On structure of homogenenous Wick ideals in Wick \(*\)-algebras with braided coefficients

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1 Introduction

In this paper we present some results on structure of Wick homogenenous ideals of quadratic algebras allowing Wick ordering, shortly Wick algebras, introduced in \[3\]. Namely, let \(\{T_{ij}^{kl}, i,j,k,l=1,\ldots,d\} \subset \mathbb{C}\) satisfy conditions \(T_{lk}^{ji} = T_{ij}^{kl}\), then Wick algebra \(W(T)\) is generated by \(a_i, a_i^*, i = 1,\ldots,d\), satisfying commutation relations of the form

\[
a_i^* a_j = \delta_{ij} 1 + \sum_{k,l=1}^d T_{ij}^{kl} a_l a_k^*, \quad i,j = 1,\ldots,d.
\]

(1)

Following \[3\] consider finite-dimensional Hilbert space \(\mathcal{H} = \mathbb{C}\langle e_1,\ldots,e_d \rangle\) and its formal dual \(\mathcal{H}^* = \mathbb{C}\langle e_1^*,\ldots,e_d^* \rangle\), where \(\{e_i, i = 1,\ldots,d\}\) form an orthonormal base of \(\mathcal{H}\). Put \(\mathcal{T}(\mathcal{H},\mathcal{H}^*)\) to be the full tensor algebra over \(\mathcal{H}\) and \(\mathcal{H}^*\), then

\[
W(T) \simeq \mathcal{T}(\mathcal{H},\mathcal{H}^*)/\langle e_i^* \otimes e_j - \sum_{k,l=1}^d T_{ij}^{kl} e_l \otimes e_k^* \rangle.
\]

(2)

Note, that in this realisation the free algebra generated by \(a_i, i = 1,\ldots,d\) coincides with \(\mathcal{T}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n \in \mathbb{N}} \mathcal{H}^\otimes_n\).

The Fock representation of \(W(T)\) is defined on \(\mathcal{T}(\mathcal{H})\) by the rules

\[
a_i^* \Omega = 0, \quad a_i e_{i_1} \otimes \cdots \otimes e_{i_k} = e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_k}, \quad i = 1,\ldots,d,
\]

the action of \(a_i^*, i = 1,\ldots,d\), on vectors other than \(\Omega\), is determined inductively using the commutation relation in \(W(T)\). It was proved in \[3\] that there exists a unique sesquilinear form \(\langle \cdot, \cdot \rangle_F\), called the Fock scalar product, on \(\mathcal{T}(\mathcal{H})\),

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such that the Fock representation becomes a $\ast$-representation with respect to this form. It is defined in such a way that the subspaces $H^\otimes n$ and $H^\otimes m$ are orthogonal if $m \neq n$ and

$$\langle X, Y \rangle_F = \langle X, P_n Y \rangle, \quad X, Y \in \mathcal{H}^\otimes n.$$  

where by $\langle \cdot, \cdot \rangle$ we denote the standard scalar product on $H^\otimes n$ and $P_n : \mathcal{H}^\otimes n \to \mathcal{H}^\otimes n$ is an operator defined in the following way (see [3]): First we introduce an operator $T : \mathcal{H}^\otimes 2 \to \mathcal{H}^\otimes 2$ given by

$$T e_k \otimes e_l = \sum_{i,j=1}^d T^k_{ij} e_i \otimes e_j. \quad (3)$$

Note that $T$ is self-adjoint with respect to the standard scalar product on $H^\otimes 2$.

Further, for any $n > 2$ consider the following extensions of $T$ to $H^\otimes n$:

$$T_i = \bigotimes_{k=1}^{i-1} 1_H \otimes T \bigotimes_{k=i+2}^n 1_H, \quad i = 1, \ldots, n-1.$$  

Then we set $P_0 = 1$, $P_1 = 1_H$, $P_2 = 1_H \otimes 1_H + T$ and

$$P_n = (1_H \otimes P_{n-1}) R_n, \quad n \geq 3, \quad (4)$$

where

$$R_n : \mathcal{H}^\otimes n \to \mathcal{H}^\otimes n, \quad R_n = 1_{\mathcal{H}^\otimes n} + T_1 + T_1 T_2 + \cdots + T_1 T_2 \cdots T_{n-1}.$$  

Remark 1. The operators $R_n, n \geq 2$, are used to obtain explicit formulas for commutation relations between generators $a_i^\ast, i = 1, \ldots, d$ and homogeneous polynomial in noncommutative variables $a_1, \ldots, a_d$. Namely, by [5], for $X \in \mathcal{H}^n$ one has the following equality in $W(T)$ (here we use the canonical realisation)

$$e_i^\ast \otimes X = \mu_0(e_i^\ast)(R_n X + \sum_{k=1}^d T_1 T_2 \cdots T_n (X \otimes e_k) \otimes e_k),$$

where $\mu_0(e_i^\ast) : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is given by

$$\mu_0(e_i^\ast)e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_s} = \delta_{i_1 i_2} \otimes \cdots e_{i_s}, \quad s \geq 1, \quad \mu_0(e_i^\ast) \Omega = 0.$$  

This allows to determine explicitly the action of $a_i^\ast$ in the Fock representation as follows

$$a_i^\ast X = \mu_0(e_i^\ast) R_n X, \quad X \in \mathcal{H}^\otimes n.$$  

Positivity of the Fock scalar product means that $P_n \geq 0$ for all $n \geq 2$. In this case the Fock representation can be extended to a $\ast$-representation of $W(T)$ on a Hilbert space, which is a completion of $\mathcal{T}(\mathcal{H})/ \bigoplus_{n \geq 2} \ker P_n$ with respect to the norm defined by the Fock scalar product. Sufficient conditions for positivity
of family \( \{ P_n, n \geq 2 \} \) can be found in [1, 2, 3]. For instance if \( T \) is braided, i.e. \( T_1 T_2 T_1 = T_2 T_1 T_2 \) on \( \mathcal{H} \otimes^3 \), and \( ||T|| \leq 1 \), then by [1] \( P_n \geq 0, n \geq 2 \). Moreover in this case for any \( n \geq 2 \)

\[
\ker P_n = \sum_{i=1}^{n-1} \ker(1_{\mathcal{H} \otimes^i} + T_i)
\]

and the kernel of the Fock representation is generated as a two-sided \(*\)-ideal by \( \ker(1_{\mathcal{H} \otimes^2} + T) \), see [2]. Furthermore, if \( T \) is braided and \( \ker(1_{\mathcal{H} \otimes^2} + T) \neq \{0\} \), the two-sided ideal \( \mathcal{I}_2 \subset \mathcal{T}(\mathcal{H}) \) generated by \( \ker(1_{\mathcal{H} \otimes^2} + T) \) is invariant with respect to multiplication by any \( a_i^*, i = 1, \ldots, d \). i.e.

\[
e_i^* \otimes \mathcal{I}_2 \subset \mathcal{I}_2 + \mathcal{I}_2 \otimes \mathcal{H}^*
\]

(5)

Ideals \( I \subset \mathcal{T}(\mathcal{H}) \) satisfying (5) are called Wick ideals, see [3]. It was shown that homogeneous Wick ideals, i.e. those ones which are generated by subspaces in \( \mathcal{H} \otimes^n \), are annihilated by the Fock representation, see [3]. In [2] the authors prove that if the operator \( T \) is braided then existence of homogeneous Wick ideals is necessary for existence of Wick ideals in general. If \( T \) is a braided contraction, then any homogeneous Wick ideal of higher degree is contained in a largest quadratic one, see [2]. Note that for some Wick algebras (e.g. Wick algebras associated with twisted canonical commutation relations of W. Pusz and S.L. Woronowicz, see [3, 7]; quonic commutation relations, see [4] and others) their quadratic Wick ideals are contained in their \(*\)-radicals, i.e. such ideals are annihilated by any bounded \(*\)-representation of the corresponding algebra.

