The Voisin Map via Families of Extensions

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Abstract

We prove that given a cubic fourfold $Y$ not containing any plane, the Voisin map $v : F(Y) \times F(Y) \rightarrow Z(Y)$ constructed in [Voi16] where $F(Y)$ is the variety of lines and $Z(Y)$ is the Lehn-Lehn-Sorger-van Straten eightfold [LLSVS17], can be resolved by blowing up the incident locus $\Gamma \subset F(Y) \times F(Y)$ endowed with the reduced scheme structure. Moreover, if $Y$ is very general, then this blowup is a relative Quot scheme over $Z(Y)$ parametrizing quotients in a heart of a Kuznetsov component of $Y$.

1 Introduction

The derived category $D^b(Y)$ of a smooth cubic fourfold $Y$ decomposes

$$D^b(Y) = \langle Ku(Y), \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$$

into a relatively simple part (an exceptional collection) and a highly nontrivial subcategory $Ku(Y)$, the Kuznetsov component, which is believed to encode the geometry of $Y$ [Kuz10]. One example of this principle is the work of [LLMS17] and [LPZ18], which shows that the variety of lines $F(Y)$ and the LLSvS eightfold $Z(Y)$ constructed in [LLSVS17] (with the assumption that $Y$ does not contain any plane) both can be naturally interpreted as moduli of stable objects in $Ku(Y)$, with respect to a Bridgeland stability condition on $Ku(Y)$ constructed in [BLMS17].

In [Voi16], C. Voisin established, in particular, a degree six rational map $v : F(Y) \times F(Y) \rightarrow Z(Y)$ using geometry of the cubic fourfold. In the light of [LLMS17] and [LPZ18], one can reinterpret the Voisin map via families of extensions in $Ku(Y)$. Indeed, there are two moduli spaces $M_\sigma(\lambda_1)$ and $M_\sigma(\lambda_1 + \lambda_2)$ that are both isomorphic to $F(Y)$, and $M_\sigma(2\lambda_1 + \lambda_2)$ is isomorphic to $Z(Y)$, where $\lambda_i$ are numerical classes (see section 2.1) and $\sigma$ is a Bridgeland stability condition on $Ku(Y)$ (see [BLMS17] and section 2.2), such that for a general point $(F, P) \in M_\sigma(\lambda_1) \times M_\sigma(\lambda_1 + \lambda_2)$, one has $Ext^1(P, F) \cong \mathbb{C}$. The unique nontrivial extension gives a stable object of class $2\lambda_1 + \lambda_2$ and therefore defines

$$v : M_\sigma(\lambda_1) \times M_\sigma(\lambda_1 + \lambda_2) \rightarrow M_\sigma(2\lambda_1 + \lambda_2),$$

which coincides with Voisin’s construction.

The purpose of this note is to study the Voisin map by addressing how the family of extensions spread along the indeterminacy locus.

Let $\Gamma$ be the incident locus $\{(L_1, L_2) \in F(Y) \times F(Y) : L_1 \cap L_2 \neq \emptyset\}$. As we will see, if $(F, P) \in \Gamma$, identifying $F(Y) \times F(Y)$ with $M_\sigma(\lambda_1) \times M_\sigma(\lambda_1 + \lambda_2)$, then $ext^1(P, F) > 1$. Thus, the map $v$ is not defined on $\Gamma$ (we notice that this agrees with a result of [Mur17]).

Denoted by $F$ and $P$ the pullbacks of two universal families on $M_\sigma(\lambda_1) \times Y$ and $M_\sigma(\lambda_1 + \lambda_2) \times Y$, respectively, to $M_\sigma(\lambda_1) \times M_\sigma(\lambda_1 + \lambda_2) \times Y$. Let $A$ be the heart of the t-structure...
associated to \( \sigma \). By considering a notion of families of extensions of \( \mathcal{P} \) by \( \mathcal{F} \) as in \cite{Lan83} (see definition 3.1), we prove:

**Theorem 1.1** (c.f. theorem 4.6). Let \( b : Bl_\Gamma(M_\sigma(\lambda_1) \times M_\sigma(\lambda_1 + \lambda_2)) \rightarrow M_\sigma(\lambda_1) \times M_\sigma(\lambda_1 + \lambda_2) \) be the blowup along the reduced scheme structure of \( \Gamma \).

(a). Suppose that \( Y \) does not contain any plane. The Voisin map can be resolved by \( b : Bl_\Gamma(M_\sigma(\lambda_1) \times M_\sigma(\lambda_1 + \lambda_2)) \rightarrow M_\sigma(2\lambda_1 + \lambda_2) \).

(b). Moreover, if \( Y \) is very general, then the blowup above is a relative Quot scheme over \( M_\sigma(2\lambda_1 + \lambda_2) \) parametrizing quotients in the heart \( \mathcal{A} \) and of class \( \lambda_1 + \lambda_2 \).

The main idea is to consider a functor of families of non-splitting extensions of \( \mathcal{P} \) by \( \mathcal{F} \). Let \( f : M_1 \times M_2 \times Y \rightarrow M_1 \times M_2 \) be the projection, define \( RHom_f := Rf_*RHom \), and \( Ext^i_f := H^i(RHom_f) \). In \cite{Lan83}, it has been shown that the set of families of extensions (definition 3.1) over a reduced scheme \( g : T \rightarrow S \) is the same as \( H^0(T, g^*Ext^1_f(\mathcal{P}, \mathcal{F})) \), provided that \( Ext^1_f(\mathcal{P}, \mathcal{F}) \) satisfies a condition called "commute with base change" (definition 3.5). However, such a condition does not hold in our case. This can be fixed; indeed the right functor to consider should be

\[
g : T \rightarrow Hom(O_T, Lg^*RHom_f(\mathcal{P}, \mathcal{F}))[1],
\]

which renders the condition "commute with base change" automatic. Meanwhile, this functor is still a sheaf on \( S \) in our case, and consequently we have a bijection between \( Hom(O_T, Lg^*RHom_f(\mathcal{P}, \mathcal{F}))[1] \) and the set of families of extensions of \( \mathcal{P}_T \) by \( \mathcal{F}_T \) (lemma 3.7).

Using this observation, we can adapt the functor of families of non-splitting extensions to our case (lemma 3.10), and then (in section 4) show that it is represented by the blowup in theorem 1.1 and obtain a morphism to \( Z(Y) \).

Once we prove part (a), the missing ingredient for part (b) is a universal relative quotient on the blowup. To obtain that, we take a global family of extensions on the blowup which splits along the exceptional divisor, and perform an elementary modification to get a nowhere-split global family. See section 4.

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2 Review on Kuznetsov components, stability conditions and constant families of t-structures

In this section, we first recall some results about Kuznetsov components of cubic fourfolds, Bridgeland stability conditions and moduli of stable objects that are crucial for us.
in section 2.3 we put together [Kuz11] and [AP06], [Pol07] to clarify a slight technical issue, namely, $\mathcal{F}, \mathcal{P}$ in our case are not families of sheaves but complexes, and we would like to consider their extensions lie in a certain (sheaf of) hearts of t-structures.

2.1 Kuznetsov components of cubic fourfolds. Let $Y$ be a smooth cubic fourfold, and $D^b(Y)$ be the bounded derived category of coherent sheaves on $Y$. There is an exceptional collection $\langle \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$ of $D^b(Y)$ (see e.g. [Kuz10]).

**Definition 2.1.** The Kuznetsov component $Ku(Y)$ of $Y$ is defined as

$$Ku(Y) := \{ E \in D^b(X) : R^i\text{Hom}(\mathcal{O}_Y, E) = 0, \ i = 0, 1, 2 \}.$$ 

It is an admissible triangulated subcategory of $D^b(Y)$, i.e., the inclusion $i : Ku(Y) \hookrightarrow D^b(Y)$ admits a left (and right) adjoint $i^* : D^b(Y) \rightarrow Ku(Y)$ (and $i^!$). The Serre functor on $Ku(Y)$ is $(-)[2]$ [Kuz17].

Let $K_{top}(Y)$ be the topological K-theory of $Y$, $\chi(\_ , \_)$ be the Euler pairing. We recall the definition of Mukai lattices of Kuznetsov components by Addington and Thomas [ATT14], see also [DLMS17].

