Analytic Solution of Strongly Coupling Schrödinger Equations

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A recently developed expansion method for analytically solving the ground states of strongly coupling Schrödinger equations by Friedberg, Lee and Zhao is extended to excited states and applied to the pedagogically important problems of power-law central forces. With the extended method, the Hydrogen atom problem is resolved and the low-lying states of Yukawa potential are approximately obtained.

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I. INTRODUCTION

The main task in application of quantum mechanics is to solve Schrödinger equations for various potentials. Unfortunately, only few of them can be solved exactly. In textbooks various approximation methods, especially the perturbation theory, are discussed with great length. While the perturbation theory, as the most important approximation method, has exhibited its power in a lot of weakly coupling cases, it is not applicable to strongly coupling problems which, however, exist widely and play important roles in the real world. How to analytically solve the Schrödinger equations in nonperturbative case remains an open question.

Recently, a new method to discuss strongly coupling potentials was presented by Friedberg, Lee and Zhao [1]. The goal is to solve the ground state of the time independent Schrödinger equation with strongly coupling potential

\[ \left( \frac{\hat{p}^2}{2m} + V \right) \psi = E \psi . \]  

(1)

Considering a boundary potential \( V(\vec{r}) \geq 0 \) with the minimum \( V(\vec{r}) = 0 \) at the origin point \( \vec{r} = 0 \), one introduces a dimensionless scalar parameter \( g \) which reflects the intensity of the potential by writing

\[ V(\vec{r}) = g^2 v(\vec{r}) . \]  

(2)

For strong potentials we are interested in, \( g \) is a large number, the ground state \( \psi \) and the corresponding ground energy \( E \) can be expanded in the inverse number of \( g \),

\[ \psi(\vec{r}) = e^{-S(\vec{r})}, \]

\[ S(\vec{r}) = g S_0(\vec{r}) + g^0 S_1(\vec{r}) + g^{-1} S_2(\vec{r}) + \cdots \]

\[ E = g E_0 + g^0 E_1 + g^{-1} E_2 + \cdots . \]  

(3)

Substituting the expansion into (1) and then comparing the coefficients of \( g^{-n} \) on the both sides, one obtains a series of first order differential equations (by contrast, the Schrödinger equation itself is of second order):

\[ (\nabla S_0)^2 = 2mv, \]

\[ \nabla S_0 \cdot \nabla S_1 = \frac{1}{2} \nabla^2 S_0 - mE_0, \]

\[ \nabla S_0 \cdot \nabla S_2 = \frac{1}{2} (\nabla^2 S_1 - \nabla S_1 \cdot \nabla S_1) - mE_1, \]

\[ \cdots \]  

(4)

where \( m \) is the particle mass. The first equation of the above hierarchy contains only \( S_0 \) as unknown term and can be solved by direct integration with boundary condition. The second equation contains only \( S_1 \) as unknown term (note that \( S_0 \) has been solved by the first equation) and can be solved in the same way as \( S_0 \), and \( E_0 \) can be fixed by considering the convergence of \( S_1 \) at the origin. With the same procedure, all \( S_i \) and \( E_i \) can be determined step by step. To illustrate the procedure clearly, one considers the familiar one dimension harmonic oscillator with \( V(x) = g^2 m^2 x^2 \). In this case the hierarchy (4) is reduced to
Using the boundary condition \( \psi(\pm \infty) = 0 \) or \( S_0(\pm \infty) = -\infty \) and the convenient renormalization \( \psi(0) = 1 \) or \( S_0(0) = 0 \), the integration of the first equation gives \( S_0(x) = -\frac{1}{\sqrt{2}} m^2 x^2 \). With the solved \( S_0 \), the second equation is reduced to \( S_1' = \left( \frac{1}{2} - \frac{E}{\sqrt{2} m^2} \right) / x \). To avoid divergence of \( S_1' \) at \( x = 0 \), the only way is to choose \( E_0 = \frac{m}{\sqrt{2}} \). With the same procedure, all \( S_i \) and \( E_i \) with \( i > 0 \) can be proven to be zero, and the final result gives precisely the same solution as that in any textbook. For details, see [1]-[3].

