A note on gaussian distributions in $\mathbb{R}^n$

B.G. MANJUNATH and K.R. PARTHASARATHY

Indian Statistical Institute, Delhi Centre, 7, S. J. S. Sansanwal Marg,
New Delhi – 110016, India
E–mail : bgmanjunath@gmail.com; krp@isid.ac.in

Abstract. Given any finite set $\mathcal{F}$ of $(n – 1)$–dimensional subspaces of $\mathbb{R}^n$ we give examples of nongaussian probability measures in $\mathbb{R}^n$ whose marginal distribution in each subspace from $\mathcal{F}$ is gaussian. However, if $\mathcal{F}$ is an infinite family of such $(n – 1)$–dimensional subspaces then such a nongaussian probability measure in $\mathbb{R}^n$ does not exist.

Key words. gaussian distribution, characteristic function, homogeneous polynomial, linear functionals, nonunimodality, Hermite polynomial

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1 Introduction

Starting with the simple example of E. Nelson as cited by W. Feller in [1] we have from the papers of B.K. Kale [3], G.G. Hamedani and M.N. Tata [2] and Y. Shao and M. Zhou [4] etc., as well as Section 10 of J. Stoyanov’s book [5], several examples of bivariate and multivariate nongaussian distributions under which many linear functionals can have a gaussian distribution on the real line. These results suggest the possibility of characterizing a gaussian distribution in $\mathbb{R}^n$ through properties of classes of linear functionals. Motivated by Nelson’s example in [1] and the bivariate construction in [2] we introduce a perturbation of the standard gaussian density function in $\mathbb{R}^n$ exhibiting the following interesting features: (1) Given any finite set $\{S_j, 1 \leq j \leq N\}$ of $(n – 1)$–dimensional subspaces it has a marginal density function which is standard gaussian in each $S_j$, $j \in \{1,2,...,N\}$; (2) There can exist linear functionals whose distributions may have nonunimodal density functions; (3) For certain choices of subspaces the nongaussian perturbation can be so chosen that any real symmetric measurable function of all the $n$ coordinates has its distribution preserved. In particular, the sum of squares of all the coordinates can have the $\chi^2$ distribution with $n$ degrees of freedom.
We also demonstrate the following characterization of the multivariate
Gaussian distribution. Suppose \{S_j, j = 1, 2, \ldots\} is a countably infinite set
of \((n-1)\)-dimensional subspaces of \(\mathbb{R}^n\) and \(\mu\) is a probability measure in
\(\mathbb{R}^n\) such that the projection of \(\mu\) in each subspace \(S_j\) is Gaussian. Then \(\mu\)
itself is Gaussian. This is a generalization of the characterization in [2] and
a more precise version of the result in [4].

Our proofs follow the steps in [2] and use some additional geometric
and topological arguments of a very elementary kind.

2 A perturbation of the Gaussian
characteristic function

We begin by examining a small perturbation of the characteristic function
of the \(n\)-variate standard Gaussian distribution with mean vector \(0\) and
covariance matrix \(I\) as follows. Choose and fix any homogeneous polynomial \(P\)
of even degree \(2k\) in \(n\) real variables \(t_1, t_2, \ldots, t_n\) and define

\[
\Phi(t; \varepsilon, \sigma, P) = e^{-\frac{1}{2}|t|^2} + \varepsilon e^{-\frac{1}{2\sigma^2}|t|^2} P(t), \quad t \in \mathbb{R}^n
\]  

(2.1)

where \(t = (t_1, \ldots, t_n)^T\), \(\varepsilon\) is a real parameter and \(\sigma\) is a parameter satisfying
\(0 < \sigma < 1\). Here

\[
|t|^2 = (t_1^2 + \ldots + t_n^2).
\]

Clearly, \(\Phi(\cdot; \varepsilon, \sigma, P)\) is a real analytic function on \(\mathbb{R}^n\) satisfying

\[
\Phi(0; \varepsilon, \sigma, P) = 1,
\]

\[
\Phi(-t; \varepsilon, \sigma, P) = \Phi(t; \varepsilon, \sigma, P).
\]  

(2.2)

Let

\[
Z_P = \{t | P(t) = 0, t \in \mathbb{R}^n\}
\]  

(2.3)

be the set of zeros of \(P\) in \(\mathbb{R}^n\).

