Vortex Solutions in Two-Higgs Systems
and \(\tan \beta\)

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Classical vortex solutions in various two-Higgs systems are studied. The systems we consider include the standard model with two Higgs doublets, in which case the vortex appears as part of a string-like object. The Higgs potentials contain several different couplings in general and the spontaneous symmetry breaking involves with two different vacuum expectation values. In particular it is shown that the existence of such a solution in general requires a specific ratio of the two Higgs vacuum expectation values, i.e. \(\tan \beta\), and some inequalities between different Higgs couplings. This ratio can be determined in terms of the couplings in the Higgs potential. The Higgs masses are also computed in this case. (1+2)-d solutions are topological so that they are topologically stable and the Bogomol’nyi bound is saturated for some couplings. Some comments on the stabilization of (1+3)-d solutions are also given. Thus, as long as such a defect can be formed in the early universe, stable or not, \(\tan \beta\) is no longer an independent free parameter in the theory.

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1. Introduction

The idea of the spontaneous symmetry breaking is one of the most important building blocks of the electroweak theory, which is one of the greatest achievements in theoretical particle physics. Although there is no doubt about the validity of the electroweak theory, the final experimental proof however is not yet completed because the wanted Higgs particle is still at large. And also there is some possibility that the detail of such spontaneous symmetry breaking may be a bit different from the minimal content originally proposed. There is growing anticipation, although indirect, the electroweak symmetry breaking may be induced by two Higgs doublets, rather than one.

For example, the recent measurements of the gauge couplings\(^1\) have led us to anticipation that the minimal supersymmetric Grand Unified Theories (GUTs)\(^2\) or supergravity GUTs\(^3\)\(^4\) with the supersymmetry scale of order 1 TeV or below may be a phenomenologically plausible unified theory of strong and electroweak interactions\(^5\). These supersymmetric GUTs in general require at least two Higgs multiplets for the electroweak symmetry breaking\(^4\)\(^6\)\(^7\)\(^8\).

Another indication is that the current observed top quark mass bound \((m_t > 91\text{GeV})\)\(^9\) has already exceeded the upper bound required by the Coleman-Weinberg mechanism of the single Higgs case to radiatively induce the electroweak symmetry breaking\(^3\), which requires \(m_t < 78\text{GeV}\)\(^8\). Although it is rather sufficient to have tree level breaking for particle physics purpose, it will be difficult to have cosmological implications without radiative breaking. However in the two-doublet case this upper bound can become sufficiently high due to the contribution of other scalar field masses\(^10\), so that we can avoid such a trouble.

In exchange of these positive points, having one more Higgs doublet will introduce further complication to the theory. Needless to say, first, we have to deal with more observable massive scalar particles. Theoretically, it also introduces more free parameters. To spontaneously break the \(SU(2) \times U(1)_Y\) symmetry down to the \(U(1)_{em}\) each Higgs gets its own vacuum expectation value (VEV), say \(v_1, v_2\). These VEVs are phenomenologically important but unfortunately they are not determined theoretically except in some no-scale models\(^11\). The geometric sum \(v^2/2 = v_1^2 + v_2^2\) can be determined in terms of the mass of the gauge boson, where \(v\) denotes the electroweak symmetry breaking scale. This however leaves the ratio of the two VEVs, \(\tan \beta \equiv v_2/v_1\), still undetermined.

If the two-doublet model would turn out to explain the electroweak symmetry break-
ing, eventually future experiments will determine \( \tan \beta \). That will however still leave us a question why that ratio should be different from others. This is fairly a common situation. We always wonder if nature selects out some particular property among many possible choices. Then curiosity drives us to look for some explanation. Pursuing such a question, we are often led to a new phenomenon in physics. Thus it is very important to look for any argument to constrain the ratio rather theoretically, if possible.

With such a motivation in mind, in this paper we shall attempt to find any relation to constrain \( \tan \beta \) in two-Higgs systems. The result is indeed positive and we find that there is a simple formula to express \( \tan \beta \) in terms of the couplings of the Higgs potential, so far as nature admits certain vacuum defects during the electroweak phase transition. Preliminary results were presented in [12].

To demonstrate how it works we shall first work on a simple (1+2)-dimensional toy model with \( U(1) \) gauge symmetry. Vortex solution in this model takes the role of a necessary vacuum defect. The generic structure however persists in the two-Higgs-doublet standard model, in which (cosmic) string-like solution takes the role.

The interest in the application of vortex was initiated in the study of the Ginzburg-Landau theory of the superconductivity [13]. Subsequently, in the relativistic field theory context, known as the Abelian-Higgs model, the vortex lines are interpreted as string-like objects [14]. Due to the difficulty of solving the nonlinear field equations any exact vortex solutions are not yet known in a general case. However, asymptotic solutions can be easily found. It is also known that if the gauge coupling constant and the coupling in the Higgs potential satisfy a special relation, one can obtain an exact solution [15]. In the Abelian-Higgs model one can naively expect that these vortices are stable because of the nontrivial topological configuration, i.e. \( \pi_1(U(1)) \neq 0 \). But more careful analysis tells us that this is true only if the above two coupling constants satisfy a certain inequality [16]. If not, solutions in the higher winding sector than one are unstable.

Such a structure persists even in more realistic models in particle physics. It was pointed out that the standard model also admits such a vortex solution (more precisely, string-like solution which forms a vortex on a plane perpendicular to the string.) [17] [18] [19].

In this case, there is no obvious topological configuration because the hypercharge \( U(1)_Y \) is not spontaneously broken, but rather \( SU(2) \times U(1)_Y \) is broken to \( U(1)_{em} \). In ref. [17], however, a solution of a pair of \( SU(2) \) magnetic monopoles connected by a string is derived. (a similar solution can also be obtained in our case. See the section 4.) It was also
pointed out that in the standard model case these monopoles can stabilize this system. One can also derive string-like solutions in the standard model with one Higgs doublet without attaching to monopoles\cite{12}. These are unstable solutions except for \(\sin\theta_W = 1\), where \(\theta_W\) is the Weinberg angle\cite{1}.

In this paper we shall extend the results to the cases with more Higgs scalars. If there are two Higgs multiplets to start with, we call it a two-Higgs system. As soon as we introduce more scalar fields, the potential to induce spontaneous symmetry breaking becomes more complicated. In particular there are more couplings including the interactions between different scalar fields. The structure of the phase transition itself may become more involved. For example, in a simpler case without gauge couplings and only two scalar couplings there are already three different critical points\cite{21}. If the two VEVs are very much different, each scalar can get its VEV one by one\cite{2}. In the two-doublet model case, if so, we can naively anticipate that the cosmic phase transition would occur in two steps. Thus perhaps a full investigation of the structure of the fixed points of two-Higgs potential may be necessary.

