Abstract. We study the minimization of the expected costs under stochastic constraint at the terminal time. Our first main result says that for a power type of costs, the value function is the minimal positive solution of a second order semi-linear ordinary differential equation (ODE). Our second main result establishes the optimal control. In addition we show that the case of exponential costs leads to a trivial optimal control. All our proofs are based on the martingale approach.

1. Introduction

In the recent years, mainly due to financial applications there is a growing interest in stochastic target problems. The main question that was studied up to date is: what is the minimal initial condition which guarantees that the controlled process in a fixed time horizon will satisfy certain condition. This question is closely related to super–hedging in incomplete financial markets. The main outcome of the established theory (see, for instance, [2, 3, 4, 5, 6, 12, 13]) is the link between stochastic target control problem and viscosity solutions of non linear PDE’s.

In this paper we deal with a different type of stochastic target problems. Formally, we study the following question. For a deterministic time horizon $T > 0$, for a random variable $\Xi$ known at time $T$, for a cost function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $x \in \mathbb{R}$, find a control $u$ with state process $X_t = x + \int_0^t u_s ds$ which minimizes the expected costs given by $\mathbb{E} \left[ \int_0^T F(u_t) dt \right]$ under the terminal constraint $X_T \geq \Xi$.

This question was inspired by a series of papers [1, 7, 8, 10, 11] which dealt with stochastic tracking problems under the terminal state constraint: $\{X_T = \Xi\}$ on a given event $A$. In this case, it is well established that the optimal control to such a problem is typically characterized by two coupled backward stochastic differential equations (BSDEs) with singular terminal constraints. Up to date, there are no examples with random $\Xi$ which lead to non trivial solution. The reason for this is that roughly speaking the equality $\{X_T = \Xi\}$ is too demanding. We relax this equality to an inequality. From financial point of view, if $X$ denotes the number of shares in a portfolio, then it is more natural to require the super–hedging condition $X_T \geq \Xi$ rather than the exact replication condition $\{X_T = \Xi\}$.

We focus on the case where $\Xi$ is the indicator function, the function $F$ is the power function and the information flow is given by the Brownian filtration. In this

\[\text{Date: July 5, 2019.}\]
\[2010 \text{ Mathematics Subject Classification.} \quad 49J15, 60H30, 93E20.\]
\[\text{Key words and phrases.} \quad \text{stochastic control, stochastic terminal constraints, stochastic target problems .}\]
case, by using the Markov and the scaling properties of Brownian motion we can reduce the problem to a one dimensional problem. Our first main result says that the value function is the minimal positive solution of a second order semi-linear ODE. Our second main result establishes the optimal control. The proof uses the martingale approach.

In addition, we show that if the cost function \( F \) is exponential, then the stochastic target problem is equivalent to the deterministic control problem with the same cost function and a terminal target \( \equiv 1 \). In other words, the exponential case leads to a trivial solution.

The rest of the paper is organized as follows. In the next section we introduce the setup and formulate the main results. In Section 3 we derive auxiliary lemmas which are essential for the proof of the main results. In Section 4 we complete the proof of the main results. Section 5 is devoted to the case of an exponential cost function. In Section 6 we provide numerical results for the quadratic loss function \( F(z) = z^2 \).

2. Preliminaries and Main Results

Consider a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) together with a standard one-dimensional Brownian motion \( W_t, t \geq 0 \) and the Brownian filtration \( \mathcal{F}_t^W = \sigma(W_u : u \leq t) \) completed by the null sets.

For any \((T, x) \in (0, \infty) \times \mathbb{R} \) and a progressively measurable processes \( u = \{u_t\}_{t=0}^T \) denote \( X_{t}^{x,u} = x + \int_0^t u_s ds, \ t \in [0, T] \). For any \((T, x, c) \in (0, \infty) \times \mathbb{R}^2 \) let \( U(T, x, c) \) be the set of all progressively measurable processes \( u = \{u_t\}_{t=0}^T \) which satisfy \( X_{T}^{x,u} \geq \mathbb{I}_{W_T > c} \) almost surely. As usual, we set \( \mathbb{I}_Q = 1 \) if an event \( Q \) occurs and \( \mathbb{I}_Q = 0 \) if not. For a given \( p > 1 \) introduce the optimization problem

\[
v(T, x, c) = \inf_{u \in U(T, x, c)} \mathbb{E}\left[ \int_0^T |u_t|^p dt \right].
\]

From the perspective of stochastic finance the process \( u \) represents the speed at which the agent trades in the risky asset. The transaction costs of such strategy are given by the integral \( \int_0^T |u_t|^p dt \). Thus, we seek to minimize the expected transaction costs subject to the constraint that the number of shares at the maturity date are non-negative and bigger or equal than 1 if the event \( \{W_T > c\} \) occurs. This corresponds to super–hedging with physical delivery of a call option with maturity date \( T \) where the stock price is modeled by a geometric Brownian motion. We will assume that the initial number of shares \( x \in [0, 1] \).

For a given \((T, x, c) \in (0, \infty) \times [0, 1] \times \mathbb{R} \) we say that \( u \in U(T, x, c) \) is optimal if \( \mathbb{E}[\int_0^T |u_t|^p dt] = v(T, x, c) \). Let \( U^+(T, x, c) \subset U(T, x, c) \) be the set of all \( u \in U(T, x, c) \) such that \( u \geq 0 \ dt \otimes \mathbb{P} \) a.s. and \( X_{T}^{x,u} \leq 1 \) a.s.

**Lemma 2.1.** Let \((T, x, c) \in (0, \infty) \times [0, 1] \times \mathbb{R} \).

(i). There exist an optimal control which is unique in the sense that if \( u, \tilde{u} \) are optimal controls then \( u = \tilde{u} \ dt \otimes \mathbb{P} \) a.s.

(ii) The optimal control belongs to the set \( U^+(T, x, c) \).

**Proof.** (i) First, let us prove uniqueness. Assume by contradiction that \( u, \tilde{u} \) are optimal controls and \( dt \otimes \mathbb{P}(u \neq \tilde{u}) > 0 \). Consider the process \( \frac{u + \tilde{u}}{2} \). Clearly, \( \frac{u + \tilde{u}}{2} \in U(T, x, c) \). Moreover, from the strict convexity of the function \( z \rightarrow |z|^p \) we
have
\[
\mathbb{E} \left[ \int_0^T \left| u_t + \tilde{u}_t \right|^p dt \right] < \frac{1}{2} \left( \mathbb{E} \left[ \int_0^T |u_t|^p dt \right] + \mathbb{E} \left[ \int_0^T |\tilde{u}_t|^p dt \right] \right) = v(T, x, c)
\]
which is a contradiction, and uniqueness follows.

