Generalizing Coordinate Non Commutativity

Abolfazl Jafari
Department of Physics, Faculty of Science, Shahrekord University, P. O. Box 115, Shahrekord, Iran
(Dated: August 8, 2017)

Abstract: In this paper, we establish and employ a local framework to the first order of Riemann’s curvature tensor in order to develop the corresponding coordinate non commutativity into general manifolds. We also exploit a new translation of function at the level of quantum mechanics to show that the final correlation result of the generalized non commutativity is a mixture of the Canonical and Quadratic formalisms and does not consist only of the Lie algebraic formalism. The basic premise of this article is that the geometry of a four-dimensional pseudo Riemann manifold representing space time, is isomorphic to Minkowski space time.

PACS numbers: 03.67.Mn, 73.23.-b, 74.45.+c, 74.78.Na

Keywords: Non commutative coordinates, Star product, Localized homomorphism, Pseudo Riemann’s manifolds

INTRODUCTION

Continuous space time makes available short distances. Today, there are indications that at very short distances we might have to go beyond differential manifolds. Nowadays, we can formulate the fundamental laws of physics, consisting of field theory, gauge field and the theory of gravity on differentiable manifolds. This is only one of the several issues that we confront in relation to changes in classical physics for very short distances\[1\],\[2\]. Physics data has forced us to admit of change in the conception of space time for very short distances and introduce non commutative coordinates\[3\],\[4\]. We define a non commutative space by replacing the local coordinates $y^\mu$ of $R^{D+1}$ with hermitian operators $\hat{y}^\mu$ obeying the commutation relations:

$$[\hat{y}^\mu, \hat{y}^\nu] = i\omega^{\mu\nu}(\hat{y}, t),$$

in addition to the standard relations: $[\hat{y}^\mu, \hat{p}^\nu] = i\delta^{\mu\nu}$ and $[\hat{p}^\mu, \hat{p}^\nu] = 0$. Where $\hat{y}$ and $\hat{p}$ stand for the coordinates and momentum operators, respectively. $\omega(y, t)$ is a real, antisymmetric four dimensional matrix. Here, the Latin and Greek indices run from 1 to 3 and 0 to 3, respectively. An important special case of the non commutativity (without time contribution) is as follows\[1\],\[3\],\[5\]:

$$[\hat{y}^k, \hat{y}^l] = i\theta^{kl},$$

where $\theta^{kl} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^{12} & \theta^{13} \\ 0 & \theta^{21} & 0 & \theta^{23} \\ 0 & \theta^{31} & \theta^{32} & 0 \end{pmatrix}$ is a constant and real tensor. Of course, there are famous structures of non commutativity such as: the Lie algebraic $[\hat{y}^k, \hat{y}^l] = \theta^{kl}_{mn}\hat{y}^m\hat{y}^n$ and Quadratic formalisms $[\hat{y}^k, \hat{y}^l] = \theta^{kl}_{mn}\hat{y}^m\hat{y}^n$ where $\theta^{kl}_{mn}$ and $\theta^{kl}_{mn}$ are constant structures. The formulation of physical theories on non commutative spaces is constructed by a very simple role, namely, replacing the ordinary products between quantities with a new product, the so-called $\star$ product named by Moyal-Weyl\[3\],\[4\],\[8\].

$$f(\hat{y})g(\hat{y}) = f(y) \star g(y) = f(y)e^{\frac{-i}{\hbar} \theta^{\mu\nu}\frac{\partial}{\partial y^\mu}\hat{y}^\nu}g(y),$$

where $\partial_\mu = \frac{\partial}{\partial \hat{y}^\mu}$.

Recently and for general space time, a new star product has been used which is presented in Ref.\[12\]. These authors proposed a new product which is given by,

$$*_{\text{extended}} = e^{\frac{\kappa}{\hbar}\partial_\nu Y_\nu \otimes \Pi_\nu \otimes \Pi_\nu},$$

where $\kappa$ is a constant and $\Pi_\nu$ is a covariant derivative. It can be seen that Eq.(4) is similar to the Moyal-Weyl mapping with slight changes such as the replacement of the momentum operators.

In this article, the canonical non commutativity can readily be reformulated to apply to general pseudo Riemann manifolds. In order for the above ideas to materialize, it is necessary to employ localized homomorphism between pseudo Riemannian manifolds. Locally, the accessibility of $\omega^{ij} \to \theta^{ij}$ follows from the theory of general relativity. Assuming that the equivalence principle holds, special relativity is available in the presence of a gravitational field. In fact, one can always construct local inertial frames at a given event belonging to space time, in which free particles would move along straight lines. In such frames, the components of the metric tensor can be expanded in terms of Riemann’s tensors. If $\eta_{\mu\nu} = (-1,1,1,1)$, then the components of the metric and the relevant affine connections (Christoffel multipliers) are given by

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \frac{1}{3}R_{\alpha\mu\beta\nu}y^\mu y^\nu,$$

and

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{3}\eta^{\mu\nu}(R_{\beta\nu\alpha\gamma} + R_{\nu\beta\alpha\gamma})y^\nu,$$

where raising of the localized Lorentz indices are done with Eqs.(5),(13). The coordinate system of such a frame...
are called Riemann’s normal coordinates \[14\ \textcircled{18}\]. It can be seen that up to the first order of \(\theta\), Eq.\[5\] and Eq.\[6\] will remain unchanged for a non commutative framework. A common tool for all approximated methods is to work with local coordinates. That is, the validity of the above statements is limited to the non covariant observer.

