Quantum backreaction (Casimir) effect
II. Scalar and electromagnetic fields

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Abstract

Casimir effect in most general terms may be understood as a backreaction of a quantum system causing an adiabatic change of the external conditions under which it is placed. This paper is the second installment of a work scrutinizing this effect with the use of algebraic methods in quantum theory. The general scheme worked out in the first part is applied here to the discussion of particular models. We consider models of the quantum scalar field subject to external interaction with “softened” Dirichlet or Neumann boundary conditions on two parallel planes. We show that the case of electromagnetic field with softened perfect conductor conditions on the planes may be reduced to the other two. The “softening” is implemented on the level of the dynamics, and is not imposed ad hoc, as is usual in most treatments, on the level of observables. We calculate formulas for the backreaction energy in these models. We find that the common belief that for electromagnetic field the backreaction force tends to the strict Casimir formula in the limit of “removed cutoff” is not confirmed by our strict analysis. The formula is model dependent and the Casimir value is merely a term in the asymptotic expansion of the formula in inverse powers of the distance of the planes. Typical behaviour of the energy for large separation of the plates in the class of models considered is a quadratic fall-off. Depending on the details of the “softening” of the boundary conditions the backreaction force may become repulsive for large separations.

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1 Introduction

This is the second of the two papers devoted to the Casimir effect in which we develop more fully what was announced in [1]. In these papers we advocate the use of the algebraic approach to the quantum systems as the most natural setting for the discussion of the Casimir effect. This approach gives a clear understanding of the sources of the difficulties one encounters in more traditional treatments, and allows a mathematically rigorous analysis of the effect. In the first paper [1] this analysis was carried out on a more general level, for a wide class of quantum systems and external conditions. We derived a general criterion for admissible models and obtained formulae for the backreaction energy and generalized force. The reader should refer to [1] for the background and our statement of the problem, and the results mentioned here. Sections 1 – 5 of that paper form prerequisites for the present one, and in what follows we assume their knowledge by the reader. Also, we refer the reader to [1] for bibliography.

The main results and outline of the paper are as follows. In Section 2 we show how the quantum scalar field with external interaction of planar symmetry fits into the scheme discussed in Sections I-3 and I-5. More precisely, the perturbing interaction is assumed to modify the \( z \)-motion only (the system is then translationally symmetric in the directions orthogonal to the \( z \)-axis). The necessary preliminary condition (ii) of Sec. I-5 for the admissibility of the model is formulated in terms of the \( z \)-motion generator. In Section 3 we obtain the necessary and sufficient condition for the equivalence of the vacuum representation and the ground state representation of the field influenced by the perturbation. In Section 4 this condition is then extended to be also necessary and sufficient for the existence of finite backreaction energy. Next, in Section 5 we propose a class of models of the field perturbation imitating the Dirichlet or Neumann conditions on two parallel planes separated by an adiabatically changeable distance. The models depend on two functions whose role is to soften the effect of the planes for high energies of the particles. For a class of these functions the admissibility conditions are satisfied and an explicit formula for the Casimir energy is obtained. We investigate this formula in Section 6. We show that the original Casimir expression constitutes a term in the expansion of the formula in inverse powers of the distance of the planes. Using a scaling property of the formula we can approximate the strict boundary conditions. In the case of Dirichlet conditions the Casimir energy becomes infinite, but the Casimir force may be interpreted to approach the Casimir value in the limit (although with qualifications). However, in the Neumann case both energy and force become meaningless in the limit. We view this as a typical situation, the Dirichlet case being exceptional. Section 7 treats the electromagnetic case with metallic boundary conditions. We show that this model may be reduced to the superposition of scalar Dirichlet and Neumann cases. Therefore, there is no strict boundary limit for this model. We find that in the class of models considered here typical fall-off of the backreaction force for large separation of the plates is by one order weaker than in the original Casimir formula, and the force may become re-
pulsive in the limit. Appendices contain more technical results needed in the main text.

Precise formulation of the central results of the paper is given in (Mod) in Section 5 (Asym) in Section 6 and in Section 7. Also, more extensive discussion of their physical significance will be found in the opening parts of Sections 5 and 6 and in the closing part of Section 7.

2 Scalar field under external conditions with planar symmetry

We apply here the formalism of Sections I-3 and I-5 to more specific models. We take the real Hilbert space \( R \) of Sec. I-3 to be the tensor product of a space \( R_\perp \) of the motion in the \( x-y \) plane, and of a space \( R_z \) of the motion in the \( z \)-direction, \( R = R_\perp \otimes R_z \). For the complexified versions of these spaces we also have \( K = K_\perp \otimes K_z \). Let a positive operator \( h_\perp \) in \( K_\perp \), with domain \( D(h_\perp) \), describe the perpendicular motion, and a positive operator \( h_z \) in \( K_z \), with domain \( D(h_z) \), describe the \( z \)-motion. (Both operators are assumed to commute with the complex conjugation.) Then the operator \( h \) in \( K \) defined in standard way by the form method as

\[
D(h) = D(h_\perp \otimes \text{id}) \cap D(\text{id} \otimes h_z), \quad h = \sqrt{(h_\perp \otimes \text{id})^2 + (\text{id} \otimes h_z)^2},
\]

(2.1)
is a densely defined, selfadjoint, positive operator. More precisely, the prescription

\[
q(\psi, \varphi) = ((h_\perp \otimes \text{id})\psi, (h_\perp \otimes \text{id})\varphi) + ((\text{id} \otimes h_z)\psi, (\text{id} \otimes h_z)\varphi)
\]

(2.2)
defines a closed quadratic form on \( D(h_\perp \otimes \text{id}) \cap D(\text{id} \otimes h_z) \). This form determines in the standard way the unique positive operator \( h \) by \( q(\psi, \varphi) = (h\psi, h\varphi) \). This operator defines a quantum model, as described in Sec. I-3.

The free quantum scalar field model fits into this scheme with the choices \( K_\perp = L^2(\mathbb{R}^2, dx \, dy) \), \( K_z = L^2(\mathbb{R}, dz) \), \( h_\perp^2 = -\Delta_\perp \), \( h_z^2 = -\partial_z^2 \), where \( \Delta_\perp \) is the two-dimensional Laplacian. We now want to introduce external conditions which modify the \( z \)-motion, while leaving the transversal motion unchanged. However, if we leave the strict translational symmetry in the \( x-y \) plane intact, then the bound (I-5.8), which gives the condition for the implementability of the modified dynamics in the original representation (technically: the implementability of an appropriate Bogoliubov transformation), cannot be satisfied. Both mathematical and physical reason is clear: this condition states that in the new state the particle number \( N_a \) is finite, which cannot be satisfied due to the translational symmetry in the \( x-y \) plane. Nevertheless, we can apply the standard “thermodynamic limit” procedure for the transversal directions: we restrict the \( x-y \) motion to some compact region, demand finiteness of particle number \( N_a \) and Casimir energy \( E_a \) (conds. (I-5.8,9)), and then aim at obtaining finite limits of these values “per unit area” when the region increases to the whole plane. We shall not investigate the problem of the
thermodynamic limit in full generality and restrict attention to the following cases.
The modified space $\mathcal{K}_\perp$ is a Hilbert space of functions on a rectangle with sides $(L_x, L_y)$, the modified operator $h_\perp$ has an orthonormal basis of eigenvectors:

$$h_\perp \psi_{kl} = \epsilon_{kl} \psi_{kl}, \quad \epsilon_{kl} = \sqrt{(k\epsilon_x)^2 + (l\epsilon_y)^2}, \quad \epsilon_x = \pi/L_x, \quad \epsilon_y = \pi/L_y,$$

and the scope of the values of $(k, l)$ is either

$$(D) \quad \mathbb{N} \times \mathbb{N}, \quad \text{or} \quad (N) \quad (\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{0\}).$$

For the scalar field the following choices lead to these two cases:

$$(D) \quad \mathcal{K}_\perp = L^2((-L_x/2, L_x/2) \times (-L_y/2, L_y/2), dx \, dy) \text{ and } -h_\perp^2 \text{ is the two-dimensional Laplace operator with the Dirichlet boundary conditions;}$$

$$(N) \quad \mathcal{K}_\perp \text{ is the orthogonal complement of constant functions in the space } L^2((-L_x/2, L_x/2) \times (-L_y/2, L_y/2), dx \, dy), \text{ and } -h_\perp^2 \text{ is the two-dimensional Laplace operator with the Neumann boundary conditions.}$$

In the case of Neumann boundary conditions we exclude constants, as they would lead to more singular case. Related, but slightly different choices, also lead to the cases (D) and (N) for the electromagnetic field (see Section 7). For this field the absence of constants appears naturally.

With these modifications we can now introduce external conditions. As a result the $z$-motion Hamiltonian $h_z$ is replaced by a new positive operator $h_{za}$, and in analogy to Eq. (2.1) we have

$$D(h_a) = D(h_\perp \otimes \text{id}) \cap D(\text{id} \otimes h_{za}), \quad h_a = \sqrt{(h_\perp \otimes \text{id})^2 + (\text{id} \otimes h_{za})^2}.$$  

A particular choice of $h_{za}$ will be proposed later, but first we want to identify all models fitting into the scheme, i.e. all perturbations which are admissible in the sense described in [I]. A preliminary condition for this is that the symplectic mapping $L_a$ determined by $h$ and $h_a$ as in Section I-5 be bounded, with bounded inverse – condition (i) in that section. In the rest of the present section we find an equivalent form of this condition in terms of the $z$-motion Hamiltonians $h_z$ and $h_{za}$, and express quantities $\mathcal{N}_a$ and $\mathcal{E}_a$ in terms of these operators. In the next two sections the conditions for finiteness of $\mathcal{N}_a$ and $\mathcal{E}_a$ and their infinite plane limits per area are obtained.

First of all we observe that each of the subspaces $\psi_{kl} \otimes \mathcal{K}_z$ is invariant under both operators $h$ and $h_a$. If we fix the basis $\{\psi_{kl}\}$ (we give up the freedom of phase factor multiplication) then each of the subspaces $\psi_{kl} \otimes \mathcal{K}_z$ is naturally unitarily isomorphic to $\mathcal{K}_z$. It is then easy to see that if we denote for $u \geq 0$

$$D(h(u)) = D(h(z)), \quad h(u) = \sqrt{h_{za}^2 + u \text{id}},$$  

$$D(h_a(u)) = D(h_{za}), \quad h_a(u) = \sqrt{h_{za}^2 + u \text{id}}.$$
then also
\[ \mathcal{D}(h^{1/2}(u)) = \mathcal{D}(h_z^{1/2}), \quad \mathcal{D}(h_a^{1/2}(u)) = \mathcal{D}(h_{za}^{1/2}) \quad (2.8) \]

and one can use the following identifications
\[ \mathcal{K} = \bigoplus_{kl} (K_z)_{kl}, \quad h^{1/2} = \bigoplus_{kl} h_z^{1/2}(\epsilon_{kl}^2), \quad h_a^{1/2} = \bigoplus_{kl} h_a^{1/2}(\epsilon_{kl}^2), \quad (2.9) \]

where \((K_z)_{kl}\) are identical copies of \(K_z\). The spectrum of both \(h_z^{1/2}(u)\) and \(h_a^{1/2}(u)\)

is contained in \([u^{1/4}, \infty)\), so both operators \(h_z^{-1/2}\) and \(h_a^{-1/2}\) are bounded, with

\[ \|h_z^{-1/2}\|, \|h_a^{-1/2}\| \leq (\min_{kl} \{\epsilon_{kl}\})^{-1/2}. \]

After these preliminaries it is now easy to show that the following conditions are equivalent:

(i) The operators \(h\) and \(h_a\) (cf. (2.4) and (2.5) resp.) satisfy the conditions
\[ \mathcal{D}(h^{\pm 1/2}) = \mathcal{D}(h_z^{\pm 1/2}), \quad (2.10) \]
\[ B_a \equiv h_a^{1/2} h^{-1/2} \text{ and } B_a^{-1} \text{ extend to bounded operators in } \mathcal{K}. \quad (2.11) \]

(i)' The operators \(h_z\) and \(h_{za}\) satisfy the conditions
\[ \mathcal{D}(h_z^{1/2}) = \mathcal{D}(h_{za}^{1/2}), \quad (2.12) \]
\[ B_a(u) \equiv h_a^{1/2}(u) h^{-1/2}(u) \text{ and } B_a^{-1}(u) \text{ are uniformly bounded on each set } u \in [v, \infty), \quad v > 0. \quad (2.13) \]

Note that the condition (i) is identical with the condition (ii) of Section I-5, which is equivalent to the symplectic mapping \(L_a\) being bounded together with its inverse (condition (i) in Sec. I-5).

To prove the equivalence suppose first that (i)' holds. Then
\[ B_a = \bigoplus_{kl} B_a(\epsilon_{kl}^2), \quad (2.14) \]

so \(\|B_a^{\pm 1}\| \leq \max_{kl} \|B_a^{\pm 1}(\epsilon_{kl})\| \leq \text{const}\). Then \(h_z^{1/2} = B_a h^{1/2}\) and \(h^{1/2} = B_a^{-1} h_a^{1/2}\), so \(h_z^{1/2}\) and \(h_a^{1/2}\) have a common domain. As \(h^{-1/2}\) and \(h_a^{-1/2}\) are bounded, this ends the proof of (i). Conversely, suppose that (i) is true. Then the restrictions of the domains of \(h_z^{1/2}\) and \(h_a^{1/2}\) to \((K_z)_{kl}\) must be equal, which implies Eq. (2.12).

Condition (2.11) implies that \(B_a^{\pm 1}(\epsilon_{kl})\) are uniformly bounded. But for \(u, w > 0\)
\[ B_a(u) = h_z^{1/2}(u) h_a^{-1/2}(w) B_a(w) h_z^{1/2}(w) h^{-1/2}(u), \quad (2.15) \]
so by spectral analysis for \(h\) and \(h_a\) one finds
\[ \|B_a^{\pm 1}(u)\| \leq \max \left\{ \left(\frac{u}{w}\right)^{1/4}, \left(\frac{w}{u}\right)^{1/4} \right\} \|B_a^{\pm 1}(w)\|. \quad (2.16) \]
Setting here \( w = \epsilon_{kl}^2 \) and \( u \) in the interval between \( \epsilon_{kl}^2 \) and a neighbouring value of \( \epsilon_{kl}^2 \), or between zero and \( \epsilon_{kl}^2 \), if the latter is the minimal value, one shows that the condition (2.13) is satisfied.

The equivalence shows that if (i) is satisfied for some \( L_x, L_y \), then it is true for all finite values of these parameters.

If the equivalent conditions (i) and (i)' are satisfied then the operators \( T_a \) and \( S_a \) defining the Bogoliubov transformation are bounded, and given by

\[
T_a = \bigoplus_{kl} T_a(\epsilon_{kl}^2), \quad S_a = \bigoplus_{kl} S_a(\epsilon_{kl}^2),
\]

where

\[
T_a(u) = \frac{1}{2} \left[ B_a^{-1}(u) + B_a^*(u) \right], \quad S_a(u) = \frac{1}{2} \left[ B_a^{-1}(u) - B_a^*(u) \right] K, \quad (2.18)
\]

and \( K \) is the operator of complex conjugation. The quantities \( N_a \) (particle number) and \( \mathcal{E}_a \) (Casimir energy), which have to be investigated, may be now written as

\[
N_a = \text{Tr}[S_a S_a^*] = \sum_{kl} \text{Tr} \left[ N_a(\epsilon_{kl}^2) \right],
\]

where \( N_a(u) = S_a(u) S_a^*(u) \),

\[
\mathcal{E}_a = \text{Tr} \left[ h^{1/2} S_a S_a^* h^{1/2} \right] = \sum_{kl} \text{Tr} \left[ E_a(\epsilon_{kl}^2) \right],
\]

where \( E_a(u) = h^{1/2}(u) S_a(u) S_a^*(u) h^{1/2}(u) \),

where we do not know yet whether these expressions are finite (but they are unambiguously defined as positive numbers, finite or not, as the operators are positive).

3 The necessary and sufficient condition for the unitary equivalence of representations and the existence of finite limit

\[ n_a \equiv \lim_{L_x, L_y \to \infty} \frac{N_a}{L_x L_y} \]

We now add the condition:

(ii) The ground state representations determined by \( h \) and \( h_a \) are unitarily equivalent, and the infinite plane limit of particle number per area is well defined, i.e.

\[
N_a < \infty \quad \text{for all} \quad L_x, L_y, \quad \exists \lim_{L_x, L_y \to \infty} \frac{N_a}{L_x L_y} < \infty. \quad (3.1)
\]
We show in this section that the conditions (i) and (ii) are satisfied if, and only if:

(Eq) The operators $h_z$ and $h_{za}$ have a common domain and $h_{za} - h_z$ extends to a Hilbert-Schmidt operator on $K_z$, i.e.

$$\text{Tr}(h_{za} - h_z)^2 < \infty.$$  \hspace{1cm} (3.2)

In this case

$$n_a \equiv \lim_{L_x, L_y \to \infty} \frac{N_a}{L_x L_y} = \frac{1}{16\pi} \text{Tr}(h_{za} - h_z)^2.$$  \hspace{1cm} (3.3)

In the case of Dirichlet boundary conditions for $h_{\perp}$ (case (D) in Eq. (2.4)) the limit in Eqs. (3.1) and (3.3) can be taken in arbitrary way, but in the case of Neumann conditions (case (N) in Eq. (2.4)) one has to keep $1/M \leq L_x/L_y \leq M$ for some arbitrary, but fixed $M > 1$ (which means that the limit takes place over the values of $(L_x, L_y)$ within a conic proper subset of $(0, \infty) \times (0, \infty)$).

3.1 (Eq) $\Rightarrow$ (i) and (ii)

Suppose first that the condition (Eq) is true, and denote $c = \|h_{za} - h_z\|$. Then for $\varphi \in D(h_{za}) = D(h_z)$ we have

$$\|h_a(u)\varphi\|^2 = \|h_{za}\varphi\|^2 + u\|\varphi\|^2 \leq (\|h_z\varphi\| + c\|\varphi\|)^2 + u\|\varphi\|^2$$

$$< \left(1 + \frac{c}{\sqrt{u}} + \frac{c^2}{u}\right)(\|h_z\varphi\|^2 + u\|\varphi\|^2) = \left(1 + \frac{c}{\sqrt{u}} + \frac{c^2}{u}\right)\|h(u)\varphi\|^2.$$  

Now we use the monotonicity of the square root: if $A$ and $B$ are positive operators, $D(A) \subseteq D(B)$ and $\|B\psi\| \leq \|A\psi\|$ for all $\psi \in D(A)$, then also $D(A^{1/2}) \subseteq D(B^{1/2})$ and $\|B^{1/2}\psi\| \leq \|A^{1/2}\psi\|$ for all $\psi \in D(A^{1/2})$ (see [2], proof of Thm. X.18). Thus we get

$$\|h_a^{1/2}(u)\varphi\| < \left(1 + \frac{c}{\sqrt{u}} + \frac{c^2}{u}\right)^{1/4}\|h^{1/2}(u)\varphi\|$$

This implies $\|B_a(u)\| < (1 + c/\sqrt{u} + c^2/u)^{1/4}$. Changing the roles of $h(u)$ and $h_a(u)$ one exhausts the condition (i)'.

