Actions of diagonalizable groups and attractors: I.
Definitions, affine case and root subgroups.

Arnaud Mayeux

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Abstract. For a diagonalizable group scheme $D(M)$ acting on an algebraic space $X$, we introduce for any subsemigroup $N$ of $M$ an attractor space $X^N$. We explain that this formalism allows to see root groups of reductive groups as attractors.

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1. Introduction

For a diagonalizable group scheme $D(M)_S$ acting on an algebraic space $X$ over a scheme $S$, we introduce for any subsemigroup $N$ of $M$ an attractor space $X^N$. If $M = \mathbb{Z}$ and $N = 0$ (resp. $N = \mathbb{N}$ and $N = -\mathbb{N}$), our construction coincides with [Ri16] and gives the fixed points (resp. algebraic attractors and repellers). Recall that [Ri16] generalizes [Dr13] in the sense that [Dr13] considers only base scheme $S = \text{Spec}(k)$ with $k$ a field. In [JS18], over fields, the authors generalize [Dr13] to actions of much more general groups than $\mathbb{G}_m$. Our construction goes in a different generality than [JS18] for two main reasons:

(i) in the case of $\mathbb{G}_m$ (i.e $M = \mathbb{Z}$) over fields, semigroups appearing in the construction of [JS18] are the same than [Dr13] or [Ri16], whereas we can also consider semigroups like $\mathbb{N}_{\geq k}$, $a\mathbb{N}$, $a\mathbb{N} + b\mathbb{N}$ or $\mathbb{N} \setminus \{0, 1, 2, 4, 5, 7, 10, 13\}$.

(ii) we allow bases that are not fields.

We refer to [Co14, Introduction] for history and pioneering works around this subject, we would like to cite here [CGP10, Lemma 2.1.4]. At the end of the document, we explain that this formalism allows to see roots groups of reductive groups as attractors, we are able to do this because our formalism allows very general semigroups.

Section 2 introduces notations and definitions about schemes associated to semigroups. Section 3 gives the definition of attractors as functors. Section 4 shows that attractors are representable in the affine case and gives an explicit formula in this case. Sections 5 and 6 state some properties of attractors. Section 7 is about the interpretation of root groups as attractors. Section 8 relates attractors to dilatations [MRR20].

2. Rings associated to semigroups and their spectra

In this article, semigroups are always abelian semigroups:

**Definition 2.1.** A semigroup is a set endowed with an associative and commutative operation.

**Definition 2.2.** A monoid is a semigroup with a neutral element.

Let $N$ be a semigroup with addition law denoted by $+_N$. To $N$ is associated a monoid $\overline{N}$ defined as follows. If $N$ is a monoid, we put $\overline{N} = N$, otherwise we put $\overline{N} = N \cup \{0\}$ with law $0 + _N x = x$ and $x + _N y = x + _N y$ for all $x, y \in N$. We now always write laws of semigroups by $+$.

**Example 2.3.** (i) We can consider the semigroup underlying an abelian group, it is a monoid.

(ii) The set of natural number $\mathbb{N}$ is a semigroup under addition. The set $\mathbb{N}_{\geq k}$ of positive integers bigger than $k$ is a semigroup, for any $k \geq 1$. We put $\mathbb{I} = \mathbb{N}_{\geq 1}$.

(iii) The group $\mathbb{Z}/n\mathbb{Z}$ is a monoid for any integer $n$.

(iv) The emptyset $\emptyset$ is a semigroup but it is not a monoid.

(v) The zero semigroup is a monoid.

**Definition 2.4.** Let $N$ and $L$ be subsemigroups in a semigroup $S$, we say that $N$ is a lower semigroup than $L$ if $\overline{N} \subseteq \overline{L}$ (for an arbitrary semigroup $D$, the notation $\overline{D}$ means $D \setminus \{0\}$ if $D$ is a monoid and $D$ otherwise). We then write $N \leq L$.

**Remark 2.5.** Let $N, L, M$ be subsemigroups in a semigroup $S$. If $N \leq L$ and $L \leq M$, then $N \leq M$. 
Example 2.6. We have $0 \leq \mathbb{I}$.

Definition 2.7. Let $R$ be a ring and let $N$ be a semigroup, then we denote by $R[N]$ the associated $R$-algebra whose underlying $R$-module is $\oplus_{n \in N} R \cdot X^n$ and multiplication is induced by the operation of the semigroup: $X^n \times X^{n'} = X^{n+n'}$. By convention we have $X^0 = 1$.

Example 2.8. Let $R$ be a ring.
(i) For any semigroup $N$, we have $R[N] = R[N]$.
(ii) $R[\emptyset] = R$
(iii) $R[N] \cong R[X]$ is the $R$-algebra of polynomials.
(iv) $R[N_{\geq k}]$ is the algebra of polynomials with monomials of degree 0 or bigger than $k$.
(v) $R[N_{\leq k}] \cong R[X_1, \ldots, X_n]$ is the algebra of polynomials with $n$-variables.
(vi) $R[N/nN] \cong R[X]/X^n - 1$.
(vii) If $N' \leq N$ are semigroups, then we have a canonical injective morphism of rings $R[N'] \rightarrow R[N]$.

Definition 2.9. Let $N$ be a semigroup then we define $A(N)$ to be the scheme $\text{Spec}(\mathbb{Z}[N])$, this is a $\mathbb{Z}$-scheme. If $S$ is a scheme, then we put $A(N)_S = A(N) \times_{\text{Spec}(\mathbb{Z})} S$. If $M$ is an abelian group, then $A(M)$ is denoted $D(M)$ and is a group scheme called the diagonalizable group scheme associated to $M$ [SGA3, Exp 1, §4.4].

Remark 2.10. For any semigroup $N$, we have $A(N)_S = A(N)_S$.

If $N' \leq N$, then we obtain morphisms of $S$-schemes $A(N)_S \rightarrow A(N')_S$, for every scheme $S$.

Remark 2.11. Let us recall the augmentation, the comultiplication and the antipode of the group scheme $D(M)$ for an abelian group $M$.

The augmentation is the projection map $\mathbb{Z}[M] \rightarrow \mathbb{Z}$ that sends $X^m$ to 0 for every $m \in M \setminus \{0\}$.

The antipode is the map $\mathbb{Z}[M] \rightarrow \mathbb{Z}[M]$ that sends $X^m$ to $X^{-m}$.

The comultiplication is the map $\mathbb{Z}[M] \rightarrow \mathbb{Z}[M] \otimes \mathbb{Z}[M]$ that sends $X^m$ to $X^m \otimes X^m$ for $m \in M$.

Let $M$ be an abelian group and $N$ be a subsemigroup of $M$. Then we have an algebraic action of $D(M)$ on $A(N)$ over $\mathbb{Z}$ given by:

$$
\mathbb{Z}[N] \xrightarrow{\delta_{N,M}} \mathbb{Z}[N] \otimes \mathbb{Z}[M] \\
X^n \mapsto X^n \otimes X^n.
$$

By base change, we obtain an action of $D(M)_S$ on $A(N)_S$ for every scheme $S$.

Proposition 2.12. Let $M$ be a semigroup and $N \subset M$ and $L \subset M$ be two subsemigroups. Then $N \cap L$ is a semigroup. Let $N + L$ be $\{n + l \in M | n \in N, l \in L\}$. Then $N + L$ is a semigroup. If $N$ or $L$ is a monoid, $N + L$ is a monoid.

Proof. Trivial. \qed

Definition 2.13. Let $M$ be a semigroup and let $\Sigma$ be a subset of $M$, then we denote by $N_\Sigma$ the smallest subsemigroup of $M$ containing $\Sigma$, this is the intersection of all semigroups of $M$ containing $\Sigma$. 


