Instantons: topological aspects

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Abstract

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Cross-references: Moduli spaces (62); Instantons in gauge theory (72); Mathematical uses of gauge theory (75); Index theorems (149); Donaldson invariants (255); Characteristic classes (354); Differential geometry (386); Finite dimensional algebras and quivers (418).

1 Introduction

Let $X$ be a closed (connected, compact without boundary) smooth manifold of dimension 4, provided with a Riemannian metric denoted by $g$. Let $\Omega^p_X$ denote space of smooth $p$-forms on $X$, i.e. the sections of $\wedge^p T X$. The Hodge operator acting on $p$-forms:

\[ * : \Omega^p_X \to \Omega^{4-p}_X \]

satisfies $*^2 = (-1)^p$. In particular, $*$ splits $\Omega^2_X$ into two sub-spaces $\Omega^2_{X,\pm}$ with eigenvalues $\pm 1$:

\[ \Omega^2_X = \Omega^2_{X,+} \oplus \Omega^2_{X,-} . \tag{1} \]

Note also that this decomposition is an orthogonal one, with respect to the inner product:

\[ \langle \omega_1, \omega_2 \rangle = \int_X \omega_1 \wedge * \omega_2 . \]

A 2-form $\omega$ is said to be self-dual if $* \omega = \omega$ and it is said to be anti-self-dual if $* \omega = -\omega$. Any 2-form $\omega$ can be written as the sum

\[ \omega = \omega^+ + \omega^- \]
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of its self-dual $\omega^+$ and anti-self-dual $\omega^-$ components.

Now let $E$ be a complex vector bundle over $X$ as above, provided with a connection $\nabla$, regarded as a $\mathbb{C}$-linear operator

$$\nabla : \Gamma(E) \to \Gamma(E) \otimes \Omega^1_X$$

satisfying the Leibnitz rule:

$$\nabla(f\sigma) = f\nabla\sigma + \sigma \otimes df$$

for all $f \in C^\infty(X)$ and $\sigma \in \Gamma(E)$. Its curvature $F_\nabla = \nabla \circ \nabla$ is a 2-form with values in $\text{End}(E)$, i.e. $F_\nabla \in \Gamma(\text{End}(E)) \otimes \Omega^2_X$, satisfying the Bianchi identity $\nabla F_\nabla = 0$.

The Yang-Mills equation is:

$$\nabla \ast F_\nabla = 0 \quad (2)$$

It is a 2nd-order non-linear equation on the connection $\nabla$. It amounts to a non-abelian generalization of Maxwell equations, to which it reduces when $E$ is a line bundle; the four components of $\nabla$ are interpreted as the electric and magnetic potentials.

An instanton on $E$ is a smooth connection $\nabla$ whose curvature $F_\nabla$ is anti-self-dual as a 2-form, i.e. it satisfies:

$$F_\nabla = 0 \quad \ast F_\nabla = -F_\nabla \quad (3)$$

The instanton equation is still non-linear (it is linear only if $E$ is a line bundle), but it is only 1st-order on the connection.

Note that if $F_\nabla$ is either self-dual or anti-self-dual as a 2-form, then the Yang-Mills equation is automatically satisfied:

$$\ast F_\nabla = \pm F_\nabla \implies \nabla \ast F_\nabla = \pm \nabla F_\nabla = 0$$

by the Bianchi identity. In other words, instantons are particular solutions of the Yang-Mills equation. Furthermore, while the Yang-Mills equation (2) makes sense over any Riemannian manifold, the instanton equation (3) is well-defined only in dimension 4.

A gauge transformation is a bundle automorphism $g : E \to E$ covering the identity. The set of all gauge transformations of a given bundle $E \to X$ form a group through composition, called the gauge group and denoted by $\mathcal{G}(E)$. The gauge group acts on the set of all smooth connections on $E$ by conjugation:

$$g \cdot \nabla = g^{-1} \nabla g$$

It is then easy to see that (3) is gauge invariant condition, since $F_{g \nabla} = g^{-1} F_\nabla g$. The anti-self-duality equation (3) is also conformally invariant: a conformal change in the metric does not change the decomposition (1), so it preserves self-dual and anti-self-dual 2-forms.
The topological charge $k$ of the instanton $\nabla$ is defined by the integral:

$$k = -\frac{1}{8\pi^2} \int_X \text{tr}(F_{\nabla} \wedge F_{\nabla})$$

$$= c_2(E) - \frac{1}{2} c_1(E)^2$$

where the second equality follows from Chern-Weil theory.

