Weighted Erdős-Burgess and Davenport constant in commutative rings

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Abstract

Let $R$ be a finite commutative unitary ring. An idempotent in $R$ is an element $e \in R$ with $e^2 = e$. Let $\Psi$ be a subgroup of the group $\text{Aut}(R)$ of all automorphisms of $R$. The $\Psi$–weighted Erdős-Burgess constant $I_{\psi}(R)$ is defined as the smallest positive integer $\ell$ such that every sequence over $R$ of length at least $\ell$ must contain a nonempty subsequence $a_1, \ldots, a_r$ such that $\prod_{i=1}^{r} \psi_i(a_i)$ is one idempotent of $R$ where $\psi_1, \ldots, \psi_r \in \Psi$. In this paper, for the finite quotient ring of a Dedekind domain $R$, a connection is established between the $\Psi$–weighted-Erdős-Burgess constant of $R$ and the $\Psi$–weighted Davenport constant of its group of units by all the prime ideals of $R$.

Key Words: Weighted Erdős-Burgess constant; Weighted Davenport constant; Zero-sum; Principal ideal rings; Dedekind domains

1 Introduction

The additive properties of sequences in finite abelian groups have been widely studied (see [8] for a survey), since K. Rogers [19] in 1963 pioneered the investigation of a combinatorial invariant associated with an arbitrary finite abelian group $G$, here denoted by $D(G)$ and called
the Davenport constant of $G$, which can be defined as the smallest $\ell \in \mathbb{N}$ such that every sequence $T$ of terms from the group $G$ of length at least $\ell$ contains a nonempty subsequence $T'$ the sum of whose terms is equal to the identity element of the group $G$. Later on, H. Davenport popularized this invariant in the study of algebraic number theory (as reported in [18]), which may be the reason why this invariant was named after Davenport instead of its pioneer, Rogers. The invariant $D(G)$ has been investigated extensively in the past almost 60 years, and also been generalized from different aspects in connection with Factorization Theory in Algebra or binary quadratic forms, etc (see [6, 13, 20, 21]). Adhikari, Chen et.al [1, 2, 4] defined the Davenport constant with integers weights, which motivates a huge amount of researches (see [3, 12, 16, 17, 22, 31], and [11] Chapter 16). Zeng and Yuan [32] further defined the Davenport constant with homomorphisms weights as follows.

**Definition A.** [32] Let $G^*$, $G$ be finite abelian groups written additively with a nontrivial homomorphism group Hom($G^*$, $G$). For a nonempty subset $\Psi$ of Hom($G^*$, $G$), let $D_\Psi(G)$ be the least positive integer $\ell$ such that any sequence over $G^*$ of length least $\ell$ must contain some terms $a_1, \ldots, a_r$ ($r \geq 1$) such that $\sum_{i=1}^r \psi_i(a_i) = 0_G$ for some $\psi_1, \ldots, \psi_r \in \Psi$.

Recently, the Davenport constant together with some other additive properties of sequences were investigated in the setting of semigroups and rings (see [7, 23–30] for example.) Among them, one of the following combinatorial invariant on idempotents of semigroups was proposed due to a question by P. Erdős (see [5, 10]).

**Definition B.** ([25], Definition 4.1) For a commutative semigroup $S$, define the **Erdős-Burgess constant** of $S$, denoted by $I(S)$, to be the least $\ell \in \mathbb{N} \cup \{\infty\}$ such that every sequence $T$ of terms from $S$ and of length $\ell$ must contain one or more terms whose product is an idempotent.

For a finite abelian group $G$, since the identity is the unique idempotent, then the Erdős-Burgess constant of $G$ reduces to the Davenport constant. That is, the Erdős-Burgess constant is the natural generalization of Davenport constant into semigroups. While, the most important class of commutative semigroups are the multiplicative semigroups of commutative rings. On commutative rings, the author [28] showed that the Erdős-Burgess constant exists only for **finite** commutative rings except for a family of infinite commutative rings with a given very
special form. That is, to study this invariant in the realm of commutative rings, we may consider it only for \textit{finite} commutative rings. The upper bound for a general finite semigroup can be given by the Gillam-Hall-Williams Theorem \cite{10}. Recently, the lower bound of Erdős-Burgess constant was obtained for some classical finite commutative principal ideal rings (see \cite{29}), and more generally, for the finite quotient rings of a Dedekind domain which can be seen as below.

\textbf{Theorem C.} \cite{15} Let $D$ be a Dedekind domain and $K$ a nonzero proper ideal of $D$ such that $R = D/K$ is a finite ring. Then $I(S_R) \geq D(U(R)) + \Omega(K) - \omega(K)$, where $\Omega(K)$ is the number of the prime ideals (repetitions are counted) and $\omega(K)$ the number of distinct prime ideals in the factorization when $K$ is factored into a product of prime ideals. Moreover, equality holds for the case when $K$ is factored into either a power of some prime ideal or a product of some pairwise distinct prime ideals.

- For any commutative unitary ring $R$, let $U(R)$ be the group of units and $S_R$ the multiplicative semigroup of the ring $R$.

The author obtained a sharp lower bound for a general finite commutative ring which generalized Theorem C.

