INVERTIBILITY OF CONVOLUTION OPERATORS ON HOMOGENEOUS GROUPS

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Abstract

We say that a tempered distribution $A$ belongs to the class $S^m(g)$ on a homogeneous Lie algebra $g$ if its Abelian Fourier transform $a = \hat{A}$ is a smooth function on the dual $g^*$ and satisfies the estimates

$$|D^\alpha a(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}.$$  

Let $A \in S^0(g)$. Then the operator $f \mapsto f \ast \tilde{A}(x)$ is bounded on $L^2(g)$. Suppose that the operator is invertible and denote by $B$ the convolution kernel of its inverse. We show that $B$ belongs to the class $S^0(g)$ as well. As a corollary we generalize Melin’s theorem on the parametrix construction for Rockland operators.

In a former paper [10] we describe a calculus of a class of convolution operators on a nilpotent homogeneous group $G$ with the Lie algebra $g$. These operators are distinguished by the conditions imposed on the Abelian Fourier transforms of their kernels similar to those required from the $L^p$-multipliers on $\mathbb{R}^n$. More specifically, a tempered distribution $A$ belongs to the class $S^m(G) = S^m(g)$ if its Fourier transform $a = \hat{A}$ is a smooth function on the dual to the Lie algebra $g^*$ and satisfies the estimates

$$|D^\alpha a(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}, \quad \xi \in g^*.$$  

In [10] we follow and extend to the setting of a general homogeneous group the ideas of Melin [14] who first introduced such a calculus on the subclass of stratified groups. The classes $S^m(g)$ of convolution operators have the expected properties of composition and boundedness (see Propositions 1.1 and 1.2 below) which is a generalization of the results of Melin [14]. However, a complete calculus should also deal with the problem of invertibility. The aim of the present paper is to fill the gap.

Suppose that $A \in S^0(g)$. Then, by the boundedness theorem (see Proposition 1.2 below), the operator

$$f \mapsto f \ast \tilde{A}(x) = \int_g f(xy)A(y) dy$$

defined initially on the Schwartz class functions extends uniquely to a bounded operator on $L^2(g)$. Furthermore, suppose that the operator $f \mapsto f \ast \tilde{A}$ is invertible on $L^2(g)$ and denote by $B$ the convolution kernel of its inverse. We show here that under these circumstances $B$ belongs to the class $S^0(g)$ as well. This is done by replacing Melin’s techniques of parametrix construction involving the more refined classes $S^{m,s}(g) \subset S^m(g)$ of convolution operators by the calculus of less restrictive classes $S^0(g)$, where no estimates in the central directions are required.

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Let us remark that the described result can be also looked upon as a close analogue of the theorem on the inversion of singular integrals, see [9] and Christ-Geller [3].

By using auxiliary convolution operators, namely accretive homogeneous kernels $P^m$ smooth away from the origin, we construct "elliptic" operators $V^m$ of order $m > 0$ and get inversion results for classes $S^m(g)$ for all $m > 0$, which enables us to generalize Melin’s theorem on the parametrix construction for Rockland operators. At the same time, however, we present a direct parametrix construction for Rockland operators which avoids the machinery of Melin and also that of the present paper and depends only on well-known properties of Rockland operators as derived in Folland-Stein [7] and the calculus of [10].

We believe that the presented symbolic calculus may be a step towards a more comprehensive pseudodifferential calculus on nilpotent Lie groups parallel to that of Christ-Geller-Głowacki-Polin [4].

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1. Symbolic calculus.

Let $g$ be a nilpotent Lie algebra endowed with a family of dilations $\{δ_t\}_{t>0}$. We identify $g$ with the corresponding nilpotent Lie group by means of the exponential map. Let

$$1 = p_1 < p_2 < \cdots < p_d$$

be the exponents of homogeneity of the dilations. Let $|\cdot|$ be a homogenous norm on $V$. Let

$$g_j = \{x \in g : tx = t^{p_j} \cdot x\}, \quad 1 \leq j \leq d.$$ Denote by $Q = \sum_k \dim g_k \cdot p_k$ the homogeneous dimension of $g$.

Let $|\cdot|$ be a homogenous norm on $g$. Let

$$\rho(x) = 1 + |x|.$$ A similar notation will be applied for the dual space $g^*$. In expressions like $D^\alpha$ or $x^\alpha$ we shall use multiindices

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d),$$

where

$$\alpha_k = (\alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{kn_k}),$$

are themselves multiindices with positive integer entries corresponding to the spaces $g_k$ or $g^*_k$. The homogeneous length of $\alpha$ is defined by

$$|\alpha| = \sum_{k=1}^d |\alpha_k|, \quad |\alpha_k| = \dim g_k \cdot p_k.$$ As usual we denote by $S(g)$ or $S(g^*)$ the Schwartz classes of smooth and rapidly vanishing functions. The Fourier transform

$$\hat{f}(\xi) = \int_g f(x) e^{-i(\xi, x)} \, dx$$
maps $S(g)$ onto $S(g^*)$ and extends to tempered distributions on $g$. Let

$$\|f\|^2 = \int_g |f(x)|^2 \, dx, \quad f \in L^2(g).$$

A similar notation will be applied to $f \in L^2(g^*)$, where the Lebesgue measure $d\xi$ on $g^*$ is normalized so that

$$\int_g |f(x)|^2 \, dx = \int_{g^*} |\hat{f}(x)|^2 \, d\xi.$$

The algebra of bounded linear operators on $L^2(g)$ will be denoted by $B(L^2(g))$.