If $X$ is a smooth, non-compact, complete Riemannian manifold, an instanton on $X$ is an anti-self-dual connection for which the integral (4) converges. Note that in this case, $k$ as above need not be an integer; however it is expected to always be quantized, i.e. always a multiple of some fixed (rational) number which depends only on the base manifold $X$.

**Summary.** This note is organized as follows. After revisiting the variational approach to the anti-self-duality equation (3), we study instantons over the simplest possible Riemannian 4-manifold, $\mathbb{R}^4$ with the flat Euclidean metric. We present ’t Hooft’s explicit solutions (Section 3), the ADHM construction (Section 4) and its dimensional reductions to $\mathbb{R}^3$, $\mathbb{R}^2$ and $\mathbb{R}$ (Section 5). We conclude in Section 6 by explaining the construction of the central object of study in gauge theory, the instanton moduli spaces.

## 2 Variational aspects of Yang-Mills equation.

Given a fixed smooth vector bundle $E \to X$, let $\mathcal{A}(E)$ be the set of all (smooth) connections on $E$. The Yang-Mills functional is defined by

$$\text{YM} : \mathcal{A}(E) \to \mathbb{R}$$

$$\text{YM}(\nabla) = \|F_{\nabla}\|^2_{L^2} = \int_M \text{tr}(F_{\nabla} \wedge *F_{\nabla})$$

The Euler-Lagrange equation for this functional is exactly the Yang-Mills equation (2). In particular, self-dual and anti-self-dual connections yield critical points of the Yang-Mills functional.

Splitting the curvature into its self-dual and anti-self-dual parts, we have

$$\text{YM}(\nabla) = \|F_{\nabla}^+\|^2_{L^2} + \|F_{\nabla}^-\|^2_{L^2}$$

It is then easy to see that every anti-self-dual connection $\nabla$ is an absolute minimum for the Yang-Mills functional, and that $\text{YM}(\nabla)$ coincides with the topological charge (4) of the instanton $\nabla$ times $8\pi^2$.

One can construct, for various 4-manifolds but most interestingly for $X = S^4$, solutions of the Yang-Mills equations which are neither self-dual nor anti-self-dual. Such solutions do not minimize (4). Indeed, at least for gauge group $SU(2)$ or $SU(3)$, it can be shown that there are no other local minima: any critical point which is neither self-dual nor anti-self-dual is unstable and must be a “saddle point” (4).
3 Instantons on Euclidean space

Let $X = \mathbb{R}^4$ with the flat Euclidean metric, and consider a hermitian vector bundle $E \to \mathbb{R}^4$. Any connection $\nabla$ on $E$ is of the form $\nabla = d + A$, where $d$ denotes the usual de Rham operator and $A \in \Gamma(\text{End}(E)) \otimes \Omega^1_{\mathbb{R}^4}$ is a 1-form with values in the endomorphisms of $E$; this can be written as follows:

$$A = \sum_{k=1}^{4} A_k dx^k, \quad A_k : \mathbb{R}^4 \to \mathfrak{u}(r).$$

In the Euclidean coordinates $x_1, x_2, x_3, x_4$, the anti-self-duality equation is given by:

$$F_{12} = F_{34}, \quad F_{13} = -F_{24}, \quad F_{14} = F_{23}$$

where

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j].$$

The simplest explicit solution is the charge 1 $SU(2)$-instanton on $\mathbb{R}^4$. The connection 1-form is given by:

$$A_0 = \frac{1}{1 + |x|^2} \cdot \text{Im}(qd\overline{q})$$

where $q$ is the quaternion $q = x_1 + x_2i + x_3j + x_4k$, while $\text{Im}$ denotes the imaginary part of the product quaternion; we are regarding $i, j, k$ as a basis of the Lie algebra $\mathfrak{su}(2)$; from this, one can compute the curvature:

$$F_{A_0} = \left(\frac{1}{1 + |x|^2}\right)^2 \cdot \text{Im}(dq \wedge d\overline{q})$$

We see that the action density function

$$|F_{A_0}|^2 = \left(\frac{1}{1 + |x|^2}\right)^2$$

has a bell-shaped profile centered at the origin and decaying like $r^{-4}$.

