Energy spectra of Hartmann and ring-shaped oscillator potentials using the quantum Hamilton–Jacobi formalism

A Gharbi and A Bouda

Laboratoire de Physique Théorique, Université de Béjaïa, Campus Targa Ouzemour, 06000 Béjaïa, Algeria
E-mail: hakimgharbi@yahoo.fr and bouda_a@yahoo.fr

Received 26 March 2013
Accepted for publication 6 September 2013
Published 26 September 2013
Online at stacks.iop.org/PhysScr/88/045007

Abstract
In the present work, we apply the exact quantization condition, introduced within the framework of Padgett and Leacock’s quantum Hamilton–Jacobi formalism, to angular and radial quantum action variables in the context of the Hartmann and the ring-shaped oscillator potentials which are separable and non-central. The energy spectra of the two systems are exactly obtained.

PACS number: 03.65.Ca

1. Introduction
Since the advent of quantum mechanics, several methods have been developed in order to find the exact energy spectra of bound states in stationary quantum systems. The knowledge of these spectra is necessary for several applications in many fields of physics and theoretical chemistry. Among these methods we may mention the factorization method [1], supersymmetric quantum mechanics [2], the integral equation method [3], the path integral formalism [4], the momentum space method [5], Ma and Xu’s method [6], the group-theoretical method [7], etc.

Leacock and Padgett proposed in 1983 [8, 9] a quantum version of the Hamilton–Jacobi formalism where the energy spectrum of stationary quantum systems is obtained using an exact quantization condition. They postulated that the quantum Hamilton–Jacobi equation (QHJE) for a one-dimensional stationary system is of the form

$$\frac{\hbar}{i} \frac{\partial^2 W(x, E)}{\partial x^2} + \left( \frac{\partial W(x, E)}{\partial x} \right)^2 = 2\mu [E - V(x)] ,$$

(1)

where $W(x, E)$ is the quantum Hamilton characteristic function, $V(x)$ is the potential, $E$ and $\mu$ are respectively the energy and the mass of the particle. By defining the quantum momentum function (QMF) as

$$p(x, E) = \frac{\partial W(x, E)}{\partial x} ,$$

(2)
equation (1) becomes

$$\frac{\hbar}{i} \frac{\partial p(x, E)}{\partial x} + p^2(x, E) = 2\mu [E - V(x)] = p^2_c(x, E) ,$$

(3)

where $p_c$ is the classical momentum function. Consequently we obtain the following boundary condition on the QMF:

$$p(x, E) \rightarrow p_c(x, E) ,$$

(4)

The previous condition can be considered as a correspondence principle [8, 9]. By analogy to the classical action variable [10], the quantum action variable is defined in the complex x-plane by the following contour integral:

$$J = \int_c dx p(x, E) ,$$

(5)

where $c$ is a counterclockwise contour that encloses the two physical turning points of $p_c(x, E)$. Padgett and Leacock [8, 9] have shown that the quantum action variable $J$ can be used to obtain the energy spectra through the following exact quantization condition:

$$J = nh , \quad n = 0, 1, 2, \ldots$$

(6)

without solving equation (3).

Kapoor and co-workers [11, 12] derived energy eigenvalues for a class of one-dimensional potentials and showed
that in addition to obtaining eigenenergies, Leacock and Padgett’s formulation can also yield the eigenfunctions of the energy [13]. Subsequently a relativistic extension of this approach was proposed by Kim and Choi [14, 15] where energy spectra of some relativistic systems have been obtained. The quantum Hamilton–Jacobi formalism (QHJF) was also successfully applied to PT symmetric Hamiltonians and non-hermitian exponential-type potentials [16, 17], supersymmetric potentials [18], the position-dependent mass model [19] and two dimensional central potentials [20] and two dimensional singular oscillator [21].

