Topological T-duality for Stacks using a Gysin Sequence

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Abstract

In this note I compare three formalisms for obtaining the Topological T-dual of a semi-free $S^1$-space.

1 Introduction

Topological T-duality is a recent theory inspired by the theory of T-duality in String Theory. Principal circle (and torus) bundles with a class in $H^3$ of the total space of the bundle possess an unusual symmetry, namely, from this data it is possible to naturally construct T-dual bundles which are also principal circle bundles over the same base with a class in $H^3$ of the total space of the T-dual bundle (See Ref. \[1,2\] for examples).

In Refs. \[3,4\], the authors generalize Topological T-duality to principal bundles of topological stacks (see Refs. \[3,5,6,7\] for an introduction) $\mathcal{E} \rightarrow \mathcal{Y}$ with an $S^1$-gerbe $\mathcal{G}$ on the stack $\mathcal{E}$. The authors show that in such a situation, the T-dual exists and is also a principal bundle of stacks together with a gerbe on it. Since this result applies to stacks, it applies to spaces with circle group actions which are not necessarily free and, in particular, to semi-free $S^1$-spaces.

A brief outline of the paper is as follows: In Ref. \[8\] the T-duals of some semi-free $S^1$-spaces were derived using $C^*$-algebraic techniques. In Ref. \[9\] a general formalism using the Borel construction was used to derive the T-dual of any semi-free $S^1$-space. Neither of these constructions used stack theory, and it would be interesting to compare the T-duals obtained using all three theories. In Sec. \[2\] below, we attempt to do this in a simple example.

We calculate Topological T-duals of semi-free spaces for the examples of Ref. \[8\] (these are the Kaluza-Klein monopole backgrounds of string theory) using the methods of Ref. \[3\] in Sec. \[3\]. We also comment on the results obtained.

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A phenomenon seen in backgrounds with Kaluza-Klein monopolies is the dyonic coordinate (See Refs. [10, 11] for details). A model for this using $C^*$-algebraic methods was developed in Ref. [8]. We point out in Sec. [4] that this phenomenon may also be obtained completely independently of the $C^*$-algebraic formalism in the stack theory formalism of Ref. [3] using the results from Ref. [12]. In Ref. [13], the authors give a Gysin sequence for a $S^1$-stack. Very roughly, this is an exact sequence of stack cohomology groups which is derived by taking classifying spaces for $E$ and $Y$, obtaining an ordinary principal bundle, and then using the ordinary (homology) Gysin sequence. It is clear that a cohomology Gysin sequence can also be obtained from this argument. Can this be used to obtain the Topological T-dual of a semi-free space in analogy to the argument in ordinary Topological T-duality? In Sec. [5] we develop this theory and calculate a few T-duals. We prove that the T-dual of a semi-free space obtained using this method is the one obtained by Mathai and Wu in Ref. [9].

In this paper we restrict ourselves to $S^1$-actions on stacks and stacks which are principal $S^1$-bundles over a stack.

## 2 Comparison of the Three Formalisms

We use the convention of Ref. [7]: If $X$ is a topological space then $\underline{X}$ is $X$ viewed as a stack using the Yoneda Lemma. Consider a space which is a point $\text{pt}$ with an $S^1$-action which fixes that point. It is interesting to compare the T-duals obtained for this space using the formalisms of Refs. [1, 3, 9]. We have the following theorem:

**Theorem 2.1.** Consider a point $\text{pt}$ with an $S^1$-action which fixes the point. Then,

1. The $C^*$-algebraic T-dual is $\mathbb{R}$ with quotient space the point.
2. The Topological T-dual in the formalism of Bunke et al. (see Refs. [2, 3, 4]) is the principal bundle of stacks $[\text{pt}/S^1] \times S^1 \to [\text{pt}/S^1]$ with a gerbe on the total space corresponding to $H$-flux.
3. The Topological T-dual in the formalism of Mathai-Wu (see Ref. [9]) is $BS^1 \times S^1 \times \text{pt}$ with $H$-flux.

**Proof.** Suppose one considers a point with a $S^1$-action which fixes the point. The quotient is still a point.

1. In the $C^*$-algebraic formalism, to this geometry we would assign the $C^*$-algebra of compact operators $\mathcal{K}$ (since the spectrum of $\mathcal{K}$ is the point). The trivial action of $S^1$ on $\mathcal{K}$ lifts to a trivial action $\alpha$ of $\mathbb{R}$ on $\mathcal{K}$. Also, any other action of $\mathbb{R}$ on $\mathcal{K}$ is exterior equivalent to this trivial one. The T-dual would be the spectrum of the crossed product $\mathcal{K} \rtimes \mathbb{R} \simeq C_0(\mathbb{R})$ i. e. $\mathbb{R}$ where $\simeq$ denotes Morita equivalence (See Ref. [16]). The group action on the T-dual $C^*$-algebra $C_0(\mathbb{R})$ is induced from the translation action of $\mathbb{R}$ on itself (See Ref. [16] for details). The quotient would be the point as expected.
2. In Ref. [3], Sec. (4.2, 4.3), Prop. (4.3), the T-dual of a stack is obtained by the following procedure: One passes to the geometric realization of the simplicial space of the groupoid associated with that stack. This gives an ordinary principal bundle with $H$-flux from the principal bundle of stacks one one began with. This principal bundle may be T-dualized in the normal way [3].

Consider the principal bundle of stacks $pt \to [pt/S^1]$ and the atlas $pt \to [pt/S^1]$. Let $Y = pt \times_{[pt/S^1]} pt$ be an atlas for $pt$ (See Ref. [3] Sec. (4.2)). We have a commutative square

\[
\begin{array}{ccc}
Y & \longrightarrow & pt \\
\downarrow & & \downarrow \\
pt & \longrightarrow & [pt/S^1].
\end{array}
\]

The groupoid associated to $pt \to [pt/S^1]$ is $pt \times S^1 \cong pt$ as there is a canonical isomorphism $Y \cong (pt \times S^1)$ since $(pt \times S^1)$ is the canonical bundle over $pt$ (See Heinloth Ex. 2.5 and following). Similarly the groupoid associated to the atlas $Y = (pt \times_{[pt/S^1]} pt) \to pt$ is $Y \times Y \cong pt$. Since the fiber product of $Y$ with itself over $pt$ is $pt \times (S^1)^2$ the associated groupoid would be $pt \times (S^1)^2 \to pt$.

It is clear that the iterated fiber product of $Y$ with itself $n$ times would be isomorphic to $pt \times (S^1)^n$. The total space of the associated simplicial bundle would then (by definition of $EG$) be $ES^1$ and the base would (by the construction above) be $BS^1$. Therefore the T-dual of the simplicial bundle would be $BS^1 \times S^1$ with a gerbe on total space (See Refs. [2, 3]). This corresponds to the T-dual bundle $[pt/S^1] \times S^1 \to [pt/S^1]$ with a gerbe on the total space of the bundle corresponding to the $H$-flux.

3. In the formalism of Mathai and Wu (See Ref. [9]), the original space would be replaced by $ES^1 \times pt$ as a principal circle bundle over $BS^1 \times pt$ and the T-dual would be $BS^1 \times S^1 \times pt$ as a principal circle bundle over $BS^1 \times pt$ with $H$-flux. The T-dual obtained here namely $BS^1 \times S^1 \times pt$ should be compared with the T-dual $[pt/S^1] \times S^1$ obtained in Part (2).

Thus, the formalisms of Bunke-Schick and Mathai-Wu give similar answers here for this example and the $C^*$-algebraic formalism gives a different one. This difference is probably due to the fact that in Ref. [1], an $S^1$-action on a principal bundle lifts to an $\mathbb{R}$-action (with $\mathbb{Z}$-stabilizers) on the $C^*$-dynamical system associated to that space while in Ref. [9], the $S^1$-action remains an $S^1$-action. That is, in the $C^*$-algebraic formalism a circle action is viewed as an $\mathbb{R}$-action with $\mathbb{Z}$-stabilizers while in the other formalisms an $S^1$-action is viewed only as an $S^1$-action.

In Topological T-duality, it is expected that the original and T-dual spaces have the same $K$-theory up to a degree shift. It is clear, from the Connes-Thom isomorphism, that the $C^*$-algebraic T-dual will have this property. Similarly, the answers obtained by the other two formalisms will also have this property, one would have to use $K$-theory twisted by the $H$-flux on the T-dual to perform the above calculation.
3 T-dual of Kaluza-Klein monopole Backgrounds

We now restrict our attention to spaces with semi-free circle actions. We may further restrict ourselves to spaces which contain Kaluza-Klein monopoles (KK-monopoles). Away from fixed points such spaces are equivariantly $S^1$-homeomorphic to the total space of a principal circle bundle.