**Definition 2.2.** The topological Mukai lattice of $Ku(Y)$ is $K_{top}(Ku(Y)) := \{ v \in K_{top}(Y) : \chi([\mathcal{O}_Y(i)], v) = 0, i = 0, 1, 2 \}$ equipped with the Mukai pairing $(-, -) := -\chi(\_ , \_).$ The (numerical) Mukai lattice $K_{num}(Ku(Y))$ is the image of the map $Ku(Y) \rightarrow K_{top}(Ku(Y)).$

For an object $E \in Ku(Y)$, refer to its class $[E] \in K_{num}(Ku(Y))$ as its Mukai vector.

For any cubic fourfold $Y$, the Mukai lattice $K_{num}(Ku(Y))$ always contains two special classes:

$$\lambda_1 := [i^*(\mathcal{O}_L(H))] \quad \text{and} \quad \lambda_2 := [i^*(\mathcal{O}_L(2H))],$$

where $L$ is a line in $Y$. They generate a sublattice of $K_{num}(Ku(Y))$ that is isomorphic to

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$ 

As shown in [ATT14] proposition 2.3], if $Y$ is generic in the moduli of cubic fourfold, then $K_{num}(Ku(Y)) = A_2$. We call such a cubic fourfold very general.

2.2 Stability conditions and Moduli

**Definition 2.3.** [Bri07] A Bridgeland stability condition on $Ku(Y)$ is a pair $(Z, A)$, where $Z : K_{num}(Ku(Y)) \rightarrow \mathbb{C}$ is a group homomorphism and $A$ is a heart of a bounded t-structure of $Ku(Y)$ satisfying the following conditions:

1. For any nonzero object $E \in A$, $Z(E) := Z([E]) = r(E)e^{i\pi \phi(E)}$, then $r(E) > 0$ and $\phi(E) \in (0, 1]$. ($\phi(E)$ is called the phase of $E$, and it defines a notion of semistability: an object $E \in A$ is semistable if for any nonzero subobject $F \hookrightarrow E$, $\phi(F) \leq \phi(E)$.)

2. For any object $E \in A$, $E$ has a Harder-Narasimhan filtration

$$0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_{n-1} \hookrightarrow E_n = E,$$

such that quotient objects $A_i \cong E_i/E_{i-1}$ are semistable with decreasing phases, i.e. $\phi(A_i) > \phi(A_{i+1})$ for $i = 1, 2, \ldots n$. 


(3) For a given norm $\| \cdot \|$ on $K_{\num}(Ku(Y))$, there exists a constant real number $C > 0$ such that
$$|Z(E)| < C\|E\|$$
for every semistable object in $E \in \mathcal{A}$.

**Definition 2.4.** An object $E \in Ku(Y)$ is $\sigma$-semistable if $E[i] \in \mathcal{A}$ is semistable for some $i$.

The following two theorems are fundamental to us.

**Theorem 2.5.** [BLMS17, theorem 1.2] $Ku(Y)$ admits a Bridgeland stability condition $\sigma$.

In the following, $\sigma$ will always be a stability condition on $Ku(Y)$ as constructed in [BLMS17] proof of theorem 1.2, and $\mathcal{A}$ always be the corresponding heart of t-structure.

Let $M_\sigma(\lambda)$ be the moduli of $\sigma$-semistable objects in $\mathcal{A}$ with Mukai vector $\lambda \in K_{\num}(Ku(Y))$.

Let $L$ denote a line in $Y$, $C$ a generalized twisted cubic curve in $Y$, and $I_L$, $I_C$ their ideal sheaves. $\mathcal{R}_{\mathcal{O}_Y}$ and $\mathcal{L}_{\mathcal{O}_Y}$ denote the right and left mutation with respect to the exceptional object $O_Y$.

**Theorem 2.6.** [LLMS17, LPZ18].

(a). $M_\sigma(\lambda_1)$ is isomorphic to the variety of line $F(Y)$, parametrizing mutations of ideal sheaves of lines $F_L := \mathcal{L}_{\mathcal{O}_Y}(I_L(1))[-1] = \ker(H^0(I_L(1)) \otimes \mathcal{O}_Y \to I_L(1))$.

(b). $M_\sigma(\lambda_1 + \lambda_2)$ is isomorphic to $F(Y)$, parametrizing double mutations of ideal sheaves of lines $P_L := \mathcal{R}_{\mathcal{O}_Y}(L_{\mathcal{O}_Y}(I_L(1))) \otimes \mathcal{O}_Y(-1)[-1]$. In particular, it fits into a non-splitting extension $0 \to \mathcal{O}_Y(-1)[1] \to P_L \to I_L \to 0$.

(c). Assume in addition that $Y$ does not contain any plane. $M_\sigma(2\lambda_1 + \lambda_2)$ is isomorphic to the LLSvS eightfold $Z(Y)$.

The following fact is shown in the proof of [BLMS17, proposition 9.11].

**Lemma 2.7.** Suppose that $F_L \in M_\sigma(\lambda_1)$ and $P_L \in M_\sigma(\lambda_1 + \lambda_2)$, then $\phi(F_L) < \phi(P_L) < \phi(F_L) + 1$.

**2.3 Constant families of t-structures on $Ku(Y)$**. We would like to consider $\mathcal{F}, \mathcal{P}$ as families of objects in the heart $\mathcal{A} \subset Ku(Y)$. The foundation for this is a combination of [Kuz11] and [AP06, Pol07], which we quickly review here.

First, we recall the following special case of a theorem of Kuznetsov:

**Definition 2.8.** [Kuz11] Let $D^{[a,b]}(Y) := \{ F \in D(Y) : H^i(F) = 0, \text{ for any } i \notin [a,b] \}$. Let $D$ be a triangulated subcategory of $D(X)$ and $\Phi : D \to D(Y)$ be a triangulated functor. We say $\Phi$ has finite amplitude if $\Phi(D \cap D^{[p,q]}(X)) \subset D^{[p+a,q+b]}(Y)$ for some finite integers $a, b$, for all $p, q \in \mathbb{Z}$.

**Theorem 2.9.** [Kuz11, theorem 5.6] Let $Y$ be a smooth projective variety with a semiorthogonal decomposition of its derived category $D^b(Y) = \langle D_1, \ldots, D_m \rangle$, such that the projection functors to $D_i$ have finite amplitude. Let $S$ be a scheme of finite type over $\mathbb{C}$, write $Y_S := Y \times S$ and $f : Y_S \to S, p : Y_S \to Y$ for the projections. Then the derived category of $Y_S$ decomposes
$$D^b(Y_S) = \langle D_{1,S}, \ldots, D_{m,S} \rangle.$$ In particular, if $i : T \subset S$ is either an open immersion or a smooth point, then the functors $Li^* : D^b(Y_S) \to D^b(Y_T)$ and $Ri_* : D^b(Y_T) \to D^b(Y_S)$ respect the decompositions.
Example 2.10. Let $Y$ be as above and suppose that $D^b(Y) = \langle Ku(Y), E_1, \ldots, E_m \rangle$, where $(E_1, \ldots, E_m)$ is an exceptional collection, and $Ku(Y) := \{ F \in D^b(Y) : R\text{Hom}(E_i, F) = 0, \text{ for all } i = 1, \ldots, m \}$, then the projection functors are compositions of mutations. Since $Y$ is smooth and projective, mutations are of finite amplitude, and thus so are the projection functors. Thus theorem [2.9] produces a triangulated subcategory $Ku(Y)_S$ of $D^b(Y_S)$, which we refer to as the family of Kuznetsov components over $S$.

Definition 2.11. [Pol07] A t-structure $(D^{\leq 0}, D^{\geq 0})$ of a triangulated category $D$ is Noetherian if its heart $D^{\leq 0} \cap D^{\geq 0}$ is a Noetherian abelian category. It is close to Noetherian if there exist a Noetherian t-structure $(D^{\leq 0}, D^{\geq 0})$ of $D$ such that $D^{\leq -1} \subset D^{\leq 0} \subset D^{\leq 0}$.

