BTZ Black Hole as Solution of
3d Higher Spin Gauge Theory

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Abstract

BTZ black hole is interpreted as exact solution of 3d higher spin gauge theory. Solutions for free massless fields in BTZ black hole background are constructed with the help of the star-product algebra formalism underlying the formulation of 3d higher spin theory. It is shown that a part of higher spin symmetries remains unbroken for special values of the BTZ parameters.

1 Introduction

An important difference of (2+1) gravity [1, 2, 3, 4, 5] from higher dimensional gravitational theories is that the vacuum theory is topological, describing no local degrees of freedom. It was shown in [6, 7] that (2+1) gravity is equivalent to the Chern-Simons gauge theory of $SL(2|\mathbb{R}) \times SL(2|\mathbb{R})$ in which the gauge potential describes dreibein and Lorentz connection. Among other things, on mass shell, this formulation allows one to treat diffeomorphisms of general relativity as gauge transformations, that essentially simplifies the quantum analysis [8].

In three dimensions, the Riemann tensor is fully represented by the Ricci tensor. As a result, $R_{mn} = 0$ implies $R_{mnpq} = 0$, i.e., any vacuum solution is locally Minkowski. Analogously, any vacuum solution of the Einstein equations with negative cosmological term is locally $AdS_3$.

BTZ black hole solution in $AdS_3$ was discovered in [9]. The “No Black Hole Theorem” [10] states that no black hole type solution with (non-zero) horizons in 2+1 dimensions exists unless negative cosmological constant is introduced.
The BTZ black hole is in many respects analogous to the four-dimensional Kerr black hole, thus providing a useful model for the study of black hole physics. The important difference is however that the BTZ black hole has no curvature singularity \([11]\). The black hole type geodesics behaviour results instead from the topological peculiarity of the BTZ solution which is locally isomorphic to \(AdS_3\). As shown in \([11]\), the BTZ solution can be obtained via factorization of \(AdS_3\) over a discrete symmetry group.

Since BTZ solution has zero \(o(2,2)\) curvature, it is also the exact solution of the nonlinear 3d higher spin (HS) gauge theory \([12, 13]\) which, for the case of vanishing matter fields, amounts to the Chern-Simons theory for the 3d HS algebra that contains \(o(2,2) \sim sp(2) \oplus sp(2)\) as a subalgebra. Until now a very few exact solutions in the nonlinear HS gauge theory are known apart from pure \(AdS_3\). One is the Lorentz invariant 3d solution found in \([13]\) and its generalization to the 4d HS theory obtained in \([14]\). Unfortunately, the physical interpretation of these solutions is still lacking although they are likely to play a fundamental role in the HS theory as the basis solutions for the application of the integrating flow machinery \([13]\). Recently, new exact solutions have been found by Sezgin and Sundell \([15]\), which may receive some interpretation in the AdS/CFT context.

The investigation of black hole solutions in higher-dimensional HS gauge theories is, of course, of primary importance. The main motivation for this work is that, although being very simple, the study of the BTZ black hole in the 3d gauge theory can be useful for the study of less trivial Schwarzschild-Kerr-type solutions at least in two respects. Firstly, we learn how the HS star-product machinery applies to the black hole physics. This is the aim of this paper. Secondly, pretty much as 4d Minkowski space-time is a slice of the flat ten-dimensional space-time with matrix coordinates \([16, 17, 18]\) \(X^{AB} = X^{BA}\) (\(A,B\) is the 4d Majorana spinor index), it is tempting to speculate that the 4d Kerr black hole can be interpreted as a slice of a BTZ-like solution associated with the group manifold \(Sp(4)\) which represents the \(AdS\)-like geometry in this framework \([19, 18, 20, 21]\). If so, the BTZ-like zero curvature solutions may shed some light on the study of the usual black hole physics from a more general perspective of higher-dimensional generalized spaces with matrix coordinates.

The modest aim of this letter is to demonstrate how the methods of HS gauge theory can be applied to reproduce the known results of the BTZ black hole physics. Namely, using the oscillator realization of the \(AdS_3\) isometry algebra \(o(2,2) \sim sp(2) \oplus sp(2)\) we find the gauge function of the BTZ solution in terms of \(Sp(2)\) group and then solve free massless field equations in the BTZ background in terms of the Fock module \([22]\) to show how solutions for massless scalar and spinor fields \([23, 24, 25, 26]\) are obtained in our approach.

The layout of the rest of the paper is as follows. In Section 2 we summarize basic facts on BTZ black hole metric, its symmetries and factorization procedure. In
Section 3 we recall the oscillator realization of $AdS_3$ algebra. In Section 4 coordinate-free description of BTZ black hole as a flat connection is given. BTZ gauge function is represented in Section 5. In Section 6 we review dynamical $Sp(4)$ covariant equations and Fock module formulation of \cite{20, 22}. Star-product realization of Killing vectors on the Fock module is obtained in Section 7. In Section 8 we find explicit solutions for dynamical fields in BTZ background using unfolded dynamics approach. In Section 9 we discuss briefly the case of extremal BTZ black hole. Finally, in Section 10 we explore symmetries of massless fields in BTZ black hole background. Some useful formulae and intermediate calculations are given in two Appendices.

\section{BTZ black hole}

In this section we briefly recall some properties of BTZ black hole. For more detail we refer the reader to the review \cite{11}.

The 3d Einstein-Hilbert action with negative cosmological constant $\Lambda = -\lambda^2$
\begin{equation}
S = \frac{1}{2\pi} \int \sqrt{-g}(R + 2\lambda^2) dt d^2x
\end{equation}
gives Einstein equations
\begin{equation}
R_{mn} - \frac{1}{2} R g_{mn} = \lambda^2 g_{mn}.
\end{equation}
In three dimensions this implies
\begin{equation}
R_{mnpq} = -\lambda^2 (g_{mp}g_{nq} - g_{np}g_{mq}).
\end{equation}
This means that, being a vacuum solution, 3d black hole is locally equivalent to $AdS_3$. In \cite{9} it was shown that the metric
\begin{equation}
ds^2 = (-M + \lambda^2 r^2 + \frac{J^2}{4r^2}) dt^2 - (-M + \lambda^2 r^2 + \frac{J^2}{4r^2})^{-1} dr^2 - r^2(d\phi - \frac{J}{2r^2} dt)^2,
\end{equation}
where $\phi \in [0, 2\pi]$, solves (2.2) and describes a rotating black hole with dimensionless mass $M$ and angular momentum $J$. It has the inner and outer horizons
\begin{equation}
r^2_{\pm} = \frac{M}{2\lambda^2} \left( 1 \pm \sqrt{1 - \frac{J^2\lambda^2}{M^2}} \right).
\end{equation}
The ergosphere (i.e., the $g_{00} = 0$ surface of infinite redshift) has $r_{\text{erg}} = \frac{1}{\lambda} M^{1/2}$.

Note that $r_{\pm}$ are complex for $|J| > M/\lambda$, in which case the horizons disappear and the metric has naked singularity at $r = 0$. Formally, one can take negative $M$ \footnote{Units are chosen so that $G = 1/8$.}
in the metric (2.4) when $J = 0$. But, except for $M = -1$ corresponding to the $AdS_3$ space-time, this leads to the naked conical singularity at $r = 0$ [11], which is easily seen by the rescaling of the radial variable $r \to \sqrt{-Mr}$. The case of $M = 0$ and $J = 0$ corresponds to “massless” black hole that does not reproduce the $AdS$ space-time (unlike the $4d$ case). So, we demand

$$M > 0, \quad |J| \leq M/\lambda.$$  \hspace{1cm} (2.6)

The limiting case of $|J| = M/\lambda$ corresponds to the extremal black hole with $r_+ = r_-$. The $AdS_3$ space-time is a quadric in the four-dimensional pseudo-Euclidian space with the metric $\eta = \text{diag}(+ + - -)$

$$ds^2 = du^2 + dv^2 - dx^2 - dy^2,$$  \hspace{1cm} (2.7)

$$u^2 + v^2 - x^2 - y^2 = \lambda^{-2}.$$  \hspace{1cm} (2.8)

The metric of generic BTZ black hole (2.4) for $r > r_+$ is conveniently parameterized by

\begin{align*}
u &= \sqrt{A(r)} \cosh(\tilde{\phi}(t, \phi)), \\
v &= \sqrt{B(r)} \sinh(\tilde{t}(t, \phi)), \\
x &= \sqrt{A(r)} \sinh(\tilde{\phi}(t, \phi)), \\
y &= \sqrt{B(r)} \cosh(\tilde{t}(t, \phi)),
\end{align*}

where

$$A(r) = \frac{1}{\lambda^2} \left( \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right), \quad B(r) = \frac{1}{\lambda^2} \left( \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right),$$  \hspace{1cm} (2.10)

$$\tilde{t} = \lambda^2 r_+ t - \lambda r_- \phi, \quad \tilde{\phi} = -\lambda^2 r_- t + \lambda r_+ \phi.$$  \hspace{1cm} (2.11)

In this paper, we will use the embedding relations (2.9) with $r > r_+$ for the case of generic black hole. (For more detail on other patches with $r \leq r_+$ as well as on the cases of extremal and vacuum black holes we refer the reader to [11]).

The properties of BTZ black hole are heavily based on its group origin. Indeed, one can combine $(u, v, x, y)$ into a $2 \times 2$ matrix $S_0 \in SL(2|\mathbb{R})$

$$S_0 = \lambda \begin{pmatrix} u + x & v - y \\ -v - y & u - x \end{pmatrix}, \quad \det(S_0) = 1.$$  \hspace{1cm} (2.12)

As shown in [11], the BTZ solution results from the $SL(2|\mathbb{R})$ group manifold, via factorization over a discrete subgroup by the identification

$$S_0 \sim \rho^+ S_0 \rho^-, \quad \rho^\pm = \begin{pmatrix} e^{\mp \lambda (r_+ \pm r_-)} & 0 \\ 0 & e^{\mp \lambda (r_+ \pm r_-)} \end{pmatrix},$$  \hspace{1cm} (2.13)
which makes cyclic the variable $\phi$ in the metric (2.4).

The isometries of $AdS_3$ are represented as elements of the group $SL(2|\mathbb{R})_L \times SL(2|\mathbb{R})_R/\mathbb{Z}_2 \sim SO(2,2)$ acting on group elements by left and right multiplication $S_0 \to P_LS_0P_R$ with the identification $(P_L, P_R) \sim (-P_L, -P_R)$. In accordance with (2.7), the $AdS_3$ space-time is invariant under the $SO(2,2)$ transformations generated by

$$J_{ab} = X_b \frac{\partial}{\partial X^a} - X_a \frac{\partial}{\partial X^b}, \quad \text{where} \quad X^a = (u, v, x, y).$$

According to [11], the isometry algebra of a general BTZ metric (2.4) is generated by the vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$. In the case $r_+^2 - r_-^2 > 0$, the Killing vector responsible for the identification (2.13) is

$$\frac{\partial}{\partial \phi} = -\lambda r_+ J_{12} + \lambda r_- J_{03},$$

whereas the time translation generator is

$$\frac{\partial}{\partial t} = \lambda^2 r_- J_{12} - \lambda^2 r_+ J_{03}.$$ 

Note that, as shown in [11], among six Killing vectors of $AdS_3$ only (2.15) and (2.16) remain globally defined upon the identification (2.13).