We begin this section by making a remark about the existence of T-dual stacks. In this paper, we will use the method of Ref. [3] to calculate the T-dual of a stack associated to a semi-free space. In the method of Ref. [3], if we restrict ourselves to $U(1)$-bundles over such stacks, the associated simplicial bundles, being circle bundles, may always be T-dualized. There is always a ‘T-duality diamond’ of Ref. [2] (see Diagram (2.14) in Lemma (2.13) of that reference) for the associated simplicial bundles, since these are only circle bundles. This gives a diagram of the form Diagram (4.1.2) in Ref. [4] for the associated stacks. Hence, any $U(1)$-bundle over such a stack may be T-dualized in the sense of Def. (4.1.4) of Ref. [4], even if the base is not an orbispace.

In the following, the stacks we dualize are not orbispaces in the sense of Ref. [3], but, due to the above, the principal bundles $p : \mathcal{E} \to \mathcal{B}$ may be completed into a diagram of the form of Diagram (4.1.2) of Ref. [4], and hence these stacks may be T-dualized (by Ref. [4], Def. (4.1.4)).

In a neighbourhood of a fixed point such spaces are equivariantly $S^1$-homeomorphic to $\mathbb{R}^4$ with an orthogonal $S^1$-action. The associated topological stacks are the stacks $[CS^3/Z_k]$. In Ref. [8], the $C^*$-algebraic approach of Ref. [1] was used to compute the T-duals of some semi-free spaces.

First, we consider the T-dual of spaces (See Thm. (3.1) below) which are the total space of KK-monopoles of charge $l \in \mathbb{N}, l > 0$. As argued above, these correspond to stacks of the form $E = [CS^3/Z_l]$ with $E/S^1 = CS^2$ for $l \geq 2$, and $E = CS^3$ and $E/S^1 = CS^2$ for $l = 1$. For $l = 1$ the associated principal bundle of stacks would be $CS^3 \to [CS^3/S^1]$. For $l \geq 2$, the associated principal bundle of stacks would be $[CS^3/Z_l] \to [CS^3/S^1]$. We consider the T-dual of spaces containing multiple KK-monopoles in Thm. (3.2) and Cor. (3.2) below.

Consider a spacetime which is a KK-monopole spacetime with charge 1. Physically, the T-dual would have a source of $H$-flux over the set in the base corresponding to the image of the singular fiber and the $H$-flux would be undefined at the location of the source. In the $C^*$-formalism of Topological T-duality the $C^*$-algebra describing the background loses the continuous-trace property exactly on this locus. In Ref. [8], it was argued that this is a model for a space with a source of $H$-flux.

In the formalisms of Refs. [3, 9], however, the T-dual $H$-flux would be everywhere defined, that is, there would be no source of $H$-flux present. In particular, for these two theories, the following would hold: The T-dual of a single NS5-brane with a background of $k$-units of (sourceless) $H$-flux would be indistinguishable from the T-dual of a space with $(k + 1)$-units of sourceless $H$-flux.

Let $\mathcal{E} \to \mathcal{Y}$ be a principal $S^1$-bundle of stacks over $\mathcal{Y}$. It is clear from the axioms

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1 In String Theory, the Taub-NUT metric (see Refs. [11]) describes a space with one KK-monopole. For more details see Ref. [8] and references therein.

2 See Ref. [8] for a detailed discussion.
of a principal $S^1$-bundle (see Ref. [7] after Remark (2.14)) that the topological stack $\mathcal{E}$ is a space with a left $S^1$-action (in the sense of Ref. [13], Sec. (3)).

**Lemma 3.1.**

1. Let $E$ be a space with a circle action. Then, $\mathcal{E} \to [E/S^1]$ is a principal bundle of topological stacks.

2. Let $p : \mathcal{E} \to \mathcal{Y}$ be a principal $S^1$-bundle of topological stacks. Then $\mathcal{E}$ is a stack with a left $S^1$-action in the sense of Ref. [13] and $\mathcal{Y} \cong [S^1 \backslash \mathcal{E}]$.

3. For a space $E$ with a circle action, $[S^1 \backslash E] \cong [E/S^1]$.

**Proof.**

1. $E$ is a space with a $S^1$-action and satisfies the conditions for a principal $S^1$-bundle described after Remark (2.14) in Ref. [7]. This is equivalent to the definition of a principal bundle of stacks using atlases by the Claim before Example (2.15) in Ref. [7].

2. That $\mathcal{E}$ is a stack with a left $S^1$-action follows from the definition of a principal bundle of stacks (See Ref. [7]).

From Ginot and Noohi (Ref. [13]), Prop. (4.8), $[S^1 \backslash \mathcal{E}]$ is a stack. Let $\bar{p} : \mathcal{E} \to [S^1 \backslash \mathcal{E}]$ be the natural map defined in Sec. (4.1) of Ref. [13].

Let $T \to \mathcal{Y}$ be an atlas for $\mathcal{Y}$. By definition of a principal bundle of stacks, (see Ref. [7]), we have a commutative square

$$
\begin{array}{ccc}
P & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow \\
T & \longrightarrow & \mathcal{Y}
\end{array}
$$

where $P$ is a principal bundle over $T$ and an atlas for $\mathcal{E}$. As noted above, we have a natural map $\bar{p} : \mathcal{E} \to [S^1 \backslash \mathcal{E}]$. By the second part of the proof of Prop. (4.8) of Ref. [13], $P$ is also an atlas for $[S^1 \backslash \mathcal{E}]$. By Ref. [13], Sec. (4.1) the stack $[S^1 \backslash \mathcal{E}]$ is the stackification of the prestack $\lfloor S^1 \backslash \mathcal{E} \rfloor$. Then, we have

$$
[S^1 \backslash \mathcal{E}] (P) \simeq S^1 \backslash (\mathcal{E}(P)), \text{ (by definition of } [S^1 \backslash \mathcal{E}], \text{ see Sec. (4.1) of Ref. [13]),}
\simeq \mathcal{Y}(P), \text{ (By definition, see Sec. (4.1) of Ref. [13]),}
\simeq \mathcal{Y}(P/S^1) \simeq \mathcal{Y}(T), \text{ (because the Diagram (1) commutes).}
$$

Hence, the stackification of $\mathcal{Y}$ is isomorphic to the stackification of $[S^1 \backslash \mathcal{E}]$. Since $\mathcal{Y}$ is a stack, this implies that $\mathcal{Y} \cong [S^1 \backslash \mathcal{E}]$.

3. This follows from Parts (1) and (2) above.

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3See Ref. [7], Def. (2.11)
Lemma 3.2. Let \( X \) be a stack with a \( S^1 \)-action in the sense of Ginot et. al (See Ref. [13], Def. (3.1)). Let \( q : X \to [S^1 \backslash X] \) be the quotient map of Ref. [13], Sec. (3.2). Then \([S^1 \backslash X]\) is a topological stack and \( q : X \to [S^1 \backslash X]\) is a principal bundle of stacks in the sense of Ref. [7].

Proof. By definition, we have an action \( \mu \) of \( S^1 \) on the stack \( X \). By Prop. (4.8) of Ref. [13], \([S^1 \backslash X]\) is also a topological stack. By Prop. (4.7) of the same reference, the map \( q \) is representable. It can be checked that the conditions for a principal bundle of stacks given after Remark (2.14) in Ref. [7] are satisfied with \( \text{act} = \mu, p = q \). \qed

Corollary 3.1. \( p : E \to Y \) is a principal bundle of stacks iff \( E \) is a stack with a left \( S^1 \)-action in the sense of Ref. [13] and \( Y \simeq [S^1 \backslash E] \).

Proof. This follows from Lemma (3.1), Part (2), and Lemma (3.2) above. \qed

In Thm. (2.1) (2) we determined the Topological T-dual of the principal bundle of stacks \( \text{pt} \to [\text{pt}/S^1] \). The calculation for the Topological T-dual of \( CS^3 \) is nearly similar to that for the Topological T-dual of \( \text{pt} \to [\text{pt}/S^1] \) for the following two reasons: Firstly, the space \( CS^3 \) is equivariantly homotopy equivalent to its vertex \( \text{pt} \). Secondly, \( CS^3 \) and \( CS^2 \) are contractible and are homeomorphic to \( \mathbb{R}^4 \) and \( \mathbb{R}^3 \) respectively. As a result, any principal bundle over \( CS^3 \) is trivial. Due to this, we can take the above proof and replace \( \text{pt} \) with \( CS^3 \) and \( [\text{pt}/S^1] \) with \( CS^2 \) and obtain a working proof.

