Determining input-to-state and incremental input-to-state stability of nonpolynomial systems

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Summary
In this study, we propose constructive ways to determine input-to-state stability (ISS) as well as incremental ISS (δISS) of nonpolynomial dynamical systems. The developed procedures are based on sums-of-square decomposition. This tool is only applicable to polynomial systems. Thus, a rational recast of the nonpolynomial system description is used. This recast generally leads to an increased system order and additional constraints. These constraints must be respected in the resulting formulations. The proposed approach gives a unique and constructive procedure to determine the ISS and the δISS property, which is normally nontrivial and needs a good understanding of the system's dynamics. The proposed approaches are illustrated on several examples.

Keywords
nonlinear feedback control, input-to-state stability, Lyapunov methods, sum-of-squares programming

INTRODUCTION

Stability is one of the fundamental properties of control engineering. In the theory of nonlinear systems, Lyapunov’s direct method emerges as a standard technique for stability analysis. Despite the diversity of possible applications, Lyapunov’s method is in its original form only applicable to autonomous systems. This is not sufficient for many applications, especially if inputs or disturbances are considered. This leads to the concept of input-to-state stability (ISS).1 Due to the importance of that concept, it has been intensively studied during the last decades, for example, References 1-5. Nevertheless, the determination of that property is not an easy task, such that generally a deep insight into the system is necessary. This particular holds if ISS is not sufficient because disturbances on a desired input signal should be considered. In this case, incremental input-to-state stability (δISS)6,7 is the concept of choice. This concept extends the idea of stability corresponding to an equilibrium point to the behavior of trajectories corresponding to each other. Hence, ISS describes how trajectories change their behavior due to an equilibrium under disturbances while δISS gives a statement of how does a disturbed input signal influences the resulting trajectory in comparison to the desired one. So ISS is generally interesting...
in a stabilizing context, where robustness against disturbances should be achieved. Whereas \( \delta \text{ISS} \) guarantees to limit the resulting error of tracking control, considering uncertainties of the initial value and disturbances of the input signal.

In a Lyapunov context, a stability property often leads to a definiteness requirement, which is generally a complex problem. More precisely, the determination of the definiteness of fourth-order polynomials is NP-hard, for example, Reference 8. To overcome that problem and organize the calculation in an automatic manner, sum-of-squares (SOS) decomposition is used. This framework is based on the idea that a polynomial, which can be decomposed in a sum of squares, is positive semidefinite. Although not every positive semidefinite polynomial can be decomposed as an SOS, the question if there exists a suitable decomposition can be computed with a semidefinite programming procedure. Thus, the computation is based on a convex optimization framework. As mentioned before, this concept is only applicable to polynomials and thus polynomial system descriptions, but there are a lot of examples containing trigonometric or exponential functions (e.g., in mechanical or chemical systems). A priori such systems cannot be analyzed using SOS decomposition. A rational recast is briefly introduced to avoid these limitations. The results given here were partly published in the PhD thesis of the author.

This article is organized as follows: The recasting process is explained in the next section. This is followed by an introduction of the Lyapunov based stability analysis of recast systems. In Sections 4 and 5, these ideas are extended to ISS as well as \( \delta \text{ISS} \), respectively. The article ends up with a concluding summary in Section 6.

2 POLYNOMIAL RECAST

Consider a nonlinear system

\[
\dot{z} = f(z)
\]

with the vector field \( f: \mathbb{R}^n \to \mathbb{R}^n \). We assume that (1) has an equilibrium point in the origin, that is, \( f(0) = 0 \). If the map \( f \) is nonpolynomial but consists of sums and products of nested elementary functions, like exponentials, logarithms, trigonometric, or hyperbolic functions, it can be transformed into a rational equivalent with a higher dimension using an algorithmic procedure. This was already shown in the 1980s. An equivalent approach is used in algorithmic differentiation.

Thus, systems of the form

\[
\dot{z}_i = \sum_j a_{ij} \prod_k f_{ijk}(z),
\]

are examined in the following considerations, where \( f_{ijk}(z) \) are nested elementary functions. Algorithm 1 shows how the rational recast is carried out. For each nonpolynomial term, a new state is introduced and afterward the time-derivative is calculated. This is repeated until the system description only contains rational terms.

**Algorithm 1** (Rational recast by References 13,18).

1. Set \( x_i = z_i \) for \( i = 1, \ldots, n \).
2. For each \( f_{ijk}(z) \) which is not of the form \( f_{ijk}(z) = z_i^a \), with \( a \in \mathbb{Z} \) and \( 1 \leq i \leq n \), introduce a new variable \( x_m \). This variable is defined by \( x_m = f_{ijk}(z) \).
3. Calculate the time derivative of \( x_m \).
4. Substitute all \( f_{ijk}(z) \) in the system description by \( x_m \).
5. Repeat 2-4 until a system in rational form is left.

