Over-barrier decay of the mixed state.

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May 28, 2018

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Classical escape in 2D Hamiltonian systems with the mixed state has been studied numerically and analytically. The wide class of potentials with the mixed state is presented by polynomial potentials. In potentials, where the mixed state could be realized, i.e. the phase space contains regions of both regular and chaotic motion, escape problem has a number of new features. In particular, some local minima become a trap with number of particles depending on energy and other values that characterize the ensemble of particles. Choosing the form of initial ensemble one chooses the set of parameters that determine the number of trapped particles.

1 Introduction

Escape of the trajectories from the localized regions of phase or configuration space is an important topic in dynamics and describes the decay of metastable states in many areas of physics, as for instance chemical and nuclear reactions, atomic ionization and others. The problem has a rich history and a number of realizations in different systems. Almost a century ago Sabine \[1\] had considered the decay of sound in concert halls and later Legrand and Sornette \[2\] had shown that problem is equivalent to the escape one. Corresponding decay rate is \[\int \alpha(s) ds\] and \[\alpha(s)\] is the absorption coefficient at coordinate \[s\] of billiard boundary, \[\alpha(s) = 1\] at the opening of width \[\Delta\] and \[\alpha(s) = 0\] elsewhere.

Another application of the escape problem links to the nondestructive monitoring of the system \[3\]. If some system is connected to the surroundings only via small opening in its boundary, it became possible to understand the dynamics of the system by exploring the escaping particles. So the natural question arises: how does escape law depend on the character of the motion? For strongly chaotic systems exponential decay is expected \[1, 5, 6\]. Bauer and Bertsch \[4\] considered the escape of the particles through the small opening in the billiards boundary. When exploring the regular billiard, i.e. rectangular without the scattering center, power law emerges at long time. Qualitative understanding of the mechanism of power tails generation is given in \[7\].

For rectangular billiard with circular scattering center inside the decay of the initial ensemble of \[N(0)\] particles is exponential and by simple considerations one could obtain the corresponding decay rate:
\[ N(t) = N(0) \exp(-\alpha t), \quad \alpha = \frac{p\Delta}{\pi A_c}, \]  

(1)

here \( p \) is the particles momentum, \( \Delta \) - the width of the opening and \( A_c \) - the area of the billiard. As it would be shown further, exponential decay is a common feature of the purely chaotic systems.

2 The mixed state

Passing from billiards to the potential systems broadens the number of possible situations. One-well potential is the simplest case for considering the escape. Zhao and Du [8] have explored the escape from Henon - Heiles potential:

\[ U_{HH}(x, y) = \frac{x^2 + y^2}{2} + xy^2 - \frac{x^3}{3} \]  

(2)

At the energies \( E > 1/6 \) trajectory could leave the potential well through one of the three openings, placed symmetrically. Numerical simulation, performed by Zhao and Du, has shown that escape follows the exponential law. At over-saddle energies phase space of the Henon - Heiles Hamiltonian is almost homogeneous and motion is chaotic. Using this, the escape rate could be derived and it fits numerical results with high accuracy. This situation is similar to the escape from chaotic billiards. Another study of the escape from Henon - Heiles potential was performed by [9].

In contrast to billiards and Henon - Heiles potential, some potentials have highly inhomogeneous phase space, that consists of macroscopically significant components of both regular and chaotic motions. One wide class of systems with inhomogeneous phase space is represented by multi-well potentials. We will focus our research on the escape in such potentials. The preliminary results are presented in [11]. Regularity-chaos transition in multi-well potentials has a distinctive feature, which consists in the difference of critical energies in different local minima. This leads to the different (either regular or chaotic) regimes of motion in different local minima at the same energy, i.e. the ratio of chaotic trajectories in some local minimum significantly differs from the ratio in other minima. Such kind of the dynamics is called the mixed state [10]. It is important to mention, that critical energies lie below the saddle energy, i.e. the energy above which local minima are no more separated.

