Self-adjoint Schrödinger and Dirac operators with Aharonov-Bohm and magnetic-solenoid fields

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Abstract

We study all the s.a. Schrödinger and Dirac operators (Hamiltonians) both with pure AB field and with magnetic-solenoid field. Then, we perform a complete spectral analysis for these operators, which includes finding spectra and spectral decompositions, or inversion formulas. In constructing the Hamiltonians and performing their spectral analysis, we respectively follow the von Neumann theory of s.a. extensions of symmetric differential operators and the Krein method of guiding functionals. The examples of similar consideration are given by us in arXiv:0903.5277, where a non-relativistic particle in the Calogero potential field is considered and in Theor. Math. Phys. 150(1) (2007) 34, where a Dirac particle in the Coulomb field of arbitrary charge is considered. However, due to peculiarities of the three-dimensional problems under consideration, we elaborated a generalization of the approach used in the study of the Dirac particle.

1 Introduction

Aharonov-Bohm (AB) effect [1] plays an important role in quantum theory refining the status of electromagnetic potentials in this theory. First this effect was discussed in relation to a study of interaction between a non-relativistic charged particle and an infinitely long and infinitesimally thin magnetic solenoid field (further AB field) which yields a magnetic flux \( \Phi \) (a similar effect was discussed earlier by Ehrenberg and Siday [2]). It was discovered that particle wave functions vanish at the solenoid line. In spite of the fact that the magnetic field vanishes out of the solenoid, the phase shift in the wave functions is proportional to the corresponding magnetic flux [3]. A non-trivial particle scattering by the solenoid is interpreted as a possibility for quantum particles to ”feel” potentials of the corresponding electromagnetic field. Indeed, potentials of AB field do not vanish out of the solenoid. For the first time, a construction of self-adjoint (s.a. in what follows) Schrödinger operators with the AB field was given in [4]. First, the need for s.a. extensions of the Dirac Hamiltonian with the AB field in 2 + 1 dimensions was recognized in [5, 6, 7]. S.a. extensions of the Dirac Hamiltonian with the AB field in 3 + 1 dimensions were found in [8], see also [9, 10]. The

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physically motivated boundary conditions for the particle scattering by the AB field and a Coulomb center were studied in [11, 12]. A splitting of Landau levels in a superposition of parallel uniform magnetic field and AB field (further magnetic-solenoid field) gives an example of AB effect for bound states. First, exact solutions of Schrödinger equation with the magnetic-solenoid field (non-relativistic case) were studied in [13]. Exact solutions of the relativistic wave equations (Klein-Gorgon and Dirac) with the magnetic-solenoid field were obtained in [14, 15, 16] and used then to study AB effect in cyclotron and synchrotron radiations, see [15, 16, 17]. Later on the problem of self-adjointness of the Dirac Hamiltonian with magnetic-solenoid field was studied in [18, 19].

In the present work, we construct systematically all the s.a. Schrödinger and Dirac operators both with pure AB field and with magnetic-solenoid field. Then, we perform a complete spectral analysis for these Hamiltonians, which includes finding spectra and spectral decompositions, or inversion formulas. In constructing the Hamiltonians and performing their spectral analysis, we respectively follow the theory of s.a. extensions of symmetric differential operators [20, 21, 25] and the Krein method of guiding functionals [20, 21]. The examples of similar consideration are given in [22] where a nonrelativistic particle in the Calogero potential field is considered and in [23] where a Dirac particle in the Coulomb field of arbitrary charge is considered. However, due to peculiarities of the three-dimensional problem under consideration, we use a necessary generalization of the approach [23].

We recall that the AB field of infinitely thin solenoid (with the constant flux Φ) along the axis \( z = x^3 \) can be described by electromagnetic potentials \( A^\mu_{\text{AB}}; \mu = 0, 1, 2, 3, \)

\[
A^0_{\text{AB}} = (0, A_{\text{AB}}), \quad A_{\text{AB}} = (A^k_{\text{AB}}, k = 1, 2, 3), \quad A^3_{\text{AB}} = 0,
\]

\[
A^1_{\text{AB}} = -\frac{\Phi \sin \varphi}{2\pi \rho}, \quad A^2_{\text{AB}} = \frac{\Phi \cos \varphi}{2\pi \rho},
\]

where \( \rho, \varphi \) are cylindrical coordinates, \( x^1 = \rho \cos \varphi, \ x^2 = \rho \sin \varphi. \) The magnetic field of AB solenoid has the form \( B_{\text{AB}} = (0, 0, B_{\text{AB}}). \) It is easy to see that outside the \( z \) axis the magnetic field \( B_{\text{AB}} = \text{rot} A_{\text{AB}} \) is equal to zero. Nevertheless, for any surface \( \Sigma \) with a boundary \( L \) being any contour (even an infinitely small one) around the \( z \) axis, the circulation of the vector potential along \( L \) does not vanish and reads \( \oint_L A_{\text{AB}} dl = \Phi. \) If one interprets this circulation as the flux of the magnetic field \( B_{\text{AB}} \) through the surface \( \Sigma, \)

\[
\int_\Sigma B_{\text{AB}} d\sigma = \oint_L A_{\text{AB}} dl = \Phi,
\]

then we obtain an expression for the magnetic field,

\[
B_{\text{AB}} = \Phi \delta(x^1) \delta(x^2),
\]

where the term infinitely thin solenoid comes from.

In the cylindrical coordinates, we have

\[
\frac{e}{c\hbar} A^1_{\text{AB}} = -\phi \rho^{-1} \sin \varphi, \quad \frac{e}{c\hbar} A^2_{\text{AB}} = \phi \rho^{-1} \cos \varphi, \quad \phi = \Phi/\Phi_0,
\]

where \( \Phi_0 \) is a fundamental unit of magnetic flux,

\[
\Phi_0 = 2\pi c\hbar/e = 4, 135 \times 10^{-7} \text{ Gauss} \cdot \text{cm}^2
\]
(we recall that $e > 0$ is the absolute value of the electron charge).

The magnetic-solenoid field is a superposition of a constant uniform magnetic field of strength $B$ directed along the axis $z$ and the AB field with the flux $\Phi$ in the same direction. The magnetic-solenoid field is given by electromagnetic potentials by potentials $A'^{\mu} = (0, A)$, $A = (A^k, k = 1, 2, 3)$ of the form

$$A^1 = A_{AB}^1 - \frac{B x^2}{2}, \quad A^2 = A_{AB}^2 + \frac{B x^1}{2}, \quad A^3 = 0. \quad (1)$$

The potentials (1) define the magnetic field $B$ of the form

$$B = (0, 0, B + B_{AB}).$$

In the cylindrical coordinates, the potentials of the magnetic-solenoid field have the form

$$\frac{e}{\hbar} A^1 = -\tilde{\phi} \rho^{-1} \sin \varphi, \quad \frac{e}{\hbar} A^2 = \tilde{\phi} \rho^{-1} \cos \varphi, \quad A^3 = 0,$$

$$\tilde{\phi} = \phi + \frac{\epsilon_B \gamma \rho^2}{2}, \quad \gamma = \frac{e |B|}{\hbar} > 0, \quad \epsilon_B = \text{sign } B. \quad (2)$$

For further consideration, it is convenient to introduce the following representation:

$$\phi = \epsilon_B (\phi_0 + \mu), \quad \phi_0 = [\epsilon_B \phi] \in \mathbb{Z}, \quad \mu = \epsilon_B \phi - \phi_0, \quad 0 \leq \mu < 1. \quad (3)$$

The quantity $\mu$ is called the mantissa of the magnetic flux and, in fact, determines all the physical effects in the AB field, see e.g. \[16\].

## 2 S.a. Schrödinger Hamiltonians

In this section, we consider two-dimensional and three-dimensional nonrelativistic motions of a particle of mass $m_e$ and charge $q = \epsilon_q e$, $\epsilon_q = \text{sign} q = \pm 1$ (positron or electron) in the magnetic-solenoid field. The canonical formulation of the problem is the following. The starting point is the “formal Schrödinger Hamiltonian $\hat{H}$” with the magnetic-solenoid field that is respectively a two- or three-dimensional s.a. differential operation well-known from physical textbooks. In three dimensions, it is given by

$$\hat{H} = \frac{1}{2m_e} \left( \hat{p} - \frac{q}{c} A \right)^2, \quad \hat{p} = -i\hbar \nabla, \quad \nabla = (\partial_x, \partial_y, \partial_z). \quad (4)$$

It is convenient to represent $\hat{H}$ as a sum of two terms, $\hat{H}^\perp$ and $\hat{H}^\parallel$,

$$\hat{H} = \hat{H}^\perp + \hat{H}^\parallel,$$

where the two-dimensional s.a. differential operation $\hat{H}^\perp$, the “formal two-dimensional Schrödinger Hamiltonian” with the magnetic-solenoid field,

$$\hat{H}^\perp = M^{-1} \hat{\mathcal{H}}^\perp, \quad \hat{\mathcal{H}}^\perp = \left( -i \nabla^\perp - \frac{q}{c\hbar} A^\perp \right)^2,$$

$$M = 2m_e \hbar^{-2}, \quad \nabla^\perp = (\partial_x, \partial_y), \quad A^\perp = (A^1, A^2), \quad (5)$$
$A^1$ and $A^2$ are given by (2), corresponds to a two-dimensional motion in the $xy$ plane perpendicular to the $z$ axis, while the one-dimensional differential operation $\hat{H}^\parallel$,

$$\hat{H}^\parallel = \hat{\mathcal{H}} = \frac{\hat{p}_z^2}{2m_e}, \quad \hat{p}_z = -i\hbar \partial_z,$$

corresponds to a one-dimensional free motion along the $z$-axis.

The problem to be solved is to construct s.a. nonrelativistic two- and three-dimensional Hamiltonians $\hat{H}^\perp$ and $\hat{H}$ associated with the respective s.a. differential operations $\hat{H}^\perp$ and $\hat{H}$ and to perform a complete spectral analysis for these operators.

We begin with the two-dimensional problem. We successively consider the case of pure AB field, with $B = 0$, and then the case of the magnetic-solenoid field, with $B \neq 0$ In the subsequent subsection, we generalize obtained results to three-dimensions.

### 2.1 Two-dimensional case

#### 2.1.1 Reduction to radial problem

In the case of two dimensions, the space of particle quantum states is the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^2)$ of square-integrable functions $\psi(\rho)$, $\rho = (x,y)$, with the scalar product

$$(\psi_1, \psi_2) = \int \overline{\psi_1(\rho)}\psi_2(\rho)d\rho, \quad d\rho = dx dy = \rho d\rho d\phi.$$

A quantum Hamiltonian $\hat{H}^\perp$ should be defined as a s.a. operator in this Hilbert space. It is more convenient to deal with a s.a. operator $\hat{H}^\perp = M\hat{H}^\perp$ associated with s.a. differential operation $\hat{\mathcal{H}}^\perp = M\hat{\mathcal{H}}^\perp$ defined in (5).

The construction is essentially based on the requirement of rotation symmetry which certainly holds in a classical description of the system. This requirement is formulated as the requirement of the invariance of a s.a. Hamiltonian under rotations around the solenoid line, the $z$ axis. As in classical mechanics, the rotation symmetry allows separating the polar coordinates $\rho$ and $\varphi$ and reducing the two-dimensional problem to a one-dimensional radial problem.

The group of rotations $SO(2)$ in $\mathbb{R}^2$ naturally acts in the Hilbert space $\mathfrak{H}$ by unitary operators: if $S \in SO(2)$, then the corresponding operator $\hat{U}_S$ is defined by the relation $(\hat{U}_S\psi)(\rho) = \psi(S^{-1}\rho)$, $\psi \in \mathfrak{H}$.

The Hilbert space $\mathfrak{H}$ is a direct orthogonal sum of subspaces $\mathfrak{H}_m$, that are the eigenspaces of the representation $\hat{U}_S$,

$$\mathfrak{H} = \sum_{m \in \mathbb{Z}} \oplus \mathfrak{H}_m, \quad \hat{U}_S \mathfrak{H}_m = e^{-im\theta} \mathfrak{H}_m,$$

where $\theta$ is the rotation angle corresponding to $S$.

It is convenient to change indexing, $m \to l$, $\mathfrak{H}_m \to \mathfrak{H}_l$, as follows

$$m = \epsilon (\phi_0 - l), \quad l = \phi_0 - \epsilon m,$$

where

$$\epsilon = \epsilon_q \epsilon_B, \quad \phi_0 = [\epsilon_B \phi]. \quad (6)$$
We define a rotationally-invariant initial symmetric operator \( \hat{\mathcal{H}}^\perp \) associated with the differential operation \( \hat{\mathcal{H}}^\perp \) in the Hilbert space \( \mathfrak{H} = L^2(\mathbb{R}^2) \) as follows:

\[
\hat{\mathcal{H}}^\perp : \begin{cases} 
D_{\hat{\mathcal{H}}^\perp} = \{ \psi(\rho) : \psi \in D(\mathbb{R}^2\setminus\{0\}) \} \\
\hat{\mathcal{H}}^\perp \psi = \hat{\rho}^2_{\perp} \psi, \quad \forall \psi \in D_{\hat{\mathcal{H}}^\perp}
\end{cases}
\]

where \( D(\mathbb{R}^2\setminus\{0\}) \) is the space of smooth and compactly supported functions vanishing in a neighborhood of the point \( \rho = 0 \). The domain \( D_{\hat{\mathcal{H}}^\perp} \) is dense in \( \mathfrak{H} \) and the symmetricity of \( \hat{\mathcal{H}}^\perp \) is obvious.

In the polar coordinates \( \rho \) and \( \varphi \), \( \hat{\mathcal{H}}^\perp \) becomes

\[
\hat{\mathcal{H}}^\perp = -\partial^2_{\rho} - \rho^{-1} \partial_{\rho} + \rho^{-2}(i\partial_{\varphi} + \epsilon_{\rho}\tilde{\phi})^2,
\]

where \( \tilde{\phi} \) is given by (2).

For every \( l \), the relation

\[
(S_l f)(\rho, \varphi) = \frac{1}{\sqrt{2\pi\rho}} e^{i\epsilon_{\rho}(\phi_0 - l)\varphi} f_i(\rho)
\]

determines a unitary operator \( S_l : L^2(\mathbb{R}_+) \rightarrow \mathfrak{H}_l \), where \( L^2(\mathbb{R}_+) \) is the Hilbert space of square-integrable functions on the semi-axis \( \mathbb{R}_+ \) with scalar product

\[
(f,g) = \int_{\mathbb{R}_+} f(\rho)g(\rho) d\rho.
\]

For every \( l \), we define the linear operator \( V_l \) from \( \mathfrak{H} \) to \( L^2(\mathbb{R}_+) \) by setting

\[
(V_l \psi)(\rho) = \sqrt{\frac{\rho}{2\pi}} \int_{0}^{2\pi} \psi(\rho, \varphi) e^{-i\epsilon_{\rho}(\phi_0 - l)\varphi} d\varphi.
\]

If \( \psi = \sum_{l \in \mathbb{Z}} \psi_l \in \mathfrak{H} \), then we have \( \psi_l = S_l V_l \psi \) for all \( l \). In other words, \( V_l = S_l^{-1} P_l \), where \( P_l \) is the orthogonal projector onto the subspace \( \mathfrak{H}_l \). However, we prefer to work with \( V_l \) rather than \( P_l \) because the latter cannot be reasonably defined in the three-dimensional case, where the Hilbert state space should be decomposed into a direct integral instead of a direct sum (see below). Clearly, \( V_l \psi \in D(\mathbb{R}_+) \) for any \( \psi \in D(\mathbb{R}^2\setminus\{0\}) \), and it follows from (7) and (9) that

\[
V_l \hat{\mathcal{H}}^\perp \psi = \hat{h}(l)V_l \psi, \quad \psi \in D(\mathbb{R}^2\setminus\{0\}),
\]

where the symmetric operators \( \hat{h}(l) \) in \( L^2(\mathbb{R}_+) \) defined on \( D_{\hat{h}(l)} = D(\mathbb{R}_+) \), where it acts as

\[
\hat{h}(l) = -\partial^2_{\rho} + \rho^{-2}\left[(l + \mu + \gamma \rho^2/2)^2 - 1/4\right],
\]

with \( \mu = \epsilon_{B\phi} - \phi_0 \) defined in (3).

In view of (10), for any \( \psi \in D(\mathbb{R}^2\setminus\{0\}) \), the \( \mathfrak{H}_l \)-component \( (\hat{\mathcal{H}}^\perp \psi)_l \) of \( \hat{\mathcal{H}}^\perp \psi \) can be written as

\[
(\hat{\mathcal{H}}^\perp \psi)_l = S_l V_l \hat{\mathcal{H}}^\perp \psi = S_l \hat{h}(l) S_l^{-1} S_l V_l \psi = S_l \hat{h}(l) S_l^{-1} \psi_l.
\]
Suppose we have a (not necessarily closed) operator \( \hat{f}_l \) in \( \mathcal{H}_l \) for each \( l \). We define the operator
\[
\hat{f} = \sum_{l \in \mathbb{Z}} \hat{f}_l
\]  
(13)
in \( \mathcal{H} \) by setting
\[
\hat{f} \psi = \sum_{l \in \mathbb{Z}} \hat{f}_l \psi_l, \quad \psi = \sum_{l \in \mathbb{Z}} \psi_l.
\]

The domain \( D_f \) of \( \hat{f} \) consists of all \( \psi = \sum_{l \in \mathbb{Z}} \psi_l \in \mathcal{H} \) such that \( \psi_l \in D_{f_l} \) for all \( l \) and the series \( \sum_{l \in \mathbb{Z}} \hat{f}_l \psi_l \) converges in \( \mathcal{H} \). The operator \( \hat{f} \) is closed (self-adjoint) if and only if all \( \hat{f}_l \) are closed (resp., self-adjoint). For every \( l \), we have \( D_{f_l} = D_f \cap \mathcal{H}_l \).

We say that a closed operator \( \hat{f} \) in \( \mathcal{H} \) is rotationally invariant if it can be represented in form (13) for some family of operators \( f_l \) in \( \mathcal{H}_l \).

By (12), the direct sum of the operators \( S_l \hat{h}(l) S_l^{-1} \) is an extension of \( \hat{H}_l^\perp \):
\[
\hat{H}_l^\perp \subset \sum_{l \in \mathbb{Z}} \oplus S_l \hat{h}(l) S_l^{-1}.
\]  
(14)

Let \( \hat{h}_\epsilon(l) \) be s.a. extensions of the symmetric operators \( \hat{h}(l) \). Then the operators
\[
\hat{H}_\epsilon^\perp(l) = S_l \hat{h}_\epsilon(l) S_l^{-1}
\]  
(15)
are s.a. extensions of \( S_l \hat{h}(l) S_l^{-1} \), and it follows from (14) that the orthogonal direct sum
\[
\hat{H}_\epsilon^\perp = \sum_{l \in \mathbb{Z}} \oplus \hat{H}_\epsilon^\perp(l),
\]  
(16)
is a rotationally invariant s.a. extension of the initial operator \( \hat{H}_\epsilon^\perp \).