Table 1 shows how many \( (k) \) new variables are needed to transform some common nonlinearities. Although there might be overlapping effects, for example, if sine and cosine are used simultaneously, not four but rather two new state variables are needed. The following example illustrates this procedure.

**Example 1** (Rational recast). The system \( \dot{z} = z^2 + u \) is in example 3.45 of Reference 17 controlled by \( u(z) = -z^2 - z\sqrt{z^2} + 1 \). This gives the closed-loop system

\[
\dot{z} = -z\sqrt{z^2} + 1.
\]
TABLE 1

| Common nonlinearities and their related ODEs |  |
|---------------------------------------------|---|
| $e^z$                                       | $f_{ijk}(z) = x_m$ | 1 |
| $\frac{1}{z}$                              | $-f_{ijk}(z) = -x^2_m$ | 1 |
| $\cos(z)$                                  | $-f_{ijk}(z) = -x_m$ | 2 |
| $\sin(z)$                                  | $-f_{ijk}(z) = -x_m$ | 2 |
| $\sqrt{z^2 + c}$                           | $-2(f'_{ijk}(z))^3 = -2x^3_m$ | 2 |

Following Algorithm 1, we get $x_1 = z$ and $x_2 = \sqrt{z^2 + 1} = \sqrt{x^2_1 + 1}$. This leads to the polynomial system

$$
\dot{x}_1 = -x_1 x_2 \\
\dot{x}_2 = \frac{x_1 x_3}{\sqrt{x^2_1 + 1}} = \frac{-x^2_2 x_2}{x_2} = -x^2_1.
$$

Thus one additional state is needed to polynomialize the system, although Table 1 shows that two (k=2) states are necessary. The cutting of the denominator leads to this simplification. Furthermore, two additional constraints are arising. The first is $x_2 - 1 \geq 0$ and the second is by definition $x_2 = \sqrt{x^2_1 + 1}$. This second constraint is not polynomial and therefore not manageable with SOS, but we can this reformulate to

$$
x_2 = \sqrt{x^2_1 + 1} \Rightarrow x^2_2 - x^2_1 - 1 = 0,
$$

which is a polynomial expression.

The proposed recasting process is the basis for the upcoming stability considerations. This leads to the three-step approach. First, the investigated system is reviewed whether it is polynomial or not. In the nonpolynomial case, the illustrated recasting process is used to generate a polynomial system with constraints. These systems can be analyzed with the following stability formulations.

3 | STABILITY ANALYSIS OF THE RECASTED SYSTEM

In this section, the recasted system

$$
\dot{x}_1 = f_1(x_1, x_2) \\
\dot{x}_2 = f_2(x_1, x_2),
$$

is analyzed. The following statements are according to Reference 18 and briefly reviewed as a basis of the upcoming considerations. As mentioned the recasting process leads to additional constraints. These constraints have several causes. The constraints directly induced by the recasting process are given by

$$
x_2 = F(x_1),
$$

and the indirect ones are

$$
G_1(x_1, x_2) = 0 \\
G_2(x_1, x_2) \geq 0,
$$

where $F(x_1)$ and $G_i(x_1, x_2)$ are given by Table 1. These constraints are used to analyze the stability of the recasted system.
where $F$, $G_1$, and $G_2$ are column vectors of functions and the equalities as well as inequalities hold element-wise. Furthermore, let $g(x_1,x_2)$ be the collective denominator of (6) and (7), which means that $g_1(x_1)$ and $g_2(x_2)$ are polynomials. Moreover we assume that $g(x_1,x_2) \geq 0$, otherwise the system is not well-posed. Furthermore, there are constraints resulting from the domains of the original nonpolynomial functions (e.g., that the square root is only defined for non-negative values). These constraints are described by the semi-algebraic set

\[ \mathbb{D}_1 \times \mathbb{D}_2 = \{(x_1,x_2) \in \mathbb{R}^n \times \mathbb{R}^m : G_D(x_1,x_2) \geq 0\}, \]

where $G_D(x_1,x_2)$ is a column vector of polynomials that fulfill the inequality entry-wise. The set

\[ S = \{ p \in \mathcal{P} | p = \sum_{i=1}^{k} q_i^2, \quad q_i \in \mathcal{P} \} \]

defines the set of all SOS polynomials, while $\mathcal{P}$ is the set of all polynomials. Based on those assumptions, we can formulate the following proposition.