Example 2.12. [Pol07, section 1.2] Let $D$ be a triangulated category with $K_{num}(D)$ being finitely generated. Then given any Bridgeland stability condition $\sigma = (A, Z)$ on $D$ with $Z : K_{num}(D) \otimes \mathbb{C} \to \mathbb{C}$, the associated heart of t-structure $A$ is close to Noetherian.

Definition 2.13. [AP06] A sheaf of hearts (or equivalently, t-structures) of $D^b(X)$ over a scheme $S$ is a functor $U \to A_U$, where $U \subset S$ is an open subset and $A_U \subset D^b(X \times U)$ is the heart of a t-structure, such that the restriction functor $D^b(X \times U) \to D^b(X \times V)$ is t-exact, for any open immersion $V \subset U$.

We would like to have the analog of the following result for $A \subset Ku(Y)$:

Theorem 2.14. [AP06, Pol07] Suppose that $Y$ is a smooth projective variety and $A \subset D^b(Y)$ is a close to Noetherian and bounded heart. Let $S$ be a finite type scheme over $\mathbb{C}$. Then there is a sheaf of hearts $A_S \subset D^b(Y_S)$, such that for any smooth point $i_s : S \to S$, $Li_s^* A_S \cong A$.

One way to obtain that is to use gluing of t-structures:

Proposition 2.15. [BBI, Pol07, lemma 3.1.1] Suppose that $D$ is a triangulated category with a semiorthogonal decomposition $D = \langle D_1, \ldots, D_m \rangle$, such that each inclusion functor $D_i \subset D$ admits left and right adjoints $pl_i, pr_i$, respectively. Then given a bounded t-structure $(D_i^{\leq 0}, D_i^{\geq 0})$ for each $D_i$, there exists a bounded t-structure on $D$ given by $D^{\leq 0} := \{ F \in D : pl_i(F) \in D_i^{\leq 0}, i = 1, \ldots, m \}$ and $D^{\geq 0} := \{ F \in D : pr_i(F) \in D_i^{\geq 0}, i = 1, \ldots, m \}$.

Lemma 2.16. Notations and assumptions as in proposition 2.15, in addition suppose that the functors $pr_i : D \to D_i$ have finite amplitude. If the t-structures $(D_i^{\leq 0}, D_i^{\geq 0})$ are all (close to) Noetherian, then there is a choice of shiftings for these t-structure, such that the shifted t-structures give rise to a (close to) Noetherian one on $D$.

Proof. This is an application of [Pol07, lemma 3.1.2]. Since $pr_i$ are assumed to be of finite amplitude, one can shift the given t-structures so that $pr_i|_{D_j} : D_i \to D_j$ is right t-exact with respect to the shifted t-structures, for every $j > i$. Then by [Pol07, lemma 3.1.2], we have

$$D^{[a,b]} = \{ F \in D : pr_i(F) \in D_i^{[a,b]} \}.$$

Now we first assume that the hearts $A_i := D_i^{\leq 0} \cap D_i^{\geq 0}$ are Noetherian. Let $A$ be the heart of the t-structure that comes from gluing $(D_i^{\leq 0}, D_i^{\geq 0})$ with appropriate shifts as above. For any $F \in A$, we have a decomposition of $F$.

$$
\begin{array}{cccc}
0 & \rightarrow & F_m & \rightarrow & F_{m-1} & \rightarrow & \ldots & \rightarrow & F_1 & \rightarrow & F \\
pr_m(F) & \leftarrow & pr_{m-1}(F) & \leftarrow & \ldots & \leftarrow & pr_1(F)
\end{array}
$$
Given an ascending chain of subobjects \( E_0 \subset E_1 \subset E_2 \subset \ldots \subset F \) in \( \mathcal{A} \), then \( pr_1(E) \) is an ascending chain of subobjects of in a Noetherian abelian category and therefore stable. We see that \( E_0 \subset E_1 \subset E_2 \subset \ldots \subset F \) is stable. Hence, \( \mathcal{A} \) is Noetherian.

Now suppose that \( (D_1^{>0}, D_1^{<0}) \) are close to Noetherian. By definition, we have a Noetherian t-structure \( (\hat{D}_1^{>0}, \hat{D}_1^{<0}) \) of \( D_1 \), such that \( D_1^{>0} \subset D_1^{<0} \subset \hat{D}_1^{>0} \). Gluing \( (\hat{D}_1^{>0}, \hat{D}_1^{<0}) \), up to appropriate shiftings, gives a Noetherian t-structure \( (\hat{D}^{>0}, \hat{D}^{<0}) \) of \( D \), with \( \hat{D}^{>0} \subset \hat{D}^{<0} \). Thus, \( (D^{>0}, D^{<0}) \) is close to Noetherian.

**Corollary 2.17.** Let \( Y \) be a smooth cubic fourfold and \( \mathcal{A} \subset \text{Ku}(Y) \) be the heart of the t-structure associated to a Bridgeland stability condition on \( \text{Ku}(Y) \). Let \( S \) be a quasi-projective variety. Then there exists a sheaf of hearts \( \mathcal{A}_S \subset \text{Ku}(Y)_S \), such that for any smooth point \( i_s : s \mapsto S \), \( Li_s^* \mathcal{A}_S \cong \mathcal{A} \).

**Proof.** Recall that \( D^b(Y) = \langle \text{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle \). Note that the triangulated subcategory \( \langle \mathcal{O}_Y(i) \rangle \cong D^b(pt) \). Thus we can glue \( \mathcal{A} \) with a choice of hearts of t-structures on \( \langle \mathcal{O}_Y(i) \rangle \) to get a heart \( \mathcal{C} \subset D^b(Y) \) that is close to Noetherian, by lemma [2.16]. Then theorem [2.14] produces a sheaf of t-structures \( \mathcal{C}_S \) of \( D^b(Y)_S \). Consider \( \mathcal{A}_S := \mathcal{C}_S \cap \text{Ku}(Y)_S \); it defines a bounded t-structure of \( \text{Ku}(Y)_S \). Moreover, since the semiorthogonal decompositions are compatible with base change as in theorem [2.9] we see that \( \mathcal{A}_S \) is indeed a sheaf of hearts and \( Li_s^* \mathcal{A}_S \cong \mathcal{A} \) for any smooth point \( i_s : s \mapsto S \).

**Definition 2.18.** [AP06, definition 3.3.1] Let \( S \) be a scheme of finite type over \( \mathcal{C} \). A family of objects in the heart \( \mathcal{A} \subset \text{Ku}(Y) \) over \( S \) is an object \( F \in \text{Ku}(Y)_S \), such that for every closed point \( i_s : s \in S \) one has \( Li_s^* F \in \mathcal{A} \).

**Proposition 2.19.** [AP06, corollary 3.3.3] Let \( S \) be a smooth quasi-projective variety, and \( E \) be a family of objects in the heart \( \mathcal{A} \) over \( S \), then \( E \in \mathcal{A}_S \).

**Example 2.20.** Let \( S \) be the moduli space \( M_s(\lambda_1) \) (or \( M_s(\lambda_1 + \lambda_2) \)), there exists a universal family \( \mathcal{F} \) (resp. \( \mathcal{P} \)) over \( Y_S \). For any \( s \in S \), we have \( \mathcal{F}_s \) (resp. \( \mathcal{P}_s \)) \( \in \mathcal{A} \subset \text{Ku}(Y) \) in the heart associated to a Bridgeland stability condition constructed in [BLMS17], thus \( \mathcal{F} \) (resp. \( \mathcal{P} \)) \( \in \mathcal{A}_S \).

**Definition 2.21.** Let \( S \) be a scheme of finite type, and \( \mathcal{A} \) be the heart of a t-structure on \( \text{Ku}(Y) \). Fix a family of objects \( \mathcal{E} \in \text{heart} \) \( \mathcal{A} \) over \( S \). A family of quotients in \( \mathcal{A} \) of \( \mathcal{E} \) over \( S \) is a morphism \( \mathcal{E} \to \mathcal{P} \), whose restriction \( \mathcal{E}_s \to \mathcal{P}_s \) to every closed point \( s \in S \) is a surjection in \( \mathcal{A} \). This defines a relative quot functor

\[
\text{Quot}_{\mathcal{A}, S}(\mathcal{E}, \lambda) : (T \to S) \mapsto \left\{ \text{families of quotients of } \mathcal{E}_T \right\} \quad \text{in } \mathcal{A} \text{ and of class } \lambda \text{ over } T. \tag{2.1}
\]

### 3 Families of extensions

In this section, we review the description of families of extensions in [Lan83], then adapt it slightly for our study of the Voisin map in next section.