**PRESENTATION OF THE THEORY**

In this article, we suppose that all the employed functions are smooth and \(C^\infty\) and the hat symbol is used for the non commutative coordinates and operators. Here, \([f(\hat{y}), g(\hat{y})] = f(\hat{y}) \ast g(\hat{y}) - g(\hat{y}) \ast f(\hat{y})\). If \(a\) is a real and continuous parameter, then we can represent an unusual interpretation of the basic vectors by

\[
<y | e^{-i \hat{a}_p y} = <\tilde{y} | \text{.}
\]

Therefore, at the quantum mechanical level, the basic vectors can have an unusual operational role. According to the above, we can consider the effect of the translation operator on the given quantum mechanical state \(\Psi\):

\[
<y | \{ \hat{T} | \Psi >\} = <\hat{T}(\hat{y}) | \Psi > = <\tilde{y} | \Psi >
\]

when \(\hat{T}\) is used as a translation operator. In the first step, the validity of the translation operators, is related to the magnitude of the translation parameters. So, in quantum mechanics, the translation parameters consist of more information about transportation. The main purpose of our work is to investigate the validity of \(n\)-dimensional Taylor series expansion of the translation operators. The compact form of a well-defined set of the second class of translators will be (Ref.\[19\]),

\[
\hat{T}(y;\Omega) = e^{-iA(y;\Omega) \cdot \hat{p}}.
\]

\(\Omega\) is a somewhere where \(A\) is small. In fact, the measure of the set of \(\Omega\) is limited by the validity of the series expansion and \(A(y)\) has yet to obey the magnitude condition in the range of \(\Omega\). That is

\[
\begin{cases} 
A(y) \to 0; \quad \forall \ y \in \Omega, \\
A(y) = 0; \quad \text{otherwise},
\end{cases}
\]

particularly, if \(y \in \partial \Omega\), then \(A(y) = 0\).

The evolution of the states occurs within the Heisenberg comprehensive formula;

\[
\hat{T}^\dag(y)\Psi(0)\hat{T}(\hat{y}) = \Psi(y),
\]

in which \(\hat{T}(\hat{y})\) as a translation operator is very famous in quantum literature. So, we need to introduce the Heisenberg picture which refers to the second class of translators (Eq.\[5\]). In the simple case, Eq.\[5\] has been assembled by:

\[
\hat{T}(y) = e^{-iy \hat{p}}.
\]

**IMPORTATION OF STAR PRODUCT**

In this section, we derive the \(*\)-product for the non commutative framework in Minkowski space time. In the special case of non commutativity, we only consider the case of \([\hat{y}^1, \hat{y}^2] = i\theta\). However, for the case of \(\omega^{ij} = 0\), we have \(T_1(y)T_2(y) = T_2(y)T_1(y)\). But, this is not always true and its validity is questioned by Eq.\[1\]. So that, Eq.\[2\] provides a set of arranged-ordered translators. Obviously, \(\omega^{\mu \nu}\) is the cause of the arrange-ordered translators. Generally,

\[
\hat{T}_1(\hat{y})\hat{T}_2(\hat{y}) \neq \hat{T}_2(\hat{y})\hat{T}_1(\hat{y}).
\]

Substituting, the unit operator in the non commutative framework, the key point of this paper appears which is to transfer the magnitude condition of \(A(y)\) to the limits of integration. That is,

\[
\hat{T}(y;\Omega) = e^{-iA(y;\Omega) \cdot \hat{p}} = \hat{T}(\hat{y};\Omega) d\hat{y} = \int_{\mu \nu} \hat{T}_1(\hat{y})\hat{T}_2(\hat{y}) \neq \hat{T}_2(\hat{y})\hat{T}_1(\hat{y}).
\]

