Semi-analytical description of clumping factor and cosmic microwave background free–free distortions from reionization

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ABSTRACT
The density contrast of the Universe, parametrized in terms of the matter power spectrum and its variance, can amplify the signal of the free–free process in the plasma. The damping of fluctuations on scales smaller than the dark matter particle free streaming scale corresponds to a suppression of the total matter power spectrum on large wavenumbers $k$. We derive the time evolution of the variance of the matter power spectrum for various cosmological models and parameters by numerically computing the power spectrum with a modified version of the Boltzmann code CAMB, for different values of the cut-off parameter $k_{\text{max}}$. Suitable analytical approximations of the numerical results are presented. We then characterize the cosmic microwave background (CMB) free–free spectral distortion accounting for the amplification effect coming from clumping factor. Indeed, the clumpiness, associated with the density contrast of the intergalactic medium, increases at decreasing redshift. The analysis is carried out for selected astrophysical and phenomenological cosmological reionization histories for which we evaluate the impact of the clumping factor on the free–free distortion and discuss the wavelength dependence of the predicted signal. Finally, we address a comparison with other classes of unavoidable CMB spectral distortions and future observational perspectives. While Comptonization from reionization is dominant at high frequencies, the free–free signal predicted in the considered models contributes to the distortion at a level of few (few tens) per cent at frequencies below $\sim 25$ GHz ($\sim 10$ GHz) and represents the main signature below $\sim 4$ GHz. The cosmological signal from the $\text{H}_21$-cm background is found to prevail over the free–free distortion in a restricted, model-dependent frequency window between $\sim 0.1$ and $\sim 0.2$ GHz.

Key words: radiation mechanisms: general – intergalactic medium – cosmic background radiation – dark ages, reionization, first stars – dark matter.

1 INTRODUCTION
After the electron pair annihilation, the evolution of the photon occupation number of cosmic microwave background (CMB) photons in the cosmic plasma is suitably described by the Kompaneets equation (Kompaneets 1956). This approximation of the generalized kinetic equation includes, in its generalized form, Compton scattering, that conserves the photon number, and all the possible photon emission or absorption processes, such as (at least) double Compton and bremsstrahlung. Compton and double Compton processes depend linearly on the baryon density in the Universe, while the bremsstrahlung term is proportional to the square of the baryon density. After the recombination epoch, an accurate description of this process should account for an inhomogeneous, evolving intergalactic medium (IGM). Indeed, the matter power spectrum depends on cosmological parameters and, obviously, it is a function of the wavenumber which is inversely proportional to the linear scale. The so-called matter transfer function, $T(k)$, defines the matter power spectrum evolution from a primordial stage, usually identified at the end of the inflationary stage, to a desired time, and is associated with the growth of the perturbations. Therefore, the matter density contrast of the Universe, related to the evolution of the matter power spectrum, amplifies the signal of the free–free process in the plasma with respect to the case of a uniform medium. The analysis presented in this work is carried out in the context of astrophysical and phenomenological cosmological reionization histories inferred from recent astronomical and cosmological data.

In this paper, we present numerical results and analytical approximations for the amplification factor of the free–free rate and
distortion parameter derived exploiting the cosmological Boltzmann code \texttt{CAMB},\textsuperscript{1} properly modified to ingest different reionization histories. This investigation, although originally designed to describe the free–free spectral distortion related to an inhomogeneous IGM, can be applied to other studies that may need an accurate estimation of the IGM density contrast itself.

In Section 2 we introduce the fundamental concepts to characterize the free–free rate in an inhomogeneous medium (see also Appendix A). Section 3 is devoted to the computation of the clumping factor in different cosmological models and to the derivation of suitable approximations of its time evolution (see also Appendix B). In Section 4 we combine the recipes of previous sections to compute the CMB free–free distortion in the considered reionization scenarios. The main results are presented in Section 5 (see also Appendix C). Finally, we draw our main conclusions in Section 6 and discuss our main results and the observational perspectives at radio to submillimetre wavelengths in Section 7.

2 THEORETICAL FRAMEWORK

During the interaction between radiation and ionized matter, the photon equilibrium distribution function is described by the Planck law, while, more in general, the evolution in time of the photon occupation number, \( n(x, t) \), is represented by the complete kinetic equation (Zeldovich & Sunyaev 1969; Illarionov & Sunyaev 1974; Danese & de Zotti 1977; Burigana, Danese & de Zotti 1991; Hu & Silk 1993):

\[
\frac{\partial n(x, t)}{\partial t} = \left( \frac{\partial n}{\partial t} \right)_{C} + \left( \frac{\partial n}{\partial t} \right)_{b} + \left( \frac{\partial n}{\partial t} \right)_{DC} + \left( \frac{\partial n}{\partial t} \right)_{\text{cyc}} + \left( \frac{\partial n}{\partial t} \right)_{\text{sources}},
\]

where the first term in the right-hand side is the collision term, hence the Compton scattering (C), and the other terms refer to photon sources, bremsstrahlung (B), double or radiative Compton (DC; Lightman 1981; Gould 1984), cyclotron process (cyc; Zullo & Burigana 2005) and other possible photon production processes. A dimensionless, redshift invariant frequency (Burigana, de Zotti & Danese 1995), \( x = \nu t / k T_0 \), is typically adopted for numerically solving this equation. Here \( a T_0^4 \) defines the current CMB radiation energy density independently of the CMB spectrum shape, \( T_0 = 2.725 \) K (Mather et al. 1999) being usually referred as the present radiation temperature.

The Kompaneets equation, a convenient approximation of the more general kinetic equation, describes the evolution of the photon occupation number. Considering Compton scattering and bremsstrahlung, it can be expressed by

\[
\frac{\partial n}{\partial t} = \frac{1}{\xi c^2 \Phi} \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial n}{\partial x} + \eta(1 + \eta) \right) \right]
+ K_0 \frac{g_B}{\Phi} e^{-x \tau} \left[ 1 - \eta (e^{-x \tau} - 1) \right],
\]

where \( g_B \) is the Gaunt factor and \( x = x(T_e) = \nu / \phi \), being \( T_e \) and \( T_r = T_e (1 + z) \) the electron and radiation temperature, respectively, \( \phi = T_e / T_r \) and \( z \) the redshift. The coefficient \( K_0 \) is given by

\[
K_0(z) = \frac{8 \pi}{3} \frac{e^4 \gamma \Phi n_e (n_e + 4 n_b)}{m (6 \pi \eta m k T_e)^{3/2} (k T_e)^2} = K_0(z) \phi^{-7/2},
\]

being

\[
K_0(z) = 2.6 \times 10^{-25} \left( \frac{T_0}{2.7 \text{K}} \right)^{-7/2} (1 + z)^{7/2} \Omega_b^2 \text{s}^{-1}.
\]

In equation (2), the kinetic equilibrium time-scale between matter and radiation is expressed by

\[
t_c = t_{c/e} \frac{mc^2}{k T_c}
\]

\[
\simeq 4.5 \times 10^28 \left( \frac{T_0}{2.7 \text{K}} \right)^{-1} \phi^{-1} \Omega_b^{-1} (1 + z)^{-4} \text{s},
\]

being \( t_{c/e} = (n_e \sigma_{te})^{-1} \) the photon electron collision time.

Here, the term \( \Omega_b \), related to the baryon density in units of the critical density and to the Hubble constant \( H_0 \), is defined as \( \Omega_b = \Omega_b (H_0 / (50 \text{km s}^{-1} \text{Mpc}^{-1}) \)^2. Equations (3) and (4) point out the proportionality of the bremsstrahlung term to the square of the baryon density. This makes necessary to take into account the density contrast in the intergalactic medium.

