Abstract. The Batalin-Vilkovisky master equations, both classical and quantum, are precisely the integrability equations for deformations of algebras and differential algebras respectively. This is not a coincidence; the Batalin-Vilkovisky approach is here translated into the language of deformation theory.

The following exposition is based in large part on work by Marc Henneaux (Bruxelles) especially and with Glenn Barnich (Penn State and Bruxelles) and Tom Lada and Ron Fulp of NCSU (The Non-Commutative State University). The first statement of the relevance of deformation theory to the construction of interactive Lagrangians, that I am aware of, is due to Barnich and Henneaux [3]:

We point out that this problem can be economically reformulated as a deformation problem in the sense of deformation theory [13], namely that of deforming consistently the master equation.

The ‘ghosts’ introduced by Fade’ev and Popov [12] were soon incorporated into the BRST-cohomology approach [7] to a variety of problems in mathematical physics. There they were reinterpreted by Stora [25] and others in terms of the Maurer-Cartan forms in the case of a finite dimensional Lie group and more generally as generators of the Chevalley-Eilenberg complex [10] for Lie algebra cohomology. This led eventually to the Batalin-Vilkovisky approach [4, 5, 6] to quantizing particle Lagrangians and then to string field theory, both classical and quantum [30]. With hindsight, the Batalin-Vilkovisky machinery can be recognized as that of homological algebra [17]. The ‘quantum’ Batalin-Vilkovisky master equation has the form of the Maurer-Cartan equation for a flat connection, while the ‘classical’ version has the form of the integrability equation of deformation theory. In the context of the present conference, my goal is to show that these are more than analogies;

1 Research supported in part by NSF grant DMS-9504871.
the master equations are indeed the integrability equations of the deformation theory of, respectively, differential graded commutative algebras and graded commutative algebras.

First I will review the jet bundle approach to Lagrangian field theory. Here we could already encounter cohomological physics in the form of the variational bicomplex of differential forms on the jet bundle, but I will omit that today and proceed instead to the anti-field, anti-bracket formalism, the rubric under which physicists reinvented homological algebra. Here the ‘standard construction’ is the Batalin-Vilkovisky complex.

Just as the Maurer-Cartan equation makes sense in the context of Lie algebra cohomology, so the Batalin-Vilkovisky master equation has an interpretation in terms of strong homotopy Lie algebras ($L_\infty$-algebras), as I explain next. After a brief recollection of deformation theory for differential graded algebras, I will look at various physical examples where free Lagrangians are deformed to interactive Lagrangians. Particularly interesting examples are provided by Zwiebach’s closed string field theory [30] and higher spin particles [8].

1. The jet bundle setting for Lagrangian field theory

Let us begin with a space $\Phi$ of fields regarded as the space of sections of some bundle $\pi : E \to M$. For expository and coordinate computational purposes, I will assume $E$ is a trivial vector bundle and will write a typical field as $\phi = (\phi^1, \ldots, \phi^k) : M \to \mathbb{R}^k$. In terms of local coordinates, we start with a trivial vector bundle $E = F \times M \to M$ with base manifold $M$, locally $\mathbb{R}^n$, with coordinates $x^i, i = 1, \ldots, n$ and fibre $\mathbb{R}^k$ with coordinates $u^a, a = 1, \ldots, k$.

We ‘prolong’ this bundle to create the associated jet bundle $J = J^\infty E \to E \to M$ which is an infinite dimensional vector bundle with coordinates $u^a_I$ where $I = i_1 \ldots i_r$ is a symmetric multi-index (including, for $r = 0$, the empty set of indices, meaning just $u^a$). The notation is chosen to bring to mind the mixed partial derivatives of order $r$. Indeed, a section of $J$ is the (infinite) jet $j^\infty \phi$ of a section $\phi$ of $E$ if, for all $r$, we have $\partial_{i_1} \partial_{i_2} \ldots \partial_{i_r} \phi^a = u^a_I \circ j^\infty \phi$ where $\phi^a = u^a \circ \phi$ and $\partial_i = \partial/\partial x^i$.

**Definition 1.1.** A local function $L(x, u^{(p)})$ is a smooth function in the coordinates $x^i$ and the coordinates $u^a_I$, where the order $|I| = r$ of the multi-index $I$ is less than or equal to some integer $p$.