For a tempered distribution $A$ on $g$, we write

$$\text{Op}(A) f(x) = f \star \tilde{A}(x) = \int_g f(xy) A(dy), \quad f \in S(g).$$

Let $m \in \mathbb{R}$. By $S^m(g) = S^m(g, \rho)$ we denote the class of all distributions $A \in S'(g)$ whose Fourier transforms $a = \hat{A}$ are smooth and satisfy the estimates

$$|D^\alpha a(\xi)| \leq C_\alpha \rho(\xi)^{m-|\alpha|},$$

where $|\alpha|$ stands for the homogeneous length of a multi-index. Let us recall that $S^m(g)$ is a Fréchet space with the family of norms

$$|a|_\alpha = \sup_{\xi \in g^*} |\rho(\xi)^{m+|\alpha|} D^\alpha a(\xi)|.$$

It is not hard to see that for every $\varphi \in C_0^\infty(g)$ equal to 1 in a neighbourhood of 0 the distribution $(1 - \varphi)A$ is a Schwartz class function. Thus

$$A = A_1 + F,$$

where $A_1$ is compactly supported and $F \in S(g)$.

It follows from (1.2) that for every $m \in \mathbb{R}$

$$\text{Op}(A) : S(g) \to S(g)$$

is a continuous mapping if $A \in S^m(g)$. Therefore, it extends to a continuous mapping denoted by the same symbol of $S'(g)$. It is also clear that for $A \in S^m(g)$ and $B \in S^n(g)$ the convolution $A \star B$ makes sense and $\text{Op}(A \star B) = \text{Op}(A) \text{Op}(B)$.

The following two propositions have been proved in [10].

**Proposition 1.1.** If $A \in S^m(g)$ and $B \in S^n(g)$, then $A \star B \in S^{m+n}(g)$ and the mapping

$$S^m(g) \times S^n(g) \ni (A, B) \mapsto A \star B \in S^{m+n}(g)$$

is continuous.

**Proposition 1.2.** If $A \in S^0(g)$, then $\text{Op}(A)$ is bounded on $L^2(g)$ and the mapping

$$S^0(g) \ni A \mapsto \text{Op}(A) \in B(L^2(g))$$

is continuous.

Let $\mathfrak{z}$ be the central subalgebra corresponding to the largest eigenvalue of the dilations. We may assume that

$$g = g_0 \times \mathfrak{z}, \quad g^* = g_0^* \times \mathfrak{z}^*,$$

(1.3)
where \( g_0 \) may be identified with the quotient Lie algebra \( g/\mathfrak{j} \). The multiplication law in \( g \) can be expressed by

\[
(x, t)(y, s) = (x \circ y, t + s + r(x, y)),
\]

where \( x \circ y \) is multiplication in \( g_0 \). Here the variable in \( g \) has been split in accordance with the given decomposition. In a similar way we also split the variable \( \xi = (\eta, \lambda) \) in \( g^* \).

Let \( m \in \mathbb{R} \). By \( S^m_0(g^*) \) we denote the class of all distributions \( A \in S'(g) \) whose Fourier transforms \( a = \hat{A} \) are smooth in the variable \( \eta \) and satisfy the estimates

\[
|D^\alpha_\eta a(\eta, \lambda)| \leq C_\alpha \rho(\eta, \lambda)^{m-|\alpha|}.
\]

(1.4)

Again, \( S^m_0(g) \) is a Fréchet space with the family of norms

\[
|a|_\alpha = \sup_{(\eta, \lambda) \in g^*} \rho(\eta, \lambda)^{-m+|\alpha|}|D^\alpha_\eta a(\eta, \lambda)|.
\]

The following result has not been stated explicitly in [10] but follows by the argument given there.

**Proposition 1.3.** If \( A \in S^m_0(g^*) \) and \( B \in S^n_0(g^*) \), then \( A \ast B \in S^{m+n}_0(g^*) \) and the mapping

\[
S^m_0(g^*) \times S^n_0(g^*) \ni (A, B) \mapsto A \ast B \in S^{m+n}_0(g^*)
\]

is continuous.

Let us introduce the following notation:

\[
\widehat{f \# g}(\xi) = \hat{f} \ast \hat{g}(\xi), \quad \xi \in g^*;
\]

for \( f, g \in S(g) \). Then, for every fixed \( \lambda \in g^* \),

\[
a \# b(\eta, \lambda) = a(\cdot, \lambda) \# b(\cdot, \lambda)(\eta),
\]

(1.5)

where

\[
\widehat{f \# g}(\eta) = (f \ast \lambda g)(\eta), \quad f \ast \lambda g(x) = \int_{g_0} f(x \circ y^{-1}) g(y) e^{i \langle r(x, y^{-1}), \lambda \rangle} dy
\]

for \( f, g \in S(g_0) \). In particular, \( f \ast_0 g \) is the usual convolution on the quotient group \( g_0 \).

