The proposal that the interaction between a macroscopic body and its environment plays a crucial role in producing the correct classical limit in the Bohm interpretation of quantum mechanics is investigated, in the context of a model of quantum Brownian motion. It is well known that one of the effects of the interaction is to produce an extremely rapid approximate diagonalisation of the reduced density matrix in the position representation. This effect is, by itself, insufficient to produce generically quasi-classical behaviour of the Bohmian trajectory. However, it is shown that, if the system particle is initially in an approximate energy eigenstate, then there is a tendency for the Bohmian trajectory to become approximately classical on a rather longer time-scale. The relationship between this phenomenon and the behaviour of the Wigner function post-decoherence (as analysed by Halliwell and Zoupas) is discussed. It is also suggested that the phenomenon may be related to the storage of information about the trajectory in the environment, and that it may therefore be a general feature of every situation in which such environmental monitoring occurs.
1. Introduction

It is well-known [1, 2, 3, 4] that the trajectories in the Bohm Interpretation of Quantum Mechanics are often highly non-classical. This gives rise to an important problem for the Bohm interpretation: namely, the question as to how the interpretation can account for existence of generically quasi-classical trajectories on the macroscopic level. In a previous paper [4] we argued that the Bohm interpretation typically fails to produce the correct classical limit if the system is isolated. The purpose of this paper is to argue that the correct classical limit does emerge once one takes into the account the effect of the environment. Our discussion extends the analysis given in Chapter 8 of the book by Bohm and Hiley [1].

We are particularly interested in the connection between the role of the environment in the Bohm interpretation, and the phenomenon of decoherence, which plays a central role in the decoherent histories approach [5, 6, 7, 8], and in Zurek’s existential interpretation [9, 10, 11]. The mechanism considered by Bohm and Hiley—the scattering of a beam of radiation or other particles—is also one of the mechanisms by which decoherence is produced [8, 12, 13]. This has suggested to some authors [14, 15] that the process by which the Bohmian trajectory becomes quasi-classical is closely related to the phenomenon of decoherence. The suggestion is certainly plausible. However, it is not entirely clear, just from the argument given by Bohm and Hiley, that the suggestion is actually correct. At the macroscopic level decoherence is an ubiquitous phenomenon, which can be produced by a wide variety of different mechanisms. By contrast, Bohm and Hiley only consider the particular case of a scattering process. One would like to know whether other kinds of interaction between a macroscopic body and its environment also have the effect of causing the Bohmian trajectory to become approximately classical. More generally, one might ask whether every process which causes decoherence also causes the Bohmian trajectory to become approximately classical, or whether it is only some of them. These are the questions which motivated our investigation.

We will focus on the models of quantum Brownian motion which have been discussed by Caldeira and Leggett, Hu, Paz and Zhang, and many others [16, 17, 18, 19]. These models have played an important role in studies of decoherence, and are therefore a natural starting point for an investigation into the role of decoherence in the Bohm interpretation. We accordingly consider a system particle, with position $\hat{\mathbf{x}}$ and momentum $\hat{\mathbf{p}}$, interacting with a heat bath consisting of $N$ other particles with positions $\hat{x}_1, \ldots, \hat{x}_N$ and momenta $\hat{p}_1, \ldots, \hat{p}_N$. The Hamiltonian is

$$\hat{H} = \left( \frac{1}{2m} \hat{\mathbf{p}}^2 + \frac{1}{2} m \omega_0^2 \hat{\mathbf{x}}^2 \right) + \sum_{r=1}^{N} \left( \frac{1}{2m_r} \hat{\mathbf{p}}_r^2 + \frac{1}{2} m_r \omega_r^2 \hat{x}_r^2 \right) + \sum_{r=1}^{N} \kappa_r \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}_r$$

(1)

Here $\omega_0$ denotes the bare frequency of the system particle. The renormalised frequency will be denoted $\omega$.

The model is characterized by the spectral density

$$I(\omega') = \sum_{r=1}^{N} \frac{\kappa_r^2}{2m_r \omega_r} \delta(\omega' - \omega_r)$$

(2)

Taking $I \propto \omega'$ (for $\omega' < \text{the cut-off frequency}$) gives the Caldeira-Leggett model [16].

If one leaves $I$ arbitrary one obtains the general class of master equations derived by Hu, Paz and Zhang [15].

At $t = 0$ the heat bath is taken to be in the thermal state with density matrix

$$\hat{\rho}_{\text{bath}} = N \exp \left( -\frac{\hat{H}_{\text{bath}}}{k_B T} \right)$$

(3)
where
\[ \hat{H}_{\text{bath}} = \sum_{r=1}^{N} \left( \frac{1}{2m} \hat{p}_r^2 + \frac{1}{2} m \omega_r^2 \hat{x}_r^2 \right) \]
and \( N \) is a constant. We assume that at \( t = 0 \) system+heat bath are in the product state \( |\psi_{\text{sys}}\rangle \langle \psi_{\text{sys}}| \otimes \hat{\rho}_{\text{bath}} \). At \( t \geq 0 \) the density matrix describing system+heat bath will consequently be
\[ \hat{\rho}(t) = e^{-i\hat{H}/\hbar} \left( |\psi_{\text{sys}}\rangle \langle \psi_{\text{sys}}| \otimes \hat{\rho}_{\text{bath}} \right) e^{i\hat{H}/\hbar} \]
In the conventional approach one now integrates out the environmental degrees of freedom, and focuses on the behaviour of the reduced density matrix. Unfortunately matters are not so simple in the Bohm interpretation.

In the Bohm interpretation a mixed state such as \( \hat{\rho}_{\text{bath}} \) is taken to describe an ensemble of pure states
\[ \hat{\rho}_{\text{bath}} = \sum_{\alpha} \rho_{\alpha} |\phi_{\alpha}\rangle \langle \phi_{\alpha}| \]  
(4)
This way of writing the density matrix is not simply a mathematical device, as in the conventional approach. Rather, one takes it that at \( t = 0 \) the heat bath actually is in one of the pure states in the ensemble, with \( \rho_{\alpha} \) being the probability that it is in the state \( |\phi_{\alpha}\rangle \). The problem we then face is, that the density matrix does not uniquely determine the ensemble, and that in the Bohm interpretation it makes a difference which ensemble we choose (for a classification of the set of all discrete ensembles corresponding to a given density matrix see Hughston et al [20]). We discuss this point further in Section 4.

Suppose that a particular ensemble has been chosen, and suppose that at \( t = 0 \) the heat bath is in the pure state \( |\phi_{\alpha}\rangle \). Then at \( t > 0 \) system+heat bath will be in the pure state
\[ |\Psi_{\alpha}(t)\rangle = e^{-i\hat{H}/\hbar} |\psi_{\text{sys}}\rangle \otimes |\phi_{\alpha}\rangle \]  
(5)
and the Bohmian velocity of the system particle will be given by
\[ v_{B}^{(\alpha)}(t, x, x_1, \ldots, x_N) = \frac{\hbar \text{Im} \langle \Psi_{\alpha}(t) | x, x_1, \ldots, x_N | \Psi_{\alpha}(t) \rangle}{m |\langle x, x_1, \ldots, x_N | \Psi_{\alpha}(t) \rangle|^2} \frac{\partial}{\partial x} \langle x, x_1, \ldots, x_N | \Psi_{\alpha}(t) \rangle \]  
(6)
We see that the Bohmian velocity depends, not only on \( x \), but also on \( x_1, \ldots, x_N \), as well as the index \( \alpha \). The reduced density matrix clearly does not provide enough information to calculate this function. Consequently, the problem of determining the effect of the interaction with the heat bath is significantly more difficult in the Bohm interpretation than it is in the conventional approach.

Nevertheless, although the reduced density matrix does not provide us with complete information regarding the Bohmian velocity of the system particle, it does tell us something. To see this, consider the effect of averaging \( v_{B}^{(\alpha)} \) over all the possible values of \( x_1, \ldots, x_N \), and of the index \( \alpha \):
\[ \bar{v}_E(t, x) = \frac{\sum_{\alpha} \rho_{\alpha} \int dx_1 \ldots dx_N |\langle x, x_1, \ldots, x_N | \Psi_{\alpha}(t) \rangle|^2 v_{B}^{(\alpha)}(t, x, x_1, \ldots, x_N)}{\sum_{\alpha} \rho_{\alpha} \int dx_1 \ldots dx_N |\langle x, x_1, \ldots, x_N | \Psi_{\alpha}(t) \rangle|^2} \]
We will refer to $\bar{v}_E$ as the ensemble-averaged velocity. The reduced density matrix elements are given by

$$\langle x | \hat{\rho}_{\text{red}}(t) | x' \rangle = \sum_\alpha \rho_\alpha \int dx_1 \ldots dx_N \langle x, x_1, \ldots, x_N | \Psi_\alpha(t) \rangle \langle \Psi_\alpha(t) | x', x_1, \ldots, x_N \rangle$$

It is then straightforward to infer

$$\bar{v}_E(t, x) = \frac{\hbar}{m} \Im \left( \frac{\partial}{\partial x} \langle x | \hat{\rho}_{\text{red}}(t) | x' \rangle \bigg|_{x' = x} \right)$$

from which we see that the reduced density matrix does provide enough information to calculate $\bar{v}_E$.

