Simple curves on hyperbolic tori

By Greg McShane and Igor Rivin*

Abstract. Let $T$ be a once punctured torus, equipped with a complete hyperbolic metric. Herein, we describe a new approach to the study of the set $S$ of all simple geodesics on $T$. We introduce a valuation on the homology $H_1(T, \mathbb{Z})$, which associates to each homology class $h$ the length $\ell(h)$ of the unique simple geodesic homologous to $h$, and show that $\ell$ extends to a norm on $H_1(T, \mathbb{R})$. We analyze the boundary of the unit ball $B(\ell)$ and the variation of the area of $B(\ell)$ over the moduli space of $T$. These results are applied to obtain sharp asymptotic estimates on the number of simple geodesics of length less than $L$.

Version française abrégée

Soit $T$ un tore troué, muni d’une métrique hyperbolique complète, d’aire finie. Nous présentons une nouvelle approche de l’étude de l’ensemble $S$ de toutes les géodésiques fermées simples (sans points doubles) de $T$. Nous introduisons une application sur l’homologie $H_1(T, \mathbb{Z})$, qui associe à chaque classe $h \in H_1(T, \mathbb{Z})$ indivisible la longueur $\ell(h)$ de l’unique géodésique simple homologue à $h$, et nous démontrons que $\ell$ s’étend en une norme sur $H_1(T, \mathbb{R})$. Nous étudions la géométrie de la sphère $\partial B(\ell)$ et la variation de l’aire de $B(\ell)(T)$ sur l’espace des modules. On utilise ces résultats pour donner des estimations asymptotiques du nombre de géodésiques fermées simples de longueur inférieure à $L$.

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Nous dirons que $m$ est plongée lorsque l'image de $m$ est une réunion disjointe de lacets simples – l'application $m$ n'est pas forcément injective. Notre premier résultat est que pour chaque classe $h \in H_1(S, \mathbb{Z})$, il y a une multi-courbe $m$ de longueur minimale, homologue à $h$. Cette multi-courbe $m$ est plongée, et toutes les composantes connexes de $m$ sont des lacets géodésiques.

Dans le cas spécial où $S$ est un tore troué, nous savons qu'il y a un seul lacet géodésique simple dans chaque $h \in H_1(S, \mathbb{Z})$, quand $h$ est indivisible, et que deux lacets géodésiques simples quelconques se rencontrent. Donc, il y a une seule multi-courbe minimale $m_h$ dans chaque classe d'homologie $h$ – si $h$ est indivisible, $m_h$ est l’unique lacet géodésique simple homologue à $h$ et, autrement, $m_h$ revêt un lacet géodésique simple.

Maintenant, nous pouvons définir une valuation $\ell : H_1(T, \mathbb{Z}) \rightarrow \mathbb{R}$, telle que $\ell(h)$ est la longueur de la (seule) multi-courbe homologue à $h$. Nous pouvons démontrer (en utilisant la discussion précédente) que $\ell$ s'étend en une norme sur $H_1(T, \mathbb{R}) \simeq \mathbb{R}^2$. Soit $B_\ell$ la boule unité de cette norme – $B_\ell$ est un convexe symétrique dans le plan $\mathbb{R}^2$. La sphère unité $\partial B_\ell$ est une représentation de l’espace projectif des laminations mesurées du tore troué.

La géométrie de la boule $B_\ell$ est assez intéressante. Le théorème suivant suggère que la sphère $\partial B_\ell$ est vraiment irrégulière:

**Théorème.** Le bord de la boule unité, $\partial B_\ell$, a un coin en tous les points de pente irrationnelle. L'angle extérieur au coin de pente $p/q$ décroît exponentiellement comme fonction de $\max(p,q)$. En tout point de pente irrationnelle le bord est infiniment plat.