We introduce a bounded selfadjoint operator on $K_z$:

$$\Delta_a(u) = h_a(u) - h(u)$$  \hspace{1cm} (3.4)

(boundedness by $\|h_a(u) - h_a(0)\| \leq \sqrt{u}$, $\|h(u) - h(0)\| \leq \sqrt{u}$ and the boundedness of $h_{za} - h_z$). In the further course of the proof we shall need several theorems on Hilbert-Schmidt properties of operators like $\Delta_a(u)$ and $S_a(u)$. As their demonstration is somewhat more technical we shift their derivation to Appendix B and in the main text use the results. We also remind the reader the following facts on
operators in Hilbert space: if $B$ is bounded and $H$ is Hilbert-Schmidt, then $BH$ and $HB$ are Hilbert-Schmidt, and if in addition $K$ is also Hilbert-Schmidt, then
\[ \text{Tr}(BHK) = \text{Tr}(HKB) = \text{Tr}(KBH). \]

Furthermore, if $B$ is in addition positive, then
\[ \text{Tr} \left( \sqrt{B}HH^*\sqrt{B} \right) = \text{Tr}(H^*BH) \leq \|B\| \text{Tr}(H^*H). \]

We use these properties extensively in what follows.

We have to show that the expression (2.19) satisfies condition (ii), and Eq. (3.3) holds. We have assumed that $\Delta a(0)$ is a HS operator. It follows then from the result (iv) of Appendix B that for all $u \geq 0$ the operators $\Delta a(u)$ are also Hilbert-Schmidt, and the function $(0, \infty) \ni u \mapsto \text{Tr} \Delta a^2(u)$ is continuous on its domain and continuously differentiable on $(0, \infty)$, decreasing, and $\lim_{u \to \infty} \text{Tr} \Delta a^2(u) = 0$.

Moreover, the differentiation on $u$ may be carried out by formally pulling the derivative under the trace sign and differentiating formally the $u$-dependent operators. Using the formal rule
\[ 2 \frac{d}{du} \Delta a(u) = -h^{-1}(u)\Delta a(u)h^{-1}(u) = -h^{-1}(u)\Delta a(u)h^{-1}(u) \quad (3.5) \]
and the identity
\[ S_a(u) = -\frac{1}{2}h^{-1/2}(u)\Delta a(u)h^{-1/2}(u)K \quad (3.6) \]
following from the formula (2.18) one finds
\[ 2 \frac{d}{du} \text{Tr} \Delta a^2(u) = -\text{Tr} \left[ h^{-1}(u)\Delta a(u)h^{-1}(u)\Delta a(u) \right] \]
\[ - \text{Tr} \left[ \Delta a(u)h^{-1}(u)\Delta a(u)h^{-1}(u) \right] = -8 \text{Tr} N_a(u). \quad (3.7) \]

It is now clear that the function $(0, \infty) \ni u \mapsto \text{Tr} N_a(u)$ is continuous, positive, and
\[ \int_0^\infty \text{Tr} N_a(u) \, du = \frac{1}{4} \text{Tr} \Delta a^2(0) < \infty. \quad (3.8) \]

Again by the result (iv) of Appendix B the function $(0, \infty) \ni u \mapsto \text{Tr} N_a(u)$ is decreasing. With the use of formulas (A.4) and (A.2) of Appendix A this is sufficient to conclude that in the case (D) of Eq. (2.4) the condition (3.1), and Eq. (3.3) are satisfied. In the case (N) of Eq. (2.4) $N_a$ is bigger by
\[ N_a^{(N)} - N_a^{(D)} = \epsilon_x \epsilon_y \sum_{l=1}^{\infty} \text{Tr} N_a((l\epsilon_y)^2) + \{x \leftrightarrow y\}. \]

If one keeps $1/M \leq \epsilon_x/\epsilon_y \leq M$ then
\[ N_a^{(N)} - N_a^{(D)} \leq M \epsilon_y^2 \sum_{l=1}^{\infty} \text{Tr} N_a((l\epsilon_y)^2) + M \epsilon^2_x \sum_{l=1}^{\infty} \text{Tr} N_a((l\epsilon_x)^2), \]

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which by (3.8) and the properties (A.5) and (A.15) of Appendix A is finite and tends to zero for \( \epsilon_x, \epsilon_y \to 0 \). Thus in both cases (D) and (N) the condition (ii) and the equation (3.3) follow from the criterion (3.2).

3.2 (i) and (ii) \( \Rightarrow \) (Eq)

We now turn to the proof that the criterion (Eq) follows from (i) and (ii). This fact is of interest, as it shows that in the context adopted in this paper there is no escape from this condition. Suppose that the first of conditions (3.1) is satisfied. By the result (iv) of Appendix B if \( \text{Tr} \ N_a(u) \) is finite for any \( u > 0 \), then it is finite for all \( u \in (0, +\infty) \), and the function \( (0, +\infty) \ni u \mapsto \text{Tr} \ N_a(u) \) is continuously differentiable and decreasing. Thus if \( N_a \) is finite for all \( L_x, L_y \), then by the first of inequalities in (A.14) we have for each \( v > 0 \):

\[
\int_v^\infty \text{Tr} \ N_a(u) \, du < \infty. \tag{3.9}
\]

Let now \( P_{\beta} = P_{(0, \beta)}(h_z) \) be the projection operator from the spectral family of the operator \( h_z \), projecting onto the spectral interval \( (0, \beta) \), and similarly \( P_{\alpha\gamma} = P_{(0, \gamma)}(h_{za}) \) for the operator \( h_{za} \). Then for each \( u > 0 \) the expression \( P_{\beta} \Delta_a(u) P_{\alpha\gamma} \) defines a bounded operator (although we do not know yet whether \( \Delta_a(u) \) alone makes sense), and

\[
\text{Tr}[P_{\beta} \Delta_a(u) P_{\alpha\gamma} \Delta_a(u) P_{\beta}] \leq 4 \sqrt{(\beta^2 + u)(\gamma^2 + u)} \text{Tr}[P_{\beta} S_a(u) P_{\alpha\gamma} S_a^*(u) P_{\beta}] \\
\leq 4 \sqrt{(\beta^2 + u)(\gamma^2 + u)} \text{Tr} \ N_a(u) \to 0 \quad (u \to \infty), \tag{3.10}
\]

the last line by the property (A.4) (note that here and in what follows all operators under the trace sign are positive). It follows from the result (iv)_B of Appendix B that the function on the l.h. side may be differentiated as that in Eq. (3.7), so one finds

\[
\frac{d}{du} \text{Tr}[P_{\beta} \Delta_a(u) P_{\alpha\gamma} \Delta_a(u) P_{\beta}] = -4 \text{Tr}[P_{\beta} S_a(u) P_{\alpha\gamma} S_a^*(u) P_{\beta}]. \tag{3.11}
\]

Integrating this identity on \( \langle v, \infty \rangle, \ v > 0 \), one has

\[
\text{Tr}[P_{\beta} \Delta_a(v) P_{\alpha\gamma} \Delta_a(v) P_{\beta}] = 4 \int_v^\infty \text{Tr}[P_{\beta} S_a(u) P_{\alpha\gamma} S_a^*(u) P_{\beta}] \, du. \tag{3.12}
\]

The integrand on the r.h. side is bounded by \( \text{Tr} \ N_a(u) \), so the double limit \( \beta, \gamma \to \infty \) exists. Thus

\[
\text{Tr} \Delta_a^2(v) = 4 \int_v^\infty \text{Tr} \ N_a(u) \, du. \tag{3.13}
\]
Now we let the second of the assumptions in (3.1) come into play. By formula (A.2) this implies that the r.h. side of the last formula has a finite limit for $v \to 0$.

As a result we have

$$\text{Tr} \Delta^2_a(v) \leq 4 \int_0^\infty \text{Tr} N_a(u) \, du < \infty.$$  
(3.14)

By the result (iv)_B this implies that $\Delta_a(0)$ is HS, which completes the proof that (A.2) is a necessary condition.

4 The necessary and sufficient condition for finite backreaction energy and its limit

$$\varepsilon_a \equiv \lim_{L_x, L_y \to \infty} \frac{\mathcal{E}_a}{L_x L_y}$$

We now add the energy condition:

(iii) The backreaction energy $\mathcal{E}_a$ is finite for all $L_x, L_y$, and the limit of Casimir energy per area is well defined for infinite plane limit:

$$\mathcal{E}_a < \infty, \quad \exists \lim_{L_x, L_y \to \infty} \frac{\varepsilon_a}{L_x L_y} < \infty.$$  
(4.1)

We show in this section that if the conditions (i) and (ii) (or criterion (Eq)) are satisfied, then (iii) is fulfilled if, and only if:

(En) The operator $(h_{za} - h_z)h_z^{1/2}$ is a Hilbert-Schmidt operator, that is

$$\text{Tr}[(h_{za} - h_z)h_z(h_{za} - h_z)] < \infty.$$  
(4.2)

If this condition is satisfied, then

$$\varepsilon_a \equiv \lim_{L_x, L_y \to \infty} \frac{\mathcal{E}_a}{L_x L_y} = \frac{1}{24\pi} \text{Tr}[(h_{za} - h_z)(2h_z + h_{za})(h_{za} - h_z)].$$  
(4.3)

The limit with respect to $L_x, L_y$ is specified as in (3.3).

4.1 (Eq) and (En) $\Rightarrow$ (iii)

The proof is closely analogous to the one in the last section. Suppose that criterions (Eq) and (En) are satisfied. We have to show that the expression (4.1) satisfies the conditions (4.1) and Eq. (4.3). We have assumed that $\text{Tr}[(\Delta_a(0))^2] < \infty$ and $\text{Tr}[(\Delta_a(0)h(0)\Delta_a(0)) < \infty$. It follows then from the result (v)_B of Appendix B that the functions $u \mapsto \text{Tr}[(\Delta_a(u)h(u)\Delta_a(u)]$ and $u \mapsto \text{Tr}[(\Delta_a(u)h(u)\Delta_a(u)]$ are continuous on $(0, \infty)$ and continuously differentiable on $(0, \infty)$, and tend to zero.
for \( u \to \infty \). Also, the formal differentiation with respect to \( u \) as in Eqs. (3.5), (3.7), yields the correct result. A direct calculation then yields

\[
\frac{1}{6} \frac{d}{du} \left\{ 2 \text{Tr}[\Delta_a(u)h(u)\Delta_a(u)] + \text{Tr}[\Delta_a(u)h_a(u)\Delta_a(u)] \right\} = -\text{Tr} E_a(u),
\]

and the integration leads to

\[
\int_0^\infty \text{Tr} E_a(u) \, du = \frac{1}{6} \text{Tr}[\Delta_a(0)(2h(0) + h_a(0))\Delta_a(0)] < \infty.
\]

By the result (iv)\( _B \) the function \( \text{Tr} E_a(u) \) is non-increasing. By a reasoning analogous to that following Eq. (3.8) one shows that \( \mathcal{E}_a^{(N)} - \mathcal{E}_a^{(D)} \) vanishes in the limit. One concludes that the conditions (4.1) and the equation (4.3) are satisfied.

**4.2 (Eq) and (iii) ⇒ (En)**

Conversely, suppose that \( h_{za} - h_z \) is a Hilbert-Schmidt operator and \( \mathcal{E}_a \) is finite. Similarly as in the case of \( \text{Tr} N_a(u) \) it follows from the result (iv)\( _B \) of Appendix B that if \( \text{Tr} E_a(u) \) is finite for any \( u > 0 \), then it is finite for all \( u \in (0, +\infty) \), and the function \( (0, +\infty) \ni u \mapsto \text{Tr} E_a(u) \) is continuously differentiable and decreasing. Thus if \( \mathcal{E}_a \) is finite, then for each \( v > 0 \)

\[
\int_v^\infty \text{Tr} E_a(u) \, du < \infty.
\]

Therefore

\[
\text{Tr}[P_\beta \Delta_a(u)P_{a\gamma}h_a(u)P_{a\gamma}\Delta_a(u)P_\beta] \\
\leq (\gamma^2 + u) \text{Tr} \left[ P_\beta \Delta_a(u) \frac{P_{a\gamma}}{h_a(u)} \Delta_a(u)P_\beta \right] \\
\leq (\gamma^2 + u) \text{Tr} E_a(u) < \infty,
\]

and

\[
\text{Tr}[P_{a\gamma}\Delta_a(u)P_\beta h(u)P_\beta \Delta_a(u)P_{a\gamma}] \\
\leq \sqrt{(\beta^2 + u)(\gamma^2 + u)} \text{Tr} \left[ \frac{P_{a\gamma}}{h_{za}^{1/2}(u)} \Delta_a(u)P_\beta \frac{P_{a\gamma}}{h_a^{1/2}(u)} \right] \\
\leq \sqrt{(\beta^2 + u)(\gamma^2 + u)} \text{Tr} E_a(u) < \infty,
\]

and by (A.4) both functions tend to zero for \( u \to \infty \). Differentiation is again allowed and gives

\[
\frac{1}{6} \frac{d}{du} \left\{ 2 \text{Tr}[P_{a\gamma}\Delta_a(u)P_\beta h(u)P_\beta \Delta_a(u)P_{a\gamma}] + \text{Tr}[P_\beta \Delta_a(u)P_{a\gamma}h_a(u)P_{a\gamma}\Delta_a(u)P_\beta] \right\} \\
= -\text{Tr} \left[ P_\beta \Delta_a(u) \frac{P_{a\gamma}}{h_a(u)} \Delta_a(u)P_\beta \right].
\]
Now we take into the consideration the second condition in (4.1) and go over the steps analogous to those following Eq. (3.11). This gives
\[ \text{Tr}[\Delta_a(v)h(v)\Delta_a(v)] \leq 3 \int_0^\infty \text{Tr} E_a(u) \, du < \infty. \] (4.10)

This implies, by (v)_B, that \( \Delta_a(0)h^{1/2}(0) \) is also HS, which ends the proof.

5 Modified Dirichlet and Neumann conditions

From now on we can restrict attention to the \( z \)-motion dynamics, and specify a class of models. We set \( K_z = L^2(\mathbb{R}) \), \( h_z = -\partial_z^2 \), where the unique selfadjoint extension of the second derivative defined for \( S(\mathbb{R}) \) is meant. We want the external conditions to be some modification of the strict boundary conditions enforced on the planes \( z = \pm b \), where \( 2b = a \). The Hamiltonian of the \( z \)-motion for such conditions is determined by \( (h_{za}^B)^2 = -\partial_z^2 \), where \( B \) stands for Dirichlet or Neumann conditions at \( z = \pm b \). Let \( F \) and \( G \) be real, non-negative, bounded, measurable functions on \( (0, \infty) \). We postulate a class of models by setting
\[ h_{za} = h_z + G(h_z) [F(h_{za}^B) - F(h_z)] G(h_z). \] (5.1)

We assume that for all \( u \geq 0 \) we have \( F(u) \leq u \) and \( G(u) \leq 1 \), which guarantees the positivity of \( h_{za} \). The strict boundary conditions are recovered by formally setting \( F(u) = u \) and \( G(u) = 1 \) for all \( u \geq 0 \). Our intention is to keep these equalities for \( u \) “not to large”, and modify \( F \) and \( G \) so that for \( u \to \infty \) \( F \) remains bounded, and \( G \) (possibly) tends to zero. This seems to model correctly the idea that the boundaries should become transparent for very energetic particles. We show in this section the following:

(Mod) Let the functions \( F \in C^2((0, \infty)) \) and \( G \in C((0, \infty)) \) satisfy the estimates
\[
0 \leq F(p) \leq \text{const}, \quad F(p) \leq p, \quad 0 \leq G(p) \leq 1, \\
G(p) \leq \text{const}(p + 1)^{-\alpha}, \quad |F^{(2)}(p)| \leq \text{const} (p + 1)^{-2-\gamma} \] (5.2)

for some \( \alpha, \gamma \in (0, 1) \) and such that \( 4\alpha + 2\gamma > 1 \).

Then \( h_{za} - h_z \), where \( h_{za} \) is the modified \( z \)-motion operator given by (5.1), satisfies conditions (Eq) and (En). In this case \( h_{za} - h_z \) is an integral operator
\[
(\psi, [h_{za} - h_z] \psi') = \int \overline{\hat{\psi}(p)} K_a^{F,G}(p, p') \hat{\psi}'(p') \, dp \, dp', \] (5.3)

with \( K_a^{F,G}(p, p') \in L^2(\mathbb{R}^2) \) and \( \sqrt{|p|} K_a^{F,G}(p, p') \in L^2(\mathbb{R}^2) \). The explicit form
of $K^F,G_a$ is obtained in (5.23 – 5.25) below. The Casimir energy is given by
\[
\varepsilon_a = \frac{1}{24\pi} \text{Tr} \left\{ (h_{za} - h_z)[3h_z + (h_{za} - h_z)](h_{za} - h_z) \right\}
\]
\[
= \frac{1}{24\pi} \left\{ \int 3|p| |K^F,G_a(p, p')|^2 dp dp' + \int K^F,G_a(p, p') K^F,G_a(p', p'') K^F,G_a(p'', p) dp dp' dp'' \right\}. \tag{5.4}
\]
Moreover, let for $\mu > 0$:
\[
F_\mu(p) = \mu F(p/\mu), \quad G_\mu(p) = G(p/\mu), \tag{5.5}
\]
and denote by $\varepsilon'_\mu$ the Casimir energy for the model with $F,G$ replaced with $F_\mu,G_\mu$. Then
\[
\varepsilon'_\mu = \mu^3 \varepsilon_\mu. \tag{5.6}
\]
As explained in the opening paragraph of this section, we believe that (5.1), with $F$ and $G$ in suitable classes, defines a class of models reasonably imitating boundary conditions for low energies, while giving the means for “softening” of the walls. More specific of the assumptions on $F$ and $G$ are technical. (Mod) then shows that indeed those models are not only admissible for the discussion of adiabatic backreaction as explained in earlier sections, but also susceptible to a rigorous treatment. Low energy behaviour $F(p) \sim p$ and $G(p) \sim 1$ for “small” $p$, important for the above interpretation, is not needed in (Mod). Once these assumptions are added in the next section, the scaling properties described in (5.5) and (5.6) provide means for investigation of the sharp boundary limit.

5.1 Spectral representation of $h^B_{za}$

The spectral representation for $h_z$ is obtained by Fourier transforming $\psi \mapsto \hat{\psi}$ ($\hat{\psi} \in L^2(\mathbb{R})$), so that in the new representation $h_z$ acts by $\hat{h}_z \hat{\psi}(p) = |p| \hat{\psi}(p)$, where the conventions for the transform are defined by
\[
\hat{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int \psi(z) e^{-ipz} dz. \tag{5.7}
\]
To prove (Mod) it is sufficient to show that in this representation $F(h_{za}^B) - F(h_z)$ is an integral operator with a kernel $K^F_a(p, p')$ such that both functions $G(|p|) K^F_a(p, p') G(|p'|)$ and $\sqrt{|p|} G(|p|) K^F_a(p, p') G(|p'|)$ are in $L^2(\mathbb{R}^2)$.

