A simple approach to predict settlement due to constant rate loading in clays

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1. Introduction

Consolidation is one of the most relevant and most studied phenomena in Geotechnical Engineering. The process involves volume change due to water flow in response to stress increase and it is particularly relevant with saturated clayey soils. Due to the extremely low permeability of clays, consolidation can last for decades.

Terzaghi and Fröhlich’s (1936) one-dimensional consolidation theory is based on Darcy’s law and uses a set of simplifying hypotheses. One of the main assumptions imposes that load is applied instantaneously. The average degree of consolidation $U$ is given by

$$U(T) = 1 - \sum_{m=0}^{\infty} \frac{2}{M^2} e^{-M/T}$$

where

$$M = (2m+1)\pi / 2$$

for $m = 1, 2, 3\ldots$ and the Time Factor $T$ is defined as

$$T = c_e t / H_s^2$$

Where $t$ is time, $H_s$ is the maximum length of the drainage path and $c_e$ is the coefficient of consolidation.

The consideration of instantaneous load is unlikely to occur in engineering practice. Loading is generally carried out gradually, in stages, during a given construction period. Therefore, consolidation takes place while loading is still in progress.

Several methods have addressed the time-dependent loading issue (Terzaghi, 1943; Schiffman, 1958; Schiffman and Stein, 1970; Olson, 1977; Zhu and Yin, 1998; Conte and Troncone, 2006; Liu and Ma, 2011, Hanna et al. 2013; Gerscovich et al., 2018). The two most worldwide known approaches for linearly increasing loading are Terzaghi’s (1943) graphical method and Olson’s (1977) theoretical solution. Terzaghi’s (1943) empirical method is a graphical procedure performed differently before and after construction. After the end of construction, the settlement curve is shifted by half the construction period $t_c$. During construction, the calculation considers only a fraction of the total load as if applied instantaneously in half the time. Despite being a graphical method, whose accuracy is subject to the operator expertise, it can be expressed by the following set of equations (Gerscovich et al., 2018):

$$U'(T) = \begin{cases} T U(T) / 2 & \text{if } T \leq T_c \\ U(T-T_c) / 2 & \text{if } T > T_c \end{cases}$$

Abstract

Classical theory of consolidation was conceived considering loads instantaneously applied. Since then, researchers have addressed this issue by suggesting graphical and/or analytical solutions to incorporate different time-depending load schemes. The simplest alternative is to assume a linearly increasing load. Another approach to predict the average degree of consolidation caused by a constant rate loading is based on instantaneous excess pore pressures during and at the end of construction. This technical note explains why and how this approach leads to substantial errors after the end of construction. A corrected solution is then proposed, based on the concept of superposition of effects. The final set of equations agree with the theoretical ones. A new simplified approximate methodology is also presented. Numerical examples using the proposed approach showed an excellent agreement with the analytical solution. The validity of this new approach was also proven by reproducing oedometer test results with a good agreement.

Keywords
Consolidation
Settlement prediction
Soft clays
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Where \( U'(T) \) is the corrected average degree of consolidation, \( U(T) \) is the corresponding value for instantaneous loading (Equation 1) and \( T_c \) is the Time Factor corresponding to the end of construction \( t_c \).

Olson (1977) developed a mathematical solution for a linearly increasing loading as an extension of Terzaghi and Fröhlich’s (1936) theory. The ramp load was discretized into small instantaneous incremental loads. Using the principle of superposition, Olson (1977) could express the average degree of consolidation \( (U') \) both during and after construction. The solution is divided into two equations:

\[
U'(T) = \begin{cases} 
\frac{T}{T_c} \left[ 1 - \frac{1}{M} \sum_{m=0}^{\infty} \frac{2}{M^m} \left( 1 - e^{-M'T} \right) \right] & \text{for } T \leq t_c \\
1 - \frac{1}{M} \sum_{m=0}^{\infty} \frac{2}{M^m} \left( e^{M'T} - 1 \right) e^{-M'T} & \text{for } T > t_c 
\end{cases}
\]

(5)

where \( M \) and \( T \) were formerly defined in Equations 2 and 3, respectively.

Mota (1996) and Hanna et al. (2013) showed that Terzaghi’s (1943) empirical method tends to overestimate the average degree of consolidation when compared with Olson’s (1977) solution; the higher the value of \( T_c \), the higher the error. The maximum divergence occurs at \( T = T_c \). The method overestimates the average degree of consolidation by approximately 10%.

For this reason, Hanna et al. (2013) proposed a new approach to construct the consolidation curve due to a ramp load. During construction, the analytical development agrees with Olson’s (1977). After construction, a simple equation is proposed, assuming that the remaining excess pore water pressure at the end of construction \( T_c \) is an instantaneous load applied at \( T = T_c \).

