Arithmetic formulas for the Fourier coefficients of Hauptmoduln of level 2, 3, and 5

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Abstract

We give arithmetic formulas for the coefficients of Hauptmoduln of higher levels as analogues of Kaneko’s formula for the elliptic modular \( j \)-invariant. We also obtain their asymptotic formulas by employing Murty-Sampath’s method.

1 Introduction

For the elliptic modular function \( j(\tau) \), let \( t_m(d) \) be the modular trace function (the precise definition will be given later) and \( c_n \) \((n \geq 1)\) the \( n \)th Fourier coefficient of \( j(\tau) \), that is, \( j(\tau) = q^{-1} + 744 + \sum_{n=1}^{\infty} c_n q^n \). Zagier \[12\] studied the traces of singular moduli and showed that the generating function of \( t_m(d) \) is a meromorphic modular form of weight \( 3/2 \) on the right group for each \( m \). Multiplying it by theta function and observing the modular forms of weight 2, Kaneko \[7\] gave the following arithmetic formula for \( c_n \) experimentally, and showed it.

\[
c_n = \frac{1}{2n} \left\{ \sum_{r \in \mathbb{Z}} t_1(n - r^2) + \sum_{r \geq 1, \text{ odd}} ((-1)^r t_1(4n - r^2) - t_1(16n - r^2)) \right\}
\]

On the other hand, by using the circle method, Petersson \[10\] and later Rademacher \[11\] independently derived the asymptotic formula for \( c_n \):

\[
c_n \sim e^{\pi \sqrt{2n/3}} \text{ as } n \to \infty.
\]

The circle method is introduced by Hardy and Ramanujan \[4\] to prove the asymptotic formula for the partition function

\[
p(n) \sim \frac{e^{\pi \sqrt{2n/3}}}{4 \sqrt{3n}} \text{ as } n \to \infty,
\]

where \( p(n) \) is defined by \( \sum_{n=0}^{\infty} p(n)/q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-1} \). In 2013, Bruinier and Ono \[2\] considered certain traces of singular moduli for weak Maass forms and derived the algebraic formula for \( p(n) \). Combining this formula with Laplace’s method, Dewar and Murty \[4, 5\] proved the asymptotic formulas for \( p(n) \) and \( c_n \) without the circle method. More recently, Murty and Sampath \[8\] derived the asymptotic formula for \( c_n \) from Kaneko’s arithmetic formula with Laplace’s method.

In this article, we generalize these formulas to Hauptmoduln (defined in section 2) for the congruence subgroups \( \Gamma_0(p) \) and \( \Gamma_0^*(p) \) (the extension of \( \Gamma_0(p) \) by the Atkin-Lehner involution) with \( p = 2, 3, \) and 5.

Let \( j_p(\tau) \) and \( j_p^*(\tau) \) be the corresponding Hauptmoduln for \( \Gamma_0(p) \) and \( \Gamma_0^*(p) \), respectively. Ohta \[9\] gave the arithmetic formulas for the Fourier coefficients of \( j_2(\tau) \) and \( j_2^*(\tau) \), and a part of those of \( j_3(\tau) \). She also
treated the cases of \( j_4(\tau) \) and \( j_6(\tau) \). Let \( c_n^{(p)} \) and \( c_n^{(ps)} \) be the \( n \)th Fourier coefficients of \( j_p(\tau) \) and \( j^*_p(\tau) \), respectively. We express these coefficients in terms of the modular trace functions \( t_n^{(ps)}(d) \).

**Theorem 1.1.** For any \( n \geq 1 \), we have

\[
\begin{align*}
c_n^{(2)} &= \frac{1}{2n} \times \left\{ -\sum_{r \in \mathbb{Z}} t_2^{(2)}(4n - r^2) + 24\sigma_1^{(2)}(n) \right\} \\
c_n^{(3)} &= \frac{1}{2n} \times \left\{ -\sum_{r \in \mathbb{Z}} t_2^{(3)}(4n - r^2) + 36\sigma_1^{(3)}(n) \right\} \\
c_n^{(5)} &= \frac{1}{2n} \times \left\{ -\sum_{r \in \mathbb{Z}} t_2^{(5)}(4n - r^2) + 18\sigma_1^{(5)}(n) \right\}
\end{align*}
\]

where \( \sigma_1(n) = \sum_{d|n} d \), and \( \sigma_1^{(p)}(n) = \sum_{d|n} d \).