### 3 Oscillator realization of $o(2,2)$

Let us describe the oscillator realization of the algebra $o(2,2)$ which will be particularly useful for our analysis. The isometry algebra of $AdS_3$ is $o(2,2) \sim sp(2) \oplus sp(2)$. It is spanned by the diagonal $sp(2)$ Lorentz generators $L_{\alpha\beta} = L_{\beta\alpha}$ and $AdS_3$ translations $P_{\alpha\beta} = P_{\beta\alpha}$ $(\alpha, \beta, \ldots = 1, 2)$. The commutation relations are

$$[L_{\alpha\beta}, L_{\gamma\delta}] = \frac{1}{2}(\epsilon_{\beta\gamma} L_{\alpha\delta} + \epsilon_{\beta\delta} L_{\alpha\gamma} + \epsilon_{\alpha\gamma} L_{\beta\delta} + \epsilon_{\alpha\delta} L_{\beta\gamma}),$$

$$[P_{\alpha\beta}, P_{\gamma\delta}] = 2\lambda^2(\epsilon_{\beta\gamma} P_{\alpha\delta} + \epsilon_{\beta\delta} P_{\alpha\gamma} + \epsilon_{\alpha\gamma} P_{\beta\delta} + \epsilon_{\alpha\delta} P_{\beta\gamma}),$$

$$[L_{\alpha\beta}, P_{\gamma\delta}] = \frac{1}{2}(\epsilon_{\beta\gamma} P_{\alpha\delta} + \epsilon_{\beta\delta} P_{\alpha\gamma} + \epsilon_{\alpha\gamma} P_{\beta\delta} + \epsilon_{\alpha\delta} P_{\beta\gamma}),$$

where

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the antisymmetric $sp(2)$ invariant form.

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3 Spinor indices are raised and lowered according to the rules $A_\alpha = A^\beta \epsilon_{\beta\alpha}$, $A^\alpha = \epsilon^{\alpha\beta} A_\beta$. 

5
Let $\hat{a}_\alpha$ and $\hat{b}_\alpha$ be oscillators with the commutation relations

$$[\hat{a}_\alpha, \hat{b}_\beta] = \delta_\alpha^\beta, \quad [\hat{a}_\alpha, \hat{a}_\beta] = 0, \quad [\hat{b}_\alpha, \hat{b}_\beta] = 0. \tag{3.2}$$

The generators of $sp(2) \oplus sp(2)$ admit the standard oscillator realization [27]

$$\hat{L}_{\alpha\beta} = \frac{1}{2} \{\hat{a}_\alpha, \hat{b}_\beta\} - \frac{1}{4} \{\hat{a}_\gamma, \hat{b}_\gamma\} \delta_\alpha^\beta, \quad \hat{P}_{\alpha\beta} = \hat{a}_\alpha \hat{a}_\beta + \lambda^2 \hat{b}_\alpha \hat{b}_\beta. \tag{3.3}$$

Instead of working with operators, it is more convenient to use the star-product operation in the algebra of polynomials of commuting variables $a_\alpha$ and $b^\alpha$

$$(f \star g)(a, b) = \frac{1}{\pi^4} \int f(a + u, b + t) g(a + s, b + v) e^{2(s_\alpha t_\alpha - u_\alpha v_\alpha)} d^2 u d^2 t d^2 s d^2 v. \tag{3.4}$$

Equivalently,

$$(f \star g)(a, b) = f(a, b) e^{\frac{1}{\pi} \left( \frac{\partial}{\partial a_\alpha} \frac{\partial}{\partial b_\alpha} - \frac{\partial}{\partial b_\alpha} \frac{\partial}{\partial a_\alpha} \right)} g(a, b).$$

The star-product defined this way (often called Moyal product) describes the associative product of symmetrized (i.e., Weyl ordered) polynomials of oscillators in terms of symbols of operators. The integral is normalized so that 1 is the unit element of the algebra. In particular,

$$1 \star 1 = \frac{1}{\pi^4} \int e^{2(s_\alpha t_\alpha - u_\alpha v_\alpha)} d^2 u d^2 t d^2 s d^2 v = 1.$$  

From (3.4) it follows that

$$a_\alpha \star f(a, b) = a_\alpha f(a, b) + \frac{1}{2} \frac{\partial}{\partial b^\alpha} f(a, b), \quad b_\alpha \star f(a, b) = b_\alpha f(a, b) + \frac{1}{2} \frac{\partial}{\partial a^\alpha} f(a, b).$$

In particular, the defining relations of the associative star-product algebra are

$$[a_\alpha, b^\beta]_* = \delta_\alpha^\beta, \quad [a_\alpha, a_\beta]_* = 0, \quad [b^\alpha, b^\beta]_* = 0, \tag{3.5}$$

where $[a, b]_* = a \star b - b \star a$. The star-product realization of the $o(2, 2)$ generators is

$$L_{\alpha\beta} = \frac{1}{2} (a_\alpha b_\beta + a_\beta b_\alpha), \quad P_{\alpha\beta} = a_\alpha a_\beta + \lambda^2 b_\alpha b_\beta. \tag{3.6}$$

For convenience, from now on we set the $AdS_3$ radius equal to unity ($\lambda=1$).
4 BTZ black hole as flat connection

Since BTZ black hole is locally equivalent to $AdS_3$ it can be described by a flat connection of $sp(2) \oplus sp(2)$. Indeed, let $w_0$ be a $sp(2) \oplus sp(2)$ valued 1-form

$$w_0(a, b|X) = \frac{1}{2} \omega^{\alpha\beta}(X) L_{\alpha\beta} + \frac{1}{4} h^{\alpha\beta}(X) P_{\alpha\beta},$$

(4.1)

where $P_{\alpha\beta}$ and $L_{\alpha\beta}$ are the $AdS_3$ generators (3.6) while $\omega_{\alpha\beta}(X)$ and $h_{\alpha\beta}(X)$ are 1-forms. Then the zero-curvature condition

$$R = dw_0 - w_0 \star w_0 = 0$$

(4.2)

is equivalent to the equations

$$d\omega_{\alpha\beta} + \frac{1}{2} \omega^\gamma_{\alpha} \wedge \omega_{\gamma\beta} + \frac{1}{2} h^\gamma_{\alpha} \wedge h_{\beta\gamma} = 0,$$

(4.3)

$$dh_{\alpha\beta} + \frac{1}{2} \omega^\gamma_{\alpha} \wedge h_{\gamma\beta} + \frac{1}{2} h^\gamma_{\beta} \wedge h_{\alpha\gamma} = 0.$$  

(4.4)

Identifying $\omega_{\alpha\beta}$ with Lorentz connection and $h_{\alpha\beta}$ with dreibein, (4.4) gives the zero torsion condition while (4.3) implies local $AdS_3$ geometry.

The equation (4.2) is invariant under the gauge transformations

$$\delta w_0 = d\epsilon - [w_0, \epsilon]_\star,$$

(4.5)

where $\epsilon(a, b|X)$ is an arbitrary infinitesimal gauge parameter. Any fixed vacuum solution $w_0$ of the equation (4.2) breaks the local symmetry to its stability subalgebra with the infinitesimal parameters $\epsilon_0(a, b|X)$ satisfying the equation

$$d\epsilon_0 - [w_0, \epsilon_0]_\star = 0.$$

(4.6)

Consistency of this equation is guaranteed by (4.2). Its generic solution has at most six independent parameters, the global symmetry parameters. How many of these survive in a locally $AdS_3$ geometry depends on its global properties (i.e., boundary conditions). The true $AdS_3$ space-time has all six symmetries which are $o(2, 2)$ motions of $AdS_3$. For the generic BTZ black hole solution only two of the six parameters survive.

Locally, the general form of the dreibein and Lorentz connection of $sp(2) \oplus sp(2)$ algebra that satisfy (4.3) and (4.4) is

$$h_{\alpha\beta} = (W_1^{-1})_\alpha^\gamma d(W_1)_\gamma\beta - (W_2)_\alpha^\gamma d(W_2^{-1})_\gamma\beta,$$

(4.7)

$$\omega_{\alpha\beta} = (W_1^{-1})_\alpha^\gamma d(W_1)_\gamma\beta + (W_2)_\alpha^\gamma d(W_2^{-1})_\gamma\beta,$$

(4.8)
where $W_{1,2 \alpha \beta}(X) \in Sp(2)$, i.e.,
\[(W_{1,2}^{-1})_{\alpha \beta} = -(W_{1,2})_{\beta \alpha} . \tag{4.9}\]
From (4.7) it follows that the metric is
\[ds^2 = \frac{1}{2} h_{\alpha \beta} h^{\alpha \beta} = \frac{1}{2} dS_{\alpha \beta} dS^{\alpha \beta} , \tag{4.10}\]
where
\[S_{\alpha \beta} = (W_{1})_{\alpha}^{\gamma} (W_{2})_{\gamma \beta} . \tag{4.11}\]
Thus, any locally AdS$_3$ metric is determined by a $Sp(2)$ matrix field $S_{\alpha \beta}(X)$. (Note that, generally, $S_{\alpha \beta} \neq S_{\beta \alpha}$.) To obtain the BTZ metric (2.4) one can use the matrix $S_0$ (2.12) with the parametrization (2.9).

A class of the dreibeins (4.7) and Lorentz connections (4.8), that are well-defined with respect to the identification $\phi \rightarrow \phi + 2\pi$, can be found using the following decomposition of the matrix $S_0$ (2.12):
\[S_0^{\alpha \beta} = (K_+ U_r K_-)^{\alpha \beta} \tag{4.12}\]
with the $Sp(2)$ matrices $K_\pm$ and $U_r$ of the form
\[K_\pm = \begin{pmatrix} e^{i \frac{\phi \pm t}{2}} & 0 \\ 0 & e^{-i \frac{\phi \pm t}{2}} \end{pmatrix}, \quad U_r = \begin{pmatrix} \sqrt{A} & -\sqrt{B} \\ -\sqrt{B} & \sqrt{A} \end{pmatrix} . \tag{4.13}\]
Note, that $K_\pm$ belong to the Abelian BTZ Killing subgroup of $Sp(2) \times Sp(2)$.

Setting $W_1 = K_+ U_1$ and $W_2 = U_2 K_-$ with $U_1 U_2 = U_r$, we reproduce (4.12) in the form (4.11). The corresponding dreibein and Lorentz connection
\[h = U_1^{-1} K_+^{-1} dK_+ U_1 - U_2 K_- dK_- U_2^{-1} + U_1^{-1} dU_1 - U_2 dU_2^{-1} , \tag{4.14}\]
\[\omega = U_1^{-1} K_+^{-1} dK_+ U_1 + U_2 K_- dK_- U_2^{-1} + U_1^{-1} dU_1 + U_2 dU_2^{-1} \tag{4.15}\]
do not depend on $t$, $\phi$ as soon as $U_{1,2} = U_{1,2}(r)$. Therefore they remain well-defined in the BTZ case upon the identification $\phi \rightarrow \phi + 2\pi$.