Le terme “infiniment plat” est défini de la manière suivante:

Après une isométrie du plan euclidien, nous pouvons supposer que le point $p$ est l’origine du plan, une droite de contact à $B_\ell$ dans $p$ est la droite $y = 0$ et la boule $B_\ell$ est située dans le demi-espace $y \geq 0$. Il existe un voisinage $|x| < \delta_0$, tel que le bord de $B_\ell$ est un graphe d’une fonction convexe $f$. Nous disons que $\partial B_\ell$ est *infiniment plat* en $p$, si pour tout entier $N$, il existe un $0 < \delta < \delta_0$, tel que $f(x) \leq |x|^N$, quand $|x| < \delta$.

Nous appliquons ces résultats pour calculer le nombre asymptotique de lacets géodésiques de longueur bornée par $L$ sur le tore troué $T$. Nous démontrons que

$$N_L = \frac{L^2}{\text{Area } B_\ell} + O(L \log L).$$

Ce résultat est assez simple modulo la discussion précédente – la quantité $N_L$ est égale au nombre des points entiers *indivisibles* – $(m,n)$ est indivisible quand le rayon de l’origine à $(m,n)$, ne contient aucun point entier plus petit que $(m,n)$; Hardy et Wright utilisent le terme “visible” – dans $LB_\ell$. Sans l’hypothèse d’indivisibilité, l’erreur est $O(L)$; l’estimation pour les points
s’ensuit après une inversion de Möbius. C’est un fait remarquable que cette estimation est presque exacte:

**Théorème.** Il y a une partie dense de l’espace des modules pour laquelle l’erreur ci-dessus est plus grande que $O(L)$.

Nous étudions aussi la géométrie de $\mathcal{B}_\ell(T)$ comme une fonction de $T$ dans l’espace des modules $\mathcal{M}_{1,1}$. Nous démontrons que l’aire de $\mathcal{B}_\ell(T)$ tend vers l’infini quand $T$ s’approche des cusps de l’espace de modules.

La structure hyperbolique la plus symétrique, et d’intérêt arithmétique maximal est le tore troué modulaire – le tore troué $T_m$ unique tel que $T_m = \Gamma \backslash \mathbb{H}^2$, où $\Gamma$ est un sous-groupe de $SL(2, \mathbb{Z})$. Nous croyons que:

**Conjecture 1.** L’aire de $\mathcal{B}_\ell(T)$ est minimale pour $T = T_m$ (dans cet cas, l’aire est égale à $A_m = 1.08...$

et aussi

**Conjecture 2.** Pour $T_m$, l’estimation n’est pas générique. C’est-à-dire:

$$N_L(T_m) = L^2 / A_m + O(L^{\frac{3}{2} + \epsilon}),$$

pour tout $\epsilon$, comme $L \rightarrow \infty$.

Nous croyons que la deuxième conjecture est très difficile.

**Introduction**

Let $T$ be a torus with one puncture, equipped with a complete hyperbolic metric of finite area. In this paper we describe a new approach to the study of the set $\mathcal{S}$ of all simple geodesics (that is, those without self-intersections) on $T$. The set $\mathcal{S}$ is of considerable interest in both geometry and number theory (see [4]).

The plan of the paper is as follows. In sections 1 and 2, we introduce a valuation on the homology group $H_1(T, \mathbb{Z})$, which associates to each homology class $h$ the length $\ell(h)$ of the unique simple geodesic homologous to $h$, and show that $\ell$ extends to a norm on $H_1(T, \mathbb{R})$. In section 2 we investigate the geometry of the norm $\ell$, by analyzing the boundary of the unit ball $\mathcal{B}(\ell)$. In section 3 we analyze the behavior of $\ell$ as $T$ varies over the moduli space $\mathcal{M}_{1,1}$ of finite area complete hyperbolic metrics on the punctured torus. Finally, in section 4 we apply these results to obtain asymptotic estimates on the number of simple geodesics with length bounded above by $L$.