The readers who are familiar with the well-known WKB approximation may ask the difference between the WKB and the hierarchy (4). While the hierarchy (4) looks similar to the WKB hierarchy [4], they are very different from each other. The most explicit difference is that in WKB the energy \( E \) appears always in the first equation of the hierarchy together with the potential \( V \), but here \( E_0 \) and \( v \) may not appear in the first equation simultaneously, see (4). In WKB, the energy is determined by the continuity condition of wave functions at the so-called "classical turning point", or equivalently by using the Bohr-Sommerfeld like quantization condition. In the approach here, the energy eigenvalue is directly determined by the boundary condition. Physically, in WKB the wave function is expanded in \( h \), it is useful only for semiclassical systems, while the wave function and energy eigenvalue here are expanded in the inverse number of the strong coupling constant, the hierarchy (4) is like the expansion in the inverse number of large \( N \) often used in nonperturpative treatment of quantum field systems, see, for instance, [5]. Also the WKB wave function \( e^{i \frac{S}{\hbar}(x)} \) is an oscillating form which corresponds to real particles, so that more applicable to semiclassical case, but the wave function in (4) is a damping form and more suitable for dealing with bound states.

Although the new method is powerful in solving strongly coupling Schrödinger equations, the step of writing the wave function \( \psi(\vec{r}) \) in the form \( \psi(\vec{r}) = e^{S(\vec{r})} \) in (3) is a strong constraint in the application of the method to more kinds of potentials. For example, the linear potential \( V = gr \) cannot be solved through expansion (3) due to the divergence encountered in dealing with (4). From the analysis in Section 2 we will see that the expansion with the form of (3) is only suitable for harmonic-oscillator-like potentials which can be expanded quadratically around the minimum. On the other hand, the expansion (3) is valid mainly for ground states, an excited state can not be expressed as a pure exponential function.

To make the expansion method more applicable, one should construct carefully the expansion form of \( \psi(\vec{r}) \) through physical analysis of different potentials and of different states. In this paper, we first investigate the coupling constant dependence of bound states solutions for power-law central forces in order to construct proper expansions of \( \psi \) and \( E \) in inverse coupling constant in Sections 2 and 3, and then apply the extended expansion method to Coulomb potential and resolve the excited states in Section 4. Finally we discuss Yukawa potential and obtain the low-lying states approximately in Section 5. The Conclusions are given in Section 6. We have chosen \( \hbar = c = 1 \) throughout the paper.

II. COUPLING CONSTANT DEPENDENCE OF BOUND STATES SOLUTIONS

As mentioned above, the dependence of bound states solutions on coupling constant is crucial since the method is based on the coupling constant expansion. Qualitatively speaking, when coupling constant \( g \) in potentials like (1) increases, the bound state energy drops down and the space extension of the corresponding wave function becomes narrower, that is, the system is more bounded. In the following we concentrate our discussion on power-law central forces \( V(r) \propto g^k r^n \), the reason is twofold: first, with those potentials the coupling constant dependence can be factorized, as we will see below; second, any boundary potential closely around its minimum point can be approximated well by power-law curve with certain value of \( n \).

A. Coupling constant dependence of energy

Let’s consider a power-law central potential
\[ \hat{H} = \frac{\hat{p}^2}{2m} + V, \]
\[ V(r) = \pm g^k m(mr)^n. \]  \hspace{1cm} (6)

In order to keep boundary condition we should choose the sign + for \( n > 0 \) and − for \( n < 0 \). Here the coupling constant \( g \) is guaranteed to be dimensionless. For instance, the harmonic oscillator \( V = \frac{1}{2}m\omega^2r^2 \) can be written as \( V = g^k m^3 r^2 \) with \( g^k = \frac{1}{2} (\frac{\omega}{m})^2 \).

