Filtered expansions in general relativity II

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Abstract This is the second of two papers in which we construct formal power series solutions in external parameters to the vacuum Einstein equations, implementing one bounce for the Belinskii-Khalatnikov-Lifshitz (BKL) proposal for spatially inhomogeneous spacetimes. Here we show that spatially inhomogeneous perturbations of spatially homogeneous elements are unobstructed. A spectral sequence for a filtered complex, and a homological contraction based on gauge-fixing, are used to do this.

Keywords First keyword · Second keyword · More

1 Introduction

This is a continuation of [2], using the homological, graded Lie algebra (gLa) framework for general relativity in [1]. Familiarity with both these papers is assumed, this introduction only gives pointers to the pertinent material.

In [1] we defined a gLa $E$ whose nondegenerate Maurer-Cartan (MC) elements are the solutions to the vacuum Einstein equations. In [2] we first defined what a filtered expansion is in general, then introduced a specific 1-index gLa filtration called BKL filtration, and derived 2- and 3-index filtrations from it. Their Rees algebras

$$P_{\text{bounce}} = \{ \sum_{p_1,p_3} x_2^{p_2} s_3^{p_1} x_{p_1 p_3} | x_{p_1 p_3} \in F_{p_1 p_3} E \}$$
$$P_{\text{free}} = \{ \sum_{p_1,p_2,p_3} x_1^{p_1} x_2^{p_2} s_3^{p_3} x_{p_1 p_2 p_3} | x_{p_1 p_2 p_3} \in F_{p_1 p_2 p_3} E \}$$

are free over $\mathbb{R}[[s_2,s_3]]$ and $\mathbb{R}[[s_1,s_2,s_3]]$, subalgebras of $E[[s_2,s_3]]$ and $E[[s_1,s_2,s_3]]$ respectively. Here $s_1,s_2,s_3$ are formal symbols. The associated graded gLa are

$$A_{\text{bounce}} = P_{\text{bounce}} / (s_2,s_3) \quad A_{\text{free}} = P_{\text{free}} / (s_1,s_2,s_3)$$

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In this paper we take certain MC-elements in $A_{\text{bounce}}$ and $A_{\text{free}}$ and calculate the homology of the associated differentials. An essential point is that we study these differentials on the associated gradeds, $A_{\text{bounce}}$ and $A_{\text{free}}$, because they control formal perturbations on $P_{\text{bounce}}$ and $P_{\text{free}}$, which is what one is actually interested in. See [1], [2]. With this understanding, this paper is set in the associated gradeds.

In [2] we constructed quite general MC-elements in $A_{\text{bounce}}$ and $A_{\text{free}}$. They depend on several function parameters, and so do their differentials. We only study the differential associated to spatially homogeneous MC-elements. These differentials control spatially inhomogeneous perturbations, not merely homogeneous ones. This keeps calculations manageable. We speculate that results would be similar starting from the spatially inhomogeneous MC-elements in [2], but we have not checked this.

The differential on the associated gradeds defines a filtered complex that is studied using two tools: a spectral sequence for a filtered complex, reviewed in [2]; and a contraction based on gauge-fixing [1], a tool specific to general relativity.

**Theorem (No obstructions - informal version)** For homogeneous MC-elements in $A_{\text{free}}$ and $A_{\text{bounce}}$, the associated differential has vanishing second homology.

Hence by [2], spatially inhomogeneous formal power series ‘free motion’ and ‘bounce’ solutions exist, implementing building blocks of the BKL proposal. We first suggested in [3] that these obstructions could vanish.

The main ingredients from [2] are:

- **(J1)** The over-parametrization lemma [2, Lemma 8].
- **(J2)** The remark about homogeneous MC-elements in $A_{\text{free}}$ [2, Remark 2].
- **(J3)** The theorem about MC-elements in $A_{\text{bounce}}$ [2, Theorem 3].

## 2 The homology in $A_{\text{free}}$

of some homogeneous Maurer-Cartan elements

Let $\gamma \in \text{MC}(A_{\text{free}})$ be a homogeneous MC-element as in Lemma (J2). Here we compute the homology $H^k(d)$ of the differential

$$d = [\gamma, -] \in \text{End}^1(A_{\text{free}})$$

**Assumption 1 (In force through Section 2)** Suppose $\gamma \in \text{MC}(A_{\text{free}})$ as above and:

- $g_1^0, g_2^0, g_3^0$ are nonzero and pairwise different. The sum of any two is negative.
- $g_1, g_2, g_3$ are nonzero.
- $S$ is an open subset of a smooth 3-dimensional Lie group containing the identity element, and $D_1, D_2, D_3$ is a basis of left-invariant vector fields.
- There is an $\varepsilon > 0$ and a diffeomorphism $(x^1, x^2, x^3) : S \to (-\varepsilon, \varepsilon)^3$ such that $(0,0,0)$ is the identity element and, with $\partial_1, \partial_2, \partial_3$ the partial derivatives for the coordinate system $x^1, x^2, x^3$, one has

$$D_2 = \partial_2 \quad D_1|_{e=0} = \partial_1$$

(1)

1. Let $\phi^\varepsilon$ be the flow associated to $D_1$ on the Lie group; this flow exists. Define the map $(x^1, x^2, x^3) \mapsto (\phi_3^\varepsilon \circ \phi_2^\varepsilon \circ \phi_1^\varepsilon)(e)$ where $e$ is the identity element. Using the inverse function theorem, we can construct a coordinate system with the required properties.
This follows from the remainder of Section 2, and the fact [2] that the last page

Table 1 Definition of $G_0e_0$ associated to a conformally orthonormal basis $\theta_0, \theta_1, \theta_2, \theta_3$. We omit wedges, so $\theta_0 \theta_1 = \theta_0 \wedge \theta_1$. The basis-dependent injection $\text{Der}^{(\infty)} \to \text{CDer}(W)$ is implicit [1]. All elements in the table are elements of $\delta'$ via the canonical surjection $\mathcal{Z} \to \delta'$.

