Abstract. We give an explicit isomorphism between the Gassner representation and the first weight level of a representation of quantum sl(2). Then we construct and provide matrices for colored versions of the BKL representation and higher Lawrence’s representations.

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1. Introduction

Definition 1.1 (Braid groups, Pure braid groups). Let \(n \in \mathbb{N}\), and let \(D_n\) be the unit disk with \(n\) punctures. It consists in the unit disk with \(n\) points denoted \(p_1, \ldots, p_n\) (considered inside and to lie on the real line) removed. The braid group on \(n\) strands is the mapping class group of the punctured disk \(D_n\).

\[
\mathcal{B}_n = \text{Mod}(D_n).
\]

It is a group generated by \(n-1\) elements satisfying the so called “braid relations”:

\[
\mathcal{B}_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ if } |i-j| \leq 2 , \text{ for } i = 1, \ldots, n-2 \right\rangle
\]

where the generator \(\sigma_i\) corresponds to the isotopy class of the half Dehn twist swapping \(p_i\) and \(p_{i+1}\).

The pure braid group on \(n\) strands \(\mathcal{PB}_n\) consists in the pure mapping class group of \(D_n\), namely mapping classes of homeomorphisms fixing punctures pointwise. Let \(\text{perm} : \mathcal{B}_n \rightarrow \mathfrak{S}_n\) be the morphism assigning the permutation it involves to a given braid, then \(\mathcal{PB}_n\) is the kernel of \(\text{perm}\).
Proposition 1.2 ([Bir, Lemma 1.8.2]). There exists a family of elements denoted $A_{i,j}$ for $1 \leq i < j \leq n$ generating the pure braid group $\mathcal{PB}_n$. Their expressions in terms of generators of $\mathcal{B}_n$ is provided in [Bir, Lemma 1.8.2].

In [Law], R. Lawrence constructs homological representations for the braid groups using the fact that they act by homeomorphisms on punctured disks.

Definition 1.3 (Configuration space of the punctured disk). Let $n, m$ be integers. The configuration space $C_{n,m}$ of $m$ unordered points in $D_n$ is defined as follows:

$$C_{n,m} = \{(z_1, \ldots, z_m) \in (D_n)^m \text{ s.t. } z_i \neq z_j \text{ for } i \neq j\}/S_m$$

Then the construction uses a local system:

$$L_m : \pi_1(C_{n,m}) \to \mathbb{Z}^2$$

that turns homology groups $H_m(C_{n,m})$ into modules over the ring of Laurent polynomials in two variables. The braid group $\mathcal{B}_n$ acts on $H_m(C_{n,m})$ and its action commutes with the action of Laurent polynomials. This provides a graded family of Lawrence representations for the braid groups (graded by $m \in \mathbb{N}$). See [Ito, Section 3.1].

The notoriety of Lawrence’s representations comes from independent work of S. Bigelow and D. Krammer ([Big, Kra]) showing the faithfulness of braid representations at the second level of the grading ($m = 2$), the one we refer to as the BKL representation. It is the first known linear and finite dimensional representation of the braid groups.

It was proved that Lawrence’s representations can be recovered as submodules of some quantum representations of quantum $\mathfrak{sl}(2)$, $U_q\mathfrak{sl}(2)$. In [J-K] for the case of BKL representation, and in [K2] for all the other level of the grading but the isomorphism does not hold on the ring of Laurent polynomials, but only generically when variables are specialized to complex values (this result is summed up in [Ito, Theorem 4.5]). In [M] Lawrence’s representations are extended to relative homology modules; this allows to recover the braid group representations on the whole (tensor product of) quantum Verma modules, while over the ring of Laurent polynomials. This result recovers Kohno’s theorem ([K2]) allowing to remove genericity conditions. A first identification with quantum representation was provided by Zinno in [Zi], where he identifies the BKL representation with a quantum algebraic object, namely the quotient of the Birman-Wenzl-Murakami algebra. In [M-W], they recover BKL representation as a submodule of some representations of quantum $\mathfrak{sl}(2|1)$.

The goal of the present paper is to provide a concrete approach to the colored Lawrence representations by providing explicit matrices for the action of the generators of the braid groups. By colored we mean versions with $n + 1$ variables instead of two. This is achieved by modifying Morphism (1) by changing the target group to $\mathbb{Z}^{n+1}$. These colored Lawrence representations are also defined and treated in [M] [K1].

The first level of Lawrence representations is known to be the Burau representation (see [J-K] for instance), whose colored version is called the Gassner representation, see [L-N, Mo]. In Section 2 we present these representations from the Magnus representations point of view (involving Fox derivatives), we provide matrices obtained from Burau matrices, and we relate them to quantum representations of braid groups arising from quantum algebra $U_q\mathfrak{sl}(2)$.

In Section 3 we construct a colored version for BKL representations (level two of the grading) using the methods and basis provided in [Big]. We compute matrices in this context, using Fox calculus and Bigelow’s forks–noodles pairing.

Section 4 gives a panorama on different bases for Lawrence representations and the relations with quantum representations. We then compute matrices for colored higher Lawrence representations ($m \geq 2$) using homology techniques developed in [M]. We provide a computation on an integral basis, namely a basis of the entire homology module (which is not the case of the forks’ basis introduced by Bigelow). This basis differs from that used in Section 3 and as explained in Section 4 it has better properties which allow to match the action of the braid group to the quantum representations while working over the ring of Laurent polynomials, see [M, Theorem 3]. In section 4.4 we explain how to operate the change of basis in the case $n = 3$, the general case is analogue but the notation is just cumbersome.
The Appendix (Section 5) explains why passing to colored representations involves restriction to the pure braid group, and suggests point of views on how to consider representations of the whole braid group or representations of the colored braid groupoid.

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2. Gassner representation recovered by \( U_q sl(2) \) representations.

In this section we present Gassner representations in Section 2.1, then some quantum representations of the braid groups in Section 2.2. Finally we show how to recover one from the other in Section 2.3.

2.1. Gassner representations. In this section we provide a survey about the Gassner representation, that is a "colored version" of the Burau representation. By colored, we mean with several variables, such that if one specializes all variables to the same one, it recovers the Burau representation. They both arises from the family of Magnus representations, defined from a representation of the braid group in the automorphism group of the free group, using Fox free differential calculus. We introduce this family following [Bir].

2.1.1. Magnus representations.

Definition 2.1 (Fox free differential calculus). For each \( j = 1, \ldots, n \) there is a map:

\[
\frac{\partial}{\partial x_j} : \mathbb{Z}F_n \to \mathbb{Z}F_n
\]

given by:

\[
\frac{\partial}{\partial x_j} (x_{\mu_1}^{\epsilon_1} \cdots x_{\mu_r}^{\epsilon_r}) = \sum_{i=1}^r \epsilon_i \delta_{\mu_i,j} x_{\mu_i}^{\epsilon_i} \cdots x_{\mu_i}^{(\epsilon_i-1)/2},
\]

and

\[
\frac{\partial}{\partial x_j} \left( \sum a_g g \right) = \sum a_g \frac{\partial}{\partial x_j} (g), \quad g \in F_n, \quad a_g \in \mathbb{Z},
\]

where \( \epsilon_i = \pm 1 \), \( \delta \) is the Kronecker symbol, and \( \mathbb{Z}F_n \) is the the group ring of \( F_n \).

Let \( \Phi \) be a homomorphism acting on \( F_n \) and \( A_{\Phi} \) be any group of automorphisms of \( F_n \) satisfying:

\[
\Phi(x) = \Phi(a(x))
\]

for each \( x \in F_n \) and \( a \in A_{\Phi} \).

Definition 2.2 (Magnus representation, [Bir] Theorem 3.9). Let \( a \in A_{\Phi} \) and \( [a]^{\Phi} \) be the following \( n \times n \) matrix:

\[
[a]^{\Phi} = \left[ \Phi \left( \frac{\partial (a(x_i))}{\partial x_j} \right) \right]_{i,j}.
\]

Then the morphism:

\[
\left\{ \begin{array}{c}
A_{\Phi} \to \mathcal{M}(n, \mathbb{Z}F_n) \\
\{ a \to [a]^{\Phi} \}
\end{array} \right.
\]

is a well defined group homomorphism, called a Magnus representation.

Let \( Z_n \) be the free abelian group of rank \( n \) with free basis \( t_1, \ldots, t_n \) and \( a \) be the following morphism:

\[
a : \left\{ \begin{array}{c}
F_n \to Z_n \\
x_i \mapsto t_i
\end{array} \right.
\]
Definition 2.3 (Gassner representation of the pure braid group). Let $1 \leq r < s \leq n$ and $A_{r,s} \in PB_n$ the corresponding generator of the pure braid group on $n$ strands. Let $[A_{r,s}]^a$ be the following matrix:

$$[A_{r,s}]^a = \left[ a \left( \frac{\partial (A_{r,s}(x_i))}{\partial x_j} \right) \right]_{i,j}.$$ 

Then the morphism:

$$PB_n \rightarrow \mathcal{M}(n, \mathbb{Z}[\mathbb{Z}_n])$$

$$A_{r,s} \mapsto [A_{r,s}]^a$$

is a Magnus representation, called the Gassner representation of the pure braid group.

Lemma 2.4 ([Bir, Lemma 3.11.1]). The Gassner representation is reducible to an $(n-1) \times (n-1)$ representation.

Sketch of proof. Let $g_i = x_1 \cdots x_i \in F_n$, this provides a change of generator basis for $F_n$. The matrices:

$$\left[ a \left( \frac{\partial (A_{r,s}(g_i))}{\partial g_j} \right) \right]_{i,j}$$

correspond to Gassner matrices given in another basis associated to the $g_i$’s. After computation one remarks that the last rows and columns for all these matrices is $(0, \ldots, 1)$ so that it can be deleted. \qed

Remark 2.5. Let $t = t_1 = \cdots = t_n$, then the Gassner representation becomes the Burau representation. See [Bir, Section 3.3].

2.1.2. Matrices. Now we give a concrete definition of the Gassner representation, with concrete matrices. They were first defined in Definition 2.3, but we follow [B-N] from now on to obtain matrices. In [B-N], the Gassner representation is built as a “multi-color” Burau one. Let $t$ be a formal variable and $U_{n,i}(t)$ be the standard Burau matrix associated to $\sigma_i$, the $i$th standard generator of $B_n$. It consists in an $n \times n$ identity matrix where one replaces the $2 \times 2$ block obtained with the $i$th and $i+1$’th rows and columns by the standard block:

$$\begin{pmatrix} 1 - t & 1 \\ t & 0 \end{pmatrix}.$$ 

The following definition for a multivariable Burau representation is due to Morton.

Definition 2.6 ([B-N, Mo]). Let $b = \prod_{\alpha=1}^k \sigma_{i_\alpha}^\alpha$ be a braid written as a product of standard generators. Let $\Gamma$ be the following product of matrices:

$$\Gamma(b) = \prod_{\alpha=1}^k U_{n,j_{\alpha}}(t_{\alpha})^{\alpha}$$

where $j_{\alpha}$ is the index of the “over passing” strand at the $\#\alpha$ crossing, and $t_1, \ldots, t_n$ are set to be formal variables.

Proposition 2.7 ([B-N]). The map:

$$\Gamma : \mathcal{B}_n \rightarrow \mathcal{M}_n(\mathbb{Z}[t_i^{\pm 1}]_{i=1,...,n})$$

is well defined.

The map $\Gamma$ is well defined but not multiplicative, i.e. not an algebra morphism. Namely, $\Gamma(ab) \neq \Gamma(a)\Gamma(b)$ when $a$ and $b$ are braids in general.

Proposition 2.8. The morphism $\Gamma$ becomes multiplicative when restricted to the pure braids, so that it yields a representation of $PB_n$. 
Let $\mathcal{R} = \mathbb{Z} \left[ t_i^{\pm 1} \right]_{i=1,...,n}$. We build an induced representation of $B_n$ over $\mathcal{R}[\mathcal{S}_n] \otimes \mathcal{R} \langle g_1, \ldots, g_n \rangle$, where $\{g_1, \ldots, g_n\}$ designates the canonical basis to write matrices in $\mathcal{M}_n(\mathbb{Z} \left[ t_i^{\pm 1} \right]_{i=1,...,n})$. We define the induced Gassner representation as follows.

**Definition 2.9 (Gassner representation of $B_n$).** The induced Gassner (see Definition [5,2]) representation of $B_n$, denoted $Gassner_n$, is defined using the following endomorphisms associated to standard generators and extended to all the braids multiplicatively.

$$Gassner_n(\sigma_i) : \begin{cases} \mathcal{R}[\mathcal{S}_n] \otimes \mathbb{R}^n & \rightarrow \mathcal{R}[\mathcal{S}_n] \otimes \mathbb{R}^n \\ \tau \otimes v & \mapsto (i, i + 1) \circ \tau \otimes U_{n, i}(t_{r - 1(i + 1)})(v) \end{cases}$$

where $\sigma_i$ is the $i^{th}$ standard generator of $B_n$, and $(i, i + 1)$ is the permutation of $i$ and $i + 1$. It’s a representation over a space of dimension $n! \times n$.

This representation contains the Gassner representation of pure braids. It also contains the Burau representation as it was already the case for $\Gamma$, we state this in the following remark.

