Matrix Generalizations of Some Dynamic Field Theories

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(Received 12 November 2021)

We introduce matrix generalizations of the Navier–Stokes (NS) equation for fluid flow, and the Kardar–Parisi–Zhang (KPZ) equation for interface growth. The underlying field, velocity for the NS equation, or the height in the case of KPZ, is promoted to a matrix that transforms as the adjoint representation of SU($N$). Perturbative expansions simplify in the $N \to \infty$ limit, dominated by planar graphs. We provide the results of a one-loop analysis, but have not succeeded in finding the full solution of the theory in this limit.

I. INTRODUCTION

The occurrence of complex probability distributions under non-equilibrium conditions is a challenging topic. Recently, a great deal of effort has been focussed on the study of the Kardar–Parisi–Zhang (KPZ) equation \[1\], a deceptively simple prototype of non-equilibrium dynamics. The evolution of the field $h(x, t)$, denoting, for example, the height of a growing interface \[3\], has a stochastic component governed by the noise $\eta(x, t)$, usually assumed to be Gaussian distributed, with zero mean, and correlations $\langle \eta(x, t)\eta(x', t') \rangle = 2D\delta^d(x - x')\delta(t - t')$. The dynamical equation can be regarded as representing an elaborate filter that converts the simple correlations of the noise $\eta$ to complex correlations of the field $h(x, t)$. For example, the two point correlations satisfy the dynamic scaling form,

$$ \langle |h(x, t) - h(x', t')|^2 \rangle = |x - x'|^{2\chi} f(|x - x'|^2/|t - t'|), $$

where $z$ and $\chi$ are the so called dynamic and roughness exponents.

For $\lambda = 0$, the linear diffusion equation gives $z = 2$ and $\chi = (2 - d)/2$. Equation (1) is rendered interesting and non-trivial due to the nonlinearity. The renormalization group (RG) flow of the effective coupling constant $g^2 \equiv \lambda^2 D/\nu^3$, is given in a one loop perturbative calculation \[3\] as

$$ \frac{dg^2}{d\lambda} = g^2 \left[ 2 - d + \frac{g^2}{2} K_d \frac{2d - 3}{d} \right], $$

where $K_d = S_d/(2\pi)^d$, and $S_d$ is the $d$-dimensional solid angle. Equation (4) suggests the presence of strong coupling behavior in all dimensions $d$. In $d = 1$ the dynamics is super-diffusive, characterized by exponents $z = 3/2$ and $\chi = 1/2$ \[2\]. At $d = 2$, the nonlinearity is marginally relevant, signaling strong coupling behavior; recent numerical studies \[4\] indicate $z \approx 1.612$, and $\chi \approx 0.386$. In dimensions $d > 2$, a critical strength of $\lambda$ separates diffusive $[z = 2$ and $\chi = (2 - d)/2]$, and non-trivial regimes. Unfortunately, the strong coupling fixed point is not accessible by perturbative RG \[4\], which has recently been extended to two loop order \[5\]. In an alternative approach, the one loop perturbation equations are converted into a set of self-consistent, so-called mode-coupling equations \[6\] \[11\]. The numerical solution of these equations remains controversial, with glassy solutions recently suggested as a possibility \[12\]. The mode-coupling equations have been shown to be exact when the field $h$ is generalized to an $N$ component vector, in a large $N$ limit.

The KPZ equation bears certain similarities (but also profound differences) with the Navier–Stokes (NS) equation for fluid flow. The fluid velocity field $v(x, t)$ evolves according to

$$ \frac{\partial v}{\partial t} + \lambda (v \cdot \nabla)v = - \frac{\nabla p}{\rho} + \nu \nabla^2 v + f(x, t), $$

where $p$ and $\rho$ are the pressure and density of the fluid, respectively.
where \( \nu \) is the viscosity, \( \rho \) is the density, and \( \lambda (=1) \) is introduced for book-keeping purposes. The pressure \( p(x, t) \) is adjusted to enforce the incompressibility condition \( \nabla \cdot v = 0 \). The qualitative difference with the KPZ equation is in the stirring force \( f(x, t) \), which has non-zero Fourier components \( f(k, \omega) \) only at small \( k \sim 1/L \), where \( L \) is the length scale at which energy is pumped into the system. The nonlinearity then transfers the energy into modes with higher wave-numbers \( k \), eventually dissipating it at the dissipative length scale. This establishes the \textit{Kolmogorov energy cascade} in the intermediate (inertial) regime of wave-vectors \cite{13}. A basic question in turbulence is the nature of this energy cascade, usually described by a scaling form for the energy spectrum, \( E(k) \propto k^{-\zeta} \). (If the velocity correlations satisfy a dynamic scaling form as in Eq.(4), then

\[
E \equiv \int_0^\infty dk E(k) \sim \int d^d k d\omega \int d^d x dt e^{i(k \cdot x - \omega t)} \langle v(x, t)v(0, 0) \rangle
\]

and thus

\[
E(k) \sim k^{d-1} \int d^d x e^{i(k \cdot x - 2\chi)}
\]

and thus, as is well known, we have the exponent relation \( \zeta = 2\chi + 1 \).) The famous argument by Kolmogorov \cite{13} gives \( \zeta = 5/3 \), in reasonable agreement with experiment.

