Physical States in $G/G$ Models and 2d Gravity

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ABSTRACT

An analysis of the BRST cohomology of the $G/G$ topological models is performed for the case of $A_1^{(1)}$. Invoking a special free field parametrization of the various currents, the cohomology on the corresponding Fock space is extracted. We employ the singular vector structure and fusion rules to translate the latter into the cohomology on the space of irreducible representations. Using the physical states we calculate the characters and partition function, and verify the index interpretation. We twist the energy-momentum tensor to establish an intriguing correspondence between the $SL(2)$ model with level $k = \frac{p}{q} - 2$ and $(p, q)$ models coupled to gravity.

† Work supported in part by the US-Israel Binational Science Foundation and the Israel Academy of Sciences.
1. Introduction

Topological Quantum field theories\cite{1} have been a center of much interest during recent years. At the same time a great emphasis was devoted to the research of non critical string theory for \( c \leq 1 \). This topic was explored via matrix models\cite{2} as well as continuum Liouville theory.\cite{3} In several cases a strong relationship between some topological models and the string models was revealed. For instance recursion relations for amplitudes in the one matrix model were found to be identical to those derived in the topological theory of gravity.\cite{4} The exploration of connections between another class of interesting topological models, the so called \( G/G \) models, and theories of \( c \leq 1 \) matter coupled to 2d gravity is the subject of the present work.

The \( G/G \) topological theories\cite{5,6} are theories based on \( G \) current algebra. Formally they are obtained by gauging the \( G \) WZW model. Following the well known \( G/G \) construction\cite{7,8} the \( G/G \) models are derived by gauging the anomaly free group \( G \) of the \( G_k \) WZW model. This recasts, using a complexified \( G^c \) algebra, the \( G/G \) action into a sum of three terms: a \( G \) WZW at level \( k \), a \( (G/G)_{k+2C_G} \) WZW model which can be viewed as a \( G \) WZW model at level \(- (k + 2C_G)\) and a \( (1,0) \) odd ghost system in the adjoint representation of the group. The \( G/G \) models have a significant overlap with other TQFT’s. For one they are related to the 2+1 dimensional Chern-Simons theory\cite{10}. In particular the amplitudes, which are given in terms of \( N_{ijk} \) fusion rule coefficients,\cite{5,6} coincide with the scalar product of wave functions of the Chern-Simons theory. Other related TQFT are the topological flat connection models.\cite{9} They share with the \( G/G \) models the correspondence to the moduli space of flat gauge connections. More interesting is the relation with models of \( c \leq 1 \) coupled to gravity. This is the main topic of the present work. In ref. \[5\] the structure of the \( G/G \) theories was investigated and some striking resemblance to the 2 dimensional gravity models was revealed. One would expect that the \( G_k \) WZW model plays the role of the matter system, while the \( (G/G)_{k+2C_G} \) WZW model is affiliated with the Liouville degrees of freedom. Both in the gravitational models\cite{11} as well
as in the SU models the full character gets a contribution only from the primary states of the matter \((G_k)\) sector whereas the matter and Liouville \((G_{-(k+2C_G)})\) oscillators are canceled out exactly by the ghost contribution. The character can be interpreted as an index which encodes the information about the BRST invariant states of the theory. This index is \(\text{Tr}[(-)^G q^N]\) and \(\text{Tr}[(-)^G q^N (e^{\pi i \theta} J_3 - \frac{1}{2})]\) for the gravity and \(A_1^{(1)}\) cases respectively, where \(G\) stands here for the ghost number, \(N\) is the level and \(J_3\) is the charge, appropriately defined\(^5\). The partition function is obtained by integrating over the moduli. In the gravity case this leaves only matter primary states while in the \(G\) model the partition function equals the number of conformal blocks. For the \(SU(2)\) case with an integer level \(k\) the result is \(k + 1\).

The information from the index interpretation of the partition function was used in ref. [5] to disentangle the spectrum. In view of possible cancelations, an index cannot a-priori give full information about the spectrum. A complete account of the physical states can be achieved only by working out the cohomology of the relevant BRST operator. This type of analysis for \(c \leq 1\) matter coupled to gravity was worked out originally in ref. [12] and latter in ref. [13]. An extremely rich spectrum of discrete states was revealed. They were found to correspond to previous matrix model calculations\(^1\)\(^7\). In the case of \(c = 1\) a “ground ring”\(^1\) of discrete states of zero ghost number and dimension was exposed with an underlying \(W_\infty\) symmetry. The corresponding Ward identities were implemented in the computations of correlators\(^1\). A different method for those calculations was suggested in ref. [16] together with an extension to the \(c < 1\) models. The counterpart of these results in the context of the \(G\) models is still missing. One of the main tasks of the present work is to take the first step in this direction. It is devoted to a BRST analysis of the gauged \(SU(2)\) WZW model\(^\dagger\) of level \(k\) defined on a sphere. We establish rigorously the result of ref. [5] that for an integer \(k\) there are \(k + 1\) zero ghost number primary states corresponding to the matter primaries. On each of these primaries there is a whole tower of descendant states, one at each ghost number.

\(^\dagger\) Throughout the paper we shall freely move between \(SU(2)\) and \(SL(2)\) cases depending on the values of the level \(k\).
The outcome of the BRST analysis follows closely that of $c < 1$ models coupled to 2d gravity.\cite{12,13} To be more precise $(p,q)$ minimal matter systems coupled to gravity correspond to $k = \frac{p}{q} - 2$ level of the $SL(2)$ case. This is related and indeed has a lot in common with the observation that $SL(2)$ WZW model is connected to the $(p,q)$ system via the Hamiltonian reduction.\cite{18} Moreover, it was proven\cite{18} that the irreducible representation of the Virasoro algebra can be extracted from the irreducible representation of the $SL(2)$ current algebra by putting a constraint. The quantum gravity coupled to $c_{p,q}$ matter was shown to be equivalent to the constrained WZW model of level $-(k + 4)$.

The paper is organized as follows. In section 2 the $G$ topological theory is briefly reviewed. This serves mainly to establish notation and put forward important topics to what follows. In section 3 a BRST analysis similar to the one employed in ref.\cite{13} is performed for $G = A_{1}^{(1)}$. The main idea is to use a free field parametrization for the currents of the $k$ and $-(k + 4)$ levels. In fact it turns out that to implement the procedure of ref.\cite{13} it is convenient to invoke “conjugate” parametrizations for the two sectors. The physical states of the theory correspond to the cohomology on another space $\mathcal{F}_{k} \otimes \mathcal{F}_{-(k+4)} \otimes \mathcal{F}_{\text{ghost}}$, where $\mathcal{F}_{k}$ denotes the irreducible representation of $A_{1}^{(1)}$ level $k$, $\mathcal{F}_{-(k+4)}$ and $\mathcal{F}_{\text{ghost}}$ the Fock space of the $-(k+4)$ sector and the ghosts respectively. In section 4 we discuss aspects of the representation theory of Kac-Moody algebra $A_{1}^{(1)}$\cite{20} for an arbitrary complex level $k$. Special attention is devoted to singular vectors associated with null states. The existence of these vectors in a given Verma module indicates that it is reducible.\cite{19} This information is crucial for the determination of the BRST cohomology on the space of irreducible representations. There is a similarity between the representations of the current algebra and those of the Virasoro algebra. The duality between representations of the latter which are characterized by $|h, c>$ and $|1 - h, 26 - c>$ is shown\cite{12} to correspond to a duality between $|J, k>$ and $|-1 - J, -(k + 4)>$ in the $A_{1}^{(1)}$ algebra. The singular vectors are used in the appendix to determine the fusion rules. The passage from the Fock space BRST cohomologies to an irreducible representation is presented in section 5. It follows
the work of Bernard and Felder\textsuperscript{[22]}. In section 6 we demonstrate the correspondence between \((p, q)\) minimal models coupled to gravity and the \(G \bar{G}\) model for \(G = SL(2)\) and \(k + 2 = \frac{p}{q}\). We start with some “numerological” indications about the relationship. Similar “numerical” observations are also the basis for the \(SL(2)\) quantum Hamiltonian reduction approach\textsuperscript{[18]}. We then show that we have the right number of “primary” operators. Next we suggest a map between the fields of the two models after introducing a twisted energy momentum tensor. This is supplemented by a computation of the partition function of the \(G \bar{G}\) model and a proof that it coincides with that of the \((p, q)\) minimal model provided a particular value of the moduli of flat \(G\) gauge connection is picked. An explicit construction of the operators which correspond to physical states, and their correlators is presented in section 7 for the simplest case of \(k = 0\).\textsuperscript{[23]} This provides a demonstration of our general considerations. We write these operators down in two “conjugate” parametrizations, prove that there is a one to one map between them and determine the non-vanishing correlation functions. A summary and a set of open questions is brought in section 8. We also present first results indicating that the generalization to \(G = SL(n)\) provides a model of \(W_N\)\textsuperscript{[24]} matter coupled to \(W_N\)\textsuperscript{[25]} gravity. A derivation of fusion rules of the rational level \(k A_1^{(1)}\) algebra using the singular vectors is presented in the appendix.

2. \textbf{The \(G \bar{G}\) Topological Field Theory}

The \(G \bar{G}\) TQFT is constructed by gauging the anomaly free diagonal \(G\) group of the \(G - WZW\) model. The classical action takes the form

\[
S_k(g, A^\mu) = S_k(g) - \frac{k}{2\pi} \int_{\Sigma} d^2 z Tr(g^{-1} \partial g \bar{A}_z + g \bar{\partial} g^{-1} A_z - \bar{A}_z g^{-1} A_z g + A_z \bar{A}_z) \tag{1}
\]

where \(g \in G\) and \(S_k(g)\) is the WZW action at level \(k\). In case that \(\Sigma\) is topologically trivial the gauge field can be parametrized as follows \(A_z = i h^{-1} \partial_z h, \bar{A}_\bar{z} = \)
\( h^* \partial_z h^{*-1} \) where \( h(z) \in G^c \). The action then reads

\[
S_k(g, A) = S_k(g) - S_k(hh^*)
\]

(2)

The Jacobian of the change of variables introduces a dimension \((1, 0)\) system of anticommuting ghosts \( \chi \) and \( \rho \) in the adjoint representation of the group. The quantum action thus takes the form of

\[
S_k(g, h, \rho, \chi) = S_k(g) - S_k + 2C_G(hh^*) - i \int d^2z Tr[\rho \partial \overline{\chi} + c.c]
\]

(3)

where \( C_G \) is the second Casimir of the adjoint representation. The second term can be viewed as \( S_{-(k+2C_G)}(h) \). Since the Hilbert space of the model decomposes into holomorphic and anti-holomorphic sectors we restrict our discussion only to the former. There are three sets of holomorphic \( G \) transformations which leave (3) invariant \( \delta J g = i[\epsilon(z), g] \delta f h = i[\epsilon(z), h] \) and \( \delta J^{(gh)} \chi^a = if_{bc}^a \chi^b \chi^c \); \( \delta J^{(gh)} \rho^a = -if_{bc}^a \rho^b \rho^c \) with \( \epsilon \) in the algebra of \( G \). The corresponding currents \( J^a, I^a \) and \( J^{(gh)} = if_{bc}^a \chi^b \rho^c \) satisfy the \( G \) Kac-Moody algebra with the levels \( k, -(k+2C_G) \) and \( 2C_G \) respectively. We define now \( J^{(tot)} = J^a + I^a + J^{(gh)} = J^a + I^a + if_{bc}^a \chi^b \rho^c \)