Thus a local function is in fact the pullback of a smooth function on some finite jet bundle $J^p E$, i.e. a composite $J \to J^p E \to \mathbb{R}$.
The space of local functions will be denoted $C(J)$.

**Definition 1.2.** A local functional

\begin{equation}
L[\phi] = \int_M L(x, \phi^p(x))dvol_M = \int_M (j^\infty \phi)^* L(x, u^{(p)})dvol_M
\end{equation}

is the integral over $M$ of a local function evaluated for sections $\phi$ of $E$. (Of course, we must restrict $M$ and $\phi$ or both for this to make sense.)

The variational approach is to seek the critical points of such a local functional. More precisely, we seek sections $\phi$ such that $\delta L[\phi] = 0$ where $\delta$ denotes the variational derivative corresponding to an ‘infinitesimal’ variation: $\phi \mapsto \phi + \delta \phi$. The condition $\delta L[\phi] = 0$ is equivalent to the Euler-Lagrange equations on the corresponding local function $L$ as follows: Let

\begin{equation}
D_i = \frac{\partial}{\partial x^i} + u^a_{Ii} \frac{\partial}{\partial u^a_I}
\end{equation}

be the total derivative acting on local functions and

\begin{equation}
E_a = (-D)_I \frac{\partial}{\partial u^a_I}
\end{equation}

the Euler-Lagrange derivatives. The notation $(-D)_I$ means $(-1)^r D_i \cdots D_{ir}$. The Euler-Lagrange equations are then

\begin{equation}
E_a(L) = 0.
\end{equation}

Two local functionals $L$ and $K$ are equivalent if and only if the Euler-Lagrange derivatives of their integrands agree, $E_a(L) = E_a(K)$, for all $a = 1, \ldots, k$. The kernel of the Euler-Lagrange derivatives is given by total divergences,

\begin{equation}
E_a(L) = 0, \text{ for all } a = 1, \ldots, k \iff L = (-1)^k D_i j^i
\end{equation}

for some local functions $j^i$. Equivalently,

\begin{equation}
E_a(L) = 0, \text{ for all } a \iff L(x, u^{(p)})dvol_M = (-1)^{i-1}d j^i dvol_M / dx^i
\end{equation}

where if $dvol_M = dx^1 \cdots dx^n$, then $dvol_M / dx^i = dx^1 \cdots dx^{i-1} dx^{i+1} \cdots dx^n$.

Since $L$ is the integral of an $n$-form on $J$, it is not surprising that this all makes sense in the deRham complex $\Omega^*(J)$, which remarkably splits as a bicomplex (though the finite level complexes $\Omega^*(J^pE)$ do not). The appropriate 1-forms in the fibre directions are not the $du^a_I$ but rather the **contact forms** $\theta^a_I = du^a_I - u^a_{Ii} dx^i$. The total differential $d$ splits as
\[ d = d_H + d_V \] where \( d_H = dx^i \partial / \partial x^i \) is the usual exterior differential on \( M \) pulled up to \( J \).

We will henceforth restrict the coefficients of our forms to be local functions, although we will not decorate \( \Omega^*(J) \) to show this.

The Euler-Lagrange operators assemble into an operator on forms:

\[ E(Ldvol_M) = E_a(L) \theta^a dvol_M. \]  (1.7)

A Lagrangian \( L \) determines a **stationary surface** or **solution surface** or shell \( \Sigma \subset J^\infty \) such that \( \phi \) is a solution of the variational problem (equivalently, the Euler-Lagrange equations) if \( j^\infty \phi \) has its image in \( \Sigma \). The corresponding algebra is the **stationary ideal** \( I \) of local functions which vanish ‘on shell’, i.e. when restricted to the solution surface \( \Sigma \).