In this paper we will investigate the behaviour of $\bar{v}_E$ and $v^{(n)}$ as functions of time for the case when the initial system state is an approximate energy eigenstate, of the form

$$|\psi_{\text{sys}}\rangle = \sum_{r = -\Delta n}^{\bar{n} + \Delta n} c_r |\bar{n} + r\rangle$$

In this expression $|n\rangle$ denotes the $n$th eigenstate of the isolated system particle Hamiltonian:

$$\hat{H}_{\text{sys}} |n\rangle = E_n |n\rangle$$

where $\hat{H}_{\text{sys}} = \hat{p}^2/(2m) + m\omega^2 \hat{x}^2/2$ and $E_n = (n + 1/2)\hbar\omega$. We assume that $\bar{n} \gg 1$ (so that the state is highly excited), and $\Delta n \ll \bar{n}$ (so that the energy distribution is sharply peaked about the mean). Classically one would therefore expect the particle to be following a well-defined orbit with energy close to $E_{\bar{n}}$ and amplitude close to $x_{\text{max}} = (2E_{\bar{n}}/(m\omega^2))^{1/2}$. In particular, when the particle is located at $x$, one would classically expect its velocity to be close to $\pm p_{\text{cl}}(x)/m$, where

$$p_{\text{cl}}(x) = m\omega \left(x_{\text{max}}^2 - x^2\right)^{1/2}$$

On the other hand it was shown in ref. [4] that, assuming the system to be isolated, there is only a high probability of this being true of the Bohmian velocity at all stages of the motion in the very special case for which $|\psi_{\text{sys}}\rangle$ is a narrowly localized wave packet. In the following we will show that the effect of the interaction with the heat bath is to make the distribution of Bohmian velocities eventually become approximately classical, whether or not this is true initially.

We begin, in Sections 2 and 3, by considering the behaviour of the ensemble-averaged velocity $\bar{v}_E$. The feature of the interaction with the environment which has probably attracted most attention is the tendency of the reduced density matrix to become approximately diagonal in the position representation. In Section 2 we show that this phenomenon is, by itself, insufficient to produce approximately classical behaviour of the Bohmian trajectory. However, the interaction has other important effects, apart from the approximate diagonalisation of $\hat{\rho}_{\text{red}}$. In particular, Halliwell and Zoupas [21] have shown that, in the case of the Caldeira-Leggett model, the Wigner function becomes non-negative after a sufficient elapse of time. In Section 3 we show that, as a consequence of this effect, $\bar{v}_E$ comes to lie approximately within the classical range

$$-p_{\text{cl}}(x)/m \leq \bar{v}_E(x) \leq p_{\text{cl}}(x)/m$$

We also derive conditions for this to occur in the case of other models of the type defined by Eq. (1) (and, in fact, for a number of models which are not of this type).
Inequalities (9) represent a necessary condition for the Bohmian trajectory to be approximately classical. However, they are clearly not sufficient. In Section 4 we accordingly calculate the function $v_B^{(a)}(t,x,x_1,\ldots,x_N)$ on the assumption that the ensemble described by $\hat{\rho}_{\text{bath}}$ consists of coherent states [see remarks following Eq. (4)]. Our calculation is based on Halliwell and Yu’s alternative derivation [22] of the Hu-Paz-Zhang master equation (also see Anglin and Habib [23]), which has the advantage (from our point of view) that, unlike the usual path integral methods, it allows us explicitly to keep track of the heat bath degrees of freedom. We show that in the case of the Caldeira-Leggett model, for sufficiently large values of $t$, there is a high probability that $v_B^{(a)}$ will be close to one of the classical values $\pm p_{cl}(t,x,x_1,\ldots,x_N)$. We also derive conditions for this to occur in the case of other models of the type defined by Eq. (1).

Finally, in the conclusion, we discuss the bearing that these results have on the questions which provided the original motivation for this investigation, and we suggest some directions for further enquiry.

2. Effect of Approximately Diagonalising $\hat{\rho}_{\text{red}}$ in the $x$-representation.

One of the most striking effects of the interaction between a macroscopic body and its environment is that the reduced density matrix tends rapidly to become approximately diagonal in the position representation. We begin by showing that this effect is not, by itself, sufficient to cause the Bohmian trajectory to become quasi-classical.

The point is most conveniently illustrated in the context of the Caldeira-Leggett model, for which the master equation is [16]

$$i\hbar \frac{\partial}{\partial t} \hat{\rho}_{\text{red}} = \left[\hat{H}_{\text{sys}}, \hat{\rho}_{\text{red}}\right] + \gamma \left[\hat{x}, \{\hat{x}, \hat{\rho}_{\text{red}}\}\right] - \frac{2im\gamma k_B T}{\hbar} \left[\hat{x}, \hat{p}_{\text{red}}\right]$$

(10)

where $\hat{H}_{\text{sys}}$ is the renormalised system particle Hamiltonian (which, for the purposes of this section, need not be assumed to be of oscillator form), and where $\{\ldots\}$ denotes an anti-commutator. It should be noted that this equation is not exact, and that it does in fact violate the positivity of $\hat{\rho}_{\text{red}}$ over very short times [24, 25]. However, it provides a good approximation over somewhat longer times.

Under conditions where the last term on the right hand side of Eq. (10) dominates, and provided that $t$ is sufficiently small [but not so small as to render the approximation of Eq. (10) invalid], one has [10, 12, 13]

$$\langle x | \hat{\rho}_{\text{red}}(t) | x' \rangle \approx \exp \left[-\Lambda t (x - x')^2\right] \langle x | \hat{\rho}_{\text{red}}(0) | x' \rangle$$

(11)

where $\Lambda = (2m\gamma k_B T) / \hbar^2$ is the localization rate. In the case of a macroscopic object $\Lambda$ is typically very large [26], even when the interaction with the environment is comparatively weak. Eq. (11) consequently plays an important role in attempts to explain the emergence of an effectively classical statistics of “facts” [27] from an underlying theory which is fully quantum mechanical [1, 3, 5, 6, 7, 8, 9, 10, 12, 13].

Substituting the expression given by Eq. (11) into Eq. (3) we find

$$\tilde{v}_E(t,x) \approx \tilde{v}_E(0,x)$$

(12)

However, at $t = 0$ system+environment are in the product state $|\psi_{\text{sys}}\rangle \langle \psi_{\text{sys}}| \otimes \hat{\rho}_{\text{bath}}$, which means that $\tilde{v}_E(0,x)$ coincides with the actual Bohmian velocity of the system particle at $t = 0$, $\tilde{v}_E(0,x) = \frac{\hbar}{m} \text{Im} \left( \langle \psi_{\text{sys}} | x \rangle \frac{\partial}{\partial x} \langle x | \psi_{\text{sys}} \rangle \right) / \left| \langle x | \psi_{\text{sys}} \rangle \right|^2$. 

It was shown in ref. [4] that, for many choices of $|\psi_{\text{sys}}\rangle$, this quantity tends to take values greatly in excess of the classical speed. In view of Eq. (12) the same must be true of $\bar{v}_E(t, x)$. It follows, that the approximate diagonalisation of $\hat{\rho}_{\text{red}}$ in the $x$-representation is not, by itself, sufficient to produce generically quasi-classical behaviour of the Bohmian trajectory.

3. THE BEHAVIOUR OF $\bar{v}_E$ AT LATER TIMES

The approximation of Eq. (11) is only valid for sufficiently small values of $t$. We now want to investigate the behaviour of the Bohmian velocity over longer time-scales, and for other models of Brownian motion, apart from the Caldeira-Leggett model. We will consider the function $\bar{v}_E(t, x)$ in this Section, and the function $v_B^{(a)}(t, x, x_1, \ldots, x_N)$ in Section 4.

The advantage of considering $\bar{v}_E(t, x)$ is that in order to calculate it one only needs to know the master equation. We will consider master equations of the form

$$i\hbar \frac{\partial}{\partial t} \hat{\rho}_{\text{red}} = \frac{1}{2} \left[ \left( h_1(t) \hat{x}^2 + h_2(t) \hat{p}^2 + h_3(t) \left[ \hat{x}, \{ \hat{p}, \hat{\rho}_{\text{red}} \} \right] \right), \hat{\rho}_{\text{red}} \right] + \gamma(t) \left[ \hat{x}, \{ \hat{p}, \hat{\rho}_{\text{red}} \} \right]$$

$$- \frac{i}{\hbar} \left( J_{11}(t) \left[ \hat{p}, \hat{\rho}_{\text{red}} \right] - 2J_{12}(t) \left[ \hat{x}, \{ \hat{p}, \hat{\rho}_{\text{red}} \} \right] + J_{22}(t) \left[ \hat{x}, \{ \hat{x}, \hat{\rho}_{\text{red}} \} \right] \right)$$

(13)

for which the right-hand side is quadratic in $\hat{x}$ and $\hat{p}$. This class includes equations of the Hu-Paz-Zhang [18] type, corresponding to Brownian motion models of the kind defined by Eq. (1). It also includes those equations of the Lindblad form [28] for which the right-hand side is quadratic in $\hat{x}$ and $\hat{p}$. In particular, it includes the equation discussed by Diósi [24, 25].

