Joint min-max distribution and Edwards-Anderson's order parameter of the circular 1/f-noise model

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Abstract – We calculate the joint min-max distribution and the Edwards-Anderson’s order parameter for the circular model of 1/f-noise. Both quantities, as well as generalisations, are obtained exactly by combining the freezing-duality conjecture and Jack-polynomial techniques. Numerical checks come with significantly improved control of finite-size effects in the glassy phase, and the results convincingly validate the freezing-duality conjecture. Application to diffusive dynamics is discussed. We also provide a formula for the pre-factor ratio of the joint/marginal Carpentier-Le Doussal tail for minimum/maximum which applies to any logarithmic random energy model.

Introduction. – The statistical physics of a particle in logarithmically correlated random potentials was initially studied as simplified models of spin glass known as logarithmic random energy models (log-REM’s) [1–3], but is now realised to be relevant to subjects ranging from multifractal wave functions [4,5], extrema of 2d Gaussian free field (GFF) [6,7] and 2d quantum gravity [8], to the value distribution of random matrix characteristic polynomials and the Riemann zeta on the critical line [9–11].

A key feature of log-REM’s is freezing, i.e., temperature independence of the free-energy density in the glassy phase. The extension of freezing to describe the free-energy fluctuation has a long history [2,3,12,13] and was recently promoted to a rigorous stage [14] by using derivative multiplicative chaos [8,15]. Yet, explicit predictions (e.g., of free-energy distribution) still require non-rigorous approaches and some “integrability” coming from log-gas integrals [16], β-random matrix theory [17], or symmetric functions [18].

From the study of the latter, accumulating evidence supports the freezing-duality conjecture (FDC), first put forward in [13]. To describe it, consider any thermodynamic observable \( \mathcal{O}(\beta) \), supposed analytical in \( \beta \) for \( 0 < \beta < \beta_c \), and analytical continued to \( \mathcal{O}_a(\beta), \beta \in (0, \infty) \). The FDC claims that, if \( \mathcal{O}_a(\beta) \) is duality-invariant, \( \mathcal{O}(\beta) \) freezes:

\[
\mathcal{O}_a(\beta) \overset{\text{duality-inv.}}{\Rightarrow} \mathcal{O}_a(\beta^2/\beta) \Rightarrow \mathcal{O}(\beta) \overset{\beta \geq \beta_c}{\Rightarrow} \mathcal{O}(\beta_c). \tag{1}
\]

\( \mathcal{O} \) can be observables not yet covered by the rigorous results, e.g., moments of the minimum position [19,20]. The theoretical understanding of the FDC is unsatisfactory and tests of its predictions remain limited in both quantity and quality (due to slow numerical convergence in the glassy phase).

This letter improves significantly the situation by studying the circular model, introduced in [12], where the distribution of the minimum was calculated. We show here that there is an infinite series of duality-invariant observables, of which the simplest is the Edwards-Anderson’s (EA) order parameter, fundamental in the spin-glass theory [21]. It provides one of the most accurate numerical tests of the FDC (cf. fig. 3).

We also calculate the joint min-max distribution. As application, we obtain the distribution of the span (the min-max difference), which is the extremal width of interfaces modeled by the log-correlated field studied in experiments [22,23]. Moreover, properties of opposite extrema are related to the dynamics of an overdamped diffusive (Langevin) particle in the 1d potential. Roughly, the span...
is the barrier that the particle should surmount to explore the whole system, and is thus related to Arrhenius passage times and to the diffusion coefficient in the periodic potential [24–26]. In the log-correlated 1d case, the freezing of log-REM’s is directly responsible for the freezing of dynamical exponents [27]. Since the opposite extrema are far apart in space and in value, they are often assumed to the independent. Our results provide correction to this approximation for the circular model. Another highlight is the modification of the amplitude of the joint Carpentier-Le Doussal (CLD) tail [3] by the max-min correlation (compared to the product of marginals). We shall give a formula (13) for the tail ratio for the whole system, and is thus related to Arrhenius passage times and to the diffusion coefficient in the periodic potential [24–26].

