Bad list assignments for non-$k$-choosable $k$-chromatic graphs with $2k + 2$-vertices

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Abstract
It was conjectured by Ohba, and proved by Noel, Reed and Wu that $k$-chromatic graphs $G$ with $|V(G)| \leq 2k + 1$ are chromatic-choosable. This upper bound on $|V(G)|$ is tight: if $k$ is even, then $K_{3*(k/2+1),1*(k/2-1)}$ and $K_{4,2*(k-1)}$ are $k$-chromatic graphs with $2k + 2$ vertices that are not chromatic-choosable. It was proved by Zhu and Zhu that these are the only non-$k$-choosable complete $k$-partite graphs with $2k + 2$ vertices. For $G \in \{K_{3*(k/2+1),1*(k/2-1)}, K_{4,2*(k-1)}\}$, a bad list assignment of $G$ is a $k$-list assignment $L$ of $G$ such that $G$ is not $L$-colourable. Bad list assignments for $G = K_{4,2*(k-1)}$ were characterized by Enomoto, Ohba, Ota and Sakamoto. In this paper, we first give a simpler proof of this result, and then we characterize bad list assignments for $G = K_{3*(k/2+1),1*(k/2-1)}$. Using these results, we characterize all non-$k$-choosable $k$-chromatic graphs with $2k + 2$ vertices.

Keywords
bad list assignment, chromatic-choosable graphs, Ohba conjecture

1 | INTRODUCTION

A proper colouring of a graph $G$ is a mapping $f: V(G) \to \mathbb{N}$ such that $f(u) \neq f(v)$ for every edge $uv$ of $E(G)$. The chromatic number $\chi(G)$ of $G$ is the minimum $k$ such that $G$ has a proper $k$- colouring, that is, a proper colouring $f$ with $|f(V(G))| \leq k$. A $k$-list assignment of a graph $G$ is a mapping $L$ which assigns to each vertex $v$ a set $L(v)$ of at least $k$ permissible colours, that is,
\(|L(v)| \geq k\). An \(L\)-\textit{colouring} of \(G\) is a proper colouring of \(G\) which colours each vertex \(v\) with a colour from \(L(v)\). We say that \(G\) is \(L\)-\textit{colourable} if there exists an \(L\)-colouring of \(G\), and \(G\) is \(k\)-\textit{choosable} if \(G\) is \(L\)-colourable for each \(k\)-list assignment \(L\) of \(G\). More generally, for a function \(f : V(G) \to \mathbb{N}\), an \(f\)-\textit{list assignment} of \(G\) is a list assignment \(L\) with \(|L(v)| \geq f(v)\) for all \(v \in V(G)\). We say \(G\) is \(f\)-\textit{choosable} if \(G\) is \(L\)-colourable for every \(f\)-list assignment \(L\) of \(G\). The \textit{choice number} \(\text{ch}(G)\) of \(G\) is the minimum \(k\) for which \(G\) is \(k\)-choosable. List colouring of graphs was introduced independently by Erdős–Rubin–Taylor [2] and Vizing [11], and has been studied extensively in the literature (cf. [10]).

It follows from the definitions that \(\chi(G) \leq \text{ch}(G)\) for any graph \(G\), and it is well known [2] that bipartite graphs can have arbitrarily large choice number. A graph \(G\) is called \textit{chromatic}-\textit{choosable} if \(\chi(G) = \text{ch}(G)\). Some families of graphs are conjectured or proven to be chromatic-choosable. For example, the list colouring conjecture, proposed independently by Albertson and Collins, Gupta, and Vizing (see [4]), asserts that line graphs are chromatic-choosable. This conjecture was proved for bipartite graphs [3], for complete graphs of odd order [4] and for complete graphs of order \(p + 1\) for primes \(p\) [9]. Also it was conjectured by Ohba [8] and proved by Noel, Reed and Wu [7] that any graph \(G\) with \(|V(G)| \leq 2\chi(G) + 1\) is chromatic-choosable. We shall refer this result as Noel–Reed–Wu Theorem in the remainder of this paper.

We denote by \(K_{k_1,k_2,\ldots,k_q}\) the complete multipartite graph with \(n_i\) partite sets of size \(k_i\), for \(i = 1, 2, \ldots, q\). If \(n_j = 1\), then the number \(n_j\) is omitted from the notation. For example, \(K_{4,2,\ldots,2,1}\) is the complete \(k\)-partite graph with one partite set of size \(4\), and \(k - 1\) partite sets of size \(2\). It was proved in [1] that if \(k\) is an even integer, then \(K_{4,2,\ldots,2,1}\) and \(K_{3,3,\ldots,3,1}\) are not \(k\)-choosable.

So Noel–Reed–Wu Theorem is tight, when \(\chi(G)\) is even. Noel [6] conjectured that if \(\chi(G)\) is odd, then the bound is not tight, that is, if \(k\) is odd, then all \(k\)-chromatic graphs with \(2k + 2\) vertices are \(k\)-choosable. This conjecture was confirmed in [12], where the following result was proved.

**Theorem 1.1.** If \(G\) is a complete \(k\)-partite non-\(k\)-choosable graph with \(2k + 2\) vertices, then \(k\) is even and \(G = K_{4,2,\ldots,2,1} \) or \(G = K_{3,3,\ldots,3,1} \).

Thus any \(k\)-chromatic non-\(k\)-choosable graph with \(2k + 2\) vertices is a spanning subgraph of \(G = K_{4,2,\ldots,2,1}\) or \(G = K_{3,3,\ldots,3,1}\), and \(k\) is an even integer.

For \(G = K_{4,2,\ldots,2,1}\) or \(G = K_{3,3,\ldots,3,1}\), a bad list assignment of \(G\) is a \(k\)-list assignment \(L\) of \(G\) such that \(G\) is not \(L\)-colourable. Bad list assignments \(L\) for \(G = K_{4,2,\ldots,2,1}\) were characterized in [1]: There are two disjoint colour sets \(A, B\) with \(|A| = |B| = k\), each of the \(k - 1\) partite sets \(P_i\) of size \(2\) has one vertex \(u_i\) with \(L(u_i) = A\), and one vertex \(v_i\) with \(L(v_i) = B\). Then the colours in \(A \cup B\) are assigned to vertices in \(P_i\) (the partite set of size \(4\)) in such a way that for any colour \(a \in A, b \in B\), there is a vertex \(v \in P_i\) such that \(a, b \notin L(v)\). (See Theorem 3.1 for detailed description of the lists for vertices in \(P_i\).) We shall give an alternate and simpler proof of this result. Then we characterize bad list assignments for \(G = K_{3,3,\ldots,3,1}\): If \(k \geq 4\), then a \(k\)-list assignment \(L\) of \(G = K_{3,3,\ldots,3,1}\) is bad if and only if

- \(\bigcup_{v \in V(G)} L(v) = \frac{3}{2}k\),
- for each 3-part \(P\) of \(G\), \(\bigcap_{v \in P} L(v) = \emptyset\).

If \(k = 2\), then a \(k\)-list assignment \(L\) of \(K_{3,3}\) is bad if and only if it is isomorphic to one of the list assignments in Figure 2.
As a consequence, we characterize all non-\(k\)-choosable \(k\)-chromatic graphs with \(2k + 2\) vertices. Let \(G^* = \overline{K_4} \lor (2K_{k-1})\) be the join of \(K_4\), an independent set of size 4, and \(2K_{k-1}\), which is the disjoint union of two copies of \(K_{k-1}\). Let \(L\) be a bad \(k\)-list assignment of \(K_{4,2* (k-1)}\) as described in Theorem 3.1. Then \(G^*\) is not \(L\)-colourable, because in an \(L\)-colouring of \(G^*\), one copy of \(K_{k-1}\) uses \(k - 1\) colours from \(A\), and the other copy of \(K_{k-1}\) uses \(k - 1\) colours from \(B\). So only one colour \(a \in A\) and one colour \(b \in B\) are left for vertices in \(P_1\), but \(P_1\) has a vertex \(v\) such that \(a, b \not\in L(v)\). We shall prove the following corollary:

**Corollary 1.2.** Assume that \(k \geq 2\) and \(G\) is a \(k\)-chromatic non-\(k\)-choosable graph with \(2k + 2\) vertices. Then \(k\) is even and one of the following holds:

- \(G\) is a subgraph of \(K_{4,2* (k-1)}\) which contains a copy of \(G^*\),
- \(G\) is isomorphic to one of the graphs in Figure 1,
- \(G = K_{3* (k/2+1),1* (k/2-1)}\).