Next, we apply the Komlos lemma for proving existence. Let \( u^{(n)} \in U(T, x, c), \ n \in \mathbb{N} \) be a sequence of progressively measurable processes such that
\[
v(T, x, c) = \lim_{n \to \infty} \mathbb{E} \left[ \int_0^T |u_t^{(n)}|^p dt \right].
\]
By Lemma A1.1 in [9] there exists a sequence \( \eta^{(n)} \in \text{conv}(u^{(n)}, u^{(n+1)}, \ldots), \ n \in \mathbb{N} \) such that \( \eta^{(n)} \to u \) almost surely to a stochastic process \( u \). Let us show that \( u \) is an optimal control. Clearly \( u \) is progressively measurable. From the convexity of the function \( z \to |z|^p \)
\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T |\eta_t^{(n)}|^p dt \right] \leq \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T |u_t^{(n)}|^p dt \right] < \infty.
\]
Hence \( \eta^{(n)} : [0, T] \times \Omega \to \mathbb{R}, \ n \in \mathbb{N} \) is uniformly integrable and so the a.s. convergence \( \eta^{(n)} \to u \) implies the \( L^1 \) convergence
\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T |u_t - \eta_t^{(n)}|^p dt \right] = 0.
\]
Thus, \( \int_0^T u_t dt = \lim_{n \to \infty} \int_0^T \eta_t^{(n)} dt \) where the limit is in \( L^1 \). We conclude that \( u \in U(T, x, c) \).

Finally, from the Fatou lemma and the convexity of the function \( z \to |z|^p \)
\[
\mathbb{E} \left[ \int_0^T |u_t|^p dt \right] \leq \liminf_{n \to \infty} \mathbb{E} \left[ \int_0^T |\eta_t^{(n)}|^p dt \right] \\
\leq \lim_{n \to \infty} \mathbb{E} \left[ \int_0^T |u_t^{(n)}|^p dt \right] = v(T, x, c).
\]

(ii) Let \( u \in U(T, x, c) \) be the optimal control. Define
\[
\theta = T \wedge \inf \{ t : X_t^{x, u} = 1 \}.
\]
Introduce the progressively measurable process
\[
\tilde{u}_t = \mathbb{I}_{t \leq \theta} \max(0, u_t), \ t \in [0, T].
\]
Clearly,
\[
X_T^{x, \tilde{u}} \geq 1 \wedge X_T^{x, u} \geq \mathbb{I}_{W_T > c}
\]
and
\[
\mathbb{E} \left[ \int_0^T |u_t|^p dt \right] \geq \mathbb{E} \left[ \int_0^T |\tilde{u}_t|^p dt \right].
\]
From uniqueness of the optimal control we conclude that \( u = \tilde{u} \ dt \otimes \mathbb{P} \). Hence, \( u \in U^+(T, x, c) \).

The following Proposition will be crucial for deriving the main result.

**Proposition 2.2.** For any \( (T, x, c) \in (0, \infty) \times [0, 1] \times \mathbb{R} \)
\[
v(T, x, c) = \frac{(1 - x)^p}{T^{p-1}} v \left( 1, 0, \frac{c}{\sqrt{T}} \right).
\]
Proof. The statement is obvious for $x = 1$. Thus assume that $x < 1$.

We use the scaling property of Brownian motion. Define the Brownian motion $B_t = \frac{W_{tT}}{\sqrt{T}}$, $t \geq 0$. Let $\mathcal{F}_t^B = \sigma\{B_u : u \leq t\}$ be the filtration generated by $B$ completed with the null sets. Clearly, $\mathcal{F}_t^B = \mathcal{F}_{tT}^W$, $t \geq 0$. Let $\tilde{U}$ be the set of all stochastic processes $u = \{\tilde{u}_t\}_{t=0}^1$ which are non negative, progressively measurable with respect to $\mathcal{F}_t^B$ and satisfy

$$I_{B_t > \frac{c}{\sqrt{T}}} \leq \int_0^1 \tilde{u}_t dt \leq 1.$$

We notice that there is a bijection $U^+(T, x, c) \leftrightarrow \tilde{U}$ which is given by

$$u_t = \frac{(1 - x)\tilde{u}_t}{T}, \quad t \in [0, T].$$

Thus, from Lemma 2.1

$$v(T, x, c) = \min_{u \in U^+(T, x, c)} \mathbb{E} \left[ \int_0^T u_t^p dt \right] = \min_{\tilde{u} \in \tilde{U}} \left( \frac{1 - x}{T} \right)^p \mathbb{E} \left[ \int_0^1 \tilde{u}_t^p dt \right] = \left( \frac{1 - x}{T} \right)^p v \left( 1, 0, \frac{c}{\sqrt{T}} \right).$$

Next, let $\Phi(\cdot) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Phi^{-1}(z)} e^{-\frac{y^2}{2}} dy$ be the cumulative distribution function of the standard normal distribution. Define the function $g : (0, 1) \to \mathbb{R}_+$ by

$$g(z) = v \left( 1, 0, \Phi^{-1}(z) \right)$$

where $\Phi^{-1}$ is the inverse function. From Proposition 2.2 we have

$$v(T, x, c) = \left( \frac{1 - x}{T} \right)^p g \left( \Phi \left( \frac{c}{\sqrt{T}} \right) \right). \quad (2.1)$$

Clearly, by choosing $u = \frac{1}{\sqrt{T}}$, we have $v(T, x, c) \leq \left( \frac{1 - x}{T} \right)^p$. Thus, $g \in [0, 1]$. Now, we are ready to state the main results which will be proved in Section 4.