**Remark 2.10.**
- If $a$ is a pure braid, $Gassner_n(a)$ is block diagonal and $\Gamma(a)$ is the matrix restricted to $\mathcal{R}([\cdot]) \otimes \mathbb{R}^n$, $\cdot$ stands for the identity permutation.
- If all the variables are set to be equal to one variable, namely $t_1 = \cdots = t_n = t$, then $\Gamma$ is the Burau representation.

2.1.3. A word about faithfulness. The Burau representation is known to be faithful for $n = 2, 3$, unfaithful for $n \geq 5$, and it remains an open question for $n = 4$. The natural question coming from the study of Burau is if the Gassner representation is faithful, as it is richer than Burau in terms of variables. It is in fact still an open question.

This question is entirely contained in the question whether $\Gamma$ is a faithful representation of $\mathcal{PB}_n$ or not. The explication is the following remark:

**Remark 2.11.** The image of $\mathcal{R}([\cdot]) \otimes \mathbb{R}^n$ under the action of a braid $a$ is contained in the space $\mathcal{R}[\text{perm}(a)] \otimes \mathbb{R}^n$. This ensures that in order to get the identity matrix from $Gassner_n$, the braid $a$ must be pure.

This remark is a direct consequence of Definition [5,2]. The faithfulness of the Gassner representation is reduced to the following open question.

**Open Question.** Is $\Gamma$ faithful as a representation of $\mathcal{PB}_n$?

We end this presentation with a word about faithfulness of Gassner representations. We recall the Birman exact sequence [F-M] Theorem 4.6] in the case of the punctured disk that involves the pure braid group $\mathcal{PB}_n$:

$$1 \rightarrow F_{n-1} \rightarrow \mathcal{PB}_n \rightarrow \mathcal{PB}_{n-1} \rightarrow 1,$$

which is called the *Fadell – Neuwirth* exact sequence. Indeed, let $D_n$ be the disk with $n$ punctures, this exact sequence is the Birman exact sequence (see [F-M] Theorem 4.6]) while remarking that the pure braid group is the pure mapping class group of $D_n$, and that the $\pi_1$ of $D_{n-1}$ is a free group in $n - 1$ generators denoted $F_{n-1}$. Moreover this pure Birman exact sequence splits so that $\mathcal{PB}_n$ is the semi direct product of $\mathcal{PB}_{n-1}$ with $F_{n-1}$. Let $\Gamma_n$ be the Gassner representation of the pure braid group $\mathcal{PB}_n$, then one can check that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{PB}_n & \xrightarrow{\text{Forget}} & \mathcal{PB}_{n-1} \\
\downarrow \Gamma_n & & \downarrow \Gamma_{n-1} \\
\Gamma_n(\mathcal{PB}_n) & \rightarrow & \Gamma_{n-1}(\mathcal{PB}_{n-1})
\end{array}$$

where the lower horizontal arrow consists in setting $t_n$ to be 1 and deleting last row and column of the matrix. This fact allows a treatment of the faithfulness question by recursion on $n$. In some sense the Gassner representation commutes with the *Forget* map so that the recursion property is reduced to the faithfulness of the induced representation of $\Gamma_n$ over $F_{n-1}$ ([Knu] Section 2.2] for a presentation of these
facts). It was used in a series of articles to refine the kernel of Gassner representations. The theorem giving the finest kernel the author knows is the following:

**Theorem 2.12 (Knu Theorem 3.4).** The kernel of the action of $\Gamma_n$ over $F_{n-1}$ lies in $[C^3F_{n-1}, C^2F_{n-1}]$ where $C^*F_{n-1}$ stands for the terms of the lower central series of $F_{n-1}$.

2.2. Quantum representations. In this section, we define some quantum representations for the braid groups, arising from the quantum group $U_q\mathfrak{sl}(2)$.

2.2.1. An integral version for $U_q\mathfrak{sl}(2)$. We give a first definition for $U_q\mathfrak{sl}(2)$.

**Definition 2.13.** The algebra $U_q\mathfrak{sl}(2)$ is the algebra over $\mathbb{Q}(q)$ generated by elements $E, F$ and $K^{\pm 1}$, satisfying the following relations:

\[
KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F
\]
\[
[E, F] = \frac{K - K^{-1}}{q - q^{-1}} \quad \text{and} \quad KK^{-1} = K^{-1}K = 1.
\]

The algebra $U_q\mathfrak{sl}(2)$ is endowed with a coalgebra structure defined by $\Delta$ and $\epsilon$ as follows:

\[
\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1
\]
\[
\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}
\]
\[
\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = \epsilon(K^{-1}) = 1
\]

and an antipode is defined as follows:

\[
S(E) = EK^{-1}, S(F) = -KF, S(K) = K^{-1}, S(K^{-1}) = K.
\]

This provides a Hopf algebra structure, so that the category of modules over $U_q\mathfrak{sl}(2)$ is monoidal.

We define quantum numbers.

**Definition 2.14.** Let $i$ be a positive integer. We define the following elements of $\mathbb{Z}[q^{\pm 1}]$.

\[
[i]_q := \frac{q^i - q^{-i}}{q - q^{-1}}, \quad [k]_q! := \prod_{i=1}^{k} [i]_q, \quad \left[ \begin{array}{c} k \\ l \end{array} \right]_q := \frac{[k]_q!}{[l]_q! [k-l]_q!}
\]

We are going to define an integral version for $U_q\mathfrak{sl}(2)$, so that we will obtain integral representations for braid groups. This integral version is similar to the one introduced by Lusztig in [Lus]. The difference is that we consider only the divided powers of $F$ as generators, not those of $E$. This version is introduced in [Hab, J-K] and [M] (with subtle differences in the definitions for divided powers for $F$). We follow [M], so that we first define the so called divided powers. Let:

\[
F^{(n)} = \frac{(q - q^{-1})^n}{[n]_q!} F^n.
\]

Let $\mathcal{R}_0 = \mathbb{Z}[q^{\pm 1}]$ be the ring of integral Laurent polynomials in the variable $q$.

**Definition 2.15** (Half integral algebra, [J-K, M]). Let $U_q^{\frac{1}{2}}\mathfrak{sl}(2)$ be the $\mathcal{R}_0$-subalgebra of $U_q\mathfrak{sl}(2)$ generated by $E, K^{\pm 1}$ and $F^{(n)}$ for $n \in \mathbb{N}^*$. We call it a half integral version for $U_q\mathfrak{sl}(2)$, the word half to illustrate that we consider only half of divided powers as generators.

**Remark 2.16.** $U_q^{\frac{1}{2}}\mathfrak{sl}(2)$ inherits a Hopf algebra structure, making its category of modules monoidal. The coproduct is given by:

\[
\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \text{and} \quad \Delta(F^{(n)}) = \sum_{j=0}^{n} q^{-j(n-j)} K^{j-n} F^{(j)} \otimes F^{(n-j)}.
\]
2.2.2. Verma modules and braiding. We define the Verma modules for $U_q^I\mathfrak{sl}(2)$, they are infinite dimensional modules depending on a parameter. To preserve the integral structure of coefficients, we let $R_1 := \mathbb{Z}\left[q^{\pm 1}, s^{\pm 1}\right]$.

**Definition 2.17** (Verma modules for $U_q^I\mathfrak{sl}(2)$, [JK] (18)). Let $V^s$ be the Verma module of $U_q^I\mathfrak{sl}(2)$. It is the infinite $R_1$-module, generated by vectors $\{v_0, v_1, \ldots\}$, and endowed with an action of $U_q^I\mathfrak{sl}(2)$, generators acting as follows:

$$K \cdot v_j = sq^{-2j}v_j, \quad E \cdot v_j = v_{j-1} \quad \text{and} \quad F^{(n)}v_j = \left(\begin{array}{c} n + j \\ j \end{array}\right) \prod_{k=0}^{n-1} (sq^{-k-j} - s^{-1}q^{i+k}) v_{j+n}.$$  

**Remark 2.18** (Weight vectors). We will often make implicitly the change of variable $s := q^a$ and denote $V^s$ by $V^a$. This choice made to use a practical and usual denomination for eigenvalues for the $K$ action (which is diagonal in the given basis). Namely, we say that vector $v_j$ is of weight $\alpha - 2j$, as $K \cdot v_j = q^{a-2j}v_j$.

The notation with $s$ shows an integral Laurent polynomials structure strictly speaking.

**Definition 2.19** ($R$-matrix, [JK] (21)). Let $s = q^a$, $t = q^{a'}$. The operator $q^{H \otimes H/2}$ is the following:

$$q^{H \otimes H/2} : \left\{ \begin{array}{rcl} V^s \otimes V^t & \to & V^s \otimes V^t \\ v_i \otimes v_j & \mapsto & q^{(\alpha - 2i)(\alpha' - 2j)}v_i \otimes v_j. \end{array} \right.$$  

We define the following $R$-matrix:

$$R : q^{H \otimes H/2} \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)}$$

which will be well defined as an operator on Verma modules, see the following proposition.

**Proposition 2.20** ([JK] Theorem 7). Let $V^s$ and $V^t$ be Verma modules of $U_q^I\mathfrak{sl}(2)$ (with $s = q^a$ and $t = q^{a'}$). Let $R$ be the following operator:

$$R : T \circ R$$

where $T$ is the twist defined by $T(v \otimes w) = w \otimes v$. Then $R$ provides a braiding for $U_q^I\mathfrak{sl}(2)$ integral Verma modules. Namely, the morphism:

$$Q : \left\{ \begin{array}{rcl} R_1[B_n] & \to & \text{End}_{R_1, U_q^I\mathfrak{sl}(2)} (V^s \otimes V^t) \\ \sigma & \mapsto & 1^{(i-1)} \otimes R^{1^{(n-i-2)}} \end{array} \right.$$  

is an $R_1$-algebra morphism. It provides a representation of $B_n$ such that its action commutes with that of $U_q^I\mathfrak{sl}(2)$.

We now pass to a colored version, which corresponds to taking different Verma modules in the tensor product instead of the same one. Let $V^0 = V^\lambda_i \otimes \cdots \otimes V^\lambda_n$, and by analogy, $V^\tau = V^{\lambda_1(\tau)} \otimes \cdots \otimes V^{\lambda_{\tau(n)}}$, for $\tau \in S_n$ (($\tau$) designates the identity permutation). Let $R = \mathbb{Z}[q^{\pm 1}, s^{\pm 1}]$, the morphism $Q$ extends as follows:

$$Q(\sigma_i) \in \text{Hom}_R(V^{(i)}, V^{(i,i+1)})$$

So that if $\beta$ is pure, $Q(\beta) \in \text{End}(V^{(i)})$, and:

$$Q : R \left[ PB_n \right] \to \text{End}_{R_1, U_q^I\mathfrak{sl}(2)} \left(V^{(i)}\right)$$

is a representation of $PB_n$. One can consider the induced (colored) representation of $B_n$ as follows:

$$Q_{\text{Quant}} : R[B_n] \to \text{End}_{R_1, U_q^I\mathfrak{sl}(2)} \left(R[S_n] \otimes V^{(i)}\right)$$

noticing the following isomorphism:

$$\left\{ \begin{array}{rcl} R[S_n] \otimes V^{(i)} & \to & \bigoplus_{\tau \in S_n} V^{\tau} \\ \tau \otimes v^{(i)} & \mapsto & v^{\tau}. \end{array} \right.$$
2.2.3. **Finite dimensional braid representations.** Although braid group representations over products of Verma modules are infinite dimensional, it turns out that they are graded by finite dimensional subrepresentations.

**Remark 2.21.** For \( r \in \mathbb{N} \), the subspace \( W_{n,r} = \text{Ker}(K - (\prod_i s_i) q^{-2r}) \) is such that \( \mathcal{R}[\mathfrak{S}_n] \otimes W_{n,r} \) provides a sub-representation of \( \mathcal{B}_n \). We usually call \( W_{n,r} \) the space of **subweight** \( r \) vectors.

Let \( \{v_0^0, v_1^0, \ldots\} \) be the standard basis of \( V^\tau \). The space \( W_{n,1}^{(i)} := \langle \rangle \otimes W_{n,1} \in V^{(i)} \) is spanned by \((f_1, f_2, \ldots, f_n)\), where the \( f_i \)'s are defined as follows:

\[
(\langle \rangle \otimes f_1) = v_1^{\lambda_1} \otimes v_0^{\lambda_2} \otimes \cdots \otimes v_0^{\lambda_r}
\]

and so on, with:

\[
(\langle \rangle \otimes f_i) = v_0^{\lambda_1} \otimes v_1^{\lambda_2} \otimes \cdots \otimes v_1^{\lambda_{i-1}} \otimes v_0^{\lambda_i}.
\]

These vectors are built as the tensor products of \( n - 1 \) maximal weight vectors plus one of weight ("sub-maximal"), namely \( v_1^{\lambda_1} \), inserted on the \( i \)-th position of the tensor product.

2.2.4. **Computation of generators’ actions.** We compute the action of braid groups generators over \( W := \bigoplus_{\tau \in \mathfrak{S}_n} W_{n,1}^\tau \).