Model and main results. – The circular model of 1/f-noise is defined as random signals $V_{j,M}$, $1 \leq j \leq M$, and their periodic extension on $\mathbb{Z}$ of period $M$, generated by independent Gaussian Fourier modes with variance proportional to inverse frequency:

$$V_{j,M} = \Re \left[ \sum_{k=-M}^{M-1} \frac{1}{|k|} (u_k + iv_k) \exp \left( \frac{2\pi ikj}{M} \right) \right].$$

(2)

Here $\{u_k, v_k\}_k$ are i.i.d. standard centered Gaussian [28]. The definition implies a logarithmically growing variance

$$\mathbb{V}^2_{j,M} = 2(\ln M + W), \quad W \to \gamma_E - \ln 2$$

(3)

characteristic of all (log)-REM’s, and the off-diagonal correlations describing the planar GFF on the unit circle. Indeed, setting

$$\xi_{j,M} = \exp \left( \frac{2\pi ij}{M} \right),$$

(4)

then for any sequence of pairs $(j_M, k_M)$ such that $(\xi_{j_M,M}, \xi_{k_M,M}) \to (\xi, \eta)$ with $\xi \neq \eta$,

$$V_{j_M,M}V_{k_M,M} \to 2 \ln |\xi - \eta|,$$

(5)

which is the defining correlator of the planar GFF.

The observable $\xi_{j,M}$ in (4) can be seen as an O(2) (XY) spin (in the $M \to \infty$ limit). Its thermal average with inverse temperature $\beta > 0$ is

$$\langle \xi \rangle = \sum_{j=1}^{M} \exp(-\beta V_{j,M}) \xi_{j,M} \over \sum_{j=1}^{M} \exp(-\beta V_{j,M}).$$

(6)

We define the modulus square of the above as the EA order parameter of the circular model. Here we obtain its full temperature dependence:

$$|\langle \xi \rangle|^2_{M=\infty} = \begin{cases} \beta^2, & \beta \leq 1, \\ \frac{1 + \beta^2}{2\beta - 1}, & \beta > 1. \end{cases}$$

(7)

The minimum and maximum, denoted as $V_{M\pm} = \pm \min_{j=1}^{M} (\pm V_{j,M})$, are known for standard log-REM’s to satisfy [3,29]

$$V_{M\pm} = \mp 2 \ln M \pm 3 \frac{2}{3} \ln \ln M + v_\pm \pm c_M,$$

(8)

where $c_M$ converges to some unknown deterministic constant as $M \to \infty$, while $v_\pm$ are the fluctuations. For the present circular model, $-v_-$ and $v_+$ have the same distribution [12]:

$$P(v_+ > y) = 2e^{\frac{y}{4}} K_1(2e^{\frac{y}{4}}) \Rightarrow \exp(v_\pm) = \Gamma^2(1 \pm t),$$

(9)

exhibiting the CLD tail [3] $P(v_+) \to \sim -v_+ e^{v_+}$ (see footnote 1). We generalise (9) to the joint $v_\pm$ distribution:

$$S_2(1, t_2) = \sum_{\lambda} \prod_{i=1}^{2} \frac{x^{\beta-1} + y^{\beta} + t_{i}}{(x + 1)^{\beta-1} + (y + 1)^{\beta} + t_{i}},$$

(10)

$$S_2(1, t_2) = \sum_{\lambda} \prod_{i=1}^{2} \frac{x^{\beta-1} + y^{\beta} + t_{i}}{(x + 1)^{\beta-1} + (y + 1)^{\beta} + t_{i}},$$

(11)

where $S_2$ is a sum over partitions, which we define here as sets of integer plane coordinates: $\lambda = \{(x, y) : x = 0, \ldots, x_0 - 1, y = 0, \ldots, l - 1\}$, determined by a decreasing sequence of integers $x_0 > \cdots > x_l > 0, l \geq 0$. The sum $\sum_{\lambda} = \sum_{l=0}^{\infty} \sum_{x_0 > \cdots > x_l = 0}$ runs over all such sequences, including the empty one $(l = 0, \lambda = \emptyset)$. Equations (10) and (11) imply the following joint/marginal CLD tail ratio:

$$R := \lim_{v_\pm \to \mp \infty} \frac{P(v_+, v_-)}{P(v_+)P(v_-)} = S_1(-1, -1) = 2.$$  

(12)

This result is a special case of the following ratio formula:

$$R = \lim_{M \to \infty} \frac{1}{M^2} \sum_{j,k=1}^{M} \exp(-\beta V_{j,M}V_{k,M}),$$

(13)

valid for any log-REM defined by the covariance matrix $[V_{j,M}V_{k,M}]_{j,k=1}^{M}$ such that the above limit exists.

