Abstract. The commutative and homological algebra of modules over posets is developed, as closely parallel as possible to the algebra of finitely generated modules over noetherian commutative rings, in the direction of finite presentations, primary decompositions, and resolutions. Interpreting this finiteness in the language of derived categories of subanalytically constructible sheaves proves two conjectures due to Kashiwara and Schapira concerning sheaves with microsupport in a given cone.

The motivating case is persistent homology of arbitrary filtered topological spaces, especially the case of multiple real parameters. The algebraic theory yields computationally feasible, topologically interpretable data structures, in terms of birth and death of homology classes, for persistent homology indexed by arbitrary posets.

The exposition focuses on the nature and ramifications of a suitable finiteness condition to replace the noetherian hypothesis. The tameness condition introduced for this purpose captures finiteness for variation in families of vector spaces indexed by posets in a way that is characterized equivalently by distinct topological, algebraic, combinatorial, and homological manifestations. Tameness serves both the theoretical and computational purposes: it guarantees finite primary decompositions, as well as various finite presentations and resolutions all related by a syzygy theorem, and the data structures thus produced are computable in addition to being interpretable.

The tameness condition and its resulting theory are new even in the finitely generated discrete setting, where being tame is materially weaker than being noetherian.

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Overview. A module over a poset is a family of vector spaces indexed by the poset elements, with a homomorphism for each poset relation. The setup is inherently commutative: the homomorphism for a poset relation $p \preceq q$ is the composite of the homomorphisms for the relations $p \preceq r$ and $r \preceq q$ whenever $r$ lies between $p$ and $q$. This paper is a study of the extent to which modules over posets behave like modules over noetherian commutative rings, particularly when it comes to finite presentations, primary decompositions, and resolutions. Interpreting this finiteness in the language of derived categories of subanalytically constructible sheaves proves two conjectures due to Kashiwara and Schapira concerning sheaves with microsupport in a given cone.

The algebraic and homological investigations can be viewed as testing the frontier of multigraded commutative algebra regarding how far one can get without a ring and with minimal hypotheses on the multigrading. However, beyond this abstract mathematical interest, the impetus lies in data science applications, where the poset consists of “parameters” indexing a family of topological subspaces of a fixed topological space. Taking homology of the subspaces in this topological filtration yields a poset module, called the persistent homology of the filtration, referring to how homology classes are born, persist for a while, and then die as the parameter moves up in the poset.

In ordinary persistent homology, the poset is totally ordered—usually the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, or a subset $\{1, \ldots, m\}$. This case is well studied (see [EH10], for example), and the algebra is correspondingly simple [Cra13]. Persistence with multiple totally ordered parameters, introduced by Carlsson and Zomorodian [CZ09], has been developed in various ways, often assuming that the poset is $\mathbb{N}^n$. That discrete framework is preferred in part because it arises frequently when filtering finite simplicial complexes, but also because settings involving continuous parameters—including the application that drives the advances here—unavoidably produce modules that fail to be finitely presented in several fundamental ways.

The foundations laid here take the lack of noetherian hypotheses head on, to open the possibility of working directly with modules over arbitrary posets. The focus is therefore on the nature and ramifications of a suitable finiteness condition to replace the noetherian hypothesis. The tameness condition introduced here appears to be the natural candidate, capturing finiteness of variation in a way that is characterized equivalently by distinct topological, algebraic, combinatorial, and homological manifestations. Tameness serves both the theoretical and computational purposes: it guarantees finite primary decompositions, as well as various finite presentations and resolutions all related by a syzygy theorem, and the data structures thus produced are computationally feasible, in addition to being topologically interpretable in terms of birth and death of homology classes. The tameness condition, and its resulting primary decompositions, syzygy theorem, and interpretable computational structures, are new even in the finitely generated discrete setting, where tame is much weaker than noetherian.
No restriction on the underlying poset is required except for primary decomposition, which needs the poset to be a group to allow analogues of prime ideals and localization. There is no difference in any of the theory for cases that are, for example, not locally finite, such as $\mathbb{R}^n$ or other partially ordered real vector spaces. Moreover, the data structures and transitions between the topological, algebraic, combinatorial, and homological perspectives take advantage of and preserve supplementary geometry, be it subanalytic, semialgebraic, or piecewise-linear, for instance, if the ambient structures—the partial orderings and the modules—have such geometry to begin with. The subanalytic and piecewise-linear cases are crucial for the applications to the conjectures of Kashiwara and Schapira.

**Acknowledgements.** First, a special acknowledgement goes to Ashleigh Thomas, who has been and continues to be a long-term collaborator on this project. She was listed as an author on earlier drafts of [Mil17] (of which this is roughly the first third), but her contributions lie more properly beyond these preliminaries (see [MT19], for example), so she declined in the end to be named as an author on this installment. Early in the development of the ideas here, Thomas put her finger on the continuous rather than discrete nature of multiparameter persistence modules for fly wings. She computed the first examples explicitly, namely those in Example 1.2, and produced the biparameter persistence diagrams there as well as some of the figures in Example 3.21.

Justin Curry pointed out connections from the combinatorial viewpoint taken here, in terms of modules over posets, to higher notions in algebra and category theory, particularly those involving constructible sheaves, which are in the same vein as Curry’s proposed uses of them in persistence [Cur14]; see Remarks 2.4, 3.2, 4.26, and 7.11.

The author is indebted to David Houle, whose contribution to this project was seminal and remains ongoing; in particular, he and his lab produced the fruit fly wing images [Hou03]. Paul Bendich and Joshua Cruz took part in the genesis of this project, including early discussions concerning ways to tweak persistent (intersection [BH11]) homology for the fly wing investigation. Ville Puuska discovered several errors in an early version of Section 4, resulting in substantial correction and alteration; see Examples 2.8 and 4.16. Pierre Schapira gave helpful comments on Section 8. Banff International Research Station provided an opportunity for valuable feedback and suggestions at the workshop there on Topological Data Analysis (August, 2017) as this research was being completed; many participants, especially the organizers, Uli Bauer and Anthea Monod, as well as Michael Lesnick, shared important perspectives and insight. Thomas Kahle requested that Proposition 6.7 be an equivalence instead of merely the one implication it had stated. Hal Schenck gave helpful comments on an earlier version of the Introduction. Some passages in Section 1.1 are based on or taken verbatim from [Mil15]. Portions of this work were funded by NSF grant DMS-1702395.

1.1. **Motivation and examples.** The developments here grew out of investigation of data structures for real multiparameter persistence modules, where both senses of
the word “real” are intended: actual—from genuine data, with a particular dataset in mind—and with parameters taking continuous as opposed to discrete values. Instead of reviewing the numerous possible reasons for considering multiparameter persistence, many already having been present from the outset [CZ09, §1.1], what follows is a description of the instance of real multiparameter persistence that arises in the biological problem that the theory here is specifically designed to serve.

**Example 1.1.** The veins in a fruit fly wing can be presented as an embedded planar graph, with a location for each vertex and an algebraic curve for each arc [Hou03]. There are many options for statistical summaries of fly wings, some of them elementary, such as a linear model taking into account a weighted sum of (say) the number of vertices and the total edge length. Whatever the chosen method, it has to grapple with the topological vein variation, giving appropriate weight to new or deleted singular points in addition to varying shape, as in the following images.

The nature of wing vein formation from gene expression levels during embryonic development (see [Bla07] for background) provides rationale for believing that continuous biparameter persistent homology models biological reality reasonably faithfully.

Let $Q = \mathbb{R}_- \times \mathbb{R}_+$ with the coordinatewise partial order, so $(r, s) \in Q$ for any nonnegative real numbers $-r$ and $s$. Let $X = \mathbb{R}^2$ be the plane in which the fly wing is embedded and define $X_{rs} \subseteq X$ to be the set of points at distance at least $-r$ from every vertex and within $s$ of some edge. Thus $X_{rs}$ is obtained by removing the union of the balls of radius $r$ around the vertices from the union of $s$-neighborhoods of the edges. In the following portion of a fly wing, $-r$ is approximately twice $s$:

The biparameter persistent homology module $M_{rs} = H_0(X_{rs})$ summarizes wing vein structure for the intended biological purposes.

Relevant properties of these modules are best highlighted in a simplified setting.

**Example 1.2.** Using the setup from Example 1.1, the zeroth persistent homology for the toy-model “fly wing” at left in Figure 1 is the $\mathbb{R}^2$-module $M$ shown at center. Each point of $\mathbb{R}^2$ is colored according to the dimension of its associated vector space in $M$,
Figure 1. Biparameter persistence module and finite encoding

namely 3, 2, or 1 proceeding up (increasing $s$) and to the right (increasing $r$). The structure homomorphisms $M_{rs} \to M_{r's'}$ are all surjective.

This $\mathbb{R}^2$-module fails to be finitely presented for three fundamental reasons. First, the three generators sit infinitely far back along the $r$-axis. (Fiddling with the sign on $r$ does not help: the natural maps on homology proceed from infinitely large radius to 0 regardless of how the picture is drawn.) Second, the relations that specify the transition from vector spaces of dimension 3 to those of dimension 2 or 1 lie along a real algebraic curve, as do those specifying the transition from dimension 2 to dimension 1. These curves have uncountably many points. Third, even if the relations are discretized—restrict $M$ to a lattice $\mathbb{Z}^2$ superimposed on $\mathbb{R}^2$, say—the relations march off to infinity roughly diagonally away from the origin. (See Example 1.4 for the right-hand image.)

1.2. Modules over posets. There are many essentially equivalent ways to think of a poset module. The definition in the first line of this Introduction is among the more elementary formulations; see Definition 2.1 for additional precision. Others include a

- representation of a poset [NR72];
- functor from a poset to the category of vector spaces (e.g., see [Cur19]);
- vector-space valued sheaf on a poset (e.g., see Section 8.1.3 or [Cur14, §4.2]);
- representation of a quiver with (commutative) relations (e.g., see [Oud15, §A.6]);
- representation of the incidence algebra of a poset [DRS72]; or
- module over a directed acyclic graph [CL18].

The premise here is that commutative algebra provides an elemental framework out of which flows corresponding structure in these other contexts, in which the reader is encouraged to interpret all of the results. Section 8 provides an example of how that can look, in this case from the sheaf perspective. Expressing the foundations via commutative algebra is natural for its infrastructure surrounding resolutions. And as the objects are merely graded vector spaces with linear maps among them—there are no rings to act—it is also the most elementary language available, to make the exposition accessible to a wide audience, including statisticians applying persistent homology in addition to topologists, combinatorialists, algebraists, geometers, and programmers.

Some of the formulations of poset module are only valid when the poset is assumed to be locally finite (see [DRS72], for instance), or when the object being acted upon satisfies a finitary hypothesis [KN09] in which the algebraic information is nonzero on only finitely many points in any interval. This is not a failing of any particular
formulation, but rather a signal that the theory has a different focus. Combinatorial formulations are built for enumeration. Representation theories are built for decomposition into and classification of irreducibles. While commutative algebra appreciates a direct sum decomposition when one is available, such as over a noetherian ring of dimension 0, its initial impulse is to relate arbitrary modules to simpler ones by less restrictive decomposition, such as primary decomposition, or by resolution, such as by projective or injective modules. That is the tack taken here.

1.3. Topological tameness. The finiteness condition introduced here generalizes the notion of topological tameness for ordinary persistence in a single parameter (see [CdS+16, §3.8], for example), reflecting the intuitive notion that given a filtration of a topological space from data, only finitely many topologies should appear. The tame condition (Definitions 2.7 and 2.12) partitions the poset into finitely many domains of constancy for a given module. Tameness is a topological concept, designed to control the size and variation of homology groups in subspaces of a fixed topological space.

Example 1.3. The $\mathbb{R}^2$-module in Example 1.2 is tame, with four constant regions: over the bottom-left region (yellow) the vector space is $k^3$; over the middle (olive) region the vector space is $k^2$; over the upper-right (blue) region the vector space is $k$; and over the remainder of $\mathbb{R}^2$ the vector space is 0. The homomorphisms between these vector spaces do not depend on which points in the regions are selected to represent them. For instance, $k^3 \to k^2$ always identifies the two basis vectors corresponding to the connected components that are the left and right halves of the horizontally infinite red strip.

It is worth noting that in ordinary totally ordered persistence, tameness means simply that the bar code (see Section 1.9) has finitely many bars, or equivalently, the persistence diagram has finitely many off-diagonal dots: finiteness of the set of constant regions precludes infinitely many non-overlapping bars (the bar code can’t be “too long”), while the vector space having finite dimension precludes a parameter value over which lie infinitely many bars (the bar code can’t be “too thick”).

In principle, tameness can be reworked to serve as a data structure for algorithmic computation, especially in the presence of an auxiliary hypothesis to regulate the geometry of the constant regions—when they are semialgebraic or piecewise linear (Definition 2.16.1 or 2.16.2), for example. The algorithms would generalize those for polyhedral “sectors” in the discrete case [HM05] (or see [MS05, Chapter 13]).

1.4. Combinatorial tameness: finite encoding. Whereas the topological notion of tameness requires little more than an arbitrary subdivision of the poset into regions of constancy (Definition 2.7), the combinatorial incarnation imposes additional structure on the constant regions, namely that they should be partially ordered in a natural way. More precisely, it stipulates that the module $M$ should be pulled back from a $P$-module along a poset morphism $Q \to P$ in which $P$ is a finite poset and the $P$-module has finite dimension as a vector space over the field $k$ (Definition 4.1).
Example 1.4. The right-hand image in Example 1.2 is a finite encoding of $M$ by a three-element poset $P$ and the $P$-module $H = \mathbb{k}^3 \oplus \mathbb{k}^2 \oplus \mathbb{k}$ with each arrow in the image corresponding to a full-rank map between summands of $H$. Technically, this is only an encoding of $M$ as a module over $Q = \mathbb{R}_- \times \mathbb{R}_+$. The poset morphism $Q \to P$ takes all of the yellow rank 3 points to the bottom element of $P$, the olive rank 2 points to the middle element of $P$, and the blue rank 1 points to the top element of $P$. (To make this work over all of $\mathbb{R}^2$, the region with vector space dimension 0 would have to be subdivided, for instance by introducing an antidiagonal extending downward from the origin, thus yielding a morphism from $\mathbb{R}^2$ to a five-element poset.) This encoding is semialgebraic (Definition 2.16): its fibers are real semialgebraic sets.

In general, constant regions need not be situated in a manner that makes them the fibers of a poset morphism (Example 4.4). Nonetheless, over arbitrary posets, modules that are tame by virtue of admitting a finite constant subdivision (Definition 2.12) always admit finite encodings (Theorem 4.22), although the constant regions are typically subdivided by the encoding poset morphism. In the case where the poset is a real vector space, if the constant regions have additional geometry (Definition 2.16), then a similarly geometric finite encoding is possible.

The framework of poset modules arising from filtrations of topological spaces is more or less an instance of MacPherson’s exit path category [Tre09, §1.1]. In that context, Lurie defined a notion of constructibility in the Alexandrov topology [Lur17, Definitions A.5.1 and A.5.2], independently of the developments here and for different purposes. It would be reasonable to speculate that tameness should correspond to Alexandrov constructibility, given that encoding of a poset module is defined by pulling back along a poset morphism (in Lurie’s language, a continuous morphism of posets), but it does not; see Remark 4.26. The difference between constant in the sense of tameness via constant subdivision (Section 2.2) and locally constant in the sheaf-theoretic sense with the Alexandrov topology makes tameness—in the equivalent finitely encoded formulation—rather than Alexandrov constructibility the right notion of finiteness for the syzygy theorem as well as for algorithmic computation and data analysis applications of multipersistance. That contrasts with the comparison between tameness and subanalytic constructibility in the usual topology on real vector spaces, which are essentially the same notion for the relevant sheaves; see Section 1.10.

Finite encoding has its roots in combinatorial commutative algebra in the form of sector partitions [HM05] (or see [MS05, Chapter 13]). Like sector partitions, finite encoding is useful, theoretically, for its enumeration of all topologies encountered as the parameters vary. However, enumeration leads to combinatorial explosion outside of the very lowest numbers of parameters. And beyond its inherent inefficiency, poset encoding lacks many of the features that have come to be expected from persistent homology, including the most salient: a description of homology classes in terms of their persistence, meaning birth, death, and lifetime.
1.5. **Discrete persistent homology by birth and death.** \(\mathbb{Z}^n\)-modules have been present in commutative algebra for over half a century [GW78], but the perspective arising from their equivalence with multipersistence is relatively new [CZ09]. Initial steps have included descriptions of the set of isomorphism classes [CZ09], presentations [CSV17] and algorithms for computing [CSZ09, CSV12] or visualizing [LW15] them, as well as interactions with homological algebra of modules, such as persistence invariants [Knu08] and certain notions of multiparameter noise [SCL+16].

Algebraically, viewing persistent homology as a module rather than (say) a diagram or a bar code, a birth is a generator. In ordinary persistence, with one parameter, a death is more or less the same as a relation. However, in multipersistence the notion of death diverges from that of relation. The issue is partly one of geometric shape in the parameter poset, say \(\mathbb{Z}^n\) (the uppermost shaded regions indicate where classes die):

If death is to be dual to birth, then a nonzero homology class at some parameter should die if it moves up along any direction in the poset. Birth is not the bifurcation of a homology class into two independent ones; it is the creation of a new class from zero. Likewise, genuine death is not the joining of two classes into one; it is annihilation. And death should be stable, in the sense that wiggling the parameter and then pushing up should still kill the homology class.

In algebraic language, death is a “cogenerator” rather than a relation. For finitely generated \(\mathbb{N}^n\)-modules, or slightly more generally for finitely determined modules (Example 4.5 and Definition 6.1), cogenerators are irreducible components, cf. [MS05, Section 5.2]. Indeed, irreducible decomposition suffices as a dual theory of death in the finitely generated case; this is more or less the content of Theorem 6.19. The idea there is that surjection from a free module covers the module by sending basis elements to births in the same (or better, dual) way that inclusion into an injective module envelops the module by capturing deaths as injective summands. The geometry of this process in the parameter poset on the injective side is as well understood as it is on the free side [MS05, Chapter 11], and in finitely generated situations it is carried out theoretically or algorithmically by finitely generated truncations of injective modules [Mil02, HM05].

Combining birth as free cover and death as injective hull leads naturally to flange presentation (Definition 6.12), which composes the augmentation map \(F \rightarrow M\) of a flat resolution with the augmentation map \(M \hookrightarrow E\) of an injective resolution to get a homomorphism \(F \rightarrow E\) whose image is \(M\). The indecomposable summands of \(F\) capture births and those of \(E\) deaths. Flange presentation splices a flat resolution to
an injective one in the same way that Tate resolutions (see [Coa03], for example) segue from a free resolution to an injective one over a Gorenstein local ring of dimension 0.

Why a flat cover $F \to M$ instead of a free one? There are two related reasons: first, flat modules are dual to injective ones (Remark 6.11), so in the context of finitely determined modules the entire theory is self-dual; and second, births can lie infinitely far back along axes, as in the toy-model fly wing from Example 1.2.

1.6. **Algebraic tameness: fringe presentation.** That multipersistence modules can fail to be finitely generated, like Example 1.2 does, in situations reflecting reasonably realistic data analysis was observed by Patriarca, Scolamiero, and Vaccarino [PSV12, Section 2]. Their “monomial module” view of persistence covers births much more efficiently, for discrete parameters, by keeping track of generators not individually but gathered together as generators of monomial ideals. Huge numbers of predictable syzygies among generators are swallowed and hence are present only implicitly. And that is good, as nothing topologically new about persistence of homology classes is taught by the well known syzygies of monomial ideals, which in this setting are merely an interference pattern from the merging of separate birth points of the same class.

Translating to the setting of continuous parameters, and including the dual view of deaths, which works just as well, suggests an uncountably more efficient way to cover births and deaths than listing them individually. In fact this urge to gather births or deaths does not really depend on the transition to continuous parameters from discrete ones. Indeed, any $\mathbb{R}^n$-module $M$ can be approximated by a $\mathbb{Z}^n$-module, the result of restricting $M$ to, say, the rescaled lattice $\varepsilon \mathbb{Z}^n$. Suppose, for the sake of argument, that $M$ is bounded, in the sense of being zero at parameters outside of a bounded subset of $\mathbb{R}^n$; think of Example 1.2, ignoring those parts of the module there that lie outside of the depicted square. Ever better approximations, by smaller $\varepsilon \to 0$, yield sets of lattice points ever more closely hugging an algebraic curve. Neglecting the difficulty of computing where those lattice points lie, how is a computer to store or manipulate such a set? Listing the points individually is an option, and perhaps efficient for particularly coarse approximations, but in $n$ parameters the dimension of this storage problem is $n - 1$. As the approximations improve, the most efficient way to record such sets of points is surely to describe them as the allowable ones on one side of an algebraic curve. And once the computer has the curve in memory, no approximation is required: just use the (points on the) curve itself. In this way, even in cases where the entire topological filtration setup can be approximated by finite
simplicial complexes, understanding the continuous nature of the un-approximated setup is both more transparent and more efficient.