**Remark 2.22.** Since \( E(v_0) = 0 \), if \( i + j \leq 1 \), then:

\[
R(v_i \otimes v_j) = q^{H \otimes H/2}(\text{Id} \otimes \text{Id} + E \otimes F^{(1)})v_i \otimes v_j.
\]

The space \( W \) fulfill the conditions of this formula.

**Lemma 2.23.** Let \( \{v_0^{\lambda_i}, v_1^{\lambda_i}, \ldots\} \) and \( \{v_0^{\lambda_j}, v_1^{\lambda_j}, \ldots\} \) be the basis of Verma modules \( V^{s_i} \) and \( V^{s_j} \) respectively \((i,j \in \{1, \ldots, n\})\). Then:

\[
\begin{align*}
R(v_0^{\lambda_1} \otimes v_0^{\lambda_2}) &= v_0^{\lambda_2} \otimes v_0^{\lambda_1} \\
R(v_1^{\lambda_1} \otimes v_0^{\lambda_2}) &= s_1 v_0^{\lambda_2} \otimes v_1^{\lambda_1} + (s_2^2 - 1) v_1^{\lambda_2} \otimes v_0^{\lambda_1} \\
R(v_0^{\lambda_1} \otimes v_1^{\lambda_2}) &= s_2 v_1^{\lambda_2} \otimes v_0^{\lambda_1}
\end{align*}
\]

**Proof.** It is a straightforward computation from Definition 2.19 and Remark 2.22. \( \square \)

**Corollary 2.24.** The representation of \( \mathcal{B}_n \) over \( W = \mathcal{R}[\mathfrak{S}_n] \otimes W_{n,1} \) is defined by the action of generators over the basis as follows:

\[
\begin{align*}
\text{Quant}_n(\sigma_k)(\tau \otimes f_k) &= (1 - s_{\tau^{-1}(k)})((k, k + 1) \tau \otimes f_k) - s_{\tau^{-1}(k+1)}((k, k + 1) \tau \otimes f_{k+1}) \\
\text{Quant}_n(\sigma_k)(\tau \otimes f_{k+1}) &= -s_{\tau^{-1}(k)}((k, k + 1) \tau \otimes f_k) \\
\text{Quant}_n(\sigma_k)(\tau \otimes f_i) &= (i, i + 1) \tau \otimes f_i \quad \text{if } i \neq k, k + 1.
\end{align*}
\]

2.3. **Gassner representation from quantum ones.** We recall the context of both representations, namely:

- from Section 2.1 that Gassner, is a representation of \( \mathcal{B}_n \):

\[
\text{Gassner}_n : \mathcal{R}[\mathcal{B}_n] \rightarrow \text{End}_\mathcal{R} \left( \mathcal{R}[\mathfrak{S}_n] \otimes \text{Span}(g_1, \ldots, g_n) \right)
\]

involving formal variables \( t_1, \ldots, t_n \).

- from Section 2.2 that Quant, is a representation of \( \mathcal{B}_n \):

\[
\text{Quant}_n : \mathcal{R}[\mathcal{B}_n] \rightarrow \text{End}_\mathcal{R} \left( \mathcal{R}[\mathfrak{S}_n] \otimes \text{Span}(f_1, \ldots, f_n) = W \right)
\]

involving formal variables \( s_1, \ldots, s_n \).
In order to relate the representations Quant and Gassner, we need first to connect variables. We use the following identification:

\[ s_i^2 = t_i \]

Then let \( \Phi \) be the following morphism relating both representations:

\[ \mathcal{R}[\mathfrak{S}_n] \otimes \text{Span}_\mathcal{R}(f_1, \ldots, f_n) \rightarrow \mathcal{R}[\mathfrak{S}_n] \otimes \text{Span}_\mathcal{R}(g_1, \ldots, g_n) \]

\[ \tau \otimes f_i \rightarrow \prod_{j=1}^{s_{\tau^{-1}(j)}-1} g_j \]

**Theorem 2.25.** Gassner representations are of quantum type, namely the morphism \( \Phi \) conjugates Quant to Gassner, in the sense that for all \( n \in \mathbb{N} \), for all \( \alpha \in \mathcal{B}_n \) the following relation holds:

\[ \text{Gassner}_n(\alpha) \circ \Phi = \Phi \circ \text{Quant}_n(\alpha) \]

**Proof.** Let \( \sigma_k \) be a standard Artin generator of \( \mathcal{B}_n \), \( \tau \in \mathfrak{S}_n \).

Remark that if \( i \) is different from \( k \) and \( k+1 \) then, as Quant and Gassner both act by identity over \( \tau \otimes f_i \), the equality is trivial on these vectors. It remains two cases.

- **Case 1:** \( i = k \). Let’s compute the two sides of the commutation equality. We begin with \( \Phi \circ \text{Quant}_n(\sigma_k)(\tau \otimes f_k) \):

\[ \Phi \circ \text{Quant}_n(\sigma_k)(\tau \otimes f_k) = \Phi \left( (1 - s_{\tau^{-1}(k)}^2)((k, k + 1) \tau \otimes f_k) + s_{\tau^{-1}(k+1)}(k, k + 1) \tau \otimes f_{k+1}) \right) \]

\[ = A \cdot (k, k + 1) \tau \otimes g_k + B \cdot (k, k + 1) \tau \otimes g_{k+1} \]

where:

\[ A = (1 - s_{\tau^{-1}(k)}^2) \prod_{j=k}^{n} s_{\tau^{-1}(j)} \]

\[ B = s_{\tau^{-1}(k+1)} \prod_{j=k}^{n} s_{\tau^{-1}(j)} \]

Now we compute \( \text{Gassner}_n(\sigma_k) \circ \Phi(\tau \otimes f_k) \):

\[ \text{Gassner}_n(\sigma_k) \circ \Phi(\tau \otimes f_k) = \prod_{j=k}^{n} s_{\tau^{-1}(j)} \text{Gassner}_n(\sigma_k)(\tau \otimes g_k) \]

\[ = \prod_{j=k}^{n} s_{\tau^{-1}(j)} \left( (1 - s_{\tau^{-1}(k+1)}^2)((k, k + 1) \tau \otimes g_k) + s_{\tau^{-1}(k+1)}^2(k, k + 1) \tau \otimes g_{k+1} \right) \]

\[ = \Phi \circ \text{Quant}_n(\sigma_k)(\tau \otimes f_k) \]

The last equality comes from the expression of \( \Phi \circ \text{Quant}_n(\sigma_k)(\tau \otimes f_k) \) obtained above, and provides the conjugation in this case.

- **Case 2:** \( i = k + 1 \). We begin with the computation of \( \Phi \circ \text{Quant}_n(\sigma_k)(\tau \otimes f_{k+1}) \):

\[ \Phi \circ \text{Quant}_n(\sigma_k)(\tau \otimes f_{k+1}) = \Phi \left( \tau \tau^{-1}(k) \otimes f_{k+1} \right) \]

\[ = s_{\tau^{-1}(k)} \prod_{j=k}^{n} s_{\tau^{-1}(j)}(k, k + 1) \tau \otimes g_k \]

\[ = \prod_{j=k+1}^{n} s_{\tau^{-1}(j)}(k, k + 1) \tau \otimes g_k \]

Now we compute \( \text{Gassner}_n(\sigma_k) \circ \Phi(\tau \otimes f_{k+1}) \):

\[ \text{Gassner}_n(\sigma_k) \circ \Phi(\tau \otimes f_{k+1}) = \text{Gassner}_n(\sigma_k) \left( \prod_{j=k+1}^{n} s_{\tau^{-1}(j)} \tau \otimes f_{k+1} \right) \]

\[ = \prod_{j=k+1}^{n} s_{\tau^{-1}(j)} \tau \otimes g_k \]

\[ = \Phi \circ \text{Quant}_n(\sigma_k)(\tau \otimes f_{k+1}) \]

The last equality coming from the expression of \( \Phi \circ \text{Quant}_n(\sigma_k)(\tau \otimes f_{k+1}) \) obtained above, and provides the desired equality.

We have proved that for any generator \( \sigma_k \) of \( \mathcal{B}_n \), its representations by \( \text{Quant}_n \) and by \( \text{Gassner}_n \) are conjugated by \( \Phi \). As \( \text{Quant}_n \) and \( \text{Gassner}_n \) are representations, the theorem is proved for all braids. \( \square \)
Remark 2.26. The morphism $\Phi$ is invertible whenever $s_i \neq \pm 1$ for all $i \in \{1, \ldots, n\}$. [M, Theorem 3] provides an isomorphism of modules preserving the integral Laurent polynomials structure of coefficients, by use of an appropriate basis for the Gassner representation.

3. Colored BKL representations

In this section, we construct BKL-like homological representations of braid groups, called colored BKL representations. We follow the construction of [K-T] and [Big] for the (uncolored) BKL-representation that inspires a generalization of it. We follow ideas from [Big] to compute the matrices of this representation. This construction corresponds to the level $r = 2$ of the one over modules $\mathcal{H}_c^{\text{abs}}$ in [M]. Although the obtained representations are the same, the following construction is different: it involves Fox calculus for the computation of the homology, and uses a pairing to compute matrices. Matrices are given in Bigelow’s basis of forks, we follow his work [Big].

3.1. Construction and Faithfulness. The general concept of Lawrence’s representations, see [Law] [M], is to make the braid group act on a homology group of a certain covering of the configuration space of several points in the punctured disk. We recall Definition 1.3 of the configuration spaces of points.

Definition 3.1 (Configuration space of the punctured disk). Let $n, m$ be integers. The configuration space $C_{n,m}$ of $m$ unordered points in $D_n$ is defined as follows:

$$C_{n,m} = \{(z_1, \ldots, z_m) \in (D_n)^m \text{ s.t. } z_i \neq z_j \text{ for } i \neq j\}/\mathfrak{S}_m$$

where $\mathfrak{S}_m$ acts by permutation on the order of coordinates.

Let $C := C_{n,2}$ for $n$ fixed (we omit $n$ in this notation whenever no confusion arises).

We denote $\{x,y\}$ an element of $C$ ($\{x,y\} = \{y,x\}$), and $c = \{d_1, d_2\}$ a base point of $C$ with the $d_i$’s lying in the boundary of $D_n$.

Proposition 3.2 ([P-P, Proposition 1.3]). The first homology group of $C$, namely $H_1(C, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^n \oplus \mathbb{Z}$.

Definition 3.3. We consider the Hurewicz morphism:

$$\text{Hurewicz} : \pi_1(C) \to H_1(C) = \mathbb{Z}^n \oplus \mathbb{Z} = \langle q_1 \rangle \oplus \cdots \oplus \langle q_n \rangle \oplus \langle t \rangle,$$

and we denote by $\tilde{C}$ the covering corresponding to the kernel of this map, namely the maximal abelian cover.

The homology group $H_2(\tilde{C})$ is turned into a $\mathbb{Z} [q_1^{\pm 1}, \ldots, q_n^{\pm 1}, t^{\pm 1}]$-module. We are going to show that, in Lawrence’s construction spirit, this homological module is acted upon by $\mathcal{PB}_n$ by $\mathbb{Z} [q_1^{\pm 1}, \ldots, q_m^{\pm 1}, t^{\pm 1}]$-module automorphisms. This action is the so called colored BKL representation; it is shown in the general case of Lawrence representations (all level of gradings) in [M, Lemma 6.34]. Here we follow [Big].

Remark 3.4. In Definition 3.3, we keep the $n + 1$ generators of the abelianized group, while the uncolored version from [Big] consists in post-composing this Hurewicz map with an augmentation morphism, sending the $n$ generators $q_i$’s to a single one.

A path $\xi : I \to C$ is a pair of paths $\xi = \{\xi_1, \xi_2\}$ where $\xi_1, \xi_2 : I \to D_n$. As we are looking to unordered pairs of points, there are two possibilities for a path $\xi$ to be a loop:

$$\xi_1(0) = \xi_1(1) \text{ and } \xi_2(0) = \xi_2(1)$$

so that both $\xi_i$’s are loops, or:

$$\xi_1(0) = \xi_2(1) \text{ and } \xi_2(0) = \xi_1(1),$$

where $\xi_1$ and $\xi_2$ permutes their endpoints (they are not loops) but the product $\xi_1 \xi_2$ is a loop.

We define invariants $w_i$ of homotopy classes of loops in $C$ for all $i \in \{1, \ldots, n\}$ and for the two cases of a loop $\xi = \{\xi_1, \xi_2\}$ of $C$:

- If $\xi_1$ and $\xi_2$ both are loops, then we define $w_i(\xi) = w_i(\xi_1) + w_i(\xi_2)$ where $w_i(\xi_k)$ is the winding number of $\xi_k$ ($k = 1, 2$) around the puncture $p_i$.
• For the case where $\xi_1$ and $\xi_2$ permute base points we define $w_i(\xi) = w_i(\xi_1 \xi_2)$ to be the winding number around the puncture $p_i$ of the loop $\xi_1 \xi_2$.

We define another invariant $u$, remarking that the map:

$$\begin{align*}
I & \to S^1/\xi_1(s) - \xi_2(s) \\
s & \mapsto \frac{S^1/\xi_1(s) - \xi_2(s)}{S^1/\xi_1(s) - \xi_2(s)}
\end{align*}$$

sends $s = 0, 1$ to the same points or to opposite ones. Hence, the square of this function provides a loop of $S^1$, $u(\xi)$ is the index of it. Note that $u(\xi)$ is even if the $\xi_i$’s are loops, odd otherwise. These classic invariants are additive with respect to product of loops and preserved under homotopy.

