GENERALIZED VOLterra TYPE INTEGRAL OPERATORS
ON LARGE BERGMAN SPACES

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ABSTRACT. Let \( \phi \) be an analytic self-map of the open unit disk \( \mathbb{D} \) and \( g \) analytic in \( \mathbb{D} \). We characterize boundedness and compactness of generalized Volterra type integral operators

\[
GI_{(\phi,g)}f(z) = \int_0^z f'(\phi(\xi)) \, g(\xi) \, d\xi
\]

and

\[
GV_{(\phi,g)}f(z) = \int_0^z f(\phi(\xi)) \, g(\xi) \, d\xi,
\]

acting between large Bergman spaces \( A^p_\omega \) and \( A^q_\omega \) for \( 0 < p, q \leq \infty \). To prove our characterizations, which involve Berezin type integral transforms, we use the Littlewood-Paley formula of Constantin and Pełaez and establish corresponding embedding theorems, which are also of independent interest. When \( \phi(z) = z \), our results for \( GV_{(\phi,g)} \) complement the descriptions of Pau and Pełaez.

1. INTRODUCTION AND MAIN RESULTS

For \( 0 < p < \infty \) and a positive function \( \omega \in L^1(\mathbb{D}, dA) \), the weighted Bergman spaces \( A^p_\omega \) and \( A^\infty_\omega \) consist of all analytic functions defined on the unit disk \( \mathbb{D} \) for which

\[
\|f\|_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)^{p/2} \, dA(z) < \infty
\]

and

\[
\|f\|_{A^\infty_\omega} = \sup_{z \in \mathbb{D}} |f(z)| \omega(z)^{1/2} < \infty,
\]

respectively, where \( dA \) is the normalized area measure on \( \mathbb{D} \).

In this paper, we study generalized Volterra type integral operators between weighted Bergman spaces for a certain class \( \mathcal{W} \) of radial rapidly decreasing weights. The class \( \mathcal{W} \), considered previously in [4] and [15], consists of the radial decreasing weights of the form \( \omega(z) = e^{-2\varphi(z)} \), where \( \varphi \in C^2(\mathbb{D}) \) is a radial function such that \( (\Delta \varphi(z))^{-1/2} \asymp \tau(z) \) for some radial positive function \( \tau(z) \) that decreases to 0 as \( |z| \to 1^- \) and satisfies \( \lim_{r \to 1^-} \tau'(r) = 0 \). Here \( \Delta \) denotes the standard Laplace operator. Furthermore, we assume that there either exists a constant \( C > 0 \) such that \( \tau(r)(1 - r)^{-C} \) increases for \( r \) close to 1 or

\[
\lim_{r \to 1^-} \tau'(r) \log \frac{1}{\tau(r)} = 0.
\]

See Section 7 of [15] for examples of weights in \( \mathcal{W} \), such as the following exponential type weight

\[
\omega_{\gamma,\alpha}(z) = (1 - |z|)^\gamma \exp \left( \frac{-b}{(1 - |z|)^\alpha} \right), \quad \gamma \geq 0, \alpha > 0, b > 0.
\]

2020 Mathematics Subject Classification. 47B38, 30H20.

Key words and phrases. Volterra type integral operators, large Bergman spaces.
For the weights \( \omega \) in \( \mathcal{W} \), the point evaluations \( L_z : f \rightarrow f(z) \) are bounded linear functionals on \( A_\omega^2 \) for each \( z \in \mathbb{D} \), and so \( A_\omega^2 \) is a reproducing kernel Hilbert space; that is, for each \( z \in \mathbb{D} \), there are functions \( K_z \in A_\omega^2 \) with \( \|L_z\| = \|K_z\|_{A_\omega^2} \) such that \( L_z f = f(z) = \langle f, K_z \rangle_\omega \), where

\[
\langle f, g \rangle_\omega = \int_\mathbb{D} f(z) \overline{g(z)} \omega(z) \, dA(z).
\]

The function \( K_z \) is called the reproducing kernel for the Bergman space \( A_\omega^2 \) and has the property that \( K_z(\xi) = K_\xi(z) \). The Bergman spaces with exponential type weights have attracted considerable attention in recent years because of novel techniques different from those used for standard Bergman spaces; see, e.g., [2, 5] and the references therein. Various estimates for the reproducing kernel play an important role in our work and we discuss them further in Section 2.1.

Let \( \phi \) and \( g \) be analytic self-maps of \( \mathbb{D} \). The generalized Volterra type integral operators \( GI_{(\phi, g)} \) and \( GV_{(\phi, g)} \) induced by the pair of symbols \( (\phi, g) \) are defined by

\[
(1.1) \quad GI_{(\phi, g)} f(z) = \int_0^z f'(\phi(\xi)) g(\xi) \, d\xi \quad \text{and} \quad GV_{(\phi, g)} f(z) = \int_0^z f(\phi(\xi)) g(\xi) \, d\xi,
\]

where \( f \in H(\mathbb{D}) \) and \( z \in \mathbb{D} \). When \( g = \phi' \), the operator \( GI_{(\phi, \phi')} \) is the composition operator \( C_\phi \) up to a certain constant—these operators acting between different large Bergman spaces were recently studied in [1]. As another special case, when \( \phi(\xi) = \xi \), we obtain the Volterra integral operator

\[
(1.2) \quad V_g f(z) := GV_{(\phi, g')} f(z) = \int_0^z f(\xi) g'(\xi) \, d\xi,
\]

and its companion integral operator

\[
(1.3) \quad J_g f(z) := GI_{(\phi, g)} f(z) = \int_0^z f'(\xi) g(\xi) \, d\xi.
\]

Previously Pau and Peláez characterized boundedness and compactness of \( V_g : A^p_\omega \rightarrow A^q_\omega \) in [15] when \( 0 < p, q < \infty \). Via (1.2) and (1.3), our characterizations extend the previous results to the full range \( 0 < p, q \leq \infty \) and to all weights in \( \mathcal{W} \), and also deal with the companion operator \( J_g \) for the first time. The generalized Volterra type integral operators \( GI_{(\phi, g)} \) and \( GV_{(\phi, g)} \) were previously studied by Mengestie [11, 12, 13] in standard Fock spaces and by Li [6] in standard Bergman spaces and Bloch type spaces.

1.1. Main results. In this paper we study boundedness and and compactness of the generalized Volterra type integral operators \( GI_{(\phi, g)} \) and \( GV_{(\phi, g)} \). Our results on Schatten class properties, compact differences, and the essential norm of these operators will be published elsewhere.

For \( 0 < p, q < \infty \), our characterizations for boundedness and compactness are given in terms of the integral transform

\[
GB_{n,p,q}^\phi(\omega)(z) = \int_\mathbb{D} |k_{p,z}(\phi(\xi))|^n \frac{(1 + \psi'(|\phi(\xi)|))^q}{(1 + \varphi'(|\phi(\xi)|))^q} |g(\xi)|^q \omega(\xi) q^{q/2} \, dA(\xi), \quad z \in \mathbb{D},
\]

where \( n = 0, 1 \) and \( k_{p,z} \) is the normalized reproducing kernel of \( A^p_\omega \).

**Theorem 1.1.** Let \( \omega \in \mathcal{W} \), \( \phi : \mathbb{D} \rightarrow \mathbb{D} \) be analytic, and \( g \in H(\mathbb{D}) \).

(A) For \( 0 < p \leq q < \infty \), the operator \( GI_{(\phi, g)} : A^p_\omega \rightarrow A^q_\omega \) is bounded if and only if

\[
GB_{1,p,1}^\phi(\omega) \in L^\infty(\mathbb{D}),
\]

and compact if and only if \( \lim_{|z| \rightarrow 1-} GB_{1,p,1}^\phi(\omega)(z) = 0 \).
(B) For $0 < p < \infty$, $G_{I(\phi,g)} : A^p_\omega \to A^\infty_\omega$ is bounded if and only if

$$MI_{g,\phi,\omega}(z) := |g(z)| \frac{(1 + \varphi' (\phi(z)))}{(1 + \varphi' (\phi(z)))} \omega(z)^{1/2} \Delta \varphi(\phi(z))^{1/p} \in L^\infty(\mathbb{D}, dA),$$

and compact if and only if $\lim_{|\phi(z)| \to 1^-} MI_{g,\phi,\omega}(z) = 0$.

(C) The operator $G_{I(\phi,g)} : A^\infty_\omega \to A^\infty_\omega$ is bounded if and only if

$$NI_{g,\phi,\omega}(z) := |g(z)| \frac{(1 + \varphi' (\phi(z)))}{(1 + \varphi' (\phi(z)))} \omega(z)^{1/2} \Delta \varphi(\phi(z))^{1/2} \in L^\infty(\mathbb{D}, dA),$$

and compact if and only if $\lim_{|\phi(z)| \to 1^-} NI_{g,\phi,\omega}(z) = 0$.

(D) For $0 < q < p \leq \infty$, both boundedness and compactness of $G_{I(\phi,g)} : A^p_\omega \to A^q_\omega$ are equivalent to the condition

$$GB_{1,p,q}(g) \in L^q(\mathbb{D}, d\lambda),$$

where $\lambda(z) = dA(z)/\tau(z)^2$ and $s = p/(p-q)$ if $p < \infty$ and $s = 1$ if $p = \infty$.

**Theorem 1.2.** Let $\omega \in W$, $\phi : \mathbb{D} \to \mathbb{D}$ be analytic, and $g \in H(\mathbb{D})$.

(A) For $0 < p \leq q < \infty$, $G_{V(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded if and only if $GB_{1,p,q}(g) \in L^\infty(\mathbb{D}, dA)$, and compact if and only if $\lim_{|\phi(z)| \to 1^-} GB_{1,p,q}(g) = 0$.

(B) For $0 < p < \infty$, the operator $G_{V(\phi,g)} : A^p_\omega \to A^\infty_\omega$ is bounded if and only if

$$MV_{g,\phi,\omega}(z) := |g(z)| \frac{\omega(z)^{1/2}}{(1 + \varphi' (\phi(z))) \omega(\phi(z))^{1/2}} \Delta \varphi(\phi(z))^{1/p} \in L^\infty(\mathbb{D}, dA),$$

and compact if and only if $\lim_{|\phi(z)| \to 1^-} MV_{g,\phi,\omega}(z) = 0$.

(C) The operator $G_{V(\phi,g)} : A^\infty_\omega \to A^\infty_\omega$ is bounded if and only if

$$NV_{g,\phi,\omega}(z) := |g(z)| \frac{\omega(z)^{1/2}}{(1 + \varphi' (\phi(z))) \omega(\phi(z))^{1/2}} \in L^\infty(\mathbb{D}, dA),$$

and compact if and only if $\lim_{|\phi(z)| \to 1^-} NV_{g,\phi,\omega}(z) = 0$.

(D) For $0 < q < p \leq \infty$, both boundedness and compactness of $G_{V(\phi,g)} : A^p_\omega \to A^q_\omega$ are equivalent to the condition

$$GB_{1,p,q}(g) \in L^q(\mathbb{D}, d\lambda),$$

where $r = p/(p-q)$ when $p < \infty$ and $r = 1$ when $p = \infty$.

We also prove the following simpler necessary conditions for boundedness and compactness.

**Proposition 1.3.** Let $\omega \in W$, $\phi : \mathbb{D} \to \mathbb{D}$ be analytic, and $g \in H(\mathbb{D})$.

(A) If $0 < p, q < \infty$ and $G_{I(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded, then

$$z \mapsto |g(z)| \frac{\tau(z)^{2/q}}{(\tau(\phi(z)))^{2/p}} \frac{(1 + \varphi' (\phi(z)))}{(1 + \varphi' (\phi(z)))} \omega(z)^{1/2} \in L^\infty(\mathbb{D}, dA);$$

and if $G_{I(\phi,g)}$ is compact, then the function in (1.8) vanishes as $|z| \to 1$.

(B) If $0 < p \leq q < \infty$ and $G_{V(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded, then

$$z \mapsto \frac{\tau(z)^{2/q}}{(\tau(\phi(z)))^{2/p}} \frac{|g(z)|}{(1 + \varphi' (\phi(z))) \omega(z)^{1/2}} \in L^\infty(\mathbb{D}, dA);$$

and if $G_{V(\phi,g)}$ is compact, then the function in (1.9) vanishes as $|\phi(z)| \to 1$. 


As a consequence of the two main theorems, when \( \phi(z) = z \), we obtain characterizations for boundedness and compactness of \( V_g \) and its companion operator \( J_g \). The results for \( J_g \) are new while the descriptions for \( V_g \) had been partially obtained before as explained in the following remark.

**Remark 1.4.** Notice that (C) and (D) of Corollary 1.5 are analogous to the descriptions given in Theorem 3 of Constantin and Peláez [3] when \( V_g \) is acting between weighted Fock spaces, but the two cases require different methods due to fundamental differences between the two types of spaces. Further, Corollary 1.6 implies Theorem 2 of Pau and Peláez [15], that is, we show that the their conditions are equivalent of those in Theorem 1.2 when \( \phi(z) = z \).

**Corollary 1.5.** Let \( \omega \in \mathcal{W} \) and \( g \in H(\mathbb{D}) \).

(A) For \( 0 < p < q \leq \infty \), \( J_g : A^p_{\omega} \to A^q_{\omega} \) is bounded if and only if \( g = 0 \).

(B) For \( p > q \), \( J_g : A^p_{\omega} \to A^q_{\omega} \) is compact if and only if \( g \in L^s(\mathbb{D}, dA) \), where \( s = pq/(p - q) \).

(C) For \( 0 < p \leq q \leq \infty \), \( V_g : A^p_{\omega} \to A^q_{\omega} \) is bounded if and only if

\[
(1.10) \quad z \mapsto \frac{|g'(z)|}{(1 + \varphi'(z))} \Delta \phi(z)^{1 - \frac{1}{q}} \in L^\infty(\mathbb{D}, dA),
\]

and \( V_g : A^p_{\omega} \to A^q_{\omega} \) is compact if the function in (1.10) vanishes as \( |z| \to 1 \).

(D) For \( 0 < q < p < \infty \), \( V_g : A^p_{\omega} \to A^q_{\omega} \) is bounded if and only if

\[
(1.11) \quad \frac{|g'(z)|}{(1 + \varphi'(z))} \in L^{\frac{pq}{p-q}}(\mathbb{D}, dA).
\]

In the next corollary, we consider the weighted Bergman space \( A^p(\omega) = A^{p}_{\omega} \), that is,

\[
A^p(\omega) = \left\{ f \in H(\mathbb{D}) : \|f\|_{A^p(\omega)} = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty \right\},
\]

where the weight \( \omega \in \mathcal{W} \) satisfies the condition

\[
\Delta \phi(z) \asymp ((1 - |z|)^t \psi_\omega(z))^{-1}, \quad z \in \mathbb{D}, \text{ for some } t \geq 1.
\]

In particular, we obtain the conditions of Theorem 2 in [15] for boundedness and compactness of the operator \( V_g : A^p(\omega) \to A^q(\omega) \).

**Corollary 1.6.** Let \( 0 < p, q < \infty \), \( \omega \in \mathcal{W} \), and \( g \in H(\mathbb{D}) \).

(I) For \( p = q \), we have the following statements

(a) \( GB^{id}_{0,p,q}(g') \in L^\infty(\mathbb{D}, dA) \) if and only if

\[
\psi_\omega(z)|g'(z)| \in L^\infty(\mathbb{D}, dA).
\]

(b) \( \lim_{|z| \to 1} GB^{id}_{0,p,q}(g') = 0 \) if and only if

\[
\lim_{|z| \to 1} \psi_\omega(z)|g'(z)| = 0.
\]

(II) Let \( \omega \in \mathcal{W} \) with

\[
(1.12) \quad \Delta \phi(z) \asymp ((1 - |z|)^t \psi_\omega(z))^{-1}, \quad z \in \mathbb{D}, \text{ for some } t \geq 1.
\]

For \( p < q \), the following statements are equivalent:

(c) \( GB^{id}_{0,p,q}(g') \in L^\infty(\mathbb{D}, dA) \).

(d) The function \( g \) is constant.