To apply the methods of dynamical RG \cite{3,4}, it is usually assumed that \( f(k, \omega) \) is Gaussian distributed with zero mean, and correlations

\[
\langle f_i(k, \omega) f_j(k', \omega') \rangle = 2D(k)P_{ij}(k)(2\pi)^{d+1} \delta^d(k + k') \delta(\omega + \omega').
\]  

The transverse projection operator \( P_{ij}(k) \equiv \delta_{ij} - k_i k_j/k^2 \) is a consequence of the condition \( \nabla \cdot f = 0 \). Thermal fluctuations in the fluid can be modelled by \( D(k) \propto k^2 \), and lead to the familiar long-time tails in the velocity correlation functions \cite{3}. More drastic forms of stirring are modelled by \( D(k) \propto k^{-p} \) with \(-p < 2 \) \cite{4}. In the problem of turbulence, the stirring at the longest scales resembles \( D(k) \propto \delta^d(k) \), which has the same scaling as \( p = d \). This observation motivates some of the more recent applications of RG to turbulence \cite{13}. An extensive review of the application of RG to turbulence in given by Mou and Weichman in Ref. \cite{16}. In particular, these authors develop a set of self-consistent equations for the problem, exact in an appropriate \( N \to \infty \) limit, for which \( \zeta = 3/2 \).

The study of large \( N \) matrix theories started with the classic work of Wigner \cite{17} and has been developed \cite{18,19} over the years. Notable advances include the formulation of large \( N \) quantum chromodynamics \cite{20}, and applications to random surfaces \cite{21}. Intensive studies of the subject have continued in recent years \cite{22}. Here we formulate and study large \( N \) matrix generalizations of the KPZ and NS equations by taking advantage of a crucial difference between vector and matrix models: the product of two vectors is not a vector while the product of two matrices is a matrix. Thus, we are able to promote the fields \( h(x, t) \) and \( v(x, t) \) to \( N \times N \) hermitian matrices and preserve the non-linear structure of the \( N = 1 \) equations. As large \( N \) matrix models are generally quite different from their vector counterparts, we may hope to obtain results that are distinct from the earlier mode-coupling equations \cite{10}. In particular, it is possible that techniques developed in large \( N \) matrix theory, such as the fact that its perturbative expansion is dominated by planar diagrams, may be brought to bear on this problem, leading to new analytic results. Although the primary interest is in the \( N = 1 \) limit, analytical insights may shed light on such controversial issues as the existence or absence of an upper critical dimension for the KPZ equation. The main focus of this paper is a perturbative analysis of the generalized KPZ equation introduced in Sec.I. A brief analysis of the matrix NS equation is presented next in Sec.II. Prospects for further developments are discussed in the final Section, IV.

**II. MATRIX KPZ EQUATION**

We promote the height \( h(x, t) \) in Eq.(1) to a hermitian matrix field \( h^\alpha_\beta(x, t) \), transforming as the adjoint representation under the \( SU(N) \) symmetry group. The indices \( \alpha \) and \( \beta \), which we call color indices, run over \( 1, \cdots, N \), with \( N \) large. Time evolution of the matrix is governed by the generalized KPZ equation,

\[
\frac{\partial h^\alpha_\beta}{\partial t} = \nu_1 \nabla^2 h^\alpha_\beta + \nu_2 \delta^\alpha_\beta \nabla^2 h^\gamma_\gamma + \frac{\lambda_1}{2} \nabla h^\alpha_\gamma \nabla h^\gamma_\beta + \frac{\lambda_2}{2} \delta^\alpha_\beta \nabla h^\gamma_\gamma \nabla h^\gamma_\gamma + \frac{\lambda_3}{2} \delta^\beta_\alpha \nabla h^\gamma_\gamma \nabla h^\gamma_\gamma + \eta^\alpha_\beta(x, t).
\]  