Conversely, let \( \hat{H}_\epsilon^\perp \) be a rotationally invariant s.a. extension of \( \hat{H}_\epsilon^\perp \). Then it has the form (16), where \( \hat{H}_\epsilon^\perp(l) \) are s.a. operators in \( \mathcal{H}_l \). Set \( \hat{h}_\epsilon(l) = S_l^{-1} \hat{H}_\epsilon^\perp(l) S_l \). For all \( l \), \( \hat{h}_\epsilon(l) \) are s.a. operators in \( L^2(\mathbb{R}_+) \). If \( f \in \mathcal{D}(+) \), then \( S_l f \in \mathcal{D}(\mathbb{R}^2 \setminus \{0\}) \cap \mathcal{H}_l \) and (14) and (16) imply that
\[
S_l \hat{h}(l) f = S_l \hat{h}_\epsilon(l) S_l^{-1} S_l f = \hat{H}_\epsilon^\perp S_l f = \hat{H}_l^\perp(l) S_l f = S_l \hat{h}_\epsilon(l) f.
\]
Hence, \( \hat{h}(l) f = \hat{h}_\epsilon(l) f \), i.e., \( \hat{h}_\epsilon(l) \) is a s.a. extension of \( \hat{h}(l) \). We thus conclude that \( \hat{H}_\epsilon^\perp \) can be represented in form (16), where \( \hat{H}_\epsilon^\perp(l) \) are given by (15) and \( \hat{h}_\epsilon(l) \) are a s.a. extensions of \( \hat{h}(l) \).

The problem of constructing a rotationally invariant s.a. Hamiltonian \( \hat{H}_\epsilon^\perp \) is thus reduced to constructing s.a. radial Hamiltonians \( \hat{h}_\epsilon(l) \).

We first consider the case of pure AB field where \( B = 0 \).

### 2.1.2 S.a. Hamiltonians with AB field

In this case, we have \( \gamma = 0 \), and s.a. radial differential operation \( \hat{h}(l) \) (11) becomes
\[
\hat{h}(l) = -\partial_l^2 + \alpha l^2 - 1/4, \quad \alpha = \kappa_l^2 - 1/4, \quad \kappa_l = |l + \mu|, \quad l \in \mathbb{Z}.
\]  
(17)
It is easy to see that this differential operation and the corresponding initial symmetric operator \( \hat{h}(l) \) are actually identical to the respective operation and operator encountered in studying the Calogero problem, see (22). We can therefore directly carry over the previously obtained results to s.a. extensions of \( \hat{h}(l) \).
First region: \( \alpha \geq 3/4 \). In this region, we have \((l + \mu)^2 \geq 1\), which is equivalent to

\[
l \geq 1 - \mu \text{ or } l \leq -1 - \mu.
\]

Because \( l \in \mathbb{Z} \) and \( 0 \leq \mu < 1 \), we have to distinguish the cases of \( \mu = 0 \) and \( \mu > 0 \):

\[
\mu = 0 : l \leq 1 \text{ or } l \geq 1, \text{ i.e., } l \neq 0,
\]

\[
\mu > 0 : l \leq -2 \text{ or } l \geq 1, \text{ i.e., } l \neq 0, -1.
\]

For such \( l \), the initial symmetric operator \( \hat{h}(l) \) has zero deficiency indices, is essentially s.a., and its unique s.a. extension is \( \hat{h}_\epsilon(l) = \hat{h}(l) = \hat{h}^+(l) \) with the domain

\[
D_{\hat{h}(l)} = \{ \psi_* : \psi_*, \psi'_* \text{ are absolutely continuous (a.c.) in } \mathbb{R}_+, \psi_*, \hat{h}(l) \psi_* \in L^2(\mathbb{R}_+) \}.
\]

The spectrum of \( \hat{h}_\epsilon(l) \) is simple and continuous and coincides with the positive semiaxis, \( \text{spec} \hat{h}_\epsilon(l) = \mathbb{R}_+ \).

The generalized eigenfunctions \( U_\mathcal{E} \),

\[
U_\mathcal{E}(\rho) = (\rho/2)^{1/2} J_{\kappa_\alpha}(\sqrt{\rho}), \quad \hat{h}_\epsilon(l) U_\mathcal{E} = \mathcal{E} U_\mathcal{E}, \quad \mathcal{E} \in \mathbb{R}_+,
\]

of \( \hat{h}_\epsilon(l) \) form a complete orthonormalized set in each Hilbert space \( \mathcal{L}_l \).

Second region: \(-1/4 < \alpha < 3/4\) In this region, we have \( 0 < (l + \mu)^2 < 1 \), which is equivalent to

\[
-\mu < l < 1 - \mu \text{ or } -1 - \mu < l < -\mu.
\]

If \( \mu = 0 \), inequalities \([13]\) have no solutions for \( l \in \mathbb{Z} \). If \( \mu > 0 \), these inequalities have two solutions \( l = l_a \), where, for brevity, we introduce the notation

\[
l_a = a, \quad a = 0, -1.
\]

So, in the second region, we remain with the case of \( \mu > 0 \).

For each \( l = l_a \) \((a = 0, -1)\) there exists a one-parameter \( U(1)\)-family of s.a. Hamiltonians \( \hat{h}_\epsilon(l_a) = \hat{h}_{\lambda_a}(l_a) \) parametrized by the real parameter \( \lambda_a \in \mathbb{S}(-\pi/2, \pi/2) \), where \( \mathbb{S}(-\pi/2, \pi/2) \) denotes an interval with identified ends (a circle). These Hamiltonians are specified by the asymptotic s.a. boundary conditions at the origin,

\[
\psi_{\lambda_a}(\rho) = c \left[ (\kappa_0 \rho)^{1/2 + \kappa_a} \cos \lambda_a + (\kappa_0 \rho)^{1/2 - \kappa_a} \sin \lambda_a \right] + O(\rho^{3/2}), \quad \rho \to 0,
\]

\[
D_{\hat{h}_{\lambda_a}(l_a)} = \{ \psi \in D_{\hat{h}(l_a)}(\mathbb{R}_+), \psi \text{ satisfies } [13] \}.
\]

where \( \kappa_a \equiv \kappa_a = |\mu + a|, 0 < \kappa_a < 1 \), and \( c \) is an arbitrary constant, whereas \( k_0 \) is a constant of dimension of inverse length.

For \( \lambda_a \not\in (-\pi/2, 0) \), the spectrum of each of \( \hat{h}_{\lambda_a}(l_a) \) is simple and continuous and \( \text{spec} \hat{h}_{\lambda_a}(l_a) = \mathbb{R}_+ \).
The generalized eigenfunctions $U_\varepsilon$,
\[
U_\varepsilon(\rho) = \sqrt{\frac{\rho}{2Q_a}} \left[ J_\kappa_a \left( \sqrt{\varepsilon} \rho \right) + \tilde{\lambda}_a \left( \sqrt{\varepsilon/2\kappa_0} \right)^{2\kappa_a} J_{-\kappa_a} \left( \sqrt{\varepsilon} \rho \right) \right],
\]
\[
Q_a = 1 + 2\tilde{\lambda}_a (E/4)^{\kappa_a} \cos(\pi\kappa_a) + \left( \tilde{\lambda}_a \right)^2 (E/4)^{2\kappa_a} > 0,
\]
\[
\tilde{\lambda}_a = \Gamma(1-\kappa_a)\Gamma^{-1}(1+\kappa_a) \tan \lambda_a, \quad \hat{h}_{\lambda_a} (l_a) U_\varepsilon = \varepsilon U_\varepsilon, \quad \varepsilon \in \mathbb{R}_+,
\]
(21)
of the Hamiltonian $\hat{h}_{\lambda_a} (l_a)$ form a complete orthonormalized set in each Hilbert space $L_{l_a}$.

For $\lambda_a \in (-\pi/2, 0)$, the spectrum of each of $\hat{h}_{\lambda_a} (l_a)$ is simple, but in addition to the continuous part of the spectrum, there exists one negative level $\varepsilon_{\lambda_a}^{(-)} = -4\kappa_0^2 |\tilde{\lambda}_a|^{1-\kappa_a}$, such that $\text{spec} \hat{h}_{\lambda_a} (l_a) = \mathbb{R}_+ \cup \{\varepsilon_{\lambda_a}^{(-)}\}$.

The generalized eigenfunctions $U_\varepsilon$ of the continuous spectrum, $\varepsilon \geq 0$, are given by the same (21), while the eigenfunction $U^{(-)}$ corresponding to the discrete level is
\[
U^{(-)}(\rho) = \frac{2\rho|\varepsilon_{\lambda_a}^{(-)}|^{\sin(\pi\kappa_a)}}{\pi\kappa_a} K_{\kappa_a} \left( \sqrt{|\varepsilon_{\lambda_a}^{(-)}|} \rho \right), \quad \hat{h}_{\lambda_a} (l_a) U^{(-)} = \varepsilon_{\lambda_a}^{(-)} U^{(-)},
\]
they together form a complete orthonormalized set in each Hilbert space $L_{l_a}$.

**Third region:** $\alpha = -1/4$  In this region, we have $l + \mu = 0$. If $\mu = 0$, this equation has a unique solution $l = l_0 = 0$, while if $\mu > 0$, there are no solutions, and we remain with the only case of $\mu = 0$.

For $l = l_0$, there exists a one-parameter $U(1)$-family of s.a. Hamiltonians $\hat{h}_\varepsilon (l_0) = \hat{h}_\lambda (l_0)$ parametrized by the real parameter $\lambda \in \mathbb{S} (-\pi/2, \pi/2)$ These Hamiltonians are specified by the asymptotic s.a. boundary conditions at the origin
\[
\psi_\lambda (\rho) = c \left[ \rho^{1/2} \ln (\kappa_0 \rho) \cos \lambda + \rho^{1/2} \sin \lambda \right] + O(\rho^{3/2} \ln \rho), \quad \rho \to 0, \quad (22)
\]
( the constants $c$ and $k_0$ are of the same meaning as in (19)), and their domains are given by
\[
D_{h_{\lambda}(l_0)} = \{ \psi : \psi \in D_{h_{\lambda}(l_0)}^* (\mathbb{R}_+), \text{\psi satisfies (22)} \}, \quad D_{h_{\lambda}(l_0)}^* (\mathbb{R}_+) = \{ \psi_+, \psi_- \text{ are a.c. in } \mathbb{R}_+, \psi_*, h_\lambda (l_0) \psi_* \in L^2(\mathbb{R}_+) \}.
\]

The spectrum of $\hat{h}_\lambda (l_0)$ is simple. For $|\lambda| = \pi/2$, the spectrum is continuous and non-negative, $\text{spec} \hat{h}_{\pm \pi/2} (l_0) = \mathbb{R}_+$. For $|\lambda| < \pi/2$, in addition to the continuous part of the spectrum, $\varepsilon \geq 0$, there exists one negative level $\varepsilon_{\lambda}^{(-)} = -4\kappa_0^2 \exp \left[2(\tan \lambda - C)\right]$, where $C$ is the Euler constant, such that
\[
\text{spec} \hat{h}_\lambda (l_0) = \{ \varepsilon_{\lambda}^{(-)} \} \cup \mathbb{R}_+ , \quad |\lambda| < \pi/2.
\]

The generalized and normalized eigenfunctions $U_\varepsilon$ of the continuous spectrum are
\[
U_\varepsilon(\rho) = \frac{\rho}{\sqrt{2 (\tilde{\lambda}^2 + \pi^2/4)}} \left[ \tilde{\lambda} J_0 \left( \sqrt{\varepsilon} \rho \right) + \frac{\pi}{2} N_0 \left( \sqrt{\varepsilon} \rho \right) \right], \quad \tilde{\lambda} = \tan \lambda - C - \ln \left( \sqrt{\varepsilon/2\kappa_0} \right), \quad \hat{h}_\lambda (l_0) U_\varepsilon = \varepsilon U_\varepsilon, \quad \varepsilon \in \mathbb{R}_+, \quad |\lambda| \leq \pi/2,
while the normalized eigenfunction \( U^{(-)} \) corresponding to the discrete level is

\[
U^{(-)}(\rho) = \sqrt{2\rho |E^{(-)}_{\lambda}|} K_0 \left( \sqrt{|E^{(-)}_{\lambda}|} \rho \right), \quad \hat{h}_\lambda(l_0) U^{(-)} = E^{(-)}_{\lambda} U^{(-)}, \quad |\lambda| < \pi/2,
\]

they together form a complete orthonormalized set in the Hilbert space \( L_{l_0} \).

**Complete spectrum and inversion formulas** In the previous subsubsecs., we constructed all s.a. radial Hamiltonians \( \hat{h}_\epsilon(l) \) associated with the s.a. differential operation \( \hat{h}(l) \) as s.a. extensions of the symmetric operator \( \hat{h}(l) \) for any \( l \in \mathbb{Z} \) and for any any \( \phi_0 \) and \( \mu \). We assemble our previous results into two groups.

For \( \mu = 0 \), we have

\[
\hat{h}_\epsilon(l) = \hat{h}_{(1)}(l), \quad l \neq l_0, \quad D_{\hat{h}_{(1)}(l)} = D^*_{\hat{h}(l)}(\mathbb{R}_+), \quad \hat{h}_\lambda(l_0) = \hat{h}_\lambda(l_0), \quad \lambda \in \mathbb{S}(-\pi/2, \pi/2),
\]

the domain \( D_{\hat{h}_{(1)}(l_0)} \) is given by \( (23) \);

For \( \mu > 0 \), we have

\[
\hat{h}_\epsilon(l) = \hat{h}_{(1)}(l), \quad l \neq l_0 = a = 0, -1, \quad D_{\hat{h}_{(1)}(l)} = D^*_{\hat{h}(l)}(\mathbb{R}_+), \quad \hat{h}_\lambda(l_0) = \hat{h}_{\lambda_\alpha}(l_0), \quad \lambda_\alpha \in \mathbb{S}(-\pi/2, \pi/2),
\]

the domain \( D_{\hat{h}_{\lambda_\alpha}(l_\alpha)} \) is given by \( (20) \).

As a final result, we find a family of all s.a. rotationally-invariant two-dimensional nonrelativistic Hamiltonians \( \hat{H}^\perp_\epsilon = M^{-1}\hat{H}^\perp_\epsilon \) associated with the s.a. differential operation \( \hat{H}^\perp_\epsilon \) with \( B = 0 \). Each set of possible s.a. radial Hamiltonians \( \hat{h}_\epsilon(l) \) generates a s.a. rotationally-invariant Hamiltonian \( \hat{H}^\perp_\epsilon \) in accordance with the relations \( (15) \) and \( (16) \).

When presenting the spectrum and inversion formulas for \( \hat{H}^\perp_\epsilon \), we also consider the case of \( \mu = 0 \) and the case of \( \mu > 0 \) separately. We let \( E \) denote the spectrum points of \( \hat{H}^\perp_\epsilon \) and let \( \Psi_E \) denote the corresponding (generalized) eigenfunctions. The spectrum points of the operators \( \hat{h}_\epsilon(l) \) and \( \hat{H}^\perp_\epsilon \) are evidently related by \( \mathcal{E} = ME \). Therefore, when writing formulas for eigenfunctions \( \Psi_E \) of the operator \( \hat{H}^\perp_\epsilon \) in terms of eigenfunctions \( U_\mathcal{E} \) of the operators \( \hat{h}_\epsilon(l) \), we have to introduce the factor \( 1/\sqrt{2\pi\rho}\ e^{i\epsilon(l_\alpha - l)} \) in accordance with eq.(8) with \( \epsilon = \epsilon_q \) (because \( \epsilon_B = 1 \)), to make the substitutions \( \mathcal{E} = ME \) and \( \mathcal{E}^{(-)} = ME^{(-)} \), \( \mathcal{E}^{(-)} = ME^{(-)} \) for the respective points of the continuous spectrum and discrete spectrum, and, in addition, to multiply eigenfunctions of the continuous spectrum of the operators \( \hat{h}_\epsilon(l) \) by the factor \( \sqrt{M} \) because of the change of the spectral measure \( d\mathcal{E} \) to the corresponding spectral measure\(^1 \) \( dE \).

For \( \mu = 0 \), there is a family of s.a. two-dimensional nonrelativistic Hamiltonians \( \hat{H}^\perp_\lambda \) parametrized by the real parameter \( \lambda \in \mathbb{S}(-\pi/2, \pi/2) \), \( \hat{H}^\perp_\lambda = \hat{H}^\perp_{\lambda_0} \),

\[
\hat{H}^\perp_{\lambda_0} = \sum_{l \in \mathbb{Z}, l \neq l_0} ^{\oplus} \hat{H}^\perp_{\lambda_0}(l) \oplus \hat{H}^\perp_{\lambda_0}(l_0), \quad \hat{H}^\perp_{\lambda_0}(l) = M^{-1}S_l \hat{h}_{(1)}(l) S_l^{-1}, \quad l \neq l_0, \quad \hat{H}^\perp_{\lambda_0}(l_0) = M^{-1}S_{l_0} \hat{h}_\lambda(l_0) S_{l_0}^{-1}.
\]

\(^1\)From the physical standpoint, the latter is related to the change of the “normalization of the eigenfunctions of the continuous spectrum to \( \delta \) function” from \( \delta(\mathcal{E} - \mathcal{E}') \) to \( \delta(E - E') \).
The spectrum of $\hat{H}_\lambda^\perp$ is given by
\[
\text{spec}\hat{H}_\lambda^\perp = \left\{ E_\lambda^{(-)} = -4M^{-1}k_0^2 \exp[2(\tan \lambda - C)], \ |\lambda| < \pi/2 \right\} \cup \mathbb{R}_+.
\]

The complete set of orthonormalized eigenfunctions of $\hat{H}_\lambda^\perp$ consists of the generalized eigenfunctions $\Psi_{l,E}(\rho)$ of the continuous spectrum, $E \geq 0$,
\[
\Psi_{l,E}(\rho) = (M/4\pi)^{1/2} e^{i\epsilon \theta_{0,-l} \varphi} J_{\lambda l} \left( \sqrt{ME\rho} \right), \ l \neq l_0,
\]
\[
\Psi_{l_0,E}^\lambda(\rho) = \sqrt{\frac{M}{4\pi (\tilde{\lambda}^2 + \pi^2/4)}} e^{i\epsilon \theta_{0,0} \varphi} \left[ \tilde{\lambda} J_0 \left( \sqrt{ME\rho} \right) + \frac{\pi}{2} N_0(\sqrt{ME\rho}) \right],
\]
\[
\tilde{\lambda} = \tan \lambda - C - \ln \left( \sqrt{ME/2k_0} \right),
\]
and the eigenfunction $\Psi_{l_0}^\lambda(\rho)$ corresponding to the discrete level $E_\lambda^{(-)}$ in the case of $|\lambda| < \pi/2$,
\[
\Psi_{l_0}^\lambda(\rho) = M \sqrt{E_\lambda^{(-)}} / \pi e^{i\epsilon \theta_{0,0} \varphi} K_0 \left( \sqrt{M \left| E_\lambda^{(-)} \right| \rho} \right),
\]
such that
\[
\hat{H}_\lambda^\perp \Psi_{l,E}(\rho) = E \Psi_{l,E}(\rho), \quad \hat{H}_\lambda^\perp \Psi_{l_0,E}^\lambda(\rho) = E \Psi_{l_0,E}^\lambda(\rho), \quad E \geq 0,
\]
\[
\hat{H}_\lambda^\perp \Psi_{l_0}^\lambda(\rho) = E_\lambda^{(-)} \Psi_{l_0}^\lambda(\rho).
\]
The corresponding inversion formulas are
\[
\Psi(\rho) = \sum_{l \in \mathbb{Z}, l \neq l_0} \int_0^\infty \Phi_l(E) \Psi_{l,E}(\rho) dE + \int_0^\infty \Phi_{l_0}(E) \Psi_{l_0,E}^\lambda(\rho) dE + \Phi_{l_0} \Psi_{l_0}^\lambda(\rho),
\]
\[
\Phi_l(E) = \int d\rho \Psi_{l,E}(\rho) \Psi(\rho), \quad \Phi_{l_0}(E) = \int d\rho \Psi_{l_0,E}^\lambda(\rho) \Psi(\rho), \quad \Phi_{l_0} = \int d\rho \Psi_{l_0}^\lambda(\rho) \Psi(\rho),
\]
\[
\int d\rho |\Psi(\rho)|^2 = \sum_{l \in \mathbb{Z}} \int_0^\infty |\Phi_l(E)|^2 dE + |\Phi_{l_0}|^2, \quad \forall \Psi \in L^2(\mathbb{R}_2),
\]
the terms including $\Phi_{l_0}$ and $\Psi_{l_0}^\lambda(\rho)$ are absent in the case of $|\lambda| = \pi/2$.

