SYMMETRY BREAKING BOUNDARIES
I. GENERAL THEORY

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Abstract
We study conformally invariant boundary conditions that break part of the bulk symmetries. A general theory is developed for those boundary conditions for which the preserved subalgebra is the fixed algebra under an abelian orbifold group. We explicitly construct the boundary states and reflection coefficients as well as the annulus amplitudes. Integrality of the annulus coefficients is proven in full generality.
1 Introduction and summary

The space of conformally invariant boundary conditions of two-dimensional conformal field theories is of interest in statistical mechanics, e.g. for the description of the Kondo effect and the theory of critical percolation, as well as in open string theory, where particular attention followed the observation [1] that string perturbation theory in solitonic sectors can be formulated in terms of world sheets with boundaries. In these applications it is crucial that the boundary conditions preserve conformal invariance; in contrast, additional symmetries that the bulk theory may possess typically need not be respected.

The special case of boundary conditions that preserve the full bulk symmetry was already considered a long time ago [2]. In this case the consistent conformal boundary conditions are in one-to-one correspondence with the irreducible representations of the fusion rule algebra of the theory, the so-called (generalized) quantum dimensions. To be precise, this result holds when the torus partition function is given by charge conjugation. More recently, it has been observed [3,4] that also in the case when the torus partition function corresponds to some simple current automorphism of the fusion rules, one can find a relative of the fusion algebra whose irreducible representations precisely correspond to the boundary conditions that preserve the full bulk symmetry. This algebra has been dubbed the classifying algebra.

The consideration of Dirichlet boundary conditions for a free boson conformal field theory brought yet another insight. Namely, for every conformal field theory, say with charge conjugation modular invariant, one should also study boundary conditions that relate left and right movers by some automorphism of the fusion rules [5,6] that preserves conformal weights. For a given fusion rule automorphism \( g^* \), respectively the corresponding automorphism \( g \) of the chiral algebra, there will typically exist several distinct conformally invariant boundary conditions. They constitute the possible Chan–Paton types for the fixed automorphism type \( g \). Again, it is natural to construct these boundary conditions as irreducible representations of some classifying algebra that generalizes the fusion rule algebra [5].

One goal of this paper is to identify these algebras; but to do so, it turns out to be convenient to solve a more general problem. The boundary conditions of automorphism type \( g \) respect only a subset of the bulk symmetries \( \mathfrak{A} \), namely the subalgebra \( \mathfrak{A}(g) \) of those elements that are fixed under \( g \). More generally, one may therefore address the following question. Given a subalgebra \( \mathfrak{A} \) of the chiral algebra \( \mathfrak{A} \), determine all those boundary conditions that preserve (at least) \( \mathfrak{A} \), but not necessarily all of \( \mathfrak{A} \). It should be appreciated that even when we ask this question for the subalgebra \( \mathfrak{A} = \mathfrak{A}(g) \) associated to some automorphism \( g \) of the chiral algebra, there will typically exist several distinct conformally invariant boundary conditions. They constitute the possible Chan–Paton types for the fixed automorphism type \( g \).

As long as \( \mathfrak{A} \) is completely arbitrary, at present this problem is still too general to be tractable. We will therefore restrict our attention to a particular subclass of (consistent) subalgebras. Namely, we require that \( \mathfrak{A} \) be the fixed algebra \( \mathfrak{A}^G \) of some group \( G \) of automorphisms of the chiral algebra \( \mathfrak{A} \). In other words, \( \mathfrak{A} = \mathfrak{A}^G \) is the chiral algebra of an orbifold of the theory that has chiral algebra \( \mathfrak{A} \). The orbifold group \( G \) need not necessarily be a finite group; one may even study orbifolds with respect to finite-dimensional Lie groups. But for the present purposes we assume that \( G \) is indeed finite, and we still specialize further to the situation that \( G \) is a finite abelian group.
In this case the original chiral algebra $\mathfrak{A}$ can be reassembled from its subalgebra $\overline{\mathfrak{A}}$ by an \textit{integer spin simple current extension}. This allows us to utilize simple current technology \cite{7, 8, 9, 10, 11, 12}. This way we have several nice structures at our disposal, which have passed various rather non-trivial checks in chiral conformal field theory (see e.g. \cite{12, 13, 14, 15}). They allow us to write down a natural candidate for a classifying algebra. We then take this ansatz and compute reflection coefficients and annulus coefficients, and afterwards show that these quantities pass the usual consistency checks. In particular, the annulus coefficients are proven to be integral; it should be noticed that this property is an outcome of our analysis rather than a requirement we impose.

For the convenience of the reader, we now present a brief summary of our main results. We assume full reducibility (which is satisfied in all known examples), i.e. that we can decompose the representation spaces $\mathcal{H}_\lambda$ for all primary fields of the $\mathfrak{A}$-theory as

$$\mathcal{H}_\lambda = \bigoplus \overline{\mathcal{V}}_{\overline{\mu}} \otimes \overline{\mathcal{H}}_{\overline{\mu}}$$

into irreducible $\mathfrak{A}$-modules $\overline{\mathcal{H}}_{\overline{\mu}}$. The degeneracy spaces $\overline{\mathcal{V}}_{\overline{\lambda}}$ introduced this way are modules of suitable subgroups $U_\lambda$ of the orbifold group $G$. We make the mild assumption that each of these $G$-modules is irreducible, so that $\overline{\mathcal{V}}_{\overline{\lambda}} \cong V_{\Psi}$ where $\Psi \in U_\lambda^\times$. As a consequence, we can label the primary fields of the orbifold theory by pairs $(\lambda, \Psi)$ where $\lambda$ is an $\mathfrak{A}$-primary and $\Psi \in U_\lambda^\times$. Actually, at this point we have somewhat oversimplified the story. Indeed, by assumption we have an action of $G$ on the chiral algebra, and thus on the vacuum primary field $\lambda = \Omega$. While this does induce an action of $U_\lambda$ on the degeneracy spaces that arise in the decomposition for other primaries as well, that action is in general only \textit{projective}. Thus in general we must allow for $V_{\Psi}$ to be only a projective module. Note that projective modules of an abelian group do not necessarily have dimension one; accordingly, additional multiplicities will occur in our analysis. That this effect is indeed realized in concrete models can already be seen for orbifolds of a free boson, compactified at self-dual radius; when one orbifoldizes by the dihedral group $D_2$, then the dihedral group acts on the primary field of conformal dimension $\Delta = 1/4$ only projectively, and those projective irreducible representations are, of course, irreducible representations of the universal central extension of $D_2$, the quaternion group (for more details see \cite{16}).

Technically, we will proceed in this work in a manner that is opposite to the orbifold philosophy, i.e. we express $\mathfrak{A}$-quantities in terms of quantities of the $\overline{\mathfrak{A}}$-theory instead of the other way round. It can be seen that under the above-mentioned non-degeneracy assumption the primaries $J = (\Omega, \Psi)$ of the $\overline{\mathfrak{A}}$-theory that come from the vacuum sector $\Omega$ of the original theory form an abelian group $\mathcal{G}$ under fusion; in other words, they are \textit{simple currents}. This group is actually isomorphic to the character group of the orbifold group $G$, i.e. $\mathcal{G} = G^\ast$.

Equipped with this information, we are then in a position to apply simple current technology. First, by its action through the fusion product, the simple current group $\mathcal{G}$ organizes the $\overline{\mathfrak{A}}$-primaries $\overline{\lambda}$ into orbits. Generically this action is not free, so one associates to every primary field $\overline{\lambda}$ its stabilizer, i.e. the subgroup $\mathcal{S}_\overline{\lambda}$ of $\mathcal{G}$ whose elements leave $\overline{\lambda}$ fixed (as $\mathcal{G}$ is abelian, the stabilizer is the same for all fields on the same $\mathcal{G}$-orbit). Further, for every simple current $J \in \mathcal{G}$ we associate to each primary field $\overline{\lambda}$ the rational number

$$Q_J(\lambda) := \Delta_{\overline{\lambda}} + \Delta_J - \Delta_{J, \overline{\lambda}} \mod \mathbb{Z},$$

(1.2)
called the monodromy charge of $\bar{\lambda}$, which is constant on $G$-orbits. In orbifold terminology, the fields whose monodromy charge vanishes for every $J \in G$ are those in the untwisted sector of the orbifold. More generally, the function $g_\lambda \equiv g_\lambda^{(Q)} : G \to \mathbb{C}$ with

$$g_\lambda(J) := \exp(2\pi i Q_J(\lambda))$$

(1.3)

for all $J \in G$ is an element of the character group $G^* = (G^*)^\ast \cong G$ and can be identified with an element of the orbifold group; $g_\lambda$ characterizes the twist sector to which the field $\bar{\lambda}$ belongs.

Based on this description one might expect that it is possible to express the $\mathfrak{A}$-primaries $\lambda$ in terms of $\bar{\mathfrak{A}}$-quantities as follows. The label $\lambda$ is interpreted as a pair $([\bar{\lambda}], \psi)$, consisting of a $G$-orbit $[\bar{\lambda}]$ and a character $\psi$ of the stabilizer $S_\lambda$. This would correspond to the decomposition

$$\mathcal{H}_\lambda \sim \bigoplus_{J \in G/S_\lambda} \mathcal{H}_{J,\lambda}$$

(1.4)

of irreducible $\mathfrak{A}$-modules, with the character $\psi \in S_\lambda^*$ accounting for the fact that inequivalent $\mathfrak{A}$-modules can be equivalent as $\bar{\mathfrak{A}}$-modules. However, as established in [12], this ansatz is too naive. The origin of the failure was actually already mentioned above; namely, the decomposition (1.4) would exclude the possibility of having only a projective action of the orbifold group on sectors other than the vacuum. In contrast, the formalism developed in [12], which is briefly reviewed in appendix A, correctly takes this effect into account. What is required as an additional ingredient is to introduce for each $\bar{\lambda}$ a certain subgroup $U_\lambda$ of $S_\lambda$, called the untwisted stabilizer of $\bar{\lambda}$. This subgroup is of quadratic index; the positive integer

$$d_\lambda := \sqrt{|S_\lambda| / |U_\lambda|}$$

(1.5)

is just the dimension of the relevant projective representation. The analysis of [12] shows that the $\bar{\mathfrak{A}}$-primaries are in fact described by pairs $[\bar{\lambda}, \hat{\psi}]$, where $\hat{\psi}$ is a character of the untwisted stabilizer rather than of the full stabilizer. The action of $G/S_\lambda$ is then implemented by an equivalence relation that also involves the character $\hat{\psi}$ (see formula (A.8)). In [17], where some of our results were announced, we have concentrated on the case where for all fields $\bar{\lambda}$ the untwisted stabilizer coincides with the full stabilizer; in the present work, the whole structure is displayed for the most general situation.

We can now exhibit the boundary conditions that preserve only the subalgebra $\mathfrak{A}$ of the bulk symmetries. Owing to factorization, boundary conditions are characterized [2, 18] by the one-point correlation functions of bulk fields on the disk. The corresponding chiral blocks are two-point blocks on the sphere. However, as only the symmetries in $\mathfrak{A}$ are preserved, these blocks are not the ordinary chiral two-point blocks of the $\mathfrak{A}$-theory; rather, we should take the chiral blocks of the $\bar{\mathfrak{A}}$-theory and combine them in a way compatible with the decomposition of the spaces $\mathcal{H}_\lambda$. Since states in different $\bar{\mathfrak{A}}$-modules that occur in such a decomposition are possibly reflected differently at the boundary, this way we arrive at an independent chiral two-point block for each pair $(\bar{\lambda}, \hat{\psi}_\lambda)$, where $\bar{\lambda}$ is a field in the untwisted sector of the orbifold theory and $\hat{\psi}_\lambda$ is a character of the untwisted stabilizer of $\bar{\lambda}$. We must still be somewhat more careful, though. The chiral blocks of our interest are linear forms

$$\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^\vee} \to \mathbb{C}.$$
However, when the degeneracy space has dimension $d_A > 1$, then we cannot simply obtain boundary blocks for the $\bar{A}$-theory by composing the corresponding boundary blocks of the $\bar{A}$-theory (which are linear forms $H^\lambda \otimes H^\lambda \to \mathbb{C}$); rather, the construction of a boundary block then requires in addition a linear form on the tensor product of the $d_A$-dimensional degeneracy spaces. There are $d_A^2 = |S^\lambda|/|U^\lambda|$ such forms.

As a consequence, for each primary $\bar{\lambda}$ in the untwisted sector of the $\bar{A}$-theory we get

$$N_{\text{block}}(\bar{\lambda}) = d_A^2 |U^\lambda| = |S^\lambda| \quad (1.7)$$

many independent chiral two-point blocks. As we will demonstrate in section 4, the labels characterizing these blocks naturally combine into a pair $(\bar{\lambda}, \psi^\lambda)$, where $\psi^\lambda$ is now a character of the full stabilizer.

Next we analyze also the way in which these blocks combine to correlation functions, whereby we effectively characterize the boundary conditions. We first observe that in the case where the full bulk symmetry $\mathfrak{A}$ is conserved and the torus partition function is given by charge conjugation, the boundary conditions correspond to the (generalized) quantum dimensions of the $\mathfrak{A}$-theory. Quantum dimensions, in turn, are related to primary fields via the modular $S$-matrix of the theory. (Actually in this simple case the structure is somewhat obscured by the fact that the modular $S$-matrix is symmetric so that there exists a natural identification between quantum dimensions and primary fields.) The fact that a modular transformation relates boundary blocks to boundary conditions has become even more apparent in the example considered in [4].

It is therefore not too surprising that also in the more general situation considered here, boundary blocks and boundary conditions are connected by a modular transformation. Let us further explore this idea heuristically. The labels $\lambda$ of the boundary blocks are subject to $g^\lambda \equiv 1$. In orbifold language, this means that we are only dealing with the untwisted sector of the orbifold. Thus along the ‘space’ direction of the torus only the twist by the identity occurs. It follows that after a modular $S$-transformation, only the identity appears as twist in the ‘time’ direction of the torus, which in turn means that the usual orbifold projection does not take place. In simple current language, the corresponding statement is that the boundary conditions are labelled by $G$-orbits $[\bar{\rho}]$ of $\bar{\mathfrak{A}}$-primaries rather than by individual primary fields. On the other hand, after the modular $S$-transformation arbitrary twists in the ‘space’ direction occur in the orbifold; this means that in the labelling of the boundary conditions all $G$-orbits $[\bar{\rho}]$ appear, not just those with vanishing monodromy charges, i.e. not just the ones in the untwisted sector. In fact in [19] we will show that the character $g^\rho \in G^* \simeq G$ furnished by the monodromy charges of $\bar{\rho}$ can be naturally identified with the automorphism type of the boundary condition. This in turn allows us to derive (rather than to assume ad hoc) that every boundary condition of the form considered here possesses a definite automorphism type. Finally, for the boundary conditions there is an additional degeneracy, too, this time governed by the untwisted stabilizer. As a matter of fact, in the structures we are going to exhibit, consistency is achieved through a rather subtle (and beautiful) interplay between the untwisted stabilizer and the full stabilizer.

For the convenience of the reader, we now collect a few explicit formulæ. They are most conveniently presented in terms of a certain square matrix $\tilde{S}$. This matrix diagonalizes the structure constants of the classifying algebra; accordingly its first index refers to a boundary block $(\lambda, \psi^\lambda)$, while the second index corresponds to a boundary condition $[\bar{\rho}, \tilde{\psi}^\rho]$. The formula
for \( \tilde{S} \) is

\[
\tilde{S}_{(\lambda,\omega,\lambda),[\rho,\hat{\psi}_1]} = \frac{|G|}{|S_X| |S_\lambda| |U_\rho|} \frac{1}{4} \sum_{J \in S_X \cap U_\rho} \psi_\lambda(J) \hat{\psi}_\rho(J)^* \tilde{S}^J_{\lambda,\beta}.
\]

(1.8)

Roughly, one has to sandwich certain matrices \( S^J \) between the characters \( \psi_\lambda \in S^*_\lambda \) and \( \hat{\psi}_\rho \in U^*_\rho \); these matrices are the modular transformation matrices for one-point chiral blocks with insertion of the simple current \( J \) on the torus \([15]\) and appear naturally in the study of simple current extensions \([12]\). In terms of the matrix \( \tilde{S} \) the one-dimensional irreducible representations of the classifying algebra which provide the reflection coefficients read

\[
R_{[\rho,\hat{\psi}_\rho]}(\tilde{\Phi}_{(\lambda,\omega,\lambda)}) = \frac{\tilde{S}_{(\lambda,\omega,\lambda),[\rho,\hat{\psi}_1]}^{\rho,\hat{\psi}_\rho}}{S_{\Omega,\rho}^{\rho,\hat{\psi}_\rho}}.
\]

(1.9)

We will also see that there is a natural conjugation on the boundary conditions, a map of order two that implements the reversal of the orientation of the boundary.

Finally, we display the annulus amplitude for an annulus with boundary conditions \([\hat{\rho}_1, \hat{\psi}_1] \) and \([\hat{\rho}_2, \hat{\psi}_2] \). As we will see, it is natural to express the annulus amplitude as a linear combination

\[
A_{[\hat{\rho}_1, \hat{\psi}_1]; [\hat{\rho}_2, \hat{\psi}_2]} = \sum_{[\sigma, \hat{\psi}_\sigma]} A_{[\hat{\rho}_1, \hat{\psi}_1]; [\hat{\rho}_2, \hat{\psi}_2]}^{[\sigma, \hat{\psi}_\sigma]} \chi_{[\sigma, \hat{\psi}_\sigma]}'(1.10)
\]

of characters \( \chi_{[\sigma, \hat{\psi}_\sigma]}' \) of the conformal field theory that is obtained by extending the \( \tilde{\Phi} \)-theory by the simple currents in the subgroup

\[
G' \equiv G'_{\rho_1, \rho_2} = \{ J \in G \mid Q_3(\rho_1) = Q_3(\rho_2) = 0 \}
\]

(1.11)

of \( G \). The annulus coefficients \( A_{[\hat{\rho}_1, \hat{\psi}_1]; [\hat{\rho}_2, \hat{\psi}_2]}^{[\sigma, \hat{\psi}_\sigma]} \) can then be written, up to a prefactor, as a sum of the fusion rule coefficients \( g^N_{[\hat{\rho}_1, \hat{\psi}_1]; [\sigma, \hat{\psi}_\sigma]} \) of the \( G' \)-extension:

\[
A_{[\hat{\rho}_1, \hat{\psi}_1]; [\hat{\rho}_2, \hat{\psi}_2]}^{[\sigma, \hat{\psi}_\sigma]} = N \sum_{\psi_1^1 \in U_\rho^1} \sum_{\psi_2^1 \in U_\rho^2} \sum_{J \in G'/G''} g^N_{[\hat{\rho}_1, \hat{\psi}_1]; [\sigma, \hat{\psi}_\sigma]} \chi_{J[J]}^{[\hat{\rho}_2, \hat{\psi}_2]^1, [\sigma, \hat{\psi}_\sigma]}'.
\]

(1.12)

Here \( G'' \) is a certain subgroup of \( G \) which is intermediate between \( G' \) and \( G \), i.e. \( G' \subseteq G'' \subseteq G \), and \( \hat{\psi}_i = \hat{\psi}_i |_{U_i \cap G'} \). As an important consistency check we will present a general proof that the prefactor \( N \equiv N_{\rho_1, \rho_2} \), which is a quite complicated ratio involving the sizes of various subgroups of \( G \) (see formula \((3.11)\)), is always a non-negative integer, so that the annulus amplitude can be consistently interpreted as a partition function for open string states.

More precisely, the number \( N \) can be written as a product of three separate integral factors, each of which possesses a natural group theoretic respectively representation theoretic interpretation (see formula \((5.50) \) and \((6.54)\)). While this definitely implies that the annulus coefficients are non-negative integers (as befits the coefficients of a partition function), the interpretation of the prefactor \( N \) should also play a role in non-chiral field-state correspondence. One expects to be able to associate to every open string state a field operator in the full conformal field theory. Now the partition function for these states is the annulus amplitude, and
the presence of additional multiplicities in the latter means that several distinct operators in
the full conformal field theory must be built from one and the same chiral vertex operator.
An explicit construction of these operators is not known for the moment, but in any case the
information that the multiplicities all possess an interpretation in terms of fusion rules and
other representation theoretic objects seems to be highly relevant.

The rest of this paper is organized as follows. We start in section 2 with a description of
our setup, i.e. boundary conditions which preserve a subalgebra of the bulk symmetries that is
fixed under a finite abelian group of automorphisms. The analysis of such boundary conditions
proceeds in two steps, where in the first step one works exclusively at the chiral level, while in
the second non-chiral quantities enter. The general features of the chiral part are collected in
section 3, while in section 4 a natural basis for the basic chiral ingredients, the boundary blocks,
is constructed. Section 5 is devoted to the non-chiral level. First, in subsection 5.1, we show
that the boundary conditions of interest to us are governed by a classifying algebra; in the rest
of this section we establish the precise form of this algebra and investigate its properties. While
we regard our arguments leading to these results as convincing, they are not mathematically
rigorous. As further evidence we therefore perform, in section 6, several additional consistency
checks based on properties of the annulus amplitudes, the most important one (subsection 6.4)
being a general proof of the integrality of the annulus coefficients that appear in the open string
channel.

In a follow-up paper [19], we will address several complementary issues which concern the
structure of the space of symmetry breaking boundary conditions and display the boundary
conditions for various classes of conformal field theories explicitly. More concretely, we start
by associating to each boundary condition its automorphism type, which arises as a direct
consequence of the general structure. Then we show that boundary conditions of definite
automorphism type can be naturally formulated with the help of certain twisted boundary
blocks, obeying twisted Ward identities, and that they carry their own individual classifying
algebra, which is a suitable quotient of the total classifying algebra. Further we study the
realization of T-duality on the space of boundary conditions, show that this space carries an
action of the orbifold group (‘boundary homogeneity’), and introduce the concept of a universal
classifying algebra for all conformally invariant boundary conditions. Finally we will exhibit in
a large number of examples the concrete realization of various structures that we have uncovered.

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2 Broken bulk symmetries

In this paper we analyze the following situation. We start with some prescribed conformal
field theory that is consistently defined on all closed orientable surfaces, and choose the charge
conjugation modular invariant as the partition function on the torus. This theory is, moreover,
assumed to be non-heterotic, i.e. for left and right movers we deal with one and the same sym-
metry algebra $\mathfrak{A}$, the chiral algebra, which contains the Virasoro algebra $\mathfrak{W}_{ir}$. (This condition
in fact refers to the oriented Schottky cover of the surface, which for a closed orientable surface
consists of two isomorphic disjoint sheets; the requirement is that we deal with one and the
same chiral conformal field theory on both sheets.) We call $\mathfrak{A}$ the algebra of bulk symmetries. For technical reasons we will assume that the theory is rational, i.e. that it contains only finitely many $\mathfrak{A}$-primaries.

Boundary conditions that respect the full bulk symmetry have been studied for quite a while [2,18]. In contrast, in the present work we are interested in boundary conditions that do not preserve all bulk symmetries, but only a subalgebra $\mathfrak{A}$ of $\mathfrak{A}$. This does not yet restrict at all the kind of boundary condition we consider, since to any arbitrary boundary condition one may associate the subalgebra of $\mathfrak{A}$ that is preserved. Further, we are interested in conformally invariant boundary conditions only, so that $\mathfrak{A}$ must in particular contain the Virasoro subalgebra of $\mathfrak{A}$. Moreover, $\mathfrak{A}$ must be ‘consistent’; by this qualification we understand that the algebra is closed under charge conjugation and allows for the definition of sheaves of chiral blocks which come with a projectively flat Knizhnik–Zamolodchikov connection and which obey consistent factorization rules.

A typical chiral algebra $\mathfrak{A}$ will, however, possess very many, if not infinitely many, consistent subalgebras $\mathfrak{A}$. Accordingly, the first step towards a classification of all conformal boundary conditions would be to classify all those subalgebras. This problem depends largely on the specific bulk conformal field theory under consideration, and (except for a discussion of a possible limiting algebra of an inductive system of classifying algebras in [19]) we will not have to say much about it. On the other hand, for sufficiently simple theories, such as the Virasoro minimal models or the free boson or its $\mathbb{Z}_2$-orbifold, all consistent subalgebras are known. More generally, once the problem of classifying the consistent subalgebras has been solved for any single model, the methods presented below provide us (possibly modulo the existence of so-called complex charges, compare [20]) with all conformal boundary conditions of that model.

Here we rather concentrate on the task of classifying all boundary conditions that preserve some specified consistent subalgebra $\mathfrak{A}$. As long as $\mathfrak{A}$ is a completely arbitrary subalgebra, this problem is still too general and cannot be solved with the methods that are available at present. We will therefore restrict our attention to a particular subclass of consistent subalgebras. Namely, we require that $\mathfrak{A}$ be the fixed algebra of some group $G$ of automorphisms of the chiral algebra $\mathfrak{A}$. In other words,

$$\mathfrak{A} = \mathfrak{A}^G$$  \hspace{1cm} (2.1)

is the chiral algebra of an orbifold of the original theory. In principle the orbifold group $G$ can be quite arbitrary; for instance, it need not even be finite, but rather could be some finite-dimensional Lie group. Still, for the purpose of the present paper we restrict our attention to the case when $G$ is finite, and when moreover it is abelian.

This situation may seem rather special compared to the general problem sketched above, but it nevertheless covers a variety of cases of practical interest. Examples are provided by the critical three-state Potts model and, more generally, by Virasoro minimal models of $(A,D_{even})$ type, by Dirichlet boundary conditions for a free boson for which only the chiral algebra of the $\mathbb{Z}_2$-orbifold of the boson theory is preserved, by D-branes in toroidal compactifications at generic positions, by charge conjugation in WZW theories, and by those boundary conditions for a free boson that correspond to a change in the compactification radius. (For a more extensive list, see the final sections in the follow-up [14] of this paper.) Moreover, already with this restriction we can gain a number of additional physical insights, e.g. concerning the relation between boundary conditions that preserve subalgebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of $\mathfrak{A}$ that are contained in
each other.

Let us briefly recall how to describe boundary conditions in conformal field theory on surfaces \( \mathcal{C} \) with boundaries. First \([5]\), one must set up a chiral conformal field theory on a closed oriented twofold covering surface \( \tilde{\mathcal{C}} \) of \( \mathcal{C} \), the Schottky cover \([21]\), from which \( \mathcal{C} \) is obtained by dividing out an anti-conformal involution. This amounts to specifying a system of chiral blocks that has a Knizhnik–Zamolodchikov connection and obeys factorization rules; as a consequence of factorization, the blocks most relevant to the boundary conditions are the chiral blocks for a single bulk field insertion on the disk, which are two-point blocks on the projective line \( \mathbb{P}^1 \). In an independent second step we have to construct correlation functions as linear combinations of these blocks that satisfy \([2,18,3,22,23]\) locality and factorization constraints. As was emphasized in \([5]\), these two conceptual levels should be carefully distinguished, and accordingly we will divide our discussion in two parts. We start by analyzing, in the next two sections, the chiral conformal field theory on the Schottky cover.

3 Chiral theory for symmetry breaking boundaries

As just pointed out, for the thorough investigation of boundary conditions it is advisable to distinguish clearly between the two conceptual levels of chiral conformal field theory and full conformal field theory. In the chiral theory, boundaries are not yet present explicitly; but as a prerequisite for analyzing the breaking of bulk symmetries by boundary conditions in the full theory various structures need to be understood already at this stage. These chiral concepts are the topic of the present and the next section.

3.1 Simple currents versus orbifolds

As already outlined above, our general situation is as follows. We are given a rational conformal field theory with chiral algebra \( \mathfrak{A} \), and we consider boundary conditions that preserve only a consistent subalgebra \( \hat{\mathfrak{A}} \) of \( \mathfrak{A} \). Let us assume that \( \mathfrak{A} \) can be obtained from its subalgebra \( \hat{\mathfrak{A}} \) by the extension with a simple current group \( \mathcal{G} \), which is some finite abelian group. Each simple current \( J \in \mathcal{G} \) corresponds to an irreducible infinite-dimensional representation space \( \hat{\mathcal{H}}_J \) of \( \hat{\mathfrak{A}} \); moreover, the vacuum module \( \mathcal{H}_\Omega \) of \( \mathfrak{A} \) and the vacuum module \( \hat{\mathcal{H}}_\Omega \) of \( \hat{\mathfrak{A}} \) are related as

\[
\mathcal{H}_\Omega \cong \bigoplus_{J \in \mathcal{G}} \hat{\mathcal{H}}_J ,
\]

where the symbol \( \cong \) stands for isomorphism as \( \hat{\mathfrak{A}} \)-modules; the identity element in \( \mathcal{G} \) corresponds to \( \hat{\mathcal{H}}_\Omega \).