Since we are interested in the inverse Fourier transform of \(\Phi\) we introduce the renormalized polynomial : \(\mathcal{P}\) : in the form of a formal definition.

**Definition 2.1.** Let

\[
\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}
\]

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and let $H_m(x)$ be the $m$-th Hermite polynomial defined by
\[
\frac{d^m}{dx^m} \mathcal{H}(x) = (-1)^m H_m(x) \mathcal{H}(x), \quad m = 0, 1, 2, ...
\]
(as in Feller [1]). For any real polynomial $\mathcal{P}$ in $n$ real variables given by
\[
\mathcal{P}(t_1, t_2, ..., t_n) = \sum_m a_{m_1,m_2,...,m_n} t_1^{m_1} t_2^{m_2} ... t_n^{m_n}
\]
itself renormalized version: $\mathcal{P}$ is defined by
\[
:\mathcal{P} : (x_1, ..., x_n) = \sum_m a_{m_1,m_2,...,m_n} H_{m_1}(x_1) H_{m_2}(x_2) ... H_{m_n}(x_n).
\]
Note that for a homogeneous polynomial, its renormalized version need not be homogeneous.

Since the function $\Phi$ in (2.1) is in $L_1(\mathbb{R}^n)$ its inverse Fourier transform $f$ is defined by
\[
f(x; \varepsilon, \sigma, \mathcal{P}) = \frac{1}{(2\pi)^n} \int e^{-it^T x} \Phi(t; \varepsilon, \sigma, \mathcal{P}) dt_1 dt_2 ... dt_n
\]
\[
= \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}x^2} \varepsilon + \frac{1}{(2\pi)^n} \int e^{-it^T x} e^{-\frac{1}{2}\sigma^2 |t|^2} \mathcal{P}(t) dt_1 ... dt_n.
\]

First, we note that
\[
\frac{1}{(2\pi)^n} \int e^{-it^T x} e^{-\frac{1}{2}\sigma^2 |t|^2} dt_1 dt_2 ... dt_n = \frac{1}{\sigma^n} \prod_{j=1}^n \mathcal{H} \left( \frac{x_j}{\sigma} \right) .
\]
Repeated differentiation with respect to $x_1, x_2, ..., x_n$ shows that for the homogeneous polynomial $\mathcal{P}$ of degree $2k$ we have
\[
\frac{1}{(2\pi)^n} \int e^{-it^T x} e^{-\frac{1}{2}\sigma^2 |t|^2} \mathcal{P}(t) dt_1 dt_2 ... dt_n
\]
\[
= \frac{1}{\sigma^n} \mathcal{P} \left( i \frac{\partial}{\partial x_1}, ..., i \frac{\partial}{\partial x_n} \right) \left\{ \prod_{j=1}^n \mathcal{H} \left( \frac{x_j}{\sigma} \right) \right\}
\]
\[
= \frac{(-1)^k}{\sigma^{n+2k}} : \mathcal{P} : \left( \frac{x_1}{\sigma}, ..., \frac{x_n}{\sigma} \right) \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2} x^2}.
\]
Thus the inverse Fourier transform (2.4) assumes the form
\[
f(x; \varepsilon, \sigma, \mathcal{P}) = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2} |x|^2} \left\{ 1 + \frac{(-1)^k \varepsilon}{\sigma^{n+2k}} \mathcal{P} \left( \left( \frac{x_1}{\sigma}, ..., \frac{x_n}{\sigma} \right) e^{-\frac{1}{2\sigma^2} |x|^2 (1-\varepsilon^2)} \right) \right\}.
\]
(2.5)

