IMAGE MEASURES OF INFINITE PRODUCT MEASURES
AND GENERALIZED BERNOULLI CONVOLUTIONS

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Abstract

We examine measure preserving mappings $f$ acting from a probability space $(\Omega, F, \mu)$ into a probability space $(\Omega^*, F^*, \mu^*)$, where $\mu^* = \mu(f^{-1})$. Conditions on $f$, under which $f$ preserves the relations “to be singular” and “to be absolutely continuous” between measures defined on $(\Omega, F)$ and corresponding image measures, are investigated.

We apply the results to investigate the distribution of the random variable $\xi = \sum_{k=1}^{\infty} \xi_k \lambda^k$, where $\lambda \in (0; 1)$, and $\xi_k$ are independent not necessarily identically distributed random variables taking the values $i$ with probabilities $p_{ik}, i = 0, 1$.

We also studied in details the metric-topological and fractal properties of the distribution of a random variable $\psi = \sum_{k=1}^{\infty} \xi_k a_k$, where $a_k > 0$ are terms of the convergent series.

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1 Introduction

Let $P_\xi$ be the probability distribution of a random variable

$$\xi = \sum_{k=0}^{\infty} \xi_k \lambda^k,$$

where $\lambda \in (0; 1)$, and $\xi_k$ are independent random variables taking values $-1$ and $+1$ with probabilities $\frac{1}{2}$. $P_\xi$ is known as "infinite Bernoulli convolution". Measures of this form have been studied since 1930’s from the pure probabilistic point of view as well as for their applications in harmonic analysis, dynamical systems and fractal analysis. We will not describe in details the history of the investigation of these measures, paper [13] contains a comprehensive survey on Bernoulli convolutions, corresponding historical notes and brief discussion of some applications, generalizations and problems.

Let $S_\perp$ be the set of those $\lambda \in [\frac{1}{2}; 1)$ for which the random variable $\xi$ has singular distribution, and let $S_{<}$ be the set of those $\lambda \in (\frac{1}{2}; 1)$ for which the measure $P_\xi$ is absolutely continuous (with respect to Lebesgue measure $L$). In 1939 P. Erd"os [5] proved that $\lambda \in S_\perp$ whenever $\lambda$ is the reciprocal of a Pisot number in $(1; 2)$. Up to now we do not know any other examples of $\lambda$ belonging to $S_\perp$. In 1940 P. Erd"os proved the existence a number $a$ independent of $\lambda$ such that almost all $\lambda$ from $(a; 1)$ belong to $S_{<}$. In 1962 A. Garsia [6] found the largest explicitly described subset of $S_{<}$ known up to now. Garsia formulated the following conjecture: almost all $\lambda \in [\frac{1}{2}; 1)$ belong to $S_{<}$. This conjecture was proved by B.Solomyak [19] in 1995, a simpler proof was found by Y. Peres and B.Solomyak in [14]. In [15] a more general nonsymmetric model was considered by Y. Peres and B.Solomyak: if the $\xi_k$ in (1) take values $-1$ and $+1$ with probabilities $1-p$ and $p$ ($p \in [\frac{1}{3}; \frac{2}{3}]$) correspondingly, then almost all $\lambda$ from $[p^p(1-p)^{(1-p)}; 1)$ belong to $S_{<}$.

The main goal of this paper is to investigate the distributions of the following random variables which are generalizations of the Benoulli convolutions:

1) $\psi = \sum_{k=1}^{\infty} \psi_k \lambda^k$, where $\lambda \in (0; 1)$, and $\psi_k$ are independent not necessarily indentically distributed random variables taking the values 0 resp.1 with probabilities $p_{0k}$ resp. $p_{1k}$, i.e., schematically:

$$\begin{array}{ccc}
\psi_k & 0 & 1 \\
\hline
p_{0k} & p_{1k}
\end{array}$$
with $p_{ik} \geq 0$, $p_{0k} + p_{1k} = 1$.