Joint $(v_+, v_-)$ distribution. – Now we derive the joint distribution (10); for this we study the thermodynamics at inverse temperature $\pm \beta$, encoded in the partition functions

$$Z_{M\pm} = \sum_{j=1}^{M} \exp(\mp \beta V_{j,M}).$$

(14)

When $\beta \to \infty$, the free energy $F_{M\pm} := \mp \beta^{-1} \ln Z_{M\pm} \to V_{M\pm}$. Let us define the regularised partition functions (cf. (3))

$$Z_\pm = Z_{M\pm}/Z_{M\mp} = \frac{Z_{M\pm}}{M^{1+\beta} e^{\beta W}}.$$  

(15)

Then, the replica averages $Z_{M\mp}^{-1} Z_{M\pm}$ converge to Coulomb-gas integrals as $M \to \infty$ if $\beta < \min(n^{-1/2}, m^{-1/2})$ (in this

$^1P(an \ event) \ denotes \ its \ probability: \ P(random \ variable(s)) \ denotes \ its \ probability \ density \ function.
work, we denote by \( \frac{1}{\alpha} \) equations that hold in the \( M \to \infty \) limit and for \( \beta \) small enough:

\[
\sum_{a,b} \left| \xi_{ab} \right|^{-2/\alpha} = \Pi_{a,b}^{(\alpha)}(\xi) \Pi_{a,b}^{(\alpha)}(\eta),
\]

and for \( \alpha = -2 \beta^2 \phi(\xi_{ab}^{-1/\alpha}) = \prod_{a,b} \xi_{ab}^{-2/\alpha}.
\]

The integrals run on the unit circle \( |\xi_{ab}| = |\eta_{ab}| = 1 \) and the product runs from \( a = 1, \ldots, n \) and \( b = 1, \ldots, m \). The notations introduced in (17) are convenient for applying the Jack-polynomial theory to calculate the integral (16), the \( m = n \) case of which appeared in [30] for studying the Kondo problem. Their approach consists of two steps that we adapt to the present case, following conventions of [30], sect. 6.10. First, one uses the Cauchy identity ([18], 6.10.4) combined with the second paragraph on p. 380:

\[
\prod_{a,b}(1 - \xi_{ab})^{-1/\alpha} = \sum_{\lambda} P^{(\alpha)}(\xi) Q^{(\alpha)}(\eta),
\]

where \( P^{(\alpha)}(\xi) \) and \( Q^{(\alpha)}(\xi) \) form dual bases of Jack polynomials. Using (18) we expand the product in (16):

\[
\prod_{a,b}(1 - \xi_{ab})^{-2/\alpha} = \sum_{\lambda,\mu} P^{(\alpha)}(\xi) Q^{(\alpha)}(\eta) \times (c.c.)
\]

Then we apply the orthogonality relation ([18], 6.10.35–6.10.37))

\[
\int \mu^{\alpha}_{(\lambda)}(\xi) \mu^{\alpha}_{(\lambda)}(\xi) \xi^{(\alpha)} d\xi = \delta_{\lambda,\mu} p^{\alpha}_{(\mu)}(\xi) c_{n}(\alpha),
\]

\[
c_{n}(\alpha) = \int \phi^{\alpha}_{(\lambda)}(\xi) \phi^{(\alpha)}_{(\lambda)}(\xi) d\xi = \frac{\Gamma(1 + n/\alpha)}{\Gamma(1 + 1/\alpha) n!},
\]

\[
p^{\alpha}_{(\lambda)} = \prod_{(x,y) \in \lambda} \frac{\alpha(x + 1) + n - (y + 1)}{\alpha(x + 1) + n - (y + 1)},
\]

where \( c_{n}(\alpha) \) is Dyson’s integral [31]. Orthogonality, combined with eqs. (19) and (16), yields the following equation:

\[
\sum_{\lambda}^{\infty} \sum_{\lambda}^{\infty} \frac{\Gamma(1 - \beta^2) \Gamma(1 - m \beta^2)}{\Gamma(1 + \beta^2)^{m+n}} \prod_{\lambda}^{\infty} p^{\alpha}_{(\lambda)}(\xi) p^{\alpha}_{(\lambda)}(\xi),
\]

