Hopf algebra structure on symplectic superspace \( SP_{q}^{2|1} \)

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MSC: 16W30, 16T20, 17B37, 17B66, 20G42, 58B32
Keywords: quantum symplectic superspace, super \( \ast \)-algebra, quantum supergroup \( SP_{q}(2|1) \)

Abstract

Super-Hopf algebra structure on the function algebra on the extended quantum symplectic superspace \( SP_{q}^{2|1} \) has been defined. The dual Hopf algebra is explicitly constructed.

1 Introduction

Quantum supergroups and quantum superalgebras are even richer mathematical subjects as compared to Lie supergroups and Lie superalgebras. A quantum superspace is a space that quantum supergroup acts with linear transformations and whose coordinates belong to a noncommutative associative superalgebra [8].

Some algebras have been considered which are covariant with respect to the quantum supergroups in [5]. Using the corepresentation of the quantum supergroup \( OSP_{q}(1|2) \), some non-commutative spaces covariant under its coaction have been constructed [8]. In the present work, we set up a super-Hopf algebra structure on an algebra which appears in both paper. We denoted this algebra by \( \mathcal{O}(OSP_{q}(1|2)) \). As is known, the matrix elements of the quantum supergroups \( SP_{q}(1|2) \) and \( SP_{q}(2|1) \) are the same and they act both quantum superspaces \( SP_{q}^{1|2} \) and \( SP_{q}^{2|1} \). But these two quantum superspaces are not the same. A study on \( SP_{q}^{1|2} \) was made in [4]. Here we will work on the quantum symplectic superspace \( SP_{q}^{2|1} \).

2 Review of quantum symplectic group

In this section, we will give some information about the structures of quantum symplectic groups as much as needed.

The algebra \( \mathcal{O}(OSP_{q}(1|2)) \) is generated by the even elements \( a, b, c, d \) and odd elements \( a, \delta \). Standard FRT construction [6] is obtained via the matrix \( R \) given in [7]. Using the RTT-relations and the \( q \)-orthosymplectic condition, all defining relations of \( \mathcal{O}(OSP_{q}(1|2)) \) are explicitly obtained in [8].
Theorem 2.1 The generators of $O(\text{SP}_q(1|2))$ satisfy the relations

\[ ab = q^2 ba, \quad ac = q^2 ca, \quad aa = qaa, \quad a\delta = q\delta a + (q - q^{-1})ac, \]
\[ ad = da + (q - q^{-1})[(1 + q^{-1})bc + q^{-1/2}a\delta], \quad bc = cb, \quad bd = q^2 db, \]
\[ b\alpha = q^{-1} ab, \quad b\delta = q\delta b, \quad cd = q^2 dc, \quad ca = q^{-1} ac, \quad c\delta = q\delta c, \]
\[ d\alpha = q^{-1} ad + (q^{-1} - q)\delta b, \quad d\delta = q^{-1} d\alpha + q^{-1/2}(q^2 - 1)bc, \]
\[ \alpha^2 = q^{1/2} (q - 1) ba, \quad \delta^2 = q^{1/2} (q - 1) dc. \]

In (1), the relations involving the elements $\gamma$, $e$ and $\beta$ are not written. They can be found in $[3]$. Other relations that we need in this study are given below:

\[ [e, \alpha]_q = q^{1/2}(q - 1)(\gamma b + \beta a), \quad [e, \beta]_q^{-1} = q^{-1/2}(q^{-1} - 1)(\delta b + a\delta), \]
\[ [e, \gamma]_q = q^{1/2}(1 - q)(\delta a + ac), \quad [e, \delta]_q^{-1} = q^{-1/2}(1 - q^{-1})(\delta + bc), \]
\[ \beta^2 = q^{1/2} (q - 1) dc, \quad \gamma^2 = q^{1/2} (q - 1) ca, \]
\[ e^2 = 1 - q^{-1/2}[\alpha, \delta]_q = 1 + q^{1/2}[\beta, \gamma]_q^{-1}. \]

where $[u, v]_q = uv - Quv$.

The quantum superdeterminant is defined by

\[ D_q = ad - qbc - q^{1/2}\alpha\delta = da - q^{-1}bc + q^{-1/2}\delta a. \]

This element of $O(\text{SP}_q(2|1))$ commutes with all elements of $O(\text{SP}_q(2|1))$.

