Another proof of Ricci flow on incomplete surfaces with bounded above Gauss curvature

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Abstract

We give a simple proof of an extension of the existence results of Ricci flow of G. Giesen and P.M. Topping [GiT1], [GiT2], on incomplete surfaces with bounded above Gauss curvature without using the difficult Shi’s existence theorem of Ricci flow on complete non-compact surfaces and the pseudolocality theorem of G. Perelman [P1] on Ricci flow. We will also give a simple proof of a special case of the existence theorem of P.M. Topping [T] without using the existence theorem of W.X. Shi [S1].

Key words: Ricci flow, incomplete surfaces, negative Gauss curvature

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Recently there is a lot of study on the Ricci flow on manifold by R. Hamilton [H1], [H2], S.Y. Hsu [Hs1–4], G. Perelman [P1], [P2], W.X. Shi [S1], [S2], L.F. Wu [W1], [W2], and others because Ricci flow is a powerful tool in the study of geometric problems. We refer the readers to the book [CLN] by B. Chow, P. Lu and L. Ni, for the basics of Ricci flow and the papers [P1], [P2], of G. Perelman and the book [Z] by Qi S. Zhang for the most recent results on Ricci flow.

In 1982 R. Hamilton [H1] proved that if $M$ is a compact manifold and $g_{ij}(x)$ is a metric of strictly positive Ricci curvature, then there exists a unique metric $g$ that evolves by the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

on $M \times (0, T)$ for some $T > 0$ with $g_{ij}(x, 0) = g_{ij}(x)$ where $R_{ij}(\cdot, t)$ is the Ricci curvature of $g(\cdot, t)$. 

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Short time existence of solutions of the Ricci flow on complete non-compact Riemannian manifold with bounded curvature was proved by W.X Shi [S1]. Global existence and uniqueness of solutions of the Ricci flow on non-compact manifold $\mathbb{R}^2$ was obtained by S.Y. Hsu in [Hs1]. Existence and uniqueness of the Ricci flow on incomplete surfaces with negative Gauss curvature was obtained by G. Giesen and P.M. Topping in [GiT1]. In [GiT1] G. Giesen and P.M. Topping proved the following theorem.

**Theorem 1.** (Theorem 1.1 of [GiT1]) Suppose $M$ is a surface (i.e. a 2-dimensional manifold without boundary) equipped with a smooth Riemannian metric $g_0$ whose Gauss curvature satisfies $K[g_0] \leq -\eta < 0$, but which need not be complete. Then exists a unique smooth Ricci flow $g(t)$ for $t \in [0, \infty)$ with the following properties:

(i) $g(0) = g_0$;
(ii) $g(t)$ is complete for all $t > 0$;
(iii) the curvature of $g(t)$ is bounded above for any compact time interval within $[0, \infty)$;
(iv) the curvature of $g(t)$ is bounded below for any compact time interval within $(0, \infty)$.

Moreover this solution satisfies $K[g(t)] \leq -\frac{\eta}{1+\eta}$ for $t \geq 0$ and $-\frac{1}{2t} \leq K[g(t)]$ for $t > 0$.

By abuse of notation we will write $K[u] = K[g]$ for the Gauss curvature of a metric of the form $g = e^{2u}\delta_{ij}$. As observed by G. Giesen and P.M. Topping [GiT1] in order to prove Theorem 1 it suffices to assume that $M = D$ is a unit disk in $\mathbb{R}^2$ and $g_0 = e^{2u_0}\delta_{ij}$ is a conformal metric on $D$. Then by scaling Theorem 1 is equivalent to the following two theorems.

**Theorem 2.** (cf. Theorem 3.1 and Lemma 2.2 of [GiT1]) Let $g_0 = e^{2u_0}\delta_{ij}$ be a smooth conformal metric on the unit disk $D$ with

$$K[u_0] \leq -1. \quad (2)$$

Then there exists a smooth solution $g(t) = e^{2u}\delta_{ij}$ of (1) in $D \times [0, \infty)$ with $g(0) = g_0$ such that $g(t)$ is complete for every $t > 0$ with the Gauss curvature $K[u(t)]$ satisfying

$$K[u(t)] \geq -\frac{1}{2t} \quad \forall t > 0, \quad (3)$$
$$K[u(t)] \leq -\frac{1}{2t+1} \quad \forall t \geq 0, \quad (4)$$
$$u(x,t) \geq \log \frac{2}{1-|x|^2} + \frac{1}{2}\log(2t) \quad \forall x \in D, t > 0, \quad (5)$$
$$u(x,t) \leq \log \frac{2}{1-|x|^2} + \frac{1}{2}\log(2t+1) \quad \forall x \in D, t \geq 0, \quad (6)$$
and
\[ u(x, t) \geq u_0(x) + \frac{1}{2} \log(2t + 1) \quad \text{in } \mathcal{D} \times [0, \infty). \]  
Moreover \( g(t) \) is maximal in the sense that if \( \tilde{g}(t) \) for \( t \in [0, \varepsilon] \) is another Ricci flow with \( \tilde{g}(0) = g_0 \) for some \( \varepsilon > 0 \), then
\[ \tilde{g}(t) \leq g(t) \quad \forall 0 \leq t \leq \varepsilon. \]  