2 | SOME NOTATION AND PRELIMINARIES

Assume that \(k\) is even and \(G = K_{4,2* (k-1)}\) or \(G = K_{3* (k/2+1),1* (k/2-1)}\), and \(L\) is a bad \(k\)-list assignment of \(G\).

For a subset \(X\) of \(V(G)\), let

\[
L(X) = \bigcup_{v \in X} L(v).
\]

Let \(C = L(V(G))\). It was proved in [5] that if \(L\) is a bad list assignment of \(G\) with \(|C|\) minimum, then \(|C| < |V(G)|\). We observe that by the same argument, without assuming the minimality of \(|C|\), the conclusion still holds for bad list assignments of \(G\). Indeed, if \(|C| \geq |V(G)|\), then we let \(X\) be a maximum subset of \(V(G)\) with \(|X| > |L(X)|\) (if no such \(X\) exists, then we define \(X = \emptyset\)). As \(|V(G)| \leq |C|\), we know that \(X \neq V(G)\). Thus \(|X| \leq 2k + 1\) and by Noel–Wu–Reed Theorem, we have \(G[X]\) is \(L\)-colourable. The maximality of \(X\) implies that for any \(Y \subseteq V(G) - X\), \(|L(Y) - L(X)| \geq |Y|\). By Hall’s Theorem, there is an injective colouring \(f\) of \(G - X\) such that for each vertex \(v\), \(f(v) \in L(v) - L(X)\). Hence \(G\) is \(L\)-colourable.

Each partite set of \(G\) is called a part of \(G\), and a part of size \(i\) (respectively, at least \(i\) or at most \(i\)) is called an \(i\)-part (respectively, \(i^+\)-part, or \(i^-\)-part).

For \(c \in C\) and \(C' \subseteq C\), let

\[
L^{-1}(c) = \{v : c \in L(v)\}, \quad L^{-1}(C') = \bigcup_{c \in C'} L^{-1}(c).
\]

**Figure 1** All non-2-choosable proper subgraphs of \(K_{3,3}\).
For a part $P$ of $G$ and integer $i$, let

$$C_{p,i} = \{ c \in C : |L^{-1}(c) \cap P| = i \}.$$  

**Definition 1.** Assume $\mathcal{S}$ is a partition of $V(G)$, in which each part $S \in \mathcal{S}$ is an independent set. We denote by $G/\mathcal{S}$ the graph obtained from $G$ by identifying each part $S \in \mathcal{S}$ into a single vertex $v_S$. Let $L_\mathcal{S}$ be the list assignment of $G/\mathcal{S}$ defined as $L_\mathcal{S}(v_S) = \bigcap_{v \in S} L(v)$.

If $S = \{v\} \in \mathcal{S}$ is a singleton part of $\mathcal{S}$, then we may also denote $v_S$ by $v$. In this case, $L_\mathcal{S}(v) = L(v)$. In the partitions $\mathcal{S}$ constructed in this paper, most parts of $\mathcal{S}$ are singleton parts. To define $\mathcal{S}$, it suffices to list its nonsingleton parts.

**Definition 2.** Let $B_\mathcal{S}$ be the bipartite graph with partite sets $V(G/\mathcal{S})$ and $C$, in which $c(v) \in L_\mathcal{S}(v) = \bigcap_{v \in S} L(v)$ is an edge if and only if $c \in \bigcup_{v \in S} L(v)$.

It is obvious that a matching $M$ in $B_\mathcal{S}$ covering $V(G/\mathcal{S})$ induces a proper $L_\mathcal{S}$-colouring of $G/\mathcal{S}$, which in turn induces a proper $L$-colouring of $G$. As $G$ is not $L$-colourable, no such matching $M$ exists. By Hall’s Theorem, there is a subset $X_\mathcal{S}$ of $V(G/\mathcal{S})$ such that $|X_\mathcal{S}| > |N_{B_\mathcal{S}}(X_\mathcal{S})|$.

In the remainder of the paper, for a given partition $\mathcal{S}$ of $V(G)$, let $X_\mathcal{S}$ be a maximum subset of $V(G/\mathcal{S})$ for which $|X_\mathcal{S}| > |N_{B_\mathcal{S}}(X_\mathcal{S})|$. Let

$$Y_\mathcal{S} = N_{B_\mathcal{S}}(X_\mathcal{S}) = \bigcup_{v_S \in X_\mathcal{S}} L_\mathcal{S}(v_S).$$

**Observation 2.1.** The following facts will be used often in this paper.

1. No two vertices in the same part of $G$ have the same list.
2. No colour is contained only in the lists of vertices in a single part. Thus for any part $P$ of $G$, $\bigcup_{v \in V(G) - P} L(v) = C$.
3. Each $2^+$-part has no common colour.

**Proof.** (1) Suppose that there is a part $P$ of $G$ such that there are two vertices $u$ and $v$ of $P$ with $L(u) = L(v)$. By Noel–Reed–Wu Theorem, $G - u$ has a proper $L$-colouring $\varphi$ (as $|V(G - u)| = 2k + 1$ and $\chi(G - u) \leq k$). It is easy to verify that $\varphi$ can extend to an $L$-colouring of $G$ by letting $\varphi(u) = \varphi(v)$, a contradiction.

(2) Suppose that there is a colour $c \in C$ and a part $P$ of $G$ such that $L^{-1}(c) \subseteq P$. Now, we colour all vertices of $L^{-1}(c)$ by colour $c$. Let $G' = G - L^{-1}(c)$ and $L'(v) = L(v) - \{c\}$ for all vertices $v \in V(G')$. Note that $|L'(v)| = |L(v)| \geq k$ for each vertex $v$ of $G'$, since $L^{-1}(c) \subseteq P$ which implies that $c \notin L(v)$ for all vertices $v \in V(G')$. Combining with $|V(G')| \leq 2k + 1$ (as $L^{-1}(c) \neq \emptyset$) and Noel–Reed–Wu Theorem, $G'$ has an $L'$-colouring $\varphi$ which can extend to an $L$-colouring of $G$ by letting $\varphi(v) = c$ for $v \in V(G - G')$, a contradiction.

(3) Assume $P$ is a $2^+$-part of $G$ with $\bigcap_{v \in P} L(v) \neq \emptyset$, say $c \in \bigcap_{v \in P} L(v)$. We colour all vertices of $P$ by colour $c$. Let $G' = G - P$ and $L'(v) = L(v) - \{c\}$ for any vertex $v$ of $G'$. As
\(|V(G')| \leq 2\chi(G') + 2\) and \(\chi(G') = k - 1\) is odd, it follows from Theorem 1.1 that \(G'\) is \(L'\)-colourable, and hence \(G\) is \(L\)-colourable, a contradiction. \(\square\)

The following lemma was proved in [12] and also will be used in this paper.

