INFINITE FAMILIES OF ISOGENY-TORSION GRAPHS

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Abstract. Let $E$ be a $\mathbb{Q}$-isogeny class of elliptic curves defined over $\mathbb{Q}$. The isogeny graph associated to $E$ is a graph which has a vertex for each element of $E$ and an edge for each $\mathbb{Q}$-isogeny of prime degree that maps one element of $E$ to another element of $E$, with the degree recorded as a label of the edge. The isogeny-torsion graph associated to $E$ is the isogeny graph associated to $E$ where, in addition, we label each vertex with the abstract group structure of the torsion subgroup over $\mathbb{Q}$ of the corresponding elliptic curve. The main result of the article is a determination of which isogeny-torsion graphs associated to $\mathbb{Q}$-isogeny classes of elliptic curves defined over $\mathbb{Q}$ correspond to infinitely many $j$-invariants.

1. Introduction

Let $E/\mathbb{Q}$ be an elliptic curve. It is well known that $E$ has a following Weierstrass model of the form

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3.$$

The Weierstrass model is usually dehomogenized. Moreover, $E$ has the structure of an abelian group with group identity $O = [0:1:0]$. By the Mordell–Weil theorem, the set of points on $E$ defined over $\mathbb{Q}$, denoted $E(\mathbb{Q})$, has the structure of a finitely generated abelian group. Thus, the set of points on $E$ defined over $\mathbb{Q}$ of finite order, denoted $E(\mathbb{Q})_{tors}$ is a finite group. By Mazur’s theorem, $E(\mathbb{Q})_{tors}$ is isomorphic to one of fifteen groups (see Theorem 2.1). Moreover, these fifteen groups occur infinitely often. Let $E'/\mathbb{Q}$ be an elliptic curve. An isogeny mapping $E$ to $E'$ is a rational morphism $\phi: E \to E'$ such that $\phi$ maps the identity of $E$ to the identity of $E'$. If there is a non-constant isogeny defined over $\mathbb{Q}$, mapping $E$ to $E'$, we say that $E$ is $\mathbb{Q}$-isogenous to $E'$. This relation is an equivalence relation and the set of elliptic curves defined over $\mathbb{Q}$ that are $\mathbb{Q}$-isogenous to $E$ is called the $\mathbb{Q}$-isogeny class of $E$.

Non-constant isogenies have finite kernels. We are particularly interested in non-constant isogenies with cyclic kernels. The isogeny graph associated to the $\mathbb{Q}$-isogeny class of $E$ is a visual description of the $\mathbb{Q}$-isogeny class of $E$. Denote the $\mathbb{Q}$-isogeny class of $E$ by $\mathcal{E}$. The isogeny graph associated to $\mathcal{E}$ is a graph which has a vertex for each element of $\mathcal{E}$ and an edge for each $\mathbb{Q}$-isogeny of prime degree that maps one element of $\mathcal{E}$ to another element of $\mathcal{E}$, with the degree recorded as a label of the edge. The isogeny-torsion graph associated to $\mathcal{E}$ is the isogeny graph associated to $\mathcal{E}$ where, in addition, we label each vertex with the abstract group structure of the torsion subgroup over $\mathbb{Q}$ of the corresponding elliptic curve.

Example 1.1. There are four elliptic curves in the $\mathbb{Q}$-isogeny class with LMFDB label 17.a which we will denote $E$, $E'$, $E''$, and $E'''$. The isogeny graph associated to 17.a is on the left and the isogeny-torsion graph associated to 17.a is on the right.
The classification of isogeny graphs associated to \( \mathbb{Q} \)-isogeny classes of elliptic curves over \( \mathbb{Q} \) was probably known in the 1980’s as it follows directly from the classification of non-cuspidal \( \mathbb{Q} \)-rational points on the modular curve \( X_0(N) \) for positive integers \( N \). Nonetheless, a proof can be found in section 6 of [1].

**Theorem 1.2.** There are 26 isomorphism types of isogeny graphs that are associated to elliptic curves defined over \( \mathbb{Q} \). More precisely, there are 16 types of (linear) \( L_k \) graphs of \( k = 1 \)-4 vertices, 3 types of (nonlinear two-primary torsion) \( T_k \) graphs of \( k = 4 \), 6, or 8 vertices, 6 types of (rectangular) \( R_k \) graphs of \( k = 4 \) or 6 vertices, and 1 (special) \( S \) graph.

In the case of a linear graph of \( L_2 \) or \( L_3 \) type or in the case of a rectangular graph of \( R_4 \) type, the degree of the maximal finite, cyclic \( \mathbb{Q} \)-isogeny of the isogeny graph is written in parentheses to distinguish it from other isogeny-torsion graphs of the same size and shape, but with different isogeny degree. For example, there are \( L_2(2) \) graphs; graphs of \( L_2 \) type generated by an isogeny of degree 2 and there are \( L_2(3) \) graphs; isogeny graphs of \( L_2 \) type generated by an isogeny of degree 3. Relying only on the size and shape of isogeny graphs of \( L_2 \) type is not enough to distinguish \( L_2(2) \) and \( L_2(3) \) isogeny graphs. The main theorem in [1] was the classification of isogeny-torsion graphs associated to \( \mathbb{Q} \)-isogeny classes of elliptic curves over \( \mathbb{Q} \).

**Theorem 1.3** (Chiloyan, Lozano-Robledo, [1]). There are 52 isomorphism types of isogeny-torsion graphs that are associated to \( \mathbb{Q} \)-isogeny classes of elliptic curves defined over \( \mathbb{Q} \). In particular, there are 23 isogeny-torsion graphs of \( L_k \) type, 13 isogeny-torsion graphs of \( T_k \) type, 12 isogeny-torsion graphs of \( R_k \) type, and 4 isogeny-torsion graphs of \( S \) type.

For each of the fifteen torsion subgroups, \( G \), there are infinitely many \( j \)-invariants, that correspond to elliptic curves \( E \) defined over \( \mathbb{Q} \) such that \( j(E) = j \) and \( E(\mathbb{Q})_{\text{tors}} \cong G \). It is natural to ask if there is an analogous result for isogeny-torsion graphs. In other words, for which isogeny-torsion graphs \( G \), do there exist infinitely many \( j \)-invariants that correspond to elliptic curves \( E \) defined over \( \mathbb{Q} \) with \( j(E) = j \) and the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E \) is \( G \)?

Let \( j(G) \) be the set of all \( j \)-invariants of all elliptic curves \( E \) defined over \( \mathbb{Q} \) such that the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E \) is \( G \). We will say that \( j(G) \) is the set of \( j \)-invariants associated to \( G \). The main result of this article is a determination of the isogeny-torsion graphs \( G \) such that \( j(G) \) is infinite. In the case that \( j(G) \) is infinite, we will say that \( G \) corresponds to an infinite set of \( j \)-invariants and in the case that \( j(G) \) is finite, we will say that \( G \) corresponds to a finite set of \( j \)-invariants.

Let \( V_G \) be a vertex of \( G \). Let \( j(V_G) \) be the set of \( j \)-invariants of all elliptic curves \( E \) defined over \( \mathbb{Q} \) such that the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E \) is \( G \) and \( E \) is represented by the vertex \( V_G \). We will say that \( j(V_G) \) is the set of \( j \)-invariants associated to the
vertex $V_G$. By Theorem 2.8, each isogeny-torsion graph has at most 8 vertices. Hence, an isogeny-torsion graph corresponds to an infinite set of $j$-invariants if and only if any one of the vertices on the isogeny-torsion graph corresponds to an infinite set of $j$-invariants.

**Example 1.4.** Consider a $\mathbb{Q}$-isogeny class of elliptic curves over $\mathbb{Q}$ that contains four elliptic curves over $\mathbb{Q}$, $E_1$, $E_2$, $E_3$, and $E_4$ such that there is an isogeny $\phi: E_1 \to E_4$ defined over $\mathbb{Q}$, which has a cyclic kernel of order 27. In other words, the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class is of $L_4$ (see the following diagram).

The $j$-invariants of $E_1$ and $E_4$ are equal to $-12288000$ and the $j$-invariants of $E_2$ and $E_3$ are equal to 0. Hence, there are two $j$-invariants associated to an isogeny-torsion graph of $L_4$ type, regardless of torsion configuration. Thus, the two isogeny-torsion graphs of $L_4$ type both correspond to a finite set of $j$-invariants.

**Example 1.5.** Let $G$ be the isogeny-torsion graph in Example 1.1.

The center vertex of the isogeny-torsion graph represents elliptic curves over $\mathbb{Q}$ which

1. have full two-torsion defined over $\mathbb{Q}$,
2. do not contain a cyclic $\mathbb{Q}$-rational subgroup of order 4,
3. are $\mathbb{Q}$-isogenous to two non-isomorphic elliptic curves over $\mathbb{Q}$ with a point of order 4 defined over $\mathbb{Q}$.

Elliptic curves over $\mathbb{Q}$ represented by the center vertex of $G$ correspond to non-cuspidal $\mathbb{Q}$-rational points on the modular curve $X_{24e}$ in the notation of [9]. Moreover, they have $j$-invariant equal to

$$J = 2^8 \cdot \frac{(t^2 + t + 1)^3 \cdot (t^2 - t + 1)^3}{t^4 \cdot (t^2 + 1)^2}$$

for some non-zero rational $t$ or $J = 1728$. An argument using Hilbert’s Irreducibility Theorem (see Section 3.1 where we will elaborate on this idea further) shows that $G$ corresponds to an infinite set of $j$-invariants.

The goal of this article is to prove the following statement:

**Theorem 1.6.** Let $G$ be an isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of an elliptic curve defined over $\mathbb{Q}$.

Then $G$ corresponds to a finite set of $j$-invariants if and only if

- $G$ is of $L_2(p)$ type with isogeny degree $p = 11, 17, 19, 37, 43, 67, 163$, or
- $G$ is of $L_4$ type, or
- $G$ is of $R_4(pq)$ type with maximal, cyclic, isogeny degree $pq = 14, 15, 21$. 
1.1. Philosophy and structure of the paper. The main ideas motivating in this paper is to think about elliptic curve theory, not necessarily from the viewpoint of individual elliptic curves over $\mathbb{Q}$ but $\mathbb{Q}$-isogeny classes of elliptic curves defined over $\mathbb{Q}$ and the groups which generate said $\mathbb{Q}$-isogeny classes.

Let $E/\mathbb{Q}$ be an elliptic curve. Then $E(\mathbb{Q})_{\text{tors}}$ is isomorphic to one of fifteen groups. The main result in [1] is the classification of the isogeny-torsion graphs associated to $\mathbb{Q}$-isogeny classes of elliptic curves defined over $\mathbb{Q}$. Originally, the authors in [1] wanted to classify the torsion subgroups of a pair of $\mathbb{Q}$-isogenous elliptic curves defined over $\mathbb{Q}$, which was extended to classifying the torsion subgroups of all elliptic curves in a $\mathbb{Q}$-isogeny class, which was extended to the main result of [1].

Consider again the isogeny graph of $L_4$ type like in Example 1.4. Let $\mathcal{E}$ be a $\mathbb{Q}$-isogeny class such that the isogeny graph associated to $\mathcal{E}$ is of $L_4$ type. By the classification of isogeny graphs, the isogeny graph of $L_4$ type is the only isogeny graph with an isogeny of degree 27. Instead of thinking about the four individual elliptic curves in the isogeny graph, this paper considers the collection of the four elliptic curves in $\mathcal{E}$ simultaneously. Actually, the best way to think about $\mathcal{E}$ is that it is a $\mathbb{Q}$-isogeny class containing an elliptic curve with a cyclic, $\mathbb{Q}$-rational subgroup $H$ of order 27. In other words, $\mathcal{E}$ contains an elliptic curve, $E$ corresponding to a non-cuspidal, $\mathbb{Q}$-rational point on $X_0(27)$; the modular curve generated by $B_{27} = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\} \subseteq \text{GL}(2,\mathbb{Z}/27\mathbb{Z})$. The group $H$ is represented by the left column, $\left\{ \begin{bmatrix} * \\ 0 \end{bmatrix} \right\}$ of $B_{27}$. In some sense, the group $H$ generates $\mathcal{E}$ because $H$ contains four distinct subgroups, each generating a unique elliptic curve that is $\mathbb{Q}$-isogenous to $E$.

On the other hand, a similar but ultimately distinct analysis can be done with the the isogeny-torsion graph in Example 1.1. Let $E/\mathbb{Q}$ be an elliptic curve represented by the center vertex in the isogeny-torsion graph in Example 1.1. Then

- the $\mathbb{Q}$-isogeny class of $E$ has four elliptic curves,
- $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$,
- $E$ is $\mathbb{Q}$-isogenous to two non-isomorphic elliptic curves over $\mathbb{Q}$, each with a point of order 4 defined over $\mathbb{Q}$

and hence, either the $j$-invariant of $E$ is equal to 1728 or $E$ is non-CM and corresponds to a non-cuspidal, $\mathbb{Q}$-rational point on the modular curve $X_24e$ using the notation in [9] and 4.24.0.7 using LMFDB notation.

Let $N$ be a positive integer such that $X_0(N)$ has finitely many non-cuspidal $\mathbb{Q}$-rational points. Then $N \in S = \{11, 14, 15, 17, 19, 21, 27, 37, 43, 67, 163\}$ and the genus of $X_0(N)$ is greater than or equal to 2. The main result of this paper is obvious: Let $\mathcal{E}$ be a $\mathbb{Q}$-isogeny class of elliptic curves over $\mathbb{Q}$ and let $G$ be its associated isogeny graph. Then $G$ corresponds to a finite set of $j$-invariants if and only if $\mathcal{E}$ contains an elliptic curve over $\mathbb{Q}$ that corresponds to a non-cuspidal $\mathbb{Q}$-rational point on $X_0(N)$ for some $N$ in $S$. One direction of the previous statement is proven in Proposition 5.1. The other direction (proving that all other isogeny graphs correspond to infinite sets of $j$-invariants) requires case by case analysis.

In section 2, we will go over the elementary algebraic properties of elliptic curves over $\mathbb{Q}$ including Mazur’s theorem and Kenku’s theorem (the classification of non-cuspidal, $\mathbb{Q}$-rational points on $X_0(N)$). In section 3, we will summarize the work done by Rouse and Zureick-Brown ([9]) in classifying the 2-adic Galois images attached to non-CM elliptic curves defined over $\mathbb{Q}$ and the work done by Sutherland and Zywina in classifying the modular curves of prime-power level with infinitely many
Q-rational points ([12]). Section 4 covers lemmas from group theory that will be helpful. Section 5 contains the proof of Proposition 5.1, one direction of our main result. Section 7 contains the proof that all isogeny-torsion graphs of $S$ type correspond to infinite sets of $j$-invariants, followed by section 8 which contains the proof that all isogeny-torsion graphs of $T_k$ type correspond to infinite sets of $j$-invariants, followed by section 9 which determines which isogeny-torsion graphs of $R_k$ type correspond to infinite sets of $j$-invariants, and concluded by section 10 which determines which isogeny-torsion graphs of $L_k$ type correspond to infinitely many $j$-invariants.

Most of the proofs follow the same recipe. We determine whether a fixed isogeny-torsion graph $G$ corresponds to an infinite set of $j$-invariants. This can be done by seeing, for example, if $G$ contains a unique point defined over $\mathbb{Q}$. For example, the isogeny-torsion graph $L_1^2(7)$ is the isogeny-torsion graph of $L_2(7)$ with torsion configuration ([7], [1]) (see below).

\[ \mathbb{Z}/7\mathbb{Z} \xrightarrow{7} \{ O \} \]

Moreover, $L_1^2(7)$ is the only isogeny-torsion graph containing a point of order 7 defined over $\mathbb{Q}$. As there are infinitely many $j$-invariants corresponding to elliptic curves over $\mathbb{Q}$ that have a point of order 7 defined over $\mathbb{Q}$, $L_1^2(7)$ corresponds to an infinite set of $j$-invariants. All isogeny-torsion graphs that are acquired from quadratic twisting an elliptic curve in $G$ also correspond to an infinite set of $j$-invariants and we use this method many times. For example, we use quadratic twists to prove that the isogeny-torsion graph $L_2^2(7)$; the isogeny graph of $L_2(7)$ type with torsion configuration ([1], [1]) corresponds to an infinite set of $j$-invariants (see below).

\[ \{ O \} \xrightarrow{7} \{ O \} \]

We may simply take an elliptic curve $E/\mathbb{Q}$ such that $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/7\mathbb{Z}$ and twist it by an appropriate integer and the isogeny-torsion graph of the quadratic twist will be $L_2^2(7)$.

We also make use of Hilbert’s irreducibility theorem to prove some isogeny-torsion graphs correspond to infinite sets of $j$-invariants. For example, the isogeny-torsion graph $L_1^2(5)$ is the “simplest” isogeny-torsion graph containing a point of order 5 defined over $\mathbb{Q}$ (see below).

\[ \mathbb{Z}/5\mathbb{Z} \xrightarrow{5} \{ O \} \]

For example, we use Hilbert’s irreducibility theorem to prove that $L_1^2(5)$ corresponds to an infinite set of $j$-invariants using the fact that the sets of $j$-invariants corresponding to the isogeny-torsion graphs that properly contain $L_1^2(5)$ constitute thin sets in the language of Serre. The isogeny graph of $L_1$ type corresponds to an infinite set of $j$-invariants. The proof of this is the final proof of the paper and again makes use of Hilbert’s irreducibility theorem.

Acknowledgements. The author would like to express his gratitude to his advisor Álvaro Lozano-Robledo for his patience and many helpful conversations on this topic. The author would also like to thank Harris Daniels for conversations about non-trivial entanglements. The author would also like to thank those, including John Voight, who initially asked which isogeny-torsion graphs correspond to an infinite set of $j$-invariants. The author would like to thank the referee for their many helpful comments.
2. Background

Let $E$ be an elliptic curve defined over $\mathbb{Q}$. Denote the set of $\mathbb{Q}$-rational points on $E$ by $E(\mathbb{Q})$. Then $E(\mathbb{Q})$ has the structure of a finitely generated abelian group. Denote the set of points on $E$ defined over $\mathbb{Q}$ of finite order by $E(\mathbb{Q})_{\text{tors}}$. Then, $E(\mathbb{Q})_{\text{tors}}$ is isomorphic to one of 15 groups.

**Theorem 2.1** (Mazur [8]). Let $E/\mathbb{Q}$ be an elliptic curve. Then

$$E(\mathbb{Q})_{\text{tors}} \simeq \begin{cases} \mathbb{Z}/M\mathbb{Z} & \text{with } 1 \leq M \leq 10 \text{ or } M = 12, \text{ or} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z} & \text{with } 1 \leq N \leq 4. \end{cases}$$

Let $N$ be a positive integer. The points on $E$ of order dividing $N$ with coordinates in $\overline{\mathbb{Q}}$ form a finite group, denoted $E[N]$ which is isomorphic to $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$. An element of $E[N]$ is called an $N$-torsion point. The group $G_Q := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $E[N]$ for all positive integers $N$. From this action, we have the mod-$N$ Galois representation attached to $E$: $\bar{\rho}_{E,N}: G_Q \to \text{Aut}(E[N])$.

After identifying $E[N] \cong \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ and fixing a set of (two) generators of $E[N]$, we may consider the mod-$N$ Galois representation attached to $E$ as $\bar{\rho}_{E,N}: G_Q \to \text{GL}(2, N)$. Let $u$ be an element of $(\mathbb{Z}/N\mathbb{Z})^\times$. By the properties of the Weil pairing, there exists an element of the image of $\bar{\rho}_{E,N}$ that has determinant $u$. A subgroup $H$ of $\text{GL}(2, \mathbb{Z}/N\mathbb{Z})$ is said to have full determinant mod-$N$ if $\det(H) = (\mathbb{Z}/N\mathbb{Z})^\times$. Moreover, the image of $\bar{\rho}_{E,N}$ contains an element that represents complex conjugation. If $E$ is non-CM, then by Lemma 2.8 in [12], complex conjugation is represented by $\left( \begin{array}{cc} 1 & 1 \\ 0 & -1 \end{array} \right)$ or $\left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$. Matrices in $\text{GL}(2, \mathbb{Z}/N\mathbb{Z})$ that are conjugate to $\left( \begin{array}{cc} 1 & 1 \\ 0 & -1 \end{array} \right)$ or $\left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$ will be called representatives of complex conjugation mod-$N$.

Let $E$ and $E'$ be elliptic curves defined over $\mathbb{Q}$. An isogeny mapping $E$ to $E'$ is a non-constant rational morphism $\phi: E \to E'$ that maps the identity of $E$ to the identity of $E'$. Isogenies are group homomorphisms with kernels of finite order. The degree of an isogeny agrees with the order of its kernel.

**Definition 2.2.** Let $E/\mathbb{Q}$ be an elliptic curve. A subgroup $H$ of $E$ of finite order is said to be $\mathbb{Q}$-rational if $\sigma(H) = H$ for all $\sigma \in G_Q$.

**Remark 2.3.** Note that for an elliptic curve $E/\mathbb{Q}$, a group generated by a point $P$ on $E$ defined over $\mathbb{Q}$ of finite order is certainly a $\mathbb{Q}$-rational group but in general, the elements of a $\mathbb{Q}$-rational subgroup of $E$ need not be fixed by $G_Q$. For example, $E[3]$ is a $\mathbb{Q}$-rational group but certainly, the group $G_Q$ fixes one or three of the nine points of $E[3]$ by Theorem 2.1.