It is convenient to use the following matrices $U_{1,2}$
\[U_1 = \left( \frac{A}{B} \right)^{\frac{1}{4}} \begin{pmatrix} 0 & -\mu(r) \sqrt{B} \\ \eta(r) \sqrt{A} & \mu(r) \sqrt{A} \end{pmatrix} , \quad U_2 = \left( \frac{A}{B} \right)^{\frac{1}{4}} \begin{pmatrix} \mu(r) & 0 \\ -\mu^{-1}(r) \eta(r) \sqrt{AB} & \eta(r) \sqrt{AB} \end{pmatrix} , \tag{4.16}\]
where $\mu(r)$, $\eta(r)$ are some functions that depend on the radial coordinate and satisfy
\[\mu(r) \eta(r) = A^{-1}(r) . \tag{4.17}\]
The resulting matrices $W_{1\alpha}^\beta = (K_+ U_1)_{\alpha}^\beta$, $W_{2\alpha}^\beta = (U_2 K_-)_{\alpha}^\beta$ are

$$W_{1\alpha}^\beta = \sqrt{\frac{u + x}{y - v}} \left( \begin{array}{cc} 0 & -\mu(y - v) \\ \eta(u - x) & \mu(u - x) \end{array} \right),$$

$$W_{2\alpha}^\beta = \sqrt{\frac{u + x}{y - v}} \left( \begin{array}{cc} \mu & 0 \\ -\mu^{-1} \eta(u - x)(y - v) & 0 \end{array} \right).$$

(4.18)

According to (4.7) and (4.8), the corresponding dreibein and Lorentz connection have the form

$$h_{11} = A\mu^2 \left( -d\tilde{t} + d\tilde{\phi} + \frac{1}{2AB} dA \right),$$

$$h_{12} = h_{21} = -d\tilde{\phi} - \frac{1}{2B} dA,$$

$$h_{22} = -\mu^{-2} \left( d\tilde{t} + d\tilde{\phi} - \frac{1}{2AB} dA \right),$$

$$\omega_{11} = A\mu^2 \left( -d\tilde{t} + d\tilde{\phi} + \frac{1}{2AB} dA \right),$$

$$\omega_{12} = \omega_{21} = -d\tilde{\phi} - \frac{1}{2A} dA - \frac{2}{\mu} d\mu,$$

$$\omega_{22} = \mu^{-2} \left( d\tilde{t} + d\tilde{\phi} - \frac{1}{2AB} dA \right),$$

(4.19)

where $A, B$ and $\tilde{\phi}, \tilde{t}$ are defined in (2.10) and (2.11), respectively. These expressions are well-defined on $S^1$ with the cyclic coordinate $\phi \sim \phi + 2\pi$.

### 5 Gauge function

Locally, the equation (4.2) admits a pure gauge solution

$$w_0(a, b|X) = -g^{-1}(a, b|X) * dg(a, b|X),$$

(5.1)

where $g(a, b|X)$ is some invertible ($g^{-1} \star g = g \star g^{-1} = 1$) element of the star-product algebra. Once the gauge function $g(a, b|X)$ is known, in the unfolded formulation this is equivalent to the full solution of the linear problem. In particular, the global symmetry parameters satisfying (4.6) have the form

$$\epsilon_0(a, b|X) = g^{-1}(a, b|X) * \xi * g(a, b|X),$$

(5.2)

where $\xi = \xi(a, b)$ is an arbitrary $X$-independent element of the star-product algebra. In Section 6 it is explained how the knowledge of $g(a, b|X)$ allows one to reconstruct...
a generic solution of free field equations. As is well-known, the pure gauge representation (5.1) is invariant under left global transformations

$$g(a, b|X) \rightarrow f(a, b) \ast g(a, b|X)$$  \hspace{1cm} (5.3)

with an $X$-independent star-invertible $f(a, b)$.

Using the results of [20] we obtain that the gauge function $g(a, b|W_1, W_2)$, that generates (4.7) and (4.8) via (5.1), is

$$g(a, b|W_1, W_2) = \frac{4}{\sqrt{\det|\left(W_1 + 1\right)\left(W_2 + 1\right)|}} \exp \left( -\frac{1}{2} \Pi^{\alpha\beta}(W_1)T_{\alpha\beta}^+ - \frac{1}{2} \Pi^{\alpha\beta}(W_2)T_{\alpha\beta}^- \right),$$

$$g^{-1}(a, b|W_1, W_2) = \frac{4}{\sqrt{\det|\left(W_1 + 1\right)\left(W_2 + 1\right)|}} \exp \left( \frac{1}{2} \Pi^{\alpha\beta}(W_1)T_{\alpha\beta}^+ + \frac{1}{2} \Pi^{\alpha\beta}(W_2)T_{\alpha\beta}^- \right)$$  \hspace{1cm} (5.4)

with\(^3\)

$$\Pi_{\alpha\beta}(W) = \Pi_{\beta\alpha}(W) = \left(\frac{W - 1}{W + 1}\right)_{\alpha\beta}$$  \hspace{1cm} (5.5)

and

$$T_{\alpha\beta}^\pm = a_\alpha a_\beta \pm b_\alpha b_\beta \pm (a_\alpha b_\beta + b_\alpha a_\beta).$$  \hspace{1cm} (5.6)

Here $T_{\alpha\beta}^\pm$ are the generators of the $sp(2)$ subalgebras of $sp(2) \oplus sp(2)$ generated by the two mutually commuting sets of oscillators $\alpha_\pm^\alpha = a_\alpha \pm b_\alpha$ satisfying the commutation relations $[\alpha_\pm^\alpha, \alpha_\pm^\beta]_\ast = \pm 2\epsilon_{\alpha\beta}$. In practice, it is often convenient to use the star-product defined in terms of mutually commuting oscillators $\alpha_\pm^\alpha$ as\(^4\)

$$(f \ast g)(\alpha^\pm) = \frac{1}{(2\pi)^2} \int f(\alpha^\pm + u)g(\alpha^\pm + v)e^{\mp u\alpha v^\dagger}d^2ud^2v.$$  \hspace{1cm} (5.7)

Taking into account that $T_{\alpha\beta}^\pm = \alpha_\pm^\alpha \alpha_\pm^\beta$, the following useful formula for the gauge function (5.4) results from (5.7) (for more detail see [20])

$$g(a, b|K_1, K_2) \ast g(a, b|U_1, U_2) = g(a, b|K_1U_1, U_2K_2)$$  \hspace{1cm} (5.8)

at the condition that the matrices $K_{1,2} + 1$ and $U_{1,2} + 1$ are non-degenerate. Owing to the equality

$$\Pi_{\alpha\beta}(W) = -\Pi_{\alpha\beta}(W^{-1}),$$

\(^3\)A matrix fraction $\frac{B}{A}$ is understood as $A^{-1}B$. Note that (5.5) is an analogue of the so-called Caley’s transformation.

\(^4\)Note that linear transformations of the generating elements of the Weyl star-product form automorphisms of the star-product algebra. This is the consequence of the definition of the Weyl star-product as resulting from the totally symmetrized ordering prescription in terms of the generating oscillators, which is insensitive to the particular choice of basis oscillators.
the transformation of the gauge function (5.4)

\[ g(a, b|W_1, W_2) \rightarrow g(a, b|W_1, W_2) \ast \Lambda^{-1}(a, b|V) \]  

(5.9)

with

\[ \Lambda(a, b|V) = \frac{4}{\det||V + 1||} \exp \left( \frac{1}{2} \Pi^{\alpha\beta}(V)(T^{+}_{\alpha\beta} - T^{-}_{\alpha\beta}) \right), \]

\[ \Lambda^{-1}(a, b|V) = \frac{4}{\det||V + 1||} \exp \left( -\frac{1}{2} \Pi^{\alpha\beta}(V)(T^{+}_{\alpha\beta} - T^{-}_{\alpha\beta}) \right), \]  

(5.10)

where \( V_{\alpha\beta}(X) \in Sp(2) \), describes the local Lorentz transformation of the dreibein (4.7)

\[ h_{\alpha\beta} \rightarrow V^\gamma_{\alpha} V^\delta_{\beta} h_{\gamma\delta}, \]  

(5.11)

that leaves invariant the metric (4.10).

Also from (5.8) it follows that the dreibein (4.14) and Lorentz connection (4.15) are reproduced by the gauge function of the form

\[ g(a, b|t, \phi, r) = \Phi(a, b|t, \phi) \ast U(a, b|r) \]  

(5.12)

with

\[ \Phi(a, b|t, \phi) = \frac{4}{\sqrt{\det||(K^+ + 1)(K^- + 1)||}} \exp \left( -\frac{1}{2} \Pi^{\alpha\beta}(K^+ T^{+}_{\alpha\beta} - \frac{1}{2} \Pi^{\alpha\beta}(K^- T^{-}_{\alpha\beta}) \right), \]

\[ U(a, b|r) = \frac{4}{\sqrt{\det||(U_1 + 1)(U_2 + 1)||}} \exp \left( -\frac{1}{2} \Pi^{\alpha\beta}(U_1 T^{+}_{\alpha\beta} - \frac{1}{2} \Pi^{\alpha\beta}(U_2 T^{-}_{\alpha\beta}) \right), \]

provided that \( U_1 U_2 = U_r \) (4.13).

Note that the metric (4.10) is invariant under global left and right group multiplications of \( S_{\gamma\delta}(X) S \rightarrow HS\tilde{H} \), where \( H \) and \( \tilde{H} \) are some \( X \)-independent elements of \( Sp(2) \). We will use this ambiguity in Section 6 to analyze the problem away from the outer horizon. For that purpose let us choose

\[ S_{\gamma\delta} = (HS_0)_{\gamma\delta} \]  

(5.13)

with the constant matrix \( H \) of the form

\[ H^{\delta}_{\gamma} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}, \]  

(5.14)

where \( \alpha^2 = A(r_0) \) and \( \beta^2 = B(r_0) \) for some \( r_0 > r_+ \). From (2.10) it follows that \( \alpha^2 - \beta^2 = 1 \). The new matrix \( S_{\gamma\delta} \) is

\[ S_{\gamma\delta} = \begin{pmatrix} \alpha(y - v) + \beta(x - u) & \alpha(x + u) - \beta(y + v) \\ \beta(y - v) + \alpha(x - u) & \beta(x + u) - \alpha(y + v) \end{pmatrix}. \]  

(5.15)
Taking into account (4.11) and (5.8), this transformation is reached by

\[ g(a, b|W_1, W_2) \rightarrow K(a, b|H) \ast g(a, b|W_1, W_2), \tag{5.16} \]

where

\[ K(a, b|H) = \frac{2}{\sqrt{\det ||H + 1||}} e^{-\frac{1}{2}H^{\gamma\delta}H_{\gamma\delta}}. \]

Thus, dreibein and Lorentz connection remain invariant under the transformation (5.13).

Note, that \( K(a, b|1) = 1 \). As explained in the next section, the case with \( K(a, b|H) \neq 1 \) plays a role of regularization that allows us to analyze the problem away from a point where a solution of interest develops singularity. After the solution is found we remove the regularization by setting \( \alpha = 1, \beta = 0 \).