Definition 2.14. Let $N$ and $L$ be semigroups such that $N \leq L$. We say that $N \leq L$ is nice if the projection map

$$Z[L] \to Z[N], X^l \mapsto 0 \text{ if } l \in \overline{L} \setminus \overline{N} \text{ and } X^l \mapsto X^l \text{ if } l \in \overline{N}$$

is a morphism of rings. If $N \subset L$ and $N \leq L$ is nice, we say that $N \subset L$ is nice.

Remark that if $N \leq L$ is nice, then the associated morphism of schemes $A(N)_S \to A(L)_S$ is $D(M)_{S}$-equivariant for any scheme $S$.

Proposition 2.15. Let $N \subset L$ be monoids. Then $N \leq L$ is nice if and only if for all $x, y \in L$, then $x + y \in N \iff x \in N$ and $y \in N$.

Proof. Let $\phi$ denote the projection and assume it is a morphism of rings. Let $x, y \in L$. Then $x + y \in N \iff \phi(x^z + y^z) = x^z + y^z = \phi(x^y)\phi(y^y)$ is not zero $\iff$ both $x$ and $y$ are in $N$. Reciprocally assume that for all $x, y \in L$, $x + y \in N \iff x \in N$ and $y \in N$, then we have $\phi(x^z y^y) = \phi(x^y)\phi(y^y)$.

Proposition 2.16. Let $N$ be a monoid in $M$. Let $N^\times = \{x \in N | \exists y \in N \text{ and } x + y = 0\}$, then $N^\times$ is a submonoid of $N$ and a group, moreover $N^\times \subset N$ is nice.

Proof. Take $x, y \in N$. Assume $x + y \in N^\times$, then there exists $z \in N$ such that $x + y + z = 0$, this shows that $x$ and $y$ are in $N^\times$. Now apply Proposition 2.15.

Remark 2.17. If $N \subset L$ is a nice inclusion of monoids, then $L \setminus N$ is a subsemigroup of $L$. The converse is wrong: $Z \setminus N$ is a subsemigroup of $Z$ but $N \subset Z$ is not nice.

Remark 2.18. We refer to [Og] for a more detailed introduction to monoids. Our notion of nice inclusion corresponds to the notion of face in [Og, Definition 1.3.1].

3. Definition of attractors associated to semigroups

Let $X$ be an algebraic space over a base scheme $S$. Let $M$ be an abelian group and let $D(M)_S$ be the associated diagonalizable $S$-group scheme. Assume that $D(M)_S$ acts on $X$, this means that we have a morphism of algebraic spaces $D(M)_S \times_S X \to X$ satisfying the usual axioms. Let $N$ be a semigroup such that $N \subset M$ and consider $A(N)_S$ with the canonical action of $D(M)_S$ as in the previous section.

We now introduce the $N$-attractor associated to the action of $D(M)_S$ on $X$.

Definition 3.1. Let $X^N$ be the functor

$$(Sch/S) \to Sets, (T \to S) \mapsto \text{Hom}^{D(M)_T}_T(A(N)_T, X_T)$$

where $\text{Hom}^{D(M)_T}_T(A(N)_T, X_T)$ is the set of $D(M)_T$-equivariant $T$-morphisms from $A(N)_T$ to $X_T = X \times_S T$.

Remark 3.2. Taking attractors commutes with base change: if $S' \to S$ is a morphism of schemes, then $X^N \times_S S' = (X \times_S S')^N$. Moreover if $X$ and $Y$ are two algebraic spaces with an action of $D(M)_S$, then $D(M)_S$ acts on $X \times_S Y$ componentwise and we have $X^N \times_S Y^N \cong (X \times_S Y)^N$.

Remark 3.3. We have an identification $X \cong X^M$ via: for all scheme $T \to S$ and $x \in X^M(T) : D(M)_T \to X_T$ we associate the morphism $T \overset{x}{\to} D(M)_T \to X_T \in X(T)$ where $\varepsilon$ is the neutral element.
Remark 3.4. If $N \subseteq L$ is an inclusion of semigroups, then we have a morphism of functors $\iota : X^N \to X^L$. See Remark 4.6 for more details.

Definition 3.5. Let $N$ and $L$ be semigroups in an abelian group $M$ and assume that $N \leq L$ is nice. Let $X$ be an $S$ scheme with an action of $D(M)_S$. Then for all $T \to S$ the morphism $A(N)_T \to A(L)_T$ induces a morphism

$$\text{Hom}_{T}^{D(M)_T}(A(L)_T, X_T) \to \text{Hom}_{T}^{D(M)_T}(A(N)_T, X_T).$$

So we obtain a morphism of functors $X^L \to X^N$, that we denote $p$.

When $N \subseteq L$ is nice, the morphism $p$ satisfies $p \circ \iota = \text{Id}.$

Remark 3.6. If $X \to Y$ is a $D(M)$ equivariant morphism of algebraic spaces, then for any subsemigroup $N$ of $M$, we have a morphism of functors $X^N \to Y^N$.

Remark 3.7. The very general work [HP19] implies that in the context of Definition 3.1, the functor $X^N$ is representable in many cases. In an up-coming work, we will study the representability and more properties of $X^N$ using more descriptive methods. In Section 4, we show that $X^N$ is representable by an $S$-affine scheme if $X$ is an $S$-affine scheme.

Remark 3.8. We have defined attractors using semigroups, as we have seen it is also possible to define attractor using only monoids. We have preferred to introduce the formalism using semigroups since it is more general and thus more flexible with notation.

4. Representability and properties of attractors in the affine case

Let us prove that attractors are representable in the affine case and state some results about them. Let $M$ be an abelian group and $N$ be a subsemigroup of $M$. Let $S$ be a scheme and $X$ an $S$-affine scheme endowed with an action of $D(M)_S$.

Proposition 4.1. Then $X^N$ is representable by a closed $S$-affine scheme whose explicit quasi-coherent ideal sheaf is given in the following proof.

Proof. Let $A$ be the $O_S$ quasi-coherent algebra of $X$. By [SGA3, SGA3.1 Corollaire 4.7.3.1], $A$ is $M$-graded. So we have a decomposition $A = \oplus_{m \in M} A_m$ with each $A_m$ quasi-coherent. Let $J$ be the ideal sheaf generated by homogeneous element in $A_m$ for $m \in M \setminus N$. Then $J$ is the image of the morphism $A \otimes \oplus_{m \in M \setminus N} A_m \to A$, so it is quasi-coherent. We claim that $X^N = \text{spec}_S A/J$. Let $p : T \to S$ be an $S$-scheme. Let $A_T = p^* A$, this is an $O_T$ quasi-coherent algebra satisfying $\text{spec}_{O_T} A_T = X \times_S T$ by [StP, 01LQ]. If $N$ is a semigroup, let $O_T[N]$ be the $O_T$ quasi-coherent algebra of the $T$-affine scheme $A(N)_T$. On a first hand, we have

$$X^N(T) = \text{Hom}_{T}^{D(M)_T}(A(N)_T, X_T)$$

$$= \left\{ \begin{array}{l}
  f \in \text{Hom}_{T}(A_T, O_T[N]) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
On the other hand, we have $(\text{Spec}_S A/J)(T) = \text{Hom}_S(T, \text{Spec} A/J) = \text{Hom}_{O_S}(A/J, p_* O_T)$.

Now take an open $U \subset S$ and put 

$$O_S(U) = B, A(U) = A, J(U) = J, p_* O_T(U) = B'.$$

It is enough to define functorial maps $\Theta$ and $\Psi$

| Hom$_B(A/J, B')$ | $a$ | $\Theta$ | $\Psi$ |
|-------------------|-----|---------|-------|
| $f \in \text{Hom}_B(A, B'[N])$ | $A \longrightarrow A \otimes_B B[M]$ | $\downarrow$ | $\downarrow$ |
| $B'[N] \longrightarrow B'[N] \otimes_B B[M]$ | commutes |

such that $\Theta \circ \Psi = \text{Id}$ and $\Psi \circ \Theta = \text{Id}$.