The Euler-Lagrange equations generate \( I \) as a differential ideal, but this means we may have not only **Noether identities**

\[ r_\alpha^a E_a(L) = 0 \]  (1.8)

but also

\[ r_\alpha^{al} D_I E_a(L) = 0. \]  (1.9)

Of course we have ‘trivial’ identities of the form

\[ D_j E_b(L) \mu_\alpha^{bjal} D_I E_a(L) = 0, \]  (1.10)

since we are dealing with a commutative algebra of functions. One can show [17] that all Noether identities, where the coefficients \( r_\alpha^a \) vanish on shell, are of the above form. We now assume we have a set of indices \{\( \alpha \)\} such that the above identities generate all the non-trivial relations in \( I \). According to Noether [22], each such identity corresponds to an **infinitesimal gauge symmetry**, i.e. an infinitesimal variation that preserves the space of solutions or, equivalently, a vector field tangent to \( \Sigma \). For each Noether identity indexed by \( \alpha \), we denote the corresponding vector field by \( \delta_\alpha \). We denote by \( \Xi \), the **space of gauge symmetries**, considered as a vector space but also as a module over \( \mathcal{C}(J) \). We can regard \( \delta_\alpha \) as a (constant) vector field on the space of fields \( \Phi \) and hence \( \delta \) as a linear map

\[ \delta : \Xi \to Vect\Phi. \]

Since the bracket of two such vector fields \([\delta_\alpha, \delta_\beta]\) is again a gauge symmetry, it agrees with something in the image of \( \delta \) when acting on solutions. If we denote that something
as \([\alpha, \beta]\), one says this bracket ‘closes on shell’. It is not in general a Lie bracket, since the Jacobi identity may hold only ‘on shell’. (Later I will address the issue that \([\delta_\alpha, \delta_\beta]\) may not be constant on \(\Phi\).)

To make this more explicit, write

\[
[\delta_\alpha, \delta_\beta] = \delta_{[\alpha, \beta]} + \nu^a_{\alpha\beta} \frac{\delta L}{\delta u^a}.
\]

(1.11)

The possible failure of the Jacobi identity results from those last terms which vanish only on shell and the fact that we are working in a module over \(\mathcal{C}(J)\). (For example, we have structure functions rather than structure constants in terms of our generators.)

All of this, including these latter subtleties, are incorporated into a remarkable complex by Batalin-Vilkovisky, which we shall describe below.

2. Deformation theory

A deformation theoretic approach to producing interactive Lagrangians is to start with a ‘free’ Lagrangian \(L_0\) on a space of fields \(\Phi\) with an abelian Lie algebra \(\Xi\) represented as gauge symmetries by a Lie map

\[
\delta_0 : \Xi \rightarrow \text{Vect}\Phi,
\]

and then study formal deformations

\[
L = \sum t^i L_i, \quad \delta = \sum t^i \delta_i,
\]

keeping \(\Phi\) and \(\Xi\) fixed as vector spaces, such that \(\delta_\xi \int L \, \text{dvol}_M = 0\) for all \(\xi \in \Xi\).

Classical deformation theory emphasizes the following two problems:

1. Consider a candidate \textit{infinitesimal} \(L_1\) and see if there exists a full formal deformation \(L\) with corresponding \(\delta\). Such candidate terms \(L_1\) are often suggested by physical descriptions of elementary interactions.

2. Classify all formal deformations up to appropriate equivalence.

A third subsequent problem, that of convergence of the power series involved, is a problem in analysis; cohomological techniques apply to the formal algebraic theory - I don’t do estimates.

The cohomological approach to deformation theory, as initiated by Gerstenhaber [13], situates the problem in an appropriate complex, the Hochschild cochain complex \(C^* (A, A)\) in the case of an associative algebra \(A\). For the Lagrangian problem, the complex is due to
Batalin and Vilkovisky [4, 5, 6] using anti-field and ghost technology and the anti-bracket of Zinn-Justin [29].

3. Anti-fields, (anti-)ghosts and the anti-bracket

Let me take you ‘through the looking glass’ and present a ‘bi-lingual’ (math and physics) dictionary.

From here on, we will talk in terms of algebra extensions of $C^\infty(E)$ and $C(J)$, but the extensions will all be free graded commutative. We could instead talk in terms of an extension of $E$ or $J$ as a super-manifold, the new generators being thought of as (super)-coordinates.

We first extend $C^\infty(E)$ by adjoining generators of various degrees to form a free graded commutative algebra $A_1$ over $C^\infty(E)$, that is, even graded generators give rise to a polynomial algebra and odd graded generators give rise to a Grassmann (= exterior) algebra. The generators (and their products) are, in fact, bigraded $(p,q)$; the graded commutativity is with respect to the total degree $p - q$.