\[\begin{array}{|c|c|}
\hline
\alpha = p_1 p_2 p_3 & G_0 e_0 \subseteq \delta' \text{ is the } C^\infty \text{-span of these elements} \\
\hline
\text{note that } G_0 e_0 = G_0 e_0 \oplus \theta_0 (G_0 e_0) & C^\infty \text{-rank} \\
\hline
000 & \text{Der}^{(\infty)}, \theta_0, \theta_0 \theta_0 + \theta_2 \theta_1, \theta_2 \theta_0 + \theta_0 \theta_2, \theta_2 \theta_0 + \theta_0 \theta_2, \\
& \theta_1 \theta_1 \sigma_1 + \theta_1 \theta_1 \sigma_2 + \theta_2 \theta_0 \sigma_0 + 2 \theta_0 \theta_0 \sigma_1 + 2 \theta_0 \theta_0 \sigma_2 + 2 \theta_0 \theta_0 \sigma_3 \\
& 9 \\
\hline
001 & + \theta_0 \sigma_1 + \theta_0 \sigma_2 + \theta_0 \sigma_3 \\
& 1 \\
\hline
002 & + \theta_0 \sigma_1 + \theta_0 \sigma_2 + \theta_0 \sigma_3 \\
& 1 \\
\hline
011 & \sigma_1, \sigma_2, \text{Der}^{(\infty)}, \theta_0 \sigma_1 + \theta_0 \sigma_0, \theta_0 \sigma_3 + \theta_0 \sigma_1, \\
& \theta_1 \sigma_1, \theta_2 \sigma_2, \theta_1 \theta_1 \sigma_2 + \theta_0 \theta_1 \sigma_2 \\
& 11 \\
\hline
101 & \sigma_2, \sigma_1, \text{Der}^{(\infty)}, \theta_0 \sigma_2 + \theta_0 \sigma_0, \theta_0 \sigma_3 + \theta_0 \sigma_1, \\
& \theta_1 \sigma_1, \theta_2 \sigma_2, \theta_1 \theta_1 \sigma_2 + \theta_0 \theta_1 \sigma_2 \\
& 11 \\
\hline
110 & \sigma_1, \sigma_2, \theta_1 \text{Der}^{(\infty)}, \theta_0 \sigma_3 + \theta_0 \sigma_0, \theta_0 \sigma_3 + \theta_0 \sigma_1, \\
& \theta_1 \sigma_1, \theta_2 \sigma_2, \theta_1 \theta_1 \sigma_2 + \theta_0 \theta_1 \sigma_2 \\
& 11 \\
\hline
211 & \sigma_0, \theta_0 \sigma_0 + \theta_0 \sigma_2, \text{Der}^{(\infty)}, \\
& \theta_0 \sigma_0 + \theta_1 \sigma_1, \theta_1 \sigma_1, \theta_0 \sigma_0 + \theta_0 \sigma_2 - 2 \theta_0 \theta_0 \sigma_3 \\
& 7 \\
\hline
121 & \theta_0 \sigma_0 + \theta_0 \sigma_2, \text{Der}^{(\infty)}, \\
& \theta_0 \sigma_0 + \theta_0 \sigma_2, \theta_0 \sigma_0 + \theta_0 \sigma_2 - 2 \theta_0 \theta_0 \sigma_3 \\
& 7 \\
\hline
112 & \sigma_2, \sigma_1, \text{Der}^{(\infty)}, \\
& \theta_0 \sigma_0 + \theta_2 \sigma_1, \theta_0 \sigma_2, \theta_0 \sigma_0 + \theta_0 \sigma_2 - 2 \theta_0 \theta_0 \sigma_3 \\
& 7 \\
\hline
222 & \sigma_1 \sigma_1 + \theta_0 \sigma_1, \text{Der}^{(\infty)}, \\
& \theta_0 \sigma_1 + \theta_0 \sigma_2, \theta_0 \sigma_0 + \theta_0 \sigma_2 + 3 \theta_0 \theta_0 \sigma_3 \\
& 6 \\
\hline
\text{else} & \text{none} \\
\hline
\end{array}\]

The last assumption is technical. An alternative would be to work not over $C^\infty(S, \mathbb{R})$ but over germs of smooth functions at the identity element of the Lie group.

**Theorem 1 (No obstructions)** With Assumption 1 we have:
- $H^0(d) \simeq \mathbb{R}^3$, the right-invariant vector fields.
- $\text{Gr} H^1(d) \simeq H^1(D_{\theta_0}) \oplus \ker \mathcal{D}_3$, see Lemmas 5 and 11.
- $H^2(d) = 0$.
- $H^3(d) = 0$.
- $H^4(d) = 0$.

Here $\simeq$ is an isomorphism as vector spaces, and $\text{Gr}$ is the associated graded for the decreasing filtration coming from the $\mathbb{Z}$-grading of $\mathcal{A}_\text{free}$ by $p_2 + p_3$.

**Proof** This follows from the remainder of Section 2, and the fact [2] that the last page of the spectral sequence is isomorphic to the associated graded of the homology. For example, this way we get $\text{Gr} H^2(d) = 0$, which implies $H^2(d) = 0$. $\Box$

2.1 Overview

Recall [2] that $\mathcal{A}_\text{free} \simeq \mathcal{U}$ is graded by tuples $\alpha = p_1 p_2 p_3$, see Table 1. There is a corresponding decreasing filtration $F_{\geq \alpha} \mathcal{A}_\text{free} \simeq F_{\geq \alpha} \mathcal{U}$ given by

$$F_{\geq \alpha} \mathcal{U} = \bigoplus_{\beta \geq \alpha} G_\beta \mathcal{U}$$
It is respected by the differential, so we have a decreasingly filtered complex. That is, we have $d(F_{\geq \alpha, \mathcal{A}_{\text{free}}}) \subseteq F_{\geq \alpha, \mathcal{A}_{\text{free}}}$. The homology is calculated in two steps:

- Via gauge fixing, we construct a contraction of $(\mathcal{A}_{\text{free}}, d)$ onto a filtered complex $(C, d_C)$, so they are quasi-isomorphic. The space $C$ is a free $C^\infty(S, \mathbb{R})$-module. The contraction is a variant of the one in [1]. Since $C^4 = 0$ we get $H^4(d) = 0$.
- We use a spectral sequence to compute the homology of $(C, d_C)$. The spectral sequence is relative to the decreasing filtration of $C$ associated to the single index $p_2 + p_3$. We found this to be more useful than the total index $p_1 + p_2 + p_3$.

We anticipate the structure of the spectral sequence. In the following diagrams, left to right corresponds to the filtration index $p_2 + p_3 = 0, 1, 2, 3, 4$. The homological degrees are suppressed; one should visualize each bullet as being a complex with arrows perpendicular to the page. A $(k)$ over a bullet means that the complex is concentrated in homological degree $k$.

2.2 A contraction and the complex $(C, d_C)$

Recall Assumption 1. The complex $(\mathcal{A}_{\text{free}}, d)$ lives in 4 dimensions, namely $\mathcal{A}_{\text{free}}$ is a free $C^\infty(\mathbb{R} \times S, \mathbb{R})$-module, but there is a contraction to a complex $(C, d_C)$ that lives in 3 dimensions, with $C$ a free $C^\infty(S, \mathbb{R})$-module. The differentials are $\mathbb{R}$-linear.
Here we define $C$, give a basis compatible with the decreasing filtration, and define $d_C$. We do not write $d_C$ down completely because the formulas would be too long. Luckily, we do not need every detail about $d_C$.