This paper is organized as follows: In sect.2 \((1 + 2)\)-dimensional \(U(1)\) gauge models are considered. One with \(U(1) \times U(1)\) global symmetry and another without any larger accidental symmetry. Then it is shown that these vortices are topologically stable and in some case the Bogomol’nyi bound is saturated. In sect.3 the two-Higgs-doublet standard model is investigated. In sect.4 it is also speculated the possibility of stabilization of the electroweak Z-string by attaching monopoles. In sect.5 the essence of this paper is summarized and further issues are discussed. Finally, in the appendix A we show that the assumption \(c_1/c_2 = v_1/v_2\) we made in previous sections is reasonable.

2. \((1+2)\)-d Models

As simplified models we shall first consider local \(U(1)\) gauge theories with two Higgs singlet scalars. In some sense these models can be viewed as a generalization of the Abelian Higgs model in the Euclidean two-dimension, but for our purpose which requires spontaneous symmetry breaking, we consider them rather as \((1 + 2)\)-dimensional systems. How-

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1 It has been recently argued that an extra scalar field can stabilize the Z-string if it forms a bound state\cite{20}.

2 The \(v_2 \gg v_1\) case was studied in ref.\cite{23} by integrating out the heavy scalar modes.
ever solving classical field equations are completely equivalent because we are interested in static solutions.

The Higgs potential we use in fact is motivated by the general two-Higgs potential that induces $SU(2) \times U(1)_Y \rightarrow U(1)_{em}$ symmetry breaking, which we shall use in the next section and is usually written in terms of two Higgs doublets\cite{7}\cite{8}. In terms of singlet Higgs scalars the potential takes the following form:

$$V(\phi_1, \phi_2) = \frac{\lambda_1}{4}(|\phi_1|^2 - v_1^2)^2 + \frac{\lambda_2}{4}(|\phi_2|^2 - v_2^2)^2 + \frac{\lambda_3}{4}(|\phi_1|^2 + |\phi_2|^2 - v^2)^2 + \frac{\lambda_5}{2} |\phi_1 |\phi_2 - v_1 v_2|^2,$$

(2.1)

where $v^2 = v_1^2 + v_2^2$. The significance of the $\lambda_5$-term resides in the symmetry structure of the Higgs potential. Thus we consider the two cases for $\lambda_5 = 0$ and $\lambda_5 \neq 0$ separately.

Note that if $\lambda_1 = \lambda_2 = 0$, this Higgs system has an accidental $SU(2)$ global symmetry if we require that $(\phi_1, \phi_2)$ form an $SU(2)$ doublet. This pattern of symmetry breaking, $SU(2)_{\text{global}} \times U(1)_{\text{local}} \rightarrow U(1)_{\text{global}}$, is known to lead to semilocal topological defects\cite{23}. If $\lambda_3 = 0$, this becomes simply a decoupled two-scalar system with global $U(1) \times U(1)$ symmetry. There is a vortex solution trivially generalized from the result in ref.\cite{14}.

In this paper we shall stick to the general case that $\lambda_3 \neq 0$ and at least one of $\lambda_1$ or $\lambda_2$ is not zero. Then we shall find that this system reveals a rather interesting result, which cannot be obtained otherwise. The key observation is that the spontaneous symmetry breaking of Eq.(2.1) leads to a vortex solution, whose existence will introduce an extra condition on the Higgs VEVs. Then we can determine them completely, which are related only by $v^2 = v_1^2 + v_2^2$ otherwise.

Case I: $\lambda_5 = 0$

This two-Higgs system has not only the local $U(1)$ symmetry but also (accidental) global $U(1)_1 \times U(1)_2$ symmetry respectively for $\phi_1$ and $\phi_2$.\footnote{Such a phenomenon that the potential has a larger global symmetry than the gauge symmetry was called accidental symmetry first by Weinberg in \cite{24}} If $\lambda_5$ did not vanish, there would not be such a global symmetry. In this case this model can also be seen from a different viewpoint. One can start from a two-scalar field theory with global $U(1) \times U(1)$ symmetry with a given scalar potential. Then gauge a $U(1)$ subgroup of the global symmetry. Such a viewpoint is common in dynamical symmetry breaking models.
When the two scalar fields get VEVs, not only the local $U(1)$ symmetry is spontaneously broken but also the global $U(1) \times U(1)$ symmetry is broken. There are two Goldstone bosons due to the global symmetry breaking. One will be absorbed to make the gauge boson massive by the Higgs mechanism of the local $U(1)$ breaking. But, as a result, there is one Goldstone boson left over. Since the $\lambda_5$-term explicitly breaks the global symmetry, we can expect that with $\lambda_5$-term it will become a pseudo-Goldstone boson whose mass is proportional to $\lambda_5$. This will be considered as Case II.

Let us now consider the Lagrangian density in (1+2)-dimensional space-time

$$\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} |D_\mu \phi_1|^2 + \frac{1}{2} |D_\mu \phi_2|^2 - V(\phi_1, \phi_2),$$  \hspace{1cm} (2.2)$$

where $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu - ieA_\mu$. For minimum energy solutions we usually solve the minimum energy condition, but in this case it is not so easy to obtain the necessary Bogomol’nyi bound. Thus we shall deal with the field equations, then look for static finite energy solutions.

The equations of motion can be obtained from the Lagrangian density as

$$D^\mu D_\mu \phi_1 + (\lambda_1 + \lambda_3)(|\phi_1|^2 - v_1^2)\phi_1 + \lambda_3(|\phi_2|^2 - v_2^2) \phi_1 = 0, \hspace{1cm} (2.3a)$$
$$D^\mu D_\mu \phi_2 + (\lambda_2 + \lambda_3)(|\phi_2|^2 - v_2^2)\phi_2 + \lambda_3(|\phi_1|^2 - v_1^2) \phi_2 = 0, \hspace{1cm} (2.3b)$$
$$\partial_\mu F_{\mu \nu} = J_\nu \equiv J_{1 \nu} + J_{2 \nu}, \hspace{1cm} (2.3c)$$
$$J_{i \nu} = -\frac{1}{2} ie(\phi_i^* \partial_\nu \phi_i - \phi_i \partial_\nu \phi_i^*) - e^2 A_\nu |\phi_i|^2, \hspace{0.5cm} i = 1, 2.$$  

For time-independent solutions we choose $A_0 = 0$ gauge, then the system effectively reduces to a two-dimensional one. In this case since we are interested in vortex solutions in $\mathbb{R}^2$, it is convenient to represent them in the polar coordinates $(r, \theta)$ such as

$$\phi_1 = e^{im\theta} f(r), \hspace{0.5cm} \phi_2 = e^{in\theta} g(r), \hspace{0.5cm} A = \hat{e}_\theta \frac{1}{r} A(r),$$  \hspace{1cm} (2.4)$$

where $m, n$ are integers identifying each winding sector. To become desired finite-energy defects located at $r = 0$ these should satisfy the following boundary conditions:

$$f(0) = 0, \hspace{0.5cm} g(0) = 0, \hspace{0.5cm} A(0) = 0, \hspace{1cm} (2.5)$$
$$f \to v_1, \hspace{0.5cm} g \to v_2, \hspace{0.5cm} A \to \text{const. as } r \to \infty.$$  

