2D FLOW OF CASSON FLUID WITH NON-UNIFORM HEAT SOURCE/SINK AND JOULE HEATING

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ABSTRACT

In this paper, two-dimensional magnetohydrodynamic (MHD) flow of Casson fluid over a fixed plate under non-uniform heat source/sink and Joule heating is analyzed by the homotopy analysis method (HAM). The governing boundary-layer equations have been reduced to the ordinary differential equations (ODEs) through the similarity variables. The current HAM-series solution is compared and successfully validated by the previous studies. Furthermore, the effects of thermo-physical parameters on the current solution are precisely examined. It is found that the skin friction coefficient and local Nusselt number are greatly affected by the Hartmann number. It can be concluded that employing the Casson fluid together with the suction effect can minimize the rate of heat and mass transfer.

Keywords: Nanoparticle, Casson fluid, Heat source/sink, Lorentz force, HAM.

1. INTRODUCTION

In general, fluid mechanics can be categorized into two main types: hydraulics and hydrodynamics which are developed through the experimental and theoretical analyses, respectively (Falkner and Skan, 1931). In recent decades, there have been many research studies concerning the hydrodynamics as well as heat and mass transfer theory. In this way, Khan and Azam (2017) investigated unsteady flow of Carreau fluid over a permeable stretching wall with the Lorentz force and suction/injection effect. They solved the governing boundary-layer equations through the bvp4c function in MATLAB and found that the skin friction coefficient increases with an increase in the Weissenberg number. They also showed that the nanoparticle concentration boundary-layer thickness is significantly affected by the Lewis number. Borrelli et al. (2017) presented a model dealt with the Oberbeck-Boussinesq approximation for three-dimensional (3D) stagnation-point flow of Newtonian fluids. They found that the skin friction coefficient increases with an increase in the Hartmann number. They also illustrated that the reversed flow without the effect of buoyancy force occurs at the minimum value of Hartmann number (i.e., 0.7583). They finally proved that if two of the findings are fully consistent with those of Ramachandran et al. (1988) and Ishak et al. (2008). Rahman et al. (2014) investigated forced-convection flow of fluids over an exponentially permeable shrinking/stretching wall based on the Buongiorno mathematical model in which the effect of thermophoresis and Brownian motion had been taken into account. They could develop those of Kuznetsov and Nield (2013) and found that the momentum boundary-layer thickness decreases with an increase in the second order slip parameter. Ranjit and Shit (2017) analytically examined the combined effects of Joule heating and zeta potential on the flow past a peristaltically induced microchannel which was supported by those of Tripathi (2013). They also employed the Debye-Hückel approximation technique and found that although the viscous dissipation increases with a decrease in the Brinkman number, the local Nusselt number is a decreasing function of this number. Besthapu et al. (2017) numerically studied the MHD mixed-convection flow of stratified nanofluids using the finite difference method (FDM). They showed that the nanoparticle concentration boundary-layer thickness increases with an increase in the resistive Lorentz force. Furthermore, they found that the effect of thermal stratification parameter can be neglected at the surface. Sheikholeslami and Ganji (2017) applied the Koo-Kleinstreuer-Li (KKL) model to investigate the MHD flow of CuO-H₂O nanofluid over a permeable annulus. They formulated the averaged Nusselt number in terms of inclination angle, Hartmann and Rayleigh numbers, and showed a consistency with those of Khanafer et al. (2003). It is to be noted that mode details are set out in Hedayati and Domairry, 2015; Khoshrouye Ghiasi and Saleh, 2018a, 2018b, 2017.

As discussed above, analytical and numerical models play a leading role in solving boundary-layer differential equations. This paper focuses on how the HAM (Liao, 1992, 2003) may be implemented to give a solution for MHD flow of Casson fluid combined with the non-uniform heat source/sink, inclined Lorentz force and Joule heating based on the Buongiorno mathematical model (Buongiorno, 2006). It is found that the current findings are in agreement with those of previous studies. In addition, some tables and graphs are provided to signify the effects of thermo-physical parameters on the current solution. To the best of the author's knowledge, there have been no reports of this problem being solved to date.