Since, by assumption, \(1 - \sigma^2 > 0\) the positive constant \(K(\sigma, \mathcal{P})\) defined by
\[
K(\sigma, \mathcal{P}) = \sup_{x \in \mathbb{R}^n} \left| \mathcal{P} \left( \left( \frac{x_1}{\sigma}, ..., \frac{x_n}{\sigma} \right) e^{-\frac{1}{2\sigma^2} |x|^2 (1-\varepsilon^2)} \right) \right|
\]
(2.6)
is finite and for all \(x \in \mathbb{R}^n\)
\[
f(x; \varepsilon, \sigma, \mathcal{P}) \geq 0 \text{ if } |\varepsilon| \leq K^{-1}(\sigma, \mathcal{P})
\]
we observe that \(\Phi(\cdot; \varepsilon, \sigma, \mathcal{P})\) is a real characteristic function of the probability density function \(f(\cdot; \varepsilon, \sigma, \mathcal{P})\) defined by (2.5) for any \(\varepsilon \in [-K^{-1}(\sigma, \mathcal{P}), K^{-1}(\sigma, \mathcal{P})]\). Here we have made use of property (2.2).

Thus we can summarize the discussion above as a theorem.

**Theorem 2.2.** Let \(0 < \sigma < 1\), \(\mathcal{P}\) be a real homogeneous polynomial in \(n\) variables of even degree \(2k\), \(K(\sigma, \mathcal{P})\) the positive constant defined by (2.6) and \(\varepsilon \in [-K^{-1}(\sigma, \mathcal{P}), K^{-1}(\sigma, \mathcal{P})]\). Then the function \(\Phi(\cdot; \varepsilon, \sigma, \mathcal{P})\) defined by (2.1) is the characteristic function of a probability density function \(f(\cdot; \varepsilon, \sigma, \mathcal{P})\) defined by (2.5). Under this density function \(f(\cdot; \varepsilon, \sigma, \mathcal{P})\) the linear functional \(x \mapsto a^T x\) with \(|a| = 1\) has characteristic function \(\varphi_a\) and probability density function \(g_a\) on the real line given respectively by
\[
\varphi_a(t) = e^{-\frac{1}{2}t^2} + \varepsilon \mathcal{P}(a) e^{-\frac{1}{2\sigma^2}t^2} t^{2k}, \quad t \in \mathbb{R}
\]
(2.7)
\[
g_a(x) = \frac{1}{\sqrt{2\pi}} \left\{ e^{-\frac{1}{2}x^2} + \frac{(-1)^k \varepsilon \mathcal{P}(a)}{\sigma^{2k+1}} H_{2k} \left( \frac{x}{\sigma} \right) e^{-\frac{1}{2\sigma^2}x^2} \right\}
\]
(2.8)

In particular, for any \(a \in \mathbb{Z}^n\), the linear functional \(a^T x\) has the normal distribution with mean 0 and variance \(|a|^2\) but \(f(\cdot; \varepsilon, \sigma, \mathcal{P})\) is a nongaussian density function for any \(\varepsilon \in [-K^{-1}(\sigma, \mathcal{P}), K^{-1}(\sigma, \mathcal{P})] \setminus \{0\}\).
Proof. The first part is immediate from the discussion preceding the statement of the theorem. To prove the second part we note that the characteristic function \( \varphi_a(t) \) of the linear functional \( a^\top x \) under the density function \( f(\cdot; \varepsilon, \sigma, \mathcal{P}) \) is \( \Phi(ta; \varepsilon, \sigma, \mathcal{P}) \) and (2.7) follows from (2.1) and the homogeneity of \( \mathcal{P} \). Now (2.8) follows from Fourier inversion of (2.7). If \( 0 \neq a \in \mathbb{Z}_\mathcal{P} \) then \( 0 = \mathcal{P}(a) = \mathcal{P}\left(\frac{a}{|a|}\right) \) and therefore

\[
\varphi_{\frac{a}{|a|}}(t) = e^{-\frac{1}{2}t^2}.
\]

Hence \( a^\top x \) is normally distributed with mean 0 and variance \( |a|^2 \).

Corollary 2.3. Let \( \{S_j, 1 \leq j \leq N\} \) be any finite set of \( (n - 1) \)-dimensional subspaces of \( \mathbb{R}^n \). Then there exists a non-gaussian analytic probability density function whose projection on \( S_j \) is gaussian for each \( j \in \{1, 2, \ldots, N\} \).