\textbf{Theorem D.} \cite{26} Let $R$ be a finite commutative ring with identity. Then

$$I(S_R) \geq D(U(R)) + \sum_M (\text{ind}(M) - 1)$$

where $M$ is taken over all distinct prime ideals of $R$ and $\text{ind}(M)$ is the least integer $t > 0$ such that $M^t = M^{t+1}$. Moreover, equality holds if $R$ is a local ring or all prime ideals of $R$ have the indices one.

The above Theorem D established a connection between the Erdős-Burgess constant of a ring $R$ and the Davenport constant of its group of units by the indices of all prime ideals of $R$. Motivated by the study of Davenport constant with weights, it would be interesting to consider the following question:

‘For a finite commutative ring with identity $R$, does the above relation between the Erdős-Burgess constant $R$ and the Davenport constant of its group of units still hold under given automorphisms of $R$?’
To consider this question, we need to generalize the Erdős-Burgess with weights in the setting of commutative semigroups, for which the definition is as follows.

**Definition 1.1.** Let $S^*$, $S$ be commutative semigroups, and let $\text{hom}(S^*, S)$ be the set of all homomorphisms from $S^*$ to $S$. Let $\Psi$ be a nonempty subset of $\text{hom}(S^*, S)$. Define the $\Psi$-weighted-Erdős-Burgess constant of $S$, denoted $I_\Psi(S)$, to be the least $\ell \in \mathbb{N} \cup \{\infty\}$, such that every sequence over $S^*$ of length at least $\ell$ must contain some terms $a_1, \ldots, a_r$ ($r \geq 1$) such that $\prod_{i=1}^r \psi_i(a_i)$ is one idempotent of $S$ where $\psi_1, \ldots, \psi_r \in \Psi$.

Let $R$ be a finite commutative ring with identity. Let $\Psi$ be a subgroup of $\text{Aut}(R)$. In this paper, we established a connection between the $\Psi$-weighted-Erdős-Burgess constant of the multiplicative semigroup $S_R$ and the $\Psi$-weighted Davenport constant of its group of units by a function of indices of all the prime ideals of $R$. Our main Theorems are stated as Theorems 3.3 and 3.6 which generalizes Theorem C.

### 2 Notations

For integers $a, b \in \mathbb{Z}$, we set $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$. For a real number $x$, we denote by $\lfloor x \rfloor$ the largest integer that is less than or equal to $x$, and by $\lceil x \rceil$ the smallest integer that is greater than or equal to $x$.

Let $R$ be a finite commutative ring with identity $1_R$. The addition and multiplication of $R$ will be denoted as $+$ and $*$ respectively. We use idempotent to mean an element $e \in R$ such that $e * e = e$. Let $\text{U}(R)$ be the group of all units of $R$. The set $\text{Aut}(R)$ of all ring automorphisms $R \to R$ forms a group under the operation of composition of functions. Let $\text{Spec } R$ be the spectrum of $R$, i.e., the set of all prime ideals. Let $\Psi$ be a subgroup of the group $\text{Aut}(R)$. Then the group $\Psi$ acts on the set $\text{Spec } R$ given by: $\psi P = \psi(P) \in \text{Spec } R$ where $\psi \in \Psi$ and $P \in \text{Spec } R$. For any $P \in \text{Spec } R$, let

$$\mathcal{O}_P = \{Q \in \text{Spec } R : Q = \psi P \text{ for some } \psi \in \Psi\}$$
be the orbit of $P$ under the action of $\Psi$ on $\text{Spec } R$. Note that

$$ |\mathcal{O}_P| = \frac{|\Psi|}{|\text{St}(P)|} $$

where

$$ \text{St}(P) = \{ \psi \in \Psi : \psi(P) = P \} $$

is the stabilizer of $P$.

We also need to introduce notation and terminologies on sequences over rings and follow the notation of A. Geroldinger, D.J. Grynkiewicz and others used for sequences over groups (cf. [11], Chapter 10] or [9], Chapter 5]). For any nonempty subset $A$ of the ring $R$ (usually $A$ is taken to be $R$ or $U(R)$ in this paper), let $\mathcal{F}(A)$ be the free commutative monoid, multiplicatively written, with basis $A$. We denote multiplication in $\mathcal{F}(A)$ by the boldsymbol $\cdot$ and we use brackets for all exponentiation in $\mathcal{F}(A)$. By $T \in \mathcal{F}(A)$, we mean $T$ is a sequence of terms from $A$ which is unordered, repetition of terms allowed. Say

