AN ASYMPTOTIC FOR SUMS OF LYAPUNOV EXPONENTS IN FAMILIES

PATRICK INGRAM, DAVID JARAMILLO-MARTINEZ, AND JORGE MELLO

Abstract. Let \( f_t \) be a meromorphic family of endomorphisms of \( \mathbb{P}_C^N \) of degree at least 2, and let \( L(f_t) \) be the sum of Lyapunov exponents associated to \( f_t \). Favre showed that

\[
L(f_t) = L(f_\eta) \log |t^{-1}| + o(\log |t^{-1}|)
\]
as \( t \to 0 \), where \( L(f_\eta) \) is the sum of Lyapunov exponents on the generic fibre, interpreted as an endomorphism of some projective Berkovich space. Under some additional constraints on the family \( f_t \), we provide an explicit error term.

1. Introduction

The sum of Lyapunov exponents of \( f : \mathbb{P}_C^N \to \mathbb{P}_C^N \) quantifies the average rate of expansion, and is defined by the integral

\[
L(f) = \int \log \| \det(Df) \| d\mu,
\]
where \( \mu \) is the measure of maximal entropy associated to \( f \), and \( \| \cdot \| \) is the norm in terms of the Fubini-Study metric. From work of Bassanelli and Bertoloot [1], it is known that \( L(f) \) varies nicely in holomorphic families, and so it is natural to ask what happens as a meromorphic family degenerates to a pole. As a sample application of a deeper result on degeneration of measures on hybrid spaces, Favre proved the following.

Theorem 1. (Favre, \([5, \text{Theorem C}]\)) Let \( f_t \) be a meromorphic family of endomorphisms of \( \mathbb{P}_C^N \) parametrized by the unit disk, and let \( f_\eta \) be the endomorphism induced by this family by the Berkovich space associated to \( \mathbb{C}((t)) \). Then

\[
L(f_t) = L(f_\eta) \log |t^{-1}| + o(\log |t^{-1}|),
\]
as \( t \to 0 \), where \( o(x)/x \to 0 \) as \( x \to 0 \).

The purpose of this note is to obtain an effective error term on this asymptotic, under some additional hypotheses, and using more elementary techniques.

Theorem 2. Let \( f_t \) be a family of endomorphisms of \( \mathbb{P}_C^N \) parametrized over \( \mathbb{P}^1_K \) for some number field \( K \subseteq \mathbb{C} \). Then

\[
L(f_t) = L(f_\eta) \log |t^{-1}| + O(\log |t^{-1}|)^{1-\epsilon_N}
\]
as \( t \to 0 \), with \( \epsilon_N = 1/(3N + 4) \).
2. Effective estimates on norms of homogeneous forms

Let $K \subseteq \mathbb{C}$ be a number field, and let $\Phi \in K[X,s]$ be homogeneous in the variables $X = X_0, \ldots, X_N$, with coefficients in $K[s]$. We will write
\[ \|\Phi\| = \max\{|c| : c \text{ is a coefficient of } \Phi\}, \]
(as a polynomial in $X$ and $s$) and for a value $t \in \mathbb{C}$ we will write $\Phi_t$ for the specialization at $s = t$, and set $\|\Phi_t\|$ to be the analogous quantity for the specialized polynomial (as a homogeneous form in $X$). We will write
\[ \deg_s(\Phi) = \max\{ \deg_s(c) : c \text{ is a coefficient of } \Phi \} \]
(as a homogeneous form in $X$ with coefficients in $K[s]$), and $\deg_X(\Phi)$ for the degree of $\Phi$ as a homogeneous form.

Let $M_K$ be the usual set of places $v$ of $K$ indexing the set of absolute values $|\cdot|_v$ of $K$. We set $n_v = [K_v : \mathbb{Q}_v]/[K : \mathbb{Q}]$, and normalize our absolute values so that
\[ \sum_{v \in M_K} n_v \log |x|_v = 0 \]
for $x \neq 0$. The height of the projective tuple $[x_0 : \cdots : x_N]$ is defined as
\[ h([x_0 : \cdots : x_N]) = \sum_{v \in M_K} n_v \log \max_i |x_i|_v, \]
which is independent of the representative coordinates by (1). For an algebraic number $\alpha$, we set $h(\alpha) := h([\alpha : 1])$. As it will be useful below, we note that the height gives a trivial lower bound on any absolute value: we have for any $v$ that
\[ n_v \log |a|_v = -\sum_{w \neq v} n_w \log |a|_w \geq -\sum_{w \neq v} n_w \log^+ |a|_w \geq -[K : \mathbb{Q}] h(a), \]
by (1) and (2), whence
\[ |a|_v \geq e^{-[K : \mathbb{Q}] h(a)/n_v} \geq e^{-[K : \mathbb{Q}] h(a)} \]
as $n_v \geq 1$.