In this situation the chiral algebra \( \hat{\mathfrak{A}} \) is necessarily an orbifold subalgebra of \( \mathfrak{A} \). Namely, we can obtain an action of the dual group \( G = \mathcal{G}^* \) on \( \hat{\mathcal{H}}_\Omega \) as follows. For every \( g \in G \), we define \( R(g) \) to act on the subspace \( \hat{\mathcal{H}}_J \) of \( \hat{\mathcal{H}}_\Omega \) as the multiple \( J(g) \) of the identity map, where \( J \) is regarded as a character on \( G \). Field-state correspondence relates the vectors in \( \mathcal{H}_\Omega \) to operators in the chiral algebra, and thus this prescription provides us with a group of automorphisms of \( \mathfrak{A} \) that is isomorphic to \( G \).

Conversely, suppose we are given an action of a finite abelian group \( G \) on the chiral algebra \( \hat{\mathfrak{A}} \) that leaves the Virasoro subalgebra \( \mathfrak{Vir} \subseteq \mathfrak{A} \) pointwise fixed. Then \( \mathfrak{A} \) contains as a subalgebra
the algebra $\mathfrak{A}^G$ of all elements that are left pointwise fixed under the action of the orbifold group $G$, and $\mathfrak{A}^G$ contains the Virasoro subalgebra of $\mathfrak{A}$. Again by field-state correspondence, we then also have an action of $G$ on the vacuum module $\mathcal{H}_\Omega$, and this action commutes with the action of $\mathfrak{A}^G$. It can then be shown \cite{24,25} that $\mathcal{H}_\Omega$ is completely reducible as an $\mathfrak{A}^G$-module, so that we can decompose $\mathcal{H}_\Omega$ into irreducible submodules of $\mathfrak{A}^G \times \mathfrak{A}^G$ as

$$\mathcal{H}_\Omega \cong \bigoplus_{\lambda} \bigoplus_{\Psi \in G^*} V_\Psi \otimes \mathcal{H}_\lambda,$$  \hspace{1cm} (3.2)

where $\mathcal{H}_\lambda$ are irreducible $\mathfrak{A}^G$-modules and $V_\Psi$ are irreducible modules of $G$ (and are thus one-dimensional). It follows in particular that all $\mathfrak{A}^G$-modules $\mathcal{H}_\lambda$ that appear in (3.2) are simple currents. This holds true because the fusion of these modules must be compatible with the decomposition of tensor products of irreducible $G$-representations. The latter decomposition, in turn, is just described by the dual group $G = G^*$, and hence we conclude that with respect to the fusion product the modules $\mathcal{H}_\lambda$ appearing in (3.2) form a simple current group isomorphic to $G$.

### 3.2 Simple current extensions

That the boundary conditions of interest to us preserve only the subalgebra $\mathfrak{A}$ implies that generically the fields corresponding to vectors in different $\mathfrak{A}$-submodules of a given $\mathfrak{A}$-module are reflected differently at the boundary. Accordingly we need to decompose every sector of the chiral conformal field theory, i.e. every irreducible representation of $\mathfrak{A}$, into irreducible $\mathfrak{A}$-representations (again we impose full reducibility with respect to $\mathfrak{A}$).

Fortunately, the decomposition of $\mathfrak{A}$-modules in terms of $\mathfrak{A}$-modules is a purely chiral issue, i.e. is in particular independent of any boundary effects, and this chiral issue is well understood in the simple current framework. We summarize some relevant information here; for more details see appendix A and \cite{9,12}. The fusion product provides an action of the simple current group $G$ on the fields $\lambda$ of the $\mathfrak{A}$-theory. To each primary field $\lambda$ one then associates a subgroup of $G$, the stabilizer

$$S_\lambda := \{ J \in G \mid J\lambda = \lambda \}.$$  \hspace{1cm} (3.3)

Stabilizer subgroups are constant on $G$-orbits (which we already anticipated by writing $S_\lambda$ in place of $S_\lambda$); conjugate $\mathfrak{A}$-fields have identical stabilizers, too.

---

1. Here we make the assumption that the action of $G$ on the vacuum module $\mathcal{H}_\Omega$ is a honest action rather than only a projective one. This condition should be regarded as part of the definition of the term orbifold. If it were not satisfied, the structure to be divided out would no longer be a group.

2. Note that $G$ commutes with the Virasoro algebra so that it preserves the grading of the infinite-dimensional space $\mathcal{H}_\Omega$. Thus each homogeneous subspace of fixed conformal weight is a finite-dimensional $G$-module, which is fully reducible. As $G$ even commutes with all of $\mathfrak{A}^G$, full reducibility with respect to $G \times \mathfrak{A}^G$ then follows from full reducibility with respect to $\mathfrak{A}^G$. Incidentally, a vertex operator algebra for which every graded representation is fully reducible possesses only finitely many inequivalent irreducible representations \cite{26}, thus giving rise to a rational conformal field theory.

A decomposition of the vacuum module of the form (3.2) is expected to hold for arbitrary finite orbifold groups $G$, and has been proven for many non-abelian groups in \cite{24,25}. Analogous decompositions are valid \cite{27} for other $\mathfrak{A}$-modules, including twisted modules.
In the decomposition of a given \( \mathfrak{A} \)-module \( \mathcal{H}_\lambda \) only \( \mathfrak{A} \)-modules on a single \( G \)-orbit appear. However, one and the same \( G \)-orbit \([\bar{\mu}]\) of primaries in the \( \mathfrak{A} \)-theory can give rise to several distinct primaries of the \( \mathfrak{A} \)-theory. In other words, inequivalent \( \mathfrak{A} \)-modules can be isomorphic as \( \bar{\mathfrak{A}} \)-modules. This effect is controlled by a subgroup of the stabilizer, the so-called untwisted stabilizer, which in turn is obtained with the help of the following structure. To each \( \bar{\mathfrak{A}} \)-primary \( \bar{\lambda} \) one can associate a bi-homomorphism

\[
F_\lambda : \ G \times G \to \mathbb{C}^\times
\]

that is alternating in the sense that \( F_\lambda(J, J) = 1 \) for all \( J \in G \), that again depends only on the orbit, and that is the same for any two conjugate orbits (for the precise definition see appendix [3]). Every alternating bi-homomorphism is the commutator cocycle for some two-cocycle \( F \) on \( G \), i.e.

\[
F_\lambda(J_1, J_2) = F_\lambda(J_1, J_2) / F_\lambda(J_2, J_1),
\]

and the cohomology class of \( F_\lambda \) is uniquely determined by \( F_\lambda \).

### 3.3 The untwisted stabilizer

The commutator cocycle \( F_\lambda \) allows us to single out the *untwisted stabilizer* \([12]\) as the subgroup

\[
\mathcal{U}_\lambda := \{ J \in S_\lambda | F_\lambda(J, K) = 1 \text{ for all } K \in S_\lambda \}
\]

of the full stabilizer \( S_\lambda \). As shown in \([12]\), those \( \mathfrak{A} \)-primaries that are isomorphic as \( \bar{\mathfrak{A}} \)-modules are naturally labelled by characters of the group \( \mathcal{U}_\lambda \). As a consequence, the \( \mathfrak{A} \)-primaries can be denoted by \( G \)-orbits \([\bar{\lambda}, \hat{\psi}_\lambda]\) of pairs consisting of a primary label \( \bar{\lambda} \) of the \( \mathfrak{A} \)-theory and a character \( \hat{\psi}_\lambda \) of the untwisted stabilizer of \( \bar{\lambda} \). (The equivalence relation that defines the classes \([\bar{\lambda}, \hat{\psi}_\lambda]\) involves both constituents of the pair, see formula (A.8).)

A second important piece of information that we can extract from the commutator cocycle \( F_\lambda \) is a collection of projective representations of \( S_\lambda \). They are characterized by the two-cocycle \( F_\lambda \), or rather its cohomology class. The theory of projective representations of finite abelian groups (for a brief summary see appendix [3]) tells us that the projective irreducible representations are labelled by the characters \( \hat{\psi} \) of the *untwisted* stabilizer \( \mathcal{U}_\lambda \) and, moreover, that they all have the same dimension \( d_\lambda \) that was defined in (1.3), i.e.

\[
d_\lambda = \sqrt{s_\lambda / u_\lambda}
\]

with

\[
s_\lambda := |S_\lambda|, \quad u_\lambda := |\mathcal{U}_\lambda|.
\]

Note that, even though not manifest from their definition, the numbers \( d_\lambda \) are indeed integral \([12, 14]\); they constitute the (additional) ground state degeneracy of the resolved fixed point in the \( \mathfrak{A} \)-theory \([12]\).

Taking all this information together, we arrive at the decomposition

\[
\mathcal{H}_\lambda \equiv \mathcal{H}_{[\bar{\lambda}, \hat{\psi}]} = \bigoplus_{J \in G / S_\lambda} \mathcal{V}_{\hat{\psi}} \otimes \mathcal{H}_J, \quad (3.9)
\]
where $V^\hat{\psi}$ is an irreducible projective $S_\lambda$-module. In the special case of the vacuum $\lambda = \Omega$ of the $A$-theory the stabilizer is trivial, $S_\Omega = \{\overline{\Omega}\}$; we then recover (3.2) with $\overline{\lambda} = J\overline{\Omega}$ and $\Psi \in G^* = G$ identified with the simple current $J$. It should also be kept in mind that on the right hand side of these decompositions only such irreducible representations $\overline{\mu}$ of the $\overline{A}$-theory arise for which the monodromy charges $Q_J(\lambda)$ (1.2) with respect to all currents $J \in G$ vanish; this holds true simply because it is satisfied [9] for all irreducible representations $\lambda$ in the extension and because monodromy charges are constant on $G$-orbits.

4 Boundary blocks

We have now collected sufficient background material so as to be able to address in more detail the basic ingredient needed for the analysis of boundaries at the chiral level. As already mentioned, as a consequence of factorization this ingredient is provided by the chiral blocks for a single bulk insertion on the disk. We will refer to these basic objects as the boundary blocks for broken bulk symmetries. (When all bulk symmetries are preserved, these blocks are also called Ishibashi states.)

By definition, such boundary blocks are two-point chiral blocks on a world sheet $\mathbb{P}^1$ with the topology of the sphere. In more technical terms, they are elements of $(\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+})^* = \mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+} \rightarrow \mathbb{C}$ on the tensor product spaces $\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+}$, which satisfy the Ward identities for $\overline{A}$, i.e. are invariant under the symmetries in the chiral algebra that are preserved. In the special case when all bulk symmetries are respected, every such two-point block is uniquely determined up to a scalar factor.

4.1 Boundary blocks for broken symmetries

We are interested in the situation where only the symmetries in the prescribed subalgebra $\overline{A}$ of the chiral algebra are preserved. From the decomposition (3.9) it follows that the linear forms we are after are forms on

$$\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+} \equiv \mathcal{H}_{[\lambda,\hat{\psi}]} \otimes \mathcal{H}_{[\lambda,\hat{\psi}]}^+ \cong V_{\psi} \otimes V_{\psi^+} \otimes \bigoplus_{J,K \in G/S_\lambda} (\mathcal{H}_{J[\lambda]} \otimes \mathcal{H}_{K[\lambda^+]}),$$

and hence they are sums of tensor products of linear forms on the tensor product spaces $V_{\psi} \otimes V_{\psi^+}$ and $\bigoplus_{J,K} \mathcal{H}_{J[\lambda]} \otimes \mathcal{H}_{K[\lambda^+]}$. Moreover, since the $\overline{A}$-symmetries are preserved, the latter forms satisfy the Ward identities of the $\overline{A}$-theory and hence are precisely the two-point blocks for the relevant $\overline{A}$-fields. These blocks in turn can be non-vanishing only when one deals with tensor products of conjugate $\overline{A}$-modules, i.e. effectively we are working with forms on the subspace

$$V_{\psi} \otimes V_{\psi^+} \otimes \bigoplus_{J \in G/S_\lambda} (\mathcal{H}_{J[\lambda]} \otimes \mathcal{H}_{J[\lambda]})^+$$

of the space (L). Moreover, when non-vanishing, then these forms on the subspaces $\mathcal{H}_{\overline{\mu}} \otimes \mathcal{H}_{\overline{\mu}^+}$ are uniquely fixed up to normalization, just as the two-point blocks for $\overline{A}$ are. Thus, in short, they are just the ordinary boundary blocks

$$\overline{B}_{\overline{\mu}} : \mathcal{H}_{\overline{\mu}} \otimes \mathcal{H}_{\overline{\mu}^+} \rightarrow \mathbb{C}$$

(4.3)
of the $\mathfrak{A}$-theory.

It follows that the boundary blocks can be written as linear combinations

$$
\sum_{I \in \{1,2,\ldots,d\}^2} \xi_{I,J} b_{\hat{\psi};(i)} \otimes \bar{B}_{I,J},
$$

(4.4)

where the maps $b_{\hat{\psi};(i)}$ constitute some basis of the linear forms

$$
\beta_{\hat{\psi}} : \mathcal{V}_{\hat{\psi}} \otimes \mathcal{V}_{\hat{\psi}^+} \to \mathbb{C}.
$$

(4.5)

The coefficients $\xi_{I,J} \in \mathbb{C}$ appearing in (4.4) are undetermined at the level of chiral conformal field theory, simply because the Ward identities for the unbroken symmetries are satisfied independently of the values of these coefficients. At the level of full conformal field theory, however, we will be able to determine them; each consistent set of values then corresponds to a boundary condition that preserves $\mathfrak{A}$.

Now for the $\mathfrak{A}$-part we are already given a natural basis of linear forms, namely the ordinary boundary blocks (4.3). On the other hand, at this point we are still lacking a concrete basis for the linear forms $\beta_{\hat{\psi}}$ on the degeneracy spaces. Therefore we now turn our attention to those forms $\beta_{\hat{\psi}}$. As already mentioned, the degeneracy spaces are projective modules of the stabilizer group $\mathcal{S}_{\lambda}$ or, more precisely, ordinary modules of that twisted group algebra $\mathbb{C}\mathcal{F}_{\lambda} \mathcal{S}_{\lambda}$ which corresponds to (the cohomology class of) the two-cocycle $\mathcal{F}_{\lambda}$ that was introduced in formula (3.5). Thus to every $J \in \mathcal{S}_{\lambda}$ is associated a linear map $R_{\hat{\psi}}(J)$ on $\mathcal{V}_{\hat{\psi}}$, and these maps represent $\mathcal{S}_{\lambda}$ projectively in the sense that

$$
R_{\hat{\psi}}(J) R_{\hat{\psi}}(J') = \mathcal{F}_{\lambda}(J,J') R_{\hat{\psi}}(JJ')
$$

(4.6)

for all $J,J' \in \mathcal{S}_{\lambda}$. When the simple current $J$ is even contained in the untwisted stabilizer $\mathcal{U}_{\lambda} \subseteq \mathcal{S}_{\lambda}$, whose group algebra coincides with the center of the twisted group algebra $\mathbb{C}\mathcal{F}_{\lambda} \mathcal{S}_{\lambda}$, then it is represented by a multiple of the unit matrix:

$$
R_{\hat{\psi}}(J) = \hat{\psi}(J) \mathbb{1}_{d_{\lambda}} \quad \text{for } J \in \mathcal{U}_{\lambda}.
$$

(4.7)

We also note that for any set $\{K\}$ of representatives of the quotient $\mathcal{S}_{\lambda}/\mathcal{U}_{\lambda}$, the matrices $R_{\hat{\psi}}(K)$ form a basis of the full matrix algebra $M_{d_{\lambda}}(\mathcal{V}_{\hat{\psi}})$ on $\mathcal{V}_{\hat{\psi}}$. (For further properties of twisted group algebras and their representations, see appendix B.)

By employing these maps $R_{\hat{\psi}}(J)$ we will now construct a natural basis for the linear forms $\beta_{\hat{\psi}}$ and thereby for the boundary blocks. To this end we first establish an underlying basis $\{\mathcal{O}_{\hat{\psi}}\}$ for the endomorphisms of $\mathcal{V}_{\hat{\psi}}$. Before doing so, however, we pause for a remark about the character $\hat{\psi}^+$ that first appeared in formula (4.1) above. It arises via the formula

$$
[\bar{\lambda},\hat{\psi}]^+ = [\bar{\lambda}^+,\hat{\psi}^+]
$$

(4.8)

for the conjugation of $\mathfrak{A}$-representations, and thus comes from the $\mathcal{G}$-orbit that is conjugate to the $\mathcal{G}$-orbit of $\lambda$. Now the commutator cocycles of conjugate orbits are just each others' complex conjugates (see relation (A.27)), so $\hat{\psi}^+$ is a character of the same group $\mathcal{U}_{\lambda}$ as $\hat{\psi}$. However, it does not, in general, coincide with the complex conjugate character $\hat{\psi}^*$ (see formula (A.23) for the precise definition). Thus in particular the irreducible projective $\mathcal{S}_{\lambda}$-module $\mathcal{V}_{\hat{\psi}^+}$ in general neither coincides with $\mathcal{V}_{\hat{\psi}}$ itself nor with the module $\mathcal{V}_{\hat{\psi}^*}$ that is dual to $\mathcal{V}_{\hat{\psi}}$ in the sense that the representation matrices are hermitian conjugate to those for $\mathcal{V}_{\hat{\psi}}$.  

13
4.2 A natural basis for $\text{End}(\mathcal{V}_\psi)$

As an intermediate step towards constructing the desired basis for the linear forms on $\mathcal{V}_\psi \otimes \mathcal{V}_\psi^*$, we introduce in this subsection a basis $\{\mathcal{O}_\psi\} \equiv \{\mathcal{O}_\psi^{(\hat{\psi})}\}$ for the linear maps on the degeneracy space $\mathcal{V}_\psi$. To this end we first introduce the following concept. For every character $\psi$ of the full stabilizer, $\psi \in S^*_\lambda$, the restriction of $\psi$ to $U_\lambda \subseteq S_\lambda$ is an element $\pi(\psi) \in U^*_\lambda$. We write

$$\psi \succ \hat{\psi} \quad \text{or} \quad \hat{\psi} = \pi(\psi) \equiv \psi|_{U_\lambda}$$

(4.9)

to characterize this situation. Each $U_\lambda$-character $\hat{\psi}$ has $d_\lambda^2$ pre-images under the projection $\pi$. We will show how these pre-images label the desired basis of the endomorphisms of $\mathcal{V}_\psi$, according to $\{\mathcal{O}_\psi | \psi \succ \hat{\psi}\}$.

We start with the observation that for $\psi \succ \hat{\psi}$ the product $\psi^*(J) R_{\hat{\psi}}(J)$ does not depend on the choice of representative $J$ of a class in the quotient $S_\lambda/U_\lambda$. For, if $K \in U_\lambda$, then

$$\psi^*(JK) R_{\hat{\psi}}(JK) = \psi^*(J) \psi^*(K) R_{\hat{\psi}}(J) R_{\hat{\psi}}(K) = \psi^*(J) \hat{\psi}^*(K) R_{\hat{\psi}}(J) \hat{\psi}(K) = \psi^*(J) R_{\hat{\psi}}(J)$$

(4.10)

owing to the identity (4.7). Therefore for each $\psi \succ \hat{\psi}$ we can introduce the endomorphism

$$\mathcal{O}_\psi := (\frac{s_\lambda}{u_\lambda})^{3/4} \sum_{J \in S_\lambda/U_\lambda} \psi(J)^* R_{\hat{\psi}}(J) \in \text{End}(\mathcal{V}_\psi),$$

(4.11)

and these maps are well-defined. The following argument shows that the matrices $\mathcal{O}_\psi$ form a basis of the full matrix algebra $M_{d_\lambda} = \text{End}(\mathcal{V}_\psi)$. We use the fact, derived in the appendix after relation (B.35), that the partial sum over characters yields a non-zero result if and only if $K \in U_\lambda$ or, more precisely,

$$\sum_{\psi \in S^*_\lambda \atop \psi \succ \hat{\psi}} \psi(K) = (\frac{s_\lambda}{u_\lambda})^{1/4} \sum_{J \in S_\lambda/U_\lambda} \delta_{KJ^{-1} \in U_\lambda} \hat{\psi}(KJ^{-1}) R_{\hat{\psi}}(J) = d_\lambda^{1/2} \hat{\psi}(KJ^{-1}) R_{\hat{\psi}}(J)$$

(4.12)

The identity (4.12) implies that

$$\sum_{\psi \in S^*_\lambda \atop \psi \succ \hat{\psi}} \psi(K) \mathcal{O}_\psi = (\frac{s_\lambda}{u_\lambda})^{1/4} \sum_{J \in S_\lambda/U_\lambda} \delta_{KJ^{-1} \in U_\lambda} \hat{\psi}(KJ^{-1}) R_{\hat{\psi}}(J) = d_\lambda^{1/2} \hat{\psi}(KJ^{-1}) R_{\hat{\psi}}(J)$$

(4.13)

for all $K \in S_\lambda$. Here $J_K$ denotes the chosen representative in $S_\lambda/U_\lambda$ that is in the same class as $K$. These sums, of course, depend on $K$, and not just on $K$ modulo $U_\lambda$. However, for any set $\{K\}$ of representatives of $S_\lambda/U_\lambda$, we recover all elements in a basis of the space of endomorphisms of $\mathcal{V}_\psi$, because the operators $R_{\hat{\psi}}(J)$ span this space.

It follows that the operators $\mathcal{O}_\psi$ span the space $\text{End}(\mathcal{V}_\psi)$ of endomorphisms; for dimensional reasons, they therefore constitute a basis of $\text{End}(\mathcal{V}_\psi)$, as claimed. Furthermore, since its construction is entirely specified in terms of the character $\hat{\psi}$ and the simple currents in $S_\lambda$, this basis is indeed completely natural.

---

3 Naively one might also expect that these matrices are (proportional to) idempotents. But the non-triviality of the cocycle $\mathcal{F}_\lambda$ spoils this property.
Later on we will also need the traces of the operators \( \mathcal{O}_\psi \) and of a product of two of them, for two \( \mathcal{S}_\lambda \)-characters \( \psi, \varphi \) that restrict to the same \( \mathcal{U}_\lambda \)-character \( \hat{\psi} \). To evaluate these traces we observe that the trace of \( \mathcal{R}_{\psi}(J) \) is given by

\[
\text{tr} \, \mathcal{R}_{\psi}(J) = \delta_{J \in \mathcal{U}_\lambda} \hat{\psi}(J) \, \text{tr} \, 1 = d_\lambda \hat{\psi}(J) \delta_{J \in \mathcal{U}_\lambda} \tag{4.14}
\]

(compare the relations (B.29) and (B.30)). We then find

\[
\text{tr} \, \mathcal{O}_\psi = d_\lambda^{-3/2} \sum_{J \in \mathcal{S}_\lambda/\mathcal{U}_\lambda} \psi(J)^* \, \text{tr} \, \mathcal{R}_{\psi}(J) = d_\lambda^{-1/2} \sum_{J \in \mathcal{S}_\lambda/\mathcal{U}_\lambda} \psi(J)^* \hat{\psi}(J) \delta_{J \in \mathcal{U}_\lambda} = d_\lambda^{-1/2} \psi(J_1)^* \hat{\psi}(J_1) = d_\lambda^{-1/2} \tag{4.15}
\]

(here \( J_1 \) is the chosen representative of the class of the unit element of \( \mathcal{S}_\lambda/\mathcal{U}_\lambda \)) as well as

\[
\text{tr} \mathcal{O}_\psi^\dagger \mathcal{O}_\varphi = d_\lambda^{-3} \sum_{J_1, J_2 \in \mathcal{S}_\lambda/\mathcal{U}_\lambda} \psi(J_1) \varphi(J_2)^* \, \text{tr} \, \mathcal{R}_{\psi}(J_1) \mathcal{R}_{\psi}(J_2) = d_\lambda^{-3} \sum_{J_1, J_2 \in \mathcal{S}_\lambda/\mathcal{U}_\lambda} \psi(J_1) \varphi(J_2)^* \, \text{tr} \, \mathcal{R}_{\psi}(J_1^{-1}J_2) \cdot \mathcal{F}_\lambda(J_1^{-1}, J_2) = d_\lambda^{-3+1} \sum_{J_1, J_2 \in \mathcal{S}_\lambda/\mathcal{U}_\lambda} \psi(J_1) \varphi(J_2)^* \mathcal{F}_\lambda(J_1^{-1}, J_2) \delta_{J_1^{-1}, J_2 \in \mathcal{U}_\lambda} \hat{\psi}(J_1^{-1}J_2) = d_\lambda^{-2} \sum_{J \in \mathcal{S}_\lambda/\mathcal{U}_\lambda} \psi(J) \varphi(J)^* \mathcal{F}_\lambda(J^{-1}, J) = \frac{\mu_\lambda}{s_\lambda} \sum_{J \in \mathcal{S}_\lambda/\mathcal{U}_\lambda} \psi(J) \varphi(J)^* = \delta_{\psi, \varphi} \tag{4.16}
\]

Thus the basis \( \{ \mathcal{O}_\psi \mid \psi > \hat{\psi} \} \) is orthonormal. Also, combining these results we learn that the endomorphisms \( d_\lambda^{-1/2} \mathcal{O}_\psi \) form a partition of unity:

\[
\sum_{\psi \in \mathcal{S}_\lambda^+} \mathcal{O}_\psi = d_\lambda^{1/2} \, 1 = d_\lambda^{1/2} \tag{4.17}
\]

### 4.3 A natural basis for the boundary blocks

Our next aim is to associate to each of the endomorphisms \( \mathcal{O}_\psi : \mathcal{V}_{\psi} \to \mathcal{V}_{\psi} \) with \( \psi > \hat{\psi} \) and to each \( J \in \mathcal{G}/\mathcal{S}_\lambda \) a linear form

\[
b_\psi \equiv b_{\psi}(\hat{\psi}, J^{\lambda}) : \mathcal{V}_{\psi} \otimes \mathcal{V}_{\psi}^+ \to \mathbb{C}, \tag{4.18}
\]

in such a way that the collection of these forms constitutes a basis. (Thus these maps are required to provide a concrete realization of the basis elements \( b_{\psi, i} \) that were introduced in

\[\text{In the last line we assume that the cocycle } \mathcal{F} \text{ has been chosen to be standard, which means (see formula (B.14)) that for elements of the basis of the twisted group algebra the operations of forming the inverse and of conjugating with an element of the center look the same as in the untwisted case. This property of } \mathcal{F} \text{ can always be achieved by a suitable choice of basis.}\]
When doing so, we still have to allow for an arbitrary over-all normalization of the blocks, pairs (blocks on $H$ for all $V$) linear forms on $H$ independent; since there are $d$ for all $J \in G$.

Of course, the scalar factor for $\beta_0$ is different for different isotypic components of $\mathcal{H}_\lambda$, so that for each $J \in \mathcal{G}/S_\lambda$ we obtain a different map $\beta_0^{(\hat{\psi},J\lambda)}$. Hence for arbitrary elements $p, q$ of $\bigoplus_{J \in \mathcal{G}/S_\lambda} \mathcal{H}_{J\lambda}$ and $\bigoplus_{J \in \mathcal{G}/S_\lambda} \mathcal{H}_{(J\lambda)^+}$, respectively, we have

$$B_\lambda(v \otimes p^{(J\lambda)} \otimes w \otimes q^{(J\lambda)}) = \beta_0^{(\hat{\psi},J\lambda)}(v \otimes w) \cdot \tilde{B}_{\lambda\lambda}(p^{(J\lambda)} \otimes q^{(J\lambda)})$$

(4.22)

for all $J \in \mathcal{G}/S_\lambda$, where $p =: \sum_{j \in \mathcal{G}/S_\lambda} p^{(j\lambda)}$ with $p^{(j\lambda)} \in \mathcal{H}_{j\lambda}$ and analogously for $q$.

By the linear independence of the endomorphisms $\mathcal{O}_\psi$, also the forms (4.20) are linearly independent; since there are $d^2$ of them, they therefore provide us indeed with a basis of the linear forms on $V_\psi \otimes V_{\psi^+}$. We now combine this basis with the $\mathfrak{A}$-blocks (1.3) with $\tilde{\mu} = J\lambda$. When doing so, we still have to allow for an arbitrary over-all normalization of the blocks, which cannot be determined at the present stage. We thus arrive at the linear forms

$$\tilde{B}_{(\tilde{\mu},\psi)} = \tilde{B}_{\lambda}(\tilde{\mu},\psi) := N_{\tilde{\mu},\psi} d^{1/2}_\lambda b_{\psi} \otimes \tilde{B}_{\tilde{\mu}}$$

(4.23)

on $\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+}$ with some non-zero $N_{\tilde{\mu},\psi} \in \mathbb{C}$, acting as

$$\tilde{B}_{(J\lambda,\psi)}(v \otimes p \otimes w \otimes q) := N_{J\lambda,\psi} d^{1/2}_\lambda b_{\psi}(v \otimes w) \cdot B_{J\lambda}(p^{(J\lambda)} \otimes q^{(J\lambda)})$$

(4.24)

for all $v \in V_\psi$, $w \in V_{\psi^+}$, $p \in \mathcal{H}_\lambda$ and $q \in \mathcal{H}_{\lambda^+}$. When $\lambda = [\hat{\lambda}, \hat{\psi}]$, then for $\tilde{\mu} = J\hat{\lambda}$ with $J$ ranging over $\mathcal{G}/S_\lambda$ and $\psi$ over all $\hat{\psi}$, the forms $\tilde{B}_{(\tilde{\mu},\psi)}$ constitute a natural basis for the boundary blocks on $\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+}$ that preserve $\mathfrak{A}$. In short, the boundary blocks are naturally labelled by pairs $(\tilde{\mu}, \psi)$, one label referring to a primary field $\tilde{\mu}$ of the $\mathfrak{A}$-theory with vanishing monodromy charge $Q_{\bar{\psi}}(\mu) = 0$, the other a character of the full stabilizer, $\psi \in S^*_\mu$. 


