I. INTRODUCTION

The concept of synthetic dimensions has recently opened the door to novel perspectives for expanding the dimensionality of well-understood physical systems. One strategy to explore synthetic dimensions consists in activating the coupling between different internal modes which under normal conditions remain uncoupled. By doing so, the resulting coupled modes exhibit lattice-like structures that exist in an abstract space which is nonetheless physical. The importance of synthetic lattices lies on the fact that they allow us to explore a variety of effects that are not available in spatial or temporal domains. To illustrate the basic idea of activating synthetic dimensions, and to set the stage for the present work, we begin by elucidating how a one-dimensional quantum harmonic oscillator generates a lattice in Fock space. The oscillator’s Hamiltonian is given as $\hat{H} = \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$, and its dynamics is governed by the Schrödinger equation $i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle$. Here, $\omega$ is the angular frequency of the oscillator, and $\hat{a}$ and $\hat{a}^\dagger$ denote, respectively, the annihilation and creation operators. Notice, we have set the reduced Planck constant and the oscillator mass to unity, i.e., $\hbar = 1$ and $m_o = 1$. When the oscillator is initially prepared in the eigenstate $|\Psi(0)\rangle = |n\rangle$, then it will remain in this state, only acquiring a time-dependent phase factor during evolution, i.e., $|\Psi(t)\rangle = e^{-i(n+\frac{1}{2})\omega t} |n\rangle$. No transitions to other eigenstates occur. However, by subjecting the oscillator to a time-dependent displacement, $\hat{x}(t) = f(t) \left( \hat{a}^\dagger + \hat{a} \right)$, the Hamiltonian acquires the form $\hat{H}(t) = \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + f(t) \left( \hat{a}^\dagger + \hat{a} \right)$. Substituting the general state vector $|\Psi(t)\rangle = \sum_{m=0}^{\infty} c_m(t) |m\rangle$ - where $c_m(t) = \langle m | \hat{U}(t) |\Psi(0)\rangle$ are the transition amplitudes from the initial state $|\Psi(0)\rangle$ to the final state $|m\rangle$ and $\hat{U}(t)$ is the time evolution operator - into the Schrödinger equation, we find that the amplitudes $c_m(t)$ obey the...
These equations clearly illustrate that the time dependent displacement $\hat{x}(t)$ activates transitions between the amplitudes $c_{m-1}(t)$, $c_m(t)$, and $c_{m+1}(t)$. This implies that in Fock space the oscillator generates a lattice, where it can "hop" from eigenstate $|m\rangle$ to the adjacent eigenstates $|m-1\rangle$ and $|m+1\rangle$ with hopping rates $f(t)\sqrt{m}$ and $f(t)\sqrt{m+1}$, respectively [8–14].

In general, applying dynamic modulations to the potentials associated with physical systems induces coupling among the supported eigenstates. Using this technique, a photonic topological insulator in synthetic dimensions has been recently implemented via modulated waveguide lattices [15, 16, 17]. Synthetic dimensions have also been explored in harmonic traps [17], optical lattices [15], cavities [14] and even in room-temperature Rydberg atoms [19].

Within the realm of optics and photonics, synthetic dimensions can be created by exploiting the spatial, temporal, polarization, and frequency degrees of freedom of light [3]. For instance, large-scale parity-time symmetric lattices have been implemented in the temporal domain using optical fiber loops endowed with gain and loss [20, 21] and a driven-dissipative analog of the four-dimensional quantum Hall effect has been observed in a spatially 3D resonator lattice [4].

In this work, we show that high-dimensional lattices emerge in photon-number space when a photonic lattice of $M$ ports is excited by $N$ indistinguishable photons, see Fig. 1. More precisely, the Fock representation of $N$-photon states in systems composed of $M$ evanescently coupled single-mode waveguides yields to a new layer of abstraction where the associated states can be visualized as the energy levels of a synthetic atom that features a number of allowed and disallowed transitions between its energy levels.

In photonic waveguide lattices, where all the waveguides are coupled to each other, the quantum optical Hamiltonian in paraxial approximation is given as $\hat{H} = \sum_{j=1}^{M} \beta_j \hat{a}_j \hat{a}_j^\dagger + \sum_{i \neq j}^{M} \kappa_{ij} \hat{a}_j \hat{a}_i$, where $\hat{a}_j$ and $\hat{a}_i$ respectively, are bosonic creation and annihilation operators for photons in the $j$-th waveguide. Further, $\beta_j$ denotes the propagation constant of the $j$-th waveguide and $\kappa_{ij}$ is the coupling coefficient between the $i$-th and $j$-th waveguide.

