MODULAR SYMBOLS FOR REDUCTIVE GROUPS AND $p$-ADIC RANKIN-SELBERG CONVOLUTIONS OVER NUMBER FIELDS

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Abstract. We give a construction of a wide class of modular symbols attached to reductive groups. As an application we construct a $p$-adic distribution interpolating the special values of the twisted Rankin-Selberg $L$-function attached to cuspidal automorphic representations $\pi$ and $\sigma$ of $\text{GL}_n$ and $\text{GL}_{n-1}$ over a number field $k$. If $\pi$ and $\sigma$ are ordinary at $p$, our distribution is bounded and yields analyticity of the associated $p$-adic $L$-function.

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Introduction

The study of special values of $L$-functions has a long history, dating back to Kummer and Euler. Iwasawa theory provides us with a strong motivation for the construction of $p$-adic $L$-functions. Unfortunately, we still don’t know how to construct a $p$-adic $L$-function for a general motive. As a result all known constructions of $p$-adic $L$-functions rely on the study of special values of complex $L$-functions and are essentially automorphic. By Langland’s philosophy this seems to be no severe restriction [13, Question 4.16 et Théorème 5.1].

In this paper we study the problem of $p$-adic interpolation of the special values of twisted Rankin-Selberg $L$-functions of two automorphic representations $\pi$ and $\sigma$ on $\text{GL}_n$ and $\text{GL}_{n-1}$ (in the sense of [25, 15]). Our approach is greatly inspired by [4, 33, 35, 36], and especially [43, 30, 44], where Kazhdan, Mazur and Schmidt treat the case $k = \mathbb{Q}$.

To be more precise, let $\pi$ and $\sigma$ be irreducible cuspidal automorphic representations of $\text{GL}_n(\mathbb{A}_k)$ and $\text{GL}_{n-1}(\mathbb{A}_k)$, where $\mathbb{A}_k$ denotes the adèle ring of a number field $k$.
field \( k \). Assuming that \( \pi \) and \( \sigma \) occur in cohomology and are ordinary at \( p \), we show the existence of a \( p \)-adic measure \( \mu \) which is characterized by the property that for a certain “period” \( \Omega \in \mathbb{C} \) we have

\[
\int \chi d\mu = \Omega \cdot L\left(\frac{1}{2}, (\pi \otimes \chi) \times \sigma \right)
\]

for any Hecke character \( \chi \) of finite order with (non-trivial) \( p \)-power conductor (cf. Theorems 4.4, 4.7 and 5.1 below).

Without the assumption of ordinarity, only assuming that \( \pi \) and \( \sigma \) are unramified at \( p \), we still get a \( p \)-adic distribution \( \mu \), whose order may be easily bounded by the \( p \)-valuations of the roots of the corresponding Hecke polynomials of \( \pi \) and \( \sigma \) at \( p \).

In the spirit of Iwasawa theory \( \mu \) corresponds to a measure (resp. a distribution) on the Galois group of the maximal abelian extension \( k^{\infty}/k \), unramified outside \( p \).

A general problem in the field is that we don’t know for general \( n \) if

\[ \Omega \neq 0. \]

Of course this assertion is vital for our theory. There are well known affirmative answers in the cases \( n = 2 \) and \( n = 3 \) for totally real \( k \), and recent work of Schmidt and Kasten [29] extend the case \( n = 3 \) to non-constant coefficients. We don’t attack this problem here, since the techniques involved are of a different nature than the problems we discuss. Furthermore we prefer to confine us to a clear treatment involving only cohomology with constant coefficients, since the methods of loc. cit. easily carry over to our setting, but come for the price of additional notation.

In those cases where \( \pi \) and \( \sigma \) arise from Hilbert modular forms, Shimura’s algebraicity result [49] may be used to relate our periods to Shimura’s periods, which may in certain cases eventually be related to Deligne’s periods [16], notably if \( \pi \) and \( \sigma \) arise from a base change in \( k/\mathbb{Q} \), [17, 18, 55]. In general it remains an open problem to relate \( \Omega \) to Deligne’s periods if \( \pi \) and \( \sigma \) arise from motives.

In our attempt to generalize the results of Kazhdan, Mazur and Schmidt (loc. cit.) to arbitrary number fields \( k \) we encountered two major problems.

First, even over \( \mathbb{Q} \), their results on \( p \)-adic interpolation are incomplete, as there is a technical restriction on the conductors of the twisting characters, depending on \( n \). This concerns the so-called generalized local Birch lemmas of loc. cit., which only hold under the hypothesis that for the Dirichlet characters \( \chi \) under consideration \( \chi, \chi^2, \ldots, \chi^{n-1} \) share the same conductor \( f \neq 1 \). The problem had been attacked before in [53], where the cases \( n = 2, 3, 4 \) were settled by computation. We solve the general problem with a new local Birch lemma (Theorem 2.1 below). As a consequence we can show the unconditional existence of \( \mu \), completing also the previously known results over \( \mathbb{Q} \). Our result also suggests a natural generalization for Rankin-Selberg convolutions on \( GL_m \times GL_n \) for general pairs \( m, n \). However, even if we believe that it might be possible to attack the \( p \)-adic interpolation problem for pairs \( (m, n) \neq (n-1, n) \) by similar methods, we content us to give a general global formula in Theorem 3.1.

Second, Kazhdan, Mazur and Schmidt’s argument for proving algebraicity and boundedness of the distribution relies on relative modular symbols, as introduced in [43]. It turns out that this specific approach does not generalize to the more general setting, especially if \( k \) is no more totally real. The reason being that the modular

1We also encountered a minor problem related to the class number of \( k \), which had hitherto been overlooked, i.e. in [3], cf. the discussion following Theorem 3.1 below.
symbols previously considered are defined by means of analytic Lie groups. To
overcome this problem we give an algebraic interpretation of these modular symbols
which enables us to give an abstract treatment of the problem. We are naturally led
to consider relative modular symbols attached to morphisms of reductive algebraic
groups.