This technical note reveals that Hanna et al.’s approach (2013) may lead to considerable errors in predicting the average degree of consolidation after the construction period. A corrected solution to the development is then proposed. Besides, a new simple approximate method is suggested, aiming to be more accurate than Terzaghi’s (1943).

2. Method based on instantaneous excess pore pressures

2.1 Hanna et al.’s (2013) proposition

Hanna et al. (2013) discretized the ramp load into infinitesimal increments at a rate of \( \lambda \) per unit of time. As shown in Figure 1, Hanna et al. (2013) assume that after an infinitesimal increment \( dt \), the loading is increased by \( \lambda dt \). At the end of construction, the total applied load \( (q_c) \) is \( \lambda t_c \). Each load increment results in an infinitesimal increase in pore pressure \( (\Delta u) \), which is assumed constant with depth. At the end of construction \( (t_c) \), only part of each increment of pore pressure will have been dissipated.

Each infinitesimal load increment is applied instantaneously. The dissipation of each excess pore pressure may be expressed by Terzaghi and Fröhlich’s (1936) theory. The dissipated excess pore water pressure at the end of loading is \( U(t_c - t)\lambda dt \). Considering all intervals, the average degree of consolidation at a time \( t \) is given by:

\[
U'(T_c) = \int_{t_c}^{T_c} U(t_c - t)\lambda dt = \int_{t_c}^{T_c} U(t_c - t)dt
\]

(6)

Due to loading linearity (Figure 1), Equation 6 can also be expressed as:

\[
U'(T_c) = \int_{t_c}^{T_c} U(t_c - t)dt = \int_{t_c}^{T_c} U(t_c - t)dt = \int_{t_c}^{T_c} \left[ U(T) - U(t) \right] dt = \int_{t_c}^{T_c} \frac{1}{T_c} \sum_{m=0}^{\infty} \frac{2}{M^m} \left( 1 - e^{-M'T} \right) \]

(7)

By taking \( T = T_c \), Hanna et al. (2013) extended Equation 7 for any time during construction. Thus, the final equation is given by:

\[
U'(T \leq T_c) = \int_{t_c}^{T_c} U(T) - U(t) dt = \int_{t_c}^{T_c} \left[ U(T) - U(t) \right] dt = \int_{t_c}^{T_c} \left[ U(T) - U(t) \right] dt = \int_{t_c}^{T_c} \left[ U(T) - U(t) \right] dt = \int_{t_c}^{T_c} \frac{1}{T_c} \sum_{m=0}^{\infty} \frac{2}{M^m} \left( 1 - e^{-M'T} \right) \]

(8)

which agrees with Olson’s (1977) solution during construction (Equation 5).

After the end of construction, Hanna et al. (2013) proposed another methodology. The remaining excess pore pressure \( \bar{u} \) at the end of construction \( T = T_c \) is expressed as a fraction of the final load by:
of pore pressure is associated with a different value of the average degree of consolidation. The dissipation of the remaining excess pore pressure at $T = T_c$ is certainly slower than if it was applied instantaneously at that moment.

### 2.2 Correcting post-construction consolidation prediction

Hanna et al. (2013)’s procedure may be corrected by the simple application of superposition principle. As shown in Figure 3, the first step consists in a loading extrapolation beyond the end of construction ($t' ≥ t_c$) to $q'$. Then, the average degree of consolidation is computed by subtracting from $U'(t')$ the corresponding value $U_2(t')$ due to the excess load; i.e.:

$$U'(t') = U_1(t') - U_2(t')$$

(11)

The first term $U_1(t')$ comprises all the real and the virtual infinitesimal load increments. The fraction of excess pore water pressure that is dissipated at time $t' ≥ t_c$ is $U_3(t' - t_c - t)dt$. Thus, for all time intervals, the average degree of consolidation at time $t'$ is given by:

$$U_1(t') = \int_{t_c}^{t'} \frac{1}{\lambda_c} \frac{dt}{t_c} U(t)dt$$

(12)

The second term contains only the virtual load increments. The average degree of consolidation $U_2(t')$ is determined similarly by shifting the origin of the Cartesian axis. The fraction of excess pore water pressure that is dissipated at time $t' ≥ t_c$ is $U_3(t' - t_c - t)dt$; so:

$$U_2(t') = \int_{t_c}^{t'} \frac{1}{\lambda_c} \frac{dt}{t_c} U(t)dt$$

(13)

Thus, the average degree of consolidation at any time after the end of loading is given by:

![Figure 2. Predictions of the average degree of consolidation due to the construction of an embankment on a 4 m thick, single-drained clay deposit.](image)

![Figure 3. Calculation scheme for $t > t_c$.](image)
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\[ U'(t) = \frac{1}{t_c} \int_0^t \left[ \frac{U(t) - c(t) - u(t)}{t_c} \right] \, dt = \frac{1}{t_c} \int_0^t U(t) \, dt \] (14)

Finally, the average degree of consolidation at any time after the end of construction is expressed by:

\[ U'(T \geq T_c) = 1 - \frac{1}{T_c} \sum_{n=0}^{\infty} \frac{2}{n!} \left[ e^{-n(t-t_c)} - e^{-nT} \right] \] (15)

This equation is analogous to Olson’s (1977) solution (Equation 5) for \( t \geq t_c \).

