Hausdorff Realization of Linear Geodesics of Gromov–Hausdorff Space.

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Abstract

We have constructed a realization of rectilinear geodesic (in the sense of [1]), lying in Gromov–Hausdorff space, as a shortest geodesic w.r.t. the Hausdorff distance in an ambient metric space.

Introduction

In [2] it was shown that the space of isometry classes of compact metric spaces, endowed with the Gromov–Hausdorff metric, is a geodesic metric space. The proof consisted of two steps: first, it was shown how an optimal correspondence $R$ between two finite metric spaces $X$ and $Y$ can be endowed with a one-parametric family of metrics generating a shortest geodesic $R_t$, $t \in [0, 1]$, connecting $X$ with $Y$, and such that its length equals the Gromov–Hausdorff distance between $X$ and $Y$. At the second step the Gromov Precompactness Criteria was used to prove that for any two compact metric spaces there exists a compact metric space which is their “midpoint”. A little bit later, in [3] and independently in [4], it was shown that a compact optimal correspondence exists between any two compact metric spaces, and if one defines a one-parametric family of metrics on it in the same manner as it was done in [2], then again a shortest geodesic with the described properties is obtained. In [1] such geodesics were called rectilinear (there are some other shortest geodesics that are called deviant in [1]).

In the present paper we show that for each pair $X, Y$ of compact metric spaces, and each compact optimal correspondence $R$ between them, the compact $R \times [0, 1]$ can be endowed with a metric in such a way that for each $t$ the restriction of the metric to $R \times \{t\}$ coincides with the metric of the compact $R_t$, and that the Hausdorff distance between $R \times \{t\}$ and $R \times \{s\}$ equals to the Gromov–Hausdorff distance between $R_t$ and $R_s$. In other words, we construct a realization of the geodesic $R_t$ as a shortest geodesic w.r.t. the Hausdorff distance defined on the subsets of the space $R \times [0, 1]$ endowed with some special metric.
1. Main Definitions and Preliminary Results

Let $X$ be a metric space. By $|xy|$ we denote the distance between points $x, y \in X$.

Let $\mathcal{P}(X)$ be the set of all nonempty subsets of the space $X$. For each $A, B \in \mathcal{P}(X)$, and $x \in X$, we put

$$|xA| = |Ax| = \inf \{ |xa| : a \in A \}, \quad |AB| = \inf \{ |ab| : a \in A, b \in B \},$$

$$d_H(A, B) = \max \{ \sup_{a \in A} |aB|, \sup_{b \in B} |Ab| \} = \max \{ \sup_{a \in A} \inf_{b \in B} |ab|, \sup_{b \in B} \inf_{a \in A} |ba| \}.$$

**Definition 1.1.** The function $d_H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}$ is called the Hausdorff distance.

The set of all nonempty closed bounded subsets of the metric space $X$ is denoted by $\mathcal{H}(X)$.

**Theorem 1.2** ([5]). The function $d_H$ is a metric on $\mathcal{H}(X)$.

**Theorem 1.3** ([5]). The space $\mathcal{H}(X)$ is compact iff $X$ is compact.

Let $X$ and $Y$ be metric spaces. A triple $(X', Y', Z)$ consisting of a metric space $Z$ and two its subsets $X'$ and $Y'$ isometric to $X$ and $Y$, respectively, is called a realization of the pair $(X, Y)$. The Gromov–Hausdorff distance $d_{GH}(X, Y)$ between $X$ and $Y$ is the infimum of the reals $r$ such that there exist realizations $(X', Y', Z)$ of the pair $(X, Y)$ with $d_H(X', Y') \leq r$.

**Theorem 1.4** ([5]). The Gromov–Hausdorff distance is a metric on the set of all isometry classes of compact metric spaces.

**Definition 1.5.** The set of all isometry classes of compact metric spaces endowed with the Gromov–Hausdorff metric is called the Gromov–Hausdorff space and is denoted by $\mathcal{M}$.

We need one more (equivalent) definition of the Gromov–Hausdorff distance. Recall that a relation between sets $X$ and $Y$ is a subset of the Cartesian product $X \times Y$. The set of all nonempty relations between $X$ and $Y$ we denote by $\mathcal{R}(X, Y)$.