6 Unfolded equations for 3d massless fields and Fock module

To formulate free dynamical equations for massless fields in the BTZ black hole background we follow the unfolded formulation of the massless field equations worked out in [28, 22, 20]. In particular, as shown in [22], the free field dynamics of massless spins \( s = 0 \) and \( s = \frac{1}{2} \) in \( AdS_3 \) can be formulated in a manifestly conformal invariant way in terms of sections of a certain Fock fiber bundle. Namely, consider space-time fields that take values in the Fock module generated by the oscillator \( b^\alpha \)

\[ |C(b|X)\rangle = C(b|X) \ast |0\rangle\langle 0|, \tag{6.1} \]

where \( C(b|X) \) is the generating function

\[ C(b|X) = \sum_{k=0}^{\infty} \frac{1}{k!} C_{\alpha_1...\alpha_k}(X)b^{\alpha_1}...b^{\alpha_k}, \tag{6.2} \]

and \( |0\rangle\langle 0| = e^{-2a_{\alpha}b^\alpha} \) is the Fock vacuum satisfying

\[ a_{\alpha} \ast |0\rangle\langle 0| = 0, \quad |0\rangle\langle 0| \ast b_{\alpha} = 0. \tag{6.3} \]

The dynamical massless scalar and spinor fields identify with the lowest components

\[ C(X) = C(b|X)|_{b=0}, \quad C_\alpha(X) = \frac{\partial}{\partial b^{\alpha}} C(b|X)|_{b=0}. \tag{6.4} \]

The dynamical equations for massless fields in a locally \( AdS_3 \) space-time can be formulated in the unfolded form

\[ d|C(b|X)\rangle - w_0(a, b|X) \ast |C(b|X)\rangle = 0; \tag{6.5} \]
where \( w_0(a, b|X) \) fulfils the zero-curvature condition (4.2). Let us show that (6.5) is equivalent to conformal Klein-Gordon and Dirac equations along with constraints that express higher multispinor components in the expansion (6.2) via higher derivatives of the dynamical fields [22]. Using (4.1), the equation (6.5) can be rewritten in the component form as

\[
DC_{\alpha_1...\alpha_k} = \frac{k(k-1)}{4} h_{(\alpha_1\alpha_2} C_{\alpha_3...\alpha_k)} + \frac{1}{4} \lambda_{\beta\lambda_1...\lambda_k} \,. \quad (6.6)
\]

where parentheses denote total symmetrization and \( D \) is the Lorentz covariant differential

\[
DC_{\alpha_1...\alpha_k} = dC_{\alpha_1...\alpha_k} + \frac{k}{2} \lambda_{\alpha_1...\alpha_k} \,. \quad (6.7)
\]

Setting \( k = 0 \) and \( k = 2 \) one gets from (6.6)

\[
D_n C = \frac{1}{4} h_{n, \alpha\beta} C_{\alpha\beta} \,, \quad (6.8)
\]

Using that \( C_{\alpha\beta\gamma\delta} \) is symmetric in its indices we obtain from (6.7), (6.8) the Klein-Gordon equation for the scalar field \( C(X) \)

\[
\Box C = D^n D_n C = \frac{3}{4} C \,. \quad (6.9)
\]

Analogously, from the equations (6.6) with \( k = 1 \) we obtain the Dirac equation

\[
h_{n, \alpha\beta} D_n C_{\alpha\beta} = 0 \,. \quad (6.10)
\]

All other fields in the multiplet (6.2) are expressed by (6.6) via derivatives of the dynamical fields (6.4).

The gauge transformation (4.5) acts on the Fock module in a natural way

\[
\delta|C(b|X)\rangle = \epsilon(a, b|X) \ast |C(b|X)\rangle \,. \quad (6.11)
\]

In particular, the Lorentz transformation (5.9) of the gauge function acts as follows

\[
|C(b|X)\rangle \rightarrow \Lambda(a, b|V) \ast |C(b|X)\rangle = |C(V_{\alpha\beta} b_\beta|X)\rangle \,, \quad (6.12)
\]

where \( \Lambda(a, b|V) \) is defined in (5.10).

Choosing \( w_0(a, b|X) \) in the pure gauge form (5.1), one obtains general local solution for \( |C(b|X)\rangle \) in the form

\[
|C(b|X)\rangle = g^{-1}(a, b|X) \ast |C(b|X_0)\rangle = g^{-1}(a, b|X) \ast C(b) \ast |0\rangle\langle 0| \,, \quad (6.13)
\]
where \( |C(b|X_0)\rangle = C(b)\star|0\rangle\langle 0| \) plays a role of initial data. The meaning of the formula (6.13) is that, for \( g(a, b|X_0) = 1 \) at some \( X = X_0 \), it gives a covariantized Taylor expansion that reconstructs a solution in terms of its on-shell nontrivial derivatives at \( X = X_0 \) parameterized by \( C(b) \). Note that this interpretation can be adjusted to any given regular point \( X_0 \) by the redefinition of the gauge function

\[
g(a, b|X) \rightarrow \tilde{g}(a, b|X) = g^{-1}(a, b|X_0) * g(a, b|X) \quad (6.14)
\]

that leaves unchanged the flat connection (4.7) and (4.8), effectively implying the redefinition of the \( C(b|X) \)

\[
|C(b|X)\rangle = \tilde{g}^{-1}(a, b|X) * \tilde{C}(b) * |0\rangle\langle 0|, \quad \tilde{C}(b) * |0\rangle\langle 0| = g^{-1}(a, b|X_0) * C(b) * |0\rangle\langle 0|.
\]

Clearly, this formalism cannot be applied to a point \( X_0 \) at which a solution \( C(b|X) \) develops a singularity. In practice, a space-time singularity at \( X_0 \) manifests itself in the nonexistence of the corresponding \( \tilde{C}(b) * |0\rangle\langle 0| \) (note that the star-product of nonpolynomial functions is not necessarily well defined). The way out is to perform some redefinition (6.14) that would correspond to the analysis at some regular point of the solution.

The unfolded form of massless field equations (6.6) is manifestly conformal invariant with the 3d conformal algebra \( sp(4) \sim o(3, 2) \) generated by various bilinears of the oscillators (3.2). It can be extended to the massive case by the replacement of the usual oscillators \( a_\alpha, b^\alpha \) with the so-called deformed oscillators along the lines of [13] (and references therein) or using the Fock module realization of the deformed oscillator algebra with the doubled number of oscillators as in [22] and in this paper. (Note that, as expected, the conformal algebra \( sp(4) \) breaks down to the \( AdS_3 \) algebra \( sp(2) \oplus sp(2) \) in the massive case because of the properties of the deformed oscillators.) As the corresponding formulation is technically more involved the case of arbitrary mass is not considered in this paper.

In the standard formulation, the case of a massive scalar field in the BTZ black hole background (2.4) was originally considered in [23, 24]. A solution of

\[
\Box C = m^2 C
\]

with definite energy \( E \) and angular momentum \( L \) has the form

\[
C(t, r, \phi) = e^{-iEt} e^{iL\phi} R(r), \quad (6.15)
\]

where

\[
R(r) = \left(1 - A(r)^{-1}\right) P \left( A(r)^{-\gamma} f(r) \right),
\]

\[
P = \frac{E - L}{2(r_+ - r_-)}, \quad Q = \frac{E + L}{2(r_+ + r_-)}, \quad m^2 = 4\gamma(1 - \gamma) \quad (6.17)
\]
and
\[ f(r) = K_1 F(P+\gamma, Q+\gamma, 2\gamma; A(r)^{-1}) + K_2 A(r)^{2\gamma-1} F(P+1-\gamma, Q+1-\gamma, 2-2\gamma; A(r)^{-1}) \]
with \( K_1, K_2 \) being integration constants. \( A(r) \) is defined in (2.10) and \( F(a, b, c; x) \) is the hypergeometric function. Note that the substitution \( \gamma \to 1 - \gamma \) interchanges the two independent basis solutions. The massless case (6.9) corresponds to \( m^2 = 3/4 \) and, consequently, \( \gamma = 1/4 \).

In the rest of this paper we show how known results for massless scalar and spinor fields in BTZ black hole background are reproduced in our approach. To single out the states with definite energy and angular momentum in the multiplet \( |C(b|X) \) we impose the following conditions
\[ \epsilon_t \ast |C(b|X) = -iE|C(b|X) \), \( \epsilon_\phi \ast |C(b|X) = iL|C(b|X) \), (6.18)\]
where \( \epsilon_t = g^{-1} \ast \xi_t \ast g \), \( \epsilon_\phi = g^{-1} \ast \xi_\phi \ast g \) are the symmetry generators of the BTZ Killing vectors \( \frac{\partial}{\partial \phi} \) (2.15) and \( \frac{\partial}{\partial t} \) (2.16). Using (6.13) we rewrite (6.18) as
\[ \xi_t \ast C(b) \ast |0\rangle\langle 0 = -iE C(b) \ast |0\rangle\langle 0, \quad \xi_\phi \ast C(b) \ast |0\rangle\langle 0 = iL C(b) \ast |0\rangle\langle 0. \] (6.19)
To analyse these equations, which define initial data \( C(b) \), we have to find the form of the generators \( \xi_t \) and \( \xi_\phi \) in the star-product algebra. This is done in the next section.

The following comment is now in order. The solution (6.15) is singular at \( r = r_+ \). Therefore, it cannot be treated in the unfolded formulation within the expansion at the horizon. Indeed, in Section 8 we will see that the equations (6.19), that correspond to the expansion at \( r = r_+ \), admit no solutions with \( C(b) \) regular in \( b^0 \), that can be interpreted in terms of the Fock module. Note, that since the gauge function (5.4) is regular on the horizon, this means that the singularity of a solution results from the condition that it carries definite energy and momentum and can be avoided by relaxing this condition.

To see that the gauge function (5.4) indeed corresponds to the expansion near horizon we observe that it can be transformed to unity by a Lorentz transformation. Actually, according to (5.9) and (6.13) the Lorentz transformation \( \Lambda(a, b|W_2) \) acts on \( g(a, b|W_1, W_2) \) as
\[ \tilde{g}(a, b|W_1, W_2) = g(a, b|W_1, W_2) \ast \Lambda^{-1}(a, b|W_2) = g(a, b|W_1 W_2, 1) \]
and thus,
\[ \tilde{g}(a, b|W_1(X_0), W_2(X_0)) = 1 \quad \text{iff} \quad W_1(X_0)W_2(X_0) = 1. \]
The choice of the gauge function (5.16) with \( H_\gamma^\delta = \delta_\gamma^\delta \) corresponds to \( S_\gamma^\delta = (HW_1W_2)^{\gamma^\delta} = \delta_\gamma^\delta \) at the point \( X_0 = \{r = r_+, t = 0, \phi = 0\} \) that belongs to the horizon. Indeed, \( S_0(X_0)^{\gamma^\delta} = \delta_\gamma^\delta \) implies \( v_0 = x_0 = y_0 = 0, u_0 = 1 \) that corresponds to \( r_0 = r_+, t_0 = \phi_0 = 0. \) To avoid this problem we apply the transformation (5.16) to achieve the redefinition (5.15). Now \( S(X_0)^{\gamma^\delta} = \delta_\gamma^\delta \) at the point
\( X_0 = \{ r_0 > r_+, t = 0, \phi = 0 \} \) which, unless \( \alpha = 1, \beta = 0 \), is regular thus allowing consistent unfolded analysis at least in some its neighbourhood. The regularization with \( \alpha \neq 1 \) and \( \beta \neq 0 \) is necessary for intermediate calculations (see Appendix B) while the limit \( \alpha \to 1, \beta \to 0 \) can be taken in the final expression for \( C(b|X) \). Recall, that the ambiguity in \( H_{\gamma \delta} \) does not affect BTZ black hole connections \((4.19), (4.20)\).

### 7 Star-product realization of \( AdS_3 \) Killing vectors

Any Killing vector \( \frac{\partial}{\partial \xi} \) of \( AdS_3 \) is a linear combination of \( J_{ab} \) \((2.14)\), i.e.,

\[
\frac{\partial}{\partial \xi} = \Omega^{ab} J_{ab},
\]

where \( \Omega^{ab} = -\Omega^{ba} \) are some constants. In the star-product algebra it corresponds to a global symmetry generator \( \xi \) that belongs to \( sp(2) \oplus sp(2) \) algebra, i.e.,

\[
\xi = (\kappa_1)^{\alpha \beta} L_{\alpha \beta} + (\kappa_2)^{\alpha \beta} P_{\alpha \beta}
\]

with some constant matrices \( \kappa_1 \) and \( \kappa_2 \). To find \( \xi_t \) and \( \xi_\phi \) associated with the BTZ Killings \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial \phi} \) let us evaluate the on-shell action of the generators \( L_{\alpha \beta} \) and \( P_{\alpha \beta} \) on the scalar field. We will use the gauge function \((5.16)\) with \( S_{\alpha \beta} \) \((5.15)\).