**Remark.** These formulas are different from those in Ohta \[9\]. In \[9\], the definition of \( t_n^{(p)}(d) \) was mixed with that of \( t_n^{(ps)}(d) \), and used the values of \( t_n^{(p)}(d) \) instead of \( t_n^{(ps)}(d) \).

Combining these formulas with Laplace’s method as in \[8\], we obtain the asymptotic formulas of \( c_n^{(p)} \).

**Theorem 1.2.** We have

\[
\begin{align*}
c_n^{(2)} &\sim \frac{e^{2\pi \sqrt{n}}}{2n^{3/4}} \times \left\{ -1 \quad (n \equiv 0 \mod 2), \\
&\quad 1 \quad (n \equiv 1 \mod 2), \\
c_n^{(3)} &\sim \frac{e^{4\pi \sqrt{n}/3}}{\sqrt[3]{6n}^{3/4}} \times \left\{ -1 \quad (n \equiv 0, 2 \mod 3), \\
&\quad 2 \quad (n \equiv 1 \mod 3), \\
c_n^{(5)} &\sim \frac{e^{4\pi \sqrt{5}/5}}{\sqrt[5]{10n}^{3/4}} \times \left\{ -1 \quad (n \equiv 0 \mod 5), \\
&\quad (3 + \sqrt{5})/2 \quad (n \equiv 1 \mod 5), \\
&\quad -1 + \sqrt{5} \quad (n \equiv 2 \mod 5), \\
&\quad -1 - \sqrt{5} \quad (n \equiv 3 \mod 5), \\
&\quad (3 - \sqrt{5})/2 \quad (n \equiv 4 \mod 5)
\right. 
\end{align*}
\]

as \( n \to \infty \).

2 Preliminaries

In this section, we shall define the Hauptmoduln and the modular trace functions.

**Definition 2.1.** Let \( \Gamma \) be a congruence subgroup of \( SL_2(\mathbb{R})/\pm I \) containing \( \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \). If the genus of \( \Gamma \) is equal to 0, there is a unique modular function \( f \) of weight 0 satisfying the following conditions. We call this \( f \) the Hauptmodul with respect to \( \Gamma \).

1. \( f \) is holomorphic in the upper half plane \( \mathfrak{H} \).
2. \( f \) has a Fourier expansion of the form \( f(\tau) = q^{-1} + \sum_{n=1}^{\infty} H_n q^n \) \( (q := e^{2\pi i \tau}) \).
3. \( f \) is holomorphic at cusps of \( \Gamma \) except i\( \infty \).

For \( \Gamma_0(p) := \{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in PSL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{p} \} \) and \( \Gamma_0^*(p) := \Gamma_0(p) \cup \Gamma_0(p)(\sqrt{p}^{-1/\sqrt{p}}) \) \( (p = 2, 3, 5) \), the corresponding Hauptmoduln \( j_p(\tau) \) and \( j^*_p(\tau) \) can be described by means of the Dedekind \( \eta \)-function.
\[ \eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n); \]

\[
j_2(\tau) = \left( \frac{\eta(\tau)}{\eta(2\tau)} \right)^{24} + 24 = \frac{1}{q} + 276q - 2048q^2 + 11202q^3 + \cdots,
\]

\[
j_2^*(\tau) = \left( \frac{\eta(\tau)}{\eta(2\tau)} \right)^{24} + 24 + 2^{12} \left( \frac{\eta(2\tau)}{\eta(\tau)} \right)^{24} = \frac{1}{q} + 4372q + 96256q^2 + 1240002q^3 + \cdots,
\]

\[
j_3(\tau) = \left( \frac{\eta(\tau)}{\eta(3\tau)} \right)^{12} + 12 = \frac{1}{q} + 54q - 76q^2 - 243q^3 + \cdots,
\]

\[
j_3^*(\tau) = \left( \frac{\eta(\tau)}{\eta(3\tau)} \right)^{12} + 12 + 3^6 \left( \frac{\eta(3\tau)}{\eta(\tau)} \right)^{12} = \frac{1}{q} + 783q + 8672q^2 + 65367q^3 + \cdots,
\]