(III) For \( q < p \),

\[
GB^{id}_{0,p,q}(g') \in L^{p/(p-q)}(\mathbb{D}, d\lambda) \implies g \in A^{pq/(p-q)}(\omega).
\]
1.2. Outline. In Section 2 we provide the basic definitions and results that are needed to deal with the weights \( \omega \) in \( \mathcal{W} \), and consider useful estimates for the reproducing kernel of \( A^\omega_p \). In Section 3 we recall known geometric characterizations of Carleson measures, and in Section 4 we establish embedding theorems of \( S^p_\omega \) into \( L^q(\mathbb{D}, d\mu) \), for \( 0 < p, q \leq \infty \) and \( \omega \in \mathcal{W} \), where

\[
S^p_\omega := \left\{ f \in H(\mathbb{D}) : \|f\|_{S^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \frac{\omega(z)^{p/2}}{(1 + \varphi'(z))^p} dA(z) < \infty \right\}
\]

and

\[
S^\infty_\omega := \left\{ f \in H(\mathbb{D}) : \|f\|_{S^\infty_\omega} = \sup_{z \in \mathbb{D}} |f(z)| \frac{\omega(z)^{1/2}}{1 + \varphi'(z)} < \infty \right\}.
\]

In Section 5 we prove Theorems 1.1 and 1.2 using the embedding theorems, the strong decay of the weights \( e^{-2\varphi} \) and the following Littlewood-Paley type formulas (see (9.3) of [3] and [13]):

\[
\|f\|_{A^\omega_p} \approx |f(0)| + \int_{\mathbb{D}} |f'(z)|^p \frac{\omega(z)^{p/2}}{(1 + \varphi'(z))^p} dA(z),
\]

\[
\|f\|_{A^\omega_\infty} \approx |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)| \frac{\omega(z)^{1/2}}{(1 + \varphi'(z))}.
\]

Finally, Proposition 1.3 and Corollaries 1.5 and 1.6 are proved in Section 6.

Throughout the paper, we use the notation \( a \lesssim b \) to indicate that there is a constant \( C > 0 \) with \( a \leq C b \). By \( a \asymp b \) we mean that \( a \lesssim b \) and \( b \lesssim a \). For simplicity, we write \( L^p_\omega \) and \( A^p_\omega \) for \( L^p(\mathbb{D}, \omega^{p/2} dA) \) and \( A^p(\mathbb{D}, \omega^{p/2} dA) \), respectively.

2. Preliminaries and Basic Properties

A positive function \( \tau \) on \( \mathbb{D} \) is said to be of class \( \mathcal{L} \) if there are two constants \( c_1 \) and \( c_2 \) such that

\[
\tau(z) \leq c_1 (1 - |z|) \quad \text{for all } z \in \mathbb{D}
\]

and

\[
|\tau(z) - \tau(\zeta)| \leq c_2 |z - \zeta| \quad \text{for all } z, \zeta \in \mathbb{D}.
\]

For such \( c_1 \) and \( c_2 \), we set

\[
m_\tau := \frac{1}{4}\min(1, c_1^{-1}, c_2^{-1}).
\]

Given \( a \in \mathbb{D} \) and \( \delta > 0 \), we denote by \( D_\delta(a) \) the Euclidean disc centered at \( a \) with radius \( \delta \tau(a) \). It follows from (2.1) and (2.2) (see [15] Lemma 2.1) that if \( \tau \in \mathcal{L} \) and \( z \in D_\delta(a) \), then

\[
\frac{1}{2}\tau(a) \leq \tau(z) \leq 2\tau(a),
\]

whenever \( \delta \in (0, m_\tau) \). These inequalities will be used frequently in what follows.

Definition 2.1. We say that a weight \( \omega \) is of class \( \mathcal{L}^* \) if it is of the form \( \omega = e^{-2\varphi} \), where \( \varphi \in C^2(\mathbb{D}) \) with \( \Delta \varphi > 0 \), and \( (\Delta \varphi(z))^{-1/2} \preceq \tau(z) \), with \( \tau \) being a function in the class \( \mathcal{L} \). Here \( \Delta \) denotes the classical Laplace operator.

It is straightforward to see that \( \mathcal{W} \subset \mathcal{L}^* \). The following result (see [15] Lemma 2.2) implies that the point evaluation functional at each \( z \in \mathbb{D} \) is bounded on \( A^2_\omega \).
Theorem A. Let $\omega \in L^p$, $0 < p < \infty$, and $z \in \mathbb{D}$. If $\beta, \gamma \in \mathbb{R}$, there exists $M \geq 1$ such that

$$ |f(z)|^p \omega(z)^{\beta} \leq \frac{M}{\delta^2 \tau(z)^2} \int_{D_\delta(z)} |f(\xi)|^p \omega(\xi)^{\beta} dA(\xi) $$

for all $f \in H(\mathbb{D})$ and all sufficiently small $\delta > 0$.

Using the preceding lemma and the fact that there exists $r_0 \in [0, 1)$ such that for all $a \in \mathbb{D}$ with $1 - |a| > r_0$, and any $\delta > 0$ small enough we have

$$ \varphi'(a) \approx \varphi'(z), \quad z \in D_\delta(a) $$

(see statement (d) in [3, Lemma 32]), one has

$$ |f(z)|^p \omega(z)^{\beta} \lesssim \frac{1}{c^2 \tau(z)^2} \int_{D_\delta(z)} |f(\xi)|^p \omega(\xi)^{\beta} (1 + \varphi'(z))^{\gamma} dA(\xi), $$

for $\beta, \gamma \in \mathbb{R}$.

The next lemma provides upper estimates for the derivatives of functions in $A^p_\omega$. Its proof is similar to the case of doubling measures $\Delta \varphi$ in Lemma 19 of [10], and it can be found in the following form in [5, 14].

Lemma B. Let $\omega \in L^p$ and $0 < p < \infty$. For any $\delta_0 > 0$ sufficiently small there exists a constant $C(\delta_0) > 0$ such that

$$ |f'(z)|^p \omega(z)^{p/2} \leq \frac{C(\delta_0)}{\tau(z)^{2+p}} \left( \int_{D(\delta_0 \tau(z)/2)} |f(\xi)|^p \omega(\xi)^{p/2} dA(\xi) \right)^{1/p} $$

for all $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

The following lemma on coverings is due to Oleinik [14].

Lemma C. Let $\tau$ be a positive function on $\mathbb{D}$ of class $L$, and let $\delta \in (0, m_\tau)$. Then there exists a sequence of points $\{z_n\} \subset \mathbb{D}$ such that the following conditions are satisfied:

1. $z_n \notin D_\delta(z_k), \quad n \neq k$.
2. $\bigcup_n D_\delta(z_n) = \mathbb{D}$.
3. $D_\delta(z_n) \subset D_\delta(z_n)$, where $D_\delta(z_n) = \bigcup_{z \in D_\delta(z_n)} D_\delta(z), \quad n \in \mathbb{N}$.
4. $\{D_\delta(z_n)\}$ is a covering of $\mathbb{D}$ of finite multiplicity $N$.

The multiplicity $N$ in the previous lemma is independent of $\delta$, and it is easy to see that one can take, for example, $N = 256$. Any sequence satisfying the conditions in Lemma C will be called a $(\delta, \tau)$-lattice. Note that $|z_n| \to 1$ as $n \to \infty$. In what follows, the sequence $\{z_n\}$ will always refer to the sequence chosen in Lemma C.

2.1. Reproducing kernel estimates. The following norm estimates for the reproducing kernel $K_z$ valid for all $z \in \mathbb{D}$ can be found in [2, 7, 15] when $p = 2$ and in [5] when $p > 0$, while for the estimate for the points close to the diagonal, see [8, Lemma 3.6].

Theorem A. Let $K_z$ be the reproducing kernel of $A^2_\omega$. Then

(a) For $\omega \in \mathcal{W}$ and $0 < p < \infty$, one has

$$ \|K_z\|_{A^p_\omega} \asymp \omega(z)^{-1/2} \tau(z)^{2(1-p)/p}, \quad z \in \mathbb{D}. $$

(b) For $\omega \in \mathcal{W}$ and $p > 0$, one has

$$ \|K_z\|_{A^\infty_\omega} \asymp \omega(z)^{-1/2} \tau(z)^{-2}, \quad z \in \mathbb{D}. $$
Lemma E. Let \( (2.12) \) and \( (2.11) \), \( \omega (2.9) \) and Lemma 3.3 in [2]. Without loss of generality, we modified the original version by taking \( z \) for test functions and some estimates.

2.2. Lemma 2.2. Let \( \omega \in W \). Then

(a) For each \( z \in \mathbb{D} \), \( 0 < p < \infty \) and \( \beta \in \mathbb{R} \), one has

\[
|K_z(\zeta)| \lesssim \|K_z\|_{A^2_z} \cdot \|K_\zeta\|_{A^2_\zeta}, \quad \zeta \in D_\delta(z).
\]

(b) For all sufficiently small \( \delta \in (0, m_\tau) \) and \( \omega \in W \), one has

\[
|K_z(\zeta)| \lesssim \|K_z\|_{A^2_z} \cdot \|K_\zeta\|_{A^2_\zeta}, \quad \zeta \in D_\delta(z).
\]

The next lemma generalizes the statement (a) of the above theorem. For the proof, see [1].

Lemma D. Let \( K_z \) be the reproducing kernel of \( A^2_\omega \) where \( \omega \) is a weight in the class \( W \). For each \( z \in \mathbb{D} \), \( 0 < p < \infty \) and \( \beta \in \mathbb{R} \), one has

\[
\int_{\mathbb{D}} |K_z(\xi)|^p \omega(\xi)^{p/2} \tau(\xi)\beta \, dA(\xi) \lesssim \omega(z)^{-p/2} \tau(z)^{2(1-p)+\beta}.
\]

The following result gives estimates for the normalized reproducing kernel \( k_{p,z} \) in \( A^p_\omega \) defined by

\[
k_{p,z} = K_z/\|K_z\|_{A^p_\omega}
\]

for \( z \in \mathbb{D} \).

Lemma 2.2. Let \( \omega \in W \). Then

(a) For each \( z \in \mathbb{D} \), \( 0 < p \leq \infty \), and \( 0 < q < \infty \),

\[
|k_{p,z}(\zeta)|^q \lesssim \tau(z)^{2(1-p)/q} |k_{q,z}(\zeta)|^q, \quad \zeta \in \mathbb{D}.
\]

(b) For \( q = \infty \),

\[
|k_{p,z}(\zeta)| \lesssim \tau(z)^{-2/p} |k_{q,z}(\zeta)|, \quad \zeta \in \mathbb{D}.
\]

(c) For all \( \delta \in (0, m_\tau) \) sufficiently small,

\[
|k_{p,z}(\zeta)|^p \omega(\zeta)^{p/2} \lesssim \tau(z)^{-2}, \quad \zeta \in D_\delta(z).
\]

Proof. The proof is immediate from Theorem[A].

2.2. Test functions and some estimates. The following result on test functions was obtained in [15] and Lemma 3.3 in [2]. Without loss of generality, we modified the original version by taking \( \omega(z)^{p/2} \) instead of \( \omega(z) \) when \( 0 < p < \infty \).

Lemma E. Let \( n \in \mathbb{N} \setminus \{0\} \) and \( \omega \in W \). There is a number \( \rho_0 \in (0, 1) \) such that for each \( a \in \mathbb{D} \) with \( |a| > \rho_0 \) there is a function \( F_{a,n} \), analytic in \( \mathbb{D} \) with

\[
|F_{a,n}(z)\omega(z)^{1/2} \approx 1 \quad \text{if} \quad |z - a| < \tau(a),
\]

and

\[
|F_{a,n}(z)\omega(z)^{1/2} \lesssim \min \left(1, \frac{\min(\tau(a), \tau(z))}{|z - a|} \right)^{3n}, \quad z \in \mathbb{D}.
\]

Moreover,

(a) For \( 0 < p < \infty \), the function \( F_{a,n} \) belongs to \( A^p(\omega) \) with

\[
\|F_{a,n}\|_{A^p_\omega} \lesssim \tau(a)^{2/p}.
\]

(b) For \( p = \infty \), the function \( F_{a,n} \) belongs to \( A^\infty_\omega \) with

\[
\|F_{a,n}\|_{A^\infty_\omega} \lesssim 1.
\]

As a consequence, we have the following pointwise estimates for the derivative of the test functions \( F_{a,n} \). Its proof is a simple application of (2.11).

Lemma 2.3. Let \( n \in \mathbb{N} \setminus \{0\} \) and \( \omega \in W \). For any \( \delta > 0 \) small enough,

\[
|F'_{a,n}(z)\omega(z)^{1/2} \approx 1 + \varphi'(z), \quad z \in D_\delta(a).
\]
The next Proposition is a partial result about the atomic decomposition on $A^p_\omega$ and its proof follows easily from Lemma [F]

**Proposition 2.4.** Let $n \geq 2$ and $\omega \in \mathcal{W}$. Let $\{z_k\}_{k \in \mathbb{N}} \subset \mathbb{D}$ be the sequence defined in Lemma [C]

(a) For $0 < p < \infty$, the function given by

$$F(z) := \sum_{k=0}^{\infty} \lambda_k \frac{F_{z_k,n}(z)}{\tau(z_k)^{2/p}}$$

belongs to $A^p_\omega$ for every sequence $\lambda = \{\lambda_k\} \in \ell^p$. Moreover,

$$\|F\|_{A^p_\omega} \lesssim \|\lambda\|_{\ell^p}.$$

(b) For $p = \infty$, the function given by

$$F(z) := \sum_{k=0}^{\infty} \lambda_k F_{z_k,n}(z)$$

belongs to $A^\infty_\omega$ for every sequence $\lambda = \{\lambda_k\} \in \ell^\infty$. Moreover,

$$\|F\|_{A^\infty_\omega} \lesssim \|\lambda\|_{\ell^\infty}.$$

**Proof.** The proof of (a) can be found in [15, Proposition 2]. To prove (b), estimate the norm of $F$ as follows

$$\|F\|_{A^\infty_\omega} = \sup_{z \in \mathbb{D}} |F(z)| \omega(z)^{1/2} \lesssim \|\lambda\|_{\ell^\infty} \sum_{k=0}^{\infty} |F_{z_k,n}(z)| \omega(z)^{1/2}$$

$$= \|\lambda\|_{\ell^\infty} \left( \sum_{z_k \in D_\delta(z)} |F_{z_k,n}(z)| \omega(z)^{1/2} + \sum_{z_k \notin D_\delta(z)} |F_{z_k,n}(z)| \omega(z)^{1/2} \right)$$

Now, using (2.11) and (iv) of Lemma [C], we have

$$\sum_{z_k \in D_\delta(z)} |F_{z_k,n}(z)| \omega(z)^{1/2} \lesssim 1. \tag{2.14}$$

It remains to show that

$$\sum_{z_k \notin D_\delta(z)} |F_{z_k,n}(z)| \omega(z)^{1/2} \lesssim 1. \tag{2.15}$$

Indeed, by Hölder’s inequality, we have

$$\sum_{z_k \notin D_\delta(z)} |F_{z_k,n}(z)| \omega(z)^{1/2} \leq I(z) \cdot II(z),$$

where

$$I(z) = \sum_{z_k \notin D_\delta(z)} \min \left( \tau(z_k), \tau(z) \right)^2 |F_{z_k,n}(z)| \omega(z)^{1/2},$$

and

$$II(z) = \sum_{z_k \notin D_\delta(z)} |F_{z_k,n}(z)| \omega(z)^{1/2} \cdot \min \left( \tau(z_k), \tau(z) \right)^2.$$
Therefore, by (2.12), we have
\[ \text{This fact together with the finite multiplicity of the covering (see Lemma C) gives} \]
\[ \square \]
Combining this and (2.16) with (2.15) completes the proof.