It is clear that \(A(y;\Omega)\) itself consists of the necessary information of translation and should still be very small, whereas on the right side of Eq.\[13\], \(A(\hat{y})\) does not consist of information and the information of translation has been delivered to the limits of integration. In other words, the direct responsibility of \(A(y;\Omega)\) can be reduced by integration. Clearly, arrange-ordered translators satisfy the cumulative properties. That is, Eq.\[13\] is true. We distinguish the momentum operators belonging to the Hilbert and Dual space, by introducing: \(\hat{p}^\nu = -\hat{p}_A^\nu\) and \(\hat{p}^\nu = \hat{p}_A^\nu\), immediately, one finds

\[
<y | \hat{p}^\nu \hat{y} > = -i\delta_{\nu \nu}(\hat{x} - \hat{y})
\]

\[
<y | \hat{p}^\nu \hat{y} > = -i\delta_{\nu \nu}(\hat{x} - \hat{y}).
\]

We know that non commutativity holds exclusively for each set of coordinates belonging to the same framework, i.e. \([\hat{x}^i, \hat{y}^j] = 0\). When Eq.\[13\] holds, hermiticity has full authority to enter the unit operator. This is also allowed only for adjacent coordinates. In particular, if
the vectors pass from the left to right (or vice versa) and if \( A(\hat{x}, \hat{y}) \) vanishes on the boundaries, then
\[
< \hat{x} | A \hat{p}_i | \hat{y} > = -i \int d^n z A(\hat{x}, \hat{z}) \delta_{.z}(\hat{z} - \hat{y}) \\
= i \int d^n z A(\hat{x}, \hat{z}) \delta_{.\hat{p}_i}(\hat{z} - \hat{y}),
\]
so,
\[
\delta_{.z}(\hat{z} - \hat{y}) = -\delta_{.\hat{p}_i}(\hat{z} - \hat{y}).
\]
Now, In the neighborhood of \( \Omega \), we can introduce:
\[
\overrightarrow{w}(\hat{y}; \Omega) = A(\hat{y}; \Omega) \cdot \overrightarrow{p},
\]
and
\[
\overleftarrow{w}(\hat{y}; \Omega) = \overrightarrow{p} \cdot A(\hat{y}; \Omega).
\]
Substituting Eq. (13) into Eq. (18), becomes
\[
< \hat{x} | \overrightarrow{w} | \hat{y} > = -\tau A^i(\hat{x}; \Omega)_{\delta_{.\hat{p}_i}(\hat{x} - \hat{y}),}
\]
and
\[
< \hat{x} | \overleftarrow{w} | \hat{y} > = -\delta_{.\hat{p}_i}(\hat{x} - \hat{y}) A^i(\hat{y}; \Omega).
\]
In this way, we show that the second class of the translators can be enabled to make Eq. (8).

**Derivation of the *-product:** In order to derive explicit expressions for the *-product, we can refer to the Heisenberg picture. Eq. (10) and the translation operator \((\hat{T} = e^{-i\hat{w}})\), provided that:
\[
< \hat{x} | \overrightarrow{w} | \hat{y} > = \overrightarrow{w}(\hat{x}, \hat{y}; \Omega),
\]
and
\[
< \hat{x} | \overleftarrow{w} | \hat{y} > = \overleftarrow{w}(\hat{x}, \hat{y}; \Omega),
\]
where \(\overrightarrow{w}(\hat{x}, \hat{y}; \Omega) \) and \(\overleftarrow{w}(\hat{x}, \hat{y}; \Omega) \) are given by Eq. (19).

Thus, every displacement on the non commutative pseudo Riemann manifolds is strongly dependent on the commutation relations given by Eq. (11). Therefore, the advent of a non trivial operator is inevitable. The central operator can be represented as,
\[
\omega_{\Omega} := \overrightarrow{w}(\hat{y}; \Omega) \overleftarrow{w}(\hat{y}; \Omega).
\]
Specifically, we will follow ”\( \omega_{\Omega} \)” in the case of non commutativity which lies on the \(xyz\)-plane.