For simplicity we restrict our subsequent analysis to the simplest scenario of (in real space) essentially one-dimensional waveguide arrays with nearest-neighbor couplings

$$\hat{H} = \sum_{j=1}^{M} \left[ \beta_j \hat{a}_j \hat{a}_j^\dagger + \kappa_{j,j-1} \hat{a}_{j-1} \hat{a}_j + \kappa_{j,j+1} \hat{a}_{j+1} \hat{a}_j \right].$$

Under these premises, the propagation of a single-photon along the waveguide can be described using the Heisenberg equations of motion for the bosonic creation operators $[25, 26]$

$$i \frac{d\hat{a}_m}{dz} = \beta_m \hat{a}_m + \kappa_{m,m-1} \hat{a}_{m-1} + \kappa_{m,m+1} \hat{a}_{m+1},$$

where $m = 1, \ldots, M$. Accordingly, the single-photon response is computed through the input-output transformation $\hat{a}_m^{\dagger}(0) = \sum_{n=1}^{M} U_{m,n}(z) \hat{a}_n^{\dagger}(z)$, where $U_{m,n}(z)$ denotes the $(m,n)$ matrix element of the evolution operator $\hat{U}(z) = \exp \left( -iz\hat{H} \right)$. Using this formalism, it is straightforward to show that an initial $N$-photon state $|n_1, n_2, \ldots, n_M\rangle$, with $N = \sum_{m=1}^{M} n_m$, will transform into the output state

$$|\Psi(0)\rangle = \left( \frac{\hat{a}_1^{\dagger}(0)}{\sqrt{n_1!}} \cdots \frac{\hat{a}_M^{\dagger}(0)}{\sqrt{n_M!}} \right) |0\rangle \rightarrow$$

$$\sum_{n_1}^{n_1!} \cdots \sum_{n_M}^{n_M!} \frac{U_{1,n_1}(z) \hat{a}_1^{\dagger}(z) \cdots U_{M,n_M}(z) \hat{a}_M^{\dagger}(z)}{\sqrt{n_1! \cdots n_M!}} |0\rangle.$$
FIG. 2: Probability distribution $|\langle m, N - m | \hat{U}(z) | \psi(0) \rangle|^2$ for the initial state $|\psi(0)\rangle = [5, 5]$ propagating through a waveguide beam splitter with (a) $\beta_1 = \beta_2 = 1$ (discrete “diffraction” in state space) and (b) $\beta_1 = 0$ and $\beta_2 = 4$ (“Bloch oscillations” in state space).

with $m = 1, \ldots, M$. By computing the matrix elements of the Hamiltonian given in (3) for $N = 1$, $H_{a,m} = \langle 1_m | \hat{H} | 1_m \rangle = \beta_n \delta_{n,m} + \kappa_{n,m-1} \delta_{n,m-1} + \kappa_{n,m+1} \delta_{n,m+1}$, one can readily see that the single-photon states are coupled to each other as displayed by the equations

$$i \frac{d}{dz} |1_m\rangle = \beta_m |1_m\rangle + \kappa_{m,m-1} |1_{m-1}\rangle + \kappa_{m,m+1} |1_{m+1}\rangle,$$

(7)

in agreement with (4).

We now consider the more interesting scenario of $N$ photons propagating through a waveguide beam splitter, $M = 2$, with propagation constants $\beta_1$ and $\beta_2$ and symmetric coupling, i.e., $\kappa_{1,2} = \kappa_{2,1} \equiv \kappa$. In this case, there exists a total of $(N + 1)$ states, namely $|0,N\rangle, |1,N-1\rangle, \ldots, |N-1,1\rangle, |N,0\rangle$, and the Hamiltonian given in (3) acquires the form

$$\hat{H} = \beta_1 \hat{a}_1^\dagger \hat{a}_1 + \beta_2 \hat{a}_2^\dagger \hat{a}_2 + \kappa \hat{a}_1^\dagger \hat{a}_2 + \kappa \hat{a}_2^\dagger \hat{a}_1.$$

(8)

Computing the matrix elements $\hat{H}_{(m,n),(p,q)} = \langle m,n \hat{H} | p,q \rangle$ reveals that the states obey the $(N + 1)$ equations of motion

$$i \frac{d}{dz} |m,n\rangle = (\beta_1 m + \beta_2 n) |m,n\rangle + C_m |m-1,n+1\rangle + C_{m+1} |m+1,n-1\rangle,$$

(9)

with $C_m = \kappa \sqrt{m(m+1)}$ and $n = N - m$. This indicates that inside a waveguide beam splitter the amplitudes of two-mode $N$-photon states evolve coupled to each other with hopping rates $C_m$, and the corresponding phases depend on both propagation constants.

For the case of two identical waveguides we have $\beta_1 = \beta_2 = \beta$ so that the first term on the r.h.s. of (9) becomes $\beta N |m, N - m\rangle$ which indicates that all the states will exhibit the same effective propagation constant. Interestingly, it has been recently shown that waveguide beam splitters produce the Discrete Fractional Fourier Transform (DFrFT) of $N$-photon states [20], as well as exceptional points of arbitrary order, provided that losses are introduced in one of the waveguides [20].