$$ T = a_1 a_2 \cdots a_\ell $$

where $a_i \in A$ for $i \in [1, \ell]$. The sequence $T$ can be also denoted as $T = \prod_{a \in A} a^{|v_a(T)|}$, where $v_a(T)$ is a nonnegative integer and means that the element $a$ occurs $v_a(T)$ times in the sequence $T$. By $|T|$ we denote the length of the sequence, i.e., $|T| = \sum_{a \in A} v_a(T) = \ell$. By $\varepsilon$ we denote the empty sequence in $\mathcal{F}(A)$ with $|\varepsilon| = 0$. We call $T'$ a subsequence of $T$ if $v_a(T') \leq v_a(T)$ for each element $a \in A$, denoted by $T' \mid T$, moreover, we write $T'' = T \cdot T'^{-1}$ to mean the unique subsequence of $T$ with $T' \cdot T'' = T$. We call $T'$ a proper subsequence of $T$ provided that $T' \mid T$ and $T' \neq T$. In particular, $\varepsilon$ is a proper subsequence of every nonempty sequence. We call $T$ a $\Psi$-idempotent-product sequence provided that there exists $\psi_1, \ldots, \psi_\ell \in \Psi$ (not necessarily distinct) such that $\prod_{i=1}^\ell \psi_i(a_i)$ is an idempotent of $R$. We call $T$ a $\Psi$-idempotent-product-free sequence provided that $T$ contains no nonempty subsequence which is $\Psi$-idempotent-product.

Note that the restriction $\psi \mid U(R)$ is an automorphism of the group $U(R)$. For convenience, in what follows we still use $\psi$ instead of $\psi \mid U(R)$ to denote the automorphism of the group $U(R)$. Then it does make a sense to say a sequence of terms from $U(R)$ to be $\Psi$–idempotent-product or $\Psi$–idempotent-product free.
To give our main result, we still need the following notation. For any pair of positive integers \((m, h)\), let
\[
T(m; h) = \max \sum_{i=1}^{h} t_i, \quad (t_i \geq 0)
\]
where
\[
\sum_{i=1}^{d} i t_i < dm \quad \text{for all} \quad d = 1, 2, \ldots, h.
\] (1)
In particular, \(T(1, h) = 0\) for any positive integer \(h\).

3 Results

We start with the following easy proposition.

**Proposition 3.1.** For any integer \(h \geq 1\), there exist \(t_1, \ldots, t_h \geq 0\) such that \(\sum_{i=1}^{d} i t_i = T(m, h)\) and (1) holds, where \(t_1, \ldots, t_h\) satisfy the following recurrence relation:

(i) \(t_1 = m - 1\);

(ii) \(t_d = \left\lfloor \frac{(dm - 1) - \sum_{i=1}^{d-1} i t_i}{d} \right\rfloor\) for all \(d = 2, \ldots, h\).

**Lemma 3.2.** Let \(R\) be a finite commutative ring with identity. Let \(\Psi\) be a subgroup of the group \(\text{Aut}(R)\). Suppose that for every prime ideal \(P\) with \(\text{Ind}(P) > 1\), there exists some element \(c_p \in P\) such that \((c_p) + P^{\text{Ind}(P)} = P\). Then \(I_{\Psi}(R) \geq D_{\Psi}(U(R)) + \sum_{P \in \text{Spec}(R)} \frac{T(\text{Ind}(P))}{\frac{\text{Ind}(P)}{\text{St}(P)}}\).

**Proof.** Denoted
\[
G_P = \{ c \in P : (c) + P^{\text{Ind}(P)} = P \},
\] (2)
by the hypothesis of the lemma, we have
\[
G_P \neq \emptyset.
\] (3)

**Claim A.** Let \(P \in \text{Spec} R\) and \(\psi \in \Psi\). Then,

(i) \(\psi(P)^t = \psi(P^t)\) for all \(t \geq 0\), and in particular, \(\text{Ind}(\psi(P)) = \text{Ind}(P)\);
(ii) \( \psi(G_p) = G_{\psi}(P) \).

**Proof of Claim A.**

(i) Trivial.

(ii) Take an arbitrary element \( a \in G_p \). By (2) and Conclusion (i), we have that \( (\psi(a)) + \psi(P)\text{Ind}(\psi(P)) = (\psi(a)) + \psi(P)\text{Ind}(P) = \psi((a)) + \psi(P)\text{Ind}(P) = \psi(P) \). Since \( \psi(a) \in \psi(G_p) \subset \psi(P) \), it follows that \( \psi(a) \in G_{\psi}(P) \). By the arbitrariness of choosing \( a \), we have \( \psi(G_p) \subset G_{\psi}(P) \). Since \( \psi^{-1} \in \Psi \), it follows that \( G_{\psi}(P) = \psi(\psi^{-1}(G_{\psi}(P))) \subset \psi(G_p) \), and thus, \( \psi(G_p) = G_{\psi}(P) \). \( \square \)

**Claim B.** Let \( P \in \text{Spec } R \) be such that \( \text{Ind}(P) > 1 \). Let \( \ell \in [1, \text{Ind}(P) - 1] \) and \( a_1, \ldots, a_\ell \in G_p \) (not necessarily distinct). Then \( \prod_{i=1}^{\ell} a_i \notin P\text{Ind}(P) \).

