KÄHLER METRICS UNIFORMLY EQUIVALENT TO GOOD REFERENCE METRICS

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Abstract. We produce complete bounded curvature solutions to Kähler-Ricci flow with existence time estimates, assuming only that the initial data is smooth Kähler metrics uniformly equivalent to a fixed complete bounded curvature Kähler metric. We also obtain related results on instantaneously complete solutions to Kähler-Ricci flow starting from degenerate initial conditions.

Keywords: Kähler Ricci flow, complete non-compact Kähler manifolds.

1. Introduction

Let $M^n$ be a non-compact complex manifold. The Kähler Ricci flow on $M^n$ starting from an initial Kähler metric $g_0$ is the evolution equation

\begin{equation}
\begin{cases}
\frac{\partial g_{ij}}{\partial t} = -R_{ij} \\
g(0) = g_0.
\end{cases}
\end{equation}

By a solution to (1.1) we mean a smooth family of Kähler metrics $g(t)$ satisfying (1.1) on $M \times [0, T)$ for some $T > 0$. A classical theorem of W.X. Shi \cite{20} says that if $g_0$ is complete with bounded curvature then (1.1) has a short time bounded curvature solution $g(t)$ which is equivalent to $g_0$ (namely, $g(t)$ has bounded curvature and is equivalent to $g_0$ for $t > 0$). In this paper we show that the same result holds assuming only that $g_0$ is equivalent to a complete Kähler metric with bounded curvature. In the general Riemannian setting, Simon \cite{21} proved a similar result for the real Ricci flow $g' = -Rc$ assuming $g_0$ is Riemannian and $\epsilon$-close to a complete bounded curvature Riemannian metric $h$ for a sufficiently small constant $\epsilon$ depending only on $n$. As a solution to Kähler Ricci flow is a special solution to the real Ricci flow, our result shows the smallness condition on $\epsilon$ can be removed in Simon’s theorem when $g_0$ and $h$ are Kähler.

Fix a complete Kähler metric $h$ on $M$ with curvature bound $|\text{Rm}(h)| \leq K$. For $c_2 > c_1 > 0$, define the following space of Kähler metrics

\begin{equation}
S(c_1, c_2, h) := \{\text{smooth Kähler metrics } g_0 \text{ on } M : c_1 h \leq g_0 \leq c_2 h\}
\end{equation}

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and for each $g_0 \in S(c_1, c_2, h)$, define
\begin{equation}
T_{g_0} = \sup \{ T : (1.1) \text{ has a solution } g(t) \text{ on } M \times [0, T) \text{ which has bounded curvature and is equivalent to } g_0 \text{ for all } t > 0 \} \tag{1.3}
\end{equation}

Then our main existence result for (1.1) can be stated as follows.

**Theorem 1.1.** If $g_0 \in S(1, c, h)$ then (1.3) has a bounded curvature solution $g(t)$ on $M \times [0, T_h)$ which is equivalent to $g_0$. Moreover, there exists positive constants $a(n, c, K), T(n, c, K), C_2(n, c, K), C_1(n)$ such that for all $t \in [0, T)$ we have
\begin{enumerate}
  \item $g(t) \in S((e^{-C_1KT}), C_2, h)$
  \item $\sup_M \| \text{Rm}(t) \|_{g(t)} \leq a/t$
\end{enumerate}

**Remark 1.1.** The existence part also holds when the reference Hermitian metric $h$ is not Kähler but with bounded torsion and Chern curvature in addition. We leave it to interested readers.

When $g_0$ is Hermitian but not Kähler on $M$ then by a smooth solution to (1.1) on $M \times (0, T)$ we mean a smooth family of Kähler metrics $g(t)$ on $M \times (0, T)$ solving the first equation in (1.1) such that $g(t) \to g_0$ pointwise on $M$ as $t \to 0$. In this sense, the conclusion of Theorem 1.1 still holds assuming only that $g_0$ is a limit of smooth Kähler metrics $g_k \in S(c, c_k, h)$ where $c_k$ is not assumed uniformly bounded above.

**Theorem 1.2.** If $g_0 \in Cl^2_0(\bigcup_{c>0} S(1, c, h))$, then there is $T(n, K) > 0$ such that (1.1) has a solution on $(0, T)$ in the sense mentioned above with $g(t) \geq e^{-C_1KT}h$ on $(0, T]$.

As another application, we have the following stability result on complex spaceform. In Riemannian case, the stability of spaceforms was first studied in [22, 23]. We show that in the Kähler category, we do not require the $\epsilon$-fairness required in those works.

**Theorem 1.3.** Suppose $(M, h)$ is a complete noncompact Kähler manifold with constant holomorphic sectional curvature $H_h = 2k \leq 0$. Then if $g_0 \in S(c_1, c_2, h)$ for some $c_2 > c_1 > 0$, then $T_{g_0} = +\infty$. Moreover,
\begin{enumerate}
  \item If $k < 0$, then $t^{-1}g(t)$ converges to $|k|(n + 1)h$ in $C^\infty_{\text{loc}}$ as $t \to \infty$;
  \item If $k = 0$, then $g(t)$ converges sub-sequentially in $C^\infty_{\text{loc}}$ to a complete flat metric as $t \to \infty$.
\end{enumerate}

On the other hand, if $g_0$ is assumed only degenerate Kähler, in other words the corresponding (1,1) form $\omega_0$ is closed and nonnegative but not necessarily positive, then we may also produce a solution to (1.1) with existence time estimates provided $\omega_0$ satisfies certain conditions as in the following

**Theorem 1.4.** Suppose $(M, \omega_h)$ is a complete noncompact Kähler manifold with bounded curvature. If $\omega_0$ is a closed nonnegative (1, 1) form such that
\begin{enumerate}
  \item $\omega_h \geq \omega_0$ on $M$;

...
(2) $\omega_0 - s\text{Ric}(\omega_h) + s\sqrt{-1}\partial\bar{\partial}f > \beta \omega_h$ for some $b, s > 0$ and $f \in C^\infty(M) \cap L^\infty(M)$.

Then there is a $C^\infty(M \times (0, s))$ solution to

$$\dot{\varphi} = \log \frac{(\omega_0 - t\text{Ric}(\omega_h) + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_h^n}$$

on $M \times (0, s)$ with $\varphi \to 0$ in $L^\infty(M)$ and for any $t \in (0, s)$, $\omega(t) = \omega_0 - t\text{Ric}(\omega_h) + \sqrt{-1}\partial\bar{\partial}\varphi$ is a solution to the Kähler-Ricci flow which is uniformly equivalent to $\omega_h$ on $M$. Moreover $\varphi$ is smooth up to $t = 0$ on $U = \{x : \omega_0 > 0\}$.

In particular, using the longtime convergence in [10], we deduce that if $h$ is a Kähler-Einstein metric with bounded curvature and negative Einstein constant, then any Kähler metric bounded from above by some multiples of $h$ can be deformed to $h$ along the normalized Kähler-Ricci flow. When $n = 1$, the existence of Ricci flow starting from incomplete metric has been studied in details by Giesen and Topping [6, 7, 8] where they do not require any boundedness on $\omega_0$. In contrast with the [10, Theorem 1.1], we remove the boundedness of $|\nabla^h g_0|$ when $h$ has bounded curvature and $g_0$ is Kähler.

There have been many other works on the existence of Ricci flow when the initial metric $g_0$ has possibly unbounded curvature, see for example [2] [3], [4], [5], [11], [16], [21], [13]. Our approach is similar in spirit to that in [16] (see also [24, 13]). We makes use of an iterative process using the Chern-Ricci flow to produce a solution $g_R(t)$ to (1.1) on $B_h(p, R) \times [0, T)$ with $g_R(0) = g_0$. Estimates for $g_R(t)$ over a compact subset $S \subset B_h(p, R)$ are then established which depend only on $n, c_1, c_2, K, S$ thus allowing us to let $R \to \infty$ to obtain a smooth limit solution $g(t)$ on $M \times [0, T)$ with the desired properties.