**Lemma 2.2.** Let \(G\) be a complete multipartite graph with parts of size at most 3. Let \(A, D, B, T\) be a partition of the set of parts of \(G\) into classes such that \(A\) and \(D\) contains only parts of size 1, while \(B\) contains all parts of size 2 and \(T\) contains all parts of size 3. Let \(k_1, d, k_2, k_3\) denote the cardinalities of classes \(A, D, B, T\), respectively. Suppose that classes \(A\) and \(D\) are ordered, that is, \(A = (A_1, ..., A_{k_1})\) and \(D = (D_1, ..., D_d)\). If \(f : V(G) \rightarrow \mathbb{N}\) is a function for which the following conditions hold:

\[
\begin{align*}
&f(v) \geq k_2 + k_3 + i \quad \text{for all } 1 \leq i \leq k_1 \text{ and } v \in A_i, \quad (a-1) \\
&f(v) \geq 2k_3 + k_2 + k_1 + i \quad \text{for all } 1 \leq i \leq d \text{ and } v \in D_i, \quad (d-1) \\
&f(v) \geq k_2 + k_3 \quad \text{for all } v \in B \in B, \quad (b-1) \\
&f(u) + f(v) \geq 3k_3 + 2k_2 + k_1 + d \quad \text{for all distinct vertices } u, v \in B \in B, \quad (b-2) \\
&f(v) \geq k_2 + k_3 \quad \text{for all } v \in T \in T, \quad (t-1) \\
&f(u) + f(v) \geq 2k_3 + 2k_2 + k_1 \quad \text{for all distinct vertices } u, v \in T \in T, \quad (t-2) \\
&\sum_{v \in T} f(v) \geq 4k_3 + 3k_2 + 2k_1 + d - 1 \quad \text{for all } T \in T, \quad (t-3)
\end{align*}
\]

then \(G\) is \(f\)-choosable.

The proof of Lemma 2.2 is by induction on the number of vertices. We choose a colour \(c\) in the following preference order:

1. \(c \in \bigcap_{v \in T} L(v)\) for some \(T \in T\),
2. \(c \in \bigcap_{v \in B} L(v)\) for some \(B \in B\),
3. \(c \in L(u) \cap L(v)\) for some \(u, v \in T \in T\) with \(f(u) + f(v) = 2k_3 + 2k_2 + k_1\),
4. \(c \in L(v)\) for some \(v \in T \in T\) with \(f(v) = k_2 + k_3\),
5. \(c \in L(v)\) for some \(v \in B \in B\) with \(f(v) = k_2 + k_3\),
6. \(c \in L(v)\) for some \(v \in A_i \in A\), where is \(i\) is minimum,
7. \(c \in L(v)\) for some \(v \in T \in T\),
8. \(c \in L(v)\) for some \(v \in D_i \in D\), where is \(i\) is minimum,
9. \(c \in L(v)\) for some \(v \in B \in B\).

It is easy to see that one of the cases occurs, provided that \(V(G) \neq \emptyset\). Then we assign colour \(c\) to the corresponding vertices. It is straightforward to verify that the remaining graph \(G'\) with the modified list assignment \(L'\) (by deleting colour \(c\) to each \(L(v)\)) satisfies the condition of Lemma 2.2. By the induction hypothesis, \(G'\) is \(L'\)-colourable. Hence \(G\) is \(L\)-colourable.

### 3 BAD LIST ASSIGNMENTS FOR \(K_{4,2} \ast (k-1)\)

The following theorem is the characterization of bad list assignments of \(K_{4,2} \ast (k-1)\) which has been proved in [1]. In this section, we give an alternate and simpler proof of this theorem.
Theorem 3.1. Assume \( L \) is a bad \( k \)-list assignment of \( G = K_{4,2^*}(k-1) \). Let \( C = \bigcup_{v \in V(G)} L(v) \) and \( P_1, P_2, ..., P_k \) be the \( k \) parts of \( G \) such that \( |P_i| = 4 \). Then there exists a labelling of \( V(G) \) so that \( P_i = \{u_i, v_i, x_i, y_i\}, P_i = \{u_i, v_i\} \) for \( 2 \leq i \leq k \) and \( C \) can be partitioned into \( A \) and \( B \) with \( |A| = |B| = k \), where \( A \) can be further partitioned into \( A_1, A_2, A_3, A_4 \) such that \( |A_1| = |A_2|, |A_3| = |A_4| \) (where \( A_3, A_4 \) maybe empty), and \( B \) can be further partitioned into \( B_1, B_2 \) with \( |B_1| = |B_2| \), and the following hold:

- \( L(u_i) = A_1 \cup A_3 \cup B_1, \ L(v_i) = A_1 \cup A_4 \cup B_2, \ L(x_i) = A_2 \cup A_4 \cup B_1, \ L(y_i) = A_2 \cup A_3 \cup B_2. \)
- \( L(u_i) = A, L(v_i) = B \) for \( 2 \leq i \leq k \).

Proof. First we show that if \( L \) is a list assignment described in the theorem, then \( G \) is not \( L \)-colourable. If \( \phi \) is an \( L \)-colouring of \( G \), then \( \{u_2, ..., u_k\} \) induces a \((k - 1)\)-clique, and hence uses \( k - 1 \) colours from \( A \), and \( \{v_2, ..., v_k\} \) uses \( k - 1 \) colours from \( B \). So only one colour \( a \in A \) and one colour \( b \in B \) is left for vertices in \( P_1 \). It is easy to see that for any colour \( a \in A \) and \( b \in B \), there is a vertex \( v \in V(G) \) such that \( a, b \notin L(v) \). Hence \( G \) is not \( L \)-colourable.

Assume \( L \) is a bad \( k \)-list assignment of \( G \). We shall show that \( L \) is as described above.

If there is a colour \( c \in C \) such that \( |L^{-1}(c) \cap P_1| \geq 3 \), then we colour vertices \( L^{-1}(c) \cap P_1 \) by colour \( c \). Let \( G' = G - (L^{-1}(c) \cap P_1) \) and \( L'(v) = L(v) - \{c\} \) for \( v \in V(G') \). It is easy to verify that \( G' \) and \( L' \) satisfy the condition of Lemma 2.2 (here we need to use the fact that any 2-part \( P \) has a vertex \( v \) with \( c \notin L(v) \), see (3) of Observation 2.1). Therefore \( G' \) is \( L' \)-colourable, and hence \( G \) is \( L \)-colourable, a contradiction.

Thus \( |L^{-1}(c) \cap P_1| \leq 2 \) for any \( c \in C \). As \( \sum_{v \in P_1} |L(v)| \geq 4k \), we have \( |C| \geq 2k \). Recall that \( |C| \leq |V(G)| - 1 = 2k + 1 \). Depending on the size of \( C \), we consider two cases.

Case 1: \( |C| = 2k + 1 \).

Since \( |C| = 2k + 1 \) and \( |L(u_i)| + |L(v_i)| + |L(x_i)| + |L(y_i)| \geq 4k \), we may assume \( L(u_i) \cap L(v_i) \neq \emptyset \). Let \( S \) be the set of \( V(G) \) with one nonsingleton part \( S = \{u_i, v_i\} \), and consider the graph \( G/S \) and list assignment \( L_S \). Let \( X_S \) and \( Y_S \) be as defined in Section 2. We denote by \( P'_1 \) the parts of \( G/S \).

It is obvious that \( X_S - \{v_S\} \neq \emptyset \), and hence \( Y_S \geq |L(v)| \geq k \) for \( v \in X_S - \{v_S\} \). Hence \( |X_S| \geq k + 1 \). If there is an index \( i \geq 2 \) such that \( P'_i \subseteq X_S \), then \( |Y_S| \geq |L_S(u_i)| + |L_S(v_i)| \geq 2k \) and hence \( |X_S| = |V(G/S)| = 2k + 1 \). But in this case, by (2) of Observation 2.1, \( C \supseteq Y_S \supseteq \bigcup_{i=2}^{k} L_S(P'_i) = \bigcup_{i=2}^{k} L(P_i) = C \) and hence \( |Y_S| = |C| = 2k + 1 = |X_S| \), a contradiction.