Considering \( g \) as a parameter in the Hamiltonian, we apply Hellmann-Feynman Theorem [4] for any arbitrary energy level \( E \) and the corresponding state \( |\psi\rangle \)

\[ \frac{\partial E}{\partial g} = \langle \psi | \frac{\partial \hat{H}}{\partial g} | \psi \rangle \]  \hspace{1cm} (7)

to (6) and obtain

\[ \frac{\partial E}{\partial g} = \frac{k}{g} \langle \psi | V | \psi \rangle. \]  \hspace{1cm} (8)

From Virial Theorem [4],

\[ \langle \psi | V | \psi \rangle = \frac{2}{n+2} \langle \psi | \hat{H} | \psi \rangle = \frac{2}{n+2} E, \]  \hspace{1cm} (9)

we get

\[ \frac{\partial E}{\partial g} = \frac{2k}{n+2} \frac{1}{g} E, \]  \hspace{1cm} (10)

and its solution

\[ E(g) = g^{\frac{2k}{n+2}} \epsilon, \]  \hspace{1cm} (11)

where \( \epsilon \) depends on \( r \) only, the coupling constant dependence is factorized. A useful consequence of this factorization is that for a given potential \( V(r) \) one can determine firstly \( \epsilon \) in some ideal case, for instance, the limit \( g \gg 1 \) or \( g \ll 1 \), and then obtain the energy \( E \) of the real system by multiplying \( \epsilon \) by the \( g \)-factor \( g^{\frac{2k}{n+2}} \).

When a potential is not in the power-law form, the factorization (11) fails. However, in strongly coupling cases the low-lying states are restricted in a small region around the origin \( r = 0 \), and the potential can be expressed in the power-law form

\[ \lim_{r \to 0} V(r) \propto \pm g^k m(mr)^n \]  \hspace{1cm} (12)
in the neighborhood of the origin, the factorization is still approximately valid, and the leading term \( E_0 \) in the expansion (3) is given by (11). This conclusion will be used to solve the Yukawa potential

\[ V(r) = -g^2 e^{-\alpha r} / r \]  \hspace{1cm} (13)
in Section 5.

B. Scale transformation

For central forces, the bound state wave functions can be written as [4]

\[ \psi(\vec{r}) = R(r)Y_{LM}(\theta, \phi), \]  \hspace{1cm} (14)

where \( Y_{LM}(\theta, \phi) \) is the spherical harmonic function carrying quantum numbers \( L \) and \( M \), and \( R(r) \) which is related to the coupling constant satisfies the radial equation

\[ \frac{d^2 R(r, g)}{dr^2} + \frac{2}{r} \frac{dR(r, g)}{dr} + 2m(E - V)R(r, g) - \frac{L(L+1)}{r^2} R(r, g) = 0. \]  \hspace{1cm} (15)
Considering a scale transformation
\[ r \rightarrow r_s = ar, \quad g \rightarrow g_s = bg, \]  
(16)
with a and b being two positive and real constants, and using the factorization (11) the radial equation (15) is transformed into
\[
\frac{1}{a^2} \frac{d^2 R(r_s, g_s)}{dr^2} + \frac{2}{a^2} \frac{dR(r_s, g_s)}{dr} + 2m(b \epsilon - b^k a^n V) R(r_s, g_s) - \frac{1}{a^2} \frac{L(L+1)}{r_s^2} R(r_s, g_s) = 0.
\]
(17)
If the constants a and b are restricted by
\[ b^{k+2} = b^k a^n = a^{-2}, \]
(18)
\[ R(r_s, g_s) \] and \[ R(r, g) \] satisfy the same radial equation and the same boundary conditions at the origin and at infinity (the transformation does not change boundary conditions), that is
\[ R(r_s, g_s) = c R(r, g), \]
(19)
where c is a \( g \) and \( r \) independent constant and can be removed by normalization.

The invariance of the equation of motion under the scale transformation (16) and (18) originates from the symmetry of the Hamiltonian (6),
\[ \hat{H} \rightarrow \hat{H}_s = \frac{1}{a^2} \hat{H}. \]
(20)
Under the transformation all eigenvalues of the Hamiltonian are rescaled by a constant factor \( 1/a^2 \) and all eigenfunctions are unchanged.