**1. Minimal multicurves**

Let $S$ be a hyperbolic surface of finite volume with at most one cusp. We define a multicurve $m$ on $S$ to be a map from a (not necessarily connected) 1-manifold $M$ to $S$. We define the length of $m$ to be the sum of the lengths of the images of components of $M$. We say that a multicurve $m$ is embedded if the image of $m$ is the union of simple closed curves $\gamma_1, \ldots, \gamma_k$ on $S$. Note
that the map $m$ may cover some of the components $\gamma_i$ multiple times, so this does not coincide with the usual meaning of embedding. A multicurve defines a singular chain, which, in turn, defines a homology class in $H_1(S, \mathbb{Z})$.

**Theorem 1.1.** Let $h \in H_1(S, \mathbb{Z})$ be a non-trivial homology class. There exists a multicurve $m$ representing $h$ of minimal length, and $m$ is embedded, with all components geodesic.

**Corollary 1.** Let $T$ be a punctured torus equipped with a hyperbolic structure. Then, the shortest multicurve representing a non-trivial homology class $h$ is a simple closed geodesic if $h$ is a primitive homology class (that is, not a multiple of another class), and a multiply covered geodesic otherwise. In addition, the shortest multicurve representing $h$ is unique.

The reader can find the proofs of these results in [6].

### 2. A norm on homology

For each primitive homology class $h \in H_1(T, \mathbb{Z})$, define $\ell(h)$ to be the length of the unique simple geodesic $\gamma_h \sim h$. The uniqueness is guaranteed by Corollary 1.1 of the previous section. For a non-primitive homology class $(m, n)$, define $\ell(h) = \gcd(m, n) \ell(h/(\gcd(m, n)))$. In this section we will show that $\ell$ can be extended to a *norm* on $H_1(T, \mathbb{R}) \cong \mathbb{R}^2$.

The results of section 1 imply, in particular, that $\ell$ satisfies a triangle inequality on $H_1(T, \mathbb{Z})$.

*Note.* The triangle inequality is *strict* (that is, if $\ell(n, m) + \ell(p, q) = \ell(n + p, m + q)$, then $(n, m)$ and $(p, q)$ are both multiples of the same class).

We can now extend $\ell$ to $H_1(T, \mathbb{R})$; first to $H_1(T, \mathbb{Q})$ by linearity, and then to $H_1(T, \mathbb{R})$ by continuity (which is an immediate consequence of the triangle inequality). It is, furthermore, clear that $\ell$ still satisfies a triangle inequality, and is, thus, a *semi-norm*. To show that it is a norm, requires some study of simple geodesics from a group-theoretic standpoint, and some elementary techniques of geometric group actions. The reader is referred to [6] for the details.

In the figure, the reader can find a computer-generated picture of the unit ball $\mathcal{B}$ of the norm $\ell$ for $T$ – the modular torus (the only one corresponding to a subgroup of $SL(2, \mathbb{Z})$, whose fundamental group is generated by the matrices

$$
\begin{pmatrix}
1 & 1 \\
1 & 2
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & -1 \\
-1 & 2
\end{pmatrix}.
$$
Several very accurate computer generated renderings of the unit ball were made by the authors. To the casual observer first seeing these pictures $B$ appears polygonal, but it is not hard to see that this is far from the truth – since the triangle inequality is strict, no three points with rational slope are collinear. To describe the true geometry of $B$ (for any complete hyperbolic structure of finite area), we first need to define some terms.

Let $B$ be a convex body in the plane, and let $p \in \partial B$. After applying a euclidian isometry we can suppose that the point $p$ is at the origin in the plane and that $B$ lies in the half space $y \geq 0$. There is a neighborhood $|x| < \delta_0$, such that the the boundary of $\partial B$ is a graph of a convex function $f$. We say that $\partial B$ is flat to order $n$ at $p$ if there exists a $\delta, 0 < \delta < \delta_0$, such that $f(x) \leq |x|^n$, when $|x| < \delta$. We say that $\partial B$ is flat to infinite order at $p$, if it is flat to order $n$, for every $n > 0$.