We start by identifying the spectral representation for $h^B_{za}$. It will be convenient to denote
\[
\delta = \begin{cases} -1 & \text{Dirichlet case}, \\ +1 & \text{Neumann case}, \end{cases} \tag{5.8}
\]

and introduce
\[ L_3^2 = \{ \chi \in L^2(\mathbb{R}) \mid \chi(-z) = \delta \chi(z) \}, \quad l_3^2 = \{ \chi \in l^2(\mathbb{Z}) \mid \chi(-k) = \delta \chi(k) \}. \] (5.9)

For \( \psi \in L^2(\mathbb{R}) \) (in the initial position representation) we denote
\[ \psi_- (z) = \theta(-z-b)\psi(z), \quad \psi_0(z) = \theta(z-b)\psi(z), \quad \psi_+(z) = \theta(z-b)\psi(z). \] (5.10)

Then the following rule defines a unitary transformation:
\[ L^2(\mathbb{R}) \ni \psi \mapsto \Psi = (\hat{\psi_0}, \hat{\psi}_0, \hat{\psi}_+ \in L_3^2 \oplus l_3^2 \oplus L_3^2; \]
\[ \hat{\psi}_0(r) = \frac{1}{\sqrt{2}} \left( e^{-ibr} \hat{\psi}_-(r) + \delta e^{ibr} \hat{\psi}_-(r) \right), \]
\[ \hat{\psi}_0(k) = \sqrt{\frac{\pi}{2a}} \left( i^k \hat{\psi}_0(kr) + \delta i^{-k} \hat{\psi}_0(-kr) \right), \quad \epsilon = \frac{\pi}{2}, \]
\[ \hat{\psi}_+ (r) = \frac{1}{\sqrt{2}} \left( e^{ibr} \hat{\psi}_+(r) + \delta e^{-ibr} \hat{\psi}_+(r) \right), \] (5.11)

where \((\hat{\psi}_-, \hat{\psi}_0, \hat{\psi}_+)\) are Fourier transforms of \((\psi_- , \psi_0, \psi_+)\) respectively. In this new representation operator \( h_{R}^\nu \) acts on \( \hat{\psi}_0 (r) \) as multiplication by \(|r| \), and on \( \hat{\psi}_0 (k) \) as multiplication by \(|k| \). This is easily seen by using an overcomplete set of eigenfunctions in the three regions
\[ \varphi\pm (r, z) = \frac{1}{2\sqrt{\pi}} e^{i(r(\pm b) + \delta e^{-i(r(\pm b))})} = \delta \varphi\pm (-r, z), \quad z \geq \pm b \]
\[ \varphi_0(k, z) = \frac{1}{2\sqrt{a}} \left( i^{-k} e^{ikz} + \delta i^k e^{-ikz} \right) = \delta \varphi_0(-k, z), \quad z \in (-b, b) \] (5.12)

normalized by
\[ \int \varphi\pm (r, z) \varphi\pm (r', z) dz = \frac{1}{2} \left[ \delta(r - r') + \delta (r + r') \right], \]
\[ \int \varphi_0(k, z) \varphi_0(k', z) dz = \frac{1}{2} \left[ \delta_{kk'} + \delta_{k,-k'} \right]. \] (5.13)

We also note for later use that for \( \psi \in \mathcal{S}(\mathbb{R}) \) by standard Fourier methods
\[ \hat{\psi}_0 (r) = \frac{1}{\pi} \int \frac{\sin b(p - r)}{p - r} \hat{\psi}(p) dp, \]
\[ \hat{\psi}_- (r) = -\frac{i}{2\pi} \int \frac{e^{ib(r-p)} \hat{\psi}(p)}{p - r - i0} dp, \quad (\hat{\psi}_- + \hat{\psi}_0)(r) = -\frac{i}{2\pi} \int \frac{e^{-ib(r-p)} \hat{\psi}(p)}{p - r - i0} dp, \]
\[ \hat{\psi}_+ (r) = \frac{i}{2\pi} \int \frac{e^{-ib(r-p)} \hat{\psi}(p)}{p - r + i0} dp, \quad (\hat{\psi}_+ + \hat{\psi}_0)(r) = \frac{i}{2\pi} \int \frac{e^{ib(r-p)} \hat{\psi}(p)}{p - r + i0} dp. \] (5.14)
5.2 The integral kernel of $h_{za} - h_z$

Let $\psi, \psi' \in \mathcal{S}(\mathbb{R})$. Using the spectral representation (5.11) one finds

$$
(\psi, F(h_{za}^B)\psi') = \int F(|r|) \left[ \hat{\psi}_-(r)\hat{\psi}'_-(r) + \hat{\psi}_+(r)\hat{\psi}'_+(r) \right] dr
$$

$$
+ \delta \int F(|r|) \left[ e^{-iar}\hat{\psi}_-(r)\hat{\psi}'_-(r) + e^{iar}\hat{\psi}_+(r)\hat{\psi}'_+(r) \right] dr
$$

$$
+ \epsilon \sum_{k \in \mathbb{Z}} F(|k|e)\hat{\psi}_0(ke)\hat{\psi}'_0(ke) + \delta \sum_{k \in \mathbb{Z}} F(|k|e)(-1)^k\hat{\psi}_0(-ke)\hat{\psi}'_0(ke)
$$

(5.15)

For further evaluation we shall need results of the Appendix C. We also use the notation introduced at the beginning of that appendix.

We represent the terms in brackets under the integral sign in the first line by

$$
\hat{\psi}_-(r)\hat{\psi}'_-(r) + \hat{\psi}_+(r)\hat{\psi}'_+(r) = \hat{\psi}(r)\hat{\psi}'(r) - \hat{\psi}_0(r)\hat{\psi}'(r)
$$

$$
- \hat{\psi}_-(r)(\hat{\psi}'_0(r) - \hat{\psi}_0(r)(\hat{\psi}'_+(r) + \hat{\psi}'_+(r))
$$

(5.16)

and use representation given in (5.14). Then for the integration of the terms in the second line of the last identity we can use identity (C.21). We get

$$
\int F(|r|) \left[ \hat{\psi}_-(r)\hat{\psi}'_-(r) + \hat{\psi}_+(r)\hat{\psi}'_+(r) \right] dr = \int F(|r|)\hat{\psi}(r)\hat{\psi}'(r) dr
$$

$$
+ \int \left[ \frac{1}{2\pi^2} \cos[b(p - p')]\chi_F(p, p') - \frac{\sin[b(p - p')]}{2\pi(p - p')} \left[ F(|p|) + F(|p'|) \right] \hat{\psi}(p)\hat{\psi}'(p') \right] dpdp'.
$$

(5.17)

For the integral in the second line in (5.15) we again use (C.21) to obtain

$$
\int F(|r|) \left[ e^{-iar}\hat{\psi}_-(r)\hat{\psi}'_-(r) + e^{iar}\hat{\psi}_+(r)\hat{\psi}'_+(r) \right] dr
$$

$$
= \int \left[ -\frac{1}{2\pi^2} \cos[b(p - p')]\chi_F(p, -p')
$$

$$
+ \frac{\sin[b(p - p')]}{2\pi(p + p')} \left[ F(|p|) - F(|p'|) \right] \hat{\psi}(p)\hat{\psi}'(p') \right] dpdp'.
$$

(5.18)

To evaluate the sums in the last line of Eq. (5.15) we represent $\hat{\psi}_0$ as in (5.14) and
use identities \((C.24)\) and \((C.25)\), which results in

\[
\begin{align*}
\epsilon \sum_{k \in Z} F(|k|) \overline{\psi_0(k \epsilon)} \tilde{\psi}_0^\prime(k \epsilon) &= \int \left[ \frac{1}{2\pi^2} \cos[b(p - p')] |x_{F,\epsilon}(p, p')| \\
&\quad + \frac{1}{2\pi^2} \cos[b(p + p')] |x_{F,\epsilon}(p, p') - x_{F,2\epsilon}(p, p')| \\
&\quad + \frac{\sin|b(p - p')|}{2\pi(p - p')} \left[ F(|p|) + F(|p'|) \right] \tilde{\psi}(p) \tilde{\psi}'(p') \, dp \, dp',
\end{align*}
\]

(5.19)

\[
\begin{align*}
\epsilon \sum_{k \in Z} F(|k|) (-1)^k \overline{\psi_0(-k \epsilon)} \tilde{\psi}_0^\prime(k \epsilon) &= \int \left[ -\frac{1}{2\pi^2} \cos[b(p + p')] |x_{F,\epsilon}(p, -p')| \\
&\quad - \frac{1}{2\pi^2} \cos[b(p + p')] |x_{F,\epsilon}(p, -p') - x_{F,2\epsilon}(p, -p')| \\
&\quad - \frac{\sin|b(p - p')|}{2\pi(p + p')} \left[ F(|p|) - F(|p'|) \right] \tilde{\psi}(p) \tilde{\psi}'(p') \, dp \, dp'.
\end{align*}
\]

(5.20)

Finally, we note that

\[
(\psi, F(h_z) \psi') = \int F(|r|) \overline{\psi(r)} \tilde{\psi}'(r) \, dr.
\]

(5.21)

Setting now expressions \((5.17) - 5.19)\) into \((5.15)\) and subtracting \((5.21)\) we find

\[
(\psi, [F(h^B_{z \alpha}) - F(h_z)] \psi') = \int \tilde{\psi}(p) K^B_{a}(p, p') \tilde{\psi}'(p') \, dp \, dp',
\]

(5.22)

and as a consequence obtain Eq. \((5.3)\), where the kernels are given by

\[
K^F_a(p, p') = \cos[b(p - p')] |k_F + k_{F,\epsilon}(p, p')| \\
\quad + \cos[b(p + p')] |k_{F,\epsilon} - k_{F,2\epsilon}(p, p')| \\
\quad = \cos[b(p - p')] |k_F + k_{F,\epsilon}(p, p') + \cos[b(p + p')] |k_{F,\epsilon}^* - k_{F,2\epsilon}^*(p, p')| \\
\quad + 2 \sin(bp) \sin(bp') k_{F,\epsilon}^*(p, p'),
\]

(5.23)

with

\[
k_F(p, p') = \frac{1}{2\pi^2} \left[ \chi_F(p, p') - \delta\chi_F(p, -p') \right],
\]

(5.24)

\[
k_{F,\epsilon}(p, p') = \frac{1}{2\pi^2} \left[ \chi_{F,\epsilon}(p, p') - \delta\chi_{F,\epsilon}(p, -p') \right],
\]

and similarly for \(k^{B,F}_0, k^{B,F}_1\), and

\[
K^F_a^{G}(p, p') = G(|p|) K^F_a(p, p') G(|p'|)
\]

(5.25)
Using the second formula in (5.23) and looking at the estimates (C.16 – C.19) one finds that both functions $K_{F,G}^a(p,p')$ and $\sqrt{|p|}K_{F,G}^a(p,p')$ are square-integrable. In more detail: square integrability on the square $(p,p') \in (-1,1)^2$ follows rather immediately from the estimates (C.16 – C.18) (the term containing $k_{F,\varepsilon}$ has to be analysed jointly with the sine functions – see (5.23)). For the square integrability on $\mathbb{R}^2 \setminus (-1,1)^2$ one uses (C.19) and (5.2); it is obvious that in this case the integration of the estimates can be harmlessly extended to $\mathbb{R}^2$, and then by symmetry narrowed to $(0,\infty)^2$. In this way one finds

$$\int_{\mathbb{R}^2 \setminus (-1,1)^2} [1 + |p|] |K_{F,G}^a(p,p')|^2 \, dpdp' \leq \text{const} \int_{0 \leq p \leq p'} \frac{1 + p + p'}{(p+1)^{2(\alpha+\gamma)}(p'+1)^{2(\alpha+1)}} \, dpdp' < \infty,$$

(5.26)

where for the integration over $p \geq p' \geq 0$ the variables have been swapped, and the condition $4\alpha + 2\gamma > 1$ has been taken into account (cf. 5.2). The formula (5.4) for the Casimir energy is now easily obtained (cf. Eq. (4.3)). Finally, to obtain the scaling behaviour it is easily checked that $K_{\mu,F,G}^a(p,p') = K_{\mu,G}^a(p/\mu,p'/\mu)$, which upon substitution into (5.4) gives the desired result.

We end this section with remarks on the relation of the present work to that of [1]. The models considered in the cited work are obtained by choosing the Dirichlet boundary conditions for $h_{za}$ and setting $G \equiv 1$. One may thus ask whether it is possible to set $G \equiv 1$ also in the case of Neumann conditions, which would allow to bring the analysis to simpler terms of [1]. It turns out that it is not, and this posed the major difficulty in extending the program to these boundary conditions. The estimates (C.16 – C.19) guarantee that $K_{F,G}^a(p,p')$ is square-integrable, but are insufficient for the square-integrability of $\sqrt{|p|}K_{F,G}^a(p,p')$ (this is easily seen by putting $\alpha = 0$ in (5.26)). However, if one takes into account that the functions $\chi$ enter the kernel only through the combinations (5.24), then it turns out that the terms in $\chi$’s decaying most slowly for $|p|, |p'| \to \infty$ subtract in the case of Dirichlet, but add in the case of Neumann conditions, and the function $\sqrt{|p|}K_{F,G}^a(p,p')$ is square-integrable in the former, but not in the latter case.

### 6 Asymptotic expansion of the Casimir energy

In this section we show how to expand the Casimir energy of the class of models defined in the previous section into inverse powers of the distance of plates $a$. We shall assume that $F$ and $G$ are as for strict boundary conditions in some neighbourhood of zero, $F(p) = p$ and $G(p) = 1$ for small $p$. This assumption is somewhat stronger then the most economic one, which demands only that a few derivatives of those functions in $p = 0$ agree with those of the above special functions. We choose the simpler version to shorten the proofs. Also, we assume
the differentiability to arbitrary order, which spares us tedious track-keeping. We note that to take $\mu$ arbitrarily large in (5.5) and (5.6) physically means to approach strict boundary conditions. The asymptotic expansion combined with the scaling determines the behaviour of the Casimir energy in that limit.

Below we prove the following.

\begin{equation}
\text{(Asym) Let the functions } F, G \in C^\infty((0, \infty)) \text{ satisfy the conditions}
\begin{align*}
0 &\leq F(p) \leq \text{const}, \quad F(p) \leq p, \quad 0 \leq G(p) \leq 1, \\
|F^{(n)}(p)| &\leq \text{const}(n)(p + 1)^{-n-\gamma}, \quad n = 1, 2, \ldots , \\
|G^{(k)}(p)| &\leq \text{const}(k)(p + 1)^{-k-\alpha}, \quad k = 0, 1, 2, \ldots
\end{align*}
\end{equation}

and let moreover $F(p) = p$ and $G(p) = 1$ in some neighbourhood of zero. Then

\begin{equation}
\varepsilon_a = \varepsilon_\infty + \left(1 + \frac{1+\delta}{2}\right) \frac{c}{a^2} - \frac{\pi^2}{1440a^3} + \varepsilon_{4,a},
\end{equation}

where $\varepsilon_\infty$ and $c$ are model $(F$ and $G)$-dependent constants and

\begin{equation}
|\varepsilon_{4,a}| \leq \frac{\text{const}}{a^4}, \quad \left|\frac{d\varepsilon_{4,a}}{da}\right| \leq \frac{\text{const}}{a^5}.
\end{equation}

The limit value $\varepsilon_\infty$ is twice the Casimir energy $\varepsilon_0$ for the configuration of one single plate,

\begin{equation}
\varepsilon_\infty = 2\varepsilon_0.
\end{equation}

The scaling limit $\mu \to \infty$ yields

\begin{equation}
\lim_{\mu \to \infty} \left(\varepsilon_\mu^\alpha - \varepsilon_\infty \mu^3 - \left(1 + \frac{1+\delta}{2}\right) \frac{c}{a^2} \mu\right) = -\frac{\pi^2}{1440a^3}.
\end{equation}

Constants $\varepsilon_\infty$ and $c$ are given in (6.3), (6.32), (6.52) and (6.53) below.

As already mentioned above we do not strive to optimize assumptions on functions $F$ and $G$, which could be substantially weakened, but rather want to simplify the proof. The assumptions on derivatives essentially mean that the functions do not have oscillatory behaviour; the stronger (then in (Mod)) bound on parameters $\alpha$ and $\gamma$ will be needed in our expansion procedure.

The usual interpretation of formula (6.5) in the Dirichlet case would be that apart from an “unimportant constant” the limit of strict boundary conditions gives the original Casimir expression. However, this is not true for the Neumann conditions. Moreover, the constant $\varepsilon_\infty$ does have physical meaning: this is the energy needed for the creation of the configuration of the field surrounding infinitely separated plates. Relation (6.3) shows that configurations surrounding each of the plates become independent in the large separation limit (at least as far as energy
is concerned). We also note, although we do not present explicit calculations, that in the sharp boundary limit also the particle number per area diverges.

One could ask how generic are the lessons drawn from (Asym). We think that the result whose generality is least prone to doubt is the nonexistence of sharp boundaries limit. We support this by two observations. First, we recall that the free quantum field and the field in presence of sharp boundaries are even not described by the same algebra of observables, not to mention their representations (this was pointed out in [1]). Second, we learn that quantities which are responsible for comparability of representations and for backreaction, \( n_a \) and \( \varepsilon_a \), become infinite. Now, note that these are positive quantities, having physical interpretation. These facts indicate very strongly to our conclusion. Similar views on the unphysical nature of sharp boundary conditions may be found in literature, but we are not aware of a rigorous discussion based on the investigation of the algebraic structure of the theory and modification of dynamics (see also discussion and bibliography in [1]).

Also, we note that our calculation offers a rigorous derivation of the original Casimir term \(-\pi^2/1440a^2\) in the backreaction energy and confirms its universality. On the other hand the derivation shows, that it cannot stand alone: the energy is the expectation value in certain state of a positive operator, so it must be positive. Even if one adds a positive, constant (\( a \)-independent) term to the Casimir term one does not get a positive expression for each separation value \( a \).

The appearance of a term quadratic in \( 1/a \) in the Neumann case stands probably more open to debate. However, we want to point out that our rigorous analysis should rather be viewed as giving rise to a converse problem: are there any models for the dynamics \( h_{\text{za}} \), for which the quadratic term would be absent?: what would distinguish them? Within our class of models there is no general reason for \( c \) to vanish, and at this point we are not aware whether the condition \( c = 0 \) has any solutions in that class. In fact, a partial result to the opposite in certain circumstances may be shown (see the end of this section), for brevity we simplify assumptions on functions \( F \) and \( G \) to a more special class then needed. Suppose that in addition to the assumptions of (Asym) we have \( G(p) = f(p-r) \) for \( p \geq 0 \), where \( r \) is a positive parameter and \( f(x) \) is a smoothed step function equal to 1 for \( x \leq -\kappa \) and to 0 for \( x \geq \kappa \), with some \( \kappa > 0 \). Then for sufficiently large \( r \) constant \( c \) is positive. Note that increasing \( r \) moves us towards better approximation of sharp boundaries. Further discussion of the physical meaning of our results will be found at the end of Section 7.

We shall assume in the proof of (Asym) that \( F(p) = p \) and \( G(p) = 1 \) for \( p \in (0, 1) \). This does not restrict generality: each case of functions satisfying the assumptions of (Asym) may be brought to this more restrictive case by rescaling (5.4).
6.1 Asymptotic value of the Casimir energy

We first consider the Casimir energy for the z-motion dynamics of modified single plate. Let $h_{z0}^2$ be determined by $(h_{z0}^2)^2 = -(\partial^2/\partial z^2)_B$ with Dirichlet or Neumann conditions on the plane $z = 0$, and define $h_{z0}$ as in Eq. (5.1). The discussion of the last section simplifies greatly in that case and shows that $h_{z0} - h_z$ is again an integral operator in the momentum representation, with the kernel

$$K_0^{F,G}(p,p') = G(|p|)k_F(p,p')G(|p'|),$$

(6.6)

with $k_F$ as defined by Eq. (5.4). The Casimir energy per unit surface $\varepsilon_0$ for this operator (formula (5.4) with this new kernel) is the energy one needs to create the configuration of the field in the lowest stationary state of the modified Hamiltonian, as discussed in [I].