In eqs. (21) through (23), \( n \) is continued to complex variable. The denominator in (23) can be absorbed by a first moment shift of the free energy:

\[
f_{\pm} := F_{\pm} \pm \frac{1}{\beta} \ln \Gamma(1 - \beta^2), \quad F_{\pm} := -\beta^{-1} \ln Z_{\pm}.
\]

Now setting \( t_1 = -n \beta, t_2 = -m \beta \) and using (17), (23) is rewritten as

\[
\exp(t_{1} f_{+} - t_{2} f_{-}) = S_{\beta}(t_1, t_2) \prod_{i=1}^{2} \Gamma(1 + \beta t_{i}),
\]

where \( S_{\beta}(t_1, t_2) \) is given by (11). Equation (25) holds actually for any \( \beta \leq 1 \) and generic complex \( t \): this claim relies on assuming analyticity in the \( \beta < 1 \) phase and is non-rigorous (however, see [32]), but can be numerically checked with high precision. Now, we observe several familiar features: the \( \Gamma(\beta + n \beta^2) / \Gamma(1 - m \beta^2) \) for \( n \geq 1 \) and \( m \geq 1 \), and the divergence of \( \Gamma(1 - \beta^2) \) (see footnote 3), suggesting that (25) is valid only until \( \beta = \beta_c = 1 \) and the \( \beta > 1 \) (glassy) phase should be described by the FDC (1). The latter can be applied because the RHS (11) is duality-invariant: transpose partition pairs give terms related by the duality transform \( \beta \rightarrow 1/\beta \). Indeed, introducing two i.i.d. standard Gumbel variables \( g_{\pm} \) jointly independent of \( f_{\pm} \), and constructing the usual duality-invariant decoration of the free energy, here at inverse temperature \( \pm \beta^{-1} \), we have

\[
\exp(t_{1} g_{+} - t_{2} g_{-}) = S_{\beta}(t_1, t_2) \prod_{i=1}^{2} \Gamma(1 + \beta t_{i}),
\]

From the FDC (1), the duality invariance of the RHS implies the freezing of the LHS:

\[
\exp(t_{1} f_{+} - t_{2} f_{-}) = S_{\beta}(t_1, t_2) \prod_{i=1}^{2} \Gamma(1 + \beta t_{i}),
\]

which yields (10) when \( \beta \to \infty \). The novelty here is the extension of the FDC to the joint distribution of opposite extrema.

A nice extension of the above result is as follows. Let \( \eta \in (0, 1) \), consider two circular models \( V_{j,M}, j = 1, \ldots, M \), correlated as \( \风暴_{j,M} V_{j,M} = -2 \ln \left| 1 - \xi_{j,M} \xi_{j,M}^{*} \right| \) (see footnote 3), and let \( f_{\pm} \) (24) be defined with respect to \( V_{j,M}^{(\pm)} \). Then a direct extension of the above derivation leads to

\[
\frac{\exp(t_{1} f_{+} - t_{2} f_{-})}{\exp(t_{1} f_{+}) \exp(\tilde{t}_{2} f_{-})} = \frac{S_{(q)}(t_1, t_2)}{S_{(q)}(t, t)}
\]

\[
S_{(q)}(t_1, t_2) = \sum_{\lambda} \prod_{(x,y) \in \lambda} \frac{q(x \beta^{2} + y \beta + t_{1})}{(x + 1) \beta^{2} + (y + 1) \beta + t_{1}},
\]

interpolating between two independent circular models \( (q \to 0) \) and \( (q \to 1) \).