2) $\varphi = \sum_{k=1}^{\infty} \varphi_k a_k$, where $\varphi_k$ are independent random variables with the following distributions

\[
\begin{array}{cc}
\varphi_k & 0 \quad 1 \\
p_{0k} & p_{1k}
\end{array}
\]

$p_{ik} \geq 0$, $p_{0k} + p_{1k} = 1$, and $a_k \geq 0$ are terms of a convergence series $\sum_{k=1}^{\infty} a_k$.

The main method we use is the method of image measures which are nonlinear projections of infinite product measures. In section 2 we consider an infinite product measure spaces and the problem of absolute continuity, singularity and discreteness of the product measures. Section 3 is devoted to measure preserving mappings $f$ acting from a probability space $(\Omega, F, \mu)$ into a probability space $(\Omega^*, F^*, \mu^*)$, where $\mu^* = \mu(f^{-1})$. Conditions on $f$, under which $f$ preserves the relations "to be singular" and "to be absolutely continuous" between measures defined on $(\Omega, F)$ and corresponding image measures, are investigated.

In section 4, by using the results of sections 2 and 3, we prove a slightly sharpened versions of the results of B.Solomyak[19], and Y.Peres and B.Solomyak[15]. We also consider an counterexample concerning a problem which was formulated in section 3.

Section 5 is devoted to the study of metric, fractal and topological properties of the distribution of the above mentioned random variable $\varphi$. We prove criteria for singularity and absolutely continuity of the distribution of the r.v. $\varphi$ in the case, when $a_k \geq \sum_{i=k+1}^{\infty} a_i$.

2 Infinite product measure spaces and absolutely continuity of product measures.

Let us consider an independent sequence of probability spaces $\{(\Omega_k, A_k, \mu_k)\}$. We will denote by $(\Omega, A, \mu)$ the infinite product of the probability spaces $(\Omega_k, A_k, \mu_k)$:

\[
(\Omega, A, \mu) = \prod_{k=1}^{\infty} (\Omega_k, A_k, \mu_k).
\]
For any element $\omega = (\omega_1, \omega_2, ..., \omega_k, ...) \in \Omega$ we have

$$\mu(\omega) = \prod_{k=1}^{\infty} \mu_k(\omega_k), \ \omega_k \in \Omega_k.$$ 

**Theorem 1.** $\mu$ is pure discrete if and only if

$$\prod_{k=1}^{\infty} \max_{\omega_k \in \Omega_k} \mu_k(\omega_k) > 0. \tag{2}$$

**Proof.** If $\prod_{k=1}^{\infty} \max_{\omega_k \in \Omega_k} \mu_k(\omega_k) = 0$, then for any point $\omega \in \Omega$ we have $\mu(\omega) \leq \prod_{k=1}^{\infty} \max_{\omega_k \in \Omega_k} p_{ik} = 0$. Therefore, condition 2 is necessary for discreteness of the measure $\mu$.

To prove the sufficiency we consider a subset $A_+ \subset \Omega$ :

$$A_+ = \left\{ \omega : \mu_k(\omega_k) > 0 \text{ and } \prod_{k=1}^{\infty} \mu_k(\omega_k) > 0 \right\}.$$ 

The set $A_+$ consists of the points $\omega = (\omega_1, \omega_2, ..., \omega_k, ...)$ such that $\mu_k(\omega_k) > 0$ and condition $\mu_k(\omega_k) \neq \max_{\omega_k \in \Omega_k} \mu_k(\omega_k)$ holds only for finite numbers of values $k$. It is easy to see that the set $A_+$ is at most countable and the event ”$\omega \in A_+$” does not depend on any finite coordinates of $\omega$. Therefore, by using the ”0 and 1” theorem of Kolmogorov, we conclude that $\mu(A_+) = 0$ or $\mu(A_+) = 1$. Since the set $A_+$ contains the point $\omega^*$ such that $\mu_k(\omega_k^*) = \max_{\omega_k \in \Omega_k} \mu_k(\omega_k)$, we have $\mu(A_+) \geq \mu(\omega^*) > 0$. Thus, $\mu(A_+) = 1$, which proves the discreteness of the measure $\mu$. □

Let $(\Omega, A, \nu) = \prod_{k=1}^{\infty} (\Omega_k, A_k, \nu_k)$ such that the measure $\nu_k$ is absolutely continuous with respect to the measure $\mu_k$ (for short $\nu_k \ll \mu_k$) for all $k \in \mathbb{N}$. By using completely analog arguments as in [9], one can prove the following sharper variant of the Kakutani theorem.