So \( |P'_1 \cap X| \leq 1 \) for \( i \geq 2 \), and hence \( |X_S| \leq k + 2 \). Since \( |X_S| \geq k + 1 \), \( X_S \cap P'_i \geq 2 \). This implies that \( |Y_S| \geq k + 1 \) and hence \( |X_S| = k + 2 \), that is, \( P'_i \subseteq X \). So \( |Y_S| \geq |L_S(v_S)| + |L_S(x_i) \cup L_S(y_i)| \geq 1 + k + 1 = k + 2 = |X_S| \) (by (1) and (3) of Observation 2.1), a contradiction.

Case 2: \( |C| = 2k \).

As \( |C_{P_1}| + 2|C_{P_1}| = \sum_{v \in P_1} |L(v)| = 4k \) and \( |C_{P_1}| + |C_{P_1}| \leq |C| \), we conclude that \( |C_{P_1}| = 2k \), that is, every colour appears in the lists of exactly two vertices of \( P_1 \).
Claim 3.2. For any 2-subset \( U \) of \( P_1 \), \( |\bigcap_{v \in U} L(v)| = |\bigcap_{v \in P_1 - U} L(v)| \).

Proof. Assume \( U \) is a 2-subset of \( P_1 \). Then \((\bigcap_{v \in U} L(v)) \cap (\bigcup_{v \in P_1 - U} L(v)) = \emptyset \) (as \( C_{P_1,2} = C \)). So

\[
2k = |C| \geq | \bigcap_{v \in U} L(v)| + 1 \bigcup_{v \in P_1 - U} L(v)| = | \bigcap_{v \in U} L(v)| + 2k - 1 \bigcap_{v \in P_1 - U} L(v)|.
\]

Hence \( |\bigcap_{v \in U} L(v)| \leq |\bigcap_{v \in P_1 - U} L(v)| \). By symmetry, \( |\bigcap_{v \in P_1 - U} L(v)| \leq |\bigcap_{v \in U} L(v)| \). So \( |\bigcap_{v \in U} L(v)| = |\bigcap_{v \in P_1 - U} L(v)| \).

Claim 3.3. Assume \( U \) is a 2-subset of \( P_1 \) with \( \bigcap_{v \in U} L(v) \neq \emptyset \). Then there is a \( k \)-subset \( Y_U \) of \( C \) and a \( (k-1) \)-subset \( S_U \) of \( V(G) \) such that for \( i = 2, 3, \ldots, k, |S_U \cap P_i| = 1 \) and

\[
L(S_U \cap P_i) = L(S_U) \cup \left( \bigcap_{v \in U} L(v) \right) \cup \left( \bigcap_{v \in P_1 - U} L(v) \right) = Y_U.
\]

Proof. Assume \( U = \{u_1, v_1\} \) and \( L(u_1) \cap L(v_1) \neq \emptyset \). By Claim 3.2, we know that \( L(x_1) \cap L(y_1) \neq \emptyset \). Let \( S \) be the partition of \( V(G) \) whose non-singleton parts are \( S_1 = \{u_1, v_1\}, S_2 = \{x_1, y_1\} \).

As \( C_{P_1,2} = C \), we have \( L'(v_s) \cap L'(v_s) = \emptyset \). This implies that \( |L'(v_s) \cup L'(v_s)| \geq 2 \) and hence \( X_S - \{v_s, v_s\} \neq \emptyset \). Hence \( |X_s| \geq |L(v)| \geq k \) for some \( v \in X_S - \{v_s, v_s\} \). This implies that \( |X_s| \geq k + 1 \). If \( |X_s| > k + 1 \), then \( P_i \subseteq X \) for some \( 2 \leq i \leq k \), and hence \( |Y_s| \geq 2k = |V(G)/S| \geq |X_s| \), a contradiction. So \( |X_s| = k + 1 \) and \( k = |Y_s| \). This implies that \( P_1 \subseteq X_S \) and \( |X_s \cap P_1| = 1 \) for \( 2 \leq i \leq k \). Let \( S_U = X_S - \{v_s, v_s\} \) and \( Y_U = Y_S \). We have \( L(S_U) \cup \left( \bigcap_{v \in U} L(v) \right) \cup \left( \bigcap_{v \in P_1 - U} L(v) \right) = Y_U \). For \( i = 2, \ldots, k \) and \( v \in S_U \cap P_i \), since \( |Y_s| = k \leq |L(v)| \), we have \( Y_U = L(v) \).

Corollary 3.4. For distinct 2-subsets \( U \) and \( U' \) of \( P_1 \) with \( \bigcap_{v \in U} L(v) \neq \emptyset \) and \( \bigcap_{v \in U'} L(v) \neq \emptyset \), either \( S_U = S_{U'} \) and \( Y_U = Y_{U'} \), or \( S_U \cap S_{U'} = \emptyset \) and \( Y_U \cap Y_{U'} = \emptyset \).

By symmetry, we may assume \( L(u_i) \cap L(v_i) \neq \emptyset \), \( S_{[u_i, v_i]} = \{u_2, u_3, \ldots, u_k\} \).

Since \( |L(u_i) \cap L(v_i)| = |L(x_i) \cap L(y_i)| \) and \( |L(u_i) \cap L(v_i)| + |L(x_i) \cap L(y_i)| \leq |Y_{[u_i, v_i]}| = k \), we have \( |L(u_i) \cap L(v_i)| = |L(x_i) \cap L(y_i)| \leq k/2 \).

As each colour \( c \in C \) appears in the lists of exactly two vertices of \( P_1 \), \( \emptyset \neq L(u_i) - L(v_i) \subseteq L(x_i) \cup L(y_i) \). By symmetry, we may assume that \( L(u_i) \cap L(x_i) \neq \emptyset \). Similarly, we have \( |L(u_i) \cap L(x_i)| = |L(v_i) \cap L(y_i)| \leq k/2 \).

Case 2(a): \( L(u_i) \cap L(y_i) = \emptyset \).

Then

\[
C_{P_1,2} = C = (L(u_i) \cap L(v_i)) \cup (L(x_i) \cap L(y_i)) \cup (L(u_i) \cap L(x_i)) \cup (L(v_i) \cap L(y_i)).
\]

Hence \( |L(u_i) \cap L(v_i)| = |L(x_i) \cap L(y_i)| = |L(u_i) \cap L(x_i)| = |L(v_i) \cap L(y_i)| = k/2 \). Then
\[(L(u_i) \cap L(v_j)) \cup (L(x_i) \cap L(y_j)) = Y_{[u_i,v_j]},
(L(u_i) \cap L(x_i)) \cup (L(v_j) \cap L(y_j)) = Y_{[u_i,x_i]}.
\]

As \(L(u_i) \cap L(v_j) \cap L(x_i) = \emptyset\), we conclude that \(Y_{[u_i,v_j]} \cap Y_{[u_i,x_i]} = \emptyset\) and \(S_{[u_i,v_j]} \cap S_{[u_i,x_i]} = \emptyset\). Thus we have \(S_{[u_i,v_j]} = \{v_2, v_3, \ldots, v_k\}\). Theorem 3.1 holds with \(A_1 = L(u_i) \cap L(v_j), A_2 = L(x_i) \cap L(y_j), A_3 = L(u_i) \cap L(y_j), A_4 = L(v_j) \cap L(x_i), B_1 = L(u_i) \cap L(x_i), B_2 = L(v_j) \cap L(y_j).

Case 2(b): \(L(u_i) \cap L(v_j) \neq \emptyset\).