Now, to obtain the joint CTD tail behaviour at \( v_{\pm} \to \pm \infty \), observe that the rightmost pole of (10) is

\[
\frac{S_{(q)}(t_{1}, t_{2})}{S_{(1)}(t_{1}, t_{2})}, \quad \text{and that only } \lambda = 0 \text{ and } \lambda = \emptyset = \{0, 0\} \text{ contribute to the sum } S_{(q)}(t_{1}, t_{2})
\]

\[
\text{pdf}(v_{\pm} \to \pm \infty) \approx -(1 + q^{2}) v_{+} e^{v_{+}} v_{-} e^{-v_{-}}
\]

where \( q \) reduces to (12) for \( q = 1 \). In fact, the value \( R = S_{1}(t_{1}, t_{2}) \) can be explained as follows. Tracking back

\[
2 \text{ This divergence was argued to be the precursor of the } \frac{1}{\alpha} \ln \ln M \text{ correction in (8). A first-principle demonstration of this point is still missing; however, see [28], sect. 3 and [33], sect. 1.}
\]

\[
3 \text{ One may see this as placing two circles at radii 1 and } q \text{ in C endowed with one GFF with correlator (5).}
\]

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the derivation, we can see that $S_β(-nβ, -mβ) \sim \frac{\beta - 1}{\beta - 2} Z_{n, m}$. Thus, $-nβ = -mβ = -1, β = 1 \Rightarrow m = n = 1$, so by definition of $Z_{±}$,

$$R = S_β(-1, -1) = \lim_{M \to \infty} \frac{1}{M^2} \sum_{j,k} \exp(-V_{j,M}V_{k,M}),$$

recovering (13) for the circular model (that RHS = 2 is elementary). Both sides of (13) are defined for general log-REM’s; moreover, its derivation here is in fine independent of the circular model context. Thus, we conjecture that the relation (13) holds for generic log-REM’s.

**Application to diffusion.** – Consider a particle hopping in 1d infinite lattice, driven by a periodic potential $V_j \equiv V_j \mod M, M$. Let the dynamics be continuous-time Markov nearest-neighbour hoppings, whose rates $W_{j \to i}$ are function of $V_j$ and $V_i$ satisfying detailed balance. The long-time dynamics is diffusive $(\langle j_i - j_0 \rangle) \sim D t_1$; we rescale the time so that when $V \equiv 0$, the diffusion constant $D_V = 1$. Using results of [24], one can show $D_V = M^2/\langle Z_{±, Z} \rangle$ (see footnote 4). Its typical value is

$$D_{typ} = \begin{cases} \Gamma^2(1 - β^2)M^{-2β^2}, & β < 1, \\ a_1 \ln(M/b_1)M^{-2}, & β = 1, \\ a_2 \ln^β(M/b_2)M^{-4β+2}, & β > 1. \end{cases}$$

where $a_{1,2}, b_{1,2}$ are unknown constants. Now, eqs. (25) and (26) describe the fluctuation of $D_V$ around $D_{typ}$ in terms of the Mellin transform

$$\left(\frac{D_V}{D_{typ}}\right)^s \sim \begin{cases} \Gamma^2(1 + sβ^2)S_β(sβ, sβ), & β \leq 1, \\ \Gamma^4(1 + sβ)S_1(sβ, sβ), & β > 1. \end{cases}$$

We remark that a closely related dynamical quantity is the sum of left and right first-passage times. The particle at 0 at $t = 0$, consider $τ_a = \min\{t : j_t = ±M\}$, then their thermal average satisfies [27] $\langle τ_+ + τ_- \rangle = D_V^{-1} M^2$, to which the above statements apply.

**Numerics on max-min correlation.** – The first consequence of eqs. (8) and (10) is the (10)-max covariance:

$$\frac{V_{M,+} - V_{M,-}}{\sqrt{\frac{C}{M}}} \sim \frac{\partial^2 S_1}{\partial t_1 \partial t_2} |_{t_1, t_2 = 0} \frac{1}{4} \prod_{x, y \neq (0, 0)} \frac{(x + y)^2}{(x + y + 2)^2} = 0.338 \ldots,$$

which is a rather small correlation compared with $v_{max} = \frac{7}{10}$ and the rescaled distribution $\tilde{y}_M = (V_{M, +} - y_{M, -})/σ_M$. As shown in fig. 2, main panel, the numerical cumulative distribution converges to the exact prediction (33) and rules out the naive prediction, eq. (34). More convincing evidence can be obtained by considering the variance, by taking into account the finite-size correction (fig. 2, bottom inset). Analytically, (33), (32) and (9) imply $σ_M = 2.6937 \ldots$, in fine agreement with the numerical value $σ_{∞} = 2.69(1)$ obtained by a quadratic finite-size Ansatz.