Let

\[
T_k F(x) = x_k F(x), \quad T_\alpha F(x) = x^\alpha F(x).
\]

For a given multiindex \( \gamma \), let

\[
k(\gamma) = \max_{1 \leq k \leq d} \{k : \gamma_k \neq 0\},
\]

and

\[
\mathcal{P}(\gamma) = \{\alpha : \alpha_k = 0, k \geq k(\gamma)\}.
\]

**Lemma 1.4.** Let \( f, g \in S(g) \). Then for every \( \gamma \),

\[
T_\gamma (f \ast g) = T_\gamma f \ast g + f \ast T_\gamma g + \sum_{\alpha, \beta \in \mathcal{P}(\gamma), |\alpha| + |\beta| = |\gamma|} c_{\alpha, \beta}^{\gamma} T_\alpha f \ast T_\beta g.
\]
By applying the Fourier transform, we obtain
\[
D^\gamma(f \# g) = D^\gamma f \# g + f \# D^\gamma g + \sum_{\alpha, \beta \in P(\gamma), |\alpha| + |\beta| = |\gamma|} c_{\alpha \beta}^\gamma D^\alpha f \# D^\beta g
\]  
(1.6)
for \( f, g \in S(g^*) \).

**Lemma 1.5.** Let \( A \in S^m(g) \). If \( B \in S^{-m}_0(g) \) is the inverse of \( A \), that is,
\[
A \star B = B \star A = \delta_0,
\]
then \( B \in S^m(g) \).

**Proof.** Let \( a = \hat{A} \), \( b = \hat{B} \). By (1.6),
\[
0 = D^\gamma(a \# b) = D^\gamma a \# b + a \# D^\gamma b + \sum c_{\alpha \beta}^\gamma D^\alpha a \# D^\beta b,
\]
where the summation extends over \( \alpha, \beta \) such that
\[
|\alpha| + |\beta| = |\gamma|, \quad |\alpha_d|, |\beta_d| < |\gamma_d|
\]
and every multiindex is split as \( \alpha = (\alpha', \alpha_d) \), \( \alpha_d \) being the part corresponding to \( g^*_d \). Therefore,
\[
D^\gamma b = -b \# D^\gamma a \# b + \sum c_{\alpha \beta}^\gamma b \# D^\alpha a \# D^\beta b,
\]
where the symbol on the right-hand side belongs to \( \hat{S}^{-m-\kappa}_0 \) provided that \( b \in \hat{S}^{-m-\kappa}_0 \) for \( \kappa < |\gamma_d| \). By induction, \( D^\gamma b \in \hat{S}^{-m-|\gamma_d|}_0(g) \), which is our assertion. \( \square \)

Let \( A_j \in S^{m_j}_0(g^*) \), where \( m_j \searrow -\infty \). Then there exists a distribution \( A \in S^{m_1}_0(g^*) \) such that
\[
A - \sum_{j=1}^N A_j \in S^{m_N+1}_0(g^*)
\]
for every \( N \in \mathbb{N} \). The distribution \( A \) is unique modulo the class
\[
S^{-\infty}_0(g^*) = \bigcap_{n<0} S^n_0(g^*).
\]

We shall write
\[
A \approx \sum_{j=1}^{\infty} A_j,
\]  
(1.7)
and call the distribution \( A \) the asymptotic sum of the series \( \sum A_j \) (cf., e.g., Hörmander [13], Proposition 18.1.3).

We say that \( A \in S^m(g) \), where \( m \geq 0 \), has a parametrix \( B \in S^{-m}(g) \) if
\[
B \star A = \delta_0 \in S(g), \quad A \star B = \delta_0 \in S(g),
\]
where \( \delta_0 \) stands for the Dirac delta at 0. If \( B_1 \) is a left-parametrix and \( B_2 \) a right one, then it is easy to see that \( B_1 = B_2 \) modulo the Schwartz class functions so both \( B_1 \) and \( B_2 \) are parametrices. In particular, if \( A \) is symmetric, then either of the conditions implies the other one.
2. Sobolev spaces

We shall say that a tempered distribution $T$ is a regular kernel of order $r \in \mathbb{R}$, if it is homogeneous of degree $-Q - r$ and smooth away from the origin. A symmetric distribution $T$ is said to be accretive, if

$$\langle T, f \rangle \geq 0$$

for real $f \in C_0^\infty(g)$ which attain their maximal value at 0. Such a $T$ is an infinitesimal generator of a continuous semigroup of subprobability measures $\mu_t$. By the Hunt theory (see, eg., Duflo [5]), $T = \text{Op}(T)$ is a positive selfadjoint operator on $L^2(g)$ with $S(g)$ as its core domain and for every $0 < m < 1$

$$\text{Op}(T)^m = \text{Op}(T^{m}), \quad \langle T^{m}, f \rangle = \frac{1}{\Gamma(-m)} \int_0^\infty t^{-1-m} \langle \delta_0 - \mu_t, f \rangle \, dt,$$

where the distribution $T^{m}$ is also accretive.