Theorem 2.3. Let $h : (0, 1) \to \mathbb{R}_+$ be given by $h(y) = \frac{\exp(-\Phi^{-1}(y)^2) - \Phi^{-1}(y)}{4\pi}$. The function $g : (0, 1) \to \mathbb{R}$ is a positive and non increasing solution of the ODE

$$h(y)g''(y) + (p - 1) \left( g(y) - g^{\frac{1}{p-1}}(y) \right) = 0, \quad y \in (0, 1) \quad (2.2)$$

with the boundary conditions

$$\lim_{y \to 0} g(y) = 1 \quad \text{and} \quad \lim_{y \to 1} g(y) = 0. \quad (2.3)$$

Moreover, the following minimality holds. If $\hat{g} : (0, 1) \to \mathbb{R}$ is a positive solution to (2.2) and satisfies

$$\lim_{y \to 0} \hat{g}(y) > 0 \quad (2.4)$$

then $\hat{g}(y) > g(y)$ for all $y \in (0, 1)$.

Theorem 2.4. Let $(T, x, c) \in (0, \infty) \times [0, 1] \times \mathbb{R}$. Define the martingale

$$M_t = \mathbb{P}(W_T < c|\mathcal{F}_t^W) = \Phi \left( \frac{c - W_t}{\sqrt{T - t}} \right), \quad t \in [0, T]. \quad (2.5)$$
The optimal control is given by

\[ u_t = (1 - x) \frac{g_t}{T - t} \exp \left( - \int_0^t \frac{g_s}{T - s} ds \right), \quad t \in [0, T). \]

Namely, for the optimal control we have the ODE:

\[ \frac{dX_t}{dt} = g_1 p - 1 \left( M_t \right) T - t. \]

**Remark 2.5.** A natural question is whether there exists a unique positive, non-increasing solution to the ODE (2.2) with the boundary conditions (2.3). Due to the singularity of \( h \) at the end points \( \{0, 1\} \) the uniqueness seems to be far from obvious and we leave it for future research.

### 3. Auxiliary Lemmas

We start with the following regularity result.

**Lemma 3.1.** The function \( g : (0, 1) \to [0, 1] \) is concave and non-increasing.

**Proof.** The fact that \( g \) is non-increasing is obvious. In particular we get that \( g \) is measurable. We move to convexity. Fix \( a_1 < a < a_2 \). Let us show that

\[ g(a) \leq g(a_1) \frac{a_2 - a}{a_2 - a_1} + g(a_2) \frac{a - a_1}{a_2 - a_1}. \]

Let \( T = 1 \), \( x = 0 \) and \( c = \Phi^{-1}(a) \). From Lemma 2.1 and (2.1) there exists \( u \in U^+(1, 0, c) \) such that

\[ (3.1) \quad g(a) = \mathbb{E} \left[ \int_0^1 |u_t|^p dt \right]. \]

Consider the martingale given by (2.5). Observe that \( M_0 = a \). Define the stopping time

\[ \tau = \inf \{ t : M_t \notin (a_1, a_2) \}. \]

Clearly, \( \tau < 1 \) a.s. and so from the equality \( \mathbb{E}[M \tau] = M_0 \) we conclude that

\[ (3.2) \quad \mathbb{P}(M_\tau = a_1) = \frac{a_2 - a}{a_2 - a_1} \quad \text{and} \quad P(M_\tau = a_2) = \frac{a - a_1}{a_2 - a_1}. \]

Next, let \( Z = \int_0^\tau u_t dt \). From the Holder inequality

\[ (3.3) \quad \int_0^\tau |u_t|^p dt \geq \frac{Z^p}{\tau^{p-1}} \quad \text{a.s.} \]

From (2.1), the fact that \( \{W_{s+\tau} - W_\tau\}_{s=0}^\infty \) is a Brownian motion independent of \( \mathcal{F}_\tau^W \), and the inequality \( Z + \int_0^\tau u_t dt \geq \mathbb{1}_{W_\tau > c - W_\tau} \) (notice that \( Z \in [0, 1] \)) we get

\[ \mathbb{E} \left[ \int_\tau^1 |u_t|^p dt \big| \mathcal{F}_\tau^W \right] \geq v (1 - \tau, Z, c - W_\tau) \]

\[ = \frac{(1-Z)^p}{(1-\tau)^{p-1}} g \left( \frac{c - W_\tau}{\sqrt{1-\tau}} \right) = \frac{(1-Z)^p}{(1-\tau)^{p-1}} g(M_\tau). \]

Thus,

\[ (3.4) \quad \mathbb{E} \left[ \int_\tau^1 |u_t|^p dt \right] \geq \mathbb{E} \left[ \frac{(1-Z)^p}{(1-\tau)^{p-1}} g(M_\tau) \right]. \]
By combining (3.1)–(3.4), the fact that $g \leq 1$ and the simple inequality \(\frac{2^p}{p^r} + \frac{(1-z)^p}{(1-y)^{p-1}} \geq 1\) for $0 < y, z < 1$ we obtain

\[
g(a) = \mathbb{E} \left[ \int_0^1 |u_t|^p dt \right] \geq \mathbb{E} \left[ \left( \frac{2^p}{p^r} + \frac{(1-z)^p}{(1-y)^{p-1}} \right) g(M_t) \right] \\
\geq \mathbb{E}[g(M_t)] = g(a_1) \frac{a_2-a}{a_2-a^1} + g(a_2) \frac{a-a_1}{a_2-a^1}.
\]

\[\square\]

**Corollary 3.2.** Let $T > 0$. The function $v(T, \cdot, \cdot) : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

**Proof.** From Lemma 3.1 it follows that $g$ is continuous and non increasing. Since $g \in [0, 1]$ we conclude that $g : (0, 1) \to [0, 1]$ is uniformly continuous. This together with (2.1) and the fact that $\Phi$ is Lipschitz continuous completes the proof. \[\square\]

Next, we derive the boundary conditions.

**Lemma 3.3.** The function $g$ satisfies the boundary conditions given by (2.3).

**Proof.** Set $T = 1$ and $x = 0$. First we show that $\lim_{y \to 0} g(y) = 1$. From the Jensen inequality it follows that for any $u \in U(1, 0, c)$

\[
\mathbb{E} \left[ \int_0^1 |u_t|^p dt \right] \geq (\mathbb{P}(W_1 > c))^p = (1 - \Phi(c))^p.
\]

Thus, $g(y) = v(1, 0, \Phi^{-1}(y)) \geq (1 - y)^p$ and we conclude that $\lim_{y \to 0} g(y) = 1$.