**Proposition 1** (SOS-Lyapunov formulation for the recast system (cf. proposition 4 of Reference 18)). *Let the system (6)-(7) as well as the functions $F$, $G_1$, $G_2$, $G_D$, and $g$ are given. Define $x_{2,0} = F(0)$. If there exists a polynomial function $V^*$, the column vectors of polynomials $\lambda_1$, $\lambda_2$ and the column vectors of SOS-polynomials $\sigma_1, \ldots, \sigma_4$ with appropriate dimensions such that*

\[ V^*(0,x_{2,0}) = 0, \]
\[ V^*(x_1,x_2) - \lambda_1^T(x_1,x_2)G_1(x_1,x_2) - \sigma_1^T(x_1,x_2)G_2(x_1,x_2) - \sigma_3^T(x_1,x_2)G_D(x_1,x_2) \]
\[ - \phi(x_1,x_2) \in S \]
\[ - g(x_1,x_2) \left( \frac{\partial V^*}{\partial x_1}(x_1,x_2)f_1(x_1,x_2) + \frac{\partial V^*}{\partial x_2}(x_1,x_2)f_2(x_1,x_2) \right) \]
\[ - \lambda_2^T(x_1,x_2)G_1(x_1,x_2) - \sigma_2^T(x_1,x_2)G_2(x_1,x_2) - \sigma_4^T(x_1,x_2)G_D(x_1,x_2) \in S, \]

*for some scalar polynomial $\phi$ with $\phi(x_1,F(x_1)) > 0$, $\forall x_1 \in \mathbb{D}_1 \setminus \{0\}$, then $z = 0$ is a globally stable equilibrium.*

**Remark 1.** By substituting $x_1 = z$ and $x_2 = F(z)$ in the Lyapunov function $V^*(x_1,x_2)$, a corresponding Lyapunov function $V(z)$ for the original system (1) results. Further note that the equilibrium $z = 0$ can determined to be globally asymptotically stable by adding a term $\phi(x_1,x_2)$ with $\phi(x_1,F(x_1)) > 0$, $\forall x_1 \in \mathbb{D}_1 \setminus \{0\}$ to Equation (15). This guaranties the negative definiteness of $V^*$ and $V$, respectively.

With Proposition 1, we can verify the stability of system (3).

**Example 2** (Stability of system (3)). As illustrated in Example 1, system (3) can be reformulated as

\[ \dot{x}_1 = f_1(x_1, x_2) = -x_1 x_2 \]
\[ \dot{x}_2 = f_2(x_1, x_2) = -x_1^2 \]

with

\[ x_2 = F(x_1) = \sqrt{x_1^2 + 1} \]
\[ G_1(x_1,x_2) = x_1^2 - x_2^2 + 1 = 0 \]
\[ G_D(x_1,x_2) = x_2 - 1 \geq 0. \]
Based on Equations (16)-(20) and the Lyapunov candidate function
\[ V^*(x) = a_1 x_1^2 + a_2 x_2 + a_3, \]  
(21)
the stability can be analyzed. To assure (13), condition \( a_2 + a_3 = 0 \) must hold. Furthermore, \( f(x_1, x_2) = e_1 x_1^2 + e_2 (x_2 - 1), e_1, e_2 \geq 0.1 \) is chosen. The calculated Lyapunov function is
\[ V^*(x) = 0.000073737 x_1^2 + 0.88884 x_2 - 0.88884, \]  
(22)
\[ V(z) = 0.000073737 z^2 + 0.88884 \sqrt{z^2 + 1} - 0.88884, \]  
(23)
with the time derivative
\[ \dot{V}(z) = -0.000147474 z^2 \sqrt{z^2 + 1} - 0.88884 z^2. \]  
(24)
Thus, system (3) is global asymptotically stable.

**Remark 2.** Please note that the Lyapunov candidate functions in the transformed coordinates do not necessarily fulfill the Lyapunov requirements (\( V \) positive definite and \( V \) negative (semi-) definite). Nevertheless, lead the additional constraints to a valid candidate in the original coordinates. For instance is Equation (22) not positive definite, but with the constraint \( x_2 = \sqrt{z^2 + 1} > 1 \), we result in a positive definite function (23).

### 4 INPUT-TO-STATE STABILITY

The ISS property is an extension of asymptotic stability to systems of the form
\[ \dot{z} = f(z, w), \]  
(25)
with \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \), the input or disturbance \( w(t) \in \mathbb{R}^m \), and the equilibrium \((0,0)\), which means that \( f(0,0) = 0 \). The definition of ISS is based on the so-called *comparison functions* given by the following two definitions.\(^{19}\)

**Definition 1** (Class \( K \) functions). A continuous function \( a : [0, a) \to [0, \infty) \) belongs to class \( K \) if it is strictly increasing and \( a(0) = 0 \). If \( a = \infty \) and \( a(r) \to \infty \) for \( r \to \infty \) then \( a \) belongs to class \( K_{\infty} \).

**Definition 2** (Class \( KL \) functions). A continuous function \( \beta : [0, a) \times [0, \infty) \to [0, \infty) \) belongs to class \( KL \), if for each fixed \( t \), the function \( \beta(., t) \) belongs to class \( K \) and for each fixed \( r \), the mapping \( \beta(r, .) \) is decreasing with \( \beta(r, t) \to 0 \) for \( t \to 0 \).