First we look for the upper bound of \( I(z) \). To do this, we need to consider the covering of \( \{ \xi \in \mathbb{D} : |z - \xi| > \delta \tau(z) \} \) given by
\[ R_j(z) = \{ \xi \in \mathbb{D} : 2^j \delta \tau(z) < |z - \xi| \leq 2^{j+1} \delta \tau(z) \}, \quad j = 0, 1, 2, \ldots \]
and observe that, using (A) of properties of \( \tau \), it is easy to see that, for \( j = 0, 1, 2, \ldots \),
\[ D_\delta(z_k) \subset D_\tau(z), \quad \text{if} \quad z_k \in D_k(z) \quad \text{with} \quad r = 5\delta 2^j \quad \text{and} \quad \tau = \delta 2^{j+1}. \]
This fact together with the finite multiplicity of the covering (see Lemma C) gives
\[ \sum_{z_k \in R_j(z)} \tau(z_k)^2 \lesssim A(D_\tau(z)) \lesssim 2^{2j} \tau(z)^2. \]
Therefore, by (2.12), we have
\[ I(z) \lesssim \sum_{z_k \notin D_\delta(z)} \tau(z_k)^2 |F_{z_k, n}(z)| \omega(z)^{1/2} \]
\[ \lesssim \sum_{z_k \notin D_\delta(z)} \tau(z_k)^2 \min \left( 1, \frac{\min(\tau(z_k), \tau(z))}{|z - z_k|} \right)^{3n} \]
\[ \lesssim \tau(z)^{3n} \sum_{j=0}^\infty \sum_{z_k \in R_j(z)} \frac{\tau(z_k)^2}{|z - z_k|^{3n}} \]
\[ \lesssim \sum_{j=0}^\infty 2^{-3nj} \sum_{z_k \in R_j(z)} \tau(z_k)^2 \]
\[ \lesssim \tau(z)^2 \sum_{j=0}^\infty 2^{(2-3n)j} \lesssim \tau(z)^2. \]
To obtain an upper estimate for (II), notice that since \( n \geq 2 \), (2.12) implies that
\[ II(z) \lesssim \sum_{z_k \notin D_\delta(z)} \min \left( \frac{\tau(z_k), \tau(z)}{|z - z_k|} \right)^{3n-2} \]
\[ \lesssim \tau(z)^{3n-4} \sum_{j=0}^\infty \sum_{z_k \in R_j(z)} \frac{\tau(z_k)^2}{|z - z_k|^{3n}} \]
\[ \lesssim \tau(z)^{-4} \sum_{j=0}^\infty 2^{-3nj} \sum_{z_k \in R_j(z)} \tau(z_k)^2 \]
\[ \lesssim \tau(z)^{-2} \sum_{j=0}^\infty 2^{(2-3n)j} \lesssim \tau(z)^{-2}. \]
Combining this and (2.16) with (2.15) completes the proof. \( \square \)

3. Geometric Characterizations of Carleson Measures

Let \( \mu \) be a positive measure on \( \mathbb{D} \). Denote by \( \hat{\mu}_\delta \) the averaging function defined as
\[ \hat{\mu}_\delta(z) = \mu(D_\delta(z)) \cdot \tau(z)^{-2}, \quad z \in \mathbb{D}, \]
and define the general Berezin transform of $\mu$ by

$$G_t(\mu)(z) = \int_D |k_{t,z}(\zeta)|^t \omega(\zeta)^{t/2} d\mu(\zeta),$$

for every $t > 0$ and $z \in \mathbb{D}$.

In this section we recall recent characterizations of $q$-Carleson measures for $A^p_\omega$ for any $0 < p, q \leq \infty$ in terms of the averaging function $\hat{\mu}_\delta$ and the general Berezin transform $G_t(\mu)$. For the proofs of all theorems in this section, see Section 3 of [1].

3.1. Carleson measures. We begin with the definition of $q$-Carleson measures.

**Definition 3.1.** Let $\mu$ be a positive measure on $\mathbb{D}$ and fix $0 < p, q < \infty$. We say that $\mu$ is a $q$-Carleson measure for $A^p_\omega$ if the inclusion $I_\mu : A^p_\omega \hookrightarrow L^q_\omega$ is bounded.

The following theorem characterizes the $q$-Carleson measures when $0 < p \leq q < \infty$.

**Theorem B.** Let $\mu$ be a finite positive Borel measure on $\mathbb{D}$. Assume $0 < p \leq q < \infty$, $s = p/q$, $0 < t < \infty$. The following conditions are equivalent:

(a) The measure $\mu$ is a $q$-Carleson measure for $A^p_\omega$.

(b) The function

$$\tau(z)^{2(1-1/s)}G_t(\mu)(z)$$

belongs to $L^\infty(\mathbb{D}, dA)$.

(c) The function

$$\tau(z)^{2(1-1/s)}\hat{\mu}_\delta(z)$$

belongs to $L^\infty(\mathbb{D}, dA)$ for any sufficiently small $\delta > 0$.

Now we characterize $q$-Carleson measures when $0 < q < p < \infty$.

**Theorem C.** Let $\mu$ be a finite positive Borel measure on $\mathbb{D}$. Assume $0 < q < p < \infty$ and $s = p/q$. The following conditions are all equivalent:

(a) The measure $\mu$ is a $q$-Carleson measure for $A^p_\omega$.

(b) For any (or some) $r > 0$, we have

$$\widehat{\mu}_r \in L^{p/(p-q)}(\mathbb{D}, dA).$$

(c) For any $t > 0$,

$$G_t(\mu) \in L^{p/(p-q)}(\mathbb{D}, dA).$$

3.2. Vanishing Carleson measures.

**Definition 3.2.** Let $\mu$ be a positive measure on $\mathbb{D}$ and fix $0 < p, q < \infty$. We say that $\mu$ is a vanishing $q$-Carleson measure for $A^p_\omega$ if the inclusion $I_\mu : A^p_\omega \hookrightarrow L^q_\omega$ is compact, or equivalently, if

$$\int_\mathbb{D} |f_n(z)|^q \omega(z)^{q/2} d\mu(z) \to 0$$

whenever $f_n$ is bounded in $A^p_\omega$ and converges to zero uniformly on each compact subsets of $\mathbb{D}$.

The following three theorems characterize vanishing $q$-Carleson measures for $A^p_\omega$ when $0 < p \leq \infty$ and $0 < q < \infty$.

**Theorem D.** Given $\tau \in L^*$, let $\mu$ be a finite positive Borel measure on $\mathbb{D}$. Assume $0 < p \leq q < \infty$, $s = p/q$, $0 < t < \infty$. The following statements are all equivalent:
Lemma 4.1. Let \( \omega \). 

Theorem 3.3. Given \( \tau \in L^s \), let \( \mu \) be a finite positive Borel measure on \( \mathbb{D} \). Assume \( 0 < q < \infty \). The following conditions are all equivalent:

(a) \( \mu \) is a q-Carleson measure for \( A^q_{\infty} \).
(b) \( \mu \) is a vanishing q-Carleson measure for \( A^q_{\infty} \).
(c) For any sufficiently small \( \delta > 0 \), we have
\[
\hat{\mu}_\delta \in L^1(\mathbb{D}, dA).
\]
(d) For any \( t > 0 \), we have
\[
G_t(\mu) \in L^1(\mathbb{D}, dA).
\]

Theorem E. Given \( \tau \in L^s \), let \( \mu \) be a finite positive Borel measure on \( \mathbb{D} \). Assume \( 0 < q < \infty \). The following statements are equivalent:

(a) \( \mu \) is a q-Carleson measure for \( A^q_{\infty} \).
(b) \( \mu \) is a vanishing q-Carleson measure for \( A^q_{\infty} \).

4. Embedding Theorems

In this section we establish embedding theorems of \( S^p_\omega \) into \( L^q(\mathbb{D}, d\mu) \) for \( 0 < p,q \leq \infty \) and \( \omega \in \mathcal{W} \), where \( S^p_\omega \) are given in \([1.13]\) and \([1.14]\). We start with the case \( 0 < p < q < \infty \).

Lemma 4.1. Let \( \omega \in \mathcal{W} \) and \( 0 < p \leq q < \infty \). Let \( \mu \) be a positive Borel measure on \( \mathbb{D} \). Then

(a) \( I_\mu : S^p_\omega \rightarrow L^q(\mathbb{D}, d\mu) \) is bounded if and only if for each \( \delta > 0 \) small enough,
\[
K_{\mu, \omega}(z) = \sup_{z \in \mathbb{D}} \frac{1}{\tau(z)^{2q/p}} \int_{D_\delta(z)} (1 + \varphi'(\xi))^q \omega(\xi)^{-q/2} \, d\mu(\xi) < \infty.
\]

(b) \( I_\mu : S^p_\omega \rightarrow L^q(\mathbb{D}, d\mu) \) is compact if and only if
\[
\lim_{|z| \rightarrow 0^+} \frac{1}{\tau(z)^{2q/p}} \int_{D_\delta(z)} (1 + \varphi'(\xi))^q \omega(\xi)^{-q/2} \, d\mu(\xi) = 0.
\]

Proof. Suppose first that the condition \((4.1)\) holds. Then, by Lemma \([6.4]\) and \((2.4)\), we get
\[
\|f\|_{L^q(\mathbb{D}, d\mu)} = \int_{\mathbb{D}} |f(z)|^q \, d\mu(z) = \sum_{k=0}^\infty \int_{D_\delta(z_k)} |f(z)|^q \, d\mu(z) \\
\leq \sum_{k=0}^\infty \int_{D_\delta(z_k)} |f(z)|^q \frac{\omega(z)^{q/2}}{(1 + \varphi'(z))^q} (1 + \varphi'(z))^q \omega(z)^{-q/2} \, d\mu(z) \\
\leq \sum_{k=0}^\infty \left( \int_{D_\delta(z_k)} |f(s)|^p \frac{\omega(s)^{p/2}}{(1 + \varphi'(s))^p} \, dA(s) \right)^{q/p} \left( \int_{D_\delta(z_k)} (1 + \varphi'(z))^q \omega(z)^{-q/2} \, d\mu(z) \right)^{q/p} \\
\leq \sum_{k=0}^\infty \left( \int_{D_\delta(z_k)} |f(s)|^p \frac{\omega(s)^{p/2}}{(1 + \varphi'(s))^p} \, dA(s) \right)^{q/p} \left( \int_{D_\delta(z_k)} (1 + \varphi'(z))^q \omega(z)^{-q/2} \, d\mu(z) \right)^{q/p} \\
\leq \sum_{k=0}^\infty \left( \int_{D_\delta(z_k)} |f(s)|^p \frac{\omega(s)^{p/2}}{(1 + \varphi'(s))^p} \, dA(s) \right)^{q/p} \left( \int_{D_\delta(z_k)} (1 + \varphi'(z))^q \omega(z)^{-q/2} \, d\mu(z) \right)^{q/p} \frac{\tau(z)^{2q/p}}{\int_{D_\delta(z_k)} \frac{\omega(s)^{q/2}}{(1 + \varphi'(s))^q} \, d\mu(z)}.
\]
for small enough $\delta > 0$. By applying our assumption, we have

$$
\int_{\mathbb{D}} |f(z)|^q \, d\mu(z) \leq K_{\mu,\omega} \sum_{k=0}^{\infty} \left( \int_{D_{3\delta}(z_k)} |f(s)|^p \frac{\omega(s)^{p/2}}{(1 + \varphi'(s))^p} \, dA(s) \right)^{q/p}.
$$

Using a similar argument as in the proof of Theorem 1.1 in [14], Minkowski’s inequality and the finite multiplicity $N$ of the covering $\{D_{3\delta}(z_k)\}$, we get

$$
\left\|f\right\|_{L^q(D, d\mu)}^q \lesssim K_{\mu,\omega} \left( \sum_{k=0}^{\infty} \int_{D_{3\delta}(z_k)} |f(s)|^p \frac{\omega(s)^{p/2}}{(1 + \varphi'(s))^p} \, dA(s) \right)^{q/p} \lesssim K_{\mu,\omega} N^{q/p} \left\|f\right\|_{S^\infty_{p}}^q.
$$

This proves that the embedding $I_\mu : S^p_{\omega} \rightarrow L^q(\mathbb{D}, d\mu)$ is bounded with $\left\|I_\mu\right\|_{L^q(\mathbb{D}, d\mu)} \leq K_{\mu,\omega}$.

Conversely, suppose that $I_\mu : S^p_{\omega} \rightarrow L^q(\mathbb{D}, d\mu)$ is bounded. Let $a \in \mathbb{D}$ with $|a| \geq \rho_0$ that is defined in Lemma $\text{[E]}$ By Lemma $\text{[2,3]}$

$$
|F'_{a,n}(z)|^q(z)^{1/2} \times (1 + \varphi'(z)), \quad z \in D_\delta(a),
$$

(where $F_{a,n}$ is the test function in Lemma $\text{[E]}$), and so

$$
\int_{D_\delta(a)} (1 + \varphi'(z))^q(z)^{-\frac{q}{2}} \, d\mu(z) \lesssim \int_{D_\delta(a)} |F'_{a,n}(z)|^q \, d\mu(z) \lesssim \int_{\mathbb{D}} |F'_{a,n}(z)|^q \, d\mu(z).
$$

Using our assumption, (a) of Lemma $\text{[E]}$ and (1.15), we obtain

$$
\int_{D_\delta(a)} (1 + \varphi'(z))^q(z)^{-\frac{q}{2}} \, d\mu(z) \lesssim \left\|I_\mu\right\|_{L^q(\mathbb{D}, d\mu)} \left\|F'_{a,n}\right\|_{S^\infty_{p}}^q.
$$

Then dividing both sides by $\tau(a)^{2q/p}$ gives

$$
\frac{1}{\tau(a)^{2q/p}} \int_{D_\delta(a)} (1 + \varphi'(z))^q(z)^{-\frac{q}{2}} \, d\mu(z) \leq \left\|I_\mu\right\|_{L^q(\mathbb{D}, d\mu)} \lesssim \left\|I_\mu\right\|_{L^q(\mathbb{D}, d\mu)} < \infty.
$$

and so

$$
\sup_{a \in \mathbb{D}} \frac{1}{\tau(a)^{2q/p}} \int_{D_\delta(a)} (1 + \varphi'(z))^q(z)^{-q/2} \, d\mu(z) \leq \left\|I_\mu\right\|_{L^q(\mathbb{D}, d\mu)} < \infty,
$$

which means that $K_{\mu,\omega} \lesssim \left\|I_\mu\right\|_{L^q(\mathbb{D}, d\mu)}$.

To prove (b), suppose that $I_\mu : S^p_{\omega} \rightarrow L^q(\mathbb{D}, d\mu)$ is compact. Consider the function

$$
f_{a,n}(z) := \frac{F_{a,n}(z)}{\tau(a)^{2q/p}}, \quad \text{for } |a| \geq \rho_0.
$$

As in the proof of Theorem 1 of [15] and using Lemma $\text{[E]}$ we can show that the function $f_{a,n}$ is bounded and converges to zero uniformly on compact subsets of $\mathbb{D}$ when $|a| \rightarrow 1^-$. Therefore, by Lemma $\text{[B]}$ $f'_{a,n}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \rightarrow 1^-$. Thus,

$$
\frac{1}{\tau(a)^{2q/p}} \int_{D_\delta(a)} (1 + \varphi'(z))^q(z)^{-\frac{q}{2}} \, d\mu(z) \lesssim \int_{D_\delta(a)} |f'_{a,n}(z)|^q \, d\mu(z)
$$

$$
\leq \int_{\mathbb{D}} |f'_{a,n}(z)|^q \, d\mu(z) = \left\|I_\mu f'_{a,n}\right\|_{L^q(\mu)}.
$$

Since $I_\mu$ is compact,

$$
\lim_{|a| \rightarrow 1^-} \left\|f'_{a,n}\right\|_{L^q(\mu)} = 0.
$$
and so
\[
\lim_{|a| \to 1} \frac{1}{\tau(a)^{2q/p}} \int_{D_\delta(a)} (1 + \varphi'(z))^q \omega(z)^{-\frac{q}{2}} d\mu(z) = 0.
\]
This shows that (4.2) holds.