Combining flange presentation with this monomial module view of births and deaths yields fringe presentation (Definition 3.16), the analogue for modules over an arbitrary poset $Q$ of flange presentation for finitely determined modules over $Q = \mathbb{Z}^n$. The role of indecomposable free or flat modules is played by upset modules (Example 4.6) which have $\mathbb{k}$ in degrees from an upset $U$ and 0 elsewhere. The role of indecomposable injective modules is played similarly by downset modules.

Fringe presentation is expressed by a monomial matrix (Definition 3.17), an array of scalars with rows labeled by upsets and columns labeled by downsets. For example,

\[
\begin{bmatrix}
\varphi_{11}
\end{bmatrix}
\]

represents a fringe presentation of $M = \mathbb{k}$ as long as $\varphi_{11} \in \mathbb{k}$ is nonzero. The monomial matrix notation specifies a homomorphism $\mathbb{k}[\mathbf{a}] \to \mathbb{k}[\mathbf{a}]$ whose image is $M$, which has $M_{\mathbf{a}} = \mathbb{k}$ over the yellow parameters $\mathbf{a}$ and 0 elsewhere. The blue upset specifies the births at the lower boundary of $M$; unchecked, the classes would persist all the way up and to the right. But the red downset specifies the deaths along the upper boundary of $M$.

When the birth upsets and death downsets are semialgebraic, or piecewise linear, or otherwise manageable algorithmically, monomial matrices render fringe presentations effective data structures for real multipersistence. Fringe presentations have the added benefit of being topologically interpretable in terms of birth and death.

Although the data structure of fringe presentation is aimed at $\mathbb{R}^n$-modules, it is new and lends insight already for finitely generated $\mathbb{N}^n$-modules (even when $n = 2$), where monomial matrices have their origins [Mil00, Section 3]. The context there is more or less that of finitely determined modules; see Definition 6.14 in particular, which is really just the special case of fringe presentation in which the upsets are localizations of $\mathbb{N}^n$ and the downsets are duals—that is, negatives—of those.

1.7. Homological tameness: the syzygy theorem. Even in the case of filtrations of finite simplicial complexes by products of intervals—that is, multifiltrations (Example 2.6) of finite simplicial complexes—persistent homology is not naturally a module over a polynomial ring in $n$ (or any finite number of) variables. This is for the same reason that single-parameter persistent homology is not naturally a module over a polynomial ring in one variable: though there might only be finitely many topological transitions, they can (and often do) occur at incommensurable real numbers. That
said, observe that filtering a finite simplicial complex automatically induces a finite encoding. Indeed, the parameter space maps to the poset of simplicial subcomplexes of the original simplicial complex by sending a parameter to the simplicial subcomplex it indexes. That is not the smallest poset, of course, but it leads to a fundamental point: one can and should do homological algebra over the finite encoding poset rather than (only) over the original parameter space.

This line of thinking culminates in a syzygy theorem (Theorem 7.12 for modules; Theorem 7.17 for complexes) to the effect that, remarkably, the topological, algebraic, combinatorial, and homological notions of tameness available respectively via

- constant subdivision (Definition 2.12),
- fringe presentation (Definition 3.16),
- poset encoding (Definition 4.1), and
- indicator resolution (Definition 7.1)

are equivalent. The moral is that the tame condition over arbitrary posets appears to be the right notion to stand in lieu of the noetherian hypothesis over $\mathbb{Z}^n$: the tame condition is robust, has multiple characterizations from different mathematical perspectives, and enables algorithmic computation in principle.

The syzygy theorem directly reflects the more usual syzygy theorem for finitely determined $\mathbb{Z}^n$-modules (Theorem 6.19), with upset and downset resolutions being the arbitrary-poset analogues of free and injective resolutions, respectively, and fringe presentation being the arbitrary-poset analogue of flange presentation.

Topological tameness via constant subdivision is a priori weaker (that is, more inclusive) than combinatorial tameness via finite encoding, and algebraic tameness via fringe presentation is a priori weaker than homological tameness via upset or downset resolution. Thus the syzygy theorem leverages relatively weak topological structure into powerful homological structure. One consequence is a proof of two conjectures due to Kashiwara and Schapira; see Section 1.10. The developments there rely on the fact that, although the characterizations of tameness require no additional structure on the underlying poset, any additional structure that is present—subanalytic, semialgebraic, or piecewise-linear—is preserved by the transitions between the characterizations of tameness in the syzygy theorem.

The proof of the syzygy theorem works by reducing to the finitely determined (Section 6) case over $\mathbb{Z}^n$. The main point is that given a finite encoding of a module over an arbitrary poset $Q$, the encoding poset can be embedded in $\mathbb{Z}^n$. The proof is completed by pushing the data forward to $\mathbb{Z}^n$, applying the more usual syzygy theorem to finitely determined modules there, and pulling back to $Q$.

1.8. Geometric algebra over partially ordered abelian groups. This paper develops commutative algebra for modules over posets from scratch. Alas, the part of the theory relating to primary decomposition is not amenable to arbitrary posets. Rather, the natural setting to carry out primary decomposition is, in the best tradition of
classical mathematics [Bir42, Cli40, Rie40], over partially ordered abelian groups (Definition 5.1). Those provide an optimally general context in which posets have a notion of “face” along which to localize without altering the ambient poset. That is, a partially ordered group $Q$ has an origin—namely its identity $0$—and hence a positive cone $Q_+$ of elements it precedes. A face of $Q$ is a submonoid of $Q_+$ that is also a downset therein (Definition 5.7). And as everything takes place inside of the ambient group $Q$, every localization of a $Q$-module along a face (Definition 5.24) remains a $Q$-module.

In persistence language, a single element in a module over a partially ordered group can a priori be mortal or immortal in more than one way. But some elements die “pure deaths” of only a single type $\tau$. These are the $\tau$-coprimary elements for a face $\tau$. In the concrete setting of a partially ordered real vector space with closed positive cone, a coprimary element is characterized (Example 5.9, Definition 5.34, and Theorem 5.35) as

1. $\tau$-persistent: it lives when pushed up arbitrarily along the face $\tau$; and
2. $\tau$-transient: it eventually dies when pushed up in any direction outside of $\tau$.

**Example 1.5.** The downset $D$ in $\mathbb{R}^2$ consisting of all points beneath the upper branch of the hyperbola $xy = 1$ canonically decomposes (Theorem 5.20) as the union

of its subsets that locally die pure deaths of some type: every red point in the

- leftmost subset on the right dies when pushed over to the right or up far enough;
- middle subset dies in the localization of $D$ along the $x$-axis (Definition 5.12 or Definition 5.24) when pushed up far enough; and
- rightmost subset dies locally along the $y$-axis when pushed over far enough.

Isolating all coprimary elements functorially requires localization that does not alter the ambient poset, after which local support functors (Definition 5.30) do the job, as in ordinary commutative algebra and algebraic geometry.

What does primary decomposition do? It expresses a given module $M$ as a submodule of a direct sum of coprimary modules. Consequently, this decomposition tells the fortune of every element: its death types are teased apart as the “pure death types” of the coprimary summands where the element lands with nonzero image.

**Example 1.6.** The union in Example 1.5 results in a canonical primary decomposition
of the downset module $k[D]$ over $\mathbb{R}^2$ (Corollary 5.21). Elements in the lower-left quadrant locally die any type of death.

It bears emphasizing that primary decomposition of downset modules, or equivalently, expressions of downsets as unions of coprimary downsets (cogenerated by the $\tau$-coprimary elements for some face $\tau$; see Definition 5.19), is canonical by Theorem 5.20 and Corollary 5.21, generalizing the canonical primary decomposition of monomial ideals in ordinary polynomial rings. However, notably lacking from primary decomposition theory over arbitrary polyhedral partially ordered abelian groups is a notion of minimality—alas, a lack that is intrinsic.

**Example 1.7.** Although three pure death types occur in $D$ in Example 1.5, and hence in the union there, the final two summands in the primary decomposition of $k[D]$ in Example 1.6 are redundant. One can, of course, simply omit the redundant summands, but for arbitrary polyhedral partially ordered groups no criterion is known for detecting a priori which summands should be omitted.

The failure of minimality here stems from geometry that can only occur in partially ordered groups more general than finitely generated free ones. More specifically, $D$ contains elements that die pure deaths of type “$x$-axis” but the boundary of $D$ fails to contain an actual translate of the face of $\mathbb{R}^2_+$ that is the positive $x$-axis. This can be seen as a certain failure of localization to commute with taking homomorphisms into $k[D]$; it is the source of much of the subtlety in the theory developed in the sequel [Mil19a] to this paper, whose purpose is partly to rectify, for real multiparameter persistence, the failure of minimality in Example 1.6.

The view toward algorithmic computation draws the focus to the case where $Q$ is polyhedral, meaning that it has only finitely many faces (Definition 5.7). This notion is apparently new for arbitrary partially ordered abelian groups. Its role here is to guarantee finiteness of primary decomposition of finitely encoded modules (Theorem 5.41).

1.9. **Bar codes.** Tame modules over the totally ordered set of integers or real numbers are, up to isomorphism, the same as “bar codes”: finite multisets of intervals. The most general form of this bijection between algebraic objects and essentially combinatorial objects over totally ordered sets is due to Crawley-Boevey [Cra13]. At its root this bijection is a manifestation of the tame representation theory of the type $A$ quiver; that is the context in which bar codes were invented by Abeasis and Del Fra, who called them “diagrams of boxes” [AD80, ADK81]. Subsequent terminology for objects equivalent to these diagrams of boxes include bar codes themselves (see [Ghr08]) and planar depictions discovered effectively simultaneously in topological data analysis, where they are called persistence diagrams [ELZ02] (see [CEH07] for attribution) and combinatorial algebraic geometry, where they are called lace arrays [KMS06].

No combinatorial analogue of the bar code can classify modules over an arbitrary poset because there are too many indecomposable modules, even over seemingly well
behaved posets like \( \mathbb{Z}^n \) [CZ09]: the indecomposables come in families of positive dimension. Over arbitrary posets, every tame module does still admit a decomposition of the Krull–Schmidt sort, namely as a direct sum of indecomposables [BC19], but again, there are too many indecomposables for this to be useful in general.

This paper makes no attempt to define bar codes or persistence diagrams for modules over arbitrary posets. Instead of decomposing modules as direct sums of elemental pieces, which could in effect be arbitrarily complicated, the commutative algebra view advocates expressing poset modules in terms of decidedly simpler modules, especially indicator modules for upsets and downsets, by way of less rigid constructions like fringe presentation, primary decomposition, or resolution. This relaxes the direct sum in a \( K \)-theoretic way, allowing arbitrary complexes instead of split short exact sequences.

Consequently, various aspects of bar codes are reflected in the equivalent concepts of tameness. The finitely many regions of constancy are seen in topological tameness by constant subdivision. The matching between left and right endpoints is seen in algebraic tameness by fringe presentation, where the left endpoints form lower borders of birth upsets and the right endpoints form upper borders of death downsets. The expressions of modules in terms of bars is seen, in its relaxed form, in homological tameness, where modules become “virtual” sums, in the sense of being formal alternating combinations rather than direct sums. Primary decomposition isolates elements that would, in a bar code, lie in bars unbounded in fixed sets of directions (see also [HOST19]).

Implicit in the notion of bar code is some concept of minimality: left endpoints must correspond to minimal generators, and right endpoints to minimal cogenerators. These are not available over arbitrary posets and are subtle to define and handle properly even for partially ordered real vector spaces [Mil19a]. When minimality is available, instead of a bijection (perfect matching) from a multiset of births to a multiset of deaths, the best one can settle for is a linear map from a filtration of the birth multiset to a filtration of the death multiset [Mil19b]. The linear map in the case of a perfect matching from left endpoints to right endpoints is represented by an identity matrix, assuming that the left and right endpoints are ordered consistently.

1.10. Derived applications. The syzygy theorem for poset modules (Theorem 7.12) is intentionally stated in the most elementary language possible, without sheaves, functors, or derived categories. But its content has deep interpretations in these enriched contexts. Section 8 demonstrates this power by proving two conjectures made by Kashiwara and Schapira. The first concerns the relationship between, on one hand, constructibility of sheaves on real vector spaces in the derived category with microsupport restricted to a cone, and on the other hand, stratification of the vector space in a manner compatible with the cone [KS17, Conjecture 3.17].\(^1\) The second concerns the case of

\[^1\]Bibliographic note: this conjecture appears in v3 (the version cited here) and earlier versions of the cited arXiv preprint. It does not appear in the published version [KS18], which is v6 on the arXiv. The published version is cited where it is possible to do so, and v3 [KS17] is cited otherwise.
piecewise linear (PL) objects in this setting, particularly the existence of polyhedrally structured resolutions that, in principle, lend themselves to explicit or algorithmic computation [KS19, Conjecture 3.20]. Both conjectures follow from Theorem 8.22, which is essentially a translation of the relevant real-vector-space special cases of the syzygy theorem for complexes of poset modules (Theorem 7.17) into the language of derived categories of constructible sheaves with conic microsupport or under a conic topology.

The theory in Sections 2–7 was developed simultaneously and independently from that in [KS18, KS19], cf. §2–5 in [Mil17]. Having made the connection between these approaches, it is worth comparing them in more detail.

The syzygy theorem here (Theorem 7.12) and its combinatorial underpinnings in Section 4 hold over arbitrary posets. When the poset is a real vector space, the constructibility encapsulated by topological tameness (Definition 2.12) has no subanalytic, algebraic, or piecewise-linear hypothesis, although these additional structures are preserved by the syzygy theorem transitions. For example, the upper boundary of a downset in the plane with the usual componentwise partial order could be the graph of any continuous weakly decreasing function, among other things, and could be present (i.e., the downset is closed) or absent (i.e., the downset is open), or somewhere in between (e.g., a Cantor set could be missing). The conic topology in [KS18] or [KS19] specializes at the outset to the case of a partially ordered real vector space, and it allows only subanalytic or polyhedral regions, respectively, with upsets having closed lower boundaries and downsets having open upper boundaries. The constructibility in [KS18, KS19] is otherwise essentially the same as tameness here, except that tameness allows constant subdivisions to be finite, whereas constructibility in the derived category allows constant subdivisions to be locally finite. That said, this agreement of constructibility with locally finite tameness that is subanalytic or PL, more or less up to boundary considerations, is visible in [KS17] or [KS19] only via conjectures, namely the ones proved in Section 8 using the general poset methods here.

The theory of primary decomposition in Section 5 requires the poset to be a polyhedral group (Definition 5.7): a partially ordered group whose positive cone has finitely many faces. Polyhedral groups can be integer or real or something in between, but the finiteness is essential for primary decomposition in any of these settings; see Example 5.42. The local finiteness allowed by the usual constructibility in [KS18] does not provide a remedy, although it is possible that the PL hypothesis in [KS19] does. Note that, in either the integer case or the real case, detailed understanding of the topology results in a stronger theory of primary decomposition than over an arbitrary polyhedral group, with much more complete supporting commutative algebra [Mil19a].

Most of the remaining differences between the developments here and those in [KS18, KS19], beyond the types of allowed functions and the shapes of allowed regions, is the behavior allowed on the boundaries of regions. That difference is accounted for by the transition between the conic topology and the Alexandrov topology, the distinction being that the Alexandrov topology has for its open sets all upsets, whereas the conic
topology has only the upsets that are open in the usual topology. This distinction is explored in detail by Berkouk and Petit [BP19]. It is intriguing that ephemeral modules are undetectable metrically [BP19, Theorem 4.22] but their presence here brings indispensable insight into homological behavior in the conic topology.

2. Tame poset modules

2.1. Modules over posets and persistence.

Definition 2.1. Let $Q$ be a partially ordered set (poset) and $\preceq$ its partial order. A module over $Q$ (or a $Q$-module) is

- a $Q$-graded vector space $M = \bigoplus_{q \in Q} M_q$ with
- a homomorphism $M_q \to M_{q'}$ whenever $q \preceq q'$ in $Q$ such that
- $M_q \to M_{q''}$ equals the composite $M_q \to M_{q'} \to M_{q''}$ whenever $q \preceq q' \preceq q''$.

A homomorphism $M \to N$ of $Q$-modules is a degree-preserving linear map, or equivalently a collection of vector space homomorphisms $M_q \to N_q$, that commute with the structure homomorphisms $M_q \to M_{q'}$ and $N_q \to N_{q'}$.

The last bulleted item is commutativity: it reflects that inclusions of subspaces induce functorial maps on homology in the motivating examples of $Q$-modules (Example 2.3).

Remark 2.2. Definition 2.1 is same as the concept of commutative module over a directed acyclic graph [CL18, Definition 2.2]. (The language of posets is equivalent as noted before [CL18, Definition 2.1].)

When the poset $Q$ is locally finite (every interval is finite), a $Q$-module is the same a module over the incidence algebra of $Q$ [DRS72]. But local finiteness, or a similarly restrictive finitary hypothesis on the module [KN09], excludes the basic, motivating examples of real multifiltrations.

Example 2.3. Let $X$ be a topological space and $Q$ a poset.

1. A filtration of $X$ indexed by $Q$ is a choice of subspace $X_q \subseteq X$ for each $q \in Q$ such that $X_q \subseteq X_{q'}$ whenever $q \preceq q'$.
2. The $i^{th}$ persistent homology of the filtered space $X$ is the associated homology module, meaning the $Q$-module $\bigoplus_{q \in Q} H_i X_q$.

Remark 2.4. There are a number of abstract, equivalent ways to phrase Example 2.3. For example, a filtration is a functor from $P$ to the category $\mathcal{S}$ of subspaces of $X$ or an $\mathcal{S}$-valued sheaf on the topological space $P$, where a base for its Alexandrov topology is the set of principal upsets (i.e., principal dual order ideals). For background on and applications of many of these perspectives, see Curry’s dissertation [Cur14], particularly §4.2 there. Some of these perspectives arise in full force in Section 8.

Convention 2.5. The homology here could be taken over an arbitrary ring, but for simplicity it is assumed throughout that homology is taken with coefficients in a field $\mathbb{k}$. 
Example 2.6. A **real multifiltration** of $X$ is a filtration indexed by $\mathbb{R}^n$, with its partial order by coordinatewise comparison. Example 1.1 is a real multifiltration of $X = \mathbb{R}^2$ with $n = 2$. The monoid $\mathbb{R}^+_n \subset \mathbb{R}^n$ of nonnegative real vectors under addition has monoid algebra $k[\mathbb{R}^+_n]$ over the field $k$, a “polynomial” ring whose elements are (finite) linear combinations of monomials $x^a$ with real, nonnegative exponent vectors $a = (a_1, \ldots, a_n) \in \mathbb{R}^+_n$. It contains the usual polynomial ring $k[\mathbb{N}^n]$ as a $k$-subalgebra.

The persistent homology of a real $n$-filtered space $X$ is a **multipersistence module**: an $\mathbb{R}^n$-graded module over $k[\mathbb{R}^+_n]$, which is the same thing as an $\mathbb{R}^n$-module [Les15, §2.1].

2.2. **Constant subdivisions.**

**Definition 2.7.** Fix a $Q$-module $M$. A **constant subdivision** of $Q$ subordinate to $M$ is a partition of $Q$ into constant regions such that for each constant region $I$ there is a single vector space $M_I$ with an isomorphism $M_I \to M_i$ for all $i \in I$ that **has no monodromy**: if $J$ is some (perhaps different) constant region, then all comparable pairs $i \preceq j$ with $i \in I$ and $j \in J$ induce the same composite homomorphism $M_I \to M_i \to M_J$.

**Example 2.8.** Consider the poset module (kindly provided by Ville Puuska [Puu18])

![Diagram](image)

in which the structure morphisms $M_a \to M_b$ are all identity maps on $k$, except for the rightmost one. This example demonstrates that module structures need not be recoverable from their **isotypic subdivision**, in which elements of $Q$ lie in the same region when their vector spaces are isomorphic via a poset relation. In cases like this, refining the isotypic subdivision appropriately yields a constant subdivision. Here, the two minimum elements must lie in different constant regions and the two maximum elements must lie in different constant regions. Any partition accomplishing these separations—that is, any refinement of a partition that has a region consisting of precisely one maximum and one minimum—is a constant subdivision. Of course, a finite poset always admits a constant subdivision with finitely many regions, since the partition into singletons works.