As above, we write
\[ \|\Phi\|_v = \max\{|c|_v : c \text{ is a coefficient of } \Phi\}, \]
(as a polynomial in $X$ and $s$) and for a value $t \in \mathbb{C}$ we will write $\Phi_t$ for the specialization at $s = t$, and set $\|\Phi_t\|_v$ to be the analogous quantity for the specialized polynomial (as a homogeneous form in $X$)

For convenience of notation, we will work around $s = \infty$, and replace $s$ by $1/s$ later to obtain results around $s = 0$. We also write $\deg^+ = \max\{\deg, 1\}$ and $\log^+ = \max\{\log, 0\}$. Finally, we set
\[ h^+(\Psi) = \sum_{v \in M_K} n_v \log^+ \|\Psi\|_v. \]
Note that this is the height of the coefficients of $\Psi$ as a point in affine space (rather than the more traditional projective height), and so is not invariant under scaling the coefficients.
Lemma 3. Let $\Psi$ be a homogeneous form in the variables $X = X_0, \ldots, X_N$, with coefficients in $K[s]$ for some number field $K$, suppose that the coefficients of $\Psi$ have no common root, and write
\[ (4) \quad \Theta(\Psi) = \max\{1, h^+(\Psi) + \deg_s(\Psi) + \deg_X(\Psi)\}. \]
Then
\[ \log \|\Psi_t\| = \deg_s(\Psi) \log^+ |t| + O \left( \deg_s(\Psi) \Theta(\Psi) \right), \]
where the implied constant depends only on $N$.

Proof. We first note the useful inequality
\[ \log^+ \|\Psi\| \leq [K: \mathbb{Q}]h^+(\Psi), \]
from the definition of $h^+$. The statement is trivially true if $\deg_s(\Psi) = 0$, since then $\|\Psi_t\| = \|\Psi\|$ is independent of $t$, and so $\log \|\Psi_t\| \leq [K: \mathbb{Q}]\Theta(\Psi)$ by the non-negativity of the other terms in (4). Hence, we can suppose that $\deg_s(\Psi) \geq 1$.

Let $D = \deg_X(\Psi)$, and let $\mathfrak{M}$ be the set of monomials of degree $D$ in $X$. In one direction, the claimed statement is easy due to the triangle inequality. In fact, we may write
\[ \Psi(X) = \sum_{m \in \mathfrak{M}} \psi_m m, \]
where $\psi_m \in K[s]$. The specialization at $s = t$ is then just
\[ \Psi_t(X) = \sum_{m \in \mathfrak{M}} \psi_m(t) m. \]

Since we have
\[ |\psi_m(t)| \leq \max\{1, |t|\}^{\deg_s(\psi_m)} \cdot (\deg_s(\psi_m) + 1)\|\psi_m\|, \]
by the triangle inequality, we may take logs and obtain
\[ \log \|\Psi_t\| = \max_{m \in \mathfrak{M}} \log |\psi_m(t)| \]
\[ \leq \left( \max_{m \in \mathfrak{M}} \deg_s(\psi_m) \right) \log^+ |t| + \max_{m \in \mathfrak{M}} \log \|\psi_m\| + \max_{m \in \mathfrak{M}} \log(\deg_s(\psi_m) + 1) \]
\[ = \deg_s(\Psi) \log^+ |t| + \log \|\Psi\| + \log(1 + \deg_s(\Psi)). \]

Note that the error term here is better than what was claimed.

In other direction, since we are assuming that the coefficients $\psi_m$ of $\Psi(X)$ have no common factor, we can use [9, Theorem 1.1] to solve
\[ (5) \quad at^{2\deg_s(\Psi) - 1} = \sum_{m \in \mathfrak{M}} \psi_m g_m \]
\[ (6) \quad a = \sum_{m \in \mathfrak{M}} \psi_m h_m \]
for polynomials $g_m, h_m$ of degree at most $\deg_s(\Psi) - 1$ in $K[s]$, and $a \neq 0$ in $K$.
Suppose that we can find a solution with
\[ \log \|g_m\|, \log \|h_m\| \leq C_1 \]
for all $m \in \mathfrak{M}$ and
\[ - \log |a| \leq C_2, \]
where $C_1$ and $C_2$ will depend on some way on $\Psi$. 