In this paper we investigate the structure of homogeneous Wick ideals of higher degrees. We present a method how to construct a homogeneous Wick ideal \( \mathcal{I}_{n+1} \) of degree \( n+1 \) out of a homogeneous Wick ideal \( \mathcal{I}_n \) of degree \( n \) so that \( \mathcal{I}_{n+1} \subset \mathcal{I}_n \). We show that in some particular cases our procedure allows to get a description of largest homogeneous Wick ideals of higher degrees having generators of the largest quadratic Wick ideal only. Finally we study classes of \(*\)-representations of Wick version of CCR annihilating certain homogeneous Wick ideals of degree higher than 2.

## 2 Wick ideals: basic definitions and properties.

The notion of Wick ideal in quadratic Wick algebra was presented in [3]. It was proposed as a natural way to introduce additional relations between generators \( a_i, i = 1, \ldots, d \), which are consistent with the basic relations of the algebra.

Following [3] we will work with the canonical realisation of \( W(T) \) as a quotient of the tensor algebra \( \mathcal{T}(\mathcal{H}, \mathcal{H}^*) \) given by (2). In this realisation the subalgebra generated by \( a_i, i = 1, \ldots, d \), is identified with \( \mathcal{T}(\mathcal{H}) \).

**Definition 1.** A two-sided ideal \( \mathcal{I} \subset \mathcal{T}(\mathcal{H}) \) is called a Wick ideal if

\[
\mathcal{T}(\mathcal{H}^*) \otimes \mathcal{I} \subset \mathcal{I} \otimes \mathcal{T}(\mathcal{H}^*).
\]
If the Wick ideal $I$ is generated by a subspace $I_0 \subset \mathcal{H}^\otimes n$, then $I$ is called a homogeneous Wick ideal of degree $n$.

It is easy to verify the following criteria for a two-sided ideal $I$ to be a Wick one, see [3].

**Proposition 1.** A two-sided ideal $I \subset \mathcal{T}(\mathcal{H})$ is Wick iff

$$\mathcal{H}^* \otimes I \subset I + I \otimes \mathcal{H}^*.$$  

**Remark 2.** If an ideal $I \subset \mathcal{T}(\mathcal{H})$ is generated by a subspace $I_0 \subset \mathcal{H}^\otimes n$, then it is Wick iff

$$\mathcal{H}^* \otimes I_0 \subset I_0 + I_0 \otimes \mathcal{H}^*.$$  

It is important from the representation theory point of view to get a precise description of generators of homogeneous Wick ideals of degrees higher than 2. The first step in this direction was done in [6]. Namely, in this paper the following statement was proved.

**Proposition 2.** Let $T$ be a braided contraction and let $I_2 \subset \mathcal{H}^\otimes 2$ generate the largest quadratic Wick ideal. Then

$$I_3 = (1_{\mathcal{H}^\otimes 3} - T_1 T_2)(I_2 \otimes \mathcal{H})$$

generates the largest Wick ideal of degree 3.

Below we will often say ”homogeneous Wick ideal of degree $n$” meaning a linear subspace in $\mathcal{H}^\otimes n$ generating this ideal.

## 3 Homogeneous Wick ideals

We start with a simple observation, showing that the product of homogeneous Wick ideals is again a homogeneous Wick ideal.

**Proposition 3.** Let $J_n$ and $J_k$ be homogeneous Wick ideals of degree $n$ and $k$ respectively, then their tensor product $J_n \otimes J_k$ is a homogeneous Wick ideal of degree $n + k$.

**Proof.** Indeed, since for a Wick ideal one has

$$\mathcal{H}^* \otimes I \subset I + I \otimes \mathcal{H}^*$$

we get

$$\mathcal{H}^* \otimes (J_n \otimes J_k) \subset (J_n + J_n \otimes \mathcal{H}^*) \otimes J_k = J_n \otimes J_k + J_n \otimes \mathcal{H}^* \otimes J_k \subset \mathcal{H}^* \otimes J_n \otimes J_k + J_n \otimes J_k \otimes \mathcal{H}^* = J_n \otimes J_k + J_n \otimes J_k \otimes \mathcal{H}^*.$$  

Thus, $J_n \otimes J_k \subset \mathcal{H}^\otimes (n + k)$ is a Wick ideal.  

\[\square\]
The following proposition was proved in [3] for quadratic Wick ideals and in [6] in general case.

**Proposition 4.** Let \( P: \mathcal{H} \otimes n \to \mathcal{H} \otimes n \) be a projection. The subspace \( \mathcal{I} = P(\mathcal{H} \otimes n) \) generates a Wick ideal iff

1. \( R_n P = 0 \) (equality in \( \mathcal{H} \otimes n \)),
2. \( [1 \otimes (1 \otimes n - P)]T_1 T_2 \cdots T_n [P \otimes 1] = 0 \) (equality in \( \mathcal{H} \otimes n + 1 \)).

Moreover, if \( T \) is braided and \( P \) is the projection onto \( \ker R_n \), the second condition holds automatically and hence \( \ker R_n \) generates the largest homogeneous Wick ideal of degree \( n \).

**Remark 3.** Note, that the second condition of Proposition 4 means

\[
T_1 T_2 \cdots T_n (\mathcal{I} \otimes \mathcal{H}) \subset \mathcal{H} \otimes \mathcal{I}.
\]

**Lemma 1.** Let \( \mathcal{I} \subset \mathcal{H} \otimes n \) generate a homogeneous Wick ideal, then

\[
(1 \otimes (n+1) - T_1 T_2 \cdots T_n)(\mathcal{I} \otimes \mathcal{H}) \subset \ker R_{n+1}.
\]

**Proof.** Let \( X \in \mathcal{I} \). Then \( X \in \ker R_n \). Note that

\[
R_{n+1} = R_n \otimes 1 + T_1 T_2 \cdots T_n = 1_{\mathcal{H} \otimes (n+1)} + T_1 (1 \otimes R_n)
\]

Then for any \( i = 1, \ldots, d \) one has

\[
R_{n+1}(1_{\mathcal{H} \otimes (n+1)} - T_1 T_2 \cdots T_n)(X \otimes e_i) =
\]

\[
= R_{n+1}(X \otimes e_i) - R_{n+1} T_1 T_2 \cdots T_n (X \otimes e_i) =
\]

\[
= (R_n \otimes 1 + T_1 T_2 \cdots T_n)(X \otimes e_i)
\]

\[
- (1_{\mathcal{H} \otimes (n+1)} + T_1 (1 \otimes R_n)) T_1 T_2 \cdots T_n (X \otimes e_i) =
\]

\[
= T_1 T_2 \cdots T_n (X \otimes e_i) - T_1 T_2 \cdots T_n (X \otimes e_i)
\]

\[
- T_1 (1 \otimes R_n) T_1 T_2 \cdots T_n (X \otimes e_i) = 0,
\]

where we used

\[
T_1 T_2 \cdots T_n (\mathcal{I} \otimes \mathcal{H}) \subset \mathcal{H} \otimes \mathcal{I} \subset \mathcal{H} \otimes \ker R_n = \ker (1 \otimes R_n).
\]

The following corollary is immediate.

