Polynomial Fermionic Forms for the Branching Functions of the Rational Coset Conformal Field Theories

\[ \widehat{su}(2)^M \times \widehat{su}(2)^N / \widehat{su}(2)^{M+N} \]

Anne Schilling *

Institute for Theoretical Physics,
State University of New York at Stony Brook, Stony Brook, NY 11794-3800

Abstract

General fermionic expressions for the branching functions of the rational coset conformal field theories \( \widehat{su}(2)^M \times \widehat{su}(2)^N / \widehat{su}(2)^{M+N} \) are given. The equality of the bosonic and fermionic representations for the branching functions is proven by introducing polynomial truncations of these branching functions which are the configuration sums of the RSOS models in regime III. The path space interpretation of the RSOS models provides recursion relations for the configuration sums. The proof of the recursion relations for the fermionic expressions is given by using telescopic expansion techniques. The configuration sums of the RSOS model in regime II which correspond to the branching functions of the \( Z_{M+N} \)-parafermion conformal field theory are obtained by the duality transformation \( q \rightarrow q^{-1} \).

1 Introduction

Recently there has been much interest in the explicit representations as fermionic \( q \)-series of the characters and branching functions of conformal field theories and many general fermionic forms have been found/conjectured for the coset models \( (G_l)_m \times (G_l)_n/(G_l)_{m+n} \) where \( G \) is a simply laced Lie algebra (and the levels are integer) and the generalization to fractional levels of the minimal models \( M(p, p') \) and the \( N = 1 \) superconformal models \( SM(p, p') \) [1]-[29]. The fermionic representations are complementary to the already known bosonic representations [30]-[37]. The two representations are related by generalized Rogers-Ramanujan identities.

In particular, Kedem et al. [3] have conjectured the characters for the coset models \( \widehat{su}(2)^M \times \widehat{su}(2)^N / \widehat{su}(2)^{M+1} \) and the identity character for the general coset model \( \widehat{su}(2)^M \times \widehat{su}(2)^N / \widehat{su}(2)^{M+N} \). The Virasoro characters for the unitary superconformal minimal models which correspond to the coset constructions \( \widehat{su}(2)^M \times \widehat{su}(2)^N / \widehat{su}(2)^{M+2} \) have been conjectured by Baver and Gepner [22]. Nakayashiki and Yamada [26] discuss the coset models \( \widehat{su}(2)^M \times \widehat{su}(2)^N / \widehat{su}(2)^{M+N} \) from the crystalline spinon basis point of view.

The proofs of the fermionic representations which have been given almost all follow either the method of Schur [38] or combinatorial methods. Schur’s method is based on finding a

*e-mail address: anne@max.physics.sunysb.edu
class of polynomials which satisfy recursion relations in terms of the degree and in the limit of infinite degree equal the characters/branching functions. It should be emphasized that the polynomial truncations of the characters/branching functions are not unique. Melzer [1] has conjectured a polynomial form for the coset models $\hat{s}u(2)_M \times \hat{s}u(2)_1/\hat{s}u(2)_{M+1}$ which has been proven by Berkovich [12] via telescopic expansion techniques and Warnaar [13] by counting weighted paths based on the Fermi gas picture. Further such polynomial recursion relation proofs have recently been given for the $N = 1$ superconformal models $SM(2, 4\nu)$ by Berkovich, McCoy and Orrick [28]. Bose-Fermi identities for the coset models $\hat{s}l(n)_1 \times \hat{s}l(n)_1/\hat{s}l(n)_2$ have been proven by Foda, Okado and Warnaar [29] using purely combinatorial methods.

The purpose of this paper is to extend these studies by 1) presenting a complete set of fermionic branching function formulas for the coset models $\hat{s}u(2)_M \times \hat{s}u(2)_N/\hat{s}u(2)_{M+N}$ and 2) to prove them via Schur’s method by introducing polynomial generalizations which satisfy the same RSOS recursion relations which were used by Andrews, Baxter and Forrester [34] and the Kyoto group [36], [37] to compute the bosonic form of the branching functions.

Our principal result for point 1) is that the fermionic form of the branching functions for the coset models $\hat{s}u(2)_{\nu-1} \times \hat{s}u(2)_N/\hat{s}u(2)_{\nu-1}$ is given by

$$q^\nu c_{\hat{\nu}, \hat{s}}^{(l)} = \sum_{m_i \geq 0, o.e. \text{ restriction } Q_{\hat{\nu}, \hat{s}, l}} q^{\frac{1}{2}m_i(Cm_i - \frac{1}{2} \hat{A}_{\hat{\nu}, \hat{s}, l} - \frac{1}{2} \hat{u}_{\hat{\nu}, \hat{s}, l})} \prod_{i=1, i \neq N}^{\nu-2} \left( \frac{1}{m_i} \right)^{\nu-2} \prod_{i=1, i \neq N}^{\nu-2} \hat{Q}_{\hat{\nu}, \hat{s}, l}.$$  (1.1)

Here the sum runs over all $m_i \geq 0$ restricted by $\hat{Q} \in (\mathbb{Z}_2)^{\nu-2}$ (i.e. if $(\hat{Q})_i = 0$ then $m_i$ even and if $(\hat{Q})_i = 1$ then $m_i$ odd), $I$ is the $(\nu - 2)$ dimensional incidence matrix $(I)_{a,b} = \delta_{a,b+1} + \delta_{a,b-1}$, $C = 2 - I$ the $(\nu - 2)$ dimensional Cartan matrix of the Lie algebra $A_{\nu-2}$ and $\hat{A}$ and $\hat{u}$ are $(\nu - 2)$ dimensional vectors with integer values. The Gaussian polynomials $\left[ \begin{array}{c} n \\ m \end{array} \right]$ are defined by

$$\left[ \begin{array}{c} n \\ m \end{array} \right] = \left\{ \begin{array}{ll} \frac{(q)_m}{(q)_{n-m}} & \text{if } 0 \leq m \leq n \\ 0 & \text{otherwise} \end{array} \right.$$  (1.2)

and

$$(q)_n = \prod_{j=1}^{n} (1 - q^j).$$  (1.3)

There are four sets of solutions for $\hat{A}, \hat{u}$ and $\hat{Q}$ which all lead to the branching function $c_{\hat{\nu}, \hat{s}}^{(l)}$. $\hat{A}, \hat{u}$ and $\hat{Q}$ are either given by

$$\hat{A}_{\hat{\nu}, \hat{s}, l} = e_{\hat{s}-1}$$

$$\hat{u}_{\hat{\nu}, \hat{s}, l} = e_{\hat{s}-1} + e_{\nu-\hat{s}} + e_{\nu-1}$$

$$\hat{Q}_{\hat{\nu}, \hat{s}, l} = -(\nu - 1) + (e_{\hat{s}-2} + e_{\hat{s}-4} + \ldots) + (e_{\nu-2} + e_{\nu-4} + \ldots) + (e_{\nu+\nu-2} + e_{\nu+3-\hat{s}} + \ldots),$$  (1.4)

$$\hat{A}_{\hat{\nu}, \hat{s}, l} = e_{\nu-\hat{s}}$$
\[
\hat{u}_{\hat{r}, \hat{s}, l} = e_{\nu-\hat{s}} + e_{\hat{r}+N-1} + e_{N-l+1}
\]
\[
\hat{Q}_{\hat{r}, \hat{s}, l} = (\hat{s} - 1)\rho + (e_{\hat{r}+N-2} + e_{\hat{r}+N-4} + \ldots) + (e_{N-l} + e_{N-l-2} + \ldots)
\]
\[
+ (e_{\nu+1-\hat{s}} + e_{\nu+3-\hat{s}} + \ldots),
\]
(1.5)
\[
\hat{A}_{\hat{r}, \hat{s}, l} = e_{\nu-\hat{s}}
\]
\[
\hat{u}_{\hat{r}, \hat{s}, l} = e_{\nu-\hat{s}} + e_{\nu-\hat{r}} + e_{l-1}
\]
\[
\hat{Q}_{\hat{r}, \hat{s}, l} = (e_{\nu-\hat{s}-1} + e_{\nu-\hat{s}-3} + \ldots) + (e_{\nu-\hat{r}-1} + e_{\nu-\hat{r}-3} + \ldots)
\]
\[
+ (e_{l-2} + e_{l-4} + \ldots)
\]
(1.6)
or
\[
\hat{A}_{\hat{r}, \hat{s}, l} = e_{\hat{s}-1}
\]
\[
\hat{u}_{\hat{r}, \hat{s}, l} = e_{\hat{s}-1} + e_{\hat{r}+N-1} + e_{N-l+1}
\]
\[
\hat{Q}_{\hat{r}, \hat{s}, l} = (e_{\hat{r}+N-2} + e_{\hat{r}+N-4} + \ldots) + (e_{N-l} + e_{N-l-2} + \ldots)
\]
\[
+ (e_{\hat{s}-2} + e_{\hat{s}-4} + \ldots)
\]
(1.7)
where \(\rho = e_1 + e_2 + \ldots + e_{\nu-2}\) and \(e_a\) is the \(\nu - 2\) dimensional unit vector in the \(a\) direction, i.e. \((e_a)_b = \delta_{a,b}\) for \(a \in \{1, 2, \ldots, \nu - 2\}\). We define \(e_a\) to be zero for \(a \notin \{1, 2, \ldots, \nu - 2\}\). The exponent \(\eta\) is given in (30) and (37) as
\[
\eta = \frac{1}{4}(\hat{r} + l - \hat{s} - 1) + \gamma(\hat{r}, l, \hat{s})
\]
(1.8)
where \(\gamma(\hat{r}, l, \hat{s}) = \frac{\hat{r}^2}{4(\nu+1-N)} + \frac{\hat{s}^2}{4(N+2)} - \frac{1}{8} - \frac{\hat{s}^2}{4(\nu+1)}\). The relation between (1.4) and (1.3) (and similarly (1.6) and (1.7)) will become clear once we introduce the polynomial expressions for the branching functions in section 3 which correspond to the RSOS configuration sums.

The polynomial generalizations of (1.1) for the \(\hat{s}u(2)_{\nu-N-1} \times \hat{s}u(2)_N/\hat{s}u(2)_{\nu-1}\) coset models are given in terms of
\[
X^N_L[A, u, Q] = \sum_{m \geq 0, \text{e.o. restr.} Q} q^{\frac{1}{2}mCm - \frac{1}{2}Am} \prod_{i=1}^{\nu-2} \left[ \frac{1}{2}(Im + u + Le_N)_i \right]^{-\frac{1}{2}}
\]
(1.9)
where the argument \(Q \in (\mathbb{Z}_2)^{\nu-2}\) and the arguments \(A\) and \(u\) are \((\nu - 2)\) dimensional vectors with integer values. We will actually find that the polynomial form for the branching functions are given as linear combinations of \(X^N_L\) of the form
\[
Y^N_L[A, u, Q; A', u', Q'] = X^N_L[A, u, Q] - q^{\frac{\nu+1}{2}} X^N_L[A', u', Q']
\]
(1.10)
where sometimes the second term is absent (the detailed results are presented in section 3 in (33) and (34)). The branching functions of the coset conformal field theories can be obtained from these polynomial expressions by taking the limit \(L \to \infty\) \([35]-[37]\). Since we always assume \(q < 1\) the second term drops out in this limit but this term will be essential for the proof of the recursion relation. Notice also that the inhomogeneous term \(L\) sits in the \(N^{th}\) slot in (1.9). Using \(\lim_{n \to \infty} \left[ \begin{array}{c} n \\ m \end{array} \right] = \frac{1}{(q)_m}\) one finds that in the limit \(L \to \infty\) the
variable $m_N$ is treated differently from the other $m$’s. This agrees with the results in [3] for $N = 1$ and [24] for $N = 2$.

The plan of the paper is as follows. In section 2 we review the general RSOS models and their relation to the conformal coset models $\hat{su}(2)_{\nu-N-1} \times \hat{su}(2)_N/\hat{su}(2)_{\nu-1}$. The path space interpretation and the recursion relation for the configuration sums of the RSOS models will be essential for the proof of the fermionic forms given in this paper. In section 3 we will state the explicit fermionic polynomial expressions of the coset models. The proof of these formulas will be presented in section 4. In subsection 4.1 we will get acquainted with the telescopic expansion technique and state three important lemmas the proof of which is reserved for the appendix. In subsection 4.2, 4.3 and 4.4 we prove the initial conditions, the equality of two forms for the “straight” characters $X(s|r,r)$ and the recursion relations respectively. We conclude in section 5 with a discussion of the transformation of the polynomial forms under the duality $q \rightarrow q^{-1}$ which correspond to the configuration sums of the RSOS model in regime II and in the critical region to the branching functions of the coset model $\hat{su}(2\nu - 2)_1/\hat{sp}(2\nu - 2)_1$ [38], [39]. These branching functions can also be identified with the $Z_{\nu-1}$ parafermion branching functions.

2 The general RSOS model

It was first observed by Huse [38] that the configuration sum of the ABF model [34] in regime III are off-critical extensions of the branching functions of the unitary minimal model $M(p,p+1)$. Since then many connections between statistical mechanical configuration sums and conformal field theory characters/branching functions have been revealed. For example, in the other regimes the ABF model corresponds to the minimal models $M(2p-1,2p+1), M(2,2p+1)$ and the $Z_p$ parafermion model. The ABF model has been generalized by the Kyoto group [35]-[37] to a two parameter RSOS model. This RSOS model in regime III corresponds to the $\hat{su}(2)_{\nu-N-1} \times \hat{su}(2)_N/\hat{su}(2)_{\nu-1}$ coset model and the RSOS model in regime II corresponds to the coset model $\hat{su}(2\nu - 2)_1/\hat{sp}(2\nu - 2)_1$ and also to the $Z_{\nu-1}$ parafermion model. The configuration sum of the RSOS model in regime III and regime II are related by the dual transformation $q \rightarrow q^{-1}$. We will use this correspondence between the RSOS model and the conformal coset models to prove the fermionic representation of the $\hat{su}(2)_{\nu-N-1} \times \hat{su}(2)_N/\hat{su}(2)_{\nu-1}$ branching functions.

The states of the general RSOS models can be described by paths and the configuration sums are weighted sums over all possible paths in the following way. Consider a square lattice $L$. To each site $i$ of $L$ one associates a state variable $l_i$ which can take the values $l_i = 1, 2, \ldots, \nu$. Two adjacent state variables $l_i$ and $l_{i+1}$ are called admissible ($l_i \sim l_{i+1}$) if they fulfill the following conditions:

$$l_i - l_{i+1} = -N, -N + 2, \ldots, N$$
$$l_i + l_{i+1} = N + 2, N + 4, \ldots, 2\nu - N$$

(2.1)

where $\nu$ and $N$ are arbitrary positive integers and $i$ runs from 1, 2, \ldots, $L+2$. In terms of the path space $N$ defines the value for the highest possible step in the vertical direction, $\nu$ and $L$ define the boundaries of the lattice in the vertical and horizontal direction, respectively.
The one dimensional configuration sum for the RSOS models is given by \[34\], \[36\], \[37\]
\[X_L(a|b, c) = \sum_{l_1} q^{\sum_{j=1}^{L} \frac{1}{2} |l_{j+2} - l_j|} \tag{2.2}\]
where \(l_1 = a, l_{L+1} = b, l_{L+2} = c\) and \(l_1 \sim l_2 \sim \cdots \sim l_{L+2}\). \(X_L\) is uniquely determined by 1) the initial condition
\[X_0(a|b, c) = \delta_{a,b} \tag{2.3}\]
and 2) the recursion relation
\[X_L(a|b, c) = \sum_{d \sim b} q^{\frac{1}{2} |d-c|} X_{L-1}(a|d, b) \tag{2.4}\]
where \(\sum^\prime\prime\) is the sum over \(d\) such that \(d \sim b\) and \(X_L\) is defined to be zero if \(a, b, c\) do not satisfy \((2.1)\).

The bosonic form for the configuration sum of the RSOS lattice models was first calculated by ABF \[34\] for the case \(N = 1\) (in particular the Ising model and the hard square gas) and for general \(N\) by the Kyoto group \[36\], \[37\]. The corresponding characters for the conformal field theories for \(N = 1\) were given by Rocha-Caridi \[30\] based on work by Feigin and Fuchs \[31\]. The characters and branching functions for \(N = 2\) have been given in \[32\] and for the general coset models \(\widehat{su}_2 \times \widehat{su}_2 \times \widehat{su}_2\) in \[33\].

Because of the equivalence between the coset conformal field theories \(\widehat{su}_2(2)_{\nu-N-1} \times \widehat{su}_2(2)_{\nu} / \widehat{su}_2(2)_{\nu-1}\) and the general RSOS models at the critical point one can interpret the expressions for the branching function of a conformal field theory also as the partition functions of the RSOS model in the limit \(L \to \infty\). The interpretation in terms of the RSOS models has the advantage that the partition functions have finite dimensional polynomial representations which obey the recursion relations \((2.4)\). The existence of these recursion relations enables us to prove the fermionic representations of the partition functions.