These invariants can equivalently be defined as follows:

$$w_i(\xi) = \frac{1}{2\pi i} \left( \int_{\xi_1} \frac{dz}{z - p_i} + \int_{\xi_2} \frac{dz}{z - p_i} \right)$$

and:

$$u(\xi) = \frac{1}{\pi i} \int_{\xi_2 - \xi_1} \frac{dz}{z} .$$

The map:

$$\phi : \xi \to q_1^{w_1(\xi)} \cdots q_n^{w_n(\xi)} t^{u(\xi)}$$

is a surjective group homomorphism from $\pi_1(C)$ to the free abelian group with $(n + 1)$ generators $q_1, \ldots, q_n, t$. It corresponds to the Hurewicz map, see the introduction of [K2].

Then $\tilde{C} \to C$ is the covering map corresponding to the kernel of $\phi$, and $\mathcal{H} = H_2^l(\tilde{C}, \mathbb{Z})$ is a module over $\mathcal{R} = \mathbb{Z}[q_1^{\pm 1}, \ldots, q_n^{\pm 1}, t^{\pm 1}]$, once we choose a lift $\tilde{c}$ of the base point $c$ to $\tilde{C}$. The letters $lf$ indicate the locally finite version of the singular homology, see [M]. Appendix for precisions.

We recall that if $f$ is a self-homeomorphism of $D_n$ (which is the identity on the boundary), it induces a homeomorphism $\hat{f} : C \to C$ by:

$$\hat{f}(\{x, y\}) = \{f(x), f(y)\}$$

Note that $\hat{f}(c) = c$ as $d_1$ and $d_2$ are picked in the boundary of $D_n$. We define the induced automorphism $f_#$ of $\pi_1(C, c)$. The following holds.

**Lemma 3.5.** Let $f$ be such a self-diffeomorphism but fixing punctures pointwise. Then $\phi \circ f_# = \phi$.

**Proof.** We need to verify that the invariants $w_i$ and $u$ are preserved by $f_#$.

For $u$, $u \circ f_# = u$ holds because this invariant does not “see” the punctures, i.e. $u$ factors through the embedding $D_n \to D^2$, where $D^2$ designates the (unpunctured) unit disk. Forgetting the punctures, all homeomorphisms are isotopic to the identity (Alexander trick [F-M]), so that $u \circ f_# = u$.

For $w_i$, it comes from the fact that the equality $w_i \circ f_# = w_i$ holds for small loops encircling the punctures, and then for arbitrary loops since it only depends on the homology class in the first homology group of $D_n$ which is generated by these small loops.

Here, the difference with construction in [Big] is that we need $f$ to fix the punctures. Otherwise, a small circle encircling a puncture transported by $f$ could count $+1$ for a different winding number before and after the application of $f$.

This lemma implies that, in the case where $f$ does not permute punctures, $\hat{f}$ uniquely lifts to a map $\hat{f} : \tilde{C} \to \tilde{C}$ fixing any lift of $c$, and that $\hat{f}$ commutes with covering deck-transformations. Therefore it induces an $\mathcal{R}$-linear automorphism $f_*$ of $\mathcal{H}$, that is an invariant of the isotopy class of $f$. Consequently, it defines a representation of the pure braid group.

**Definition 3.6** (Colored BKL representation). The colored Bigelow–Krammer–Lawrence representation of the pure braid group is:

$$\mathcal{R}[\mathcal{PB}_n] \to Aut(\mathcal{H}) , \; [f] \mapsto f_*$$
where $\mathcal{PB}_n$ refers to the pure mapping class group of the punctured disk, which corresponds exactly to (isotopy classes of) homeomorphisms fixing punctures pointwise.

What follows immediately, is that by specializing every variables $q_i$ to the same variable $q$ we obtain the BKL representation of $\mathcal{PB}_n$ as a subgroup of $\mathfrak{S}_n$, so that the following holds.

**Proposition 3.7** ([Kra] Big). The colored BKL representation of $\mathcal{PB}_n$ is faithful.

To get a representation of the whole braid group, one has to consider $\mathcal{H} \otimes \mathcal{R}[\mathfrak{S}_n]$, see Section 3.

**Definition 3.8** (Colored BKL representation). The colored Bigelow–Krammer–Lawrence representation of the braid group is:

$$\mathcal{R}[\mathfrak{S}_n] \to \text{Aut}(\mathcal{H} \otimes \mathcal{R}[\mathfrak{S}_n]).$$

3.2. **Pairing between forks and noodles.** We recall definitions of *forks* and *noodles*, together with a pairing between these objects. Everything is adapted from [Big].

**Definition 3.9** (Fork, $m = 2$). A fork is an embedded tree $F \in D_n$ with four vertices $d_1, p_i, p_j$, and $z$ such that $F \cap \partial D_n = \{d_1\}$, $F$ intersects the punctures only in $p_i, p_j$, and all three edges have $z$ as a vertex.

- The edge containing $d_1$ is called the handle of $F$ and denoted $H(F)$.
- The union of other two edges is called the tine of $F$ and denoted $T(F)$.
- The tine is oriented in such a way that it has the handle lying on its right.

For any fork $F$ we construct an associated surface $\tilde{\Sigma}$ in $\tilde{C}$ as follows. First let $F'$ be the parallel fork of $F$ with a parallel tine with same endpoints and parallel handle based on $d_2$. We define the following surface of $C$:

$$\Sigma(F) = \{\{x, y\} \text{ s.t. } x \in T(F) \setminus \{p_1, \ldots, p_n\}, y \in T(F') \setminus \{p_1, \ldots, p_n\}\}.$$ 

In order to get a surface of $\tilde{C}$ we need to choose a lift of $\Sigma(F)$. We use the handle to do so. Let $\tilde{\beta}$ be the lift beginning at $\tilde{c}$ of $\{\beta_1, \beta_2\}$ where $\beta_1, \beta_2$ are respectively the handle of $F$ and $F'$ starting on $d_1$ and $d_2$. Let $\tilde{\Sigma}(F)$ be the lift of $\Sigma(F)$ which contains $\tilde{\beta}(1)$.

**Definition 3.10** (Noodle). A Noodle is an arc embedded in $D_n$ going from $d_1$ to $d_2$.

We construct a surface associated to $N$ as follows:

$$\Sigma(N) = \{\{x, y\} \in C \text{ s.t. } x, y \in N\},$$

and then we choose $\tilde{\Sigma}(N)$ to be the lift of $\Sigma(N)$ which contains $\tilde{c}$.

Let $F$ be a fork and $N$ a noodle, and let $\tilde{\Sigma}(F)$ and $\tilde{\Sigma}(N)$ the associated surfaces of $\tilde{C}$. Suppose that $T(F)$ and $N$ intersect transversely in some points $z_1, \ldots, z_l$, and $T(F')$ and $N$ intersect transversely in $z'_1, \ldots, z'_l$ such that $z_i$ and $z'_i$ are joint by a short piece of $N$ not containing any other intersection point. Surfaces $\tilde{\Sigma}(F)$ and $\tilde{\Sigma}(N)$ do not intersect necessarily because of the choice of the lift, but there exists a unique monomial $m_{i,j} = \prod_{k \in \{1, \ldots, l\}} q_k^{w_k(\epsilon_{i,j})} \nu_{i,j}$ such that $m_{i,j} \tilde{\Sigma}(N)$ intersects $\tilde{\Sigma}(F)$ at a point lying over $\{z_i, z'_i\}$. Let $\epsilon_{i,j}$ be the sign of the intersection. We define the pairing as follows:

$$\langle N, F \rangle = \sum_{i=1}^l \sum_{j=1}^l \epsilon_{i,j} m_{i,j}. \quad (3)$$

To compute explicitly $m_{i,j}$ we define a path of $\tilde{C}$ using composition of the following arcs:

- $\alpha_1$ from $d_1$ to the handle of $F$, $\alpha_2$ from $d_2$ to the handle of $F'$,
- $\beta_1$ from $z$ to $z_1$ along $T(F)$, $\beta_2$ from $z'$ to $z'_1$ along $T(F')$,
- $\gamma_1$ from $z_i$ to one of the $d_i$’s in such a way that it doesn’t cross $z'_j$,
- $\gamma_2$ from $z'_j$ to one of the $d_i$’s in such a way that it doesn’t cross $z_i$. 


Then we define the loop $\delta_{i,j}$ of $C$:
\[
\delta_{i,j} = \{\alpha_1, \alpha_2\} \{\beta_1, \beta_2\} \{\gamma_1, \gamma_2\}
\]
Let $\tilde{\delta}_{i,j}$ be the lift of $\delta_{i,j}$ to $\tilde{C}$ beginning at $\tilde{c}$. This path goes first from $\tilde{c}$ to $\tilde{\Sigma}(F)$ then to the lift of $\{z_i, z'_j\}$ lying over $\tilde{\Sigma}(F) \cap m_{i,j} \tilde{\Sigma}(N)$, so that it ends in $m_{i,j} \tilde{c}$. It is a path from $\tilde{c}$ to $m_{i,j} \tilde{c}$, so that we have:
\[
m_{i,j} = \phi(\delta_{i,j}).
\]
From $\text{[Big]}$ Claim 3.3, we have:
\[
\epsilon_{i,j} = -\epsilon_{i,j,m_{i,j}m_{i,j}}(q = 1, t = 1).
\]
as the intersection sign is computable in $C$ (does not depend on which covering one lifts the surfaces to), it is the same as for BKL representations, see $\text{[Big]}$ Equation (1)].

Bigelow’s proof of the faithfulness involves the following two lemmas.

**Lemma 3.11** (Basic Lemma, $\text{[Big]}$). The noodle-fork pairing is well defined. Furthermore, if $[\sigma]$ lies in the kernel of the Bigelow Krammer-Lawrence representation, then
\[
\langle N, F \rangle = \langle N, \sigma(F) \rangle
\]
for any fork $F$ and noodle $N$.

This will allow us to use this pairing for computing matrices, see next section. The following key lemma is stated for informal reasons.

**Lemma 3.12** (Key Lemma, $\text{[Big]}$). Let $N$ be a noodle and let $F$ be a fork. Then $\langle N, F \rangle = 0$ if and only if $N$ and $T(F)$ do not intersect (up to isotopy).

### 3.3. Matrices for colored BKL representations

Inspired by Part 4 of $\text{[Big]}$ we give explicit matrices for colored BKL representations.

**Proposition 3.13.** $\mathcal{H}$ is a free $\mathcal{R}$-module. It has a basis:
\[
\{v_{j,k} : 1 \leq j \leq k \leq n\}
\]
The group $\mathcal{B}_n$ acts on $\mathcal{H} \otimes \mathcal{R}[\mathcal{G}_n]$ by the induced action from $\mathcal{PB}_n$. We give the action of the standard generators $\sigma_i$ on $\mathcal{H} \otimes 1$, let $\tau = (i, i + 1)$.

\[
\sigma_i(v_{j,k} \otimes \tau) =
\begin{cases}
  v_{j,k} \otimes \tau, & i \notin \{j - 1, j, k - 1, k\} \\
  q_{j-1}v_{i,j} \otimes \tau + (q_j^2 - q_{j-1})v_{i,j} \otimes \tau + \tau + 1 - q_{j-1}v_{j,k} \otimes \tau, & i = j - 1 \\
  v_{j+1,k} \otimes \tau, & i = j \neq k - 1 \\
  q_{k-1}v_{i,j} \otimes \tau + (1 - q_{k-1})v_{j,k} \otimes \tau + q_k^2 - q_{k-1}t v_{i,k} \otimes \tau, & i = k - 1 \neq j \\
  v_{j,k+1} \otimes \tau, & i = k \\
  tq_jv_{j,k} \otimes \tau, & i = j = k - 1
\end{cases}
\]

One remarks that the only variable appearing above is $q_i$. We wanted to emphasize the links with indexes of vectors, which could be useful to follow permutations of punctures and variables.

**Proof.** First we compute the homology using the Cayley complex defined in $\text{[Big]}$ Section 4. For $j = 1, \ldots, n$, let $\zeta_j$ be the loop based at $d_1$ and running once counterclockwise around $p_j$. Let $x_j$ be the loop $\{\zeta_j, d_2\}$ of $C$. Let $\tau_1$ be an arc from $d_1$ to $d_2$ and $\tau_2$ from $d_2$ to $d_1$ such that $\tau_1\tau_2$ is a simple closed curve oriented counterclockwise and enclosing no puncture points, and let $y$ be the loop $\{\tau_1, \tau_2\}$ of $C$. We define the set $\mathcal{G}$:
\[
\mathcal{G} = \{x_1, \ldots, x_n, y\}.
\]
Now we define some relations, for $j \in \{1, \ldots, n\}$:
\[
r_{j,j} = [x_j, yx_jy],
\]
and for $1 \leq j < k \leq n$:
\[
r_{j,k} = [x_j, yx_ky^{-1}]
\]
where the bracket refers to the commutator, and we define the set $\mathcal{R} = \{r_{j,k} \text{ for } 1 \leq j \leq k \leq n\}$.
Proposition 3.14 ([Big Section 4]). Let $K$ be the Cayley Complex of $(\mathcal{G}|\mathcal{R})$. Then $C$ is homotopically equivalent to $K$. It follows that a presentation of $\pi_1(C)$ is given by: $(\mathcal{G}|\mathcal{R})$.