(4)

which obeys a Kac-Moody algebra of level

\[
k^{(tot)} = k -(k+2C_G) + 2C_G = 0.
\]

(5)

The energy-momentum tensor \( T(z) \) is a sum of Sugawara terms of the \( J^a \) and \( I^a \) currents and the usual contribution of a \((1, 0)\) ghost system, namely

\[
T(z) = \frac{1}{k + c_G} : J^a J^a : - \frac{1}{k + c_G} : I^a I^a : + \rho^a \partial \chi^a.
\]

(6)

The corresponding Virasoro central charge vanishes

\[
c^{(tot)} = \frac{kd_G}{k + c_G} - \frac{(k + 2C_G)d_G}{-(k+2C_G) + c_G} - 2d_G = 0
\]

(7)

This last property is a first indication that the \( G/G \) model is a TCFT. In fact it is easy to realize that the basic algebraic structure of a TCFT is obeyed by the model.
This is expressed in terms of two bosonic and two fermionic operators. The former are $T(z)$ and the “ghost number current” $J^\# = \chi_a \rho^a$. The fermionic currents are a dimension one current which is the BRST current $J^{(BRST)}$ and a dimension two operator $G$. These holomorphic symmetry generators are given by

$$J^{(BRST)} = \chi_a [J^a + I^a + \frac{1}{2}J^{(gh)}^a]$$
$$G = \frac{1}{k + c_G} \rho_a [J^a - I^a]$$

(8)

The TCFT algebraic structure now reads

$$T(z) = \{ Q, G(z) \}$$
$$J^{(BRST)} = \{ Q, J^\#(z) \}$$

(9)

where $Q = \int J^{(BRST)}(z)$ is the BRST charge. In addition to $T(z)$ and $J^{(BRST)}(z)$, the total current $J^{(tot)}_a$ is also BRST exact,

$$J^{(tot)}_a(z) = \{ Q, \rho^a \}.$$  

(10)

The extraction of physical states as elements of the BRST cohomology will be the subject of the next section. We summarize here, following ref. [5], the picture emerging from the investigation of the torus partition function. The latter is expressed as[7]

$$Z_G = c \tau_2^{-r} \int du Z^g(\tau, u) Z^{hh^*}(\tau, u) Z^{gh}(\tau, u)$$

(11)

where $du$ is the measure over the flat gauge connections on the torus and $r$ is the rank of $G$. $Z^g(\tau, u)$ is the torus partition function of the $G_k$ WZW model

$$Z^g(\tau, u) = (q\bar{q}) \sum_{\lambda_L, \lambda_R} \chi_{k, \lambda_L}(\tau, u) \bar{\chi}_{k, \lambda_R}(\tau, u) N_{\lambda_R, \lambda_L}$$

(12)

where $q = e^{2i\pi \tau}$, $\lambda_R, \lambda_L$ denote the $G_k$ highest weights, and for the diagonal
invariant $N_{\lambda R, \lambda L} = \delta_{\lambda R, \lambda L}$. The character can be written as

$$\chi_{k,\lambda}(\tau, u) = \frac{M_{k,\lambda}(\tau, u)}{M_{0,0}(\tau, u)},$$

(13)

with $M_{k,\lambda}$ defined explicitly for the $SU(2)$ case below, $Z_{hh^\ast}(\tau, u)$ in eqn. (11) is the contribution of $h \in \frac{C_G}{G}$ at level $k + 2C_G$ or equivalently $h \in G$ at level $-(k + 2C_G)$. This was calculated in ref. [7] using the iterative Gaussian path integration technique. The outcome is that $Z_{hh^\ast}(\tau, u) \sim |M_{0,0}(\tau, u)|^{-2}$ indicating that $\frac{C_G}{G}$ contains just one conformal block. It is straightforward to calculate $Z_{gh}(\tau, u)$, the ghost contribution to the partition function $Z_{gh}(\tau, u) \sim |M_{0,0}(\tau, u)|^4$. The cancelation of the $|M_{0,0}(\tau, u)|$ factors in eqn.(11) is similar to the cancelation of the $\eta$ factors in the torus partition function of $c \leq 1$ models coupled to $2d$ gravity.[11] In both cases the resulting character is given by the numerator of the character of the “matter” sector. In the $\frac{G}{G}$ model it is $M_{k,\lambda}$. This cancelation property leads to an index interpretation for $M_{k,\lambda}$. For $G = SU(2)$ this amounts to expressing

$$M_{k,j}(\tau, \theta) = \sum_{l=-\infty}^{\infty} q^{(k+2)(l+\frac{j+1}{2})^2} \sin \{\pi \theta[(k + 2)l + j + \frac{1}{2}]\}$$

(14)

as

$$M_{k,j}(\tau, \theta) = \frac{1}{2i} q^{\frac{(j+\frac{1}{2})^2}{\theta}} e^{i\pi \theta(j+\frac{1}{2})} Tr[(\mathcal{L}_0 e^{i\pi \theta \mathcal{J}^0_{(tot)}})]$$

(15)

where $\theta$ is the holonomy in the $\tau_2$ direction, $G$ is the ghost number, $\mathcal{L}_0$ is the excitation level and $\mathcal{J}^0_{(tot)}$ is the $J^0_{(tot)}$ eigenvalue of the excitation. The prefactor in front of the trace was chosen to agree with the definition of the vacuum we will use in section 3 where the cohomology is worked out and eqn. (15) is verified. Note that $M_{k,j}(\tau, \theta)$ is obtained from the torus $M_{k,j}(\tau, u)$[7] by restricting to just one angle.[5] This amounts to consider the propagation along a cylinder rather than around the torus. As long as we are interested in the spectrum it is sufficient to consider $M_{k,j}(\tau, \theta)$. This index interpretation enables us to read important
information about the physical spectrum from eqn. (14). For a positive integer
$k$, $2j = 0, \ldots, k$. Hence there are $k + 1$ zero ghost number primary states which
correspond to the first term in the $q$ expansion of the different $M_{k,j}$’s i.e the term
corresponding to $l = 0$ with $\hat{L}_0 = \frac{j(j+1)}{k+2}$. On each of these states there is a whole
tower of states corresponding to the higher terms in the $q$ expansion, we will refer
to these states as descendants. In ref. [5] it was argued that those states appear for
all ghost number, and for a given $j$ and a given ghost number there is precisely
one physical state. The $G_G$ characters are orthonormal in the $du$ measure\textsuperscript{[7]}
\[ \int du M_{k,j}(\tau, u) \bar{M}_{k,j}(\bar{\tau}, \bar{u}) = \delta_{j,j'} \] (16)
where $j$ and $j'$ are $SU_k(2)$ multiplets. Thus eqn. (11) yields for $G = SU_k(2) \mathbb{Z}_G = k + 1$. For integer $k$ this is precisely the number of conformal blocks.

3. $A_1^{(1)}$ level $k$ BRST COHOMOLOGY ON THE Fock space

We now proceed to extract the physical states of the $G$ theory for $G = A_1^{(1)}$. We use the BRST procedure to quantize the system and thus the physical states are in the cohomology of $Q$, the BRST charge, $|\text{phys} \rangle \in H^*(Q)$. Expanding the currents $J^a$, $I^a$ and the $(1,0)$ ghost fields $\rho^a$, $\chi^a$ in modes and inserting them into eqn. (8) we obtain the following BRST charge
\[ Q = \sum_{n=-\infty}^{\infty} [g_{ab} \chi_n^a (J_n^b + I_n^b) - \frac{1}{2} f_{abc} \sum_{m=-\infty}^{\infty} : \chi_n^a \chi_m^b \chi_{n+m}^c :] \] (17)
where $:$ : denotes normal ordering namely putting modes with negative subscripts
to the left of those with positive ones and $\rho_0^a$ to the right of $\chi_0^a$. Since both $J^{(\text{tot})}_n^a$ and $L_n$ are $Q$ exact namely
\[ \{ Q, \rho_n^a \} = J^{(\text{tot})}_n^a \quad \{ Q, G_n \} = L_n \] (18)
it follows that
\[ L_0 |\text{phys} > = 0 \quad J^{(\text{tot})}_0^a |\text{phys} > = 0 \] (19).
For non-vanishing eigenvalues of $L_0$ and $J^{(\text{tot})}_0^a$ it is easy to see that $|\text{phys} >$ is in
the image of $Q$ which cannot be true for a non-trivial $|\text{phys} > \in H^*(Q)$.

Let us now select a sub-space $\mathcal{F}(J,I)$ of the space of physical states on which $\rho_0^0 = 0$ in addition to $J^{(\text{tot})}_0 = L_0 = 0$. On this sub-space $Q$ which may be written as

$$Q = \chi_0^0 J^{(\text{tot})}_0 + M \rho_0^0 + \hat{Q},$$

$$M = -\frac{1}{2} f_{0bc} \sum_{n \neq 0} \chi^b_{-n} \chi^c_n : -\frac{1}{2} f_{0bc} : \chi^b_0 \chi^c_0 :,$$

equals $\hat{Q}$. We thus start by deducing $H^*(\hat{Q})$ the cohomology of $\hat{Q}$. The states which correspond to the latter are built on a vaccum $|J,I >$ obeying the following relations

$$J_{n>0}^a |J,I >= 0 \quad J_0^+ |J,I >= 0 \quad J_0^0 |J,I >= J|J,I >$$

$$I_{n>0}^a |J,I >= 0 \quad I_0^+ |J,I >= 0 \quad I_0^0 |J,I >= I|J,I >$$

$$\chi_{n>0}^a |J,I >= 0 \quad \rho_{n\geq0}^a |J,I >= 0.$$  

A convenient way to handle the $J^a$ and $I^a$ currents is to invoke the following “bosonization” \cite{26}

$$J_n^+ = \beta_n$$

$$J_n^0 = \sum_m \beta_m \gamma_{n-m} + \frac{a}{\sqrt{2}} \phi_n$$

$$J_n^- = -\sum_{k,m} \gamma_m \gamma_k \beta_{n-m-k} - \sqrt{2a} \sum_m \phi_m \gamma_{n-m} + kn \gamma_n$$

