A HIERARCHICAL FINITE ELEMENT METHOD
FOR QUANTUM FIELD THEORY

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Abstract. We study a model of scalar quantum field theory in which space-
time is a discrete set of points obtained by repeatedly subdividing a triangle
into three triangles at the centroid. By integrating out the field variable at
the centroid we get a renormalized action on the original triangle. The exact
renormalization map between the angles of the triangles is obtained as well. A
fixed point of this map happens to be the cotangent formula of Finite Element
Method which approximates the Laplacian in two dimensions.

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1. Introduction

The most successful regularization method in understanding non-perturbative
Quantum Field Theory (QFT) is the lattice method,[1, 2] which replaces space-
time by a periodically arranged finite set of points. Numerical simulations based on
this are becoming increasingly accurate, therefore any attempt at a mathematical
formulation of quantum field theory must build on this success and aim to improve
upon it.

The classical analogue of the problem would be the solution of Partial Differential
Equations (PDEs). In the early days a lattice with identically shaped fundamental
regions was used in numerical solutions of PDEs. Later it was realized that using
meshes adapted to the boundary conditions makes more economical use of comput-
ing resources by adding more points where the field varies rapidly and fewer where
it varies slowly. The Finite Element Method[3, 4] was developed in the seventies:
it allows fundamental cells to have different shapes and sizes and use sophisticated
interpolation methods to model the field in the interior of each cell. Some of the
mathematical ideas were anticipated by Whitney[5] in his work in topology and
extended by Patodi[6]. The Whitney elements have provided a basis for a discrete
formulation of geometry. This Discrete Differential Geometry is useful not only to
solve PDEs, but also to model shapes for use in computer graphics[7, 8].

The analogue in Quantum Field Theory is to replace the periodic lattice with
a mesh that contains different length scales. This approach has been looked at by
groups in the past and met with varying degrees of success. The first approach in
this direction was by Christ, Friedberg and Lee[9] (except they proposed to average
over all locations of lattice points as a way to restore rotation invariance, which did
not turn out to be helpful). There is also some early work by Bender, Guralnik and
Sharp[10]. Patodi’s FEM to solve the eigenvalue problem for Laplacians was not
noticed by physicists at this time. Since then much of the work on Lattice Gauge
Theories has been computational along with some analytic work[11].
We propose to adapt existing methods of QFT and develop new Finite Element Methods to understand the essential problem from Wilson’s point of view: how to integrate out some variables and get an effective theory for the remaining degrees of freedom (for a recent review, see the volume[12]). The simplest case is the one dimensional lattice (the set of integers), which has a natural subdivision into even and odd numbered elements. By integrating out the odd sites and leaving only the even sites we are left with an identical lattice with a different separation between field points. Unfortunately there is no simple procedure to extend this into higher dimensions.

A natural idea would be to divide space into triangles (simplices in higher dimensions) and to fit them together to form larger ones, allowing us to integrate out the interior vertices and obtain an effective large scale theory. An advantage of our regularization method is that the renormalization map can be calculated exactly. The transformation between the angles of the triangles from subsequent generations is obtained at each stage of the subdivision. The first examples[13] we constructed this way ignored the shape (information contained in the angles) of the triangles. The Finite Element Method used by engineers leads to a “cotangent formula”[14]. It approximates the Laplacian in two dimensions on one hand and also happens to be a fixed point[15] of the renormalization dynamics. We determine this dynamics explicitly.

We expect this fixed point to be a continuum limit on a fractal, analogous to the Bethe lattice for which the renormalization group can be exactly calculated. Such QFTs can serve as approximations to theories on Euclidean spaces. Or perhaps at short distances, space-time really is not Euclidean.

If generalized to the Ising model, nonlinear sigma models or to four dimensional field theories, we could get interesting examples of Discrete Conformal Field Theory[16]. In our approach we do not average over triangulations. Such an average has been proposed as an approach to quantum gravity[17] and as a way to restore translation invariance[9].

2. The Cotangent Formula

In the early days of computational engineering, Duffin[14] derived a formula for the discrete approximation for the energy of an electrostatic field on a planar domain. In this Finite Element Method the plane is divided into triangles where the field is specified at each vertex and the energy of the field is the sum of contributions from each triangle. An approximation for the energy of a triangle is obtained by linear interpolation of the field to the interior.