Next, we establish the equality $\lim_{y \to 0} g(y) = 0$. Choose $C > 0$. Let $t_k = 1 - 2^{1-k}$ and $b_k = C \sum_{i=0}^{k}(3/4)^i$, $k \in \mathbb{N}$. Introduce the stochastic process

\[
u_t = \sum_{k=1}^{\infty} 2^k I_{t \in (t_k, t_{k+1})} I_{W_{t_{k+1}} < b_{k-1}} I_{W_{t_k} \geq b_k}, \quad t \in [0, T].
\]

Clearly, $b_k \uparrow 4C$ as $k \to \infty$. Observe that due to the continuity of the Brownian motion, on the event $\{W_1 > 4C\}$ there exists (random) $k$ such that $W_{t_{k+1}} < b_{k-1}$ and $W_{t_k} \geq b_k$. Thus, on this event

\[
\int_0^1 u_t dt \geq 2^k (t_{k+1} - t_k) = 1.
\]

We conclude that $u \in U(1, 0, 4C)$. This together with (2.1) and the inequality

\[
1 - \Phi(z) \leq \int_z^{\infty} s e^{-s^2/2} ds = e^{-z^2/2}, \quad z > 1,
\]

gives

\[
g(\Phi(4C)) \leq \mathbb{E} \left[ \int_0^1 u_t^p dt \right] \\
\leq \sum_{k=1}^{\infty} 2^{kp}(t_{k+1} - t_k) \mathbb{P}(W_{t_{k+1}} - W_{t_k} > b_k - b_{k-1}) \\
\leq \sum_{k=1}^{\infty} 2^k \frac{(b_k - b_{k-1})^2}{\sigma^2 k_{k+1}} \\
\leq \sum_{k=1}^{\infty} 2^{k(p-1)} e^{c^2(b_k/b_{k-1})^2}.
\]

By taking $C \to \infty$ we complete the proof. \[\square\]

We end this section with providing martingale properties of the optimal control.
Lemma 3.4. Let \((T, x, c) \in (0, \infty) \times [0, 1] \times \mathbb{R}\) and \(u \in U^+(T, x, c)\). The following statements hold,

(i). For any \(t < T\),
\[
\frac{(1-x)^p}{T^{p-1}} g(M_0) \leq \mathbb{E} \left[ \int_0^t u_s^p ds + \frac{(1-X_{x,u}^{T,c})^p}{(T-t)^{p-1}} g(M_t) \right]
\]
where \(M\) is the martingale given by (2.5).

(ii). The process
\[
Y_{x,u}^{T}: = \int_0^t u_s^p ds + \frac{(1-X_{x,u}^{T,c})^p}{(T-t)^{p-1}} g(M_t), \quad 0 \leq t < T
\]
is a sub–martingale.

(iii). If \(u \in U(T, x, c)\) is an optimal control then \(Y_{x,u}^{T}\) is the following martingale
\[
Y_{x,u}^{T} = \mathbb{E} \left( \int_0^T u_s^p ds \mid \mathcal{F}_t^W \right), \quad 0 \leq t < T.
\]

Proof. (i). Let \(t \in (0, T)\). Choose \(n \in \mathbb{N}\).

For any \(i \in \mathbb{Z}\) and \(j = 0, 1, ..., n-1\) consider the disjoint sets
\[
A_{i,j} = \left\{ \frac{i}{n} \leq W_t < \frac{i+1}{n}, \frac{j}{n} \leq X_{x,u}^{T,c} < \frac{j+1}{n} \right\}.
\]
Clearly, \(\bigcup_{i,j} A_{i,j} = \Omega \setminus \{X_{x,u}^{T,c} = 1\}\). From Lemma 2.1 it follows that for any \(i, j\) there exists a stochastic process \(u^{i,j} = \{u^{i,j}_s\}_{s=0}^{T-t}\) which is progressively measurable with respect to the (completed) filtration generated by the Brownian motion \(W_{t+s} - W_t\), \(s \geq 0\) and satisfy
\[
\frac{j}{n} + \int_0^{T-t} u_s^{i,j} ds > \mathbb{I}_{W_{T-W_t} > c - \frac{i+1}{n}}
\]
and
\[
\mathbb{E} \left[ \int_t^T |u_s^{i,j}|^p ds \right] = v \left( T - t, \frac{j}{n}, c - \frac{i}{n} \right).
\]

Corollary 3.2 implies that there exists a modulus of continuity (continuous function which is vanishing at zero) \(m : [0, 1] \rightarrow \mathbb{R}\) such that for any \(i, j\)
\[
v \left( T - t, \frac{j}{n}, c - \frac{i+1}{n} \right) - v \left( T - t, \frac{j+1}{n}, c - \frac{i}{n} \right) < m \left( \frac{1}{n} \right).
\]

Define the progressively measurable process \(\hat{u} = \{\hat{u}_s\}_{s=0}^T\) by
\[
\hat{u}_s = \mathbb{I}_{s < t} u_s + \mathbb{I}_{s \geq t} \sum_{i,j} \mathbb{I}_{A_{i,j}} u_s^{i,j}.
\]
From (3.5) we obtain that for any \(i, j\), on the event \(A_{i,j}\) we have
\[
X_{x,u}^{T} = x + \int_0^T \hat{u}_s ds \geq \frac{j}{n} + \int_0^{T-t} \hat{u}_s ds \geq \mathbb{I}_{W_{T-W_t} > c - \frac{i+1}{n}} \geq \mathbb{I}_{W_T > c}.
\]
Thus, $\hat{u} \in U(T, x, c)$, and so from (2.1) and (3.6)-(3.7)

$$\frac{(1-x)^p}{T-t} g(M_0) = v(x, T, c) \leq E\left[\int_0^T |\hat{u}_s|^p ds\right]$$

$$= E\left[\int_0^T |\hat{u}_s|^p ds + E\left[\int_0^T |\hat{u}_s|^p ds | F_t^W\right]\right]$$

$$\leq E\left[\int_0^T |\hat{u}_s|^p ds + \frac{(1-X_t^{s,u})^p}{(T-t)^{p-1}} g(M_t)\right] + m\left(\frac{1}{n}\right).$$

By letting $n \to \infty$ we complete the proof.

