Spaces of self-equivalences and free loops spaces

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Abstract

Let $M$ be a simply-connected closed oriented $N$-dimensional manifold. We prove that for any field of coefficients there exists a natural homomorphism of commutative graded algebras $\Psi : H_*(\Omega \text{aut}_1 M) \to H_{*+N}(M^{S^1})$ where $H_*(M^{S^1})$ is the loop algebra defined by Chas-Sullivan, [1]. As usual $\text{aut}_1 X$ (resp. $\Omega X$) denotes the monoid of the self-equivalences homotopic to the identity map (resp. the space of based loops) of the space $X$. Moreover, if $\mathbb{k}$ is of characteristic zero, $\Psi$ yields isomorphisms $\pi_n(\Omega \text{aut}_1 M) \otimes \mathbb{k} \cong HH_{n+1}(M^{S^1})$ where $\oplus_{i=1}^\infty HH_{(i)}^n$ denotes the Hodge decomposition on $H^*(M^{S^1})$.

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1. Introduction. Let $X$ be a path connected space with base point $x_0$. We denote by: $X^{S^1}$ the space of free loops on $X$, $\Omega X$ the space of based loops of $X$ at $x_0$, $\text{aut}_X X$ the monoid of self equivalence of $X$ pointed by $Id_X$, $\text{aut}_1 X$ the connected component of $Id_X$ in $\text{aut}_X X$, $\mathbb{k}$ a field and $H_*(X)$ the singular homology of the space $X$ with coefficients in $\mathbb{k}$. Composition of loops or composition of self-equivalences induce the same commutative graded algebra structure on $H_*(\Omega \text{aut}_1 X)$.

Let $M$ be a simply connected $N$-dimensional closed oriented manifold. It is convenient to write

$$H_*(M) = H_*(M^{S^1})$$

Indeed, $H_*(M)$ (resp. $H_*(M^{S^1})$) becomes a graded commutative algebra with the intersection product (resp. with the loop product defined by M. Chas and D. Sullivan, [1]).

Our first result reads:

**Theorem 1.** The natural map

$$\Phi : X \times \Omega \text{aut}_1 X \to X^{S^1}, \quad (x, \gamma) \mapsto \Phi(x, \gamma) : t \mapsto \gamma(t)(x).$$

induces a homomorphism of commutative graded algebras

$$H(\Phi) : H_*(M) \otimes H_*(\Omega \text{aut}_1 M) \to H_*(M^{S^1}).$$

Let $1 \in H_0(M) = H_0(M) \cong \mathbb{k}$ be the unit of the algebra $H_*(M)$. The homomorphism $H(\Phi)$ restricts to a homomorphism of commutative graded algebras

$$\Psi : H_*(\Omega \text{aut}_1 M) \to H_*(M^{S^1}), \quad \Psi(a) = H(\Phi)(1 \otimes a).$$
Let us denote by,
\[ h : \pi_*(\Omega_{aut} M) \otimes \mathbb{K} \to H_*(\Omega_{aut} M), \]
the homomorphism of (abelian) graded Lie algebras induced by the Hurewicz homomorphism. Thus, composition of the two homomorphisms \( \Psi \) and \( h \) defines a homomorphism of graded vector spaces:
\[ \pi_*(\Omega_{aut} M) \otimes \mathbb{K} \to H_{*+N}(M^{S^1}), \]
which, in turn, induces the dual homomorphism:
\[ \theta : H^{*+N}(M^{S^1}) \to (\pi_*(\Omega_{aut} M \otimes \mathbb{K}))^\vee. \]

If \( \mathbb{K} \) is a field of characteristic zero, the Hodge decomposition of the Hochschild homology of a commutative graded algebra \( A \):
\[ HH_*(A; A) = \bigoplus_{l \geq 0} HH_*(A; A) \]
was introduced by M. Gerstenhaber and S.D. Schack, \[8\]. See also \[10\] \[11\] \[14\] and \[17\] for alternative approaches. From the existence of a commutative model \( A \) for \( C^*(M) \) and the isomorphisms: \( HH_*(C^*(M); C^*(M)) \cong HH_*(A; A) \cong H^*(M^{S^1}) \), it results a Hodge decomposition
\[ \mathbb{H}^*(M^{S^1}) = \bigoplus_{l \geq 0} \mathbb{H}^*_l(M^{S^1}). \]