**Lemma 2.4** (Proposition III.4.12, [11]). Let $E$ be an elliptic curve over $\mathbb{Q}$ and let $N$ be a positive integer. Then for each cyclic, $\mathbb{Q}$-rational subgroup $H$ of $E$ of order $N$, there is a unique elliptic curve defined over $\mathbb{Q}$ up to isomorphism denoted $E/H$, and an isogeny $\phi_H: E \to E/H$ with kernel $H$.

**Remark 2.5.** Note that it is only the elliptic curve that is unique (up to isomorphism) but the isogeny is not. For any isogeny $\phi$, the isogeny $-\phi$ has the same domain, codomain, and kernel as $\phi$.

Moreover, for any integer $N$ the isogeny $\phi$ and the isogeny $[N] \circ \phi$ have the same domain and the same codomain. This is why the bijection in Lemma 2.4 is with cyclic, $\mathbb{Q}$-rational subgroups of an elliptic curve instead of with all $\mathbb{Q}$-rational subgroups of an elliptic curve.
Let $E/Q$ be an elliptic curve and let $N$ be a positive integer. Then there is a one to one bijective correspondence between the non-cuspidal, $Q$-rational points on the modular curve $X_0(N)$ and elliptic curves over $Q$ up to isomorphism with a cyclic, $Q$-rational subgroup of order $N$. Work by Fricke, Kenku, Klein, Kubert, Ligozat, Mazur and Ogg, among others contributed to the classification of the non-cuspidal, $Q$-rational points on $X_0(N)$ (see the summary tables in [7]).

**Theorem 2.6.** Let $N \geq 2$ be a positive integer such that $X_0(N)$ has a non-cuspidal, $Q$-rational point. Then:

1. $2 \leq N \leq 10$, or $N = 12, 13, 16, 18$ or 25. In this case $X_0(N)$ is a curve of genus 0 and its $Q$-rational points form an infinite 1-parameter family, or
2. $N = 11, 14, 15, 17, 19, 21, \text{ or } 27$. In this case $X_0(N)$ is a curve of genus 1, i.e., $X_0(N)$ is an elliptic curve over $Q$, but in all cases the Mordell-Weil group $X_0(N)(Q)$ is finite, or
3. $N = 37, 43, 67$ or 163. In this case $X_0(N)$ is a curve of genus $\geq 2$ and (by Faltings’ theorem) there are only finitely many $Q$-rational points, which are known explicitly.

Kenku’s theorem is a reformulation of Theorem 2.6 and gives a clear way to classify the isogeny graphs associated to $Q$-isogeny classes of elliptic curves over $Q$. But before we state Kenku’s theorem, we need a definition.

**Definition 2.7.** Let $E/Q$ be an elliptic curve. We define $C(E)$ to be the number of finite, cyclic, $Q$-rational subgroups of $E$ (including the trivial subgroup), and we define $C_p(E)$ similarly to $C(E)$ but only counting cyclic, $Q$-rational subgroups of order a power of $p$ (like in the definition of $C(E)$, this includes the trivial subgroup), for each prime $p$.

**Theorem 2.8** (Kenku, [5]). There are at most eight $Q$-isomorphism classes of elliptic curves in each $Q$-isogeny class. More concretely, let $E/Q$ be an elliptic curve. Then $C(E) = \prod_p C_p(E) \leq 8$ and each factor $C_p(E)$ is bounded as follows:

$$
\begin{array}{c|cccccccccccc}
 p & 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 & 37 & 43 & 67 & 163 & \text{else} \\
 \hline
 C_p(E) & 8 & 4 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
\end{array}
$$

Moreover:

1. If $C_p(E) = 2$ for a prime $p$ greater than 7, then $C_q(E) = 1$ for all other primes $q$.
2. If $C_7(E) = 2$, then $C(E) \leq 4$. Moreover, we have $C_3(E) = 2$, or $C_2(E) = 2$, or $C(E) = 2$.
3. If $C_5(E) \leq 3$ and if $C_5(E) = 3$, then $C(E) = 3$.
4. If $C_5(E) = 2$, then $C(E) \leq 4$. Moreover, either $C_3(E) = 2$, or $C_2(E) = 2$, or $C(E) = 2$.
5. If $C_3(E) \leq 4$ and if $C_3(E) = 4$, then $C(E) = 4$.
6. If $C_3(E) = 3$, then $C(E) \leq 6$. Moreover, $C_2(E) = 2$ or $C(E) = 3$.
7. If $C_3(E) = 2$, then $C_2(E) \leq 4$.

**Remark 2.9.** If the rest of the paper, if an elliptic curve $E/Q$ has full two-torsion defined over $Q$, then we will say that $E[2] = E[2](Q) = \langle P_2, Q_2 \rangle$ for some $P_2, Q_2 \in E[2]$. For an integer $M \geq 2$, let $P_{2M} \in E[2^M]$ such that $[2]P_{2M} = P_{2M-1}$ and let $Q_{2M} \in E[2^M]$ such that $[2]Q_{2M} = Q_{2M-1}$.

If an elliptic curve $E/Q$ has full two-torsion defined over $Q$ and contains a cyclic, $Q$-rational subgroup of order 4, then we may assume without loss of generality, that the cyclic groups $\{O\}$, $\langle P_2 \rangle$, $\langle Q_2 \rangle$, $\langle P_2 + Q_2 \rangle$, $\langle Q_4 \rangle$, and $\langle P_2 + Q_4 \rangle$ are $Q$-rational. If an elliptic curve $E/Q$ has full two-torsion defined over $Q$ and $C_2(E) = C(E) = 8$, then either $P_4, P_4 + Q_4, Q_4$, and $P_2 + Q_4$ generate distinct $Q$-rational subgroups of order 4; or $Q_8$ and $P_2 + Q_8$ generate distinct $Q$-rational subgroups of order 8. Thus, after a relabeling of the elliptic curves in the $Q$-isogeny class of $E$, the $Q$-rational subgroups
of $E$ are $\langle \mathcal{O} \rangle$, $\langle P_2 \rangle$, $\langle Q_2 \rangle$, $\langle P_2 + Q_2 \rangle$, $\langle Q_4 \rangle$, $\langle P_2 + Q_4 \rangle$, $\langle Q_8 \rangle$, and $\langle P_2 + Q_8 \rangle$. As a consequence, if $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, then $E(\mathbb{Q})_{\text{tors}} = \langle P_2, Q_4 \rangle$ and if $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, then $E(\mathbb{Q})_{\text{tors}} = \langle P_2, Q_8 \rangle$.

**Definition 2.10.** Let $\mathcal{E}$ be a $\mathbb{Q}$-isogeny class of elliptic curves defined over $\mathbb{Q}$. The isogeny graph associated to $\mathcal{E}$ is a graph, which has a vertex for each elliptic curve in $\mathcal{E}$ and an edge for each $\mathbb{Q}$-isogeny of prime degree that maps one element of $\mathcal{E}$ to another element of $\mathcal{E}$, with the degree recorded as the label of the edge.

The classification of isogeny graphs associated to $\mathbb{Q}$-isogeny classes of elliptic curves defined over $\mathbb{Q}$ follows from Theorem 2.8. A detailed proof is outlined in Section 6 of [1].

**Theorem 2.11.** There are 26 isomorphism types of isogeny graphs associated to $\mathbb{Q}$-isogeny classes of elliptic curves defined over $\mathbb{Q}$. More precisely, there are 16 types of (linear) $L_k$ graphs, 3 types of (non-linear, two-primary torsion) $T_k$ graphs, 6 types of (rectangular) $R_k$ graphs, and 1 (special) $S$ graph. The degree of the maximal, finite, cyclic $\mathbb{Q}$-isogeny is written in parentheses for the $L_2$, $L_3$, and $R_4$ isogeny graphs.

**Definition 2.12.** Let $\mathcal{E}$ be a $\mathbb{Q}$-isogeny class of elliptic curves defined over $\mathbb{Q}$. The isogeny-torsion graph associated to $\mathcal{E}$ is the isogeny graph associated to $\mathcal{E}$ with the vertices labeled with the abstract group structure of the torsion subgroup over $\mathbb{Q}$ of the corresponding elliptic curve.

See Example 1.1 for an example of an isogeny graph and an isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of an elliptic curve over $\mathbb{Q}$. In [1] the authors classified the isogeny-torsion graphs associated to $\mathbb{Q}$-isogeny classes of elliptic curves over $\mathbb{Q}$.

**Theorem 2.13 ([1]).** There are 52 isomorphism types of isogeny-torsion graphs that are associated to $\mathbb{Q}$-isogeny classes of elliptic curves over $\mathbb{Q}$. In particular, there are 23 isogeny-torsion graphs of $L_k$ type, 13 isogeny-torsion graphs of $T_k$ type, 12 isogeny-torsion graphs of $R_k$ type, and 4 isogeny-torsion graphs of $S$ type.

**Remark 2.14.** An application of Hilbert’s Irreducibility Theorem (see Proposition 3.4 and Remark 3.5) can be used to prove that, in terms of density, the wide majority of $\mathbb{Q}$-isogeny classes of elliptic curves over $\mathbb{Q}$ have an associated isogeny graph of $L_1$ type. In other words, an elliptic curve over $\mathbb{Q}$ with a finite, cyclic $\mathbb{Q}$-rational subgroup of order $\geq 2$ is rare.

2.1. Quadratic Twists. Let $E : y^2 = x^3 + Ax + B$ be an elliptic curve defined over $\mathbb{Q}$. Let $d$ be a non-zero, square-free integer. The quadratic twist of $E$ by $d$ is the elliptic curve $E^{(d)} : dy^2 = x^3 + Ax + B$. Note that $j(E) = j(E^{(d)})$.

Let $N$ be a positive integer. Let $\rho_N : G_{\mathbb{Q}} \rightarrow \text{Aut}(E[N])$ and $\rho_N^d : G_{\mathbb{Q}} \rightarrow \text{Aut}(E^d[N])$ be the mod-$N$ Galois representations attached to $E$ and $E^{(d)}$ respectively. There is a quadratic character $\chi_d$ such that $\chi_d \circ \rho_N = \rho_N^d$. Let $G_1$ and $G_2$ be subgroups of $\text{GL}(2, \mathbb{Z}/N\mathbb{Z})$. We will say that $G_1$ is a quadratic twist of $G_2$ if $\langle G_1, -\text{Id} \rangle = \langle G_2, -\text{Id} \rangle$.

2.2. Maps to the $j$-line. For more information, on this following subsection, see Section 2 in [12].

Let $N$ be a positive integer. Let $\mathcal{F}_N$ denote the field of meromorphic functions of the Riemann surface $X(N)$ whose $q$-expansions have coefficients in $K_N := \mathbb{Q}(\zeta_N)$. For $f \in \mathcal{F}_N$ and $\gamma \in \text{SL}(2, \mathbb{Z})$, let $f|_N \in \mathcal{F}$ denote the modular function satisfying $f|_N(\gamma \tau) = f(\tau)$. For each $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, let $\sigma_d$ be the automorphism of $K_N$ such that $\sigma_d(\zeta_N) = \zeta_N^d$. 


Proposition 2.15. The extension \( F_N \) of \( F_1 = \mathbb{Q}(j) \) is Galois. There is a unique isomorphism
\[
\theta_N : \text{GL}(2, \mathbb{Z}/N\mathbb{Z})/\{\pm I\} \to \text{Gal}(F_N/\mathbb{Q}(j))
\]
such that the following hold for all \( f \in F_N \):

1. For \( g \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z}) \), we have \( \theta_N(g)f = f|_{\gamma t} \) where \( \gamma \) is any matrix in \( \text{SL}(2, \mathbb{Z}) \) that is congruent to \( g \) modulo \( N \).

2. For \( g = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z}/N\mathbb{Z}) \), we have \( \theta_N(g)f = \sigma_d(f) \).

A subgroup \( G \) of \( \text{GL}(2, \mathbb{Z}/N\mathbb{Z}) \) containing \(-\text{Id}\) and with full determinant mod-\( N \) acts on \( F_N \) by \( g \cdot f = \theta_N(g)f \) for \( g \in G \) and \( f \in F_N \). Let \( F_N^G \) denote the subfield of \( F_N \) fixed by the action of \( G \). The modular curve \( X_G \) associated to \( G \) is the smooth projective curve with function field \( F_N^G \). The inclusion of fields \( F_1 = \mathbb{Q}(j) \subseteq F_N \) induces a non-constant morphism
\[
\pi_G : X_G \to \text{Spec} \mathbb{Q}[j] \cup \{\infty\} = \mathbb{P}^1_{\mathbb{Q}}
\]
of degree \([\text{GL}(2, \mathbb{Z}/N\mathbb{Z}) : G] \). If there is an inclusion of groups, \( G \subseteq G' \subseteq \text{GL}(2, \mathbb{Z}/N\mathbb{Z}) \), then there is an inclusion of fields \( \mathbb{Q}(j) \subseteq F_N^{G'} \subseteq F_N^G \) which induces a non-constant morphism \( g : X_G \to X_{G'} \) of degree \([G' : G] \) such that the following diagram commutes.

\[
\begin{array}{ccc}
X_G & \xrightarrow{g} & X_{G'} \\
\downarrow{\pi_G} & & \downarrow{\pi_{G'}} \\
\mathbb{P}^1 & & \\
\end{array}
\]

For an elliptic curve \( E/\mathbb{Q} \) with \( j(E) \neq 0,1728 \) the group \( p_{E,N}(G_{\mathbb{Q}}) \) is conjugate in \( \text{GL}(2, \mathbb{Z}/N\mathbb{Z}) \) to a subgroup of \( G \) if and only if \( j(E) \) is an element of \( \pi_G(X_G(\mathbb{Q})) \).

3. Work by Rouse–Zureick-Brown and Sutherland-Zywina

Theorem 3.1 (Rouse, Zureick-Brown, [9]). Let \( E \) be an elliptic curve over \( \mathbb{Q} \) without complex multiplication. Then, there are exactly 1208 possibilities for the 2-adic image \( \overline{\rho}_{E,2}\infty(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \), up to conjugacy in \( \text{GL}(2, \mathbb{Z}_2) \). Moreover, the index of \( \rho_{E,2}\infty(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \) in \( \text{GL}(2, \mathbb{Z}_2) \) divides 64 or 96.

The authors of [1] made extensive use of the database in [9] to eliminate non-examples of isogeny-torsion graphs of type \( T_4 \), \( T_6 \), and \( T_8 \). They also used the parametrization of the modular curve \( X_{24e} \) in the database compiled in [9] in a step to eliminate two “elusive” non-examples of isogeny-torsion graphs of \( S \) type (see Section 13 in [1]). In a reversal of styles, the author of this paper used one-parameter families of curves in the database from [9] to prove each of the isogeny-torsion graphs of \( T_4 \), \( T_6 \), and \( T_8 \) type correspond to infinite sets of \( j \)-invariants.

Theorem 3.2 (Sutherland, Zywina, [12]). Up to conjugacy, there are 248 open subgroups \( G \) of \( \text{GL}(2, \mathbb{Z}) \) of prime-power level containing \(-\text{Id}\), a representative of complex conjugation mod-\( N \), and having full determinant mod-\( N \) for which \( X_G \) has infinitely many non-cuspidal \( \mathbb{Q} \)-rational points. Of these 248 groups, there are 220 of genus 0 and 28 of genus 1.

The work in [12] classified the modular curves of prime-power level with infinitely many \( \mathbb{Q} \)-rational points. Additionally, it gave a formulization of Hilbert’s Irreducibility Theorem that proved key in classifying which isogeny-torsion graphs correspond to infinitely many \( j \)-invariants.
3.1. Hilbert’s Irreducibility Theorem.

**Theorem 3.3** (Hilbert). Let \( f_1(X_1, \ldots, X_r, Y_1, \ldots, Y_s), \ldots, f_n(X_1, \ldots, X_r, Y_1, \ldots, Y_s) \) be irreducible polynomials in the ring \( \mathbb{Q}(X_1, \ldots, X_r)[Y_1, \ldots, Y_s] \). Then there exists an \( r \)-tuple of rational numbers \( (a_1, \ldots, a_r) \) such that \( f_1(a_1, \ldots, a_r, Y_1, \ldots, Y_s), \ldots, f_n(a_1, \ldots, a_r, Y_1, \ldots, Y_s) \) are irreducible in the ring \( \mathbb{Q}[Y_1, \ldots, Y_s] \).

In Section 7 of [12] there is the following statement:

Let \( G \) be a subgroup of prime-power level such that the modular curve, \( X_G \) generated by \( G \) is a curve of genus 0. Define the set

\[ S_G := \bigcup_{G'} \pi_{G',G}(X_{G'}(\mathbb{Q})) \]

where \( G' \) varies over the proper subgroups of \( G \) that are conjugate to one of the groups that define a modular curve of prime-power level with infinitely many non-cuspidal, \( \mathbb{Q} \)-rational points from [12] and \( \pi_{G',G} : X_{G'} \to X_G \) is the natural morphism induced by the inclusion \( G' \subseteq G \).

Then \( X_G \cong \mathbb{P}^1 \) and \( S_G \) is a thin subset (see Chapter 3 of [10] for the definition and properties of thin sets) of \( X_G(\mathbb{Q}) \) in the language of Serre. The field \( \mathbb{Q} \) is Hilbertian and in particular, \( \mathbb{P}^1(\mathbb{Q}) \cong X_G(\mathbb{Q}) \) is not thin; this implies that \( X_G(\mathbb{Q}) \setminus S_G \) cannot be thin and is infinite.

**Proposition 3.4.** Let \( N \) be a positive integer. Let \( G \) be a subgroup of \( \text{GL}(2, \mathbb{Z}/N\mathbb{Z}) \) such that

1. \( G \) contains \(-\text{Id}\),
2. \( G \) contains a representative of complex conjugation modulo \( N \),
3. \( G \) has full determinant modulo \( N \).

If the modular curve \( X_G \) defined by \( G \) is a genus 0 curve that contains infinitely many non-cuspidal, \( \mathbb{Q} \)-rational points, then there are infinitely many \( j \)-invariants corresponding to elliptic curves over \( \mathbb{Q} \) such that the image of the mod-\( N \) Galois representation is conjugate to \( G \) itself (not a proper subgroup of \( G \)).

**Remark 3.5.** The proof of Proposition 3.4 follows directly from the discussion in Section 7 of [12]. In the case that \( X_G \) is genus 0 and contains infinitely many non-cuspidal \( \mathbb{Q} \)-rational points, the set \( X_G(\mathbb{Q}) \setminus S_G \) is infinite. Hence, the set of non-cuspidal, \( \mathbb{Q} \)-rational points on \( X_G \) that are not on the modular curves defined by any proper subgroup of \( G \) is infinite.

4. Some Lemmas

**Lemma 4.1** ([1], Lemma 5.5a). Let \( E/\mathbb{Q} \) and \( E'/\mathbb{Q} \) be elliptic curves. Let \( \phi : E \to E' \) be an isogeny such that the kernel of \( \phi \) is a finite, cyclic, \( \mathbb{Q} \)-rational group, \( H \). Then, for an arbitrary \( P \in E \), the point \( \phi(P) \in E' \) is defined over \( \mathbb{Q} \) if and only if \( \sigma(P) - P \in H \) for all \( \sigma \in G_\mathbb{Q} \).

**Lemma 4.2** ([1], Lemma 5.6). Let \( E/\mathbb{Q} \) be an elliptic curve with a point of order 2 defined over \( \mathbb{Q} \). Then, every elliptic curve over \( \mathbb{Q} \) that is \( \mathbb{Q} \)-isogenous to \( E \) also has a point of order 2 defined over the rationals.

**Lemma 4.3** ([1], Lemma 5.11). Let \( E/\mathbb{Q} \) and \( E'/\mathbb{Q} \) be elliptic curves and let \( P \) be a point of \( E \) of order \( 2^M \) with \( M \geq 1 \). Suppose \( P \) generates a \( \mathbb{Q} \)-rational group and the two cyclic groups of order \( 2^{M+1} \) that contain \( P \) are not \( \mathbb{Q} \)-rational. Let \( \phi : E \to E' \) be an isogeny with kernel \( \langle P \rangle \). Then, \( E'(\mathbb{Q})_{\text{tors}} \) is cyclic.
Corollary 4.8. Let $H$ be a single subgroup of index $p$ of order $p^m$ containing $P_1$. For $0 \leq j \leq m$, let $P_j \in \langle P_m \rangle$ such that $P_0 = O$ and $|P|P_j = P_{j-1}$, and let $\phi_j: E \to E_j$ be an isogeny with kernel $\langle P_j \rangle$.

(1) If $1 \leq j \leq m - 1$, then $E_j$ has a point of order $p$ defined over $Q$. Further, for $\phi_{m-1}: E \to E_{m-1}$, the curve $E_{m-1}$ has a point of order $p$ defined over $Q$ but no points of order $p^2$ defined over $Q$.

(2) If $j = m$, the curve $E_m$ has no points of order $p$ defined over $Q$.

Definition 4.5. Let $N$ be a positive integer. The subgroup of $GL(2, \mathbb{Z}/N\mathbb{Z})$ consisting of all matrices of the form $\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$ will be denoted $\mathfrak{B}_N$.

Lemma 4.6. Let $p$ be an odd prime and $u$ a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$. Then up to conjugation, the only proper subgroup of $\mathfrak{B}_p$ which has an element with determinant $u$ is the subgroup of diagonal matrices of $\mathfrak{B}_p$.

