Classification of supersymmetric spacetimes in eleven dimensions

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Abstract

We derive, for spacetimes admitting a Spin(7) structure, the general local bosonic solution of the Killing spinor equation of eleven dimensional supergravity. The metric, four form and Killing spinors are determined explicitly, up to an arbitrary eight-manifold of Spin(7) holonomy. It is sufficient to impose the Bianchi identity and one particular component of the four form field equation to ensure that the solution of the Killing spinor equation also satisfies all the field equations, and we give these conditions explicitly.

1 Introduction and results

Supersymmetric spacetimes are of central importance to M- and string theory. They have played a key role in the discovery, verification and subsequent generalisation of gauge theory-gravity duality, through the AdS/CFT correspondence. They have also provided one of the most fruitful areas of overlap between pure mathematics and theoretical physics, through studies of special holonomy manifolds, G-structures and mirror symmetry. The problem of classifying supersymmetric solutions of supergravity theories has therefore attracted much attention, and numerous distinct approaches have
been employed. It was shown in [1] that the notion of a G-structure allows for the deduction of an explicit set of necessary and sufficient algebraic conditions on the spin connection and fluxes for the existence of a single Killing spinor in any supergravity, and this approach has since been applied in numerous contexts, to classify minimally supersymmetric spacetimes in various supergravity theories. In particular, this analysis was applied to $d=11$ supergravity in [2], [3]. In [4], the standard G-structure formalism was refined to give a universally applicable formalism for the complete classification of all supersymmetric spacetimes, admitting any desired number of arbitrary Killing spinors, in any supergravity. The complete and explicit nature of this formalism make it an appropriate tool to apply to the exhaustive study of supersymmetric spacetimes in M- and string theory. The end result of this study will be a complete catalogue of the geometry and matter of all such spacetimes. Such a catalogue will certainly be of relevance to the AdS/CFT correspondence, and will be likely to produce many more applications.

Thus, an obvious target for the refined formalism is $d=11$ supergravity. In [5], all structure groups arising as subgroups of the isotropy group of a single null Killing spinor were classified, and the spaces of spinors fixed by these groups were constructed. These results can now be used to undertake the refined classification of all supersymmetric spacetimes in eleven dimensions admitting at least one null Killing spinor. In this letter we initiate this classification, by deriving the general local bosonic solution of the Killing spinor equation for two Killing spinors sharing a common Spin(7) isotropy group.

Very recently, in [6], the method of [4] was reformulated (using more abstract notation), and its application to the complete classification of all supersymmetric eleven-dimensional spacetimes admitting at least one timelike Killing spinor was initiated.

As was shown in [5], the existence of an arbitrary Spin(7) structure in eleven dimensions is equivalent to the existence of a pair of Killing spinors

$$\epsilon, \ (f + u_i \Gamma^i + g \Gamma^-)\epsilon,$$

where $f$, $u_i$ and $g$ are (a priori) arbitrary real functions, $g \neq 0$, and the spinor $\epsilon$ satisfies the projections

$$\Gamma_{1234}\epsilon = \Gamma_{3456}\epsilon = \Gamma_{5678}\epsilon = \Gamma_{1357}\epsilon = -\epsilon, \quad \Gamma^+\epsilon = 0,$$

in the spacetime basis

$$ds^2 = 2\epsilon^+\epsilon^- + \delta_{ij}\epsilon^i\epsilon^j + (\epsilon^9)^2,$$

where $i, j = 1, ..., 8$. The eight-manifold spanned by the $e^i$ will be referred to as the base. Our result is that assuming the existence of this pair of Killing spinors, and nothing else, the general bosonic solution of the Killing spinor equation admitting a Spin(7) structure
is determined locally as follows. We may take the Killing spinors to be $\epsilon, H^{-1/3}(x)\Gamma^\epsilon$, with metric
\[
ds^2 = H^{-2/3}(x) \left(2[du + \lambda(x)Mdx^M][dv + \nu(x)Ndx^N] + [dz + \sigma(x)Mdx^M]^2\right) + H^{1/3}(x)h_{MN}(x)dx^Mdx^N, \tag{4}\]

where $h_{MN}$ is a metric of Spin(7) holonomy and $d\lambda, dv$ and $d\sigma$ are two-forms in the 21 of Spin(7). Observe that there are three Killing vectors. Defining the elfbeins
\[
e_+ = H^{-2/3}(du + \lambda),
\]
\[
e_- = dv + \nu,
\]
\[
e^9 = H^{-1/3}(dz + \sigma),
\]
\[
e^i = H^{1/6}\hat{e}^i(x)Mdx^M, \tag{5}\]

where $\hat{e}^i$ are the achtbeins for $h$, the four-form is
\[
F = e^+ \wedge e^- \wedge e^9 \wedge d\log H + H^{-1/3}e^+ \wedge e^- \wedge d\sigma - e^+ \wedge e^9 \wedge dv + H^{-2/3}e^- \wedge e^9 \wedge d\lambda + \frac{1}{4!}F_{ijkl}^2\hat{e}^i \wedge \hat{e}^j \wedge \hat{e}^k \wedge \hat{e}^l. \tag{6}\]