More precisely let \( s : H \to G \) be a \( \mathbb{Q} \)-morphism of connected reductive groups
over \( \mathbb{Q} \) with \( \mathbb{R} \)-anisotropic kernel defined over \( \mathbb{R} \). Denote by \( \mathcal{W} \) and \( \mathcal{W}^1 \)
symmetric spaces associated to \( G(\mathbb{R}) \) and \( H^{\text{ad}}(\mathbb{R}) \) respectively, and let \( \Gamma \) and \( \Gamma^1 \) be arithmetic
subgroups of the latter groups. Using reduction theory for arithmetic groups, we
construct for any subring \( A \subseteq \mathbb{C} \) a pairing
\[
\phi^q_{s,g} : H^q_c(\Gamma \backslash \mathcal{W}, A) \times H^{\dim \mathcal{W}^1 - q}_c(\Gamma^1 \backslash \mathcal{W}^1, A) \to A,
\]
which is functorial in \( A \). Specializing to \( G = \text{Res}_{k/\mathbb{Q}} \text{GL}_n \), \( H = \text{Res}_{k/\mathbb{Q}} \text{GL}_{n-1} \),
and \( s = \text{Res}_{k/\mathbb{Q}}(\text{ad} \circ j) \) for \( \text{ad} \circ j : \text{GL}_{n-1} \to \text{PGL}_n \), we find our special values in
the image of this topological symbol, yielding algebraicity and boundedness of the
(a priori \( \mathbb{C} \)-valued) distribution \( \mu \).

Actually many more topological modular symbols may be constructed with our
technique. We expect that our method may serve to approach other cases as well.

To give an explicit application of our methods, we deduce the existence of the
\( p \)-adic symmetric cube \( L \)-function of a modular elliptic curve \( E \) over a totally real
number field \( k \). Assuming Langland’s functoriality conjecture, we get the existence
of all \( p \)-adic odd symmetric power \( L \)-functions along the same lines.

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Notation

For a set \( M \) we write \( \#M \) for its cardinality. If a family \( (M_i)_{i \in I} \) of sets is given, let
\[
\Bigcup_{i \in I} M_i
\]
denote the disjoint union of the \( M_i, i \in I \). For a ring extension \( R/S \) and an \( S \)-module \( M \) we let \( M_R := M \otimes_S R \). \( \mathbb{A}_k \) (resp. \( \mathbb{A}_k^1 \)) denotes the (finite) adèle ring of
a global field \( k \), \( M_k \) denotes the set of places of \( k \). If \( S \) is a scheme over a ring \( R \)
and if \( A \) is any commutative \( R \)-algebra, we denote by \( S(A) = \text{Hom}_{\text{Spec } R}(\text{Spec } A, S) \)
the set of \( A \)-points of \( S \).

Let \( G \) be a topological group. Then \( G^0 \) denotes the connected component of
the unit \( 1 \in G \), the same notation applies to algebraic groups with respect to the
Zariski topology.

For a finite separable field extension \( k/l \), \( \text{Res}_{k/l} G \) denotes the restriction of
scalars (à la Weil) of \( k \) to \( l \) of a (linear) algebraic group \( G \). This is a (linear)
algebraic group over \( l \), such that for any commutative \( l \)-algebra \( A \) we have
\( \text{Res}_{k/l} G(A) = G(A \otimes_l k) \). An isogeny of algebraic groups is an epimorphism with
finite kernel. Denote by \( G^{\text{der}} \) the commutator group of a linear algebraic group \( G \)
and by \( G^{\text{ad}} \) the adjoint group respectively. So \( G^{\text{ad}} \) is the image of \( G \) under the
adjoint representation \( \text{Ad} : G \to \text{Lie}(G) \). Here and in the sequel \( \text{Lie}(G) \) is the Lie
algebra of $G$. The differential of a morphism $f : G \to H$ of linear algebraic (or of Lie) groups is denoted by $L(f)$. $\mathcal{R}(G)$ (resp. $\mathcal{R}_s(G)$) is the (unipotent) radical of $G^0$. The $k$-rank of a reductive group over $k$ is the dimension of a maximal $k$-split torus in $G^0$. Note that all maximal $k$-tori are conjugate over $k$ (cf. [9, Théorème 4.21]) and any $k$-torus $T$ in $G^0$ is contained in a maximal torus of $G^0$ defined over $k$ (cf. [9, 2.15 d])

A morphism $f : G \to H$ of algebraic groups is central if the commutator map $[\cdot , \cdot] : G \times G \to G$ factors as a morphism (and as a map) over $f(G) \times f(G)$. Note that kernels of morphisms defined over $k$ need not be defined over $k$. If a $k$-morphism $f$ is (quasi-)central, then $\ker f$ is always defined over $k$ [10, Corollaire 2.12].

1. Modular symbols for reductive groups

Throughout this section $G$ is a connected linear algebraic group over $\mathbb{Q}$. We denote by $X_k(G)$ the $\mathbb{Z}$-module of characters $\alpha : G \to \mathbb{G}_m$ defined over a field $k/\mathbb{Q}$. We define the group

$$0^G := \bigcap_{\alpha \in X_k(G)} \ker \alpha^2.$$

The square on the right hand side is motivated by the fact that $\mathbb{R}^\times$ has two connected (topological) components. The group $0^G$ is normal in $G$ and defined over $\mathbb{Q}$. The restriction of any $\alpha \in X_k(G)$ on $0^G$ is at most of order 2, and consequently trivial on $(0^G)^0$. Therefore we have

$$0^G = \bigcap_{\alpha \in X_k(G)} \ker \alpha^0.$$

Let $L$ be a Levi subgroup of $G$ over $\mathbb{Q}$. Then

$$G = \mathcal{R}_s G \rtimes L,$$

which implies

$$0^G = \mathcal{R}_s G \rtimes 0^L,$$

since every character on $G$ vanishes on the unipotent radical of $G$. Our interest in $0^G$ is motivated by

**Proposition 1.1** ([8, Proposition 1.2]). Denote by $S$ a maximal $\mathbb{Q}$-split torus in the radical of $G$. Then

$$G(\mathbb{R}) = 0^G(\mathbb{R}) \rtimes S(\mathbb{R})^0.$$

The group $0^G(\mathbb{R})$ contains every compact subgroup as well as every arithmetic subgroup of $G(\mathbb{R})$.

Furthermore we have

**Proposition 1.2.** Let $G$ be reductive. Then $0^G$ is reductive and we have

$$X_\mathbb{Q} \bigl(0^G\bigr) \otimes _\mathbb{Z} \mathbb{Q} = 0.$$

**Proof.** The group $(0^G)^0$ is generated by the semi-simple group $G^{\text{der}}$ and a central torus, which implies that $0^G$ is reductive.

To prove the second assertion, we note that the canonical inclusion $(0^G)^0 \to G$ induces a map $X_\mathbb{Q}(G) \to X_\mathbb{Q} \bigl(0^G\bigr)$ with finite cokernel and trivial image by equation (11). This concludes the proof. 