Theorem 3.1.

1. The Topological T-dual of the principal bundle of stacks \( CS^3 \to [CS^3/S^1] \) (associated to a KK-monopole) in the formalism of [3] is the principal bundle of stacks \( [CS^3/S^1] \times S^1 \to [CS^3/S^1] \) with a gerbe on the stack \([CS^3/S^1] \times S^1\).

2. Consider the \( S^1 \)-action on a point which fixes the point. This gives a principal bundle of stacks \( \text{pt} \to [\text{pt}/S^1] \). Consider the subgroup \( \mathbb{Z}_k \hookrightarrow S^1 \) for any natural number \( k > 1 \). Then, \([\text{pt}/\mathbb{Z}_k] \to [\text{pt}/S^1] \) is also a principal bundle of stacks. This bundle with no \( H \)-flux has as T-dual the principal bundle of stacks \( [\text{pt}/S^1] \times S^1 \to [\text{pt}/S^1] \) with \( H \)-flux of \( k \) units.

3. For \( k \) any natural number larger than \( 1 \), there is a principal bundle of stacks \( [CS^3/\mathbb{Z}_k] \to [CS^3/S^1] \) (corresponding to a KK-monopole of charge \( k \)). The Topological T-dual of this bundle with no \( H \)-flux, in the formalism of [3], is the principal bundle of stacks \( ([CS^3/S^1] \times S^1) \to [CS^3/S^1] \) with \( k \) units of \( H \)-flux.

Proof.

1. Consider the principal bundle of stacks \( CS^3 \to [CS^3/S^1] \). The space \( \mathbb{R}^4 \simeq CS^3 \) is an atlas for the stack \([CS^3/S^1] \). Let \( Y = CS^3 \times_{[CS^3/S^1]} CS^3 \) be an atlas for \( CS^3 \) (See Ref. [3] Sec. (4.2)) induced by the atlas for the stack \([CS^3/S^1] \). We have a commutative square

\[
\begin{array}{ccc}
Y & \longrightarrow & CS^3 \\
\downarrow & & \downarrow \\
CS^3 & \longrightarrow & [CS^3/S^1].
\end{array}
\]
The atlas associated to $CS^3$ is $CS^3 \times S^1 \rightarrow CS^3$. There is a canonical isomorphism $Y \simeq (CS^3 \times S^1)$, since, due to the contractibility of $CS^3 \simeq \mathbb{R}^4$, $(CS^3 \times S^1)$ is the canonical bundle over $CS^3$ (See Heinloth Ex. 2.5 and following).

The groupoid associated to the atlas $Y = \left( \begin{array}{c} CS^3 \\ [CS^3/S^1] \end{array} \right) \rightarrow CS^3$ is $Y \times_{CS^3} CS^3$. The fiber product $Y \times_{CS^3} CS^3$ is $CS^3 \times (S^1)^n$. The groupoid is $CS^3 \times (S^1)^2 \rightarrow CS^3$. It is clear from the definition that the iterated fiber product of $Y$ with itself $n$ times is $CS^3 \times (S^1)^n$.

The associated simplicial space in each degree would be $CS^3 \times (S^1)^n$. The simplicial space is thus the fiber product of $CS^3 \rightarrow CS^2 \sim *$ with $* \times (S^1)^n \rightarrow *$. Therefore by Ref. [14], Cor. (11.6), the simplicial space is the fiber product $(A \times ES^1)$ where $A$ is the geometric realisation of the simplicial space which is $CS^3$ in each degree. Since $CS^3$ is contractible, the space $(A \times ES^1)$ would be homotopic to $ES^1$.

We had noted above that the total space of the bundle is the simplicial space which in each degree is $CS^3 \times (S^1)^n$. Hence, the base of the simplicial bundle would be the simplicial space which in each degree $> 1$ is $CS^2 \times (S^1)^{n-1}$ and at degree 1 is $CS^2 \times pt$. Let $B$ be the simplicial space which is $CS^2$ in each degree. By the above argument the base is the fiber product $(B \times BS^1)$. Since $B$ is contractible, the base has the homotopy type of $BS^1$. Thus, the simplicial bundle associated to the bundle of stacks $CS^3 \rightarrow [CS^3/S^1]$ is $(A \times ES^1) \rightarrow (B \times BS^1)$.

Therefore the T-dual of the simplicial bundle would be $(B \times BS^1) \times S^1$ with a gerbe on total space (See Refs. [2, 3]). The class in $H^3$ of the total space associated to the gerbe would correspond to the generator of $H^3((B \times BS^1) \times S^1, \mathbb{Z}) \simeq H^3(BS^1 \times S^1, \mathbb{Z})$. It is clear from the above that this is the simplicial bundle associated to $[CS^2/S^1] \times S^1$. Also, the $H$-flux on the total space of the bundle is 1. Therefore the $T$-dual stack has a gerbe on it.

2. Consider the $S^1$-action on a point which fixes the point. This gives a $S^1$-bundle of stacks $pt \rightarrow [pt/S^1]$. Then $\mathbb{Z}_k \rightarrow S^1$ acts on $pt$ as well. The $S^1$-action on $pt$ gives an action of $S^1$ on $[pt/\mathbb{Z}_k]$. The quotient is still $[pt/S^1]$. Therefore, $[pt/\mathbb{Z}_k] \rightarrow [pt/S^1]$ is a principal bundle of stacks by Lemmas (3.1,3.2) above.

The groupoid associated to the stack $[pt/\mathbb{Z}_k]$ is $\mathbb{Z}_k \rightarrow pt$ and the associated simplicial space is $B\mathbb{Z}_k$ (See Ref. [3] Sec. (5.1) ). The simplicial space associated to $[pt/S^1]$ is $BS^1$. The associated simplicial principal bundle is $B\mathbb{Z}_k \rightarrow BS^1$. This is $ES^1/\mathbb{Z}_k \rightarrow BS^1$. Thus the $T$-dual simplicial bundle is $BS^1 \times S^1$ with $H$-flux $k$. This is the simplicial bundle associated to the stack $[pt/S^1] \times S^1$ with a gerbe on the stack.

3. For every $k > 1$, there is an $S^1$-action on $CS^3/\mathbb{Z}_k$ induced by the $S^1$-action on the $S^1$-space $CS^3$ with quotient $CS^3/S^1$. Thus there is an $S^1$-action on $[CS^3/\mathbb{Z}_k]$ such that the quotient under the $S^1$-action is $[CS^3/S^1]$. Therefore, by Lemmas (3.1,3.2), there is a principal bundle of stacks $[CS^3/\mathbb{Z}_k] \rightarrow [CS^3/S^1]$. 


The simplicial space associated to \([CS^3/Z_k]\) is determined in a manner similar to Part (1). Consider the principal bundle of stacks \(CS^3 \to [CS^3/Z_k]\). Let \(W\) be an atlas for the stack \(CS^3\) induced by the the atlas \(CS^3\) for the stack \([CS^3/Z_k]\). Then, \(W\) is a principal \(Z_k\)-bundle over \(CS^3\). We have a commutative square
\[
\begin{array}{ccc}
W & \longrightarrow & CS^3 \\
\downarrow & & \downarrow \\
CS^3 & \longrightarrow & [CS^3/Z_k].
\end{array}
\]

There is a canonical isomorphism \(W \simeq (CS^3 \times Z_k)\) by the same argument as in Part (1) with \(S^1\) replaced by \(Z_k\). The groupoid associated to the atlas \(W\) is \(CS^3 \times (Z_k)^n\). Let \(A\) be the simplicial space which is \(CS^3\) in each degree. By an argument similar to that in Part (1), the simplicial space associated to \(CS^3\) by this atlas is then \((A \times E\mathbb{Z}_k)\). This has a natural action of \(Z_k\) on each factor. It is a principal \(Z_k\)-bundle over the simplicial space associated to \([CS^3/Z_k]\). (By the above diagram, \(W\) is a principal \(Z_k\) bundle over \(CS^3\), the result follows from the definition of the simplicial space associated to a groupoid.) Hence, the simplicial space associated to \([CS^3/Z_k]\) by this atlas is \((A/\mathbb{Z}_k \times B\mathbb{Z}_k)\). It is also a principal circle bundle over the simplicial space associated to \([CS^3/S^1]\).