**Example 2.9.** Constant subdivisions need not refine the isotypic subdivision in Example 2.8, one reason being that a single constant region can contain two or more incomparable isotypic regions. For a concrete instance with a single constant region comprised of uncountably many incomparable isotypic regions, let $M$ be the $\mathbb{R}^2$-module that has $M_a = 0$ for all $a \in \mathbb{R}^2$ except for those on the antidiagonal line spanned by $[-1,1] \in \mathbb{R}^2$, where $M_a = k$. There is only one such $\mathbb{R}^2$-module because all of the degrees of nonzero graded pieces of $M$ are incomparable, so all of the structure homomorphisms $M_a \to M_b$ with $a \neq b$ are zero. Every point on the line is a singleton isotypic region.
The direction of the line in Example 2.9 is important: antidiagonal lines, whose points form an antichain in $\mathbb{R}^2$, behave radically differently than diagonal lines.

**Example 2.10.** Let $M$ be an $\mathbb{R}^2$-module with $M_a = k$ whenever $a$ lies in the closed diagonal strip between the lines of slope 1 passing through any pair of points. The structure homomorphisms $M_a \to M_b$ could all be zero, for instance, or some of them could be nonzero. But the length $|a - b|$ of any nonzero such homomorphism must in any case be bounded above by the Manhattan (i.e., $\ell^\infty$) distance between the two points, since every longer structure homomorphism factors through a sequence that exits and re-enters the strip.

In particular, the structure homomorphism between any pair of points on the upper boundary line of the strip is zero because it factors through a homomorphism that points upward first; therefore such pairs of points lie in distinct regions of any constant subdivision. The same conclusion holds for pairs of points on the lower boundary line of the strip. When the strip has width 0, so the upper and lower boundary coincide, the module is supported along a diagonal line whose uncountably many points must all lie in distinct constant regions.

The reference to “no monodromy” in Definition 2.7 agrees with the usual notion.

**Lemma 2.11.** Fix a constant region $I$ subordinate to a poset module $M$. The composite isomorphism $M_I \to M_i \to \cdots \to M_{i'} \to M_I$ is independent of the path from $i$ to $i'$ through $I$, if one exists. In particular, when $i = i'$ the composite is the identity on $M_I$.

**Proof.** The second claim follows from the first. When the path has length 0, the claim is that $M_I \to M_I \to M_I$ is the identity on $M_I$, which follows by definition. For longer paths the result is proved by induction on path length. \qed

Now it is time to introduce the central finiteness concept of the paper.

**Definition 2.12.** Fix a poset $Q$ and a $Q$-module $M$.

1. A constant subdivision of $Q$ is *finite* if it has finitely many constant regions.
2. The $Q$-module $M$ is *$Q$-finite* if its components $M_q$ have finite dimension over $k$.
3. The $Q$-module $M$ is *tame* if it is $Q$-finite and $Q$ admits a finite constant subdivision subordinate to $M$. 
Remark 2.13.
1. The tameness condition here includes but is much less rigid than the compact tameness condition in [SCL+16], the latter meaning more or less that the module is finitely generated over a scalar multiple of $\mathbb{Z}^n$ in $\mathbb{Q}^n$.
2. Some literature calls Definition 2.12 pointwise finite dimensional (PFD). The terminology here agrees with that in [Mil00], on which Section 6 here is based.

Remark 2.14. Data analysis should always produce tame persistence modules. Indeed, from limited experience, in data analysis the isotypic subdivision (Example 2.8) appears consistently to be a finite constant subdivision. It is unclear what kind of data situation might produce a nonconstant isotypic subdivision, but it seems likely that in such cases a constant subdivision can always be obtained by subdividing—into contractible pieces—isotypic components that have nontrivial topology and then gathering incomparable contractible pieces into single constant regions. Exploring these assertions is left open.

Lemma 2.15. Any refinement of a constant subdivision subordinate to a $\mathbb{Q}$-module $M$ is a constant subdivision subordinate to $M$.

Proof. Choosing the same vector space $M_I$ for every region of the refinement contained in the constant region $I$, the lemma is immediate from Definition 2.7.

2.3. Auxiliary hypotheses.
Effectively computing with real multifiltered spaces requires keeping track of the shapes of various regions, such as constant regions. (In later sections, other regions along these lines include upsets, downsets, and fibers of poset morphisms.) The fact that applications of persistent homology often arise from metric considerations, which are semi-algebraic in nature, or are approximated by piecewise linear structures, suggests the following auxiliary hypotheses for algorithmic developments. The subanalytic hypothesis is singled out for the theoretical purposes surrounding conjectures of Kashiwara and Schapira in Section 8.

Definition 2.16. Fix a subposet $Q$ of a partially ordered real vector space of finite dimension (see Definition 5.1, or take $Q = \mathbb{R}^n$ for now). A partition of $Q$ into subsets is
1. semi-algebraic if the subsets are real semi-algebraic varieties;
2. piecewise linear (PL) if the subsets are finite unions of convex polyhedra, where a convex polyhedron is an intersection of finitely many closed or open half-spaces;
3. subanalytic if the subsets are subanalytic varieties;
4. of class $\mathcal{X}$ if the subsets lie in a family $\mathcal{X}$ of subsets of $Q$ that is closed under complements, finite intersections, negation, and Minkowski sum with the positive cone $Q_+$, namely the set of vectors $q \in Q$ such that $0 \preceq q$.

A module over $Q$ is semi-algebraic, or PL, subanalytic, or of class $\mathcal{X}$ if it is tamed by a subordinate finite constant subdivision of the corresponding type.
Remark 2.17. Subposets of partially ordered real vector spaces are allowed in Definition 2.16 to be able to speak of, for example, piecewise linear sets in rational vector spaces, or semialgebraic subsets of \( \mathbb{Z}^n \) (see Example 5.42 for an instance of the latter). When \( Q \) is properly contained in the ambient real vector space, subsets of \( Q \) are semialgebraic, PL, or subanalytic when they are intersections with \( Q \) of the corresponding type of subset of the ambient real vector space.

Proposition 2.18. Fix a partially ordered real vector space \( Q \).

1. The classes of semialgebraic, PL, and subanalytic subsets of \( Q \) are each closed under complements, finite intersections, and negation.
2. The Minkowski sum \( S + Q_+ \) of a semialgebraic set \( S \) with the positive cone is semialgebraic if \( Q_+ \) is semialgebraic.
3. The Minkowski sum \( S + Q_+ \) of a PL set with the positive cone is semialgebraic if \( Q_+ \) is polyhedral.
4. The Minkowski sum \( S + Q_+ \) of a bounded subanalytic set \( S \) with the positive cone is subanalytic if \( Q_+ \) is subanalytic.

Proof. See [Shi97] (for example) to treat the semialgebraic and subanalytic cases of item 1. The PL case reduces easily to checking that the complement of a single polyhedron is PL, which in turn follows because a real vector space is the union of the (relatively open) faces in any finite hyperplane arrangement, so removing a single one of these faces leaves a PL set remaining.

For item 2, use that the image of a semialgebraic set under linear projection is a semialgebraic set, and then express \( S + Q_+ \) as the image of \( S \times Q_+ \) under the projection \( Q \times Q \to Q \) that acts by \( (q, q') \mapsto q + q' \). The same argument works for item 3. The same argument also works for item 4 but requires that the restriction of the projection to the closure of \( S \times Q_+ \) be a proper map, which always occurs when \( S \) is bounded. □

3. Fringe presentation by upsets and downsets

Presentations are the core structures of computational ring theory. However, the default notions of free or projective presentation—as well as more generally and dually the notions of flat or injective presentation—are too restrictive for modules over arbitrary posets. These notions all still make formal sense, but it is too much to ask for presentations that are finite direct sums of indecomposables in such generality, as demonstrated by the formidable infinitude of such objects in the case of \( \mathbb{R}^n \)-modules like fly wing modules (see Example 1.2). The idea here, both for theoretical and computational purposes, is to allow arbitrary upset and downset modules instead of only flat and injective ones. The use of indicator modules instead of free or injective modules gathers the generators and cogenators, known as births and deaths in persistent homology, into finitely many groups (Definition 3.16), paving the way for effective data structures in the form of monomial matrices (Definition 3.17).
3.1. Upsets and downsets.

**Definition 3.1.** The vector space $k[Q] = \bigoplus_{q \in Q} k$ that assigns $k$ to every point of the poset $Q$ is a $Q$-module with identity maps on $k$. More generally,

1. an **upset** (also called a dual order ideal) $U \subseteq Q$, meaning a subset closed under going upward in $Q$ (so $U + \mathbb{R}_+^n = U$, when $Q = \mathbb{R}^n$) determines an indicator submodule or upset module $k[U] \subseteq k[Q]$; and
2. dually, a **downset** (also called an order ideal) $D \subseteq Q$, meaning a subset closed under going downward in $Q$ (so $D - \mathbb{R}_+^n = D$, when $Q = \mathbb{R}^n$) determines an indicator quotient module or downset module $k[Q] \rightarrow k[D]$.

When $Q$ is a subposet of a partially ordered real vector space, an indicator module of either sort is semialgebraic, PL, subanalytic, or of class $\mathcal{X}$ if the corresponding upset or downset is of the same type (Definition 2.16).

**Remark 3.2.** Indicator submodules $k[U]$ and quotient modules $k[D]$ are $Q$-modules, not merely $U$-modules or $D$-modules, by setting the graded components indexed by elements outside of the ideals to 0. It is only by viewing indicator modules as $Q$-modules that they are forced to be submodules or quotients, respectively. For relations between these notions and those in Remark 2.4, again see Curry’s thesis [Cur14]. For example, upsets form the open sets in the topology from Remark 2.4.

**Example 3.3.** Ising crystals at zero temperature, with polygonal boundary conditions and fixed mesh size, are semialgebraic upsets in $\mathbb{R}^n$. That much is by definition: fixing a mesh size means that the crystals in question are (staircase surfaces of finitely generated) monomial ideals in $n$ variables. Remarkably, such crystals remain semialgebraic in the limit of zero mesh size; see [Oko16] for an exposition and references.

**Example 3.4.** Monomial ideals in polynomial rings with real exponents, which correspond to upsets in $\mathbb{R}_+^n$, are considered in [ASW15], including aspects of primality, irreducible decomposition, and Krull dimension. Upsets in $\mathbb{R}^n$ are also considered in [MMc15], where the combinatorics of their lower boundaries, and topology of related simplicial complexes, are investigated in cases with locally finite generating sets.

**Definition 3.5.** A poset $Q$ is

1. connected if every pair of elements $q,q' \in Q$ is joined by a path in $Q$: a sequence $q = q_0 \preceq q_0' \preceq q_1 \preceq q_1' \preceq \cdots \preceq q_k \preceq q_k' = q'$ in $Q$;
2. upper-connected if every pair of elements in $Q$ has an upper bound in $Q$;
3. lower-connected if every pair of elements in $Q$ has a lower bound in $Q$; and
4. strongly connected if $Q$ is upper-connected and lower-connected.

**Example 3.6.** $\mathbb{R}^n$ is strongly connected. The same is true of any partially ordered abelian group (see Section 5.1 for basic theory of those posets).
Example 3.7. A poset $Q$ is upper-connected if (but not only if, cf. Example 3.6) it has a maximum element—one that is preceded by every element of $Q$. Similarly, $Q$ is lower-connected if it has a minimum element—one that precedes every element of $Q$.

Remark 3.8. The relation $q \sim q'$ defined by the existence of a path joining $q$ to $q'$ as in Definition 3.5.1 is an equivalence relation.

Definition 3.9. Fix a poset $Q$. For any subset $S \subseteq Q$, write $\pi_0 S$ for the set of connected components of $S$: the maximal connected subsets of $S$, or equivalently the classes under the relation from Remark 3.8.

Proposition 3.10. Fix a poset $Q$.

1. For an upset $U$ and a downset $D$,
   \[ \text{Hom}_Q(\mathbb{k}[U], \mathbb{k}[D]) = \mathbb{k}^{\pi_0(U \cap D)}, \]
   a product of copies of $\mathbb{k}$, one for each connected component of $U \cap D$.

2. For upsets $U$ and $U'$,
   \[ \text{Hom}_Q(\mathbb{k}[U'], \mathbb{k}[U]) = \mathbb{k}^{\{S \subseteq \pi_0 U' | S \subseteq U\}}, \]
   a product of copies of $\mathbb{k}$, one for each connected component of $U'$ contained in $U$.

3. For downsets $D$ and $D'$,
   \[ \text{Hom}_Q(\mathbb{k}[D], \mathbb{k}[D']) = \mathbb{k}^{\{S \subseteq \pi_0 D' | S \subseteq D\}}, \]
   a product of copies of $\mathbb{k}$, one for each connected component of $D'$ contained in $D$.

Proof. For the first claim, the action $\varphi_q$ of $\varphi : \mathbb{k}[U] \to \mathbb{k}[D]$ on the copy of $\mathbb{k}$ in any degree $q \in U \cap D$ is 0 because $\mathbb{k}[D]_q = 0$, so assume $q \in U \cap D$. Then $\varphi_q = \varphi_{q'} : \mathbb{k} \to \mathbb{k}$ if $q \preceq q' \in U \cap D$ because $\mathbb{k}[U]_q \to \mathbb{k}[U]_{q'}$ and $\mathbb{k}[D]_q \to \mathbb{k}[D]_{q'}$ are identity maps on $\mathbb{k}$. Similarly, $\varphi_q = \varphi_{q'}$ if $q \geq q' \in U \cap D$. Induction on the length of the path in Definition 3.5.1 shows that $\varphi_q = \varphi_{q'}$ if $q$ and $q'$ lie in the same connected component of $U \cap D$. Thus $\text{Hom}_Q(\mathbb{k}[U], \mathbb{k}[D]) \subseteq \mathbb{k}^{\pi_0(U \cap D)}$. On the other hand, specifying for each component $S \in \pi_0(U \cap D)$ a scalar $\alpha_S \in \mathbb{k}$ yields a homomorphism $\varphi : \mathbb{k}[U] \to \mathbb{k}[D]$, if $\varphi_q$ is defined to be multiplication by $\alpha_S$ on the copies of $\mathbb{k} = \mathbb{k}[U]_q$ indexed by $q \in S$ and 0 for $q \in U \cap D$; that $\varphi$ is indeed a $\mathbb{Q}$-module homomorphism follows because $\mathbb{k}[D]_{q'} = 0$ (that is, $q' \notin D$) whenever $q' \succeq q \in D$ but $q'$ does not lie in the connected component of $U \cap D$ containing $q$. Said another way, pairs of elements of $U \cap D$ either lie in the same connected component of $U \cap D$ or they are incomparable.

The proofs of the last two claims are similar (and dual to one another), particularly when it comes to showing that a homomorphism of indicator modules of the same type—that is, source and target both upset or both downset—is constant on the relevant connected components. The only point not already covered is that if $U'$ is a connected upset and $U' \not\subseteq U$ then every homomorphism $\mathbb{k}[U'] \to \mathbb{k}[U]$ is 0 because $q' \in U' \setminus U$ implies $\mathbb{k}[U']_{q'} \to 0 = \mathbb{k}[U]_{q'}$. \qed
The cases of interest in this paper and its sequels [Mil19a, Mil19b], particularly real and discrete polyhedral partially ordered groups (Example 5.3, Example 5.2, and Definition 5.7) such as $\mathbb{R}^n$ and $\mathbb{Z}^n$, have strong connectivity properties, thereby simplifying the conclusions of Proposition 3.10. First, here is a convenient notation.

**Corollary 3.11.** Fix a poset $Q$ with upsets $U, U'$ and downsets $D, D'$.

1. $\text{Hom}_Q(\mathbb{k}[U], \mathbb{k}[D]) = \mathbb{k}$ if $U \cap D \neq \emptyset$ and either $U$ is lower-connected as a subposet of $Q$ or $D$ is upper-connected as a subposet of $Q$.

2. If $U$ and $U'$ are upsets and $Q$ is upper-connected, then $\text{Hom}_Q(\mathbb{k}[U'], \mathbb{k}[U]) = \mathbb{k}$ if $U' \subseteq U$ and 0 otherwise.

3. If $D$ and $D'$ are downsets and $Q$ is lower-connected, then $\text{Hom}_Q(\mathbb{k}[D], \mathbb{k}[D']) = \mathbb{k}$ if $D \supseteq D'$ and 0 otherwise. \[\square\]

**Example 3.12.** Consider the poset $\mathbb{N}^2$, the upset $U = \mathbb{N}^2 \setminus \{0\}$, and the downset $D$ consisting of the origin and the two standard basis vectors. Then $\mathbb{k}[U] = \mathbb{m} = \langle x, y \rangle$ is the graded maximal ideal of $\mathbb{k}[\mathbb{N}^2] = \mathbb{k}[x, y]$ and $\mathbb{k}[D] = \mathbb{k}[\mathbb{N}^2]/\mathbb{m}^2$. Now calculate

$$\text{Hom}_{\mathbb{N}^2}(\mathbb{k}[U], \mathbb{k}[D]) = \text{Hom}_{\mathbb{N}^2}(\mathbb{m}, \mathbb{k}[\mathbb{N}^2]/\mathbb{m}^2) = \mathbb{k}^2,$$

a vector space of dimension 2: one basis vector preserves the monomial $x$ while killing the monomial $y$, and the other basis vector preserves $y$ while killing $x$.

**Example 3.13.** For an extreme case, consider the poset $Q = \mathbb{R}^2$ with $U$ the closed half-plane above the antidiagonal line $y = -x$ and $D = -U$, so that $U \cap D$ is totally disconnected: $\pi_0(U \cap D) = U \cap D$. Then $\text{Hom}_Q(\mathbb{k}[U], \mathbb{k}[D]) = \mathbb{k}^\mathbb{R}$ is a vector space of beyond continuum dimension, the copy of $\mathbb{R}$ in the exponent being the antidiagonal line.

The proliferation of homomorphisms in Examples 3.12 and 3.13 is undesirable for both computational and theoretical purposes; it motivates the following concept.

**Definition 3.14.** Let each of $S$ and $S'$ be a nonempty intersection of an upset in a poset $Q$ with a downset in $Q$, so $\mathbb{k}[S]$ and $\mathbb{k}[S']$ are subquotients of $\mathbb{k}[Q]$. A homomorphism $\varphi : \mathbb{k}[S] \to \mathbb{k}[S']$ is **connected** if there is a scalar $\lambda \in \mathbb{k}$ such that $\varphi$ acts as multiplication by $\lambda$ on the copy of $\mathbb{k}$ in degree $q$ for all $q \in S \cap S'$.

The cases of interest in the rest of this paper concern three situations: both $S$ and $S'$ are upsets, or both are downsets, or $S = U$ is an upset and $S' = D$ is downset with $U \cap D \neq \emptyset$. However, the full generality of Definition 3.14 is required in the sequel to this work [Mil19a].

**Remark 3.15.** Corollary 3.11 says that homomorphisms among indicator modules are automatically connected in the presence of appropriate upper- or lower-connectedness.
3.2. Fringe presentations.

**Definition 3.16.** Fix any poset $Q$. A fringe presentation of a $Q$-module $M$ is

- a direct sum $F$ of upset modules $\mathbb{k}[U]$,
- a direct sum $E$ of downset modules $\mathbb{k}[D]$, and
- a homomorphism $F \to E$ of $Q$-modules with
  - image isomorphic to $M$ and
  - components $\mathbb{k}[U] \to \mathbb{k}[D]$ that are connected (Definition 3.14).

A fringe presentation

1. is *finite* if the direct sums are finite;
2. *dominates* a constant subdivision of $M$ if the subdivision is subordinate to each summand $\mathbb{k}[U]$ of $F$ and $\mathbb{k}[D]$ of $E$; and
3. is *semialgebraic, PL, subanalytic, or of class $X$* if $Q$ is a subposet of a partially ordered real vector space of finite dimension and the fringe presentation dominates a constant subdivision of the corresponding type (Definition 2.16).

Fringe presentations are effective data structures via the following notational trick. Topologically, it highlights that births occur along the lower boundaries of the upsets and deaths occur along the upper boundaries of the downsets, with a linear map over the ground field to relate them.

**Definition 3.17.** Fix a finite fringe presentation $\varphi : \bigoplus_p \mathbb{k}[U_p] = F \to E = \bigoplus_q \mathbb{k}[D_q]$. A monomial matrix for $\varphi$ is an array of scalar entries $\varphi_{pq}$ whose columns are labeled by the birth upsets $U_p$ and whose rows are labeled by the death downsets $D_q$:

$$
\begin{bmatrix}
D_1 & \cdots & D_l \\
U_1 & \varphi_{11} & \cdots & \varphi_{1l} \\
\vdots & \vdots & \ddots & \vdots \\
U_k & \varphi_{k1} & \cdots & \varphi_{kl}
\end{bmatrix}
$$

$$
\mathbb{k}[U_1] \oplus \cdots \oplus \mathbb{k}[U_k] \xrightarrow{\varphi} E = \mathbb{k}[D_1] \oplus \cdots \oplus \mathbb{k}[D_l].
$$

**Proposition 3.18.** With notation as in Definition 3.17, $\varphi_{pq} = 0$ unless $U_p \cap D_q \neq \emptyset$. Conversely, if an array of scalars $\varphi_{pq} \in \mathbb{k}$ with rows labeled by upsets and columns labeled by downsets has $\varphi_{pq} = 0$ unless $U_p \cap D_q \neq \emptyset$, then it represents a fringe presentation.