We prove:

**Theorem 2.** If \( \mathbb{K} \) is a field of characteristic zero and with notation above, then \( \theta \) induces an isomorphism of graded vector spaces
\[ (\pi_*(\Omega_{aut} M)) \otimes \mathbb{K} \cong \mathbb{H}^*_l(M^{S^1}), \quad n \geq 0. \]

The remaining of this paper is organized as follows:
2. Proof of theorem 1.
3. Proof of theorem 2.
4. Examples and further comments..

2. **Proof of theorem 1.** The proof decomposes in two steps.

First step. We denote by \( q : M^{S^1} \to M \), the free loop space fibration and by \( \text{Sect}(q) \) the space of sections of \( q \) which are homotopic to the trivial section \( \sigma_0 \). The composition of loops makes \( \text{Sect}(q) \) into a monoid, with multiplication \( \mu \) defined by
\[ \mu(\sigma, \tau)(m)(t) = \begin{cases} 
\sigma(m)(2t), & t \leq \frac{1}{2}, \\
\tau(m)(2t - 1), & t \geq \frac{1}{2}, 
\end{cases} \quad \sigma, \tau \in \text{Sect}(q), \quad t \in [0, 1], m \in M. \]

Clearly the map
\[ \psi : \Omega(aut M_1, id_M) \to \text{Sect}(q), \quad f \mapsto \Psi(f), \quad \text{such that } \psi(f)(m)(t) = f(t)(m) \]
is a homeomorphism of momoids. To prove theorem 1, it suffices to establishes that the evaluation map:
\[ ev : M \times \text{Sect}(q) \to M^{S^1}, \quad (m, \sigma) \mapsto \sigma(m). \]
induces a homomorphism of algebras $\mathbb{H}_*(M) \otimes H_*(\text{Sect}(q)) \to \mathbb{H}_*(M^{S^1})$. This is the purpose of the second step.

Second step. We recall roughly the definition of the loop product, by Chas-Sullivan, see also [3] for another description.

Let $\alpha : \Delta^r \to M^{S^1}$ and $\beta : \Delta^m \to M^{S^1}$ be singular simplices of $M^{S^1}$ and assume that the singular simplices of $M$ and $q \circ \alpha : \Delta^m \to M$ and $q \circ \beta : \Delta^m \to M$, are transverse. Then the intersection product $(q \circ \alpha) \cdot (q \circ \beta)$ makes sense and at each point $(s,t) \in \Delta^m \times \Delta^m$ such that $q \circ \sigma(s) = q \circ \tau(t)$, the composition of the loops $\alpha(s)$ can be performed. This gives a chain $\sigma \cdot \tau \in C_{m+n-d}(M^{S^1})$ and leads to a commutative and associative multiplication, 

\[ \mathbb{H}_k(M^{S^1}) \otimes \mathbb{H}_l(M^{S^1}) \to \mathbb{H}_{k+l}(M^{S^1}), \quad a \otimes b \mapsto a \cdot b. \]

Let $a_1 \otimes b_1 \in \mathbb{H}_k(M) \otimes H_*(\text{Sect}(q))$ and $a_2 \otimes b_2 \in \mathbb{H}_l(M) \otimes H_*(\text{Sect}(q))$ then, by definition $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{r \cdot l}(a_1 \cdot a_2) \otimes (b_1 \cdot b_2) \in \mathbb{H}_{k+l}(M) \otimes H_{r+l}(\text{Sect}(q))$ and we have to prove that:

\[ \mathbb{H}_*(ev)((a_1 \otimes b_1) \cdot (a_2 \otimes b_2)) = \mathbb{H}_*(ev)((a_1 \otimes b_1)) \cdot \mathbb{H}_*(ev)((a_2 \otimes b_2)). \]

First case: $b_1 = b_2 = 1 \in H_0(\text{Sect}(q))$. M. Chas and D.Sullivan, [1]- Proposition 3.4, prove that $H_*(\sigma_0) : \mathbb{H}_*(M) \to \mathbb{H}_*(M^{S^1})$ is a homomorphism of graded algebras. Therefore the restriction of $H_*(ev)$ to $\mathbb{H}_*(M) \otimes H_0(\text{Sect}(q))$ is a homomorphism of graded algebras.