On the other hand, when considering two non-identical waveguides, $\beta_1 \neq \beta_2$, the first term on the r.h.s. of (9) acquires the form $[(\beta_1 - \beta_2)m + \beta_2 N] |m, N - m\rangle$. Remarkably, the term $[(\beta_1 - \beta_2)m]$ indicates that the state evolution will be influenced by an effective ramping potential in the same fashion as in the case of classical waves in Bloch oscillator systems [12, 31, 32]. Consequently, we can tailor the dynamics of $N$-photon states by simply adjusting the Bloch slope $(\beta_1 - \beta_2)$ in order to suppress and/or create certain output states. As an illustration, we depict in FIG. 2 the probability evolution for the initial state $|5, 5\rangle$ in a waveguide beam splitter with coupling coefficient $\kappa = 1$ (a) for $\beta_1 = \beta_2 = 1$ and (b) for $\beta_1 = 0, \beta_2 = 4$. While case (a) corresponds to discrete “diffraction” of the initial state in state space, case (b) corresponds to “Bloch oscillations” in state space. Note, that throughout this work we present all simulations using the normalized propagation coordinate $z = \kappa Z$, where $Z$ is the actual propagation distance and $\kappa$ stands for the nearest-neighbor coupling coefficient. After the above introductory examples, we now proceed to consider the most interesting case where multiple photons $N > 1$ excite more than two waveguides $M > 2$. In order to motivate the concept of pseudo-energy we first examine the simplest case of a waveguide trimer, $M = 3$, that is excited by $N = 2$ photons and then move on to the general case.

For a waveguide trimer and two identical photons, the Hamiltonian takes the form

$$\hat{H} = \beta_1 \hat{a}_1^\dagger \hat{a}_1 + \beta_2 \hat{a}_2^\dagger \hat{a}_2 + \beta_3 \hat{a}_3^\dagger \hat{a}_3 + \kappa_1 (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \kappa_2 (\hat{a}_2^\dagger \hat{a}_3 + \hat{a}_3^\dagger \hat{a}_2).$$

(10)

In this scenario, we have a total of 6 photon-number states obeying the following coupled set of equations of motion

$$i \frac{d}{dz} \begin{cases} |00\rangle = 2\beta_1 |200\rangle + \sqrt{2}\kappa_1 |110\rangle & \text{(11)} \\ |10\rangle = (\beta_1 + \beta_2) |110\rangle + \kappa_2 |101\rangle & \text{(12)} \\ & + \sqrt{2}\kappa_1 (|200\rangle + |020\rangle) \end{cases}$$

$$i \frac{d}{dz} \begin{cases} |02\rangle = 2\beta_2 |020\rangle + \sqrt{2}\kappa_1 |110\rangle + \sqrt{2}\kappa_2 |011\rangle & \text{(14)} \\ |101\rangle = (\beta_1 + \beta_3) |101\rangle + \kappa_1 |011\rangle + \kappa_2 |110\rangle & \text{(15)} \\ & + \sqrt{2}\kappa_1 (|002\rangle + |020\rangle) \end{cases}$$

$$i \frac{d}{dz} \begin{cases} |011\rangle = (\beta_2 + \beta_3) |011\rangle + \kappa_1 |101\rangle & \text{(16)} \\ & + \sqrt{2}\kappa_2 (|002\rangle + |020\rangle) \end{cases}$$

(17)
TABLE I: Possible lattice configurations for states arising in a waveguide trimer exited by two photons.

\begin{align*}
|2,0,0| & |1,1,0| |0,2,0| |0,1,1| |0,0,2| \\
|2,0,0| & |1,1,0| |0,1,1| |0,2,0| |0,1,1| |0,0,2|
\end{align*}

\[ \frac{id}{dz} |002\rangle = 2\beta_3 |002\rangle + \sqrt{2}\kappa_2 |011\rangle. \] (18)

As in the earlier examples, here we also have the possibility of molding the state dynamics via tuning the propagation constants and coupling coefficients. For instance, for equal coupling coefficients \( \kappa_1 = \kappa_2 = 1 \) and identical waveguides \( \beta_1 = \beta_2 = \beta_3 = 0 \), we observe a periodic spreading and contraction of the two-photon wave function, as illustrated in FIG. 3 (a). In contrast, choosing a different propagation constant for the central waveguide, \( \beta_2 = 2 \), leads to a quasi-periodic evolution, FIG. 3 (b). Indeed, this quasi-periodic evolution occurs because the ratios between the eigenvalues of the coupling matrix are irrational numbers. We would like to emphasize that at the propagation distance indicated by the dashed line in FIG. 3 (b), the input state \(|101\rangle\) evolves into a quasi-two-photon N00N state in state space, which is reminiscent of the Hong-Ou-Mandel effect [32].

To describe the photon dynamics in the waveguide trimer, we have obtained an even number of equations. At this point, the way in which the states should be arranged into a synthetic lattice is not at all clear. To be precise, the six states representing the sites of the synthetic lattice can be sorted into at least two distinct natural sequences as shown in Table I. Clearly, arranging the states into a lattice (i.e., sorting and analyzing the corresponding equations of motion becomes rather cumbersome when considering higher photon numbers in multiple coupled waveguides. In the following section, we, therefore, introduce a concise and universal method that facilitates studying the general case of \( N > 1 \) photons propagating in arrays formed by \( M > 2 \) waveguides. The resulting structures follow from physical and mathematical considerations that eventually allow us to describe multi-photon processes in waveguide arrays in a surprising and remarkable way that resembles the quantum-mechanical description of multi-level atoms.