The constant for the asymptotic value of $A$ will be determined properly later.
In the polar coordinates the equations of motion Eq. (2.3a–d) can be rewritten as

\[
\begin{align*}
-\frac{1}{r} \partial_r (r \partial_r f) + \frac{1}{r^2} f(m - eA)^2 + (\lambda_1 + \lambda_3)(f^2 - v_1^2)f + \lambda_3(g^2 - v_2^2)f &= 0, \\
-\frac{1}{r} \partial_r (r \partial_r g) + \frac{1}{r^2} g(n - eA)^2 + (\lambda_2 + \lambda_3)(g^2 - v_2^2)g + \lambda_3(f^2 - v_1^2)g &= 0, \\
-\partial_r^2 A + \frac{1}{r} \partial_r A - e [(m - eA)f^2 + (n - eA)g^2] &= 0.
\end{align*}
\]

(2.6a, b, c)

In general it will be a formidable task to solve these equations exactly, but it is good enough to find approximate solutions for large \( r \) to show the existence of the vortex solutions. Imposing the boundary conditions at large \( r \), Eqs. (2.6a, b, c) become consistent only if \( m = n \) and that it fixes the asymptotic value \( A \to \frac{n}{e} \) as \( r \to \infty \). This implies that there is no vortex solution of different winding numbers for different Higgs fields. This is in fact an anticipated result because the vortex solution we are interested in is due to the spontaneous symmetry breaking of the local \( U(1) \). With this condition of winding numbers we can solve Eq. (2.6d) for large \( r \) to obtain

\[
A \to \frac{n}{e} - n \sqrt{\frac{\pi v}{2e}} \sqrt{r} e^{-r/\lambda} + \cdots,
\]

(2.7)

where \( \lambda = 1/ev \) is the characteristic length of the gauge field. Note that the characteristic length defines the region over which the field becomes significantly different from the value at the location of the defect.

Now let us determine the characteristic lengths for \( \phi_1 \) and \( \phi_2 \) again for large \( r \) as follows. For simplicity we consider \( n = 1 \) case, but the result does not really depend on \( n \). Asymptotically we look for solutions of the form

\[
f - v_1 \sim c_1 e^{-r/\xi_1}, \quad g - v_2 \sim c_2 e^{-r/\xi_2},
\]

(2.8)

where the constant coefficients \( c_1 \) and \( c_2 \) are in principle calculable. The dimensionless part of the exact values of \( c_1 \) and \( c_1 \) should be at most related to the dimensionless parameters of the system or may be pure numerical constants. Thus the essential part of the following argument would not be much changed even though we had exact solutions. Also for our purpose only the ratio is relevant. Therefore here we shall assume these constants as \( c_1 = -v_1 \) and \( c_2 = -v_2 \), which is in a good approximation based on the numerical study in the Abelian Higgs model [25]. Later when we derive the Bogomol’nyi bound, we shall also confirm that this is reasonable. (Also see the appendix A.)
Then from Eq. (2.6a, b) with Eq. (2.8), in the leading order we obtain

\[ 0 = v_1 e^{-r/\xi_1} \left[ \frac{1}{\xi_1} + 2(\lambda_1 + \lambda_3) v_1^2 \right] + 2\lambda_3 v_1 v_2^2 e^{-r/\xi_2} + \cdots, \]  
(2.9a)

\[ 0 = v_2 e^{-r/\xi_2} \left[ \frac{1}{\xi_2} + 2(\lambda_2 + \lambda_3) v_2^2 \right] + 2\lambda_3 v_1 v_2 e^{-r/\xi_1} + \cdots, \]  
(2.9b)

where the ellipses include terms which vanish more rapidly as \( r \to \infty \).

If \( \xi_1 \neq \xi_2 \), then \( \lambda_3 \) should vanish. Therefore to have any nontrivial solution for \( \lambda_3 \neq 0 \) we are forced to identify

\[ \xi \equiv \xi_1 = \xi_2, \]

and that we get the desired result by demanding the vanishing coefficient of \( e^{-r/\xi} \) in Eqs. (2.9a, b) as

\[ \tan \beta \equiv \frac{v_2}{v_1} = \sqrt{\frac{\lambda_1}{\lambda_2}}, \quad \lambda_3 \neq 0. \]  
(2.10)

Thus we have determined the ratio of the two Higgs VEVs in terms of the couplings in the Higgs potential. This tells us that although different Higgs field gets different VEVs, their characteristic lengths should be the same to form a single defect. Both Higgs should reach the true vacuum at the same distance. To do that the two VEVs should satisfy a proper relation, which is Eq. (2.10).

Furthermore, together with \( v \), we can completely determine the VEVs as

\[ v_1 = v \cos \beta = v \sqrt{\frac{\lambda_2}{\lambda_1 + \lambda_2}}, \quad v_2 = v \sin \beta = v \sqrt{\frac{\lambda_1}{\lambda_1 + \lambda_2}}. \]  
(2.11)

The characteristic lengths \( \xi_1, \xi_2 \) now read

\[ \xi \equiv \xi_1 = \xi_2 = \frac{1}{\sqrt{2}v} \sqrt{\frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1}}. \]  
(2.12)

Note that, although \( \tan \beta \) does not depend on \( \lambda_3 \), it is crucial to have nonvanishing \( \lambda_3 \) coupling to obtain such a result. The gauge boson mass is \( M_A = 1/\lambda = ev \) after spontaneous symmetry breaking. In the next section when a similar structure is applied to a realistic model, this gauge field in fact can be identified with a massive neutral gauge boson, e.g. \( Z^0 \).

In this case one can compute the two Higgs masses to obtain

\[ m_1 = 2v^2 \frac{\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1}{\lambda_1 + \lambda_2}, \quad m_2 = 2v^2 \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}. \]  
(2.13)
Note that although the characteristic lengths are the same, the masses of the mass eigenstates of the Higgs fields are different. The difference of the two masses is $2\lambda_3 v^2$.

The role of the Goldstone boson in this case perhaps need more investigation, which we shall leave for future study.

**Case II: $\lambda_5 \neq 0$**

The $\lambda_5$-term explicitly breaks the global $U(1) \times U(1)$ symmetry, so that now the Higgs potential has the same symmetry as local $U(1)$ symmetry. Thus there should not be any Goldstone boson at all. In fact one can easily see there is another massive scalar field whose mass is proportional to $\lambda_5$, signalling that it becomes a Goldstone boson at $\lambda_5 = 0$ limit. This is a pseudo-Goldstone boson.