Suppose the vertices \(x_0, x_1, x_2\) correspond to field values \(\phi_0, \phi_1, \phi_2\). Each point \(x\) in the interior of the triangle divides it into three sub-triangles with vertices \(\{x, x_0, x_1\}, \{x, x_1, x_2\}\) and \(\{x, x_2, x_0\}\) respectively.

If the ratio of the area of a sub-triangle opposite to \(x_0\) to the larger triangle is

\[ u_0 = \frac{\Delta(x, x_1, x_2)}{\Delta(x_0, x_1, x_2)} \]

then

\[ x = u_0 x_0 + u_1 x_1 + u_2 x_2, \quad u_0 + u_1 + u_2 = 1, \quad u_0, u_1, u_2 > 0. \]

We can use the pair \(u_0, u_1\) as co-ordinates instead of the cartesian components of \(x\). The linear interpolation of the field values to the point \(x\) is then

\[ \phi(x) = u_0 \phi_0 + u_1 \phi_1 + u_2 \phi_2. \]
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Figure 1. A triangle with labelled vertices and cotangents of the angles.

The energy of the interpolated field inside a triangle on calculation turns out to be

\[
S = \frac{1}{4} \left[ a_2 (\phi_0 - \phi_1)^2 + a_1 (\phi_2 - \phi_0)^2 + a_0 (\phi_1 - \phi_2)^2 \right]
\]

where \(a_0, a_1, a_2\) are the cotangents of the angles at the vertices.

**Proof.** Define the vectors along the sides of the triangle (see Fig. 1),

\[
e_1^\alpha = x_2^\alpha - x_0^\alpha, \quad e_2^\alpha = x_2^\alpha - x_0^\alpha.
\]

Using \(u^a\) for \(a = 1, 2\) as co-ordinates,

\[
x^\mu = u^a e_a^\mu \implies \partial_\alpha x^\mu = e_a^\mu
\]

Then the metric tensor of the plane in these co-ordinates has as components the dot products of the sides:

\[
g_{ab} = e_a^\mu e_b^\nu \delta_{\mu\nu}, \quad g = \left( \begin{array}{cc} |e_1|^2 & e_1 \cdot e_2 \\ e_1 \cdot e_2 & |e_2|^2 \end{array} \right)
\]

Also, \(\sqrt{\det g} = e_1 \times e_2\) is twice the area of the triangle. The cotangents are

\[
a_0 = \frac{e_1 \cdot e_2}{e_1 \times e_2}, \quad a_1 = \frac{(e_2 - e_1) \cdot e_1}{(e_2 - e_1) \times e_1}, \quad a_2 = \frac{e_2 \cdot (e_2 - e_1)}{e_2 \times (e_2 - e_1)}
\]

Then,

\[
a_0 + a_1 = \frac{|e_1|^2}{e_1 \times e_2}, \quad a_0 + a_2 = \frac{|e_2|^2}{e_1 \times e_2}
\]

and

\[
\sqrt{\det g} g^{ab} = \left( \begin{array}{cc} \frac{|e_2|^2}{e_1 \times e_2} - \frac{e_1 \cdot e_2}{e_1 \times e_2} & \frac{e_1 \cdot e_2}{e_1 \times e_2} \\ -\frac{e_1 \cdot e_2}{e_1 \times e_2} & \frac{|e_1|^2}{e_1 \times e_2} \end{array} \right) = \left( \begin{array}{cc} a_0 + a_2 & -a_0 \\ -a_0 & a_0 + a_1 \end{array} \right)
\]

Thus, using \(\int d^2u = \frac{1}{2}\),

\[
S = \frac{1}{2} \int \sqrt{\det g} g^{ab} \partial_\alpha \phi \partial_\beta \phi d^2\tau
\]

\[
= \frac{1}{4} \left[ (a_0 + a_2)(\phi_1 - \phi_0)^2 - 2a_0(\phi_1 - \phi_0)(\phi_2 - \phi_0) + (a_0 + a_1)(\phi_2 - \phi_0)^2 \right]
\]