By Part (1), the simplicial space associated to \([CS^3/S^1]\) is \((B \times BS^1)\). Therefore the simplicial circle bundle associated to the principal bundle of stacks \([CS^3/Z_k] \to [CS^3/S^1]\) is \((A/\mathbb{Z}_k \times B\mathbb{Z}_k) \to (B \times BS^1)\).

Thus, the T-dual would be \((B \times BS^1) \times S^1 \rightarrow (B \times BS^1)\) with \(H\)-flux. This is the principal bundle associated to \(((CS^3/S^1) \times S^1) \to [CS^3/S^1]\) with a gerbe on it.

\(\Box\)

It is interesting to observe that if a property holds for T-duality pairs in the sense of Ref. [2], it can be applied to the pair consisting of the simplicial bundle and \(H\)-flux. Sometimes, this has interesting consequences for spaces with a non-free \(S^1\)-action. We present two examples below using this property: In the first we calculate the T-dual of a semi-free space with countably many isolated fixed sets. In the second example done in the next Section, we develop a model for the Dyonic coordinate of Ref. [8] using the stack-theoretic approach.

Note the following property of T-dual principal \(S^1\)-bundles: Let \(p : E \to B\) be a principal \(S^1\)-bundle. Let \(h \in H^3(E, \mathbb{Z})\) be an \(H\)-flux. Let \(W_1, \ldots, W_k\) be open subsets of \(B\) such that \(W_1 \cup \ldots \cup W_k = B\). Let \(E_i = p^{-1}(W_i)\) be the induced open cover of \(E\). Let \(h_i\) be the \(H\)-flux restricted to \(E_i\). Let \(q : E^\# \to B\) be the T-dual of the principal bundle \(E \to B\) with \(H\)-flux \(h^\#\). Let \(E^\#_i = q^{-1}(W_i)\) and let \(h^\#_i = h^\#|_{E_i}\). Then we claim that \((E^\#_i, h^\#_i)\) is the T-dual of \((E_i, h_i)\). This follows from the existence of a classifying space for a pair and the properties of the T-duality map on it in Ref. [8].
[2]: If $R$ is the classifying space of pairs of Ref. [2], $T : R \to R$ the T-duality map and $\phi : B \to R$ the classifying map for the pair $(E, h)$, we have, on restriction to $W_i$, $(T \circ \phi)|_{W_i} = T \circ (\phi|_{W_i})$. However, by definition, $(T \circ \phi)|_{W_i}$ classifies $(E_i^#, h_i^#)$ while $\phi|_{W_i}$ classifies $(E_i, h_i) = (E|_{W_i}, h|_{E_i})$.

A similar result holds for semi-free spaces:

**Theorem 3.2.**

1. Let $E$ be a semi-free $S^1$-space with at most countably many isolated fixed sets of the $S^1$-action $F_1, \ldots, F_k, \ldots$. Suppose we are given disjoint neighbourhoods $U_i$ of $F_i$. Let $V_i = [U_i/S^1]$. Then these data determine the T-dual of $E$.

2. The T-dual of a semi-free space $E$ with at most countably isolated fixed sets is the principal bundle of stacks $[E/S^1] \times S^1 \to [E/S^1]$ with $H$-flux. There will be $H$-flux on the T-dual coming from an NS5-brane if the T-duals of any of the $U_i$ (see previous part) possess $H$-flux.

**Proof.**

1. Let $E$ be a semi-free $S^1$-space with at most countably many isolated fixed sets of the $S^1$-action $F_1, \ldots, F_k, \ldots$. Let $W = E/S^1$ and let $U_i, l = 1, \ldots, k, \ldots$ be open disjoint subsets of $E$, such that each $F_i \subseteq U_i$. Since the fixed sets are isolated, we may always assume that $U_i \cap U_k = \phi$ for every $l \neq k$. Then, by the classification theorem for spaces with finitely many orbit types (see proof of Cor. [3.2]), $P = (E - \bigcup_{i=1}^k U_i)$ is a principal circle bundle over $V = (W - \bigcup_{i=1}^k V_i)$ where $V_i = (U_i/S^1)$. Also, $E$ is determined by $P$ and the gluing data for the $U_i$. Since there is no $H$-flux on $E$, these data determine the T-dual of $E$.

2. Given the data of the previous part, for every $i$, suppose we are given atlases $V_i$ for $V_i$ and induced atlases $Q_i$ for $U_i$ in the sense of Ref. [3]. Let $SV$ be the simplicial space associated to $V$, and, for every $i$, $SV_i$ the simplicial space associated to $V_i$ by the above atlases. Similarly, let $SP$ be the simplicial space associated to $P$ and, for every $i$, $SU_i$ the simplicial space associated to $U_i$ by the above atlasses.

Consider the principal bundle of stacks $E \to W$. Consider the atlas $X = V \cup V_i$ for $W$. Then $V_i \cap V_j = \phi$ for all $i \neq j$. Also, $V_i \cap V$ need not be empty, but, $V_i \cap V \subseteq V_i$. Now $X \times X \simeq V \cup V_i \cup (V \cap V_i)$. Similarly, for the same reason, the $n$-fold fiber product $X \times \cdots \times X \simeq V \cup \cdots \cup (V \cap V_i)$. However, since $(V \cap V_i) \subseteq V_i$, $X \times X$ may always be written as $V \cup V_i$. Also, in the associated simplicial space, $W$ is always glued to each $V_i$ while $V_i$ glue to themselves. Then, the simplicial space associated to $X$ is $SV \cup SV_i \cup \cdots \cup SV_k \cup \cdots$ for some gluing maps $f, g_i$.

Consider the atlas $Y = P \cup Q_i$ for $E$. Here also, we have that $Q_i \cap Q_j = \phi$ for all $i \neq j$. Also, $Q \cap Q_i \subseteq Q_i$ for every $i$. This implies that $Y \times Y$ may always be written as $P \cup Q_i$ by the intersection property of $P$ and $Q_i$, described above. Similarly the $n$-fold fiber product $Y \times \cdots \times Y$ may always be written as $P \cup Q_i$ by the intersection property described above. Also, in the associated simplicial
space, $P$ is always glued to each $P_i$ while the $P_i$ glue to themselves. Then, the simplicial space associated to $Y$ is $SP \cup f' SQ_1 \cup g'_1 \cdots \cup g'_k SQ_k \cdots$ for some gluing maps $f', g'_i$.

Therefore we have the associated principal bundle $(SP \cup f' SQ_1 \cup g'_1 \cdots \cup g'_k SQ_k \cdots) \to (SV \cup f' SV_1 \cup g_1 \cdots \cup g_k SV_k \cdots)$ where $f', f, g$ are defined above.

By the remark before this theorem, the T-dual will be

$$E^\# = (SP \cup f SQ_1 \cup g_1 \cdots \cup g_k SQ_k \cdots) \times S^1)$$

as a principal bundle over

$$B^\# = (SV \cup f SV_1 \cup g_1 \cdots \cup g_k SV_k \cdots).$$

Note that this is the principal simplicial bundle associated to $((E/S^1) \times S^1) \to [E/S^1]$. There will be nonzero $H$-flux on $E^\#$ due to the fact that the original bundle $E$ had nontrivial topology. There will be additional $H$-flux on $E^\#$ due to NS5-branes if there is nonzero $H$-flux on the T-dual of any of the bundles $SQ_i$ when T-dualized by themselves: By the remark before this Theorem, the $H$-flux on the total space of the simplicial bundle associated to $E^\#$ must restrict to this $H$-flux on the subspace $SV_i \times S^1$.

This Theorem lets us determine the T-dual of any semi-free space with countably many isolated fixed sets. This covers most of the semi-free spaces that would occur in a physical context. In particular, we may now determine the T-dual of a space with at most countably many isolated $KK$-monopoles.

**Corollary 3.2.** Let $E$ be a semi-free $S^1$-space with at most countably many Kaluza-Klein monopoles $p_1, \ldots, p_k, \ldots$. Then, the T-dual is a trivial principal bundle glued to spaces of the form $([CS^3/S^1] \times S^1)$. There is $H$-flux present on the T-dual.