If $A$ and $B$ are $\mathbb{Z}_2$-graded algebras, then their tensor product $A \otimes B$ is the $\mathbb{Z}_2$-graded algebra whose underlying space is $\mathbb{Z}_2$-graded tensor product of $A$ and $B$. The following definition gives the product rule for tensor product of algebras. Let us denote by $\tau(a)$ the grade (or degree) of an element $a ∈ A$.

**Definition 1** If $A$ is a $\mathbb{Z}_2$-graded algebra, then the product rule in the $\mathbb{Z}_2$-graded algebra $A \otimes A$ is defined by

\[ (a_1 \otimes a_2)(a_3 \otimes a_4) = (-1)^{\tau(a_2)\tau(a_3)} a_1a_3 \otimes a_2a_4 \]

where $a_i$’s are homogeneous elements in the algebra $A$.

**Definition 2** A super-Hopf algebra is a vector space $A$ over $K$ with three linear maps $\Delta : A → A ⊗ A$, called the coproduct, $\epsilon : A → K$, called the counit, and $S : A → A$, called the coinverse, such that

\[ (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta, \]
\[ m \circ (\epsilon \otimes \text{id}) \circ \Delta = \text{id} = m \circ (\text{id} \otimes \epsilon) \circ \Delta, \]
\[ m \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\text{id} \otimes S) \circ \Delta, \]

together with $\Delta(1) = 1 ⊗ 1$, $\epsilon(1) = 1$, $S(1) = 1$ and for any $a, b ∈ A$

\[ \Delta(ab) = \Delta(a)\Delta(b), \quad \epsilon(ab) = \epsilon(a)\epsilon(b), \quad S(ab) = (-1)^{\tau(a)\tau(b)} S(b)S(a), \]

where $m : A ⊗ A → A$ is the product map, $\text{id} : A → A$ is the identity map and $\eta : K → A$. 

\[ 2 \]
The super-Hopf algebra structure of $\mathbb{O}(\text{SP}_q(2|1))$ is given as usual in the following theorem.

**Theorem 2.2** There exists a unique super-Hopf algebra structure on the super-algebra $\mathbb{O}(\text{SP}_q(2|1))$ with co-maps $\Delta$, $\epsilon$ and $S$ such that

$$
\Delta(t_{ij}) = \sum_{k=1}^{3} t_{ik} \otimes t_{kj}, \quad \epsilon(t_{ij}) = \delta_{ij}, \quad S(T) = T^{-1}
$$

where

$$
T = \begin{pmatrix} a & \alpha & b \\ \gamma & e & \beta \\ c & \delta & d \end{pmatrix} = (t_{ij}), \quad T^{-1} = \begin{pmatrix} d & q^{-1/2} \beta & -q^{-1}b \\ -q^{1/2} \delta & e & q^{1/2} \alpha \\ -qc & -q^{1/2} \gamma & a \end{pmatrix}.
$$

### 3 Quantum symplectic superspace $\text{SP}_q^{2|1}$

In this section, we define a super-Hopf algebra structure on the extended function algebra of the quantum superspace $\text{SP}_q^{2|1}$.

#### 3.1 The algebra of polynomials on the quantum superspace $\text{SP}_q^{2|1}$

The elements of the symplectic superspace are supervectors generated by two even and an odd components. We define a $\mathbb{Z}_2$-graded symplectic space $\text{SP}_q^{2|1}$ by dividing the superspace $\text{SP}_q^{2|1}$ of $3 \times 1$ matrices into two parts $\text{SP}_q^{2|1} = V_0 \oplus V_1$. A vector is an element of $V_0$ (resp. $V_1$) and is of grade 0 (resp. 1) if it has the form

$$
\begin{pmatrix} x \\ 0 \\ y \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} 0 \\ \theta \\ 0 \end{pmatrix}.
$$

While the even elements commute to everyone, the odd element satisfies the relation $\theta^2 = 0$.

In [4], the quantum superspace $\text{SP}_q^{2|1}$ is considered as the dual space of quantum superspace $\text{SP}_q^{1|2}$ and then relations (7) below are obtained by interpreting the coordinates as differentiations.

**Definition 3** Let $\mathbb{K}\langle x, \theta, y \rangle$ be a free associative algebra generated by $x$, $\theta$, $y$ and $I_q$ be a two-sided ideal generated by $x\theta - q\theta x$, $xy - q^2yx$, $y\theta - q^{-1}\theta y$, $\theta^2 - q^{1/2}(q-1)yx$. The quantum superspace $\text{SP}_q^{2|1}$ with the function algebra

$$
\mathbb{O}(\text{SP}_q^{2|1}) = \mathbb{K}\langle x, \theta, y \rangle / I_q
$$

is called $\mathbb{Z}_2$-graded quantum symplectic space (or quantum symplectic superspace).
Here the coordinates $x$ and $y$ with respect to the $\mathbb{Z}_2$-grading are of grade 0 (or even), the coordinate $\theta$ with respect to the $\mathbb{Z}_2$-grading is of grade 1 (or odd).

