Energy-momentum for a charged nonsingular black hole solution with a nonlinear mass function

I. Radinschi*1, Th. Grammenos**2, Farook Rahaman***3, A. Spanou****4, Sayeedul Islam*****5, Surajit Chattopadhyay******6, and Antonio Pasqua*******7

1Department of Physics
“Gh. Asachi” Technical University, Iasi, 700050, Romania
2Department of Civil Engineering, University of Thessaly, 383 34 Volos, Greece
3Department of Mathematics, Jadavpur University, Kolkata 700 032, West Bengal, India
4School of Applied Mathematics and Physical Sciences, National Technical University of Athens, 157 80 Athens, Greece
5Department of Mathematics, Jadavpur University, Kolkata 700 032, West Bengal, India
6Department of Mathematics, Amity Institute of Applied Sciences, Amity University, Major Arterial Road, Action Area II, Rajarhat, New Town, West Bengal 700135, India
7Department of Physics, University of Trieste, Via Valerio, 2, 34127 Trieste, Italy

*radinschi@yahoo.com, **thgramme@civ.uth.gr, ***rahaman@iucaa.ernet.in, ****aspanou@central.ntua.gr, *****sayeedul.jumath@gmail.com, ******surajcha@associates.iucaa.in, *******toto.pasqua@gmail.com

Abstract

The energy-momentum of a new four-dimensional, charged, spherically symmetric and nonsingular black hole solution constructed in the context of general relativity coupled to a theory of nonlinear electrodynamics is investigated,
whereby the nonlinear mass function is inspired by the probability density function of the continuous logistic distribution. The energy and momentum distributions are calculated by use of the Einstein, Landau-Lifshitz, Weinberg and Møller energy-momentum complexes. In all these prescriptions it is found that the energy distribution depends on the mass $M$ and the charge $q$ of the black hole, an additional parameter $\beta$ coming from the gravitational background considered, and on the radial coordinate $r$. Further, the Landau-Lifshitz and Weinberg prescriptions yield the same result for the energy, while in all the aforesaid prescriptions all the momenta vanish. We also focus on the study of the limiting behavior of the energy for different values of the radial coordinate, the parameter $\beta$, and the charge $q$. Finally, it is pointed out that for $r \to \infty$ and $q = 0$ all the energy-momentum complexes yield the same expression for the energy distribution as in the case of the Schwarzschild black hole solution.

**Keywords:** Energy-Momentum Complexes; Nonsingular Black Holes.

**PACS Numbers:** 04.20.-q, 04.20.Cv, 04.70.Bw

**MSC Numbers:** 83C57

1 Introduction

The problem of the energy-momentum localization in General Relativity has been investigated over the years by using various and different powerful tools such as superenergy tensors [1]-[4], quasi-local expressions [5]-[9], and the mostly known pseudo-tensorial energy-momentum complexes introduced by Einstein [10],[11], Landau and Lifshitz [12], Papapetrou [13], Bergmann and Thomson [14], Møller [15], Weinberg [16], and Qadir and Sharif [17].

As it is well-known, the main difficulty which arises consists in developing a properly defined expression for the energy density of the gravitational background. Until today, no generally accepted meaningful definition for the energy of the gravitational field has been established. However, despite this difficulty, many physically reasonable results have been obtained by applying the aforesaid definitions for the energy-momentum localization. At this point, one cannot but notice the existing agreement between the pseudo-tensorial prescriptions and the quasi-local mass definition elaborated by Penrose [18] and further developed by Tod [19].

Although the dependence on the coordinate system continues to be the main “weakness” of these tools, a number of physically interesting results have been obtained for gravitating systems in $(3 + 1)$, $(2 + 1)$, and $(1 + 1)$ spacetime dimensions by using the energy-momentum complexes [20]-[50]. In fact, the Møller energy-momentum complex is the only computational tool independent of coordinates. In the context of other pseudo-tensorial prescriptions, in order to calculate the energy and momentum distributions one introduces Schwarzschild Cartesian and Kerr-Schild coordinates.

An alternative option for avoiding the problem of the coordinate system dependence is provided by the teleparallel theory of gravity [51]-[52], whereby one notices the con-
siderable similarity of results obtained by this approach with results achieved by using
the energy-momentum complexes [53]–[57].

Finally, closing this short introduction to the topic of the energy-momentum local-
ization, it is necessary to point out the broadness of the ongoing attempts in order to
define properly and, actually, rehabilitate the concept of the energy-momentum complex
[58]–[61].

The outline of the present paper is the following. In Section 2 we introduce the new
static and charged, spherically symmetric, nonsingular black hole solution under study.
Section 3 is devoted to the presentations of the Einstein, Landau-Lifshitz, Weinberg,
and Møller energy-momentum complexes used for the calculations. In Section 4 the
computations of the energy and momentum distributions are presented. In the Discussion
given in Section 5, we comment on our results and explore some limiting and particular
cases. We have used geometrized units \( (c = G = 1) \), while the signature is \((+,-,-,-)\).
The calculations for the Einstein, Landau-Lifshitz, and Weinberg energy-momentum
complexes are performed by use of the Schwarzschild Cartesian coordinates. Greek
indices range from 0 to 3, while Latin indices run from 1 to 3.