**Proof.** Proposition 3.10.1 and Definition 3.14.

**Example 3.19.** Fringe presentation in one parameter reflects the usual matching between left endpoints and right endpoints of a module, once it has been decomposed as
a direct sum of bars. A single bar, say an interval \([a, b]\) that is closed on the left and open on the right, has fringe presentation

\[
\begin{array}{c}
\ldots \\
\downarrow \\
\ldots
\end{array}
\]

with image

\[
\begin{array}{c}
a \\
\circ \\
\ldots \\
b
\end{array}
\]

in which a subset \(S \subseteq \mathbb{R}\) is drawn instead of writing \(k[S]\). With multiple bars, the bijection from left to right endpoints yields a monomial matrix whose scalar entries form the identity, with rows labeled by positive rays having the specified left endpoints (the ray is the whole real line when the left endpoint is \(-\infty\)) and columns labeled by negative rays having the corresponding right endpoints (again, the whole line when the right endpoint is \(+\infty\)). In practical terms, the rows and columns can be labeled simply by the endpoints themselves, with (say) a bar over a closed endpoint and a circle over an open one. Thus a standard bar code, in monomial matrix notation, has the form

\[
\begin{pmatrix}
\overline{a}_1 & \ldots & \overline{a}_k \\
\vdots & \ddots & \vdots \\
\overline{a}_k & & 1
\end{pmatrix}
\]

**Example 3.20.** Although there are many opinions about what the multiparameter analogue of a bar code should be, the analogue of a single bar is generally accepted to be some kind of interval in the parameter poset—that is, \(k[U \cap D]\), where \(U\) is an upset and \(D\) is a downset—sometimes with restrictions on the shape of the interval, depending on context. This case of a single bar explains the terminology “birth upset” and “death downset”. For instance, a fringe presentation of the yellow interval

\[
\begin{array}{c}
\rightarrow \\
\rightarrow
\end{array}
\]

locates the births along the lower boundary of the blue upset and the deaths along the upper boundary of the red downset. The scalar entries relate the births to the deaths. In this special case of one bar, the monomial matrix is \(1 \times 1\) with a single nonzero scalar entry; choosing bases appropriately, this nonzero entry might as well be 1.
Example 3.21. Consider an $\mathbb{N}^2$-filtration of the following simplicial complex.

Each simplex is present above the correspondingly colored rectangular curve in the following diagram, which theoretically should extend infinitely far up and to the right.

Each little square depicts the simplicial complex that is present at the parameter occupying its lower-left corner. Taking zeroth homology yields an $\mathbb{N}^2$-module that replaces the simplicial complex in each box with the vector space spanned by its connected components. A fringe presentation for this $\mathbb{N}^2$-module is

\[
\begin{pmatrix}
1 & 0 & 1 \\
-1 & 1 & 1 \\
0 & -1 & 1
\end{pmatrix},
\]

where the grey square atop the third column represents the downset that is all of $\mathbb{N}^2$. This fringe presentation means that, for example, the connected component that is the blue endpoint of the simplicial complex is born along the union of the axes with the origin removed but the point $[1]$ appended. The purple downset, corresponding to the left edge, records the death—along the upper purple boundary—of the homology class represented by the difference of the blue (left) and gold (middle) vertices. Computations and figures for this example were kindly provided by Ashleigh Thomas.
Remark 3.22. The term “fringe” is a portmanteau of “free” and “injective” (that is, “frinj”), the point being that it combines aspects of free and injective resolutions while also conveying that the data structure captures trailing topological features at both the birth and death ends.

4. Encoding poset modules

Sections 2 and 3 introduce two finiteness conditions: a topological one (tameness, Definition 2.12), which is the intuitive control of topological variation in a filtration, and an algebraic one (fringe presentation, Definition 3.16), which provides effective data structures. To interpolate between them, a third finiteness condition, this one combinatorial in nature (finite encoding, Definition 4.1), serves as a theoretical tool whose functorial essence supports much of the development in this paper.

4.1. Finite encoding.
The main result of Section 4, namely Theorem 4.22, says that tame $Q$-modules can be encoded in the following manner.

Definition 4.1. Fix a poset $Q$. An encoding of a $Q$-module $M$ by a poset $P$ is a poset morphism $\pi : Q \to P$ together with a $P$-module $H$ such that $M \cong \pi^* H = \bigoplus_{q \in Q} H_{\pi(q)}$, the pullback of $H$ along $\pi$, which is naturally a $Q$-module. The encoding is finite if

1. the poset $P$ is finite, and
2. the vector space $H_p$ has finite dimension for all $p \in P$.

Example 4.2. Example 1.2 shows a constant isotypic subdivision of $\mathbb{R}^2$ which happens to form a poset and therefore produces an encoding.

Example 4.3. A finite encoding of the module in Example 3.21 is as follows.
Example 4.4. There is no natural way to impose a poset structure on the set of regions in a constant subdivision. Take, for example, $Q = \mathbb{R}^2$ and $M = k_0 \oplus k[\mathbb{R}^2]$, where $k_0$ is the $\mathbb{R}^2$-module whose only nonzero component is at the origin, where it is a vector space of dimension 1. This module $M$ induces only two isotypic regions, namely the origin and its complement, and they constitute a constant subdivision.

Neither of the two regions has a stronger claim to precede the other, but at the same time it would be difficult to justify forcing the regions to be incomparable.

Example 4.5. Take $Q = \mathbb{Z}^n$ and $P = \mathbb{N}^n$. The convex projection $\mathbb{Z}^n \to \mathbb{N}^n$ sets to 0 every negative coordinate. The pullback under convex projection is the Čech hull [Mil00, Definition 2.7]. More generally, suppose $a \preceq b$ in $\mathbb{Z}^n$. The interval $[a, b] \subseteq \mathbb{Z}^n$ is a box (rectangular parallelepiped) with lower corner at $a$ and upper corner at $b$. The convex projection $\pi : \mathbb{Z}^n \to [a, b]$ takes every point in $\mathbb{Z}^n$ to its closest point in the box. A $\mathbb{Z}^n$-module is finitely determined if it is finitely encoded by $\pi$.

Example 4.6. The indicator module $k(Q)$ is encoded by the morphism from $Q$ to the one-point poset with vector space $H = k$. This generalizes to other indicator modules.

1. Any upset module $k[U] \subseteq k(Q)$ is encoded by a morphism from $Q$ to the chain $P$ of length 1, consisting of two points $0 < 1$, that sends $U$ to 1 and the complement $\overline{U}$ to 0. The $P$-module $H$ that pulls back to $k[U]$ has $H_0 = 0$ and $H_1 = k$.
2. Dually, any downset module $k[D]$ is also encoded by a morphism from $Q$ to the chain $P$ of length 1, but this one sends $D$ to 0 and the complement $\overline{D}$ to 1, and the $P$-module $H$ that pulls back to $k[D]$ has $H_0 = k$ and $H_1 = 0$.

Definition 4.7. Fix a poset $Q$ and a $Q$-module $M$.

1. A poset morphism $\pi : Q \to P$ or an encoding of a $Q$-module (perhaps different from $M$) is subordinate to $M$ if there is a $P$-module $H$ such that $M \cong \pi^* H$.
2. When $Q$ is a subposet of a partially ordered real vector space, an encoding of $M$ is semialgebraic, PL, subanalytic, or of class $\mathfrak{X}$ if the partition of $Q$ formed by the fibers of $\pi$ is of the corresponding type (Definition 2.16).

Example 4.8. The “antidiagonal” $\mathbb{R}^2$-module $M$ in Example 2.9 has a semialgebraic poset encoding by the chain with three elements, where the fiber over the middle element is the antidiagonal line, and the fibers over the top and bottom elements are the open half-spaces above and below the line, respectively. In contrast, using the diagonal line spanned by $[1 \ 1] \in \mathbb{R}^2$ instead of the antidiagonal line yields a module with no finite encoding; see Example 2.10.

Lemma 4.9. An indicator module is constant on every fiber of a poset morphism $\pi : Q \to P$ if and only if the module is the pullback along $\pi$ of an indicator $P$-module.
Proof. The “if” direction is by definition. For the “only if” direction, observe that if $U \subseteq Q$ is an upset that is a union of fibers of $P$, then the image $\pi(U) \subseteq P$ is an upset whose preimage equals $U$. The same argument works for downsets. 

Example 4.10 (Pullbacks of flat and injective modules). An indecomposable flat $\mathbb{Z}^n$-module $k[b + \mathbb{Z} \tau + \mathbb{N}^n]$ is an upset module for the poset $\mathbb{Z}^n$. Pulling back to any poset under a poset map to $\mathbb{Z}^n$ therefore yields an upset module for the given poset. The dual statement holds for any indecomposable injective module $k[b + \mathbb{Z} \tau - \mathbb{N}^n]$: its pullback is a downset module.

Pullbacks have particularly transparent monomial matrix interpretations.

Proposition 4.11. Fix a poset $Q$ and an encoding of a $Q$-module $M$ via a poset morphism $\pi : Q \to P$ and $P$-module $H$. Any monomial matrix for a fringe presentation of $H$ pulls back to a monomial matrix for a fringe presentation that dominates the encoding by replacing the row labels $U_1, \ldots, U_k$ and column labels $D_1, \ldots, D_\ell$ with their preimages, namely $\pi^{-1}(U_1), \ldots, \pi^{-1}(U_k)$ and $\pi^{-1}(D_1), \ldots, \pi^{-1}(D_\ell)$. 

4.2. Uptight posets.
Constructing encodings from constant subdivisions uses general poset combinatorics.

Definition 4.12. Fix a poset $Q$ and a set $\Upsilon$ of upsets. For each poset element $a \in Q$, let $\Upsilon_a \subseteq \Upsilon$ be the set of upsets from $\Upsilon$ that contain $a$. Two poset elements $a, b \in Q$ lie in the same uptight region if $\Upsilon_a = \Upsilon_b$.

Remark 4.13. The partition of $Q$ into uptight regions in Definition 4.12 is the common refinement of the partitions $Q = U \cup (Q \setminus U)$ for $U \in \Upsilon$.

Remark 4.14. Every uptight region is the intersection of a single upset (not necessarily one of the ones in $\Upsilon$) with a single downset. Indeed, the intersection of any family of upsets is an upset, the complement of an upset is a downset, and the intersection of any family of downsets is a downset. Hence the uptight region containing $a$ equals $(\bigcap_{U \in \Upsilon_a} U) \cap (\bigcap_{U \notin \Upsilon_a} \overline{U})$, the first intersection being an upset and the second a downset.

Proposition 4.15. In the situation of Definition 4.12, the relation on uptight regions given by $A \preceq B$ whenever $a \preceq b$ for some $a \in A$ and $b \in B$ is reflexive and acyclic.

Proof. The stipulated relation on the set of uptight regions is

- reflexive because $a \preceq a$ for any element $a$ in any uptight region $A$; and
- acyclic because going up from $a \in Q$ causes the set $\Upsilon_a$ in Definition 4.12 to (weakly) increase, so a directed cycle can only occur with a constant sequence of sets $\Upsilon_a$. 

Example 4.16. The relation in Proposition 4.15 makes the set of uptight regions into a directed acyclic graph, but the relation need not be transitive. An example in the poset $Q = \mathbb{N}^2$, kindly provided by Ville Puuska [Puu18], is as follows. Notationally, it is easier to work with monomial ideals in $k[x, y] = k[\mathbb{N}^2]$, which correspond to upsets (see Example 3.12). Let $\Upsilon = \{U_1, \ldots, U_4\}$ consist of the upsets with indicator modules
\[ k[U_1] = \langle x^2, y \rangle, \quad k[U_2] = \langle x^3, y \rangle, \quad k[U_3] = \langle xy \rangle, \quad k[U_4] = \langle x^2y \rangle. \]
Identifying each monomial $x^a y^b$ with the corresponding pair $(a, b) \in \mathbb{N}^2$, it follows that $\Upsilon_{x^2} = \{U_1\}$, and $\Upsilon_{x^3} = \Upsilon_y = \{U_1, U_2\}$, and $\Upsilon_{xy} = \{U_1, U_2, U_3\}$ represent three distinct uptight regions; call them $A, B,$ and $C$. They satisfy $A \prec B \prec C$ because $x^2 \prec x^3$ and $y \prec xy$. However, $A \not\prec C$ because $A = \{x^2\}$ while $U_4$ forces $C = xyk[y]$ to consist of all lattice points in a vertical ray starting at $xy$.

Definition 4.17. In the situation of Definition 4.12, the uptight poset is the transitive closure $P_\Upsilon$ of the directed acyclic graph of uptight regions in Proposition 4.15.

4.3. Constant upsets.

Definition 4.18. Fix a constant subdivision of $Q$ subordinate to $M$. A constant upset of $Q$ is either
1. an upset $U_I$ generated by a constant region $I$ or
2. the complement of a downset $D_I$ cogenerated by a constant region $I$.

Theorem 4.19. Let $\Upsilon$ be the set of constant upsets from a constant subdivision of $Q$ subordinate to $M$. The partition of $Q$ into uptight regions for $\Upsilon$ forms another constant subdivision subordinate to $M$.

Proof. Suppose that $A$ is an uptight region that contains points from constant regions $I$ and $J$. Any point in $I \cap A$ witnesses the containments $A \subseteq D_I$ and $A \subseteq U_I$ of $A$ inside the constant upset and downset generated and cogenerated by $I$. Any point $j \in J \cap A$ is therefore sandwiched between elements $i, i' \in I$, so $i \preceq j \preceq i'$, because $j \in U_I$ (for $i$) and $j \in D_I$ (for $i'$). By symmetry, switching $I$ and $J$, there exists $j' \in J$ with $i' \preceq j'$. The sequence
\[ M_I \rightarrow M_i \rightarrow M_j \rightarrow M_I \rightarrow M_J \rightarrow M_I, \]
where the first and last isomorphisms come from Definition 2.7 and the homomorphisms in between are $Q$-module structure homomorphisms, induces isomorphisms $M_I \rightarrow M_V$ and $M_J \rightarrow M_J$ by definition of constant region. Elementary homological algebra implies that $M_I \rightarrow M_J$ is an isomorphism. The induced isomorphism $M_I \rightarrow M_J$ is independent of the choices of $i, j, i'$, and $j'$ (in fact, merely considering independence of the choices of $i$ and $j'$ would suffice) because constant subdivisions have no monodromy.

The previous paragraph need not imply that $I = J$, but it does imply that all of the vector spaces $M_J$ for constant regions $J$ that intersect $A$ are—viewing the data of the original constant subdivision as given—canonically isomorphic to $M_I$, thereby allowing
the choice of $M_A = M_I$. This, plus the lack of monodromy in constant subdivisions, ensures that $M_A \to M_a \to M_b \to M_B$ is independent of the choices of $a \in A$ and $b \in B$ with $a \leq b$. Thus the uptight subdivision is also constant subordinate to $M$. □

Example 4.20. Theorem 4.19 does not claim that $I = U_I \cap D_I$, and in fact that claim is often not true, even if the isotypic subdivision (Example 2.8) is already constant. Consider $Q = \mathbb{R}^2$ and $M = \mathbb{k}_0 \oplus \mathbb{k}[\mathbb{R}^2]$, as in Example 4.4, and take $I = \mathbb{R}^2 \setminus \{0\}$. Then $U_I = D_I = \mathbb{R}^2$, so $U_I \cap D_I$ contains the other isotypic region $J = \{0\}$. The uptight poset $P_M$ has precisely four elements:
1. the origin $\{0\} = U_I \cap D_I$;
2. the complement $U_J \setminus \{0\}$ of the origin in $U_J$;
3. the complement $D_J \setminus \{0\}$ of the origin in $D_J$; and
4. the points $\mathbb{R}^2 \setminus (U_J \cup D_J)$ lying only in $I$ and in neither $U_I$ nor $D_I$.
Oddly, uptight region 4 has two connected components, the second and fourth quadrants $A$ and $B$, that are incomparable: any chain of relations from Definition 2.7 that realizes the equivalence $a \sim b$ for $a \in A$ and $b \in B$ must pass through the positive quadrant or the negative quadrant, each of which accidentally becomes comparable to the other isotypic region $J$ and hence lies in a different uptight region.

4.4. Finite encoding from constant subdivisions.

Definition 4.21. If $Q$ is a subposet of a partially ordered real vector space, then a $Q$-module $M$ has compact support if $M$ has nonzero components $M_q$ only in a bounded set of degrees $q \in Q$. A constant subdivision subordinate to such a module is compact if it has exactly one unbounded constant region (namely those $q \in Q$ for which $M_q = 0$).

Theorem 4.22. Fix a $Q$-finite module $M$ over a poset $Q$.

1. $M$ admits a finite encoding if and only if there exists a finite constant subdivision of $Q$ subordinate to $M$. More precisely,
2. the uptight poset of the set of constant upsets from any constant subdivision yields an uptight encoding of $M$ that is finite if the constant subdivision is finite.
3. If $Q$ is a subposet of a partially ordered real vector space and the constant subdivision in the previous item is
   • semialgebraic, with $Q_+$ also semialgebraic; or
   • PL, with $Q_+$ also polyhedral; or
   • compact and subanalytic, with $Q_+$ also subanalytic; or
   • of class $X$,
then the relevant uptight encoding is semialgebraic, PL, subanalytic, or class $X$.

Proof. One direction of item 1 is easy: a finite encoding induces a constant subdivision almost by definition. Indeed, if $\pi : Q \to P$ is a poset encoding of $M$ by a $P$-module $H$, then each fiber $I$ of $\pi$ is a constant region with $M_I = H_{\pi(I)}$. If $i \leq j$ with $i \in I$ and $j \in J$, then the composite homomorphism $M_I \to M_i \to M_j \to M_J$ is merely the structure morphism $H_{\pi(I)} \to H_{\pi(J)}$ of the $P$-module $H$.
The hard direction is producing a finite encoding from a constant subdivision. For that, it suffices to prove item 2. Let \( \Upsilon \) be the set of constant upsets from a constant subdivision of \( Q \) subordinate to \( M \). Consider the quotient map \( Q \to P_\Upsilon \) of sets that sends each element of \( Q \) to the uptight region containing it. Proposition 4.15 and Definition 4.17 imply that this map of sets is a morphism of posets. By Definition 2.7 the vector spaces \( M_A \) indexed by the uptight regions \( A \in P_\Upsilon \) constitute a \( P_\Upsilon \)-module \( H \) that is well defined by Theorem 4.19. The pullback of \( H \) to \( Q \) is isomorphic to \( M \) by construction. The claim about finiteness follows because the number of uptight regions is bounded above by \( 2^{2r} \), where \( r \) is the number of constant regions in the original constant subdivision: every element of \( Q \) lies inside or outside of each constant upset and inside or outside of each constant downset.

For claim 3, every constant upset is a Minkowski sum \( I + Q_+ \) or the complement of \( I - Q_+ = -(-I + Q_+) \) by Definition 4.18. These are semialgebraic, PL, subanalytic, or of class \( \mathfrak{X} \), respectively, by Proposition 2.18 (or Definition 2.16 for class \( \mathfrak{X} \)). Note that in the compact subanalytic case, the unique unbounded constant region \( I \) afforded by Definition 4.21 has \( I + Q_+ = I - Q_+ = Q \), which is subanalytic. \( \square \)

**Example 4.23.** For the “antidiagonal” \( \mathbb{R}^2 \)-module \( M \) in Examples 2.9 and 4.8, every point on the line is a singleton isotypic region, but these uncountably many isotypic regions can be gathered together: the finite encoding there is the uptight poset for the two upsets that are the closed and open half-spaces bounded below by the antidiagonal.

**Example 4.24.** In any encoding of the “diagonal strip” \( \mathbb{R}^2 \)-module \( M \) in Example 4.4, the poset must be uncountable by Theorem 4.22.