II. PSEUDO ENERGY REPRESENTATION

We now introduce a concept analogous to the concept of energy and we, therefore, refer to it as the pseudo-energy. As we show below, the concept of pseudo-energy is rather useful since it facilitates a unique sorting of multi-photon Fock states in a physically meaningful way and allows for establishing a correspondence between Fock states and the energy levels of a synthetic atom. Concurrently, we identify pseudo-energy ladder operators along with pseudo-exchange-energies in order to define the corresponding selection rules in Fock space for transitions between the pseudo-energy levels of the synthetic atom.

We consider \( N \) indistinguishable photons propagating in an array of \( M \) lossless evanescently coupled waveguides that give rise to \( N_F = (N + M - 1)!/N!(M - 1)! \) Fock states \(|n_1, \ldots, n_M\rangle\), fulfilling the condition \( \sum_{m=1}^{M} n_m = N \). The first issue to be addressed is to determine a way of sorting the multi-photon states in Fock space in a meaningful way. To do so, we associate a unique numerical value to every state \(|n_1, \ldots, n_M\rangle\) as follows

\[ |n_1, \ldots, n_M\rangle \Rightarrow |n_1, \ldots, n_M\rangle_{N+1} = n_1 \times (N + 1)^0 + \ldots + n_M \times (N + 1)^{M-1}. \] (19)

Here, the subscript \( N + 1 \) indicates that the numbers in the square brackets have to be expressed in base \( N + 1 \), and the least significant digit is the left-most number \( n_1 \). Observing that \(|n_1, \ldots, n_M\rangle_{N+1} = \sum_{m=1}^{M} (N + 1)^{m-1} n_m\rangle, \) allows us to define the pseudo-energy operator

\[ \hat{K}^{(N,M)} = \sum_{m=1}^{M} (N + 1)^{m-1} \hat{n}_m, \] (20)

such that its action on the \( N \)-photon-\( M \)-mode Fock states \(|n_1, \ldots, n_M\rangle\) yields

\[ K^{(N,M)} |n_1, \ldots, n_M\rangle = K(n_1, \ldots, n_M) |n_1, \ldots, n_M\rangle, \] (21)

with eigenspectrum \( K(n_1, \ldots, n_M) = \sum_{m=1}^{M} (N + 1)^{m-1} n_m \). From (21), we readily infer the smallest and largest eigenvalues...
\[ K_{\min} = K(N,0,...,0) = [N,0,...,0]_{N+1} = N \]
and \[ K_{\max} = K(0,0,...,0,N) = [0,...,0,N]_{N+1} = N(N+1)^{M-1} \], respectively. Accordingly, the eigenvalues are bounded by \[ K_{\min} \leq K_{\nu} \leq K_{\max} \].

As a result, in order to sort the associated Fock states, we have to compute the corresponding \( K_{\nu} \)'s and arrange them in ascending order. The resulting ladder of \( K_{\nu} \)'s then defines the synthetic lattices formed by the states. We refer to this ordering as the pseudo-energy representation of the \( N \)-photon-\( M \)-mode Fock states.

For illustration, we revisit the above case of \( N = 2 \) photons propagating in an array of \( M = 3 \) waveguides. Accordingly, there are \( N_F = 6 \) states and the spectrum of the pseudo-energy operator \( K^{(2,3)} \) comprises 6 integers

\[ \{[2,0,0]_3, [1,1,0]_3, [1,0,2]_3, [0,2,0]_3, [0,1,1]_3, [0,0,2]_3 \} = \{2,4,6,10,12,18\}. \]  

(22)

Using these numbers we readily obtain the pseudo-energy representation of the 2-photon-3-mode Fock space

\[ |\nu,0,0\rangle = |[2,0,0]_3\rangle = |K_1\rangle, \]
\[ |1,1,0\rangle = |[1,1,0]_3\rangle = |K_2\rangle, \]
\[ |0,2,0\rangle = |[0,2,0]_3\rangle = |K_3\rangle, \]
\[ |1,0,1\rangle = |[1,0,1]_3\rangle = |K_4\rangle, \]
\[ |0,1,1\rangle = |[0,1,1]_3\rangle = |K_5\rangle, \]
\[ |0,0,2\rangle = |[0,0,2]_3\rangle = |K_6\rangle. \]  

(23)

Consequently, we designate \( K_{\nu} \) as the pseudo-energy of the \( \nu \)-th Fock state in the \( N \)-photon-\( M \)-mode Fock space

\[ |K_{\nu}\rangle = \left[ n^{(\nu)}_1, ..., n^{(\nu)}_M \right]_{N+1} = |n^{(\nu)}_1, ..., n^{(\nu)}_M\rangle, \]  

(24)

with \( \nu = 1,...,N_F \). In general, for any given \( N, M \) and pseudo-energy \( K_{\nu} \), the inverse mapping onto the mode-occupation numbers is

\[ n^{(\nu)}_m = (K_{\nu} \div (N+1)^{m-1}) \#(N+1), \]  

(25)

where the symbol \( \div \) corresponds to integer division and \( \# \) is the modulo operator.