According to the above definition, if $(x, \theta, y)^t \in \text{SP}_q^{2|1}$ then we have

$$x\theta = q\theta x, \quad \theta y = qy\theta, \quad yx = q^{-2}xy, \quad \theta^2 = q^{1/2}(q - 1)yx$$

(7)

where $q$ is a non-zero complex number. This associative algebra over the complex numbers is known as the algebra of polynomials over quantum $(2+1)$-superspace.

It is easy to see the existence of representations that satisfy (7); for instance, there exists a representation $\rho: \mathbb{O}(\text{SP}_q^{2|1}) \to \text{M}(3, \mathbb{C})$ such that matrices

$$\rho(x) = \begin{pmatrix} q & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(\theta) = \begin{pmatrix} 0 & q - 1 & 0 \\ 0 & 0 & 0 \\ q^{1/2} & 0 & 0 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

representing the coordinate functions satisfy the relations (7).

Note that the last two relations in (7) can be also written as a single relation. Therefore, we say that $\mathbb{O}(\text{SP}_q^{2|1})$ is the superalgebra with generators $x_\pm$ and $\theta$ satisfying the relations [5]

$$x_\pm \theta = q^{\pm 1} \theta x_\pm, \quad [x_+, x_-] = q^{-1/2}(q + 1)\theta^2.$$  

(8)

where $x_+ = x$ and $x_- = y$.

**Definition 4 [5]** The quantum supersphere on the quantum symplectic superspace is defined by

$$r = q^{1/2}x_-x_+ + \theta^2 - q^{-1/2}x_+x_-.$$

### 3.2 A $\star$-structure on the algebra $\mathbb{O}(\text{SP}_q^{2|1})$

Here we define a $\mathbb{Z}_2$-graded involution on the algebra $\mathbb{O}(\text{SP}_q^{2|1})$.

**Definition 5** Let $\mathbb{A}$ be an associative superalgebra. A $\mathbb{Z}_2$-graded linear map $\star: \mathbb{A} \to \mathbb{A}$ is called a superinvolution (or $\mathbb{Z}_2$-graded involution) if

$$(ab)^\star = (-1)^{\tau(a)\tau(b)}b^\star a^\star, \quad (a^\star)^\star = a$$

for any elements $a, b \in \mathbb{A}$. The pair $(\mathbb{A}, \star)$ is called a $\mathbb{Z}_2$-graded $\star$-algebra.

If the parameter $q$ is real, then the algebra $\mathbb{O}(\text{SP}_q^{2|1})$ becomes a $\star$-algebra with involution determined by the following proposition.

**Proposition 3.1** If $q > 0$ then the algebra $\mathbb{O}(\text{SP}_q^{2|1})$ supplied with the $\mathbb{Z}_2$-graded involution determined by

$$x_+^\star = q^{1/2}x_-, \quad \theta^\star = i\theta, \quad x_-^\star = q^{-1/2}x_+$$

becomes a super $\star$-algebra where $i = \sqrt{-1}$.
Theorem 3.2

We must show that the relations (8) are invariant under the star operation. If $q$ is a positive number, we have

$$(x_+ \theta - q^{\pm 1} x_\pm^*) = (i \theta)(q^{\pm 1/2} x_\pm) - q^{\pm 1}(q^{\pm 1/2} x_\pm)(i \theta)$$

$$= q^{\pm 1/2} i (\theta x_\pm - q^{\pm 1} x_\pm)$$

and since $[x_+, x_-]^* = [x_+, x_-]

$$[x_+, x_-] = [x_+, x_-]^* = q^{-1/2}(q + 1)(-\theta^* \theta^*) = q^{-1/2}(q + 1)\theta^2.$$ 

Hence the ideal $(x_+ \theta - q^{\pm 1} x_\pm, [x_+, x_-] - q^{-1/2}(q + 1)\theta^2)$ is $*$-invariant and the quotient algebra $\mathbb{K}(x_+, \theta, x_-)/(x_+ \theta - q^{\pm 1} x_\pm, [x_+, x_-] - q^{-1/2}(q + 1)\theta^2)$ becomes a $*$-algebra. 