Let $T$ be a fixed symmetric accretive regular kernel of order $0 < m \leq 1$. Then there exists a symmetric nonnegative function $\Omega \in C^\infty(g \setminus \{0\})$ which is homogeneous of degree 0 such that

$$\langle T, f \rangle = cf(0) + \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \left( f(0) - f(x) \right) \frac{\Omega(x) \, dx}{|x|^{Q+1}},$$

where $c \geq 0$. If $c = 0$, $T$ is an infinitesimal generator of a continuous semigroup of probability measures with smooth densities. For every $0 < a < 1$, $T^a$ is also a symmetric regular kernel of order $am$.

Let

$$\langle P, f \rangle = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{f(0) - f(x) \, dx}{|x|^{Q+1}}$$

be a fixed symmetric accretive distribution of order 1. Let us warn the reader that the distributions $P^m$ do not belong to any of the classes $S^m(g)$ as they do not vanish rapidly at infinity which is a certain technical complication. That is why we introduce the truncated kernels

$$V_0 = I, \quad V_m = \varphi P^m, \quad m > 0,$$

where $\varphi$ is a symmetric nonnegative $[0, 1]$-valued smooth function with compact support and equal to 1 on the unit ball. Thus defined $V_m \in S^m(g)$ is also accretive and it differs from $P^m$ by a finite measure. Therefore, for every $0 < m \leq 1$, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \| (I + \text{Op}(P))^m f \| \leq \| (I + \text{Op}(V_m)) f \| \leq C_2 \| (I + \text{Op}(P))^m f \|, \quad (2.1)$$

for $f \in S(g)$.

**Proposition 2.1.** For every $0 < m \leq 1$, there exists a constant $C_m > 0$ such that

$$\| f \ast V_m \| \geq C_m \| f \|, \quad f \in S(g).$$
Proof. In fact, let \( f \in \mathcal{S}(g) \) and \( F = \tilde{f} \ast f \). Then

\[
\langle f \ast V_m, f \rangle = \langle T, \tilde{F} \rangle = \lim_{\varepsilon \to 0} \int_{|x| \leq \varepsilon \leq 1} \left( F(0) - \varphi(x)F(x) \right) \frac{\Omega_m(x)}{|x|^{Q+1}} \, dx + \int_{|x| \geq 1} \frac{\Omega_m(x)}{|x|^{Q+1}} \, dx \geq C_m^2 F(0) = C_m^2 \| f \|^2
\]

since the first integral is nonnegative.

It follows from (2.1) and Proposition 2.1 that there exist new constants \( C_1 > 0 \) and \( C_2 > 0 \) such that

\[
C_1 \|(I + \text{Op}(P))^m f\| \leq \|\text{Op}(V_m)f\| \leq C_2 \|(I + \text{Op}(P))^m f\|,
\]

(2.2)

for \( f \in \mathcal{S}(g) \) and \( 0 \leq m \leq 1 \).

Recall from [8] that \( P \) is maximal, that is, for every regular symmetric kernel \( T \) of arbitrary order \( m > 0 \) there exists a constant \( C > 0 \) such that

\[
\|f \ast \tilde{T}\| \leq C \|f \ast P^m f\|, \quad f \in \mathcal{S}(g).
\]

(2.3)

We introduce a scale of Sobolev spaces. For every \( m \in \mathbb{R} \)

\[
H(m) = \{ f \in L^2(g) : (I + \text{Op}(P))^m f \in L^2(g) \}
\]

with the usual norm \( \|f\|_{(m)} = \|(I + \|\text{Op}(P)^m f\|_2 \). The dual space to \( H(m) \) can be identified with \( H(-m) \). By (2.2), the norms defined by \( V_m \) for \( 0 < m \leq 1 \) are equivalent. It follows that for every \( 0 \leq m \leq 1 \)

\[
V_m : H(m) \to H(0)
\]

is an isomorphism.

3. Main step

Here comes a preliminary version of our theorem.

**Proposition 3.1.** Let \( 0 \leq m \leq 1 \). Let \( A = A^* \in S^m(g) \) and let \( \text{Op}(A) : H(m) \to H(0) \) be an isomorphism. If \( A^* V_m = V_m A \), then there exists \( B \in S^{-m}(g) \) such that

\[
A \ast B = B \ast A = \delta_0.
\]

In particular \( \text{Op}(B) = \text{Op}(A)^{-1} \).

By hypothesis, \( A \) is invertible in \( \mathcal{B}(L^2(g)) \). There exists a symmetric distribution \( B \) such that

\[
\text{Op}(A)^{-1} f = f \ast B, \quad f \in \mathcal{S}(g).
\]

We have to show that \( B \in S^{-m}(g) \).

Let \( S_1(g) \) denote the subspace of \( \mathcal{S}(g) \) consisting of those functions whose Fourier transform is supported where \( 1 \leq |\lambda| \leq 2 \). Note that this subspace is invariant under convolutions.