**Lemma 1 (The space $C$ and basis elements)** Define

$$C = \mathcal{U}_G / i^* \mathcal{U}_G \cong \mathcal{A}_{\text{free},G} / i^* \mathcal{A}_{\text{free},G}$$

which is a free module over

$$C^\infty(\mathbb{R} \times S, \mathbb{R}) / i^* C^\infty(\mathbb{R} \times S, \mathbb{R}) \cong \mathcal{C}^\infty(S, \mathbb{R})$$

of rank 72. A basis is given by the elements in Table 1. We denote by $b_{\alpha}^{i}$ the $i$-th element listed for $G_{\alpha} \in G_{1,2,3,4}$. Each occurrence of $\text{Der}(\mathcal{C}^\infty)$ is replaced by the ordered list $D_{0,1,2,3}$. We denote by $b_{\alpha}^{i}$ the $i$-th element in $G_{\alpha} \in G_{1,2,3,4}$. Examples:

- $b_3^{000} = D_2$
- $b_3^{011} = s_3^{-1} \theta_1 D_0$
- $b_3^{1000} = \theta_0 \sigma_0 + \theta_3 \sigma_3$
- $b_6^{2411} = s_1 s_3^{-1} (\theta_0 \sigma_1 + \theta_3 \sigma_3 - 2 \theta_2 \sigma_12)$

This module comes with a grading $C = \bigoplus_{p_1,p_2,p_3} G_{p_1,p_2,p_3} \mathcal{C}$, and the corresponding decreasing filtration is denoted $F_{\geq \alpha} C$.

**Proof** Omitted.

**Lemma 2 (The contraction and the differential $d_C$)** Let $K$ be the composition

$$K : \mathcal{A}_{\text{free},G} \rightarrow \mathcal{A}_{\text{free},G} / \mathcal{A}_{\text{free},G}$$

an $\mathbb{R}$-linear map. Then:

- $K$ has an $\mathbb{R}$-linear right inverse ($K$ is surjective).
- There is a homological contraction of the complex $(\mathcal{A}_{\text{free},G}, d)$ onto the subcomplex $(\ker K, d|_{\ker K})$. In particular, they have the same homology.
- The canonical map $\ker K \rightarrow C$, namely the composition $\ker K : \mathcal{A}_{\text{free},G} \rightarrow \mathcal{A}_{\text{free},G} / \mathcal{A}_{\text{free},G} \cong C$

is a vector space isomorphism. The differential $d|_{\ker K}$ induces a differential $d_C \in \text{End}_1(C)$ that respects the filtration, $d_C(F_{\geq \alpha} C) \subseteq F_{\geq \alpha} C$.

**Proof** The proof mimics the one for the contraction in [1]. The map $K$ is surjective because, relative to a basis compatible with the filtration as in the table, $K$ is lower block triangular with each square diagonal block of the form $D_{01} + M$ with $M$ a square matrix whose entries are functions. Surjectivity follows from global existence for linear ordinary differential equations. Hence $\ker K \rightarrow C$ is an isomorphism, with $C$ the vector space of initial data. The surjectivity yields a contraction for abstract reasons, just as in [1]. The argument in [1] differs only in that it uses a symmetric hyperbolic system of PDE rather than an ODE.
Example 1  Under the isomorphism \( \ker K \to C \), the preimage of \( fB_4^{0,000} = fD_2 \) with \( f \in C^\infty(S, \mathbb{R}) \) is \( fD_2 \) with \( f \) extended to an element of \( C^\infty(\mathbb{R} \times S, \mathbb{R}) \) by \( D_0(f) = 0 \). In fact, \( d(fD_2) \in \mathcal{A}_{\text{free},G} \).

Example 2  We discuss one example in more detail. Calculations like this can be automated. We claim that the preimage under \( \ker K \to C \) of \( fB_5^{0,000} = f\sigma_0 \) is

\[
f \sigma_0 + t(fD_0 + s_2 s_3 D_1(f)\sigma_1 + s_3 s_1 D_2(f)\sigma_2 + s_1 s_2 D_3(f)\sigma_3)
\]

with \( D_0(f) = 0 \) and \( s_i = s_i e_i^0 \); the data in the exponent belongs to the homogeneous MC-element \( \gamma \in \mathcal{A}_{\text{free}}^1 \) as in Assumption 1, whose homology we are studying. Since (2) is in \( \mathcal{A}_{\text{free},G}^0 \) and modulo \( t \) yields \( f\sigma_0 \), we only have to check that applying \( d \) yields zero in \( \mathcal{A}_{\text{free}}^1/\mathcal{A}_{\text{free},G}^1 \). Then this is the preimage by Lemma 2. The following brackets are in \( \mathcal{A}_{\text{free}}^1 \), the entry in the first slot always coming from \( \gamma \).

- \( G_{000}\mathcal{A}_{\text{free}}^1 \): One calculates,

\[
[\theta_iD_0 - \sum_i g_i^0(\theta_i \sigma_i + \theta_0 \sigma_0), f \sigma_0 + t fD_0] = f \sum_i g_i^0(\theta_i \sigma_i + \theta_0 \sigma_0) \in G_{000}\mathcal{A}_{\text{free},G}^1
\]

- \( G_{011}\mathcal{A}_{\text{free}}^1 \): One calculates,

\[
[\theta_iD_0 - \sum_i g_i^0(\theta_i \sigma_i + \theta_0 \sigma_0), s_2 s_3 D_1(f)\sigma_1] + [s_2 s_3 \theta_i D_1, f \sigma_0 + t fD_0]
\]

\[
= s_2 s_3 (1 + t(g_2^0 + g_3^0)\theta_1 \sigma_1 + t D_1(f)(g_2^0 \theta_2 \sigma_1 - g_3^0 \theta_3 \sigma_1)) + D_1(f)(1 + t(g_1^0 + g_2^0 + g_3^0))(\theta_0 \sigma_1 + \theta_1 \sigma_0) \in G_{011}\mathcal{A}_{\text{free},G}^1
\]

Similar for \( G_{101}, G_{110} \). Since \( \mathcal{A}_{\text{free}}^1/\mathcal{A}_{\text{free},G}^1 \) only has these four pieces, we are done.

Definition 1  (Other gradings of \( C \)) Recall \( C = \bigoplus_{p_1 + p_2 = p} G_{p_1 p_2} C \). Define the coarser \( G_{p_2} C = \bigoplus_{p_1} G_{p_1 p_2} C \), explicitly

\[
\begin{pmatrix}
G_{00} C & G_{01} C & G_{02} C \\
G_{10} C & G_{11} C & G_{12} C \\
G_{20} C & G_{21} C & G_{22} C \\
\end{pmatrix} =
\begin{pmatrix}
G_{000} C & G_{200} C & G_{101} C & G_{002} C \\
G_{110} C & G_{011} C & G_{211} C & G_{112} C \\
G_{020} C & G_{120} C & G_{220} C & G_{222} C
\end{pmatrix}
\]

The ordering of the direct sums gives ordered bases, so the basis for \( G_{00} C \) is the concatenation of the ordered bases for \( G_{000} C \) and \( G_{200} C \). Same for \( G_{00} C \) for every \( k \). We also define \( V_p = \bigoplus_{p_2 + p_3 = p} G_{p_2 p_3} C \), explicitly

\[
\begin{align*}
V_0 &= G_{00} C \\
V_1 &= G_{10} C \oplus G_{01} C \\
V_2 &= G_{20} C \oplus G_{11} C \oplus G_{02} C \\
V_3 &= G_{21} C \oplus G_{12} C \\
V_4 &= G_{22} C
\end{align*}
\]

and the corresponding ordered bases.
Relative to $C = V_0 \oplus \ldots \oplus V_4$ the differential has the form block form

$$d_C = \begin{pmatrix}
* & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & * & 0 & 0 \\
* & * & * & 0 & 0 \\
* & * & * & * & *
\end{pmatrix}$$

based on which we will construct a spectral sequence. The next lemma gives a first qualitative description of $d_C$, later we need more details.