If $|t| > 1$, we then estimate, using the upper bound $\#\mathcal{N} \leq (D + 1)^N$ and (6),
\[2 \deg_s(\Psi) - 1) \log |t| + \log |a| \leq \max_{m \in \mathcal{N}} \log |\psi_m(t)| + \max_{m \in \mathcal{N}} \log |g_m(t)| + N \log(\deg_X(\Psi)) + 1
\]
\[
\leq \log \|\Psi_t\| + \max_{m \in \mathcal{N}} \deg(g_m) \log^+ |t| + \max_{m \in \mathcal{N}} \log \|g_m\|
\]
\[+ \max_{m \in \mathcal{N}} \log(\deg(g_m) + 1)
\]
\[+ N \log(\deg_X(\Psi)) + 1
\]
\[
\leq \log \|\Psi_t\| + (\deg_s(\Psi) - 1) \log^+ |t| + C_1 + \log \deg_s(\Psi)
\]
\[+ N \log(\deg_X(\Psi)) + 1).
\]

Thus,
\[\deg_s(\Psi) \log |t| \leq C_1 + C_2 + \log \|\Psi_t\| + \log(\deg_s(\Psi)) + N \log(\deg_X(\Psi) + 1)),
\]
On the other hand, if $|t| < 1$, we use (5) to get
\[
\log |a| \leq \log \|\Psi_t\| - \max_{m \in \mathcal{N}} \deg(g_m) \log |t|
\]
\[+ \max_{m \in \mathcal{N}} \log \|h_m\| + \max_{m \in \mathcal{N}} \log(\deg(h_m) + 1) + N \log(\deg_X(\Psi) + 1).
\]

This also yields
\[\deg_s(\Psi) \log |t| \leq C_1 + C_2 + \log \|\Psi_t\| + \log(\deg_s(\Psi)) + N \log(\deg_X(\Psi) + 1)).
\]

Therefore, we obtained
\[\deg_X(\Psi) \log^+ |t| \leq C_1 + C_2 + \log \|\Psi_t\| + \log(\deg_s(\Psi) + 1) + N \log(\deg_X(\Psi) + 1)),
\]
and all that remains is to describe suitable constants $C_1$ and $C_2$.

The key observation is that (5) and (6) define a system of linear equations in the coefficients of the powers of $t$. If this system has rank $r$, then there is a solution in which $a \neq 0$ and the coefficients of $g_m$ and $h_m$ are all determinants of $r \times r$ submatrices of the coefficient matrix (see [10] Lemma 4). The entries are just the coefficients of $\Psi$, so we have
\[\log |a|, \log \|g_m\|, \log \|h_m\| \leq r \log \|\Psi\| + r \log r
\]
where $r \leq 4 \deg_s(\Psi)$ just by counting equations. If we replace $\log \|\Psi\|$ with $\log^+ \|\Psi\|$, we may safely replace $r$ with this upper bound.

For a lower bound on $\log |a|$ we have to be a little more careful. Note that $\Psi$ has coefficients in a number field $K$, so $a \in K$. We then have
\[-\log |a| = \log |a|^{-1} \leq \log^{-1} |a|^{-1} \leq [K : \mathbb{Q}] h(a^{-1}) = [K : \mathbb{Q}] h(a).
\]
Working as in (3) for every $|_v$ and $v \in M_K$, we have that
\[\log^+ |a|_v \leq r \log^+ \|\Psi\|_v + r \log^+ \|r|_v
\]
for every $v \in M_K$, and thus that (summing over all places and combining with (3))
\[-\log |a| \leq [K : \mathbb{Q}] (rh^+(\Psi) + r \log r).
\]
This shows that we may take $C_1 = \deg_s(\Psi)(\log^+ \|\Psi\| + \log \deg_s(\Psi))$ and $C_2 = \deg_s(\Psi)(h^+(\Psi) + \log \deg_s(\Psi))$. □
We now treat the case in which the coefficients of \( \Psi \) do have a nontrivial common factor.

**Lemma 4.** Let \( \alpha \) be a monic polynomial in \( s \) which divides every coefficient of \( \Psi \). Then \( \Theta(\Psi/\alpha) \leq 2N\Theta(\Psi) \)

**Proof.** Note that
\[
\log \| \Psi/\alpha \| + \log \| \alpha \| \leq \log \| \Psi \| + (N \deg_X(\Psi) + \deg_s(\Psi)) \log 2
\]
along with \( \log \| \alpha \| \geq 0 \) implies
\[
\log \| \Psi/\alpha \| \leq \log \| \Psi \| + (N \deg_X(\Psi) + \deg_s(\Psi)) \log 2.
\]
By the corresponding non-archimedean estimate, we also have
\[
h^+(\Psi/\alpha) \leq h^+(\Psi) + (N \deg_X(\Psi) + \deg_s(\Psi)) \log 2.
\]
It follows that
\[
\Theta(\Psi/\alpha) = \max\{1, h^+(\Psi/\alpha) + \deg_s(\Psi) + \deg_X(\Psi)\}
\leq \max\{1, h^+(\Psi) + (2 \log 2 + 1) \deg_s(\Psi) + (2N \log 2 + 1) \deg_X(\Psi)\}
\leq 2N\Theta(\Psi),
\]
since \( N \geq 2 \).