The paper is organize as follows. In §2 we prove our main local estimates for Kähler Ricci flow. In §3 we recall some basic results for Chern Ricci flow from [16]. Then in §4, 5, 6 we prove Theorems 1.2, 1.3 and 1.4 respectively.

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2. estimates for Kähler Ricci flow

In this section we establish basic estimates for solutions to (1.1). Here, $(M, g_0)$ is a smooth Kähler manifold and $h$ is another smooth Kähler metric on $M$ with bisectional curvatures bounded by 1. Theorem 2.1 Lemma 2.1 and Lemma 2.2 are purely local in nature and $B_{g_0}(p, r)$ there refers to an arbitrary ball of radius $r$ relative to $g_0$. Then in Proposition 2.1 we establish a global estimate for smooth solutions to (1.1) which are assumed to be uniformly equivalent to $h$ for all $t$.

Given a solution $g(t)$ to (1.1) we will denote by $\omega(t)$ the corresponding family of Kähler forms. We will also let $\omega_h$ denote the Kähler form corresponding to $h$. We will let
\[ \varphi(t) = \int_0^t \log \frac{\omega^n(s)}{\omega^n_h} \, ds. \]

In particular, taking \( \sqrt{-1} \partial \bar{\partial} \) both sides and using (1.1) and the local formula for the Ricci tensor of a Kähler metric allows us to write
\[ \omega(t) = \omega_0 - t \text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} \varphi. \] (2.1)

We also recall the following evolution equations which can be found in [27] when \( g(t) \) is a solution to the Kähler-Ricci flow. Here \( \Psi_{ij}^k = \Gamma(g(t))_{ij}^k - \Gamma(h)_{ij}^k \). For any \( \partial^t \), we have
\[ \left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_g h = -g^{ij} g^{pq} g_{kl} \Psi_{pi}^k \Psi_{qj}^l + g^{ij} g^{pq} R_{i,jkl}^h; \] (2.2)
\[ \left( \frac{\partial}{\partial t} - \Delta \right) |\Psi|^2 = -|\nabla \Psi|^2 - |\nabla \Psi|^2 - 2 \Re \left( g^{r,s} g^{ij} g_{kl} \Psi_{rj}^t \nabla_r \tilde{R}^k_{isp} \right); \]
\[ \left( \frac{\partial}{\partial t} - \Delta \right) |\text{Rm}|^2 \leq -|\nabla \text{Rm}|^2 - |\nabla \text{Rm}|^2 + C_n |\text{Rm}|^3. \]

For notational convenience, we adopt the notation \( a \wedge b = \min\{a, b\} \) for any real numbers \( a, b \) below. The main estimate in this section is the following

**Theorem 2.1.** For any \( a, \lambda > 1 \), there is \( \bar{T}(n, a, \lambda) > 0 \) and \( C_0(n), c_1(n) > 1 \) such that the following is true. Let \( g(t) \) be a solution to the Kähler-Ricci flow on \( B_{g_0}(p, 1) \times [0, T] \) so that
(1) \( |\text{Rm}(h)| \leq 1 \) on \( B_{g_0}(p, 1) \);
(2) \( \lambda^{-1} h \leq g_0 \leq \lambda h \) on \( B_{g_0}(p, 1) \);
(3) \( |\text{Rm}(g(t))| \leq at^{-1} \) on \( B_{g_0}(p, 1) \times (0, T] \);
(4) \( |\varphi| \leq at \) on \( B_{g_0}(p, 1) \times [0, T] \).

Then for all \( t \in [0, T \wedge \bar{T}] \) we have \( B_t(p, 1/4) \subset B_0(p, 1) \), and for all \( x \in B_t(p, 1/4) \) we have
\[ C_0^{-1} \lambda^{-1} h \leq g(t) \leq C_0 \lambda c_1 h. \]

**Proof.** We begin by constructing an appropriate barrier functions for our arguments. By the shrinking ball lemma [25, Lemma 3.2], for \( t \in [0, T] \)
\[ B_{g(t)}(p, 1 - \beta_n \sqrt{at}) \subset B_{g_0}(p, 1), \] (2.3)
for some constant \( \beta_n \). By [19, Lemma 8.3] with \( K = \frac{a}{t} \) and \( r = \sqrt{\frac{t}{a}} \) there, we may infer that
\[ \left( \frac{\partial}{\partial t} - \Delta \right) \left[ d_t(x, p) + c_1(n) \sqrt{at} \right] \geq 0 \] (2.4)
in the sense of barriers whenever \( d_t(x, p) \geq \sqrt{a^{-1} t} \) (see [25, Section 7] for a detailed exposition). Moreover, we may assume \( d_t(x, p) \) to be smooth when we apply maximum principle. Denote \( \eta(x, t) = d_t(x, p) + \beta_n \sqrt{at} \) where \( \beta_n \geq \).
max\{c_1(n), \beta_n\}. Let \( \phi \) be a smooth function on \([0, +\infty)\) so that \( \phi \equiv 1 \) on \([0, \frac{3}{4}]\), vanishes outside \([0, 1]\) and satisfies \(-100\sqrt{\phi} \leq \phi' \leq 0, \phi'' \geq -100\phi\).

We will make use of the evolving barrier \( \log \Phi^2(x, t) := \log \phi^2(\eta(x, t)) \). By (2.3) and (2.4), we may choose \( \tilde{T} \) sufficiently small depending on \( n, a \) such that for all \( t \in T \wedge \tilde{T} \)

\[
\begin{align*}
(1) & \quad \text{Domain}(\log \Phi^2(x, t)) \subset B_{g_0}(p, 1) \\
(2) & \quad B_{g(t)}(p, 1/2) \subset \{ x : \log \Phi^2(x, t) = 0 \} \\
(3) & \quad \left( \frac{\partial}{\partial t} - \Delta \right) (\log \Phi^2) \leq C_n \Phi^2
\end{align*}
\]

We will also make use of the cut off function \( \tilde{\Phi}(x, t) = \phi^2(2\eta(x, t)) \). Then similarly we may choose \( \tilde{T} \) sufficiently small depending on \( n, a \) such that for all \( t \in T \wedge \tilde{T} \)

\[
\begin{align*}
(1) & \quad B_{g(t)}(p, 1/2) \subset \text{Domain}(\tilde{\Phi}(x, t)) \subset \{ x : \log \Phi^2(x, t) = 0 \} \\
(2) & \quad B_{g(t)}(p, 1/4) \subset \{ x : \tilde{\Phi}(x, t) = 0 \} \\
(3) & \quad \left( \frac{\partial}{\partial t} - \Delta \right) (\tilde{\Phi}^2) \leq 2 \frac{\|\partial \tilde{\Phi}\|^2}{\Phi}
\end{align*}
\]

**Claim 2.1.** If \( \tilde{T}(n, a, \lambda) \) is sufficiently small, then for all \( x \in B_t(p, 1/2), t \leq T \wedge \tilde{T} \),

\[
\frac{1}{c(n)\lambda} h \leq g(t).
\]

**proof of Claim 2.1.** Consider the function

\[
F(x, t) = \log \text{tr}_g h + 2 \log \Phi - L\varphi + nL t (\log t - 2)
\]

for times \( t \in [0, T \wedge \tilde{T} \wedge (nL)^{-1}] \) where \( \Phi, \tilde{T} \) is as above and \( L = 6\lambda \). By the properties of the barrier \( \log \Phi, F(x, t) \) attains a maximum value at some point \((x_0, t_0)\) in its domain. If \( t_0 = 0 \), then the Claim immediately follows from the definition of \( F \) and the hypothesis of the Theorem. Now suppose \( t_0 > 0 \). Then using

\[
\omega(t) = \omega_0 - t \text{Ric}(h) + \sqrt{-1}\partial\bar{\partial}\varphi \geq \frac{1}{2\lambda} \omega_h + \sqrt{-1}\partial\bar{\partial} \varphi,
\]

the evolution equation of \( \log \text{tr}_h g \) from (2.2) then gives the following at \((x_0, t_0)\)

\[
\left( \frac{\partial}{\partial t} - \Delta \right) F \leq \text{tr}_g h + \frac{C_n}{\Phi^2} + L \log \frac{\det h}{\det g} + L \Delta \varphi + nL \log t - nL
\]

\[
\leq \text{tr}_g h \cdot \left( 1 - \frac{1}{2\lambda} L \right) + nL \log(t \text{tr}_g h) + \frac{C_n}{\Phi^2}
\]

\[
\leq - \text{tr}_g h + \frac{C_n}{\Phi^2}
\]

Here we have assumed that \( \text{tr}_g h(x_0, t_0) \geq 1 \) (without loss of generality) in the second inequality, and we have used the elementary inequality \( \log x \leq x \) for
all $x > 0$ and $L = 6\lambda$ in the last inequality. By the maximum principle we then conclude that $0 \leq -\tr g h + \frac{C_n}{\Phi} \Big|_{(x_0, t_0)}$ and thus

\begin{equation}
F(x_0, t_0) \leq C_n + a L t_0 + n L t_0 (\log t_0 - 2) \leq \tilde{C}(n)
\end{equation}

for some constant $\tilde{C}(n)$ provided we further shrink $\tilde{T}$ is necessary depending only on $n, a, \lambda$.