**Proof of Claim B.** It suffices to consider the case that \( \ell = \text{Ind}(P) - 1 \). Since \( P\text{Ind}(P) \subsetneq P\text{Ind}(P) - 1 \), we can take some element \( x \) of \( P\text{Ind}(P) - 1 \setminus P\text{Ind}(P) \). Since \( x \) is a finite sum of products of the form \( b_1 \cdot b_2 \cdot \ldots \cdot b_{\text{Ind}(P)-1} \) where \( b_1, b_2, \ldots, b_{\text{Ind}(P)-1} \in P \), it follows that

\[
\{ T \in \mathcal{F}(P) : |T| = \text{Ind}(P) - 1 \text{ and } \pi(T) \in P\text{Ind}(P) - 1 \setminus P\text{Ind}(P) \neq \emptyset. \tag{4}\]

Then we take a sequence \( y_1 \cdots y_{\text{Ind}(P)-1} \in \mathcal{F}(P) \) in the set given as (4) such that the number of common terms of sequences \( y_1 \cdots y_{\text{Ind}(P)-1} \) and \( a_1 \cdots a_{\text{Ind}(P)-1} \) is maximal, say

\[
y_i = a_i \text{ for each } i \in [1, s] \tag{5}\]

(with \( s \in [0, \text{Ind}(P) - 1] \) being maximal), moreover, both sequences \( y_{s+1} \cdots y_{\text{Ind}(P)-1} \) and \( a_{s+1} \cdots a_{\text{Ind}(P)-1} \) have no common terms. To prove Claim B, we need only to show that \( s = \text{Ind}(P) - 1 \). Assume to the contrary that

\[
s < \text{Ind}(P) - 1.
\]

Let \( z = \prod_{i \in [1, \text{Ind}(P)-1] \setminus [s+1]} y_i \). Since \( y_{s+1} \ast z = y_{s+1} \ast \prod_{i \in [1, \text{Ind}(P)-1] \setminus [s+1]} y_i = \prod_{i \in [1, \text{Ind}(P)-1]} y_i \notin P\text{Ind}(P) \), it follows that \( y_{s+1} \in P \setminus (P\text{Ind}(P) : z) \), equivalently,

\[
(P\text{Ind}(P) : z) \cap P \subseteq P. \tag{6}\]
Since \( a_{s+1} \in G_p \), then \( (a_{s+1} + p^{\text{Ind}(P)}) = P \). Combined with (6), we conclude that \( a_{s+1} \notin (p^{\text{Ind}(P)} : z) \). It follows from (5) that
\[
\left( \prod_{i \in \{1, s+1\}} a_i \right) \ast \left( \prod_{i \in \{s+2, \text{Ind}(P) - 1\}} y_i \right) = a_{s+1} \ast \left( \prod_{i \in \{1, s\}} a_i \right) \ast \left( \prod_{i \in \{s+2, \text{Ind}(P) - 1\}} y_i \right) =
\left( \prod_{i \in \{1, s+1\}} a_i \right) \ast \left( \prod_{i \in \{s+2, \text{Ind}(P) - 1\}} y_i \right) \in p^{\text{Ind}(P) - 1},
\]
and so
\[
\left( \prod_{i \in \{1, s+1\}} a_i \right) \ast \left( \prod_{i \in \{s+2, \text{Ind}(P) - 1\}} y_i \right) \in p^{\text{Ind}(P) - 1} \setminus p^{\text{Ind}(P)} \quad \text{which implies that the sequence}
\]
\[
a_1 \cdot \ldots \cdot a_{s+1} \cdot y_{s+2} \cdot \ldots \cdot y_{\text{Ind}(P) - 1} \in \{ T \in F(P) : |T| = \text{Ind}(P) - 1 \} \quad \text{and} \quad \pi(T) \in p^{\text{Ind}(P) - 1} \setminus p^{\text{Ind}(P)}.
\]
Then we derive a contradiction with the choosing of the sequence \( y_1 \cdot \ldots \cdot y_{\text{Ind}(P) - 1} \). This proves Claim B.

Let \( \mathcal{O} \) be an arbitrary orbit of \( \text{Spec}R \) under the action of \( \Psi \). Say,
\[
\mathcal{O} = \{ P_1, P_2, \ldots, P_h \} \quad \text{where} \quad h = \left\lfloor \frac{|\Psi|}{|\text{St}(P_1)|} \right\rfloor. \tag{7}
\]
For any nonempty subset \( X \subset [1, h] \), we define
\[
H_{\mathcal{O}, X} = \left\{ a \in R : a \equiv 1_R \pmod{Q^{\text{Ind}(Q)}} \right\} \cap \left( \bigcap_{i \in X} G_{P_i} \right). \tag{8}
\]
For any \( t \in [1, h] \), let
\[
\mathcal{H}_{\mathcal{O}, t} = \bigcup_{X \subseteq [1, t]} H_{\mathcal{O}, X}. \tag{9}
\]

In the following, we shall give three claims on the properties of \( H_{\mathcal{O}, X}, \mathcal{H}_{\mathcal{O}, t} \) given as above.

**Claim C.** For any element \( b \in H_{\mathcal{O}, X} \), we have \( \{ i \in [1, h] : b \in P_i \} = X \).

**Proof of Claim C.** By (2) and (8), we see that \( b \in P_i \) for each \( i \in X \). For any \( Q \in \text{Spec}R \setminus \{ P_i : i \in X \} \), since \( b \equiv 1_R \pmod{Q^{\text{Ind}(Q)}} \), then \( b \equiv 1_R \pmod{Q} \) and so \( b \notin Q \), done. \hfill \Box

**Claim D.** \( H_{\mathcal{O}, X} \neq \emptyset \) for any nonempty set \( X \subset [1, h] \). In particular, \( \mathcal{H}_{\mathcal{O}, t} \neq \emptyset \) for each \( t \in [1, h] \).