**Lemma 4.2.** Let $a_m \in A_m$ for some $m \in M$ and let $f$ in the right side set above. Then $f(a_m) = 0$ if $m \in M \setminus \overline{N}$ and $f(a_m) = \lambda_m X^m$ for some $\lambda_m \in B'$ if $m \in \overline{N}$. In other words, a $D(M)$-equivariant morphism preserves $M$-graduations.

**Proof.** Easy and well-known. \hfill $\square$

Take $A/J \xrightarrow{F} B'$ on the left side and define a map $f = \Theta(F)$ on the right side as

$$A = \bigoplus_{m \in M} A_m \xrightarrow{f} B'[N]$$

$$a_m \in A_m \mapsto F([a_m]) X^m$$

Let us check that this map $\Theta$ is well-defined. Since $F([a_m]) = 0$ if $m \in M \setminus \overline{N}$, the element $F([a_m]) X^m$ belongs to $B'[N]$. Now we have to explain that $f$ is a morphism of $B$-algebras. This is a consequence of the identity

$$f(a_m a_m') = F([a_m a_{m'}]) X^{m + m'} = F([a_m][a_{m'}]) X^m X^{m'} = f(a_m) f(a_{m'})$$

Now we check that the diagram of the right side condition commutes. Let $a_m \in A_m$ for some $m \in M$, if $m \in M \setminus \overline{N}$, $f(a_m) = 0$ and there is nothing to prove. Otherwise $m \in \overline{N}$ and $a_m$ is sent to $F([a_m]) X^m \otimes X^m$ by the upper way or the lower way in the diagram. So $\Theta$ is well-defined.

Now take $A \xrightarrow{f} B'[N]$ on the right side. Then $f(a_m) = 0$ for all $a_m \in A_m$ for all $m \in M \setminus \overline{N}$ by Lemma 4.2, so $f$ vanishes on $J$, i.e. $f$ factors through $A \to A/J \xrightarrow{f} B'[N]$. Now we define $F = \Psi(f)$ as the composition $A/J \xrightarrow{f} B'[N] \xrightarrow{X^n \to 1} B'$, this is a morphism of $B$-algebras. Now let us prove that $\Theta \circ \Psi = \text{Id}$. Let $f$ be a morphism on the right side. Let $a_n \in A_n$ for $n \in \overline{N}$, we have $f(a_n) = \lambda_n X^n$ by Lemma 4.2. Then

$$((\Theta \circ \Psi)(f)) (a_n) = (\Theta(\Psi(f)))(a_n) = (\Psi(f))(a_n) \cdot X^n = (f(a_n)) |_{X^n = 1} \cdot X^n = \lambda_n X^n = f(a_n)$$

Now let $a_m \in M \setminus \overline{N}$, then

$$((\Theta \circ \Psi)(f))(a_m) = (\Theta(\Psi(f)))(a_m) = (\Psi(f))(a_m) \cdot X^m = 0 = f(a_m)$$

This proves that $\Theta \circ \Psi = \text{Id}$. Now let us prove that $\Psi \circ \Theta = \text{Id}$. Let $F$ be a morphism on the left side, and let us look at the image of $[a_n]$ for some $a_n \in A_n$ with $n \in \overline{N}$ under $(\Psi \circ \Theta)(F) = \Psi(\Theta(F))$.
\[
A/J \xrightarrow{\Theta(F)} B'[N] \to B'
\]
\[
[a_n] \mapsto F([a_n])X^n \mapsto F([a_n]).
\]

This finishes the proof of the Proposition. \qed

**Definition 4.3.** Let \( \emptyset \) be the semigroup consisting in the emptyset and 0 be the trivial monoid, then \( X^\emptyset = X^0 \) is the invariant space.

**Remark 4.4.** The ideal defining \( X^M \) is generated by the emptyset, so it is zero, this gives the canonical isomorphism \( X^M \cong X \) of Remark 3.3.

Let \( X \) be a \( D(M)_S \)-space over a scheme \( S \) and \( N \) be a subsemigroup of \( M \). Since \( X^M \cong X \), we obtain a closed immersion \( X^N \hookrightarrow X \).

**Definition 4.5.** If \( N \) and \( L \) are semigroups with \( N \leq L \leq M \), then the quasi-coherent ideal \( J_L \) associated to \( L \) embeds in \( J_N \), so we get a closed immersion \( X^N \hookrightarrow X^L \). This equals to the closed immersion \( \iota_{N,L} \) of Definition 4.5 if \( X \) is \( S \)-affine.

**Proposition 4.7.** Let \( X \) be an \( S \)-affine space with an action of \( D(M)_S \). Let \( N, L \) be two submonoids of \( M \). Then \( X^{N \cap L} = X^N \cap X^L \).

**Proof.** Let \( J_{N \cap L} \) be the quasi-coherent ideal of the quasi-coherent algebra of \( X \) defining \( X^{N \cap L} \). It is generated by homogeneous elements of degree belonging in \( M \setminus (N \cap L) \). Then \( J_{N \cap L} = J_N + J_L \) where \( J_N \) and \( J_L \) are the quasi-coherent ideals defining \( X^N \) and \( X^L \). Moreover know that the quasi-coherent ideal of \( X^N \cap X^N \) equals \( J_N + J_L \). This finishes the proof. \qed

**Lemma 4.8.** Let \( S \) be a scheme and let \( f : X \to Y \) be a \( D(M)_S \) equivariant injective morphism of \( D(M) \) affine \( S \)-schemes. Let \( N \) be a semigroup in \( M \). Then we have an injective morphism of schemes \( X^N \to Y^N \).

**Proof.** We have a commutative diagram of morphisms of schemes

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\uparrow & & \uparrow \\
X^N & \longrightarrow & Y^N
\end{array}
\]

where the upper arrow is injective. The left arrow is injective so the lower arrow is injective. \qed

**Lemma 4.9.** Let \( S \) be a scheme and let \( f : X \to Y \) be a \( D(M)_S \) equivariant closed immersion of \( D(M)_S \) affine \( S \)-schemes. Let \( N \) be a semigroup in \( M \). Then

(i) we have a closed immersion \( X^N \to Y^N \)

(ii) we have \( f(X^N) \cong f(X) \cap Y^N \).
**Proof.** This is local on $S$ so we assume $S = \text{Spec}(R)$ is affine, moreover we identify $X$ with a closed subscheme of $Y$. It is enough to prove (ii). Let $A$ be the $R$-algebra of $Y$ and $I$ be the ideal of $A$ defining $X$. Let $J$ be the ideal of $A$ defining $Y^N$, cf. Proof of Proposition 4.1. Then the ideal of $A/I$ defining $X^N$ is $I + J/I$ using the proof of Proposition 4.1. So the ideal of $A$ defining $X^N$ is $I + J$, moreover this shows that $X^N$ is the closed subscheme of $Y^N$ associated to the ideal $I + J/J$ of $A/J$. This shows that $X^N = X \cap Y^N$. \hfill \Box

**Lemma 4.10.** Let $f : M \to Z$ be a morphism of abelian groups, and let $D(Z)_S \to D(M)_S$ be the corresponding morphism of group schemes. Assume that $X = \text{Spec}(A)$ is an $S$-affine scheme with a $D(M)_S$-action, then we can see it as a $D(Z)_S$-space. Let $Y$ be a subsemigroup in $Z$. Let $N$ be $f^{-1}(Y)$, this is a submonoid in $M$. Then we have an isomorphism of $S$-schemes $X^N \simeq X^Y$, where on the left side $X$ is seen as a $D(M)$-scheme and on the right side as a $D(Z)$-scheme.