Recently, the study of non-central and separable potentials has sparked a renewed interest of the research community, given their wide application in quantum physics. The paper is organized as follows. In section 2, the QHJE of a separable and non-central potential is exposed in spherical polar coordinates. In sections 3 and 4, the quantization condition is applied for angular and radial quantum action variables in the context of the Hartmann and ring shaped potentials respectively, and then the energy spectra of the two systems are exactly obtained. In section 5, concluding remarks are given.

2. Quantum Hamilton–Jacobi equation for a separable non-central potential

The three-dimensional Schrödinger equation for a stationary system is given by

$$-\frac{\hbar^2}{2\mu} \Delta \psi (r) + V (r) \psi (r) = E \psi (r),$$

where $\Delta$ is the Laplacian operator. The most general expression of a potential for which equation (7) is separable in spherical polar coordinates is

$$V (r, \theta, \phi) = V_1 (r) + \frac{V_2 (\theta)}{r^2} + \frac{V_3 (\phi)}{r^2 \sin^2 \theta},$$

where $V_1 (r)$, $V_2 (\theta)$, $V_3 (\phi)$ are respectively arbitrary functions of $r$, $\theta$ and $\phi$ [24–26]. Writing the wave function as

$$\psi = \frac{R (r)}{r} H (\theta) \frac{K (\phi)}{(\sin \theta)^2},$$

equation (7) is separated (in units of $2 \mu = 1$) into

$$\frac{d^2 R (r)}{dr^2} + \frac{1}{\hbar^2} \left( E - V_1 (r) - \frac{\hbar^2}{2\mu} \left( \frac{l^2 - \frac{1}{2}}{r^2} \right) \right) R (r) = 0,$$

where $l^2$ and $m^2$ are separation constants. We note that the three previous equations themselves are Schrödinger-like.

The generalization of the QHJE to three dimensions is given by (in units of $2 \mu = 1$)

$$\frac{\hbar}{i} \Delta \psi (\vec{r}) + \frac{\nabla V (\vec{r})}{i} \psi (\vec{r}) = E \psi (\vec{r}),$$

where $\nabla$ is the gradient operator. By writing the characteristic function in spherical polar coordinates as

$$W (\vec{r}) = W_r (r) + W_\theta (\theta) + W_\phi (\phi),$$

we obtain from (13) and (8) the following equations:

$$\frac{\hbar}{i} \left( \frac{\partial^2 W_r}{\partial r^2} + \frac{2}{r} \frac{\partial W_r}{\partial r} \right) + \left( \frac{\partial W_r}{\partial r} \right)^2 = E - V_1 (r) - \frac{a}{r^2},$$

$$\frac{\hbar}{i} \left( \frac{\partial^2 W_\theta}{\partial \theta^2} + \frac{1}{\tan \theta} \frac{\partial W_\theta}{\partial \theta} \right) + \left( \frac{\partial W_\theta}{\partial \theta} \right)^2 = a - \frac{b}{\sin^2 \theta} - V_2 (\theta),$$

$$\frac{\hbar}{i} \left( \frac{\partial^2 W_\phi}{\partial \phi^2} + \frac{\partial W_\phi}{\partial \phi} \right)^2 = b - V_3 (\phi),$$

where $a$ and $b$ are separation constants. Using the definitions [8, 9]

$$p_r = \frac{\partial W_r}{\partial r}, \quad p_\theta = \frac{\partial W_\theta}{\partial \theta}, \quad p_\phi = \frac{\partial W_\phi}{\partial \phi},$$

equations (18)–(20) can be written as

$$\frac{\hbar}{i} \left( \frac{\partial p_r}{\partial r} + \frac{2}{r} p_r \right) + p_r^2 = E - V_1 (r) - \frac{a}{r^2},$$

$$\frac{\hbar}{i} \left( \frac{\partial p_\theta}{\partial \theta} + \frac{p_\theta}{\tan \theta} \right) + p_\theta^2 = a - \frac{b}{\sin^2 \theta} - V_2 (\theta),$$