### 4.5. The category of tame modules.

**Example 4.25.** The kernel of a homomorphism of tame modules need not be tame. The upset \( U \subset \mathbb{R}^2 \) that is the closed half-space above the antidiagonal line \( L \) given by \( a + b = 1 \) has interior \( U^\circ \), also an upset. The quotient module \( N = \mathbb{k}[U]/\mathbb{k}[U^\circ] \) is the translate by one unit (up or to the right) of the antidiagonal module in Examples 2.9, 4.8, and 4.23. Both \( M = \mathbb{k}[U] \oplus \mathbb{k}[U] \) and \( N \) are tame. The surjection \( \varphi : M \to N \) that acts in every degree \( a = \begin{bmatrix} a \\ b \end{bmatrix} \) along \( L \) by sending the basis vectors of \( M_a = \mathbb{k}^2 \) to \( b \) and \(-a\) in \( N_a = \mathbb{k} \) has kernel \( K = \ker \varphi \) that is the submodule of \( M \) with

- \( \mathbb{k}^2 \) in every degree from \( U^\circ \), and
- the line in \( \mathbb{k}^2 \) through \( \begin{bmatrix} 0 \\ a \end{bmatrix} \) and \( \begin{bmatrix} a \\ 0 \end{bmatrix} \) in every degree from \( L \).
That is, $K_a$ agrees with $M_a$ for degrees $a$ outside of $L$, and $K_a$ is the line in $M_a$ of slope $b/a$ through the origin when $a$ lies on $L$. This kernel $K$ is not tame. Indeed, if $a$ and $a'$ are distinct points on $L$, then the homomorphisms $K_a \to K_{a/a'}$ and $K_{a'} \to K_{a/a'}$ have different images, so $a$ and $a'$ are forced to lie in different constant regions in every constant subdivision of $\mathbb{R}^2$ subordinate to $L$. (Note the relation between this example and Proposition 3.10 for $Q = U \subset \mathbb{R}^2$ and $D = L \subset Q$.)

**Remark 4.26.** Encoding of a $Q$-module $M$ by a poset morphism is related to viewing $M$ as a sheaf on $Q$ with its Alexandrov topology that is constructible in the sense of Lurie [Lur17, Definitions A.5.1 and A.5.2]. The difference is that poset encoding requires constancy (in the sense of Definition 2.7) on fibers of the encoding morphism, whereas Alexandrov constructibility requires only local constancy in the sense of sheaf theory. This distinction is decisive for Example 4.25, where the kernel $K$ is constructible but not finitely encoded.

Because of Remark 4.26, allowing arbitrary homomorphisms between tame modules would step outside of the tame class. More formally, inside the category of $Q$-modules, the full subcategory generated by the tame modules contains modules that are not tame. To preserve tameness, it is thus necessary to restrict the allowable morphisms.

**Definition 4.27.** A homomorphism $\varphi : M \to N$ of $Q$-modules is tame if $Q$ admits a finite constant subdivision subordinate to both $M$ and $N$ such that for each constant region $I$ the composite isomorphism $M_I \to M_i \to N_i \to N_I$ does not depend on $i \in I$. The map $\varphi$ is semialgebraic, PL, subanalytic, or class $\mathfrak{X}$ if this constant subdivision is.

**Lemma 4.28.** The kernel and cokernel of any tame homomorphism of $Q$-modules are tame morphisms of tame modules. The same is true when tameness is replaced by semialgebraic, PL, subanalytic, or class $\mathfrak{X}$.

**Proof.** Any constant subdivision as in Definition 4.27 is subordinate to both the kernel and cokernel of $M \to N$, with the vector spaces associated to any constant region $I$ being $\ker(M_I \to N_I)$ and $\text{coker}(M_I \to N_I)$. □

**Definition 4.29.** The category of tame modules is the subcategory of $Q$-modules whose objects are the tame modules and whose morphisms are the tame homomorphisms.

**Remark 4.30.** To be precise with language, a morphism of tame modules is required to be tame, whereas a homomorphism of tame modules is not. That is, morphisms in the category of tame modules are called morphisms, whereas morphisms in the category of $Q$-modules are called homomorphisms. To avoid confusion, the set of tame morphisms from a tame module $M$ to another tame module $N$ is denoted $\text{Mor}(M, N)$ instead of $\text{Hom}(M, N)$.

**Proposition 4.31.** Over any poset $Q$, the category of tame $Q$-modules is abelian. If $Q$ is a subposet of a partially ordered real vector space of finite dimension, then the category of semialgebraic, PL, subanalytic, or class $\mathfrak{X}$ modules is abelian.
Proof. Over any poset, the category in question is a subcategory of the category of $Q$-modules, which is abelian. The subcategory is not full, but $\text{Mor}(M, N)$ is an abelian subgroup of $\text{Hom}(M, N)$; this is most easily seen via Theorem 4.22, for if $\varphi : M \to N$ and $\varphi' : M \to N'$ are finitely encoded by $\pi : Q \to P$ and $\pi' : Q \to P'$, respectively, then $\varphi + \varphi'$ is finitely encoded by $\pi \times \pi' : Q \to P \times P'$. The same construction, but with the source of $\pi'$ being a new module $M'$ instead of $M$, shows that the ordinary product and direct sum of a pair of finitely encoded modules serves as a product and coproduct in the tame category. Kernels and cokernels of morphisms in the tame category exist by Lemma 4.28, which also implies that every monomorphism is a kernel (it is the kernel of its cokernel in the category of $Q$-modules) and every epimorphism is a cokernel (it is the cokernel of its kernel in the category of $Q$-modules).

The semialgebraic, PL, and class $\mathcal{X}$ cases have the same proof, noting that $\pi \times \pi'$ has fibers of the desired type if $\pi$ and $\pi'$ both do. The subanalytic case only follows from this argument when restricted to the category of modules whose nonzero graded pieces lie in a bounded subset of $Q$ (the subset is allowed to depend on the module). However, the argument in the previous paragraph can be done directly with common refinements of pairs of constant subdivisions, so reducing to Theorem 4.22 is not necessary. $\Box$

5. Primary decomposition over partially ordered groups

In the context of persistent homology, multiparameter features can die in many ways, persisting indefinitely as some of the parameters increase without limit but dying when any of the others increase sufficiently. The essence of this phenomenon is captured in elementary language by Theorem 5.35. In the ordinary situation of one parameter, the only distinction being made here is that a feature can be mortal or immortal. Beyond the intrinsic mathematical value, decomposing a module according to these distinctions has concrete benefits for statistical analysis using multipersistence [MT19].

5.1. Polyhedral partially ordered groups.

The next definition, along with elementary foundations surrounding it, can be found in Goodearl’s book [Goo86, Chapter 1].

Definition 5.1. An abelian group $Q$ is partially ordered if it is generated by a submonoid $Q_+$, called the positive cone, that has trivial unit group. The partial order is: $q \preceq q' \iff q' - q \in Q_+$. All partially ordered groups in this paper are assumed abelian.

Example 5.2. The finitely generated free abelian group $Q = \mathbb{Z}^n$ can be partially ordered with any positive cone $Q_+$, polyhedral or otherwise, resulting in a discrete partially ordered group. The free commutative monoid $Q_+ = \mathbb{N}^n$ of integer vectors with nonnegative coordinates is the most common instance and serves as a well behaved, well known foundational case (see Section 6) to which substantial parts of the general theory reduce. For notational clarity, $\mathbb{Z}^n_+$ always means the nonnegative orthant in $\mathbb{Z}^n$, which induces the standard componentwise partial order on $\mathbb{Z}^n$. Other partial orders can be specified using notation $Q \cong \mathbb{Z}^n$ with arbitrary positive cone $Q_+$. 

Example 5.3. The group $Q = \mathbb{R}^n$ can be partially ordered with any positive cone $Q_+$, polyhedral or otherwise, closed, open (away from the origin $0$) or anywhere in between, resulting in a real partially ordered group. The orthant $Q_+ = \mathbb{R}_+^n$ of vectors with nonnegative coordinates is most useful for multiparameter persistence. For notational clarity, $\mathbb{R}_+^n$ always means the nonnegative orthant in $\mathbb{R}^n$, which induces the standard componentwise partial order on $\mathbb{R}^n$. Other partial orders can be specified using notation $Q \cong \mathbb{R}^n$ with arbitrary positive cone $Q_+$.

Example 5.4. Definition 5.1 allows the group to have torsion. Thus the submonoid of $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a positive cone in the group. There is a continuous version in which the resulting partial order is easier to see geometrically:

$Q = \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ with $Q_+$ generated by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. In the figure, the blue center line is the first factor $\mathbb{R}$, with origin $0$ at the fat blue dot. The positive cone $Q_+$ is shaded.

The following allows the free use of the language of either $Q$-modules or $Q$-graded $k[Q_+]$-modules, as appropriate to the context.

Lemma 5.5. A module over a partially ordered abelian group $Q$ is the same thing as a $Q$-graded module over $k[Q_+]$. \hfill \Box

Example 5.6. When $Q = \mathbb{Z}^n$ and $Q_+ = \mathbb{N}^n$, the relevant monoid algebra is the polynomial ring $k[\mathbb{N}^n] = k[x]$, where $x = x_1, \ldots, x_n$ is a sequence of $n$ commuting variables.

Primary decomposition of $Q$-modules depends on certain finiteness conditions. In ordinary commutative algebra, where $Q = \mathbb{Z}^n$, the finiteness comes from $Q_+$, which is assumed to be finitely generated (so it is an affine semigroup). This condition implies that finitely generated $Q$-modules are noetherian: every increasing chain of submodules stabilizes. Primary decomposition is then usually derived as a special case of the theory for finitely generated modules over noetherian rings. But the noetherian condition is stronger than necessary: in the presence of a tame hypothesis, it suffices for the positive cone to have finitely many faces, in the following sense. To the author’s knowledge, the notion of polyhedral partially ordered group is new, and there is no existing literature on primary decomposition in this setting.

Definition 5.7. A face of the positive cone $Q_+$ of a partially ordered group $Q$ is a submonoid $\sigma \subseteq Q_+$ such that $Q_+ \setminus \sigma$ is an ideal of the monoid $Q_+$. Sometimes it is simpler to say that $\sigma$ is a face of $Q$. Call $Q$ polyhedral if it has only finitely many faces.
Polyhedrality suffices to prove existence of (finite) primary decomposition (Theorem 5.41). However, many of the ingredients, such as localization along or taking support on a face (Definition 5.12), make sense also under a different sort of hypothesis.

**Definition 5.8.** Let $Q$ be a partially ordered group.

1. A *ray* of the positive cone $Q_+$ is a face that is totally ordered as a partially ordered submonoid of $Q$.
2. The partially ordered group $Q$ is *closed* if the complement $Q_+ \setminus \tau$ of each face $\tau$ is generated as an upset (i.e., as an ideal) of $Q_+$ by $\rho \setminus \{0\}$ for the rays $\rho \not\subseteq \tau$.

**Example 5.9.** Any real partially ordered group (Example 5.3) $Q$ whose positive cone $Q_+$ is closed in the usual topology on $Q$ is a closed partially ordered group by the Krein–Milman theorem: $Q_+$ is the set of nonnegative real linear combinations of vectors on extreme rays of $Q_+$. For instance, a non-polyhedral closed partial order on $Q = \mathbb{R}^3$ results by taking $Q_+$ to be a cone over a disk, such as either half of the cone $x^2 + y^2 \leq z^2$. In contrast, if $Q_+$ is an intersection of finitely many closed half-spaces, then there are only finitely many extreme rays. (This case is crucial in applications—see [KS19, Mil19a], for instance.) Even in the polyhedral case the cone need not be rational; that is, the vectors that generate it—or the linear functions defining the closed half-spaces whose intersection is $Q_+$—need not have rational entries.

**Example 5.10.** Any discrete partially ordered group (Example 5.2) whose positive cone is a finitely generated submonoid is automatically both polyhedral and closed; see [MS05, Lemma 7.12]. A discrete partially ordered group can also have a positive cone that is not a finitely generated submonoid, such as $Q = \mathbb{Z}^2$ with $Q_+ = C \cap \mathbb{Z}^2$ for the cone $C \subseteq \mathbb{R}^2$ generated by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \pi \end{bmatrix}$. This particular irrational cone yields a partially ordered group that is polyhedral but not closed. Indeed, there are fewer than the expected faces, because only some of the faces of $C$ result in faces of $Q_+$ itself. The image of $Q$ is not discrete in the quotient of $Q \otimes \mathbb{R}$ modulo the subgroup spanned by the irrational real face, which can have unexpected consequences for the algebra of poset modules under localization along such a face.

**Example 5.11.** The cylindrical group $Q$ in Example 5.4 has two faces: the origin $0$ (the fat blue dot) and $\mathbb{R}_+$ (the rightmost half of the horizontal blue center line).

### 5.2. Primary decomposition of downsets.

**Definition 5.12.** Fix a face $\tau$ of the positive cone $Q_+$ in a polyhedral or closed partially ordered group $Q$ and a downset $D \subseteq Q$. Write $\mathbb{Z}\tau$ for the subgroup of $Q$ generated by $\tau$.

1. The *localization* of $D$ along $\tau$ is the subset
   \[ D_\tau = \{ q \in D \mid q + \tau \subseteq D \}. \]

2. An element $q \in D$ is *globally supported on* $\tau$ if $q \not\in D_{\tau'}$ whenever $\tau' \not\subseteq \tau$. 
3. The part of $D$ globally supported on $\tau$ is
$$\Gamma_{\tau}D = \{ q \in D \mid q \text{ is globally supported on } \tau \}. $$

4. An element $q \in D$ is locally supported on $\tau$ if $q$ is globally supported on $\tau$ in $D_{\tau}$.

5. The local $\tau$-support of $D$ is the subset $\Gamma_{\tau}(D_{\tau}) \subseteq D$ consisting of elements globally supported on $\tau$ in the localization $D_{\tau}$.

6. The $\tau$-primary component of $D$ is the downset
$$P_{\tau}(D) = \Gamma_{\tau}(D_{\tau}) - Q_+$$
cogenerated by the local $\tau$-support of $D$.

Example 5.13. The local $\tau$-supports of the under-hyperbola downset in Example 1.5 are the subsets depicted on the right-hand side there, for the faces $\tau = 0$, $x$-axis, and $y$-axis, respectively. The corresponding primary components are depicted in Example 1.6. In contrast, the global support on (say) the $y$-axis consists of the part of the local support that sits strictly above the $x$-axis, and the global support at $0$ is the part of $D$ strictly in the positive quadrant.

This example demonstrates that the $\tau$-primary component of $D$ in Definition 5.12 need not be supported on $\tau$. Indeed, $D = P_0(D)$ here, and points outside of $Q_+$ are not supported at the origin, being instead locally supported at either the $x$-axis (if the point is below the $x$-axis) or the $y$-axis (if the point is behind the $y$-axis).

Remark 5.14. Definition 5.12 makes formal sense in any partially ordered group, but extreme caution is recommended without the closed or polyhedral assumptions. Indeed, without such assumptions, faces can be virtually present, such as a missing face in a real polyhedron that is not closed or the irrational face in Example 5.10. In such cases, aspects of Definition 5.12 might produce unintended output. That said, the natural generality for the concepts in Definition 5.12 is unclear.

Example 5.15. The coprincipal downset $a + \tau - Q_+$ inside of $Q = \mathbb{Z}^n$ cogenerated by $a$ along $\tau$ is globally supported along $\tau$. It also equals its own localization along $\tau$, so it equals its local $\tau$-support and is its own $\tau$-primary component. Note that when $Q_+ = \mathbb{N}^n$, faces of $Q_+$ correspond to subsets of $[n] = \{1, \ldots, n\}$, the correspondence being $\tau \leftrightarrow \chi(\tau)$, where $\chi(\tau) = \{ i \in [n] \mid e_i \in \tau \}$ is the characteristic subset of $\tau$ in $[n]$. (The vector $e_i$ is the standard basis vector whose only nonzero entry is 1 in slot $i$.)

Remark 5.16. The localization of $D$ along $\tau$ is acted on freely by $\tau$. Indeed, $D_{\tau}$ is the union of those cosets of $\mathbb{Z}\tau$ each of which is already contained in $D$. The minor point being made here is that the coset $q + \mathbb{Z}\tau$ is entirely contained in $D$ as soon as $q + \tau \subseteq D$ because $D$ is a downset: $q + \mathbb{Z}\tau = q + \tau - \tau \subseteq q + \tau - Q_+ \subseteq D$ if $q + \tau \subseteq D$.

Remark 5.17. The localization of $D$ is defined to reflect localization at the level of $Q$-modules: enforcing invertibility of structure homomorphisms $k[D]_q \to k[D]_{q+f}$ for $f \in \tau$ results in a localized indicator module $k[D][\mathbb{Z}\tau] = k[D_{\tau}]$; see Definition 5.24.
Example 5.18. Fix a downset $D$ in a partially ordered group $Q$ that is closed (Definition 5.8 and subsequent examples). An element $q \in D$ is globally supported on $\tau$ if and only if it lands outside of $D$ when pushed far enough up in any direction outside of $\tau$—that is, every $f \in Q_+ \setminus \tau$ has a nonnegative integer multiple $\lambda f$ with $\lambda f + q \notin D$.

One implication is easy: if every $f \in Q_+ \setminus \tau$ has $\lambda f + q \notin D$ for some $\lambda \in \mathbb{N}$, then any element $f' \in \tau' \setminus \tau$ has a multiple $\lambda f' \in \tau'$ such that $\lambda f' + q \notin D$, so $q \notin D_{\tau'}$. For the other direction, use Definition 5.8: $q \in \Gamma_{\tau}D \Rightarrow q \notin D_\rho$ for all rays $\rho$ of $Q_+$ that are not contained in $\tau$, so along each such ray $\rho$ there is a vector $v_\rho$ with $v_\rho + q \notin D$. Given $f \in Q_+ \setminus \tau$, choose $\lambda \in \mathbb{N}$ big enough so that $\lambda f \geq v_\rho$ for some $\rho$.

Definition 5.19. Fix a downset $D$ in a polyhedral partially ordered group $Q$.

1. The downset $D$ is coprimary if $D = P_\tau(D)$ for some face $\tau$ of the positive cone $Q_+$. If $\tau$ needs to be specified then $D$ is called $\tau$-coprimary.
2. A primary decomposition of $D$ is an expression $D = \bigcup_{i=1}^r D_i$ of coprimary downsets $D_i$, called components of the decomposition.

Theorem 5.20. Every downset $D$ in a polyhedral partially ordered group $Q$ is the union $\bigcup_{\tau} \Gamma_{\tau}(D_{\tau})$ of its local $\tau$-supports for all faces $\tau$ of the positive cone.

Proof. Given an element $q \in D$, finiteness of the number of faces implies the existence of a face $\tau$ that is maximal among those such that $q \in D_{\tau}$; note that $q \in D = D_0$ for the trivial face $0$ consisting of only the identity of $Q$. It follows immediately that $q$ is supported on $\tau$ in $D_{\tau}$. \qed

Corollary 5.21. Every downset $D$ in a polyhedral partially ordered group $Q$ has a canonical primary decomposition $D = \bigcup_{\tau} P_\tau(D_{\tau})$, the union being over all faces $\tau$ of the positive cone with nonempty support $\Gamma_{\tau}(D_{\tau})$.

Remark 5.22. The union in Theorem 5.20 is not necessarily disjoint. Nor, consequently, is the union in Corollary 5.21. There is a related union, however, that is disjoint: the sets $(\Gamma_{\tau}D) \cap D_{\tau}$ do not overlap. Their union need not be all of $D$, however; try Example 5.13, where the negative quadrant intersects none of the sets $(\Gamma_{\tau}D) \cap D_{\tau}$.

Algebraically, $(\Gamma_{\tau}D) \cap D_{\tau}$ should be interpreted as taking the elements of $D$ globally supported on $\tau$ and then taking their images in the localization along $\tau$, which deletes the elements that aren’t locally supported on $\tau$. That is, $(\Gamma_{\tau}D) \cap D_{\tau}$ is the set of degrees where the image of $\Gamma_{\tau}k[D] \rightarrow k[D]_{\tau}$ is nonzero.

Example 5.23. The decomposition in Theorem 5.20—and hence Corollary 5.21—is not necessarily minimal: it might be that some of the canonically defined components can be omitted. This occurs, for instance, in Example 1.6. The general phenomenon, as in this hyperbola example, stems from geometry of the elements in $D_{\tau}$ supported on $\tau$, which need not be bounded in any sense, even in the quotient $Q/\mathbb{Z}\tau$. In contrast, for (say) quotients by monomial ideals in the polynomial ring $k[\mathbb{N}^n]$, only finitely many elements have support at the origin, and the downset they cogenerate is consequently artinian.
5.3. Localization and support.

**Definition 5.24.** Fix a face $\tau$ of a partially ordered group $Q$. The *localization* of a $Q$-module $M$ *along* $\tau$ is the tensor product

$$M_\tau = M \otimes_{k[Q_+]^+} k[Q_+ + \mathbb{Z}_\tau],$$

viewing $M$ as a $Q$-graded $k[Q_+]$-module. The submodule of $M$ *globally supported on* $\tau$ is

$$\Gamma_\tau M = \bigcap_{\tau' \not\subseteq \tau} \ker(M \to M_{\tau'}) = \ker(M \to \prod_{\tau' \not\subseteq \tau} M_{\tau'}).$$

**Example 5.25.** Definition 5.12 says that $1_q \in k[D] = k$ lies in $\Gamma_\tau k[D]$ if and only if $q \in \Gamma_\tau D$, because $q \not\in D_\tau$ if and only if $1_q \mapsto 0$ under localization of $k[D]$ along $\tau'$.