Since
\[
C_{P,2} = C = (L(u_i) \cap L(v_j)) \cup (L(x_i) \cap L(y_j)) \cup (L(u_i) \cap L(x_i)) \cup (L(v_j) \cap L(y_j)) \\
\cup (L(u_i) \cap L(y_j)) \cup (L(v_j) \cap L(x_i)),
\]
we may assume that \(S_{[u_i,v_j]} = S_{[u_i,x_i]} = Y_{[u_i,y_j]} = Y_{[u_i,v_j]} \cap S_{[u_i,x_i]} = \emptyset\) and \(Y_{[u_i,v_j]} \cap Y_{[u_i,x_i]} = \emptyset\). Then Theorem 3.1 holds with \(A_1 = L(u_i) \cap L(v_j), A_2 = L(x_i) \cap L(y_j), A_3 = L(u_i) \cap L(y_j), A_4 = L(v_j) \cap L(x_i), B_1 = L(u_i) \cap L(x_i), B_2 = L(v_j) \cap L(y_j).

This completes the proof of Theorem 3.1.

4 \quad BAD LIST ASSIGNMENTS FOR \(K_{3\ast(k/2+1),1\ast(k/2-1)}\)

**Theorem 4.1.** Any bad 2-list assignment of \(K_{3,3}\) is isomorphic to one of the list assignments in Figure 2.

**Proof.** Assume \(L\) is a bad 2-list assignment of \(K_{3,3}\). Assume the two parts of \(G\) are \(P_1 = \{u_i, v_i, w_i\}\) for \(i = 1, 2\). By (3) of Observation 2.1, \(C_{P,2} = \emptyset\) for \(i = 1, 2\). As \(|C| \leq |V(G)| - 1 = 5\), we know that \(C_{P,2} \neq \emptyset\). Assume \(1 \in L(u_i) \cap L(v_j)\). If \(1 \notin C_{P,2}\), then we colour \(u_i, v_1\) by colour 1, which greedily extends to an \(L\)-colouring of \(K_{3,3}\), a contradiction.

Thus we may assume that \(1 \in L(u_i) \cap L(v_j) \cap L(u_2) \cap L(v_2)\). If \(L(w_1) \subseteq (L(u_2) \cup L(v_2))\), then we colour \(\{u_i, v_1\}\) by colour 1, and colour \(w_1\) by a colour in \((L(w_1) - (L(u_2) \cup L(v_2)))\), which extends to a proper \(L\)-colouring of \(K_{3,3}\). Thus we must have \(L(w_1) = (L(u_2) \cup L(v_2)) - \{1\}\), and by symmetry, \(L(w_2) = (L(u_1) \cup L(v_1)) - \{1\}\). Thus \(L(u_i) = \{1, a\}, L(v_i) = \{1, b\}, L(w_1) = \{c, d\}, L(u_2) = \{1, c\}, L(v_2) = \{1, d\}, L(w_2) = \{a, b\}\), where \(a \neq b\) and \(c \neq d\). Depending on whether \(|\{a, b\} \cap \{c, d\}|\) is equal to 0, 1 or 2, we have three bad 2-list assignments for \(K_{3,3}\) as depicted in Figure 2. \(\Box\)

**Lemma 4.2.** Assume \(k \geq 4\) is even, \(G = K_{3\ast(k/2+1),1\ast(k/2-1)}\) and \(L\) is the \(k\)-list assignment of \(G\) with \(|C| > 3k/2\). Then \(G\) is \(L\)-colourable.

**Proof.** For each 3-part \(P\) of \(G\), let \(t_P = \max\{|L(u) \cap L(v)| : u \neq v, u, v \in P\}\).

For an ordering \(\pi = (P_1, P_2, \ldots, P_{k/2+1})\) of the 3-parts, let
Let $\pi$ be an ordering of the 3-parts such that

1. $\ell(\pi)$ is maximum.
2. Subject to (1), $|L(P_{k/2+1})|$ is maximum.

For $1 \leq i \leq \ell(\pi)$, choose a 2-subset $S_i$ of $P_i$ with $\bigcap_{v \in S_i} L(v) = t_{P_i}$. Let $S$ be the partition of $V(G)$ whose nonsingleton parts are $S_1, S_2, \ldots, S_{\ell(\pi)}$. Let $G/S, L_S, X_S, Y_S$ be as defined in Section 2. Similarly, let $P_i'$ be the parts of $G/S$.

Let $Z = \{v_{S_i} : 1 \leq i \leq \ell(\pi)\}$. Subject to the condition that $\bigcap_{v \in S_i} L(v) = t_{P_i}$, we choose $S_i$ so that $L_S(Z)$ is maximum.

**Claim 4.3.** $X_S - Z \neq \emptyset$, $\ell(\pi) \leq k/2$ and $|X_S \cap P_i| \geq 2$ for some $\ell(\pi) + 1 \leq i \leq k/2 + 1$.

**Proof.** If $X_S - Z = \emptyset$, then let $i$ be the maximum index such that $v_{S_i} \in X_S$. This implies that $|Y_{S_i}| \geq i \geq |X_{S_i}|$, a contradiction.

So $|Y_{S_i}| \geq |L(v)| = k$ for $v \in X_S - Z$. Hence $|X_{S_i}| \geq k + 1$. Thus for some part $P_i'$ of $G/S$, $|P_i'| \cap X_{S_i} \geq 2$.

If $|X_S \cap P_i| \leq 1$ for each $i \geq \ell(\pi) + 1$, then let $i$ be the maximum index such that $P_i' \subseteq X_S$. Then $|X_{S_i}| \leq k + i$. But $|Y_{S_i}| \geq |L_{S_i}(P_i')| \geq k + i$ (by (3) of Observation 2.1), a contradiction. Thus $\ell(\pi) \leq k/2$ and $X_S \cap P| \geq 2$ for some $\ell(\pi) + 1 \leq i \leq k/2 + 1$. □

**Claim 4.4.** $|Y_{S_i}| \geq 3k/2$, $|V(G/S) - X_{S_i}| \leq 1$ and $\ell(\pi) = t_{P_{k/2+1}} = k/2$.

**Proof.** By Claim 4.3, there exists $\ell(\pi) + 1 \leq i \leq k/2 + 1$ such that $|X_S \cap P_i| \geq 2$. Assume $x, y$ are distinct vertices in $X_S \cap P_i$. Then

$$|Y_{S_i}| \geq |L(x) \cup L(y)| \geq 2k - t_{P_i} \geq 2k - \ell(\pi).$$

Hence

$$2k + 2 - \ell(\pi) = |V(G/S)| \geq |X_{S_i}| \geq 2k - \ell(\pi) + 1.$$
This implies that either \( X_S = G/S \) or \( X_S = G/S - v^* \) for some vertex \( v^* \). Also, since \( \ell(\pi) \leq k/2 \), we have \( Y_S \geq 2k - \ell(\pi) \geq 3k/2 \) and \( 3k/2 + 2 \geq |X_S| \geq 3k/2 + 1 \).

If \( \ell(\pi) \leq k/2 - 1 \), then \( X_S \) contains a 3-part \( P_i \), where \( i \geq \ell(\pi) + 1 \). By the maximality of \( \ell(\pi) \), we have \( |L(x) \cap L(y)| \leq \ell(\pi) \) for any \( x, y \in P_i \).

Hence

\[
3\ell(\pi) \geq |C_{R,i}| \geq 3k - |L(P_i)| \geq 3k - |Y_S| \geq k + \ell(\pi) - 1,
\]

a contradiction. Hence \( \ell(\pi) \geq k/2 \), and by Claim 4.3, we have \( \ell(\pi) = k/2 \).

Now we show that \( t_{P_{k/2+1}} = k/2 \). It follows from the definition of \( \ell(\pi) \) that \( t_{P_{k/2+1}} \leq k/2 \). Assume to the contrary that \( t_{P_{k/2+1}} > k/2 \). Then \( |Y_S| \geq |L(x) \cup L(y)| \geq 2k - t_{P_{k/2+1}} \geq 3k/2 + 1 \). Hence \( |X_S| \geq 3k/2 + 2 \), which implies that \( X_S = V(G/S) \).