## 2 The New Charged Nonsingular Black Hole Solution with a nonlinear mass function

The determination of nonsingular black hole solutions by coupling gravity to nonlinear
electrodynamics has attracted interest long ago (for a review see, e.g., [62] for spherically
symmetric solutions, or [63] and references therein for charged axisymmetric solutions).
Recently, L. Balart and E. C. Vagenas [64] constructed a number of new charged, non-
singular and spherically symmetric, four-dimensional black hole solutions with a nonlinear
electrodynamics source. Indeed, starting with the static and spherically symmetric
spacetime geometry described by the line element

\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

with the metric function

\[
f(r) = 1 - \frac{2M}{r} \left[ \frac{\sigma(\beta r)}{\sigma_\infty} \right]^\beta,
\]

where the distribution function \( \sigma(\beta r) \) depends on the mass \( M \), the charge \( q \), the radial
coordinate \( r \), and the parameter \( \beta \in \mathbb{R}^+ \), and \( \sigma_\infty = \sigma(r \to \infty) \) is a normalization factor.
Thus one has a nonlinear mass function of the form

\[
m(r) = M \left[ \frac{\sigma(\beta r)}{\sigma_\infty} \right]^\beta,
\]

which, at infinity, becomes \( M \). It is shown in [64] that for specific distribution functions
\( \sigma(r) \) the curvature invariants \( (R, R_{\mu\nu} R^{\mu\nu}, R_{\kappa\lambda\mu\nu} R^{\kappa\lambda\mu\nu}) \) and the associated nonlinear
electric field are nonsingular everywhere.
Based on these results, we have already studied the problem of the localization of energy for a function $\sigma(r)$ resembling the form of the Fermi-Dirac distribution. Here, following up that work, we adopt another distribution function given in (4) that is inspired by the form of the probability density function of the continuous logistic distribution (5), such that

$$f(r) = 1 - \frac{2M}{r} \left\{ \frac{4 \exp \left( -\frac{2q^2}{\beta Mr} \right)}{1 + \exp \left( -\frac{2q^2}{\beta Mr} \right)^2} \right\}^\beta, \quad \beta \in \mathbb{R}^+$$

(4)

with the nonlinear mass function given by (3).

The associated nonsingular spacetime exhibits two horizons, while the nonlinear and nonsingular electric field that asymptotically goes to $q/r^2$ reads now

$$E(r) = \frac{q}{8r^2} \left\{ \text{sech} \left( \sqrt{\frac{q^2}{2\beta Mr}} \right) \right\}^{2(1+\beta)} \times \left[ (1 + \beta) - \beta \cosh \sqrt{\frac{2q^2}{\beta Mr}} + 7 \sqrt{\frac{\beta Mr}{2q^2}} \sinh \sqrt{\frac{2q^2}{\beta Mr}} \right].$$

(5)

When $\beta \to 0$, the metric function (4) takes the form $f(r) = 1 - 2M/r$, i.e. we get the Schwarzschild black hole geometry.

Thus, in what follows we are going to investigate the problem of energy-momentum localization for a charged and nonsingular black hole solution with the spacetime described by (4), (5) and the nonlinear mass function

$$m(r) = M \left\{ \frac{4 \exp \left( -\frac{2q^2}{\beta Mr} \right)}{1 + \exp \left( -\frac{2q^2}{\beta Mr} \right)^2} \right\}^\beta.$$

(6)

### 3 Einstein, Landau-Lifshitz, Weinberg and Møller Energy-Momentum Complexes

The definition of the Einstein energy-momentum complex (10), (11) for a (3 + 1) dimensional gravitating system is given by

$$\theta^\mu_\nu = -\frac{1}{16\pi} h^{\mu\lambda}_{\nu\lambda},$$

(7)

where the von Freud superpotentials $h^{\mu\lambda}_{\nu\lambda}$ are given as

$$h^{\mu\lambda}_{\nu\lambda} = \frac{1}{\sqrt{-g}} g^{\nu\sigma}[g^{\mu\sigma}g^{\lambda\kappa} - g^{\lambda\sigma}g^{\mu\kappa}].$$

(8)
and satisfy the required antisymmetric property
\[ h_{\nu}^{\mu \lambda} = -h_{\nu}^{\lambda \mu}. \] (9)
The components \( \theta_0^0 \) and \( \theta_0^i \) correspond to the energy and the momentum densities, respectively. In the Einstein prescription the local conservation law holds:
\[ \theta_{\nu, \mu} = 0. \] (10)
Thus, the energy and the momenta can be computed by
\[ P_{\nu} = \int \int \int \theta_0^0 \, dx^1 \, dx^2 \, dx^3. \] (11)
Applying Gauss' theorem, the energy-momentum is
\[ P_{\nu} = \frac{1}{16\pi} \int \int h_{\nu}^{0i} n_i \, dS, \] (12)
where \( n_i \) represents the outward unit normal vector on the surface \( dS \).
The Landau-Lifshitz energy-momentum complex \[12\] is defined as
\[ L_{\mu \nu} = \frac{1}{16\pi} S_{\mu \nu \rho \sigma}, \] (13)
where the Landau-Lifshitz superpotentials are given by:
\[ S_{\mu \nu \rho \sigma} = -g(g^{\mu \nu} g^{\rho \sigma} - g^{\mu \rho} g^{\nu \sigma}). \] (14)
The \( L^{00} \) and \( L^{0i} \) components represent the energy and the momentum densities, respectively. In the Landau-Lifshitz prescription the local conservation law reads:
\[ L_{i, \nu} = 0. \] (15)
By integrating \( L_{\mu \nu} \) over the 3-space one obtains for the energy-momentum:
\[ P^\mu = \int \int \int L^{00} \, dx^1 \, dx^2 \, dx^3. \] (16)
By using Gauss' theorem we have
\[ P^\mu = \frac{1}{16\pi} \int \int S^{000} n_i \, dS = \frac{1}{16\pi} \int \int U^{00} n_i \, dS. \] (17)
The Weinberg energy-momentum complex \[16\] is given by the expression
\[ W^{\mu \nu} = \frac{1}{16\pi} D^\lambda_{\mu \nu}, \] (18)
where \( D^{\lambda \mu \nu} \) are the corresponding superpotentials:

\[
D^{\lambda \mu \nu} = \frac{\partial h^{\kappa}}{\partial x_\lambda} \eta^{\mu \nu} - \frac{\partial h^{\kappa}}{\partial x_\mu} \eta^{\lambda \nu} - \frac{\partial h^{\kappa}}{\partial x_\nu} \eta^{\lambda \mu} + \frac{\partial h^{\lambda \nu}}{\partial x_\mu} - \frac{\partial h^{\lambda \mu}}{\partial x_\nu},
\]

(19)

and

\[
h_{\mu \nu} = g_{\mu \nu} - \eta_{\mu \nu}.
\]

(20)

Here the \( W^{00} \) and \( W^{0i} \) components correspond to the energy and the momentum densities, respectively. In the Weinberg prescription the local conservation law reads:

\[
W^{\mu \nu} = 0.
\]

(21)

The integration of \( W^{\mu \nu} \) over the 3-space yields for the energy-momentum:

\[
P^{\mu} = \iiint W^{\mu 0} \, dx^1 \, dx^2 \, dx^3.
\]

(22)

Applying Gauss’ theorem and integrating over the surface of a sphere of radius \( r \), one obtains for the energy-momentum distribution the expression:

\[
P^{\mu} = \frac{1}{16 \pi} \iiint D^{0 \mu i} n_i \, dS.
\]

(23)

The Møller energy-momentum complex [15] is given by the expression

\[
\mathcal{J}^{\mu}_\nu = \frac{1}{8 \pi} M^{\mu \lambda}_{\nu, \lambda},
\]

(24)

where the Møller superpotentials \( M^{\mu \lambda}_{\nu, \lambda} \) are

\[
M^{\mu \lambda}_{\nu, \lambda} = \sqrt{-g} \left( \frac{\partial g_{\nu \sigma}}{\partial x^\kappa} - \frac{\partial g_{\nu \kappa}}{\partial x^\sigma} \right) g^{\mu \kappa} g^{\lambda \sigma}
\]

(25)

and satisfy the necessary antisymmetric property:

\[
M^{\mu \lambda}_{\nu, \lambda} = -M^{\lambda \mu}_{\nu, \lambda}.
\]

(26)

Møller’s energy-momentum complex also satisfies the local conservation law

\[
\frac{\partial \mathcal{J}^{\mu}_\nu}{\partial x^\mu} = 0,
\]

(27)

with \( \mathcal{J}^{0}_\nu \) and \( \mathcal{J}^i_\nu \) representing the energy and the momentum densities, respectively. In the Møller prescription, the energy-momentum is given by

\[
P^{\nu}_\nu = \iiint \mathcal{J}^{0}_\nu \, dx^1 \, dx^2 \, dx^3.
\]

(28)

With the aid of Gauss’ theorem one gets

\[
P^{\nu}_\nu = \frac{1}{8 \pi} \iiint M^{0 i}_{\nu, i} n_i \, dS.
\]

(29)
4 Energy and Momentum Distributions for the New Charged Nonsingular Black Hole Solution

In order to calculate the energy and momenta by using the Einstein energy-momentum complex, it is required to transform the metric given by the line element (1) in Schwarzschild Cartesian coordinates. We obtain the line element in the following form:

\[ ds^2 = f(r)dt^2 - (dx^2 + dy^2 + dz^2) - \frac{f^{-1}(r) - 1}{r^2}(x dx + y dy + z dz)^2. \]  

(30)

The components of the superpotential \( h^0_{\nu} \) in Schwarzschild Cartesian coordinates are:

\[ h^0_1 = h^0_2 = h^0_3 = 0, \]
\[ h^0_1 = h^0_2 = h^0_3 = 0, \]
\[ h^0_1 = h^0_2 = h^0_3 = 0. \]

(31)

The remaining, non-vanishing, components of the superpotentials in the Einstein prescription are:

\[ h^0_1 = \frac{2x}{r^2} \frac{2M}{r} \left\{ \frac{4 \exp \left( -\sqrt{\frac{g^2}{\beta r^2}} \right)}{1 + \exp \left( -\sqrt{\frac{g^2}{\beta r^2}} \right)^2} \right\}^\beta, \]
\[ h^0_2 = \frac{2y}{r^2} \frac{2M}{r} \left\{ \frac{4 \exp \left( -\sqrt{\frac{g^2}{\beta r^2}} \right)}{1 + \exp \left( -\sqrt{\frac{g^2}{\beta r^2}} \right)^2} \right\}^\beta, \]
\[ h^0_3 = \frac{2z}{r^2} \frac{2M}{r} \left\{ \frac{4 \exp \left( -\sqrt{\frac{g^2}{\beta r^2}} \right)}{1 + \exp \left( -\sqrt{\frac{g^2}{\beta r^2}} \right)^2} \right\}^\beta. \]