The $\lambda_5$-term in Eq.(2.11) only modifies Eqs.(2.3a,b). Eq.(2.3a) picks up $\lambda_5 (\phi_2^+ \phi_1 - v_1 v_2) \phi_2$ as Eq.(2.3b) does $\lambda_5 (\phi_1^+ \phi_2 - v_1 v_2) \phi_1$. Thus for $\lambda_5 \neq 0$ we obtain the asymptotic equations similar to Eqs.(2.9a,b) as

$0 = v_1 e^{-r/\xi_1} \left[ -\frac{1}{\xi_1^2} + 2(\lambda_1 + \lambda_3) v_1^2 \right]$

$+ 2\lambda_3 v_1 v_2^2 e^{-r/\xi_2} + \lambda_5 v_1^2 v_2^2 \left( e^{-r/\xi_1} + e^{-r/\xi_2} \right) + \cdots$, \hspace{1cm} (2.14a)

$0 = v_2 e^{-r/\xi_2} \left[ -\frac{1}{\xi_2^2} + 2(\lambda_2 + \lambda_3) v_2^2 \right]$

$+ 2\lambda_3 v_1^2 v_2 e^{-r/\xi_1} + \lambda_5 v_1^2 v_2^2 \left( e^{-r/\xi_1} + e^{-r/\xi_2} \right) + \cdots$. \hspace{1cm} (2.14b)

Since $2\lambda_3 + \lambda_5 \neq 0$ (recall that we assumed all the couplings are nonnegative to obtain the minimum at $v_1$ and $v_2$), again we have to identify the two characteristic lengths $\xi \equiv \xi_1 = \xi_2$ to have nontrivial solutions. Thus the generic structure does not really depend on the detail of the Higgs potential so far as $\lambda_3 > 0$. However, there is one more condition on $\lambda_5$ we have to impose. From Eqs.(2.14a,b) for $\xi_1 = \xi_2 \equiv \xi$ we obtain

$$(\lambda_1 - \lambda_5) v_1^2 = (\lambda_2 - \lambda_5) v_2^2. \hspace{1cm} (2.15)$$

Both sides must have the same sign to be consistent. This implies that if $\lambda_5$ is either bigger than $\lambda_1, \lambda_2$ or smaller than them, which can be denoted by an inequality

$$(\lambda_5 - \lambda_1)(\lambda_5 - \lambda_2) > 0. \hspace{1cm} (2.16)$$

From the symmetry’s point of view it is more natural to assume the latter case so that the $\lambda_5$-term would break the global symmetry rather softly. This will also lead to a light pseudo-Goldstone boson.
The above condition Eq. (2.15) is again nothing but the constraint on the ratio of the two VEVs such that
\[ \tan \beta = \sqrt{\frac{\lambda_1 - \lambda_5}{\lambda_2 - \lambda_5}}. \] (2.17)
Again it does not depend on \( \lambda_3 \) as before. We can also compute the VEVs and the Higgs mass in this case, but the results are equivalent to the doublet case, which will be discussed in the next section.

**Remarks on the stability**

Let us first check if the solutions we obtained in this section are topological. This can be understood by investigating the topology of the vacuum manifold. In the Abelian Higgs model, where there is no accidental symmetry, it is simple to identify the vacuum manifold by just looking at the manifold of the equivalent vacuum states of the scalar field. Then each topological sector can be identified by measuring the \( U(1) \) flux around the vortex.

In our case if \( \lambda_5 \neq 0 \) we can also adopt the same philosophy because there is no accidental symmetry. If we represent \( \phi_1 = |\phi_1|e^{i\theta_1} \) and \( \phi_2 = |\phi_2|e^{i\theta_2} \), then the \( \lambda_5 \)-term’s vacuum condition \( \phi_1^\dagger \phi_2 - v_1 v_2 = 0 \) forces that \( \theta_1 = \theta_2 \). Thus the vacuum manifold is still topologically \( S^1 \). For vortex solutions \( \pi_1(S^1) = \mathbb{Z} \) so that these are topological solutions.

If \( \lambda_5 = 0 \), the topology of the vacuum manifold is a torus, \( S^1 \times S^1 \). One may think that since \( \pi_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z} \), the solutions in this case should be also topological. Without the local \( U(1) \) symmetry this could be true. However, due to the local symmetry the detail is different. Recall that Eqs. (2.6a, b) are consistent only if the two winding numbers are the same. Thus the topology relevant to the vortex solutions is in fact the diagonal \( S^1 \) of \( S^1 \times S^1 \). Although the solutions are topological, but the topology is not completely dictated by the Higgs potential. They are dictated by the topology of the local symmetry, \( \pi_1(U(1)) = \mathbb{Z} \). This is why the two winding numbers must be the same.

Anyhow, since these solutions are topological, they are topologically stable. However as usual it does not seem to be easy to derive the Bogomol’nyi bound for the arbitrary couplings. However if the couplings satisfy some relation, we can saturate the Bogomol’nyi bound.

This can be understood more easily if we use the condition \( \xi_1 = \xi_2 \), which can be rewritten as
\[ \frac{\phi}{v} \equiv \frac{\phi_1}{v_1} = \frac{\phi_2}{v_2}. \]
Although we have only obtained this condition asymptotically, we shall see this indeed
leads to vortex solution. We shall also find out that it is a necessary condition to have monopole solution in sect.4. Thus we believe this will survive as a condition for exact vortex solutions.

We will only check the Case I explicitly, but in all the other cases it can be done straightforwardly. With the above condition as we can easily see, the field equations Eqs. (2.3a – b) reduce to the field equations of a single Higgs with the VEV \( v = \sqrt{v_1^2 + v_2^2} \) and the Higgs potential

\[
V(\phi) = \frac{1}{4} \lambda (|\phi|^2 - v^2),
\]

where

\[
\lambda \equiv \frac{\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1}{\lambda_1 + \lambda_2}.
\]

In the single Higgs case, as pointed out in [10], if \( \beta_B = 2\lambda/e^2 = 1 \), it saturates the Bogomol’nyi bound. Thus we have the critical point of the two-Higgs case as

\[
\beta_B = \frac{2}{e^2} \frac{\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1}{\lambda_1 + \lambda_2} = 1.
\] (2.18)

Thus these are stable solutions except for \( n \geq 2 \) if \( \beta_B > 1 \).

This completes the proof of the existence of vortex solutions in the two-Higgs system with the potential Eq.(2.1). The consistency condition of such existence led us to be able to determine the ratio of the two Higgs VEVs, which otherwise are not completely determined. Much work is needed to find exact vortex solutions, but at this moment we still get nontrivial physical implication with approximate solutions.

Note that such a vortex solution in the \((1 + 2)\)-dimensional space is nothing but the cylindrically symmetric string-like solution in the \((1 + 3)\)-dimensional space-time. Thus we can expect that similar structure should exist in \((1 + 3)\)-dimensional models. This will be investigated in the next section.

3. Two-Higgs-Doublet Standard Model

One of the most mysterious parts of the electroweak theory lies in the Higgs sector. Higgs was introduced to achieve the electroweak symmetry breaking without spoiling the consistency of the theory. In this section we shall investigate the structure of a string-like defect (so-called “Z-string”), which can be formed during the electroweak phase transition,
in the two-Higgs-doublet standard model, expecting a similar structure will appear as in the $U(1)$ case considered in the previous section.

Note that the topology of the local $SU(2) \times U(1)/U(1)$ is the same as that of $SU(2)$, so $\pi_1(SU(2) \times U(1)/U(1)) = 0$. As is well known, the string-like solitonic solutions we get would not be topological. This is different from the $U(1)$ case. However it is known that there are nontopological solitons in the standard model [17] [18] [19].