Now we can compute $\mathcal{H}$ using the Fox derivatives (see Definition 2.1). We let $C_1$ and $C_2$ be the free $\mathcal{R}$-modules with basis $\{e_g : g \in \mathcal{G}\}$ and $\{f_r : r \in \mathcal{R}\}$ respectively. For any word in $\mathcal{G}$, we define $[w] \in C_1$ according to these rules:

$$[gw] = e_g + \phi(g)[w]$$

$$[g^{-1}w] = \phi(g)^{-1}([w] - e_g)$$

for $g \in \mathcal{G}$. Then $H_2(\tilde{C})$ is the kernel of the map $\partial : C_2 \to C_1$ defined by $\partial f_r = [r]$. The computation gives:

$$\partial f_r = \begin{cases} (q_j t + 1)((1 - t)[x_j] + (q_j - 1)[y]) & \text{if } r = r_{j,i} \\ (1 - q_k)[x_j] + (1 - q_k)(q_j - 1)[y] + t(q_j - 1)[x_k] & \text{if } r = r_{j,k} \end{cases}$$

If we restrict the morphism to the space $\text{Span}(f_{j,j}, f_{j,k}, f_{k,k})$, we get the matrix:

$$\begin{pmatrix}
(q_j - 1)(q_j t + 1) & (1 - q_k) \\
(1 - q_k)(q_j t + 1) & (q_k - 1)(q_k t + 1) \\
0 & (1 - t)(q_k t + 1)
\end{pmatrix}$$

which corresponds to the only non-vanishing block of the application $\partial$. Each block has a rank one kernel generated by the vector:

$$v_{j,k} = -(1 - q_k)(q_k t + 1)f_{j,j} + (1 - t)(q_k t + 1)(q_j t + 1)f_{j,k} - t(q_j - 1)(q_j t + 1)f_{k,k}$$

so that we get a basis of $H_2(\tilde{C})$, namely $\{v_{j,k} : 1 \leq j < k \leq n\}$.

Now a nice way to compute the matrices for the action of $\sigma_i$, is to find forks $F_{j,k}$ which correspond to the vector $v_{j,k}$, and to use the pairing with some noodles to get the expression of vectors in the fork basis, knowing the basic lemma (Lemma 3.11). In what follows we still abusively use $F$ to designate both the fork and the associated homology class of $\Sigma(F)$.

Let’s fix $d_1$ and $d_2$ lying in the lower half plane of the boundary of $D_n$.

Definition 3.15 (Standard fork). For each $1 \leq j < k \leq n$, let $F_{j,k}$ be the fork that lies entirely in the lower half of $D_n$ such that the endpoints of $T(F_{j,k})$ are the punctures $p_j$ and $p_k$, we usually call it a standard fork.

Remark 3.16. In [M Section 7.1] one can find a generalization of forks to higher configuration spaces of points ($m \geq 2$), that can also be found in [It0]. We want to warn the reader that Bigelow’s standard forks defined above are not multiforks according to the definitions from [M It0]. Indeed in the multiforks the segments only connect consecutive punctures (but maybe different ones) while Bigelow’s standard forks can join any two different punctures (but the second configuration point is then taken in the parallel fork, having same punctures as ends).

Remark 3.17. There exists $\lambda \in \mathcal{R}$ such that for all $j, k \in \{1, \ldots, n\}$:

$$F_{j,k} = \lambda v_{j,k}$$

(in terms of homology classes). The proof of this fact is exactly the same as the one for the uncolored version, see [Big proof of Theorem 4.1]. This is achieved noticing that it is sufficient to consider the homology module restricted to the disk containing $F_{j,k}$, its endpoints, and no other puncture.

By Remark 3.17, we compute the braid action over standard forks. There are cases where $\sigma_i(F_{j,k})$ is directly a standard fork, namely:

- $i \notin \{j - 1, j, k - 1, k\}$
- $i = j \neq k - 1$
- $i = k$
In the case \(i = j = k - 1\), the fork \(\sigma_1(F_{j,k})\) has the tine edges as \(F_{j,k}\) with opposite orientations:

\[
\sigma_1(F_{j,k}) = \begin{pmatrix}
p_k & p_j
\end{pmatrix}
\]

where in red is represented the handle, and in black the tine. The handle joins the boundary in \(d_1\) outside the parenthesis. It follows that it represents the same surface in \(C\) as \(F_{j,k}\) with the same orientation (both intervals are reversed). Then the classes in \(C\) differ by a covering transformation. We obtain that \(\sigma_1(v_{j,k}) = tq_jq_i^{-1}v_{j,k}\).

A similar computation is made in [M], in Example 4.2, using the handle rule introduced in Remark 4.1 and recalled in the present paper in Remark 4.1, that deals with a change of handle for forks. The remaining cases are \(i = j - 1\) and \(i = k - 1 \neq j\). The following claim restricts the linear combination, and is proved exactly the same way as Claim 4.2 of [Big].

**Claim 3.18** ([Big Claim 4.2]). \(\sigma_1(v_{j,k})\) is a linear combination of \(v_{j',k'}\) with \(j', k' \in \{i, i + 1, j, k\}\)

In the case \(i = j - 1\) for instance, this claim implies that there exists \(A, B, C \in \mathcal{R}\) s.t. :

\[
\sigma_1(F_{j,k}) = AF_{i,j}^\tau + BF_{j,k}^\tau + CF_{i,k}^\tau
\]

for \(\tau = (i, i + 1) \in \mathfrak{S}_n\), where we denote \(F_{j,k}^\tau := F_{\tau(i), \tau(k)}\). One remarks that punctures are permuted after the application of \(\sigma_1\), and for the colored version issue following the permutation of punctures is important. To get \(A, B, C\) we pair with noodles. As it only depends on homological class of the surface associated to fork, by pairing some appropriate noodles with the studied forks on one hand and with the standard fork involved in its decomposition on the other, we are able to compute the coefficients of the linear combination. In following Example 3.19 we perform this computation in one of the two remaining cases. One can adapt the computation to the last case. \qed

**Example 3.19.** Let \(F\) be the fork corresponding to the image of \(F_{2,4}\) after applying the homeomorphism corresponding to the generator \(\sigma_1\) of \(\mathfrak{B}_n\). Considering the Claim 3.18 we can restrict ourselves to \(B_4\) and the study of \(D_4\) with only four punctures. This example is enough to deduce the general expression of the action of \(\sigma_1\) on the vector \(v_{j,k}\) in the case \(i = j - 1\), which is one of the two remaining cases not entirely treated in the proof of Proposition 3.13.

First we use Claim 3.18 to deduce that the class in \(H_2(\widetilde{C})\) associated with \(F\) has a linear decomposition in terms of standard forks \(F_{1,2}^\tau, F_{1,4}^\tau\) and \(F_{2,4}^\tau\), for \(\tau = (1, 2) \in \mathfrak{S}_4\). We use the following notations:

\[
F = AF_{1,2}^\tau + BF_{1,4}^\tau + CF_{2,4}^\tau
\]

where \(A, B, C \in \mathcal{R}\) are the coefficients we are looking for. We compute \(A, B, C\) using Pairing 3.3.

**Remark 3.20.** In order to compute invariants of loops \(\delta_i\)’s (see Subsection 3.2), it is useful to draw both paths \((\xi_1, \xi_2)\) to observe the value of the invariants \(u_i\) while for the invariant \(u\) the parametrization is crucial, so that we need to think about the movie of the loop. We draw some in Figure 2.

Let \(N_i\) be the noodle starting at \(d_1\) and passing once clockwise around the puncture \(p_i\) before coming back to \(d_2\). We get the easy following computation of the pairing with standard forks:

**Remark 3.21.**

\[
\langle N_i, F_{j,k} \rangle = \begin{cases} 
-q_i^{-1}t^{-1} & \text{if } i = j \\
q_i^{-1}t^{-1} - t^{-1} + 1 + q_i & \text{if } j < i < k \\
0 & \text{if } i = k
\end{cases}
\]

Similarly we compute:

\[
\langle N_1, F \rangle = -q_2q_1^2 \\
\langle N_1, F \rangle = q_4^{-1}t^{-1}
\]
We detail the computation of the pairing of $F$ with $N_3$ (one can realize that it involves exactly the same paths as for $\langle N_i, F_{j,k} \rangle$ above with $j < i < k$). The situation is depicted in Figure 1.

$F$ and $N_3$ have two intersection points, the pairing involves four terms:

- for $\delta_{1,1}$ we get $m_{1,1} = q_3^{-1}t^{-1}$ so that $u_{1,1} = -1$ and that $e_{1,1} = 1$,
- for $\delta_{2,2}$ we get $m_{2,2} = q_3$ so that $u_{2,2} = 0$ and that $e_{1,1} = -1$,
- for $\delta_{1,2}$ we get $m_{1,2} = 1$ so that $u_{1,2} = 0$ and that $e_{1,2} = -(-1)^{u_{1,1}+u_{2,2}+u_{1,2}} = 1$,
- for $\delta_{2,1}$ we get $m_{2,1} = t^{-1}$ so that $u_{2,1} = -1$ and that $e_{2,1} = -(-1)^{u_{1,1}+u_{2,2}+u_{2,1}} = -1$.

Beside $\delta_{1,2}$ which is trivial, we draw $\delta_{1,1}$, $\delta_{2,2}$ and $\delta_{2,1}$ in Figure 2 from which above computations are immediate. Finally:

$$\langle N_3, F \rangle = q_3^{-1}t^{-1} - t^{-1} + 1 + q_3.$$  

Replacing the computations above in the expression:

$$\langle N_i, F \rangle = A\langle N_i, F_{1,2}^\tau \rangle + B\langle N_i, F_{1,4}^\tau \rangle + C\langle N_i, F_{2,4}^\tau \rangle$$  

with $i = 1$ we get the condition:

$$A + B = q_2^2$$  

and with $i = 3$:

$$B + C = 1.$$  

We need one more condition. We obtain it by pairing with the noodle $N_{2,3}$ defined as the noodle starting at $d_1$ and running around the punctures $p_2$ and $p_3$ before coming back to $d_2$ (see Figure 3 noodle oriented from left to right).

We get the pairings:

- $\langle N_{2,3}, F_{1,2}^\tau \rangle = (q_1q_3)^{-1}t^{-1}$,
- $\langle N_{2,3}, F_{1,4}^\tau \rangle = (q_1q_3)^{-1}t^{-1} - t^{-1} + 1 - q_1q_3$,
- $\langle N_{2,3}, F_{2,4}^\tau \rangle = -q_1q_3$.  

**Figure 1.** Intersection of fork $F$ with noodle $N_3$.  

**Figure 3.** Noodle oriented from left to right.
\[ \langle N_{2,3}, F \rangle = q_1 q_3^{-1} t^{-1} - q_1 t^{-1} + q_1 - q_1 q_3 = q_1 (1 - q_3) (q_3^{-1} t^{-1} + 1) . \]

By identification, we finally obtain:

\[ A = q_1^2 - q_1 , \quad B = q_1 , \quad C = 1 - q_2 1. \]

Proposition 3.13 allows computation of matrices. In the above proposition we only provide the action on vectors \( v \otimes 1 \) for \( v \in H_2(C) \). To deal with permuted vectors, one can transport punctures and permute variables consistently in the expressions of the proposition. As the only variable involved in the proposition is \( q_i \), we end this section by a computational approach to these matrices, that indicates how to transport it.

Let \( BKL(q, t) \) be the matrix representing the action of \( \sigma_i \) in the (uncolored) Bigelow-Krammer-Lawrence
representation written in the basis \( \{ F_{j,k} \} \) using the lexicographic order on \((i,k)\). See \[Big\] Section 4 for matrices, or use fonction \( LKB_{\text{matrix}}() \) from the braid package of SageMath to obtain them. We consider the ones obtained from the above proposition with \( q = q_1 = \cdots = q_n \), which corresponds to \[Big\] Section 4. It has entries in \( \mathbb{Z}[q^\pm 1, t^\pm 1] \).

Then it’s a basic matrix computation that verifies the following remark.

**Remark 3.22.** Let \( q_1, \ldots, q_n \) be variables. Then:

\[
BKL_i(q_i, t)BKL_j(q_j, t) = BKL_j(q_j, t)BKL_i(q_i, t) \quad \text{for} \quad |i - j| \geq 2
\]

\[
BKL_{i+1}(q_i, t)BKL_i(q_i, t)BKL_{i+1}(q_{i+1}, t) = BKL_i(q_{i+1}, t)BKL_{i+1}(q_i, t)BKL_i(q_i, t).
\]

One can check this by a matrix computation.

Now we can define the colored BKL matrix associated to a braid.