**Proof.** This is an elementary application of Thm. (3.2). In $E$, since the $KK$-monopoles which are the fixed points of the $S^1$-action are isolated, it is possible to enclose each one in an open set homeomorphic to a ball $CS^3$. Thus, as topological spaces, each $U_i$ is equivariantly homeomorphic to $CS^3$. The atlases $U_i$ and $V_i$ may be chosen as in Thm. (3.1). This construction is always possible by the classification theorem for spaces with finitely many orbit types since there are only two orbit types (fixed points and free orbits) and the fixed points are at most countably many and isolated (See Ref. [15] Chap. V Sec. (5)).

Given this, the T-dual may be found. Note that $SV_i$ are simplicial bundles associated to spaces of the form $[CS^3/S^1] \times S^1$ with $H$-flux. Thus, the T-dual of $E$ is a stack which is a trivial principal bundle glued to stacks of the form $([CS^3/S^1] \times S^1)$.

There is $H$-flux present on the T-dual. There will be nonzero $H$-flux on the T-dual bundle due to the fact that the original bundle had nontrivial topology. There will be additional $H$-flux on this bundle due to NS5-branes if there is nonzero $H$-flux on the T-dual of any of the $U_i$. This is because these will then contribute to a nonzero $H$-flux on the associated simplicial bundle: By Thm. (3.1), there is nonzero $H$-flux on
the T-dual of any of the $SU_i \subseteq E$, i.e. there is a $H$-flux on $(SV_i \times S^1) \subseteq E^\#$. By the remark before Thm. \((3.2)\), the $H$-flux on $(SV_i \times S^1)$ is the restriction of the $H$-flux on the T-dual to $SV_i \times S^1$. However, by the above, this restriction is nonzero hence the T-dual $H$-flux cannot be zero.

Note that unlike the $C^*$-algebraic case \((8)\) there is no source of $H$-flux on the T-dual. However, the T-dual does possess $H$-flux. We make this precise in the following:

**Corollary 3.3.**

1. The T-dual $E^\#$ of a semi-free $S^1$-space $E$ with at most countably many $KK$-monopoles is the principal bundle of stacks $[E/S^1] \times S^1 \to [E/S^1]$ with $H$-flux.
2. $E^\#$ is a topological stack which is not equivalent to a topological space if and only if the $S^1$-action on $E$ has fixed sets.
3. The natural map $\phi_{\text{mod}} : E^\# \to E^\#_{\text{mod}}$ is a homeomorphism iff the $S^1$-action on $E$ has no fixed sets.

**Proof.**

1. It follows from the proof of Thm. \((3.2)\) that the simplicial bundle associated to the T-dual stack is the trivial bundle over the base with $H$-flux. This is the simplicial bundle associated to the principal bundle of stacks $[E/S^1] \times S^1 \to [E/S^1]$ with $H$-flux.

2. First note the following: If $X$ is a stack equivalent to a topological space $X$, then, restricting the equivalence to a substack shows that every substack of $X$ is equivalent to a topological space. Suppose the action had no fixed sets, i.e. none of the $U_i$ was present in $E$, then, from the proof of Thm. \((3.2)\) $E$ would be a topological space and so would $E^\#$. Now suppose the $S^1$-action on $E$ had fixed sets. Then one of the $U_i$ would be present in $E$, then, from the same proof, the T-dual would contain substacks of the form $[U_i/S^1] \times S^1$. Here, by the classification theorem for spaces with finitely many orbit types (see proof of Cor. \((3.2)\) and by the proof of Thms. \((3.1,3.2)\)), each of these would be equivalent to stacks of the form $[CS^3/S^1] \times S^1$. These are not equivalent to topological spaces \(^4\). As a result, the T-dual could not be equivalent to a topological space.

3. Suppose the $S^1$-action on $E$ had fixed points and the map $\bar{\phi}_{\text{mod}} : E^\# \to E^\#_{\text{mod}}$ induced by $\bar{\phi}_{\text{mod}}$ was an equivalence of stacks. Then, by the proof of the previous part choosing suitable neighbourhoods of the fixed points will give an inclusion of stacks $[CS^3/S^1] \to E^\#$. The above would imply that the map $\bar{\phi}_{\text{mod}} : [CS^3/S^1] \to CS^2 \times S^1$ would be an equivalence of stacks. Since the stack cohomology groups of $CS^2 \times S^1$ are different from the stack cohomology groups of $[CS^3/S^1] \times S^1$, this is impossible.

Conversely, suppose the $S^1$-action on $E$ had no fixed points. Then, by the previous part of the theorem, the T-dual stack would be equivalent to a space and so $\bar{\phi}_{\text{mod}}$ would give an equivalence $\bar{\phi}_{\text{mod}} : E^\# \to E^\#_{\text{mod}}$.

\[^4\]The space $CS^2$ is the coarse moduli space of the stack $[CS^3/S^1]$ (see below) and, as spaces, $CS^3/S^1 \simeq CS^2$. The stack $CS^2$ has different stack cohomology groups to the stack $[CS^3/S^1]$, hence they cannot be equivalent.
Consider the T-dual of $CS^3$: The coarse moduli space of $[CS^3/S^1]$ is $CS^2$ (See Ref. [5] Example (4.13), $[CS^3/S^1]$ is the quotient stack of the transformation groupoid $((CS^3 \times S^1) \to CS^3)$). However, $H^3(CS^2 \times S^1, \mathbb{Z}) = 0$, so there can be no $H$-flux on the topological space $CS^2 \times S^1$. By the above, however, the stack $[CS^3/S^1] \times S^1$ possesses $H$-flux. This is because the simplicial bundle associated to this stack (see proof of Thm. (3.1)) is nontrivial, and so the stack cohomology groups of $[CS^3/S^1]$ are nontrivial.

Since the $H$-flux on the T-dual stack would vanish (see Cor. 3.2) if there were no fixed points of the $S^1$-action on the original space, presumably this $H$-flux is the flux generated by the T-dual NS5-brane. Note that this also happens for the T-dual of $[CS^3/Z_k]$ for $k > 1$ since the T-duals are the same as the case above only the $H$-flux changes.

This should also happen in the example in Cor. (3.2) above: The T-dual is a principal bundle $P$ glued to copies of $([CS^3/S^1] \times S^1)$. By the proof of Ref. ([2]), the T-dual is a topological stack. As a space, the coarse moduli space of the T-dual will be $P \times S^1$ glued to $CS^2 \times S^1$. Also, by Cor. (3.3) the T-dual coarse moduli space will be a trivial principal circle bundle. The $CS^2$ factor is contractible and the resulting space cannot have nonzero $H$-flux coming from an NS5-brane. The space will have $H$-flux only due to the $H$-flux on $P^\#$. However, the T-dual stack does have $H$-flux coming from this source.

Note that in all these T-duals (see also Cor. (3.3)) above, the reason the T-dual has a nontrivial $H$-flux is due to the fact that the stack cohomology groups of $[CS^3/S^1]$ are different from those of the coarse moduli space $CS^2$.

### 4 The Dyonic Coordinate

In String Theory backgrounds which contain $KK$-monopoles possess a dyonic coordinate. (See Ref. [10] for details. See also Ref. [11]). Roughly speaking, a large gauge transformation of the $B$-field on a $KK$-monopole background under T-duality corresponds to a rotation of the T-dual NS5-brane around its circle fiber. A model for this was constructed for $KK$-monopole backgrounds using $C^*$-algebraic methods in Ref. [8].

Large gauge transformations of a gerbe on a space $X$ are given by a class in $H^2(X, \mathbb{Z})$ (See Ref. [12]). We would like to understand the behaviour of these classes under Topological T-duality for semi-free spaces. As we have argued earlier, the $S^1$-spaces underlying $KK$-monopole spacetimes are semi-free spaces.

Using the results of Ref. [12], we show below that for these semi-free spaces $X$, an automorphism of a trivial gerbe on $X$ gives a class in $H^2(X^\#, \mathbb{Z})$ under Topological T-duality.

**Theorem 4.1.**

1. Consider the principal bundle of stacks $[pt/Z_k] \to [pt/S^1]$ for $k = 2$. Consider a trivial gerbe on this stack. Each cyclic subgroup of the group of automorphisms
of the gerbe on $[pt/\mathbb{Z}_k]$ gives rise to a cyclic subgroup of $H^2([pt/\mathbb{S}^1] \times \mathbb{S}^1, \mathbb{Z})$. For $k = 2$, this may be calculated explicitly.