Let us introduce the generating parameters \( \xi^L = (\kappa_1)^{\alpha \beta} L_{\alpha \beta} \) for a Lorentz generator and \( \xi^P = (\kappa_2)^{\alpha \beta} P_{\alpha \beta} \) for a \( AdS \)-translation generator. Using \((5.2), (5.4), (6.11)\) and the equations of motion it is not hard to obtain (see Appendix A for details)

\[
\delta^L C(X) = \frac{1}{2} (\kappa_1)^{\alpha \beta} \mathcal{L}_{\alpha \beta} \nabla n C(X)
\]

and

\[
\delta^P C(X) = (\kappa_2)^{\alpha \beta} \mathcal{P}_{\alpha \beta} \nabla n C(X),
\]

where

\[
\mathcal{P}_{\alpha \beta} = \partial_n S_{\alpha \gamma} S_{\beta}^{\gamma} - \partial_n S_{\gamma \alpha} S_{\gamma}^{\beta}, \quad \mathcal{L}_{\alpha \beta} = \frac{1}{2} (\partial_n S_{\alpha \gamma} S_{\beta}^{\gamma} + \partial_n S_{\gamma \alpha} S_{\gamma}^{\beta}).
\]

Substituting \((5.15)\) into \((7.5)\) and comparing the resulting expression with the \( AdS \) Killing vectors \((2.14)\) we obtain

\[
\mathcal{L}_{\gamma \delta} = \begin{pmatrix}
\alpha \beta (J_{12} - J_{03}) - \alpha^2 J_{23} - \beta^2 J_{01} + J_{02} & -\alpha \beta (J_{01} + J_{03}) - \alpha^2 J_{03} + \beta^2 J_{12} \\
-\alpha \beta (J_{01} + J_{23}) - \alpha^2 J_{03} + \beta^2 J_{12} & \alpha \beta (J_{12} - J_{03}) - \alpha^2 J_{23} - \beta^2 J_{01} - J_{02}
\end{pmatrix},
\]

\[
\mathcal{P}_{\gamma \delta} = 2 \begin{pmatrix}
\alpha \beta (J_{12} - J_{03}) - \beta^2 J_{23} - \alpha^2 J_{01} + J_{13} & -\alpha \beta (J_{01} + J_{23}) - \alpha^2 J_{03} + \alpha^2 J_{12} \\
-\alpha \beta (J_{01} + J_{23}) - \beta^2 J_{03} + \alpha^2 J_{12} & \alpha \beta (J_{12} - J_{03}) - \beta^2 J_{23} - \alpha^2 J_{01} - J_{13}
\end{pmatrix}.
\]
From here we find that the components \( J_{03} \) and \( J_{12} \) which contribute to the BTZ Killing vectors (2.15) and (2.16) are

\[
J_{03} = -\frac{1}{4} \tau_1^{\gamma\delta} P_{\gamma\delta} - \frac{1}{2} \tau_2^{\gamma\delta} L_{\gamma\delta}, \\
J_{12} = \frac{1}{4} \tau_2^{\gamma\delta} P_{\gamma\delta} + \frac{1}{2} \tau_1^{\gamma\delta} L_{\gamma\delta},
\]

(7.7)

where

\[
\tau_1^{\gamma\delta} = \begin{pmatrix} -\alpha & \beta^2 \\ \beta^2 & \alpha \end{pmatrix}, \quad \tau_2^{\gamma\delta} = \begin{pmatrix} -\alpha & \alpha^2 \\ \alpha^2 & -\alpha \end{pmatrix}.
\]

(7.9)

Note that the matrices \( \tau_1 \) and \( \tau_2 \) satisfy

\[
\frac{1}{2} \tau_1^{\gamma\delta} \tau_2^{\gamma\delta} = \beta^2, \quad \frac{1}{2} \tau_2^{\gamma\delta} \tau_2^{\gamma\delta} = \alpha^2, \quad \tau_1^{\gamma\delta} \tau_2^{\gamma\delta} = 0,
\]

\[
\tau_2^{\gamma\delta} - \tau_1^{\gamma\delta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Thus, the oscillator realization of BTZ Killing vectors on the Fock module is

\[
\xi_t = \frac{1}{2} (r_+ \tau_1^{\gamma\delta} + r_- \tau_2^{\gamma\delta}) L_{\gamma\delta} + \frac{1}{4} (r_+ \tau_1^{\gamma\delta} + r_- \tau_2^{\gamma\delta}) P_{\gamma\delta}, \\
\xi_\phi = -\frac{1}{2} (r_+ \tau_1^{\gamma\delta} + r_- \tau_2^{\gamma\delta}) L_{\gamma\delta} - \frac{1}{4} (r_+ \tau_1^{\gamma\delta} + r_- \tau_2^{\gamma\delta}) P_{\gamma\delta}.
\]

(7.10)

(7.11)

8 Explicit solutions for massless fields

Having found the oscillator realization of BTZ Killing vectors (7.10), (7.11), we can rewrite the equations (6.19) on the generating function for a field with definite energy and angular momentum in the following form

\[
(r_2 - r_1)^{\gamma\delta} (L - \frac{1}{2} P)^{\gamma\delta} C(b) \ast |0\rangle \langle 0| = -4PC(b) \ast |0\rangle \langle 0|,
\]

(8.1a)

\[
(r_2 + r_1)^{\gamma\delta} (L + \frac{1}{2} P)^{\gamma\delta} C(b) \ast |0\rangle \langle 0| = -4QC(b) \ast |0\rangle \langle 0|,
\]

(8.1b)

where \( P \) and \( Q \) are given in (6.17). Let

\[
b^\alpha = (p,q).
\]

(8.2)

Then the system (8.1) amounts to the two second-order differential equations

\[
(p \partial_p - q \partial_q + pq - \partial_p \partial_q) C(p,q) = -4PC(p,q),
\]

(8.3)
\[-\alpha \beta (\partial_p \partial_p + \partial_q \partial_q + p^2 + q^2 + 2p \partial_q - 2q \partial_p) C(p, q) + (\alpha^2 + \beta^2) (p \partial_p - q \partial_q + \partial_p \partial_q - p q) C(p, q) = -4QC(p, q). \quad (8.4)\]

Note that the case of \(\alpha = 1\) and \(\beta = 0\) is degenerate reducing the sum of the equations (8.3) and (8.4) to the first order equation. As a result, the system (8.3), (8.4) at \(\alpha = 1, \beta = 0\) admits no solutions regular in \(b^\alpha\). Indeed, in this case from (8.3) and (8.4) it follows that
\[(p \partial_p - q \partial_q) C(p, q) = -2(P + Q)C(p, q)\]
and, therefore, \(C(p, q) = p^{-2(P+Q)} \chi(pq)\) is not regular in the oscillators \(b^\alpha\) for physical values of \(P\) and \(Q\). The substitution \(C(p, q) = e^{pq} f(p, q)\) reduces (8.3) to
\[(\partial_p \partial_q + 2q \partial_q) f(p, q) = (4P - 1) f(p, q), \quad (8.5)\]
which can be solved as
\[f(p, q) = \int_{-\infty}^{\infty} e^{-\frac{\alpha}{\beta} s^2} g(s)e^{ps} \left(\frac{s}{2} + q\right)^{2P-\frac{1}{2}} ds, \quad (8.6)\]
where \(g(s)\) is still arbitrary. Plugging this into (8.4) leads to the differential equation for \(g(s)\)
\[\alpha \beta g''(s) - \frac{1}{2} sg'(s) - (Q + \frac{1}{4}) g(s) = 0, \quad (8.7)\]
which is the confluent hypergeometric equation. Its general solution can be expressed in the integral form as a superposition of the following two basis solutions
\[\int_{0}^{\infty} w^{2Q-\frac{1}{2}} e^{-\alpha \beta w^2 + sw} dw \quad \text{and} \quad \int_{0}^{\infty} w^{2Q-\frac{1}{2}} e^{-\alpha \beta w^2 - sw} dw. \quad (8.8)\]
The integrals are convergent since \(\alpha \beta > 0\) and \(\text{Re} \ Q > -\frac{1}{4}\).
Abusing notation, we denote general solution of (8.7) as
\[\int_{0}^{\infty} w^{2Q-\frac{1}{2}} e^{-\alpha \beta w^2 + sw} dw, \quad (8.9)\]
assuming by this a linear combination of the integrals
\[\int_{0}^{\infty} w^{2Q-\frac{1}{2}} e^{-\alpha \beta w^2 + sw} dw \quad \text{and} \quad \int_{-\infty}^{0} w^{2Q-\frac{1}{2}} e^{-\alpha \beta w^2 - sw} dw. \quad (8.10)\]

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Note that although the second integral in (8.10) is infinitely-valued, the ambiguity is modulo an arbitrary constant phase factor that can be absorbed into an integration constant.

Using \( g(s) \) from (8.9) and changing the integration variable \( s \to s - 2q \) in (8.6), we obtain the generating function in the form

\[
C(p, q) = e^{-pq} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dw \, e^{-\frac{q}{2s}(s-2q)^2 - \alpha \beta w^2 + (s-2q)w + sp} s^{2P - \frac{1}{2}} w^{2Q - \frac{1}{2}}
\]

\[
= \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dw \, e^{m_{\alpha} b^1 b^\gamma + n_{\gamma} w} e^{-\alpha \beta w^2 + s w - \frac{4\pi s^2}{2\beta} w^2} s^{2P - \frac{1}{2}} w^{2Q - \frac{1}{2}},
\]  

(8.11)

where we use the notation

\[
m_{\alpha} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{2}{3} & \end{pmatrix}, \quad n_{\gamma} = (s, \frac{\alpha}{\beta} s - 2w).
\]

Using (8.11) and (5.16) one can calculate generating function (6.13), redefining the integration variable \( \beta w \to w \) and setting then \( \alpha = 1, \beta = 0 \). We obtain the following integral representation for the generating function \( C(b|X) \) up to a constant factor (see Appendix B)

\[
C(b|t, r, \phi) = e^{-iEt} e^{iL\phi} A(r)^{-\frac{1}{2}} (1 - A(r)^{-1}) \frac{d^2}{dz} e^{-b^1 b^2} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dw \, s^{2P - \frac{1}{2}} w^{2Q - \frac{1}{2}}
\]

\[
\times \exp \left( -\frac{s^2}{4} - \frac{w^2}{4A(r)} + \frac{sw}{2A(r)} + \mu(r) sb^1 - \eta(r) w b^2 \right),
\]  

(8.12)

where \( A(r) \) and \( \mu(r), \eta(r) \) are defined in (2.10) and (4.17), respectively. Note that, as discussed in Section 6 and in the beginning of this section, the formalism does not allow to set \( \alpha = 1, \beta = 0 \) in \( C(b) \) before completing its star multiplication with \( g^{-1}(a, b|W_1, W_2) \).

By construction, the generating function (8.12) gives solutions of free massless equations in the BTZ background along with all derivatives of the massless fields as coefficients of the expansion in powers of \( b^\alpha \). Using the standard integral representation for the hypergeometric function (see, e.g., [29]) the generating function (8.12) can be written in the form

\[
C(b|t, r, \phi) = e^{-iEt} e^{iL\phi} (1 - A^{-1}) \frac{d^2}{dz} A^{-\frac{1}{2}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mu(-\eta)^n}{m!n!} A^m (b^1)^m (b^2)^n e^{-b^1 b^2}
\]

\[
\times \left[ K_1 F \left( P + \frac{2m + 1}{4}, Q + \frac{2n + 1}{4}, \frac{1}{2}, A^{-1} \right) + K_2 A^{-\frac{1}{2}} F \left( P + \frac{2m + 3}{4}, Q + \frac{2n + 3}{4}, \frac{3}{2}, A^{-1} \right) \right],
\]  

(8.13)
where $K_1, K_2$ are arbitrary integration constants and $P, Q$ are defined in (6.17).

