A note on the zeros of generalized Hurwitz zeta functions

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Abstract

Given a function \( f(n) \) periodic of period \( q \geq 1 \) and an irrational number \( 0 < \alpha \leq 1 \), Chatterjee and Gun (cf. [4]) proved that the series \( F(s, f, \alpha) = \sum_{n=0}^{\infty} \frac{f(n)}{(n+\alpha)^s} \) has infinitely many zeros for \( \sigma > 1 \) when \( \alpha \) is transcendental and \( F(s, f, \alpha) \) has a pole at \( s = 1 \), or when \( \alpha \) is algebraic irrational and \( c = \frac{\max f(n)}{\min f(n)} < 1.15 \). In this note, we prove that the result holds in full generality.

1 Introduction

Let \( 0 < \alpha \leq 1 \) be a real number, the Hurwitz zeta function is defined as

\[
\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s},
\]

for \( s = \sigma + it \in \mathbb{C} \) with \( \sigma > 1 \). It is known that it admits a meromorphic continuation to \( \mathbb{C} \) with a simple pole at \( s = 1 \). In their paper [5], Davenport and Heilbronn proved that if \( \alpha \notin \{1, \frac{1}{2}\} \) is either rational or transcendental, then \( \zeta(s, \alpha) \) has infinitely many zeros for \( \sigma > 1 \). The same result when \( \alpha \) is algebraic irrational was proved by Cassels in [3].

Let now \( f(n) \) be a periodic function of period \( q \geq 1 \). For \( \sigma > 1 \), we define the generalized Hurwitz zeta function as

\[
F(s, f, \alpha) = \sum_{n=0}^{\infty} \frac{f(n)}{(n+\alpha)^s}.
\]

As for \( \zeta(s, \alpha) \), \( F(s, f, \alpha) \) is absolutely convergent for \( \sigma > 1 \) and it admits a meromorphic continuation to the whole complex plane (see e.g. [4]).

In [4], Chatterjee and Gun assume that \( f(n) \) is positive valued and prove that \( F(s, f, \alpha) \) has infinitely many zeros in the half-plane \( \sigma > 1 \) if \( \alpha \) is transcendental and \( F(s, f, \alpha) \) has a pole at \( s = 1 \), or if \( \alpha \) is algebraic irrational and

\[
c := \frac{\max f(n)}{\min f(n)} < 1.15.
\]

In this note we show that these assumptions can be removed, proving the result in full generality, also including the case of \( \alpha \) rational, which can be easily deduced from [6].

**Theorem 1.** Let \( f(n) \) be a non-identically zero periodic function with period \( q \geq 1 \) and let \( 0 < \alpha \leq 1 \) be a real number. If \( \alpha \notin \{1, \frac{1}{2}\} \), or if \( \alpha \in \{1, \frac{1}{2}\} \) and \( F(s, f, \alpha) \) is not of the form \( P(s) L(s, \chi) \), where \( P(s) \) is a Dirichlet polynomial and \( L(s, \chi) \) is the \( L \)-function associated to a Dirichlet character \( \chi \), then \( F(s, f, \alpha) \) has infinitely many zeros with \( \sigma > 1 \).

Observe that if \( \alpha = 1 \), \( F(s, f, 1) \) reduces to a Dirichlet series with periodic coefficients. By the result of Saias and Weingartner [6, Corollary], we know that it does not vanish in the half-plane \( \sigma > 1 \) if and only if it is the product of a Dirichlet polynomial and a Dirichlet \( L \)-function.
Remark 1. Examples of functions $f(n)$ giving rise to non-vanishing series in the right half-plane are $f(n) = \chi(n + 1)$, where $\chi$ is a Dirichlet character mod $q$, or $f(n) = (-1)^n$.

If $0 < \alpha < 1$ is rational, $F(s, f, \alpha)$ can be written as a linear combination of Dirichlet $L$-function,

$$F(s, f, \alpha) = \sum_{\chi \in \mathcal{C}} P_\chi(s)L(s, \chi),$$

where $\mathcal{C}$ is a set of primitive characters and $P_\chi(s)$ is a Dirichlet polynomial. Again by [4], expression \ref{eq:2} does not vanish in the half-plane $\sigma > 1$ if and only if the sum reduces to a single term. Let now $\alpha = \frac{a}{b} \in \mathbb{Q}$, with $(a, b) = 1$, $1 < a < b$. Then,

$$F(s, f, a/b) = b^s \sum_{n=0}^{\infty} \frac{f(n)}{(bn + a)^s} = b^s \sum_{m \equiv a \pmod{b}} \frac{g(m)}{m^s},$$

where $g(m)$ is periodic of period $bq$. We prove the following lemma.