**Theorem 2** Let $\nu_k$ be absolutely continuous with respect to the measure $\mu_k$. Then the measure $\mu$ is either purely absolutely continuous with respect to the measure $\mu$ or purely singular (including the discreteness). Moreover,

$$\nu \ll \mu \text{ if and only if } \prod_{k=1}^{\infty} \rho(\mu_k, \nu_k) > 0, \tag{3}$$

4
where \( \rho(\mu_k, \nu_k) = \int_{\Omega_k} \sqrt{\frac{d\nu}{d\mu_k}} d\mu_k. \)

**Remark.** The expression \( \int_{\Omega_k} \sqrt{\frac{d\nu}{d\mu_k}} d\mu_k \) coincides with the Hellinger's integral [9].

### 3 Measure-preserving mappings of probability spaces.

Let \( (\Omega, F, \eta) \) and \( (\Omega, F, \tau) \) be abstract probability spaces.

Let us consider a measurable mapping \( f : \)

\[
(\Omega, F, \eta) \xrightarrow{f} (\Omega^*, F^*, \eta^*), \\
(\Omega, F, \tau) \xrightarrow{f} (\Omega^*, F^*, \tau^*),
\]

where the measures \( \eta^* \) and \( \tau^* \) are defined as follows:

\[
\eta^*(E) = \eta(f^{-1}(E)), \\
\tau^*(E) = \tau(f^{-1}(E)), \text{ for any subset } E \in F^*,
\]

and \( f^{-1}(E) = \{ \omega : \omega \in \Omega \text{ and } f(\omega) \in E \} \).

The mapping \( f \) is measure-preserving by definition (for details see, e.g., [3],[22]).

**Theorem 3.** If \( \eta \ll \tau \), then \( \eta^* \ll \tau^* \).

**Proof.** Suppose that \( \tau^*(E) = 0 \) for some subset \( E \in F^* \). Then \( \tau(f^{-1}(E)) = \tau^*(E) = 0 \). Since \( \eta \ll \tau \), we conclude that \( \eta(f^{-1}(E)) = \eta^*(E) = 0 \), which proves an absolute continuity of the measure \( \eta^* \) with respect to the measure \( \tau^* \).

**Remark** In general the inverse result to theorem 3 does not hold. In fact, it is possible to construct probability spaces \( (\Omega, F, \eta) \) and \( (\Omega, F, \tau) \), and a mapping \( f \) such that \( \eta \perp \tau \), but \( \eta^* \ll \tau^* \). A corresponding example will be considered later in section 4.

**Theorem 4.** If \( \eta^* \perp \tau^* \), then \( \eta \perp \tau \).

**Proof.** If \( \eta^* \perp \tau^* \), then there exists a subset \( E \in F^* \) such that \( \eta^*(E) = 0 \) and \( \tau^*(\Omega^\ast \setminus E) = 0 \). By the definitions of the measures \( \eta^* \) and \( \tau^* \), we have \( \eta(f^{-1}(E)) = \eta^*(E) = 0 \) and \( \tau(f^{-1}(\Omega^\ast \setminus E)) = \tau^*(\Omega^\ast \setminus E) = 0 \). Since
\( f^{-1}(\Omega^* \setminus E) \cap f^{-1}(E) = \emptyset \), we conclude for the singularity of the measures \( \eta \) and \( \tau \).

