PROCYCLIC COVERINGS OF COMMUTATORS IN PROFINITE GROUPS

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Abstract. We consider profinite groups in which all commutators are contained in a union of finitely many procyclic subgroups. It is shown that if $G$ is a profinite group in which all commutators are covered by $m$ procyclic subgroups, then $G$ possesses a finite characteristic subgroup $M$ contained in $G'$ such that the order of $M$ is $m$-bounded and $G'/M$ is procyclic. If $G$ is a pro-$p$ group such that all commutators in $G$ are covered by $m$ procyclic subgroups, then $G'$ is either finite of $m$-bounded order or procyclic.

1. Introduction

A covering of a group $G$ is a family $\{S_i\}_{i \in I}$ of subsets of $G$ such that $G = \bigcup_{i \in I} S_i$. If $\{H_i\}_{i \in I}$ is a covering of $G$ by subgroups, it is natural to ask what information about $G$ can be deduced from properties of the subgroups $H_i$. In the case where the covering is finite actually quite a lot about the structure of $G$ can be said. In particular, as was first pointed out by Baer (see [11, p. 105]), a group covered by finitely many cyclic subgroups is either cyclic or finite. Fernández-Alcober and Shumyatsky proved that if $G$ is a group in which the set of all commutators is covered by finitely many cyclic subgroups, the derived group $G'$ is either finite or cyclic [5]. Later, in [3], Cutolo and Nicotera showed that if $G$ is a group in which the set of all $\gamma_j$-commutators is covered by finitely many cyclic subgroups, then $\gamma_j(G)$ is finite-by-cyclic. They also showed that $\gamma_j(G)$ can be infinite and not cyclic. It is still unknown whether a similar result holds for the derived words $\delta_j$.

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Recall that a profinite group is a topological group that is isomorphic to an inverse limit of finite groups. The textbooks [10] and [12] provide a good introduction to the theory of profinite groups. In the context of profinite groups all the usual concepts of group theory are interpreted topologically. In particular, the derived group \( G' \) of a profinite group \( G \) is the closed subgroup generated by all commutators in \( G \).

In this paper we examine profinite groups in which all commutators are covered by finitely many procyclic subgroups. Our natural expectation was that the derived subgroup in such a group should either be finite or procyclic, but this turned out to be false. Indeed, let \( A \) be a finite group such that \( A' \) is noncyclic of order four, and let \( B \) be a pro-\( p \) group such that \( B' \) is infinite procyclic, where \( p \) is an odd prime. Then it is easy to see that \( G = A \times B \) is a profinite group in which \( G' \) is infinite, not procyclic, and can be covered by 3 procyclic subgroups.

However, we can prove the following result.

**Theorem A.** Let \( m \) be a positive integer and let \( G \) a profinite group in which all commutators are covered by \( m \) procyclic subgroups. Then \( G \) possesses a finite characteristic subgroup \( M \) contained in \( G' \) such that the order of \( M \) is \( m \)-bounded and \( G'/M \) is procyclic.

As usual, we use the expression “\( a \)-bounded” to mean “bounded from above by some function depending only on the parameter \( a \)”.

Further, we concentrate on pro-\( p \) groups in which all commutators are covered by finitely many procyclic subgroups. In this case our initial expectation that \( G' \) is either finite or procyclic has been confirmed.

**Theorem B.** Let \( p \) be a prime and let \( G \) be a pro-\( p \) group such that all commutators in \( G \) are covered by \( m \) procyclic subgroups. Then \( G' \) is either finite or \( m \)-bounded order or procyclic.

The above results are not the first that deal with coverings of word-values in profinite groups. For a family of group words \( w \) it was shown in [1] that if \( G \) is a profinite group in which all \( w \)-values are contained in a union of finitely many closed subgroups with a prescribed property, then the verbal subgroup \( w(G) \) has the same property as well. More recently the results obtained in [1] have been extended to profinite groups in which all \( w \)-values are contained in a union of countably many closed subgroups [4]. Quite possibly, for profinite groups in which the commutators are covered by countably many procyclic subgroups some analogues of Theorem A and Theorem B hold true.