Conversely, suppose that (4.2) holds. Let \( \{f_n\} \subset S^p_\omega \) be a bounded sequence converging to zero uniformly on compact subsets of \( \mathbb{D} \) and \( \{z_k\} \) be a \((\delta, \tau)\)-lattice. To prove that \( I_\mu \) is compact, it suffices to show that \( \|f_n\|_{L^q(\mu)} \to 0 \). By the assumption, given any \( \varepsilon > 0 \), there exists \( 0 < r_1 < 1 \) with
\[
(4.3) \quad \frac{1}{\tau(a)^{2q/p}} \int_{D_\delta(a)} (1 + \varphi'(z))^q \omega(z)^{-\frac{q}{2}} d\mu(z) < \varepsilon, \quad r_1 < |a| < 1.
\]
Observe that there is \( r_1 < r_2 < 1 \) such that if a point \( z_j \) of the sequence \( \{z_k\} \) belongs to \( \{z \in \mathbb{D} : |z| \leq r_1\} \), then \( D_\delta(z_j) \subset \{z \in \mathbb{D} : |z| \leq r_2\} \). Therefore, since \( \{f_n\} \) converges to zero uniformly on compact subsets of \( \mathbb{D} \), there exists an integer \( n_0 \) such that
\[
|f_n(z)| < \varepsilon, \quad \text{for } |z| \leq r_2 \text{ and } n \geq n_0.
\]
We split the integration of this function into two parts: the first integration is over \( |z| \leq r_2 \) and the other integration is over \( |z| \geq r_2 \). On the one hand,
\[
(4.4) \quad \int_{|z| \leq r_2} |f_n(z)|^q d\mu(z) < \varepsilon^q.
\]
On the other hand, by Lemma[C] and Lemma[B] we obtain
\[
\int_{|z| > r_2} |f_n(z)|^q d\mu(z) \leq \sum_{|z_k| > r_1} \int_{D_\delta(z_k)} |f_n(z)|^q d\mu(z)
\leq \sum_{|z_k| > r_1} \int_{D_\delta(z_k)} \left( \frac{1}{\tau(z_k)^{2q/p}} \int_{D_\delta(z)} |f_n(s)|^p \frac{\omega(s)^{p/2}}{(1 + \varphi'(s))^p} dA(s) \right)^{q/p} (1 + \varphi'(z))^q \omega(z)^{-\frac{q}{2}} d\mu(z)
\leq \sum_{k=0}^\infty \left( \int_{D_\delta(z_k)} |f_n(s)|^p \frac{\omega(s)^{p/2}}{(1 + \varphi'(s))^p} dA(s) \right)^{q/p} \int_{D_\delta(z_k)} (1 + \varphi'(z))^q \omega(z)^{-\frac{q}{2}} d\mu(z)
\leq \varepsilon \|f_n\|_{S^p_\omega} \sup_{|z_k| > r_1} \frac{1}{\tau(z_k)^{2q/p}} \int_{D_\delta(z_k)} (1 + \varphi'(z))^q \omega(z)^{-\frac{q}{2}} d\mu(z) \leq \varepsilon \|f_n\|_{S^p_\omega} \leq \varepsilon.
\]
These together with (4.4) show that \( I_\mu : S^p_\omega \to L^q(\mathbb{D}, d\mu) \) is compact. \( \square \)

To characterize boundedness and compactness of \( I_\mu : S^p_\omega \to L^q(\mathbb{D}, d\mu) \) with \( 0 < q < p < \infty \), consider the function \( F_{\delta, \mu}(\varphi) \) defined by
\[
(4.5) \quad F_{\delta, \mu}(\varphi)(z) := \frac{1}{\tau(z)^2} \int_{D_\delta(z)} (1 + \varphi'(\xi))^q \omega(\xi)^{-q/2} d\mu(\xi).
\]
We use Luecking’s approach in [9] based on Khinchine’s inequality. Recall that Rademacher functions \( R_n \) are defined by
\[
R_0(t) = \begin{cases} 
1 & \text{if } 1 \leq t - [t] < 1/2, \\
-1 & \text{if } 1/2 \leq t - [t] < 1; 
\end{cases}
\]
\[
R_n(t) = R_0(2^n t), \quad n \geq 1,
\]
where \([t]\) denotes the largest integer not exceeding \( t \).
Lemma F (Khinchine’s inequality \[9\]). For \(0 < p < \infty\), there exists a positive constant \(C_p\) such that
\[
C_p^{-1} \left( \sum_{k=1}^{n} |\lambda_k|^2 \right)^{p/2} \leq \int_0^1 \left| \sum_{k=1}^{n} \lambda_k R_k(t) \right|^p dt \leq C_p \left( \sum_{k=1}^{n} |\lambda_k|^2 \right)^{p/2},
\]
for all \(n \in \mathbb{N}\) and \(\{\lambda_k\}_{k=1}^{n} \subset \mathbb{C}\).

Lemma 4.2. Let \(\omega \in \mathcal{W}\) and \(0 < q < p < \infty\). Let \(\mu\) be a finite positive Borel measure on \(\mathbb{D}\). Then, the following statements are equivalent:

(a) The operator \(I_{\mu} : L^p(\mathbb{D}, d\mu) \to L^q(\mathbb{D}, d\mu)\) is bounded.
(b) The operator \(I_{\mu} : L^p(\mathbb{D}, d\mu) \to L^q(\mathbb{D}, d\mu)\) is compact.
(c) The function
\[
F_{\delta,\mu}(\varphi) \in L^{p/(p-q)}(\mathbb{D}, dA).
\]

Proof. The implication (b) \(\Rightarrow\) (a) is obvious. To prove that (a) implies (c), suppose that the operator \(I_{\mu} : S^p_{\omega} \to L^q(\mathbb{D}, d\mu)\) is bounded. Let \(\{z_k\}\) be a \((\delta, \tau)\)-lattice on \(\mathbb{D}\). Corresponding to each \(\lambda = \{\lambda_m\}_{m} \in \ell^p\), we consider
\[
f(z) = \sum_{|z_m| \geq \rho_0} \lambda_m f_{z_m,n}(z),
\]
where \(f_{z_m,n}(z) = \frac{F_{z_m,n}(z)}{\tau(z_m)^{2/p}}\) and \(0 < \rho_0 < 1\) as in Lemma 4. By Proposition 2.4 and (1.15),
\[
\|f'||_{S^p_{\omega}} \lesssim \|f||_{A^p_{\omega}} \lesssim \|\lambda||_{\ell^p}.
\]

Note that as an application of Khinchine’s inequality (Lemma F), replace \(\lambda_m\) with the Rademacher functions \(R_m(t)\lambda_m\), and then integrate with respect to \(t\) from 0 and 1, which yields
\[
\left( \sum_{|z_m| \geq \rho_0} |\lambda_m|^2 \right)^{q/2} \leq \int_0^1 \sum_{|z_m| \geq \rho_0} R_m(t) |f_{z_m,n}(z)|^q dt
\]
and so
\[
\int_\mathbb{D} \left( \sum_{|z_m| \geq \rho_0} |\lambda_m|^2 |f_{z_m,n}(z)|^2 \omega(z) \right)^{q/2} d\mu(z)
\]
\[
\lesssim \int_\mathbb{D} \int_0^1 \sum_{|z_m| \geq \rho_0} R_m(t) |\lambda_m f_{z_m,n}(z)|^q \omega(z)^{n/2} dt d\mu(z)
\]
\[
= \int_0^1 \int_{\mathbb{D}} \sum_{|z_m| \geq \rho_0} R_m(t) |\lambda_m f_{z_m,n}(z)|^q \omega(z)^{n/2} d\mu(z) dt
\]
\[
\lesssim \int_0^1 \|f'||_{S^p_{\omega}}^q dt = \|f'||_{S^p_{\omega}}^q \lesssim \|f||_{A^p_{\omega}}^q \lesssim \|\lambda||_{\ell^p}^q.
\]
By Lemmas C and 2.3
\[
\sum_{|z_m| \geq \rho_0} \frac{|\lambda_m|^q}{\tau(z_m)^{2/p}} \int_{D_{3\delta}(z_m)} (1 + \varphi'(\xi))^q \omega(\xi)^{-\frac{q}{2}} \, d\mu(\xi)
\]
\[
\lesssim \sum_{|z_m| \geq \rho_0} |\lambda_m|^q \int_{D_{3\delta}(z_m)} |f'_{z_m,n}(\xi)|^q \, d\mu(\xi)
\]
\[
= \int_D \sum_{|z_m| \geq \rho_0} |\lambda_m|^q |f'_{z_m,n}(\xi)|^q \chi_{D_{3\delta}(z_m)}(\xi) \, d\mu(\xi),
\]
where \( \chi_{D_{3\delta}(z_m)}(\xi) \) denotes the characteristic function of the set \( D_{3\delta}(z_m) \). Now, by the fact that \( \sum_k^{\infty} z_m^k \leq (\sum_k^{\infty} z_m)^k \), \( k \geq 1, z_m \geq 0 \) for \( q \geq 2 \), we get
\[
\int_D \sum_{|z_m| \geq \rho_0} |\lambda_m|^q |f'_{z_m,n}(\xi)|^q \chi_{D_{3\delta}(z_m)}(\xi) \, d\mu(\xi)
\]
\[
= \int_D \sum_{|z_m| \geq \rho_0} \left( |\lambda_m|^2 |f'_{z_m,n}(\xi)|^2 \chi_{D_{3\delta}(z_m)}(\xi) \right)^{q/2} \, d\mu(\xi)
\]
\[
\lesssim \int_D \left( \sum_{|z_m| \geq \rho_0} |\lambda_m|^2 |f'_{z_m,n}(\xi)|^2 \right)^{q/2} \, d\mu(z).
\]
For \( q < 2 \), by Hölder’s inequality and Lemma C we get
\[
\int_D \sum_{|z_m| \geq \rho_0} |\lambda_m|^q |f'_{z_m,n}(\xi)|^q \chi_{D_{3\delta}(z_m)}(\xi) \, d\mu(\xi)
\]
\[
\leq \int_D \left( \sum_{|z_m| \geq \rho_0} |\lambda_m|^2 |f'_{z_m,n}(\xi)|^2 \right)^{q/2} \left( \sum_{|z_m| \geq \rho_0} \chi_{D_{3\delta}(z_m)}(\xi) \right)^{1-\frac{q}{2}} \, d\mu(z)
\]
\[
\lesssim N^{1-\frac{q}{2}} \int_D \left( \sum_{|z_m| \geq \rho_0} |\lambda_m|^2 |f'_{z_m,n}(\xi)|^2 \right)^{q/2} \, d\mu(z).
\]
Therefore, for \( q < 2 \) and \( q \geq 2 \), we have
\[
\sum_{|z_m| \geq \rho_0} \frac{|\lambda_m|^q}{\tau(z_m)^{2/p}} \int_{D_{3\delta}(z_m)} (1 + \varphi'(\xi))^q \omega(\xi)^{-\frac{q}{2}} \, d\mu(\xi)
\]
\[
\lesssim \int_D \sum_{|z_m| \geq \rho_0} |\lambda_m|^q |f'_{z_m,n}(\xi)|^q \chi_{D_{3\delta}(z_m)}(\xi) \, d\mu(\xi)
\]
\[
\lesssim \max(1, N^{1-\frac{q}{2}}) \int_D \left( \sum_{|z_m| \geq \rho_0} |\lambda_m|^2 |f'_{z_m,n}(\xi)|^2 \right)^{q/2} \, d\mu(z).
\]
By applying 4.7, we have
\[
\sum_{|z_m| \geq \rho_0} \frac{|\lambda_m|^q}{\tau(z_m)^{2/p}} \int_{D_{3\delta}(z_m)} (1 + \varphi'(\xi))^q \omega(\xi)^{-\frac{q}{2}} \, d\mu(\xi) \lesssim \|\lambda\|^q_{L^p}.
\]
Thus, taking $|b_m| = |\lambda_m|^q \in \ell^{p/q}$ and using the duality $(\ell^p)^* = \ell^q$, we see that the sequence
\[
\left\{ \frac{1}{\tau(z_m)^{2/p}} \int_{D_{\delta}(z_m)} (1 + \varphi'(\xi)) q \omega(\xi) - \frac{q}{2} d\mu(\xi) \right\}_m \in \ell^{p/(p-q)}.
\]

Observe that there is $r_0 < r_1 < 1$ such that if a point $z_j$ of the sequence $\{z_j\}$ belongs to \( z \in \mathbb{D} : |z| \leq r_0 \), then $D_{\delta}(z_j) \subset \{ z \in \mathbb{D} : |z| \leq r_1 \}$. Thus, by Lemma 3 and (2.3), we get
\[
\int_{|z| \geq r_1} \left( \frac{1}{\tau(z)^2} \int_{D_{\delta}(z)} (1 + \varphi'(\xi)) q \omega(\xi) - \frac{q}{2} d\mu(\xi) \right)^{p/(p-q)} dA(z)
\]
\[
\lesssim \sum_{|z_m| \geq r_0} \left( \frac{1}{\tau(z_m)^{2/p}} \int_{D_{\delta}(z_m)} (1 + \varphi'(\xi)) q \omega(\xi) - \frac{q}{2} d\mu(\xi) \right)^{p/(p-q)} dA(z)
\]
\[
\lesssim \sum_{|z_m| \geq r_0} \left( \frac{1}{\tau(z_m)^{2/p}} \int_{D_{\delta}(z_m)} (1 + \varphi'(\xi)) q \omega(\xi) - \frac{q}{2} d\mu(\xi) \right)^{p/(p-q)} < \infty.
\]

Therefore, since
\[
\int_{|z| \leq r_1} \left( \frac{1}{\tau(z)^2} \int_{D_{\delta}(z)} (1 + \varphi'(s)) q \omega(s) - \frac{q}{2} d\mu(s) \right)^{p/(p-q)} d\mu(z) < \infty,
\]
we obtain
\[
\int_{\mathbb{D}} F_{\delta, \mu}(\varphi)(z)^{p/(p-q)} dA(z) = \int_{\mathbb{D}} \left( \frac{1}{\tau(z)^2} \int_{D_{\delta}(z)} (1 + \varphi'(\xi)) q \omega(\xi) - \frac{q}{2} d\mu(\xi) \right)^{p/(p-q)} dA(z)
\]
\[
\lesssim \int_{|z| < r_1} \left( \frac{1}{\tau(z)^2} \int_{D_{\delta}(z)} (1 + \varphi'(s)) q \omega(s) - \frac{q}{2} d\mu(s) \right)^{p/(p-q)} dA(z)
\]
\[
+ \int_{|z| > r_1} \left( \frac{1}{\tau(z)^2} \int_{D_{\delta}(z)} (1 + \varphi'(s)) q \omega(s) - \frac{q}{2} d\mu(s) \right)^{p/(p-q)} dA(z) < \infty.
\]

This proves the desired result.