Let $t_{\lambda,y} : \mathbb{R}^4 \to \mathbb{R}^4$ be the isometry given by the composition of a translation by $y \in \mathbb{R}^4$ with a homothety by $\lambda \in \mathbb{R}^+$. The pullback connection $t_{\lambda,y}^* A_0$ is still anti-self-dual; more explicitly:

$$A_{\lambda,y} = t_{\lambda,y}^* A_0 = \frac{\lambda^2}{\lambda^2 + |x - y|^2} \cdot \text{Im}(qd\overline{q})$$

and

$$F_{A_{\lambda,y}} = \left(\frac{\lambda^2}{\lambda^2 + |x - y|^2}\right)^2 \cdot \text{Im}(dq \wedge d\overline{q}).$$

Note that the action density function $|F_A|^2$ has again a bell-shaped profile centered at $y$ and decaying like $r^{-4}$; the parameter $\lambda$ measures the concentration of
the energy density function, and can be interpreted as the “size” of the instanton $A_{\lambda,y}$.

Instantons of topological charge $k$ can be obtained by “superimposing” $k$ basic instantons, via the so-called ‘t Hooft ansatz. Consider the function $\rho: \mathbb{R}^4 \to \mathbb{R}$ given by:

$$\rho(x) = 1 + \sum_{j=1}^{k} \frac{\lambda_j^2}{(x - y_j)^2},$$

where $\lambda_j \in \mathbb{R}$ and $y_j \in \mathbb{R}^4$. Then the connection 1-form $A = A_\mu dx_\mu$ with coefficients

$$A_\mu = i \sum_{\nu=1}^{4} \overline{\sigma}_{\mu\nu} \frac{\partial}{\partial x_\nu} \ln(\rho(x))$$

is anti-self dual; here, $\overline{\sigma}_{\mu\nu}$ are the matrices given by $(\mu, \nu = 1, 2, 3)$:

$$\overline{\sigma}_{\mu\nu} = \frac{1}{4i} [\sigma_\mu, \sigma_\nu], \quad \overline{\sigma}_{\mu4} = \frac{1}{2} \sigma_\mu$$

where $\sigma_\mu$ are the Pauli matrices.

The connection (8) correspond to $k$ instantons centered at points $y_i$ with size $\lambda_i$. The basic instanton (6) is exactly (modulo gauge transformation) what one obtains from (8) for the case $k = 1$. The ‘t Hooft instantons form a $5k$ parameter family of anti-self-dual connections.

$SU(2)$-instantons are also the building blocks for instantons with general structure group $G$. Let $G$ be a compact semi-simple Lie group, with Lie algebra $\mathfrak{g}$. Let $\phi: \mathfrak{su}(2) \to \mathfrak{g}$ be any injective Lie algebra homomorphism. If $A$ is an anti-self-dual $SU(2)$-connection 1-form, then it is easy to see that $\phi(A)$ is an anti-self-dual $G$-connection 1-form. Using (8) as an example, we have that:

$$A = i \sum_{\mu,\nu} \phi(\overline{\sigma}_{\mu\nu}) \frac{\partial}{\partial x_\nu} \ln(\rho(x)) dx_\mu$$

is a $G$-instanton on $\mathbb{R}^4$.

While this guarantees the existence of $G$-instantons on $\mathbb{R}^4$, note that the instanton (8) might be reducible (e.g. $\phi$ can simply be the obvious inclusion of $\mathfrak{su}(2)$ into $\mathfrak{su}(n)$ for any $n$) and that its charge depends on the choice of representation $\phi$. Furthermore, it is not clear whether every $G$-instanton can be obtained in this way, as the inclusion of a $SU(2)$-instanton through some representation $\phi: \mathfrak{su}(2) \to \mathfrak{g}$.