**Proof.** We reduce to the case where $S$ is affine and $X = \text{Spec}(A)$ is also affine. We have two compatible gradings on $A$, one given by $Z$ and one given by $M$. For any $y \in Y$, we have $A_y = \oplus_{n \in f^{-1}(y)} A_n$. So $\oplus_{z \in Z \cap Y} A_z = \oplus_{n \in M \setminus N} A_n$. Then the ideal defining $X^N$ equals the ideal defining $X^Y$, cf. the proof of Proposition 4.1. \hfill \Box

**Lemma 4.11.** Let $X$ be an affine $S$-scheme and let $D(M)_S$ a diagonalizable group acting on it. Let $N$ be a subsemigroup of $M$. Then $X^N$ is $D(M)_S$ stable and $(X^N)^N = X^N$.

**Proof.** We reduce to the case $S = \text{Spec}(R)$. Let $A$ be the $R$-algebra of $X$. Let $x \in X^N(R)$ and $g \in D(M)(R)$. By definition $g.y$ is the composition morphism

$$A \xrightarrow{\mu} A \otimes R[M] \xrightarrow{X^n \otimes 1} R.$$  

We identify this with

$$A \xrightarrow{\mu} A \otimes R[M] \xrightarrow{x \otimes 1} R[N] \otimes R[M] \xrightarrow{1 \otimes g} R[N].$$  

So $g.y$ is the composition

$$A \xrightarrow{\mu} A \otimes R[M] \xrightarrow{x \otimes 1} R[N] \otimes R[M] \xrightarrow{1 \otimes g} R[N].$$  

Now since $x \in X^N$, we get that $g.y$ is the composition

$$A \xrightarrow{x} R[N] \xrightarrow{\delta} R[N] \otimes R[M] \xrightarrow{1 \otimes g} R[N].$$  

Now the following commutative diagram, where $\delta$ and $\mu$ are the coactions,

$$\begin{array}{ccc}
A & \xrightarrow{\mu} & A \otimes R[M] \\
\downarrow x & \downarrow x \otimes 1 & \downarrow x \otimes 1 \\
R[N] & \xrightarrow{\delta} & R[N] \otimes R[M] \\
\downarrow \delta & \downarrow \delta \otimes 1 & \downarrow \delta \otimes 1 \\
R[N] \otimes R[M] & \xrightarrow{1 \otimes g} & R[N] \otimes R[M] \\
\downarrow 1 \otimes g & \downarrow 1 \otimes g \otimes 1 & \downarrow 1 \otimes g \otimes 1 \\
R[N] & \xrightarrow{\delta} & R[N] \otimes R[M]
\end{array}$$
proves that \( g.x \in X^N(R) \). Note that this diagram is commutative because the upper and lower rectangles are commutative. We prove similarly that \( X^N(R') \) is \( D(M)(R') \) stable for any \( R' \)-algebra \( R' \).

The formula \( (X^N)^N = X^N \) is a direct consequence of the construction of \( X^N \) as a closed subscheme of \( X \) defined by a certain ideal related to \( N \).

\[ \square \]

**Remark 4.12.** Even if \( X/S \) is not affine, we have an action of \( D(M)_S \) on \( X^N \) given as follows. For any \( S \)-scheme \( T \), we have an action of \( D(M)_S(T) \) on \( X_T \) (for any \( t \in D(M)_S(T) \), we have an arrow \( X_T \xrightarrow{\ell} X_T \)). Now let \( f \in X^N(T) = \text{Hom}^{D(M)_S}(A(N)_T, X_T) \) and \( t \in D(M)_S(T) \). We define \( t \cdot f \) to be \( A(N)_T \xrightarrow{f} X_T \xrightarrow{t} X_T \). The morphism \( X^N \to X \) is \( D(M)_S \)-equivariant.

**Example 4.13.**

(i) Let \( M = \mathbb{Z} \) and \( N = \mathbb{N} \), then \( D(M)_S = \mathbb{G}_m,S \) and \( A(N)_S = (\mathbb{A}^1)^+ \) is \( \mathbb{A}^1 \) endowed with the \( \mathbb{G}_m \)-action \( \lambda.x = \lambda x \). For a \( \mathbb{G}_m \)-space \( X/S \), the space \( X^N \) is the attractor as introduced by [CGP10, Lemma 2.14], [Dr13] and [Ri16].

(ii) Consider the action of \( \mathbb{G}_m = D(\mathbb{Z}) \) on \( X = \mathbb{A}^1 \) given by \( \lambda.x = \lambda x \) for an integer \( s \geq 1 \). For \( n \geq 1 \), consider the semigroup \( \mathbb{N}_{\geq s'} \) of integers bigger than \( s' \). Then the attractor \( X^{N_{\geq s'}} \) equals 0 if \( s' > s \) and equals \( X \) if \( s' \leq s \). More generally, given a \( \mathbb{G}_m \)-action on a space \( X \), we interpret \( X^{N_{\geq s'}} \) as the space of attracted points that converge with speed at least \( s' \).

(iii) Let \( M = \mathbb{Z}/n\mathbb{Z} \). Let \( s \) be a divisor of \( n \). Consider the action of \( D(M) = \text{Spec}(\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]) = \text{Spec}(\mathbb{Z}[X]/X^n - 1) \) on \( X = A(\mathbb{N}) \) defined by the formula:

\[
\mathbb{Z}[\mathbb{N}] \to \mathbb{Z}[\mathbb{N}] \otimes \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]
\]

\[
X \mapsto X \otimes X^s.
\]

Let \( s' \) be a divisor of \( n \), then \( X^{s'/\mathbb{Z}/n\mathbb{Z}} \) equals 0 if \( s' \) does not divide \( s \) and equals \( X \) if \( s'|s \).

(iv) Consider \( \mathbb{G}^2_{\mathbb{Z}} = D(\mathbb{Z}^2) \) and \( X = \mathbb{A}^1 \) endowed with the action \( (\lambda, \beta).x = \lambda^2 \beta x \). Then \( X^{N \times N_{\geq 2}} = X^{N \times 0} = X^{N,(1,1)} = 0 \) and \( X^{N,(2,1)} = X^{N_{\geq 2} \times N} = X \).

5. Action by group automorphisms

**Proposition 5.1.** Assume \( D(M)_S \) acts on an affine \( S \)-group scheme \( G \) by group automorphisms. Let \( N \) be a semigroup in \( M \), then the attractor \( G^N \) is a group scheme.