Throughout this section, let \( X \to S \) be a projective and flat morphism between noetherian schemes, \( \mathcal{C} \subset D^b(X) \) be a sheaf of heart over \( S \), \( \mathcal{F} \) and \( \mathcal{P} \) be two families of objects in \( \mathcal{C} \). Given \( \eta \in R^1 \text{Hom}(\mathcal{P}, \mathcal{F}) \) that corresponds to an extension in \( \mathcal{C} \):

\[
0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{P} \to 0, \tag{3.1}
\]
use \( \eta(s) \in R^1\text{Hom}_{X_s}(P_s,F_s) \) to denote the extension class represented by
\[
0 \to F_s \to E_s \to P_s \to 0,
\]
the restriction of eq. (3.1) to \( s \).

**Definition 3.1.** [Lan83, definition 2.1] A family of extensions of \( P \) by \( F \) over \( S \) is a collection \( \{ \eta_s \in R^1\text{Hom}_{X_s}(P_s,F_s) \}_{s \in S} \) such that there exists an open cover \( (U_i)_{i \in I} \) of \( S \) with \( \eta_i \in R^1\text{Hom}_{f^{-1}(U_i)}(P_{f^{-1}(U_i)},F_{f^{-1}(U_i)}) \) for each \( i \in I \), satisfying \( \eta_i(s) = \eta_s \) for all \( s \in U_i \).

As in [Lan83], we will be considering two functors from the category of noetherian schemes over \( S \) to the category of sets:

\[
(g : T \to S) \mapsto \left\{ \begin{array}{l}
\text{families of extensions} \\
of P_T by F_T over T
\end{array} \right\}, 
\]

and

\[
(g : T \to S) \mapsto \left\{ \begin{array}{l}
\text{families of nonsplitting extensions} \\
of P_T by F_T over T
\end{array} \right\} / H^0(T,O^n_T). 
\]

In particular, the representability of the second functor will be crucial for us to resolve the Voisin map via extensions. The following definition is the key to this question:

**Definition 3.2.** [Lan83, section 1] Define

\[
R\text{Hom}_f(P,F) := Rf_*R\text{Hom}(P,F), \quad \text{Ext}^i_f(P,F) := H^i(R\text{Hom}_f(P,F)). 
\]

**Remark 3.3.** Suppose in addition that \( S \) is affine, then \( \text{Ext}^i_f(P,F) \cong \text{RHom}(P,F) \).

Therefore, \( \text{Ext}^i_f(P,F) \) is the sheaf on \( S \) associated to the presheaf

\[
U \to R^i\text{Hom}_{f^{-1}(U)}(P_{f^{-1}(U)},F_{f^{-1}(U)}).
\]

**Lemma 3.4.** [Lan83, corollary 1.2]

(a). \( R^i\text{Hom}_f(P,F) \) is quasi-isomorphic to a locally free complex \( W \).

(b). Given a Cartesian diagram, where \( T \) is a Noetherian scheme,

\[
\begin{array}{ccc}
X_T & \xrightarrow{g} & X \\
\downarrow f_T & & \downarrow f \\
T & \xrightarrow{f} & S,
\end{array}
\]

and any sheaf \( \mathcal{G} \) on \( T \), we have \( H^i(W, \otimes \mathcal{G}) \cong \text{Ext}^i_f(Lg^*P, \mathcal{F} \otimes^L \mathcal{G}) \).

**Proof.** (a). Let \( I \) be an injective replacement of \( F \) that has finitely many non-zero terms, \( L \) a \( f \)-ample line bundle on \( X \). Then for sufficient large \( n \), we have \( R\text{Hom}_f(L^{-n},I_k) = f_*\text{Hom}(L^{-n},I_k) \) locally free for all \( k \). Now take a locally free replacement of \( P \) with each term being sufficiently negative in the above sense, we obtain a double complex whose entries are all locally free sheaves. \( R\text{Hom}_f(P,F) \) is represented by its total complex \( W \).
(b). \( \mathcal{H}^i(\mathcal{W} \otimes \mathcal{G}) \cong \mathcal{H}^i(L^gRf_!R\text{Hom}(\mathcal{P}, \mathcal{F}) \otimes^L \mathcal{G}) \cong \mathcal{E}xt^i_f(L^g\mathcal{P}, \mathcal{F} \otimes^L \mathcal{G}) \). For the last isomorphism, we use a base change theorem: as we have \( T \) being Noetherian, \( f \) flat and projective, and \( R\text{Hom}_f(\mathcal{P}, \mathcal{F}) \) quasi-isomorphic to a locally free complex of finite length.

By lemma \ref{3.4} we have a natural base change morphism \( \mathcal{E}xt^i_f(\mathcal{P}, \mathcal{F}) \otimes k(s) \rightarrow R^i\text{Hom}(\mathcal{P}_s, \mathcal{F}_s) \) for each \( s \in S \), where \( k(s) \) is the residue field.

**Definition 3.5.** We say \( \mathcal{E}xt^i_f(\mathcal{P}, \mathcal{F}) \) commutes with base change if the maps \( \mathcal{E}xt^i_f(\mathcal{P}, \mathcal{F}) \otimes k(s) \rightarrow R^i\text{Hom}(\mathcal{P}_s, \mathcal{F}_s) \) are isomorphisms for all points \( s \in S \).

**Proposition 3.6.** \([\text{Lan83}, \text{proposition 2.3}]\) Let \( f : X \rightarrow S, \mathcal{P}, \mathcal{F} \) be as before. Suppose that \( \mathcal{E}xt^i_f(\mathcal{P}, \mathcal{F}) \) commutes with base change, then there is a canonical bijection between the set of families of extensions of \( \mathcal{P} \) by \( \mathcal{F} \) over \( S \) and \( H^0(S, \mathcal{E}xt^1_f(\mathcal{P}, \mathcal{F})) \).

However, the condition that \( \mathcal{E}xt^1_f(\mathcal{P}, \mathcal{F}) \) ”commutes with base change” will not hold in our case. We observe the following:

**Lemma 3.7.** Suppose that \( R^i\text{Hom}_f(\mathcal{P}, \mathcal{F}) = 0 \) for all \( i \leq 0 \). Then for any morphism \( g : T \rightarrow S \), where \( T \) is a reduced Noetherian scheme, there is a canonical bijection between the set of families of extensions of \( \mathcal{P}_T \) by \( \mathcal{F}_T \) over \( T \) and \( \text{Hom}(\mathcal{O}_T, L^gR^i\text{Hom}_f(\mathcal{P}, \mathcal{F})[1]) \).

**Proof.** Given \( \phi \in \text{Hom}(\mathcal{O}_T, L^gR^i\text{Hom}_f(\mathcal{P}, \mathcal{F})[1]) \), and a point \( t \in T \), we get \( \phi_t := \phi \otimes^L k(t) : k(t) \rightarrow L^gR^i\text{Hom}_f(\mathcal{P}_t, \mathcal{F}_t) \otimes k(t)[1] \). Note that by lemma \ref{3.3} \( L^gR^i\text{Hom}_f(\mathcal{P}, \mathcal{F}) \otimes k(t)[1] \cong \bigoplus (R^i\text{Hom}_{X_t}(\mathcal{P}_t, \mathcal{F}_t)[1-i]) \), thus we obtain a collection \( \{ \phi_t \in R^i\text{Hom}_{X_t}(\mathcal{P}_t, \mathcal{F}_t) \}_{t \in T} \).