We will demonstrate the mixed state in two representative examples of 2D multi-well potentials: the lower umbilic catastrophe \( D_5 \),

\[ U_{D_5} = 2ay^2 - x^2 + xy^2 + \frac{1}{4}x^4 \]  

for \( a = 1.1 \) and the quadrupole nuclear oscillations potential (QO),

\[ U_{QO}(x, y, W) = \frac{(x^2 + y^2)}{2W} + xy^2 - \frac{1}{3}x^3 + (x^2 + y^2)^2 \]  

(4)

for \( W = 18 \). \( D_5 \) potential has two local minima and three saddles and it is the simplest potential, where where mixed state is observed. Fig.1 shows the Poincare sections for different energies in the considered potentials. It demonstrates the evolution of dynamics in
different local minima. At low energies motion has well marked quasiperiodic character in both minima. As energy grows, gradual regularity-chaos transition is observed. However, changes in features of trajectories, localized in different minima, are sharply distinct. In

Figure 1: Level lines and Poincare sections for $D_5$ (upper row) and $QO$ potentials at different energies

the left minimum significant fraction of trajectories becomes chaotic already at about a half of the saddle energy, and at near saddle energy almost all initial conditions result in chaotic trajectories. In the right minimum at the same energy the motion remains regular and this situation is preserved up to the saddle energy (we will call this minimum the regular one for simplicity). Moreover, at energies above the saddle one the phase space is still divided on chaotic and regular components, but they are not separated in configuration space.

Earlier we have shown that the mixed state opens new possibilities for investigations of quantum manifestation of classical stochasticity [12]. Aim of the present work is to study the classical escape from separated local minima, realizing the mixed state. We show that escape from such local minima has all above mentioned properties of the decay of chaotic systems and also a diversity of principally new features, representing an interesting topic for conceptual understanding of chaotic dynamics and for applications as well. We are interested in both the ”first passage” effects and dynamical equilibrium setup for the finite motion (for example, in $QO$ potential). It is important to stress, that though we study the process of escape from concrete local minimum, the over-barrier case of the mixed state has specific memory: general phase space structure at supersaddle energies is determined by the characteristics of the motion in all other local minima.
3 Decay of the uniformly distributed ensemble

At the energies above the saddle, i.e. $E > E_S$, different components are not separated in the configuration space. Fig. 2 represents the Poincare section for $D_5$ potential at supersaddle energy. The "chaotic sea" stretches on whole accessible area, while regular island in the right well is localized. This means that, been initially localized in right well, chaotic trajectories could leave the well and regular ones remain trapped, i.e. the decay of the mixed state occurs. Therefore we will explore the escape of the particles from the right well of $D_5$ and peripheral wells of $QO$ potential.

Numerical simulation of the escape process in these potentials implies three steps. At the first stage we select initial distribution of the particles inside the well. Then direct numerical integration of the equations of motion for all particles is performed and we extract $N(t)/N(0)$—relative number of particles in the well. Using this function one can calculate escape rate and part of the trapped trajectories. One remark should be made about the first step of the numerical simulation. Initial distribution, in general, determines the ratio between regular and chaotic trajectories and hence it should be physically motivated. One chance is to distribute particles uniformly in all classically allowed configuration space and another—to put all particles at the same point. Second case emulates injection of particles to the well. In both cases momentum is calculated using the energy conservation (in the second it will be the same for all particles) and its direction is uniformly distributed in $[0, 2\pi]$. These initial distributions present quite simple extreme cases of real distributions. Uniform distribution will be illustrated with
$U_{D_5}$ and "point" distribution - with $U_{QO}$.

Phase space density for uniform initial distribution is

$$\rho(E) = \frac{1}{2\pi S(E)}$$

where $S(E)$ is the area of classically allowed space:

$$S(E) = \int_{x \gg x_S} dx dy \Theta(E - U(x,y))$$

Numerical simulation reveals some substantial features of escape process for this initial distribution. Fig. 3 demonstrates the normalized number of particles in the well as a function of time. Decay law has three important features:

- **saturation** at $t \to \infty$:

  $$N(t \to \infty) = \rho_\infty N_0$$

  Because of uniform initial distribution, quantity $\rho_\infty$ is a relative phase volume, occupied by trapped trajectories.