**Theorem 5.** If \( f \) is a bijective mapping, then
\[
\eta \ll \tau \text{ if and only if } \eta^* \ll \tau^*, \\
\eta \perp \tau \text{ if and only if } \eta^* \perp \tau^*.
\]

**Proof.** If \( f \) is a bijective mapping, then \( A = f^{-1}[f(A)] \) and \( \eta(A) = \eta^*(f(A)) \). Using the same arguments, we have \( \tau(A) = \tau^*(f(A)) \) for all measurable subsets \( A \subset \Omega \).

To prove the first statement of the theorem it is sufficient to prove that the condition \( \eta^* \ll \tau^* \) implies the condition \( \eta \ll \tau \). Suppose that \( \tau(A) = 0 \) for some measurable subset \( A \subset \Omega \). Then \( \tau^*(f(A)) = \tau(A) = 0 \). Since \( \eta^* \ll \tau^* \), we conclude that \( \eta(f(A)) = \eta(A) = 0 \), which proves the absolute continuity of the measure \( \eta \) with respect to the measure \( \tau \).

The second assertion of the theorem can be prove in a similar way. \( \square \)

**Theorem 6.** If there exists a measurable subset \( \Omega_0 \subset \Omega \) such that \( \eta(\Omega_0) = \tau(\Omega_0) = 0 \) and the mapping \( f : (\Omega \setminus \Omega_0) \rightarrow \Omega^* \) is a bijection, then
\[
\eta \ll \tau \text{ if and only if } \eta^* \ll \tau^*, \\
\eta \perp \tau \text{ if and only if } \eta^* \perp \tau^*.
\]

**Proof.** For any measurable subset \( A \subset \Omega \) we have \( A \subset f^{-1}[f(A)] \). Since \( f : (\Omega \setminus \Omega_0) \rightarrow [0; 1] \) is a bijective mapping, we conclude that \( f^{-1}[f(A)] = A \cup A_0 \), where \( A_0 \subset \Omega_0 \). Therefore,
\[
\eta(A) = \eta(A \cup A_0) = \eta(f^{-1}[f(A)]) = \eta^*(f(A)), \\
\tau(A) = \tau^*(f(A)).
\]

We complete the proof by arguments which are completely similar to those used in the proof of the previous theorem. \( \square \)

## 4 Generalized Bernoulli convolutions I

Let us consider a random variables
\[
\psi = \sum_{k=1}^{\infty} \psi_k \lambda^k,
\]
where $\lambda \in (0; 1)$, and $\psi_k$ are independent not necessarily indentically distributed random variables with the following distributions

\[
\begin{pmatrix}
\psi_k & 0 & 1 \\
p_{0k} & p_{1k}
\end{pmatrix},
\]

with $p_{ik} \geq 0$, $p_{0k} + p_{1k} = 1$.

**Theorem 7.** If

\[
\sum_{k=1}^{\infty} \left( \frac{1}{2} - p_{0k} \right)^2 < \infty,
\]

(4)

then for almost all $\lambda \in [\frac{1}{2}; 1)$ the r.v. $\psi$ has an absolute continuous distribution.

**Proof.** Let us consider two sequences of probability spaces: \{$(\Omega_k, A_k, \mu_k)$\} and \{$(\Omega_k, A_k, \nu_k)$\}, where $\Omega_k = \{0; 1\}$, $A_k$ consists of all subsets of $\Omega_k$ and measures $\mu_k$ and $\nu_k$ are defined as follows:

\[
\begin{align*}
\nu_k(0) &= p_{0k}, \quad \nu_k(1) = p_{1k}; \\
\mu_k(0) &= \frac{1}{2}, \quad \mu_k(1) = \frac{1}{2}.
\end{align*}
\]

It is easy to see that $\nu_k \ll \mu_k$ for all $k$, and $\rho(\mu_k, \nu_k) = \int_{\Omega_k} \sqrt{\frac{d\nu_k}{d\mu_k}} d\mu_k = \sqrt{\frac{1}{2}p_{0k} + \frac{1}{2}p_{1k}}$. Let $(\Omega, A, \mu) = \prod_{k=1}^{\infty} (\Omega_k, A_k, \mu_k)$ and $(\Omega, A, \nu) = \prod_{k=1}^{\infty} (\Omega_k, A_k, \nu_k)$. By using theorem 2, we can deduce that the measure $\nu$ is absolute continuous with respect to the measure $\mu$ if and only if

\[
\prod_{k=1}^{\infty} \left( \sqrt{\frac{1}{2}p_{0k} + \frac{1}{2}p_{1k}} \right) < \infty.
\]

(5)

It is easy to check that the product (5) converges if and only if condition (4) holds.