**Proof of Claim D.** By (3), we take an element
\[
c_i \in G_{P_i} \quad \text{for each} \quad i \in X. \tag{10}
\]
Note that \( \{P_i^{\text{Ind}(P)} : P \in \text{Spec}R\} \) is a family of ideals which are pairwise coprime. By the Chinese Remainder Theorem, we can find one element \( a \) such that

\[
a \equiv c_i \pmod{P_i^{\text{Ind}(P)}} \quad \text{for each } i \in X \tag{11}
\]

and

\[
a \equiv 1_R \pmod{Q^{\text{Ind}(Q)}} \quad \text{for each } Q \in \text{Spec}R \setminus \{P_i : i \in X\}. \tag{12}
\]

To derive \( H_{\tilde{\mathcal{O}};X} \neq \emptyset \), by \( \text{(2)} \) we need only to show that \( a \in \bigcap_{i \in X} G_{P_i} \).

Let \( i \in X \). It follows from \( \text{(11)} \) that

\[
c_i = a + v_i \quad \text{for some } v_i \in P_i^{\text{Ind}(P_i)}. \tag{13}
\]

Since \( G_{P_i} \subset P_i \), it follows from \( \text{(10)} \) and \( \text{(13)} \) that \( a \in P_i \). Then it suffices to show that \( (a) + P_i^{\text{Ind}(P_i)} = P_i \). Take an arbitrary element \( b \in P_i \). By \( \text{(2)} \) and \( \text{(10)} \), we derive that \( b = r_b c_i + u_b \) where \( r_b \in R \) and \( u_b \in P_i^{\text{Ind}(P_i)} \). Combined with \( \text{(13)} \), we have that \( b = r_b(a + v_i) + u_b = ra + (r_b v_i + u_b) \in (a) + P_i^{\text{Ind}(P_i)} \). By the arbitrariness of choosing \( b \), we proved \( (a) + P_i^{\text{Ind}(P_i)} = P_i \) and so \( H_{\tilde{\mathcal{O}};X} \neq \emptyset \). Then \( \mathcal{H}_{\tilde{\mathcal{O}};X} \neq \emptyset \) follows from \( \text{(9)} \) trivially. \( \square \)

**Claim E.** For any \( t \in [1, h] \) and any \( \psi \in \Psi \), we have \( \psi(\mathcal{H}_{\tilde{\mathcal{O}};X}) = \mathcal{H}_{\tilde{\mathcal{O}};X} \).

**Proof of Claim E.** Take an arbitrary element \( a \in \mathcal{H}_{\tilde{\mathcal{O}};X} \), equivalently, \( a \in H_{\tilde{\mathcal{O}};X} \) for some \( X \subset [1, h] \) with \( |X| = t \). Since \( \psi \) acts on \( \text{Spec}R \), it follows from \( \text{(7)} \) that there exists \( X' \subset [1, h] \) of cardinality \( |X'| = |X| = t \) such that

\[
\{\psi(P_i) : i \in X\} = \{P_j : j \in X'\}, \tag{14}
\]

and follows from Claim A (i) that

\[
\{\psi(Q)^{\text{Ind}(\psi(Q))} : Q \in \text{Spec}R \setminus \{P_i : i \in X\}\} = \{Q'^{\text{Ind}(Q')} : Q' \in \text{Spec}R \setminus \{P_j : j \in X'\}\}. \tag{15}
\]

By \( \text{(3)} \) and Claim A (i), for all \( Q \in \text{Spec}R \setminus \{P_i : i \in X\} \), we have that \( \psi(a) - 1_R = \psi(a) - \psi(1_R) = \psi(a - 1_R) \in \psi(Q)^{\text{Ind}(Q)} = \psi(Q')^{\text{Ind}(Q')} = \psi(Q)^{\text{Ind}(\psi(Q))} \), i.e., \( \psi(a) \equiv 1_R \pmod{\psi(Q)^{\text{Ind}(\psi(Q))}} \). Combined with \( \text{(15)} \), we conclude that

\[
\psi(a) \equiv 1_R \pmod{Q'^{\text{Ind}(Q')}} \quad \text{for all } Q' \in \text{Spec}R \setminus \{P_j : j \in X'\}. \]
Since \( a \in \bigcap_{i \in X} G^i \), it follows from (14) and Claim B that \( \psi(a) \in \psi(\bigcap_{i \in X} G^i) = \bigcap_{j \in X'} G^j \). Then we conclude that \( \psi(a) \in H_{\Theta; X'} \subset H_{\Theta; X} \), completing the proof of Claim D. \( \square \)

Let \( \Theta \) be the orbit given as (7). By \( B_{\Theta} \) we denote the sequence associated with the orbit \( \Theta \) which are given as below. By Claim D, we choose an element
\[
b_i \in H_{\Theta; j} \text{ for each } t \in [1, h].
\] (16)

Let \( d_1, d_2, \ldots, d_h \) be positive integers such that
\[
\sum_{i=1}^{h} d_i = T(\text{Ind}(P_1); h)
\] (17)
and
\[
\sum_{i=1}^{n} id_i < n \text{ Ind}(P_1) \text{ for all } n = 1, 2, \ldots, h.
\] (18)

Then we set
\[
B_{\Theta} = b_1^{d_1} \cdot \ldots \cdot b_h^{d_h}.
\] (19)

One thing worth remarking is that all the quantities \( h, d_1, d_2, \ldots, d_h \) in (19) varies with \( \Theta \) taking distinct orbits, and thus, the length \( |B_{\Theta}| \) varies accordingly. Precisely, we have the following.