For $\mu > 0$, there is a family of s.a. two-dimensional nonrelativistic Hamiltonians $\hat{H}_\lambda^\perp$ parametrized by two real parameters $\lambda_a \in \mathbb{S}(-\pi/2, \pi/2)$, $\hat{H}_\lambda^\perp = \hat{H}_\lambda^\perp(\lambda_a)$, $a = 0, -1$,
\[
\hat{H}_\lambda^{\perp}(\lambda_a) = \sum_{l \in \mathbb{Z}, l \neq l_a} \uplus \hat{H}_\perp^{\perp}(l) \uplus \sum_{a} \uplus \hat{H}_\lambda^{\perp}(l_a),
\]
\[
\hat{H}_\perp^{\perp}(l) = M^{-1} S_l \hat{h}_\perp(l) S_l^{-1}, \ l \neq l_a,
\]
\[
\hat{H}_\lambda^{\perp}(l_a) = M^{-1} S_l \hat{h}_\lambda(l_a) S_l^{-1}.
\]
The spectrum of $\hat{H}_{\lambda_a}^l$ is
\[ \text{spec} \hat{H}_{\lambda_a}^l = \left\{ E_{\lambda_a}^{(-)} = -4M^{-1}k_0^2|\lambda_a|^{-\kappa_a}, \ \lambda_a \in (-\pi/2, 0) \right\} \cup \mathbb{R}_+, \]
where $\kappa_a = |\mu + a|$, $\lambda_a = \Gamma(1 - \kappa_a)^{-1}(1 + \kappa_a)\tan\lambda_a$.

The complete set of orthonormalized eigenfunctions of $\hat{H}_{\lambda_a}^l$ consists of the generalized eigenfunctions $\Psi_{l,E}(\rho)$, $l \neq l_a$, and $\Psi_{l_a,E}^\lambda(\rho)$ of the continuous spectrum, $E \geq 0$,
\[ \Psi_{l,E}(\rho) = \left(\frac{M}{4\pi}\right)^{1/2} e^{i\kappa_a(\phi_0 - l)\rho} J_{\kappa_a} \left(\sqrt{ME}\rho\right), \ \kappa_l = |l + \mu|, \ l \neq l_a, \]
\[ \Psi_{l_a,E}^\lambda(\rho) = \sqrt{\frac{1}{4\pi Q_a}} e^{i\kappa_a(\phi_0 - l_a)\rho} \left[ J_{\kappa_a} \left(\sqrt{ME}\rho\right) + \lambda_a \left(\sqrt{ME/2\kappa_0}\right)^{2\kappa_a} J_{-\kappa_a} \left(\sqrt{ME}\rho\right) \right], \]
\[ Q_a = 1 + 2\lambda_a (M/4)^{\kappa_a} \cos(\pi\kappa_a) + \left(\frac{\lambda_a}{2}(M/4)^{2\kappa_a} \right), \]
and the eigenfunctions $\Psi_{l_a,E}^\lambda(\rho)$ corresponding to the discrete levels $E_{\lambda_a}^{(-)}$ in the case of $\lambda_a \in (-\pi/2, 0)$
\[ \Psi_{l_a}^\lambda(\rho) = \sqrt{\frac{M^2 |E_{\lambda_a}^{(-)}| \sin(\pi\kappa_a)}{\pi^2 \kappa_a}} e^{i\kappa_a(\phi_0 - l_a)\rho} K_{\kappa_a} \left(\sqrt{ME_{\lambda_a}^{(-)}}\rho\right), \]
such that
\[ \hat{H}_{l_a}^l \Psi_{l,E}(\rho) = E \Psi_{l,E}(\rho), \ l \neq l_a, \ \hat{H}_{l_a}^l \Psi_{l_a,E}^\lambda(\rho) = E \Psi_{l_a,E}^\lambda(\rho), \ E \geq 0, \]
\[ \hat{H}_{\lambda_a}^l \Psi_{l_a}^\lambda(\rho) = E_{\lambda_a}^{(-)} \Psi_{l_a}^\lambda(\rho), \ b = 0, -1. \]

The corresponding inversion formulas are
\[ \Psi(\rho) = \sum_{l \in \mathbb{Z}, \ l \neq l_a} \int_0^\infty \Phi_l(E) \Psi_{l,E}(\rho) dE + \sum_a \left[ \int_0^\infty \Phi_{l_a}(E) \Psi_{l_a,E}^\lambda(\rho) dE + \Phi_{l_a} \Psi_{l_a}^\lambda(\rho) \right], \]
\[ \Phi_l(E) = \int d\rho \Psi_{l,E}(\rho) \Psi(\rho), \ l \neq l_a, \]
\[ \Phi_{l_a}(E) = \int d\rho \Psi_{l_a,E}^\lambda(\rho) \Psi(\rho), \ \Phi_{l_a} = \int d\rho \Psi_{l_a}^\lambda(\rho) \Psi(\rho), \]
\[ \int d\rho |\Psi(\rho)|^2 = \sum_{l \in \mathbb{Z}} \int_0^\infty |\Phi_l(E)|^2 dE + \sum_a |\Phi_{l_a}|^2, \ \forall \Psi \in L^2(\mathbb{R}^2), \]
the terms with $\Phi_{l_a}$ and $\Psi_{l_a}^\lambda(\rho)$ are absent in the case of $\lambda_a \notin (-\pi/2, 0)$.

We now consider the case of the magnetic-solenoid field where $B \neq 0$.

### 2.1.3 S.a. Hamiltonians with magnetic-solenoid field

In this case, the radial differential operation $\hat{h}(l)$ is given by (11) with $\gamma = e^{\frac{\epsilon_B}{\hbar}} \neq 0$, or
\[ \hat{h}(l) = -\partial_{\rho}^2 + g_1 \rho^{-2} + g_2 \rho^2 + \epsilon_l^{(0)}, \]
\[ g_1 = \kappa_a^2 - 1/4, \ \kappa_a = |l + \mu|, \ g_2 = \gamma^2/4, \ \epsilon_l^{(0)} = \gamma(l + \mu). \]
Up to the constant term $E_i^{(0)}$, this s.a. differential operation is identical to the one-dimensional Schrödinger operation $-d_x^2 + g_1 x^{-2} + g_2 x^2$ studied by us recently (to be published). We therefore directly carry over the obtained results to s.a. extensions of $\hat{h}(l)$. We note that as in the case of pure AB field, a division to different regions of $g_1$ is actually determined by the same term $g_1 \rho^{-2} = (\kappa^2 - 1/4)/\rho^{-2}$ singular at the origin and independent of the value of $B$.

**First region: $g_1 \geq 3/4$** In this region, we have $(l + \mu)^2 \geq 1$ and as before, we distinguish the cases of $\mu = 0$ and $\mu > 0$:

\[
\begin{align*}
\mu &= 0 : l \leq -1 \text{ or } l \geq 1, \text{ i.e., } l \neq l_0 , \\
\mu &> 0 : l \leq -2 \text{ or } l \geq 1, \text{ i.e., } l \neq l_a .
\end{align*}
\]

For such $l$, the initial symmetric operator $\hat{h}(l)$ has zero deficiency indices, is essentially s.a., and its unique s.a. extension is $\hat{h}_\epsilon(l) = \hat{h}(l) = \hat{h}^+(l)$ with the domain $D^*_\hat{h}(\mathbb{R}_+)$. The spectrum of $\hat{h}(l)$ is simple and discrete,

\[
\text{spec} \hat{h}(l) = \{ E_{l,m} = \gamma (1 + |l + \mu| + (l + \mu) + 2m) , \ m \in \mathbb{Z}_+ \} . 
\] (26)

Eigenfunctions $U_{l,m}^{(1)}$ of the Hamiltonian $\hat{h}(l)$ are

\[
\begin{align*}
U_{l,m}^{(1)}(\rho) &= Q_{l,m} (\gamma/2)^{1/4+\kappa_l/2} \rho^{1/2+\kappa_l} e^{-\gamma \rho^2/4} \Phi(-m, 1 + \kappa_l; \gamma \rho^2/2) , \\
Q_{l,m} &= \left( \frac{\sqrt{2\gamma} \Gamma(1 + \kappa_l + m)}{m! \Gamma^2(1 + \kappa_l)} \right)^{1/2} , \\
\hat{h}(l) U_{l,m}^{(1)} = E_{l,m} U_{l,m}^{(1)}, \ m \in \mathbb{Z}_+ ,
\end{align*}
\] (27)

they form a complete orthonormalized set in each Hilbert space $L_l$.

**Second region: $-1/4 < \alpha < 3/4$** In this region, we have $0 < (l + \mu)^2 < 1$, or equivalently (13). We know that if $\mu = 0$, these inequalities have no solutions for $l \in \mathbb{Z}$, while if $\mu > 0$ there are the two solutions $l = l_a = a$, $a = 0, -1$. Therefore, we again remain with the case of $\mu > 0$.

For each $l = l_a$, there exists a one-parameter $U(1)$-family of s.a. Hamiltonians $\hat{h}_\epsilon(l_a) = \hat{h}_{\lambda_a}(l_a)$ parametrized by the real parameter $\lambda_a \in \mathbb{S}(-\pi/2, \pi/2)$ These Hamiltonians are specified by the asymptotic s.a. boundary conditions at the origin

\[
\psi_{\lambda_a}(\rho) = c \left[ (\sqrt{\gamma/2 \rho})^{1/2+\kappa_a} \sin \lambda_a + (\sqrt{\gamma/2 \rho})^{1/2-\kappa_a} \cos \lambda_a \right] + O(\rho^{3/2}) , 
\] (28)

where $\kappa_a = |\mu + a|$, $0 < \kappa_a < 1$, and $c$ is an arbitrary constant\footnote{In comparison with (19), we fix the dimensional parameter $k_0$ by $k_0 = \sqrt{\gamma/2}$.}, and their domains are given by

\[
\begin{align*}
D_{\hat{h}_{\lambda_a}}(\mathbb{R}_+) &= \{ \psi \in D^*_\hat{h}(l_a) (\mathbb{R}_+), \ \psi \text{ satisfies (28)} \} , \\
D^*_\hat{h}(l_a) (\mathbb{R}_+) &= \{ \psi_*, \psi_{\lambda_a}^* : \psi_*, \psi_{\lambda_a}^* \text{ are a.c. in } \mathbb{R}_+, \ \psi_*, \hat{h}(l_a) \psi_* \in L^2(\mathbb{R}_+) \} .
\end{align*}
\] (29)
The spectrum of \( \hat{h}_{\lambda_a}(l_a) \) is simple and discrete and is bounded from below,

\[
\text{spec}\hat{h}_{\lambda_a}(l_a) = \left\{ \mathcal{E}_{a,m} = \tau_{a,m} + \mathcal{E}_{l_a}^{(0)}, \ m \in \mathbb{Z}_+ \right\},
\]

where \( \tau_{a,m} \) are solutions of the equation \( \omega_{\lambda_a}(\tau_{a,m}) = 0 \),

\[
\omega_{\lambda_a}(W) = \omega_+(W) \sin \lambda_a + \omega_-(W) \cos \lambda_a,
\]

\[
\omega_\pm(W) = \Gamma(1 \pm \kappa_a)/\Gamma(1/2 \pm \kappa_a/2 - W/2\gamma).
\] (30)

The eigenfunctions \( U_{\lambda_a,m}^{(2)} \) of the Hamiltonian \( \hat{h}_{\lambda_a}(l_a) \) are

\[
U_{\lambda_a,m}^{(2)}(\rho) = Q_{a,m} [u_+(\rho; \tau_{a,m}) \sin \lambda_a + u_-(\rho; \tau_{a,m}) \cos \lambda_a],
\]

\[
Q_{a,m} = \left( \frac{\bar{\omega}_{\lambda_a}(\tau_{a,m})}{\sqrt{2\gamma \kappa_a \omega_{\lambda_a}(\tau_{a,m})}} \right)^{1/2}, \quad \bar{\omega}_{\lambda_a}(W) = \omega_+(W) \cos \lambda_a - \omega_-(W) \sin \lambda_a,
\]

\[
u_\pm(\rho; W) = (\gamma/2)^{1/4 \pm \kappa_a/2} \rho^{1/2 \pm \kappa_a} e^{-\gamma \rho^2/4} \Phi(1/2 \pm \kappa_a/2 - W/2\gamma, 1 \pm \kappa_a; \gamma \rho^2/2),
\]

\[
\hat{h}_{\lambda_a}(l_a) U_{\lambda_a,m}^{(2)} = \mathcal{E}_{a,m} U_{\lambda_a,m}^{(2)},
\] (31)

they form a complete orthonormalized set in each Hilbert space \( \mathcal{L}_{l_a} \).

Explicit expressions for the spectrum and eigenfunctions can be easily obtained in the cases of \( \lambda_a = \pm \pi/2 \) and \( \lambda_a = 0 \). In the case of \( \lambda_a = \pm \pi/2 \), they are given by the respective formulas (26) and (27) with the substitutions \( l \rightarrow l_a \) and \( \kappa_l \rightarrow \kappa_a \). In the case of \( \lambda_a = 0 \), these formulas are modified by the additional substitution \( \kappa_a \rightarrow -\kappa_a \).

**Third region:** \( \alpha = -1/4 \) In this region, we have \( l + \mu = 0 \). As we know, we remain with the only case of \( \mu = 0 \) and with \( l = l_0 = 0 \).

For \( l = l_0 \), there exists a one-parameter \( U(1) \)-family of s.a. Hamiltonians \( \hat{h}_\kappa(l_0) = \hat{h}_{\lambda_a}(l_0) \), parametrized by the real parameter \( \lambda \in \mathbb{S}(-\pi/2, \pi/2) \). These Hamiltonians are specified by the asymptotic s.a. boundary conditions at the origin

\[
\psi_\lambda(\rho) = c \left[ \rho^{1/2} \ln \left( \sqrt{\gamma/2\rho} \right) \cos \lambda + \rho^{1/2} \sin \lambda \right] + O(\rho^{3/2} \ln \rho), \ \rho \to 0,
\] (32)

where \( c \) is an arbitrary constant, and their domains are given by

\[
D_{\hat{h}_\kappa(l_0)} = \{ \psi : \psi \in D_{\hat{h}_\kappa(l_0)}^*(\mathbb{R}_+) \text{, } \psi \text{ satisfies } (32) \}, \quad D_{\hat{h}_\kappa(l_0)}^*(\mathbb{R}_+) = \{ \psi_* : \psi_* \text{ are a.c. in } \mathbb{R}_+, \ \psi_* \in L^2(\mathbb{R}_+) \}. \quad (33)
\]

The spectrum of \( \hat{h}_{\lambda_a}(l_0) \) is simple and discrete and is bounded from below, \( \text{spec}\hat{h}_{\lambda_a}(l_0) = \{ \mathcal{E}_m, \ m \in \mathbb{Z}_+ \} \), where \( \mathcal{E}_m \) are solutions of the equation \( \omega_{\lambda}(\mathcal{E}_m) = 0 \),

\[
\omega_{\lambda}(W) = \cos \lambda (\psi(\alpha_0) - 2\psi(1)) - \sin \lambda, \ \alpha_0 = 1/2 - W/2\gamma.
\] (34)

In the case of \( \lambda \neq \pm \pi/2 \), the limit \( \lambda \rightarrow \pm \pi/2 \) in these solutions yields the spectrum in the case of \( \lambda = \pm \pi/2 \).
The eigenfunctions $U_{\lambda,m}^{(3)}$ of the Hamiltonians $h_{\lambda}(l_0)$ are

$$
U_{\lambda,m}^{(3)} = Q_{\lambda,m} [u_1(\rho; \mathcal{E}_m) \sin \lambda + u_3(\rho; \mathcal{E}_m) \cos \lambda],
$$

$$
u_1(\rho; W) = (\gamma/2)^{1/4} \rho^{1/2} e^{-\gamma \rho^2/4} \Phi(\alpha_0, 1; \gamma \rho^2/2),$$

$$
u_3(\rho; W) = u_1(\rho; W) \ln \left(\sqrt{\gamma/2}\rho\right) + (\gamma/2)^{1/4} \rho^{1/2} e^{-\gamma \rho^2/4} \partial_\rho \Phi(1/2 + \mu - W/2\gamma, 1 + 2\mu; \gamma \rho^2/2) |_{\mu=0}.
$$

$$
Q_{\lambda,m} = \left[-\frac{\bar{\omega}_\lambda(\mathcal{E}_m)}{\sqrt{2\gamma \omega_\lambda(\mathcal{E}_m)}}\right]^{1/2}, \quad \bar{\omega}_\lambda(W) = \sin \lambda [\psi(\alpha_0) - 2\psi(1)] + \cos \lambda,
$$

$$
\hat{h}_\lambda(l_0) U_{\lambda,m}^{(3)} = \mathcal{E}_m U_{\lambda,m}^{(3)}, \quad m \in \mathbb{Z},
$$

they form a complete orthonormalized set in the Hilbert space $\mathcal{L}_{l_0}$.

We note that spectrum and eigenfunctions in the case of $\lambda = \pm \pi/2$ can be obtained from the respective formulas for the first region in the formal limit $l \to 0$.

**Complete spectrum and inversion formulas** In the previous subsubsecs., we constructed all s.a. radial Hamiltonians $\hat{h}_\ell(l)$ associated with the s.a. differential operation $\hat{h}(l)$ as s.a. extensions of the symmetric operator $\hat{h}(l)$ for any $l \in \mathbb{Z}$ and for any any $\phi_0, \mu$, and $B$. We assemble our previous results into two groups.

For $\mu = 0$, we have

$$
\hat{h}_\ell(l) = \hat{h}_{(1)}(l), \quad l \neq l_0 = 0 , \quad D_{h_{(1)}(l)} = D_{h_{(1)}(l)}^*(\mathbb{R}_+),
$$

$$
\hat{h}_\ell(l_0) = \hat{h}_\lambda(l_0), \quad \lambda \in \mathbb{S}(-\pi/2, \pi/2),
$$

(36)

and the domain $D_{h_{\lambda}(l_0)}$ is given by eq. (33);

For $\mu > 0$, we have

$$
\hat{h}_\ell(l) = \hat{h}_{(1)}(l), \quad l \neq l_a = a = 0, -1 , \quad D_{h_{(1)}(l)} = D_{h_{(1)}(l)}^*(\mathbb{R}_+),
$$

$$
\hat{h}_\ell(l_a) = \hat{h}_{\lambda_a}(l_a), \quad \lambda_a \in \mathbb{S}(-\pi/2, \pi/2),
$$

(37)

and the domain $D_{h_{\lambda_a}(l_a)}$ is given by eq. (29).