On the other hand, we also have $F(0) \leq \log \lambda + \log n$. Thus in summary, we conclude that for $t \in [0, T \wedge \tilde{T} \wedge (nL)^{-1}]$ we have

\begin{equation}
F(x, t) \leq \max \{ C_n, \log \lambda + \log n \}
\end{equation}

and the claim follows from this, the definition of $F$ and the properties of the barrier $\log \Phi$. \hfill \Box

To prove the Theorem, it suffices only to obtain a local upper bound for the volume form of $g(t)$ in view of Claim 2.1. We do this in the following

\textbf{Claim 2.2.} There is $c(n) > 0$ such that if $\tilde{T}(n, a, \lambda)$ is sufficiently small, then for all $x \in B_t(p, 1/4)$, $t \leq T \wedge \tilde{T}$,

\begin{equation}
\dot{\varphi} = \log \frac{\omega^n(t)}{\omega_h^n} \leq c(n) \log \lambda.
\end{equation}

\textbf{proof of Claim 2.2.} Consider the function

$$G = \bar{\Phi} \frac{(\dot{\varphi})^2}{1 + L \varphi}$$

for $t \leq T \wedge \tilde{T}$ where $L > 0$ is a constant to be chosen later. Due to the cutoff function, $G$ attains its maximum at some point $(x_0, t_0)$ with $t_0 \leq T \wedge \tilde{T}$. If $t_0 = 0$, then the upper bound in the claim is trivial. If $t_0 > 0$ we calculate the following at $(x_0, t_0)$ where we denote $\tilde{g} = g_0 - t \Ric(h)$ and further assume, as we may, that $\tilde{T} \leq 1/2\lambda$ and in particular $t_0 \leq 1/2\lambda$ in (2.7):

\begin{align*}
\left( \frac{\partial}{\partial t} - \Delta \right) \frac{(\dot{\varphi})^2}{1 + L \varphi} &= -\frac{L(\dot{\varphi})^2}{(1 + L \varphi)^2} (\dot{\varphi} - n + \tr g \tilde{g}) - \frac{2 L^2 \dot{\varphi}^2}{(1 + L \varphi)^2} g^{i\bar{j}} \dot{\varphi}_i \varphi_j \\
&\leq -\frac{L(\dot{\varphi})^2}{(1 + L \varphi)^2} (\dot{\varphi} - n + \tr g \tilde{g}) + \frac{2 |\dot{\varphi}|}{1 + L \varphi} \tr g h \\
&\leq -\frac{L \dot{\varphi}^3}{(1 + L \varphi)^2} + \frac{Ln \dot{\varphi}^2}{(1 + L \varphi)^2} + \left[ -\frac{L \dot{\varphi}^2}{2\lambda(1 + L \varphi)^2} + \frac{2 |\dot{\varphi}|}{1 + L \varphi} \right] \tr g h.
\end{align*}

(2.12)
By Claim 2.1 and the properties of the cut off function $\tilde{\Phi}$ we have the lower bound

$$\dot{\phi} = \log \frac{\det g}{\det h} \geq -n \log(2n\lambda).$$

(2.13)

on the support of $G$. Therefore, we will assume that $\dot{\phi}(x_0, t_0) \geq 1$ as the otherwise the Claim follows. Now we choose $L = 16\lambda$ and further assume, as we may, that $\tilde{T}$ is sufficiently small depending only on $n, a, \lambda$ so that the last term in (2.12) will be bounded above by 0. Then using $\nabla G(x_0, t_0) = 0$ and the properties of $\tilde{\Phi}$ we have the following at $(x_0, t_0)$

$$0 \leq \left( \frac{\partial}{\partial t} - \Delta \right) G$$

(2.14)

$$\leq -\frac{L\dot{\phi}}{(1 + L\phi)} G + \frac{Ln}{(1 + L\phi)} G + C_n G + 2\frac{|\tilde{\Phi}|^2}{\Phi^2} G$$

$$\leq -\frac{L\dot{\phi}}{(1 + L\phi)} G + \frac{Ln}{(1 + L\phi)} G + C_n G + CG$$

which in turn implies $G(x_0, t_0) \leq C(n)$ for some constant $C(n)$ provided we further assume, as we may, that $\tilde{T}$ is sufficiently small depending only on $n, a, \lambda$.

We thus conclude that $G \leq C_n$ and in particular, by the properties of the cut off $\tilde{\Phi}$, if $x \in B_r(p, 1/4)$ and $t \leq T \wedge \tilde{T}$ we have

$$\dot{\phi} \leq C_n \log \lambda.$$

(2.15)

□

The Theorem follows from combining Claim 2.1 and Claim 2.2 together with the following elementary inequality

$$\text{tr}_h g \leq \frac{\det g}{\det h} \left( \text{tr}_h g \right)^{n-1}.$$}

□

The next lemma shows that we have a local curvature estimate $g(t)$ stays uniformly equivalent to a good reference metric. In [20], Sherman-Weinkove showed a related estimate but with less detail on the dependence of the various constants in the Lemma.

**Lemma 2.1.** For any $a, \Lambda > 1$, there is $C_1(n, \Lambda), \tilde{T}(n, a, \Lambda) > 0$ so that the following holds. Suppose $g(t)$ is a solution to the Kähler-Ricci flow on $B_{g_0}(p, 1) \times [0, T]$ so that

1. $|\nabla \text{Rm}(h)| + |\text{Rm}(h)|^{3/2} \leq 1$ on $B_{g_0}(p, 1)$;
2. $|\text{Rm}(g(t))| \leq at^{-1}$ on $B_{g_0}(p, 1) \times (0, T]$;
3. $\Lambda^{-1} h \leq g(t) \leq \Lambda h$ on $B_{g_0}(p, 1)$, $t \in [0, T]$. 


Then for all \( x \in B_t(p, 1/4) \), \( t \leq \hat{T} \wedge T \),
\[
t|\operatorname{Rm}(x, t)| + t^{3/2} |\nabla \operatorname{Rm}| \leq C_1.
\]

**Proof.** We follow the proof in [26]. We will use \( C_i \) to denote constants depending only on \( n, \Lambda \) but not \( a \). As in the proof of Theorem 2.1 we may choose \( \hat{T}(n, a, \Lambda) \) sufficiently small so that the evolving cut off function \( \Phi(x, t) = \phi(\eta(x, t)) \) there satisfies the following for all \( t \leq T \wedge \hat{T} \):

1. Domain(\( \Phi(x, t) \)) \( \subset B_{g_0}(p, 1) \)
2. \( B_{\hat{g}(t)}(p, 1/2) \subset \{ x : \tilde{\Phi}(x, t) = 1 \} \)
3. \( (\frac{\partial}{\partial t} - \Delta) \Phi \leq 100 \Phi \) in the sense of barriers.

The dependence of \( a \) will in fact only appear in \( \Phi \) and hence \( \hat{T} \).