This can be rewritten as

\[
S(\phi_0, \phi_1, \phi_2|a_0, a_1, a_2) = \frac{1}{4} \left[ a_0(\phi_1 - \phi_2)^2 + a_1 (\phi_2 - \phi_0)^2 + a_2 (\phi_0 - \phi_1)^2 \right]
\]

as claimed. \(\Box\)
3. The Geometry of Triangles

The space $S$ of similarity classes of triangles (with marked vertices) is a hyperboloid [15]. This can be understood in several ways. A pair of sides of a triangle forms a basis, thus the space of marked triangles may be identified with $GL(2,\mathbb{R})$: this group acts transitively and without a fixed point on the space of bases. Quotienting by rotation, scaling and reflection around a side gives

$$S = GL(2,\mathbb{R})/ (SO(2,\mathbb{R}) \times \mathbb{R}^+ \times \mathbb{Z}_2) = SL(2,\mathbb{R})/SO(2,\mathbb{R}).$$

which is a hyperboloid. This argument generalizes to $n$ dimensions: the similarity classes of marked simplices is $GL(n,\mathbb{R})/\mathbb{R}^+ \times SO(n,\mathbb{R}) \times \mathbb{Z}_2 = SL(n,\mathbb{R})/SO(n,\mathbb{R})$.

An equivalent point of view is that $S$ is the space of symmetric tensors of determinant one: a pair of sides of the triangles define a symmetric tensor through their inner products. By scaling we can choose this symmetric tensor to have determinant one. It is clear that $SL(2,\mathbb{R})$ acts on the space of such tensors transitively, with $SO(2,\mathbb{R})$ as the isotropy group at one point. Again this generalizes to $n$ dimensions.

A more explicit point of view will be useful in what follows. A similarity class of marked triangles is determined by the angles at the vertices (or, for convenience, the cotangents of the angles). Since the angles $(\theta_0, \theta_1, \theta_2)$ of a triangle add up to $\pi$, the cotangents satisfy

$$a_0a_1 + a_1a_2 + a_2a_0 = 1, \quad a_i = \cot \theta_i$$

This can be written as

$$a^T \eta a = 1, \quad a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}, \quad \eta = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Since $\eta$ has signature $(1, -1, -1)$, this is the equation for a time-like hypersurface in Minkowski space $\mathbb{R}^{1,2}$. Setting

$$p_0 = \frac{a_1 + a_2 + a_0}{\sqrt{3}}, \quad p_1 = \frac{a_2 - a_1}{2}, \quad p_2 = \frac{2a_0 - a_1 - a_2}{2\sqrt{3}}$$

the “cotangent identity” becomes the equation for a hyperboloid

$$p_0^2 - p_1^2 - p_2^2 = 1.$$ 

The quantity $4(a_0 + a_1 + a_2)$ is the ratio of the sum of squares of the sides to the area of the triangle. It is a minimum for an equilateral triangle and becomes large for a flat triangle (one with small area or large perimeter).

So far we discussed triangles with marked vertices but we should also consider invariant transformations of the vertices. The group $S_3$ of permutations of vertices is generated by the cyclic permutation

$$\sigma : 012 \mapsto 120$$

and the interchange of a pair of vertices

$$\tau : 012 \mapsto 021$$

$$S_3 = \langle \sigma, \tau | \sigma^3 = 1, \tau^2 = 1, \tau \sigma \tau = \sigma^2 \rangle.$$ 

These permutations act on the cotangents through the matrices

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

We can also parametrize $S$ by the complex number

$$z = \frac{a_1 + i}{a_1 + a_2}.$$
By a translation, we can choose the first vertex \( x_1 = 0 \) and by a rotation and scaling we may choose \( x_2 = 1 \). \( z \) is then the co-ordinate of the remaining vertex. Then the permutation of the vertices becomes

\[
\sigma(z) = \frac{1}{1-z}, \quad \tau(z) = 1 - \bar{z}.
\]

By a reflection around the side 12, we can choose \( a_0 + a_1 + a_2 > 0 \); equivalently \( \text{Im}(z) > 0 \). Note that \( \tau \) is the reflection around the perpendicular from vertex 0 to the opposite side 12 of the triangle.

3.1. Subdivision of a triangle. We can subdivide a triangle into three sub-triangles of equal area by connecting the centroid \( x_3 = \frac{x_0 + x_1 + x_2}{3} \) to the vertices \( x_0, x_1, x_2 \) by straight lines. (If we subdivide at some other interior point, we get similar results).