Lemma 1. Let $\alpha = \frac{a}{b}$, with $(a, b) = 1$, $1 \leq a < b$. If $\frac{a}{b} \neq \frac{1}{r}$, then $F(s, f, \frac{a}{b})$ is not of the form $P(s)L(s, \chi)$, where $P$ is a Dirichlet polynomial and $L(s, \chi)$ is the Dirichlet $L$-function associated to the character $\chi$.

Proof. Consider a Dirichlet polynomial $P(s) = \sum_{n \in \mathcal{N}} \frac{a(n)}{n^s}$, where $\mathcal{N}$ is a non-empty finite set of positive integers, and let $\chi$ be a Dirichlet character mod $k$. Then,

$$P(s)L(s, \chi) = \sum_m \frac{b(m)}{m^s}, \quad \text{where} \quad b(m) = \sum_{n \in \mathcal{N}} a(n)\chi\left(\frac{m}{n}\right),$$

and the coefficients $b(m)$ are periodic of period $k \prod_{n \in \mathcal{N}} n$. Assume that there exist two coprime integers $h < r$, such that $b(m) \neq 0$ only if $m \equiv h \pmod{r}$. Let $n_1 := \min \mathcal{N}$, then $b(n_1) = a(n_1) \neq 0$ and so $n_1 \equiv h \pmod{r}$. On the other hand, $b(-n_1) = \chi(-1)a(n_1) \neq 0$, then $-n_1 \equiv h \pmod{r}$. It follows that $2h \equiv 0 \pmod{r}$, which implies $r = 2$. Thus, we conclude that expression \ref{eq:3} can be of the form $P(s)L(s, \chi)$ only if $\alpha = \frac{1}{2}$. \hfill \Box

Observe that if $\alpha = \frac{1}{2}$, the sum \ref{eq:2} reduces to a single term for instance if $g(m) = c\chi(m)$, where $\chi$ is a Dirichlet character mod $2q$ and $c$ is a non-zero constant (i.e. $f(n) = c\chi(2n + 1)$). In this case, $F(s, f, \frac{1}{2}) = c^{2s}L(s, \chi) \neq 0$ in $\sigma > 1$.

If $\alpha$ is transcendental, the argument of Davenport and Heilbronn (cf. [5]) for the Hurwitz zeta function applies also to $F(s, f, \alpha)$. Indeed, we have

$$\sum_{n=0}^{\infty} \frac{|f(n)|}{(n + \alpha)^\sigma} \to +\infty \quad \text{as} \quad \sigma \to 1^+.$$  \hfill \ref{eq:4}

Then, the assumption on the existence of the pole can be avoided and one can proceed as in [5] or [4]. Thus, we focus on the case of $\alpha$ algebraic irrational. The proof of the theorem in this case is based on a modification of Cassels’ original lemma (see [3]). A suitable decomposition over the residue classes allows us to remove the assumption \ref{eq:1}.

2 Proof of the theorem

As observed, we can assume that $\alpha$ is algebraic irrational. Let $K = \mathbb{Q}(\alpha)$ and let $O_K$ be its ring of integers. Denote by $\mathfrak{a}$ the denominator ideal of $\alpha$, i.e. $\mathfrak{a} = \{ r \in O_K \mid r \cdot (\alpha) \subseteq O_K \}$, where $(\alpha)$ is the principal fractional ideal generated by $\alpha$. Then for any integer $n \geq 0$, $(n + \alpha)\mathfrak{a}$ is an integral ideal. The following result holds.
Lemma 2. Let $0 < \alpha < 1$ be an algebraic irrational number and let $K = \mathbb{Q}(\alpha)$. Given a positive integer $q$, fix $b \in \{0, \ldots, q-1\}$. There exists an integer $N_0 > 10^6q$, depending on $\alpha$ and $q$, satisfying the following property:

for any integer $N > N_0$ put $M = \lfloor 10^{-6}N \rfloor$, then at least $0.51\frac{M}{q}$ of the integers $n \equiv b \pmod{q}$, $N < n \leq N + M$ are such that $(n + \alpha)a$ is divisible by a prime ideal $p_n$ for which

$$p_n \mid \prod_{m \leq N + M, m \neq n} (m + \alpha)a.$$

In the following sections, we first show how to complete the proof of Theorem 1 assuming the above lemma and then we give a proof of the lemma itself.