We now consider the limit value $\varepsilon_\infty$ in our model of modified parallel plates. For $a \to \infty$, i.e. $\epsilon \to 0$, one has for $p \neq 0$ the point-wise limit $\lim_{\epsilon \to 0} \Lambda_{F,\epsilon}(p) = \Lambda_F(p)$. This is easily shown with the use of formula (5.12) and the estimates (C.12) and (C.13). Then for $p, p' \neq 0$, $p \neq p'$, there is $\lim_{\epsilon \to 0} \chi_{F,\epsilon}^*(p,p') = \chi_F(p,p')$, and $\lim_{\epsilon \to 0} \chi_{F,\epsilon}^0(p,p') = 0$. At the same time referring back to the estimates discussed towards the end of Section 6 one realizes that the functions $G(|p|)|\chi_{F,\epsilon}^* - \chi_F(p,p')G(|p'|)|$, $\sqrt{|p|}G(|p|)|\chi_{F,\epsilon}^* - \chi_F(p,p')G(|p'|)|$, $\sin(bp)\sin(bp')G(|p|)|\chi_{F,\epsilon}^0(p,p')G(|p'|)|$ and $\sqrt{|p|}G(|p|)|\sin(bp)\sin(bp')G(|p|)\chi_{F,\epsilon}^0(p,p')G(|p'|)|$ remain bounded in modulus by square-integrable functions independent of $\epsilon$ (when it is small). Hence for $\epsilon \to 0$ we have

$$\|G(|p|)|\chi_{F,\epsilon}^* - \chi_F(p,p')G(|p'|)|\|_{L^2} \to 0,$$
$$\|\sqrt{|p|}G(|p|)|\chi_{F,\epsilon}^* - \chi_F(p,p')G(|p'|)|\|_{L^2} \to 0,$$
$$\|\sin(bp)\sin(bp')G(|p|)|\chi_{F,\epsilon}^0(p,p')G(|p'|)|\|_{L^2} \to 0,$$
$$\|\sqrt{|p|}G(|p|)|\sin(bp)\sin(bp')G(|p|)\chi_{F,\epsilon}^0(p,p')G(|p'|)|\|_{L^2} \to 0,$$

so for the purpose of calculating the limit $a \to \infty$ one can replace the kernel $K_a^{F,G}(p,p')$ with $\cos[b(p - p')]\cos[b(p' - p'')]$ in the formula (6.6). Using now the identities:

$$\cos[b(p - p')] = \frac{1}{2} \left(1 + \cos[2b(p - p')]\right),$$

(6.7)
$$\cos[b(p - p')] \cos[b(p' - p'')] = \frac{1}{4} \left(1 + \cos[2b(p - p')] + \cos[2b(p' - p'')] + \cos[2b(p'' - p)]\right),$$

(6.8)
together with the Riemann-Lebesgue lemma we find

\[
\varepsilon_\infty = \frac{1}{24\pi} \left\{ \int |p| |K_0 F,G(p, p')|^2 dp dp' + \int K_0 F,G(p, p') K_0 F,G(p', p'') K_0 F,G(p'', p) dp dp' dp'' \right\} = 2\varepsilon_0. \quad (6.9)
\]

### 6.2 Decomposition \( \varepsilon_a = \varepsilon_a^{(i)} + \varepsilon_a^{(ii)} \)

We now turn to the derivation of further leading terms of the asymptotic expansion of \( \varepsilon_a \) for \( a \to \infty \). It will be convenient to denote

\[
D_a^F = F(h_{za}) - F(h_z)
\]

and

\[
g(u) = 1 - G^2(u). \quad (6.10)
\]

We note that the operators \( D_a^F \) and \( \sqrt{h} G(h_z) D_a^F \) are HS – this is shown by methods used towards the end of Section 5 for the estimation of kernels (and here for the second one of these operators we need the strengthened condition on \( \alpha \) and \( \gamma \), see (6.1)). Using this notation and manipulating the operators under the trace sign we obtain

\[
\begin{align*}
\text{Tr}(h_{za} - h_z)^3 &= \text{Tr} \left[ (D_a^F)^3 (1 - 3g(h_z)) \right] \\
&\quad + 3 \text{Tr} \left[ (D_a^F)^2 g(h_z) D_a^F g(h_z) \right] - \text{Tr} \left[ D_a^F g(h_z) D_a^F g(h_z) D_a^F g(h_z) \right], \\
\text{Tr} \left[ (h_{za} - h_z) h_z(h_{za} - h_z) \right] &= \text{Tr} \left[ \sqrt{h_z} G(h_z) (D_a^F)^2 G(h_z) \sqrt{h_z} \right] \\
&\quad - \text{Tr} \left[ D_a^F G^2(h_z) h_z D_a^F g(h_z) \right]. \quad (6.12)
\end{align*}
\]

We set these expressions into the formula for the Casimir energy, in addition we add and subtract the term \( \text{Tr} \left[ D_a^F F(h_z) D_a^F (1 - 3g(h_z)) \right] \), and write the result in the form

\[
\begin{align*}
\varepsilon_a &= \varepsilon_a^{(i)} + \varepsilon_a^{(ii)}, \\
\varepsilon_a^{(i)} &= \frac{1}{24\pi} \text{Tr} \left\{ [(D_a^F)^3 + D_a^F F(h_z) D_a^F] (1 - 3g(h_z)) \\
&\quad + 3 \sqrt{h_z} G(h_z) (D_a^F)^2 G(h_z) \sqrt{h_z} - (D_a^F)^2 F(h_z) \right\}, \\
\varepsilon_a^{(ii)} &= \frac{1}{24\pi} \text{Tr} \left\{ 3D_a^F F(h_z) - G^2(h_z) D_a^F g(h_z) \\
&\quad + 3(D_a^F)^2 g(h_z) D_a^F g(h_z) - D_a^F g(h_z) D_a^F g(h_z) \right\}. \quad (6.16)
\end{align*}
\]

This decomposition of \( \varepsilon_a \) is purely technical: different techniques will be now employed for the calculation of the asymptotic expansion in each of the two cases \( \varepsilon_a^{(i)} \)
and $\varepsilon_a^{(ii)}$. However, it could help to observe that if it was possible to put $G \equiv 1$ (as can, in fact, be done in the Dirichlet case – see remarks towards the end of Section 5), then the second term in (6.14) would vanish (as then $g \equiv 0$). More comments on this point will be found below.

### 6.3 Integral representation of $\varepsilon_a^{(i)}$

By elementary manipulations one finds the identities

\[
\begin{align*}
(Da_F^2)^2 &= D_a^2 - F(h_z) Da^2 - D_a^F F(h_z), \\
(Da_F^3)^3 + D_a^F F(h_z) Da_F^2 &= D_a^F - F(h_z) Da^2 - D_a^2 F(h_z) + F(h_z) Da^F F(h_z).
\end{align*}
\]

(6.17) (6.18)

Operators $D_a^2$ and $D_a^3$ are HS, with the kernels in momentum representation $K_a^F, K_a^F \in L^2(\mathbb{R}^2)$. The above two identities, when written in terms of kernels, take the form

\[
\int K_a^F(p,q) K_a^F(q,q') dq = K_a^F(p,p') - \left[ F(|p|) + F(|p'|) \right] K_a^F(p,p'),
\]

(6.19)

\[
\int K_a^F(p,q) K_a^F(q,q') K_a^F(q',p') dq' dq + \int K_a^F(p,q) K_a^F(q,p') dq = K_a^F(p,p') - \left[ F(|p|) + F(|p'|) \right] K_a^F(p,p') + F(|p|) K_a^F(p,p') F(|p'|).
\]

(6.20)

These identities hold in the $L^2$-sense. However, as both sides are continuous functions of $(p,p')$ for $p,p' \neq 0$, they are also valid point-wise, in particular also for $p = p'$. Using this fact for the calculation of (6.15) one finds

\[
\varepsilon_a^{(i)} = \frac{1}{24\pi} \int \left\{ [1 - 3g(|p|)] K_a^F(p,p) \right. \\
+ 3 \left[ G^2(|p|)|p| - F(|p|) + 2F(|p|)g(|p|) \right] K_a^F(p,p) \right. \\
+ 3G^2(|p|)F(|p|) \left[ F(|p|) - 2|p| \right] K_a^F(p,p) \left. \right\} dp.
\]

(6.21)

We note that this expression engages only the diagonal values $K_a(p,p)$ of the kernels. In fact, as part $\varepsilon_a^{(i)}$ is concerned, one could bypass the calculation of the kernels and use a simpler technique employed in Ref. 1. But once we have the kernels (needed for the calculation of $\varepsilon_a^{(ii)}$), it is convenient to use them also here.

Looking at eq. (6.22) we see that

\[
K_a^F(p,p) = [k_F + kf_e](p,p) + \cos(ap)[k_F, k_F]_e(p,p).
\]

(6.22)
Using the definitions (C.4) and (C.7) we find $\chi_F(p, p) = \Lambda_F^{(1)}(p)$ and $\chi_F(p, -p) = \Lambda_F(p)/p$ and then recalling (5.22) we have

$$k_F(p, p) = \frac{1}{4\pi^2} \int \left\{ \left( \frac{1}{p-q} + \frac{\delta}{p+q} \right)^2 \left[ F(|q|) - F(|p|) \right] - \frac{2|p|F^{(1)}(|p|)}{(q-p)(q+p)} \right\} dq. \tag{6.23}$$

In a similar way one finds

$$k_{F,\varepsilon}(p, p) = \frac{\varepsilon}{4\pi^2} \sum_{k \in \mathbb{Z}} \left\{ \left( \frac{1}{p-k\varepsilon} + \frac{\delta}{p+k\varepsilon} \right)^2 \left[ F(|k\varepsilon|) - F(|p|) \right] - \frac{2|p|F^{(1)}(|p|)}{(k\varepsilon-p)(k\varepsilon+p)} \right\}. \tag{6.24}$$

Also, analogous formulas for $F^2$ or $F^3$ replacing $F$ in (6.22 – 6.24) are valid. Setting all these formulas into Eq. (6.21) one finds that the terms proportional to $F^{(1)}(|p|)$ cancel out and one is left with

$$\varepsilon_a^{(i)} = \frac{1}{24\pi^3} \int \rho(q, p) \, dq \, dp \tag{6.25}$$

and

$$\rho(q, p) = \lambda_F^2(q, p) \left[ \frac{1+\delta}{2} p^2 + \frac{1-\delta}{2} q^2 \right] \times \left[ F(|q|)[1 - 3g(|p|)] - F(|p|) + 3G^2(|p||p|) \right]. \tag{6.26}$$

All integrals and sums are absolutely convergent. Both here and in further expressions below involving $\rho$ and its derivatives one uses the estimates found in Appendix C to prove convergence. We do not go into easy but tedious details.

### 6.4 Expansion of $\varepsilon_a^{(i)}$

As $\rho(q, p)$ is even in each of the variables, one can reduce integration and summation to nonnegative values. We evaluate the sum $\varepsilon \sum_{k \in \mathbb{Z}} \rho(k\varepsilon, p)$ for fixed $p > 0$ with the use of formula (A.12) for $n = 4$:

$$\varepsilon \sum_{k \in \mathbb{Z}} \rho(k\varepsilon, p) = 2 \int_0^\infty \rho(q, p) \, dq - \varepsilon^2 B_2 \rho^{(1,0)}(0, p) - 2\varepsilon^4 R_{\rho,4,e}(p), \tag{6.27}$$

where

$$R_{\rho,4,e}(p) = \frac{1}{4!} \int_0^\infty \rho^{(4,0)}(q, p) b_4(q/\varepsilon) \, dq \tag{6.28}$$

and $\rho^{(1,0)}(0, p)$ is the limit value of $\rho^{(1,0)}(q, p)$ for $q \to 0$. To obtain the other sum in the integrand of Eq. (6.25) one needs only to replace $\epsilon$ by $2\epsilon$. The integrability
of \(|R_{\rho,4,\epsilon}(p)|\) at infinity is easily shown with the use of estimates to be found in Appendix C. On the other hand, these estimates do not guarantee the integrability for small \(p\). However, this property will be obvious from what follows below. Setting the above expansion into Eq. (6.25) we have

\[
\varepsilon^{(i)} = \frac{1}{3\pi^3} \int_0^\infty \rho(q,p) \, dq \, dp - \frac{\varepsilon^2}{72\pi^3} \int_0^\infty \rho^{(1,0)}(0,p)[1 - 3 \cos(ap)] \, dp \\
+ \frac{1}{6\pi^3} \int_0^\infty \left\{ -[1 + \cos(ap)] \varepsilon R_{\rho,4,\epsilon}(p) + \cos(ap) (2\varepsilon)^4 R_{\rho,4,2,\epsilon}(p) \right\} \, dp.
\]

(6.29)

The first term on the r.h. side is \(a\)-independent and it contributes to \(\varepsilon_\infty\). To evaluate the second term we find explicitly

\[
\rho^{(1,0)}(0,p) = -3(1+\delta)\eta(p), \quad \eta(p) = G^2(p) \frac{F(p)(2p - F(p))}{p^2} > 0.
\]

(6.30)

Using this one finds

\[
-\frac{\varepsilon^2}{72\pi^3} \int_0^\infty \rho^{(1,0)}(0,p)[1 - 3 \cos(ap)] \, dp \\
= \left(\frac{1+\delta}{24\pi a^2}\right) \int_0^\infty \eta(p) \, dp + \frac{1}{8\pi a^3} \int_0^\infty \eta^{(1)}(p) \sin(ap) \, dp,
\]

(6.31)

where integration by parts in the second term has been performed. From (6.30) we find that \(\eta(p) = 1\) for \(p \in (0,1)\), so \(\eta^{(1)}(p) = 0\) in this interval. It is easily seen by standard Fourier transform properties that the second integral in the last formula and its derivative with respect to \(a\) vanish faster than any inverse power of \(a\) for \(a \to \infty\) (by repeated integration by parts and use of estimates (6.1)). The first integral contributes only to the constant \(c\) in (6.2), and yields

\[
\varepsilon^{(i)} = \frac{1}{24\pi} \int_0^\infty G^2(p) \frac{F(p)(2p - F(p))}{p^2} \, dp.
\]

(6.32)

Next, to evaluate the second line on the r.h. side of Eq. (6.29) we split \(R_{\rho,4,\epsilon} = R_{\rho,4,\epsilon} + R_{\rho,4,1,\epsilon}\), where

\[
\rho = \rho_r + \rho_s, \quad \rho_s(q,p) = \frac{p^3}{(p+q)^2} + \delta \frac{p^2}{p+q}.
\]

(6.33)

For \(q,p \in (0,1)\) we have \(\rho(q,p) = \rho_s(q,p) + (1-\delta)q/2\), so \(\rho_r\) is regular in a neighbourhood of zero. Thus using (6.25) we find the estimate

\[
\left| \frac{1}{6\pi^3} \int_0^\infty \left\{ -[1 + \cos(ap)] \varepsilon R_{\rho,4,\epsilon}(p) + \cos(ap) (2\varepsilon)^4 R_{\rho,4,2,\epsilon}(p) \right\} \, dp \right| \\
\leq \frac{\text{const}}{a^4} \int_0^\infty |\rho_r^{(4,0)}(q,p)| \, dq \, dp < \infty.
\]

(6.34)
For the integrability of $|\rho_{4,0}^4(q,p)|$ one uses the assumption (6.1), relation (6.33) and the results of Appendix C. Furthermore, using the relations

\[
\int_0^\infty \frac{\partial \cos(ap)}{\partial a} R_{p+4,\epsilon}(p) \, dp = \frac{1}{a} \int_0^\infty \frac{\partial \cos(ap)}{\partial p} p R_{p+4,\epsilon}(p) \, dp
\]

\[
= -\frac{1}{a} \int_0^\infty \cos(ap)(pR_{p+4,\epsilon}^{(1)}(p) + R_{p+4,\epsilon}(p)) \, dp ,
\]

\[
R_{p+4,\epsilon}^{(1)}(p) = \frac{1}{4!} \int_0^\infty \rho_{p+4,\epsilon}^{(4,1)}(q,p) b_4(q/\epsilon) \, dq ,
\]

\[
\frac{\partial}{\partial \epsilon} [\epsilon^4 R_{p+4,\epsilon}(p)] = \frac{\partial}{\partial \epsilon} \left[ \epsilon^5 \int_0^\infty \rho_{p+4,\epsilon}^{(4,0)}(t\epsilon,p) b_4(t) \, dt \right]
\]

\[
= 5\epsilon^3 R_{p+4,\epsilon}(p) + \frac{\epsilon^3}{4!} \int_0^\infty q \rho_{p+4,\epsilon}^{(5,0)}(q,p) b_4(q/\epsilon) \, dq ,
\]

one also finds

\[
\left| \frac{d}{da} \frac{1}{6\pi^2} \int_0^\infty \left\{ -[1 + \cos(ap)] \epsilon^4 R_{p+4,\epsilon}(p) + \cos(ap)(2\epsilon)^4 R_{p+4,2\epsilon}(p) \right\} \, dp \right|
\]

\[
\leq \frac{\text{const}}{a^3} \int_0^\infty \left[ |\rho_{p+4,\epsilon}^{(4,0)}(q,p)| + p|\rho_{p+4,\epsilon}^{(4,1)}(q,p)| + q|\rho_{p+4,\epsilon}^{(5,0)}(q,p)| \right] dq \, dp < \infty .
\]

Integrability of $p|\rho_{p+4,\epsilon}^{(4,1)}(q,p)|$ and of $q|\rho_{p+4,\epsilon}^{(5,0)}(q,p)|$ is shown as for $|\rho_{p+4,\epsilon}^{(4,0)}(q,p)|$ above.

The second term resulting from the split of $\rho$, that involving $R_{p+4,\epsilon}$, can be explicitly evaluated. We have

\[
\epsilon^4 R_{p+4,\epsilon}(p) = \epsilon^2 R_{p+4,1}(p/\epsilon) ,
\]

\[
R_{p+4,1}(t) = (-t^3 \partial_t + \delta t^2) \frac{1}{4!} \int_0^\infty \left( \partial_u^4 \frac{1}{t+u} \right) b_4(u) \, du .
\]

With the use of formula (A.8) with $n = 4$ and $\epsilon = 1$, applied to the function $f(u) = 1/(t+u)$ (for fixed $t$) we find

\[
\frac{1}{4!} \int_0^\infty \left( \partial_u^4 \frac{1}{t+u} \right) b_n(u) \, du = \lim_{N \to \infty} \left\{ \int_0^N \frac{du}{t+u} - \sum_{k=0}^{N} \frac{1}{t+k} \right\} + \frac{1}{2t} + \frac{B_2}{2t^2}
\]

\[
= \psi(t) - \log t + \frac{1}{2t} + \frac{B_2}{2t^2} = \frac{w_2(t)}{t^2} ,
\]

where we used formula (D.1) and notation introduced in (D.6), and then

\[
R_{p+4,1}(t) = -tw_{2}^{(1)}(t) + (2 + \delta)w_2(t) .
\]
Setting this into (6.39) and using the identities (D.8) and (D.9) we find

\[
\frac{1}{6\pi^3} \int_0^\infty \left\{ - [1 + \cos(ap)] \epsilon^4 R_{\rho,4,\epsilon}(p) + \cos(ap) (2\epsilon)^4 R_{\rho,4,2\epsilon}(p) \right\} dp
\]

\[
= \frac{1}{6a^3} \int [1 + \cos(\pi t) - 8 \cos(2\pi t)] [tw^{(1)}(t) - (2 + \delta)w(t)] dt
\]

(6.43)

\[
= - \frac{1}{16\pi^2} \zeta(4) = - \frac{\pi^2}{1440a^3}.
\]

To arrive at the last line in this formula one has to take into account on the r.h. side of (D.9) that

\[
\sum_{k=1}^{\infty} \frac{1}{(k + \frac{3}{2})^s} = (2^s - 1) \sum_{k=1}^{\infty} \frac{1}{k^s} - 2^s.
\]

6.5 Integral representation and expansion of \( \varepsilon_a^{(ii)} \)

We now turn to \( \varepsilon_a^{(ii)} \) – the second term in (6.14). We use identity (6.17) in formula (6.16) and calculate the trace in the momentum representation, which yields

\[
\varepsilon_a^{(ii)} = \frac{1}{8\pi} \int K_a^F(p,p')K_a^{s^2}(p',p)g(|p|)g(|p'|) dp dp' - \frac{1}{8\pi} \int \left[ K_a^F(p,p') \right]^2 g(|p|) \left\{ G^2(|p'|)|p'| - F(|p'|) + 2F(|p'|)g(|p'|) \right\} dp dp' - \frac{1}{24\pi} \int K_a^F(p,p')K_a^F(p',p'')K_a^F(p'',p)g(|p|)g(|p'|)g(|p''|) dp dp' dp''.
\]

(6.44)

This expression looks much more complicated than the integral formula (6.21) for \( \varepsilon_a^{(i)} \) (multiple integrals and full kernels \( K_a(p,p') \)). However, a substantial simplification of the asymptotic expansion will be due to the fact, that the integrations here stay away from the singularities of the kernels \( K_a(p,p') \), which occur in the neighbourhood of \( p = 0 \) or \( p' = 0 \) (as for \( u \in (0,1) \) there is \( G(u) = 1 \), \( g(u) = 0 \) and \( F(u) = u \) according to our assumptions) and the integrands are, consequently, infinitely differentiable. Namely, products of kernels \( K_a \) contain products of cosine functions, which may be expressed as linear combinations of other cosine functions and, possibly, unity (as in Eqs. (6.7), (6.8)). Those terms in the integrals above which contain one of the resulting cosines yield functions of \( a \) which vanish faster than any inverse power of \( a \) for \( a \to \infty \). This is shown by standard Fourier methods: integrate by parts and use decay properties of the integrand and its derivatives – estimates (6.1) and (C.20); in the process the decay rate of the integrand increases. One also shows that the \( a \)-derivative of those terms has similar decay properties. Indeed, the derivative when acting on trigonometric functions does not change the decay; if it acts on a sum of the form \( \epsilon \sum f(k\epsilon) \) it
yields \(-\frac{1}{a}\epsilon \sum [f(\epsilon k) + k \epsilon f^{(1)}(\epsilon k)]\). These facts are used for the proof of the last statement, but we do not go into further straightforward details.