**Definition 1.6.** If $X$ and $Y$ are metric spaces, then the distortion $\text{dis} \sigma$ of a relation $\sigma \in \mathcal{R}(X, Y)$ is the value

$$\text{dis} \sigma = \sup \left\{ |xx'| - |yy'| : (x, y), (x', y') \in \sigma \right\}.$$

A relation $R \subset X \times Y$ between sets $X$ and $Y$ is called a correspondence, if the restrictions to $R$ of the canonical projections $\pi_X : (x, y) \mapsto x$ and $\pi_Y : (x, y) \mapsto y$ are surjective. The set of all correspondences between $X$ and $Y$ we denote by $\mathcal{R}(X, Y)$. 

Theorem 1.7 ([5]). For any metric spaces $X$ and $Y$ we have

$$d_{GH}(X, Y) = \frac{1}{2} \inf \{ \text{dis } R : R \in \mathcal{R}(X, Y) \}.$$ 

For topological spaces $X$ and $Y$, we consider $X \times Y$ as the topological space with the standard Cartesian product topology. Then it makes sense to talk about closed relations and correspondences. The set of all closed correspondences between $X$ and $Y$ we denote by $\mathcal{R}^c(X, Y)$.

Corollary 1.8 ([4]). For metric spaces $X$ and $Y$ we have

$$d_{GH}(X, Y) = \frac{1}{2} \inf \{ \text{dis } R : R \in \mathcal{R}^c(X, Y) \}.$$ 

Definition 1.9. A correspondence $R \in \mathcal{R}(X, Y)$ is called optimal if $d_{GH}(X, Y) = \frac{1}{2} \text{dis } R$. The set of all optimal correspondences between $X$ and $Y$ is denoted by $\mathcal{R}_{\text{opt}}(X, Y)$. The subset of $\mathcal{R}_{\text{opt}}(X, Y)$ consisting of all closed optimal correspondences is denoted by $\mathcal{R}^c_{\text{opt}}(X, Y)$.

Theorem 1.10 ([4]). For any $X, Y \in \mathcal{M}$ we have $\mathcal{R}^c_{\text{opt}}(X, Y) \neq \emptyset$.

Theorem 1.11 ([3], [4]). For any $X, Y \in \mathcal{M}$ and each $R \in \mathcal{R}^c_{\text{opt}}(X, Y)$ the family $R_t$, $t \in [0, 1]$, of compact metric spaces such that $R_0 = X$, $R_1 = Y$, and for $t \in (0, 1)$ the space $R_t$ is the set $R$ with the metric

$$\|(x, y), (x', y')\|_t = (1 - t)|xx'| + t|yy'|,$$

is a shortest curve in $\mathcal{M}$ connecting $X$ and $Y$, and its length is equal to $d_{GH}(X, Y)$.

In [1] the curve $R_t$ was called a rectilinear geodesic corresponding to $R \in \mathcal{R}^c_{\text{opt}}(X, Y)$. In the same paper it was noted that the Gromov–Hausdorff space has non rectilinear shortest geodesics.

The main result of the present paper follows from more general theorem describing a special construction of metric on the Cartesian product of a metric space and a segment of a Euclidean line.

2 A special extension of a metric to Cartesian product

Let $Z$ be an arbitrary set, and $\rho_t$, $t \in [a, b]$, a one-parametric family of metrics on $Z$. For convenience, we put $|zz'|_t = \rho_t(z, z')$ and $Z_t = Z \times \{t\}$. Fix some $c > 0$, and define a distance function on $Z \times [a, b]$ as follows:

$$\|(z_1, t_1)(z_2, t_2)\| = \inf_{z \in Z} (|z_1z|_{t_1} + |zz_2|_{t_2}) + c|t_1 - t_2|, \quad (1)$$

It is clear that this function is non negative and symmetric, i.e., it is a distance function such that its restriction to each section $Z \times \{t\}$ coincides with $\rho_t$, and on the section $\{z\} \times [a, b]$ with the Euclidean distance $c|t - s|$, $t, s \in [a, b]$ (notice that the latter one does not depend on the choice of $z$); also, $|Z_tZ_s| = c|t - s|$.