All repeated indices are summed over; the contractions guarantee the \( SU(N) \) invariance. The stochastic noise is assumed to have zero mean, with correlations

\[
\langle \eta^\alpha_\beta(x, t)\eta^{\alpha'}_{\beta'}(x', t') \rangle = 2 \left[ D_1 \delta^\alpha_{\alpha'} \delta^\beta_{\beta'} + D_2 \delta^\alpha_{\alpha'} \delta^\beta_{\beta'} \right] \delta^d(x - x')\delta(t - t').
\]
Although constrained by $SU(N)$ invariance, the equation still admits $8$ parameters; four at the linear level $(\nu_1, D_1)$ and four nonlinearities $(\lambda_{1,2,3,4})$. Due to this large number of parameters, Eqs. $(\text{10})$ are quite formidable. It would be convenient if we could focus on a subset of parameters such as $(\nu_1, \lambda_1, D_1)$. To check if such a simplification is possible, we appeal to a graphical perturbation expansion. The diagrammatic approach $(\text{11})$ to the KPZ equation may be generalized to the matrix $h^\alpha_\beta$ represented by two points: The propagator is a double line that keeps track of the flow of the color indices $\alpha, \beta$, as well as momenta and frequencies. The nonlinearities are represented as triplets of double lines, while averaging over noise joins pairs of such lines. These diagrammatic entities are depicted in Fig. $(\text{12})$. By substituting these entities for the corresponding elements, each term in the original perturbative expansion now has several counterparts in the matrix theory. Summing over the internal labels generates a factor of $N$ for every closed loop. Keeping only the diagrams with the largest power of $N$ will hopefully lead to some simplification.

To gain insight into the structure of the perturbation series, we started by setting all parameters except $(\nu_1, \lambda_1, D_1)$ to zero, and performing a one loop RG analysis. From the perturbative expansion of the propagator, and the correlation function, we obtain respectively

\[
\frac{d\nu_1}{d\ell} = (z - 2) \nu_1 - \frac{K_d(d - 2)}{2d} \frac{N\lambda_1^2 D_1}{\nu_1^3}, \quad \frac{d\nu_2}{d\ell} = (z - 2) \nu_2 + \frac{K_d(d - 2)}{2d} \frac{\lambda_1^2 D_1}{\nu_1^3},
\]

\[
\frac{dD_1}{d\ell} = (z - 2\chi - d) D_1 + \frac{K_d}{4} \frac{N\lambda_1^2 D_1^2}{\nu_1^3}, \quad \frac{dD_2}{d\ell} = (z - 2\chi - d) D_2 + \frac{K_d}{4} \frac{\lambda_1^2 D_2^2}{\nu_1^3}.
\]

(9)

The equations for $\nu_1$ and $D_1$ are similar to those of the scalar KPZ, with $2\lambda^2$ replaced by $N\lambda_1^2$. We also see that $\nu_2$ and $D_2$ are already generated at this order, although these corrections are smaller by a factor of $1/N$. The reductions by powers of $1/N$ arise because to create the pairing of indices (the ‘U’ turns) corresponding to $\nu_2$, etc., out of $(\nu_1, \lambda_1, D_1)$, the lines have to cross at some point. However, as is well known in matrix theory, such non-planar diagrams carry smaller powers of $N$, and can be neglected in the $N \to \infty$ limit $(\text{13})$. The next question is whether these generated parameters, albeit small, can feed back into the recursion relations for $(\nu_1, \lambda_1, D_1)$ significantly, so that their inclusion is necessary. Clearly, replacing any of the original graphical elements with the above generated ones results in a smaller power of $N$ from their smaller magnitude. The question is whether the rearrangements of the indices can generate compensating powers of $N$. For all simple graphs that we examined the answer was negative.

We thus assume that the subspace $(\lambda_1 \equiv \lambda/\sqrt{N}, \nu_1 = \nu, D_1 \equiv D)$ is closed up to order $1/N$, and study the matrix equation

\[
\frac{\partial h^\alpha_\beta}{\partial t} = \nu \nabla^2 h^\alpha_\beta + \frac{\lambda}{2\sqrt{N}} \nabla h^\alpha_\gamma \nabla h^\gamma_\beta + \eta^\alpha_\beta(x, t),
\]

(10)

with a noise term described by

\[
\langle \eta^\alpha_\beta(x, t) \eta^{\beta\gamma}(x', t') \rangle = 2D\delta^\alpha_\gamma \delta^\beta_\gamma \delta^\delta_\delta (x - x') \delta(t - t').
\]

(11)