**Corollary 1.** If the operator \( T \) is braided, then

\[
(1 \otimes (n+1) - T_1 T_2 \cdots T_n)(\ker R_n \otimes \mathcal{H}) \subset \ker R_{n+1}.
\]

Below we will use the following simple observation.
Lemma 2. Let $T$ be braided. Then for any $n \geq 2$ and $k \leq n - 1$

$$(T_1 T_2 \cdots T_n)(T_1 T_2 \cdots T_k) = (T_2 T_3 \cdots T_{k+1})(T_1 T_2 \cdots T_n).$$

Proof. Evidently it is enough to check that

$$T_1 T_2 \cdots T_n T_j = T_{j+1} T_1 T_2 \cdots T_n, \quad 1 \leq j \leq n - 1.$$  

Indeed, since $T_j T_i = T_i T_j$ when $|i-j| \geq 2$ and $T_j T_{j+1} T_j = T_{j+1} T_j T_{j+1}$ we get

$$T_1 T_2 \cdots T_n T_j = T_1 T_2 \cdots T_{j-1} T_{j+1} T_j T_{j+2} \cdots T_n = T_1 T_2 \cdots T_{j-1} T_{j+1} T_j T_{j+2} \cdots T_n = T_{j+1} T_1 T_2 \cdots T_n.$$  

\[ \square \]

The following proposition gives a procedure to compute generators of certain homogeneous Wick ideals of degree $n + 1$ out of generators of Wick ideals of degree $n$ when $T$ is braided.

Proposition 5. Let $T$ be braided and $\mathcal{I}_n \subset \mathcal{H}^\otimes n$ generate a homogeneous Wick ideal of degree $n$. Then

$$\mathcal{I}_{n+1} = (1_{\mathcal{H}^\otimes (n+1)} - T_1 T_2 \cdots T_n)(\mathcal{I}_n \otimes \mathcal{H})$$

generates a homogeneous Wick ideal of degree $n + 1$.

Proof. According to Lemma \[\] 

$$(1_{\mathcal{H}^\otimes (n+1)} - T_1 T_2 \cdots T_n)(\mathcal{I}_n \otimes \mathcal{H}) \subset \ker R_{n+1}$$

so, it remains to prove that

$$T_1 T_2 \cdots T_{n+1}(\mathcal{I}_{n+1} \otimes \mathcal{H}) \subset \mathcal{H} \otimes \mathcal{I}_{n+1}. \quad (6)$$

Indeed

$$T_1 T_2 \cdots T_{n+1}(\mathcal{I}_{n+1} \otimes \mathcal{H}) = T_1 T_2 \cdots T_{n+1}(1_{\mathcal{H}^\otimes (n+1)} - T_1 T_2 \cdots T_n)(\mathcal{I}_n \otimes \mathcal{H} \otimes \mathcal{H}) =$$

$$= (T_1 T_2 \cdots T_{n+1} - T_1 T_2 \cdots T_{n+1}) (\mathcal{I}_n \otimes \mathcal{H} \otimes \mathcal{H}) =$$

$$= (T_1 T_2 \cdots T_{n+1} - T_2 T_3 \cdots T_{n+1} T_1 T_2 \cdots T_n) (\mathcal{I}_n \otimes \mathcal{H} \otimes \mathcal{H}) =$$

$$= (1_{\mathcal{H}^\otimes (n+1)} - T_2 T_3 \cdots T_{n+1} T_1 T_2 \cdots T_n) (\mathcal{I}_n \otimes \mathcal{H} \otimes \mathcal{H}) \subset$$

$$\subset (1_{\mathcal{H}^\otimes (n+1)} - T_2 T_3 \cdots T_{n+1}) T_1 T_2 \cdots T_n (\mathcal{I}_n \otimes T(\mathcal{H} \otimes \mathcal{H})) \subset$$

$$\subset (1_{\mathcal{H}^\otimes (n+1)} - T_2 T_3 \cdots T_{n+1} T_1 T_2 \cdots T_n) (\mathcal{I}_n \otimes \mathcal{H} \otimes \mathcal{H}) =$$

$$= \mathcal{H} \otimes (1_{\mathcal{H}^\otimes n} - T_1 T_2 \cdots T_n) (\mathcal{I}_n \otimes \mathcal{H}) =$$

$$= \mathcal{H} \otimes \mathcal{I}_{n+1}.$$  

\[ \square \]
Next aim is to describe largest Wick ideals.

**Lemma 3.** Let $T$ satisfy the braid relation. Then

$$R_{n+1}T_1T_2\cdots T_n = T_1T_2\cdots T_n + T_1^2T_2\cdots T_n (R_n \otimes 1_\mathcal{H}) \quad (7)$$

**Proof.** Indeed

$$R_{n+1}T_1T_2\cdots T_n =$$

$$= T_1T_2\cdots T_n + T_1(1_\mathcal{H} \otimes n + 1 + T_2T_3 + \cdots + T_2T_3\cdots T_n)T_1T_2\cdots T_n =$$

$$= T_1T_2\cdots T_n + T_1^2T_2\cdots T_n (1_\mathcal{H} \otimes n + 1 + T_1T_2 + \cdots + T_1T_2\cdots T_{n-1}) =$$

$$= T_1T_2\cdots T_n + T_1^2T_2\cdots T_n (R_n \otimes 1_\mathcal{H}).$$

$\square$

**Lemma 4.** Let $T$ be braided. Then

$$R_{n+1}(1_\mathcal{H} \otimes (n+1) - T_1T_2\cdots T_n) = (1_\mathcal{H} \otimes (n+1) - T_1^2T_2\cdots T_n) (R_n \otimes 1_\mathcal{H}).$$

**Proof.** By the previous Lemma

$$R_{n+1} - R_{n+1}T_1T_2\cdots T_n =$$

$$= R_n \otimes 1_\mathcal{H} + T_1T_2\cdots T_n - T_1T_2\cdots T_n - T_1^2T_2\cdots T_n (R_n \otimes 1_\mathcal{H}) =$$

$$= (1 - T_1^2T_2\cdots T_n) (R_n \otimes 1_\mathcal{H}).$$

$\square$

Let $\mathcal{K}_2 = \ker R_2$ and

$$\mathcal{K}_{m+1} = (1_\mathcal{H} \otimes (m+1) - T_1T_2\cdots T_m)(\mathcal{K}_m \otimes \mathcal{H}), \quad m \geq 2.$$