2. GOVERNING EQUATIONS

In rheology, one of the well-known non-Newtonian models is the Casson fluid which is defined by the following constitutive equation for an isotropic incompressible flow (Casson, 1959):

\[
\tau_{ij} = \begin{cases} 
2 \left( \mu_d + \frac{p_c}{\sqrt{2} \pi} \right) \varepsilon_{ij}, & \pi > \pi_c, \\
2 \left( \mu_d + \frac{p_d}{\sqrt{2} \pi} \right) \varepsilon_{ij}, & \pi < \pi_c
\end{cases}
\]
where $\tau_{ij}$ is the shear stress tensor, $\mu_B$ is the plastic dynamic viscosity of the fluid, $p_y$ is the yield stress, $e_{ij}$ is the $(i,j)$th component(s) of the strain rate, $\pi = (e_{ij} e_{ij})$ is the product of strain rate component(s) and $\pi_c$ is the critical value of $\pi$.

For the 2D flow in the Cartesian coordinate system, the velocity, temperature and nanoparticle concentration fields can be expressed as follows:

$$\mathbf{V} = [u(x,y), v(x,y)], T = T(x,y), C = C(x,y),$$

where $u$ and $v$ are the velocity components along $x$ and $y$ directions, respectively, $T$ is the temperature and $C$ is the nanoparticle concentration.

Utilizing the aforementioned assumptions, the governing continuity, momentum, energy and nanoparticle concentration equations yield:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u \left(1 + \frac{\nu}{\sigma} \right) \frac{\partial^2 u}{\partial y^2} - \frac{\nu}{\sigma} \sin^2 \omega u,$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \alpha \frac{\partial^2 v}{\partial y^2} + \gamma \left[ \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \right]^{\frac{1}{2}}  \left[ q_n + \alpha B_n u^2 \right],$$

$$\frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{\partial \theta}{\partial y} \frac{\partial C}{\partial x},$$

where $v$ is the kinematic viscosity, $\beta$ is the Casson fluid parameter, $\sigma$ is the electrical conductivity, $B_0$ is the magnetic field strength, $\rho$ is the density, $\omega$ is the inclination angle of magnetic field, $\alpha$ is the thermal diffusivity, $\gamma(=\frac{\rho C_p}{c_p})$ is the ratio of effective heat capacity of the nanoparticle to effective heat capacity of the base fluid, $D_B$ is the Brownian diffusion coefficient, $\nu$ is the thermophoresis diffusion coefficient, $T_{\infty}$ is the ambient temperature, $C_p$ is the specific heat at constant pressure and $q_n$ is the non-uniform heat source/sink. The associated boundary conditions are given by:

$$\begin{align*}
    &u|_{y=0} = 0, v|_{y=0} = v_w(x), T|_{y=0} = T_w, C|_{y=0} = C_w, \\
    &\lim_{y \to \infty} u = 0, \lim_{y \to \infty} T = T_{\infty}, \lim_{y \to \infty} C = C_{\infty},
\end{align*}$$

where $v_w(x)$ is the rate of mass transfer, $T_w$ is the wall temperature, $C_w$ is the nanoparticle concentration at the wall, $U_{\infty}$ is the free stream velocity and $C_{\infty}$ is the ambient nanoparticle concentration.

In order to derive the similarity solution of equation (3), the following variables can be utilized:

$$\phi = \sqrt{U_{\infty} u} \psi, \eta = y \sqrt{U_{\infty} \psi} \theta = \frac{T - T_{\infty}}{T_w - T_{\infty}}, \psi = \frac{C - C_{\infty}}{C_w - C_{\infty}}$$

where $\phi$ is the stream function which is governed by the continuity equation (i.e., $\frac{\partial \phi}{\partial y} = 0$), $f$ is the similarity function, $\eta$ is the similarity parameter, $\theta$ is the non-dimensional temperature and $\phi$ is the non-dimensional nanoparticle concentration.

The non-uniform heat source/sink involved in equation (3) is given by (Abu-Eldahab and Aziz, 2004; Nandeppanavar et al., 2010; Subbas et al., 2007):

$$q_n = \frac{kw}{x} \left[ A(T_w - T_{\infty}) \frac{\partial \theta}{\partial \eta} + B(T - T_{\infty}) \right],$$

where $A$ and $B$ are the coefficients of space and temperature-dependent heat source/sink, respectively.