2.1 Expansion time

The evolution of the background quantities of the Universe depends on the contributions of the different types of energy densities. After a radiation or, more in general, relativistic particle energy density dominated phase of the Universe, (cold) matter energy density starts to dominate at \( z \lesssim 3000 \), followed, at \( z \) less than few units, by a dark energy or cosmological constant dominated epoch. Considering a cosmic scale factor, \( \omega = \Omega_b (H_0 / (50 \text{km s}^{-1} \text{Mpc}^{-1}) \)^2, the CMB temperature was \( k_B T_e = m_e c^2 \) (Silk & Stebbins 1983), and taking into account the recent acceleration of the Universe, parametrized by the cosmological constant \( \Omega_\Lambda \), and the curvature term \( \Omega_K \), the expansion of the Universe is governed by the equation for \( \Omega \) (Procopio & Burigana 2009):

\[
\frac{\dot{\Omega}}{\Omega} = \frac{1}{\omega} = \frac{\tau_{\Omega}}{\rho_{\Omega}} = \left[ 1 + \beta \omega \left( 1 + \frac{\Omega_b}{\Omega_{b0}} \frac{\Omega_{\Lambda0}}{\Omega_{\Lambda0}} \right)^{2/3} + \frac{\Omega_b}{\Omega_{b0}} \frac{\Omega_{\Lambda0}}{\Omega_{\Lambda0}} \right]^{1/2},
\]

where \( \beta \) is the initial ratio between matter and photon energy densities and \( \tau_{\Omega} \) can be seen as an initial gravitational time-scale, defined as

\[
\frac{1}{\tau_{\Omega}} = \left( \frac{8 \pi G \rho_{\Omega}}{3} \right)^{1/2} = \left[ \frac{8 \pi G a}{3} \left( \frac{m_e c^2}{k} \right) \right]^{1/2}
\]

\[
= 0.0763 \text{s}^{-1}.
\]

Accounting also for the relativistic neutrinos contribution to the Universe’s dynamic, the above gravitational time-scale term becomes \( \tau_{\Omega} = 13.11 / (\kappa_v)^{1/2} \), where \( \kappa_v \) is the present ratio of neutrino to photon energy densities:

\[
\kappa_v = \frac{1}{2} \left( g_\nu + \frac{7}{8} g_\nu N_s \left( \frac{N_e}{N_e} \right)^{4/3} \right) = 1 + \frac{7}{8} \frac{N_s}{N_e} \left( \frac{4}{11} \right)^{4/3},
\]

being \( g_\nu \) the effective number of photons spin states, \( N_s \) the number of relativistic, two-component neutrino species, \( N_e^{\text{eff}} \) the effective number of species at the decoupling epoch. Typically, for three species of massless neutrinos and for two massless photon spin states, \( \kappa_v \simeq 1.68 \).

\textsuperscript{1} \url{http://camb.info/}

\[ \]
2.2 Bremsstrahlung process

Defining the density as the contribution of a mean term plus a small perturbation, \( \rho = \langle \rho \rangle (1 + \delta \rho / \rho) \), and averaging over a representative volume of the Universe:

\[
\frac{\langle \rho^2 \rangle}{\langle \rho \rangle^2} = 1 + \frac{(\delta \rho^2)}{\langle \rho \rangle^2} = 1 + \sigma^2 > 1.
\]

(10)

This defines the clumping factor, that, in this context, is a time-dependent multiplicative factor to be accounted to modify the bremsstrahlung rate from the case of homogeneous medium to the case of inhomogeneous medium. In Section 3 we will derive numerical and semi-analytical estimates of \( 1 + \sigma^2 \) for a set of different cosmological models. We then get the complete free–free term in an inhomogeneous medium multiplying the usual baryonic density parameter in equation (4) by this factor, i.e.

\[
\Omega_b^2(z) = \Omega_{b,\text{homog}}^2(1 + \sigma^2(z)).
\]

(11)

where \( \Omega_{b,\text{homog}}^2 \) corresponds to the (standard) homogeneous case. Note that the density contrasts of free electrons and atoms in different ionization states could be in principle different. Also, inhomogeneities in the medium temperature could be in principle taken into account. On the other hand, being weak the dependence on temperature of \( K_B(x) \propto \phi^{-1/2} \) and of the term e\(^{-x_0} \approx 1 - x_0 \) at low frequencies, where free–free process is particularly relevant, we can neglect to first order this further correction. Higher order corrections could be searched evaluating the variances of the various terms in equation (3). Although an accurate analysis of these effects is out of the scope of this paper, in Section 7 we will present some numerical estimates based on currently available studies on IGM temperature.

The separation approach exploited in this work, based on the factorization of the square of the baryon density, allows us to get the main effect from matter density contrast and to couple it with all the other relevant aspects in the thermal and ionization history, evaluated considering averaged functions, with versatility regarding the underlying cosmological model.

The total Gaunt factor (see Rybicki & Lightman 1979) appearing in equation (2), \( g_B(x, \phi) \), has been evaluated following the simple formulas in Burigana et al. (1995) (see Itoh et al. 2000 for higher accuracy expressions) that, when properly account for the ion specie, approximate to per cent level the results by Karzas & Latter (1961), an accuracy compatible with, or better than, the precision coming from currently unavoidable uncertainties in astrophysical models considered in this work. A few implementation allows us to ingest the adopted fractions of hydrogen and helium. Indeed, the free–free rate defined in equation (3) holds in a fully ionized medium. Actually, being this analysis focused at intermediate and low redshifts, resulting into late spectral distortions, this condition is not reasonable at all since the plasma, during this period, stands in different ionization states. The initial heating redshift is set in fact in the code at \( z_h = 30 \). Precisely, the electrons, hydrogen and helium number densities become \( n_e(n_{\text{HI}} + 4n_{\text{HeI}}) = n_e(n_{\text{HI}} + 4n_{\text{HeI}} + n_{\text{HeII}}) \), being the free electron fraction \( n_e^2 = \chi_{e}n_e \), with \( n_e \) the total electron fraction, \( n_e = n_e(1 + f_{\text{He}})/2 \), and where \( n_b \) is the baryon number density, provided by

\[
\rho_b = \frac{\rho_b}{m_b} = 2.8 \times 10^{-6} \Omega_b (1 + z)^3.
\]

(12)

The ionization fraction, \( \chi_{e} \), is determined by the reionization history. The reparation of atoms in different ionization states has been evaluated, at equilibrium, by solving the Saha equations, a system that describes the ratio between two different ionization states of an element (see Appendix A). The weighted total Gaunt factor is

\[
\frac{g_B(x)}{\chi_{e}} = \left( \chi_{\text{HI}} + \chi_{\text{HeI}} \right) g_B^\text{H}(x) + \chi_{\text{HeII}} g_B^\text{He}(x).
\]

(13)

The photon occupation number at a time \( t \), assuming an initial blackbody spectrum, \( \eta_{\text{BB}} \), under the only effect of bremsstrahlung as photon emission–absorption process can be well represented by the relation (Burigana et al. 1995)

\[
\eta \approx \eta_{\text{BB}} + \frac{\eta_B}{x} - u \frac{2}{x} \phi,
\]

(14)

with \( u = u(t) \) the Comptonization parameter, and \( \phi \) the initial electron temperature necessary for the evaluation of the distortion parameter, related to the fractional amount of energy exchanged between matter and radiation by \( \phi = T_e / T = (1 + \Delta e / s) \approx 1 - u \) (where \( u \) is computed at the final epoch).

Note that we are interested here at wavelengths larger than few millimetres where the Comptonization spectral shape is well approximated by the usual brightness temperature decrement characterizing the Rayleigh–Jeans region. At shorter wavelengths it should be replaced by the complete Comptonization formula.

At very low frequencies, the free–free distortion becomes self-absorbed. A formal, complete solution in the case of small late distortions under the combined effect of Compton scattering and bremsstrahlung (Danese & de Zotti 1980), including spontaneous and induced emission and absorption, can be found in section 3.3 of Burigana et al. (1995). On the other hand, in the frequency range \( x_0 \ll x \ll 1 \) (being \( x_0 \) the frequency at which the Universe optical depth for free–free absorption, \( \eta_{b,\text{BB}} \), Burigana et al. 1995, is unit de Zotti 1986) the expression in equation (14) very well approximates the complete solution. The validity of the assumption \( x \gg x_0 \) for the range of wavelengths considered in this work has been numerically checked for the suppression model at \( k_{\text{max}} = 100 \) including the details of ionization and thermal histories, but it is certainly satisfied by other cut-off values and reionization histories, since they exhibit not so different clumping factors and free–free rates. In particular, we find \( \eta_{b,\text{BB}} \approx 1.157 \times 10^{-22} \) at \( \lambda = 0.01 \) cm and, obviously, a much greater value \( \eta_{b,\text{BB}} = 1.947 \times 10^{-9} \) at \( \lambda = 10 \) m.

Finally, the free–free parameter \( y_{B}(t) \) turns out to be

\[
y_B = \int_{t_h}^{t} (\phi - \phi_0) \frac{\rho}{x} g_B(x, \phi) K_{0B} \frac{dz}{1 + z}.
\]

(15)

2.3 Cold and warm dark matter models

The matter density contrast depends on the cosmological model and parameters. In particular, the nature of dark matter (DM) particles affects the power spectrum of density perturbations at small scales, with implications for the clumping factor.