Combining these, we can obtain a version of Lemma 3 even in the case that the coefficients of \( \Psi \) have common roots.

**Lemma 5.** Let \( \Psi \) be a homogeneous form in \( X_0, \ldots, X_N \) with coefficients in \( K[s] \), let \( \alpha(s) \) be the (monic) greatest common divisor of the coefficients of \( \Psi \). There exists a constant \( C \) depending just on the set of roots of \( \alpha \) such that for \( |t| > C \),
\[
\log \| \Psi_t \| = \deg_s(\Psi) \log^+ |t| + O(\deg_s^+(\Psi)\Theta(\Psi) + \log \| \alpha \|),
\]
where the implied constant depends just on \( N \).

**Proof.** By the triangle inequality,
\[
\log \| \Psi_t \| - \deg_s(\Psi) \log^+ |t| \leq \log \| \Psi_t/\alpha_t \| - \deg_s(\Psi/\alpha) \log^+ |t| + \log |\alpha_t| - \deg(\alpha) \log^+ |t|.
\]
Using Lemmas 3 and 4, the first term in the upper bound satisfies
\[
|\log \| \Psi_t/\alpha_t \| - \deg_s(\Psi/\alpha) \log^+ |t| | = O(\deg_s^+(\Psi/\alpha)\Theta(\Psi/\alpha)) = O(\deg_s^+(\Psi)\Theta(\Psi)),
\]
where the implied constant still depends only on \( N \). On the other hand, if \( |t| > 2|\gamma| \) for every root \( \gamma \), then the triangle inequality gives
\[
|\log |e_t| - \deg(\alpha) \log^+ |t| | \leq \log \| \alpha \| + \deg(\alpha) \log 2
\leq \log \| \alpha \| + \deg_s^+(\Psi)\Theta(\Psi),
\]
simply because \( \deg(\alpha) \leq \deg_s(\Psi) \) and \( \Theta(\Psi) \geq 1 \). Since both terms are non-negative, we may replace both bounds by large multiples to conclude the proof of the lemma.

\( \square \)
3. Lifts and pushforwards of divisors

Here we choose a lift $F$ of $f : \mathbb{P}^N \to \mathbb{P}^N$, that is, a choice of homogeneous forms representing $f$, which we can suppose that has degree $d \geq 2$. For a homogeneous form $\Phi$, we set

$$F_* \Phi(X) = \prod_{F(Y) = X} \Phi(Y).$$

A priori we have that $F_* \Phi$ is an element of the field obtained from $\mathbb{C}(X)$ by adjoining all roots of $F(Y) = X$, but it is not hard to check that $F_* \Phi$ is in fact a homogeneous form in $X$ of degree $d \deg(\Phi)$. (See page 4 in [8])

Lemma 6.

$$G_F(\Phi) := \lim_{k \to \infty} \frac{\log \| F^k_* \Phi \|}{d_k(N+1)} = \int_{\mathbb{P}^N} \log \| \Phi \|_{G\mu_f},$$

where $\mu_f$ is the measure of maximal entropy associated to $f$.

Proof. See [8] Proposition 14]

First, we need a lemma that shows that any finite collection of polynomials can be “sampled” such that the output is comparable to the norm of the polynomial.

Lemma 7. Let $\mathbb{C}_v$ be a complete, non-archimedean field, and fix $q_1, \ldots, q_m \in \mathbb{C}_v[s]$. Then there exist infinitely many $t \in \mathbb{C}_v$ with $|t|_v = 1$ such that

$$|q_i(t)|_v - \|q_i\|_v \leq \kappa_v,$$

where $\kappa_v = 0$ for nonarchimedean places, and

$$\kappa_v = (\max\{\deg(q_i)\} + 1) \log \left(2 \sum_{i=1}^m \deg(q_i)\right)$$

for archimedean places.

Proof. For non-archimedean places, if $|t|_v = 1$, then we have $|q_i(t)|_v \leq \|q_i\|_v$ immediately. Since the Gauss norm is multiplicative, it suffices (for the other direction) to treat the case where the $q_i$ are all linear. Now, if $q_i(s) = \alpha_i s - \beta_i$, with $|\alpha_i| \neq |\beta_i|$, we can take any $t$ with $|t| = 1$. If $|\alpha_i| = |\beta_i|$, then there are infinitely many $t$ with $|t| = 1$ and $|t - \beta_i/\alpha_i| = 1$. Intersecting these set for $i = 1, \ldots, m$ still gives us infinitely many choices with $|t| = 1$.