Based on these types of comparison functions, we can define the ISS property.\(^1\)

**Definition 3** (Input-to-state stability). A system (25) is called ISS, if there exist two functions \( \beta \in KL \) and \( \gamma \in K_{\infty} \), such that for every initial value \( z_0 = z(0) \) and each measurable essentially bounded input function \( w \), the corresponding solution \( z(t, z_0, w) \) exists on the entire real axis and the inequality
\[ |z(t, z_0, w)| \leq \beta(|z_0|, t) + \gamma(|w|_{\infty}), \]  
(26)
holds for \( t \geq 0 \), where \( |\cdot| \) denotes the Euclidean norm and \( ||\cdot||_{\infty} \) denotes the norm of the Lebesgue space \( L_{\infty} \), respectively.

The ISS condition (26) means that for an ISS system every trajectory remains in a ball with the radius \( \beta(|z_0|, t) + \gamma(|w|_{\infty}). \) For \( t \to \infty \), this leads to the smaller ball with the radius \( \gamma(|w|_{\infty}). \) If the input is identically zero, we see that ISS implies global asymptotic stability. An equivalent characterization of this behavior can be formulated using ISS-Lyapunov functions.\(^{2,20}\)
**Definition 4** (ISS-Lyapunov function). A smooth function \( V : \mathbb{R}^n \to [0, \infty) \) is called ISS-Lyapunov function of (25) if the following conditions hold for all \( x \) and \( w \)

\[
\alpha(|z|) \leq V(z) \leq \overline{\alpha}(|z|),
\]

(27)

\[
|x| \geq \gamma(|w|) \Rightarrow \dot{V}(z, w) \leq -\alpha(|z|),
\]

(28)

with \( \alpha, \overline{\alpha} \in K_\infty \) and \( \alpha, \gamma \in K \).

Alternatively to condition (28), the ISS-Lyapunov function can be defined in a so called “dissipation” type of characterization (see remark 2.4 of Reference 2).

**Remark 3.** A smooth function \( V : \mathbb{R}^n \to [0, \infty) \) is an ISS-Lyapunov function of (25) if and only if the following conditions are fulfilled for all \( x \) and \( w \)

\[
\alpha(|z|) \leq V(z) \leq \overline{\alpha}(|z|),
\]

(29)

\[
\dot{V}(z, w) \leq -\alpha(|z|) + \gamma(|w|),
\]

(30)

with \( \alpha, \overline{\alpha}, \alpha, \gamma \in K_\infty \).

Equations (27) and (29) ensure the positive definiteness as well as the radially unboundedness of the function \( V \), while (28) and (30) guarantee the negative definiteness of \( \dot{V} \) for all input magnitudes if \(|x|\) is large enough. Furthermore is an ISS-Lyapunov function \( V(z) \) of the System \( f(z, w) \), a Lyapunov function for the system \( f(z, 0) = f(z) \) and guaranties the stability of the autonomous system.

### 4.1 Verification of ISS for polynomial systems

In this subsection, we consider a polynomial vector fields \( f(x, w) \). The ISS analysis for such systems has been done by Ichihara\(^3\) and is briefly reviewed here. The first task is to find SOS compatible formulations for the comparison functions, because every ISS condition is based on them. The following lemma\(^3\) yields a possible SOS condition for \( K_\infty \) functions.

**Lemma 1.** A univariate real even polynomial without constant term

\[
\alpha^*(s) = \sum_{i=1}^{N} c_{2i}s^{2i},
\]

(31)

with at least one coefficient \( c_{2i} \neq 0 \), belongs to class \( K_\infty \) if and only if

\[
s \cdot \frac{d\alpha^*(s)}{ds} \geq 0
\]

(32)

holds for all \( s \in \mathbb{R} \).

**Remark 4.** It seems unusual that the condition (32) should be fulfilled for all \( s \), since functions of norms are considered. This requirement arises from the SOS procedure. So (32) is checked to be a sum of squares and thus it must hold for all \( s \).

Using the above lemma and the reformulation of Equations (29) and (30) by

\[
V(z) - \alpha(|z|) \in S
\]

(33)

\[
\overline{\alpha}(|z|) - V(z) \in S
\]

(34)
with $S$ being the set of all SOS polynomials, the ISS property of a polynomial system can be determined with the software package SOSTOOLS.

TABLE 2  Common nonlinearities and their upper bounds

| $|F(z)|$ | Upper bounds in $|z|$ |
|---|---|
| $e^z$ | $e^z$ |
| $|\cos(z_2)|$ | $1$ |
| $|\sin(z_2)|$ | $|z|$ |
| $|\sqrt{z_2} + c|$ | $\sqrt{|z|} + c$ |

\[ - \frac{\partial V}{\partial z} f(z, w) - \alpha(|z|) + \gamma(|w|) \in S, \quad (35) \]

This tackles the following proposition.