Since by $\mathbb{R}$

$$\mathcal{K}_{m+1} \subset \mathcal{H} \otimes \mathcal{K}_m + \mathcal{K}_m \otimes \mathcal{H},$$

the Wick ideals generated by $\mathcal{K}_m$, $m \geq 2$, form a nested sequence

$$\langle \mathcal{K}_2 \rangle \supset \langle \mathcal{K}_3 \rangle \supset \cdots \supset \langle \mathcal{K}_m \rangle \supset \cdots$$

**Proposition 6.** Suppose that $T$ is braided and for any $m \geq 2$

$$\ker (1_\mathcal{H} \otimes (m+1) - T_1T_2\cdots T_m) = \{0\} \quad \text{and} \quad \ker (1_\mathcal{H} \otimes (m+1) - T_1^2T_2\cdots T_m) = \{0\}.$$

Then

$$\mathcal{K}_m = \ker R_m, \quad m \geq 2,$$

and hence $\mathcal{K}_m$ generates the largest homogeneous Wick ideals of degree $m$ for any $m \geq 2$.  

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Proof. Suppose that \( \dim \mathcal{H} = d \). If \( 1_{\mathcal{H}^\otimes (m+1)} - T_1 T_2 \cdots T_m, m \geq 2 \) are invertible, by the definition of \( \mathcal{K}_m \) we have

\[
\dim \mathcal{K}_m = d \cdot \dim \mathcal{K}_{m-1} = d^{m-2} \cdot \dim \ker R_2.
\]

As \( \mathcal{K}_m \subset \ker R_m \) (by Lemma 1) it remains to see that for any \( m \geq 2 \) one has

\[
\dim \ker R_m = d \cdot \dim \ker R_{m-1} = \ldots = d^{m-2} \dim \ker R_2
\]

But this immediately follows from the equality

\[
R_{m+1}(1_{\mathcal{H}^\otimes (m+1)} - T_1 T_2 \cdots T_m) = (1_{\mathcal{H}^\otimes (m+1)} - T_1^2 T_2 \cdots T_m)(R_m \otimes 1_{\mathcal{H}})
\]

and invertibility of the operators \( 1_{\mathcal{H}^\otimes (m+1)} - T_1 T_2 \cdots T_m \) and \( 1_{\mathcal{H}^\otimes (m+1)} - T_1^2 T_2 \cdots T_m \).

Hence, \( \dim \ker R_m = \dim \ker \mathcal{K}_m \) and \( \mathcal{K}_m = \ker R_m, \ m \geq 2. \)

\[\square\]

Lemma 5. Let \( T \) be braided and \( \|T_1 T_2 T_1\| = q < 1, \|T\| = 1. \) Then \( \ker R_m = \mathcal{K}_m \) for any \( m \geq 2. \)

Proof. By Proposition 6 it is enough to see that

\[1 \notin \sigma(T_1 T_2 \cdots T_n) \quad \text{and} \quad 1 \notin \sigma(T_1^2 T_2 \cdots T_n).
\]

Indeed, since \( T_i T_j = T_j T_i, |i - j| \geq 2, \) and \( \|T_i\| = 1, i = 1, \ldots, n, \) we get

\[(T_1 T_2 T_3 \cdots T_n)^2 = (T_1 T_2 T_1)(T_3 T_4 \cdots T_n T_2 T_3 \cdots T_n)
\]

implying

\[\|(T_1 T_2 \cdots T_n)^2\| \leq q < 1
\]

and hence \( 1 \notin \sigma(T_1 T_2 \cdots T_n). \)

Analogously,

\[(T_1^2 T_2 \cdots T_n)^2 = T_1(T_1 T_2 T_1)(T_3 T_4 \cdots T_n T_1 T_2 \cdots T_n)
\]

and \( \|(T_1^2 T_2 \cdots T_n)^2\| \leq q < 1 \) giving \( 1 \notin \sigma(T_1^2 T_2 \cdots T_n). \)

In what follows we shall often say ideal \( \mathcal{K}_m \) meaning the ideal generated by \( \mathcal{K}_m. \)

In general, see Section 3 and Section 4, largest homogeneous Wick ideals do not coincide with the ideals \( \mathcal{K}_m. \) However a direct calculations in Mathematica shows that for some Wick algebras, including Wick versions of CCR, twisted CCR, twisted CAR and quonic commutation relations, see [3], the following conjecture is true.

Conjecture 1. If \( T \) is braided then

\[
\ker R_{n+1} = (1_{\mathcal{H}^\otimes (n+1)} - T_1 T_2 \cdots T_n)(\ker R_n \otimes \mathcal{H}) + \ker R_{n-2} \otimes \ker R_2.
\]
4 Homogeneous ideals of Wick version of quon commutation relations

Here we apply results of the previous section to get a description of homogeneous ideals the Wick algebra, $A_q^2$, associated with quon commutation relations with two degrees of freedom, see \cite{4}. Recall that $A_q^2$ is a $*$-algebra generated by elements $a_i$, $a_i^*$, $i=1,2$, satisfying commutation relations of the form
\[
\begin{align*}
    a_i^*a_i &= 1 + qa_i a_i^*, \quad i = 1,2, \\
    a_1^* a_2 &= \lambda a_2 a_1^*,
\end{align*}
\]
where $q$, $\lambda$ are parameters such that $0 < q < 1$, $|\lambda| = 1$. In this case $\dim \mathcal{H} = 2$ and the operator $T$ is given by
\[
\begin{align*}
    T e_i \otimes e_i &= q e_i \otimes e_i, \quad i = 1,2, \\
    T e_2 &= \lambda e_2, \quad T e_1 = e_1, \quad T e_2 \otimes e_1 = \lambda e_1 \otimes e_2
\end{align*}
\]
It is easy to verify that $T$ is braided, $||T|| = 1$ for any $q \in (0,1)$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and $\ker(1_{\mathcal{H} \otimes 2} + T) = \mathbb{C} \langle A = e_2 \otimes e_1 - \lambda e_1 \otimes e_2 \rangle$.

**Proposition 7.** Let $T: \mathcal{H}^{\otimes 2} \rightarrow \mathcal{H}^{\otimes 2}$ be defined by \cite{8} and $\dim \mathcal{H} = 2$. Then for any $m \geq 2$, $\ker R_m = \mathcal{K}_m$ is the largest homogeneous Wick ideal of degree $m$.