Thus it is sufficient to consider only those terms which take a constant from the product of cosines. One finds easily that the only products of cosines in the above integrals which do give constants are those in which cosines with “+” sign between the variables do not occur, or occur twice. However, we shall see below that these parts of the kernels \(K\) which multiply the cosines with the “+” sign are of order \(\epsilon^2\) at least, so the quadratic terms of that type do not contribute to the orders \(\leq 3\) in \(\epsilon\) in the energy, and to the orders \(\leq 4\) in the force. Thus we shall be left only with the products \((6.7)\) and \((6.8)\).

To expand the kernels \(K_a\) we use the definition \((6.5)\) and the formula \((6.12)\) to get

\[
\begin{align*}
\Lambda_{F,\epsilon}(p) &= \Lambda_F(p) + \frac{\epsilon^2}{6p} - \epsilon^4 \Lambda_{F,4,\epsilon}(p), \\
\Lambda_{F^2,\epsilon}(p) &= \Lambda_{F^2}(p) - \epsilon^4 \Lambda_{F^2,4,\epsilon}(p),
\end{align*}
\]

\(\tag{6.45}\)

where

\[
\Lambda_{F,4,\epsilon}(p) = \frac{1}{4!} 2p \int_0^\infty \Lambda_{F,4}(q, p) b_4(q/\epsilon) \, dq, \quad i = 1, 2.
\]

\(\tag{6.47}\)

Setting these expressions into \((6.8)\) and \((6.24)\) one finds

\[
\begin{align*}
k_{F,\epsilon}(p, p') &= k_F(p, p') - \frac{1 + \frac{\epsilon^2}{12p^2}}{p + p'} k_{F,4,\epsilon}(p, p'), \\
k_{F^2,\epsilon}(p, p') &= k_{F^2}(p, p') - \epsilon^4 k_{F^2,4,\epsilon}(p, p'),
\end{align*}
\]

\(\tag{6.48}\)

where for \(i = 1, 2:\)

\[
k_{F^{i,4},\epsilon}(p, p') = \frac{1}{2p^2} \left[ \frac{\Lambda_{F,4,\epsilon}(p) - \Lambda_{F,4,\epsilon}(p')}{p - p'} - \frac{\delta \Lambda_{F,4,\epsilon}(p) + \Lambda_{F,4,\epsilon}(p')}{p + p'} \right].
\]

\(\tag{6.49}\)

Note that \(|p|, |p'| \geq 1\) for our present purposes, so there are no singularities. Then with the use of bounds similar to those in \((6.20)\) one shows that the terms containing \(k_{F^2,4,\epsilon}\) indeed do not contribute in the given orders. Thus omitting the last term in both formulas \((6.48)\) and taking into account the result of the discussion following Eq. \((6.44)\) we find that disregarding terms higher than \((1/a)^3\) in \(\epsilon\) we can make the following replacements in \((6.44)\) (recall the formula \((6.26)\)):

\[
\begin{align*}
K_a^F(p, p')K_a^F(p''p) &\to \frac{1}{2} \left[ 2k_F(p, p') - \frac{1 + \frac{\epsilon^2}{12p^2}}{pp'} \right] 2k_{F^2}(p', p), \\
[K_a^F(p, p')]^2 &\to \frac{1}{2} \left[ 2k_F(p, p') - \frac{1 + \frac{\epsilon^2}{12p^2}}{pp'} \right]^2, \\
K_a^F(p, p')K_a^F(p'', p) &\to \frac{1}{4} \left[ 2k_F(p, p') - \frac{1 + \frac{\epsilon^2}{12p^2}}{pp'} \right] \left[ 2k_F(p', p') - \frac{1 + \frac{\epsilon^2}{12p^2}}{pp'} \right] \left[ 2k_F(p'', p) - \frac{1 + \frac{\epsilon^2}{12p''p}}{pp'} \right].
\end{align*}
\]

\(\tag{6.50}\)
In this way we obtain

$$
\varepsilon^{(ii)}_a = \left(\frac{1+a^2}{2}\right) c^{(ii)} + \text{terms independent of } a + O(a^{-4}), \quad (6.51)
$$

where

$$
c^{(ii)} = -\frac{1}{12\pi} \int_0^\infty k_{F^2}(p, p') \frac{g(p)g(p')}{pp'} dp dp' + \frac{1}{6\pi} \int_0^\infty k_F(p, p') \frac{g(p)}{p} \left[ G^2(p') - \frac{F(p')}{p'} + 2 \frac{F(p')}{p'} g(p') \right] dp dp'. \quad (6.52)
$$

The constant $c$ in (6.52) is now

$$
c = c^{(i)} + c^{(ii)}. \quad (6.53)
$$

This ends the proof of Eqs. (6.2) and (6.3). Equation (6.5) is their simple consequence.

Finally, we want to prove the statement on positivity of $c$ in circumstances described towards the end of the discussion following the formulation of (Asym). We note that the term $c^{(i)}$ obtained in (6.32) is then a positive, increasing function of $r$ tending to the limit value

$$
\lim_{r \to \infty} c^{(i)}(r) = \frac{1}{24\pi} \int_0^\infty \frac{F(p)[2p - F(p)]}{p^2} dp. \quad (6.54)
$$

At the same time one shows with the use of the present assumptions on $G(p)$ and estimates on functions $k$ that $|c^{(ii)}(r)| \leq \text{const}/r^7$, so

$$
\lim_{r \to \infty} c^{(ii)}(r) = 0, \quad (6.55)
$$

which ends the proof.

7 Electromagnetic field

When dealing with a scalar field we have first formulated the free field model in terms of the value of the field, and its time-derivative, on a Cauchy hyperplane. Then we formulated the dynamics in presence of the external influence in terms of the same variables, and took over for the unperturbed energy of the field itself (at a given time) the old expression built with the use of them. In this way a simple model could be obtained in which the “sources” of the field were supplied by the field variables themselves. A similar procedure for the electromagnetic field needs some caution. The main problem lies in the fact that the dynamics of the field
is constrained, and one of the constraints depends on sources (Gauss law). The evolution is governed by a first order equation and one could think of the electric and magnetic parts of the field as the analogues of the scalar field variables at a given time, but then the initial data for the electric part are differently constrained in free and interacting case. Therefore one first has to solve constraints and identify independent field variables. The modification of the dynamics of these variables must then produce an imitation of the perfect conductor boundary conditions of the electromagnetic field. To achieve this, one has to adjust the choice of the independent fields to the geometry of the problem. We shall first describe our choice in the case of the classical free field restricted to the region $\Omega \times \mathbb{R}$, where $\Omega$ is a compact convex region in the $x$-$y$ plane, and $\mathbb{R}$ is the $z$-axis. The following informal discussion serves to motivate the precise formulation of the model, which starts after Eq. (7.10).

For a vector $\vec{A}$ we denote by $\vec{A}_\perp$ its part perpendicular to the $z$-axis, and by $A$ its $z$-component. By $^*\vec{A}_\perp$ we denote the dual vector in the $x$-$y$ plane, that is in a Cartesian basis $^*A^1 = A^2$, $^*A^2 = -A^1$. The free Maxwell equations are then

$$\vec{\nabla}_\perp \cdot \vec{B}_\perp + \partial_z B = 0, \quad \vec{\nabla}_\perp \cdot \vec{E}_\perp + \partial_z E = 0,$$  \hspace{1cm} (7.1)

$$\vec{\nabla}_\perp \cdot ^*\vec{B}_\perp - \partial_t E = 0, \quad \vec{\nabla}_\perp \cdot ^*\vec{E}_\perp + \partial_t B = 0,$$  \hspace{1cm} (7.2)

$$\partial_z \vec{B}_\perp - \partial_t ^*\vec{E}_\perp = \vec{\nabla}_\perp B, \quad \partial_z \vec{E}_\perp + \partial_t ^*\vec{B}_\perp = = \vec{\nabla}_\perp E.$$  \hspace{1cm} (7.3)

Locally, of course, the wave equation follows for each of the components, in particular

$$[\partial_t^2 - \partial_z^2 - \Delta_\perp]B = 0, \quad [\partial_t^2 - \partial_z^2 - \Delta_\perp]E = 0.$$  \hspace{1cm} (7.4)

Global extension to the whole region demands some boundary conditions for $\Delta_\perp$ in each case. We assume that the Dirichlet extension for the electric case and the Neumann extension for the magnetic case have been chosen. Assume moreover that $\int_{\Omega} B \, dx \, dy = 0$, that is $B$ is orthogonal to constants in $L^2(\Omega)$. Then one can represent these fields as

$$B = -\Delta_\perp \Psi_m, \quad E = -\Delta_\perp \Psi_e,$$  \hspace{1cm} (7.5)

where $\Psi_m$ is assumed to be orthogonal to constants, and the fields $\Psi_m$ and $\Psi_e$ are thus uniquely determined by $B$ and $E$. Equations (7.4) can be now expressed as

$$[\partial_t^2 - \partial_z^2 - \Delta_\perp]\Psi_m = 0, \quad [\partial_t^2 - \partial_z^2 - \Delta_\perp]\Psi_e = 0,$$  \hspace{1cm} (7.6)

with appropriate boundary conditions. It is now easy to see that setting

$$\vec{B}_\perp = \vec{\nabla}_\perp \partial_z \Psi_m + ^*\vec{\nabla}_\perp \partial_t \Psi_e, \quad \vec{E}_\perp = \vec{\nabla}_\perp \partial_z \Psi_e - ^*\vec{\nabla}_\perp \partial_t \Psi_m$$  \hspace{1cm} (7.7)

one solves the complete set of Maxwell equations. The transversal fields satisfy boundary conditions

$$\vec{n} \cdot \vec{B}_\perp = 0, \quad ^*\vec{n} \cdot \vec{E}_\perp = 0,$$  \hspace{1cm} (7.8)
where \( \vec{n} \) is a vector in the \( x\)-\( y \) plane, orthogonal to \( \partial \Omega \). Let us now add to \( B \) the omitted part \( B' \) independent of \( \vec{x}_\perp \), and to \( \vec{B}_\perp \) and \( \vec{E}_\perp \) additional fields \( \vec{B}'_\perp \) and \( \vec{E}'_\perp \) respectively in order to investigate uniqueness. We want to keep boundary conditions (7.8). One finds then by integrating the first equation in (7.1) and the second equation in (7.2) over \( \Omega \) that \( B' \) is in fact a constant, which we exclude. Then it is easy to show that the only solution for the remaining fields \( \vec{B}'_\perp \) and \( \vec{E}'_\perp \) is zero.

Our boundary conditions are those of a perfect conductor, but this is not necessary in all its details. For instance, one could interchange the roles of the electric and magnetic fields. More generally, the boundary conditions should eliminate \( \vec{x}_\perp \)-independent fields. However, for definiteness we keep our choice.

The electromagnetic potential \( (\varphi, \vec{A}_\perp, A) \) producing fields (7.5), (7.7) may be chosen as

\[
\varphi = -\partial_z \Psi_e, \quad \vec{A}_\perp = *\vec{\nabla}_\perp \Psi_m, \quad A = \partial_t \Psi_e.
\]

(7.9)

However, the model is defined in terms of gauge-independent quantities and the potential is a purely auxiliary field.

Using the boundary conditions one finds the total energy of the field in terms of \( \Psi_e \) and \( \Psi_m \):

\[
\frac{1}{8\pi} \int \left[ \vec{E}^2 + \vec{B}^2 \right](t, \vec{x}) \, d^3x = \frac{1}{8\pi} \sum_{s=e,m} \int \left[ \partial_t \Psi_s(-\Delta_\perp)\partial_t \Psi_s + \partial_z \Psi_s(-\Delta_\perp)\partial_z \Psi_s + (-\Delta_\perp \Psi_s)^2 \right](t, \vec{x}) \, d^3x.
\]

(7.10)

This brings us to the following formulation of the classical dynamics of our system. We use for this formulation variables \( V_e \) and \( V_m \) introduced below, which should be thought of as supplying Cauchy data for fields \( \Psi_e \) and \( \Psi_m \) respectively, i.e. \( V_s = (\Psi_s, \partial_t \Psi_s)|_{t=0}, \ s = e, m \). The electromagnetic field is derived from these variables.

Let \( L^2_0(\Omega) \) denote the subspace of \( L^2(\Omega) \) orthogonal to constants, \( h^D_\perp = \sqrt{-\Delta^D_\perp} \) and \( h^N_\perp = \sqrt{-\Delta^N_\perp} \), where \( \Delta^D_\perp \) is the Dirichlet selfadjoint Laplacian on \( L^2(\Omega) \), and \( \Delta^N_\perp \) is the Neumann selfadjoint Laplacian on \( L^2(\Omega) \). Generally, let \( C \) be a selfadjoint operator on a Hilbert space \( \mathcal{H} \). By using a spectral representation of \( C \) it is then easy to see the following: if \( C \) is positive and has a bounded inverse, then \( \mathcal{H}_C \equiv \mathcal{D}(C) \) is a Hilbert space with respect to the scalar product \( (\varphi, \psi)_C = (C\varphi, C\psi) \). Moreover, the restriction of \( C \) to \( \mathcal{D}(C^2) \) is a selfadjoint, positive operator in \( \mathcal{H}_C \), with spectrum equal to that of \( C \). We apply this statements
to the operators $h_D^\perp$ and $h_N^\perp$ defined above, and denote

$$R_e^\perp = D(h_D^\perp), \quad (\varphi, \psi)_{\perp e} = \frac{1}{4\pi} (h_D^\perp \varphi, h_D^\perp \psi),$$

$$R_m^\perp = D(h_N^\perp), \quad (\varphi, \psi)_{\perp m} = \frac{1}{4\pi} (h_N^\perp \varphi, h_N^\perp \psi),$$

$$R_z = L^2_R(\mathbb{R}), \quad R^e = R^e_{\perp} \otimes R_z, \quad R^m = R^m_{\perp} \otimes R_z,$$

and the scalar product in $R^s$ is denoted by $(.,.)_s$. Moreover, we denote by $h^\perp_e$ and $h^\perp_m$ the selfadjoint operators in $R^e_{\perp}$ and $R^m_{\perp}$ determined in the above way by $h_D^\perp$ and $h_N^\perp$ respectively, and then set

$$h^e = \sqrt{(h^e_{\perp} \otimes \text{id})^2 + (\text{id} \otimes h_z)^2}, \quad h^m = \sqrt{(h^m_{\perp} \otimes \text{id})^2 + (\text{id} \otimes h_z)^2},$$

where $h_z = \sqrt{-\partial^2}$. The space of our model is now

$$L = L^m \oplus L^e, \quad \mathcal{L}^s = \mathcal{D}(h^s) \oplus R^s, \quad s = m, e.$$}

We write $V_s = v_s \oplus u_s \in L^s, s = m, e$. The spaces $\mathcal{L}^s$ are equipped with symplectic forms

$$\sigma^s(V_s, V'_s) = (v'_s, u_s)_s - (v_s, u'_s)_s.$$}

The Hamiltonian of the system is then

$$H(V_m, V_e) = H^m(V_m) + h^e(V_e), \quad H^s(V_s) = \frac{1}{2} [(u_s, u_s)_s + (h^s v_s, h^s v_s)_s],$$

and the evolution in each of the spaces $\mathcal{L}^s$ is independently determined by $H^s$ as a symplectic transformation, as discussed in Sec. I-3. One can now show that for sufficiently regular fields this reproduces the evolution equations (7.6) and the total energy of the field (7.10).

It is now evident that our classical system is described by the direct sum of two systems of the type discussed in Section 2. If we put $\Omega = (-L_x, L_x) \times (-L_y, L_y)$, then $h^e_{\perp}$ and $h^m_{\perp}$ have the spectrum of the type (D) and (N) respectively. The quantization of each independent part of the system follows the same lines as before. The algebra of quantum variables of the entire system is then the $C^*$-tensor product of the two Weyl algebras. The modification of the dynamics brought about by the modified boundary conditions is implemented by the replacement of (7.14) by

$$h^e_a = \sqrt{(h^e_{\perp} \otimes \text{id})^2 + (\text{id} \otimes h_{2a})^2}, \quad h^m_a = \sqrt{(h^m_{\perp} \otimes \text{id})^2 + (\text{id} \otimes h_{2a})^2},$$

where $h_{2a}, s = e, m$, are constructed as in (5.1), with $h_{2a}^B$ determined by Neumann conditions for the case $s = e$ and by the Dirichlet conditions for the case $s = m$ (which imitates perfect conductor conditions). The representations needed for the discussion of the Casimir effect are constructed as tensor products of the two
subsystems, and then the energy observable is the sum of the energies of the subsystems. Thus, in particular, the Casimir energy is the sum of the Casimir energies of the two subsystems, which proves our claim formulated in Introduction.