3.3 The super-Hopf algebra structure on $\text{SP}_q^{2|1}$

We define the extended $\mathbb{Z}_2$-graded quantum symplectic space to be the algebra containing $\text{SP}_q^{2|1}$, the unit and $x_+^{-1}$, the inverse of $x_+$, which obeys $x_+ x_+^{-1} = 1 = x_+^{-1} x_+$. We will denote the unital extension of $\mathbb{U}(\text{SP}_q^{2|1})$ by $\mathbb{F}(\text{SP}_q^{2|1})$.

The following theorem asserts that the superalgebra $\mathbb{F}(\text{SP}_q^{2|1})$ is a super-Hopf algebra:

**Theorem 3.2** The algebra $\mathbb{F}(\text{SP}_q^{2|1})$ is a $\mathbb{Z}_2$-graded Hopf algebra. The definitions of a coproduct, a counit and a coinverse on the algebra $\mathbb{F}(\text{SP}_q^{2|1})$ are as follows

(i) the coproduct $\Delta : \mathbb{F}(\text{SP}_q^{2|1}) \rightarrow \mathbb{F}(\text{SP}_q^{2|1}) \otimes \mathbb{F}(\text{SP}_q^{2|1})$ is defined by

$$\Delta(x_+) = x_+ \otimes x_+, \quad \Delta(\theta) = \theta \otimes 1 + 1 \otimes \theta, \quad \Delta(x_-) = x_-^{-1} \otimes x_- + x_- \otimes x_-^{-1}. \quad (9)$$

(ii) the counit $\epsilon : \mathbb{F}(\text{SP}_q^{2|1}) \rightarrow \mathbb{C}$ is given by

$$\epsilon(x_+) = 1, \quad \epsilon(\theta) = 0, \quad \epsilon(x_-) = 0.$$

(iii) the algebra $\mathbb{F}(\text{SP}_q^{2|1})$ admits a $\mathbb{C}$-algebra antihomomorphism (coinverse)

$S : \mathbb{F}(\text{SP}_q^{2|1}) \rightarrow \mathbb{F}(\text{SP}_q^{2|1})$ defined by

$$S(x_+) = x_+^{-1}, \quad S(\theta) = -\theta, \quad S(x_-) = -x_- x_- x_+.$$

**Proof** The axioms (2)–(5) are satisfied automatically. It is also not difficult to show that the co-maps preserve the relations (8). In fact, for instance,

$$\Delta([x_+, x_-]) = \Delta(x_+ x_- - x_- x_+) = 1 \otimes [x_+, x_-] + [x_+, x_-] \otimes 1$$

$$= q^{-1/2}(q + 1)(1 \otimes \theta^2 + \theta^2 \otimes 1)$$

$$\Delta(\theta^2) = 1 \otimes \theta^2 + \theta^2 \otimes 1.$$
and

\[ S([x_+, x_-]) = -[x_+, x_-], \quad S(\theta^2) = -\theta^2. \]

Since \( S^2(a) = \text{id}(a) \) for all \( a \in \mathbb{F}(\text{SP}_q^{2|1}) \), the coinverse \( S \) is of second order. \( \square \)

The set \( \{x^k \theta^l y^m : k, l, m \in \mathbb{N}_0\} \) form a vector space basis of \( \mathbb{F}(\text{SP}_q^{2|1}) \). The formula (9) gives the action of the coproduct \( \Delta \) only on the generators. The action of \( \Delta \) on product on generators can be calculated by taking into account that \( \Delta \) is a homomorphism.

**Corollary 3.3** For the quantum supersphere \( r \), we have

\[ \Delta(r) = r \otimes 1 + 1 \otimes r, \quad \epsilon(r) = 0, \quad S(r) = -r. \]

**Proof** Using the definition of \( \Delta \), as an algebra homomorphism, on the generators of \( \mathbb{F}(\text{SP}_q^{2|1}) \) in (9), it is easy to see that the element \( r \in \mathbb{F}(\text{SP}_q^{2|1}) \) is a primitive element, that is,

\[
\Delta(r) = q^{1/2}(x_+^{-1} \otimes x_- + x_- \otimes x_+^{-1})(x_+ \otimes x_+) + (\theta \otimes 1 + 1 \otimes \theta)(\theta \otimes 1 + 1 \otimes \theta) \\
- q^{-1/2}(x_+ \otimes x_+)(x_-^{-1} \otimes x_- + x_- \otimes x_-^{-1}) \\
= q^{1/2}(1 \otimes x_-x_+ + x_-x_+ \otimes 1) + \theta^2 \otimes 1 + 1 \otimes \theta^2 \\
- q^{-1/2}(1 \otimes x_+x_- + x_+x_- \otimes 1) \\
= 1 \otimes (q^{1/2}x_-x_+ + \theta^2 - q^{-1/2}x_+x_-) + (q^{1/2}x_-x_+ + \theta^2 - q^{-1/2}x_+x_-) \otimes 1.
\]