Let us consider a mapping $f : \Omega \to [0; \frac{\lambda}{1-\lambda}]$ defined as follows: for any $\omega = (\omega_1\omega_2...\omega_k...) \in \Omega$,

\[
f(\omega) = \sum_{k=1}^{\infty} \omega_k\lambda^k.
\]

(6)
It is easy to see that $f$ is a measurable mapping. The image measure $\mu^* = \mu(f^{-1})$ is the above classical mentioned Bernoulli measure $\mu_\lambda$, the image measure $\nu^* = \nu(f^{-1})$ is the probability measure corresponding to the distribution of the random variable $\psi$. By using theorem 2 and theorem 3, we conclude that condition (4) implies the absolute continuity of the measure $\nu^*$ with respect to measure $\mu^*$. B.Solomyak in [19] proved that for almost all $\lambda \in [\frac{1}{2}; 1)$ the measure $\mu^*$ is absolutely continuous with respect to Lebesgue measure. Moreover, since for $\lambda \in [\frac{1}{2}; 1)$ the support of the measure $\mu^*$ coincides with the whole closed interval $[0; \frac{1}{1-\lambda}]$, we conclude that $\mu^*$ is equivalent to Lebesgue measure (for short $\mu^* \sim L$) for almost all $\lambda \in [\frac{1}{2}; 1)$. Therefore, $\nu^* \ll L$ for almost all $\lambda \in [\frac{1}{2}; 1)$.

Theorem 8. If there exist a number $p \in [\frac{1}{3}; \frac{2}{3}]$ such that

$$\sum_{k=1}^{\infty} (p_0 - p_{0k})^2 < \infty,$$

then for almost all $\lambda \in [p^p \cdot (1-p)^{(1-p)}; 1)$ the r.v. $\psi$ has an absolutely continuous distribution.

Proof. The proof of this theorem is analogous to the previous one, but we define the measures $\mu_k$ as follows:

$$\mu_k(0) = 1 - p, \quad \mu_k(1) = p,$$

and we use the following result from [13]: if $p \in [\frac{1}{3}; \frac{2}{3}]$, then for almost all $\lambda \in [p^p \cdot (1-p)^{(1-p)}; 1)$ the measure $\mu$ is absolutely continuous with respect to Lebesgue measure.

Proposition There exist product measures $\nu$ and $\mu$, and a measure preserving mapping $f$ such that $\nu \perp \mu$, but $\nu^* \ll \mu^*$.

Proof. Let $\Omega_k = \{0; 1\}$, $A_k$ consists of all subsets of $\Omega_k$ and measures $\mu_k$ and $\nu_k$ defined as follows:

$$\nu_k(0) = p_{0k} = 1 - p, \quad \nu_k(1) = p_{1k} = p \in \left[\frac{1}{3}; \frac{2}{3}\right], p \neq \frac{1}{2};$$

$$\mu_k(0) = \frac{1}{2}, \quad \mu_k(1) = \frac{1}{2}.$$

Since $\sum_{k=1}^{\infty} (\frac{1}{2} - p_{0k})^2 = \infty$, the corresponding product measures $\nu$ and $\mu$ are mutually singular. Let the mapping $f$ be defined by (6). As mentioned
above, the image measure $\mu^* = \mu(f^{-1})$ is equivalent to Lebesgue measure for almost all $\lambda \in \left[\frac{1}{2}; 1\right)$, and the image measure $\nu^* = \nu(f^{-1})$ is equivalent to Lebesgue measure for almost all $\lambda \in \left[p^{p} \cdot (1-p)^{(1-p)}; 1\right)$. Therefore, for almost all $\lambda \in \left[p^{p} \cdot (1-p)^{(1-p)}; 1\right)$, the image measures $\nu^*$ and $\mu^*$ are equivalent. □