We will prove the recursion relations in the spirit of \[12\] using telescopic expansions. The telescopic expansions are based on the recursion relations of the Gaussian polynomials
\[
\begin{bmatrix} n \\ m \end{bmatrix} = q^n \begin{bmatrix} n-1 \\ m \end{bmatrix} + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \tag{2.5}
\]
\[
\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \tag{2.6}
\]
Notice that the recursion relations do not hold for \(m = n = 0\). In this case one gets the contradiction \(1 = 0\). This leads to some complications for nonunitary models \[28\] but won’t be encountered in this paper.

### 3 Fermionic form of the configuration sums of the RSOS model and branching functions

In this section we present the fermionic polynomial expression for the configuration sums of the RSOS model in regime III in the form of \([14]\). To this end let us define
\[A_{r,s}^{\text{down}} = e_{s-1}\]
and

\begin{align*}
A_{r,s}^{\text{up}} &= e_{\nu-s} \\
u_{r,s}^{\text{up}} &= e_{\nu-s} + e_{\nu-r} \\
Q_{r,s}^{\text{up}} &= (s-1) \rho + (e_{\nu-2} + e_{\nu-4} + \ldots + (e_{\nu+1-s} + e_{\nu+3-s} + \ldots) \\
&\quad+ L(e_{N-1} + e_{N-3} + \ldots) \tag{3.2}
\end{align*}

where again \( \rho = e_1 + e_2 + e_3 + \ldots + e_{\nu-2} \) and \( e_a \) is the \( \nu-2 \) dimensional unit vector in the \( a \) direction, i.e \( (e_a)_b = \delta_{a,b} \) for \( a \in \{1, 2, \ldots, \nu-2\} \). We define \( e_a \) to be zero for \( a \not\in \{1, 2, \ldots, \nu-2\} \).

Our result for the fermionic form of the RSOS configuration sum is given by

\begin{align*}
X_L(s|r, r + N) &= X_L^N [A_{r,s}^{\text{down}}, u_{r,s}^{\text{down}}, Q_{r,s}^{\text{down}}] \\
X_L(s|r, r - N) &= X_L^N [A_{r,s}^{\text{up}}, u_{r,s}^{\text{up}}, Q_{r,s}^{\text{up}}] \tag{3.3}
\end{align*}

and

\begin{align*}
X_L(s|r, r + N - 2n) &= X_L^N [A_{r-n,s}^{\text{down}}, u_{r-n,s}^{\text{down}} + e_n, Q_{r-n,s}^{\text{down}} + (e_{n-1} + e_{n-3} + \ldots)] \\
&\quad- \theta(r > n + 1) q^{\frac{L+1}{2}} X_L^N [A_{r-n-1,s}^{\text{down}}, u_{r-n-1,s}^{\text{down}} + e_{n-1}, Q_{r-n-1,s}^{\text{down}} + (e_{n-2} + e_{n-4} + \ldots)] \tag{3.4}
\end{align*}

\begin{align*}
X_L(s|r, r - N + 2n) &= X_L^N [A_{r+n,s}^{\text{up}}, u_{r+n,s}^{\text{up}} + e_n, Q_{r+n,s}^{\text{up}} + (e_{n-1} + e_{n-3} + \ldots)] \\
&\quad- \theta(r < \nu - n) q^{\frac{L+1}{2}} X_L^N [A_{r+n+1,s}^{\text{up}}, u_{r+n+1,s}^{\text{up}} + e_{n-1}, Q_{r+n+1,s}^{\text{up}} + (e_{n-2} + e_{n-4} + \ldots)]
\end{align*}

for \( n = 1, 2, \ldots, \lfloor \frac{N}{2} \rfloor \) where \([x]\) denotes the integer part of \( x \) and \( \theta(x > a) = \begin{cases} 1 & \text{for } x > a \\ 0 & \text{otherwise} \end{cases} \).

For \( N = 1 \) the polynomials in (3.3) with \( A, u, Q \) as given in (3.1) and (3.2) coincide with Melzer’s polynomials \( \{p\} \) (the up and down cases in this paper are encoded by \( L \not\equiv s-r \mod 2 \) and \( L \equiv s-r \mod 2 \) in Melzer’s paper). Melzer gives a second set of fermionic expressions for the configuration sum. This set can also be generalized to higher \( N \)

\begin{align*}
A_{r,s}^{\text{down}} &= e_{\nu-s} \\
u_{r,s}^{\text{down}} &= e_{\nu-s} + e_{\nu-r} \\
Q_{r,s}^{\text{down}} &= (e_{\nu-s-1} + e_{\nu-s-3} + \ldots) + (e_{\nu-r-1} + e_{\nu-r-3} + \ldots) \\
&\quad+ L(e_{N-1} + e_{N-3} + \ldots) \tag{3.5}
\end{align*}
These two sets lead exactly to the same polynomials as the sets (3.1) and (3.2) and are merely a rewriting of the fermionic expressions. The equivalence of both fermionic representations readily follows from lemma 1 (see below). Hence we will focus in the following on the set given in (3.1) and (3.2).

Notice that there exists a symmetry between the “up” and the “down” expression for the configuration sum which implies the mirror symmetry of the conformal grid. One may verify that

\[ A_{r,s}^{\text{up}} = e_{s-1} \]
\[ u_{r,s}^{\text{up}} = e_{s-1} + e_{r-1} \]
\[ Q_{r,s}^{\text{up}} = (e_{r-2} + e_{r-4} + \ldots) + (e_{s-2} + e_{s-4} + \ldots) + L(e_{N-1} + e_{N-3} + \ldots) \] (3.6)

for both sets (3.1), (3.2) and (3.5), (3.6) where \( \tilde{r} = \nu + 1 - r \) and \( \tilde{s} = \nu + 1 - s \). Hence it follows that

\[ X_L^{N-2n \downarrow}(s|r,r+N-2n) = X_L^{N-2n \uparrow}(\tilde{s}|\tilde{r},(N-2n)) \] (3.8)

for \( n = 0, 1, 2, \ldots, \left[ N/2 \right] \).

It is also important to notice that for even \( N \) one gets two different expressions for the “straight” character. In this case \( n = N/2 \) and \( X_L(s|r,r) \) is given once in terms of \( A^{\text{up}}, u^{\text{up}}, Q^{\text{up}} \) and on the other hand in terms of \( A^{\text{down}}, u^{\text{down}}, Q^{\text{down}} \). We will prove the equality of both expressions in section 4.3. Therefore in the proof of the recursion relations we can choose whichever form is more convenient.

One obtains the branching functions of the rational coset conformal field theories \( \hat{su}(2)_{\nu-N-1} \times \hat{su}(2)_{N}/\hat{su}(2)_{\nu-1} \) by taking the limit \( L \to \infty \)[36, 37]. Since we always assume \( q < 1 \) the second term in (3.4) drops out in the limit. If one further uses

\[ \lim_{n \to \infty} \left[ \begin{array}{c} n \\ m \end{array} \right] = \frac{1}{(q)_m} \] (3.9)

one gets

\[ \lim_{L \to \infty} X_L(s|r,r+N-2n) = \sum_{m \geq 0, \text{e.o. restr. } Q} q^{\frac{1}{2}mc - \frac{1}{2}A_m} \frac{1}{(q)_{mN}} \prod_{i=1}^{\nu-2} \left[ \frac{1}{(q)_{m_i}} i \right] \] (3.10)

for \( n = 0, 1, \ldots, \left[ N/2 \right] \) where from (3.1)

\[ A = e_{s-1} \]
\[ u = e_{s-1} + e_{\nu - r + n} + e_n \]
\[ Q = (r - n - 1)\rho + (e_{s-2} + e_{s-4} + \ldots) + (e_{\nu + 1 - r + n} + e_{\nu + 3 - r + n} + \ldots) \]
\[ + (e_{n-1} + e_{n-3} + \ldots) \]  
(3.11)

or from (3.5)

\[ A = e_{\nu - s} \]
\[ u = e_{\nu - s} + e_{\nu - r + n} + e_n \]
\[ Q = (e_{\nu - s - 1} + e_{\nu - s - 3} + \ldots) + (e_{\nu - r + n - 1} + e_{\nu - r + n - 3} + \ldots) \]
\[ + (e_{n-1} + e_{n-3} + \ldots) \]  
(3.12)

Similarly for \( X_L(s|r, r - N + 2n) \) one gets the same form as in (3.10) for the limit \( L \to \infty, L \text{ even} \) but with

\[ A = e_{s-1} \]
\[ u = e_{s-1} + e_{r+n-1} + e_n \]
\[ Q = (e_{r+n-2} + e_{r+n-4} + \ldots) + (e_{s-2} + e_{s-4} + \ldots) \]
\[ + (e_{n-1} + e_{n-3} + \ldots) \]  
(3.13)

It has been established in [36] and [37] that the limit of the configuration sum of the RSOS model equals the branching functions of the coset conformal field theory

\[ q^{\eta} c_{\hat{r}, \hat{s}}^{(l)} = \lim_{L \to \infty, L \text{ even}} X_L(a|b, c) \]  
(3.15)

where

\[ \hat{r} = \frac{1}{2}(b + c - N) \]
\[ \hat{l} = \frac{1}{2}(b - c + N) + 1 \]
\[ \hat{s} = a \]
\[ \eta = \frac{1}{4}(b - a) - \gamma(\hat{r}, \hat{l}, \hat{s}) \]

and

\[ \gamma(j_1, j_2, j_3) = \frac{j_1^2}{4m_1} + \frac{j_2^2}{4m_2} - \frac{1}{8} - \frac{j_3^2}{4m_3} \]  
(3.16)

where in turn \( m_1 = \nu + 1 - N, m_2 = N + 2, m_3 = \nu + 1 \). Hence the polynomial forms (3.3) and (3.4) exactly lead to (1.1) with (1.4) or (1.6) for the “down” case and (1.5) or (1.7) for the “up” case in the limit \( L \to \infty, L \text{ even} \).
4 The proof of the fermionic configuration sums

4.1 The telescopic expansion technique

In this section we set up the tools needed for the proof of the fermionic sum formulas. We use the telescopic expansion technique first developed in [1, 2]. We introduce a shorthand notation (see (4.1)), and start by deriving important identities. At the end of this subsection we state the general lemmas needed for the proof of (3.3)-(3.4). The proof of the lemmas is reserved for the appendix.

Denote
\[
\left\{ \begin{array}{c} A \\ B \end{array} \right\} \equiv q^{\frac{1}{4}mCm - \frac{1}{2}Am} \prod_{i=1}^{\nu-2} \left[ \frac{1}{2}(Im + u + LeN + A)_{i} \right] \quad (4.1)
\]

where \( u \) and \( A \) should be specified before each use of the abbreviation. This notation is useful since it avoids rewriting symbols that do not change during the calculations. The entries \( A \) and \( B \) can arise either from \( u^{\text{up}} \) and \( u^{\text{down}} \), variable changes \( m_{i} \to m_{i} + \text{constant} \) or the use of recursion relations (2.5) and (2.6).

For example for \( \nu = 8, N = 2, s = 1, r = 5, L \) even and \( A = u = 0 \) we have

\[
X^{2}_{L}[A^{\text{up}}_{r,s}, u^{\text{up}}_{r,s}, Q^{\text{up}}_{r,s}] = \sum_{m_{1}, m_{3} \text{ odd}} \sum_{m_{2}, m_{4}, m_{5}, m_{6} \text{ even}} \left\{ \begin{array}{c} e_{4} \\ 0 \end{array} \right\}
\]

where we have made the variable changes \( m_{1} \to m_{1} + 1, m_{3} \to m_{3} + 1 \) to obtain the second line. \( n_{i} \) is defined by

\[
n_{i} = \frac{1}{2}(m_{i-1} + m_{i+1}) - m_{i}. \quad (4.3)
\]

This combination always arises from the variable change in the exponent \( \frac{1}{4}mCm \) and also as the phase in the recursion relation (2.6). We now perform our first telescopic expansion. Use the recursion relation (2.6) in the 4th slot in (4.2)

\[
\sum_{m \text{ even}} \left( q^{-n_{1}-n_{3}+1} \left\{ \begin{array}{c} 2e_{2} \\ e_{1} + e_{3} \end{array} \right\} 
+ q^{-n_{1}-n_{3}+n_{4}+2} \left\{ \begin{array}{c} 2e_{2} \\ e_{1} + e_{3} - e_{4} \end{array} \right\} \right) \quad (4.4)
\]

Change variables \( m_{4} \to m_{4} + 2 \) in the second term

\[
\sum_{m \text{ even}} \left( q^{-n_{1}-n_{3}+1} \left\{ \begin{array}{c} 2e_{2} \\ e_{1} + e_{3} \end{array} \right\} 
+ q^{-n_{1}-n_{3}+n_{4}+2} \left\{ \begin{array}{c} 2e_{2} + 2e_{3} + 2e_{5} \\ e_{1} + e_{3} + e_{4} \end{array} \right\} \right) \quad (4.5)
\]
(Notice that the 2 in the exponent becomes 1 since one gets 2 from $\frac{1}{2}mCm$, $-2$ from $n_4$ and $-1$ from $-n_3$). Repeating this procedure for the $5^{th}$ slot etc. one finally obtains

$$
\sum_{m \text{ even}} \left( q^{-n_1-n_3+1} \begin{cases} 2e_2 \\ e_1 + e_3 \end{cases} 
+ q^{-n_1-n_3-n_4+1} \begin{cases} 2e_2 + 2e_3 \\ e_1 + e_3 + e_4 \end{cases} 
+ q^{-n_1-n_3-n_4-n_5+1} \begin{cases} 2e_2 + 2e_3 + 2e_4 \\ e_1 + e_3 + e_4 + e_5 \end{cases} 
+ q^{-n_1-n_3-n_4-n_5-n_6+1} \begin{cases} 2e_2 + 2e_3 + 2e_4 + 2e_5 \\ e_1 + e_3 + e_4 + e_5 + e_6 \end{cases} \right)
$$

(4.6)

In general there is an extra phase $q^{-\frac{1}{2}m_{\nu-s}}$ or $q^{-\frac{1}{2}m_{s-1}}$ and an additional entry $e_{\nu-s}$ or $e_{s-1}$ in the “up” or “down” case respectively. Then one gets an extra phase $\frac{1}{2}$ when using the recursion relation in the $(\nu-s)^{th}$ or $(s-1)^{th}$ slot which will be turned into $-\frac{1}{2}$ by doing the variable change $m_{\nu-s} \to m_{\nu-s} + 2$ or $m_{s-1} \to m_{s-1} + 2$. Hence it is useful to define

$$
n_i' = n_i + \frac{1}{2}(A)_i.
$$

(4.7)

A general telescopic expansion to the right of length $M$ starting from $i$ with $A = u$ corresponding to recursion relation (2.6) is then given by

$$
\sum_{Q} q^{-n_i'} \left( A_{\leq i} + A_{>i+M} + 2e_{i+1}' \right) 
= \sum_{l=i}^{i+M} \sum_{Q} q^{-\sum_{k=i}^{i+l} n_k'} \left( A_{\leq i} + A_{>i+M} + 2\sum_{k=i}^{i+l-1} e_k + 2e_{i+M+1}' \delta_{i+M}' \right) 
$$

(4.8)

for $i + M < \nu - 2$. Throughout the whole paper we define empty sums to be zero (i.e. $\sum_{k=i}^{i} = 0$ if $l < i$) which avoids writing lots of $\theta$-functions. Further $\sum_{Q}$ stands for $\sum_{m \geq 0, e.o.\ restr. Q}$. This abbreviation will also be used in the rest of the paper. $Q \in (Z_2)^{\nu-2}$ is arbitrary and $A_{\leq i}, A_{>i+M}, B_{<i}$ and $B_{>i+M}$ are $\nu - 2$ dimensional vectors with nonzero entries only in the slots $\leq i, > i + M, < i$ and $> i + M$ respectively. Notice that (4.8) is also valid for $i + M = \nu - 2$ that is the telescopic expansion hits the end of the product of the Gaussians. The term in the top row $2e_{i+M+1}' \delta_{i+M}'$ simply drops out. (4.6) is an example of this case.

One can derive (4.8) in a similar fashion to (4.6) by using repeatedly (2.6) in the slots $k = i + 1, i + 2, \ldots, i + M$ followed by a variable change $m_k \to m_k + 2$. Another form of telescopic expansion corresponding to recursion relation (2.6) is given by

$$
\sum_{Q} q^{-\sum_{k=i}^{i+M} n_k'} \left( A_{\leq i} + A_{>i+M} + 2\sum_{k=i}^{i+M} e_k \right) 
= \sum_{l=i}^{i+M} \sum_{Q} q^{-\sum_{k=i}^{i+l} n_k'} \left( A_{\leq i} + A_{>i+M} + 2\sum_{k=i}^{i+l-1} e_k \right)
$$

(4.9)
where we use a similar notation to (4.8). (4.9) is derived via repeated application of (2.6) starting from slot $i + M$. There are no variable changes involved.