**Claim F.** \( |B_{\Theta}| = \sum_{P \in \Theta} \frac{T(\text{Ind}(P); \frac{\ell}{T(\text{Ind}(P); P)})}{|P|} \).  

**Proof of Claim F.** Note that \( \text{St}(P_1) = \ldots = \text{St}(P_h) \). By Claim A, we have \( \text{Ind}(P_1) = \ldots = \text{Ind}(P_h) \). Then it follows from (7), (17) and (19) that \( |B_{\Theta}| = T(\text{Ind}(P_1); h) = \sum_{i=1}^{h} \frac{T(\text{Ind}(P_1); n)}{h} = \sum_{i=1}^{h} \frac{T(\text{Ind}(P_1); \frac{\ell}{T(\text{Ind}(P_1); P)})}{|P|} \). This proves Claim F. \( \square \)

**Claim G.** For any nonempty subsequence \( W \) of \( B_{\Theta} \), say \( W = a_1 \cdot \ldots \cdot a_\ell \), and for any \( \psi_1, \ldots, \psi_\ell \in \Psi \) (not necessarily distinct), there exists \( r \in [1, h] \) such that \( \ell_{r} \psi(a_i) \in P_r \setminus P_{r^{\text{Ind}(P_1)}} \).

**Proof of Claim G.** By (16), (19) and Claim E, we have that
\[
\psi_m(a_m) \in \bigcup_{i=1}^{h} H_{\Theta; j} \text{ for each } m \in [1, \ell].
\] (20)

Let \( n = \#\{s \in [1, h] : \ell \psi_m(a_m) \in P_s\} \). Obviously, \( 1 \leq n \leq h \). By arranging the indices of \( (P_1, \ldots, P_h) \) if necessary, we can assume without loss of generality that
\[
\{s \in [1, h] : \ell \psi_m(a_m) \in P_s\} = [1, n].
\] (21)
Then
\[ \{ s \in [1, h] : \psi_m(a_m) \in P_s \} \subset [1, n] \text{ for each } m \in [1, \ell]. \] (22)

By (20), (22) and Claim C, we derive that
\[ \psi_m(a_m) \in \bigcup_{t=1}^{n} \mathcal{H}_t \text{ for each } m \in [1, \ell]. \] (23)

By (18), (19), (22), (23), Claim C and Claim E, we conclude that
\[ \sum_{s=1}^{n} \# \{ m \in [1, \ell] : \psi_m(a_m) \in P_s \} \]
\[ = \sum_{s=1}^{n} \# \{ m \in [1, \ell] : \psi_m(a_m) \in \mathcal{H}_t \} \]
\[ = \sum_{m=1}^{\ell} \sum_{t=1}^{n} (t \times \# \{ m \in [1, \ell] : \psi_m(a_m) \in \mathcal{H}_t \}) \]
\[ \leq \sum_{t=1}^{n} td_t \]
\[ < n \text{ Ind}(P_1). \]

Combined with Claim A, we derive that there exists
\[ r \in [1, n] \] (24)
such that \( \# \{ m \in [1, \ell] : \psi_m(a_m) \in P_r \} < \text{ Ind}(P_1) = \text{ Ind}(P_r). \) Moreover, since \( P_r \) is prime, it follows from (21) that \( \# \{ m \in [1, \ell] : \psi_m(a_m) \in P_r \} \geq 1, \) and so,
\[ \# \{ m \in [1, \ell] : \psi_m(a_m) \in P_r \} \in [1, \text{ Ind}(P_r) - 1]. \]

By arranging the indices of \([1, \ell]\) if necessary, we can assume without loss of generality that there exists some
\[ u \in [1, \text{ Ind}(P_r) - 1] \] (25)
such that
\[ \left\{ \begin{array}{ll}
\psi_m(a_m) \in P_r, & \text{if } m = 1, \ldots, u; \\
\psi_m(a_m) \notin P_r, & \text{if } m = u + 1, \ldots, \ell.
\end{array} \right. \] (26)