- **initial linear decrease** - from 0 to some $\tau(E)$:

  $$N(t)/N_0 = 1 - \alpha(t) t$$

Figure 3: Decay law $N(t)/N(0)$ for $D_5$ potential at different energies. $E_i = 0.1 + 0.2k$, $k = 0.4$
Figure 4: Poincare section for trapped trajectories. These trajectories form regular island, i.e. they are regular

- exponential decrease at $t > \tau(E)$:

$$N(t)/N_0 = \rho_\infty + C \exp(-\alpha(E) t)$$ \hspace{1cm} (9)

Fig. 4 presents the Poincare section for trapped trajectories. Obviously, regular island in the right well is formed by trapped, i.e regular, trajectories. Function $\rho_\infty(E)$ is demonstrated on Fig. 5. This is decreasing function, as expected from physical considerations—number of regular trajectories decreases as energy increases.

It is interesting to mention, that there exists correlation between $\rho_\infty(E)$ and relative area of regular island in Poincare section $\rho$. While relative part of trapped trajectories is linked to the volume of four-dimensional phase space, occupied by regular trajectories, Poincare section is two-dimensional and thus there is no argument to expect the precise coincidence of $\rho$ and $\rho_\infty(E)$.

To calculate the relative area of regular island in Poincare section we first determine the border of the island through numerical integration of equation of motion and calculate the area inside it. Then this area was divided by total area, determined by the conditions $x > 0$, $p^2 > 0$. In spite of topological inequivalence, $\rho$ and $\rho_\infty(E)$ are very close to each other. This means that one could determine and control the part of trapped trajectories using only the Poincare section.

Linear part of the decay law is more pronounced in comparison with pure ensemble $\mathbb{S}$. In the time interval between 0 and $\tau(E)$ decay has the form $\mathbb{S}$. Thus, there are two quantities describing linear decay—its duration $\tau(E)$ and corresponding escape rate $\alpha^{(l)}$.

After linear part, at $t > \tau(E)$, decay law has exponential form $\mathbb{S}$. It’s important that $\mathbb{S}$ is not a linear approximation of $\mathbb{S}$, and linear decay has independent nature. This
Figure 5: $\rho$, the relative area of the regular island in Poincare section, is shown by squares and $\rho_{\infty}(E)$ is shown by circles.

means that $\rho_{\infty} + C \neq 1$. Moreover, decay law of the form (8) precisely works up to time $t \sim \tau(E)$ and in this interval it differs substantially from corresponding exponential law $\rho_{\infty} + (1 - \rho_{\infty}) \exp(-\alpha(E)t)$.

Let’s now calculate $\tau(E)$. From analysis of numerical calculation one can derive that this time corresponds to the time of one-dimensional, along $y = 0$, motion from the saddle to opposite side of the well and back. Thus $\tau(E)$ has the form:

$$\tau_{D_{\infty}}(E) = 2 \int_0^{\sqrt{2(1+E)}} \frac{dx}{|v|} = \sqrt{2} K\left(\sqrt{\frac{1 + \sqrt{E}}{2}}\right)$$

(10)

$$\tau_{QO}(E) = 12 \left(\frac{E_{S}}{E}\right)^{\frac{1}{4}} \left(\sqrt{\frac{1 + \sqrt{E}}{2}}\right) = 6\sqrt{2} \tau_{D_{\infty}}\left(\frac{E_{S}}{E}\right)$$

(11)

where $K(k)$ is full elliptic integral of the first type and $E_s = 1/12^4$—saddle energy in $QO$ potential at $W = 18$.

Such nature of $\tau(E)$ and analysis of linearly escaping trajectories allow to conclude that linear decay corresponds to the escape of trajectories that move along $y = 0$ and cross the well not more than two times. Correspondingly, trajectories which initially move toward saddle along $y = 0$ escape first, and then escape occur for trajectories moving to the opposite part of the well.