**Lemma 3.2.** \( \text{Op}(B) \) maps continuously \( \mathcal{S}(g) \) into \( \mathcal{S}(g) \). The same applies to the invariant space \( S_1(g) \).
Proof. Being a convolution operator bounded on $L^2(\mathfrak{g})$, $\text{Op}(B)$ commutes with right-invariant vector fields $Y$ and hence maps $\mathcal{S}(\mathfrak{g})$ into $L^2(\mathfrak{g}) \cap C^\infty(\mathfrak{g})$. Therefore, by Lemma 1.4,

$$T_\gamma \text{Op}(B) = \text{Op}(B)T_\gamma + \text{Op}(B)[T_\gamma, \text{Op}(A)]\text{Op}(B)$$

$$+ \sum_{\alpha, \beta \in P(\gamma), |\alpha| + |\beta| = |\gamma|} c_{\alpha, \beta} \cdot \text{Op}(B)\text{Op}(A_\alpha)T_\beta \text{Op}(B),$$

(3.1)

where $A_\alpha = T_\alpha A$. Note that $A_\alpha \in S^{m-|\alpha|} \subset S^0$ so, by Proposition 1.2, $\text{Op}(A_\alpha)$ is bounded on $L^2(\mathfrak{g})$. By induction it follows that $\text{Op}(B)$ maps $\mathcal{S}(\mathfrak{g})$ into the space of functions vanishing rapidly at infinity. Since $\mathcal{S}(\mathfrak{g})$ is invariant under $\text{Op}(B)$, the operators $\text{Op}(A)$ and $\text{Op}(B) = \text{Op}(A)^{-1}$ are isomorphisms of $\mathcal{S}(\mathfrak{g})$ and $\mathcal{S}_1(\mathfrak{g})$.

For $n \in \mathbb{Z}$, let

$$\langle A_n, f \rangle = 2^{-nm} \int_{\mathfrak{g}} f(2^n x) A(dx), \quad \langle B_n, f \rangle = 2^{-nm} \int_{\mathfrak{g}} f(2^n x) B(dx).$$

COROLLARY 3.3. The operators $\text{Op}(B_n)$ are equicontinuous on $\mathcal{S}_1(\mathfrak{g})$.

Proof. By Proposition 1.2, the mapping

$$S^m(\mathfrak{g}) \ni A \rightarrow \text{Op}(B) \in \mathcal{B}(L^2(\mathfrak{g}))$$

is continuous. Since the family $\{A_n\}$ is bounded in $S^m(\mathfrak{g})$ so is $\{\text{Op}(B_n)\}$ in $\mathcal{B}(L^2(\mathfrak{g}))$. Hence our assertion follows by induction using (3.1). \qed

Let $a = \hat{A}$, and let

$$\tilde{A}_\lambda(\eta) = a_\lambda(\eta) = a(\eta, \lambda), \quad \lambda \in \mathfrak{g}^\ast.$$

LEMMA 3.4. For every $f \in \mathcal{S}(\mathfrak{g}_0^\ast)$ the function

$$\lambda \rightarrow \|f \#_\lambda a_\lambda\|^2$$

is continuous.

Proof. Let $0 < h \in \mathcal{S}(\mathfrak{g}^\ast)$ and $h(0) = 1$. Then $F = (f \otimes h)\# a \in \mathcal{S}(\mathfrak{g}^\ast)$ and

$$\lambda \rightarrow \int_{\mathfrak{g}_0^\ast} |F(\eta, \lambda)|^2 d\eta = |h(\lambda)|^2 \|f \#_\lambda a_\lambda\|^2$$

is continuous, which implies our claim. \qed

From now on we shall proceed by induction. The assertion is obviously true in the Abelian case. Let us assume that it holds for $\mathfrak{g}_0 = \mathfrak{g}/\mathfrak{z}$.

LEMMA 3.5. The distribution $A_0$ satisfies the hypothesis of the theorem on $\mathfrak{g}_0$.

Proof. Observe that under the remaining assumptions of Proposition 3.1 the condition that $\text{Op}(A) : H(m) \rightarrow H(0)$ is an isomorphism is equivalent to the estimate

$$\|f * A\| \geq C\|f * V_m\|, \quad f \in \mathcal{S}(\mathfrak{g}).$$
Now, since $A \star V_m = V_m \star A$, we also have
$$A_0 \star (V_m)_0 = (V_m)_0 \star A,$$
where $(V_m)_0$ is the counterpart of $V_m$ on $g_0$. Furthermore, we have
$$\|f \star A\| \geq C \|f \star V_m\|$$
so, by Lemma 3.4,
$$\|f_0 \star A_0\| \geq C \|f_0 \star (V_m)_0\|, \quad f \in \mathcal{S}(g),$$
which implies
$$\|f \star A_0\| \geq C \|f \star (V_m)_0\|, \quad f \in \mathcal{S}(g_0).$$

Let $b = \hat{B}$ and $b_n = \hat{B}_n$. Of course, $b_n \in \mathcal{S}'(g^*)$.