Proof. Let $H$ be a subgroup of $\mathfrak{B}_p$ of full determinant mod-$p$. Then $H$ contains an element of the form $h = \begin{pmatrix} 1 & x \\ 0 & d \end{pmatrix}$ for some $x \in \mathbb{Z}/p\mathbb{Z}$. The matrix $h$ has order $p - 1$ because $h$ is conjugate to $d = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$ by the matrix $\begin{pmatrix} 1 & \frac{1}{u} \\ 0 & 1 \end{pmatrix}$. Notice that $d$ generates the subgroup of $\mathfrak{B}_p$ of diagonal matrices. The group $\mathfrak{B}_p$ is of order $(p - 1)p$ and $\langle h \rangle$ is a subgroup of $\mathfrak{B}_p$ of index $p$. As $p$ is prime, $\langle h \rangle$ is a maximal subgroup of $\mathfrak{B}_p$. The group $H$ contains $\langle h \rangle$, and either $H = \langle h \rangle$ and hence, is conjugate to $\langle d \rangle$ or $H$ properly contains $\langle h \rangle$ and hence, $H = \mathfrak{B}_p$. \qed

Lemma 4.7. Let $p$ be an odd prime. Then $\mathfrak{B}_p$ and the subgroup of diagonal matrices of $\mathfrak{B}_p$ both have a single subgroup of index 2.

Proof. Let $u$ be a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$. Let $d = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$ and let $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $\mathfrak{B}_p = \langle d, t \rangle$. Let $H$ be a subgroup of $\mathfrak{B}_p$ of index 2. Then $H$ is a normal subgroup of $\mathfrak{B}_p$ and all squares of $\mathfrak{B}_p$ are elements of $H$. Thus, $t^2$ is an element of $H$. The matrix $t$ has order $p$ which is an odd prime and so, the group generated by $t$ is equal to the group generated by $t^2$. Hence, $t$ is an element of $H$. Thus, $H = \langle t^2, t \rangle$.

Note that the group of diagonal matrices of $\mathfrak{B}_p$ is generated by $d$. As $\langle d \rangle$ is a cyclic group, it has a single subgroup of index 2. \qed

Corollary 4.8. Let $p$ be an odd prime and let $E$ be an elliptic curve defined over $Q$ that has a point of order $p$ defined over $Q$. Then when $p \equiv 1 \mod 4$, $Q(\sqrt{p})$ is the only quadratic subfield of $Q(E[p])$ and when $p \equiv 3 \mod 4$, $Q(\sqrt{-p})$ is the only quadratic subfield of $Q(E[p])$.

Proof. The $p$-division field $E[p]$ contains the field generated by the $p$-th roots of unity, $Q(\zeta_p)$. Moreover, $Q(\zeta_p)$ contains $\sqrt{(-1)^{(p-1)/2} \cdot p}$. By the fundamental theorem of Galois theory, subfields of $Q(E[p])$ of degree 2 correspond to subgroups of the image of the mod-$p$ Galois representation attached to $E$ of index 2. By Lemma 4.6 the image of the mod-$p$ Galois representation attached to $E$ is conjugate to $\mathfrak{B}_p$ or the group of diagonal matrices of $\mathfrak{B}_p$. By Lemma 4.7, $\mathfrak{B}_p$ and the group of diagonal matrices of $\mathfrak{B}_p$ both contain a single subgroup of index 2. \qed
Lemma 4.9. Let $p$ be an odd prime. Then the group $\langle B_p, -\text{Id} \rangle$ contains three subgroups of index 2.

Proof. Let $G = \langle B_p, -\text{Id} \rangle$. As $-\text{Id}$ is not an element of $B_p$, $B_p$ is an index-2 subgroup of $G$ and hence, the order of $G$ is equal to $2(p - 1)p$. Let $u$ be a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$. Let $d = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$ and $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $G = \langle d, t, -\text{Id} \rangle$. Let $H_1 = B_p = \langle t, d \rangle$, let $H_2 = \langle t, -d \rangle$, and let $H_3 = \langle t, d^2, -\text{Id} \rangle$.

First we will prove that the orders of $H_1$, $H_2$, and $H_3$ are all equal to $(p - 1)p$. It is clear to see that the order of $H_1 = B_p$ is equal to $(p - 1)p$. We will now prove that the order of $H_2$ is equal to $(p - 1)p$.

Note that $\langle t \rangle$ is a normal subgroup of $H_2$ and intersects $\langle -d \rangle$ trivially. Thus, $H_2$ is isomorphic to the product $\langle t \rangle \cdot \langle -d \rangle$. The group $\langle -d \rangle$ contains $\langle d^2 \rangle$ as a subgroup of index 2 and hence, the order of $\langle -d \rangle$ is equal to $p - 1$. Now we will prove that the order of $H_3$ is equal to $(p - 1)p$. Clearly, $-\text{Id}$ is not contained in $H_3$. Suppose that $H$ does not contain $d$. If $H$ contains $-\text{Id}$, then $H = H_3$. If $H$ does not contain $-\text{Id}$ and $H$ does not contain $d$, then it contains their product $-d$ and hence, $H = H_2$.

Corollary 4.10. Let $p$ be an odd prime and let $E$ be an elliptic curve defined over $\mathbb{Q}$ such that the image of the mod-$p$ Galois representation attached to $E$ is conjugate to $\langle B_p, -\text{Id} \rangle$. Then there are three quadratic subfields of $\mathbb{Q}(E[p])$. Moreover, in the case that $p \equiv 3 \pmod{4}$, two of these quadratic subfields are totally imaginary and one of them is totally real.

Proof. By the fundamental theorem of Galois theory, subfields of $\mathbb{Q}(E[p])$ of degree 2 correspond to subgroups of the image of the mod-$p$ Galois representation attached to $E$ of index 2. The three subgroups of the image of the mod-$p$ Galois representation attached to $E$ of index 2 are the groups $H_1, H_2, H_3$ in the proof of Lemma 4.9. Hence, there are three quadratic subfields of $\mathbb{Q}(E[p])$. Let $p \equiv 3 \pmod{4}$. Note that $\mathbb{Q}(\sqrt{-p})$ is a totally imaginary quadratic subfield of $\mathbb{Q}(E[p])$. This is enough to see that two of the three quadratic subfields of $\mathbb{Q}(E[p])$ are totally imaginary.

Lemma 4.11. Let $p = 3$ or 5 and let $E/\mathbb{Q}$ be an elliptic curve with a point of order $p$ defined over $\mathbb{Q}$. If $E$ contains two distinct $\mathbb{Q}$-rational subgroups of order $p$, then the image of the mod-$p$ Galois representation attached to $E$ is conjugate to

$$D = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} : z \in (\mathbb{Z}/p\mathbb{Z})^\times \right\}.$$ 

Proof. Let $p$ and $E$ be as in the hypothesis. Let $P$ be a point of $E$ of order $p$ defined over $\mathbb{Q}$ and let $Q$ be a point of $E$ of order $p$, not defined over $\mathbb{Q}$, that generates a $\mathbb{Q}$-rational group. Then $E[p] = \langle P, Q \rangle$ and with this basis, the image of the mod-$p$ Galois representation attached to $E$ is a subgroup of $D$.

Noting that the image of the mod-$p$ Galois representation attached to $E$ must have full determinant mod-$p$, we can conclude that the image of the mod-$p$ Galois representation attached to $E$ is equal to $D$. 

□
Lemma 4.12. Let \( p = 3, 5, \) or \( 7 \). Let \( E/\mathbb{Q} \) be an elliptic curve that contains an element \( P \) that generates a \( \mathbb{Q} \)-rational subgroup of order \( p \). Suppose the image of the mod-\( p \) Galois representation attached to \( E \) contains \(-\text{Id}\). Then neither \( E \) nor \( E/(P) \) have a point of order \( p \) defined over \( \mathbb{Q} \).

Proof. Let \( Q \) be a point on \( E \) of order \( p \). The matrix \(-\text{Id}\) is in the center of \( \text{GL}(2, \mathbb{Z}/p\mathbb{Z}) \) so a subgroup \( H \) of \( \text{GL}(2, \mathbb{Z}/p\mathbb{Z}) \) contains an element conjugate to \(-\text{Id}\) if and only if \( H \) contains \(-\text{Id}\). Thus, the image of the mod-\( p \) Galois representation attached to \( E \) contains \(-\text{Id}\) and there exists a Galois automorphism \( \tau \in G_Q \) such that \( \tau(Q) = -Q \). As \( Q \) has odd order, \( Q \neq -Q \). Hence, \( Q \) is not defined over \( \mathbb{Q} \).

Now let \( \phi: E \to E/\langle P \rangle \) be an isogeny with kernel \( \langle P \rangle \). Let \( R \) be a point on \( E \) of order \( p \). Let \( \tau \in G_Q \) such that \( \tau(R) = -R \). If \( \phi(R) \) is defined over \( \mathbb{Q} \), then \( \tau(R) - R = [-2] R \in \langle P \rangle \) by Lemma 4.1. If this is the case, then \( [2]\phi(R) = \phi([2]R) = \mathcal{O} \) and so, \( \phi(R) \) has order equal to \( 1 \) or \( 2 \), contradicting the fact that \( \phi(R) \) has order \( p \).

\( \square \)

Corollary 4.13. Let \( p = 3, 5, \) or \( 7 \). Let \( E/\mathbb{Q} \) be an elliptic curve such that the image of the mod-\( p \) Galois representation attached to \( E \) contains \(-\text{Id}\). Then none of the elliptic curves over \( \mathbb{Q} \) in the \( \mathbb{Q} \)-isogeny class of \( E \) have a point of order \( p \) defined over \( \mathbb{Q} \).

Proof. By Lemma 4.12, at least two elliptic curves over \( \mathbb{Q} \) in the \( \mathbb{Q} \)-isogeny class of \( E \) do not have points of order \( p \) defined over \( \mathbb{Q} \). By the classification of isogeny-torsion graphs, no elliptic curve over \( \mathbb{Q} \) in the \( \mathbb{Q} \)-isogeny class of \( E \) has a point of order \( p \) defined over \( \mathbb{Q} \) (see the tables in Section 2 in \( [1] \)).

\( \square \)

Lemma 4.14. Let \( E/\mathbb{Q} \) be an elliptic curve with full two-torsion defined over \( \mathbb{Q} \). Let \( M \) be the positive integer such that \( 2^M \) is the order of the largest finite, cyclic, \( \mathbb{Q} \)-rational subgroup of \( E \) of \( 2 \)-power order. Suppose the image of the mod-\( 2^M+1 \) Galois representation attached to \( E \) contains \(-\text{Id}\). Then no elliptic curve over \( \mathbb{Q} \) that is \( \mathbb{Q} \)-isogenous to \( E \) has a point of order \( 4 \) defined over \( \mathbb{Q} \).

Proof. The elliptic curve \( E \) has full two-torsion defined over \( \mathbb{Q} \) and so \( E \) has a \( 2 \)-isogeny and \( C_2(E) \geq 4 \). As \( 2^M \geq 2 \), we have \( 2^{M+1} \geq 4 \), and the image of the mod-\( 4 \) Galois representation attached to \( E \) contains \(-\text{Id}\). Let \( R \) be a point on \( E \) of order \( 4 \). There is a Galois automorphism \( \tau \in G_Q \) such that \( \tau(R) = -R \). As \( R \) is a point of order \( 4 \), \( \tau(R) \neq R \) and so, \( R \) is not defined over \( \mathbb{Q} \). We break up the rest of the proof into three cases, depending on \( C_2(E) \).

- \( C_2(E) = 4 \).
  Suppose that \( C(E) = 4 \). Then \( E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and the cyclic, \( \mathbb{Q} \)-rational subgroups of \( E \) are the groups \( \langle P_2 \rangle, \langle Q_2 \rangle, \langle P_2 + Q_2 \rangle \), and the trivial group. Let \( A \) be any element of \( E \) of order \( 2 \) and \( \phi: E \to E/\langle A \rangle \) be an isogeny with kernel generated by \( A \). We claim that \( E/\langle A \rangle(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \).

  Indeed, \( E/\langle A \rangle(\mathbb{Q})_{\text{tors}} \) is cyclic by Lemma 4.3. Let \( A' \in E \) such that \( [2]A' = A \). Let \( B \) be a point of \( E \) of order \( 2 \) not equal to \( A \) and let \( B' \in E \) such that \( [2]B' = B \). By our hypothesis, \( E \) has full two-torsion defined over \( \mathbb{Q} \) and so, \( B \) is defined over \( \mathbb{Q} \), and \( \phi(B) \) is the point of \( E/\langle A \rangle \) of order \( 2 \) defined over \( \mathbb{Q} \) by Lemma 4.1. The two cyclic groups of order \( 4 \) containing \( \phi(B) \) are \( \langle \phi(B') \rangle \) and \( \langle \phi(A' + B') \rangle \). Let \( C \) be \( B' \) or \( A' + B' \). Note that \( C \) is an element of \( E \) of order \( 4 \). By the fact that the image of the mod-\( 4 \) Galois representation attached to \( E \) contains \(-\text{Id}\), there is a Galois automorphism \( \tau \in G_Q \) such that \( \tau(C) = -C \).
Hence, \( \tau(C) - C = [-2]C = B \) or \( A + B \). Both \( B \) and \( A + B \) are elements of \( E \) that are not contained in \( \langle A \rangle \). Thus, \( \phi(C) \) is not defined over \( \mathbb{Q} \) by Lemma 4.1. Hence, all groups \( E/\langle P_2 \rangle \langle \mathbb{Q} \rangle_{\text{tors}}, E/\langle Q_2 \rangle \langle \mathbb{Q} \rangle_{\text{tors}}, \text{and } E/\langle P_2 + Q_2 \rangle \langle \mathbb{Q} \rangle_{\text{tors}} \) are isomorphic to \( \mathbb{Z}/2\mathbb{Z} \).

If we suppose that \( C(E) = 8 \), then \( C_3(E) = 2 \) and the isogeny graph associated to the \( \mathbb{Q} \)-isogeny class of \( E \) is of type \( S \). In this case, the finite, cyclic, \( \mathbb{Q} \)-rational subgroups of \( E \) are the ones generated by \( \mathcal{O}, P_2, Q_2, P_2 + Q_2, D_3, D_3 + P_2, D_3 + Q_2, \) and \( D_3 + P_2 + Q_2 \) where \( D_3 \) is a point on \( E \) of order 3 that generates a \( \mathbb{Q} \)-rational group. As none of \( E/\langle P_2 \rangle, E/\langle Q_2 \rangle, E/\langle P_2 + Q_2 \rangle \) have points of order 4 defined over \( \mathbb{Q} \), neither do \( E/\langle D_3 + P_2 \rangle, E/\langle D_3 + Q_2 \rangle \), or \( E/\langle D_3 + P_2 + Q_2 \rangle \) as passing through a 3-isogeny defined over \( \mathbb{Q} \) preserves points of order 4 defined over \( \mathbb{Q} \).

- \( C_3(E) = 6 \).

We may assume without loss of generality that the finite, cyclic, \( \mathbb{Q} \)-rational subgroups of \( E \) are the groups \( \langle P_2 \rangle, \langle Q_2 \rangle, \langle P_2 + Q_2 \rangle, \langle Q_4 \rangle, \langle P_2 + Q_4 \rangle, \langle Q_8 \rangle, \langle P_2 + Q_8 \rangle, \) and the trivial group by Remark 2.9. Thus, the largest finite, cyclic, \( \mathbb{Q} \)-rational subgroup of \( E \) is of order 4 and so, by the hypothesis, the image of the mod-8 Galois representation attached to \( E \) contains \(-\text{Id}\). Using a similar proof in the case of \( C_2(E) = 4 \), we can conclude that the groups \( E/\langle P_2 \rangle \langle \mathbb{Q} \rangle_{\text{tors}}, E/\langle P_2 + Q_2 \rangle \langle \mathbb{Q} \rangle_{\text{tors}} \) and \( E/\langle P_2 + Q_2 \rangle \langle \mathbb{Q} \rangle_{\text{tors}} \) are of order 2.

Let \( A = Q_4 \) or \( P_2 + Q_4 \). We claim that \( E/\langle A \rangle \langle \mathbb{Q} \rangle_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \). Let \( \phi: E \to E/\langle A \rangle \) be an isogeny with kernel \( \langle A \rangle \). By Lemma 4.3, the group \( E/\langle A \rangle \langle \mathbb{Q} \rangle_{\text{tors}} \) is cyclic. Let \( B \) be a point on \( E \) of order 2 not equal to \( Q_2 \). The point of \( E/\langle A \rangle \) of order 2 defined over \( \mathbb{Q} \) is \( \phi(B) \) by Lemma 4.1. Let \( A' \) be a point on \( E \) such that \( [2]A' = A \) and let \( B' \) be a point on \( E \) such that \( [2]B' = B \). The two cyclic groups of order 4 that contain \( \phi(B) \) are \( \langle \phi(B') \rangle \) and \( \langle \phi(A' + B') \rangle \). Note that \( B' \) and \( A' + B' \) are points on \( E \) of order 4 and 8 respectively. As the image of the mod-8 Galois representation attached to \( E \) contains \(-\text{Id}\), there is a Galois automorphism \( \tau \in G_{\mathbb{Q}} \) such that \( \tau(B') = -B' \) and \( \tau(A') = -A' \). Hence, \( \tau(B') - B' = [-2]B' = B \notin \langle A \rangle \) and \( \tau(A' + B') - (A' + B') = [-2](A' + B') = -A + B \notin \langle A \rangle \). Thus, both \( \phi(B') \) and \( \phi(A' + B') \) are not defined over \( \mathbb{Q} \) by Lemma 4.1 and hence, \( E/\langle Q_4 \rangle \langle \mathbb{Q} \rangle_{\text{tors}} \) and \( E/\langle P_2 + Q_4 \rangle \langle \mathbb{Q} \rangle_{\text{tors}} \) are groups of order 2. By the classification of isogeny-torsion graphs of \( T_6 \) type, (see Table 4) we have proven that no elliptic curve over \( \mathbb{Q} \) in the \( \mathbb{Q} \)-isogeny class of \( E \) has a point of order 4 defined over \( \mathbb{Q} \).

- \( C_2(E) = 8 \).

We may assume without loss of generality that the finite, cyclic, \( \mathbb{Q} \)-rational subgroups of \( E \) are the groups \( \langle P_2 \rangle, \langle Q_2 \rangle, \langle P_2 + Q_2 \rangle, \langle Q_4 \rangle, \langle P_2 + Q_4 \rangle, \langle Q_8 \rangle, \langle P_2 + Q_8 \rangle, \) and the trivial group. By our hypothesis, the image of the mod-16 Galois representation attached to \( E \) contains \(-\text{Id}\). Using a similar proof in the case of \( C_2(E) = 4 \), we can conclude that the groups \( E/\langle P_2 \rangle \langle \mathbb{Q} \rangle_{\text{tors}}, E/\langle P_2 + Q_2 \rangle \langle \mathbb{Q} \rangle_{\text{tors}} \) are of order 2. Using a similar argument as in the case of \( C_2(E) = 6 \) with \( A \) replaced with \( Q_8 \) or \( P_2 + Q_8 \) we can prove that \( E/\langle Q_8 \rangle \langle \mathbb{Q} \rangle_{\text{tors}}, \text{and } E/\langle P_2 + Q_8 \rangle \langle \mathbb{Q} \rangle_{\text{tors}} \) are groups of order 2. Making note of the classification of isogeny-torsion graphs of \( T_8 \) type (see Table 5), we can conclude that no elliptic curve over \( \mathbb{Q} \) in the \( \mathbb{Q} \)-isogeny class has a point of order 4 defined over \( \mathbb{Q} \).

\[\square\]

Remark 4.15. Lemma 4.12 and Lemma 4.14 show that
For any elliptic curve $E/\mathbb{Q}$ with full two-torsion defined over $\mathbb{Q}$, there is a quadratic twist that eliminates all the points of order 4 defined over $\mathbb{Q}$ from all the elliptic curves over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class of $E$.

For an elliptic curve $E/\mathbb{Q}$ that has a cyclic, $\mathbb{Q}$-rational subgroup of even order, there is a quadratic twist that eliminates all the points defined over $\mathbb{Q}$ of order 3 and 5 but not points of order 4 of the elliptic curves over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class of $E$.

For an elliptic curve $E/\mathbb{Q}$ that has a non-trivial, cyclic, $\mathbb{Q}$-rational subgroup of odd order, there is a quadratic twist that eliminates all the points of order $\geq 3$ defined over $\mathbb{Q}$ from the elliptic curves over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class of $E$.

5. Isogeny-Torsion Graphs Corresponding to Finite Sets of $j$-Invariants

In this section, we determine the isogeny-torsion graphs that correspond to sets of finitely many $j$-invariants (without having to mention torsion configuration). This classification follows directly from Theorem 2.6. Each elliptic curve over $\mathbb{Q}$ represented by a vertex in the isogeny-torsion graphs in Table 1 corresponds to a non-cuspidal, $\mathbb{Q}$-rational point on a modular curve $X_0(N)$ of genus $\geq 1$ for some positive integer $N$ which has finitely many non-cuspidal, $\mathbb{Q}$-rational points (see Theorem 2.6). The $j$-invariants in Table 1 came from the work in [7].