Denote \( \Psi = \Gamma_{\hat{g}(t)} - \Gamma_h \). Consider the function \( F = t \Phi Q + L \cdot \operatorname{tr}_g h \) where \( L \) is a constant to be chosen later and \( Q = |\Psi|^2 \). Using the evolution equation (2.2), we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) F = \Phi Q + t \left[ \Phi \left( \frac{\partial}{\partial t} - \Delta \right) Q + Q \left( \frac{\partial}{\partial t} - \Delta \right) \Phi \right] \\
- 2t \operatorname{Re} \left( g^{ij} \Phi_i Q_j \right) + L \left( \frac{\partial}{\partial t} - \Delta \right) \operatorname{tr}_g h \\
\leq \Phi Q + t\Phi [-|D\Psi|^2 + C_2 Q] \\
+ 100t \Phi Q + C_n t Q^2 \frac{1}{2} |D\Psi||D\Phi| + L (-\Lambda^{-1} Q + C_2) \\
\leq Q (-L \Lambda^{-1} + C_3) + C_3 \\
\leq -Q + C_3.
\]

where the last inequality holds provided we choose \( L = \Lambda(C_3 + 1) \). By the maximum principle, we conclude that \( Q \leq C_3 \) at a point where \( F \) is maximal for \( t \leq T \wedge \hat{T} \), and we conclude from this, the definition of \( F \) and the properties of the cutoff \( \Phi \) that we have

\[
tQ(x, t) \leq C_2.
\]

on \( B_t(p, 1/4) \) for all \( t \leq T \wedge \hat{T} \)

Now we consider the function

\[
G = \frac{\Phi t^2 |\operatorname{Rm}|^2}{A - \bar{F}}
\]

where \( \bar{F} := tQ + L \operatorname{tr}_g h \) and \( \Phi = \phi(2 \cdot \eta(x, t)) \). By (2.17) and our previous estimates, we may assume \( A \) is sufficiently large so that \( 2A \geq A - \bar{F} \geq \frac{1}{2} A \) on the support of \( \bar{\Phi} \). We also may choose \( \hat{T}(n, a, \Lambda) \) smaller still so that \( \bar{\Phi}(x, t) = 1 \) on \( B_t(p, 1/4) \). On the support of \( \bar{\Phi} \) we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \bar{F} \leq -t |\nabla \Psi|^2 - t |\nabla \Psi|^2 + C_4.
\]
Thus we have
\[
\left(\frac{\partial}{\partial t} - \Delta\right) [t^2|\text{Rm}|^2(A - \tilde{F})^{-1}]
\leq (A - \tilde{F})^{-1} [C_6 t^2|\text{Rm}|^3 - t^2|\nabla \text{Rm}|^2 - t^2|\nabla \Psi|^2 + 2t|\text{Rm}|^2] + t^2|\text{Rm}|^2(A - \tilde{F})^{-2} [-t|\nabla \Psi|^2 - t|\nabla \Psi|^2 + C_5]
- 2(A - \tilde{F})^{-3} t^2|\text{Rm}|^2 |\nabla \tilde{F}|^2 - 2t^2(A - \tilde{F})^{-2} \text{Re} (\tilde{h}^2|\text{Rm}|^2 \cdot \tilde{F})
\leq -C_6^{-1} t^3|\text{Rm}|^4 + C_6 t^{-1}.
\]

Now suppose \((x_0, t_0)\) is the point where \(G\) is maximal for \(t \leq T \wedge \hat{T}\). Then either \(t_0 = 0\) and then \(G(t)\) thus \(|\text{Rm}(g(t))|\) vanishes for all \(t\), or else \(t_0 > 0\) and from (2.19) we get
\[
0 \leq \left(\frac{\partial}{\partial t} - \Delta\right) G \leq C_7 \tilde{\Phi}^{-1} G + C_7 t^{-1} - C_7^{-1} \tilde{\Phi}^3|\text{Rm}|^4
\]
and hence \(G(x_0, t_0) \leq C_8\). In particular, we have shown that if \(t \leq T \wedge \hat{T}\), \(x \in B_t(p, 1/4)\), then
\[
t|\text{Rm}(g(t))| \leq C_9.
\]
The first order estimate \(|\nabla \text{Rm}(g(t))|\) follows from the Shi’s estimate, see for example [1]. \(\square\)

We also need the following local estimate from [17, Proposition A.1].

**Lemma 2.2.** For any \(A_1, n > 0\), there exists \(B(n, A_1)\) depending only on \(n, A_1\) such that the following holds: For any Kähler manifold \((N^n, g_0)\) (not necessarily complete), suppose \(g(t), t \in [0, T]\) is a solution of Kähler Ricci flow on \(B_0(x_0, r) \subset N\) such that
\[
|\text{Rm}_{g_0}| \leq A_1 r^{-2} \quad \text{and} \quad |\nabla_{g_0} \text{Rm}(g_0)| \leq A_1 r^{-3}.
\]
on \(B_0(x_0, r)\) and
\[
A_1^{-1} g_0 \leq g(t) \leq A_1 g_0
\]
on \(B_0(x_0, r) \times [0, T]\). Then we have
\[
|\text{Rm}|(g(t)) \leq Br^{-2}
\]
on \(B_0(x_0, \frac{r}{8}) \times [0, T]\).

The next proposition shows that the global lower bound of a complete solution is continuous in the following sense.

**Proposition 2.1.** Let \(g(t)\) be a smooth solution to (1.1) on \(M \times [0, T]\) such that \(g(t) \in S(c_1, c_2, h)\) for all \(t \in [0, T]\) and some constants \(c_i > 0\). If \(g(0) \geq h\) then we have
\[
g(t) \geq e^{-nKt/c_1} h
\]
on \(M \times [0, T]\).
Proof. Suppose the curvatures of $h$ are bounded by $K$ in absolute values. In particular, there exists an exhaustion function $\tilde{\rho}$ of $M$ with $\tilde{\rho} \geq 1$ where $|\partial \tilde{\rho}| + |\sqrt{-1} \partial \bar{\partial} \tilde{\rho}|$ bounded on $M$. We will let $\rho := e^{At} \tilde{\rho}$ where the constant $A$ is large enough such that
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \rho \geq 1
\]
on $M \times [0, T]$. Let $\epsilon > 0$ be given and consider the family of tensors
\[
\alpha(t) = (1 + \epsilon \rho)g(t) - (e^{-cKt})h
\]
on $M$ where $c = n/c_1$. Clearly, $\alpha(0) > 0$ and $\alpha(t) \to +\infty$ as $x \to \infty$. We will show that $\alpha(t) > 0$ on $M \times [0, T]$. The proposition will then follow by letting $\epsilon \to 0$.

Suppose on the contrary that $\alpha$ is not positive on $M \times [0, T]$. Due to the presence of $\rho$, there is $(x_0, t_0) \in M \times (0, T]$ and $X \in T_{x_0}^1, 0$ with $|X|_{g(t_0)} = 1$ such that
\[
\alpha_{X\bar{X}} = 0.
\]
We may choose $t_0 > 0$ such that for all $t < t_0$, $\alpha > 0$ on $M$. Extend $X$ by parallel transport using metric $g(t_0)$ so that at $(x_0, t_0)$, $\partial_t X = \nabla X = \Delta X = 0$. At $(x_0, t_0)$, we have
\[
0 \geq \left( \frac{\partial}{\partial t} - \Delta \right) \alpha_{X\bar{X}}
\]
\[
= -(1 + \epsilon \rho)R_{X\bar{X}} + e^{-cKt} \Delta h_{X\bar{X}} + cK e^{-cKt} h_{X\bar{X}} + \epsilon \Box \rho
\]
on the other hand, using the fact that $\alpha_{X\bar{Y}} = 0$ for all $Y \in T^{1,0} M$, we have
\[
\Delta h_{XX} = g^{pq}h_{kl} \Psi^k_p \Psi^l_q - g^{pq} \tilde{R}_{pqXX} + \frac{1}{2} \left( h^i_j h_{Xl} + R^i_l h_{kX} \right)
\]
\[
= g^{pq}h_{kl} \Psi^k_p \Psi^l_q - g^{pq} \tilde{R}_{pqXX} + cK (1 + \epsilon \rho) h_{\bar{X}X}.
\]
where we denote $\tilde{R}$ to be the curvature of $h$. Hence, we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \alpha_{X\bar{X}} \geq cK e^{-cKt} h_{X\bar{X}} - e^{-cKt} g^{pq} \tilde{R}_{pqXX} + \epsilon
\]
\[
\geq e^{-cKt} (cK - \frac{n}{c_1} K) + \epsilon
\]
\[
> 0.
\]
But this contradicts (2.23).