The cold dark matter (CDM) standard cosmological model forecasts the existence of primordial cold and collisionless particles with negligible small velocity dispersion at the epoch of radiation–matter equality and a derived matter power spectrum supporting small-scale structure formation. The DM velocity distribution suppresses fluctuations below its free streaming scale, proportional to the mean velocities–mass ratio. In this framework, galaxy formation is a hierarchical process, resulting in the typical bottom up scenario where denser clumps give rise to satellite galaxies (Boylankoby & Wu 2011), leading to the well known overprediction of these galaxies with respect to those observed in the Milky Way.
In this context, the idea of warm dark matter (WDM) particles was proposed as a possible solution of the small scales CDM and cuspy halo problems, the latter related to the density profile in virialized DM haloes.

WDM candidates have intrinsic thermal velocities, so exhibit larger velocity dispersions, compared to CDM particles, and present characteristic free streaming scales, which contribute to determine clustering properties, such that small-scale structures formation is suppressed.

Typical particle masses are in the range of the keV scale, which is intermediate between cold and hot DM masses, \( m_{\text{warm}} \sim (1-10) \text{keV} \) (Boyarskovsky, de Vega & Sanchez 2008), being \( m_{\text{cold}} \sim (10-10^2) \text{ GeV} \) and \( m_{\text{hot}} \sim \text{few eV} \). Sterile neutrinos or gravitinos are possible candidates for WDM.

The suppression of fluctuations due to WDM particles on scales smaller than their free-streaming scale, corresponding to a cut-off of the total matter power at large \( k \), slows down the growth of structure (Viel et al. 2005) and can be described by means of the transfer function for different cut-off values.

For a given cosmological model, the transfer function describes the effect induced by the free-streaming length on the matter distribution such that

\[
T(k) = \left[ \frac{P(k)_{\text{WDM}}}{P(k)_{\text{CDM}}} \right]^{1/2}.
\] (16)

While the formation of the first generation of stars is affected by the adopted cosmological model, large-scale structure distributions are independent of the model (Gao & Theuns 2007). Thus, in this analysis we assumed a CDM model with a cut-off \( k_{\text{max}} \) in the range (20, 1175) which approximately mimics the power spectrum drop for WDM models with different particle properties.

### 2.4 Reionization histories

The epoch of reionization, driven by the formation of the first luminous objects at \( z \sim 7-12 \), as probed by the discovery of galaxies and quasars at \( z \gtrsim 6 \) and by the Gunn–Peterson test (Fan et al. 2006), is one of the main tools to unravel the initial stages of structure formation. During this phase, ionizing radiation was transferred to the inhomogeneous cosmic gas in several distinct steps. The physical properties of the astrophysical sources responsible for this process largely affect the evolution of the medium in the reionization era (Barkana & Loeb 2001). In particular, during the hydrogen reionization phase it is possible to identify an initial pre-overlap stage and an intermediate and quick enough overlap period which ends up in the reionization process. The same ionizing sources should induce the first ionization of helium, while the second one should be delayed to smaller redshift, being therefore in principle accessible to observations (through Ly\( \alpha \) absorption lines).

The intergalactic ionizing radiation field is typically characterized by the so called escape fraction, the amount of ionizing radiation supplied by stars and quasars, which depends on the radiative efficiency of these sources, their emission spectrum and their density distribution. In principle, it is possible to distinguish two contributions, one coming from galactic gas and one from galactic dust, both of them able to absorb and re-emit electromagnetic radiation at lower frequencies (Benson et al. 2001). Hence, the global escape fraction is strictly related to galaxy mass and redshift, so it should be tracked within the densest regions in the Universe, where the transfer of ionizing radiation is more efficient.

Usually, reionization era is associate to the Thomson optical depth, \( \tau \), an integrated cosmological parameter which conveys information on the physical conditions of the Universe due to recent events, opening an observational window on the physics of ancient cosmic objects. Furthermore, it is clearly linked to later non-linear evolution.

In order to investigate the effects triggered by the reionization process and amplified by a non-negligible IGM density contrast on CMB free–free spectral distortions, we focalized our analysis on three well determined Universe’s reionization models. Two of them, namely suppression (S) (Choudhury & Ferrara 2006) and filtering (F) (Gnedin & Fan 2006), are astrophysical motivated, the other is a phenomenological, late double peaked (L) history (Naselsky & Chiang 2004), considered here for comparison (see Trombetti & Burigana 2012 and references therein for further details about these models and their parameters). The S and F reionization scenarios can be tuned to match various kinds of astronomical observations (see Schneider et al. 2008). Their optical depths are then well determined: \( \tau_S = 0.1017 \) and \( \tau_F = 0.0631 \), respectively. The L scenario can be modelled by means of free quantities describing the electron ionization fraction and electron temperature. We set its main parameters in order to have \( \tau_L = \tau_S \).

Finally, we exploited also the reionization history produced with the standard \textsc{CAMB} assuming the cosmological parameters derived by the Planck Collaboration\(^2\) (Planck Collaboration et al. 2013), presented with the recent first release of cosmological data.

### 3 SEMI-ANALYTICAL DESCRIPTION OF CLUMPING FACTOR

The clumping factor, related to the density of the IGM, affects the escape of radiation from an inhomogeneous medium (Barkana & Loeb 2001). Typically, assuming a uniform ionized medium, the clumping quantifies the number of spatially averaged recombinations in the IGM relative to the ones in the cosmic gas, per time and volume unit (Pawlik, Schaye & van Scherpenzeel 2009). Commonly, the clumping is supposed to be roughly spatially uniform providing an ionized volume much greater than the clumping scale. This parameter also has an impact on the gas recombination timescale, such that the higher is the clumping value, the faster is the recombination process.

Free–free emission from the ionized medium, with the interplay of cosmic radiative feedback of the sources responsible for Universe reionization, can produce a potentially remarkable distortion in the CMB spectrum, particularly at long wavelengths. Salvaterra et al. (2009) estimated lower limits of this effect for the filtering and the suppression model explicitly assuming a diffuse, averaged density, i.e. neglecting clumping, but suggesting the relevance of including it in the computation.

In order to derive the variance of the matter power spectrum we run \textsc{CAMB}, the code for the anisotropies in the CMB. It allows in fact a precise computation of the power spectrum of all the relevant components of the cosmic fluid at the desired grid of redshifts. The variance of the baryonic matter can be then calculated integrating over a suitable wavenumber interval the corresponding power spectrum:

\[
\sigma^2(z) = \frac{1}{2\pi^2} \int P(k, z) k^2 \, dk.
\] (17)

Clearly, the most accurate choice is to perform this computation for the interesting set of cosmological parameters for the

\(^2\) www.rssd.esa.int/Planck
considered models. On the other hand, this is highly demanding in terms of computational time (and, possibly, solution storage) for the exploration of a wide set of cosmological models.

In this section we provide a set of suitable approximations able to represent \( \sigma^2(z) \) at an accuracy level of a few per cent for the models considered here, exploiting also changes in two parameters crucial in this context, the amplitude and the small-scale cut-off of perturbations, and some other cosmological models. The found formulas can be also adopted as guidelines to derive approximations valid for other models.

In the context of this work, this analysis represents an intermediate step for the evaluation of the free–free distortion, but, more in general, the found results can be used to improve the treatment of clumping factor in other cosmological applications, as, for example, the study of alternative reionization scenarios (see e.g. Salvaterra, Ferrara & Dayal 2011).

### 3.1 Suppression model

We start from the cosmological parameters adopted in the suppression reionization history (see Salvaterra et al. 2009). With reference to CAMB notation, they are

\[
\tau = 0.1017, \quad n_s = 0.95, \quad \sigma_8 = 0.74, \\
\Omega_b = 0.0413, \quad \frac{\Omega_{cdm}}{\Omega_b} = 0, \quad \text{Amp} = 2.018 \times 10^{-9}, \\
\Omega_{cdm} = 0.1987, \quad r = 0.1, \quad z_{\text{ion}} = 10, \\
\Omega_{\Lambda} = 0.76, \quad h = 73, \quad w = -1.
\] (18)

The CAMB code allows us to save a file of the evaluated matter power spectrum in \( h \) Mpc\(^{-1}\) units, normalized for baryons, CDM particles and massive neutrinos, and the transfer function in the synchronous gauge, given a unit primordial curvature perturbation on superhorizon scales, for each requested redshift and for each parameter that represents the cut-off in the estimate of the variance, the transfer \( k \) maximum. We selected a redshift interval between 0 and 30 with an increasing step of 0.1, and an initial \( k_{\text{max}} \) to be then able to truncate the integral at the desired wavenumber.

For the computation of the variance we used the routine DOLGAF of the NAG LIBRARIES that evaluates the integral of a function (defined numerically at four or more points) over the whole specified range, using third-order finite difference formulae with error estimates, according to a method discussed by Gill & Miller (1972).