\[
j_5(\tau) = \left( \frac{\eta(\tau)}{\eta(5\tau)} \right)^6 + 6 = \frac{1}{q} + 9q + 10q^2 - 30q^3 + \cdots,
\]

\[
j_5^*(\tau) = \left( \frac{\eta(\tau)}{\eta(5\tau)} \right)^6 + 6 + 5^3 \left( \frac{\eta(5\tau)}{\eta(\tau)} \right)^6 = \frac{1}{q} + 134q + 760q^2 + 3345q^3 + \cdots.
\]

For \( p = 2, 3, \) and 5, let \( d \) be a positive integer such that \(-d\) is congruent to a square modulo \( 4p \), and \( \mathcal{Q}_{d,p} \), the set of positive definite binary quadratic forms \( Q(X, Y) = [a, b, c] = aX^2 + bXY + cY^2 \) \((a, b, c \in \mathbb{Z})\) of discriminant \(-d\) with \( a \equiv 0 \) \((\text{mod } p)\). Moreover, we fix an integer \( \beta \) \((\text{mod } 2p)\) with \( \beta^2 \equiv -d \) \((\text{mod } 4p)\) and denote by \( \mathcal{Q}_{d,p,\beta} \) the set of quadratic forms \([a, b, c] \in \mathcal{Q}_{d,p} \) such that \( b \equiv \beta \) \((\text{mod } 2p)\). For every positive integer \( m \), let \( \varphi_m(j^*_p) \) be a unique polynomial of \( j^*_p \) satisfying \( \varphi_m(j^*_p(\tau)) = q^{-m} + O(q) \). We define two modular trace functions:

\[
t_m^{(p)}(d) := \sum_{Q \in \mathcal{Q}_{d,p,\beta} / \Gamma_0(p)} \frac{1}{|\Gamma_0(p)q|} \varphi_m(j^*_p(\alpha_Q)),
\]

\[
t_m^{(p^*)}(d) := \sum_{Q \in \mathcal{Q}_{d,p,\beta} / \Gamma_0(p)} \frac{1}{|\Gamma_0(p)q|} \varphi_m(j^*_p(\alpha_Q)),
\]

where \( \alpha_Q \) is the root of \( Q(X, 1) = 0 \) in \( \mathcal{Q} \). The definition of \( t_m^{(p)}(d) \) is independent of \( \beta \). In addition, we set \( t_2^{(2^s)}(0) := 5, t_2^{(3^s)}(0) := 3, t_2^{(3^s)}(-1) := -1, t_2^{(3^s)}(-4) := -2, t_2^{(3^s)}(d) := 0 \) for \( d < -4 \) or \( d \not\equiv \) square \((\text{mod } 4p)\) \((p = 2, 3, 5)\). For the relation between two modular trace functions, see [3].

Remark. For \( p = 1 \), we put \( j_1^*(\tau) := j(\tau) - 744 = \{ (\eta(\tau)/\eta(2\tau))^8 + 2^8(\eta(2\tau)/\eta(\tau))^16 \}^3 - 744 \) and \( t_m(d) := t_m^{(1^*)}(d) \).

## 3 Proof of Theorem 1.1

We give a proof only for the case \( p = 3 \); the other cases are proved in the same way.

**Definition 3.1.** For every positive integer \( t \), we define the operator \( U_t \) by

\[
\left( \sum a_nq^n \right) U_t := \sum a_{tn}q^n.
\]

Then \( U_t \) sends a modular form to a modular form of the same weight but raises the level in general. To prove Theorem 1.1, we need the following theorem, which is a special case \( f = \varphi_m(j^*_p(\tau)) \) of Theorem 1.1 in [4].