For each variable $u^a$, adjoin an anti-field $u^*_a$ and for each $r_\alpha$, adjoin a corresponding ghost $C^\alpha$ and a corresponding anti-ghost $C^{*\alpha}$. Here is a table showing the corresponding math terms and the bidegrees.

| Physics Term | Math Term | Ghost Degree | Anti-ghost Degree | Total Degree |
|--------------|-----------|--------------|-------------------|--------------|
| field        | section   | 0            | 0                 | 0            |
| anti-field   | Koszul generator | 0      | 1                 | -1           |
| ghost        | Cartan-Eilenberg generator | 1      | 0                 | 1            |
| anti-ghost   | Tate generator  | 0            | 2                 | -2           |

Note that the anti-field coordinates depend on $E$ but the ghosts and anti-ghosts depend also on the specific Lagrangian.

This algebra in turn can be given an anti-bracket $( , )$ of degree $-1$ which, remarkably, combines with the product we began with to produce precisely an ‘up-to-homotopy’ analog
of a Gerstenhaber algebra [21, 18], though this was not recognized until quite recently.

**Definition 3.1.** A **Gerstenhaber algebra** is a graded commutative and associative algebra $A$ together with a bracket $[,] : A \otimes A \to A$ of degree $-1$, such that for all homogeneous elements $x, y,$ and $z$ in $A$,

$$[x, y] := -(-1)^{(|x|-1)(|y|-1)}[y, x],$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{(|x|-1)(|y|-1)}[y, [x, z]],$$

and

$$[x, yz] = [x, y]z + (-1)^{(|x|-1)|y|}y[x, z].$$

In the field-anti-field formalism (without jet coordinates), the anti-bracket looks like

$$(\phi_0^a(x), \phi_\beta^b(y)) = \delta_\beta^a \delta(x - y)$$

where $\delta(x - y)$ is the Dirac delta ‘function’ (distribution). In the corresponding jet bundle formalism, the anti-bracket is defined on generators and then extended to polynomials by applying the graded Leibniz identity so that $(\psi, )$ is a graded derivation for any $\psi$ in this algebra. The only non-zero anti-brackets of generators are

$$(u^a, u_b^*) = \delta_b^a \quad \text{and} \quad (C_\alpha^a, C^*_\beta) = \delta_\alpha^\beta.$$ 

Now we further extend $C(J)$ with corresponding jet coordinates $u_a^I, C_\alpha^I$ and $C^*_\alpha I$ with the corresponding pairings giving the extended anti-bracket. The resulting Batalin-Vilkovisky algebra we denote $\mathcal{A}$.

Define an operator $s_0$ of degree $-1$ on $\mathcal{A}$ as $(L, )$.

We call the antifields Koszul generators because

$$(u^a, u_a^*) = \delta_b^a \quad \text{and} \quad (C_\alpha^a, C^*_\beta) = \delta_\alpha^\beta.$$ 

Now consider the extended Lagrangian

$$(3.4) \quad L_1 = L_0 + u_a^* r^a_\alpha C^\alpha$$
and \( s_1 = (L_1, \ ) \), so that

\[(3.5) \quad s_1 C_a^\alpha = u_a r_a^\alpha \]

and \( H^{0,1} \) is now 0, as in Tate’s extension of the Koszul complex of the ideal to produce a resolution [27]. That is why we refer to the anti-ghosts as Tate generators. (If needed, Tate tells us to add further generators in bidegree \( (0, q) \) for \( q > 2 \) so that \( H^{0,q} = 0 \) for \( q > 0 \).

Further extend \( L_1 \) to

\[(3.6) \quad L_2 = L_1 + C_\alpha^\alpha C_\beta^\beta C_\gamma^\gamma, \]

so that

\[
\begin{align*}
    s_2 C_\alpha^\alpha &= c_\beta^\alpha C_\beta^\beta C_\gamma^\gamma \\
    s_2 u_a &= r_a^\alpha C_\alpha^\alpha
\end{align*}
\]

which is how the Chevalley-Eilenberg coboundary looks in terms of bases for a Lie algebra and a module and corresponding structure constants. However, we may not have \( (s_2)^2 = 0 \) since \( r_a^\alpha \) and \( c_\beta^\alpha \) are functions. Batalin and Vilkovisky prove that all is not lost. First, they add to \( L_2 \) a term involving the functions \( r_a^\alpha \).