We shall use the CP invariant two-doublet Higgs potential that induces $SU(2) \times U(1)_Y \rightarrow U(1)_{em}$ symmetry breaking [1] [5]:

$$V(\phi_1, \phi_2) = \frac{1}{2} \lambda_1 (|\phi_1|^2 - v_1^2)^2 + \frac{1}{2} \lambda_2 (|\phi_2|^2 - v_2^2)^2 + \frac{1}{2} \lambda_3 (|\phi_1|^2 + |\phi_2|^2 - v_1^2 - v_2^2)^2$$

$$+ \left( |\phi_1|^2 |\phi_2|^2 - |\phi_1^\dagger \phi_2|^2 \right) + \lambda_5 \left| \phi_1^\dagger \phi_2 - v_1 v_2 \right|^2,$$

(3.1)

where $\phi_1$, $\phi_2$ are $SU(2)$ doublets. In this section we shall stick to the general case that $\lambda_i \neq 0$ for $i = 1, 2, 3$ and also assume that all $\lambda_j$, $j = 1, \ldots, 5$ are nonnegative. This potential shows $\phi_1, \phi_2 \leftrightarrow -\phi_1, -\phi_2$ discrete symmetry, which is necessary to suppress the flavor changing neutral current. Then we shall find that this system reveals a rather interesting result, which cannot be obtained otherwise.

Note that if $\lambda_4 = 0 = \lambda_5$, the Higgs potential has a global $U(2) \times U(2)$ symmetry. The symmetry breaking will lead to global $U(1) \times U(1)$ unbroken so that there will be two Goldstone bosons left over. If only $\lambda_5$ is vanishing, due to $|\phi_1^\dagger \phi_2|^2$ in the $\lambda_4$-term the global symmetry now becomes $U(2) \times U(1) \times U(1)$. There still is one Goldstone boson after symmetry breaking to global $U(1) \times U(1)$.

There however are good reasons to keep $\lambda_5 \neq 0$. First, we don’t want Goldstone bosons which may not be welcomed phenomenologically. Secondly, though not directly related, in the supersymmetric models $\lambda_5$ is related to the supersymmetry breaking parameter so that $\lambda_5 \neq 0$ to have the supersymmetry broken. Note that for $\lambda_5 \neq 0$ there is a global $U(2)$ symmetry, which is locally isomorphic to $SU(2) \times U(1)$.

Let us consider the bosonic sector of the standard model described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} \text{tr} G_{\mu\nu} G^{\mu\nu} - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + |D_\mu \phi_1|^2 + |D_\mu \phi_2|^2 - V(\phi_1, \phi_2),$$

(3.2)

where $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, $G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g e^{abc} W_\mu^b W_\nu^c$, and $D_\mu = \partial_\mu - ig Y/2 B_\mu - ig^{abc} W_\mu^a$. Both Higgs’ have hypercharge $Y = 1$.

Then the equations of motion for the scalar fields are

$$0 = D^\mu D_\mu \phi_1 + \lambda_1 (|\phi_1|^2 - v_1^2) \phi_1 + \lambda_3 (|\phi_1|^2 + |\phi_2|^2 - v_1^2 - v_2^2) \phi_1$$
\[0 = D^\mu D_\mu \phi_2 + \lambda_2 \left( |\phi_2|^2 - v_2^2 \right) \phi_2 + \lambda_3 \left( |\phi_1|^2 + |\phi_2|^2 - v_1^2 - v_2^2 \right) \phi_2 + \lambda_4 \left( |\phi_1|^2 \phi_2 - (\phi_1^\dagger \phi_2^+) \phi_1 \right) + \lambda_5 (\phi_1^\dagger \phi_2 - v_1 v_2) \phi_1, \quad (3.3a)\]

and for the gauge fields we have

\[- \partial^\mu F_{\mu \nu} = j_\nu \equiv J_{1\nu} + J_{2\nu}, \quad (3.4a)\]

\[j_\nu \equiv \frac{i}{2} g' \left[ \phi_1^\dagger \partial_\nu \phi_1 - (\partial_\nu \phi_1)^\dagger \phi_1 \right] + \frac{1}{2} g'^2 B_\nu |\phi_1|^2 + \frac{1}{2} g' g W_{\nu}^2 \phi_1^\dagger \tau^a \phi_1, \quad i = 1, 2, \]

\[- \partial^\mu G_{\mu \nu} - g e^{abc} W^{b\mu} G_{\mu \nu} = J_{1\nu}^a \equiv J_{1\nu}^a + J_{2\nu}^a, \quad (3.4b)\]

\[J_{1\nu}^a \equiv \frac{i}{2} g \left[ \phi_1^\dagger \tau^a \partial_\nu \phi_1 - (\partial_\nu \phi_1)^\dagger \tau^a \phi_1 \right] + \frac{1}{2} g g' B_\nu \phi_1^\dagger \tau^a \phi_1 + \frac{1}{2} g^2 W_{\nu}^2 \phi_1^\dagger \tau^a \tau^b \phi_1, \quad i = 1, 2, \]

For time-independent solutions we choose \( B_0 = 0 = W_0^a \) gauge and impose the cylindrical symmetry around the string, then the system effectively reduces to a two-dimensional one. In this case the string solutions in the \((1 + 3)\)-dimensional spacetime correspond to the vortex solutions in \( \mathbb{R}^2 \). When Higgs gets VEV, the false vacuum region forms vacuum defects. As usual, we redefine the neutral gauge fields as

\[A_\mu = \cos \theta_W B_\mu + \sin \theta_W W_3^\mu, \quad Z_\mu = \sin \theta_W B_\mu - \cos \theta_W W_3^\mu, \quad (3.5)\]

where \( \theta_W \) is the Weinberg angle defined by \( \tan \theta_W = g'/g \). We shall also use \( \tilde{g} \equiv \frac{1}{2} \sqrt{g^2 + g'^2} \) for convenience.

For vortex solutions it is convenient to represent them in the polar coordinates \((r, \theta)\) such as

\[\phi_1 = \begin{pmatrix} 0 \\ e^{im \theta} f_1(r) \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 \\ e^{in \theta} f_2(r) \end{pmatrix}, \quad \tilde{Z} = \tilde{e}_\theta \frac{1}{r} Z(r), \quad (3.6)\]

where \( m, n \) are integers identifying each “winding” sector (we shall come back to this point later again.). Here we are mainly interested in the case of \( W_1^1 = 0 = W_2^1 \), but we expect there are other solutions similar to the case of ref.\[13\].