**Proof.** We reduce to the case where \( S = \text{Spec}(R) \). Let \( A \) be the Hopf algebra of \( A \). Let \( \Delta \) and \( \tau \) be the comultiplication and the antipode of \( A \). We prove now that \( G^N(R) \) is stable by the group law of \( G \). Let \( g, h \in G^N(R) \). Consider the diagram of morphisms of algebras where \( \delta \) and \( \mu \) are
the coactions

\[
\begin{array}{cccccc}
  A & \xrightarrow{\Delta} & A \otimes R[M] \\
  \downarrow{\Delta} & & \downarrow{\Delta \otimes \text{Id}} \\
  A \otimes A & \xrightarrow{\mu \otimes \mu} & A \otimes A \otimes R[M] \otimes R[M] & \xrightarrow{\text{Id} \otimes \text{Id} \otimes \text{m}} & A \otimes A \otimes R[M] \\
  \downarrow{g \otimes h} & & \downarrow{g \otimes h \otimes \text{Id} \otimes \text{Id}} & & \downarrow{g \otimes h \otimes \text{Id}} \\
  R[N] \otimes R[N] & \xrightarrow{\delta \otimes \delta} & R[N] \otimes R[N] \otimes R[M] \otimes R[M] & \xrightarrow{\text{Id} \otimes \text{Id} \otimes \text{m}} & R[N] \otimes R[N] \otimes R[M] \\
  \downarrow{m} & & \downarrow{m \otimes \text{Id}} & & \downarrow{g \otimes h \otimes \text{Id}} \\
  R[N] & \xrightarrow{\delta} & R[N] \otimes R[M]. 
\end{array}
\]

It is enough to prove that the biggest rectangle is commutative. The upper rectangle is commutative because \(D(M)_S\) acts by group automorphisms. The left middle rectangle commutes because \(g\) and \(h\) are in the attractor. The right middle rectangle and the lower one commute by definitions. So the biggest rectangle commutes. One proves similarly that \(G^N(R')\) is stable for any \(R\)-algebra \(R'\). Stability of \(G^N(R)\) under inversion follows from the commutativity of the diagram

\[
\begin{array}{cccccc}
  A & \xrightarrow{\mu} & A \otimes R[M] \\
  \downarrow{\gamma} & & \downarrow{\gamma \otimes \text{Id}} \\
  A & \xrightarrow{\mu} & A \otimes R[M] \\
  \downarrow{g} & & \downarrow{g \otimes \text{Id}} \\
  R[N] & \xrightarrow{\delta} & R[N] \otimes R[M]. 
\end{array}
\]

We show similarly that \(G^N(R')\) is stable under inversion for any algebra \(R'\).

\[\square\]

6. Cartesian Squares

In this section, we show that we can obtain Cartesian squares using the formalism of attractors.

**Proposition 6.1.** Let \(S\) be a scheme. Let \(M\) be an abelian group and let \(X/S\) be a \(D(M)_S\) scheme. Let \(N, N', L\) and \(L'\) be subsemigroups in \(M\) such that \(L \subset L', N' \subset L', N = L \cap N'\) and \(L' = L + N'\). We assume that \(X^E\) is representable by a scheme for all \(E \in \{N, N', L, L'\}\). We assume that the inclusion \(L \subset L'\) is nice. Then the inclusion \(N \subset N'\) is nice, moreover assume that one of the following conditions holds

(i) we have an equality \(N' = L \setminus (L \setminus N)\)

(ii) \(S = \text{Spec}(R)\) and \(X = \text{Spec}(A)\) are affine, \(A_lA_{n'} = A_{l+n'}\) for all \(l \in L \setminus N\) and \(n' \in N'\) (as usual \(A_m\) denote the \(m\)-graded part of \(A\)
then the following diagram is a cartesian square in the category of schemes

\[
\begin{array}{ccc}
X^{N'} & \xrightarrow{\iota_{N',L'}} & X^N \\
\downarrow{p_{N,N'}} & & \downarrow{p_{L,L'}} \\
X^L & \xleftarrow{\iota_{N,L}} & X^L
\end{array}
\]

Proof. The inclusion \((N' \cap L) \subset N'\) is nice because for any ring \(R\), the projection \(R[N'] \to R[N' \cap L]\) is the restriction of the projection \(R[L'] \to R[L] \to R[N']\), now \(R[L'] \to R[L]\) is a morphism of rings because \(L \subset L'\) is nice, and so \(R[N'] \to R[N' \cap L]\) is a morphism of rings.

(i) Assume (i) is satisfied. Let \(R\) be a ring. Since \(f : R[L'] \to R[L]\) is surjective, we know that the fiber product \(R[L'] \times_{R[L]} R[N]\) exists in the category of rings by [StP, 0ET0]. Let \(g : R[N] \to R[L]\) be the morphism associated to \(N \subset L\). We have \(R[L'] \times_{R[L]} R[N] \simeq \{(x, y) \in R[L'] \times R[N] | f(x) = g(y)\} \simeq R[N']\), indeed an element \(x \in R[L']\) maps to an element in \(R[N]\) under the projection \(R[L'] \to R[L]\) if and only if \(x \in R[(L' \setminus L) \sqcup N]\). The map \(R[N'] \to R[N]\) is the projection morphism associated to the nice inclusion \(N \subset N'\). The map \(R[N'] \to R[L']\) is the morphism associated to the inclusion \(N' \subset L'\). So by [StP, 0ET0] the scheme \(A(N')_R\) is the push-out, in the category of schemes, of the diagram

\[
\begin{array}{ccc}
A(L)_R & \xrightarrow{A(L')_R} & A(N)_R \\
\downarrow{A(L')_S} & & \downarrow{A(N)_S} \\
A(L')_S & \xleftarrow{A(L)_S} & A(N)_S
\end{array}
\]

We need the following Lemma.

**Lemma 6.2.** For any scheme \(S\), \(A(N')_S\) is the push-out, in the category of schemes, of

\[
\begin{array}{ccc}
A(L)_S & \xrightarrow{A(L')_S} & A(N)_S \\
\downarrow{A(L')_Y} & & \downarrow{A(N)_Y} \\
A(L')_Y & \xleftarrow{A(L)_Y} & A(N)_Y
\end{array}
\]

Proof. Let \(S = \cup_{i \in I} U_i\) be an affine open covering and write \(U_i = \text{Spec}(R_i)\). Let \(Y\) be a scheme and let \(A(L')_S \to Y\) and \(A(N)_S \to Y\) be two morphisms such that the following diagram commutes

\[
\begin{array}{ccc}
A(L)_S & \xrightarrow{A(L')_S} & A(N)_S \\
\downarrow{A(L')_Y} & & \downarrow{A(N)_Y} \\
A(L')_Y & \xleftarrow{A(L)_Y} & A(N)_Y
\end{array}
\]
We then obtain, for any $i \in I$, a commutative diagram

\[
\begin{array}{ccc}
A(L)_{U_i} & \overset{\cdot}{\longrightarrow} & A(N)_{U_i} \\
\bigg\uparrow & & \bigg\uparrow \\
A(L')_{U_i} & \longrightarrow & A(N')_{U_i} \\
\bigg\downarrow & & \bigg\downarrow \downarrow \downarrow \\
Y & \longrightarrow & Y
\end{array}
\]

Now since $U_i$ is affine, we obtain a unique morphism $f_i: A(N')_{U_i} \rightarrow Y$ such that the following diagram commutes

\[
\begin{array}{ccc}
A(L)_{U_i} & \overset{\cdot}{\longrightarrow} & A(N)_{U_i} \\
\bigg\uparrow & & \bigg\uparrow \\
A(L')_{U_i} & \longrightarrow & A(N')_{U_i} \\
\bigg\downarrow & & \bigg\downarrow \downarrow \downarrow \\
Y & \longrightarrow & Y
\end{array}
\]

For $i, j \in I$, we have $U_i \times_S U_j = U_i \cap U_j$. Let $U_i \cap U_j = \cup_{q \in Q} V_q$ be an affine open covering. We have $f_i|_{A(N')_{V_q}} = f_j|_{A(N')_{V_q}}$ for all $q \in Q$ by the affine case done before the statement of Lemma 6.2. So we have $f_i|_{A(N')_{U_i \cap U_j}} = f_j|_{A(N')_{U_i \cap U_j}}$ by [GW, Prop. 3.5]. Thus using [GW, Prop. 3.5] again, we obtain a unique morphism $f : A(N')_S \rightarrow Y$ such that the following diagram commutes

\[
\begin{array}{ccc}
A(L)_S & \overset{\cdot}{\longrightarrow} & A(N)_S \\
\bigg\uparrow & & \bigg\uparrow \\
A(L')_S & \longrightarrow & A(N')_S \\
\bigg\downarrow & & \bigg\downarrow \downarrow \downarrow \\
Y & \longrightarrow & Y
\end{array}
\]

Now let $T$ be a scheme and let $T \rightarrow X^{L'}$, $T \rightarrow X^N$ be two morphisms of schemes such that the following diagram commutes

\[
\begin{array}{ccc}
T & \overset{\cdot}{\longrightarrow} & X^N \\
\bigg\uparrow & & \bigg\uparrow \\
X^{L'} & \longrightarrow & X^N \\
\bigg\downarrow & & \bigg\downarrow \downarrow \downarrow \\
X^L & \longrightarrow & X^L
\end{array}
\]
This corresponds to a diagram

\[
\begin{array}{ccc}
A(L)_T & & A(N)_T \\
\downarrow & & \downarrow \\
A(L')_T & & X_T \\
\end{array}
\]

where all arrows are \( D(M)_T \)-equivariant.