The first type of telescopic expansion (4.8) goes from left to right whereas (4.9) goes from right to left. Analogous telescopic expansions of each type in the opposite direction hold as well. There are also telescopic expansions corresponding to recursion relation (2.5).

All of these formulas are given in appendix A.

There again exists a mirror identity for recursion relation (2.5) which is given in appendix A.

For the proof of (3.3)-(3.4) we need further identities. One of them is the mirror identity

$$
\sum_{l=1}^{i+M} q^{-\sum_{k=i}^{l} n_k} \left\{ \begin{array}{ll}
A_{\leq i} + A_{> i + M} + 2 \sum_{k=i}^{l-1} e_k \\
B_{\leq i} + B_{> i + M} + \sum_{k=i}^{l} e_k
\end{array} \right\}
$$

(4.10)

There again exists a mirror identity for recursion relation (2.5) which is given in appendix A.

A mirror identity similar to (4.10) has already been proven in [12]. Since we use a different notation in this paper we repeat the main steps of the proof. Define for $t = 0, 1, \ldots, M + 1$

$$
Z_t = \sum_{l=1}^{i+M} q^{-\sum_{k=i}^{l} n_k} \left\{ \begin{array}{ll}
A_{\leq i} + A_{> i + M} + 2 \sum_{k=i}^{l-1} e_k \\
B_{\leq i} + B_{> i + M} + \sum_{k=i}^{l} e_k
\end{array} \right\}
$$

(4.11)

where again empty sums are defined to be zero. Notice that $Z_0 =$lhs of (4.10) and $Z_{M+1} =$rhs of (4.10). To prove that $Z_0 = Z_1 = \ldots = Z_{M+1}$ expand the term with $l = i + M - t$ of the second sum in (4.11) via (4.9) from $i + M - t - 1$ to $i$ to the left. One then obtains

$$
\sum_{Q} q^{-\sum_{k=i}^{l} n_k} \left\{ \begin{array}{ll}
A_{\leq i} + A_{> i + M} + 2 \sum_{k=i}^{l-1} e_k + 2 \sum_{k=i}^{l} e_k \\
B_{\leq i} + B_{> i + M} + \sum_{k=i}^{l} e_k
\end{array} \right\}
$$

(4.12)
The last term in (4.12) is already the $l' = i + M - t$ term of the first sum in $Z_{l+1}$. The first term in (4.12) can be combined with the remaining terms in the second sum in (4.11) in the $(i + M - t)$th slot

$$
\sum_{l=i}^{i+M-t-1} \sum_{Q} g^{-l} \sum_{k=i}^{l} q_{k}^{-l} - \sum_{k=i+M-t}^{i+M} q_{k}^{-l}
$$

$$
\left\{ \begin{array}{l}
A_{\leq i} + A_{>i+M} + 2 \sum_{k=i+M}^{l} e_{k} + 2 \sum_{k=i+M-t}^{l} e_{k} \\
B_{<i} + B_{>i+M} + \sum_{k=i}^{l} e_{k} + \sum_{k=i+M-t}^{l} e_{k}
\end{array} \right\}
$$

which is the second sum in $Z_{l+1}$. Hence (4.11) is proven.

Further we will need the extended mirror identity.

$$
\sum_{Q} \sum_{l=i+2}^{i+M} q^{-l} \sum_{k=i}^{l} q_{k}^{-l} \left\{ \begin{array}{l}
A_{\leq i} + A_{>i+M} + 2 \sum_{k=i+M}^{l} e_{k} + 2 \sum_{k=i+M-t}^{l} e_{k} \\
B_{<i} + B_{>i+M} + e_{i} + \sum_{k=i+M-t}^{l} e_{k}
\end{array} \right\}
$$

The proof of this formula goes as follows. Telescopically expand to different lengths each term in the sum on the lhs via (4.13) as follows. Expand the $l$th term to the right from $i$ to $l-2$. Hence there is no expansion for the term $l = i + 2$, however we expand the term $l = i + 3$ once, term $l = i + 4$ twice etc. We then obtain

$$
\sum_{Q} \sum_{l=i+2}^{i+M} \sum_{l' = i}^{l-2} q^{-l} \sum_{k=i}^{l'} q_{k}^{-l'} - \sum_{k=i+M}^{l} q_{k}^{-l'}
$$

$$
\left\{ \begin{array}{l}
A_{\leq i} + A_{>i+M} + 2 \sum_{k=i+M}^{l} e_{k} + 2 \sum_{k=i+M}^{l-2} \delta_{l', l-2} + 2 \sum_{k=i+M-t}^{l} e_{k} \\
B_{<i} + B_{>i+M} + \sum_{k=i+M}^{l} e_{k} + \sum_{k=i+M-t}^{l} e_{k}
\end{array} \right\}
$$

Exchanging the sums yields $\sum_{l' = i}^{l+M-2} \sum_{l'' = i}^{l+M-2}$. We can now recombine the $l$-sums via telescopic expansions to the left and obtain the rhs of (4.14).

Finally, we need a special telescopic expansion. It will be needed when either $r$ or $s$ is even.

$$
\sum_{Q} \sum_{l = i}^{\nu-3} q^{-l} \sum_{k=i}^{l} q_{k}^{-l} \left\{ \begin{array}{l}
A_{\leq i} + 2 \sum_{k=i}^{l-1} e_{k} + e_{\nu-2} \\
B_{<i} + \sum_{k=i}^{l-1} e_{k}
\end{array} \right\}
$$

$$
= \sum_{Q} q^{-l} \sum_{k=i+2}^{\nu-2} q_{k}^{-l} \left\{ \begin{array}{l}
A_{\leq i} + 2 \sum_{k=i+1}^{\nu-3} e_{k} + e_{\nu-2} \\
B_{<i} + e_{i} + \sum_{k=i+2}^{\nu-2} e_{k}
\end{array} \right\}
$$

The proof of this formula is similar to the one of the extended mirror identity. To this end we telescopically expand each term on the lhs to the left starting from $\nu - 2$. The $l$th term should be expanded to slot $l + 2$

$$
\sum_{Q} \sum_{l = i}^{\nu-3} \sum_{l' = i+2}^{\nu-1} q^{-l} \sum_{k=i}^{l} q_{k}^{-l} - \sum_{k=i+2}^{\nu-2} q_{k}^{-l'} + \frac{1}{2} \delta_{l' < \nu-1}
$$

$$
\left\{ \begin{array}{l}
A_{\leq i} + 2 \sum_{k=i}^{l-1} e_{k} + 2 \sum_{k=i+2}^{\nu-2} e_{k} + 2 e_{l+1} \delta_{l', l+2} - e_{\nu-2} \\
B_{<i} + \sum_{k=i}^{l} e_{k} + \sum_{k=i+2}^{\nu-2} e_{k}
\end{array} \right\}
$$

(4.17)
Exchanging again the sums $\sum_{l' = i + 1}^{\nu - 1} \sum_{l = i}^{\nu - 2}$ and recombining the $l$-sum via inverse telescopic expansion results in

$$
\sum_{Q} \sum_{l' = i + 2}^{\nu - 1} q^{- \nu - 2} \sum_{k = i}^{\nu - 2} n_k' + \frac{1}{2} \theta (l' < \nu - 1) \left\{ \begin{array}{ll}
A_{l' i} + 2 e_{l' + 1} + 2 \sum_{l' = i + 1}^{\nu - 2} e_{l' - e_{l' - 2}} \\
B_{l' i} + e_i + \sum_{l' = i + 1}^{\nu - 2} e_{l' - e_{l' - 2}}
\end{array} \right. 
$$

(4.18)

Using then (4.9) for the remaining sum in a slightly modified version for slot $\nu - 2$ yields the rhs of (4.16). The analogue of (4.16) in opposite direction is given in appendix A.

With the help of the telescopic expansions and identities discussed above we can prove three lemmas which enable us to give the proof of the fermionic configuration sums. The first lemma makes it possible to transform the phase $q^{- \frac{1}{2} m_{s - 1}}$ and the entry $e_{s - 1}$ into $q^{- \frac{1}{2} m_{\nu - s}}$ and $e_{\nu - s}$ respectively and vice versa. Since recursion relation (2.4) involves both $X^{\text{up}}$ and $X^{\text{down}}$ we will be able to transform them into comparable formulas. This lemma also establishes the equivalence between the two different sets (3.1), (3.2) and (3.5), (3.6).

Hence we can restrict ourselves to the proof of (3.3) and (3.4) with $A, u, Q$ given by (3.1) and (3.2) in the following.

The configuration sums have the entry $\frac{L}{\nu}$ in the $N^\text{th}$ slot. Since the telescopic expansions are sensitive to the entries in the top line one has to treat the part to the right of the $N^\text{th}$ slot and the part to the left of the $N^\text{th}$ slot separately. Lemmas 2 and 3 give the necessary tools to handle this and will be frequently used in the following.

We now state the three lemmas:

**Lemma 1** With $A = u = 0$ we have in the notation of (4.4)

$$
\sum_{Q + (e_{s - 2} + e_{s - 4} + \ldots)} q^{- \frac{1}{2} m_{s - 1}} \left\{ \begin{array}{ll}
A + e_{s - 1} \\
0
\end{array} \right. 
= \sum_{Q + (s - 1) \rho + (e_{\nu + 1 - s} + e_{\nu + 3 - s} + \ldots)} q^{- \frac{1}{2} m_{\nu - s}} \left\{ \begin{array}{ll}
A + e_{\nu - s} \\
0
\end{array} \right. 
$$

(4.19)

where $A$ is an $\nu - 2$ dimensional vector with integer values and $Q \in (\mathbb{Z}_2)^{\nu - 2}$ arbitrary.

Part a) and b) of the next lemma are useful for treating the slots to the left of the $N^\text{th}$ slot excluding and including the $N^\text{th}$ slot respectively for $N > 2$.

**Lemma 2 a)** With $A = u$ we have in the notation of (4.4)

$$
\sum_{Q + (e_{a - 1} + e_{a - 3} + \ldots)} \left\{ \begin{array}{ll}
e_{a} + B \geq N \\
0
\end{array} \right. 
= \sum_{Q + (e_{a} + e_{a + 2} + \ldots)_{\leq N} + a \rho_{N}} \left\{ \begin{array}{ll}
e_{a - 1} + e_{N - 1} + B \geq N \\
0
\end{array} \right. 
- \sum_{Q + (e_{a - 1} + e_{a + 1} + \ldots)_{\leq N} + a \rho_{N}} q^{\frac{1}{2}} \left\{ \begin{array}{ll}
e_{a - 2} + e_{N} + B \geq N \\
0
\end{array} \right. 
$$

(4.20)

where $Q \in (\mathbb{Z}_2)^{\nu - 2}$ arbitrary, $\rho_{N} = e_{1} + e_{2} + \ldots + e_{N - 1}$, $B \geq N$ is a $(\nu - 2)$ dimensional vector that has non-negative values only for the entries bigger or equal to $N$, i.e. $B \geq N; =
\[
\begin{cases}
0 & \text{for } 1 \leq i < N \\
\in \mathbb{N} \cup \{0\} & \text{for } N \leq i \leq \nu - 2
\end{cases}
\]

, \ a \in \{0, 1, 2, \ldots, N-2\} \text{ and } \bar{a} = N - a. (\ldots)_{<N} \text{ denotes the projection onto the entries smaller than } N, \ i.e. (\ldots)_{<N} = \begin{cases}
e_n & \text{for } n < N \\
0 & \text{for } n \geq N
\end{cases}.

\textbf{b) Under the same conditions as part a) we have}

\[
\sum_{Q+(e_{r+a}+e_{r+a-2}+\ldots)\geq N} \left\{ \begin{array}{ll}
e_{r+a+1} + B_{>N} \\
0
\end{array} \right. \\
= \sum_{Q+(e_{r+a}+e_{r+a-2}+\ldots) < N+(a-1)\rho < N} \left\{ \begin{array}{ll}
e_{r+\bar{a}-1} + e_{N+1} + B_{\leq N} \\
0
\end{array} \right. \\
- \sum_{Q+(r-a)\rho \geq N + e_N + (e_{r-r+\bar{a}} + e_{r-r+\bar{a}+2} + \ldots \ )} q^{1+0} \left\{ \begin{array}{ll}
e_{r+\bar{a}} + e_{N+1} + B_{\leq N} \\
0
\end{array} \right. \\
\text{(4.21)}
\]

where \( B_{>N} \) is a \((\nu-2)\) dimensional vector that has non-negative values only for the entries bigger than \( N \).

The next lemma is the analogue of lemma (2) for the slots to the right of the \( N \)th slot.

\textbf{Lemma 3 a) With } \( A = u \) \( \text{we have in the notation of (4.1) }

\[
\sum_{Q+(e_{r+a}+e_{r+a-2}+\ldots)\geq N} \left\{ \begin{array}{ll}
e_{r+a+1} + B_{\leq N} \\
0
\end{array} \right. \\
= \sum_{Q+(e_{r+a}+e_{r+a-2}+\ldots) < N+(a-1)\rho \geq N} \left\{ \begin{array}{ll}
e_{r+\bar{a}-1} + e_{N+1} + B_{\leq N} \\
0
\end{array} \right. \\
- \sum_{Q+(r-a)\rho \geq N + e_N + (e_{r-r+\bar{a}} + e_{r-r+\bar{a}+2} + \ldots \ )} q^{1+0} \left\{ \begin{array}{ll}
e_{r+\bar{a}} + e_{N+1} + B_{\leq N} \\
0
\end{array} \right. \\
\text{(4.22)}
\]

where \( Q \in (\mathbb{Z}_2)^{\nu-2} \) arbitrary, \( B_{\leq N} \) is a \((\nu-2)\) dimensional vector that has non-negative values only for the entries smaller or equal to \( N \), \( \rho_{\geq N} = e_N + e_{N+1} + \ldots + e_{\nu-2} \), \( \bar{a} = N - a \) and \( r + a > N \). (\ldots)_{\geq N} \text{ denotes the projection onto the entries bigger or equal to } N.

\textbf{b) Under the same condition as in part a) we have}

\[
\sum_{Q+(e_{r+a}+e_{r+a-3}+\ldots)\geq N} \left\{ \begin{array}{ll}
e_{r+a} + B_{\leq N} \\
0
\end{array} \right. \\
= \sum_{Q+(r-a)\rho \geq N + e_N + (e_{r-r+\bar{a}} + e_{r-r+\bar{a}+2} + \ldots \ )} q^{1+0} \left\{ \begin{array}{ll}
e_{r+\bar{a}} + e_{N+1} + B_{\leq N} \\
0
\end{array} \right. 
\]

14
− \sum_{q \geq N} \left\{ \begin{array}{ll} e_{\nu - r + \tilde{a}} + e_{N-1} + B_{< N} \\ 0 \end{array} \right\} (4.23)

The proof of the three lemmas is given in appendix 3.

4.2 Initial conditions

In this section we prove the initial conditions for the configuration sums

\[ X_0(s|r, r - N + 2n) = \delta_{r,s} \]
\[ X_0(s|r, r + N - 2n) = \delta_{r,s} \]

for \( n = 0, 1, \ldots, \left[ \frac{N}{2} \right] \).