(32)-(34)

From the line element (30), the expression (12) and the superpotentials (32)-(34), we obtain for the energy distribution in the Einstein prescription (see Figure 1):

\[ E_E = M \left\{ \frac{4 \exp \left( -\sqrt{\frac{g^2}{\beta r^2}} \right)}{1 + \exp \left( -\sqrt{\frac{g^2}{\beta r^2}} \right)^2} \right\}^\beta. \]

(35)

By using (12) and (31) we find that all the momenta vanish:

\[ P_x = P_y = P_z = 0. \]

(36)
In order to apply the Landau-Lifshitz energy-momentum complex, we use the $U^\mu_0\nu$ quantities defined in (17) and we find the following nonvanishing components:

$$U^{001} = \frac{2x}{r^2} \left\{ \frac{2M}{r} \left\{ \frac{4 \exp \left( -\sqrt{\frac{2y^2}{\beta M r}} \right)}{1 + \exp \left( -\sqrt{\frac{2y^2}{\beta M r}} \right)} \right\}^\beta \right\},$$

$$U^{002} = \frac{2y}{r^2} \left\{ \frac{2M}{r} \left\{ \frac{4 \exp \left( -\sqrt{\frac{2y^2}{\beta M r}} \right)}{1 + \exp \left( -\sqrt{\frac{2y^2}{\beta M r}} \right)} \right\}^\beta \right\},$$

$$\frac{4 \exp \left( -\sqrt{\frac{2y^2}{\beta M r}} \right)}{1 + \exp \left( -\sqrt{\frac{2y^2}{\beta M r}} \right)} \right\}^\beta.$$
Now, replacing (37)-(39) in (17), we obtain the energy distribution in the Landau-Lifshitz prescription:

\[
E_{LL} = \frac{2M}{r^2} \left( 1 - \frac{2M}{r} \right) \left\{ \frac{4 \exp \left(-\sqrt{\frac{2q^2}{\beta M r}}\right)}{\left[ 1 + \exp \left(-\sqrt{\frac{2q^2}{\beta M r}}\right) \right]^2} \right\}^\beta. 
\]

(39)

while all the momenta vanish

\[ P^x = P^y = P^z = 0. \]

(41)

In the Weinberg prescription the non-vanishing superpotential components are:

\[
D_{100} = \frac{2M}{r^2} \left( 1 - \frac{2M}{r} \right) \left\{ \frac{4 \exp \left(-\sqrt{\frac{2q^2}{\beta M r}}\right)}{\left[ 1 + \exp \left(-\sqrt{\frac{2q^2}{\beta M r}}\right) \right]^2} \right\}^\beta, 
\]

(42)

\[
D_{200} = \frac{2M}{r^2} \left( 1 - \frac{2M}{r} \right) \left\{ \frac{4 \exp \left(-\sqrt{\frac{2q^2}{\beta M r}}\right)}{\left[ 1 + \exp \left(-\sqrt{\frac{2q^2}{\beta M r}}\right) \right]^2} \right\}^\beta. 
\]

(43)
Substituting the expressions obtained in (42)-(44) into (23) we get for the energy distribution inside a 2-sphere of radius \( r \) the expression:

\[
E_W = \frac{2M}{r^2} \frac{\frac{4 \exp \left( -\sqrt{\frac{2q^2}{\beta M r}} \right)}{1 + \exp \left( -\sqrt{\frac{2q^2}{\beta M r}} \right)^2}}^{\beta}.
\] (44)

Substituting the expressions obtained in (42)-(44) into (23) we get for the energy distribution inside a 2-sphere of radius \( r \) the expression:

\[
E_W = \frac{M}{1 - 2M r \left\{ \begin{array}{c} 4 \exp \left( -\sqrt{\frac{2q^2}{\beta M r}} \right) \\ 1 + \exp \left( -\sqrt{\frac{2q^2}{\beta M r}} \right)^2 \end{array} \right\}^{\beta}}.
\] (45)

In the Weinberg prescription all the momenta vanish

\[ P^x = P^y = P^z = 0. \] (46)

One can see from (40) and (45) that the energy in the Landau-Lifshitz prescription is identical with the energy in the Weinberg prescription.

Finally, in the Møller prescription, we find only one non-vanishing superpotential:

\[
M_0^{\beta} = 2M \sin \theta \left\{ \begin{array}{c} \frac{4 \exp \left( -\sqrt{\frac{2q^2}{\beta M r}} \right)}{1 + \exp \left( -\sqrt{\frac{2q^2}{\beta M r}} \right)^2} \\ 1 - \frac{q^2}{M r^3} \frac{1 - \exp \left( -\sqrt{\frac{2q^2}{\beta M r}} \right)}{1 + \exp \left( -\sqrt{\frac{2q^2}{\beta M r}} \right)} \end{array} \right\}^{\beta}.
\] (47)