Second case: $a_1 = a_2 = 1 \in \mathbb{H}_0(M) = H_N(M) = lk\omega$. Let $\alpha : \Delta^r \to \text{Sect}(q)$ and $\beta : \Delta^s \to \text{Sect}(q)$ be simplices and consider the maps of $M^{S^1}$:

\[ \alpha' : M \times \Delta^r \xrightarrow{id \times f} M \times \text{Sect}(q) \xrightarrow{ev} M^{S^1}, \quad \beta' : M \to \Delta^s \xrightarrow{id \times g} M \times \text{Sect}(q) \xrightarrow{ev} M^{S^1} \]

Since the simplices, $q \circ \alpha$ and $q \circ \beta$, are transverse in $M$ the product

\[ f' \cdot g' : M \times \Delta^r \times \Delta^s \xrightarrow{id \times f \times g} M \times \text{Sect}(q) \times \text{Sect}(q) \xrightarrow{(ev, ev)} M^{S^1} \times_M M^{S^1} \xrightarrow{c} M^{S^1}, \]

is well defined. Here, $c$ denotes pointwise composition of loops.

As the multiplication $\mu$ makes commutative the diagram

\[
\begin{array}{ccc}
M \times \text{Sect}(q) \times \text{Sect}(q) & \xrightarrow{(ev, ev)} & M^{S^1} \times_M M^{S^1} \\
\downarrow{id \times \mu} & & \downarrow{c} \\
M \times \text{Sect}(q) & \xrightarrow{ev} & M^{S^1},
\end{array}
\]

the map $f' \cdot g'$ is equal to $\mu(f, g)'$.

Last case: $a_2 = 1 \in \mathbb{H}_0(M) = H_N(M) = lk\omega$ and $b_1 = 1 \in H_0(\text{Sect}(q))$. If $\alpha : \Delta^r \to M$ (resp. $\beta : \Delta^s \to \text{Sect}(q)$) is a simplex of $M$ (resp. of $\text{Sect}(q)$) then the chain $ev \circ (\alpha \times \beta) : \Delta^r \times \Delta^s \to M^{S^1}$ induced $(ev \otimes (\alpha \times \beta))'$ coincides with the product of $(\sigma_0 \circ \alpha \circ \sigma_2 : M \times \Delta^r \xrightarrow{f_2} \Delta^r \xrightarrow{\sigma_0} M^{S^1}$ with $\beta' : M \times \Delta^s \to M^{S^1}$. Therefore $H_*(ev)([f \otimes [g]]) = H_*(\sigma_0)([g]) \cdot H_*(ev)(\omega \otimes [f])$. \hfill \qed

3. Proof of theorem 2. Since, $Q \subset \mathbb{R}$, we may as well suppose that $k = Q$. Hereafter we make intensive use of the theory minimal models, in the sense of Sullivan [15], for which refer systematically to [3]-§12. We denote by $(\land V, d)$ the minimal model of $M$. D. Sullivan and M. Vigué, [16], have proved that a relative minimal model for the fibration $q : M^{S^1} \to M$ is given by:

\[ (\land V, d) \mapsto (\land V \otimes \land sV, D), |sv| = |v| - 1, D(v) = d(v), D(sv) = -s(dv), \]
where \( s : \wedge V \to \wedge V \otimes \wedge sV \) is the unique derivation defined by: \( s(v) = sv \), (see also [3]-§15(e)). The cochain complex \( \wedge V \otimes \wedge sV, D \) decomposes into a direct sum of complexes
\[
(\wedge V \otimes \wedge sV, D) = \oplus_{k \geq 0} (\wedge V \otimes \wedge^k sV, D).
\]
This induces a new a graduation on \( H^*(M^{S^1}) \),
\[
H^*(M^{S^1}) = \oplus_k H^*_k(M^{S^1}), \quad H^*_k(M^{S^1}) = H^*(\wedge V \otimes \wedge^k sV, D).
\]
We denote by \( H^*_k(M^{S^1}) \) the dual graduation on \( H_*(M^{S^1}) \). M. Vigué has proved, [17], that the Hodge decomposition of the Hochschild homology
\[
HH_*(((\wedge V, d); (\wedge V, d)) = \oplus_{i \geq 0} HH_1^*((\wedge V, d); (\wedge V, d)),
\]
is determined by the isomorphisms \( H^*(\wedge V \otimes \wedge^k sV, D) \cong HH_1^k((\wedge V, d); (\wedge V, d)). \)