**Theorem 3.** (cf. Theorem 4.1 of [GiT1]) Let \( e^{2u_0} \delta_{ij} \) be a smooth metric on the unit disk \( \mathcal{D} \) satisfying the upper curvature bound (2). Let \( e^{2u} \delta_{ij} \) be a solution of (1) in \( \mathcal{D} \times (0, \infty) \) with \( u(\cdot, 0) = u_0 \) which satisfies (3) and (4). Then \( u \) is unique among solutions that satisfy (3) and (4).

The proof of Theorem 3.1 and Lemma 2.2 of [GiT1] uses the results of [1], the Schwartz Lemma of S.T. Yau [Y], and the difficult existence theorem of W.X. Shi [S1] for Ricci flow on complete non-compact manifolds. In this paper we will give a simple proof of Theorem 2 using the results of K.M. Hui in [Hu3] and [Hu4]. We will assume that \( M = \mathcal{D} \subset \mathbb{R}^2 \) is a unit disk for the rest of the paper. Note that for a metric \( g \) on a 2-dimensional manifold, \( \text{Ric}[g] = K[g]g \). We will also give simple proofs of the following extension of the existence results of G. Giesen and P.M. Topping [GiT2] and a special case of Theorem 1.1 of [T] without using the existence theorem of W.X. Shi [S1] and the pseudolocality Theorem of G. Perelman [P1] on Ricci flow.

**Theorem 4.** (cf. Theorem 3.1 of [GiT2] and Theorem 1.1 of [T]) Let \( g_0 = e^{2u_0} \delta_{ij} \) be a smooth (possible incomplete) Riemannian metric on \( \mathcal{D} \). Then there exists a maximal instantaneous smooth complete Ricci flow \( g(t) = e^{2u} \delta_{ij} \) on \( \mathcal{D} \) for all time \( t \in [0, \infty) \) with \( g(0) = g_0 \) which satisfies (3) and (5). Suppose in addition the Gauss curvature satisfies
\[ K[g_0] \leq K_0 \]  
for some constant \( K_0 \geq 0 \). Then the following holds.

(i) If \( K_0 > 0 \), then
\[ K[u(t)] \leq \frac{1}{K_0^2 - 2t} \quad \forall 0 \leq t < (2K_0)^{-1} \]  
and
\[ u(x, t) \geq u_0(x) + \frac{1}{2} \log(1 - 2K_0 t) \quad \forall 0 \leq t < (2K_0)^{-1}. \]  
(ii) If \( K_0 = 0 \), then
\[ K[u(t)] \leq 0 \quad \forall t \geq 0 \]  
and
\[ u(x, t) \geq u_0(x) \quad \forall t \geq 0. \]
Theorem 5. (cf. [DPI], [Hu2], [Hs1], Theorem 1.1 of [T] and Theorem 3.2 of [GiT2]) Let 
\[ g_0 = e^{2u_0} \delta_{ij} \] be a smooth metric on \( \mathbb{R}^2 \) which need not be complete. Then there exists a maximal instantaneous smooth complete Ricci flow 
\[ g(t) = e^{2u(t)} \delta_{ij} \] on \( \mathbb{R}^2 \) for \( t \in [0, T) \) with 
\[ g(0) = g_0 \] which satisfies (3) and for any \( 0 < T_1 < T \) and \( r_0 > 1 \) there exists a constant \( C > 0 \) such that 
\[ u(x, t) \geq -C - \log(|x| \log |x|) + \frac{1}{2} \log(2t) \quad \forall |x| \geq r_0, 0 \leq t \leq T_1 \] 
(14)

where 
\[ T = \begin{cases} \frac{\text{Vol}_{g_0}(\mathbb{R}^2)}{4\pi} & \text{if } \text{Vol}_{g_0}(\mathbb{R}^2) < \infty \\ \infty & \text{if } \text{Vol}_{g_0}(\mathbb{R}^2) = \infty \end{cases} \] 
(15)

and 
\[ \text{Vol}_{g(t)}(\mathbb{R}^2) = \begin{cases} 4\pi(T - t) & \forall 0 \leq t < T \quad \text{if } \text{Vol}_{g_0}(\mathbb{R}^2) < \infty \\ \infty & \forall t > 0 \quad \text{if } \text{Vol}_{g_0}(\mathbb{R}^2) = \infty \end{cases} \] 
(16)

If in addition the Gauss curvature satisfies (9) for some constant \( K_0 \geq 0 \), then the following holds.

