D4-branes on Complete Intersection in Toric Variety

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Abstract

We consider D4-branes on toric Calabi-Yau spaces. The quiver gauge theory that describes several D4-branes on the Calabi-Yau has a Higgs branch, that describes configurations of a single large D4-brane with the same charges. We propose that the world volume of such a D4-brane is described by a determinantal variety. We discuss a description of the Higgs branch of the moduli space in terms of a quiver with twice as many nodes and only bifundamental fields, arising from a $D6 - \overline{D6}$ system. We recast the tachyon condensation of the $D6 - \overline{D6}$ system in the language of open string gauge linear sigma model.

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1. Introduction

BPS black holes in IIA compactification on a Calabi-Yau have a dual description in terms of wrapped D-branes. The world volume theory of D-branes wrapping different cycles of the Calabi-Yau is a quiver gauge theory with each node corresponding to the wrapped D-branes and bifundamental fields corresponding to massless open strings stretched between them \([1-10]\). Ultimately, it would be interesting to understand the microscopic degeneracies of BPS black holes from features of the moduli space of the gauge theory. Toward this end, in \([10]\) we studied an example where we have N D4-branes and M D0-branes on the quintic. We proposed that the quantum mechanics of N D4-branes and M D0-branes on the quintic is described by a dimensional reduction of a \(U(N) \times U(M)\) quiver gauge theory with the appropriate superpotential. We showed that the moduli space of the Higgs branch reproduces the moduli space of hypersurfaces with a certain flux on it. In this note, we generalize the classical geometry analysis in \([10]\) for toric Calabi-Yau spaces. In section 2, we again find that the Higgs branch of N D4-branes is described by a determinantal variety. In section 3, we revisit the construction of the D4-brane moduli space in \([10]\) and recover the D4-brane moduli space from tachyon condensation of the \(D6 - \overline{D6}\) system \([11][12]\). In section 4, we comment on the open string gauged linear sigma model description of D-branes. We recover our determinantal equation for the D4-branes by solving the F-term constraint in this formalism.

2. Generalizing the quintic story

The construction developed for the quintic can be generalized straightforwardly to Calabi-Yau manifolds arising from complete intersections in toric varieties.

Let us consider a toric variety constructed as a symplectic quotient \(\mathcal{T} = \mathbb{C}^m \ln dictatorship \mathbb{C}^n\).

Let \(z_i\) be the coordinates on \(\mathbb{C}^m\), and the \(\mathbb{C}^n\)'s acts by

\[
z_i \rightarrow \lambda^{q_i^\alpha} z_i, \quad \alpha = 1, \cdots, n.
\]

The quotient space has \(n\) Kahler moduli, corresponding to blown up \(\mathbb{P}^1\)'s. In physical terms they are controlled by Fayet-Illiopoulos parameters in the D term constraints for the corresponding gauged linear sigma model \([13]\). The Calabi-Yau \(X\) is given as a complete intersection

\[
P_1(z) = 0, \cdots, P_{m-n-3}(z) = 0,
\]

(2.2)
where the \( P_k \) are polynomials in \( z \) that are homogeneous with respect to the \( \mathbb{C}^* \) actions, i.e. they scale as
\[
P_k(z) \rightarrow \lambda^{Q^k_\alpha} P_k(z)
\]
under (2.1). The vanishing of \( c_1(X) \) requires
\[
\sum_i q^i_\alpha = \sum_k Q^k_\alpha
\]
for all \( \alpha \).

The toric variety \( T \) is naturally equipped with \( n \) line bundles \( \mathcal{L}_\alpha \), whose chern classes \( x^\alpha = c_1(\mathcal{L}_\alpha) \) restrict to the Calabi-Yau \( X \) to give the generators of \( H^{1,1}(X) \). We will denote by \( J^\alpha \) the corresponding integral harmonic \((1,1)\)-forms. A holomorphic surface \( P \) in the Calabi-Yau can be described by one polynomial equation
\[
A(z) = 0
\]
For generic \( A \), the only integral harmonic \((1,1)\) forms on \( P \) are the \( J^\alpha \)'s.

As in the last section of [10], we want to consider a set of D4-branes, labelled by an index \( I \), wrapped on the cycles \( n^I_\alpha \) and carrying gauge field flux \( F^I = k^I_\alpha J^\alpha \). We will give the precise description of the corresponding quiver gauge theory in the next section, and justify some of the following claims. It is natural to propose that the world volume of the D4-brane corresponding to a generic point on the Higgs branch of the quiver gauge theory is described by the following equation
\[
\det[A_{IJ}(z)] = 0
\]
Here \( A_{IJ}(z) \) are polynomials in \( z_i \)'s with homogeneous scaling
\[
A_{IJ}(z) \rightarrow \lambda^{\frac{n^I_\alpha + n^J_\alpha}{2} + k^I_\alpha - k^J_\alpha} A_{IJ}(z)
\]
under (2.1). Such surfaces are special in that they contain an extra integral harmonic \((1,1)\) form dual to the curve \( C \) given by
\[
C : \sum_I v^I(z) \bar{A}_{IJ}(z) = 0
\]
where \( \bar{A}_{IJ}(z) \) are minors of the matrix \( A(z) \) and \( v(z) \) are polynomials in \( z \) of homogeneous degree \( \frac{1}{2} n^I_\alpha + k^I_\alpha \). Note that \( C \) is not a complete intersection and is not homologous to linear combinations of \( J^\alpha \)'s.
As in [10], charge conservation requires an appropriate flux on the D4-brane world volume $P$. It is given by