**Theorem 3.2.** The function

\[
g_m^{(p^*)}(\tau) := \sum_{d>0} t_m^{(p^*)}(d)q^d + (\sigma_1(m) + p\sigma_1(m/p)) - \sum_{k|m} kq^{-k^2}
\]

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(where \( \sigma_1(x) = 0 \) if \( x \notin \mathbb{Z} \)) is a meromorphic modular form of weight \( 3/2 \), holomorphic outside the cusps, with respect to \( \Gamma_0(4p) \), that is,

\[
g_m^{(m)}(\tau) \in M_{3/2}^{\text{mer}}(\Gamma_0(4p)).
\]

Here \( M_k^{\text{mer}}(\Gamma) \) denotes the space of meromorphic modular forms of weight \( k \) with respect to \( \Gamma \).

We prove Theorem \[\text{[1]}\] For the modular form \( f(\tau) = \sum a_n q^n \), we define the functions \( \tilde{f}_0, \tilde{f}_1 \) and \( \tilde{f}_2 \) by

\[
\tilde{f}_0(\tau) := \frac{1}{3} \left\{ f(\tau) + f(\tau + \frac{1}{3}) + f(\tau + \frac{2}{3}) \right\},
\]

\[
\tilde{f}_1(\tau) := \frac{1}{3} \left\{ f(\tau) + \zeta^{-1} f(\tau + \frac{1}{3}) + \zeta f(\tau + \frac{2}{3}) \right\},
\]

\[
\tilde{f}_2(\tau) := \frac{1}{3} \left\{ f(\tau) + \zeta f(\tau + \frac{1}{3}) + \zeta^{-1} f(\tau + \frac{2}{3}) \right\}
\]

where \( \zeta = e^{2\pi i/3} \). For each \( k \) (mod 3), then \( \tilde{f}_k \) has a Fourier expansion of the form \( \tilde{f}_k(\tau) = \sum_{n \equiv k(3)} a_n q^n \), and it is also a modular form of the same weight. By Theorem \[\text{[5.2]}\] we have

\[
g_2^{(3\ast)}(\tau) = \sum_{d = -4}^{\infty} t_2^{(3\ast)}(d) q^d \in M_2^{\text{mer}}(\Gamma_0(12)).
\]

Now consider the modular form \( g_2^{(3\ast)}(\tau) \cdot \theta_0(\tau) \) where \( \theta_0(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2} \in M_{1/2}(\Gamma_0(4)) \). This form is of weight 2 and we have

\[
g_2^{(3\ast)}(\tau) \cdot \theta_0(\tau) = \left( \sum_{d = -4}^{\infty} t_2^{(3\ast)}(d) q^d \right) \left( \sum_{r \in \mathbb{Z}} q^{r^2} \right) = \sum_{n = -4}^{\infty} \left( \sum_{r \in \mathbb{Z}} t_2^{(3\ast)}(n - r^2) \right) q^n \in M_2^{\text{mer}}(\Gamma_0(12)).
\]

Similarly, the product \( g_2^{(3\ast)}(\tau) \cdot \theta_0(9\tau) \) is also a modular form of weight 2 and its Fourier expansion is

\[
g_2^{(3\ast)}(\tau) \cdot \theta_0(9\tau) = \sum_{n = -4}^{\infty} \sum_{r \in \mathbb{Z}} t_2^{(3\ast)}(n - (3r)^2) q^n = \sum_{n = -4}^{\infty} \left( \sum_{r \equiv 0(3)} t_2^{(3\ast)}(n - r^2) \right) q^n \in M_2^{\text{mer}}(\Gamma_0(36)).
\]

We put

\[
F(\tau) := \left( g_2^{(3\ast)}(\tau) \cdot \theta_0(\tau) \right) |_{U_4} = \sum_{n = -1}^{\infty} \left( \sum_{r \in \mathbb{Z}} t_2^{(3\ast)}(4n - r^2) \right) q^n \in M_2^{\text{mer}}(\Gamma(12)),
\]

\[
G(\tau) := \left( g_2^{(3\ast)}(\tau) \cdot \theta_0(9\tau) \right) |_{U_4} = \sum_{n = -1}^{\infty} \left( \sum_{r \equiv 0(3)} t_2^{(3\ast)}(4n - r^2) \right) q^n \in M_2^{\text{mer}}(\Gamma(36)).
\]

