Definable Orthogonality Classes in Accessible Categories are Small

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Joint work with J. Bagaria, A. R. D. Mathias and J. Rosický

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Herzliche Glückwünsche und vielen Dank
an Professor Rüdiger Göbel!
Summary

The assumptions needed to infer \textit{reflectivity} or \textit{smallness} of \textit{orthogonality classes} in \textit{accessible categories} may depend on the \textit{complexity} of the formulas in the language of set theory defining those classes.
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- **Σ_{n+1}-formulas** are those of the form ∃x φ where φ is Πₙ.
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Accessible and Locally Presentable Categories

Let $\lambda$ denote a regular cardinal and let $\mathcal{C}$ be a category.

- An object $X$ in $\mathcal{C}$ is $\lambda$-presentable if the functor $\mathcal{C}(X, -)$ preserves $\lambda$-filtered colimits.

$\mathcal{C}$ is $\lambda$-accessible if $\lambda$-filtered colimits exist in $\mathcal{C}$ and there is a set $X$ of $\lambda$-presentable objects in $\mathcal{C}$ such that every object of $\mathcal{C}$ is a $\lambda$-filtered colimit of objects from $X$.

A category is accessible if it is $\lambda$-accessible for some $\lambda$.

A cocomplete accessible category is locally presentable.

For every theory $T$ with any signature $\Sigma$, the category of models of $T$ (i.e., $\Sigma$-structures satisfying all sentences of $T$) is accessible.

Conversely, every accessible category is a category of models of some theory [Adámek–Rosický, 1994].

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Orthogonality Classes

For a class $\mathcal{M}$ of morphisms in a category $\mathcal{C}$, the **orthogonal class** $\mathcal{M}^\perp$ consists of those objects $X$ such that $\mathcal{C}(f, X): \mathcal{C}(B, X) \cong \mathcal{C}(A, X)$ for all $f: A \to B$ in $\mathcal{M}$.

\[
\begin{array}{c}
A \\
\downarrow \forall g \\
X \\
\uparrow \exists ! h \\
B
\end{array}
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![Diagram](attachment:orthogonality_classes_diagram.png)

We say that a class $\mathcal{X}$ of objects is

- an **orthogonality class** if $\mathcal{X} = \mathcal{M}^\perp$ for some class of morphisms $\mathcal{M}$;
- a **small-orthogonality class** if $\mathcal{X} = \mathcal{M}^\perp$ for some set of morphisms $\mathcal{M}$. 
Orthogonality Subcategory Problem

Under which assumptions on a category \( \mathcal{C} \) and a class of morphisms \( \mathcal{M} \) does it follow that the orthogonal class \( \mathcal{M}^\perp \) is reflective? [Freyd–Kelly, 1972]
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**Vopěnka’s Principle** is equivalent to the statement that all orthogonality classes in locally presentable categories are small-orthogonality classes. (This cannot be proved in ZFC.)
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This is in fact an axiom schema, since it is a statement about classes. In the language of set theory, it is formulated as follows, for all formulas $\varphi(x, y)$ of the language of set theory with two free variables:

$$\forall x [(\forall y \forall z (\varphi(x, y) \land \varphi(x, z) \rightarrow y \text{ and } z \text{ are } \Sigma\text{-structures for some } \Sigma) \land \forall \alpha \in \text{Ord } \exists y (\text{rank}(y) > \alpha \land \varphi(x, y))) \rightarrow \exists y \exists z (\varphi(x, y) \land \varphi(x, z) \land y \neq z \land \exists e (e: y \rightarrow z \text{ is elementary}))].$$
Some well-known consequences of Vopěnka’s Principle (VP):

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- If VP holds, then every full limit-closed subcategory of a locally presentable category is reflective and every full colimit-closed subcategory is coreflective. [Adámek–Rosický, 1994]
More consequences of Vopěnka’s Principle (VP):

- If VP holds, then for every homotopical localization $L$ on simplicial sets there is a map $f$ such that $L \simeq L_f$. [C–Scevenels–Smith, 2005]
- If VP holds, then cohomological localizations of simplicial sets exist. [C–Scevenels–Smith, 2005]
- If VP holds, then every localizing subcategory of a triangulated category with combinatorial models is coreflective, and every colocalizing subcategory is reflective. [C–Gutiérrez–Rosický, 2011]