Later on, we will use the above equations in the application of the exact quantization condition (6) for each degree of freedom and that for the Hartmann and the ring shaped potentials.
3. Hartmann potential

Hartmann potential was introduced in quantum chemistry as a model for some ring shaped molecules. Its expression is

\[ V(r, \theta) = \frac{\alpha}{r} + \frac{\beta}{r^2 \sin^2 \theta}, \]

where \( \alpha \) and \( \beta \) are real constants [22, 24, 25]. Hence, according to (8) we have

\[ V_1(r) = \frac{\alpha}{r}, \quad V_2(\theta) = -\frac{\beta}{\sin^2 \theta}, \quad V_3(\phi) = 0. \]

Let us start with the \( \phi \) variable. Equation (24) becomes

\[ \frac{\hbar}{i} \frac{\partial \phi}{\partial \theta} + p_\phi^2 = b. \]  

(27)

We can easily show that the solution of (12) is

\[ K(\phi) = e^{im\phi} \]

and from the periodic boundary condition

\[ K(\phi) = K(\phi + 2\pi), \]

(29)

one can find that

\[ m = 0, \quad \pm 1, \quad \pm 2, \ldots. \]

(30)

So, using (14), (17), (21) and (27) we obtain

\[ b = \hbar^2 m^2 \to \sqrt{b} = \hbar m. \]

(31)

For the \( \theta \) variable, equation (23) takes the form

\[ \frac{\hbar}{i} \left( \frac{\partial p_\theta}{\partial \theta} + \frac{p_\theta}{\tan \theta} \right) + p_\theta^2 = a - \frac{b + \beta}{\sin^2 \theta}. \]

(32)

The \( \theta \) quantum action variable is given by

\[ J_\theta = \frac{1}{2\pi} \oint c_v d\theta p_\theta, \]

(33)

where \( c_\theta \) [8, 9] is a counterclockwise contour which encloses the two turning points defined by the vanishing of the right-hand side of (32). After the change of variable \( y = -\cot \theta \), relations (32) and (33) become respectively

\[ \frac{\hbar}{i} \left( y^2 + 1 \right) \frac{\partial p_y}{\partial y} - y p_y + p_y^2 = a - b - y^2 (\beta + b) \]

(34)

and

\[ J_\theta = \frac{1}{2\pi} \oint c_v (y+i)(y-i), \]

(35)

where the integration is around the counterclockwise contour \( c_v \) enclosing the two turning points which are solutions of \( a - b - y^2 (\beta + b) = 0 \), and the section of Re \( y \) axis between them. The previous integral can be evaluated by distorting the contour \( c_v \) to enclose the first order poles of the integrand at \( y = \pm i \) and \( y = \infty \) [8, 9]. Then we write

\[ J_\theta = J_1 + J_\infty. \]

(36)

To evaluate \( J_\infty \) at the pole \( y = \infty \), we define a new variable \( s = 1/y \), and we deduce

\[ J_\infty = \frac{1}{2\pi} \oint c_v \frac{p_s}{s} ds \]

(37)

where \( c_s \) is a counterclockwise contour that encloses only the pole at \( s = 0 \). Equation (34) becomes

\[ \frac{\hbar}{i} \left( 1 + s^2 \right) \frac{\partial p_s}{\partial s} + \frac{p_s}{s} + p_s^2 = a - \beta - b - \frac{(\beta + b)}{s^2}. \]

(38)

We write \( p_s \) near \( s = 0 \) as [8, 9]

\[ p_s = \frac{b_1}{s} + a_0 + a_1 s + \cdots. \]

(39)

Substituting this expression into equation (38), we deduce that

\[ b_1 = \pm i \sqrt{\beta + b}. \]

(40)