**Proposition 2.** If the function $\alpha(x_1, x_2)$ is a suitable comparison function in the transformed coordinates fulfilling (33), then $\alpha(0, z)$ is a suitable comparison function in the original coordinates fulfilling (33).

**Proof.** If $V^*(x_1, x_2) - \alpha(x_1, x_2) \geq 0$ holds then $V(z) - \alpha(z, F(z)) \geq 0$ holds as well. With $|(z, F(z))| = \sqrt{|z|^2 + |F(z)|^2} \geq \sqrt{|z|^2}$, it follows due to the monotony property of $\alpha$ that $\alpha(z, F(z)) \geq \alpha(z, 0) = \alpha(|z|)$. Thus $V(z) - \alpha(|z|) \geq 0$ holds as well, with the $K_\infty$ function $\alpha(|z|)$.

The same argumentation holds for the function $\alpha$ in (35). We need an upper bound for the function $\alpha$ in (34). This estimation is not as generalizable as the lower bound. Nevertheless, the determination of an upper bound is unproblematic for common nonpolynomial terms, see Table 2. With the estimations of Table 2 and Equation (36), the monotony condition as well as $z = 0$ hold.

Using these results, a theorem for the ISS analysis of the recasted system (6)-(7) can be formulated.

**Theorem 1** (ISS of the recasted system). Let the system (6)-(7) as well as the functions $F$, $G_1$, $G_2$, $G_D$, and $g$ are given. If there exists a polynomial function $V^*$, the column vectors of polynomials $\lambda_1$, $\lambda_2$ and the column vectors of SOS-polynomials $\sigma_1$, $\sigma_2$, $\sigma_3$ with appropriate dimensions such that

\[ V^*(x_1, x_2) - \lambda_1^T(x_1, x_2)G_1(x_1, x_2) - \lambda_2^T(x_1, x_2)G_2(x_1, x_2) - \sigma_3^T(x_1, x_2)G_D(x_1, x_2) \]

\[ - \alpha(x_1, x_2) \in S \quad (37) \]

\[ \tilde{\alpha}(x_1, x_2) - V^*(x_1, x_2) + \lambda_1^T(x_1, x_2)G_1(x_1, x_2) + \sigma_1^T(x_1, x_2)G_2(x_1, x_2) \]

\[ + \sigma_3^T(x_1, x_2)G_D(x_1, x_2) \in S \quad (38) \]
\[-g(x_1, x_2) \left( \frac{\partial V^*}{\partial x_1}(x_1, x_2)f_1(x_1, x_2) + \frac{\partial V^*}{\partial x_2}(x_1, x_2)f_2(x_1, x_2) \right) - \lambda_2^T(x_1, x_2)G_1(x_1, x_2) - \sigma_2^T(x_1, x_2)G_2(x_1, x_2) - \sigma_4^T(x_1, x_2)G_D(x_1, x_2) - \bar{a}(x_1, x_2) + \gamma(w) \in S, \]

for \( \bar{\xi} = \xi^*(x_1, x_2) - \xi^*(0, x_2, y) \), \( \forall \xi^* \in \{ \bar{a}^*, a^*, \alpha^* \} \) with class \( K_{\infty} \) functions \( \bar{a}^*, a^*, \alpha^* \). Then, the original system is ISS, if an appropriate upper bound for \( \bar{a} \) exists.

**Proof.** Let \( V(z) = V(z, F(z)) \) and \( \xi = \bar{\xi}(z, F(z)) \), \( \forall \xi^* \in \{ \bar{a}, a, \alpha \} \). Using (9)-(11), and the SOS-property of \( \sigma_1, \ldots, \sigma_4 \), Equation (37) can be transformed into

\[
V^*(x_1, x_2) \geq \lambda_1^T(x_1, x_2)G_1(x_1, x_2) + \sigma_1^T(x_1, x_2)G_2(x_1, x_2) + \sigma_3^T(x_1, x_2)G_D(x_1, x_2) + \bar{a}(x_1, x_2)
\]

(40)

\[
\geq \bar{a}(x_1, x_2)
\]

(41)

\[
\geq \bar{a}(x_1, 0).
\]

(42)

Equation (33) holds, since \( \bar{a}(z) = \bar{a}(0, z) \leq \bar{a}(z, F(z)) \) is a class \( K_{\infty} \) function. With the same argumentation, we can conclude that (38) satisfies Equation (34). Furthermore, we can conclude that with (9)-(11), the SOS-property of \( \sigma_1, \ldots, \sigma_4 \), considering \( g(x_1, x_2) > 0 \) and the chain-rule

\[
\frac{\partial V}{\partial z}f(z) = \frac{\partial V^*}{\partial x_1}f_1(z, F(z)) + \frac{\partial V^*}{\partial x_2}f_2(z, F(z)),
\]

that (39) implies (35). Such that (33)-(35) hold and system (25) is ISS.