Though profinite groups constitute the main topic of the present study, the above results also have a bearing on the case of abstract groups. As we have already mentioned, the main result in [5] says that
if $G$ is an abstract group whose commutators are covered by finitely many cyclic subgroups, then $G'$ is either finite or cyclic. Now we can deduce the following additional information.

**Theorem C.** Let $G$ be a group that possesses $m$ cyclic subgroups whose union contains all commutators of $G$. Then $G$ has a characteristic subgroup $M$ contained in $G'$ such that the order of $M$ is $m$-bounded and $G'/M$ is cyclic.

Of course, the information provided by Theorem C is meaningful only in the case where $G'$ is finite, since otherwise $G'$ is cyclic.

2. **Finite groups with commutators covered by few cyclic subgroups**

We start with some elementary lemmas.

**Lemma 2.1.** Let $n \geq 1$ be a positive integer and let $H$ be a characteristic finite nilpotent subgroup of a group $G$. Assume that for every prime $p$ dividing the order of $H$ the Sylow $p$-subgroup $P$ of $H$ has a characteristic subgroup $M_p$ of order at most $n$, such that $P/M_p$ is cyclic. Then $G$ possesses a characteristic subgroup $M$ contained in $H$ such that the order of $M$ is $n$-bounded and $H/M$ is cyclic.

**Proof.** We take $M$ to be the product of all $M_p$, where $p$ ranges through the set of all prime divisors of the order of $H$. Obviously, $M_p = 1$ whenever $p \geq n + 1$ and therefore the order of $M$ is less than $n^n$. It is clear that, being the product of characteristic subgroups, $M$ is characteristic. The quotient $H/M$ is a nilpotent group with cyclic Sylow subgroups and therefore $H/M$ is cyclic. The proof is complete. □

**Lemma 2.2.** Let $H$ be a characteristic subgroup of an abstract (resp. profinite) group $G$. Suppose that $H$ possesses a normal finite subgroup $N$ such that $H/N$ is cyclic (resp. procyclic). Then $G$ has a characteristic subgroup $M$ contained in $H$ such that the order of $M$ is at most $|N|^2$ and $H/M$ is cyclic (resp. procyclic).

**Proof.** Take $M$ to be the subgroup generated by all elements of $H$ of order dividing $|N|$. It is clear that $M$ is a characteristic subgroup in $G$ containing $N$. Since the quotient $M/N$ is cyclic (resp. procyclic) and generated by elements of order dividing $|N|$, it has order dividing $|N|$. It follows that the order of $M$ is at most $|N|^2$, as required. □

The following lemma is taken from [2]. It will play an important role in our arguments.
Lemma 2.3. Let $G$ be a finite noncyclic $p$-group that can be covered by $m$ cyclic subgroups. Then the order of $G$ is $m$-bounded.

Recall that in a group $G$ the subgroup $\gamma_\infty(G)$ is the intersection of all $\gamma_i(G)$ for $i \in \mathbb{N}$. Clearly, a finite group $G$ is nilpotent if and only if $\gamma_\infty(G) = 1$. It is an easy exercise to show that if $G$ is a finite group, then $\gamma_\infty(G)$ is generated by the commutators $[x, y]$ such that $x, y$ are elements of $G$ having mutually coprime orders. The following theorem was proved in [2].

Theorem 2.4. Let $G$ be a finite group that possesses $m$ cyclic subgroups whose union contains all commutators $[x, y]$ such that $x, y$ are elements of $G$ having mutually coprime orders. Then $\gamma_\infty(G)$ has a subgroup $\Delta$ such that

(i) $\Delta$ is normal in $G$;
(ii) $|\Delta|$ is $m$-bounded;
(iii) $\gamma_\infty(G)/\Delta$ is cyclic.

Further, we will require the following special case of a result of Guralnick [6, Theorem A].

Theorem 2.5. Let $G$ be a finite group in which $G'$ is an abelian $p$-group generated by at most two elements. Then every element of $G'$ is a commutator.

We will now start our analysis of finite groups in which commutators are covered by at most $m$ cyclic subgroups. We recall that a finite group $G$ has rank $r$ if $r$ is the least integer such that every subgroup of $G$ can be generated by at most $r$ elements.