It is most convenient to work in terms of the reduced Wigner function,

$$W_{\text{red}}(t, x, p) = \frac{1}{\hbar} \int dy \exp \left( \frac{i}{\hbar} py \right) \langle x - \frac{y}{2} | \hat{\rho}_{\text{red}}(t) | x + \frac{y}{2} \rangle$$

(14)

Expressing Eq. (13) in terms of $W_{\text{red}}$ we find

$$\frac{\partial}{\partial t} W_{\text{red}}(t, \eta) = \sum_{r,s=1}^{2} \left( K_{rs}(t) \frac{\partial}{\partial \eta_r} (\eta_s W_{\text{red}}(t, \eta)) + J_{rs}(t) \frac{\partial^2}{\partial \eta_r \partial \eta_s} W_{\text{red}}(t, \eta) \right)$$

(15)

where

$$\eta = \left( \begin{array}{c} x \\ p \end{array} \right)$$

$$K(t) = \left( \begin{array}{cc} -h_3(t) & -h_2(t) \\ h_1(t) & 2\gamma(t) + h_3(t) \end{array} \right)$$

$$J(t) = \left( \begin{array}{cc} J_{11}(t) & J_{12}(t) \\ J_{12}(t) & J_{22}(t) \end{array} \right)$$

It is straightforward to verify that the solution to Eq. (15) may be written [24, 29, 30]

$$W_{\text{red}}(t, \eta) = \frac{1}{\pi \det A(t) \sqrt{\det M(t)}}$$

$$\times \int d^2 \eta' \exp \left[ - \left( \eta' - A^{-1}(t)\eta \right)^T M^{-1}(t) \left( \eta' - A^{-1}(t)\eta \right) \right] W_{\text{red}}(0, \eta')$$

(16)

where the matrices $A$ and $M$ are defined by the equations

$$\frac{\partial}{\partial t} A = -KA$$

$$\frac{\partial}{\partial t} M = 4A^{-1}J \left( A^{-1} \right)^T$$

(17)

(18)
(superscript “T” signifying “transpose”) subject to the initial conditions

\[ A(0) = 1 \quad M(0) = 0 \]

Before proceeding further it will be useful to relate this equation to the discussion in the last section. Specialising to the case of the Caldeira-Leggett master equation, Eq. (10), with \( \hat{H} \) taking the oscillator form \( \hat{p}^2/(2m) + m\omega^2\hat{x}^2/2 \), one has

\[ K = \left( \begin{array}{c}
\frac{1}{m\omega^2} \\
-\frac{1}{2\gamma}
\end{array} \right) \]

\[ J = \left( \begin{array}{c}
0 \\
0
\end{array} \right)
\]

where \( D = 2m\gamma k_B T \). If \( \omega t, \gamma t \ll 1 \) Eqs. (17) and (18) then imply

\[ A \approx \left( 1 - \frac{t}{m\omega^2} \right) \]

\[ M \approx 4Dt \left( \frac{\gamma^2}{2m} - \frac{t}{2m} \right) \]

so that

\[ W_{\text{red}}(t, x, p) \approx \sqrt{\frac{3m}{2\pi D t^2}} \int dx' dp' \exp \left\{ -3m^2 \frac{D}{Dt^2} (x' - x)^2 ight. \\

- \left. \frac{3m}{Dt^2} (x' - x)(p' - p) \right\} W_{\text{red}}(0, x', p') \]

(20)

where we have made the further approximation \( A^{-1}(t) \approx 1 \). In this expression the width of the Gaussian convolution in the \( p \)-direction is \( \propto t \), whereas the width in the \( x \)-direction is \( \propto t^2 \). It follows that, if \( t \) is sufficiently small, there will be a significant degree of smoothing in the \( p \)-direction, but no significant smoothing in the \( x \)-direction. To be specific, suppose that the initial system state \( |\psi_{\text{sys}}\rangle \) is of the form specified by Eq. (8), and suppose that \( t \ll t_c \), where

\[ t_c = \left( \frac{3m^2\lambda_B^2}{D} \right)^{\frac{1}{4}} \]

(21)

where \( \lambda_B = \hbar/(m\omega_{\text{max}}) \) is the minimum value of the de Broglie wavelength. In that case \( W_{\text{red}} \) will be nearly constant over the width of the Gaussian in the \( x \)-direction, and we can approximately write

\[ W_{\text{red}}(t, x, p) \approx \sqrt{\frac{3m}{2\pi D t^2}} \int dp' \left( \int dx' \exp \left[ -3m^2 \frac{D}{Dt^2} (x' - x)^2 ight. \right. \\

- \left. \left. \frac{3m}{Dt^2} (x' - x)(p' - p) \right] \right\} \exp \left[ -\frac{1}{4Dt}(p' - p)^2 \right] W_{\text{red}}(0, x, p') \\

= \left( \frac{1}{4\piDt} \right)^{\frac{1}{4}} \int dp' \exp \left[ -\frac{1}{4Dt}(p' - p)^2 \right] W_{\text{red}}(0, x, p') \]

If this result is re-expressed in terms of the density matrix one recovers Eq. (11) (with \( \Lambda = D/\hbar^2 \)). It follows that the discussion in the last section only applies to the situation when \( t \ll t_c \), before there has been any significant degree of smoothing in the \( x \)-direction. The question we now have to consider is whether there is a tendency for the distribution of Bohmian velocities to become generically quasi-classical when \( t > t_c \).

The result established by Halliwell and Zoupas [24] provides some preliminary indication that such an outcome might be expected. Halliwell and Zoupas show
that, in the case of the Caldeira-Leggett model as applied to a free particle, with negligible dissipation, the Wigner function becomes strictly non-negative once
\[ t \geq \left( \frac{3}{16} \right)^{1/4} t_{\text{loc}}, \]
where \( t_{\text{loc}} \) is the localization time given by
\[ t_{\text{loc}} = \left( \frac{\hbar}{\gamma k_B T} \right)^{1/2}. \] (22)

It is easily seen that this is also true for the case of a harmonically bound particle considered here (provided \( \omega t_{\text{loc}}, \gamma t_{\text{loc}} \ll 1 \), so that the approximation of Eq. (20) is still valid at \( t = t_{\text{loc}} \)). More generally, what is essentially the same argument shows that, whenever the Wigner function propagator takes the form specified by Eq. (16), \( W_{\text{red}} \) becomes strictly non-negative once
\[ \det \mathbf{M}(t) \geq \hbar^2. \]

It follows from Eqs. (1) and (14) that
\[
\bar{v}_E(t, x) = \int dp \int dp' W_{\text{red}}(t, x, p) \times \{ \exp \left[ \frac{i}{\hbar} S(x - y) \right] g_-(x - y) - \exp \left[ \frac{i}{\hbar} S(x) \right] g_+(x) \} \]
\[
\times \{ \exp \left[ \frac{i}{\hbar} S(x + y) \right] g_+(x + y) - \exp \left[ \frac{i}{\hbar} S(x) \right] g_-(x) \}
\]

provided that \( x \) is not close to one of the classical turning points at \( x = \pm x_{\text{max}} \). In this expression
\[ S(x) = \int_{-x_{\text{max}}}^{x} dx' \frac{p_{\text{cl}}(x')}{m} + \frac{\hbar}{8} \]
and
\[ g_\pm(x) = \left\{ \begin{array}{ll}
\left( \frac{2 \pi p_{\text{cl}}(x)}{\pi} \right)^{1/2} \sum_r \Delta p_r^\pm e^{i r} \exp \left[ \pm i r \sin^{-1} \left( \frac{x}{x_{\text{max}}} \right) \right] & \text{if } -x_{\text{max}} < x < x_{\text{max}} \\
0 & \text{otherwise}
\end{array} \right. \]

This gives, for the reduced Wigner function at \( t = 0 \),
\[
W_{\text{red}}(0, x, p) \approx \frac{1}{\hbar} \int dy \exp \left( \frac{i}{\hbar} py \right) \times \left\{ \exp \left[ \frac{i}{\hbar} S \left( x - \frac{y}{2} \right) \right] g_-(x - \frac{y}{2}) - \exp \left[ \frac{i}{\hbar} S \left( x + \frac{y}{2} \right) \right] g_+(x + \frac{y}{2}) \right\} \times \left\{ \exp \left[ \frac{i}{\hbar} S \left( x + \frac{y}{2} \right) \right] g_+(x + \frac{y}{2}) - \exp \left[ \frac{i}{\hbar} S \left( x - \frac{y}{2} \right) \right] g_-(x - \frac{y}{2}) \right\}
\]