Now consider archimedean places. In one direction, if $|t|_v = 1$, then the triangle inequality gives

$$\log |q_i(t)|_v \leq \log \|q_i\|_v + \log(\deg(q_i) + 1).$$

In the other direction, let $N = \sum_{i=1}^m \deg(q_i)$ and let $\epsilon = 2/N$. Note that, given any $N$ points in $\mathbb{C}_v = \mathbb{C}$, disks of radius $\epsilon$ around these points cover at most a subset of the unit circle of arc length

$$N \frac{\pi}{\sqrt{2}^2} \epsilon = 2 \pi \sqrt{2} < 2\pi,$$

and so there exists a point with $|t|_v = 1$ which is not within $\epsilon$ of any of these points.

Now, if $|t|_v = 1$ and $|t - \gamma| \geq \epsilon$, we have three cases to consider. First, if $|\gamma| \leq 1$, then $\|s - \gamma\| = 1$, so we have $|t - \gamma|_v \geq \epsilon \|s - \gamma\|$. On the other hand, if $1 < |\gamma|_v \leq 2$, then...
we have $\|s - \gamma\|_v \leq 2$, and so $|t - \gamma|_v \geq \frac{\varepsilon}{2} \log \|s - \gamma\|$. Finally, if $|\gamma|_v > 2$, we have from $|t|_v < 1$ that $|t - \gamma|_v \geq \frac{\varepsilon}{2} |s - \gamma|_v$. Either way,

$$|t - \gamma|_v \geq \frac{\varepsilon}{2} \|s - \gamma\|,$$

and this holds for all roots of $q_i$ (and for all $i$). Assuming $q_i$ is monic (without loss of generality, looking at the difference we’re bounding), we have

$$\log |q(t)| = \sum_{j=1}^{\text{deg}(q_i)} \log |t - \gamma_j|$$

$$\geq \sum_{j=1}^{\text{deg}(q_i)} (\log \|s - \gamma\| - \log N)$$

$$\geq \|q_i\| - (\text{deg}(q_i) + 1) \log N - \text{deg}(q_i) \log 2$$

by Gelfond’s Lemma ([3, p. 22, p. 27]). Note that $N$ is the sum of degrees, so the error should take this into account.

\[ \square \]

Lemma 8. Let $\Phi$ be a homogeneous form in $X$ with coefficients in $K(s)$. Then

$$\log \| (F^k)\Phi \| = \text{deg}_s(F^k\Phi) \log^+ |t| + O(d^{\delta k(N+1)})$$

as $k \to \infty$ and $t$ sufficiently large, independent of $k$. The implied constant now depends on $F$, $\Psi$, and $N$, but not $t$ or $k$.

**Proof.** For each $k$, let $\alpha_k$ be the (monic) greatest common divisor (in $K[s]$) of the coefficients of $F^k\Phi$. We will show that the roots of $\alpha_k$ are contained in a finite set that does not depend on $k$, and that

$$\log \| \alpha_k \| \leq C_0 d^{k(N+1)}$$

$$\text{deg}_s(F^k\Phi) \leq C_1 d^{k(N+1)}$$

and

$$\Theta(F^k\Phi) \leq C_2 d^{2k(N+1)},$$

which, in light of Lemma [3] will prove the lemma.

First, for each $\beta \in \mathbb{C}$, let $\| \cdot \|_\beta$ be the corresponding absolute value on $\mathbb{C}[s]$, defined by

$$\|w\|_\beta = e^{-\text{ord}_{s=\beta}(w)},$$

with

$$\|w_0, ..., w_m\|_\beta = \max\{\|w_0\|_\beta, ..., \|w_m\|_\beta\}$$

as expected. We also write $\|w\|_\infty = e^{\text{deg}(w)}$.

Now, by Lemma 12 in [8], there is a finite set $S$ depending on $F$ but not $\Phi$ such that

$$\log \| F^k\Phi \|_\beta = d^{N+1} \log \|\Phi\|_\beta$$

for all $\beta \notin S$. Enlarging $S$ to include all common roots of the coefficients of $\Phi$, we then have

$$\log \| F^k\Phi \|_\beta = 0$$

for all $k$ and all $\beta \notin S$. In other words, for every $k$ and every $\beta \notin S$, some coefficient of $F^k\Phi$ is non-vanishing at $\beta$. It follows that the roots of $\alpha_k$ lie in $S$ for all $k$, proving the first claim.
Next, note that we have
\[ \alpha_k(s) = \prod_{\beta \in S} (s - \beta)^{-\log \| F^k \Phi \|_\beta}. \]
Thus, by Gelfond’s Lemma, and [Lemmas 7 and 9, Ing22] applied to \( \| \cdot \|_\beta \), we have
\[ 0 \leq \log \| \alpha_k \| \leq \sum_{\beta \in S} -\log \| F^k \Phi \|_\beta (\log \| s - \beta \| + \log 2) \]
\[ = \sum_{\beta \in S} -\log \| F^k \Phi \|_\beta (\log^+ |\beta| + \log 2) \]
\[ \leq \sum_{\beta \in S} (-d^{k(N+1)} G_{F, \beta}(\Phi) + C_\beta)(\log^+ |\beta| + \log 2) \]
with \( G_{F, \beta}(\Phi) = \lim_{k \to \infty} \log \| F^k \Phi \|_\beta / d^k (N+1) \), and \( C_\beta \) some constant depending on \( F \) and \( \beta \).
In particular, this proves (9).