**Theorem 3.1.** \( L_2 \) can be further extended by terms of higher degree in the anti-ghosts to \( L_\infty \) so that \( (L_\infty, L_\infty) = 0 \) and hence the corresponding \( s_\infty \) will have square zero.

With hindsight, we can see that the existence of these terms of higher order is guaranteed because the antifields and antighosts provide a resolution of the stationary ideal.

We refer to this complex \( (A, s_\infty) \) as the **Batalin-Vilkovisky complex**.

What is the significance of \( (s_\infty)^2 = 0 \) in our Lagrangian context, or, equivalently, of the **Master Equation** \( (L_\infty, L) = 0 \)? There are three answers: in higher homotopy algebra, in deformation theory and in mathematical physics. It is the deformation theory that provides the transition between the other two.

**4. The Master Equation and Higher Homotopy Algebra**

If we expand \( s = s_0 + s_1 + \ldots \) where the subscript indicates the change in the ghost degree \( p \), the individual \( s_i \) do not correspond to \( (x, \ ) \) for any term \( x \) in \( L_\infty \) but do have the following graphical description:

standard diagram of differentials of a spectral sequence goes here
so that we see the B-V-complex as a multi-complex. The differential $s_1$ gives us the Koszul-Tate differential $d_{KT}$ and part of $s_2$ looks like that of Chevalley-Eilenberg. That is, $C^*_{\alpha} C^\alpha C^\gamma$ describes the (not-quite-Lie) bracket on $\Xi$. Further terms with one anti-ghost $C^*_{\alpha}$ and three ghosts $C^\beta C^\gamma C^\delta$ describe a tri-linear $[ , , ]$ and so on for multi-brackets of possibly arbitrary length. Moreover, the graded commutativity of the underlying algebra of the B-V-complex implies appropriate symmetry of these multi-brackets. The condition that $s^2_\infty = 0$ translates to the following identities, which are the defining identities for a strong homotopy Lie algebra or $L_\infty$ algebra.

\begin{equation}
\begin{split}
d[v_1, \ldots, v_n] + \sum_{i=1}^{n} \epsilon(i)[v_1, \ldots, dv_i, \ldots, v_n] \\
= \sum_{k+l=n+1} \sum \epsilon(\sigma)[[v_{i_1}, \ldots, v_{i_k}], v_{j_1}, \ldots, v_{j_l}],
\end{split}
\end{equation}

(4.1)

where $\epsilon(i) = (-1)^{|v_1|+\cdots+|v_{i-1}|}$ is the sign picked up by taking $d$ through $v_1, \ldots, v_{i-1}$ and, for the unshuffle $\sigma : \{1,2,\ldots,n\} \mapsto \{i_1,\ldots,i_k,j_1,\ldots,j_l\}$, the sign $\epsilon(\sigma)$ is the sign picked up by the elements $v_i$ passing through the $v_j$’s during the unshuffle of $v_1, \ldots, v_n$, as usual in superalgebra.

**Remark 4.1.** Here we follow the physics grading and sign conventions in our definition of a strong homotopy Lie algebra [28, 30]. These are equivalent to but different from those in the existing mathematics literature, cf. Lada and Stasheff [20], in which the $n$-ary bracket has degree $2 - n$. (The correspondence and additional insights are presented in full detail in Kjeseth’s dissertation [19].) With those mathematical conventions, $L_\infty$–algebras occur naturally as deformations of Lie algebras. If $L$ is a Lie algebra and $V$ is a complex with a homotopy equivalence to the trivial complex $0 \to L \to 0$, then $V$ is naturally a homotopy Lie algebra, see Schlessinger and Stasheff [23], Barnich, Fulp, Lada and Stasheff [2] and Getzler and Jones [14].

Realized in the Batalin-Vilkovisky complex, these defining identities tell us, for small values of $n$, that $d_{KT}$ is a graded derivation of the bracket, that the bracket may not satisfy the graded Jacobi identity but that we do have (with the appropriate signs)

\begin{equation}
[[v_1, v_2], v_3] \pm [[v_1, v_3], v_2] \pm [[v_2, v_3], v_2] = \\
- d[v_1, v_2, v_3] \pm [dv_1, v_2, v_3] \pm [v_1, dv_2, v_3] \pm [v_1, v_2, dv_3].
\end{equation}

(4.2)
i.e. the Jacobi identity holds *up to homotopy* or, for closed forms, the Jacobi identity holds modulo an exact term - the tri-linear bracket.