**Lemma 3.6.** There exist $p \in \hat{S}_0^{-m}(g^*)$ and $q \in \mathcal{S}(g^*)$ such that
$$p\#a(\eta, \lambda) = 1 - q(\eta, \lambda), \quad 1 \leq \lambda \leq 2. \quad (3.2)$$

**Proof.** Let $u \in C_0^\infty([0, \infty)$ be equal to 1 in a neighbourhood of $[0, 1]$ and supported in $[0, 2]$. Then
$$\psi(\eta, \lambda) = u\left(\frac{\rho(0, \lambda)}{\rho(\eta, 0)}\right)$$
is an element of $\hat{S}_0^0(g^*)$. By Lemma 3.5 and the induction hypothesis, there exists $b_0 \in \hat{S}_0^{-m}(g^*)$ on $a$ such that
$$b_0\#_0a_0 = 1.$$

Let
$$p(\eta, \lambda) = \psi(\eta, \lambda)b_0(\eta).$$
Then $p \in \hat{S}_0^{-m}(g^*)$ and
$$p\#a(\eta, \lambda) = p\#(a - a_0)(\eta, \lambda) + b_0\#_0a_0(\eta) + (1 - \psi)(\cdot, \lambda)b_0\#_0a_0(\eta)$$
$$= 1 - q_0(\eta, \lambda),$$
where for every $\varphi \in C_c^\infty(g^*)$, $\varphi(\lambda)q_0(\eta, \lambda)$ is in $\hat{S}_0^{-1}(g^*)$. Therefore we take $\varphi \in C_c^\infty(g^*)$ which equals 1 where $1 \leq |\lambda| \leq 2$ and modify $p_0$ and $q_0$ by letting
$$p_1(\eta, \lambda) = p_0(\eta, \lambda)\varphi(\lambda), \quad q_1(\eta, \lambda) = q_0(\eta, \lambda)\varphi(\lambda).$$

Now, $p_1 \in \hat{S}_0^{-m}(g^*)$, $q_1 \in \hat{S}_0^{-1}(g^*)$, and
$$p_1\#a = 1 - q_1, \quad 1 \leq |\lambda| \leq 2.$$

Let
$$p \approx \sum_{k=1}^\infty q_k^k \#p_1,$$
where the infinite sum is understood as in (1.7). Then $p \in S_0^{-m}$ and
$$p\#a = 1 - q, \quad 1 \leq \lambda \leq 2,$$
where $q \in \mathcal{S}(g^*)$. \qed
Now we are in a position to conclude the proof of Proposition 3.1. By acting with $b$ on the right on both sides of (3.2), we get

$$ b = p + q \# b, \quad 1 \leq |\lambda| \leq 2, $$

where $q \# b \in S(g)$. Consequently,

$$ |D_\eta^\alpha b(\eta, \lambda)| \leq C_\alpha \rho(\eta, \lambda)^{-m - |\alpha|}, \quad 1 \leq |\lambda| \leq 2. $$

However, the same applies to $b_n$ for every $n \in \mathbb{Z}$ with the same constants $C_\alpha$.

Therefore, $B \in S^{-m}(g)$. Finally, by Lemma 1.5, we conclude that $B \in S^{-m}(g)$.

**Corollary 3.7.** Let $A \in S^0(g)$ and let

$$ \|f \ast A\| \geq C\|f\|, \quad f \in S(g). $$

There exists $B \in S^0(g)$ such that $B \ast A = \delta_0$.

**Proof.** It is not hard to see that

$$ \|\text{Op}(A^* \ast A)f\| \geq C\|f\|, \quad f \in S(g), $$

so $\text{Op}(A^* \ast A) : L^2(g) \rightarrow L^2(g)$ is an isomorphism. By Proposition 3.1 there exists $B_1 \in S^0(g)$ such that $B_1 \ast A^* \ast A = \delta_0$. Therefore $B_1 \ast A^*$ is the left-inverse for $A$.

**Corollary 3.8.** For every $0 \leq m \leq 1$, there exists $V_{-m} \in S^{-m}(g)$ such that

$$ V_{-m} \ast V_{-m} = V_{-m} \ast V_{m} = \delta_0. $$

**4. The operator Op($V_1$)**

In this section we show that the role of the family of distributions $V_m \in S^m(g)$ in defining the Sobolev spaces can be taken over by the family of fractional powers of one single distribution $V_1$. This will enable the final step towards our theorem.

Recall that if a positive selfadjoint operator $A : L^2(g) \rightarrow L^2(g)$ is invertible, then

$$ A^{-k}f = \frac{\sin k\pi}{\pi} \int_0^{\infty} t^{-k}(tI + A)^{-1}f \, dt $$

(4.1)

for $0 < k < 1$ (see, e.g., Yosida [18], IX.11).

The operator $\text{Op}(V_1)$ is positive selfadjoint and invertible. In the proof of the next proposition we follow Beals [2], Theorem 4.9.

**Proposition 4.1.** For every $m \in \mathbb{R}$, $\text{Op}(V_1)^m = \text{Op}(V_1^m)$, where $V_1^m \in S^m(g)$.

**Proof.** It is sufficient to prove the proposition for $-1 < m < 0$. For $t \geq 0$ let

$$ R_t = (V_1 + t\delta_0)^{-1}, \quad r_t = R_t. $$
The operators $\text{Op} (V_1) + tI$ satisfy the hypothesis of Proposition 3.1 with the exponent $m = 1$ uniformly so there exist constants $C'_\alpha$ independent of $t$ such that
\begin{equation}
|D^\alpha r_t| \leq C'_\alpha \rho^{-1 - |\alpha|}.
\end{equation}

On the other hand
\begin{equation}
t R_t = \delta_0 - R_t * V_1 \in S^0 (\mathfrak{g})
\end{equation}
uniformly in $t$ so that
\begin{equation}
t |D^\alpha r_t| \leq C''_\alpha \rho^{-\alpha}.
\end{equation}
Combining (4.2) with (4.3) we get
\begin{equation}
|D^\alpha r_t| \leq C_\alpha (t + \rho)^{-1} \rho^{-\alpha}
\end{equation}
with $C_\alpha$ independent of $t \geq 0$.