Our strategy in the above analysis was to formulate the classical dynamics in terms of unconstrained and gauge-independent variables, and then quantize. For our purposes we did not need the quantum version of the electromagnetic potential or field itself. However, for the completeness we briefly sketch their formulation. We obtain the electromagnetic field from the potential, and for the latter we keep the quantum version of the formulas (7.9). Recall that the fields independent of $\vec{x}_\perp$ are to be excluded. Moreover, we do not expect the formulas for potential (or electromagnetic field) to extend to the boundary of $\Omega$. Thus we take for our test function space $\Delta_\perp D_\Omega$, where $D_\Omega$ denotes the space of infinitely differentiable functions with compact support contained in $\Omega \times \mathbb{R}$. Denote by $v^q_s$ and $u^q_s$ heuristic quantized versions of $v_s$ and $u_s$ respectively. Then motivated by (7.9) we put for $f \in D_\Omega$

$$\varphi(\Delta_\perp f) = \int v^q_s \Delta_\perp \partial_z f \, d^3x = -4\pi (v^q_e, \partial_z f)_e,$$

$$A(\Delta_\perp f) = \int u^q_s \Delta_\perp f \, d^3x = -4\pi (u^q_s, f)_e,$$

$$\vec{A}_\perp(\Delta_\perp f) = -\int v^q_m * \vec{\nabla}_\perp \Delta_\perp f \, d^3x = 4\pi (v^q_m, * \vec{\nabla}_\perp f)_m. \quad (7.19)$$

Now, the precise meaning of $v^q_s$ and $u^q_s$ is given by

$$\Phi_s(V_s) = (v^q_s, u_s)_s + (u^q_s, v_s)_s, \quad (7.20)$$

where $\Phi_s(V_s)$ are quantum fields as described in Section I-3. Therefore the precise meaning of (7.19) is

$$\varphi(\Delta_\perp f) = -4\pi \Phi_e(0, \partial_z f), \quad A(\Delta_\perp f) = -4\pi \Phi_e(f, 0), \quad (7.21)$$

$$\vec{A}_\perp(\Delta_\perp f) = 4\pi \Phi_m(0, * \vec{\nabla}_\perp f). \quad (7.22)$$

The electromagnetic field is then easily found:

$$E(\Delta_\perp f) = 4\pi \Phi_e(0, [\Delta_\perp - (h^z_s \Delta_\perp - h^2_s)] f),$$

$$B(\Delta_\perp f) = 4\pi \Phi_m(0, \Delta_\perp f),$$

$$\vec{E}_\perp(\Delta_\perp f) = -4\pi \Phi_e(0, \vec{\nabla}_\perp \partial_z f) - 4\pi \Phi_m(* \vec{\nabla}_\perp f, 0), \quad (7.23)$$

$$\vec{B}_\perp(\Delta_\perp f) = 4\pi \Phi_e(* \vec{\nabla}_\perp f, 0) - 4\pi \Phi_m(0, \vec{\nabla}_\perp \partial_z f).$$

The formula for $E$ depends on dynamics; for the free field $h^z_s$ should be replaced by $h_z$, and then $E(\Delta_\perp f) = \Phi_e(0, \Delta_\perp f)$. Using the commutation relations

$$[\Phi_s(V_s), \Phi_s(V'_s)] = i\sigma^s(V_s, V'_s) \text{id} \quad (7.24)$$
one finds equal time commutators

\[
[E^i_\perp (\Delta f), B^j_\perp (\Delta g)] = -4\pi i\epsilon^{ij} \int (\Delta f) \partial_z (\Delta g) \, d^3 x,
\]

\[
[B(\Delta f), \vec{E}^\perp (\Delta g)] = 4\pi i \int (\Delta f)^* \vec{\nabla}^\perp (\Delta g) \, d^3 x, \tag{7.25}
\]

\[
[E(\Delta f), \vec{B}^\perp (\Delta g)] = -4\pi i \int (\Delta f)^* \vec{\nabla}^\perp (\Delta g) \, d^3 x
\]

\[
+ 4\pi i \int (\langle h_{za}^2 \rangle - h_{z}^2 f)^* \vec{\nabla}^\perp (\Delta g) \, d^3 x,
\]

where \( \epsilon^{ij} \) is the antisymmetric symbol with \( \epsilon^{12} = 1 \), and all other commutators vanish. For the free field the last term in the third relation vanishes and the commutators reproduce the usual quantization scheme. Also, the free Maxwell equations then hold for fields \( \ref{7.23} \). In the presence of the modified plates the commutators cannot remain unchanged, as this would violate the constraints; the last term in the third commutator above takes care of that. The modification is nonlocal – this reflects the nonlocality of our model \( (\langle h_{za}^2 \rangle)^2 \) is nonlocal). The pair of homogeneous Maxwell equations is still satisfied for fields \( \ref{7.23} \) – by definition, which may be regarded as gauge invariance – while the r.h. sides of the other two give the sources.

We note, moreover, that the identification of the full electromagnetic field in the interacting case is subject to some arbitrariness. The sources are linear functionals of the same fields, so one could shift some part of the electromagnetic fields to the r.h. sides of inhomogeneous equations to contribute to sources (but provided the homogeneous equations are conserved under the operation). Our choice of the interpretation is the most simple one.

We end with a summary of the physical meaning of our results. We believe that our analysis places the Casimir effect firmly within standard quantum theory. On the other hand it also shows why less conscious traditional quantum field formulations suffer from difficulties. The models analysed here are well defined and are free from the usual anomalies. The Casimir energy can be rigorously calculated for them as the expectation value of the positive free field energy operator in the ground state enforced by the environment (in accordance with the analysis presented in \[I\]).

In the case of the electromagnetic field our results may have direct physical application. It follows from the discussion following (Asym) in Section \[G\] that one should not expect a universal law for the total Casimir force between modified parallel plates. The original Casimir expression \(-\pi^2/720a^3\) constitutes the third order term of the expansion of the energy in inverse powers of separation \(a\), and cannot give a dominant contribution to the force for all values of this parameter. In particular, energy is positive, so the term is dominated by other nonconstant contributions for small \(a\). On the other end of the range of \(a\) we have found that in our models typical fall-off of Casimir energy is governed by a term of order \(1/a^2\). This prediction may be of more special character, but we believe that if it is indeed,
then disappearance of such a term in any other models needs further explanation. What we find in our setting is that for a wide range of models the coefficient at this term is positive, so the force at sufficiently large $a$ becomes repulsive.

Eventually, these predictions must be checked against experiment. However, although we have witnessed impressive progress in precision measurements of the Casimir force in recent years, and the existence of the force is now beyond doubt, its detailed form seems to us less certain. Due to experimental difficulties a measurement of the force between parallel plates has been reported only relatively recently [5]. The results seem to confirm the original Casimir formula. However, we think it is to early to accept this as “the whole truth” on this force. First of all, the measurements span a limited range of the distance parameter $a$. As explained above we have very strong reasons to believe that the Casimir formula cannot hold for $a$ tending to zero. The behaviour of the force for large $a$ could also bring deviations. Second, the data analysis seems to be oriented at testing the coefficient at the $1/a^4$ term of the measured force. To a large extent it involves peeling off other influences, while keeping the $\sim 1/a^4$ formula for the backreaction force; the best fit then confirms the original Casimir coefficient at this term. When seen from that angle the result does not contradict our predictions. We think that more extensive experimental tests of the functional form of the force are needed.

Appendices

A Sums and integrals

In this appendix we gather a handful of approximation formulas connecting sums with integrals. Some of these results are well-known, but we give them the form needed in the main text. In particular, the Euler-Maclaurin expansion is usually formulated for analytical functions only, with no estimates on the rest.

(i)\textsubscript{A} Let $f : (0, \infty) \mapsto (0, \infty)$ be a non-increasing, continuous function. Then

\begin{equation}
\lim_{\varepsilon \to 0} \varepsilon \sum_{k=1}^{\infty} f(k\varepsilon) = \int_0^{\infty} f(u) \, du , \tag{A.1}
\end{equation}

\begin{equation}
\lim_{\varepsilon_x, \varepsilon_y \to 0} \varepsilon_x \varepsilon_y \sum_{k, l=1}^{\infty} f(k^2 \varepsilon_x^2 + l^2 \varepsilon_y^2) = \frac{\pi}{4} \int_0^{\infty} f(u) \, du , \tag{A.2}
\end{equation}

where the equalities hold both for finite and infinite integral on the right.
hand side. If in addition
\[ \int_0^1 f(u) \, du < \infty \quad \text{then} \quad \lim_{u \to 0} uf(u) = 0, \quad (A.3) \]
\[ \int_1^\infty f(u) \, du < \infty \quad \text{then} \quad \lim_{u \to \infty} uf(u) = 0, \quad (A.4) \]
\[ \int_0^\infty f(u) \, du < \infty \quad \text{then} \quad \lim_{\varepsilon \to 0} \varepsilon^2 \sum_{k=1}^\infty f(k^2\varepsilon^2) = 0. \quad (A.5) \]

(ii) Euler-Maclaurin expansion

Let \( f \) be a complex function in \( C^n((0, \infty)) \) for some \( n \in \mathbb{N} \) and such that
\[ \int_0^\infty |f^{(n)}(u)| \, du < \infty, \quad (A.6) \]
\[ \lim_{u \to \infty} f^{(m)}(u) = 0 \quad \text{for} \quad m = 0, 1, 3, 5, \ldots, \leq n - 2. \quad (A.7) \]

Then the following identity holds
\[ \lim_{N \to \infty} \left\{ \frac{\varepsilon}{2} f(0) + \varepsilon \sum_{k=1}^N f(k\varepsilon) - \int_0^N f(u) \, du \right\} \]
\[ = -\sum_{m=2}^{n-1} \frac{\varepsilon^m}{m!} B_m f^{(m-1)}(0) - \varepsilon^n R_{f,n,\varepsilon}, \quad (A.8) \]
where
\[ R_{f,n,\varepsilon} = \frac{1}{n!} \int_0^\infty f^{(n)}(u) b_n(u/\varepsilon) \, du, \quad (A.9) \]
\[ b_n(k + s) = b_n(s) \equiv B_n(1 - s) - \frac{1}{2} B_n(0) - \frac{1}{2} B_n(1) \quad \text{for} \quad s \in (0, 1), \; k \in \mathbb{N}, \quad (A.10) \]
\[ |R_{f,n,\varepsilon}| \leq c_n \int_0^\infty |f^{(n)}(s)| \, ds. \quad (A.11) \]

Here \( B_n(.) \) are Bernoulli polynomials, \( B_n = B_n(0) \) are Bernoulli constants [3], and \( c_n = (n!)^{-1} \max_{s \in (0, 1)} |B_n(s) - B_n| \).

In particular, if in addition the integral \( \int_0^\infty f(u) \, du \) converges then also the sum \( \varepsilon \sum_{k=1}^\infty f(k\varepsilon) \) does (neither needs to converge absolutely), and in that case the identity can be written as
\[ \frac{\varepsilon}{2} f(0) + \varepsilon \sum_{k=1}^\infty f(k\varepsilon) = \int_0^\infty f(u) \, du - \sum_{m=2}^{n-1} \frac{\varepsilon^m}{m!} B_m f^{(m-1)}(0) - \varepsilon^n R_{f,n,\varepsilon}. \quad (A.12) \]

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A.1 Proof of (i)

The first identity (A.1) follows immediately from Riemann approximations for a positive, non-increasing, continuous function

$$
\int_0^\infty f(u) \, du \leq \epsilon \sum_{k=1}^\infty f(k\epsilon) \leq \int_0^\infty f(u) \, du.
$$

The application of these inequalities to $f(\alpha^2 + \beta^2)$ as a function of $\alpha$ and $\beta$ leads to

$$
\int_0^\infty \int_0^\infty f(\alpha^2 + \beta^2) \, d\alpha \, d\beta \leq \epsilon x \epsilon y \sum_{k,l=1}^{\infty} f(k^2 \epsilon^2 x + l^2 \epsilon^2 y) 
\leq \int_0^\infty f(u) \, du,
$$

which proves the identity (A.2).

The function $f$ is non-increasing, so $uf(u) \leq 2 \int_{u/2}^u f(v) \, dv$, which implies the properties (A.3) and (A.4). By the first of these limits the first term in the sum in Eq. (A.5) tends to zero. For the rest of the sum, for $\epsilon < 1$, we have

$$
\epsilon^2 \sum_{k=2}^{\infty} f(k^2 \epsilon^2) \leq \epsilon \int_\epsilon^\infty f(u^2) \, du = \frac{\epsilon}{2} \int_0^\infty \frac{f(u)}{\sqrt{u}} \, du
\leq \frac{1}{2} \int_\epsilon^\infty f(u) \, du + \frac{\sqrt{\epsilon}}{2} \int_\epsilon^\infty f(u) \, du,
$$

where the first inequality follows by the Riemann approximation of the integral. This is sufficient to conclude that Eq. (A.5) holds.

A.2 Proof of (ii)

One first notes that for $n \in \mathbb{N}$ the following identity holds

$$
\frac{\epsilon}{2} f(0) + \epsilon \sum_{k=1}^{N-1} f(k\epsilon) + \frac{\epsilon}{2} f(N\epsilon) - \int_0^{N\epsilon} f(s) \, ds
= - \sum_{m=2}^{n-1} \epsilon^m \frac{B_m}{m!} [f^{(m-1)}(0) - f^{(m-1)}(N\epsilon)]
- \sum_{k=0}^{N-1} \int_0^\epsilon f^{(n)}(k\epsilon + s) [B_n(1 - (s/\epsilon)) - \frac{1}{2} B_n(0) - \frac{1}{2} B_n(1)] \, ds
$$

This is shown by induction with respect to $n$, with the use of integration by parts in the integrals in the second sum on the r.h. side (remember that
\( \frac{d}{du}B_n(u) = nB_{n-1}(u) \). The change of integration variable \( u = k\epsilon + s \) in each of the integrals in this sum puts the second line on the r.h. side into the form
\[
-\frac{\epsilon^n}{n!} \int_0^{N\epsilon} f^{(n)}(u) b_n(u/\epsilon) du.
\]
Functions \( b_n \) are measurable and bounded, so the estimate (A.11) follows. Therefore, if the assumptions are satisfied then the r.h. side of (A.16), and the term \((\epsilon/2)f(N\epsilon)\) on the l.h. side converge for \( N \to \infty \). All the statements of the thesis are now readily seen.

### B Hilbert-Schmidt properties of operators \( \Delta_a(u) \)

In this appendix \( h_z \) and \( h_{za} \) are arbitrary selfadjoint, positive operators with domains \( \mathcal{D}(h_z) \) and \( \mathcal{D}(h_{za}) \) respectively, and operators \( h(u) \) and \( h_a(u) \) are defined by Eqs. (2.6) and (2.7) respectively (we keep this notation to make application of the results in the main text obvious, but we do not need any further general restrictions). Let \( f \) and \( g \) be real (or complex) measurable functions on \( \langle 0, \infty \rangle \) such that \( f(u), g(u), uf(u) \) and \( ug(u) \) are all bounded on the domain. Then the expressions
\[
f(h_{za})\Delta^p,q_a(u)g(h_z), \quad \text{where formally}
\]
\[
\Delta^p,q_a(u) = h_a^{-p}(u)\Delta_a(u)h^{-q}(u), \quad \Delta_a(u) = h_a(u) - h(u),
\]
define bounded operators for all \( p, q \geq 0 \) and all \( u \in (0, \infty) \) \((u \in (0, \infty) \text{ if } p = q = 0)\), although in general expressions \( \Delta^p,q(u) \) need not make sense by themselves. We denote by \( P_B \) and \( P_{aB} \) the spectral projectors determined by \( h_z \) and \( h_{za} \) respectively, projecting onto the set \( B \) indicated in the subscripts. In particular, the intervals in the symbols \( P_{(\beta_1, \beta_2)} \) and \( P_{a(\gamma_1, \gamma_2)} \) below are chosen inside \( (0, \infty) \) (positivity of operators). Symbol \( \|A\|_{\text{HS}} \) will be used for the Hilbert-Schmidt norm of a HS operator \( A \), i.e. \( \|A\|_{\text{HS}} = \sqrt{\text{Tr}[A^*A]} \).

We shall show in this appendix that under these assumptions the following identities and implications hold. The principal results, which are needed in the main text, are contained in (iv)\( \text{B} \) and (v)\( \text{B} \) below.

(i)\( \text{B} \) For each \( u > 0 \) and \( v \geq 0 \) the following integral representation holds in the uniform sense
\[
f(h_{za})\Delta_a(u)g(h_z) = \frac{1}{\pi} \int_u^{\infty} \frac{f(h_{za})}{h_{za}^2 + t} \left[ h_a(v)\Delta_a(v) + \Delta_a(v)h(v) \right] \frac{g(h_z)}{h_z^2 + t} \sqrt{t - u} dt.
\]

(ii)\( \text{B} \) If \( P_{a(\gamma_1, \gamma_2)}\Delta^p,q_a(u)P_{(\beta_1, \beta_2)} \) is a HS operator for a given \( u = v \geq 0 \), then it is also a HS operator for \( u \in (0, \infty) \), and
\[ \|P_a\Delta_{a}^{p,q}(u)P_{(\beta_1, \beta_2)}\|_{HS} \leq \|P_a\Delta_{a}^{p,q}(v)P_{(\beta_1, \beta_2)}\|_{HS} \]
\[ \times \frac{\sqrt{\gamma_2^2 + v} + \sqrt{\gamma_3^2 + u}}{\sqrt{\gamma_1^2 + u} + \sqrt{\beta_2^2 + u}} \times \begin{cases} 1 & \text{for } u > v , \\ \left(\frac{u}{v}\right)^{(p+q)/2} & \text{for } u \leq v . \end{cases} \quad (B.3) \]

(iii) If \( P_a\Delta_{a}^{1/2}(u)\Delta_{a}(v)P_{(\beta_1, \beta_2)} \) and \( P_a\Delta_{a}(v)h^{1/2}(v)P_{(\beta_1, \beta_2)} \) are HS operators for a given \( v \geq 0 \), then \( P_a\Delta_{a}(v)P_{(\beta_1, \beta_2)} \) is HS for \( u \in (0, \infty) \), and
\[ \|P_a\Delta_{a}(u)P_{(\beta_1, \beta_2)}\|_{HS} \leq \left(\frac{\sqrt{\gamma_2^2 + v} + \sqrt{\gamma_3^2 + u}}{\sqrt{\gamma_1^2 + u} + \sqrt{\beta_2^2 + u}}\right) \times \max\left\{\|P_a\Delta_{a}^{1/2}(v)\|_{HS}, \|P_a\Delta_{a}^{1/2}(v)h^{1/2}(v)\|_{HS}\right\} . \]

(B.4)

(iv) Let \( \Delta_{a}^{p,q}(u) \) be a HS operator for a given \( u = v \geq 0 \). Then it is also a HS operator for \( u \in (0, \infty) \), and
\[ \|\Delta_{a}^{p,q}(u)\|_{HS} < \|\Delta_{a}^{p,q}(v)\|_{HS} \quad \text{for } u > v , \quad (B.5) \]
\[ \|\Delta_{a}^{p,q}(u)\|_{HS} \leq \left(\frac{v}{u}\right)^{(p+q+1)/2} \|\Delta_{a}^{p,q}(v)\|_{HS} \quad \text{for } u \leq v , \quad (B.6) \]
\[ \lim_{u \to \infty} \|\Delta_{a}^{p,q}(u)\|_{HS} = 0 . \quad (B.7) \]

The operator function \( (0, \infty) \ni u \mapsto \Delta_{a}^{p,q}(u) \) is continuously differentiable in the HS-norm sense and the formal differentiation yields the correct result. The operator \( \Delta_{a}^{p,q}(0) \) is HS iff \( \|\Delta_{a}^{p,q}(u)\|_{HS} \leq \text{const} \) for \( u \in (0, \infty) \) and then
\[ \|\Delta_{a}^{p,q}(0)\|_{HS} = \lim_{u \to 0} \|\Delta_{a}^{p,q}(u)\|_{HS} . \]

The theorem remains valid upon replacement of the operator \( \Delta_{a}^{p,q}(u) \) with \( P_{a,B}\Delta_{a}^{p,q}(u)P_C \), for any measurable sets \( B, C \).