Finally, it remains to prove that (c) implies (b). Suppose that (4.6) holds and let $\{ f_n \}$ be a bounded sequence of functions belonging to $S^p_\rho$ that converges uniformly to zero on compact subsets of $\mathbb{D}$. Since the function $\tau$ is decreasing and converges to zero as $|z| \to 1$, there is $r' > 0$ such that
\[
D_{\delta/2}(z) \subset \left\{ \xi \in \mathbb{D} : |\xi| > r/2 \right\}, \quad \text{if } |z| > r > r'.
\]

On the other hand, it also follows from (2.4) that
\[
|f_n(z)|^q \frac{\omega(z)^{q/2}}{(1 + \varphi'(z))^q} \lesssim \frac{1}{\tau(z)^2} \int_{D_{\delta}(z)} |f_n(s)|^q \frac{\omega(s)^{q/2}}{(1 + \varphi'(s))^q} dA(s).
\]
Integrate with respect to $d\mu$, and use (4.8) and (2.3) to obtain
\[
\int_{|z| \geq r} |f_n(\xi)|^q d\mu(\xi)
\]
\[
\lesssim \int_{|\xi| \geq r/2} |f_n(\xi)|^q \frac{\omega(\xi)^{q/2}}{(1 + \varphi'(\xi))^q} \left( \frac{1}{\tau(\xi)^2} \int_{D_{\delta}(\xi)} (1 + \varphi'(z))^q \omega(z)^{-q/2} d\mu(z) \right) dA(\xi)
\]
By (c), for each \( \varepsilon > 0 \), there is an \( r_0 > r' \) such that
\[
\int_{|\xi| \geq r_0/2} \frac{1}{\tau(\xi)^2} \int_{D_{\delta}(\xi)} (1 + \varphi'(z))^{q/2} \omega(z)^{-q/2} d\mu(z) \geq r_0/2 \]
\[
\left( \frac{1}{\tau(\xi)^2} \int_{D_{\delta}(\xi)} (1 + \varphi'(z))^{q/2} \omega(z)^{-q/2} d\mu(z) \right)^{p/(p-q)} dA(\xi) < \varepsilon^{p/(p-q)}.
\]
Combining this with Hölder’s inequality, we have
\[(4.10) \int_{|\xi| \geq r_0} |f_n(\xi)|^q d\mu(\xi) \lesssim \|f_n\|_{S^\infty}^q \left( \int_{|\xi| \geq r_0/2} \frac{1}{\tau(\xi)^2} \int_{D_{\delta}(\xi)} (1 + \varphi'(z))^{q/2} \omega(z)^{-q/2} d\mu(z) \right)^{p/(p-q)} dA(\xi) \lesssim \varepsilon.
\]
This together with the fact that
\[
\lim_{n \to \infty} \int_{|\xi| \leq r_0} |f_n(\xi)|^q d\mu(\xi) = 0
\]
gives \( \lim_{n \to \infty} \|f_n\|_{L^q(\mu)} = 0 \), which completes the proof.

We finish this section with the case \( 0 < q < \infty \) and \( p = \infty \).

**Lemma 4.3.** Let \( \omega \in \mathcal{W} \), \( 0 < q < \infty \), and \( \mu \) be a finite positive Borel measure on \( \mathbb{D} \). Then the following statements are equivalent:

(a) The operator \( I_\mu : S^\infty_\omega \to L^q(\mathbb{D}, d\mu) \) is bounded.

(b) The operator \( I_\mu : S^\infty_\omega \to L^q(\mathbb{D}, d\mu) \) is compact.

(c) The function
\[(4.11) F_{\delta,\mu}(\varphi) \in L^1(\mathbb{D}, dA).
\]

**Proof.** Suppose first that the operator \( I_\mu : S^\infty_\omega \to L^q(\mathbb{D}, d\mu) \) is bounded. Let \( \{z_m\}_m \) be a \((\delta, \tau)\)-lattice on \( \mathbb{D} \). Corresponding to each \( \lambda = \{\lambda_m\}_m \in \ell^\infty \), we consider again
\[
f(z) = \sum_{|z_m| \geq \rho_0} \lambda_m F_{z_m,n}(z),
\]
where \( F_{z_m,n}(z) \) is in Lemma 4. By Proposition 2.4 and (1.16), we have
\[
\|f'\|_{S^\infty} \leq \|f\|_{A^\infty} \lesssim \|\lambda\|_{\ell^\infty}.
\]

By our assumption, we get
\[
\int_{\mathbb{D}} \left| \sum_{|z_m| \geq \rho_0} \lambda_m F'_{z_m,n}(z) \right|^q \omega(z)^{q/2} d\mu(z) \lesssim \|\lambda\|_{\ell^\infty}^q
\]
and so
\[
\int_{\mathbb{D}} \left( \sum_{|z_m| \geq \rho_0} |\lambda_m|^2 |F'_{z_m,n}(z)|^2 \omega(z) \right)^{q/2} d\mu(z) \lesssim \|\lambda\|_{\ell^\infty}^q.
\]
This together with Lemma 2.3 and Hölder’s inequality imply that
\[
\sum_{|z_m| \geq \rho_0} |\lambda_m|^q \int_{D_{3\delta}(z_m)} \left( 1 + \varphi'(\xi) \right)^q \omega(\xi)^{-\frac{q}{2}} \, d\mu(\xi)
\leq \max(1, N^{1 - \frac{q}{2}}) \int_{D} \left( \sum_{|z_m| \geq \rho_0} |\lambda_m|^2 |F_{z_m,n}(\xi)|^2 \omega(\xi) \right)^{q/2} \, d\mu(z) \lesssim |\lambda|^q_{\ell^\infty}.
\]
Then, taking $|\lambda_m| = 1$ gives
\[
(4.12) \quad \sum_{|z_m| \geq \rho_0} \int_{D_{3\delta}(z_m)} \left( 1 + \varphi'(\xi) \right)^q \omega(\xi)^{-\frac{q}{2}} \, d\mu(\xi) \lesssim 1.
\]
As in the previous proof, by Lemma 2.3 and (4.12), we get
\[
\int_{|z| \geq r_1} \left( \frac{1}{\tau(z)^2} \int_{D_\delta(z)} \left( 1 + \varphi'(\xi) \right)^q \omega(\xi)^{-q/2} \, d\mu(\xi) \right) \, dA(z)
\leq \sum_{|z_m| \geq \rho_0} \int_{D_{3\delta}(z_m)} \left( \frac{1}{\tau(z)^2} \int_{D_{\delta}(z)} \left( 1 + \varphi'(\xi) \right)^q \omega(\xi)^{-q/2} \, d\mu(\xi) \right) \, dA(z)
\leq \sum_{|z_m| \geq \rho_0} \int_{D_{3\delta}(z_m)} \left( 1 + \varphi'(\xi) \right)^q \omega(\xi)^{-q/2} \, d\mu(\xi) < \infty.
\]
Combining this with the fact that
\[
\int_{|z| \leq r_1} \left( \frac{1}{\tau(z)^2} \int_{D_\delta(z)} \left( 1 + \varphi'(\xi) \right)^q \omega(\xi)^{-q/2} \, d\mu(\xi) \right) \, dA(z) < \infty,
\]
we have the desired result—see (4.5).

It remains to show that (c) implies (b). Let $\{f_n\}$ be a bounded sequence of functions in $S_{\Omega}^\infty$ converging uniformly to zero on compact subsets of $\mathbb{D}$. Since the function $\tau(z)$ is decreasing and converges to zero as $|z| \to 1$, there is $r' > 0$ such that
\[
(4.13) \quad D_{\delta/2}(z) \subset \left\{ \xi \in \mathbb{D} : |\xi| > r/2 \right\}, \quad \text{if} \quad |z| > r > r'.
\]
On the other hand, it follows from (2.4) that
\[
|f_n(z)|^q \frac{\omega(z)^{q/2}}{(1 + \varphi'(z))^q} \lesssim \frac{1}{\tau(z)^2} \int_{D_{\delta}(z)} |f_n(s)|^q \frac{\omega(s)^{q/2}}{(1 + \varphi'(s))^q} \, dA(s).
\]
Integrate with respect to $d\mu$, and use (4.13), (2.3), and (2.4), to obtain
\[
\begin{aligned}
\int_{|z| \geq r} |f_n(\xi)|^q \, d\mu(\xi)
\lesssim \int_{|\xi| \geq r/2} |f_n(\xi)|^q \frac{\omega(\xi)^{q/2}}{(1 + \varphi'(\xi))^q} \left( \frac{1}{\tau(\xi)^2} \int_{D_{\delta}(\xi)} \left( 1 + \varphi'(\xi) \right)^q \omega(\xi)^{-q/2} \, d\mu(\xi) \right) \, dA(\xi)
\lesssim \|f_n\|^q_{S_{\Omega}^\infty} \int_{|\xi| \geq r/2} \frac{1}{\tau(\xi)^2} \int_{D_{\delta}(\xi)} \left( 1 + \varphi'(\xi) \right)^q \omega(\xi)^{-q/2} \, d\mu(\xi) \, dA(\xi).
\end{aligned}
\]
Now the rest follows as in the previous proof. □
5. PROOFS OF THEOREMS [1.1] AND [1.2]

5.1. Proof of Theorem [1.1] (A). Let \(0 < p \leq q < \infty\). By (1.15),

\[
\|G_I(\phi,g)f(z)\|_{A_q^\infty} \leq \int_D |f'(\phi(z))| |g(z)| \frac{\omega(z)^{q/2}}{(1 + \varphi'(z))^q} dA(z)
\]

\[
= \int_D |f'(z)|^q d\mu_{\phi,\omega,g}(z) = \|f'\|^q_{L^q(D,d\mu_{\phi,\omega,g})}.
\]

Therefore, \(G_I(\phi,g) : A_p^q \to A_q^\infty\) is bounded if and only if \(I_{\mu_{\phi,\omega,g}} : S_p^q \to L^q(\mu_{\phi,\omega,g})\) is bounded. Using (a) of Lemma [4.1], this is equivalent to

\[
\sup_{z \in D} \frac{1}{\tau(z)^{2q/p}} \int_{D_{\delta}(z)} (1 + \varphi'(\xi))^q \omega(\xi)^{-q/2} d\mu_{\phi,\omega,g}(\xi) < \infty,
\]

which, by Theorem [13] is equivalent to

\[
\sup_{z \in D} \tau(z)^{2(1-q/p)} \int_D |k_{q,z}(\xi)|^q \omega(\xi)^{q/2} d\nu_{\phi,\omega,g}(\xi) < \infty.
\]

Now, by (a) of Lemma [2.2] we get

(5.1)

\[
\tau(z)^{2(1-q/p)} \int_D |k_{q,z}(\xi)|^q \omega(\xi)^{q/2} d\nu_{\phi,\omega,g}(\xi) \leq \int_D |k_{p,z}(\xi)|^q \omega(\xi)^{q/2} d\nu_{\phi,\omega,g}(\xi)
\]

\[
= \int_D |k_{p,z}(\xi)|^q (1 + \varphi'(\xi))^q d\mu_{\phi,\omega,g}(\xi)
\]

\[
= \int_D |k_{p,z}(\xi)|^q |g(z)|^q \frac{(1 + \varphi'(\xi))^q}{(1 + \varphi'(\xi))^q} \omega(z)^{q/2} dA(\xi)
\]

\[
= GB_{1,p,q}^\phi.
\]

Thus, \(G_I(\phi,g)\) is bounded if and only if \(GB_{1,p,q}^\phi(g(z)) \in L^\infty(D,dA)\). Compactness can be proved similarly using (b) of Lemma [4.1].

5.2. Proof of Theorem [1.1] (B). Boundedness. Let \(0 < p < q = \infty\) and suppose first that (1.4) holds. Then by (1.16) and our assumption, we have

\[
\|G_I(\phi,g)f\|_{A_q^\infty} \leq \sup_{z \in D} |f'(\phi(z))| |g(z)| \frac{\omega(z)^{1/2}}{(1 + \varphi'(z))}
\]

\[
\leq \sup_{z \in D} M_{g,\phi,\omega}(z) \sup_{z \in D} \frac{|f'(\phi(z))| \omega(\phi(z))^{1/2}}{(1 + \varphi'(\phi(z)))} \Delta \varphi(\phi(z))^{-1/p}
\]

\[
\leq \sup_{z \in D} M_{g,\phi,\omega}(z) \sup_{z \in D} \frac{|f'(\phi(z))| \omega(\phi(z))^{1/2}}{(1 + \varphi'(\phi(z)))} \tau(\phi(z))^{2/p}.
\]

Therefore, by (2.4),

\[
\|G_I(\phi,g)f\|_{A_q^\infty} \leq \sup_{z \in D} \left( \int_{D_{\delta}(\phi(z))} \frac{|f'(\xi)|^p \omega(\xi)^{p/2}}{(1 + \varphi'(\xi))^p} dA(\xi) \right)^{1/p}
\]

\[
\leq \left( \int_D \frac{|f'(\xi)|^p \omega(\xi)^{p/2}}{(1 + \varphi'(\xi))^p} dA(\xi) \right)^{1/p} \lesssim \|f\|_{A_p^q},
\]
which implies that $GI_{\phi,g} : A^p_\omega \to A^p_\omega$ is bounded.

Conversely, suppose that the operator $GI_{\phi,g} : A^p_\omega \to A^\omega_\omega$ is bounded. Choose $\xi \in \mathbb{D}$ so that $|\phi(\xi)| > \rho_0$, and consider the function $f_{\phi(\xi),n,p}$ given by

$$f_{\phi(\xi),n,p} := \frac{F_{\phi(\xi),n,p}}{\tau(\phi(\xi))^{2/p}},$$

where $F_{\phi(\xi),n,p}$ is the test function in Lemma 2. Notice that $f_{\phi(\xi),n,p}$ is in $A^p_\omega$ and $\|f_{\phi(\xi),n,p}\|_{A^p_\omega} \approx 1$. By our assumption, we get

$$\|GI_{\phi,g}(f_{\phi(\xi),n,p})\|_{A^\omega_\omega} \geq \sup_{z \in B} \frac{|f'_{\phi(\xi),n,p}(z)||g(z)|}{(1 + \varphi'(z))} \omega(z)^{1/2} \geq \sup_{z \in B} \frac{|F'_{\phi(\xi),n,p}(z)||g(z)|}{\tau(\phi(\xi))^{2/p}(1 + \varphi'(z))} \omega(z)^{1/2} \geq \sup_{\xi \in D} \frac{|F'_{\phi(\xi),n,p}(\phi(\xi))||g(\xi)|}{\tau(\phi(\xi))^{2/p}(1 + \varphi'(\xi))} \omega(\xi)^{1/2}.$$

By Lemma 2, we have

$$|F'_{\phi(\xi),n,p}(\xi)|\omega(\xi)^{1/2} \approx (1 + \varphi'(\xi)), \quad z \in D(\phi(\xi)),$$

and so

$$\infty > \|GI_{\phi,g}(f_{\phi(\xi),n,p})\|_{A^\omega_\omega} \geq |g(\xi)| \frac{(1 + \varphi'(\phi(\xi)))}{(1 + \varphi'(\xi))} \times \frac{\omega(\xi)^{1/2}}{\omega(\phi(\xi))^{1/2}} \tau(\phi(\xi))^{-2/p} \times \frac{\omega(\xi)^{1/2}}{\omega(\phi(\xi))^{1/2}} \Delta \varphi(\phi(\xi))^{1/p} = M_{g,\phi,\omega}(\xi).$$

On the other hand, by taking $f(z) = z$ and using the boundedness of the operator $GI_{\phi,g} : A^p_\omega \to A^\omega_\omega$, we obtain

$$\|GI_{\phi,g}f\|_{A^\omega_\omega} = \sup_{z \in B} \frac{|g(z)|}{(1 + \varphi'(z))} \omega(z)^{1/2} \lesssim \|f\|_{A^p_\omega} < \infty.$$

Therefore, in the case of $|\phi(\xi)| \leq \rho_0$, $\xi \in \mathbb{D}$, we have

$$|M_{g,\phi,\omega}(\xi)| = |g(\xi)| \frac{(1 + \varphi'(\phi(\xi)))}{(1 + \varphi'(\xi))} \times \frac{\omega(\xi)^{1/2}}{\omega(\phi(\xi))^{1/2}} \Delta \varphi(\phi(\xi))^{1/p} \times \frac{\omega(\xi)^{1/2}}{\omega(\phi(\xi))^{1/2}} \tau(\phi(\xi))^{-2/p} \leq C_1 \frac{|g(\xi)|}{(1 + \varphi'(\xi))} \omega(\xi)^{1/2} < \infty,$$

where

$$C_1 = \sup_{|\phi(\xi)| \leq \rho_0} \left\{ \frac{(1 + \varphi'(\phi(\xi)))\omega(\phi(\xi))^{1/2}}{\tau(\phi(\xi))^{-2/p}} \right\} < \infty.$$

Combining this with (5.2) completes the proof of boundedness.
Compactness. Suppose now that $GI_{(\phi,g)} : A^p_\omega \to A^\omega_\infty$ is compact. Then, since $f_{\phi(z),n,p}$ converges to zero uniformly on compact subsets of $D$ as $|\phi(z)| \to 1$ (see Lemma 3.1 of (15), it follows that

$$\|GI_{(\phi,g)}(f_{\phi(z),n,p})\|_{A^\omega_\infty} \to 0$$

as $|\phi(z)| \to 1$. Thus, by (5.2),

$$0 = \lim_{|\phi(z)| \to 1^-} \|GI_{(\phi,g)}(f_{\phi(z),n,p})\|_{A^\omega_\infty} \geq \lim_{|\phi(z)| \to 1^-} M_{g,\phi,\omega}(z).$$

To prove the converse, let $\{f_n\}$ be a bounded sequence of functions in $A^p_\omega$ converging uniformly to zero on compact subsets of $D$. Since (1.4) holds, for each $\varepsilon > 0$, there exists an $r_0 > 0$ such that

$$M_{g,\phi,\omega}(z) = |g(z)| \frac{1 + \varphi'(z)}{1 + \varphi(z)} \omega(z) \frac{1}{\omega(z)} \Delta \varphi(z)^{1/2} \Delta \varphi(z) < \varepsilon,$$

whenever $|\phi(z)| > r_0$. In addition, by (2.3),

$$\frac{|f_n'(z)|}{1 + \varphi'(z)} \omega(z) \frac{1}{\Delta \varphi(z)} \Delta \varphi(z)^{1/2} \Delta \varphi(z) < \varepsilon,$$

whenever $|\phi(z)| > r_0$.