For this purpose, we implemented a Fortran routine which derives the variance for the above set of redshifts and stores it in a file. At each run we adopted different values of the input parameter \( k_{\text{max}} \) in the NAG routine, such as 20, 50, 100, 150, 200, 250, 300, 350, 400, 500, 600, 700, 800, 850, 900, 1000, 1175, to explore the differences in the shape of the curves, and looking for an analytical description of the results valid for the whole set of parameters. With this aim, we adopted the case \( k_{\text{max}} = k_{\text{ref}} = 100 \) as a reference. We found for it an analytical relation with the fitting functions of the program IGOR PRO and searched for a general function able to reproduce the other curves of variance just depending on the variation of the cut-off parameter.

In all cases, the ratio between the variance’s reference case and the one at a generic cut-off was best fitted quite well by a linear function of the redshift, for which the intercept and the slope, say \( a(k) \) and \( b(k) \), depend only on the chosen \( k_{\text{max}} \), as shown in Fig. 1.

To generalize also this behaviour, we looked for a universal law for these parameters, able to reproduce the results for any other cut-off’s value \( k = k_{\text{max}} \). The two new parametric quantities, \( a(k) \) and \( b(k) \), were extremely good represented in terms of the function double exponential offset (see Appendix B for the fit coefficients):

\[
\begin{align*}
  a(k) &= a_0 + a_1 \exp \left( \frac{a_2 - k}{a_3} \right) + a_4 \exp \left( \frac{a_5 - k}{a_6} \right), \\
  b(k) &= b_0 + b_1 \exp \left( \frac{b_2 - k}{b_3} \right) + b_4 \exp \left( \frac{b_5 - k}{b_7} \right),
\end{align*}
\] (19)

so that we could establish the relation

\[
\frac{\sigma^2(k_{\text{ref}})}{\sigma^2(k)}(z) = \sigma^2_{\text{ref}}(z) = a(k) + b(k)z.
\] (20)

When fitting the case \( k_{\text{max}} = 100 \), now dividing the range of redshift in two subintervals, \( z \in [0; 9] \) and \( z \in [9; 30] \), the most accurate representation is, again, in terms of the function Double Exponential X Offset (see also Appendix B):

\[
\begin{align*}
  \sigma^2_{\text{low}}(k_{\text{ref}})(z) &= l_0 + l_1 \exp \left( \frac{z - l_3}{l_4} \right) + l_5 \exp \left( \frac{z - l_6}{l_7} \right), \\
  \sigma^2_{\text{high}}(k_{\text{ref}})(z) &= h_0 + h_1 \exp \left( \frac{z - h_3}{h_4} \right) + h_5 \exp \left( \frac{z - h_6}{h_7} \right).
\end{align*}
\] (21)

Finally, we extrapolated a generalized analytic relation for the evaluation of the variance for each desired cut-off parameter from the reference one, as

\[
\sigma^2_{\text{analytic}}(k) = \frac{\sigma^2_{\text{low}}(k_{\text{ref}})}{\sigma^2_{\text{ref}}(z)},
\] (22)

where \( \sigma^2_{\text{low}}(k_{\text{ref}}) \) comes from the concatenation of \( \sigma^2_{\text{low}}(k_{\text{ref}})(z) \) and \( \sigma^2_{\text{low}}(k_{\text{ref}})(z) \). Fig. 2 shows the variance of the matter power spectrum derived from CAMB code and from this analytical fit. From the plot emerges that, to first approximation, the curves are reproduced with good accuracy also in the case of a cut-off value much greater than the reference one. The relative difference between simulated and fitted data, displayed in Fig. 3, is at most of 8 per cent for the highest cut-off value\(^3\) and typically below a few per cent.

\(^3\) The entity of these errors are the consequence that the higher is the chosen cut-off value, the bigger is the deviation of the ratios \( \sigma^2(k_{\text{ref}})/\sigma^2(k) \) from an ideal linear profile, mostly at low redshift (see Fig. 1 for comparison).
Figure 2. Variance for different values of the cut-off parameter $k_{\text{max}}$ with the relative fitting functions. The curve reference fit is the fit to the $k_{\text{max}} = 100$ curve, while fit other $k_{\text{max}}$ are the other curves obtained with analytical functions derived from the reference one as explained in the text.

Figure 3. Variance relative errors for the fitting functions of the previous plot.

3.2 Exploiting other cosmological models

3.2.1 Constant cut-off parameter

We explore here various alternative cosmological models, looking for an analytical approximation of the variance first assuming fixed cut-off values of the matter power spectrum. To this aim, we run the original CAMB code with different combinations of cosmological parameters compatible with Wilkinson Microwave Anisotropy Probe (WMAP) results$^4$ (Komatsu et al. 2011; Larson et al. 2011). We have taken into account two classes of theoretical models, adopting four different prescriptions for each class, whose parameters are specified in Tables 1 and 2.

(1) ΛCDM+run+tens, the standard cosmological model with the inclusion of the Sunyaev–Zel’dovich (SZ) effect, a dark energy component, the tensors, the gravitational lensing and the running, the latter being the variation of the scalar spectral index with respect to the wavenumber $k$.

(2) wCDM, in which the dark energy equation of state is allowed to vary, hence with $w \neq -1$.

In the analysis, the cut-off parameter has been set to $k_{\text{max}} = 1175$, to allow then the truncation at the desired wavenumber.

As in the previous section, the variance of these models has been first evaluated with $k_{\text{max}} = 100$. The matter power spectra at $z = 0$ and 30 for the considered models are displayed in Fig. 4. In both cases the trend is very similar: differences between models are more evident at small scales and tend to be smaller on intermediate scales.

![Figure 4. Matter power spectrum at $z = 0$ and 30 for the two different cosmologies and their alternative set of parameters.](https://academic.oup.com/mnras/article-abstract/437/3/2507/1027707)

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$^4$ http://lambda.gsfc.nasa.gov/product/map/
Table 3. Parameters of the ΛWDM cosmologies. In table, the amplitude (Amp) is in $10^{-9}$ units and there is no running (dln/dln k = 0).

| Model      | σ_{corr} |
|------------|----------|
| ΛCDM0     | 1.131    |
| ΛCDM1     | 1.603    |
| ΛCDM2     | 3.354    |
| ΛCDM3     | 1.889    |
| wCDM0     | 1.447    |
| wCDM1     | 0.611    |
| wCDM2     | 1.705    |
| wCDM3     | 1.968    |

We assumed that, for the determination of the $σ^2$, we could take into account the same reference curve at $k_{max} = 100$ we already adopted in the case of the suppression model, since we are studying a generalized limiting case $k_{max} = 100$ and the coefficients $a(k)$ and $b(k)$ are $k$-independent constants. To reproduce $σ^2(z)$ with a reasonable accuracy in each considered model, we found that the reference curve can be simply multiplied by a model-dependent correction factor, $σ_{corr}$, provided in Table 3.

In Fig. 5 we show the comparison between the variance computed numerically and the fitting relations for the two considered cosmological models, while the relative errors of fitting formulas are reported in Fig. 6. The accuracy of the above fits is typically better than a few per cent (except, for some models, at $z \gtrsim 2$) and always better than 15 per cent.

The power spectrum shape depends on the cosmological model and the full set of parameters. Fixing the other parameters, the perturbation amplitude mainly defines the $P(k)$ overall level. Thus, the variance scales essentially linearly with the perturbation amplitude as the power spectrum does. To verify that our computations satisfy this scaling, we considered the case ΛCDM0 (first column of Table 1), varying the amplitude from $A_{s1} = 1.885 \times 10^{-9}$ to $A_{s2} = 2.011 \times 10^{-9}$. The ratio of the two amplitudes is $A_{s1}/A_{s2} = 0.93735$, while $(σ^2_1(z)/σ^2_2(z)) = 0.93742$. This test confirms the expected general scaling law and the good accuracy level achieved by the numerical integration.

3.2.2 Variable cut-off parameter

We simply extend here the results found in the previous section, allowing for a variable cut-off parameter. We analysed the cases: $k_{max} = 200, 400, 600, 800$ and 1000. Combining the results found in Sections 3.1 and 3.2.1, we find the following fitting formulas:

$$
\begin{align*}
σ^2_{mod}(k_{ref}) &= σ^2_{ref}(k_{ref})σ_{corr}, \\
σ^2_{mod}(k) &= σ^2_{mod}(k_{ref})/σ^2_{ratio}(z),
\end{align*}
$$

where $σ^2_{mod}(k_{ref})$ and $σ^2_{ref}(k_{ref})$ are, respectively, the variances of the generic model and of the reference one, and $σ^2_{ratio}(z)$ is the fitting function of equation (20).