We remark that with the help of the identity (4.17) one checks that the ordinary boundary blocks of the $\mathfrak{A}$-theory can be expressed in terms of the blocks $\tilde{B}_{(\bar{\mu},\psi)}$ as

$$\tilde{B}_\lambda = \bigoplus_{\psi \in S_\lambda} \bigoplus_{J \in \mathcal{G}/S_\lambda} (\mathcal{N}_{J\lambda,\psi})^{-1} \tilde{B}_{(J\lambda,\psi)}.$$  

(4.25)

Moreover, it is easy to see that the form $\beta_\circ$ satisfies a 'degeneracy space Ward identity', i.e.

$$\beta_\circ \circ (y \otimes 1 - 1 \otimes y) = 0$$  

(4.26)

for every $y \in \text{End}(V_\hat{\psi})$. This result, in turn, when combined with the Ward identities of the $\bar{\mathfrak{A}}$-theory, immediately implies that the linear combination (4.25) indeed satisfies the Ward identities of the $\mathfrak{A}$-theory, i.e.

$$\tilde{B}_\lambda \circ (Y_n \otimes 1 + (-1)^{A_Y} 1 \otimes Y_{-n}) = 0$$  

(4.27)

for all $Y \in \mathfrak{A}$ ($\Delta_Y$ denotes the conformal weight of $Y$).

### 4.4 Scalar products

For the computation of annulus amplitudes we need to deal with suitable scalar products of the boundary blocks. As a matter of fact, the boundary blocks are not normalizable; but for every $t > 0$ there exists a modified inner product

$$\langle \tilde{B}_{(\lambda,\psi)} | e^{-(2\pi/t)(L_0 \otimes 1 + 1 \otimes L_0 - c/12)} | \tilde{B}_{(\bar{\mu},\bar{\psi})} \rangle$$  

(4.28)

with respect to which they become normalizable. To substantiate this statement and perform the concrete calculation, we compare it to the analogous computation for the boundary blocks of the $\bar{\mathfrak{A}}$-theory. As usual, we normalize the ordinary boundary blocks of the $\mathfrak{A}$-theory by prescribing the over-all factor in their modified inner product, according to

$$\langle B_\lambda | e^{-(2\pi/t)(L_0 \otimes 1 + 1 \otimes L_0 - c/12)} | B_\mu \rangle = \frac{1}{S_{\lambda,\Omega}} \chi_\lambda(2i/t) \delta_{\lambda,\mu},$$  

(4.29)

and analogously for the boundary blocks of the $\bar{\mathfrak{A}}$-theory:

$$\langle \tilde{B}_\lambda | e^{-(2\pi/t)(L_0 \otimes 1 + 1 \otimes L_0 - c/12)} | \tilde{B}_{\bar{\mu}} \rangle = \frac{1}{\bar{S}_{\lambda,\bar{\Omega}}} \bar{\chi}_{\lambda}(2i/t) \delta_{\lambda,\bar{\mu}}.$$  

(4.30)

---

5 The introduction of the explicit factor of $d_\lambda^{-1/2}$ in (4.24) was chosen with hindsight, so as to cancel the corresponding factor in (4.17).

6 Note that in the two terms $y$ acts on different spaces, i.e. in more pedantic notation the identity reads $\beta_\circ \circ [R_{\hat{\psi}}(y) \otimes 1 - 1 \otimes R_{\hat{\psi}}(y)] = 0$. Just like in the usual Ward identities, in (4.26) the representation symbols are suppressed.

7 While at this point this observation is a mere peculiarity without any particular application, these modified inner products indeed appear in the computation of annulus amplitudes, see subsection 6.2 below. In that context, $t$ is the modular parameter of the annulus.
What we need as an additional new ingredient is to construct an inner product on the space of linear maps (4.20); this is achieved as follows. First we construct a scalar product on the degeneracy space $\mathcal{V}_\psi$, i.e. a sesquilinear map

$$\kappa_\psi : \mathcal{V}_\psi \times \mathcal{V}_\psi \to \mathbb{C}. \quad (4.31)$$

The construction uses the invariant scalar products on the modules of $\mathfrak{A}$ and $\bar{\mathfrak{A}}$. Namely, on $\mathfrak{A}$ we have an antilinear conjugation map $c : Y \mapsto Y^\dagger$, and on each $\mathfrak{A}$-module $\mathcal{H}_\lambda$ there is a scalar product $\kappa_\lambda : \mathcal{H}_\lambda \times \mathcal{H}_\lambda \to \mathbb{C}$ which is invariant in the sense that

$$\kappa_\lambda(Y_n p, p') + \kappa_\lambda(p, (Y_n)^\dagger p') = 0 \quad (4.32)$$

for all $p, p' \in \mathcal{H}_\lambda$ and all $Y_n \in \mathfrak{A}$. Such a scalar product on an irreducible module is unique up to a scalar. Moreover, since the subalgebra $\bar{\mathfrak{A}}$ must be consistent, it is closed under $c$, and hence there is an analogous structure on $\bar{\mathfrak{A}}$-modules.

The scalar product on $\mathcal{V}_\psi$ can now be constructed as follows. The subspace $\mathcal{V}_\psi \otimes \bar{\mathcal{H}}_\lambda$ of $\mathcal{H}_\lambda$ inherits a scalar product from the scalar product $\kappa_\lambda$ of $\mathcal{H}_\lambda$. For any two fixed vectors $v, v'$ in the degeneracy space $\mathcal{V}_\psi$, the mapping $(p, p') \mapsto \kappa_\lambda(v \otimes p, v' \otimes p')$ for $p, p' \in \bar{\mathcal{H}}_\lambda$ provides a sesquilinear form. This sesquilinear form is still unitary with respect to the restriction of the conjugation $c$ to the subalgebra $\bar{\mathfrak{A}}$. It must thus be proportional to the standard scalar product $\bar{\kappa}_\lambda$ on $\bar{\mathcal{H}}_\lambda$. We call the constant of proportionality $\kappa_\psi(v, v')$. In formulae,

$$\kappa_\psi(v, v') = \frac{\kappa_\lambda(v \otimes p, v' \otimes p')}{\bar{\kappa}_\lambda(p, p')}, \quad (4.33)$$

where any pair $p, p' \in \bar{\mathcal{H}}_\lambda$ of vectors can be chosen that obeys $\bar{\kappa}_\lambda(p, p') \neq 0$. One verifies that $\kappa_\psi$ constitutes a non-degenerate scalar product on the degeneracy space $\mathcal{V}_\psi$.

The scalar product $\kappa_\psi$ possesses an invariance property as well. Consider the elements of $\mathfrak{A}$ that commute with the subalgebra $\bar{\mathfrak{A}}$. Since the conjugation $c$ is an automorphism of $\mathfrak{A}$, this commutant is mapped by $c$ to itself, so that the commutant, too, comes with its own conjugation. The scalar product is now invariant in the sense that

$$\kappa_\psi(yv, v') = \kappa_\psi(v, c(y)v') \quad (4.34)$$

for all $y \in \text{End}(\mathcal{V}_\psi)$. (In case the commutant should be smaller than $\text{End}(\mathcal{V}_\psi)$, one simply extends $c$ to the rest of $\text{End}(\mathcal{V}_\psi)$.)

Now since the degeneracy spaces $\mathcal{V}_\psi$, and analogously also $\mathcal{V}_{\psi^+}$, carry an invariant scalar product, also the space of linear forms on $\mathcal{V}_\psi \otimes \mathcal{V}_{\psi^+}$ and the space of endomorphisms $\text{End}(\mathcal{V}_\psi) \cong (\mathcal{V}_\psi)^* \otimes \mathcal{V}_\psi$ have a scalar product. For the latter, it is given by $\kappa_{\text{End}(\mathcal{V}_\psi)}(y, y') = \text{tr}(y^\dagger y')$. (Notice that the trace is independent of the scalar product on $\mathcal{V}_\psi$; the latter does enter, however, through the hermitian conjugation.) On the space of linear forms, the scalar product reads

$$\kappa_b(b_\psi, b_\varphi) := \sum_{i,j=1}^{d_\lambda} b_\psi(v_i \otimes w_j) \cdot b_\varphi(v_i \otimes w_j), \quad (4.35)$$

where $\{v_i\}$ and $\{w_j\}$ are orthonormal bases of $\mathcal{V}_\psi$ and $\mathcal{V}_{\psi^+}$ with respect to the scalar products $\kappa_\psi$ and $\kappa_{\psi^+}$, respectively.
The non-degenerate form $\beta_\circ$ defined in (4.21) provides us with an isomorphism

$$y \mapsto \beta_\circ \circ (y \otimes id)$$

between $\text{End}(V_\hat{\psi})$ and the space of linear forms on $V_\hat{\psi} \otimes V_\hat{\psi}^+$. We would like to check that this isomorphism is a homothety, i.e. that it preserves angles. With the orthonormal bases introduced above we need to show that

$$\sum_{i,j=1}^{d_\lambda} \beta_\circ(v_i \otimes w_j)^* \beta_\circ(y'v_i, w_j) = \xi \text{ tr } y^\dagger y'$$

for some non-zero number $\xi \equiv \xi_{[\lambda, \hat{\psi}]} \in \mathbb{C}$, implying in particular that

$$\langle b_\varphi | b_\psi \rangle \equiv \kappa_b(b_\psi, b_\varphi) = \xi \text{ tr } O_\psi^\dagger O_\varphi = \xi \delta_{\psi,\varphi}.$$

In components with respect to the two chosen orthonormal bases, the relation (4.37) amounts to

$$\sum_{j=1}^{d_\lambda} \beta_\circ(v, w_j)^* \beta_\circ(v', w_j) = \xi \kappa_\tilde{\psi}(v, v'),$$

where the sum is over any arbitrary orthonormal basis $\{w_j\}$ of $V_\hat{\psi}^+$. The validity of this relation can be established by showing that $\tilde{\kappa}_\lambda(p, p') \sum_j \beta_\circ(v \otimes w_j)^* \beta_\circ(v' \otimes w_j)$ provides a non-degenerate and invariant scalar product on $\mathcal{H}_\lambda$. This is indeed possible; the details are presented in appendix C.

### 4.5 The normalization of the boundary blocks

We are now finally in a position to determine the value of the over-all normalization constant $\mathcal{N}_{\lambda, \psi}$ that was left undetermined in the definition (4.23) of the boundary blocks $\tilde{B}_{(\tilde{\lambda}, \hat{\psi})}$. To this end we have to prescribe some normalization of the modified inner product of these blocks, much as was done in (4.29) and (4.30) for the ordinary boundary blocks. As it turns out, a convenient prescription is

$$\langle \tilde{B}_{(\lambda, \psi)} | e^{-2(\pi/t) L_0 \otimes 1 + 1 \otimes (L_0 - c/12)} | \tilde{B}_{(\mu, \varphi)} \rangle = \frac{1}{(|G|/u_\lambda) \mathcal{S}_{\lambda, \Omega}} \tilde{x}_{\lambda} (2i/t) \delta_{\lambda, \mu} \delta_{\psi, \varphi}. $$

We also observe that relation (4.16) amounts to the statement that the operators $\mathcal{O}_\psi$ with $\psi \succ \hat{\psi}$ form an orthonormal basis of the space of endomorphisms $\text{End}(V_\hat{\psi})$. It follows that the constants $d_\lambda^{1/2} (\mathcal{N}_{\lambda, \psi})^{-1}$ are precisely the constants of proportionality between the scalar product on $\text{End}(V_\hat{\psi})$ and on the space of linear forms that appear in the relation (4.37). This implies in particular that $\mathcal{N}_{\lambda, \psi}$ actually depends only on the $\mathcal{U}_\lambda$-character $\hat{\psi}$ and not on the particular $\psi$ that extends it to a character of $\mathcal{S}_\lambda$. 
To proceed, we combine formula (4.40) with the decomposition (4.25) of the ordinary boundary blocks $B_\lambda$ of the $\mathcal{A}$-theory. We then find
\[
\langle B_\lambda | e^{-(2\pi/\ell)(L_0 \otimes 1 \otimes L_0 - c/12)} | B_\mu \rangle = \sum_{\psi \in S^+_\lambda} \sum_{\varphi \in S^+_{\lambda'}} \sum_{J \in G/S_\lambda} \sum_{J' \in G/S_\mu} (\mathcal{N}^*_{J\lambda,\psi}) (\mathcal{N}^{J'}_{\bar{\mu},\varphi})^{-1} \langle \tilde{B}_{(J\lambda,\psi)} | e^{-(2\pi/\ell)(L_0 \otimes 1 \otimes L_0 - c/12)} | \tilde{B}_{(J'\mu,\varphi)} \rangle
\]
\[
= \delta_{\psi,\bar{\psi}} \delta_{\lambda,\bar{\mu}} \sum_{\psi \in S^+_\lambda} \sum_{J \in G/S_\lambda} |\mathcal{N}_{J\lambda,\psi}|^{-2} \frac{1}{(|G|/|u_\lambda|) S_{J\lambda,\Omega}} \bar{x}_{J\lambda}(\frac{2i}{T})
\]
\[
= \delta_{\psi,\bar{\psi}} \delta_{\lambda,\bar{\mu}} d_\lambda \frac{1}{(|G|/|u_\lambda|) S_{\lambda,\Omega}} |\mathcal{N}_{\bar{\lambda},\bar{\psi}}|^{-2} \chi_{\lambda}(\frac{2i}{T})
\]
(4.41)

Here in the last step we have used the information that according to formula (4.29) the result must be proportional to the $\mathcal{A}$-character $\chi_{\lambda}$, so that the relation (A.31) between the $\mathcal{A}$-characters and those of the $\mathcal{A}$-theory tells us in particular that the normalizations $\mathcal{N}_{\bar{\lambda},\bar{\psi}}$ in fact do not depend on $J$, and hence only on $\lambda = [\bar{\lambda}, \bar{\psi}]$. Moreover, by inspection of formula (A.10) for the modular matrix $S$ of the $\mathcal{A}$-theory we have
\[
S_{[\lambda,\psi],\Omega} = \frac{|G|}{\sqrt{A_{\lambda,\Omega}}} S_{\lambda,\Omega}.
\]
(4.42)

Thus the normalization condition (4.29) also allows us to determine the explicit value of the constants $\mathcal{N}_{\bar{\lambda},\bar{\psi}}$, namely $|\mathcal{N}_{\bar{\lambda},\bar{\psi}}|^{-2} d_\lambda u_\lambda/|G| = \sqrt{s_\lambda u_\lambda}/|G|$, and hence simply
\[
|\mathcal{N}_{\bar{\lambda},\bar{\psi}}| = 1.
\]
(4.43)

Note that we determine these constants only up to a phase. Manifestly, we cannot do better, because the relations (4.29) and (4.30) determine the boundary blocks also only up to a phase.

To conclude this section, we summarize our results about the natural basis of boundary blocks for symmetry breaking boundary conditions. For every primary $\lambda = [\bar{\lambda}, \bar{\psi}]$ of the $\mathcal{A}$-theory we have $|G|/|S_\lambda| \cdot d_\lambda^2$ basis elements $\tilde{B}_{(J,\lambda,\psi)}$ which are labelled by those $\mathcal{A}$-primaries $\bar{\mu}$ that are on the $G$-orbit of $\bar{\lambda}$ and by the $S_\lambda$-characters $\psi$ that restrict to $\bar{\psi}$. These boundary blocks obey the normalization condition (4.40).

5 The classifying algebra

In this section we turn to the level of full conformal field theory. We explain how the representation theory of a classifying algebra allows us to determine the boundary conditions for a given conformal field theory, and we explicitly construct the classifying algebra for boundary conditions that preserve a prescribed subalgebra $\mathcal{A}$ of the bulk symmetries.

\[\text{We admit that our conventions were chosen with some hindsight.}\]
5.1 Boundary conditions and reflection coefficients

Because of factorization, a boundary condition should essentially be characterized by a consistent collection of one-point correlation functions for all bulk fields $\phi_{\lambda,\bar{\lambda}}$ on the disk \[2,18,29,3\]. Thus in order to classify the boundary conditions, one needs to find all consistent one-point correlation functions $\langle \phi_{\lambda,\bar{\lambda}} \rangle$ for those fields. As explained in [5], the correlation functions on a surface with boundaries are linear combinations of blocks on its Schottky cover; for the disk this oriented cover has the topology of the sphere, so that the relevant chiral blocks are those studied in section 4. Our task is now to determine the coefficients that give the correct physical correlators.

For a more detailed discussion it is convenient to use the language of vertex operators and operator products. For every vector $v \otimes \tilde{v} \in \mathcal{H}_\lambda \otimes \mathcal{H}_{\bar{\lambda}}$ we have a vertex operator $\phi_{\lambda,\bar{\lambda}}(v \otimes \tilde{v} ; z)$. Such vertex operators are suitable linear combinations of pairs of chiral vertex operators, the correlators of which are nothing but the boundary blocks discussed above. In view of the description (4.4) of the boundary blocks and their precise definition (4.23), we are thus looking for coefficients $\xi_{\mu,\psi}$ such that the value of the one-point correlator for $\phi_{\lambda,\bar{\lambda}}(v \otimes \tilde{v} ; z)$ on the disk at $z = 0$ is

$$\langle \phi_{\lambda,\bar{\lambda}}(v \otimes \tilde{v} ; z=0) \rangle = \sum_{\psi \in S_\lambda^*} \sum_{J \in \mathcal{G}/S_\lambda} \xi_{J,\lambda,\psi} B_{J,\lambda,\psi}(v \otimes \tilde{v}) . \tag{5.1}$$

Moreover, it follows from the results at the chiral level that this correlator can be non-vanishing only for $\tilde{\lambda} = \lambda^+$, which we therefore assume from now on.

Note, however, that the index structure of the vertex operator in formula (5.1) is tailored to the case of a closed orientable surface, where the bulk fields $\phi_{\lambda,\bar{\lambda}}$ are the only fields present. In contrast, for surfaces with boundaries, where there are also boundary fields $\Psi(x)$, and allowing for boundary conditions that break part of the bulk symmetries, the index structure can actually be more complicated. Accordingly we have to be careful when interpreting formula (5.1). What we have to implement correctly is the fact that, while chiral vertex operators can definitely be extracted from the three-point chiral blocks on $\mathbb{P}^1$, their concrete form does depend on which chiral symmetries are preserved. In the situation of interest to us we are not allowed to employ all symmetries of the bulk, but rather we must take the three-point blocks of the orbifold theory with symmetry $\hat{\mathcal{A}}$ for extracting the chiral vertex operators. In other words, we must take into account that states in different $\hat{\mathcal{A}}$-submodules of $\mathcal{H}_\lambda = \bigoplus_{J \in \mathcal{G}/S_\lambda} \mathcal{V}_\psi \otimes \mathcal{H}_{(J,\lambda,\tilde{\psi})}$ can cause different excitations on the boundary and can thus be reflected differently. Note that, unlike in formula (3.9), here we have attached the label $\tilde{\psi}$ also to the $\hat{\mathcal{A}}$-modules $\mathcal{H}$, so as to indicate that the reflection at the boundary may also depend on the particular $\hat{\mathcal{A}}$-module into which a given $\hat{\mathcal{A}}$-module is embedded. In addition, we have to account for the dimensionality of the projective $S_\lambda$-module $\mathcal{V}_\psi$, which amounts to using characters $\psi \in S_\lambda^*$ instead of $\tilde{\psi} \in \mathcal{U}_\lambda^*$.

Accordingly, when studying the behavior of bulk fields close to the boundary, for vectors $v \otimes \tilde{v} \in (\mathcal{V}_\psi \otimes \mathcal{H}_{(\lambda,\tilde{\psi})}) \otimes (\mathcal{V}_{\psi^+} \otimes \mathcal{H}_{(\lambda^+,\tilde{\psi}^+)}) \subset \mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+}$ we must work with vertex operators that are labelled as

$$\phi_{(\lambda,\psi),(\lambda^+,\psi^+)}(v \otimes \tilde{v} ; z) . \tag{5.2}$$

As for the correlation functions, this means that in place of (3.1) we are interested in the
individual summands

\[ \langle \phi(\bar{\lambda},\psi), (\bar{\lambda}+,\psi+) (v \otimes \tilde{v}; z=0) \rangle = \xi_{\bar{\lambda},\psi} \tilde{B}(\bar{\lambda},\psi)(v \otimes \tilde{v}) . \]  

(5.3)

In order to determine the coefficients \( \xi_{\bar{\mu},\psi} \) in this relation, we study the operator product expansion describing the excitation that a bulk field causes on the boundary when it approaches the boundary; this operator product reads

\[ \phi(\bar{\lambda},\psi), (\bar{\lambda}+,\psi+) (r e^{i\sigma}) = \sum_{\bar{\mu}} (1-r^2)^{-2\Delta_{\bar{\lambda}}+\Delta_{\bar{\mu}}} R_{(\bar{\lambda},\psi);\bar{\mu}}^a \Psi^a_{\bar{\mu}} (e^{i\sigma}) + \text{descendants for } r \to 1 . \]  

(5.4)

Comparing this expansion with relation (5.3) we learn that

\[ \xi_{\bar{\mu},\psi} = R_{(\bar{\mu},\psi);\bar{\Omega}}^a \langle \Psi^a_{\bar{\Omega}} \rangle . \]  

(5.5)

In words, up to a normalization given by the (constant) one-point correlator of a boundary vacuum field \( \Psi^a_{\bar{\Omega}} \), the coefficients \( \xi_{\bar{\mu},\psi} \) are equal to the reflection coefficients \( R_{(\bar{\mu},\psi);\bar{\Omega}}^a \).

We pause to comment on the index structure of the boundary fields \( \Psi^{ab}_{\bar{\mu}}(x) \). The underlying three-point blocks for the operator product (5.4) are those of the orbifold theory, because boundary fields are involved and the latter only need to preserve the symmetries in \( \tilde{\mathfrak{A}} \). As a consequence, the boundary field carries a chiral label \( \bar{\mu} \) of the orbifold theory. In addition, there are two labels \( a,b \) which account for the fact that the insertion of a boundary field can change the boundary condition. (And finally, in order to account for annulus coefficients that are bigger than one – which can appear for \( \bar{\mu} \neq \bar{\Omega} \) – one must allow for an additional degeneracy label, which we suppress.) The presence of these boundary labels on the right hand side of (5.4) tells us that, in contrast to conformal field theory on surfaces that are closed and orientable, on surfaces with boundaries the locality and factorization constraints for the correlation functions do not, in general, possess a unique solution. Rather, there are several consistent collections of reflection coefficients \( R_{(\bar{\lambda},\psi);\bar{\Omega}}^a \), and as a consequence there are several solutions

\[ \langle \phi(\bar{\lambda},\psi), (\bar{\lambda}^+,\psi^+) \rangle = \langle \phi(\bar{\lambda},\psi), (\bar{\lambda}^+,\psi^+) \rangle_a \]  

(5.6)

which are indexed by the boundary conditions.

Note that up to this point it was not necessary to specify the values that the boundary label \( a \) can take. To determine the possible boundary conditions, we analyze the factorization of bulk-bulk-boundary correlators in much the same manner as [29,20,3] for boundary conditions that preserve all of \( \mathfrak{A} \). This is possible because, by the requirement that \( \mathfrak{A} \) is a consistent chiral algebra, the orbifold chiral blocks obey the usual factorization rules. Concretely, we consider two different factorization limits of the disk correlation function

\[ \langle \phi(\lambda_1,\psi_1), (\lambda_1^+,\psi_1^+) (z_1) \phi(\lambda_2,\psi_2), (\lambda_2^+,\psi_2^+) (z_2) \rangle_a \]  

(5.7)

involving two bulk fields. On one hand we can use the operator product between bulk fields (this is an operator product respecting the full \( \mathfrak{A} \)-symmetry, although for fields that are only \( \tilde{\mathfrak{A}} \)-primaries but may be \( \mathfrak{A} \)-descendants) and afterwards the operator product (5.4), so as to express the correlator in terms of bulk operator product coefficients and a single reflection
coefficient $R^a_{(\lambda_3,\psi_3)\Omega}$. On the other hand, applying the expansion (5.4) twice expresses the correlator in terms of two bulk-boundary operator products, i.e. two reflection coefficients. The latter are to be understood as prefactors of a four-point block on the projective line $\mathbb{P}^1$, and since boundary insertions are involved, these are four-point blocks of the orbifold theory. The two different factorizations correspond to such blocks in different channels, so that for their comparison one must relate them through fusing matrices or, to be precise, through fusing matrices of the $\mathfrak{A}$-theory. Such matrices exist because by assumption the orbifold chiral blocks come with a Knizhnik–Zamolodchikov connection.

Taking everything together we arrive at a relation of the form

$$R^a_{(\lambda_1,\psi_1)\Omega} R^a_{(\lambda_2,\psi_2)\Omega} = \sum_{\lambda_3,\psi_3} \tilde{N}_{(\lambda_1,\psi_1),(\lambda_2,\psi_2)}^{(\lambda_3,\psi_3)} R^a_{(\lambda_3,\psi_3)\Omega},$$

(5.8)

where the numbers $\tilde{N}$ are combinations of bulk operator product coefficients and fusing matrices of the $\mathfrak{A}$- and of the $\mathfrak{A}$-theory. Notice that the fact that quantities of both the original and the orbifold theory appear is in accordance with the index structure of the vertex operators (5.2), in which there appear individual fields $\lambda$ of the orbifold theory rather than $G$-orbits of such fields, but also a character $\psi$ that keeps track of the information as submodule of which $\mathfrak{A}$-module a given $\mathfrak{A}$-module occurs.

None of the constituents of the numbers $\tilde{N}$ depends on the boundary label $a$. The result (5.8) can therefore be interpreted as follows. The conformally invariant boundary conditions that preserve $\mathfrak{A}$ correspond to one-dimensional representations of some algebra, which we call the classifying algebra and denote by $C(\mathfrak{A})$. It is expected [5] that the algebra $C(\mathfrak{A})$ shares most properties of fusion algebras, i.e. it should be a commutative associative semisimple algebra, so that in particular all its irreducible representations are one-dimensional. These properties imply the existence of a diagonalizing matrix $\tilde{S}$ through which the structure constants of $C(\mathfrak{A})$ are expressible via an analogue of the Verlinde formula.

It is worth stressing that the two labels of the diagonalizing matrix $\tilde{S}$ are on a rather different footing: the row index labels the basis of the classifying algebra $C(\mathfrak{A})$ which is given by the allowed boundary blocks, while the column index labels the irreducible representations of $C(\mathfrak{A})$. In the case of boundary conditions that preserve the full bulk symmetry (and where the pairing for the labels of the bulk fields is given by charge conjugation, i.e. $\lambda = \lambda^+$), it has already been argued long ago [2] that $\tilde{S}$ is the modular matrix that implements the modular transformation $\tau \mapsto -1/\tau$ on the characters. In this case the classifying algebra is just the fusion rule algebra and the reflection coefficients are the generalized quantum dimensions; in particular there is a natural correspondence between the two types of labels.

In the general case, this natural correspondence does not persist. But it has been seen that even in more general situations (see [3] for an example) nevertheless the two sets of labels are still related by modular transformations. Moreover, it can be expected that the boundary conditions are labelled by orbits of fields rather than individual fields, as in [4]. That this is indeed the case can be seen as follows. As for the labels $\lambda$ of the boundary blocks, only those occur which appear in the decomposition of some $\mathfrak{A}$-module, which means that they satisfy $Q_J(\lambda) = 0$ for every $J \in G$. In orbifold terminology, we are only dealing with the untwisted sector of the orbifold or, in other words, along the ‘space’ direction of the torus only the trivial twist by the identity occurs. This implies that after a modular $S$-transformation, only the
identity appears as a twist in the ‘time’ direction of the torus, which in turn tells us that we must not perform the usual orbifold projection in the twisted sector. Translating this back into simple current terminology, we arrive at the statement that the boundary conditions must not be labelled by individual primary fields of the $\mathcal{A}$-theory, but rather by $G$-orbits of $\mathcal{A}$-primaries. On the other hand, in the ‘time’ direction we start with arbitrary twists, since the labelling is by individual primary fields; it follows that after the $S$-transformation arbitrary twists in the ‘space’ direction occur in the orbifold. Thus in the labelling of the boundary conditions all $G$-orbits appear, not just those with vanishing monodromy charges, i.e. not just the ones in the untwisted sector. Moreover, by comparison with the $S$-transformation of the $\mathcal{A}$-characters one is led to expect that these orbits are to be combined with the characters of the relevant untwisted stabilizer; as we will see below, this provides us indeed with a consistent ansatz for the classifying algebra.