(ii). Choose $t_1 < t_2 < T$. From statement (i) and the fact that $\{W_{t+s} - W_{t_1}\}_{s=0}^{T-t_1}$ is a Brownian motion which is independent of $F_t^W$ we obtain (replace $c$ with $c - W_{t_1}$)

$$\frac{(1-X_t^{s,u})^p}{(T-t_1)^{p-1}} g(M_{t_1}) \leq E\left[\int_{t_1}^{t_2} |u_s|^p ds + \frac{(1-X_t^{s,u})^p}{(T-t_2)^{p-1}} g(M_{t_2}) | F_{t_1}^W\right].$$

Thus, by adding $\int_{0}^{t_1} |u_s|^p ds$ to both sides of the above inequality we get

$$Y_{t_1}^{s,u} \leq E\left[Y_{t_2}^{s,u} | F_{t_1}^W\right]$$

and the result follows.

(iii). Let $t < T$. Again, from the Markov property of Brownian motion and (2.1) it follows that

$$\frac{(1-X_t^{s,u})^p}{(T-t)^{p-1}} g(M_t) \leq E\left[\int_t^{T} |u_s|^p ds | F_t^W\right].$$

Thus the sub-martingale $Y_t^{s,u} = \int_0^t |u_s|^p ds + \frac{(1-X_t^{s,u})^p}{(T-t)^{p-1}} g(M_t)$, $0 \leq t < T$ satisfies

$$\int_0^t |u_s|^p ds + \frac{(1-X_t^{s,u})^p}{(T-t)^{p-1}} g(M_t) \leq E\left[\int_0^T |u_s|^p ds | F_t^W\right].$$

Since $u \in U(T, x, c)$ is an optimal control then from (2.1) and (3.8) we obtain

$$E\left[\int_0^T |u_s|^p ds\right] = \frac{(1-x)^p}{(T-t)^{p-1}} g(M_0)$$

$$\leq E\left[\int_0^T |u_s|^p ds + \frac{(1-X_t^{s,u})^p}{(T-t)^{p-1}} g(M_t)\right]$$

$$\leq E\left[\int_0^T |u_s|^p ds | F_t^W\right] = E\left[\int_0^T |u_s|^p ds\right].$$

We conclude that the inequality in (3.8) is in fact an equality.

\qed

4. PROOF OF MAIN RESULTS

In this section we complete the proof of Theorems 2.3–2.4.

Proof. Proof of Theorem 2.4.

The theorem is obvious for $x = 1$, and so we assume that $x < 1$. Choose $(T, x, c) \in (0, \infty) \times [0, 1] \times \mathbb{R}$ and let $M = \{M_t\}_{t=0}^{T}$ be the martingale given by (2.5). Since $g : [0, 1] \to [0, 1]$ is concave, then $g(M_t), t \in [0, T]$ is a uniformly integrable super-martingale. Thus, there exists a decomposition

$$g(M_t) = N_t - A_t, \quad t \in [0, T]$$

where $N = \{N_t\}_{t=0}^{T}$ is a martingale and $A = \{A_t\}_{t=0}^{T}$ is a continuous, non decreasing process with $A_0 = 0$. 

\begin{thebibliography}{9}

\end{thebibliography}
Let \( u \in U(T, x, c) \) be an optimal control and let \( \hat{u} \in U^+(T, x, c) \) be a control of the form

\[
\hat{u}_t = (1 - x) \frac{\hat{w}_t}{T - t} \exp \left( -\int_0^t \frac{\hat{w}_s}{T - s} \, ds \right), \quad t \in [0, T)
\]

for an arbitrary progressively measurable, non-negative process \( \hat{w} = \{\hat{w}_t\}_{t=0} \). Observe that

\[
X^x,u_t = 1 - (1 - x) \exp \left( -\int_0^t \frac{\hat{w}_s}{T - s} \, ds \right) \in [0, 1].
\]

Thus, indeed \( \hat{u} \in U^+(T, x, c) \). Set \( \theta = T \land \inf \{ t : X^x,u_t = 1 \} \). We recall from Lemma 2.1 that the optimal control is non-negative and \( u \equiv 0 \) on \( (\theta, T] \). Define the stochastic process

\[
w_t = \mathbb{I}_{t < \theta} \frac{(T - t)u_t}{1 - X^x,u_t}, \quad t \in [0, T].
\]

Next, we apply the maximum principle and Lemma 3.4. Recall the martingale \( Y^x,u \) and the sub-martingale \( Y^x,\hat{u} \) from Lemma 3.4. From the product rule

\[
dY^{x,u}_{t\land\theta} = \mathbb{I}_{t < \theta} \frac{M_s(X^x,u)}{T - s} \times \left[ (T - t)(dN_t - dA_t) + (w_t^{p} + (p - 1)g(M_t) - pw_t g(M_t)) \, dt \right]
\]

and

\[
dY^{x,\hat{u}}_{t\land\theta} = \mathbb{I}_{t < \theta} \frac{M_s(X^x,\hat{u})}{T - s} \times \left[ (T - t)(dN_t - dA_t) + (\hat{w}_t^{p} + (p - 1)g(M_t) - p\hat{w}_t g(M_t)) \, dt \right].
\]

Since \( Y^x,u \) is a martingale then the drift in (4.2) is vanishing, and so

\[
A_{t\land\theta} = \int_0^{t\land\theta} \frac{w_s^{p} + (p - 1)g(M_s) - pw_s g(M_s)}{T - s} \, ds, \quad \forall t \quad \mathbb{P} \text{ a.s.}
\]

In particular

\[
\frac{dA_{t\land\theta}}{dt} = \mathbb{I}_{t < \theta} \frac{w_t^{p} + (p - 1)g(M_t) - pw_t g(M_t)}{T - t} \, dt \otimes \mathbb{P} \text{ a.s.}
\]

On the other hand, \( Y^x,\hat{u} \) is a sub-martingale and so the drift in (4.3) is non-negative. Hence

\[
\frac{dA_{t\land\theta}}{dt} \leq \mathbb{I}_{t < \theta} \frac{\hat{w}_t^{p} + (p - 1)g(M_t) - \hat{p}\hat{w}_t g(M_t)}{T - t} \, dt \otimes \mathbb{P} \text{ a.s.}
\]