**Proposition 5.1.** Let $\mathcal{G}$ be an isogeny-torsion graph of one of the following types (regardless of torsion configuration)

1. $L_2(p)$ where $p = 11, 17, 19, 37, 43, 67, \text{ or } 163$,
2. $L_4$,
3. $R_4(pq)$ where $pq = 14, 15, \text{ or } 21$.

Then $\mathcal{G}$ corresponds to a finite set of $j$-invariants.

5.1. Isogeny-Torsion Graphs of $L_2(p)$ Type Corresponding to Finite Sets of $j$-Invariants.

Let $E/\mathbb{Q}$ be an elliptic curve and let $p$ equal 11, 17, 19, 37, 43, 67, or 163. If $E$ has a $\mathbb{Q}$-rational subgroup of order $p$, then $C(E) = C_p(E) = 2$ by Theorem 2.8. In other words, the $\mathbb{Q}$-isogeny class of $E$ contains two elliptic curves over $\mathbb{Q}$ and both of those elliptic curves have a $\mathbb{Q}$-rational subgroup of order $p$ and no other non-trivial, cyclic, $\mathbb{Q}$-rational subgroups. As a consequence, both elliptic curves have trivial rational torsion. By Theorem 2.6, there are finitely many $j$-invariants corresponding to such elliptic curves.

5.2. Isogeny-Torsion Graphs of $L_4$ Type. Let $E/\mathbb{Q}$ be an elliptic curve such that either $E$ has a cyclic, $\mathbb{Q}$-rational subgroup of order 27 or $E$ has a cyclic, $\mathbb{Q}$-rational subgroup of order 9 and an independent $\mathbb{Q}$-rational subgroup of order 3. In both cases, the isogeny-torsion graph attached to the $\mathbb{Q}$-isogeny class of $E$ is of $L_4$ type. In the first case, the $j$-invariant of $E$ is $-12288000$ and in the second case, the $j$-invariant is 0 (both of which are CM $j$-invariants).

**NB** Not every elliptic curve of $j$-invariant 0 fits into an isogeny graph of $L_4$ type; only the elliptic curves with Weierstrass model $y^2 = x^3 + 16t^3$ for some non-zero, square-free, rational $t$.

5.3. Isogeny-Torsion Graphs Of $R_4(pq)$ Type Corresponding to Finite Sets of $j$-Invariants.

Let $E/\mathbb{Q}$ be an elliptic curve and let $pq = 14, 15, \text{ or } 21$. Suppose $E$ has a cyclic, $\mathbb{Q}$-rational subgroup of order $pq$. By Theorem 2.8, there are four elliptic curves over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class of $E$ and moreover, every elliptic curve over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class has a cyclic, $\mathbb{Q}$-rational subgroup of...
order \(pq\). There are finitely many \(j\)-invariants corresponding to elliptic curves over \(\mathbb{Q}\) with a cyclic, \(\mathbb{Q}\)-rational subgroup of order 14, 15, or 21 by Theorem 2.6. This concludes the proof of Proposition 5.1.

6. ISOGENY-TORSION GRAPHS CORRESPONDING TO INFINITE SETS OF \(j\)-INVARANTS

6.1. Methodology Of The Remaining Proofs. Table 1 lists the \(j\)-invariants corresponding to the 15 isogeny-torsion graphs corresponding to finite sets of \(j\)-invariants. It remains to prove that each of the other 37 isogeny-torsion graphs correspond to infinite sets of \(j\)-invariants.

The following proofs are of a mostly group theoretic flavor. To prove that an isogeny-torsion graph \(\mathcal{G}\) corresponds to an infinite set of \(j\)-invariants, we will focus on proving a nicely chosen vertex on \(\mathcal{G}\) corresponds to an infinite set of \(j\)-invariants. To do this, we establish the necessary algebraic properties for an elliptic curve in a \(\mathbb{Q}\)-isogeny class to be represented by the ideal vertex of \(\mathcal{G}\). By algebraic properties, we also mean the data afforded by the images of the mod-\(N\) Galois representation for some prime powers \(N\).

Once we have the necessary mod-\(N\) data corresponding to the ideal vertex on \(\mathcal{G}\), we make note if the modular curve that parametrizes such an elliptic curve has infinitely many non-cuspidal, \(\mathbb{Q}\)-rational points. Here is where the results of Rouse–Zureick-Brown [9] and Sutherland–Zywina [12] come in to determine some of the isogeny-torsion graphs that correspond to infinite sets of \(j\)-invariants. At times, the results of [9] and [12] are not enough to conclude that \(\mathcal{G}\) corresponds to an infinite set of \(j\)-invariants. But other methods like the ones described in Section 3.1 or Theorem 2.6 can be

| Graph  | Isogeny Graph | Torsion       | \(j\)       |
|--------|---------------|---------------|------------|
| \(E_1\) \(\overset{p}{\rightarrow}\) \(E_2\) | \(L_4(11)\) | \([1],[3]\) | \(-11 \cdot 131^3\) |
|        | \(L_5(17)\) | \([1],[4]\) | \(-17^2 \cdot 101^{1/2}\) |
|        | \(L_5(19)\) | \([1],[6]\) | \(-17 \cdot 37^3 / 2^2\) |
|        | \(L_5(37)\) | \([1],[3]\) | \(-17^2 \cdot 11^2\) |
|        | \(L_6(43)\) | \([1],[3]\) | \(-2^2 \cdot 3^3 \cdot 5^3\) |
|        | \(L_6(67)\) | \([1],[3]\) | \(-213 \cdot 3^3 \cdot 5^3 \cdot 13^3\) |
|        | \(L_6(163)\) | \([1],[4]\) | \(-2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3\) |

| Graph  | Isogeny Graph | Torsion       | \(j\)       |
|--------|---------------|---------------|------------|
| \(E_1\) \(\overset{3}{\rightarrow}\) \(E_2\) \(\overset{3}{\rightarrow}\) \(E_3\) \(\overset{3}{\rightarrow}\) \(E_4\) | \(L_4\) | \([6],[3],[3],[1]\) | \(-2^{15} \cdot 3^3 \cdot 5^3\) |
| \(q\)  | \(R_4(14)\) | \([2],[2],[2],[2]\) | \(-3^3 \cdot 5^3\) |
|        | \([5],[5],[5],[1]\) | \(-2^5 / 2\) |
| \(R_4(15)\) | \([3],[3],[3],[1]\) | \(-5^3 \cdot 241^{1/2}\) |
| \(E_3\) \(\overset{p}{\rightarrow}\) \(E_4\) | \(R_4(21)\) | \([3],[3],[3],[1]\) | \(-3^3 \cdot 5^3 / 2\) |
|        | \([1],[1],[1],[1]\) | \(-3^3 \cdot 5^3 \cdot 101^{1/2}\) |
|        | \([1],[1],[1],[1]\) | \(-3^3 \cdot 5^3 \cdot 383^{1/2}\) |

Table 1. Isogeny-torsion graphs corresponding to finitely many \(j\)-invariants
used if the results of [9] and [12] are insufficient. Once we have that a vertex of an isogeny-torsion graph with some torsion configuration corresponds to an infinite set of \( j \)-invariants, we take quadratic twists. Taking quadratic twists of all elliptic curves over \( \mathbb{Q} \) in a \( \mathbb{Q} \)-isogeny class does not change the isogeny graph but it very likely changes the isogeny-torsion graph. Usually, a well chosen quadratic twist will toggle between the possible torsion configurations of an isogeny graph. From this fact, it should be noted that one \( j \)-invariant may correspond to more than one isogeny-torsion graph. As the isogeny-torsion graph with one type of torsion configuration corresponds to an infinite set of \( j \)-invariants, the isogeny-torsion graph with torsion configuration that comes from quadratic twisting the elliptic curve represented by the ideal vertex also corresponds to an infinite set of \( j \)-invariants. Finally, at times, it will be necessary to exclude 0, 1728, and the \( j \)-invariants corresponding to the isogeny graphs from Table 1. As the complement of an infinite set with a finite subset is still an infinite set, no issues arise from this exclusion.

The philosophy of this article has been a group theoretic approach to solving questions regarding elliptic curves. Theorems 2.6 and 2.8, and work in [9] and [12] surely make use of the arithmetic and geometric properties of elliptic curves defined over \( \mathbb{Q} \). But aside from that, our proofs appeal only to Galois theory.

7. ISOGONY-TORSION GRAPHS OF \( S \) TYPE

In this section we prove that each of the isogeny-torsion graphs of \( S \) type correspond to infinite sets of \( j \)-invariants.

**Proposition 7.1.** Let \( G \) be one of the four isogeny-torsion graphs of \( S \) type (regardless of torsion configuration). Then \( G \) corresponds to an infinite set of \( j \)-invariants.

Once we prove that the isogeny-torsion graph corresponding to an elliptic curve over \( \mathbb{Q} \) with rational 12-torsion corresponds to an infinite set of \( j \)-invariants, we can use quadratic twists to prove the other three isogeny-torsion graphs of \( S \) type correspond to infinite set of \( j \)-invariants. Thus, let us first talk about quadratic twists of elliptic curves over \( \mathbb{Q} \) with rational 12-torsion.

### Table 2. Isogeny-Torsion Graphs of \( S \) type

| Isogeny Graph | Type | Isomorphism Types | Label |
|---------------|------|------------------|-------|
| \( E_3 \) - 3 \( E_4 \) \( \begin{array}{c} 2 \end{array} \) \( E_1 \) - 3 \( E_2 \) \( \begin{array}{c} 2 \end{array} \) \( E_5 \) - 3 \( E_6 \) - 3 \( E_8 \) \( \begin{array}{c} 2 \end{array} \) | \( S \) | \(([2,6],[2,2],[12],[4],[6],[2],[6],[2])\) | \( S^1 \) |
| \( ([2,6],[2,2],[6],[2],[6],[2],[6],[2]) \) | \( S^2 \) |
| \( ([2,2],[2,2],[4],[2],[4],[2],[2],[2]) \) | \( S^3 \) |
| \( ([2,2],[2,2],[2],[2],[2],[2],[2],[2]) \) | \( S^4 \) |

7.1. **Quadratic Twists of Elliptic Curves with Rational 12-Torsion.** To start, we note that there are infinitely many \( j \)-invariants corresponding to elliptic curves over \( \mathbb{Q} \) with a point of order 12 defined over \( \mathbb{Q} \). For example, let \( E_{a,b}(t) : y^2 + (1-a)xy - by = x^3 - bx^2 \) such that

\[
a = \frac{t(1-2t)(3t^2-3t+1)}{(t-1)^3} \quad \text{and} \quad b = -a \frac{2t^2-2t+1}{t-1}.
\]
Let \( t \in \mathbb{Q} \) such that \( E_{a,b}(t) \) is an elliptic curve. Then \( E_{a,b}(t)(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/12\mathbb{Z} \) (see Appendix E of [6]). There is only a single isogeny-torsion graph containing a point of order 12, namely, \( S^1 \) (see Table 2). Hence, \( S^1 \) corresponds to an infinite set of \( j \)-invariants. Let \( X \) be the modular curve that parametrizes elliptic curves defined over \( \mathbb{Q} \) with a point of order 12 defined over \( \mathbb{Q} \). In other words, \( X \) is generated by \( H = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix} \subseteq \text{GL}(2, \mathbb{Z}/12\mathbb{Z}) \). The genus of \( X \) is equal to 0 and \( X \) has infinitely many points defined over \( \mathbb{Q} \). Reducing \( H \) modulo 4, we get the group \( \mathcal{B}_4 \) which is the group of all elements of \( \text{GL}(2, \mathbb{Z}/4\mathbb{Z}) \) of the form \( \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} \) and reducing \( H \) modulo 3, we get the group \( \mathcal{B}_3 \) which is the group of all elements of \( \text{GL}(2, \mathbb{Z}/3\mathbb{Z}) \) of the form \( \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} \). Let \( K_8 \) be the full lift of \( \mathcal{B}_4 \) to level 8.

By Hilbert’s irreducibility theorem, there are infinitely many \( j \)-invariants corresponding to elliptic curves \( E/\mathbb{Q} \) such that the image of the mod-24 Galois representation attached to \( E \) is conjugate to \( H = K_8 \times \mathcal{B}_3 \). In such a case, \( \mathbb{Q}(E[8]) \cap \mathbb{Q}(E[3]) = \mathbb{Q} \). For more on non-trivial entanglement of division fields, see work in [2] and [3]. In the case that \( p_{E,12}(G_{\mathbb{Q}}) \) is conjugate to \( H \), the quadratic subfield of \( \mathbb{Q}(E[3]) \) is \( \mathbb{Q}(\sqrt{-3}) \) by Corollary 4.8.

Now we will prove that the isogeny-torsion graphs \( S^2 \), \( S^3 \), and \( S^4 \) each correspond to infinite sets of \( j \)-invariants.

- \( S^2 \)
  - Let \( E/\mathbb{Q} \) be an elliptic curve such that \( E \) has a point of order 12 defined over \( \mathbb{Q} \) and the image of the mod-24 Galois representation attached to \( E \) is conjugate to \( H \). Let \( E^{(-3)} \) be the quadratic twist of \( E \) by \(-3\). Twisting by \(-3\) shifts the points of order 3 to the other side of the graph. Hence, four of the elliptic curves over \( \mathbb{Q} \) in the isogeny-torsion graph have points of order 3 defined over \( \mathbb{Q} \). As \( \sqrt{-3} \) is not in \( \mathbb{Q}(E[8]) \), we have that the image of the mod-8 Galois representation attached to \( E^{(-3)} \) contains \(-\text{Id}\). Let \( E'/\mathbb{Q} \) be the elliptic curve that is 2-isogenous to \( E \). Then \( E' \) has full two-torsion defined over \( \mathbb{Q} \) and the image of the mod-4 Galois representation attached to \( E' \) contains \(-\text{Id}\). By Lemma 4.14, no elliptic curve over \( \mathbb{Q} \) in the \( \mathbb{Q} \)-isogeny class of \( E' \) has a point of order 4 defined over \( \mathbb{Q} \). Thus, the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E^{(-3)} \) is \( S^2 \).

- \( S^3 \)
  - Let \( E/\mathbb{Q} \) be an elliptic curve such that \( E \) has a point of order 12 defined over \( \mathbb{Q} \) and the image of the mod-24 Galois representation attached to \( E \) is conjugate to \( H \). Let \( E^{(-1)} \) be the quadratic twist of \( E \) by \(-1\). Twisting by \(-1\) does not change the position of the points of order 4 on the isogeny-torsion graph. As \( \sqrt{-1} \) is not in \( \mathbb{Q}(E[-3]) \), we have that the image of the mod-3 Galois representation attached to \( E^{(-1)} \) contains \(-\text{Id}\). By Corollary 4.13, no elliptic curve in the \( \mathbb{Q} \)-isogeny class of \( E^{(-1)} \) contains a point of order 3 defined over \( \mathbb{Q} \). Thus, the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E^{(-1)} \) is \( S^3 \).

- \( S^4 \)
  - Let \( E/\mathbb{Q} \) be an elliptic curve such that \( E \) has a point of order 12 defined over \( \mathbb{Q} \) and the image of the mod-24 Galois representation attached to \( E \) is conjugate to \( H \). Let \( q \) be a prime number such that \( \sqrt{q} \) is not contained in \( \mathbb{Q}(E[8]) \) and \( \sqrt{q} \) is not contained in \( \mathbb{Q}(E[3]) \). Let \( E^{(q)} \) be the quadratic twist of \( E \) by \( q \). Then the image of the mod-8 Galois representation attached to \( E^{(q)} \) contains \(-\text{Id}\) and the image of the mod-3 Galois representation attached to \( E^{(q)} \) contains \(-\text{Id}\). Thus, no elliptic curve that is \( \mathbb{Q} \)-isogenous to \( E^{(q)} \) contains a point of order 12.
3 or 4 defined over \( \mathbb{Q} \). Hence, the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E^{(q)} \) is \( S^4 \).

All isogeny-torsion graphs of type \( S \) are simply twists of \( S^1 \). As \( S^1 \) corresponds to an infinite set of \( j \)-invariants, so do \( S^2 \), \( S^3 \), and \( S^4 \). This proves Proposition 7.1.

8. Isogeny-Torsion Graphs of \( T_k \) Type

8.1. Isogeny-Torsion Graphs of \( T_4 \) Type. In this subsection, we prove that each of the three isogeny-torsion graphs of \( T_4 \) type correspond to infinite sets of \( j \)-invariants.

Proposition 8.1. Let \( G \) be any one of the three isogeny-torsion graphs of \( T_4 \) type (regardless of torsion configuration). Then \( G \) corresponds to an infinite set of \( j \)-invariants.

We will prove this proposition case by case. The methodology is finding the image of the \( \mathbb{Q} \)-adic Galois representation attached to the elliptic curve in the \( \mathbb{Q} \)-isogeny class with full two-torsion defined over \( \mathbb{Q} \) for each of the three isogeny-torsion graphs of \( T_4 \) type. Using the RZB database, we show that each of these subgroups of \( \text{GL}(2,\mathbb{Z}/4\mathbb{Z}) \) define modular curves with infinitely many non-cuspidal, \( \mathbb{Q} \)-rational points. Examples of three such modular curves from the RZB database, SZ database, and LMFDB are provided when possible.

### Table 3. Isogeny-Torsion Graphs of \( T_4 \) Type

| Graph Type | Type | Isomorphism Types | Label | RZB, LMFDB, SZ |
|------------|------|------------------|-------|----------------|
| \( T_4^1 \) | \( T_4 \) | \( ([2, 2], [4], [4], [2]) \) | \( T_{24}^1 \) | \( H_{24e}, 4.24.0.7, — \) |
| \( T_4^2 \) | \( T_4 \) | \( ([2, 2], [4], [2], [2]) \) | \( T_{24}^2 \) | \( H_{24d}, 4.24.0.4, — \) |
| \( T_4^3 \) | \( T_4 \) | \( ([2, 2], [2], [2], [2]) \) | \( T_{24}^3 \) | \( H_{24}, 4.12.0.4, 4E^0 - 4b \) |

- \( T_4^1 \)

Let \( E/\mathbb{Q} \) be an elliptic curve with full two-torsion defined over \( \mathbb{Q} \). Suppose the image of the mod-4 Galois representation attached to \( E \) is conjugate to

\[
H_{24e} = \left\{ \text{Id}, \left( \begin{array}{cc} 3 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 2 \\ 2 & 3 \end{array} \right), \left( \begin{array}{cc} 3 & 2 \\ 2 & 3 \end{array} \right) \right\}.
\]

We claim that the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E \) is \( T_4^1 \).

A Magma computation reveals that \( H_{24e} \) is not conjugate to a subgroup of \( \text{GL}(2,\mathbb{Z}/4\mathbb{Z}) \) consisting of upper-triangular matrices. Hence \( E \) does not have a cyclic, \( \mathbb{Q} \)-rational subgroup of order 4. In the course of the proof in Section 13 of [1] to eliminate the two “elusive” non-examples of isogeny-torsion graphs of \( S \) type, it was shown that if the image of the mod-4 Galois representation attached to \( E \) is conjugate to a subgroup of \( \langle H_{24e}, -\text{Id} \rangle \), then \( E \) does.
not contain a $\mathbb{Q}$-rational subgroup of order 3. Thus, the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is of type $T_4$.

We must prove the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $T_4^1$. To do this, we will show that $E$ is $\mathbb{Q}$-isogenous to two distinct elliptic curves over $\mathbb{Q}$ with rational 4-torsion (see Table 3). Let us pick a basis $\{P_4, Q_4\} \in E[4]$ such that the image of the mod-4 Galois representation attached to $E$ is $H_{24e}$. Let $\phi_1: E \to E/\langle P_2 + Q_2 \rangle$ and $\phi_2: E \to E/\langle P_2 \rangle$ be isogenies with kernel generated by $P_2 + Q_2$ and $P_2$ respectively. Note that for each $\sigma \in G_4$, $\sigma(Q_4) = Q_4$ or $P_2 + [3]Q_4$. Hence $\sigma(Q_4) - Q_4 \in \langle P_2 + Q_2 \rangle$ for all $\sigma \in G_4$. By Lemma 4.1, $\phi_1(Q_4)$ is a point of order 4 defined over $\mathbb{Q}$. Also, adding the two column vectors in each element of $H_{24e}$, we see that for each $\sigma \in G_4$, $\sigma(P_4 + Q_4) = P_4 + Q_4$ or $[3]P_4 + Q_4$. Hence, $\sigma(P_4 + Q_4) - (P_4 + Q_4) \in \langle P_2 \rangle$ for all $\sigma \in G_4$. By Lemma 4.1, $\phi_2(P_4 + Q_4)$ is a point of order 4 defined over $\mathbb{Q}$. This is enough to prove that the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $T_4^1$.