To prove (10), we again cite [8]. In particular, since the right-hand-side is unbounded, we may consider just \( \deg_s(F^k \Phi) \) and compute
\[ \deg_s(F^k \Phi) = \log \| F^k \Phi \|_\infty \]
\[ \leq d^{k(N+1)} G_{F, \infty}(\Phi) + C_\infty. \]

Finally, to prove (11), we claim that
\[ \log \| F_* \Phi \|_v \leq d^{N+1} \log \| \Phi \|_v + C_v \]
Note that we cannot simply apply the results of [8], though, since \( \| \Phi \|_v \) is the largest modulus of a coefficient of \( \Phi \) as a polynomial in both \( X \) and \( s \). Assuming (12), and assuming that \( C_v = 0 \) for almost all \( v \), we can then sum over all places to obtain
\[ h^+(F_* \Phi) \leq d^{N+1} h^+(\Phi) + C, \]
which by summing a geometric series gives
\[ h^+(F_* \Phi) \leq d^{k(N+1)} (h^+(\Phi) + C'). \]

Now fix a place (nonarchimedean for now), and by Lemma 8(6) of [8] we have, for any \( t \in C_v \),
\[ \log \| (F_t)_*, \Phi_t \|_v \leq d^{N+1} \log \| \Phi_t \|_v + \deg(\Phi_t) C_t, \]
where \( C_t \) depends on \( N, d, \) and the coefficients of \( F \). We have \( \deg(\Phi_t) = \deg(\Phi) \) for all but finitely many \( t \), so by specializing we can ensure that \( \| q(t) \|_v = \| q \|_v \) for all of these coefficients, and the coefficients of \( \Phi \) and \( F_*, \Phi \). This shows that we have the same error term on the generic fibre:
\[ \log \| F_* \Phi \|_v = \log \| (F_t)_*, \Phi_t \|_v \]
\[ \leq d^{N+1} \log \| \Phi_t \|_v + \deg(\Phi_t) C_t \]
\[ = d^{N+1} \log \| \Phi \|_v + \deg(\Phi) C. \]
We want to show that the error is as maximal as it could be, independent of \( t \).
Specifically,
\[ C_t = d^N \lambda_{\text{Hom}^0_t}(f_t) - d^N \log \| F_t \|_v + d^N c_3 + d^{N+1} N \log^+ |2|, \]
with $c_3$ depending only on $N$ and $d$, and
\[
\lambda_{\text{Hom}^N}(f_t) = -\log |\text{Res}(F_t)|_v + (N + 1)d^N \log \|F_t\|_v.
\]
In particular, if $T$ is the set of coefficients of $F$, coefficients of $\Phi(F)$, and $\text{Res}(F)$, then for all $t$ as in the sampling lemma (Lemma 7) above, $C_t$ will be the same, and will depend only on $d$, $N$, $\|\text{Res}(F)\|_v$, and $\|F\|_v$.