where $a^2 = k + 2$. The fields $\beta$ and $\gamma$ form a bosonic $(1,0)$ system with $[\gamma_m,\beta_n] = \delta_{m+n}$. The modes $\phi_n$ correspond to the dimension one operator $i\partial \phi$ and they obey $[\phi_m,\phi_n] = m \delta_{m+n}$. In the $I$ sector a similar parametrization of the currents in terms of $\tilde{\beta}, \tilde{\gamma}$ and $\tilde{\phi}$ is performed but now with $I_n^0 \leftrightarrow -I_n^0$, $I_n^+ \leftrightarrow I_n^-$, which is an automorphism of the Kac-Moody algebra. In the $I$ sector we take $k \rightarrow -k - 4$ and therefore $a \rightarrow ia$. It is easy to realize that the conditions of eqn.(21) are obeyed
only provided $\beta_0|J,I >= \tilde{\gamma}_0|J,I >= 0$. The normal ordering, however, is with respect to the usual $SL(2,R)$ invariant vacuum $\beta_0|J,I >= \tilde{\beta}_0|J,I >= 0$. In terms of the new variables, $\hat{Q}$ takes the form

\[
\hat{Q} = \sum_{l,m,n} \chi_{l-m}[\beta_n - a(\phi^+_m - \phi^-_m)\tilde{\gamma}_{n-m} - i\gamma_m \tilde{\gamma}_{l} \tilde{\beta}_{n-m-l} : -(k + 4)n\tilde{\gamma}_n] \\
+ 2 \sum_{n \neq 0, m} \chi_{-n}[\beta_m \gamma_{n-m} : - : \tilde{\beta}_m \tilde{\gamma}_{n-m} : +a\phi^-_n] \\
+ \sum_{k,m,n} \chi_{-n}[\tilde{\beta}_n - a(\phi^+_m + \phi^-_m)\gamma_{n-m} : - \gamma_m \gamma_k \tilde{\beta}_{n-m-k} : +kn\gamma_n] \\
- \sum_{m,n} f^{abc}_{lmn} \chi_{l-m}^a \chi_{m+n}^b \beta_{m+n}^c,
\]

where $\phi^\pm_n = \frac{1}{\sqrt{2}}(\phi_n \pm i\tilde{\phi}_n)$ and $\sum'$ denotes a sum over $m$ and $n$ which does not include $\chi^0_0$ and $\rho^0_0$ modes.

We now proceed following ref. [13] to assign a degree to the various fields. The idea is to decompose $\hat{Q}$ into terms of different degrees in such a way that there is a nil-potent operator that carries the lowest degree which is zero.

An assignment that obeys this requirement is the following

\[
deg(\chi) = \deg(\gamma) = \deg(\tilde{\gamma}) = \deg(\phi^+) = 1 \\
deg(\rho) = \deg(\beta) = \deg(\tilde{\beta}) = \deg(\phi^-) = -1
\]

The decomposition of $\hat{Q}$ to different degrees now reads
\[ \hat{Q} = Q^{(0)} + Q^{(1)} + Q^{(2)} + Q^{(3)} \]

\[ Q^{(0)} = \sum_n \chi_n \beta_n + 2a \sum_{n \neq 0} \chi_n \phi_n + \sum_n \chi_n \tilde{\beta}_n \]

\[ Q^{(1)} = a \sum_{m,n} \chi_n \phi_n \gamma_{m-n} + \sum_{n \neq 0} 2\chi_n \sum (\beta_n \gamma_{n-m} - \tilde{\beta}_n \tilde{\gamma}_{n-m}) \]

\[ - a \sum_{m,n} \chi_n \phi_n \gamma_{n-m} - \sum_{m,n} \frac{1}{2} f_{abc} \chi_n \phi_n \rho_{m+n} \]

\[ Q^{(2)} = - \sum_n \chi_n \left[ \sum_{k,m} \gamma_{m} \tilde{\gamma}_{n-m-k} + (a(\phi_0 + \phi_0^2) + (k+4)n) \tilde{\gamma}_n \right] \]

\[ - \sum_n \chi_n \left[ \sum_{k,m} \gamma_{m} \tilde{\gamma}_{n-m-k} + (a(\phi_0^2 + \phi_0^2) - kn) \gamma_n \right] \]

\[ Q^{(3)} = - a \sum_{m,n} (\chi_n \phi_n \gamma_{m-n} + \chi_n \phi_n \gamma_{n-m}) \].

From the fact that terms of different degree in \((\hat{Q})^2\) vanish separately it follows\(^{[13]}\) that on \(\mathcal{F}(I,J)\) \(Q^i\) obey the following relations

\[ (Q^{(0)})^2 = (Q^{(3)})^2 = 0 \quad \{Q^{(0)}, Q^{(1)}\} = \{Q^{(2)}, Q^{(3)}\} = 0 \]

\[ Q^{(1)} + \{Q^{(0)}, Q^{(2)}\} = (Q^{(2)})^2 + \{Q^{(1)}, Q^{(3)}\} = \{Q^{(0)}, Q^{(3)}\} + \{Q^{(1)}, Q^{(2)}\} = 0 \]

In fact \((Q^{(0)})^2 = 0\) holds on the entire Fock space. We want now to apply the results of ref. [13] which hold only provided that there is a finite number of degrees for each ghost number. Recall that states in \(\mathcal{F}(I,J)\) are annihilated by both

\[ L_0 = \hat{L}_0 + \sum_n \left[ : \beta_n \gamma_n : + : \tilde{\beta}_n \tilde{\gamma}_n : + g_{ab} \chi_n \rho_{m+n} : \right] + \sum_{n \neq 0} \phi_{m+n} \phi_n \]

and

\[ J^{(tot)\, 0}_0 = J + \sum_n \left[ : \beta_n \gamma_n : - : \tilde{\beta}_n \tilde{\gamma}_n : - f_{bc} \chi_n \rho_{m+n} : \right] \]

It is clear from the expression of \(L_0\) that for a given \(|J, I\rangle\) the amount of excitations and thus the degree they carry is limited. The restriction of vanishing \(J^{(tot)\, 0}_0\) further
limits the contribution of the zero modes $\gamma_0$ and $\tilde{\beta}_0$. This proves that on $\mathcal{F}(J,I)$ and in particular for each ghost number the degree carried by any state is bounded from both sides. Hence we can proceed and use the lemmas proven in ref. [13]. The next step is to find the cohomology of $Q^{(0)}$ on $\mathcal{F}(J,I)$. It is not difficult to realize that $\hat{L}_0$, the contribution to $L_0$ of the exitations, is $Q^{(0)}$ exact

$$\hat{L}_0 = \{Q^{(0)}, \hat{G}^{(0)}\}$$

$$\hat{G}_0 = -\sum_n n[\rho_n^+ \gamma_n + \rho_n^- \tilde{\gamma}_n] + \frac{1}{a} \sum_{n \neq 0} \rho_n^0 \phi_n^+$$

(29)

The consequence of the last relation is that $\hat{L}_0$ annihilates the states in the cohomology of $Q^{(0)}$ on $\mathcal{F}(J,I)$ and thus there are no excitations in $H^*(Q^0)$. Moreover, since $L_0 = 0$, states in the latter must have either $I = J$ or $I = -(J + 1)$. Let us now extract the zero modes contributions to the cohomology. The general structure of these states is

$$|n_{\gamma}, n_{\tilde{\beta}}, n_+, n_- >= (\gamma_0)^{n_+}(\tilde{\beta}_0)^{n_-}(\chi_0^+)^{n_+}(\chi_0^-)^{n_-}|I, J >$$

(30)

where obviously $n_+, n_- = 0, 1$ and $n_{\gamma}, n_{\tilde{\beta}}$ are non-negative integers. Using the following commutation relations

$$[Q^{(0)}, \gamma_0] = -\chi_0^- \quad [Q^{(0)}, \tilde{\beta}_0] = \{Q^{(0)}, \chi_0^+\} = \{Q^{(0)}, \chi_0^-\} = 0$$

(31)

and the relation $Q^{(0)}|I, J >= \chi_0^+ \tilde{\beta}_0|I, J >$, one finds that the result of operating with $Q^{(0)}$ on (30) is

$$Q^{(0)}|n_{\gamma}, n_{\tilde{\beta}}, n_+, n_- >= (-1)^{n_+ + 1}n_{\gamma}|n_{\gamma} - 1, n_{\tilde{\beta}}, n_+, n_- + 1 > + (-1)^{n_-}n_{\gamma}, n_{\tilde{\beta}} + 1, n_+ + 1, n_- >$$

(32)

where states with $n_\pm > 1$ are identified as zero. It is now straightforward to deduce
the Kernel and the Image of $Q(0)$. The former takes the form

$$\text{Ker}Q(0) = \sum_{n_\gamma, n_\beta} A_{n_\gamma, n_\beta} |n_\gamma, n_\beta, 1, 1 >$$

$$+ \sum_{n_\beta \geq 0, n_\gamma > 0} B_{n_\gamma, n_\beta} |n_\gamma, n_\beta, 1, 0 > + B_{0,0} |0, 0, 1, 0 >$$

$$- \sum_{n_\beta, n_\gamma \geq 0} (n_\gamma + 1)B_{n_\gamma+1, n_\beta+1} |n_\gamma, n_\beta, 0, 1 >$$

and the latter has the same terms apart from the $|0, 0, 1, 0 >$ term. Therefore the only possible state in the relative cohomology of $Q(0)$ is $\chi_0^+ |I, J >$, and from the condition $J^{(\text{tot})} |0 = 0$ we find a state only provided that $I = -J - 1$ and then

$$H_{\text{rel}}^\text{rel}(Q(0)) = \{ \chi_0^+ | - (J + 1), J > \}.$$ 

(34)

The passage from the relative cohomology to the absolute one is then given by

$$H_{\text{abs}}(Q(0)) \simeq H_{\text{rel}}^\text{rel}(Q(0)) \oplus \chi_0^0 H_{\text{rel}}^\text{rel}(Q(0))$$

(35)

in the same way as in ref. [13]. Since the derived cohomology includes a single degree, $H^*(Q(0)) \simeq H^*(Q)$ in a complete analogy to the Liouville theory discussed in ref. [13]. We conclude that the cohomology of $Q$ on the full Fock space includes states of arbitrary $J$ with a corresponding $I = -(J + 1)$ and with ghost number $G = 1, 2$, one state at each ghost number. We shift from here on the definition of the ghost-number so that the state $\chi_0^+ |I, J >$ is at $G = 0$. We should note here that since there is a symmetry between the $I$ and $J$ sectors we could also have used an “inverse bosonization” in the $J$ sector and the ordinary one in the $I$ sector with the same resulting cohomology. So far we have analyzed the cohomology on the whole Fock Space. Now one has to pass to the space of irreducible representations of the $J$ sector. For that we need certain results related to the singular vectors of arbitrary level $k$ of the $A_1^{(1)}$ algebra. These results are presented in the next section. The reader who is not interested in the details can skip it and move directly to section 5.
4. REPRESENTATION THEORY OF $A_1^{(1)}$

We give in this section a short review of the representation theory of $A_1^{(1)}$ and of its singular vectors. The results presented in this section will be heavily used in what follows. We follow the presentation of Malikov, Feigin and Fuks [20] but we use the language of current algebra. We concentrate on these results which will be relevant for what follows. Let $g$ be the $A_1^{(1)}$ algebra defined by the commutation relations

$$
\begin{align*}
[J^0_n, J^\pm_m] &= \pm J^\pm_{n+m} \\
[J^+_n, J^-_m] &= 2J^0_{n+m} + nk\delta_{n+m,0} \\
[J^0_n, J^0_m] &= \frac{1}{2}nk\delta_{n+m,0} \\
[k, J^a_n] &= [J^\pm_n, J^\pm_m] = 0
\end{align*}
$$

The universal enveloping algebra $U(g)$ can be written

$$
U(g) = N_- \otimes H \otimes N_+
$$

where $N_+$ is the subalgebra generated by $J^+_0$ and $J^-_1$, $H$ is the Cartan subalgebra generated by $J^0_0$ and $k$, and $N_-$ is the subalgebra generated by $J^-_0$ and $J^+_1$.