By Lemma 6.2, we obtain a unique arrow \( A(N')_T \rightarrow X_T \) such that the following diagram commutes

\[
\begin{array}{ccc}
A(L)_T & & A(N)_T \\
\downarrow & & \downarrow \\
A(L')_T & & X_T \\
\end{array}
\]

It is enough to show that the arrow \( A(N')_T \rightarrow X_T \) is \( D(M)_T \)-equivariant. Consider the diagram

\[
\begin{array}{ccc}
A(L)_T & & A(N)_T \\
\downarrow & & \downarrow \\
A(L')_T & & X_T \\
\end{array}
\]

obtained by fiber product with \( D(M)_T \). We have

\[
A(E)_T \times_T D(M)_T = (A(E) \times_{\text{Spec}(\mathbb{Z})} T) \times_T D(M)_T = A(E) D(M)_T
\]

for \( E \in \{N, N', L, L'\} \), so the left diamond is a push-out by Lemma 6.2. Now we want to show that the lower rectangle is commutative. Consider the upper right composition in this rectangle and precompose it with the right part of the left diamond, denote this arrow by \( a_1 \). Consider the lower left composition in the rectangle and precompose it with the left part of the left diamond, denote this arrow by \( a_2 \). Now using the commutative diagrams coming from the \( D(M)_T \) equivariant morphisms on the right, we see that \( a_1 \) and \( a_2 \) are both equal to the arrow

\[
A(L)_T \times_T D(M)_T \rightarrow A(L)_T \rightarrow A(N)_T \rightarrow X_T.
\]

Using the left push-out diamond, this now implies that the lower rectangle is commutative. So the arrow \( A(N')_T \rightarrow X_T \) is \( D(M)_T \)-equivariant. This finishes the proof.
(ii) Let \( x \in X^{N'}(R) \). Then \( x \) is a morphism \( A \to R[N'] \). Now we have
\[
(A \to R[N'] \to R[L'] \to R[L]) = (A \to R[N'] \to R[N' \cap L] \to R[L]).
\]
This shows that the diagram is commutative. Now let us prove that it is cartesian. Let \( Y = \text{Spec}(B) \) be an affine \( R \)-scheme and let \( f : Y \to X^{L'/L} \) and \( g : Y \to X^N \) be two morphisms such that \( p_{L,L'} \circ f = \iota_{L,N} \circ g \). So \( f \) is a morphism of graded algebras \( A \to B[L'] \) and \( g \) is a morphism of graded algebras \( A \to B[N] \). Let \( m \in L' = L + N' \) and let \( A_m \) be the \( m \)-graded part of \( A \). Let \( x \in A_m \) and let \( \lambda_m \) such that \( f(x) = \lambda_m \). Then since \( p_{L,L'} \circ f = \iota_{L,N} \circ g \), we obtain that \( \lambda_m = 0 \) for all \( l \in L \setminus (N' \cap L) \). So we get \( f(x) = 0 \) for all \( x \in A_m \) for all \( m \in L' \setminus N' \) (we use that \( A_l A_{n'} = A_{l+n'} \) for all \( l \in L \setminus (N' \cap L) \) and \( n' \in N' \)). So we obtain a unique morphism \( h \) from \( Y \) to \( X^{N'} \) with the cartesian property.

\[\square\]

**Remark 6.3.** Let \( L' \) be a monoid and \( N \subset L \subset L' \) be submonoids. Assume \( L \leq L' \) is nice. Then \( N' := L' \setminus (L \setminus N) \) is a monoid, moreover \( N' \cap L = N \) and \( N' \cup L = N' + L = L' \).

### 7. Root groups, parabolic and Levi subgroups in reductive groups

Let \( G \) be a split connected reductive group scheme over a field \( R \). Let \( T \) be a maximal split torus and choose a Borel \( B \) containing \( T \). Let \( \Phi = \Phi(G, T) \subset X^*(T) \) denote the set of roots associated to \( (G, T) \) and \( \Phi^+ = \Phi(B, T) \) the roots in \( B \). Let \( B \) be a basis of \( \Phi^+ \). For \( \alpha \in \Phi \), let \( U_\alpha \) be the associated unipotent root group and \( u_\alpha \) be the root group associated to \( \alpha \) in the Lie algebra of \( G \). We refer to [SGA3, Exp. XXII] for the definition of \( U_\alpha \) and \( u_\alpha \). Let \( U \) be the unipotent radical of \( B \) and let \( u \subset b \subset g \) be the Lie algebras of \( U, B \) and \( G \). Consider the adjoint action of \( T \) on \( G, U \) and \( u \).

**Proposition 7.1.** There exists a \( T \)-equivariant isomorphism of \( R \)-schemes \( u \simeq U \).

*Proof.* This is a direct consequence of [SGA3, Exp. XXII Th. 1.1, Exp. XXVI Prop. 1.12], indeed these results imply the following assertions. For each root \( \alpha \in \Phi^+ \), we have a \( T \)-equivariant isomorphism \( U_\alpha \simeq u_\alpha \). We have \( T \)-equivariant isomorphisms of schemes \( u = \Pi_{\alpha \in \Phi^+} u_\alpha \) and \( U = \Pi_{\alpha \in \Phi^+} U_\alpha \). This finishes the proof.

Let us now fix \( \alpha \in \Phi^+ \). Let \( \mathbb{I} \subset X^*(T) \) be the subsemigroup generated by \( \alpha \).

**Proposition 7.2.** We have a canonical isomorphism \( u_\alpha \simeq u^{\alpha} \) (resp. \( U_\alpha \simeq U^{\alpha} \)), between root group and \( \mathbb{I} \alpha \)-attractor for the action of \( T = \text{Spec}(R[X^*(T)]) \) on \( u \) (resp. \( U \)).

*Proof.* Since \( G \) is split, \( \Phi \) is reduced. Since we have a \( T \)-equivariant isomorphism \( u \simeq U \) by Proposition 7.1, it is enough to prove it for \( u \). We have a decomposition \( u = \oplus_{\alpha \in \Phi^+} u_\alpha \). We identify for each \( \alpha \in \Phi^+ \) the scheme \( u_\alpha \) to \( \text{Spec}(R[\{X_\alpha\}]) \). So we identify \( u \) with \( \text{Spec}(R[\{X_\alpha\}_{\alpha \in \Phi^+}]) \).

Moreover the adjoint action of \( T \) on \( u \) is given by the coaction
\[
R[\{X_\alpha\}_{\alpha \in \Phi^+}] \to R[\{X_\alpha\}_{\alpha \in \Phi^+}] \otimes R[X^*(T)]
\]
\[
X_\alpha \mapsto X_\alpha \otimes X^\alpha \quad \text{for all } \alpha \in \Phi^+
\]
The scheme \( u^{\alpha} \) is the closed subscheme of \( u \) whose ideal \( J_\alpha \) is generated by homogeneous element in \( R[\{X_\beta\}_{\beta \in \Phi^+}] \chi \) for \( \chi \in X^*(T) \setminus \mathbb{I} \alpha \). The formula for the coaction shows that \( J_\alpha = (\{X_\beta\}_{\beta \in \Phi^+ \setminus \{\alpha\}}) \). This ends the proof.

\[\square\]
Proposition 7.3. We have a bijection between parabolic subgroups of \( G \) containing \( B \) and subsets of \( B \).

