Abstract

We perform a two-loop calculation of the effective Lagrangian for the low–energy modes of the quantum mechanical system obtained by dimensional reduction from 4D, \( N = 1 \) supersymmetric QED. The bosonic part of the Lagrangian describes the motion over moduli space of vector potentials \( A_i \) endowed with a nontrivial conformally flat metric

\[
g_{ij} = \delta_{ij} \left( 1 + \frac{1}{2|A|^3} - \frac{3}{4|A|^6} + \ldots \right). \]

For the matrix model obtained from Abelian 4D, \( N = 2 \) theory, the two–loop correction \( \propto 1/|A|^6 \) vanishes as it should.

1 Introduction

Back in 1987 we determined the one–loop corrections to the effective Born–Oppenheimer Hamiltonian of 4D, \( N = 1 \) supersymmetric QED reduced to \((0 + 1)\) dimensions \([1]\). The corresponding effective Lagrangian is expressed in terms of the real supervariable carrying vector index \( k \) \([2]\),

\[
\Phi_k = -\frac{1}{4} \epsilon^{\beta\gamma}(\sigma_k)_\gamma^\alpha (D_\alpha \bar{D}_\beta + D_\beta \bar{D}_\alpha)V, \quad (1.1)
\]

where \( D_\alpha, \bar{D}_\alpha \) are supersymmetric covariant derivatives and \( V \) is a scalar real supervariable, the quantum–mechanical descendant of the 4D vector superfield. The lowest component of \( \Phi_k \) is the vector potential \( A_k \). Remarkably,
$A_k$ and $\Phi_k$ are gauge invariant in the QM limit. A generic supersymmetric Lagrangian involving $\Phi_k$ is

$$L = \int d^2\theta d^2\bar{\theta} \ F(\Phi), \quad (1.2)$$

Its bosonic part describes the motion along a 3-dimensional manifold with the conformally flat metric

$$ds^2 = 2\Delta F(A)dA^2. \quad (1.3)$$

Explicit one-loop calculations give the result

$$2\Delta F(A) = 1 + \frac{1}{2|A|^3} + \ldots \quad (1.4)$$

We noted back in [1] that the coefficient of $1/|A|^3$ in Eq. (1.4) is related to the first coefficient of the 4-dimensional $\beta$ function of supersymmetric QED. In recent [4] we reproduced the result (1.4), obtained originally in the framework of the Hamiltonian Born–Oppenheimer procedure, by Lagrangian methods and showed that the term $\propto 1/|A|^3$ is determined by exactly the same graph as the 1-loop correction to the effective Lagrangian in 4 dimensions.

The effective Lagrangian both in (0+1) and in (3+1) theories accept also higher loop corrections. It is natural to expect that they are related to each other, as the one-loop corrections do. In particular, higher-order corrections vanish both in $\mathcal{N} = 2$ (3+1) theories and in their quantum-mechanical counterparts. It is unconceivable for us that this is a purely accidental coincidence [4].

However, to see a relationship between the corrections at the two-loop level or higher is not an easy task. At the one-loop level the basic rule of correspondence is

$$\frac{1}{4|A|^3} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + A^2)^2} \rightarrow \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + \mu^2)^2} = \frac{1}{16\pi^2} \ln \frac{\Lambda_{UV}^2}{\epsilon \mu^2}. \quad (1.5)$$

This analysis has been extended to non-Abelian case in [3]. For example, for $SU(2)$ $\mathcal{N} = 1$ theory, the effective Lagrangian is given, again, by Eq.(1.2) with

$$2\Delta F(A) = 1 - \frac{3}{2|A|^3} + \ldots$$
The correspondence between multiple loop integrals in different dimensions is much more obscure. The basic motivation of the present study was the desire to establish such a correspondence. We have to say right away that we failed to do it. The correction to $\mathcal{L}_{\text{eff}}^{(0+1)}$ turned out to be

$$
\mathcal{L}_{\text{eff}}^{(0+1)} = \frac{\dot{A}^2}{2} \left( 1 + \frac{1}{2|A|^3} - \frac{3}{4|A|^6} + \ldots \right).
$$

This does not look similar to

$$
\mathcal{L}_{\text{eff}}^{(3+1)} = -\frac{F_{\mu\nu}^2}{4e^2(\mu)} = -\frac{F_{\mu\nu}^2}{4} \left[ \frac{1}{e_0^2} + \frac{1}{4\pi^2} \ln \frac{\Lambda}{\mu} + \frac{e_0^2}{16\pi^4} \ln \frac{\Lambda}{\mu} + \ldots \right].
$$

As we will see, there is also no obvious correspondence between the individual contributions in $\mathcal{L}_{\text{eff}}^{(0+1)}$ and $\mathcal{L}_{\text{eff}}^{(3+1)}$. To be precise, the diagrams determining the corrections in these two cases are identical, but the results of their evaluation are not similar.