We now proceed to show how the pseudo-energy representation of Fock states allows us to express the equations of motion of \( N \) photons in \( M \) waveguides in a concise way. To do so, we take a closer look at the action of the operator \( \hat{a}^\dagger \hat{a}_j \) on a Fock state

\[ \hat{a}^\dagger \hat{a}_j |n_1, ..., n_M\rangle = \sqrt{(n_j+1)n_j} |n_1, ..., n_j+1, \]
\[ ..., n_j-1, ..., n_M\rangle. \]  

(26)

If the state \( |n_1, ..., n_M\rangle \) corresponds to the pseudo-energy \( K_{\nu} \), then the resulting state on the r.h.s. of (26) must have the pseudo-energy

\[ K_{\nu} = [n_1, ..., n_j+1, ..., n_M]_{N+1} = K_{\nu} + (N + 1)^{i-1} - (N + 1)^{j-1}. \]  

(27)

Therefore, the action of \( \hat{a}^\dagger \hat{a}_j \) changes the pseudo-energy of Fock states by the amount

\[ \Delta K_{ij} = (N+1)^{i-1} - (N+1)^{j-1} = -\Delta K_{ji}, \]  

(28)

which we denote as the pseudo-exchange energy associated with the tunneling process taking place between waveguides \( i \) and \( j \). In this sense the operators \( \hat{a}^\dagger \hat{a}_j \) can be thought of as pseudo-energy ladder operators, which raise or lower the pseudo-energy of Fock states. Consequently, we can write

\[ \langle K_{\mu}\rangle \hat{a}^\dagger \hat{a}_j |K_{\nu}\rangle = \kappa_{ij} \sqrt{(n^{(\nu)}_i+1) n^{(\nu)}_j} \delta_{K_{\mu}, K_{\nu}+\Delta K_{ij}}. \]  

(29)

The physical significance of (29) is that a direct transition between the states \( |K_{\mu}\rangle \) and \( |K_{\nu}\rangle \) is only possible if there exists a pseudo-exchange energy \( \Delta K_{ij} \) such that

\[ |\Delta K_{ij}| = |K_{\mu} - K_{\nu}|. \]  

(30)

Obviously, (30) defines the selection rules in Fock space. Together with the action of the photon number operators \( \hat{n}_m \), the full system of coupled equations governing the propagation of \( N \) photons through \( M \) coupled waveguides in the pseudo-energy representation is given by

\[ i \frac{d}{dz} |K_{\nu}\rangle = \sum_{m=1}^{M} \beta_m n^{(\nu)}_m |K_{\nu}\rangle \]
\[ + \sum_{\nu=1}^{N_F} \kappa_{ij} \sqrt{(n^{(\nu)}_i+1) n^{(\nu)}_j} \delta_{K_{\mu}, K_{\nu}+\Delta K_{ij}} |K_{\nu}\rangle. \]  

(31)

For the case of nearest-neighbour coupled, identical waveguides, where all the propagation constants are the same, the relevant pseudo-exchange energies are \( \Delta K_{i} = \Delta K_{i+1,j} = N(N+1)^{i-1} \) and the set of coupled equations reduces to

\[ i \frac{d}{dz} |K_{\nu}\rangle = N\beta |K_{\nu}\rangle \]
\[ + \sum_{\nu=1}^{N_F} \sum_{i=1}^{M-1} \kappa_i \sqrt{n^{(\nu)}_i+1) n^{(\nu)}_{i+1} \delta_{K_{\mu}, K_{\nu}+\Delta K_{i}} |K_{\nu}\rangle. \]  

(32)

To further illustrate the resulting coupling system in Fock space, we revisit the case of a single photon \( N = 1 \) propagating in \( M = 3 \) waveguides. The effective coupling behavior - of allowed and forbidden transitions in Fock space - can now be visualized within a pseudo-energy term diagram, as illustrated in FIG.3(a). In this particular case the nearest-neighbour coupling of the waveguides is retained in Fock space and any given Fock state \( |K_{\nu}\rangle \)
only couples to its nearest neighbours $|K_{\nu \pm 1}\rangle$.
A similar picture arises in the case of two waveguides $M = 2$ and $N = 2$ photons, as depicted in FIG. 4 (b). Here, we obtain a term-diagram that is essentially isomorphic to FIG. 4 (a), where – again – only nearest-neighbour Fock states are coupled to each other. The nearest neighbour picture radically changes when applying the pseudo-energy approach to the case of $N = 2$ photons and $M = 3$ waveguides as displayed in the corresponding term-diagram in FIG. 4 (c). Importantly, even when the waveguides are – in real space – only coupled to their nearest neighbors, in photon number space certain states become coupled to next-nearest neighbor states. For instance, in FIG. 4 (c) we observe that the state $|K_2\rangle = |4\rangle = |1,1,0\rangle$ not only couples to its neighbors $|K_1\rangle = |2\rangle = |2,0,0\rangle$ and $|K_3\rangle = |6\rangle = |0,2,0\rangle$, but also to the next-nearest neighbour state $|K_{4}\rangle = |10\rangle = |1,0,1\rangle$. For illustrative purposes, we present in FIG. 5 the coupling matrix for this particular set of states when the three-waveguide system is formed by identical waveguides, $\beta_1 = \beta_2 = \beta_3 = 0$, and balanced coupling coefficients $\kappa_1 = \kappa_2 = 1$.

At this point it is rather evident that the richness and complexity of the emerging synthetic configurations will become more prominent when higher number of photons and waveguides are considered. Moreover, it is worth stressing that in order to generate the present synthetic structures we did not require any modulation of the system parameters as the states naturally couple due to the system’s internal dynamics.