**Lemma:** The \( \omega_{\Omega} \) enables us to create Eq. (8).

**Proof:** We note that the second class of the translators up to the first order of \( \theta \), obey the equation: \((e^{-iA(\hat{y};\Omega)} \overrightarrow{p}) = e^{-i\hat{p}} A(\hat{y};\Omega)\) therefore, we can state the following:
\[
\omega_{\Omega} = e^{-i\hat{p}} A(\hat{y};\Omega) e^{-iA(\hat{y};\Omega)} \overrightarrow{p}.
\]
By dropping the redundant indices, the above equation consists of: \([\overrightarrow{p} j A^j, \overrightarrow{p} i] = [\overrightarrow{p} j A^j, \overrightarrow{p} i]\), in which the second term should to be clarified. In order to explain the second term, we can show,
\[
[\overrightarrow{p} j A^j, \overrightarrow{p} i] = \overrightarrow{p} j [A^j, \overrightarrow{p} i],
\]
thus, Eq. (23) becomes
\[
\omega_{\Omega} = e^{-i\hat{p}} A(\hat{y};\Omega) e^{-iA(\hat{y};\Omega)} \overrightarrow{p} - \frac{1}{2} [\overrightarrow{p} A(\hat{y};\Omega), A(\hat{y};\Omega) \overrightarrow{p}]
\]
Subsequently, to obtain the *-product, we can take \( A^i = y^i; i = 1, 2 \) and \( \Omega \) as a neighborhood of the origin. In this way, Eq. (24) gives:
\[
\omega_{\Omega} = \exp(\delta_{\Omega}) + \frac{1}{2} \int \int d^n \hat{x} d^n \hat{y} d^n \hat{z} | x > \times \delta_{.\hat{p}_i}(\hat{x} - \hat{y}) | y^i, \hat{y} ^i > \delta_{.\hat{p}_i}(\hat{y} - \hat{z}) < z |.
\]
But, for a small \( \Omega \), \( \Omega \) is locally homomorphic to the pseudo Riemann manifolds. This allows us to employ Eq. (9). Therefore, assuming
\[
\lim_{\Omega \rightarrow 0} [\hat{y}^i, \hat{y}^j] \sim \theta^{ij}
\]
and up to the first order of the small parameters, \( \omega_{\Omega} \) reduces to the *-product and Eq. (26) will be an analog of the Moyal-Weyl mapping. However, up to the first order of \( \theta \), \( \ast = \omega_{\Omega} \) that is
\[
\ast = e^{-i\hat{p}} f_{\Omega} f_{\Omega} d^n x d^n y | \chi > \times \frac{\partial}{\partial \hat{y}^i}(\hat{x} - \hat{y}) \frac{\partial}{\partial \hat{x}^i}(\hat{x} - \hat{y}) < y |.
\]
Integration can be removed by refunding information to the transition function. In this case and for a neighborhood of the origin, Eq. (27) can be written in the momentum representation as the operator:
\[
\ast = e^{-i\hat{p}}[\overrightarrow{p} j, \overrightarrow{p} i].
\]
So, our conclusions leads us to the canonical non commutativity,
\[
[\hat{y}^i, \hat{y}^j] = i \theta^{ij} \hat{1}.
\]

**Another Minkowski space time**

We now generalize the previous process for a space time other than the Minkowski space time. Since, in quantum mechanics, the time coordinate does not manifest in the role of operator, "time" does not contribute to coordinate non commutativity. However, from the Heisenberg picture, we can introduce time dependent vectors:
\[
| y; \ell >
\]
We also assume that all the operators and vectors will be defined at time \( \tau \). This means that, with good approximation, we can suppose that the framework is falling along a geodesic \( G \) of space time. In Riemann’s normal coordinates, each space-like hypersurface of constant \( \tau \) is normal to this geodesic and contains the set of space like geodesics normal to \( G \). The time \( \tau \) of an event in a hypersurface is the proper time along \( G \) at which the every point intersects the hypersurface \([13]\). Consequently, the unit operator is made in compliance with the principle of symmetrization:

\[
\hat{1}(\hat{t}) = \int d^{n-1}y \mid y; \hat{t} > \sqrt{-g(y; \hat{t})} < y; \hat{t} \mid. \tag{31}
\]

Without loss of generality, we can replace \( \mid y; \hat{t} > \mid \) with \( \mid y > \mid \) which is a symbolic notation. In addition, by introducing \( d^n y = d^{n-1}y d\hat{t} \), the unit operator becomes:

\[
\hat{1} = \int d^n y \mid y > \sqrt{-g(y)} \mid y > . \tag{32}
\]

According to Ref. \([13]\) and in commutative algebra, the generalized momentum operator, \( \hat{\Pi} \) is given by:

\[
\langle x \mid \hat{\Pi}_k \mid y > = -i \frac{\partial}{\partial x_k} \delta(x, y) - \frac{i}{4} \frac{\partial}{\partial x_k} \ln \left(-g(x)\right) \delta(x, y), \tag{33}
\]

with,

\[
\langle x \mid y > = \delta(x, y) = \frac{\delta(x-y)}{-g(y)}. \tag{34}
\]

Eq. \((33)\) is not always true and its validity is questioned by Eq. \((1)\). For non commutative coordinates, we can generalize the rotation rule as:

\[
\sqrt{-g(\hat{x})} \delta(\hat{x}, \hat{y}) = \delta(\hat{x}, \hat{y}) \sqrt{-g(y)}. \tag{35}
\]

**Proof-** Since \( \hat{x} \) and \( \hat{y} \) do not belong to the same space, we deduce that the functions of \( \hat{x} \) and \( \hat{y} \) are obviously commutative. We are allowed to use the unit operator (with an arbitrary number of the unit operator). So, we can write

\[
\langle h \mid f >= \int d^n \hat{x} d^n \hat{y} \times \langle h \mid \hat{x} > \sqrt{-g(\hat{x})} \delta(\hat{x}, \hat{y}) \sqrt{-g(y)} < \hat{y} \mid f >, \tag{36}
\]

this implies that:

\[
\int d^n \hat{y} \delta(\hat{x}, \hat{y}) \sqrt{-g(y)} < \hat{y} \mid = < \hat{x} \mid ,
\]

and

\[
\int d^n \hat{x} \mid \hat{x} > \sqrt{-g(\hat{x})} \delta(\hat{x}, \hat{y}) = \mid \hat{y} >. \tag{37}
\]