For $|\phi(z)| \geq r_0$, we have

$$\sup_{|\phi(z)| \leq r_0} \frac{|f_n'(z)|}{1 + \varphi'(z)} \omega(z) \frac{1}{\Delta \varphi(z)} \Delta \varphi(z)^{1/2} \Delta \varphi(z) \to 0,$$

as $n \to \infty$ because the sequence of functions $f_n'$ also converges uniformly to zero on compact subsets of $D$ (see Lemma (3)). This together with (5.3) yields

$$\|GI_{(\phi,g)}(f_n)\|_{A^\omega_\infty} \times \sup_{z \in D} \frac{|f_n'(z)|}{1 + \varphi'(z)} \omega(z) \frac{1}{\Delta \varphi(z)} \Delta \varphi(z)^{1/2} \Delta \varphi(z) \to 0, \quad n \to \infty,$$

which shows the compactness of the operator $GI_{(\phi,g)} : A^p_\omega \to A^\omega_\infty$.

5.3. Proof of Theorem (1.1)(C). Boundedness. Let $p = q = \infty$ and suppose that (1.5) holds. Using (1.16), we get

$$\|GI_{(\phi,g)}(f)\|_{A^\omega_\infty} \geq \sup_{z \in D} \frac{|f'(z)|}{1 + \varphi'(z)} \omega(z) \frac{1}{\Delta \varphi(z)} \Delta \varphi(z)^{1/2} \Delta \varphi(z) \leq \sup_{z \in D} N_{g,\phi,\omega}(z) \sup_{z \in D} \frac{|f'(z)|}{1 + \varphi'(z)} \omega(z) \frac{1}{\Delta \varphi(z)} \Delta \varphi(z)^{1/2} \Delta \varphi(z) \leq \sup_{z \in D} N_{g,\phi,\omega}(z) \sup_{z \in D} \frac{|f(z)|}{1 + \varphi'(z)} \omega(z) \frac{1}{\Delta \varphi(z)} \Delta \varphi(z)^{1/2} \Delta \varphi(z) \leq \|f\|_{A^\omega_\infty}$$

which shows that $GI_{(\phi,g)}$ is bounded.

Conversely, suppose that $GI_{(\phi,g)} : A^\omega_\infty \to A^\omega_\infty$ is bounded. Let $\xi \in \mathbb{D}$ be such that $|\phi(z)| > \rho_0$. Then $F_{\phi(z),n,p} \in A^\omega_\omega$ and $\|F_{\phi(z),n,p}\|_{A^\omega_\omega} \sim 1$, and hence

$$\|GI_{(\phi,g)}(F_{\phi(z),n,p})\|_{A^\omega_\infty} = \sup_{z \in \mathbb{D}} \frac{|F_{\phi(z),n,p}'(z)|}{1 + \varphi'(z)} \omega(z) \frac{1}{\Delta \varphi(z)} \Delta \varphi(z)^{1/2} \Delta \varphi(z) \geq \frac{|F_{\phi(z),n,p}'(\phi(z))|}{1 + \varphi'(z)} \omega(z) \frac{1}{\Delta \varphi(z)} \Delta \varphi(z)^{1/2} \Delta \varphi(z),$$
By Lemma 2.3,
\[ |F'_{\phi (\xi),n,p}(z)| \omega (z)^{1/2} \sim (1 + \varphi '(z)), \quad z \in D_{\delta}(\phi (\xi)), \]
so
\[
\begin{align*}
\|G I_{\phi ,g}(F_{\phi (\xi),n,p})\|_{A^{\infty}} & \sim |g(\xi)| \frac{(1 + \varphi '(\phi (\xi)))}{(1 + \varphi ' (\xi))} \frac{\omega (\xi)^{1/2}}{\omega (\phi (\xi))^{1/2}} = N I_{g,\phi ,\omega} (\xi).
\end{align*}
\]
(5.4)

To deal with the case \(|\phi (\xi)| \leq \rho_0\), take \(f(z) = z\) and use the boundedness of the operator \(G I_{\phi ,g}\) to obtain
\[
\|G I_{\phi ,g} f\|_{A^{\infty}} = \sup_{z \in \mathbb{D}} \frac{|g(z)|}{(1 + \varphi '(z))} \omega (z)^{1/2} \leq \|f\|_{A^{\infty}} < \infty.
\]
Therefore, when \(|\phi (\xi)| \leq \rho_0\), \(\xi \in \mathbb{D}\), we have
\[
|g(\xi)| \frac{(1 + \varphi '(\phi (\xi)))}{(1 + \varphi ' (\xi))} \frac{\omega (\xi)^{1/2}}{\omega (\phi (\xi))^{1/2}} \leq C_2 \frac{|g(\xi)|}{(1 + \varphi '(\xi))} \omega (\xi)^{1/2} < \infty,
\]
where
\[ C_2 = \sup_{|\phi (\xi)| \leq \rho_0} \left\{ \frac{(1 + \varphi '(\phi (\xi))\omega (\phi (\xi))^{-1}} \right\} < \infty. \]
Combining this with (5.4) completes the proof of boundedness.

**Compactness.** If \(G I_{\phi ,g} : A^{\infty}_{\omega} \rightarrow A^{\infty}_{\omega}\) is compact, then, using (5.4) again, we get
\[
\lim_{|\phi (\xi)| \rightarrow 1^-} N I_{g,\phi ,\omega} (\xi) \leq \lim_{|\phi (\xi)| \rightarrow 1^-} \|G I_{\phi ,g}(f_{\phi (\xi),n,p})\|_{A^{\infty}} = 0.
\]
To prove the converse, let \(\{f_n\}\) be a bounded sequence of functions in \(A^{\infty}_{\omega}\) converging uniformly to zero on compact subsets of \(\mathbb{D}\). By assumption, for any \(\varepsilon > 0\), there exists \(r_0 > 0\) such that
\[
N I_{g,\phi ,\omega} (\xi) = |g(z)| \frac{(1 + \varphi '(\phi (\xi)))}{(1 + \varphi ' (\xi))} \frac{\omega (\xi)^{1/2}}{\omega (\phi (\xi))^{1/2}} < \varepsilon,
\]
whenever \(|\phi (\xi)| > r_0\). The rest follows as in the proof of (B).

### 5.4. Proof of Theorem 1.1 (D).
Let \(0 < q < p < \infty\) and suppose that \(G I_{\phi ,g} : A^{p}_{\omega} \rightarrow A^{q}_{\omega}\) is bounded. If \(\{f_n\} \subset A^{p}_{\omega}\) is a bounded sequence converging to zero uniformly on compact subsets of \(\mathbb{D}\), then
\[
\|G I_{\phi ,g} f_n\|_{A^{q}_{\omega}} \sim \int_{\mathbb{D}} \frac{|f_n'(\phi (\xi))|^q |g(z)|^q}{(1 + \varphi ' (\xi))^q} \omega (z)^{q/2} dA(z) = \|f_n'\|_{L^q(\mu_{\phi ,\omega ,g})},
\]
which goes to zero as \(n \rightarrow \infty\) because of the compactness of the embedding \(I_{\mu_{\phi ,\omega ,g}}\).

We next prove that (a) and (c) are equivalent. By (5.7) and Lemma 4.2, we get \(G I_{\phi ,g} : A^{p}_{\omega} \rightarrow A^{q}_{\omega}\) is bounded if and only if \(I_{\mu_{\phi ,\omega ,g}} : S^{p}_{\omega} \rightarrow L^q(\mu_{\phi ,\omega ,g})\) is bounded if and only if \(I_{\mu_{\phi ,\omega ,g}} : S^{p}_{\omega} \rightarrow L^q(\mu_{\phi ,\omega ,g})\) is compact if and only if the function
\[
F_{\delta,\mu_{\phi ,\omega ,g}} (\varphi) (z) := \frac{1}{\tau(z)^2} \int_{D_{\delta}(z)} (1 + \varphi ' (\xi))^{q} \omega (\xi)^{-q/2} d\mu_{\phi ,\omega ,g} (\xi)
\]
belongs to $L^{p/(p-q)}(\mathbb{D}, dA)$. By Theorem [C] this is equivalent to
\[
\int_{\mathbb{D}} |k_{q,z}(\xi)|^q \omega(\xi)^{q/2} d\nu_{\phi,\omega,g}(\xi) \in L^{p/(p-q)}(\mathbb{D}, dA),
\]
which is as well equivalent to $GB_{1,p,q}^\phi(g)(z) \in L^{p/(p-q)}(\mathbb{D}, d\lambda)$, where $d\lambda(z) = dA(z)/\tau(z)^2$, because of
\[
\int_{\mathbb{D}} G_q(\nu_{\phi,\omega,g})^{p/p-q} dA(z) = \int_{\mathbb{D}} \left( \frac{1}{\tau(z)^{2(1-q/p)}} G_q(\nu_{\phi,\omega,g}) \right)^{\frac{p}{p-q}} d\lambda(z)
= \int_{\mathbb{D}} \left( \frac{1}{\tau(z)^{2(1-q/p)}} \int_{\mathbb{D}} |k_{q,z}(\xi)|^q \omega(\xi)^{q/2} d\nu_{\phi,\omega,g}(\xi) \right)^{\frac{p}{p-q}} d\lambda(z)
= \int_{\mathbb{D}} GB_{1,p,q}^\phi(g)(z)^{p/p-q} d\lambda(z).
\]
This completes the proof of (D) when $0 < q < p < \infty$.

Suppose that $0 < q < p = \infty$. If $GI_{(\phi,g)} : A_q^\omega \to A_q^\omega$ is bounded and $\{f_n\} \subset A_q^\omega$ is a bounded sequence converging to zero uniformly on compact subsets of $\mathbb{D}$, then
\[
(5.8) \quad \|GI_{(\phi,g)} f_n\|_{A_q^\omega}^q \propto \int_{\mathbb{D}} \frac{|f_n'(\phi(z))|^q |g(z)|^q}{(1 + \varphi'(z))^q} \omega(z)^{q/2} dA(z) = ||f_n'||_{L^q(\mu_{\phi,\omega,g})} \to 0,
\]
where we used again the compactness of the embedding $I_{\mu_{\phi,\omega,g}}$, and so $GI_{(\phi,g)}$ is compact.

It remains to prove that (a) and (c) are equivalent. By (5.8) and Lemma [L.3] we get $GI_{(\phi,g)} : A_q^\omega \to A_q^\omega$ is bounded if and only if $I_{\mu_{\phi,\omega,g}} : S_q^\infty \to L^q(\mu_{\phi,\omega,g})$ is bounded if and only if $I_{\mu_{\phi,\omega,g}} : S_q^\infty \to L^q(\mu_{\phi,\omega,g})$ is compact if and only if the function
\[
F_{\delta,\mu_{\phi,\omega,g}}(\varphi)(z) := \frac{1}{\tau(z)^{2q/p}} \int_{D_{\delta}(z)} (1 + \varphi'(z))^q \omega(\xi)^{q/2} d\mu_{\phi,\omega,g}(\xi)
\]
belongs to $L^1(\mathbb{D}, dA)$. By Theorem [D] this is equivalent to
\[
\int_{\mathbb{D}} |k_{q,z}(\xi)|^q \omega(\xi)^{-q/2} d\nu_{\phi,\omega,g}(\xi) \in L^1(\mathbb{D}, dA),
\]
which is in turn equivalent to $GB_{1,p,q}^\phi(g)(z) \in L^1(\mathbb{D}, d\lambda)$, where $d\lambda(z) = dA(z)/\tau(z)^2$, because of
\[
GB_{1,p,q}^\phi(g)(z) \propto \tau(z)^{2(1-q/p)} \int_{\mathbb{D}} |k_{q,z}(\xi)|^q \omega(\xi)^{q/2} d\nu_{\phi,\omega,g}(\xi).
\]

5.5. Proof of Theorem [L.2](A). Boundedness. Let $0 < p \leq q < \infty$. By (1.15),
\[
(5.9) \quad \|GV_{(\phi,g)} f\|_{A_q^\omega}^q \propto \int_{\mathbb{D}} \frac{|f(\phi(z))|^q |g(z)|^q}{(1 + \varphi'(z))^q} \omega(z)^{q/2} dA(z) = \int_{\mathbb{D}} |f(z)|^q \omega(z)^{q/2} d\nu_{\phi,\omega,g}(z)
\]
Therefore, $GV_{(\phi,g)} : A_q^\omega \to A_q^\omega$ is bounded if and only if the measure $\nu_{\phi,\omega,g}$ is a $q$-Carleson measure for $A_q^\omega$. According to Theorem [D], this is equivalent to
\[
\sup_{z \in \mathbb{D}} \tau(z)^{2(1-q/p)} \int_{\mathbb{D}} |k_{q,z}(\xi)|^q \omega(\xi)^{q/2} d\nu_{\phi,\omega,g}(\xi) < \infty.
\]
Now, using (a) of Lemma 2.2, we get
\[
\tau(z)^{2(1-q/p)} \int_D |k_{q,z}(\xi)|^q \omega(\xi)^{q/2} d\nu_{\phi,\omega,q}(\xi) \\
= \int_D |k_{p,z}(\phi(\xi))|^q \frac{|g(z)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi)^{q/2} dA(\xi) \\
= GB_{0,p,q}^\phi.
\]
Thus, $GV_{(\phi,g)}$ is bounded if and only if $GB_{0,p,q}^\phi(g) \in L^\infty(D, dA)$.

**Compactness.** By above, $GV_{(\phi,g)} : A_p^\infty \to A_\omega^\infty$ is compact if and only if the measure $\nu_{\phi,\omega,g}$ is a vanishing $q$-Carleson measure for $A_\omega^0$. This is equivalent to
\[
\lim_{|z| \to 1^-} \tau(z)^{2(1-q/p)} \int_D |k_{q,z}(\xi)|^q \omega(\xi)^{q/2} d\nu_{\phi,\omega,g}(\xi) = 0.
\]
Now, using (a) of Lemma 2.2 we get
\[
\tau(z)^{2(1-q/p)} \int_D |k_{q,z}(\xi)|^q \omega(\xi)^{q/2} d\nu_{\phi,\omega,q}(\xi) \\
= \int_D |k_{p,z}(\phi(\xi))|^q \frac{|g(z)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi)^{q/2} dA(\xi) \\
= GB_{0,p,q}^\phi.
\]
Therefore, $\lim_{|z| \to 1^-} GB_{0,p,q}^\phi(g) = 0$ if and only if $GV_{(\phi,g)}$ is compact.