**Proof.** By Lemma \cite{3} it is enough to show that $||T_1 T_2 T_1|| = 1$. Indeed, it is easy to see that for the standard orthonormal basis of $\mathcal{H}^{\otimes 3}$ one has
\[
\begin{align*}
    T_1 T_2 T_1 e_i \otimes e_i \otimes e_i &= q^2 e_i \otimes e_i \otimes e_i, \quad i = 1,2 \\
    T_1 T_2 T_1 e_2 \otimes e_1 \otimes e_1 &= q^2 e_2 \otimes e_1 \otimes e_1, \\
    T_1 T_2 T_1 e_2 \otimes e_2 \otimes e_1 &= q^2 e_2 \otimes e_2 \otimes e_2, \\
    T_1 T_2 T_1 e_2 \otimes e_2 \otimes e_2 &= q^2 e_2 \otimes e_2 \otimes e_2, \\
    T_1 T_2 T_1 e_2 \otimes e_2 \otimes e_2 &= q^2 e_2 \otimes e_2 \otimes e_2.
\end{align*}
\]
Hence $||T_1 T_2 T_1|| = q < 1$. \qed

**Remark 4.** 1. For Wick quonic relations with three generators Lemma \cite{5} cannot be applied, since in this case $||T_1 T_2 T_1|| = 1$. However, since $T$ is a braided contraction we have by Proposition \cite{2}
\[
\ker R_3 = (1_{\mathcal{H} \otimes 3} - T_1 T_2)(\ker R_2 \otimes \mathcal{H})
\]
and one can apply Proposition \cite{6} to show that in this case $\mathcal{K}_m = \ker R_m$, $m \geq 2$ as well.

2. Computations in Mathematica show that for Wick quonic relations with four or more generators the ideals $\mathcal{K}_m$ do not coincide with $\ker R_m$ for $m > 3$. 

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4.1 \(*\)-Representations of \(A_q^2\), annihilating homogeneous ideals

In this section we show that any \(*\)-representation of the Wick quonic relations annihilating \(\mathcal{K}_m\) for some fixed \(m \geq 2\) annihilates the ideal \(\mathcal{K}_2\).

First we recall that for any bounded \(*\)-representation \(\pi\) of \(A_q^2\) one has \(\pi(\mathcal{K}_2) = 0\), see [5]. Indeed, it easy to verify, that if \(A = a_2a_1 - \lambda a_1a_2\), then

\[
\begin{align*}
a_1^*A &= \lambda q Aa_1^*, \\
a_2^*A &= \lambda q Aa_2^*
\end{align*}
\]

implying that \(A^*A = q^2 AA^*\). Evidently, the only bounded operator \(A\) satisfying such relation is the zero one.

**Proposition 8.** Let \(\pi\) be an irreducible \(*\)-representation (possibly unbounded) of \(A_q^2\) such that \(\pi(\mathcal{K}_m) = \{0\}\) for some \(m \geq 3\). Then \(\pi(A) = 0\) and hence \(\pi(\mathcal{K}_2) = 0\).

**Proof.** By Propositions 7 for any \(m \geq 3\) the ideal \(\mathcal{K}_m\) coincides with the largest homogeneous ideal of degree \(m\).

Let \(m = 2k\), for some \(k > 1\). Then, since the product of homogeneous Wick ideals is a homogeneous Wick ideal, we get

\[
(\ker R_2)^{\otimes k} \subset \ker R_{2k} = \mathcal{K}_m.
\]

So if \(\pi(\mathcal{K}_m) = \{0\}\), then \(\pi(A^k) = 0\) and hence \(\ker \pi(A) \neq \{0\}\). Further, \(A^*A = q^2 AA^*\) implies that \(\ker \pi(A) = \ker \pi(A^*)\) and from

\[
\begin{align*}
Aa_1^* &= \lambda q^{-1} a_1^* A, \\
Aa_2^* &= \lambda q^{-1} a_2^* A, \\
A^*a_1 &= \lambda qa_1^* A, \\
A^*a_2 &= \lambda qa_2^* A
\end{align*}
\]

we obtain that \(\ker \pi(A) = \ker \pi(A^*)\) is invariant with respect to \(\pi(a_i)\) and \(\pi(a_i^*)\), \(i = 1, 2\). Thus if \(\pi\) is irreducible, \(\pi(A) = \{0\}\).

Suppose now that \(\pi(\mathcal{K}_m) = \{0\}\) and \(m = 2k + 1\), for fixed \(k \geq 1\). Then as above \(A^{k+1} \in \mathcal{K}_{m+1}\). Since \(\langle \mathcal{K}_{m+1} \rangle \subset \langle \mathcal{K}_m \rangle\) we get \(\pi(A^{k+1}) = 0\) and repeating the arguments from the previous paragraph we obtain \(\pi(A) = 0\).

We refer the reader to [6] for definitions and facts about unbounded \(*\)-representations of \(*\)-algebras. Note that such representations can be rather complicated and one usually restricts oneself to a subclass of "well-behaved" representations. For Lie algebras natural well-behaved representations are integrable representations i.e. those which can be integrated to a unitary representation of the corresponding Lie group (see for example [6, Section 10]).

5 \(*\)-Representations of Wick version of CCR annihilating homogeneous ideals

In this Section we consider a Wick version of CCR, denoted below by \(A_d^0\), and given by

\[
A_d^0 = C \langle a_i, a_i^* | a_i^*a_j = \delta_{ij}1 + a_ja_i^*, \ i, j = 1, \ldots, d \rangle.
\]
In this case $T$ is the flip operator

$$T e_i \otimes e_j = e_j \otimes e_i, \ i, j = 1, \ldots, d$$

and the largest quadratic ideal $K_2 = \ker R_2$ is generated by the elements

$$A_{ij} = e_j \otimes e_i - e_i \otimes e_j, \ i \neq j, \ i, j = 1, \ldots, d.$$ 

The action of the operator $T_1 T_2 \cdots T_k$ on a product of the form $B \otimes e_i$, $B \in \mathcal{H}^k$, $i = 1, \ldots, d$, is the following

$$(T_1 T_2 \cdots T_k)(B \otimes e_i) = e_i \otimes B, \ i = 1, \ldots, d.$$ 

Thus if the homogeneous Wick ideal $K_m$ is generated by a family $\{B_j, j \in J\}$, then

$$K_{m+1} = \langle e_i \otimes B_j - B_j \otimes e_i, \ i = 1, \ldots, d, \ j \in J \rangle$$

Recall that

$$e^*_i \otimes B_j = \mu_0(e^*_i)(R_mB_j + \sum_{k=1}^d T_1 T_2 \cdots T_m(B_j \otimes e_k) \otimes e^*_k), \ i = 1, \ldots, d, \ j \in J.$$ 

Since

$$T_1 T_2 \cdots T_m(B_j \otimes e_k) = e_k \otimes B_j, \ R_mB_j = 0,$$

and $\mu_0(e^*_i)e_k \otimes X = \delta_{ik}X$ for any $X \in \mathcal{T}(\mathcal{H})$, we get

$$e^*_i \otimes B_j = B_j \otimes e^*_i, \ i = 1, \ldots, d, \ j \in J.$$ 

In other words if we consider the quotient of $\mathcal{A}^0_d$ by the homogeneous Wick ideal $K_{m+1}$ we obtain the following commutation relations between generators of the algebra and generators of the ideal $K_m$

$$a^*_i B_j = B_j a^*_i, \ a_i B_j = B_j a_i, \ i = 1, \ldots, d, \ j \in J.$$ 

We intend to study representations of $\mathcal{A}^0_2$ annihilating the ideals $K_m$, $m = 2, 3, 4$.

