Always be Two Steps Ahead of Your Enemy
Maintaining a Routable Overlay under Massive Churn in Networks with an Almost Up-to-date Adversary

Thorsten Götte
Department of Computer Science,
Paderborn University,
33102 Paderborn, Germany
thgoette@mail.upb.de
https://orcid.org/0000-0001-9798-6993

Vipin Ravindran Vijayalakshmi
School of Business and Economics,
RWTH Aachen,
52072 Aachen, Germany
vipin.rv@oms.rwth-aachen.de

Christian Scheideler
Department of Computer Science,
Paderborn University,
33102 Paderborn, Germany
scheidel@mail.upb.de

Abstract
We investigate the maintenance of overlay networks under massive churn, i.e. nodes joining and leaving the network. We assume an adversary that may churn a constant fraction $\alpha n$ of nodes over the course of $\mathcal{O}(\log n)$ rounds. In particular, the adversary has an almost up-to-date information of the network topology as it can observe an only slightly outdated topology that is at least 2 rounds old. Other than that, we only have the provably minimal restriction that new nodes can only join the network via nodes that have taken part in the network for at least one round.

Our contributions are as follows: First, we show that it is impossible to maintain a connected topology if adversary has up-to-date information about the nodes’ connections. Further, we show that our restriction concerning the join is also necessary. As our main result present an algorithm that constructs a new overlay- completely independent of all previous overlays - every 2 rounds. Furthermore, each node sends and receives only $\mathcal{O}(\log^3 n)$ messages each round. As part of our solution we propose the Linearized DeBruijn Swarm (LDS), a highly churn resistant overlay, which will be maintained by the algorithm. However, our approaches can be transferred to a variety of classical P2P Topologies where nodes are mapped into the $[0,1)$-interval.

2012 ACM Subject Classification Theory of computation → Distributed algorithms;

Keywords and phrases Overlay networks

1 Introduction
Peer-To-Peer (P2P) has proven to be a useful technique to construct resilient decentralized systems. In a P2P architecture the nodes are connected via the Internet and form a logical network topology, a so-called overlay network, on top of it. Within the overlay each node has a logical address and logical links that allows it to search and store information in the network.

As in every large-scale system, errors and attacks are a rule rather than the exception, so nodes in P2P networks are constantly failing. At the same time there is usually no or only little admission control, resulting in new nodes frequently joining. This implies a massive amount of churn, i.e.
nodes joining and leaving the network, at any given time. In fact, empirical studies have shown that 50% of all nodes are subjected to churn over the course of an hour [1].

We want to investigate overlays under massive churn controlled by a singular adversary. The adversary’s goal is to partition the network into two or more subnetworks that cannot communicate with one another. A powerful technique to prevent this is to frequently rearrange the network and change the nodes’ connections. Previous theoretical works, e.g. [2, 3, 4], often considered models where the adversary has slightly outdated information about the network. In particular, the adversary could access all information that is at least $\Theta(\log \log n)$ rounds old, where $n$ is minimal amount of nodes in the network. This includes the nodes’ internal states, random decisions, and the content of all messages etc. However, the techniques presented in these papers cannot be used if one wants to grant the adversary access to even more recent information. Thus, in this paper we propose a trade-off as we present the $(a, b)$-late omniscient adversary that has almost up-to-date information about the topology, but is more outdated with regard to all other aspects. In particular, it has full knowledge of the topology after $a$ rounds and complete knowledge after $b$ rounds.

In this work we try to minimize the adversary’s lateness with respect to the overlay’s topology and present an overlay maintenance algorithm that handles a $(2, \Theta(\log n))$-late adversary. Our approach uses several structural properties of the overlay as well as a careful analysis of non-independent events to ensure the fast reconfiguration of the network.

In the real world, an adversary with similar properties could, e.g., be an agency eavesdropping at Internet exchange points. They can see who communicates based on the involved IP-addresses but they are either unable to decrypt the messages or take a long time to decrypt them.

1.1 Model

We assume that time proceeds in synchronous rounds. Furthermore, we observe a dynamic set of nodes $\mathcal{V} := \{V_1, V_2, \ldots\}$ determined by an adversary such that $V_t$ is the set of nodes in round $t$. For all $V_t$ it holds that $|V_t| \in [n, \kappa n]$ where $\kappa \in \mathbb{R}$ is a small constant. In other words, the number of nodes stays within $\Theta(n)$. This reflects the fact that the number of nodes in a real-world P2P system is (relatively) stable over time [1].

Further, each node is identified by a unique and immutable ID. In a real-world network these IDs could, e.g., be the nodes’ IP addresses. A node can create an edge and thus send a message to another node only if it knows its ID. Creating an edge may be compared to creating a TCP connection to the desired receiver. We assume that a node can create edges to $\Theta(\log n)$ different nodes in each round and can send $\Theta(\text{polylog } n)$ bits via each edge. Note that we assume that an ID is of size $\Theta(\log n)$. This results in series of graphs $\mathcal{G} := \{G_1, G_2, \ldots\}$ with $G_t = (V_t, E_t)$ and $E_t = \{(u, v) | u \text{ send message to } v \text{ in round } t\}$. Observe that each $G_t$ is a directed graph.

Next, we will explain the structure of a (synchronous) round in our model. For each node, a round consists of the following three phases: First, a node receives all messages sent in the previous round. Second, a node can perform calculations on its local variables and the received messages. Third, it can send messages to other nodes, given it knows their ID. Note that sending a message to another node implicitly creates an edge. Every message sent in round $t$ is received in $t + 1$.

In the remainder of this section we describe the adversary’s capabilities and limitations: First, we assume that until a round $B \in \Theta(\log n)$ the adversary is inactive and no churns happens. We call this the bootstrap phase. Note that several other works also assume a bootstrap phase to prepare the random sampling (cf. [2, 3, 4]).

After the bootstrap phase, the adversary may begin churning nodes. For a suitable value $T \in \Theta(\log n)$ we assume that $V_{t+T} \cap V_t \geq (1 - \alpha)n$ where $\alpha \in \mathbb{R}$ is a constant. This allows the churn to be $\Theta(n)$ in each round as long as there is a stable set of size $\Theta(n)$ that remains in the network for at least $T$ rounds. In particular, this set may change over time and can be selected by the adversary.
In principle our algorithm in Section 5 could be extended to tolerate \(\Theta(\log n)\) joins per node as in [2]. We chose a constant because a higher number of joins would only increase the number of messages by a polylogarithmic factor and does not introduce further algorithmic challenges.

### Table 1 Overview of different models in the literature

| Paper | Lateness\(^a)\) | Churn Rate\(^b)\) | Immediate |
|-------|----------------|-------------------|-----------|
| [2]   | \(\Theta(\log n), \Theta(\log \log )\) | \(O(n, \Theta(\log n))\) | Yes       |
| [4]   | \(\Theta(\log n), \Theta(\log n)\) | \(\Theta(n - \frac{n}{\log n}, \Theta(\log n))\) | No\(^c)\) |
| [5]   | \(\Theta(\log n), \Theta(\log n)\) | \(\Theta(\frac{n}{\log n}), \Theta(\log n)\) | Yes       |
| This  | \(\Theta(\log n)\) | \((\Theta, \Theta(\log n))\) | Yes       |

\(\text{a) An adversary is } (a,b)\text{-late if it has full knowledge of the topology after } a \text{ rounds and complete knowledge after } b \text{ rounds.}\)

\(\text{b) The churn rate is } (C,T) \text{ if the adversary can perform } C \text{ join/leaves in } T \text{ rounds.}\)

\(\text{c) Nodes remain in the network for additional } \Theta(\log n) \text{ rounds.}\)

This model incorporates observation from [1] that new nodes join and leave very frequently but there is a (relatively) stable set of older nodes. Note that to best of our knowledge this is the most flexible model compared to the related work.

Further, we assume that a new node \(v \in V_t \setminus V_{t-1}\) can only join via a node \(w \in V_t \cap V_{t-2}\). In this case we say that \(v\) joins via \(w\) in round \(t\). In Section 2 we show that this is a necessary condition. Last, the number of nodes that join the network via the same node \(v \in V_t\) is constant\(^d)\).

Next, we describe the adversary’s knowledge. As mentioned in the introduction, the adversary is \((2, \Theta(\log n))\)-late omniscient. That means it has slightly outdated knowledge of the topology, i.e., the series of graphs \(G := (G_0, G_1, \ldots)\) created through the communication between nodes. In particular, since our adversary is 2-late in round \(t\) the adversary has full knowledge of all graphs until \(G_{t-2}\). Further, it has no knowledge of the nodes’ internal states and the contents of messages for \(\Theta(\log n)\) rounds. That means it learns the content of message sent in round \(t\) only in round \(t + \Theta(\log n)\).

For the adversary, a round unfolds as follows: At the beginning of each round (that is - before - the messages are received) the adversary can select of a set of nodes \(O_t \subset V_{t-1}\) that leave the network. These nodes do not receive any messages and leave the network immediately. However, the messages that nodes sent will be delivered if the receiving node remains in the network. Further, the adversary may propose a set of nodes \(J_t\) that joins in round \(t\). For each node \(v \in J_t\) the adversary selects a bootstrap node \(w \in V_t \setminus J_t\) that receives a reference to \(v\) in round \(t\).

### 1.2 Related Work

There has been extensive work on analyzing overlay networks under high adversarial churn. As already mentioned in the introduction, these works had a variety of different model assumptions. See [2] for a comprehensive survey on previous results. In the following, we only concentrate on works closely related to our work.

First, there was a variety of works (cf. [4, 7, 8]) that assumed only a subset of nodes is subjected to adversarial churn. However, these nodes could also act arbitrarily byzantine and try to sabotage the overlay’s maintenance and the routing. A general assumption was that up to a constant fraction of nodes would behave byzantine. In [6] Scheideler presented a protocol that would spread these over the network such that each group of logarithmic size would have a constant fraction of non-byzantine nodes in them. Fiat et al. [7] build upon this work and presented a full overlay maintenance algorithm.
that provided a robust DHT. In their approach, each virtual address is maintained by a committee of \( O(\log n) \) nodes. An idea which we will reuse.