3. A new simple approach to predict the average degree of consolidation

Alternatively, a new simple procedure is proposed to calculate the average degree of consolidation. Its goal is to be as simple and accurate as possible.

3.1 During construction

An approximate solution for the definite integral in Equation 7 can be obtained by numerical integration methods, such as Simpson’s rule (Davis and Rabinowitz, 1984). Given three points, Simpson’s rule approximates the integrand into a quadratic function.

Applying Simpson’s rule on Equation 7, the approximated average degree of consolidation can be expressed by the function values at the lower limit, midpoint, and upper limit:

\[ \int_{t_c}^{t} U(t) \, dt \approx \frac{t_c - T_c}{6} \left[ U(T_c) + 4U \left( \frac{T_c + T}{2} \right) + U(T) \right] \] (16)

The loading process initiates at \( t_c = 0 \). Thus, at any Time Factor \( T \) during construction, the approximate value of the average degree of consolidation is given by:

\[ U'(T \leq T_c) = \frac{1}{T_c} \int_0^T U(T) \, dT = \frac{T}{T_c} \left[ U(0) + 4U \left( \frac{T_c}{2} \right) + U(T) \right] \] (17)

It is worth noting that Equation 17 incorporates an error that increases with the decrease in the rate of loading. If \( T_c \) tends to infinity, the average degree of consolidation at the end of construction should be 100%, since consolidation and loading would occur simultaneously. However, Equation 17 gives \( U'(T_c) = 5/6(83.3\%) \), since \( U(0) = 0 \).

To overcome this inherent error, a slight adjustment on the first term in Equation 16 is recommended, as shown in Equation 18. This correction improves the accuracy of Equation 17 and it has no significant influence on predicting the average degree of consolidation for any speed of construction.

\[ U'(T \leq T_c) = \frac{T}{T_c} \left[ U \left( \frac{T}{2} \right) + 4U \left( \frac{T}{2} \right) + U(T) \right] \] (18)

It is worth noting that Equation 18 is close to Terzaghi’s (1943) graphical method for \( T \leq T_c \) (Equation 4), but it provides average degrees of consolidation that are always lower. As shown in Figure 4, the higher the value of \( T_c \), the higher the difference between Terzaghi’s (1943) curve and both Olson’s (1977) and the method herein proposed.

The error of the approximate propositions relative to Olson’s (1977) theoretical solution can be expressed by:

\[ \text{Relative error} = \frac{\Delta U'}{U_{\text{theoretical}}} = \frac{U'_{\text{approximate}} - U'_{\text{theoretical}}}{U'_{\text{theoretical}}} \] (19)

Figure 5 compares the relative error of the two approximate methods. The relative error of Terzaghi’s (1943) method reaches magnitudes that cannot be neglected, exceeding 10% for Time Factors \( T_c \) higher than 1.0. The proposed approach is less sensitive to the construction duration, with an acceptable error close to 1%.

Figure 4. Influence of the construction period on the average degree of consolidation prediction at the end of construction \( U'(T_c) \) for ramp loads.

Figure 5. Comparison of the relative error of the two approximate methods.
After nine months, at the end of construction ($T_c = 0.0938$), one has:

$$U'(0.0938) = \frac{0.0938}{0.0625} = 14.9\%$$

And after two years ($T = 0.25$), one has:

$$U'(0.25) = \frac{0.25}{0.0938} = 26.2\%$$

Olson’s (1977) exact solution for $T_c = 0.0938$ gives $U'(0.0625) = 12.5\%$, $U'(0.0938) = 23.0\%$ and $U'(0.25) = 50.7\%$.

### 4. Predicting laboratory test results

The accuracy of the proposed method was also verified in its ability to predict the experimental oedometer test curves. Sivakugan et al. (2014) carried out laboratory oedometer tests with ramp loading on an artificially mixed kaolinite/sand blend. The ramp loading was performed by filling a bucket, located on the loading arm, with sand scoops over varying periods.