Lemma 2.6. Let $G$ be a finite nilpotent group of class 2 that possesses $m$ cyclic subgroups whose union contains all commutators of $G$. Then $G'$ has a characteristic subgroup $M$ such that $|M|$ is $m$-bounded and $G'/M$ is cyclic.

Proof. By Lemma 2.1 it is sufficient to show that the claim is correct for each Sylow subgroup of $G$. Therefore we can assume that $G$ is a $p$-group for some prime $p$. Since $G$ is of class 2, it follows that for each element $y \in G$ the subgroup $[G, y]$ consists entirely of commutators. By Lemma 2.3 there exists a bound $\beta$ such that either $[G, y]$ is cyclic or $|[G, y]| \leq \beta$. Let $M$ be the product of all subgroups of $G'$ whose order is at most $\beta$. Since $G'$ is an abelian group with at most $m$ generators, the rank of $G'$ is at most $m'$. It follows that the order of $M$ is bounded as well. We pass to the quotient $G/M$ and we obtain that $[G, y]$ is cyclic for all $y \in G$. Suppose that $G'$ is not cyclic. Passing to $G/\Phi(G')$,
we assume that $G'$ is elementary abelian. We can choose $x, y \in G$ such that $[G, x]$ and $[G, y]$ are both nontrivial and $[G, x] \neq [G, y]$. Now choose any element $t \in G$ which does not belong to $C_G(x) \cup C_G(y)$. Such an element $t$ exists because a group cannot be the union of two proper subgroups. Then $[G, t] = [G, x]$ because these are both cyclic groups of order $p$ containing $[x, t] \neq 1$. Similarly $[G, t] = [G, y]$, a contradiction.

In what follows we write $O_{p'}(X)$ to denote the largest normal $p'$-subgroup of a finite group $X$.

**Lemma 2.7.** Let $G$ be a finite metabelian group that possesses $m$ cyclic subgroups whose union contains all commutators of $G$. Then $G'$ has a characteristic subgroup $M$ such that $|M|$ is $m$-bounded and $G'/M$ is cyclic.

**Proof.** By Lemma 2.1 it is sufficient to show that each Sylow subgroup of $G'$ possesses a characteristic subgroup with the required properties. Let $p$ be a prime divisor of the order of $G'$. Passing to the quotient $G/O_{p'}(G')$ we can assume that $G'$ is a $p$-group. Let $a \in G'$ and $b \in G$. It is clear that each element of $\langle a, b \rangle$ has form $[a^i, b]$. If $x \in G$ we have $[xa^i, b] = [x, b]a^i[a^i, b] = [x, b][a^i, b]$ and so every element in the coset $[x, b]\langle [a, b] \rangle$ is a commutator. Thus, the coset $[x, b]\langle [a, b] \rangle$ is covered by $m$ cyclic subgroups. It follows that for some $1 \leq i \neq j \leq m + 1$ one of the elements $[x, b][a^i, b]$ and $[x, b][a^j, b]$ is a power of the other. For simplicity, assume that $C_1 = \langle [x, b][a^i, b] \rangle$ contains $[x, b][a^j, b]$. Then also $[a^{j-i}, b] \in C_1$. Therefore the subgroup $\langle C_1, [a, b] \rangle$ decomposes as a direct product $C_2 \times D_2$, where $C_2$ is cyclic and $D_2$ is cyclic of order at most $m$. Let $D$ be the product of all subgroups of $G'$ whose order is at most $m$. Since $G'$ is an abelian group with at most $m$ generators, the rank of $G'$ is at most $m$. It follows that the order of $D$ is at most $m^m$. We pass to the quotient $G/D$ and we see that

\begin{equation}
\langle [a, b], [x, b] \rangle \text{ is cyclic for all } a \in G' \text{ and } b, x \in G.
\end{equation}

Note that (2.1) implies in particular that $\langle G', y \rangle$ is cyclic for all $y \in G$.

Let us now show that $\gamma_3(G)$ is cyclic. We can pass to $G/\Phi(\gamma_3(G))$ and assume that $\gamma_3(G)$ is elementary abelian. Choose $y \in G$ such that $\langle G', y \rangle \neq 1$ and $x$ outside $C_G(y)$. Since $\langle G', y \rangle$ is of order $p$, by (2.1) we have $\langle G', y \rangle \leq \langle [x, y] \rangle$. The same argument shows that $\langle G', x \rangle \leq \langle [x, y] \rangle$. Using that $\langle G', x \rangle$ is of order at most $p$, we conclude that $\langle G', x \rangle \leq \langle G', y \rangle$. This happens for all $x$ outside $C_G(y)$. Since the set of all such $x$ outside $C_G(y)$ generates the whole group $G$, it follows that $\gamma_3(G) = [G', y]$. 