In more recent works all of the nodes are subjected to churn and not only a fixed set. However, they do not consider byzantine behavior. The adversary in these papers can be described by three properties: The lateness, the churn rate, and if it is immediate. We say adversary is \((a, b)\)-late if it has full knowledge of the topology after \( a \) rounds and complete knowledge after \( b \) rounds. The churn rate is \((C, T)\) if the adversary can perform \( C \) join/leaves in \( T \) rounds. Last, an adversary is immediate if churned out nodes have to leave the network immediately and without the possibility to send and receive more messages. Table 1 shows an overview over the different models. Note that the table is only for comparison as it simplifies some of the models and does not depict all of their respective nuances. However, these simplifications do not weaken the adversary.

Augustine et al. [3] present an algorithm that builds and maintains an overlay in the presence of a nearly completely oblivious adversary. Here, the overlay no longer has a fixed structure but is an unstructured expander graph of constant degree. Note that this overlay has no virtual addressing. However, in [5] the authors present a scheme that allows to quickly search for data in these networks.

Further, Drees et al. [4] build a structured expander, a so-called \( H_d \)-Graph, which is the union of \( d \) random rings. Their adversary is not only \( O(\log \log n) \)-late with regard to communication, it also has access to all nodes’ memory and all sent messages after \( O(\log \log n) \). Nodes that are churned out in round \( t \) may remain in the network until some round \( T \in O(t + \log \log n) \). Thus, it is not immediate.

Last, the SPARTAN framework presented in [2] probably bears the greatest resemblance with our work. In SPARTAN the nodes maintain a logical overlay resembling a butterfly network. To ensure robustness each of the butterfly’s virtual nodes is simulated by \( O(\log n) \) nodes. The key difference between our work and SPARTAN is the adversary’s lateness. Just as [4] SPARTAN assumes the adversary to be \( (O(\log \log n), O(\log \log n)) \)-late, but in return allows the churn to be as high as \( \alpha n \) in \( O(\log \log n) \) rounds. However, other than [4] SPARTAN allows the adversary to be immediate.

### 1.3 Our Contribution & Organization of this Paper

In this work we deal with the problem of maintaining a routable network in spite of adversarial churn. A network is routable, if each node in every round is able to send a message to a given logical address \( p \in [0, 1) \). That means, given a dynamic set of nodes \( V := (V_1, V_2, \ldots) \) chosen by an adversary we propose an algorithm that creates a series of graphs \( \mathcal{G} := (G_1, G_2, \ldots) \) with \( G_i := (V_i, E_i) \) such that with high probability (w.h.p.) it holds for \( O(n^k) \) rounds that each \( G_i \) is routable.

This paper is organized as follows.

- In Section 2 we show that our model assumptions are necessary in order the solve the problem. In particular, we show that any adversary can partition a network where nodes can join via nodes that themselves just joined one round ago. Further, we prove that our model requires the adversary to be at least 1-late with regard to the topology.
- In Section 3 we introduce the Linearized DeBruijn Swarm (LDS). This graph topology is based on the linearized DeBruijn Graph presented by Richa et al. [9] and the concept if swarms used by Fiat et al. for the Chord overlay network [7].
- In Section 4 we present a routing algorithm for the LDS, which optimizes the congestion if we want guaranteed message delivery. In this section, we also define when a dynamic overlay is routable.

---

2 Throughout this paper w.h.p. means \( \Theta(1 - \frac{1}{n^k}) \) with a tunable constant \( k \).
Section 5 contains this paper’s main contribution, an algorithm that rearranges the graph topology such that it is completely rebuilt every 2 rounds but still allows routing. The message complexity is $O(\log^3 n)$ messages per node and round w.h.p.\(^3\)

1.4 Preliminaries

In this section we present some definitions and results from probability theory that we will use in the analysis of our algorithms. During our analysis, we deal with both dependent and independent random variables. The following general class of random variables will prove to be useful:

**Definition 1 (Negative Association).** Let $X := \{X_i\}_{i \in I}$ be a set of random variables. Further, let $f, g$ be monotonically increasing (or decreasing) functions defined on disjoint subsets of $X$. Then is $X$ is negatively associated (NA) if it holds:

$$E[f(X), g(X)] \leq E[f(X)] \cdot E[g(X)]$$

Note that all independent random variables are NA.

Last, we make use of a standard trick from randomized algorithms, the Chernoff Bounds, defined as follows,

**Lemma 2 (Chernoff Bounds).** Let $X := \sum X_i$ be the sum of negatively correlated random variables with $X_i \in \{0, 1\}$ and let $E[X] = \mu$. Then it holds

$$P[X \geq (1 + \delta)\mu] \leq e^{\frac{-\delta^2\mu}{2}} \quad \text{and} \quad P[X \geq (1 - \delta)\mu] \leq e^{\frac{-\delta^2\mu}{2}}$$

2 Impossibility Results and Lower Bounds

In this section, we present two fundamental impossibilities with regard to our model. First, we show that it is impossible to maintain an overlay under massive churn and a $(0, \infty)$-late adversary that always has up-to-date information about the topology, but is oblivious of everything else. Second, we show the necessity that new nodes can only join via bootstrap nodes that are in the network for at least 2 rounds.

We begin with the $(0, \infty)$-lateness. The result is stated in the following lemma.

**Lemma 3.** A $(0, \infty)$-late adversary with churn rate $(\alpha n, O(\log n))$ can disconnect any overlay in $O(\log n)$ rounds.

Proof. Let the execution start at round 0 and let $V_0$ be the initial set of nodes with $|V_0| := n$. Now consider the following strategy: Let a node $v$ join the network in round 0. Further, let a node $w$ join via $v$ in round 2. We will show that within $O(\log^2 n)$ rounds, a 0-late adversary can separate $w$ from the network. Denote $D_2 \subset V_0$ as all node $v$ communicates with in round 2. As $v$ is the only node that knows $w$ in round 2, it holds that $w$ can only be known by nodes from $D' := D_2 \cup \{v\}$ in round 3. On the other hand, $v$ may have sent a set of IDs $D_w \subset V_0$ to $w$ in round 2. These are the only nodes that $w$ knows.

Let now all nodes $D'$ be churned out in round 3, i.e. before they can communicate with any more nodes. Then, it holds that no node in the network knows $w$. Further, $w$ knows only the IDs received from $v$ upon it join. Now continue as follows. In each round until all nodes from $V_0$ are gone:

1. Churn out each node $w$ communicates with. This ensures that no new node will learn $w$’s ID.

\(^3\) Note that we do not seek to optimize message complexity
2. Churn out as much nodes from \( V_0 \) as possible and churn in the same amount of new nodes. Note that the total number of nodes does not change.

One can easily verify that within \( O(\log^2 n) \) rounds all nodes from \( V_0 \) are gone. Since all ID that \( w \) knows belong to nodes from \( V_0 \) and no node in the network knows \( w \), it is separated from the network. This was to be shown.

We continue with the restrictions for the joining nodes. The result is stated in the following lemma.

\[ \textbf{Lemma 4.} \text{ Let } v \in V \text{ be a node that joined in round } t. \text{ Now assume a model where in round } t+1 \text{ a new node } w \in V \text{ can join the network via } v. \text{ Then a } (\alpha, \infty) \text{-late adversary with churn rate } (cn, O(\log n)) \text{ can disconnect any overlay after } O(\log n) \text{ rounds.} \]

\[ \textbf{Proof.} \text{ For simplicity we assume that the adversary has no informations whatsoever about the communication between the nodes. Further, let the execution start at round } 0 \text{ and let } V_0 \text{ be the initial set of nodes with } |V_0| := n. \text{ The proof consists of two parts. First, we show that the adversary can create a situation that a node } v \text{ joins the network in round } T \in O(\log n \log\log n) \text{ and only receives IDs of nodes } w \in V_0. \text{ Second, we present a strategy that churns out all nodes form } V_0 \text{ within } O(\log n \log\log n) \text{ rounds. We now present the strategy for either claim:} \]

1. Consider the following strategy: Let \( V' := v_0, \ldots, v_T \) be a set of nodes such that each \( v_i \in V' \) with \( i > 1 \) joins the network in round \( i \) via \( v_{i-1} \). Further, each \( v_i \in V' \) is churned out between round \( t+1 \) and \( t+2 \), i.e., immediately \( v_{i+1} \) joined.

Further, let \( D_i \subset V \) be set of all IDs that \( v_i \) it knows in round \( i+1 \). We now claim that it holds \( D_i \subset D_1 \cup \{v_{i-1} \} \) for all \( i > 1 \).

We proof the claim via induction: For the induction’s beginning consider \( v_1 \). Here, the claim trivially holds as it just the definition of \( D_i \). For the induction’s step assume that the claim holds for \( v_i \). Now consider the join of \( v_{i+1} \) in round \( i+1 \). Any message that reaches \( v_{i+1} \) in round \( i+2 \) must be sent in round \( t+1 \). However, in this round only \( v_i \) knows \( v_{i+1} \) and therefore only \( v_i \) may share the references with \( v_{i+1} \). Thus, \( D_{i+1} \) can only be a subset of \( D_i \cup \{v_i \} \). By the induction’s hypothesis we know that \( D_0 \subset D_1 \). This proves the claim.

2. Now consider the following strategy: Every \( O(\log n) \) rounds the adversary churns out \( cn \) nodes \( w \in V_0 \). The adversary may choose these nodes uniformly at random. Further, For every node that is churned out, a new node is churned in. Recall that in the first strategy \( v_{i-1} \) is churned out when \( v_{i+1} \) is churned in. This way, the number of nodes in the system stays exactly \( n \). One can easily verify that after \( O(\log n \log\log n) \) many rounds all nodes of \( V_0 \) are churned out. This was to be shown.

The lemma now simply follows from the combination of the two strategies: Let a node \( v_T \) join in round \( T \in O(\log n \log\log n) \) and let it only know references to churned out nodes in round \( T+1 \). Now let a new node \( v_{T+1} \) join via \( v_T \). Then \( v_T \) cannot introduce \( v_{T+1} \) to any node currently in the network. This was to be shown.

Note that this impossibility is different from the similar statement in [3] because we allow a node to communicate with \( O(\log n) \) different nodes instead of constantly many.