Substituting this expression into Eq. (17) and carrying out the \( p' \)-integration gives
\[ W_{\text{red}}(t, x, p) \]
\[ \approx \frac{\sqrt{\Delta}}{h\sqrt{\pi b}} \int dx' dy' \times \exp \left[ -\frac{1}{4\hbar^2} y'^2 - \frac{\Delta}{b} (x' - x)^2 + \frac{i}{\hbar} \left( p - \frac{c}{b} (x' - x) \right) y' \right] \times \left\{ \exp \left[ -\frac{i}{\hbar} S \left( x' - \frac{y'}{2} \right) \right] g_- \left( x' - \frac{y'}{2} \right) - \exp \left[ -\frac{i}{\hbar} S \left( x' + \frac{y'}{2} \right) \right] g_+ \left( x' + \frac{y'}{2} \right) \right\} \times \left\{ \exp \left[ \frac{i}{\hbar} S \left( x' + \frac{y'}{2} \right) \right] g_+ \left( x' + \frac{y'}{2} \right) - \exp \left[ \frac{i}{\hbar} S \left( x' - \frac{y'}{2} \right) \right] g_- \left( x' - \frac{y'}{2} \right) \right\} \] 

(24)

where we have assumed that \( t \) is sufficiently small to justify the approximation \( A^{-1}(t) \approx 1 \), and where we have set

\[ M^{-1} = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \quad \text{det} \ M^{-1} = \Delta \] 

(25)

In order to evaluate this expression we note, first of all, that the functions \( g_\pm \) are slowly-varying \([4]\). We may therefore write

\[ g_+ \left( x' \pm \frac{y'}{2} \right) \approx g_+(x) \quad \text{and} \quad g_- \left( x' \pm \frac{y'}{2} \right) \approx g_-(x) \]

provided that the Gaussian peaks are sufficiently narrow, and provided that \( x \) is not too close to one of the classical turning points. It will also be convenient to write these functions in modulus-argument form:

\[ g_\pm = \sqrt{\rho_\pm} e^{i\phi_\pm} \]

Finally we make the approximation

\[ \frac{1}{\hbar} S \left( x' \pm \frac{y'}{2} \right) \approx \frac{1}{\hbar} S(x) + \frac{1}{\hbar} p_{cl}(x) \left( x' \pm \frac{y'}{2} - x \right) + \frac{1}{2\hbar^2} p'_{cl}(x) \left( x' \pm \frac{y'}{2} - x \right)^2 \]

(26)

This approximation will be justified provided

\[ \frac{1}{\hbar} |p''_{cl}(x)| \left( \frac{b}{\Delta} \right)^{\frac{3}{2}} \ll 1 \quad \text{and} \quad \frac{1}{\hbar} |p''_{cl}(x)| \left( 4\hbar^2 b \right)^{\frac{3}{2}} \ll 1 \] 

(27)

or, using the fact that \( |p''_{cl}(x)| \sim m/(\omega x_{\text{max}}) \) everywhere except in the vicinity of the classical turning points,

\[ \frac{m\omega b^{\frac{3}{2}}}{\hbar \Delta^{\frac{3}{2}}} \ll x_{\text{max}} \quad \text{and} \quad 8m\omega b^{\frac{3}{2}} \ll x_{\text{max}} \] 

(28)

Making these approximations in Eq. (24) and carrying out the Gaussian integrations gives, after a certain amount of algebra,

\[ W_{\text{red}}(t, x, p) \approx W_{\text{cl}}(t, x, p) + W_{\text{osc}}(t, x, p) \] 

(29)

where

\[ W_{\text{cl}}(t, x, p) = \frac{\sigma_-}{\sqrt{\pi}} \exp \left[ -\sigma_-^2 (p + p_{cl}(x))^2 \right] \rho_-(x) + \frac{\sigma_+}{\sqrt{\pi}} \exp \left[ -\sigma_+^2 (p - p_{cl}(x))^2 \right] \rho_+(x) \]
and

\[ W_{\text{osc}}(t, x, p) = \left( \frac{4\hbar \sigma_2 \sqrt{\Delta}}{\pi} \right)^\frac{\dagger}{2} \exp \left[ -\sigma_2^2 (p + \beta p_{\text{cl}}(x))^2 - \sigma_1^2 (p_{\text{cl}}(x))^2 \right] \]

\[ \times \cos \left[ \frac{2}{\hbar} S(x) + \chi(t, x, p) \right] \left( \rho_-(x) \rho_+(x) \right)^\frac{\dagger}{4} \]

(30)

where we have set

\[ \rho_\pm = |g_\pm|^2 \]

\[ \sigma_\pm^2 = \frac{\Delta}{a \pm 2cp_{\text{cl}}'(x) + b(p_{\text{cl}}'(x))^2} \]

\[ \sigma_1^2 = \frac{\Delta}{\hbar^2 a \Delta + b(p_{\text{cl}}'(x))^2} \]

\[ \sigma_2^2 = \frac{\hbar^2 a \Delta + b(p_{\text{cl}}'(x))^2}{\hbar^2 a^2 + (1 - 2\hbar^2 c^2 + \hbar^4 \Delta^2)(p_{\text{cl}}'(x))^2 + \hbar^2 b^2 (p_{\text{cl}}'(x))^4} \]

\[ \beta = \frac{c (1 + \hbar^2 \Delta)}{\hbar^2 a \Delta + b(p_{\text{cl}}'(x))^2} \]

\[ \chi(t, x, p) \] is a phase whose functional form is unimportant for present purposes.

\[ W_{\text{cl}} \] is non-negative, and it is concentrated on the classical energy surface at \( p = \pm p_{\text{cl}} \). It is therefore a possible classical phase space probability distribution describing a particle of energy \( E_{\text{in}} \), with \( \rho_+ \) (respectively \( \rho_- \)) being the probability density function for the particle to be located at \( x \) and moving to the right (respectively left). On the other hand the term \( S/\hbar \) in the argument of the cosine means that \( W_{\text{osc}} \) is very rapidly oscillating. \( W_{\text{osc}} \) is the term responsible for the tendency of the Wigner function to swing negative. It may therefore be regarded as the quantum mechanical correction to the classical distribution.

It can be seen from Eq. (30) that \( W_{\text{osc}} \) will become negligible once

\[ \sigma_1 p_{\text{cl}}(x) \gg 1 \]

In that case \( W_{\text{red}} \approx W_{\text{cl}} \) and, in view of Eq. (23),

\[ \bar{v}_E(x) \approx \frac{p_{\text{cl}}(x)}{m} \left( \frac{\rho_+(x) - \rho_-(x)}{\rho_+(x) + \rho_-(x)} \right) \]

from which it follows that \( \bar{v}_E \) lies within the classical range

\[ \frac{p_{\text{cl}}(x)}{m} \leq \bar{v}_E(x) \leq \frac{p_{\text{cl}}(x)}{m} \]

Specialising to the case of the Caldeira-Leggett model it can be seen from Eqs. (19), (25) and (28) that the condition for the approximation of Eq. (26) to be valid is

\[ \left( \frac{\omega t_{\text{loc}}}{9} \right)^\frac{\dagger}{2} \ll \frac{t}{t_c} \ll \left( \frac{x_{\text{max}}}{\lambda_B} \right)^\frac{\dagger}{4} \]

(31)

where \( t_c \) is the time at which the smearing in the \( x \)-direction becomes significant [see Eq. (21)], \( t_{\text{loc}} \) is the localisation time [see Eq. (22)], and \( \lambda_B = \hbar/(m \omega x_{\text{max}}) \), as before. We also have

\[ \sigma_1^2 p_{\text{cl}}^2 = \frac{D \hbar^2 p_{\text{cl}}^2}{3m^2 \hbar^2} \left( 1 + \frac{4D^2 \hbar^2 p_{\text{cl}}^2}{9m^4 \hbar^2} \right)^{-1} \]
Provided that $x$ is not too close to one of the classical turning points at $x = \pm x_{max}$ we have $p_{cl}(x) \sim m \omega x_{max}$ and $p'_{cl}(x) \sim m \omega$. Consequently

$$\sigma^2_{p_{cl}} \sim \frac{(\frac{t}{t_c})^3}{1 + 4 \left( \frac{\lambda_B}{x_{max}} \right)^2 \left( \frac{t}{t_c} \right)^6}$$

The fact that $|\psi_{sys}\rangle$ is highly excited means that $x_{max} \gg \lambda_B$. Taking into account inequalities (31) we conclude that, in the case of the Caldeira-Leggett model, $W_{osc}$ is negligible, and $\bar{v}_E$ is approximately within the classical range of values, once $t \gg t_c$.