Since \( \epsilon(1) = 1 \) and

\[ m(\text{id} \otimes \epsilon)\Delta(r) = r\epsilon(1) + \epsilon(r)1 = r = m(\epsilon \otimes \text{id})\Delta(r), \]

we obtain \( \epsilon(r) = 0 \). Finally, using the fact that \( S \) is an anti-homomorphism we get

\[
S(r) = q^{1/2}x_+^{-1}(-x_+x_-x_+) - (-\theta)(-\theta) - q^{-1/2}(-x_+x_-x_+)x_-^{-1} \\
= -(q^{1/2}x_-x_+ + \theta^2 - q^{-1/2}x_+x_-),
\]

as desired. \( \square \)

### 3.4 Coactions on the quantum symplectic superspace

Let \( a, b, c, d, e, \gamma, \alpha, \delta, \beta \) be elements of an algebra \( \mathbb{A} \). Assuming that the generators of \( \mathbb{O}(\text{SP}_q^{2|1}) \) super-commute with the elements of \( \mathbb{O}(\text{SP}_q^{2|1}) \), define the components of the vectors \( X' = (x', \theta', y')^t \) and \( X'' = (x'', \theta'', y'')^t \) using the following matrix equalities

\[ X' = TX \quad \text{and} \quad (X'')^t = X^t T \quad (10) \]

where \( X = (x, \theta, y)^t \in \text{SP}_q^{2|1} \). If we assume that \( q = 1 \) then we have the following theorem proving straightforward computations.
Theorem 3.4  If the couples \((x', \theta', y')\) and \((x'', \theta'', y'')\) in (10) satisfy the relations (7), then the generators of \(\mathcal{O}(\text{SP}_q(2|1))\) together with \(q\)-orthosymplectic condition fulfill the relations (1).

A left quantum space (or left comodule algebra) for a Hopf algebra \(H\) is an algebra \(X\) together with an algebra homomorphism (left coaction) \(\delta_L : X \rightarrow H \otimes X\) such that

\[
(id \otimes \delta_L) \circ \delta_L = (\Delta \otimes id) \circ \delta_L \quad \text{and} \quad (\epsilon \otimes id) \circ \delta_L = id.
\]

Right comodule algebra can be defined in a similar way.

Theorem 3.5  (i) The algebra \(\mathcal{O}(\text{SP}_q^2)\) is a left and right comodule algebra of the Hopf algebra \(\mathcal{O}(\text{SP}_q(2|1))\) with left coaction \(\delta_L\) and right coaction \(\delta_R\) such that

\[
\delta_L(X_i) = \sum_{k=1}^{3} t_{ik} \otimes X_k, \quad \delta_R(X_i) = \sum_{k=1}^{3} X_k \otimes t_{ki}. \quad (11)
\]

(ii) The quantum supersphere \(r\) belongs to the center of \(\mathcal{O}(\text{SP}_q^2)\) and satisfies \(\delta_L(r) = 1 \otimes r\) and \(\delta_R(r) = r \otimes 1\).

Proof (i): These assertions are obtained from the relations in (1) and (2) together with (7).

(ii): That \(r\) is a central element of \(\mathcal{O}(\text{SP}_q^2)\) is shown by using the relations in (7). To show that \(\delta_L(r) = 1 \otimes r\) and \(\delta_R(r) = r \otimes 1\) we use the definitions of \(\delta_L\) and \(\delta_R\) in (11) and the relations (1) and (2) with \(D_q = 1\). \(\square\)

4 Dual pairings of Hopf algebras

Let us consider the algebra \(A\) as a Hopf algebra. Then its dual \(U = A'\) is a Hopf algebra as well. Using the coproduct \(\Delta\) in \(A\), one defines a product in \(U\). Using the product in the Hopf algebra \(A\), one defines a coproduct in \(U\).