5.2 The matrix $\tilde{S}$

As advocated above, the boundary blocks are in one-to-one correspondence with the elements of a basis of the classifying algebra $C(\mathcal{A})$, while the $\mathcal{A}$-preserving boundary conditions are in one-to-one correspondence with the (isomorphism classes of) one-dimensional irreducible representations of $C(\mathcal{A})$. Thus a basis of $C(\mathcal{A})$ is labelled by pairs $(\bar{\lambda}, \psi_\lambda)$ consisting of a primary label $\bar{\lambda}$ of the $\mathcal{A}$-theory in the untwisted sector (i.e. $Q_J(\lambda) = 0$ for all $J \in G$) and a character $\psi_\lambda \in S_{\bar{\lambda}}^\ast$ of the stabilizer of $\bar{\lambda}$, while the arguments at the end of the previous subsection tell us that the one-dimensional irreducible $C(\mathcal{A})$-representations are labelled by $G$-orbits $[\bar{\rho}, \hat{\psi}_\rho]$ of pairs consisting of an arbitrary primary label $\bar{\rho}$ of the $\mathcal{A}$-theory and a character $\hat{\psi}_\rho$ of the untwisted stabilizer of $\bar{\rho}$.

According to our general expectations the classifying algebra $C(\mathcal{A})$ should possess most properties of fusion algebras, in particular there should exist a diagonalizing square matrix $\tilde{S}$. Note that according to the previous remarks the row and column labels of this matrix are on a rather different footing, so that at this point it is still far from obvious that the two sets of labels indeed have equal size. Our strategy is now to start by making an educated ansatz for the matrix $\tilde{S}$ and then develop the classifying algebra and its representation theory along analogous lines as one may study fusion algebras by starting from the modular $S$-matrix $S$. We stress that, unlike the considerations in the previous section, here we are indeed making an ansatz, and it will be necessary to support this ansatz by performing various consistency checks, the most basic one being that $\tilde{S}$ is manifestly a square matrix. But once one accepts this ansatz, the arguments presented in the previous subsection allow us to learn more about how the fusing matrices in an integer spin simple current extension are related to the fusing matrices of the original theory.

As a matter of fact, by combining the considerations that relate symmetry breaking, orbifolds and integer spin simple current extensions with the results about simple current extensions obtained in [12], it follows that, up to normalizations, there is a natural candidate for the matrix $\tilde{S}$. It reads

$$
\tilde{S}_{(\bar{\lambda}, \psi_\lambda), [\bar{\rho}, \hat{\psi}_\rho]} := \frac{|G|}{\sqrt{|S_{\bar{\lambda}} \cap S_{\bar{\rho}}|}} \sum_{J \in S_{\bar{\lambda}} \cap S_{\bar{\rho}}} \psi_\lambda(J) \hat{\psi}_\rho(J)^* S_{\bar{\lambda}, \bar{\rho}}^J.
$$

(5.9)

Here the matrices $S^J$ are those which appear [12] in the modular $S$-matrix of the simple current
extension; upon a canonical normalization of the one-point chiral blocks with insertion $J$ on the torus, the matrix $S^J$ also represents the modular S-transformation on these blocks [13]. (For the convenience of readers who are not familiar with the pertinent results of [12], we summarize them in appendix A. For a brief account, see also section 3 of [17].)

Note that at the chiral level, where one deals with conformal field theory on a complex curve, there is no direct influence of boundaries [5]. The chiral conformal field theory structures that are related to the matrices $S^J$ which enter the discussion here are thus logically independent of any boundary data; they have passed independent tests [12, 13] in the context of closed conformal field theory. In the present situation, where the $\mathfrak{A}$-theory can be regarded as an orbifold of the $\mathfrak{A}$-theory, as compared to [12] there is actually even further evidence for the existence of the matrices $S^J$. Namely, under a finiteness assumption on the codimension of a certain subspace of the vacuum module, it has been proven in [30] (see also [31, 32]) that one can associate a modular S-transformation matrix to the chiral blocks on the torus for arbitrary descendants of the vacuum. In our case, we are precisely concerned with one-point blocks for descendants of the vacuum of the $\mathfrak{A}$-theory (which are not descendants of the vacuum in the $\mathfrak{B}$-theory, though). In connection with the reasoning of [12] one may say that if an extension by integer spin simple currents is possible at all, then such matrices $S^J$ must necessarily exist in order to comply with the general result of [30].

As a first consistency check, we consider the special case where $Q(\rho) \equiv 0$. These boundary conditions correspond precisely to orbits that furnish primary fields in the extended theory. On the other hand, the boundary conditions that respect the full bulk symmetry should also be recovered from our ansatz, since a fortiori they preserve the subalgebra $\bar{\mathfrak{A}}$. According to [2], these boundary conditions correspond to primary fields of the $\mathfrak{A}$-theory. Indeed, the following consideration shows that for $Q(\rho) \equiv 0$ we recover the modular S-matrix of the $\mathfrak{A}$-theory. The latter can be expressed through the matrices $S^J$ as in (A.10), i.e.

$$S_{[\bar{\lambda}, \hat{\psi}_\lambda],[\bar{\rho}, \hat{\psi}_\rho]} = \frac{|G|}{\sqrt{|S^\lambda U^\lambda| u_{\lambda}}} \sum_{J \in U^\lambda \cap U_\rho} \hat{\psi}_\lambda(J) \hat{\psi}_\rho(J)^* S^J_{\bar{\lambda}, \bar{\rho}} \cdot (5.10)$$

where both $\bar{\lambda}$ and $\bar{\rho}$ have monodromy charge zero. Because of the latter property, we know that for every $J \in S^\lambda \setminus U^\lambda$ there exists at least one $K \in S^\lambda$ such that

$$S^J_{\bar{\lambda}, \bar{\rho}} = S^J_{K\bar{\lambda}, \bar{\rho}} = F_{\lambda}(K, J) \cdot 1 \cdot S^J_{\bar{\lambda}, \bar{\rho}} \quad (5.11)$$

with $F_{\lambda}(K, J) \neq 1$, from which we conclude that $S^J_{\bar{\lambda}, \bar{\rho}} = 0$ for all $J \in S^\lambda \setminus U^\lambda$. It follows that for $Q_G(\rho) = 0$ the J-summations in the two expressions (5.9) and (5.10) actually extend over the same range; moreover, we then have $\hat{\psi}_\lambda(J) = \psi_\lambda(J)$ for all $J$ that appear in the sum, so the two expressions indeed are equal, i.e.

$$\bar{S}_{[J_{\bar{\lambda}}, \hat{\psi}_\lambda],[J_{\bar{\rho}}, \hat{\psi}_\rho]} = S_{[J_{\bar{\lambda}}, \hat{\psi}_\lambda],[J_{\bar{\rho}}, \hat{\psi}_\rho]} \quad (5.12)$$

for all $J \in G$ and all $\psi_\lambda \succ \hat{\psi}_\lambda$.

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9 The choice of canonical basis still leaves some residual freedom in the normalization, which remains to be clarified.
5.3 Properties of $\tilde{S}$

Let us now establish further properties of the matrix $\tilde{S}$ that we defined in (5.9). As a matter of fact, we first need to check that $\tilde{S}$ is well-defined, i.e. does not depend on the choice of representative of the $G$-orbit of the pair $(\bar{\rho}, \hat{\psi}_\rho)$. To do so, we need the explicit form of the equivalence relation, which reads $(\bar{\rho}, \hat{\psi}_\rho) \sim J'(\bar{\rho}, \hat{\psi}_\rho) = (J'\bar{\rho}, J\hat{\psi}_\rho)$ (see formulae (A.8) and (A.9)). We observe that

$$\lambda$$

satisfies the standard simple current relation, too, i.e. we have

$$(\tilde{S}_\lambda)_{\bar{\rho}, \hat{\psi}_\rho} = e^{2\pi i Q_{\bar{\rho}}(\lambda)} \tilde{S}_\lambda F_{\bar{\rho}}(J) S_{\lambda, \hat{\psi}_\rho}^J,$$

where we used the simple current property (A.12) of $S^J$ and the fact that $\tilde{\lambda}$ is in the untwisted sector. This tells us that the corresponding part of formula (5.9), and hence the whole matrix $\tilde{S}$, is indeed independent of the choice of representative.

We may define an analogous transformation as in this equivalence relation also when dealing with characters of full stabilizers, i.e. also for the row index of $\tilde{S}$, namely

$$J'(\bar{\lambda}, \hat{\psi}_\lambda) := (J\bar{\lambda}, J\hat{\psi}_\lambda)$$

for all $J' \in G$, with

$$J\hat{\psi}_\lambda(J) := F_{\lambda}(J, J^* \psi_\lambda(J)), \tag{5.15}$$

By the bi-homomorphism property of $F$, $J\hat{\psi}_\lambda$ is again a character of $S_\lambda$. It then follows that $\tilde{S}$ satisfies the standard simple current relation, too, i.e. we have

$$\tilde{S}_{J'(\bar{\lambda}, \hat{\psi}_\lambda), [\bar{\rho}, \hat{\psi}_\rho]} = e^{2\pi i Q_{J'}(\rho)} \tilde{S}_{(\bar{\lambda}, \hat{\psi}_\lambda), [\bar{\rho}, \hat{\psi}_\rho]} \tag{5.16}$$

for every $J' \in G$. This holds because in each term in the J-summation on the right hand side of (5.9) the factor of $F_{\lambda}(J', J)^*$ that comes from the action of $J'$ on $\psi_\lambda$ cancels against the factor $F_{\lambda}(J', J)$ that accompanies the phase $e^{2\pi i Q_{J'}(\rho)}$ in the simple current relation for $S^J$.

Next we note that the matrix $\tilde{S}$ is (in general) not symmetric; in fact it does not even make sense to talk about invertibility and unitarity. A direct calculation shows that $\tilde{S}$ is invertible, the inverse being given by

$$(\tilde{S}^{-1})_{[\bar{\rho}, \hat{\psi}_\rho], (\bar{\lambda}, \hat{\psi}_\lambda)} = \frac{|G|}{|S_{\lambda, \hat{\psi}_\lambda}|} \tilde{S}_{(\bar{\lambda}, \hat{\psi}_\lambda), [\bar{\rho}, \hat{\psi}_\rho]}^* \tag{5.17}$$

That (5.17) is a right-inverse is seen by

$$\sum_{[\bar{\rho}]} \sum_{\hat{\psi}_\rho \in U_{\mu}^J} \tilde{S}_{(\bar{\lambda}, \hat{\psi}_\lambda), [\bar{\rho}, \hat{\psi}_\rho]} (\tilde{S}_{(\bar{\mu}, \hat{\psi}_\mu), [\bar{\rho}, \hat{\psi}_\mu]}^*)^* = \frac{|G|^2}{|S_{\lambda, \hat{\psi}_\lambda}|} \sum_{[\bar{\rho}]} \sum_{J \in S_\lambda \cap S_{\mu} \cap U_{\mu}} \frac{1}{s_{\bar{\rho}}} \psi_\lambda(J) \psi_\mu(J)^* S^J_{\lambda, \hat{\psi}_\rho} (S^J_{\bar{\mu}, \hat{\psi}_\rho})^*$$

$$= \frac{|G|}{|S_{\lambda, \hat{\psi}_\lambda}|} \sum_{J \in S_\lambda \cap S_{\mu}} \psi_\lambda(J) \psi_\mu(J)^* \sum_{\bar{\rho}} S^J_{\lambda, \hat{\psi}_\rho} (S^J_{\bar{\mu}, \hat{\psi}_\rho})^*$$

$$= \frac{|G|}{s_{\lambda, \hat{\psi}_\lambda}} \delta_{\lambda, \bar{\mu}} \sum_{J \in S_\lambda} \psi_\lambda(J) \psi_\mu(J)^* = \frac{|G|}{s_{\lambda, \hat{\psi}_\lambda}} \delta_{(\bar{\lambda}, \hat{\psi}_\lambda), (\bar{\mu}, \hat{\psi}_\mu)}.$$ 

(5.18)
Here in the first step we inserted the definition (5.19) and performed the \( \hat{\psi}_\rho \)-summation, while in the second step we replaced \( \sum_\rho \) by \( \sum_\rho (s_\rho/|G|) \), which is possible owing to the fact that \( Q_J(\lambda) = 0 = Q_J(\mu) \) and that \( J \in \mathcal{U}_\rho \), and furthermore we dropped taking the intersection with \( \mathcal{U}_\rho \) in the \( J \)-summation. To see that this change in the summation range is allowed, let first \( J \not\in \mathcal{S}_\rho \); then according to (A.11) we simply have \( S^J_{\lambda,\bar{\rho}} = 0 = S^J_{\mu,\bar{\rho}} \). Otherwise, i.e. when \( J \in \mathcal{S}_\rho \setminus \mathcal{U}_\rho \), there must exist a \( J' \in \mathcal{S}_\rho \) with \( F_\rho(J', J) \neq 1 \), and we have

\[
S^J_{\lambda,\bar{\rho}} = \sum_{(J', J)} F_\rho(J', J)^* e^{2\pi i Q_{J'}(\lambda)} S^J_{\lambda,\bar{\rho}} = \sum_{(J', J)} F_\rho(J', J)^* S^J_{\lambda,\bar{\rho}},
\]

where again we use that \( Q(\lambda) = 0 \); thus in this case we have \( S^J_{\lambda,\bar{\rho}} = 0 \) as well.

To check that the matrix (5.17) is also a left-inverse of \( \tilde{S} \), we start by calculating

\[
\sum_\lambda \sum_{Q_{\mathcal{G}}(\lambda)=0} \tilde{S}_{\lambda,\bar{\rho}} \cdot \tilde{S}_{\lambda,\bar{\rho}}^* = \frac{|G|}{|\bar{\rho}|} \sum_\lambda \sum_{J \in \mathcal{U}_\rho \cap \mathcal{S}_\lambda} \hat{\psi}_\rho(J)^* \hat{\psi}_\rho(J) S^J_{\lambda,\bar{\rho}} (S^J_{\lambda,\bar{\rho}})^* = \frac{|G|}{|\bar{\rho}|} \sum_\lambda \sum_{J \in \mathcal{U}_\rho \cap \mathcal{S}_\lambda} \hat{\psi}_\rho(J)^* \hat{\psi}_\rho(J) S^J_{\lambda,\bar{\rho}} (S^J_{\lambda,\bar{\rho}})^*.
\]

(5.20)

Here after first performing the \( \psi_\lambda \)-summation, we dropped taking the intersection with \( \mathcal{S}_\lambda \) in the \( J \)-summation, which is allowed for the same reason as above. Next we extend the \( \lambda \)-summation to all sectors by inserting a projector and use the unitarity of \( S^J \) (see formula (A.34)), so as to obtain

\[
\sum_\lambda \sum_{Q_{\mathcal{G}}(\lambda)=0} \tilde{S}_{\lambda,\bar{\rho}} \cdot \tilde{S}_{\lambda,\bar{\rho}}^* = \sum_{J' \in \mathcal{G}} \frac{\delta_{\bar{\rho}, J' \bar{\sigma}}}{|\bar{\rho}|} \sum_{J \in \mathcal{U}_\rho} \hat{\psi}_\rho(J)^* F_\sigma(J', J)^* \hat{\psi}_\sigma(J) = \frac{1}{s_\rho} \sum_{J' \in \mathcal{G}} \delta_{\bar{\rho}, J' \bar{\sigma}} \hat{\psi}_\rho(J') \hat{\psi}_\sigma(J') = \frac{1}{s_\rho} \sum_{J' \in \mathcal{G}} \delta_{\bar{\rho}, J' \bar{\sigma}} \hat{\psi}_\rho(J') \hat{\psi}_\sigma(J') = \sum_{J' \in \mathcal{G}} \delta_{\bar{\rho}, J' \bar{\sigma}} = |\mathcal{U}_\rho|.
\]

(5.21)

(Here we also used the fact that \( \bar{\rho} = J' \bar{\sigma} \) already implies \( \mathcal{U}_\rho = \mathcal{U}_\sigma \) and \( \mathcal{S}_\rho = \mathcal{S}_\sigma \).)

The fact that \( \tilde{S} \) has a two-sided inverse means in particular that \( \tilde{S} \) is a square matrix. This implies the sum rule

\[
|\mathcal{S}_\lambda| = \sum_{|\bar{\rho}|} |\mathcal{U}_\rho|.
\]

(5.22)

In words: the number of primary fields in the untwisted (i.e., charge zero) sector, counted with their (full) stabilizer, is the same as the number of orbits in all sectors, counted with their untwisted stabilizer.

It is also worth pointing out that \( \tilde{S} \) is (in general) not unitary. Of course, we could redefine the matrix \( \tilde{S} \) so as to make it unitary; however, this would spoil some other nice properties of \( \tilde{S} \) and hence we refrain from doing so.

10 Compare also the remarks before eq. (C.2) in [12].

11 Note, however, that in the present case we would not be allowed to drop the untwisted stabilizer \( \mathcal{U}_\lambda \) if it were present, because for currents in \( \mathcal{S}_\lambda \setminus \mathcal{U}_\lambda \) the above reasoning would not go through: in (5.13) one would now have a factor of \( e^{2\pi i Q_{J'}(\rho)} \), which unlike \( e^{2\pi i Q_{J}(\lambda)} \) is not necessarily equal to one, since \( \bar{\rho} \) can be in any twist sector.
5.4 Conjugation

Since the row and column labels of $\tilde{S}$ are on different footings, there are two distinct matrices which are candidates for conjugations, namely

\[ C^B := \tilde{S} \tilde{S}^t \]  \hspace{1cm} (5.23)

and

\[ C^B := \tilde{S}^t U \tilde{S} ; \]  \hspace{1cm} (5.24)

here the superscripts $B$ and $B$ indicate that the entries of these two matrices are labelled by boundary blocks and boundary states (i.e. boundary conditions), respectively. The presence of the diagonal matrix $U$, defined as

\[ U_{(\lambda,\psi)\,(\bar{\mu},\psi_\lambda)} := \frac{u_{\lambda}}{|G|} \delta(\bar{\lambda},\psi_\lambda), \quad (5.25) \]

in (5.24) accounts for the natural weight of the boundary blocks, cf. for instance formula (5.17).

Both matrices are manifestly symmetric. To establish further properties, we write them in the form

\[ C^B_{(\lambda,\psi)\,(\bar{\mu},\psi_\lambda)} = \frac{|G|}{\sqrt{s_{\lambda} u_{\mu} u_{\bar{\mu}}}} \sum_{J \in S_{\lambda} \cap S_{\mu}} \psi_\lambda(J) \psi_{\bar{\mu}}(J^{-1}) S^J_{\lambda,\bar{\mu}} S^{J^{-1}}_{\lambda,\bar{\mu}} \]

\[ = \frac{|G|}{\sqrt{s_{\lambda} u_{\mu} u_{\bar{\mu}}}} \sum_{J \in S_{\lambda} \cap S_{\mu}} \psi_\lambda(J) \psi_{\bar{\mu}}(J^{-1}) \delta_{\lambda,\bar{\mu}} \eta^{-1}_J = \frac{|G|}{u_{\lambda}} C^B_{(\lambda,\psi)\,(\bar{\mu},\psi)} \delta_{\lambda,\bar{\mu}} \delta_J^{\lambda,\bar{\mu} +} ; \]  \hspace{1cm} (5.26)

\[ C^B_{(\bar{\rho},\hat{\psi}_{\rho})\,(\bar{\sigma},\hat{\psi}_{\sigma})} = \frac{|G|}{\sqrt{s_{\rho} u_{\sigma} u_{\bar{\sigma}}}} \sum_{\lambda} \sum_{J \in U_{\rho} \cap U_{\sigma}} \hat{\psi}_{\rho}(J^{-1}) \hat{\psi}_{\sigma}(J) \delta_{\lambda,\bar{\rho}} \eta^{-1}_J \]

\[ = \frac{1}{\sqrt{s_{\rho} u_{\sigma} u_{\bar{\sigma}}}} \sum_{\lambda} \sum_{J \in U_{\rho} \cap U_{\sigma}} \hat{\psi}_{\rho}(J) \hat{\psi}_{\sigma}(J) \delta_{\lambda,\bar{\rho}} \delta_{\lambda,\bar{\sigma}} \]

\[ = \frac{1}{s_{\rho}} \sum_{K \in \mathcal{G}} C^B_{(\rho,\hat{\psi}_{\rho})\,(\sigma,\hat{\psi}_{\sigma})} \delta_{\rho,\sigma} \delta_{\lambda,\bar{\rho}} \delta_{\lambda,\bar{\sigma}} \]

Here we used the identities (A.13) and (A.14), and introduced

\[ C^B_{\psi,\psi'} := \frac{1}{s_{\lambda}} \sum_{J \in S_{\lambda}} \psi(J) \eta^{-1}_J \psi'(J) \]  \hspace{1cm} (5.27)

as well as

\[ C^B_{\psi,\psi'} := \frac{1}{u_{\rho}} \sum_{J \in U_{\rho}} \hat{\psi}(J) \eta^{-1}_J \hat{\psi}'(J) \]  \hspace{1cm} (5.28)

for any two characters $\psi, \psi' \in S_{\lambda}^*$, respectively $\hat{\psi}, \hat{\psi}' \in U_{\rho}^*$.\hspace{1cm}12 Compare formula (C.3) of [12].
To proceed, we need several properties of the matrices $C^B(\bar{\lambda})$ and $C^B(\bar{\rho})$. First, with the help of the identity \( \text{(A.20)} \) we have
\[
(C^B(\bar{\lambda})^*) = \frac{1}{s_\lambda} \sum_{J \in S_\lambda} \psi(J)^* (\eta^j_\lambda)^* \psi'(J) = \frac{1}{s_\lambda} \sum_{J \in S_\lambda} \psi(J^{-1}) \eta^j_\lambda^{-1} \psi'(J^{-1})^* = C^B(\lambda),
\]
i.e. $C^B(\bar{\lambda})$ is real. Second, combining this reality property with the identity \( \text{(A.21)} \) we see that
\[
C^B(\bar{\lambda}^+) = \frac{1}{s_\lambda} \sum_{J \in S_\lambda} \psi(J) \eta^j_\lambda \psi'(J)^* = \frac{1}{s_\lambda} \sum_{J \in S_\lambda} \psi'(J)^* (\eta^j_\lambda)^* \psi(J) = (C^B(\bar{\lambda}))^* = C^B(\lambda),
\]
i.e.
\[
C^B(\bar{\lambda}^+) = (C^B(\lambda))^t.
\]
(5.31)

Analogous computations yield
\[
(C^B(\bar{\rho}))^* = C^B(\bar{\rho}) \quad \text{and} \quad C^B(\bar{\rho}^+) = (C^B(\bar{\rho}))^t.
\]
(5.32)

Finally, implementing the identity \( \text{(A.22)} \), i.e. the fact that $\eta^j_\lambda$ is a character of $U_\rho$, we have
\[
\sum_{\psi' \in U^*_\rho} C^B(\bar{\rho})_{\psi,\psi'} \psi'(J') = \frac{1}{s_\rho} \sum_{J \in U_\rho} \hat{\psi}(J) \eta^j_\rho \sum_{\dot{\psi}' \in U^*_\rho} \hat{\psi}'(J)^* \hat{\psi}'(J') = \hat{\psi}(J') \eta^j_\rho = \hat{\psi}^+(J')
\]
(5.33)

with the character $\hat{\psi}^+ \in U^*_\rho$ (not to be mixed up with the complex conjugate character $\hat{\psi}^* \in U^*_\rho$) as defined by \( \text{(A.23)} \), which means that the map $C^B(\bar{\rho})$ on the boundary conditions is a permutation. We can thus write
\[
C^B(\bar{\rho})_{\psi,\psi'} = \delta_{\psi',\hat{\psi}^+} = \delta_{\psi',\pi\rho(\hat{\psi})}
\]
(5.34)

with some permutation $\pi\rho$. Because of \( \text{(5.32)} \) we also have
\[
\pi\rho^+ = (\pi\rho)^{-1}.
\]
(5.35)

Having obtained these properties of $C^B(\bar{\rho})$, we can finally conclude that $C^B$ is a \textit{conjugation}, i.e. it is symmetric and each column and row contains just a single non-zero element:
\[
C^B_{\psi,\psi'} = \frac{1}{s_\rho} \sum_{K \in G} \delta_{\hat{\psi},(K\sigma)^+} C^B_{\hat{\psi},\hat{\psi}} = \delta_{\hat{\psi},\hat{\psi}} = \delta_{\psi,(K\sigma)^+} = \delta_{\psi,\pi\rho(\hat{\psi})^+}
\]
(5.36)

with $[\sigma, \hat{\psi}]^+ = [\sigma^+, \hat{\psi}^+]$ (and $\hat{\psi}^+(J) \equiv \hat{\psi}(J) \eta^j_\sigma$). In particular $C^B$ is an involution, i.e. we have
\[
(C^B)^2 = 1.
\]
(5.37)

(The crucial property of $\pi\rho$ entering here is \( \text{(5.33)} \); that relation is not tied to the order of $\pi\rho$, so that in particular $\pi\rho$ need not have order two itself.)

For the map $C^B$ on boundary blocks, the conclusions are somewhat different. We still have
\[
\sum_{\psi' \in S_\lambda}^\pi C^B_{\psi,\psi'} \psi'(J') = \psi(J') \eta^j_\lambda
\]
(5.38)

29
similarly to (5.33); but since $\eta_{\lambda}$ is only a character of $U_{\lambda}$, but not necessarily of $S_{\lambda}$, the expression on the right hand side, seen as a function on $S_{\lambda}$, is not a character any more. Therefore $C^{B}$ in general no longer constitutes a permutation. (Of course, when $U_{\lambda}$ coincides with $S_{\lambda}$, it still does. In this case the arguments are completely parallel to those above, leading to the conclusion that $UC^{B}$ is a conjugation as well.) As a consequence, the matrix $C^{B}$ is not, in general, a (weighted) conjugation. Nevertheless we can again conclude that $C^{B}$ is (weighted) involutive. Namely, we find

$$
\sum_{\psi'' \in S_{\lambda}^{*}} \left( \sum_{\psi'' \in S_{\lambda}^{*}} C^{B}(\bar{\lambda}) C^{B}(\bar{\lambda}) \right) \psi''(J) = \frac{1}{u_{\lambda}} \sum_{\psi'' \in S_{\lambda}^{*}} \eta_{\lambda}^{J'} \eta_{\lambda}^{J''-1} \sum_{\psi'' \in S_{\lambda}^{*}} \psi(J') \psi''(J')^{*} \psi''(J)
$$

(5.39)

where in the last step we used the identity (A.23), and hence

$$
((C^{B})^{2})_{(\bar{\lambda},\psi_{\lambda}), (\bar{\rho},\psi_{\rho})} = \frac{|G|^{2}}{u_{\lambda}^{\psi}} \delta_{\bar{\lambda},\bar{\rho}} \sum_{\psi_{\mu} \in S_{\mu}^{*}} C^{B}((\bar{\lambda}) \psi_{\lambda}, \psi_{\mu}) C^{B}((\bar{\lambda}) \psi_{\mu}, \psi_{\rho}) = \frac{|G|^{2}}{u_{\lambda}^{\psi}} \delta_{\bar{\lambda},\bar{\rho}} \delta_{\psi_{\lambda}, \psi_{\rho}} = \frac{|G|^{2}}{u_{\lambda}^{\psi}} \delta_{(\bar{\lambda},\psi_{\lambda}), (\bar{\rho},\psi_{\rho})}. \quad (5.40)
$$

We can also deduce that the inverse of $C^{B}$ is given in terms of the inverse of $\tilde{S}$ as

$$
(C^{B})^{-1} = (\tilde{S}^{-1})^{\dagger} \tilde{S}^{-1}. \quad (5.41)
$$

Let us point out that the existence of a conjugation $C^{B}$ on the boundary conditions does not come as a big surprise. Indeed, it precisely implements what one heuristically expects as the result of changing the orientation of the boundary. The latter manipulation is required e.g. when one wants to glue surfaces along boundaries. In contrast, for the boundary blocks such a manipulation would not make any sense; accordingly it is not too surprising that in the most general case a genuine conjugation on the boundary blocks does not exist.