□
Now let $G$ be an arbitrary connected isotropic linear algebraic group over $\mathbb{Q}$ with $X_{\mathbb{Q}}(G) = 1$. Denote by $P$ a minimal parabolic $\mathbb{Q}$-subgroup of $G$ with unipotent radical $U$. Furthermore let $K$ be a maximal compact subgroup of $G(\mathbb{R})$ and denote by $\theta$ the Cartan involution associate to $K$ in the sense of [8 Proposition 1.6, Definition 1.7]. Then there exists a unique $\theta$-invariant $\mathbb{Q}$-Levi-subgroup $M$ in $P$ [8 Corollary 1.9]. We denote by $S := M \cap \mathbb{A}_d(P)$ the maximal $\theta$-invariant $\mathbb{Q}$-split torus in the radical of $P$.

Finally let $\Phi_\mathbb{Q}(S, P) \subseteq X_\mathbb{Q}(S)$ be the set of the relative roots of $P$ with respect to $S$. Let $\Delta \subseteq \Phi_\mathbb{Q}(S, P)$ denote the set of simple roots in $\Phi_\mathbb{Q}(S, P)$. Then $\Delta$ is a basis of the relative root system $\Phi_\mathbb{Q}(S, G)$ and as such defines an ordering $\geq$ on the latter, such that $\Phi_\mathbb{Q}(S, P)$ can be identified with the positive roots with respect to $\geq$.

Let $\Gamma \subseteq G(\mathbb{R})$ be an arithmetic subgroup and we write $\mathcal{X} = G(\mathbb{R})/K$ for the associated symmetric space. As a reductive Lie group $G(\mathbb{R})$ possesses an Iwasawa decomposition [7 Section 1.11]. In order to study arithmetic quotients, we modify this decomposition as in [20, 8, 5]. The decomposition we choose is compatible with the left action of $G(\mathbb{R})$ on $\mathcal{X}$, contrary to the right action in the works cited above.

$A := S(\mathbb{R})^0$ is $\theta$-invariant. Define for $t > 0$

$$A_t := \{ a \in A \mid \forall \alpha \in \Delta : \alpha(a) \geq t \}.$$  

According to Proposition 1.1, we have as in [5, Section 4.2]

$$P(\mathbb{R}) = {}^0 P(\mathbb{R}) \rtimes A.$$  

Furthermore

$${}^0 P = U \ltimes {}^0 M,$$

and as a consequence we get the decomposition

$$(2) \quad U(\mathbb{R}) \langle {}^0 M(\mathbb{R}) \rangle AK = G(\mathbb{R}).$$

Here $a \in A$ is uniquely determined through $g \in G(\mathbb{R})$ and the map $g \mapsto a$ is real analytic [8 Proposition 1.5].

$$K_P := K \cap P(\mathbb{R}) = K \cap {}^0 M(\mathbb{R})$$

is maximal compact in $P(\mathbb{R})$ and in $^0 M(\mathbb{R})$ as well. With

$$Z := {}^0 M(\mathbb{R})/K_P$$

we get a canonical diffeomorphism

$$^0 P(\mathbb{R})/K_P \cong U(\mathbb{R}) \times Z.$$  

Now $K$ defines a point $o$ in $\mathcal{X}$ (we may choose any point fixed by $K$) and we end up with an isomorphism

$$\mu_o : Y := U(\mathbb{R}) \times Z \times A \to \mathcal{X},$$

$$(u, z, a) \mapsto uza \cdot o.$$  

In the sequel we identify $U(\mathbb{R}) \times Z \times A$ with its image under $\mu_o$.

**Definition 1.3** ([6, section 3.2]). A $\phi \in \mathcal{C}^\infty(\mathcal{X})$ is of moderate growth if for any compact set $\omega \subseteq U(\mathbb{R}) \times Z$ and any $t > 0$ there exist $C > 0$ and $\lambda \geq 0$ such that

$$(3) \quad \forall (x, a) \in \omega \times A_t : |\phi(xa)| \leq C \cdot \lambda(a).$$

If for any $\omega, t$ and $\lambda \in X_{\mathbb{Q}}(S)$ we may find a $C > 0$ such that (3) is fulfilled, then $\phi$ is fast decreasing.
θ defines a Cartan decomposition

\[ \text{Lie}(G(\mathbb{R})) =: g = \mathfrak{k} \oplus \mathfrak{p}. \]

More precisely let \( \mathfrak{k} \) (resp. \( \mathfrak{p} \)) be the 1- (resp. \( -(1) \)-) eigenspace of the action of \( \theta \) on \( g \). We write \( d := \dim_{\mathbb{R}} \mathfrak{p} = \dim_{\mathbb{R}} \mathfrak{X} \) and we choose a basis \( \omega_1, \ldots, \omega_d \) of the invariant 1-forms \( \mathfrak{p}^* \) consisting of Maurer-Cartan forms. Furthermore we have for any subset \( I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, d\} \) of cardinality \( r \) an \( r \)-form

\[ \omega_I := \omega_{i_1} \wedge \cdots \wedge \omega_{i_r}, \]

where \( i_1 < i_2 < \cdots < i_r \). The form

\[ \eta = \sum_I \omega_I \otimes \phi_I \in \Omega^r(\mathfrak{X}) \]

is of moderate growth (resp. fast decreasing) if all \( \phi_I \) share this property.

Following Borel we let \( \Omega_{mg}^\bullet(\Gamma \backslash \mathfrak{X}) \) (resp. \( \Omega_{id}^\bullet(\Gamma \backslash \mathfrak{X}) \)) denote the subcomplex of \( \Omega^\bullet(\Gamma \backslash \mathfrak{X}) \) consisting of forms \( \eta \) which are, together with their differential \( d\eta \), of moderate growth (resp. fast decreasing).

We call a function \( \phi \) on \( G(\mathbb{R}) \) fast decreasing (resp. of moderate growth), if there is a \( \mathbb{Q} \)-morphism \( \rho : G \to \text{GL}_n \) with finite kernel, such that for all \( \mu \in \mathbb{Z} \) (resp. for one \( \mu \geq 0 \)) we may find a \( C > 0 \), such that for any \( x \in G(\mathbb{R}) \)

\[ |\phi(x)| \leq C \cdot |x|^\mu \]

with

\[ |x| := \text{tr}(\rho(x)^t \cdot \rho(x))^{\frac{1}{2}}. \]

It is well known (cf. [9, Proposition 3.10], [20, §3, Lemmas 5 and 6]) that a \( \phi \) on \( \Gamma \backslash \mathfrak{X} \) is of moderate growth (resp. fast decreasing) if and only if the same applies to its pullback on \( G(\mathbb{R}) \). In particular these growths conditions are actually intrinsic and do not depend on our choices in Definition 1.3.