(i) If \( K_0 > 0 \), then (10) and (11) holds on \( \mathbb{R}^2 \) for any \( 0 \leq t < (2K_0)^{-1} \) and \( T \geq (2K_0)^{-1} \).

(ii) if \( K_0 = 0 \), then (12) and (13) holds on \( \mathbb{R}^2 \) for any \( t \geq 0 \) and \( T = \infty \).

We start with some definitions. For any \( r_1 > 0, T_1 > 0 \), let \( B_{r_1} = \{ x \in \mathbb{R}^2 : |x| < r_1 \} \), 
\( Q_{r_1} = B_{r_1} \times (0, \infty), \) \( Q_{r_1}^{T_1} = B_{r_1} \times (0, T_1) \), and 
\( \partial_B Q_{r_1} = (\partial B_{r_1} \times [0, \infty)) \cup (\overline{B_{r_1}} \times \{0\}) \). For any set \( A \subset \mathbb{R}^2 \), let \( \chi_A \) be the characteristic function of the set \( A \). Note that for any metric of the form \( e^{2u(x,t)} \delta_{ij} \) in \( Q_{r_1}^{T_1} \), we have 
\[ K[u] = -\frac{\Delta u}{e^{2u}} \]
and \( e^{2u(x,t)} \delta_{ij} \) is a solution of (11) in \( Q_{r_1}^{T_1} \) if any only if 
\[ \frac{\partial u}{\partial t} = e^{-2u} \Delta u \quad \text{in } Q_{r_1}^{T_1}. \]
(17)
where \( \Delta \) is the Euclidean Laplacian on \( \mathbb{R}^2 \). Let 
\[ v = e^{2u}. \]
(18)

Then (17) is equivalent to 
\[ \frac{\partial v}{\partial t} = \Delta \log v \quad \text{in } Q_{r_1}^{T_1}. \]
(19)

Existence and various properties of (19) were studied by P. Daskalopoulos and M.A. Del Pino [DPI], S.H. Davis, E. Dibenedetto, and D.J. Diller [DDD], J.R. Esteban, A. Rodriguez, and J.L. Vazquez [ERV1], [ERV2], S.Y. Hsu [Hs1], K.M. Hui [Hu2], etc. We refer the readers to the book [DK] by P. Daskalopoulos and C.E. Kenig and the book [V] by J.L. Vazquez for the recent results on the equation (19).
For any $0 \leq v_0 \in L^1_{\text{loc}}(B_{r_1})$, we say that $v$ is a solution of
\begin{align*}
\begin{cases}
v_t = \Delta \log v & \text{in } Q_{r_1} \\
v > 0 & \text{in } Q_{r_1} \\
v(x, t) = \infty & \text{on } \partial B_{r_1} \times (0, \infty) \\
v(x, 0) = v_0(x) & \text{in } B_{r_1}
\end{cases}
\end{align*}
(20)
if $v$ is a classical solution of (19) in $Q_{r_1}$,\[\lim_{y \to x} v(y, t) = \infty \quad \forall x \in \partial B_{r_1}, t > 0,\]
\[
\inf_{B_{r_1} \times (t_1, t_2)} v(x, t) > 0 \quad \forall t_2 > t_1 > 0,
\]
and\[
\lim_{t \to 0} \|v(\cdot, t) - v_0\|_{L^1(K)} = 0
\]
(21)
for any compact set $K \subset B_{r_1}$. We say that $v$ is a solution of (19) in $\mathbb{R}^2 \times (0, T)$ with initial value $v_0$ if $v$ is a classical solution of (19) in $\mathbb{R}^2 \times (0, T)$, $v > 0$ in $\mathbb{R}^2 \times (0, T)$, and (21) holds for any compact set $K \subset \mathbb{R}^2$.

Note that for any solution $v$ of (19) the equation (19) is uniformly parabolic on $K$ for any compact set $K \subset Q^T_{t_1}$. Hence by the Schauder estimates [LSU] and a bootstrap argument $v \in C^\infty(Q^T_{t_1})$ for any solution $v$ of (19). We first observe that by the same argument as the proof of Theorem 1.6 of [Hu3] we have the following result.

**Theorem 6.** (cf. Theorem 1.6 of [Hu3]) Let $r_1 > 0$. Suppose $v_0 \geq 0$ satisfies\[v_0 \in L^p_{\text{loc}}(B_{r_1}) \quad \text{for some constants } p > 1.\]
Then exists a solution $v$ of (20) which satisfies\[v_t \leq \frac{v}{t}\]
in $Q_{r_1}$ and\[v(x, t) \geq C \frac{t}{\phi(x)} \quad \text{in } Q_{r_1}\]
for some constant $C > 0$ depending on $\phi$ and $\lambda$ where $\phi$ and $\lambda$ are the first positive eigenfunction and the first positive eigenvalue of the Laplace operator $-\Delta$ on $B_{r_1}$ with $\|\phi\|_{L^2(B_{r_1})} = 1$.