(We shall shortly demonstrate that this choice of scaling the parameters with $N$, which is in line with the one loop result, leads to a consistent large $N$ theory at all orders.) The reduced matrix equation is invariant under the transformation

\[
h^\alpha_\beta(x, t) = h^\alpha_\beta \left( x + \frac{\lambda}{\sqrt{N}} u t, t \right) + \delta^\alpha_\xi \frac{\lambda}{2\sqrt{N}} u \cdot x + \delta^\alpha_\gamma \frac{\lambda}{2\sqrt{N}} u^2 t, \]

(12)

which generalizes the so called Galilean invariance $(\text{8})$ of the scalar KPZ equation. Referring to Eq. $(\text{6})$, we see that the dynamic and roughness exponents are defined by requiring that the effective equation preserves its form under the transformation $x \to b x$, $t \to b^\gamma t$, and $h \to b^\delta h$. Under these rescalings, and with $b = e^\ell$, the non-linear coupling $\lambda$ evolves according to

\[
\frac{d\lambda}{d\ell} = (\chi + z - 2) \lambda,
\]

(13)

quite generally for any $N$. To preserve the invariance in Eq. $(\text{12})$ under the above rescalings, the coefficient $\lambda$ must remain constant, leading to the exponent relation

\[
\chi + z = 2.
\]

(14)

This identity holds at any fixed point with finite $\lambda$. 

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In $d = 1$, Eq. (10) also satisfies a fluctuation–dissipation condition [24] which implies that the probability distribution

$$P_0[h] \propto \exp \left[ -\frac{\nu}{2D} \int dx \tau H(\partial_x h)^2 \right],$$

is a stationary solution of the Fokker–Planck equation for $P(|h|, t)$ [24]. In deriving the matrix version of the Fokker–Planck equation we must be careful about the non-commutativity of matrices. Going through the usual formal steps, we obtain

$$\frac{\partial}{\partial t} P[h, t] = -\int d^dx \left\{ \text{tr} \left[ \frac{\delta}{\delta h} \left[ \left( \nu \nabla^2 h + \frac{\lambda}{\sqrt{2N}} \nabla h \right)^2 P \right] - D \nabla^2 \left( \frac{\delta^2}{\delta h^2} P \right) \right] \right\}$$

The nesting of the parentheses in the first term of this equation requires some explanation: the large round brackets contain all the matrix quantities to be traced over ($P$ is of course not a matrix), while the square parenthesis indicates that $\delta/\delta h$ should also act on $P$. Inserting Eq. (15), we find that the right hand side of Eq. (13) becomes, for a general dimension $d$, proportional to $\int d^dx \text{tr}[\nabla^2 h \nabla^2 h]$. For $d = 1$ the integrand is a total divergence, indicating that $P_0$ is a stationary solution. For this stationary state $\chi = 1/2$ (and $z = 3/2$ from Eq. (14)), also in agreement with the RG results of Eq. (10). Thus, in $d = 1$ the exponents for $N = 1$ and $N \to \infty$ are identical; also a feature of the vector equations [10]. In this context, it is interesting to note a recent study of a trimer deposition model on a line that is claimed to be equivalent to an $N = 2$ matrix model [25]. Numerical results on this model appear to suggest $z \approx 2.5$.

By integrating over the noise in a functional integral description, we can also obtain a more traditional field theoretical formulation of the problem as

$$\langle Z \rangle = \int Dh \int D\eta J[h] \delta \left( \frac{\partial h^2}{\partial t} - \nu \nabla^2 h^2 - \frac{\lambda}{\sqrt{2N}} \nabla h \nabla h - \eta^2 \right) \exp \left( -\frac{1}{2D} \text{tr} \eta^2 \right) \equiv \int Dh e^{-S(h)},$$

where $J[h]$ is the Jacobian associated with the transformation from $h$ to $\eta$. The value of $J[h]$ depends on the discretization of the evolution equation. In an Ito choice of discretization in which the noise $\eta(x, t)$ only affects the field at $h(x, t + \tau)$, the Jacobian associated with $[\partial h(x, t + \tau)/\partial h(x, t)]$ is simply a constant [26]. We shall thus ignore this factor henceforth. The $\delta$–functions in Eq. (17) can be implemented by using a conjugate field $\bar{h}$ as in Ref. [26]. Integrating over this field, as well as the noise $\eta$, then leads to the action

$$S(h) = \frac{1}{2D} \int d^dx dt \text{tr} \left( \frac{\partial h}{\partial t} - \nu \nabla^2 h - \frac{\lambda}{\sqrt{2N}} \nabla h \right)^2.$$  

(18)

We can now remove two of the remaining parameters by the rescalings, $t \to t/\nu$ and $h \to \sqrt{ND}/\nu h$, leading to

$$S(h) = \frac{N}{2} \int d^dx dt \text{tr} \left( \left( \frac{\partial h}{\partial t} - \nabla^2 h - \frac{g}{2} \nabla^2 h \right)^2 \right),$$

(19)

with $g$ as defined after Eq. (14). With its cubic and quartic interactions, the above action is reminiscent of that of a non-abelian gauge theory such as quantum electrodynamics without quarks.