Then \( F(\tau) \) and \( G(\tau) \) are meromorphic modular forms of weight 2. Moreover, for

\[
j_3'(\tau) = \sum_{n = -1}^{\infty} n c_n^{(3)} q^n \in M_2^{\text{mer}}(\Gamma_0(3))
\]

\[
E_2^{(3)}(\tau) := \frac{1}{2} (3E_2(3\tau) - E_2(\tau)) = 1 + 12 \sum_{n = 1}^{\infty} \sigma_1^{(3)}(n) q^n \in M_2(\Gamma_0(3)),
\]

(where the prime denotes \((2\pi i)^{-1} d/d\tau\) and \( E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \) is the Eisenstein series of weight 2), we put

\[
H(\tau) := j_3'(\tau) - \frac{3}{2} E_2^{(3)}(\tau) = -\frac{1}{q} - \frac{3}{2} + \sum_{n = 1}^{\infty} (nc_n^{(3)} - 18\sigma_1^{(3)}(n)) q^n \in M_2^{\text{mer}}(\Gamma_0(3)).
\]
Then, the theorem in the case of $p = 3$ is equivalent to the following identities of modular forms:

$$2\tilde{H}_0(\tau) = -\tilde{G}_0(\tau), \quad 2\tilde{H}_1(\tau) = \tilde{F}_1(\tau), \quad 2\tilde{H}_2(\tau) = \tilde{F}_2(\tau).$$

Since these modular forms are of weight 2 on $\Gamma(36)$, we see that, by the Riemann-Roch theorem, it is enough to check the coincidence of Fourier coefficients on both sides of the equalities up to $q^{3960}$. We checked this by using Mathematica and Pari-GP.

Similarly, we can show the equation $j_3^*(\tau) = j_3(\tau) - 3(j_3(U_3))(\tau)$, and we obtain $c_n^{(3*)} = c_n^{(3)} - 3e_n^{(3)}$.

4 Proof of Theorem 1.2

In this section, we give an overview of a proof. Since we can prove any case in the same way as [8], we give a proof only for the case $p = 3$. First, we prepare for a proof.

Definition 4.1. The binary quadratic forms

$$\begin{align*}
\begin{cases}
[3, 0, d/12] & (-d \equiv 0 \pmod{12}), \\
[3, 1, (d + 1)/12] & (-d \equiv 1 \pmod{12}), \\
[3, 2, (d + 4)/12] & (-d \equiv 4 \pmod{12}), \\
[3, 3, (d + 9)/12] & (-d \equiv 9 \pmod{12})
\end{cases}
\end{align*}$$

are forms with discriminant $-d$ and are called the principal form of discriminant $-d$.

Lemma 4.2. The following conditions are equivalent for a form $Q \in \mathcal{Q}_{d,3}$:

1. There are $x, y \in \mathbb{Z}$ such that $Q(x, y) = 3$.
2. $Q$ is $\Gamma_0(3)$-equivalent to $[3, B, C]$ for some $B, C \in \mathbb{Z}$.
3. $Q$ is $\Gamma_0(3)$-equivalent to a principal form of discriminant $-d$.

This lemma can be proved in the same way as Lemma 2.2 in [8]. The key theorem for the proof of Theorem 1.2 is the following.

Theorem 4.3. (Laplace’s method). Suppose that $h(t)$ is a real-valued $C^2$-function defined on the interval $(a, b)$ (with $a, b \in \mathbb{R}$). If we further suppose that $h$ has a unique maximum at $t = c$ with $a < c < b$ so that $h'(c) = 0$ and $h''(c) < 0$, then, we have

$$\int_a^b e^{\lambda h(t)} dt \sim e^{\lambda h(c)} \left(\frac{-2\pi}{\lambda h''(c)}\right)^{1/2}$$

as $\lambda \to \infty$.