Our calculation concludes that:

\[
\delta(\hat{x}, \hat{y}) \sqrt{-g(\hat{y})} = \delta(\hat{x} - \hat{y}) = \sqrt{-g(\hat{x})} \delta(\hat{x}, \hat{y}). \tag{38}
\]

In commutative algebra, \( \partial_\mu (\partial_\nu (g(y))) = (\partial_\nu (g(y)})^{-\frac{1}{2}} \Gamma_{\nu \alpha}^\beta (y) \)[20] and the metric functions obey:

\[
\partial_\mu (g(y))^{-\frac{1}{2}} = (\partial_\nu (g(y))^{-\frac{1}{2}} \Gamma_{\nu \alpha}^\beta (y) = 1. \tag{39}
\]

One can see that the corresponding non commutative version of the metric functions obeys the same equation:

\[
g(\hat{y})g^{-1}(\hat{y}) = 1, \tag{40}
\]

because we can ignore \((\Gamma_{\nu \alpha}^{\beta})^2\) in

\[
g(\hat{y})g^{-1}(\hat{y}) = g(y)g^{-1}(y) + \frac{1}{2} \partial_\mu \Gamma_{\nu \alpha}^{\beta} \Gamma_{\alpha \beta} + ... ,
\]

therefore, the metric functions will be commutative. Thus, we have

\[
(\partial_\mu (g(\hat{y})))^\beta_\nu = -(\partial_\nu (g(\hat{y})))^\beta_\nu \Gamma_{\nu \alpha}^\beta (\hat{y}), \tag{41}
\]

also, due to Eq. \((9)\) and Eq. \((10)\), we can write

\[
(\partial_\mu (g(\hat{y})))^\beta_\nu = \partial_\mu \left(1 - \frac{1}{3} (R_{\nu k} - 2 R_{\nu k 0} \hat{y}^l \hat{y}^k)\right), \tag{42}
\]

hereafter, \( -\frac{1}{3} \partial_\mu \partial_\nu \hat{y}^l \hat{y}^k \) as a four vector will be shown by \( \Gamma_\mu (\hat{y}) \). Considering the commutators of the metric functions and Eq. \((\tau)\), we have

\[
\delta_{\nu k}(\hat{x}, \hat{y}) = \delta_{\nu k}(\hat{x}, \hat{y}) - \Gamma_k(\hat{x})(-g(\hat{x}))^{\frac{1}{2}} \delta(\hat{x} - \hat{y}), \tag{43}
\]

also

\[
\delta_{\nu k}(\hat{x}, \hat{y}) = \delta_{\nu k}(\hat{x}, \hat{y}) - \delta(\hat{x} - \hat{y})(-g(\hat{y}))^{-\frac{1}{2}} \Gamma_k(\hat{y}), \tag{44}
\]

finally,

\[
\delta_{\nu k}(\hat{x}, \hat{y}) + \delta_{\nu k}(\hat{x}, \hat{y}) = -\delta(\hat{x}, \hat{y}) \Gamma_k(\hat{y}) = -\Gamma_k(\hat{x}) \delta(\hat{x}, \hat{y}). \tag{45}
\]

Our calculations conclude that:

\[\Gamma_k(\hat{x}) \delta(\hat{x}, \hat{y}) = \delta(\hat{x}, \hat{y}) \Gamma_k(\hat{y}). \]

Now, we introduce the following non commutative version of \( \hat{\Pi} \):

\[< \hat{x} \mid \hat{D}_k \mid \hat{y} >= -i \delta_{\nu k}(\hat{x}, \hat{y}) = - \frac{1}{2} \Gamma_k(\hat{x}) \delta(\hat{x}, \hat{y}), \tag{46}
\]

and

\[< \hat{x} \mid \hat{D}_k \mid \hat{y} >= -i \delta_{\nu k}(\hat{x}, \hat{y}) = - \frac{1}{2} \delta(\hat{x}, \hat{y}) \Gamma_k(\hat{y}), \tag{47}
\]

which operate from the right and left. Such that,

\[< \hat{x} \mid \hat{D}_k \mid f >= -i \delta_{\nu k}(\hat{x}, \hat{y}) = - \frac{1}{2} \delta(\hat{x}, \hat{y}) \Gamma_k(\hat{x}) f(\hat{x}), \tag{48}
\]
and
\[ < h | \hat{D}_k | \hat{y} > = -i h_k(y) - \frac{1}{2} \hat{n}_h(y) \Gamma_k(y), \]  
where \(< h | \hat{y} > = \hat{h}(y)\). It can be easily shown that:
\[ < h | \hat{D}_\mu | f > = - < h | \hat{D}_\mu | f > . \]  