Clearly, for a positive constant \( \mu > 0 \) the function \( z \to z^p - \mu pz \) has a unique minimum at \( z = \mu^{\frac{1}{p-1}} \) and the corresponding value is \( -(p - 1)\mu^{\frac{1}{p-1}} \). From (4.5)–(4.6) we conclude that

\[
w_{t\land\theta} = g^{\frac{1}{p-1}}(M_{t\land\theta}) \, dt \otimes \mathbb{P} \text{ a.s.}
\]

This together with (4.1) and solving the ODE

\[
\frac{dX^x,u_t}{dt} = w_t \frac{1 - X^x,u_t}{T - t}, \quad t < \theta
\]

gives (recall that \( X^x,u \) is constant after \( \theta \))

\[
X^x,u_t = 1 - (1 - x) \exp \left( -\int_0^{t\land\theta} g^{\frac{1}{p-1}}(M_s) \, ds \right) \quad \forall t \quad \mathbb{P} \text{ a.s.}
\]
In particular $X_t^{x,a} < 1$ for $t < T$. Thus,
\begin{equation}
\theta = T \quad \text{a.s.}
\end{equation}

By combining (4.8)–(4.9) we complete the proof. \qed

Next, we move to the proof of Theorem 2.3.

**Proof.** Proof of Theorem 2.3.
The boundary conditions were proved in Lemma 3.3. Thus, the proof will be divided into two steps. In the first step we show that (2.2) holds true. In the second step we prove the minimality property.

**First Step:** Choose $a \in (0, 1)$. Let $x = 0$, $T = 1$ and $c = \Phi^{-1} \left( \frac{a}{3} \right)$. From (4.4), (4.7) and (4.9) we obtain that there exists a martingale \{N_t\}_{t=0}^{1} such that
\begin{equation}
g(M_t) = N_t - (p - 1) \int_0^t \frac{g(M_s) - g^\frac{p-1}{p}(M_s)}{1 - s} ds \quad \forall t \in [0, 1] \quad \mathbb{P} \quad \text{a.s.}
\end{equation}
where $M$ is the martingale given by (2.5). Observe that $M_0 = \frac{a}{2}$.

On the closed interval $\left[ \frac{a}{3}, \frac{1+a}{2} \right]$ define the function
\begin{equation*}
f(y) = -(p - 1) \int_0^y \int_{\beta = a/3}^{\beta} \int_{\alpha = y/3}^{\beta} \frac{g(\alpha) - g^\frac{p-1}{p}(\alpha)}{h(\alpha)} d\alpha d\beta.
\end{equation*}
Notice that $f \in C^2 \left[ \frac{a}{3}, \frac{1+a}{2} \right]$ and $f''(y) = -(p - 1) 2(y-a)^{p-1} \frac{g^\frac{p-1}{p}(y)}{h(y)}$, $y \in \left[ \frac{a}{3}, \frac{a+1}{2} \right]$.
For any $y \in (M_0, \frac{1+a}{2})$ consider the stopping time $\tau_y = \inf \{ t : M_t \notin \left( \frac{a}{3}, \frac{a+1}{2} \right) \}$.
Clearly, $\tau_y < T \quad \text{a.s.}$
We notice that $\frac{d(M_t)}{dt} = \frac{2h(M_t)}{1-t}$, and so from the Ito Formula and (4.10) we obtain
\begin{equation}
g(M_{\tau_y}) - f(M_{\tau_y}) = N_{\tau_y} - f(M_0) - \int_0^{\tau_y} f'(M_t) dM_t.
\end{equation}
Hence,
\begin{equation}
\mathbb{E}[g(M_{\tau_y})] - \mathbb{E}[f(M_{\tau_y})] = g(M_0) - f(M_0).
\end{equation}
Similarly, to (3.2)
\begin{equation*}
\mathbb{P}(M_{\tau_y} = y) = \frac{M_0 - \frac{a}{3}}{y - \frac{a}{3}} \quad \text{and} \quad \mathbb{P}(M_{\tau_y} = \frac{a}{3}) = \frac{y - M_0}{y - \frac{a}{3}}.
\end{equation*}
This together with (4.11) yields that $g(y) - f(y)$ is a linear function on the interval $(M_0, \frac{1+a}{2})$. In particular
\begin{equation*}
g''(a) = f''(a) = -(p - 1) \frac{g(a) - g^\frac{p-1}{p}(a)}{h(a)}.
\end{equation*}
Since $a$ was arbitrary the first step is completed.

**Second Step:** Assume that there exists a positive function $\hat{g} \neq g$ which satisfies (2.2) and (2.4).
Choose $y \in (0, 1)$. Let $x = 0$, $T = 1$ and $c = \Phi^{-1}(y)$. Introduce the progressively measurable process
\begin{equation*}
\hat{u}_t = \frac{\hat{g}^\frac{p-1}{p}(M_t)}{1-t} \exp \left( - \int_0^t \frac{\hat{g}^\frac{p-1}{p}(M_s)}{1-s} ds \right), \quad t \in [0, 1]
\end{equation*}
and the stochastic process

\[ \hat{Y}_t = \int_0^t \hat{u}_s ds + \frac{(1 - X^{0, \hat{u}}_t)_p}{(1 - t)^{p-1}} \hat{g}(M_t), \quad t \in [0, 1). \]

As before the martingale \( M \) is given by (2.5).

Similarly to (4.8) we have

\[ X^{x, \hat{u}}_t = 1 - \exp \left( - \int_0^t \hat{g}^{1/p} (M_s) \frac{1 - X^{0, \hat{u}}_s}{1 - s} ds \right), \quad t \in [0, 1]. \]

This together with (2.4) gives that on the event \( \{ W_1 > c \} = \{ M_1 = 0 \} \) we have \( X^{x, \hat{u}}_1 = 1 \). Thus, \( u \in U(1, 0, c) \). From (2.1)

(4.12) \[ g(y) = g(M_0) \leq \mathbb{E} \left[ \int_0^1 \hat{u}_s ds \right]. \]

Next, from the equality

\[ \hat{u}_t = \frac{dX^{0, \hat{u}}_t}{dt} = \hat{g}^{1/p} (M_t) \frac{1 - X^{0, \hat{u}}_t}{1 - t}, \quad t \in [0, 1), \]

the Ito formula, and the fact that \( \hat{g} \) solves (2.2) we get

\[ \hat{Y}_t = \hat{g}(M_0) + \int_0^t \frac{(1 - X^{0, \hat{u}}_s)_p}{(1 - s)^{p-1}} \hat{g}'(M_s) dM_s, \quad t \in [0, 1). \]

Thus, \( \{ \hat{Y}_t \}_{0 \leq t < 1} \) is a local martingale. Since it is non negative we conclude that it is a super–martingale. In particular for any \( t < 1 \)

\[ \mathbb{E} \left[ \int_0^t \hat{u}_s ds \right] \leq \hat{g}(M_0) = \hat{g}(y). \]

This together with the monotone convergence theorem and (4.12) gives \( g(y) \leq \hat{g}(y) \).