}\]
Thus, indeed $\gamma_3(G)$ is cyclic (we no longer assume that $\gamma_3(G)$ is elementary abelian). By Lemma 2.6, $G'/\gamma_3(G)$ has a characteristic subgroup $K/\gamma_3(G)$ such that $|K/\gamma_3(G)|$ is $m$-bounded and $G'/K$ is cyclic. We see that $K$ is an abelian subgroup containing a cyclic subgroup of bounded index. The subgroup of $K$ generated by all elements of order at most the exponent of $K/\gamma_3(G)$ is characteristic and has bounded order. By factoring this subgroup we may assume that $K/\gamma_3(G)$ is cyclic and so $G'/\gamma_3(G)$ has rank at most two. Now Theorem 2.5 tells us that every element of $G'/\gamma_3(G)$ is a commutator. Therefore Lemma 2.3 shows that $G'/\gamma_3(G)$ either is cyclic or has $m$-bounded order.

Suppose that $G'/\gamma_3(G)$ is cyclic. Taking into account that $\gamma_3(G)$ is cyclic as well, we conclude that $G'$ is 2-generator. By Theorem 2.5 every element of $G'$ is a commutator. Therefore Lemma 2.3 shows that $G'$ either is cyclic or has $m$-bounded order, whence the lemma follows.

Suppose now that $G'/\gamma_3(G)$ has $m$-bounded order. We argue as above. Let $X$ be the product of all subgroups of $G'$ of order at most $|G'/\gamma_3(G)|$. As $G'$ has rank at most 3, the subgroup $X$ has bounded order and $G'/X$ is cyclic. The proof is complete. □

**Theorem 2.8.** Let $G$ be a finite group that possesses $m$ cyclic subgroups whose union contains all commutators of $G$. Then $G'$ has a characteristic subgroup $M$ such that the order of $M$ is $m$-bounded and $G'/M$ is cyclic.

**Proof.** Let $\Delta$ have the same meaning as in Theorem 2.4. By Lemma 2.2 we may assume that $\Delta$ is characteristic in $G$. We can pass to the quotient $G/\Delta$ and suppose that $G$ is soluble with $\gamma_\infty(G)$ cyclic. The group $G$ acts on $\gamma_\infty(G)$ by conjugation and as the automorphism group of a cyclic group is abelian, it follows that $G'$ centralizes $\gamma_\infty(G)$. Therefore $G'$ is nilpotent. By Lemma 2.1 it is sufficient to show that each Sylow subgroup of $G'$ possesses a characteristic subgroup with the required properties. Let $p$ be a prime divisor of the order of $G'$. Passing to the quotient $G/O_p'(G')$ we can assume that $G'$ is a $p$-group.

Next we remark that since $G'$ is an $m$-generator $p$-group, the Burnside Basis Theorem [9, III.3.15] shows that $G'$ is generated by $m$ commutators. Therefore we can choose at most $2m$ elements in $G$ such that $G'$ is generated by commutators in the chosen elements. Without loss of generality we can assume that $G$ is generated by the chosen elements.

Let $x$ be a commutator. Then any conjugate of $x$ is again a commutator and so it belongs to at least one of the $m$ cyclic subgroups covering the commutators of $G$. Since any finite cyclic subgroup has at most one subgroup of any given order, it follows that the subgroup $\langle x \rangle$...
has at most $m$ conjugates. Thus, $N_G((x))$ has index at most $m$. Set $T = \cap N_G((x))$, where $x$ ranges over all commutators in $G$. Since $G$ can be generated by $2m$ elements, it has only boundedly many subgroups of any given index [8, Theorem 7.2.9] and so $T$ has $m$-bounded index in $G$. Also, $T' \leq C_G(x)$ for every commutator $x$, and so $T'$ centralizes $G'$. Therefore $\widetilde{T}$ is metabelian and the derived length of $G$ is bounded.