Proof. By adding one more \( (n - 1) \)-dimensional subspace to the collection \( \{S_j, 1 \leq j \leq N\} \), if necessary, we may assume without loss of generality that \( N \) is even. Choose a unit vector \( a^{(j)} \in S_j^\perp \) for each \( j \) and define the homogeneous real polynomial \( \mathcal{P} \) of degree \( N \) by

\[
\mathcal{P}(t) = \prod_{j=1}^{N} a^{(j)\top} t, \ t \in \mathbb{R}^n.
\]

Clearly,

\[
\mathcal{P}(t) = 0 \text{ if } t \in \bigcup_{j=1}^{N} S_j.
\]

In other words

\[
\bigcup_{j=1}^{N} S_j \subset \mathbb{Z}_\mathcal{P}.
\]

If we choose \( \mu \) to be the probability measure with the density function \( f(\cdot; \varepsilon, \sigma, \mathcal{P}), 0 \neq \varepsilon \text{ in } [-K^{-1}(\sigma, \mathcal{P}), K^{-1}(\sigma, \mathcal{P})] \) in Theorem 2.2 it follows immediately from the last part of the theorem that every linear functional of the form \( b^\top x \) has a normal distribution with mean 0 and variance \( |b|^2 \) whenever \( b \in \mathbb{Z}_\mathcal{P} \). This completes the proof.
Remark 2.4. In the context of understanding the modes of the density function $g_a(x)$ given by (2.8) it is of interest to note that

$$\left\{ x \bigg| x \neq 0, g'_a(x) = 0 \right\} =$$

$$\left\{ x \bigg| x \neq 0, e^{\frac{x^2}{2\sigma^2}} + \frac{(-1)^k \epsilon P(a) H_{2k+1}(\frac{x}{\sigma})}{\sigma^{2k+2}} = 0 \right\}.$$ 

Indeed, this is obtained by straightforward differentiation and using the recurrence relation $H_{2k+1}(x) = xH_{2k}(x) - H_{2k-1}(x)$.

Example 2.5. Let $n$ be even,

$$P(t_1, t_2, ..., t_n) = t_1 t_2 ... t_n \prod_{i > j} (t_i^2 - t_j^2)$$

$$= t_1 t_2 ... t_n \begin{vmatrix} 1 & 1 & ... & 1 \\ t_1^2 & t_2^2 & ... & t_n^2 \\ t_1^4 & t_2^4 & ... & t_n^4 \\ ... & ... & ... & ... \\ t_1^{2(n-1)} & t_2^{2(n-1)} & ... & t_n^{2(n-1)} \end{vmatrix}$$

(2.9)

Then $P$ is a polynomial of even degree $n^2$, which is antisymmetric in the variables $t_1, t_2, ..., t_n$. The renormalized version $: P :$ of $P$ is given by

$$: P : (x_1, x_2, ..., x_n) = \begin{vmatrix} H_1(x_1) & H_1(x_2) & ... & H_1(x_n) \\ H_3(x_1) & H_3(x_2) & ... & H_3(x_n) \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ H_{2n+1}(x_1) & H_{2n+1}(x_2) & ... & H_{2n+1}(x_n) \end{vmatrix}$$

(2.10)

In particular, $: P :$ is antisymmetric in the variables $x_1, x_2, ..., x_n$. Fixing $0 < \sigma < 1$ we get for each $\epsilon \in [ -K^{-1}(\sigma, P), K^{-1}(\sigma, P) ]$, with $K(\sigma, P)$ being determined by (2.6), (2.10) and $k = \frac{1}{2} n^2$, the probability density function $f(\cdot; \epsilon, \sigma, P)$ given by
\[
f(x; \varepsilon, \sigma, \mathcal{P}) = \frac{1}{(\sqrt{2\pi})^n} \left\{ e^{-\frac{1}{2}|x|^2} + \frac{\varepsilon}{\sigma^{n(n+1)}} \begin{vmatrix} H_1\left(\frac{\varepsilon}{\sigma}\right) & H_1\left(\frac{\varepsilon}{\sigma}\right) & \cdots & H_1\left(\frac{\varepsilon}{\sigma}\right) \\ H_3\left(\frac{\varepsilon}{\sigma}\right) & H_3\left(\frac{\varepsilon}{\sigma}\right) & \cdots & H_3\left(\frac{\varepsilon}{\sigma}\right) \\ \vdots & \vdots & \ddots & \vdots \\ H_{2n+1}\left(\frac{\varepsilon}{\sigma}\right) & H_{2n+1}\left(\frac{\varepsilon}{\sigma}\right) & \cdots & H_{2n+1}\left(\frac{\varepsilon}{\sigma}\right) \end{vmatrix} \right\}.
\]