### 5.6. Proof of Theorem 1.2 (B). Boundedness.

Let $0 < p < q = \infty$ and suppose that (1.6) holds. Then, by (1.16), we have
\[
\|GV_{(\phi,g)} f\|_{A_\omega^\infty} \leq \sup_{z \in D} \frac{|f(\phi(z))||g(z)|}{(1 + \varphi'(z))} \omega(z)^{1/2} \\
\leq \sup_{z \in D} MV_{g,\phi,\omega}(z) \sup_{z \in D} |f(\phi(z))| |\omega(\phi(z))^{1/2} \Delta \varphi(\phi(z))^{-1/p} \\
\leq \sup_{z \in D} MV_{g,\phi,\omega}(z) \sup_{z \in D} |f(\phi(z))| |\omega(\phi(z))^{1/2} \tau(\phi(z))^{2/p}.
\]
By (2.4) for $f \in A_p^\infty$, we obtain
\[
\|GV_{(\phi,g)} f\|_{A_\omega^\infty} \leq \sup_{z \in D} \left( \int_{D_\delta(\phi(z))} |f(\xi)|^p \omega(\xi)^{1/2} dA(\xi) \right)^{1/p} \\
\leq \left( \int_D |f(\xi)|^p \omega(\xi)^{1/2} dA(\xi) \right)^{1/p} = \|f\|_{A_p^\infty}.
\]
Therefore, the operator $GV_{(\phi,g)}$ is bounded.

Conversely, suppose that the operator $GV_{(\phi,g)} : A_p^\infty \to A_\omega^\infty$ is bounded. Taking $\xi \in D$ such that $|\phi(\xi)| > \rho_0$, we consider the function $f_{\phi(\xi),n,p}$ given by $f_{\phi(\xi),n,p} := \frac{F_{\phi(\xi),n,p}}{\tau(\phi(\xi))^{2/p}}$ where $F_{\phi(\xi),n,p}$ is the test function defined in Lemma E. These functions $f_{\phi(\xi),n,p}$ belong to $A_\omega^0$ with $\|f_{\phi(\xi),n,p}\|_{A_\omega^0} \leq 1$. By
as
which proves the compactness of the operator 
This, (1.6) holds.
where
Therefore, in the case of (1.16),
On the other hand, if we define
which are in \( \mathbb{D} \) \([81,133]\).
In this case, by (2.11),
(5.10)
\[ \begin{align*}
\infty > \| GV_{(\phi, g)} (f_\phi (\xi)) \|_{A^\infty} & \geq \sup_{z \in \mathbb{D}} \frac{|f_{\phi (\xi), n, p}(z)||g(z)|}{(1 + \varphi'(z))} \omega(z)^{\frac{1}{2}} \\
& \geq \sup_{z \in \mathbb{D}} \frac{|F_{\phi (\xi), n, p}(z)||g(z)|}{\tau(\phi(\xi))^{2/p}(1 + \varphi'(z))} \omega(z)^{\frac{1}{2}} \\
& \geq \frac{|F_{\phi (\xi), n, p}(\phi(\xi))||g(\xi)|}{\tau(\phi(\xi))^{2/p}(1 + \varphi'(\xi))} \omega(\xi)^{\frac{1}{2}} \omega(\phi(\xi))^{\frac{1}{2}}.
\end{align*} \]
On the other hand, if we define \( f(z) = z \) and use the boundedness of the operator \( GV_{(\phi, g)} : A^p_\omega \to A^\infty_\omega \), we obtain
(5.11)
\[ \| GV_{(\phi, g)} f \|_{A^\infty} \geq \sup_{z \in \mathbb{D}} \frac{|g(z)|}{(1 + \varphi'(z))} \omega(z)^{\frac{1}{2}} \| f \|_{A^p_\omega} < \infty. \]
Therefore, in the case of \( |\phi(\xi)| \leq \rho_0, \xi \in \mathbb{D} \), we have
\[ \frac{|g(\xi)|}{(1 + \varphi'(\xi))} \omega(\phi(\xi))^{\frac{1}{2}} \Delta \varphi(\phi(\xi))^{1/p} \leq C_1 \frac{|g(\xi)|}{(1 + \varphi'(\xi))} \omega(\phi(\xi))^{\frac{1}{2}} < \infty, \]
where
\[ C_1 = \sup_{|\phi(\xi)| \leq \rho_0} \left\{ \frac{1}{2} \tau(\phi(\xi))^{-2/p} \right\} < \infty. \]
This, (1.6) holds.

**Compactness.** Suppose that the operator \( GV_{(\phi, g)} : A^p_\omega \to A^\infty_\omega \) is compact and define
\[ f_{\phi (\xi), n, p} := \frac{F_{\phi (\xi), n, p}}{\tau(\phi(\xi))^{2/p}}, \quad \text{for } |\phi(\xi)| > \rho_0, \]
which are in \( A^p_\omega \) and converge uniformly to zero on compact subsets of \( \mathbb{D} \) as \( |\phi(\xi)| \to 1 \). Thus,
\[ \| GI_{(\phi, g)} (f_{\phi (\xi), n, p}) \|_{A^\infty} \to 0 \]
as \( |\phi(\xi)| \to 1 \). Thus, (5.10) shows that
\[ \lim_{|\phi(\xi)| \to 1^-} MV_{g, \phi, \omega}(\xi) \leq \lim_{|\phi(\xi)| \to 1^-} \| GV_{(\phi, g)} (f_{\phi (\xi), n, p}) \|_{A^\infty} = 0. \]
Conversely, if \( \{ f_n \} \) is a bounded sequence of functions in \( A^p_\omega \) converging uniformly to zero on compact subsets of \( \mathbb{D} \), then, as for \( GI_{(\phi, g)} \), it follows that
\[ \| GV_{(\phi, g)} (f_n) \|_{A^\infty} = \sup_{\xi \in \mathbb{D}} \frac{|f_n(\phi(\xi))||g(\xi)|}{1 + \varphi'(\xi)} \omega(\xi)^{\frac{1}{2}} \omega(\phi(\xi))^{\frac{1}{2}} \to 0, \quad n \to \infty, \]
which proves the compactness of the operator \( GV_{(\phi, g)} : A^p_\omega \to A^\infty_\omega \).
5.7. Proof of Theorem 1.2 (C). Boundedness. Let \( p = q = \infty \). Suppose first that (1.7) holds. Then, by (1.16),
\[
\| \mathcal{G}_\nu f \|_{A^\infty} \leq C \left( \sup_{z \in \mathbb{D}} |f(\phi(z))| \right) \left( \sup_{z \in \mathbb{D}} |\nu(\phi(z))| \right)^{\frac{1}{q}} \leq C \left( \sup_{z \in \mathbb{D}} |f(\phi(z))| \right) \left( \sup_{z \in \mathbb{D}} |\nu(\phi(z))| \right)^{\frac{1}{q}} \leq \| f \|_{A^\infty},
\]
that is, \( \mathcal{G}_\nu f \) bounded.

Conversely, suppose that \( \mathcal{G}_\nu f : A^\infty_\omega \to A^\infty_\omega \) is bounded and show that (1.7) holds. As before, if \( \xi \in \mathbb{D} \) is such that \( |\phi(\xi)| > \rho_0 \), we use the test functions \( F_{\phi(\xi), n, p} \) to obtain
\[
\| \mathcal{G}_\nu f \|_{A^\infty} \geq \sup_{z \in \mathbb{D}} \left( \frac{|f(\phi(z))|}{1 + |\nu(\phi(z))|} \right) \left( \frac{|\nu(\phi(z))|}{1 + |\nu(\phi(z))|} \right)^{\frac{1}{q}} \| f \|_{A^\infty_\omega} < \infty.
\]

Now
\[
\| \mathcal{G}_\nu f \|_{A^\infty} \geq \sup_{z \in \mathbb{D}} \left( \frac{|f(\phi(z))|}{1 + |\nu(\phi(z))|} \right) \left( \frac{|\nu(\phi(z))|}{1 + |\nu(\phi(z))|} \right)^{\frac{1}{q}} \| f \|_{A^\infty_\omega} < \infty.
\]

If \( f(z) = z \), the boundedness of the operator \( \mathcal{G}_\nu f : A^\infty_\omega \to A^\infty_\omega \) implies that
\[
\| \mathcal{G}_\nu f \|_{A^\infty_\omega} < \infty.
\]

Therefore, in the case of \( |\phi(\xi)| \leq \rho_0 \), \( \xi \in \mathbb{D} \), we have
\[
\| \mathcal{G}_\nu f \|_{A^\infty_\omega} \leq C_2 \| f \|_{A^\infty_\omega} < \infty,
\]
where
\[
C_2 = \sup_{|\phi(\xi)| \leq \rho_0} \left\{ \frac{\| \nu(\phi(\xi)) \|_{A^\infty_\omega}}{1 + |\nu(\phi(\xi))|} \right\} < \infty.
\]

Combining this with (5.12) shows that (1.7) holds.

Compactness. This is similar to the proof of (C) of Theorem 1.1

5.8. Proof of Theorem 1.2 (D). Let \( 0 < q < p < \infty \) and suppose that \( \mathcal{G}_\nu f : A^p_\omega \to A^q_\omega \) is bounded. According to (5.9), the measure \( \nu_{\phi, \omega, g} \) is a \( q \)-Carleson measure for \( A^p_\omega \). Thus, by Theorem 3.3, \( \nu_{\phi, \omega, g} \) is a vanishing \( q \)-Carleson measure for \( A^p_\omega \). In this case, we have
\[
\| \mathcal{G}_\nu f_n \|_{A^q_\omega}^q \to 0, \quad n \to \infty,
\]
for any sequence \( \{ f_n \} \subset A^p_\omega \) converges to zero uniformly on compact subsets of \( \mathbb{D} \). By Lemma 3.7 of [16], \( \nu_{\phi, \omega, g} \) is compact.
Next we show that (a) and (c) are equivalent. Suppose first that (c) holds. Then

\[
\int_{\mathbb{D}} G_q(v_{\phi,\omega,q})(z)^{p/(p-q)} dA(z) = \int_{\mathbb{D}} \left( \tau(z)^{2(1-\frac{q}{p})} G_q(v_{\phi,\omega,q})(z) \right)^{p/(p-q)} d\lambda(z)
\]
\[
= \int_{\mathbb{D}} GB_{0,p,q}^G(g)^{p/(p-q)} d\lambda(z) < \infty.
\]
(5.14)

Thus, according to Theorem \[C\], \(\nu_{\phi,\omega,q}\) is a \(q\)-Carleson measure for \(A^p_\omega\). Then, by (1.15),

\[
\|GV_{(\phi,g)} f_n\|_{A^q_\omega}^q \leq \int_{\mathbb{D}} |f(z)|^q \omega(z)^{q/2} d\nu_{\phi,\omega,q}(z) \leq \|f\|_{A^p_\omega}^q,
\]

for any function \(f \in A^p_\omega\).

Conversely, suppose the operator \(GV_{(\phi,g)} : A^p_\omega \to A^q_\omega\) is bounded. Then, for each function \(f \in A^p_\omega\), by (1.15),

\[
\|GV_{(\phi,g)} f\|_{A^q_\omega}^q \leq \int_{\mathbb{D}} |f(z)|^q \omega(z)^{q/2} d\nu_{\phi,\omega,q}(z).
\]

Thus, the measure \(\nu_{\phi,\omega,q}\) is a \(q\)-Carleson measure for \(A^p_\omega\). According to Theorem \[C\], \(\nu_{\phi,\omega,q}\) belongs to \(L^{p/(p-q)}(\mathbb{D},dA)\). Combining this with (5.14) yields that \(GB_{0,p,q}^G(g) \in L^{p/(p-q)}(\mathbb{D},d\lambda)\).

Let \(0 < q < p = \infty\) and suppose that \(GV_{(\phi,g)} : A^{\infty}_\omega \to A^q_\omega\) is bounded. Then, by (1.15),

\[
\|GV_{(\phi,g)} f\|_{A^q_\omega}^q \leq \int_{\mathbb{D}} |f(z)|^q \omega(z)^{q/2} d\nu_{\phi,\omega,q}(z) \leq \|f\|_{A^{\infty}_\omega}^q,
\]

and it follows from Theorem \[B\] that the measure \(\nu_{\phi,\omega,q}\) is a \(q\)-Carleson measure for \(A^{\infty}_\omega\). Thus, by Theorem \[3.3\], \(\nu_{\phi,\omega,q}\) is a vanishing \(q\)-Carleson measure for \(A^{\infty}_\omega\). As in the previous case, this shows the compactness of the operator \(GV_{(\phi,g)}\).

It remains to prove that (1) and (3) are equivalent when \(p = \infty\). Assume first that (3) holds. Then

\[
\int_{\mathbb{D}} G_q(v_{\phi,\omega,q})(z) dA(z) = \int_{\mathbb{D}} \left( \tau(z)^2 G_q(v_{\phi,\omega,q})(z) \right) d\lambda(z)
\]
\[
= \int_{\mathbb{D}} GB_{0,p,q}^G(g)(z) d\lambda(z).
\]
(5.15)

Thus, according to Theorem \[E\], \(\nu_{\phi,\omega,q}\) is a \(q\)-Carleson measure for \(A^{\infty}_\omega\). Then for any function \(f \in A^{\infty}_\omega\), we have

\[
\|GV_{(\phi,g)} f_n\|_{A^q_\omega}^q \leq \int_{\mathbb{D}} |f(z)|^q \omega(z)^{q/2} d\nu_{\phi,\omega,q}(z) \leq \|f\|_{A^{\infty}_\omega}^q.
\]

Conversely, suppose the operator \(GV_{(\phi,g)} : A^{\infty}_\omega \to A^q_\omega\) is bounded. Then, for any function \(f \in A^{\infty}_\omega\), we have

\[
\|GV_{(\phi,g)} f\|_{A^q_\omega}^q \leq \int_{\mathbb{D}} |f(z)|^q \omega(z)^{q/2} d\nu_{\phi,\omega,q}(z).
\]

By assumption, this implies that the measure \(\nu_{\phi,\omega,q}\) is a \(q\)-Carleson measure for \(A^{\infty}_\omega\). According to Theorem \[E\], \(\nu_{\phi,\omega,q}\) belongs to \(L^1(\mathbb{D},dA)\). Combining this with (5.15) implies that \(GB_{0,p,q}^G(g) \in L^1(\mathbb{D},d\lambda)\).
6. Proofs of Proposition 1.3 and Corollary 1.5

6.1. Proof of Proposition 1.3 (A). Suppose that the operator $GI_{(\phi, g)} : A^p_0 \rightarrow A^q_0$ is bounded. Let $\xi \in \mathbb{D}$ be such that $|\phi(\xi)| > \rho_0$. Using the test function of Lemma 2.3 and Lemma 2.4, we get

$$
\|F_{\phi(\xi), n, p}\|_{A^q_0}^q \gtrsim \|GI_{(\phi, g)}F_{\phi(\xi), n, p}\|_{A^q_0}^q \asymp \int_{\mathbb{D}} \left| \frac{f'_{\phi(\xi), n, p}(\phi(z))}{(1 + \varphi'(z))^q} \right|^q |g(z)|^q \omega(z)^q dA(z)
$$

while Lemma 2.3 implies that

$$
\|F_{\phi(\xi), n, p}\|_{A^q_0}^q \gtrsim \tau(\xi)^2 |g(\xi)|^q \left(1 + \varphi'(\xi)\right)^q \omega(\xi)^{2q} \omega(\phi(\xi))^{2q}.
$$

By Lemma 3, we have

$$
1 \gtrsim |g(\xi)| \frac{\tau(\xi)^{2q}}{\tau(\phi(\xi))^{2q}} \frac{\left(1 + \varphi'(\phi(\xi))\right)^q}{\left(1 + \varphi'(\xi)\right)^q} \frac{\omega(\xi)^{2q}}{\omega(\phi(\xi))^{2q}}.
$$

When $|\phi(\xi)| \leq \rho_0$, we have

$$
\sup_{\phi(\xi) \leq \rho_0} \left| g(\xi) \right| \frac{\tau(\xi)^{2q}}{\tau(\phi(\xi))^{2q}} \frac{\left(1 + \varphi'(\phi(\xi))\right)^q}{\left(1 + \varphi'(\xi)\right)^q} \frac{\omega(\xi)^{2q}}{\omega(\phi(\xi))^{2q}} < \infty.
$$

Thus, (1.8) holds.