We are now ready to define our modular symbols. Let \( H \) be a (connected) reductive group over \( \mathbb{Q} \). Let \( T \) be the maximal \( \mathbb{Q} \)-split central torus in \( H \). Then Proposition 1.1 says that

\[ H(\mathbb{R}) = 0H(\mathbb{R}) \times T(\mathbb{R})^0. \]

We write \( \mathcal{Y} \) for the symmetric space of \( H(\mathbb{R}) \) associated to a maximal compact subgroup \( K' \), so that we get the canonical decomposition

\[ \mathcal{Y} = 0\mathcal{Y} \times T(\mathbb{R})^0, \]

where

\[ 0\mathcal{Y} := 0H(\mathbb{R})/K', \]

since \( K' \) is contained in \( 0H(\mathbb{R}) \) (cf. Proposition 1.1). Appealing to the same proposition we see that we get a decomposition

\[ \Gamma' \backslash \mathcal{Y} = \Gamma' \backslash 0\mathcal{Y} \times \mathbb{R}^r \]

for any arithmetic subgroup \( \Gamma' \) of \( H(\mathbb{R}) \).
Proposition 1.4. Let $G$ and $H$ be connected reductive groups over $Q$ with $X_Q(G) = 1$, $\Gamma$ and $\Gamma'$ arithmetic subgroups of $G(\mathbb{R})$ and $H(\mathbb{R})$ and $s : H \to G$ be a central morphism with $R$-anisotropic kernel. Let furthermore $\mathcal{X}$ and $\mathcal{Y}$ be associated symmetric spaces and fix $g \in G(\mathbb{Q})$, such that $s$ induces a map 
\[ \mathcal{Y} \to \mathcal{X}, \; y \mapsto s(y) \]
of symmetric spaces which itself induces a map 
\[ s_g : \Gamma'\mathcal{Y} \to \Gamma\mathcal{X}, \; \Gamma'y \mapsto \Gamma g \cdot s(y) \]
on arithmetic quotients.

Then for any $\eta \in \Omega^q_{\text{id}}(\Gamma, \mathcal{X})$ we may integrate $\delta(s_g)(\eta)$ along the fibers of 
\[ \Gamma'\mathcal{Y} = \Gamma'\mathcal{Y} \times \mathcal{Y}, \]
where $r = \text{rank}_Q \mathcal{Y}(Y)$. This integration yields a form 
\[ s_{g,*}(\eta) \in \Omega^{q-r}_{\text{ng}}(\Gamma'\mathcal{Y}), \]
and the map $s_{g,*}$ is a chain map.

Proof. We keep the notation of the preceding paragraph. With $f$ also $h \mapsto f(gh)$ is fast decreasing and since the $1$-forms $\omega_1, \ldots, \omega_d$ are left-invariant, we may assume that $g = 1$.

$K \cap s(H(\mathbb{R}))$ is maximal compact in $s(H(\mathbb{R}))$. We conclude that the Cartan involution $\theta'$ to $K'$ and $\theta$ induce Cartan involutions on the image of $s$ which actually coincide [S Proposition 1.6]. Hence $s$ is compatible with $\theta$ and $\theta'$, i.e. $\theta \circ s = s \circ \theta'$.

$s(T)$ is contained in a maximal $Q$-split torus $T'$ of $H$. Now $T'$ is contained in a minimal parabolic $Q$-subgroup $P$ of $G$. Due to the $\theta'$-invariance of $T(\mathbb{R})$ we see that $s(T(\mathbb{R}))$ is $\theta$-invariant and therefore contained in the unique $\theta$-invariant Levi-subgroup $M/Q$ of $P$. Then $S := M \cap \mathcal{X}(P)$ is a maximal $Q$-split $\theta$-invariant torus in the radical of $P$ containing $s(T)$.

Remember the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and fix similarly the Cartan decomposition $\mathfrak{h} = \mathfrak{i} \oplus \mathfrak{q}$ of $\text{Lie}(H(\mathbb{R}))$. As before, choose a basis $\omega_1, \ldots, \omega_d$ of $\mathfrak{p}^*$ of Maurer-Cartan forms and similarly a Maurer-Cartan basis $\omega'_{1}, \ldots, \omega'_{d'}$ for $\mathfrak{q}^*$. We may assume that for $1 \leq i \leq d'$
\[ \delta(s)(\omega_i) = L(s)^*(\omega_i) = \omega'_{i} \]
and for $d' < j \leq d$
\[ \delta(s)(\omega_j) = L(s)^*(\omega_j) = 0, \]
since $L(s)$ is injective. Now let
\[ \eta = \sum_{I} \omega_I \otimes f_I \in \Omega^q_{\text{id}}(\Gamma, \mathcal{X}). \]
Then
\[ \delta(s)(\eta) = \sum_{I'} \omega_{I'} \otimes (\phi_{I'} \circ s), \]
where $I'$ runs through the same subsets of $\{1, \ldots, d'\}$ as $I$ does. We fix integration parameters $a_1^{-1}da_1, \ldots, a_r^{-1}da_r$, corresponding to $\omega_{d-r+1}, \ldots, \omega_d$ and parametrize the fibers by means of a dual basis $\Delta^* \subseteq \Phi_Q(T,H)^*$. Integration of $\delta(s)(\eta)$ along the fibers then means that we integrate all $\phi_{I'} \circ s$ where $\{d' - r + 1, \ldots, d'\} \subseteq I'$. 

Assume that $I'$ satisfies this condition. For any $h \in H(R)$ we have to consider the integral
\[
\int_{\mathbb{R}_{\geq 0}^r} \phi_{I'}(s(h)s(\lambda_1^*(a_1) \cdots \lambda_r^*(a_r))) \frac{da_1}{a_1} \cdots \frac{da_r}{a_r},
\]
where $\lambda_1^*, \ldots, \lambda_r^* \in \Delta^*$ run through the corresponding basis of cocharacters of $T$. We have
\[
s(\lambda_1^*(a_1) \cdots \lambda_r^*(a_r)) \in S(R)^0
\]
and since $s$ induces an isogeny of $T$ onto a subtorus $S'$ of $S$, we get, modulo orientation and the finite kernel of this isogeny, the integral
\[
\int_{\mathbb{R}_{\geq 0}^r} \phi_{I'}(s(h)\lambda_1^*(a_1) \cdots \lambda_r^*(a_r)) \frac{da_1}{a_1} \cdots \frac{da_r}{a_r},
\]
where $\mathbb{R}_{\geq 0}^r$ parametrizes $S'(R)^0$ via the dual simple roots $\lambda_1^*, \ldots, \lambda_r^* \in (\Phi_Q(S', P) \cap \Delta)^*$. Let
\[
a(h) := \mu_o^{-1}(s(h)K) \in S(R)^0
\]
be the projection of $s(h)$ onto the factor $A = S(R)^0$ of the decomposition \cite{[2]}.