To illustrated the result, let us analyze system (3). For that purpose, the input or disturbance \( w \) is added.

**Example 3** (Example ISS). Considering the system

\[
\dot{z} = -z \sqrt{\dot{z}^2 + 1} + w
\]

(43)

and the candidate functions

\[
V^* = a_1z_1^2 + a_2z_2^2 + a_3z_2 + a_4
\]

(44)

\[
\bar{a}(x) = c_{a,1}|x|^2 - 1
\]

(45)

\[
\bar{a}(x) = c_{a,1}|x|^2 - 1 + c_{a,2}|x|^4 - 1
\]

(46)

\[
\bar{a}(x) = c_{a,1}|x|^2 - 1
\]

(47)

\[
\gamma(w) = c_{g,1}w^2.
\]

(48)

Applying Theorem 1, it results in

\[
V^*(x) = -0.0204z_1^2 + 0.2439z_2^2 + 0.2439z_1 - 0.4877
\]

(49)

\[
V(z) = 0.2235z^2 + 0.2439\sqrt{(z^2 + 1)} - 0.2439
\]

(50)
Using all decimals it is the exact half, so $V(z = 0) = 0$. In our case, we do not need to estimate the comparison functions because they are automatically fulfill the conditions of Definition 1.

To confirm the results, we need to check if (29) and (30) hold. This implies that

$$V(z) - a(z) \geq 0$$

is fulfilled. Furthermore

$$\tilde{a}(z) - V(z) \geq 0$$

holds, because $(z^2 + 1) \geq 1$ and thus $(z^2 + 1) \geq \sqrt{z^2 + 1}$. The last condition for ISS we need to verify is that $\gamma(|w|) - \dot{V}(z, w) - a(|z|) \geq 0$ holds. Analyzing $\dot{V}(z, w)$ leads to

$$\dot{V}(z, w) = 0.4470wz - 0.4469z^2 \sqrt{z^2 + 1} - 0.2439z^2 + \frac{0.2439wz}{\sqrt{z^2 + 1}}$$

which is

$$\geq 0.2235z^2 + 0.2439\sqrt{z^2 + 1} - 0.2439 - 0.1364z^2 \geq 0$$

and

$$0.0871z^2 + 0.2439\sqrt{z^2 + 1} - 0.2439 \geq 0$$

leaves

$$0.8894z^4 + 0.4684z^2 - 0.2235z^2 - 0.2439\sqrt{z^2 + 1} + 0.2439 \geq 0$$

and

$$0.8894z^4 + 0.2449z^2 - 0.2439\sqrt{z^2 + 1} + 0.2439 \geq 0$$

which results in

$$0.8894z^4 + 0.001z^2 + 0.2439\left(z^2 + 1 - \sqrt{z^2 + 1}\right) \geq 0$$

Thus, inequality (30) is satisfied as well. Consequently, all conditions are fulfilled and the system is ISS.

In the following section, these results are extended to the incremental ISS property.
5 | VERIFICATION OF INCREMENTAL ISS

The classical stability concepts like stability in the sense of Lyapunov, asymptotic stability and ISS describe the behavior of trajectories corresponding to an equilibrium or a special trajectory. The generalization of that idea leads to the concept of incremental stability and if we consider inputs to incremental ISS (δISS). This concept characterizes the behavior of the trajectories to each other. This gives a framework to analyze the influence of errors or disturbances in the input signal as well as the initial conditions. Based on those considerations, we can define δISS.\footnote{7}

**Definition 5** (Incremental Input-to-state stability). A system (25) is called δISS, if there exist two functions $\beta \in KL$ and $\gamma \in K_{\infty}$, such that for every initial values $z_{10}, z_{20}$ and each measurable essentially bounded input functions $w_1, w_2 \in M_w$, with $M_w$ being closed and convex, the corresponding solutions $z(t, z_{10}, w_1), z(t, z_{20}, w_2)$ exist on the entire real axis and the inequality

$$|z(t, z_{10}, w_1) - z(t, z_{20}, w_2)| \leq \beta(|z_{10} - z_{20}|, t) + \gamma(||w_1 - w_2||_{\infty}).$$

(67) holds for $t \geq 0$.

The inequality (67) describes the following behavior. For $t \to \infty$ vanishes the initial value depending $KL$ function $\beta$ such that the influence of an error of the initial value vanishes for large $t$ as well. This means that for $t \to \infty$ only the term $\gamma(||w_1 - w_2||_{\infty})$ remains and the resulting distance of the trajectories only depends on $||w_1 - w_2||_{\infty}$. If we set $w_2$ and $z_{20}$ identically zero and assume that $f(0,0) = 0$, it yields the condition for ISS. Thus, ISS is a special case of δISS and therefore every δISS system is ISS as well.