## III. NON-PLANAR SYNTHETIC LATTICES: FOCK GRAPHS

In this section we introduce a more convenient way of representing the Hamiltonian matrix of $N$-photons exciting $M$-waveguides. To do so, we interpret the states as vertices of a graph (Fock graph) where the allowed inter-state transitions represent the edges. A practical representation of finite graphs is the so-called adjacency matrix whose entries indicate whether pairs of vertices are adjacent or not. In the present context, the effective Hamiltonian $H_{\mu \nu} = \langle K_\mu | H | K_\nu \rangle$ in the $N$-photon-$M$-mode pseudo-energy representation determines such an adjacency matrix

$$ A_{\mu \nu}^{(N,M)} = \Theta(H_{\mu \nu}), \quad (33) $$

where $\Theta$ is the step-function and $A_{\mu \nu}^{(N,M)} = 1(0)$ indicates a connection (or no connection) between the vertices $\mu$ and $\nu$. In what follows, we assume identical waveguides with $\beta_1 = \ldots = \beta_M = 0$ in order to omit self-loops in the graph representation. As an example, in FIG. 5 we depict the Fock graph arising from the effective Hamiltonian of FIG. 6 which we have already discussed in the previous section. In FIG. 6 (a), we depict further examples for photon numbers up to $N = 5$ and up to $M = 6$ waveguides. The first row, which corresponds to single photon graphs, simply reflects the one-dimensional spatial configuration of the waveguides. By introducing a second photon, we observe that the Fock graphs become two-dimensional, FIG. 6 (b), except for the case $M = 2$. The inclusion of more photons leads to non-planar graphs, i.e. graphs that cannot be drawn in 2D without intersecting edges, which exhibit a layered structure in three dimensions as indicated by the different coloring of the nodes in different layers. A prominent feature to highlight is the symmetry observed among graphs emerging for the combinations $(M,N)$ and $(M - l, N + l)$ and for $(M,N)$ and $(M + l, N - l)$, where $l$ is an integer. In other words, every Fock graph has an isomorphic partner graph

$$ A_{\mu \nu}^{(N,M)} = A_{\mu \nu}^{(M-1,N+1)} \forall N, M, \quad (34) $$
with an identical adjacency matrix, up to a trivial permutation of the node labels. In FIG. 7 (b), we depict the smallest non-trivial pair of Fock graphs and the corresponding adjacency matrices that are induced by the pseudo-energy representation. If we were to start from $A_{\mu\nu}^{(3,3)}$ and permute its rows and columns according to $(1, \ldots, 10) \rightarrow (1, 2, 4, 7, 3, 5, 8, 6, 9, 10)$ we will exactly obtain $A_{\mu\nu}^{(2,4)}$.

Indeed, this underlying symmetry in the space of possible Fock graphs has very interesting implications. For instance, in Ref. [29] we have shown that it is possible to implement the number-resolved $N + 1$-dimensional Discrete Fractional Fourier Transform (DFrFT) with a single waveguide beam splitter by launching $N$ indistinguishable photons. Furthermore, using the same photon-number-resolved mapping in Ref. [30] we have shown how to attain so-called exceptional points of $N + 1$ order, by way of exciting a semi-lossy waveguide beam splitter with high photon number states. In fact, it is now clear that these results emerge as special cases of [34], which pertains to the identity of the first row and column in FIG. 7 (a). Thus, by following similar ideas it is possible, in principle, to find the corresponding effects for waveguide systems with $M \geq 3$ excited by $N \geq 2$ photons.

Additionally, by exploiting the graph symmetry it becomes apparent that a specific transformation which requires $N$ photons and $M$ waveguides could likewise be implemented with $M - 1$ photons and $N + 1$ waveguides. Of course, such alternative pathways of implementing a transformation are not always guaranteed because of the different dimensions of the experimentally accessible parameter spaces. Nonetheless, this may serve as a useful Ansatz to overcome concrete experimental difficulties. Quite interestingly, synthetic Fock lattices have been explored previously in the context of QED circuits by Wang et al. [14]. In such a study, the joint excitation states of an atom coupled to the $N$-photon 3-cavity Fock space form a two-dimensional, hexagonal Haldane-like synthetic lattice, which facilitates the generation of high photon-number NOON states. Crucially, the realization of this scheme demands the judicious implementation of the coupling between atom and cavity, as well as the precise modulation of the cavity resonance frequencies. In contrast, the multi-photon synthetic dimensions explored in the present work are intrinsically active by virtue of the indistinguishability of the photons, and as such they do not require any external driving of the system’s parameters.

The Fock graphs offer a rich variety of synthetic coupled structures. This variety can be further enhanced by considering different spatial arrangements of the waveguides, for instance, ring- or star-shaped structures instead of the simple planar configuration studied here. Importantly, the evolution of multi-photon states in synthetic lattices and graphs can be dynamically reconfigured by using programmable photonic chips [34]. That is, integrated optical devices where the waveguides’ refractive index and coupling coefficients can be modified externally. Nevertheless, even with this simple one-dimensional arrangement comprising a few waveguides, small photon numbers, and a time-independent Hamiltonian, one encounters interesting effects that are only possible due to the multi-dimensionality of the corresponding Fock graphs.