As a result, we find a family of all s.a. rotationally-invariant two-dimensional nonrelativistic Hamiltonians $\hat{H}_\ell^\perp = M^{-1}\hat{R}_\ell^\perp$ associated with the s.a. differential operation $\hat{R}_\ell^\perp$ with $B \neq 0$. Each set of possible s.a. radial Hamiltonians $\hat{h}_\ell(l)$ generates a s.a. rotationally-invariant Hamiltonian $\hat{H}_\ell^\perp$ in accordance with the relations (15) and (16). As in the case of pure AB field where $B = 0$, we let $E$ denote the spectrum points of $\hat{H}_\ell^\perp$.

It is convenient to change the indexing $l, m$ of the spectrum points and eigenfunctions to $l, n$, as follows:

$$
n = n(l, m) = \begin{cases} n, & l \leq -1 \\ m + l, & l \geq 0 \end{cases}, \quad m \in \mathbb{Z}, \quad l \in \mathbb{Z};
$$

$$
m = m(n, l) = \begin{cases} n, & l \leq -1 \\ n - l, & 0 \leq l \leq n \end{cases}, \quad n \in \mathbb{Z}, \quad l \in \mathbb{Z},
$$

(38)

and then to interchange their position, such that finally, the indices $l, m$ are replaced by indices $n, l$. 

When writing formulas for eigenfunctions $\Psi_{n,l}(\rho)$ of the operator $\hat{H}_\lambda^\perp$ in terms of eigenfunctions $U_{l,m}$ of the operators $\hat{h}_l(l)$, we have to introduce the factor $1/\sqrt{2\pi\rho}$ $e^{i\epsilon(\phi_0-l)\phi}$ in accordance with eq. (8) and to make the substitution $E_{l,m} = M E_{n,l}$ for the corresponding spectrum points.

The final result is the following.

There is a family of s.a. two-dimensional nonrelativistic Hamiltonians $\hat{H}_\lambda^\perp$ parametrized by real parameters $\lambda_*$, $\hat{H}_\lambda^\perp = \hat{H}_\lambda^\perp$, 
\[
\hat{H}_\lambda^\perp = \sum_{l^*\neq l_*, l_\perp} \hat{H}_\lambda^\perp(l) + \sum_{l_\perp \neq l_*} \hat{H}_\lambda^\perp(l_\perp),
\]
\[
\hat{H}_\lambda^\perp(l) = M^{-1}S_{l}H_{l}(l)S_{l}^{-1}, \ l \neq l_*,
\]
\[
\hat{H}_\lambda^\perp(l_*) = M^{-1}S_{l_*}H_{l_*}(l_*)S_{l_*}^{-1},
\]
\[
l_* = \begin{cases} 0, \mu = 0 \\ l_\perp, \mu > 0 \end{cases}, \lambda_* = \begin{cases} \lambda \in \mathbb{S}(-\pi/2, \pi/2), \mu = 0 \\ \lambda_* \in \mathbb{S}(-\pi/2, \pi/2), \mu > 0. \end{cases}
\]

(39)

The spectrum of $\hat{H}_\lambda^\perp$ is given by (remember that $E_{n,l} = M^{-1}E_{l,m(n,l)}$)
\[
\text{spec} \hat{H}_\lambda^\perp = \{ \cup_{l \in \mathbb{Z}, l \neq l_*} \{ E_{n,l}, n \in \mathbb{Z} \} \} \cup \{ \cup_{l_*} \{ E_{n}^{(\lambda)}, n \in \mathbb{Z} \} \} ;
\]
\[
E_{n,l} = \gamma M^{-1}[1 + \sqrt{4n^2 + 4\theta(l)\mu}], \ l \leq n, l \neq l_*, \theta(l) = \begin{cases} 1, l \geq 0 \\ 0, l < 0 \end{cases}, \quad E_{n}^{(\lambda)} = \gamma M^{-1}(1 + 2n), \ \mu = 0,
\]
\[
\begin{cases} E_{n}^{(\lambda_\ast)} = M^{-1}[\tau_{a,n} + \gamma(a + \mu)], \ \omega_{\lambda_\ast}(\tau_{a,n}) = 0 \\ E_{n}^{(\perp)} = \gamma M^{-1}[1 + \sqrt{4n^2 + 4\theta(a)\mu}], \ n \in \mathbb{N} \end{cases}, \mu > 0,
\]

(41)

where $\omega_{\lambda}(W)$ and $\omega_{\lambda_*}(W)$ are given by the respective eqs. (34) and (30).

The complete set of orthonormalized eigenfunctions of $\hat{H}_\lambda^\perp$ consists of the functions $\Psi_{n,l}(\rho)$, $l \neq l_*$, and $\Psi_{n,l_*}^{\lambda_*}(\rho)$,
\[
\Psi_{n,l}(\rho) = \frac{1}{\sqrt{2\pi\rho}}e^{i\epsilon(\phi_0-l)\phi}U_{l,m(n,l)}^{(1)}(\rho),
\]

(42)

where $U_{l,m}^{(1)}(\rho)$ are given by eqs. (27), and (we note that $m(n,l_*) = n$)
\[
\Psi_{n,0}(\rho) = \frac{1}{\sqrt{2\pi\rho}}e^{i\epsilon\phi_0\phi}U_{n,0}(\rho), \ \mu = 0,
\]
\[
\Psi_{n,a}(\rho) = \frac{1}{\sqrt{2\pi\rho}}e^{i\epsilon(\phi_0-l_0)\phi}U_{n,a}(\rho), \ \mu > 0,
\]

where $U_{n,a}(\rho)$ and $U_{n,a}(\rho)$ are given by the respective eqs. (35) and (31), such that
\[
\hat{H}_\lambda^\perp \Psi_{n,l}(\rho) = E_{n,l}\Psi_{n,l}(\rho), l \neq l_*, \ \hat{H}_\lambda^\perp \Psi_{n,l_*}^{\lambda_*}(\rho) = E_{n}^{(\lambda_\ast)}\Psi_{n,l_*}^{\lambda_*}(\rho).
\]

We note, that for the case of $\lambda = \pm\pi/2$ where $l = l_0 = 0$ and for the case of $\lambda_\ast = \pm\pi/2$ where $l = l_\ast = a = 0, -1$, the energy eigenvalues $E_{n}^{(\lambda)}$ and $E_{n}^{(\lambda_\ast)}$ and corresponding eigenfunctions $\Psi_{n}^{\lambda}$ and $\Psi_{n}^{\lambda_*}$ are given by the respective eqs. (40) and (42) extended to all values of $l.$
The corresponding inversion formulas are

\[ \Psi(\rho) = \sum_{l \in \mathbb{Z}, l \neq l_\ast} \Phi_{n,l} \Psi_{n,l}(\rho) + \sum_{l = l_\ast, n \in \mathbb{Z}_+} \Phi_{n,l_\ast} \Psi_{n,l_\ast}(\rho), \]

\[ \Phi_{n,l} = \int d\rho \overline{\Phi_{n,l}(\rho)} \Psi(\rho), \quad l \neq l_\ast, \quad \Phi_{n,l_\ast} = \int d\rho \overline{\Phi_{n,l_\ast}(\rho)} \Psi(\rho), \]

\[ \int d\rho |\Psi(\rho)|^2 = \sum_{l \in \mathbb{Z}} \sum_{n \in \mathbb{Z}_+} |\Phi_{n,l}|^2, \quad \forall \Psi \in L^2(\mathbb{R}^2). \]

2.2 Three-dimensional case

In three dimensions, we start with the differential operation \( \hat{H}^{(4)} \). The initial symmetric operator \( \hat{H} \) associated with \( \hat{H} \) is defined in the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^3) \) by

\[ D_H = \mathcal{D}(\mathbb{R}^3 \setminus \mathbb{R}_z), \quad \hat{H} \psi = \hat{H} \psi, \quad \forall \psi \in D_H, \]

where \( \mathcal{D}(\mathbb{R}^3 \setminus \mathbb{R}_z) \) is the space of smooth and compactly supported functions vanishing in a neighborhood of the \( z \)-axis. The domain \( D_H \) is dense in \( \mathcal{H} \), and the symmetricity of \( \hat{H} \) is obvious. A s.a. quantum Hamiltonian must be defined as a s.a. extension of \( \hat{H} \).

There is an evident space symmetry in the classical description of the system, the symmetry with respect to rotations around the \( z \)-axis and translations along this axis, which is manifested as the invariance of the classical Hamiltonian under these space transformations. The key point in constructing a quantum description of the system is the requirement of the invariance of the quantum Hamiltonian under the same transformations.

Namely, let \( G \) be the group of the above space transformations \( S \): \( r \rightarrow Sr \). This group is unitarily represented in \( \mathcal{H} \): if \( S \in G \), then the corresponding operator \( U_S \) is defined by

\[ (U_S \psi)(r) = \psi(S^{-1}r), \quad \forall \psi \in \mathcal{H}. \]

The operator \( \hat{H} \) evidently commutes\(^3\) with \( U_S \) for any \( S \). We search only for s.a. extensions \( \hat{H} \) of \( H \) that also commute with \( U_S \) for any \( S \). This condition is the explicit form of the invariance, or symmetry, of a quantum Hamiltonian under the space transformations. As in classical mechanics, this symmetry allows separating the cylindrical coordinates \( \rho, \varphi, \) and \( z \) and reducing the three-dimensional problem to a one-dimensional radial problem. Let \( L^2(\mathbb{R} \times \mathbb{R}_+) \) denote the space of square-integrable functions with respect to the Lebesgue measure \( dp_z dp \) on \( \mathbb{R} \times \mathbb{R}_+ \), and let \( V: \sum_{l \in \mathbb{Z}} L^2(\mathbb{R} \times \mathbb{R}_+) \rightarrow \mathcal{H} \) be the unitary operator defined by the relation

\[ (V f)(\rho, \varphi, z) = \frac{1}{2\pi \sqrt{p}} \int_{\mathbb{R}_+} dp_z \sum_{l \in \mathbb{Z}} e^{i(\phi - l) \varphi + p_z z} f(l, p_z, \rho). \]

In the two-dimensional case considered in the preceding subsection, all s.a. Hamiltonians were represented as direct sums over \( l \) of s.a. extensions of suitable radial symmetric

\(^3\)We remind the reader of the notion of commutativity in this case (where one of the operators, \( U_S \), is bounded and defined everywhere): we say that the operators \( \hat{H} \) and \( U_S \) commute if \( U_S \hat{H} \leq \hat{H} U_S \), i.e., if \( \psi \in D_H \), then also \( U_S \psi \in D_H \) and \( \hat{H} U_S \psi = U_S \hat{H} \psi \).
operators. In the three-dimensional case, a new continuous parameter $p_z$ arises in addition to the discrete parameter $l$, and analogous representations should be written in terms of direct integrals rather than direct sums. More precisely, any s.a. Hamiltonian $\hat{H}_e$ can be represented in the form

$$\hat{H}_e = V \int_{\mathbb{R}_+} dp_z \sum_{l \in \mathbb{Z}} \hat{h}_e(l, p_z) V^{-1},$$

where $\hat{h}_e(l, p_z)$ for fixed $l$ and $p_z$ is a s.a. extension of the symmetric operator $\hat{h}(l, p_z) = \hat{h}(l) + p_z^2/2m_e$ in $L^2(\mathbb{R}_+)$, and the operator $\hat{h}(l)$ in $L^2(\mathbb{R}_+)$ is given by

$$D_{h(l)} = D(\mathbb{R}_+),$$

$$\hat{h}(l)f(\rho) = \left(-\partial_\rho^2 + \rho^{-2} \left[ (l + \mu + \gamma \rho^2/2)^2 - 1/4 \right] \right) f(\rho), \ f \in D_{h(l)}.$$ (43)

A detailed derivation of the above representation for $\hat{H}_e$ will be published in the short run.

The inversion formulas in three dimensions are obtained by the following modifications in the two-dimensional inversion formulas:

1) $\sum_{m \in \mathbb{Z}} \int dE - \int dp_z \sum_{l \in \mathbb{Z}} \int dE^\perp$, where $E^\perp$ are spectrum points of two-dimensional s.a. Hamiltonians $\hat{H}_e^\perp$, whereas the eigenvalues (spectrum points) $E$ of three-dimensional s.a. Hamiltonians $\hat{H}_e$ are $E = E^\perp + p_z^2/2m \left( \int dp_z = \int_{-\infty}^\infty dp_z \right)$.

2) The contribution of discrete spectrum points of two-dimensional s.a. Hamiltonian $\hat{H}_e^\perp$ have to be multiplied by $\int dp_z$.

3) Eigenfunctions of two-dimensional s.a. Hamiltonians $\hat{H}_e^\perp$ have to be multiplied by $(2\pi\hbar)^{-1/2} e^{ip_z z/\hbar}$ in order to obtain eigenfunctions of three-dimensional s.a. Hamiltonians $\hat{H}_e$.

4) The extension parameters $\lambda_a$ and $\lambda$ have to be replaced by functions $\lambda_a(p_z)$ and $\lambda(p_z)$.

### 2.2.1 S.a. nonrelativistic Hamiltonians with AB field

For the case of $\mu = 0$, there is a family of s.a. three-dimensional Hamiltonians parametrized by a real-valued function $\lambda(p_z)$ $(\lambda(p_z) \in \mathbb{S}(-\pi/2, \pi/2), \ p_z \in \mathbb{R})$.

The corresponding spectrum of $\hat{H}_{\lambda(p_z)}$ is

$$\text{spec} \hat{H}_{\lambda(p_z)} = \left\{ \frac{p_z^2}{2m_e} - 4M^{-1}\kappa_0^2 \exp \left[ 2(\tan \lambda(p_z) - C) \right], \ |\lambda(p_z)| < \pi/2 \right\} \cup \mathbb{R}_+.$$

The complete set of orthonormalized generalized eigenfunctions of $\hat{H}_{\lambda(p_z)}$ consists of functions $\Psi_{\ell, p_z, E^\perp}(r)$, $l \neq l_0$, and $\Psi_{l_0, p_z, E^\perp}(r)$,

$$\Psi_{\ell, p_z, E^\perp}(r) = \left(8\pi^2 \hbar/M \right)^{-1/2} e^{ip_z z/\hbar + i\kappa_0 \phi_0 l} J_{\kappa_0} \left( \sqrt{ME^\perp} \rho \right),$$

$$\Psi_{l_0, p_z, E^\perp}(r) = \left(8\pi^2 \hbar \left( \lambda^2 + \pi^2/4 \right) / M \right)^{-1/2} e^{ip_z z/\hbar + i\kappa_0 \phi_0 \varphi} \left[ \tilde{\lambda} J_0 \left( \sqrt{ME^\perp} \rho \right) + \frac{\pi}{2} N_0 \left( \sqrt{ME^\perp} \rho \right) \right],$$

$$\tilde{\lambda} = \tan \lambda(p_z) - C - \ln \left( \sqrt{ME^\perp} / 2\kappa_0 \right),$$

and functions $\Psi_{l_0, p_z}(r)$,
\[\Psi_{l_0,p_z}(r) = \frac{1}{2\pi\sqrt{\hbar}} e^{ip_zz/\hbar+i\phi_0\varphi} \left\{ \sqrt{2M^2 \left| E_{\lambda(p_z)}^{\perp(-)} \right|} K_0 \left( \sqrt{M \left| E_{\lambda(p_z)}^{\perp(-)} \right| \rho} \right), \quad |\lambda(p_z)| < \pi/2 \right.\]

\[E_{\lambda(p_z)}^{\perp(-)} = -4M^{-1} \kappa_0^2 \exp (2(\tan \lambda(p_z) - C)),\]

such that

\[\hat{H}\Psi_{l_0,p_z,E^{\perp}}(r) = (p_z^2/2m_e + E^{\perp}) \Psi_{l_0,p_z,E^{\perp}}(r), \quad E^{\perp} \geq 0,\]

\[\hat{H}\Psi_{l_0,p_z,E^{\perp}}^\lambda(r) = (p_z^2/2m_e + E^{\perp}) \Psi_{l_0,p_z,E^{\perp}}^\lambda(r), \quad E^{\perp} \geq 0,\]

\[\hat{H}\Psi_{l_0,p_z}^\lambda(r) = (p_z^2/2m_e + E_{\lambda(p_z)}^{\perp(-)}) \Psi_{l_0,p_z}^\lambda(r).\]

The corresponding inversion formulas are

\[\Psi(r) = \int dp_z \left[ \sum_{l \neq 0} \int_0^\infty \Phi_{l_0,p_z,E^{\perp}}(r) \Psi_{l_0,p_z,E^{\perp}}(r) dE^{\perp} + \int_0^\infty \Phi_{l_0,p_z}(E^{\perp}) \Psi_{l_0,p_z}(r) dE^{\perp} + \Phi_{l_0,p_z} \Psi_{l_0,p_z}(r) \right],\]

\[\Phi_{l_0,p_z}(E^{\perp}) = \int \Psi_{l_0,p_z,E^{\perp}}(r) \Psi(r) dr, \quad \Phi_{l_0,p_z} = \int \Psi_{l_0,p_z}(r) \Psi(r) dr,\]

\[\int |\Psi(r)|^2 dr = \int \left[ \sum_{l \neq 0} \int_0^\infty |\Phi_{l_0,p_z}(E^{\perp})|^2 dE^{\perp} + |\Phi_{l_0,p_z}|^2 \right], \quad \forall \Psi \in L^2(\mathbb{R}^3).\]

For the case of \(\mu > 0\), there is a family of s.a. three-dimensional Hamiltonians \(\hat{H}_{(\lambda_a(p_z))}\) parametrized by two real-valued functions \(\lambda_a(p_z) (\lambda_a \in \mathbb{S}(-\pi/2, \pi/2)), \quad a = 0, -1, \quad p_z \in \mathbb{R}\).

The spectrum of \(\hat{H}_{(\lambda_a(p_z))}\) is

\[\text{spec} \hat{H}_{(\lambda_a(p_z))} = \left\{ p_z^2/2m_e - 4M^{-1} \kappa_0^2 \left| \tilde{\lambda}_a \right|^{-\kappa_0^2}, \quad \lambda_a(p_z) \in (-\pi/2, 0), \right\} \cup \mathbb{R}_+,\]

\[\tilde{\lambda}_a = |\mu + a|, \quad \lambda_a = \Gamma(1 - \kappa_a) \Gamma^{-1}(1 + \kappa_a) \tan \lambda_a(p_z), \quad \kappa_a = |\mu + a|\].