First of all it suffices to show that the initial conditions hold for \( X_{0}^{up} \) because by symmetry (3.8) it follows that

\[ X_{0}^{down}(s| r, r + N - 2n) = X_{0}^{up}(s| \tilde{r}, \tilde{r} - N + 2n) = \delta_{\tilde{r}, \tilde{s}} = \delta_{r,s}. \]

For \( n = 1, 2, \ldots, \left[ \frac{N}{2} \right] \) we have

\[ X_0(s|r, r - N + 2n) = X_0^{N}[A_{r+n,s}^{up}, u_{r+n,s}^{up} + e_{r,s}, Q_{r+n,s}^{up} + (e_{n-1} + e_{n-3} + \ldots)] - \theta(r < \nu - n)q^{\frac{1}{2}}X_0^{N}[A_{r+n+1,s}^{up}, u_{r+n+1,s}^{up} + e_{n-1}, Q_{r+n+1,s}^{up} + (e_{n-2} + e_{n-4} + \ldots)] \]

\[ = \sum_{Q + (e_{r+n-2} + e_{r+n-4} + \ldots) + (e_{n-2} + e_{n-4} + \ldots)} q^{\frac{1}{2}} \left\{ \begin{array}{ll} e_{r+n-1} + e_{n} \\ 0 \end{array} \right\} (4.26) \]

where we used the abbreviation (4.1) with \( A = u = e_{\nu-s} \) and \( Q = (s-1)\rho + (e_{\nu+1-s} + e_{\nu+3-s} + \ldots) \). For \( r = \nu - n \) (4.26) equals to

\[ \sum_{Q + (r-1)\rho + (e_{\nu+1-r} + e_{\nu+3-r} + \ldots)} \left\{ \begin{array}{ll} e_{\nu-r} \\ 0 \end{array} \right\} = X_0^{1}[A_{r,s}^{down}, u_{r,s}^{down}, Q_{r,s}^{down}] \]

(4.27)

(the equal sign follows by lemma 3 since it allows us to transform the phase \( q^{-\frac{1}{2}m_{\nu-s}} \) into \( q^{-\frac{1}{2}m_{\nu-s-1}} \) and the entry \( e_{\nu-s} \) into \( e_{s-1} \)). For \( 2 < r < \nu - n \) (4.24) becomes

\[ \sum_{Q + (e_{r-2} + e_{r-4} + \ldots)} \left\{ \begin{array}{ll} e_{r-1} \\ 0 \end{array} \right\} = X_0^{1}[A_{r,s}^{up}, u_{r,s}^{up}, Q_{r,s}^{up}] \]

(4.28)
To prove this let us first consider the case that \( r \) is odd. In this case (4.26) becomes

\[
\sum_{Q+(e_{n+1}+e_{n+3}+\ldots+e_{r+n-2})} \left\{ \begin{array}{ll} e_{r+n-1} + e_n \\ 0 \end{array} \right\} - \sum_{Q+(e_{n}+e_{n+2}+\ldots+e_{r+n-1})} q^{\frac{r}{2}} \left\{ \begin{array}{ll} e_{r+n} + e_{n-1} \\ 0 \end{array} \right\}.
\] (4.29)

Change variables \( m_i \to m_i + 1 \) for \( i = n + 1, n + 3, \ldots, r + n - 2 \) in the first term and \( i = n, n + 2, \ldots, r + n - 1 \) in the second term. Performing then a telescopic expansion of type (4.8) to the left and using the mirror identity (4.10) followed by the extended mirror identity (4.14) \( \frac{r-3}{2} \) times for the first and second term we obtain

\[
\sum_{l=r-2}^{r+n-2} \sum_{Q} q^{-n'_1-n'_3-\ldots-n'_{r-4}-\sum_{k=r-2}^{l} n'_k + \frac{r-3}{4}} \left\{ \begin{array}{ll} 2e_2 + 2e_4 + \ldots + 2e_{r-3} + 2 \sum_{k=r-2}^{l-1} e_k + 2e_{r+n-1} \\ e_1 + e_3 + \ldots + e_{r-4} + \sum_{k=r-2}^{l} e_k \end{array} \right\}
\]

\[
- \sum_{l=r-2}^{r+n-3} \sum_{Q} q^{-n'_1-n'_3-\ldots-n'_{r-4}-\sum_{k=r-2}^{l} n'_k - n'_{r+n-1} + \frac{r+3}{4}} + \frac{1}{2} \left\{ \begin{array}{ll} 2e_2 + 2e_4 + \ldots + 2e_{r-3} + 2 \sum_{k=r-2}^{l-1} e_k + 2e_{r+n-2} + 2e_{r+n} \\ e_1 + e_3 + \ldots + e_{r-4} + \sum_{k=r-2}^{l} e_k + e_{r+n-1} \end{array} \right\}
\] (4.30)

Changing variables \( m_{r+n-1} \to m_{r+n-1} - 2 \) in the second term we can combine the two sums in (4.30) in the \((r + n - 1)^{th}\) slot

\[
\sum_{l=r-2}^{r+n-3} \sum_{Q} q^{-n'_1-n'_3-\ldots-n'_{r-4}-\sum_{k=r-2}^{l} n'_k + \frac{r+3}{4}} \left\{ \begin{array}{ll} 2e_2 + 2e_4 + \ldots + 2e_{r-3} + 2 \sum_{k=r-2}^{l-1} e_k \\ e_1 + e_3 + \ldots + e_{r-4} + \sum_{k=r-2}^{l} e_k \end{array} \right\}
\]

\[
+ \sum_{Q} q^{-n'_1-n'_3-\ldots-n'_{r-4}-\sum_{k=r-2}^{r+n-2} n'_k + \frac{r+3}{4}} \left\{ \begin{array}{ll} 2e_2 + 2e_4 + \ldots + 2e_{r-3} + 2 \sum_{k=r-2}^{r+n-3} e_k + 2e_{r+n-1} \\ e_1 + e_3 + \ldots + e_{r-4} + \sum_{k=r-2}^{r+n-2} e_k \end{array} \right\}
\] (4.31)

Using now the telescopic expansion (4.8) and changing variables \( m_i \to m_i - 1 \) for \( i = 1, 3, \ldots, r - 2 \) gives exactly (4.28).

The proof of (4.28) for \( r \) even is very similar. In this case one uses the telescopic expansion corresponding to (A.3), applies the mirror and extended mirror identities, changes variables and combines the two sums again in the \((r + n - 1)^{th}\) slot and uses (4.8) to obtain (4.28).

The case \( r = 2 \) is special and can only appear for \( N = 2, n = 1 \) assuming \( 0 < n < \left[ \frac{N}{2} \right] \) since otherwise \((r, r - N + 2n)\) is not admissible. For \( N = 2, n = 1 \) (4.26) becomes \( X^1_0[A_{2,s}^{\downarrow}, u_{2,s}^{\downarrow}, Q_{2,s}^{\downarrow}] \) the proof of which is left to the reader.
Hence we have shown that for \( n = 0, 1, \ldots, \left[ \frac{N}{2} \right] \)

\[
X_0^{\text{up}}(s|r, r - N + 2n) = \begin{cases} 
X^1_{r,s} [A^{\text{down}}_{r,s}, u^{\text{down}}_{r,s}, Q^{\text{down}}_{r,s}] & \text{for } r = \nu - n \text{ or } r = 2 \text{ and } n > 1 \\
X^2_{r,s} [A^{\text{up}}_{r,s}, u^{\text{up}}_{r,s}, Q^{\text{up}}_{r,s}] & \text{otherwise}
\end{cases}
\]  

(4.32)

These are exactly the \( L = 0 \) “up” and “down” characters for \( N = 1 \). Hence the proof of (4.24) reduces to the proof of

\[
X_0^{\text{up}}[A^{\text{up}}_{r,s}, u^{\text{up}}_{r,s}, Q^{\text{up}}_{r,s}] = \delta_{r,s} 
\]  

(4.33)

\[
X_0^{\text{down}}[A^{\text{down}}_{r,s}, u^{\text{down}}_{r,s}, Q^{\text{down}}_{r,s}] = \delta_{r,s}.
\]  

(4.34)

Explicitly we have

\[
X_0^{\text{up}}[A^{\text{up}}_{r,s}, u^{\text{up}}_{r,s}, Q^{\text{up}}_{r,s}] = \sum_{Q + (s - 1)\rho + (e_{r-2} + e_{r-4} + \ldots) + (e_{\nu+s} + e_{\nu+3} + \ldots)} q \frac{m_{\nu-s}}{2} \begin{cases} 
\{ e_{\nu-s} + e_{r-1} \\
0
\end{cases}
\]  

(4.35)

where we used the abbreviation (4.4) with \( A = u = 0 \). \( \{ e_{\nu-s} + e_{r-1} \} \) is only nonzero if there exists a solution for the \( m \)'s of the following set of inequalities

\[
\begin{align*}
0 & \leq m_1 \leq \frac{1}{2}(m_2 + \delta_{r-1,1} + \delta_{\nu-s,1}) \\
0 & \leq m_k \leq \frac{1}{2}(m_{k-1} + m_{k+1} + \delta_{r-1,k} + \delta_{\nu-s,k}) \\
0 & \leq m_{\nu-2} \leq \frac{1}{2}(m_{\nu-3} + \delta_{r-1,\nu-2} + \delta_{\nu-s,\nu-2})
\end{align*}
\]  

(4.36)

Adding all inequalities we get for \( 1 < s < \nu \) and \( 1 < r < \nu \) yields

\[
m_1 + m_{\nu-2} \leq 2
\]  

(4.37)

For even \( L \) (hence especially for \( L = 0 \)) \( r - s \) is even. It follows that \( m_1 \) is always odd and hence \( m_1 = 1 \). Adding all but the first inequality yields \( m_2 + m_{\nu-2} \leq 3 \). We can further deduce from the first inequality that \( m_1 = 1 \leq \frac{1}{2}m_2 \) and since \( m_2 \) is even it follows that \( m_2 = 2 \). Proceeding this way for the case \( r - 1 < \nu - s \) finally yields

\[
m_k = \begin{cases} 
k & \text{for } k = 1, 2, \ldots, r - 1 \\
r - 1 & \text{for } k = r, \ldots, \nu - s \\
\nu + r - s - k - 1 & \text{for } k = \nu - s, \ldots, \nu - 2
\end{cases}
\]  

(4.38)

The last inequality requires \( 0 \leq r - s + 1 \leq \frac{1}{2}(r - s + 2) \) from which follows that \( r = s \). One may verify that \( \frac{1}{2}mCm = \frac{e-1}{\delta} \) for the above determined values for the \( m \)'s. Hence \( q^{\frac{1}{2}mCm - \frac{1}{2}m_{\nu-s}} = q^0 = 1 \) and we have proven (4.33) for \( 1 < s < \nu \) and \( 1 < r \leq \nu - s \). All other cases can be proven in a very similar fashion. Equation (4.34) follows again by the symmetry argument (3.8).
4.3 Proof of the equality of the two forms for the “straight” configuration sum

In this section we want to show that the two forms for the “straight” characters are equal. The “straight” characters can only occur for \( N = 2n \) even and \( n > 0 \) and are according to (3.4) given by

\[
X_{L}^{(\text{down})}(s|r, r) = X_{L}^{N}[A_{r-n}^{\text{down}}, u_{r-n}^{\text{down}} + e_{n}, Q_{r-n}^{\text{down}} + (e_{n-1} + e_{n-3} + \ldots)]
\]

\[-\theta(r > n + 1)q^{\frac{r+n}{2}}X_{L}^{N}[A_{r-n-1}^{\text{down}}, u_{r-n-1}^{\text{down}} + e_{n-1}, Q_{r-n-1}^{\text{down}} + (e_{n-2} + e_{n-4} + \ldots)] \tag{4.39}
\]

and

\[
X_{L}^{(\text{up})}(s|r, r) = X_{L}^{N}[A_{r+n}^{\text{up}}, u_{r+n}^{\text{up}} + e_{n}, Q_{r+n}^{\text{up}} + (e_{n-1} + e_{n-3} + \ldots)]
\]

\[-\theta(r < n - n)q^{\frac{r+n}{2}}X_{L}^{N}[A_{r+n+1}^{\text{up}}, u_{r+n+1}^{\text{up}} + e_{n-1}, Q_{r+n+1}^{\text{up}} + (e_{n-2} + e_{n-4} + \ldots)] \tag{4.40}
\]

Using abbreviation (1.1) with \( A = u = e_{s-1} \) and \( Q = L(e_{N-1} + e_{N-3} + \ldots) + (e_{s-2} + e_{s-4} + \ldots) \) and lemma 1 to convert the phase \( q^{-\frac{m_{s-1}}{2}} \) and the entry \( e_{\nu-s} \) into \( q^{-\frac{m_{s-1}}{2}} \) and \( e_{s-1} \) respectively in (4.40) we have explicitly

\[
X_{L}^{(\text{down})}(s|r, r) = \sum_{Q + (r-n-1)\rho + (e_{n-1} + e_{n-3} + \ldots) + (e_{\nu+1-r+n} + e_{\nu+3-r+n} + \ldots)} q^{\frac{r+n}{2}} \begin{cases} e_{n} + e_{\nu-r+n} \\ 0 \end{cases} \tag{4.41}
\]

\[-\theta(r > n + 1)\sum_{Q + (r-n)\rho + (e_{n-2} + e_{n-4} + \ldots) + (e_{\nu+2-r+n} + e_{\nu+4-r+n} + \ldots)} q^{\frac{r+n}{2}} \begin{cases} e_{n-1} + e_{\nu-r+n+1} \\ 0 \end{cases} \]

and

\[
X_{L}^{(\text{up})}(s|r, r) = \sum_{Q + (e_{n-1} + e_{n-3} + \ldots) + (e_{r+n-2} + e_{r+n-4} + \ldots)} q^{\frac{r+n}{2}} \begin{cases} e_{n} + e_{r+n-1} \\ 0 \end{cases} \tag{4.42}
\]

\[-\theta(r < \nu - n)\sum_{Q + (e_{n-2} + e_{n-4} + \ldots) + (e_{r+n-1} + e_{r+n-3} + \ldots)} q^{\frac{r+n}{2}} \begin{cases} e_{n-1} + e_{r+n} \\ 0 \end{cases} \]
Let us start with the easy case \( r = n + 1 \). In this case (4.41) and (4.42) reduce to
\[
X_L^{(\text{down})}(s|\tau, r) = \sum_{Q + (\tau_{n-1} + \tau_{n-3} + \cdots)} \begin{cases} e_n \\ 0 \end{cases}
\]
(4.43)
and
\[
X_L^{(\text{up})}(s|\tau, r) = \sum_{Q + (\tau_{n-1} + \tau_{n-3} + \cdots)} \begin{cases} e_n + e_N \\ 0 \end{cases} - q^{L+1} \sum_{Q + (\tau_{n-1} + \tau_{n-3} + \cdots)} \begin{cases} e_{n-1} + e_{N+1} \\ 0 \end{cases}
\]
(4.44)

The restrictions in the first term of (4.44) can be rewritten as \((\tau_{n+1} + \tau_{n+3} + \cdots) < N + n\rho < N\) and the one for the second term as \((\tau_{n} + \tau_{n+2} + \cdots) < N + n\rho < N\). The equality of (4.43) and (4.44) hence follows by lemma 2b) with \( \bar{a} = n + 1 \) and \( a = n - 1 \).

The equality of (4.42) and (4.41) for \( r = \nu - n \) follows by the symmetry (3.8) from the former case.

Let us now come to the proof for the case \( n + 1 < r < \nu - n \). To this end we split the first term in (4.44) via lemma 3b) with \( \bar{a} = n + 1 \) and \( a = N - n - 1 = n - 1 \).

\[
X_L^{(\text{down})}(s|\tau, r)
\]
\[
= \sum_{Q + (r - n - 1)\rho < N + (\tau_{n-1} + \tau_{n-3} + \cdots) + (\tau_{n+2} + \tau_{n+4} + \cdots) \geq N} \begin{cases} e_n + e_{r+n-1} - e_N \\ 0 \end{cases}
\]
\[
+ \sum_{Q + (r - n - 1)\rho < N + (\tau_{n-1} + \tau_{n-3} + \cdots) + (\tau_{n+2} + \tau_{n+4} + \cdots) \geq N} q^{L} \begin{cases} e_n + e_{\nu-r+n+1} - e_N + e_{N-1} \\ 0 \end{cases}
\]
\[
- \sum_{Q + (r - n - 1)\rho < N + (\tau_{n-1} + \tau_{n-3} + \cdots) + (\tau_{n+2} + \tau_{n+4} + \cdots) \geq N} q^{L+1} \begin{cases} e_{n-1} + e_{\nu-r+n+1} \\ 0 \end{cases}
\]
(4.45)

Using lemma 2a) with \( \bar{a} = n + 1 \), \( a = n - 1 \) and \( B_{\geq N} = -e_N + e_{\nu-r+n+1} \) to combine the second and third term in (4.45) we get
\[
X_L^{(\text{down})}(s|\tau, r)
\]
\[
= \sum_{Q + (r - n - 1)\rho < N + (\tau_{n-1} + \tau_{n-3} + \cdots) + (\tau_{n+2} + \tau_{n+4} + \cdots) \geq N} \begin{cases} e_n + e_{r+n-1} - e_N \\ 0 \end{cases}
\]
\[ + \sum_{Q + (r - n)\rho \geq N + (e_{n-2} + e_{n-4} + \ldots)} q^\frac{L}{2} \left\{ e_{n-1} + e_{\bar{r} + n+1} - e_N \right\} \quad (4.46) \]

We can now combine the second term in (4.46) minus the second term in (4.42) where we use \( \bar{r} = \nu + 1 - r \)

\[ q^\frac{L}{2} \sum_{Q + (r - n)\rho < N + (e_{n-2} + e_{n-4} + \ldots)} \left\{ e_{n-1} + e_{\nu+1-\bar{r}+n} - e_N \right\} \]

\[ + q^\frac{L+1}{2} \sum_{Q + (r - n)\rho \geq N + (e_{n-2} + e_{n-4} + \ldots)} \left\{ e_{n-1} + e_{\nu+1-\bar{r}+n} - e_N + e_N \right\} \quad (4.47) \]

where we obtained the last line by using lemma 3a) with \( a = n - 1 \) and \( \tilde{a} = n + 1 \). Hence we have

\[ X_L^{(\text{down})}(s|r, r) - X_L^{(\text{up})}(s|r, r) \]

\[ = \sum_{Q + (r - n - 1)\rho < N + (e_{n-1} + e_{n-3} + \ldots)} \left\{ e_n + e_{r+n-1} - e_N \right\} \]

\[ + \sum_{Q + (r - 1)\rho < N + (e_n + e_{n+2} + \ldots)} q^\frac{L}{2} \left\{ e_{N+1} + e_{r+n-1} - e_N + e_n \right\} \]

\[ - \sum_{Q + (r - 1)\rho < N + (e_{n+1} + e_{n+3} + \ldots)_N} \left\{ e_{r+n-1} + e_n \right\} \quad (4.48) \]

which is zero via lemma 2b) with \( a = n - 1 \) and \( \tilde{a} = n + 1 \).
4.4 Recursion relations

We now turn to the main part of the proof, the recursion relations. We have to show that (3.3)-(3.4) satisfy the recursion relations (2.4) which reads for the “down”-character

\[ X_L(s|r, r + N - 2n) = \sum_{i=0}^{n} q^{(n-i)\frac{1}{2}} X_{L-1}(s|r + N - 2i, r) + \sum_{i=1}^{N-n} q^{i\frac{1}{2}} X_{L-1}(s|r + N - 2n - 2i, r) \]  

(4.49)

where \( n = 0, 1, 2, \ldots, \left[ \frac{N}{2} \right] \). Using the symmetry (3.8) equation (4.49) becomes

\[ X_L(\tilde{s}|\tilde{r}, \tilde{r} - N + 2n) = \sum_{i=0}^{n} q^{(n-i)\frac{1}{2}} X_{L-1}(\tilde{s}|\tilde{r} - N + 2i, \tilde{r}) + \sum_{i=1}^{N-n} q^{i\frac{1}{2}} X_{L-1}(\tilde{s}|\tilde{r} - N + 2n + 2i, \tilde{r}) \]  

(4.50)

Hence the recursion relations for the “up”-characters follow automatically from the proof of (4.49) for all admissible \( r \) and \( s \). It is essential that we know the equality of the two forms of \( X(s|r, r) \) when using the symmetry argument here since via (3.8) \( X_{\text{down}}(s|r, r) \rightarrow X_{\text{up}}(\tilde{s}|\tilde{r}, \tilde{r}) \). If \( X_{\text{up}}(s|r, r) \neq X_{\text{down}}(s|r, r) \) (4.50) would not follow from (4.49).