In this section, in order to obtain the dual of the Hopf algebra \(A\) defined in Section 3, we have applied to \(A = F(\text{SP}_q^2)\) the approach given in [9]. We are going to use the relations (7) in order to facilitate the operations in this section. Let us begin with the definition of the duality (1).

Definition 6  A dual pairing of two super bialgebras \(U\) and \(A\) is a bilinear mapping

\[
\langle \cdot, \cdot \rangle : U \times A \rightarrow \mathbb{K}, \quad \langle u, a \rangle \mapsto \langle u, a \rangle, \quad u \in U, \ a \in A \quad (12)
\]

such that

\[
\langle u, ab \rangle = (\Delta_U(u), a \otimes b), \quad \langle uv, a \rangle = \langle u \otimes v, \Delta_A(a) \rangle, \quad \langle u, 1_A \rangle = \epsilon_U(u), \quad \langle 1_U, a \rangle = \epsilon_A(a) \quad (13)
\]

for all \(u, v \in U\) and \(a, b \in A\).
We say that the pairing is non-degenerate if
\[ \langle u, a \rangle = 0 \quad (\forall a \in \mathbb{A}) \implies u = 0 \quad \text{and} \quad \langle u, a \rangle = 0 \quad (\forall u \in \mathbb{U}) \implies a = 0. \]

Two super Hopf algebras \( \mathbb{U} \) and \( \mathbb{A} \) are said to be in duality if they are in duality as bialgebras and if
\[ \langle S\mathbb{U}(u), a \rangle = \langle u, S\mathbb{A}(a) \rangle, \quad \forall u \in \mathbb{U}, \ a \in \mathbb{A}. \]

Such a pairing can be extended to a pairing of \( \mathbb{U} \otimes \mathbb{U} \) and \( \mathbb{A} \otimes \mathbb{A} \) by
\[ \langle u \otimes v, a \otimes b \rangle = (-1)^{\tau(v)\tau(a)}\langle u, a \rangle\langle v, b \rangle. \]

It is enough to define the pairing (12) between the generating elements of the two algebras. Pairing for any other elements of \( \mathbb{U} \) and \( \mathbb{A} \) follows from (13) and the bilinear form inherited by the tensor product.

As a super Hopf algebra \( \mathbb{A} = \mathbb{F}(\text{SP}_{q}^{2|1}) \) is generated by the elements \( x, \theta, \) and \( y \) and a basis is given by all monomials of the form
\[ f = x^{k}\theta^{l}y^{n}, \]
where \( k, l, n \in \mathbb{N}_{0} \). Let us denote the dual algebra by \( \mathbb{U} \) and its generating elements by \( K, \nabla, \) and \( N \).

**Theorem 4.1** The commutation relations between the generators of the algebra \( \mathbb{U} \) dual to \( \mathbb{A} \) are the following:
\[ K\nabla = \nabla K, \quad KN = NK, \quad \nabla N = N\nabla, \quad \nabla^{2} = 0. \]

**Proof** The pairing is defined through the tangent vectors as follows
\[ \langle K, f \rangle = k\delta_{l,0}\delta_{n,0}, \quad \langle \nabla, f \rangle = \delta_{l,1}, \quad \langle N, f \rangle = \delta_{n,1}. \]

We also have
\[ \langle 1_{\mathbb{U}}, f \rangle = \epsilon_{\mathbb{A}}(f) = \delta_{k,0}. \]

Using the defining relations one gets
\[ \langle KN, f \rangle = (k-1)(\delta_{l,0}\delta_{n,1} + \delta_{l,1}\delta_{n,0}) = \langle NK, f \rangle, \]
where differentiation is from the right as this is most suitable for differentiation in this basis. Thus one obtains
\[ \langle KN - NK, f \rangle = 0. \]

The other relations can be obtained similarly. \( \square \)
Theorem 4.2 The Hopf algebra structure of the algebra \( U \) is given by:

(i) the comultiplication is given by

\[
\Delta_U(K) = K \otimes 1_U + 1_U \otimes K, \\
\Delta_U(\nabla) = \nabla \otimes 1_U + q^K \otimes \nabla, \\
\Delta_U(N) = N \otimes q^{-2K} + 1_U \otimes N,
\]

(ii) the counity is given by

\[
\epsilon_U(K) = 0, \quad \epsilon_U(\nabla) = 0, \quad \epsilon_U(N) = 0,
\]

(iii) the coinverse is given by

\[
S_U(K) = -K, \quad S_U(\nabla) = -q^{-K} \nabla, \quad S_U(N) = -q^{2K} N.
\]