By (8), (9), (22), (26) and Claim C, we conclude that
\[ \left\{ \begin{array}{ll}
\psi_m(a_m) \in G_{P_r}, & \text{if } m = 1, \ldots, u; \\
\psi_m(a_m) \equiv 1 \text{ (mod } P_r^{\text{Ind}(P_r)}) & \text{if } m = u + 1, \ldots, \ell.
\end{array} \right. \]
It follows that
\[ \prod_{m=1}^{\ell} \psi_m(a_m) = \left( \prod_{m=1}^{u} \psi_m(a_m) \right) \cdot \left( \prod_{m=u+1}^{\ell} \psi_m(a_m) \right) \equiv \prod_{m=1}^{u} \psi_m(a_m) \cdot 1_R = \prod_{m=1}^{u} \psi_m(a_m) \pmod{P^{\text{Ind}(P_r)}}, \]
and follows from (25) and Claim B that
\[ \prod_{m=1}^{u} \psi_m(a_m) \equiv 0 \pmod{P^{\text{Ind}(P_r)}}. \]
Therefore, it follows from (21) and (24) that \( \prod_{m=1}^{\ell} \psi_m(a_m) \in P_r \setminus P_r^{\text{Ind}(P_r)} \). This proves Claim G.

Take a \( \Psi \)-product-one free sequence \( V \) of terms from the group \( U(R) \) with length
\[ |V| = D_{\Psi}(U(R)) - 1. \] (27)
Suppose that \( O_1, \ldots, O_k \) are all distinct orbits of Spec \( R \) under the action of \( \Psi \). Let
\[ T = B_{O_1} \cdot \ldots \cdot B_{O_k} \cdot V. \] (28)

**Claim H.** The sequence \( T \) is \( \Psi \)-idempotent-product free.

**Proof of Claim H.** Suppose to the contrary that there exists a nonempty subsequence \( T' \) of \( T \), say \( T' = a_1 \cdot \ldots \cdot a_\ell \), such that \( T' \) is a \( \Psi \)-idempotent-product sequence. Since \( 1_R \) is the unique idempotent in \( U(R) \) and \( V \) is \( \Psi \)-product-one free, it follows that at least one term of the sequence \( B_{O_1} \cdot \ldots \cdot B_{O_k} \) appears in the sequence \( T' \). Then by rearranging the indices \( i \in [1, \ell] \) we may assume without loss of generality that
\[ a_1 \cdot \ldots \cdot a_s \mid B_{O_1}, \]
\[ a_{s+1} \cdot \ldots \cdot a_\ell \mid B_{O_2} \cdot \ldots \cdot B_{O_k} \cdot V, \]
where \( 1 \leq s \leq \ell \). Since \( T' \) is a \( \Psi \)-idempotent-product sequence, there exists \( \psi_1, \ldots, \psi_\ell \in \Psi \) such that \( \prod_{i=1}^{s} \psi_i(a_i) \) is an idempotent. By Claim G, there exists some prime ideal \( P \in O_1 \) such that \( \prod_{i=1}^{s} \psi_i(a_i) \in P \setminus P^{\text{Ind}(P)} \). By (8), we see that \( \psi_i(a_i) \equiv 1_R \pmod{P^{\text{Ind}(P)}} \) or \( \psi_i(a_i) \in U(R) \) according to \( a_i \mid B_{O_2} \cdot \ldots \cdot B_{O_k} \) or \( a_i \mid V \) respectively, where \( i \in [s+1, \ell] \). Then \( \prod_{i=1}^{\ell} \psi_i(a_i) \in \)
$P \setminus P^{\text{Ind}(P)}$ still holds. This is a contradiction with $\prod_{i=1}^{\ell} \psi_i(a_i)$ being idempotent, since $\prod_{i=1}^{\ell} \psi_i(a_i) = (\prod_{i=1}^{\ell} \psi_i(a_i))^{\text{Ind}(P)} \in P^{\text{Ind}(P)}$. This proves Claim H. □

By (27), (28), Claim F and Claim H, we conclude that $I_{\Psi}(S_R) \geq 1 + |T| = 1 + |V| + \sum_{i=1}^{k} |B_i| = D_{\Psi}(U(R)) + \sum_{P \in \text{Spec}(R)} T\left(\text{Ind}(P); \frac{\psi|_{\text{St}(P)}}{\text{St}(P)}\right)$. This complete the proof of the lemma. □

**Theorem 3.3.** Let $R$ be a finite commutative principal ideal ring with identity. Let $\Psi$ be a subgroup of the group $\text{Aut}(R)$. Then

$$I_{\Psi}(S_R) \geq D_{\Psi}(U(R)) + \sum_{P \in \text{Spec}(R)} T\left(\text{Ind}(P); \frac{\psi|_{\text{St}(P)}}{\text{St}(P)}\right).$$

It is not hard to check that the following proposition holds, i.e., the bound in Theorem 3.3 is best possible in general.

**Proposition 3.4.** Let $L$ be a finite commutative local P.I.R, and let $R = L \times L$. Let $\varphi : R \to R$ be an automorphism given as $\varphi : (a, b) \mapsto (b, a)$ for any $(a, b) \in R$. Let $H = \langle \varphi \rangle$ be the group of automorphisms of order two generated by $\varphi$. Then $I_{\Psi}(S_R) = D_{\Psi}(U(R)) + \sum_{P \in \text{Spec}(R)} T\left(\text{Ind}(P); \frac{\psi|_{\text{St}(P)}}{\text{St}(P)}\right)$.