We still believe that the correspondence between higher order corrections to $\mathcal{L}_{\text{eff}}$ in different dimensions will eventually be unravelled, but at the moment we can only report our accurate calculation of the term $-3/(4|A|^6)$ in Eq.(1.6).

Before proceeding with it, let us make a remark on the title of our paper. The term “matrix models” usually refers to quantum-mechanical versions of non-Abelian gauge theories (the dynamic variables in such theories are matrices). Matrix models (in the first place, maximally supersymmetric matrix models obtained by dimensional reduction of 4D $\mathcal{N} = 4$ theories) attracted recently a considerable attention due to their implications for strings/branes/M–theory dynamics. In particular, the corrections $\propto \dot{A}^2$ to the effective Lagrangian in the maximally supersymmetric matrix models vanish, while the nontrivial corrections $\propto (\dot{A}^2)^2$ and of still higher order in derivatives can be related to the scattering amplitudes of gravitons in 11–dimensional space [5]-[8].

In the Abelian models discussed in this paper, gauge and fermion fields do not have matrix form. In addition, these models probably do not have stringy or gravity implications (our primary interest here is the effective Lagrangian itself). But their kinship to non–Abelian matrix models is obvious.
2 Calculational set-up

Supersymmetric QED involves the gauge field $A_\mu$, the photino (Majorana fermion) field $\lambda$, two charged scalars $\phi, \chi$, and a charged Dirac fermion $\psi$. We assume that the charged fields are massless. In the gauge $A_0 = 0$, the Lagrangian of the dimensionally reduced theory has the form

$$L = \frac{1}{2} \dot{A}_k^2 + \dot{\phi} \ddot{\phi} + \dot{\chi} \ddot{\chi} + i[\ddot{\lambda} \lambda + \ddot{\psi} \psi]$$

$$-e^2(\ddot{\phi} \phi + \ddot{\chi} \chi) A_k^2 - \frac{1}{2}e^2(\ddot{\phi} \phi - \ddot{\chi} \chi)^2 + ieA_k \ddot{\psi} \gamma_k \psi +$$

$$e\sqrt{2} \left[ \chi \ddot{\psi}_L \lambda + \ddot{\chi} \lambda \psi_L - \ddot{\phi} \ddot{\psi}_R \lambda - \ddot{\phi} \ddot{\lambda} \psi_R \right]$$

(2.1)

$[\psi_{L,R} = \frac{1}{2}(1 \mp \gamma^0)\psi; \gamma_k$ are Euclidean, i.e. Hermitian. We will set in what follows $e \equiv 1]$. Consider the system in a constant gauge field background $A(t) = C$ [in (0+1) dimensions, not only the field strength $E = \dot{A}$, but also the potential $A$ cannot be disposed of by a gauge transformation and has direct physical meaning]. Then the charged fields $\phi, \chi, \psi$ acquire the mass $|A|$. If this mass is much larger than the energy scale set up by the gauge kinetic term $(1/2)\partial^2/(\partial A)^2$ in the effective Hamiltonian, $E_{\text{char}} \sim 1/A^2$, we can treat the charged variables as “fast” (in the Born–Oppenheimer framework) and integrate them over. Actually, in a constant background all corrections to the effective potential vanish, this is guaranteed by supersymmetry.

Nontrivial corrections to $L_{\text{eff}}$ are obtained if considering a slowly changing background

$$A(t) = C + Et. \quad (2.2)$$

The leading correction (the one which vanishes in the $\mathcal{N} = 4$ case) is proportional to $E^2$. The calculation of the 1–loop contributions to $L_{\text{eff}}$ was described in [4], and our task here is to tackle the two-loop graphs drawn in Fig. 1 in the background (2.2).

The calculation to be done is very similar in spirit to the calculation of charge renormalization in 4D SQED performed in Ref. [5]. In 4 dimensions, one also has to evaluate the graphs in Fig. 1 substituting there the Green’s functions in the constant field strength background

$$A_\mu = -\frac{1}{2} F_{\mu\nu} x_\nu. \quad (2.3)$$

\footnote{The corresponding technique was developed in Ref. [10].}
Figure 1: Graphs contributing to $L_{\text{eff}}^{(2)}$. Thin solid lines describe fermions, bold lines – scalars, dashed lines - photon and dotted line - photino.