\[ \Box \]

3. A global existence result for Chern Ricci flow

We recall here a global existence result for the Chern Ricci flow from [16]. A family of hermitian metrics $g(t)$ on $M$ is said to be a solution to the Chern
If \( \frac{\partial}{\partial t} g_{ij} = -R^C_{ij} \) (3.1)

where \( R^C_{ij}(g(t)) \) is the Chern-Ricci curvature of \( g(t) \). If \( g(t) \) is a family of Kähler metrics, then (3.1) and (1.1) coincide since in this case we have the Chern-Ricci curvature \( R^C_{ij} \) of \( g(t) \) is the same as the Ricci curvature \( R_{ij} \) of \( g(t) \). In a local holomorphic coordinate, the Chern Ricci curvature is given by

\[
R^C_{ij} = -\partial_i \partial_j \log \det g(t) = g^{kl} R^C_{ijk\ell} R_{\ell j}
\]

where \( R^C \) denotes the curvature tensor with respect to the Chern connection.

Similar to the Kähler-Ricci flow, if \( g(t) \) is a Hermitian family solving (3.1) then we may write the corresponding family of forms \( \omega(t) \) as

\[
\omega(t) = \omega_0 - t \text{Ric}^C(h) + \sqrt{-1} \partial \bar{\partial} \varphi
\]

where \( h \) is any Hermitian background metric on \( M \) and the evolving potential \( \varphi \) is given by

\[
\varphi(t) = \int_0^t \log \frac{\det g(s)}{\det h} ds
\]

In particular, the Chern-Ricci flow will preserve Kählerity on open set \( U \) provided \( g_0 \) is initially Kähler on \( U \). From now on we will simply use Ric to denote the Chern-Ricci curvature of a Hermitian metric \( g \), which will coincide with the Levi Civita Ricci curvature at any point \( g \) is Kähler. We have the following fundamental existence theorem for the Chern-Ricci flow.

**Theorem 3.1.** [16, Theorem 4.2] Let \( (M^n, g_0) \) be a complete noncompact Hermitian manifold and let \( \partial \omega_0 \) be the torsion of \( g_0 \). Suppose the Chern curvature of \( g_0 \), \( |\partial \omega_0|_{g_0}^2 \) and \( |\nabla g_0 \partial \omega_0|_{g_0}^2 \) are uniformly bounded by \( K > 0 \). Suppose also the Riemannian curvature of \( g_0 \) is bounded. Then there is a constant \( \alpha(n) > 0 \) depending only on \( n \) such that the Chern-Ricci flow has a smooth solution \( g(t) \) on \( M \times [0, \alpha K^{-1}] \) such that \( \alpha g_0 \leq g(t) \leq \alpha^{-1} g_0 \) on \( M \times [0, \alpha K^{-1}] \).

4. **Proof of Main Theorem 1.1**

We begin by proving a result on the existence of local solutions to (1.1) satisfying certain estimates (Lemma 4.1). For our purposes later on, it will be convenient to state the result in terms of the following constants.

**Definition 4.1.** Let \( h \) be a complete Kähler metric satisfying \( |\nabla \text{Rm}(h)| + |\text{Rm}(h)|^{3/2} \leq 1 \) on \( M \) and let \( g_0 \) be a Kähler metric with \( \lambda^{-1} h \leq g_0 \leq \lambda h \). Define the following numbers depending only on \( n, \lambda \):

- Let \( \Lambda(n, \lambda) = \max\{\lambda, 2C_0 \lambda c_1 \} \) where \( C_0(n) \) and \( c_1(n) \) be the constants obtained from Theorem 2.1.
- Let \( C_1(n, \Lambda) \) be the constant from Lemma 2.1 with \( \Lambda \) from above;
- Let \( B(n, C_1 + \alpha(n)^{-1}) \) be the constant obtained from Lemma 2.2 with \( A_1 = C_1 + \alpha(n)^{-1} \).
Let $\mu(n, \lambda) = \sqrt{\left(1 + \frac{\alpha(n)}{C_1 + \beta_n \lambda^2} \right) - 1}$ where $\alpha(n)$ is from Lemma 3.1, $C_1$ is from above, $b$ is from Lemma 4.2 with $\kappa = 0.1$ and $\beta_n$ is a large dimensional constant to be specified in the proof;

- Let $a(n, \lambda) = \max\left\{2B(1 + \mu)^2, n \log \frac{\lambda}{\alpha} \right\}$;
- Let $\tilde{T}(n, a, \lambda)$ be the constant from Theorem 2.1;
- Let $\hat{T}(n, a, \Lambda)$ be the constant from Lemma 2.1;
- Let $\sigma(n, \lambda) = \max\left\{1, \frac{\tilde{T} - 1}{2}, \frac{\hat{T} - 1}{2} \right\}$.

By the bound on the curvature of $h$ and its gradient in Definition 4.1, by [28] there is an exhaustion function $\rho$ on $M$ with
\[
|\partial \rho|^2 + |\sqrt{-1} \partial \bar{\partial} \rho|_h \leq 1.
\]
Let
\[
U_s = \{x \in M : \rho(x) < s\} \subset M.
\]

Our result on existence of local solutions can then be stated as

**Lemma 4.1.** Under Definition 4.1, given any $R >> 1$ there is a solution $g(t)$ to Kähler-Ricci flow (1.1) defined on $U_{R-4\Lambda \mu(1+\mu)^{-1}} \times [0, \sigma^{-2}(1 + \mu)^{-2}]$ with
\[
(1) \quad |Rm(g(t))| \leq at^{-1};
(2) \quad |\varphi(t)| \leq at.
\]

**Proof.** We begin with the following

**Claim 4.1.** There is a short-time solution to the Kähler-Ricci flow on $U_R \times [0, t_0]$ for some $t_0 > 0$ sufficiently small and possibly depending on $R$, so that
\[
(1) \quad |Rm(x, t)| \leq at^{-1};
(2) \quad |\varphi(x, t)| \leq at.
\]

**proof of Claim 4.1.** Let $F(x) = F(\rho(x))$ with $0 < \kappa < 1/8$ such that $(1+R)(1-\kappa) \geq R$ where $F, \kappa$ are from Lemma 4.2. Consider the complete Hermitian metric $\tilde{g}_0 = e^{2F}g_0$ on $U_{R+1}$. It follows from Lemma 4.2 that $\tilde{g}_0$ will satisfy the hypothesis of Theorem 3.1 (see [16]) and so we obtain a solution $\tilde{g}(t)$ to Chern Ricci flow (3.1) on $U_{R+1} \times [0, t_0]$ for some $t_0 > 0$ with initial condition $\tilde{g}_0$. Moreover, the restriction $g(t) = \tilde{g}(t)|_{U_R}$ solves Kähler-Ricci flow (1.1) with initial condition $g_0$ because $\tilde{g}_0 = g_0$ on $U_R$. Let $\hat{h} = e^{2F}h$ and define
\[
\varphi = \int_0^t \log \frac{\det \tilde{g}(s)}{\det \hat{h}} ds.
\]
Then we may write $\tilde{\omega}(t) = \tilde{\omega}_0 - tRic(\hat{h}) + \sqrt{-1} \partial \bar{\partial} \varphi$.

By smoothness of the solution, the curvature estimate in the claim follows by shrinking $t_0$ if necessary to be sufficiently small. Moreover by initial Lipschitz constant, we may choose $t_0$ sufficiently such that
\[
\frac{1}{2\lambda^n} \leq \frac{\det \tilde{g}(t)}{\det \hat{h}} \leq 2\lambda^n
\]
on $U_R \times [0, t_0]$ which implies the second conclusion in the claim. □
Now we provide the construction of an auxiliary function used in our constructions of local solutions to Kähler-Ricci flow.