In particular, \( P_{k/2+1} \subseteq X_S \) and hence \( |Y_S| \geq |L(P_{k/2+1})| \geq 3k - 3t_{P_{k/2+1}} \geq 3k/2 + 3 > |X_S| \), a contradiction.

\begin{corollary}
For \( i = 1, 2, ..., k/2 + 1 \), \( t_P = k/2 \) and \( |L(P_i)| \geq 3k/2 \).
\end{corollary}

\begin{proof}
Interchange the positions of \( P_i \) and \( P_{k/2+1} \), we obtain an ordering \( \pi' \) of the parts of \( G \) with \( \ell(\pi') = k/2 \). Apply Claims 4.3 and 4.4 to \( \pi' \), we conclude that \( t_P = k/2 \). (Note that the proofs of these two claims only used the fact that \( \pi \) is an ordering with \( \ell(\pi) \) maximum among all orderings of the parts of \( G \).)

For \( i = 1, 2, ..., k/2 + 1 \),

\[
2|L(P_i)| \geq |C_{R,i}| + 2|C_{R,i}| = \sum_{v \in P_i} |L(v)| \geq 3k
\]

and hence \( |L(P_i)| \geq 3k/2 \).
\end{proof}

\begin{corollary}
If \( 1 \leq i \leq k/2 + 1 \) and \( |L(P_i)| = 3k/2 \), or \( k/2 + 2 \leq i \leq k \), then \( L(P_i) = L_S(P_i') \).
\end{corollary}

\begin{proof}
If \( k/2 + 2 \leq i \leq k \), then \( P_i \) is a singleton part of \( G \) and hence \( P_i = P_i' \) and \( L(P_i) = L_S(P_i') \). Assume that \( 1 \leq i \leq k/2 + 1 \) and \( |L(P_i)| = 3k/2 \). By Corollary 4.5, we have \( t_P = k/2 \) and hence \( L(P_i) \) can be partitioned into three parts \( C_1, C_2 \) and \( C_3 \) with \( |C_1| = |C_2| = |C_3| = k/2 \) and there is a labelling of \( V(P_i) = \{u_i, v_i, w_i\} \) such that \( L(u_i) = C_1 \cup C_2, L(v_i) = C_1 \cup C_3 \) and \( L(w_i) = C_2 \cup C_3 \). Therefore, \( L(P_i) = L_S(P_i') \).
\end{proof}

In the following, we assume that \( X_S = G/S \) or \( X_S = G/S - v^* \) for some vertex \( v^* \). Note that for any \( i \leq k/2 \), if \( \pi' \) is the ordering of the parts of \( G \) obtained from \( \pi \) by interchanging the position of \( P_i \) and \( P_{k/2+1} \), we have \( \ell(\pi') = k/2 = \ell(\pi) \). Thus by the choice of \( \pi \), we have

\[
|L(P_i)| \leq |L(P_{k/2+1})|, \forall i \leq k/2.
\]

Combining with Corollary 4.6, we can obtain the following corollary.

\begin{corollary}
If \( |L(P_{k/2+1})| = 3k/2 \), then \( L(P) = L_S(P') \) for any part \( P \) of \( G \).
\end{corollary}
Claim 4.8. If $P_{k/2+1} \subseteq X_\mathcal{S}$, then $|Y_\mathcal{S}| > 3k/2$.

Proof. Assume $P_{k/2+1} \subseteq X_\mathcal{S}$. By Corollary 4.5, $|Y_\mathcal{S}| \geq |L_\mathcal{S}(P_{k/2+1})| = |L(P_{k/2+1})| \geq 3k/2$. Assume to the contrary that $|Y_\mathcal{S}| = 3k/2$.

Combined with $|L(P_{k/2+1})| \leq |Y_\mathcal{S}| = 3k/2$, we have $|L(P_{k/2+1})| = 3k/2$ and hence for each part $P$ of $G$, $L(P) = L_\mathcal{S}(P')$, by Corollary 4.7.

Assume $v^* \in P_{i_0}$ (if $v^*$ exists) for some $1 \leq i_0 \leq k/2$. By (2) of Observation 2.1, we have $C = \bigcup_{v \in V(G) - P_{i_0}} L(v)$. By the definition of $v^*$, we know that $P'_i \subseteq X_\mathcal{S}$ if $i \neq i_0$. Hence

$$C = \bigcup_{v \in V(G) - P_{i_0}} L(v) = \bigcup_{i \neq i_0} L(P_i) = \bigcup_{i \neq i_0} L_\mathcal{S}(P'_i) \subseteq Y_\mathcal{S}.$$  

This is a contradiction, as $|C| > 3k/2$. □

In the remaining part of the proof, we consider two cases.

Case 1: $X_\mathcal{S} = G/S - v^*$.

In this case, $|X_\mathcal{S}| = 3k/2 + 1$ and $|Y_\mathcal{S}| = 3k/2$. By Claim 4.8, we conclude that $P_{k/2+1} \not\subseteq X_\mathcal{S}$ and hence $v^* \in P_{k/2+1}$. By the maximality of $X_\mathcal{S}$, we know that

$$L(v^*) - Y_\mathcal{S}| \geq 2. \quad (2)$$

Assume $P_{k/2+1} = \{u_{k/2+1}, v_{k/2+1}, w_{k/2+1}\}$ and $v^* = v_{k/2+1}$. Since $L(u_{k/2+1}) \cup L(w_{k/2+1}) \leq |Y_\mathcal{S}| = 3k/2$, we have $|L(u_{k/2+1}) \cap L(w_{k/2+1})| \geq k/2$. As $\ell(\pi) = k/2$, we have $|L(u_{k/2+1}) \cap L(w_{k/2+1})| = k/2$.

As $2|C_{P_{i,2}}| + |C_{P_{i,1}}| \geq 3k$ and $|C_{P_{i,2}}| + |C_{P_{i,1}}| \leq |C| \leq 2k + 1$, we have $|C_{P_{i,2}}| \geq k - 1 > k/2$ (as $k \geq 4$). Hence there is a 2-subset $S'_1$ of $P_i$ such that

$$\bigcap_{v \in S'_1} L(v) \subseteq L(u_{k/2+1}) \cap L(w_{k/2+1}).$$

Let $S_{k/2+1} = \{u_{k/2+1}, w_{k/2+1}\}$ and let $S' = (S - \{S_{i}\}) \cup \{S'_1, S_{k/2+1}\}$.

Let $P''_i$ be the part of $G/S'$ and let $Z' = (Z - \{v_{S_i}\}) \cup \{v_{S'_1}, v_{S_{k/2+1}}\}$.

We have

$$X_\mathcal{S} - Z' \neq \emptyset,$$

for otherwise, if $|X_\mathcal{S} \cap Z'| \leq k/2$, then $|X_\mathcal{S}| \leq k/2 \leq |Y_\mathcal{S}|$ (as it is obvious that $|X_\mathcal{S}| \geq 2$), a contradiction, and if $X_\mathcal{S} = Z'$, then $|X_\mathcal{S}| = k/2 + 1 \leq |L_\mathcal{S}(v_{S_i})| \cup L_\mathcal{S}(v_{S_{k/2+1}})| \leq |Y_\mathcal{S}|$, a contradiction.

Hence, there is a vertex $v \in X_\mathcal{S} - Z'$ such that $|L_\mathcal{S}(v)| \geq k$ and this implies that $|X_\mathcal{S}| \geq k + 1$. So, we know that there exists an index $1 \leq j \leq k/2 + 1$ such that $P_j \subseteq X_\mathcal{S}$.  