### 3 The DeBruijn Swarm

In this section, we present our overlay, the Linearized DeBruijn Swarm (LDS). For this, we combine a well-analyzed graph class of low degree, i.e. the Linearized DeBruijn Graph (LDG) presented in [9] and [10] with techniques from robust overlays, i.e., the usage of logarithmic size quorums called swarms. The concept of these swarms was proposed in [7].
Given two nodes \( v \) and \( w \), denote their closest neighbor of \( v \) and \( w \) by \( \lambda \), respectively. Recall that in a classical DeBruijn Graph it would only be connected to the nodes left and right of \( v \). For convenience we assume throughout this work that each node knows \( n \) and \( \kappa \), i.e, the upper and lower bound for the nodes currently in the network. We make this simplification due to space constraints.

Note that throughout this work we assume that each node knows \( n \) and \( \kappa \), i.e the upper and lower bound for the nodes currently in the network. We make this simplification due to space constraints and the fact that the number of nodes stays relatively stable. Furthermore, we denote \( \lambda := \log \kappa n \) for easier notation. All of our presented algorithms may be adapted to work with close estimates of \( \lambda \) and \( \kappa \). For this one could use the approaches presented in [9, 7, 11, 12].

In the remainder of this section we present our overlay’s general topology and show some of its basic properties: To define the edges of the LDS, each node \( v \in V \) chooses a position in \( p_v \in [0,1) \) independently and uniformly at random. Whenever we want to use the position of a node \( v \in V \), we just write \( v \) instead of \( p_v \) for convenience. It should always be clear from the context if we mean the node, its ID, or its (current) position. If it is important to distinguish between these properties, we will make it clear.

All nodes can calculate the distance to another node via the distance function \( d : V^2 \rightarrow [0,1) \). Given two nodes \( v, w \in V \), the function \( d \) returns the shortest distance between \( v \) and \( w \) in the \([0,1)\)-ring. Formally, \( d \) is defined as follows.

\[
d(v,w) := \begin{cases} |v-w| & \text{if } |v-w| \leq \frac{1}{2} \\ 1-|v-w| & \text{otherwise} \end{cases}
\]

For convenience we introduce the following notions: Consider two nodes \( u, v \) with with \( |u-v| \leq \frac{1}{2} \). Then \( u \) is left of \( v \) if \( u < v \) and right otherwise. If it holds \( |u-v| > \frac{1}{2} \) the relation is reversed. Further, the set \( \langle v, w \rangle \subset V \) contains all nodes which are left of \( v \) and also right of \( w \). Further, given some node \( w \) and two nodes \( u, v \), we say that \( u \) is closer to \( w \) than \( v \) if \( d(u,w) < d(v,w) \). We call a node \( u \) the closest neighbor of \( v \) if there are no other nodes that are closer to \( v \) than \( u \).

For a given point \( p \in [0,1) \) we call \( S(p) \subset V \) the swarm of \( p \). It holds \( v \in S(p) \) if and only if \( d(v,p) \leq \frac{c_1}{n} \). Note that \( c > 1 \) is a robustness parameter which should be chosen as small as possible. These swarms (and not the nodes) will be the building blocks of our overlay. We call the swarms \( S(p) \) adjacent to \( S(p') \) if there is an edge \( \langle v,w \rangle \) between each node \( v \in S(p) \) and \( w \in S(p') \). Formally linearized DeBruijn Swarm is then defined as follows:

**Definition 5 (Linearized DeBruijn Swarm).** Let \( V \subset [0,1) \) be a set points with \( |V| = n \) and \( \lambda := \log n \). Then, the LDS \( G_C := (V,E_L \cup E_{DB}) \) with parameter \( C \in \mathbb{N} \) has the following properties:

\[
\begin{align*}
& (v,w) \in E_L \leftrightarrow w \in V \text{ and } d(v,w) \leq \frac{2\lambda}{n}.
& (v,w) \in E_{DB} \leftrightarrow w \in V \text{ and } d(\frac{v+i}{2}, w) \leq \frac{3\lambda}{2n} \text{ with } i \in \{0,1\}.
\end{align*}
\]

\(4\) For convenience we assume throughout this work that \( \lambda \) is an integer.

![Figure 1 Sketch of a Linearized DeBruijn Swarm. A node \( v \) is connected each node in red areas. Note that the swarms \( S(v), S(\frac{v}{2}), \) and \( S(\frac{v+1}{2}) \) are real subsets. These sets are chosen such that they contain \( O(\log n) \) nodes w.h.p. Recall that in a classical DeBruijn Graph it would only be connected to the nodes left and right of \( v, \frac{v}{2} \) and \( \frac{v+1}{2} \).](image)
In other words, in a LDS each node in the classical LDG also knows all nodes within an interval of size $\frac{1}{2} \frac{\lambda}{n}$ to the left and right. This is illustrated in Figure 1. Over the course of this paper we will refer to the edges in $E_L$ as list edges, whereas the edges in $E_{DB}$ are long-distance edges. Note that according to the definition each node has connections to all nodes in an interval that is bigger than its swarm. We see the reason for this in the following lemma:

**Lemma 6 (Swarm Property).** Consider any point $p \in [0, 1)$ and its swarm $S(p) \subset V$. Then $S(p)$ is adjacent to $S(\frac{p}{2})$ and $S(\frac{p + 1}{2})$

**Proof.** W.l.o.g. we only observe $S(\frac{p}{2})$. The other case is analogous. Let $p \in [0, 1)$ be any point and $v \in S(p)$ be a node in $p$’s swarm. This implies that $d(p, v) \leq \frac{c \lambda}{n}$. We now wish to show that $v$ holds a connection to each node in $S(\frac{p}{2})$. Therefore we distinguish between two cases:

1. If $|p - v| \leq \frac{1}{2}$ it holds:

   $$d\left(\frac{p}{2}, v\right) = \frac{1}{2} |x - v| = \frac{1}{2} |v - v| \leq \frac{c \lambda}{2n}$$

   Let now $u$ be any node in $S(\frac{p}{2})$ then the distance between $u$ and $\frac{p}{2}$ is at most $c \lambda$. Given our observations from before, it holds the distance between $u$ and $\frac{p}{2}$ is at most $\frac{1}{2c \lambda}$ following the triangle inequality. Since $v$ has an edge to each node in distance $\frac{1}{2c \lambda}$ or less to $\frac{p}{2}$ the lemma holds.

2. Otherwise, it holds $|p - v| > \frac{1}{2}$. This case can only appear if either $p \in [0, \frac{1}{2}]$ or $p \in \left[1 - \frac{1}{2c \lambda}, 1\right]$, i.e. the point $p$ is close to 0 or 1 and $v$ lies on the opposite of the interval. Now we distinguish between these two cases

   a. If $p \in [0, \frac{1}{2}]$ then it also holds that $\frac{p}{2} \in [0, p]$. Thus, it holds $d(\frac{p}{2}, v) \leq d(v, p) \leq \frac{c \lambda}{n}$. By the triangle inequality, it holds for every node $u \in S(\frac{p}{2})$ that $d(u, v) \leq d(u, \frac{p}{2}) + d(\frac{p}{2}, v) \leq \frac{c \lambda}{n}$. Thus, the node $u$ is a list neighbor of $v$.

   b. Otherwise $p \in \left[1 - \frac{1}{2c \lambda}, 1\right]$ and it hold that $\frac{p}{2} \in \left[\frac{1}{2} - \frac{1}{2c \lambda}, \frac{1}{2}\right]$. Now consider the distance between $\frac{p}{2}$ and $\frac{p + 1}{2}$ (and not $\frac{1}{2}$). Here, it holds

   $$d\left(\frac{p + 1}{2}, \frac{p}{2}\right) = \frac{1}{2} |v + 1| - \frac{1}{2} |v - 1| - p$$

   Observe that since $v < p$ and $p < v + 1$ it holds that $|(1 + v) - p|$ is equivalent to $1 - |v - p|$. Therefore, the the inequality simplifies to

   $$d\left(\frac{p + 1}{2}, \frac{p}{2}\right) = \frac{1}{2} (1 - |v - p|) = \frac{1}{2} d(v, p)$$

   Since $\frac{1}{2} d(v, p) \leq \frac{c \lambda}{n}$ we can again apply the triangle inequality and the lemma follows.

**Proof of Lemma 13** This lemma directly follows from the routing’s correctness and the overlay’s topology. We prove each statement seperately:

1. Given our routing algorithm described in section X works correctly, in round $t$ each node in $S_t(p_v)$ received $(v, p_v)$ w.h.p.

2. For the second statement w.l.o.g. only observe $S_t(\frac{p_v}{2})$. The other case is analogous. According to the swarm property, each node in $S_t(p_v)$ has a connection to each node in $S_t(\frac{p_v}{2})$. Note that w.h.p. at least $C \lambda$ nodes $S_t(p_v)$ are not churned out and will forward the message. This implies that in round $t + 1$ it holds that each node in $S_t(p_v)$ received a reference to $v$. 
3. Let \( l \in S_t(p_v) \) any node of \( p_v \)'s swarm that is a. left of \( p_v \), and b. survives round \( 2t \). According to Lemma 17 there are at least \( \frac{C_f}{4} \) such nodes \( w.h.p. \). Recall that \( l \) knows all nodes in \( [l + \frac{2C_k}{n}] \), which includes all nodes in \( [p_v + \frac{2C_k}{n}] \) since \( d(p_v, l) \leq \frac{C_f}{4} \). Thus, in round \( 2t \) the node \( l \) will forward \((v, p_v)\) to all nodes in \([p_v + \frac{2C_k}{n}]\). We can make an analogous proof for the right side.