**Example 5.26.** The global supports of the indicator subquotient for the interval

in $\mathbb{R}^2$ on the left-hand side of this display are the indicator subquotients for the intervals on the right-hand side, each labeled by the relevant face $\tau$. Caution: this example is not to be confused with Examples 1.5, 1.6, 5.13, and 5.23, where the curve is a hyperbola whose asymptotes are the two axes. In contrast, here upper boundary of the interval has the vertical axis as an asymptote, whereas the horizontal axis is exactly parallel to the positive end of the upper boundary.

**Lemma 5.27.** The kernel of any natural transformation between two exact covariant functors is left-exact. In more detail, if $\alpha$ and $\beta$ are two exact covariant functors $\mathcal{A} \to \mathcal{B}$ for abelian categories $\mathcal{A}$ and $\mathcal{B}$, and $\gamma_X : \alpha(X) \to \beta(X)$ naturally for all objects $X$ of $\mathcal{A}$, then the association $X \mapsto \ker \gamma_X$ is a left-exact covariant functor $\mathcal{A} \to \mathcal{B}$.

**Proof.** This can be checked by diagram chase or spectral sequence. \hfill \Box

**Proposition 5.28.** The global support functor $\Gamma_\tau$ is left-exact.

**Proof.** Use Lemma 5.27: global support is the kernel of the natural transformation from the identity to a direct product of localizations. \hfill \Box

**Proposition 5.29.** For modules over a polyhedral partially ordered group, localization commutes with taking support: $(\Gamma_\tau M)_\tau = \Gamma_\tau(M_\tau)$, and both sides are 0 unless $\tau' \supseteq \tau$. 


Proof. Localization along $\tau$ is exact, so
\[
\ker(M \to M_{\tau''})_\tau = \ker(M_\tau \to (M_{\tau''})_\tau) = \ker(M_\tau \to (M_\tau)_{\tau''}).
\]
Since localization along $\tau$ commutes with finite intersections of submodules, $(\Gamma_\tau M)_\tau$ is the intersection of the leftmost of these modules over the faces $\tau'' \not\subseteq \tau'$, of which there are only finitely many by the polyhedral hypothesis. But $\Gamma_{\tau'}(M_\tau) = \Gamma_{\tau'}(M)$ equals the same intersection of the rightmost of these modules by definition. And if $\tau' \not\supseteq \tau$ then one of these $\tau''$ equals $\tau$, so $M_\tau \to (M_\tau)_{\tau''} = M_\tau$ is the identity map, whose kernel is 0. \qed

**Definition 5.30.** Fix a $Q$-module $M$ for a polyhedral partially ordered group $Q$. The local $\tau$-support of $M$ is the module $\Gamma_\tau M$ of elements globally supported on $\tau$ in the localization $M_\tau$, or equivalently (by Proposition 5.29) the localization along $\tau$ of the submodule of $M$ globally supported on $\tau$.

**Definition 5.31.** A module $M$ over a polyhedral partially ordered group is coprimary if for some face $\tau$, the localization map $M \hookrightarrow M_\tau$ is injective and $\Gamma_\tau M$ is an essential submodule of $M_\tau$: every nonzero submodule of $M_\tau$ intersects $\Gamma_\tau M$ nontrivially.

**Remark 5.32.** It is easy to check that over any polyhedral partially ordered group, if a module $E$ is coprimary then it is $\tau$-coprimary for a unique face $\tau$ of $Q$.

**Remark 5.33.** It is an interesting exercise to check that every element of a coprimary module is coprimary when the polyhedral partially ordered group is discrete (Example 5.10) and closed (Definition 5.8).

The coprimary concept has an elementary, intuitive formulation in the language of persistence, when the ambient partially ordered group is polyhedral and closed.

**Definition 5.34.** Fix a face $\tau$ of the positive cone $Q_+$ in a partially ordered group $Q$. A homogeneous element $y \in M_q$ in a $Q$-module $M$ is
\begin{enumerate}
\item $\tau$-persistent if it has nonzero image in $M_{q'}$ for all $q' \in q + \tau$;
\item $\tau$-transient if, for each $f \in Q_+ \setminus \tau$, the image of $y$ vanishes in $M_{q'}$ whenever $q' = q + \lambda f$ for $\lambda \gg 0$;
\item $\tau$-coprimary if it is $\tau$-persistent and $\tau$-transient.
\end{enumerate}

**Theorem 5.35.** Fix a $Q$-module $M$ and a face $\tau$ of the positive cone $Q_+$ in a closed polyhedral partially ordered group $Q$. The module $M$ is $\tau$-coprimary if and only if every homogeneous element divides a $\tau$-coprimary element, where $y \in M_q$ divides $y' \in M_{q'}$ if $q \leq q'$ and $y$ has image $y'$ under the structure morphism $M_q \to M_{q'}$.

Proof. If $M$ is $\tau$-coprimary and $y \in M_q$ is a nonzero homogeneous element, then $y$ is $\tau$-persistent because $M$ is a submodule of $M_\tau$ on which $k[\mathbb{Z}\tau]$ acts freely. On the other hand, $y$ divides a $\tau$-transient element because $\Gamma_\tau M_\tau$ is an essential submodule of $M_\tau$: the submodule of $M_\tau$ generated by $y$ intersects $\Gamma_\tau M_\tau$ nontrivially. The closed hypothesis on $Q$ implies that an element supported on $\tau$ is $\tau$-transient, as in Example 5.18.
The other direction does not require the closed hypothesis. Assume that every homogeneous element of $M$ divides a $\tau$-coprimary element. The graded component of the localization $M_\tau$ in degree $q \in Q$ is the direct limit of $M'_q$ over $q' \in q + \tau$. If $y \in M_q$ lies in $\ker(M \to M_\tau)$, then the image of $y$ must vanish in some $M_{q'}$, whence $y = 0$ to begin with, by $\tau$-persistence. On the other hand, that $\Gamma_\tau M_\tau$ is an essential submodule of $M_\tau$ follows because every $\tau$-transient element is supported on $\tau$. □

5.4. Primary decomposition of modules.

Definition 5.36. Fix a $Q$-module $M$ for a polyhedral partially ordered group $Q$. A primary decomposition of $M$ is an injection $M \hookrightarrow \bigoplus_{i=1}^r M/M_i$ into a direct sum of coprimary quotients $M/M_i$, called components of the decomposition.

Remark 5.37. Primary decomposition is usually phrased in terms of primary submodules $M_i \subseteq M$, which by definition have coprimary quotients $M/M_i$, satisfying $\bigcap_{i=1}^r M_i = 0$ in $M$. This is equivalent to Definition 5.36.

Example 5.38. A primary decomposition $D = \bigcup_{i=1}^r D_i$ of a downset $D$ yields a primary decomposition of the corresponding indicator quotient, namely the injection $k[D] \hookrightarrow \bigoplus_{i=1}^r k[D_i]$ induced by the surjections $k[D] \twoheadrightarrow k[D_i]$. See, e.g., Example 1.6.

Example 5.39. The interval module in Example 5.26 has a primary decomposition

in which the global support along each face is extended downward so as to become a quotient instead of a submodule of the original interval module.

The existence of primary decomposition in Theorem 5.41 is intended for tame modules, but because it deals with essential submodules and not generators, it only requires the downset half of a fringe presentation.

Definition 5.40. A downset hull of a module $M$ over an arbitrary poset is an injection $M \hookrightarrow \bigoplus_{j \in J} E_j$ with each $E_j$ being a downset module. The hull is finite if $J$ is finite. The module $M$ is downset-finite if it admits a finite downset hull.

Theorem 5.41. Every downset-finite module over a polyhedral partially ordered group admits a primary decomposition.
Proof. If \( M \hookrightarrow \bigoplus_{j=1}^k E_j \) is a downset hull of the module \( M \), and \( E_j \hookrightarrow \bigoplus_{i=1}^\ell E_{ij} \) is a primary decomposition for each \( j \) afforded by Corollary 5.21 and Example 5.38, then let \( E^\tau \) be the direct sum of the downset modules \( E_{ij} \) that are \( \tau \)-coprimary. Set \( M^\tau = \ker(M \to E^\tau) \). Then \( M/M^\tau \) is coprimary, being a submodule of a coprimary module. Moreover, \( M \to \bigoplus_{\tau} M/M^\tau \) is injective because its kernel is the same as the kernel of \( M \to \bigoplus_{ij} E_{ij} \), which is a composite of two injections and hence injective by construction. Therefore \( M \to \bigoplus_{\tau} M/M^\tau \) is a primary decomposition. \( \square \)

Example 5.42. The finiteness of primary decomposition depends on the polyhedral condition that posits finiteness of the number of faces of the positive cone (Definition 5.7). When the positive cone has infinitely many faces, such as the positive half \( Q_+ \) of the right circular cone \( x^2 + y^2 \leq z^2 \) in \( Q = \mathbb{R}^3 \), the \( Q \)-module

\[
\mathbb{k}[\partial Q_+] = \mathbb{k}[Q_+]/\mathbb{k}[Q_+]^\circ
\]

does not admit a finite primary decomposition. The module \( M = \mathbb{k}[\partial Q_+] \) has a vector space of dimension 1 on the boundary of the positive cone and 0 elsewhere. Every face of the positive cone must get its own summand \( M/M_i \) in Definition 5.36 for the homomorphism \( M \to \bigoplus_{i=1}^r M/M_i \) there to be injective, and in that case the infinite number of faces would force the direct sum to become a direct product. This particular example, with the right circular cone, works as well in the discrete partially ordered group \( \mathbb{Z}^3 \) because the circle has infinitely many rational points.

6. Finitely determined \( \mathbb{Z}^n \)-modules

Unless otherwise stated, this section is presented over the discrete polyhedral partially ordered group \( Q = \mathbb{Z}^n \) with \( Q_+ = \mathbb{N}^n \). It begins by reviewing the structure of finitely determined \( \mathbb{Z}^n \)-modules, including (minimal) injective and flat resolutions. These are the foundation underlying the syzygy theorem for tame modules (Section 7.2), including the existence of fringe presentations. They also serve as models for the concepts of socle, cogenerator, and downset hull over real polyhedral groups, covered in the sequel to this work [Mil19a], as well as their dual notions of top, generator, and upset covers.

The main references for \( \mathbb{Z}^n \)-modules used here are [Mil00, MS05]. The development of homological theory for injective and flat resolutions in the context of finitely determined modules is functorially equivalent to the development for finitely generated modules, by [Mil00, Theorem 2.11], but it is convenient to have on hand the statements in the finitely determined case directly. The characterization of finitely determined modules in Proposition 6.7 and (hence) Theorem 6.19 is apparently new.

6.1. Definitions.

The essence of the finiteness here is that all of the relevant information about the relevant modules should be recoverable from what happens in a bounded box in \( \mathbb{Z}^n \).
Definition 6.1. A \( \mathbb{Z}^n \)-finite module \( N \) is \textit{finitely determined} if for each \( i = 1, \ldots, n \) the multiplication map \( :x_i : N_b \to N_{b+e_i} \) (see Example 5.6 for notation) is an isomorphism whenever \( b_i \) lies outside of some bounded interval.

Remark 6.2. This notion of finitely determined is the same notion as in Example 4.5. A module is finitely determined if and only if, after perhaps translating its \( \mathbb{Z}^n \)-grading, it is \textit{a-determined} for some \( a \in \mathbb{N}^n \), as defined in [Mil00, Definition 2.1].

Remark 6.3. For \( \mathbb{Z}^n \)-modules, the finitely determined condition is weaker—that is, more inclusive—than finitely generated, but it is much stronger than tame or (equivalently, by Theorem 4.22) finitely encoded. The reason is essentially Example 4.5, where the encoding has a very special nature. For a generic sort of example, the restriction to \( \mathbb{Z}^n \) of any \( \mathbb{R}^n \)-finite \( \mathbb{R}^n \)-module with finitely many constant regions of sufficient width is a tame \( \mathbb{Z}^n \)-module, and there is simply no reason why the constant regions should be commensurable with the coordinate directions in \( \mathbb{Z}^n \). Already the toy-model fly wing modules in Examples 1.2 and 1.4 yield infinitely generated but tame \( \mathbb{Z}^n \)-modules, and this remains true when the discretization \( \mathbb{Z}^n \) of \( \mathbb{R}^n \) is rescaled by any factor.

Example 6.4. The local cohomology of an affine semigroup rings is tame but usually not finitely determined; see [HM05] and [MS05, Chapter 13], particularly Theorem 13.20, Example 13.17, and Example 13.4 in the latter.

6.2. Injective hulls and resolutions.

Remark 6.5. Every \( \mathbb{Z}^n \)-finite module that is injective in the category of \( \mathbb{Z}^n \)-modules is a direct sum of downset modules \( \mathbb{k}[D] \) for downsets \( D \) cogenerated (Example 5.15) by vectors along faces. This statement holds over any polyhedral discrete partially ordered group (Definition 5.7 and Example 5.2) by [MS05, Theorem 11.30].

Minimal injective resolutions work for finitely determined modules just as they do for finitely generated modules. The standard definitions are as follows.

Definition 6.6. Fix a \( \mathbb{Z}^n \)-module \( N \).

1. An \textit{injective hull} of \( N \) is an injective homomorphism \( N \to E \) in which \( E \) is an injective \( \mathbb{Z}^n \)-module (see Remark 6.5). This injective hull is
   - \textit{finite} if \( E \) has finitely many indecomposable summands and
   - \textit{minimal} if the number of such summands is minimal.

2. An \textit{injective resolution} of \( N \) is a complex \( E^* \) of injective \( \mathbb{Z}^n \)-modules whose differential \( E^i \to E^{i+1} \) for \( i \geq 0 \) has only one nonzero homology \( H^0(E^*) \cong N \) (so \( N \leftarrow E^0 \) and \( \text{coker}(E^{i-1} \to E^i) \leftarrow E^{i+1} \) are injective hulls for all \( i \geq 1 \)). \( E^* \)
   - has \textit{length} \( \ell \) if \( E^i = 0 \) for \( i > \ell \) and \( E^\ell \neq 0 \);
   - is \textit{finite} if \( E^* = \bigoplus_i E^i \) has finitely many indecomposable summands; and
   - is \textit{minimal} if \( N \leftarrow E^0 \) and \( \text{coker}(E^{i-1} \to E^i) \leftarrow E^{i+1} \) are minimal injective hulls for all \( i \geq 1 \).
Proposition 6.7. The following are equivalent for a $\mathbb{Z}^n$-module $N$.
1. $N$ is finitely determined.
2. $N$ admits a finite injective resolution.
3. $N$ admits a finite minimal injective resolution.
Any finite minimal resolution is unique up to isomorphism and has length at most $n$.

Proof. The proof is based on existence of finite minimal injective hulls and resolutions for finitely generated $\mathbb{Z}^n$-modules, along with uniqueness and length $n$ given minimality, as proved by Goto and Watanabe [GW78].

First assume $N$ is finitely determined. Translating the $\mathbb{Z}^n$-grading affects nothing about existence of a finite injective resolution. Therefore, using Remark 6.2, assume that $N$ is a-determined. Truncate by taking the $\mathbb{N}^n$-graded part of $N$ to get a positively a-determined—and hence finitely generated—module $N_{\geq 0}$; see [Mil00, Definition 2.1]. Take any minimal injective resolution $N_{\geq 0} \to E^\bullet$. Extend backward using the Čech hull [Mil00, Definition 2.7], which is exact [Mil00, Lemma 2.9], to get a finite minimal injective resolution $\check{C}(N_{\geq 0} \to E^\bullet) = (N \to \check{C}E^\bullet)$, noting that $\check{C}$ fixes indecomposable injective modules whose $\mathbb{N}^n$-graded parts are nonzero and is zero on all other indecomposable injective modules [Mil00, Lemma 4.25]. This proves $1 \Rightarrow 3$.

That $3 \Rightarrow 2$ is trivial. The remaining implication, $2 \Rightarrow 1$, follows because every indecomposable injective is finitely determined and the category of finitely determined modules is abelian. (The category of $\mathbb{Z}^n$-modules each of which is nonzero only in a bounded set of degrees is abelian, and constructions such as kernels, cokernels, or direct sums in the category of finitely determined modules are pulled back from there.) □

6.3. Flat covers and resolutions.
Minimal flat resolutions are not commonplace, but the notion is Matliss dual to that of minimal injective resolution. In the context of finitely determined modules, flat resolutions work as well as injective resolutions. The definitions are as follows.

Definition 6.8. Fix a $\mathbb{Z}^n$-module $N$.
1. A flat cover of $N$ is a surjective homomorphism $F \to N$ in which $F$ is a flat $\mathbb{Z}^n$-module (see Remark 6.11). This flat cover is
   - finite if $F$ has finitely many indecomposable summands and
   - minimal if the number of such summands is minimal.
2. A flat resolution of $N$ is a complex $F_\bullet$ of flat $\mathbb{Z}^n$-modules whose differential $F_{i+1} \to F_i$ for $i \geq 0$ has only one nonzero homology $H_0(F_i) \cong N$ (so $F_0 \to N$ and $F_{i+1} \to \ker(F_i \to F_{i-1})$ are flat covers for all $i \geq 1$). The flat resolution $F_\bullet$ has length $\ell$ if $F_i = 0$ for $i > \ell$ and $F_\ell \neq 0$;
   - is finite if $F_\bullet = \bigoplus_i F_i$ has finitely many indecomposable summands; and
   - is minimal if $F_0 \to N$ and $F_{i+1} \to \ker(F_i \to F_{i-1})$ are minimal flat covers for all $i \geq 1$. 
Definition 6.9. The Matlis dual of a \( \mathbb{Z}^n \)-module \( M \) is the \( \mathbb{Z}^n \)-module \( M^\vee \) defined by \[ (M^\vee)_a = \text{Hom}_k(M_{-a}, k), \]
so the homomorphism \( (M^\vee)_a \to (M^\vee)_b \) is transpose to \( M_{-b} \to M_{-a} \).

Lemma 6.10. \( (M^\vee)^\vee \) is canonically isomorphic to \( M \) for any \( \mathbb{Z}^n \)-finite module \( M \). \( \square \)

Remark 6.11. By the adjunction between \( \text{Hom} \) and \( \otimes \), a module is flat if and only its Matlis dual is injective (see [Mil00, §1.2], for example). The Matlis dual of Remark 6.5 therefore says that every \( \mathbb{Q} \)-finite flat module over a discrete polyhedral partially ordered group \( \mathbb{Q} \) is isomorphic to a direct sum of upset modules \( k[U] \) for upsets of the form \( U = b + \mathbb{Z} \tau + \mathbb{Q} \). These upset modules are the graded translates of localizations of \( k[\mathbb{Q}] \) along faces.

6.4. Flange presentations.

Definition 6.12. Fix a \( \mathbb{Z}^n \)-module \( N \).

1. A flange presentation of \( N \) is a \( \mathbb{Z}^n \)-module morphism \( \varphi : F \to E \), with image isomorphic to \( N \), where \( F \) is flat and \( E \) is injective in the category of \( \mathbb{Z}^n \)-modules.

2. If \( F \) and \( E \) are expressed as direct sums of indecomposables, then \( \varphi \) is based.

3. If \( F \) and \( E \) are finite direct sums of indecomposables, then \( \varphi \) is finite.

4. If the number of indecomposable summands of \( F \) and \( E \) are simultaneously minimized then \( \varphi \) is minimal.

Remark 6.13. The term flange is a portmanteau of flat and injective (i.e., “flainj”) because a flange presentation is the composite of a flat cover and an injective hull.

The same notational trick to make fringe presentations effective data structures (Definition 3.17) works on flange presentations.

Definition 6.14. Fix a based finite flange presentation \( \varphi : \bigoplus_p F_p = F \to E = \bigoplus_q E_q \).

A monomial matrix for \( \varphi \) is an array of scalar entries \( \varphi_{pq} \) whose columns are labeled by the indecomposable flat summands \( F_p \) and whose rows are labeled by the indecomposable injective summands \( E_q \):

\[
\begin{array}{cccc}
E_1 & \cdots & E_\ell \\
F_1 & \varphi_{11} & \cdots & \varphi_{1\ell} \\
\vdots & \vdots & \ddots & \vdots \\
F_k & \varphi_{k1} & \cdots & \varphi_{k\ell} \\
\end{array}
\]

\[
F_1 \oplus \cdots \oplus F_k = F \quad \xrightarrow{\varphi} \quad E = E_1 \oplus \cdots \oplus E_\ell.
\]

The entries of the matrix \( \varphi_{\cdot \cdot} \) correspond to homomorphisms \( F_p \to E_q \).