By substituting equation (5) into equation (3), the following system of equations can be stated as:
\( L_f[ C_1 e^{\eta} + C_2 e^{-\eta} ] = 0, \quad L_\theta[ C_1 e^{\eta} + C_2 e^{-\eta} ] = 0, \quad L_\psi[ C_1 e^{\eta} + C_2 e^{-\eta} ] = 0, \quad (13) \)

where \( C_1, C_2, \ldots, C_r \) are the arbitrary constants.

Using Liao's theorem (1992), the following zeroth-order deformation equations can be constructed:

\[
(1 - p)L_f[f(\eta, p) - f_0(\eta)] = ph_Nf[f(\eta, p)], \\
(1 - p)L_\theta[\theta(\eta, p) - \theta_0(\eta)] = ph_N\theta[f(\eta, p), \theta(\eta, p), \psi(\eta, p)], \\
(1 - p)L_\psi[\psi(\eta, p) - \psi_0(\eta)] = ph_N\psi[f(\eta, p), \theta(\eta, p), \psi(\eta, p)],
\]

in which,

\[
N_f[f(\eta, p), \theta(\eta, p), \psi(\eta, p)] = \left( 1 + \frac{1}{p} \right) \frac{\partial^2 f(\eta, p)}{\partial \eta^2} + f(\eta, p) \frac{\partial^3 f(\eta, p)}{\partial \eta^3} - \frac{\partial f(\eta)}{\partial \eta} \left( \frac{\partial f(\eta)}{\partial \eta} + Ha^2 \sin^2 \omega \right), \\
N_\theta[f(\eta, p), \theta(\eta, p), \psi(\eta, p)] = Pr^{-1} \frac{\partial^2 \theta(\eta, p)}{\partial \eta^2} + \frac{1}{2} f(\eta, p), \\
N_\psi[f(\eta, p), \theta(\eta, p), \psi(\eta, p)] = \frac{\partial^2 \psi(\eta, p)}{\partial \eta^2} + \frac{1}{2} L\psi(\eta, p) \frac{\partial^2 \psi(\eta, p)}{\partial \eta^2} + \frac{\partial \psi(\eta, p)}{\partial \eta} \left( \frac{\partial \psi(\eta, p)}{\partial \eta} + \frac{\partial \psi(\eta, p)}{\partial \eta} + \frac{\partial \psi(\eta, p)}{\partial \eta} + \frac{\partial \psi(\eta, p)}{\partial \eta} \right), \quad (15)
\]

with the boundary conditions:

\[
\begin{align*}
\text{at } \eta = 0: & \quad f(\eta, p) = S \frac{\partial f(\eta, p)}{\partial \eta} = 0, \theta(\eta, p) = 1, \\
& \psi(\eta, p) = 1, \\
\text{lim } & \frac{\partial \psi(\eta, p)}{\partial \eta} \to 1, \text{ lim } \theta(\eta, p) \to 0, \text{ lim } \psi(\eta, p) \to 0,
\end{align*}
\]

where \( 0 \leq p \leq 1 \) is an embedding parameter, \( h_f, h_\theta \) and \( h_\psi \) are the auxiliary parameters, and \( N_f, N_\theta \) and \( N_\psi \) are the nonlinear operators. \( \Delta f, \Delta \theta \) and \( \Delta \psi \) are auxiliary linear operators and auxiliary parameters are properly chosen, one would get at \( p = 1 \):

\[
\begin{align*}
\Delta f(\eta, p) &= \sum_{n=0}^{\infty} f(\eta, n), \\
\Delta \theta(\eta, p) &= \sum_{n=0}^{\infty} \theta(\eta, n), \\
\Delta \psi(\eta, p) &= \sum_{n=0}^{\infty} \psi(\eta, n),
\end{align*}
\]

If the initial guesses, auxiliary linear operators and auxiliary parameters are properly chosen, one would get at \( p = 1 \):

\[
\begin{align*}
\Delta f(\eta, p) &= \sum_{n=0}^{\infty} f(\eta, n), \\
\Delta \theta(\eta, p) &= \sum_{n=0}^{\infty} \theta(\eta, n), \\
\Delta \psi(\eta, p) &= \sum_{n=0}^{\infty} \psi(\eta, n),
\end{align*}
\]

where \( i = 20 \) and \( \delta \eta = 0.5 \).