For $\kappa \in (0, 1)$, let $f : (-\infty, 1) \to [0, \infty)$ be the function:

$$f(s) = \begin{cases} 
0, & s \in (-\infty, 1 - \kappa]; \\
-\log \left[ 1 - \left( \frac{s - 1 + \kappa}{\kappa} \right)^2 \right], & s \in (1 - \kappa, 1).
\end{cases}$$

Let $\varphi \geq 0$ be a smooth function on $\mathbb{R}$ such that $\varphi(s) = 0$ if $s \leq 1 - \kappa + 2\kappa^2$

$$\varphi(s) = \begin{cases} 
0, & s \in (-\infty, 1 - \kappa + 2\kappa^2]; \\
1, & s \in (1 - \kappa + 2\kappa^2, 1).
\end{cases}$$

such that $\frac{2}{\kappa^2} \geq \varphi' \geq 0$. Define

$$\mathfrak{F}(s) := \int_0^s \varphi(\tau)f'(\tau)d\tau.$$ 

**Lemma 4.2.** Suppose $0 < \kappa < \frac{1}{8}$. Then the function $\mathfrak{F} \geq 0$ defined above is smooth and satisfies the following:

(i) $\mathfrak{F}(s) = 0$ for $0 \leq s \leq 1 - \kappa$.

(ii) $\mathfrak{F}' \geq 0$ and $\sum_{k=1}^{2} \exp(-k\mathfrak{F})|\mathfrak{F}^{(k)}| \leq b(\kappa)$.

Now we define $t_k$ and $R_k$ inductively as follows

(a) $t_0$ is from claim 4.1 and $R_0 = R$.

(b) $t_k = t_{k-1}(1 + \mu)^2 = t_0(1 + \mu)^{2k}$.

(c) $R_k = R_{k-1} - 4\Lambda\sigma\sqrt{t_{k-1}} = R - 4\Lambda\sigma(1 + \mu)\mu^{-1}\sqrt{t_k}[1 - (1 + \mu)^{-k}]$.

Consider the following statement

$P(k)$: There is a solution of the Kähler-Ricci flow $g(t)$ on $U_{R_k} \times [0, t_k]$ with $g(0) = g_0$ such that

(1) $|\text{Rm}(g(t))| \leq at^{-1}$;

(2) $|\varphi(t)| \leq at$

on $U_{R_k} \times [0, t_k]$ with $t_k \leq \sigma^{-2}$.

Clearly we see that $P(0)$ is true, while $t_k \to +\infty$ and $R_k \to -\infty$ as $k \to \infty$.

Let $k$ be the largest integer such that $P(k)$ is true. Then at $k + 1$, we have the following possibilities.

**Case 1:** $t_{k+1} = (1 + \mu)^2 t_k \geq \sigma^{-2} > t_k$. Then

$$R_k = R_{k-1} - 4\Lambda\sigma\sqrt{t_{k-1}}$$

$$= R - 4\Lambda\sigma\sqrt{t_k} \cdot \sum_{i=1}^{k} \frac{1}{(1 + \mu)^i}$$

$$\geq R - \frac{4\Lambda\mu}{1 + \mu}.$$
and the Lemma holds in this case.

**Case 2:** \( t_{k+1} < \sigma^{-2} \). We will show that this is not possible. Let \( g(t) \) be the solution to \((1.1)\) on \( U_{R_k} \times [0, t_k] \) from the statement \( P(k) \). Using \( \lambda^{-1}h \leq g_0 \leq \lambda h \), for \( x \in U_{R_k - \Lambda \sigma \sqrt{t_k}} \) we have

\[
B_{g_0}(x, \sigma \sqrt{t_k}) \subset U_{R_k}.
\]

Let \( r = \sigma \sqrt{t_k} < 1 \). By the choice of \( \sigma \) from definition \( 4.1 \), we can apply Theorem \( 2.1 \) to Kähler-Ricci flow \( \hat{g}(t) = r^{-2}g(r^2t) \) on \( B_{\hat{g}(0)}(x, 1) \times [0, t_kr^{-2}] \) with reference metric \( \hat{h} = r^{-2}h \) to deduce that for all \((x, t) \in U_{R_k} \times [0, t_k] \),

\[
(4.6) \quad \Lambda^{-1}h \leq g(t) \leq \Lambda h.
\]

Here we use the fact that \( r < 1 \) so that the rescaled reference metric \( \hat{h} \) satisfies the curvature assumptions made on \( h \) in definition \( 4.1 \). Thus for \( x \in U_{R_k - 2\Lambda \sigma \sqrt{t_k}} \) we apply Lemma \( 2.1 \) to \( \hat{g}(t) = r^{-2}g(r^2t) \) on \( B_{\hat{g}(0)}(x, 1) \times [0, t_kr^{-2}] \) with reference metric \( \hat{h} \) to show that for \((x, t) \in U_{R_k - 2\Lambda \sigma \sqrt{t_k}} \times [0, t_k] \),

\[
(4.7) \quad t|\text{Rm}(x, t)| + t^{3/2}|\nabla \text{Rm}| \leq C_1.
\]

At \( t = t_k \), we extend the Kähler-Ricci flow as follows. Fix \( \kappa \in (0, \frac{1}{10}] \), let

\[
F(x) = \hat{g}(1 - \kappa + \frac{\kappa}{\Lambda \sigma \sqrt{t_k}} (\rho(x) - R_k + 3\Lambda \sigma \sqrt{t_k}))
\]

and consider the conformal change of metric \( \hat{g}_0 = e^{2F}g(t_k) \) and \( \hat{h} = e^{2F}h \). By \( [16] \) Lemma 4.3, \( \hat{g}_0 \) and \( \hat{h} \) are complete Hermitian metric with bounded geometry of \( \infty \) order on \( U_{R_k - 2\Lambda \sigma \sqrt{t_k}} \) with \( \hat{g}_0 = g(t_k) \) on \( U_{R_k - 3\Lambda \sigma \sqrt{t_k}} \). Moreover, by \( [16] \) Lemma B1], \( 4.1 \], \( 4.6 \) and the fact that \( r < 1 < \Lambda \), there is \( C'(n) > 0 \) such that

\[
(4.8) \quad |\text{Rm}^C(\hat{g}_0)| + |\partial \omega_0|^2 + |\nabla \partial \omega_0| \leq \frac{C_1 + C'(n)\Lambda^4}{t_k} = K.
\]

We choose \( \beta(n) \) in definition \( 4.1 \) to be \( C'(n) \) here. By Theorem \( 3.1 \) there is a solution to the Kähler-Ricci flow \( \hat{g}(t) \) starting from \( \hat{g}_0 \) on \( U_{R_k - 2\Lambda \sigma \sqrt{t_k}} \times [0, \alpha(n)K^{-1}] \) with

\[
(4.9) \quad \alpha \hat{g}_0 \leq \hat{g}(t) \leq \alpha^{-1} \hat{g}_0.
\]

Restrict \( \hat{g}(t) \) to \( U_{R_k - 3\Lambda \sigma \sqrt{t_k}} \) provides solution \( g(t) = \hat{g}(t - t_k) \) to Kähler-Ricci flow on \( U_{R_k - 3\Lambda \sigma \sqrt{t_k}} \times [0, t_{k+1}] \).

Moreover using \( 4.9 \), \( 4.7 \) we apply Lemma \( 2.2 \) on \( B_{\hat{g}(t_k)}(x, \sqrt{t_k}) \) where \( x \in U_{R_k - 4\Lambda \sigma \sqrt{t_k}} \) to deduce that for all \( t \in [0, t_{k+1}] \),

\[
(4.10) \quad |\text{Rm}(x, t)| \leq \frac{B}{t_k} < \frac{a}{t}.
\]
The last inequality is by our choice of \( a \) in definition \ref{definition} and properties from \( P(k) \). On the other hand, using \((4.6)\) and \((4.9)\) for \( t \in [t_k, t_{k+1}] \),
\[
|\varphi(t)| \leq at + n(t - t_k) \log \Lambda \alpha < at.
\]
(4.11)
We have thus shown that in this case \( P(k + 1) \) is true, which contradicts the maximality of \( k \).