Finally, we remark that it follows from Eqs. (21) and (22) that

$$t_c/t_{loc} = \left( \frac{3 \hbar \gamma kT}{8 E^2} \right)^\frac{1}{2}$$

where $E_n = (\bar{n} + 1/2) \hbar \omega = (1/2) m \omega^2 x_{max}^2$ is the mean energy. We see from this that, in the case of a macroscopic body, $t_c$ is typically $\ll t_{loc}$. As we mentioned above, $t_{loc}$ is the time at which the Wigner function becomes strictly non-negative, for every possible choice of initial state [2]. However, the argument just given shows that, for states of the form specified by Eq. (8), the Wigner function typically becomes approximately non-negative very much sooner than this, and approximate non-negativity is enough to ensure that $\bar{v}_E$ lies approximately within the classical range of values.

4. Calculation of $v_B^{(\alpha)}(t, x, x_1, \ldots, x_N)$

The requirement that $|\bar{v}_E| \leq p_{cl}/m$ is a necessary condition for the Bohmian trajectories to be quasi-classical. However, it is clearly not sufficient. We therefore need to turn from the ensemble-averaged quantity $\bar{v}_E(t, x)$ to the Bohmian velocity itself, $v_B^{(\alpha)}(t, x, x_1, \ldots, x_N)$.

In order to calculate $v_B^{(\alpha)}$ it is necessary to resolve the ambiguity mentioned in the Introduction, arising from the fact that the density matrix $\hat{\rho}_{bath}$ [see Eq. (3)], describing the initial state of the heat bath, does not uniquely determine a corresponding ensemble [see the discussion in the paragraph following Eq. (4)]. One obvious choice is to represent $\hat{\rho}_{bath}$ in terms of eigenstates of the heat bath Hamiltonian:

$$\hat{\rho}_{bath} = \sum_E p_E |E\rangle \langle E|$$

where $|E\rangle$ is the eigenstate of $\hat{H}_{bath}$ with eigenvalue $E$ and $p_E = N \exp [-E/(k_B T)]$ [c.f. Eq. (3)]. However, we will find it more convenient to use the coherent state representation

$$\hat{\rho}_{bath} = \int d\bar{x}_1 d\bar{p}_1 \ldots d\bar{x}_N d\bar{p}_N P(\bar{x}_1, \bar{p}_1, \ldots, \bar{x}_N, \bar{p}_N)$$

$$\times |\bar{x}_1, \bar{p}_1, \ldots, \bar{x}_N, \bar{p}_N\rangle \langle \bar{x}_1, \bar{p}_1, \ldots, \bar{x}_N, \bar{p}_N|$$

where $P$ is the thermal Glauber-Sudarshan $P$-function [32, 33, 34, 35, 36]

$$P(\bar{x}_1, \bar{p}_1, \ldots, \bar{x}_N, \bar{p}_N) = \prod_{r=1}^N \left( \frac{e^{\beta_r} - 1}{\hbar} \right) \exp \left[ -\frac{1}{2} \left( e^{\beta_r} - 1 \right) \left( \frac{1}{\lambda_r^2} \bar{x}_r^2 + \frac{\lambda_r^2}{\hbar^2} \bar{p}_r^2 \right) \right]$$
Eqs. (6), (34) and (35) imply that this equation is of a similar form to Eq. (23). It should, however, be noted that the Wigner function propagates in the same way as a classical phase space distribution.

The fact that, because the Hamiltonian is quadratic in the positions and momenta, the only an average.

Let \[ \rho(x, p) \] and \[ \eta(x, p) \] be solutions to the classical equations of motion, which result from the classical analogue of the Hamiltonian of Eq. (1). Since the Hamiltonian is
respectively. We have \[32, 33, 34\]

where \(D\) rather complicated, due to the fact that Appendix A, gives an exact, closed form expression for \(\tilde{\psi}\). Carrying out the Gaussian integrations in Eq. (44), and using the results in Appendix A, we show how the matrix \(M\) appearing in Eq. (44) can be expressed in terms of them.

We can use these matrices to propagate \(W_\alpha\) forward in time \[32, 33\]:

\[W_\alpha(t, \eta, \eta_1, \ldots, \eta_N) = W_\alpha(0, \eta, \eta_1, \ldots, \eta_N) = W_{\text{sys}}(\eta)W_{\text{bath}}^{(\alpha)}(\eta_1, \ldots, \eta_N)\]  

where \(W_{\text{sys}}, W_{\text{bath}}^{(\alpha)}\) are the Wigner functions corresponding to \(|\psi_{\text{sys}}\rangle, |\tilde{\eta}_1, \ldots, \tilde{\eta}_N\rangle\) respectively. We have \[32, 33, 34\]

\[W_{\text{bath}}^{(\alpha)}(\eta_1, \ldots, \eta_N) = \frac{2^N}{\hbar^N} \exp \left[ \sum_{r=1}^N (\eta_r - \bar{\eta}_r)^T \Lambda_r (\eta_r - \bar{\eta}_r) \right]\]  

where

\[\Lambda_r = \frac{1}{\hbar} \begin{pmatrix} m_r \omega_r & 0 \\ 0 & m_r \omega_r \end{pmatrix}\]  

Using Eqs. (35), (39), (40), (41) we deduce

\[\tilde{W}_\alpha(t, \eta, x_1, \ldots, x_N) = \int d^2 \eta' \tilde{G}(t, \eta, x_1, \ldots, x_N|\eta')W_{\text{sys}}(\eta')\]  

where

\[\tilde{G}(t, \eta, x_1, \ldots, x_N|\eta')\]

\[= \frac{2^{N-2}}{\pi^2 \hbar^N} \int d^2 \xi dp_1 \ldots dp_N \exp \left[ i \xi^T \left( \eta' - A(-t)\eta - \sum_{r=1}^N B_r(-t)\eta_r \right) \right. \]

\[\left. - \sum_{r, r', r''=1}^N \left( C_r(t)\eta + D_{rr'}(t)\eta_{r'} - \bar{\eta}_{r'} \right)^T \Lambda_r \left( C_{r''}(t)\eta + D_{rr''}(t)\eta_{r''} - \bar{\eta}_{r''} \right) \right]\]

Carrying out the Gaussian integrations in Eq. (44), and using the results in Appendix A, gives an exact, closed form expression for \(\tilde{G}\). However, the expression is rather complicated, due to the fact that \(D_{rr'}\) couples together the different oscillators constituting the heat bath. We will therefore confine ourselves to the case when the interaction between heat bath and system is weak. It is shown in Appendix A that we may then approximate

\[A(t) \approx \begin{pmatrix} \cos \omega t & \frac{1}{m} \sin \omega t \\ -m \omega \sin \omega t & \cos \omega t \end{pmatrix}\]  

(45)
where
\[ h_r^{(0)}(t) = -\frac{\omega_r \sin \omega t - \omega \sin \omega_r t}{\omega_r^2 - \omega^2} \]
and
\[ D_r(t) = \left( \begin{array}{c} \cos \omega_r t \\ -m_r \omega_r \sin \omega_r t \\ \cos \omega_r t \end{array} \right) \]

If we also assume that \( \omega t \ll 1 \), then we can further approximate
\[ A(t) \approx 1 \quad \text{and} \quad h_r^{(0)}(t) \approx -\frac{\omega (\omega_r t - \sin \omega_r t)}{\omega_r^2} \]

Using these results in Eq. (44) and carrying out the integrations gives
\[ \tilde{G}(t, \eta, x_1, \ldots, x_N | \eta') \approx \text{const.} \times \exp \left[ -\sum_{r=1}^{N} \frac{m_r \omega_r}{\hbar} (x_r - q_r(\eta))^2 \right] \]
\[ \times \exp \left[ - (\eta' - \eta - \delta)^T \tilde{M}^{-1}(t) (\eta' - \eta - \delta) \right] \]

where
\[ q_r(\eta) = \left( D_r(t) \left( \bar{\eta}_r - C_r(-t) \eta \right) \right) \]
\[ \delta = \sum_{r=1}^{N} B_r(-t) \left( D_r(t) \left( \bar{\eta}_r - C_r(-t) \eta \right) \right) \]
\[ \tilde{M}(t) = 2\hbar \int_0^\infty d\omega' I(\omega') \left( \frac{(\omega' t - \sin \omega')^2}{m_r \omega_r^2 \sin \omega' t} \right)^2 \left( \frac{(\omega' t - \sin \omega')^2}{m_r \omega_r^2 \sin \omega' t} \right)^2 \]

and \( I(\omega') \) is the spectral density function defined by Eq. (3). Eqs. (13) and (43) then imply
\[ \tilde{W}_\alpha (t, \eta, x_1, \ldots, x_N) \approx \left( \frac{m_1 \omega_1 \ldots m_N \omega_N}{\pi^{N/2} \hbar^N \det \tilde{M}(t)} \right)^{\frac{1}{2}} \exp \left[ -\sum_{r=1}^{N} \frac{m_r \omega_r}{\hbar} (x_r - q_r(\eta))^2 \right] \]
\[ \times \int d^2 \eta' \exp \left[ - (\eta' - \eta)^T \tilde{M}^{-1}(t) (\eta' - \eta) \right] W_{\text{sys}}(\eta') \]

where we have set \( \delta \approx 0 \), which will be justified if the coupling between system and heat bath is sufficiently weak. Define
\[ \tilde{M}^{-1} = \left( \begin{array}{c} \tilde{a} \\ \tilde{c} \\ \tilde{b} \end{array} \right) \quad \text{and} \quad \det \tilde{M}^{-1} = \tilde{\Delta} \]