Elliptic curves over $\mathbb{Q}$ whose transpose of the image of the mod-4 Galois representation is conjugate to a subgroup of $H_{24e}$ are in one-to-one correspondence with non-cuspidal, $\mathbb{Q}$-rational points on the modular curve $X_{24e}$ from the list compiled in [9]. The modular curve $X_{24e}$ is a genus 0 curve and has infinitely many non-cuspidal, $\mathbb{Q}$-rational points. By Proposition 3.4, there are infinitely many $j$-invariants corresponding to elliptic curves over $\mathbb{Q}$ such that the image of the mod-4 Galois representation is conjugate to $H_{24e}$ (not a proper subgroup of $H_{24e}$). Thus $T_4^1$ corresponds to an infinite list of $j$-invariants.

\[ T_4^2 \]

Let $E/\mathbb{Q}$ be an elliptic curve over $\mathbb{Q}$ such that the image of the mod-4 Galois representation attached to $E$ is conjugate to

\[ H_{24d} = \left\{ I, \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 2 & 1 & 2 \end{array} \right), \left( \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 2 \end{array} \right) \right\}. \]

In the course of the proof in Section 13 of [1] to eliminate the two “elusive” non-examples of isogeny-torsion graphs of $S$ type, it was shown that if the image of the mod-4 Galois representation attached to $E$ is conjugate to a subgroup of $\langle H_{24d}, \text{Id} \rangle = \langle H_{24e}, \text{Id} \rangle$, then $E$ does not contain a $\mathbb{Q}$-rational subgroup of order 3. A Magma computation reveals that $H_{24d}$ is not conjugate to a subgroup of $\text{GL}(2, \mathbb{Z}/4\mathbb{Z})$ consisting of upper-triangular matrices. Hence, $E$ does not have a cyclic, $\mathbb{Q}$-rational subgroup of order 4. Thus, the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is of type $T_4$. It remains to prove that $E$ is $\mathbb{Q}$-isogenous to one and only one elliptic curve over $\mathbb{Q}$ with rational 4-torsion.

Let $\{P_4, Q_4\}$ be a basis of $E[4]$ such that the image of the mod-4 Galois representation attached to $E$ is $P_2$. Let $\phi_1: E \to E/\langle P_2 \rangle$, $\phi_2: E \to E/\langle Q_2 \rangle$, and $\phi_3: E \to E/\langle P_2 + Q_2 \rangle$ be isogenies with kernels generated by $P_2, Q_2, \text{ and } P_2 + Q_2$ respectively. Note that for each $\sigma \in G_3$, $\sigma(P_4) = P_4$ or $P_4 + Q_4$. Hence, $\sigma(P_4) - P_4 \in \langle Q_2 \rangle$ for all $\sigma \in G_4$. By Lemma 4.1, $\phi_2(P_4)$ is defined over $\mathbb{Q}$ and thus, $E/\langle Q_2 \rangle (\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z}$. By Lemma 4.1, $\phi_1(Q_4)$ and $\phi_3(Q_4)$ are points of order 2 defined over $\mathbb{Q}$. The point $\phi_1(Q_4)$ lives in two cyclic subgroups of order 4, namely, $\langle \phi_1(Q_4) \rangle$ and $\langle \phi_1(P_4 + Q_4) \rangle$. Let $C_1$ be a Galois automorphism that maps $Q_4$ to $[3]Q_4$. Then $C_1(Q_4) - Q_4 = [3]Q_4 - Q_4 = [2] \notin \langle P_2 \rangle$. Hence, $\phi_1(Q_4)$ is not defined over $\mathbb{Q}$. Let $C_3$ be a Galois automorphism that maps $P_4$ to $P_4$ and $Q_4$ to $P_2 + Q_4$. Then $C_3(P_4 + Q_4) - (P_4 + Q_4) = P_2 + Q_2 \notin \langle P_2 \rangle$. Hence, $\phi_1(P_4 + Q_4)$ is not defined over $\mathbb{Q}$ and thus, $E/\langle P_4 \rangle (\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z}$. The point $\phi_3(Q_2)$ lives in two cyclic, $\mathbb{Q}$-rational groups.
of order 4, namely \(\langle \phi_3(Q_4) \rangle\) and \(\langle \phi_3(P_4) \rangle\). Note that \(\sigma_1(Q_4) - Q_4 = Q_2 \notin \langle P_2 + Q_2 \rangle\) and \(\sigma_3(P_4) - P_4 = P_2 + Q_2 - P_4 = Q_2 \notin \langle P_2 + Q_2 \rangle\). Hence, \(E/ \langle P_2 + Q_2 \rangle (\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z}\).

Elliptic curves over \(\mathbb{Q}\) such that the transpose of the image of the mod-4 Galois representation is conjugate to \(H_{24d}\) correspond to non-cuspidal, \(\mathbb{Q}\)-rational points on the modular curve \(X_{24d}\) in [9] which is a genus 0 curve with infinitely many non-cuspidal, \(\mathbb{Q}\)-rational points. Using 3.4, there are infinitely many \(j\)-invariants corresponding to elliptic curves over \(\mathbb{Q}\) such that the image of the mod-4 Galois representation is conjugate to \(H_{24d}\) (not a proper subgroup of \(H_{24d}\)). Hence, \(T_4^2\) corresponds to an infinite set of \(j\)-invariants.

\(\bullet\) \(T_4^3\)

Let \(E/\mathbb{Q}\) be an elliptic curve such that the image of the mod-4 Galois representation attached to \(E\) is conjugate to \(H_{24} = \langle H_{24e}, -\text{Id} \rangle = \langle H_{24d}, -\text{Id} \rangle\). Note that again, as proven in [1], that \(E\) does not have a \(\mathbb{Q}\)-rational subgroup of order 3. Also, \(E\) has full two-torsion defined over \(\mathbb{Q}\) and does not have a cyclic, \(\mathbb{Q}\)-rational subgroup of order 4. Thus, the isogeny-torsion graph associated to the \(\mathbb{Q}\)-isogeny class of \(E\) is of \(T_4\) type.

By Lemma 4.14, no elliptic curve over \(\mathbb{Q}\) that is \(\mathbb{Q}\)-isogenous to \(E\) has a point of order 4 defined over \(\mathbb{Q}\). Hence, the isogeny-torsion graph associated to \(E\) is \(T_4^3\). Note that elliptic curves over \(\mathbb{Q}\) such that the image of the mod-4 Galois representation is conjugate to a subgroup of \(H_{24}\) correspond to non-cuspidal, \(\mathbb{Q}\)-rational points on the modular curve defined by the group with label 4E\(^0\) - 4b in [12]. Elliptic curves over \(\mathbb{Q}\) such that the image of the mod-4 Galois representation is conjugate to a subgroup of the transpose of \(H_{24}\) correspond to non-cuspidal, \(\mathbb{Q}\)-rational points on the modular curve \(X_{24}\) in [9]. Both of these modular curves are genus 0 curves with infinitely many non-cuspidal, \(\mathbb{Q}\)-rational points. Applying Proposition 3.4, we can conclude that there are infinitely many \(j\)-invariants corresponding to elliptic curves over \(\mathbb{Q}\) such that the image of the mod-4 Galois representation is conjugate to \(H_{24}\). Thus, \(T_4^3\) corresponds to an infinite set of \(j\)-invariants.

This concludes the proof of Proposition 8.1.

**N.B.** This is not the only way to generate isogeny-torsion graphs of \(T_4\) type nor prove that they correspond to infinite sets of \(j\)-invariants. It is possible for example, to have an elliptic curve over \(\mathbb{Q}\) with Galois image mod-4 conjugate to the full inverse image of the trivial subgroup of \(\text{GL}(2, \mathbb{Z}/2\mathbb{Z})\) via the mod-4 reduction map from \(\text{GL}(2, \mathbb{Z}/4\mathbb{Z})\) and a surjective mod-3 Galois representation. The isogeny-torsion graph associated to a \(\mathbb{Q}\)-isogeny class containing such an elliptic curve is \(T_4^3\).

### 8.2. Isogeny-Torsion Graphs of \(T_6\) Type

In this subsection, we prove that each of the four isogeny-torsion graphs of \(T_6\) type correspond to infinite sets of \(j\)-invariants.

**Proposition 8.2.** Let \(G\) be an isogeny-torsion graph of type \(T_6\) (regardless of torsion configuration). Then \(G\) corresponds to an infinite set of \(j\)-invariants.

We will prove this proposition case by case. The methodology is finding the image of the mod-8 Galois representation attached one of the two elliptic curves in the \(\mathbb{Q}\)-isogeny class with full two-torsion defined over \(\mathbb{Q}\) for each of the four isogeny-torsion graphs of \(T_6\) type. Using the RZB database, we show that each of these subgroups of \(\text{GL}(2, \mathbb{Z}/8\mathbb{Z})\) define modular curves with infinitely many non-cuspidal, \(\mathbb{Q}\)-rational points. Examples of four such modular curves from the RZB database, LMFDB, and SZ database are provided when possible.
Let $E/\mathbb{Q}$ be an elliptic curve such that $E(\mathbb{Q})_{\text{tors}} = \langle P_2, Q_4 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. By Theorem 2.8, $E$ does not contain a non-trivial, cyclic, $\mathbb{Q}$-rational subgroup of odd order. Suppose the isogeny-torsion graph associated to $E$ is of type $T_6$. Then $E$ is represented in the isogeny-torsion graph in Table 4 by $E_1$. The isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $T_6^1$ if $E$ is $\mathbb{Q}$-isogenous to an elliptic curve over $\mathbb{Q}$ with a point of order 8 defined over $\mathbb{Q}$ and the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $T_6^2$ otherwise.

- $T_6^1$

  We may assume $E/\langle P_2 \rangle(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/8\mathbb{Z}$. Let $\phi: E \to E/\langle P_2 \rangle$ be an isogeny with kernel $\langle P_2 \rangle$. By Lemma 4.1, $\phi(Q_4)$ is a point of order 4 defined over $\mathbb{Q}$. The cyclic, $\mathbb{Q}$-rational subgroups of $E/\langle P_2 \rangle$ of order 8 that contain $\phi(Q_4)$ are $\langle \phi(Q_8) \rangle$ and $\langle \phi(P_4 + Q_8) \rangle$. Let us say that $\phi(Q_8)$ is defined over $\mathbb{Q}$. Then $\sigma(Q_8) - Q_8 \in \langle P_2 \rangle$ for all $\sigma \in G_{\mathbb{Q}}$ by Lemma 4.1. 

  For each Galois automorphism $\sigma \in G_{\mathbb{Q}}$, there exist integers $a, b, c \in \mathbb{Z}$ (depending on $\sigma$) such that $\sigma(P_8) = [2a + 1]P_8 + [2b]Q_8$ and $\sigma(Q_8) = [c]P_2 + Q_8$. Thus, the image of the mod-8 Galois representation attached to $E$ is conjugate to a subgroup of

  $$H_{98e} = \left\langle \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \right\rangle.$$ 

  Elliptic curves over $\mathbb{Q}$ whose transpose of the image of the mod-8 Galois representation is conjugate to a subgroup of $H_{98e}$ correspond to non-cuspidal, $\mathbb{Q}$-rational points on the modular curve in the list compiled in [9] labeled $X_{98e}$ which is a genus 0 modular curve with infinitely many non-cuspidal, $\mathbb{Q}$-rational points. By Proposition 3.4, there are infinitely many $j$-invariants corresponding to elliptic curves over $\mathbb{Q}$ such that the image of the mod-8 Galois representation is conjugate to $H_{98e}$ itself (not a proper subgroup of $H_{98e}$). The image of the mod-8 Galois representation being conjugate to $H_{98e}$ guarantees that the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is precisely $T_6^1$. Thus, $T_6^1$ corresponds to an infinite set of $j$-invariants.

- $T_6^2$

  Let $E/\mathbb{Q}$ be an elliptic curve such that the image of the mod-8 Galois representation associated to $E$ is conjugate to

  $$H_{98h} = \left\langle \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \right\rangle.$$ 

  Note that $H_{98h}$ is a quadratic twist of the group $H_{98e}$ (multiply the first generator of $H_{98e}$ by $-\text{Id}$ to get $H_{98h}$). The image of the mod-8 Galois representation attached to $E$ is generated by $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in G_{\mathbb{Q}}$ such that $\sigma_i$ is represented by the $i$-th generator in $H_{98h}$. For example,
σ₁(P₈) = [5]P₈ and fixes Q₈. We claim that $E/\langle Q₄ \rangle_{\text{tors}} \cong E/\langle P₂ + Q₄ \rangle_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z}$ and that would be enough to conclude that the isogeny-torsion graph associated to the Q-isogeny class of E is $T₆^2$ (see Table 4).

To generalize, let $A$ equal $Q₄$ or $P₂ + Q₄$. We will prove that $E/\langle A \rangle_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z}$. Let $φ: E \to E/\langle A \rangle$ be an isogeny with kernel $\langle A \rangle$. Note that $σ_i(P₄)ₙ = 0$ or $σ_i(P₄)ₙ = P₂$ for all $i = 1, 2, 3, 4$ and thus $σ_i(P₄)ₙ - P₄ \in \langle Q₂ \rangle$ for all $σ \in Gₘ$. By Lemma 4.1, $φ(P₄)$ is a point of order 4 defined over $\mathbb{Q}$. The group $E/\langle A \rangle_{\text{tors}}$ is cyclic of order 4 or 8 by Lemma 4.2 and Lemma 4.3. We need to prove that it is not of order 8.

Let $B \in E$ such that $B = Q₈$ if $A = Q₄$ and $B = P₄ + Q₈$ if $A = P₂ + Q₄$. Note that $[2]B = A$. By Lemma 4.3, $φ(B)$ is a point of order 2 that is not defined over $\mathbb{Q}$. The cyclic groups of order 8 that contain $φ(P₄)$ are $\langle φ(P₄) \rangle$ and $\langle φ(P₈ + B) \rangle$. Note that $σ₁(P₈)ₙ - P₈ = [4]P₂ = P₂ \not\in \langle A \rangle$. Also, $σ₁$ fixes $B$, so we have that $σ₁(P₈ + B) = P₂ \not\in \langle A \rangle$. Thus, both $φ(P₈)$ and $φ(P₈ + B)$ are not defined over $\mathbb{Q}$ by Lemma 4.1. Thus, $E/\langle Q₄ \rangle_{\text{tors}} \cong E/\langle P₂ + Q₄ \rangle_{\text{tors}} = \mathbb{Z}/4\mathbb{Z}$ and so, the isogeny-torsion graph associated to the Q-isogeny class associated to $E$ is $T₆^2$.

Elliptic curves over $\mathbb{Q}$ such that the transpose of the image of the mod-8 Galois representation is conjugate to a subgroup of $H_{g₈ₖ}$, correspond to non-cuspidal, Q-rational points on the modular curve that appears in the list compiled in [9] with label $X_{g₈ₖ}$ which is a genus 0 curve with infinitely many non-cuspidal, Q-rational points. By Proposition 3.4, there are infinitely many j-invariants corresponding to elliptic curves over $\mathbb{Q}$ such that the image of the mod-8 Galois representation is conjugate to $H_{g₈ₖ}$ itself (not a proper subgroup of $H_{g₈ₖ}$). The image of the mod-8 Galois representation being conjugate to $H_{g₈ₖ}$ guarantees that the isogeny-torsion graph associated to the Q-isogeny class of E is precisely $T₆^2$. Thus, $T₆^3$ corresponds to an infinite set of j-invariants.

Now let $E/\mathbb{Q}$ be an elliptic curve such that $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the isogeny-torsion graph associated to $E$ is of type $T₆$ but this time, $E$ is not Q-isogenous to an elliptic curve over $\mathbb{Q}$ with torsion subgroup of order 8. Again, we may assume without loss of generality that $E$ is represented in the isogeny-torsion graph in Table 4 by $E₁$. If $E$ is Q-isogenous to an elliptic curve over $\mathbb{Q}$ with a point of order 4 defined over $\mathbb{Q}$, then the isogeny-torsion graph associated to the Q-isogeny class of $E$ is $T₆^3$. If $E$ is not Q-isogenous to an elliptic curve over $\mathbb{Q}$ with a point of order 4 defined over $\mathbb{Q}$, then the isogeny-torsion graph associated to the Q-isogeny class of $E$ is $T₆^4$.

• $T₆^3$

Let $E/\mathbb{Q}$ be an elliptic curve such that the image of the mod-8 Galois representation attached to $E$ is conjugate to

$H_{g₈₆} = \left\{ \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 4 \\ 0 & 7 \end{pmatrix} \right\}$.

Note that $H₃$ is a quadratic twist of $H_{g₈₆}$ (multiply the fourth generator of $H$ by -Id). The image of the mod-8 Galois representation attached to $E$ is generated by $σ₁, σ₂, σ₃, σ₄ \in Gₘ$ such that $σ₁(P₈) = [3]P₈$, etc. We claim that the isogeny-torsion graph associated to the Q-isogeny class of $E$ is $T₆^3$.

Let $φ: E \to E/\langle P₂ + Q₄ \rangle$ be an isogeny with kernel $\langle P₂ + Q₄ \rangle$. Note that the group $E/\langle P₂ + Q₄ \rangle_{\text{tors}}$ is cyclic of even order by Lemma 4.3 and Lemma 4.2. We claim that $φ(Q₈)$ is a point of order 4 defined over $\mathbb{Q}$. Note that $σ₄(Q₈) - Q₈ = P₂ + [7]Q₈ − Q₈ = P₂ + [3]Q₄ = [3](P₂ + Q₄) \in \langle P₂ + Q₄ \rangle$. The other generators fix $Q₈$ and hence, $φ(Q₈)$ is a
point of order 4 defined over \( \mathbb{Q} \) by Lemma 4.1 and so \( E/\langle P_2 + Q_4 \rangle (\mathbb{Q})_{\text{tors}} \) is cyclic of order 4 or 8.

Now let \( \phi': E \to E/\langle Q_4 \rangle \) be an isogeny with kernel \( \langle Q_4 \rangle \). The group \( E/\langle Q_4 \rangle (\mathbb{Q})_{\text{tors}} \) is cyclic of even order by Lemma 4.2 and Lemma 4.3. The point \( \phi'(P_2) \) is of order 2 and is defined over \( \mathbb{Q} \) by Lemma 4.1. The cyclic subgroups of \( E/\langle Q_4 \rangle \) of order 4 that contain \( \phi'(P_2) \) are \( \langle \phi'(P_3) \rangle \) and \( \langle \phi'(P_4 + Q_8) \rangle \). We need to prove both of these groups of order 4 are not generated by points defined over \( \mathbb{Q} \).

First we prove that \( \phi'(P_4) \) is not defined over \( \mathbb{Q} \). Observe that \( \sigma_1(P_8) = [3]P_8 \) and hence, \( \sigma_1(P_4) = [3]P_4 \). From this, we can say \( \sigma_1(P_4) - P_4 = [3]P_4 - P_4 = P_2 \notin \langle Q_4 \rangle \). Thus, \( \phi'(P_4) \) is not defined over \( \mathbb{Q} \) by Lemma 4.1. Note that \( \sigma_1 \) fixes \( Q_8 \). Thus, \( \sigma_1(P_4 + Q_8) - (P_4 + Q_8) = P_2 \notin \langle Q_4 \rangle \). From Lemma 4.1, we can conclude that \( \phi'(P_4 + Q_8) \) is not defined over \( \mathbb{Q} \). Thus, \( E/\langle P_2 + Q_4 \rangle (\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \) and this forces \( E/\langle Q_2 \rangle (\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z} \) and for the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E \) to be \( T_6^3 \) (see Table 4).

Elliptic curves over \( \mathbb{Q} \) whose transpose of the image of the mod-8 Galois representation is conjugate to a subgroup of \( H_3 \) correspond to non-cuspidal, \( \mathbb{Q} \)-rational points on the modular curve that appears in the list compiled in [9] with label \( X_{98g} \), which is a genus 0 curve with infinitely many non-cuspidal, \( \mathbb{Q} \)-rational points. By Proposition 3.4, there are infinitely many \( j \)-invariants corresponding to elliptic curves over \( \mathbb{Q} \) such that the image of the mod-8 Galois representation is conjugate to \( H_3 \) (not a proper subgroup of \( H_3 \)). The image of the mod-8 Galois representation being conjugate to \( H_3 \) guarantees that the isogeny-torsion graph attached to the \( \mathbb{Q} \)-isogeny class of \( E \) is \( T_6^3 \). Thus \( T_6^3 \) corresponds to an infinite set of \( j \)-invariants.

\( \bullet \ T_6^4 \)

Let \( E/\mathbb{Q} \) be an elliptic curve such that the image of the mod-8 Galois representation attached to \( E \) is conjugate to \( H_{98g} = \langle H_{98g}, -\text{Id} \rangle \). Then by Lemma 4.14, no elliptic curve over \( \mathbb{Q} \) in the \( \mathbb{Q} \)-isogeny class of \( E \) has a point of order 4 defined over \( \mathbb{Q} \). Hence, the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E \) is \( T_6^4 \).

The group \( H_{98} \) is conjugate to the group that appears in the list compiled in [12] with label \( 8.f^0 - 8c \) which defines a modular curve of genus 0 with infinitely many non-cuspidal, \( \mathbb{Q} \)-rational points. Elliptic curves over \( \mathbb{Q} \) such that the transpose of the image of the mod-8 Galois representation is conjugate to \( H_{98} \) correspond to non-cuspidal, \( \mathbb{Q} \)-rational points on the modular curve in [9] with label \( X_{98} \) which is a modular curve of genus 0 with infinitely many non-cuspidal, \( \mathbb{Q} \)-rational points. By Proposition 3.4 there are infinitely many \( j \)-invariants corresponding to elliptic curves over \( \mathbb{Q} \) such that the transpose of the image of the mod-8 Galois representation is conjugate to \( H_{98} \) (not a proper subgroup of \( H_{98} \)). The transpose of the image of the mod-8 Galois representation being conjugate to \( H_{98} \) guarantees that the isogeny-torsion graph attached to the \( \mathbb{Q} \)-isogeny class of \( E \) is \( T_6^4 \). Thus, \( T_6^4 \) corresponds to an infinite set of \( j \)-invariants.