To ensure the triangle inequality, one needs two more conditions.
Theorem 2.1. Under the notations introduced above, suppose also that the following conditions hold:

1. For any \( z, z' \in Z \) the function \( f(t) = |zz'|_t, t \in [a, b] \) is monotonic on \( t \);
2. For any \( (z, t), (z', s) \in Z \times [a, b] \) we have
   \[
   |zz'|_t \leq c|t - s| + |zz'|_s + c|t - s|.
   \]

Then the distance function defined by Equation (1) satisfies the triangle inequality, i.e., it is a metric; moreover, for any \( t, s \in [a, b] \) we have \( d_H(Z_t, Z_s) = |Z_tZ_s| = c|t - s| \), where \( d_H \) is the Hausdorff distance in \( \mathcal{H}(Z \times [a, b]) \).

Proof. We illustrate the proof in Figure 1.

![Figure 1: Triangle inequality verification.](image)

To verify the triangle inequality, let us choose three arbitrary points \((z_i, t_i), \) \(i = 1, 2, 3\) and show that

\[
|(z_1, t_1)(z_2, t_2)| + |(z_2, t_2)(z_3, t_3)| \geq |(z_1, t_1)(z_3, t_3)|.
\]

The proof depends on the ordering of the values \( t_i \) (in the figure the \( t_i \) correspond to the height of the sections \( Z_{t_i} \)): we get three cases up to symmetry and degeneration (when some \( t_i \) equal to each other). Now we proceed each of the cases analytically. Notice that the solid polygonal lines connecting \( z_i \) and \( z_j \) code, in a natural way, the distances between the points \((z_i, t_i)\) and \((z_j, t_j)\); the dashed lines show how one could decrease the value \( |(z_1, t_1)(z_2, t_2)| + |(z_2, t_2)(z_3, t_3)| \) to obtain a one greater or equal to \( |(z_1, t_1)(z_3, t_3)| \).

The initial steps are the same in all the three cases, and they can be obtained
from the following triangle inequality: \( |zz_2|_{t_2} + |zz'|_{t_2} \geq |zz'|_{t_2} \):

\[
|z_1, t_1)(z_2, t_2)| + |(z_2, t_2)(z_3, t_3)| = \inf_{z \in \mathbb{Z}} \left( |z_1 z|_{t_1} + c|t_1 - t_2| + \inf_{z' \in \mathbb{Z}} \left( |zz'|_{t_2} + |z' z_3|_{t_3} \right) + c|t_3 - t_2| \right) \geq \inf_{z, z' \in \mathbb{Z}} \left( |z_1 z|_{t_1} + |zz'|_{t_2} + |z' z_3|_{t_3} \right) + c|t_1 - t_2| + c|t_3 - t_2|.
\]

Then some differences between the Cases appear.

**Case (a).** We have

\[
\inf_{z, z' \in \mathbb{Z}} \left( |z_1 z|_{t_1} + |zz'|_{t_2} + |z' z_3|_{t_3} \right) + c|t_1 - t_2| + c|t_3 - t_2| = \left| (z_1, t_1)(z_3, t_3) \right|,
\]

where the first equality, according to the given order between \( t_i \), follows from \( c|t_1 - t_2| = c|t_1 - t_3| + c|t_3 - t_2| \); the first inequality follows from condition (2) of Theorem under consideration, that gives \( c|t_3 - t_2| + |zz'|_{t_2} + c|t_1 - t_3| \geq |zz'|_{t_3}; \) the last inequality follows from the triangle inequality \( |zz'|_{t_3} + |z' z_3|_{t_3} \geq |zz'|_{t_3} \).

**Case (b).** In this case we use Condition (1) of Theorem, and thus we conclude that \( |zz'|_{t_2} \) is more or equal than either \( |zz'|_{t_1} \), or \( |zz'|_{t_1} \). Without loss of generality, let us suppose that \( |zz'|_{t_2} \geq |zz'|_{t_1} \), then

\[
\inf_{z, z' \in \mathbb{Z}} \left( |z_1 z|_{t_1} + |zz'|_{t_2} + |z' z_3|_{t_3} \right) + c|t_1 - t_2| + c|t_3 - t_2| \geq \inf_{z, z' \in \mathbb{Z}} \left( |z_1 z|_{t_1} + |zz'|_{t_2} + |z' z_3|_{t_3} \right) + c|t_1 - t_3| + c|t_3 - t_2| \geq \inf_{z \in \mathbb{Z}} \left( |z_1 z|_{t_1} + |zz'|_{t_2} + |z' z_3|_{t_3} \right) + c|t_1 - t_3| = \left| (z_1, t_1)(z_3, t_3) \right|,
\]