Figs 7 and 8 show the relative errors of the variance for two values of the cut-off wavenumber. As expected, the found approximations are more accurate for a cut-off parameter closer to the reference one. In spite of the wide range explored for the cut-off parameter, the relative errors of these simple formulas are always within $\pm 20$ per cent for all scenarios, but for the ΛCDM2 prescription, for which the error is about 40 per cent for $k_{max} = 1000$. Finally, Fig. 8 shows that the relative error, $E_r$, of the above fitting formulas is not strongly dependent on redshift. Therefore, even for high values of the cut-off parameter, a simple multiplicative factor $1/(1 + E_r)$
allows to correct the above formulas keeping the relative error within \(\sim 10^{-15}\) per cent.

### 3.3 Planck clumping factor

Given the recent delivery of the first *Planck* cosmological data and results, it is interesting to derive suitable approximations of the clumping factor for cosmological parameters in agreement with those found by the *Planck* Collaboration. To this aim, we assumed the *Planck* cosmological parameters set and derived the variance (see Fig. 9) using the standard *CAMB*. In this analysis, we took into account the cases \(k_{\text{max}} = 100, 200, 1000\), i.e. the reference starting point and some remarkable cases. We plot also for comparison the results found before for few other cosmologies. In the plot, the ‘fit models’ curves (dashed lines) refer to the variance of these scenarios numerically computed, while the ‘fit Planck’ curves (dashed–dotted lines) describe the approximations for \(\sigma^2(z)\) found assuming *Planck* cosmological parameters. We found that they can be relatively well represented by the formulas in Section 3.1, but corrected with a multiplicative factor \(\sigma_{\text{corr}} = 1.57\). In the figure, the inset shows the relative error in this approximation. A unique correction factor leads to a certain underestimation (overestimation) of \(\sigma^2(z)\) for \(k_{\text{max}} = 1000\) (respectively for the reference cut-off wavenumber \(k_{\text{max}} = 100\)), but the found discrepancy is typically \(\lesssim 4\) per cent.

### 4 FREE–FREE DISTORTION

To estimate the spectral distortions produced by the free–free process and amplified by the clumping, we developed a dedicated *FOTRAN*90 tool, which consists of different modules. The output of the code, the free–free parameter \(y_B\) of equation (15) has been evaluated in the redshift interval \([0, 30]\) with the *DOIF* routine of the *NAG LIBRARIES*, based on the Gauts 10-point and Kronrod 21-point rules. At the same time, we tested the accuracy of the free *FORTRAN* Numerical Recipes (NR) libraries (Press et al. 1992) against the *NAG* routines, by implementing the code also with the NR Gaussian quadrature.

In Fig. 10, we compare the time varying ionization fraction for all the scenarios accounted in the study (top panel), and the evolution of the electron temperature (top middle panel), of the bremsstrahlung rate (bottom middle panel) and, lastly, of the clumping factor (bottom panel) for S, F and L models. From the first panel we can see how the *Planck* cosmology (at the basis of the standard *CAMB*) give rise to an ionization fraction which is highly comparable with the S model. Concerning the second panel, since *CAMB* assumes a mathematical representation for the evolution of the cosmic plasma in order to track the CMB anisotropies, we were not able to derive, at a glance, the electron temperature, variable required in the Saha equations. Thus, since \(\tau_F < \tau_{\text{Pl}} < \tau_S\), we hypothesized that \(T^P_e\) history could have been between two limits, the F and S prescriptions. Thereby, we evaluated the spectral distortions produced with *Planck* data in the two cases.

Globally, as expected, the clumping factor gives an important contribution at low redshift, when the plasma is characterized by a high ionization fraction, as highlighted by the comparison of the panels.

The CMB spectrum, in terms of the brightness temperature, can be described by the relation

\[
T_b(x) \simeq \left( \frac{y_B(x)}{x^2} - 2 u \phi_i + \phi_i \right) T_r, \tag{24}
\]

which depends on the frequency only (and holds at any redshift, provided that \(y_B\) and \(u\) are integrated over the corresponding interval \([z, z_h]\)).
where the lines denoted with the term $\text{Avg}_k$ ($k_{\text{max}}$ = 1000) in agreement with an increasing averaged over $\lambda_{\text{Planck}} = 6$ when the two scenarios predict almost $\lambda_{\text{Planck}} = 200$ (red line) and $\lambda_{\text{Planck}} = 500$ and the case of $\lambda_{\text{Planck}} = 1$, and then by a significant ionization increase only at $\lambda_{\text{Planck}} = 1000$. From this plot, it is easy to remark that the IGM density contrast noticeably affects the CMB brightness temperature evolution especially at decimetre wavelengths.

Basically, in order to describe the effect induced by free–free mechanism in terms of temperature excess at different frequencies, we can rewrite equation (24) as

$$\frac{\Delta T_{\text{ff}}(x)}{T_r} \simeq \frac{y_B(x)}{x^2},$$

where we have defined $\Delta T_{\text{ff}} = (T_{\text{ff}} - T_r \phi)$. This is illustrated in Fig. 12 where the lines denoted with the term $\text{Avg}_{k_{\text{max}}}$ refer to a temperature variation derived assuming a value of $y_B$ averaged over a suitable wavelength range.

In all cases, the steeper lines refer to the complete computation, while the flatter lines are derived using the averaged $\langle y_B(x) \rangle$ (for which the exact value depends also on the considered frequency range) and thus show the simple wavelength dependence $\alpha x^2$. The slope derived including all the effects is $\sim$0.2 steeper than that derived using $\langle y_B(x) \rangle$.

We finally point out that, although a power law with a spectral index slightly larger than 2 represents a certain improvement with respect to the simple assumption of constant $y_B(\lambda)$, a dependence $y_B(\lambda) \simeq a \log \lambda + b$ well approximates $y_B(\lambda)$ at $\lambda \gtrsim 1.5$ cm for all the considered models, but with slightly different values of $a$ and $b$. This behaviour simply reflects the bremsstrahlung Gaunt factor this reason, in the phenomenological model the effective outcome of the clumping is much less outstanding, although we can again appreciate a tiny deviation of the CMB temperature between the case with (solid blue line) and without (dashed–triple dotted green line) the inclusion of the clumping factor for an equivalent value of $k_{\text{max}}$. Finally, last panel in figure makes a comparison between the spectral distortions induced by these histories, reporting the (non-preferential) case $k_{\text{max}} = 1000$. From this plot, it is easy to remark that the IGM density contrast noticeably affects the CMB brightness temperature evolution especially at decimetre wavelengths.

5 RESULTS

The global free–free spectral distortions are shown in Fig. 11, where each panel displays one of the possible scenarios accounted in the study, i.e. suppression (S), filtering (F) and late double peaked (L), and for three independent maximum wavenumbers $k_{\text{max}}$. We also report a curve, $k_{\text{max}}(nc)$, in which we do not account for the clumping amplification factor, aiming at directly unravel the impact of a primordial IGM density contrast on CMB spectral distortions.

The free–free distortions induced by the Planck cosmology, as reported in ‘S’ and ‘F’ panels of figure, has been evaluated adopting the astrophysical electron temperatures for the numerical solution of the Saha equations. Indeed, since the standard CAMB does not track the evolution of the electron temperature, providing the ionization fraction as a rough parametrization which maps the fraction into the recombination residuals during the matter-dominated era, we assumed the suppression and filtering electron temperature as upper and lower limits. The reason for this choice resides in their corresponding reionization optical depth values, for which $\tau_{\text{Planck}} = 0.0949$ lies almost in the middle of the models optical depth. The bottom panel in figure shows almost indistinguishable curves (dashed and dot–dashed lines) meaning that the spectral distortions generated by the two methods are significantly comparable, thus confirming the independence of this kind of distortion from the ionization history.

The entity of the distortions is, as expected, stronger in the case of the astrophysical models, where the cosmic plasma becomes fully ionized since redshift $z \simeq 10$, in agreement with an increasing clumping factor, in comparison with the phenomenological scenario induced distortions (see Fig. 10). Also, the two astrophysical models predict signal with relatively low differences, much more smaller when clumping is included, since it introduces the most relevant amplification at $z \lesssim 6$ when the two scenarios predict almost complete ionization. Differently, the late history, in spite of being characterized by a first ionization peak taking place at $z \simeq 10$, where the clumping factor is almost negligible, followed by a rapid ionization decrease resulting into an almost neutral medium till recent epochs ($z \simeq 1$), and then by a significant ionization increase only at low redshift when the clumping factor significantly increases. For

\[ \frac{\Delta T_{\text{ff}}(x)}{T_r} \simeq \frac{y_B(x)}{x^2}, \]

where we have defined $\Delta T_{\text{ff}} = (T_{\text{ff}} - T_r \phi)$. This is illustrated in Fig. 12 where the lines denoted with the term $\text{Avg}_{k_{\text{max}}}$ refer to a temperature variation derived assuming a value of $y_B$ averaged over a suitable wavelength range.