**Lemma 7.** Let $r_1 > 0$. Suppose $v_0$ is a positive smooth function on $B_{r_1}$ and $v$ is a solution of (20). Then $v \in C^\infty(B_{r_1} \times [0, \infty))$.

**Proof:** We will first use a modification of the technique of Lemma 1.6 of [Hu1] to show that $v$ is uniformly bounded below by a positive constant on $\overline{B}_{r_3} \times [0, 1]$ for any $0 < r_3 < r_1$. Let $0 < r_3 < r_2 < r_1$, $\delta_1 = r_3 - r_2$, and\[\varepsilon = \frac{1}{2} \min_{\overline{B}_{r_2}} v_0.\]
Since \( v_0 \) is a smooth positive function on \( B_{r_1}, \varepsilon > 0 \). Let \( w \) be the maximal solution of the equation (19) in \( \mathbb{R}^2 \times (0, T) \) constructed in \([DP1]\) and \([Hu2]\) with initial value \( w(x, 0) = \varepsilon \chi_{B_{\delta_1}} \) and \( T = \varepsilon |B_{\delta_1}| / 4\pi \) which satisfies
\[
\int_{\mathbb{R}^2} w(x, t) \, dx = \int_{\mathbb{R}^2} w(x, 0) \, dx - 4\pi t \quad \forall 0 \leq t < T.
\]
For any \( \alpha > 0 \), let
\[
w_\alpha(x, t) = w(\alpha x, \alpha^2 t) \quad \forall x \in \mathbb{R}^2, 0 \leq t < T / \alpha^2.
\]
and
\[
v_{\alpha, p}(x, t) = v(\alpha x + p, \alpha^2 t) \quad \forall p \in \overline{B}_{r_3}, |x| < \delta_1 / \alpha, t > 0.
\]
Since \( v_0(x + p) \geq w(x, 0) \) for any \( p \in \overline{B}_{r_3}, x + p \in B_{r_1} \), and \( v = \infty \) on \( \partial B_{r_1} \times (0, \infty) \), by Corollary 1.8 and Lemma 2.9 of \([Hu4]\),
\[
v(x + p, t) \geq w(x, t) \quad \forall p \in \overline{B}_{r_3}, x + p \in B_{r_1}, 0 \leq t < T
\]
\[
\Rightarrow \quad v_{\alpha, p}(x, t) \geq w_{\alpha}(x, t) \quad \forall p \in \overline{B}_{r_3}, |x| < \delta_1 / \alpha, 0 \leq t < T / \alpha^2.
\] (23)
Since \( w_\alpha(x, 0) = w(\alpha x, 0) = \varepsilon \chi_{B_{\delta_1 / \alpha}} \),
\[
0 \leq w_{\alpha_1}(x, 0) \leq w_{\alpha_2}(x, 0) \leq \varepsilon \quad \forall x \in \mathbb{R}^2, \alpha_1 \geq \alpha_2 > 0.
\] (24)
By (24) and the comparison theorems in \([DP1]\) and \([Hu2]\),
\[
0 < w_{\alpha_1}(x, t) \leq w_{\alpha_2}(x, t) \leq \varepsilon \quad \forall x \in \mathbb{R}^2, 0 < t < \varepsilon |B_{\delta_1}| / (4\pi \alpha_1^2), \alpha_1 \geq \alpha_2 > 0.
\] (25)
We choose \( 0 < \alpha_0 < \delta_1 \) such that \( \varepsilon |B_{\delta_1}| / (4\pi \alpha_0^2) > 1 \). By (25) the equation (19) for \( w_\alpha \),
\[
0 < \alpha \leq \alpha_0 \), is uniformly parabolic on every compact set \( K \subset \mathbb{R}^2 \times (0, 1) \). Then by the Schauder estimates \([LSU]\) \( \{w_\alpha\}_{0 < \alpha \leq \alpha_0} \) are equi-Holder continuous in \( C^2(K) \) for any compact set \( K \subset \mathbb{R}^2 \times (0, 1) \). Hence by (25) \( w_\alpha \) increases and converges uniformly on every compact subset \( K \) of \( \mathbb{R}^2 \times (0, 1) \) to the constant function \( \varepsilon \) in \( \mathbb{R}^2 \times (0, 1] \) as \( \alpha \to 0 \). Then there exists \( \alpha_1 \in (0, \alpha_0) \) such that
\[
w_\alpha(x, 1) \geq \frac{\varepsilon}{2} \quad \forall |x| \leq 1, 0 < \alpha \leq \alpha_1.
\] (26)
By (23) and (20),
\[
v(\alpha x + p, \alpha^2) = v_{\alpha, p}(x, 1) \geq \frac{\varepsilon}{2} \quad \forall p \in \overline{B}_{r_3}, |x| \leq 1, 0 < \alpha \leq \alpha_1
\]
\[
\Rightarrow \quad v(p + y, t) \geq \frac{\varepsilon}{2} \quad \forall p \in \overline{B}_{r_3}, |y| \leq \sqrt{t}, 0 < t \leq \alpha_1^2.
\] (27)
Let \( \lambda \) and \( \phi \) be the first positive eigenvalue and the first positive eigenfunction of \( -\Delta \) on \( B_{r_2} \)
with \( \|\phi\|_{L^2(B_{r_2})} = 1 \). Let \( C_1 = \max(2\lambda, \|\phi\|_{L^\infty}, 12\|\nabla \phi\|_{L^\infty}, \|v_0\|_{L^\infty(B_{r_1})} + 1) \) and
\[
\psi(x, t) = C_1(t + 1)e^{1/\phi(x)}.
\]
By the computation on P.784-785 of [Hu3], $\psi$ is a supersolution of (19). Then an argument similar to the proof Theorem 2.9 of [Hu4],