Now, we can refer to the Heisenberg picture. We assume again the second class of the translator: \( \hat{T}(\hat{y}, \Omega) = e^{-i \hat{w}(\hat{y}, \Omega)}, \) in which \( \hat{w}(\hat{y}, \Omega) = \Lambda(\hat{y}, \Omega) \hat{D}. \) Similar to the previous section, the advent of a non trivial operator is still inevitable, i.e.
\[ \hat{\phi}_\Omega = \hat{T}(\hat{x}, \Omega) \hat{T}(\hat{y}, \Omega) = \exp\{-i \hat{D}_\mu \Lambda^\mu - i \Lambda^\nu \hat{D}_\nu - \frac{1}{2} [\hat{D}_\mu \Lambda^\mu, \Lambda^\nu \hat{D}_\nu]\}. \]  

In order to explain the latest term and for the sake of simplicity, we denote \( \partial_\mu \) by \( \partial_k \), (it is used exclusively for \( \hat{y} \) as a middle coordinate) and have \( \frac{\partial}{\partial \nu} \Lambda = \Lambda_{,\mu}. \) Since, \( \Lambda_{,\mu} = i\Lambda_1 + \frac{i}{2}[\Gamma_\mu, \Lambda]_1, \) and \( [\hat{D}_\mu, \Lambda] = i\Lambda_1 + \frac{i}{2}[\Lambda, \Gamma_\mu], \) we can use
\[ \hat{\phi}_\Omega = \exp\{-i \hat{D}_\mu \Lambda^\mu - i \Lambda^\nu \hat{D}_\nu - \frac{1}{2} [\hat{D}_\mu \Lambda^\mu, \Lambda^\nu \hat{D}_\nu]\}. \]  

where, \( \Gamma_{i,j} = \Gamma_{i,j} + \Gamma_{j,i} \), to calculate
\[ [\hat{\phi}_\Omega \Lambda^\mu, \Lambda^\nu \hat{D}_\nu] = \hat{\phi}_\Omega [\Lambda^\mu, \Lambda^\nu] \hat{D}_\nu + [\hat{\phi}_\Omega \Lambda^\mu, \hat{\phi}_\Omega \Lambda^\nu] \hat{D}_\nu + \Lambda^\nu [\hat{\phi}_\Omega \Lambda^\mu, \hat{\phi}_\Omega \Lambda^\nu] \hat{D}_\nu, \]  

Additionally, it is easily shown that, \( \hat{\phi}_\Omega \Lambda^\mu = i\Lambda^\mu - \Lambda^\nu \hat{D}_\nu + \frac{i}{2} [\Gamma_\mu, \Lambda^\nu] \) and \( \hat{\phi}_\Omega (\Lambda^\nu \hat{D}_\mu) = i(\Lambda^\nu \Lambda^\mu)_{,\mu} - (\Lambda^\nu \Lambda^\mu)_{,\mu} \hat{D}_\mu + \frac{i}{2} [\Gamma_\mu, \Lambda^\nu \Lambda^\mu]. \) Thus, Eq. (49) reads
\[ \hat{\phi}_\Omega = \exp\{A^\mu_{,\mu} + \frac{i}{2}[\Gamma_\mu, \Lambda^\mu] - \frac{1}{2} [\hat{\phi}_\Omega \Lambda^\mu, \Lambda^\nu] \hat{D}_\nu + \frac{i}{2} \hat{\phi}_\Omega \Lambda^\nu [\Gamma_\mu, \Lambda^\mu] \hat{D}_\nu + \frac{i}{2} \hat{\phi}_\Omega [\Gamma_\mu, \Lambda^\nu \Lambda^\mu] \hat{D}_\nu, \]  

Eq. (49) must be satisfied by Moyal-Weyl mapping, in order to achieve the non commutativity of Minkowski space time. We are now ready to develop the Moyal-Weyl mapping. It can be seen that: \( \ast = \circ^{NC}/\circ^{CL}. \) Substituting the unit operator in Eq. (52), \( \ast \)-product becomes:
\[ \ast^{\text{extended}} = \phi^{\lim_{\Omega \rightarrow \infty}} \int_0^1 \int_0^1 d^n x \ d^n \hat{y} \ d^n \hat{z} \ d^n \hat{\theta} \ d^n \hat{\theta} (\hat{z} \hat{\theta} \sqrt{\hat{v}^\dagger(\hat{z})} \hat{v}^\dagger(\hat{z}) \sqrt{\hat{v}(\hat{z})} < \hat{z} < \hat{\theta} < \hat{\theta}), \]  