By Theorem 2.2 and its Corollary we conclude that the projection of this density function on the \((n - 1)\)-dimensional hyperplanes \(\{x| x_j = 0\}, 1 \leq j \leq n\); \(\{x|x_i - x_j = 0\}, 1 \leq i \leq j \leq n\) and \(\{x|x_i + x_j = 0\}, 1 \leq i \leq j \leq n\) are all \((n - 1)\)-dimensional gaussian densities.

If \(g(x_1, x_2, \ldots, x_n)\) is any bounded continuous function which is symmetric in the variables \(x_1, x_2, \ldots, x_n\) then the function \(g : \mathcal{P} :\) is an antisymmetric function in \(\mathbb{R}^n\) and therefore

\[
\int_{\mathbb{R}^n} g(x_1, x_2, \ldots, x_n) f(x; \varepsilon, \sigma, \mathcal{P}) \, dx_1 dx_2 \ldots dx_n = 0.
\]

Thus

\[
\int_{\mathbb{R}^n} g(x_1, \ldots, x_n) f(x; \varepsilon, \sigma, \mathcal{P}) \, dx_1 dx_2 \ldots dx_n = \int_{\mathbb{R}^n} g(x_1, x_2, \ldots, x_n) \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}|x|^2} \, dx_1 dx_2 \ldots dx_n.
\]

In other words, for \(0 \neq \varepsilon \in [-K^{-1}(\sigma, \mathcal{P}), K^{-1}(\sigma, \mathcal{P})]\), any symmetric measurable function \(g\) on \(\mathbb{R}^n\) has the property that its distribution under the nongaussian density function \(f(x; \varepsilon, \sigma, \mathcal{P})\) in (2.11) is the same as its distribution under the standard gaussian density function with mean \(0\) and covariance matrix \(I\).

**Example 2.6.** We now specialize Example 2.5 to the case \(n = 2, \sigma = 2^{-1/2}\) when

\[
\mathcal{P}(t_1, t_2) = t_1 t_2 (t_1^2 - t_2^2),
\]

: \(\mathcal{P} : (x_1, x_2) = H_3(x_1) H_1(x_2) - H_1(x_1) H_3(x_2)
\]

\[
= x_1^3 x_2 - x_2^3 x_1.
\]
A simple computation shows that

\[ K(σ, P) = 8 \sup |x_1^3 x_2 - x_2^3 x_1| e^{-\frac{1}{2} (x_1^2 + x_2^2)} = 128 \ e^{-2}. \]

This supremum is easily evaluated by switching over to the polar coordinates \( x_1 = r \cos \theta, \ x_2 = r \sin \theta \). Then

\[
\begin{align*}
f(x; ε, σ, P) &= \frac{1}{2\pi} e^{-\frac{1}{2} (x_1^2 + x_2^2)} \left\{ 1 + 32 ε (x_1^3 x_2 - x_2^3 x_1) e^{-\frac{1}{2} (x_1^2 + x_2^2)} \right\} \\
&= \frac{1}{2\pi} e^{-\frac{1}{2} (x_1^2 + x_2^2)} \left\{ 1 + \frac{32}{128} ε \left( x_1^3 x_2 - x_2^3 x_1 \right) e^{-\frac{1}{2} (x_1^2 + x_2^2)} \right\} \quad \text{(2.12)}
\end{align*}
\]

which is a probability density function whenever

\[ |ε| ≤ \frac{e^2}{128}. \]