Then Eqs.\((3.4a, 4)\) become

\[0 = - \frac{1}{r} \partial_r (r \partial_r B_\theta) + \frac{1}{r^2} B_\theta \]

\[- \frac{g'}{r} \left[ (m - \frac{1}{2} (g' B_\theta - g W_3^3)) f_1^2 + (n - \frac{1}{2} (g' B_\theta - g W_3^3)) f_2^2 \right], \quad (3.7a)\]

\[0 = - \frac{1}{r} \partial_r (r \partial_r W_3^3) + \frac{1}{r^2} W_3^3 \]

\[+ \frac{g}{r} \left[ (m - \frac{1}{2} (g' B_\theta - g W_3^3)) f_1^2 + (n - \frac{1}{2} (g' B_\theta - g W_3^3)) f_2^2 \right]. \quad (3.7b)\]
As we can easily see, $A_\mu$ satisfies a trivial equation so that we can set $A_\mu = 0$. Thus from the rest of the equations of motion we obtain

$$0 = -\frac{1}{r} \partial_r (r \partial_r f_1) + \frac{1}{r^2} f_1 (m - \tilde{g}Z)^2 + (\lambda_1 + \lambda_3) (f_1^2 - v_1^2) f_1 + \lambda_5 \left( f_1 f_2 - v_1 v_2 e^{i(n-m)\theta} \right) f_2, \tag{3.8a}$$

$$0 = -\frac{1}{r} \partial_r (r \partial_r f_2) + \frac{1}{r^2} f_2 (n - \tilde{g}Z)^2 + (\lambda_2 + \lambda_3) (f_2^2 - v_2^2) f_2 + \lambda_3 \left( f_1 f_2 - v_1 v_2 e^{i(m-n)\theta} \right) f_1, \tag{3.8b}$$

$$0 = -\partial_r^2 Z + \frac{1}{r} Z - 2\tilde{g} \left[ (m - \tilde{g}Z) f_1^2 + (n - \tilde{g}Z) f_2^2 \right]. \tag{3.8c}$$

Note that $\lambda_4$ coupling does not take part in this structure classically.

To become desired finite-energy defects located at $r = 0$ the solutions we are looking for should satisfy the following boundary conditions:

$$f_1(0) = 0, \quad f_2(0) = 0, \quad Z(0) = 0,$$

$$f_1 \to v_1, \quad f_2 \to v_2, \quad Z \to \text{const. as } r \to \infty. \tag{3.9}$$

The constant for the asymptotic value of $Z$ will be determined properly later.

Again we shall look for asymptotic solutions. Imposing the boundary conditions at large $r$, Eqs.(3.8a, b) become consistent only if $m = n$ and that it fixes the asymptotic value $Z \to n/\tilde{g}$ as $r \to \infty$. This implies that there is no vortex solution of different “winding” numbers for different Higgs fields. With this condition of winding numbers we can solve Eq.(3.8c) for large $r$ to obtain

$$Z \to \frac{n}{\tilde{g}} - n \sqrt{\frac{\pi v}{2\tilde{g}}} \sqrt{r e^{-r/\lambda}} + \cdots, \tag{3.10}$$

where $\lambda = 1/\tilde{g}v$ is the characteristic length of the gauge field. Note that the characteristic length defines the region over which the field becomes significantly different from the value at the location of the defect.

The asymptotic solutions for $\phi_1$ and $\phi_2$ can be found as follows: For simplicity we consider $n = 1$ case, but the result does not really depend on $n$. Besides, since these are nontopological, it is not really necessary to consider other $n$ sector. Asymptotically we look for solutions of the form

$$f_1 - v_1 \sim c_1 e^{-r/\xi_1}, \quad f_2 - v_2 \sim c_2 e^{-r/\xi_2}, \tag{3.11}$$
where \( \lambda_1 \) and \( \lambda_2 \) are the characteristic lengths of \( \phi_1 \) and \( \phi_2 \) respectively and the constant coefficients \( c_1 \) and \( c_2 \) are in principle calculable, thus they are not free parameters. Note that we can normalize any dimensionless constants in \( c_i \) to be the same. Furthermore, for our purpose only the ratio is relevant. Therefore these constants can be taken as \( c_1 = -v_1 \) and \( c_2 = -v_2 \) in a good approximation. Even though the exact results differed from these, the essential argument of the following is much the same. Then in the leading order we obtain

\[
\begin{align*}
v_1 e^{-r/\xi_1} & \left[ -\frac{1}{\xi_1^2} + 2(\lambda_1 + \lambda_3)v_1^2 + \lambda_5v_2^2 \right] + (2\lambda_3 + \lambda_5)v_1 v_2^2 e^{-r/\xi_2} + \cdots = 0, \quad (3.12a) \\
v_2 e^{-r/\xi_2} & \left[ -\frac{1}{\xi_2^2} + 2(\lambda_2 + \lambda_3)v_2^2 + \lambda_5v_1^2 \right] + (2\lambda_3 + \lambda_5)v_1^2 v_2 e^{-r/\xi_1} + \cdots = 0, \quad (3.12b)
\end{align*}
\]

where the ellipses include terms which vanish more rapidly as \( r \to \infty \).

Recall that \( \lambda_3 > 0 \) and \( \lambda_5 \geq 0 \) so that \( 2\lambda_3 + \lambda_5 \neq 0 \). Thus to have any vortex solution we are forced to identify the two characteristic lengths of the scalar fields such that

\[ \xi = \xi_1 = \xi_2. \]

We shall find in the next section that to attach monopoles at each side of this string we should require that \( \phi_1/v_1 = \phi_2/v_2 \). Thus this is also a necessary condition to stabilize by attaching monopoles.

To be consistent, as in the \( U(1) \) case we also obtain an inequality

\[ (\lambda_5 - \lambda_1)(\lambda_5 - \lambda_2) > 0. \quad (3.13) \]

However in the two-doublet standard model case \( \lambda_5 \) does not need to be small to be natural. Then we get the desired result by demanding the vanishing coefficients of \( e^{-r/\xi} \) in Eqs.\((3.12a,b)\) as

\[ \tan \beta \equiv \frac{v_2}{v_1} = \sqrt{\frac{\lambda_1 - \lambda_5}{\lambda_2 - \lambda_5}}, \quad \lambda_3 \neq 0 \text{ or } \lambda_5 \neq 0. \quad (3.14) \]

Thus we have determined the ratio of the two Higgs VEVs in terms of the couplings in the Higgs potential. This tells us that although different Higgs field gets different VEVs, their characteristic lengths should be the same to form a single defect. Both Higgs should reach the true vacuum at the same distance. To do that the two VEVs should satisfy a proper relation, which is Eq.\((3.14)\).
Furthermore, together with \( v \), we can completely determine the VEVs as

\[
v_1 = \frac{v}{\sqrt{2}} \cos \beta = \frac{v}{\sqrt{2}} \sqrt{\frac{\lambda_2 - \lambda_5}{\lambda_1 + \lambda_2 - 2\lambda_5}}, \quad v_2 = \frac{v}{\sqrt{2}} \sin \beta = \frac{v}{\sqrt{2}} \sqrt{\frac{\lambda_1 - \lambda_5}{\lambda_1 + \lambda_2 - 2\lambda_5}}. \tag{3.15}
\]

The characteristic lengths \( \xi_1, \xi_2 \), now satisfy

\[
\xi \equiv \xi_1 = \xi_2 = \frac{1}{v} \sqrt{\frac{\lambda_1 + \lambda_2 - 2\lambda_5}{\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 - \lambda_5(2\lambda_3 + \lambda_5)}}. \tag{3.16}
\]

Note that, although \( \tan \beta \) does not depend on \( \lambda_3 \), it is crucial to have nonvanishing \( \lambda_3 \) or \( \lambda_5 \) coupling to obtain such a result. The gauge boson mass is \( M_Z = 1/\lambda = \tilde{g}v \) after spontaneous symmetry breaking.