We prove Theorem 1.2. By definition,

$$t_2^{(3*)}(d) := \sum_{Q \in \mathcal{Q}_{d,3}/\Gamma_0(3)^*} \frac{1}{|\Gamma_0(3)Q|} \varphi_2(j_3^*(\alpha Q)).$$

If $Q = [a, b, c]$ is the element of $\mathcal{Q}_{d,3}$, we have

$$e^{2\pi i c} = \exp\left(2\pi i \left(-\frac{b + i\sqrt{d}}{2a}\right)\right) = \exp\left(-\frac{\pi i b}{a}\right) \exp\left(-\frac{\pi \sqrt{d}}{a}\right)$$

and consequently,

$$\varphi_2(j_3^*(\alpha Q)) = q^{-2} + O(q) = \exp\left(2\pi i \frac{b}{a}\right) \exp\left(2\pi \frac{\sqrt{d}}{a}\right) + O\left(\exp\left(-\frac{\pi \sqrt{d}}{a}\right)\right).$$
By this calculation, the contribution to $t_2(3^*) (d)$ comes only from classes of forms with $a = 3$. By Lemma 4.2, any such form is equivalent to a principal form, so that we have

$$t_2(3^*) (d) = O \left( \exp \left( -\frac{\pi \sqrt{d}}{3} \right) \right) + \exp \left( \frac{2\pi \sqrt{d}}{3} \right) \times \left\{ \begin{array}{cl} 1 & (d \equiv 0, 3 \mod 12), \\ -1 & (d \equiv 8, 11 \mod 12). \end{array} \right.$$ 

Combining this formula with Theorem 4.1 we obtain

$$c_n(3) \sim \frac{1}{2n} \times \left\{ \begin{array}{l} -\sum_{r \equiv 0(3)} \exp \left( \frac{2\pi \sqrt{4n - r^2}}{3} \right) & (n \equiv 0 \mod 3), \\ \sum_{r \equiv 1, 2(3)} \exp \left( \frac{2\pi \sqrt{4n - r^2}}{3} \right) & (n \equiv 1 \mod 3), \\ -\sum_{r \equiv 0(3)} \exp \left( \frac{2\pi \sqrt{4n - r^2}}{3} \right) & (n \equiv 2 \mod 3). \end{array} \right.$$ 

For each $k = 0, 1, 2$, we consider the sum

$$S_n^{(k)} := \frac{3}{2\sqrt{n}} \sum_{r \equiv k(3)} e^{\frac{4\pi \sqrt{n}}{3} \sqrt{1 - \frac{r^2}{4n}}} = \frac{3}{2\sqrt{n}} \sum_{\frac{4n \geq r^2}{4n \geq r^2}} e^{\frac{4\pi \sqrt{n}}{3} \sqrt{1 - \frac{4n + k^2}{4n}}} ,$$

and view this sum as a Riemann sum for the function $t \mapsto e^{\frac{4\pi \sqrt{n}}{3} \sqrt{1 - \frac{t^2}{3}}} : (-1, 1) \to \mathbb{R}$. We can show that $S_n^{(k)}$ is asymptotic to the corresponding Riemann integral $J_n$ where

$$J_n := \int_{-1}^{1} e^{\frac{4\pi \sqrt{n}}{3} \sqrt{1 - \frac{t^2}{3}}} dt.$$ 

(For further detail, see [6]). Moreover, applying Laplace’s method to the case $\lambda = \sqrt{n}$ and $h(t) = 4\pi \sqrt{1 - \frac{t^2}{3}}$ on $(-1, 1)$, we have

$$J_n \sim e^{\frac{\sqrt{n}}{4} \pi/3} \cdot \left( \frac{-2\pi}{-4\pi \sqrt{n/3}} \right)^{1/2} = \frac{\sqrt{3}}{\sqrt{2n}^{1/4}} e^{\frac{4\pi \sqrt{n}}{3}}.$$ 

Putting these asymptotic formulas together, we obtain

$$c_n^{(3)} \sim \frac{1}{3\sqrt{n}} \times \left\{ \begin{array}{l} -S_n^{(0)} & (n \equiv 0 \mod 3), \\ S_n^{(1)} + S_n^{(2)} & (n \equiv 1 \mod 3), \\ -S_n^{(0)} & (n \equiv 2 \mod 3), \end{array} \right.$$ 

$$\sim e^{\frac{4\pi \sqrt{n}}{3}} \cdot \left\{ \begin{array}{l} -1 & (n \equiv 0 \mod 3), \\ 2 & (n \equiv 1 \mod 3), \\ -1 & (n \equiv 2 \mod 3) \end{array} \right.$$ 

as $n \to \infty$. 