A heuristic explanation of the negative $v_{±}$ correlation is that every term in $D_V$ is a plane wave that pushes $V_{M, ±}$ to opposite directions. For comparison, in the case of the Cayley tree (or branching Brownian motion, BBM) model [2], the $v_{±}$ correlation is positive, since it originates from their common ancestor. In both BBM and circular model, although persisting in the thermodynamic limit, the correlation is vanishingly weak compared to $V_{M, ±}$. In this respect, let us mention the strong max-max correlation exhibited by Ramola et al. [35] in a generalised BBM with particles dying and splitting at tunable rates. We are not aware of any non-hierarchical analogue.

As another numerical check, we consider the span, i.e. the difference between the two extrema. Its distribution is inferred from eqs. (8) and (10):

$$y_M := V_{M, +} - V_{M, -} \sim -4 \ln M + 3 \ln \ln M + y + 2c_M, \quad \exp(\tilde{y}_M) = \Gamma^4(1 + t)S_1(t, t).$$

The naive approximation can be obtained by discarding the non-trivial sum $S_1$ encoding the $v_{±}$ correlation:

$$\exp(\tilde{y}_M) ≃ \Gamma^4(1 + t).$$

Now we compare predictions (33) and (34) against numerical measures of $y_M$. We consider separately the variance $\sigma^2_M = \frac{y^2_M}{\Gamma^4(1 + t)}$ and the rescaled distribution $\tilde{y}_M = (y_M - \frac{y}{M})/\sigma_M$. As shown in fig. 2, main panel, the numerical cumulative distribution converges to the exact prediction (33) and rules out the naive prediction, eq. (34). More convincing evidence can be obtained by considering the variance, by taking into account the finite-size correction (fig. 2, bottom inset). Analytically, (33), (32) and (9) imply $σ_M = 2.6937 \ldots$, in fine agreement with the numerical value $σ_{∞} = 2.69(1)$ obtained by a quadratic finite-size Ansatz.
In both cases, simple finite-size Ansätze improve the quality of numerical evidence of freezing by giving very good agreement at zero temperature. Yet, it is still hard to check conclusively the CLD tail ratio (12), or its consequence on the span: (33) implies \( P(y \rightarrow -\infty) \approx -2^{1/6} e^y \), while the approximation (34) would miss the factor 2. Nonetheless, we hope that the more general formula (13) be tested in physical or numerical parameters.

**Edwards-Anderson’s order parameter.** — Sums over partitions here and elsewhere [20,36] share the duality-invariance structure, yet to be interpreted. Here we show that the first term of the sum in [36] encodes the Edwards-Anderson’s (EA) order parameter (6), whose glassy phase behaviour is a non-trivial consequence of the FDC.

Let us recall the partition sum studied in [36]:

\[
\alpha_n^{-1} \int \mu_n^a(\xi) \prod_{a,b} (1 - q \xi_a \xi_b)^{-1/2} = \sum_{\lambda} q^{|\lambda|} p_n^\lambda(\alpha), \quad (35)
\]

where \( \mu, c \) and \( p \) are defined in eqs. (17), (21) and (22), \( q \in (-1, 1) \), and \( |\lambda| \) is the size of the partition. Its derivation is similar to that of (23), [36], it was used to calculate the minimum of the circular model deformed by the presence of Dirichlet boundary condition. Here we will see it as a \( q \)-power series of observables on the non-deformed model.