Using the above expression for the superpotential and with the aid of the metric coefficient (3) and the expression for energy given by (29), we obtain the energy distribution in the Møller prescription (see Figure 2):

\[
E_M = M \left\{ \begin{array}{c} \frac{4 \exp \left( -\sqrt{\frac{2q^2}{\beta M r}} \right)}{1 + \exp \left( -\sqrt{\frac{2q^2}{\beta M r}} \right)^2} \\ 1 - \frac{\sqrt{q^2 \beta}}{2M r} \frac{1 - \exp \left( -\sqrt{\frac{2q^2}{\beta M r}} \right)}{1 + \exp \left( -\sqrt{\frac{2q^2}{\beta M r}} \right)} \end{array} \right\}^{\beta},
\] (48)

while all the momenta vanish

\[ P_r = P_\theta = P_\phi = 0. \] (49)
Discussion

Following up our previous results [65], we have studied the problem of energy-momentum localization for a new charged, nonsingular and spherically symmetric, static black hole solution in (3+1) dimensions with a nonlinear mass function recently constructed by L. Balart and E.C. Vagenas [64]. To this purpose, we have applied the Einstein, Landau-Lifshitz, Weinberg, and Møller pseudotensorial prescriptions. The calculations have shown that, in all the four prescriptions, the momenta vanish, while the energy depends (see, Table 1) on the black hole’s mass and charge, the radial coordinate, and a parameter \( \beta \in \mathbb{R}^+ \) that is a scaling factor of \( r \) (in essence a dilation factor, as it is always positive) inspired by the form of the logistic distribution and marks the spacetime geometry considered. In fact, for each value of \( \beta \) one can numerically determine the values of the two existing horizon radii (an inner and an outer) obtained from the metric function. It is pointed out that the Landau-Lifshitz and the Weinberg prescriptions yield the same energy distribution.

We have also examined the limiting behavior of the energy in the cases \( r \to \infty, q = 0, \beta \to 0, \) and \( \beta \to \infty \). For \( \beta \to 0 \), the metric function \( f(r) \) becomes \( 1 - 2M/r \), while for \( \beta \to \infty \), \( f(r) \) becomes \( 1 - (2M/r)[\exp(-q^2/2Mr)] \). The corresponding energies are presented in Table 2, where one can see that for \( q = 0 \) as well as at infinity the Einstein and Møller prescriptions yield the same result which is also obtained for the classical Schwarzschild black hole solution, namely the ADM mass \( M \). In the case of
| Prescription       | Energy                                                                 |
|-------------------|------------------------------------------------------------------------|
| Einstein          | \[ E_E = M \left\{ \frac{4 \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right)}{\left[ 1 + \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right) \right]^2} \right\}^\beta \] |
| Landau-Lifshitz   | \[ E_{LL} = \frac{M \left\{ \frac{4 \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right)}{\left[ 1 + \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right) \right]^2} \right\}^\beta}{1 - \frac{2M}{r} \left\{ \frac{4 \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right)}{\left[ 1 + \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right) \right]^2} \right\}^\beta} \] |
| Weinberg          | \[ E_W = \frac{M \left\{ \frac{4 \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right)}{\left[ 1 + \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right) \right]^2} \right\}^\beta}{1 - \frac{2M}{r} \left\{ \frac{4 \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right)}{\left[ 1 + \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right) \right]^2} \right\}^\beta} \] |
| Møller            | \[ E_M = M \left[ \frac{4 \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right)}{\left[ 1 + \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right) \right]^2} \right]^\beta \times \left\{ 1 - \sqrt{\frac{q^2 \beta}{2Mr}} \left[ \frac{1 - \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right)}{\left[ 1 + \exp \left( -\sqrt{\frac{2q^2}{\beta Mr}} \right) \right]} \right] \right\} \] |

Table 1: Energy distributions calculated by use of the energy-momentum complexes of Einstein, Landau-Lifshitz, Weinberg, and Møller

the Landau-Lifshitz and Weinberg prescriptions the energy equals the ADM mass \( M \) at infinity \( r \to \infty \), while in the chargeless case \( q = 0 \) these two energy-momentum complexes give for energy the expression \( M/(1 - \frac{2M}{r}) \). These results coincide with those
Table 2: Limiting cases for the energy

| Prescription   | Energy for $r \to \infty$ | Energy for $q = 0$ | Energy for $\beta \to 0$ | Energy for $\beta \to \infty$ |
|---------------|--------------------------|------------------|--------------------------|--------------------------|
| Einstein      | $M$                      | $M$              | $M$                      | $M \exp \left( -\frac{q^2}{2Mr} \right)$ |
| Landau-Lifshitz | $M$                      | $\frac{M}{1 - \frac{2M}{r}}$ | $\frac{M}{1 - \frac{2M}{r}}$ | $\frac{M \exp \left( -\frac{q^2}{2Mr} \right)}{1 - \frac{2M}{r} \exp \left( -\frac{q^2}{2Mr} \right)}$ |
| Weinberg      | $M$                      | $\frac{M}{1 - \frac{2M}{r}}$ | $\frac{M}{1 - \frac{2M}{r}}$ | $\frac{M \exp \left( -\frac{q^2}{2Mr} \right)}{1 - \frac{2M}{r} \exp \left( -\frac{q^2}{2Mr} \right)}$ |
| Møller        | $M$                      | $M$              | $M$                      | $[M - \frac{q^2}{2r}] \exp \left( -\frac{q^2}{2Mr} \right)$ |

obtained by Virbhadra’s approach in Schwarzschild Cartesian coordinates [67]. As it is pointed out at the end of Section 2, for $\beta \to 0$ we get the classical Schwarzschild black hole metric. Hence, the corresponding energies given in Table 2 for $\beta \to 0$ are those obtained in the four prescriptions for the Schwarzschild black hole. It is noteworthy that for $\beta \to \infty$ a factor of 2 appears in the exponential in all four prescriptions. In fact, the same dependence on this factor of 2 is obtained for the Einstein and Møller energies in [65], where a different metric function $f(r)$ is considered.