Moreover, if we choose an affine open cover \( \{ U_i \} \) of \( T \), then by remark \ref{3.3}(c), \( \phi_t := \phi_{u_t} \in \text{Hom}(\mathcal{O}_{U_t}, L^gR^i\text{Hom}_f(\mathcal{P}, \mathcal{F})[1]) = R^1\text{Hom}_{X_t}(\mathcal{P}_t, \mathcal{F}_t) \), where \( X_t \) (resp. \( \mathcal{F}_t, \mathcal{P}_t \)) is the base change of \( X \) (resp. \( \mathcal{F}, \mathcal{P} \)) to \( U_t \). Suppose that \( \phi_t \) corresponds to an extension \( 0 \rightarrow \mathcal{F}_t \rightarrow \mathcal{E}_t \rightarrow \mathcal{P}_t \rightarrow 0 \), and for any \( t \in U_i \), the class \( \phi_i(t) \) represents the restriction \( 0 \rightarrow \mathcal{F}_t \rightarrow \mathcal{E}_t \rightarrow \mathcal{P}_t \rightarrow 0 \). Then \( \phi_i(t) \) is the image of

\[
\begin{array}{c}
\kappa(t) \\
\downarrow \phi_i(t) \otimes^L k(t)
\end{array}
\rightarrow \mathcal{E}xt^1_f(\mathcal{P}_t, \mathcal{F}_t) \otimes k(t) \rightarrow R^1\text{Hom}_{X_t}(\mathcal{P}_t, \mathcal{F}_t),
\]

where \( f_t \) is the base change of \( f \) to \( U_i \). Note that \( \phi_i(t) = \phi_t \). Hence, \( \{ \phi_t \in R^1\text{Hom}_{X_t}(\mathcal{P}_t, \mathcal{F}_t) \}_{t \in T} \) is a family of extensions.

Conversely, given a family of extensions over \( T \), we have \( \{ U_i, \eta_i \} \) as in definition \ref{3.1} without loss of generality, we may assume all \( U_i \) are affine. Again by remark \ref{3.3} and lemma \ref{3.4} \( \mathcal{E}xt^1_f(\mathcal{P}_{U_i}, \mathcal{F}_{U_i}) \cong \mathcal{E}xt^1_f(\mathcal{P}_{U_i}, \mathcal{F}_{U_i}) \cong H^0(L^gR^i\text{Hom}_f(\mathcal{P}, \mathcal{F})[1]) \). Thus \( \eta_t \in \text{Hom}(\mathcal{O}_{U_t}, H^0(L^gR^i\text{Hom}_f(\mathcal{P}, \mathcal{F})[1])) \cong \text{Hom}(\mathcal{O}_{U_t}, L^gR^i\text{Hom}_f(\mathcal{P}, \mathcal{F})[1]) \). By definition of a family of extensions, we have \( \eta_t \otimes^L k(t) = \eta_j \otimes^L k(t) \), for all \( t \in U_{ij} \).

Now the assumption that \( R^i\text{Hom}_f(\mathcal{P}, \mathcal{F}) = 0, i \leq 0 \) means \( R\text{Hom}(\mathcal{P}, \mathcal{F})[1] \in D^b(S)^{\geq 0} \). Then according to \ref{BBD} corollary 2.1.22 (see also \ref{Lie05} proposition 2.1.10),

\[
U \rightarrow \text{Hom}(\mathcal{O}_U, L^gR^i\text{Hom}_f(\mathcal{P}, \mathcal{F})[1])
\]

is a sheaf on \( T \). Thus the classes \( \eta_t \) glue, giving an element \( \eta \in \text{Hom}(\mathcal{O}_T, L^gR^i\text{Hom}_f(\mathcal{P}, \mathcal{F})[1]) \).

**For later use, we recall a well-known theorem:**
Theorem 3.8. [LK79, theorem A.5 (i)] Suppose that the base change map \( \text{Ext}^j_f(P, F) \otimes k(s) \to R^i\text{Hom}_{X_S}(P_S, F_s) \) is surjective for some \( s \in S \), then there is an open neighborhood \( U \) of \( s \) such that \( \text{Ext}^j_f(P, F) \otimes k(s') \to R^i\text{Hom}_{X_S}(P'_S, F'_s) \) is isomorphic for all \( s' \in U \).

Remark 3.9. The assumption that \( R^i\text{Hom}_f(P, F) = 0, i \leq 0 \) in lemma 3.7 can be obtained, for example, by taking \( P, F \) to be families of stable objects with slopes \( \phi(P_s) > \phi(F_s) \). Because in that case we have \( R^i\text{Hom}(P_s, F_s) = 0 \) for all \( i \leq 0 \) and all \( s \in S \), then by theorem 3.8 the claim follows.

Next, we characterize the set of families of extensions whose restriction to any closed point does not split.

Lemma 3.10. With the same assumption and notations as in lemma 3.7, let \( \mathcal{W} \) denote \( R^i\text{Hom}_f(P, F)[1] \), then there is a canonical bijection:

\[
\begin{align*}
\psi : Lg^*(\mathcal{W}^\vee) & \to \mathcal{L}_T, \text{where } \mathcal{L}_T \in \text{Pic}(T) \\
\text{such that } \mathcal{H}^0(\psi) : L^0g^*(\mathcal{W}^\vee) & \to \mathcal{L}_T
\end{align*}
\]

\[
\left\{ \psi : Lg^*(\mathcal{W}^\vee) \to \mathcal{L}_T, \text{where } \mathcal{L}_T \in \text{Pic}(T) \right\}
\leftrightarrow
\left\{ \text{families of nonsplitting extensions of } P_T \text{ by } F_T \text{ over } T \right\}
\left/ H^0(T, \mathcal{O}_T^*) \right.
\]

(3.4)

Proof. By lemma 3.7, the right hand side of the bijection above is the same as

\[
\left\{ \psi : \mathcal{L}_T \to Lg^*(\mathcal{W}) : \psi \otimes L k(t) \neq 0, \forall t \in T \right\}
\]

We want to show that this is equivalent to the left hand side. By lemma 3.4, we have \( \psi := \phi^\vee : Lg^*(\mathcal{W}^\vee) \to \mathcal{L}_T^\vee \). Applying \( -\otimes L k(t) \) to the triangle \( \mathcal{L}_T \xrightarrow{\phi} Lg^*(\mathcal{W})[1] \to \text{cone}(\phi) \), we see that \( \phi \otimes L k(t) \neq 0 \) if and only if \( \mathcal{H}^{-1}(\text{cone}(\phi) \otimes L k(t)) \xrightarrow{\sim} \mathcal{H}^{-1}(Lg^*(\mathcal{W}) \otimes L k(t)) \cong 0 \), because \( \mathcal{W} \in D^b(S)^{\geq 0} \) as we have seen in lemma 3.7. On the other hand, we have \( Lg^*(\mathcal{W}^\vee) \xrightarrow{\psi^\vee} \mathcal{L}_T^\vee \xrightarrow{\sim} \text{cone}(\psi) \), and \( \mathcal{H}^0(\psi) \) is surjective if and only if \( \mathcal{H}^0(\text{cone}(\psi) \otimes L k(t)) = 0 \). Note that \( \text{cone}(\phi)^\vee[1] \cong \text{cone}(\psi) \), so \( \mathcal{H}^{-1}(\text{cone}(\phi) \otimes L k(t)) \cong \mathcal{H}^0(\text{cone}(\psi) \otimes L k(t))^\vee \cong \mathcal{H}^0(\text{cone}(\psi) \otimes L k(t))^\vee \), Thus, the conditions on \( \phi \) and \( \psi \) are equivalent and the bijection is given by dualizing.

Therefore, we can rewrite the functor 3.8 as follow:

Definition 3.11. The functor of families of non-splitting extensions of \( P \) by \( F \) is

\[
\Psi : (g : T \to S) \mapsto \left\{ \psi : Lg^*(\mathcal{W}^\vee) \to \mathcal{L}_T, \text{where } \mathcal{L}_T \in \text{Pic}(T) \right\}
\]

\[
\left\{ \psi : Lg^*(\mathcal{W}^\vee) \to \mathcal{L}_T, \text{where } \mathcal{L}_T \in \text{Pic}(T) \right\}
\]

\[
\left\{ \psi : Lg^*(\mathcal{W}^\vee) \to \mathcal{L}_T, \text{where } \mathcal{L}_T \in \text{Pic}(T) \right\}, \quad (3.5)
\]

from the category of Noetherian schemes over \( S \) to the category of sets, where \( \mathcal{W} := R^i\text{Hom}_f(P, F)[1] \).