Lemma 6.15. If \( F = k[a + \mathbb{Z}\tau' + \mathbb{N}^n] \) is an indecomposable flat \( \mathbb{Z}^n \)-module and \( E = k[b + \mathbb{Z}\tau - \mathbb{N}^n] \) is an indecomposable injective \( \mathbb{Z}^n \)-module, then \( \text{Hom}_{\mathbb{Z}^n}(F, E) = 0 \) unless \( (a + \mathbb{Z}\tau' + \mathbb{N}^n) \cap (b + \mathbb{Z}\tau - \mathbb{N}^n) \neq \emptyset \), in which case \( \text{Hom}_{\mathbb{Z}^n}(F, E) = k \).
**Proof.** Corollary 3.11.1.

**Definition 6.16.** In the situation of Lemma 6.15, write $F \preceq E$ if their degree sets have nonempty intersection: $(a + \mathbb{Z}\tau + \mathbb{N}^n) \cap (b + \mathbb{Z}\tau - \mathbb{N}^n) \neq \emptyset$.

**Proposition 6.17.** With notation as in Definition 6.14, $\varphi_{pq} = 0$ unless $F_p \preceq E_q$. Conversely, if an array of scalars $\varphi_{qp} \in \mathbb{k}$ with rows labeled by indecomposable flat modules and columns labeled by indecomposable injectives has $\varphi_{pq} = 0$ unless $F_q \preceq E_q$, then it represents a flange presentation.

**Proof.** Lemma 6.15 and Definition 6.16.

The unnatural hypothesis that a persistence module be finitely generated results in data types and structure theory that are asymmetric regarding births as opposed to deaths. In contrast, the notion of flange presentation is self-dual: their duality interchanges the roles of births ($F$) and deaths ($E$).

**Proposition 6.18.** A $\mathbb{Z}^n$-module $N$ has a finite flange presentation $F \to E$ if and only if the Matlis dual $E^\vee \to F^\vee$ is a finite flange presentation of the Matlis dual $N^\vee$.

**Proof.** Matlis duality is an exact, contravariant functor on $\mathbb{Z}^n$-modules that takes the subcategory of finitely determined $\mathbb{Z}^n$-modules to itself (these properties are immediate from the definitions), interchanges flat and injective objects therein, and has the property that the natural map $(N^\vee)^\vee \to N$ is an isomorphism for finitely determined $N$ (Lemma 6.10); see [Mil00, §1.2] for a discussion of these properties.

6.5. **Syzygy theorem for $\mathbb{Z}^n$-modules.**

**Theorem 6.19.** A $\mathbb{Z}^n$-module is finitely determined if and only if it admits one, and hence all, of the following:

1. a finite flange presentation; or
2. a finite flat presentation; or
3. a finite injective copresentation; or
4. a finite flat resolution; or
5. a finite injective resolution; or
6. a minimal one of any of the above.

Any minimal one of these objects is unique up to noncanonical isomorphism, and the resolutions have length at most $n$.

**Proof.** The hard work is done by Proposition 6.7. It implies that $N$ is finitely determined $\iff N^\vee$ has a minimal injective resolution $\iff N$ has a minimal flat resolution of length at most $n$, since the Matlis dual of any finitely determined module $N$ is finitely determined. Having both a minimal injective resolution and a minimal flat resolution is stronger than having any of items 1–3, minimal or otherwise, so it suffices to show that $N$ is finitely determined if $N$ has any of items 1–3. This follows, using that the category of finitely determined modules is abelian as in the proof of Proposition 6.7, from the fact that every indecomposable injective or flat $\mathbb{Z}^n$-module is finitely determined. \qed
Remark 6.20. Conditions 1–6 in Theorem 6.19 remain equivalent for $\mathbb{R}^n$-modules, with the standard positive cone $\mathbb{R}_+^n$, assuming that the finite flat and injective modules in question are finite direct sums of localizations of $\mathbb{R}^n$ along faces and their Matlis duals. (The equivalence, including minimality, is a consequence of the generator and cogenerator theory over real polyhedral groups [Mil19a].) The equivalent conditions do not characterize $\mathbb{R}^n$-modules that are pulled back under convex projection from arbitrary modules over an interval in $\mathbb{R}^n$, though, because all sorts of infinite things can happen inside of a box, such as having generators along a curve.

7. Homological algebra of poset modules

7.1. Indicator resolutions.

Definition 7.1. Fix any poset $Q$ and a $Q$-module $M$.

1. An upset resolution of $M$ is a complex $F_\bullet$ of $Q$-modules, each a direct sum of upset submodules of $\mathbb{k}[Q]$, whose differential $F_i \to F_{i-1}$ decreases homological degrees, has components $\mathbb{k}[U] \to \mathbb{k}[U']$ that are connected (Definition 3.14), and has only one nonzero homology $H_0(F_\bullet) \cong M$.

2. A downset resolution of $M$ is a complex $E_\bullet$ of $Q$-modules, each a direct sum of downset quotient modules of $\mathbb{k}[Q]$, whose differential $E_i \to E_{i+1}$ increases cohomological degrees, has components $\mathbb{k}[D'] \to \mathbb{k}[D]$ that are connected, and has only one nonzero homology $H^0(E_\bullet) \cong M$.

An upset or downset resolution is called an indicator resolution if the up- or down-nature is unspecified. The length of an indicator resolution is the largest (co)homological degree in which the complex is nonzero. An indicator resolution

3. is finite if the number of indicator module summands is finite,

4. dominates a constant subdivision or encoding of $M$ if the subdivision or encoding is subordinate to each indicator summand, and

5. is semialgebraic, PL, subanalytic, or of class $\mathcal{X}$ if $Q$ is a subposet of a real partially ordered group and the resolution dominates a constant subdivision or encoding of the corresponding type.

Definition 7.2. Monomial matrices for indicator resolutions are defined similarly to those for fringe presentations in Definition 3.17, except that for the cohomological case the row and column labels are source and target downsets, respectively, while in the homological case the row and column labels are target and source upsets, respectively:
As in Proposition 4.11, pullbacks have transparent monomial matrix interpretations.

**Proposition 7.3.** Fix a poset $Q$ and an encoding of a $Q$-module $M$ by a poset morphism $\pi : Q \to P$ and $P$-module $H$. Monomial matrices for any indicator resolution of $H$ pull back to monomial matrices for an indicator resolution of $M$ that dominates the encoding by replacing the row and column labels with their preimages under $\pi$. □

**Definition 7.4.** Fix any poset $Q$ and a $Q$-module $M$.

1. An **upset presentation** of $M$ is an expression of $M$ as the cokernel of a homomorphism $F_1 \to F_0$ such that each $F_i$ is a direct sum of upset modules and every component $k[U'] \to k[U]$ of the homomorphism is connected (Definition 3.14).

2. A **downset copresentation** of $M$ is an expression of $M$ as the kernel of a homomorphism $E_0 \to E_1$ such that each $E_i$ is a direct sum of downset modules and every component $k[D] \to k[D']$ of the homomorphism is connected.

These indicator presentations are finite, or dominate a constant subdivision or encoding of $M$, or are semialgebraic, PL, subanalytic, or of class $\mathfrak{X}$ as in Definition 7.1.

**Example 7.5.** In one parameter, the bar $[a, b)$ in Example 3.19, has upset presentation

```
with cokernel
```

isomorphic to the single bar. When there are multiple bars, the bijection from left to right endpoints yields a monomial matrix whose scalar entries again form an identity matrix, with rows labeled by positive rays having the specified left endpoints (the ray is the whole real line when the left endpoint is $-\infty$) and columns labeled by positive rays having the corresponding right endpoints—but with their open or closed nature reversed—as left endpoints (the ray is empty when the specified right endpoint is $+\infty$).

**Example 7.6.**

```
is the cokernel of
```

**Lemma 7.7.** The homomorphisms in indicator presentations and resolutions are tame, so their kernels and cokernels are tame. If the indicator modules in question are semialgebraic, PL, subanalytic, or of class $\mathfrak{X}$ then the morphisms are, as well.
Proof. Any connected homomorphism among indicator modules is tame—and satisfies one of the auxiliary hypotheses, if the source and target do—by Definition 4.27, so the conclusion follows from Proposition 4.31. □

Example 7.8. The poset module in Example 2.8 has an upset presentation

\[
\begin{array}{cc}
T & B \\
L & 2 & 1 \\
R & -1 & -1 \\
\end{array}
\]

\[\mathbb{k}[L] \oplus \mathbb{k}[R] \leftarrow \mathbb{k}[T]\]

in which the monomial matrix has row and column labels

- \(L\), the upset generated by the leftmost element;
- \(R\), the upset generated by the rightmost element;
- \(T\), the upset consisting solely of the maximal element depicted on top; and
- \(B\), the upset consisting solely of the maximal element depicted on the bottom.

Although the disjoint union of \(T\) and \(B\) is an upset, and there is a homomorphism \(\varphi : \mathbb{k}[T \cup B] \to \mathbb{k}[L] \oplus \mathbb{k}[R]\) whose cokernel is the desired poset module, there is no way to arrange for the homomorphism \(\varphi\) to be connected.

Remark 7.9. It is tempting to think that a fringe presentation is nothing more than the concatenation of the augmentation map of an upset resolution (that is, the surjection at the end) with the augmentation map of a downset resolution (that is, the injection at the beginning), but there is no guarantee that the components \(F_i \to E_j\) of the homomorphism thus produced are connected (Definition 3.14). In contrast, a flange presentation (Definition 6.12) is in fact nothing more than the concatenation of the augmentation maps of a flat resolution and an injective resolution, since connected homomorphisms are forced by Lemma 6.15.

7.2. Syzygy theorem for modules over posets.

Proposition 7.10. For any inclusion \(\iota : P \to Z\) of posets and \(P\)-module \(H\) there is a \(Z\)-module \(\iota_* H\), the pushforward to \(Z\), whose restriction to \(\iota(P)\) is \(H\) and is universally repelling: \(\iota_* H\) has a canonical map to every \(Z\)-module whose restriction to \(\iota(P)\) is \(H\).

Proof. At \(z \in Z\) the pushforward \(\iota_* H\) places the colimit \(\varinjlim H_{\leq z}\) of the diagram of vector spaces indexed by the elements of \(P\) whose images precede \(z\). The universal property of colimits implies that \(\iota_* H\) is a \(Z\)-module with the desired universal property. □

Remark 7.11. With perspectives as in Remark 2.4, the pushforward is a left Kan extension [Cur14, Remark 4.2.9]. This instance is a special case of [Cur19, Example 4.4].
Theorem 7.12 (Syzygy theorem). A module $M$ over a poset $Q$ is tame if and only if it admits one, and hence all, of the following:

1. a finite constant subdivision of $Q$ subordinate to $M$; or
2. a finite poset encoding subordinate to $M$; or
3. a finite fringe presentation; or
4. a finite upset presentation; or
5. a finite downset copresentation; or
6. a finite upset resolution; or
7. a finite downset resolution; or
8. any of the above dominating any given finite encoding; or
9. a finite encoding subordinate to any given one of items 1–7; or
10. a finite constant subdivision subordinate to any given one of items 1–7.

The statement remains true over any subposet of a real partially ordered group if “tame” and all occurrences of “finite” are replaced by “semialgebraic”, “PL”, or “class $\mathfrak{X}$”. Moreover, any tame or semialgebraic, PL, or class $\mathfrak{X}$ morphism $M \to M'$ lifts to a similarly well behaved morphism of presentations or resolutions as in parts 3–7. All of these results except item 9 hold in the subanalytic case if $M$ has compact support.

Proof. Tame is equivalent to item 1 without auxiliary hypotheses by Definition 2.12 and with auxiliary hypotheses by Definition 2.16. Tame is equivalent to item 2 by Theorem 4.22.1. With auxiliary hypotheses, $1 \Rightarrow 2$ by Theorem 4.22.3; to apply that result in the subanalytic case starting from an arbitrary subanalytic finite constant subdivision subordinate to $M$, construct a compact such subdivision by keeping the bounded constant regions as they are and taking the union of all unbounded constant regions to get a single unbounded one. The implication $2 \Rightarrow 1$ holds because the fibers of the encoding poset morphism form a constant subdivision of the relevant type.

The necessity to construct an auxiliary compact subdivision from the given one is the reason to exclude item 9 from the subanalytic case, as the upcoming argument produces constant subdivisions, not directly encodings. For all of the other cases, item 9 proceeds via item 10, given the uptight constructions in the previous paragraph. For item 10, to produce a subordinate finite constant subdivision given a finite fringe presentation, take the common refinement of the canonical constant subdivision subordinate to each of its indicator summands. The same construction works if indicator presentations or resolutions are given, and it preserves auxiliary hypotheses by Proposition 2.18.1.

What remains is item 8: a finitely encoded $Q$-module $M$ has finite upset and downset resolutions and (co)presentations, as well as a finite fringe presentation, all dominating the given encoding. (As noted in the first paragraph, the fibers of the encoding morphism are already a constant subdivision of the relevant type.) The domination takes care of the cases with auxiliary hypotheses by Definitions 3.16.3, 4.7.2, 7.1.5, and 7.4.

Fix a $Q$-module $M$ finitely encoded by a poset morphism $\pi : Q \to P$ and $P$-module $H$. The finite poset $P$ has order dimension $n$ for some positive integer $n$; as such $P$ has an
embedding \( \iota : P \hookrightarrow \mathbb{Z}^n \). The pushforward \( \iota_* H \) (Proposition 7.10) is finitely determined (Definition 6.1; see also Example 4.5) as it is pulled back from any box containing \( \iota(P) \). The desired presentation or resolution is pulled back to \( Q \) (via \( \iota \circ \pi : Q \to \mathbb{Z}^n \)) from the corresponding flange, flat, or injective presentation or resolution of \( \iota_* H \) afforded by Theorem 6.19. These pullbacks are finite indicator resolutions of \( M \) dominating \( \pi \) by Example 4.10 and Lemma 4.9. The component homomorphisms are connected because, by Corollary 3.11 and Example 3.6 (see Definition 3.5), components of flange presentations, flat resolutions, and injective resolutions over \( \mathbb{Z}^n \) are automatically connected.

The preceding argument proves the claim about a morphism \( M \to M' \), as well, since

- only one poset morphism is required to encode the morphism \( M \to M' \);
- the push-pull constructions are functorial; and
- morphisms of finitely determined modules can be lifted to the relevant presentations and resolutions, since the relevant covers, presentations, and resolutions are free or injective in the category of finitely determined modules. \( \square \)

**Remark 7.13.** Comparing Theorems 7.12 and 6.19, what happened to minimality? It is not clear in what generality minimality can be characterized. The sequel [Mil19a] to this paper can be seen as a case study for posets arising from abelian groups that are either finitely generated and free (the closed discrete polyhedral case) or real vector spaces of finite dimension (the closed real polyhedral case). The answer is much more nuanced in the real case, obscuring how minimality might generalize beyond these cases.

**Remark 7.14.** In the situation of the proof of Theorem 7.12, composing two applications of Proposition 4.11—one for the encoding \( \pi : Q \to P \) and one for the embedding \( \iota : P \hookrightarrow \mathbb{Z}^n \)—yields a monomial matrix for a fringe presentation of \( M \) directly from a monomial matrix for a flange presentation.

**Remark 7.15.** Lesnick and Wright consider \( \mathbb{R}^n \)-modules [LW15, §2] in finitely presented cases. As they indicate, homological algebra of such \( \mathbb{R}^n \)-modules is no different than finitely generated \( \mathbb{Z}^n \)-modules. This can be seen by finite encoding: any finite poset in \( \mathbb{R}^n \) is embeddable in \( \mathbb{Z}^n \), because a product of finite chains is all that is needed.

### 7.3. Syzygy theorem for complexes of modules.

Theorem 7.12 is stated for individual modules, but the proof works just as well for complexes, in a sense recorded here for reference during the proof of Theorem 8.22.

**Definition 7.16.** Fix a complex \( M^* \) of modules over a poset \( Q \).

1. \( M^* \) is *tame* if its modules and morphisms are tame (Definitions 2.12 and 4.27).
2. A constant subdivision or poset encoding is *subordinate* to \( M^* \) if it is subordinate to all of the modules and morphisms therein, and in that case \( M^* \) is said to *dominate* the subdivision or encoding.
3. An *upset resolution* of \( M^* \) is a complex of \( Q \)-modules in which each \( F_i \) is a direct sum of upset modules and the components \( \mathbb{k}[U] \to \mathbb{k}[U'] \) are connected, with a homomorphism \( F^* \to M^* \) of complexes inducing an isomorphism on homology.
4. A downset resolution of $M^\bullet$ is a complex of $Q$-modules in which each $E_i$ is a direct sum of downset modules and the components $k[D] \to k[D']$ are connected, with a homomorphism $M^\bullet \to E^\bullet$ of complexes inducing an isomorphism on homology. These resolutions are finite, or dominate a constant subdivision or encoding, or are semialgebraic, PL, subanalytic, or of class $\mathcal{X}$ as in Definition 7.1.

**Theorem 7.17** (Syzygy theorem for complexes). Theorem 7.12 holds verbatim for a bounded complex $M^\bullet$ in place of the module $M$ as long as items 3, 4, and 5 are ignored.

**Proof.** As already noted, the proof is the same. It bears mentioning that finite injective and flat resolutions of complexes exist in the category of finitely determined $\mathbb{Z}^n$-modules because finite injective resolutions do (Proposition 6.7): any of the standard constructions that produce injective resolutions of complexes given that modules have injective resolutions works in this setting, and then Matlis duality (Definition 6.9) produces finite flat resolutions (see Remark 6.11). □

8. Derived applications: conjectures of Kashiwara and Schapira

The syzygy theorem for poset modules (Theorem 7.12) enhances an arbitrary finite constant subdivision (the tame condition) to a more structured subdivision (a finite encoding)—or even an algebraic presentation or resolution—whose pieces play well with the ambient combinatorial structure. In the context of partially ordered real vector spaces, this enhancement produces a $Q_+$-stratification from an arbitrary subanalytic triangulation. If the triangulation is subordinate to a given constructible derived $Q_+$-sheaf, meaning an object in the bounded derived category of constructible sheaves with microsupport contained in the negative polar cone of $Q_+$, then this enhancement produces a $Q_+$-structured resolutions of the given sheaf. These two instances of the syzygy theorem are the crucial ingredients for proofs of two conjectures due to Kashiwara and Schapira. Since the mathematical context is more sophisticated than the rest of the paper, these notions require review to make precise statements and proofs.

To avoid endlessly repeating hypotheses, and so readers can quickly identify when the same hypotheses are in effect, the blanket assumption in this section is for $Q$ to satisfy the following, where a positive cone is full if it has nonempty interior. Real partially ordered groups are partially ordered real vector spaces of finite dimension (Example 5.3).

**Hypothesis 8.1.** $Q$ is a real partially ordered group with closed, full, subanalytic $Q_+$.

8.1. Stratifications, topologies, and cones. This subsection collects the relevant definitions and theorems from the literature. The sizeable edifice on which the subject is built makes it unavoidable that readers seeing some of these topics for the first time will need to consult the cited sources for additional background. The goal here is to bring readers as quickly as possible to a general statement (Theorem 8.22) while circumscribing the ingredients necessary for its proof in such a way that those already familiar with the conjectures of Kashiwara and Schapira, specifically [KS17,
Conjecture 3.17] and [KS19, Conjecture 3.20], can skip seamlessly to Section 8.2 after skimming Section 8.1 for terminology.

**Remark 8.2.** Some basic notions are used freely without further comment.

1. The notion of simplicial complex here is the one in [KS90, Definition 8.1.1]: a collection $\Delta$ of subsets (called *simplices*) of a fixed vertex set that is closed under taking subsets (called *faces*), contains every vertex, and is locally finite in the sense that every vertex of $\Delta$ lies in finitely many simplices of $\Delta$. Any simplicial complex $\Delta$ has a realization $|\Delta|$ as a topological space, with each relatively open simplex $|\sigma|$ being an open convex set in an appropriate affine space.

2. The notion of subanalytic set in an analytic manifold is as in [KS90, §8.2].

3. The term *sheaf* on a topological space here means a sheaf of $k$-vector spaces. Sometimes in the literature this word is used to mean an object in the bounded derived category of sheaves of $k$-vector spaces; for clarity here, the term *derived sheaf* is always used when an object in the derived category is intended.

8.1.1. *Subanalytic triangulation.*

**Definition 8.3.** Fix a real analytic manifold $X$.

1. A *subanalytic triangulation* of a subanalytic set $Y \subseteq X$ is a homeomorphism $|\Delta| \cong Y$ such that the image in $Y$ of the realization $|\sigma|$ of the relative interior of each simplex $\sigma \in \Delta$ is a subanalytic submanifold of $X$.