4 Routing and Sampling in the LDS under Churn

In this section, we present a low-congestion routing algorithm \( \mathcal{R}_{\text{Routing}} \). The algorithm delivers each message \( w.h.p. \) even in the presence of churn and a changing communication structure. We furthermore present a sampling algorithm \( \mathcal{R}_{\text{Sampling}} \) that allows each node to send a message to a uniformly picked random node. The underlying technique is adapted from King et al. [11] [12]

4.1 Preliminaries

Before we go into the details of our algorithm, we will first recall the classical LDG’s routing algorithm (cf. [9] [10]), which our algorithm is based on. Routing in the LDG works by the bitwise adaption of the target address: Recall that each node knows \( \lambda \). Given any destination \( p \in [0, 1) \) a node can calculate the first \( \lambda \) bits \((p_1, \ldots, p_\lambda)\) of \( p \). Then, starting with the least significant bit \( p_\lambda \), the node \( v \) sends the message to the node closest to \( v' := \frac{v + p_\lambda}{2} \). For this, it uses the corresponding long-distance edge. After that, the message is sent to the node closest to \( v'' := \frac{v' + p_\lambda - 1}{2} \). This goes on until the first bit \( p_1 \). After that, there are \( w.h.p. \) only \( O(\log n) \) hops over list edges left to \( d \). Formally this path can be described by the so-called trajectory, which is defined as follows:
Definition 7 (Trajectory). Let \( v \in V \) be a node and \( p \in [0,1) \) be an arbitrary point. Further let \((v_1, \ldots, v_\lambda) \in [0,1)^\lambda\) and \((p_1, \ldots, p_\lambda) \in [0,1)^\lambda\) be the \( \lambda \) most significant bits of \( v \) and \( p \) respectively. Then the trajectory \( \tau(v,p) := x_0, \ldots, x_{\lambda+1} \in [0,1)^{\lambda+1} \) is a series of points defined as follows:

\[
x_i := \begin{cases} v & \text{if } i = 0 \\ (p_{i-\lambda+1}, \ldots, p_i, v_{0}, \ldots, v_i) & \text{if } i \leq \lambda \\ p & \text{if } i = \lambda + 1 \end{cases}
\]

Given this definition, the classical Linearized DeBruijn Routing can described as follows: For each point \( x_i \) the trajectory, forward the message to the node closest to it. Then, forward the message along list edges until it reaches the target. We now wish to adapt this algorithm for a dynamic series of graphs \( D := (D_1, D_2, \ldots) \) where each \( D_i \) is a LDS. The obvious solution would be to send the message not only to the closest node of each trajectory point but to the whole swarm. However, there are two problems we need to address, the churn and the dynamic reconfiguration of the overlays.

First, one can easily see that this routing algorithm fails in the presence of churn. Given that any node on a message’s trajectory can be churned out, a fraction of routing requests may never reach their destinations. In particular, if the adversary is aware of the topology, it could even churn out the whole swarm for a given trajectory point. Thus, we introduce the notion of a good swarm adapted from \[7\]. A swarm is good if at least a constant \( \frac{3}{4} \)-fraction of its nodes take part in the next round. We call these nodes good. Further, a LDS is good if all its swarms are good. This property implies that there is always at a constant fraction of good nodes in each swarm that can forward the message. Note that the value \( \frac{3}{4} \) is only chosen for an easy analysis.

Besides the churn there is the problem of the dynamically rearranging overlay. In particular, our algorithm will create a series of overlays \( (D_0, D_1, \ldots) \), which will persist for only 2 rounds each. That means a node changes its position every 2 rounds. If now every node would keep all its routing requests and forward them from its new position, they would lose all the progress they made so far. This also needs to be reflected in our routing algorithm: Therefore, we define the so-called handover using a helper graph \( H_i \). For any point \( p \in [0,1) \) let \( S_i(p) \) be the swarm of \( p \) in \( D_i \) and likewise let \( S_{i+1}(p) \) be the swarm of \( p \) in \( D_{i+1} \). We assume that during the change from \( D_i \) to \( D_{i+1} \) each node from \( S_i(p) \) can send a message to any set of nodes from \( S_{i+1}(p) \), i.e., the nodes from a helper graph \( H_i \) where the swarms \( S_i(p) \) and \( S_{i+1}(p) \) are adjacent. Therefore, the switch can be handled like every other routing step from one swarm to another. Later, in Section 5 we will see how to implement the handover in an algorithm whereas here we just treat it as a property for a simpler description. Last, note that we call a graph \( H_i \) good if for each \( p \in [0,1) \) a \( \frac{3}{4} \)-fraction of all nodes in \( S_{i+1}(p) \) is not churned out in the next round.

We summarize our observations in following definition for a routable series of graphs:

Definition 8 (Routable Graphs). Let \( \mathcal{D} := (D_1, H_1, D_2, H_2, \ldots) \) be series of graphs defined on nodes \( \mathcal{V} := (V_1, V_2, \ldots) \). Then we call \( \mathcal{D} \) routable, if

1. each \( D_i \) is a LDS,
2. each \( H_i \) enables is a handover from each \( D_i \) to \( D_{i+1} \), and
3. each \( D_i \) (and \( H_i \)) is good, i.e. it holds \(|S_i(p) \cap V_{i+2}| \geq \frac{3}{4} |S_i(p)| \) for all \( p \in [0,1) \).

4.2 The Routing Algorithm \( \mathcal{D} \)-Routing

We now present a routing algorithm \( \mathcal{D} \)-Routing for a dynamic series of routable graphs \( \mathcal{D} := (D_1, D_2, \ldots) \). With the routing algorithm sketched in the previous section, in each step the message is either sent to the whole swarm of each trajectory point or to each node in the handover swarm. However, forwarding a message to a whole swarm would require \( \mathcal{O}(\log^2 n) \) messages to be sent in each step. We
wish to limit this to only $O(\log n)$. Therefore, we adapt this approach as follows: Say, a node $v \in V_t$ wants to route a message $m$. First, $m$ is forwarded to all nodes in its swarm $S(v)$. Then, each node in the swarm picks $r \in \Theta(1)$ nodes in the next swarm $S(x_0)$ and sends $m$ to them. These $r$ nodes are picked uniformly and independently at random. Then, each node that received $m$ at least once, forwards it to $r$ nodes in $S(x_1)$ and so on. Only in the last step, the message is forwarded to all nodes of the target swarm to ensure that the whole swarm receives the message.

Listing 1 depicts the pseudocode for $A\mathcal{R}\mathcal{OUTING}$. There we see that the round in which the message was started is also transmitted to the target. A message that was sent in an even round but would be received in an odd round is held back one round. This ensures that messages started in an even/odd round are also received in an even/odd round. Note that this is not needed for the algorithm’s correctness. However, our maintenance algorithm in Section 5 makes use of this property for an easier description.

**Analysis**

In this section, we analyze $A\mathcal{R}\mathcal{OUTING}$. In particular, we observe the dilation, i.e. the number of steps until a message reaches its target, and the congestion, i.e. the number of messages handled by each node in a round. This greatly depends on how many messages are sent each round and how their destinations are chosen. In the following only consider that destinations are chosen uniformly at random. In the remainder of this section we will show the following result:

**Lemma 9.** Let $D := (D_1, H_1, \ldots)$ be a routable series of LDS defined on nodes $V := (V_1, V_2, \ldots)$. Further, let each node $v \in V_1$ start $k$ messages to random targets $p \in (0, 1)$. Then $A\mathcal{R}\mathcal{OUTING}$ with a suitable parameter $r \in \Theta(1)$ delivers each message with dilation exactly $2\lambda + 2$ and congestion $O(k \log n)$ w.h.p.

Note that due to space constraints we will only sketch the analysis. We refer to [?] for the proofs.

We begin with the following lemma:

**Lemma 10.** Let $D := (D_1, H_1, \ldots)$ be a routable series of LDS defined on nodes $V := (V_1, V_2, \ldots)$. Let $v$ be any node in $V_1$, which sends a message to point $p$ using $A\mathcal{R}\mathcal{OUTING}$. Further, let $\tau(v, p)$ be the message’s trajectory. Then it holds w.h.p. that the message is at $S(i)$ after exactly $2i$ steps (if it is not dropped due to churn).

The proof follows from an induction. In each step, the message is either forwarded along the trajectory or is handed over. Lemma 6 and the handover property imply that the nodes have necessary connections for each step but the last. Since the trajectory’s last point is with distance $\frac{2\lambda}{n}$ to $p$ w.h.p. for a big enough $c$, the lemma follows.

In the proof we assumed that the message is forwarded in each step. We omitted the fact that not all nodes of a swarm forward the message as they may be churned out before they can do that. Of course, if a complete swarm is churned out, the message surely can’t be forwarded. As stated before we assume that all swarms are good so that at most a constant fraction of each swarm is malicious and does not forward the message. Under this assumption it holds:

**Lemma 11.** Let $m$ be a message that is routed along $\tau(v, p) := x_1, \ldots, x_\lambda$ using $A\mathcal{R}\mathcal{OUTING}$. Then, it all nodes in $S(x_\lambda)$ receive $m$ w.h.p. after exactly $2\lambda + 2$ rounds for a suitable $r \in \Theta(1)$.

---

5 Given that $\alpha n$ nodes may fail in single round and we want to route each message on the first try, it reasonable that one needs $O(\log n)$ copies of a message each round to ensure the survival of at least one w.h.p.
The idea is to view the forwarding step as a balls-into-bins experiment where the messages are balls and the good nodes in the next interval are bins. One can easily verify that the number of bins that receive at least one ball is NA (see e.g. [13] for a proof). Thus, for $r$ big enough (but still constant) more than half of all good nodes in the swarm receive the message $w.h.p$. A simple induction then yields the lemma.

**Proof.** We proof the statement via induction on the steps $x_0, \ldots, x_{k+1}$. For the induction’s beginning consider $S(x_0)$. Since initially $v$ sends $m$ to all $w \in S(x_0)$ the hypothesis holds.