Adding all contributions, one obtains

$$\beta^{(2)} = \frac{1}{\pi^2}$$

for the second coefficient in the $\beta$ function, which leads to (1.7).

The calculation in the QM limit is much more difficult, however, because (i) Lorentz invariance is lost and (ii) in contrast to the background (2.3), the background (2.2) is genuinely noninvariant with respect to time translation. A similar (actually, more complicated) calculation was performed for non-Abelian theories [6, 7]. Unfortunately, these papers are not written in a "user-friendly" way and, for the benefit of future users, we took pain to describe the technical details of the calculation at some length.

3 Boring technicalities.

First of all, we go over into Euclidean space, $t \to -i\tau$. The background is still

$$A(\tau) = C + E\tau$$

(3.1)
with Euclidean $E_E$ which differs from the physical Minkowskian $E_M$ by the factor $i$. We will calculate the effective action in the Euclidean background \( \text{(3.1)} \) assuming $E_E$ real, not forgetting to change the sign of the term $\propto E^2$ in the end of the day. In the background \( \text{(2.2)} \), the scalar and fermion Green’s functions $D$ and $G$ depend on both initial and final times $\tau, \tau'$ in a nontrivial way. The exact expressions can be found in Refs. [6, 7]. Instead of working with them directly, it is convenient to make a Fourier transform over the variable $\tau_- = \tau' - \tau$ and represent

\[
D(\tau, \tau') = \int \frac{d\epsilon}{2\pi} e^{i\epsilon\tau_-} D(\epsilon, \tau_+),
\]

\[
G(\tau, \tau') = \int \frac{d\epsilon}{2\pi} e^{i\epsilon\tau_-} G(\epsilon, \tau_+), \tag{3.2}
\]

where $\tau_+ = (\tau + \tau')/2$. Next, we expand $D(\epsilon, \tau_+)$ and $G(\epsilon, \tau_+)$ in $E$ at fixed $A(\tau_+) = C + E\tau_+$. The leading terms are

\[
D^{(0)}(\epsilon, \tau_+) = \frac{1}{\epsilon^2 + A^2(\tau_+)},
\]

\[
G^{(0)}(\epsilon, \tau_+) = \frac{1}{\epsilon\gamma^0 + A(\tau_+)\gamma}, \tag{3.3}
\]

where $\gamma^0$ and $\gamma$ are Euclidean $\gamma$ matrices, $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^\mu_\nu$. For our purposes, we only need linear and quadratic terms in the expansion of $D(\epsilon, \tau_+)$, $G(\epsilon, \tau_+)$ in $E$. One can work them out from the general expressions of Refs. [6, 7] or, alternatively, use the results of Shuryak and Vainshtein [11] who calculated the 4D scalar and fermion Green’s functions in a generic Euclidean constant field strength background, \cite{11}

\[
D(p) = \frac{1}{\hat{p}^2 - \frac{2}{p^8} \left[ p_\alpha F_{\alpha\beta} F_{\beta\gamma} p_\gamma + \frac{1}{4} F_{\mu\nu}^2 p^2 \right]},
\]

\[
G(p) = \frac{1}{\hat{p}^4} + \frac{i F_{\alpha\beta}}{4p^4} (\hat{p}\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\hat{p}) - \frac{2p_\alpha}{p^8} F_{\alpha\beta} F_{\beta\gamma}(\hat{p}p_\gamma - p^2 \gamma_\gamma). \tag{3.4}
\]

\text{If performing the average $F_{\alpha\beta} F_{\beta\gamma} \rightarrow -\frac{1}{4} \delta_{\alpha\gamma} \langle F^2 \rangle$ as is usually done in most (though not all, hence the paper \cite{11}) QCD applications, the quadratic terms in Eq. (3.4) vanish.}
In our case, we have to set \( p = (\epsilon, A) \), \( F_{0j} = E_j, F_{jk} = 0 \). We obtain

\[
D(\epsilon, \tau_+) = \frac{1}{p^2} + \frac{1}{p^8} \left[ \mathbf{E}^2(\epsilon^2 - A^2) + 2(\mathbf{E}A)^2 \right],
\]

\[
G(\epsilon, \tau_+) = \frac{1}{p} + \frac{iE_k}{2p^4}(\phi\gamma^0\gamma^k + \gamma^0\gamma_k\phi)
+ \frac{2}{p^8} \left[ \epsilon\mathbf{E}^2(\epsilon\phi - p^2\gamma^0) + p(\mathbf{E}A)^2 - p^2(\mathbf{E}A)(\mathbf{E}\gamma) \right],
\]

(3.5)