$\alpha(E)$ could be calculated via averaging of the flow through the saddle:

$$\alpha(E) = \rho(E) \int_{x=x_s} dy \int_{-\pi/2}^{\pi/2} d\theta |v| \cos \theta$$

(12)
This procedure is the same as in [8]. Using the density (5) and integrating the (12) we obtain the expression for linear escape rate:

$$\alpha^{(l)}_{Dr}(E) = \frac{E}{2\sqrt{aS_{Dr}(E)}}$$  \hspace{1cm} (13)

$$\alpha^{(l)}_{QO}(E) = \frac{\sqrt{\varepsilon}}{12\pi S_{QO}(E)} \times \left\{ (16\sqrt{\varepsilon} + 1) K \left( \frac{1 - \frac{1}{16\sqrt{\varepsilon}}}{2} \right) - 2E \left( \frac{1 - \frac{1}{16\sqrt{\varepsilon}}}{2} \right) \right\}$$  \hspace{1cm} (14)

Exponential decrease of $N(t)/N(0)$, as it could be understood from the analysis of escaped trajectories, corresponds to the leaving of sticking orbits, i.e. those chaotic trajectories which moved in the vicinity of the regular island in the Poincare section.

Energy is the parameter which determines the part of trapped trajectories for the uniform ensemble. Changing the energy of ensemble one could trap the given number of particles.

![Figure 6](image-url)

**Figure 6:** Initial ensemble for extraction of asymptotic distribution.

To illustrate the splitting of initial ensemble on regular and chaotic components one needs to plot the ensemble in $(x, y)$ plane at time $t \gg \tau(E)$. It is convenient to plot the asymptotic evolution of the initial ensemble in $QO$ potential because of finite character of motion in it. At $t = 0$ particles are equally distributed between peripheral minima. Fig. 6 represents this initial ensemble.

The ensemble splits during the evolution in time—regular trajectories remain trapped in peripheral minima, while chaotic trajectories cover almost entire accessible configuration space. Integrating equations of motion for all trajectories in the initial ensemble for enough long time (in fact—for time much greater than the typical escape time) we obtain the particles positions corresponding to asymptotic distribution. We have calculated the asymptotic distribution for energy $E = 1.5E_S$ ($E_S = 1/12^4$ - saddle energy in $QO$ potential), while integration was performed for $t = 150$ (for this energy $\tau(E) = 28.395$). Fig. 7 shows the asymptotic distribution of trapped particles. At enough large time these particles tend to accumulate closer to the center of the well.
The corresponding distribution of free particles is represented at fig. 8. As it was mentioned above, free particles cover entire central minimum and, according to their character, the area near the plane \( y = 0 \) in peripheral minima. If free particles are removed in some way after their leave the peripheral minima (this is, of course, correct for right well of \( D_5 \) and any well of same topology), we obtain pure regular ensemble inside the well, i.e. the initial mixed ensemble splits.

### 4 Point ensemble

For simplicity we will consider the point distribution with \( y = 0 \). Thus the governing parameter is \( x_0 \)—\( x \)-coordinate of the point of injection. Corresponding density has the form:

\[
\rho (x, y, p, \varphi) = \frac{\delta \{x - x_0\} \delta \{y\} \delta \{p - \sqrt{2(E - U(x_0, 0))}\}}{2\pi \sqrt{2(E - U(x_0, 0))}}
\]  

(15)
Numerical procedure is identical to that for uniform ensemble, but for distribution (15) one could made general conclusion about the character of decay even without numerical integration.

For this one needs to consider the representation of initial distribution on the Poincare section:

$$\rho_{PS}(x, p_x) = \frac{\delta \{ x - x_0 \} \Theta \left\{ p_x - \sqrt{2(E - U(x_0, 0))} \right\} \Theta \left\{ \sqrt{2(E - U(x_0, 0))} + p_x \right\}}{2\sqrt{2(E - U(x_0, 0))}}$$  \hspace{1cm} (16)

where $\Theta \{ a \}$ is a step function. Lets denote:

$$p_x^{(\text{max})} = \sqrt{2(E - U(x_0, 0))}.$$  \hspace{1cm} (17)