W.l.o.g consider the step from $S_j(x_{j-1})$ to $S_j(x_j)$, i.e. a forwarding step. For a handover step the proof is completely analogous. We can derive from Lemma[17] that it holds $S := \|S(x_{j-1})\| \geq 1/4|S(x_j)|$ w.h.p. Furthermore, by the induction hypothesis we know that at least half of all nodes in $S_j(x_{j-1})$ forward $r$ copies of $m$. Thus, in total there are at least $K := \lceil r/|S_j(x_j)| \rceil$ copies of $m$ sent to $S_j(x_j)$. We enumerate these copies $m_1, \ldots, m_K$. For any node $v \in S(x_j)$ and any copy $m_i$ let $X_{m,v} \in \{0, 1\}$ be the random variable that $m_i$ is send to $v$. Now consider the set of variables $X_m := \{X_{m,v} : v \in S(x_j)\}$. Surely, each message $m_i$ can only be send to at most one node. Therefore, at most one $X_{m,v}$ is. Since all copies of $m$ are sent independently, it holds that $X := \{X_m : m \in K\}$ is also NA. For each node $v \in S(x_j)$ we now define the variable $Y_v \in \{0, 1\}$ that indicates whether $v$ received at least one message. Our goal is to show that $P\left[\sum Y_v \leq \frac{|S(x_j)|}{2}\right] \leq \frac{1}{2}$. We can define $Y_v$ as a monotone function $f$, such that

$$f(x_1, \ldots, x_K) = \begin{cases} 0 & \text{if } (x_1, \ldots, x_K) \leq (0, \ldots, 0) \\ 1 & \text{else} \end{cases}$$

Surely, this function is monotonically increasing and therefore $Y_v$ is NA by the closure properties. Since for two distinct nodes $v, w \in S(x_j)$ the corresponding function $f_v, f_w$ depend on disjoint subsets of $X$, the whole set $Y := \{Y_v : v \in S(x_j)\}$ is NA. Based on these observations we can view the forwarding step as a balls-into-bins experiment where $K$ balls are thrown into $|S_j(x_j)|$ bins. One can easily verify that the number of bins that receive at least one ball is NA (see e.g. for a proof). Thus, we see that the Chernoff Bound applies to $Y$ and in particular to any subset of it. Consider now all nodes in $S(x_j)$ that are not churned out. As per assumption there is a $(1 - \epsilon)$ fraction of good nodes in $S_j(x_j)$.

Thus, the expected number of good nodes that receive at least one message is

$$E[G] := \sum_{v \in G(x_j)} E[Y_v] \geq \sum_{v \in G(x_j)} \left(1 - \left(1 - \frac{1}{G}\right)^{|G(x_j)|}\right)$$

$$\geq \sum_{v \in G(x_j)} \left(1 - \frac{1}{e^{r/8}}\right) = \left(1 - \frac{1}{e^{r/8}}\right) 3/4S$$

Thus, for $r$ big enough at least half of all the good nodes get a message $w.h.p$. This can be shown via the Chernoff bound. This proofs the lemma.

We conclude the analysis by observing each node’s congestion. Therefore, we first bound the expected number of trajectories that cross an interval in each round. Note that a trajectory is defined on points in $[0, 1)$ and not on actual nodes except the first and last element. It holds:

**Lemma 12.** Let $X^f_j$ be the random variable that counts how many trajectories have their $f^{th}$ step $x_j \in I$. Then, it holds:

$$E[X^f_j] = knI, \text{ and}$$
\[ X_j^i \text{ is the sum of independent, binary RVs.} \]

**Proof.** We proof both parts of the lemma separately:

1. We proof the the first statement by induction over all steps.

   For the induction’s beginning consider the only the first step \( j = 0 \). Observe that a trajectory’s first step \( x_0 \) is always the message’s starting node. Since the expected number of nodes in \( I \) is \( In \) and each nodes starts exactly \( k \) messages, the statement follows easily.

   Next, consider the induction’s step. Therefore, assume that each interval \( J \in [0,1) \) contains -in expectation- \( knI \) tractors in step \( j - 1 \).

   Let now \( I := [a,b] \) be a arbitrary interval. Further, let \( I_0 := I \cap [0,1/2) \) and \( I_1 := I \cap [1/2, 1) \) be the parts of \( I \) that lie in the first and second half of \([0,1)\) respectively. Note that the bit representation of each point in \( I_i \) starts with \( i \). Thus, any point in \( I_j \) may be reached if a \( i \) is pushed in the \( j \)-th step of the trajectory.

   W.l.o.g. we only observe \( I_0 \) in the following. If any point \( x_j \in I_0 \) is the \( j \)-th step of some trajectory, then it must hold \( x_{j-1} := 2x_j \). This follows directly from the trajectories definition. Therefore, all trajectories that can potentially cross \( I \) in their \( j \)-th step, have their \( j-1 \)-th step in \( I_0 := [2a,2b] \). This interval’s length is

   \[ J_0 := 2b - 2a := 2(b-a) = 2I_0 \]

   By the induction’s hypotheses \( 2knI_0 \) trajectories have their \( j-1 \)-th step in \( J_0 \). As the probability to push a 0 in the \( j \)-th step is \( 1/2 \) the statement follows.

2. The second statement is straightforward. Let \( X_{vi} \in \{0,1\} \) be the RV that the \( j \)-th message started by \( v \) crosses \( I \) in its \( j \)-th step. Then the number of messages with their \( j \)-th step in \( I \) is defined as:

   \[ X_j = \sum_{v \in V} \sum_{j \leq k} X_{vi} \]

   We now wish to show that all \( X_{vi} \) are independent. A message’s trajectory is uniquely defined by starting node \( v \) and end point \( p \). Both of these values are chosen uninformly and independently at random. Thus, two variables \( X_{vi} \) and \( X_{vl} \) with \( v \neq w \) or \( i \neq l \) are independent. This was to be shown.

   Using this fact, we can now simply proof the congestion bound by application the Chernoff Bound.

   Let \( v \in V \) be any node and let \( [v \pm \frac{2\lambda}{n}] := I_v \subset [0,1) \) be an interval that contains all points \( p \) with \( v \in S(p) \). A message may routed via \( v \) only if its trajectory passes \( I_v \). Based on \( I_v \)’s size a simple application of the Chernoff Bound yields that \( O(k \log n) \) messages pass \( I_v \) w.h.p.

   In the worst case, each message is sent \( rS \) times where \( S \geq |S(v_j)| \) is an upper bound on nodes in the message’s previous swarm. Thus, the number of all copies is at most \( O(k \log nrS) \). In the remainder assume the number of copies to be \( K \).

   Let \( (X_i)_{1 \leq k} \) be a set of random variables that denote whether copy \( m_i \) is sent to \( v \). It holds \( P[X_i = 1] \in \theta(\frac{1}{k}) \) since all swarms are roughly the same size w.h.p and message is sent with equal probability to all nodes and therefore \( E[\sum X_i] \in \theta((k \log nr)) \). Thus, the lemma follows from the Chernoff bound since \( r \) is a constant.
4.3 The Random Sampling Algorithm $\mathcal{S}_{\text{sampling}}$

Besides routing to a random swarm $S(p)$ for some $p \in [0, 1)$ the algorithm $\mathcal{S}_{\text{routing}}$ can also be extended to send a message to a node chosen uniformly at random. We call this algorithm $\mathcal{S}_{\text{sampling}}$. The underlying approach is adapted from King et al. in [11, 12] and works as follows. A node picks a random destination $p \in [0, 1)$ and a random number $\Delta \in [0, 2\lambda \lambda]$. The sampling then proceeds in two steps: The node first routes the message to the swarm $S(p)$ using algorithm. Then, the message is only delivered to the node $w \in S(p)$ for which it holds $|\{u | u \in \langle p, w \rangle\}| = \Delta$. If there is no such node, the message is discarded. Listing 2 depicts the pseudocode for $\mathcal{S}_{\text{sampling}}$ if executed on a routable series of LDS. Formally, it holds:

**Lemma 13** (see [11, 12]). Let $\mathcal{G}$ be routable. Assume, a node $v \in V_i$ starts a message $m$ using $\mathcal{S}_{\text{sampling}}$. Then it holds:

1. $\forall v, w \in V_{i+2\lambda+2}: P[v \text{ receives } m] = P[w \text{ receives } m]$
2. $P[m \text{ is discarded}] \leq \frac{1}{4}$

5 The Maintenance Algorithm

In this section we present our main contribution, an algorithm that constructs a routable dynamic overlay $\mathcal{G} = (D_0, H_0, D_1, \ldots)$. In particular, we assume a $(2, 2\lambda + 7)$-late adversary with a churn rate \( \left(\frac{\alpha}{n}, 4\lambda + 14\right) \). Further, we assume that the number of nodes is at most $(1 + 1/16)n$. Note that the values $\alpha = 1/16$ and $\kappa = (1 + 1/16)$ are chosen for convenience.

In this section we assume that the system starts in an initial LDS $D_0$ in round 0. This assumption is made for convenience as the initial overlay can be easily constructed in the churn-free bootstrap phase using algorithms from [14]. Using these techniques this can be archived in $O(\log^2 n)$ rounds with a deterministic algorithm. Furthermore, the congestion and degree of each node is polylogarithmic, so it fits into our computational model. Since our focus lies on fast reconfiguration and not on optimizing the bootstrap phase we omit the algorithmic details and their analysis. For ease of notation we will refer to round $t + B$ simply as $t$. Last, we assume the bootstrap phase to be $2\lambda + 7$ rounds long.

If we want to refer to all nodes except the newly joined in a round $t$ we write $V_t := V_t \cap V_{t-1}$. Over the course of this chapter we distinguish between two types of nodes in each round $t$: First, there are all nodes that are in the network for more than $\lambda' = 2\lambda + 4$ rounds (or $2\lambda + 5$ if they joined in an even round). We call these nodes matured. We denote these as $\mathcal{M}_t \subseteq V_t$. Second, we have the remaining nodes that joined the network less than $\lambda'$ rounds ago. We call these nodes fresh. We denote these as $\mathcal{F}_t \subseteq V_t$. Note that $V_t := \mathcal{M}_t \cup \mathcal{F}_t$.

---

### Listing 2: Random Sampling $\mathcal{S}_{\text{sampling}}$

1. **Desci**: This algorithm is executed on a routable graph $\mathcal{G} := (D_t, H_t, \ldots)$. It routes a message $m$ from any node $u \in V_t$ to a node $v$ uniformly picked from $V_{i+\lambda}$ (or discards it with prob. $p \leq \frac{1}{2}$).
2. **Upon sending a message $m$ to a random node $v \in V_{i+\lambda}$**
3. $p \leftarrow $ Uniformly chosen from $[0, 1)$
4. $\lambda \leftarrow $ Uniformly chosen from $[0, 2\lambda\lambda]$
5. Route message $(m, \lambda, p)$ to target $S(p)$ using $\mathcal{S}_{\text{routing}}$
6. Upon receiving $(m, \lambda, p)$ from $\mathcal{S}_{\text{routing}}$
7. $P \leftarrow $ $\{w \in S(p) | p_w \in (p, w)\}$
8. **if** $|P| = \lambda$ **then**
9. Deliver $m$
10. **else**
11. Discard $m$
5.1 Algorithm Description

On a high level our algorithm has two building blocks, the algorithms $\mathscr{A}_{\text{LDS}}$ and $\mathscr{A}_{\text{RANDOM}}$ which are executed in parallel. The two algorithms ensure the following two properties:

1. After the bootstrap phase, $\mathscr{A}_{\text{LDS}}$ creates a series of overlays $\mathcal{D} = (D_0, H_0, D_1, H_1, \ldots)$ that contains all mature nodes in a given round.