We restrict ourselves to order \( q^1 \) of (35) (cf. (22)):

\[
\frac{1}{\alpha_n(\alpha)} \int \mu_n^a(\xi) \sum_{a,b} \xi_a \xi_b = p_n^a(\alpha) = \frac{n}{n-1+\alpha}. \quad (36)
\]

The RHS is duality-invariant upon the usual change of variables:

\[
n = -t/\beta, \quad 1/\alpha = -\beta^2 \Rightarrow p_n^a(\alpha) = \frac{t}{t+\beta+\beta^{-1}}. \quad (37)
\]

This suggests the possibility of applying FDC. But in order to meaningfully do so, we need to interpret the LHS as the \( M \rightarrow \infty \) expression of some observable in the \( \beta < 1 \) phase. This requires a standard replica calculation involving the EA parameter (6), similar in spirit to (16):

\[
\frac{1}{Z_n^a}\langle \xi_1 \xi_2 \rangle = \int \mu_n^a(\xi) \xi_1 \xi_2. \quad (38)
\]

Using this, plus replica permutation symmetry, we obtain

\[
\int \mu_n^a(\xi) \sum_{a,b} \xi_a \xi_b = n c_n(n) + n(n-1) \int \mu_n^a(\xi) \xi_1 \xi_2
\]

\[
= n Z_n^a + n(n-1) Z_n^a(\xi) = -\frac{t}{\beta^2} \exp(\beta -(t+\beta)|\xi|). \quad (39)
\]

Combining (36), (37), (39) and (42) gives

\[
e^{-t \beta} \left( \beta -(t+\beta)|\xi| \right) = e^{-t} \left( \Gamma(1+\beta) \right) \frac{1}{(t+\beta+\beta^{-1})}. \quad (40)
\]

The usual decoration \( y_\beta := f - \beta^{-1} g \), where \( g \) is an independent Gumbel is again valid here: indeed (40) implies (recall \( e^{-t} \sigma = e^{-t} \Gamma(1+t/\beta) \))

\[
e^{-t \beta} \left( \beta -(t+\beta)|\xi| \right) = e^{-t} \left( \Gamma(1+\beta) \right) \frac{1}{(t+\beta+\beta^{-1})}. \quad (41)
\]

The duality invariance of RHS triggers the freezing of the LHS, yielding

\[
e^{-t \beta} \left( \beta -(t+\beta)|\xi| \right) = e^{-t} \left( \Gamma(1+\beta) \right) \frac{1}{(t+\beta+\beta^{-1})}. \quad (42)
\]

Using the known result \( e^{-t} \sigma = \Gamma(1+t/\beta) \), the EA order parameter conditioned on the free energy \( f = f_+ \) for the \( \beta < 1 \) case, we have

\[
\frac{(\beta^2 + 1 - \beta \theta f) \delta(f_+ - f)}{\delta |\xi|^2} = \frac{1}{\beta^2} e^{-t \beta} \Gamma(1+\beta), \quad (43)
\]

Setting \( t = 0 \), we retrieve (7). The last prediction is conclusively confirmed by the numerics, see fig. 3.

We discuss briefly other information contained in (43). As a series of \( t \), it contains all the joint moments \( f_+^b \langle |\xi|^2 \rangle \). Applying the inverse Fourier transform, we obtain also \( \langle |\xi|^2 \rangle_f \), the EA order parameter conditioned on the free energy \( f_+ = f \). For the \( \beta < 1 \) case, we have

\[
(\beta^2 + 1 - \beta \theta f) \delta(f_+ - f) |\xi|^2 = 1 - e^{-t \beta + \epsilon f} \Gamma(1-\beta^2, e^{f/\beta}), \quad (45)
\]

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where $\Gamma(s, z) = \int_3^{\infty} e^{-x} x^{-s-1} dx$ is the incomplete Gamma function. This yields the asymptotics $\langle |\xi |^2 f \rangle \approx e^{\beta f} \Gamma(1 - \beta^2)$ and $\langle |\xi |^2 f \rangle \approx \beta^2 e^{-f/\beta}$. The $\beta > 1$ phase calculation follows the same principle and will be omitted.

**Conclusion.** – We calculated the joint min-max distribution and the Edwards-Anderson’s order parameter of the circular $1/f$-noise model, as well as generalisations. Each of them provides a numerically convincing test of the freezing-duality conjecture. Its implementations are variants of the usual decoration of the free-energy distribution; it would be interesting to see how the mathematical treatment [14] can be adapted to cover these cases. The treatment on the EA order parameter is an example to be generalised to further terms, which provide an infinite series of duality-invariant observables, indexed by (pairs of) partitions, and hopefully a clarification on the origin and generality of the duality invariance.