Now we can state:

**Theorem 2.1.** The boundary of the unit ball $B_\ell$ has a corner at each point of rational slope. The exterior angle at such a corner at the slope $p/q$ decreases exponentially as a function of $\max(p,q)$ (the height). At a point of irrational slope $\theta$, the boundary of $B_\ell$ is flat, to infinite order.
The proof of Theorem 2.1 is somewhat involved (and presented in full in [7]). Among the main ingredients are the following:

**Fricke trace relation.** This says that for every pair of elements $A, B$ of $SL(2, \mathbb{R})$

$$2 + \text{tr}[A,B] = \text{tr}^2 A + \text{tr}^2 B + \text{tr}^2 AB - \text{tr} A \text{ tr} B \text{ tr} AB,$$

where $[A,B](= ABA^{-1}B^{-1})$ is the commutator of $A$ and $B$. (See, eg, [2]).

When $A$ and $B$ are associated generators of the fundamental group of a cusped torus, their commutator is a parabolic element (a simple loop around the cusp), and so the left hand side of this relation vanishes.

**Elliptic involution.** The punctured torus is elliptic, and has four Weierstrass points, preserved by the elliptic involution. One of these is the cusp, the other three have the property that every simple closed geodesic passes through two of them, and further if $\alpha$ passes through $W_1, W_2$ and $\beta$ passes through $W_1, W_3$, then $\alpha \beta$ passes through $W_2$ and $W_3$.

The trace relation can be used to compute the angle between the geodesics $\alpha$ and $\beta$ at the point $W_1$ in terms of their lengths, as follows:

**Theorem 2.2.**

$$\sinh \ell(\alpha) \sinh \ell(\beta) \sin \angle \alpha, \beta = 1.$$  

This identity can then be used to show that the angle between $\alpha$ and $\beta$ goes to $0$ (or $\pi$) exponentially fast with the greater of the two lengths, and this, in turn leads to a proof that the triangle inequality is exponentially close to equality, again, as a function of the greater of the two lengths. This, in turn, is the main tool in the proof of Theorem 2.1.

### 3. Variation over moduli space

In this section we describe the variation of $\mathcal{B}_\ell$ over the moduli space of complete finite area hyperbolic metrics on the punctured torus. First, we note that points on $\partial \mathcal{B}_\ell$ correspond to projective measured laminations on the punctured torus – thus $\mathcal{B}_\ell$ can be viewed as an explicit picture of the projective lamination space $\mathcal{P}L$. It should be noted that a somewhat similar combinatorial picture of that space is constructed in [8]. It is not hard to show that the topology of convergence in $\ell$ is exactly the topology on $\mathcal{P}L$. Thus we can use Kerckhoff’s results [5] to show the following:

**Theorem 3.1.** Every point of $\partial \mathcal{B}_\ell(T)$ varies analytically over moduli space.
**Corollary 1.** The area of $B_\ell(T)$ varies analytically over moduli space.

**Theorem 3.2.** The area of $B_\ell(T)$ goes to infinity as $T$ approaches the cusps of the moduli space.

**Proof.** The cusps of the moduli spaces are characterized by the condition that the length of some simple geodesic $\gamma$ approaches to 0. Pick a point $T_0$ of moduli space, pick a simple geodesic $\gamma$, and let $\gamma'$ be the shortest associated generator of the fundamental group. Since $\gamma'$ is shortest, it follows that $\ell(\gamma'\gamma) \geq \ell(\gamma')$ and $\ell(\gamma'\gamma^{-1}) \geq \ell(\gamma')$. Keeping in mind that $\cosh \frac{\ell(\gamma)}{2} = \text{tr} \gamma/2$, (where we abuse notation by using $\gamma$ both for the geodesic and the corresponding element of $SL(2, \mathbb{R})$, the above inequality and the Fricke trace relation imply:

\begin{equation}
\cosh \frac{\ell(\gamma')}{2} \leq \cosh \frac{\ell(\gamma'\gamma^{-1})}{2} = \cosh \frac{\ell(\gamma')}{2} \cosh \frac{\ell(\gamma)}{2} \left(1 - \sqrt{1 - \frac{1}{\cosh^2 \frac{\ell(\gamma)}{2}} - \frac{1}{\cosh^2 \frac{\ell(\gamma)}{2}}}ight).
\end{equation}