Another useful property is given in the following proposition.\footnote{7}

**Proposition 3.** For every δISS system (25) with arbitrary $z_{10}, z_{20} \in R^n$, there holds $|z(t, z_{10}, u_1) - z(t, z_{20}, u_2)| \to 0$ if the condition $\lim_{t \to +\infty} |u_1(t) - u_2(t)| = 0$ is fulfilled.

For a proof of that proposition, see Reference 7. In other words, the states of a δISS system converge, if the inputs converge. Using this property, we can show with the following example that not every ISS system is δISS.

**Example 4** (Counterexample). The system

$$\dot{z} = -z + u^3$$

(68)

is ISS and incremental global asymptotically stable but not δISS. To show that the system is ISS, we choose $V(z) = z^2$, which automatically fulfills the condition (29). The time derivative of the candidate function $V$ is given by

$$V(z, u) = 2z(-z + u^3) = -2z^2 + 2zu^3$$

(69)

$$\leq -2z^2 + z^2 + u^6 = -z^2 + u^6.$$  

(70)

Hence, condition (30) holds as well and the system is ISS. Following the ideas presented in Reference 7, we can show that the system is not δISS. Considering that the system (68) is into $z = 1$. This leads to the inputs $u_1 = (z(t, z_{10}, u_1) + 1)^3$ and $u_2 = (z(t, z_{20}, u_2) + 1)^3$ as well as the trajectories $z(t, z_{10}, u_1) = t + z_{10}$ and $z(t, z_{20}, u_2) = t + z_{20}$. Then is the difference of the both trajectories constant $z(t, z_{10}, u_1) - z(t, z_{20}, u_2) = z_{10} - z_{20}$ for all $t$, but the inputs converging for $t \to \infty \Rightarrow u_1(t) - u_2(t) \to 0$. This contradicts Proposition 3. Thus system (68) is not δISS.

As in the case of ISS exists an equivalent Lyapunov condition to characterize the δISS property.\footnote{7}

**Definition 6** (δISS-Lyapunov function). A smooth function $V : R^n \times R^n \to [0, \infty)$ is called an δISS-Lyapunov function of (25) if the following conditions hold

$$\bar{g}(|z_1 - z_2|) \leq V(z_1, z_2) \leq \bar{V}(|z_1 - z_2|).$$

(71)
\[
\frac{\partial V}{\partial z_1} f(z_1, w_1) + \frac{\partial V}{\partial z_2} f(z_2, w_2) \leq \gamma |w_1 - w_2| - \kappa V(z_1, z_2), \tag{72}
\]

with $\alpha, \bar{\alpha}, \gamma \in K_{\infty}$ and $\kappa > 0$.

The term $\kappa V(z_1, z_2)$ can without limitation to the generality be replaced with a $K_{\infty}$ function $\alpha(|z_1 - z_2|)$.\textsuperscript{21} Using this relation and the conditions of Definition 6, the equations

\[
V(z_1, z_2) - \alpha(|z_1 - z_2|) \in S
\tag{73}
\]

\[
\bar{\alpha}(|z_1 - z_2|) - V(z_1, z_2) \in S
\tag{74}
\]

\[
- \frac{\partial V}{\partial z_1} f(z_1, w_1) - \frac{\partial V}{\partial z_2} f(z_2, w_2) - \alpha(|z_1 - z_2|) + \gamma (|w_1 - w_2|) \in S. \tag{75}
\]

arise as sufficient SOS conditions. Equations (73)-(75) are applied in the following example.

**Example 5** (Tank system). This example is adapted from Reference 14. To illustrate the procedure, we analyze the system shown in Figure 1. Using Torricelli’s law,\textsuperscript{22} the system can be described with

\[
\dot{z} = \begin{bmatrix}
\frac{1}{A_1} \left( w_1 - A_V \sqrt{2g(z_1 - z_2)} \right) \\
\frac{1}{A_2} \left( w_2 + A_V \sqrt{2g(z_1 - z_2)} - A_V \sqrt{2g(z_2)} \right)
\end{bmatrix}. \tag{76}
\]

The roots in (76) are nonpolynomial and thus make SOS not applicable. Thus we want to approximate these roots with appropriate Taylor polynomials. The resulting polynomials with different order can been seen in Figure 2, developed at 0.5 m. The third-order polynomial gives a well trade-off between accuracy and complexity. Thus it results in

\[
\sqrt{2g(\Delta z)} \approx \sqrt{gA_V} + \sqrt{gA_V(\Delta z - 0.5)} - \frac{\sqrt{g}}{2} A_V(\Delta z - 0.5)^2 + \frac{\sqrt{g}}{2} A_V(\Delta z - 0.5)^3, \tag{77}
\]

\[
\sqrt{2g(z_2)} \approx \sqrt{gA_V} + \sqrt{gA_V(z_2 - 0.5)} - \frac{\sqrt{g}}{2} A_V(z_2 - 0.5)^2 + \frac{\sqrt{g}}{2} A_V(z_2 - 0.5)^3. \tag{78}
\]