Thus Case 2 above cannot occur, leaving only Case 1 in which case the Lemma holds. Thus completes the proof of the Lemma.

\[ \square \]

We can now prove Theorem \ref{Theorem} using Lemma \ref{Lemma}. 

\textbf{Proof of Theorem \ref{Theorem}.} By the results in \cite{20} and by scaling, there is a Kähler metric \( \tilde{h} \) having curvature and gradient of curvature bounded on \( M \) by 1. Also we may have \( C^{-1}(n, c, K)\tilde{h} \leq g_0 \leq C(n, c, K)\tilde{h} \) on \( M \) for some constant \( C(n, c, K) \) where \( c, K \) are from the hypothesis of Theorem \ref{Theorem}. It follows by Lemma \ref{Lemma} that there is a solution \( g_k(t) \) to Kähler-Ricci flow \((1.1)\) defined on \( B_{g_0}(k) \times [0, S(n, c, K)] \) for all \( k \) sufficiently large satisfying the conditions in the Lemma. Theorem \ref{Theorem} and local Evans Krylov theory \cite{14, 18} (see \cite{20} for a maximum principle proof) then imply the existence of \( \tilde{T}(n, h) > 0 \) so that \( (1.1) \) has a solution \( g(t) \) on \( M \times (0, T] \) such that \( g(t) \in S(c_1, c_2) \) for all \( t \) and some \( c_1 \) and satisfying the conditions in the claim. The existence of the constants \( a, T, C_2, C_1 \) in Theorem \ref{Theorem} follows from this, and Proposition \ref{Proposition}. The fact that \( g(t) \) extends as a bounded curvature solution to \((1.1)\) on \( M \times [0, T_h) \) follows from condition (1) in the Theorem and \cite{5, Theorem 2.2} (see also \cite{16, Theorem 1.3}).

\[ \square \]

5. Non-smooth Initial Metric

Theorem \ref{Theorem} will follow from the following slightly more general existence theorem.

\textbf{Theorem 5.1.} Let \( g_0 \) be a continuous Hermitian metric. Suppose there is a sequence of smooth Kähler metric \( h \) having curvature and gradient of curvature bounded on \( M \) by 1. Also we may have \( C^{-1}(n, c, K)h \leq g_0 \leq C(n, c, K)h \) on \( M \) for some constant \( C(n, c, K) \) where \( c, K \) are from the hypothesis of Theorem \ref{Theorem}. It follows by Lemma \ref{Lemma} that there is a solution \( g_k(t) \) to Kähler-Ricci flow \((1.1)\) defined on \( B_{g_0}(k) \times [0, S(n, c, K)] \) for all \( k \) sufficiently large satisfying the conditions in the Lemma. Theorem \ref{Theorem} and local Evans Krylov theory \cite{14, 18} (see \cite{20} for a maximum principle proof) then imply the existence of \( \tilde{T}(n, h) > 0 \) so that \( (1.1) \) has a solution \( g(t) \) on \( M \times (0, T] \) with \( g(t) \geq \frac{1}{2n}h \).

We begin by recalling the following Lemma from \cite{4} which basically says that if a local solution \( h(t) \) to \((1.1)\) is a priori uniformly equivalent to a fixed metric \( \hat{g} \) in space time, and close to \( \hat{g} \) at time \( t = 0 \), then it remains close to \( \hat{g} \) in a uniform space time region.
Lemma 5.1. Let \( h(t) \) be a smooth solution to (1.1) on \( B(1) \times [0, T) \) with \( h(0) = h_0 \) where \( B(1) \) is the unit Euclidean ball in \( \mathbb{C}^n \). Let \( \hat{g} \) be a smooth Kähler metric on \( B(1) \). Suppose
\[
N^{-1} \hat{g} \leq h(t) \leq N \hat{g}
\]
on \( B(1) \times [0, T) \) for some \( N > 0 \), and that
\[
(1 - d) \hat{g} \leq h_0 \leq (1 + d) \hat{g}
\]
on \( B(1) \). Then there exists a positive continuous function \( a(t) : [0, T) \to \mathbb{R} \) depending only on \( \hat{g}, N, d \) and \( n \) such that
\[
\frac{(1 - d)(1 - a(t))}{(1 + d)} h_0 \leq h \leq (1 + a(t)) h_0
\]
on \( B(1/2) \times [0, T) \), where \( \lim_{t \to 0} a(t) = n \sqrt{2d(1 + d)/(1 - d)} \).

Proof of Theorem 1.2. Let \( g_0 \) be as in Theorem 1.2. Thus there is a sequence \( \{h_k\} \subset S(1, c_k, h) \) for some sequence \( c_k \) such that \( h_k \to g_0 \) in \( C^0_{\text{loc}}(M) \). By Theorem 1.1, there is a sequence of solutions \( h_k(t) \to (1.1) \) on \( M \times [0, T_k] \) and stays uniformly equivalent to \( h \) for \( t \leq T(k) \). By applying maximum principle on equation (2.2) of \( \text{tr}_{h_k(t)} \), there is \( \epsilon_n > 0 \) such that for all \( (x, t) \in M \times [0, \epsilon_n K^{-1}] \),
\[
h_k(t) \geq \frac{1}{2n} h.
\]
By [1] Lemma 3.3, we also have local uniform upper bound for \( h_k(t) \) with respect to \( h \) on \( [0, \epsilon_n K^{-1}] \). In particular, by the Evans-Krylov theory [14, 18] or Kähler-Ricci flow local estimates [26], we may conclude subsequence convergence \( h_k(t) \to g(t) \) in \( C^\infty_{\text{loc}}(M \times (0, \epsilon_n K^{-1}]) \) where \( g(t) \) solves (1.1). By applying Proposition 2.1 on each \( h_k(t) \), it is easy to see that
\[
g(t) \geq e^{-C_n K t} h.
\]

It remains to prove \( g(t) \to g_0 \) in \( C^0_{\text{loc}}(M) \). Fix some holomorphic coordinate ball \( B(1) \) on \( M \) and let \( \delta > 0 \) be given. Then we may choose some constant \( k_0 \) sufficiently large and some constant \( C'(n, c_k, K) > 0 \) so that on \( B(1) \) we have the following
\[
(1 - \delta) g_0 \leq h_{k_0} \leq C' h_{k_0} \leq h_{k} \leq (1 + \delta) g_0
\]
for some constant \( C' \) depends only on \( n, c, K \). Then by Lemma 5.1 and (5.4) there is a function \( a(t) : [0, T) \to \mathbb{R} \) depending only on \( h_{k_0}, C'(n, c, K), \delta, n \) such that
\[
\lim_{t \to 0} a(t) = n \sqrt{2\delta(1 + \delta)/(1 - \delta)}
\]
and on $B(1) \times [0, T)$ we have
\[
\frac{(1-\delta)(1-a(t))}{(1+\delta)}(1-\delta)^2 g_0 \leq h_k(t) \leq (1+a(t))(1+\delta)^2 g_0
\]
for all $k \geq k_0$.

Using (5.5) and (5.6), we conclude that given any $\epsilon > 0$ we may choose $k_0, t_0 > 0$ so that for all $k \geq k_0$ and $0 < t < t_0$ we have $|h_k(t) - g_0|_{g_0} \leq \epsilon$ on $B(1)$. It follows that $g(t) \rightarrow g_0$ in $C_{loc}^0(M)$ which completes the proof of Theorem 1.2.

\[\Box\]

6. Proof of Theorem 1.3

Proof of Theorem 1.3. Let $g_0 \in S(c_1, c_2, h)$, by Theorem 1.1 and By [16, Theorem 1.3] there is a longtime solution $g(t)$ on $M \times [0, +\infty)$ with $g(0) = g_0$ such that for any $[a, b] \subset [0, +\infty)$, $g(t)$ is uniformly equivalent to $h$. If $k < 0$, the longtime convergence follows from [9, Theorem 1.1] and the uniqueness of complete Kähler-Einstein metric. It suffices to discuss the case of $k = 0$.