5 Generalized Bernoulli convolutions II

Let us consider a random variable

$$\varphi = \sum_{k=1}^{\infty} \varphi_k a_k,$$

where $\varphi_k$ are independent random variables with the following distributions

$$\begin{pmatrix} 0 & 1 \\ p_{0k} & p_{1k} \end{pmatrix},$$

$p_{0k} \geq 0$, $p_{0k} + p_{1k} = 1$, and $a_k \geq 0$ are terms of a convergent series. Without loss of generality we shall assume that

$$\sum_{k=1}^{\infty} a_k = 1. \quad (7)$$

One can proved that the support $S_{\varphi}$ of the r.v. $\varphi$, i.e., the smallest closed set supported the distribution, is a perfect set of the following form:

$$S_{\varphi} = \left\{ x : x = \sum_{k=1}^{\infty} \gamma_k(x) a_k, \gamma_k(x) \in \{0; 1\} \text{ and } p_{\gamma_k(x),k} \neq 0 \right\}. \quad (8)$$

In other words, $S_{\varphi}$ is a set of all possible ”incomplete sums” of the series (7). The metric and topological properties of the set $S_{\varphi}$ directly depend on the properties of the series (7). Let us consider three examples illustrating this dependence.

**Example 1.** If $a_k = \frac{2}{3^k}$, then $S_{\varphi}$ coincides with the classical Cantor set $C_0$.

**Example 2.** If $a_k = \frac{1}{2^k}$, then $S_{\varphi} = [0; 1]$.

**Example 3.** If $a_k = \frac{1}{2^k} + \frac{3\varepsilon_0}{4^k}, \varepsilon_0 \in (0; 1)$, then, by using theorem 9, one proves that $S_{\varphi}$ is a nowhere dense set of positive Lebesgue measure.
Let \( r_k = \sum_{i=k+1}^{\infty} a_i \) and \( \delta_k = \frac{a_k}{r_k} \). If the condition \( \delta_k < 1 \) holds for all \( k > k_0 \), then the support \( S_\varphi \) is a finite union of closed intervals. Moreover, the distribution of the r.v. \( \varphi \) is the so-called distribution with big overlaps, because almost all points of the support have an essentially nonunique representation of the form (8). The following theorem is proven in ([17]).

**Theorem 9.** If condition \( \delta_k > 1 \) holds for all \( k > k_0 \), then \( S_\varphi \) is a nowhere dense set, and \( S_\varphi \) has a positive Lebesgue measure if and only if
\[
\sum_{k=1}^{\infty} (\delta_k - 1) < \infty. \tag{9}
\]

If condition (9) does not hold, then the r.v. \( \varphi \) has a fractal distribution with the following Hausdorff-Besicovitch dimension of the support:
\[
\alpha_0(S_\varphi) = \lim_{k \to \infty} \frac{k \ln 2}{\sum_{i=1}^{k} (\delta_i + 1)}.
\]

In addition, if condition \( \delta_k > 1 \) holds for all \( k \in \mathbb{N} \), then all points of the support \( S_\varphi \) have a unique representation of the form (8).

Our main goal in this section is to investigate the topological properties of the support \( S_\varphi \) in a general situation, i.e., when there exist sequences \( \{k'_n\} \) and \( \{k''_n\} \) such that \( \delta_{k'_n} > 1 \) and \( \delta_{k''_n} < 1 \) for all \( n \in \mathbb{N} \).

**Theorem 10.** Let \( p_k > 0 \) and the sequence \( \{a_k\} \) be monotonically non-increasing, i.e., \( a_k \geq a_{k+1}, k \in \mathbb{N} \). Then the support \( S_\varphi \) is a nowhere dense set if and only if condition \( \delta_k > 1 \) holds for infinitely many number of \( k \).