The complete set of orthonormalized generalized eigenfunctions of \(\hat{H}_{(\lambda_a(p_z))}\) consists of functions \(\Psi_{l_0,p_z,E^{\perp}}(r), \quad l \neq l_a, \) and \(\Psi_{l_0,p_z}^\lambda(r), \)

\[\Psi_{l_0,p_z,E^{\perp}}(r) = (8\pi^2 \hbar/M)^{-1/2} e^{ip_zz/\hbar+i\phi_0(\phi_0 - l_0)\varphi} J_{\kappa_a} \left( \sqrt{ME^{\perp}} \rho \right),\]

\[\Psi_{l_0,p_z}^\lambda(r) = (8\pi^2 \hbar Q_a)^{-1/2} e^{ip_zz/\hbar+i\phi_0(\phi_0 - l_a)\varphi} \left[ J_{\kappa_a} \left( \sqrt{ME^{\perp}} \rho \right) + \left( \sqrt{ME^{\perp}} / 2\kappa_0 \right)^{2\kappa_a} \tilde{\lambda}_a J_{-\kappa_a} \left( \sqrt{ME^{\perp}} \rho \right) \right]^{2\kappa_a},\]

\[Q_a = 1 + 2 \left( ME^{\perp}/4 \right)^{2\kappa_a} \tilde{\lambda}_a \cos(\pi \kappa_a) + (ME/4)^{2\kappa_a} \tilde{\lambda}_a^2,\]

and functions \(\Psi_{l_a,p_z}(r),\)
\[ \Psi_{l_a,p_z}(r) = (2\pi^2\hbar)^{-1} e^{ip_zz/h+i(l_a+\epsilon\phi_0)\varphi} \left\{ \sqrt{M^2|E_{\lambda_0(p_z)}^{(-)}|^2\sin(\pi\kappa_0)} K_{\kappa_0} \left( \sqrt{M|E_{\lambda_0(p_z)}^{(-)}|} \rho \right) , \lambda_0(p_z) \in (-\pi/2, 0) \right. \]

\[ E_{\lambda_0(p_z)}^{(-)} = -4M^{-1}\kappa_0^2 \exp 2(\tan \lambda_0(p_z) - C), \]

such that

\[ \hat{H}\Psi_{l,p_z,E^\perp}(r) = \left( p_z^2/2m_e + E^\perp \right) \Psi_{l,p_z,E^\perp}(r), \quad E^\perp \geq 0, \]

\[ \hat{H}\Psi_{l_a,p_z,E^\perp}(r) = \left( p_z^2/2m_e + E^\perp \right) \Psi_{l_a,p_z,E^\perp}(r), \quad E^\perp \geq 0, \]

\[ \hat{H}\Psi_{l_a,p_z}(r) = \left( p_z^2/2m_e + E_{\lambda_0(p_z)}^{(-)} \right) \Psi_{l_a,p_z}(r). \]

The corresponding inversion formulas are

\[ \Psi(r) = \int dp_z \left[ \sum_{l \in \mathbb{Z}, \ l \neq l_a} \int_0^\infty \Phi_{l,p_z}(E^\perp)\Psi_{l,p_z,E^\perp}(r)dE^\perp \right] \]

\[ + \sum_a \int_0^\infty \Phi_{l_a,p_z}(E^\perp)\Psi_{l_a,p_z,E^\perp}(r)dE^\perp + \Phi_{l_a,p_z}(r), \quad l \neq l_a, \]

\[ \Phi_{l,p_z}(E^\perp) = \int d\mathbf{r}\Psi_{l,p_z,E^\perp}(\mathbf{r})\Psi(\mathbf{r}), \quad E^\perp \geq 0, \quad \Phi_{l_a,p_z} = \int d\mathbf{r}\Phi_{l_a,p_z}(\mathbf{r})\Psi(\mathbf{r}), \]

\[ \int d\mathbf{r} |\Psi(\mathbf{r})|^2 = \int dp_z \left[ \sum_{l \in \mathbb{Z}} \int_0^\infty |\Phi_{l,p_z}(E^\perp)|^2 dE^\perp + \sum_a |\Phi_{l_a,p_z}|^2 \right], \quad \forall \Psi \in L^2(\mathbb{R}^3). \]

### 2.2.2 S.a. nonrelativistic Hamiltonians with magnetic-solenoid field

There is a family of s.a. three-dimensional Hamiltonians \( \hat{H}_{\lambda_0(p_z)} \) parametrized by real-valued functions \( \lambda_0(p_z) \) \( \left( \lambda_0(p_z) \in \mathbb{S}(-\pi/2, \pi/2), \ p_z \in \mathbb{R} \right) \), \( \lambda_0 \) are defined by eq. (39).

The spectrum of \( \hat{H}_{\lambda_0(p_z)} \) is

\[ \text{spec} \hat{H}_{\lambda_0(p_z)} = \left\{ p_z^2/2m_e + E_{n\lambda_0(p_z)}^{\perp}, \ n \in \mathbb{Z}^+ \right\} \cup \left[ \gamma M^{-1}, \infty \right), \]

where \( E_{n\lambda_0(p_z)}^{\perp} \) are defined by eqs. (40) and (41) with the substitution \( \lambda \rightarrow \lambda_0(p_z) \).

The complete set of generalized orthonormalized eigenfunctions of \( \hat{H}_{\lambda_0(p_z)} \) consists of functions \( \Psi_{p_z,l,n}(\mathbf{r}), \ l \neq l_*, \) and \( \Psi_{p_z,l_*,n}(\mathbf{r}), \ n \in \mathbb{Z}^+, \)

\[ \Psi_{p_z,l,n}(\mathbf{r}) = \frac{1}{2\pi\sqrt{\hbar\rho}} e^{ip_zz/h+i(\phi_0-l)\varphi} U^{(1)}_{l,m(n,l)}(\rho), \ l \neq l_*, \]

where \( l_* \) are defined by eq. (39), \( m(n,l) \) is given by (38), and \( U^{(1)}_{l,m}(\rho) \) are given by eqs. (27),

\[ \Psi_{p_z,l_*,n}(\mathbf{r}) = \frac{1}{2\pi\sqrt{\hbar\rho}} e^{ip_zz/h+i\epsilon\phi_0\varphi} U^{(3)}_{\lambda_0(p_z),n}(\rho), \quad \mu = 0, \]

\[ \Psi_{p_z,l_*,n}(\mathbf{r}) = \frac{1}{2\pi\sqrt{\hbar\rho}} e^{ip_zz/h+i\epsilon\phi_0-l_0)\varphi} U^{(2)}_{\lambda_0(p_z),n}(\rho), \quad \mu > 0, \]

\[ \Psi_{p_z,l_0,n}(\mathbf{r}) = \frac{1}{2\pi\sqrt{\hbar\rho}} e^{ip_zz/h+i\epsilon\phi_0-l_0)\varphi} U^{(2)}_{\lambda_0(p_z),n}(\rho), \quad \mu > 0, \]
where \( U_{\lambda(p_z),n}^{(3)}(\rho) \) and \( U_{\lambda_0(p_z),n}^{(2)}(\rho) \) are given by the respective eqs. (35) and (31) with the substitution \( \lambda_* \rightarrow \lambda_*(p_z) \), such that

\[
\hat{H}\Psi_{p_z,l,n}(\mathbf{r}) = \left( p_z^2/2m_e + E_{n,l}^{\perp} \right) \Psi_{p_z,l,n}(\mathbf{r}), \quad \text{l} \neq l_* \quad \text{and} \quad \hat{H}\Psi_{p_z,l+n,n}^{\lambda_*(p_z)}(\mathbf{r}) = \left( p_z^2/2m_e + E_{n,l}^{\perp,\lambda_*(p_z)} \right) \Psi_{p_z,l+n,n}^{\lambda_*(p_z)}(\mathbf{r}),
\]

where

\[
E_{n,l}^{\perp} = \gamma M^{-1}[1 + 2n + 2\theta(l)\mu], \quad l \leq n, \quad l \neq l_*, \quad n \in \mathbb{Z}_+.
\]

We note that for \( \lambda(p_z) = \lambda_0(p_z) = \pm \pi/2 \), the energy eigenvalues and corresponding eigenfunctions \( \Psi_{p_z,l,n}(\mathbf{r}) \) are given by eqs. (15), (16), and (14) extended to all values of \( l \).

The corresponding inversion formulas are

\[
\Psi(\mathbf{r}) = \int dp_z \sum_{n \in \mathbb{Z}_+} \left[ \sum_{l \in \mathbb{Z}, l \neq l_*} \Phi_{p_z,l,n} \Psi_{p_z,l,m(n,l)}(\mathbf{r}) + \sum_{l = l_*} \Phi_{p_z,l,n} \Psi_{p_z,l,n}^{\lambda_*(p_z)}(\mathbf{r}) \right],
\]

\[
\Phi_{p_z,l,n} = \int d\mathbf{r} \overline{\Psi}_{p_z,l,n}(\mathbf{r}) \Psi(\mathbf{r}), \quad l \neq l_*, \quad \Phi_{p_z,l,n} = \int d\mathbf{r} \overline{\Psi}_{p_z,l+n,n}^{\lambda_*(p_z)}(\mathbf{r}) \Psi(\mathbf{r}),
\]

\[
\int d\mathbf{r} |\Psi(\mathbf{r})|^2 = \int dp_z \sum_{l \in \mathbb{Z}} \sum_{n \in \mathbb{Z}_+} |\Phi_{p_z,l,n}|^2, \quad \forall \Psi \in L^2(\mathbb{R}^3).
\]

3 S.a. Dirac Hamiltonians with magnetic-solenoid field

3.1 Generalities

In this section, we set \( c = \hbar = 1 \). Written in the form of the Schrödinger equation, the Dirac equation with the magnetic-solenoid field for a relativistic particle of mass \( m_e \), spin 1/2, and charge \( q = e_q e \) (positron or electron) is

\[
i \frac{\partial \Psi(x)}{\partial t} = \hat{H}\Psi(x), \quad x = (x^0, \mathbf{r}), \quad \mathbf{r} = (x^k, \quad k = 1, 2, 3), \quad x^0 = t,
\]

where \( \Psi(x) = \{ \psi_\alpha(x), \quad \alpha = 1, ..., 4 \} \) is a four-spinor and \( \hat{H} \) is the s.a. Dirac differential operation, the ” formal Dirac Hamiltonian”,

\[
\hat{H} = \alpha \left( \mathbf{\nabla} - e_q e \mathbf{A} \right) + m_e \beta,
\]

where the vector potential \( \mathbf{A} \) is given by (2), \( \alpha = (\gamma^0 \gamma^k, \quad k = 1, 2, 3), \quad \beta = \gamma^0, \quad \text{and} \quad \gamma^\mu, \mu = 0, 1, 2, 3, \) are the Dirac \( \gamma \) matrices.

The space of quantum states for a particle is the Hilbert space \( \mathfrak{H} = L^2(\mathbb{R}^3) \) of square-integrable four-spinors \( \Psi(\mathbf{r}) \) with the scalar product

\[
(\Psi_1, \Psi_2) = \int d\mathbf{r} \overline{\Psi}_1^+(\mathbf{r}) \Psi_2(\mathbf{r}), \quad d\mathbf{r} = dx^1 dx^2 dx^3 = \rho d\rho \varphi dz.
\]

The Hilbert space \( \mathfrak{H} \) can be presented as

\[
\mathfrak{H} = \bigoplus_{\alpha=1}^{4} \mathfrak{H}_\alpha, \quad \mathfrak{H}_\alpha = L^2(\mathbb{R}^3).\]
Our first aim is to construct a quantum Dirac Hamiltonian that is a s.a. operator associated with differential operation $\tilde{H}$ in this Hilbert space. A construction is based on the known spatial symmetry in the problem, which allows separating the cylindric coordinates $\rho, \varphi$, and $z$, and on the s.a. extension theory.

It is convenient to choose the following representation for $\gamma$ matrices:

$$
\gamma^0 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix},
$$
$$
\gamma^3 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \Sigma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}.
$$

Written in the cylindric coordinates, the differential operation $\tilde{H}$ then becomes

$$
\tilde{H} = \begin{pmatrix} Q \sigma^3 \partial_\rho + \rho^{-1} \left( i\partial_\varphi + \epsilon_q \bar{\varphi} \right) + m_e \sigma^3 & -i\sigma^3 \partial_z \\ -i\sigma^3 \partial_z & Q \sigma^3 \partial_\rho + \rho^{-1} \left( i\partial_\varphi + \epsilon_q \bar{\varphi} \right) - m_e \sigma^3 \end{pmatrix},
$$

$$
\epsilon_q \bar{\varphi} = \epsilon(\phi_0 + \mu + \gamma \rho^2/2), \quad \epsilon_B \varphi = \phi_0 + \mu, \quad \phi_0 = [\epsilon_B \varphi], \quad 0 \leq \mu < 1, \quad \gamma = e |B| > 0,
$$
$$
Q = \sigma^1 \sin \varphi - \sigma^2 \cos \varphi = -\text{iantidiag}(e^{i\varphi}, -e^{-i\varphi}) = -i \begin{pmatrix} 0 & -e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}, \quad Q^2 = 1.
$$

The formal Dirac Hamiltonian commutes with the s.a. differential operations

$$
\tilde{p}_z = -i\partial_z, \quad \tilde{S}_z = \gamma^5 \left( \gamma^3 - m_e^{-1} \tilde{p}_z \right),
$$
$$
\tilde{J}_z = \tilde{L}_z + \frac{1}{2} \Sigma^3 = -i\partial_\varphi + \frac{1}{2} \Sigma^3 = \text{diag}(\tilde{j}_z, j_z), \quad \tilde{j}_z = -i\partial_\varphi + \sigma^3/2.
$$

We pass to the $p_z$ representation for four-spinors, $\Psi(r) \rightarrow \tilde{\Psi}(p_z, \rho)$,

$$
\Psi(r) = \frac{1}{\sqrt{2\pi}} \int e^{ip_z z} \tilde{\Psi}(p_z, \rho) dp_z, \quad \tilde{\Psi}(p_z, \rho) = \frac{1}{\sqrt{2\pi}} \int e^{-ip_z z} \Psi(x) dz.
$$

In this representation, the operation $\tilde{J}_z$ is the same, while the differential operation $\tilde{H}$ and the operation $\tilde{S}_z$ respectively become

$$
\tilde{H} \rightarrow \tilde{H}(p_z) = \begin{pmatrix} Q \sigma^3 \partial_\rho + \rho^{-1} \left( i\partial_\varphi + \epsilon_q \bar{\varphi} \right) + m_e \sigma^3 \sigma^3 p_z & \sigma^3 p_z \\ \sigma^3 p_z & Q \sigma^3 \partial_\rho + \rho^{-1} \left( i\partial_\varphi + \epsilon_q \bar{\varphi} \right) - m_e \sigma^3 \end{pmatrix},
$$
$$
\tilde{S}_z \rightarrow \tilde{S}_z(p_z) = \begin{pmatrix} I & m_e^{-1} p_z \\ m_e^{-1} p_z & -I \end{pmatrix}.
$$

We decompose the four-spinor $\tilde{\Psi}(p_z, \rho)$ for a fixed $p_z$ into two orthogonal components that are the eigenvectors for the spin matrix $\tilde{S}_z(p_z)$:

4By the spatial symmetry, we mean the invariance under rotations around the solenoid axis and under the translations along this axis.
The space of four-spinors $\tilde{\Psi}(p_z, \rho)$ with fixed $p_z$ is the direct orthogonal sum of two eigenspaces of $\hat{S}_z(p_z)$,

$$\tilde{\Psi}(p_z, \rho) = \tilde{\Psi}_1(p_z, \rho) + \tilde{\Psi}_{-1}(p_z, \rho),$$

where

$$\hat{S}_z(p_z) \tilde{\Psi}_s(p_z, \rho) = s \frac{M}{m_e} \tilde{\Psi}_s(p_z, \rho), \quad M = \sqrt{m^2 + p_z^2}, \quad s = \pm 1,$$

$$\tilde{\Psi}_1(p_z, \rho) = \left( \frac{M + m_e}{2M} \right)^{1/2} \left( \begin{array}{c} \chi_1 \\ p_z (m_e + M)^{-1} \chi_1 \end{array} \right) = \chi_1(p_z, \rho) \otimes e_1(p_z),$$

$$\tilde{\Psi}_{-1}(p_z, \rho) = \left( \frac{M + m_e}{2M} \right)^{1/2} \left( \begin{array}{c} -p_z (m_e + M)^{-1} \chi_{-1} \\ \chi_{-1} \end{array} \right) = \chi_{-1}(p_z, \rho) \otimes e_{-1}(p_z),$$

$e_{s}(p_z), s = \pm 1,$ are two orthonormalized two-spinors,

$$e_1(p_z) = \left( \begin{array}{c} (2M)^{-1/2} (m_e + M)^{1/2} \\ p_z [2M (m_e + M)]^{-1/2} \end{array} \right), \quad e_{-1}(p_z) = -i \sigma^2 e_1(p_z),$$

and $\chi_{s}(p_z, \rho)$ are some doublets. We thus obtain a one-to-one correspondence between four-spinors $\Psi(r)$ and pairs of doublets $\chi_{s}(p_z, \rho)$,

$$\Psi(r) \leftrightarrow \tilde{\Psi}_{s}(p_z, \rho) \leftrightarrow \chi_{s}(p_z, \rho),$$

such that $||\Psi||^2 = \sum_s ||\chi_{s}||^2 = \sum_s \int dp_z d\rho \chi_{s}^* (p_z, \rho) \chi_{s}(p_z, \rho)$.

The differential operations $\hat{H}$ and $\hat{J}_z$ induce the differential operations $\hat{h}$ and $\hat{j}_z$ in the space of doublets $\chi_{s}(p_z, \rho)$:

$$\hat{H} (p_z) \tilde{\Psi}_s = \tilde{\Psi}_s (s, p_z) \chi_{s} \otimes e_{s}, \quad \hat{J}_z (p_z) \tilde{\Psi}_s = j_z \chi_{s} \otimes e_{s},$$

$$\hat{h} (s, p_z) = Q \left[ \sigma^3 \partial_\rho + \rho^{-1} \left( i \partial_\varphi + \epsilon_q \dot{\varphi} \right) \right] + sM \sigma^3.$$

The s.a. operator $\hat{j}_z$ associated with the differential operation $\hat{j}_z$ has a discrete spectrum, its eigenvalues are all half-integers labelled here by integers $l$ as $\epsilon (\phi_0 - l + 1/2)$. It is convenient to represent vectors $\xi_l(\varphi) \equiv \xi_l(p_z, \rho, \varphi)$ of the corresponding eigenspaces,

$$\hat{j}_z \xi_l(\varphi) = [\epsilon (\phi_0 - l + 1/2)] \xi_l(\varphi), \quad l \in \mathbb{Z},$$

as

$$\xi_l(\varphi) = (2\pi)^{-1/2} e^{i (\phi_0 - l + 1/2) - \sigma^3 \phi} \vartheta_l = S_l(\varphi) \frac{1}{\sqrt{2\pi \rho}} F(l, p_z, \rho),$$

$$S_l(\varphi) = e^{i (\phi_0 - l + 1/2) \varphi} \text{antidiag} \left( i e^{i \varphi/2}, -e^{-i \varphi/2} \right) \otimes S_l^+(\varphi), \quad S_l(\varphi) S_l(\varphi) = I,$$

(48) where $\vartheta_l = \vartheta_l(p_z, \rho)$ and $F(l, p_z, \rho)$ are arbitrary doublets independent of $\varphi$.

The space of each of doublets $\chi_{s}(p_z, \rho)$ is a direct orthogonal sum of the eigenspaces of the operator $\hat{j}_z$, which means that the doublets allows the representations

$$\chi_{s}(p_z, \rho) = \sum_{l \in \mathbb{Z}} \frac{1}{\sqrt{2\pi \rho}} S_l(\varphi) F(s, l, p_z, \rho),$$

where
the factor $1/\sqrt{2\pi\rho}$ is introduced for further convenience.