The problem (73)-(75) is not solvable if we just use all monomials until order 2 as comparison and Lyapunov function. However, if the function candidates

\[
V(x, \dot{x}) = a|\dot{x} - \ddot{x}|^4 \tag{79}
\]
Approximated flow velocity with Taylor polynomial of order two, three, and seven (adapted from Ref. 14)

**FIGURE 2**

\[ \alpha(|x - \hat{x}|) = c_2 |x - \hat{x}|^4 \]  \hspace{1cm} (80)

\[ \tilde{\alpha}(|x - \hat{x}|) = c_7 |x - \hat{x}|^4 \]  \hspace{1cm} (81)

\[ \alpha(|x - \hat{x}|) = c_7 |x - \hat{x}|^4 \]  \hspace{1cm} (82)

\[ \gamma(|u - \hat{u}|) = c_7 |u - \hat{u}|^2, \]  \hspace{1cm} (83)

are used, the resulting SOS formulation leads to the coefficients \( a = 0.0136, \ c_\pi = 0.006801, \ c_\sigma = 0.3129, \ c_\epsilon = 0.0003413, \) and \( c_\gamma = 5.17 \) as a result. Hence, the system is incremental input-to-state-stable.

Following the considerations of Section 4.2, the calculation procedure for \( \delta\text{ISS} \) can be extended to the nonpolynomial case. This leads to the conditions

\[ V^*(x_1, x_2) - \lambda_1^T(x_1) G_1(x_1) - \sigma_1^T(x_1) G_2(x_1) - \sigma_2^T(x_1) G_D(x_1) \]
\[ - \lambda_2^T(x_2) G_1(x_2) - \sigma_3^T(x_2) G_2(x_2) - \sigma_4^T(x_2) G_D(x_2) - \gamma(|x_1 - x_2|) \in S \]  \hspace{1cm} (84)

\[ \tilde{\alpha}(|x_1 - x_2|) - V^*(x_1, x_2) + \lambda_1^T(x_1) G_1(x_1) + \sigma_1^T(x_1) G_2(x_1) + \sigma_2^T(x_1) G_D(x_1) \]
\[ + \lambda_2^T(x_2) G_1(x_2) + \sigma_3^T(x_2) G_2(x_2) + \sigma_4^T(x_2) G_D(x_2) \in S \]  \hspace{1cm} (85)

\[ - g(x_1) \frac{\partial V^*}{\partial x_1} f(x_1, w_1) - g(x_2) \frac{\partial V^*}{\partial x_2} f(x_2, w_2) - \lambda_2^T(x_1) G_1(x_1) - \sigma_2^T(x_1) G_D(x_1) \]
\[ - \sigma_3^T(x_2) G_1(x_2) - \lambda_4^T(x_2) G_2(x_2) - \sigma_4^T(x_2) G_D(x_2) - \gamma(|x_1 - x_2|) + \gamma(|w_1 - w_2|) \in S, \]  \hspace{1cm} (86)

with \( x_1 = (x_{11}, x_{12})^T \) and \( x_2 = (x_{21}, x_{22})^T. \) The states \( x_{11}, x_{12}, x_{21}, x_{22} \) denote the states of the original system \( (x_{11}, x_{21}) \) and them resulting from the recasting process \( (x_{12}, x_{22}) \).

6 | CONCLUSION

The proposed approach gives a numeric procedure to determine the ISS and the \( \delta\text{ISS} \) property for polynomial as well as nonpolynomial vector fields. This is done using SOS-decomposition techniques and is exemplarily shown in some
examples. To overcome the restriction of polynomial systems, a rational recast process is used. This generally results in additional constraints that need to be considered in the analysis. Thus extensions to the Lyapunov ISS and δISS conditions are presented. These additional constraints $F, G_1, G_2$ are respected in using the polynomials $\lambda_i$ and $\sigma_i$. This approach seems complicated and is deeply rooted in the SOS procedure since we are just able to verify if a given polynomial is decomposable in a sum of squares. So the direct consideration of the constraints is not possible.

Alternatively, quantifier elimination (QE) techniques can be used to include the given constraints.23,24 In that approach, uncertain or design parameters can be considered. The main disadvantage is the enormous computational effort, which is needed to solve such problems computationally. Nevertheless is some significant progress in the QE algorithms and implementations done.25 So it seems fruitful for further applications.

Approximation yields another possibility to analyze nonpolynomial systems. Considering higher order terms can be used to respect certain phenomena. Using a second-order approximation, for instance, Coriolis and centrifugal terms can be taken into account. Indeed, these approximations are not exact and may produce errors.

In each of these options, the most challenging issues are the choice of the initially chosen Lyapunov candidate as well as comparison functions and the inherent computational barriers coming along with the subordinated semidefinite program or QE method. Nevertheless, the proposed procedure can easily be applied to other properties or control techniques.

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