### 5.1 Representations of $\mathcal{A}^0_2$ annihilating quadratic and cubic ideals

Below we assume $d = 2$. The quadratic ideal $K_2$ is generated by $a_1 \otimes a_2 - a_2 \otimes a_1$ and the quotient $\mathcal{A}^0_2/K_2$ is the Weyl algebra with two degrees of freedom. Note that it is a quotient of the universal enveloping of the Heisenberg algebra. The unique irreducible well-behaved representation of the Weyl algebra (by well-behaved we mean a representation which can be integrated to a unitary representation of the Heisenberg Lie group), is the Fock representation: the space of the representation is $\mathcal{H} = l_2(\mathbb{Z}_+) \otimes l_2(\mathbb{Z}_+)$ and

$$a_1 = a \otimes 1, \ a_2 = 1 \otimes a,$$
where \(ae_n = \sqrt{n+1}e_{n+1}, n \in \mathbb{Z}_+\), and \(\{e_n, n \in \mathbb{Z}_+\}\) is the standard orthonormal basis in \(l_2(\mathbb{Z}_+)\).

Now we study irreducible representations of \(A_0^2\) which annihilate the ideal \(K_3\). The ideal \(K_3\) is generated by the elements

\[Aa_1 - a_1A, \quad Aa_2 - a_2A\]

with \(A = a_2a_1 - a_1a_2\).

Since \(a_i^*A = Aa_i^*, \quad i = 1, 2\), we conclude that \(A\) belongs to the center of the quotient \(A_0^2/K_3\).

Proposition 9. The \(*\)-algebras \(A_{2,x}\) and \(A_{2,0}\) are isomorphic for any \(x \in \mathbb{C}\).

Proof. For any fixed \(x \in \mathbb{C}\) let

\[d_1 = a_1 \quad \text{and} \quad d_2 = \left(1 + |x|^2\right)^{-\frac{1}{2}}a_2 - xa_1^*.
\]

Then it is easy to verify that \(d_1, d_2\) generate \(A_{2,x}\) and

\[d_i^*d_i - d_id_i^* = 1, \quad i = 1, 2, \quad d_1d_2 = d_2d_1^* = d_2d_1 = d_1d_2. \quad (10)
\]

Conversely, let \(c_1, c_2\) be generators of \(A_{2,0}\) satisfying (10). Put

\[b_1 = c_1, \quad b_2 = \left(1 + |x|^2\right)^{\frac{1}{2}}c_2 + xc_1^*.
\]

Then \(b_1, b_2\) satisfy (9) and generate \(A_{2,0}\). Hence \(A_{2,x} \simeq A_{2,0}\).\[\square\]

It follows from the uniqueness of irreducible well-behaved representation of CCR with two degrees of freedom that there exists a unique, up to a unitary equivalence, irreducible representation of (10) defined on \(l_2(\mathbb{Z}_+)^{\otimes 2}\) by the formulas

\[d_1 = a \otimes 1, \quad d_2 = 1 \otimes a.
\]

Below by well-behaved representation of \(A_{2,x}\) we mean a well-behaved representation of \(A_{2,0} \simeq A_{2,x}\). Applying Proposition 9 we get the following result.

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Theorem 1. For any \( x \in \mathbb{C} \) there exists a unique, up to unitary equivalence, irreducible well-behaved representation of \( A_{2,x} \) given by

\[
\begin{align*}
    a_1 &= a \otimes 1, \\
    a_2 &= \sqrt{1 + |x|^2} \otimes a + xa^* \otimes 1.
\end{align*}
\]

Evidently in the case \( x = 0 \) we get the Fock representation, annihilating \( K_2 \).

5.2 Representations annihilating \( K_4 \)

Let us describe representations of \( A^{0}_2 \) which annihilate the ideal \( K_4 \). Recall that

\[
K_4 = \langle B_i a_j - a_j B_i, \quad i, j = 1, 2 \rangle,
\]

where \( B_i = Aa_i - a_i A, \ i = 1, 2, \) are generators of \( K_3 \). Since

\[
a_j^* B_i = B_i a_j^*, \quad i = 1, 2,
\]

the elements \( B_1, B_2 \) belong to the center of the quotient \( A^{0}_2/K_4 \). Identifying again the elements with their images in a representation \( \pi \) annihilating \( K_4 \) we require that for a well-behaved irreducible representation

\[
B_1 = Aa_1 - a_1 A = x_1 1, \quad B_2 = Aa_2 - a_2 A = x_2 1
\]

for some \( x_1, x_2 \in \mathbb{C} \). Note also that in \( A^{0}_2 \) we have \( a_i^* A = Aa_i^*, \ i = 1, 2. \)

5.2.1 Representations with \( x_1 \neq 0 \).

Fix \( (x_1, x_2) \in \mathbb{C}^2 \) with \( x_1 \neq 0 \) and consider the \( \ast \)-algebra \( A_{x_1,x_2} \), generated by elements \( a_1, a_2, A \) satisfying the following commutation relations

\[
\begin{align*}
    a_i^* a_i - a_i a_i^* &= 1, \\
    a_i^* a_2 &= a_2 a_i^*, \quad A = a_2 a_1 - a_1 a_2, \\
    A a_i - a_i A &= x_i 1, \quad a_i^* A = A a_i^*, \quad i = 1, 2.
\end{align*}
\]

Let

\[
\begin{align*}
    d_1 &= a_1 \\
    d_2 &= |x_1|^{-1} (A - x_1 a_1^*) \\
    d_3 &= \left( 1 + \frac{|x_2|^2}{|x_1|^2} \right)^{-\frac{3}{2}} \left( a_2 + \frac{x_2}{|x_1|} d_2^* d_2 - \frac{x_1}{2} d_2^2 - |x_1| d_1^* d_2 - \frac{x_1 (d_1^*)^2}{2} \right)
\end{align*}
\]

Below we show that the elements \( d_i, \ i = 1, 2, 3, \) generate \( A_{x_1,x_2} \) and satisfy CCR with three degrees of freedom.