### 5.5 Structure constants

According to our general expectations, the structure constants of the classifying algebra are to be defined through a Verlinde-like formula via the matrix $\tilde{S}$. We first introduce a corresponding quantity

$$
\tilde{N}_{(\bar{\lambda}_{1},\psi_{\lambda_{1}}), (\bar{\lambda}_{2},\psi_{\lambda_{2}}), (\bar{\lambda}_{3},\psi_{\lambda_{3}})} := \frac{\tilde{S}_{(\bar{\lambda}_{1},\psi_{\lambda_{1}}), (\bar{\rho},\psi_{\rho})} \tilde{S}_{(\bar{\lambda}_{2},\psi_{\lambda_{2}}), (\bar{\rho},\psi_{\rho})} \tilde{S}_{(\bar{\lambda}_{3},\psi_{\lambda_{3}}), (\bar{\rho},\psi_{\rho})}}{\tilde{S}_{\Omega, (\bar{\rho},\psi_{\rho})}} \quad (5.42)
$$

with only lower indices and then raise the third index with the inverse (5.41) of $C^{B}$, so as to arrive at the expression

$$
\tilde{N}_{(\bar{\lambda}_{1},\psi_{\lambda_{1}}), (\bar{\lambda}_{2},\psi_{\lambda_{2}}), (\bar{\lambda}_{3},\psi_{\lambda_{3}})} = \sum_{[\bar{\rho}, \psi_{\rho} \in U_{\rho}^{*}} \sum_{[\bar{\rho}, \psi_{\rho} \in U_{\rho}^{*}} \frac{\tilde{S}_{(\bar{\lambda}_{1},\psi_{\lambda_{1}}), (\bar{\rho},\psi_{\rho})} \tilde{S}_{(\bar{\lambda}_{2},\psi_{\lambda_{2}}), (\bar{\rho},\psi_{\rho})} \tilde{S}_{(\bar{\lambda}_{3},\psi_{\lambda_{3}}), (\bar{\rho},\psi_{\rho})}}{\tilde{S}_{\Omega, (\bar{\rho},\psi_{\rho})}} \quad (5.43)
$$
for the structure constants. More explicitly, we find

\[
\tilde{N}_{(\lambda_1, \psi_1), (\lambda_2, \psi_2)}^{(\lambda_3, \psi_3)} = \frac{|G|}{\sqrt{S_{11} S_{22} S_{33}}} \sum_{\substack{j_1, j_2, j_3 = 1, 2, 3 \atop j_3 = j_1, j_2}} \psi_1^* (J_1) \psi_2 (J_2) \psi_3 (J_3) \bar{\Omega}^{-1} S_{\lambda_1, \rho}^1 S_{\lambda_2, \rho}^2 (S_{\lambda_3, \rho}^3)^* / \bar{\Omega}_{\lambda, \rho}.
\]

(5.44)

\[\tilde{N}_{(\lambda_1, \psi_1), (\lambda_2, \psi_2)}^{(\lambda_3, \psi_3)} = \frac{|G|}{\sqrt{S_{11} S_{22} S_{33}}} \sum_{\substack{j_1, j_2, j_3 = 1, 2, 3 \atop j_3 = j_1, j_2}} \psi_1^* (J_1) \psi_2^* (J_2) \psi_3 (J_3) \bar{\Omega}^{-1} S_{\lambda_1, \rho}^1 S_{\lambda_2, \rho}^2 (S_{\lambda_3, \rho}^3)^* / \bar{\Omega}_{\lambda, \rho}.
\]

5.6 Semisimplicity and irreducible representations

The following properties of the structure constants and of the classifying algebra \( C(\tilde{A}) \) now follow directly:

- The structure constants (5.43) are manifestly symmetric in the first two indices. Thus \( C(\tilde{A}) \) is commutative.
- We have

\[
(\tilde{\Phi}_{(\lambda_1, \psi_1)} \circ \tilde{\Phi}_{(\lambda_2, \psi_2)}) \circ \tilde{\Phi}_{(\lambda_3, \psi_3)} = \sum_{\rho} \sum_{\psi \in \mathcal{H}_{\tilde{A}^*}} \sum_{\lambda_4 \in \mathcal{L}^*} \sum_{\rho \in \mathcal{L}^*} \tilde{S}_{(\lambda_1, \psi_1), (\rho, \psi_{[\rho, \psi]})} \tilde{S}_{(\lambda_2, \psi_2), (\rho, \psi_{[\rho, \psi]})} \tilde{S}_{(\lambda_3, \psi_3), (\rho, \psi_{[\rho, \psi]})} (\tilde{S}^{-1})_{\tilde{A}^*} (\lambda_4, \psi_{[\rho, \psi]}) \tilde{S}_{(\lambda_4, \psi_4), (\rho, \psi_{[\rho, \psi]})}.
\]

(5.45)

This is totally symmetric in the three labels \((\lambda_i, \psi_i)\) for \(i = 1, 2, 3\). It follows that \( C(\tilde{A}) \) is associative.
- It is also immediately verified that \( C(\tilde{A}) \) is unital. The unit element is \( \tilde{1} = \Omega \).
- By construction, the matrix \( \tilde{S} \) simultaneously diagonalizes all matrices \( \tilde{N}_{(\lambda_1, \psi_1), (\lambda_2, \psi_2)} \). Explicitly,

\[
\sum_{\lambda_2, \lambda_3, \psi_2, \lambda_3, \psi_3} (\tilde{S}^{-1})_{\tilde{A}^*} (\lambda_2, \psi_2) \tilde{N}_{(\lambda_1, \psi_1), (\lambda_2, \psi_2)} \tilde{S}_{(\lambda_3, \psi_3), (\rho, \psi_{[\rho, \psi]})} = \sum_{\lambda_2, \lambda_3, \psi_2, \lambda_3, \psi_3} \sum_{\lambda_4, \psi_4} (\tilde{S}^{-1})_{\tilde{A}^*} (\lambda_2, \psi_2) \tilde{S}_{(\lambda_3, \psi_3), (\rho, \psi_{[\rho, \psi]})} (\tilde{S}^{-1})_{\tilde{A}^*} (\lambda_4, \psi_4) \tilde{S}_{(\lambda_4, \psi_4), (\rho, \psi_{[\rho, \psi]})}.
\]

(5.46)

Thus the regular representation of \( C(\tilde{A}) \) is fully reducible.
- Together these properties imply in particular that the associative algebra \( C(\tilde{A}) \) is semisimple.
- The matrix \( C^B \) can be expressed through the structure constants as

\[
C^B_{(\lambda_1, \psi_{[\rho, \psi]})} = \tilde{N}_{(\lambda_1, \psi_{[\rho, \psi]})} \tilde{S}_{(\lambda_2, \psi_{[\rho, \psi]})} \tilde{S}_{(\lambda_3, \psi_{[\rho, \psi]})} = |G| \tilde{N}_{(\lambda_1, \psi_{[\rho, \psi]})} \tilde{S}_{(\lambda_2, \psi_{[\rho, \psi]})} \tilde{S}_{(\lambda_3, \psi_{[\rho, \psi]})}.
\]

(5.47)

Note, however, that generically this is not a conjugation. (Recall that while the matrix \( C^B \) provides a conjugation on the boundary conditions, \( C^B \) is in general only an involution, but not a conjugation on the boundary blocks; for a given pair \((\tilde{\lambda}_1, \psi_1)\) the matrix element \( C^B_{(\tilde{\lambda}_1, \psi_{[\rho, \psi]})} \) can be non-vanishing for several pairs \((\tilde{\lambda}_2, \psi_{[\rho, \psi]})\).)
The calculation in (5.48) implies that each equivalence class $[\rho, \hat{\psi}_\rho]$ furnishes a one-dimensional irreducible representation $R_{(\rho, \hat{\psi}_\rho)}$ of $\mathcal{C}(\mathfrak{A})$. According to the result (5.8) these irreducible representations are, in turn, precisely the reflection coefficients. We thus have

$$R_{(\lambda, \varphi), \Omega}^{[\rho, \hat{\psi}_\rho]} = R_{[\rho, \hat{\psi}_\rho]}(\Phi_{(\lambda, \varphi)}) = \frac{\tilde{S}_{(\lambda, \varphi), [\rho, \hat{\psi}_\rho]}}{S_{\Omega, [\rho, \hat{\psi}_\rho]}}.$$  

(5.48)

Moreover, due to the sum rule (5.22) and the result that the algebra is semisimple we can conclude that in fact the reflection coefficients (5.48) even provide all inequivalent irreducible representations and that these are all one-dimensional.

### 5.7 Relation with chiral blocks

Our next aim is to write the structure constants of the classifying algebra in terms of quantities related to chiral blocks. In connection with blocks, the natural quantities are the structure constants with only lower indices. We find that

$$\tilde{N}_{(\lambda_1, \psi_1), (\lambda_2, \psi_2), (\lambda_3, \psi_3)} = d_1 d_2 d_3 |\mathcal{G}| \sum_{(J_1, J_2, J_3) \in S_1 \times S_2 \times S_3} \prod_{i=1}^{3} \psi_i(J_i) s_i \rho \sum_{\rho} \tilde{S}_{(\lambda_1, \rho), (\lambda_2, \rho), (\lambda_3, \rho)}.$$  

(5.49)

where $d_1 \equiv d_\lambda$ etc., and where in the second step we introduced projected fusion coefficients $\tilde{N}$; also, $S_1 \cdot S_2 \cdot S_3$ is by definition the subgroup of $\mathcal{G}$ that is generated by the three subgroups $S_i$.

To rewrite the coefficients $\tilde{N}$ in a more convenient form, we consider the group homomorphism $p$: $S_1 \times S_2 \times S_3 \rightarrow \mathcal{G}$ that is defined by taking the product in $\mathcal{G}$ of three elements,

$$p: \quad S_1 \times S_2 \times S_3 \ni (s_1, s_2, s_3) \mapsto p(s_1, s_2, s_3) := s_1 s_2 s_3.$$  

(5.50)

By the homomorphism theorem, the number of elements in the kernel of $p$ is

$$|\ker(p)| = s_1 s_2 s_3 / |S_1 \cdot S_2 \cdot S_3|.$$  

(5.51)

Thus we can replace the product over the $s_i$ by the product of $|\ker(p)|$ and the number of elements in the group generated by all $S_i$, so that

$$\tilde{N}_{(\lambda_1, \psi_1), (\lambda_2, \psi_2), (\lambda_3, \psi_3)} = \frac{1}{|\ker(p)|} \sum_{(J_1, J_2, J_3) \in S_1 \times S_2 \times S_3} \prod_{i=1}^{3} \psi_i(J_i) \sum_{\rho} \tilde{S}_{(\lambda_1, \rho), (\lambda_2, \rho), (\lambda_3, \rho)}.$$  

(5.52)

We remark that $|\ker(p)|$ is precisely the number of elements of the group over which we have to perform a Fourier transformation in order to attain a projection on the chiral blocks. Accordingly $|\ker(p)|$ indeed needs to be absorbed into the definition of the coefficients $\tilde{N}_{(\lambda_1, \psi_1), (\lambda_2, \psi_2), (\lambda_3, \psi_3)}$.

These relations suggest that the structure constants of the classifying algebra should be related to an appropriate action of the simple current group $\mathcal{G}$ on the space of chiral blocks. This feature is familiar from the situation studied in [4]. However, owing to the fact that the action of $\mathcal{G}$ is only projective, here its precise realization is more involved and remains to be uncovered.
6 Annulus coefficients

6.1 The $G'$-extension

In this section we compute the annulus amplitude $A_{\rho_1, \rho_2}$ for two arbitrary boundary conditions

$$\rho_i \equiv [\bar{\rho}_i, \hat{\psi}_i]$$

($i = 1, 2$) and study its properties. One way to obtain the annulus amplitude is to evaluate it in the closed string channel, where it can be regarded as factorizing into two disk one-point functions and a sphere two-point function, so that it corresponds to propagation between the boundary states $\langle B_{[\bar{\rho}_2, \hat{\psi}_2]} | B_{[\bar{\rho}_1, \hat{\psi}_1]} \rangle$ according to

$$A_{[\bar{\rho}_1, \hat{\psi}_1]} [\bar{\rho}_2, \hat{\psi}_2] (t) = \langle B_{[\bar{\rho}_2, \hat{\psi}_2]} | e^{-(2\pi t/(L_0 + 1 - \epsilon/12)} | B_{[\bar{\rho}_1, \hat{\psi}_1]} \rangle.$$  

(6.2)

Here by a boundary state [2, 33, 34, 35, 36, 37, 5, 6] one means a linear form $B_{[\bar{\rho}, \hat{\psi}]}$ on the space $\bigoplus \bar{\mu} \bar{\mathcal{H}}_{\bar{\mu}} \otimes \bar{\mathcal{H}}^{-}_{\bar{\mu}}$ of all closed string states (which correspond to bulk fields) that is characterized by the property that when applied to an element $v \otimes \tilde{v} \in \mathcal{V}_{\bar{\lambda}} \otimes \mathcal{H}_{(\bar{\lambda}, \hat{\phi})} \otimes \mathcal{V}_{\bar{\lambda}^+} \otimes \mathcal{H}_{(\bar{\lambda}^+, \hat{\phi}^+)}$ it yields the one-point correlator of the corresponding bulk field on the disk, i.e.

$$B_{[\bar{\rho}, \hat{\psi}]} (v \otimes \tilde{v}) = \langle \phi_{(\bar{\lambda}, \hat{\phi})}, (\bar{\lambda}^+, \hat{\phi}^+)(v \otimes \tilde{v}; z=0) | [\bar{\rho}, \hat{\psi}] \rangle.$$  

(6.3)

It follows that $B_{[\bar{\rho}, \hat{\psi}]}$ can be written as the specific linear combination

$$B_{[\bar{\rho}, \hat{\psi}]} = \bigoplus_{Q \in \mathcal{S}_\lambda} \bigoplus_{\bar{\phi} \in \mathcal{S}_\bar{\lambda}} R_{[\bar{\rho}, \hat{\psi}]} [\bar{\phi}, \tilde{\phi}] \langle \Psi_{[\bar{\rho}, \hat{\psi}]} [\bar{\phi}, \tilde{\phi}] | \bar{B}_{(\bar{\lambda}, \hat{\phi})} \rangle$$  

(6.4)

of boundary blocks $\bar{B}_{(\bar{\lambda}, \hat{\phi})}$.

To each boundary condition $\rho_i$ we associate the character

$$g_i \equiv g^{[Q]}_{\rho_i} \in G^* = (G^*)^* \cong G$$

(6.5)

of $G$ that maps every simple current to the value of the corresponding monodromy charge [1,2], i.e.

$$g_i (J) := \exp(2\pi i Q_{\lambda}(\rho_i))$$

(6.6)

for all $J \in \mathcal{G}$. As already mentioned in the introduction, this quantity can be regarded as an element of the orbifold group $G$, and indeed it coincides with the so-called automorphism type of the boundary condition (for details, see [19]). Inspection shows that in the amplitude (6.2) one deals with linear combinations of characters of that orbifold theory in which the cyclic subgroup $\langle g_1^{-1} g_2 \rangle$ of $G$ that is generated by $g_1^{-1} g_2 \in G$ is broken. (These combinations are known as twining characters [11,12] of the $\bar{A}$-theory.) This orbifold theory can equivalently be described as an integer spin simple current extension of the $\bar{A}$-theory by a subgroup of $G$ which is a proper subgroup when $g_1 \neq g_2$.

In more precise terms the situation is described as follows. The exact sequence

$$0 \to \langle g_1^{-1} g_2 \rangle \to G \to G/\langle g_1^{-1} g_2 \rangle \to 0$$  

(6.7)
of finite abelian groups implies the exact sequence

\[ 0 \to (\mathcal{G}/(g_1^{-1}g_2))^* \to \mathcal{G} \to (g_1^{-1}g_2)^* \to 0 \]  

(6.8)
of their character groups. Therefore we can extend the \(\hat{\mathfrak{A}}\)-theory by \((\mathcal{G}/(g_1^{-1}g_2))^*\). This is the subgroup of those characters of \(G\) which descend to characters of the quotient, and these are precisely those which are the identity on \(\langle g_1^{-1}g_2 \rangle\), i.e. those simple currents \(J\) which obey \(g_1^{-1}g_2(J) = 1\), which in turn is the same as \(Q_1(\rho_1) = Q_1(\rho_2)\).

Accordingly, one expects that the annulus amplitude can be expressed as a linear combination of characters of the extension of the \(\hat{\mathfrak{A}}\)-theory by the subgroup

\[ \mathcal{G}^o \equiv \mathcal{G}_{\rho_1,\rho_2}^o := \{ J \in \mathcal{G} | Q_1(\rho_1) = Q_1(\rho_2) \} \]  

(6.9)
of \(\mathcal{G}\). As we will see, this is indeed possible; still, as it turns out, this is not the most natural choice for the following reason. Ultimately, our goal is to express the annulus amplitude as a linear combination

\[ A_{\rho_1,\rho_2} = \sum_{\sigma} A_{\rho_1,\rho_2}^\sigma \mathcal{X}_\sigma^{(K)} \]  

(6.10)
of irreducible characters \(\mathcal{X}_\sigma^{(K)}\) in some extension \(K\) of the \(\hat{\mathfrak{A}}\)-theory. Now the interpretation of the annulus amplitude as an open string partition function imposes the requirement that when we expand \(A_{\rho_1,\rho_2}(t)\) as a function of \(q = \exp(2\pi i(t/2))\), then the coefficients in this expansion are non-negative integers. While this does not necessarily imply that already all the numbers \(A_{\rho_1,\rho_2}^\sigma\) are integers, it has been observed in many situations \([4,38]\) that the multiplicities \(A_{\rho_1,\rho_2}^\sigma\) possess a natural interpretation as the rank of (a subsheaf of) a sheaf of chiral blocks. (This interpretation also enables one to establish the integrality property in full generality.) In order to relate \(A_{\rho_1,\rho_2}^\sigma\) to such a rank of a chiral block, we need to work with an extended theory in which both \(\rho_1\) and \(\rho_2\) are allowed fields, which is the case if their monodromy charges both vanish. The \(\mathcal{G}^o\)-extension does not meet this condition in general; rather, we need to consider the extension by the subgroup

\[ \mathcal{G}' \equiv \mathcal{G}_{\rho_1,\rho_2}' := \{ J \in \mathcal{G} | Q_1(\rho_1) = 0 = Q_1(\rho_2) \} \]  

(6.11)
of \(\mathcal{G}\), which is the largest subgroup of \(\mathcal{G}^o\) that has the desired property.

Thus we define the annulus coefficients \(A_{\rho_1,\rho_2}^\sigma\) as the multiplicities of characters in the extension of the \(\hat{\mathfrak{A}}\)-theory by \(\mathcal{G}'\); in more precise notation,

\[ A_{[\rho_1,\hat{\psi}_1],[\rho_2,\hat{\psi}_2]}(t) = \sum_{[\sigma]} A_{[\rho_1,\hat{\psi}_1],[\rho_2,\hat{\psi}_2]}^{[\sigma,\hat{\psi}_\sigma]} \mathcal{X}_{[\sigma,\hat{\psi}_\sigma]}' \left( \frac{it}{2} \right) , \]  

(6.12)
where \([\sigma,\hat{\psi}_\sigma]'\) is the \(\mathcal{G}'\)-orbit of \((\sigma,\hat{\psi}_\sigma)\). We will demonstrate that these quantities can be expressed as a sum of fusion rule coefficients in the \(\mathcal{G}'\)-extension, and check that various consistency requirements are satisfied. Our first task is to make sure that the characters of the \(\mathcal{G}'\)-extension indeed appear in the expression of the annulus partition function in the closed string channel. These characters read (compare formula (A.31))

\[ \mathcal{X}_{[\lambda,\hat{\psi}_\lambda]}' = \frac{1}{\sqrt{s_{\lambda}^I}} \sum_{J \in \mathcal{G}'} \tilde{X}_{(J\lambda,\hat{\psi}_\lambda)} \equiv \frac{1}{\sqrt{s_{\lambda}^I}} \left( \sum_{J \in \mathcal{G}'} \tilde{X}_{(J\lambda)} \right) \hat{\psi}_\lambda. \]  

(6.13)
Here \( s'_\lambda = |S'_\lambda| \) and \( u'_\lambda = |U'_\lambda| \) are the cardinalities of the full and untwisted stabilizer \( S'_\lambda \) and of
\[
U'_\lambda = \{ J \in S'_\lambda \mid F_\lambda(J, K) = 1 \text{ for all } K \in S'_\lambda \}, \tag{6.14}
\]
respectively, which are relevant in the \( G' \)-extension, and \( \hat{\psi}'_\lambda \) is a character of \( U'_\lambda \). Note that while one has
\[
S'_\lambda = S_\lambda \cap G', \tag{6.15}
\]
there is no simple relation between \( U'_\lambda \) and \( U_\lambda \). In particular, \( U'_\lambda \) differs, in general, from the intersection \( U_\lambda \cap G' \), though it always contains it as a subgroup:
\[
U_\lambda \cap G' \subseteq U'_\lambda; \tag{6.16}
\]
in fact, already in simple examples it happens that \( U'_\lambda \) is larger than \( U_\lambda \).

### 6.2 Expressions for the annulus coefficients

To proceed, we need more explicit expressions for the annulus coefficients. To this end we insert the result (4.40) for the regulated inner product (4.28) of the boundary blocks and the relation (5.3) between one-point correlators and boundary blocks into formula (6.2). We also substitute for the coefficients in the latter relation the explicit expressions (5.5) and (5.48) as well as the normalization
\[
\langle \Psi_{\bar{\rho}, \hat{\psi}_\rho} [\bar{\rho}, \hat{\psi}_\rho] | \bar{\rho}, \hat{\psi}_\rho \rangle = \tilde{S}_{\bar{\rho}, \hat{\psi}_\rho} \tag{6.17}
\]
for the vacuum boundary fields. This yields
\[
A_{[\bar{\rho}_1, \hat{\psi}_1]} [\bar{\rho}_2, \hat{\psi}_2] (t) = \sum_{\lambda, \psi_\lambda \in S'_\lambda} \sum_{\mu; \psi_\mu \in S'_\mu} (R_{\mu, \psi_\mu} [\bar{\rho}_2, \hat{\psi}_2])^* (\tilde{R}_{\mu, \psi_\mu} [\bar{\rho}_1, \hat{\psi}_1]) \langle \bar{\Omega}, \hat{\psi}_\rho \rangle \langle \bar{\Omega}, \hat{\psi}_\mu \rangle \tilde{S}^* (\lambda, \psi_\lambda, \bar{\rho}_1, \hat{\psi}_1) \tilde{S} (\lambda, \psi_\lambda, \bar{\rho}_2, \hat{\psi}_2) \frac{1}{(|G|/u_\lambda) S_{\lambda, \bar{\rho}} \tilde{X}(\lambda, \psi_\lambda) (2i/t)}. \tag{6.18}
\]

This result is in agreement with the requirement that in the closed channel only those fields \((\lambda, \psi_\lambda)\) are exchanged whose monodromy charges with respect to the currents in \( G' \) vanish; indeed, the summation even extends only over those fields for which all monodromy charges of currents in the larger group \( G \) are zero. Our next goal is to rewrite (6.18) entirely in terms of

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13 An example occurs for the case of the \( D_4 \) level 2 WZW theory. In this case for the fixed point with stabilizer \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) the untwisted stabilizer is trivial, but when the second boundary condition is taken to be in a twisted sector, then both \( S' \) and \( U' \) are equal to the corresponding \( \mathbb{Z}_2 \) under which the twisted sector is fixed.
\(G'\)-quantities; first we arrive at a sum over \(G'\)-orbits \([\lambda]'\):

\[
A_{[\rho_1,\psi_1],[\rho_2,\psi_2]}(t) = \sum_{Q'_{G'}(\lambda'\psi_2,\rho_1\psi_1)} \sum_{\psi_2 \in \bar{S}'_{\lambda'} \cdot \psi_2} \tilde{S}^*_{(\lambda,\psi_2),[\rho_1,\psi_1]} \tilde{S}_{(\lambda,\psi_2),[\rho_2,\psi_2]} \frac{1}{(|G'|/u_\lambda) S_{\lambda,\psi_2}(\frac{2i}{t})} \bar{X}_{(\lambda,\psi_2)}(\frac{2i}{t}).
\]

In (6.19) we are still dealing with the characters \(\psi_\lambda \in \bar{S}'_{\lambda}\) and \(\psi_i \in \bar{U}'_{\lambda}\). To express the amplitude through the correct quantities \(\tilde{\psi}_\lambda \in \bar{U}'_{\lambda}\) and \(\tilde{\psi}_i \in \bar{U}'_{\lambda}\), analogously to (4.9) we write

\[
\psi_\lambda \succ \psi'_\lambda
\]

when the restriction of the \(G\)-character \(\psi_\lambda\) to the subgroup \(G'\) of \(G\) is equal to the \(G'\)-character \(\psi'_\lambda\), and similarly when we deal with other embedded pairs of groups, e.g. stabilizers and untwisted stabilizers (however, for the \(\psi_i\) we will have to be careful because in general \(\bar{U}'_{\lambda}\) is not a subgroup of \(\bar{U}_\lambda\)). We then arrive at

\[
A_{[\rho_1,\psi_1],[\rho_2,\psi_2]}(t) = \sum_{Q'_{G'}(\lambda'\psi_2,\rho_1\psi_1)} \sum_{\psi_2 \in \bar{U}'_{\lambda}} \left( \sum_{\psi_\lambda \in \bar{S}'_{\lambda} \psi_\lambda \succ \tilde{\psi}_\lambda} \tilde{S}^*_{(\lambda,\psi_2),[\rho_1,\psi_1]} \tilde{S}_{(\lambda,\psi_2),[\rho_2,\psi_2]} \right) \frac{1}{(d_{\lambda'} |G'|/u_\lambda) S_{\lambda,\psi_2}(\frac{2i}{t})} \bar{X}'_{(\lambda,\psi_\lambda)}(\frac{2i}{t}).
\]

We are now in a position to perform a modular transformation that involves the S-matrix \(S'\) of the \(G'\)-extension; we get

\[
A_{[\rho_1,\psi_1],[\rho_2,\psi_2]}(t) = \sum_{Q'_{G'}(\lambda'\psi_2,\rho_1\psi_1)} \sum_{\psi_2 \in \bar{U}'_{\lambda}} \left( \sum_{\psi_\lambda \in \bar{S}'_{\lambda} \psi_\lambda \succ \tilde{\psi}_\lambda} \tilde{S}^*_{(\lambda,\psi_2),[\rho_1,\psi_1]} \tilde{S}_{(\lambda,\psi_2),[\rho_2,\psi_2]} \right) \frac{1}{(d_{\lambda'} |G'|/u_\lambda) S_{\lambda,\psi_2}(\frac{2i}{t})} \sum_{[\sigma']} \sum_{\tilde{\psi}_\sigma' \in \bar{U}'_{\lambda}} S'_{[\lambda,\psi_\lambda'][\sigma,\tilde{\psi}_\sigma'] \bar{X}'_{[\sigma,\tilde{\psi}_\sigma']}(\frac{2i}{t})},
\]

from which we finally can read off the annulus coefficients as the coefficients of the \(G'\)-characters \(X'_{[\sigma,\tilde{\psi}_\sigma']}.\) Thus we finally see that the annulus coefficients are given by

\[
A_{[\sigma,\tilde{\psi}_\sigma'],[\rho_1,\psi_1],[\rho_2,\psi_2]} = \sum_{Q'_{G'}(\lambda'\psi_2,\rho_1\psi_1)} \frac{|G'|}{|G'|} u_\lambda \sum_{\psi_2 \in \bar{U}'_{\lambda}} \sum_{\psi_\lambda \in \bar{S}'_{\lambda} \psi_\lambda \succ \tilde{\psi}_\lambda} \tilde{S}^*_{(\lambda,\psi_2),[\rho_1,\psi_1]} \tilde{S}_{(\lambda,\psi_2),[\rho_2,\psi_2]} S'_{[\lambda,\psi_\lambda'] [\sigma,\tilde{\psi}_\sigma'] / S'_{[\lambda,\psi_\lambda']} }, \Omega'.
\]

As a check of the normalization of the annulus coefficients, let us specialize to the case of boundary conditions that preserve the full bulk symmetry, in which case \(G' = G\) and the annulus

\[\text{In the product of the two } \tilde{S}\text{-elements, one is supposed to choose a representative of the orbit } [\lambda]'\;\text{; it does not matter which one, because the monodromy charges vanish. If one worked with the larger group } G^'\text{ instead of } G', \text{ one would again be able to show that only characters of the } G^'\text{-extension appear, as the monodromy charges of } \rho_1 \text{ and } \rho_2 \text{ are equal and hence cancel due to the complex conjugation. However, the two factors } \tilde{S} \text{ separately would depend on the choice of representative on the } G^'\text{-orbits, and for having a well-defined expression one would have to choose one and the same representative in both matrix elements.}\]
coefficients coincide with the structure constants of the fusion rule algebra. As seen after (6.11), in this case the matrix elements of \( \tilde{S} \) coincide with those of \( S' = S \) where one takes the orbit corresponding to \( \lambda \). In addition we then have \( \hat{\psi}' = \hat{\psi} \), so that the summation over \( \psi \succ \hat{\psi}' \) just amounts to a factor of \( s_{\lambda}/u_{\lambda} \), and we can use (compare (4.12))

\[
S'_{[\lambda,\hat{\psi}_\lambda]'\Omega} = \frac{|G|}{\sqrt{s_{\lambda}'u_{\lambda}^*}} \tilde{S}_{\lambda,\Omega}.
\]  (6.24)

Thus in the case \( G' = G \) the result (6.22) reduces to

\[
A^{[\rho,\hat{\psi}_{\rho}]_{[\rho_1,\hat{\psi}_{\rho_1}],[\rho_2,\hat{\psi}_{\rho_2}]} = \sum_{[\lambda]} \sum_{\psi_\lambda \in U_{\lambda}} S^*_{[\lambda,\hat{\psi}_\lambda],[\rho_1,\hat{\psi}_{\rho_1}]} S_{[\lambda,\hat{\psi}_\lambda],[\rho_2,\hat{\psi}_{\rho_2}]} = S_{[\lambda,\hat{\psi}_\lambda],[\sigma,\hat{\psi}_{\sigma}]} / S_{[\lambda,\hat{\psi}_\lambda],\Omega};
\]  (6.25)

from which by comparison with the Verlinde formula for the \( \mathfrak{A} \)-theory we learn that the annulus coefficients indeed coincide with the structure constants of the fusion rule algebra.