Proof. This is [Co14, Page 35, lines 4-5]

Recall that if \( \Sigma \) is a subset of \( X^*(T) \), we define \( N_\Sigma \) to be the semigroup generated by \( \Sigma \) in \( X^*(T) \).

Proposition 7.4. Let \( \zeta \subset B \).

(i) Let \( \Theta \) be \( \zeta \cup -\zeta \), then \( G^{N_\Theta} \) is the Levi subgroup \( L_\Theta \) such that \( \Phi(L_\Theta, T) = N_\Theta \cap \Phi \).

(ii) Let \( \Sigma \) be \( B \cup -\zeta \), then \( G^{N_\Sigma} \) is the associated parabolic subgroup (cf. Proposition 7.3), moreover \( L_\Theta \) is a Levi component of \( P \).

Proof. Let \( P_\Sigma \) be the parabolic corresponding to \( \Sigma \) by [CGP10]. By [CGP10, 2.2.8, 2.2.9], there exists a \( \lambda \in X_*(T) \) such that \( P_\Sigma \) is the attractor associated to the semigroup \( N \) relatively to the action of \( G_m = D(Z) \) on \( G \) via \( x.\theta = \text{ad}(\lambda(x))\theta \) and such that \( \lambda(\beta) \succeq 0 \) for all \( \beta \in \Sigma \) and \( \lambda(\beta) = 0 \) for all \( \beta \in \Theta \). The Levi subgroup \( L_\Theta \) corresponding to \( \Theta \) is the fixed point in \( G \) of the action of \( \lambda \) by conjugation, i.e. \( L_\Theta = G^\Theta \). Now we prove the Proposition.

(i) Assume first that \( \zeta = B \). Then \( L_\Theta = G \) and \( N_\Theta = N_{\Phi(G,T)} \). Using Prop. 7.2, we deduce that the big cell \( \Omega = \Pi_{\alpha \in \Phi} U_{\alpha} \times T \times \Pi_{\alpha \in \Phi^+} U_{\alpha} \) is in \( G^{N_\Theta} \). Now we have inclusions \( \Omega \subset G^{N_\Theta} \subset G \) with \( \Omega \) dense in \( G \) and \( G^{N_\Theta} \) closed in \( G \). This implies \( G^{N_\Theta} = G \). Let us now prove the general case, let \( \zeta \subset B \).

By Lemma 4.10, we have \( G^f = G^0 = L_\Theta \). We have \( \Theta \subset f^{-1}(0) \), and so \( N_\Theta \subset f^{-1}(0) \), consequently \( G^{N_\Theta} \subset G^f \). So we have proved that \( G^{N_\Theta} \subset L \) and let us now prove that this is an equality. We remark that \( N_\Theta = \Phi(L_{\Theta}, T) \). Now since \( L \subset G \) and using the first case done before, we have \( L = L^{N_{\Phi(L_{\Theta}, T)}} \subset G^{N_\Theta} \). This finishes the proof.

(ii) Recall that \( \lambda : G_m \rightarrow T \) corresponds to the morphism of abelian groups \( f : X^*(T) \rightarrow \mathbb{Z}, \chi \mapsto (\lambda, \chi) \). Now we see \( G \) as a \( G_m \) space and as a \( T \) space. By Lemma 4.10, we have \( G^N = G^{f^{-1}(0)} \). Since, for all \( \beta \in \Sigma \), \( f(\beta) = \lambda(\beta) \succeq 0 \), we have \( f(\beta) \in N \) and so \( \Sigma \subset f^{-1}(N) \), and so \( N_\Sigma \subset f^{-1}(\Phi) \). Consequently, \( P_\Sigma = G^N = G^{f^{-1}(0)} \supset G^{N_\Sigma} \). Let us prove that \( P_\Sigma \subset G^{N_\Sigma} \). We have \( P_\Sigma = L_\Theta \times R_{u}(P_\Sigma) \) where \( R_u(P_\Sigma) \) is the unipotent radical of \( P_\Sigma \). Using (i), we have \( L_\Theta = L^{N_{\Phi(L_{\Theta}, T)}} \subset L^{N_\Sigma} \subset L_{\Theta} \), and so \( L^{N_\Sigma} = L_{\Theta} \). Using Proposition 7.2, one has \( (R_u(P_\Sigma))^{N_{\Sigma}} = R_u(P_\Sigma) \). So \( P_\Sigma^{N_{\Sigma}} = P_\Sigma \), and so \( P_\Sigma \subset G^{N_\Sigma} \).

Remark 7.5. Proposition 7.4 implies that any parabolic or Levi subgroup of \( G \) containing \( T \) can be obtained as an attractor under the conjugation action of \( T \) on \( G \). Moreover, assume that \( B \) is a parabolic subgroup in a parabolic \( P \) and \( L \) is a Levi component of \( P \). We assume that \( B, P \) and \( L \) contain \( T \). Let \( M \) be \( B \cap L \), this is a parabolic subgroup in \( L \). Then one has a cartesian square

\[
\begin{array}{ccc}
P & \searrow & M \\
\swarrow & & \\
B & & L
\end{array}
\]
This square can be obtained using Proposition 6.1. Indeed let $L'$ be the subsemigroup generated by $\Phi(P, T)$, let $L$ be the subsemigroup generated by $\Phi(L, T)$ and $N$ be the subsemigroup generated by $\Phi(M, T)$. Using Proposition 2.16, we deduce that $L \leq L'$ is nice. Now let $N'$ be $L' \setminus (L \setminus N)$. We have $G^{L'} = P$, $G^N = M$ and $G^L = L$. We have $G^{N'} \subset G^{L'} = P$, and similar arguments as in the proof of Proposition 7.4 show that $G^{N'} = B$.

For any root $\alpha \in \Phi^+$ we denote by $H_\alpha \subset G$ the semidirect product $T \rtimes U_\alpha$, this is a group scheme whose unipotent radical equals $U_\alpha$.

**PROPOSITION 7.6.** We have a canonical isomorphism $H_\alpha \simeq G^{\tilde{\alpha}}$.

**Proof.** Since $\tilde{\alpha} \subset \Sigma_{\Phi^+}$ and by Proposition 7.4, we have a closed immersion $G^{\tilde{\alpha}} \subset B = G^{\Sigma_{\Phi^+}}$ where $B$ is the Borel subgroup. So by Lemma 4.9 and Lemma 4.11, we get a closed immersion $G^{\tilde{\alpha}} \subset B^\tilde{\alpha}$ and thus an equality $G^{\tilde{\alpha}} = B^\tilde{\alpha}$. Now we have a $T$-equivariant isomorphism of schemes $B \simeq T \times U$.

Using Proposition 7.2, we get

$$U_\alpha = U^{\tilde{\alpha}}.$$  \hspace{1cm} (7.1)

It is obvious that

$$T = T^{\tilde{\alpha}}.$$  \hspace{1cm} (7.2)

Now equations 7.1 and 7.2 and Remark 3.2 imply that $B^{\tilde{\alpha}} = H_\alpha$. This finishes the proof. \hfill $\Box$

**COROLLARY 7.7.** We have a canonical isomorphism $U_\alpha = R_u(G^{\tilde{\alpha}})$ where $R_u$ means the unipotent radical.

**REMARK 7.8.** We plan to study similar results for more general bases than fields, as it is clearly non optimal. We also plan to study what happens for non-split reductive groups. We plan to study Grassmannians using attractors.