**Lemma 3** (*dc is a first order matrix differential operator*) There is a $72 \times 72$ lower block triangular matrix $B$ and three strictly lower block triangular matrices $A_1, A_2, A_3$, all with constant entries, such that $d_C = A_1D_1 + A_2D_2 + A_3D_3 + B$.

**Proof** The diagonal blocks of $A_1, A_2, A_3$ are zero since the derivatives $D_1, D_2, D_3$ appear in the Maurer-Cartan element with $s_2s_3, s_3s_1, s_1s_2$.

**Lemma 4** (*Blocks of $d_C$*) We denote by

$$d_{k,p_1p_2p_3 \rightarrow q_1q_2q_3} \in \text{Hom}_\mathbb{R}(G_{p_1p_2p_3}^k, G_{q_1q_2q_3}^{k+1})$$

the blocks of $d_C$ relative to the different gradings of $C$. If the index $k$ is omitted, then a direct sum over $k$ is understood. We have:

- If $p_1 > q_1$ for at least one $i$, then the block $d_{p_1p_2p_3 \rightarrow q_1q_2q_3}$ is zero.
- Each diagonal block $d_{p_1p_2p_3 \rightarrow p_1p_2p_3}$ is $C^\infty(S, \mathbb{R})$-linear (entries in $C^\infty(S, \mathbb{R})$).

Analogous statements hold for $d_{p_2p_3 \rightarrow q_2q_3}$ and $d_{p \rightarrow q}$.

**Proof** The linearity over $C^\infty(S, \mathbb{R})$, is because $D_1, D_2, D_3$ appear only on the subdiagonals, see Lemma 3.

2.3 Spectral sequence: the 0th page

Recall Assumption 1. The 0th-page is

$$\bigcup_{D_{00}} V_0 \bigcup_{D_{01}} V_1 \bigcup_{D_{22}} V_2 \bigcup_{D_{33}} V_3 \bigcup_{D_{44}} V_4$$

with differentials $D_{pp} = d_{p \rightarrow p}$ that are linear over $C^\infty(S, \mathbb{R})$. The differentials are also diagonal relative to the decompositions (3), therefore:

$$H(D_{00}) = H(d_{00 \rightarrow 00})$$
$$H(D_{11}) = H(d_{10 \rightarrow 01}) \oplus H(d_{01 \rightarrow 01})$$
$$H(D_{22}) = H(d_{20 \rightarrow 20}) \oplus H(d_{11 \rightarrow 11}) \oplus H(d_{02 \rightarrow 02})$$
$$H(D_{33}) = H(d_{21 \rightarrow 21}) \oplus H(d_{12 \rightarrow 12})$$
$$H(D_{44}) = H(d_{22 \rightarrow 22})$$

(4)
For example, $D_{11}$ is diagonal because the off-diagonal $d_{10\rightarrow 01}$ and $d_{01\rightarrow 10}$ vanish by Lemma 4. We now give these differentials in matrix form, all calculated using the computer.

- Consider the block $d_{00\rightarrow 00}$. Note that
  
  $$\text{rank}_{C_{\sim (5,2)}} V_0^k = 5, 4, 1, 0$$
  
  with ordered bases $b_1^{\ldots, 000}$, $b_2^{\ldots, 000}$, $b_3^{\ldots, 000}$, $b_4^{\ldots, 000}$, $b_5^{\ldots, 000}$ respectively $b_1^{\ldots, 100}$, $b_2^{\ldots, 100}$, $b_3^{\ldots, 1200}$, $b_1^{\ldots, 2000}$, $b_2^{\ldots, 2000}$ respectively $b_1^{\ldots, 2000}$ relative to which
  
  $$d_{0,00\rightarrow 00} = \begin{pmatrix} 0 & 0 & 0 & 0 & g_1^0 \\ 0 & 0 & 0 & 0 & g_2^0 \\ 0 & 0 & 0 & 0 & g_3^0 \\ -2g_1^0 & 0 & 0 & 0 & -g_1^0 \end{pmatrix}$$
  
  and
  
  $$d_{1,00\rightarrow 00} = (-\frac{1}{3}(g_2^0 + g_3^0) - \frac{1}{3}(g_3^0 + g_1^0) - \frac{1}{3}(g_1^0 + g_2^0) 0)$$
  
  and $d_{2,00\rightarrow 00} = 0$. By construction, $d_{00\rightarrow 00}$ must itself be a differential. To check this explicitly, use the algebraic constraint $g_2^0g_3^0 + g_3^0g_1^0 + g_1^0g_2^0 = 0$, see [2].

- Consider $d_{01\rightarrow 10}$ and $d_{01\rightarrow 01}$. Then rank$_{C_{\sim G_{10}C^k}} = \text{rank}_{C_{\sim G_{01}C^k}} = 2, 8, 1, 0$ and

  $$d_{0,10\rightarrow 10} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & g_2^0 + g_3^0 & 0 \\ 0 & 0 & 0 \\ g_1^0 + g_2^0 + g_3^0 & 0 & 0 \end{pmatrix}$$
  
  and
  
  $$d_{0,01\rightarrow 01} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & g_1^0 + g_2^0 + g_3^0 & 0 \\ 0 & 0 & 0 \\ g_1^0 + g_2^0 + g_3^0 & 0 & 0 \end{pmatrix}$$
  
  and
  
  $$d_{1,10\rightarrow 10} = (0 0 0 0 -g_1^0 - g_2^0 0 g_3^0)$$
  
  and
  
  $$d_{1,01\rightarrow 01} = (0 0 0 0 -g_1^0 - g_3^0 0 g_2^0)$$
  
  and $d_{2,10\rightarrow 10} = 0$ and $d_{2,01\rightarrow 01} = 0$.

- The block $d_{20\rightarrow 20}$ is zero because rank$_{C_{\sim G_{20}C^k}} = 0, 1, 0, 0$. So is $d_{02\rightarrow 02}$.