2. Consider the principal bundle of stacks corresponding to a $KK$-monopole of charge $k$. Consider a trivial gerbe on the total space of the principal bundle. Each cyclic subgroup of automorphisms of the trivial gerbe on the $KK$-monopole of charge $k > 1$ gives rise to a cyclic subgroup of the (second) cohomology of the $T$-dual $H^2(\mathbb{C}S^3/\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{Z})$.

Proof.

1. Consider a cyclic subgroup of the group of automorphisms of the trivial gerbe on $[pt/\mathbb{Z}_k]$. It is enough to prove the result for the generator of this subgroup. An automorphism of the trivial gerbe on $[pt/\mathbb{Z}_k]$ gives rise to a class in $H^2([pt/\mathbb{Z}_k], \mathbb{Z})$. Consider the proof of Part (2) of Thm. (3.1). The simplicial bundle associated to the principal bundle of stacks $[pt/\mathbb{Z}_k] \to [pt/\mathbb{S}^1] = \mathbb{Z}$ is $p : \mathbb{Z}_k \to \mathbb{S}^1$. Since we have a class in $H^2([pt/\mathbb{Z}_k], \mathbb{Z})$, we obtain a cohomology class on the simplicial space associated to this stack (See Ref. [7], the proof of Prop. (4.7)). In turn this gives a cohomology class on its geometric realization $\mathbb{Z}_k/H^2(\mathbb{Z}_k, \mathbb{Z})$.

By the argument in Ref. [12], Thm. (6.3), this class gives rise to a natural class in the second cohomology group of the $T$-dual bundle $q : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$. For all natural numbers $k > 1$, $H^2(\mathbb{Z}_k, \mathbb{Z}) \approx k/\mathbb{Z}$. Under $T$-duality an element of $H^2(\mathbb{Z}_k, \mathbb{Z})$ gives a class of the form $kq^*(x) \approx k(a \times 1) \in H^2(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{Z})$ for some unknown integer $k$ (where $a$ is the generator of $H^2(\mathbb{S}^1, \mathbb{Z})$ (See Ref. [12], Thm. (6.3)). Thus, an automorphism of the gerbe on $[pt/\mathbb{Z}_k]$ with this characteristic class gives rise to a cohomology class on the $T$-dual stack $[pt/\mathbb{S}^1] \times \mathbb{S}^1$.

2. This part of the proof is very similar to the previous part. Consider a cyclic subgroup of the group of automorphisms of the trivial gerbe on $[CS^3/\mathbb{Z}_k]$. It is enough to prove the result for the generator of this subgroup. The proof is similar to the proof of the previous part, with the principal bundle changed. An automorphism of the trivial gerbe on $[CS^3/\mathbb{Z}_k]$ gives rise to a class in $H^2([CS^3/\mathbb{Z}_k], \mathbb{Z})$. Consider the proof of Part (3) of Thm. (3.1). The simplicial bundle associated to the principal bundle of stacks $[CS^3/\mathbb{Z}_k] \to [CS^3/\mathbb{S}^1]$ is $(A \times \mathbb{Z}_k) \to (A \times B) (B \times \mathbb{S}^1)$ (where $A$ and $B$ are defined in Thm. (3.1)). Let $P_{k} = (A \times \mathbb{Z}_k)$ and $W_{k} = (B \times \mathbb{S}^1)$.

Since we have a class in $H^2([CS^3/\mathbb{Z}_k], \mathbb{Z})$, this gives a cohomology class on the simplicial space associated to this stack (See Ref. [7], the proof of Prop. (4.7)) and this, in turn, gives a cohomology class on its geometric realization, that is, a class in $H^2(P_{k}, \mathbb{Z})$. The $T$-dual bundle is $W_{k} \times \mathbb{S}^1 \to \mathbb{S}^1$ with $H$-flux, and, by Ref. [12], a class in $H^2(P_{k}, \mathbb{Z})$ under Topological $T$-duality naturally gives rise to a class in $H^2(W_{k} \times \mathbb{S}^1, \mathbb{Z})$.

Thus a class in $H^2(X, \mathbb{Z})$ naturally gives rise to a class in $H^2(X^{\#}, \mathbb{Z})$. For a $KK$-monopole background, a large gauge transformation of the $B$-field gives rise to a class in $H^2(X, \mathbb{Z})$. By the above this induces a class in $H^2(X^{\#}, \mathbb{Z})$. For the analogy with the dyonic coordinate to be complete, the induced class in $H^2(X^{\#}, \mathbb{Z})$ should be viewed as
an automorphism of the T-dual semi-free space $X^\#_k$ which rotates each fiber through $2\pi$. However, it is not clear how to prove this in the stack picture.

A similar construction was made in the $C^*$-algebraic picture of topological T-duality in Ref. [8] where such a rotation did correspond to a nontrivial spectrum-fixing automorphism of the T-dual $C^*$-algebra: The T-dual automorphism obtained there was of this type.

5 The Gysin Sequence

Consider a principal circle bundle of stacks $p : \mathcal{E} \to \mathcal{F}$. From Ref. [13], a Gysin sequence may be constructed for this principal bundle in homology. Can a cohomology Gysin sequence be so constructed? We construct this Gysin sequence below. We also show that for stacks $\mathcal{E}$ which are the stacks associated to a semi-free $S^1$-space $E$ a ‘T-dual’ stack may be constructed using this Gysin sequence. We show that that this T-dual stack is the stack associated to the T-dual of $E$ which would be obtained in the formalism of Mathai and Wu (See Ref. [9]). We show at the end of this section that the stack T-dual defined in this section agrees with the T-dual of Bunke et al. in the examples of Thm. (3.1).

Lemma 5.1. Let $q : \mathcal{E} \to \mathcal{Y}$ be a principal $S^1$-bundle of stacks. There is a cohomology Gysin sequence for this bundle if $q$ is adequate in the sense of Behrend et al. [17] Def. (7.4).

Proof. Suppose $q$ is adequate, then the cohomology transfer map $T_{S^1}$ (See Definition (8.2) of Ref. [13]) is well defined since the product $x \cdot u^*(\theta_{\frac{1}{2}})$ in that definition is defined (See Ref. [17] paragraph after Ex. (7.5)).

The proof of Prop. (8.4) of Ref. [13] may now be followed with $H_*$ replaced by $H^*$. It is clear that the full proof goes through. Let $Z \to [S^1 \backslash \mathcal{E}]$ be a classifying space for $[S^1 \backslash \mathcal{E}]$ and $Y \to \mathcal{E}$ be the classifying space for $\mathcal{E}$ obtained by pullback along $q$. Let $c$ be the Euler class of disk bundle associated to the principal bundle $Y \to Z$, then we obtain the following Gysin Sequence:

$$\ldots \to H^{i-1}_{S^1}((\mathcal{E})) \xrightarrow{q^*} H^{i-1}(\mathcal{E}) \xrightarrow{T_{S^1}} H^{i-2}_{S^1}(\mathcal{E}) \xrightarrow{\cup c} H^i_{S^1}(\mathcal{E}) \xrightarrow{q^*} H^i(\mathcal{E}) \to \ldots$$

(2)

from the cohomology Gysin sequence associated to $Y \to Z$ under the identifications $H^i(Z) \simeq H^i([S^1 \backslash \mathcal{E}]) \simeq H^i_{S^1}(\mathcal{E})$, and $H^i(Y) \simeq H^i(\mathcal{E})$ exactly as in Ref. [13], Prop. (8.4).

Recall from Sec. (3) above that the semi-free spaces we consider are all total spaces of $KK$-monopoles. In particular, they are all oriented orbifolds. By Prop. (8.35) of Ref. [17], they are all strongly oriented in the sense of Ref. [17] Def. (8.21).

Lemma 5.2. Let $\mathcal{E}$ be a stack associated to a semi-free space. Let $\mathcal{Y} = [S^1 \backslash \mathcal{E}]$ and let $p : \mathcal{E} \to \mathcal{Y}$ be the quotient map. Let $V$ be the associated vector bundle to the principal bundle $p : \mathcal{E} \to \mathcal{Y}$. Suppose $V$ is metrizable. Further, suppose $\mathcal{Y}$ is strongly oriented in the sense of Ref. [17], Def. (8.21). Then $p$ is strongly oriented in the sense of Ref. [17], Def. (8.21). Also, $p$ is adequate.
Proof. Firstly, $E$ is strongly oriented due to Prop. (8.35) of Ref. [17]. Secondly, $Y$ is strongly oriented by assumption. Thirdly, we will argue that $p$ is strongly proper and normally nonsingular. Hence, $p : E \to Y$ has a strong orientation class by Prop. (8.32) of Ref. [17].