Of particular interest is the behavior of the energy near the origin, namely for $r \to 0$. In the case of the Einstein prescription, the energy tends to zero (see Figure 3). However, the energy obtained by the application of the Landau-Lifshitz, the Weinberg and the Møller energy-momentum complexes shows a rather pathological behavior. In the first two cases the energy is jumping between infinitely positive and negative values at some radial distances in the range $0 < r < 5$ for different values of the parameter $\beta$, while for significantly larger values of $r$ ($r > 50$) the energy falls off rapidly to a constant value for various values of the parameter $\beta$. The energy calculated by the Møller prescription becomes clearly negative in the range $0 < r < 0.2$ for different values of the parameter $\beta$, while the position where the energy retains a positive value and then keeps increasing monotonically is shifted nearer to the origin as $\beta$ becomes smaller. This strange behavior of the energy distribution may enhance the argumentation (see, e.g. [67]) according to which the Einstein energy-momentum complex is indeed a more reliable tool for the study of the gravitational energy-momentum localization as it yields physically meaningful results. Thus, it may gain a preference among the different pseudotensorial
Figure 3: Energy distribution near zero obtained in the Einstein prescription for different values of $\beta$.

Figure 4: Energy distribution near zero obtained in the Møller prescription for different values of $\beta$. 
energy-momentum complexes.

Acknowledgements

The authors thank the anonymous referees for their valuable comments and suggestions. Farook Rahaman is grateful to the Inter-University Centre for Astronomy and Astrophysics (IUCAA), India, for providing Associateship Programme. Farook Rahaman and Sayeeful Islam are thankful to DST, Government of India for providing financial support under the SERB and INSPIRE programme.

Statement on conflict of interests

The authors declare that there is no conflict of interest regarding the publication of this paper. Also, there is no conflict of interest regarding the received funding mentioned in Acknowledgement section. Also, there are no other possible conflicts of interests in the manuscript.

References

[1] L. Bel, “Définition d’une densité d’énergie et d’un état de radiation totale généralisée”, Comptes Rendus de l’Académie des Sciences, vol. 246, pp. 3015-3018, 1958.

[2] I. Robinson, “On the Bel-Robinson tensor”, Classical and Quantum Gravity, vol. 14, no. 1, pp. A331-A333, 1997.

[3] M. A. G. Bonilla and J. M. M. Senovilla, “Some properties of the Bel and Bel-Robinson tensors”, General Relativity and Gravitation, vol. 29, no. 1, pp. 91-116, 1997.

[4] J. M. Senovilla, “Super-energy tensors”, Classical and Quantum Gravity, vol. 17, no. 14, pp. 2799-2841, 2000.

[5] J. D. Brown and J. W. York, “Quasilocal energy and conserved charges derived from the gravitational action”, Physical Review D, vol. 47, no. 4, p. 1407, 1993.

[6] S. A. Hayward, “Quasilocal gravitational energy”, Physical Review. D., vol. 49, no. 2, pp. 831-839, 1994.

[7] C.-M. Chen and J.M. Nester, “Quasilocal quantities for general relativity and other gravity theories”, Classical and Quantum Gravity, vol. 16, no. 4, pp. 1279-1304, 1999.
[8] C.-C. M. Liu and S.-T. Yau, “Positivity of quasilocal mass”, *Physical Review Letters*, vol. 90, no. 23, Article ID 231102, 4 pages, 2003.

[9] L. Balart, “Quasilocal energy, Komar charge and horizon for regular black holes”, *Physics Letters B*, vol. 687, no. 4-5, pp. 280-285, 2010.

[10] A. Einstein, “On the general theory of relativity”, *A. Einstein, Sitzungsber. Preuss. Akad. Wiss.*, vol. 47, pp. 778-786, 1915, Addendum: *Preuss. Akad. Wiss.*, vol. 47, p. 799, 1915.

[11] A. Trautman, “Conservation laws in general relativity”, in *Gravitation: an Introduction to Current Research*, L. Witten, Ed., p. 169, John Wiley & Sons, New York, NY, USA, 1962.

[12] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Pergamon Press, New York, NY, USA, 1987.

[13] A. Papapetrou, “Einstein’s theory of gravitation and flat space”, *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences*, vol. 52, pp. 11-23, 1948.

[14] P.G. Bergmann and R. Thomson, “Spin and angular momentum in general relativity”, *Physical Review Letters*, vol. 89, pp. 400-407, 1953.

[15] C. Møller, “On the localization of the energy of a physical system in the general theory of relativity”, *Annals of Physics*, vol. 4, no. 4, pp. 347-371, 1958.

[16] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of General Theory of Relativity*, John Wiley & Sons, New York, NY, USA, 1972.

[17] A. Qadir and M. Sharif, “General formula for the momentum imparted to test particles in arbitrary spacetimes”, *Physics Letters A*, vol. 167, no. 4, pp. 331-334, 1992.