By our hypothesis $\phi_{I'}$ is fast decreasing, therefore we find for any $a_1, \ldots, a_r \in \mathbb{Z}$ and any $t > 0$ a $C > 0$ such that $\forall a_1, \ldots, a_r \geq t$
\[
|\phi_{I'}(s(h)\lambda_1^*(a_1) \cdots \lambda_r^*(a_r))| \leq C \cdot a_1^{\alpha_1} \cdots a_r^{\alpha_r} \cdot \lambda_1(a(h))^{\alpha_1} \cdots \lambda_r(a(h))^{\alpha_r}.
\]
We may assume that $C$ is constant on a (compact) neighborhood of $h$. Hence our integral is absolutely convergent and bounded as well on the $2^r$ segments $X_1 \times \cdots \times X_r$ with $X_i \in \{(0; \lambda_i(a(h))^{-1}], (\lambda_i(a(h))^{-1}; \infty]\}$. The resulting map $\phi_{I'}$ is also bounded and therefore of moderate growth. We end up with a form
\[
s_{g,*}(\eta) := \sum_{\#I = q - r} \omega_{\hat{I}} \otimes \hat{\phi}_{I \cup \{1, \ldots, r\}} \in \Omega_{mg}^{\eta - r}(\Gamma \setminus \mathcal{Y}),
\]
where the sum ranges over the indices such that $\{1, \ldots, r\} \cap \hat{I} = \emptyset$.

Finally we observe that
\[
d(s_{g,*}(\eta)) = s_{g,*}(d\eta),
\]
and this if of moderate growth, concluding the proof of the lemma. \hfill \Box

\textbf{Remark 1.5.} The assumption that $s$ is central simplifies our formulation. It guarantees that the kernel of $s$ is already defined over $Q$. The proposition, as well as its proof are still valid word by word, if we only assume that the kernel of $s$ is defined over $R$ and $R$-anisotropic.

For later use we prove the following

\textbf{Lemma 1.6.} Assume $G$ reductive over $Q$ and $X_Q(G) = 1$. Let $\Gamma$ be an arithmetic and $K$ be a maximal compact subgroup of $G(R)$, $\Gamma^1$ an arithmetic and $K^1$ a maximal compact subgroup of $G^m(R)$ with $\text{ad}(\Gamma) \subseteq \Gamma^1$ and $\text{ad}(K) \subseteq K^1$. Then the map
\[
p : \Gamma \setminus \mathcal{Y} \rightarrow \Gamma^1 \setminus \mathcal{Y}^1
\]
induced by $\text{ad}$ on the associated arithmetic quotients is proper.
Proof. Note that ad(\(\Gamma\)) is of finite index in \(\Gamma^1\) since ad(\(\Gamma\)) is arithmetic as well. Now \(G\) is an isogenous image of \(G^{\text{der}} \times Z\) with \(Z := \mathcal{C}(G)^0\). On the other hand \(G^{\text{ad}}\) is an isogenous image of \(G^{\text{der}}\). Hence by defining \(\Gamma_1 := \Gamma \cap G^{\text{der}}(R), K_1 := K \cap G^{\text{der}}(R), \Gamma_2 := \Gamma \cap Z(R), K_2 := K \cap Z(R),\) we get a canonical map

\[
p' : \Gamma_1 \backslash G^{\text{der}}(R)/K_1 \times \Gamma_2 \backslash Z(R)/K_2 \to \Gamma^1 \backslash \mathcal{Y}^{-1},
\]

\[
(\Gamma_1 x K_1, \Gamma_2 y K_2) \mapsto \Gamma^1 \text{ad}(xy)K^1.
\]

This naturally factorizes over \(\Gamma \backslash \mathcal{Y}\). More precisely we have \(p' = p \circ p''\), where \(p''\) is surjective and defined analogously.

Now \(\Gamma_1 \times \Gamma_2\) is an arithmetic subgroup of \(G^{\text{der}} \times Z\), hence \(p''\) is proper. It remains to see that \(p'\) is proper. Obviously the canonical map

\[
p'_1 : \Gamma_1 \backslash G^{\text{der}}(R)/K_1 \to \Gamma^1 \backslash \mathcal{Y}^{-1}
\]

is proper. Finally we show that \(\Gamma_2 \backslash Z(R)\) is compact, which will conclude the proof. Our argument is nothing but a natural generalization of Dirichlet’s classical unit theorem. \(Z\) decomposes over \(R\) into an almost product of an \(R\)-split torus \(T_d\) and an \(R\)-anistropic torus \(T_s\). Now \(T_d(R)\) is compact and \(\Gamma_d := \Gamma_2 \cap T_d(R)\) is a discrete subgroup in \(T_d(R)\). Our assumption that \(X_Q(G) = 1\) implies \(X_Q(Z) = 1\), hence \(\Gamma_2 \backslash Z(R)\) has finite invariant measure. Therefore \(\Gamma_d \backslash T_d(R)\) has finite invariant measure, so \(\Gamma_d\) is a lattice in \(T_d(R)\) and \(\Gamma_d \backslash T_d(R)\) is compact. This proves the lemma.

Remark 1.7. We get an analogous statement by replacing \(G^{\text{ad}}(R)^0\) by an arbitrary real Lie group \(G'\), such that \(G(R)^0 \to G^{\text{ad}}(R)^0\) factorizes over \(G'\) and \(G(R)^0 \to G'\) is an epimorphism.

We may regard \(\Gamma \backslash \{0\} \mathcal{Y}\) as an arithmetic quotient of \(\{0\} G^0\). Hence Proposition 1.2 implies that it has finite invariant measure and by \([10\) Theorem 5.2] the canonical inclusion \(\Omega^\bullet_\mathcal{Y} \to \Omega^\bullet_{\mathcal{Y}^1}\) induces an isomorphism in cohomology. Proposition 1.4 and Lemma 1.5 may be used to construct many different modular symbols. With our application in mind we will stick only to one case. We write \(p : \{0\} \mathcal{Y} \to \mathcal{Y}^1\) for the projection induced by \(\text{ad}\), where \(\mathcal{Y}^1\) is a symmetric space for \(H^{\text{ad}}(R)\). We define the following modular symbol

\[
\mathcal{Y}^q_{s, g} : H^q_\mathcal{Y}(\Gamma, \mathcal{X}, C) \times H^q_{\text{dim}\mathcal{Y}-q}(\Gamma^1 \backslash \mathcal{Y}^{-1}, C) \to C,
\]

\[
([\eta], [\eta']) \mapsto \int_{\Gamma \backslash \{0\} \mathcal{Y}} s_{g, \cdot}(\eta) \wedge p^*(\eta').
\]

Due to its topological construction this pairing respects the natural \(Z\)-structure on cohomology.