We will now use induction on the derived length of $G$. By induction, we can assume that $G''$ has a characteristic subgroup $M_1$ such that $|M_1|$ is $m$-bounded and $G''/M_1$ is cyclic. Passing to the quotient over $M_1$, we assume that $G''$ is cyclic. The group $G$ induces by conjugation an abelian group of automorphisms of $G''$. Hence, $G'$ centralizes $G''$ and thus the nilpotency class of $G'$ is at most 2.

By Lemma 2.7 (applied to $G/G''$) the derived group $G'$ has a characteristic subgroup $M$ containing $G''$, and such that $M/G''$ has $m$-bounded order while $G'/M$ is cyclic. As $|M : Z(M)| \leq |M : G''|$, by the Schur Theorem [11, Theorem 4.12] $M'$ has $m$-bounded order as well. Factoring $M'$ out we can assume that $M$ is abelian. We can write $M = R \times M_2$, where $R$ is a cyclic group and $M_2$ is a subgroup of $m$-bounded order. It follows from Lemma 2.2 that $M_2$ is contained in a characteristic subgroup of $G$ of $m$-bounded order. Factoring it out, we can assume that $M$ is cyclic. Moreover, $G$ acts on $M$ by conjugation so $[G', M] = 1$. It follows that $G'/Z(G')$ is cyclic. We conclude that $G'$ is abelian and the theorem follows from Lemma 2.7.

It is easy to see that under the hypothesis of Theorem 2.8 the order of $G'$ cannot be bounded even if we know that $G'$ is noncyclic. Indeed, let $A$ be a finite group such that $A'$ is noncyclic of order four and let $B$ be a finite group such that $B'$ has odd prime order $p$. Set $G = A \times B$. Then $G'$ is noncyclic and covered by 3 cyclic subgroups. The order of $G'$ is $4p$ and this tends to infinity when $p$ does so.

However, our next result shows that if $G$ is a $p$-group satisfying the hypothesis of Theorem 2.8 and having noncyclic derived group $G'$, then the order of $G'$ is $m$-bounded.

**Theorem 2.9.** Let $p$ be a prime and let $G$ be a finite $p$-group in which all commutators can be covered by $m$ cyclic subgroups. Then either $G'$ is cyclic or the order of $G'$ is $m$-bounded.

**Proof.** Let us assume that $G'$ is not cyclic. By Theorem 2.8 the derived group $G'$ contains a characteristic subgroup $M$ of $m$-bounded order such that $G'/M$ is cyclic.

We choose in $G$ a normal subgroup $N$ of minimum possible order subject to the condition that $(G/N)'$ is cyclic. Then $1 \neq N \subseteq G'$, and
the order of $N$ is $m$-bounded. Since $G$ is a finite $p$-group, there exists a normal subgroup $L$ in $G$ which is contained in $N$ and has index $p$ in $N$. By the assumption on $N$, the derived group of $G/L$ is not cyclic. Thus by factoring out $L$ we may assume that $N$ has order $p$ and therefore $G'$ is 2-generator. Now Theorem 2.5 tells us that every element of $G'$ is a commutator. Hence, it follows from Lemma 2.3 that $G'$ has $m$-bounded order. The proof is complete. □

3. Proofs of the main results

We are now ready to complete the proofs of the theorems stated in the introduction.

Proof of Theorem A. By Lemma 2.2 it suffices to find a normal subgroup of $G$ inside $G'$ with the desired properties. Let $\mathcal{N}$ be the family of all open normal subgroups of $G$, and observe that $G \cong \varprojlim_{N \in \mathcal{N}} G/N$. Consider an arbitrary $N \in \mathcal{N}$, and put $Q = G/N$. Let us write $\mathcal{M}(N)$ for the set of all subgroups $R$ of $Q'$ which are normal in $Q$, of order at most $f(m)$, and satisfy the condition that $Q'/R$ is cyclic. By Theorem 2.8, $\mathcal{M}(N)$ is not empty.