[18] R. Penrose, “Quasi-local mass and angular momentum in general relativity”, *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 381, no. 1780, pp. 53-63, 1982.

[19] K. P. Tod, “Some examples of Penrose’s quasilocal mass construction”, *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 388, no. 1795, pp. 457-477, 1983.

[20] K. S. Virbhadra, “Energy associated with a Kerr-Newman black hole”, *Physical Review D*, vol. 41, no. 4, pp. 1086-1090, 1990.

[21] K. S. Virbhadra, “Energy distribution in Kerr-Newman spacetime in Einstein’s as well as Møller’s prescriptions”, *Physical Review D*, vol. 42, no. 8, pp. 2919-2921, 1990.
[22] N. Rosen and K. S. Virbhadra, “Energy and momentum of cylindrical gravitational waves”, General Relativity and Gravitation, vol. 25, no. 4, pp. 429-433, 1993.

[23] J. M. Aguirregabiria, A. Chamorro, and K. S. Virbhadra, “Energy and angular momentum of charged rotating black holes”, General Relativity and Gravitation, vol. 28, no. 11, pp. 1393-1400, 1996.

[24] S. S. Xulu, “Møller energy for the Kerr-Newman metric”, Modern Physics Letters A, vol. 15, no. 24, pp. 1511-1517, 2000.

[25] S. S. Xulu, “Total energy of the Bianchi type I universes”, International Journal of Theoretical Physics, vol. 39, no. 4, pp. 1153-1161, 2000.

[26] S. S. Xulu, “Bergmann-Thomson energy-momentum complex for solutions more general than the Kerr-Schild class”, International Journal of Theoretical Physics, vol. 46, no. 11, pp. 2915-2922, 2007.

[27] P. K. Sahoo, K. L. Mahanta, D. Goit, A.K. Sinha, S.S. Xulu, U.R. Das, A. Prasad and R. Prasad, “Einstein energy-momentum complex for a phantom black hole metric”, Chinese Physics Letters, vol. 32, no. 2, Article ID 020402, 2015.

[28] S. K. Tripathy, B. Mishra, G. K. Pandey, A. K. Singh, T. Kumar, and S. S. Xulu, “Energy and momentum of Bianchi type VIh universes”, Advances in High Energy Physics, vol. 2015, Article ID 705262, 8 pages, 2015.

[29] I.-C. Yang and I. Radinschi, “On the difference of energy between the Einstein and Møller prescription”, Chinese Journal of Physics, vol. 42, no. 1, p. 40, 2004.

[30] I. Radinschi and T. Grammenos, “Møller’s energy-momentum complex for a space-time geometry on a noncommutative curved D3-brane”, International Journal of Theoretical Physics, vol. 47, no. 5, pp. 1363-1372, 2008.

[31] I.-C. Yang, C.-L. Lin, and I. Radinschi, “The energy of a regular black hole in general relativity coupled to nonlinear electrodynamics”, International Journal of Theoretical Physics, vol. 48, no. 1, pp. 248-255, 2009.

[32] I. Radinschi, F. Rahaman, and A. Ghosh, “On the energy of charged black holes in generalized dilaton-axion gravity”, International Journal of Theoretical Physics, vol. 49, no. 5, pp. 943-956, 2010.

[33] I. Radinschi, F. Rahaman, and A. Banerjee, “On the energy of Hořava-Lifshitz black holes”, International Journal of Theoretical Physics, vol. 50, no. 9, pp. 2906-2916, 2011.

[34] I. Radinschi, F. Rahaman, and A. Banerjee, “The energy distribution of Hořava-Lifshitz black hole solutions”, International Journal of Theoretical Physics, vol. 51, no. 5, pp. 1425-1434, 2012.
[35] M. Abdel-Megied and R. M. Gad, “Møller’s energy in the Kantowski-Sachs space-time”, Advances in High Energy Physics, vol. 2010, Article ID 379473, 6 pages, 2010.

[36] T. Bringley, “Energy and momentum of a stationary beam of light”, Modern Physics Letters A, vol. 17, no. 3, pp. 157-161, 2002.

[37] I. Radinschi, F. Rahaman, and U. F. Mondal, “Energy distribution for non-commutative radiating Schwarzschild black holes”, International Journal of Theoretical Physics, vol. 52, no. 1, pp. 96-104, 2013.

[38] M. Saleh, B.B. Thomas and T.C. Kofane, “Quasi-Local Energy Distribution and Thermodynamics of Reissner-Nordström Black Hole Surrounded by Quintessence”, Communications in Theoretical Physics, vol. 55 no. 2, pp. 291-295, 2011.

[39] M. Saleh, B.B. Thomas, and T.C. Kofane, “Energy distribution and thermodynamics of the quantum-corrected Schwarzschild black hole”, arXiv:1701.06929.

[40] M. Sharif and M. Azam, “Energy-momentum distribution: some examples”, International Journal of Modern Physics A, vol. 22, no. 10, pp. 1935-1951, 2007.

[41] P. Halpern, “Energy of the Taub cosmological solution”, Astrophysics and Space Science, vol. 306, no. 4, pp. 279-283, 2006.

[42] E. C. Vagenas, “Energy distribution in 2d stringy black hole backgrounds”, International Journal of Modern Physics A, vol. 18, no. 31, pp. 5781-5794, 2003.