4 The ADHM construction

All $SU(r)$-instantons on $\mathbb{R}^4$ can be obtained through a remarkable construction due to Atiyah, Drinfeld, Hitchin and Manin. It starts by considering hermitian vector spaces $V$ and $W$ of dimension $c$ and $r$, respectively, and the following data:

$$B_1, B_2 \in \text{End}(V), \quad i \in \text{Hom}(W,W), \quad j \in \text{Hom}(V,W),$$
so-called ADHM data. Assume moreover that \((B_1, B_2, i, j)\) satisfy the ADHM equations:

\[
[B_1, B_2] + ij = 0 \tag{10}
\]

\[
[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j = 0 \tag{11}
\]

Now consider the following maps

\[
\alpha : V \times \mathbb{R}^4 \to (V \oplus V \oplus W) \times \mathbb{R}^4
\]

\[
\beta : (V \oplus V \oplus W) \times \mathbb{R}^4 \to V \times \mathbb{R}^4
\]

given as follows (\(1\) denotes the appropriate identity matrix):

\[
\alpha(z_1, z_2) = \begin{pmatrix} B_1 + z_1 \mathbf{1} \\ B_2 + z_2 \mathbf{1} \\ j \end{pmatrix} \tag{12}
\]

\[
\beta(z_1, z_2) = \begin{pmatrix} -B_2 - z_2 \mathbf{1} \\ B_1 + z_1 \mathbf{1} \\ i \end{pmatrix} \tag{13}
\]

where \(z_1 = x_1 + ix_2\) and \(z_2 = x_3 + ix_4\) are complex coordinates on \(\mathbb{R}^4\). The maps (12) and (13) should be understood as a family of linear maps parameterized by points in \(\mathbb{R}^4\).

A straightforward calculation shows that the ADHM equation (10) imply that \(\beta \alpha = 0\) for every \((z_1, z_2) \in \mathbb{R}^4\). Therefore the quotient \(E = \ker \beta / \im \alpha = \ker \beta \cap \ker \alpha^\dagger\) forms a complex vector bundle over \(\mathbb{R}^4\) or rank \(r\) whenever \((B_1, B_2, i, j)\) is such that \(\alpha\) is injective and \(\beta\) is surjective for every \((z_1, z_2) \in \mathbb{R}^4\).

To define a connection on \(E\), note that \(E\) can be regarded as a sub-bundle of the trivial bundle \((V \oplus V \oplus W) \times \mathbb{R}^4\). So let \(\iota : E \to (V \oplus V \oplus W) \times \mathbb{R}^4\) be the inclusion, and let \(P : (V \oplus V \oplus W) \times \mathbb{R}^4 \to E\) be the orthogonal projection onto \(E\). We can then define a connection \(\nabla\) on \(E\) through the projection formula

\[
\nabla s = P \bar{d}s
\]

where \(\bar{d}\) denotes the trivial connection on the trivial bundle \((V \oplus V \oplus W) \times \mathbb{R}^4\).

To see that this connection is anti-self-dual, note that projection \(P\) can be written as follows

\[
P = \mathbf{1} - \mathcal{D}^\dagger \Xi^{-1} \mathcal{D}
\]

where

\[
\mathcal{D} : (V \oplus V \oplus W) \times \mathbb{R}^4 \to (V \oplus V) \times \mathbb{R}^4
\]

\[
\mathcal{D} = \begin{pmatrix} \beta \\ \alpha^\dagger \end{pmatrix}
\]

and \(\Xi = \mathcal{D} \mathcal{D}^\dagger\). Note that \(\mathcal{D}\) is surjective, so that \(\Xi\) is indeed invertible. Moreover, it also follows from (11) that \(\beta \beta^\dagger = \alpha^\dagger \alpha\), so that \(\Xi^{-1} = (\beta \beta^\dagger)^{-1} \mathbf{1}\).
The curvature $F_\nabla$ is given by:

\[ F_\nabla = P \left( d (1 - D^\dagger) \Xi^{-1} D \right) = P (dD^\dagger \Xi^{-1} (dD)) = P \left( (dD^\dagger) \Xi^{-1} (dD) + D^\dagger d(\Xi^{-1} (dD)) \right) = (dD^\dagger) \Xi^{-1} (dD) \]

for $P (dD^\dagger (\Xi^{-1} (dD))) = 0$ on $E = \ker D$. Since $\Xi^{-1}$ is diagonal, we conclude that $F_\nabla$ is proportional to $dD^\dagger \wedge dD$, as a 2-form.