**Definition 3.23.** Let \( \alpha \) be a braid having the following word decomposition in the standard generators:

\[
\alpha = \prod_{m=1}^{k} \sigma_{i_m}^{s_m}
\]

where \( s_m \) are signs. Let \( j_m \) be the index of the “over” strand at the \( m \)’th crossing in \( \alpha \), braids read from right to left. Let the matrix \( cBKL(\alpha) \) associated to the braid \( \alpha \) be:

\[
cBKL(\alpha) := \prod_{m=1}^{k} BKL_{j_m}(q_{j_m}, t)^{s_m}.
\]

Remark 3.22 shows that \( cBKL \) is a well defined map between the braid group and the matrix group, but it is not multiplicative. For pure braids, \( cBKL \) becomes a homomorphism and what we get is a representation of \( PB_n \):

\[
cBKL : \begin{cases} 
PB_n \rightarrow GL(\mathbb{Z}[q_1^\pm 1, \ldots, q_n^\pm 1, t]) \\
\alpha \rightarrow cBKL(\alpha).
\end{cases}
\]

Remark 3.22 is a computational proof that this is a representation, i.e. that it satisfies braid relations. From Proposition 3.13 we remark that it is the colored BKL representation, corresponding to the initial homological definition (Proposition 3.13). One remarks that the only variable involved in the action of \( \sigma_i \) in Proposition 3.13 corresponds to the over passing strand. Specializing all \( q_i \)’s to a single variable \( q \) recovers the uncolored BKL-representation.

**Remark 3.24.** In Section 2.1 we have presented a construction of the Gassner representation as a generalization of the Burau representation. Namely we used the standard Burau block of matrix but one has to use the variable \( t_i \) if the strand \( i \) is passing above, i.e. the coloring follows strands. Here the conclusion is the same: the colored BKL representation uses the BKL standard block but with formal variables following the index of the strands (it is clear in Formula (4)).

4. **Higher Lawrence representations**

Section 4.1 is a discussion about bases for computations of Lawrence representations. It emphasizes the importance in the choice of the basis in order to respect the Laurent polynomials structure of coefficients, and it discusses the general relation between Lawrence representations and quantum representations of braid groups. Section 4.2 contains Proposition 4.5 computing the Lawrence representation in the basis of code sequences previously introduced. Section 4.3 provides details of the computation in the case of four punctures, which proves Proposition 4.5. Section 4.4 gives explicit matrices in the case of 3 punctures and \( m = 2 \), and how to recover matrices from previous Section 3 in this precise case.

4.1. **Panorama: Bases and quantum relations.**
4.1.1. Bases and ring of coefficients. In the proof of Proposition 3.13 we used Bigelow’s technique to find a basis \((\{u_{i,j,k}\})\) of the homology by Fox calculus computations. In Remark 3.17 we used Bigelow’s argument saying that there exists a diagonal matrix sending this basis of the homology to the forks basis. It turns out that this matrix is not invertible in \(R\) but only generically when variables are specialized to complex values, so that forks are not a basis of the homology as an \(R\)-module (we say that it is not an integral basis), see [M Proposition 7.2] or [P-P]. In fact, in [Big] as in Proposition 3.13 we are computing the action of the braid group on the sub-module generated by forks.

Bases (said “integral”) of this Lawrence homological modules are given in [M], two of them are provided one is the so called code sequences basis. By expressing the change of bases from forks to code sequences, one obtains the morphism relating the representation on the entire homology to the one restricted to the forks module. This morphism is not invertible in \(R\) but under generic conditions that are explicitly given in [M Proposition 7.2], the fact that the module of forks is a strict sub-module of the entire homology as an \(R\)-module was first shown in [P-P].

In [M], colored Lawrence representations are defined for all levels of the grading, see [M Lemma 6.34]. Homological rules are provided, so that one can compute matrices for these representations using integral bases (so to get the representation on the entire homology, defined as an \(R\)-module). In the present paper we used a different approach to compute actions (in Sections 2 with Gassner representations and 3 with BKL representations), the purpose was to emphasize the appearance of Fox differential calculus in the computation of these two first levels of Lawrence representations grading. Such Fox calculus technique for computing the homology seems hard to be generalized for higher Lawrence representations \((m > 2)\), although it would be very interesting to obtain such a generalization.

4.1.2. Links with quantum representations. In [J-K] the authors prove that the BKL representation is isomorphic to the restriction of the braid representation on \(W_{n,2}\) (see 2.21) to the space of highest weight vectors denoted \(Y_{n,2}\). In [M] graded Lawrence representations of the braid groups (over modules denoted \(H^n_{r}^{\text{bs}}\), for \(r \in \mathbb{N}\)) are extended to graded relative homology modules (denoted \(H^n_{r}^{\text{rel}}\)). These extended representations are shown to be isomorphic to quantum representation of the braid group on \(W_{n,r}\), [M Theorem 3]. It generalizes Theorem 2.25 and the result from [J-K] for BKL representations.

4.2. Matrices in the basis of code sequences. In the spirit of what we did in Section 3 there exists a notion of multiforks to designate elements of \(H^n_{r}^{\text{bs}} := H^n_{r}(C_{n,m}, L_m)\), see [Ho2] or [M Section 7.1] for a colored version (Notation \(H^n_{r}^{\text{bs}}\) is taken from [M] with index \(m\) instead of \(r\)). In this section we suggest a computation of matrices for higher Lawrence representations \((m \geq 2)\), in the colored version but in the basis of code sequences presented in [M Section 3.1]. It constitutes a basis of homology modules, see [Big] Lemma 3.1 for a proof, or [M Proposition 3.1] which is an extended version of the latter. The morphism sending multiforks to code sequences is given in [M Proposition 7.2]. We recall the colored version for the local system \(L_m\) used in this section, involving a choice of base point for \(C_{n,m}\):

\[
L_m : \begin{cases}
\mathbb{Z}[\pi_1(C_{n,m}, d)] & \rightarrow \mathcal{R} := \mathbb{Z}[s_i^{\pm 1}, t_i^{\pm 1}]_{i=1, \ldots, n} \\
\xi & \mapsto \prod_i s_i^{u_i(\xi)} f_i(\xi) w(\xi)
\end{cases}
\]

where \(d = \{d_1, \ldots, d_m\}\) is a base point chosen so that coordinates lie on the boundary of the disk, \(u_i\) is the invariant computing the total winding number of path \(\xi\) around puncture \(p_i\), \(w_i\) is the winding number between configuration points, these invariants of loops are defined by analogy as in Section 3.1 in the case of \(m = 2\). For a more precise definition of this local system, using generators of the fundamental group, refer to [M Definition 2.4], the correspondence between present variables and ones from [M] is \(s_i = q^{-2u_i}\) and \(t = t_i\).

In first subsection we present code sequences basis for \(H^n_{r}^{\text{bs}}\) while in the second one we state propositions providing expressions for matrices of colored Lawrence representations in the basis of code sequences. In next subsection we will prove these propositions by performing a computation in the case of 4 punctures.

4.2.1. Code sequences.
Definition 4.1. We define the set of partitions of $n - 1$ in $m$ integers as follows:

$$E_{n,m} = \{(k_1, \ldots, k_{n-1}) \in \mathbb{N}^{n-1} \text{ s.t. } \sum k_i = m\}.$$ 

We now define two families of topological objects indexed by $E_{n,m}$, that will correspond to classes in $\mathcal{H}_m^{\text{abs}}$.

Definition 4.2 (Code sequences diagrams). We draw topological objects inside the punctured disk, the gray color is used to draw the punctured disk. Red arcs are going from a coordinate of the base point $d$ of $C_{n,m}$ lying in its boundary to a dashed black arc. Dashed black arcs are oriented from left to right. Finally the red arcs will end up going like in the following picture inside the dashed box, so that all families of red arcs are attached to the base point $\{d_1, \ldots, d_m\}$ of $C_{n,m}$ (here, $m' = m - k_1$), and that we will sometimes omit their ends in what follows when no confusion is possible.

Let $k = (k_1, \ldots, k_{n-1}) \in E_{n,m}$, we define the code sequence $U_k = U(k_1, \ldots, k_{n-1})$ to be the following drawing.

The indexes $k_i$’s stand to illustrate the fact that $k_i$ configuration points are embedded in the corresponding dashed segment, as we explain in what follows. We have attached to an indexed $k_i$ dashed arc a red arc called a $(k_i)$-handle. It is represented by a little red tube which is a simpler representation used to represent $k_i$ parallel red arcs that are called handles, in the spirit of what we did for forks. We let $\mathcal{U} = \{U(k_1, \ldots, k_{n-1})\}_{k \in E_{n,m}}$ designate the family of code sequences. The definition of these objects comes from [Big1].

Now we explain how to assign a class in $\mathcal{H}_m^{\text{abs}}$ from this diagram, by analogy of what we did for Forks in Section 3.2. Let $k \in E_{n,m}$ and for all $i = 1, \ldots, n - 1$, let:

$$\phi_i : I_i \to D_n$$

be the embedding of the dashed black arc number $i$ of $U(k_1, \ldots, k_{n-1})$ indexed by $k_i$, where $I_i$ is a unit interval. Let $\Delta^k$ be the standard (open) $k$ simplex:

$$\Delta^k = \{0 < t_1 < \cdots < t_k < 1\}$$
for $k \in \mathbb{N}$. It can also be thought as the configuration of $k$ points inside the unit interval so that, for all $i$, the map $\phi^{k_i}$:

$$\phi^{k_i} : \Delta^{k_i} \rightarrow C_{n,k_i} \ni \{\phi_i(t_1), \ldots, \phi_i(t_{k_i})\}$$

is a well defined map. It is a locally finite cycle. These two last facts are detailed in [M Section 3.1]. We use handles to get a class in the local system homology as we did for forks. To get a cycle in the local system homology, one has to choose a lift of the chain to the maximal abelian cover of $C_{n,m}$ associated with the morphism $L_m$. The way to do so is using the red handles to which is canonically associated a path:

$$h = \{h_1, \ldots, h_m\} : I \rightarrow C_{n,m}$$

joining the base point $d$ and the $m$-chain assigned to dashed arcs. At the cover level there is a unique lift $\hat{h}$ of $h$ that starts at $d$. The lift of $U(k_1, \ldots, k_{n-1})$ passing by $\hat{h}(1)$ defines a cycle, so a class in $\mathcal{H}^{abs}_m$, that we still call $U(k_1, \ldots, k_{n-1})$.

The above construction of class is made in [M, Section 3.1], but in some relative homology case, which involves adding one dashed arc going to the boundary. The following is proved in [Bigl] Lemma 3.1], and rephrased in the relative homology formalism in [M Proposition 3.1].

**Proposition 4.3.** The group $\mathcal{H}^{abs}_m$ is a free $\mathcal{R}$-module for which the family $U$ is a basis.

4.2.2. Matrices for colored Lawrence representations. We compute the action of the braid group $\mathcal{B}_n$ on $\mathcal{H}^{abs}_m$ in the basis of code sequences. Actually, the order of punctures $p_i$'s is of importance as they can be permuted by braids. We designate by $\mathcal{H}^{abs}_m$ the space built from $C_{n,m}$ with punctures ordered from 1 to $n$ (1 refers to the identity permutation). For $\tau \in \mathcal{S}_n$, we designate by $\mathcal{H}^{abs}_m^{\tau}$ the one obtained from $C_{n,m}^{\tau}$ with punctures permuted by $\tau$. As for previous part, the action of $\mathcal{B}_n$ is over $\bigoplus_{\tau \in \mathcal{S}_n} \mathcal{H}^{abs}_m^{\tau}$, see the appendix in Section 3.

We need new quantum numbers to perform homology computations.

**Definition 4.4.** Let $i$ be a positive integer. We define the following elements of $\mathbb{Z}[t^{\pm 1}] \subset \mathcal{R}$.

$$(i)_t := (1 + t + \cdots + t^{i-1}) = \frac{1 - t^i}{1 - t}, \quad (k)_t! := \prod_{i=1}^{k} (i)_t, \quad \text{and} \quad {k \choose t} := \frac{(k)_t!}{(k - l)_t!(l)_t!}.$$ We also define quantum trinomials as follows:

$$\begin{pmatrix} n \\ i,j,k \end{pmatrix}_t = \frac{(n)_t!}{(i)_t(j)_t(k)_t!}.$$

**Proposition 4.5** (Colored Lawrence action). Let $n \in \mathbb{N}$, and $(k_1, \ldots, k_{n-1}) \in E_{n,m}$. The action of standard generators of $\mathcal{B}_n$ on $\mathcal{H}^{abs}_m$ is computed on a standard code sequence as follows:

$$\sigma_1 \cdot U^{(i)}(k_1, \ldots, k_{n-1}) = (-1)^{k_1} t^{k_1(k_1 + 1)/2} \sum_{l=0}^{k_2} s_{1}^{k_1+l} \begin{pmatrix} k_1 + l \\ k_1 \end{pmatrix}_t U^{(1,2)}(k_1 + l, k_2 - l, k_3, \ldots, k_{n-1}),$$

$$\sigma_{n-1} \cdot U^{(i)}(k_1, \ldots, k_{n-1}) = (-1)^{k_{n-1}} t^{\frac{k_{n-1}(k_{n-1} + 1)}{2}} \sum_{l=0}^{k_{n-2}} s_{n-1}^{k_{n-1} - l} \begin{pmatrix} k_{n-1} + l \\ k_{n-1} \end{pmatrix}_t U^{(n-2,n-1)}(k_1, \ldots, k_{n-2} - l, k_{n-1} + l),$$

$$\sigma_i \cdot U^{(i)}(k_1, \ldots, k_{n-1}) = (-1)^{k_i} t^{\frac{k_i(k_i + 1)}{2}} \sum_{l_i=0}^{k_{i-1}} \sum_{l_2=0}^{k_{i+1}} s_i^{l_i+l_2} \begin{pmatrix} k_i + l_i + l_2 \\ k_i, l_i, l_2 \end{pmatrix}_t U^{(i,i+1)}_{i;i_1;i_2}$$

for $i = 2, \ldots, n - 2$, where:

$$U^{(i,i+1)}_{i;i_1;i_2} = U^{(i,i+1)}(k_1, \ldots, k_{i-1} - l_1, k_i + l_1 + l_2, k_{i+1} - l_2, \ldots, k_{n-1}).$$
Proof. As a half Dehn twist is a "local move" in the sense that it only involves arcs reaching the swapped
two points, the computation of generators' action in all cases of punctures, is a straightforward consequence of
the 4 punctured case. We perform this computation in Example 4.10 of next section, from which it suffices
to replace 2 by $i$ in the action of $\sigma_2$ to obtain that of $\sigma_i$, and to replace 3 by $n - 1$ in the action of $\sigma_3$ to
deduce that of $\sigma_{n-1}$. □

The above proposition is sufficient to get matrices for colored Lawrence representations by considering
the following remark.