Since (4.50) is implied by (4.49) we can restrict ourselves to the proof (4.49). \( r \) must fulfill \( \nu - N + n \geq r \geq n + 1 \) such that \( (r, r + N - 2n) \) is admissible. Let us first consider the case that \( \nu - N + n \geq r > \left[ \frac{N+3}{2} \right] \) where \([x]\) denotes the integer part of \( x \). The characters \( N \) “up” \( X_{L-1}(s|r + N, r) \) to \( N - 2n \) “up” \( X_{L-1}(s|r + N - 2n, r) \) on the rhs of (4.49) cancel pairwise (the first term of \( X_{L-1}(s|r + N - 2i, r) \) with the second term of \( X_{L-1}(s|r + N - 2(i+1)) \)) except for the first term of \( X_{L-1}(s|r + N - 2n) \). Similarly all “down” terms cancel \( (X_{L-1}(s|r - N + 2i, r) \) with \( i = 0, 1, \ldots, \left[ \frac{N}{2} \right] \) except for the first term of \( X_{L-1}(s|r - N + 2 \left[ \frac{N}{2} \right] , r) \). The characters \( X_{L-1}(s|r + N - 2n - 2, r) \) to \( X_{L-1}(s|r + 2, r) \) for \( N \) even or \( X_{L-1}(s|r + 1, r) \) for \( N \) odd are all nonzero. Hence (4.49) becomes

\[
X_L^N[A_{r-n,s}, u_{r-n,s}, e_n, Q_{r-n,s}, \frac{1}{2}] = q^{\frac{1}{2}} \theta(n > 0) X_L^N[A_{r-n-1,s}, u_{r-n-1,s}, e_{n-1}, Q_{r-n-1,s}, \frac{1}{2}]
\]

(4.51)
Further if \( n + 1 < r \leq \left\lfloor \frac{N-3}{2} \right\rfloor \) all “down” characters are zero since the endpoints are not admissible and \( X_{L-1}^{N-2k} (s| r + N - 2k, r) \) is only nonzero for \( k \leq r - 1 \) since otherwise the pair \((r + N - 2k, r)\) is not admissible. Again the sum \( \sum_{i=0}^{n} q^{(n-i)} \bar{b} X_{L-1}^{(s| r + N - 2i, r)} \) cancels except for the first term in \( X_{L-1}^{(s| r + N - 2, r)} \). Hence (4.49) becomes

\[
X_{L}^{N} \left[ A_{r-n,s}^{\downarrow}, u_{r-n,s}^{\downarrow} + e_n, Q_{r-n,s}^{\downarrow} + (e_{n-1} + e_{n-3} + \ldots) \right] \\
- q \frac{b-1}{b} \theta(n > 0) X_{L}^{N} \left[ A_{r-n-1,s}^{\downarrow}, u_{r-n-1}^{\downarrow} + e_n, Q_{r-n-1}^{\downarrow} + (e_{n-2} + e_{n-4} + \ldots) \right] \\
= \sum_{k=n}^{r-1} q^{(k-n)} \frac{L}{2} X_{L-1}^{N} \left[ A_{r+N-k,s}^{\uparrow}, u_{r+N-k,s}^{\uparrow} + e_n, Q_{r+N-k,s}^{\uparrow} + (e_{k-1} + e_{k-3} + \ldots) \right] \\
- \sum_{k=n}^{r-2} q^{(k-n)} \frac{L}{2} + L X_{L-1}^{N} \left[ A_{r+N-k,s}^{\uparrow}, u_{r+N-k,s}^{\uparrow} + e_n, Q_{r+N-k,s}^{\uparrow} + (e_{k-1} + e_{k-3} + \ldots) \right]
\]

(4.52)

for \( n = 0, 1, 2, \ldots, \left\lfloor \frac{N}{2} \right\rfloor \).

Finally for \( r = n + 1 \) (4.49) reduces to

\[
X_{L}^{N} \left[ A_{1,s}^{\downarrow}, u_{1,s}^{\downarrow} + e_n, Q_{1,s}^{\downarrow} + (e_{n-1} + e_{n-3} + \ldots) \right] \\
= X_{L-1}^{N} \left[ A_{N+1,s}^{\uparrow}, u_{N+1,s}^{\uparrow} + e_n, Q_{N+1,s}^{\uparrow} + (e_{n-1} + e_{n-3} + \ldots) \right].
\]

(4.53)

Equation (4.51)-(4.53) are the recursion relations we ought to prove.

We start with the proof of (4.51). To this end we define

\[
F_{b} = \sum_{k=n}^{\left\lfloor \frac{N-1}{2} \right\rfloor - b} q^{(k-n)} \frac{L}{2} X_{L-1}^{N} \left[ A_{r+N-k,s}^{\uparrow}, u_{r+N-k,s}^{\uparrow} + e_n, Q_{r+N-k,s}^{\uparrow} + (e_{k-1} + e_{k-3} + \ldots) \right] \\
- \sum_{k=n}^{\left\lfloor \frac{N-3}{2} \right\rfloor - b} q^{(k-n)} \frac{L}{2} + L X_{L-1}^{N} \left[ A_{r+N-k,s}^{\uparrow}, u_{r+N-k,s}^{\uparrow} + e_n, Q_{r+N-k,s}^{\uparrow} + (e_{k-1} + e_{k-3} + \ldots) \right] \\
+ q^{\left(\left\lfloor \frac{N+1}{2} \right\rfloor - n - b\right)} \frac{L}{2} X_{L-1}^{N} \left[ A_{r-N+1+s}^{\downarrow} + b, u_{r-N+1+s}^{\downarrow} + b, Q_{r-N+1+s}^{\downarrow} + (e_{2}^{n} + e_{n+b-1} + e_{n+b-3} + \ldots) \right]
\]

(4.54)

for \( b = 0, 1, \ldots, \left\lfloor \frac{N-3}{2} \right\rfloor - n + 1 \). For \( b = \left\lfloor \frac{N-3}{2} \right\rfloor - n + 1 \) the second sum is empty and hence it drops out. We are now going to show that

\[
F_{0} = F_{1} = \cdots = F_{\left\lfloor \frac{N-3}{2} \right\rfloor - n + 1}.
\]

(4.55)

Notice that \( F_{0} = \) the rhs of (4.51). Equation (4.55) follows if we can prove

\[
X_{L-1}^{N} \left[ A_{r-N+s}^{\uparrow} + b, u_{r-N+s}^{\uparrow} + b, Q_{r-N+s}^{\uparrow} + (e_{2}^{n} + e_{n+b-1} + e_{n+b-3} + \ldots) \right]
\]
\[-q^{\frac{L}{2}} X_{L-1}^N \left[ A_{r+}^u \left[ \begin{array}{c} N+1 \\ \frac{1}{2} \end{array} \right] + b^* u_{r-}^u \left[ \begin{array}{c} N+1 \\ \frac{1}{2} \end{array} \right] + e \left[ \begin{array}{c} N+1 \\ b^* \end{array} \right] - b \right] \]
\[+ q^{\frac{L}{2}} X_{L-1}^N \left[ A_{r-}^d \left[ \begin{array}{c} N+1 \\ \frac{1}{2} \end{array} \right] + b^* u_{r-}^d \left[ \begin{array}{c} N+1 \\ \frac{1}{2} \end{array} \right] + e \left[ \begin{array}{c} N+1 \\ b^* \end{array} \right] - b \right] \]
\[= X_{L-1}^N \left[ A_{r-}^d \left[ \begin{array}{c} N+1 \\ \frac{1}{2} \end{array} \right] + b^* u_{r-}^d \left[ \begin{array}{c} N+1 \\ \frac{1}{2} \end{array} \right] + e \left[ \begin{array}{c} N+1 \\ b^* \end{array} \right] - b \right] \]

where we dropped the subindex \(s\). To prove this formula we define \(a = \left[ \begin{array}{c} N+1 \\ \frac{1}{2} \end{array} \right] + b, \bar{a} = N - a = \left[ \begin{array}{c} N+1 \\ \frac{1}{2} \end{array} \right] - b, Q = (e_{s-2} + e_{s-4} + \ldots) + (L - 1)(e_{N-1} + e_{N-3} + \ldots)\) and

\[\begin{pmatrix} A \\ B \end{pmatrix} \equiv q^{\frac{L}{2}} m C m - \frac{1}{2} m_{s-1} \prod_{i=1}^{\nu} \left[ \frac{1}{2} (I_m + (L - 1)e_N + e_{s-1} + A)_i \right] \frac{1}{(m + B)_i}. \tag{4.57}\]

(Notice that we changed \(Le_N \rightarrow (L - 1)e_N\) in the definition which is more convenient since we deal with \(X_{L-1}\)). Hence using lemma 1 on the first two terms in (4.56) to transform the phase \(q^{\frac{m_{s-2}}{2}}\) into \(q^{\frac{m_{s-1}}{2}}\) and the entry \(e_{\nu-s}\) into \(e_{s-1}\) the lhs of (4.56) becomes

\[\sum_{Q + (e_{r+a-1} + e_{r+a-3} + \ldots)\atop + (e_{a-2} + e_{a-4} + \ldots)} e_{r+a} + e_{\bar{a}-1} \]
\[-q^{\frac{L}{2}} \sum_{Q + (e_{r+a} + e_{r+a-2} + \ldots)\atop + (e_{a-3} + e_{a-5} + \ldots)} e_{r+a+1} + e_{\bar{a}-2} \]
\[+ q^{\frac{L}{2}} \sum_{Q + (r - \bar{a}) (e_{r+a} + e_{r+a-1} + e_{r+a-3} + \ldots)\atop + (e_{a-1} + e_{a-3} + \ldots)} e_{\nu-r+\bar{a}} + e_{a} \] \tag{4.58}

Using lemma 2a) on the last term and splitting the restrictions for the first two terms into restrictions \(\geq N\) and \(< N\) \tag{4.58} becomes

\[\sum_{Q + (r - N - 1)_{\rho < N} + (e_{a} + e_{a+2} + \ldots)_{< N} + (e_{r+a-1} + e_{r+a-3} + \ldots)_{\geq N}} e_{r+a} + e_{\bar{a}-1} \]
\[-q^{\frac{L}{2}} \sum_{Q + (r - N - 1)_{\rho < N} + (e_{a-1} + e_{a+1} + \ldots)_{< N} + (e_{r+a} + e_{r+a-2} + \ldots)_{\geq N}} e_{r+a+1} + e_{\bar{a}-2} \]

23
Notice that (4.62) immediately follows from (4.51) for $N$. Hence (4.62) can be thought of as a generalization of the

begin by showing that (4.62) holds for $n < N$ and hence (4.56) is proven.

with the help of (4.55) equation (4.51) reduces to

\[
X_L^n \left[ A_{r-N}^{\downarrow}, u_{r-N}^{\downarrow} + e_{n-1}, Q_{r-N}^{\downarrow} + (e_{n-1} + e_{n-3} + \ldots) \right]
\]

\[
- q^{\frac{L}{2}} \theta(n > 0) X_L^n \left[ A_{r-N}^{\downarrow}, u_{r-N}^{\downarrow} + e_{n-1}, Q_{r-N}^{\downarrow} + (e_{n-2} + e_{n-4} + \ldots) \right]
\]

\[
= X_L^n \left[ A_{r+N-n}^{\uparrow}, u_{r+N-n}^{\uparrow} + e_{n}, Q_{r+N-n}^{\uparrow} + (e_{n-1} + e_{n-3} + \ldots) \right]
\]

\[
+ q^{\frac{L}{2}} X_L^n \left[ A_{r-N}^{\downarrow}, u_{r-N}^{\downarrow} + e_{N-n-1}, Q_{r-N}^{\downarrow} + (e_{N-n-2} + e_{N-n-4} + \ldots) \right]
\]

and hence (4.58) is proven.

Using lemma 3b) to combine the first and third term and lemma 3a) to combine the second and fourth term we get (notice that we use the lemmas with $L \to L - 1$)

\[
\sum \left\{ e_{\bar{a}-1} + e_{N-1} + e_{\nu - r + \bar{a}} \right\} \]

\[
Q + (r - N - 1) \rho_{N} + (e_{a} + e_{a+2} + \ldots)_{N}
\]

\[
+ (r - a) \rho_{N} + (e_{\nu - r + \bar{a}} + e_{\nu - r + \bar{a} + 2} + \ldots)
\]

\[
= X_L^{N-1} \left[ A_{r-N}^{\downarrow}, u_{r-N}^{\downarrow} + e_{a+1}, Q_{r-N}^{\downarrow} + (e_{a} + e_{a-2} + \ldots) \right]
\]

which is via lemma 2b)

\[
\sum \left\{ e_{\nu - r + \bar{a} - 1} + e_{a+1} \right\}
\]

\[
Q + (r - a) \rho_{N} + (e_{a} + e_{a+2} + \ldots)
\]

\[
(r - a) \rho_{N} + (e_{\nu - r + \bar{a}} + e_{\nu - r + \bar{a} + 2} + \ldots)
\]

\[
= X_L^{N-1} \left[ A_{r-N}^{\downarrow}, u_{r-N}^{\downarrow} + e_{a+1}, Q_{r-N}^{\downarrow} + (e_{a} + e_{a-2} + \ldots) \right]
\]

and hence (4.58) is proven.