Now, the operator $\text{Op} (V_1)$ is positive and invertible so, by (4.1), $\text{Op} (V_1)^m = \text{Op} (V_1^m)$, where
\begin{equation}
(V_1^m)^\wedge = - \frac{\sin m \pi}{\pi} \int_0^\infty t^m r_t dt,
\end{equation}
where $-1 < m < 0$. Therefore
\begin{equation}
|D^\alpha (V_1^m)^\wedge| \leq \frac{C_\alpha}{\pi} \int_0^\infty t^m (t + \rho)^{-1} dt \rho^{-|\alpha|}
\end{equation}
\begin{equation}
\leq C''_\alpha \rho^{-|\alpha|},
\end{equation}
which proves our case.

\begin{lemma}
Let $K$ be a distribution on $\mathfrak{g}$ smooth away from the origin and satisfying the estimates
\begin{equation}
|D^\alpha K(x)| \leq C_\alpha |x|^{m - Q - |\alpha|}, \quad x \neq 0,
\end{equation}
for some $m > 0$. Then,
\begin{equation}
K = R + \nu,
\end{equation}
where $R \in S^{-m}(\mathfrak{g})$ and $\partial \mu \in L^1(\mathfrak{g})$ for every left-invariant differential operator on $\mathfrak{g}$.
\end{lemma}

\textbf{Proof.} It is sufficient to observe that (4.4) implies that $\hat{K}$ is smooth away from the origin and
\begin{equation}
|D^\alpha \hat{K}(\xi)| \leq C_\alpha |\xi|^{-m - |\alpha|}, \quad \xi \neq 0,
\end{equation}
and let $R = \varphi K$, $\nu = K - R$, where $\varphi \in C_\infty^\infty(\mathfrak{g})$ is equal to 1 in a neighbourhood of 0.

Recall that
\begin{equation}
P^m = V_m + \mu,
\end{equation}
where $V_m \in S^m(\mathfrak{g})$ and $\partial \mu \in L^1(\mathfrak{g})$ for every invariant differential operator $\partial$ on $\mathfrak{g}$.

\textbf{Proposition 4.3.} Let $m > 0$. Then
\begin{equation}
(P^m + \delta_0)^{-1} = R + \nu,
\end{equation}
where \( R \in S^{-m}(g) \) and \( \partial \nu \in L^1(g) \) for every invariant differential operator \( \partial \) on \( g \).

**Proof.** Since the kernel \( P^m \) is maximal (see (2.3) above), it follows (see Dziubański [6], Theorem 1.13) that the semigroup generated by \( P^m \) consists of operators with the convolution kernels

\[
h_t(x) = t^{-Q/m} h_1(t^{-1/m} x), \quad t > 0,
\]

which are smooth functions satisfying the estimates

\[
|D^\alpha h_t(x)| \leq C_\alpha t^{-1/(m+|\alpha|)+1}, \quad x \in g.
\]

Therefore,

\[
(P^m + \delta_0)^{-1}(x) = \int_0^\infty e^{-t} h_t(x) dt,
\]

and consequently satisfies the estimates (4.4).

We know that there exists a constant \( C > 0 \) such that

\[
C^{-1} \|f * V_1^m\| \leq \|f * P^m\| + \|f\| \leq C \|f * V_1^m\|,
\]

whence

\[
\|f * V_1^m\| \geq C_m \|f\|, \quad f \in S(g),
\]

(4.5)

for \( m > 0 \).

Now we have much more.

**Corollary 4.4.** For every \( m > 0 \) there exists a constant \( C > 0 \) such that

\[
C^{-1} \|f * V_1^m\| \leq \|f * P^m\| + \|f\| \leq C \|f * V_1^m\|.
\]

(4.6)

**Proof.** In fact, we have

\[
V_1^m = V_1^m * (P^m + \delta_0)^{-1} * (P^m + \delta_0) = (V_1^m * R + V_1^m * \nu) * (P^m + \delta_0),
\]

where \( R \) and \( \nu \) are as in Proposition 4.3. Then \( V_1^m * R \in S^0(g) \) and \( V_1^m * \nu \in L^1(g) \) so

\[
\|f * V_1^m\| \leq C_1(\|f * P^m\| + \|f\|).
\]

The proof of the opposite inequality uses the identity

\[
f * P^m = f * V_1 * V_1^{-m} * V_1^m + f * \mu
\]

and (4.5). \[\square\]

### 5. Main theorem

Here comes our main theorem and the conclusion of its proof.