Differentiating equation (14) \( n \) times with respect to \( p \), setting \( p = 0 \) and dividing them by \( n! \) gives the following \( n \)-th order deformation equations:

\[
\begin{align*}
\Delta f(\eta, n) &= \left( 1 + \frac{1}{p} \right) \frac{\partial^n f_{n-1}(\eta)}{\partial \eta^n} + \frac{\partial^2 f(\eta, n)}{\partial \eta^2} - \frac{\partial f(\eta)}{\partial \eta} \left( \frac{\partial f(\eta)}{\partial \eta} + Ha^2 \sin^2 \omega \right), \\
\Delta \theta(\eta, n) &= Pr^{-1} \frac{\partial^n \theta_{n-1}(\eta)}{\partial \eta^n} + \frac{1}{2} f(\eta, n), \\
\Delta \psi(\eta, n) &= \frac{\partial^n \psi(\eta, n)}{\partial \eta^n} + \frac{1}{2} L\psi(\eta, n) \frac{\partial^n \psi(\eta, n)}{\partial \eta^n} + \frac{\partial \psi(\eta, n)}{\partial \eta} \left( \frac{\partial \psi(\eta, n)}{\partial \eta} + \frac{\partial \psi(\eta, n)}{\partial \eta} + \frac{\partial \psi(\eta, n)}{\partial \eta} + \frac{\partial \psi(\eta, n)}{\partial \eta} \right), \quad (23)
\end{align*}
\]

with the boundary conditions:

\[
\begin{align*}
\text{at } \eta = 0: & \quad f(\eta, n) = 0, \frac{\partial f(\eta, n)}{\partial \eta} = 0, \theta(\eta, n) = 0, \psi(\eta, n) = 0, \\
& \text{lim } \frac{\partial \psi(\eta, n)}{\partial \eta} \to 0, \text{ lim } \theta(\eta, n) \to 0, \text{ lim } \psi(\eta, n) \to 0.
\end{align*}
\]

The general solutions for equation (21) in terms of particular solutions (i.e., \( f^*, \theta^* \) and \( \psi^* \)) can be expressed as follows:

\[
\begin{align*}
\Delta f(\eta, n) &= f_n^*(\eta) + C_1 e^{\eta} + C_2 e^{-\eta}, \\
\Delta \theta(\eta, n) &= \theta_n^*(\eta) + C_3 e^{\eta} + C_4 e^{-\eta}, \\
\Delta \psi(\eta, n) &= \psi_n^*(\eta) + C_5 e^{\eta} + C_6 e^{-\eta},
\end{align*}
\]

in which,

\[
\begin{align*}
C_2 &= C_4 = C_6 = 0, C_1 = C_3 = -\left( C_3 + f_n^*(0) \right), \\
C_3 &= \frac{\partial f_n^*(0)}{\partial \eta}, C_5 = \theta_n^*(0), C_7 = \psi_n^*(0).
\end{align*}
\]

The square residual errors can be defined as (Liao, 2010):

\[
\begin{align*}
\Delta f_{n, n} &= \frac{1}{i+1} \sum_{i=0}^{i} \left\{ N_f \left[ \sum_{n=0}^{f_{n-1}(\eta)} f(\eta, n) \right] \right\}^2, \\
\Delta \theta_{n, n} &= \frac{1}{i+1} \sum_{i=0}^{i} \left\{ N_\theta \left[ \sum_{n=0}^{\theta_{n-1}(\eta)} \theta(\eta, n) \right] \right\}^2, \\
\Delta \psi_{n, n} &= \frac{1}{i+1} \sum_{i=0}^{i} \left\{ N_\psi \left[ \sum_{n=0}^{\psi_{n-1}(\eta)} \psi(\eta, n) \right] \right\}^2,
\end{align*}
\]