For simplicity, we work on the universal cover $\mathbb{C}^n$. According to the parabolic Schwarz Lemma (2.2) with $h$ being the flat metric, we can apply maximum principle to deduce that for all $t > 0$, $g(t) \geq n c_1 h$. Also, since $\left(\frac{\partial}{\partial t} - \Delta\right) \log \det g = 0$ and $\log \det g(t)$ is bounded from above within finite time, we can further infer that $\exists C > 1$ such that for all $(x, t) \in \mathbb{C}^n \times [0, +\infty)$,
\[
C^{-1} h \leq g(t) \leq Ch. \tag{6.1}
\]

By Evans-Krylov estimates [14, 18] or Kähler-Ricci flow local estimate [26], $g(t_k)$ converges to some Kähler metric $g_\infty$ on $\mathbb{C}^n$ for some subsequence $t_k \rightarrow +\infty$ in $C_{loc}^\infty$.

On the other hand, by considering the function $F = t|\Psi|_{g(t)}^2 + L \text{tr}_g h$ on $\mathbb{C}^n \times [0, +\infty)$ where $L$ is sufficiently large so that
\[
\left(\frac{\partial}{\partial t} - \Delta\right) F \leq 0. \tag{6.2}
\]

Since $F$ is bounded, we may apply maximum principle to show that $|\partial g(t)| \leq C't^{-1/2}$ on $\mathbb{C}^n \times [0, +\infty)$ for some $C' > 0$. In particular, $\partial g_\infty = 0$ implying $\omega_\infty = \sqrt{-1} \partial \bar{\partial} f$ for some quadratic polynomial $f$ and hence $\omega_\infty$ is a flat metric.

\[\Box\]

7. Proof of Theorem 1.4

Proof. The proof here is similar to that in [10] except that we have a Kähler approximation directly. For the sake of completeness, we present the proof here. For $\epsilon > 0$, let $g_{\epsilon, 0} = g_0 + \epsilon h$ be a complete Kähler metric uniformly equivalent to $h$. By our main Theorem 1.1 there is a short-time solution to the Kähler-Ricci flow $g_{\epsilon}(t)$ starting from $g_{\epsilon, 0}$. Let $T_\epsilon$ be the maximal of $s > 0$ such that $g_{\epsilon}(t)$ is uniformly equivalent to $h$ on $[0, s]$. Consider the
corresponding potential flow \( \varphi_\epsilon(t) = \int_0^t \log \frac{\det g_\epsilon(s)}{\det h} \, ds \) where we can rewrite the Kähler form of \( g_\epsilon(t) \) to be \( \omega_\epsilon(t) = \omega_0 - t \text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon \).

**Claim 7.1.** There is \( C(n, s, K, \beta, f) > 0 \) such that for all \( \epsilon > 0 \) and \( t \in [0, s \wedge T_\epsilon) \),

(i) \( \varphi_\epsilon \leq Ct \);
(ii) \( \varphi_\epsilon \geq nt \log t - Ct \);
(iii) \( \dot{\varphi}_\epsilon \leq C \).

**proof of claim 7.1.** Since \( h \) has bounded curvature and \( g_\epsilon(t) \) is uniformly equivalent to \( h \), there is an exhaustion function \( \rho > 0 \) on \( M \) such that \( |\sqrt{-1} \partial \bar{\partial} \rho|_h \leq C \). For any \( \delta > 0 \), consider \( \tilde{\varphi} = \varphi_\epsilon - \delta \rho \) on \( [0, s \wedge T_\epsilon - \delta] \). Apply maximum principle on \( \tilde{\varphi} - \tilde{C}t \) where \( \tilde{C} \gg 1 \) so that at the maximum point \((x_0, t_0)\). If \( t_0 > 0 \),

\[
0 \leq \tilde{\varphi}' = \varphi_\epsilon' - \tilde{C} \leq \log \left( \frac{\omega_0 + \epsilon \omega_h - t \text{Ric}(h) + \delta \sqrt{-1} \partial \bar{\partial} \rho}{\omega^n_h} \right) - \tilde{C} < 0.
\]

This is impossible. Hence \( t_0 = 0 \). Here we have used the \( |\text{Rm}(h)| \leq K \) and \( \epsilon, \delta \ll 1 \). By letting \( \delta \to 0 \), we have the first conclusion.

For the lower bound, we consider \( \hat{\varphi} = \varphi_\epsilon + \delta \rho - tf - nt(log t - 1) + \tilde{C}t \) on \([0, s \wedge T_\epsilon - \delta]\) where \( \tilde{C} \) is sufficiently large. Then at its minimum point \((x_0, t_0)\), if \( t_0 > 0 \),

\[
0 \geq \hat{\varphi}' = \varphi_\epsilon' - f - n \log t + \tilde{C} \geq \log \left( \frac{\omega_0 + \epsilon \omega_h - t \text{Ric}(h) + t \sqrt{-1} \partial \bar{\partial} f + \delta \sqrt{-1} \partial \bar{\partial} \rho}{\omega^n_h} \right) - f - n \log t + \tilde{C} \\
\geq -n \log s + n \log \beta - \sup_M |f| + \tilde{C} > 0.
\]

This is impossible. Hence \( t_0 = 0 \). By letting \( \delta \to 0 \), we have the second conclusion. The upper bound bound on \( \varphi_\epsilon \) can be obtained by applying maximum principle on function \( t \dot{\varphi}_\epsilon - \varphi_\epsilon - nt - \delta \rho \) on \([0, s \wedge T_\epsilon - \delta]\) and followed by letting \( \delta \to 0 \). \( \square \)

We now establish the \( C^2 \) estimate of \( \varphi_\epsilon(t) \). Recall that the function \( v = (s - t) \dot{\varphi}_\epsilon + \varphi_\epsilon - sf + nt \) satisfies

\[
\left( \frac{\partial}{\partial t} - \Delta \right) v = \text{tr}_{g_\epsilon} \dot{g}_s \geq \beta \text{tr}_{g_\epsilon} h.
\]

(7.3)
Then for sufficiently large \( L >> 1 \), the function \( F = \log \text{tr}_{g_t} h - Lv + (ns + 1) \log t \) will satisfy

\[
(7.4) \quad \left( \frac{\partial}{\partial t} - \Delta \right) F < \frac{ns + 1}{t} - \text{tr}_{g_t} h.
\]

Apply maximum principle to function \( F_\delta = F - \delta e^{At} \rho \) where \( A \) is large enough so that \( \left( \frac{\partial}{\partial t} - \Delta \right) (e^{At} \rho) \geq 0 \). Then at the maximum of \( F_\delta \), we have \( t \text{tr}_{g_t} h \leq m \).

We may assume \( \text{tr}_{g_t} h \geq 1 \) at that point, otherwise the conclusion trivially holds. Then

\[
(7.5) \quad F_\delta \leq (ns + 1) \log t + \log \text{tr}_{g_t} h + (s - t) \log \frac{\det g_t}{\det h} + C_1(n, s, f, K)
\]

\[
\leq (ns + 1) \log t + (ns + 1) \log \text{tr}_{g_t} h + C_2
\]

\[
\leq C_3.
\]

By letting \( \delta \to 0 \), we conclude that for any \( [a, b] \subset (0, s \land T) \), there is \( C_4(n, s, f, K, a, b) > 1 \) such that

\[
C_4^{-1} h \leq g_\varphi(t) \leq C_4 h.
\]

Therefore, \( T \geq s \). Moreover, by diagonal subsequence argument and local estimate \([14] [18] [26]\), we can let \( \epsilon \to 0 \) to obtain a smooth solution \( \varphi \) to \((1.4)\) on \( M \times (0, s) \). By Claim 7.1 \( \varphi(t) \to 0 \) uniformly on \( M \). The smoothness on \( U = \{ x : g_0 > 0 \} \) follows from \([10] \) Theorem 1.2 since the argument there is purely local once we have the estimates from claim 7.1. One can also argue using the pseudolocality of Ricci flow \([19] [12]\), see \([10] \) Remark 3.1.

\[\square\]
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