This concludes the proof of Proposition 8.2.

8.3. Isogeny-Torsion Graphs of \( T_8 \) Type. In this subsection, we prove that each of the six isogeny-torsion graphs of \( T_8 \) type correspond to infinite sets of \( j \)-invariants.

**Proposition 8.3.** Let \( \mathcal{G} \) be an isogeny-torsion graph of type \( T_8 \) (regardless of torsion configuration). Then \( \mathcal{G} \) corresponds to an infinite set of \( j \)-invariants.
We will prove this proposition case by case. The methodology to proving $\mathcal{T}_8^1$ and $\mathcal{T}_8^2$ correspond to infinite sets of $j$-invariants is finding the image of the mod-8 Galois representation attached to one of the three elliptic curves in the $\mathbb{Q}$-isogeny class with full two-torsion defined over $\mathbb{Q}$. Using the RZB database, we show that each of these subgroups of $\text{GL}(2, \mathbb{Z}/8\mathbb{Z})$ define modular curves with infinitely many non-cuspidal, $\mathbb{Q}$-rational points.

For the isogeny-torsion graphs $\mathcal{T}_S^3$, $\mathcal{T}_S^4$, $\mathcal{T}_S^5$, and $\mathcal{T}_S^6$, it is most convenient to compute the image of the mod-16 Galois representation attached to one of the elliptic curves over $\mathbb{Q}$ with full two-torsion defined over $\mathbb{Q}$. Then, again, we will use the RZB database to show that each of these subgroups of $\text{GL}(2, \mathbb{Z}/16\mathbb{Z})$ define modular curves with infinitely many non-cuspidal, $\mathbb{Q}$-rational points. Examples of six such modular curves from the RZB database, LMFDB, and SZ database are provided when possible.

| Isogeny Graph | Type | Isomorphism Types | Label | RZB, LMFDB, SZ |
|---------------|------|------------------|-------|----------------|
| $E_3$ | $E_5$ | $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ | $\mathcal{T}_8^1$ | $H_{193n}$, 8.96.0.40, — |
| $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \end{pmatrix}$ | $\mathcal{T}_8^2$ | $H_{194l}$, 8.96.0.39, — |
| $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \end{pmatrix}$ | $\mathcal{T}_8^3$ | $H_{215c}$, 16.96.0.7, — |
| $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \end{pmatrix}$ | $\mathcal{T}_8^4$ | $H_{225l}$, 16.96.0.9, — |
| $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \end{pmatrix}$ | $\mathcal{T}_8^5$ | $H_{215b}$, 16.96.0.61, — |
| $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \end{pmatrix}$ | $\mathcal{T}_8^6$ | $H_{215c}$, 16.48.0.3, 16$G^{16} - 16b$ |

Table 5. $T_8$ Type Isogeny Graphs

Let $E/\mathbb{Q}$ be an elliptic curve such that the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is of $T_8$ type. By Theorem 2.8, $E$ does not contain a $\mathbb{Q}$-rational subgroup of odd prime order. There are six possibilities for the torsion configuration of the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ (see Table 5).

- $T_8^1$
  
  It is clear to see that the isogeny-torsion graph associated to a $\mathbb{Q}$-isogeny class of elliptic curves over $\mathbb{Q}$ is $T_8^1$ if and only if there is an elliptic curve over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class with a torsion subgroup of order 16 (see Table 5). Let $E/\mathbb{Q}$ be an elliptic curve such that $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$. The image of the mod-8 Galois representation attached to $E$ is conjugate to a subgroup of

$$H_{193n} = \left\{ \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\}.$$

Elliptic curves over $\mathbb{Q}$ such that the transpose of the image of the mod-8 Galois representation is conjugate to a subgroup of $H_{193n}$ are parametrized by the modular curve that appears in the list compiled in [9] with label $X_{193n}$ which is a genus 0 curve with infinitely many non-cuspidal, $\mathbb{Q}$-rational points. Thus, $T_8^1$ corresponds to an infinite set of $j$-invariants.

- $T_8^2$
  
  Let $E/\mathbb{Q}$ be an elliptic curve such that the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $T_8^2$. We may assume without loss of generality that $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and $E/\langle P_2 \rangle(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/8\mathbb{Z}$.
Let \( \phi: E \to E/\langle P_2 \rangle \) be an isogeny with kernel \( \langle P_2 \rangle \). Clearly, we have \( \phi(Q_4) \) is a point of order 4 defined over \( \mathbb{Q} \) by Lemma 4.1. The cyclic subgroups of \( E/\langle P_2 \rangle \) of order 8 that contain \( \phi(Q_4) \) are \( \langle \phi(Q_8) \rangle \) and \( \langle \phi(P_1 + Q_8) \rangle \). The group \( \langle Q_8 \rangle \) is \( \mathbb{Q} \)-rational. If \( \phi(Q_8) \) is defined over \( \mathbb{Q} \), then \( \sigma(Q_8) - Q_8 \in \langle P_2 \rangle \) for all \( \sigma \in G_\mathbb{Q} \) by Lemma 4.1. This would force \( Q_8 \) to be defined over \( \mathbb{Q} \), a contradiction. Hence, \( \phi(P_4 + Q_8) \) is defined over \( \mathbb{Q} \) and so, \( \sigma(P_4 + Q_8) - (P_4 + Q_8) \in \langle P_2 \rangle \) for all \( \sigma \in G_\mathbb{Q} \) by Lemma 4.1. In this case, the image of the mod-8 Galois representation attached to \( E \) is conjugate to a subgroup of

\[
H_{194l} = \left\langle \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 5 \end{pmatrix} \right\rangle.
\]

Elliptic curves over \( \mathbb{Q} \) such that the transpose of the image of the mod-8 Galois representation is conjugate to a subgroup of \( H_{194l} \) correspond to non-cuspidal, \( \mathbb{Q} \)-rational points on the modular curve that appears in the list compiled in [9] with label \( X_{194l} \) which is a genus 0 curve with infinitely many non-cuspidal, \( \mathbb{Q} \)-rational points. By Proposition 3.4, there are infinitely many \( j \)-invariants corresponding to elliptic curves over \( \mathbb{Q} \) such that the image of the mod-8 Galois representation is conjugate to \( H_{194l} \) itself (not a proper subgroup of \( H_{194l} \)). The image of the mod-8 Galois representation being conjugate to \( H_{194l} \) guarantees that the isogeny-torsion graph attached to the \( \mathbb{Q} \)-isogeny class of \( E \) is \( T_8^2 \). Thus, \( T_8^2 \) corresponds to an infinite set of \( j \)-invariants.

**\( T_8^3 \)**

Let \( E/\mathbb{Q} \) be an elliptic curve such that the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E \) is \( T_8^3 \). Then we may assume without loss of generality that \( E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \) and that \( E/\langle P_2 + Q_4 \rangle (\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/8\mathbb{Z} \). Let \( \phi: E \to E/\langle P_2 + Q_4 \rangle \) be an isogeny with kernel \( \langle P_2 + Q_4 \rangle \). By Lemma 4.1, \( \phi(Q_8) \) is a point of order 4 defined over \( \mathbb{Q} \). The cyclic subgroups of \( E/\langle P_2 + Q_4 \rangle \) of order 8 that contain \( \phi(Q_8) \) are \( \langle \phi(Q_{16}) \rangle \) and \( \langle \phi(P_4 + Q_{16}) \rangle \). Let us say \( \phi(Q_{16}) \) is defined over \( \mathbb{Q} \), so \( \sigma(Q_{16}) - Q_{16} \in \langle P_2 + Q_4 \rangle \) for all \( \sigma \in G_\mathbb{Q} \) by Lemma 4.1. Hence, the image of the mod-16 Galois representation attached to \( E \) is conjugate to a subgroup of

\[
H_{215c} = \left\langle \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 8 \\ 0 & 5 \end{pmatrix} \right\rangle.
\]

Elliptic curves over \( \mathbb{Q} \) such that the transpose of the image of the mod-16 Galois representation is conjugate to a subgroup of \( H_{215c} \) correspond to non-cuspidal, \( \mathbb{Q} \)-rational points on the modular curve that appears in the list compiled in [9] with label \( X_{215c} \) which is a genus 0 curve with infinitely many non-cuspidal, \( \mathbb{Q} \)-rational points. By Proposition 3.4, there are infinitely many \( j \)-invariants corresponding to elliptic curves over \( \mathbb{Q} \) with image of the mod-16 Galois representation conjugate to \( H_{215c} \) itself (not a proper subgroup of \( H_{215c} \)). The image of the mod-16 Galois representation attached to \( E \) being conjugate to \( H_{215c} \) guarantees the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E \) is \( T_8^3 \). Thus, \( T_8^3 \) corresponds to an infinite set of \( j \)-invariants.

**\( T_8^4 \)**

Let \( E/\mathbb{Q} \) be an elliptic curve such that the image of the mod-16 Galois representation attached to \( E \) is conjugate to

\[
H_{215l} = \left\langle \begin{pmatrix} 13 & 0 \\ 0 & 15 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 15 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 8 \\ 0 & 5 \end{pmatrix} \right\rangle.
\]
Note that $H_4$ is a quadratic twist of $H_{215c}$ (multiply the first two generators of $H_{215c}$ by -Id). We claim that the isogeny-torsion graph attached to the $\mathbb{Q}$-isogeny class of $E$ is $T_8^4$. Let $\sigma_i$ denote the $i$-th generator of $H_{215l}$ for $1 \leq i \leq 4$. Reducing the image of the mod-16 Galois representation attached to $E$ by 4, we get

$$\overline{H_{215l}} = \left\{ \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right) \right\}$$

and hence, $\sigma(P_4) - P_4 \in \langle Q_2 \rangle$ for all $\sigma \in G_4$. Let $\phi_1 : E \to E/\langle Q_8 \rangle$ be an isogeny with kernel generated by $Q_8$ and let $\phi_2 : E \to E/\langle P_2 + Q_8 \rangle$ be an isogeny with kernel generated by $P_2 + Q_8$. Let $\psi : E \to E/\langle P_2 + Q_4 \rangle$ be an isogeny with kernel generated by $P_2 + Q_4$. Note that $E/\langle Q_8 \rangle, E/\langle P_2 + Q_8 \rangle$, and $E/\langle P_2 + Q_4 \rangle$, are elliptic curves over $\mathbb{Q}$ with cyclic torsion subgroups of even order by Lemma 4.2 and Lemma 4.3. If we can prove that their respective torsion subgroups are all cyclic of order 4, then we would prove that the isogeny-torsion graph attached to the $\mathbb{Q}$-isogeny class of $E$ is $T_8^4$ (see Table 5).

Let us prove that $E/\langle Q_8 \rangle (\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z}$. As $\sigma(P_4) - P_4 \in \langle Q_2 \rangle$ for all $\sigma \in G_4$, $\phi_1(P_4)$ is a point of order 4 defined over $\mathbb{Q}$ by Lemma 4.1. The point $\phi_1(P_4)$ lives in the two cyclic groups of order 8, $\langle \phi_1(P_4) \rangle$ and $\langle \phi_1(P_8 + Q_16) \rangle$. Note that $\sigma_1(P_8) - P_8 = [13]P_8 - P_8 = [12]P_8 \notin \langle Q_8 \rangle$. Hence, $\phi_1(P_8)$ is not defined over $\mathbb{Q}$ by Lemma 4.1. Note that $\sigma_4(P_8 + Q_16) - (P_8 + Q_16) = P_8 + P_2 + [5]Q_16 - (P_8 + Q_16) = P_2 + [4]Q_16 \notin \langle Q_8 \rangle$. Thus, $\phi_1(P_8 + Q_16)$ is not defined over $\mathbb{Q}$ by Lemma 4.1. Hence, $E/\langle Q_8 \rangle (\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z}$. A similar computation shows that $E/\langle P_2 + Q_4 \rangle (\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z}$.

Now we will prove that $E/\langle P_2 + Q_4 \rangle (\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z}$. Note that again, $\psi(P_4)$ is a point of order 4 defined over $\mathbb{Q}$ by Lemma 4.1. The point $\psi(P_4)$ lives in two cyclic groups of order 8, $\langle \psi(P_4) \rangle$ and $\langle \psi(P_8 + Q_8) \rangle$. As $\sigma_1(P_8) - P_8 = [13]P_8 - P_8 = [12]P_8 \notin \langle P_2 + Q_4 \rangle$, we have that $\psi(P_8)$ is not defined over $\mathbb{Q}$ by Lemma 4.1. As $\sigma_2(P_8 + Q_8) - (P_8 + Q_8) = [9]P_8 + [15]Q_8 - (P_8 + Q_8) = [14]Q_8 = [3]Q_4 \notin \langle P_2 + Q_4 \rangle$, we have that $\psi(P_8 + Q_8)$ is not defined over $\mathbb{Q}$ by Lemma 4.1. This is enough to prove that the isogeny-torsion graph attached to the $\mathbb{Q}$-isogeny class of $E$ is $T_8^4$ (see Table 5).

The set of elliptic curves over $\mathbb{Q}$ such that the image of the mod-16 Galois representation is conjugate to a subgroup of the transpose of $H_{215l}$ corresponds to the non-cuspidal, $\mathbb{Q}$-rational points on the modular curve found in [9] with label $X_{215l}$. This modular curve is a genus 0 curve with infinitely many non-cuspidal, $\mathbb{Q}$-rational points. By Proposition 3.4, there are infinitely many $j$-invariants corresponding to elliptic curves over $\mathbb{Q}$ such that the image of the mod-16 Galois representation is conjugate to $H_{215l}$ (not a proper subgroup of $H_{215l}$). The image of the mod-16 Galois representation being conjugate to $H_{215l}$ guarantees that the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $T_8^4$. Thus, $T_8^4$ corresponds to an infinite set of $j$-invariants.

- $T_8^4$

Let $E/\mathbb{Q}$ be an elliptic curve such that the image of the mod-16 Galois representation attached to $E$ is conjugate to

$$H_{215k} = \left\{ \left( \begin{array}{cc} 13 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & 15 \end{array} \right), \left( \begin{array}{cc} 15 & 8 \\ 0 & 11 \end{array} \right) \right\}.$$
The group $H_{215k}$ is a quadratic twist of $H_{215c}$ (multiply the first, second, and fourth generators of $H_{215c}$ by -Id). We claim that the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $T_8^{5}$.

Let $\phi: E \rightarrow E/\langle Q_8 \rangle$ be an isogeny with kernel $\langle Q_8 \rangle$. The group $E/\langle Q_8 \rangle (\mathbb{Q})_{\text{tors}}$ is cyclic of even order by Lemma 4.3 and Lemma 4.2. The point of $E/\langle Q_8 \rangle$ of order 2 defined over $\mathbb{Q}$ is $\phi(P_2)$ by Lemma 4.1. The point $\phi(P_2)$ is contained in two cyclic groups of order 4, $\langle \phi(P_3) \rangle$ and $\langle \phi(P_3 + Q_{16}) \rangle$. The point $\phi(P_4 + Q_{16})$ is defined over $\mathbb{Q}$ if and only if $\sigma(P_4 + Q_{16}) - (P_4 + Q_{16}) \in \langle Q_8 \rangle$ for all $\sigma \in G_{\mathbb{Q}}$ by Lemma 4.1. In other words, for an arbitrary $\sigma \in G_{\mathbb{Q}}$

$$\sigma(P_4) = P_4 \text{ or } P_4 + Q_2 \Leftrightarrow \sigma(Q_{16}) = \{2b + 1\}Q_{16} \text{ for some } b \in \mathbb{Z}$$

$$\sigma(P_4) = [3]P_4 \text{ or } [3]P_4 + Q_2 \Leftrightarrow \sigma(Q_{16}) = P_2 + [2b + 1]Q_{16} \text{ for some } b \in \mathbb{Z}.$$  

Lifting $P_4$ to level 16, we have satisfy the following algebraic relations:

$$\sigma(P_{16}) = [1 + 4a]P_{16} + [2c]Q_{16} \Leftrightarrow \sigma(Q_{16}) = \{2b + 1\}Q_{16} \text{ for some } a,b,c \in \mathbb{Z}$$

$$\sigma(P_{16}) = [3 + 4a]P_{16} + [2c]Q_{16} \Leftrightarrow \sigma(Q_{16}) = P_2 + [2b + 1]Q_{16} \text{ for some } a,b,c \in \mathbb{Z},$$

That is how the elements of $H_{215k}$ behave! Hence, $\phi(P_4 + Q_{16})$ is defined over $\mathbb{Q}$, making $E/\langle Q_8 \rangle (\mathbb{Q})_{\text{tors}}$ a cyclic group of order 4 or 8.

Now let $\phi': E \rightarrow E/\langle P_2 + Q_8 \rangle$ be an isogeny with kernel $\langle P_2 + Q_8 \rangle$. Then the group $E/\langle P_2 + Q_8 \rangle (\mathbb{Q})_{\text{tors}}$ is a cyclic group of even order by Lemma 4.3 and Lemma 4.2. The point of $E/\langle P_2 + Q_8 \rangle$ of order 2 defined over $\mathbb{Q}$ is $\phi'(P_2)$ by Lemma 4.1. The point $\phi'(P_2)$ lives in two cyclic groups of order 4, $\langle \phi'(P_3) \rangle$ and $\langle \phi'(Q_{16}) \rangle$. We have that $\phi'(P_4)$ is defined over $\mathbb{Q}$ if and only if $\sigma(P_4) - P_4 \in \langle P_2 + Q_8 \rangle$ for all $\sigma \in G_{\mathbb{Q}}$ by Lemma 4.1. Let $\tau \in G_{\mathbb{Q}}$ such that $\tau(P_{16}) = [15]P_{16}$. Then $\tau(P_4) = [15]P_4 = [3]P_4$ and hence, $\tau(P_4) - P_4 = [2]P_4 = P_2$. As $P_2 \notin \langle P_2 + Q_8 \rangle$, $\phi'(P_4)$ is not defined over $\mathbb{Q}$ by Lemma 4.1. Similarly, we have that $\phi'(Q_{16})$ is defined over $\mathbb{Q}$ if and only if $\sigma(Q_{16}) - Q_{16} \in \langle P_2 + Q_8 \rangle$ for all $\sigma \in G_{\mathbb{Q}}$ by Lemma 4.1. Let $\tau \in G_{\mathbb{Q}}$ such that $\tau(Q_{16}) = [15]Q_{16}$. Then $\tau(Q_{16}) - Q_{16} = [14]Q_{16} = [7]Q_{8}$. As $[7]Q_8 \notin \langle P_2 + Q_8 \rangle$, we have that $\phi'(Q_{16})$ is not defined over $\mathbb{Q}$ by Lemma 4.1. This is enough to prove that $E/\langle P_2 + Q_8 \rangle (\mathbb{Q})_{\text{tors}}$ is cyclic of order 2. And thus, we may conclude that the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $T_8^{5}$ (see Table 5).

Elliptic curves over $\mathbb{Q}$ such that the transpose of the image of the mod-16 Galois representation is conjugate to a subgroup of $H_{215k}$ correspond to non-cuspidal, $\mathbb{Q}$-rational points on the modular curve found in [9] with label $X_{215k}$ which is a genus 0 curve with infinitely many non-cuspidal, $\mathbb{Q}$-rational points. By Proposition 3.4, there are infinitely many $j$-invariants corresponding to elliptic curves over $\mathbb{Q}$ such that the image of the mod-16 Galois representation is conjugate to $H_{215k}$ itself (not a proper subgroup of $H_{215k}$). The image of the mod-16 Galois representation being conjugate to $H_{215k}$ guarantees that the isogeny-torsion graph attached to the $\mathbb{Q}$-isogeny class of $E$ is $T_8^{5}$. Thus, $T_8^{5}$ corresponds to an infinite set of $j$-invariants.

$\ast$ $T_8^{6}$

Let $E/\mathbb{Q}$ be an elliptic curve such that the image of the mod-16 Galois representation attached to $E$ is conjugate to $H_{215} = (H_{215k}, -Id)$. Then by Lemma 4.14, there are no elliptic curves over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class of $E$ that have a point of order 4 defined over $\mathbb{Q}$. Hence, the isogeny-torsion graph of $E$ is $T_8^{6}$.

Elliptic curves over $\mathbb{Q}$ such that the image of the mod-16 Galois representation is conjugate to a subgroup of $H_{215}$ correspond to non-cuspidal, $\mathbb{Q}$-rational points on the modular curve defined by the group that appears in the list compiled in [12] with label $16G^0 - 16b$ which is a
genus 0 curve with infinitely many non-cuspidal, \( \mathbb{Q} \)-rational points. This is the same modular curve that appears in the list \([9]\) with label \(X_{215}\). By Proposition 3.4, there are infinitely many \(j\)-invariants corresponding to elliptic curves over \( \mathbb{Q} \) such that the image of the mod-16 Galois representation is conjugate to \(H_{215}\) itself (not a proper subgroup of \(H_{215}\)). The image of the mod-16 Galois representation attached to \(E\) being conjugate to \(H_{215}\) guarantees that the isogeny-torsion graph associated to the \(\mathbb{Q}\)-isogeny class of \(E\) is \(T_8^6\). Thus, \(T_8^6\) corresponds to an infinite set of \(j\)-invariants.

