$$  \hspace{1cm} (5.17)

$$\hat{H} = \frac{1}{2} \eta_{ij} \hat{P}^i \hat{P}^j - \frac{1}{2} \eta_{ab} \hat{Y}^a \hat{Y}^b + \frac{1}{16} R_{ijkl}(\gamma^i \gamma^j \otimes 1) + \frac{1}{16} G_{ijkl}(\gamma^i \otimes \gamma^{ab})$$

$$+ \frac{1}{24} \epsilon_{[i}^\mu \nabla_{[k} \epsilon_{j]a} \gamma^{kl} \otimes 1) - \frac{1}{8} \epsilon_{[i}^\mu \nabla_{[k} m_{j]a} \gamma^{kl} \otimes \gamma^{ab})$$

$$+ \frac{1}{8} \epsilon_{[i}^\mu \nabla_{[k} n_{j]a} \gamma^{kl} \otimes \gamma^{ab}) - \frac{1}{24} \epsilon_{[i}^\mu \nabla_{[k} l_{j]abc} (\gamma^i \gamma^{d+1} \otimes \gamma^{abc})$$

$$+ \frac{1}{2} \epsilon_{[i}^\mu \nabla_{[k} A_{j]a} (\gamma^i \otimes 1) + \frac{1}{2} m_{[i} \nabla_{[k} s_a (\gamma^i \gamma^{d+1} \otimes \gamma^a)$$

$$- \frac{1}{2} \Gamma_{\nu \rho \sigma} v^{\nu \rho} \frac{\partial}{\partial q^\mu} - \frac{1}{8} \omega_{m}^{mi} \nabla_{m} \omega_{kl} (\gamma^i \otimes 1)$$

$$- \frac{1}{2} \Gamma_{\nu \rho \sigma} v^{\nu \rho} \frac{\partial}{\partial q^\mu} + \frac{1}{24} \epsilon_{ijk} c_{ijkl} - \frac{1}{24} \epsilon_{abc} l_{abc}$$

$$- \frac{1}{16} c_{ijk} \omega_{m}^{mi} (\gamma^j \otimes 1) - \frac{1}{8} n_{ija} \omega_{m}^{mi} (\gamma^j \gamma^{d+1} \otimes \gamma^a) + \frac{1}{8} m_{iab} \omega_{m}^{mi} (1 \otimes \gamma^a)$$  \hspace{1cm} (5.18)

The new terms which appear here include a Riemann tensor which vanishes classically because it is contracted with four fermions, and several terms involving a trace of the connection which arise because of the way we have chosen to order the equation.
In (5.17) and (5.18) we have defined as usual
\[ \hat{P}_i = -\epsilon^a \frac{\partial}{\partial q^a} - \frac{1}{4} \omega_{ijk}(\gamma^j \otimes 1) - \frac{1}{4} \Omega_{iab}(1 \otimes \gamma^b) \]
\[ - \frac{1}{4} c_{ijk}(\gamma^j \otimes 1) + \frac{1}{4} m_{ab}(1 \otimes \gamma^b) - \frac{1}{2} n_{ija}(\gamma^j \gamma^{d+1} \otimes \gamma^a) - A_i \] (5.19)

and
\[ \hat{Y}_a = \frac{1}{2} m_{ab}(\gamma^i \gamma^{d+1} \otimes \gamma^b) - \frac{1}{4} n_{ija}(\gamma^j \otimes 1) + \frac{1}{4} l_{abc}(1 \otimes \gamma^b) - m_s(1 \otimes 1) \] (5.20)

It can be checked that
\[ \{ \hat{Q}, \hat{Q} \} = 2\hat{H} \] (5.21)

the familiar result that the square of the Dirac operator gives the Klein Gordon equation.

Finally we note that it is not always the case that the system will have physical states. This is because manifolds exist for which the Dirac-like operators do not have zero modes. However it is expected that most models will have a physical Hilbert space which is non-empty.

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