\( (p^2 \equiv \epsilon^2 + A^2, \ \dot{\phi} \equiv \epsilon\gamma^0 + A\gamma \) and \( A \) is evaluated at \( \tau_+ \). The photon Green’s function can be chosen in the gauge \( A_0 = 0 \) (which is equivalent to the Landau gauge \( \partial_\mu A_\mu = 0 \) in the QM limit),

\[
D_{jk}(\tau - \tau') = \delta_{jk} \int \frac{d\omega}{2\pi\omega^2} e^{i\omega(\tau - \tau')} .
\]

(3.6)

The 3–point scalar–photon vertex entering the graph in Fig. 1b appears due to nonvanishing background:

\[
\Gamma_{\phi\phi A_k} = \Gamma_{\chi\chi A_k} = -2i(C_k + E_k\tau) .
\]

(3.7)

The graphs in Fig. 1 determine the correction to the effective action \( S_{\text{eff}} \). The corresponding analytic expressions involve the integrals over the time moments referring to the vertices — the integral \( \int d\tau \) for the figure-eight graphs in Fig. 1c,e and the double integral over \( d\tau d\tau' \) for other graphs. Let us represent \( d\tau d\tau' = d\tau d\tau_+ \) and postpone the integration over \( d\tau_+ \) (over \( d\tau \) for the figure-eight graphs). The effective action is expressed as

\[
S_{\text{eff}} = \int d\tau \tilde{\mathcal{L}}(\tau) .
\]

(3.8)

Let us call the integrand \( \tilde{\mathcal{L}}(\tau) \) pseudo-Lagrangian. The pseudo-Lagrangian depends on \( \tau \) only via \( A(\tau) \). \( \tilde{\mathcal{L}} \) can be calculated using standard Feynman rules. One only has to take into account the modification of the charged propagators as written in Eq.(3.3) and the appearance of the new vertex
We obtain

\[ \tilde{\mathcal{L}} = -\frac{1}{2} \int \frac{d\epsilon}{2\pi} \int \frac{d\omega}{2\pi \omega^2} \text{Tr}\{\gamma_\beta G(\epsilon) \gamma_\beta G(\epsilon + \omega)\} + \]

\[ \int \frac{d\epsilon}{2\pi} \int \frac{d\omega}{2\pi \omega^2} \left[ A^2 D(\epsilon) D(\epsilon + \omega) + E^2 D''(\epsilon) D(\epsilon + \omega) \right] - 
\]

\[ 6 \int \frac{d\epsilon}{2\pi} D(\epsilon) \int \frac{d\omega}{2\pi \omega^2} - 2 \int \frac{d\epsilon}{2\pi} \int \frac{d\omega}{2\pi \omega} \text{Tr}\{\gamma^0 G(\epsilon)\} D(\epsilon + \omega) 
\]

\[ - \left[ \int \frac{d\epsilon}{2\pi} D(\epsilon) \right]^2, \quad (3.9) \]

where the five terms above correspond to the five graphs in Fig. 1. In the case of constant external field, different contributions in Eq.(3.9) exactly cancel each other as they should.

When \( E \neq 0 \), \( \tilde{\mathcal{L}} \) does not vanish. Note first of all that certain terms in Eq.(3.9) involve a power infrared divergence

\[ \int \frac{d\omega}{2\pi \omega^2} \rightarrow \int \frac{d\omega}{2\pi (\omega^2 + \mu^2)} = \frac{1}{2\mu} \quad (3.10) \]

(\( \mu \) is a fictitious photon mass). This divergence vanishes as everything else does when the background is constant, but the divergence survives when \( E \neq 0 \). This divergences have actually a rather transparent physical meaning which will be clarified a little bit later. For the time being let us just ignore the infrared divergent terms and present the result of our calculation for the finite pieces in \( \tilde{\mathcal{L}} \).

The fermion loop in Fig. 1a gives

\[ \tilde{\mathcal{L}}_{1a} = \frac{7[E^2 A^2 - (EA)^2]}{16 A^8} \quad (3.11) \]

(\( A = |A| \)). The scalar loop of Fig. 1b gives

\[ \tilde{\mathcal{L}}_{1b} = \frac{E^2}{2 A^6} - \frac{7(EA)^2}{8 A^8}. \quad (3.12) \]

The graph in Fig. 1c does not give a finite contribution (only an infrared divergent one). The photino graph in Fig. 1d gives

\[ \tilde{\mathcal{L}}_{1d} = -\frac{5E^2}{8 A^6} + \frac{5(EA)^2}{4 A^8}. \quad (3.13) \]
And finally the scalar self-interaction graph in Fig. 1e gives
\[ \tilde{\mathcal{L}}_{\text{e}} = \frac{E^2}{8A^6} - \frac{5(EA)^2}{16A^8}. \] (3.14)