2. A general local Birch lemma

Fix a nonarchimedean local field \(F\). So \(F\) is a finite extension of the \(p\)-adic prime field \(\mathbb{Q}_p\) or of \(F_p((T))\). Denote by \(\mathcal{O}_F\) its maximal compact subring, by \(p\) its maximal ideal and by \(v\) the place associated to \(F\). We fix a prime \(\pi \in \mathcal{O}_F\) and the absolute value \(\|\cdot\|\) of \(F\) such that \(\|\pi\| = \mathfrak{N}(p)^{-1}\), where \(\mathfrak{N}(a)\) is the absolute norm of a fractional ideal in a local (or a global) field. Fix a character \(\psi : F \to \mathbb{C}^\times\) with conductor \(\mathcal{O}_F\). For a generic automorphic representation \(\pi_v\) of \(\text{GL}_n(F)\) we write \(\mathcal{Y}(\pi_v, \psi)\) for the \(\psi\)-Whittaker space of \(\pi_v\). The same notation applies to global
Whittaker spaces of global representations. The choice of $\psi$ also fixes the Gauß sum

$$G(\chi) := \sum_{x+f \in (\mathcal{O}_F/l)^\times} \chi(x)\psi \left( \frac{x}{f} \right).$$

for any quasi-character $\chi : F^\times \to \mathbb{C}^\times$ of conductor $f = f\mathcal{O}_F$. By abuse of notation we also write $\chi(g)$ instead of $\chi(\det(g))$ for $g \in \text{GL}_n(F)$. We will make repeated use of the following elementary fact. Let $0 \neq g \in \mathcal{O}_F$, then with $h := f \cap g\mathcal{O}_F$ we have

$$\sum_{x+b \in (\mathcal{O}_F/h)^\times} \chi(x)\psi \left( \frac{x}{h} \right) = \begin{cases} \chi(g/f) \cdot G(\chi), & \text{if } f = g\mathcal{O}_F, \\ 0, & \text{otherwise}. \end{cases}$$

We denote by $I_n$ the Iwahori subgroup of $\text{GL}_n(\mathcal{O}_F)$, i.e. the group of matrices $g \in \text{GL}_n(\mathcal{O}_F)$ that become upper triangular modulo $p$. $B_n$ is the standard Borel subgroup of $\text{GL}_n$ of upper triangular matrices, $U_n$ denotes its unipotent radical. We extend $\psi$ to $U_n(F)$ by the rule

$$\psi(u) := \prod_{i=1}^{n-1} \psi(u_{ii+1})$$

for $u = (u_{ij}) \in U_n(F)$. We write $B_n^-$ for the subgroup of $\text{GL}_n$ of lower triangular matrices.

$W_n$ denotes the Weyl group of $\text{GL}_n$ realized as the subgroup of permutation matrices. To each $\omega \in W_n$ we associate a permutation $\sigma \in S_n$ ($S_n$ is the symmetric group on $\{1, \ldots, n\}$) via

$$\omega \cdot b_k = b_{\sigma^{-1}(k)}$$

for $1 \leq k \leq n$. Here $b_k$ denotes the $k$-th standard basis vector of $F^n$. Then for any $a = (a_i)_{1 \leq i \leq n} \in F^n$ we have $\omega a = (a_{\sigma(i)})_{1 \leq i \leq n}$. The map $\omega \mapsto \sigma^{-1}$ is an isomorphism $W_n \to S_n$. Let $w_n \in W_n$ denote the longest element. We have

$$w_n = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 &&&& \end{pmatrix}.$$
be the inclusion
\[ g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1_{m-n} \end{pmatrix}. \]
For a fixed \( 0 \neq f \in \mathcal{O}_F - \mathcal{O}_F^\times \) we define the matrix
\[ D_n := \text{diag}(f^{-(n-1)}, f^{-(n-3)}, \ldots, f^{n-3}, f^{n-1}) \in \text{GL}_n(F) \]
and the linear form
\[ \lambda_n : F^{n \times n} \to F, \quad g \mapsto b_n^t \cdot g \cdot \phi_n, \]
where
\[ \phi_n := (f^{n-1}, f^{-(n-1)}, \ldots, f^{-1})^t. \]

The main result of this section is

**Theorem 2.1** (general local Birch lemma). Let \( w \) and \( v \) be Iwahori invariant \( \psi \)-resp. \( \psi^{-1} \)-Whittaker functions on \( \text{GL}_m(F) \) and \( \text{GL}_n(F) \). For any quasi-character \( \chi : F^\times \to \mathbb{C}^\times \) with nontrivial conductor \( f = f\mathcal{O}_F \) we have the explicit formula
\[
\int_{U_n(F) \setminus \text{GL}_n(F)} \psi(\lambda_n(g))w(\langle j(gD_nw_n) \rangle v(g)\chi(\det(g))\|\det(g)\|^{-\frac{m+n}{2}} \, dg = \\
\prod_{\nu=1}^n (1 - \mathfrak{N}(p)^{-\nu})^{-1} \cdot \mathfrak{N}(f)^{-\sum_{i=1}^k (n+1-k)} \cdot G(\chi)^{\frac{n(n+1)}{2}} \cdot w(1) \cdot v(1) .
\]

In the case \( m = n + 1 \) our formula appears simpler than the generalized local Birch lemma of [40] and contains [43, 44] as a special case. Our proof differs substantially from the previous approaches. A crucial observation is that the integrand of the theorem is constant on the classes of a refinement of the Iwasa decomposition \([5]\). This enables us to evaluate the integral as a sum. It will turn out that for \( e \neq 0 \) or \( \omega \neq 1 \) the corresponding partial sums vanish so that our integral eventually becomes a finite sum (cf. Lemma 2.7), giving rise to the formula of the theorem.

For the proof of Theorem 2.1 we may assume \( m = n \) and \( j = 1 \) without loss of generality. The final argument is inductive and we need some preparation for the induction step. We start by giving the refined double coset decomposition of \( \text{GL}_n(F) \).