2.1 Proof of the main result

We rearrange Cassels’ argument with some suitable small modifications. As in [3], or directly by Bohr’s theory (see [1, Theorem 8.16]), it suffices to show that for any $0 < \delta < 1$ there exist a $\sigma$, with $1 < \sigma < 1 + \delta$, and a completely multiplicative function $\varphi(n) := \varphi((n + \alpha)a)$ of absolute value 1, such that

$$\sum_{n=0}^{\infty} \frac{f(n)\varphi(n)}{(n + \alpha)^\sigma} = 0.$$  

Notice that it is enough to define $\varphi(p)$, with $|\varphi(p)| = 1$, on the prime ideals $p$ dividing $(n + \alpha)a$.

Let $0 < \delta < 1$, $N_1 = \max(N_0, 10^2q)$ and consider $\sigma$ such that $1 < \sigma < 1 + \delta$ and

$$\sum_{n=0}^{N_1} \frac{|f(n)|}{(n + \alpha)^\sigma} < \frac{1}{100} \sum_{n=N_1+1}^{\infty} \frac{|f(n)|}{(n + \alpha)^\sigma}.  \quad (5)$$

Observe that such a $\sigma$ exists by [3]. Now, for $p \mid a$ or $p \mid (n + \alpha)a$ with $n \leq N_1$ we choose $\varphi(p) = 1$.

Proceeding by induction, for $j \geq 1$, we put $M_j = \lfloor 10^{-6}N_j \rfloor$ and $N_{j+1} = N_j + M_j$. Suppose we have defined $\varphi(p)$ for any $p \mid (n + \alpha)a$ with $n \leq N_j$ in such a way that

$$\left| \sum_{n=0}^{N_j} \frac{f(n)\varphi(n)}{(n + \alpha)^\sigma} \right| < \frac{1}{100} \sum_{n=N_{j+1}}^{\infty} \frac{|f(n)|}{(n + \alpha)^\sigma}.  \quad (6)$$

We want to define $\varphi(p)$ for any prime ideal

$$p \mid \prod_{n \leq N_{j+1}} (n + \alpha)a  \quad (7)$$

in such a way that (6) holds for $j + 1$ in place of $j$. For any $b \in \{0, \ldots, q-1\}$, we divide the integers $N_j < n \leq N_{j+1}$, with $n \equiv b \pmod{q}$ into two sets $\mathfrak{A}(b)$ and $\mathfrak{B}(b)$ according to whether a prime ideal $p_n$ as in Lemma 2 exists or not for $N = N_j$ and $M = M_j$. We can easily notice that $|\mathfrak{A}(b)| \geq 5$, since

$$|\mathfrak{A}(b)| \geq \frac{54}{100} \cdot \frac{M_j}{q} = \frac{54}{100} \cdot \frac{10^{-6}N_j}{q},$$

and $N_j \geq 10^6q$. We have then divided the integers $N_j < n \leq N_{j+1}$ into the disjoint sets $\mathfrak{A} = \bigcup_{n=0}^{q-1} \mathfrak{A}(b)$ and $\mathfrak{B} = \bigcup_{n=0}^{q-1} \mathfrak{B}(b)$. As in Cassels’, given a prime ideal as in (7), we distinguish three cases:

1. $p \mid \prod_{n \leq N_j} (n + \alpha)a$: in this case $\varphi(p)$ is fixed by the inductive hypothesis.
2. $p = p_n$ for some $n \in \mathfrak{A}$
3. the remaining $p$ with property (7). In this case, we fix arbitrarily $\varphi(p) = 1$. 
In particular, \( \varphi(n) \) is defined for any \( n \in \mathcal{B} \), whereas if \( n \in \mathcal{A} \), we have that \( \varphi(n) = c_n \varphi(p_n) \), with \( c_n \) fixed of modulus 1. Now assume \( n \in \mathcal{A} \) and \( n \equiv b \mod {q} \) with \( b \in \{ 0, \ldots, q - 1 \} \). Since \( f(n) \) is periodic of period \( q \) and \( |\mathcal{A}(b)| \geq 5 \), by Bohr’s results on addition of convex curves (cf. [2]), for an appropriate choice of \( \varphi(p_n) \) for all \( n \in \mathcal{A}(b) \), we have that

\[
\sum_{n \in \mathcal{A}(b)} \frac{f(n)\varphi(n)}{(n + \alpha)^\sigma} = \sum_{n \in \mathcal{A}(b)} \frac{f(n)c_n\varphi(p_n)}{(n + \alpha)^\sigma} = f(b) \sum_{n \in \mathcal{A}(b)} \frac{c_n\varphi(p_n)}{(n + \alpha)^\sigma}
\]
takes any given value \( z \) satisfying

\[
|z| \leq S_{3,b} := |f(b)| \sum_{n \in \mathcal{A}(b)} \frac{1}{(n + \alpha)^\sigma}.
\]

Let now

\[
\Lambda(b) := f(b) \left( \sum_{n \leq N_j} \frac{\varphi(n)}{(n + \alpha)^\sigma} + \sum_{n > N_j} \frac{\varphi(n)}{(n + \alpha)^\sigma} \right),
\]

and define \( \varphi(p_n) \) for \( n \in \mathcal{A}(b) \) so that

\[
\sum_{n \in \mathcal{A}(b)} \frac{f(n)\varphi(n)}{(n + \alpha)^\sigma} = \begin{cases} -\Lambda(b) & \text{if } \left| \Lambda(b) \right| \leq S_{3,b} \\ -S_{3,b} & \text{if } \left| \Lambda(b) \right| > S_{3,b} \end{cases}
\]

With this choice, it is easy to verify that

\[
\left| \sum_{n \leq N_{j+1}} \frac{f(n)\varphi(n)}{(n + \alpha)^\sigma} \right| \leq \max(0, \left| \Lambda(b) \right| - S_{3,b}). \tag{8}
\]

We introduce the notation

\[
S_{1,b} = \sum_{n=0}^{N_j} \frac{f(n)\varphi(n)}{(n + \alpha)^\sigma}, \quad S_{4,b} = |f(b)| \sum_{n > N_{j+1}} \frac{1}{(n + \alpha)^\sigma}, \quad S_{2,b} = |f(b)| \sum_{n \in \mathcal{A}(b)} \frac{1}{(n + \alpha)^\sigma}.
\]

Now, recalling that \( \mathcal{B}(b) \) contains at most \( 0.46 \frac{M}{q} \) elements and \( \mathcal{A}(b) \) at least \( 0.54 \frac{M}{q} \), we have

\[
S_{3,b} \geq \frac{54}{46} \frac{(N_j + \alpha)^\sigma}{(N_{j+1} + \alpha)^\sigma} > \frac{101}{99}.
\]

Thus, we deduce

\[
S_{3,b} - S_{2,b} > \frac{1}{100} (S_{3,b} + S_{2,b}). \tag{9}
\]

Now, by the equations (9), (8) and (9) we get

\[
\left| \sum_{n=0}^{N_{j+1}} \frac{f(n)\varphi(n)}{(n + \alpha)^\sigma} \right| < \frac{1}{100} S_{4,b} = \frac{1}{100} \sum_{n > N_{j+1}} \frac{|f(n)|}{(n + \alpha)^\sigma}.
\]

Summing over the classes modulo \( q \), we finally get that

\[
\left| \sum_{n=0}^{N_{j+1}} \frac{f(n)\varphi(n)}{(n + \alpha)^\sigma} \right| < \frac{1}{100} \sum_{b=0}^{q-1} S_{4,b} < \frac{1}{100} \sum_{n > N_{j+1}} \frac{|f(n)|}{(n + \alpha)^\sigma}.
\]