Similar flux terms to those arising in our general solution have been recognised before in different contexts. The $F^27$ term was used in [7] to construct resolved membrane solutions. Similar fibrations to those arising here were used to construct rotating and/or wrapped M2 brane solutions with a Calabi-Yau transverse space in, for example, [2], [8]. In the case of a Spin(7) structure, the terms given above exhaust all possibilities.

Typically, the Killing spinor equation provides a first integral of some (but not all) of the field equations and Bianchi identities. We will show below that it is sufficient to impose the Bianchi identity and the $+ - 9$ component of the four-form field equation $\star(d \star F + \frac{1}{2} F \wedge F) = 0$ on our solution of the Killing spinor equation to ensure that all field equations are satisfied. The Bianchi identity reduces to
\[
dF^27 = 0, \tag{7}\]

which implies that $F^27 = F^27(x)$, and $\tilde{d}F^27 = 0$, where $\tilde{d}$ is the exterior derivative restricted to the base. The field equation is
\[
\tilde{\nabla}^2 H = -\frac{1}{2}d\sigma_{MN}d\sigma^{MN} - d\lambda_{MN}d\nu^{MN} - \frac{1}{2 \times 4!}F^27_{MNQP}F^27_{MNQP}, \tag{8}\]

where $\tilde{\nabla}^2$ is the Laplacian on the eight-manifold with metric $h_{MN}$, and in this equation all indices are raised with $h^{MN}$. Our general solution of the Killing spinor equation is determined explicitly, up to an arbitrary eight-manifold with Spin(7) holonomy. We
have nothing new to say about classifying Spin(7) holonomy manifolds here. Rather, we regard eight-manifolds of Spin(7) holonomy as the input for our results, with the output, for a particular Spin(7) manifold, being the general $\mathcal{N} = 2$ solution of the Killing spinor equation induced by that manifold. More precisely, our results, together with those of [3], may usefully be thought of as explicitly providing the most general map from the space of eight-manifolds with Spin(7) holonomy to the space of solutions of the Killing spinor equation in eleven dimensions. We expect that qualitatively similar results will be obtained for all other G-structures in eleven dimensions.

2 Deriving the solution

Let us briefly discuss the method we use to derive the solution; more details may be found in [4], [5]. We assume that the null spinor $\epsilon$ is Killing. The constraints associated with its existence were derived in [3]. A useful set of projections satisfied by $\epsilon$, together with miscellaneous definitions and identities for Spin(7) forms which we use throughout, are given in [3]. We adopt all the conventions and notation of this paper, so that we may readily incorporate their results. We may construct a basis for spinor space by acting on $\epsilon$ with a subset of the Clifford algebra; it was shown in [5] that a basis is given by the thirty-two spinors

$$\epsilon, \; \Gamma^i \epsilon, \; J^A_{ij} \Gamma^{ij} \epsilon, \; \Gamma^{-i} \epsilon, \; J^A_{ij} \Gamma^{-ij} \epsilon,$$

where the forms $J^A_{ij}$, $A = 1, \ldots, 7$ furnish a basis for the 7 of Spin(7). The most general additional Killing spinor compatible with the existence of a Spin(7) structure is $\eta = (f + u_i \Gamma^i + g \Gamma^-) \epsilon$, $g \neq 0$. We may simplify $\eta$ by acting on it with an element of the isotropy group of $\epsilon$, thus leaving the constraints on the intrinsic torsion derived in [3] invariant. By acting with with the $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ element

$$1 + g^{-1} u_i \Gamma^{+i} + g^{-1} f \Gamma^{+9},$$

we may always set $\eta = g \Gamma^- \epsilon$, which we do in what follows. Now, since $\epsilon$ is Killing, $g \Gamma^- \epsilon$ is Killing if and only if

$$[D_\mu, g \Gamma^-] \epsilon = 0,$$

where $D_\mu$ is the supercovariant derivative. We may impose the projections satisfied by $\epsilon$ to write each spacetime component of $[D_\mu, g \Gamma^-] \epsilon$ as a manifest sum of the basis spinors (9). Then by the linear independence of the basis, the coefficient of each basis spinor in each spacetime component must vanish separately.