With the help of (4.55) equation (4.51) reduces to

Notice that (4.62) immediately follows from (4.51) for $N = 1, 2$ without any derivation. Hence (4.62) can be thought of as a generalization of the $N = 1, 2$ recursion relations. Let us begin by showing that (4.62) holds for $n = 0$. Using lemma 4 again as before (4.62) reads explicitly for $n = 0$

\[
\sum \left\{ e_{\nu - r} + e_{N} \right\}
\]

\[
Q + (r - 1) \rho + (e_{N-1} + e_{N-3} + \ldots)
\]

\[
+ (e_{\nu + 1 - r} + e_{\nu + 3 - r} + \ldots)
\]

\[
= X_L^{N-1} \left[ A_{r-N}^{\downarrow}, u_{r-N}^{\downarrow} + e_{N-1}, Q_{r-N}^{\downarrow} + (e_{N-2} + e_{N-4} + \ldots) \right]
\]

\[
= X_L^{N-1} \left[ A_{r+N-N-n}^{\uparrow}, u_{r+N-N-n}^{\uparrow} + e_{n}, Q_{r+N-N-n}^{\uparrow} + (e_{n-1} + e_{n-3} + \ldots) \right]
\]

\[
+ q^{\frac{L}{2}} X_L^{N-1} \left[ A_{r-N}^{\downarrow}, u_{r-N}^{\downarrow} + e_{N-n-1}, Q_{r-N}^{\downarrow} + (e_{N-n-2} + e_{N-n-4} + \ldots) \right]
\]

\[
(4.62)
\]
where \(Q\) and \(\frac{A}{B}\) are defined as in (4.57). But (4.63) holds via lemma 3b) with \(a = N - 1\). It remains to show that (4.62) is true for \(n > 0\). In this case we have explicitly

\[
\sum_{Q + (r - n - 1)\rho + (e_{n-1} + e_{n-3} + \ldots) + (e_{r+n-2} + e_{r+N-n-4} + \ldots)} e_{\nu-r+n} + e_N + e_n
\]

\[
\sum_{Q + (r - n)\rho + (e_{n-1} + e_{n-3} + \ldots) + (e_{r+n-2} + e_{r+N-n-4} + \ldots)} e_{\nu-r+n+1} + e_N + e_{n-1}
\]

\[
\sum_{Q + (e_{n-1} + e_{n-3} + \ldots) + (e_{r+n-2} + e_{r+N-n-4} + \ldots)} e_n + e_{r+N-n-1}
\]

\[
\sum_{Q + (e_{n-1} + e_{n-3} + \ldots) + (e_{r+n-2} + e_{r+N-n-4} + \ldots)} e_{\nu-r+n+1} + e_{N-n-1}
\]

We can combine the second term on the lhs with the second term on the rhs via lemma 2a) with \(a = N - n - 1\) and rewrite the restrictions. We then obtain for (4.64)

\[
\sum_{Q + (r - n - 1)\rho_{\geq N} + (e_{r+n+1} + e_{r+n+3} + \ldots) + (N + r - 1)\rho_{< N} + (e_{n+1} + e_{n+3} + \ldots)_{< N}} e_{\nu-r+n} + e_N + e_n
\]

\[
\sum_{Q + (e_{n-1} + e_{n-3} + \ldots) + (e_{r+n-2} + e_{r+N-n-4} + \ldots)} e_n + e_{r+N-n-1}
\]

\[
\sum_{Q + (r - n)\rho_{\geq N} + (e_{r+n+1} + e_{r+n+3} + \ldots) + (N + r - 1)\rho_{< N} + (e_{n+1} + e_{n+3} + \ldots)_{< N}} e_{\nu-r+n+1} + e_{N-n-1} + e_n
\]

which is again true due to lemma 3b) with \(a = N - n - 1\). Hence (4.51) is proven.
We now turn to the proof of recursion relation (4.52). Again we use the notation \( \{ A \} B \) and the definition of \( Q \) as in (4.57) and apply lemma 1 to write (4.52) explicitly as

\[
\sum Q + (r - n - 1)\rho + (e_{\nu + n + 1} + e_{\nu + n + 3} + \ldots) + (e_{n - 1} + e_{n - 3} + \ldots)
\]

and

\[
-\theta(n > 0) \sum q^{L+1} \frac{L+1}{2} \left\{ e_{\nu - r + n + 1} + e_{n - 1} + e_N \right\}
\]

We can combine the first term on the lhs with the term \( k = n \) of the first sum on the rhs via lemma 3b) with \( a = N - n - 1 \)

\[
\sum Q + (r - n - 1)\rho < N + (e_{n+1} + e_{n+3} + \ldots) < N
\]

\[
(r - n - 1)\rho \geq N + (e_{\nu - r + n + 1} + e_{\nu - r + n + 3} + \ldots)
\]

\[
- \sum Q + (r - n - 1)\rho < N + (e_{n+1} + e_{n+3} + \ldots) < N
\]

\[
(r - n)\rho \geq N + (e_{\nu + 2 - r + n} + e_{\nu + 4 - r + n} + \ldots)
\]

\[
= q^{\frac{L}{2}} \sum Q + (r - n)\rho + (e_{N - n - 2} + e_{N - n} + \ldots)
\]

\[
+ (e_{\nu - r + n + 2} + e_{\nu - r + n + 4} + \ldots)
\]

In the case that \( n > 0 \) we can further combine this result with the second term on the lhs of (4.66) via lemma 2a) with \( a = N - n - 1 \) and obtain

\[
q^{\frac{L}{2}} \sum Q + (r - n)\rho + (e_{N - n + 2} + e_{N - n - 4} + \ldots)
\]

\[
+ (e_{\nu - r + n + 2} + e_{\nu - r + n + 4} + \ldots)
\]

\[
(4.68)
\]
Hence with the definition
\[ K_n \equiv \sum_{Q + (r - n)\rho + (e_{N-n-2} + e_{N-n-4} + \ldots) + (e_{\rho+r+n+2} + e_{\rho+r+n+4} + \ldots)} \left\{ \begin{array}{l} e_{k+1} + e_{r+N-k-2} \\ 0 \end{array} \right\} \]
\[ - \sum_{k=n}^{r-2} q^{(k-n)\frac{b}{2}} \sum_{Q + (e_k + e_{k-2} + \ldots) + (e_{r+N-k-3} + e_{r+N-k-5} + \ldots)} \left\{ \begin{array}{l} e_{k} + e_{r+N-k-1} \\ 0 \end{array} \right\} \]
\[ - \frac{L}{q} \sum_{Q + (e_{k-1} + e_{k-3} + \ldots) + (e_{r+N-k-2} + e_{r+N-k-4} + \ldots)} \left\{ \begin{array}{l} e_{k} + e_{r+N-k-1} \\ 0 \end{array} \right\} \] (4.69)
for \( n = 0, 1, \ldots, r-2 \) we need to prove that \( K_n = 0 \). First we show that similar to (4.55)
\[ K_0 = K_1 = \cdots = K_{r-2}. \] (4.70)

To this end we need to derive for \( b = 0, 1, 2, \ldots, r-3 \)
\[ \sum_{Q + (r-b)\rho + (e_{N-b-2} + e_{N-b-4} + \ldots) + (e_{r+2+b-r} + e_{r+4+b-r} + \ldots)} \left\{ \begin{array}{l} e_{N-b-1} + e_{\rho-r+b+1} \\ 0 \end{array} \right\} \]
\[ - \sum_{Q + (e_b + e_{b-2} + \ldots) + (e_{r+N-b-3} + e_{r+N-b-5} + \ldots)} \left\{ \begin{array}{l} e_{b+1} + e_{r+N-b-2} \\ 0 \end{array} \right\} \]
\[ + \sum_{Q + (e_{b-1} + e_{b-3} + \ldots) + (e_{r+N-b-2} + e_{r+N-b-4} + \ldots)} q^{b\frac{L}{2}} \left\{ \begin{array}{l} e_{b} + e_{r+N-b-1} \\ 0 \end{array} \right\} \]
\[ = \sum_{Q + (r+b+1)\rho + (e_{N-b-3} + e_{N-b-5} + \ldots) + (e_{r+3+b-r} + e_{r+5+b-r} + \ldots)} \left\{ \begin{array}{l} e_{N-b-2} + e_{\rho-r+b+2} \\ 0 \end{array} \right\} \] (4.71)

We can apply lemma 2b) on the first term on the lhs with \( a = N - b - 2 \) and rewrite the restriction for the second and third term and get
\[ \text{lhs of (4.71)} \]
\[ = \sum_{Q + (N + r - 1)\rho < N + (e_{b+2} + e_{b+4} + \ldots) < N + (r+b)\rho_{\geq N} + (e_{r+2+b-r} + e_{r+4+b-r} + \ldots)} \left\{ \begin{array}{l} e_{\rho-r+1+b} + e_{b+1} + e_{N} \\ 0 \end{array} \right\} \]
Combining now the first and third term via lemma 3b) and the second and fourth term via lemma 3a) with \(a = N - b - 2\) in both cases we get

\[
\text{lhs of (4.71)} = \sum Q \{ e_{b+1} + e_{N+2} \} + \sum Q \{ e_{b+1} + e_{r+b+2} \} - \sum Q \{ e_{r-1} + e_N \} - \sum Q \{ e_{r-2} + e_{N+1} \}
\]

Via lemma 2a) with \(a = N - b - 2\) this is equal to the rhs of (4.71).

Hence it remains to show that \(K_{r-2} = 0\). This equation reads explicitly

\[
\sum Q \{ e_{N-r+1} \} + \sum Q \{ e_{r-1} + e_N \} - \sum Q \{ e_{r-2} + e_{N+1} \} = 0
\]

and is true due to lemma 2b) with \(a = N - r\). Therefore the proof of recursion relation (4.52) is complete.
Finally the recursion relation for \( r = n + 1 \) follows since due to lemma \([1]\) both sides of equation \((4.53)\) are equal to

\[
\sum_{Q + (e_{N-1} + e_{N-3} + \ldots) + (e_{n-1} + e_{n-3} + \ldots)} \left\{ \begin{array}{c}
e_n + e_{N} \\ 0 \end{array} \right\}.
\]  
\((4.75)\)

This concludes the proof of all recursion relations.

5  **Polynomial Expressions under the duality** \( q \rightarrow q^{-1} \)

In this section we obtain the configuration sums of the RSOS model in regime II by the duality transformation \( q \rightarrow q^{-1} \) from our results \((3.3)\) and \((3.4)\) in regime III. The RSOS model in regime II corresponds to the coset conformal field theory \( \hat{su}(2\nu - 2)/\hat{sp}(2\nu - 2) \) \([36], [37]\). We will also identify our results with the branching functions of the \( Z_{\nu - 1} \)-parafermion model.

Under duality the Gaussian polynomials transform as follows

\[
\left[ \begin{array}{c} n \\ m \end{array} \right]_{q^{-1}} = q^{m(m-n)} \left[ \begin{array}{c} n \\ m \end{array} \right]_{q}.
\]  
\((5.1)\)

Hence the object

\[
X_L[A, u, Q, q] = \sum_{Q} q^{\frac{1}{2}mCm} \frac{1}{2} Am \prod_{i=1}^{\nu-2} \left[ \frac{1}{2}(Im + u + Le_N)_i \right]_q
\]

transforms as

\[
X_L[A, u, Q, q^{-1}] = \sum_{Q} q^{\frac{1}{2}mCm + \frac{1}{2}(A-u)m} \frac{1}{2} LmN \prod_{i=1}^{\nu-2} \left[ \frac{1}{2}(Im + u + Le_N)_i \right]_q.
\]  
\((5.3)\)

To take the limit \( L \rightarrow \infty \) we have to take out a factor \( q^{f(L)} \) where \( f(L) \) is some function of \( L \). To determine the appropriate factor we make the variable change to the dual variable \( \tilde{n}_i \)

\[
m_i + \tilde{n}_i = \frac{1}{2}(Im + u + Le_N)_i.
\]  
\((5.4)\)

This equation is equivalent to the following partition problem

\[
\sum_{i=1}^{\nu-1} \tilde{n}_i = \frac{1}{2}(NL + \sum_{i=1}^{\nu-2} iw_i + (\nu - 1)Q_{\nu-2})
\]  
\((5.5)\)

where \( \tilde{n}_i \) are non-negative integers \( (i = 1, 2, \ldots, \nu - 1) \) and \( \tilde{n}_{\nu-1} \equiv \frac{1}{2}(m_{\nu-2} + Q_{\nu-2}) \).

We can also solve equation \((5.4)\) for \( m \) and obtain

\[
m = C^{-1}(u + Le_N - 2\tilde{n})
\]  
\((5.6)\)
where $C^{-1}$ is the inverse of the Cartan matrix and is given by

$$C^{-1}_{ij} = \begin{cases} \frac{(\nu - 1)i}{\nu} & \text{for } j \leq i \\ \frac{-1}{\nu} & \text{for } j > i \end{cases} \quad (5.7)$$

Hence (5.3) reads in the dual variables $\tilde{n}_i$

$$q^{f(L)} X_L[A, u, Q, q^{-1}] = \sum q^n C^{-1}_{\tilde{n}} - AC^{-1}_{\tilde{n}} - \frac{1}{4} u C^{-1}_{u} + \frac{1}{2} A C^{-1}_{u} \prod_{i=1}^{\nu-2} \left[ \tilde{n}_i + m_i - \frac{1}{\nu} \tilde{n}_i \right] \quad (5.8)$$

where $m$ as in (5.4) and $f(L) = \frac{1}{4} L^2 N \nu - N + 1 + \frac{1}{4} L(u - A) C^{-1} e_N$. The sum $\sum$ runs over all $\tilde{n}_i \geq 0$ satisfying (5.3). Notice that all the $L$ dependence in the exponent is independent of the variable $\tilde{n}$ and hence could be factored out.

Let us now calculate $\frac{1}{4} L(u - A) C^{-1} e_N$ for the different cases. Let us start with the “down” configuration sums $X_L(s|r, r + N - 2n)$. For the pair $(r, r + N - 2n)$ to be admissible it must fulfill $N + 2 \leq 2r + N - 2n \leq 2\nu - N$ which implies $N \leq \nu - r + n$. Hence we conclude that for the first term in $X_L(s|r, r + N - 2n)_{q^{-1}}$ (see (3.3) and (3.4))

$$\frac{1}{2} L(u - A) C^{-1} e_N = \frac{1}{2} L(e_{\nu - r + n} + e_n) C^{-1} e_N$$

$$= \frac{L}{2(\nu - 1)} ((r - 1 - n) N + (\nu - 1 - N) n). \quad (5.9)$$

Similarly for the second term in (3.4)

$$\frac{1}{2} L(u - A) C^{-1} e_N + \frac{L}{2} = \frac{1}{2} L(e_{\nu - r + n + 1} + e_{n - 1}) C^{-1} e_N + \frac{L}{2}$$

$$= \frac{L}{2(\nu - 1)} ((r - 2 - n) N + (\nu - 1 - N) (n - 1) + \nu - 1)$$

$$= \frac{L}{2(\nu - 1)} ((r - 1 - n) N + (\nu - 1 - N) n). \quad (5.10)$$

Hence both terms have the same function $f$

$$f^{\text{down}}(L) = \frac{1}{4} \frac{L^2}{\nu - 1} N(\nu - 1 - N) + \frac{1}{2} \frac{L}{\nu - 1} ((\nu - 1 - N) n + (r - n - 1) N). \quad (5.11)$$

Similarly one can calculate the function $f$ for the “up” character $X_L(s|r, r - N + 2n)_{q^{-1}}$

$$f^{\text{up}}(L) = \frac{1}{4} \frac{L^2}{\nu - 1} N(\nu - 1 - N) + \frac{1}{2} \frac{L}{\nu - 1} ((\nu - 1 - N) n + (\nu - r - n) N). \quad (5.12)$$

Hence if we define

$$\tilde{X}_L[A, u, Q] = \sum q^n C^{-1}_{\tilde{n}} - AC^{-1}_{\tilde{n}} - \frac{1}{4} u C^{-1}_{u} + \frac{1}{2} A C^{-1}_{u} \prod_{i=1}^{\nu-2} \left[ \tilde{n}_i + C^{-1}(u + L e_N - 2n) \right] \quad (5.13)$$

we have

$$q^{f^{\text{down}}(L)} X_L(s|r, r + N)_{q^{-1}} = \tilde{X}_L[A_{r,s}^{\text{down}}, u_{r,s}^{\text{down}}, Q_{r,s}^{\text{down}}]$$

$$q^{f^{\text{up}}(L)} X_L(s|r, r - N)_{q^{-1}} = \tilde{X}_L[A_{r,s}^{\text{up}}, u_{r,s}^{\text{up}}, Q_{r,s}^{\text{up}}] \quad (5.14)$$
and

\[
q_{\downarrow}^{(L)} X_L(s|r, r + N - 2n)_{q^{-1}} = \tilde{X}_L[A_{r-n,s}^{\downarrow}, u_{r-n,s}^{\downarrow} + e_n; Q_{r-n,s}^{\downarrow}]
\]

\[
-\theta(r > n + 1) \tilde{X}_L[A_{r-n-1,s}^{\downarrow}, u_{r-n-1,s}^{\downarrow} + e_n; Q_{r-n-1,s}^{\downarrow}]
\]

\[
q_{\uparrow}^{(L)} X_L(s|r, r - N + 2n)_{q^{-1}} = \tilde{X}_L[A_{r+s,n}^{\uparrow}, u_{r+s,n}^{\uparrow} + e_n; Q_{r+s,n}^{\uparrow}]
\]

\[
-\theta(r < \nu - n) \tilde{X}_L[A_{r+s,n+1,s}^{\uparrow}, u_{r+s,n+1,s}^{\uparrow} + e_n; Q_{r+s,n+1}^{\uparrow}]
\]  

(5.15)

for \(n = 1, 2, \ldots, \left\lfloor \frac{N}{2} \right\rfloor\). \(A_{r,s}^{\downarrow}, u_{r,s}^{\downarrow}, Q_{r,s}^{\downarrow}\) and \(Q_{r,s}^{\uparrow}\) are defined as in (3.1) and (3.2). The dependence on \(Q_{r,s}^{\downarrow}\) is only implicit through (5.5). Since the dependence is only on the \((\nu - 2)^{th}\) component of \(Q\) we could drop the terms \((e_{n-1} + e_{n-3} + \ldots)\) and \((e_{n-2} + e_{n-4} + \ldots)\).

Since \((C^{-1}e_N)_{i} \neq 0\) for all \(i \in \{1, 2, \ldots, \nu - 2\}\) all Gaussians in the product in (5.13) turn into \(\frac{1}{(q)_{n_i}}\) in the limit \(L \rightarrow \infty\) and we get

\[
\tilde{X}_\infty[A, u, Q] \equiv \lim_{L \rightarrow \infty} \tilde{X}_L[A, u, Q]
\]

\[
= \sum q^\nu C^{-1} \nu - AC^{-1} \nu - b \nu C^{-1} u + \frac{1}{2} \nu C^{-1} u \prod_{i=1}^{\nu-2} \frac{1}{(q)_{\tilde{n}_i}}.
\]

(5.16)

where \(\sum\) runs over \(\tilde{n}_i \geq 0\) such that \((C^{-1}(\tilde{n} - \frac{1}{2} u) - \frac{1}{2} Q)_{\nu-2} \in \mathbb{Z}\). Notice that the \(N\) dependence drops out.