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REFERENCES

[1] Derrida B., Phys. Rev. Lett., 45 (1980) 79.
[2] Derrida B. and Spohn H., J. Stat. Phys., 51 (1988) 817.
[3] Carpentier D. and Le Doussal P., Phys. Rev. E, 63 (2001) 062110.
[4] Chamon C., Mudry C. and Wen X., Phys. Rev. Lett., 77 (1996) 4194.
[5] Castillo H. E., de C. Chamon C., Fradkin E., Goldbart P. M. and Mudry C., Phys. Rev. B, 56 (1997) 10668.
[6] Daviaud O., Ann. Probab., 34 (2006) 962.
[7] Deng J. and Zeitouni O., Ann. Probab., 42 (2014) 1480.
[8] Duplantier B., Rhodes R., Sheffield S. and Vargas V., Ann. Probab., 42 (2014) 1769.
[9] Fyodorov Y. V., Hiary G. A. and Keating J. P., Phys. Rev. Lett., 108 (2012) 170601.
[10] Fyodorov Y. V. and Keating J. P., Philos. Trans. R. Soc. London, Ser. A, 372 (2014) 20120503.
[11] Arguin L.-P., Belius D. and Harper A. J., arXiv:1506.00629 (2015).
[12] Fyodorov Y. V. and Bouchaud J.-P., J. Phys. A: Math. Theor., 41 (2008) 372001.
[13] Fyodorov Y. V., Le Doussal P. and Rosso A., J. Stat. Mech. (2009) P10005.
[14] Madaule T., Rhodes R. and Vargas V., Ann. Appl. Probab., 26 (2016) 643.
[15] Duplantier B., Rhodes R., Sheffield S. and Vargas V., arXiv:1407.5605 (2014).
[16] Forrester P. J., Log-gases and Random Matrices (LMS-34) (Princeton University Press) 2010.
[17] Dumitriu I. and Edelman A., J. Math. Phys., 43 (2002).
[18] Macdonald I. G., Symmetric Functions and Hall Polynomials (Oxford University Press) 1995.
[19] Fyodorov Y. V., Le Doussal P. and Rosso A., EPL, 90 (2010) 60004.
[20] Fyodorov Y. V. and Le Doussal P., J. Stat. Phys., 164 (2016) 190.
[21] Edwards S. F. and Anderson P. W., J. Phys. F: Met. Phys., 5 (1975) 965.
[22] Aarts D. G., Schmidt M. and Lekkerkerker H. N., Science, 304 (2004) 847.
[23] De Villeneuve V., Van Leeuwen J., Van Saarloos W. and Lekkerkerker H., J. Chem. Phys., 129 (2008) 164710.
[24] Derrida B., J. Stat. Phys., 31 (1983) 433.
[25] Le Doussal P. and Vinokur V. M., Physica C: Supercond., 254 (1995) 63.
[26] Dean D. S., Gupta S., Oshanin G., Rosso A. and Schehr G., J. Phys. A: Math. Theor., 47 (2014) 372001.
[27] Castillo H. E. and Le Doussal P., Phys. Rev. Lett., 86 (2001) 4850.
[28] Fyodorov Y., Le Doussal P. and Rosso A., J. Stat. Phys., 149 (2012) 898.
[29] Bramson M. and Zeitouni O., Commun. Pure Appl. Math., 65 (2012) 1.
[30] Fendley P. and Saleur H., Phys. Rev. Lett., 75 (1995) 4492.
[31] Dyson F. J., J. Math. Phys., 3 (1962) 140.
[32] Ostrovsky D., Comm. Math. Phys., 288 (2009) 287.
[33] Fyodorov Y. V. and Giraud O., Chaos, Solitons Fractals, 74 (2015) 15.
[34] Le Doussal P. and Machta J., Phys. Rev. B, 49 (1989) 9427.
[35] Ramola K., Majumdar S. N. and Schehr G., Phys. Rev. E, 91 (2015) 042131.
[36] Cao X., Rosso A. and Santachiara R., J. Phys. A: Math. Theor., 49 (2016) 02LT02.