where,
\[ \hat{Q}(\hat{x}, \hat{z}) = \int d^n \hat{y} \]  
\[ (\hat{D}_\mu(\hat{x}, \hat{y})[\Lambda^\mu(\hat{y}); \Lambda^\nu(\hat{y}) \hat{D}_\nu(\hat{y}, \hat{z}) + \frac{i}{2} \hat{D}_\mu(\hat{x}, \hat{y}) \Lambda^\nu(\hat{y}) [\Gamma_\mu(\hat{y}), \Lambda^\mu(\hat{y})] + \frac{i}{2} \Lambda^\nu(\hat{y}) [\Gamma_\mu(\hat{y}), \Lambda^\mu(\hat{y})] \hat{D}_\nu(\hat{y}, \hat{z}) + \frac{i}{2} \Lambda^\nu(\hat{y}) [\Gamma_\mu(\hat{y}), \Lambda^\mu(\hat{y})] \Lambda^\nu(\hat{y}). \]  

The integration of Eq. (53) can be removed by restoring the transition’s information, so that Eq. (52) becomes:
\[ \ast^{\text{extended}} = \exp\{-i \hat{D}_\mu \Lambda^\mu - i \Lambda^\nu \hat{D}_\nu - \frac{1}{2} [\hat{D}_\mu \Lambda^\mu, \Lambda^\nu \hat{D}_\nu] - \frac{1}{4} [\Lambda^\nu, \Gamma_\mu] \Lambda^\mu \}, \]  

in which,
\[ [A ; B] = A \sqrt{g} B - B \sqrt{g} A = A[B, \sqrt{g}] + B[A, \sqrt{g}] + [A, B] \sqrt{g}, \]  

therefore, \( [\Lambda^\nu(\hat{y}) ; \Lambda^\nu(\hat{y})] \) can be expressed explicitly as follows:
\[ [\Lambda^\nu(\hat{y}) ; \Lambda^\nu(\hat{y})] = [\Lambda^\nu(\hat{y}) ; \Lambda^\nu(\hat{y})] \sqrt{-g}(\hat{y}) \]  

\[ + \Lambda^\nu(\hat{y}) \sqrt{-g}(\hat{y}) + \Lambda^\nu(\hat{y}) \Lambda^\nu(\hat{y}) \sqrt{-g}(\hat{y}) \]  

**LOCALITY**

Due to the principles of general relativity, special relativity is possible on tangent space time. So, up to the first order of the Riemann curvature tensors, one finds
\[ \omega^{ij}(\hat{y}) \rightarrow \theta^{ij} + \hat{\theta}^{ij}(\hat{y}), \]  

where, \( \theta \) is a constant part and \( \hat{\theta}(\hat{y}) \) is a coordinate dependent part of non commutativity. The second term includes the first order of the Riemann curvature tensors. There is a homomorphism of local pseudo Riemann manifolds with Minkowski space time; so, we can employ Eq. (1). Also, in the case study,
\[ [\hat{y}^1, \hat{y}^2] = i\theta^{12}, \]  

we have
\[ [\hat{y}^\mu, \hat{y}^\nu] = 0, \mu, \nu \neq 1, 2, \]
thus, all Greek indices which appear in the above equations take the values 1 and 2. Therefore, Eq. (43) becomes,

\[ T(\Omega)T^\dagger(\Omega) |[\hat{y}',\hat{y}]=\Theta\hat{y} \]

Now, we can again set: \( \Lambda' = \hat{y}' \) and \( \Omega \) as a neighbor of the origin is substituted in Eq. (54), the Moyal-Weyl mapping becomes available on other than Minkowski space time:

\[ A(\hat{y})B(\hat{y}) = A(y)e^{\frac{i}{\hbar}\mathbf{\hat{Q}} B(y)}, \]  

(61)

where \( \mathbf{\hat{Q}} \), is given by Eq. (54) or Eq. (55). According to Eq. (55), the generalized Moyal-Weyl mapping will be:

\[ \hat{\mathbf{\Gamma}}_{ extended} = e^{-\frac{i}{\hbar} \mathbf{D}, \mathbf{N}^{ij} \mathbf{D}_j, -\frac{i}{\hbar} \mathbf{D}_i \mathbf{G}^{ij} + \frac{i}{\hbar} \mathbf{G}^{ij} \mathbf{D}_i, -\frac{i}{\hbar} \mathbf{G}^{ij} \Lambda'}, \]