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This implies that \( |Y_S| \geq k + 1 \) (by (3) of Observation 2.1), and hence \( |X_{S'}| \geq k + 2 \). This in turn implies that there exists an index \( 2 \leq j \leq k/2 + 1 \) such that \( P_j' \subseteq X_{S'} \). This implies that \( |Y_{S'}| \geq |L_{S'}(P_j')| = |L_S(P_j')| \geq k + t P = 3k/2 \) by Corollary 4.5. Hence \( |X_{S'}| = 3k/2 + 1 \), that is, \( X_{S'} = V(G/S') \) and \( Y_{S'} = 3k/2 \). But we know that \( |Y_{S'}| \geq |Y_S \cup L(v^*)| \geq 3k/2 + 2 \). As \( L_S(P_j') = L_{S'}(P_j') \subseteq Y_{S'} \) and \( |L_S(P_j')| = 3k/2 \) for \( 1 \leq i \leq k/2 \), and \( |L(v^*)| = 2 \geq 2 \), we have \( |Y_{S'}| \geq 3k/2 + 2 \), a contradiction.

**Case 2: \( X_S = V(G/S) \).**

In this case, \( |X_{S'}| = 3k/2 + 2 \). In particular,

\[
P_{k/2+1} \subseteq X_S, L(P_{k/2+1}) \subseteq Y_S.
\] (3)

By Claim 4.8, \( |Y_{S'}| = 3k/2 + 1 \).

**Claim 4.9.** \( |L_S(Z)| \geq k/2 + 1 \) and \( L(P_i) \subseteq Y_S \) for \( 1 \leq i \leq k/2 + 1 \).

**Proof.** By Corollary 4.5, \( t_P = k/2 \) for all \( 1 \leq i \leq k/2 + 1 \). By 3, \( |L(P_{k/2+1})| \leq |Y_{S'}| = 3k/2 + 1 \). Combining with 1, \( 3k/2 + 1 \geq |L(P_{k/2+1})| \geq |L(P)| \). This implies that \( P_i \) has at least two \( \ell_i \) subsets \( S_i \) with \( \bigcap_{v \in S_i} L(v) = k/2 \). As \( \ell(\pi) = k/2 \geq 2 \) (by Claim 4.4), we can make a new choice of \( S_2 \) so that \( \bigcap_{v \in S_1} L(v) \neq \bigcap_{v \in S_2} L(v) \). Hence

\[
|L_S(Z)| \geq 1 \bigcap_{v \in S_1} L(v) \cup 1 \bigcap_{v \in S_2} L(v) \geq k/2 + 1.
\]

If \( |L(P_{k/2+1})| = 3k/2 + 1 \), then it follows from Corollary 4.7 that \( L(P) = L_S(P) \) for all parts \( P \) of \( G \). Therefore, \( L(P) = L_S(P) \subseteq Y_S \), and the claim is true.

Assume \( |L(P_{k/2+1})| = 3k/2 + 1 \). Then \( Y_S = L(P_{k/2+1}) \).

Assume to the contrary that \( L(P) \not\subseteq Y_S \) for some \( 1 \leq i \leq k/2 \). Since \( t_P = k/2 \) for \( 1 \leq i \leq k/2 \), by a re-ordering of the parts of \( G \) if needed, we may assume that \( P_i = \{u_1, v_1, w_1\} \) and \( L(u_1) - Y_S \neq \emptyset \) (the resulting ordering \( \pi' \) will still have \( \ell(\pi') = k/2 \)). Let \( S'_1 = \{v_1, w_1\} \) and \( S' = (S - \{S_1\}) \cup \{S'_1\} \). The argument remains valid for \( X_{S'} \). Hence \( |X_{S'}| = |V(G/S')| = 3k/2 + 2 \). However, \( Y_S \cup (L(u_1) - Y_S) \subseteq Y_{S'} \). Hence \( |Y_{S'}| \geq |Y_{S'}| + 1 = 3k/2 + 2 \), a contradiction. \( \square \)

By (2) of Observation 2.1, \( C = \bigcup_{i \in [k-(k/2+1)]} L(P_i) \). By Claim 4.9, \( \bigcup_{i \in [k-(k/2+1)]} L(P_i) \subseteq Y_S \). Therefore, \( \bigcup_{i \in [k-(k/2+1)]} L(P_i) = Y_S = C \).

(a) If \( |L(P_{k/2+1})| = 3k/2 + 1 \) and \( |L(P)| = 3k/2 \) for \( 1 \leq i \leq k/2 \), then there is a vertex \( u \in P_{k/2+1} \) such that \( L(u) - L(S) = L(u) - L(P) \neq \emptyset \). Let \( S_{k/2+1} = P_{k/2+1} - \{u\} \).

(b) Otherwise, let \( S_{k/2+1} \) be any 2-subset of \( P_{k/2+1} \).

Let \( S' = S \cup \{S_{k/2+1}\} \). Let \( P''_i \) be the part of \( G/S' \) and let \( Z' = Z \cup \{v_{S_{k/2+1}}\} \). Note that \( L_{S'}(v_S) = t_P = k/2 \) for \( 1 \leq i \leq k/2 \) (by Corollary 4.5) and \( |L_{S'}(v_{S_{k/2+1}})| \geq 1 \) (as \( |L(P_{k/2+1})| \leq 3k/2 + 1 \)). Then \( |Y_{S'}| \geq k/2 \) (as it is obvious that \( |X_{S'}| \geq 2 \) and hence...
\[|X_{S'}| \geq k/2 + 1\]. As \(|L_{S'}(Z')| \geq |L_{S'}(Z)| \geq k/2 + 1\), we know that \(X_{S'} - Z' \neq \emptyset\). Therefore \(Y_{S'} \geq |L(v)| \geq k\) for some \(v \in X_{S'} - Z'\) and hence \(|X_{S'}| \geq k + 1\). So, we know that \(P_i^\ast \subseteq X_{S'}\) for some \(1 \leq i \leq k/2 + 1\).

This implies that \(|Y_{S'}| \geq k + 1\) (by (3) of Observation of 2.1) and hence \(|X_{S'}| \geq k + 2\). This in turn implies that \(P_i^\ast \subseteq X_{S'}\) for some \(1 \leq i \leq k/2\). So \(|Y_{S'}| \geq |L_{S'}(P_i^\ast)| \geq |L_{S'}(P_i)| \geq 3k/2\) and \(|X_{S'}| \geq 3k/2 + 1\), that is, \(X_{S'} = V(G/S')\). This implies that \(Y_{S'} < |X_{S'}| = 3k/2 + 1 = |Y_{S'}|\).

In Case (a),

\[Y_{S'} \geq |L_{S'}(P_{k/2+1})| \cup L_{S'}(P_i^\ast)| \geq |L_{S'}(P_i)| + 1 = |L_{S'}(P_i)| + 1 = 3k/2 + 1,
\]
a contradiction.

In Case (b), \(|L(P_{k/2+1})| = 3k/2\) and hence \(L(P_i) = L_{S'}(P_i)|\) for \(1 \leq i \leq k/2 + 1\), by Corollary 4.7. Therefore \(Y_S = \bigcup_{i \in [k]} L(P_i) = \bigcup_{i \in [k]} L_{S'}(P_i) = \bigcup_{i \in [k]} L_{S'}(P_i) \subseteq Y_{S'}\), which implies that \(|Y_{S'}| = 3k/2 + 1\), again a contradiction.

This completes the proof of Lemma 4.2.

\[\square\]

**Theorem 4.10.** Assume \(k \geq 4\) and \(L\) is a \(k\)-list assignment of \(G = K_{3\ast(k/2+1),1\ast(k/2-1)}\). Then \(L\) is bad if and only if

- \(\bigcup_{v \in V(G)} L(v) = 3k/2\),
- \(\bigcap_{v \in V(P)} L(v) = \emptyset\) for each 3-part \(P\) of \(G\).