2.1 Calculating $[D_\mu, g \Gamma^-] \epsilon$

Let us evaluate

$$[D_\mu, g \Gamma^-] \epsilon,$$
using throughout the results of [3], and using the algebraic relationships derived in that paper to express the components of the four form in terms of the spin connection, wherever possible. The + component of (12) is

\[
- g\omega_{++}^9 - g\omega_{++i} \Gamma^i + \partial_+ g\Gamma^- + \frac{g}{3}(\omega_{++ij} \Gamma^{-ij} - \frac{2}{3}\omega_{++i} \Gamma^{-i}) \epsilon. \tag{13}
\]

The − component is

\[
\left[ g(\omega_{-9} - \omega_{-9}) + \frac{g}{3}(2\omega_{99i} - \omega_{i-} - 3\omega_{-+i}) \Gamma^i + \frac{2g}{3}(2\omega_{ij} \Gamma^{ij} + \partial_+ g) \right] \epsilon. \tag{14}
\]

The 9 component is

\[
\left[ -g\omega_{9+} - \frac{g}{3}(3\omega_{9+i} + \omega_{+9}) \Gamma^i + \frac{g}{3}(2\omega_{ij} \Gamma^{ij} + \partial_+ g) \right] \epsilon. \tag{15}
\]

The i component is

\[
\left[ g(\omega_{+-i} + \omega_{9i}) \Gamma^i + \frac{g}{3}(2\omega_{ij} \Gamma^{ij} + \partial_+ g) \right] \epsilon. \tag{16}
\]

2.2 Constraints for a Spin(7) structure

Now we impose

\[
[D_\mu, g\Gamma^-] \epsilon = 0. \tag{17}
\]

Setting the coefficient of each basis spinor in each spacetime component of (12) to zero, and using the N = 1 constraints of [3], we find that the only non-zero components of the spin connection are

\[
\begin{align*}
\omega_{++} & = \omega_{--} = \omega_{99i} = 0, \\
\omega_{++i} & = \partial_+ \log g, \\
\omega_{ij+} & = 2\omega_{ij} \Gamma^{ij} = -2F_{+ij}, \\
\omega_{ij-} & = 2F_{-ij}, \\
\omega_{i9j} & = 2F_{i-9j}, \\
\omega_{ijk} & = -\frac{1}{4}\delta_{[i} \partial_{j]} \log g + \frac{1}{8}\phi_{ij} \partial_l \log g + \omega_{ij} \epsilon. \tag{18}
\end{align*}
\]
together with $\omega_{+jk}^{21}$, $\omega_{-jk}^{21}$ and $\omega_{ijk}^{21}$, which drop out of the Killing spinor equations for $\epsilon$ and $g\Gamma^\epsilon$ and are unconstrained. In the above, $\omega_{ijk}^{21}$ denotes the 21 projection of $\omega_{ijk}$ on $j, k$. The four-form is required to be

$$F = -3e^+ \wedge e^- \wedge e^9 \wedge d\log g + e^+ \wedge e^- \wedge (\omega_{ij9}^{21} e^i \wedge e^j) - e^+ \wedge e^9 \wedge (\omega_{ij}^{21} e^i \wedge e^j) + e^- \wedge e^9 \wedge (\omega_{ij}^{21} - e^i \wedge e^j) + \frac{1}{4!} F_{ijkl}^{27} e^i \wedge e^j \wedge e^k \wedge e^l. \tag{19}$$

$F_{ijkl}^{27}$ drops out of the Killing spinor equations for $\epsilon$ and $g\Gamma^\epsilon$ and is unconstrained. The function $g$ is required to satisfy

$$\partial_\tau g = \partial_\tau g = \partial_\theta g = 0. \tag{20}$$

### 2.3 Solving the constraints

Given our constraints on the spin connection, by repeating the arguments of [3], we see that we may consistently introduce their local coordinates. Thus we take

$$e^+ = L^{-1}(du + \lambda),$$
$$e^- = dv + \frac{1}{2} F du + Bdz + \nu,$$
$$e^9 = C(dz + \sigma),$$
$$e^i = e^i_M dx^m. \tag{21}$$

The one-forms $\lambda, \nu$ and $\sigma$ have components only on the base, and $L, F, B, C, \lambda, \nu, \sigma$ and $e^i_M$ are independent of the coordinate $v$. Note that with this choice of coordinates, (20) implies that

$$g = g(x). \tag{22}$$

Now, using $de^\mu = \omega_{\nu\sigma}^{\mu} e^\nu \wedge e^\sigma$, and employing our constraints from the previous subsection, we find

$$de^+ = 2d\log g \wedge e^+ + \omega_{ij}^{21} e^i \wedge e^j,$$
$$de^- = \omega_{ij}^{21} e^i \wedge e^j,$$
$$de^9 = d\log g \wedge e^9 + \omega_{ij}^{21} e^i \wedge e^j. \tag{23}$$