**Proof.** **Necessity** is obvious.

**Sufficiency.** To obtain a contradiction we suppose that the condition \( \delta_k > 1 \) holds for infinitely many number of \( k \), but there exist an closed interval \([\alpha^*; \beta^*]\) such that \([\alpha^*; \beta^*] \subset S_\varphi \). Let
\[
\alpha = \inf \{ x : [x; \beta^*] \subset S_\varphi \},
\]
and
\[
\beta = \sup \{ x : [\alpha^*; x] \subset S_\varphi \}.
\]
Since \( S_\varphi \) is a closed set, \( \alpha \in S_\varphi \) and \( \beta \in S_\varphi \). Let us prove that \( \alpha \neq 0 \). If we suppose that \( \alpha = 0 \), then we should conclude that \([0; \beta] \subset S_\varphi \). We can choose
a number $k_1$ such that $a_{k_1} < \beta$ and $a_{k_1} > r_{k_1}$. Then, the interval $(r_{k_1}; a_{k_1})$ does not contain any point from the support. Indeed, if $\varphi_1 = 1$ or $\varphi_2 = 1$ or ... or $\varphi_{k_1} = 1$, then $\varphi \geq a_{k_1}$; if $\varphi_1 = 0$ and $\varphi_2 = 0$ and ... and $\varphi_{k_1} = 0$, then $\varphi \leq r_{k_1}$. In a similar way we prove that $\beta \neq 1$. Since $S_\varphi$ is a perfect set, from the construction of the set $[\alpha; \beta]$ there follows the existence of a positive number $\varepsilon_0$ such that both of the intervals $(\alpha - \varepsilon_0; \alpha)$ and $(\beta; \beta + \varepsilon_0)$ do not contain any points from the support $S_\varphi$. Let $k_2$ be a natural number such that $r_{k_2} < \varepsilon_0$ and let us consider set

$$S_{k_2} = \left\{ x : x = \sum_{k=1}^{k_2} \gamma_k a_k, \gamma_k \in \{0; 1\} \right\}.$$ 

It is easy to see that $S_{k_2}$ consists of at most $2^{k_2}$ elements. We define $S_{k_2}' = S_{k_2} \cap [\alpha; \beta]$, and $S_{k_2}'' = S_{k_2} \setminus S_{k_2}'$. If a point $x_0 = \sum_{k=1}^{k_2} \gamma_k(x_0) a_k$ belongs to $S_{k_2}''$, then any point $x$ of the form

$$x = \sum_{k=1}^{k_2} \gamma_k(x_0) a_k + \sum_{k=k_2+1}^{\infty} \gamma_k a_k$$

does not belong to $[\alpha; \beta]$, since $x_0 \leq \alpha - \varepsilon_0$ or $x_0 \geq \beta + \varepsilon_0$, and $r_{k_2} < \varepsilon_0$.

In the set $S_{k_2}'$ there exist a minimal point $x_1$. It is easy to understand that $x_1 = \alpha$, otherwise the interval $[\alpha; x_1)$ does not contain any points from $S_\varphi$. Therefore

$$\alpha = \sum_{k=1}^{k_2} \gamma_k(\alpha) a_k + \sum_{k=k_2+1}^{\infty} 0 \cdot a_k = 0, \gamma_1(\alpha) \gamma_2(\alpha) \ldots \gamma_{k_2}(\alpha) 000 \ldots \ (10)$$

Let $x_2$ be the second minimal point of the set $S_{k_2}'$, i.e.,

$$x_2 = \min \left\{ S_{k_2}' \setminus x_1 \right\}.$$ 

Finally, let us consider a natural number $k_3$ such that $k_3 > k_2$, $a_{k_3} < (x_2 - x_1)$ and $a_{k_3} > r_{k_3}$. If $\gamma_{k_2+1}(x) = 1$ or $\gamma_{k_2+2}(x) = 1$ or ... or $\gamma_{k_3}(x) = 1$, then $x \geq \alpha + a_{k_3}$; if $\gamma_{k_2+1}(x) = 0$ and $\gamma_{k_2+2}(x) = 0$ and ... and $\gamma_{k_3}(x) = 0$, then $x \leq \alpha + r_{k_3}$. Therefore the interval $(\alpha + r_{k_3}; \alpha + a_{k_3})$ does not contain any point from the support. This contradicts our assumption.\[\square\]
Remark. The condition $a_k \geq a_{k+1}, k \in N$ is not restrictive, since the terms in the series $(7)$ are positive, and we can choose the order of the terms of the series to fulfilled this condition.