The highest weight vector is characterized by two parameters: the spin $J$ and the central charge $k$. It is convenient, though, to work with the variable $t = k + 2$. We denote the highest weight vector by $|J, t\rangle$. The conditions it satisfies are

$$
\begin{align*}
N_+ |J, t\rangle &= 0 \\
J^0_0 |J, t\rangle &= J|J, t\rangle \\
K|J, t\rangle &= (t-2)|J, t\rangle
\end{align*}
$$

We recall that the Sugawara construction gives us another diagonal operator

$$
L_0 |J, t\rangle = h(J)|J, t\rangle \equiv \frac{1}{t} J(J + 1)|J, t\rangle.
$$

On this highest weight $U(g)|J, t\rangle \sim N_-|J, t\rangle$. We call this module the Verma module and denote it $V_{J,t}$. The Verma module is an infinite dimensional representation.
of $A_1^{(1)}$. The Verma module $V_{J,t}$ is naturally graded with respect to $L_0$ (the level) and to $J_0^0$ (the spin). The homogeneous subspaces are finite dimensional. Kac and Kazhdan\cite{19} studied the question of reducibility of these modules and gave the following conditions for reducibility: $V_{J,t}$ is reducible if and only if there exist two positive integers $r$ and $s$ such that the value of $J$ is at least one of the following

$$2J_{r,s,+} + 1 = r - (s - 1)t$$
$$2J_{r,s,-} + 1 = -r + st$$

or if $t = 0$. A singular vector is a state $|\chi\rangle \in V_{J,t}$ such that $|\chi\rangle \neq |J,t\rangle$ and such that $|\chi\rangle$ is a highest weight vector. It is clear that a Verma module is reducible if and only if it contains a singular vector. An interesting question is what are the level and spin of a singular vector in the module $J = J_{r,s,\pm}$ and what is its explicit form. These questions were first studied in ref. \[20\]. It was found that if $J = J_{r,s,+}$ for a couple $(r,s)$ then

$$|\chi_{r(s-1),r}\rangle = (J_0^--)^{r+(s-1)t} (J_{-1}^+)^{r+(s-2)t} (J_0^--)^{r+(s-3)t} \cdots (J_{-1}^-)^{-r-(s-2)t} (J_0^+)^{-r-(s-1)t} |J_{r,s,+},t\rangle$$

The subindices of $|\chi\rangle$ keep track of the level and spin of the singular vector

$$L_0 |\chi_{r(s-1),r}\rangle = (h(J) + r(s - 1))|\chi_{r(s-1),r}\rangle$$
$$J_0^0 |\chi_{r(s-1),r}\rangle = (J - r)|\chi_{r(s-1),r}\rangle.$$  

What does this expression mean algebraically and geometrically?

Algebraically, MFF\cite{20} found an infinite set of positive integer $t$ such that this expression makes sense and it can be written as

$$|\chi_{r(s-1),r}\rangle = \sum_{p,q} P_{p,q}(g([J_0^- , J_{-1}^+]^t), t) (J_0^-)^{rs-p(J_{-1}^-)^{r(s-1)-q} |J_{r,s,+},t\rangle}$$

where $P_{p,q}$ depend polynomially on $t$ and, thus, can be continued analytically to the whole complex plane. Take for example $r = 1, s = 2$ then we use analytically
continued commutation relations

\[
[(J^{-}_0)^x, J^{+}_{-1}] = -2 x J^{0}_1 (J^{-}_0)^{x-1} - x(x-1) J^{-}_0 (J^{-}_0)^{x-2}
\]  \hspace{1cm} (44)

in order to solve

\[
|\chi_{1,1}\rangle = (J^{-}_0 J^{+}_{-1} J^{-}_0 - t J^{-}_0 J^{0}_{-1} - t J^{0}_1 J^{-}_0 - t^2 J^{-}_{-1}) |J_{1,2,,t}\rangle
\]  \hspace{1cm} (45)

If \( J = J_{r,s,-} \) then \( J^{-}_0 \leftrightarrow J^{+}_{-1} \) and \( |\chi\rangle = |\chi_{rs,-r}\rangle \). We remark that in practice there is another way to find the singular vector which is more efficient and is briefly described in the appendix.

In order to see the geometrical meaning of (41) we denote \( \lambda_1 = 2J \) and \( \lambda_2 = k - 2J \). Using (44) we see that formally

\[
(J^{-}_0)^{\lambda_1+1} |J, t\rangle \simeq | -J - 1, t\rangle
\]

\[
(J^{+}_{-1})^{\lambda_2+1} |J, t\rangle \simeq |k - J + 1, t\rangle
\]

are singular vectors for any value of \( J \). The geometrical meaning of these operators is clear. In the \( \lambda \) plane (see Fig. 1) they conserve the line \( \lambda_1 + \lambda_2 + 2 = t \).

Fig. 1- The \( \lambda \) plane.

In fact the operators \( (J^{-}_0)^{\lambda_1+1} \) and \( (J^{+}_{-1})^{\lambda_2+1} \) are reflections of this line on itself by the points \((-1, t - 1)\) and \((t - 1, -1)\) respectively. They are the generators of the Weyl group. MFF\[20\] assert that for the values of \( J \) that were given by Kac and Kazhdan\[19\] we get after finite number of reflections a meaningful expression.
If $\lambda_1 + 1$ ($\lambda_2 + 1$) is positive then $(J_0^-)^{\lambda_1+1}|J,t\rangle$ $((J_+^{-1})^{\lambda_2+1}|J,t\rangle)$ is a formal singular vector in the Verma module $V_{J,t}$ ($V_{J,t}$). If on the other hand $\lambda_1 + 1$ ($\lambda_2 + 1$) is negative then $V_{J,t} \simeq (J_0^-)^{-\lambda_1-1}|J - 1, t\rangle$ $(V_{J,t} \simeq (J_+^{-1})^{-\lambda_2-1}|J - 1, t\rangle)$ is a formal singular vector in the Verma module $V_{-J-1,t}$ ($V_{-J-1,t}$). The only reducible modules which can not be generated by (formal) singular vectors are those that satisfy $\lambda_1, \lambda_2 \geq -1$. These conditions for the case $t = p/q$ read

$$0 \leq rq - p(s - 1) \leq p \quad (47)$$

For such $J_{r,s}$ we can calculate the inclusion diagram

**Fig.-2 J inclusion diagram**

Where

$$a_{r,s}^{p,q}(l) = J_{r-2p,s}^{p,q} = \frac{r - 1}{2} - \frac{s - 1}{2} \frac{p}{q} - lp$$
$$b_{r,s}^{p,q}(l) = J_{-r-2p,s}^{p,q} = \frac{r - 1}{2} - \frac{s - 1}{2} \frac{p}{q} - lp - r \quad (48)$$

are the spins of the singular vectors. An arrow from $x$ to $y$ means that $y$ is a singular vector of $x$. The solid arrows are those obtained by the Weyl reflections. The dashed arrows inclusion was proved by other method in ref. [39] as well as the fact that this list is exhaustive. It is interesting to notice that there exists a kind of “mirror symmetry” between the representations $|J, t\rangle$ and the representations $|-J-1, -t\rangle$. In fact for $J_{r,s,\pm}$ that satisfy (47), $|-J_{r,s,\pm} - 1, -t\rangle$ are irreducible modules which are included in an infinite number of Verma modules. The inclusion
diagram (Fig. 3) is similar to that of Fig. 2, only the direction of the arrows is inverted.

Fig. 3-I inclusion diagram

It is even more interesting to notice the striking similarity of this structure to the Virasoro representations. In the Virasoro case there exists also a “mirror symmetry” between $|h,c\rangle$ and $|1-h,26-c\rangle$.\cite{12} We will see in the following chapters that this similarity is the reason for the similarity between the spectra of $\frac{SL(2)}{SL(2)}$ and that of the minimal models coupled to Liouville. We refer to this mirror symmetry as “duality”. Note also that for $r = p$ eqn.(48) gives $a(l) = b(l + 1)$. Hence, in this case, we have just one set of singular vectors and the two branches of the embedding diagram of Figs. 2,3 degenerate into a single branch diagram.

5. Irreducible representation and the BRST cohomology

The next step in the extraction of the physical states is to pass from the cohomology on the Fock space to the irreducible representations of the level $k$ $A_1^{(1)}$ Kac-Moody algebra. In general a representation $L$ is reducible iff \[2L + 1 = r - (s - 1)(k + 2)\] where $r$ and $s$ are integers with either $r, s \geq 1$ or $r < 0, s \leq 0$.\cite{19} In the $G$ model with $G = A_1^{(1)}$ we therefore have

\[2J_{r,s} + 1 = r - (s - 1)(k + 2) \quad 2I_{r,s} + 1 = r + (s - 1)(k + 2) \quad (49)\]

Note that $J_{r,s} = -I_{-r,s} - 1$. Completely irreducible representations, which have infinitely many null vectors, appear provided that $k + 2 = \frac{p}{q}$ for $p$ and $q$ positive.
integers which can be chosen with no common divisor. In this case \( I_{r,s} = I_{r+p,s-q} \) and \( J_{r,s} = J_{r+p,s+q} \). It is, thus, enough to analyze the domain \( 1 \leq s \leq q \), and we will choose \( 1 \leq r \leq p - 1 \). This choice corresponds to the double line embedding diagram of Fig 2. The states corresponding to \( r = p \) have a single line embedding diagram. It was found that\(^{[22]}\) for the case of the double line one can construct the irreducible representation which is contained in \( \mathcal{F}_{r,s} \), the Fock space built on \( |J_{r,s}> \). This is achieved via the cohomology of an operator \( Q_J \) which acts on \( \mathcal{F}_{r,s} \) the union of the Fock spaces that correspond to \( J_{r+2lp,s} \) and \( J_{r-2lp,s} \) for every integer \( l \). It turns out\(^{[22]}\) that the relevant information is encoded in \( H^0(\mathcal{F}_{r,s}, Q_J) \) and all other levels of the cohomology vanish. The cohomology is only in the \( J \) sector and not in the \( I \) sector just as there is no use of the cohomology of the Liouville sector in models of \( C < 1 \) matter coupled to gravity\(^{[13]}\). Thus, the space of physical states of ghost number \( n \) is given by

\[
H^{(n)}_{rel}[H^{(0)}(\mathcal{F}_{r,s}, Q_J) \times \mathcal{F}_I \times \mathcal{F}_G, Q_J] \quad (50)
\]

where \( \mathcal{F}_G \) is the ghosts’ Fock space built on the new vaccum \( |0>_G \). Since \( Q_J \) acts only in the \( J \) sector we can rewrite \( H^{(n)}_{rel} \) as

\[
H^{(n)}_{rel}[H^{(0)}(\mathcal{F}_{r,s} \times \mathcal{F}_I \times \mathcal{F}_G, Q_J), Q_J]. \quad (51)
\]