$$F = C - \frac{1}{2} \sum_I n_I^I J^\alpha$$  \hspace{1cm} (2.9)

To verify the agreement of D2 and D0-brane charges one needs to compute the intersection numbers $C \cdot J^\alpha$ and $C \cdot C$, along the same lines as for the quintic. The final expressions are

$$C \cdot J^\alpha = C^{\alpha\beta\gamma} \left[ \frac{1}{2} \left( \sum_{\beta} n_{\beta}^I \left( \sum_{\gamma} n_{\gamma}^I \right) + \sum n_{\beta}^I k_{\gamma}^I \right) \right]$$

$$C \cdot C = C^{\alpha\beta\gamma} \left[ \frac{1}{6} \left( \sum_{\alpha} n_{\alpha}^I \left( \sum_{\beta} n_{\beta}^I \left( \sum_{\gamma} n_{\gamma}^I \right) + \frac{1}{12} \sum n_{\alpha}^I n_{\beta}^I n_{\gamma}^I + \sum n_{\alpha}^I k_{\beta}^I k_{\gamma}^I \right) \right] \right. \hspace{1cm} (2.10)
$$

where $C^{\alpha\beta\gamma} = \int J^\alpha \wedge J^\beta \wedge J^\gamma$. As we will see the line bundle with curvature $F$ is essentially the bundle of zero eigenvectors of $A_{IJ}$.

3. D4-brane moduli space from tachyon condensation

3.1. The quintic

In [10] the moduli space of a stack of N D4-branes wrapped on a hypersurface in the quintic with minimal flux on each was obtained by solving the D-term constraints of a $U(N)$ gauge theory with four adjoint chiral matter. It is well known from Sen’s work [14] that a D4-brane can be realized as a topologically nontrivial tachyon configuration on the $D6 - \overline{D6}$-brane system with the appropriate fluxes on it. On a Calabi-Yau manifold, the tachyonic open string going between a D6-brane and a $\overline{D6}$-brane with different $U(1)$ gauge fluxes can condense and give rise to a D4-brane, whose charge is determined by the fluxes. Let us consider the case of the quintic. A D4-brane wrapped on a degree $d$ surface D, and with world volume flux $kJ$ can be interpreted as a (derived category of) coherent sheaf $\mathcal{O}_D$, with locally free resolution $\mathcal{O}_{k,d}$

$$0 \rightarrow \mathcal{O}(k - \frac{d}{2}) \rightarrow \mathcal{O}(k + \frac{d}{2}) \rightarrow \mathcal{O}_D \rightarrow 0$$  \hspace{1cm} (3.1)

where $\mathcal{O}(k - \frac{d}{2})$ and $\mathcal{O}(k + \frac{d}{2})$ are the gauge bundle of a $\overline{D6}$-brane and a D6-brane with $k - \frac{d}{2}$ and $k + \frac{d}{2}$ units of fluxes, respectively. The zero modes of the open string “tachyons” $T$, are $^1$k is an integer or a half-integer depending on whether $d$ is even or odd, due to Freed-Witten anomaly. [18]
given by the holomorphic sections of the gauge bundle $\mathcal{O}(d)$, that is they are polynomials in $z_i$ of homogeneous degree $d$, up to multiples of the quintic equation.

The D4-brane world volume is at the zero locus of the tachyon profile. We would like to propose a supersymmetric quiver gauge theory describing the bound state of the brane/anti-brane system, with gauge group $U(1) \times U(1)$ and bifundamental matter $T_a$ coming from the tachyons, where $a = 1, \ldots, \dim H^0(X, \mathcal{O}(d))$. There is no gauge invariant superpotential for this theory since there are no loops in the quiver diagram. Solving the D-term constraints give a moduli space of vacua which is the projectivization of $H^0(X, \mathcal{O}(d))$. This matches the moduli of the degree $d$ surface. A similar discussion can be found in [19].

It might appear unexpected to have a supersymmetric quiver theory that describes the $D6 - \overline{D6}$ system. However, from the four dimensional point of view they are two BPS objects with generically different central charges. Along a wall in the Kähler moduli space of $X$ they can be mutually supersymmetric, i.e. their central charges are aligned. Deforming the Kähler moduli corresponds to turning on FI parameter in the gauge theory, and then the bifundamental open string fields become tachyonic.