This concludes the proof of Proposition 8.3. Also, this concludes the proof that all isogeny-torsion graphs containing an elliptic curve over \(\mathbb{Q}\) with full two-torsion defined over \(\mathbb{Q}\) correspond to infinite sets of \(j\)-invariants.

9. Isogeny Graphs of \(R_6\) Type

9.1. Isogeny-Torsion Graphs of \(R_6\) Type. In this subsection, we prove that the two isogeny-torsion graphs of \(R_6\) type correspond to infinite sets of \(j\)-invariants.

**Proposition 9.1.** Let \(G\) be an isogeny-torsion graph of \(R_6\) type (regardless of torsion configuration). Then \(G\) corresponds to an infinite set of \(j\)-invariants.

Let \(E/\mathbb{Q}\) be an elliptic curve such that \(E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/6\mathbb{Z}\) and the isogeny-torsion graph associated to the \(\mathbb{Q}\)-isogeny class of \(E\) is \(R_6^1\). Let \(d\) be a non-zero, square-free integer not equal to 1 or \(-3\) and let \(E^{(d)}\) be the quadratic twist of \(E\) by \(d\). Then the isogeny-torsion graph associated to \(E^{(d)}\) is \(R_6^2\). Hence, if we prove that \(R_6^1\) corresponds to an infinite set of \(j\)-invariants, we just have to take quadratic twists to show that \(R_6^2\) corresponds to an infinite set of \(j\)-invariants.

| Graph Type | Type | Isomorphism Types | Label |
|------------|------|------------------|-------|
| \(E_1 \sim E_5 \sim E_3\) | \(R_6\) | \(([6], [6], [6], [2], [2])\) | \(R_6^1\) |
| \(E_2 \sim E_4 \sim E_6\) | \(R_6\) | \(([2], [2], [2], [2], [2])\) | \(R_6^2\) |

**Table 6.** Isogeny Graphs of \(R_6\) Type

Let \(E'/\mathbb{Q}\) be an elliptic curve. Note that the isogeny-torsion graph associated to the \(\mathbb{Q}\)-isogeny class of \(E'\) is of \(R_6\) type if and only if \(E'\) is \(\mathbb{Q}\)-isogenous to an elliptic curve \(E\) over \(\mathbb{Q}\) such that \(E\) has a point of order 2 defined over \(\mathbb{Q}\) and has two \(\mathbb{Q}\)-rational subgroups of order 3.

- \(R_6^1\)

Let \(E/\mathbb{Q}\) be an elliptic curve. Then the isogeny-torsion graph associated to the \(\mathbb{Q}\)-isogeny class of \(E\) is \(R_6^1\) if and only if (up to relabeling) \(E\) has a point of order 6 defined over \(\mathbb{Q}\) and two \(\mathbb{Q}\)-rational subgroups of order 3. In this case, the image of the mod-2 Galois representation attached to \(E\) is conjugate to \(B_2\), the subgroup of \(\text{GL}(2, \mathbb{Z}/2\mathbb{Z})\) consisting of upper-triangular matrices. By Lemma 4.11, the image of the mod-3 Galois representation attached to \(E\) is conjugate to \(D_3 = \left\langle \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right) \right\rangle \). Suppose the image of the mod-12 Galois representation is conjugate to \(B_2 \times D_3 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\).

Work in \([4]\) has parametrized elliptic curves over \(\mathbb{Q}\) such that the image of the mod-6 Galois representation is isomorphic to \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\). Let \(E_t : y^2 = x^3 + A(t)x + B(t)\) where
$A(t) = -27t^{12} + 216t^9 - 6480t^6 + 12528t^3 - 432$ and
$B(t) = 54t^{18} - 648t^{15} - 25920t^{12} + 166320t^9 - 651888t^6 + 222912t^3 + 3456.$

Using the infinite one-parameter family $E_t$, we see that $\mathcal{R}_6^1$ corresponds to an infinite set of $j$-invariants.

- $\mathcal{R}_6^2$

Let $E/\mathbb{Q}$ be an elliptic curve such that $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/6\mathbb{Z}$ and $E$ has two independent $\mathbb{Q}$-rational subgroups of order 3. Then the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $\mathcal{R}_6^4$. Moreover, $\mathbb{Q}(E[3]) = \mathbb{Q}(\sqrt{-3})$. Let $E'/\mathbb{Q}$ be the quadratic twist of $E$ by a square-free integer not equal to 1 or $-3$. The image of the mod-2 Galois representation attached to $E'$ is the same as the image of the mod-2 Galois representation attached to $E$ and the image of the mod-3 Galois representation attached to $E'$ is conjugate to $\left\{ \left( \begin{array}{cc} * & 0 \\ 0 & * \end{array} \right) \right\}$. By lemma 4.12, no elliptic curve over $\mathbb{Q}$ that is $\mathbb{Q}$-isogenous to $E'$ has a point of order 3 defined over $\mathbb{Q}$. Thus, the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E'$ is $\mathcal{R}_6^5$. As $\mathcal{R}_6^1$ corresponds to an infinite set of $j$-invariants, $\mathcal{R}_6^2$ also corresponds to an infinite set of $j$-invariants.

This concludes the proof of Proposition 9.1.

9.2. Isogeny-Torsion Graphs of $R_4$ Type. In this subsection, we prove that each of the isogeny-torsion graphs of $R_4(6)$ type and $R_4(10)$ type correspond to infinite sets of $j$-invariants.

**Proposition 9.2.** Let $\mathcal{G}$ be an isogeny-torsion graph of $R_4(10)$ or $R_4(6)$ type (regardless of torsion configuration). Then $\mathcal{G}$ corresponds to an infinite set of $j$-invariants.

Proving both of the isogeny-torsion graphs of $R_4(10)$ type correspond to infinite sets of $j$-invariants will be relatively easy. We just need to find a one-parameter family of elliptic curves over $\mathbb{Q}$ with a point of order 10 defined over $\mathbb{Q}$ and then use an appropriate twist. On the other hand, the proof that both of the isogeny-torsion graphs of $R_4(6)$ type correspond to infinite sets of $j$-invariants will be relatively messy, relying completely on Hilbert’s Irreducibility Theorem.

| Graph Type | Type     | Isomorphism Types | Label  |
|------------|----------|------------------|--------|
| $E_1 \xrightarrow{a} E_2$ | $R_4(10)$ | ([10], [10], [2], [2]) | $\mathcal{R}_4^1(10)$ |
| $\downarrow p$ | | ([2], [2], [2], [2]) | $\mathcal{R}_4^2(10)$ |
| $E_3 \xrightarrow{a} E_4$ | $R_4(6)$ | ([6], [6], [2], [2]) | $\mathcal{R}_4^1(6)$ |
| | | ([2], [2], [2], [2]) | $\mathcal{R}_4^2(6)$ |

**Table 7. Isogeny Graphs of $R_4$ Type**

- Isogeny-torsion Graphs of $R_4(10)$ Type
  1. $\mathcal{R}_4^1(10)$

  The isogeny-torsion graph of a $\mathbb{Q}$-isogeny class of elliptic curves over $\mathbb{Q}$ is $\mathcal{R}_4^1(10)$ if and only there is an elliptic curve over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class with rational 10-torsion. Let $t \in \mathbb{Q}$ and let $E_t : y^2 + (1 - a)xy - by = x^3 - bx^2$ with
  
  \[
  a = \frac{(t-1)(2t-1)}{t^2-3t+1} \quad \text{and} \quad b = \frac{t^3(t-1)(2t-1)}{(t^2-3t+1)^2}.
  \]
If $E_t$ is a smooth elliptic curve, then $E_t(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/10\mathbb{Z}$ (see Appendix E of [6]). Note that in this case, the image of the mod-2 Galois representation attached to $E_t$ is conjugate to $\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$. By Lemma 4.6, the image of the mod-5 Galois representation attached to $E_t$ is conjugate to $B_5$, the subgroup of $\text{GL}(2, \mathbb{Z}/5\mathbb{Z})$ consisting of matrices of the form $\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}$. Using the one-parameter family of elliptic curves $E_t$, we can conclude that $R_4^1(10)$ corresponds to an infinite set of $j$-invariants.

(2) $R_4^2(10)$

By Corollary 4.8, the only quadratic subfield of $\mathbb{Q}(E_t[5])$ is $\mathbb{Q}(\sqrt{5})$. Let $d$ be a non-zero, square-free integer not equal to 1 or 5 and let $E_t^{(d)}/\mathbb{Q}$ be the quadratic twist of $E_t$ by $d$. Then the image of the mod-5 Galois representation attached to $E_t^{(d)}$ is conjugate to $\langle B_5, \text{-Id} \rangle$. By Lemma 4.12, none of the elliptic curves that are $\mathbb{Q}$-isogenous to $E_t^{(d)}$ have a point of order 5 defined over $\mathbb{Q}$. The elliptic curve $E_t^{(d)}$ still has a point of order 2 defined over $\mathbb{Q}$ and a $\mathbb{Q}$-rational subgroup of order 5. The isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E_t^{(d)}$ is $R_4^2(10)$. As $R_4^1(10)$ corresponds to an infinite set of $j$-invariants, $R_4^2(10)$ also corresponds to an infinite set of $j$-invariants.

- **Isogeny-Torsion Graphs of $R_4(6)$ Type**

Let $E/\mathbb{Q}$ be an elliptic curve such that the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is of $R_4(6)$ type. There are two possibilities for the isogeny-torsion graph, namely, $R_4^1(6)$ and $R_4^2(6)$. In both cases, all the elliptic curves over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class have a point of order 2 defined over $\mathbb{Q}$. In the case that an elliptic curve over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class has a point of order 6 defined over $\mathbb{Q}$, we may assume without loss of generality that $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/6\mathbb{Z}$. In this case, by Lemma 4.6, the image of the mod-3 Galois representation attached to $E$ is conjugate to $B_3$; the subgroup of $\text{GL}(2, \mathbb{Z}/3\mathbb{Z})$ consisting of matrices of the form $\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}$. In the case that no elliptic curve over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class has a point of order 6 defined over $\mathbb{Q}$, the image of the mod-3 Galois representation of every elliptic curve over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class is conjugate to $\langle B_3, \text{-Id} \rangle$. In either case of torsion configuration, the image of the mod-2 Galois representation attached to the elliptic curves over $\mathbb{Q}$ in the $\mathbb{Q}$-isogeny class is conjugate to $B_2$; the subgroup of $\text{GL}(2, \mathbb{Z}/2\mathbb{Z})$ consisting of upper-triangular matrices.

Isogeny-torsion graphs of $R_4(6)$ type are proper subgraphs of isogeny-torsion graphs of $S$ type and isogeny-torsion graphs of $R_6$ type. To prove there are infinitely many $j$-invariants corresponding to isogeny-torsion graphs of $R_4(6)$ type, it suffices to prove that there are infinitely many $j$-invariants corresponding to elliptic curves over $\mathbb{Q}$ with a $\mathbb{Q}$-rational subgroup of order 6 and

1. without full two-torsion (to avoid an isogeny-torsion graph of type $S$),
2. without a cyclic $\mathbb{Q}$-rational subgroup of order 12 (to avoid an isogeny-torsion graph of type $S$),
3. without a cyclic $\mathbb{Q}$-rational subgroup of order 9 (to avoid an isogeny-torsion graph of type $R_6$),
4. without two distinct $\mathbb{Q}$-rational subgroups of order 3 (to avoid an isogeny-torsion graph of type $R_6$).
Let $G$ be the full inverse image of the subgroup $\mathfrak{B}_2 \times \langle \mathfrak{B}_3, -\text{Id} \rangle$ of $\text{GL}(2, \mathbb{Z}/2\mathbb{Z}) \times \text{GL}(2, \mathbb{Z}/3\mathbb{Z})$ via the mod-6 reduction map from $\text{GL}(2, \mathbb{Z}/4\mathbb{Z}) \times \text{GL}(2, \mathbb{Z}/9\mathbb{Z})$. The group $G$ has order $2^7 \cdot 3^5$ and defines a modular curve of genus 0 with infinitely many non-cuspidal, $\mathbb{Q}$-rational points. Elliptic curves over $\mathbb{Q}$ with a $\mathbb{Q}$-rational subgroup of order 6 correspond to non-cuspidal, $\mathbb{Q}$-rational points on the modular curve $X_0(6)$. Note that the modular curve defined by $G$ is precisely the modular curve defined by $\mathfrak{B}_2 \times \langle \mathfrak{B}_3, -\text{Id} \rangle$, which is precisely the modular curve $X_0(6)$.

(1) Let $G_1$ be the full inverse image of the subgroup

$$K_1 = \{I\} \times \langle \mathfrak{B}_3, -\text{Id} \rangle$$

of $\text{GL}(2, \mathbb{Z}/2\mathbb{Z}) \times \text{GL}(2, \mathbb{Z}/3\mathbb{Z})$ via the mod-6 reduction map from the group $\text{GL}(2, \mathbb{Z}/4\mathbb{Z}) \times \text{GL}(2, \mathbb{Z}/9\mathbb{Z})$. Elliptic curves over $\mathbb{Q}$ with full two-torsion defined over $\mathbb{Q}$ and a $\mathbb{Q}$-rational subgroup of order 3 correspond to non-cuspidal $\mathbb{Q}$-rational points on the modular curve defined by $K_1$ which is precisely the modular curve defined by $G_1$. The group $G_1$ defines a modular curve of genus 0 with infinitely many non-cuspidal, $\mathbb{Q}$-rational points and $G_1$ is an index-2 subgroup of $G$.

(2) Let $G_2$ be the full inverse image of the subgroup

$$K_2 := \left\langle \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \\ \end{array} \right), \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \\ \end{array} \right), -\text{Id} \right\rangle \times \langle \mathfrak{B}_3, -\text{Id} \rangle$$

of $\text{GL}(2, \mathbb{Z}/4\mathbb{Z}) \times \text{GL}(2, \mathbb{Z}/3\mathbb{Z})$ via the mod-3 reduction map from the matrix group $\text{GL}(2, \mathbb{Z}/4\mathbb{Z}) \times \text{GL}(2, \mathbb{Z}/9\mathbb{Z})$. Elliptic curves over $\mathbb{Q}$ with a $\mathbb{Q}$-rational subgroup of order 12 correspond to non-cuspidal, $\mathbb{Q}$-rational points on the modular curve defined by $K_2$ which is precisely the modular curve defined by $G_2$. The group $G_2$ is an index-2 subgroup of $G$ and $G_2$ defines a modular curve of genus 0 with infinitely many non-cuspidal, $\mathbb{Q}$-rational points.

(3) Let $G_3$ be the full inverse image of the subgroup

$$K_3 := \mathfrak{B}_2 \times \left\langle \left( \begin{array}{cc} 2 & 0 \\ 0 & 1 \\ \end{array} \right), \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \\ \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \\ \end{array} \right) \right\rangle$$

of $\text{GL}(2, \mathbb{Z}/2\mathbb{Z}) \times \text{GL}(2, \mathbb{Z}/9\mathbb{Z})$ via the mod-2 reduction map from the matrix group $\text{GL}(2, \mathbb{Z}/4\mathbb{Z}) \times \text{GL}(2, \mathbb{Z}/9\mathbb{Z})$. Elliptic curves over $\mathbb{Q}$ with a $\mathbb{Q}$-rational subgroup of order 18 correspond to non-cuspidal $\mathbb{Q}$-rational points on the modular curve defined by $K_3$ which is precisely the modular curve $X_0(18)$. The group $G_3$ is an index-3 subgroup of $G$ and $G_3$ defines a modular curve of genus 0 with infinitely many non-cuspidal, $\mathbb{Q}$-rational points.

(4) Finally, let $G_4$ be the full inverse image of the subgroup $\mathfrak{B}_2 \times \langle \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \\ \end{array} \right), -\text{Id} \rangle$ of $\text{GL}(2, \mathbb{Z}/2\mathbb{Z}) \times \text{GL}(2, \mathbb{Z}/3\mathbb{Z})$ via the mod-6 reduction map from the group $\text{GL}(2, \mathbb{Z}/4\mathbb{Z}) \times \text{GL}(2, \mathbb{Z}/9\mathbb{Z})$. Elliptic curves over $\mathbb{Q}$ with a point of order 2 defined over $\mathbb{Q}$ and two $\mathbb{Q}$-rational subgroups of order 3 correspond to non-cuspidal, $\mathbb{Q}$-rational points on the modular curve defined by $G_4$. The group $G_4$ is an index-3 subgroup of $G$ and $G_4$ defines a modular curve of genus 0 with infinitely many non-cuspidal, $\mathbb{Q}$-rational points.
Now let
\[ S_G := \bigcup_{i=1}^{4} \pi_{G_i,G}(X_G_i(Q)) \]
where \( \pi_{G_i,G}: X_{G_i} \to X_G \) is the natural morphism induced by the inclusion \( G_i \subseteq G \). The degree of \( \pi_{G_i,G} \) is equal to \([G : G_i] = 2\) or \(3\). Note that \( X_G \cong \mathbb{P}^1 \) and that \( S_G \) is a thin subset in the language of Serre because all the degrees of the maps \( \pi_{G_i,G} \) are at least 2. The field \( \mathbb{Q} \) is Hilbertian and \( \mathbb{P}^1_\mathbb{Q} \cong X_G(\mathbb{Q}) \) is not thin. This implies that the complement \( X_G(\mathbb{Q}) \setminus S_G \) is not thin and must be infinite.

This proves that there are infinitely many \( j \)-invariants corresponding to elliptic curves over \( \mathbb{Q} \) such that the image of mod-2 Galois representation is conjugate to \( \mathfrak{B}_2 \) and the image of the mod-3 Galois representation is conjugate to one of three subgroups of \( GL(2, \mathbb{Z}/3\mathbb{Z}) \), namely,
\[ B_3, \left\{ \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right) \right\}, \text{ or } \langle B_3, -\text{Id} \rangle. \]
For the first two groups, the isogeny-torsion graph is \( R^1_4(6) \). For the third group, the isogeny-torsion graph is \( R^2_4(6) \). Thus, we have proven that \( R^1_4(6) \) and \( R^2_4(6) \) together correspond to infinitely many \( j \)-invariants. We must prove that both correspond to infinitely many \( j \)-invariants individually.

Let \( E/\mathbb{Q} \) be an elliptic curve such that \( E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/6\mathbb{Z} \) and the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E \) is \( R^1_4(6) \). Let \( E^{(r)} \) be the quadratic twist of \( E \) by a non-zero, square-free integer \( r \) not equal to 1 or \(-3\). The only quadratic subfield of \( \mathbb{Q}(E[3]) \) is \( \mathbb{Q}(\sqrt{-3}) \) and thus, \( \mathbb{Q}(\sqrt{-3}) \) does not contain \( \sqrt{7} \). The image of the mod-3 Galois representation attached to \( E^{(r)} \) is conjugate to \( \langle B_3, -\text{Id} \rangle \) and thus, the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E^{(r)} \) is \( R^2_4(6) \).

On the other hand, let \( E/\mathbb{Q} \) be an elliptic curve such that the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E \) is \( R^2_4(6) \). Then the image of the mod-3 Galois representation attached to \( E \) is conjugate to \( \langle B_3, -\text{Id} \rangle \). By Lemma 4.9, there are three subgroups of \( \langle B_3, -\text{Id} \rangle \) of index 2 and by Corollary 4.10, these three subgroups of index 2 correspond to three subfields of \( \mathbb{Q}(E[3]) \) of degree 2, one totally real and two totally imaginary. The three quadratic subfields of \( \mathbb{Q}(E[3]) \) are \( \mathbb{Q}(\sqrt{-3}) \), \( \mathbb{Q}(\sqrt{-d}) \), and \( \mathbb{Q}(\sqrt{-3d}) \) for some positive, square-free integer \( d \) not equal to 1. Let \( E^{(d)} \) be the quadratic twist of \( E \) by \( d \) and let \( E^{(-3d)} \) be the quadratic twist of \( E \) by \(-3d \). Then the image of the mod-3 Galois representation attached to \( E^{(d)} \) or the image of the mod-3 Galois representation attached to \( E^{(-3d)} \) is conjugate to \( B_3 \). Hence, either the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E^{(d)} \) is \( R^1_4(6) \) or the isogeny-torsion graph associated to the \( \mathbb{Q} \)-isogeny class of \( E^{(-3d)} \) is \( R^1_4(6) \).

From the fact that one of \( R^1_4(6) \) or \( R^2_4(6) \) corresponds to an infinite set of \( j \)-invariants, and the graphs are interchangeable by quadratic twists, both \( R^1_4(6) \) and \( R^2_4(6) \) correspond to infinite sets of \( j \)-invariants.

This concludes the proof of Proposition 9.2.

10. Linear Graphs

10.1. Isogeny-Torsion Graphs of \( L_3 \) Type. In this subsection, we prove that the isogeny-torsion graphs of \( L_3(25) \) and \( L_3(9) \) type correspond to infinite sets of \( j \)-invariants.