Remark 3.12. Note that we do not restrict the functor to the category of reduced Noetherian schemes, even though lemma 3.7 requires reducedness of \( T \). Indeed, we can take \( \text{Hom}(O_T, Lg^* R^i\text{Hom}_f(P, F)[1]) \) as our definition of families of extension over \( T \).

4 The Voisin map

Now we specialize to our case: suppose that \( Y \) is a cubic fourfold not containing any plane, \( Ku(Y) \) is the Kuznetsov component of \( Y \), \( M_1 := M_o(\lambda_1) \) and \( M_2 := M_o(\lambda_1 + \lambda_2) \) are the
Proof. (a) By lemma 3.4, we have a locally free replacement \( M_1 \times M_2, X := M_1 \times M_2 \times Y, f \) to be the projection, and \( F (\text{resp.} \, P) \) to be the pullback of a universal families on \( M_1 \times Y \) (resp. \( M_2 \times Y \)). Recall that \( F, P \) are families of objects in a heart \( \mathcal{A} \subset Ku(Y) \) associated to the stability condition \( \sigma \) constructed in [BLMS17] theorem 1.2. Also we specialize the functor \( \Psi \) (definition 3.11) to this setting. We may use the notation \( Ext^i(\cdot, \cdot) := R^i\text{Hom}(\cdot, \cdot) \).

Via a canonical identification \( M_1 \times M_2 \cong F(Y) \times F(Y) \), we define the incident locus \( \Gamma := \{(L_1, L_2) \in M_1 \times M_2 : L_1 \cap L_2 \neq \emptyset \} \) and the type II locus \( \Delta_2 := \{ L \in \Delta \cong F(Y) : N_{L/Y} \cong \mathcal{O}(1)^2 \oplus \mathcal{O} \} \) of the diagonal \( \Delta \subset M_1 \times M_2 \) (see e.g. [Deb13]).

Lemma 4.1. Recall \( F_L \) and \( P_L \) from theorem 2.6, then \( Ext^i(F_{L_1}, P_{L_2}) = 0 \) unless \( i = 0 \) or 1, and

\[
Ext^i(F_{L_1}, P_{L_2}) \cong \begin{cases}
\mathbb{C}, & (L_1, L_2) \in M_1 \times M_2 \setminus \Gamma, \\
\mathbb{C}^2, & (L_1, L_2) \in \Gamma \setminus \Delta_2, \\
\mathbb{C}^3, & (L_1, L_2) \in \Delta_2.
\end{cases}
\]  

(4.1)

Proof. First, recall the defining sequences of \( F_L \) and \( P_L \) respectively:

\[
F_L \rightarrow H^0(I_{L/Y}(H)) \otimes \mathcal{O}_Y \rightarrow I_{L/Y}(H), \quad \mathcal{O}_Y(-H)[1] \rightarrow P_L \rightarrow I_{L/Y}.
\]

One can easily verify that \( \text{Hom}^{-1}(F_{L_1}, \mathcal{O}_Y(-H)[1]) = 0 \), and thus by Serre duality on \( Ku(Y) \) (since both \( F_L, P_L \in Ku(Y) \) and \( Ku(Y) \) is a full subcategory of \( D^b(Y) \)), we get \( Ext^i(F_{L_1}, P_{L_2}) = 0 \) unless \( i = 0, 1 \) or 2. Also, according to [LLMS17], \( F_{L_1} \) and \( P_{L_2} \) are stable with respective to a stability condition with phases \( \phi(F_{L_1}) < \phi(P_{L_2}) \), therefore \( Ext^2(F_{L_1}, P_{L_2}) = \text{Hom}(P_{L_2}, F_{L_1}) = 0 \).

Next, recall that \( \chi(F_{L_1}, P_{L_2}) = -(\lambda_1, \lambda_1 + \lambda_2) = -1 = \text{hom}(F_{L_1}, P_{L_2}) - e\text{ext}^1(F_{L_1}, P_{L_2}) \), so we may just compute \( \text{Hom}(F_{L_1}, P_{L_2}) \). Using long exact sequences derived from the two defining sequences, we obtain:

\[
\text{Hom}(F_{L_1}, P_{L_2}) \cong \text{Hom}(I_{L_1/Y}(H), \mathcal{O}_{L_2}) \cong \begin{cases}
0, & (L_1, L_2) \in M_1 \times M_2 \setminus \Gamma, \\
\mathbb{C}, & (L_1, L_2) \in \Gamma \setminus \Delta_2, \\
\mathbb{C}^2, & (L_1, L_2) \in \Delta_2.
\end{cases}
\]

\[
\square
\]

Corollary 4.2. We have the following two exact sequences:

(a) \( 0 \rightarrow V_0 \xrightarrow{\alpha} V_1 \rightarrow \mathcal{E}xt^1(F, P) \rightarrow 0 \), where \( V_0, V_1 \) are locally free sheaves and \( \mathcal{E}xt^1(F, P) \) commutes with base change;

(b) \( 0 \rightarrow \mathcal{E}xt^1(P, F) \rightarrow W_1 \xrightarrow{\beta} W_2 \rightarrow \mathcal{E}xt^2(P, F) \rightarrow 0 \), where \( W_1, W_2 \) are locally free sheaves, \( \mathcal{E}xt^2(P, F) \) commutes with base change and \( \mathcal{E}xt^1(P, F) \) is a line bundle.

Proof. (a). By lemma 3.3, we have a locally free replacement \( V \), of \( R^i\text{Hom}(F, P) \). By theorem 3.8 and lemma 4.1, \( V \), at most have cohomologies at degree 0 or 1. Thus we have

\[
0 \rightarrow \text{Hom}(F, P) \rightarrow V_0 \rightarrow V_1 \rightarrow \mathcal{E}xt^1(F, P) \rightarrow 0.
\]

\( \text{Hom}(F, P) \) must be zero as well, because otherwise it would be a torsion sheaf again by theorem 3.8 and lemma 4.1 but \( V_0 \) is locally free.

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Similarly, we have a locally free replacement $W$ of $R\mathcal{H}om_f(\mathcal{P}, \mathcal{F})$ and moreover an exact sequence

$$0 \to \mathcal{E}xt^1_f(\mathcal{P}, \mathcal{F}) \to W_1 \to W_2 \to \mathcal{E}xt^2_f(\mathcal{P}, \mathcal{F}) \to 0.$$ 

The Grothendieck spectral sequence this time yields

$$\mathcal{E}xt^2_f(\mathcal{P}, \mathcal{F}) \otimes k(s) \cong \mathcal{E}xt^2(\mathcal{P}_s, \mathcal{F}_s).$$

Thus, $\mathcal{E}xt^2_f(\mathcal{P}, \mathcal{F})$ commutes with base change. Note that $\mathcal{E}xt^1_f(\mathcal{P}, \mathcal{F})$ is of rank one and reflexive, and thus a line bundle.

\[\square\]

**Remark 4.3.** On the open subset $U := M_1 \times M_2 \setminus \Gamma$, $\mathcal{E}xt^1_f(\mathcal{P}, \mathcal{F})_U$ commutes with base change since $\mathcal{E}xt^1_f(\mathcal{P}, \mathcal{F}) = 0$, then by [Lan83, corollary 4.5] (which in addition needs the fact that $\mathcal{H}om_f(\mathcal{P}, \mathcal{F}) = 0$) we have a universal nonsplitting extension

$$0 \to \mathcal{F}_U \boxtimes (\mathcal{E}xt^1_f(\mathcal{P}, \mathcal{F})_U)^{\vee} \to \mathcal{E}_U \to \mathcal{P}_U \to 0.$$ 

By results of [LLMS17] and [LPZ18], $\mathcal{E}_s$ is stable with respect to a Bridgeland stability condition $\sigma$ for all closed points $s \in U$. As $M_1 \times M_2$ is reduced, this $\mathcal{E}$ defines a rational map $v : M_1 \times M_2 \dashrightarrow M_\sigma(2\lambda_1 + \lambda_2)$. We refer to this as the Voisin map.