- The block $d_{11\rightarrow 11}$ is given by

  $$d_{0,11\rightarrow 11} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_2^0 + g_3^0 & 0 \\ 0 & g_1^0 - g_2^0 & g_3^0 & g_1^0 \end{pmatrix}$$
and

\[ d_{1,11 \rightarrow 11} = \begin{pmatrix}
0 & 0 & 0 & -g_2^0 & -g_3^0 & 0 & g_1^0 & -g_3^0 & g_2^0 - g_1^0 & 0 \\
2g_1^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 2g_1^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2g_1^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2g_1^1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -g_1^1 & 0 & 0 & 0 & g_1^0 + g_2^0 + g_3^0 \\
-2g_1^0g_1^1 & 0 & 0 & -g_1^1 & 0 & -g_1^1 & g_1^0 & g_2^0 + g_3^0 & -g_1^0 \\
\end{pmatrix} \]

and \( d_{2,11 \rightarrow 11} = 0. \)

- The blocks \( d_{21 \rightarrow 21} \) and \( d_{12 \rightarrow 12} \) are given by

\[ d_{1,21 \rightarrow 21} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ g_1^0 + g_2^0 + g_3^0 \\ -g_2^0 \end{pmatrix} \quad \text{and} \quad d_{1,12 \rightarrow 12} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ g_1^0 + g_2^0 + g_3^0 \\ -g_3^0 \end{pmatrix} \]

whereas \( d_{0,21 \rightarrow 21} = d_{2,21 \rightarrow 21} = 0 \) and \( d_{0,12 \rightarrow 12} = d_{2,12 \rightarrow 12} = 0. \)

- The block \( d_{22 \rightarrow 22} \) is given by

\[ d_{2,22 \rightarrow 22} = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \\ -2(g_1^0 + g_2^0 + g_3^0) \end{pmatrix} \]

whereas \( d_{0,22 \rightarrow 22} = d_{1,22 \rightarrow 22} = 0. \)

Lemma 5 (Homology of the 0th page) The \( H(d_{p_1,p_2 \rightarrow p_1,p_2}) \) are free \( C^\infty(S, \mathbb{R}) \)-modules with ranks:

| \( H^k(d_{00 \rightarrow 00}) \) | \( k = 0 \) | \( k = 1 \) | \( k = 2 \) | \( k = 3 \) |
|---|---|---|---|
| \( H^k(d_{10 \rightarrow 10}), H^k(d_{01 \rightarrow 01}) \) | (5) | (4) | (1) |
| \( H^k(d_{20 \rightarrow 20}), H^k(d_{02 \rightarrow 02}) \) | (2) | (5) | (8) | (1) |
| \( H^k(d_{11 \rightarrow 11}) \) | (2) | (1) |
| \( H^k(d_{21 \rightarrow 21}), H^k(d_{12 \rightarrow 12}) \) | (1) | (5) | (6) |
| \( H^k(d_{22 \rightarrow 22}) \) | (1) | (4) | (5) |

Accordingly, the \( H(D_{pp}) \) are free \( C^\infty(S, \mathbb{R}) \)-modules with ranks:

| \( H^k(D_{00}) \) | \( k = 0 \) | \( k = 1 \) | \( k = 2 \) | \( k = 3 \) |
|---|---|---|---|
| \( H^k(D_{11}) \) | (3) | (5) | (4) | (1) |
| \( H^k(D_{21}) \) | (1) | (4) | (16) | (2) |
| \( H^k(D_{22}) \) | (2) | (10) | (11) | (7) |
| \( H^k(D_{33}) \) | (2) | (7) | (12) |
| \( H^k(D_{44}) \) | (1) | (4) | (5) |
The small numbers in brackets are the ranks before taking homology.

**Proof** Use Assumption 1. The claim follows from the $d_{P_1P_2\rightarrow P_1P_3}$. For freeness it suffices to exhibit bases, which are in Lemma 6.

**Lemma 6 (Ordered bases of representatives)** As always, Assumption 1 is in force. The row vectors in this lemma are relative to the ordered bases for the $G_{P_2P_3}C^k$ in Lemma 1 and Definition 1. In each case, an ordered basis for the module to the left of the colon is given by the vectors to the right of the colon:

- $H^0(d_{00\rightarrow 00})$: $(0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0)$.
- $H^1(d_{00\rightarrow 00})$: $(g_2^0 - g_3^0, g_3^0 - g_1^0, g_1^0 - g_2^0, 0)$.
- $H^1(d_{10\rightarrow 10})$: $(1, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0, 0, 0, 0)$.
- $H^1(d_{01\rightarrow 01})$: $(0, 0, 0, 1, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, g_1^0 - g_3^0, g_3^0 - g_2^0)$.
- $H^1(d_{20\rightarrow 20})$ and $H^1(d_{02\rightarrow 02})$: $(1)$.
- $H^2(d_{21\rightarrow 21})$ and $H^2(d_{12\rightarrow 12})$: $(0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0)$.
- $H^3(d_{22\rightarrow 22})$: $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$.

**Proof** By inspection of the matrices for the $d_{P_2P_3\rightarrow P_2P_1}$.

**Lemma 7 (Choice of complements of the kernel)** In $G_{P_2P_3}C^k$ a complement of $Ker d_{P_2P_3\rightarrow P_2P_1}$ is spanned by all elements of the form:

$$
G_{00}C^0: \quad (*, 0, 0, 0, *)
$$

$$
G_{01}C^1: \quad (1, 1, 1, 0)
$$

$$
G_{10}C^1: \quad (0, 0, 0, 0, 0, *, 0, 0, 0)
$$

$$
G_{01}C^1: \quad (0, 0, 0, 0, 0, 0, *, 0, 0, 0)
$$

$$
G_{11}C^1: \quad (0, 0, *, 0, 0, 0, 0, 0, *, 0, 0, 0)
$$

For all other $G_{P_2P_3}C^k$ the kernel is zero or all, so there is a unique complement.

**Proof** By inspection of the matrices for the $d_{P_2P_3\rightarrow P_2P_1}$.

**Lemma 8 (Contractions)** There are unique linear maps

$$
\begin{align*}
&G_{P_2P_3}C \\ &i_{P_2P_3} \quad \bigcup \quad p_{P_2P_3} \\ &H(d_{P_2P_3\rightarrow P_2P_1}) \\
&H(d_{P_2P_3\rightarrow P_2P_1}) \\
&H(p_{P_2P_3})
\end{align*}
$$

such that $i_{P_2P_3}$ associates to each element of the homology the unique representative in the span of the basis in Lemma 6; $p_{P_2P_3}$ associates to each element of $Ker d_C$ the
corresponding element in the homology, and \( \ker p_{p, s} \) contains the complements in Lemma 7; and \( h_{p, p} \) associates to each element of image \( d_c \) the unique \( d_c \)-preimage in the complement in Lemma 7, and \( \ker h_{p, p} \) contains the elements in Lemma 6 and the complements in Lemma 7. Then \( di = pd = hi = ph = h^2 = 0 \) and \( pi = 1 \) and \( ip = 1 - hi - dh \) with obvious abbreviations. Analogous statements for \( i_p, p_p, h_p \).

They are block-diagonal with entries \( i_{p, p}^{-1}, p_{p, p}, h_{p, p} \), with \( p_2 + p_1 = p \). We denote by \( i^k_{p, p}, p^k_p, h^k_p \) the corresponding maps at homological degree \( k \).