   In particular, in each even round $2r$ the algorithm constructs a LDS $D_r$ which consists of all nodes $V_{2r-\lambda}$. In each odd round $2r + 1$ the algorithm creates an intermediate graph $H_r$ in which for each $p \in [0, 1)$ it holds that $S_r(p)$ and $S_{r+1}(p)$ are adjacent. Note that this ensures that in the even rounds the forwarding step from $\mathscr{A}_{\text{ROUTING}}$ can be performed and the handover in the odd rounds. Thus, $\mathscr{A}_{\text{LDS}}$ maintains a routable overlay.

2. $\mathscr{A}_{\text{RANDOM}}$ ensures that (after the bootstrap phase) each fresh node $v \in F_r$ is known to $\delta \in O'(\log(n))$ mature nodes which are part of $\mathcal{D}$. This ensures that these node can route a message on behalf of the fresh nodes via the overlay $\mathcal{D}$.

We now describe the two subroutines separately.

Maintaining a Routable Overlay via $\mathscr{A}_{\text{LDS}}$

In this section we describe the maintenance algorithm $\mathscr{A}_{\text{LDS}}$ for the dynamic graph $\mathcal{D} = (D_1, H_1, \ldots)$. The algorithm’s basic idea is to construct a new overlay $D_t$ in every even round $2r$ after the bootstrap phase. In an odd round $2r + 1$ a handover between $D_r$ and $D_{r+1}$ is performed using a helper graph $H_r$. In particular, each $D_t$ is implied by the $ID$s of all nodes $V_{2t-\lambda}$.

On a high level the algorithm works as follows. In each even round $2r$ (this includes even rounds in the bootstrap phase) each node $v \in V_{2r}$ starts a message $(v, p_v^{\lambda+1})$ containing its $ID$ to a random destination $p_v^{\lambda+1} \in [0, 1]$ using algorithm $\mathscr{A}_{\text{ROUTING}}$. We call such a tuple $(v, p_v^{\lambda+1})$ a join request. The point $p_v^{\lambda+1}$ will be node’s new position in the overlay $D_{t+\lambda+1}$ in round $2r + 2\lambda + 2$. Note that

---

**Listing 3: Overlay Maintenance Algorithm $\mathscr{A}_{\text{LDS}}$**

- **Upon receiving $\text{CREATE}(v, p_v^2)$ from $u'$**
  - $D_v^2 \leftarrow D_v^2 \cup \{(v, p_v^2)\}$
  - $u'$ creates edges to these nodes

- **Upon receiving $\text{JOIN}(v, p_v^3)$ from $\mathscr{A}_{\text{ROUTING}}$**
  - Send $\text{JOIN}(v, p_v^3)$ to all nodes $w \in D_v^2$ with $p_v^3 \in \{p_v^2 + 2\lambda, p_v^2 + \frac{2\lambda}{3}, p_v^2 + \frac{\lambda}{3}, p_v^2 - \frac{\lambda}{3}, p_v^2 - \frac{2\lambda}{3}\}$
  - Perform Forwarding Step from $\mathscr{A}_{\text{ROUTING}}$ using edges created from $D_v$
  - $C \leftarrow$ All fresh nodes known by $v$ (provided through $\mathscr{A}_{\text{RANDOM}}$)
  - $\forall v \in C \cup \{u\}$ do
    - $p_v \leftarrow h(v, t)$
  - Route message $\text{JOIN}(v, p_v^{t+1})$ to target $p_v^{t+1}$ using Algorithm $\mathscr{A}_{\text{ROUTING}}$

- **Upon receiving $\text{JOIN}(v, p_v^2)$ from a node $u$**
  - $u$ creates edges to these nodes

- **Upon receiving $\text{JOIN}(v, p_v^2)$ from $\mathscr{A}_{\text{ROUTING}}$**
  - Send $\text{CREATE}(v, p_v^3)$ to all nodes $(w, p_w^3) \in H_v$ with $p_w^3 \in \{p_v^2 + 2\lambda, p_v^2 + \frac{2\lambda}{3}, p_v^2 + \frac{\lambda}{3}, p_v^2 - \frac{\lambda}{3}, p_v^2 - \frac{2\lambda}{3}\}$

---

**Note:** The following code is executed by each node $v \in M$, every even and odd round respectively. The messages are handled in the given order. The last block of commands in each phase is executed after all messages have been handled.
algorithm $\mathcal{A}_{\text{ROUTING}}$ ensures that all these messages arrive in the same round. In other words, all nodes concurrently re-join the network. Note that both the fresh and the mature nodes send out the join requests. Therefore, we assume that each fresh node is known by least one mature node, which is part of $D_t$. This, however, will be maintained by $\mathcal{A}_{\text{RANDOM}}$ and explained in the next section.

In the following assume that the mature nodes currently form $D_t$ in round $2t$ (after the bootstrap phase). Then the construction of the next overlay $D_{t+1}$ works as follows: Each node that receives a message $(v, p_v^t)$ that contains a node $v$ and its new position $p_v^{t+1}$ forwards to its all its neighbors in the current overlay $D_t$. These messages arrive at their targets in round $2t+1$. Thus, in round $2t+1$ each node in $D_t$ knows a set of nodes and their new positions in the next overlay $D_{t+1}$. Now all nodes iterate over all received tuples $(v, p_v^{t+1})$ and introduces it to all its neighbors in $D_{t+1}$ it knows.

By introduction, we mean that the neighbor’s $ID$ and position is sent to $v$. Then, in round $2t+2$ each node know its neighbors in new overlay $D_{t+1}$ and creates an edge to them. Thus, starting in round $2t+2$ all mature nodes from the overlay $D_{t+1}$.

In parallel the $\mathcal{A}_{\text{LDS}}$ also performs the forwarding and the routing step from $\mathcal{A}_{\text{ROUTING}}$. In each round where the nodes form $D_t$ they perform the forwarding step. Further, recall that in the odd steps all nodes in a swarm $S_t(p)$ receive the references of nodes in $S_{t+1}(p)$. Thus, the nodes form a helper graph $H_t$ and can perform a handover.

Observe that in the first $2\lambda - 2$ rounds of the bootstrap phase, the nodes perform nothing in the odd rounds since the join request did not yet reach their target. However, since the nodes form the overlay $D_0$ and there is no churn the handover step is not needed for routing.

Last, note that after round $2t+1$ no edge of $D_t$ is ever used again and the nodes’ positions in $D_t$ and $D_{t+1}$ are in no relation each other. Therefore, the adversary stays oblivious of all nodes’ current positions.

Listing 3 presents the pseudocode for the algorithm. Each node has the two variables $D_v^t$ and $H_v^t$. $D_v^t$ stores $u$’s neighborhood in $D_t$ whereas $H_v^t$ stores the references for the handover. Both variables may be reset at the end of each round. We have two types of messages, $JOIN(v, p_v^t)$ represents a join request and $CREATE(v, p_v^t)$ is used to construct $D_t$. Both contain the $ID$ of a node $v$ and its position in $p_v^t$ in $D_t$. Further, we see how the pick their random value. It is is determined by a uniform hash function $h: V \times N \rightarrow [0, 1]$ known to all nodes. This hash function takes the node’s $ID$ and the current round as an input and computes a random value $p_v$. The mature nodes send out requests on behalf of each fresh node $u \in F_t$ known to them. Note that each node can compute $h(u, t)$ if it knows $u$’s ID. The $IDs$ of these nodes are stored in the variable $C$. This variable is set by $\mathcal{A}_{\text{RANDOM}}$. Details on how it set can be found in the next section.

**Maintaining a Random Overlay via $\mathcal{A}_{\text{Random}}$**

In this section we present $\mathcal{A}_{\text{Random}}$, an algorithm that ensures that each fresh node is known by $\delta \in \Theta(\log n)$ randomly chosen mature nodes each round w.h.p. This algorithm also handles the adversarial join of nodes.

On a high level $\mathcal{A}_{\text{Random}}$ works as follows: In every round each mature node sends out $\tau \in \Theta(\log n)$ messages containing the node’s $ID$. We call such these messages tokens for short. The tokens are sent to random nodes using a variation of $\mathcal{A}_{\text{SAMPLING}}$. In particular, this ensures that each node $v \in V_t \cap V_{t-2}$ receives $\Theta(\tau)$ token w.h.p. in every round after the bootstrap phase. These tokens are then used by the fresh nodes to send their $ID$ to $\delta \in \Theta(\log n)$ random mature nodes. We distinguish between two cases:

1. Each fresh nodes $v \in F_t$ that is in the network for at least one round uniformly picks $\delta$ tokens it received in past rounds and sends its $ID$ to the corresponding nodes.
2. For each fresh node $v$ that just joined the network in round $t$, the bootstrapping node uniformly picks $\delta$ tokens it received and sends $v$’s ID to the corresponding nodes. Furthermore, the newly joined is supplied with $\delta$ tokens for the next round. Recall that we require $w \in V_t \cap V_{t-2}$.

Last, all unused tokens are discarded at the end of the round.

The distribution of tokens to the nodes works in three steps:

1. In the first step, the token is to send to a mature node picked uniformly at random. For this, the mature nodes use $\mathcal{A}_{\text{Random}}$.

2. Recall that the ID of each fresh node is sent to $\delta$ mature nodes. A mature node that receives an ID assigns it a unique number in $[0, 2\delta]$ such that each number is assigned to at most one ID. If more than $2\delta$ send their ID, a set of $2\delta$ IDs is picked at random and assigned to $[0, 2\delta]$.

3. For each token that a mature node $v \in M_t$ receives in step 1 a fair coin is flipped. With probability $p = 0.5$ $v$ keeps the token and uses it for the newly joined nodes. Otherwise, the algorithm uniformly picks a random number from $[0, 2\delta]$. If there was an ID assigned this number in step 2, the token forwarded is forwarded there. If not, the token is dropped. Note that this preserves the independence of each token as the coin is independently flipped for each token and choice of the random number is also independent.