where \( i = 20 \) and \( \delta \eta = 0.5 \).
4. RESULTS AND DISCUSSION

This section deals with the previously outlined HAM-series solution for heat and mass transfer analysis in the MHD flow of Casson fluid subjected to inclined Lorentz force and Joule heating. In this way, the comparisons and parametric studies are made to investigate the validity and accuracy of the current solution. In this paper the pertinent parameters, unless stated otherwise, are listed as follows:

\[
\begin{align*}
\beta &= 0.4, Ha = 1, \omega = 45^\circ, S = 1, Pr = 0.7, \\
Nb &= Nt = 0.5, A = 0.05, B = -0.05, Ec = 0.1, Le = 1.3.
\end{align*}
\] (28)

Table 1 represents the variation of auxiliary parameters and its square residual errors at any order of approximation. From this table, it is seen that the allowable values of auxiliary parameters can be chosen by minimizing the square residual errors. Therefore, the current findings are provided using the optimized 20th-order of approximation (i.e., \( h_f = -0.8169, h_\theta = -0.8246 \) and \( h_\phi = -1.0962 \)).

Effect of the Hartmann number \( Ha \) on the skin friction coefficient \(-C_f Re_x^{\frac{1}{2}}\) is depicted in Fig. 1 for different values of inclination angle of magnetic field \( \omega \). As this figure shows, \(-C_f Re_x^{\frac{1}{2}}\) increases with an increase in \( \omega \) which is due to the presence of resistive Lorentz force. It should be noted that the Casson fluid is affected by the viscous force while the Lorentz force tends to decelerate flow of the fluid and retards its motion (Khoshrouye Ghiasi and Saleh, 2018c). Moreover, unlike \( \omega = 0^\circ \), \(-C_f Re_x^{\frac{1}{2}}\) increases with an increase in \( Ha \). Since \( \omega = 0^\circ \) the effect of Hartmann number on the momentum boundary-layer thickness is negligible.

Table 1 selection of auxiliary parameters

| n   | \( h_f \)     | \( \Delta f,n \) | \( h_\theta \) | \( \Delta \theta,n \) | \( h_\phi \) | \( \Delta \phi,n \) |
|-----|--------------|-----------------|--------------|-----------------|--------------|-----------------|
| 2   | -0.7614      | 6.14×10^{-8}   | -0.8006      | 5.29×10^{-5}   | -1.0070      | 4.54×10^{-7}   |
| 4   | -0.7698      | 3.03×10^{-8}   | -0.8025      | 4.68×10^{-5}   | -1.0195      | 2.19×10^{-7}   |
| 6   | -0.7740      | 9.26×10^{-9}   | -0.8049      | 3.90×10^{-5}   | -1.0296      | 9.80×10^{-8}   |
| 8   | -0.7788      | 5.99×10^{-9}   | -0.8082      | 3.51×10^{-5}   | -1.0399      | 7.22×10^{-8}   |
| 10  | -0.7939      | 1.04×10^{-9}   | -0.8107      | 3.08×10^{-5}   | -1.0498      | 4.93×10^{-8}   |
| 12  | -0.7983      | 8.06×10^{-10}  | -0.8131      | 2.79×10^{-5}   | -1.0585      | 2.30×10^{-8}   |
| 14  | -0.8032      | 5.12×10^{-10}  | -0.8159      | 2.32×10^{-5}   | -1.0694      | 9.79×10^{-9}   |
| 16  | -0.8070      | 2.97×10^{-10}  | -0.8190      | 2.01×10^{-5}   | -1.0790      | 6.95×10^{-9}   |
| 18  | -0.8111      | 8.25×10^{-11}  | -0.8214      | 1.71×10^{-5}   | -1.0883      | 3.93×10^{-9}   |
| 20  | -0.8169      | 4.89×10^{-11}  | -0.8246      | 1.46×10^{-5}   | -1.0962      | 1.09×10^{-9}   |

Fig. 2 Variation of \( \frac{\partial f}{\partial \eta} \) for \( \beta = 1, \beta = 2 \) and \( \beta = 3 \)