Proof We only will obtain the actions of \( \Delta_U, \epsilon_U \) and \( \kappa_U \) on \( \nabla \). The others can be obtained similarly. So, we assume that the action of \( \Delta_U \) on \( \nabla \) is \( \Delta_U(\nabla) = \nabla \otimes c_1 + c_2 \otimes \nabla \). Then the commutation relations in \( \mathcal{A} \) (or comultiplication in \( U \)) will imply that \( c_1 = q^{-K} c_2 \).

\[
\langle \Delta_U(\nabla), \theta \otimes x^k \rangle = \langle \nabla \otimes c_1 + c_2 \otimes \nabla, \theta \otimes x^k \rangle = \langle \nabla, \theta \rangle \langle c_1, x^k \rangle - \langle c_2, \theta \rangle \langle \nabla, x^k \rangle = \langle c_1, x^k \rangle
\]

and

\[
\langle \Delta_U(\nabla), x^k \otimes \theta \rangle = \langle \nabla \otimes c_1 + c_2 \otimes \nabla, x^k \otimes \theta \rangle = \langle \nabla, x^k \rangle \langle c_1, \theta \rangle + \langle c_2, x^k \rangle \langle \nabla, \theta \rangle = \langle c_2, x^k \rangle
\]

so that, considering \( \langle \nabla, x^k \theta \rangle \) and \( \langle \nabla, \theta x^k \rangle \) and taking \( c_1 = 1_U \) we have found the action of \( \Delta_U \) on \( \nabla \).

The action of \( \epsilon_U \) on \( \nabla \) is

\[
\epsilon_U(\nabla) = \langle \nabla, 1_A \rangle = \langle \nabla, x^0 \rangle = 0.
\]

This can also be found from the identities

\[
\mu \circ (\epsilon_U \otimes 1_U) \Delta_U(\nabla) = 1_U(\nabla) = \mu \circ (1_U \otimes \epsilon_U) \Delta_U(\nabla).
\]

The action of \( S_U \) on \( \nabla \) is

\[
\langle S_U(\nabla), \theta x^k \rangle = \langle \nabla, S_A(\theta x^k) \rangle = \langle \nabla, x^{-k}(-\theta) \rangle = \langle \nabla, -q^{-k} \theta x^{-k} \rangle = -q^{-k} \\
\implies S_U(\nabla) = -q^{-K} \nabla.
\]

This can also be found from the identities

\[
\mu \circ (S_U \otimes 1_U) \Delta_U(\nabla) = \epsilon_U(\nabla) = \mu \circ (1_U \otimes S_U) \Delta_U(\nabla).
\]
5 An $h$-deformation of the superspace $\text{SP}^{2|1}$

In this section, we introduce an $h$-deformation of the superspace $\text{SP}^{2|1}$ from the $q$-deformation via a contraction following the method of [2]. Consider the $q$-deformed algebra of functions on the quantum superspace $\text{SP}^{2|1}_q$ generated by $x_{\pm}$ and $\theta$ with the relations (8).

We introduce new coordinates $X_{\pm}$ and $\Theta$ by

$$x = \begin{pmatrix} x_+ \\ \theta \\ x_- \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{h}{q-1} & 0 & 1 \end{pmatrix} \begin{pmatrix} X_+ \\ \Theta \\ X_- \end{pmatrix} = g X.$$  

When the relations (8) are used, taking the limit $q \to 1$ we obtain the following exchange relations, which define the $h$-superspace $\text{SP}^{2|1}_h$:

**Definition 7** Let $\mathcal{O}(\text{SP}^{2|1}_h)$ be the algebra with the generators $X_{\pm}$ and $\Theta$ satisfying the relations

$$X_+ \Theta = \Theta X_+, \quad X_- \Theta = \Theta X_- - 2h \Theta X_+, \quad X_+ X_- = X_- X_+ + 2\Theta^2 \quad (14)$$

where the coordinates $X_{\pm}$ are even and the coordinate $\Theta$ is odd. We call $\mathcal{O}(\text{SP}^{2|1}_h)$ the algebra of functions on the $\mathbb{Z}_2$-graded quantum space $\text{SP}^{2|1}_h$.

$h$-deformed supersphere on the symplectic $h$-superspace is given by

$$r_h = X_- X_+ + \Theta^2 + h X_+^2 - X_+ X_- = h X_+^2 - \Theta^2.$$  

It is easily seen that the quantum supersphere $r_h$ belongs to the center of the superalgebra $\mathcal{O}(\text{SP}^{2|1}_h)$.