According to [36] and [37] the branching functions \(e_{jk}^l\) for the coset model \(\tilde{s}u(2\nu - 2)/\tilde{sp}(2\nu - 2)\) satisfy

\[
q^\tilde{n} e_{r-1,s-1}^{\nu-1} = \lim_{L \rightarrow \infty} q_{\downarrow}^{(L)} X_L(s|r, r + N)_{q^{-1}}
\]

(5.17)

where \(\tilde{n} = \frac{1}{4(\nu-1)} \left(\frac{\nu+1}{2} - r\right)^2 - \frac{1}{4(\nu+1)} \left(\frac{\nu+1}{2} - s\right)^2 - \frac{1}{24}\). Hence the fermionic representation for the branching functions is

\[
q^\tilde{n} e_{r-1,s-1}^{\nu-1} = \tilde{X}_\infty[A_{r,s}^{\downarrow}, u_{r,s}^{\downarrow}, Q_{r,s}^{\downarrow}]
\]

(5.18)

where \(A_{r,s}^{\downarrow}, u_{r,s}^{\downarrow}\) and \(Q_{r,s}^{\downarrow}\) are defined as in (3.1).

Finally, we would like to notice that (5.18) is also related to the branching functions for the \(Z_{\nu-1}\)-parafermion model [1], [4], [11]. The fermionic form of these branching functions is given by [1], [2], [3], [29]

\[
q^\frac{\nu-1}{2} q^{\frac{(s+1)(r-s)}{2(\nu-1)(\nu+1)}} b_{2p-s-1}^{r-1} = \sum_{m_1, \ldots, m_{\nu-2} \geq 0, \text{restr.}} q^{\nu C^{-1} m - c \nu - 1, C^{-1} m} \prod_{i=1}^{\nu-2} \frac{1}{(q)_{m_i}}
\]

(5.19)

where the \(m_i\)'s are subject to the restriction \((C^{-1}m)_{\nu-2} - \frac{1}{\nu-1} \nu - \frac{1}{2} \nu \nu \in \mathbb{Z}\) and \(c = \frac{2(\nu-2)}{\nu+1}\) is the central charge. A bosonic form of the branching function \(b^{r}_{m}\) is given in terms of a double
Identifying $p$ with $(\nu-1)/2(C^{-1}u^{\text{down}}_{r,s} + Q^{\text{down}}_{r,s})_{\nu-2}$ and taking out the factor

\[ q^{-\frac{1}{2}u^{\text{down}}_{r,s}C^{-1}u^{\text{down}}_{r,s} + C^{-1}A_{r,s}} = q^{-\frac{1}{2}((\nu-s)(s-1)-(r-1)(\nu-r))} \tag{5.20} \]

establishes the relation between (5.18) and (5.19), namely

\[ e_{r-1,s-1}^{\nu-1} = b^{s-1}_{r-1}. \tag{5.21} \]

The branching functions of the $Z_{\nu-1}$-parafermion model correspond to the coset models $\tilde{su}(\nu-1)_1 \times \tilde{su}(\nu-1)_1/\tilde{su}(\nu-1)_2$ \cite{1, 40, 41, 44} and by level rank duality \cite{16} to the ones of $\tilde{su}(2)_\nu/U(1)$ \cite{12, 13, 15}.

**Acknowledgements**

I would like to thank B. McCoy for suggesting this problem to me and also for many helpful discussions and comments. I would further like to thank A. Waldron and W. Orrick for discussions. This work was partially supported by NSF grants PHY-9309888 and DMR-9404747.

**Appendix**

**A Further Identities**

In this appendix we give further useful identities in addition to the ones already discussed in section 4.1. Let us first give the telescopic expansion to the left of length $M$ of the same type as \cite{48}

\[
\sum_q q^{-n'_{i+M}} \begin{cases}
A_{<i} + A_{\geq i+M} + 2e_{i+M-1} \\
B_{<i} + B_{>i+M} + e_{i+M}
\end{cases}
= \sum_{l=i}^{i+M} \sum_q q^{-\sum_{k=l}^{i+M} n'_{k}} \begin{cases}
A_{<i} + A_{\geq i+M} + 2\sum_{k=l}^{i+M} e_k + 2e_{i-1}\delta_{i,i} \\
B_{<i} + B_{>i+M} + \sum_{k=l}^{i+M} e_k
\end{cases} \tag{A.1}
\]

where we used a similar notation to \cite{48}. In the case that $i = 1$ the term $2e_{i-1}\delta_{i,i}$ just drops out.

Next we give the telescopic expansion from left to right of the same type as \cite{49}

\[
\sum_q q^{-\sum_{k=1}^{i+M} n'_{k}} \begin{cases}
A_{<i} + A_{\geq i+M} + 2\sum_{k=1}^{i+M} e_k \\
B_{<i} + B_{>i+M} + \sum_{k=1}^{i+M} e_k
\end{cases}
= \sum_{l=1}^{i+M} \sum_q q^{-\sum_{k=l}^{i+M} n'_{k}} \begin{cases}
A_{<i} + A_{\geq i+M} + 2\sum_{k=l+1}^{i+M} e_k \\
B_{<i} + B_{>i+M} + \sum_{k=l}^{i+M} e_k
\end{cases}
+ \sum_q q^{\frac{1}{2}(A_{\geq i+M})_{i+M}} \begin{cases}
A_{<i} + A_{\geq i+M} \\
B_{<i} + B_{>i+M}
\end{cases}. \tag{A.2}
\]
The analogue of \((4.10)\) in opposite direction is given by
\[
\sum_{Q} \sum_{l=2}^{i} q^{-\sum_{k=l}^{i} n'_k} \left\{ \frac{A_{\geq l} + 2 \sum_{k=l+1}^{i} e_k + e_1}{B_{>l} + \sum_{k=l}^{i} e_k} \right\}
\]
\[= \sum_{Q} q^{-n'_i-\sum_{k=1}^{i-2} n'_k + \frac{i}{2}} \left\{ \frac{A_{\geq i} + 2 \sum_{k=i}^{i-2} e_k + e_1}{B_{>i} + e_i + \sum_{k=1}^{i-2} e_k} \right\} \tag{A.3}
\]

Now we come to the telescopic expansions corresponding to recursion relation (2.3). We first give the analogue of (4.8) (expansion to the right from \(i\) to \(i + M\))
\[
\sum_{Q} q^{m_i-m_{i+1}} \left\{ \frac{A + 2e_{i+1}}{B_{<i} + B_{>i+M} + e_i} \right\}
\]
\[= \sum_{l=i}^{i+M} \sum_{Q} q^{m_i-m_{i+1}} \left\{ \frac{A + 2 \sum_{k=i}^{i+M} e_k + 2i_{i+M} + 1 \delta_{i+M}}{B_{<i} + B_{>i+M} + \sum_{k=i}^{i+M} e_k} \right\} \tag{A.4}
\]

Since the phase in (2.3) is independent of the top entries \(A\) is just any vector with integer values.

The analogous telescopic expansion to (4.9) corresponding to (2.3) is
\[
\sum_{Q} q^{m_i-m_{i+1}} \left\{ \frac{A + 2 \sum_{k=i}^{i+M} e_k}{B_{<i} + B_{>i+M} + \sum_{k=i}^{i+M} e_k} \right\}
\]
\[= \sum_{l=i}^{i+M} \sum_{Q} q^{m_i+1} \left\{ \frac{A + 2 \sum_{k=i}^{l-1} e_k}{B_{<i} + B_{>i+M} + \sum_{k=i}^{l} e_k} \right\}
\]
\[+ \sum_{Q} q^{m_i} \left\{ \frac{A}{B_{<i} + B_{>i+M}} \right\} \tag{A.5}
\]

The proof of (A.4) and (A.3) is analogous to the proofs of (4.8) and (4.9) respectively. Instead of using recursion relation (2.6), one simply uses (2.5). There are again analogous formulas to (A.4) and (A.5) in the opposite direction the derivation of which is left to the reader.

Next we have the mirror identity due to recursion relation (2.5)
\[
\sum_{l=i}^{M+i} \sum_{Q} q^{m_i} \left\{ \frac{A + 2 \sum_{k=i}^{l-1} e_k}{B_{<i} + B_{>i+M} + \sum_{k=i}^{l} e_k} \right\}
\]
\[= \sum_{l=i}^{i+M} \sum_{Q} q^{m_i} \left\{ \frac{A + 2 \sum_{k=i+1}^{i+M} e_k}{B_{<i} + B_{>i+M} + \sum_{k=i}^{i+M} e_k} \right\} \tag{A.6}
\]

The proof is again similar to the one for (4.10).

Finally we have the extended mirror identity corresponding to (2.5)
\[
\sum_{l=i}^{M+i} \sum_{Q} q^{m_i-m_{i+1}+m_i} \left\{ \frac{A + 2e_{i+1} + 2 \sum_{k=i+1}^{i+M} e_k}{B_{<i} + B_{>i+M} + e_i + \sum_{k=i+1}^{i+M} e_k} \right\}
\]
\[= \sum_{l=i}^{i+M-2} \sum_{Q} q^{m_i+m_{i+M}-m_{i+M}-1} \left\{ \frac{A + 2e_{i+M-1} + 2 \sum_{k=i}^{i-1} e_k}{B_{<i} + B_{>i+M} + e_i + \sum_{k=i}^{i+M-1} e_k} \right\} \tag{A.7}
\]
the proof of which is similar to the one for (4.14).

For the proof of lemma 1 we need two further identities. They are special telescopic expansions using recursion relation (2.5).

\[ \sum_{Q} q^{\frac{1}{2} m_{1}} \left\{ \begin{array}{c} A + 2e_{1} \\ 0 \end{array} \right\} = \sum_{Q} q^{\frac{1}{2} m_{1}} \left\{ \begin{array}{c} A + 2 \sum_{k=1}^{\nu-2} e_{k} \\ \sum_{k=1}^{\nu-2} e_{k} \end{array} \right\} \] (A.8)

where \( A \) is an arbitrary vector with integer values and \( Q \in (\mathbb{Z}_{2})^{\nu-2} \) arbitrary. The derivation of this identity goes as follows. Use recursion relation (2.5) in the first slot of the lhs of (A.8)

\[ \sum_{Q} q^{\frac{1}{2} m_{1}} \left\{ \begin{array}{c} A \\ 0 \end{array} \right\} + \sum_{Q} q^{-\frac{1}{2} m_{1}} \left\{ \begin{array}{c} A \\ -e_{1} \end{array} \right\} = \sum_{Q} q^{\frac{1}{2} m_{1}} \left\{ \begin{array}{c} A \\ 0 \end{array} \right\} + \sum_{Q} q^{\frac{1}{2} m_{1} - m_{2} + 1} \left\{ \begin{array}{c} A + 2e_{2} \\ e_{1} \end{array} \right\} \] (A.9)

where we made the variable change \( m_{1} \rightarrow m_{1} + 2 \) in the second term to get the second line. We can now use (A.4) with \( i = 1 \) and \( i + M = \nu - 2 \) on the second term (the extra exponent \( \frac{1}{2} m_{1} \) does not matter since one uses only variable changes for \( m_{2} \) and higher in the derivation of (A.4)) and obtain

\[ q^{\frac{1}{2} m_{1}} \sum_{Q} \left\{ \begin{array}{c} A \\ 0 \end{array} \right\} + \sum_{l=1}^{\nu-2} q^{m_{l} + 1} \left\{ \begin{array}{c} A + 2 \sum_{k=1}^{l-1} e_{k} \\ \sum_{k=1}^{l-1} e_{k} \end{array} \right\} \] (A.10)

which is the rhs of (A.8) via (A.5).

And last but not least we have the following identity

\[ \sum_{Q} q^{\frac{1}{2} m_{1} + m_{i} - m_{i-1}} \left\{ \begin{array}{c} A + 2 \sum_{k=1}^{l-1} e_{k} \\ B_{>i} + \sum_{k=1}^{l-2} e_{k} + e_{i} \end{array} \right\} = \sum_{l=2}^{i} \sum_{Q} q^{-\frac{1}{2} m_{1} + m_{i}} \left\{ \begin{array}{c} A + 2e_{1} + 2 \sum_{k=1}^{l-1} e_{k} \\ B_{>i} + \sum_{k=1}^{l-1} e_{k} \end{array} \right\} \] (A.11)

For the derivation of this identity we use recursion relation (2.5) in the first slot on the rhs of (A.11) for \( l > 2 \).

\[ q^{\frac{1}{2} m_{1}} \sum_{Q} \left\{ \begin{array}{c} A + 2e_{1} + 2 \sum_{k=1}^{\nu} e_{k} \\ B_{>i} + \sum_{k=1}^{\nu} e_{k} \end{array} \right\} + \sum_{l=3}^{i} q^{m_{l}} \left\{ \begin{array}{c} A + 2 \sum_{k=1}^{l} e_{k} \\ B_{>i} + \sum_{k=1}^{l} e_{k} \end{array} \right\} + \sum_{l=3}^{i} q^{m_{1} - m_{2} + 1 + m_{l}} \left\{ \begin{array}{c} A + 2e_{2} + 2 \sum_{k=l+1}^{l} e_{k} \\ B_{<i} + e_{1} + \sum_{k=l}^{l} e_{k} \end{array} \right\} \] (A.12)

We can combine the first two terms via the analogue of (A.4) in the opposite direction. In the last sum we can proceed analogously to the derivation of the extended mirror identity.
corresponding to recursion relation (2.5): expand the \( l^{th} \) term from 1 to \( l - 2 \) which leads to another summation over \( l' \). Swapping the two sums and recombining the \( l \) sum yields for (A.12)

\[
q^{\frac{2}{d}m_1} \sum_{Q} \left( q^{-m_{i-1}+m_i} \begin{cases} A + 2e_{i-1} \\ B_{>1} + e_i \end{cases} \right) + \sum_{l'=1}^{i-2} q^{m_i-m_{i-1}+m_l} \left( A + 2 \sum_{k=1}^{l'} e_k + 2e_{i-1} \right)
\]

which is the lhs of (A.11) due to (A.3).

Again analogous formulas to (A.8) and (A.11) hold in opposite direction. The derivation of these is left to the reader.

**B  Proof of the Lemmas**

In this appendix we give the proofs of the three lemmas stated in section 4.1. We begin with the proof of lemma 1.

**Proof of lemma 1**

We first consider the case that \( s \) is odd. The lemma is trivially true for \( s = 1 \). For \( s \geq 3 \) we make the variable change \( m_i \rightarrow m_i + 1 \) for \( i = 1, 3, \ldots, s - 2 \) and get for the lhs of (4.19)

\[
\sum_{Q+(e_{s-2}+e_{s-4}+\ldots)} q^{-\frac{1}{2}m_{s-1}} \left( \begin{array}{c} A + e_{s-1} \\ 0 \end{array} \right) = \sum_{Q} q^{-n_1-n_3-\ldots-n_{s-2}+\frac{1}{2}m_{s-1}} \left( \begin{array}{c} A + 2e_2 + 2e_4 + \ldots + 2e_{s-1} \\ e_1 + e_3 + \ldots + e_{s-2} \end{array} \right). \tag{B.1}
\]

Next notice that we can rewrite the exponent as

\[
-n_1 - n_3 - \ldots - n_{s-2} - \frac{1}{2}m_{s-1} = m_1 - m_2 + m_3 - m_4 + \ldots + m_{s-2} - m_{s-1}. \tag{B.2}
\]

Now using (A.4) with \( i = s - 2 \) and \( i + M = \nu - 2 \), (A.6), \( \frac{1}{2^{\nu-\nu}} \) times (A.7) and the analogue of (A.4) in the opposite direction we obtain for (B.1)

\[
\sum_{Q} q^{-n_1-n_3-\ldots-n_{\nu-4}-\ldots-n_{\nu-s+1}+\frac{1}{2}m_{\nu-s}} \left( \begin{array}{c} A + 2e_{\nu-s} + 2e_{\nu-s+2} + \ldots + 2e_{\nu-3} \\ e_{\nu-s+1} + e_{\nu-s+3} + \ldots + e_{\nu-2} \end{array} \right) \tag{B.3}
\]

which in turn is the rhs of (4.19) after making the variable change \( m_i \rightarrow m_i - 1 \) for \( i = \nu - s + 1, \nu - s + 3, \ldots, \nu - 2 \).