**Remark 4.6.** The above proposition compute representation of a braid $\beta \in B_n$ as an element of

$$\text{Hom}_R \left( \mathcal{H}_{m}^{\text{abs}}(\epsilon), \mathcal{H}_{m}^{\text{abs,perm}}(\beta) \right).$$

To be able to write matrices for the representation of $B_n$ on $\bigoplus_{\tau \in S_n} \mathcal{H}_{m}^{\text{abs,\tau}}$ one has to compute the action of
braids on elements $U^\tau$ for $\tau \in S_n$ instead of $U^{(l)}$. To do so, one has to take formulas from Proposition 4.5
transporting variables $s_i$'s by $\tau^{-1}$, namely replacing $s_i$ by $s_{\tau^{-1}(i)}$ for all $i = 1, \ldots, n$.

Matrices for the (uncolored) Lawrence representation of braid groups is an immediate corollary of Propo-
sition 4.5 by equalizing variables $s_i$'s to a single one.

**Corollary 4.7** ((uncolored) Lawrence action). Let $n \in \mathbb{N}$, $s := s_1 = \cdots = s_n$, and $(k_1, \ldots, k_{n-1}) \in E_{n,m}$. The representation of $B_n$ on $\mathcal{H}_{m}^{\text{abs}}$ is given by the action of its generators on the standard code sequences basis as follows:

$$\sigma_1 \cdot U(k_1, \ldots, k_{n-1}) = (-1)^{k_1} t^{-\frac{k_1(k_1-1)}{2}} \sum_{l=0}^{k_2} s^{k_1+l} \binom{k_1+l}{k_1}_{t^{-1}} U(k_1+l, k_2-l, k_3, \ldots, k_{n-1}),$$

$$\sigma_{n-1} \cdot U(k_1, \ldots, k_{n-1}) = (-1)^{k_{n-1}} t^{-\frac{k_{n-1}(k_{n-1}-1)}{2}} \sum_{l=0}^{k_{n-1}} s^{k_{n-1}+l} \binom{k_{n-1}+l}{k_{n-1}}_{t^{-1}} U(k_1, \ldots, k_{n-2} - l, k_{n-1} + l),$$

$$\sigma_i \cdot U(k_1, \ldots, k_{n-1}) = (-1)^{k_i} t^{-\frac{k_i(k_i-1)}{2}} \sum_{l_1=0}^{k_i} \sum_{l_2=0}^{k_{i+1}} s^{k_i+l_2} \binom{k_i+l_2}{k_i}_{t^{-1}} U_{i,l_1,l_2}$$

for $i = 2, \ldots, n - 2$, where:

$$U_{i,l_1,l_2}^{(i,i+1)} = U_{i,i+1}^{(i,i+1)}(k_1, \ldots, k_{i-1} - l_1, k_i + l_1 + l_2, k_{i+1} - l_2, \ldots, k_{n-1}).$$

Then the following remark tells one how to compute matrices for the colored version (as a representation
of the pure braid group) out of matrices of Lawrence representation given in the above corollary.

**Remark 4.8** (Coloring the Lawrence representation). Let $L_i(s)$ be the matrix associated with $\sigma_i \in B_n$
by the uncolored Lawrence representation over $\mathcal{H}_{m}^{\text{abs}}$ written in the code sequence basis (given in the above
Corollary 4.7). Let $\beta \in B_n$ such that:

$$\beta = \prod_{m=1}^{k} \sigma_i^{\epsilon_m}$$

where $\epsilon_m$ are signs ($\pm 1$). Let $j_m$ be the index of the "over" passing strand at the $m$'th crossing of $\beta$, braids
read from right to left. Then:

$$cL(\beta) := \prod_{m=1}^{k} L_i(s_{j_m})^{\epsilon_m}$$

is a well defined matrix associated with $\beta$. For pure braids, it provides the colored representation of $PB_n$ on
$\mathcal{H}_{m}^{\text{abs}}(\epsilon)$

**Remark 4.9.** The matrices provided in this section work also for the case $m = 2$, and provide matrices
for the BKL representations. In this case there is a change of basis relating them to those from Proposition 3.13
This is detailed in Section 4.4 below in the case of the three strands braid group, which should allow
the reader to understand the general case.
4.3. **Computation with four strands.** We compute the action of generators in the case of $B_4$. We use homological techniques developed in [M, Section 4], not reproving them. Still we give subtle examples of handle rules computation at the end of present section, in Example 4.12.

4.3.1. **Computation.**

**Example 4.10 (Computation of $B_4$ representations).** Let $n = 4$. We compute the action of $\sigma_2 \in B_4$ on the code sequence $U := U(k_1, k_2, k_3)$ such that $\sum k_i = n$.

\[
\sigma_2 \cdot U = (-1)^{k_2} \left( \begin{array}{c}
  p_1 \\
  p_3 \\
  k_1
\end{array} \right)
\]

\[
\left( \begin{array}{c}
  p_1 \\
  p_3 \\
  k_1
\end{array} \right)
\]

\[
\frac{k_2(k_2-1)}{2} \sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_3} t^{l_2(l_2-1)}
\]

\[
\sigma_2 \cdot U(k_1, k_2, k_3) = (-1)^{k_2} t^{k_2(k_2-1)/2} \sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_3} t^{l_2(l_2-1)}/2
\]

\[
= (-1)^{k_2} t^{k_2(k_2-1)/2} \sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_3} t^{k_2+l_2}
\]

The coefficient $(-1)^{k_2}$ shows up for reversing the orientation of the indexed $k_2$ dashed arc (that has been reversed by the half Dehn twist $\sigma_2$) so that all dashed arcs are still oriented from left to right. The coefficient $s_{k_2} t^{k_2(k_2-1)/2}$ stands for preserving a straight $(k_2)$-handle: after the application of $\sigma_2$, the $(k_2)$-handle runs once around puncture $p_2$, we use the handle rule introduced in [M, Remark 4.2], and recalled in Remark 4.11, that precises the coefficient appearing while modifying a handle. This precise handle rule coefficient is given in below Example 4.12 (i).
dashed arcs, presented in [M Corollary 4.10], to transform middle theta (dashed) diagram into one dashed arc and to recover a code sequence, with the apparition of quantum trinions in exchange.

\[
(5) \sigma_2 \cdot U^{(1)}(k_1, k_2, k_3) = (-1)^{k_2}t^{-\frac{k_2(k_2-1)}{2}} \sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_3} s_{l_2}^{l_2+l_2} \binom{k_2 + l_1 + l_2}{k_2, l_1, l_2} U^{(2,3)}(k_1 - l_1, k_2 + l_1 + l_2, k_3 - l_2)
\]

for \((2, 3) \in \mathfrak{S}_3\), where we used notations \(U^{(i)}\) and \(U^{\tau}\) to distinguish elements in \(\mathcal{H}^{\text{abs}}\) and \(\mathcal{H}^{\text{abs}^{\tau}}\) respectively. By using same part of computation (but simpler as we now deal with leftmost and rightmost generators), one can compute the action of the two other generators to obtain the following formulas.

\[
(6) \sigma_1 \cdot U^{(1)}(k_1, k_2, k_3) = (-1)^{k_1}t^{-\frac{k_1(k_1-1)}{2}} \sum_{l=0}^{k_2} s_{k_1+l}^{k_1+l} \binom{k_1 + l}{k_1} U^{(1, 2)}(k_1 + l, k_2 - l, k_3),
\]

\[
(7) \sigma_3 \cdot U^{(1)}(k_1, k_2, k_3) = (-1)^{k_3}s_3^{k_3} t^{-\frac{k_3(k_3-1)}{2}} \sum_{l=0}^{k_2} \binom{k_3 + l}{k_3} U^{(3,4)}(k_1, k_2 - l, k_3 + l).
\]

4.3.2. Handle rule. We recall the handle rule, and we give two subtle examples used several times in the previous computation.

**Remark 4.11** (Handle rule, [M Remark 4.1]). Let \(B\) be a singularly locally finite \(r\)-cycle of \(C_m(C_{n,m}, \mathbb{Z})\). We’ve seen a process to choose a lift of \(B\) to the homology with local coefficients in \(L_m\), using a handle which is a path joining \(d\) to \(x \in B\). Let \(\alpha\) and \(\beta\) be two different paths joining \(d\) and \(B\). Let \(\bar{B}^\alpha\) and \(\bar{B}^\beta\) be the lifts of \(B\) chosen using \(\alpha\) and \(\beta\) respectively. By the handle rule we have the following relation in \(\mathcal{H}^{\text{abs}}\).

\[
\bar{B}^\alpha = L_m(\beta \alpha^{-1}) \bar{B}^\beta
\]

where \(L_m\) is the representation of \(\pi_1(C_{n,m})\) used for the local system. This expresses how the local system coordinate of a homological class is translated after a change of handle.

**Example 4.12** (Examples of handle rules). We provide two examples of handle rules applications in the case of a modification of a \((k)\)-handle, standing for \(k\) parallel handles.

(i) A fist application of the handle rule is the following:

\[
\begin{pmatrix}
  p_i & \bullet & \bullet & \bullet & p_j \\
  k
\end{pmatrix} = s_j^{k} t^{-\frac{k(k+1)}{2}} \begin{pmatrix}
  p_i & \bullet \bullet \bullet & p_j
\end{pmatrix}
\]

where drawings are the same outside parenthesis. This is a generalization of [M Example 4.2]. One can deduce the above coefficient from it, by replacing the simple handle \(\beta\) by a \((k)\)-handle that is a ribbon of \(k\) parallel handles. By replacing a simple strand (see [M Figure 3]) by a ribbon of \((k)\)-strands running once around puncture \(p_j\), there is a coefficient \(s_j^k\) appearing for the total winding number around \(p_j\). There is also a coefficient \(t^{-\frac{k(k+1)}{2}}\) appearing for the twist of the ribbon necessary for the ribbon to encircle the puncture. Twisting the ribbon is assimilated to a framed Reidemeister I move, involving a ribbon effect.

(ii) As in (i), we have:

\[
\begin{pmatrix}
  p_{i-1} & \bullet & \bullet & p_i & \bullet & \bullet & p_{i+1} \\
  k
\end{pmatrix} = t^{-\frac{k(k+1)}{2}} \begin{pmatrix}
  p_{i-1} & \bullet & \bullet & p_i & \bullet & \bullet & p_{i+1}
\end{pmatrix}
\]

where drawings are the same outside parenthesis. This is a generalization of [M Example 4.2]. One can deduce the above coefficient from it, by replacing the simple handle \(\beta\) by a \((k)\)-handle that is a ribbon of \(k\) parallel handles. By replacing a simple strand (see [M Figure 3]) by a ribbon of \((k)\)-strands running once around puncture \(p_j\), there is a coefficient \(s_j^k\) appearing for the total winding number around \(p_j\). There is also a coefficient \(t^{-\frac{k(k+1)}{2}}\) appearing for the twist of the ribbon necessary for the ribbon to encircle the puncture. Twisting the ribbon is assimilated to a framed Reidemeister I move, involving a ribbon effect.
with the coefficient $t^{\frac{k(k-1)}{2}}$ coming from twisting the ribbon of $k$ parallel handles. So that:

$$
\begin{pmatrix}
\bullet
\vline \quad \bullet
\end{pmatrix}
= \sum_{l=0}^{k} t^{\frac{l(l-1)}{2}}
\begin{pmatrix}
\bullet
\vline \quad \bullet
\end{pmatrix}
\begin{pmatrix}
\bullet
\vline \quad \bullet
\end{pmatrix}
= \sum_{l=0}^{k} s^{l}
\begin{pmatrix}
\bullet
\vline \quad \bullet
\end{pmatrix}
\begin{pmatrix}
\bullet
\vline \quad \bullet
\end{pmatrix}.
$$

The first equality comes from breaking a dashed arc, see [M, Example 4.6]. The second one is an application of (i), so that powers of $t$ are simplified.