We then have, by essentially the same argument as the one leading to Eq. (29), that if
\[ \frac{m_0 \omega \hbar}{\hbar \Delta} \ll x_{\text{max}} \quad \text{and} \quad 8m_0 \omega \hbar \tilde{b} \ll x_{\text{max}} \]
then
\[
\tilde{W}_\alpha \approx \tilde{W}_- + \tilde{W}_+ + \tilde{W}_{\text{osc}} \tag{55}
\]
where
\[
\tilde{W}_\pm(t, x, p, x_1, \ldots, x_N)
= \text{const.} \times \exp \left[ -\sum_{r=1}^{N} \frac{m_r \omega_r}{\hbar} (x_r - q_r(x, p))^2 - \hat{\sigma}_\pm^2 (p + \hat{p}_c(x))^2 \right] \rho_\pm(x)
\]
and
\[
\tilde{W}_{\text{osc}}(t, x, p, x_1, \ldots, x_N)
= \text{const.} \times \exp \left[ -\sum_{r=1}^{N} \frac{m_r \omega_r}{\hbar} (x_r - q_r(x, p))^2 - \hat{\sigma}_\pm^2 (p + \hat{p}_c(x))^2 - \hat{\sigma}_1^2 (p_c(x))^2 \right]
\times \cos \left[ \frac{2}{\hbar} S(x) + \tilde{\chi}(t, x, p) \right] (\rho_- (x) \rho_+(x))^{\frac{1}{2}}
\]

with
\[
\rho_\pm = |g_\pm|^2
\]
\[
\hat{\sigma}_\pm^2 = \frac{\Delta}{\hat{a} \pm 2 \hat{\sigma}'_c(x) + \hat{b}(p'_c(x))^2}
\]
\[
\hat{\sigma}_1^2 = \frac{\Delta}{\hbar^2 \Delta + \hat{b}(p'_c(x))^2}
\]
\[
\hat{\sigma}_2^2 = \frac{\hbar^2 \hat{a} \Delta + \hat{b}(p'_c(x))^2}{\hbar^2 \Delta + \left( 1 - 2 \hbar^2 \gamma^2 + \hbar^4 \Delta^2 \right) (p'_c(x))^2 + \hbar^2 \hat{b}^2 (p'_c(x))^4}
\]
\[
\hat{\beta} = \frac{\hat{\epsilon} \left( 1 + \hbar^2 \Delta \right) p'_c(x)}{\hbar^2 \Delta + \hat{b}(p'_c(x))^2}
\]

\(\tilde{\chi}(t, x, p)\) is a phase.

If \(\hat{\sigma}_1 p_c \gg 1\), then \(\tilde{W}_{\text{osc}}\) will be negligible and
\[
\tilde{W}_\alpha \approx \tilde{W}_- + \tilde{W}_+
\]

For a given value of \(x\) the Gaussian in the expression for \(\tilde{W}_\pm\) is peaked at the point \(p = \pm p_c(x), x_r = q_r(x, \pm p_c(x))\). Suppose that \(x_r = q_r(x, p_c(x))\) for \(r = 1, \ldots, N\). Then
\[
\tilde{W}_- = \text{const.} \times \exp \left[ -\hat{\sigma}_3^2 (p - p_c(x))^2 - \hat{\sigma}_2^2 (p + p_c(x))^2 \right]
\]
where \(\text{c.f. Eq. (50)}\)
\[
\hat{\sigma}_3^2 = \frac{2}{\hbar m^2} \int_0^\infty d\omega' I(\omega') \frac{(\sin \omega' t - \omega' t \cos \omega' t)^2}{\omega'^4}
\]

It follows that, if \(\hat{\sigma}_- p_c, \hat{\sigma}_3 p_c \gg 1\), then \(\tilde{W}_-\) will be negligible, so that
\[
\tilde{W}_\alpha \approx \tilde{W}_+
\]

Referring to Eq. (36) we see that this implies \(v^{(\alpha)}_B \approx +p_c/m\). Suppose, on the other hand, that \(x_r = q_r(x, -p_c(x))\) for \(r = 1, \ldots, N\). We then find that \(\tilde{W}_+\) will be negligible if \(\hat{\sigma}_+ p_c, \hat{\sigma}_3 p_c \gg 1\), in which case \(v^{(\alpha)}_B \approx -p_c/m\). The configuration space probability density function is obtained from \(\tilde{W}_\alpha\) by integrating out the momentum.
It follows, that with probability close to 1, either \( x_r \approx q_r(x, p_{cl}(x)) \) for \( r = 1, \ldots, N \), or \( x_r \approx q_r(x, -p_{cl}(x)) \) for \( r = 1, \ldots, N \). We conclude that there will be a high probability of \( v_B^{(a)} \) being close to one of the two classical values \( \pm p_{cl}/m \) provided

\[
\tilde{\sigma}_{1p_{cl}} > 1 \quad \tilde{\sigma}_{2p_{cl}} > 1 \quad \tilde{\sigma}_{3p_{cl}} > 1
\]  

(62)

Let us now specialise to the case of the Caldeira-Leggett model, for which the spectral density defined by Eq. (2) takes the form

\[
I(\omega') = \begin{cases} 
\frac{2m\gamma\omega'}{\pi} & \text{if } 0 \leq \omega' < \Omega \\
0 & \text{if } \Omega < \omega'
\end{cases}
\]  

(63)

for some cut-off frequency \( \Omega \). Substituting this expression in Eq. (52) gives

\[
\bar{M}(t) \approx \frac{4\hbar m\gamma}{\pi} \int_0^{\Omega t} \text{du} \left( \frac{t^2(u - \sin u)^2}{m^2 u^3 (1 - \cos u)^2} - \frac{t^2(u - \sin u)(1 - \cos u)}{m u^4} \right)
\]

\[
= \frac{4\hbar m\gamma}{\pi} \left( \frac{\ln \Omega t - \frac{1}{2} \ln 2 - 2 - \gamma_E}{\ln \Omega t - \ln 2 + \gamma_E} - \frac{\ln \Omega t}{2} \ln \Omega t - \frac{1}{2} \ln 2 + \frac{3}{2} \gamma_E \right) + O(1)
\]

where \( \gamma_E \) is Euler’s constant. Referring to Eqs. (53) we deduce that, if \( \ln \Omega t \gg 1 \),

\[
\left( \frac{\hat{a}}{\hat{b}} \right) \approx \frac{\pi}{2\hbar m\gamma \ln \Omega t} \left( \frac{3m^2}{\pi} \frac{m}{1} \right) \quad \text{and} \quad \hat{\Delta} \approx \frac{\pi^2}{8h^2(\gamma t \ln \Omega t)^2}
\]

It can be seen that \( \hat{a}, \hat{b}, \hat{c} \) and \( \hat{\Delta} \) are cut-off dependent (unlike the quantities \( a, b, c \) and \( \Delta \) considered in the last Section).

With these values inequalities (54) become

\[
(\omega t)^2 \left( \frac{\lambda_B}{x_{\max}} \right)^\frac{1}{\gamma} \ll \omega \gamma t^2 \ln \Omega t \ll \left( \frac{x_{\max}}{\lambda_B} \right)^\frac{1}{\gamma}
\]  

(64)

where \( \lambda_B = \hbar/(m\omega x_{\max}) \).

Eqs. (61) and (63) imply

\[
\tilde{\sigma}_3^2 = \frac{4\gamma t^2}{\hbar m \pi} \int_0^{\Omega t} \text{du} \left( \frac{\sin u - u \cos u}{u^3} \right)^2 \approx \frac{2\gamma t^2 \ln \Omega t}{\pi \hbar m}
\]

if \( \ln \Omega t \gg 1 \) (where \( \gamma_E \) is Euler’s constant, as before). If \( x \) is not too close to one of the classical turning points \( p_{cl}(x) \sim m\omega x_{\max} \) and \( p'_{cl}(x) \sim m\omega \). Consequently

\[
\tilde{\sigma}_3^2 p_{cl}(x) \sim \frac{2x_{\max}}{\pi \lambda_B} \omega \gamma t^2 \ln \Omega t
\]

and [c.f. Eq. (58)]

\[
\tilde{\sigma}_3^2 p'_{cl}(x) \sim \frac{4x_{\max}}{3\pi \lambda_B} \omega \gamma t^2 \ln \Omega t
\]

Taking into account the fact that these equations assume that \( t \) is in the range specified by inequalities (54), we see that \( \tilde{\sigma}_3 p_{cl} \) and \( \tilde{\sigma}_3 p_{cl} \) will be \( \gg 1 \) provided \( \omega \gamma t^2 \ln \Omega t \gg \lambda_B/x_{\max} \).