The operation $\hat{h}(s,p_\perp)$ induces an operation $\hat{h}(s,l)$ ("radial Hamiltonian" depending on parameter $p_\perp$ as well) in the space of doublets $F$,

$$\hat{h}(s,p_\perp) \chi_s = \sum_{l \in \mathbb{Z}} \frac{1}{\sqrt{2\pi\rho}}S_l(\varphi)\hat{h}(s,l) F(s,l,p_\perp; \rho),$$

$$\hat{h}(s,l) = i\sigma^2 \partial_\rho + \epsilon(\gamma\rho/2 + \rho^{-1}\varkappa)\sigma^1 - sM\sigma^3,$$  \hspace{1cm} (49)

where $\varkappa = l + \mu - 1/2$.

In the Hilbert space $L^2(\mathbb{R}_+) = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$ of doublets $F(\rho)$ (with $p_\perp$ fixed), we define the initial symmetric radial Hamiltonian $\hat{h}(s,l)$ associated with the s.a. differential operation $\hat{h}(s,l)$ by

$$\hat{h}(s,l) = \begin{cases} 
D_{h(s,l)} = \mathcal{D}(\mathbb{R}_+) = \mathcal{D}(\mathbb{R}_+) \oplus \mathcal{D}(\mathbb{R}_+), \\
\hat{h}(s,l) F(\rho) = \hat{h}(s,l) F(\rho).
\end{cases}$$  \hspace{1cm} (50)

### 3.2 Solutions of radial equations

I. We first consider the homogeneous equation

$$\left[\hat{h}(s,l) - W\right] F(\rho) = 0$$  \hspace{1cm} (51)

and some of its useful solutions.

We let $f$ and $g$ denote the respective upper and lower components of doublets $F$, $F = (f/g)$. Then eq.(51) is equivalent to the system of radial equations for the doublet components

$$f' - \epsilon(\gamma\rho/2 + \rho^{-1}\varkappa)f + (W - sM)g = 0,$$

$$g' + \epsilon(\gamma\rho/2 + \rho^{-1}\varkappa)g - (W + sM)f = 0,$$  \hspace{1cm} (52)

where the prime denotes the derivatives with respect to $\rho$.

We start with the case of $\epsilon = 1$.

The system (52) can be reduced to second-order differential equations both for $f$ and $g$. For example, we have the following system equivalent to (52):

$$f'' - \left((\gamma\rho/2)^2 + \frac{\varkappa(\varkappa - 1)}{\rho^2} - w + \gamma\left(\varkappa + \frac{1}{2}\right)\right) f = 0,$$

$$g = (W - sM)^{-1} \left[-f' + (\gamma\rho/2 + \rho^{-1}\varkappa)f\right], \hspace{1cm} w = W^2 - M^2.$$  \hspace{1cm} (53)

By the substitution

$$f(\rho) = z^{a/2}e^{-z/2}p(z), \hspace{0.5cm} z = \gamma\rho^2/2, \hspace{0.5cm} a = 1/2 \pm (\varkappa - 1/2),$$

we reduce the first equation (53) to the equation for $p(z)$ that is the equation for confluent hypergeometric functions,

$$z\partial_z^2 p + (\beta - z)\partial_z p - \alpha p = 0, \hspace{0.5cm} \beta = a + 1/2,$$

$$\alpha = a/2 + \varkappa/2 + 1/2 - w/2\gamma.$$  \hspace{1cm} (54)
Known solutions of eq. (54) allows obtaining solutions of eqs. (51).

In what follows, we use the following solutions of equation (51):

\[ F_1 = \rho^{1/2 - l - \mu} e^{-z/2} \left( - (2\beta_1)^{-1} (W - sM) \rho \Phi(\alpha_1 + 1, \beta_1 + 1; z) \right), \]

\[ F_2 = \rho^{l + \mu - 1/2} e^{-z/2} \left( \Phi(\alpha_2, \beta_2; z) (2\beta_2)^{-1} (W + sM) \rho \Phi(\alpha_2, \beta_2 + 1; z) \right), \]

\[ F_3 = \rho^{1/2 - l - \mu} e^{-z/2} \left( 2^{-1}(W - sM) \rho \Psi(\alpha_1 + 1, \beta_1 + 1; z) \right) = \omega_2 F_1 - \omega_1 F_2, \] 

(55)

where

\[ \beta_1 = 1 - l - \mu, \quad \alpha_1 = -w/2\gamma, \quad \beta_2 = l + \mu, \quad \alpha_2 = l + \mu - w/2\gamma, \]

\[ \omega_1 = \omega_1(s, W) = \frac{2(\gamma/2)^l \Gamma(\beta_1)}{(W + sM)\Gamma(\alpha_1)}, \quad \omega_2 = \omega_2(W) = \frac{\Gamma(\beta_2)}{\Gamma(\alpha_2)}. \]

All the solutions \( F_1, F_2, \) and \( F_3 \) are real-entire in \( W \).

The solutions (55) have the following asymptotic behavior at the origin and at infinity. As \( \rho \to 0 \), we have:

\[ F_1 = \rho^{1/2 - l - \mu} \left( - (2\beta_1)^{-1} (W - sM) \rho \right) \left( 1 + O(\rho^2) \right), \]

\[ F_2 = \rho^{l + \mu - 1/2} \left( (2\beta_2)^{-1} (W + sM) \rho \right) \left( 1 + O(\rho^2) \right), \]

\[ f_3 = \frac{(W - sM)\Gamma(\beta_1)}{2(\gamma/2)^l \Gamma(\alpha_1 + 1)} \rho^{l + \mu - 1/2} \times \begin{cases} (1 + O(\rho^2)), & l \leq -1 \\ (1 + O(\rho^{2-\mu})), & l = 0, \mu > 0 \\ (1 + O(\rho^2 \ln \rho)), & l = 0, \mu = 0 \end{cases}, \]

\[ g_3 = \frac{\Gamma(\beta_2)}{\Gamma(\alpha_2)} \rho^{1/2 - l - \mu} \times \begin{cases} (1 + O(\rho^2)), & l \geq 1 \\ (1 + O(\rho^{2\mu})), & l = 0, \mu > 0 \end{cases}. \] 

(56)

where \( F_3 = (f_3/g_3) \).

As \( \rho \to \infty \), we have:

\[ F_1 = \frac{(\gamma/2)^{\alpha_1 - \beta_1} \Gamma(\beta_1)}{\Gamma(\alpha_1)} \rho^{-\alpha_1 + 2\alpha_1 - 2\beta_1} e^{z/2} \left( \gamma \rho (W + sM)^{-1} / 1 \right) \left( 1 + O(\rho^{-2}) \right), \]

\[ F_2 = \frac{(\gamma/2)^{\alpha_1} \Gamma(\beta_2)}{\Gamma(\alpha_2)} \rho^{\alpha_1 - 2\alpha_1} e^{z/2} \left( (\gamma \rho)^{-1} (W + sM) \right) \left( 1 + O(\rho^{-2}) \right), \]

\[ F_3 = (\gamma/2)^{-\alpha_1} \rho^{\alpha_1 - 2\alpha_1} e^{-z/2} \left( (\gamma \rho)^{-1} (W - sM) / 1 \right) \left( 1 + O(\rho^{-2}) \right). \]

We define the Wronskian \( \text{Wr}(F, \tilde{F}) \) of two doublets \( F = (f/g) \) and \( \tilde{F} = (\tilde{f}/\tilde{g}) \) by

\[ \text{Wr}(F, \tilde{F}) = f\tilde{g} - g\tilde{f} = iF\sigma^2 \tilde{F}. \]
If \((\tilde{h} - W)F = (\tilde{h} - W)\tilde{F} = 0\), then \(\text{Wr}(F, \tilde{F}) = C = \text{const}\). Solutions \(F\) and \(\tilde{F}\) are linearly independent iff \(C \neq 0\). It is easy to see that \(\text{Wr}(F_1, F_2) = -1\).

If \(\text{Im} W > 0\), the solutions \(F_1, F_2,\) and \(F_4\) are pairwise linearly independent,

\[
\text{Wr}(F_1, F_3) = \omega_1(W), \quad \text{Wr}(F_2, F_3) = \omega_2(W).
\]

Taking the asymptotics of the linearly independent solutions \(F_1\) and \(F_3\) into account, we obtain that there are no square integrable solutions of eq. \((51)\) with \(\text{Im} W \neq 0\) and \(|l| \geq 1\) or \(l = 0, \mu = 0\). This implies that in these cases, the deficiency indices of \(h(s, l)\) are zero. In the case of \(l = 0, \mu > 0\), the solution \(F_3\) is square integrable, which implies that the deficiency indices of \(h(s, 0)\) are equal to \((1, 1)\).

For any \(l\) and \(\mu\), the asymptotic behavior of any solution \(F\) of eq. \((51)\) at the origin, as \(\rho \to 0\), is not more singular than \(\rho^{-|s\mu|}, F(\rho) = O(\rho^{-|s\mu|})\).

II. We now consider the inhomogeneous equation

\[(\tilde{h}(s, l) - W)F(\rho) = \Psi(\rho), \quad \forall \Psi \in L^2(\mathbb{R}_+).\]

Its general solution allows the representations

\[
F(\rho) = c_1 F_d(\rho; W) + c_2 F_3(\rho; W)
+ \omega_d^{-1} \left[ F_d(\rho; W) \int_\rho^\infty F_3(r; W)\Psi(r)dr + F_3(\rho; W) \int_0^\rho F_d(r; W)\Psi(r)dr \right],
\]

\[
\omega_d = \text{Wr}(F_d, F_3), \quad d = 1, 2, \quad l \leq 0 \text{ for } d = 1, \quad l \geq 1 \text{ for } d = 2. \quad (57)
\]

A simple estimate of the integral terms in the r.h.s. of \((57)\) using the Cauchy-Bunyakovskii inequality shows that they are bounded as \(\rho \to \infty\). It follows that \(F \in L^2(\mathbb{R}_+)\) implies \(c_1 = 0\).

For \(|s\mu| \geq 1/2\), an evaluation shows that as \(\rho \to 0\), the integral terms are of the order of \(O(\rho^{1/2})\) (up to the factor \(\ln \rho\) for \(|s\mu| = 1/2\)) . In this case, \(F \in L^2(\mathbb{R}_+)\) implies \(c_2 = 0\), and we find

\[
F(\rho) = \omega_d^{-1} \left[ F_d(\rho; W) \int_\rho^\infty F_3(r; W)\Psi(r)dr + F_3(\rho; W) \int_0^\rho F_d(r; W)\Psi(r)dr \right]. \quad (58)
\]

For \(|s\mu| \leq 1/2\), the doublet \(F_3(\rho; W)\) is square-integrable, and a solution \(F(\rho) \in L^2(\mathbb{R}_+)\) allows the representation

\[
F(\rho) = b\omega_1^{-1} F_1(\rho; W) + c_2 F_3(\rho; W)
+ \omega_1^{-1} \left[ F_3(\rho; W) \int_0^\rho F_1(r; W)\Psi(r)dr - F_1(\rho; W) \int_0^\rho F_3(r; W)\Psi(r)dr \right],
\]

\[
F(\rho) = b\omega_1^{-1} F_1(\rho; W) + c_2 F_3(\rho; W) + O(\rho^{1/2}), \quad \rho \to 0, \quad (59)
\]

where

\[
b = \int_0^\infty F_3(r; W)\Psi(r)dr.
\]

We use representations. \((57)-(59)\) to determine the Green functions for s.a. radial Hamiltonians.
3.3 S.a. radial Hamiltonians

3.3.1 Generalities

We proceed to constructing s.a. radial Hamiltonians $\hat{h}_s (s, l)$ in the Hilbert space $L^2(\mathbb{R}_+)$ of doublets as s.a. extensions of the initial symmetric radial operators $\hat{h}_s (s, l)$ [49] associated with the differential operations $\hat{h}_s (s, l)$ [49] and analyze the corresponding spectral problems.

The r.h.s. of this identity has a limit (finite or infinite) as $\rho \to \infty$ doublets as s.a. extensions of the initial symmetric radial operators $\hat{h}_s (s, l)$, therefore we cite only their domains.

We begin with the adjoint $\hat{h}_s (s, l)$ of the initial symmetric operator $\hat{h}_s (s, l)$. Its domain $D_{h^+}$ is the natural domain for $\hat{h}_s (s, l)$,

$$ D_{h^+} = D_{\hat{h}_s (s, l)}(\mathbb{R}_+) = \{ F_s(\rho) : F_s \text{ a.c. in } \mathbb{R}_+, F_s, \hat{h}_s (s, l) F_s \in L^2(\mathbb{R}_+) \}. $$

The quadratic asymmetry form $\Delta_{h^+} (F_s)$ of $\hat{h}_s (s, l)$ is expressed in terms of the local quadratic form

$$ [F_s, F_s^*(\rho)] = \bar{g}(\rho) f(\rho) - \bar{f}(\rho) g(\rho), \quad F_s = (f / g), $$

as follows:

$$ \Delta_{h^+} (F_s) = \left( F_s^* \hat{h}^+ F_s - \hat{h}^+ F_s, F_s \right) = - |F_s, F_s^* (\rho)|^2. $$

We can prove that $\lim_{\rho \to \infty} F_s(\rho) = 0$ for any $F_s \in D_{h^+}(\mathbb{R}_+)$. Indeed, because $F_s$ and $\hat{h}_s (s, l) F_s$ are square integrable at infinity, the combination

$$ F_s' - (\gamma \rho / 2) \sigma^3 F_s = -i a^2 [\hat{h}_s (s, l) F_s - (\chi_1 / \rho) \sigma^1 F_s + s M \sigma^3 F_s] $$

is also square-integrable at infinity. This implies that $f$ and $f' - (\gamma \rho / 2) f$, together with $g$ and $g' + (\gamma \rho / 2) g$, are square-integrable at infinity. We consider the identity

$$ |f(\rho)|^2 = \int_a^\rho \sqrt{\partial f(r) f(r) + \bar{f}(r) \partial f(r)} dr + \gamma \int_a^\rho r |f(r)|^2 dr + |f(a)|^2, \quad \partial = \partial_\rho - \gamma \rho / 2. $$

The r.h.s. of this identity has a limit (finite or infinite) as $\rho \to \infty$. Therefore, $|f(\rho)|$ also has a limit as $\rho \to \infty$. This limit has to be zero because $f(\rho)$ is square-integrable at infinity. In the same way, we can verify that $g(\rho) \to 0$ as $\rho \to \infty$.

To analyze the behavior of $F_s$ at the origin, we consider the relation

$$ \Psi = \hat{h}_s (s, l) F_s, \quad \Psi, F_s \in L^2(\mathbb{R}_+), $$

or

$$ f' - (\gamma \rho / 2 + \rho^{-1} \chi_2) f = -\chi_2, \quad g' + (\gamma \rho / 2 + \rho^{-1} \chi_2) g = \chi_1, $$

$$ \chi = (\chi_1 / \chi_2) = \Psi + s M \sigma^3 F_s \in L^2(\mathbb{R}_+), $$

as an equation for $F_s$ at a given $\chi$. The general solution of these equations allows the representation

$$ f(\rho) = \rho^{\chi_2} e^{-\gamma \rho^2 / 4} \left[ c_1 + \int_0^\infty r e^{-\gamma r^2 / 4} \chi_2(r) dr \right], $$

$$ g(\rho) = \rho^{-\chi_2} e^{-\gamma \rho^2 / 4} \left[ c_2 + \int_{\rho_0}^\rho r e^{\gamma r^2 / 4} \chi_1(r) dr \right]. $$

(61)
It turns out that the asymptotic behavior of the functions $f$ and $g$ at the origin crucially depends on the value of $l$. Therefore, our exposition is naturally divided into subsections related to the corresponding regions. We distinguish three regions of $l$.

### 3.3.2 First region: $\kappa_l \leq -1/2$

In this region, we have

$$l \leq \begin{cases} -1, & \mu > 0 \\ 0, & \mu = 0 \end{cases}.$$  

The representation (61) allows estimating an asymptotic behavior of doublets $F_* \in D_{h(s,l)}^* (\mathbb{R}_+)$ at the origin for the first region.

$$f(\rho) = \rho^{-|s|} e^{\gamma r^2/4} \left( \tilde{c}_1 - \int_0^\rho r |\rho| e^{-\gamma r^2/4} \chi_2(r) dr \right) = \tilde{c}_1 \rho^{-|s|} + O(\rho^{1/2}), \; \rho \to 0,$$

$$\tilde{c}_1 = c_1 + \int_0^\infty r |\rho| e^{-\gamma r^2/4} \chi_2(r) dr.$$

The condition $f \in L^2 (\mathbb{R}_+)$ implies $\tilde{c}_1 = 0$, and therefore, $f(\rho) = O(\rho^{1/2})$ as $\rho \to 0$. As to $g(\rho)$, we find

$$g(\rho) = \begin{cases} O(\rho^{1/2}), & \kappa_l < -1/2 \\ O(\rho^{1/2} \ln \rho), & \kappa_l = -1/2 \; (l = 0, \mu = 0) \end{cases}, \; \rho \to 0.$$

We thus obtain that $F_* (\rho) \to 0$ as $\rho \to 0$, which implies that $\Delta_{h^+} (F_*) = 0, \forall F_* \in D_{h(s,l)}^* (\mathbb{R}_+)$.

This means that the deficiency indices of each of the symmetric operators $\hat{h} (s,l)$ in the first region are zero. Therefore, there exists only one s.a. extension $\hat{h}_e (s,l,p_z) = \hat{h}_{(1)} (s,l) = \hat{h}^+ (s,l)$ of $\hat{h} (s,l)$, i.e., a unique s.a. radial Hamiltonian with given $s$ and $l$, its domain is the natural domain, $D_{\hat{h}_{(1)}(s,l)} = D_{\hat{h}(s,l)}^* (\mathbb{R}_+)$.

The representation (58) with $d = 1$ implies that the Green function for the s.a. Hamiltonian $\hat{h}_{(1)} (s,l)$ is given by

$$G(\rho, \rho'; W) = \frac{1}{\omega_1(W)} \left\{ \begin{array}{l} F_3(\rho; W) \otimes F_1(\rho'; W), \; \rho > \rho' \\ F_1(\rho; W) \otimes F_3(\rho'; W), \; \rho < \rho' \end{array} \right.$$

Unfortunately, we can not use representation (55) for $F_3$ as a sum of two terms directly for all values of $\mu$ because the both are singular at $\mu = 0$ (although the sum is not).

To cover the total range of $\mu$, we use another representation for $F_3$.