**Lemma 4.4.** $(\mathcal{V}_0 \xrightarrow{\alpha} \mathcal{V}_1)^{\vee}[-2] \cong (\mathcal{W}_1 \xrightarrow{\beta} \mathcal{W}_2)$.

**Proof.** Recall that $(\mathcal{V}_0 \xrightarrow{\alpha} \mathcal{V}_1) \cong Rf_\ast R\mathcal{H}om(\mathcal{F}, \mathcal{P})$ and $(\mathcal{W}_1 \xrightarrow{\beta} \mathcal{W}_2) \cong Rf_\ast R\mathcal{H}om(\mathcal{P}, \mathcal{F})$. By Grothendieck-Verdier duality,

$$R\mathcal{H}om(Rf_\ast R\mathcal{H}om(\mathcal{F}, \mathcal{P}), \mathcal{O}_{M_1 \times M_2}) \cong Rf_\ast R\mathcal{H}om(R\mathcal{H}om(\mathcal{F}, \mathcal{P}), \pi^\ast \omega_Y[4])$$

$$\cong Rf_\ast R\mathcal{H}om(\mathcal{P}, S_Y(\mathcal{F}))$$

$$\cong Rf_\ast R\mathcal{H}om(\mathcal{P}, i^\ast S_Y(\mathcal{F}))$$

$$\cong Rf_\ast R\mathcal{H}om(\mathcal{P}, \mathcal{F})[2],$$

where $\pi : M_1 \times M_2 \times Y \to Y$ is the projection, $S_Y$ (resp. $S_{K_n}$) denotes the Serre functor on $M_1 \times M_2 \times Y$ (resp. $Ku(M_1 \times M_2 \times Y)$), and $i^\ast$ (resp. $i^\ast$) is the right (resp. left) adjoint to the inclusion $Ku(M_1 \times M_2 \times Y) \hookrightarrow D^b(M_1 \times M_2 \times Y)$. For the last isomorphism, we use that $i^\ast \circ S_Y \cong S_{K_n} \circ i^\ast$.

\[\square\]

**Proposition 4.5.** $R\mathcal{H}om_f(\mathcal{F}, \mathcal{P})[1] \cong \mathcal{E}xt^1_f(\mathcal{F}, \mathcal{P}) \cong I_\Gamma$, where $I_\Gamma$ is the ideal sheaf of the reduced scheme structure on $\Gamma$. 

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Proof. Since $\mathcal{E}xt^2_T(P, F)$ commutes with base change, it is a torsion sheaf supported on $\Gamma$, by the computation in lemma [X]. Meanwhile $\mathcal{E}xt^2_T(P, F)$ is the cokernel of $\beta : W_1 \to W_2$, thus the degeneracy locus of $\beta$ is exactly $\Gamma$. Note that we have the Eagon-Northcott resolution $0 \to W_2^\vee \xrightarrow{\nu} W_1^\vee \to I_\Gamma \to 0$ of the ideal sheaf of the reduced scheme structure on $\Gamma$ (see e.g. [BVSS, theorem 2.11 & 2.16]). By lemma [1.3] this resolution is exactly $0 \to V_0 \xrightarrow{\sigma} V_1 \to \mathcal{E}xt^1_T(F, P) \to 0$.

**Theorem 4.6.** Let $b : Bl_\Gamma(M_1 \times M_2) \to M_1 \times M_2$ be the blowup of $M_1 \times M_2$ along $I_\Gamma$.

(a). The functor $\Psi$ is represented by $Bl_\Gamma(M_1 \times M_2)$. Consequently, the Voisin map can be resolved by $Bl_\Gamma(M_1 \times M_2)$:

$$
\begin{array}{c}
Bl_\Gamma(M_1 \times M_2) \\
\downarrow b \\
M_1 \times M_2 \\
\downarrow q \\
\Gamma(2\lambda_1 + \lambda_2)
\end{array}
$$

(b). If $K_{num}(Ku(Y)) \cong A_2$, then $q : Bl_\Gamma(M_1 \times M_2) \to M_\sigma(2\lambda_1 + \lambda_2)$ is a relative Quot scheme of objects in $A_\sigma$, with a universal quotient $E \to P$.

Proof. (a). By lemma [6.10] the functor $\Psi$ can be interpreted as

$$
\Psi(g : T \to S) = \left\{ \psi : Lg^*(I_\Gamma) \to \mathcal{L}_T, \text{where } \mathcal{L}_T \in \{ \text{Pic}(T), \text{ s.t. } H^0(\psi) : g^*(I_\Gamma) \to \mathcal{L}_T \} \right\}.
$$

(4.2)

We want to show that this is represented by $b : Bl_\Gamma(M_1 \times M_2) \to M_1 \times M_2$.

First recall that on $Bl_\Gamma(M_1 \times M_2)$ there is a universal quotient $b^*I_\Gamma \to \mathcal{O}(-E)$, where $E$ is the exceptional divisor. Thus, given any $g : T \to M_1 \times M_2$ that factors through $T \xrightarrow{\tilde{b}} Bl_\Gamma(M_1 \times M_2) \xrightarrow{b} M_1 \times M_2$, we have $g^*I_\Gamma \to \tilde{g}^*\mathcal{O}(-E)$.

Conversely, given $Lg^*I_\Gamma \to g^*I_\Gamma \to \mathcal{L}_T$, since $Lg^*I_\Gamma \cong (g^*V_0 \to g^*V_1)[1]$, we have $g^*V_1 \to g^*I_\Gamma \to \mathcal{L}_T$, which then induces an embedding $\iota_1 : T \hookrightarrow \mathbb{P}_{M_1 \times M_2}(V_1)$. Also we have $\iota_2 : Bl_\Gamma(M_1 \times M_2) \hookrightarrow \mathbb{P}_{M_1 \times M_2}(V_1)$ induced by $V_1 \to I_\Gamma$. For any $t \in T$, because $V_1 \otimes k(t) \to I_\Gamma \otimes k(t) \to \mathcal{L}_T \otimes k(t)$, $\iota_1(t)$ lies in the image of $\iota_2$, and therefore $\iota_1$ factors through a morphism $\iota_3 : T \to Bl_\Gamma(M_1 \times M_2)$.

Thus, $Bl_\Gamma(M_1 \times M_2)$ represents the functor $\Psi$, admitting a universal quotient $b^*I_\Gamma \to \mathcal{O}(-E)$. By lemma [6.7] this corresponds to a universal family of nonsplitting extensions of $P$ by $\mathcal{O}(-E) \boxtimes F$. In particular, there is a open covering of $Bl_\Gamma(M_1 \times M_2)$ with extension classes $\{U_i, \eta_i\}$ represents this family. On each $U_i$ we have an extension

$$
0 \to O_{U_i}(-E) \boxtimes F_i \to E_i \to P_i \to 0.
$$

For any closed point $s \in U_i$, $(E_i)_s$ is a $\sigma$-stable object in $Ku(Y)$ of class $2\lambda_1 + \lambda_2$, according to results in [LLMS17]. Moreover, $E_i$ is flat over $U_i$. Since $Bl_\Gamma(M_1 \times M_2)$ is reduced and so is $U_i$, $E_i$ defines a morphism $q_i : U_i \to M_\sigma(2\lambda_1 + \lambda_2)$. As $(E_i)_s \cong (E_j)_s$, for any $s \in U_{ij}$, these morphisms glue, giving a morphism $q : Bl_\Gamma(M_1 \times M_2) \to M_\sigma(2\lambda_1 + \lambda_2)$ that resolves the Voisin map.