Thus in the Poincare section initial ensemble occupies the interval $x = x_0, p_x \in [-p_x^{(\text{max})}, p_x^{(\text{max})}]$. Edges of this interval correspond to the momentum directions $\varphi = \pi, 0$. In the over–barrier case this interval crosses the regular island in the points $(x_0, p_x^{(\text{reg})})$ and $(x_0, -p_x^{(\text{reg})})$. Obviously, particles with $p_x$ in the interval $[-p_x^{(\text{reg})}, p_x^{(\text{reg})}]$ could not leave the well. Value

$$\varphi_{\text{max}}(E, x_0) = 2 \arccos \left( \frac{p_x^{(\text{reg})}}{p_x^{(\text{max})}} \right)$$  \hspace{1cm} (18)

defines the cone of directions which could leave the well. In other words, particles with $p_y$ which is greater than some maximum value are trapped. Thus first conclusion about decay of point ensemble implies the existence of escape cone and this feature reveals the role of transversal momenta in the escape process.

The second feature consists in the fact that decay begins at the time $\tau_1$, which corresponds to the time of motion of particle with momentum $p_x = -p$ from point $x_0$ to the saddle:

$$\tau_1 = \int_{x_S}^{x_0} \frac{dx}{p_x} = \int_{x_S}^{x_0} \frac{dx}{\sqrt{2(E - U(x, 0))}}$$  \hspace{1cm} (19)

Moreover, escape is a two-stage process due to existence of escape cone. At the second stage the particles, moving toward the well boundary in the escape cone, leave.

Numerical procedure implies the determination of the $\varphi_{\text{max}}(x_0, E)$, $\tau_1(x_0, E)$, and $N(t = \infty)$, i.e. the number of particles in the well. Due to uniform distribution of momenta directions one has the relation:

$$\rho_\infty(x_0, E) = \frac{N(\infty)}{N(0)} = 1 - \frac{\varphi_{\text{max}}(x_0, E)}{\pi}$$  \hspace{1cm} (20)

We will illustrate the above consideration for $QO$ potential. Energy is normalized to the saddle energy, $E_S = 1/144^2$: $E = \varepsilon E_S$. Initial ensemble is localized in peripheral minimum with $x_{\text{min}} = 1/6$ and corresponding saddle $x_S = 1/12$. Fig. 9 represents the normalized number of particles in the well as a function of time for different values of energy and $x_0 = 0.16$. 

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Figure 9: Normalized particle number in the well for point ensemble at different energy and $x_0 = 0.16$

Numerically obtained $\tau_1$ could be compared to analytical value:

$$\tau_1 = \int_{1/12}^{x_0} \frac{dx}{p_x} = \int_{1/12}^{x_0} \frac{dx}{\sqrt{2(\varepsilon_E - U_{QO}(x, 0))}}$$

(21)

Numerical and analytical value of first escape time are presented at fig. 10.

The most interesting question is the correspondence between escape cone angle, number of trapped particles and linear part of regular island in Poincare section. Fig. 11 represents the quantities

$$\rho_{PS} = 1 - \frac{p^{(reg)}(x, \varepsilon)}{p^{(max)}(x, \varepsilon)}$$

(22)

and

$$\rho_{\varphi} = 1 - \cos(\varphi_{\text{max}}(x, \varepsilon)/2)$$

(23)

First is a linear part of regular island in the section and second is the corresponding expression through the escape cone angle.

This angle could be determined during numerical simulation. Thus, Poincare section could be used to determine the angle of the escape cone. On the other hand, this angle is connected with the part of trapped trajectories. To demonstrate this connection one needs to compare

$$\rho_{\varphi}^{(N)} = 1 - \frac{\varphi_{\text{max}}(x, \varepsilon)}{\pi}$$

(24)

with $\rho_\infty$. Corresponding data are presented at fig. 12. This analysis allows to conclude that point ensemble differs substantially from the uniform when considering the escape.