At \( ε = 0 \), it is the standard normal density function with mean 0 and covariance matrix \( I \). We write \( η = 32 \ ε \) and express the density function (2.12) as

\[
\begin{align*}
f_η(x_1, x_2) &= \frac{1}{2\pi} e^{-\frac{1}{2} (x_1^2 + x_2^2)} \left\{ 1 + η (x_1^3 x_2 - x_2^3 x_1) e^{-\frac{1}{2} (x_1^2 + x_2^2)} \right\} \\
&= \frac{1}{2\pi} e^{-\frac{1}{2} (x_1^2 + x_2^2)} \left\{ 1 + \frac{32}{128} η \left( x_1^3 x_2 - x_2^3 x_1 \right) e^{-\frac{1}{2} (x_1^2 + x_2^2)} \right\} \quad \text{(2.13)}
\end{align*}
\]

where (See Fig. (1).)

\[ |η| ≤ \frac{e^2}{4}. \]

When \( \mathbf{a} = (\sin \theta, \cos \theta) \) the density function \( g_θ \) of the linear functional

\[ x \mapsto -x_1 \sin \theta + x_2 \cos \theta, \]

under \( f_η \) is given by the formula (2.8) of Theorem 2.2 as

\[
\begin{align*}
g_θ(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} \left\{ 1 - \frac{\sqrt{2} \ η \sin(4θ)}{32} (4x^4 - 12x^2 + 3) e^{-\frac{1}{2} x^2} \right\} \quad \text{(2.14)}
\end{align*}
\]

Thus

\[ g_θ'(x) = -\frac{x}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} \left\{ e^{\frac{1}{2} x^2} - \frac{\sqrt{2} \ η \sin(4θ)}{16} (4x^4 - 20x^2 + 15) e^{-\frac{1}{2} x^2} \right\}. \]

It is not difficult to find values of \( η \) in the range \((0, \frac{1}{4} e^2]\) and angle \( θ \) for which

\[ \left\{ x \bigg| e^{\frac{1}{2} x^2} - \frac{\sqrt{2} \ η \sin(4θ)}{16} (4x^4 - 20x^2 + 15) = 0, \ x \neq 0 \right\} \neq \emptyset. \quad \text{(2.15)} \]

This reveals the possibility of nonunimodality of the density of some linear functionals under the joint density \( f_η \). For an illustration cf. Fig. (2).
Figure 1: Bivariate density $f_{\eta}(x_1, x_2)$ at $\eta = e^2/4$.

3 A characterization of gaussian distributions in $\mathbb{R}^n$

In the context of the Corollary to Theorem 2.2 we have the following characterization of a gaussian distribution in $\mathbb{R}^n$ when the number $N$ of $(n-1)$–dimensional subspaces in the corollary is countably infinite.

Theorem 3.1. Let $\{S_j, j = 1, 2, \ldots\}$ be a countably infinite set of $(n-1)$–dimensional subspaces of $\mathbb{R}^n$ and let $\mu$ be a probability measure in $\mathbb{R}^n$ whose projection on $S_j$ is gaussian for each $j = 1, 2, \ldots$. Then $\mu$ is gaussian.

Proof. The fact that the projection of $\mu$ on the two distinct $(n-1)$–dimensional subspaces $S_1$ and $S_2$ are gaussian implies that the multivariate Laplace transform $\tilde{\mu}$ of $\mu$ given by

$$\tilde{\mu}(z_1, \ldots, z_n) = \int \exp(z_1x_1 + \ldots + z_nx_n)\mu(dx_1dx_2\ldots dx_n)$$

(3.1)
is well-defined for \( z \in \mathbb{C}^n \) and analytic in each of the complex variables \( z_j, j = 1, \ldots, n \). Let \( \mu \) and \( \Sigma \) be respectively the mean vector and covariance matrix of the \( \mathbb{R}^n \) valued random variable \( x \) with distribution \( \mu \).