Let
\[ J_{l,n} := \ker \left( \text{GL}_n(\mathcal{O}_F) \to \text{GL}_n(\mathcal{O}_F/\mathfrak{p}^l) \right). \]
From now on we assume \( l \geq 2n \). This guarantees that
\[ J_{l,n} \subseteq I_n \cap w_nD_n^{-1}I_nD_nw_n. \]
Choose a system \( R_l \) of representatives of \( \mathcal{O}_F/\mathfrak{p}^l \) and let \( R_l^\times \subseteq R_l \) be a system of representatives of \( (\mathcal{O}_F/\mathfrak{p}^l)^\times \). This enables us to define
\[ \mathfrak{R}_{l,n} := \{ (r_{ij}) \in I_n \mid r_{ij} \in R_l \}. \]
Then \( \mathfrak{R}_{l,n} \) is a system of representatives for \( I_n/J_{l,n} \) and as such may be endowed with the natural group structure which is induced by matrix multiplication modulo \( \mathfrak{p}^l \). We may assume that
\[ 0, \pm 1, \pm f, \ldots, \pm f^{l-1} \in R_l \]
holds, which simplifies notation. This allows us to define for any \( \omega \in W_n \) with corresponding \( \sigma \in S_n \)
\[ \mathfrak{R}_{l,n}^\omega := \{ (r_{ij}) \in \mathfrak{R}_{l,n} \mid \forall i, j : i < j \Rightarrow r_{\sigma(i)\sigma(j)} = 0 \}. \]
Note that for \( r = (r_{ij})_{ij} \) we have
\[
\omega r \omega^{-1} = (r_{\sigma(i)\sigma(j)})_{ij}.
\]
Therefore
\[
\mathcal{R}_l = \mathcal{R}_l \cap \omega^{-1} B_n^- (\mathcal{O}_F) \omega.
\]
Consequently for any \( r \in \mathcal{R}_l \) we see that
\[
\det(r) = \prod_{k=1}^{n} r_{kk}
\]
and that furthermore \( \mathcal{R}_l \) is a subgroup of \( \mathcal{R}_n \). Our interest in \( \mathcal{R}_l \) is motivated by

**Proposition 2.2.** The set \( \varpi \mathcal{R}_l \) is a system of representatives of the double cosets
\[
U_n(F) \varpi \mathcal{R}_l \varpi U_n(F), \quad s \in I_n
\]
in \( U_n(F) \varpi \mathcal{R}_l \varpi U_n(F) \). Fix an \( l(e) \geq 2n \) for any \( e \in \mathbb{Z}^n \). Then
\[
\prod_{e \in \mathbb{Z}^n} U_n(F) \varpi e J_{l(e),n}.
\]

**Proof.** Since \( \varpi U_n(F) \varpi^{-1} = U_n(F) \) we may assume that \( e = 0 \). Define
\[
U^n := U_n(F) \cap \omega I_n \omega^{-1}.
\]
As a system of representatives for \( \omega I_n / J_{l,n} \) the set \( \omega \mathcal{R}_l \) contains a system of representatives for the double cosets
\[
U^n \omega s J_{l,n}, \quad s \in I_n
\]
in \( U^n \omega I_n \). We need to show that \( \omega \mathcal{R}_l \) is a system of representatives of these double cosets.

For any \( r = (r_{ij}) \in I_n \) we have
\[
\omega \cdot r \cdot \omega^{-1} = (r_{\sigma(i)\sigma(j)})_{ij}.
\]
The action of \( U_n(F) \) from the left allows us to add multiples of a row \( i \) with entries \( r_{\sigma(i)\sigma(j)} \) to a row \( k < i \) with entries \( r_{\sigma(k)\sigma(j)} \). Since \( r_{\sigma(i)\sigma(i)} \) are units we may use them to annihilate all entries \( r_{\sigma(k)\sigma(i)} \) with \( k < i \) of the same column. If we proceed inductively, starting with \( i = n \), this process corresponds to a multiplication with a matrix
\[
u_{ij} \in U_n(F).
\]
We show inductively that \( u \in U^n \) and that
\[
\omega \cdot r \omega^{-1} \in \omega I_n \omega^{-1} \cap B_n^- (\mathcal{O}_F).
\]
To see this iteratively define matrices \( u^{(\nu)} = (u^{(\nu)}_{ij}) \in U^n, r^{(\nu)} \in I_n \) as follows. Let
\[
u^{(0)} := 1_n \text{ and } r^{(0)} := r.
\]
For \( \nu \geq 0 \) let \( u^{(\nu+1)} = (u^{(\nu+1)}_{ij}) \in U_n(F) \) be given by
\[
u^{(\nu+1)} := \delta_{ij} \text{ (Kronecker delta) if } j \neq n - \nu \text{ or } i \geq j.
\]
For \( j = n - \nu \) and \( i < j \) let
\[
u^{(\nu+1)}_{i,n-\nu} := \frac{r^{(\nu)}_{\sigma(i)\sigma(n-\nu)}}{r^{(\nu)}_{\sigma(n-\nu)\sigma(n-\nu)}}.
\]
Finally define
\[ r^{(ν+1)} := ω^{-1} u^{(ν+1)} ω \cdot r^{(ν)}. \]
We see inductively that \( r^{(ν)} \) is \( u^{(ν)} \) \( I_n \) if \( j > n - ν \) and \( i < j \). Furthermore we show that \( u^{(ν)} \in U^ω \) implies \( u^{(ν+1)} \in U^ω \). Indeed, the former implies \( r^{(ν)} \in I_n \).

The definition of \( u^{(ν+1)} \) shows that there are two cases. If \( \sigma(i) < \sigma(n - ν) \), then \( r^{(ν)} \) \( I_n \) if \( \sigma(i) > \sigma(n - ν) \), then \( r^{(ν)} \) is lower triangular, so \( u^{(ν)} \in U^ω \).

With \( u := u^{(n-1)} \cdot u^{(n-2)} \cdot \cdots \cdot u^{(1)} \in U^ω \)
we conclude that
\[ u \cdot ωrω^{-1} \in ω \cdot I_n \cdot ω^{-1} \cap B_n^-(O_F). \]
Now let \( s \in K_{l,n} \) be the representative of the double coset
\[ ω^{-1}uω \cdot r \cdot J_{l,n}. \]
Then \( ωs \) represents the same double coset as \( ωr \) does and
\[ ωsω^{-1} \in u \cdot ωrω^{-1} \cdot J_{l,n} \subseteq B_n^-(O_F)J_{l,n} \]
yields with \( \boxtimes \) that
\[ s \in K_{l,n}^ω. \]
Consequently \( ωK_{l,n}^ω \) contains a system of representatives.
To finish the proof, we assume
\[ u \cdot ωr \in ωs \cdot J_{l,n} \]
for any \( u \in U^ω \), \( r \), \( s \in K_{l,n}^ω \). Since \( J_{l,n} \) is normal in \( GL_n(O_F) \) this is equivalent to
\[ u \in ωsω^{-1} \cdot ωr^{-1} \cdot J_{l,n}. \]
Now \( ωsω^{-1} = (s_{σ(i)σ(j)})_{ij} \) and \( ωr^{-1} \) are lower triangular, so
\[ u \in U_n^-(O_F)J_{l,n}. \]
We observe that this means
\[ u \equiv 1_n \pmod{J_{l,n}}, \]
concluding the proof of the proposition. \( \square \)