The scale transformation property of the radial function implies that we can replace the original two variables \( r \) and \( g \) by one dimensionless scale variable
\[ s = g^k (mr)^{n+2} \]
(21)
which is invariant under the scale transformation (16) and (18). With properly selected normalization scheme, the radial functions \( R(s_1) \) and \( R(s_2) \) with two different coupling constants \( g_1 \) and \( g_2 \) are related by
\[
R(g_1^k (mr)^{n+2}) = R \left( g_2^k \left( \frac{g_1}{g_2} \right)^k (mr)^{n+2} \right) = R \left( g_2^k \left( \frac{g_1}{g_2} \right)^{k/n+2} mr \right)^{n+2}. 
\]
(22)
Therefore, the wave function at point \( r \) with coupling constant \( g_1 \) equals the wave function at point \( (g_1/g_2)^{k/n+2} r \) with \( g_2 \). The wave function is contracted in a smaller region when the coupling becomes stronger while extended to a larger region when the coupling becomes weaker. With the relation (22), the behavior of a weakly coupling system at the place far from the origin is equivalent to that of a strongly coupling system near the origin.

The conclusion of this section can be summarized as
\[
V = \pm g^k m (mr)^n, \quad E = g^{\frac{2k+2}{n+2}} \epsilon, \quad R(r, g) = R(g^k (mr)^{n+2}). 
\]
(23)
For the potentials with power-law behavior (12) around the origin, the leading terms of the energy and wave function are also given by (23). This conclusion tells us how to write down the expansion in \( g \) for a power-law or a power-law-like central potential. For example, the expansion (3) is only suitable for harmonic oscillator and the similar potentials.
III. EXTENDED EXPANSION METHOD

The key point of the method by Friedberg, Lee and Zhao is to expand the energy and wave function of a strongly coupling system in the inverse number of the coupling constant \( g \). As mentioned above, the original expansion (3) is 1) only suitable for the potentials with harmonic oscillator behavior around the minimum, and 2) only valid for ground states. To solve (1) by the expansion method, we should proceed three steps: First, write a proper g-power expansion for energy and wave function according to the scaling property discussed in previous section; Second, substitute the expansion back into (1) and obtain a hierarchy of first-order differential equations; Last, integrate the hierarchy one by one with the help of the boundary and convergence conditions, as described in the example of one dimension harmonic oscillator in Section 1.

As a pedagogical goal, let’s extend the method to solve three dimensional Coulomb and harmonic oscillator problems in quantum mechanics.

In [1] and [2], the expansion and solution of the ground state are given by

\[
V = -g^2 \frac{1}{r} \\
E = g^4 E_0 + g^2 E_1 + g^0 E_2 + \cdots \\
S(r) = g^2 S_0(r) + g^0 S_1(r) + g^{-2} S_2(r) + \cdots
\]

for Coulomb potential, and

\[
V = \frac{1}{2} g^2 m^3 r^2 \\
E = gE_0 + g^0 E_1 + g^{-1} E_2 + \cdots \\
S(r) = gS_0(r) + g^0 S_1(r) + g^{-1} S_2(r) + \cdots
\]

for harmonic oscillator. It is clear that the expansions for the two potentials are in accord with the scaling law shown in (23).

Since any excited state is not so bounded like the ground state, we assume that the excited states differ from the ground state \( e^{-S(r)} \) by a smooth function \( P(r) \),

\[
R(r) = P(r) e^{-S(r)}.
\]

When \( P(r) = 1 \) we go back to the ground state. From the spirit of the expansion method we express \( P(r) \) and \( S(r) \) as finite polynomials of \( 1/g \). With the help of the scaling law (23), they can be further rewritten as polynomials of a dimensionless scaling variable \( g^2 mr \) for Coulomb potential

\[
S(r) = \sum_{j=1}^{\alpha} b_{1-j}(g^2 mr)^j,
\]

\[
P(r) = \sum_{i=\beta}^{\gamma} a_i(g^2 mr)^i,
\]

\[
E = g^4 \epsilon,
\]

or \( gm^2 r^2 \) for harmonic oscillator

\[
S(r) = \sum_{j=1}^{\alpha} b_{1-j}(gm^2 r^2)^j.
\]
\[ P(r) = \sum_{i=\beta}^{\gamma} a_i (g m^2 r^2)^i, \]
\[ E = g \epsilon. \]  

(28)

Here \( \alpha, \beta, \gamma \) are nonnegative integers, and \( a_i, b_{i-j} \) are \( g \) and \( r \) independent constants. Note that the expressions (27) and (28) obtained from the scaling analysis guarantee a wider spread of excited states than the ground state. The expansions here are also in accordance with the discussion with algebra method [6].