The Feynman rules corresponding to Eq. (14) are given in Fig. 2. Each propagator line has a factor of $1/N$, while each vertex is proportional to $N$, and each loop generates a factor of $N$. Thus, a given diagram with $E$ external lines, $I$ internal lines, $V_3$ three-point vertices, $V_4$ four-point vertices, and $L$ loops, is associated with a factor of $N^P$ with $P = L + V_3 + V_4 - I$. Using the standard topological identity $L = I - V_3 - V_4 + 1$, we obtain $P = 1$. For $E = 3$ and $E = 4$; this certainly coincides with the bare scalings of three and four point vertices in Eq. (19). Thus the choice of scaling the nonlinearity with $\sqrt{N}$ indeed leads to a consistent large $N$ expansion.

We also carried out a one–loop perturbative RG, directly on the action in Eq. (19). The basic idea is to integrate out short wavelength modes, with $\Lambda/b < k < \Lambda$ where $\Lambda$ is a cutoff. The effective action for the remaining modes then has a leading gradient expansion of the form

$$\hat{S}(h) = \frac{N}{2} \int d^dx dt \text{tr} \left( \alpha \frac{\partial h}{\partial t} - \beta \nabla^2 h \right) - \alpha \frac{g}{2} \nabla^2 h \right)^2 ,$$

(20)

depending on two parameters $\alpha$ and $\beta$. The same coefficient $\alpha$ multiplies both $\partial_t h$ and the nonlinearity, as required by the Galilean symmetry in Eq. (12). To calculate the effective action, we only have to look at terms quadratic in $h$. Thus, it suffices to evaluate the renormalized propagator.
Note that a term proportional to \( \text{tr}(\nabla h)^2 \) is also allowed by symmetries, and is in fact generated. However, such a term can be eliminated by transforming to a moving coordinate frame \( h \rightarrow h + ct \). Thus, in calculating the one-particle-irreducible self-energy function \( \Sigma(k, \omega) \), we only have to extract the coefficients of the \( \omega^2 \) and \( k^4 \) terms in a low frequency and wave-number expansion. The one-loop corrections to the propagator are indicated in Fig. (3). In fact, the two diagrams are identical, except that one is smaller by a factor of 1/\( \Lambda^d \) (2/\( \pi^d \)). A simplifying feature is that the quartic interaction term \( \sim g^2(\nabla h)^4 \) in Eq. (19) does not enter into the calculation to this order. Evaluating the frequency and momentum dependence of the remaining self-energy diagram is cumbersome, but straightforward, and leads to

\[
\Sigma(k, \omega) = \frac{1}{4} \left[ \omega^2 + \frac{4d^2 - d - 6}{d(d + 2)} k^4 \right] \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \cdot \frac{1}{q^2}.
\]

Adding the self-energy to the bare propagator, we obtain the renormalization parameters,

\[
\alpha = 1 - \frac{g^2}{16} \left( 1 + \frac{1}{N} \right) K_d d\ell,
\]

\[
\beta = 1 - \frac{g^2}{16} \left( 1 + \frac{1}{N} \right) \frac{4d^2 - d - 6}{d(d + 2)} K_d d\ell.
\]

We have evaluated the integral in Eq. (21) over an infinitesimal shell by setting \( b = 1 + d\ell \) and writing \( \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \cdot \frac{1}{q^2} = K_d d\ell \), with \( K_d = S_d \Lambda^{d-2}/(2\pi)^d \), where \( S_d \) is the \( d \)-dimensional solid angle.

The cutoff in the effective action can be restored to its original value by setting \( x = bx' \), accompanied by \( t = b^2 t' \) and \( h = bh' \). The parameters in the quadratic part of the renormalized action can be reset to unity by choosing the exponents

\[
z = 2 + \frac{g^2}{16} K_d \left( 1 + \frac{1}{N} \right) \frac{3(d - 2)(d + 1)}{d(d + 2)},
\]

\[
\chi = \frac{2d - 3}{2} + \frac{g^2}{32} K_d \left( 1 + \frac{1}{N} \right) \frac{(d - 1)(5d + 6)}{d(d + 2)}.
\]