For the archimedean places, again let $T$ be the set of coefficients of $F$, $\Phi$, $F_\ast \Phi$, and $\text{Res}(F)$, and suppose that all of these polynomials have degree at most $B$. Since
\[
m(\Phi_t) - \frac{N}{2} \log(\deg(\Phi_t) + 1) \leq \log \|\Phi_t\| \leq m(\Phi_t) + N \deg(\Phi_t) \log 2
\]
For all but finitely many $t$, we have $\deg(\Phi_t) = \deg(\Phi)$ and same for $F_\ast \Phi$. We use denote by $C(S)$ a constant depending on a sequence of numbers $S$. Note that $C_{t,v} = C(\|F_t\|_v, |\text{Res}(F_t)|_v)$ is linear in the logarithms of the relevant quantities, 
so that we have
\[
C(\|F_t\|_v, |\text{Res}(F_t)|_v) \leq C(\|F\|_v, |\text{Res}(F)|_v) + O((B + 1) \log(\#T \cdot B)) = O_F((B + 1) \log(\#T \cdot B)).
\]
Thus, by the sampling lemma (Lemma 7) there is a $t$ with
\[
\log \|F_\ast \Phi\|_v \leq \log (\|F_t\|_v \Phi_t) + (B + 1) \log(\#T \cdot B)
\leq m(\|F_t\|_v \Phi_t) + N \deg(\Phi) \log 2 + (B + 1) \log(\#T \cdot B)
\leq d^{N+1} m(\Phi_t) + \deg(\Phi) C(\|F_t\|_v, |\text{Res}(F_t)|_v, d, N) + (B + 1) \log(\#T \cdot B)
\leq d^{N+1} \log \|\Phi\|_v + \frac{N}{2} \log(\deg(F, \Phi) + 1) + \deg(\Phi) C(\|F_t\|_v, |\text{Res}(F_t)|_v, d, N) + (B + 1) \log(\#T \cdot B)
\leq d^{N+1} \log \|\Phi\|_v + (d^{N+1} + 1)(B + 1) \log(\#T \cdot B) + \frac{N}{2} \log(\deg(F, \Phi) + 1) + \deg(\Phi) C(\|F_t\|_v, |\text{Res}(F_t)|_v, d, N) + (B + 1) \log(\#T \cdot B)
\leq d^{N+1} \log \|\Phi\|_v + (d^{N+1} + 1)(B + 1) \log(\#T \cdot B) + \frac{N}{2} \log(d^N \deg(\Phi) + 1) + O_F(\deg(\Phi)(B + 1) \log(\#T \cdot B)).
\]
We now estimate
\[
B = \max\{\log \|F\|_v, \|\text{Res}\|_v, \log \|\Phi\|_v, \|F, F, \Phi\|_v\}
= \max\{\log \|\Phi\|_v, \log \|F, F, \Phi\|_v\} + O_F(1)
= O_F(\log \|\Phi\|_v + 1)
\]
by previous estimates. Similarly, $\#T$ is one more than the number of coefficients of $F$, $\Phi$, and $F_\ast \Phi$, and so we have
\[
\#T = O_F(\deg(\Phi)^N + 1),
\]
so we have
\[
\log \|F_\ast \Phi\|_v \leq d^{N+1} \log \|\Phi\|_v + O_F(\deg(\Phi) \log \|F\|_v \log(\deg(\Phi)^N \log \|\Phi\|_v)).
\]
Since $\deg(F^k_\ast \Phi) = d^k \deg(\Phi)$ and $\log \|F^k_\ast \Phi\|_v \leq d^{k(N+1)} (\log \|\Phi\|_v + O_F(1))$, we obtain
\[
\log \|F^{k+1}_\ast \Phi\|_v \leq d^{N+1} \log \|F^k_\ast \Phi\|_v + O_F(\deg(\Phi)(d^{k(2N+1)} k)).
\]
It follows by induction that
\[ \log \| F_*^k \Phi \|_v = O_{F_*}(d^{k(2N+1+\delta)}) \]
This then gives \( h^+(F_*^k(\Phi)) = O(d^{k(2N+1+\delta)}) \) for any \( \delta > 0 \), or just \( \Theta(F_*^k(\Phi)) = O(d^{2k(N+1)}) \).

To obtain (11), we also notice that
\[ \deg_X(F_*^k \Phi) = d^k \deg_X(\Phi) \]
and
\[ \deg_s(F_*^k \Phi) = \log \| F_*^k \Phi \|_\infty = O(d^{k(N+1)}(\log \| \Phi \|_\infty + 1)), \]
both of which are negligible with respect to the claimed bounds for \( \Theta(F_*^k \Phi) \).

\[ \square \]

4. Variation of the escape rate

Using again the function field absolute value \( \| \cdot \|_\infty \) defined by
\[ \| \alpha \|_\infty = e^{-\text{ord}_{t=\infty}(\alpha)}, \]
we have that
\[ \deg_t(\Phi) = -\text{ord}_{t=\infty}(\Phi) := -\min\{\text{ord}_{t=\infty}(\Phi)\} = \log \| \Phi \|_\infty \]
Recall that
\[ G_F(\Phi) = \lim_{k \to \infty} \frac{\log \| F_*^k \Phi \|}{d^{k(N+1)}}, \]
so that \( G_F(F_* \Phi) = d^{N+1}G_F(\Phi) \), and \( G_F(\Phi) = \log \| \Phi \| + O(\deg(\Phi)) \). From the second property, we also get
\[ -\text{ord}_{t=\infty}(F_*^k \Phi) = G_F(\Phi) + O(\deg(F_*^k \Phi)) = G_F(\Phi) + O(d^{kN}) \]
for fixed \( \Phi \). It is also true that
\[ \log \| F_*^k \Phi_t \| = G_{F_t}(\Phi_t) + O(d^{kN} \log^+ |t|) \]