We note that it is only the term proportional to \( b_1 \) in (39) which has a non-vanishing contribution in (37). To remove the ambiguity in the sign of \( b_1 \), we apply the boundary condition (4). Indeed, according to [8, 9, 11], the classical momentum function \( p_c^2 \) is defined as the branch of the square root which is positive just below the branch cut joining the two classical turning points given by the solution of the equation \( p_c^2 = 0 \). Therefore, from (38), the classical momentum function satisfies \( p_c^2 = +i(\beta + b)^{1/2}/s \) near \( s = 0 \ (y = \infty) \). It follows from (39) that \( b_1 = +i(\beta + b)^{1/2} \). Applying the residue theorem to (37), we find

\[ J_\infty = \frac{1}{2\pi} 2\pi i b_1 = -\sqrt{\beta + b}. \]

(41)

To evaluate \( J_1 \) we make a change of variable \( y = s+i \), and we obtain

\[ J_1 = \frac{1}{2\pi} \oint c_v \frac{p_s}{(s+2i)s} ds, \]

(42)

where \( c_s \) is a clockwise contour that encloses only the pole at \( s = 0 \). Equation (34) becomes

\[ \frac{\hbar}{i} \left( s^2 + 2i s \right) \frac{\partial p_s}{\partial s} - \frac{\hbar}{i} (s+i) p_s + p_s^2 = a - (b+\beta) s^2 - 2i (\beta+b) s. \]

(43)

We expand \( p_s \) near \( s = 0 \) as

\[ p_s = \frac{b_1}{s} + a_0 + a_1 s + \cdots \]

(44)

and note that the only contributing coefficient in (44) to the contour integral (42) is \( a_0 \). Substituting (44) in (43) and matching terms of same power of \( s \), we obtain

\[ a_0 = \frac{\hbar}{2} \pm \sqrt{a + \left( \frac{\hbar}{2} \right)^2}. \]

(45)

According to [8, 9], the ambiguity in the sign of \( a_0 \) can be removed by using the boundary condition (4). Then we find

\[ a_0 = \frac{\hbar}{2} - \sqrt{a + \left( \frac{\hbar}{2} \right)^2}. \]

(46)
So, using the residue theorem we find
\[ J_i = -\frac{a_0}{2} = -\frac{\hbar}{4} + \frac{1}{2} \sqrt{a + \left(\frac{\hbar}{2}\right)^2}. \]  
(47)

The contribution \( J_{-i} \) at the pole \( y = -i \) is evaluated in a similar way as \( J_i \) and we obtain
\[ J_{-i} = \frac{1}{2} \sqrt{a + \left(\frac{\hbar}{2}\right)^2} - \frac{\hbar}{4}. \]  
(48)

Then from (41), (47) and (48), we find
\[ J_0 = \sqrt{\left(\frac{\hbar}{2}\right)^2 + a - \sqrt{b + \beta} - \frac{\hbar}{2}}. \]  
(49)

Using the quantization condition for \( \theta \) variable, we obtain
\[ J_0 = \sqrt{\left(\frac{\hbar}{2}\right)^2 + a - \sqrt{b + \beta} - \frac{\hbar}{2}} = h\epsilon_0, \quad n_0 = 0, 1, 2, \ldots \]
(50)

For \( r \) variable, equation (22) takes the form
\[ \frac{h}{i} \left( \frac{\partial p_r}{\partial r} + \frac{2}{r} p_r \right) + p_r^2 = E - \frac{\alpha}{r} - \frac{a}{r^2}. \]  
(51)

The \( r \) quantum action variable is given by [8, 9]
\[ J_r = \frac{1}{2\pi} \oint_{c_r} dr p_r, \]  
(52)

where \( c_r \) is a counterclockwise contour that encloses the two turning points which are solutions of
\[ E - \frac{\alpha}{r} - \frac{a}{r^2} = 0 \]
and the section of Re \( r \) axis between them. The integral (52) may be evaluated by distorting \( c_r \) to enclose the poles of the integrand at \( r = 0 \) and \( r = \infty \). We call \( J_0 \) and \( J_\infty \) the contributions to \( J_r \) at \( r = 0 \) and \( r = \infty \) respectively. Then, we write
\[ J_r = J_0 + J_\infty. \]  
(53)