In uniform ensemble Poincare section gives only estimate (nevertheless very accurate) of trapped particles number, while in the point ensemble one could calculate not only part
Figure 10: Numerically obtained $\tau_1$ and analytically calculated $-\tau(E)$ and $-\tau_{exp}$

Figure 11: $\rho_{PS}$ and $\rho_\varphi$ for different energies and injection points
of trapped particles, but also the escape cone angle, using only Poincare section. Point ensemble allows dual control of trapped particles part using the energy and injection point. One even needs not to compute the entire Poincare section, but only the boundary of regular island, \( p^{(\text{reg})}_x(x_0, \varepsilon) \), and then use the relation:

\[
\rho_\infty = 1 - \frac{2 \arccos \left( \frac{p^{(\text{reg})}_x}{p^{(\text{max})}_x} \right)}{\pi}
\]  

The procedure of calculating the part of trapped particles, thus, implies three steps. First of all one needs to calculate the boundary of the regular island in the Poincare section. After that, the \( p^{(\text{reg})}_x \) and \( p^{(\text{max})}_x \) should be determined. Now value \( \rho_\infty \) could be calculated.

5 Quantum escape problem

Now we consider the over-barrier decay of the mixed state from the quantum-mechanical point of view. In semiclassical limit, temporal evolution of an initial state in form of the minimum uncertainty Gaussian wave packet

\[
\Psi_G(x, y; x_0, y_0, p_{x0}, p_{y0}) = \frac{1}{\sqrt{\pi \sigma_x \sigma_y}} e^{-\frac{(x-x_0)^2}{2\sigma_x^2} - \frac{(y-y_0)^2}{2\sigma_y^2} + \frac{i p_{x0}(x-x_0)}{\hbar} + \frac{i p_{y0}(y-y_0)}{\hbar}}
\]

represents the quantum analogue for classical motion of a point particle with initial condition \((x_0, y_0, p_{x0}, p_{y0})\). The quantum-classical correspondence between the wave packet motion and the classical trajectory preserves quite a long time until the wave packet spreads.
We made numerical simulations for time evolution of the Gaussian wave packets for three different initial conditions in the $D_5$ potential (3) with $a = 1.1$. The following wave packets parameters

$$\bar{\hbar} = 0.01, \sigma_x = \sqrt{\frac{\hbar}{2}} \simeq 0.07, \sigma_y = \sqrt{\frac{\hbar}{2\sqrt{a + 1}}} \simeq 0.06, x_0 = \sqrt{2}, y_0 = 0$$

and $p_0 = \sqrt{p_{x0}^2 + p_{y0}^2} = 2$ were the same for all the three initial states, so all of them started from the right local minimum of the potential with initial energy $E = 1$, which is twice higher than the barrier height. (Recall that in the $D_5$ potential (3) $E_{\text{min}} = -1$ and $E_S = 0$). The only difference was in the direction of initial momentum: we considered the cases $\varphi_0 = 0, \pi, \pi/2$.

The initial condition with $\varphi_0 = \pi/2$ falls near the center of the principal stability island and therefore it corresponds to regular classical trajectory trapped in the right potential well. Quantum autocorrelation function

$$P(t) = \int dx dy \Psi(t = 0)\Psi(t)$$

manifests quasiperiodic nature of the corresponding classical trajectory: the sharp peaks indicate periodically repeated recurrences of the wave packet to the initial state, and rather high amplitude of the peaks shows that there is almost no spreading of the wave packet. The reason of that slow spreading is the fact that even for highly over-barrier energies the regular classical motion near the potential minimum is still pretty close to that in two-dimensional harmonic potential, as it is seen for example in the characteristic structure of the stability island on the Poincaré section. For the considered case of the right local minimum in the $D_5$ potential the corresponding quadratic (harmonic) approximation reads

$$U(x, y) \simeq \frac{\omega_x^2(x - \sqrt{2})^2 + \omega_y^2y^2}{2} - 1$$

with $\omega_x = 2$ and $\omega_y = 2\sqrt{a + 1}/\sqrt{2} \simeq 2.7$ for $a = 1.1$. Therefore the trapped quasiperiodic trajectories actually are very close to the Lissajoux figures — simple superpositions of harmonic oscillations in perpendicular directions. Remarkably, the considered gaussian wave packet coincides with the coherent state for the harmonic oscillator potential (27) — the exact non-spreading solution of time depending Schrödinger equation — that is why it is ”almost” non-spreading near the minimum of the $D_5$ potential. As the considered initial condition $\varphi_0 = \pi/2$ leads to almost one-dimensional motion (along the $y$-axis), only one frequency $\omega_y \simeq 2.7$ is manifested in the autocorrelation function $P(t)$ (26).