**Lemma 9.** We have
\[ G_{F_t}(\Phi_t) = G_F(\Phi) \log^+ |t| + O(\log^+ |t|^{(3N+3)/(3N+4)}). \]

**Proof.** We proceed as in Silverman and Call-Silverman (11). Let \( M = 3N + 3 \), choose \( k = k(t) \) so that
\[ d^{k(M+1)} \leq \log^+ |t| < d^{(k+1)(M+1)}, \]
and estimate
\[ |G_{F_t}(\Phi_t) - G_F(\Phi) \log^+ |t|| = d^{-k(N+1)} |G_{F_t}((F_t)^k \Phi_t) - G_F(F_*^k \Phi) \log^+ |t|| \]
\[ \leq d^{-k(N+1)} \left| G_{F_t}((F_t)^k \Phi_t) - \log \| (F_*^k \Phi_t) \| \right| + d^{-k(N+1)} \log \| (F_*^k \Phi) \| + \text{ord}_{t=\infty}(F_*^k \Phi) \log^+ |t| \]
\[ + d^{-k(N+1)} | -\text{ord}_{t=\infty}(F_*^k \Phi) - G_F(F_*^k \Phi) \log^+ |t|. \]
We bound each term separately. It is well known from previous calculations that
\[ G_{F_t}(H) = \log \| H \| + O((\deg H) \log \| F_t \|) = \log \| H \| + O((\deg H) \log^+ |t|). \]
For $H = (F_t)^k \Phi_t$, this implies that
\[
|G_{F_t}((F_t)^k \Phi_t) - \log \| (F_t)^k \Phi_t \| | = O(\deg((F_t)^k \Phi_t) \log^+ |t|) = O(d^{kN} \log^+ |t|).
\]
Thus, the first term will easily be $O(d^{-k} \log^+ |t|)$. The last term is automatically
\[
d^{-k(N+1)} \cdot O(\deg_t(\Phi)(d^{kN} + d^{k(N+1)} \log^+ |t|)) = O(d^{-k} \log^+ |t|)).
\]
The middle term, based on Lemma 8, is $O(d^{kM})$. Since
\[
d^{-k} \leq (\log^+ |t|)^{-\frac{3}{M+1}} \text{ and } d^{kM} \leq (\log^+ |t|)^{\frac{M}{M+1}},
\]
we thus bound all the three referred terms with a desired error term of magnitude size at most $(\log^+ |t|)^{1-\epsilon_N}$, where $M = 3N + 3$.
\[
\Box
\]

Proof of the main result. Replace $s$ by $s^{-1}$ so that we are considering $L(f_s)$ as $s \to \infty$. It follows from Lemma 6 and [1] that for $J_s = \det(DF_s)$, we have
\[
L(f_s) = G_{F_t}(J_s),
\]
and so from Lemma 9 we have
\[
L(f_t) = G_F(J_F) \log^+ |t| + O((\log^+ |t|)^{1-\epsilon_N}),
\]
with $\epsilon_N = 1/(3N + 4)$, as $t \to \infty$. Note that one can deduce that $L(f) = G_F(J_F)$ on the generic fibre from the machinery in [21], but in any case Favre’s asymptotic [4]
\[
L(f_t) = L(f) \log^+ |t| + o(\log^+ |t|)
\]
combined with (13) ensures this.
\[
\Box
\]

Remark. Gauthier, Okuyama, and Vigny [6] have proven the analogue of (14) in the case of non-archimedean fields. Although our focus is on the complex case, the arguments in this paper apply over non-archimedean fields as well.

References

[1] Giovanni Bassanelli and François Berteloot. Bifurcation currents in holomorphic dynamics on $\mathbb{P}^k$. *J. Reine Angew. Math.* **608** (2007), pp. 201–235.
[2] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Solution to a non-Archimedean Monge-Ampère equation. *J. Amer. Math. Soc.* **28** (2015), no. 3, pp. 617–667.
[3] Enrico Bombieri and Walter Gubler. *Heights in Diophantine geometry*. New Mathematical Monographs, 4. Cambridge University Press, Cambridge, 2006.
[4] Gregory S. Call and Joseph H. Silverman. Canonical heights on varieties with morphisms. *Compositio Math.* **89** no. 2 (1993), pp. 163–205.
[5] Charles Favre. Degeneration of endomorphisms of the complex projective space in the hybrid space. *J. Inst. Math. Jussieu* **19** no. 4 (2020), pp. 1141–1183.
[6] Thomas Gauthier, Yûsuke Okuyama, and Gabriel Vigny. Approximation of non-archimedean Lyapunov exponents and applications over global fields. *Trans. Amer. Math. Soc.* **373** no. 12 (2020), pp. 8963–9011.
[7] Patrick Ingram. Minimally critical endomorphisms of $\mathbb{P}^N$. (2020), preprint arXiv:2006.12869
[8] Patrick Ingram. Explicit canonical heights for divisors relative to endomorphisms of $\mathbb{P}^N$. (2022), preprint arXiv:2207.07266
[9] Zbigniew Jelonek. On the effective Nullstellensatz. *Invent. Math.* **162** no. 1 (2005), pp. 1–17.
[10] D. W. Masser and G. Wüthrich. Fields of large transcendence degree generated by values of elliptic functions. *Invent. Math.* **72** no. 3 (1983), pp. 407–464.