The definition of dual $q$-deformed symplectic superspace is given as follows [3].

**Definition 8** Let $\mathbb{K}\{\varphi_+, z, \varphi_-\}$ be a free associative algebra generated by $z, \varphi_+, \varphi_-$ and $I_q$ be a two-sided ideal generated by $z \varphi_{\pm} - q^{\pm 1} \varphi_{\pm} z, \varphi_- \varphi_+ + q^{-2} \varphi_+ \varphi_- + q^{-2} Q z^2$ and $\varphi_{\pm}^2$. The quantum superspace $\text{SP}^{1|2}_q$ with the function algebra

$$\mathcal{O}(\text{SP}^{1|2}_q) = \mathbb{K}\{\varphi_+, z, \varphi_-\}/I_q$$

is called $\mathbb{Z}_2$-graded quantum symplectic space (or quantum symplectic superspace) where $Q = q^{1/2} - q^{3/2}$ and $q \neq 0$.

In case of exterior $h$-superspace, we use the transformation

$$\mathbf{x} = g \hat{X}$$

with the components $\varphi_+, z$ and $\varphi_-$ of $\hat{X}$. The definition is given below.
Definition 9 Let $\Lambda(\text{SP}^2_{h})$ be the algebra with the generators $\Phi_{\pm}$ and $Z$ satisfying the relations
\begin{align*}
\Phi_{+}Z &= Z\Phi_{+}, \quad Z\Phi_{-} = \Phi_{-}Z - 2h\Phi_{+}Z, \quad \Phi_{-}\Phi_{+} = -\Phi_{+}\Phi_{-}, \\
\Phi_{+}^2 &= 0, \quad \Phi_{-}^2 = h(2\Phi_{-}\Phi_{+} - Z^2)
\end{align*}
where the coordinate $Z$ is even and the coordinates $\Phi_{\pm}$ are odd. We call $\Lambda(\text{SP}^2_{h})$ the quantum exterior algebra of the $\mathbb{Z}_2$-graded quantum space $\text{SP}^2_{h}$.

6 A new algebra derived from $\mathbb{F}(\text{SP}^2_{q})$

It is known that an element of a Lie group can be represented by exponential of an element of its Lie algebra. By virtue of this fact, one can define the generators of the algebra $\mathbb{F}(\text{SP}^2_{q})$ as
\begin{align*}
x_{+} &:= e^u, \quad \theta := q^{-1/2} \xi, \quad x_{-} := e^{-u}v.
\end{align*}
Then, the following lemma can be proved by direct calculations using the relations
\begin{align*}
x^{k}_{\pm} \theta = q^{\pm k} \theta x^{k}_{\pm}, \quad [x^{k}_{+}, x^{k}_{-}] = q^{-1/2} \frac{q^2 - 1}{q - 1} \theta^2 x^{k-1}_{+}, \quad \forall k \geq 1
\end{align*}
whose the proof follows from induction on $k$.

Lemma 6.1 The generators $u, \xi, v$ have the following commutation relations (Lie (anti-)brackets)
\begin{align*}
[u, \xi] &= h \xi, \quad [\xi, v] = 0, \quad [u, v] = \frac{2h}{1-e^{-h}} \xi^2, \quad (16)
\end{align*}
where $q = e^h$ and $h \in \mathbb{R}$.

We denote the algebra for which the generators obey the relations (16) by $\mathbb{L}_h := \mathbb{L}(\text{SP}^2_{q})$. The $\mathbb{Z}_2$-graded Hopf algebra structure of $\mathbb{L}_h$ can be read off from Theorem 3.2:

Theorem 6.2 The algebra $\mathbb{L}_h$ is a $\mathbb{Z}_2$-graded Hopf algebra. The definitions of a coproduct, a counit and a coinverse on the algebra $\mathbb{L}_h$ are as follows:
\begin{align*}
\Delta(u) &= u \otimes 1 + 1 \otimes u, \quad \epsilon(u) = 0, \quad S(u) = -u
\end{align*}
for $u \in \{u, \xi, v\}$.

The following proposition can be easily proved by using the Proposition 3.1 together with (15).

Proposition 6.3 The algebra $\mathbb{L}_h$ supplied with the $\mathbb{Z}_2$-graded involution determined by
\begin{align*}
u^* &= \frac{1}{2} h + \ln(e^{-u}v), \quad \xi^* = i \xi, \quad v^* = v
\end{align*}
becomes a super Lie $\star$-algebra.
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