$$\int_{B_r} (v - \psi)_+(x, t) \, dx \leq \int_{B_r} (v - \psi)_+(x, t_1) \, dx \quad \forall t \geq t_1 > 0. \quad (28)$$

Since

$$\int_{B_r} (v - \psi)_+(x, t_1) \, dx \leq \int_{B_r} |v(x, t_1) - v_0(x)| \, dx + \int_{B_r} (v_0(x) - \psi(x, t_1))_+ \, dx \to 0 \quad \text{as } t_1 \to 0,$$

letting $t_1 \to 0$ in (28),

$$\int_{B_r} (v - \psi)_+(x, t) \, dx = 0 \quad \forall t > 0.$$

Hence

$$v(x, t) \leq \psi(x, t) = C_1(t + 1)e^{1/\phi(x)} \quad \forall |x| < r_2, t \geq 0.$$

Thus

$$v(x, t) \leq C'(t + 1) \quad \forall |x| \leq r_3, t \geq 0 \quad (29)$$

for some constant $C' > 0$. By (27) and (29), the equation (19) is uniformly parabolic on $B_{r_3} \times [0, 1]$ for any $0 < r_3 < r_1$. By standard parabolic estimates [LSU] and a bootstrap argument, $v \in C^\infty(B_{r_1} \times [0, \infty))$ and the lemma follows.

We will now prove Theorem 2.

**Proof of Theorem 2.** Let $v_0(x) = e^{2u_0(x)}$. For any $k = 2, 3, \ldots$, by Theorem 6 there exists a solution $v_k$ of (20) with $r_1 = 1 - (1/k)$ and initial value $v_0$ which satisfies (22) in $B_{1-(1/k)} \times (0, \infty)$ and

$$v_k(x, t) \geq C_k \frac{t}{\phi_k(x)} \quad \text{in } B_{1-(1/k)} \times (0, \infty) \quad (30)$$

for some constant $C_k > 0$ depending on $\phi_k$ and $\lambda_k$ where $\phi_k$ and $\lambda_k$ are the first positive eigenfunction and the first positive eigenvalue of the Laplace operator $-\Delta$ on $B_{1-(1/k)}$ with $\|\phi\|_{L^2(B_{1-(1/k)})} = 1$. By Lemma 7 $v_k \in C^\infty(B_{1-(1/k)} \times [0, \infty))$. By Corollary 1.8 and Theorem 2.9 of [Hu4],

$$v_k \geq v_{k+1} \quad \text{in } B_{1-(1/k)} \times (0, \infty) \quad \forall k \geq 2. \quad (31)$$

Since $\phi_k$ and $\lambda_k$ converges to $\phi$ and $\lambda$ as $k \to \infty$ where $\phi$ and $\lambda$ are the first positive eigenfunction and the first positive eigenvalue of the Laplace operator $-\Delta$ on $B_1$ with $\|\phi\|_{L^2(B_1)} = 1$, by the proof of Theorem 1.2 and Theorem 1.6 of [Hu3] there exists a constant $C > 0$ such that

$$C_k \geq C \quad \forall k \geq 2. \quad (32)$$

Let $k_0 \geq 2$. Then by (30) and (32) there exists a constant $C_0 > 0$ such that

$$v_k(x, t) \geq C_0 t \quad \text{in } B_{1-(1/k_0)} \times (0, \infty) \quad \forall k > k_0. \quad (33)$$
By Corollary 1.8 of [Hu4] for any $0 < t_1 < t_2$ there exists a constant $C' > 0$ such that
\[ v_k(x,t) \leq C' \quad \forall |x| \leq 1 - (1/k_0), t_1 \leq t \leq t_2, k > k_0. \]  
(34)