[43] E. C. Vagenas, “Effective mass of a radiating charged particle in Einstein’s universe”, Modern Physics Letters A, vol. 19, no. 3, pp. 213-222, 2004.

[44] E. C. Vagenas, “Energy distribution in a BTZ black hole spacetime”, International Journal of Modern Physics D, vol. 14, no. 03n04, pp. 573-585, 2005.

[45] E. C. Vagenas, “Energy distribution in the dyadosphere of a Reissner-Nordström black hole in Møller’s prescription”, Modern Physics Letters A, vol. 21, no. 25, pp. 1947-1956, 2006.

[46] T. Multamaki, A. Putaja, E. C. Vagenas, and I. Vilja, “Energy-momentum complexes in f(R) theories of gravity”, Classical and Quantum Gravity, vol. 25, no. 7, Article ID 075017, 2008.

[47] L. Balart, “Energy distribution of (2+1)-dimensional black holes with nonlinear electrodynamics”, Modern Physics Letters A, vol. 24, no. 34, pp. 2777-2785, 2009.

[48] A. M. Abbassi, S. Mirshekari, and A. H. Abbassi, “Energy-momentum distribution in static and nonstatic cosmic string space-times”, Physical Review D, vol. 78, no. 6, Article ID 064053, 2008.
[49] J. Matyjasek, “Some remarks on the Einstein and Møller pseudotensors for static and spherically-symmetric configurations”, Modern Physics Letters A, vol. 23, no. 8, pp. 591-601, 2008.

[50] I-Ching Yang, “Some characters of the energy distribution for a charged wormhole”, Chinese Journal of Physics, vol. 53, no. 6, 110108, pp. 1-4, 2015.

[51] R. M. Gad, “On teleparallel version of stationary axisymmetric solutions and their energy contents”, Astrophysics and Space Science, vol. 346, no. 2, pp. 553-557, 2013.

[52] R. Aldrovandi and J. G. Pereira, Teleparallel Gravity: An Introduction, Springer, 2013.

[53] G. G. L. Nashed, “Kerr-Newman solution and energy in teleparallel equivalent of Einstein theory”, Modern Physics Letters A, vol. 22, no. 14, pp. 1047-1056, 2007.

[54] G. G. L. Nashed, “Energy and angular momentum of general four-dimensional stationary axi-symmetric space-time in teleparallel geometry”, International Journal of Modern Physics A, vol. 23, no. 12, pp. 1903-1918, 2008.

[55] G. G. L. Nashed, “Braneworld black holes in teleparallel theory equivalent to general relativity and their Killing vectors, energy, momentum and angular momentum”, Chinese Physics B, vol. 19, no. 2, Article ID020401, 2010.

[56] G. G. L. Nashed, “Energy of spherically symmetric space-times on regularizing teleparallelism”, International Journal of Modern Physics A, vol. 25, no. 14, pp. 2883-2895, 2010.

[57] M. Sharif and A. Jawad, “Energy contents of some well-known solutions in teleparallel gravity”, Astrophysics and Space Science, vol. 331, no. 1, pp. 257-263, 2011.

[58] C.-C. Chang, J. M. Nester, and C.-M. Chen, “Pseudotensors and quasi-local energy-momentum”, Physical Review Letters, vol. 83, no. 10, pp. 1897-1901, 1999.

[59] C. M. Chen and J. M. Nester, “A Symplectic Hamiltonian derivation of quasilocal energy-momentum for general relativity”, Gravitation and Cosmology, vol. 6, pp. 257-270, 2000.

[60] L. L. So, J. M. Nester, and H. Chen, “Classical pseudotensors and positivity in small regions”, in Gravitation and Astrophysics. Proceedings, 7th Asia-Pacific International Conference, J. M. Nester, C.-M. Chen, and J.-P. Hsu, Eds., p. 356, World Scientific, 2007.

[61] J. M. Nester, C. M. Chen, J.-L. Liu, and G. Sun, “A reference for the covariant Hamiltonian boundary term”, in Relativity and Gravitation-100 years after Einstein in Prague, J. Bičák and T. Ledvinka (eds.), pp. 177-184, Springer, 2014.
[62] S. Ansoldi, “Spherical black holes with regular center”, *Proceedings of “BH2, Dynamics and Thermodynamics of Black holes and Naked Singularities”*, May 10-12 2007, Milano, Italy (arXiv:0802.0330).

[63] I. Dymnikova and S. Galaktionov, “Regular rotating electrically charged black holes and solitons in nonlinear electrodynamics minimally coupled to gravity”, *Classical and Quantum Gravity* vol. 32, p. 165015, 2015.

[64] L. Balart and E. C. Vagenas, “Regular black holes with a nonlinear electrodynamics source”, *Physical Review D*, vol. 90, no. 12, Article ID 124045, 2014.

[65] I. Radinschi, F Rahaman, Th. Grammenos and S. Islam, “Einstein and Müller energy-momentum complexes for a new regular black hole solution with a nonlinear electrodynamics source”, *Advances in High Energy Physics*, vol. 2016, Article ID 9049308, 2016.

[66] N. Balakrishnan (ed.), *Handbook of the Logistic Distribution*, NY: Dekker, 1992.

[67] K. S. Virbhadra, “Naked singularities and Seifert’s conjecture”, *Physical Review D*, vol. 60, no. 10, Article ID 104041, 1999.