Moreover, since \( \{Q, Q_J\} = 0 \) one can use theorems\(^{[21]}\) about double cohomologies and write this as isomorphic to

\[
H^{(n)}[H^{(0)}_{rel}(\mathcal{F}_{r,s} \times \mathcal{F}_I \times \mathcal{F}_G, Q_J), Q_J]. \quad (52)
\]

The theorems\(^{[21]}\) apply only provided that each cohomology separately is different from zero only for one single degree. In the present case this was shown in section 3. In fact we have already calculated \( H^{(0)}_{rel}(\mathcal{F}_{r,s} \times \mathcal{F}_I \times \mathcal{F}_G, Q_J) \) since \( \mathcal{F}_{r,s} \) is the union of free Fock spaces. Hence the result is that the latter has one state if the Fock space of \( J = -I - 1 \) is in \( \mathcal{F}_{r,s} \), and it is empty otherwise. For each \( J_{r,s} \) we get
states at $I = -J_{r+2lp,s} - 1$ and $I = -J_{r+2lp,s} - 1$ where their ghost number is equal to the corresponding degree in the complex of ref. [22]. For each such $J_{r,s}$ there is an infinite set of states with $I = I_{r-2lp,s}$, $G = -2l$ and $I = I_{r-2lp,s}$, $G = 1 - 2l$ for every integer $l$. An example is the case of $k = 0$ which is discussed in detail in section 7. There $J_{r,s} = 0$ since the only possible values of $r$ and $s$ are $r = s = 1$. Thus the possible states are at $I_{r-2lp,s} = -2l - 1$ and $I_{r-2lp,s} = -2l$ which implies that the possible values of the ghost number are $G = I + 1$ for every integer $I$.

Though the general derivation of the cohomology does not give an explicit construction of the physical states, it is clear that they involve null states. (Recall that originally there is only one physical state in the Fock space.) In our construction an irreducible representation of the $J$ sector was used namely null states were eliminated, while the $I$ sector was left as a Fock space. In the latter case, it can be shown that half of the null states vanish identically on the Fock space. For instance in the $k = 0$ case, for negative ghost number one has $Q|\psi\rangle = |null\rangle$, where by $|null\rangle$ we mean a null state or it’s descendant. Hence, these states are in the cohomology provided the corresponding null states vanish, which indeed is the case. For positive ghost number there exist states $|\psi\rangle$ for which $Q|\psi\rangle = |phys\rangle + |null\rangle$ and therefore those states are in the cohomology only when the corresponding null states are non-zero. The next step is to compute the level $\hat{L}_0$ and $\hat{J}^{(tot)}_0$ for the excitations on all states. The results are summarized as follows

\begin{align*}
J &= J_{r,s}, \quad I = I_{r-2lp,s}, \\
G &= -2l \\
\hat{L}_0 &= l^2pq + l(qr - sp) + lp \\
\hat{J}^{(tot)}_0 &= lp
\end{align*}

\begin{align*}
J &= J_{r,s}, \quad I = I_{r-2lp,s}, \\
G &= 1 - 2l \\
\hat{L}_0 &= l^2pq - l(qr + sp) + r(s - 1) + lp \\
\hat{J}^{(tot)}_0 &= lp - r
\end{align*}

where $\hat{L}_0$ is given in eqn. (27) and $\hat{J}^{(tot)}_0 = -I - J - 1$ is the total charge of the excitations since the total $J_0$ vanishes and our ghost vacuum has $J_0 = 1$. Again we should note that had we used the reversed parametrization, the results would have
been analogous to those just derived. Since the direction of the cohomology in the complex of ref. [22] would have been reversed, we would have obtained the same states but with opposite ghost numbers, namely, $G = 2l$ and $G = 2l - 1$ instead of $G = -2l$ and $G = 1 - 2l$ respectively. The partition function that is computed below is not affected by those changes.

The last step in the deduction of the space of physical states is the passage to the absolute cohomology which is the same as in ref. [13] since, as we have just shown, there is a single state for each $I$ and $J$,

$$H_{abs}^{(n)} \simeq H_{rel}^{(n)} \oplus \chi_0 H_{rel}^{(n-1)}.$$ (54)

Let us now examine whether we can verify the index interpretation of the torus partition function which was discussed in section 2. For integer $k = 0, \ldots, k/2$. The partition function in terms of the characters $M_{k,j}$ was given in eqn. (14). We want to check now whether it can be rewritten as a trace over the space of the physical states. One has to insert the values of $\hat{L}_0$ and $\hat{J}_0 \to \hat{J}_0$ of eqn. (53) into eqn. (15), with $\hat{J}_0 \to \hat{J}_0$ and $G$ shifted to the values defined for an $SL(2)$ invariant vacuum. Inserting these values for every $l$ one gets exactly the expression of eqn.(15). We can obviously add the values of the level and the eigenvalue of $J_{(tot)}^0$ of the $|J >$ vacuum namely $\hat{L}_0 \to \hat{L}_0 + \frac{J(J+1)}{k+2}$ and $J_{(tot)}^0 \to J_{(tot)}^0 + J$ to derive a simpler expression

$$M_{k,j}(\tau, \theta) = \frac{1}{2i} q^{\frac{k}{2(k+2)}} e^{i\pi\frac{q^2}{2}} Tr[(-)^G q^{\hat{L}_0} e^{i\pi\theta \hat{J}_0 \to \hat{J}_0}].$$ (55)

### 6. The Correspondence to $c < 1$ Models Coupled to Gravity

Let us now raise the question of whether one can map the $\frac{SL(2)}{2\pi} \times \frac{SL(2)}{2\pi}$ model to minimal models coupled to gravity. Or differently to what extent are the two types of TCFT equivalent. The minimal models can be formulated either in a Liouville approach [3] or in a world-sheet light-cone gauge [28]. Since these two gauges
are believed to be equivalent\textsuperscript{[27]}: it is enough to demonstrate the correspondence of the $G$ model with one of the two. Nevertheless, we discuss here the relations of the $G$ with the two 2d gravity pictures.

Motivated by the comparison of the partition functions of the $G$ model of level $k = \frac{p}{q} - 2$ and that of a $(p, q)$ model coupled to gravity, which is discussed below, we define now a twisted energy-momentum tensor $\tilde{T}$ as follows:

$$T(z) \to \tilde{T}(z) = T(z) + \partial J^{(tot)}_0(z)$$  \hfill (56)

Since both $T(z)$ and $J^{(tot)}_0(z)$ are BRST exact so is $\tilde{T}(z)$. Hence, this twist does not introduce a Virasoro anomaly. However, both the dimensions and the contributions of the various currents to the central charge are now altered. In addition it is clear that $\tilde{T}$ is not an $A_{1}^{(1)}$ invariant operator. The contributions to $c$ of $A_{1}^{(1)}$ currents at level $k$ are shifted from $\frac{3k}{k+2} \to \frac{3k}{k+2} - 6k$. Hence, for the case of $k + 2 = \frac{p}{q}$ one gets the following anomalies in the $J$, $I$ and $J^{(gh)}$ sectors

$$\tilde{c}_J = 2 + c_{p,q} \quad \tilde{c}_I = 2 + (26 - c_{p,q}) \quad \tilde{c}_{gh} = -30$$ \hfill (57)

where $c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$. One can rewrite the contribution to the total central charge in the form they appear in the Liouville and the light-cone formulations

\begin{align*}
c^{(tot)} & = c_{p,q} + (26 - c_{p,q}) - 26 \\
c^{(tot)} & = c_{p,q} + (\frac{3\kappa}{\kappa + 2} - 6\kappa) - 28 \hfill (58)
\end{align*}

where $\kappa = -k - 4$. Obviously so far it is only a rewriting of zero and by itself it does not prove much. However, we want to check whether one may provide a map between the set of fields of the $A_{1}^{(1)}$ $G$ model and those of minimal models coupled to gravity. In particular eqn. (58) may suggest the following correspondence. The $J$ sector contains the “matter” degrees of freedom and a bosonic $(1, 0)$ system which compensates a similar anticommuting system from the ghost sector. The $I$ sector is
the $SL(2, R)$ gravity part in the approach of ref.[28] or it is the Liouville sector plus an additional commuting system of $c = 2$ which again pairs with a ghost partner. The $\rho, \chi$ ghosts translate into the $b, c$ system (or to the ghost system of ref [29]) plus additional anticommuting $(1, 0)$ degrees of freedom. This implies that the ghost vacuum transformation $|0 \rangle \rightarrow \chi_0^+ |0 \rangle$ corresponds to the usual $|0 \rangle \rightarrow c_1 |0 \rangle$.

Before discussing these kind of relations let us observe another “numerological correspondence” between KPZ[29] and the $A_{1}^{(1)} \frac{G}{G}$ model. Recall that the relation between the “undressed” and “dressed” dimensions is

$$\lambda + \Delta^{(0)} - \frac{\lambda(1+\lambda)}{k+2} = 0.$$  

If we identify $J$ with $\lambda$ and take for it $J = J_{r,s}$ then we find that

$$\Delta^{0} = \frac{J(1+J)}{k+2} - J = \frac{1}{4t} [(ts-r)^2 - (t-1)^2] = \Delta_{r,s}^{(0)}$$  

for $t = \frac{p}{q}$.

The modified $T$ of eqn. (56) introduces the following modified dimensions:

$$(\rho^-, \chi^+ ) \rightarrow (2, -1) \leftarrow (\bar{\beta}, \bar{\gamma})$$

$$(\chi^-, \rho^+ ) \rightarrow (1, 0) \leftarrow (\gamma, \beta)$$

The conformal dimension of $\chi^0, \rho^0$ and $\phi, \tilde{\phi}$ remains the same.