Finally, the flow of the coupling constant is given by \( dg/d\ell = (\chi + z - 2) g \), resulting in

\[
\frac{dg^2}{d\ell} = g^2 \left[ 2 - d + \frac{g^2}{16} K_d \left( 1 + \frac{1}{N} \right) \frac{11d^2 - 5d - 18}{d(d + 2)} \right].
\]

Surprisingly, this flow equation looks quite different from that obtained by standard dynamic RG methods \([5]\), which generalize Eq. (4) to

\[
\frac{dg^2}{d\ell} = g^2 \left[ 2 - d + \frac{g^2}{4} K_d \left( 1 + \frac{1}{N} \right) \frac{2d - 3}{d} \right].
\]

The resolution to this discrepancy may be due to the fact that the flow equations are not fundamental. The two equations in fact agree for both \( d = 2 \) and \( d = 1 \). In \( d = 1 \), the RG is constrained to give the correct exponents due to the symmetries embodied by Eqs. (12) and (13). It is indeed easy to check that by substituting \( K_d (1 + 1/N) g^2 = 4 \) in Eqs. (24) we recover \( z = 3/2 \) and \( \chi = 1/2 \). The exponents at the phase transition between the weak and strong coupling phases in \( d = 2 + \epsilon \) dimensions are also correctly given by \( z = 2 + O(\epsilon^2) \), \( \chi = 0 + O(\epsilon^2) \). (The correlation length at this transition diverges with an exponent \( \nu = \epsilon^{-1} + O(1) \).) Thus all physical quantities calculated from Eqs. (26) and (27) appear to be identical. The coefficient of the quadratic term in Eq. (26) changes sign at \( d = 1.5265 \) as opposed to \( d = 3/2 \) for Eq. (23). However, there is probably no physical significance to this dimension, as it disappears in the two loop calculation of Ref. \([6]\). Even simpler flow equations (without any higher order terms) are obtained in an RG of the KPZ equation that proceeds from a mapping to directed polymers \([28]\).

### III. MATRIX NS EQUATION

For the Navier-Stokes equation we promote the velocity vector field \( \mathbf{v} = (v_1, \ldots, v_d) \) to a set of \( N \times N \) hermitian matrices \( \mathbf{v}^{(2)}(\mathbf{x}, t) \), imposing the incompressibility \( \partial_t v_i = 0 \) as before. The non-linear term has four matrix counterparts
\[ v_j(\partial_j v_i), (\partial_j v_i) v_j, v_j(\partial_i v_j), \text{ and } (\partial_i v_j) v_j. \] We will suppress the matrix indices henceforth. (We shall neglect various linear and nonlinear terms that can be constructed using combinations of the unit matrix and/or trv as in Eq.(7). They are again expected to be irrelevant in the large $N$ limit.) We further require the matrix generalization of Eq.(1) to satisfy two important physical conditions: (1) a suitable generalization of Galilean invariance, and (2) conservation of energy in the absence of dissipation. Subject to the above limitations, we arrive at

\[ \frac{\partial v_i}{\partial t} + \frac{\lambda}{2\sqrt{N}} [v_j, \partial_j v_i] + \frac{\kappa}{2\sqrt{N}} [v_j, \partial_j v_i + \partial_i v_j] = -\frac{\partial p_i}{\rho} + \nu \partial_j^2 v_i + f_i(x, t), \] (28)

where the curly $\{\ldots\}$ and square $\ldots$ brackets indicate, respectively, matrix anti-commutation and commutation. The matrix products ensure $SU(N)$ invariance. The commutator term proportional to $\kappa$ has no scalar counterpart. The ‘pressure’ matrix $p(x, t)$, is again adjusted to enforce the incompressibility condition $\partial_i v_i = 0$. Finally, it is assumed that the random (matrix) force is Gaussian distributed with zero mean, and correlations that in Fourier space are given by

\[ \left\langle (f'_\beta, (k, \omega) \left( f'_{\alpha'} \right)_j \right\rangle = 2D(k) \delta'_{\alpha \alpha'} \delta'_{\beta j} P_{ij}(k)(2\pi)^{d+1} \delta^d(k + k') \delta(\omega + \omega'). \] (29)

The transverse projection operator again follows from incompressibility, and $\partial_j f_k = 0$.