Let us mention one immediate consequence of the result (6.23). Relabelling the summation variable \( (\lambda,\hat{\psi}_\lambda) \) to \( (\bar{\lambda},\hat{\psi}_\lambda) \) for an arbitrary current \( J \in G \) and inserting the simple current symmetries (5.14) and (A.15) of \( \tilde{S} \) and \( S' \), respectively, one finds that

\[
A^{[\sigma,\hat{\psi}_{\sigma}]_{[\rho_1,\hat{\psi}_{\rho_1}],[\rho_2,\hat{\psi}_{\rho_2}]} = e^{2\pi i Q_3(\rho_2) - Q_3(\rho_1) + Q_3(\sigma)} A^{[\sigma,\hat{\psi}_{\sigma}]_{[\rho_1,\hat{\psi}_{\rho_1}],[\rho_2,\hat{\psi}_{\rho_2}]}.
\]  (6.26)

It follows that \( A^{[\sigma,\hat{\psi}_{\sigma}]_{[\rho_1,\hat{\psi}_{\rho_1}],[\rho_2,\hat{\psi}_{\rho_2}]} \) vanishes unless \( Q_3(\sigma) = Q_3(\rho_1) - Q_3(\rho_2) \) for all \( J \in G \). In short, the annulus coefficients are graded by the monodromy charge.

### 6.3 Relation with fusion coefficients of the \( G' \)-extension

To proceed we insert the formulæ (5.9) for \( \tilde{S} \) and (the analogue for the \( G' \)-extension of) (A.10) for \( S' \) into (6.23), leading to

\[
A^{[\sigma,\hat{\psi}_{\sigma}]_{[\rho_1,\hat{\psi}_{\rho_1}],[\rho_2,\hat{\psi}_{\rho_2}]} = \sum_{[\lambda]} \sum_{\psi_\lambda \in U_{\lambda}} \frac{s_{\lambda}u_{\lambda}}{|G|^2 |G'|} \sum_{\psi_\lambda \in U_{\lambda}} \sum_{\psi_\lambda \in U_{\lambda}} \sum_{\psi_{\lambda'} \in U_{\lambda'}} \sum_{\psi_{\lambda''} \in U_{\lambda''}} \sum_{\psi_{\lambda'''}} \psi_{\lambda}(J_1)^* \hat{\psi}_1(J_1) (S_{J_1,\rho_1})^* \psi_\lambda(J_2) \hat{\psi}_2(J_2)^* S_{J_2,\rho_2} \psi_\sigma(J_3)^* \tilde{S}_{\lambda,\Omega}. \]  (6.27)

This somewhat unwieldy expression simplifies a lot when one performs the \( \hat{\psi}_\lambda \)-summation (obtained by combining the \( \hat{\psi}_\lambda'^* \) and \( \hat{\psi}_{\lambda''} \)-summation) and implements the fact that \( S_{\lambda,\rho} \) is non-zero only if \( Q_3(\rho) = 0 \) (which follows from the identities (A.11) and (A.5)), i.e. only if \( J \in G' \):

\[
A^{[\sigma,\hat{\psi}_{\sigma}]_{[\rho_1,\hat{\psi}_{\rho_1}],[\rho_2,\hat{\psi}_{\rho_2}]} = \sum_{J_1 \in U \cap G'} \frac{1}{s_{\lambda}} \sum_{J_2 \in U} \sum_{J_3 \in U} \psi_{\lambda}(J_1)^* \hat{\psi}_1(J_1) (S_{J_1,\rho_1})^* \psi_\lambda(J_2) \hat{\psi}_2(J_2)^* S_{J_2,\rho_2} \psi_\sigma(J_3)^* \tilde{S}_{\lambda,\Omega}. \]  (6.28)
Next we insert the analogue of \((4.12)\) for the embedding \(U_i \cap G' \subseteq U_i'\) to arrive at an expression with \(U_i'\)-characters:

\[
A_{[\bar{\rho}', \bar{\psi}]'}^{[\bar{\rho}, \bar{\psi}]} = \frac{|G'| |G'|}{\sqrt{s_{\rho_1} u_{\rho_1} s_{\rho_2} u_{\rho_2} s_\sigma u_\sigma}} \sum_{|\lambda|'} \sum_{\psi_1' \in U_i'} \sum_{\psi_2' \in U_i'} \sum_{\psi_2'' \in U_i''} \sum_{J_i' \in U_i'} \sum_{J_i'' \in U_i''} \sum_{J_i''' \in U_i'''} \frac{1}{u_{\psi_2'}} \frac{1}{u_{\psi_2''}} \frac{1}{u_{\psi_2'''}} \frac{1}{u_{\psi_1'}} \frac{1}{u_{\psi_1''}} \frac{1}{u_{\psi_1'''}} J_i' J_i'' J_i''' \phi_i' (J_i')^* \phi_i'' (J_i'')^* \phi_i''' (J_i''')^* (S_{\lambda, \bar{\rho}_1}^1)^* S_{\lambda, \bar{\rho}_2}^2 S_{\lambda, \sigma}^3 S_{\bar{\lambda}, \bar{\rho}_1}^1 S_{\bar{\lambda}, \bar{\rho}_2}^2 S_{\bar{\lambda}, \sigma}^3 \phi_i \phi_i'' \phi_i''' / \phi_i'.
\]

(6.29)

Here \(\tilde{\psi}_i\) denotes the character

\[
\tilde{\psi}_i := \psi_i |_{U_i \cap G'}
\]

of \(U_i \cap G'\).

The \(\bar{\lambda}\)-summation in formula \((4.23)\) is still over all \(G'\)-orbits that are even \(G\)-allowed. We now rewrite it such that we sum over all \(G'\)-orbits that are just \(G'\)-allowed; we first convert the summation to a sum over all orbits by inserting the projector \((A.33)\), and then restrict again to \(G'\)-allowed orbits, which means that the factor \(e^{2\pi i Q_3(\lambda)}\) in \((A.33)\) is equal to 1 for \(J \in G'\) and constant on the cosets of \(G\) with respect to \(G'\). This amounts to using the projector

\[
\frac{|G'|}{|G'|} \sum_{J' \in G/G'} e^{2\pi i Q_3(\lambda)}.
\]

(6.31)

Afterwards we get rid of the phase factor \(e^{2\pi i Q_3(\lambda)}\) by exploiting the simple current symmetry \((A.12)\) of the \(S^1\)-matrices; this yields

\[
A_{[\bar{\rho}', \bar{\psi}]'}^{[\bar{\rho}, \bar{\psi}]} = \sqrt{s_{\rho_1} u_{\rho_1} s_{\rho_2} u_{\rho_2} s_\sigma u_\sigma} \sum_{|\lambda|'} \sum_{\psi_1' \in U_i'} \sum_{\psi_2' \in U_i'} \sum_{\psi_2'' \in U_i''} \sum_{J_i' \in U_i'} \sum_{J_i'' \in U_i''} \sum_{J_i''' \in U_i'''} \frac{1}{u_{\psi_2'}} \frac{1}{u_{\psi_2''}} \frac{1}{u_{\psi_2'''}} \frac{1}{u_{\psi_1'}} \frac{1}{u_{\psi_1''}} \frac{1}{u_{\psi_1'''}} J_i' J_i'' J_i''' \phi_i' (J_i')^* \phi_i'' (J_i'')^* \phi_i''' (J_i''')^* (S_{\lambda, \bar{\rho}_1}^1)^* S_{\lambda, \bar{\rho}_2}^2 S_{\lambda, \sigma}^3 S_{\bar{\lambda}, \bar{\rho}_1}^1 S_{\bar{\lambda}, \bar{\rho}_2}^2 S_{\bar{\lambda}, \sigma}^3 \phi_i \phi_i'' \phi_i''' / \phi_i'.
\]

(6.32)

where \([J]'\) denotes the surviving simple current of the \(G'\)-extension that comes from the simple current \(J\) of the \(\bar{\mathfrak{A}}\)-theory, and where

\[
g' N_{[\bar{\rho}, \bar{\psi}]}^{[\bar{\rho}', \bar{\psi}]'} = \frac{|G'|^2}{\sqrt{s_{\rho_1} u_{\rho_1} s_{\rho_2} u_{\rho_2} s_\sigma u_\sigma}} \sum_{|\lambda|'} \sum_{\psi_1' \in U_i'} \sum_{\psi_2' \in U_i'} \sum_{\psi_2'' \in U_i''} \sum_{J_i' \in U_i'} \sum_{J_i'' \in U_i''} \sum_{J_i''' \in U_i'''} \frac{1}{u_{\psi_2'}} \frac{1}{u_{\psi_2''}} \frac{1}{u_{\psi_2'''}} \frac{1}{u_{\psi_1'}} \frac{1}{u_{\psi_1''}} \frac{1}{u_{\psi_1'''}} J_i' J_i'' J_i''' \phi_i' (J_i')^* \phi_i'' (J_i'')^* \phi_i''' (J_i''')^* (S_{\lambda, \bar{\rho}_1}^1)^* S_{\lambda, \bar{\rho}_2}^2 S_{\lambda, \sigma}^3 S_{\bar{\lambda}, \bar{\rho}_1}^1 S_{\bar{\lambda}, \bar{\rho}_2}^2 S_{\bar{\lambda}, \sigma}^3 \phi_i \phi_i'' \phi_i''' / \phi_i'.
\]

(6.33)

The numbers \((6.33)\) are precisely the fusion coefficients of the \(G'\)-extension, as can be seen by inserting (the analogue for \(S^0\) of \((A.10)\)) into the Verlinde formula.

We thus have succeeded in writing the annulus coefficients as a linear combination of fusion coefficients of the \(G'\)-extension. Still we would like to manipulate our result further. To this end we observe that in \((6.32)\) we are free to let \([J]'\) act on the label of the \(G'\)-fusion coefficients
where we like it most. In particular for suitable $[J]'$ the action will then be trivial. To determine these currents, consider first the requirement

$$[J]' \ast [\bar{\sigma}, \hat{\psi}_\sigma]' \equiv [J\bar{\sigma}, J\hat{\psi}_\sigma]' \equiv [\bar{\sigma}, \hat{\psi}_\sigma]'$$.

(6.34)

This implies, first, that we need $J\bar{\sigma} = J'\bar{\sigma}$ for some $J' \in \mathcal{G}'$, which is solved by $J \in S_\sigma \cdot \mathcal{G}'$. In addition we then need $J\hat{\psi}_\sigma = J'\hat{\psi}_\sigma$, which is equivalent to

$$F_\sigma(J(J')^{-1}, J_3') = 1 \quad \text{for all } J_3' \in \mathcal{U}_\sigma$$.

(6.35)

because of $\mathcal{U}_\sigma' \subseteq S_\sigma$, for the latter equality it is sufficient (though not necessary, in general)\(^\text{15}\) that

$$J \in \mathcal{U}_\sigma \cdot \mathcal{G}'$$.

(6.36)

Similar arguments apply to $\rho_1$ or $\rho_2$, but now we can also take into account the summations over the $\hat{\psi}'_i$ which satisfy $\hat{\psi}'_i \succ \tilde{\psi}_i$; accordingly, while the first part of the argument is identical, leading to the requirement that

$$J \in S_i \cdot \mathcal{G}'$$.

(6.37)

in the second part the equality between characters only needs to hold for the restriction of the $\mathcal{U}_i$-characters to $\mathcal{U}_i \cap \mathcal{G}'$, so that the analogue of (6.35) gets relaxed to

$$F_{\rho_i}(J(J')^{-1}, J_3') = 1 \quad \text{for all } J_3' \in \mathcal{U}_i \cap \mathcal{G}'$$.

(6.38)

which in turn is satisfied for every $J \in S_i \cdot \mathcal{G}'$. Thus we conclude that whenever $J$ is in the group

$$\mathcal{G}'' \equiv \mathcal{G}''_{\rho_1\rho_2\sigma} \equiv S_\sigma' \cdot S_{\rho_1} \cdot S_{\rho_2} \cdot \mathcal{G}' \cdot \mathcal{U}_\sigma$$

(6.39)

then $[J]'$ acts trivially. It follows that we can rewrite (6.32) as

$$A_{[\bar{\sigma}, \hat{\psi}_\sigma]' [\bar{\rho}_1, \hat{\psi}_1]' [\bar{\rho}_2, \hat{\psi}_2]}' = N \sum_{\hat{\psi}'_1 \in \mathcal{U}_1'} \sum_{\hat{\psi}'_2 \in \mathcal{U}_2'} \sum_{J \in \mathcal{G}''} \mathcal{G}_[\bar{\rho}_1, \hat{\psi}_1]' J \mathcal{G}_[\bar{\rho}_2, \hat{\psi}_2]' J'$$.

(6.40)

with

$$N := \frac{[\mathcal{G}']'}{[\mathcal{G}'][\mathcal{U}_1 \cap \mathcal{G}'][\mathcal{U}_2 \cap \mathcal{G}']^{1/2} \sqrt{^s_{\rho_1}^{u_{\rho_1}^{s_{\rho_2}^{u_{\rho_2}}}}}$$.

(6.41)

Note that both the fusion coefficients and the prefactor $N$ are manifestly non-negative, and hence the result (6.40) shows that the annulus coefficients are non-negative. For the interpretation of the annulus amplitude as a partition function they must even be non-negative integers. To establish this stronger property will require some more work. As the fusion coefficients are manifestly integral, we only have to show integrality for the prefactor $N$. As a preparation we rewrite this number as a product

$$N = N'' \cdot N_{\rho_1} \cdot N_{\rho_2}$$

(6.42)

\(^\text{15}\) It is of course also sufficient that $J$ is in $S_{\rho_1}'$, which in general is not a subgroup of $\mathcal{U}_\sigma$. However, we have $S_{\rho_1}' \subseteq \mathcal{G}'$, and hence because of the explicit appearance of $\mathcal{G}'$ on the right hand side of (6.39) this is in fact not relevant.
of three factors
\[ N'' := \frac{|G'| s_1' s_2'}{|G| s_{p_1} s_{p_2}} \] (6.43)
and \( N_{p_i} \), where
\[ N_p := \sqrt{\frac{s_p |\mathcal{U}_p \cap G'|}{\sqrt{u_p s_p u_p'}}} = \frac{d_p}{d_p'} \frac{|\mathcal{U}_p \cap G'|}{w_p}. \] (6.44)

As we will see in the next subsection, actually each of the three factors is already integral individually; furthermore, those integers possess a natural representation theoretic interpretation.

### 6.4 Integrality

We first show the integrality of \( N'' \) (6.43). Consider the map \( p: S_{p_1} \times S_{p_2} \times (\mathcal{U}_\sigma \cap G') \to G'' \) that is defined by
\[ p: \ (J_1, J_2, J_3) \mapsto J := J_1^{-1} J_2 J_3, \] (6.45)
which of course we can also interpret as a map to the subgroup
\[ \mathcal{I} := p(S_{p_1} \times S_{p_2} \times \mathcal{U}_\sigma \cap G') \subseteq G'' \] (6.46)
on which (6.43) is a surjection. We would like to determine when the image \( J \) is already in \( G' \subseteq \mathcal{I} \). Let us look at the monodromy charges for \( J \). Using the fact that the monodromy charge of a fixed point vanishes (see (A.3)) and using the gradation property (6.26) of the annulus coefficients, we conclude that
\[
\begin{align*}
Q_{J_1}(\rho_1) &= 0, \quad Q_{J_1}(\rho_2) = -Q_{J_1}(\sigma), \\
Q_{J_2}(\rho_2) &= 0, \quad Q_{J_2}(\rho_1) = Q_{J_2}(\sigma), \\
Q_{J_3}(\sigma) &= 0, \quad Q_{J_3}(\rho_1) = 0 = Q_{J_3}(\rho_2).
\end{align*}
\] (6.47)

Additivity of monodromy charges then implies
\[
\begin{align*}
Q_{J_1}(\rho_1) &= Q_{J_2}(\rho_1), \quad Q_{J_1}(\rho_2) = -Q_{J_1}(\rho_2), \quad Q_{J_1}(\sigma) = Q_{J_1}(\rho_2) + Q_{J_2}(\rho_1),
\end{align*}
\] (6.48)
which tells us that in order to have \( J \in G' \), i.e. \( Q_{J_1}(\rho_1) = 0 = Q_{J_2}(\rho_2) \), it is necessary and sufficient that \( Q_{J_1}(\rho_2) = 0 = Q_{J_2}(\rho_1) \), which in turn is equivalent to \( J_1, J_2 \in G' \). We conclude that the kernel of the map \( (J_1, J_2, J_3) \mapsto [J_1^{-1} J_2 J_3] \in \mathcal{I}/G' \) is the subgroup \( S_{p_1}' \times S_{p_2}' \times (\mathcal{U}_\sigma \cap G') \) of \( S_{p_1} \times S_{p_2} \times (\mathcal{U}_\sigma \cap G') \). By the homomorphism theorem this in turn implies that
\[
[\mathcal{I}] s_1' s_2' = |G'| s_{p_1} s_{p_2}. \] (6.49)

Moreover, \( \mathcal{I} \) is a subgroup (not just a subset) of \( G'' \), so \( |G''|/|\mathcal{I}| \) is integral, and hence also
\[
N'' = \frac{|G''| s_1' s_2'}{|G| s_{p_1} s_{p_2}} = |G' : \mathcal{I}| \in \mathbb{Z}_{>0}. \] (6.50)

This proves the integrality of the number \( N'' \); it also provides us with a simple reason for the integrality property, namely that \( N'' \) is the index of the subgroup \( \mathcal{I} = p(S_{p_1} \times S_{p_2} \times \mathcal{U}_\sigma \cap G') \) in the subgroup \( G'' \) of \( G \).
Note that for the case where all untwisted stabilizers are equal to the full stabilizers (which immediately implies $N_\rho = 1$), this already settles the integrality problem. To establish integrality of $N_\rho$ as defined in (6.44) in the general case, we use information about the representation theory of twisted group algebras (see appendix B for an introduction to twisted group algebras).

Concretely, we just need to observe that the twisted group algebra $C_{F'}S'_{\rho}$ is a semisimple subalgebra of $C_FS_{\rho}$, where $F$ is the two-cocycle (determined uniquely up to a coboundary) whose commutator cocycle is $F|_{S_{\rho} \times S_{\rho}}$, while $F'$ is the analogous two-cocycle whose commutator cocycle is $F'|_{S'_{\rho} \times S'_{\rho}}$.

The results (B.47) and (B.52) about the decomposition of $C_FS_{\rho}$-representations into irreducible $C_{F'}S'_{\rho}$-representations then tell us that the number $N_\rho$ has the natural interpretation as the multiplicity $\beta$ (B.52) that occurs in those branching rules, and hence in particular that

$$N_\rho = \beta(S'_\rho \subseteq S_\rho) \in \mathbb{Z}_{\geq 0},$$

(6.51)
as announced.

### 6.5 Further consistency checks

We would like to stress once more that the natural annulus coefficients are the quantities $A_{[\bar{\sigma}, \hat{\psi}_{\sigma}]}^{[\bar{\rho}_1, \hat{\psi}_{\rho_1}][\bar{\rho}_2, \hat{\psi}_{\rho_2}]}$ that were defined in subsection 6.1, for which the upper and lower indices are, in general, of different type. (In fact, they differ in a rather subtle way, as even the very meaning of the upper index depends, via the definition of the group $G' \equiv G'_{\rho_1, \rho_2}$, on the value of the two lower indices.) In particular, it is the integrality of these numbers that guarantees that the coefficients in an expansion of $A_{[\bar{\rho}_1, \hat{\psi}_{\rho_1}][\bar{\rho}_2, \hat{\psi}_{\rho_2}]}(t)$ in powers of $q = e^{-\pi t}$ are integral and therefore allows for the interpretation of the annulus amplitude as a partition function. On the other hand, for certain purposes it is also desirable to have at one’s disposal some closely related numbers $A^\circ$ for which all three labels are on an equal footing, which means that the upper index should be of the same form, i.e. $[\bar{\sigma}, \hat{\psi}_{\sigma}]$, as the labels for the boundary conditions. In this subsection we show that numbers of the latter form can indeed be introduced, and that they satisfy two interesting systems of relations, see (6.64) and (6.66) below. (Further inspection shows that in many cases the $A^\circ$-coefficients are just multiples of the annulus coefficients, although the constants of proportionality are generically non-integral.)

Let us start by inspecting the formula (6.23) for the annulus coefficients. It may be noticed that to derive that result, no other property of $G'$ was used than that it is a subgroup of $G$. Accordingly, analogous expressions are obtained when any other subgroup of $G$ is used. In particular, let us introduce the quantities $A_{[\bar{\rho}_1, \hat{\psi}_{\rho_1}][\bar{\rho}_2, \hat{\psi}_{\rho_2}]}^{[\bar{\sigma}, \hat{\psi}_{\sigma}]}_\circ$ as the coefficients of the annulus amplitude in the expansion

$$A_{[\bar{\rho}_1, \hat{\psi}_{\rho_1}][\bar{\rho}_2, \hat{\psi}_{\rho_2}]}(t) = \sum_{[\bar{\sigma}]} A_{[\bar{\rho}_1, \hat{\psi}_{\rho_1}][\bar{\rho}_2, \hat{\psi}_{\rho_2}]}^{[\bar{\sigma}, \hat{\psi}_{\sigma}]}_\circ A_{[\bar{\sigma}, \hat{\psi}_{\sigma}]}(\frac{it}{2})$$

(6.52)

with respect to the characters $X^\circ$ of the extension of the $\hat{\mathfrak{A}}$-theory by the simple currents in
the group $G^\circ \subseteq G$ that was defined in (1.9). We then have

$$A^0[\bar{\sigma}, \bar{\psi}_\lambda] = \sum_{[\bar{\lambda}]^0 \in \mathcal{G}^0} \frac{|{\lambda}|}{|{\lambda}|} \sum_{\bar{\psi}_\lambda \in \mathcal{S}^0_{\lambda}} \tilde{S}^*_{(\bar{\lambda}, \bar{\psi}_{\lambda}),[\bar{\rho}_1, \bar{\psi}_{\lambda}]} \tilde{S}_{(\bar{\lambda}, \bar{\psi}_{\lambda}),[\bar{\rho}_2, \bar{\psi}_{\lambda}]} S^0_{[\bar{\lambda}, \bar{\psi}_{\lambda}]^0, [\bar{\sigma}, \bar{\psi}_{\lambda}]} / S^0_{[\bar{\lambda}, \bar{\psi}_{\lambda}]^0, {\bar{\Omega}}},$$

(6.53)

where

$$S^0_{[\bar{\lambda}, \bar{\psi}_{\lambda}]^0, [\bar{\mu}, \bar{\psi}_{\lambda}]} := \frac{|{\lambda}|}{\sqrt{s^0_{\lambda} u^0_{\lambda} s^0_{\mu} u^0_{\mu}}} \sum_{J \in \mathcal{U}^0_{\lambda} \cap \mathcal{U}^0_{\mu}} \hat{\psi}^0_{\lambda}(J) \hat{\psi}^0_{\mu}(J)^* S^J_{\lambda, \mu}$$

(6.54)

is the modular S-matrix of the $G^0$-extension. While by construction the upper index of the numbers $A^0$ is a priori again of a type different from the two lower ones, we will now show that actually their values only depend on full $G$-orbits $[\bar{\sigma}, \bar{\psi}_{\lambda}]$.

To this end we compare the expressions (6.54) for $S^0$ and (6.9) for $\tilde{S}$ and take into account the specific way in which $S^0$ appears in (6.53). Our aim is then to show that up to numerical factors, we are allowed to replace $S^0$ by $\tilde{S}$. To this end we observe that apart from the different prefactors, we have to deal with the presence of different group characters and with the different summation range for the simple currents. As for the characters, we simply need to implement their restriction properties. Concerning the simple current summation, the following reasoning shows that in the expression (6.53) only terms with J in the intersection of the two groups $\mathcal{U}^0_\lambda \cap \mathcal{U}^0_\sigma$ and $S^0_\lambda \cap \mathcal{U}_\sigma$ give non-vanishing contributions.

- For $J \in S_\lambda \setminus \mathcal{U}^0_\lambda$, we distinguish between two cases. First, when $J \in S_\lambda \setminus S^0_\lambda$, we deduce from the definition (6.9) of $G^0$ that $Q_3(\rho_1) \neq Q_3(\rho_2)$, so that by the gradation property of the numbers (6.53) (which follows by a consideration analogous to that for the annulus coefficients) we can assume that $Q_3(\sigma) \neq 0$; but then $\bar{\sigma}$ cannot be a fixed point of J, and hence $S^J_{\lambda, \sigma} = 0$, so that the corresponding contribution to the annulus tensor vanishes. Second, when $J \in S^0_\lambda \setminus S^0_\lambda$, then there exists a $K \in S^0_\lambda$ such that $F_\lambda(J, K) \neq 0$, and hence from

$$S^J_{\lambda, \sigma} = S^J_{K, \lambda, \sigma} = e^{2\pi i Q_K(\sigma)} F_\lambda(J, K) S^J_{\lambda, \sigma} = F_\lambda(J, K) S^J_{\lambda, \sigma}$$

(6.55)

we can again conclude that $S^J_{\lambda, \sigma}$ vanishes. Here in the last equality we have also used the fact that $K \in G^0$, so that we can again invoke the grading property so as to set $Q_K(\sigma) = 0$.

- For $J \in \mathcal{U}_\sigma \setminus \mathcal{U}_\sigma$, the same reasoning as in the first part of the previous case applies.

- For $J \in \mathcal{U}^0_\sigma \setminus \mathcal{U}_\sigma$, there exists a $K \in S_\sigma \setminus S^0_\sigma$ with $F_\lambda(J, K) \neq 0$, so that now

$$S^J_{\lambda, \sigma} = S^J_{\lambda, K, \sigma} = e^{2\pi i Q_K(\lambda)} F_\sigma(J, K) S^J_{\lambda, \sigma} = F_\sigma(J, K) S^J_{\lambda, \sigma}$$

(6.56)

tells us that $S^J_{\lambda, \sigma}$ must be zero.

Furthermore, we can combine the summations over $\hat{\psi}^0_{\lambda}$ and $\psi_{\lambda} \psi_{\lambda}$ to a summation over all $\psi_{\lambda}$, while the $[\bar{\lambda}]^0$-summation can be converted to a sum over all $\bar{\lambda}$ times a factor of $s^0_{\lambda}/|G^0|$. It follows that

$$A^0[\bar{\sigma}, \bar{\psi}_\lambda] = a_\sigma \sum_{[\bar{\lambda}]^0 \in \mathcal{G}^0} \sum_{\bar{\psi}_{\lambda} \in \mathcal{S}^0_{\lambda}} \frac{\tilde{S}^*_{(\bar{\lambda}, \bar{\psi}_{\lambda}),[\bar{\rho}_1, \bar{\psi}_{\lambda}]} \tilde{S}_{(\bar{\lambda}, \bar{\psi}_{\lambda}),[\bar{\rho}_2, \bar{\psi}_{\lambda}]} S^0_{[\bar{\lambda}, \bar{\psi}_{\lambda}]^0, [\bar{\sigma}, \bar{\psi}_{\lambda}]} / S^0_{[\bar{\lambda}, \bar{\psi}_{\lambda}]^0, \bar{\Omega}}},$$

(6.57)
with
\[ a_\sigma \equiv a_{\rho_1,\rho_2;\sigma} := \frac{\sqrt{s_\sigma u_{\sigma}}}{\sqrt{s_\sigma u_{\rho_2}}}, \]
(6.58)

Note that, as indicated by the notation \( a_{\rho_1,\rho_2;\sigma} \), this prefactor not only depends on the upper label \( \sigma \) of the coefficient (6.57), but implicitly on the values of the two lower labels \( \rho_1 \) and \( \rho_2 \) as well, namely through the relevant subgroup \( G^0 \equiv G_{\rho_1,\rho_2}^0 \) of \( G \). What is more interesting, however, is that the prefactor is constant on the \( G \)-orbit of \( \sigma \); this implies that we can replace the upper label according to
\[ A^0_{[\sigma,\hat{\psi}_\sigma]} \equiv A^0_{[\sigma,\hat{\phi}_\sigma]} \equiv A^0_{[\sigma,\hat{\phi}_\sigma]} \]
(6.59)

where \( \hat{\phi}_\sigma \) is any \( U_\sigma \)-character satisfying
\[ \hat{\phi}_\sigma|_{U_\sigma \cap U_\bar{\sigma}} = \hat{\psi}_\sigma|_{U_\sigma \cap U_\bar{\sigma}}. \]
(6.60)

Thus, as announced, we are dealing with quantities where all three labels are on the same footing. By comparison with formula (5.43) for the structure constants of \( C(\mathfrak{A}) \), the coefficients \( A^0 \) are, up to the prefactor (6.58), just the ‘opposite structure constants’, i.e. those obtained when summing over the other index of the non-symmetric \( \check{S} \)-matrix.