(v) Let \( \Delta_{a}(0) \) be a HS operator. If \( \Delta_{a}(u)h^{1/2}(u) \) is HS for a given \( u = v \geq 0 \), then all values of the operator functions \( (0, \infty) \ni u \mapsto \Delta_{a}(u)h^{1/2}(u) \) and \( (0, \infty) \ni u \mapsto h^{1/2}(u)\Delta_{a}(u) \) are also HS operators. Both functions are continuous in the HS-norm on their domain and continuously differentiable in HS-norm sense on \( (0, \infty) \). Moreover,
\[ \lim_{u \to \infty} \|h^{1/2}(u)\Delta_{a}(u)\|_{HS} = 0 , \quad \lim_{u \to \infty} \|\Delta_{a}(u)h^{1/2}(u)\|_{HS} = 0 . \]

The theorem remains valid upon replacement of the operator \( \Delta_{a}(u) \) with \( P_{a,B}\Delta_{a}(u)P_C \), for any measurable sets \( B, C \).
B.1 Proof of (i)\textsubscript{B}

For a positive real number \( a \) one has \( 1 = \frac{1}{\pi} \int_0^\infty \frac{a}{a^2 + t} \, dt \). Using this in the spectral representation of \( h_z \) one shows that for \( u > 0 \)

\[
h(u)g(h_z) = \frac{1}{\pi} \int_u^\infty \frac{(h_z^2 + u)g(h_z)}{h_z^2 + t} \, dt \sqrt{t-u},
\]

and the integral on the r.h. side converges uniformly (in norm). With a similar representation of \( f(h_z)\Delta_a(u) \) we have a uniformly convergent representation

\[
f(h_{za})\Delta_a(u)g(h_z) = \frac{1}{\pi} \int_u^\infty f(h_{za}) \left[ \frac{h_{za}^2 + u}{h_{za}^2 + t} - \frac{h_z^2 + u}{h_z^2 + t} \right] g(h_z) \frac{dt}{\sqrt{t-u}}.
\]

Using the formal relation

\[
\frac{h_{za}^2 + u}{h_{za}^2 + t} - \frac{h_z^2 + u}{h_z^2 + t} = (t-u)\frac{1}{h_{za}^2 + t} [h_a(v)\Delta_a(v) + \Delta_a(v)h(v)] \frac{1}{h_z^2 + t},
\]

which becomes a correct identity when placed between \( f(h_{za}) \) and \( g(h_z) \), one arrives at Eq. (B.2).

B.2 Proof of (ii)\textsubscript{B}

If one multiplies Eq. (B.2) by \( h^{-p_a}_a(u) \) from the left and by \( h^{-q_a}(u) \) from the right \( (u > 0) \), one obtains a similar identity with \( \Delta^{p,q}_a \) replacing \( \Delta_a \) on both sides, and the integrand on the r.h. side multiplied by \( h^p(v)h^{-p_a}_a(u) \) from the left and by \( h^q(v)h^{-q_a}(u) \) from the right. We put \( f(h_{za}) = P_{a(\gamma_1, \gamma_2)}(h_{za}) \) and \( g(h_z) = P_{(\beta_1, \beta_2)}(h_z) \) in this identity, and estimate the HS norm of the l.h. side. We find

\[
\|P_{a(\gamma_1, \gamma_2)}\Delta^{p,q}_a(u)P_{(\beta_1, \beta_2)}\|_{\text{HS}} \leq \|P_{a(\gamma_1, \gamma_2)}\Delta^{p,q}(v)P_{(\beta_1, \beta_2)}\|_{\text{HS}} \times \|P_{a(\gamma_1, \gamma_2)}(h_{za})h^{-p_a}_a(u)\| \times \|P_{(\beta_1, \beta_2)}(h_z)h^q(v)\| \times \frac{1}{\pi} \int_u^\infty \left\| \frac{P_{a(\gamma_1, \gamma_2)}(h_{za})}{h_{za}^2 + t} \right\| \left\| \frac{P_{(\beta_1, \beta_2)}(h_z)}{h_z^2 + t} \right\| \sqrt{t-u} \, dt,
\]

where we have pulled the HS norm sign under the integral and used the fact that \( \|ABC\|_{\text{HS}} \leq \|A\| \|B\|_{\text{HS}} \|C\| \). The second line in this estimate is bounded by \( \sqrt{\gamma_1^2 + v + \sqrt{\beta_2^2 + v}} \), the first factor in the third line by \( \max\{1, (v/u)^{p/2}\} \), the second factor in the third line by \( \max\{1, (v/u)^{q/2}\} \), and the fourth line by

\[
\frac{1}{\pi} \int_u^\infty \frac{\sqrt{t-u} \, dt}{(\gamma_1^2 + t)(\beta_2^2 + t)} = \frac{1}{\sqrt{\gamma_1^2 + u + \sqrt{\beta_2^2 + u}}},
\]

which ends the proof of (ii)\textsubscript{B}. 

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B.3 Proof of (iii)\textsubscript{B} 

We put \(f(h_{za}) = P_{a(\gamma_1, \gamma_2)}(h_{za})\) and \(g(h_z) = P_{(\beta_1, \beta_2)}(h_z)\) in the identity \(B.2\). Using the method of the last proof we find

\[
\|P_{a(\gamma_1, \gamma_2)} \Delta_a(u) P_{(\beta_1, \beta_2)}\|_{HS} \\
\leq \left( \|P_{a(\gamma_1, \gamma_2)} h_a^{1/2}(v)\| \|P_{a(\gamma_1, \gamma_2)} h_a^{1/2}(v) \Delta_a(v) P_{(\beta_1, \beta_2)}\|_{HS} \\
+ \|P_{a(\gamma_1, \gamma_2)} \Delta_a(v) h_a^{1/2}(v) P_{(\beta_1, \beta_2)}\|_{HS} \|P_{(\beta_1, \beta_2)} h_a^{1/2}(v)\| \right) \\
\times \frac{1}{\pi} \int_{u}^{\infty} \left\| \frac{P_{a(\gamma_1, \gamma_2)}(h_{za})}{h_{za}^2 + t} \right\| \left\| \frac{P_{(\beta_1, \beta_2)}(h_z)}{h_z^2 + t} \right\| \sqrt{t-u} dt ,
\]

which leads to the estimate \(B.4\).

B.4 Proof of (iv)\textsubscript{B} 

We set in the estimate \(B.3\):

\[
\gamma_1 = \sqrt{k\tau}, \quad \gamma_2 = \sqrt{(k+1)\tau}, \quad \beta_1 = \sqrt{l\tau}, \quad \beta_2 = \sqrt{(l+1)\tau},
\]

where \(k, l = 0, 1, \ldots\), and denote

\[
a_{k,l}(\tau, v, u) = \frac{\sqrt{k\tau + \tau + v} + \sqrt{l\tau + \tau + v}}{\sqrt{k\tau + u} + \sqrt{l\tau + u}}.
\]

Also, we introduce \(P_{a,k,\tau} = P_{a(\sqrt{k\tau}, \sqrt{k\tau+\tau})}, \quad P_{l,\tau} = P_{(\sqrt{l\tau}, \sqrt{l\tau+\tau})}\). Then from the bound \(B.3\) we have

\[
\|P_{a,k,\tau} \Delta_a^{p,q}(u) P_{l,\tau}\|_{HS} \\
\leq \|P_{a,k,\tau} \Delta_a^{p,q}(v) P_{l,\tau}\|_{HS} a_{k,l}(\tau, v, u) \times \begin{cases} 
1 & \text{for } u > v, \\
\left(\frac{u}{v}\right)^{(p+q)/2} & \text{for } u \leq v .
\end{cases}
\]

If \(v < u\) then we choose \(\tau\) such that \(v + \tau < u\), and then \(a_{k,l}(\tau, v, u) < \left(\frac{v + \tau}{u}\right)^{1/2}\). Moreover, there is \(\lim_{u \to \infty} a_{k,l}(\tau, v, u) = 0\). It is now sufficient to observe that

\[
\|\Delta_a^{p,q}(t)\|_{HS}^2 = \sum_{k,l=0}^{\infty} \|P_{a,k,\tau} \Delta_a^{p,q}(t) P_{l,\tau}\|_{HS}^2 ,
\]

to be able to conclude that Eqs. \(B.3\) and \(B.7\) hold. For \(u \leq v\) one obtains \(\|\Delta_a^{p,q}(u)\|_{HS}^2 < \|\Delta_a^{p,q}(v)\|_{HS}^2 \left(\frac{v + \tau}{u}\right)^{p+q}\) for each \(\tau > 0\), which leads to the estimate \(B.6\). Also, it should be clear that the replacement \(\Delta_a^{p,q}(\cdot) \to P_a^B \Delta_a^{p,q}(\cdot) P_C\) poses no difficulties.
To save space we prove the remaining statements only in the case $p = q = 0$, the generalization to nonzero $p, q$ and/or added projections $P_A, P_C$ is then easily obtained. We note first that

$$h(t) - h(u) = \frac{t - u}{h(t) + h(u)},$$

and similarly for $h_a(.)$, are bounded operators. Thus if $\Delta_a(t), \Delta_a(u)$ are bounded then

$$\Delta_a(t) - \Delta_a(u) = -(t - u) \frac{1}{h_a(t) + h_a(u)} (\Delta_a(t) + \Delta_a(u)) \frac{1}{h(t) + h(u)}.$$

If on top of that $\Delta_a(t)$ and $\Delta_a(u)$ are HS, then

$$\|\Delta_a(t) - \Delta_a(u)\|_{\text{HS}} \leq \frac{|t - u|}{(\sqrt{t} + \sqrt{u})^2} \left(\|\Delta_a(t)\|_{\text{HS}} + \|\Delta_a(u)\|_{\text{HS}}\right).$$

In a similar way one shows now that for $t, u > 0$ there is

$$\left\|\Delta_a(t) - \Delta_a(u)\right\|_{\text{HS}} \leq \frac{1}{2} h_a^{-1}(u) \Delta_a(u) h_a^{-1}(u) \left(\|\Delta_a(t)\|_{\text{HS}} + \|\Delta_a(u)\|_{\text{HS}}\right) \leq \text{const} \left|\frac{t - u}{u^2}\right| \left(\|\Delta_a(t)\|_{\text{HS}} + \|\Delta_a(u)\|_{\text{HS}}\right).$$

Therefore, if the assumption is satisfied (with $v \geq 0$), then the operator function $(0, \infty) \ni u \mapsto \Delta_a(u)$ is HS-differentiable, and its derivative is the HS-continuous operator function $-\frac{1}{2} h_a^{-1}(u) \Delta_a(u) h_a^{-1}(u)$.

If $\Delta_a(0)$ is HS, then the function $(0, \infty) \ni u \mapsto \|\Delta_a(u)\|_{\text{HS}}$ is decreasing, hence $\|\Delta_a(0)\|_{\text{HS}} \geq \lim_{t \to 0} \|\Delta_a(t)\|_{\text{HS}} \geq \|\Delta_a(u)\|_{\text{HS}}$ for $u \in (0, \infty)$. Conversely, if the decreasing function $(0, \infty) \ni u \mapsto \|\Delta_a(u)\|_{\text{HS}}$ is bounded, then

$$\infty > \lim_{t \to 0} \|\Delta_a(t)\|_{\text{HS}}^2 \geq \|\Delta_a(u)\|_{\text{HS}}^2 \geq \sum_{n=1}^{N} \|\Delta_a(u) \varphi_n\|^2,$$

where $\{\varphi_n\}$ is any orthonormal basis. But

$$\|\Delta_a(u) - \Delta_a(0)\| = \left\|\frac{u}{h_a(u) + h_{2a}} - \frac{u}{h(u) + h_{2z}}\right\| \leq 2\sqrt{u},$$

so taking the limit $u \searrow 0$, followed by $N \to \infty$, we have $\lim_{t \searrow 0} \|\Delta_a(t)\|_{\text{HS}} \geq \|\Delta_a(0)\|_{\text{HS}}$. In conjunction with the opposite inequality obtained above this becomes the desired equality.
B.5 Proof of (v)B

If \( \Delta_a(0) \) is HS, then by (iv)B also \( \Delta_a(v) \) is HS. Thus if in addition \( \Delta_a(v)h^{1/2}(v) \) is HS, then by observing the identity

\[
\Delta_a(v)h_a(v)\Delta_a(v) = \Delta_a^3(v) + \Delta_a(v)h(v)\Delta_a(v)
\]

we learn that \( h_a^{1/2}(v)\Delta_a(v) \) is also HS. The proofs of the theorem for \( \Delta_a(u)h^{1/2}(u) \) and \( h_a^{1/2}(u)\Delta_a(u) \) are similar, we take the first of these functions. Using the estimate (B.4) we have

\[
\|P_{a,k,\tau}\Delta_a(u)h^{1/2}(u)P_{l,\tau}\|^2_{\text{HS}} \leq (l\tau + \tau + u)^{1/2} \|P_{a,k,\tau}\Delta_a(u)P_{l,\tau}\|^2_{\text{HS}}
\]

\[
\leq c_{kl}(\tau, v, u) \left( \|P_{a,k,\tau}h_a^{1/2}(v)\Delta_a(v)P_{l,\tau}\|^2_{\text{HS}} + \|P_{a,k,\tau}\Delta_a(v)h^{1/2}(v)P_{l,\tau}\|^2_{\text{HS}} \right),
\]

where

\[
c_{kl}(\tau, v, u) = (l\tau + \tau + u)^{1/4} \frac{(k\tau + \tau + v)^{1/4} + (l\tau + \tau + v)^{1/4}}{\sqrt{k\tau + u + \sqrt{l\tau + u}}}
\]

\[
\leq 2(l\tau + \tau + u)^{1/4} \frac{(k\tau + \tau + v)^{1/4} + (l\tau + \tau + v)^{1/4}}{(k\tau + u)^{1/4} + (l\tau + u)^{1/4}}
\]

\[
\leq 2\left( \frac{\tau + u}{u} \right)^{1/4} \max \left\{ 1, \left( \frac{\tau + v}{u} \right)^{1/4} \right\}.
\]

Thus \( \Delta_a(u)h^{1/2}(u) \) is a HS operator for all \( u \in (0, \infty) \). Also, \( \lim_{u \to \infty} c_{kl}(\tau, v, u) = 0 \) and \( c_{kl}(\tau, v, u) < 2^{5/4} \) for \( u > \tau + v \), so \( \|\Delta_a(u)h^{1/2}(u)\|_{\text{HS}} \to 0 \) for \( u \to \infty \).

The HS-differentiability on \((0,\infty)\) is proved similarly as in (iv)B. It remains to investigate \( \Delta_a(0)h^{1/2}(0) \). Using the methods of the previous proof we find

\[
\Delta_a(u)h^{1/2}(u) - \Delta_a(0)h^{1/2}(0) = -\frac{u}{h_a(u) + h_a(0)}(\Delta_a(u) + \Delta_a(0))\frac{h^{1/2}(u)}{h(u) + h(0)}
\]

\[
+ \Delta_a(0)\frac{u}{[h^{1/2}(u) + h^{1/2}(0)] [h(u) + h(0)]}.
\]

As the r.h. side and the first term on the l.h. side are HS operators, the operator \( \Delta_a(0)h^{1/2}(0) \) is also HS. Estimating the HS-norm of the r.h. side we find

\[
\|\Delta_a(u)h^{1/2}(u) - \Delta_a(0)h^{1/2}(0)\|_{\text{HS}} \leq u^{1/4} (2\|\Delta_a(0)\|_{\text{HS}} + \|\Delta_a(u)\|_{\text{HS}}),
\]

which proves the missing HS-continuity at zero. Finally, it is now easy to convince oneself that the replacement of \( \Delta_a(.) \) with \( P_aB\Delta_a(.)P_C \) poses no difficulties.
C Definitions, estimates and identities

In this appendix we introduce some denotations and prove some results needed for the evaluation of the integral kernel of the operator $h_{za} - h_{z}$ as defined by (5.1).

Let $F$ be a real (or complex), bounded function in $C^{N+2}((0, +\infty))$ for some $N \geq 0$, such that $|F^{(N+2)}(p)| \leq \text{const} (p+1)^{-(N+2+\gamma)}$ for some $\gamma \in (0, 1)$. Then also

$$|F^{(n)}(p)| \leq \text{const} (p+1)^{-(n+\gamma)} \quad 1 \leq n \leq N + 2,$$

$$|F(p) - F_\infty| \leq \text{const} (p+1)^{-\gamma},$$

where $F_\infty$ is a constant (limit value at infinity). We define the following functions. For $(q,p) \in \mathbb{R}^2 \setminus \{0, 0\}$ we denote

$$\lambda_F(q,p) = \frac{F(|q|) - F(|p|)}{(q-p)(q+p)},$$

so that

$$\lambda_F(q,p) = \lambda_F(p,q) = \lambda_F(-q,p) = \lambda_F(q,-p).$$

We denote for $p \in \mathbb{R}$

$$\Lambda_F(p) = p \int_{-\infty}^{+\infty} \lambda_F(q,p) \, dq = 2p \int_{0}^{\infty} \lambda_F(q,p) \, dq = -\Lambda_F(-p),$$

$$\Lambda_{F,e}(p) = pe \sum_{k=-\infty}^{+\infty} \lambda_F(k\epsilon,p) = pe\lambda_F(0,p) + 2pe \sum_{k=1}^{\infty} \lambda_F(k\epsilon,p) = -\Lambda_{F,e}(-p),$$

$$\Lambda^0_{F,e}(p) = pe\lambda_F(0,p) = \frac{F(|p|) - F(0)}{p}, \quad \Lambda^*_{F,e}(p) = 2pe \sum_{k=1}^{\infty} \lambda_F(k\epsilon,p),$$

and for $p \neq p'$:

$$\chi_F(p,p') = \frac{\Lambda_F(p) - \Lambda_F(p')}{p - p'} = \chi_F(p',p) = \chi_F(-p,-p'),$$

$$\chi_{F,e}(p,p') = \frac{\Lambda_{F,e}(p) - \Lambda_{F,e}(p')}{p - p'} = \chi_{F,e}(p',p) = \chi_{F,e}(-p,-p'),$$

$$\chi^0_{F,e}(p,p') = \frac{\Lambda^0_{F,e}(p) - \Lambda^0_{F,e}(p')}{p - p'}, \quad \chi^*_{F,e}(p,p') = \frac{\Lambda^*_{F,e}(p) - \Lambda^*_{F,e}(p')}{p - p'}.$$

With these assumptions and denotations we have the following results.