By Lemma. (3.1) above, $p : E \to Y$ is a principal $S^1$-bundle. By Ref. [13], it is also representable (By Prop. (4.7) of Ref. [13]). Let $q : V \to Y$ be the associated vector bundle. $V$ is metrizable by assumption. Let $s : E \to V$ be the embedding of $E$ as the unit sphere bundle in $V$. Then, the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{id} & \mathcal{V} \\
\uparrow s & & \downarrow q \\
E & \xrightarrow{p} & Y
\end{array}
$$

This is the required normally nonsingular diagram for $p$.

It is clear that $p$ is bounded proper from Def. (6.1) of Ref. [17]. It remains to prove that $p$ is strongly proper. Let $w : C \to E$ be an orientable metrizable vector bundle on $E$. Choose classifying spaces $Y \to Y$ and $E \to E$, so we obtain a principal bundle $E \to Y$. Pulling $V$ back to $Y$ we obtain a bundle $V \times Y \to Y$. The pullback of this bundle along the map $E \to Y$ is trivial since the bundle has the same Euler class as the bundle of stacks $V \to Y$ (this follows from the definition of the Euler class, see Behrend et al. Ref. [17], Ex. (8.26)). It follows that $p^*(V)$ is a trivial bundle. Hence, taking charts, there is an integer $n > 0$ such that $C$ is a direct summand of $(p^*(V))^n \simeq p^*(V^n)$. Now, $V$ is the vector bundle associated to $E$, and, since $p$ is orientable, so is $V$. Further, $V$ is metrizable by assumption and hence, so is $V^n$. Hence, $p$ is strongly proper. By the above $p$ is normally nonsingular. Hence, $p$ has a strong orientation class by Prop. (8.32) of Ref. [17]. By Ex. (7.5) (1) of Ref. [17], $p$ is adequate.

For all the examples of $KK$-monopoles in Thm. (3.1) above, the total space is the stack $[CS^3/Z_k]$ associated to an oriented orbifold and hence strongly oriented by Prop. (8.35) of Ref. [17]. From Thm. (3.1) above, the quotient stack by the $S^1$-action is always $[CS^3/S^1]$ and any vector bundle $V$ over $[CS^3/S^1]$ is metrizable by Ex. (3.3) of Ref. [17]. Also $[CS^3/S^1]$ is strongly oriented by Prop. (8.33) and Def. (8.21) of Ref. [17]. Hence, in all the examples of $KK$-monopoles calculated above, the bundle maps $p_k : [CS^3/Z_k] \to [CS^3/S^1]$ and $p : CS^3 \to [CS^3/S^1]$ are adequate. Thus, we may use the Gysin sequence argument above to obtain the T-dual.

Let $E$ be a non-free $S^1$-space. Let $E$ be the underlying stack. Consider the stack $[E/S^1]$. This has a natural presentation as the transformation groupoid $E = [(E \times S^1) \rightrightarrows E]$. This stack has a natural classifying space $BE$ (the Haefliger-Milnor Classifying Space) associated to this groupoid which is given by the Borel construction $E \times_{S^1} ES^1$. The principal bundle of stacks $p : E \to [E/S^1]$ gives a principal bundle of

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5See Ref. [19], Sec. (4.3), also the proof of Thm. (6.3) for the definition and properties of classifying spaces.
spaces \( EE \simeq (E \times EG) \to B\mathbb{E} \simeq (E \times S^1 E S^1) \) by pullback and a 2-cartesian square:  

\[
\begin{array}{ccc}
Y = (E \times ES^1) & \xrightarrow{f} & E \\
\downarrow & & \downarrow \\
Z = (E \times S^1 ES^1) & \xrightarrow{\phi} & [E/S^1]
\end{array}
\]

where the space in each row is a classifying space. Also, the map \( \phi \) is natural (See Ref. \[19\], before Sec. (4.2)).

In Ref. \[9\] Mathai and Wu obtain the T-dual of a non-free \( S^1 \)-space \( E \) using the Borel construction: To the space \( E \) they associate the same principal circle bundle of spaces \( E \times ES^1 \to E \times S^1 ES^1 \) as described above.

Given a principal bundle of stacks \( p : \mathcal{E} \to \mathcal{Y} \), for which a Gysin sequence exists, one could obtain a T-dual principal bundle of stacks by the following procedure: To \( p \) we associate the principal bundle of classifying spaces above \( Y \to Z \) (See Ref. \[13\], proof of Thm. (8.4)). Note that even if \( \mathcal{E} \) was the stack associated to a space with a non-free \( S^1 \)-action, the principal bundle of classifying spaces would be a principal circle bundle.

Given a \( S^1 \)-gerbe on \( \mathcal{E} \), we place an \( H \)-flux with characteristic class equal to the characteristic class of this gerbe (See Ref. \[7\], Prop. (5.8)) on the total space of this principal bundle of classifying spaces. We can then calculate a Topological T-dual principal circle bundle \( Y^\# \to Z \) for the associated principal bundle of classifying spaces (possibly with \( H \)-flux) using the usual Gysin sequence argument (See Ref. \[13\]). Note that this is similar to the argument of Mathai and Wu in Ref. \[9\] for the T-dual of a non-free circle action and \( Y^\# \) is the T-dual space that they obtain.

If a principal bundle of stacks \( p^\# : \mathcal{E}^\# \to \mathcal{Y} \) existed which had the spaces \( Y^\#, Z \) in this bundle as classifying spaces, then we would say that \( p^\# \) (possibly with a gerbe on it corresponding to the the T-dual \( H \)-flux above, if any) was the Topological T-dual stack.

Note that the result of the procedure above is the same as calculating the T-dual of \( p \) using the argument for principal circle bundles with the ordinary Gysin sequence replaced by the stack Gysin sequence. The only problem is that if the stack Gysin sequence is used directly, unlike the case of a principal circle bundle, there is no guarantee that \( p^\# \) exists.

It turns out that such a dual bundle of stacks always exists for semi-free spaces because the above T-dual is connected to the T-dual of Ref. \[9\]:

**Theorem 5.1.** Let \( E \) be a semi-free space with a \( S^1 \)-action for which the results of Ref. \[9\] hold. Suppose \( p : E \to [E/S^1] \) is adequate in the sense of Ref. \[17\]. Then, the T-dual of \( E \) using the stack Gysin sequence above exists and is \( E^\# \) where \( E^\# \) is the T-dual of \( E \) obtained using the formalism of Ref. \[9\]. There will be \( H \)-flux on the T-dual if the T-dual of Ref. \[9\] had \( H \)-flux.

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6See Ref. \[19\] Sec. (4.3) and cartesian square after Lemma (4.1).

7See Prop. (6.1) of Ref. \[19\].
Proof. Consider a semi-free space $E$. Let $p : \mathcal{E} \to [E/S^1]$ be the associated principal bundle of stacks. By Ref. [9], after Prop. (1), we have a principal circle bundle $\hat{E} \to E$ and the T-dual is $\hat{p} : E^\# \simeq E/S^1 \to \hat{E}/S^1$.

The map $\hat{E} \to E$ can be replaced by the associated map of stacks $\hat{\mathcal{E}} \to \mathcal{E}$. The $S^1$-action on the stacks in this map gives rise to a quotient map of stacks $\hat{p} : [\hat{E}/S^1] \to [E/S^1]$. We claim that $\hat{p}$ is a principal bundle of stacks and that it is the principal bundle of stacks obtained from $p$ using the Gysin sequence argument described above. Also, Eq. (1) of Ref. [9] is a principal bundle of classifying spaces associated to this principal bundle of stacks.

By the proof of Thm. (6.3) of Ref. [19] we have classifying spaces $E \times_{S^1} ES^1 \to [E/S^1]$ and $\hat{\mathcal{E}} \times_{S^1} ES^1 \to [\hat{E}/S^1]$ which are atlases for these stacks. Hence, pulling back the map $\hat{p}$ along the atlas $E \times_{S^1} ES^1 \to [E/S^1]$ gives the principal circle bundle $\hat{E} \times_{S^1} ES^1 \to E \times_{S^1} ES^1$ on the atlas. By definition, (see Def. (2.11) of Ref. [7]) this implies that $[\hat{E}/S^1] \to [E/S^1]$ is a principal bundle of stacks. It is clear that Eq. (1) of Ref. [9] is a principal bundle of classifying spaces associated to this principal bundle of stacks.