Comparing these expressions with the exterior derivatives of (21), we find that locally we may take $L^{-1/2} = C = g$, $F = B = 0$, $\lambda = \lambda(x)$, $\nu = \nu(x)$, $\sigma = \sigma(x)$, and $d\lambda, d\nu, d\sigma \in \Lambda^2_{21}$. Next we want to solve the constraints on the base space. Defining

$$M_{ij} = \delta_{ik}(\partial_\theta e^k)_j,$$
$$A_{ij} = \delta_{ik}(\partial_\tau e^k)_j, \tag{24}$$
we find on comparing our constraints on the spin connection (given the form of $e^+, e^-, e^9$ we have just derived) with the explicit expressions worked out in Appendix D of [3], that

$$M_{ij} = M_{ij}^{21},$$
$$\Lambda_{ij} = \Lambda_{ij}^{21},$$

and hence that

$$\omega^7_{ijk} = \tilde{\omega}^7_{ijk} = -\frac{1}{4} \delta_{[j} \partial_{k]} \log g + \frac{1}{8} \phi_{ijk} \partial_l \log g,$$
$$\omega^{21}_{ijk} = \tilde{\omega}^{21}_{ijk} + \sigma_i \Lambda_{jk} + \lambda_i M_{jk},$$

where $\tilde{\omega}$ denotes the spin connection of the base. Conformally rescaling the base according to $e^i = g^{-1/2} \tilde{e}^i$, we find that $\tilde{\omega}^7_{ijk} = 0$, where $\tilde{\omega}$ denotes the spin connection of the conformally rescaled base. Thus the conformally rescaled metric is a metric of Spin(7) holonomy. In fact, we may take the base to be independent of $u$ and $z$. To see this, note that we have the freedom to perform Spin(7) transformations on the base preserving $\epsilon, g \Gamma^\epsilon$, and thus the intrinsic torsion. Under such a transformation, the basis transforms as

$$\tilde{e}^i \to (\tilde{e}^i)' = Q^i \tilde{e}^j.$$  

By performing a $v$-independent Spin(7) transformation, we may choose a $u$-independent basis $(\tilde{e}^i)'$ if we can find a Spin(7) matrix $Q$ such that

$$M = (\partial_u Q^{-1})Q$$

Since $M$ is required to be in the adjoint of Spin(7), we may always find such a $Q$ locally. Repeating the argument for a $u, v$ independent Spin(7) transformation we find that we can also take the $\tilde{e}^i$ to be independent of $z$. Finally defining $g = H^{-1/3}$, and expressing the four-form in terms of $\lambda, \nu$ and $\sigma$, we obtain the general Spin(7) solution in the form given in the introduction.

### 2.4 Integrability conditions

Let us assume that we impose the four-form Bianchi identity, $dF = 0$, on our solution of the Killing spinor equation. We want to use the integrability conditions for the Killing spinor equation to determine which of the field equations we must impose to ensure that all are satisfied. Given that the Bianchi identity is satisfied, the (contracted) integrability condition for an arbitrary Killing spinor $\rho$ is

$$\Gamma^\nu [D_\mu, D_\nu] \rho = (E_{\mu\nu} \Gamma^\nu + Q_{\nu\sigma\tau} \Gamma^\nu_{\mu\sigma\tau} - 6 Q_{\mu\nu\sigma} \Gamma^\nu_{\mu\nu\sigma}) \rho = 0,$$  

7
where \( E_{\mu\nu} = 0 \) and \( Q_{\mu\nu\sigma} = 0 \) are the Einstein and four-form field equations. Taking \( \rho = \epsilon \), by writing each spacetime component of (29) as a manifest sum of basis spinors, we find the following algebraic relationships between the components of the field equations:

\[
\begin{align*}
E_{++} &= E_{99} = 12Q_{+9}, \\
E_{+i} &= 18Q_{+i9}, \\
E_{ij} &= -6Q_{+9} \delta_{ij}.
\end{align*}
\]

The components \( E_{++} \) and \( Q^{21}_{+ij} \) drop out, and are unconstrained by the integrability condition for \( \epsilon \), but all other components of the field equations are required to vanish identically. Next taking \( \rho = g \Gamma^{-}\epsilon \), we find the additional conditions

\[
E_{++} = E_{+i} = Q_{+9} = Q^{21}_{+ij} = 0.
\]

Thus given the existence of the pair of Killing spinors \( \epsilon, g \Gamma^{-}\epsilon \), it is sufficient to impose the four-form Bianchi identity and the single equation \( Q_{+9} = 0 \) to ensure that all field equations are satisfied.

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