To evaluate \( J_\infty \) we make the change of variable \( s = 1/r \), and find
\[ J_\infty = \frac{1}{2\pi} \oint_{s=0} \frac{p_r}{s^2} ds, \]  
(57)

where \( c_r \) is a counterclockwise contour enclosing the pole at \( s = 0 \). Equation (51) becomes
\[ \frac{1}{i} \left( -s^2 \frac{\partial p_r}{\partial s} + 2sp_r \right) + p_r^2 = E - \alpha s - a s^2. \]  
(58)

We expand \( p_r \) near \( s = 0 \) as \( p_r = b_1/s + a_0 + a_1 s + \cdots \) and substitute it in the last equation. Collecting the coefficients of the zeroth and the first power of \( s \) terms gives \( a_0^2 = E \) and \( 2a_0 b_1 - 2a_0 a_1 = \alpha \). Using the boundary condition (4), we find \( a_1 = i\hbar - i\alpha /2\sqrt{-E} \) [8, 9, 20]. Applying the residue theorem to evaluate (57) we obtain
\[ J_\infty = \frac{1}{2\pi} 2i \pi a_1 = \frac{\alpha}{2\sqrt{-E}} - \hbar. \]  
(59)

From (56) and (59) and using the quantization condition for \( r \) variable, we can write
\[ J_r = \frac{\alpha}{2\sqrt{-E}} - \frac{\hbar}{2} - \sqrt{\left(\frac{\hbar}{2}\right)^2 + a} = h\epsilon_r, \quad n_r = 0, 1, 2, \ldots \]  
(60)

From (31), (50) and (60) we obtain
\[ E_{n_r,n_0} = -\frac{\alpha^2}{4\hbar^2} \left( n_r + n_0 + \sqrt{\frac{\beta^2}{\hbar^4} + 1} \right)^2, \]  
(61)

where \( n_r, n_0 = 0, 1, 2, \ldots \) and \( m = 0, \pm 1, \pm 2, \ldots \). Our result (61) is identical to the one obtained in [24, 25] for energy levels of the Hartmann potential.

Now we will point out how to obtain the energy eigenfunctions. For the \( \phi \) variable, we have already established in (28) and (30) that
\[ K (\phi) = e^{i\omega_\phi}, \quad m = 0, \pm 1, \pm 2, \ldots \]  
(62)

For the \( \theta \) variable, according to relations (9), (14) and (17), we can write
\[ H (\theta) = (\sin \theta)^2 \exp \left( \frac{W_\theta (\theta)}{\hbar} \right). \]  
(63)

Substituting the above relation in (11), and comparing with (19), we find that
\[ \bar{l}^2 = \frac{1}{\hbar^2} \left( a + \left(\frac{\hbar}{2}\right)^2 \right) \]  
(64)

and from (31) and (50) we obtain
\[ \bar{l}^2 = \left( n_0 + \sqrt{m^2 + \frac{\beta^2}{\hbar^2} + \frac{1}{2}} \right)^2. \]  
(65)

Replacing the above result in (11) and after a suitable change of variable, equation (11) is transformed into a standard hypergeometric equation and then we can write the solution \( H (\theta) \) in terms of a hypergeometric function (for more details see [25]).
For \( r \) variable, we substitute (65) and (61) in the radial Schrödinger equation given by (10), and through an appropriate change of variable, we obtain a differential equation which is satisfied by Laguerre polynomials \( L_{2n}^{\lambda}(z) \). So, the solution \( R(r) \) can be written in terms of Laguerre polynomials (see [25]).