**Proof.** Assume \(L\) is a bad \(k\)-list assignment of \(G\). By (3) of Observation 2.1, for any 3-part \(P\) of \(G\), \(\bigcap_{v \in P} L(v) = \emptyset\). Thus \(2|\bigcup_{v \in V(G)} L(v)| \geq \sum_{v \in P} |L(v)| \geq 3k/2\). Hence \(|\bigcup_{v \in V(G)} L(v)| \geq 3k/2\). Combing with Lemma 4.2, \(|\bigcup_{v \in V(G)} L(v)| = 3k/2\).

Assume \(L\) satisfies the above two conditions. Then \(G\) is not \(L\)-colourable. Assume to the contrary that \(f\) is an \(L\)-colouring of \(G\). Then \(|f(P)| \geq 2\) for each 3-part \(P\) and \(|f(P)| = 1\) for each 1-part \(P\). As \(f(P) \cap f(P') = \emptyset\) for distinct parts \(P\) and \(P'\), we have \(|\bigcup_{v \in V(G)} L(v)| \geq \sum |f(P)| \geq 2 \times (k/2 + 1) + (k/2 - 1) = 3k/2 + 1\), a contradiction. \(\square\)

### 5 CHARACTERIZATION OF NON-\(k\)-CHOOSABLE \(k\)-CHROMATIC GRAPHS WITH \(2k + 2\)-VERTICES

This section proves Corollary 1.2. By Theorem 1.1 and Noel–Reed–Wu Theorem, it suffices to consider proper spanning subgraphs of \(K_{3\ast(k/2+1),1\ast(k/2-1)}\) and \(K_{4,2\ast(k-1)}\), where \(k\) is an even integer.

Let \(G^*\) be the subgraph of \(K_{4,2\ast(k-1)}\) obtained by deleting all the edges \(\{u_i v_j : 2 \leq i, j \leq k\}\), that is, \(G^* = K_4 \vee (2K_{k-1})\). As explained in Section 1, \(G^*\) is not \(k\)-choosable. Hence any subgraph of \(K_{4,2\ast(k-1)}\) containing a copy of \(G^*\) is not \(k\)-choosable. The following lemma shows that any other subgraph of \(K_{4,2\ast(k-1)}\) is \(k\)-choosable.
Theorem 5.1. Assume $G$ is a subgraph of $K_{4,2^{*}(k-1)}$ which does not contain a copy of $G^*$. Then $G$ is $k$-choosable.

Proof. Assume $L$ is a $k$-list assignment of $G$. We may assume $L$ is the $k$-list assignment as described in Theorem 3.1, for otherwise, $K_{4,2^{*}(k-1)}$ is $L$-colourable, and hence $G$ is $L$-colourable. By the description of $L$ in Theorem 3.1, we may assume that $u_i v_j \notin E(G)$ for any $2 \leq i, j \leq k$ (as $L(u_i) \cap L(v_j) = \emptyset$).

If there are $2 \leq i < j \leq k$ such that $u_i u_j$ is not an edge of $G$, then since $L(u_i) = L(u_j)$, we identify $u_i$ and $u_j$ into a single vertex and denote the resulting graph by $G'$. It is easy to see that $\chi(G') = k$ and $|V(G')| = 2k + 1$. It follows from Noel–Reed–Wu Theorem that $G'$ is $L$-colourable and hence $G$ is $L$-colourable by colouring $u_i$ and $u_j$ by the same colour. Thus we may assume that $\{u_2, u_3, \ldots, u_k\}$ induces a clique. Similarly, $\{v_2, v_3, \ldots, v_k\}$ induces a clique.

As $G$ does not contain $G^*$ as a subgraph, we know that $uv \notin E(G)$ for some $u \in P_1$ and $v \in P_1$ for some $2 \leq i \leq k$. By symmetry, assume $u_i u_2 \notin E(G)$. We obtain an $L$-colouring of $G$ by colouring $u_1, u_2$ by a colour $c \in L(u_1) \cap L(u_2) \subseteq A$ (which exists by the description of $L$ in Theorem 3.1), colouring $x_i$ by another colour from $A$, colouring $v_1$ and $y_1$ by a colour $c' \in B_2$, colouring the $(k-2)$-clique induced by $\{u_3, \ldots, u_k\}$ by the remaining $k-2$ colours from $A$, and colouring the $(k-1)$-clique induced by $\{v_2, \ldots, v_k\}$ by the remaining $k-1$ colours from $B$.

For $k = 2$, it is easy to verify (and also follows from the characterization of 2-choosable graphs in [2]) that, up to isomorphism, $K_{3,3}$ has two proper subgraphs that are not 2-choosable, as shown in Figure 1. In the following, we show that for $k \geq 4$, all proper subgraph of $K_{3^{*}(k/2+1),1^{*}(k/2-1)}$ is $k$-choosable.

Theorem 5.2. If $k \geq 4$ and $G$ is a proper subgraph of $K_{3^{*}(k/2+1),1^{*}(k/2-1)}$, then $G$ is $k$-choosable.

Proof. Let $P_i = \{u_i, v_i, w_i\}$ for $1 \leq i \leq k/2 + 1$ and $P_i = \{u_i\}$ for $k/2 + 2 \leq i \leq k$. Assume $G$ is a proper subgraph of $K_{3^{*}(k/2+1),1^{*}(k/2-1)}$. If $|V(G)| \neq V(K_{3^{*}(k/2+1),1^{*}(k/2-1)})$, then $|V(G)| \leq 2k + 1$, $G$ is $L$-colourable by Noel–Reed–Wu Theorem. Hence we may assume that $G = K_{3^{*}(k/2+1),1^{*}(k/2-1)} - \{e\}$ for some edge $e = xy$, say $x \in P_1$ and $y \in P_j$, $i < j$. If $j \geq k/2 + 2$, then $G$ is a subgraph of $K_{3^{*}(k/2+1),1^{*}(k/2-1)}$ which is $k$-choosable by Theorem 1.1. Hence, $G$ is $L$-colourable, a contradiction.

Thus we may assume $x = u_1$ and $y = u_2$. Since $C = 3k/2$, we have $|L(u_1) \cap L(u_2)| \geq k/2$ and $|L(v_i) \cap L(w_i)| \geq k/2$ for $i = 1, 2$. Let $L(u_1) \cap L(u_2) = A_1$, $L(v_1) \cap L(w_1) = A_2$ and $L(v_2) \cap L(w_2) = A_3$.

Note that $L(u_1) \subseteq C - A_2$ and $L(u_2) \subseteq C - A_3$. Hence $|A_2| = |A_3| = k/2$, $L(u_1) = C - A_2$ and $L(u_2) = C - A_3$. This implies that $|A_1 \cup A_2 \cup A_3| = |(C - A_2) \cap (C - A_3)| + |A_2 \cup A_3| = |C - (A_2 \cup A_3)| + |A_2 \cup A_3| = |C| > k$.

So there exists $c_i \in A_1$ such that $|L(u_k) - \{c_1, c_2, c_3\}| \geq k - 2$.

Now, we colour $u_1, u_2$ by colour $c_1$, colour $v_1, w_1$ by colour $c_2$ and colour $v_2, w_2$ by colour $c_3$. Let $L'(v) = L(v) - \{c_1, c_2, c_3\}$ for any vertex $v$ of $G' = G - P_1 - P_2$. 


It easy to verify that $G'$ and $L'$ satisfy the condition of Lemma 2.2 (for the verification of condition (t-2), if $k \geq 6$, then it is trivial and if $k = 4$, then we need to use the fact that $|L(u) \cap L(v)| \leq 2$ for any $u, v$ of 3-part $P$.) and hence $G'$ is $L'$-colourable, a contradiction. □

Corollary 1.2 follows from Theorems 5.1 and 5.2.

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