We now substitute the expansions into the corresponding radial equations and discuss the convergence in the limit \( r \to 0 \). It is easy to prove that when \( \alpha > 0 \) the most divergent term in the radial equation is \( \alpha^2 g^{-2 \alpha} b_{\alpha+1}^2 r^{-2(\gamma+2\alpha+2)} \) for Coulomb potential and \( 4 \alpha^2 g^{-2 \alpha} b_{\alpha+1}^2 r^{-2(\gamma+2\alpha+1)} \) for harmonic oscillator. Since there is only one most divergent term, it is impossible to cancel it by other terms. The only way is to take \( \alpha = 0 \). In this case, the expansion of \( S(r) \) in (27) or (28) is reduced to two terms, and the constant term can be absorbed by the normalization of the wave function. Moreover, the terms with negative powers \( (\gamma > 0) \) in the expansion of \( P(r) \) must vanish also in order to keep the whole radial wave function \( R(r) = P(r) e^{-S(r)} \) from divergence at the origin \( r = 0 \). Finally we obtain for any bound state the extended expansion

\[ S(r) = b_0 g^2 m r, \]
\[ P_N(r) = \sum_{i=0}^{N-1} a_i (g^2 m r)^i \]

(29)

for Coulomb potential, and

\[ S(r) = b_0 g^2 m^2 r^2, \]
\[ P_N(r) = \sum_{i=0}^{N-1} a_k (g m^2 r^2)^i \]

(30)

for harmonic oscillator with \( N \geq 1 \).

IV. SOLUTIONS FOR COULOMB POTENTIAL

We now turn to the second step, namely substitute the convergent expansion (29) into the radial equation for Coulomb potential, and get a series of equations in different orders of \( g \):

\[ g^{2N+2} : \quad b_0^2 m + 2 \epsilon = 0 \]
\[ g^{2N} : \quad 1 - b_0 N = 0 \]
\[ g^{2k}(0 < k < N) : (k(k+1) - L(L+1)) a_k + \frac{2}{N}(N-k)a_{k-1} = 0 \]
\[ g^0 : \quad L(L+1)a_0 = 0 \]

(31)

From the first two equations we derive immediately the energy level \( \epsilon_N = -\frac{m}{2N^2} \) and the coefficient \( b_0 = \frac{1}{N} \) in \( S \).

To obtain the coefficients \( a_i \) in the polynomial \( P \), we need to discuss the relation between the two quantum numbers \( N(\geq 1) \) and \( L(\geq 0) \), they come from the polynomial \( P_N(r) \) and the spherical harmonic function \( Y_{LM}(\theta, \phi) \). For \( N = 1 \), the requirement \( P_1(r) = a_0 \neq 0 \) leads to \( L = 0 \) from the last equation of (31), the general expression (29) is then reduced to the ground state solution (24).

For \( N > 1 \), we discuss three cases for the quantum number \( L \) separately: 1) \( L = 0 \), the third equation of (31) determines the recursion relation between the coefficients \( a_k \)

\[ R_{N0} = \left( \sum_{k=0}^{N-1} a_k (g^2 m r)^k \right) e^{-g^2 m r / N} \]
\[ a_k = -\frac{2(N-k)}{Nk(k+1)} a_{k-1}, \quad 0 < k < N, \]

(32)

the only unknown coefficient \( a_0 \neq 0 \) is determined by the normalization. 2) \( 0 < L < N \), the last two equations of (31) lead to the coefficients \( a_0 = a_1 = a_2 = \cdots = a_{L-1} = 0 \), and then the radial solutions are written as
\[ R_{NL} = \left(\sum_{k=L}^{N-1} a_k (g^2 m r)^k \right) e^{-g^2 mr/N} \]
\[ a_k = \frac{2}{N L(L+1)} \frac{N-k}{k(k+1)} a_{k-1}, \quad L < k < N \] (33)

\( a_L \neq 0 \) is determined by the normalization too. 3) \( L \geq N \), the last two equations of (31) require \( a_0 = a_1 = a_2 = \cdots = a_{N-1} = 0 \), and there is no nonzero solution in this case.