(i) The function $\Lambda_F$ is in $C^{N+1}(\mathbb{R}_+^2)$, all derivatives $\lambda_F^{(m,n)}$ for $0 \leq m + n \leq N + 1$ extend to continuous functions on $(0, \infty)^2 \setminus \{0, 0\}$ and satisfy the estimates

$$\text{const} (p+1)^{-(n+\gamma)} \quad 1 \leq n \leq N + 2,$$

$$\text{const} (p+1)^{-\gamma}.$$
for \( q + p \leq 1 \):  
\[
|\Lambda_F^{(m,n)}(q,p)| \leq \text{const} \frac{1}{(p+q)^{m+n+1}},
\]  
(C.10)  

for \( q + p \geq 1 \):  
\[
|\Lambda_F^{(m,n)}(q,p)| \leq \text{const} \begin{cases} 
\frac{1}{(q+1)^{m+2(p+1)^{n+\gamma}},} & q \geq p, \\
\frac{1}{(q+1)^{m+\gamma(p+1)^{n+\gamma}}}, & q \leq p.
\end{cases}
\]  
(C.11)  

The integrals \( \int_0^\infty \Lambda_F^{(m,0)}(q,p) \, dq \) are in \( C^{N+1-m}((\mathbb{R}_+)) \), the differentiation may be carried out under the integral sign, and one has the estimates:

for \( p \leq 1 \):  
\[
\int_0^\infty |\Lambda_F^{(m,n)}(q,p)| \, dq \leq \text{const} \begin{cases} 
\frac{1}{p^{m+n}}, & m+n \geq 1, \\
\frac{1}{p^{n+1}}, & m = 0,
\end{cases}
\]  
(C.12)  

for \( p \geq 1 \):  
\[
\int_0^\infty |\Lambda_F^{(m,n)}(q,p)| \, dq \leq \text{const} \begin{cases} 
\frac{1}{p^{n+1}}, & m = 0, \\
\frac{1}{p^{n+2}}, & m \geq 1.
\end{cases}
\]  
(C.13)  

(ii) The functions \( \Lambda_F, \Lambda_F^\epsilon \) and \( \Lambda_F^0 \) are in \( C^{N+1}(\mathbb{R}_+) \) and satisfy the estimates:

for \( p \leq 1 \):  
\[
|\Lambda_F^{(n)}(p)|, \, |\Lambda_F^\epsilon^{(n)}(p)| \leq \text{const} \begin{cases} 
\frac{p(|\log p|+1)}, & n = 0, \\
\frac{(|\log p|+1)}{p^{n-1}}, & n > 1,
\end{cases}
\]  
(C.14)  

\[
|\Lambda_F^0(p)| \leq \text{const} \epsilon,
\]  

for \( p \geq 1 \):  
\[
|\Lambda_F^{(n)}(p)|, \, |\Lambda_F^\epsilon^{(n)}(p)| \leq \frac{\text{const} \epsilon}{p^{n+\gamma}}, \quad |\Lambda_F^0(p)| \leq \frac{\text{const} \epsilon}{p^{n+1}}.
\]  
(C.15)  

The functions \( \chi_F \) and \( \chi_F^\epsilon \) are in \( C^N((\mathbb{R} \setminus \{0\})^2) \) and in this domain satisfy the estimates:

for \( 0 < |p| \leq 1, \, 0 < |p'| \leq 1 \):
\[
|\chi_F(p,p')|, \, |\chi_F^\epsilon(p,p')| \leq \text{const} \left( |\log |p|| + |\log |p'|| + 1 \right),
\]  
(C.16)  

for \( 0 < |p| \leq 1, \, 0 < |p'| \leq 1, \, pp' > 0 \):
\[
|\chi_F^0(p,p')| \leq \text{const} \epsilon,
\]  
(C.17)  

for \( 0 < |p| \leq 1, \, 0 < |p'| \leq 1, \, pp' < 0 \):
\[
|\chi_F^\epsilon(p,p')| \leq \frac{\text{const} \epsilon}{|p| + |p'|},
\]  
(C.18)  

for \( (p,p') \in \mathbb{R}^2 \setminus (-1,1)^2 \):
\[
|\chi_F(p,p')|, \, |\chi_F^\epsilon(p,p')| \leq \text{const} \begin{cases} 
\frac{1}{(|p|+1)^{\gamma(|p'|+1)}}, & |p| \leq |p'|, \\
\frac{1}{(|p|+1)(|p'|+1)^\gamma}, & |p| \geq |p'|.
\end{cases}
\]  
(C.19)
Moreover, for $|p|, |p'| \geq 1$ there is

$$|\chi_{F,e}^{(m,n)}(p,p')|, \ |\chi_{F,e}^{(m,n)}(p,p')| \leq \text{const} \begin{cases} \frac{1}{(|p| + 1)^{m+\gamma}} & |p| \leq |p'|, \\ \frac{1}{(|p| + 1)^{n+\gamma}} & |p| \geq |p'|. \end{cases}$$

(C.20)

(iii) The following identity is satisfied in the distributional sense:

$$\int_{-\infty}^{+\infty} \frac{F(|q|)}{(p - q \pm i0)(p' - q \pm i0)} \, dq = \chi_{F}(p,p') \pm i\pi \frac{F(|p|) - F(|p'|)}{p - p'},$$

(C.21)

On the l.h.s. the distribution $[(p - q \pm i0)(p' - q \pm i0)]^{-1}$ is first applied to a smooth function of compact support $f(p,p')$, and the result integrated as indicated; the r.h.s. side is multiplied by $f(p,p')$ and integrated over $dp \, dp'$.

Note that in the integral on the l.h.s. side the signs in front of $\pi$ must match.

For $p, p' \neq k \epsilon$ there is

$$\epsilon \sum_{k=-\infty}^{+\infty} \frac{F(|k|\epsilon)}{(p - k\epsilon)(p' - k\epsilon)} = \chi_{F,e}(p,p') - \frac{\pi}{p - p'} [F(|p|) \cot(\pi p/\epsilon) - F(|p'|) \cot(\pi p'/\epsilon)],$$

(C.22)

$$\epsilon \sum_{k=-\infty}^{+\infty} \frac{(-1)^k F(|k|\epsilon)}{(p - k\epsilon)(p' - k\epsilon)} = \chi_{F,2e}(p,p') - \chi_{F,e}(p,p') - \frac{\pi}{p - p'} \left[ \frac{F(|p|)}{\sin(\pi p/\epsilon)} - \frac{F(|p'|)}{\sin(\pi p'/\epsilon)} \right].$$

(C.23)

In consequence further identities follow for all $p, p' \in \mathbb{R}$:

$$2\epsilon \sum_{k=-\infty}^{+\infty} \frac{F(|k|\epsilon) \sin[b(p - k\epsilon)] \sin[b(p' - k\epsilon)]}{(p - k\epsilon)(p' - k\epsilon)} = \cos[b(p - p')] \chi_{F,e}(p,p') + \cos[b(p + p')] [\chi_{F,e}(p,p') - \chi_{F,2e}(p,p')]$$

$$+ \frac{\pi \sin[b(p - p')]}{(p - p')} [F(|p|) + F(|p'|)],$$

(C.24)

$$2\epsilon \sum_{k=-\infty}^{+\infty} F(|k|\epsilon)(-1)^k \frac{\sin[b(p + k\epsilon)] \sin[b(p' - k\epsilon)]}{(p + k\epsilon)(p' - k\epsilon)}$$

$$= -\cos[b(p - p')] \chi_{F,e}(p,-p') - \cos[b(p + p')] [\chi_{F,e}(p,-p') - \chi_{F,2e}(p,-p')]$$

$$- \frac{\pi \sin[b(p - p')]}{(p + p')} |F(|p|) - F(|p'|)|,$$

(C.25)
C.1 Proof of (i)

For \( q, p > 0 \) one has

\[
\frac{F(q) - F(p)}{q - p} = \int_0^1 F^{(1)}(qt + p(1 - t)) \, dt,
\]

so by (C.26) the differential properties of \( \lambda^{(m,n)}_F \) are satisfied. For \( q + p \leq 1 \) the derivatives of \( F \) are bounded, so the estimate (C.10) is also true. To prove the estimate (C.11) it is sufficient to assume that \( q + p \geq 1 \) and \( q \geq p \) (due to the symmetry of \( \lambda_F \)), which implies \( q + p \geq (1 + q)/2 \) and

\[
\left| \partial_x^k \partial_y^l \frac{1}{q + p} \right| \leq \frac{\text{const}}{(q + 1)^{k+l+1}}.
\]

We consider two cases \( q \leq 3p \) and \( q > 3p \) separately. In the first of these regions one has \( p \geq q/3 \geq (q + p)/6 \geq (q + 1)/12 \) and

\[
\left| \partial_x^r \partial_y^s F(q) - F(p) \frac{1}{q - p} \right| \leq \int_0^1 |F^{(r+s+1)}(qt + p(1 - t))| \, dt
\]

\[
\leq \text{const} \int_0^1 \frac{dt}{(p + (q - p)t)^{r+s+1+\gamma}} \leq \frac{\text{const}}{(q + 1)^{r+s+1+\gamma}},
\]

so \( |\lambda^{(m,n)}(q,p)| \leq \text{const} \,(q + 1)^{-(m+n+2+\gamma)} \) in this region, which complies with the estimate (C.11). In the second region one has \( q - p = (q + p - 3p)/2 \geq (q + 1)/4 \), so taking into account Eq. (C.27) one has in that region:

\[
\left| \partial_x^k \partial_y^l \frac{1}{(q - p)(q + p)} \right| \leq \frac{\text{const}}{(q + 1)^{k+l+2}}.
\]

Taking also into account the bounds (C.11), and similar ones with the argument \( p \) replaced by \( q \), one confirms the estimate (C.11).

The statements following estimates (C.10) and (C.11) are their simple consequences. The next two estimates (C.12) and (C.13) are obtained from the preceding two by elementary integration.

C.2 Proof of (ii)

For \( \Lambda_F \) the differentiability properties and the estimates (C.14) and (C.15) are simple applications of the properties stated in the last sentence of (i). To prove the same facts for \( \Lambda_{F, \epsilon} \) one has to estimate the sums by integrals, which can be done rather easily by observing that for each \( p > 0 \) the r.h. sides of the bounds (C.11) and (C.11) can be glued together into a continuous decreasing function of \( q \).

For \( \Lambda_{F, \epsilon} \) the estimates (C.14) and (C.15) follow from (C.26) and (C.11) respectively. The estimates (C.20) in the enlarged region \(|p|, |p'| \geq 1/2 \) follow from the
bounds (C.15) (which stay valid for $|p| > 1/2$) with the use of the method similar to that applied in the proof of (i)$_C$. The estimates (C.19) are then easily extended to the whole region of application with the use of bounds (C.14), (C.15) (with $n = 0$). The remaining estimates also follow from these bounds by cutting the square $(-1,1)^2$ into four squares according to the signs of $p$ and $p'$.

### C.3 Proof of (iii)$_C$

To prove (C.21) we first note that for complex $z, z'$ with $\Im z, \Im z' \neq 0$ there is

$$z \int_{-\infty}^{+\infty} \frac{F(|q|) dq}{(q-z)(q+z)} - z' \int_{-\infty}^{+\infty} \frac{F(|q|) dq}{(q-z')(q+z')} = (z - z') \int_{-\infty}^{+\infty} \frac{F(|q|) dq}{(q-z)(q-z')}$$

(see evenness of the integrand on the r.h. side). Let $1 > \alpha > \alpha' > 0$. Setting $z = p \pm i\alpha$, $z' = p' \pm i\alpha'$, denoting

$$\Lambda_F^\pm\alpha(p) = (p \pm i\alpha) \int_{-\infty}^{+\infty} \frac{F(|q|) - F(|p|)}{(q - (p \pm i\alpha))(q + p \pm i\alpha)} dq,$$

and using the identity

$$\int_{-\infty}^{+\infty} \frac{dq}{(q - (p \pm i\alpha))(q + p \pm i\alpha)} = \pm \frac{\pi i}{p \pm i\alpha}$$

we get

$$\int_{-\infty}^{+\infty} \frac{F(|q|) dq}{(p - q \pm i\alpha)(p' - q \pm i\alpha')} = \Lambda_F^\pm\alpha(p) - \Lambda_F^\pm\alpha'(p') \pm i\pi \frac{F(|p|) - F(|p'|)}{p - p' \pm i(\alpha - \alpha')}$$

(C.30)

We multiply this identity by a Schwartz function $f(p,p')$, integrate, and take successive limits $\alpha' \rightarrow 0$, $\alpha \rightarrow 0$. Now, for the l.h. side we observe that the order of integration may be changed and the result represented as an integral over $q$ of $F(|q|)$ multiplied by

$$\int \frac{f(p,p')}{(p - q \pm i\alpha)(p' - q \pm i\alpha')} dp dp'.$$

By the well known techniques one shows that the consecutive limit exists and, moreover, the above expression is bounded by const $(q^2 + 1)^{-1}$. This is sufficient to conclude that l.h. side of Eq. (C.21) is obtained in the limit. For the r.h. side of (C.30) one easily notes that its consecutive point-wise limit yields for $p \neq p'$ the function on the r.h. side of (C.21). Thus to complete the proof one only needs to show that in the limiting process the r.h. side of (C.30) stays bounded by a function defining a distribution. This is immediate for the second term, as

$$\left| \frac{F(|p|) - F(|p'|)}{p - p' \pm i(\alpha - \alpha')} \right| \leq |p + p'| \left| \Lambda_F(p,p') \right|.$$
Also, the property is rather obvious for the limiting process \( \alpha' \to 0 \) in the first term (note that \( |\Lambda_F^{\pm \alpha}(p)| \leq (|p|+1) \int |\lambda_F(q,p)| dq \)). Thus we are left with the function

\[
\frac{\Lambda_F^{\pm \alpha}(p) - \Lambda_F(p')}{p - p' \pm i\alpha} = \frac{\Lambda_F^{\pm \alpha}(p) - \Lambda_F(p)}{p - p' \pm i\alpha} + \frac{\Lambda_F(p) - \Lambda_F(p')}{p - p' \pm i\alpha}.
\]

The second term on the r.h. side of this equality is bounded by \( |\chi_F(p, p')| \), which is sufficient for our purpose, while for the first one it is sufficient to estimate the function

\[
\frac{\Lambda_F^{\pm \alpha}(p) - \Lambda_F(p)}{p - p' \pm i\alpha} = \int \frac{[F(|q|) - F(|p|)] \, dq}{(q - (p \pm i\alpha))(q + p \pm i\alpha)} \pm i\pi \frac{2p \pm i\alpha}{p \pm i\alpha} \frac{\lambda_F(p, p)}{q - (p \pm i\alpha)}(q + p \pm i\alpha) \, dq.
\]

(C.31)

The first two terms on the r.h. side of this identity are bounded respectively by \( \int |\lambda_F(q, p)| dq \) and \( 2\pi |p| \lambda_F(p, p) \), which fulfills our demands. For the third term we assume that \( p > 0 \) (the case \( p < 0 \) needs only obvious modifications), represent the integral as twice the integral over \((0, \infty)\), and consider the integration sets \((0, p+1)\) and \((p+1, \infty)\) separately. The term containing integration over the second set is bounded by

\[
(p + 1) \log(1 + 2p) \left( |\lambda_F(p, p)| + \max_{q \in (p+1, \infty)} |\lambda_F(q, p)| \right) \leq \text{const} \frac{\log(1 + 2p)}{(p + 1)^{1+\gamma}}.
\]

The remaining part is bounded by

\[
2p \int_0^{p+1} \left| \frac{\lambda_F(q, p) - \lambda_F(p, p)}{q - p} \right| \, dq \leq 2p \int_0^{p+1} \frac{1}{q - p} \int_{r<} \left| \lambda_F^{(1,0)}(s, p) \right| \, ds \, dq,
\]

where \( r< = \min \{q, p\} \), \( r> = \max \{q, p\} \). For \( p > 1 \) there is \( |\lambda_F^{(1,0)}(s, p)| \leq \text{const} p^{-2} \) (cf. (C.11)), which gives a sufficient estimate in this region. For \( p < 1 \) and \( q < p + 1 \) one has \( |\lambda_F(s, p)| \leq \text{const} (s + p)^{-2} \) (by (C.10), which may be used in this extended region). Thus the r.h. side is bounded by \( \text{const} \log((2p+1)/p) \), which ends the proof of (C.21).

To prove (C.22) we first note that for \( p, p' \neq k\epsilon \), there is

\[
(p-p') \sum_{k \in \mathbb{Z}} \frac{F(|k|\epsilon)}{(p-\epsilon)(p'-\epsilon)} = p \sum_{k \in \mathbb{Z}} \frac{F(|k|\epsilon)}{(k\epsilon-p)(k\epsilon+p)} - p' \sum_{k \in \mathbb{Z}} \frac{F(|k|\epsilon)}{(k\epsilon-p')(k\epsilon+p')}.
\]

But

\[
p \sum_{k \in \mathbb{Z}} \frac{F(|k|\epsilon)}{(k\epsilon-p)(k\epsilon+p)} = \Lambda_{F, \epsilon}(p) - F(|p|) \cot(\pi p/\epsilon),
\]

and

\[
p' \sum_{k \in \mathbb{Z}} \frac{F(|k|\epsilon)}{(k\epsilon-p')(k\epsilon+p')} = \Lambda_{F, \epsilon}(p') - F(|p'|) \cot(\pi p'/\epsilon),
\]
where we used the following identity

\[ x \sum_{k \in \mathbb{Z}} \frac{1}{x^2 - k^2} = \pi \cot(\pi x), \quad (x \notin \mathbb{Z}) \]

(cf. [1], formula 1.421(3)). This ends the proof of (C.22).

Formula (C.23) follows easily from the preceding one if one observes that its l.h. side may be written as

\[ 2\epsilon \sum_{F} (|k| \epsilon) \left( p - k \epsilon \right) \left( p' - k \epsilon \right) - \epsilon \sum_{F} (|k| \epsilon) \left( p - k \epsilon \right) \left( p' - k \epsilon \right). \]

Finally, to prove (C.24) and (C.25) one notes that

\[ 2 \sin[b(p - k \epsilon)] \sin[b(p' - k \epsilon)] = \cos[b(p + p') - (-1)^{k} \cos[b(p + p')]] \]

and uses (C.22) and (C.23).

D Some identities for the logarithmic derivative of the Gamma function \( \psi \)

Two particular textbook representations of the function \( \psi \) are of importance for us:

\[ \psi(z) = -\frac{1}{z} + \lim_{N \to \infty} \left\{ \log N - \sum_{k=1}^{N} \frac{1}{k + z} \right\}, \quad (D.1) \]

\[ \psi(z) - \log z + \frac{1}{z} = \int_{0}^{\infty} v(s) e^{-zs} ds, \quad \Re z > 0, \quad (D.2) \]

where

\[ v(s) = \frac{1}{s} - \frac{1}{e^s - 1}. \quad (D.3) \]

The function \( v \) is analytical in the complex plane outside the points \( z = 2k\pi i, \quad k \in \mathbb{Z} \setminus \{0\} \), where it has poles with principal values \(-1\). Moreover,

\[ v^{(k)}(0) = -\frac{B_{k+1}}{k+1}, \quad |v^{(k)}(s)| \leq \frac{c_k}{(|s| + 1)^{k+1}}, \quad s \in \mathbb{R}, \quad k = 0, 1, \ldots . \quad (D.4) \]

Using these properties in the representation (D.2) one finds by induction for \( m \in \mathbb{N} \) and \( \Re z > 0 \) the expansion:

\[ \psi(z) - \log z = -\frac{1}{2z} - \sum_{k=1}^{m} \frac{B_{2k}}{2k} \frac{1}{z^{2k}} + \frac{1}{2z} \int_{0}^{\infty} v^{(2m)}(s) e^{-zs} ds \quad (D.5) \]

(remember that \( B_{2k+1} = 0 \) for \( k \in \mathbb{N} \)). We denote

\[ w_{2m}(z) = z^{2m} \left\{ \psi(z) - \log z + \frac{1}{2z} + \sum_{k=1}^{m} \frac{B_{2k}}{2k} \frac{1}{z^{2k}} \right\}. \quad (D.6) \]
Then for $\lambda \geq 0$, $m = 1, 2, \ldots$, one has the identity
\[
\int_0^\infty \cos(2\pi \lambda t)w_{2m}(t) \, dt = (-1)^{m-1} \frac{(2m)!}{2(2\pi)^{2m}} \sum_{k=1}^\infty \frac{1}{(k + \lambda)^{2m+1}}.
\] (D.7)

In particular, \[\int_0^\infty \cos(2\pi \lambda t)w_2(t) \, dt = \frac{1}{4\pi^2} \sum_{k=1}^\infty \frac{1}{(k + \lambda)^3},\] (D.8)
and by the application of the operator $\lambda \partial_\lambda$:
\[
\int_0^\infty \cos(2\pi \lambda t)w_2^{(1)}(t) \, dt = \frac{1}{4\pi^2} \left\{ 3\lambda \sum_{k=1}^\infty \frac{1}{(k + \lambda)^4} - \sum_{k=1}^\infty \frac{1}{(k + \lambda)^3} \right\}.
\] (D.9)

To prove Eq. (D.7) one notes that $v^{(2m)}(-s) = -v^{(2m)}(s)$ and uses Eq. (D.5) to obtain for $\lambda > 0$
\[
\int_0^\infty \cos(2\pi \lambda t)w_{2m}(t) \, dt = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{v^{(2m)}(s)}{s + i2\pi \lambda} \, ds = \frac{(2m)!}{2} \int_{-\infty}^{+\infty} \frac{v(s)}{(s + i2\pi \lambda)^{2m+1}} \, ds.
\] (D.10)

The proof is now completed for $\lambda > 0$ by integration in the complex $s$-plane along the rectangular contour with vertices at points $\pm \xi$, $\pm \xi + (2k + 1)\pi i$ followed by the successive limits $\xi \to \infty$ and $k \to \infty$. Finally, one uses the continuity in $\lambda$ of both sides of (D.7) to extend the formula to $\lambda = 0$.

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