Let us investigate criteria for singularity and absolutely continuity of the distribution of the random variable $\varphi$.

**Theorem 12.** Let $\delta_k > 1$ holds for all $k \in N$. Then the distribution of the random variable $\varphi$ has pure type, and:

1) is pure discrete if and only if

$$\prod_{k=1}^{\infty} \max_{0 \leq i \leq s-1} \{p_{0k}, p_{1k}\} > 0;$$

2) is pure absolutely continuous if and only if both

$$\sum_{k=1}^{\infty} (\delta_k - 1) < \infty,$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{2} - p_{0k}\right)^2 < \infty$$

converge;

3) is pure singular in all other cases.

**Proof.**

Purity of the distribution of the r.v. $\varphi$ follows from the Jessen-Wintner theorem [8].

Condition (12) is necessary for absolutely continuity because of theorem 9. Let us consider two infinite product measure spaces $\Omega, A, \mu$ and $(\Omega, A, \nu)$ defined as in the proof of theorem 7, i.e.,

$$\nu_k(0) = p_{0k}, \quad \nu_k(1) = p_{1k};$$

$$\mu_k(0) = \frac{1}{2}, \quad \mu_k(1) = \frac{1}{2}.$$  

It is easy to see that $\nu_k \ll \mu_k$ for all $k$, and $\rho(\mu_k, \nu_k) = \int_{\Omega_k} \sqrt{\frac{d\nu_k}{d\mu_k}} d\mu_k = \sqrt{\frac{1}{2} p_{0k}} + \sqrt{\frac{1}{2} p_{1k}}$. By using theorem 2, we conclude that the measure $\nu$ is absolute continuous with respect to measure $\mu$ if and only if

$$\prod_{k=1}^{\infty} \left(\sqrt{\frac{1}{2} p_{0k}} + \sqrt{\frac{1}{2} p_{1k}}\right) < \infty.$$
It is easy to check that the product (14) converges if and only if condition (13) holds.

Let us consider a mapping $g : \Omega \to [0; 1]$ defined as follows: for any $\omega = (\omega_1 \omega_2 \ldots \omega_k \ldots) \in \Omega$,

$$g(\omega) = \sum_{k=1}^{\infty} \omega_k a_k. \tag{15}$$

It is easy to see that $g$ is a measurable mapping. Moreover, $g$ is bijective, since $\delta_k > 1$ holds for all $k \in N$. The image measure $\mu^* = \mu(g^{-1})$ is a probability measure which is uniformly distributed on the support $S_\varphi$ of positive Lebesgue measure $L(S_\varphi) > 0$, i.e., for all Borel subset $E \subset [0; 1]$ we have

$$\mu^*(E) = \frac{L(E \cap S_\varphi)}{L(S_\varphi)}. $$

The image measure $\nu^* = \nu(g^{-1})$ is the distribution of the random variable $\varphi$. Since $g$ is a bijective mapping, the measure $\nu^*$ is absolutely continuous with respect to the measure $\mu^*$ if and only if the measure $\nu$ is absolutely continuous with respect to the measure $\mu$. Since $\mu^* \ll L$, we conclude that the measure $\nu^*$ is absolute continuous with respect to Lebesgue measure if and only if both (12) and (13) hold.

The criterium for discreteness follows from the Levi theorem [10] or can be deduced as a corollary from theorem 1 and the bijectivity of $g$.

Assertion 3) follows from the purity of the distribution of the r.v. $\varphi$. □

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