It remains to prove that this principal bundle of stacks is the one obtained from $p$ using the Gysin sequence argument described above.

From Ref. [9] second paragraph after Prop. (1), the bundle $\hat{E} \to E$ defines an equivariant characteristic class $c^3_{S^1}(\hat{E}) \in H^2_{S^1}(E, \mathbb{Z})$ which is equal to $p_*([H])$. By Ref. [17] before Prop. (2.5), $H^2_{S^1}(E, \mathbb{Z}) \simeq H^2([\hat{E}/S^1])$, the stack cohomology of the moduli space. Hence, we obtain a class in $H^2([\hat{E}/S^1])$ from the principal bundle $\hat{E} \to E$ above. We claim that this class is the class of the T-dual bundle $\hat{p} : [\hat{E}/S^1] \to [E/S^1]$ : By definition and the discussion above, the characteristic class of $\hat{p}$ is the characteristic class of the bundle of classifying spaces $\hat{E} \times_{S^1} ES^1 \to E \times_{S^1} ES^1$. By the discussion above, this can only be the class $c^3_{S^1} \in H^2_{S^1}(E, \mathbb{Z}) \simeq H^2(E \times_{S^1} ES^1, \mathbb{Z})$, the equivariant characteristic class of the bundle $\hat{E} \to E$.

By item (1) before Thm. (1) of Ref. [9], the T-dual $H$-flux has the property $\hat{p_*}([H]) = e_{S^1} \in H^2_{S^1}(E, \mathbb{Z})$. By the above $H^2_{S^1}(E, \mathbb{Z}) \simeq H^2([E/S^1])$ and the class $e_{S^1}$ by definition maps to the characteristic class of the bundle of stacks $p : \mathcal{E} \to [E/S^1]$. (By Ref. [17], see Sec. (2.6), paragraph before Eq. (2.1)) the singular cohomology of a stack is the singular cohomology of its classifying space. Also, the characteristic class of a bundle of stacks $\mathcal{Y} \to \mathcal{X}$ is the characteristic class of its associated bundle of classifying spaces $Y \to X$ in $H^2(X, \mathbb{Z})$. For the bundle of stacks $p : \mathcal{E} \to [E/S^1]$ we obtain a bundle of classifying spaces $E \times ES^1 \to E \times_{S^1} ES^1$. The class $e_{S^1}$ defined above is, by definition (see Ref. [9] before Prop. (1)) the class of this bundle. This gives the result.) Thus we obtain that $p_*([H])$ is the class of the T-dual bundle $\hat{p}$ and $\hat{p_*}([H])$ is the class of the bundle $p$. This implies that the T-dual bundle and $H$-flux are identical to those obtained from the Gysin sequence argument above.

\[\square\]

It is not clear what the relation is between the stack T-dual obtained using classifying spaces in this section and that obtained from the formalism of Bunke et al. (Ref. [3]). Note first that the simplicial space associated to a groupoid associated to a stack is a classifying space (See Ref. [19], Sec. (4.2)). Roughly, to a stack the formalism of
Bunke et al. associates a principal bundle of classifying spaces each space of which is the associated simplicial space of a stack in that bundle. The T-dual is obtained by T-dualizing this bundle.

In addition to the simplicial space, it was argued above that one could also associate to a a stack another classifying space the Haefliger-Milnor classifying space. For a principal bundle of stacks, one could pick a groupoid presentation of each stack and pass to the Haefliger-Milnor classifying space of each groupoid. One could define a T-dual stack by T-dualizing this bundle. However, in general this classifying space depends on the groupoid presentation. For the quotient stack of a semi-free space, there is a natural choice of the associated groupoid and hence the associated classifying space.

The two stack T-duals are closely related, and, in most interesting cases, identical. This is because the stack T-dual of Ref. [2] is connected to the simplicial space associated to a stack while the stack T-dual defined in this section is connected to the Haefliger-Milnor classifying space also associated to that stack (See Ref. [19], Sec. (4.2)).

We show that for the T-duals of Thm. (3.1), the two formalisms give the same result:

**Corollary 5.1.** For every \( k > 1 \), \( k \) a natural number, the T-dual of the principal bundle of stacks \( p_k : [CS^3/Z_k] \to [CS^3/S^1] \) using classifying spaces is the principal bundle of stacks \( q : ([CS^3/S^1] \times S^1) \to [CS^3/S^1] \) with \( H \)-flux.

**Proof.** The Borel construction \( CS^3 \times S^1 \cdot E S^1 \) is a classifying space for the stack \([CS^3/S^1]\). Also, the Borel construction gives \((CS^3/S^1) \times BZ_k\) as the classifying space for the stack \([CS^3/Z_k]\). By definition, the principal bundle of classifying spaces associated to the principal bundle of stacks \( p_k \) above is the principal circle bundle \((CS^3/S^1) \times BZ_k \to (CS^3/S^1) \times BS^1\). Using the Gysin Sequence for this principal bundle of spaces, the T-dual would be the principal circle bundle \( q : ((CS^3/S^1) \times BS^1) \times S^1 \to ((CS^3/S^1) \times BS^1) \) with \( H \)-flux. Now, by the above, \((CS^3/S^1) \times BS^1\) is the classifying space associated to the stack \([CS^3/S^1]\). From the proof of Thm. (5.1) above, the principal circle bundle \( q \) above is the bundle of classifying spaces associated to the principal bundle of stacks \( q : ([CS^3/S^1] \times S^1) \to [CS^3/S^1] \). The fact that there is a \( H \)-flux present on the bundle of classifying spaces implies that there is a gerbe on this bundle of stacks. \( \square \)

By the discussion in the paragraph after Lemma (5.2) above, each of the maps \( p_k \) above is adequate. Hence, one may use the Gysin sequence to calculate the T-dual. It is clear by inspecting the argument that the T-dual will be exactly the same as that obtained in Cor. (5.1) above.

### 6 Final Remarks

In this paper we have studied the Topological T-dual of spaces containing \( KK \)-monopoles. In Secs. (2,3) of the paper above, we have explicitly calculated the T-dual of several spaces with \( KK \)-monopoles. In Sec. (4) we have attempted to model the ‘Dyonic Coordinate’ associated with \( KK \)-monopoles within the stack theory formalism. In Ref. [8]
we had obtained a model for the same phenomenon using the $C^*$-algebraic formalism of Topological T-duality. It is interesting that the same phenomenon appears in two completely independent approaches to Topological T-duality.

We now make a few remarks concerning the relation between the three formalisms of Topological T-duality for semi-free spaces with particular reference to the above calculations involving spaces containing $KK$-monopoles: The formalism of Bunke et al. \([2, 3, 4]\) obtains the topological T-dual of the stack associated to a $KK$-monopole by passing to the associated simplicial bundle and taking the Topological T-dual of the simplicial bundle. Obviously, the Topological T-dual of the associated simplicial bundle agrees with the $C^*$-algebraic T-dual of that bundle. If the stack which one was T-dualizing was the stack associated to the total space of a principal circle bundle, the formalism of Bunke et al. would give the same answer as the $C^*$-algebraic formalism. Thus, the formalism of Bunke ‘regularizes’ the neighbourhood of the fixed point by passing to the associated simplicial bundle.

Also, as has been argued above, the $C^*$-algebraic T-dual lifts the $S^1$-action on the space to a $\mathbb{R}$-action on the $C^*$-algebra. The formalism of Bunke et al. does not lift the $S^1$-action to a $\mathbb{R}$-action. This causes a difference in T-duals when fixed points are encountered. The formalism of Mathai-Wu \([9]\) also does not lift the $S^1$-action to a $\mathbb{R}$-action.

As has been argued in section Sec. \((5)\) before Cor. \((5.1)\), the Topological T-dual obtained from the Gysin sequence formalism and that obtained from the formalism of Bunke et al. should agree for most spaces. Since the T-dual obtained from the formalism of Mathai-Wu agrees with the T-dual obtained from the Gysin sequence formalism by Thm. \((5.1)\), the T-dual obtained by Bunke et al. should agree with the T-dual obtained by Mathai-Wu in most cases. It should differ from the $C^*$-algebraic T-dual for the reasons discussed above.

It is interesting to note that the calculation of Topological T-duals for $U(1)$-gerbes on a principal $T^n$-bundle over an arbitrary topological groupoid using the theory of crossed products of groupoid $C^*$-algebras has also been done by Daenzer in Ref. \([20]\). The formalism can T-dualize non-free group actions. It would be interesting to compare the results of that formalism with the results of this paper.

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