We let $F_{d}(\rho; W)$ denote the functions $F_d(\rho; W)$, $d = 1, 2, 3$, with a fixed $l$ and represent $F_{3l}$ as

$$F_{3l} = \omega_1 [A_{l1} F_{l1} + F_{4l}], \; A_{l1} = A_{l1}(W) = \Omega_1(W) - \Gamma(\beta_2) P_{l1}(W),$$

$$F_{4l} = F_{4l}(\rho; W) = \Gamma(\beta_2) P_{l1}(W) F_{1l}(\rho; W) - F_{2l}(\rho; W), \; \Omega_1(W) = \frac{\omega_2(W)}{\omega_1(W)},$$

$$P_{l1}(W) = \frac{(W + s M)(\gamma/2)^l |l|^l \Gamma(\alpha_1)}{2 |l|! \Gamma(\alpha_1 - |l|)}.$$
Using the relation (see [24])

\[
\lim_{\beta \to -n} \frac{1}{\Gamma(\beta)} \Phi(\alpha, \beta; x) = \frac{x^{n+1} \Gamma(\alpha + n + 1)}{(n + 1)! \Gamma(\alpha)} \Phi(\alpha + n + 1, n + 2; x),
\]

we can verify that

\[
\Gamma^{-1}(\beta_2) F_{2\mu}(\rho; W)|_{\mu=0} = P_{1\mu}(W) F_{1\mu}(\rho; W)|_{\mu=0}.
\]

Taking the latter relation into account, it is easy to see that in the first region, \(A_{1\mu}\) and \(F_{4\mu}\) are finite for \(\mu \geq 0\), as well as \(\omega_1\) and \(F_{1\mu}\), and also that \(P_{1\mu}(E)\) and \(F_{4\mu}(\rho; E)\) are real.

The Green function is then represented as

\[
G(\rho, \rho'; W) = A_{1\mu}(W) F_{1\mu}(\rho; W) \otimes F_{1\mu}(\rho'; W) + \left\{ \begin{array}{ll}
F_{4\mu}(\rho; W) \otimes F_{1\mu}(\rho'; W), & \rho > \rho' \\
F_{1\mu}(\rho; W) \otimes F_{4\mu}(\rho'; W), & \rho < \rho',
\end{array} \right.
\]

for all \(\mu \geq 0\).

We choose the guiding functional \(\Phi_1(F; W)\) for the s.a. operator \(\hat{h}_{(1)} (s, l)\) in the form

\[
\Phi_1(F; W) = \int_0^\infty F_1(\rho; W) F(\rho), \quad F(\rho) \in \mathbb{D} = D_\sigma(\mathbb{R}_+) \cap D_{\hat{h}_{(1)}(s, l)}.
\]

It is easy to prove that the guiding functional is simple. It follows that the spectrum of \(\hat{h}_{(1)} (s, l)\) is simple.

Using representation [64] for the Green function, we obtain that the derivative \(\sigma'(E) = [\pi F_1^2(\rho; W)]^{-1} \text{Im} G(\rho, \rho; E + i0)\) of the spectral function is given by

\[
\sigma'(E) = \pi^{-1} \text{Im} A_{1\mu}(E + i0).
\]

It is easy to prove that \(\text{Im} A_{1\mu}(E + i0)\) is continuous in \(\mu\) for \(\mu \geq 0\), such that it is sufficient to find \(\sigma'(E)\) only for the case of \(\mu > 0\) where eq. (65) is more simple,

\[
\sigma'(E) = \left. \frac{(W + sM) (\gamma/2)^{-\beta_2} \Gamma(\beta_2)}{2\pi \Gamma(\beta_1) \Gamma(\alpha_2)} \right|_{W=E} \text{Im} \Gamma(\alpha_1)|_{W=E+i0}.
\]

It is easy to see that \(\sigma'(E)\) may differ from zero only at the points \(E_k\) defined by the relation \(\alpha_1 = -k (\Gamma(\alpha_1) = \infty)\), or \(M^2 - E_k^2 = -2\gamma k\), which yields

\[
E_k = \pm M_k, \quad M_k = \sqrt{M^2 + 2\gamma k}, \quad M_0 = M, \quad k \in \mathbb{Z}_+.
\]

The presence of the factor \((E + sM)\) in the r.h.s. of (66) implies that the points \(E = -sM = -sM_0\) do not belong to the spectrum of \(\hat{h}_{(1)} (s, l)\). In what follows it is convenient to change the numeration of the spectrum points. Introduce an index \(n(s)\):

\[
n(s) \in \mathcal{Z}(s) = \{n_\sigma(s)\}, \quad \sigma = \pm, \quad n_+(s) \in \left\{ \begin{array}{ll}
\mathbb{Z}_+, & s = 1 \\
\mathbb{N}, & s = -1
\end{array} \right., \quad n_-(s) \in \left\{ \begin{array}{ll}
-s, & s = 1 \\
\mathbb{Z}_-, & s = -1
\end{array} \right..
\]
Then we have
\[ E_k = \pm M_k \implies E_{n(s)} = \sigma M_{[n(s)]}, \ n(s) \in \mathcal{Z}(s). \]

Finally, we obtain
\[
\sigma'(E) = \sum_{n(s) \in \mathcal{Z}(s)} \frac{Q_{n(s)}^2 \delta(E - E_{n(s)})}{\sqrt{2(\gamma/2)^{\beta_1}} \Gamma(\beta_1 + |n(s)|) \left(1 + sM^2E_k^{-1}\right)}, \quad \beta_1 = 1 + |l| - \mu.
\]

Thus, the spectrum of the s.a. Hamiltonian \( \hat{h}_{(1)}(s, l) \) is simple and discrete, \( \spec \hat{h}_{(1)}(s, l) = \{ E_{n(s)}, \ n(s) \in \mathcal{Z}(s) \} \).

The eigenvectors
\[
U_{n(s)}^I = U_{n(s)}^I(s, l, p_z; \rho) = Q_{n(s)}F_1(\rho; E_{n(s)}), \ n(s) \in \mathcal{Z}(s),
\]

of the Hamiltonian \( \hat{h}_{(1)}(s, l) \) form a complete orthonormalized set in the space \( L^2(\mathbb{R}_+) \) of doublets \( F(\rho) \).

### 3.3.3 Second region: \( \nu_1 \geq 1/2 \)

In this region, we have \( l \geq 1 \).

The representation \( [61] \) yields the following estimates for an asymptotic behavior of doublets \( F_s \in D_{\hat{h}_{(s,l)}}^*(\mathbb{R}_+) \) at the origin for the second region:

\[
\begin{align*}
f(\rho) &= \begin{cases} O(\rho^{1/2}), & \nu_1 > 1/2 \\ O(\rho^{1/2} \ln \rho), & \nu_1 = 1/2 \quad , \rho \to 0. \end{cases} \\
g(\rho) &= O(\rho^{1/2})
\end{align*}
\]

It follows that \( F_s(\rho) \to 0 \) as \( \rho \to 0 \), which implies that \( \Delta_{h^+}(F_s) = 0, \forall F_s \in D_{\hat{h}_{(s,l)}}^*(\mathbb{R}_+) \).

This means that the deficiency indices of each of the symmetric operators \( \hat{h}_{(s,l)} \) in the second region are also zero. Therefore, there exists only one s.a. extension \( \hat{h}_s(s, l, p_z) = \hat{h}_{(2)}(s, l) = \hat{h}^+(s, l) \) of \( \hat{h}(s, l) \), i.e., a unique s.a. radial Hamiltonian with given \( s \) and \( l \), its domain is the natural domain, \( D_{\hat{h}_{(s,l)}} = D_{\hat{h}_{(s,l)}}^*(\mathbb{R}_+) \).

The representation \( [58] \) with \( d = 2 \) implies that the Green function for the s.a. Hamiltonian \( \hat{h}_{(2)}(s, l) \) is given by

\[
G(\rho, \rho'; W) = \omega_2^{-1}(W) \left\{ \begin{array}{ll}
F_{3l}(\rho; W) \otimes F_{2l}(\rho'; W), & \rho > \rho' \\
F_{2l}(\rho; W) \otimes F_{3l}(\rho'; W), & \rho < \rho'.
\end{array} \right.
\]

Again, the representation \( [55] \) for \( F_3 \) as a sum of two terms is not applicable directly for \( \mu = 0 \) We therefore use the following representation for \( F_3 \):

\[
F_{3l} = \omega_2[F_{5l} - A_{2l}F_{2l}], \quad A_{2l} = A_{2l}(W) = \Omega_2(W) + \Gamma(\beta_1)P_{2l}(W),
\]

\[
F_{5l} = F_{5l}(\rho; W) = F_{1l}(\rho; W) + \Gamma(\beta_1)P_{2l}(W)F_{2l}(\rho; W), \quad \Omega_2(W) = \frac{\omega_1(W)}{\omega_2(W)},
\]

\[
P_{2l}(W) = \frac{(W - sM)(\gamma/2)^{l-1} \Gamma(\alpha_1 + l)}{2(l - 1)! \Gamma(\alpha_1 + 1)}.
\]
Using relation (63), we can verify that
\[
\Gamma^{-1}(\beta_1)F_{1l}(\rho; W)|_{\mu=0} = - P_{2l}(W)F_{2l}(\rho; W)|_{\mu=0}
\]
Taking the latter relation into account, it is easy to see that \(A_{2l}\) and \(F_{5l}\) are finite for \(\mu \geq 0\), as well as \(\omega_2\) and \(F_{2l}\), in the second region, and \(P_{2l}(E)\) and \(F_{5l}(\rho; E)\) are real.

The Green function is then represented as
\[
G(\rho, \rho'; W) = -A_{2l}(W)F_{2l}(\rho; W) \otimes F_{2l}(\rho'; W)
\]
\[
= \begin{cases} 
  F_{5l}(\rho; W) \otimes F_{2l}(\rho'; W), & \rho > \rho' \\
  F_{2l}(\rho; W) \otimes F_{5l}(\rho'; W), & \rho < \rho'
\end{cases}
\] (68)

for all \(\mu \geq 0\).

We choose the guiding functional \(\Phi_2(F; W)\) for the s.a. operator \(\hat{h}_{(2)}(s, l)\) in the form
\[
\Phi_2(F; W) = \int_0^\infty F_2(\rho; W)F(\rho), \ F(\rho) \in D = D_r(\mathbb{R}_+) \cap D_{h_{(2)}(s, l)}.
\]
It is easy to prove that the guiding functional is simple. It follows that the spectrum of \(\hat{h}_{(2)}(s, l)\) is simple.

Using representation (63) for the Green function, we obtain that the derivative \(\sigma'(E)\) of the spectral function is given by
\[
\sigma'(E) = -\pi^{-1} \text{Im} A_{2l}(E + i0).
\] (69)

It is easy to prove that \(\text{Im} A_{2l}(E + i0)\) is continuous in \(\mu\) for \(\mu \geq 0\), such that it is sufficient to find \(\sigma'(E)\) only for the case of \(\mu > 0\) where eq. (69) is more simple,
\[
\sigma'(E) = \frac{(W - sM) (\gamma/2)^{\beta_2} \Gamma(\beta_1)}{\pi \Gamma(\beta_2) \Gamma(1 + \alpha_1)} \left| \text{Im} \Gamma(\alpha_2) \right|_{E=E+i0}.
\]

It is easy to see that \(\sigma'(E)\) may differ from zero only at the points \(E_k\) defined by the relation \(\alpha_2 = -\sqrt{M^2 - 2\gamma(l + \mu)} = -2\gamma k, \ k \in \mathbb{Z}_+\),

which yields
\[
E_k = \pm \sqrt{M^2 + 2\gamma(k + l + \mu)} = \pm M_{k+l+\mu}, \ k \in \mathbb{Z}_+.
\]

All the points \(E_k\) are the spectrum points.

It is convenient to change indexing \(k\) for \(n(s)\),
\[
E_k \rightarrow E_{n(s)} = \sigma M_{|n(s)|+\mu}; \ \{n(s) \in \mathbb{Z}(s), \ |n(s)| \geq l\} \ (n_\sigma(s) = \sigma(k + l), \ k \in \mathbb{Z}_+)\).

We finally obtain that
\[
\sigma'(E) = \sum_{n \in \mathbb{Z}, |n| \geq l} Q_{n(s)}^2 \delta(E - E_n),
\]
\[
Q_{n(s)} = \sqrt{\frac{(\gamma/2)^{\mu} \Gamma(|n(s)| + \mu)(1 - sME_{n(s)}^{-1})}{(|n(s)| - l)!\Gamma^2(l + \mu)}}.
\]
So, the spectrum of the s.a. Hamiltonian $\hat{h}_{(2)}(s,l)$ is simple and discrete, $\text{spec} \hat{h}_{(2)}(s,l) = \{ E_{n(s)} : n(s) \in \mathcal{Z}, |n(s)| \geq l \}$. The eigenvectors

$$U^{II}_{n(s)}(s,l,p;\rho) = Q_{n(s)}F_{2}(\rho;E_{n(s)}), \quad n(s) \in \mathcal{Z}(s),$$

(70)

of the Hamiltonian $\hat{h}_{(2)}(s,l)$ form a complete orthonormalized set in the space $L^{2}(\mathbb{R}_{+})$ of doublets $F(\rho)$.

### 3.3.4 Third region: $|\varkappa| < 1/2$

In this region, we have $l = l_{0} = 0$, and $\varkappa$ reduces to $\varkappa_{0} = \mu - 1/2$, $\mu > 0$.

The representation (61) yields the following asymptotic behavior of doublets $F_{s} \in D^{*}_{h(s,l_{0})}(\mathbb{R}_{+})$ at the origin:

$$\left\{ \begin{array}{l}
    f(\rho) = c_{1}(m_{e}\rho)^{\varkappa_{0}} + O(\rho^{1/2}), \rho \to 0, \\
    g(\rho) = c_{2}(m_{e}\rho)^{-\varkappa_{0}} + O(\rho^{1/2}), \rho \to 0.
\end{array} \right.$$  

It follows that

$$\Delta_{h+}(F_{s}) = c_{2}c_{1} - c_{1}c_{2}, \quad \forall F_{s} \in D^{*}_{h(s,l_{0})}(\mathbb{R}_{+}).$$

Up to the factor $i$, the r.h.s. is a quadratic form in $c_{1}$ and $c_{2}$ with the inertia indices $(1,1)$, which implies that the deficiency indices of the initial symmetric operator $\hat{h}(s,l_{0})$ are $(1,1)$.

The additional asymptotic boundary conditions

$$F(\rho) = c \left( \frac{(m_{e}\rho)^{\varkappa_{0}} \cos \lambda}{(m_{e}\rho)^{-\varkappa_{0}} \sin \lambda} \right) + O(\rho^{1/2}), \rho \to 0,$$

(71)

with a fixed $\lambda \in \mathbb{S}(-\pi/2, \pi/2)$ (note that $\lambda$ depend on $s$ and $p_{z}$), define a maximum subspace in $D^{*}_{h(s,l_{0})}(\mathbb{R}_{+})$ where $\Delta_{h+} = 0$. This subspace is the domain of a s.a. operator that is a s.a. extension of $\hat{h}(s,l_{0})$.

We thus obtain that there exists a one-parameter $U(1)$ family of s.a. radial Hamiltonians $\hat{h}_{\lambda}(s,l_{0},p_{z}) = \hat{h}_{\lambda}(s,l_{0})$ parametrized by the real parameter $\lambda \in \mathbb{S}(-\pi/2, \pi/2)$. These Hamiltonians are specified by asymptotic s.a. boundary conditions (71), and their domains are given by

$$D_{h_{\lambda}(s,l_{0})} = \left\{ F(\rho) : F(\rho) \in D^{*}_{h_{\lambda}(s,l_{0})}(\mathbb{R}_{+}), \ F \text{ satisfies (71)} \right\}.$$  

(72)

According to representation (52), which certainly holds for the doublets $F$ belonging to $D_{h_{\lambda}(s,l_{0})}$, and (56), the asymptotic behavior of $F$ at the origin is given by

$$F = \left( -\frac{c_{2}\omega_{1}\rho^{\varkappa_{0}}}{(b\omega_{1}^{-1} + c_{2}\omega_{2})} \rho^{-\varkappa_{0}} \right) + O(\rho^{1/2}), \rho \to 0.$$

On the other hand, $F$ satisfies boundary conditions (71), whence it follows that there must be

$$c_{2} = -\frac{b\cos \lambda}{\omega_{1}\omega(\lambda)}, \quad \omega(\lambda) = \omega_{2} \cos \lambda + m_{e}^{-2\varkappa_{0}}\omega_{1} \sin \lambda.$$  

(73)
It is easy to prove that the guiding functional is simple. It follows that the spectrum of \( F \) of doublets of the spectral function is given by

\[
G(\rho, \rho'; W) = \Omega^{-1}(W)F(\rho; W) \otimes F(\rho'; W)
\]

\[
+ \begin{cases} 
\tilde{F}(\rho; W) \otimes F(\rho'; W), & \rho > \rho' \\
F(\rho; W) \otimes \tilde{F}(\rho'; W), & \rho < \rho'
\end{cases}
\]

(74)

where

\[
F(\rho; W) = m_e^{-\rho_\lambda}F_1(\rho; W) \sin \lambda + m_e^{\rho_\lambda}F_2(\rho; W) \cos \lambda,
\]

\[
\tilde{F}(\rho; W) = m_e^{-\rho_\lambda}F_1(\rho; W) \cos \lambda - m_e^{\rho_\lambda}F_2(\rho; W) \sin \lambda,
\]

\[
\Omega(W) = \frac{\omega_0(\lambda)(W)}{\tilde{\omega}_0(\lambda)(W)}, \quad m_e^{\rho_\lambda} = \tilde{\omega}_0(\lambda)F(\lambda) + \omega_0(\lambda)\tilde{F}(\lambda),
\]

\[
\tilde{\omega}_0(\lambda)(W) = \omega_2 \sin \lambda - m_e^{-2\rho_\lambda}\omega_1 \cos \lambda.
\]

We choose the guiding functional \( \Phi_\lambda(F; W) \) for the s.a. operator \( \hat{h}_\lambda(s, l_0) \) in the form

\[
\Phi_\lambda(F; W) = \int_0^\infty F(\rho; W)F(\rho), \quad F(\rho) \in \mathbb{D} = D_r(\mathbb{R}_+) \cap D_{\lambda}(s, l_0).
\]

It is easy to prove that the guiding functional is simple. It follows that the spectrum of \( \hat{h}_\lambda(s, l) \) is simple.

Using the representation (74) for the Green function, we obtain that the derivative \( \sigma'(E) \) of the spectral function is given by

\[
\sigma'(E) = \pi^{-1} \text{Im } \Omega^{-1}(E + i0).
\]

Because \( \Omega(E) \) is real, \( \sigma'(E) \) differs from zero only at the zero points \( E_k \) of the function \( \Omega(E), \Omega(E_k) = 0 \), and we find

\[
\sigma'(E) = \sum_k Q_k^2 \delta(E - E_k), \quad Q_k = [-\Omega'(E)]^{-1/2},
\]

\[
\text{spec } \hat{h}_\lambda(s, l_0) = \{ E_k \}, \quad k \in \mathbb{Z}.
\]

The eigenvectors

\[
\hat{U}_k = \hat{U}_k(\lambda, s, p_z; \rho) = Q_kF(\lambda)(\rho; E_k), \quad k \in \mathbb{Z},
\]

(75)

of the Hamiltonian \( \hat{h}_\lambda(s, l_0) \) form a complete and orthonormalized set in the space \( L^2(\mathbb{R}_+) \) of doublets \( F(\rho) \).