It is then a straightforward calculation to show that each entry of $dD^\dagger \wedge dD$ belongs to $\Omega^2$. The extraordinary accomplishment of Atiyah, Drinfeld, Hitchin and Manin was to show that every instanton, up to gauge equivalence, can be obtained in this way; see e.g. [4]. For instance, the basic $SU(2)$-instanton (6) is associated with the following data ($c = 1, r = 2$):

\[ B_1, B_2 = 0 \ , \ i = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \ , \ j = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \]

Remark. The ADHM data $(B_1, B_2, i, j)$ is said to be stable if $\beta$ is surjective for every $(z_1, z_2) \in \mathbb{R}^4$, and it is said to be costable if $\alpha$ is injective for every $(z_1, z_2) \in \mathbb{R}^4$. $(B_1, B_2, i, j)$ is regular if it is both stable and costable. The quotient:

\[ \left\{ \text{regular solutions of (10) and (11)} \right\} / U(V) \]

coincides with the moduli space of instantons of rank $r = \dim W$ and charge $c = \dim V$ on $\mathbb{R}^4$ (see below). It is also an example of a quiver variety (see the article "Finite dimensional algebras and quivers" by Alistair Savage), associated to the quiver consisting of two vertices $V$ and $W$, two loop-edges on the vertex $V$ and two edges linking $V$ to $W$, one in each direction.

5 Dimensional reductions of the anti-self-dual Yang-Mills equation.

As pointed out above, a connection on an hermitian vector bundle $E \to \mathbb{R}^4$ of rank $r$ can be regarded as 1-form

\[ A = \sum_{k=1}^{4} A_k(x_1, \cdots, x_4) dx^k \ , \ A_k : \mathbb{R}^4 \to u(r) \ . \]

Assuming that the connection components $A_k$ are invariant under translation in one direction, say $x_4$, we can think of

\[ A = \sum_{k=1}^{3} A_k(x_1, x_2, x_3) dx^k \]

as a connection on a hermitian vector bundle over $\mathbb{R}^3$, with the fourth component $\phi = A_4$ being regarded as a bundle endomorphism $\phi : E \to E$, called a Higgs
field. In this way, the anti-self-duality equation \( \text{reduces to the so-called Bogomolny (or monopole) equation:} \)

\[
F_A = *d\phi 
\]

where \( * \) is the Euclidean Hodge star in dimension 3.

Now assume that the connection components \( A_k \) are invariant under translation in two directions, say \( x_3 \) and \( x_4 \). Consider

\[
A = \sum_{k=1}^{2} A_k(x_1, x_2) dx^k 
\]

as a connection on a hermitian vector bundle over \( \mathbb{R}^2 \), with the third and fourth components combined into a complex bundle endomorphism:

\[
\Phi = (A_3 + i \cdot A_4)(dx_1 - i \cdot dx_2) 
\]

taking values on 1-forms. The anti-self-duality equation is then reduced to the so-called Hitchin’s equations:

\[
\begin{cases}
F_A = [\Phi, \Phi^*] \\
\partial A\Phi = 0
\end{cases} 
\]

(15)

Conformal invariance of the anti-self-duality equation means that Hitchin’s equations are well-defined over any Riemann surface.

Finally, assume that the connection components \( A_k \) are invariant under translation in three directions, say \( x_2, x_3 \) and \( x_4 \). After gauging away the first component \( A_1 \), the anti-self-duality equations reduce to the so-called Nahm’s equations:

\[
\frac{dT_k}{dx_1} + \frac{1}{2} \sum_{j,l} \epsilon_{kjl}[T_j, T_l] = 0, \quad j, k, l = \{2, 3, 4\} 
\]

(16)

where each \( T_k \) is regarded as a map \( \mathbb{R} \to u(\mathbb{R}) \).

Those interested in monopoles and Nahm’s equations are referred to the survey and the references therein. The best source for Hitchin’s equations still are Hitchin’s original papers. A beautiful duality know as Nahm transform relates the various reductions of the anti-self-duality equation to periodic instantons; see the survey article.