Whereas it is obvious that $\phi$ corresponds to the Fegin-Fuks[40] free field representation of the matter part, since their central charges are the same, one cannot simply assign $\tilde{\phi}$ to the Liouville degree of freedom. This is due to the fact that its background charge is $\frac{1}{2}(\sqrt{k+2} - \frac{1}{\sqrt{k+2}})$ which leads to a contribution to the Virasoro central charge $2 - c_{p,q}$ rather than $26 - c_{p,q}$. The role of $\partial \phi_L$, where $\phi_L$ is the Liouville mode, is played here by a combination of $\partial \tilde{\phi}$ and $\bar{\beta} \bar{\gamma}$. Notice that for $k = -1$ which formally corresponds to $c = 1$ both $\phi$ and $\tilde{\phi}$ have a vanishing background charge. In the reversed parametrization mentioned in section 3, $\tilde{\phi}$ would have had the central charge of the Liouville mode but then the central charge of $\phi$ would have been $c_{p,q} - 24$ instead of $c_{p,q}$. Obviously using the same parametrization for both the $J$ and $I$ sectors implies that $\phi$ and $\tilde{\phi}$ have the background charges
of the two scalars in the \((p, q)\) models coupled to gravity. Moreover, in this case the systems \(\beta, \gamma\) and \(\tilde{\beta}, \tilde{\gamma}\) systems carry \((1, 0)\) dimensions and pair with \(\chi^- \rho^+\) and \(\chi^0 \rho^0\).

Next we want to compare the partition function of the \((p, q)\) model to that of \(\mathcal{G}\) for \(G = SL(2)\) and \(k = \frac{q}{q} - 2\). Comparing eqn. (15) to the numerator of the character of the minimal model it is clear that correspondence might be achieved only provided one shifts \(\tau \rightarrow \tau - \frac{1}{2} \theta\) or equivalently taking \(e^{i \pi \theta} = q^{-1}\). In this case the numerator of the character in the minimal model which is proportional to \(Tr[(-1)^G q \hat{L}_0]\) is mapped into \(Tr[(-1)^G q \hat{L}_0 - \hat{J}^{(tot)}_0]\) in the \(\mathcal{G}\) model. That is the origin of the twisted \(T\) defined in eqn. (56). We thus need to compare the number of states at a given level and ghost number in the minimal models with the corresponding numbers at the same ghost number and “twisted level”. From eqn. (53) we read

\[
I = I_{-r-2lp, s} \quad G = -2l \quad \hat{L}_0 - \hat{J}^{(tot)}_0 = l^2 pq + l(qr - sp) \\
I = I_{r-2lp, s} \quad G = 1 - 2l \quad \hat{L}_0 - \hat{J}^{(tot)}_0 = l^2 pq - l(qr + sp) + rs
\]

In the minimal models we have states built on vacua labeled by the pair \(r, s\) with \(1 \leq r \leq p - 1\) and \(1 \leq s \leq q - 1\) with \(ps > qr\) which have dimension \(h_{r, s} = \frac{(qr-sp)^2-(p-q)^2}{4pq}\). The levels of the excitations are \(\hat{L}_0 = \Delta - h_{r, s}\). For \(G = 2l + 1\) one has \(\Delta = A(l) = \frac{[(2pql+qr+sp)^2-(q-p)^4]}{4pq}\) and for \(G = 2l\) \(\Delta = B(l) = \frac{[(2pql-qr+sp)^2-(q-p)^2]}{4pq}\). Hence, the the contribution of the various levels to the partition function are identical to those of \(\hat{L}_0 - \hat{J}^{(tot)}_0\) in eqn. (61) for the same ghost numbers and the respective vacua satisfy \(J = \sqrt{\frac{p}{2q}} p_m\) and \(I = \sqrt{\frac{m}{2q}} p_L\) where \(p_m\) and \(p_L\) are the matter and Liouville momenta respectively. It is thus clear that for a given \(r, s\) we get the same number of states with the same ghost number parity in the two models and thus the two partition functions on the torus are in fact identical. The relation between \(a_{r, s}(l)\) \((b_{r, s}(l))\) of eqn. (48) and the dimensions \(A_{r, s}(l)\) \((B_{r, s}(l))\) of the null states appearing on the minimal models.
embedding diagram is given by

\[ \frac{1}{\ell} a_{r,s}^{p,q}(l)(a_{r,s}^{p,q}(l) + 1) - a_{r,s}^{p,q}(l) = A_{r,s}^{q,p}(l) = A_{s,r}^{p,q} \]  \hspace{1cm} (62)

with a similar expression relating \( b_{r,s}(l) \) to \( B_{r,s}(l) \). To obtain the partition function of the \((p,q)\) models coupled to 2d gravity we have restricted \( r \) and \( s \) as follows

\[ 1 \leq s \leq q - 1 \text{ and } 1 \leq r \leq p - 1. \]

It is interesting to note that we could include in the sum over \( r \) and \( s \) which appears in the partition function also the terms with \( r = p \). Those terms arise from the states which have a single line as their embedding diagram. The \( r = p \) terms cancel between themselves and do not change the result for the partition function. This cancelation follows from the observation that \( a_{r,s}(l) = b_{r,s}(l + 1) \) which translates via eqn. (62) into \( A_{r,s}(l) = B_{r,s}(l + 1) \).

The \((p,s)\) states appear at the boundary of the Kac table. Further discussion of these states appear in the last section.

In fact there are 4 different possible identifications of the \( \mathcal{G}_G \) states with those of the minimal models. There are the two possible parametrizations discussed in the previous sections, and there is in each of them the possibility to assign \( h_{r,s} \) either to \( J_{r,s} \) or \( J_{p-r,q-s} \) with the appropriate ghost numbers. The complete identification of a minimal model coupled to gravity and its \( \mathcal{G}_G \) counterpartner requires obviously the identification of all physical states and all non-trivial correlators in both theories. This question is under current investigation.

7. Physical states and correlation functions in the \( SU_{k=0}(2) \) case\[^{23}\]

As an explicit demonstration of the general results obtained in the previous section we consider the simplest non-trivial case namely \( \mathcal{G}_G \) model for \( SU(2) \) at level \( k = 0 \). Here we present the calculation of the cohomology in the free field parametrization of section 3 as well as in a scheme where both the \( J \) and the \( I \) sectors are parametrized according to eqn.(22) and show how the equivalence between the two methods is utilized in the determination of non-trivial correlation.
functions. The matter sector of the Hilbert space consists only of the \(|J = 0>\) highest weight state since all other states are nulls. Therefore, in this case the choice of the bosonization in the \(J\) sector does not affect the results. The \(I\) sector is associated with the \(k = -4\) Kac-Moody algebra and in addition there is the usual \((1, 0)\) ghost system. Following the general analysis of sections 3 and 5, the operators which furnish the cohomology of the \(k = 0\) case are found to be

\[
\begin{align*}
\tilde{V}_{n-1} &= \chi^0 \chi^+ \partial \chi^- \ldots \partial^{n-1} \chi^- e^{-n\tilde{\delta}^+} \delta(\tilde{\gamma}) \\
\tilde{V}_1 &= \chi^0 \chi^+ \delta(\tilde{\gamma}) \\
\tilde{V}_{-n-1} &= \chi^0 \chi^+ \rho^+ \partial \rho^+ \ldots \partial^{n-1} \rho^+ e^{n\tilde{\delta}^+} \delta(\tilde{\gamma}) + \text{corrections}
\end{align*}
\]

for \(n > 0\). In the second scheme that we have used, the same parametrization for both the \(J\) and \(I\) currents is used. It is the one given in eqn.(22) where now \(a = \sqrt{2}\) and \(-i\sqrt{2}\) for the \(k = 0\) and \(k = -4\) sectors respectively. As mentioned earlier the passage between the two prescriptions is via the automorphism \(I^0 \leftrightarrow -I^0\) and \(I^\pm \leftrightarrow I^{\mp}\). The other difference is the properties of the vacuum. The operators in section 3 were constructed with respect to the \(\tilde{\gamma}_0|0>\), whereas now we use the \(SL(2, R)\) invariant vacuum which is annihilated by \(\tilde{\beta}_0\). Formally the relation between the two vacua can be written as \(|\tilde{\gamma}_0 = 0> = \delta(\tilde{\gamma})|\tilde{\beta}_0 = 0>\)

First it is straightforward to check that the following states are in the cohomology in this parametrization.

\[
\begin{align*}
|V_0> &= |I = 0> \\
|V_n> &= \chi^+_0 \chi^+_1 \ldots \chi^+_n |I = -n>
\end{align*}
\]

Next we want to extract states with negative ghost number. The steps in the construction are the following. First we search for states \(|V_{-n}>\) which are BRST closed up to a null state namely

\[
Q|V_{-n}> = |null >_n
\]

where by \(|null >_n\) we denote a null state or a descendant of a null which is built
on the state $|I = n>$ as depicted in the embedding diagram. In addition we pick states which carry the lowest ghost number possible for the given level and $\hat{J}^{(tot)}$, so that they cannot be BRST exact. It is easy to verify that states of the form $|V_n >= (\rho^-_{-1}\rho^-_{-2}..\rho^-_{-n} + \text{corrections})|I = n>$ obey these conditions. For instance the first state which has the explicit form

$$|V_{-1} >= (\rho^-_{-1} + \rho^0_{-1}I_0^- - \frac{1}{2}\rho^+_{-1}(I_0^-)^2)|I = 1>$$

leads to the null state of $r = 1, s = 2$ when acted with Q namely

$$Q|V_{-1} >= (I_{-1}^- + I^0_{-1}I_0^- - \frac{1}{2}I^+_{-1}(I_0^-)^2)|I = 1> = |\chi_{1,1} >\equiv \hat{\chi}_{1,1}|I = 1>,$$

From the general structure of singular vectors described in section 4 we can read the eigenvalues of $|\chi_{1,1} >$, $L_0|\chi_{1,1} > = I^0_0|\chi_{1,1} > = 0$. The next null state $\hat{\chi}_{2,1}|I = 2 > = |\chi_{2,1} >$ corresponds to $|V_{-2} >$ as follows $Q|V_{-2} >= (\rho^-_{-1} + \text{correction})\hat{\chi}_{2,1}|I = 2 >$ with the correction terms carrying $L_0 = 1$ and $J^{(tot)}_0 = -1$. Notice that since $|\chi_{2,1} >$ has $L_0 = 1$ and $J^{(tot)}_0 = +1$, it is its descendant which is in the image of $Q$. It is easy to verify that the general structure of the BRST charge acting on the states $V_{-n}$ takes the form $Q|V_{n} >= (\rho^-_{-1}\rho^-_{-2}..\rho^-_{-(n-1)} + \text{correction})\hat{\chi}_{n,1}|I = n>$.

It can be proven \cite{22} that all the states $|\chi_{r(s-1),r} >$ with positive $r, s$ vanish upon using the bosonization eqn.(22) Since we are interested in the space which includes the Fock space of the $I$ sector, it is obvious that $|V_{-n} >\in H^*(Q)$. Thus there is no need to perform the procedure described in ref.[22] in this sector. Clearly restricting ourselves to the Verma modules of the current algebra gives rise to a different cohomology. It is presumably a general feature of the embedding diagram that half of the null states vanish upon invoking the parametrization of eqn.(22). This is proven in ref. [22] for any $k > 0$. The generalization to $k \leq 0$ seems to follow essentially the same steps. In the present case the null states which carry positive values of $I$, namely those which are on the right handed branch of the dual embedding diagram (Fig 3.) vanish. The situation is reversed once one uses the
parametrization of section 3. Writing the state \( |I = n > \) in terms of the operator \( e^{n\tilde{\phi}} \) acting on the \( SL(2, R) \) invariant vacuum, we can now write down the complete set of operators that span the cohomology.

\[
V_n = \chi^+ \partial^+ \cdots \partial^{n-1} \chi^+ e^{n\tilde{\phi}} \\
V_0 = 1 \\
V_{-n} = \rho^- \partial \rho^- \cdots \partial^{n-1} \rho^- e^{n\tilde{\phi}} + \text{corrections}
\]

(68)

Note that all these operators carry zero dimension and thus they close upon a ring. The multiplication operation is just the OPE, namely

\[
V_m(z)V_n(\omega) = V_{m+n}(\omega) + \{Q, O'\}
\]

(69)

where \( O' \) is a dimension zero and ghost number \( n + m \) operator. The \( Q \) exact term does not always appear. For instance there is no such a term for \( n, m > 0 \). Using OPE’s we can, in principle, starting from eqn. (66) calculate all the \( V_{-n} \).