IV. ALL-OPTICAL DARK STATES AND PARALLEL QUANTUM RANDOM WALKS

To show possible applications of the pseudo-energy synthetic lattices we discuss the generation of all-optical dark states [35] and multi-photon quantum. The simplest dark states are encountered in 3-level atomic- or molecular systems, where radiative transitions between, e.g., $|1\rangle \leftrightarrow |2\rangle \leftrightarrow |3\rangle$ are allowed but the transition $|1\rangle \leftrightarrow |3\rangle$ is forbidden. In this simple scenario, a dark state is a superposition of the uncoupled states $|D\rangle = \cos(\theta)|1\rangle - \sin(\theta)|3\rangle$, where $\theta$ is given in terms of the Rabi frequencies of the allowed transitions [35]. Once the system is in such a state, adiabatic changes in the Rabi frequencies allow for the tuning of the populations of the states $|1\rangle$ and $|3\rangle$, while the probability of $|2\rangle$ remains 0. This interesting behavior, which seemingly evades the radiative selection rules, can be mimicked in the pseudo-energy representation of Fock states using our all-optical setup.

To do so, we revisit one more time the case of $M = 3$ waveguides, with equal propagation constants $\beta_1 = \beta_2 = \beta_3 = 0$ and balanced coupling coefficients $\kappa_1 = \kappa_2 = \frac{1}{\sqrt{2}}$, excited by $N = 2$ photons. The pseudo-energy representation of the effective Hamiltonian takes the form

$$H_{\mu\nu} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1 & \frac{1}{\sqrt{2}} & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}. \quad (35)$$

With this choice of parameters the spectrum of $H_{\mu\nu}$ is integer-valued

$$\lambda_1, \ldots, \lambda_6 = (-2, -1, 0, 0, 1, 2), \quad (36)$$

FIG. 6: Two-dimensional Fock graph for $M = 3$ waveguides excited by $N = 2$ indistinguishable photons. The corresponding adjacency matrix is induced by the effective Hamiltonian in FIG. 5 according to [33].

\[\begin{array}{c}
N = 2 \\
M = 3 \\
|0, 0, 2\rangle \\
|1, 0, 1\rangle \\
|0, 1, 1\rangle \\
|2, 0, 0\rangle \\
|1, 1, 0\rangle \\
|0, 2, 0\rangle
\end{array}\]
FIG. 7: (a) Overview of several two- and three-dimensional embeddings of Fock graphs $\mathcal{A}_{N,M}^{(N,M)}$ for $M = 2, \ldots, 6$ waveguides excited by $N = 1, \ldots, 5$ indistinguishable photons. Different node colors indicate layer-like structures that emerge for $N \geq 3, M \geq 4$ (all nodes in the same layer feature the same color). For the sake of readability we have omitted the node labels as well as the graphs for $M \geq 5, N \geq 4$. (b) The smallest example of an isomorphic pair of planar Fock graphs with $N = 2, M = 4$ and $N = 3, M = 3$ respectively.

which indicates that the third and fourth eigenstates are degenerate with eigenvalues $\lambda_3 = \lambda_4 = 0$. We now consider the evolution of a coherent superposition $|\psi\rangle$ of the eigenstates

$$
|\phi_3\rangle = \begin{pmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
0 \\
\frac{1}{\sqrt{2}}
\end{pmatrix}
$$

and

$$
|\phi_5\rangle = \frac{1}{2} \begin{pmatrix}
1 \\
1 \\
0 \\
0 \\
-1 \\
-1
\end{pmatrix}
$$

with corresponding eigenvalues $\lambda_3 = 0$ and $\lambda_5 = 1$, specifically

$$
|\psi\rangle = \frac{1}{\sqrt{2}} (|\phi_3\rangle + |\phi_5\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
-\frac{1}{\sqrt{2}} \\
0
\end{pmatrix}.
$$

(37)

FIG. 8: Evolution of the probabilities $|\langle K_\nu | \hat{U}(z) | \psi \rangle|^2$ of the state $|\psi\rangle$ as defined in (39).

In the standard Fock representation $|\psi\rangle$ reads as

$$
|\psi\rangle = \frac{1}{\sqrt{2}} \left( |200\rangle + \frac{1}{2} |110\rangle - \frac{1}{\sqrt{2}} |101\rangle - \frac{1}{2} |011\rangle \right).
$$

(39)