First we establish some commutation relations between \( a_i \) and \( d_j, \ i, j = 1, 2. \)
Lemma 6. The elements \( a_1, a_2, d_1, d_2 \) satisfy the following relations
\[
\begin{align*}
  d_1^* a_2 &= a_2 d_1^*, \\
  a_2 d_1 - d_1 a_2 &= |x_1| d_2 + x_1 d_1^*, \\
  a_2^* d_2 &= d_2 a_2^* + x_1 d_2 + |x_1| d_1, \\
  a_2 d_2 &= d_2 a_2 - \frac{x_2}{|x_1|}.
\end{align*}
\]

Proof. The first two relations follow directly from the definition of \( d_1, d_2 \) and (11). Further
\[
|x_1| a_2 d_2 = a_2 A - x_1 a_2^* a_1 = A a_2 - x_2 - x_1 a_1^* a_2 = (A a_2 - x_1 a_1^*) a_2 - x_2 = |x_1| a_2 d_2 - x_2,
\]
and
\[
|x_1| a_2^* d_2 = a_2^* A - x_1 a_2^* a_1^* = A^* a_2^* - x_1 (a_1^* a_2^* - A^*) = (A - x_1 a_1^*) a_2^* + x_1 A^* = |x_1| d_2 a_2^* + x_1 (|x_1| d_2^* + \bar{x}_1 d_1) = |x_1| (d_2 a_2^* + x_1 d_2^* + |x_1| d_1).
\]
\[\square\]

Lemma 7. The elements \( d_i, d_i^*, i = 1, 2, 3 \), generate \( A_{x_1, x_2} \) and satisfy CCR with three degrees of freedom, i.e. for any \( i = 1, 2, 3 \) and \( i \neq j \)
\[
d_i^* d_i - d_i d_i^* = 1, \quad d_i^* d_j = d_j d_i^*, \quad d_i d_j = d_j d_i.
\]

Proof. It easily follows from (12) that
\[
\begin{align*}
  a_1 &= d_1, \\
  A &= |x_1| d_2 + x_1 d_1^*, \\
  a_2 &= \left(1 + \frac{|x_2|^2}{|x_1|^2}\right) d_3 - \frac{x_2}{|x_1|} d_2^* + \frac{\bar{x}_1}{2} d_2^* + |x_1| d_1^* d_2 + \frac{\bar{x}_1}{2} (d_1^*)^2
\end{align*}
\]
proving that \( A_{x_1, x_2} \) is generated by \( d_1, d_2, d_3 \).

Further
\[
|x_1| d_2 d_1 = (A - x_1 a_1^*) a_1 = A a_1 - x_1 (1 + a_1 a_1^*) = a_1 A + x_1 - x_1 a_1 a_1^* = a_1 (A - x_1 a_1^*) = |x_1| d_1 d_2
\]
\[
|x_1| d_2 d_1 = a_1^* (A - x_1 a_1^*) = A a_1^* - x_1 (a_1^*)^2 = (A - x_1 a_1^*) a_1^* = |x_1| d_2 d_1^*
\]
Now let us check that \( d_2^* d_2 - d_2 d_2^* = 1 \)
\[
|x_1|^2 d_2^* d_2 = (A^* - \bar{\tau}_1 a_1)(A - x_1 a_1^*) = A^* A - x_1 A^* a_1^* - \bar{\tau}_1 a_1 A + |x_1|^2 a_1 a_1^* = AA^* - x_1 (a_1^* A^* - \bar{\tau}_1) - \bar{\tau}_1 (A a_1 - x_1) + |x_1|^2 (a_1^* a_1 - 1) = AA^* - x_1 a_1^* A^* - \bar{\tau}_1 A a_1 + |x_1|^2 a_1 a_1^* + |x_1|^2 = (A - x_1 a_1^*)(A^* - \bar{\tau}_1 a_1) + |x_1|^2 = |x_1|^2 (1 + d_2 d_2^*).
\]
Here we use the evident fact that $AA^* = A^*A$.

The relation $d_1^* d_3 = d_3^* d_1$ follows immediately from the definition of $d_3$ and the commutation relations between $d_1^*$ and $d_2$, $d_2^*$. Using this commutation again as well as relations (13) we get

$$\sqrt{1 + \frac{|x_2|^2}{|x_1|^2}(d_1^* d_3 - d_3 d_1)} = d_1 a_2 - a_2 d_1 + |x_1|(d_1^* d_1 - d_1 d_1^*)d_2 +$$

$$+ \frac{x_1}{2} ((d_1^* d_1 - d_1 d_1^*)^2) =$$

$$- |x_1|d_2 - |x_1|d_1^* + x_1 d_2 + \frac{x_1}{2} 2d_1^* = 0,$$

$$\sqrt{1 + \frac{|x_2|^2}{|x_1|^2}(d_2^* d_3 - d_3 d_2)} = d_2 a_2 - a_2 d_2 -$$

$$- \frac{x_1}{2}((d_2^* d_2 - d_2 d_2^*) - |x_1|(d_2^* d_2 - d_2 d_2^*)d_1^* =$$

$$= x_1 d_2 + |x_1|d_1^* - \frac{x_1}{2} 2d_2 - |x_1|d_1^* = 0$$

and

$$\sqrt{1 + \frac{|x_2|^2}{|x_1|^2}(d_2^* d_3 - d_3 d_2)} = d_2 a_2 - a_2 d_2 + \frac{x_2}{|x_1|} (d_2 d_2^* - d_2^* d_2) =$$

$$= \frac{x_2}{|x_1|} = 0.$$

Finally, since $d_3 d_i = d_i d_3$, $d_i^* d_3 = d_3 d_i^*$, $i = 1, 2$ one has

$$1 = a_2^* a_2 - a_2 a_2^* = (1 + \frac{|x_2|^2}{|x_1|^2})(d_3^* d_3 - d_3 d_3) - \frac{|x_2|^2}{|x_1|^2}(d_2^* d_2 - d_2 d_2^*) +$$

$$+ \frac{|x_1|^2}{4} ((d_2^* d_2 - d_2 d_2^*)^2) + \frac{|x_1|^2}{4} (d_1^* d_1 - d_1 d_1^*)^2 +$$

$$+ |x_1|^2 (d_3^* d_3 - d_3 d_3^*) - |x_1|^2 (d_2^* d_2 - d_2 d_2^*) +$$

$$+ \frac{|x_1|^2}{2} d_3^* d_1^* - |x_1|^2 d_3^* d_1^* + \frac{|x_1|^2}{2} d_2^* d_3^* - |x_1|^2 d_2^* d_3^* +$$

$$= (1 + \frac{|x_2|^2}{|x_1|^2})(d_3^* d_3 - d_3 d_3^*) - \frac{|x_2|^2}{|x_1|^2} +$$

$$+ \frac{|x_1|^2}{4} (2 + 4d_2 d_2^*) - \frac{|x_1|^2}{4} (2 + 4d_1 d_1^*) + |x_1|^2 (d_1^* d_1^* - d_2^* d_2^*) +$$

$$+ \frac{|x_1|^2}{2} d_3^* d_1^* - \frac{|x_1|^2}{2} d_3^* d_1^* + \frac{|x_1|^2}{2} d_2^* d_3^* - \frac{|x_1|^2}{2} d_2^* d_3^* =$$

$$= (1 + \frac{|x_2|^2}{|x_1|^2})(d_3^* d_3 - d_3 d_3^*) - \frac{|x_2|^2}{|x_1|^2}.$$
showing that $d_3^3d_3 - d_3^3d_3^3 = 1$. □

Denote by $A_3$ the $*$-algebra generated by CCR with 3 degrees of freedom and denote by $c_1, c_2, c_3$ the canonical generators of $A_3$. Construct elements $b_1, b_2, B$ of $A_3$ using formulas (15).