For \( \lambda = 0 \) and \( \lambda = \pm \pi/2 \), we can evaluate the spectrum explicitly.

a) Let \( \lambda = 0 \), and we consider the s.a. Hamiltonian \( \hat{h}_0(s, l_0) \). We have

\[
F(0)(\rho; W) = m_e^{\rho_0}F_2(\rho; W), \quad \Omega(W) = -\frac{m_e^{\rho_0}\omega_2(W)}{\omega_1(W)}
\]
and find
\[
\sigma'(E) = - \frac{2m_e^{-2\sigma_0} (\gamma/2)^\beta_2 \Gamma(\beta_1)}{\pi (W + sM) \Gamma(\alpha_1) \Gamma(\beta_2)} \left. \frac{\Im \Gamma(\alpha_2)|_{W=E}}{|W=E+i0}. \right|
\]

As in the second region, \(\sigma'(E)\) differs from zero only at the points \(E_k\), defined by the relation \(\alpha_2 = -k \ (\Gamma(\alpha_2) = \infty)\) or
\[
M^2 - E_k^2 + 2\gamma + \mu = -2\gamma k, \ k \in \mathbb{Z}_+,
\]
which yields
\[
E_k = \pm M_{k+\mu}, \ k \in \mathbb{Z}_+.
\]
All the points \(E_k\) are the spectrum points.

It is convenient to introduce an index \(n\),
\[
n \in \mathcal{Z} = \{ n_{\sigma} \in \bar{\sigma} \mathbb{Z}_+, \ \bar{\sigma} = \pm \},
\]
where \(n_+ = 0\) and \(n_- = 0\) are considered different elements of the set \(\mathcal{Z}\). Then we can write
\[
E_k \mapsto E_n = \bar{\sigma} M_{|n|+\mu}, \ n \in \mathcal{Z}.
\] (76)

We finally obtain that
\[
\sigma'(E) = \sum_{n \in \mathcal{Z}} m_e^{-2\sigma_0} Q_n^2 \delta(E - E_n), \ Q_n = \sqrt{\frac{(\gamma/2)^\mu \Gamma(|n|+\mu)(1 - sM E_n^{-1})}{|n|! \Gamma^2(\mu)}}.
\]

So, the spectrum of the s.a. Hamiltonians \(\hat{h}_0 (s, l_0)\) is simple and discrete, \(\text{spec} \hat{h}_0 (s, l_0) = \{E_n, \ n \in \mathcal{Z}\}\). The eigenvectors \(U_n = U_n (0, s, l, p_+; \rho) = Q_n F_2 (\rho; E_n), \ n \in \mathcal{Z}\) of the Hamiltonian \(\hat{h}_0 (s, l_0)\) form a complete orthonormalized set in the space \(L^2(\mathbb{R}_+)\) of doublets \(F(\rho)\).

We note that the spectrum, spectral function and eigenfunctions of \(\hat{h}_0 (s, l_0)\) are obtained from the respective expressions for the second region, \(\sigma_l \geq 1/2\), by the substitution \(l = 0\). We also note that for \(\mu > 1/2\), the function \(F_{(0)} (\rho; W) = m_e^{-2\sigma_0} F_2 (\rho; W)\) is minimally singular at the origin among the functions \(F_{(\lambda)} (\rho; W)\), in fact, it is nonsingular.

b) Let \(\lambda = \pi/2\), which is equivalent to \(\lambda = -\pi/2\), and we consider the s.a. Hamiltonian \(\hat{h}_{\pi/2} (s, l_0)\). We have
\[
F_{(\pi/2)} (\rho; W) = m_e^{-2\sigma_0} F_1 (\rho; W), \ \Omega(W) = \frac{m_e^{-2\sigma_0} \omega_1(W)}{\omega_2(W)}
\]
and find
\[
\sigma'(E) = \frac{m_e^{2\sigma_0} \Gamma(\beta_2)(W + sM)}{2\pi (\gamma/2)^{\beta_2} \Gamma(\beta_1) \Gamma(\alpha_2)} \left. \frac{\Im \Gamma(\alpha_1)|_{W=E+i0}}{|W=E+i0}. \right|
\] (77)

As in the first region, \(\sigma'(E)\) differs from zero only at the points \(E_k\), defined by the relation \(\alpha_1 = -k \ (\Gamma(\alpha_1) = \infty)\), or
\[
\frac{M^2 - E_k^2}{2\gamma} = -k, \ E_k = \pm \sqrt{M^2 + 2\gamma k}, \ k \in \mathbb{Z}_+.
\]
The presence of the factor \((E + sM)\) in the r.h.s. of (77) implies that the points \(E = -sM = -sM_0\) do not belong to the spectrum of \(\hat{h}_{\pi/2}(s, l_0)\). We change the indexing of the spectrum points:

\[
E_k \rightarrow E_n = \sigma M|n|, \; n \in \mathcal{Z}(s).
\]

\[
E_k = (\text{sign}k)M|k|, \; |k| \geq 1 \; E_0 = sM ; \; k \in \mathbb{Z}.
\]

We finally obtain that

\[
\sigma'(E) = \sum_{k \in \mathbb{Z}} m_e^2 \delta_0 \gamma E - E_n, \; Q_n = \sqrt{\frac{\Gamma(|n| + 1 - \mu) \left(1 + sME^{-1}_n\right)}{(\gamma/2)^{2\eta} |n|!\Gamma^2(1 - \mu)}}.
\]

So, the spectrum of the s.a. Hamiltonian \(\hat{h}_{\pi/2}(s, l_0)\) is simple and discrete, \(\text{spec} \hat{h}_{\pi/2}(s, l_0) = \{E_n, \; n \in \mathcal{Z}(s)\}\).

The eigenvectors \(U_n = U_n(\pi/2, s, l_0, p_z; \rho) = Q_nF_1(\rho; E_n), \; n \in \mathcal{Z}(s)\), of the Hamiltonian \(\hat{h}_{\pi/2}(s, l_0)\) form a complete orthonormalized set in the space \(L^2(\mathbb{R}^3)\) of doublets \(F(\rho)\).

We note that the spectrum, spectral function, and eigenfunctions of \(\hat{h}_{\pi/2}(s, l_0)\) can be obtained from the respective expressions for the first region, \(z_\rho \leq -1/2\), by the substitution \(l = 0\). We also note that for \(\mu < 1/2\), the function \(F_{\pi/2}(\rho; W) = m_e^{-\mu_0}F_1(\rho; W)\) is minimally singular at the origin among the functions \(F_{\pi/2}(\rho; W)\); in fact, it is nonsingular.

### 3.4 Complete spectrum and inversion formulas for Dirac spinors

In the previous subsubsec., we constructed all s.a. radial Hamiltonians \(\hat{h}_\pi(s, l, p_z)\) as s.a. extensions of the symmetric operators \(\hat{h}(s, l, p_z)\) for any \(s, l\), and \(p_z\) and for any values of \(\phi_0, \mu,\), and \(\gamma\). The total s.a. operators \(\hat{H}_\pi\) associated with the Dirac differential operator \(\hat{H}\) in the Hilbert space \(\mathfrak{F}_3 = L^2(\mathbb{R}^3)\) of Dirac spinors are constructed from the sets of \(\hat{h}_\pi(s, l, p_z)\) by means of a procedure of "direct summation over \(s, l\) and direct integration over \(p_z\)."

Each set of possible s.a. radial Hamiltonians \(\hat{h}_\pi(s, l, p_z)\) generates a spatially invariant s.a. Hamiltonian \(\hat{H}_\pi\). Namely, let \(\mathbb{G}\) be the group of the above space transformations \(S: \mathbf{r} \rightarrow S\mathbf{r}\). This group is unitarily represented in \(\mathfrak{F}_3\): if \(S \in \mathbb{G}\), then the corresponding operator \(U_S\) is defined by

\[(U_S \psi)(\mathbf{r}) = e^{-i\theta\mathbf{\Sigma}/2}\psi(S^{-1}\mathbf{r}), \forall \psi \in \mathfrak{F}_3,
\]

where \(\theta\) is the rotation angle of vector \(\mathbf{\rho}\) around the z-axis. The operator \(\hat{H}\) evidently commutes with \(U_S\) for any \(S\). We search only for s.a. extensions \(\hat{H}_\pi\) of \(\hat{H}\) that also commutes with \(U_S\) for any \(S\). This condition is the explicit form of the invariance, or symmetry, of a quantum Hamiltonian under the space transformations. As in classical mechanics, this symmetry allows separating the cylindrical coordinates \(\rho, \varphi,\) and \(z\) and reducing the three-dimensional problem to a one-dimensional radial problem. Let \(V\) be the unitary operator defined by the relation

\[(V f)(\rho, \varphi, z) = \frac{1}{2\pi \sqrt{\rho}} \int_{\mathbb{R}_z} dp_z \sum_{l \in \mathbb{Z}} e^{ipz} [S_l(\varphi)F(s, l, p_z; \rho)] \otimes e_s(p_z),
\]

\(\text{i.e., invariant under rotations around the z axis and under translations along the z axis.}\)
where $S_l(\varphi)$ and $e_s(p_z)$ are given by the respective (47) and (48). Similarly to the considerations in subsec. 2 and 3 of sec. 2, it is natural to expect that any s.a. Hamiltonian $\hat{H}_\varepsilon$ can be represented in the form of the type

$$\hat{H}_\varepsilon = V \int_{\mathbb{R}_z} dp_z \sum_{s = \pm 1} \sum_{l \in \mathbb{Z}} \hat{h}_\varepsilon(s, l, p_z) V^{-1},$$

where $\hat{h}_\varepsilon(s, l, p_z)$ for fixed $s, l,$ and $p_z$ is s.a. extension of symmetric operator $\hat{h}(s, l, p_z)$,

$$\hat{h}(s, l, p_z) = \begin{cases} D_{h(s,l,p_z)} = D(\mathbb{R}_+) \subset \mathbb{L}^2(\mathbb{R}_+,dp), \\ \hat{h}(s, l, p_z) = \hat{h}(s, l, p_z)F(s, l, p_z, \rho), \forall F(s, l, p_z, \rho) \in D_{h(s,l,p_z)}, \end{cases}$$

acting in the Hilbert space $\mathbb{L}^2(\mathbb{R}_+,dp)$ of the functions $f(\rho,l,p_z)$ with the scalar product $(F(s, l, p_z), G(s, l, p_z)) = \int_{\mathbb{R}_+} F(s, l, p_z, \rho)G(s, l, p_z, \rho)dp,$ $\hat{h}(s, l, p_z)$ is given by eq. (49). A correct expression for $\hat{H}_\varepsilon$ is

$$\hat{H}_\varepsilon = V \int_{\mathbb{R}_z} dp_z \sum_{s = \pm 1} \sum_{l \in \mathbb{Z}} \hat{h}_\varepsilon(s, l, p_z) V^{-1}.$$

A detailed exposition of the procedure will be published in the short run.

The inversion formulas in the Hilbert space $\mathfrak{F}$ are correspondingly obtained from the known radial inversion formulas by a procedure of summation over $s, l$ and integration over $p_z$. And we now must consider the extension parameter $\lambda$ a function of $s$ and $p_z$, $\lambda = \lambda(s, p_z)$. In what follows, $\int dp_z$ means $\int_{-\infty}^{\infty} dp_z$.

For $\mu = 0$, there is a unique s.a. Dirac Hamiltonian $\hat{H}_\varepsilon$.

The spectrum of $\hat{H}_\varepsilon$ is

$$\text{spec} \hat{H}_\varepsilon = (-\infty, -m_e] \cup [m_e, \infty).$$

A complete set of generalized eigenfunctions of the s.a. Hamiltonian $\hat{H}_\varepsilon$ is the set

$$\{ \Psi_{s,p_z,n(s),l}(\mathbf{r}), \ n(s) \in \mathcal{Z}(s), \ l \leq |n(s)| \},$$

$$\Psi_{s,p_z,n(s),l}(\mathbf{r}) = \frac{1}{2\pi \sqrt{\rho}} e^{ip_zz} S_l(\varphi) F_{n(s)}(s, l, p_z; \rho) \otimes e_s(p_z),$$

$$F_{n(s)}(s, l, p_z; \rho) = \begin{cases} U_{n(s)}^I(s, l, p_z; \rho), \ l \leq 0, \\ U_{n(s)}^H(s, l, p_z; \rho), \ 1 \leq l \leq |n(s)|, \end{cases}$$

the doublets $U_{n(s)}^I(s, l, p_z; \rho)$ and $U_{n(s)}^H(s, l, p_z; \rho)$ are given by the respective (67) and (70), such that

$$\hat{H}\Psi_{s,p_z,n(s),l}(\mathbf{r}) = E_{s,p_z,n(s),l}\Psi_{s,p_z,n(s),l}(\mathbf{r}),$$

where

$$E_{s,p_z,n(s),l} = \sigma \sqrt{m_e^2 + p_z^2 + 2\gamma|n(s)|}, \ n(s) \in \mathcal{Z}(s), \ l \leq |n(s)|.$$
The corresponding inversion formulas are

\[ \Psi(r) = \int dp_z \sum_{s=\pm 1} \sum_{n(s) \in Z(s)} \sum_{l \leq |n(s)|} \Phi_{s,p_z,n(s),l}(r) \Psi_{s,p_z,n(s),l}(r), \]

\[ \Phi_{s,p_z,n(s),l} = \int \frac{\Psi_{s,p_z,n(s),l}(r)}{\Psi(r)} \Psi(r) dr, \]

\[ \int |\Psi(r)|^2 dr = \int dp_z \sum_{s=\pm 1} \sum_{n(s) \in Z(s)} \sum_{l \leq |n(s)|} |\Phi_{s,l,p_z,n}|^2, \forall \Psi \in L^2(\mathbb{R}^3). \]

For \( \mu > 0 \), there is a family of s.a. Dirac Hamiltonians \( \hat{H}_{\{\lambda(s,p_z)\}} \) parametrized by two real-valued functions \( \lambda(s,p_z), \lambda \in \mathbb{S}(-\pi/2, \pi/2) \), \( s = \pm 1 \).

The spectrum of \( \hat{H}_{\{\lambda(s,p_z)\}} \) is

\[ \text{spec} \hat{H}_{\{\lambda(s,p_z)\}} = (-\infty, -m_e] \cup [m_e, \infty). \]

A complete set of generalized eigenfunctions of the s.a. Hamiltonian \( \hat{H}_{\{\lambda(s,p_z)\}} \) is the set

\[ \{ \Psi_{s,p_z,n(s),l}(r), n(s) \in Z(s), l \leq |n(s)|, l \neq 0 \} \cup \{ \Psi_{s,p_z,k,l0}(r), k \in \mathbb{Z} \}, \]

\[ \Psi_{s,p_z,n(s),l}(r) = \frac{1}{2\pi \sqrt{\rho}} e^{ip_z z_s} \mathcal{S}_l(\varphi) F_{n(s)}(s,l,p_z; \rho) \otimes e_s(p_z), \]

\[ F_{n}(s,l,p_z; \rho) = \begin{cases} \hat{U}_n (s,l,p_z; \rho), & l \leq -1, \\ H \hat{U}_n (s,l,p_z; \rho), & 1 \leq l \leq |n(s)|, \end{cases} \]

\[ \Psi_{s,p_z,k,l0}^{\lambda(s,p_z)}(r) = \frac{1}{2\pi \sqrt{\rho}} e^{ip_z z_s} \mathcal{S}_{l0}(\varphi) \hat{U}_k (\lambda(s,p_z), s,p_z; \rho) \otimes e_s(p_z), \]

where \( \hat{U}_{\lambda}(\lambda(s,p_z), s,p_z; \rho) \) are given by (73) with the substitution \( \lambda \rightarrow \lambda(s,p_z) \), such that

\[ \hat{H} \Psi_{s,p_z,n(s),l}(r) = E_{s,p_z,n(s),l} \Psi_{s,p_z,n(s),l}(r), \]

where

\[ E_{s,p_z,n(s),l} = \sigma \sqrt{m_e^2 + p_z^2 + 2\gamma(|n(s)| + \theta(l))}, l \leq |n(s)|, l \neq 0, \]

\[ \theta(l) = \begin{cases} 0, & l \leq 0 \\ 1, & l \geq 1 \end{cases}, \]

and

\[ \hat{H} \Psi_{s,p_z,k,l0}^{\lambda(s,p_z)}(r) = E_{s,p_z,k,l0}^{\lambda(s,p_z)} \Psi_{s,p_z,k,l0}^{\lambda(s,p_z)}(r), \]

where

\[ E_{s,p_z,k,l0}^{\lambda(s,p_z)} : \Omega(\lambda, E_{s,p_z,k,l0}^{\lambda(s,p_z)}) = 0, \Omega(\lambda, W) = \frac{\cos \lambda + a(W) \sin \lambda}{\sin \lambda - a(W) \cos \lambda}, \]

\[ a(W) = \frac{2m_e^{2\mu-1}(\gamma/2)^{1-\mu} \Gamma(\mu) \Gamma(1-\mu-w/2\gamma)}{(W+sM) \Gamma(1-\mu) \Gamma(-w/2\gamma)}. \]
We recall that for \( \lambda(s,p_z) = 0 \) and \( \lambda(s,p_z) = \pm \pi/2 \), the eigenvalues \( E_{s,l,p_z,n}^{\lambda(s,p_z)} \) can be found explicitly, see the respective eqs. (76) and (78).

The corresponding inversion formulas are

\[
\Psi(r) = \int dp_z \sum_{s=\pm 1} \left[ \sum_{n(s) \in Z(s)} \sum_{l \leq n(s), l \neq 0} \Phi_{s,p_z,n(s),l} \Psi_{s,p_z,n(s),l}(r) + \sum_k \Phi_{s,p_z,k,l_0} \Psi_{s,p_z,k,l_0}^{\lambda(s,p_z)}(r) \right],
\]

\[
\Phi_{s,p_z,n(s),l} = \int \Psi_{s,p_z,n(s),l}(r) \Psi(r) dr, \quad l \neq 0, \quad \Phi_{s,p_z,k,l_0} = \int \Psi_{s,p_z,k,l_0}^{\lambda(s,p_z)}(r) \Psi(r) dr,
\]

\[
\int |\Psi(r)|^2 dr = \int dp_z \sum_{s=\pm 1} \left[ \sum_{n(s) \in Z(s)} \sum_{l \leq n(s), l \neq 0} \left| \Phi_{s,p_z,n(s),l} \right|^2 + \sum_k \left| \Phi_{s,p_z,k,l_0} \right|^2 \right], \quad \forall \Psi \in L^2(\mathbb{R}^3),
\]

### 3.5 The case \( \epsilon = -1 \).

We let \( \hat{h}_+ = \hat{h}_+(s,l) \) and \( \hat{h}_- = \hat{h}_-(s,l) \) denote the differential operation \( \hat{h} \) with the respective \( \epsilon = 1 \) and \( \epsilon = -1 \). We then have

\[
\hat{h}_-(s,l) = i \sigma^2 \partial_\rho - (\gamma \rho/2 + \rho^{-1} \kappa_1) \sigma^1 - sM\sigma^3 =
\]

\[
= i \sigma^2 \left[ i \sigma^2 \partial_\rho + (\gamma \rho/2 + \rho^{-1} \kappa_1) \sigma^1 + sM\sigma^3 \right] (i \sigma^2)^+ = i \sigma^2 \hat{h}_+(s,l) (i \sigma^2)^+.
\]

It follows that solutions \( F_- = F_-(s,l,E_-;\rho) \) of the equation \( (\hat{h}_- - E_-) F_- = 0 \) are bijectively related to solutions \( F_+ = F_+(s,l,E_+;\rho) \) of the equation \( (\hat{h}_+ - E_+) F_+ = 0 \) by

\[
F_-(s,l,E_-;\rho) = i \sigma^2 F_+(s,l,E_+;\rho), \quad E_- = E_+.
\]

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