In all cases, the steeper lines refer to the complete computation, while the flatter lines are derived using the averaged $\langle y_B(x) \rangle$ (for which the exact value depends also on the considered frequency range) and thus show the simple wavelength dependence $\alpha x^2$. The slope derived including all the effects is $\sim$0.2 steeper than that derived using $\langle y_B(x) \rangle$.

We finally point out that, although a power law with a spectral index slightly larger than 2 represents a certain improvement with respect to the simple assumption of constant $y_B(\lambda)$, a dependence $y_B(\lambda) \simeq a \log \lambda + b$ well approximates $y_B(\lambda)$ at $\lambda \gtrsim 1.5$ cm for all the considered models, but with slightly different values of $a$ and $b$. This behaviour simply reflects the bremsstrahlung Gaunt factor

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\[ \frac{\Delta T_{\text{ff}}(x)}{T_r} \simeq \frac{y_B(x)}{x^2}, \]
dependence on \(\ln(x_e)\) at low frequencies, but with coefficients \(a\) and \(b\) globally depending on the selected power spectrum cut-off parameter and reionization scenario, in the relevant redshift range. Appendix C reports the values of \(y_B(\lambda)\) computed for the considered models at some representative wavelengths. From the values at the two longest wavelengths we derived the above parameters \(a\) and \(b\) and their dependence on \(k_{\text{max}}\) that allow us to find the relation \(y_B(\lambda) \simeq A k_{\text{max}}^m (\log \lambda + B)\), with \(A\) and \(B\), respectively, dependent and independent of the model. Remarkably, \(y_B(\lambda)\) can be approximated by a simple power-law dependence on \(k_{\text{max}}\) with a slope, \(m\), weakly dependent on the model.

6 CONCLUSION

We have developed a method able to characterize the variance of the matter power spectrum for different cosmologies and different values of the cut-off parameter \(k_{\text{max}}\), the maximum wavenumber adopted to integrate the power spectrum for the variance computation, exploiting the output of the Boltzmann code CAMB. These numerical results and the found approximations may be used for a wide set of cosmological applications. An interesting example is the investigation of the reionization process possibly driven at high redshifts ([5, 10]) by galaxy populations with active star formation, as recently explored in the model by Salvaterra et al. (2011) where the recombination rate density in the IGM is proportional to the H II clumping factor (Madau, Haardt & Rees 1999), a parameter substantially identical to that studied in this work. The suitable approximations presented here could be implemented in such formalism to improve the precision and the astrophysical motivations of the predictions.

The main scope of this paper is the evaluation of the efficiency of the bremsstrahlung process and the free–free distortion of the CMB spectrum when the density contrast of the IGM in an inhomogeneous Universe is taken into account. We developed a numerical code for the evaluation of the free–free distortion parameter and applied it to some reionization histories well defined in literature: two astrophysical models, suppression and filtering; a phenomenological description, late double peaked; an ionization history compatible with the recent derivation of cosmological parameters by the Planck mission. In the code, we implemented a routine aimed at finding a numerical solution of the Saha equations for a variable mixture of hydrogen and helium primordial abundances, providing a precise evaluation of their corresponding ionization states. We also performed a dedicated routine to weight on their Gaunt factors.

We showed, in a redshift interval \(z \in [0, 30]\), the electron ionization fraction and the electron temperature for the histories accounted in the code, the corresponding bremsstrahlung rate and the clumping, the correction factor to the free–free term which accounts for the IGM density contrast. We compare the effect on free–free distortion coming from different values of \(k_{\text{max}}\) for the same reionization history and for different scenarios. Focusing on the wavelength dependence on the free–free excess, we compare the signal slope derived from the complete computation and assuming an \(y_B(\lambda)\) value averaged over a suitable wavelength range (from 1 cm to 1 m), obtaining a somewhat larger slope in the former case than in the latter.

The two astrophysical histories show much less relative different free–free distortions when the clumping factor is included, because an almost full ionization state is achieved in both cases when the clumping factor significantly amplifies the signal. The late double peaked model turns out to be characterized by a significantly smaller free–free distortion, in spite of the assumption of a Thomson optical depth equal to that of the suppression model. This is related to the peculiar ionization fraction history, from a fully ionized medium at \(z \sim 10\) to an almost nearly neutral phase in the range \(1 \lesssim z \lesssim 10\) and then again to a fully ionization only at lower redshifts where the clumping factor becomes significant.

7 DISCUSSION

It is interesting to compare our results with those found in Ponente et al. (2011), where the average free–free emission is obtained from N-body simulations (see also their discussion about limitations of numerical simulations and semi-analytical treatments, given the complexity of the astrophysical processes involved). The authors computed free–free fluxes projected along the line of sight in slices of pixelized 2D map and then derived the global signal adding over the slices. In particular, we note that the results shown in Fig. 13 for the astrophysical models well agree with the level of distortion found by Ponente et al. (2011) when the free–free emissivity has been computed with the temperature of gas particles derived from the simulation (compare with the solid line in their fig. 5).

In order to identify the epochs giving the major contributions to \(y_B\), where it will be particularly relevant to focus observational and theoretical efforts, we consider a redshift bin \(\Delta z = 0.2\) and evaluated the partial contribution of each bin to \(y_B(\lambda)\) for some characteristic wavelengths. Each scenario has been investigated with and without the inclusion of clumping factor. As evident from Fig. 14, the IGM clumping factor largely affects the evolution of the free–free term, particularly at low redshift where the contribution increases of about an order of magnitude. The redshift dependence of \(\Delta y_B(z)/y_B\) shows two epochs of particular relevance, as a consequence of the behaviours displayed in Fig. 10. The first occurs at high redshift when the ionization fraction significantly raises or, in the case of the L history, at the redshift corresponding to the peak in the ionization fraction. The second occurs at low redshift when full ionization is achieved and the clumping factor becomes particularly high.

Fig. 14 explains also the (weak) dependence on the model of the slope, \(m\), characterizing the power-law dependence of \(y_B(\lambda)\) on the cut-off wavenumber \(k_{\text{max}}\): \(m\) increases with the increase of \(\Delta y_B(z)/y_B\) at low redshift, as a result of the larger relevance of

![Figure 13](https://academic.oup.com/mnras/article-abstract/437/3/2507/1027707)

Figure 13. Free–free distortion in terms of brightness temperature computed from combining Boltzmann and free–free CMB distortion numerical codes produced by the two astrophysical reionization histories and a phenomenological model at \(k_{\text{max}} = 1000\). CMB temperature distortions induced by free–free accounting for the clumping factor are found to be approximately 10 times larger than the ones derived neglecting IGM density contrast. The inset allows to appreciate the excess at low frequencies.
clumping. When the astrophysical scenario will be well understood, free–free signatures could be in principle used to further test DM properties through their effect on the power spectrum at small scales.

In principle, a complete analysis of the problem requires the proper treatment of the temperature (spatial) fluctuations of the ionized IGM, as mentioned in Section 2.2. To clarify the possible relevance of this aspect, it is interesting to compare the terms $K_{\phi}/x_e^2 \propto \phi^{-1}/2$ and $e^{-x_e} \simeq 1 - x_e$, estimated adopting an averaged IGM temperature or averaged over a representative volume. When the contribution to free–free distortion is remarkable, the electron temperature is significantly larger than the CMB temperature, i.e. $\phi \gg 1$. For example, even at $\sim 1$ cm and for $\phi \sim 10$ a temperature fluctuation of $\sim 30$ per cent implies a change of only $\sim 1.6$ per cent in the term $1 - x_e$ and the effect clearly decreases at increasing temperature as well as at longer wavelengths where free–free distortion is more important. Let us consider the term $K_{\phi}/x_e^2 \propto \phi^{-1}/2$. The ratio $(\phi^{-1/2})/\phi^{-1/2}$, where $\phi = (\phi)$, obviously depends on the level of IGM temperature fluctuations, $\delta\phi$, and on the shape of their distribution function. In general, to second order in $\delta\phi$, $(\phi^{-1/2})/\phi^{-1/2} = 1 + (3/8)(\sigma_{\phi}/\delta\phi)^2$, where $\sigma_{\phi}^2 = \langle (\delta\phi)^2 \rangle$ is the IGM temperature fluctuation variance, while in the particular case of a Gaussian distribution of $\delta\phi$ one gets $(\phi^{-1/2})/\phi^{-1/2} = 1 + (3/8)(\sigma_{\phi}/\delta\phi)^2 + (105/128)(\sigma_{\phi}/\delta\phi)^4 + (3465/1024)(\sigma_{\phi}/\delta\phi)^6 + \cdots$. A compilation of temperature evaluations through hydrodynamical simulations and line-of-sight radiative transfer approaches is presented in Bolton et al. (2010) and estimates of the IGM temperature from a seminumerical model are given by Raskutti et al. (2012) for some representative redshift values and intervals. The authors also compared theoretical predictions with direct measurement of the IGM temperature around quasars. Temperature standard deviations are quoted to be always less than $\simeq 50$ per cent and typically less than $\simeq 30$ per cent. Also, Ciardi et al. (2012) reported IGM volume averaged temperature for various ionizing emissivity models, focusing on a redshift range $z = [6, 14]$. Again, the relative differences between the considered model predictions are less than $\simeq 30$ per cent. Therefore, even assuming as a generous upper limit for the whole redshift range relevant for free–free distortion $\sigma_{\phi}/\phi \simeq 30$ per cent, we find $(\phi^{-1/2})/\phi^{-1/2} - 1 \simeq 4$ per cent.