As explained in Section 6, a solution for the scalar field is given by $C (0 | X)$ (6.4). So, from (8.13) we obtain
\[
C(t, r, \phi) = e^{-iEt} e^{iL\phi} (1 - A^{-1})^{P+Q \over 2} A^{-1 \over 4} \times \left[ K_1 F \left( P + {1 \over 4}, Q + {1 \over 4}, {1 \over 2}; A^{-1} \right) + K_2 A^{-1 \over 2} F \left( P + {3 \over 4}, Q + {3 \over 4}, {3 \over 2}; A^{-1} \right) \right],
\]
\[(8.14)\]

This formula coincides with the solution of massless scalar equation (6.9) in the BTZ background originally obtained in [23, 24] for any value of mass.

Also it is now straightforward to find from (8.13) the solution for the spinor field $C_{\alpha} (X)$ (6.4)
\[
C_{\alpha} (t, r, \phi) = e^{-iEt} e^{iL\phi} (1 - A^{-1})^{P+Q \over 2} A^{-1 \over 4} (K_1 \psi_{1\alpha} + K_2 \psi_{2\alpha}),
\]
\[(8.15)\]

with
\[
\psi_1 = \begin{pmatrix}
\mu F \left( P + {3 \over 4}, Q + {1 \over 4}, {1 \over 2}; A^{-1} \right) \\
-(Q + {1 \over 2}) \eta F \left( P + {3 \over 4}, Q + {3 \over 4}, {3 \over 2}; A^{-1} \right)
\end{pmatrix}
\]
and
\[
\psi_2 = \begin{pmatrix}
(P + {1 \over 4}) \mu A^{-1 \over 2} F \left( P + {3 \over 4}, Q + {3 \over 4}, {3 \over 2}; A^{-1} \right) \\
-\eta A^{-1 \over 2} F \left( P + {1 \over 4}, Q + {3 \over 4}, {1 \over 2}; A^{-1} \right)
\end{pmatrix}.
\]

Different choices of the functions $\mu (r), \eta (r)$ (4.17) correspond to different Lorentz gauges in the general solution of Dirac equation (6.10) with definite energy $E$ and angular momentum $L$ in BTZ black hole background. Note that our Lorentz gauge differs from that of [25, 26].

### 9 Extremal BTZ black hole

Exact solutions for the Klein-Gordon and Dirac equations in the extremal BTZ background were found in [30] and [31]. In the extremal case with $M = |J|$ the two horizons coincide and the parametrization (2.9) cannot be used. As before, black hole connection $w_0 (a, b | X)$ is expressed via the gauge function $g (a, b | W_1, W_2)$ but now the ambient coordinates $X^a$ are parameterized differently (see [11]). In the extremal case the Killing vector responsible for the identification (2.13) has additional terms that cannot be removed by a $SO(2, 2)$ transformation
\[
\partial \over \partial \phi = -\lambda r_+ J_{12} + \lambda r_- J_{03} + J_{13} - J_{23}.
\]
\[(9.1)\]

Consequently, the system of equations (8.1) changes its form. Fortunately, to obtain the solutions in the extremal case it is not necessary to solve the equations again.
As pointed out e.g. in [32] one can simply take the limit for the solutions (8.14) and (8.15). Namely,

\[ A^{-1} P = \kappa = i \frac{(E - L)(r_+ + r_-)}{2(r^2 - r_e^2)} \]

is regular in the limit \( r_+ \to r_- \) (\( P \) is defined in (6.17)). Now let us substitute \( A^{-1} = \frac{\kappa}{P} \) in (8.14), (8.15) and consider the limit \( r_+ \to r_- \) or, equivalently, \( P \to \infty \). The result can be written in terms of Whittaker functions \( M_{p,q}(x) \) [33].

For the massless scalar we obtain

\[ C(t, r, \phi) = e^{-iEt} e^{iL\phi} \left( K_1 M_{-Q, -\frac{1}{4}}(\kappa_e) + K_2 M_{-Q, \frac{1}{4}}(\kappa_e) \right), \quad (9.2) \]

where

\[ \kappa_e = i \frac{(E - L) r_e}{r^2 - r_e^2} \quad (9.3) \]

and \( r_e \) is the horizon of the extremal black hole. One can easily check that this solution indeed satisfies the conformal Klein-Gordon equation written in extremal black hole background.

For the massless spinor we have

\[ C_{\alpha}(t, r, \phi) = e^{-iEt} e^{iL\phi} (K_1 \psi_1 + K_2 \psi_2), \quad (9.4) \]

with

\[ \psi_1 = \begin{pmatrix} \mu M_{-Q, -\frac{1}{4}}(\kappa_e) \\ -(Q + \frac{1}{4}) \eta \kappa_e^{-\frac{1}{4}} M_{Q, -\frac{1}{4}}(\kappa_e) \end{pmatrix} \]

and

\[ \psi_2 = \begin{pmatrix} \mu M_{-Q, \frac{1}{4}}(\kappa_e) \\ -\eta \kappa_e^{\frac{1}{4}} M_{Q, \frac{1}{4}}(\kappa_e) \end{pmatrix}, \]

where \( K_1, K_2 \) are arbitrary constants.

10 Symmetries of massless fields in BTZ black hole background

Any fixed vacuum solution (4.1) of (4.2) breaks local HS symmetries to the global symmetries associated with the stability subalgebra with the parameter \( \epsilon_0(a, b|X) \) satisfying (4.6). The BTZ boundary condition (2.13) restricts the space of solutions of (4.6) thus providing a (non-local) mechanism of spontaneous symmetry breaking. Namely, only those symmetries remain well-defined upon the factorization (2.13) that commute to the Killing vector \( \xi_{\phi} \) responsible for the angle identification

\[ [\xi(a, b), \xi_{\phi}]_\ast = 0, \quad (10.1) \]
where $\xi(a, b)$ is the generating parameter in (5.2). The spaces of solutions of (10.1) are different for generic and extremal black holes. We, therefore, consider these cases separately.

Let us start with the case of generic black hole $r_+^2 - r_-^2 > 0$ and $\frac{\partial}{\partial \phi}$ given by (2.15) with its star-product realization $\xi_{\phi}$ (7.11). To make contact with the gauge function (5.4), we set $\alpha = 1$ and $\beta = 0$, thus

$$\xi_{\phi} = -\frac{1}{2} r_- \tau^{\gamma \delta} L_{\gamma \delta} - \frac{1}{4} r_+ \tau^{\gamma \delta} P_{\gamma \delta}, \quad \tau^{\gamma \delta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (10.2)$$

To solve the equation (10.1) it is convenient to introduce the new set of oscillators $p_\alpha, q_\beta$

$$p_1 = \frac{1}{\sqrt{2}} (a_1 + b_1), \quad p_2 = \frac{1}{\sqrt{2}} (a_1 - b_1),$$

$$q_1 = \frac{1}{\sqrt{2}} (a_2 - b_2), \quad q_2 = \frac{1}{\sqrt{2}} (a_2 + b_2) \quad (10.3)$$

that satisfy the commutation relations

$$[p_\alpha, p_\beta]_* = [q_\alpha, q_\beta]_* = 0, \quad [p_\alpha, q_\beta]_* = \delta_\alpha^\beta \quad (10.4)$$

and are chosen so that $\xi_{\phi}$ takes the following simple form

$$\xi_{\phi} = -\frac{1}{2} A_{\alpha \beta} p_\beta q^\alpha, \quad A_{\alpha \beta} = \begin{pmatrix} r_+ + r_- & 0 \\ 0 & r_- - r_+ \end{pmatrix}. \quad (10.5)$$

Note, that since the oscillator commutation relations remain unchanged, the same star-product formula (3.4) is valid with $a$ and $b$ replaced by $p$ and $q$, respectively.

The equation (10.1) gives (cf (B.1))

$$A_{\alpha \beta} (p_\beta \frac{\partial}{\partial p_\alpha} + q^\alpha \frac{\partial}{\partial q_\beta}) \xi(p, q) = 0. \quad (10.6)$$

Infinitesimal HS symmetries we are interested in correspond to local transformations with a finite number of space-time derivatives. The corresponding symmetry generating parameters $\xi(p, q)$ are described by polynomial functions of the oscillators. A class of polynomial solutions of (10.1) depends on the parameter

$$\sigma = \frac{r_+ + r_-}{r_+ - r_-}. \quad (10.7)$$

There are following different cases:

- $\sigma \notin \mathbb{N}$
For any positive non-integer $\sigma$ the general solution of (10.6) is
\[
\xi(p, q) = \sum R_{mn}(q_1 p_2)^m (q_2 p_1)^n \sim \sum \tilde{R}_{mn}(\xi_0)^m (\xi_t)^n,
\] (10.8)
where $R_{mn}$ are arbitrary constants. Note, that the conformal algebra $sp(4)$, spanned by various bilinears of oscillators (10.3), is broken to the $u(1) \oplus u(1)$ subalgebra spanned by the BTZ Killing vectors $\xi_\phi$ and $\xi_t$ (equivalently, $q_1 p_2$ and $q_2 p_1$).

- $\sigma = 2, 3, \ldots$

In the interesting case of positive integer $\sigma$ a larger class of HS symmetries survives. General solution of (10.6) is
\[
\xi(p, q) = \sum R_{n_1 n_2 m_1 m_2} (q_1 p_2)^{n_1} (q_2 p_1)^{n_2} (p_1 p_2^\sigma)^{m_1} (q_1 q_2^\sigma)^{m_2}.
\] (10.9)

The conformal algebra $sp(4)$ is still broken to $u(1) \oplus u(1)$. The condition $\sigma = 2, 3, \ldots$ imposes specific quantization of the mass $M$ in terms of the angular momentum $J$ since $\sigma = \sqrt{\frac{M+J}{M-J}}$. For this case it follows that $\rho^+ = (\rho^-)^\sigma$ which means that one of the holonomy operators involved in the factorization of BTZ black hole is the integral power of the other.\footnote{We are grateful to S. Carlip for drawing our attention to this fact.}

- $\sigma = 1$

This is the case of non-rotating black hole with $J = 0$. Polynomial solutions for $\xi(p, q)$ are
\[
\xi(p, q) = \sum R_{m_1 m_2 n_1 n_2} (q_1 p_2)^{m_1} (q_2 p_1)^{m_2} (p_1 p_2)^{n_1} (q_1 q_2)^{n_2}.
\] (10.10)

The distinguishing property of the non-rotating black hole is that in this case a larger part of the conformal symmetry survives. It is generated by the bilinears $q_1 p_2$, $q_2 p_1$, $p_1 p_2$, $q_1 q_2$ and is isomorphic to $gl(2)$. In addition to BTZ Killing vectors, it has two generators of special conformal transformations associated with $b_1 b_1$ and $b_2 b_2$.