Note that the identity is the Jacobi identity if $d_{KT} = 0$ and all the other brackets vanish and that the identity has content even if only one $n$-linear bracket is non-zero and all the others vanish. Precisely that situation has recent been studied quite independently of my work and of each other by Hanlon and Wachs [16] (combinatorial algebraists), by Gnedbaye [15] (of Loday’s school) and by Azcarraga and Bueno [11] (physicists). At the Ascona conference, Flato brought to my attention that Takhtajan’s identity [26] for his trilinear ‘bracket’ is not the one above, but suitably symmetrized does agree with it. (This identity was also known to Flato and Fronsdal in 1992, though unpublished.)

5. The Master Equation and Deformation Theory

The Master Equation \((L_\infty, L_\infty) = 0\) has precisely the form of Gerstenhaber’s condition for \(L_\infty\) to be a deformation of \(L_0\). Classical (formal) algebraic deformation theory uses a differential graded Lie algebra (dgla) \(\mathbb{L}\) (e.g. the Hochschild cochain complex with the Gerstenhaber bracket) to study the problem of ‘integrating’ an infinitesimal deformation \(\theta\) to a full formal deformation \(\theta_t\). The primary obstruction, regarding \(\theta\) as a class in \(H^*\) of this dgla \(\mathbb{L}\), is \([\theta, \theta]\) and further obstructions can be described in terms of multi-brackets on this cohomology. Alternatively, the formal deformation \(\theta_t\) itself as a cochain must satisfy \([\theta_t, \theta_t] = 0\). The analogy with the Master Equation is manifest.

Following the historical pattern in algebraic deformation theory, we could hope to calculate this homology to be 0 in the relevant dimensions in certain cases, thus obtaining results of unobstructedness for the integrability question or of rigidity for the classification problem. Such calculations are highly non-trivial, however, and to my knowledge have been carried out only in the case of electricity and magnetism (Maxwell’s equations), Yang-Mills and gravity.

6. The Master Equation in Field Theory

In the Lagrangian setting, we wish to deform not just the local functional, but rather the underlying local function \(L\). In the case of electricity and magnetism (Maxwell’s equations), Yang-Mills and gravity, the relevant algebra of gauge symmetries is described by a finite dimensional Lie algebra which, moreover, holds off shell. In terms of an appropriate basis
and in the notation of section 1, we have

\[ [\delta_\alpha, \delta_\beta] = c^\gamma_{\alpha\beta} \delta_\gamma \]  

for structure constants \( c^\gamma_{\alpha\beta} \) and acting on all fields, not just on solutions. This allows the extended Lagrangian to be no more than quadratic in the ghosts.

As field theories, the electron can be described by a field of spin 1, as can a Yang-Mills particle, while the graviton can be described by a field of spin 2. Somehow this is related to the strict Lie algebra structures just described. For higher spin particles, however, we have quite a different story, which first caught my attention in the work of Burgers, Behrends and van Dam \[8, 9\], though I have since learned there was quite a history before that and major questions still remain open. By higher spin particle Lagrangians, I mean that the fields are symmetric \( s \)-tensors (sections of the symmetric \( s \)-fold tensor product of the tangent bundle). If the power is \( s \), the field is said to be of spin \( s \) and represents a particle of spin \( s \). Burgers, Behrends and van Dam start with a free theory with abelian gauge symmetries and calculate all possible infinitesimal interaction terms up to the appropriate equivalence (effectively calculating the appropriate homology group). They then sketch the problem of finding higher order terms for the Lagrangian, but do not carry out the full calculation. In fact, according to the folklore in the subject, a consistent theory for \( s \geq 3 \) will require additional fields of arbitrarily high spin \( s \). For \( s = 3 \), the conjecture is that all higher integral spins are needed. From the deformation theory point of view, this suggests the following attack: Compute the primary obstructions and discover that all infinitesimals are obstructed. Add additional fields to kill the obstructions and calculate that indeed additional fields of arbitrarily high spin \( s \) are needed. In one memorable phrase, this would be ‘doing string field theory the hard way’.