Given $L, N \in \mathcal{N}$ with $L \leq N$, the natural map $\pi_{LN}$ from $G/L$ to $G/N$ induces a map $\varphi_{LN}$ from $\mathcal{M}(L)$ to $\mathcal{M}(N)$. This way we get an inverse system $\{\mathcal{M}(N), \varphi_{LN}, \mathcal{N}\}$ of finite sets. By [10, Proposition 1.1.4], the corresponding inverse limit is not empty. If $(M_N/N)_{N \in \mathcal{N}}$ is an element of that inverse limit, then $\pi_{LN}(M_L/L) = M_N/N$ for all $L, N \in \mathcal{N}$ such that $L \leq N$. Hence we can form the inverse limit $\varprojlim_{N \in \mathcal{N}} M_N/N$, which corresponds to a normal closed subgroup $M$ of $G$. Since $|M_N/N| \leq f(m)$ for every $N \in \mathcal{N}$, we also have $|M| \leq f(m)$. Finally, observe that

$$(G/M)' = G'/M \cong \varprojlim_{N \in \mathcal{N}} \frac{(G/N)'}{M_N/N}$$

is an inverse limit of cyclic subgroups, and so procyclic. □

Proof of Theorem B. Let $G$ be a pro-$p$ group such that all commutators in $G$ are covered by $m$ procyclic subgroups. Choose an open normal subgroup $N$ of $G$. The quotient $Q = G/N$ is a finite $p$-group satisfying the hypotheses of Theorem 2.9. Therefore either $Q'$ is cyclic or the order of $Q'$ is at most some $m$-bounded number $k$. Suppose now that the derived group $G'$ is not of order at most $k$. Then there exists an open normal subgroup $N$ in $G$ such that the order of the derived group $(G/N)'$ is larger than $k$ and hence $(G/N)'$ is cyclic. Then $(G/H)'$ is
cyclic for any open normal subgroup $H$ contained in $N$. It follows that $G'$ is procyclic, as required. □

Proof of Theorem C. By the main result of [5] mentioned in the introduction, we know that $G'$ is either cyclic or finite. So it is sufficient to concentrate on the case where $G'$ is finite. There exists a finitely generated subgroup of $G$ whose derived subgroup coincides with $G'$, and consequently we may assume that $G$ is finitely generated. As $G'$ is finite, the centralizer $C_G(G')$ has finite index in $G$ and so it is also finitely generated. Moreover, $C_G(G')$ is nilpotent of class at most 2 and thus it is residually finite (see [7]). We conclude that $G$ is residually finite as well. Since $G'$ is finite, there exists a normal subgroup $N$ of $G$ of finite index in $G$ such that $G' \cap N = 1$. As $G'$ is isomorphic to $G'N/N = (G/N)'$ and in the finite group $G/N$ the result holds by Theorem 2.2, the conclusion follows. □

References

[1] C. Acciarri, P. Shumyatsky, On profinite groups in which commutators are covered by finitely many subgroups, Math. Z. 274 (2013), 239–248.
[2] C. Acciarri, P. Shumyatsky, On finite groups in which coprime commutators are covered by few cyclic subgroups, J. Algebra 407 (2014), 358–371.
[3] G. Cutolo, C. Nicotera, Verbal sets and cyclic coverings, J. Algebra 324 (2010), 1616–1624.
[4] E. Detomi, M. Morigi, P. Shumyatsky, On countable coverings of word values in profinite groups, J. Pure Appl. Algebra, to appear.
[5] G.A. Fernández-Alcober, P. Shumyatsky, On groups in which commutators are covered by finitely many cyclic subgroups, J. Algebra 319 (2008), 4844–4851.
[6] R. Guralnick, Commutators and commutator subgroups, Adv. Math. 45 (1982), 319-330.
[7] K.A. Hirsch, On infinite soluble groups, III, Proc. London Math. Soc. (2) 49 (1946), 184–194.
[8] M. Hall, Jr., The Theory of Groups, The Macmillan Co., New York, 1959.
[9] B. Huppert, Endliche Gruppen, Springer-Verlag, Berlin, 1967.
[10] L. Ribes, P. Zalesski, Profinite Groups, Springer-Verlag, Berlin, 2000.
[11] D.J.S. Robinson, Finiteness conditions and generalized soluble groups, Part 1, Springer-Verlag, New York-Berlin, 1972.
[12] J.S. Wilson, Profinite Groups, Clarendon Press, Oxford, 1998.