By the Milnor-Moore Theorem ([12]), \( H_*(\text{Sect} \,(q); \mathbb{Q}) \) is isomorphic as an Hopf algebra to the universal enveloping algebra on the graded homotopy Lie algebra \( \pi_*(\Omega \text{aut} M) \otimes \mathbb{Q} \).

Thus theorem 2 in the introduction is a direct consequence of theorem 3 below.

**Theorem 3.** The restriction of \( H_*(ev) \) to \( \mathbb{H}_0(M) \otimes (\pi_* \text{Sect} \,(q)) \otimes \mathbb{Q} \),
\[
\Phi_1 : \pi_* \text{Sect} \,(q) \otimes \mathbb{Q} \to \mathbb{H}_*(M^{S^1}; \mathbb{Q})
\]
is an injective homomorphism whose image is isomorphic to \( H^{(1)}_{\geq N}(M^{S^1}; \mathbb{Q}) \).

**Proof.** We first construct a quasi-isomorphism \( \rho : (\wedge V, d) \to (A, d) \) with \( (A, d) \) a commutative differential graded algebra satisfying
\[
\begin{cases}
A^0 = \mathbb{Q}, A^1 = 0, \\
A^N > 0, A^N = \mathbb{Q}\Omega, \\
\dim A^i < \infty, \text{ for all } i.
\end{cases}
\]
For this we put,
\[
\begin{aligned}
Z^k &= \text{Ker}(d : (\wedge V)^k \to (\wedge V)^{k+1}), \\
(\wedge V)^k &= Z^k \oplus S^k, \\
I &= S^{N-1} \oplus dS^{N-1} \oplus S^N \oplus (\wedge V)^{>N}.
\end{aligned}
\]
Then, the quotient \( (\wedge V)^N / (S^N \oplus dS^{N-1}) \cong H^N(M) \) has dimension one. Since \( V^1 = 0 \), the subcomplex \( I \) is an ideal of \( (\wedge V, d) \). The acyclicity of \( I \) implies that the natural projection \( \rho : (\wedge V, d) \to (A, d) = (\wedge V/I, d) \) is a quasi-isomorphism of differential graded algebras.

The homomorphism \( \rho \) extends to a quasi-isomorphism \( \rho \otimes 1 : (\wedge V \otimes \wedge sV, D) \to (A \otimes \wedge sV, D) \) with \( D(a \otimes sv) = d(a) \otimes sv - (-1)^{|a|} a \cdot (\rho \otimes 1)(sv) \).

The complex \( (A \otimes \wedge sV, D) \) also decomposes into the direct sum of the complexes \( (A \otimes \wedge^k sV, D) \).

Denote by \( (a_i), i = 1, \ldots, n \), an homogeneous linear basis of \( A \), such that \( a_n = \Omega \), and by \( (a_i^\vee) \) the dual basis i.e. the linear basis of \( A^\vee = \text{Hom}(A, \mathbb{Q}) \) such that \( \langle a_i^\vee, a_j \rangle = \delta_{ij} \).

A. Haefliger, [1], has proved that a model for the evaluation map \( ev : M \times \text{Sect} \,(q) \to M^{S^1} \) is given by the morphism
\[
\theta : (A \otimes \wedge sV, D) \to (A, d) \otimes (A^\vee \otimes sV), \delta \quad a \otimes v \mapsto \theta(a \otimes sv) = \sum_i aa_i \otimes (a_i^\vee \otimes sv).
\]
Since \( D(sv) \subset A \otimes sV \), and \( \theta \) is a morphism of differential graded algebras, \( \delta(A^\vee \otimes sV) \subset A^\vee \otimes sV \). We denote by:
\( \rho_1 : (\wedge (A^V \otimes sV), \delta) \to (A^V \otimes sV, \delta) \) the projection onto the complex of indecomposable elements.