Next we proceed to the calculation of correlation functions. To calculate the expectation value of products of operators which are in the BRST cohomology, one has to define the scalar product or equivalently the notion of the conjugate to a given state. Due to appearance of zero modes of the ghosts \( \chi_0^+, \chi_0^0, \chi_0^- \), the zero mode of the commuting field \( \tilde{\gamma}_0 \) and the background charge \(-1\) of \( \tilde{\phi} \), it is clear that the vacuum is not self-conjugate, \(< 0|0 >= 0 \). Instead, \(< \tilde{0}| \) the conjugate to \( |0 > \) is given by

\[
< \tilde{0}| = < 0|\chi_0^+ \chi_0^0 \chi_0^- e^{-\tilde{\phi}} \delta(\tilde{\gamma}), \quad < \tilde{0}|0 >= 1
\]

(70)

where we have introduced formally \( \delta(\tilde{\gamma}) \) to absorb the zero mode of \( \tilde{\gamma}_0 \). To compute correlators we thus define formally another operator in the cohomology

\[
\tilde{1}(z) = \chi^+ \chi_0^0 \chi^- e^{-\tilde{\phi}} \delta(\gamma) = \tilde{V}_0.
\]

(71)
and denote by $\tilde{V}_n$ the result of the OPE of the $\tilde{I}$ with $V_n$ namely

$$\tilde{I}(z)V_n(\omega) = \tilde{V}_n(\omega) + O(z - \omega). \quad (72)$$

We see that the $\tilde{V}_n$ are exactly the operators appearing in the second parametrization up to an interchange $\chi^+ \leftrightarrow \chi^-$ and $\rho^+ \leftrightarrow \rho^-$. Using these new definitions we can write the correlators of the model as follows

$$G(z, z_1, \ldots, z_n) = <\tilde{0}|V_n(z) \prod_{i=1}^{N} V_{n_i}(z_i)|0> = <0|\tilde{I}V_n(z) \prod_{i=1}^{N} V_{n_i}(z_i)|0> = <0|\tilde{V}_n(z) \prod_{i=1}^{N} V_{n_i}(z_i)|0> \quad (73)$$

In particular it is obvious that $<\tilde{V}_{-n}|V_n> = 1$. This proves that the states $|V_n>$ given above are not exact, since otherwise their correlations with closed states would vanish. Notice that since all the operators in the correlators are of zero dimension the result is a number which is independent on $z, z_i$, as it is expected for a topological model. The ghost number of $V_n, \tilde{V}_n$ are $n, n + 3$ and the momentum $P_\phi$ are $-n, -(n + 1)$ respectively. The conditions for a non-vanishing correlator on the sphere are

$$\sum G = 3 \quad \sum P_\phi = -1 \quad (74)$$

which translate into a single condition for $G(z, z_1, \ldots, z_n)$ namely $n + \sum_i n_i = 0$. Usually in TCFT there are in addition to the correlators of zero dimension operators also those of (1, 1) forms. The general derivation of the latter from the zero forms follows the general construction of ref. [30,31]

$$V_n^{1,1} = \int d^2z G_{-1} \tilde{G}_{-1} V_n \tilde{V}_n = \int d^2z W_n \tilde{W}_n \quad (75)$$

The holomorphic part of the lowest operator takes the form

$$W_{-1} = 2\rho^- \rho^+ I_0^{-} e^{\tilde{\phi}} \quad (76)$$

The computation of non-trivial correlators, and the relation with the (2, 1) model, namely pure topological gravity will appear in a future publication.
8. **Summary and Discussion**

In the present work we have worked out the space of physical states of the $G$ models for the case of $A_1^{(1)}$. Strictly speaking only for integer levels we could have used $SU(2)$ gauged WZW model. For non integer $k$ one has to adopt the $SL(2,R)$ counterpart. The extraction of these states was done in two stages. At first the BRST cohomology on a Fock space based on a “free fields” parametrization was derived. In order to apply a method developed in ref. [13] for the Virasoro case, we had to apply a conjugate parametrization in the $J$ and $I$ sectors. The second stage was the translation of physical states from the Fock space into the irreducible representation of the corresponding Kac-Moody algebras. For this procedure we implemented the structure of the singular vectors of the $A_1^{(1)}$ algebra for arbitrary $k$. A “duality” between the singular vectors associated with the Verma module of $|J,k>$ and that of $|(J+1),-k-4>$ is expressed in the embedding diagrams (Fig 2,3). This is analogous to the “duality” in the Virasoro algebra between $|\delta,c>$ and $|1-\delta,26-c>$. 

The physical states are built on the highest state vacua $|J,I>$ where $J = J_{r,s}$ and $I = I_{r-2l,p,s}$ or $I = I_{r-2l,p,s}$ for every integer $l$ with $G = 1 - 2l$ and $-2l$ respectively. The $I$ values are those of points in the dual embedding diagram (Fig 3). They are related to values of the points on the $J$ diagram (Fig 2.) by $I = -(J+1)$. The ghost number $G$ is in fact the “distance” between the point on the $I$ diagram and the top of this diagram. Using the entire space of physical states we constructed explicitly the characters of the $G$ theory and verified the index interpretation of those characters.

Perhaps the most intriguing and interesting observation of this work is the intimate connection of the $SL(2)$ models at $k = \frac{p}{q} - 2$ and the $(p,q)$ minimal models coupled to gravity. The set of primary fields correspond to the $|J_{r,s}>$ in the $J$ sector. The fusion rules of the latter are discussed in the appendix and arguments are given in favor of the conjecture$^{[41]}$ that they are closed under their O.P.E.. The primaries and their $I$ descendants at all ghost numbers carry conformal dimensions
which match, after twisting, those of the minimal models. Moreover, the partition
function coincides with that of the latter provided that a particular value of the
moduli of the flat $G$ connection i.e $u = q^{-1}$ is chosen. This amounts to shift
$\hat{L}_0 \to \hat{L}_0 - J^{(\text{tot})}_0$ which would follow from twisting the energy momentum tensor,
$T \to T + \partial J^{(\text{tot})}_0$. The conformal dimension of the various fields with respect to this
modified operator are shown to correspond to those of the minimal models with
an addition of two “topological” $(1,0)$ systems. A complete isomorphism between
the theories would be established when one compares positively the correlation
function in the two theories. A relation between correlators in the $A_1^{(1)}$ WZW
model at level $k = \frac{p}{q} - 2$ and those of the $(p,q)$ models have been worked out
within the Hamiltonian reduction approach.\[^{33}\] We expect similar relations to hold
in the topological version which we study in this paper. Alternatively, one would
like to identify the TCFT algebra of the two theories. These topics are under
current investigation. Notice that there is an apparent difference in the boundaries
of the Kac table between the $G$ models and the $(p,q)$ models. While in the latter
there is a double line embedding diagram and hence states at every ghost number
for $1 \leq r \leq p - 1$, $1 \leq s \leq q - 1$, in our case this happens also for $s = q$. Single-
line diagrams, which correspond to physical states only at a finite set of ghost
numbers, appear in the minimal models on both boundaries of the Kac table namely,
$r = p$ and $s = q$ whereas in the $G$ models they exist only for $r = p$. As was discussed
in section 6 those differences do not affect the equivalence of the torus partition
functions of the two types of models. Certainly the states on the boundary of the
Kac table deserve further investigation. An explicit construction of the physical
states was demonstrated for the case of $k = 0$. The zero dimension operators which
correspond to the states produced can further generate $(1,1)$ forms which are in
the BRST cohomology, $V^{(1,1)} = \int dzd\bar{z}G_{-1}\bar{G}_{-1}V^{(0)}(z)\bar{V}^{(0)}(\bar{z})$. That hierarchy of
operators, which is typical to all TCFT, holds for every value of $k$. The zero forms
establish a structure of a ring. It is expected that for the general case the zero ghost
number operators form a fusion ring;\[^{5,36,34,35}\] i.e. $V_{J_1}(z)V_{J_2}(\omega) = N_{J_1,J_2,J_3}V_{J_3}(\omega)$
where the $J_i$ refer to the matter primaries and the $N_{J_1,J_2,J_3}$ are the fusion algebra

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coefficients. It would be instructive to establish this result once the operators are explicitly constructed.

Correlators in topological 2d gravity are known to obey recursion relations.\cite{4,30} We expect, therefore, similar recursion relations involving correlators defined on higher genera to be derivable in the context of the $G$ models. Recently, such recursion relations were derived using Ward identities related to the $W_\infty$ symmetry of the ground ring of $c=1$\cite{15} as well as using “contact relations”\cite{16}. It will be very interesting to recast these results in the gauged WZW framework. In particular for $c<1$ the Ward identities approach is missing in the Liouville approach. It is still not clear what will be the fate of its $G$ model counterpartner. A long standing problem of TCFTs is to write down a theory which is isomorphic to the $c=1$ model. It is not difficult to realize that the $k=-1$ model has the right numerologics to play the role of this theory especially when the bosonization of eqn.(22) is chosen in both sectors. This model as well as some others are under current investigation. In fact it is easy to check using eqn.(57) that the level which corresponds to a given $c$ of the matter sector is $k = -\frac{11}{12}(11 + c \mp \sqrt{(c - 1)(c - 25)}).$

Not surprisingly a $G$ model with real $k$ has a forbidden domain which is the familiar range of $c$, $1 < c < 25$.

This work was entirely devoted to the $A_1^{(1)}$ case. It is pretty clear that a great part of the results achieved here could be extended to other $A_{n-1}^{(1)}$ cases and maybe other algebras. Generalizing the twisting of $T$ in the form $T \rightarrow T + \sum_i \partial J^{(tot)}_i$ where $i$ runs over the Cartan subalgebra, one gets for $SL(N)$ in analogy to eqn.(57) the following contribution of the “matter” sector to $c$, $c = (N-1)/[(2N^2 + 2N + 1) - N(N+1)(t+\frac{1}{2})]$ where now $t = k + N$. Here we have assumed a ghost system of $W_N$ gravity. This is identical to the $c$ of $W_N$ models. We note that the relationship between $SL(N)$ and $W_N$ was established also via the Hamiltonian reduction.\cite{18,38} Explicit discussion of these models will appear in a future publication. Another obvious generalization is to the case of super Lie algebras. In particular for $G = SL(N, N-1)$ we expect to obtain the Kazama- Suzuki $\frac{SU(N)}{SU(N-1) \times U(1)}$ matter, which has been shown\cite{37} to have super $W_N$ algebra as its chiral algebra, coupled to super
W_N gravity. In this respect we also recall the work of ref. [35] where the embedding of $G/G$ models into topological matter theories was obtained by twisting hermitian symmetric $N = 2$ supersymmetric coset models.