4.4. A concrete case and relation with Section 3. We recall that there exists a family of multiforks generating a submodule of $H_{m}^{abs}$ for all $m$, and that in the case $m = 2$, Bigelow’s standard forks provide another basis (of the multiforks submodule). The definition of multiforks is given in [M, Section 7.1]. The module generated by (multi)-forks is a strict submodule of $H_{m}^{abs}$, see Corollary 7.2 from [M]; this is a consequence of the fact that there exists a diagonal matrix sending the family of code sequences to that of multiforks, but with (non invertible) quantum factorials on the diagonal terms, see [M, Corollary 7.2] for the precise coefficients. Nevertheless, matrices from Proposition 3.13 can be recovered by those from Proposition 4.5. In this section we study the example with 3 punctures, we compute matrices from both set-ups and we discuss how they are related. This should help the reader understanding notations and dealing with higher cases.

From now on, $m = 2$ so that we study the representation over $H_{2}^{abs}$. The braid group $B_{3}$ has two generators $\sigma_{1}$ and $\sigma_{2}$. Let $L_{1}$ and $L_{2}$ be their representations from Proposition 4.7, the uncolored version for Lawrence representations, matrices names are introduced in Remark 4.8. Then:

$$
L_{1}(s, t) = \begin{pmatrix}
{s^2t^{-1}} & -s^2(1 + t^{-1}) & s^2 \\
0 & -s & s \\
0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
L_{2}(s, t) = \begin{pmatrix}
1 & 0 & 0 \\
1 & -s & 0 \\
1 & -s(1 + t^{-1}) & s^2t^{-1}
\end{pmatrix}.
$$

Let $BKL_{1}$ and $BKL_{2}$ the generators’ representations from Proposition 3.13; matrices notations from Remark 3.22. Then:

$$
BKL_{1}(q, t) = \begin{pmatrix}
q^2t & 0 & q^2 - q \\
0 & q & q \\
0 & 1 & 1 - q
\end{pmatrix}
\quad \text{and} \quad
BKL_{2}(q, t) = \begin{pmatrix}
0 & q & 0 \\
1 & 1 - q & 0 \\
0 & t(q^2 - q) & q^2t
\end{pmatrix}.
$$

First, one can check that the two following relations hold, they correspond to Remarks 3.22 and 4.8 adapted to this case.

$$
BKL_{1}(q_2, t)BKL_{2}(q_1, t)BKL_{1}(q_1, t)BKL_{1}(q_1, t)BKL_{2}(q_2, t),
\quad \text{and}
$$

$$
L_{1}(s_2, t)L_{2}(s_1, t)L_{1}(s_1, t) = L_{2}(s_1, t)L_{1}(s_1, t)L_{2}(s_2, t).
$$

This relations should help one with computation of colored version matrices for pure braid groups elements. Then let:

$$
P := \begin{pmatrix}
1 + t & 1 + t & 0 \\
0 & 1 + t & 0 \\
0 & 1 + t & 1 + t
\end{pmatrix},
$$

such that one can check:

$$
P^{-1}L_{1}(s, t)P = BKL_{1}(s, t^{-1}) \quad \text{while} \quad P^{-1}L_{2}(s, t)P = BKL_{1}(s, t^{-1}).
$$

This emphasizes how to recover matrices from Proposition 3.13 out of those from Proposition 4.7, namely by the change of bases given by $P$ and with the relations between variables $q = s$ and $t = t^{-1}$ (the second...
one is due to different choices for winding numbers in the literature). One notices that the matrix $P$ is not diagonal as announced in [M, Corollary 7.2], for passing from code sequences to forks. This is because forks generators used in Section 3 (taken from [Big]) do not fit perfectly with multiforks for higher Lawrence representations, see Section 7.1 in [M], and Remark 3.16 of the present paper. In the present case, the fork denoted $F_{1,3}$ in Section 3 has the following decomposition:

$$F_{1,3} = F(2, 0) + (1 + t)F(1, 1) + F(0, 2)$$

where $F(i, j)$ are standard multiforks from [M, Section 7.1]. This decomposition can be computed from [M, Example 4.5] and the handle rule, Remark 4.11.

5. Appendix: colored vs pure

We have seen in Section 2.2.2 as in Definition 3.6 that to pass to colored version for representations, one needs to associate one variable per puncture in such a way that if the punctures are permuted variables have to be transported. It seems that the first way to handle this issue is to restrict to a representation of the pure braid group for which punctures are fixed pointwise. Then by means of induced representation one can obtain a representation of the entire braid group for which generators are simpler. This section is devoted to define this induced representation and then to present an object appropriate to this set-up, the colored braid groupoid.

5.1. Induced representation. Passing from a representation of $\mathcal{PB}_n$ to one of $\mathcal{B}_n$ uses the concept of induced representation that we present in this section.

Definition 5.1 (Representation of the braid group). A representation of $\mathcal{B}_n$ is an algebra morphism:

$$A[\mathcal{B}_n] \rightarrow \text{End}_A(V)$$

where $A$ is a ring and $V$ is an $A$-module.

Definition 5.2 (Induced representation from the pure braid group). Let $r$ be a representation of $\mathcal{PB}_n$:

$$r : A[\mathcal{PB}_n] \rightarrow \text{End}_A(V)$$

There exists a natural induced representation $\text{Ind}(r)$ of $\mathcal{B}_n$ over the space:

$$\text{Ind}(V) = A[\mathcal{B}_n] \otimes_A [\mathcal{PB}_n] V$$

where the action of $\mathcal{PB}_n$ is given by product on the left of the tensor product and by $r$ on the right. Its dimension is $n! \times \dim(V)$.

With this notation, the action of $\mathcal{PB}_n$ on $\text{Ind}(V)$ stabilizes $1 \otimes V$, which recovers initial representation $r$.

Example 5.3. Let $\text{perm} : \mathbb{C}[\mathcal{B}_n] \rightarrow \mathbb{C}[\mathcal{S}_n]$ be the representation that assigns to a braid the permutation it involves on punctures. It is the induced representation from the trivial representation of $\mathcal{PB}_n$.

Remark 5.4. In the present paper we deal with three families of modules parametrized by the symmetric group.

- In Section 2.2.2 we introduce a family $V^\tau$ (for $\tau \in \mathcal{S}_n$).
- By analogy one can define $\mathcal{H}^\tau$ to be $\mathcal{H} \otimes\tau$ from Definition 3.8. In Section 4 we introduced modules $\mathcal{H}_{\alpha\beta}^{n\tau}$ for $\tau \in \mathcal{S}_n$ and $n \in \mathbb{N}$.

Let $X$ designates the letter $\mathcal{H}$ or the letter $V$. There exists a representation of the pure braid group on $X^{(1)}$ such that:

$$\text{Ind}(X) = \bigoplus_{\tau \in \mathcal{S}_n} X^\tau.$$  

The induced representation is more convenient for computation as generators of $\mathcal{B}_n$ are simpler than those of $\mathcal{PB}_n$, although vectors of the modules involves more complicated notations and formulas and the dimension is bigger. We state a remark about the faithfulness of the induced representation.

Remark 5.5. For a matrix associated with a braid to be the identity on $\text{Ind}(X)$, it must stabilize $X^{(1)}$, hence be a pure braid.
Subsection 2.1.2 is probably the easiest to read as it concerns the Gassner representation, and the links between representations of the pure braid groups and induced representations of the braid groups. Next sections apply same protocols in construction of matrices.

5.2. Colored braid groupoid. We present a point of view allowing to see the colored representations not as an induced representation but as a representation of a single object. Namely it involves the generalization of the notion of representation of groups to that of representation of groupoids.

Definition 5.6 (Groupoid). A groupoid $G$ is a category inside which every morphism is invertible.

Definition 5.7 (Representation of a groupoid). A representation of a groupoid $G$ is a functor from $G$ to the category $\text{Vect}$ of vector spaces.

Example 5.8. This notion of groupoid is used in topology to generalize the one of fundamental group.

(i) The fundamental groupoid $\Pi_1(M)$ of a topological space $M$ is the groupoid whose set of objects is $M$ and whose morphisms from $x$ to $y$ are the homotopy-classes $[\gamma]$ of continuous maps $\gamma: [0,1] \to M$ with endpoints map to $x$ and $y$ (which the homotopies are required to fix). Composition is by concatenation (and reparametrization) of representative maps.

(ii) Let $O$ be a subset of a topological space $M$, there exists a sub-groupoid of the fundamental groupoid:

$$G = \bigcup_{\alpha \in O} G_{\alpha} \subset \Pi_1(M),$$

it consists in the groupoid of paths having endpoints in $O$.

(iii) When $O = \{x\}$ is a single point, then the corresponding sub-groupoid is the fundamental group based in $x$.

All the background regarding links between fundamental groupoid and topology can be found in [Br], where one can find the correspondence between topological coverings and the fundamental groupoid.

Definition 5.9 (Colored braid groupoid). Let $N \in \mathbb{N}^*$. The colored braid groupoid on $N$ strands is the groupoid whose set of objects is $\mathcal{G}_N$ and morphisms between $\tau_1$ and $\tau_2 \in \mathcal{G}_N$ are braids $\beta$ satisfying:

$$\tau_1 \text{ perm}(\beta) = \tau_2$$

where perm is the morphism that sends a braid to its induced permutation.

Remark 5.10. The braid group is the fundamental group of a configuration space, for $x$ an arbitrary base point:

$$\mathcal{B}_N = \pi_1(C_{n,0}, x)$$

using Definition 1.3 for configuration spaces. See [Br].

Remark 5.11. Let $\sigma_i, i = 1 \ldots, N$ be the standard generators of $\mathcal{B}_N$. Then the system:

$$\sigma_i^\alpha : \alpha \to \text{perm}(\sigma_i)\alpha$$

of morphisms for $\alpha \in \mathcal{G}_N$, provides generating morphisms of $G$.

Using the letter $X$ to designate either the letter $\mathcal{H}$ or $V$ from Remark 5.4, we define a representation of the colored braid groupoid as follows. Let $\sigma_i^\alpha$ be a generating morphism of the colored braid category ($\alpha \in \mathcal{G}_n$), we define its representation:

$$\begin{cases} 
X^\alpha &\rightarrow X^{(i,i+1)\alpha} \\
v &\mapsto \sigma_i, v.
\end{cases}$$

This is a way to deal with colored representations by consideration of the colored braid groupoid.
References

[B-N] D. Bar-Natan, A Note on the Unitarity Property of the Gassner Invariant, Bulletin of Chelyabinsk State University (Mathematics, Mechanics, Informatics) 3-358-17 (2015), 22–25.

[Big] S. Bigelow, Braid groups are linear, J. Amer. Math. Soc. 14 (2000), 471–486.

[Br] R. Brown, Topology and Groupoids, Booksurge, (2006).

[F-M] B. Farb, D. Margalit, A Primer on Mapping Class Groups, Princeton University Press (2012).

[Hab] K. Habiro, An integral form of the quantized enveloping algebra of sl2 and its completions, J. Pure Appl. Alg. 211 (2007), 265–292.

[Ito] T. Ito, Reading the dual Garside length of braids from homological and quantum representations, Comm. Math. Phys. 335 (2015) 345–367.

[Ito2] T. Ito, Topological formula of the loop expansion of the colored Jones polynomials, Trans. Amer. Math. Soc., (2019).

[J-K] C. Jackson and T. Kerler, The Lawrence-Krammer-Bigelow representations of the braid groups via $U_q sl(2)$, Adv. Math., 228, (2011), 1689–1717.

[K-T] C. Kassel, V. Turaev, Braid Groups, Springer (2008).

[Knu] K. Knudson, On the kernel of the Gassner representation, Archiv der Mathematik 85, (2005), 108–117.

[K1] T. Kohno, Hyperplane arrangements, local system homology and iterated integrals, Arrangements of Hyperplanes—Sapporo 2009, 157–174, Mathematical Society of Japan, Tokyo, Japan, 2012.

[K2] T. Kohno, Quantum and homological representations of braid groups, Configuration Spaces - Geometry, Combinatorics and Topology, Edizioni della Normale (2012), 355–372.

[Kra] D. Krammer, Braid groups are linear, Ann. Math. 155 (2002), 131–156.

[Law] R. Lawrence, Homological representations of the Hecke algebra, Comm. Math. Phys. 135 (1990), 141–191.

[Lus] G. Lusztig, Finite dimensional Hopf algebras arising from quantum groups, J. Am. Mat. Soc. 3 (1990), 257–296.

[M-W] I. Marin, E. Wagner, A cubic defining algebra for the Links-Gould polynomial, Advances in Mathematics, 248, (2013), 1332–1365.

[M] J. Martel, A homological model for $U_q sl(2)$ Verma-modules and their braid representations, arXiv:2002.08785 math.GT.

[Mo] H.R. Morton, The Multivariable Alexander Polynomial for a Closed Braid, Contemporary Mathematics 233, Amer. Math. Soc. (1999), 167–172.

[P-P] L. Paoluzzi, L. Paris, A note on the Lawrence-Krammer-Bigelow representation, Algebr. Geom. Topol., 2 (2002), 499–518.

[Zi] M. Zinno, On Krammer’s representation of the braid group, Math. Ann. 321, (2001), 197 – 211.