Zwiebach [30] does indeed have a consistent closed string field theory (CSFT), but produced in an entirely different way. Recall one of the earliest examples of deformation quantization, the Moyal bracket. Moyal was able to produce a non-trivial deformation of a commutative algebra \( C^\infty(M) \) on a symplectic manifold, with infinitesimal given by the Poisson bracket, by writing down the entire formal power series. Similarly, Zwiebach is able to describe the entire CSFT Lagrangian (at tree level) by giving it in terms of the differential geometry of the moduli space of punctured Riemann spheres (tree level = genus 0). In fact,
Zwiebach has the following structure: a differential graded Hilbert space \((\mathcal{H}, <, >, Q)\) related to the geometry of the moduli spaces from which he deduces \(n\)-ary operations \([,\ldots ,]\) which give an \(L_\infty\) structure:

\[
d[\phi_1,\ldots,\phi_N] + \sum_{i=1}^{N} \pm [\phi_1,\ldots,d\phi_i,\ldots,\phi_N] = \sum_{Q=2}^{N-1} \pm [[\phi_{i_1},\ldots,\phi_{i_Q}],\phi_{i_{Q+1}},\ldots,\phi_{i_N}]
\]

with \([\phi] = Q\phi\).

The deformed Lagrangian (still classical) and hence the Master Equation is satisfied for

\[
S(\Psi) = \frac{1}{2}<\Psi,Q\Psi> + \sum_{n=3}^{\infty} \frac{\kappa^{n-2}}{n!}\{\Psi,\ldots,\Psi\}.
\]

The expression \(\{\Psi,\ldots,\Psi\}\) contains \(n\)-terms and will be abbreviated \(\{\Psi^n\}\); it is given in terms of the brackets by \(\{\Psi,\ldots,\Psi\} = <\Psi,[\Psi,\ldots,\Psi]>\).

The field equations follow from the classical action by simple variation:

\[
\delta S = \sum_{n=2}^{\infty} \frac{\kappa^{n-2}}{n!}\{\delta\Psi,\Psi^{n-1}\}
\]

with gauge symmetries given by

\[
\delta_A \Psi = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!}[\Psi^n,\Lambda].
\]

### 7. Quantization

So far our description of the anti-field, anti-bracket formalism has been in the context of deformations of ‘classical’ Lagrangians. Batalin and Vilkovisky (as well as much of the work on BRST cohomology) were motivated by problems in quantum theory. The quantum version of the anti-field, anti-bracket formalism involves a further ‘second order’ differential operator \(\Delta\) of square 0 on the B-V complex relating the graded commutative product and the bracket - namely, the bracket is the deviation of the operator \(\Delta\) from being a derivation of the product. This has led to the abstract definition of a BV-algebra.

**Definition 7.1.** A **BV-algebra** is a Gerstenhaber algebra with an operator (necessarily of degree \(-1\) for a bracket of degree \(-1\)) such that

\[
[A,B] = \Delta(AB) - \Delta(A)B + (-1)^A\Delta(B).
\]

Alternatively, a definition can be given in terms of a graded commutative algebra with an appropriate operator \(\Delta\) \([1, 24]\).
The quantization of Zwiebach’s CSFT involves further expansion of the Lagrangian in terms of (the moduli space of) Riemann surfaces of genus $g \geq 0$. Here the operator $\Delta$ is determined by the self-sewing of a pair of pants (a Riemann sphere with 3 punctures). Now Zwiebach’s CSFT provides a solution of the ‘quantum Master Equation’ which in the context of a BV-algebra, is

$$(S, S) = \Delta S.$$  

Again we see an analog of the Maurer-Cartan equation or of a flat connection, but why?