It is also worth mentioning the book by Mason & Woodhouse, where other interesting dimensional reductions of the anti-self-duality equation are discussed, providing a deep relation between instantons and the general theory of integrable systems.

6 The instanton moduli space

Now fix a rank \( r \) complex vector bundle \( E \) over a 4-dimensional Riemannian manifold \( X \). Observe that the difference between any two connections is a linear
operator:
\[
(\nabla - \nabla')(f\sigma) = f\nabla\sigma + \sigma \cdot df - f\nabla'\sigma - \sigma \cdot df = f(\nabla - \nabla')\sigma.
\]

In other words, any two connections on \( E \) differ by an endomorphism valued 1-form. Therefore, the set of all smooth connections on \( E \), denoted by \( \mathcal{A}(E) \), has the structure of an affine space over \( \Gamma(\text{End}(E)) \otimes \Omega^1_M \).

The gauge group \( \mathcal{G}(E) \) acts on \( \mathcal{A}(E) \) via conjugation:
\[
g \cdot \nabla := g^{-1}\nabla g.
\]

We can form the quotient set \( \mathcal{B}(E) = \mathcal{A}(E)/\mathcal{G}(E) \), which is the set of gauge equivalence classes of connections on \( E \).

The set of gauge equivalence classes of anti-self-dual connections on \( E \) is a subset of \( \mathcal{B}(E) \), and it is called the moduli space of instantons on \( E \to X \). The subset of \( \mathcal{M}_X(E) \) consisting of irreducible anti-self-dual connections is denoted \( \mathcal{M}_X^+(E) \).

Since the choice of a particular vector bundle within its topological class is immaterial, these sets are usually labeled by the topological invariants (Chern or Pontrjagyn classes) of the bundle \( E \). For instance, \( \mathcal{M}(r, k) \) denotes the moduli space of instantons on a rank \( r \) complex vector bundle \( E \to X \) with \( c_1(E) = 0 \) and \( c_2(E) = k > 0 \).

It turns out that \( \mathcal{M}_X(E) \) can be given the structure of a Hausdorff topological space. In general, \( \mathcal{M}_X(E) \) will be singular as a differentiable manifold, but \( \mathcal{M}_X^+(E) \) can always be given the structure of a smooth Riemannian manifold.

We start by explaining the notion of a \( L^2_p \) vector bundle. Recall that \( L^2_p(\mathbb{R}^n) \) denotes the completion of the space of smooth functions \( f : \mathbb{R}^n \to \mathbb{C} \) with respect to the norm:
\[
\|f\|_{L^2_p}^2 = \int_X \left( |f|^2 + |df|^2 + \cdots |d^{(p)}f|^2 \right).
\]

In dimension \( n = 4 \) and for \( p > 2 \), by virtue of the Sobolev embedding theorem, \( L^2_p \) consists of continuous functions, i.e. \( L^2_p(\mathbb{R}^n) \subset C^0(\mathbb{R}^n) \). So we define the notion of a \( L^2_p \) vector bundle as a topological vector bundle whose transition functions are in \( L^2_p \), where \( p > 2 \).

Now fixed a \( L^2_p \) vector bundle \( E \) over \( X \), we can consider the metric space \( \mathcal{A}_p(E) \) of all connections on \( E \) which can be represented locally on an open subset \( U \subset X \) as a \( L^2_{p}(U) \) 1-form. In this topology, the subset of irreducible connections \( \mathcal{A}_p^+(E) \) becomes an open dense subset of \( \mathcal{A}_p(E) \). Since any topological vector bundle admits a compatible smooth structure, we may regard \( L^2_p \) connections as those that differ from a smooth connection by a \( L^2_p \) 1-form. In other words, \( \mathcal{A}_p(E) \) becomes an affine space modeled over the Hilbert space of \( L^2_p \) 1-forms with values in the endomorphisms of \( E \). The curvature of a connection in \( \mathcal{A}_p(E) \) then becomes a \( L^2_{p-1} \) 2-form with values in the endomorphism bundle \( \text{End}(E) \).