Figure 6 shows the oedometer test data for a 2 hours loading test. The specimen was $18.241$ mm thick, and the coefficient of consolidation was $c_v = 0.6$ m$^2$/year, determined by conventional oedometer tests on the same soil with instantaneous loading. Total stress increase was $215.1$ kPa, and settlement at the end of loading was $p_c = 0.22$ mm. Experimental normalized settlement was defined as the ratio of vertical displacement $p$ to final settlement $p_c$. Estimated normalized settlement, based on the estimations from the distinct propositions shown in previous sections, was calculated as the ratio of estimated average degree of consolidation $U'$ to $U'_c$ at the end of loading ($T_c = 1.6$). It worth noting that the average degree of consolidation $U'_c$ is around $80\%$ for both methods.

There is a reasonable agreement, despite the slight difference between the experimental and numerical results. Olson’s (1977) theory and the proposed method only include primary consolidation. If the specimen develops secondary consolidation, the final settlement is higher than the primary compression value. As a result, the experimental normalized settlement becomes lower than the predicted ones. The maximum difference is $5.25\%$, at $T/T_c = 0.45$.

Mota (1996) performed laboratory tests with ramp loading on $2$ cm thick specimens of very soft clay from Barra da Tijuca, Rio de Janeiro, Brazil. Soil characterization revealed natural water content ranging from $132\%$ to $626\%$. Liquid and plastic limits range from $64\%$ to $488\%$ and $36\%$ to $214\%$, respectively. The ramp loading test was performed...
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Figure 6. Predicted and measured normalized settlements versus normalized time.

Figure 7 compares the experimental data with Olson’s (1977) theoretical solution and the proposed method for a 2 hours loading test ($T_c = 0.63$) and a total stress increase of 100 kPa. The analytical and proposed methods agreed with the experimental results, although small deviations are observed after approximately 60% of primary consolidation. The differences are attributed mainly due to secondary consolidation. At a Time Factor of 2.5, the experimental curve reaches an average degree of consolidation of 106%.

5. Conclusions

This technical note revisited some approximate methods for predicting consolidation settlements due to ramp loading. Terzaghi’s method (1943) has shown to be accurate only for small values of $T_c$ values ($T_c < 0.2$). Hanna et al.’s (2013) approach provides exact results during construction, but it leads to significant errors after the end of construction.

Two procedures have been proposed herein to overcome these issues. The first one improved Hanna et al.’s (2013) approach by combining the applied load discretization with the concept of superposition effects. The correction solved the inaccuracies. The new set of equations was identical to Olson’s (1977) solution for any construction time.

Finally, a new approximate method was developed. Simple and easy to apply, it revealed to be much more accurate than Terzaghi’s (1943) method when compared to Olson’s (1977) solution. Numerical examples have shown that the difference between the proposed method and Olson’s (1977) theory is negligible for the whole time range.

The approximate method was also validated by very good reproductions of oedometer tests results in clayey soils subjected to ramp loading. The differences were mainly due to secondary consolidation.

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Declaration of interest

The authors declare no conflict of interest.

Author’s contributions

Raphael F. Carneiro: conceptualization, data curation, formal analysis, investigation, methodology, validation, writing – original draft. Denise M. S. Gerscovitch: investigation, validation, visualization, writing – review & editing. Bernadete R. Danziger: validation, writing – review & editing.
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List of symbols

\[ H_d \quad \text{maximum length of drainage path}; \]
\[ M \quad \text{count parameter}; \]
\[ T \quad \text{Time factor}; \]
\[ T' \quad \text{Time factor at time } t'; \]
\[ T_c \quad \text{Time factor at the end of construction}; \]
\[ T^* \quad \text{a Time Factor such that } U(T^*) = U'(T_c); \]
\[ U \quad \text{average degree of consolidation (instantaneous loading)}; \]
\[ U' \quad \text{average degree of consolidation (non-instantaneous loading)}; \]
\[ U_c' \quad \text{average degree of consolidation at the end of loading}; \]
\[ c_v \quad \text{coefficient of consolidation}; \]
\[ d t \quad \text{time increment}; \]
\[ m_v \quad \text{coefficient of volume change}; \]
\[ q_c \quad \text{total load}; \]
\[ t \quad \text{time}; \]
\[ t_a \quad \text{lower integral limit}; \]
\[ t_b \quad \text{upper integral limit}; \]
\[ t_c \quad \text{time at the end of construction}; \]
\[ t' \quad \text{any time after the end of construction}; \]
\[ u_e \quad \text{remaining excess pore pressure at the end of construction}; \]
\[ \lambda \quad \text{rate of loading}; \]
\[ \rho \quad \text{settlement}; \]
\[ \rho_c \quad \text{settlement at the end of loading}. \]