Table 2 Values of the skin friction coefficient for \( Ha \) and \( S \)

| \( Ha \) | \( S \) | \( \frac{\partial f}{\partial \eta} \) \( \eta=0 \) |
|---------|--------|-----------------|
| 1       | 1.1665 |
| 2       | 1.2796 |
| 3       | 1.3481 |
| 4       | 1.4195 |
| 5       | 0.6    | 0.9228          |
| 6       | 0.7    | 0.9396          |
| 7       | 0.8    | 0.9451          |
| 8       | 0.9    | 0.9517          |

Due to the effect of yield stress \( p_y \) on the Casson fluid given in equation (1), one can observe from Fig. 2 that the velocity distribution decreases with an increase in the Casson fluid parameter \( \beta \). This is because, an increase in \( \beta \) leads to a decrease in \( p_y \), which is replaced by the Newtonian fluid. This fact is also illustrated in Aziz (2016) and Khoshrouye Ghiasi and Saleh (2019). It is worth mentioning that some previous studies (Raju et al., 2017; Raju and Sandeep, 2017) suggest ascending behavior of the velocity distribution with an increase in \( \beta \) which is largely due to the domination of buoyancy force.

Table 2 tabulates the effect of Hartmann number \( Ha \) and mass suction parameter \( S \) on the values of \( \frac{\partial f}{\partial \eta} \) \( \eta=0 \). According to the results reported in this table, it is observed that \( \frac{\partial f}{\partial \eta} \) is an increasing function of \( Ha \) and \( S \) simultaneously. It should be noted that for large values of mass suction parameter, the axial velocity decreases.

Table 3 shows a comparison between the current solution and those of Khan and Pop (2010) and Wang (1989) to determine the values of \( -\frac{\partial \theta}{\partial \eta} \) \( \eta=0 \). The results in this table are provided by \( \beta \to \infty, Ha = 0.1, Ec = 0.1, Le = 10 \) and \( \omega = S = Nt = Nt = A = B = 0 \). It is seen from Table 3 that \( -\frac{\partial \theta}{\partial \eta} \) \( \eta=0 \) increases with an increase in \( Pr \). Moreover, the relative error between the current solution and those of Khan and Pop (2010) and Wang (1989) does not exceed 0.047% and 0.111%, respectively. Hence, the reliability of the current solution is verified.

As mentioned earlier, the irreversibility of the Joule heating process can be measured by the magnetic entropy generation (Bejan, 1982). This fact is illustrated in Fig. 3 for the variation of local Nusselt number \( Nu_x/Re_x^{\frac{1}{2}} \) versus \( Ha \). From this figure, it can be seen that \( Nu_x/Re_x^{\frac{1}{2}} \) decreases with an increase in the Eckert number \( Ec \) which is due to the stored energy in the fluid. This observation can also be considered as an
optimization criterion for minimizing the entropy generation (Bejan, 1995).

To investigate the effect of non-uniform heat source/sink on the temperature distribution, the variation of \( \theta(\eta) \) for different values of \( A \) and \( B \) is depicted in Fig. 4. As this figure shows, \( \theta(\eta) \) increases with an increase in \( A \) or \( B \) which is due to an increase in the thermal boundary-layer thickness. Furthermore, Fig. 4 emphasizes that the surface temperature \( \theta(0) \) increases with an increase in \( A \).

**Table 3** Values of \(-\left( \frac{\partial \theta}{\partial \eta} \right)_{\eta=0}\) compared with those of Khan and Pop (2010) and Wang (1989)

| \( Pr \) | Khan and Pop (2010) | Wang (1989) | Current solution |
|------|-----------------|-------------|------------------|
| 0.07 | 0.0663          | 0.0656      | 0.0661           |
| 0.2  | 0.1691          | 0.1691      | 0.1691           |
| 0.7  | 0.4539          | 0.4539      | 0.4538           |
| 2    | 0.9113          | 0.9114      | 0.9114           |
| 7    | 1.8954          | 1.8954      | 1.8954           |
| 20   | 3.3539          | 3.3539      | 3.3539           |
| 70   | 6.4621          | 6.4622      | 6.4621           |

![Fig. 3](image)

**Fig. 3** Variation of \( \frac{Nu_x}{Re_y^2} \) versus \( Ha \) for \( Ec = 0.1, Ec = 0.2, Ec = 0.3 \) and \( Ec = 0.4 \)