By (33) and (34) the equation (19) for $v_k$ are uniformly parabolic on $\overline{Q}_{1-(1/k_0)}$ for all $k > k_0$. By the parabolic Schauder estimates [LSU] $v_k$ are equi-Holder continuous in $C^2(K)$ for every compact set $K \subset \overline{Q}_{1-(1/k_0)}$ and all $k > k_0$. Hence by (31) $v_k$ decreases and converges uniformly on $K$ to a solution $v$ of (19) in $D \times (0, \infty)$ as $k \to \infty$ for any compact set $K \subset D \times (0, \infty)$. Since $v_k$ satisfies (22) in $Q_{1-(1/k)}$ for any $k \in \mathbb{Z}^+$, by an argument similar to the proof of Theorem 2.4 of [Hu3] $v$ has initial value $v_0$ and satisfies (22) in $D \times (0, \infty)$. Letting $k \to \infty$ in (30), by (32),
\[ v(x,t) \geq C \frac{t}{\phi(x)} \quad \text{in } D \times (0, \infty) \]
\[ \Rightarrow \lim_{x \to x_0} v(x,t) = \infty \quad \forall |x_0| = 1, t > 0. \]

Hence $v$ is a solution of (20) with $B_{r_1} = D$. By Lemma 7, $v \in C^\infty(D \times [0, \infty))$.

Let $u$ be given by (18) and $g(t) = e^{2u \delta_j}$. Then $g(t)$ is a smooth solution of (11) in $D \times [0, \infty)$ with initial value $e^{2u_0 \delta_j}$. By (19) and (22), we get (3). By (3),
\[ \Delta u(x,t) \leq \frac{1}{2t} e^{2u(x,t)} \quad \text{in } D \quad \forall t > 0. \]  
(35)

For any $0 < \delta < 1$, let
\[ \psi_\delta(x,t) = \log \frac{2(1+\delta)}{(1+\delta)^2 - |x|^2} + \frac{1}{2} \log(2t) \quad \forall x \in D, t > 0. \]

Then $\psi_\delta$ satisfies
\[ \Delta \psi_\delta(x,t) = \frac{1}{2t} e^{2\psi_\delta(x,t)} \quad \text{in } D \quad \forall t > 0. \]  
(36)

Let $\Omega(t) = \{ x \in D : u(x,t) < \psi_\delta(x,t) \}$. By (35) and (36),
\[ \Delta (u(x,t) - \psi_\delta(x,t)) \leq \frac{1}{2t} (e^{2u(x,t)} - e^{2\psi_\delta(x,t)}) < 0 \quad \text{in } \Omega(t) \quad \forall t > 0. \]
(37)

Since $u(x,t) - \psi_\delta(x,t) \to \infty$ as $|x| \to 1, \Omega(t) \subset D$. Hence by (37) and the maximum principle [GT],
\[ u(x,t) - \psi_\delta(x,t) \geq \min_{x \in \partial\Omega(t)} (u(x,t) - \psi_\delta(x,t)) = 0 \quad \text{in } \Omega(t) \quad \forall t > 0 \]
\[ \Rightarrow u(x,t) \geq \psi_\delta(x,t) \quad \text{in } D \quad \forall t > 0. \]
(38)

Letting $\delta \to 0$ in (38), (3) follows. We now let $v_{k,m}$ be the solution of
\[
\begin{cases}
  v_t = \Delta \log v & \text{in } Q_{1-(1/k)} \\
  v > 0 & \text{in } Q_{1-(1/k)} \\
  v(x,t) = v_0(x) e^{mt^2+2t} & \text{on } \partial B_{1-(1/k)} \times (0, \infty) \\
  v(x,0) = v_0(x) & \text{in } B_{1-(1/k)}
\end{cases}
\]  
(39)
for any $k \geq 2$, $m \in \mathbb{Z}^+$. By an argument similar to the proof of [Hu3] $v_{k,m}$ increases and converges uniformly to $v_k$ on every compact subset $K$ of $Q_{1-(1/k)}$ as $m \to \infty$ and $v_{k,m}$ satisfies

$$0 < \min_{|x| \leq 1-k^{-1}} v_0(x) \leq v_{k,m}(x,t) \leq e^{mT_1^2 + 2T_1} \max_{|x| \leq 1-k^{-1}} v_0(x) \quad \forall |x| \leq 1 - k^{-1}, 0 \leq t \leq T_1, k \geq 2,$$

for any $T_1 > 0$. By (40) the equation (19) for $v_{k,m}$ is uniformly parabolic on $Q_{1-(1/k)}^1$ for any $T_1 > 0$. Hence by (39), (40), and parabolic estimates [LSU], $v_{k,m} \in C^\infty(\overline{Q}_{1-(1/k)})$. Let $p_{k,m} = \partial_t v_{k,m}/v_{k,m}$. Then $p_{k,m} \in C^\infty(\overline{Q}_{1-(1/k)})$. By (2) and (39) $p_{k,m}$ satisfies

$$p_t = e^{-v_{k,m}} \Delta p - p^2 \quad \text{in } Q_{1-(1/k)}$$

and

$$\begin{cases} p = 2mt + 2 & \text{on } \partial B_{1-(1/k)} \times [0, \infty) \\ p(x,0) \geq 2 & \forall |x| \leq 1 - k^{-1}. \end{cases}$$

Note that the function $2/(2t + 1)$ satisfies (41) and by (42), $p_{k,m} \geq 2/(2t + 1)$ on $\partial Q_{1-(1/k)}$.