In this two-Higgs-doublet model there are five physical Higgs bosons: \( H^\pm, A^0, H^0, h^0 \). \( A^0 \) is a CP-odd neutral scalar, while \( H^0, h^0 \) are CP-even scalars. \( h^0 \) denotes the lightest Higgs. Using Eq. (3.15), we can compute the masses of all these physical Higgs bosons in terms of the couplings in the Higgs potential and \( v \), where \( v = 247 \text{ GeV} \). \( \lambda_5 \) is related to \( M_{A^0} \) and \( M_{H^0, h^0} \) can be determined in terms of \( \lambda_1, \lambda_2, \lambda_3, \lambda_5 \) and \( v \). Thus we only have five free parameters, if such an electroweak Z-string exists.

In particular the neutral Higgs masses become impressively simple:

\[
m_{H^0} = v^2 \left[ \lambda_3 + \frac{\lambda_1\lambda_2 - \lambda_5^2}{\lambda_1 + \lambda_2 - 2\lambda_5} \right],
\]

\[
m_{h^0} = v^2 \frac{2\lambda_1\lambda_2 - \lambda_5(\lambda_1 + \lambda_2)}{2(\lambda_1 + \lambda_2 - 2\lambda_5)}. \tag{3.17}
\]

Note that \( m_{h^0} \) does not depend on \( \lambda_3 \).

The appearance of integers in the solutions, which we still call “winding” number, is rather intriguing because there is no explicit \( U(1) \) symmetry to be broken which should determine the necessary topological sector. If our vortex solutions are nontopological as in ref. [20], there should not be such a parameter. This however can be explained as follows: If we regard \( W_1^\mu = 0 = W_2^\mu \) as gauge fixing conditions, then effectively we can view the symmetry of the system as \( U(1) \times U(1)_Y \). When we twist this symmetry to obtain \( U(1)_{em} \), the remaining twisted \( U(1)_{\tilde{g}} \) is spontaneously broken to lead to the winding sector.

Since \( SU(2) \) is a simple group, \( U(1) \times U(1)_Y \) is not an invariant subgroup of \( SU(2) \times U(1)_Y \). Furthermore, \( \pi_1 (SU(2) \times U(1)_Y / U(1)_{em}) = 0 \) implies that these winding sectors would not provide any topological stability. In other words, they must be gauge equivalent to \( n = 1 \) solution via deformable gauge transformation.
Even for \( n = 1 \) solution it is most likely that this solution would not saturate the Bogomol’nyi bound. Although it obviously is a finite energy solution, it does not seem to be a classically stable solution. It is argued that there is a case of quantum stabilization of a classically unstable solution\[27\]. It however cannot be applied in this case unless one of \( \lambda_4 \) or \( \lambda_5 \) vanishes because there is no tree level Goldstone boson. If \( \lambda_5 = 0 \), then there is one Goldstone boson. But the argument still does not apply because \( \pi_1(G_{\text{gauge}}/H_{\text{gauge}}) \) is still trivial.

Nevertheless, there is one more possibility to stabilize such an electroweak \( Z \)-string, which will be presented in the next section.

4. Monopole-String-Antimonopole

In the one-doublet standard model it is claimed that such a string solution can be stabilized by attaching monopoles at each end and keeping them sufficiently far apart\[17\]. The rationale behind the stabilization is that these monopoles are genuine \( SU(2) \) monopoles so that they can be topologically stable. In other words, the string can be constructed as a string connecting two Wu-Yang type monopoles.

In fact we can apply the same argument here. As we shall see soon, the two-doublet model also admits such a monopole solution. The consistency condition for the existence is again \( \phi_1/v_1 = \phi_2/v_2 \), which is an equivalent condition to \( \xi_1 = \xi_2 \) in the string case. Thus it seems to us that this condition is not just accidental but may have more profound significance in the spontaneous symmetry breaking of the two-Higgs systems.

In the static case the energy density of the system can be easily derived from the Lagrangian density Eq.(3.2). Then the minimum energy conditions are

\[
D_\mu \phi_1 = 0, \quad D_\mu \phi_2 = 0, \quad (4.1)
\]

and

\[
V(\phi_1, \phi_2) = 0. \quad (4.2)
\]

Eq.(1.2) implies that

\[
|\phi_1|^2 = v_1^2, \quad |\phi_2|^2 = v_2^2, \quad \phi_1^\dagger \phi_2 = v_1v_2, \quad (4.3)
\]

where the third condition does not follow from the two other conditions.

From Eq.(1.1) by multiplying \( \phi_i^\dagger \tau^a \) and subtracting its hermitean conjugate we obtain

\[
g v_i^2 W^a_\mu + g' B_\mu (\phi_i^\dagger \tau^a \phi_i) = -i \left( \phi_i^\dagger \tau^a \partial_\mu \phi_i - \partial_\mu \phi_i^\dagger \tau^a \phi_i \right), \quad i = 1, 2. \quad (4.4)
\]
These two simultaneous equations need to be consistent to have solutions. Comparing them, we are required to identify
\[
\frac{\phi_1}{v_1} = \frac{\phi_2}{v_2}. \tag{4.5}
\]
Thus the two equations become identical. Then using the Nambu’s method we can solve Eq. (4.4) to obtain \( W^a_\mu \) and \( B_\mu \) in terms of \( \phi_1 \) and \( \phi_2 \).

For a monopole solution with a singularity along the negative \( z \)-direction, the Higgs fields would be
\[
\phi_i = v_i \left( \frac{\cos \frac{1}{2} \theta}{\sin \frac{1}{2} \theta e^{i \varphi}} \right), \quad r \neq 0, \quad 1 = 1, 2, \tag{4.6}
\]
where the singularity is identified by the ill-defined phase at \( \theta = \pi \). Thus we have
\[
\phi_i^\dagger \tau^a \phi_i = v_i^2 x^a \frac{x}{r}. \tag{4.7}
\]
Since we expect that \( B_\mu \) behaves like a monopole with a Dirac string, we use
\[
g' B_a = -\epsilon^{abc} \frac{x^c}{r(r+z)}, \tag{4.8}
\]
where we have identified the spatial indices and the gauge group indices as \( a, b, ... = 1, 2, 3 \) and \( x^1 = x, \ x^2 = y, \ x^3 = z \). In the spherical coordinates we can easily see that only \( \varphi \)-component is nonvanishing. \( B_0 = 0 \) by the gauge choice.

Then plugging into Eq. (4.4) we can solve for \( W^a_\mu \) to get
\[
g W^b_a = -\epsilon^{abc} \frac{x^c}{r^2}, \tag{4.9}
\]
which is nothing but the \( SU(2) \) monopole solution\[28\]. Note that \( \phi_i^\dagger \tau^a \phi_i \) takes the role of the adjoint Higgs field in the usual non-Abelian Higgs model. Since \( W^b_a \) denotes a genuine \( SU(2) \) monopole, the string carries only \( U(1) \) returning flux but does not carry \( SU(2) \) returning flux.