(62)

where, \( \mathbf{N}^{ij} = \theta^{ij} + \theta^{ki} \hat{y}' \Gamma_k + \theta^{lk} \Gamma_k \hat{y}' \), \( \mathbf{D}_i = -i \partial_i + \frac{1}{2} \mathbf{\Gamma}_i \), \( \mathbf{\hat{J}}_i = -i \mathbf{\hat{J}}_j + \frac{1}{2} \mathbf{\Gamma}_j \) and \( \mathbf{\hat{G}}^{ij} = \frac{1}{2} [\mathbf{\Gamma}_j, \mathbf{\Lambda}^i] \). Now, our calculations give us:

\[ [\hat{y}',\hat{y}']_{ extended} = +i\theta^{ij} + \frac{1}{2} \theta^{ik} \Gamma_k \hat{y}' - \frac{1}{2} \theta^{ik} \Gamma_k \hat{y}' + \frac{1}{2} \theta^{kj} \Gamma_k \hat{y}' + \frac{1}{4} \theta^{ikj} \Gamma_k \hat{y}' \Gamma_{l,k} \hat{y}'. \]

(63)

Based on Eq. (60) and for a spherically symmetric universe, \([13, 15, 18]\), one has:

\[ \Gamma_{\mu,\nu} = \Gamma_{\nu,\mu}, \quad \Gamma_{\mu,\nu} \hat{y}' = \Gamma_{\mu}; \]

(64)

in order to satisfy Eq. (60), Eq. (55) becomes,

\[ [\hat{y}',\hat{y}']_{ extended} = +i\theta^{ij} + \frac{1}{2} \theta^{ik} \Gamma_k \hat{y}' - \frac{1}{2} \theta^{ik} \Gamma_k \hat{y}'. \]

(65)

CONCLUSION

At the level of quantum mechanics, we have found a way to generalize coordinate non commutativity to a general space time other than the Minkowski space time. For the case of Eq. (54), we made the local operators which were arranged-order and satisfied the commutation relation of Eq. (12). The translation information was entrusted into the range of integration, by substituting the unit operator in the second class of translators. Also, we showed that the new operator (named \( \circ \)) can be made by employing the evolution operator (in the Heisenberg picture), homomorphism of local pseudo Riemann manifolds with Minkowski space time and the assumption of Eq. (59). \( \circ \) enabled us to generalize coordinate non commutativity for a more general space time. It can be seen that if we substitute Eq. (54) in Eq. (55) then the obtained non commutativity:

\[ [\hat{y}',\hat{y}']_{ extended} = +i\theta^{ij} + \frac{1}{2} \theta^{ik} \Gamma_k \hat{y}' - \frac{1}{2} \theta^{ik} \Gamma_k \hat{y}'. \]

(65)

ACKNOWLEDGMENTS

The author thanks Shahrekord University for supporting this work with a research grant.

[1] N. Seiberg, E. Witten, JHEP 9909 (1999) 032.
[2] A. Connes, Marcolli, "non commutative Geometry, Quantum Fields and Motives", (Academic Press, INC. London) 1994.
[3] J. Wess, "non commutative Space times: Symmetries in non commutative Geometry and Field Theory", Lect. Notes Phys. 774 (Springer, Berlin Heidelberg, DOI 10.1007/978-3-540-89793-4) 2009.
[4] D. J. Gross, N. A. Nekrasov, JHEP 0103 (2001) 044 .
[5] R. J. Szabo, Phys. Rept. 378 (2003) 207.
[6] A. Fischer, R. J. Szabo, JHEP, 0902 (2009) 031.
[7] M. Chaichian, P. Presnajder, M.M. Sheikh- Jabbari and A. Tureanu, Eur.Phys.J. C29 (2003) 413.
[8] M. M. Sheikh-Jabbari, J. High Energy Phys. 9906 (1999) 015.
[9] R. J. Szabo, Phys. Rept. 378, 207-299, 2003.
[10] O. Bertolami, C. A. D. Zorro, Phys. Lett. B673 (2009) 83.
[11] A. Jafari, Eur. Phy. J. C 73 (2013) 2271.
[12] R. Amarim, J. Barcelos-Neto, J. Phys. A 34 (2001) 8851.
[13] L. Parker, Phys. Rev. Lett. 44 (1980) 1559 and Phys. Rev. D22 (1980) 1922.
[14] A. I. Nestrov, Class. Quant. Grav. 16 (1999) 465-477.
[15] C. W. Misner, K. S. Thorne, J. A. Wheeler "Gravitation", (Freeman Publishing Company, San Francisco) 1973.
[16] J. Weber, “General Relativity and Gravitational Waves”, (Interscience Publisher INC, New York) 1961, Dover Edition, 2004 and “Gravitational Radiation and Relativity”, edited by J. Weber and T. M. Karade, Vol. 3 (Proceedings of the Sir Arthur Eddington Centenary Symposium, Nagpur, India) 1984.
[17] M. Maggiore, “Gravitational Waves”, (Oxford University Press INC, New York) 2008.
[18] R. D’inverno, “Introducing Einstein’s Relativity”, (Oxford University Press Inc, New York) 1993.
[19] B. s. DeWitt, Rev. Mod. Phys. 29 (1957).
[20] H. Kleinert "Gauge Fields in Condensed Matter", (World Scientific Publisher Company, Inc) 1987.