In general, the bremsstrahlung rate involves a product proportional to $\rho^2\phi^{-3/2}$. We should then include in the treatment also the mixing between density and temperature fluctuations. By expanding in Taylor’s series, one can see that a further second-order term, not included in the previous discussion, appears: $-(\delta\rho^2\phi^{-3/2}\delta\rho\delta\phi)$. Its precise evaluation requires the treatment of possible physical correlations between the evolution of density and temperature fluctuations. We can write $\delta\rho\delta\phi$ as the sum of an uncorrelated and a correlated term, $\delta\rho\delta\phi = (\delta\rho\delta\phi)_u + (\delta\rho\delta\phi)_c$. The average of the uncorrelated term clearly vanishes. We are then driven to consider the term $-(\delta\rho^2\phi^{-3/2}(\delta\rho\delta\phi)_u)$, that, divided by the homogeneous term $\rho^2\phi^{-3/2}/\sigma^2$, introduces the further correction term $-\langle\sigma_\rho/\rho\rangle\langle\sigma_\phi/\phi\rangle\langle(\delta\rho/\rho)(\delta\phi/\phi)\rangle$ to be included in the global correction factor. Clearly, $\langle\sigma_\rho/\rho\rangle = \sigma$ is much smaller than the main clumping correction term, $(1 + \sigma^2)$, object of this work, while $\langle\sigma_\phi/\phi\rangle \lesssim 0.3$. Therefore, unless the average of the correlated density–temperature fluctuations is much larger than the product of their root mean squares, the mixing term is subleading.

We then deduce that IGM temperature fluctuations could likely introduce only a very small modification of the global free–free distortion.

7.1 Comparison with other spectral distortions and future observational perspectives

Within this context, it is important to compare the amplitude of free–free distortion predicted in these models with that expected from other kinds of spectral distortions. We consider here few examples of unavoidable spectral distortions, in particular the Compton scattering between CMB photons and electrons. Actually, the reionization process produces an electron heating which causes a distortion proportional to the fractional amount of energy exchanged during the interaction, the so-called Comptonization parameter $\kappa \simeq (1/4)\Delta e/k_i$.

For the astrophysical reionization models considered here, the parameter is $\kappa \sim (0.965–1.69) \times 10^{-7}$ (Burigana et al. 2008). The ratio between the brightness temperature excess by free–free distortion and decrement by Comptonization can be approximated by

$$\frac{\Delta T_B}{\Delta T_C} \simeq 1.79 \times 10^{-2} \left(\frac{\lambda}{\text{cm}}\right)^2 \left(\frac{y_B/10^{-9}}{u/10^{-7}}\right). \tag{26}$$

Therefore, assuming $y_B \sim 5 \times 10^{-9}$ (see Appendix C), we find that the free–free distortion exceeds the Comptonization decrement for $\lambda \lesssim 3–5$ cm ($\nu \lesssim 6–10$ GHz), below the frequency range proposed for both PIXIE (Kogut et al. 2011) and PRISM,\textsuperscript{6} spanning from 30 GHz to 6 THz.

Other kinds of unavoidable spectral distortions are Bose–Einstein (BE)-like distorted spectra produced by the dissipation of primordial perturbations at small scales (Sunyaev & Zeldovich 1970; Daly 1991; Hu, Scott & Silk 1994; Chluba, Khatri & Sunyaev 2012b; Pajer & Zaldarriaga 2013) which produces a positive dimensionless chemical potential, $\mu_0$, and BE condensation of CMB by colder electrons (Chluba & Sunyaev 2012; see also Khatri, Sunyaev & Chluba (2012) and Sunyaev & Khatri (2013) for recent reviews), which gives a negative chemical potential. The two kinds of distortions are characterized by an amplitude, respectively, in the

\textsuperscript{6}It could weakly reduce or amplify the free–free signal according to the dominance of correlation or anticorrelation of $\delta\rho$ and $b\phi$, respectively.

\textsuperscript{7}http://www.prism-mission.org/
range $^8 \sim 1.5 \times 10^{-6} - 10^{-7}$ (and in particular $\sim 2.52 \times 10^{-8}$ for $n_S = 0.96$, without running) and $\sim 3.08 \times 10^{-9}$. It is interesting to compare the free–free distortion with the BE-like distorted spectrum in two particular regions: at the wavelength $\lambda_m$, dependent on $\Omega_b$, where the modified BE spectrum shows the maximum distortion, $(\Delta T/T)_m$, and at $\lambda \lesssim 3$ cm where it is very weakly dependent on $\Omega_b$. Adopting, respectively, the approximations by Burigana et al. (1991) for $\lambda_m$ and $(\Delta T/T)_m$ and the $x \gg \mu_0$ limit in the BE formula, we find

$$\frac{\Delta T_{\text{ff}}}{\Delta T_{\text{BE}}} \simeq 91.0 \left(\frac{\Omega_b}{0.1}\right)^{-2/3} \left(\frac{\gamma_0/10^{-9}}{\mu_0/10^{-9}}\right)$$

(27)

at $\lambda \simeq \lambda_m$, and

$$\frac{\Delta T_{\text{ff}}}{\Delta T_{\text{BE}}} \simeq 1.89 \left(\frac{\lambda}{\text{cm}}\right) \left(\frac{\gamma_0/10^{-9}}{\mu_0/10^{-9}}\right)$$

(28)

at $\lambda \lesssim 3$ cm.

Therefore, at $\lambda \simeq \lambda_m$, the amplitude of free–free distortion always exceeds, or even overwhelms, that of BE-like distortion predicted in these models. Instead, assuming as reference $\gamma_0 \sim 5 \times 10^{-9}$ and $\mu_0 \simeq 2.52 \times 10^{-8}$, the predicted BE-like distortion exceeds the free–free excess for $\lambda \lesssim 2.7$ cm.

To provide an estimate of negative free–free distortion generated by colder electrons in the case of BE condensation, we exploited the temperature evolution reported in Chluba & Sunyaev (2012) (see their fig. 2), coupled with the ionization history computed with CAMB assuming cosmological parameters by Planck (see Section 3.3). A certainly generous upper limit to $\gamma_0$ amplitude, computed in different redshift ranges, is obtained by equation (15) integrating over the interval $10 < z < 10^5$, before the IGM temperature increase by reionization, when clumping is negligible. The result is reported in Fig. 12, showing that the distortion is two (or more) orders of magnitude less than free–free signal generated in any reionization model.

We finally compare the predicted free–free distortion with the cosmological signal expected from the HI 21-cm background in the same reionization scenarios. Looking at fig. 5 of Schneider et al. (2008) it is found to prevail over the free–free distortion only in a restricted frequency window between $\sim 90$ and $\sim 130$ MHz or $\sim 110$ and $\sim 250$ MHz for the suppression and filtering model, respectively. On the other hand, the two signals display very different frequency dependences that can be exploited to distinguish them.