Hence by the maximum principle (cf. [A]),

$$\frac{\partial_t v_{k,m}}{v_{k,m}} = p_{k,m}(x,t) \geq \frac{2}{2t + 1} \quad \text{in } Q_{1-(1/k)} \forall k \geq 2, m \geq 1$$

$$\Rightarrow \quad \frac{v_t}{v} \geq \frac{2}{2t + 1} \quad \text{in } \mathcal{D} \times [0, \infty) \text{ as } m \to \infty, k \to \infty,$$

and (4) follows. By (4) and an argument similar to the proof of (5) we get (6). By (5) $g(t)$ is complete for any $t > 0$. Now by (4),

$$K[u(t)] \leq -\frac{1}{2t + 1} \quad \text{in } \mathcal{D} \times [0, \infty)$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} \geq \frac{1}{2t + 1} \quad \text{in } \mathcal{D} \times [0, \infty)$$

$$\Rightarrow \quad u(x,t) \geq u_0(x) + \frac{1}{2} \log(2t + 1) \quad \text{in } \mathcal{D} \times [0, \infty)$$

and (7) follows. Suppose $\tilde{g}(t)$, $0 \leq t \leq \varepsilon$, is a solution of (11) in $\mathcal{D} \times (0, \varepsilon)$ with $g(0) = g_0$. As in [GiTT] we can write $\tilde{g} = e^{2u}\delta_{ij}$. Let $\tilde{v} = e^{2u}$. Then $\tilde{v}$ is a solution of (20) with $r_1 = 1$. Hence by Corollary 1.8 and Theorem 2.9 of [Hu4],

$$\tilde{v}(x,t) \leq v_k(x,t) \quad \forall |x| \leq 1 - (1/k), 0 \leq t \leq \varepsilon$$

$$\Rightarrow \quad \tilde{v}(x,t) \leq v(x,t) \quad \forall |x| < 1, 0 \leq t \leq \varepsilon \quad \text{as } k \to \infty$$

and (8) follows.
Proof of Theorem 4 Let \( v_0 = e^{2u_0}, v_k, v_{k,m}, v, u, g(t), p_{k,m} \) be as in the proof of Theorem 3. By the proof of Theorem 3 \( v \in C^\infty(D \times [0, \infty)) \) satisfies (3) and (5) and \( g(t) \) is a smooth maximal instantaneous complete solution of (1) for all time \( t \geq 0 \).

Suppose now (9) holds for some constant \( K_0 \geq 0 \). We will show that the curvature \( K[u(t)] \) satisfies (10). Let \( T_0 = (2K_0)^{-1} \). By (9) and (39), \( p_{k,m} \) satisfies

\[
\begin{cases}
    p(x, t) = 2mt + 2 \geq -\frac{1}{(2K_0)^{-1} - t} & \forall |x| = 1 - k^{-1}, t \geq 0 \\
    p(x, 0) \geq -2K_0 & \forall |x| \leq 1 - k^{-1}.
\end{cases}
\] (43)

Since both \( p_{k,m} \) and \( -1/((2K_0)^{-1} - t) \) satisfy (11), by (43) and the maximum principle,

\[
\frac{\partial}{\partial t} v_{k,m} = p_{k,m}(x, t) \geq -\frac{1}{(2K_0)^{-1} - t} \quad \forall |x| < 1 - (1/k), 0 \leq t < (2K_0)^{-1}
\]

\[
\Rightarrow \frac{v_t}{v} \geq -\frac{1}{(2K_0)^{-1} - t} \quad \forall |x| < 1 - (1/k), 0 \leq t < (2K_0)^{-1} \quad \text{as } m \to \infty, k \to \infty
\]

and (10) follows. By (10) and (17),

\[
\frac{u_t}{u} \geq -\frac{1}{K_0^{-1} - 2t} \quad \forall |x| < 1, 0 < t < (2K_0)^{-1}.
\]

Integrating the above equation with respect to \( t \) and (11) follows.

Suppose now (9) holds with \( K_0 = 0 \). Then by (9) and (39),

\[
p_{k,m} \geq 0 \quad \text{on } \partial p Q_{1-(1/k)}.
\]

Hence by the maximum principle,

\[
\frac{\partial}{\partial t} v_{k,m} = p_{k,m}(x, t) \geq 0 \quad \forall |x| < 1 - k^{-1}, t \geq 0
\]

\[
\Rightarrow v_t \geq 0 \quad \forall |x| < 1, t \geq 0 \quad \text{as } m \to \infty, k \to \infty
\]

and (12) follows. By (12) and (18),

\[
u_t(x, t) \geq 0 \quad \forall |x| < 1, t > 0
\]

and (13) follows.