### 4. Ring-shaped potential

The ring shaped potential was proposed for the first time by Queene in 1988 [23, 25, 26]. Its expression in spherical coordinates is given by

\[
V(r, \theta) = \alpha r^2 + \frac{\beta}{r^2 \sin^2 \theta},
\]

where \( \alpha \) and \( \beta \) are real constants. Comparing to (8) we have

\[
V_1(r) = \alpha r^2, \quad V_2(\theta) = \frac{\beta}{\sin^2 \theta}, \quad V_3(\phi) = 0.
\]

Therefore equations (18)–(20) become

\[
\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\beta}{r^2 \sin^2 \theta} \right) + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial \phi^2} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial \theta^2} = \left( E - \alpha r^2 \right) \frac{a}{r^2},
\]

(66)

\[
\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\beta}{r^2 \sin^2 \theta} \right) + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial \phi^2} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial \theta^2} = \left( E - \alpha r^2 \right) \frac{a}{r^2}.
\]

(67)

We note that equations (69) and (70) are respectively identical to equations (32) and (27). Thus, the obtained results for \( \phi \) and \( \theta \) variables for the Hartmann potential remain valid for the ring-shaped potential. Accordingly, we apply the quantization condition (6) for the ring-shaped potential only for the \( r \) variable.

The \( r \) quantum action variable is given by [8, 9]

\[
J_r = \frac{1}{2\pi} \oint_{c_r} \frac{\hbar}{i} \, dr \, p_r,
\]

(71)

where \( c_r \) is a counterclockwise contour enclosing the two physical turning points which are solutions of

\[
E - \alpha r^2 \frac{a}{r^2} = 0
\]

(72)

and the section of \( \text{Re} \ r \text{ axis} \) between them. We note that about the four solutions of (72)

\[
r_1 = \left( \frac{1}{2\alpha} \left( E + \sqrt{-4\alpha a + E^2} \right) \right)^{1/2}, \quad r_3 = -r_1,
\]

\[
r_2 = \left( \frac{1}{2\alpha} \left( E - \sqrt{-4\alpha a + E^2} \right) \right)^{1/2}, \quad r_4 = -r_2,
\]

we retain only the two positive ones, \( r_1 \) and \( r_2 \), the two others are unphysical [8, 9, 20].

To calculate \( J_r \), the contour \( c_r \) is distorted to enclose the poles at \( r = 0 \) and \( r = \infty \) as well as the additional poles of \( p_r \) located on the negative \( \text{Re} \ r \text{ axis} \) between the unphysical turning points \( r_3 \) and \( r_4 \). Then we write \( J_r = J_r + J_{\infty} + J_c \) where \( J_c \) is the contribution to \( J_r \) from the poles between \( r_3 \) and \( r_4 \). Since the effective potential \( V_{\text{eff}} = \alpha r^2 + a/r^2 \) is symmetric with respect to \( r = 0 \), we can associate to each pole of \( p_r \) between \( r_1 \) and \( r_2 \) a symmetrical pole relative to \( r = 0 \) with residue \(-i\hbar \) located between \( r_3 \) and \( r_4 \). In other words, \( p_r \) is symmetrical with respect to \( r = 0 \) in number and location of poles with residue \(-i\hbar \). Because the distorted contour enclosing poles on the negative \( \text{Re} \ r \text{ axis} \) is clockwise, we have \( J_r = -J_c \). Consequently we obtain \( J_r = (J_0 + J_{\infty})/2 \) [8, 9, 20]. \( J_0 \) and \( J_{\infty} \) are evaluated by following the same method used above in the case of radial quantum action variable of Hartmann potential. We obtain \( J_0 = \frac{1}{2} \hbar - \sqrt{\left( \frac{h}{\alpha} \right)^2 + a} \) and \( J_{\infty} = -\frac{1}{2} \hbar + \frac{1}{2} \sqrt{\frac{E}{\alpha}} \).