Suppose next that the operator $GI_{(\phi, g)} : A^p_0 \rightarrow A^q_0$ is compact. Let $\xi \in \mathbb{D}$ be such that $|\phi(\xi)| > \rho_0$ and define

$$
f_{\phi(\xi), n, p} = \frac{F_{\phi(\xi), n, p}}{\tau(\phi(\xi))^{2q/p}}, \quad \text{for } |\phi(\xi)| > \rho_0,
$$

which belongs to $A^p_0$ and converges uniformly to zero on compact subsets of $\mathbb{D}$ as $|\phi(\xi)| \rightarrow 1$. By (2.4) and Lemma 2.3, we get

$$
\|GI_{(\phi, g)}f_{\phi(\xi), n, p}\|_{A^q_0}^q \asymp \int_{\mathbb{D}} \left| \frac{f'_{\phi(\xi), n, p}(\phi(z))}{(1 + \varphi'(z))^q} \right|^q |g(z)|^q \omega(z)^q dA(z)
$$

$$
\gtrsim \tau(\xi)^2 \left| \frac{f'_{\phi(\xi), n, p}(\phi(z))}{(1 + \varphi'(z))^q} \right|^q |g(z)|^q \omega(z)^q
$$

$$
\gtrsim |g(\xi)|^q \frac{\tau(\xi)^2}{\tau(\phi(\xi))^{2q/p}} \frac{\left(1 + \varphi'(\phi(\xi))\right)^q}{\left(1 + \varphi'(\xi)\right)^q} \frac{\omega(\xi)^{2q}}{\omega(\phi(\xi))^{2q}}.
$$

Using the compactness of the operator $GI_{(\phi, g)}$, we have the desired conclusion and the proof is complete.
6.2. Proof of Proposition [1.3] (B). Suppose that $GV_{(\phi, q)} : A^p_\omega \to A^q_\omega$ is bounded. By Theorem [1.2] (A), this is equivalent to $GB^\phi_{0,p,q}(g) \in L^\infty(\mathbb{D}, dA)$. By (2.4) and (2.10), we have

$$GB^\phi_{0,p,q}(g)(\phi(z)) = \int_{\mathbb{D}} |k_{p,\phi}(z)(\phi(\xi))|^q \frac{|g(\xi)|^q}{1 + \varphi'(\xi)^q} \omega(\xi)^{q/2} dA(\xi)$$

(6.2)

which proves that [1.9] holds. If $GV_{(\phi, q)}$ is compact, then it follows from Theorem [1.2] (A) that $GB^\phi_{0,p,q}(g)(\phi(z)) \to 0$ as $|z| \to 1$, which completes the proof.

6.3. Proof of Corollary [1.5] (A) Let $p < q$ and suppose that $GI_{(id, g)}$ is bounded. By (2.10) and Lemma [A] we have

$$|g(z)|^q \leq \frac{\tau(z)^2}{\tau(z)^2} \int_{D_\delta(z)} \frac{|g(s)|^q |k_{p,\phi}(s)|^q \omega(s)^{q/2} dA(s)}{\tau(z)^2} GB^{id}_{1,p,q}(g)(z).$$

Then, using the boundedness of $GI_{(id, g)}$, we obtain

$$\sup_{z \in \mathbb{D}} |g(z)|^q \tau(z)^{2(1-q/p)} \leq \sup_{z \in \mathbb{D}} GB^{id}_{1,p,q}(g)(z) < \infty.$$ 

Since $\tau(z)^{2(1-q/p)} \to \infty$, as $|z| \to 1$, the function $g$ must be zero.

6.4. Proof of Corollary [1.5] (B) Let $q < p$. Using

$$\|GI_{(\phi, g)} f\|_{A^q_\omega} \leq \|f\|_{L^q(\mu_{\phi,\omega,g})} \frac{\tau(z)^2}{\tau(z)^{2}} \int_{D_\delta(z)} \frac{|f'(\phi(z))|^q |g(z)|^q}{(1 + \varphi'(z))^q} \omega(z)^{q/2} dA(z) = \|f\|_{L^q(\mu_{\phi,\omega,g})}$$

(see (1.15)) and Lemma [4.2] we get $GI_{(\phi, g)} : A^p_\omega \to A^q_\omega$ is bounded if and only if $I_{\mu_{\phi,\omega,g}} : S^p_\omega \to L^q(\mu_{\phi,\omega,g})$ is bounded if and only if $I_{\mu_{\phi,\omega,g}} : S^p_\omega \to L^q(\mu_{\phi,\omega,g})$ is compact if and only if the function $\phi = id$, we have

$$d\mu_{\phi,\omega,g}(z) = \frac{|g(z)|^q}{(1 + \varphi'(z))^q} \omega(z)^{q/2} dA(z)$$

and invoking this in the condition (6.4), it becomes exactly

$$\frac{1}{\tau(z)^2} \int_{D_\delta(z)} |g(\xi)|^q dA(\xi) \in L^{p/(p-q)}(\mathbb{D}, dA).$$

Applying Lemma [A] we get that $g \in L^r(\mathbb{D}, dA)$, with $r = pq/(p - q)$. 

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Conversely, suppose that \( g \in L^r(\mathbb{D}, dA) \). By Hölder's inequality and (1.15), we obtain

\[
\|G_i(t, g)\|_{A^q}^q \leq \int_D \frac{|f'(z)|^q |g(z)|^q}{(1 + \varphi'(z))^q} \omega(z)^{\frac{q}{r}} dA(z)
\]

which proves boundedness and completes the proof.

6.5. Proof of Corollary 1.5(C). Let \( 0 < p \leq q < \infty \). We characterize boundedness using Theorem 1.2. Suppose that \( GB_{0,p,q}^0(g') \in L^{\infty}(\mathbb{D}, dA) \). It follows from (6.2) (changing \( g \) by \( g' \) and \( \phi = id \)),

\[
GB_{0,p,q}^0(g')(z) = \int_D |k_{q,z}(\xi)|^q \frac{|g'(\xi)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi)^{q/2} dA(\xi)
\]

Thus,

\[
\frac{|g'(z)|}{(1 + \varphi'(z))} \Delta \varphi(z)^{\frac{1}{p} - \frac{1}{q}} \in L^{\infty}(\mathbb{D}, dA).
\]

Conversely, suppose that

\[
T(g, \varphi)(z) := \frac{|g'(z)|}{(1 + \varphi'(z))} \Delta \varphi(z)^{\frac{1}{p} - \frac{1}{q}} \in L^{\infty}(\mathbb{D}, dA).
\]

By (2.9), we have

\[
GB_{0,p,q}^0(g')(z) \leq \int_D |k_{q,z}(\xi)|^q \frac{|g'(\xi)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi)^{q/2} dA(\xi)
\]

Since \( \Delta \varphi(z) \propto \tau(z)^{-2} \),

\[
GB_{0,p,q}^0(g')(z) \leq \left( \tau(z)^2 \int_D |k_{q,z}(\xi)|^q \omega(\xi)^{q/2} dA(\xi) \right) \sup_{z \in \mathbb{D}} (T(g, \varphi)(z))^q.
\]

This finishes the proof of boundedness.

The characterization for compactness follows from Theorem 1.2, (6.7), and (6.6).

6.6. Proof of Corollary 1.5(D). Let \( 0 < q < p < \infty \). We first suppose that \( V_{q} : A_{\omega}^p \rightarrow A_{\omega}^q \) is bounded, that is, \( GB_{0,p,q}^0(g') \in L^{\infty}(\mathbb{D}, d\lambda) \) (see Theorem 1.2). Then, by (2.4), we have

\[
GB_{0,p,q}^0(g')(z) = \int_D |k_{q,z}(\xi)|^q \frac{|g'(\xi)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi)^{q/2} dA(\xi)
\]

which proves boundedness and completes the proof.
By Lemma 2.2, we obtain
\[ GB_{0,p,q}^{id}(g')(z) \geq \frac{\tau(z)^2}{\tau(z)^{2q/p}} \cdot \frac{|g'(z)|^q}{(1 + \varphi'(z))^q} \cdot \left( \frac{|g'(z)|}{(1 + \varphi'(z))} \cdot \Delta \varphi(z)^{\frac{1}{p} - \frac{1}{q}} \right)^q. \]

In this case, we extract that
\[ \tau(z)^{2(\frac{2}{p} - 1)}GB_{0,p,q}^{id}(g')(z) \geq \left( \frac{|g'(z)|}{(1 + \varphi'(z))} \right)^q. \]

By our assumption and the fact that \( \tau(z)^{2q/p} \) is bounded, it follows that (1.11) holds. Conversely, put \( r = \frac{pq}{p-q} \). By Hölder’s inequality, we obtain
\[ GB_{0,p,q}^{id}(g')(z)^{p/(p-q)} = \left( \int_{\mathbb{D}} |k_{p,z}(\xi)|^{\frac{p}{r}} \cdot \frac{|g'(\xi)|^q}{(1 + \varphi'(\xi))^q} \cdot \omega(\xi)^{q/2} \cdot dA(\xi) \right)^{\frac{p}{p-q}} \]
\[ \leq \|K_z\|_{\mathbb{A}_p}^{\frac{r}{p}} \left( \int_{\mathbb{D}} |K_z(\xi)|^{\frac{r}{2}} \cdot \omega(\xi)^{\frac{q}{2}} \cdot dA(\xi) \right) \cdot \left( \int_{\mathbb{D}} |K_z(\xi)|^{\frac{q}{2}} \cdot \omega(\xi)^{\frac{p}{2}} \cdot dA(\xi) \right) \]
\[ = \frac{\|K_z\|_{\mathbb{A}_p}^{\frac{r}{p}}}{\|K_z\|_{\mathbb{A}_p}^{\frac{q}{p}}} \cdot \left( \int_{\mathbb{D}} |K_z(\xi)|^{\frac{r}{2}} \cdot \omega(\xi)^{\frac{q}{2}} \cdot dA(\xi) \right) \cdot \left( \int_{\mathbb{D}} |K_z(\xi)|^{\frac{q}{2}} \cdot \omega(\xi)^{\frac{p}{2}} \cdot dA(\xi) \right). \]

By Theorem A, \( \|K_z\|_{\mathbb{A}_p}^{\frac{r}{p}} \leq \omega(z)^\frac{r}{2} \cdot \tau(z)^r \), and Fubini’s theorem implies that
\[ \int_{\mathbb{D}} GB_{0,p,q}^{id}(g')(z)^{p/(p-q)} \cdot dA(z) \]
\[ \leq \int_{\mathbb{D}} \left( \frac{|g'(\xi)|}{(1 + \varphi'(\xi))} \right)^r \cdot \omega(\xi)^\frac{q}{2} \cdot \left( \int_{\mathbb{D}} |K_z(\xi)|^{\frac{q}{2}} \cdot \omega(z)^\frac{p}{2} \cdot \tau(z)^{r-2} \cdot dA(z) \right) \cdot dA(\xi). \]

Since
\[ \omega(\xi)^\frac{q}{2} \cdot \left( \int_{\mathbb{D}} |K_z(\xi)|^{\frac{q}{2}} \cdot \omega(z)^\frac{p}{2} \cdot \tau(z)^{r-2} \cdot dA(z) \right) \leq 1 \]
(see Lemma D), the proof is complete.

6.7. Proof of Corollary 1.6 (I) Let \( 0 < p = q < \infty \). By (c) of Lemma 32 in [3].
\[ \psi_{\omega}(r) \asymp (1 + \varphi'(r))^{-1} \text{ for } r \in [0, 1). \]

Therefore,
\[ GB_{0,p,p}^{id}(g')(z) = \int_{\mathbb{D}} |k_{p,z}(\xi)|^p \cdot \frac{|g'(\xi)|^p}{(1 + \varphi'(\xi))^p} \cdot \omega(\xi)^{p/2} \cdot dA(\xi) \]
\[ \asymp \sup_{\xi \in \mathbb{D}} \left( \psi_{\omega}(\xi)|g'(\xi)| \right)^p \cdot \left( \int_{\mathbb{D}} |k_{p,z}(\xi)|^p \cdot \omega(\xi)^{p/2} \cdot dA(\xi) \right) \]
\[ = \sup_{\xi \in \mathbb{E}} \left( \psi_{\omega}(\xi)|g'(\xi)| \right)^p \cdot |k_{p,z}|_{\mathbb{A}_p}^p = \sup_{\xi \in \mathbb{E}} \left( \psi_{\omega}(\xi)|g'(\xi)| \right)^p. \]

The other assertion follows easily from (6.9).
6.8. Proof of Corollary 1.6 (II) Let $0 < p < q < \infty$. Note that the weighted Bergman space $A^p(\omega)$, defined in [15], is the same as the Bergman spaces $A^p_W$, with $W = \omega^{2/p}$. Moreover,

$$GB_{0,p,q}^{id}(g')(z) = \int_{D_h(z)} |k_{p,z}(\xi)|^q \frac{|g'(\xi)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi) \, dA(\xi),$$

and (2.10) is transformed to

$$(6.10) |k_{p,z}(\xi)|^q \omega(\xi)^{q/p} \lesssim \tau(z)^{-2q/p}, \quad \zeta \in D_h(z),$$

where $k_{p,z}(\xi) = K_z(\xi)/\|k_{p,z}\|_{A^p(\omega)}$.

Let $s = \frac{2}{p} - \frac{2}{q}$. Then, by (6.8) and successively (2.4), (2.3) and (6.10), we get

$$\left(\|K_z\|_{A^2(\omega)}^{2s} |g'(z)|\right)^q \lesssim \frac{\|K_z\|_{A^2(\omega)}^{2q}}{\tau(z)^2 \omega(z)^{1-p}} \int_{D_h(z)} \frac{|g'(\xi)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi)^{1-\frac{q}{p}} \, dA(\xi)$$

$$\lesssim \frac{1}{\tau(z)^{2q/p}} \int_{D_h(z)} \frac{|g'(\xi)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi)^{1-\frac{q}{p}} \, dA(\xi)$$

$$\lesssim \int_{D_h(z)} |k_{p,z}(\xi)|^q \frac{|g'(\xi)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi) \, dA(\xi)$$

$$\lesssim GB_{0,p,q}^{id}(g')(z) < \infty.$$

Thus, to prove that the function $g'$ vanishes on $\mathbb{D}$, it is enough to show that $\|K_z\|_{A^2(\omega)}^{2s} |\psi(\omega)|$ goes to infinity as $|z| \to 1$. Indeed, by (2.3) and (1.12), we have

$$\|K_z\|_{A^2(\omega)}^{2s} |\psi(\omega)| \gg \frac{\tau(z)^{2(1-s)}}{(1-|z|)^s} \omega(z)^s,$$

and so,

$$\lim_{|z| \to 1} \|K_z\|_{A^2(\omega)}^{2s} |\psi(\omega)(z)| = \infty$$

because of Lemma 2.3 in [15].

6.9. Proof of Corollary 1.6 (III) Let $q < p$, and suppose that $GB_{0,p,q}^{id}(g') \in L^{p/(p-q)}(\mathbb{D}, d\lambda)$. Then

$$GB_{0,p,q}^{id}(g')(z) \gtrsim \int_{D_h(z)} |k_{p,z}(\xi)|^q \frac{|g'(\xi)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi) \, dA(\xi)$$

$$\gtrsim \tau(z)^{-2q/p} \int_{D_h(z)} \frac{|g'(\xi)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi)^{\frac{n-a}{p}} \, dA(\xi),$$

and so it follows from the assumption that

$$\int_{\mathbb{D}} \left(\tau(z)^{-2} \int_{D_h(z)} \frac{|g'(\xi)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi)^{\frac{n-a}{p}} \, dA(\xi)\right)^{\frac{p}{p-q}} dA(z)$$

$$\lesssim \int_{\mathbb{D}} \left(GB_{0,p,q}^{id}(g')(z)^{\frac{p}{p-q}} d\lambda(z) < +\infty.$$

Thus, using (1.11), we get

$$\|g\|_{A^{p/(p-q)}(\omega)} \lesssim \int_{\mathbb{D}} \left(\tau(z)^{-2} \int_{D_h(z)} \frac{|g'(\xi)|^q}{(1 + \varphi'(\xi))^q} \omega(\xi)^{\frac{n-a}{p}} \, dA(\xi)\right)^{\frac{p}{p-q}} dA(z).$$
This completes the proof of Corollary 1.6.

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