To see how the numbers \( A^0_{[\sigma,\hat{\psi}_\sigma]} \) are related to the annulus coefficients \( A^0_{[\sigma,\hat{\psi}_\sigma]} \), we recall the definition (A.31) of extended characters. Consider first the situation where the relevant untwisted stabilizer groups are related as \( U'_\sigma \subseteq U^{0}_{\sigma} \); then we can immediately conclude that
\[ A^0_{[\sigma,\hat{\psi}_\sigma]} = \frac{1}{\sqrt{s_\sigma} u_{\sigma}} \sum_{J \in G^0} \check{X}(J,\hat{\psi}_\sigma) = \frac{\sqrt{s_\sigma} u_{\rho_2}}{\sqrt{s_\sigma} u_{\rho_2}} \sum_{[K] \in G^0/G^0} A^0_{[K]\times[\sigma,\hat{\psi}_\sigma]|_{U'_\sigma}}, \]
(6.61)

where \( \hat{\psi}_\sigma|_{U'_\sigma} \) is the \( U'_\sigma \)-character that is obtained by restricting the \( U^{0}_{\sigma} \)-character \( \hat{\psi}_\sigma \) to the subgroup \( U'_\sigma \). This implies that the corresponding coefficients of the annulus amplitude are linearly related as well,
\[ A^0_{[\sigma,\hat{\phi}_\sigma]} = \frac{\sqrt{s_\sigma} u_{\rho_2}}{\sqrt{s_\sigma} u_{\rho_2}} \cdot A^0_{[\sigma,\hat{\phi}_\sigma]|_{U'_\sigma}} \]
(6.62)

In contrast, in the case where \( U'_\sigma \) is not a subgroup of \( U^{0}_{\sigma} \) (which, in spite of \( S'_\sigma \subseteq S^{0}_{\sigma} \), can happen for the same reasons as in the case of \( U'_\sigma \) versus \( U^{0}_{\sigma} \), see the remarks after formula (6.13)), more complicated linear combinations arise that mix those characters of the \( G' \)- and of the \( G^{0} \)-extensions for which the group characters \( \hat{\psi}_\sigma \in U^{\sigma}_{\sigma} \) and \( \hat{\phi}_\sigma \in U^{\sigma}_{\sigma} \) have common restrictions to the intersection \( U^{0}_{\sigma} \cap U^{\sigma}_{\sigma} \). As the precise form of this relation does not seem to play any particular role, we refrain from writing it out here.

Having arrived at sensible coefficients \( \check{X} \) with three labels of equal type, we are now in a position to perform a few additional consistency checks. We first compute the product of two of these quantities, regarding them as matrices in their lower indices. By direct computation
we find
\[
\sum_{[\rho]} \sum_{\psi_\rho \in U_\rho^\sigma} A^{0}_{[\bar{\rho}, \hat{\psi}_1]} [\bar{\rho}, \hat{\psi}_1] \rho_1 \rho_3 | \rho_2, \hat{\psi}_2 \rangle a_{\rho_1, \rho_3, \sigma_1}^{-1} a_{\rho_2, \rho_3, \sigma_2}^{-1} \tilde{A}^{0}_{[\bar{\rho}, \hat{\psi}_1]} \rho_1 \rho_3 | \bar{\rho}, \hat{\psi}_2 \rangle
\]
\[
= \sum_{[\sigma_3, \hat{\psi}_3]} \sum_{Q_{G}(\lambda)=0} \sum_{\psi_\lambda \in S^*_\lambda} \frac{u_\lambda}{|\mathcal{G}|} [\bar{\Sigma}(\lambda, \psi_\lambda), \Omega]^{-2} \tilde{\Sigma}(\lambda, \psi_\lambda), [\sigma_3, \hat{\psi}_3] \tilde{\Sigma}(\lambda, \psi_\lambda), [\rho_1, \hat{\psi}_1] \tilde{\Sigma}(\lambda, \psi_\lambda), [\bar{\rho}, \hat{\psi}_2]
\]
\[
= \sum_{[\sigma_3, \hat{\psi}_3]} \sum_{Q_{G}(\lambda)=0} \sum_{\psi_\lambda \in S^*_\lambda} \frac{u_\lambda}{|\mathcal{G}|} \tilde{\Sigma}(\lambda, \psi_\lambda), [\sigma_3, \hat{\psi}_3] \tilde{\Sigma}^*(\lambda, \psi_\lambda), [\rho_1, \hat{\psi}_1] \tilde{\Sigma}^*(\lambda, \psi_\lambda), [\bar{\rho}, \hat{\psi}_2] [\tilde{\Sigma}(\lambda, \psi_\lambda), \Omega]^{-1}
\]
\[
= \sum_{[\sigma_3, \hat{\psi}_3]} \sum_{Q_{G}(\lambda)=0} \sum_{\psi_\lambda \in S^*_\lambda} \frac{u_\lambda}{|\mathcal{G}|} \tilde{\Sigma}(\mu, \psi_\mu), [\sigma_3, \hat{\psi}_3] \tilde{\Sigma}^*(\mu, \psi_\mu), [\rho_1, \hat{\psi}_1] \tilde{\Sigma}^*(\mu, \psi_\mu), [\bar{\rho}, \hat{\psi}_2] [\tilde{\Sigma}(\lambda, \psi_\lambda), \Omega]^{-1}.
\]

(6.63)

Here we have introduced a weight factor \(a_{\sigma_1}^{-1}a_{\sigma_2}^{-1}\) into the summation, which correctly accounts for the number of chiral boundary labels (i.e. boundary blocks) that is subsumed in the upper index \(\sigma\); because of the dependence of \(a_{\sigma}\) on the lower labels, this weight factor need not be constant. (Of course one could avoid the presence of a weight factor by simply considering the quantities \(a_{\sigma}^{-1}A^{0}_{[\bar{\rho}, \hat{\psi}_1]} [\bar{\rho}, \hat{\psi}_1]\) instead, but this appears to be less natural.) From formula (6.63) we can read off that
\[
\sum_{[\rho]} A^{0}_{[\bar{\rho}, \hat{\psi}_1]} [\bar{\rho}, \hat{\psi}_1] \rho_1 \rho_3 | \rho_2, \hat{\psi}_2 \rangle a_{\rho_1, \rho_3, \sigma_1}^{-1} a_{\rho_2, \rho_3, \sigma_2}^{-1} \tilde{A}^{0}_{[\bar{\rho}, \hat{\psi}_1]} [\bar{\rho}, \hat{\psi}_2] = \sum_{[\sigma_3, \hat{\psi}_3]} \sum_{Q_{G}(\lambda)=0} \sum_{\psi_\lambda \in S^*_\lambda} \frac{u_\lambda}{|\mathcal{G}|} [\bar{\Sigma}(\lambda, \psi_\lambda), \Omega]^{-2} \tilde{\Sigma}(\lambda, \psi_\lambda), [\sigma_3, \hat{\psi}_3] \tilde{\Sigma}(\lambda, \psi_\lambda), [\rho_1, \hat{\psi}_1] \tilde{\Sigma}(\lambda, \psi_\lambda), [\bar{\rho}, \hat{\psi}_2]
\]
\[
= \sum_{[\sigma_3, \hat{\psi}_3]} \sum_{Q_{G}(\lambda)=0} \sum_{\psi_\lambda \in S^*_\lambda} \frac{u_\lambda}{|\mathcal{G}|} \tilde{\Sigma}(\mu, \psi_\mu), [\sigma_3, \hat{\psi}_3] \tilde{\Sigma}^*(\mu, \psi_\mu), [\rho_1, \hat{\psi}_1] \tilde{\Sigma}^*(\mu, \psi_\mu), [\bar{\rho}, \hat{\psi}_2] [\tilde{\Sigma}(\lambda, \psi_\lambda), \Omega]^{-1}.
\]

(6.64)

with
\[
M^{0}_{[\sigma_3, \hat{\psi}_3]} [\sigma_3, \hat{\psi}_3] = \frac{a_{\rho_1, \rho_3, \sigma_1}^{-1} a_{\rho_2, \rho_3, \sigma_2}^{-1}}{a_{\sigma_3 \sigma_2 \rho_3} a_{\rho_1 \rho_2 \sigma_3}} \tilde{A}^{0}_{[\bar{\rho}, \hat{\psi}_1]} [\bar{\rho}, \hat{\psi}_1] [\bar{\rho}, \hat{\psi}_2].
\]

(6.65)

Thus the coefficients \(A^{0}\) can be regarded as the basis elements of a finite-dimensional algebra with structure constants (6.63). The presence of such an algebraic structure is often interpreted as a ‘completeness relation’ for the boundary conditions.

It is already apparent from the fact that the structure constants (6.63) are essentially equal to suitable numbers \(A^{0}\) that in addition some kind of ‘associativity relation’ holds, where one sums over the upper index of these objects. Indeed, by the same kind of calculation as above one checks that
\[
\sum_{[\rho]} \sum_{\psi_\rho \in U_\rho^\sigma} A^{0}_{[\bar{\rho}, \hat{\psi}_1]} [\bar{\rho}, \hat{\psi}_1] \rho_1 \rho_3 | \rho_2, \hat{\psi}_2 \rangle a_{\rho_1, \rho_3, \sigma_1}^{-1} a_{\rho_2, \rho_3, \sigma_2}^{-1} A^{0}_{[\bar{\rho}, \hat{\psi}_4]} [\bar{\rho}, \hat{\psi}_4] = \sum_{[\rho]} \sum_{\psi_\rho \in U_\rho^\sigma} A^{0}_{[\bar{\rho}, \hat{\psi}_1]} [\bar{\rho}, \hat{\psi}_1] \rho_1 \rho_3 | \rho_2, \hat{\psi}_2 \rangle a_{\rho_1, \rho_3, \sigma_{\rho_1 \rho_2, \sigma}}^{-1} A^{0}_{[\bar{\rho}, \hat{\psi}_4]} [\bar{\rho}, \hat{\psi}_4] + A^{0}_{[\bar{\rho}, \hat{\psi}_4]} [\bar{\rho}, \hat{\psi}_4] + A^{0}_{[\bar{\rho}, \hat{\psi}_4]} [\bar{\rho}, \hat{\psi}_4].
\]

(6.66)

Relations of the form (6.64) and (6.66) are expected on the basis of factorization arguments [23, 24, 25]. But a rigorous derivation of these identities from factorization, in particular for boundary conditions that do not preserve the full bulk symmetry, still remains to be established. Moreover, such relations are technically rather difficult to exploit in non-trivial theories. In our opinion, they do not constitute an optimal starting point for the classification of boundary conditions.
A Simple current extensions

A.1 The spectrum of primary fields

This appendix summarizes the results about integer spin simple current extensions \[9, 12\] that are needed in the main text. When some conformal field theory with chiral algebra \(\hat{A}\) is obtained as an extension of a theory with chiral algebra \(\bar{A}\) by a group \(G\) of integer spin simple currents, then its primary fields are labelled by pairs \([\bar{\lambda}, \hat{\psi}_\lambda]\), where \([\bar{\lambda}]\) is a \(G\)-orbit with vanishing monodromy charge \(Q_J(\bar{\lambda})\) for all \(J \in G\), \(\hat{\psi}_\lambda\) is a certain group character (see below), and the square brackets refer to classes with respect to the equivalence relation that will be given in (A.8). Here by the monodromy charge of \(\bar{\lambda}\) with respect to \(J\) we mean the combination

\[
Q_J(\bar{\lambda}) := \Delta_\bar{\lambda} + \Delta_J - \Delta_{J \bar{\lambda}} \mod \mathbb{Z}
\]  

of conformal weights; we write \(\lambda\) rather than \(\bar{\lambda}\) for the argument because this quantity is constant on \(G\)-orbits. The monodromy charges also satisfy

\[
Q_{J^{-1}}(\mu) = -Q_J(\mu),
\]

and for every \(\bar{\lambda}\) the map

\[
J \mapsto \exp(2\pi i Q_J(\lambda))
\]

furnishes a character of the simple current group \(G\).

To explain the meaning of the character \(\hat{\psi}_\lambda\), we first need to introduce the stabilizer of \(\bar{\lambda}\). This is the subgroup

\[
S_\lambda := \{J \in G \mid J \bar{\lambda} = \bar{\lambda}\} \subseteq G,
\]

which is again constant on \(G\)-orbits. (The symbol ‘\(*\)' stands for the fusion product, and for brevity the basis elements of the fusion ring are just denoted by their labels \(\bar{\lambda}\).) When \(J \in S_\lambda\), then one says that \(\bar{\lambda}\) is a fixed point of \(J\); fixed points of an integer spin simple current \(J\) clearly have vanishing monodromy charge:

\[
J \in S_\lambda \Rightarrow Q_J(\bar{\lambda}) = \Delta_\bar{\lambda} + \Delta_J - \Delta_{J \bar{\lambda}} \mod \mathbb{Z} = 0.
\]  

Now \(\hat{\psi}_\lambda\) is a character of a particular subgroup \(U_\lambda\) of the full stabilizer \(S_\lambda\), \(\hat{\psi}_\lambda \in U_\lambda^*\). This subgroup is called the untwisted stabilizer of \(\bar{\lambda}\); it is obtained as the subset

\[
U_\lambda := \{J \in S_\lambda \mid F_\lambda(K, J) = 1 \text{ for all } K \in S_\lambda\}
\]

on which a certain alternating bi-homomorphism

\[
F_\lambda : \quad G \times G \to U(1)
\]

is trivial. The map \(F_\lambda\), in turn, is determined by the matrices \(S_J^J\) described in the next subsection through relation (A.12).

We remark that in \[12\] the notation \(([\bar{\rho}], \hat{\psi}_\rho)\) was chosen in place of \([\bar{\rho}, \hat{\psi}_\rho]\). This is actually slightly misleading. Namely, in the equivalence relation that defines the classes \([\cdots]\), the simple currents \(J\) act both on the primary label \(\bar{\lambda}\) and on the character \(\hat{\psi}_\lambda\):

\[
(\bar{\lambda}, \hat{\psi}_\lambda) \sim J(\bar{\lambda}, \hat{\psi}_\lambda) = (J \bar{\lambda}, J \hat{\psi}_\lambda)
\]  

(A.8)
for all $J \in \mathcal{G}$, with
\[ J \hat{\psi}_\lambda(J') := F_\lambda(J, J')^* \hat{\psi}_\lambda(J') \] (A.9)

for all $J' \in \mathcal{U}_\lambda$; by the multiplicative property of the $F_\lambda$’s, the quantity $J \hat{\psi}_\lambda$ is again a character of $\mathcal{U}_\lambda$. (The crucial point here is that we consider $F_\lambda$ also for currents $J$ that are not in the stabilizer. When $J$ is in the stabilizer, then $F_\lambda$ is equal to one by definition of the untwisted stabilizer.)

**A.2 The modular S-matrix**

The modular transformation matrix $S$ of the $\mathfrak{A}$-theory is given by

\[
S_{[\psi_\lambda, \hat{\psi}_\mu]} := \frac{|\mathcal{G}|}{\sqrt{s_\lambda u_\lambda s_\mu u_\mu}} \sum_{J \in \mathcal{U}_\lambda \cap \mathcal{U}_\mu} \hat{\psi}_\lambda(J) \hat{\psi}_\mu(J)^* S_{\lambda, \mu}^{J},
\] (A.10)

where $s_\lambda \equiv |\mathcal{S}_\lambda|$ and $u_\lambda \equiv |\mathcal{U}_\lambda|$, and where $\{S^J | J \in \mathcal{G}\}$ is a set of matrices which satisfy the following relations.

- $S^J$ is non-vanishing only on fixed points:
  \[ S_{\lambda, \mu}^{J} = 0 = S_{\mu, \lambda}^{J} \text{ for } J \notin \mathcal{S}_\mu. \] (A.11)

- The restriction of $S^J$ to the fixed points of $J$ is unitary, and together with the restriction $T^J$ of the T-matrix it obeys the usual relations $(S^J T^J)^3 = (S^J)^2$ and $(S^J)^4 = \mathbb{I}$ of the two-fold cover $\text{SL}(2, \mathbb{Z})$ of the modular group.

- For every element $J' \in \mathcal{G}$, $S^J$ satisfies the simple current relations
  \[ S_{\lambda, \mu}^{J} = F_\lambda(J', J) e^{2\pi i Q_\mu(J')} S_{\lambda, \mu}^{J}, \quad S_{\lambda, \mu}^{J} = F_\mu(J', J)^* e^{2\pi i Q_\mu(J')}, \] (A.12)

- The matrices for inverse currents are transposed to each other:
  \[ S_{\mu, \lambda}^{J} = S_{\mu, \lambda}^{J}. \] (A.13)

- The space of one-point chiral blocks with insertion of the simple current $J$ on the torus has a natural basis labelled by the fixed points $\lambda$ of $J$. Upon a suitable canonical choice of normalization of the basis elements, $S^J$ plays the role of the modular S-transformation matrix for those blocks \[\text{[13]}\]. In particular, $S^\Omega$ is the ordinary modular S-matrix of the $\mathfrak{A}$-theory, $S^\Omega = \tilde{S}$. There is some freedom left in the canonical basis choice, which is irrelevant for the formula \[\text{[A.10]}\] but does play a role for the definition of the matrix $\tilde{S}$ in \[\text{[5.9]}\]. We expect that the prescription given in \[\text{[13]}\] can be refined in such a manner that the remaining freedom in the normalization of the blocks constitutes a character of the full stabilizer $\tilde{S}_\lambda$.

- In the case of WZW theories, $S^J$ coincides, up to possibly a fourth root of unity, with the ordinary S-matrix of another WZW theory that is determined by $\mathfrak{A}$ and $J$ \[\text{[11, 12]}\].

- The square of $S^J$ obeys
  \[ (S^J)^2 = \eta^J C^J = C^J (\eta^J)^*, \] (A.14)
where \( C^J \) is the restriction of the charge conjugation of the \( \text{Cl} \)-theory to the fixed points of \( J \) and \( \eta^J \) is a diagonal matrix, with properties to be specified below.

When the extended theory has a surviving simple current \([J']\) with \( J' \not\in G\), then the relations (A.12) are still valid for that current. It follows that the matrix (A.10) satisfies
\[
S_{[\nu][\lambda,\vec{\psi}_\lambda],[\mu,\vec{\psi}_\mu]} = e^{2\pi i Q_{J'}(\mu)} \cdot S_{[\lambda,\vec{\psi}_\lambda],[\mu,\vec{\psi}_\mu]}.
\]
In words, the monodromy charges with respect to \([J']\) in the extended theory are the same as those with respect to \( J' \) in the original theory.

The properties (A.3) of fixed points and (A.12) of \( S^J \) can e.g. be employed to derive the alternating property of the bi-homomorphisms \( F^\lambda \). Indeed, for every \( J \) and every fixed point \( \lambda \) of \( J \) there exists at least one \( \bar{\mu} \) such that \( S^J_{\lambda,\bar{\mu}} \neq 0 \). The fact that \( Q_J(\bar{\mu}) = 0 \) then implies
\[
S^J_{\lambda,\bar{\mu}} = S^J_{\lambda,\bar{\mu}} \cdot e^{2\pi i Q_J(\mu)} F^\lambda(J, J) = F^\lambda(J, J) S^J_{\lambda,\bar{\mu}} ,
\]
from which it follows that
\[
F^\lambda(J, J) = 1.
\]
By the homomorphism property, one then concludes that the bi-homomorphism is trivial even within the whole cyclic group generated by \( J \), i.e.
\[
F^\lambda(J^m, J^n) = 1
\]
for all \( m, n \).

### A.3 Properties of the matrices \( \eta^J \)

The entries \( \eta^J_{\lambda,\bar{\mu}} \delta_{\lambda,\bar{\mu}} \) of the matrix \( \eta^J \) which appears in (A.14) are constant on \( G \)-orbits,
\[
\eta^J_{K\lambda} = \eta^J_{\lambda}
\]
for all \( K \in G \). Further, these matrices satisfy
\[
\eta^J_{\lambda}^{-1} = (\eta^J_{\lambda})^* \quad \text{(A.20)}
\]
as well as
\[
\eta^J_{\lambda}^* = (\eta^J_{\lambda})^* \quad \text{(A.21)}
\]
Moreover, for every fixed \( \lambda \), \( \eta^J_{\lambda} \) constitutes a character of \( \mathcal{U}_\lambda \), i.e.
\[
\eta^J_{\lambda} \eta^K_{\lambda} = \eta^K_{\lambda} \quad \text{(A.22)}
\]
for all \( J, K \in \mathcal{U}_\lambda \); it is not necessarily a character of the full stabilizer \( S_\lambda \), though.

Combining (A.13), (A.14) and the fact that \( C^J \) is the restriction of a permutation matrix and hence has order two, it follows that
\[
\eta^J^{-1} = (\eta^J)^* \quad \text{(A.23)}
\]
Finally, based on results [33] about mapping class group representations, for self-conjugate fixed points \( \lambda \) of \( J \) one can show that
\[
\eta^J_{\lambda} = \epsilon_{\lambda} \sum_{\mu, \nu} Q_J(\mu) \bar{S}_{\Omega_{\mu} \bar{\nu}} \bar{S}_{\Omega_{\mu} \bar{\nu}} e^{4\pi i (\Delta_{\mu} - \Delta_{\nu})} \bar{N}_{\mu, \nu} \bar{\lambda} ,
\]
where \( \epsilon_{\lambda} \) is the Frobenius–Schur indicator for \( \bar{\lambda} \) [15].
A.4 Conjugation properties

The relations (A.13), (A.14) and (A.20), (A.21) imply the conjugation properties

\[ S^j_{\lambda, \mu} = \eta^j_\lambda (S^j_{\mu, \lambda})^*, \quad S^j_{\lambda, \mu} = \eta^j_\mu (S^j_{\mu, \lambda})^* \]  

(A.25)

of the matrices \( S^j \). When combined with the simple current symmetry (A.12), the result (A.25) implies in particular that

\[ e^{2\pi i Q_K(\mu)} F_{\lambda+}(K, J) S^j_{\lambda+} \mu, \mu = S^j_{K\lambda+} \mu, \mu = S^j_{(K^{-1}\lambda)+, \mu} = \eta^j_\lambda (e^{2\pi i Q_K^{-1}(\mu)} F_{\lambda}(K^{-1}, J)^* S^j_{\mu, \lambda})^*, \]  

(A.26)

from which with the help of the homomorphism property of \( F \), the identity (A.2) for monodromy charges, and once more formula (A.25) it follows that the bi-homomorphisms for conjugate orbits are complex conjugate to each other,

\[ F_{\lambda+}(K, J) = F_{\lambda}(K, J)^* \]  

(A.27)

for all \( J, K \in G \).

The conjugation of the primary labels of the extended theory is defined by

\[ \bar{\lambda}, \hat{\psi}_\lambda] = [\bar{\lambda}, \hat{\psi}_\lambda] \]  

(A.28)

with

\[ \hat{\psi}_\lambda(J) := \hat{\psi}_\lambda(J) \eta^j_\lambda. \]  

(A.29)

Because of the character property (A.22) of \( \eta_\lambda \), the quantity \( \hat{\psi}_\lambda^+ \) introduced this way is again a character of \( \mathcal{U}_\lambda \). That the prescription (A.28) is the right one can be checked by inserting the relations (A.29) and (A.25) into the definition (A.10) of \( S \), which leads to the correct conjugation property

\[ S_{[\bar{\lambda}, \hat{\psi}_\lambda], [\mu, \hat{\psi}_\mu]} = S_{[\lambda, \hat{\psi}_\lambda], [\mu, \hat{\psi}_\mu]}^* \]  

(A.30)

of the modular S-matrix.

A.5 Characters

The irreducible characters \( \chi_{[\bar{\lambda}, \hat{\psi}] \lambda} \) of the \( \mathfrak{A} \)-theory are expressed in terms of the irreducible characters \( \bar{\chi}_{\lambda} \) of the \( \mathfrak{A} \)-theory as

\[ \chi_{[\bar{\lambda}, \hat{\psi}] \lambda} = d_\lambda \sum_{J \in G/S_\lambda} \bar{\chi}_{(J \lambda, \hat{\psi})} = \frac{d_\lambda}{|S|} \sum_{J \in G} \bar{\chi}_{(J \lambda, \hat{\psi})} = \frac{1}{\sqrt{|S| |U_\lambda|}} \sum_{J \in G} \bar{\chi}_{(J \lambda, \hat{\psi})}. \]  

(A.31)

The factor

\[ d_\lambda = \sqrt{s_\lambda/u_\lambda}, \]  

(A.32)

which is different from 1 when there is a genuine untwisted stabilizer, accounts for the presence of the degeneracy space \( \mathcal{V}_\psi \) in the decomposition (3.9).
We close this exposition with a remark on the $\mathfrak{A}$-theory. One may always extend a summation over the untwisted sector to a summation over all sectors by inserting the projection operator

$$\frac{1}{|G|} \sum_{J \in G} e^{2\pi i Q_J(\lambda)}.$$  

(A.33)

For instance, we have

$$\sum_{\bar{\mu} \in \mathfrak{A}} Q_{G}(\mu) = 0$$

$$S_{J}^{\bar{\mu}, \rho}(S_{\bar{J}, \sigma})^* F_{\sigma}(J', J)^* = \frac{1}{|G|} \sum_{J' \in G} F_{\sigma}(J', J)^* \delta_{\bar{\rho}, J' \bar{\sigma}}.$$  

(A.34)

B Twisted group algebras and bi-homomorphisms

B.1 Two-cocycles

Let $G$ be a finite group (not necessarily abelian), and $\mathcal{F}$ a two-cocycle on $G$ with values in $\mathbb{C}^\times$, i.e. a map

$$\mathcal{F} : G \times G \to \mathbb{C}^\times$$  

(B.1)

such that

$$\mathcal{F}(g_1, g_2) \mathcal{F}(g_1 g_2, g_3) = \mathcal{F}(g_1, g_2 g_3) \mathcal{F}(g_2, g_3)$$  

(B.2)

for all $g_1, g_2, g_3 \in G$. The cohomologically trivial two-cocycles, i.e. the coboundaries, are of the form

$$\mathcal{F}(g, h) = \epsilon(g) \epsilon(h) / \epsilon(gh)$$  

(B.3)

with $\epsilon$ an arbitrary function $\epsilon : G \to \mathbb{C}^\times$. When $G$ is cyclic, all two-cocycles are coboundaries.

By setting $g_1 = e = g_2$, respectively $g_2 = e = g_3$ (with $e$ the unit element of $G$), we learn that

$$\mathcal{F}(e, g) = \mathcal{F}(g, e) = \mathcal{F}(e, e) =: f$$  

(B.4)

for every $g \in G$. Let now $\mathcal{F}$ be some given two-cocycle; we change $\mathcal{F}$ by multiplying it with a coboundary obtained from any function $\epsilon$ such that $\epsilon(e) = f^{-1}$. The so obtained cohomologous cocycle $\mathcal{F}'$ satisfies

$$\mathcal{F}'(g, e) = \mathcal{F}(g, e) \epsilon(g) \epsilon(e) \epsilon(ge)^{-1} = \mathcal{F}(g, e) f^{-1} = 1.$$  

(B.5)

We will from now on assume this property, but for simplicity denote this cohomologous cocycle $\mathcal{F}'$ just by $\mathcal{F}$; doing so, we have to keep in mind that we are now only allowed to modify it by coboundaries coming from such $\epsilon$ which in addition fulfil $\epsilon(e) = 1$. (Sometimes this is also taken as part of the definition of a cocycle.) Further, by looking at the triple $g, g^{-1}, g$ we learn that

$$\mathcal{F}(g, g^{-1}) = \mathcal{F}(g^{-1}, g)$$  

(B.6)

for every $g \in G$. 

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B.2 Twisted group algebras

To any finite group $G$ one associates its group algebra $\mathbb{C}G$, which is an associative unital algebra over the complex numbers. The dimension of $\mathbb{C}G$ is $|G|$, and it has a basis $\{b_g\}$ that is labelled by group elements $g \in G$ and has multiplication

$$b_g b_{g'} = b_{gg'}.$$  \hfill (B.7)

The elements $b_g$ are units of $\mathbb{C}G$. The group algebra is commutative if and only if $G$ is abelian.

Given a two-cocycle $\mathcal{F}$ on the finite group $G$, one can also define the $\mathcal{F}$-twisted group algebra $\mathbb{C}_\mathcal{F}G$ by modifying the multiplication (B.7) to

$$b_g b_{g'} = \mathcal{F}(g, g') b_{gg'}.$$  \hfill (B.8)

Relation (B.2) ensures that $\mathbb{C}_\mathcal{F}G$ is still associative. Twisted group algebras are unital; the unit element is given by $\mathcal{F}(e, e)^{-1} b_e$. In particular, for every cocycle with property (B.5), $b_e$ is still a unit element. We will assume from now on that $\mathcal{F}$ is such a cocycle.