Note that at the end of round $t$ resets the assignment of numbers to IDs. Last, note that the bootstrap phase only executes the first tokens reach their target.

Listing 4 depicts the pseudocode for $\mathcal{A}_{\text{Random}}$. We use two types of messages, $\text{TOKEN}(v)$ and $\text{CONNECT}(v)$. Both messages contain a nodes $v$’s ID. The former is used to spread the mature nodes’ IDs, the latter is used send a fresh node’s ID for sampling. Note that all that are ready to create an are stored in the variable $T$. Further, the array $(c_1, \ldots, c_{2\delta})$ stores the assignment.
of numbers to IDs. It holds $c_i = v$ if $v$’s ID is assigned to $i$. If no ID is assigned to $i$ we write $c_i = \perp$. Note that the set $C$ mentioned in Listing 3 consists of all $c_i \neq \perp$. Last, note that a node can distinguish whether it received a $\text{TOKEN}(v)$ message through Algorithm $\mathcal{S}_{\text{SAMPLING}}$, i.e., in step 1 of the sampling process sketched above, or directly from a node, i.e., in step 3.

The main result of this section (and this paper) is stated in the following theorem:

► **Theorem 14.** Algorithms $\mathcal{A}_{\text{LDS}}$ and $\mathcal{A}_{\text{RANDOM}}$ maintain a series of overlays $\mathcal{D} := (D_0, H_0, D_1, \ldots)$ such that for $O(n^{3})$ rounds

1. the mature nodes form a routable series of graphs $\mathcal{D}$, and
2. each fresh node is known by $\Theta(\delta)$ mature nodes.

Further, the congestion is $O(\log^3 n)$ per node and round.

We divide the analysis into three sections: First, we show that $\mathcal{A}_{\text{LDS}}$ maintains the first invariant. Second, we show that $\mathcal{A}_{\text{RANDOM}}$ ensures the other one. Last, we analyze the algorithms’ congestion. Due to space constraints we only present an outline of the analysis. All detailed proofs can be found in the full version [?].

### 5.1.1 Analysis of $\mathcal{A}_{\text{LDS}}$

In this section we show that $\mathcal{A}_{\text{ROUTING}}$ maintains a dynamic overlay with the properties needed for routing. In its essence the proof is an induction over each round $t$ showing that this holds in each round w.h.p. if it held in the previous. Throughout this section we assume that $\mathcal{A}_{\text{RANDOM}}$ works correctly and each fresh node is connected to $\Theta(\delta)$ mature nodes. Thus, for each fresh node a join request is started.

In the remainder of this section we will prove the induction step and show that if the algorithm maintained a routable dynamic graph $\mathcal{D}_t := (D_0, H_0, D_1, \ldots)$ until round $2t$ it will continue to do so w.h.p.. To guarantee the correctness of $\mathcal{A}_{\text{LDS}}$, we need to show that w.h.p.

► **Lemma 15.** Let $\mathcal{D}_t := (D_0, H_0, \ldots, D_t)$ be routable graph until round $2t$. Then it holds w.h.p.

1. $\mathcal{A}_{\text{LDS}}$ successfully forwards all messages in round $2t$,
2. successfully performs a handover from $D_t$ to $D_{t+1}$ in round $2t+1$, and
3. constructs a new graph $D_{t+1}$ in round $2(t+1)$.

For the induction beginning consider the bootstrap phase. Since in the bootstrap phase there is no churn the algorithm can trivially build the good graph $D_0$ and route all messages. Since in the bootstrap phase no handover and no reconfiguration happens, the other two points also follow trivially.

In the following, we assume that the algorithm is currently in round $2t$ (after the bootstrap phase). Further, the mature nodes know all neighbors in $D_t$, all messages started $\lambda$’ ago are delivered, and the nodes are ready to perform forwarding step from $\mathcal{A}_{\text{ROUTING}}$ for the remaining messages.

We first show that the routing steps are successful w.h.p.. In particular, we must show that $D_t$ is good during the forwarding step and also that $H_t$ is good in handover step, i.e., a $\frac{3}{4}$-fraction of nodes that will not be churned out. In the following, we focus on $D_t$, but the results can analogously be applied to to show that $H_t$ is good. We observe that the adversary is oblivious of the nodes’ positions in each $D_t$ before round $2t + 1$. Formally:

► **Lemma 16.** A $(2, 2\lambda + 7)$-late adversary is oblivious of $D_t$ until the end of round $2t + 1$. 
Lemma 17. Assume the swarm size to be $\frac{c}{2}$ with $c \geq 36k$. Let now $S_i(p) \subset V_{2t-\lambda}$ be a swarm in $D_t$. Further, let $G_i := S_i(p) \cap M_{2t+1}$ be the set of good mature nodes in $S_i(p)$. Then it holds:

$$ P \left( |G_i| \geq \frac{14}{17} |S_i(p)| \right) \leq \frac{1}{n^2} $$

Proof of Lemma 17. Let $|V_{2t-\lambda}| = m$. Consider an interval $I \subset [0, 1)$. For each node $v \in V_{2t-\lambda}$, let $X_v \in \{0, 1\}$ be the RV that indicates if $p_v \in I$. Clearly, all $X_v$ are independent and it holds $E[X_v] = I$. Thus, the expected number of nodes in an interval is $mI$.

For a lower bound on the number of nodes in $I$ consider an interval of size $\frac{c}{m}$ and choose $\delta = 1/2$. Then the Chernoff Bound yields:

$$ P(X_I \leq (1 - 1/2)e\lambda) \leq e^{\frac{(1/2)^2}{2e}\lambda} \leq e^{\frac{\lambda^2}{8}} \leq \frac{1}{n^2} $$

Let there be $l$ nodes in swarm. We enumerate these nodes $1, \ldots, l$ and let $Y_1, \ldots, Y_l$ be the binary RVs that the corresponding nodes are good. Given our bounds on the churn, at least a $\frac{15}{16}$-fraction of the nodes is good. Thus, we can view a realization of $Y_1, \ldots, Y_l$ as drawing without replacement and in expectation there are $\frac{15}{16}l$ good nodes. This distribution is known to be NA and thus the Chernoff bound applies here, too. Since we know that $l \in O(\log n)$ w.h.p. for the right choice of $c$ the lemma follows.

Thus, for an appropriate choice of the constants $k$ each interval and thus the graph $D_t$ is good w.h.p. Note that the lemma holds analogous for $H_t$ and all swarms $S_{t+1}(p)$.

Since $D_t$ is good, it can successfully perform the forwarding step of $\mathcal{A}_{\text{ROUTING}}$ w.h.p.. To show that $\mathcal{A}_{\text{LDS}}$ also performs the handover and constructs $D_{t+1}$ we need to show that each node gets the necessary references. Thus, we continue with a straightforward lemma that formalizes the algorithm’s actions. It follows directly from the routing’s correctness and the swarm property.

Lemma 18. Let $\mathcal{G}_t := (D_t, H_1, \ldots, D_t)$ be routable graph until round $2t$. Then it holds:

1. In round $2t$ each node in $S_i(p_v)$ receives $(v, p_v^{t+1})$.
2. In round $2t + 1$ each node in $S_i(p_v^{t+1})$ with $i \in \{0, 1\}$ receives $(v, p_v^{t+1})$.
3. In round $2t + 1$ each node in $(p_v \pm \frac{2\lambda}{n})$ receives $(v, p_v)$.

Proof. This lemma directly follows from the routing’s correctness and the overlay’s topology. We prove each statement separately:
1. Given $\mathcal{A}_{\text{ROUTING}}$ works correctly, in round $t$ each node in $S_t(p_v)$ received $(v, p_v)$ w.h.p.

2. For the second statement w.l.o.g. only observe $S_t(\frac{t}{2})$. The other case is analogous. According to the swarm property, each node in $S_t(p_v)$ has a connection to each node in $S_t(\frac{t}{2})$. Note that w.h.p. at least $C\lambda$ nodes $S_t(p_v)$ are not churned out and will forward the message. This implies that in round $t + 1$ it holds that each node in $S_t(p_v)$ received a reference to $v$.

3. Let $\ell \in S_t(p_v)$ any node of $p_v$’s swarm that is a. left of $p_v$, and b. survives round $2t$. According to Lemma $17$ there are at least $\frac{C\lambda}{12}$ such nodes w.h.p. Recall that $\ell$ knows all nodes in $[\ell + \frac{3C\lambda}{n}]$, which includes all nodes in $[p_v + \frac{3C\lambda}{n}]$ since $d(p_v, \ell) \leq \frac{C\lambda}{n}$. Thus, in round $2t$ the node $\ell$ will forward $(v, p_v)$ to all nodes in $[p_v + \frac{3C\lambda}{n}]$. We can make an analogous proof for the right side.

Next, we show that the algorithm successfully performs a handover from $D_t$ to $D_{t+1}$ using the helper graph $H_t$. Therefore, it has to hold that for a given point $p \in [0, 1]$ each node in $S_t(p)$ knows the reference of each node in $S_{t+1}(p)$. However, this follows directly from the third statement of Lemma $18$. Further, we must show that too many nodes from each swarm $S_{t+1}(p)$ are churned out during the handover. This follows from lemma $16$ and $17$ as the adversary is oblivious of all swarms $S_{t+1}(p)$.

We conclude with the construction of $D_{t+1}$. In particular, we wish to show that every mature node $v$ creates an edge to each of its new neighbors in $D_{t+1}$. We can divide the neighbors into 1. the list neighbors left and right of $p_l^{t+1}$, and 2. the long distance neighbors left and right of $p_l^{t+1}$ and $p_l^{t+1}$. The proof’s idea is straightforward: For all nodes $v$ and $v'$ which will be neighbors in $D_{t+1}$ we show that there is a node $w$ that receives the messages $(v, p_l^{t+1})$ and $(v', p_l^{t+1})$ in round $2t + 1$ and thus introduces the nodes. For that we need the following lemma:

**Lemma 19.** Consider round $2t + 1$. Let $v, v' \in V_{2^t + 2}$ be any two neighbors in $D_{t+1}$ then $|\{w \in D_{2t+1} | w \text{ receives } (v, p_l^{t+1}) \text{ and } (v', p_l^{t+1})\}| \geq 1$ w.h.p.