We summarize the solution of the bound states for Coulomb potential:

\[ E_N = -\frac{1}{2N^2 g^4 m} \]
\[ \psi_{NLM}(r, \theta, \phi) = \left(\sum_{k=L}^{N-1} a_k (g^2 m r)^k \right) e^{-g^2 mr/N} Y_{LM}(\theta, \phi) \]
\[ a_k = \frac{2}{N L(L+1)} \frac{N-k}{k(k+1)} a_{k-1}, \quad N > 0, \quad L = 0, 1, 2, \cdots, N - 1 \] (34)

with \( a_L \neq 0 \) determined by the normalization of the wave function. (34) is exactly the same as what obtained by solving the second-order Schrödinger equation directly in normal textbooks [4], but here by using the expansion in coupling constant and with the help of the scaling law the complicated second-order differential equations are replaced by simple algebra equations.

In a similar way one can also derive all the bound states for harmonic oscillator by substituting the convergent expansion (30) into the radial equation.

V. YUKAWA POTENTIAL

The potential (13) was first introduced into physics in 1930s by Yukawa in the study of strong interaction between nucleons through meson exchange [7]. It is also known as Debye-Hückel potential in plasma physics and Thomas-Fermi potential in solid-state physics [8]. While the Yukawa potential is important in physics, the corresponding Schrödinger equation can not be solved analytically and exactly. A lot of publications have been contributed to the problem using different approximations (see [9] - [12] and references therein). Here we use the extended expansion method to obtain its ground state and low-lying excited states. Comparison with strict but numerical calculation will be made.

We first consider the proper \( 1/g \) expansion for Yukawa potential. Since Yukawa potential is not a power-law central force, the \( g \) dependence of its bound states can not be simply factorized like (23). In strong coupling case with \( g >> 1 \), however, the low-lying wave functions are mainly distributed in the vicinity of the origin. Since in the limit of \( r \to 0 \), Yukawa potential approaches to Coulomb potential, we can thus write the expansion of the low-lying states of Yukawa potential by properly modifying the expansion of Coulomb potential. The inverse of the parameter \( \alpha \) in the Yukawa potential (13) represents the mean potential range. To make our calculation based on Coulomb potential( whose potential range approaches infinity) more effective, we should require the mean potential range to be much larger than the "Bohr radius" of Coulomb potential, that is, \( \frac{1}{\alpha} >> \frac{1}{g \xi m} \), or equivalently \( \frac{\alpha}{g^2 m} << 1 \). We will show below how the dimensionless number \( \frac{1}{g \xi m} \) controls the expanding series.

We now deal with the ground state. In the light of (24) for Coulomb potential, we expand the energy and wave function as

\[ E = g^4 E_0 + g^2 E_1 + g^0 E_2 + \cdots \]
\[ R = e^{-S(r)} \]
\[ S(r) = g^2 S_0(r) + g^0 S_1(r) + g^{-2} S_2(r) + \cdots \] (35)

Note that the expansions of \( E \) and \( S \) are finite for strict Coulomb potential but infinite for Coulomb-like potentials.