Similarly we can describe the quiver theory for a stack of $N$ D4-branes wrapped on a degree $d$ surface. This is a $U(N) \times U(N)$ gauge theory with bifundamental matter $T_a$ in $(N, \overline{N})$. Again there is no superpotential and the moduli space of vacua is given by a symplectic quotient of the $H^0(X, \mathcal{O}(d)) \otimes N^2$ by $GL(N) \times GL(N)$. Now the D4-brane is given by the sheaf of the zero eigenvector of the tachyon matrix. It is supported on the locus $\det(T_{(A_1, \ldots, A_N)}z^{A_1} \cdots z^{A_N}) = 0$. This is precisely the special surface corresponding to $N$ joined D4-branes as in [10].

### 3.2. Toric Calabi-Yau Threefolds

Let us now return to the case of the Calabi-Yau $X$ given by a complete intersection in toric variety $\mathbb{C}^m / \mathbb{C}^n$ as in section 2. A set of $N$ D4-branes wrapped on cycles $n^I_\alpha$ with fluxes $k^I_\alpha J^\alpha$ can be described in terms of a complex of vector bundles, as a $D6 - \overline{D6}$ system

$$0 \to \bigoplus_{I=1}^{N} \bigotimes_{\alpha} \mathcal{L}^{k^I_\alpha - n^J_\alpha} \to \bigoplus_{J=1}^{N} \bigotimes_{\alpha} \mathcal{L}^{k^J_\alpha + n^I_\alpha} \to 0$$

(3.2)

The tachyon $T$ has components $T_{IJ}$ living in the line bundle $\bigotimes_{\alpha} \mathcal{L}^{\frac{n^I_\alpha + n^J_\alpha}{2} + k^I_\alpha - k^J_\alpha}$. The quiver theory has gauge group $U(1)^N \times U(1)^N$ and bifundamental fields $T_{IJ}$. Again, the

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2 This is an $N \times N$ matrix with whose elements are in $H^0(X, \mathcal{O}(d))$. 

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D4-brane is given by the zero eigenvectors of $T$, which is indeed supported along the surface (2.3). The moduli space of the D4-branes is again determined by the D-term constraints of this quiver theory.

Starting from the derived category description of D-branes, a quiver gauge theory can be built along the same lines from the physical setup of several different $D_6$ and $\overline{D_6}$ branes. These more complicated setups will generally allow for superpotentials. Understanding the properties of these quiver gauge theories may lead to an account of the microscopic degeneracies of BPS black holes.

4. Gauged linear sigma model

The world sheet theory of closed strings on the Calabi-Yau space $X$ can be described as the IR limit of a (2, 2) gauged linear sigma model with gauge group $U(1)^n$ and chiral fields $Z^i$ and $\Lambda^k$, where $Z^i$ are assigned charges $q^i_\alpha$ under the $\alpha$th $U(1)$ [13]. $\Lambda^k$'s are Lagrangian multipliers of charge $-Q^k_\alpha$ so that the superpotential

$$W = \sum_k \Lambda^k P_k(Z^i)$$

is gauge invariant.

Following [11][12][3, the D4-brane (3.2) can be described by the boundary couplings

$$\int_{\partial \Sigma} d\theta \rho^I T_{IJ}(Z^i) \beta^J \quad (4.2)$$

where $\rho^I$ and $\beta^J$ are bosonic and fermionic fields living on the boundary with charges $-\frac{1}{2}n^I_\alpha - \kappa^I_\alpha$ and $-\frac{1}{2}n^J_\alpha + \kappa^J_\alpha$, respectively. $T_{IJ}(Z^i)$ are polynomials in $Z^i$'s with homogeneous charges $\frac{n^I_\alpha + n^J_\alpha}{2} + \kappa^I_\alpha - \kappa^J_\alpha$. There is also a $U(1)$ gauge symmetry acting on the boundary fields

$$\rho^I \rightarrow e^{-i\alpha} \rho^I, \quad \beta^I \rightarrow e^{i\alpha} \beta^I \quad (4.3)$$

This is the world sheet analogue of the previous section of building D4-branes from $D_6$-$\overline{D_6}$ branes.

The F-term constraints set $\rho^I T_{IJ} = 0$. Additionally, projecting onto charge one states of the boundary theory requires $\rho$ to be nonzero and therefore

$$\det T(Z^i) = 0 \quad (4.4)$$

4 Also see [20][21][22].

5
This again is the geometry of the D4-brane in the Higgs branch. Generically $T$ has one zero eigenvector $\beta_0^I(Z)$,

$$T_{IJ}\beta_0^J = 0 \quad (4.5)$$

$\beta_0^I$ spans a sub-line bundle $L$ of $\bigoplus_I L_I$, i.e. $L = \text{Ker} T$. It is clear that $L$ is dual to $C - \sum n_\alpha^I J^\alpha = F - \frac{1}{2} \sum n_\alpha^I J^\alpha$.

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