**Lemma 8.** The elements $b_1, b_2, B$ satisfy (11) and generate $A_3$.

**Proof.** It is evident that one can express $c_1, i = 1, 2, 3$ via $b_1, b_2, B$ using (12) with $b_1, b_2, B$ instead of $a_1, a_2, A$. So $A_3$ is generated by $b_1, b_2, b_3$.

Let us show that $b_1, b_2, B$ satisfy (11).

Indeed, it is a moment of reflection to see that $b_1^*b_2 = b_2b_1^*$ and $b_1^*b_1 - b_1b_1^* = 1$. Further

$$b_2b_1 - b_1b_2 = |x_1|c_1^*c_2c_1 + \frac{x_1}{2}(c_1^*)^2c_1 - |x_1|c_1^*c_1c_2 - \frac{x_1}{2}c_1(c_1^*)^2 =$$

$$= |x_1|(c_1^*c_1 - c_1c_1^*)c_2 + \frac{x_1}{2}((c_1^*)^2c_1 - c_1(c_1^*)^2) =$$

$$= |x_1|c_2 + \frac{x_1}{2}2c_1^* = |x_1|c_2 + x_1c_1^* = B,$$

and

$$Bb_1 = |x_1|c_2c_1 + x_1c_1^*c_1 = |x_1|c_2c_1 + x_1(1 + c_1c_1^*) =$$

$$= |x_1|c_1c_2 + x_1c_1c_1^* + x_1 = c_1(|x_1|c_2 + x_1c_1^*) + x_1 = b_1B + x_1,$$

$$Bb_2 - b_2B = -\frac{x_2}{|x_1|}|x_1|c_2c_2^* + \frac{x_2}{|x_1|x_1}x_1c_2c_2 = x_2(c_2^*c_2 - c_2c_2^*) = x_2.$$

Thus it remains to check that $b_1^*b_2 - b_2b_1^* = 1$. But in fact this was done in Lemma (7) when we checked that the relation $d_3^3d_3 - d_3^3d_3^3 = 1$ is satisfied. □

Using Lemma (7) and Lemma (8) it is easy to see that the $*$-algebras $A_{x_1,x_2}$ and $A_3$ are isomorphic.

**Proposition 10.** The $*$-algebra $A_{x_1,x_2}$ is isomorphic to the $*$-algebra $A_3$.

**Proof.** Let $\phi: A_{x_1,x_2} \rightarrow A_3$ be a homomorphism defined by

$$\phi(a_i) = b_i, \ i = 1, 2, \ \phi(A) = B,$$

where $b_1, b_2$ and $B$ are the generators constructed in Lemma (8). Similarly define $\psi: A_3 \rightarrow A_{x_1,x_2}$ by

$$\psi(c_i) = d_i, \ i = 1, 2, 3,$$

where $d_i$ are taken from Lemma (7).

Then $\psi \circ \phi = \text{id}_{A_{x_1,x_2}}$ and $\phi \circ \psi = \text{id}_{A_3}$. □
Therefore in order to study irreducible representations of \( A_{x_1,x_2} \) we can work with the generators \( d_1, d_2, d_3 \). As for the case of representations annihilating \( K_3 \), we say that a representation of \( A_{x_1,x_2} \) with \( x_1 \neq 0 \) is well-behaved if the corresponding representation of \( A_3 \cong A_{x_1,x_2} \) is well-behaved. Then from the uniqueness of irreducible well-behaved \(*\)-representation of CCR with finite degrees of freedom we get that the space of representation is \( \mathcal{H} = l_2(\mathbb{Z}_+)^3 \) and

\[
d_1 = a \otimes 1 \otimes 1, \quad d_2 = 1 \otimes a \otimes 1, \quad d_3 = 1 \otimes 1 \otimes a.
\]

Returning to the generators \( a_1, a_2, a_3 \) using (12) we get the following result.

**Theorem 2.** For any \((x_1, x_2) \in \mathbb{C}^2 \) with \( x_1 \neq 0 \) there exists a unique, up to a unitary equivalence, well-behaved irreducible representation of \( A_{x_1,x_2} \) defined on the generators by the following formulas

\[
a_1 = a \otimes 1 \otimes 1,
\]

\[
a_2 = \sqrt{1 + \frac{|x_2|^2}{|x_1|^2}} \: 1 \otimes 1 \otimes a - \frac{x_2}{|x_1|} a^* \otimes 1 \otimes 1 + \frac{1}{2} |x_1|^2 a^2 \otimes 1 \otimes 1 + |x_1| a^* \otimes a \otimes 1 + \frac{x_1}{2} (a^*)^2 \otimes 1 \otimes 1,
\]

\[
A = |x_1| 1 \otimes a \otimes 1 + x_1 a^* \otimes 1 \otimes 1.
\]

**5.2.2 Representations with \( x_1 = 0 \)**

Let \( x_1 = 0 \) and \( x_2 \neq 0 \). As in the previous case we have \( A_{0,x_2} \cong A_3 \). To see this we express the generators \( a_1, a_2, a_3 \) via the generators \( d_1, d_2 \) and \( d_3 \) of CCR using formulas (13) with \( a_2, -a_1 \) instead of \( a_1, a_2 \) respectively, exchanging \( x_1 \) with \( x_2 \) and letting then \( x_1 = 0 \). For this we observe that \((-a_1)a_2 - a_2(-a_1) = A \), and \( Aa_2 - a_2A = x_2 \). Hence we get the following result.

**Theorem 3.** For any \( x_2 \in \mathbb{C} \), \( x_2 \neq 0 \), there exists a unique, up to a unitary equivalence, irreducible well-behaved \(*\)-representation of \( A_{0,x_2} \), defined by the following formulas

\[
a_2 = a \otimes 1 \otimes 1,
\]

\[
a_1 = -(1 \otimes 1 \otimes a + \frac{x_2}{2} 1 \otimes a^2 \otimes 1 + |x_2| a^* \otimes a \otimes 1 + \frac{x_2}{2} (a^*)^2 \otimes 1 \otimes 1)
\]

\[
A = |x_2| 1 \otimes a \otimes 1 + x_2 a^* \otimes 1 \otimes 1
\]

If both \( x_1 = 0 \) and \( x_2 = 0 \), then \( Aa_i = a_iA_i \), \( i = 1, 2 \) and hence the cubic ideal \( K_3 \) is annihilated. In this case irreducible well-behaved representations are described in Theorem 1.
5.3 Concluding remarks

Note that our result shows in particular that in the case of $A_2^0$ the ideal $K_4$ does not coincide with $I_4$, the largest homogeneous ideal of degree 4. Indeed, as noted above $K_2 \otimes K_2 \subset I_4$. So, if a representation $\pi$ annihilates $I_4$, then $\pi(A^2) = 0$. Since $A$ is a normal element we immediately have $\pi(A) = 0$. However the representation that we constructed above has the property that $\pi(K_4) = \{0\}$ but $\pi(A) \neq 0$. Thus $K_4 \neq I_4$.

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