![Fig. 4](image)

**Fig. 4** Variation of \( \theta(\eta) \) for \( A = -1, A = 0 \) and \( A = 1, \) and, \( B = -0.5, B = 0 \) and \( B = 0.5 \)

Table 4 shows the effect of Brownian motion parameter \( Nb \) and thermophoresis parameter \( Nt \) on the variation of \(-\left( \frac{\partial \theta}{\partial \eta} \right)_{\eta=0}\). It can be observed from this table that \(-\left( \frac{\partial \theta}{\partial \eta} \right)_{\eta=0}\) increases with an increase in \( Nb \), which is due to an interaction between the Brownian motion and thermal conductivity. Also, Table 4 shows that \(-\left( \frac{\partial \theta}{\partial \eta} \right)_{\eta=0}\) clearly decreases with an increase in \( Nt \). Therefore, it is essential to account for the effect Brownian diffusion and thermophoresis in the Buongiorno mathematical model.

A comparison between the current solution and those of Afify and Elgazery (2016) is reported in Table 5 to determine the effect of \( Nt \) and \( Nb \) on the variation of \(-\left( \frac{\partial \theta}{\partial \eta} \right)_{\eta=0}\). It is noted that the pertinent parameters utilized by Afify and Elgazery (2016) (i.e., \( \beta \to \infty, Pr = Le = 10 \) and \( Ha = \omega = S = A = B = Ec = 0 \)) agree with those presented in this table. It is found that the relative error between the current solution and those of Afify and Elgazery (2016) equals to 0.331% in all cases.

**Fig. 5** Variation of nanoparticle concentration \( \psi(\eta) \) for different values of Lewis number \( Le \). From this figure, it is seen that \( \psi(\eta) \) decreases with an increase in \( Le \) which is due to the effect of \( D_B \).

**Table 4** Values of the local Nusselt number for \( Nb \) and \( Nt \)

| \( Nb \) | \( Nt \) | \(-\left( \frac{\partial \theta}{\partial \eta} \right)_{\eta=0}\) |
|------|------|------------------|
| 0.1  | 0.5  | 0.5059           |
| 0.2  | 0.5  | 0.5625           |
| 0.3  | 0.5  | 0.6148           |
| 0.4  | 0.5  | 0.6809           |
| 0.1  | 0.6  | 0.4730           |
| 0.7  | 0.7  | 0.4415           |
| 0.8  | 0.8  | 0.4120           |
| 0.9  | 0.9  | 0.4796           |

**Table 5** Values of \(-\left( \frac{\partial \psi}{\partial \eta} \right)_{\eta=0}\) compared with those of Afify and Elgazery (2016)

| \( Nt \) | \( Nb \) | Afify and Elgazery (2016) | Current solution |
|------|------|--------------------------|------------------|
| 0.1  | 0.1  | 2.2774                   | 2.2637           |
| 0.2  | 0.1  | 2.2490                   | 2.2388           |
| 0.3  | 0.1  | 2.2229                   | 2.2181           |
| 0.4  | 0.2  | 2.1992                   | 2.1907           |
| 0.1  | 0.2  | 2.3110                   | 2.3089           |
| 0.3  | 0.3  | 2.3299                   | 2.3216           |
| 0.4  | 0.4  | 2.3458                   | 2.3389           |
| 0.5  | 0.5  | 2.3560                   | 2.3501           |

![Fig. 5](image)

**Fig. 5** Variation of \( \psi(\eta) \) for \( Le = 0, Le = 1 \) and \( Le = 2 \)
5. CONCLUSIONS

This paper aimed to study the effect of non-uniform heat sink/source, Joule heating and inclined Lorentz force on flow of Casson fluid based on the Buongiorno mathematical model. The governing boundary-layer equations correspond to the continuity, momentum, energy and nanoparticle concentration equations are derived, and solved through the HAM. It was found that the rate of heat transfer is affected by the Brownian diffusion and thermophoresis. Moreover, the HAM findings were compared and validated by those previous studies available in the literature. One can also conclude that this paper would be worthwhile to further explore this problem with different boundary conditions, solution methodologies and geometries.

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