A form of Galilean invariance is satisfied by the generalized equation, if in a frame of reference moving with velocity $u$, the generalized velocity transforms to

\[ v'_\beta(x, t) = v_\beta \left( x + \frac{\lambda}{\sqrt{N}} ut, t \right) + u \delta_\beta. \] (30)

It is easy to check that Eq.(28) is invariant under this transformation. A natural choice for the generalized energy of the fluid is $E = \rho/2 \int d^d x \text{tr}(v_i v_i)$. In the absence of dissipation ($\nu = 0$) and forcing ($f = 0$), the change in energy is governed by

\[ \frac{dE}{dt} = \rho \int d^d x \text{tr}(v_i \partial_i v_i) = -\int d^d x \text{tr} \left( \frac{\lambda \rho}{2\sqrt{N}} v_i [v_j, \partial_j v_i] + \frac{\kappa \rho}{2\sqrt{N}} v_i [v_j, \partial_j v_i + \partial_i v_j] + v_i \partial_i p \right). \] (31)

Using the incompressibility condition $\partial_j v_j = 0$, and reordering the matrices inside the trace, we can transform the above expression to

\[ \frac{dE}{dt} = -\int d^d x \partial_j \text{tr} \left[ \frac{\lambda \rho}{2\sqrt{N}} v_i v_j + v_j p \right] = 0, \] (32)

i.e. energy is conserved as required. In what follows we will set $\kappa$ to zero for simplicity. It would be interesting to study how $\kappa$ flows under the renormalization group.

The above conditions are sufficient to give the scaling exponents for Eq.(28) under most forcing conditions. Just as before, we define the exponents $\chi$ and $z$ through the correlation function

\[ \langle v_i(x, t) v_j(x', t') \rangle = \delta_{ij} |x - x'|^{2\chi} f(|x - x'|^2/|t - t'|), \] (33)

As in Eq.(1), the requirement of generalized Galilean invariance leads to the exponent relation $\chi + z = 1$. We need one more exponent relation to determine $\chi$ and $z$ separately.

Let us first examine thermal noise with $D(k) = Dk^2$ (model A in the language of Ref. [3]). In this case, the linearized equation with $\lambda = 0$ has a steady state probability distribution

\[ P_0[\nu] \propto \exp \left[ -\frac{\nu}{2D} \int d^d x \text{tr}(\nu^2) \right]. \] (34)

It is then straightforward to show that the contribution of the non-linear term ($\lambda \neq 0$) to the probability current of the Fokker–Planck equation is the integral of a divergence. The manipulations establishing this result are almost identical to those in Eqs.(31) and (32). Thus by ensuring energy conservation, we have also set up a fluctuation–dissipation condition [23]. From Eq.(34) we can read off the dimension of the velocity as $\chi = -d/2$, leading to $z = 1 + d/2$ from Galilean invariance. (Note that in $d = 1$, Eq.(34) for model A is simply the derivative of Eq.(10).)

Now consider a more general stirring force with $D(k) = D_p k^{-p}$. Because the nonlinearity is proportional to the gradient, the perturbative series generates only powers of $k^2$. Thus the coefficient $D_p$, the most relevant component
of noise as \( k \to 0 \), is not renormalized to any order in perturbation theory. This non-renormalization leads to the exponent relation \( \chi = (z - d + p)/2 \). Together with the constraint from Galilean invariance, we obtain \( \chi = (1 - d + p)/3 \) and \( z = (2 + d - p)/3 \), and a Kolmogorov exponent of \( \zeta = 2\chi + 1 = (5 - d + p)/3 \). For the case of large scale stirring \( (p = d) \), the Kolmogorov result of \( \zeta = 5/3 \) is recovered. However, by appealing to this scaling analysis, we have simply reproduced previously known RG results, and gained nothing new from the matrix equation. By contrast, the large \( N \) equations introduced by Mou and Weichman \([11]\) can be summed exactly in a large \( N \) limit. These authors find that the perturbative expansions break down for \(-p \leq 1 - d\), when the exponent \( z \) sticks to 1. The non-linear term now dominates \( \partial_t \nu \), and equating the bare dimensions of \( \nu \cdot \nabla \nu \) and the noise leads to \( \chi = (1 - d + p)/4 \). According to Ref. \([11]\), for \( p \geq d \), this leads to the exact result of \( \zeta = 3/2 \) for the Kolmogorov exponent in their large \( N \) generalization of the NS equation. Unfortunately, we have not yet succeeded in finding the exact behavior of our matrix generalization as \( N \to \infty \). Mou and Weichman \([16]\) do consider an adjoint \( SU(N) \) generalization, presumably equivalent to Eq.\((28)\). However, we believe that the construction of the equation, and its analysis, is more transparent in our matrix formulation.