Choose and fix a unit vector \( a^{(j)}_j \in S_j^\perp \) for each \( j = 1, 2, \ldots \). Suppose

\[
\alpha_j = \max_{1 \leq r \leq n} |a_{jr}|.
\]

Since

\[
\sum_{r=1}^{n} a_{jr}^2 = 1, \ \forall j
\]

it follows that \( \alpha_j \geq n^{-1/2}, \forall j \). There exists an \( r_0 \) such that \( a_{jr_0} = \alpha_j \) for infinitely many values of \( j \). Restricting ourselves to this infinite set of \( j \)'s

Figure 2: Nonunimodality of \( g_\theta \).
and assuming $r_0 = 1$ without loss of generality we may as well assume that

\[a^{(j)} = (a_{j1}, ..., a_{jn})^T,\]

\[|a_j| = \max_{1 \leq r \leq n} |a_{jr}| \quad \forall \ j = 1, 2, ..., \]

\[|a_j| \geq n^{-1/2} \quad \forall j.\]

Now consider the $(n - 1)$-dimensional vector $b^{(j)}$ defined by

\[b^{(j)} = \left(\frac{a_{j2}}{a_{j1}}, \frac{a_{j3}}{a_{j1}}, ..., \frac{a_{jn}}{a_{j1}}\right), \quad j = 1, 2, ....\]

where

\[\left|\frac{a_{jr}}{a_{j1}}\right| \leq 1 \quad \forall \ r = 2, 3, ..., n.\]

Thus all the vectors $b^{(j)}$ are distinct and they constitute a bounded countable set in $\mathbb{R}^{(n-1)}$. Define the set

\[\mathcal{D} = \bigcap_{j<i} \left\{s \in \mathbb{R}^{(n-1)}, (b^{(j)} - b^{(i)})^T s \neq 0\right\}.\]

Being a countable intersection of dense open sets it follows from the Baire category theorem that $\mathcal{D}$ is dense in $\mathbb{R}^{(n-1)}$. Let now

\[s = (s_2, s_3, ..., s_n)^T \in \mathbb{R}^{(n-1)}\]

be any fixed point in $\mathcal{D}$. Define

\[s_{j1} = -b^{(j)^T} s, \quad j = 1, 2, ... .\]

By the definition of $\mathcal{D}$, \{s_{j1}, j = 1, 2, ... \} is a bounded and countably infinite set of distinct points on the real line. Furthermore

\[a_{j1}s_{j1} + a_{j2}s_2 + ... + a_{jn}s_n = 0 \quad \forall \ j.\]

In other words, \((s_{j1}, s_2, ..., s_n)^T \in S_j\) for each $j$. By hypothesis the linear functional $s_{j1}x_1 + s_2x_2 + ... + s_nx_n$ has a normal distribution with mean $s_{j1}m_1 + s_2m_2 + ... + s_nm_n$ and variance \((s_{j1}, s_2, ..., s_n)^T \Sigma (s_{j1}, s_2, ..., s_n)^T\). Defining

\[\psi(z_1, ..., z_n) = \exp(m^T z + \frac{1}{2}z^T \Sigma z), \quad z \in \mathbb{C}^n\]
we conclude that the Laplace transform $\hat{\mu}$ defined by (3.1) and the function $\psi$ satisfy the relation

$$\hat{\mu}(s_{j1}, s_{j2}, ..., s_{jn}) = \psi(s_{j1}, s_{j2}, ..., s_{jn})$$

for $j = 1, 2, ...$. Since $\hat{\mu}(z, s_{j2}, ..., s_{jn})$ and $\psi(z, s_{j2}, ..., s_{jn})$ are analytic functions of $z$ in the whole complex plane and they agree on the infinite bounded set $\{s_{j1}, j = 1, 2, ...\}$ it follows that

$$\hat{\mu}(z, s_{j2}, ..., s_{jn}) = \psi(z, s_{j2}, ..., s_{jn}) \forall z \in \mathbb{C}.$$ 

Since this holds for all $(s_{j2}, ..., s_{jn})^T \in D$ which is dense in $\mathbb{R}^{(n-1)}$ and both sides of the equation are continuous on $\mathbb{R}^n$ we have

$$\hat{\mu}(s_1, s_{j2}, ..., s_{jn}) = \psi(s_1, s_{j2}, ..., s_{jn})$$

for all $(s_1, s_{j2}, ..., s_{jn})^T \in \mathbb{R}^n$. This implies that $\mu$ is a gaussian measure with mean vector $m$ and covariance matrix $\Sigma$. □

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References

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