Adding all pieces, we obtain
\[ \tilde{\mathcal{L}}^2 \text{ loops} = \frac{7E^2}{16A^6} - \frac{3(EA)^2}{8A^8}. \] (3.15)

This cannot be the correct result for the effective Lagrangian, however. As was mentioned before, supersymmetry requires the metric to have a conformally flat form and the terms \( \propto (EA)^2 \) are not allowed. The paradox is solved if noting that by calculating the graphs in time-dependent background we cannot calculate directly the effective Lagrangian, but only the effective action which is equal to the integral of the pseudo-Lagrangian \( \tilde{\mathcal{L}} \) according to the definition (3.5). When doing this integral, we have to restore the time-dependence in Eq. (3.15). It is convenient (though not at all necessary) to assume that the constant and linear term are orthogonal to each other, \( CE = 0 \). Then
\[ A^2 \rightarrow C^2 + E^2\tau^2, \quad EA \rightarrow E^2\tau. \]

The integrals are easily done:
\[ \int_{-\infty}^{\infty} d\tau \frac{E^2}{(C^2 + E^2\tau^2)^3} = \frac{3\pi E}{8C^5}, \]
\[ \int_{-\infty}^{\infty} d\tau \frac{E^4\tau^2}{(C^2 + E^2\tau^2)^4} = \frac{\pi E}{16C^5}. \] (3.16)

In other words, the contribution of the structure \( (EA)^2/A^8 \) in \( \tilde{\mathcal{L}} \) into \( S_{\text{eff}} \) is exactly 6 times less than the contribution of the structure \( E^2/A^6 \). To find the effective Lagrangian, we should take into account the requirement of supersymmetry \( \mathcal{L} \propto E^2/A^6 \) and also require that \( S_{\text{eff}} = \int d\tau \mathcal{L}_{\text{eff}}(\tau) \). This finally gives
\[ \mathcal{L}_{\text{eff}}^2 \text{ loops} = \frac{3E^2_E}{8A^6} = \frac{3E^2_M}{8A^6}, \] (3.17)
from which the result (1.10) follows.
Exactly this procedure [(i) calculating the full time integral for $S_{\text{eff}}$ and (ii) restoring from this $\mathcal{L}_{\text{eff}}(\tau)$ using supersymmetry requirements] was used (if not spelled out explicitly) in Ref.[6]. We convinced ourselves in it after partially reproducing their calculations (see Appendix).

The situation may seem somewhat paradoxical. After all, the effective Lagrangian has a well defined meaning: one can derive it by canonical rules from the effective Born–Oppenheimer Hamiltonian calculated using the philosophy of Ref.[1]. On the other hand, we cannot calculate it directly in the Lagrangian approach. In particular, one can ask what happens in the nonsupersymmetric case where both structures $E^2/A^6$ and $(EA)^2/A^8$ are allowed. The answer to the last question is clear, however. The matter is that, in nonsupersymmetric case, one just cannot consistently define what is an effective Lagrangian (or Hamiltonian). Indeed, the zero point energy contribution to a nonsupersymmetric effective potential is $\sim A$ which is of the same order as the characteristic energy of the fast charged field excitations and the necessary separation of scales is absent.

On the contrary, in supersymmetric case the notion of $H_{\text{eff}}$ is well defined. At the same time, supersymmetry imposes constraints on the allowed form of the effective Lagrangian which is thereby restored unambiguously. If you will, you may consider this as an indirect proof that the metric on supersymmetric moduli space should be conformally flat.

Let us return now to the question of infrared divergences. They are present in the graphs in Fig. 1(a,b,c). Introducing the photon mass as in Eq.(3.10), we derive

$$
\hat{\mathcal{L}}_{1a}^{IR} = \frac{5[(EA)^2 - E^2A^2]}{16\mu A^7},
$$

$$
\hat{\mathcal{L}}_{1b}^{IR} = \frac{35(EA)^2 - 17E^2A^2}{32\mu A^7},
$$

$$
\hat{\mathcal{L}}_{1c}^{IR} = \frac{3E^2}{8\mu A^5} - \frac{15(EA)^2}{16\mu A^7}.
$$

We note, however, that such calculation is going to be rather difficult. The contribution $\sim E^2/A^6$ in $H_{\text{eff}}$ involves two extra orders in the Born–Oppenheimer parameter $\gamma \sim x_{\text{fast}}/x_{\text{slow}} \sim 1/\sqrt{A^3}$ compared to that calculated in [1].
and

\[
\tilde{\mathcal{L}}_{IR}^{tot} = \frac{15[(EA)^2 - E^2A^2]}{32\mu A^7}
\]

for the sum. Integrating this over \(d\tau\), we obtain

\[
-\frac{E}{2\mu C^4}.
\]