\( P : (A, d) \to (\mathbb{Q} \Omega, 0) \) the homogeneous projection onto the component of degree \( N \),

\( \pi_1 : (A \otimes \wedge sV, D) \to (A \otimes sV, D) \) the canonical projection on the subcomplex.

The dual of \( \Phi_1 \),

\[ \Phi_\vee^1 : H^{*+d}(M^{S^1}; \mathbb{Q}) \to \text{Hom}(\pi_*(\text{Sect}(q)) \otimes \mathbb{Q}, \mathbb{Q}) \]

coincides therefore with \( H^* (P \circ \rho_1 \circ \theta) \):

\[ (A \otimes \wedge sV, D) \xrightarrow{\theta} (A, d) \otimes (\wedge (A^V \otimes sV), \delta) \xrightarrow{P \circ \rho_1} \mathbb{Q} \Omega \otimes (A^V \otimes sV, \delta). \]

**Lemma.** The duality map

\[ Du : A \to A^\vee \text{, such that } \langle Du(a), b \rangle = P(ab) \in \mathbb{Q} \Omega \cong \mathbb{Q}, \]

induces a quasi-isomorphism of complexes

\[ Du \otimes 1 : (A \otimes sV, D) \to (A^\vee \otimes sV, \delta). \]

**Proof.** Denote by \( \alpha_{ij}^k \) and \( \beta_{ij}^l \) rational numbers defined by the relations

\[
\begin{align*}
\alpha_{ij}^k &= \frac{a_i \cdot a_j}{\sum_k \alpha_{ij}^k a_k} \\
\beta_{ij}^l &= \frac{d(a_i)}{\sum_j \beta_{ij}^l a_j}
\end{align*}
\]

A straightforward computations show that

- \( d(a_\gamma^i) = -(-1)^{|a_i|} \sum_j \beta_{ij}^j a_\gamma^j \)
- \( \sum_r \alpha_{ij}^t \alpha_{tk}^r = \sum_s \alpha_{js}^s \alpha_{is}^t \), for \( i, j, k, t = 1, \ldots, n \) (associativity)
- \( \sum_r \alpha_{ij}^t \beta_{rt}^s = \sum_i \beta_{ij}^s + (-1)^{|a_i|} \sum_l \beta_{il}^s \alpha_{il}^s \), for \( i, j, l = 1, \ldots, n \) (compatibility of the differential with the multiplication).
- \( \delta(a_\gamma^i \otimes sv) = (-1)^{|a_i|} \left[ \sum_i \alpha_{a_i}^i (a_\gamma^i \otimes sv_i) - \sum_r \beta_{ij}^r (a_\gamma^i \otimes sv) \right] \)
- \( Du(a_i) = \sum_j \alpha_{ij}^n a_\gamma^j \).

The duality morphism has degree \( N \). A standard computation shows then that

\[ \delta \circ (Du \otimes 1) = (-1)^N (Du \otimes 1) \circ d. \]

Since \( H^*(M) \) is a Poincaré duality algebra and since \( H^*(Du) : H^*(M) \to H_*(M) \) is the Poincaré duality, \( Du \otimes 1 \) is a quasi-isomorphism.

It is easy to check the commutativity of the following diagram of complexes

\[
\begin{array}{ccc}
(A \otimes \wedge sV, D) & \xrightarrow{\theta} & (A, d) \otimes (\wedge (A^V \otimes sV), \delta) \\
\downarrow \pi_1 & & \downarrow \pi_1 \\
(A \otimes sV, D) & \xrightarrow{\sigma} & \mathbb{Q} \Omega \otimes (A \otimes sV, D),
\end{array}
\]

with \( \sigma(a \otimes sv) = \Omega \otimes a \otimes sv \). Thus, the surjectivity of \( H^* ((1 \otimes (Du \otimes 1)) \circ \sigma \circ \pi_1) \) implies surjectivity of \( \Phi_\vee^1 \). Therefore \( \Phi_1 \) is an injective homomorphism whose image is isomorphic to \( H_{>N}^{(1)}(M^{S^1}) \). \( \square \)
4. Examples and further comments.