**Proof** This is a contraction in standard form.

---

2.4 Spectral sequence: the 1st page

By Section 2.3, and of course with Assumption 1 always in force, the 1st page is the disconnected complex, abbreviating \( H^{d-1}(-) = H^d(-) \oplus H^{d-1}(-) \).

\[
\begin{align*}
H^0(D_{00}) & \xrightarrow{D_{10}} H^1(D_{11}) & H^1(D_{22}) & \xrightarrow{D_{32}} H^2(D_{33}) & \xrightarrow{D_{43}} H^3(D_{44}) \\
\end{align*}
\]

where \( D_{10} \) annihilates \( H^1(D_{00}) \) and we denote by \( D'_{10} : H^0(D_{00}) \to H^1(D_{11}) \) the other, nontrivial part. In block notation using the decompositions (4) we have

\[
D'_{10} = p_1^1 d_{0,0,0} - 10 i_{0}^0 = \begin{pmatrix} p_{10}^1 d_{0,0,0} - 10 i_{0}^0 & p_{00}^1 d_{0,0,0} - 0 i_{0}^0 \end{pmatrix}
\]

\[
D_{32} = p_2^2 d_{1,2,3} - 13 i_{0}^2 = \begin{pmatrix} p_{21}^2 d_{1,2,0} - 21 i_{20}^1 & 0 & 0 \\
0 & p_{12}^1 d_{1,0,2} - 12 i_{12}^0 & 0 \\
0 & 0 & p_{00}^1 d_{0,0,0} - 0 i_{0}^0 \end{pmatrix}
\]

\[
D_{43} = p_3^3 d_{2,3,4} - 14 i_{0}^3 = \begin{pmatrix} p_{32}^3 d_{2,1,3} - 23 i_{21}^2 & p_{21}^2 d_{2,1,2} - 23 i_{21}^2 \end{pmatrix}
\]

The leftmost and rightmost zeros in \( D_{32} \) are due to \( d_{20,12} = d_{02,21} = 0 \), the zeros in the middle are due to \( i_{11} = 0 \), equivalently \( H(d_{11,1}) = 0 \).

The spaces are free \( C^\infty(S, R) \)-modules with bases chosen, and \( D'_{10}, D_{32}, D_{43} \) are matrix differential operators. By direct calculation,

\[
D_{10}' = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
-2 g_{1} & -2 g_{1} & D_{1} & 0 & 0 & 0 \\
0 & 0 & D_{1} & 0 & 0 & 0 \\
0 & 0 & 0 & D_{2} & 0 & 0 \\
2 g_{1} & 0 & 0 & 0 & D_{2} & 0 \\
0 & 0 & 0 & 0 & 0 & D_{2} \end{pmatrix}
\]

\[
D_{32} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
2 g_{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

and

\[
D_{43} = \begin{pmatrix}
D_{2} & 0 & -2 g_{1} & 0 & 0 & 0 \\
0 & D_{2} & 0 & 0 & 0 & 0 \\
0 & D_{2} & 0 & 0 & 0 & 0 \\
2 g_{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D_{1} & 0 \\
0 & 0 & 0 & 0 & 0 & D_{1} \end{pmatrix}
\]

Necessarily \( D_{43} D_{32} = 0 \), which can also be checked directly.
Lemma 9

- \( \ker D'_{10} \cong \mathbb{R}^3 \) as vector spaces.
- \( \ker D_{32} = 0 \).
- \( \coker D_{43} = 0 \).

Proof By inspection of their matrices. For \( D'_{10} \) note that the \( D_i \) are the left-invariant vector fields on an open subset of a Lie group; every left-invariant vector field commutes with every right-invariant vector field; and the space of right-invariant vector fields is \( \cong \mathbb{R}^3 \). The injectivity of \( D_{32} \) is clear. The map \( D_{43} \) is surjective because already its leftmost \( 4 \times 4 \) block is surjective by (1) and the global existence theorem for linear ordinary differential equations. \( \square \)

2.5 Spectral sequence: the 2nd page

By Section 2.4, and of course with Assumption 1 always in force, the 2nd page is the following complex, with \( \mathbb{R}^3 \) in homological degree zero:

\[
\begin{array}{ccc}
\mathbb{R}^3 \oplus H^1(D_{00}) & \xrightarrow{\text{image } D_{00}} & H^1(D_{11}) \xrightarrow{\ker D_{11} \text{ image } D_{32}} 0 \\
\downarrow \Phi_3 & & \\
0 & & 0
\end{array}
\]

Our main goal is to show that \( \Phi_3 \) is surjective. By the way the spectral sequence is constructed, \( \Phi_3 \) is induced by, with \( h_2 \) as in Lemma 8:

\[
d_1 \to 3 - d_2 \to 3 h_2 d_1 \to 2 : V_1 \to V_3
\]

The last map induces a map \( H(D_{11}) \to H(D_{33}) \), because \( H(D_{22}) = 0 \) as witnessed by \( h_2 \) via \( h_2 D_{22} + D_{22} h_2 = 1 \). More specifically, it induces a map \( \delta : H^1(D_{11}) \to H^2(D_{33}) \) with image \( \delta \subseteq \ker D_{43} \). We emphasize \( \delta \) because it is a map between free \( \mathbb{C}^\infty(\mathbb{S}, \mathbb{R}) \)-modules for which bases have been chosen, and it can be written as a matrix differential operator, of size \( 10 \times 10 \). Clearly

\[
\text{coker } \Phi_3 = 0 \iff \text{image } \delta + \text{image } D_{32} = \ker D_{43} \tag{5}
\]

In block notation using the decompositions (4) we have

\[
\delta = \begin{pmatrix}
p_{21} & 0 & d_{1,10} & d_{1,01} & i_1^1 \\
0 & p_{12} & d_{1,10} & d_{1,01} & i_0^1
\end{pmatrix} - \begin{pmatrix}
p_{21} & 0 & d_{1,11} & d_{1,01} & i_1^1 \\
0 & p_{12} & d_{1,11} & d_{1,01} & i_0^1
\end{pmatrix} h_{11}^2 (d_{1,10} & d_{1,01}) \begin{pmatrix}
i_1^1 & 0 \\
0 & i_0^1
\end{pmatrix}
\]

where \( h_{11}^2 \) is that part of \( h_{11} \) in Lemma 8 that maps from homological degree \( k = 2 \) back to \( k = 1 \), a \( 9 \times 7 \) matrix with constant entries. Now:

- Let \( \Phi_{43} \) be the \( 1 \times 5 \) matrix operator obtained from \( D_{43} \) by deleting rows 1, 2, 3 and columns 2, 3, 5, 8, 10. Explicitly \( \Phi_{43} = (0 \ 0 \ 0 \ D_3) \).
- Let \( \delta \) be the \( 5 \times 10 \) matrix obtained from \( \delta \) by deleting rows 2, 3, 5, 8, 10.
By construction image $\delta \subseteq \ker D_{43}$. We claim that

$$coker D_3 = 0 \iff \text{image } \delta = \ker D_{43} \quad (6)$$

This follows from (5) by deriving a sequence of equivalent conditions, as follows. Column 5 of $D_{43}$ has a single nonzero entry, in row 2, so we get an equivalent condition if we delete column 5 and row 2. Similarly column 10 and row 3. Column 3 now only contains a single nonzero non-deleted entry, in row 1, so we can also delete column 3 and row 1. Accordingly we must delete rows 3, 5, 10 in $\delta$, and rows 1, 3, 4 of $D_{43}$. This follows from (5) by deriving a sequence of equivalent conditions, as follows.