These estimates imply that a future CMB experiment at millimetre wavelengths with a sensitivity to absolute temperature as in the PRISM proposal (PRISM Collaboration et al. 2013), having the ambitious goal of detecting primordial BE like and later Comptonization distortions, will be likely affected only weakly by late free–free distortion contribution, at least for not extreme reionization models. Future accurate CMB spectrum observations at longer wavelengths, extending and improving the recent TRIS (Gervasi et al. 2008) and Absolute Radiometer for Cosmology, Astrophysics, and Diffuse Emission 2 (ARCade 2) results8 (Seiffert et al. 2011; Singal et al. 2011), and observations in the radio domain are particularly interesting for the detection of the cosmological reionization free–free distortion. The amplitude of the signal predicted in this work (and, obviously, in more extreme models such that considered for example by Oh 1999) will be accessible to the sensitivity of the Square Kilometre Array (SKA)10 (Carilli et al. 2004) and of the next-generation radio telescopes [e.g. Low Frequency Array (LOFAR), Murchison Widefield Array (MWA), Australian SKA Pathfinder (ASKAP), MeerKAT]. An accurate interferometry absolute calibration and a dedicated data analysis strategy able to estimate the background signal and its spectral shape in interferometric observations will be an important step to firmly detect and study this cosmological imprint.

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8 Since very small scales not explored by current CMB anisotropy data are relevant in this context, a wide range of primordial spectral index needs to be exploited. A wider range of chemical potentials is found by Chluba, Erickcek & Ben-Dayan (2012a) allowing also for variations of the amplitude of primordial perturbations at very small scales, as motivated by different inflation models.

9 http://asd.gsfc.nasa.gov/archive/arcade/

10 http://www.skatelescope.org/
A1 Hydrogen ionization fraction

In the case of H, taking into account the Saha equation and the charge conservation law, the system to be solved is

\[
\begin{align*}
\chi_H + \chi_{H^+} &= 1, \\
\frac{\chi_{H^+}}{\chi_H} &= \frac{m_b}{m_h} \frac{2a}{a_1} \frac{1}{\chi_H} \frac{1}{\Lambda_1 e^{\frac{\Delta_1}{k_B T}}},
\end{align*}
\]

being \(2g_{H^+}/g_H = 1\). Defining \(\Delta_1 = \epsilon_{H^+} - \epsilon_H\) and

\[
C_1 = \frac{2}{1+f_{H^+}} \frac{m_b}{m_h} \frac{1}{\chi_H} \frac{1}{\Lambda_1 e^{\frac{\Delta_1}{k_B T}}},
\]

the solution is

\[
\begin{align*}
\chi_H &= \frac{1}{1+C_1}, \\
\chi_{H^+} &= \frac{C_1}{1+C_1}.
\end{align*}
\]

A2 Helium ionization fraction

For He, being the ratio \(2g_{He^+}/g_{He} = 4\) and \(2g_{He^{++}}/g_{He^+} = 1\) the system is

\[
\begin{align*}
\chi_{He} + \chi_{He^+} + \chi_{He^{++}} &= 1, \\
\chi_{He^{++}} &= \frac{8}{1+f_{He^+}} \frac{m_b}{m_h} \frac{1}{\chi_{He}} \frac{1}{\Lambda_1} e^{\frac{\epsilon_{He^{++}}-\epsilon_{He}}{k_B T}},
\end{align*}
\]

Analogously to the hydrogen case, assuming \(\Delta\epsilon_{He} = \epsilon_{He^+} - \epsilon_{He}\) and \(\Delta\epsilon_{He^{++}} = \epsilon_{He^{++}} - \epsilon_{He^+}\):

\[
\begin{align*}
C_2 &= \frac{2}{1+f_{He^+}} \frac{m_b}{m_h} \frac{1}{\chi_{He}} \frac{1}{\Lambda_1} e^{\frac{-\Delta\epsilon_{He}}{k_B T}}, \\
C_3 &= \frac{C_2}{1+C_2}.
\end{align*}
\]

the solution for the Helium is

\[
\begin{align*}
\chi_{He} &= \frac{1}{1+C_1+C_2}, \\
\chi_{He^+} &= \frac{C_2}{1+C_1+C_2}, \\
\chi_{He^{++}} &= \frac{C_3}{1+C_2+C_3}.
\end{align*}
\]

APPENDIX A: SAHA EQUATIONS

The process of reionization can be studied on the basis of the Saha equation that describes the ratio between two different ionization states of an element. Known the free electron number density and the temperature, we can write

\[
n_{i+1} / n_i = \frac{2}{\Lambda_1} g_i e^{-\frac{\epsilon_i}{k_B T}},
\]

where \(n_i\) is the density of atoms in the \(i\)th state of ionization, \(n_e\) the electron density, \(g_i\) the degeneracy of states for the \(i\) ions, \(\epsilon_i\) the energy required to remove \(i\) electrons from a neutral atom and \(\Lambda\) the thermal de Broglie wavelength of an electron, defined by

\[
\Lambda = \sqrt{\frac{h^2}{2\pi m_e k_B T}}.
\]

From the nuclei conservation law for H and He we also know that

\[
n_{H^+} = \frac{\rho_h}{m_h} \left[ f_{H^+} (\chi_H + \chi_{H^+}) \right],
\]

\[
n_{He^{++}} = \frac{\rho_h}{m_h} \left[ 1 - f_{He^+} (\chi_{He} + \chi_{He^+} + \chi_{He^{++}}) \right],
\]

where the sum of the ionization fraction of each species is equal to unity.
Table C1. Values of the free–free parameter $y_B(\lambda)/10^{-9}$ for some characteristic wavelengths; $\lambda = 1.5$ cm is the minimum wavelength from which the relation between $y_B$ and $\log(\lambda)$ is linear.

| $\lambda$ (cm) | 0.1 | 0.3 | 1   | 1.5 | 30  |
|----------------|-----|-----|-----|-----|-----|
| $k_{100}^s$    | 1.293 | 2.240 | 2.971 | 3.106 | 8.128 |
| $k_{200}^s$    | 1.462 | 2.533 | 3.360 | 3.513 | 9.192 |
| $k_{500}^s$    | 1.860 | 3.221 | 4.273 | 4.468 | 11.69 |
| $k_{1000}^s$   | 2.155 | 3.733 | 4.953 | 5.178 | 13.55 |
| $k_{100}^f$    | 1.115 | 1.932 | 2.563 | 2.680 | 7.012 |
| $k_{200}^f$    | 1.278 | 2.214 | 2.937 | 3.071 | 8.035 |
| $k_{500}^f$    | 1.651 | 2.859 | 3.793 | 3.965 | 10.38 |
| $k_{1000}^f$   | 1.924 | 3.332 | 4.420 | 4.621 | 12.09 |
| $k_{100}^l$    | 0.218 | 0.377 | 0.500 | 0.523 | 1.369 |
| $k_{200}^l$    | 0.262 | 0.453 | 0.601 | 0.629 | 1.645 |
| $k_{500}^l$    | 0.340 | 0.589 | 0.781 | 0.817 | 2.137 |
| $k_{1000}^l$   | 0.380 | 0.660 | 0.874 | 0.914 | 2.392 |

APPENDIX C: $y_B$ VALUES

Table C1 reports the values of the free–free parameter at various wavelengths numerically computed for the suppression, filtering and late reionization models and for different cut-off parameters.

The associated numerical error is always completely negligible in the case of quadrature performed with the NAG routine. The quadrature with NR is found to introduce a negligible underestimation (absolute relative error $\lesssim$ few $\times 10^{-6}$).

The results at the 1.5 and 30 cm can be directly rewritten in terms of the parameters $a$ and $b$ characterizing $y_B(\lambda) \simeq a \log \lambda + b$ at $\lambda \gtrsim 1.5$ cm. Finally, we found that $a$ and $b$ can approximated by a log–log linear dependence on $k_{max}$, i.e. $\log a \simeq m_a \log k_{max} + n_a$ and $\log b \simeq m_b \log k_{max} + n_b$, with $m_b \simeq m_a = m$, slightly dependent on the model, and with $10^{m_n-m_a}$ independent of the model. Thus, $y_B(\lambda)$ can be approximated by a simple power-law dependence on $k_{max}$. We find $y_B(\lambda) \simeq Ak_{max}^m (\log \lambda + B)$ with $B = 0.6288$ and $(A, m) = (1.337 \times 10^{-9}, 0.2275)$ for the S model (respectively $(1.078 \times 10^{-9}, 0.2424)$ or $(2.096 \times 10^{-10}, 0.2482)$ for the F and L models).\(^{11}\)

\(^{11}\)This approximation has a relative error always $\lesssim 3$ per cent for $k_{max} \gtrsim 100$, but tends to overestimate $y_B$ up to $\sim 14$ per cent at $k_{max} \lesssim 20$. Including in the fit also the results found at $k_{max} = 20$, we find the same value for $B$ but the coefficients $(A, m)$ slightly change to $(1.149 \times 10^{-9}, 0.2527)$, $(8.835 \times 10^{-10}, 0.2755)$ and $(1.632 \times 10^{-10}, 0.2897)$, respectively, for the S, F and L model, providing a fit relative error $\lesssim 6$ per cent.

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