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More precisely, we define, for each $n < \omega$, what we call \textit{C(n)-extendible cardinals}, with the following properties:

- If there is a proper class of $C(n)$-extendible cardinals, where $n \geq 1$, then each $\Sigma_{n+1}$ orthogonality class in an accessible category is a small-orthogonality class.
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- If there is a proper class of $C(n)$-extendible cardinals, where $n \geq 1$, then each $\Sigma_{n+1}$ orthogonality class in an accessible category is a small-orthogonality class.
- Vopěnka’s Principle is equivalent to the claim that there is a proper class of $C(n)$-extendible cardinals for each $n < \omega$. 
**C(n)-Extendible Cardinals**

Let $C(n)$ denote the class of cardinals $\alpha$ such that $V_\alpha$ is a $\Sigma_n$-elementary submodel of the universe $V$. A cardinal $\kappa \in C(n)$ is **C(n)-extendible** if, for all $\lambda > \kappa$ in $C(n)$, there is an elementary embedding $j: V_\lambda \rightarrow V_\mu$ for some $\mu \in C(n)$ with critical point $\kappa$, such that $j(\kappa) \in C(n)$ and $j(\kappa) > \lambda$. 
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A class $S$ is \textit{definable with sufficiently low complexity} if

- it is $\Sigma_1$, or
- there is a proper class of \textbf{supercompact cardinals} and $S$ is $\Sigma_2$, or
- there is a proper class of $C(n)$-extendible cardinals and $S$ is $\Sigma_{n+2}$, with $n \geq 1$. 
Main Results

- Every full subcategory of an accessible category definable with sufficiently low complexity is bounded.

- If $L$ is a reflection on an accessible category with coproducts, and the class of $L$-equivalences is definable with sufficiently low complexity, then $L \cong L_f$ for some $f$.

- In a locally presentable category, every full limit-closed subcategory definable with sufficiently low complexity is reflective, and every full colimit-closed subcategory definable with sufficiently low complexity is coreflective.

Recall: A class $S$ is **definable with sufficiently low complexity** if
  - it is $\Sigma_1$, or
  - there is a proper class of supercompact cardinals and $S$ is $\Sigma_2$, or
  - there is a proper class of $C(n)$-extendible cardinals and $S$ is $\Sigma_{n+2}$, with $n \geq 1$. 
Implications in Algebraic Topology

For a spectrum $E$, the class of $E_\ast$-acyclic simplicial sets is $\Sigma_1$, while the class of $E^\ast$-acyclic simplicial sets is $\Sigma_2$.

A simplicial set $X$ is $E^\ast$-acyclic if and only if the following formula is true:

$$X \in \mathsf{sSets}_\ast \land (\forall n < \omega) \exists M \ [M \in \mathsf{sSets}_\ast$$

$$\land (\forall k < \omega) \ [(\forall f \in M_k) \ f \in \mathsf{sSets}_\ast(X \land \Delta[k]_+, E_n)$$

$$\land \forall g \ (g \in \mathsf{sSets}_\ast(X \land \Delta[k]_+, E_n) \rightarrow g \in M_k)]$$

$$\land M \text{ is weakly contractible}].$$

The formula states that the mapping space $\text{Map}(X, E_n)$ is weakly contractible for all $n$, where the (fibrant) spectrum $E = \{E_n\}_{n \geq 0}$ is treated as a parameter.
Thus, it follows from our results that the existence of $E^*$-localizations can be proved in ZFC —this was known since [Bousfield, 1975]— and that the existence of $E^*$-localizations follows from the existence of a proper class of supercompact cardinals.
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- The latter improves [C–Scevenels–Smith, 2005], where the existence of $E^*$-localizations was inferred from VP.
- More generally, if $S$ is any (possibly proper) class of maps between simplicial sets, then an $S$-localization exists if $S$ is definable with sufficiently low complexity.

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Please come to:

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Advanced Course and Workshop

Institut de Matemàtica, Universitat de Barcelona

1–8 September 2011

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