Here the contribution of the term \((EA)^2\) in the integral is \(\text{five}\) (rather than six) times smaller than the contribution of the term \(E^2A^2\). This gives

\[
\mathcal{L}^{IR} = \frac{3E^2}{8\mu A^5}
\]

in Minkowski space. It is clear, however, that the divergence is due to the virtual photons with low energies while the true effective Lagrangian is related to the graphs with only heavy degrees of freedom (with energy \(\sim A\)) circulating in the loops. The contribution \((3.20)\) should be actually interpreted as the iteration of \(\mathcal{L}^{1\text{ loop}}_{\text{eff}}\), as shown in Fig. 2, and should not be included into the definition of \(\mathcal{L}^{2\text{ loops}}_{\text{eff}}\). Indeed, evaluating the diagram in Fig. 2, we obtain

\[
\Delta \mathcal{L}^{IR} = \frac{1}{2} \left. \frac{\partial^2}{\partial A_j \partial A_j} \mathcal{L}^{1\text{ loop}}_{\text{eff}} \right| \int \frac{id\omega_M}{2\pi \omega_M^2} = \frac{E^2}{2} \Delta \left( \frac{1}{4A^3} \right) \int \frac{d\omega_E}{2\pi \omega_E^2}
\]

\((\omega_M \text{ and } \omega_E \text{ are the photon frequencies before and after Wick’s rotation}), \text{ which coincides with Eq. } (3.20)\).

Note that the infrared divergencies of the discussed type cancel out in the two–loop contributions to the quartic terms \(\propto (E^2)^2\) in \(\mathcal{L}_{\text{eff}}\) in \(\mathcal{N} = 4\) theories.
This simply follows from the fact that the one–loop contribution in $\mathcal{L}_{\text{eff}}$ is proportional to $(E^2)^2/A^7$, where $A$ is now a 9–dimensional vector and that the 9–dimensional Laplacian of $1/A^7$ vanishes for $A \neq 0$. In the $\mathcal{N} = 1$ case, the infrared divergences survive in the sum of the contributions in Fig. 1, but, as we explained, they anyway should not be taken into account in $\mathcal{L}_{\text{eff}}$.

4 $\mathcal{N} = 2$ Electrodynamics

This calculation can be easily generalized to the case $4D$, $\mathcal{N} = 2$ QED, which is equivalent to $6D$, $\mathcal{N} = 1$ QED in the quantum–mechanical limit. Thinking in the (3+1)-dimensional terms, we have now two (rather than one) photino fields $\lambda_1, \lambda_2$ and two extra real neutral scalars $a$ and $b$ [in the (5+1)–
dimensional language they correspond to the components $A_{4,5}$ of the gauge field]. As in the $\mathcal{N} = 1$ case, we have a Dirac spinor and two complex scalars in the charged sector.

The form of the reduced Lagrangian can be easily derived from the known 4–dimensional expression [13]. We have

$$\mathcal{L} = \frac{1}{2} \left( \dot{A}_k^2 + \dot{a}^2 + \dot{b}^2 \right) + \dot{\phi} \dot{\phi} + \dot{\chi} \dot{\chi} + i \sum_{f=1,2} \bar{\lambda}_f \dot{\lambda}_f + i \bar{\psi} \dot{\psi}$$

$$- (\bar{\varphi} \varphi + \bar{\chi} \chi) \left( A_k^2 + a^2 + b^2 \right) - \frac{1}{2} (\bar{\varphi} \varphi + \bar{\chi} \chi)^2 + i A_k \bar{\psi} \gamma_k \psi + \bar{\psi} (a + i b \gamma^5) \psi$$

$$+ \sqrt{2} \left[ \chi (\bar{\psi}_L \lambda_1 + \bar{\psi}_R \lambda_2) - \phi (\bar{\psi}_R \lambda_1 + \bar{\psi}_L \lambda_2) + H.c. \right]. \quad (4.1)$$

The effective Lagrangian described the motion over 5–dimensional moduli space $(A_k, a, b)$. Let us calculate it assuming as before that the background has the form (2.2) (i.e. $a$ and $b$ vanish). This is convenient because we can use the same expressions (3.5) for the Green’s functions as before. For a generic background, $\mathcal{L}_{\text{eff}}$ can be restored from rotational $O(5)$ invariance and supersymmetry.