The cotangents of the angles of the sub-triangle opposite vertex 0 are given by

\[
cot(x_2x_1x_3) = 2a_1 + a_2, \quad \cot(x_3x_2x_1) = 2a_2 + a_1, \quad \cot(x_1x_3x_2) = \frac{a_0 - 2a_1 - 2a_2}{3}
\]
as shown in Fig. 2. To see this, choose a co-ordinate system with \( x_1 = (0, 0), \ x_2 = (1, 0), \ x_0 = (x, y) \) so that \( x_3 = \left( \frac{1(x+y)}{3}, \frac{1(x+y)}{3} \right) \). Then,

\[
a_1 = \frac{x}{y}, \quad a_2 = \frac{1-x}{y}, \quad a_0 = \frac{1-a_1a_2}{a_1 + a_2}.
\]

By dropping a perpendicular from \( x_3 \) to the side \( x_1x_2 \) we get

\[
\cot(x_2x_1x_3) = \frac{1+x}{y} = 2a_1 + a_2, \quad \cot(x_3x_2x_1) = \frac{1 - \frac{1+x}{y}}{y} = 2a_2 + a_1.
\]

The remaining angle is given by solving the cotangent formula:

\[
\cot(x_1x_3x_2) = \frac{1 - (2a_1 + a_2)(2a_2 + a_1)}{(2a_1 + a_2) + (2a_1 + a_2)} = \frac{1 - a_1a_2 - 2(a_1 + a_2)^2}{3(a_1 + a_2)} = \frac{a_0 - 2a_1 - 2a_2}{3}.
\]

We can thus express the cotangents of this sub-triangle as \( \Lambda \) where

\[
\Lambda = \begin{pmatrix}
\frac{1}{3} & \frac{-x}{y} & \frac{-2}{y} \\
0 & 2 & 1 \\
0 & 1 & 2 \\
5
\end{pmatrix}
\]
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Figure 3. Action of $\Lambda$ and $\sigma$ matrices to produce subdivision of a triangle.

Note that
\[ \Lambda^T \eta \Lambda = \eta \]
since the cotangent identity is preserved. Thus subdivisions are represented by Lorentz transformations in $\mathbb{R}^{1,2}$. Note the symmetry under the interchange of 1 and 2:
\[ \Lambda \tau = \tau \Lambda \]
The cotangents of the remaining sub-triangles are given by cyclic permutations $\Lambda \sigma$ and $\Lambda \sigma^2$ (see Fig. 3). In this convention, the central angle is listed first.

In the complex parametrization $z = x + iy = \frac{a_1 + a_2}{a_1 + a_2}$, the subdivision $\Lambda$ corresponds to
\[ \Lambda(z) = \frac{1 + z}{3} \]
which is the complex co-ordinate of the centroid when $x_1 = 0$, $x_2 = 1$, $x_0 = z$. Recall that in this parametrization $\sigma(z) = \frac{1}{1 + z}$. Clearly, both $\Lambda$ and $\sigma$ map the upper half plane to itself.

The semi-group generated by $(\Lambda, \Lambda \sigma, \Lambda \sigma^2)$ describe repeated subdivisions of a triangle. After many iterations, most of the triangles are flat: they have small area and large perimeter [19, 20]. The dynamics generated by this semi-group is the renormalization group of real space decimations.

4. Renormalization Dynamics

Consider a Gaussian scalar field with values $\phi_0, \phi_1, \phi_2$ at the vertices of a triangle with cotangents $a_0, a_1, a_2$. The most general quadratic form for the discrete approximation to the action will be
\[ S(\phi_0, \phi_1, \phi_2|a) = P(a)\phi_0^2 + Q(a)\phi_1\phi_2 + P(\sigma a)\phi_1^2 + Q(\sigma a)\phi_2\phi_0 + P(\sigma^2 a)\phi_2^2 + Q(\sigma^2 a)\phi_0\phi_1 \]
The coefficients $P(a), Q(a)$ are functions of the cotangents satisfying the symmetry
\[ P(a) = P(\tau a), \quad Q(a) = Q(\tau a). \]
For example, the cotangent formula corresponds to the choice
\[ P(a) = \frac{a_1 + a_2}{4}, \quad Q(a) = -\frac{a_0}{2}. \]
If we subdivide the triangle and associate a field $\phi_3$ at the central vertex, the action will be the sum of contributions from each triangle.
\[ S_{\text{sub}}(\phi_0, \phi_1, \phi_2, \phi_3|a) = S(\phi_3, \phi_1, \phi_2|\Lambda a) + S(\phi_3, \phi_2, \phi_0|\Lambda \sigma a) + S(\phi_3, \phi_0, \phi_1|\Lambda \sigma^2 a) \]
\[ = A\phi_3^2 + B\phi_3 + C \]
On comparing coefficient of $\phi$ is also a fixed point [15] of the above dynamics. It is not hard to verify that

$$Z = \phi_0 \left\{ Q(\sigma \Lambda \sigma a) + Q(\sigma^2 \Lambda^2 a) \right\} + \phi_1 \left\{ Q(\sigma^2 \Lambda a) + Q(\sigma \Lambda \sigma^2 a) \right\} + \phi_2 \left\{ Q(\sigma \Lambda a) + Q(\sigma^2 \Lambda^2 a) \right\} + \phi_1 \phi_2 Q(\Lambda a) + \phi_2 \phi_0 Q(\Lambda \sigma a) + \phi_0 \phi_1 Q(\Lambda \sigma^2 a)$$