Let us proceed to the extremal case. The Killing vector of the extremal black hole with $r_- = r_+ = r_e$ responsible for the angle identification is defined in (9.1). Using (7.6) and setting $\alpha = 1$, $\beta = 0$, the expression for $\xi_\phi$ in $p, q$ oscillators reads
\[
\xi_\phi = -r_e p_1 q_2 + \frac{1}{2} ((p_1)^2 - (q_1)^2).
\] (10.11)

Performing simple star-product calculations, we rewrite (10.1) in the form
\[
r_e \left( p^2 \frac{\partial}{\partial p^2} - q^1 \frac{\partial}{\partial q^1} \right) \xi - p^2 \frac{\partial \xi}{\partial q^1} + q^2 \frac{\partial \xi}{\partial p^1} = 0.
\] (10.12)

The cases with $r_e \neq 0$ and $r_e = 0$ (i.e., $M = J = 0$) require different consideration.
The general polynomial solution of (10.12) is
\[ \xi(p, q) = \sum R_{mn}(p_1)^m(2r_eq_2 - p_1)^n(q_1)^n. \] (10.13)

One observes that, in addition to the usual \( u(1) \oplus u(1) \) algebra generated by Killing vectors \( \xi_t \) and \( \xi_\phi \), extremal black hole has one Killing spinor generated by \( q_1 \). This is in accordance with [34] where supersymmetry of an extremal BTZ black hole was found.

The vacuum case of \( M = J = 0 \) provides the black hole background with the maximal number of supersymmetries and generic \( \xi(p, q) \) of the form
\[ \xi(p, q) = \sum R_{mn}(p_1)^m(q_1)^n(q_1q_2 + p_1p_2)^k. \] (10.14)

It has two exact supersymmetries [34] generated by \( p_1 \) and \( q_1 \) and a part of conformal algebra spanned by \( p_1p_1, q_1q_1, q_1p_1, q_1q_2 + p_1p_2 \) which is isomorphic to \( E_2 \oplus u(1) \), where \( E_2 \) is the algebra of motions of a two-dimensional Euclidian plane.

Note that in our approach it is elementary to obtain explicit formulae for the symmetry transformation laws. The corresponding symmetry parameter (5.2) for any generating parameter \( \xi(a, b) \) results from the differentiation of the generating parameter (A.1) from Appendix A with respect to the sources \( \mu_\alpha, \eta_\beta \).

11 Conclusion

We have shown that the BTZ black hole can be concisely formulated in terms of the star-product formalism underlying the present day formulations of nonlinear HS gauge theories. Satisfying the \( o(2, 1) \oplus o(2, 1) \) zero-curvature condition, the BTZ black hole is automatically an exact solution of the nonlinear 3d HS gauge theory. It is shown how the star-product formulation allows one to solve free field equations in the black hole background.

The leftover higher spin and lower spin symmetries of massless fields in the BTZ black hole background are found. In the case of \( M > 0 \) non-extremal BTZ black hole, the conformal algebra \( o(3, 2) \sim sp(4) \) turns out to be broken to the \( u(1) \oplus u(1) \) subalgebra generated by BTZ Killing vectors and to \( gl(2) \) in the cases of \( J > 0 \) and \( J = 0 \), respectively. For \( \sigma = \sqrt{\frac{M + J\lambda}{M - J\lambda}} = 1, 2, \ldots \) the leftover HS symmetries get enhanced. A physical interpretation of this enhancement remains to be understood. Our analysis of extremal BTZ black hole reproduces the previously known lower spin (super)symmetries and determines their HS extensions.
We hope to extend the obtained results in the following two, most likely related, directions. Firstly, to Kerr solutions of nonlinear HS gauge theories in four and higher dimensions and, secondly, to BTZ-like solutions in the generalized space-times with matrix coordinates which are $Sp(M)$ group manifolds in the $AdS$-like case.

Acknowledgement

We are grateful to S. Carlip, R. Metsaev and I. Tyutin for useful comments. This research was supported in part by INTAS Grants No 03-51-6346 and 05-7928, RFBR Grant No 05-02-17654, LSS No 4401.2006.2. V.D. and A.M. acknowledges financial support from Landau Scholarship Foundation. V.D. and A.M. are grateful to Dynasty Foundation for financial support, to “Ettore Majorana” Foundation and Center for Scientific Culture for kind hospitality where a part of the work was done, and to Museo Storico della Fisica e Centro Studi e Ricerche Enrico Fermi for Enrico Fermi Junior award. M.V. is grateful to J. Azcarraga for hospitality at the Physics Department of Valencia University, Spain, where a part of this work was done.

Appendix

A. Action of (angular) momentum operator on a scalar field

To find the on-shell action of $L_{\alpha\beta}$ and $P_{\alpha\beta}$ generators on the scalar field let us consider the generating parameter

$$\xi = \exp(a_{\alpha}\mu^{\alpha} + b_{\alpha}\eta^{\alpha})$$

with constant sources $\mu_{\alpha}$ and $\eta_{\alpha}$. As shown in [20], the global symmetry generators that result from (5.2) read as

$$\epsilon = \exp(a_{\alpha}\hat{\mu}^{\alpha} + b_{\alpha}\hat{\eta}^{\alpha}),$$

(A.1)

where

$$\hat{\mu}_{\alpha} = \frac{1}{2}(W^{-1}_{1} + W_{2})_{\alpha}^{\beta}\mu_{\beta} + \frac{1}{2}(W^{-1}_{1} - W_{2})_{\alpha}^{\beta}\eta_{\beta},$$ \hspace{1cm} (A.2a)

$$\hat{\eta}_{\alpha} = \frac{1}{2}(W^{-1}_{1} + W_{2})_{\alpha}^{\beta}\eta_{\beta} + \frac{1}{2}(W^{-1}_{1} - W_{2})_{\alpha}^{\beta}\mu_{\beta}.$$ \hspace{1cm} (A.2b)
The differentiation with respect to the sources \( \mu_\alpha, \eta_\alpha \) gives the generators of global \( AdS_3 \) symmetries

\[
\epsilon^P_{\alpha\beta} = \left( \frac{\partial^2}{\partial \mu^\alpha \partial \mu^\beta} + \frac{\partial^2}{\partial \eta^\alpha \partial \eta^\beta} \right) \epsilon \bigg|_{\mu=0, \eta=0}, \quad (A.3a)
\]

\[
\epsilon^L_{\alpha\beta} = \frac{1}{2} \left( \frac{\partial^2}{\partial \mu^\alpha \partial \eta^\beta} + \frac{\partial^2}{\partial \eta^\alpha \partial \mu^\beta} \right) \epsilon \bigg|_{\mu=0, \eta=0}. \quad (A.3b)
\]

Using (6.11) and performing the star-products one obtains

\[
\delta |C(b|X)\rangle = \epsilon \ast |C(b|X)\rangle = \exp \left( b_\alpha \tilde{\eta}^\alpha + \frac{1}{2} \tilde{\mu}_\alpha \tilde{\eta}^\alpha \right) C(b + \tilde{\mu}|X) \ast |0\rangle \langle 0|,
\]

so that

\[
\delta C(b|X) = C(b + \tilde{\mu}|X) \exp \left( b_\alpha \tilde{\eta}^\alpha + \frac{1}{2} \tilde{\mu}_\alpha \tilde{\eta}^\alpha \right). \quad (A.5)
\]

As a result, from (A.3) and (A.5) one obtains the following action of \( AdS_3 \) symmetries on the scalar field \( C(X) \)

\[
\delta^P_{\alpha\beta} C(X) = \left( \frac{\partial^2}{\partial \mu^\alpha \partial \mu^\beta} + \frac{\partial^2}{\partial \eta^\alpha \partial \eta^\beta} \right) \left( C(\tilde{\mu}|X) e^{\frac{1}{2} \tilde{\mu}_\gamma \tilde{\eta}^\gamma} \right) \bigg|_{\mu=0, \eta=0}, \quad (A.6a)
\]

\[
\delta^L_{\alpha\beta} C(X) = \frac{1}{2} \left( \frac{\partial^2}{\partial \mu^\alpha \partial \eta^\beta} + \frac{\partial^2}{\partial \eta^\alpha \partial \mu^\beta} \right) \left( C(\tilde{\mu}|X) e^{\frac{1}{2} \tilde{\mu}_\gamma \tilde{\eta}^\gamma} \right) \bigg|_{\mu=0, \eta=0}. \quad (A.6b)
\]

that gives

\[
\delta^P_{\alpha\beta} C(X) = \frac{1}{2} \left( W_1 \alpha \mu \beta ^\delta + (W_2^{-1}) \alpha \gamma (W_2^{-1}) \beta ^\delta \right) \frac{\partial^2 C(\mu|X)}{\partial \mu^\gamma \partial \mu^\delta} \bigg|_{\mu=0}, \quad (A.7a)
\]

\[
\delta^L_{\alpha\beta} C(X) = \frac{1}{4} \left( W_1 \alpha \mu \beta ^\delta - (W_2^{-1}) \alpha \gamma (W_2^{-1}) \beta ^\delta \right) \frac{\partial^2 C(\mu|X)}{\partial \mu^\gamma \partial \mu^\delta} \bigg|_{\mu=0}. \quad (A.7b)
\]

Using the equation of motion (6.10), for the scalar field we have

\[
dC(X) = \frac{1}{4} h^{\alpha\beta} \frac{\partial^2 C(\mu|X)}{\partial \mu^\alpha \partial \mu^\beta} \bigg|_{\mu=0}. \quad (A.8)
\]

or, equivalently,

\[
2 h^{\alpha\beta} \partial_n C(X) = \frac{\partial^2 C(\mu|X)}{\partial \mu^\alpha \partial \mu^\beta} \bigg|_{\mu=0}. \quad (A.9)
\]

Using (4.7) and taking (4.11) into account, the substitution of (A.9) into (A.7) yields the following action of \( AdS_3 \) isometry generators on the scalar field

\[
\delta^P_{\alpha\beta} C(X) = \left( \partial_m S_\alpha \gamma S_\beta ^\gamma - \partial_m S_\gamma \alpha S_\gamma ^\gamma \right) g^{mn} \partial_n C(X), \quad (A.10a)
\]

\[
\delta^L_{\alpha\beta} C(X) = \frac{1}{2} \left( \partial_m S_\alpha \gamma S_\beta ^\gamma + \partial_m S_\gamma \alpha S_\gamma ^\gamma \right) g^{mn} \partial_n C(X). \quad (A.10b)
\]
B. Star-product calculus

Here we collect some useful star-product formulae used throughout this paper.

\[ a_\alpha a_\beta \star f(a, b) = a_\alpha a_\beta f(a, b) + \frac{1}{2} \left( a_\alpha \frac{\partial}{\partial b^\beta} + a_\beta \frac{\partial}{\partial b^\alpha} \right) f(a, b) + \frac{1}{4} \frac{\partial^2}{\partial b^\alpha \partial b^\beta} f(a, b), \]

\[ b_\alpha b_\beta \star f(a, b) = b_\alpha b_\beta f(a, b) + \frac{1}{2} \left( b_\alpha \frac{\partial}{\partial a^\beta} + b_\beta \frac{\partial}{\partial a^\alpha} \right) f(a, b) + \frac{1}{4} \frac{\partial^2}{\partial a^\alpha \partial a^\beta} f(a, b), \]  

\[ a_\alpha b_\beta \star f(a, b) = a_\alpha b_\beta f(a, b) + \frac{1}{2} \left( a_\alpha \frac{\partial}{\partial a^\beta} + b_\beta \frac{\partial}{\partial b^\alpha} \right) f(a, b) + \frac{1}{4} \frac{\partial^2}{\partial a^\alpha \partial b^\beta} f(a, b). \]  

Let us now calculate the generating function (8.12). According to the prescription given in Section 6,

\[ |C(b|X)\rangle = g^{-1}(a, b|HW_1, W_2) \star C(b) \star |0\rangle_{\beta=0} \bigg|_{\alpha=1}. \]  

where \( C(b) \) is defined in (8.11). The direct calculations are quite cumbersome. In practice, it is convenient to eliminate Lorentz term in the gauge function \( g(a, b|HW_1, W_2) \).