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Tables of $t_{m}^{(p^*)}(d)$ and $t_{m}^{(p)}(d)$ ($-4 \leq d \leq 50$)

| $d$ | $t_{1}^{(2*)}(d)$ | $t_{2}^{(2*)}(d)$ | $t_{1}^{(2)}(d)$ | $t_{2}^{(2)}(d)$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| -4  |  0              |    -2           |     0           |    -4           |
| -1  |  -1             |     -1          |     -1          |     -1          |
| 0   |  1              |      5          |      2          |     10          |
| 4   |  -26            |     518         |     -52         |    1036         |
| 7   |  -23            |    -8215        |    -23          |    -8215        |
| 8   |   76            |     7180        |     152         |    14360        |
| 12  |  -248           |     52760       |     -496        |    105520       |
| 15  |   -1            |   -385025       |     -1          |   -385025       |
| 16  |    518          |     287710      |     1036        |    575420       |
| 20  |  -1128          |   1263640       |     -2256       |   2527280       |
| 23  |  -94            |  -6987870       |     -94         |  -6987870       |
| 24  |   2200          |    4831256      |     4400        |    9662512      |
| 28  |  -4096          |   16572370      |     -8192       |   33144740      |
| 31  |    93           |  -78987171      |     93          |  -78987171      |
| 32  |   7180          |   52263100      |     14360       |   104526200     |
| 36  |  -12418         |  153553438      |     -24836      |   307108876     |
| 39  |  -236           |  -663068908     |     -236        |  -663068908     |
| 40  |   20632         |   425670680     |     41264       |   851341360     |
| 44  |  -33512         |  1122593352     |     -67024      |   2245186704    |
| 47  |   235           |  -4515675925    |     235         |  -4515675925    |
| 48  |   53256         |  2835914280     |    106512       |   5671828560    |

| $d$ | $t_{1}^{(3*)}(d)$ | $t_{2}^{(3*)}(d)$ | $t_{1}^{(3)}(d)$ | $t_{2}^{(3)}(d)$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| -4  |  0              |    -2           |     0           |    -2           |
| -1  |  -1             |     -1          |     -1          |     -1          |
| 0   |  1              |      5          |      2          |     10          |
| 11  |   22            |    -1082        |     22          |   -1082         |
| 12  |   26            |    1428         |     52          |    2856         |
| 15  |   -69           |    3195         |    -138         |    6390         |
| 20  |  -116           |   -11892        |    -116         |   -11892        |
| 23  |   115           |   -22797        |    115          |   -22797        |
| 24  |   174           |    28710        |     348         |    57420        |
| 27  |   -241          |    53223        |    -482         |   106446        |
| 32  |  -410           |   -140222       |    -410         |   -140222       |
| 35  |   492           |   -240500       |    492          |   -240500       |
| 36  |   492           |   287244        |     984         |    57488        |
| 39  |  -705           |   477567        |   -1410         |   955134        |
| 44  |  -1060          |   -1081096      |    -1060        |   -1081096      |
| 47  |   1272          |   -1718792      |    1272         |   -1718792      |
| 48  |   1442          |   2004918       |    2884         |   4009836       |
| $d$ | $t_1^{(5)}(d)$ | $t_2^{(5)}(d)$ | $t_1^{(5)}(d)$ | $t_2^{(5)}(d)$ |
|-----|----------------|----------------|----------------|----------------|
| −4  | 0              | −2             | 0              | −2             |
| −1  | −1             | −1             | −1             | −1             |
| 0   | 1              | 3              | 2              | 6              |
| 4   | −8             | −6             | −8             | −6             |
| 11  | −12            | −124           | −12            | −124           |
| 15  | −19            | 93             | −38            | 186            |
| 16  | −6             | −270           | −6             | −270           |
| 19  | 20             | 132            | 20             | 132            |
| 20  | 6              | 268            | 12             | 536            |
| 24  | −44            | 216            | −44            | 216            |
| 31  | −39            | −1863          | −39            | −1863          |
| 35  | −44            | 1668           | −88            | 3336           |
| 36  | 20             | −3054          | 20             | −3054          |
| 39  | 53             | 1653           | 53             | 1653           |
| 40  | 56             | 2868           | 112            | 5736           |
| 44  | −136           | 2416           | −136           | 2416           |

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