Thus, \( \hat{g} \geq g \).

Finally, we argue strict inequality. Indeed, assume by contradiction that there is \( y_0 \in (0, 1) \) for which \( \hat{g}(y_0) = g(y_0) \), then clearly \( y_0 \) is a minimum point for the function \( \hat{g} - g \). Hence, \( \hat{g}'(y_0) = g'(y_0) \). Since \( h(y) \) is bounded away from zero if \( y \) is bounded away rom the end points \( \{ 0, 1 \} \), then from standard uniqueness for initial value problems we conclude that \( \hat{g} = g \) on the interval \( (0, 1) \). This is a contradiction and the proof is completed. \( \square \)

5. The exponential case

Let \( \lambda > 0 \) and consider the optimization problem

(5.1) \[ w(T, x, c) = \inf_{u \in U(T, x, c)} \mathbb{E} \left[ \int_0^T (e^{\lambda |u_t|} - 1) dt \right]. \]

Namely, the cost function is given by \( z \to e^{\lambda |z|} - 1 \). The following result says that for any \( (T, x, c) \) the optimal control is targeting towards 1 with a constant speed.

**Theorem 5.1.** Let \( (T, x, c) \in (0, \infty) \times \mathbb{R}^2 \). Then

\[ w(T, x, c) = T \left( e^{\lambda(1-x)^+} - 1 \right), \]

and the unique optimal control is given by \( u = \frac{(1-x)^+}{T} \) \( dt \otimes \mathbb{P} \) a.s.
Proof. Choose \((T, x, c) \times (0, \infty) \times \mathbb{R}^2\). The statement is obvious for \(x \geq 1\). Hence, without loss of generality we assume that \(x < 1\). The cost function is strictly convex, and so, by using the same arguments as in Lemma 2.1 we obtain that the optimal control is unique. Thus, in order to prove the theorem it is sufficient to show that the value function satisfies the inequality

\[
(5.2) \quad w(T, x, c) \geq T \left( e^{\frac{\lambda(1-x)}{\lambda}} - 1 \right).
\]

Let \(\mathcal{C}\) be the set of all adapted, continuous and uniformly bounded processes. Let \(\mathcal{M}\) the set of all strictly positive and uniformly bounded martingales \(\mathbb{M} = \{M_t\}_{t=0}^T\) with \(M_0 = 1\).

Applying the standard technique of Lagrange multipliers we obtain

\[
\begin{align*}
&\inf_{C \in \mathcal{C}} \sup_{\alpha > 0} \sup_{M \in \mathcal{M}} \mathbb{E} \left[ \int_0^T (e^{|C_t|} - 1) dt - \alpha M_T \left( x + \int_0^T C_t dt - I_{W_T > c} \right) \right] \\
&\geq \sup_{\alpha > 0} \sup_{M \in \mathcal{M}} \inf_{C \in \mathcal{C}} \mathbb{E} \left[ \int_0^T (e^{|C_t|} - 1) dt - \alpha M_T \left( x + \int_0^T C_t dt - I_{W_T > c} \right) \right] \\
&= \sup_{\alpha > 0} \sup_{M \in \mathcal{M}} \inf_{C \in \mathcal{C}} \mathbb{E} \left[ \int_0^T (e^{\lambda C_t} - 1) dt - \alpha x - \alpha \int_0^T M_t C_t dt + \alpha M_T I_{W_T > c} \right].
\end{align*}
\]

Observe that for a given \(\alpha > 0\) and a martingale \(M\) the minimum of the above expression is obtained by taking \(C_t = \frac{\ln(\alpha M_t / \lambda)}{\lambda}, t \in [0, T]\). Hence,

\[
\begin{align*}
&\inf_{C \in \mathcal{C}} \sup_{\alpha > 0} \sup_{M \in \mathcal{M}} \mathbb{E} \left[ \int_0^T (e^{|C_t|} - 1) dt - \alpha M_T \left( x + \int_0^T C_t dt - I_{W_T > c} \right) \right] \\
&\geq \sup_{\alpha > 0} \sup_{M \in \mathcal{M}} \inf_{C \in \mathcal{C}} \mathbb{E} \left[ \int_0^T (e^{\lambda C_t} - 1) dt - \alpha x - \alpha \int_0^T M_t C_t dt + \alpha M_T I_{W_T > c} \right] \\
&\geq \inf_{C \in \mathcal{C}} \sup_{\alpha > 0} \sup_{M \in \mathcal{M}} \mathbb{E} \left[ \int_0^T (e^{\lambda C_t} - 1) dt - \alpha x - \alpha \int_0^T M_t C_t dt + \alpha M_T I_{W_T > c} \right] \\
&\geq -T + \exp \left( \frac{1}{T} \mathbb{E} \left[ \lambda M_T I_{W_T > c} - \int_0^T M_t \ln M_t dt \right] \right).
\end{align*}
\]

We conclude that

\[
\begin{align*}
&\inf_{C \in \mathcal{C}} \sup_{\alpha > 0} \sup_{M \in \mathcal{M}} \mathbb{E} \left[ \int_0^T (e^{|C_t|} - 1) dt - \alpha x - \alpha \int_0^T M_t C_t dt + \alpha M_T I_{W_T > c} \right] \\
&\geq -T + T e^{-\frac{\lambda}{\lambda}} \sup_{M \in \mathcal{M}} \exp \left( \frac{1}{T} \mathbb{E} \left[ \lambda M_T I_{W_T > c} - \int_0^T M_t \ln M_t dt \right] \right).
\end{align*}
\]

and (5.2) follows from the following lemma.}

\[\square\]

**Lemma 5.2.** For any \(\epsilon > 0\) there exists \(M \in \mathcal{M}\) such that

\[
(5.3) \quad \mathbb{E} [M_T I_{W_T > c}] > 1 - \epsilon
\]
and

\begin{equation}
\mathbb{E} \left[ \int_0^T M_t \ln M_t dt \right] < \epsilon.
\end{equation}