**Theorem 5.1.** Let \( A \in S^m(g) \), where \( m \geq 0 \). If \( A \) satisfies the estimate

\[
\|f * A\| \geq C(\|f * P^m\| + \|f\|), \quad f \in S(g),
\]

then there exists \( B \in S^{-m}(g) \) such that

\[
B * A = \delta_0
\]
Proof. Let \( A \in S^m(g) \) satisfy the hypothesis of our theorem. Then \( A \ast V_1^m \)
satisfies the hypothesis of Corollary 3.7 so there exists \( B_1 \in S^0(g) \) such that
\[
B_1 \ast A \ast V_1^m = \delta_0.
\]
By acting by convolution with \( V_1^m \) on the right and with \( V_1^{-m} \) on the left, we see
that \( B = V_1^{-m} \ast B_1 \) is the left-inverse for \( A \).

\[\PageIndex{5.2}\]
Let \( A = A^* \in S^m(g) \) for some \( m \geq 0 \). The following conditions are equivalent:

(i) There exists \( B \in S^{-m} \) such that \( B \ast A = A \ast B = \delta_0 \),

(ii) For every \( k \in \mathbb{R} \), \( \text{Op}(A) : H(k + m) \to H(k) \) is an isomorphism,

(iii) \( \text{Op}(A) : H(m) \to H(0) \) is an isomorphism,

(iv) There exists \( C > 0 \) such that
\[
\|f \ast A\| \geq C(\|f \ast P^m\| + \|f\|), \quad f \in S(g).
\]

\[\PageIndex{5.3}\]
Let \( A \in S^m(g) \), where \( m > 0 \), and let \( \text{Op}(A) \) be positive in \( L^2(g) \). Then \( A \) has a parametrix if and only if there exists \( C > 0 \) such that
\[
\|f \ast A\| + \|f\| \geq C\|f \ast P^m\|. \tag{5.1}
\]

Proof. Let \( B \in S^{-m}(g) \) be a parametrix for \( A \). Then
\[
B \ast A = \delta_0 + h,
\]
where \( h \in S(g) \). Consequently,
\[
P^m = V_1^m \ast B \ast A + g
\]
where \( g \in L^1(g) \). Now, \( V_1^m \ast B \in S^0(g) \) so it is easy to see that the estimate (5.1)
holds.

Suppose now that (5.1) holds true. Then
\[
\|f \ast P^m\| \leq C_1\|f \ast (A + \delta_0)\|,
\]
which, by Corollary 5.2, implies that \( A + \delta_0 \in S^m(g) \) has an inverse \( B_1 \in S^{-m} \).

Thus
\[
B_1 \ast A = \delta_0 - B_1,
\]
and the parametrix \( B \) can be found as an asymptotic series
\[
B \approx \sum_{k=1}^{\infty} B_1^k.
\]

6. Rockland operators

A left-invariant homogeneous differential operator \( R \) is said to be a Rockland
operator if for every nontrivial irreducible unitary representation \( \pi \) of \( g \), \( \pi_R \) is
injective on the space of \( C^\infty \)-vectors of \( \pi \).
Let \( R \) be a left-invariant differential operator homogeneous of degree \(-Q-m\), that is,

\[
R(f \circ \delta_t) = t^m Rf, \quad f \in \mathcal{S}(g), \quad t > 0.
\]

It is well-known that the following conditions are equivalent:

1. \( R \) is a Rockland operator,
2. \( R \) is hypoelliptic,
3. For every regular kernel \( T \) of order \( m \), there exists a constant \( C > 0 \) such that

\[
\|\text{Op}(T)f\| \leq C\|Rf\|, \quad f \in \mathcal{S}(g).
\]

That (1) is equivalent to (2) was proved by Helffer-Nourrigat [12] with a contribution from Beals [1] and Rockland [16]. Helffer-Nourrigat [12] also contains the proof of equivalence of (1)-(3) for \( \text{Op}(T) \) being a differential operator. The remaining part was obtained by the present author in [8] and [11].

It has been proved by Melin [14] that a Rockland operator on a stratified homogeneous group has a parametrix. We are going to show that in fact this is so on any homogeneous group.

**Corollary 6.1.** A Rockland operator on \( g \) has a parametrix.

**Proof.** Without any loss of generality we may assume that \( R \) is positive. Then the assertion follows from (3) and Corollary 5.3.

Thus we have one more condition equivalent to (1)-(3). However, the techniques of the present paper can be applied directly to Rockland operators rendering unnecessary any reference to Theorem 5.1 or Corollary 5.3. What is needed are well-known properties of Rockland operators and the symbolic calculus of Proposition 1.1. Here is a brief sketch of a direct parametrix construction for a Rockland operator \( R \).

We may assume that \( R \) is positive. By Folland-Stein [7], Chapter 4.B, \( R \) is essentially selfadjoint on \( L^2(g) \) with \( \mathcal{S}(g) \) for its core domain. Moreover, the semigroup generated by it consists of convolution operators with kernels

\[
p_t(x) = t^{-Q/m} p_1(t^{-1/m} x),
\]

where \( p_1 \) is a Schwartz class function. Note that \( R = \text{Op}(R\delta_0) \). Let \( S = (\delta_0 + R\delta_0)^{-1} \). It follows that

\[
\hat{S}(\xi) = \int_0^\infty e^{-t} \hat{p_1}(t^{1/m} \xi) \, dt
\]

is a smooth function satisfying the estimates which show that \( S \in S^{-m}(g) \). Moreover,

\[
S \ast R\delta_0 = \delta_0 - S,
\]

and by the usual argument the asymptotic series

\[
S_1 \approx \sum_{k=1}^{\infty} S^k
\]

defines a parametrix for \( R \) (cf. Melin [14]).
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