We proceed similarly for even \( s \). For \( s = 2 \) use (A.8) with \( \tilde{A} = A - e_1 \) and make the variable change \( m_i \rightarrow m_i - 1 \) for \( i = 1, 2, \ldots, \nu - 2 \). For \( s > 2 \) we make the variable change \( m_i \rightarrow m_i + 1 \) for \( i = 2, 4, \ldots, s - 2 \) and obtain for the lhs of (4.19)

\[
\sum_{Q} q^{-n_2-n_4-\ldots-n_{s-2}+\frac{1}{2}m_{s-1}+\frac{1}{2}} \left( \begin{array}{c} A + e_1 + 2e_3 + 2e_5 + \ldots + 2e_{s-1} \\ e_2 + e_4 + \ldots + e_{s-2} \end{array} \right). \tag{B.4}
\]
The exponent can again be rewritten as
\[-n_2 - n_4 - \ldots - n_{s-2} - \frac{1}{2} m_{s-1} = -\frac{m_1}{2} + m_2 - m_3 + m_4 - m_5 \pm \ldots + m_{s-2} - m_{s-1}. \quad (B.5)\]

Using (A.4) with \(i = s - 2\) and \(i + M = \nu - 2\), (A.6), \(\frac{\nu - s}{2}\) times (A.7) and (A.11) we obtain for (B.4)
\[
\sum_{Q} q^{\frac{1}{2} m_1 + m_{v-2} - m_{v-3} - \ldots + m_{v-s+2} - m_{v-s+1} + \frac{s - 2}{2}}
\begin{cases}
A + e_1 + 2e_2 + 2e_3 + \ldots + 2e_{\nu-s+1} + 2e_{\nu-s+2} + \ldots + 2e_{\nu-3} \\
e_1 + e_2 + e_3 + \ldots + e_{\nu-s} + e_{\nu-s+2} + \ldots + e_{\nu-2}
\end{cases}
\]
\[
= \sum_{Q} q^{-n_{v-2} - n_{v-4} - \ldots - m_{v-s} - m_{v-s-1} - \ldots - n_{1} + \frac{s - 2}{2}}
\begin{cases}
A + e_1 + 2e_2 + 2e_3 + \ldots + 2e_{\nu-s+1} + 2e_{\nu-s+2} + \ldots + 2e_{\nu-3} \\
e_1 + e_2 + e_3 + \ldots + e_{\nu-s} + e_{\nu-s+2} + \ldots + e_{\nu-2}
\end{cases}
\quad (B.6)
\]

which is the rhs of \((4.19)\) after making the variable change \(m_i \to m_i - 1\) for \(i = 1, 2, 3, \ldots, \nu - s, \nu - s + 2, \ldots, \nu - 2\).

**Proof of lemma 2a**

For the proof of this lemma we need to distinguish between \(\bar{a} = N - a\) even and odd. Let us start with the case that \(\bar{a}\) even. We make the variable change \(m_i \to m_i + 1\) for \(i = a + 1, a + 3, \ldots, N - 1\) on the lhs of \((4.20)\) and get
\[
\sum_{Q+(e_{N-1} + e_{N-3} + \ldots)} q^{-n_{a+1}' - n_{a+3}' - \ldots - n'_{N-1} + \frac{5}{4}}
\begin{cases}
B_{\geq N} + 2e_a + 2e_{a+2} + \ldots + 2e_{N-2} + e_N \\
e_{a+1} + e_{a+3} + \ldots + e_{N-1}
\end{cases}
\quad (B.7)
\]

We can now use the telescopic expansion \((4.8)\) to the left from \(a + 1\) to 1, flip the expansion via the mirror identity \((4.10)\) and apply the extended mirror identity \((4.14)\). Repeat this \(\frac{\nu - s}{2}\) times and recombine via the telescopic expansion \((A.2)\) which yields
\[
\sum_{Q+(e_{N-1} + e_{N-3} + \ldots)} q^{-n_1' + n_3' - \ldots - n_{a-1}' - n_{a-3}' + \frac{5}{4}}
\begin{cases}
B_{\geq N} + 2e_2 + 2e_4 + \ldots + 2e_{\bar{a}-2} + 2e_{\bar{a}-1} + \ldots + 2e_{N-1} + e_N \\
e_1 + e_3 + \ldots + e_{\bar{a}-1} + e_{\bar{a}} + \ldots + e_{N-1}
\end{cases}
\]
\[
- \sum_{Q+(e_{N-1} + e_{N-3} + \ldots)} q^{-n_1' + n_3' - \ldots - n_{a-1}' + \frac{5}{4}}
\begin{cases}
B_{\geq N} + 2e_2 + 2e_4 + \ldots + 2e_{\bar{a}-2} + e_N \\
e_1 + e_3 + \ldots + e_{\bar{a}-3}
\end{cases}
\]
\[
= \sum_{Q+a \rho < N+(e_{a} + e_{a+2} + \ldots ) < N} \left\{ B_{\geq N} + e_{\bar{a}-1} + e_{N-1} \right\} \\
- \sum_{Q+a \rho < N+(e_{a-1} + e_{a+1} + \ldots ) < N} q^{\frac{1}{2}} \left\{ B_{\geq N} + e_{\bar{a}-2} + e_N \right\}
\quad (B.8)
\]
where we got the equal sign by making the variable changes \( m_i \rightarrow m_i - 1 \) for \( i = 1, 3, \ldots, \bar{a} - 1, \bar{a}, \ldots, N - 1 \) for the first term and \( i = 1, 3, \ldots, \bar{a} - 3 \) for the second term.

The proof for \( \bar{a} \) odd is very similar. This time we make the variable change \( m_i \rightarrow m_i + 1 \) for \( i = 1, 2, \ldots, a, a + 2, \ldots, N - 1 \) on the lhs of (4.20) and get

\[
\sum_{Q+(e_{N-1}+e_{N-3}+\ldots)} q^{-n'_1-n'_3-\ldots-n'_{a-2}-n'_{N-1}+\frac{\bar{a}+1}{2}} q^{B_{\geq N} + e_1 + 2e_2 + 2e_3 + \ldots + 2e_{a-1} + e_{N-1} + e_N} \quad \text{(B.9)}
\]

Instead of using (4.8) one can now apply the special telescopic expansion (A.3). After that one proceeds exactly as before: flip the expansion via the mirror identity (4.10) followed by the extended mirror identity (4.14) instead of (4.9) with

\[
\sum_{Q+(e_{N-1}+e_{N-3}+\ldots)} q^{-n'_2-n'_4-\ldots-n'_{\bar{a}-1}-n'_{\bar{a}}-n'_{N-1}+\frac{\bar{a}-1}{2}} q^{B_{\geq N} + e_1 + 2e_3 + 2e_5 + \ldots + 2e_{\bar{a}-2} + e_{N-1} + e_N} \quad \text{(B.10)}
\]

Making the variable change \( m_i \rightarrow m_i - 1 \) for \( i = 2, 4, \ldots, \bar{a} - 1, \bar{a}, \ldots, N - 1 \) for the first term and \( i = 2, 4, \ldots, \bar{a} - 3 \) for the second term gives exactly the rhs of (4.20).

**Proof of lemma 2b**

The proof is similar to the proof of part a). We start this time with the case \( \bar{a} \) odd. Making the variable change \( m_i \rightarrow m_i + 1 \) for \( i = a + 2, a + 4, \ldots, N - 1 \) we get for the lhs of (4.21)

\[
\sum_{Q+(e_{N-1}+e_{N-3}+\ldots)} q^{-n'_{a+2}-n'_{a+4}-\ldots-n'_{N-1}+\frac{\bar{a}-1}{2}} q^{B_{\geq N} + 2e_{a+1} + 2e_{a+3} + \ldots + 2e_{N-2} + e_N} \quad \text{(B.11)}
\]

Again we telescopically expand to the left from \( a + 2 \) to 1, use the mirror identity (4.10) followed by the extended mirror identity (4.14) \( \frac{\bar{a}-3}{2} \) times. The only difference to part a) is that we now recombine the expansion via (4.8) instead of (4.9) with \( i = \bar{a} - 2 \) and \( i + M = N \)

\[
\sum_{Q+(e_{N-1}+e_{N-3}+\ldots)} q^{-n'_1-n'_3-\ldots-n'_{a-2}+\frac{\bar{a}-1}{4}} q^{B_{\geq N} + 2e_2 + 2e_4 + \ldots + 2e_{\bar{a}-1} + e_{N-1} + e_N} \quad \text{(B.12)}
\]

\[
-\sum_{Q+(e_{N-1}+e_{N-3}+\ldots)} q^{-n'_1-n'_3-\ldots-n'_{a-2}-n'_{a-1}-\ldots-n'_{N-1}+\frac{\bar{a}-1}{4}+\frac{\bar{a}-1}{4}} q^{B_{\geq N} + e_1 + e_3 + \ldots + e_{\bar{a}-2}}
\]
\[
\left\{ B_{>N} + 2e_2 + 2e_4 + \ldots + 2e_{\bar{a}-3} + 2e_{\bar{a}-2} + \ldots + 2e_{N-1} + e_N + 2e_{N+1} \right\} 
\]

where the second term is the term \( l = i + M = N \) in (13) and the extra phase \( \frac{L+1}{2} \) comes from the fact that there is the entry \( \frac{1}{2}(L + 1) \) in the \( N \)th slot. Changing variables \( m_i \rightarrow m_i - 1 \) for \( i = 1, 3, \ldots, \bar{a} - 2 \) for the first term and \( i = 1, 3, \ldots, \bar{a} - 2, \bar{a} - 1, \ldots, N \) for the second term yields the rhs of (12). 

The case \( \bar{a} \) even goes the same with the only difference that the first expansion is via the special telescopic expansion (A.3) instead of (A.8).

**Proof of lemma 3a**

The proof of this lemma is analogous to the proof of lemma 2a). We distinguish between \( r - \bar{a} - 1 \) even and odd. For \( r - \bar{a} - 1 \) even one changes variables \( m_i \rightarrow m_i + 1 \) for \( i = N + 1, N + 3, \ldots, r + a \) for the lhs of (4.22). Then once again one telescopically expands this time to the right from \( r + a \) to \( \nu - 2 \), uses the mirror identity followed by the extended mirror identity \( \frac{r - \bar{a} - 1}{2} \) times and recombines via (4.9). The appropriate variable changes then yield the rhs of (4.22).

For \( r - \bar{a} - 1 \) odd again telescopically expand to the right from \( r + a \) to \( \nu - 2 \), use the mirror identity followed by the extended mirror identity \( \frac{r - \bar{a} - 2}{2} \) times and combine via (4.8). The appropriate variable changes then yield the rhs of (4.22).

**Proof of lemma 3b**

Again we first consider \( r - \bar{a} - 1 \) even. Change variables \( m_i \rightarrow m_i + 1 \) for \( i = N, N + 2, \ldots, r + a - 1 \) on the lhs of (4.23), telescopically expand to the left from \( r + a - 1 \) to \( \nu - 2 \), use the mirror identity and the extended mirror identity \( \frac{r - \bar{a} - 1}{2} \) times and combine via (4.9). For \( r - \bar{a} - 1 \) odd change variables \( m_i \rightarrow m_i + 1 \) for \( i = N + 1, N + 3, \ldots, r + a - 1 \) on the lhs of (4.23), telescopically expand to the left from \( r + a - 1 \) to \( \nu - 2 \), use the mirror identity and the extended mirror identity \( \frac{r - \bar{a} - 2}{2} \) times and combine via (4.8).

**References**

[1] J. Lepowsky and M. Prime, Structure of the standard modules for the affine Lie algebra \( A_1^{(1)} \), Contemporary Mathematics, Vol. 46 (AMS, Providence, 1985)

[2] R. Kedem, T.R. Klassen, B. McCoy and E. Melzer, Phys. Lett. B304 (1993) 263

[3] R. Kedem, T. Klassen, B. McCoy, E. Melzer, Phys. Lett. B307 (1993) 68

[4] S. Dasmahapatra, R. Kedem, T. Klassen, B. McCoy and E. Melzer, Int. J. Mod. Phys B7 (1993) 3617

[5] R. Kedem, B. McCoy, J. Stat. Phys 71 (1993) 865

[6] E. Melzer, Lett. in Math. Phys. 31 (1994) 233

[7] S. Dasmahapatra, R. Kedem, B. McCoy, E. Melzer, J. Stat. Phys. 74 (1994) 239

[8] R. Kedem, B. McCoy and E. Melzer, The sums of Rogers, Schur and Ramanujan and the Bose-Fermi correspondence in 1+1 dimensional quantum field theory, Stony Brook preprint, hep-th/9304056
[9] J. Kellendonk, M. Rösgen and R. Varnhagen, Int. J. Mod. Phys. A9 (1994) 1009
[10] M. Terhoeven, Mod. Phys. Lett. A9 (1994) 133
[11] E. Melzer, Int. J. Mod. Phys. A9 (1994) 1115
[12] A. Berkovich, Nucl. Phys. B431 (1994) 315
[13] S.O. Warnaar, Fermionic Solutions of the Andrews-Baxter-Forrester model I: Unification of TBA and CMT methods, hep-th/9501134
[14] S.O. Warnaar and P. Pearce, J. Phys. A27 (1994) L891
[15] S.O. Warnaar and P. Pearce, A-D-E Polynomial Identities and the Rogers-Ramanujan Identities, hep-th/9411009
[16] O. Foda and Y-H. Quano, Int. J. of Mod. Phys. 10 (1995) 2291
[17] O. Foda and Y-H. Quano, Virasoro character Identities from the Andrews-Bailey Construction hep-th/9408086
[18] E. Melzer, Supersymmetric Analogs of the Gordon-Andrews Identities and Related TBA Systems, hep-th/9412154
[19] A. Berkovich and B. McCoy, Continued Fraction and Fermionic Representation for Characters of \( M(p, p') \) Minimal Models, hep-th/9412053, Letts. in Math. Phys. (in press)
[20] O. Foda and S.O. Warnaar, A bijection which implies Melzer’s polynomial identities: the \( \chi_{1,1}^{(p,p+1)} \) case, hep-th/9501088, to appear in Lett. Math. Phys.
[21] D. Gepner, Phys. Lett. B348 (1995) 377
[22] E. Baver and D. Gepner, Fermionic Sum Representation for the Virasoro Characters of the Unitary Superconformal Minimal Models, hep-th/9502118
[23] P. Bouwknegt, A. Ludwig and K. Schoutens, Spinon basis for higher level \( SU(2) \) WZW models, hep-th/9412108
[24] P. Bouwknegt, A. Ludwig and K. Schoutens, Spinon basis for \( \hat{sl}_2 \) integrable highest weight modules and new character formulas, hep-th/9504074
[25] A. Nakayashiki and Y. Yamada, Crystallizing the spinon basis, hep-th/9504052
[26] A. Nakayashiki and Y. Yamada, Crystalline spinon basis for RSOS models, hep-th/9505083
[27] T. Arakawa, T. Nakanishi, K. Oshima and A. Tsuchiya, Spectral Decomposition of Path Space in Solvable Lattice Model, q-alg/9507027
[28] A. Berkovich, B. McCoy, W. Orrick, Polynomial Identities, Indices, and Duality for the \( N = 1 \) Superconformal Model \( SM(2, 4\nu) \), hep-th/9507072

39
[29] O. Foda, M. Okado and S.O. Warnaar, A proof of polynomial identities of type $\hat{sl}(n)_1 \times \hat{sl}(n)_1/\hat{sl}(n)_2$, [q-alg/9507014]

[30] A. Rocha-Caridi, In: Vertex operators in mathematics and physics. J. Lepowsky et al. (eds.) MSRI publications No.3, p. 451, Berlin, Heidelberg, New York: Springer 1984

[31] B.L. Feigin and D.B. Fuchs, Funct. Anal. Appl. 17 (1983) 241

[32] P. Goddard, A. Kent and D. Olive, Commun. Math. Phys. 103 (1986) 105

[33] D. Kastor, E. Martinec and Z. Qiu, Phys. Lett. B200 (1988) 434
   J. Bagger, D. Nemeshansky and S. Yankielowicz, Phys. Rev. Lett. 60 (1988) 389
   F. Ravanini, Mod. Phys. Lett. A3 (1988) 397

[34] G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193

[35] E. Date, M. Jimbo, T. Miwa and M. Okado, Phys. Rev. B35 (1987) 2105

[36] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Nucl. Phys B290 (1987) 231

[37] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Adv. Stud. in Pure Math. 16 (1988) 17

[38] I. Schur, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. (1917) 302 (Reprinted in I. Schur, Gesammelte Abhandlungen, vol. 2, pp 117-136, Springer, Berlin, 1973)

[39] D. Huse, Phys. Rev. B30 (1984) 3908

[40] J. Lepowsky and R.L. Wilson, Proc. Natl. Acad. Sci. USA 78 (1981) 7254

[41] A.B. Zamolodchikov and V.A. Fateev, Sov. Phys. JETP 62 (1985) 215

[42] V.G. Kac and D.H. Peterson, Adv. in Math. 53 (1984) 125

[43] M. Jimbo and T. Miwa, Adv. Stud. in Pure Math. 4 (1984) 97

[44] P. Christe and F. Ravanini, Int. J. Mod. Phys. A4 (1989) 897

[45] J. Distler and Z. Qiu, Nucl. Phys. B336 (1990) 533

[46] D. Altschuler, M. Bauer and H. Saleur, J. Phys. A23 (1990) L789