Using the quantization condition for \( r \) variable, we find

\[
J_r = \frac{1}{2} \left( \frac{1}{2} \sqrt{\frac{E}{\alpha}} - h - \sqrt{\left( \frac{h}{\alpha} \right)^2 + a} \right) = n_r \hbar, \quad n_r = 0, 1, 2, \ldots
\]

(73)

From (31), (50) and (73) we reproduce the exact energy spectrum for the ring shaped oscillator

\[
E_{n_r, n_\theta} = 2\hbar \sqrt{\alpha} \left( 2n_r + n_\theta + \frac{3}{2} + \sqrt{\frac{\beta}{h^2 + m^2}} \right),
\]

(74)

where \( n_r, n_\theta = 0, 1, 2, \ldots \) and \( m = 0, \pm 1, \pm 2, \ldots \) [25, 26].

In order to obtain the energy eigenfunctions of the ring shaped oscillator, we proceed in the same way as explained for the Hartmann potential (see [25]).

### 5. Conclusion

In this work, we have obtained the exact energy spectra of Hartmann and ring shaped oscillator potentials in the framework of Padegtt and Leacock’s formalism. After separation of variables for the tridimensional QHJE, and using the definition of the QMF for \( r, \theta \) and \( \phi \) variables, we apply the exact quantization condition to radial and angular quantum action variables. This yields the energy levels for the two studied systems.

An important feature of this method resides in its ease of use. Indeed, to arrive at eigenenergies, it does not require an analytical solution of the QMF equation but merely the knowledge of its singularity structure.

As a follow-up, we intend to apply this method to other separable and non-central potentials in the context of relativistic equations such as Klein–Gordon or Dirac equations.

### References

[1] Infeld L and Hull T E 1951 Rev. Mod. Phys. 23 21–68
[2] Cooper F, Khare A and Sukhatme U 1995 Phys. Rep. 251 267–385
[3] Muñoz G 1998 Am. J. Phys. 66 254–6
[4] Bentag B and Chetouani L 2000 Czech J. Phys. 50 593–606
[5] Holstein B R 1995 Am. J. Phys. 63 710–6
[6] Ma Z Q and Xu B W 2005 Europhys. Lett. 69 685–91
[7] Yahiaoui S-A and Bentarba M 2009 Int. J. Theor. Phys. 48 315–22
[8] Leacock R A and Padgett M J 1983 Phys. Rev. Lett. 50 3–6
[9] Leacock R A and Padgett M J 1983 Phys. Rev. D 28 2491–502
[10] Goldstein H, Poole C and Safko J 2001 Classical Mechanics 3rd edn (Reading, MA: Addison Wesley)
[11] Bhalla R S, Kapoor A K and Panigrahi P K 1997 Am. J. Phys. 65 1187–94
[12] Bhalla R S, Kapoor A K and Panigrahi P K 1997 Mod. Phys. Lett. A 12 295–306
[13] Ranjani S S, Geojo K G, Kapoor A K and Panigrahi P K 2004 Mod. Phys. Lett. A 19 1457–68
[14] Kim M-J and Choi C-K 1993 J. Korean Phys. Soc. 26 97–100
[15] Kim M-J and Choi C-K 1993 Phys. Lett. A 182 184–90
[16] Ranjani S S, Kapoor A K and Panigrahi P K 2005 Int. J. Mod. Phys. A 20 4067–77
[17] Yeşiltaş O and Sever R 2008 J. Math. Chem. 43 921–31
[18] Bhalla R S, Kapoor A K and Panigrahi P K 1995 arXiv:hep-th/9507154
[19] Yeşiltaş O 2010 J. Phys. A: Math. Theor. 43 095305
[20] Chen G, Xuan P and Wang J-L 2006 Phys. Scr. 73 443–6
[21] Yeşiltaş Ö and Demircioğlu B 2008 Chin. Phys. Lett. 25 1935–8
[22] Hartmann H 1972 Theor. Chim. Acta 24 201–6
[23] Quesne C 1988 J. Phys. A: Math. Gen. 21 3093–103
[24] Chen G 2004 Chin. Phys. 13 144–6
[25] Qian S W, Huang B W, Wang D Y and Gu Z Y 2002 Commun. Theor. Phys. 38 139–42
[26] Chen G and Chen Z-D 2004 Chin. Phys. 13 445–7