IV. DISCUSSION

In this paper, we introduced and analyzed matrix generalizations of the KPZ and NS equations. It is clearly possible to construct similar generalizations of other dynamical equations, such as for the time dependent Landau–Ginzburg process. Our original motivation was to find a set of closed form expressions that are exact in the large \( N \) limit, and which hopefully yield different results from those based on other \( N \to \infty \) generalizations of these equations \([11,12,16]\). The large \( N \) limit does indeed lead to certain simplifications of perturbative expansions; most notably in the dominance of planar diagrams \([20]\). However, we have yet to succeed in taking advantage of this fact to reduce the perturbation series to a closed set of self-consistent equations. Nevertheless, we are hopeful that progress towards this goal can be made. A number of techniques for dealing with large \( N \) matrix theories have been developed over the years. For instance, consider the model where a single \( N \times N \) hermitian matrix \( \varphi \) is taken randomly from the probability ensemble \( P(\varphi) \propto \exp[-N\text{Tr}V(\varphi)] \). While difficult to evaluate diagrammatically, quite remarkably, this model can be solved for any \( V \), by using the orthogonal polynomial approach \([29]\). It is an intriguing possibility that some analogous approach may be developed for the type of problems discussed here, and for large \( N \) quantum chromodynamics (QCD). Another approach towards solving large \( N \) QCD involves the concept of a master field. In the present context, this amounts to finding a master field \( h_{\text{master}} \) which dominates the functional integral in Eq.\((17)\). Certainly, the KPZ problem is considerably simpler than QCD, since we have one master field instead of the four associated with \( A_\mu, \mu = t, x, y, z \). However, we do not have gauge invariance, which is powerful enough to constrain the master fields in QCD to be independent of space-time coordinates. Yet another approach is to consider an RG flow \([20]\) in \( N \), attempting to relate the theory for \( N + 1 \) to the one for \( N \). The hope is that the model will flow towards a fixed point as \( N \to \infty \). In any case, the field of large \( N \) matrix theories is evolving rapidly with recent advances in pure mathematics \([31,32]\). Some of these advances may ultimately prove fruitful in our context.

The calculations in this paper were limited to one loop perturbation theory. Surprisingly, in the case of the KPZ equation, distinct flow equations for the effective coupling constant were obtained from two different starting points (the dynamic equations, and an effective field theory). However, the two RG schemes are equivalent at the critical flow [30] in \( N \), attempting to relate the theory for \( N + 1 \) to the one for \( N \). The hope is that the model will flow towards a fixed point as \( N \to \infty \). In any case, the field of large \( N \) matrix theories is evolving rapidly with recent advances in pure mathematics [31,32]. Some of these advances may ultimately prove fruitful in our context.

Finally, we conclude by noting yet another mapping of the KPZ equation; to directed polymers in random media [33]. For a scalar field \( h(\mathbf{x}, t) \), the transformation \( U(\mathbf{x}, t) = \exp[gh(\mathbf{x}, t)/2] \), changes the action in Eq.\((19)\) into a field theory. The complicating new feature is the presence of the new interaction term proportional to \( \kappa \). We don’t know of any symmetry condition that will keep \( \kappa = 0 \) under renormalization. We also note that the interaction terms, while still quartic in the velocity fields, now contain only two derivatives, rather than the four in the KPZ case. It will interesting to study the RG flows in the \((\lambda, \kappa)\) parameter space. The previous scaling analysis will be invalid at fixed points where either parameter is infinite (or both are zero).

For a scalar field \( h(\mathbf{x}, t) \), the transformation \( U(\mathbf{x}, t) = \exp[gh(\mathbf{x}, t)/2] \), changes the action in Eq.\((19)\) to

\[
S(U) = \frac{2N}{g^2} \int d^d\mathbf{x} dt \text{ tr } (U^{-1}\partial_t U - U^{-1}\partial_2 U)^2. \tag{35}
\]

Promoting \( U \) to an \( N \times N \) matrix leads to another possible generalization. We note that this is not, strictly speaking, equivalent to Eq.\((19)\) due to the non-commutativity between matrices. However, we can take this expression as another generalization, no less valid than Eq.\((19)\), of the KPZ equation. The above action resembles a non-relativistic version of non-linear sigma models, and principal chiral field models, discussed in the field theory literature, with the
crucial difference that the matrix $U$ is not unitary. Perhaps some of the methods developed in this area can also be brought to bear on the problem at hand.

ACKNOWLEDGMENTS

One of us (AZ) thanks E. Brézin for a discussion on coupled integral equations in field theory. This work was supported in part by the National Science Foundation through Grant Nos. DMR-93-03667 (at MIT), and PHY-89-04035 (at the ITP).

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FIG. 1. The elements of a diagrammatic expansion starting with the dynamical equation.
FIG. 2. The elements of a diagrammatic expansion starting with the field theory action.

FIG. 3. The diagrams contributing to the one loop correction of the propagator in the action.