It is shown [14] that the following proposition holds. For the reader’s convenience, we provide a short proof of this proposition.

**Proposition 3.5.** Any finite commutative principal ideal ring $R$ is a quotient ring some P.I.D.

**Proof.** By the fundamental theorem for Noetherian rings, we know that $R \cong R_1 \times \cdots \times R_k$ where $R_1, \ldots, R_k$ are finite commutative local rings. It is shown (see [14], Definition 9 and Corollary 11) that a finite commutative local P.I.R is a quotient ring of some P.I.D. Say $D_i$ is a P.I.D, $J_i \triangleleft D_i$ and $R_i \cong D_i/J_i$ where $i \in [1, k]$. Then $R \cong (D_1/J_1) \times \cdots \times (R_k/J_k) \cong (D_1 \times \cdots \times D_k)/(J_1 \times \cdots \times J_k)$, where $(J_1 \times \cdots \times J_k) \triangleleft (D_1 \times \cdots \times D_k)$. Note that $D_1 \times \cdots \times D_k$ is a P.I.D. The conclusion is proved. □
Since a P.I.D is a Dedekind domain, therefore, any finite P.I.R is a quotient ring of some Dedekind domain. In the following, we shall show the result of Theorem 3.3 holds true for a more general setting, i.e., for finite quotient rings of any Dedekind domain.

**Theorem 3.6.** Let $R$ be a finite quotient ring of some Dedekind domain. Let $\Psi$ be a subgroup of the group $\text{Aut}(R)$. Then $I_\Psi(S_R) \geq D_\Psi(U(R)) + \sum_{P \in \text{Spec}(R)} \frac{T(\text{Ind}(P); \Psi)}{|\Psi|} |\text{St}(P)|$.

**Proof.** Let $Q \in \text{Spec}R$ be such that $\text{Ind}(Q) > 1$. Then there exists no ideal $A$ of $R$ such that $Q^2 \subseteq A \subseteq Q$. Let $R = D/J$ where $D$ is a Dedekind domain and $J \triangleleft D$. Let $\varphi: D \to R$ be the canonical epimorphism. Then

$$J = \prod_{i=1}^t P_i$$

where $t \geq 1, \alpha_1, \ldots, \alpha_t \geq 1$, and $P_1, \ldots, P_t$ are distinct prime ideals of $D$. It follows from (29) that

$$Q = \varphi(P)$$

for some $P \in \text{Spec}D$ with

$$J \subset P.$$  (31)

Then $P = P_i$ for some $i \in [1, t]$, say

$$P = P_1.$$

**Claim I.** There exists no ideal $N \triangleleft R$ such that $Q^2 \subseteq N \subseteq Q$.

**Proof of Claim I.** Assume to the contrary that there exists some ideal $N \triangleleft R$ such that $Q^2 \subseteq N \subseteq Q$. By (30) and (31), we have that $\varphi(P^2 + J) = \varphi(P^2) \subseteq \varphi(P) \ast \varphi(P) = Q^2$, and thus, $P^2 \subseteq P^2 + J = \varphi^{-1}(\varphi(P^2 + J)) \subset \varphi^{-1}(Q^2) \subseteq \varphi^{-1}(N) \subseteq \varphi^{-1}(Q) = P$. Then we derive a contradiction, since $D$ is a Dedekind domain implying that there exists no ideal $M \triangleleft D$ such that $P^2 \subseteq M \subseteq P$. This proves Claim I.

Take an arbitrary $Q \in \text{Spec}R$ such that $\text{Ind}(Q) > 1$. Take an element $x \in Q \setminus Q^2$. Since $Q^2 \subseteq (x) + Q^2 \subseteq Q$, it follows from Claim I that $Q = (x) + Q^2$. Then we have that

$$(x) + Q^k = (x) + ((x) + Q^2)^k = (x) + \left(\sum_{i=0}^{k-1}(x)^{k-i} \ast Q^2\right) + Q^{2k} = (x) + Q^{2k} \text{ for any } k \geq 1.$$  (32)
Fix an integer \( m > \ln \text{Ind}(Q) \). It follows from (32) that 
\[
Q = (x) + Q^2 = (x) + Q^4 = \cdots = (x) + Q^{2m} = (x) + Q^{\text{Ind}(Q)}.
\]
Then the conclusion follows from Lemma 3.2 readily. \( \square \)

Then we close the paper with the following two conjectures.

**Conjecture 3.7.** Let \( R \) be a finite ring with identity. Let \( \Psi \) be a subgroup of the group \( \text{Aut}(R) \).
Then \( I_{\Psi}(S_R) \geq D_{\Psi}(U(R)) + \sum_{P \in \text{spec}(R)} T\left(\text{Ind}(P); \frac{|\Psi|}{|\text{St}(P)|}\right) \).

**Conjecture 3.8.** Let \( R \) be a finite P.I.R with identity. Let \( \Psi \) be a subgroup of the group \( \text{Aut}(R) \).
Then \( I_{\Psi}(S_R) = D_{\Psi}(U(R)) + \sum_{P \in \text{spec}(R)} T\left(\text{Ind}(P); \frac{|\Psi|}{|\text{St}(P)|}\right) \).

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