Consider the submatrix of $\delta$ containing only columns 4, 5, 9. Using Assumption 1, specifically $g_1^* \neq 0$ and (1) and $g_0^0 - g_0^3 \neq 0$, this submatrix’ image is already all vectors of the form $(+, 0, +, +, 0)$. Therefore we get a condition equivalent to (6) by deleting columns 1, 3, 4 of $D_{43}$ and rows 1, 3, 4 of $\delta$. Only a $2 \times 2$ submatrix of the non-deleted part of $\delta$ remains, all other entries are zero. We get

$$coker D_3 = 0 \iff \text{image } \begin{pmatrix} 
D_1D_2 + 2g_1^0 D_1 & -D_1D_3 - 4g_1^1 g_1^3 \\
-D_2D_2 - 4g_1^1 g_1^3 & D_2D_3 - 2g_1^1 D_1 
\end{pmatrix} = \ker (D_2D_3) \quad (7)$$

Here $\subseteq$ is by construction, so $\supseteq$ is what we show. We show that the first column of the $2 \times 2$ matrix suffices, using coordinates as in (1). Given any $(f_2, f_3) \in \ker (D_2D_3)$, it suffices to find a function $h$ such that

$$((D_1D_2 + 2g_1^0 D_1)h)\Sigma = f_2|\Sigma$$

$$(-D_2D_2 - 4g_1^1 g_1^3)h = f_3$$
where $\Sigma = \{x^2 = 0\}$. The second equation has a unique solution for every choice of $h|_{\Sigma}$ and $(\partial_2 h)|_{\Sigma}$ by (1), and we fix $(\partial_2 h)|_{\Sigma} = 0$. The first equation is now equivalent to $2g_1^1\partial_1 h = f_2 + a_2(f_3 + 4g_1^1\partial_1 h)$ as an equation on $\Sigma$ for $h|_{\Sigma}$, with $a_2$ as in $D_3 = a_1\partial_1 + a_2\partial_2 + a_3\partial_3$. Since $g_1^1 \neq 0$, a solution $h|_{\Sigma}$ exists, again by (1). We have proved:

**Lemma 10**\[ \text{coker } D_3 = 0. \]

By similar arguments we have $\text{image } D_{10} \subseteq \ker \delta$, which one can also check directly using the explicit formulas given before. Every $x$ satisfies $\delta x \in \text{image } D_{32}$ if and only if $\delta x = 0$; here $\Rightarrow$ is trivial, $\Leftarrow$ uses image $\delta \subseteq \ker D_{43}$. Hence (see Theorem 1):

**Lemma 11**\[ \ker D_3 = \ker \delta / \text{image } D_{10}. \]

This could be analyzed further. Informally counting $C^\infty(S, \mathbb{R})$-ranks, and knowing image $\delta \subseteq \ker D_{43}$, we expect that $\ker \delta$ has rank $10 - 5 + 1 = 6$, and $\text{image } D_{10}$ has rank 3, so $\ker D_3$ should have rank $6 - 3 = 3$. But these are not free $C^\infty(S, \mathbb{R})$-modules, so the counting does not really make sense, and in any case is modulo lower dimensional data. One can write down a parametrization of $\ker D_3$, but we will not.

### 3 The homology in $\mathcal{A}_{\text{bounce}}$

of some homogeneous Maurer-Cartan elements

This section is closely analogous to Section 2. In fact, the differences are so small that we only say what has to be modified relative to Section 2. For a homogeneous $\gamma \in \text{MC}(\mathcal{A}_{\text{bounce}})$ we compute the homologies $H^k(d)$ of

$$d = [\gamma, -] \in \text{End}^1(\mathcal{A}_{\text{bounce}})$$

**Assumption 2** (In force through Section 3) We assume $\gamma \in \text{MC}(\mathcal{A}_{\text{bounce}}) \cap (\theta_0 D_0 + \mathcal{A}^1_{\text{bounce}, G})$

is given, using the over-parametrization in Lemma (J1), by

- \[ \mu_1 = -\frac{1}{\partial} \log(2 \cosh t) \]
- \[ \mu_2 = -\frac{m}{\partial} + \frac{1}{2} \log(2 \cosh t) \]
- \[ \mu_3 = -\frac{1}{\partial} + \frac{1}{2} \log(2 \cosh t) \]
- \[ \gamma_1^0 = \frac{1}{2} - \chi \]
- \[ \gamma_2^0 = -\frac{1}{\partial}(1 + a) + \chi \]
- \[ \gamma_3^0 = -\frac{1}{\partial}(1 + \frac{1}{2}) + \chi \]

where $\chi = \frac{1}{\partial}(1 + \tanh t)$, a special case of the solution in Theorem (J3), with:

- $u > 0$ is constant and $u \neq 1$.
- $g_1^1$ and $g_1^3$ are constant and nonzero.
- $S$ is an open subset of a smooth Lie group containing the identity element, and $D_1, D_2, D_3$ is a basis of left-invariant vector fields, with $[D_2, D_3] = -2D_1$ and $[D_3, D_1] = -2g_1^1 D_2$ and $[D_1, D_2] = -2g_1^3 D_3$. 

There is an \( \varepsilon > 0 \) and a diffeomorphism \((x^1, x^2, x^3) : S \rightarrow (-\varepsilon, \varepsilon)^3\) such that \((0,0,0)\) is the identity element and, with \(\partial_1, \partial_2, \partial_3\) the partial derivatives for the coordinate system \(x^1, x^2, x^3\), we have
\[
D_2 = \partial_2 \quad D_1|_{x^2=0} = \partial_1
\]

**Theorem 2 (No obstructions)** With Assumption 2 we have:
- \(H^0(d) \simeq \mathbb{R}^3\).
- \(\text{Gr}H^1(d) \simeq H^1(D_{(0)}) \oplus \ker \partial_3\), with symbols defined afresh, see Lemmas 16, 23.
- \(H^2(d) = 0\).
- \(H^3(d) = 0\).
- \(H^4(d) = 0\).

Here \(\simeq\) is an isomorphism as vector spaces, and \(\text{Gr}\) is the associated graded for the decreasing filtration coming from the \(\mathbb{Z}\)-grading of \(A_{\text{bounce}}\) by \(p_2 + p_3\).

**Proof** A spectral sequence calculation analogous to that for Theorem 1. \(\square\)

### 3.1 Overview

The analogy with Section 2 is via the module identification \(A_{\text{bounce}} \simeq U \simeq A_{\text{free}},\) see [2], and the fact that \(p_1\) played a passive role. The bracket on \(A_{\text{bounce}}\) only respects \(p_2p_3\). The general structure of the spectral sequence is the same. The differential \(d\) is new, and so are the objects derived from it. Beware that we will use the same symbols as before, even though they have a different meaning.

### 3.2 Contraction and the complex \((C, d_C)\)

**Lemma 12 (The space \(C\) and basis elements)** Define
\[
C = U_G/1U_G \simeq A_{\text{bounce}},G/1A_{\text{bounce}},G
\]
A basis is given as in Lemma 1. In this section we set \(s_1 = 1\) because the bracket on \(A_{\text{bounce}}\) does not respect the grading that \(s_1\) keeps track of.