The probability evolution for this state is shown in
As one can see, this state displays the characteristic behavior of a dark state. That is, the initial state evolves exhibiting oscillating transitions between the states \(|200\rangle\) and \(|002\rangle\) with period \(2\pi/\chi = 2\pi\). These transitions occur in spite of the fact that direct transition \(|200\rangle \rightarrow |002\rangle\) is forbidden \((200) H |002\rangle = 0\), and those states have the maximum possible distance within the graph, that is, at least 4 single-photon tunneling processes are required to transform one state into the other. All probabilities of the intermediate states remain constant and, in a way, assist the simultaneous tunneling of two photons between the outermost waveguides. We stress that this 6-level dark-state is induced by a time-independent Hamiltonian and it occurs naturally without the need of adiabatic fine-tuning of the parameters. We would further like to note, that the state \(|020\rangle\) exhibits zero probability for all \(z\), further attesting a multi-photon tunneling (in this case co-tunneling) effect taking place between the two waveguides. Geometrically speaking, this effect arises due to destructive interference taking place in the two-way branching of the Fock graph shown in FIG. 8. This branching effectively allows for the flow of the amplitudes to take a ‘detour’ around the \(|020\rangle\) node. As an alternative, one may attempt to implement a real space structure in one or two dimensions consisting of six coupled waveguides in order to emulate an equivalent Hamiltonian for just a single photon. However, this would be topologically impossible, since there always exist additional cross-talk between the waveguides representing the nodes at the center of the graph. In other words, our Fock-graph based analysis of multi-photon propagation in waveguide arrays allows the realization of functionalities beyond what can be realized with linear (single-photon) based networks.

Quite interestingly, by exciting waveguide lattices with multi-photon states comprising infinite coherent superpositions, e. g. coherent states \(|\alpha\rangle = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} (\alpha^n/\sqrt{n!}) |n\rangle\) or two-mode squeezed vacuum states \(|\xi\rangle = \sqrt{1-|\xi|^2} \sum_{n=0}^{\infty} \xi^n |n, n\rangle\), opens a route to generating, in principle, an infinite number of lattices or graphs with different numbers of lattice sites and many dimensions simultaneously. This possibility is very appealing for realizing parallel quantum random walks where the corresponding walkers can perform different numbers of steps that depend on the number of photons involved in each process. We stress that the observation of these quantum walks is nowadays possible utilizing bright parametric down-conversion sources in combination with photon-number-resolving detectors.

\[ |\phi_n\rangle = \sum_{m=1}^{M} u^{(n)}_{m} |\lambda_m\rangle = \sum_{m=1}^{M} u^{(n)}_{m} |K_m\rangle, \quad (40) \]

\[ \hat{H} |\phi_n\rangle = \lambda_n |\phi_n\rangle, \quad (41) \]

where \(n = 1, \ldots, M\). In the above equation, \(u^{(n)}_{m}\) is the \(m\)-th component of the \(n\)-th eigenvector of the matrix \(\hat{H}_{m,n} = \langle K_m | \hat{H} | K_n \rangle\) and it defines the single-particle eigenstates

\[ \hat{\phi}_n^\dagger = \sum_{m=1}^{M} u^{(n)}_{m} \hat{a}_m^\dagger. \quad (42) \]

When the same waveguide system is excited by \(N > 1\) photons, it is clear that the many-particle eigenstates arise from the tensor-products of the single-particle eigenstates. Formally, we may write the resulting states as

\[ |\tilde{n}_1, \ldots, \tilde{n}_M\rangle = \prod_{m=1}^{M} \hat{\phi}_m^\dagger |\tilde{n}_m\rangle |0\rangle, \quad (43) \]

but now the occupation numbers \(\tilde{n}_m\) pertain to the number of photons occupying the \(m\)-th single-particle eigenmode. Consequently, we can apply the pseudo-energy ordering to the \(N\)-particle eigenstates by defining \(\tilde{K}_\nu = [\tilde{n}_1^{(\nu)} \ldots \tilde{n}_M^{(\nu)}]_{\nu+1} \). The \(\nu\)-th eigenstate of the \(N\)-photon system is then given by

\[ |\tilde{K}_\nu\rangle = \prod_{m=1}^{M} \left( \sum_{k=1}^{\tilde{n}_m^{(\nu)}} u^{(m)}_{k} \hat{a}_k^\dagger \right) |0\rangle. \quad (44) \]

Note, that in most cases it is necessary to normalize the resulting expression on the r.h.s. of (44). By requiring

\[ |\tilde{K}_\nu\rangle^2 = \sum_{\mu=1}^{N_F} \langle \tilde{K}_\nu | K_{\mu}\rangle, \]

where \(|K_{\mu}\rangle\) denotes \(N\)-photon-\(M\)-waveguide Fock states, we find for the components...
It is now rather straightforward to show, that the \( N \)-particle eigenvalues are given as the sum of the eigenvalues of the involved single-particle eigenstates

\[
\lambda_\nu = \sum_{m=1}^{M} \tilde{n}_m^{(\nu)} \lambda_m.
\]  

(46)

Using (44) and (45) it is straightforward to find the \( N \)-photon-M-waveguide time-evolution operator

\[
\hat{U}(t) = \sum_{\nu=1}^{N_P} e^{-i\lambda_\nu t} |\tilde{K}_\nu\rangle \langle \tilde{K}_\nu|.
\]

We would like emphasize that the numerical evaluation of (44) is far more efficient than the direct diagonalization of the full matrix representation of \( \hat{H} \) in \( N \)-photon-M-waveguide Fock space. Due to the size and highly non-trivial structure of the resulting matrices, general eigensystem-solvers produce a significant amount of overhead, which we avoid in our approach. Essentially, we do not even require a calculation of the full matrix representation \( \hat{H}_{\mu\nu} \). Instead, knowledge of the single-particle eigensystem and the bosonic nature of photons suffices.

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