A computation transforms the above into

\begin{equation}
1 - \frac{1}{\cosh^2 \frac{\ell(\gamma)}{2}} \geq \frac{1}{\cosh^2 \frac{\ell(\gamma')}{2}} \geq \frac{2}{\cosh \frac{\ell(\gamma)}{2}} - \frac{2}{\cosh^2 \frac{\ell(\gamma)}{2}},
\end{equation}

where the first inequality stems from the positivity of the expression inside the radical in equation 3.1

When $\gamma$ is very short, we can expand $\cosh \frac{\ell(\gamma)}{2}$ in a Taylor series, to obtain

\[
\left| \frac{1}{\cosh^2 \frac{\ell(\gamma)}{2}} - \left( \frac{\ell(\gamma)}{2} \right)^2 \right| = O(\ell^4(\gamma)) \implies \ell(\gamma') \sim \log \ell(\gamma).
\]

Since

\[
\text{Area } B_\ell \geq \frac{1}{\ell(\gamma)\ell(\gamma')},
\]

it follows that $\lim_{\ell(\gamma) \to 0} \text{Area } B_\ell = \infty$. \qed

The above proof actually shows a little more – by combining the estimates with Theorem 2.2, we have

**Corollary 1.** The angle between a very short simple geodesic $\gamma$ and the shortest conjugate geodesic $\gamma'$ approaches $\pi/2$, as $\ell(\gamma) \to 0$.

**Definition.** The systole $\text{sys} T$ is the shortest closed geodesic on $T$.

A modification of the proof of Theorem 3.2 can be used to obtain the following result:

**Theorem 3.3.** The modular torus is isosystolic:

\[
\text{length } \text{sys } T_{\text{mod}} = \max_{T \in \mathcal{M}^{1,1}} \text{length } \text{sys } T.
\]
We would like to propose the following conjecture:

**Conjecture 3.4.** The area of $\mathcal{B}_L(T)$ is minimized when $T$ is the modular torus.

4. Number of simple geodesics with length bounded above

The results of section 2 tell us that the number of simple geodesics of length bounded by $L$ is just the number of primitive lattice points in $L\mathcal{B}_L$, which, by the usual trivial lattice point estimate combined with Möbius inversion tells us that

**Theorem 4.1.** The number of simple geodesics of length bounded by $L$ is

$$N(L) = \frac{1}{\zeta(2)} \text{Area}(\mathcal{B}_L) L^2 + O(L \log L),$$

where $\zeta$ is the Riemann zeta function.

A similar estimate for the modular torus was obtained (in a totally different language, and by number-theoretic methods) by D. Zagier in [9]. He conjectured that the error bound was essentially sharp. Indeed, we can show:

**Theorem 4.2.** The error term in Theorem 4.1 is at least $O(L)$ for a dense set in moduli space.

The difference between $O(L)$ and $O(L \log L)$ is caused by the difference between counting all lattice points and primitive lattice points. To show theorem 4.2, it is enough to observe that since the direction of the support line at any fixed slope varies analytically over moduli space, its slope will be rational at a point of irrational slope at a dense set of points in moduli space. Call one of these points $T_r$. Now, consider a convex set $S$ in the plane, such that there is a $p \in \partial S$, such that the curvature of $\partial S$ at $p$ vanishes to infinite order, and the slope of the normal is rational. It can be shown (see [3]) that the number of lattice points in $tS$ is $t^2 \text{Area} S + O(t)$, and the error term is sharp. It follows from the proof of that fact, that the error term for the primitive lattice point problem is at least linear. Thus, the error term for $T_r$ in theorem 4.2 is seen to be somewhere between $O(L)$ and $O(L \log L)$. While it is not at all clear whether the error term is better than linear anywhere in moduli space, we put forward the following conjecture:

**Conjecture 4.3.** The error term in Theorem 4.2 for the modular torus is of order $O(L^{\frac{3}{2} + \epsilon})$. 
This is contrary to Zagier’s conjecture which was also stated for the modular torus.

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