**Remark 1.** The morphism $H_*(ev)$ is in general not injective. Let us consider the intersection homomorphism $I : \mathbb{H}_*(M^{S^1}) \rightarrow H_*(\Omega M)$ defined in [1]-Proposition 3.4. M. Chas and D. Sullivan have proved that $I$ is a morphism of algebras. Together with M. Vigué-Poirrier,[6], we prove that the image of $I$ is in the center of $H_*(\Omega M)$ and that the kernel of $I$ is a nilpotent ideal with nilpotency index less than or equal to $N$. In particular, if the center of $H_*(\Omega M)$ is trivial, then $\mathbb{H}_{>0}(M^{S^1})$ is a nilpotent ideal. On the other hand $H_*(\text{Sect}(q)) \cong H_*(\text{aut}_1 M)$ is a free commutative graded algebra that we believe generated, in general, by infinitely many generators, see [1].

**Remark 2.** In [2] Cohen and Jones have proved that $\mathbb{H}_*(M^{S^1})$ is isomorphic as an algebra to the Hochschild cohomology $HH^*(C^*(M),C^*(M))$. On the other hand, in [3], J.B. Gatsinzi establishes, for any space $M$ (not necessarily a manifold), an algebraic isomorphism between $\pi_1(\text{aut} M, id_M) \otimes \mathbb{Q}$ and a subvector space of $HH^*(C^*(M),C^*(M))$. The coherence between the two results is given by our Theorem 2.

**Problem.** We would like to know if the homomorphism

$$H_*(\Phi) : \mathbb{H}_*(M) \otimes H_*(\text{aut}_1 M) \rightarrow \mathbb{H}_*(M^{S^1})$$

is surjective.

It is the case, for example if $M = CP^{N'}, N = 2N'$. Other examples can be checked using for instance [4].

For spaces, such that $H_*(\Phi)$ is surjective, we obtain a strong connection between the behaviour of the sequences of Betti numbers $\dim H_i(M^{S^1})$ and $\dim \pi_1(\text{aut} M, id_M) \otimes \mathbb{Q}$.

**Example 1.** Let $G$ be a Lie group. The minimal model of $G$ is $(\land V, 0)$ with $V$ finite dimensional and concentrated in odd degrees, (Cf. [3]-§12(a)). Therefore a model of the free loop space $G^{S^1}$ is $(\land V \otimes \land sV, 0)$ and the Haefliger model for the space $\text{Sect}(q)$ is $(\land(\land V)^V \otimes sV, 0)$. Since the model $\theta$ of the evaluation map $ev$ is injective, $H_*(ev) : H_*(M) \otimes H_*(\text{Sect}(q)) \rightarrow H_*(M^{S^1})$ is surjective. This implies that there exists an isomorphism of graded algebras.

$$\mathbb{H}_*(M^{S^1}) \cong \mathbb{H}_*(M) \otimes H_*(\Omega M),$$

where the multiplication on the right is simply the intersection product on $\mathbb{H}_*(M)$ and the usual Pontryagin product on $H_*(\Omega M)$.

**Example 2.** Let us assume that $M$ is a $\mathbb{Q}$-hyperbolic space satisfying either $(H^+(M))^3 = 0$ or else $(H^+(M))^4 = 0$ and $M$ is a coformal space.

For recall, [3]-§35, a space is $\mathbb{Q}$-hyperbolic if $\dim \pi_*(M) \otimes \mathbb{Q} = \infty$. A space is coformal, [12], if the differential graded algebras $C_*(\Omega M)$ and $(H_*(\Omega M), 0)$ are quasi-isomorphic.

Under the above hypothesis, M. Vigué proves, [13], that there exist an integer $n_0$ and some constants $C_1 \geq C_2 > 1$ such that

$$C_2^n \leq \sum_{i=1}^{n} \dim H^i(X^{S^1}) \leq C_1^n \text{ all } n \geq n_0.$$

We directly deduce from Theorem 3 that the same relations hold for the sequence $\dim \pi_*(\text{aut} M) \otimes \mathbb{Q}$ in place of $\dim H^i(X^{S^1})$, that is: in both case the sequences of Betti numbers have exponential growth, [3]-§33(a).
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