**Proof.** According to Lemma $17$ in an interval of length $\frac{C\lambda}{2^{2t}}$ there are at least one node that is not churned out w.h.p. Let $p, p'$ be two points such that the corresponding nodes are neighbors. By Lemma $18$ the these nodes $ID$s are known by all nodes in the intervals $S_t(p), S_t(\frac{t}{2}), S_t(\frac{t}{2} + 1)$. Further, $v'$ is known to all nodes in $I(p') := [p' - \frac{3C\lambda}{n}, p']$ is at least of length $\frac{C\lambda}{2^{2t}}$.

It suffices to show that the overlap of $S(p)$ and $I(p')$ is at least of length $\frac{C\lambda}{n}$. The other two cases are analogous. However, this easily follows from the fact that $d(p, p') \leq \frac{3C\lambda}{n}$. We distinguish between two cases:

1. Let $d(p, p') \leq \frac{C\lambda}{n}$. Now observe any point $p^* \in [0, 1)$ with $d(p, p^*) \leq \frac{C\lambda}{n}$. In this case the triangle inequality yields that the distance $d(p', p^*)$ is at most $d(p', p) + d(p, p^*) \leq \frac{2C\lambda}{n}$. Thus, it holds $[p + \frac{C\lambda}{n}] \subset [p' + \frac{C\lambda}{n}]$.

2. Let $d(p, p') \geq \frac{C\lambda}{n}$. W.l.o.g let $p'$ be left of $p$ and observe any point $p^* \in [0, 1)$ with $d(p, p^*) \leq \frac{2C\lambda}{n}$ that is also left of $p$. Clearly, it holds that $d(p, p^*) \leq d(p, p')$ and therefore it holds that $[p, p + \frac{C\lambda}{n}] \subset [p, p^*] \subset [p' + \frac{2C\lambda}{n}]$.

We see that in both cases the overlap between intervals is at least of length $\frac{C\lambda}{n}$ and thus the lemma holds.

The proof’s remainder follows from the combination of lemmas $18, 19$ and $17$. This concludes the analysis of the maintenance algorithm.
### 5.1.2 Analysis of $\mathcal{A}_{\text{Random}}$

In this section we show that each fresh node is able is send its ID to $\Theta(\delta)$ mature nodes each round w.h.p. One can see that the statement holds if the following three properties are fulfilled. First, each node gets enough tokens every round. Consider that a fresh node $v \in F_t$ needs at least $\delta$ tokens for itself. Further, each node $v \in V_t$ needs $2\delta$ tokens for each new node that joins via $v$ in $t$. Since only a constant number of nodes join via $v$ some big enough $\tau \in \Theta(\log n)$ is sufficient. Second, it must hold that enough of those tokens are of nodes which are still in the network and will stay in the network for at least one more round, i.e., the tokens belong to good nodes. Otherwise, they cannot supply the node with new tokens. Third, even a good node can deny a connection if more than $2\delta$ other nodes are already connected to it. Thus, we must show that the good mature nodes receive less than $2\delta$ IDs. If the latter two conditions hold we say that $v$ successfully connects to $w$.

Just as before, the proof is an induction that shows that these properties hold in round $t$ w.h.p. if held until $t-1$.

**Lemma 20 (Random Overlay Lemma).** Assume that until round $t$ each fresh node was connected to $\Theta(\delta)$ alive nodes each round. Then it holds w.h.p. that

1. each $v \in V_t \cap V_{t-2}$ receives $\Theta(\tau)$ tokens, and
2. each $v \in F_t$ successfully connects to $\Theta(\delta)$ mature nodes.

First, we calculate the probability that a token reaches a fresh node. The proof can be found. It holds:

**Lemma 21.** Assume Lemma 20 held until round $t$. Then:

1. Each token (regardless of its origin) reaches a node $v \in V_t$ with the same probability $p_v \in [0, 1)$.
2. For each $v \in V_t$ it holds $p_v \in \Theta(\frac{1}{n})$

**Proof.** The proof’s main steps are as follows. First, we extend Lemma [13] to fresh nodes and show all token reach a node $v \in V_t$ with the same (but not necessarily uniform) probability. Assume two nodes $u, w$ send a token to $v$, then the following two events must happen:

1. They must both send a token to any mature nodes that knows $v$’s ID.
2. The token must be forwarded to $v$ from the mature node.

We can easily show that both these have the same probability for two nodes $u, w$. The uniformity of the first event directly follows from Lemma [13]. The uniformity of the second event follows from the fact that each token is forwarded to $v$ with probability $\frac{1}{2\delta}$. The fact that $p_v \in \Theta(\frac{1}{n})$ then follows from three facts: First, a token reaches a given mature node with probability $\Theta(\frac{1}{2\delta})$. This follows from Lemma [13] Second, each fresh node - by assumption - is connected to $\Theta(\delta)$ mature nodes. Last, a mature node forwards a token to a connected node with probability $\frac{1}{2\delta}$. Combining these three facts yields the result.

We continue with the proof sketch for the main lemma. First, we show that each node receives $\Theta(\tau)$ tokens w.h.p. To prove the first statement we make use of a simple balls-into-bins argument. The second statement requires a more nuanced approach. Here, we need to show that $\Theta(\delta)$ connections are successful. Recall that a connection can fail for two reasons. First, the target has received more than $2\delta$ connection requests. Second, the target has been churned out (or will be in the next round). We begin by showing that the first case will never occur w.h.p.

**Lemma 22.** Each mature node receives at most $2\delta$ connections from fresh nodes w.h.p.
Lemma 23. A connection is successful with probability at least $\frac{14}{17}$. Further, the number of successful connections is NA.

Proof. The proof is analogous to the proof of Lemma 17. The adversary is oblivious of the random edges because they only persist for 2 rounds. One round to connect and one round for to reply with tokens. Further, tokens can only point to edges $V_{i-\lambda}$ because unused tokens are discarded after each round. Thus, tokens can assume that tokens are randomly drawn from $V_{i-\lambda}$. This implies the lemma.

Given that that all tokens reach a given node with the same probability (cf. Lemma 22), each node receives a constant fraction of good tokens w.h.p if $\tau$ is big enough. If we pick $\delta$ of these tokens uniformly at random without replacement, a constant fraction will point to good nodes in expectation. Now we observe a hyper-geometric distribution, which is known to be NA (cf. [13]). Thus, a constant fraction points to living nodes w.h.p.

Now we can prove the rest of Lemma 23 by showing that also constant fraction of connection attempts will succeed w.h.p. though an application of the Chernoff bound.
5.1.3 Congestion

Finally we show that it holds:

Lemma 24. Algorithms $A_{\text{LDS}}$ and $A_{\text{RANDOM}}$ have congestion of $O(\log^3 n)$ per node and round w.h.p.

Proof. The lemma can be shown by observing the algorithm’s actions:

1. In Algorithm 3, each round every mature starts at most $\delta + 1 \cdot O(\delta)$ routing requests w.h.p. One for itself and one for every fresh node it knows.
2. In Algorithm 4, each round every mature starts $\tau \in O(\log n)$ random samplings.

Thus, there are $O(\log n)$ routing requests started each round and routing requests are in the system for $O(\log n)$ rounds.

Further, during an introduction in algorithm each mature introduces $O(\log n)$ nodes to their $O(\log n)$ neighbors. Thus, there are $O(\log^3 n)$ additional messages.

Together, this sums up to a complexity of $O(\log^3 n)$.

Theorem 1 now follows from lemmas 15, 20 and 24.

6 Future Work & Conclusion

We presented an algorithm that maintains a structured overlay in presence of a $(2, O(\log n))$-late adversary. We permit $\alpha n$ deletions/additions over the course of $O(\log n)$ rounds. Note that this is exponentially higher than in [2] and [4]. However, both their algorithms are not possible if the adversary has more recent knowledge of topology. This suggests a strong connection between an adversaries lateness with regard to the topology and permitted churn. For future work, one could consider finding an algorithm that tolerates a $(1, O(\log n))$-late adversary. Also one could consider a hybrid model where the adversary has almost up-to-date information about some nodes but is more outdated with regard to others.

References

1. D. Stutzbach and R. Rejaie, “Understanding churn in peer-to-peer networks,” in SIGCOMM, 2006, pp. 189–202.
2. J. Augustine and S. Sivasubramaniam, “Spartan: A framework for sparse robust addressable networks,” in IPDPS, 2018, pp. 1060–1069.
3. J. Augustine, G. Pandurangan, P. Robinson, S. T. Roche, and E. Upfal, “Enabling robust and efficient distributed computation in dynamic peer-to-peer networks,” in FOCS, 2015, pp. 350–369.
4. M. Drees, R. Gmyr, and C. Scheideler, “Churn- and dos-resistant overlay networks based on network reconfiguration,” in SPAA, 2016, pp. 417–427.
5. J. Augustine, A. R. Molla, E. Morsy, G. Pandurangan, P. Robinson, and E. Upfal, “Storage and search in dynamic peer-to-peer networks,” in SPAA, 2013, pp. 53–62.
6. C. Scheideler, “How to spread adversarial nodes?: rotate!” in STOC, 2005.
7. A. Fiat, J. Saia, and M. Young, “Making chord robust to byzantine attacks,” in ESA, 2005, pp. 803–814.
8. B. Awerbuch and C. Scheideler, “Towards scalable and robust overlay networks,” in IPTPS, 2007.
9. A. W. Richa, C. Scheideler, and P. Stevens, “Self-stabilizing de bruijn networks,” in SSS, 2011, pp. 416–430.
10. M. Feldmann and C. Scheideler, “A self-stabilizing general de bruijn graph,” in SSS, 2017, pp. 250–264.
11 V. King and J. Saia, “Choosing a random peer,” in PODC, 2004, pp. 125–130.
12 V. King, S. Lewis, J. Saia, and M. Young, “Choosing a random peer in chord,” Algorithmica, vol. 49, no. 2, pp. 147–169, 2007.
13 D. Dubhashi and D. Ranjan, “Balls and bins: A study in negative dependence,” Random Structures & Algorithms, vol. 13, no. 2, pp. 99–124, 1998.
14 R. Gmyr, K. Hinnenthal, C. Scheideler, and C. Sohler, “Distributed monitoring of network properties: The power of hybrid networks,” in ICALP, 2017.