The effective action is described as before by the graphs in Fig. 1, where the dashed lines stand now for the gauge fields and also the neutral scalars $a, b$. It is obvious that the photino contribution of Fig. 1d is now multiplied by 2. A simple combinatorics tells us that the contribution due to scalar self–interactions of Fig. 1e is multiplied by the factor 3 compared to the $\mathcal{N} = 1$ case due to the positive relative sign in the scalar potential in Eq. (1.1).
The contribution of the scalar loop in Fig. 1b is the same as before (there is no contribution from $a, b$ exchange because the background was chosen 3–dimensional). The contribution of the graph in Fig. 1c is multiplied by $5/3$ (it is just the counting of degrees of freedom). Extra degrees of freedom $a, b$ also give a nontrivial contribution in the fermion loop in Fig. 1a, 
\[
\Delta \tilde{\mathcal{L}}_{\text{ferm}} = \int \frac{d\epsilon}{2\pi} \int \frac{d\omega}{2\pi \omega^2} \text{Tr} \{G(\epsilon)G(\epsilon + \omega)\} = -\left[\frac{E^2 A^2 - (EA)^2}{8A^8}\right] + \frac{5\left[E^2 A^2 - (EA)^2\right]}{16\mu A^7},
\]
(4.2)

The sum of all infrared divergent pieces gives 
\[
\frac{3E^2}{32\mu A^5} - \frac{15(EA)^2}{32\mu A^7}
\]
(4.3)

As was explained in the previous section, we have to substitute here $(EA)^2/A^7 \rightarrow E^2/(5A^5)$, after which the contribution vanishes. This cancellation has the same origin as the cancellation of the infrared divergences $\propto (E^2)^2/(\mu A^9)$ in $\mathcal{N} = 4$ theory mentioned above and follows from the fact that the 5–dimensional Laplacian of $1/(A^2 + a^2 + b^2)^{3/2}$ vanishes.

The finite contribution to the pseudo-Lagrangian is 
\[
\tilde{\mathcal{L}}_{\mathcal{N}=2}^{\text{2 loops}}(\tau) = -\frac{E^2}{16A^6} + \frac{3(EA)^2}{8A^8}.
\]
(4.4)

The coefficient of the second term in Eq. (4.4) is exactly 6 times greater than that for the first term and the integral of this expression over $d\tau$, from which the effective Lagrangian is extracted, just vanishes. This should have been expected, of course: supersymmetry and rotational invariance require that all higher–loop corrections to the effective Lagrangian of $\mathcal{N} = 2$ theory vanish [14].

5 Conclusions

The main result of this paper is quoted in the abstract. We also verified that the two–loop corrections vanish in the $\mathcal{N} = 2$ case. We did not achieve our goal, however, and did not establish an operative relationship between the
two–loop corrections to $L_{\text{eff}}$ in the matrix models and in 4D theories. For example, the figure-eight graph gives a nontrivial contribution (3.14) in the $(0 + 1)$ case, but its contribution to the 4D $\beta$ function vanishes (this follows from the vanishing of the corrections to the scalar Green’s function in the Lorentz–invariant situation).

On the other hand, one can note that the 4–dimensional $\beta$ function and effective Lagrangian are not, strictly speaking, uniquely defined beyond one loop. There is the Wilsonean definition, according to which the higher loops vanish not only for $\mathcal{N} = 2$, but also for $\mathcal{N} = 1$ theories. Eq.(1.7) is written in the conventional definition where $\beta$ function involves the large-distance contribution associated with anomalous dimensions of the charged fields. [9, 16]. Perhaps, the metric in Eq.(1.6) (defined quite unambiguously) is related to the 4–dimensional $\beta$ function defined in some particular physically relevant way? Further studies in this direction are welcome.

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Appendix. A sample non-Abelian calculation.

For pedagogical purposes, we sketch here a sample calculation of the “figure-eight” gauge boson graph for the $SU(2)$ $\mathcal{N} = 4$ matrix model in some more details than it was done in Ref.[3].

Choose the Abelian background $A_{0}^{\text{cl}} = 0$ and

$$A^{\text{cl}}(\tau) = \frac{1}{2}(b + v\tau)\sigma^3,$$

(A.1)

where $b$ and $v$ are two orthogonal 9–dimensional vectors. We are using the notations of Ref.[3] ($r = b + v\tau$ is interpreted as the distance between two $D0$ particles, $b$ is their impact parameter, and $v$ is their respective velocity), bearing in mind the identifications $r \equiv C, v \equiv E$.