**Proof.** Choose \( \epsilon > 0 \). First, assume that we found a strictly positive martingale \( M \) with \( M_0 = 1 \) which satisfy (5.3)–(5.4). Then for any \( N \in \mathbb{N} \) define \( M^{(N)} \) by \( M^{(N)}_t = M_t \wedge \sigma_N, \ t \in [0, T] \) where \( \sigma_N = \inf \{ t : M_t = N \} \). Clearly,

\[ M^{(N)}_t = \mathbb{E} \left[ M_t | \mathcal{F}^W_t \right], \ t \in [0, T]. \]

Thus, for the Jensen inequality for the function \( z \to z \ln z \) and the Fubini theorem

\[ \mathbb{E} \left[ \int_0^T M^{(N)}_t \ln M^{(N)}_t dt \right] \leq \mathbb{E} \left[ \int_0^T M_t \ln M_t dt \right] < \epsilon. \]

Next, from the Fatou Lemma and the fact that \( \sigma_N \uparrow T \) as \( n \to \infty \)

\[ \mathbb{E} [M_T I_{W_T > c}] \leq \lim inf_{N \to \infty} \mathbb{E} \left[ M^{(N)}_T I_{W_T > c} \right]. \]

We conclude that in order to prove the statement, it is sufficient to find a strictly positive martingale which satisfy (5.3)–(5.4).

To this end, consider a strictly positive martingale of the form

\[ M_t = e^{\int_0^t \zeta_u \, dW_u - \frac{1}{2} \zeta_u^2 \, du}, \ t \in [0, T] \]

where \( \{ \zeta_t \}_{t=0}^1 \) is continuous deterministic functions. There exists a probability measure \( Q \) such that \( \frac{dQ}{dP} | \mathcal{F}_t = M_t \). Moreover, from Girsanov theorem the process \( W_t = W_0 - \int_0^t \zeta_u du, \ t \in [0, T] \) is a Brownian motion under \( Q \). Thus,

\begin{equation}
\mathbb{E} [M_T I_{W_T > c}] = Q (W_T > c) = Q \left( \hat{W}_T + \int_0^T \zeta_t \, dt > c \right)
\end{equation}

and

\begin{align}
\mathbb{E} \left[ \int_0^T M_t \ln M_t dt \right] &= \mathbb{E}_Q \left[ \int_0^T \ln M_t \, dt \right] \\
&= \mathbb{E}_Q \left[ \int_0^T \left( \int_0^t \zeta_u \, dW_u + \frac{1}{2} \int_0^t \zeta_u^2 \, du \right) \, dt \right] \\
&= \frac{1}{2} \int_0^T \int_0^T \zeta_u^2 \, dt \, du = \frac{1}{2} \int_0^T \zeta_u^2 (T - t) \, dt.
\end{align}

Observe that for the sequence of continuous functions \( \zeta^{(n)} : [0, T] \to \mathbb{R}, \ n \in \mathbb{N} \) given by

\[ \zeta^{(n)}_t = \frac{n^{-\frac{3}{2}}}{(T + \frac{1}{n^3} - t)^{1-\frac{1}{n}}}, \ t \in [0, T] \]

we have

\[ \lim_{n \to \infty} \int_0^T \zeta^{(n)}_t \, dt \geq \lim_{n \to \infty} n^{\frac{1}{2}} \left( T^{\frac{1}{2}} - \frac{1}{n} \right) = \infty \]

and

\[ \lim_{n \to \infty} \int_0^T [\zeta^{(n)}_t]^2 (T - t) \, dt \leq \lim_{n \to \infty} n^{-\frac{1}{2}} \int_0^T \frac{dt}{(T - t)^{1-\frac{1}{n}}} = 0. \]

This together with (5.5)–(5.6) yields that for sufficiently large \( n \) the martingale given by \( M_t = e^{\int_0^t \zeta^{(n)}_u \, dW_u - \frac{1}{2} (\zeta^{(n)}_u)^2 \, du}, \ t \in [0, T] \) satisfies (5.3)–(5.4). \( \square \)
6. Numerical Results

In this section we focus on the case of quadratic costs (i.e. \( p = 2 \)) and provide numerical results for the value function and simulations for the optimal control.

From (2.1) we have

\[
g\left(\frac{1}{2}\right) = \inf_{u \in U(1,0,0)} \mathbb{E}\left[ \int_0^1 u_t^2 dt \right].
\]

By approximating the Brownian motion with scaled random walks we compute numerically the right hand side of the above equality. The result is \( g\left(\frac{1}{2}\right) = 0.88 \).

Then, we apply the shooting method and look for the correct value of the derivative \( g'\left(\frac{1}{2}\right) \). Namely we look for a real number \( \gamma \) such that the unique (\( h \neq 0 \) in the interval \( (0,1) \)) solution of the initial value problem

\[
h(y)g''(y) + g(y) - g^2(y) = 0, \quad g\left(\frac{1}{2}\right) = 0.88 \quad \text{and} \quad g'\left(\frac{1}{2}\right) = \gamma
\]

will satisfy the boundary conditions \( g(0) = 1 \) and \( g(1) = 0 \). We get (numerically) a unique value \( \gamma = -0.21 \). The result is illustrated in Figure 1.

Next, for \( T = 1 \) and \( x = c = 0 \) we simulate a path of the optimal control \( u \in U(1,0,0) \) and the corresponding strategy \( X_t^{0,u} = \int_0^t u_s ds, \quad t \in [0,1] \). This is done by simulating a Brownian path and applying Theorem 2.4.

Acknowledgments

The authors would like to thank Peter Bank and Ross Pinsky for valuable discussions. This research was partially supported by the ISF grant no 160/17 and the ISF grant no 1707/16.

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Figure 2. This simulation corresponds to the case where $W_1 < 0$. The Brownian path is in blue, the optimal control $u \in U(1, 0, 0)$ is in orange and $X_t^{0,u} = \int_0^t u_s ds$ is in red.

Figure 3. This simulation corresponds to the case where $W_1 > 0$. The Brownian path is in blue, the optimal control $u \in U(1, 0, 0)$ is in orange and $X_t^{0,u} = \int_0^t u_s ds$ is in red.

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