The initial condition $\varphi_0 = 0$ also corresponds to a regular trajectory — the periodic one-dimensional motion along the $x$-axis. The autocorrelation function (26) for the corresponding wave packet clearly shows the same periodicity of recurrences, but decreasing amplitude of the corresponding peaks reveals much faster spreading than in the former case $\varphi_0 = \pi/2$. Clear explanation for it is that any harmonic approximation is no more valid for the trajectory, and the Gaussian wave packet is not already a good approxima-
Figure 13: Survival probabilities and autocorrelation functions for considered initial conditions
tion for the exact time dependent solution. It is worth noting that the wave packet for the case $\varphi_0 = 0$ undergoes only one-dimensional spreading — along the $x$-axis, while it remains well localized in the $y$-direction.

The case with $\varphi_0 = \pi/4$ sharply differs from the two formers, as it corresponds to a chaotic trajectory. Accordingly the autocorrelation function shows almost absent recurrences and even faster spreading of the wave packet. This time the wave packet spreads already in two dimensions — over all the chaotic sea.

Survival probability for the right potential well

$$p_R(t) = \int_{x>0} dx dy |\Psi|^2$$

is a natural quantum analogue of the classical quantity $N(t)/N(0)$. The results of numerical simulations for all the three considered cases are presented on fig.13. Naturally, $p_R \equiv 1$ for the trapped state ($\varphi_0 = 0$), as well as for the two others for $t < t_{cl} \simeq 2.5$ — the classical escape time which is almost the same for both the wave packets. Periodicity of the trajectory with $\varphi_0 = 0$ perfectly manifests again in the periodic character of the $p_R(t)$ for the corresponding wave packet. Graduate decay of the oscillations amplitude is due to the spreading. Saturation of the survival probability at value $p_r \simeq 0.5$ confirms that the spreading is one-dimensional in that case: the probability density $|\Psi|^2$ gets uniformly distributed along the one-dimensional trajectory, which lies between the points $x_{1,2} = \pm \sqrt{2}\sqrt{1 + \sqrt{2}}$ (recall that $E = 1$), so exactly one half of $|\Psi|^2$ falls into each local minimum.

On the case of the chaotic motion $\varphi_0 = \pi/4$ the survival probability is aperiodic and quickly saturates at the value $p_R \simeq 0.3$. It is in accordance with our arguments of fast spreading of the wave packet to uniform two-dimensional distribution of the probability density $|\Psi|^2$ over the chaotic sea — indeed, exactly about 30% of the chaotic sea amounts for the right minimum, as it can be seen from the Poincaré section. If we calculate the ratio of the chaotic sea in the right minimum to the whole area of the sea in Poincaré section, the exact value would be 0.39.

Our preliminary considerations of three principal types of initial conditions show generally good coincidence of the quantum and classical results for the decay of the mixed state. Specifically quantum effects in the escape problem, such as the resonance barrier penetration and the chaos-assisted dynamical tunneling require further analysis, which will be published elsewhere.

6 Conclusions

Investigation of the escape from localized areas of configuration space in the existence of the mixed state is presented. When the mixed state is present in the system, it is possible to ”trap” given number of particles in the well. We have considered two possible initial distributions.

For uniform distribution escape law splits into three sections. First section corresponds to linear decay, second - to exponential and third forms the plato, which corresponds to trapped particles. Number of trapped particles depends only on energy.
In the point ensemble case the escape is a two-stage process and the number of trapped particles depends not only on energy, but on the coordinate of injection point too. Only trajectories with direction of initial momenta in some cone could escape. Angle of escape cone is connected with linear part of regular island in the Poincare section and number of trapped trajectories. Moreover, only edge of regular island is necessary to compute the number of trapped particles.

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