**Lemma 13 (The contraction and the differential \(d_C\))** Like Lemma 2, with
\[
w : A_{\text{bounce}},G \twoheadrightarrow A_{\text{bounce}} \xrightarrow{\partial} A_{\text{bounce}} \twoheadrightarrow A_{\text{bounce}}/A_{\text{bounce}},G
\]

**Example 3** In comparison to Section 2, it is a little harder to explicitly write down elements of \(\ker w\). However this is not required to calculate the differential \(d_C\), it suffices to make finite order Taylor expansions of elements of \(\ker w\) about \(t = 0\).

**Definition 2 (Other gradings of \(C\))** Like Definition 1.

**Lemma 14 (\(d_C\) is a first order matrix differential operator)** Like Lemma 3.

**Lemma 15 (Blocks of \(d_C\))** Define the blocks \(d_{p_2p_3 \rightarrow q_2q_3}\) and \(d_{p \rightarrow q}\) just like in Lemma 4. The same statements hold here.
3.3 Spectral sequence: the 0th page

Analogously to Section 2.3, we have:

- The block $d_{00\to00}$ is given by

$$d_{0,0\to0,0} = \begin{pmatrix}
-\frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & -\frac{1}{2}
\end{pmatrix}$$

and $d_{1,0\to0} = \left( \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right)$ and $d_{2,0\to0} = 0$.

- The blocks $d_{10\to10}$ and $d_{01\to01}$ are given by

$$d_{0,1\to1,0} = \begin{pmatrix}
-1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
-\frac{1}{2}u & 0 \\
\frac{1}{2}u & -1 \\
0 & -1
\end{pmatrix} \quad d_{0,0\to1,1} = \begin{pmatrix}
-1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
-\frac{1}{2}u & 0 \\
\frac{1}{2}u & -1 \\
0 & 1
\end{pmatrix}$$

and

$$d_{1,1\to1,0} = \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & -\frac{1}{2}u & \frac{1}{2}u \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and $d_{2,1\to1,0} = 0$ and $d_{2,0\to01} = 0$.

- The block $d_{20\to20}$ is zero because $\text{rank}_{C^2} G_{20} C^2 = 0,1,0,0$.

The block $d_{02\to02}$ is zero because $\text{rank}_{C^2} G_{02} C^2 = 0,1,0,0$.

- The block $d_{11\to11}$ is given by

$$d_{0,1\to1,1} = \begin{pmatrix}
-1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
-\frac{1}{2}u^2 & 0 \\
0 & -\frac{1}{2}u^2 \\
\frac{1}{2}u & 0 \\
\frac{1}{2}u & 0
\end{pmatrix} \quad d_{1,1\to1,1} = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and $d_{2,1\to1,1} = 0$.  

– The blocks \( d_{21 \rightarrow 21} \) and \( d_{12 \rightarrow 12} \) are given by

\[
d_{1,21 \rightarrow 21} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -\frac{1+u^2}{\sqrt{g}} \end{pmatrix}, \quad d_{1,12 \rightarrow 12} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -\frac{1+u^2}{\sqrt{g}} \end{pmatrix}
\]

whereas \( d_{0,21 \rightarrow 21} = d_{2,21 \rightarrow 21} = 0 \) and \( d_{0,12 \rightarrow 12} = d_{2,12 \rightarrow 12} = 0 \).

– The block \( d_{22 \rightarrow 22} \) is given by

\[
d_{2,22 \rightarrow 22} = \begin{pmatrix} 6 \\ 0 \\ 0 \\ \frac{1}{1 + u} \end{pmatrix}
\]

whereas \( d_{0,22 \rightarrow 22} = d_{1,22 \rightarrow 22} = 0 \).

**Lemma 16** (Homology of the 0th page) Like Lemma 5.

**Lemma 17** (Ordered bases of representatives)

– \( H^0(d_{00 \rightarrow 00}) \): \((0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0)\).
– \( H^1(d_{00 \rightarrow 00}) \): \((\frac{2u}{1+u^2}, 0, 0, 1)\).
– \( H^1(d_{10 \rightarrow 10}) \): \((0, 0, 0, 0, -\frac{1}{u^2}, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, 0)\).
– \( H^1(d_{01 \rightarrow 01}) \): \((0, 0, 0, 0, 1 - u^2, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, 1, -\frac{u^2}{1+u^2})\).
– \( H^1(d_{20 \rightarrow 20}) \) and \( H^1(d_{02 \rightarrow 02}) \): \((1)\).
– \( H^2(d_{21 \rightarrow 21}) \): \((0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (1 + \frac{1}{u^2}, 0, 0, 0, -\frac{1+u^2}{\sqrt{g}}, 0, -\frac{u}{\sqrt{g}}, 0)\).
– \( H^2(d_{12 \rightarrow 12}) \): \((0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (1 + u^2, 0, 0, 0, -\frac{u}{\sqrt{g}}(1 + u^2), 0, -4u, 0, 0, 0, 1 + u^2, 0)\).
– \( H^3(d_{22 \rightarrow 22}) \): \((0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0)\).

**Lemma 18** (Choice of complements of the kernel) Like Lemma 7.

**Lemma 19** (Contractions) Like Lemma 8.

3.4 Spectral sequence: the 1st page

Like Section 2.4, but here we work relative to new bases. Nevertheless, things have been arranged so that:

**Lemma 20** The matrices for \( D^1_{10}, D_{32}, D_{43} \) are exactly as in Section 2.4, with the understanding that \( g_1^1 \) must be replaced by 1.

**Lemma 21** Like Lemma 9.
Like Section 2.5. We claim that \( \operatorname{image} \delta \subseteq \ker \mathcal{D}_{43} \) and (6) still hold. The formula for \( \mathcal{D}_{43} \) remains the same. Here however

\[
\delta = \begin{pmatrix}
-\frac{2(2+u^2)}{u(1+u^2)}(D_3 D_2 + 2 D_1) & -4 D_1 - D_2 D_3 & -2 D_2 & 2 D_3 & 2 u^2 \\
0 & 0 & 0 & 0 & 0 \\
0 & D_2 D_2 & 0 & -4 D_2 & 0 \\
\frac{2}{u(1+u^2)} D_2 & 0 & D_2 D_2 + 4 g_3^1 & 0 & 0 \\
0 & \frac{2}{u(1+u^2)} (D_2 D_2 + 4 g_3^1) & 0 & 0 & 0 \\
-\frac{2 u (1-u^2)}{1+u^2} (D_3 D_3 + 4 g_3^1) & 0 & 0 & 0 & 0 \\
\frac{2}{2(2u^2 - 1)} D_3 & 2 D_3 & -2 D_2 & 2 D_3 & -\frac{2}{1-u^2} \\
D_3 & 2 g_2^3 D_2 & 2 D_1 - D_3 D_3 & 4 g_2^1 & 0 \\
\frac{2 u (1-u^2)}{1+u^2} (D_3 D_3 - 2 D_1) & 0 & 0 & 0 & 0
\end{pmatrix}
\]

By the same arguments, we again get (7) and then:

**Lemma 22** \( \operatorname{coker} \delta_3 = 0 \).

**Lemma 23** \( \ker \delta_3 = \ker \delta / \operatorname{image} \mathcal{D}_{10} \).

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