### 8. Attractors and dilatations

In this section we show how under suitable hypothesis, dilatations commute with attractors.

Let $S$ be a scheme and let $S'$ be a locally principal scheme. Let $X$ be a scheme over $S$ with a $D(M)_S$ action where $M$ is an abelian group. Then $D(M)_{S'}$ acts on $X_{S'}$. Let $Y$ be a closed subscheme of $X_{S'}$. Then by [MRR20], we get a scheme $\text{Bly}X$ called the dilatation of $X$ with center $Y$, and a morphism of schemes $\text{Bly}X \to X$. Assume now that $Y$ is stable under the action of $D(M)_{S'}$ on $X_{S'}$, then under suitable hypothesis, we obtain an action of $D(M)_S$ on $\text{Bly}X$ and the morphism $\text{Bly}X \to X$ is $D(M)_S$ equivariant. Moreover, we have a diagram

$$\begin{array}{ccc}
\text{Bly}X|_{S'} & \to & X|_{S'} \\
\downarrow & & \downarrow \\
Y|_{S'} & \to & X|_{S'}
\end{array}$$

Let $N$ be a subsemigroup in $M$. We obtain the following two diagrams

$$(\text{Bly}X)^N \to X^N,$$
We have
\[
\text{(Bl}_Y \text{X})^N|_{\mathcal{S}'} = (\text{Bl}_Y \text{X}|_{\mathcal{S}'})^N \quad \text{and} \quad X^N|_{\mathcal{S}'} = (X|_{\mathcal{S}'})^N
\]
\[
Y^N|_{\mathcal{S}'} = (Y|_{\mathcal{S}'})^N
\]
By the universal property of dilatations, we get a canonical morphism of schemes
\[
(\text{Bl}_Y \text{X})^N \to \text{Bl}_Y X^N.
\]
We plan to study more the connection between attractors and dilatations.

**Example 8.1.** Let \( S = \text{Spec}(\mathbb{Z}_p) \) be the \( p \)-adic integers. Let \( S' = \text{Spec}(\mathbb{Z}_p/p^r\mathbb{Z}_p) \) where \( r \) is a positive integer. Let \( X = G = \text{SL}_3 \) over \( \mathbb{Z}_p \) and \( Y = e \), where \( e \) is the trivial group of \( \text{SL}_3 \) over \( \mathbb{Z}_p/p^r\mathbb{Z}_p \). Then
\[
(\text{Bl}_e G)(\mathbb{Z}_p) = \left\{ M = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \in \begin{pmatrix} 1 + p^r\mathbb{Z}_p & p^r\mathbb{Z}_p & p^r\mathbb{Z}_p \\ p^r\mathbb{Z}_p & 1 + p^r\mathbb{Z}_p & p^r\mathbb{Z}_p \\ p^r\mathbb{Z}_p & p^r\mathbb{Z}_p & 1 + p^r\mathbb{Z}_p \end{pmatrix} \mid M \in \text{SL}_3(\mathbb{Z}_p) \right\}
\]
is the \( r \)-congruence group. Let \( T \) be the diagonal torus of \( \text{SL}_3 \), we have \( T = \text{Spec}(\mathbb{Z}_p)[X^*(T)] \).

We have
\[
T(\mathbb{Z}_p) = \left\{ t = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1}\beta & 0 \\ 0 & 0 & \beta^{-1} \end{pmatrix} \mid \lambda, \beta \in \mathbb{Z}_p^\times \right\}.
\]
The torus acts by conjugation on \( G \) and on \( \text{Bl}_e G \):
\[
\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1}\beta & 0 \\ 0 & 0 & \beta^{-1} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1}\beta^{-1} & 0 \\ 0 & 0 & \beta \end{pmatrix} = \begin{pmatrix} x_{11} & \lambda^2\beta^{-1}x_{12} & \lambda\beta x_{13} \\ \lambda^{-1}\beta x_{21} & x_{22} & \lambda^{-1}\beta^2 x_{23} \\ \lambda^{-1}\beta^{-1} x_{31} & \lambda\beta^{-2} x_{32} & x_{33} \end{pmatrix}.
\]

Let \( N \subset X^*(T) \) be the subsemigroup generated by the root \( \alpha \in X^*(T) \) corresponding to \( t \mapsto \lambda\beta \). Then \( G^N(\mathbb{Z}_p) = \left\{ M = \begin{pmatrix} x_{11} & 0 & x_{13} \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{pmatrix} \mid M \in \text{SL}_3(\mathbb{Z}_p) \right\} \). Obviously \( e^N = e \). Now
\[
(\text{Bl}_e G^N)(\mathbb{Z}_p) = \left\{ M = \begin{pmatrix} x_{11} & 0 & x_{13} \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{pmatrix} \in \begin{pmatrix} 1 + p^r\mathbb{Z}_p & 0 & p^r\mathbb{Z}_p \\ 0 & 1 + p^r\mathbb{Z}_p & 0 \\ 0 & 0 & 1 + p^r\mathbb{Z}_p \end{pmatrix} \mid M \in \text{SL}_3(\mathbb{Z}_p) \right\},
\]
this equals \( (\text{Bl}_e G)^N(\mathbb{Z}_p) \).

**References**

Co14 B. Conrad: *Reductive group schemes*, Autour des schémas en groupes. Vol. I, Panor. Synthèses, 2014, [http://math.stanford.edu/~conrad/papers/luminysga3.pdf](http://math.stanford.edu/~conrad/papers/luminysga3.pdf) 1, 2, 15

CGP10 B. Conrad, O. Gabber, G. Prasad: *Pseudo-reductive groups*, Cambridge University Press, 2010 2, 9, 15

AHL16 Alper, Hall, Rydh: *The étale local structure of algebraic stacks*, [https://arxiv.org/abs/1912.06162](https://arxiv.org/abs/1912.06162)

JS18 J. Jelisiejew, L. Sienkiewicz: *Bialynicki-Birula decomposition for reductive groups*, J. Math. Pures Appl. (9), 2019, [https://arxiv.org/abs/1805.11558](https://arxiv.org/abs/1805.11558)
Jelisiejew, L. Sienkiewicz: Białyńcki-Birula decomposition for reductive groups in positive characteristic, J. Math. Pures Appl. (9), 2021, https://arxiv.org/abs/2006.02315

Braden: Hyperbolic localization of intersection cohomology, Transform. Groups, 2003, https://arxiv.org/abs/math/0202251

Drinfeld: On algebraic spaces with an action of $\mathbb{G}_m$, https://arxiv.org/abs/1308.2604

Ogus: Lectures on Logarithmic Algebraic Geometry, Cambridge UP, 2006

Görtz, Wedhorn: Algebraic Geometry I Scheme With Examples and Exercises, book, 2010

Mayeux, Richarz, Romagny: Néron blowups and low-degree cohomological applications https://arxiv.org/abs/2001.03597

Richarz: Spaces with $\mathbb{G}_m$-action, hyperbolic localization and nearby cycles, J. Algebraic Geom., 2019, https://arxiv.org/abs/1611.01669

Halpern-Leistner, Preygel: Mapping stacks and categorical notions of properness https://arxiv.org/abs/1402.3204v2

Demazure, Grothendieck: Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 - Schémas en groupes - (SGA 3), Lecture notes in mathematics 151 (1970), Berlin; New York: Springer-Verlag. pp. xv+564. https://webusers.imj-prg.fr/patrick.polo/SGA3/3

The Stacks project, https://stacks.math.columbia.edu

Arnaud Mayeux
Université Clermont Auvergne, CNRS, LMBP, F-63000 Clermont-Ferrand, France
E-mail address: arnaud.mayeux@uca.fr