where the first inequality can be obtained from \( |zz'|_{t_2} \geq |zz'|_{t_1} \) and, according to the given order between \( t_i \), from \( c|t_1 - t_3| = c|t_1 - t_2| + c|t_3 - t_2| \); the second inequality follows from the triangle inequality \( |z_1 z|_{t_1} + |z' z|_{t_1} \geq |z_1 z'|_{t_1} \).

**Case (c).** We have

\[
\inf_{z, z' \in \mathbb{Z}} \left( |z_1 z|_{t_1} + |zz'|_{t_2} + |z' z_3|_{t_3} \right) + c|t_1 - t_2| + c|t_3 - t_2| = \inf_{z, z' \in \mathbb{Z}} \left( |z_1 z|_{t_1} + |zz'|_{t_2} + |z' z_3|_{t_3} \right) + c|t_1 - t_3| + c|t_3 - t_2| \geq \inf_{z, z' \in \mathbb{Z}} \left( |z_1 z|_{t_1} + |zz'|_{t_2} + |z' z_3|_{t_3} \right) + c|t_1 - t_3| \geq \inf_{z \in \mathbb{Z}} \left( |z_1 z|_{t_1} + |zz'|_{t_2} + |z' z_3|_{t_3} \right) + c|t_1 - t_3| = \left| (z_1, t_1)(z_3, t_3) \right|,
\]

where the first equality, according to the given order between \( t_i \), follows from the condition \( c|t_3 - t_2| = c|t_3 - t_1| + c|t_1 - t_2| \); the first inequality can be obtained from Condition (2) of Theorem, that gives \( c|t_1 - t_2| + |zz'|_{t_2} + c|t_1 - t_3| \geq |zz'|_{t_1}; \) the last inequality follows from the triangle inequality \( |z_1 z|_{t_1} + |z' z|_{t_1} \geq |z_1 z'|_{t_1} \). □
As a consequence from Theorem 2.1, we construct a realization of a rectilinear geodesic in Gromov–Hausdorff space, as a shortest geodesic in the sense of Hausdorff metric.

3 Realization of Rectilinear Geodesics

Choose arbitrary $X, Y \in \mathcal{M}$, $R \in \mathcal{R}_{op}^c(X, Y)$, and construct the corresponding rectilinear geodesic $R_t$, $t \in [0, 1]$. For convenience reason, the distance in $R_t$ between points $(x, y)$ and $(x', y')$ is denoted by $|(x, y), (x', y')|_t$. Put $c = \frac{1}{2} \text{dis } R$, and define the distance on $R \times [0, 1]$ by Formula (1).

**Corollary 3.1.** For nonisometric $X$ and $Y$, the distance function on $R \times [0, 1]$ defined above is a metric such that $d_H(R_t, R_s) = d_{GH}(X, Y)|t - s|$, thus, $R_t$, being considered as a curve in the space $\mathcal{H}(R \times [0, 1])$, is a shortest curve.

**Proof.** It suffices to verify that the conditions of Theorem 2.1 hold in the case under consideration.

Since $X$ and $Y$ are nonisometric, then $\text{dis } R > 0$, thus $c > 0$.

Further, for any $(x, y)$ and $(x', y')$ from $R$, the function $f(t)$ from Condition (1) equals $(1 - t)|xx'| + t|yy'|$, therefore, it is linear on $t$, thus the Condition (1) holds.

At last, let us check Condition (2). To do that, choose arbitrary $(x, y), (x', y') \in R$, and arbitrary $t, s \in [0, 1]$, then

$$|(x, y), (x', y')|_t - |(x, y), (x', y')|_s =$$

$$= (1 - t)|xx'| + t|yy'| - (1 - s)|xx'| - s|yy'| =$$

$$= (t - s)(|yy'| - |xx'|) \leq |t - s| \text{dis } R = 2c|t - s|,$$

that completes the proof. □

**References**

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