The isomorphism type of $\mathbb{C}_\mathcal{F}G$ as an algebra over $\mathbb{C}$ depends only on the cohomology class of $\mathcal{F}$. The $\mathcal{F}$-twisted group algebra $\mathbb{C}_\mathcal{F}G$ is isomorphic to the ordinary group algebra $\mathbb{C}G$ if and only if $\mathcal{F}$ is a coboundary, and if and only if there is a homomorphism of complex algebras from $\mathbb{C}_\mathcal{F}G$ to $\mathbb{C}$ (i.e. if and only if $\mathbb{C}_\mathcal{F}G$ has a one-dimensional representation). Further, a twisted group algebra is abelian if and only if $G$ is abelian and the cocycle is cohomologically trivial.

Every twisted group algebra is semisimple. Let us list a few general properties of semisimple associative algebras $A$ over $\mathbb{C}$:

- Every $A$-representation is fully reducible.
- As an algebra over $\mathbb{C}$, $A$ is isomorphic to a direct sum

$$A \cong \bigoplus_i M_{d_i}(\mathbb{C})$$  \hfill (B.9)

of full matrix algebras, where the $d_i$ are the dimensions of the inequivalent irreducible representations of $A$.

- The number of (equivalence classes of) irreducible $A$-representations equals the dimension of the center of $A$.

- The dimensions of the inequivalent irreducible representations are the square roots $d_i$ of the dimensions of the simple summands $M_{d_i}(\mathbb{C})$ that appear in the decomposition (B.9).

B.3 Representation theory of twisted group algebras

The representation theory of $\mathbb{C}_\mathcal{F}G$ is governed by the center of $\mathbb{C}_\mathcal{F}G$. For an explicit description of the center some additional concepts are required. First, for a given cocycle $\mathcal{F}$, a group element $g \in G$ is called $\mathcal{F}$-regular if and only if

$$\mathcal{F}(g, h) = \mathcal{F}(h, g)$$  \hfill (B.10)

for all $h$ in the centralizer $C_g(G)$ of $g$. This is equivalent to saying that the two elements $b_g$ and $b_h$ of $\mathbb{C}_\mathcal{F}G$ commute:

$$b_g b_h = b_h b_g \quad \text{for all} \quad h \in G \quad \text{with} \quad gh = hg.$$  \hfill (B.11)
The set of all $\mathcal{F}$-regular elements of $C_{\mathcal{F}}G$ will be denoted by $G^{\text{reg}} \equiv G_{\mathcal{F}}^{\text{reg}}$. With $g$ every conjugate element $hgh^{-1} \in G$ is $\mathcal{F}$-regular, too; accordingly we also call a conjugacy class of $\mathcal{F}$-regular elements $\mathcal{F}$-regular. Further, if $\mathcal{F}'$ is a two-cocycle cohomologous to $\mathcal{F}$, then every $\mathcal{F}$-regular element is also $\mathcal{F}'$-regular.

Second, for many purposes it is convenient to choose special cocycles within a cohomology class. A cocycle $\mathcal{F}$ is called standard if it satisfies both

$$\mathcal{F}(g, g^{-1}) = 1 \quad \text{for all } g \in G \quad \text{(B.12)}$$

and

$$\mathcal{F}(g, h) \mathcal{F}(gh, g^{-1}) = 1 \quad \text{for all } h \in G^{\text{reg}}, g \in G. \quad \text{(B.13)}$$

In terms of the twisted group algebra this means that $b_{g^{-1}}$ is the inverse of $b_g$ and that conjugation of $\mathcal{F}$-regular elements works without additional factors, i.e.

$$b_{g^{-1}} = (b_g)^{-1} \quad \text{for all } g \in G, \quad \text{(B.14)}$$

$$b_gb_h(b_g)^{-1} = b_{ghg^{-1}} \quad \text{for all } h \in G^{\text{reg}}, g \in G.$$  

By a suitable diagonal change of the basis of the twisted group algebra, the validity of (B.14) can always be achieved [40]. Finally, a left transversal for a subgroup $H$ of $G$ is a set of representatives for $H \backslash G$, i.e. a subset of $G$ that contains precisely one element from each left coset $Hx$.

The center of the twisted group algebra then corresponds to $\mathcal{F}$-regular classes as follows [40]. Let $\{g_1, g_2, ..., g_\ell\}$ be a set of representatives for the $\mathcal{F}$-regular classes, and, for each $i$, $T_i$ a left transversal for the conjugacy class $C_{g_i}$ of $g_i$. Then the $\ell$ elements

$$\sum_{h \in T_i} b_h b_{g_i} (b_h)^{-1} \quad \text{(B.15)}$$

form a basis of the center. In particular, when $\mathcal{F}$ is standard, then the $\ell$ sums

$$\sum_{h \in C_{g_i}} b_h \quad \text{(B.16)}$$

over the $\mathcal{F}$-regular conjugacy classes constitute a basis for the center.

The number of inequivalent irreducible representations of a twisted group algebra is equal to the dimension of the center (since the algebra is semisimple), and hence to the number of regular conjugacy classes. Notice, though, that there is no canonical correspondence between irreducible representations and conjugacy classes.

### B.4 Commutator cocycles for abelian groups

We now consider finite groups $G$ that are abelian. Then the $\mathcal{F}$-regular elements are characterized by

$$\mathcal{F}(g, h) = \mathcal{F}(h, g) \quad \text{for all } h \in G. \quad \text{(B.17)}$$
Further, in the abelian case the subset $G^{\text{reg}}$ of $\mathcal{F}$-regular elements of $G$ is actually a subgroup. Namely, if both $g_1$ and $g_2$ are regular, i.e. if $b_{g_i} b_h = b_h b_{g_i}$ for $i = 1, 2$ and all $h \in C_{g_i} = G$, then one has

$$\mathcal{F}(g_1, g_2) b_{g_1 g_2} b_h = b_{g_1} b_{g_2} b_h = b_{g_1} b_{g_2} = b_h \mathcal{F}(g_1, g_2) b_{g_1 g_2}$$ (B.18)

for all $h \in G$, and hence $b_{g_1 g_2}$ is regular as well.

We will use the term bi-homomorphism for every function

$$F : G \times G \to \mathbb{C}^\times$$ (B.19)

that satisfies

$$F(g_1 g_2, g_3) = F(g_1, g_3) F(g_2, g_3) \quad \text{and} \quad F(g_1, g_2 g_3) = F(g_1, g_2) F(g_1, g_3)$$ (B.20)

for all $g_1, g_2, g_3 \in G$. It may be noted that every bi-homomorphism obeys (B.2) and hence constitutes a two-cocycle on the abelian group $G$. But this property will not be important to us. Also, we are interested in definite bi-homomorphisms rather than their cohomology classes. In particular one may have to deal with bi-homomorphisms that are non-trivial even though cohomologically trivial.

A bi-homomorphism $F$ on $G$ is called alternating if

$$F(g, g) = 1 \quad \text{for all } g \in G,$$ (B.21)

which via the bi-homomorphism property implies (without using abelianness of $G$) the antisymmetry property

$$F(h, g) = (F(g, h))^{-1} \quad \text{for all } g, h \in G.$$ (B.22)

Every alternating bi-homomorphism $F$ of an abelian group $G$ can be written (see e.g. [41, p. 127] and [42]) as the commutator cocycle of some two-cocycle $\mathcal{F} \equiv \mathcal{F}_F$ of $G$, i.e.

$$F(g, h) = \mathcal{F}(g, h) / \mathcal{F}(h, g)$$ (B.23)

for all $g, h \in G$. Note that $F$ determines only the cohomology class of $\mathcal{F}$; put differently, $F$ depends on $\mathcal{F}$ only through the cohomology class of $\mathcal{F}$. Moreover, since only for cohomologically trivial cocycles the twisted group algebra is isomorphic to the ordinary group algebra, only for such cocycles the commutator cocycle is trivial (i.e. $\equiv 1$). It follows that distinct cohomology classes of cocycles also possess distinct commutator cocycles.

The subgroup of $\mathcal{F}$-regular elements can be expressed in terms of the commutator cocycle as

$$G_F^{\text{reg}} = G^F,$$ (B.24)

where

$$G^F := \{ h \in G \mid F(h, g) = 1 \quad \text{for all } g \in G \}.$$ (B.25)

This is indeed a subgroup of $G$, and its definition is symmetric in the two arguments of $F$. (In our application in the main text, $G^F = U$ is the untwisted stabilizer for the stabilizer subgroup $\mathcal{S} \subseteq \mathcal{G}$.)

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16 An example is provided by $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, with the bi-homomorphism $F$ obeying $F(1, J) = 1 = F(J, J)$ for all $J \in \mathbb{Z}_2 \times \mathbb{Z}_2$, while all other values $F(J, J')$ are $-1$. This situation is e.g. realized in the $D_4$ level 2 WZW theory.

17 The function $F$ that is present in the simple current relation (A.13) constitutes an alternating bi-homomorphism of the abelian simple current group $\mathcal{G}$. In the application to simple current extensions, the antisymmetry property holds for the reason described around (A.17).
B.5 Representation theory and traces

The irreducible representations of $\mathbb{C}_F G$ are labelled by the characters $\hat{\psi}$ of the center $G^\text{reg}_F = G^F$. Now let $R$ be a $d$-dimensional irreducible representation of the twisted group algebra $\mathbb{C}_F G$ of an abelian group $G$, and let $R_\circ$ be an irreducible representation of $G$ (in particular $R_\circ$ is one-dimensional). Then $R \otimes R_\circ$, which maps $b_g \in \mathbb{C}_F G$ to
\[
(R \otimes R_\circ)(b_g) := R(b_g) R_\circ(g),
\]
is again an irreducible representation of $\mathbb{C}_F G$ of the same dimension $d$. It can be shown that all irreducible representations of $\mathbb{C}_F G$ are related this way, and thus in particular they all have the same dimension. By the Artin-Wedderburn theorem we then have
\[
|G| = \dim(\mathbb{C}_F G) = d^2 \cdot N_{\text{irr.rep.}}.
\]
Now the number $N_{\text{irr.rep.}}$ of inequivalent irreducible representations equals the dimension of the center. We thus learn that when $G$ is abelian, then for all $d_i$ in (B.9) one has
\[
d_i = d = \sqrt{|G : G^\text{reg}_F|}.
\]
In particular, the index of $G^\text{reg}_F$ in $G$ is a complete square.

The elements of the center $G^\text{reg}$ act as multiples of the identity in every irreducible representation $R_{\hat{\psi}}$:
\[
R_{\hat{\psi}}(b_g) = \hat{\psi}(g) \mathbb{1}_d \quad \text{for all } h \in G^\text{reg}.
\]
Next we show that if $g \in G$ is not regular, then it has vanishing trace in every irreducible representation $R_{\hat{\psi}}$ of $\mathbb{C}_F G$, i.e.
\[
\tr_{R_{\hat{\psi}}} b_g = 0 \quad \text{for all } g \notin G^\text{reg}.
\]
We first consider the regular representation $R$ of $\mathbb{C}_F G$. In this representation $g \in G$ acts by mapping $b_h$ to a multiple of $b_{gh}$. Hence the only group element that has a trace in the regular representation is the identity element $e$. We thus have
\[
\tr_R b_g = |G| \delta_{g,e}.
\]
Now since $G^\text{reg}$ is a subgroup of $G$, for $g \notin G^\text{reg}$ also $gh$ is not in $G^\text{reg}$ for all $h \in G^\text{reg}$; in particular, $gh$ is not the identity element, and therefore $\tr_R b_{gh} = 0$. Now for a semisimple algebra the regular representation is a direct sum of all inequivalent irreducible representations, each appearing with multiplicity 1. Therefore the identity
\[
R_{\hat{\psi}}(b_{gh}) = R_{\hat{\psi}}(b_g) R_{\hat{\psi}}(b_h) = \hat{\psi}(b_h) R_{\hat{\psi}}(b_g),
\]
which follows with the help of (B.29), allows us to compute
\[
0 = \tr_R b_{gh} = \sum_{\hat{\psi} \in G^\text{reg}*} \tr_{R_{\hat{\psi}}} b_{gh} = \sum_{\hat{\psi} \in G^\text{reg}*} \hat{\psi}(h) \tr_{R_{\hat{\psi}}} b_g
\]
for all $h \in G^{\text{reg}}$. Fourier-transforming this relation over $G^{\text{reg}}$ then finally yields $\text{tr}_{\hat{R}_\psi} b_g = 0$ for all $\hat{\psi} \in G^{\text{reg}}^*$ and all $g \notin G^{\text{reg}}$, thus proving (B.30).

We also have
\[ \sum_{\psi \in G^*} \psi(g) = d \delta_{g \in G^{\text{reg}}} \hat{\psi}(g) \quad \text{(B.34)} \]
for every $g \in G$ and every $\hat{\psi} \in G^{\text{reg}}^*$. This is derived as follows. For $g \in G^{\text{reg}}$ the result follows immediately from the fact that in any irreducible representation central elements are represented by multiples of the unit matrix. Suppose then that $g \notin G^{\text{reg}}$; then the completeness of the $G$-characters implies that the sums
\[ \sum_{\psi \in G^*} \psi(h) = 0 \quad \text{(B.35)} \]
over all $G$-characters vanish for every $h \in G^{\text{reg}}$. This means that
\[ \sum_{\hat{\psi} \in G^{\text{reg}}^*} \hat{\psi}(h) \sum_{\psi \in G^*} \psi(g) = 0. \quad \text{(B.36)} \]
We now multiply the relation (B.35) with $\hat{\varphi}^*_h(h)$ for some $\hat{\varphi} \in G^{\text{reg}}^*$ and sum over $h \in G^{\text{reg}}$; by the properties of $G^{\text{reg}}$-characters we then arrive at
\[ 0 = \sum_{\hat{\psi} \in G^{\text{reg}}^*} \hat{\psi}(h) \sum_{\psi \in G^*} \psi(g) = \sum_{\hat{\psi} \in G^{\text{reg}}^*} \hat{\psi}(h) \sum_{\psi \in G^*} \psi(g). \quad \text{(B.37)} \]
This finishes the proof of (B.34).

When considering the tensor product of two (finite-dimensional) projective representations $R_1$ and $R_2$ of a finite abelian group $G$, one should allow for the possibility that the cohomology classes of the two relevant two-cocycles $\mathcal{F}_i$ are different. The tensor product representation $R_1 \otimes R_2$ is again endowed with the structure of a projective $G$-representation via
\[ (R_1 \otimes R_2)(b_g) := R_1(b_g) \otimes R_2(b_g), \quad \text{(B.38)} \]
and one immediately checks that the cocycle relevant to the tensor product representation is the product $\mathcal{F}_1 \mathcal{F}_2$. The most interesting case is the one where the cohomology classes of $\mathcal{F}_1$ and $\mathcal{F}_2$ are complex conjugate (i.e. when they contain representatives that are each others’ complex conjugates). Then the product $\mathcal{F}_1 \mathcal{F}_2$ is cohomologically trivial, so that the tensor product is a honest representation of $G$ and hence fully reducible into a direct sum of one-dimensional irreducible $G$-representations.

One easily verifies that for complex conjugate cocycles the set of regular elements coincide, $G^{\text{reg}}_1 = G^{\text{reg}}_2 =: G^{\text{reg}}$. It follows that in the case of irreducible projective modules $V_1 \equiv V_{\hat{\psi}_1}$ and $V_2 \equiv V_{\hat{\psi}_2}$, both $V_1$ and $V_2$ have dimension $d := \sqrt{|G|/|G^{\text{reg}}|}$, so that the tensor product module $V_1 \otimes V_2$ has dimension $d^2$. Further, for every $g \in G^{\text{reg}}$ one has
\[ (R_1 \otimes R_2)(b_g) = \hat{\psi}_1 \hat{\psi}_2 \mathbb{1}, \quad \text{(B.39)} \]
which means in particular that only those irreducible G-representations appear in the decomposition whose restriction to $G^{\text{reg}}$ is the irreducible $G^{\text{reg}}$-representation with character $\hat{\psi} = \hat{\psi}_1 \hat{\psi}_2 \in G^{\text{reg}*}$. There are $d^2$ inequivalent G-representations $V_\psi$ with this property. To determine their multiplicity in the tensor product, we use the results (B.29) and (B.31) to compute
\[
\text{mult}_{\hat{\psi}_1 \hat{\psi}_2}(\psi) = \sum_{g \in G} \psi^*(g) \text{tr} (R_1 \otimes R_2)(b_g) = \sum_{g \in G^{\text{reg}}} \psi^*(g) \text{tr} (R_1 \otimes R_2)(b_g),
\]
which only depends on the restriction of $\psi$ to $G^{\text{reg}}$. Thus all of the $d^2$ G-representations $V_\psi$ with the same restriction $\psi|_{G^{\text{reg}}} = \hat{\psi}_1 \hat{\psi}_2$ have the same multiplicity, and by comparing dimensions we learn that this multiplicity is equal to one, i.e. each of these representations appears precisely once in the decomposition of $R_1 \otimes R_2$:
\[
R_{\hat{\psi}_1} \otimes R_{\hat{\psi}_2} \cong \bigoplus_{\psi \sim \hat{\psi}_1 \hat{\psi}_2} R_\psi. \tag{B.41}
\]
We conclude in particular that the trivial irreducible representation of G appears in the tensor product of two projective irreducible G-representations if and only if the two cocycles are complex conjugate and the two characters are complex conjugate as well, and then it appears precisely once.

### B.6 Branching rules

Consider now the situation where we are given a two-cocycle $F$ on the abelian group $G$ and in addition a subgroup $G' \subset G$. Manifestly, the restriction
\[
F' := F|_{G' \times G'}
\]
of $F$ to the subgroup constitutes a two-cocycle on $G'$. Likewise, the commutator cocycles are related by restriction as well,
\[
F' \equiv F_{F'} = F_F|_{G' \times G'}. \tag{B.43}
\]
The twisted group algebra $\mathbb{C}_F G'$ is a semisimple subalgebra of $\mathbb{C}_F G$. Hence every irreducible representation $R_{\hat{\psi}}$ of $\mathbb{C}_F G$ is fully reducible into irreducible representations $R_{\hat{\psi}'}$ of $\mathbb{C}_F G'$. Our goal in this subsection is to determine the corresponding branching rules
\[
R_{\hat{\psi}} \cong \bigoplus_{\hat{\psi}' \in G'^{\text{reg}*}} \beta_{\hat{\psi}'} \hat{\psi} R_{\hat{\psi}'}, \tag{B.44}
\]
where the symbol ‘$\cong$’ stands for isomorphy of $\mathbb{C}_F G'$-representations, explicitly for all $\hat{\psi} \in G^{\text{reg}*}$. For ease of notation from now on we will use the abbreviations
\[
U := G^{\text{reg}} \equiv G_F^{\text{reg}} \quad \text{and} \quad U' := G'^{\text{reg}} \equiv G_{F'}^{\text{reg}} \tag{B.45}
\]
for the subgroups of regular elements of $G$ and $G'$, respectively. We remark that in general $U'$ is not a subgroup of $U$. On the other hand, $U \cap G'$ is a subgroup of $U'$. 

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We first observe that \( U \cap G' \) is contained both in \( U \) and in \( U' \), so that the representation matrices \( R_\psi(b_g) \) and \( R_\psi'(b_g') \) are diagonal and act with the same eigenvalue (compare (3.29)). Thus a necessary condition for \( R_\psi' \) to appear in the branching of \( R_\psi \) is that the restrictions of \( \hat{\psi} \) and \( \hat{\psi}' \) to \( U \cap G' \) coincide:

\[
\beta_{\hat{\psi} \hat{\psi}'} \neq 0 \Rightarrow \hat{\psi}|_{U \cap G'} = \hat{\psi}'|_{U \cap G'}. \tag{B.46}
\]

For every \( \hat{\psi} \in U^* \) there are \(|U'|/|U \cap G'| \) many \( C_\mathcal{F}'-G' \)-characters \( \hat{\psi}' \in U'^* \) that satisfy the criterion (B.46). We claim that each of the corresponding irreducible representations \( R_\psi' \) appears with the same multiplicity in the branching of \( R_\psi \) and that this multiplicity is the same for all irreducible \( C_\mathcal{F}'-G \)-representations, i.e. that

\[
R_\psi \cong \beta \cdot \bigoplus_{\hat{\psi}' \in G'^{\text{reg}}} R_\psi'. \tag{B.47}
\]

with \( \beta \) independent of \( \hat{\psi} \) and \( \hat{\psi}' \).

To verify this assertion we consider an arbitrary \( C_\mathcal{F}'-G' \)-representation \( R' \) and define

\[
P_{\psi'} := \frac{1}{|U'|} \sum_{g' \in G'} \hat{\psi}'(g')^* R'(b_g') \tag{B.48}
\]

for every \( \hat{\psi}' \in U'^* \). Direct computation shows that the operators \( P_{\psi'} \) are projectors,

\[
P_{\psi'} P_{\psi'} = \delta_{\psi,\psi'} P_{\psi'} \tag{B.49}
\]

for all \( \hat{\psi}', \hat{\psi} \in U'^* \). Moreover, combining the orthogonality relation for \( U' \)-characters with the result (B.30) about traces one finds that

\[
\text{tr}_{R_\psi} P_{\psi'} = d' \delta_{\hat{\psi},\hat{\psi}'} \tag{B.50}
\]

for every irreducible \( C_\mathcal{F}'-G' \)-representation \( R_\psi' \) and every \( \hat{\psi}' \in U'^* \), where, as usual, \( d' = \sqrt{|G'|/|U'|} \).

Together it follows that the multiplicity of \( R_\psi' \) in \( R' \) is given by \( \text{tr}_{R_\psi} P_{\psi'}/d' \). Applying now this result to the irreducible \( C_\mathcal{F}-G \)-representation \( R_\psi \) (considered as a, generically reducible, \( C_\mathcal{F}-G' \)-representation) we find

\[
\beta_{\hat{\psi} \hat{\psi}'} = \frac{1}{d} \text{tr}_{R_\psi} P_{\psi'} = \frac{1}{|U'|} \frac{d}{d'} \sum_{g' \in U \cap G'} \hat{\psi}'(g')^* \hat{\psi}(g') = \frac{|U \cap G'|}{|U'|} \frac{d}{d'} \delta_{\hat{\psi}'|_{U \cap G'}, \hat{\psi}|_{U \cap G'}.} \tag{B.51}
\]

This means that only the restriction of \( \hat{\phi}' \) to \( U \cap G' \) matters, which finally proves our claim (B.47). It also shows that the multiplicity \( \beta \) in the branching rule (B.47) is given by

\[
\beta \equiv \beta^{(G,G')} = \frac{|U \cap G'|}{|U'|} \frac{d}{d'}. \tag{B.52}
\]
C  The homothety property of \( \beta_\circ \)

In this appendix we derive the identity \((4.39)\) that is equivalent to the homothety property of the mapping \((4.36)\) and hence enters crucially in the calculation of inner products of the boundary blocks. To this end we define

\[
\tilde{\kappa}_\lambda(v \otimes p, v' \otimes p') := \tilde{\kappa}_\lambda(p, p') \sum_{j=1}^{d_\lambda} \beta_\circ(v \otimes w_j)^* \beta_\circ(v' \otimes w_j)
\]

(C.53)

for every pair \(p, p' \in \mathcal{H}_\lambda\), where \(\{w_j\}\) is an orthonormal basis of \(\mathcal{V}_{\psi}^+\). Note that by definition the form \(\tilde{\kappa}_\lambda\) is sesquilinear and independent of the choice of orthonormal basis \(\{w_j\}\).

The relation \((4.39)\) is equivalent to the assertion that

\[
\tilde{\kappa}_\lambda(v \otimes p, v' \otimes p') = \xi \kappa_\lambda(v \otimes p, v' \otimes p')
\]

for all \(p, p' \in \mathcal{H}_\lambda\). This relation, in turn, is proven once we have shown that \(\tilde{\kappa}_\lambda\) is a non-degenerate and invariant scalar product on \(\mathcal{H}_\lambda\), since such scalar products are unique up to a scalar. Non-degeneracy is immediate. Indeed, since the scalar product \(\tilde{\kappa}_\lambda\) is non-degenerate, for every \(v \otimes p \in \mathcal{V}_{\psi} \otimes \mathcal{H}_\lambda\) one can find a \(p' \in \mathcal{H}_\lambda\) such that \(\tilde{\kappa}_\lambda(p, p') \neq 0\). Furthermore, for \(v' = v\) the expression \(\sum_j \beta_\circ(v \otimes w_j)^* \beta_\circ(v' \otimes w_j)\) is non-vanishing.

To establish invariance, we first choose a basis \(\{y_\phi\}\) of \(\text{End}(\mathcal{V}_{\psi})\) that consists of unitary elements, which is always possible. Then we expand any \(\tilde{Y} \in \mathcal{A}\) with respect to this basis, i.e. we write

\[
\tilde{Y} = \sum_{\psi \in S'_\lambda} y_\psi \otimes \tilde{Y}^{(\psi)},
\]

(C.55)

where the first tensor factor acts on \(\mathcal{V}_{\psi}\) and the second on \(\mathcal{H}_\lambda\). To be precise, we also have to account for the fact that \(\tilde{Y}\) is in general not an endomorphism of \(\mathcal{H}_\lambda\), but rather a map

\[
\tilde{Y} : \mathcal{H}_\lambda \to \bigoplus_{j \in G/S_\lambda} \mathcal{H}_{j\lambda}.
\]

(C.56)

By linearity we can concentrate on the individual summands in the expression \((C.55)\), i.e. on elements of \(\mathcal{A}\) of the form \(Y = y_\psi \otimes \tilde{Y}\). For the second tensor factor we just invoke invariance of the scalar product \(\tilde{\kappa}_\lambda\) and are done.\footnote{Strictly speaking, because of \((C.56)\) we cannot directly work with \(\tilde{\kappa}_\lambda\), but must consider a scalar product \(\tilde{\kappa}_\text{tot} = \bigoplus_{j \in G/S_\lambda} \tilde{\kappa}_{j\lambda}\), where \(\tilde{\kappa}_{j\lambda}\) is a scalar product on \(\mathcal{H}_{j\lambda}\). Note that \(\tilde{\kappa}_\text{tot}\) is defined for the reducible module \(\bigoplus_{j \in G/S_\lambda} \mathcal{H}_{j\lambda}\) and hence is not unique up to multiplication. But this is irrelevant, because in our considerations always at least one of its arguments is a vector in \(\mathcal{H}_\lambda\) and because the two operations of restricting to a submodule and taking the hermitian conjugate commute, so that effectively we still only work with \(\tilde{\kappa}_\lambda\). Accordingly we will refrain from using the notation \(\tilde{\kappa}_\text{tot}\) below, but rather just assume that \(\tilde{Y}\) only has a component in the endomorphisms of \(\mathcal{H}_\lambda\).}

Concerning the first factor we note that the grade of \(y_\phi\) is zero, just because the grade of the degeneracy space \(\mathcal{V}_{\psi}\) is zero. Further we use the Ward identity \((1.27)\) for \(B_\lambda\) and write \((-1)^{d_\lambda-1} =: \zeta_Y\) so as to obtain

\[
|\tilde{B}_\lambda(p_o, q_o)|^2 \tilde{\kappa}_\lambda(Y(v \otimes p), v' \otimes p') = -\zeta_Y \sum_{j=1}^{d_\lambda} B_\lambda(v \otimes p_o \otimes (y_\phi w_j) \otimes q_o)^* B_\lambda(v' \otimes p_o \otimes w_j \otimes q_o) \tilde{\kappa}_\lambda(\tilde{Y} p, p')
\]

(C.57)
as well as

\[ |\tilde{B}_\lambda(p_o, q_o)|^2 \tilde{\kappa}_\lambda(v \otimes p, Y^\dagger(v' \otimes p')) = -\zeta_Y \sum_{j=1}^{d_\lambda} B_\lambda(v \otimes p_o \otimes w_j \otimes q_o)^* B_\lambda(v' \otimes p_o \otimes (y^\dagger w_j) \otimes q_o) \tilde{\kappa}_\lambda(p, \tilde{Y}^\dagger p'), \]  

(C.58)

where \( p_o \) and \( q_o \) are the elements of \( \tilde{H}_\lambda \) and \( \tilde{H}_\lambda^\dagger \) used in the definition (1.21) of \( \beta_o \). Next we implement the fact that \( \tilde{\kappa}_\lambda(p, \tilde{Y}^\dagger p') = \tilde{\kappa}_\lambda(\tilde{Y} p, p') \). The final step in establishing the invariance relation then consists in using the identity

\[ \sum_{j=1}^{d_\lambda} B_\lambda(v \otimes p_o \otimes (y^\dagger w_j) \otimes q_o)^* B_\lambda(v' \otimes p_o \otimes w_j \otimes q_o) = \sum_{j=1}^{d_\lambda} B_\lambda(v \otimes p_o \otimes w_j \otimes q_o)^* B_\lambda(v' \otimes p_o \otimes (y^\dagger w_j) \otimes q_o), \]  

(C.59)

which holds because \( w_j \mapsto y^\dagger w_j \) is a basis transformation between two orthonormal bases of \( V_{\psi^+} \) (recall that \( y^\dagger \) was chosen to be unitary).
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