**Proposition 10.1.** Let \( \mathcal{G} \) be an isogeny-torsion graph of \( L_3(9) \) type or \( L_3(25) \) (regardless of torsion configuration). Then \( \mathcal{G} \) corresponds to an infinite set of \( j \)-invariants.
To prove that the isogeny-torsion graphs of $L_3(25)$ type correspond to infinite sets of $j$-invariants, we start by proving one of the isogeny-torsion graphs correspond to an infinite set of $j$-invariants. Then we use a quadratic twist to prove the other isogeny-torsion graph corresponds to an infinite set of $j$-invariants. Proving the isogeny-torsion graphs of $L_3(9)$ correspond to infinite sets of $j$-invariants is done the same way.

| Isogeny Graph | Type | Torsion Configuration | Label |
|---------------|------|-----------------------|-------|
| $E_1 - E_2 - E_3$ | $L_3(25)$ | ([5], [5], [1]) | $L_3^1(25)$ |
| | | ([1], [1], [1]) | $L_3^2(25)$ |
| | $L_3(9)$ | ([9], [3], [1]) | $L_3^1(9)$ |
| | | ([3], [3], [1]) | $L_3^2(9)$ |
| | | ([1], [1], [1]) | $L_3^2(9)$ |

Table 8. Isogeny Graphs of $L_3$ Type

- Isogeny-Torsion Graphs of $L_3(25)$ Type

1. $L_3^1(25)$

   Let $E/\mathbb{Q}$ be an elliptic curve such that $E$ contains two $\mathbb{Q}$-rational subgroups of order 5. Then the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $L_3^1(25)$ if and only if $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/5\mathbb{Z}$ or in other words, if and only if the image of the mod-5 Galois representation attached to $E$ is conjugate to

   $H = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle.$

   In this case, the image of the mod-5 Galois representation attached to $E$ is an abelian extension of $\mathbb{Q}$. There is a parametrization of elliptic curves over $\mathbb{Q}$ such that the image of the mod-5 Galois representation is conjugate to $H$ in [4]; $E_t : y^2 = x^3 + A(t)x + B(t)$ where

   $A(t) = \frac{-2^{20} - 228t^{15} + 494t^{10} + 228t^5 + 1}{48}$

   and $B(t) = \frac{t^{30} + 522t^{25} - 1005t^{20} - 1005t^{10} - 522t + 1}{864}.

   Conversely, let $t \in \mathbb{Q}$ such that $E_t$ is an elliptic curve. Then the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E_t$ is $L_3^1(25)$. Using the infinite one-parameter family of elliptic curves over $\mathbb{Q}$, $E_t$, we prove that $L_3^1(25)$ corresponds to an infinite set of $j$-invariants.

2. $L_3^2(25)$

   Let $E/\mathbb{Q}$ be an elliptic curve such that $E$ contains two $\mathbb{Q}$-rational subgroups of order 5. Then the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $L_3^2(25)$ if and only if $E$ has trivial rational torsion. As $H$ is a cyclic group of order 4, $E_t$ has a single subgroup of index 2. Thus, $\mathbb{Q}(E_t[5])$ has a single quadratic subfield, namely $\mathbb{Q}(\sqrt{5})$. Let $E_t^{(d)}$ be a quadratic twist of $E_t$ by $d$ where $d$ is any non-zero, square-free integer not equal to 1 or 5. Then the image of the mod-5 Galois representation attached to $E_t^{(d)}$ is conjugate to $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle$ and the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E_t^{(d)}$ is $L_3^2(25)$ by Lemma 4.12. As $L_3^1(25)$ corresponds to an infinite set of $j$-invariants, $L_3^2(25)$ also corresponds to an infinite set of $j$-invariants.
• Isogeny-Torsion Graphs of $L_3(9)$ Type

(1) $L_3^1(9)$

Let $E/\mathbb{Q}$ be an elliptic curve. Then the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $L_3^1(9)$ if and only if $E$ is $\mathbb{Q}$-isogenous to an elliptic curve over $\mathbb{Q}$ with rational 9-torsion (see Table 8). Let us say that $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/9\mathbb{Z}$. A Magma computation reveals that the image of the mod-9 Galois representation attached to $E$ is conjugate to

$$H = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle.$$ 

There are infinitely many $j$-invariants corresponding to elliptic curves over $\mathbb{Q}$ with rational 9-torsion. One such one-parameter family of elliptic curves is

$$E_{a,b}(t) : y^2 + (1-a)xy - by = x^3 - bx^2$$

where $a = t^2(t-1)$ and $b = t^2(t-1)(t^2 - t + 1)$ (see appendix E of [6]). Thus, $L_3^1(9)$ corresponds to an infinite set of $j$-invariants.

(2) $L_3^2(9)$

Let $E_{a,b}^{(-3)}(t)$ be the quadratic twist of $E_{a,b}(t)$ by $-3$. Then the image of the mod-3 Galois representation attached to $E_{a,b}^{(-3)}(t)$ is conjugate to

$$H' = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle.$$ 

Lifting $H'$ to level 9, we see that the image of the mod-9 Galois representation attached to $E_{a,b}^{(-3)}(t)$ is conjugate to

$$H'' = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 8 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle.$$ 

We will prove that the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E = E_{a,b}^{(-3)}(t)$ is $L_3^2(9)$. Let $P_9$ and $Q_9$ be points on $E$ such that $\langle P_9 \rangle \cap \langle Q_9 \rangle = \{O\}$ (equivalently, $P_9, Q_9$ form a base of $E[9]$) and the image of the mod-9 Galois representation attached to $E$ is $H''$. Let $\sigma_1$ and $\sigma_2$ be a pair of Galois automorphisms such that $\sigma_1(P_9) = [8]P_9$ and $\sigma_2$ fixes $P_9$ and $\sigma_2(Q_9) = P_9 + Q_9$ and $\sigma_1(Q_9) = [7]Q_9$. Then $\sigma_1$ and $\sigma_2$ generate the image of the mod-9 Galois representation attached to $E$. Note that the group $\langle P_9 \rangle$ is $\mathbb{Q}$-rational. Let $\phi: E \to E/\langle P_9 \rangle$ be an isogeny with kernel $\langle P_9 \rangle$. Denote $Q_3 = [3]Q_9$. We claim that $\phi(Q_3)$ is a point of order 3 defined over $\mathbb{Q}$. Note that $\sigma_1$ does not fix $Q_9$ but it does fix $Q_3$ as $\sigma_1(Q_3) = \sigma_1([3]Q_9) = [3]\sigma_1(Q_9) = [3][[7]Q_9] = [21]Q_9 = [3]Q_9 = Q_3.$

Also, $\sigma_2(Q_9) = P_9 + Q_9$ and hence,$\sigma_2(Q_3) = \sigma_2([3]Q_9) = [3]\sigma_2(Q_9) = [3](P_9 + Q_9) = [3]P_9 + Q_3.$

Thus, $\sigma_2(Q_3) - Q_3 = [3]P_9$. By Lemma 4.1, $\phi(Q_3)$ is a point of order 3 defined over $\mathbb{Q}$. There are three cyclic subgroups of $E/\langle P_9 \rangle$ of order 9 that contain $\phi(Q_3)$, namely, $\langle \phi(Q_9) \rangle$, $\langle \phi(P_9 + Q_9) \rangle$, and $\langle \phi([2]P_9 + Q_9) \rangle$. We will test to see if $E/\langle P_9 \rangle$ has a point of order 9 defined over $\mathbb{Q}$ by seeing whether or not $\phi(Q_9)$, $\phi(P_9 + Q_9)$, or $\phi([2]P_9 + Q_9)$ is defined over $\mathbb{Q}$. Note that $\sigma_1(Q_9) = Q_9 = [7]Q_9 - Q_9 = [6]Q_9 \notin \langle P_9 \rangle$. By Lemma 4.1,
graphs of Proposition 10.2. Let Hilbert’s Irreducibility Theorem.

The group $H$, defined by

$$H \equiv \langle \phi(Q_9) \rangle$$

corresponds to an infinite set of $j$-invariants. Hence, $E/\langle P_9 \rangle (Q)_{\text{tors}} \cong \mathbb{Z}/3\mathbb{Z}$ and we can conclude that the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E_{a,b}^{(-3)}(t)$ is $L_3^2(9)$ (see Table 8). As $E_{a,b}(t)$ and $E_{a,b}^{(-3)}(t)$ are quadratic twists and $L_3^1(9)$ corresponds to an infinite set of $j$-invariants, so does $L_3^2(9)$.

(3) $L_3^2(9)$

Let $E_{a,b}^{(d)}(t)$ be the quadratic twist of $E_{a,b}(t)$ by a non-zero, square-free integer $d$ not equal to 1 or $-3$. Then the image of the mod-9 Galois representation attached to $E_{a,b}^{(d)}(t)$ is conjugate to $\langle H, \text{Id} \rangle$. The group $\langle H, \text{Id} \rangle$ is conjugate to the group found in [12] with label $9I^0 - 9c$. By Lemma 4.12, $E_{a,b}^{(d)}(t)$ is not $\mathbb{Q}$-isogenous to an elliptic curve over $\mathbb{Q}$ with a point of order 3 defined over $\mathbb{Q}$. Thus, we can conclude that the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E_{a,b}^{(d)}(t)$ is $L_3^2(9)$. As $E_{a,b}(t)$ and $E_{a,b}^{(d)}(t)$ are quadratic twists and $L_3^1(9)$ corresponds to an infinite set of $j$-invariants, so does $L_3^2(9)$.

This concludes the proof of Proposition 10.1.

10.2. Isogeny-Torsion Graphs of $L_2$ Type. In this subsection, we prove that the isogeny-torsion graphs of $L_2(13)$, $L_2(7)$, $L_2(5)$, $L_2(3)$, and $L_2(2)$ type each correspond to infinite sets of $j$-invariants.

Proposition 10.2. Let $\mathcal{G}$ be an isogeny-torsion graph of $L_2(13)$, $L_2(7)$, $L_2(5)$, $L_2(3)$, or $L_2(2)$ type (regardless of torsion configuration). Then $\mathcal{G}$ corresponds to an infinite set of $j$-invariants.

We break the proof down by cases. The proof that the isogeny-torsion graph of $L_2(13)$ type corresponds to an infinite set of $j$-invariants is a matter of looking up groups in [12]. The proof that the isogeny-torsion graphs of $L_2(7)$ type correspond to infinitely many $j$-invariants will be done by proving one such isogeny-torsion graph of $L_2(7)$ type corresponds to an infinite set of $j$-invariants and then using an appropriate quadratic twist to prove the other isogeny-torsion graph of $L_2(7)$ type corresponds to an infinite set of $j$-invariants. The proof that the isogeny-torsion graphs of $L_2(2)$, $L_2(3)$, and $L_2(5)$ type each correspond to infinite sets of $j$-invariants is an application of Hilbert’s Irreducibility Theorem.

- $L_2(13)$

Let $E/Q$ be an elliptic curve. Then $E$ has a $\mathbb{Q}$-rational subgroup of order 13 if and only if the image of the mod-13 Galois representation attached to $E$ is conjugate to a subgroup of

$$H = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}. $$

The group $H$ appears in the list compiled in [12] as $13A^0 - 13a$. Note that the modular curve defined by $13A^0 - 13a$ is isomorphic to $X_0(13)$, which is a genus 0 curve with infinitely many non-cuspidal, $\mathbb{Q}$-rational points (see Theorem 2.6). Elliptic curves over $\mathbb{Q}$ with a $\mathbb{Q}$-rational subgroup of order 13 correspond to non-cuspidal, $\mathbb{Q}$-rational points on the modular curve defined by $H$. As there is a single isogeny-torsion graph of $L_2(13)$ type, it must correspond to an infinite set of $j$-invariants.
| Isogeny Graph | Type     | Isomorphism Types | Label | Label   |
|--------------|----------|------------------|-------|---------|
| $E_1$ $p$ $E_2$ | $L_2(13)$ | ([1], [1]) | $L_2(13)$ |
|              | $L_2(7)$  | ([7], [1]) | $L_2^1(7)$ |
|              |          | ([1], [1]) | $L_2^2(7)$ |
|              | $L_2(5)$  | ([5], [1]) | $L_2^1(5)$ |
|              |          | ([1], [1]) | $L_2^2(5)$ |
|              | $L_2(3)$  | ([3], [1]) | $L_2^1(3)$ |
|              |          | ([1], [1]) | $L_2^2(3)$ |
|              | $L_2(2)$  | ([2], [2]) | $L_2(2)$ |

Table 9. Isogeny Graphs of $L_2$ Type

- **Isogeny-Torsion Graphs of $L_2(7)$ Type**
  1. $L_2^1(7)$
     
     Let $E/\mathbb{Q}$ be an elliptic curve such that the isogeny-torsion graph of $E$ is of $L_2(7)$ type. There are two possible torsion configurations. The isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $L_2^1(7)$ if and only if $E$ is $\mathbb{Q}$-isogenous to an elliptic curve over $\mathbb{Q}$ with rational 7-torsion.
     
     Let $E_{a,b}(t) : y^2 + (1 - a)xy - by = x^3 - bx^2$ where
     
     $$a = t^2 - t$$ and
     $$b = t^3 - t^2.$$ 
     
     If $E_{a,b}(t)$ is an elliptic curve, then $E_{a,b}(t)(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/7\mathbb{Z}$ (see appendix E of [6]). Using this one-parameter family of elliptic curves over $\mathbb{Q}$ with rational 7-torsion, we can conclude that $L_2^1(7)$ corresponds to an infinite set of $j$-invariants.

  2. $L_2^2(7)$
     
     By Lemma 4.6 the image of the mod-7 Galois representation attached to $E_{a,b}(t)$ is conjugate to $B_7$, the subgroup of $\text{GL}(2, \mathbb{Z}/7\mathbb{Z})$ of matrices of the form
     
     $$\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}.$$ By Corollary 4.8, $\mathbb{Q}(E[7])$ contains a single quadratic subfield, $\mathbb{Q}(\sqrt{-7})$. Let $E_{a,b}^{(d)}(t)$ be a quadratic twist of $E_{a,b}(t)$ by a non-zero, square-free integer $d$ not equal to 1 or $-7$. Then the image of the mod-7 Galois representation attached to $E_{a,b}^{(d)}(t)$ is conjugate to $(B_7, -\text{Id})$.
     
     By Corollary 4.13, none of the elliptic curves over $\mathbb{Q}$ that are $\mathbb{Q}$-isogenous to $E_{a,b}^{(d)}(t)$ have a point of order 7 defined over $\mathbb{Q}$. Hence, the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E_{a,b}^{(d)}(t)$ is $L_2^2(7)$. As $E_{a,b}(t)$ and $E_{a,b}^{(d)}(t)$ are quadratic twists and $L_2^1(7)$ corresponds to an infinite set of $j$-invariants, $L_2^2(7)$ also corresponds to an infinite set of $j$-invariants.

- **Isogeny-Torsion Graphs of $L_2(5)$ Type**
  
  An isogeny-torsion graph of $L_2(5)$ type is a proper subgraph of an isogeny-torsion graph of $L_3(25)$ type, an isogeny graph of $R_4(15)$ type, and an isogeny-torsion graph of $R_4(10)$ type. As there are finitely many $j$-invariants corresponding to elliptic curves over $\mathbb{Q}$ with isogeny graph of $R_4(15)$ type (see Proposition 5.1 and Theorem 2.8), we can just focus on isogeny-torsion graphs of $L_2(5)$ type, $L_3(25)$ type, and $R_4(10)$ type.
After appropriate modifications, a very similar analysis that we did in subsection 9.2 to determine the two isogeny-torsion graphs of $R_4(6)$ type correspond to infinite sets of $j$-invariants can be used to prove that one of the two isogeny-torsion graphs of $L_2(5)$ type, $L_1^3(5)$ or $L_2^3(5)$, correspond to infinite sets of $j$-invariants. Now we must prove that both $L_1^2(5)$ and $L_2^2(5)$ correspond to infinitely many $j$-invariants using quadratic twists.

Let $E/\mathbb{Q}$ be an elliptic curve such that $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/5\mathbb{Z}$ and $C_5(E) = C(E) = 2$. Then the isogeny-torsion graph associated to the $\mathbb{Q}$-isogeny class of $E$ is $L_1^2(5)$. By Lemma 4.6, the image of the mod-5 Galois representation is conjugate to $B_5$, the subgroup of $GL(2, \mathbb{Z}/5\mathbb{Z})$ consisting of matrices of the form $\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}$ and by Lemma 4.7, $B_5$ contains a single subgroup of index 2. Using an application of Hilbert’s irreducibility theorem, we can show that there are infinitely many $j$-invariants that correspond to elliptic curves $E/\mathbb{Q}$ such that the image of the mod-5 Galois representation attached to $E$ is conjugate to $B_5$ and $C_5(E) = C(E) = 2$. Thus, $L_1^2(5)$ corresponds to an infinite set of $j$-invariants. Now let $d$ be a non-zero, square-free integer not equal to 1 or 5 and let $E^{(d)}$ be the quadratic twist of $E$ by $d$. Then, the image of the mod-5 Galois representation attached to $E^{(d)}$ is conjugate to $(B_5, -\text{Id})$ and by Corollary 4.13, the isogeny-torsion graph associated to $E^{(d)}$ is $L_2^2(5)$.

We proved we can toggle amongst the two isogeny-torsion graphs of $L_2(5)$ type. Thus, both correspond to an infinite set of $j$-invariants.

- Isogeny-Torsion Graphs of $L_2(3)$ Type

An isogeny-torsion graph of $L_2(3)$ type is a proper subgraph of an isogeny-torsion graph of $L_3(9)$ type, an isogeny-torsion graph of $L_4$ type, an isogeny-torsion graph of $R_4(6)$ type, an isogeny-torsion graph of $R_4(15)$ type, an isogeny-torsion graph of $R_4(21)$ type, an isogeny-torsion graph of $R_6$ type, and an isogeny-torsion graph of $S$ type. As there are finitely many $j$-invariants corresponding to isogeny-torsion graphs of $L_4$, $R_4(15)$, and $R_4(21)$ type (see Proposition 5.1 and Theorem 2.8), we can just focus on the isogeny-torsion graphs of $L_2(3)$ type, isogeny-torsion graphs of $L_3(9)$ type, isogeny-torsion graphs of $R_4(6)$ type, isogeny-torsion graphs of $R_6$ type, and isogeny-torsion graphs of $S$ type.

After appropriate modifications, a very similar analysis that we did earlier to prove that both $L_1^3(5)$ and $L_2^3(5)$ correspond to infinite sets of $j$-invariants can be used to prove that both $L_1^2(3)$ and $L_2^2(3)$ correspond to infinite sets of $j$-invariants.

- $L_1^1(2)$

The isogeny-torsion graph of $L_1^1(2)$ type is a proper subgraph of an isogeny-torsion graph of $R_4(6)$ type, an isogeny-torsion graph of $R_4(10)$ type, the isogeny-torsion graph of $R_4(14)$ type, an isogeny-torsion graph of $T_4$ type, an isogeny-torsion graph of $T_6$ type, an isogeny-torsion graph of $T_8$ type, and an isogeny-torsion graph of $S$ type. As there are finitely many $j$-invariants corresponding to the isogeny-torsion graphs of $R_4(14)$ type (see Proposition 5.1 and Theorem 2.8), we only need to focus on the other isogeny-torsion graphs that properly contain the isogeny-torsion graph of $L_2(2)$ type.

After appropriate modifications, a very similar analysis that we did to prove that both isogeny-torsion graphs of $L_2(5)$ type correspond to infinite sets of $j$-invariants can be used to determine that the isogeny-torsion graph of $L_2(2)$ type corresponds to an infinite set of $j$-invariants.

This concludes the proof of Proposition 10.2.
10.3. The Isogeny-Torsion Graph of $L_1$ Type.

**Proposition 10.3.** The isogeny-torsion graph of $L_1$ type corresponds to an infinite set of $j$-invariants.

Let $E/Q$ be an elliptic curve. Then the isogeny-torsion graph associated to the $Q$-isogeny class of $E$ is of $L_1$ type if and only if the $Q$-isogeny class of $E$ consists only of $E$. In other words, $E$ does not have any non-trivial, finite, cyclic, $Q$-rational subgroups. We need to prove there are infinitely many $j$-invariants corresponding to elliptic curves over $Q$ with no non-trivial, finite, cyclic, $Q$-rational subgroups. By Theorem 2.8 and Proposition 5.1, it suffices to consider the cases when $E$ has a $Q$-rational subgroup of order $2, 3, 5, 7,$ or $13$.

For each $p = 2, 3, 5, 7, 13$, let $j_p$ be the $j$-invariant of some elliptic curve over $Q$ such that the image of the mod-$p$ Galois representation is conjugate to a subgroup of the group of upper triangular matrices of $GL(2, \mathbb{Z}/p\mathbb{Z})$. For each prime $p = 2, 3, 5, 7, 13$, we have $j_p = \frac{F_p(x)}{G_p(x)}$ for some $F_p(x), G_p(x) \in \mathbb{Z}[x]$ with $\deg(F) - \deg(G) \geq 2$. For each prime $p = 2, 3, 5, 7, 13$, let $H_p$ denote the set of all rational numbers $t$, such that $G_p(x)t - F_p(x)$ is irreducible. If the $j$-invariant of an elliptic curve $E/Q$ is an element of $H_p$, then $E$ does not have a $Q$-rational subgroup of order $p$. For each $p = 2, 3, 5, 7, 13$, $H_p$ is a basic Hilbert set and thus, $H = \bigcap_{2 \leq p \leq 13} H_p$ is a Hilbert set and is dense in $Q$. Thus, there are infinitely many $j$-invariants (elements of $H$) that correspond to elliptic curves over $Q$ which do not have $Q$-rational subgroups of order $2, 3, 5, 7,$ or $13$. Hence, the isogeny-torsion graph of $L_1$ type corresponds to an infinite set of $j$-invariants. This concludes the proofs of Proposition 10.3 and Theorem 1.6.

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