With the aid of (5.9), (5.8) one can rewrite (B.2) in the form

\[ |C(b|X)\rangle = \Lambda^{-1}(a, b|V) \star g^{-1}(a, b|HW_1 V, V^{-1} W_2) \star C(b) \star |0\rangle_{\beta=0} \bigg|_{\alpha=1}. \]

Let us choose Lorentz transformation matrix \( V_{\gamma \delta} \) such that

\[ HW_1 V = V^{-1} W_2 = \sqrt{S}. \]

Then the gauge function takes the form

\[ g_0(a, b|S) = g(a, b|\sqrt{S}, \sqrt{S}) = \frac{4}{\det ||\sqrt{S} + 1||} \exp \left( -\Pi^\alpha_\beta(\sqrt{S})(a_\alpha a_\beta + b_\alpha b_\beta) \right), \]

where \( S_{\gamma \delta} \) is given in (5.15). Thus using (6.12), the generating function can be calculated as

\[ |C_0(b|X)\rangle = |C_0((V_0^{-1})_{\gamma \delta} b_{\delta}|X)\rangle, \]

where \( V_0 \) is the Lorentz transformation matrix at \( \alpha = 1, \beta = 0 \) which has the following form

\[ V_{0 \gamma \delta} = \sqrt{\frac{(u+x)(y-v)}{2(u+1)}} \begin{pmatrix} \mu(r) & \mu(r) \frac{y-v-1}{y-v} \\ \eta(r)(u-x) & \eta(r) \frac{(u-x)(y-v)}{y-v} \end{pmatrix} \]

and

\[ |C_0(b|X)\rangle = g_0^{-1}(a, b|S) \star C(b) \star |0\rangle_{\beta=0} \bigg|_{\alpha=1}. \]
To evaluate star-products of Gaussian exponentials of the form

\[
F(b) \star |0\rangle\langle 0| = \sqrt{\det ||1 - f^2||} e^{f_{\alpha\beta}(a_\alpha b_\beta + b_\alpha a_\beta)} \star e^{m_\gamma b_\gamma b_\delta + n_\gamma b_\gamma \star |0\rangle\langle 0|}
\]  

(B.7)

one uses \(3.4\) to obtain by simple Gaussian integration

\[
F(b) = \sqrt{\det ||1 - f^2||} \frac{\det(A)}{\det(A)} \exp \left( \left[ (f^{-1} + 2m) \frac{1}{f + f^{-1} + 4m} (f + 2m) - m \right]_{\alpha\beta} b^\alpha b^\beta \\
+ \left[ \frac{1}{f + f^{-1} + 4m} (f^{-1} - f) \right]_{\alpha\beta} n^\alpha b^\beta + \left[ \frac{1}{f + f^{-1} + 4m} \right]_{\alpha\beta} n^\alpha n^\beta \right),
\]

(B.8)

where

\[ (f^{-1})_{\alpha\beta} f^\gamma_{\beta} = \delta_{\alpha\gamma} \]

and

\[ A_{\alpha\beta} = \varepsilon_{\alpha\beta} + f^\gamma_{\alpha} f_{\gamma\beta} + 4 f^\gamma_{\alpha} m_{\gamma\beta}. \]

Using \(8.11\), \(B.4\) and \(B.8\) we obtain the following result

\[
F(b) = \left( \frac{\beta}{y - v} \right)^{\frac{1}{2}} \frac{1}{\sqrt{2(\alpha u - \beta y + 1)}} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dw \ s^{2P-\frac{1}{2}} w^{2Q-\frac{1}{2}} e^{-\frac{u + x}{4(y - v)} s^2 - \frac{\beta}{y - v} w^2 + \frac{\beta}{y - v} s w} \exp \left( B_{\gamma\delta} b^\gamma b^\delta + \frac{A_\gamma b^\gamma}{(y - v) \sqrt{2(\alpha u - \beta y + 1)}} \right)
\]

(B.9)

with

\[ A_\gamma = \begin{pmatrix} 2\beta(\alpha(y - v) + \beta(x - u)) w + (\beta + y - v)s \\ 2\beta(\alpha(x - u) + \beta(y - v) - 1) w + (u + x + \alpha)s \end{pmatrix} \]

and

\[ B_{\gamma\delta} = \frac{\beta}{y - v} m_{\gamma\delta} + \frac{y - v + \beta}{2(y - v)(\alpha u - \beta y + 1)} S_{(\gamma\delta)}, \]

where parentheses denote index symmetrization.

By redefining the integration variable \(\beta w \rightarrow w\) (reabsorbing a \(\beta\)-dependent factor into an integration constant) and setting then \(\alpha = 1, \beta = 0\) we obtain

\[
C_0(b|X) = (y - v)^{-\frac{1}{2}} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dw \ s^{2P-\frac{1}{2}} w^{2Q-\frac{1}{2}} e^{-\frac{u + x}{4(y - v)} s^2 - \frac{\beta}{y - v} w^2 + \frac{\beta}{y - v} s w} \cdot e^{B_{\gamma\delta} b^\gamma b^\delta} \\
\times \exp \left( \frac{b^1(2w + s)(y - v) + b^2(s(u + x + 1) + 2w(x - u - 1))}{(y - v) \sqrt{2(u + 1)}} \right).
\]

(B.10)
with
\[
\hat{B}_{\gamma\delta} = \frac{1}{2} \left( \frac{y-v}{u+1} - \frac{x}{u+1} - \frac{2}{y-v} \right).
\]
Performing Lorentz transformation (B.5) and using BTZ coordinates (2.9) we finally obtain (8.12).

Let us also note, that a convenient parametrization of a $Sp(2)$-valued matrix $S_{\alpha\beta}$ is
\[
S_{\alpha\beta} = \cosh(p)\varepsilon_{\alpha\beta} + \sinh(p)\kappa_{\alpha\beta},
\]
where $\kappa_{\alpha\beta} = \kappa_{\beta\alpha}$ and $\frac{1}{2}\kappa_{\alpha\beta}\kappa^{\alpha\beta} = -1$. One can see that the $n^{th}$ power of $S$ is
\[
(S^n)_{\alpha\beta} = \cosh(np)\epsilon_{\alpha\beta} + \sinh(np)\kappa_{\alpha\beta}.
\]
As a result,
\[
(\sqrt{S})_{\alpha\beta} = \cosh \left( \frac{p}{2} \right) \varepsilon_{\alpha\beta} + \sinh \left( \frac{p}{2} \right) \kappa_{\alpha\beta}.
\]
Also one finds that the matrix $\Pi = \sqrt{\frac{S-1}{S+1}}$ is
\[
\Pi_{\alpha\beta} = \tanh \left( \frac{p}{4} \right) \kappa_{\alpha\beta}.
\]

References

[1] A. Staruszkiewicz, *Acta Phys. Polon.* 24 (1963) 734
[2] H. Leutwyler, *Nuovo Cim.* 42 (1966) 159
[3] S. Deser, R. Jackiw and G. ’t Hooft, *Annals Phys.* 152 (1984) 220
[4] S. Deser and R. Jackiw, *Annals Phys.* 153 (1984) 405; *Commun.Math.Phys.* 118 (1988) 495
[5] G. ’t Hooft, *Commun.Math.Phys.* 117 (1988) 685
[6] A. Achúcarro and P. K. Townsend, *Phys.Lett.* B180 (1986) 89
[7] E. Witten, *Nucl.Phys.* B311 (1988) 46; *Nucl. Phys.* B323 (1989) 113; *Commun.Math.Phys.* 137 (1991) 29
[8] S. Carlip, J.*Korean Phys.Soc.* 28 (1995) S447-S467, gr-qc/9503024
[9] M. Banados, C. Teitelboim and J. Zanelli, *Phys.Rev.Lett.* 69 (1992) 1849, hep-th/9204099
[10] D. Ida, *Phys.Rev.Lett.* 85 (2000) 3758, gr-qc/0005129
[11] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, *Phys.Rev.* **D48** (1993) 1506, gr-qc/9302012

[12] M.A. Vasiliev, *Mod.Phys.Lett.* **A7** (1992) 3689-3702

[13] S.F. Prokushkin and M.A. Vasiliev, *Nucl.Phys.* **B545** (1999) 385, hep-th/9806236

[14] M.A. Vasiliev, *Higher Spin Gauge Theories: Star Product and AdS Space*, Contributed article to Golfand’s Memorial Volume “Many faces of the superworld”, ed. by M. Shifman, World Scientific Publishing Co Pte Ltd, Singapore, 2000, hep-th/9910096.

[15] E. Sezgin and P. Sundell, *An Exact Solution of 4D Higher-Spin Gauge Theory*, hep-th/0508158

[16] C. Fronsdal, Preprint UCLA/85/TEP/10, in Essays on Supersymmetry, Reidel, 1986 (Mathematical Physics Studies, v.8)

[17] I. Bandos, J. Lukierski and D. Sorokin, *Phys.Rev.* **D61** (2000) 045002, hep-th/9904109

[18] M.A. Vasiliev, *Phys.Rev.* **D66** (2002) 066006, hep-th/0106149

[19] I. Bandos, J. Lukierski, C. Preitschopf and D. Sorokin, *Phys.Rev.* **D61** (2000) 065009, hep-th/9907113

[20] V.E. Didenko and M.A. Vasiliev, *J.Math.Phys.* **45** (2004) 197-215, hep-th/0301054

[21] M. Plyushchay, D. Sorokin and M. Tsulaia, *JHEP* **0304** (2003) 013, hep-th/0301067

[22] O.V. Shaynkman and M.A. Vasiliev, *Theor.Math.Phys.* **128** (2001) 1155-1168; (*Teor.Mat.Fiz.* **128**(2001) 378-394), hep-th/0103208

[23] K. Ghoroku and A. L. Larsen, *Phys.Lett.* **B328** (1994) 28-35, hep-th/9403008

[24] I. Ichinose and Y. Satoh, *Nucl.Phys.* **B447** (1995) 340, hep-th/9412144

[25] S. Das and A. Dasgupta, *JHEP* **9910** (1999) 025, hep-th/9907116

[26] D. Birmingham, I. Sachs and S.N. Solodukhin, *Phys.Rev.Lett.* **88** (2002) 151301, hep-th/0112055

[27] I. Bars and M. Günaydin, *Commun.Math.Phys.* **91** (1983) 31
[28] M.A. Vasiliev, *Class. Quant. Grav.* **11** (1994) 649-664

[29] I.S. Gradshtein and I.M. Ryzhik, *Tables of integrals, sums, series and products*, Fiz.Mat.Lit., Moscow, 1963.

[30] J. Gamboa and F. Méndez, *Class. Quant. Grav.* **18** (2001) 225-232, [hep-th/0006020](https://arxiv.org/abs/hep-th/0006020)

[31] S. Lepe, F. Méndez, J. Saavedra and L. Vergara, *Class. Quant. Grav.* **20** (2003) 2417-2428, [hep-th/0302035](https://arxiv.org/abs/hep-th/0302035)

[32] J. Troost, *JHEP* **0209** (2002) 041, [hep-th/0206118](https://arxiv.org/abs/hep-th/0206118)

[33] N.A Vilenkin, *Special functions and the theory of group representations*, Providence, American Mathematical Society, 1968

[34] O. Coussaert and M. Henneaux, *Phys. Rev. Lett.* **72** (1994) 183-186, [hep-th/9310194](https://arxiv.org/abs/hep-th/9310194)