Proof of Theorem 5 We will use a modification of the technique of [DP2] and [Hu3] to prove the theorem. Let \( v_0 = e^{2u_0} \), by the results of [DP1], [Hu2], and [Hs1], there exists a unique maximal solution \( v \) of (19) in \( \mathbb{R}^2 \times (0, T) \) which satisfies (16) and (22) in \( \mathbb{R}^2 \times (0, T) \) where \( T \) is given by (15). By Theorem 3.4 of [ERV2] and the result of [Hs1] for any \( 0 < T_1 < T \) and \( r_0 > 1 \) there exists a constant \( C > 0 \) such that

\[
v(x, t) \geq \frac{Ct}{|x|^2(\log |x|)^2} \quad \forall |x| \geq r_0, 0 \leq t < T.
\] (44)
Let \( u \) be given by (18) and \( g(t) = e^{2u} \delta_{ij} \). Then by (22) and (44), (3) and (14) hold.

For any \( k \in \mathbb{Z}^+ \), by the same argument as the proof of Theorem 4 there exists a solution \( \tilde{v}_k \) of (20) with \( r_1 = k \) which satisfies (22) in \( B_k \times (0, \infty) \). By Corollary 1.8 and Theorem 2.9 of [Hu4],

\[
\tilde{v}_k \geq \tilde{v}_{k+1} \geq v \quad \text{in} \quad B_k \times (0, T) \quad \forall k \in \mathbb{Z}^+.
\] (45)

Let \( k_0 \in \mathbb{Z}^+ \) and \( 0 < t_1 < t_2 < T \). By Corollary 1.8 of [Hu4] there exists a constant \( C > 0 \) such that

\[
\tilde{v}_k \leq C \quad \text{in} \quad B_{k_0} \times [t_1, t_2] \quad \forall k > k_0.
\] (46)

Hence by (45) and (46) the equation (19) for \( \tilde{v}_k \) is uniformly parabolic on \( B_{k_0} \times [t_1, t_2] \) for any \( k > k_0 \). By the Schauder estimates [LSU] the sequence \( \{\tilde{v}_k\}_{k>k_0} \) is equi-Holder continuous in \( C^2(B_{k_0} \times [t_1, t_2]) \). Hence by (45) as \( k \to \infty \), \( \tilde{v}_k \) decreases and converges uniformly on every compact subset \( K \) of \( \mathbb{R}^2 \times (0, T) \) to a solution \( \tilde{v} \) of (19) in \( \mathbb{R}^2 \times (0, T) \). Similar to the proof of Theorem 1.2 of [Hu3] \( \tilde{v} \) has initial value \( v_0 \). By (45), \( \tilde{v} \geq v \) in \( \mathbb{R}^2 \times (0, T) \quad \forall k \in \mathbb{Z}^+ \).

(47)

Since \( v \) is the maximal solution of (19) in \( \mathbb{R}^2 \times (0, T) \) with initial value \( v_0 \), by (47), \( \tilde{v} \equiv v \) in \( \mathbb{R}^2 \times [0, T) \).

Since \( v_0 > 0 \) on \( \mathbb{R}^2 \), by (45) and an argument similar to the proof of Lemma 7

\[
0 < \inf_{B_{r_1} \times [0,1]} v \leq \sup_{B_{r_1} \times [0,1]} v < \infty \quad \forall r_1 > 0
\]

and \( v \in C^\infty(\mathbb{R}^2 \times [0, T]) \). Then \( g(t) \) is the smooth maximal solution of (1) in \( 0 \leq t < T \) with initial value \( g_0 \). By (13), \( g(t) \) is complete for any \( 0 < t < T \).

Finally if in addition the Gauss curvature satisfies (9) for some constant \( K_0 \geq 0 \), then by an argument similar to the proof of Theorem 4 we get that (10) and (11) hold on \( \mathbb{R}^2 \) for any \( 0 \leq t < \min\{(2K_0)^{-1}, T\} \) if \( K_0 > 0 \) and (12) and (13) hold on \( \mathbb{R}^2 \) for any \( 0 \leq t < T \) if \( K_0 = 0 \). By (13), \( T = \infty \) if \( K_0 = 0 \).

If \( K_0 > 0 \), we claim that \( T \geq (2K_0)^{-1} \). Suppose not. Then \( T < (2K_0)^{-1} \). Hence by (11),

\[
\text{Vol}_{g(T)}(\mathbb{R}^2) = \int_{\mathbb{R}^2} v(x, T) \, dx \geq (1 - 2K_0 T) \int_{\mathbb{R}^2} v_0(x) \, dx > 0.
\] (48)

This contradicts (16). Hence \( T \geq (2K_0)^{-1} \) and the theorem follows.

\[ \square \]

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