By Substituting the expansions (35) into the corresponding stationary Schrödinger equation, we obtain an infinity hierarchy of first-order differential equations (5). With similar procedure in dealing (5), we can solve them one by one to any order we want. The energy obtained to \( \mathcal{O}(g^{-4}) \) and the wave function to \( \mathcal{O}(g^{-2}) \) read:
\[ E = -\frac{1}{2}g^4m + g^2\alpha - \frac{3\alpha^2}{4m} + g^{-2}\frac{\alpha^3}{2m^2} + \mathcal{O}(g^{-4}) = g^4m[-\frac{1}{2} + \frac{\alpha}{g^2m} - \frac{3}{4}(\frac{\alpha}{g^2m})^2 + \frac{1}{2}(\frac{\alpha}{g^2m})^3 + \mathcal{O}((\frac{\alpha}{g^2m})^4)] , \]

\[ S(r) = g^2mr + \int_0^r dr' \frac{1}{r}(1 - e^{-ar'}) - \alpha + \mathcal{O}(g^{-2}) = g^2m[r + \frac{\alpha}{g^2m} \int_0^r dr' \frac{1}{\alpha r'} (1 - e^{-ar'}) - 1] + \mathcal{O}((\alpha/g^2m)^2)]. \]  

We see clearly that the dimensionless parameter \( \alpha/g^2m \) controls the degree of convergence of the expansion. When \( \alpha \) vanishes, Yukawa potential is reduced to Coulomb potential, and therefore, the above solution is reduced to the strict solution (24).

We now consider the lowest excited state. By modifying the expansion (29) with \( N = 2 \) for Coulomb potential, we write the following expansion

\[ E = g^4E_0 + g^2E_1 + g^0E_2 + \cdots \]

\[ \psi = P(r)e^{-\frac{S(r)}{2}}Y_L(\theta, \phi) \]

\[ P(r) = g^2b_1(r) + g^0b_0(r) \]

\[ S(r) = g^2S_0(r) + g^0S_1(r) + g^{-2}S_2(r) + \cdots \]  

(37)

Unlike the expansion for strict Coulomb potential, here \( E \) and \( S(r) \) have infinite terms, and the \( r \)-dependence of \( b_0 \) and \( b_1 \) in \( P(r) \) is unknown and only in the limit case we have:

\[ \lim_{r \to 0} b_1(r) = \lim_{\alpha \to 0} b_1(r) \propto mr , \]

\[ \lim_{r \to 0} b_0(r) = \lim_{\alpha \to 0} b_0(r) \propto 1 . \]  

(38)

Substituting the expansions (37) into the radial equation with \( N = 2 \), we get again a series of first-order differential equations. The first two which are respectively proportional to \( g^6 \) and \( g^4 \) read

\[ \frac{dS_0}{dr} = \sqrt{-2mE_0} , \]

\[ \frac{dS_0}{dr} \frac{dS_1}{dr} = \frac{1}{b_1} \frac{dS_0}{dr} + \frac{1}{r} \left( \frac{dS_0}{dr} - me^{-ar} \right) - mE_1 . \]  

(39)

By using (38) the condition to keep convergence of \( dS_0/dr \) and \( dS_1/dr \) at \( r = 0 \) result in

\[ E_0 = -\frac{m}{8} , \]

\[ \frac{dS_0}{dr} = \frac{m}{2} , \]

\[ \frac{dS_1}{dr} = \frac{1 - 2e^{-ar}}{r} + \frac{1}{b_1} \frac{db_1}{dr} - 2E_1 . \]  

(40)

In a similar way we can solve the equations proportional to \( g^2 \) and \( g^0 \) to determine \( E_1 \) and \( S_1 \)

\[ E_1 = \frac{6 + L(L + 1)}{8} \alpha , \]

\[ b_0 = 1 - L , \]

\[ b_1 = -\frac{m}{2\alpha}(1 - e^{-ar}) . \]  

(41)

Put the constituents obtained above together we finally get approximately the analytical solution of the lowest excited state for Yukawa potential as follows

\[ E = -\frac{m}{8}g^4 + \frac{6 + L(L + 1)}{8} g^2\alpha + \mathcal{O}(g^0) = -\frac{g^4m}{8} \left[ 1 - \frac{\alpha}{g^2m} (L^2 + L + 6) + \mathcal{O}((\frac{\alpha}{g^2m})^2) \right] , \]

\[ S(r) = g^2m \left[ \frac{r}{2} + \frac{\alpha}{g^2m} \int_0^r dr' \frac{1 - 2e^{-ar'}}{ar'} + \frac{e^{-ar'}}{1 - e^{-ar}} - \frac{L^2 + L + 6}{4} \right] + \mathcal{O}((\frac{\alpha}{g^2m})^2) , \]

\[ P(r) = a_0 \left( 1 - L \right) - \frac{1}{2} \frac{g^2m}{\alpha}(1 - e^{-ar}) , \]

\[ N = 2, \quad L = 0, 1 . \]  