Now we can patch a pair of monopole and antimonopole, which leads to
\[
\phi_i = v_i \left( \frac{\cos \frac{1}{2} \Theta}{\sin \frac{1}{2} \Theta e^{i \varphi}} \right), \tag{4.10}
\]
where \( \cos \Theta = \cos \theta_1 - \cos \theta_2 + 1 \) and \( \theta_1 (\theta_2) \) is measured from the position of the monopole (antimonopole). If the distance between the monopole and the antimonopole is sufficiently far apart, the solution we derived can be used for each side so that this can be viewed as a monopole-string-antimonopole system. One can easily obtain the relevant solutions by generalizing Eqs. (4.8)(4.9) in this case. Thus by the same reasoning as in \[17\], this would be stable.
5. Discussion

We have shown that two-Higgs systems in general admit vortex solutions, which requires a specific ratio of the two VEVs. This can be understood more easily if we use the condition

\[ \hat{\phi} \equiv \frac{\phi_1}{v_1} = \frac{\phi_2}{v_2}. \]

Recall that this condition not only shows up as a consistency condition of the vortex solutions but also shows up as that of the monopole solution considered in sect.4.

Then as we can easily see, the field equations reduce to the field equations of a single Higgs with the VEV \( v \) and the Higgs potential \( V(\phi) = \lambda(|\phi|^2 - v^2/2)^2 \), where \( \phi = \sqrt{2} \hat{\phi} \) (in the 2-d case replace \( v^2/2 \) by \( v^2 \)) and

\[ \lambda \equiv \frac{\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 - \lambda_5 (2\lambda_3 + \lambda_5)}{\lambda_1 + \lambda_2 - 2\lambda_5}. \]

All such results can be obtained if

\[ \tan \beta \equiv \frac{v_2}{v_1} = \sqrt{\frac{\lambda_1 - \lambda_5}{\lambda_2 - \lambda_5}}, \quad \lambda_3 \neq 0. \]

These results also apply to the cases of \( \lambda_5 = 0 \).

Although there are characteristic differences, in fact we have observed that so far as vortex solution is concerned, such a structure is quite generic for both cases: the (1+2)-dimensional \( U(1) \) system and the (1+3)-dimensional \( SU(2) \times U(1) \). Two big characteristic differences are that the former is topological and saturates Bogomol’nyi bound (in some case), whilst the latter is nontopological but can be stabilized by attaching monopoles.

In this paper we have dealt with the tree level potential. It is usually believed that tree level solution reappears in loop-corrected effective potential, although there might be quantitative differences. This is in fact relevant to study the property of vacuum defects in the context of the cosmological phase transition.

For the unstable solutions we need further investigation to find out what kind of cosmological trace they could leave. But one certain thing is that as soon as we find out how nature selects out \( \tan \beta \) (if nature prefers the two-doublet model), we shall understand its mystery if \( \tan \beta \) turns out to meet our claim. The ultimate proof of the existence of such vacuum defects should be determined by experiments or by observations.

It will be also important and interesting to find out if there is any other reason nature prefers a specific \( \tan \beta \) theoretically. This will be still a question to be answered after we determine experimentally. In this sense, we hope this work can provide a clue to future investigation.
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Appendix A.

In this appendix we shall show explicitly why \( \frac{c_1}{c_2} = \frac{v_1}{v_2} \) is a reasonable assumption in the case I of the section 1. In other cases it can also be generalized straightforwardly so that we leave them as simple exercises. The main rationale behind our assumption is that these constants are not usually determined by solving nonlinear differential equations asymptotically so that one should use them as input parameters. Of course they will be determined by solving equations exactly, however we are not yet able to solve the equations in question exactly. As we shall see, these will be ill-defined if we try to determine otherwise and the assumption we made is the only reasonable consistent choice.

In eq. (2.8) we can conveniently redefine \( c_i = \tilde{c}_i v_i \), where \( \tilde{c}_i \) are dimensionless. Also let \( \tilde{c} \equiv \frac{\tilde{c}_1}{\tilde{c}_2} \), then the condition for \( \xi_1 = \xi_2 \) reads as

\[
\tan \beta = \sqrt{\frac{\lambda_1 + \lambda_3 - \tilde{c} \lambda_3}{\lambda_2 + \lambda_3 - \frac{1}{\tilde{c}} \lambda_3}}. \tag{A.1}
\]

Suppose \( \tilde{c} \) is not an input parameter but to be determined by this equation for any \( \tan \beta \) to nullify our claim that this equation relates \( \tan \beta \) and other couplings, then one should be able to determine \( \tilde{c} \) for given \( \tan \beta \). In this case eq. (A.1) becomes a quadratic equation for \( \tilde{c} \) as

\[
\lambda_3 \tilde{c}^2 + (\tan^2 \beta (\lambda_2 + \lambda_3) - (\lambda_1 + \lambda_3)) \tilde{c} - \lambda_3 \tan^2 \beta = 0 \tag{A.2}
\]

and one can obtain

\[
\tilde{c}_\pm = \frac{1}{2 \lambda_3} \left[ -\tan^2 \beta (\lambda_2 + \lambda_3) + (\lambda_1 + \lambda_3) \pm \sqrt{(\tan^2 \beta (\lambda_2 + \lambda_3) - (\lambda_1 + \lambda_3))^2 + 4 \lambda_3^2 \tan^2 \beta} \right]. \tag{A.3}
\]

First, note that \( \tilde{c}_- < 0 \) so that one of the Higgs fields approaches to the true vacuum from the wrong direction. Furthermore, \( \tilde{c}_- \to 0 \) for any \( \lambda \)’s as \( \tan \beta \to 0 \), so \( \tilde{c}_- \) leads to ill-defined coefficients. Thus we are left with \( \tilde{c}_+ > 0 \).

Second, if \( \lambda_3 \to 0 \), then any \( \tilde{c} \) is ill-defined unless the coefficient of the linear term in eq. (A.2) vanishes. This vanishing condition is nothing but eq. (2.10) as \( \lambda_3 \to 0 \). Otherwise,
\( \tilde{c} \) either increases indefinitely or approaches to 0 indefinitely. This is unreasonable because for vortex solutions with the same characteristic length we should expect \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are of the similar order for any \( \lambda_3 \). Without loss of generality, say, \( \tilde{c}_1 \ll \tilde{c}_2 \sim 1 \), then the notion of characteristic length in the asymptotic formula for \( \phi_1 \) field we assumed fails to make sense.

Therefore, eq. \((A.2)\) cannot be solved for plausible \( \tilde{c} \) consistently for \( \tan \beta \) as an input parameter. This leads us to a conclusion that we should treat \( \tilde{c} \) as an input parameter and the assumption we make in the previous sections is reasonable. Thus \( \tilde{c} = 1 \), which leads to eq. \((2.10)\) consistently, is a well-defined reasonable assumption to present our argument.
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