Moreover, let \( \mathcal{G}_{p+1}(E) \) be defined as the topological group of all \( L^2_{p+1} \) bundle automorphisms. By virtue of the Sobolev multiplication theorem, \( \mathcal{G}_{p+1}(E) \) has the structure of an infinite dimensional Lie group modeled on a Hilbert space; its Lie algebra is the space of \( L^2_{p+1} \) sections of \( \text{End}(E) \).
The Sobolev multiplication theorem is once again invoked to guarantee that the action $G_{p+1}(E) \times A_p(E) \to A_p(E)$ is a smooth map of Hilbert manifolds. The quotient space $B_p(E) = A_p(E)/G_{p+1}(E)$ inherits a topological structure; it is a metric (hence Hausdorff) topological space. Therefore, the subspace $M_X(E)$ of $B_p(E)$ is also a Hausdorff topological space; moreover, one can show that the topology of $M_X(E)$ does not depend on $p$.

The quotient space $B_p(E)$ fails to be a Hilbert manifold because in general the action of $G_{p+1}(E)$ on $A_p(E)$ is not free. Indeed, if $A$ is a connection on a rank $r$ complex vector bundle $E$ over a connected base manifold $X$, which is associated with a principal $G$-bundle. Then the isotropy group of $A$ within the gauge group:

$$
\Gamma_A = \{ g \in G_{p+1}(E) \mid g(A) = A \}
$$

is isomorphic to the centralizer of the holonomy group of $A$ within $G$.

This means that the subspace of irreducible connections $A^*_p(E)$ can be equivalently defined as the open dense subset of $A_p(E)$ consisting of those connections whose isotropy group is minimal, that is:

$$
A^*_p(E) = \{ A \in A_p(E) \mid \Gamma_A = \text{center}(G) \}.
$$

Now $G_{p+1}(E)$ acts with constant isotropy on $A^*_p(E)$, hence the quotient $B^*_p(E) = A^*_p(E)/G_{p+1}(E)$ acquires the structure of a smooth Hilbert manifold.

**Remark.** The analysis of neighborhoods of points in $B_p(E) \setminus B^*_p(E)$ is very relevant for applications of the instanton moduli spaces to differential topology. The simplest situation occurs when $A$ is an $SU(2)$-connection on a rank 2 complex vector bundle $E$ which reduces to a pair of $U(1)$ and such $[A]$ occurs as an isolated point in $B_p(E) \setminus B^*_p(E)$. Then a neighborhood of $[A]$ in $B_p(E)$ looks like a cone on an infinite dimensional complex projective space.

Alternatively, the instanton moduli space $M_X(E)$ can also be described by first taking the subset of all anti-self-dual connections and then taking the quotient under the action of the gauge group. More precisely, consider the map:

$$
\rho : A_p(E) \to L^2_p(\text{End}(E) \otimes \Omega^{2+}_{X})
$$

Thus $\rho^{-1}(0)$ is exactly the set of all anti-self-dual connections. It is $G_{p+1}(E)$-invariant, so we can take the quotient to get:

$$
M_X(E) = \rho^{-1}(0)/G_{p+1}(E).
$$

It follows that the subspace $M^*_X(E) = B^*_p(E) \cap M_X(E)$ has the structure of a smooth Hilbert manifold. Index theory comes into play to show that $M^*_X(E)$ is finite-dimensional. Recall that if $D$ is an elliptic operator on a vector bundle over a compact manifold, then $D$ is Fredholm (i.e. ker $D$ and coker $D$ are finite dimensional) and its index

$$
\text{ind} \ D = \dim \ker D - \dim \coker D
$$
can be computed in terms of topological invariants, as prescribed by the Atiyah-Singer index theorem. The goal here is to identify the tangent space of $\mathcal{M}_X^\ast(E)$ with the kernel of an elliptic operator.

It is clear that for each $A \in \mathcal{A}_p(E)$, the tangent space $T_A \mathcal{A}_p(E)$ is just $L^2_p(\text{End}(E) \otimes \Omega^1_X)$. We define the pairing:

$$\langle a, b \rangle = \int_X a \wedge \ast b \quad (18)$$

and it is easy to see that this pairing defines a Riemannian metric (so-called $L^2$-metric) on $\mathcal{A}_p(E)$.