(42)
the only constant $a_0$ is determined by normalization. Again, the dimensionless parameter $\alpha/g^2m$ dominates the convergence. The analytic result $E$ for ground and lowest excited states is compared with the strict but numerical result $E_N$ and with the Coulomb result $E_C$ in Table (1) for different values of the dimensionless parameter $\alpha/g^2m$. All the energy values have been scaled by $g^4m$. Since the ground state energy is calculated to the fourth order, it always agrees well with the strict solution. The Coulomb energy is just the leading order of the Yukawa energy, its deviation from the strict result is very strong, especially for the excited states. then relatively large. When the parameter $\alpha/g^2m$ decreases, the analytic result to the second order for the excited states becomes more and more close to the strict solution.

We also compared our energy eigenvalues with the often cited numerical calculations in [12]. The deviation of our analytical result from the high accurate numerical result is less than 2.5% for 1s state with $\frac{\alpha}{g^2m} < 1/3$, 5% for 2s state with $\frac{\alpha}{g^2m} < 1/30$, and 5% for 2p state with $\frac{\alpha}{g^2m} < 1/20$. The deviation decreases when the dimensionless parameter $\alpha/g^2m$ becomes smaller.

The other excited states ($N = 3, 4, \cdots$) can be obtained with the similar method.

| $\frac{\alpha}{g^2m}$ | 1/5 | 1/10 | 1/50 |
|------------------------|-----|------|------|
| state                  | 1s  | 2s   | 2p   | 1s  | 2s   | 2p   | 1s  | 2s   | 2p   |
| $E_N/g^4m$             | -0.3287 | -0.0121 | -0.0041 | -0.4071 | -0.0476 | -0.0465 | -0.4803 | -0.1062 | -0.1050 |
| $E_C/g^4m$             | -0.5000 | -0.1250 | -0.1250 | -0.5000 | -0.1250 | -0.1250 | -0.5000 | -0.1250 | -0.1250 |
| $E/g^4m$               | -0.3260 | 0.0250 | 0.0750 | -0.4070 | -0.0500 | -0.0250 | -0.4803 | -0.1100 | -0.1050 |

FIG. 1. Comparison of our analytic result $E$ of Yukawa potential to the fourth order for the ground state and to the second order for the lowest excited states with the strict but numerical result $E_N$ and with the Coulomb result $E_C$ for three values of the dimensionless parameter $\alpha/g^2m$.

The solutions (36) and (42) have the following features:
1) The leading orders are just the solutions of Coulomb potential, and therefore are invariant under the scale transformation (16). The high order corrections break the scale invariance explicitly. When $\alpha \rightarrow 0$ or $r \rightarrow 0$, the high order corrections approach to zero and the scale invariance is restored.
2) Different from the accidental degeneracy for Coulomb potential, the energy level for Yukawa potential depends on the quantum number $L$.
3) The expanding series are controlled by the combined dimensionless parameter $\frac{\alpha}{g^2m}$. For sufficiently small values, namely for very strong interaction, the expansions (36) and (42) converge very fast, one can consider the lower-order contributions only.

VI. SUMMARY

The recently developed expansion method [1] - [3] provides an alternative way to solve analytically Schrödinger equations with strong coupling. The key point of the method is to construct proper expansions for different potentials and different quantum states. Through investigating the scale transformation invariance of the bound states for power-law central forces, we have extended the expansion method from ground state to any excited state for power-law central forces and to low-lying states for power-law-like central forces. With the extended expansion method, we obtained analytically the strict solutions of all bound states for Coulomb potential and the approximate solutions of low-lying states for Yukawa potential. Further application of the method to other physical potentials is of interest and value.

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