Referring to Eq. (57) we see that, away from the classical turning points,

\[
\tilde{\sigma}_{\pm p_{cl}}^2 \sim \frac{\pi x_{\max}}{6\lambda_B} \frac{\omega}{\gamma \ln \Omega t} \left( 1 \pm \frac{4}{3} \omega t + \frac{2}{3}(\omega t)^2 \right)^{-1} \approx \frac{\pi x_{\max}}{6\lambda_B} \frac{\omega}{\gamma \ln \Omega t}
\]

(since we are assuming \( \omega t \ll 1 \)). Consequently \( \tilde{\sigma}_{\pm p_{cl}} \gg 1 \) if \( \ln \Omega t \ll \omega x_{\max}/(\gamma \lambda_B) \).

In the case of weak coupling (so that \( \gamma \ll \omega \)) and large quantum numbers (so that \( \lambda_B \ll x_{\max} \)) this inequality is automatically satisfied, for all physically reasonable
values of $\Omega t$. The classicality conditions then reduce to the single requirement $t \gg \tilde{t}_c$, where $\tilde{t}_c$ is the solution to

$$\omega_\gamma \theta^2 \ln \Omega \tilde{t}_c = \frac{\lambda_B}{x_{\text{max}}}$$

Finally, let us consider the relation between $\tilde{t}_c$ and the quantity $t_c$ discussed in the last section [see Eq. (21)]. We clearly ought to have $\tilde{t}_c \geq t_c$; however, it is not immediately apparent that this is necessarily the case. In fact, the seeming difficulty disappears once it is recalled that, in deriving the Caldeira-Leggett master equation from a Brownian motion model, it is assumed that $k_B T \gg \hbar \Omega$ and $\Omega t \gg 1$. In particular, the discussion in the last Section tacitly assumes $\Omega \tilde{t}_c \gg 1$. Hence

$$\omega_\gamma \theta^2 \ln \Omega t_c \ll \omega_\gamma \theta^2 \ln \Omega \tilde{t}_c \ll \frac{\omega_\gamma k_B T \theta^2}{\hbar} = \frac{3\lambda_B}{2x_{\text{max}}}$$

which implies that $t_c \ll \tilde{t}_c$.

5. Conclusion

The motive for this investigation was the question, whether it is true that any process which tends to produce decoherence also tends to make the Bohmian trajectory of a macroscopic object approximately classical. The results we have obtained provide some support for this hypothesis. However, it would clearly require more work to settle the question. We have only considered a particular, idealised model of the interaction between a macroscopic body and its environment. Moreover, our results were obtained on the assumption that the ensemble described by $\hat{\rho}_{\text{bath}}$ consists of coherent states [see Eq. (33), and discussion in paragraphs following]. It would clearly be desirable to see if similar results hold in the case of other models, and for other choices of ensemble.

However, realistic models of the interaction between a macroscopic object and its environment are very complicated, so that detailed calculations, of the kind carried out in this paper, are not usually feasible. What one needs is a general principle, or mechanism, which can be shown to be operative even in those cases where the complexity of the problem makes detailed calculation impracticable. The most promising candidate for such a mechanism is the process of environmental monitoring.

Particularly relevant in this respect is a recent paper by Halliwell. Halliwell analyses Brownian motion models of the kind considered in this paper, and he shows that the positions and momenta of the heat bath oscillators constitute a store of information about the trajectory of the system particle. He also shows that there is a relationship between the amount of information stored in the environment and the amount of decoherence. One may plausibly speculate that a similar principle holds true with regard to the Bohm Interpretation: namely, that there is a direct relationship between the degree to which the Bohmian trajectory is approximately classical, and the amount of information about the trajectory which is stored in the environment. It would also be interesting to know whether such a principle applies to some of the other interpretations which have been proposed in which the particles follow determinate trajectories.

1 In this connection we should mention a recent paper by Geiger et al., in which the authors attempt to derive approximately classical Bohmian trajectories by making certain postulates regarding the form of the many-body wave function describing a macroscopic object.
APPENDIX A. CLOSED FORM EXPRESSIONS FOR $A$, $B_r$, $C_r$ AND $D_{rr'}$

The analysis in Section 4 is based on Halliwell and Yu [22]. However, Halliwell and Yu do not give explicit expressions for the matrices $A$, $B_r$, $C_r$ and $D_{rr'}$. The purpose of this appendix is to derive such expressions. Define

$$\sigma_0 = \begin{pmatrix} 0 & 0 \\ -m \omega_0 & \frac{1}{m \omega_0} \end{pmatrix}, \quad \sigma_r = \begin{pmatrix} 0 & 0 \\ -m_r \omega_r & \frac{1}{m_r \omega_r} \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Referring to Eqs. (37) and (38), and to the classical analogue of Eq. (0), we see that $A$, $B_r$, $C_r$ and $D_{rr'}$ satisfy

$$\frac{d}{dt} A(t) = \omega_0 \sigma_0 A(t) - \sum_{r=1}^{N} \kappa_r \sigma_- C_r(t) \quad \frac{d}{dt} B_r(t) = \omega_0 \sigma_0 B_r(t) - \sum_{r'=1}^{N} \kappa_{r'} \sigma_- D_{rr'}(t)$$

$$\frac{d}{dt} C_r(t) = \omega_r \sigma_r C_r(t) - \kappa_r \sigma_- A(t) \quad \frac{d}{dt} D_{rr'}(t) = \omega_r \sigma_r D_{rr'}(t) - \kappa_r \sigma_- B_r(t)$$

subject to the initial conditions $A(0) = 1$, $B_r(0) = C_r(0) = 0$ and $D_{rr'}(0) = \delta_{rr'} I$.

It is convenient to re-write these equations in integral form:

$$A(t) = e^{\omega_0(t-0)} - \sum_{r=1}^{N} \kappa_r \int_0^t dt' e^{\omega_0(t-t')} \sigma_0 \sigma_- C_r(t')$$

$$B_r(t) = \sum_{r'=1}^{N} \kappa_{r'} \int_0^t dt' e^{\omega_0(t-t')} \sigma_0 \sigma_- D_{rr'}(t')$$

$$C_r(t) = -\kappa_r \int_0^t dt' e^{\omega_r(t-t')} \sigma_0 \sigma_- A(t')$$

$$D_{rr'}(t) = \delta_{rr'} e^{\omega_r(t-0)} - \kappa_r \int_0^t dt' e^{\omega_r(t-t')} \sigma_0 \sigma_- B_r(t')$$

Eqs. (65) and (67) imply

$$A(t) = e^{\omega_0(t-0)} + \int_0^t dt' L(t-t') A(t')$$

while Eqs. (66) and (68) give

$$B_r(t) = -\kappa_r \int_0^t dt' e^{\omega_0(t-t')} \sigma_0 \sigma_- e^{\omega_r(t-t')} + \int_0^t dt' L(t-t') B_r(t')$$

where

$$L(t) = \sum_{r=1}^{N} \left( \kappa_r^2 \int_0^t dt' e^{\omega_0(t-t')} \sigma_0 \sigma_- e^{\omega_r(t-t')} \sigma_- \right)$$