The effective action after integrating out the central field variable is given by

$$e^{-\tilde{S}(\phi_0, \phi_1, \phi_2 | a)} = Z \int e^{-S_{ah}(\phi_0, \phi_1, \phi_2, \phi_3 | a)} d\phi_3$$

where $Z = \sqrt{\frac{N_{\phi_0 + \phi_1 + \phi_2}}{2^N}}$ is a normalization constant.

On comparing coefficient of $\phi_1^2$, we get

$$\tilde{P}(a) = P(\sigma^2 \Lambda \sigma a) + P(\sigma \Lambda \sigma^2 a) = \frac{1}{4} \sqrt{\frac{Q(\sigma \Lambda \sigma a) + Q(\sigma^2 \Lambda^2 a)}{P(\Lambda a) + P(\sigma \Lambda a) + P(\Lambda \sigma^2 a)}}$$

On comparing coefficient of $\phi_1 \phi_2$, we get

$$\tilde{Q}(a) = Q(\Lambda a) - \frac{1}{2} \frac{Q(\sigma^2 \Lambda a) + Q(\sigma \Lambda \sigma^2 a)}{P(\Lambda a) + P(\sigma \Lambda a) + P(\Lambda \sigma^2 a)}$$

Using $A \tau = \tau A, \tau \sigma = \sigma^2$ we can verify that $\tilde{P}(\tau a) = \tilde{P}(a), \tilde{Q}(\tau a) = \tilde{Q}(a)$ as needed for symmetry. The denominator

$$A(a) = P(\Lambda a) + P(\sigma \Lambda a) + P(\Lambda \sigma^2 a)$$

is invariant under $\sigma, \tau$ and hence, under all permutations.

The semi-group generated by the map $R : (P, Q) \mapsto (\tilde{P}, \tilde{Q})$ on the space of pairs of functions on the hyperboloid is the renormalization dynamics ("renormalization group"). This explicit example should help understand such dynamics. For example, is there a notion of entropy that increases monotonically? Its fixed points correspond to some sort of continuum limit (which could be fractal [21, 22]).

5. Fixed Points

An obvious fixed point consists of constant $P, Q$. This corresponds to the "Apol- lonian subdivisions" considered in an earlier paper [13].

We now show that the cotangent formula of the FEM

$$P(a) = \frac{a_1 + a_2}{4}, \quad Q(a) = -\frac{a_0}{2}$$

is also a fixed point [15] of the above dynamics. It is not hard to verify that

$$A = P(\Lambda a) + P(\Lambda \sigma a) + P(\Lambda \sigma^2 a) = \frac{3}{2} (a_0 + a_1 + a_2)$$

$$Q(\sigma \Lambda \sigma a) + Q(\sigma^2 \Lambda \sigma^2 a) = -(a_0 + a_1 + a_2)$$

$$P(\sigma^2 \Lambda \sigma a) + P(\sigma \Lambda \sigma^2 a) = \frac{1}{12} [2a_0 + 5(a_1 + a_2)]$$

so that $\tilde{P}(a) = \frac{a_1 + a_2}{4}$.  

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Similarly,

\[
Q(\Lambda a) = -a_0 + 2a_1 + 2a_2
\]

\[
Q(\sigma^2 \Lambda a) + Q(\sigma \Lambda \sigma^2 a) = -(a_0 + a_1 + a_2)
\]

\[
Q(\sigma \Lambda a) + Q(\sigma^2 \Lambda a) = -(a_0 + a_1 + a_2)
\]

from which \( \bar{Q}(a) = -\frac{a}{2} \) follows.

This fixed point describes some sort of continuum limit of two dimensional scalar field theory. As in the examples of Ref. [13] it is likely to be a fractal of dimension less than two; but we have not been able to determine this dimension yet. An extension of this method to higher dimensions and to gauge theories would be interesting. We hope to return to these issues in the future.

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