The derivative of the map $\rho$ in (17) at the point $A$ is given by:

$$d^+_A : L^2_p(\text{End}(E) \otimes \Omega^1_X) \to L^2_{p-1}(\text{End}(E) \otimes \Omega^2_X)$$

$$a \mapsto (d_A a)^+,$$

so that for each $A \in \rho^{-1}(0)$ we have:

$$T_A \rho^{-1}(0) = \{ a \in L^2_p(\text{End}(E)) \otimes \Omega^1_X \mid d^+_A a = 0 \}.$$

Now for a gauge equivalence class $[A] \in \mathcal{B}^\ast_p(E)$, the tangent space $T[A] \mathcal{B}^\ast_p(E)$ consists of those 1-forms which are orthogonal to the fibers of the principal $G_{p+1}(E)$ bundle $A^\ast_p(E) \to \mathcal{B}^\ast_p(E)$. At a point $A \in \mathcal{A}_p(E)$, the derivative of the action by some $g \in G_{p+1}(E)$ is

$$-d_A : L^2_{p+1}(\text{End}(E)) \to L^2_p(\text{End}(E) \otimes \Omega^1_X).$$

Usual Hodge decomposition gives us that there is an orthogonal decomposition:

$$L^2_p(\text{End}(E) \otimes \Omega^1_X) = \text{im } d_A \oplus \ker d_A^\ast,$$

which means that:

$$T[A] \mathcal{M}^\ast_X(E) = \{ a \in L^2_p(\text{End}(E) \otimes \Omega^1_X) \mid d_A^\ast a = d^+_A a = 0 \}. \quad (19)$$

Thus the pairing (18) also defines a Riemannian metric on $\mathcal{B}^\ast_p(E)$.

Putting these together, we conclude that the space $T[A] \mathcal{M}^\ast_X(E)$ tangent to $\mathcal{M}^\ast_X(E)$ at an equivalence class $[A]$ of anti-self-dual connections can be described as follows:

$$T[A] \mathcal{M}^\ast_X(E) = \{ a \in L^2_p(\text{End}(E) \otimes \Omega^1_X) \mid d_A^\ast a = d^+_A a = 0 \}.$$

It turns out that the so-called deformation operator $\delta_A = d^+_A \oplus d_A$:

$$\delta_A : L^2_p(\text{End}(E) \otimes \Omega^1_X) \to L^2_{p+1}(\text{End}(E)) \oplus L^2_{p-1}(\text{End}(E) \otimes \Omega^2_X)$$

is elliptic. Moreover, if $A$ is anti-self-dual then $\text{coker } \delta_A$ is empty, so that $T[A] \mathcal{M}^\ast_X(E) = \ker \delta_A$. The dimension of the tangent space $T[A] \mathcal{M}^\ast_X(E)$ is then
simply given by the index of the deformation operator $\delta_A$. Using the Atiyah-Singer index theorem, we have for $SU(r)$-bundles with $c_2(E) = k$:

$$\dim \mathcal{M}_X^*(E) = 4rk - (r^2 - 1)(1 - b_1(X) + b_+(X)) .$$

The dimension formula for arbitrary gauge group $G$ can be found at [4]. For example, the moduli space of $SU(2)$ instantons on $\mathbb{R}^4$ of charge $k$ is a smooth Riemannian manifold of dimension $8k - 3$. These parameters are interpreted as the $5k$ parameters describing the positions and sizes of $k$ separate instantons, plus $3(k - 1)$ parameters describing their relative $SU(2)$ phases.

The detailed construction of the instanton moduli spaces can be found at [4]. An alternative source is Morgan’s lecture notes in [5].

It is interesting to note that $\mathcal{M}_X^*(E)$ inherits many of the geometrical properties of the original manifold $X$. Most notably, if $X$ is a Kähler manifold, then $\mathcal{M}_X^*(E)$ is also Kähler; if $X$ is a hyperkähler manifold, then $\mathcal{M}_X^*(E)$ is also hyperkähler. One expects that other geometric structures on $X$ can also be transferred to the instanton moduli spaces $\mathcal{M}_X^*(E)$.

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