$B_r$ can be expressed in terms of $A$:

$$B_r(t) = -\kappa_r \int_0^t dt' A(t') \sigma_- e^{\omega_r(t-t')}$$

as can be verified by substituting this expression into Eq. (70) and using Eq. (68). Carrying out the integration in the expression for $L$ we find

$$L(t) = \begin{pmatrix} \chi(t) & 0 \\ m \chi(t) & 0 \end{pmatrix}$$
where
\[
\chi(t) = \frac{2}{m\omega_0} \int_0^\infty d\omega' I(\omega') \frac{\omega' \sin \omega_0 t - \omega_0 \sin \omega' t}{\omega'^2 - \omega_0^2}
\]
\(I(\omega')\) being the spectral density defined by Eq. (2). Let \(g\) be the solution to the integral equation
\[
g(t) = \sin \omega_0 t + \int_0^t dt' \chi(t - t') g(t')
\]
In terms of this function the solution to Eq. (39) is
\[
A(t) = \frac{1}{m\omega_0} \begin{pmatrix} mg(t) & g(t) \\ m^2 g(t) & m g(t) \end{pmatrix}
\]
Eqs. (67), (68) and (71) then imply
\[
B_r(t) = \frac{\kappa_r}{mm_r\omega_0 \omega_r} \begin{pmatrix} m_r \hat{h}_r(t) & h_r(t) \\ mm_r \hat{h}_r(t) & m \hat{h}_r(t) \end{pmatrix}
\]
\[
C_r(t) = \frac{\kappa_r}{mm_r\omega_0 \omega_r} \begin{pmatrix} m \hat{h}_r(t) & h_r(t) \\ mm_r \hat{h}_r(t) & m \hat{h}_r(t) \end{pmatrix}
\]
\[
D_{rr'}(t) = \delta_{rr'} e^{\omega_r t \sigma_r} + \frac{\kappa_r \kappa_r'}{mm_r mm_r'} \begin{pmatrix} m_r \hat{f}_{rr'}(t) & f_{rr'}(t) \\ m_r \hat{f}_{rr'}(t) & m \hat{f}_{rr'}(t) \end{pmatrix}
\]
where
\[
h_r(t) = -\int_0^t dt' g(t - t') \sin \omega_r t'
\]
\[
f_{rr'}(t) = \int_0^t dt' g(t - t') \frac{\omega_r \sin \omega_r t' - \omega_r' \sin \omega_r t'}{\omega_r^2 - \omega_r'^2}
\]
We see that all four matrices may be expressed in terms of the single function \(g\).
Eqs. (72), (73) are exact. Let us now consider the case of weak coupling. Working to first order in the \(\kappa_r\) we have
\[
g(t) = \sin \omega_0 t + O(\kappa^2)
\]
and consequently
\[
A(t) = e^{\omega_0 t \sigma_0} + O(\kappa^2)
\]
\[
B_r(t) = \frac{\kappa_r}{mm_r\omega_0 \omega_r} \begin{pmatrix} m_r \hat{h}_r(0) & h_r(0) \\ mm_r \hat{h}_r(0) & m \hat{h}_r(0) \end{pmatrix} + O(\kappa^3)
\]
\[
C_r(t) = \frac{\kappa_r}{mm_r\omega_0 \omega_r} \begin{pmatrix} \hat{m} \hat{h}_r(0) & \hat{h}_r(0) \\ mm_r \hat{h}_r(0) & m \hat{h}_r(0) \end{pmatrix} + O(\kappa^3)
\]
\[
D_{rr'}(t) = \delta_{rr'} e^{\omega_r t \sigma_r} + O(\kappa^2)
\]
where
\[
h_r^{(0)}(t) = -\int_0^t dt' \sin \omega_0 (t - t') \sin \omega_r t' = -\frac{\omega_r \sin \omega_0 t - \omega_0 \sin \omega_r t}{\omega_r^2 - \omega_0^2}
\]
We also note that the frequency counterterm is \(O(\kappa^2)\), so \(\omega_0 = \omega\) to this order of approximation. This proves Eqs. (49-51).
2. Expression for $\mathbf{M}$ in Terms of $\mathbf{A}$ and $\mathbf{B}_r$

The purpose of this appendix is to derive the relationship between the matrices $\mathbf{A}$, $\mathbf{B}_r$, $\mathbf{C}_r$, $\mathbf{D}_{r,r'}$, and the matrix $\mathbf{M}$ which appears in the integrated form of the Master Equation, Eq. (16). We will also show that the matrix $\mathbf{A}$ appearing in Eq. (16) is the same as the matrix $\mathbf{A}$ derived in Appendix A.

We begin by noting that $\mathbf{D}_{r,r'}$, regarded as a $2N \times 2N$ matrix, is invertible. In fact, it follows from the time-reversibility of the classical equations of motion that

$$
\mathbf{A}(t)\mathbf{A}(-t) + \sum_{r=1}^{\infty} \mathbf{B}_r(t)\mathbf{C}_r(-t) = 1 \quad (76)
$$

$$
\mathbf{C}_r(t)\mathbf{B}_r(-t) + \sum_{r'=1}^{\infty} \mathbf{D}_{r,r'}(t)\mathbf{D}_{r',r}(-t) = \delta_{r,r'}\mathbf{1} \quad (77)
$$

$$
\mathbf{A}(t)\mathbf{B}_r(-t) + \sum_{r'=1}^{\infty} \mathbf{B}_r'(t)\mathbf{D}_{r',r}(-t) = 0 \quad (78)
$$

$$
\mathbf{C}_r(t)\mathbf{A}(-t) + \sum_{r'=1}^{\infty} \mathbf{D}_{r,r'}(t)\mathbf{C}_r(-t) = 0 \quad (79)
$$

It is then straightforward to verify that

$$
\sum_{r'=1}^{\infty} \mathbf{D}_{r,r'}(t)\mathbf{D}_{r',r}^{-1}(t) = \delta_{r,r'}\mathbf{1} \quad (80)
$$

where

$$
\mathbf{D}_{r',r}^{-1}(t) = \mathbf{D}_{r,r'}(-t) - \mathbf{C}_r(-t)\mathbf{A}^{-1}(-t)\mathbf{B}_r(-t) \quad (81)
$$

As before, we assume that at time $t$ system+environment are described by the density matrix

$$
\hat{\rho}(t) = e^{-i\hat{H}/\hbar} (|\psi_{\text{sys}}\rangle \langle \psi_{\text{sys}}| \otimes \hat{\rho}_{\text{bath}}) e^{i\hat{H}/\hbar}
$$

where $\hat{\rho}_{\text{bath}}$ is the thermal state defined by Eq. (3). Let $W(t, \eta, \eta_1, \ldots, \eta_N)$ be the corresponding Wigner function. Then

$$
W_{\text{red}}(t, \eta) = \int d^2\eta_1 \ldots d^2\eta_N W_{\hat{\rho}}(t, \eta, \eta_1, \ldots, \eta_N)
$$

We have, by essentially the same argument as the one leading to Eq. (83),

$$
W_{\text{red}}(t, \eta) = \int d^2\eta' G(t, \eta|\eta') W_{\text{red}}(0, \eta')
$$

where

$$
G(t, \eta|\eta') = K \int d^2\xi d^2\eta_1 \ldots d^2\eta_N \exp \left[ -\sum_{r=1}^{N} \left( \tanh\left( \frac{\beta_r}{2} \right) \left( \mathbf{C}_r(-t)\eta + \sum_{r'=1}^{N} \mathbf{D}_{r,r'}(-t)\eta_{r'} \right)^T \mathbf{A}_r \left( \mathbf{C}_r(-t)\eta + \sum_{r'=1}^{N} \mathbf{D}_{r,r'}(-t)\eta_{r'} \right) + i\xi^T \left( \eta' - \mathbf{A}(-t)\eta - \sum_{r'=1}^{N} \mathbf{B}_r(-t)\eta_{r'} \right) \right] \right]
$$

and where $\mathbf{A}_r$ is the matrix defined by Eq. (82), $\beta_r$ denotes the ratio $\hbar\omega_r/(k_BT)$ and $K$ is a normalisation constant.
Making the substitution $\eta'_r = C_r(t) + \sum_{r'=1}^N D_{rr'}(-t) \eta_{r'}$ in the integral on the right hand side of Eq. (82) we obtain

$$G(t, \eta|\eta') = \text{const.} \int d^2\xi d^2\eta' \ldots d^2\eta'_{N}$$

$$\times \exp \left[ -\sum_{r=1}^{N} \left( \tanh \left( \frac{\beta_r}{2} \right) \eta'^{T}_r A_r \eta'_{r} - i \xi^{T} A^{-1}(t) B_r(t) \eta'_{r} \right) \right]$$

$$+ i \xi^{T} (\eta' - A^{-1}(t) \eta)$$

where we have used Eqs. (80) and (81), together with the relations [which are easily seen to follow from Eqs. (76–81)]

$$\sum_{r'=1}^{N} B_{rr'}(-t) D_{rr'}^{-1}(-t) = -A^{-1}(t) B_r(t)$$

$$A(-t) - \sum_{r, r'=1}^{N} B_{rr'}(-t) D_{rr'}^{-1}(-t) C_r(-t) = A^{-1}(t)$$

Carrying out the Gaussian integrations we deduce

$$G(t, \eta|\eta') = \frac{1}{\pi \det A \sqrt{\det M}} \exp \left[ -\left( \eta' - A^{-1}(t) \eta \right)^{T} M^{-1}(t) \left( \eta' - A^{-1}(t) \eta \right) \right]$$

where

$$M(t) = A^{-1}(t) \left( \sum_{r=1}^{N} \coth \left( \frac{\beta_r}{2} \right) B_r(t) A_r^{-1}(t) \right) (A^{-1}(t))^{T} \quad (83)$$

and where the normalisation constant is fixed by the requirement

$$\int d\eta W_{\text{red}}(t, \eta) = 1$$

Comparing with Eq. (16) we see that the matrix $M$ appearing in Eq. (16) is the same as the matrix given by Eq. (83).

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