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(b). Suppose that we have a morphism \( \phi : B \to Z(Y) \), with a family of quotients \( \mathcal{E}_B \to \mathcal{P}_B \) over \( B \), where \( \mathcal{E}_B \) is a family of objects in \( M_s(2\lambda_1 + \lambda_2) \) defining the map \( \phi \). \( \mathcal{P}_B \) is a family of objects of class \( \lambda_1 + \lambda_2 \), and both are flat with respect to \( A \). If \( Y \) is generic, then according to Mukai’s lemma and [LLMS17, proposition 6.4 & 6.5], fibers of \( \mathcal{P}_B \) are in fact stable. Thus we have a family of nonsplitting extensions over \( B \) and therefore a morphism \( B \to Bl_B \) over \( Z(Y) \). The only thing left to show is the existence of a universal quotient on \( Bl_B \).

Note that the quotients \( \mathcal{E}_i \to \mathcal{P}_i \) over \( U_i \) in part (a) does not necessarily glue; it is a twisted sheaf a priori. However, we use elementary modification (see e.g. [AB12]) to show that in this case it does in fact come from a global quotient over \( Z(Y) \).

Because \( I_\Gamma \cong R^i \text{Hom}_f(\mathcal{F}, \mathcal{P})[-1] \cong R^i \text{Hom}_f(\mathcal{P}, \mathcal{F})^\vee [1] \), then by lemma [5.7], the inclusion \( I_\Gamma \hookrightarrow \mathcal{O}_{M_1 \times M_2} \) corresponds to a family of extensions of \( \mathcal{P} \) by \( \mathcal{F} \), whose restriction to any point in \( \Gamma \) splits. Using the Grothendieck spectral sequence \( H^i(\text{Ext}_f^j(\mathcal{P}, \mathcal{F})) \Rightarrow \text{Ext}^{i+j}_f(\mathcal{P}, \mathcal{F}) \) and \( \text{Hom}_f(\mathcal{P}, \mathcal{F}) = 0 \), we see that \( \text{Ext}^i(\mathcal{P}, \mathcal{F}) \cong H^i(\text{Ext}_f^j(\mathcal{P}, \mathcal{F})) \cong H^0(\mathcal{H}^0(I_\Gamma^j)) \cong C \), thus the family of extensions above is in fact global.

Similarly, the pullback \( b^*I_\Gamma \to \mathcal{O}_{Bl} \) corresponds to a global family of extensions

\[
0 \to b^*\mathcal{F} \to \mathcal{E}' \to b^*\mathcal{P} \to 0
\]

on the blowup \( Bl := Bl_B(M_1 \times M_2) \), such that \( \mathcal{E}'_\mathcal{E} = b^*\mathcal{P}_E \oplus b^*\mathcal{F}_E \), where \( E \) is the exceptional divisor and we abuse notation \( b : Bl \times Y \to M_1 \times M_2 \times Y \). Consider the short exact sequence in the heart \( \mathcal{A}_\mathcal{P}(Bl) \)

\[
0 \to \mathcal{E} \to \mathcal{E}' \to b^*\mathcal{F}_E \to 0,
\]

where \( \mathcal{E} \) is by definition the kernel of the composition \( \mathcal{E}' \to \mathcal{E}'_\mathcal{E} \to b^*\mathcal{F}_E \). Restrict \( \mathcal{E} \) to \( E \) and use the octahedral axiom of triangulated categories, we obtain another short exact sequence

\[
0 \to b^*\mathcal{F}(-E)_E \to \mathcal{E}_E \to b^*\mathcal{P}_E \to 0,
\]

which is non-splitting because by construction of \( \mathcal{E} \) we have \( \text{Hom}(\mathcal{E}_E, b^*\mathcal{F}_E) = 0 \). This also implies that the composition \( \mathcal{E} \to \mathcal{E}' \to b^*\mathcal{P} \) is surjective, with kernel \( b^*\mathcal{F}(-E) \), i.e. we have another global extension

\[
0 \to b^*\mathcal{F}(-E) \to \mathcal{E} \to b^*\mathcal{P} \to 0
\]

Recall that the lower row in the above diagram corresponds to \( b^*I_\Gamma \to \mathcal{O}_{Bl} \), considered as an element in \( \text{Hom}(\mathcal{O}, R^i \text{Hom}_f(b^*\mathcal{P}, b^*\mathcal{F})) \). Therefore, the upper row corresponds to \( b^*I_\Gamma \to \mathcal{O}_{Bl}(-E) \), considered as an element in \( \text{Hom}(\mathcal{O}, R^i \text{Hom}_f(b^*\mathcal{P}, b^*\mathcal{F}(-E))) \), which get mapped to \( b^*I_\Gamma \to \mathcal{O}_{Bl} \) under the map induced by natural inclusion \( \mathcal{O}(-E) \hookrightarrow \mathcal{O} \). By part (a), the upper row is a global family of non-splitting extensions and universal. In particular, we obtain a universal quotient \( \mathcal{E} \to b^*\mathcal{P} \). Thus \( Bl_B(M_1 \times M_2) \) is a relative Quot scheme over \( M_s(2\lambda_1 + \lambda_2) \).

\[ \Box \]
Remark 4.7. The map \( q : Bl_{\mathcal{I}}(M_1 \times M_2) \to M_\sigma(2\lambda_1 + \lambda_2) \) is surjective and of degree six. These follow from Voisin’s argument \[\text{[Voi16]}\]. Recall that \( M_\sigma(2\lambda_1 + \lambda_2) \) and \( M_\sigma(\lambda_1) \) generically parametrize \( F_C := L_{\mathcal{O}_V}(I_C/S(2H))[1] \) and \( F_l := L_{\mathcal{O}_V}(I_l/Y(H))[1] \) respectively. Thus, \( \text{Hom}(F_L, F_C) = \text{Hom}(I_l/Y, I_c/S(H)) \). The latter ideal sheaf is indeed \( \mathcal{O}_S(l_1 - l_2) \), where \( l_1, l_2 \) are two skew lines in \( S \), provided that \( F_C \) generic. Therefore, we have a nontrivial morphism from \( F_1 \) to \( F_C \). By \[\text{[LLMS17]}\], \( F_C \) is a nontrivial extension of some \( P_C \) by \( F_1 \). This shows the Voisin map is dominant and thus \( q \) is surjective. Moreover, the degree is six, as also shown in \[\text{[Voi16]} \text{ proposition 4.8}.\]

Remark 4.8. We record here a computation suggesting that part (b) of theorem 4.6, namely, the Voisin map being resolved by a degree six relative Quot scheme, may still be true without the assumption that \( Y \) is very general.

Suppose that \( Y \in \mathcal{C}_d \) with \( d = 2r(r-1) + 2 \), where \( \mathcal{C}_d \) is the Hassett divisor parametrizing Hodge special cubic fourfolds of discriminant \( d \), see \[\text{[Has00]}\]. In particular, \( Y \) is not very general. If \( Y \) is moreover general within \( \mathcal{C}_d \), then according to \[\text{[AT14]}\], \( K_u(Y) \cong D^b(S) \) for some K3 surface \( S \) admitting a polarization \( L \) with \( L^2 = d \). Consider two Mukai vectors of \( S \):

\[
(1, 0, -1), \ (r, -L, r - 1).
\]

Note that these two Mukai vectors also generate a sublattice that is isomorphic to \( A_2 \). By a lattice-theoretic result \[\text{[LPS0]} \text{ Theorem 2.4} \] and the derived Torelli theorem for K3, there exists an auto-equivalent \( \Phi \) of \( D^b(S) \), such that \( \Phi(\lambda_1) = (r, -L, r - 1) \) and \( \Phi(\lambda_1 + \lambda_2) = (1, 0, -1) \).

On the other hand, we can use a method by D. Johnson \[\text{[Joh17]}\] to compute the length of the finite Quot scheme \( \text{Quot}(V, (1, 0, -1)) \), where \( V \) is a general stable sheaf of class \((r + 1, -L, r - 2)\) and the quotients are in the category of coherent sheaves of \( S \) (which can never be the heart of any Bridgeland stability condition though). Roughly speaking, the number of surjections to ideal sheaves of two points is given by a top Chern class \( c_3(V^{[2]}) \), where \( V^{[2]} \) is the tautological bundle on the Hilbert scheme \( S^{[2]} \). This top Chern class can be computed by using a close formula due to \[\text{[EGL99]} \text{ theorem 4.2} \], and it turns out to be six, independent of the choice of \( V \). See \[\text{[Joh17]}\] for details.

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