2. A subanalytic triangulation of $Y$ is *subordinate* to a (derived) sheaf $\mathcal{F}$ on $X$ if $Y$ contains the support of $\mathcal{F}$ and (every homology sheaf of) $\mathcal{F}$ restricts to a constant sheaf on the image in $Y$ of every cell $|\sigma|$.

8.1.2. *Subanalytic constructibility.*

**Definition 8.4.** A (derived) sheaf on a real analytic manifold is *subanalytically weakly constructible* if there is a subanalytic triangulation subordinate to it. The word “weakly” is omitted if, in addition, the stalks have finite dimension as $k$-vector spaces.

**Remark 8.5.** Readers less familiar with constructibility can safely take Definition 8.4 at face value. For readers familiar with constructibility by other definitions, this characterization is a nontrivial theorem, which rests on the triangulability of subanalytic sets [KS90, Proposition 8.2.5] and other results concerning subanalytic stratification; see [KS90, §8.4] for the full proof of equivalence, especially Theorem 8.4.2, Definition 8.4.3, and part (a) of the proof of Theorem 8.4.5(i) there. Note that the modifier “subanalytically” in Definition 8.4 does not appear in [KS90], because the context there is subanalytic throughout. Also note that it makes no difference whether one takes constructible objects in the derived category or the derived category of constructible objects, since they yield the same result [KS90, Theorem 8.4.5]: every constructible derived sheaf is represented by a complex of constructible sheaves.
The reason to use subanalytic triangulation instead of arbitrary subanalytic stratification is the following, which is a step on the way to a constant subdivision.

**Lemma 8.6** ([KS90, Proposition 8.1.4]). *For a simplex σ in a subanalytic triangulation subordinate to a constructible sheaf \(\mathcal{F}\), there is a natural isomorphism \(\Gamma(|σ|, \mathcal{F}) \cong \mathcal{F}_x\) from the sections over \(|σ|\) to the stalk at every point \(x \in σ\).*

The reason for specifically including the piecewise linear (PL) condition in previous sections is for its application here, as one of the conjectures is in that setting. For this purpose, the sheaf version of this particularly strong type of constructibility is needed.

**Definition 8.7.** Fix \(Q\) satisfying Hypothesis 8.1.

1. A subanalytic subdivision (Definition 2.16.3) of \(Q\) is *subordinate* to a (derived) sheaf \(\mathcal{F}\) on \(Q\) if the restriction of \(\mathcal{F}\) to every *stratum* (meaning subset in the subdivision) is constant of finite rank.
2. If the subanalytic subdivision is PL (Definition 2.16.2) and \(Q\) is polyhedral (Definition 5.7), then \(\mathcal{F}\) is said to be *piecewise linear*, abbreviated PL.

**Remark 8.8.** Definition 8.7.2 is not verbatim the same as [KS19, Definition 2.3], which only requires \(Q\) to be a (nondisjoint) union of finitely polyhedra on which \(\mathcal{F}\) is constant. However, the notion of PL (derived) sheaf thus defined is the same, since any finite union of polyhedra can be refined to a finite union that is disjoint—that is, a partition. This refinement can be done, for example, by expressing \(Q\) as the union of (relatively open) faces in the arrangement of all hyperplanes bounding halfspaces defining the given polyhedra, of which there are only finitely many.

8.1.3. *Conic and Alexandrov topologies.*

**Definition 8.9.** Fix a real partially ordered group \(Q\) with closed positive cone \(Q_+\).

1. The *conic topology* on \(Q\) induced by \(Q_+\) (or induced by the partial order) consists of the upsets that are open in the ordinary topology on \(Q\).
2. The *Alexandrov topology* on \(Q\) induced by \(Q_+\) (or induced by the partial order) consists of all the upsets in \(Q\).

To avoid confusion when it might occur, write

1. \(Q_{\text{con}}\) for the set \(Q\) with the conic topology induced by \(Q_+\),
2. \(Q_{\text{ale}}\) for the set \(Q\) with the Alexandrov topology induced by \(Q_+\), and
3. \(Q_{\text{ord}}\) for the set \(Q\) with its ordinary topology.

**Remark 8.10.** The conic topology in Definition 8.9 is also known as the \(γ\)-topology, where \(γ = Q_+\) [KS90, KS18, KS19]. The Alexandrov topology makes just as much sense on any poset.

The type of stratification Kashiwara and Schapira specify [KS17, Conjecture 3.17] is not quite the same as subanalytic subdivision in Definition 2.16.3. To be precise, first recall two standard topological concepts.
Definition 8.11. A subset of a topological space $Q$ is \textit{locally closed} if it is the intersection of an open subset and a closed subset. A family of subsets of $Q$ is \textit{locally finite} if each compact subset of $Q$ meets only finitely many members of the family.

Definition 8.12 ([KS17, Definition 3.15]). Fix $Q$ satisfying Hypothesis 8.1.

1. A \textit{conic stratification} of a closed subset $S \subseteq Q$ is a locally finite family of pairwise disjoint subanalytic subsets, called \textit{strata}, which are locally closed in the conic topology and have closures whose union is $S$.

2. The stratification is \textit{subordinate} to a (derived) sheaf $\mathcal{F}$ on $Q$ if $S$ equals the support of $\mathcal{F}$ and the restriction of (each homology sheaf of) $\mathcal{F}$ to every stratum is locally constant of finite rank.

Remark 8.13. A conic stratification is called a $\gamma$-stratification in [KS17, Definition 3.15], with $\gamma = Q_+$. The only differences between conic stratification and subanalytic partition of a subset $S$ in Definition 2.16.3 are that

- conic stratifications are only required to be locally finite, not necessarily finite;
- conic strata are required to be locally closed in the conic topology (that is, an intersection of an open upset in $Q^\text{ord}$ with a closed downset in $Q^\text{ord}$); and
- the union need not actually equal all of $S$, because the only the union of the stratum closures is supposed to equal $S$.

Proposition 8.14. Fix a real partially ordered group $Q$ with closed positive cone $Q_+$.

1. The identity on $Q$ yields continuous maps of topological spaces

\[ \iota : Q^\text{ord} \to Q^\text{con} \quad \text{and} \quad \jmath : Q^\text{ale} \to Q^\text{con}. \]

2. Any sheaf $\mathcal{F}$ on $Q^\text{ord}$ pulled back from $Q^\text{con}$ has natural maps

\[ \mathcal{F}_q \to \mathcal{F}_{q'} \quad \text{for} \quad q \preceq q' \quad \text{in} \quad Q \]

on stalks that functorially define a $Q$-module $\bigoplus_{q \in Q} \mathcal{F}_q$. This functor from sheaves on $Q^\text{ale}$ to $Q$-modules is an equivalence of categories.

3. Similarly, any sheaf $\mathcal{G}$ on $Q^\text{ale}$ has natural maps

\[ \mathcal{G}_q \to \mathcal{G}_{q'} \quad \text{for} \quad q \preceq q' \quad \text{in} \quad Q \]

on stalks that functorially define a $Q$-module $\bigoplus_{q \in Q} \mathcal{G}_q$. This functor from sheaves on $Q^\text{con}$ to $Q$-modules is an equivalence of categories.

4. If sheaves $\mathcal{F}$ on $Q^\text{ord}$ and $\mathcal{G}$ on $Q^\text{ale}$ are both pulled back from the same sheaf $\mathcal{E}$ on $Q^\text{con}$, then the $Q$-modules in items 2 and 3 are the same.

5. The pushforward functor $\jmath_*$ is exact, and $\jmath_* \jmath^{-1} \mathcal{E} \cong \mathcal{E}$.

Proof. The maps in item 1 are continuous by definition: the inverse image of any open set is open because the ordinary topology refines each of the target topologies.

For item 2, if $\mathcal{F} = \iota^{-1} \mathcal{E}$ is pulled back to $Q^\text{ord}$ from a sheaf $\mathcal{E}$ on $Q^\text{con}$, then $\mathcal{F}$ has the same stalks as $\mathcal{E}$ (as a sheaf pullback in any context does), so the natural morphisms are induced by the restriction maps of $\mathcal{E}$ from open neighborhoods of $q$ to those of $q'$.
The result in 3 holds for arbitrary posets; for an exposition in a context relevant to persistence, see [Cur14, Theorem 4.2.10 and Remark 4.2.11] and [Cur19].

For item 4, the stalks $\mathcal{F}_q = \mathcal{E}_q = \mathcal{G}_q$ are the same.

For item 5, exactness is proved in passing in the proof of [BP19, Lemma 3.5], but it is also elementary to check that a surjection $\mathcal{G} \twoheadrightarrow \mathcal{G}'$ of sheaves on $Q_{\text{ale}}$ yields a surjection of stalks for the pushforwards to $Q_{\text{con}}$ because direct limits (filtered colimits) are exact. That $j_\ast j^{-1} \mathcal{E} \cong \mathcal{E}$ is because the natural morphism is the identity on stalks. □

8.1.4. Conic microsupport.

The microsupport of a (derived) sheaf on an analytic manifold $X$ is a certain closed conic isotropic subset of the cotangent bundle $T^*X$. The notion of microsupport is a central player in [KS90], to which the reader is referred for background on the topic. However, although the main result in this section (Theorem 8.22) is stated in terms of microsupport, the next theorem allows the reader to ignore it henceforth, as pointed out by Kashiwara and Schapira themselves [KS18, Remark 1.9], by immediately translating to the more elementary context of sheaves in the conic topology in Section 8.1.3.

**Theorem 8.15** ([KS18, Theorem 1.5 and Corollary 1.6]). Fix $Q$ satisfying Hypothesis 8.1. The pushforward $\iota_\ast$ of the map $\iota$ from Proposition 8.14.1 induces an equivalence from the category of sheaves with microsupport contained in the negative polar cone $Q_\vee^+$ to the category of sheaves in the conic topology. The pullback $\iota^{-1}$ is a quasi-inverse. The same assertions hold for the bounded derived categories.

**Remark 8.16.** The pushforward $\iota_\ast$ and the pullback $\iota^{-1}$ have concrete geometric descriptions. Since $\iota$ is the identity on $Q$, the pushforward of a sheaf $\mathcal{F}$ on $Q$ has sections

$$\Gamma(U, \iota_\ast \mathcal{F}) = \Gamma(U, \mathcal{F})$$

for any open upset $U$, where “open upset” means the same things as “upset that is open in the usual topology” and “subset that is open in the conic topology”. On the other hand, over any convex ordinary-open set $O$, the pullback to the ordinary topology of a sheaf $\mathcal{E}$ in the conic topology has sections

$$\Gamma(O, \iota^{-1} \mathcal{E}) = \Gamma(O + Q_+, \mathcal{E}),$$

namely the sections of $\mathcal{E}$ over the upset generated by $O$ [KS90, (3.5.1)].

**Remark 8.17.** What Theorem 8.15 does in practice is allow a given (derived) sheaf with microsupport contained in the negative polar cone $Q_\vee^+$ to be replaced with an isomorphic object that is pulled back from the conic topology induced by the partial order. The reason for mentioning the notion of microsupport at all is to emphasize that constructibility in the sense of Definition 8.4 requires the ordinary topology. This may seem a fine distinction, but the conjectures of Kashiwara and Schapira proved in Section 8.3 entirely concern the transition from the ordinary to the conic topology, so it is crucial to be clear on this point.
8.2. Resolutions of constructible sheaves.

**Definition 8.18.** Fix $Q$ satisfying Hypothesis 8.1.

1. A subanalytic upset sheaf on $Q$ is the extension by zero of the rank 1 constant sheaf on an open subanalytic upset in $Q^{\text{ord}}$.
2. A subanalytic downset sheaf on $Q$ is the pushforward of the rank 1 locally constant sheaf on a closed subanalytic downset in $Q^{\text{ord}}$.
3. A subanalytic upset resolution of a complex $\mathcal{F}^\bullet$ of sheaves on $Q^{\text{ord}}$ is a homomorphism $\mathcal{U}^\bullet \to \mathcal{F}^\bullet$ of complexes inducing an isomorphism on homology, with each $\mathcal{U}^i$ being a direct sum of subanalytic upset sheaves.
4. A subanalytic downset resolution of a complex $\mathcal{F}^\bullet$ of sheaves on $Q^{\text{ord}}$ is a homomorphism $\mathcal{F}^\bullet \to \mathcal{D}^\bullet$ of complexes inducing an isomorphism on homology, with each each $\mathcal{D}^i$ being a direct sum of subanalytic downset sheaves.

Either type of resolution is
- finite if the total number of summands across all homological degrees is finite;
- PL if $Q$ is polyhedral and the upsets or downsets are PL.

**Proposition 8.19.** Fix an upset $U$ in a real partially ordered group $Q$ with closed positive cone. If $U^\circ$ is the interior of $U$ in $Q^{\text{ord}}$, then the sheaves on $Q^{\text{ale}}$ corresponding to $k[U]$ and $k[U^\circ]$ push forward to the same sheaf on $Q^{\text{con}}$.

**Proof.** The stalk at $q$ of any sheaf on $Q^{\text{con}}$ is the direct limit over points $p \in q - Q^\circ_+$ of the sections over $p + Q^\circ_+$. In the case of the pushforward of the sheaf on $Q^{\text{ale}}$ corresponding to an upset module, these sections are $k$ if $p$ lies interior to the upset and 0 otherwise. The result holds because the upsets $U$ and $U^\circ$ have the same interior, namely $U^\circ$. □

**Proposition 8.20.** Fix a downset $D$ in a real partially ordered group $Q$ with closed positive cone. If $\overline{D}$ is the closure of $D$ in $Q^{\text{ord}}$, then the sheaves on $Q^{\text{ale}}$ corresponding to $k[D]$ and $k[\overline{D}]$ push forward to the same sheaf on $Q^{\text{con}}$.

**Proof.** Calculating stalks as in the previous proof, in the case of the pushforward of the sheaf on $Q^{\text{ale}}$ corresponding to a downset module, the sections over $p + Q^\circ_+$ are $k$ if $p$ lies interior to the downset and 0 otherwise. The result holds because the downsets $D$ and $\overline{D}$ have the same interior. □

**Remark 8.21.** The fundamental difference between Alexandrov and conic topologies reflected by Propositions 8.19 and 8.20 is explored in detail by Berkouk and Petit [BP19].

Here is the main result of Section 8. It is little more than a restatement of the relevant part of Theorem 7.17 in the language of sheaves.

**Theorem 8.22.** Fix $Q$ satisfying Hypothesis 8.1. If $\mathcal{F}^\bullet$ is a complex of compactly supported constructible sheaves on $Q^{\text{ord}}$ with microsupport in the negative polar cone $Q^\circ_-$ then $\mathcal{F}^\bullet$ has a finite subanalytic upset resolution and a finite subanalytic downset resolution. If $Q$ is polyhedral and $\mathcal{F}^\bullet$ is PL, then $\mathcal{F}^\bullet$ has PL such resolutions.
Proof. Using Theorem 8.15, assume that $\mathcal{F}^\bullet$ is pulled back to $Q^{\text{ord}}$ from $Q^{\text{con}}$, say $\mathcal{F}^\bullet = \iota^*\mathcal{E}^\bullet$. Since $\mathcal{F}^\bullet$ has compact support, any subordinate subanalytic triangulation (Definition 8.3) afforded by Definition 8.4 is necessarily finite because it is locally finite. The complex $F^\bullet = \bigoplus_{q \in Q} \mathcal{F}_q^\bullet$ of $Q$-modules that comes from Proposition 8.14 is tamed by the triangulation, which is a constant subdivision (Definition 2.7) because

- simplices are connected, so locally constant sheaves on them are constant, and
- $\Gamma(|\sigma_p|, \mathcal{F}^i_p) \to \mathcal{F}^i_p \to \mathcal{F}^i_q \leftarrow \Gamma(|\sigma_q|, \mathcal{F}^i_q)$ is locally constant—and hence constant, as simplices are connected—when $p \preceq q$ in $Q$. Here $\sigma_x$ is the simplex containing $x$, the middle arrow is from Proposition 8.14, and the outer arrows are the natural isomorphisms from Lemma 8.6.

Hence the complex $F^\bullet$ of $Q$-modules has resolutions of the desired sort by Theorem 7.17. Viewing any of these resolutions as a complex of sheaves on $Q^{\text{ale}}$ via Proposition 8.14, push it forward from the Alexandrov topology to the conic topology via the exact functor $j_*$ in Proposition 5.8.14. The resulting complex of sheaves on $Q^{\text{con}}$ is a resolution of a complex isomorphic to $\mathcal{E}^\bullet$ by Proposition 8.14 and 8.15. The upsets or downsets in the summands of the resolution may as well be assumed open or closed, respectively, by Propositions 8.19 or 8.20. The proof is concluded by pulling back the resolution from $Q^{\text{con}}$ to $Q^{\text{ord}}$ via the equivalence of Theorem 8.15. \hfill $\square$

Remark 8.23. Theorem 8.22 assumes compact support to get finite instead of locally finite subdivisions. The application in Section 8.3 to constructible sheaves without any assumption of compact support yields a locally finite subdivision by reducing to the case of compact support.

Remark 8.24. The final sentence of Theorem 8.22 is true with “polyhedral” and “PL” all replaced by “semialgebraic”, with the same proof, as long as the definitions of these semialgebraic concepts in the constructible sheaf setting are made appropriately. The semialgebraic constructible sheaf versions are not treated here because they are not relevant to the conjectures proved in Section 8.3.

8.3. Stratifications from constructible sheaves.

Corollary 8.25 ([KS19, Conjecture 3.20]). Fix $Q$ satisfying Hypothesis 8.1 with $Q_+$ polyhedral. If $\mathcal{F}^\bullet$ is a PL object in the derived category of compactly supported constructible sheaves on $Q^{\text{ord}}$ with microsupport contained in the negative polar cone $Q_+^\vee$ then the isomorphism class of $\mathcal{F}^\bullet$ is represented by a complex that is a finite direct sum of constant sheaves on bounded polyhedra that are locally closed in the conic topology.

Proof. The statement would directly be a special case of Theorem 8.22 were it not for the boundedness hypothesis on the polyhedra, since either a PL upset or PL downset resolution would satisfy the conclusion. That said, boundedness is easy to impose: since $\mathcal{F}^\bullet$ has compact support, and the resolution has vanishing homology outside of the support of $\mathcal{F}^\bullet$, each upset or downset sheaf can be restricted to the support of $\mathcal{F}^\bullet$ and extended by 0. \hfill $\square$
Corollary 8.26 ([KS17, Conjecture 3.17]). Fix $Q$ satisfying Hypothesis 8.1. If a compactly supported derived sheaf with microsupport in the negative polar cone $Q_+^\vee$ is subanalytically constructible, then its support has a subordinate conic stratification.

Proof. Part (ii) in the proof of [KS18, Theorem 3.17] reduces to the case where the support of the given derived sheaf is compact. The argument is presented in the case where $Q$ is polyhedral and the derived sheaf is PL, but the argument works verbatim for $Q$ satisfying Hypothesis 8.1, without any polyhedral or PL assumptions, because the requisite lemma, namely [KS18, Lemma 3.5]—and indeed, all of [KS18, §3.1]—is stated and proved in this non-polyhedral generality. So henceforth assume the given derived sheaf has compact support.

Remark 8.5 allows the assumption that the given derived sheaf is represented by a complex $\mathcal{F}^\cdot$ of constructible sheaves. Theorem 8.22 produces a subanalytically indicator resolution, which for concreteness may as well be an upset resolution. Each upset that appears as a summand in the resolution partitions $Q$ into the upset itself, which is open subanalytic, and its complement, which is a closed subanalytic downset. The common refinement of the partitions induced by the finitely many open subanalytic upsets in the resolution and their closed subanalytic downset complements is a partition of $Q$ into finitely many strata such that

- each stratum is subanalytic and locally closed in the conic topology, and
- the restriction of $\mathcal{F}^\cdot$ to each stratum has constant homology.

The strata with nonvanishing homology form the desired conic stratification. □

Remark 8.27. The reference in [KS17, Conjecture 3.17] to a cone $\lambda$ contained in the interior of the positive cone union the origin appears to be unnecessary, since (in the notation there) any $\gamma$-stratification is automatically a $\lambda$-stratification by [KS17, Definition 3.15] and the fact that $\lambda \subseteq \gamma$.

Remark 8.28. One main point of Corollary 8.26 is that, while the notion of a sheaf with microsupport contained in the negative polar cone of $Q_+$ is equivalent to the notion of a sheaf in the conic topology, the notion of constructibility has until now only been available on the microsupport side, where simplices from arbitrary subanalytic triangulations achieve constancy of the sheaves in question. One way to interpret Corollary 8.26 is that constructibility can be detected entirely with the more rigid conic topology, without the flexibility of appealing to arbitrary subanalytic triangulations.

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