We represent $A_{\mu}^{\text{eq}} = A_{\mu}^{\text{cl}} + a_\mu$ and impose the Feynman background gauge [13] adding to the Lagrangian the term $-\frac{1}{2}(D_\mu a_\mu)^2$, where $D_\mu$ is the covariant derivative associated with the background. We obtain for the quadratic and quartic part of the gauge field Euclidean Lagrangian

$$\mathcal{L} = -\text{Tr} \left\{ a_\mu \left( D^2 a_\mu - 2i[F^{\text{cl}}_{\mu\nu}, a_\nu] \right) \right\} - \frac{1}{2} \text{Tr} \left\{ [a_\mu, a_\nu]^2 \right\} + \text{other terms} \quad \text{(A.2)}$$
The Abelian fluctuation components \( a_{\mu}^{ab} \propto \sigma^3 \) do not interact with the background and remain massless. The corresponding figure-eight graphs are infrared divergent (like the graph in Fig. 1c above). Following Ref. [4], we ignore them here. On the other hand, the components \( \propto \sigma^{1,2} \) provide a nontrivial finite contribution in \( \mathcal{L}_{\text{eff}} \). It is convenient to represent \( a_{\mu} = (\bar{\phi}_\mu \sigma^+ + \phi_\mu \sigma^-)/\sqrt{2} \), after which Eq. (A.2) is rewritten as

\[
\dot{\bar{\phi}}_\mu \dot{\phi}_\mu + (b^2 + v^2 \tau^2) \bar{\phi}_\mu \phi_\mu - 2iF_{\mu\nu} \bar{\phi}_\mu \phi_\nu - \frac{1}{2} (\bar{\phi}_\mu \phi_\nu - \bar{\phi}_\nu \phi_\mu)^2
\]

(A.3)

\((F_{0j} = v_j, \ F_{jk} = 0)\). The contribution of the figure-eight graph to the effective pseudo-Lagrangian \( \tilde{\mathcal{L}}_{\text{eff}} \) has the form

\[
\Delta_{\mu\nu} \Delta_{\mu\nu} - \frac{1}{2} \Delta_{\mu\nu} \Delta_{\mu\nu} - \frac{1}{2} \Delta_{\mu\nu} \Delta_{\mu\nu}
\]

(A.4)

where \( \Delta_{\mu\nu} \) is the Green’s function \( \langle \bar{\phi}_\mu(\tau) \phi_\nu(\tau) \rangle \). Now, if the term \( \propto F_{\mu\nu} \) in Eq. (A.3) were absent, \( \Delta_{\mu\nu} \) would exactly coincide with the scalar Green’s function in Eqs. (3.2), (3.5), multiplied by \( \delta_{\mu\nu} \). In this case, the quadratic in \( E \) terms in the pseudo-Lagrangian would have the same form as in Eq. (3.14) up to the overall factor \( [(\delta_{\mu\mu})^2 - \delta_{\mu\nu} \delta_{\mu\nu}] / 2 = 45 \). Doing the integral over \( d\tau \) and assuming that \( \int d\tau \tilde{\mathcal{L}}_{\text{eff}}(\tau) = \int d\tau \mathcal{L}_{\text{eff}}(\tau) \) and that the effective Lagrangian does not involve the terms \( \propto (v \tau)^2 \), this gives the correction

\[
\frac{105v^2}{32r^6}
\]

(A.5)

in \( \mathcal{L}_{\text{eff}} \). Let us now take into account the modification of \( \Delta_{\mu\nu} \) due to the presence of the term \( \propto F_{\mu\nu} \) in the quadratic part of the Lagrangian. As we are interested only in the terms \( \propto v^2 \), we can neglect now \( v \)-dependent terms in \( D^2 \) (or, better to say, treat \( r^2 = b^2 + v^2 \tau^2 \) as a constant). We obtain

\[
\Delta_{\mu\nu} \sim \int \frac{d\omega}{2\pi} \left( \frac{\delta_{\mu\nu}}{\omega^2 + r^2} \right) - 2iF_{\mu\nu} \right|^{-1} =
\]

\[
\int \frac{d\omega}{2\pi} \left[ \frac{\delta_{\mu\nu}}{\omega^2 + r^2} + \frac{2iF_{\mu\nu}}{(\omega^2 + r^2)^2} - \frac{4F_{\mu\alpha}F_{\alpha\nu}}{(\omega^2 + r^2)^3} + \ldots \right]
\]

(A.6)

Substituting this into (A.4) and adding the term (A.3), we obtain

\[
- \frac{45}{4r^2} - \frac{15v^2}{2r^6} + \frac{105v^2}{32r^6} = \frac{45}{4r^2} - \frac{135v^2}{32r^6}
\]

(A.7)

which coincides with the sum of the contributions (5.6) and (5.7) in Ref. [3] (with the terms \( \propto v^4/r^{10} \) neglected).
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