So, equation (4) also holds for \( j + 1 \) in place of \( j \), as desired. By induction, it then holds for all \( j \geq 1 \). Since \( F(s, f, \alpha) \) is absolutely convergent for \( \sigma > 1 \), the right-hand side goes to zeros as \( j \to +\infty \). It then follows that \( \sum_{n=0}^{\infty} \frac{|f(n)|}{(n + \alpha)^\sigma} = 0 \) and the proof is complete, since by almost periodicity and Rouché’s theorem we can conclude the existence of infinitely many zeros for \( F(s, f, \alpha) \) with \( \sigma > 1 \).
2.2 Proof of Lemma 2

Let \( \mathfrak{P} \) be the set of the prime ideals \( \mathfrak{p} \) of \( O_K \) defined as in Cassels’, with the added condition that \((p, q) = 1\), where \( p := \text{Norm}(\mathfrak{p}) \). Then, for any integer \( n \) we write

\[
(n + \alpha)\mathfrak{a} = b \prod_{\mathfrak{p}} p^{u(p)},
\]

(10)

where \( u(p) \) is an integer and \( b \) contains all the prime factors of \((n + \alpha)\mathfrak{a}\) which are not in \( \mathfrak{P} \).

Consider now an integer \( N > 10^6 q \) and let \( M = \lfloor 10^{-6} N \rfloor \). We define \( \mathfrak{G} = \mathfrak{G}(N, q, b) \) as the set of the integers \( N < n \leq N + M \), \( n \equiv b \pmod{q} \) such that, for all the primes \( \mathfrak{p} \in \mathfrak{P} \) in the factorization \( \text{(10)} \) one has \( p^{u(p)} < M \). Let \( S = S(N, q, b) = |\mathfrak{G}| \). We want an upper bound for \( S \).

For any prime \( \mathfrak{p} \in \mathfrak{P} \) and any integer \( v \), let \( \phi(p^v, n) \) and \( \sigma(n) \) be defined as in [3]. Thus, the same argument gives, as \( N \to \infty \),

\[
\sum_{n \in \mathfrak{G}} \sigma(n) \geq (2 + o(1)) S \log M.
\]

Moreover, by the definition of \( \mathfrak{P} \), if \( \mathfrak{p}^v \mid (n_1 + \alpha)\mathfrak{a} \) and \( \mathfrak{p}^v \mid (n_2 + \alpha)\mathfrak{a} \) for some integer \( v \) then

\[
n_1 \equiv n_2 \pmod{p^v}.
\]

(12)

Since we assumed \((p, q) = 1\), by the Chinese remainder theorem \( n_1 \equiv n_2 \pmod{p^v q} \). As in [3], we get

\[
\sum_{n \in \mathfrak{G}} \phi(p^v, n) \leq \sum_{N < n \leq N + M \atop n \equiv b \pmod{q}} \phi(p^v, n) \leq \left( \frac{M}{p^v q} + 1 \right) \log p,
\]

(13)

and, assuming \( p_1 \neq p_2 \),

\[
\sum_{n \in \mathfrak{G}} \phi(p_1^v, n) \phi(p_2^v, n) \leq \sum_{N < n \leq N + M \atop n \equiv b \pmod{q}} \phi(p_1^v, n) \phi(p_2^v, n) \leq \log p_1 \log p_2 \left( \frac{M}{p_1 p_2 q} + 1 \right).
\]

(14)

Writing \( \sigma(n) = \sigma_1(n) + \sigma_2(n) + \sigma_3(n) \), with the same notation of [3], using the prime ideal theorem, partial summation and equations \( \text{(13), (14)} \), we get

\[
\sum_{n \in \mathfrak{G}} \sigma_2(n) \leq \left( \frac{1}{2} + o(1) \right) \frac{M}{q} \log M,
\]

\[
\sum_{n \in \mathfrak{G}} (\sigma_3(n))^2 \leq \left( \frac{3}{8} + o(1) \right) \frac{M}{q} \log^2 M,
\]

and

\[
\sum_{n \in \mathfrak{G}} \sigma_1(n) = O(M) = o(M \log M).
\]

We define \( \rho := \frac{qS}{M} \) and the proof now proceeds exactly as in Cassels’. The better numerical result simply follows by a more precise choice of \( \rho \) in expression \( \text{(37)} \) of [3].

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