Acknowledgements: We are indebted to M. Bershadsky, D. Levy, N. Marcus, Y. Oz and M. Spigelglass for many fruitful discussions. One of us N.S would like to thank the School of Physics in the University of Tel-Aviv for its kind hospitality. He also would like to thank M. Bauer, P. Di Francesco, I. Kostov, M. Petropoulos and J.-B. Zuber for discussions.

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APPENDIX

Fusion rules

We introduce a chiral primary field $\phi^j_m(z)$ w.r.t the Virasoro algebra as well as w.r.t $A_{1}^{(1)}$. It transforms as a vector under the horizontal algebra (the zero modes algebra):

$$[J^a_0, \phi^j_m(z)] = R^a_{mn} \phi^j_n(z)$$

Following Zamolodchikov and Fateev\cite{1} we introduce an auxiliary parameter in order to have

$$[J^a_0, \phi^j(z, x)] = R^a(x) \phi^j(z, x)$$

where $R^a(x)$ is a differential operator.

The correspondence fields-states is given by $\lim_{x \to 0} \lim_{z \to 0} \phi^j(z, x) |\Omega, t > = |j, t >$. Here $|\Omega, t >$ is the vacuum which is characterized as a highest weight state that is annihilated by the whole horizontal algebra.

The Virasoro algebra and $A_{1}^{(1)}$ are related by the Sugawara construction

$$L_n = \sum \frac{1}{t} \left( : J^0_{-n-m} J^0_m : + \frac{1}{2} : J^+_n J^-_m : + \frac{1}{2} : J^-_{-n-m} J^+_m : \right)$$

In this formulation $L_{-1}$ and $J^0_0$ generate translations in $z$ and in $x$ respectively. Thus, we can write

$$\phi^j(z, x) = e^{x J^-_0 + z L_{-1}} \phi^j(0, 0) e^{-x J^-_0 - z L_{-1}}$$

It follows that

$$[J^a_n, \phi^j(z, x)] = z^n \{(a + 1)^{[a+2]} x^{a+1} \frac{d}{dx} + (a + 1) x^a j\} \phi^j(z, x) \quad (A.1)$$

Let us look now at the short distance operator product expansion for these chiral
primary fields. For this aim it is more convenient to write

$$\phi^j(z, x) = z^{-h+L_0} x^{j-J_0^0} \phi^j(1, 1) x^{j_0} z^{-L_0}$$

which is a consequence of eq. A.1 with $a = 0$. Imagine that we are interested only in the $j$ sector in the fusion of $j_0$ and $j_1$ then

$$\phi^{j_0}(z, x)|_{j_1, t} > = \phi^{j_0}(z, x)\phi^{j_1}(0, 0)|\Omega, t >$$

in sector $j \sum_{n=0}^{\infty} \sum_{m=-n}^{\infty} z^{h-h_0-h_1+n} x^{-j+j_0+j_1+m} \psi_{n,m}^j(0, 0)|\Omega, t >$$

(A.2)

$\psi_{n,m}^j$ are fixed by the requirement that the two sides of (A.2) transform in the same manner under the Vir and $A_1^{(1)}$ algebras. It is clear also that $\psi_{0,0}^j(0, 0)$ is proportional to $\phi^j(0, 0)$. Using the Sugawara construction we can write the following recursion relation.

$$(nt + m(2j + 1 - m))\psi_{n,m}^j = (-j + j_0 + j_1 + m + 1) \sum_{k+l=n \atop k \geq 1} J^+_k \psi_{l,m+1}^j$$

$$+2(j - j_0 - m) \sum_{k+l=n \atop k \geq 1} J^0_k \psi_{l,m}^j$$

(A.3)

$$+(j - j_0 + j_1 - m + 1) \sum_{k+l=n \atop k \geq 0} J^-_k \psi_{l,m-1}^j$$

We define $A_j(n, m) = nt + m(2j + 1 - m)$ and introduce an order $(n_0, m_0) > (n, m)$ if $n_0 > n$ or if $n_0 = n$ and $m_0 > m$. We see that (A.3) is a recursion relation since $(n, m)$ of the left hand side is bigger then all the pairs of indices in the right hand side. The $\psi$'s are well defined as long as $A_j(n, m) \neq 0$. In the case where $A_j(n, m) = 0$ and $m$ divides $n$ the L.H.S of (A.3) vanishes and we find in the R.H.S $P(j, j_0, j_1) \times$ (singular vector). The equation $P(j, j_0, j_1) = 0$ is then a fusion rule since $P$ should vanish for the fusion to be possible.
We use results and notations of ref.[41] reviewed briefly above as a framework for discussion of the fusion rules. The constraints on the possible operator content of a given theory are a direct consequence of the fact that we work in an irreducible representation. In these representations we put the singular vectors (if they exist) to zero.

\[(\text{sing. vector})|j_1, t\rangle = 0.\]

We use the relation between states and operators and multiply from the left by \(\phi^{j_0}(z, x)\) to get

\[\phi^{j_0}(z, x)(\text{sing. vector})\phi^{j_1}(0, 0)|\Omega, t\rangle = 0\]

we commute the Verma module operators to the left using eq.(A.1) and we act with the operator which is the outcome of this manipulation on the fused \(\phi^{j_0}\) and \(\phi^{j_1}\). Since \(\psi_{0,0}^{j_1}\) is a highest weight state different from zero its coefficient in the expansion should vanish. This coefficient is a polynomial in the \(j, j_0\) and \(j_1\). Its vanishing condition is the fusion rule. We demonstrate this procedure by an example: take \(j_1 = j_{1,1,+} = 0\) and a generic \(j_0\). The singular vector in the Verma module \(V_{j_{1,1,+}}\) is

\[|\chi_{0,1}\rangle = J_0^- |j_{1,1,+}, t\rangle\]

We have then the following equality

\[0 = \phi^{j_0}(z, x)J_0^- \phi^{j_1}(0, 0)|\Omega, t\rangle\]

\[= (J_0^- - \frac{d}{dx})\phi^{j_0}(z, x)\phi^{j_1}(0, 0)|\Omega, t\rangle\]

We change \(x \rightarrow -x, z \rightarrow -z\) and we multiply from the left by \(e^{xJ_0^- + zL_{-1}}\) to get

\[0 = \frac{d}{dx}\phi^{j_1}(z, x)\phi^{j_0}(0, 0)|\Omega, t\rangle\]

\[= \frac{d}{dx} \sum_{n=0}^{\infty} \sum_{m=-n}^{\infty} z^{h_0 - h_1 + n} x^{-j + j_0 + j_1 + m} \psi_{n,m}^{j_0}(0, 0)|\Omega, t\rangle\]
from which the constraint

\((-j + j_0 + j_1)\psi_{0,0}^j = 0\)

follows. In our case \(j_1 = 0\) and we have the fusion rule \(j = j_0\). Another example is \(j_0 = j_{r,s,+}\) and \(j_1 = j_{1,2,+}\). The singular vector in the Verma module \(V_{j_{1,2,+}}\) is

\(|\chi_{1,1}\rangle = (J_0^- J_{-1}^+ - t J_0^- J_{-1}^0 - t J_{-1}^0 J_0^- - t^2 J_{-1}^-) |j_{1,2,+}; t\rangle\)

The same analysis as above gives the fusion rule:

\((j + j_0 + j_1 + 1)(-j + j_0 - j_1)(j - j_0 - j_1) = 0\)

from which we conclude

\(j_{r,s,+} \otimes j_{1,2,+} = j_{r,s-1,+} + j_{r,s+1,+} + j_{r,s,-}\)

for the case \(s > 1\). For \(s = 1\) we can apply this method to the singular vector in \(V_{j_{r,1}}\) to get

\(j_{r,1,+} \otimes j_{1,2,+} = j_{r,2,+}\)

It is difficult to give the general fusion rules since we don’t have an explicit formula for the singular vectors. Nevertheless we have a conjecture for a special class of modules\(^{[41]}\). For \(j_1 = -\frac{n}{2} t\) we conjecture the condition

\[\prod_{l=-n+2}^{n} (l j_1 + n(j + j_0 + 1)) \prod_{l=-n}^{n} (l j_1 + n(j - j_0)) = 0\]

from which we conclude the fusion rule

\(j_{r,s,+} \otimes j_{1,n+1,+} = \sum_{l=-n}^{n} \frac{1}{2}((-1)^l + (-1)^n) j_{r,s+l,+} + \sum_{l=-n+2}^{n} \frac{1}{2}((-1)^l + (-1)^n) j_{r,s+l-1,-}.\)  

\((A.4)\)

As we mentioned above this constraint is due to the singular vector in \(j_1\). The singular vector in \(j_0\) may give further restrictions. For example it is easy to show
that if $j_0 = j_{r,1,+}$ then

$$j_{r,1,+} \otimes j_{1,n,+} = j_{r,n,+}$$

Using this result and the associativity of the OPE we finally obtain for rational $t = p/q$ such that $j_{r,s} \equiv j_{r,s,+} = j_{p-r,q-s+1,-}$

$$j_{r,s} \otimes j_{r',s'} = \sum_{l=-s+1}^{s-1} \sum_{i=0}^{r-1} j_{r+r'-2i-1,s'+l} + \sum_{l=-s+3}^{s-1} \sum_{i=0}^{r-1} j_{r-r'-2i-1,-s'-l+2}$$

Let us further remark that these fusion rules can be related to the Virasoro fusion rules for the minimal models through the hamiltonian reduction procedure. Recall that in the hamiltonian reduction scheme $j$ and $h$ are related by

$$h = \frac{1}{t} j(j+1) - j$$

for $h_{r,s}$ and $t$ rational we find that both $j_{r,s,+}$ and $j_{r,s+1,-}$ are solutions. If we apply hamiltonian reduction to (A.4) (that is, we replace each $j$ by the appropriate $h$) we get the minimal model fusion rule. This gives us also a clue on the way our operator should be related to the models of minimal matter coupled to gravity. Since

$$\langle \phi^{j_0}(z,x) \phi^{j_1}(0,0) \rangle = \frac{\delta_{j_0,j_1}}{z^{j(j+1)/t}} \xrightarrow{z \to x} \frac{\delta_{h_0,h_1}}{z^{h}}$$

We expect that

$$\lim_{x \to z} \phi^{j_{r,s}}(z,x) e^{\alpha \varphi} \equiv \phi_{r,s}(z)e^{\alpha \varphi_L}$$

in the sense of insertions in correlation functions.