References

1. F. Akman, *On some generalizations of Batalin-Vilkovisky algebras*, preprint, Cornell University, 1995, to appear in JPAA q-alg/9506027.
2. G. Barnich, R. Fulp, T. Lada, and J Stasheff, *Two examples of sh lie algebras in field theories : The poisson bracket and the algebra of gauge symmetries*, pre-preprint, ULB,NCSU,UNC-CH, 1996.
3. G. Barnich and M. Henneaux, *Consistent couplings between fields with a gauge freedom and deformations of the master equation*, Phys. Lett. B 311 (1993), 123–129.
4. I.A. Batalin and G.S. Vilkovisky, *Quantization of gauge theories with linearly dependent generators*, Phys. Rev. D 28 (1983), 2567–2582.
5. , *Closure of the gauge algebra, generalized Lie equations and Feynman rules*, Nucl. Phys. B 234 (1984), 106–124.
6. , *Existence theorem for gauge algebra*, J. Math. Phys. 26 (1985), 172–184.
7. C. Becchi, A. Rouet, and R. Stora, *Renormalization of the abelian Higgs-Kibble model*, Commun. Math. Phys. 42 (1975), 127–162.
8. F.A. Berends, G.J.H. Burgers, and H. van Dam, *On the theoretical problems in constructing interactions involving higher spin massless particles*, Tech. report, UNC-CH, 1984, preprint IFP234-UNC.
9. G.J.H. Burgers, *On the construction of field theories for higher spin massless particles*, Ph.D. thesis, Rijksuniversiteit te Leiden, 1985.
10. C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. 63 (1948), 85–124.
11. J. A. de Azcarraga and J. C. Perez Bueno, *Higher-order simple Lie algebras*, preprint, Univ. de Valencia, 1996, q-alg/9605213.
12. L. D. Fade’ev and V. N. Popov, *Feynman diagrams for the yang-mills field*, Phys. Lett. 25B (1967), 29–30.
13. M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. of Math. 79 (1964), 59–103.
14. E. Getzler and J.D.S. Jones, *n-algebras and Batalin-Vilkovisky algebras*, preprint, 1993.
15. V. Gnedbaye, *Operads of k-ary algebras*, Operads: Proceedings of Renaissance Conferences (J.-L. Loday, J. Stasheff, and A. A. Voronov, eds.), Amer. Math. Soc., 1996, pp. 83–114.

16. P. Hanlon and M. L. Wachs, *On Lie k-algebras*, Adv. in Math. **113** (1995), 206–236.

17. M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton Univ. Press, 1992.

18. T. Kimura, A.A. Voronov, and G. Zuckerman, *Homotopy Gerstenhaber algebras and topological field theory*, Operads: Proceedings of Renaissance Conferences (J.-L. Loday, J. Stasheff, and A. A. Voronov, eds.), Amer. Math. Soc., 1996, in this volume, pp. ?–?

19. L. Kjeseth, *BRST cohomology and homotopy Lie-Rinehart pairs*, Dissertation, UNC-CH, 1996.

20. T. Lada and J.D. Stasheff, *Introduction to sh Lie algebras for physicists*, Intern’l J. Theor. Phys. **32** (1993), 1087–1103.

21. B. H. Lian and G. J. Zuckerman, *New perspectives on the BRST-algebraic structure of string theory*, Commun. Math. Phys. **154** (1993), 613–646, hep-th/9211072.

22. E. Noether, *Invariante variationsprobleme*, Bachr. könig. Gesell. Wissen. Göttingen, Math.-Phys. Kl. (1918), 235–257, in English: Transport Theory and Stat. Phys. 1 (1971),186-207.

23. M. Schlessinger and J. D. Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, J. of Pure and Appl. Alg. **38** (1985), 313–322.

24. A. Schwarz, *Geometry of Batalin-Vilkovisky quantization*, CMP **155** (1993), 249–260.

25. R. Stora, *Continuum gauge theories*, New Developments in Quantum Field Theory (M.Levy and P.Mitter, eds.), Plenum, 1977, pp. 201–224.

26. L. Takhtajan, *On foundation of the generalized Nambu mechanics*, CMP **160** (1994), 295–315.

27. J. Tate, *Homology of Noetherian rings and local rings*, Ill. J. Math. **1** (1957), 14–27.

28. E. Witten and B. Zwiebach, *Algebraic structures and differential geometry in two-dimensional string theory*, Nucl. Phys. B **377** (1992), 55–112.

29. J. Zinn-Justin, *Renormalization of gauge theories*, Trends in Elementary Particle Theory (H. Rollnick and K. Dietz, eds.), vol. 37, Springer, 1975.

30. B. Zwiebach, *Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation*, Nucl. Phys. B **390** (1993), 33–152.

**Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA**

*E-mail address: jds@math.unc.edu*