**Proposition 2.3.** For any \( e \in Z^n \), \( ω \in W_n \) and \( r \in K_{l,n}^ω \) the measure
\[ \int_{U_n(O_F)^ω \cdot ωr \cdot J_{l,n}} dg \]
is independent of \( ω \) and \( r \). If \( e = 0 \) and \( l = 2n \) we have
\[ \int_{U_n(O_F)^ω \cdot r \cdot J_{2n,n}} dg = \prod_{ν=1}^n (1 - O(p)^{-ν})^{-1} \cdot O(f)^{-n^3-n^2}. \]
Proposition 2.4. Its projection $\sigma$ under the action $t$ of the compact torus $T$ on $\text{GL}_{l,n}$.

Proof. Since $J_{l,n}$ is normal in $\text{GL}_n(O_F)$ we have for $\omega \in W_n$ and $r \in \mathfrak{A}_{l,n}^\omega$  

$$U_n(O_F)\omega r J_{l,n} = U_n(O_F)\omega J_{l,n} \cdot \omega r.$$  

By the right invariance of $dg$ the measure of this set is independent of $r$ and $\omega$.

It is well known that  

$$(\text{GL}_n(O_F) : J_{l,n}) = \# \text{GL}_n(O_F/\langle t \rangle) = \mathfrak{M}(f)^{ln^2} \cdot \prod_{\nu=1}^{n}(1 - \mathfrak{M}(p)^{-\nu}).$$

Furthermore $U_n(O_F)J_{l,n}$ is a group and  

$$(U_n(O_F)J_{2n,n} : J_{2n,n}) = (U_n(O_F) : U_n(O_F) \cap J_{2n,n}) = \# U_n(O_F/\langle t^{2n} \rangle) = \mathfrak{M}(f)^{2n \nu(n-1)} = \mathfrak{M}(f)^{n^2 - n^2}.$$  

We finish the proof with the conclusion  

$$(\text{GL}_n(O_F) : U_n(O_F)J_{2n,n}) = \prod_{\nu=1}^{n}(1 - \mathfrak{M}(p)^{-\nu}) \cdot \mathfrak{M}(f)^{n^2 + n^2}.$$  

□

The action of the compact torus  

$$T_n := (O_F^\times)^n$$

on $\text{GL}_n(F)$, for $\gamma = (\gamma_1, \ldots, \gamma_n) \in T_n$ given by  

$$\gamma : \text{GL}_n(F) \to \text{GL}_n(F), \; g \mapsto g \cdot \text{diag}(\gamma_1, \ldots, \gamma_n)$$

is vital for our argument. The torus $T_n$ operates naturally on the set of representatives $\mathfrak{A}_{l,n}^\omega$ via its action on the quotient $T_n/J_{l,n}$. This action factorizes via the finite torus  

$$\mathcal{T}_{l,n} := T_n/(1 + f)^n$$

and $\mathcal{T}_{l,n}$ acts faithfully on the orbit of any $r \in \mathfrak{A}_{l,n}^\omega$.

For the inductive step we define under the assumption $\sigma(n) = n$ the set  

$$\mathfrak{A}_{l,n}^\omega := \{(r_{ij}) \in \mathfrak{A}_{l,n}^\omega \mid r_{n1} = f^{n-1}, \; r_{nj} = -f^{n-j}, \; 2 \leq j \leq n\}.$$  

Note that $\sigma(n) = n$ implies $n \geq 1$.

Proposition 2.4. If $\sigma(n) = n$ we have for any $r \in \mathfrak{A}_{l,n}^\omega$  

$$\# \left(\mathcal{T}_{l,n} : r \cap \mathfrak{A}_{l,n}^\omega\right) = \mathfrak{M}(f)^{n(n-1)}.$$  

In other words the orbit of $r$ under the action of $T_n$ on $\mathfrak{A}_{l,n}^\omega$ contains precisely  

$\mathfrak{M}(f)^{n(n-1)}$ elements of $\mathfrak{A}_{l,n}^\omega$.

Proof. The stabilizer of  

$$(f^{n-1} + \langle t \rangle, -f^{n-2} + \langle t \rangle, \ldots, -1 + \langle t \rangle) \in O_F^n/\langle t \rangle^n$$

under the action $s \mapsto s \cdot \gamma_1$ of $T_n$ on $O_F^n/\langle t \rangle^n$ is  

$$(1 + \langle t \rangle^{n+1}) \times (1 + \langle t \rangle^{n+2}) \times \cdots \times (1 + \langle t \rangle^{n+1}) \times (1 + \langle t \rangle).$$

Its projection $\mathcal{T}_{l,n}$ has cardinality  

$$\prod_{j=1}^{n} \mathfrak{M}(f)^{n-j} = \mathfrak{M}(f)^{n(n-1)/2}.$$
This concludes the proof. □

Define the matrix
\[
C_n := \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
-f^{n-1} & -f^{n-2} & \cdots & -f & -1
\end{pmatrix}
\]
and the projection
\[
p : F^{n \times n} \to F^{n-1 \times n-1}, \quad (g_{ij}) \mapsto (g_{ij})_{1 \leq i,j \leq n-1}
\]
with its set theoretic section
\[
\tilde{j} : F^{n-1 \times n-1} \to F^{n \times n}, \quad \tilde{g} \mapsto \begin{pmatrix}
\tilde{g} & 0 \\
0 & 1
\end{pmatrix}.
\]

**Proposition 2.5.** Under the assumption \(\sigma(n) = n\) we have for \(\tilde{\omega} := p(\omega), \tilde{r} := p(r)\),
\[
\# \tilde{\mathcal{R}}_{t,n}^\omega = \# \mathcal{R}_{t,n}^\omega.
\]
More precisely the projection \(p\) induces a bijection
\[
p : \tilde{\mathcal{R}}_{t,n}^\omega \to \mathcal{R}_{t,n}^\omega
\]
and
\[
\text{GL}_{n-1}(F) \to \text{GL}_n(F), \quad \tilde{g} \mapsto \tilde{j}(\tilde{g}) \cdot C_n
\]
induces the inverse of \(p\).

**Proof.** Note that the projection \(p : F^{n \times n} \to F^{n-1 \times n-1}\) induces a map
\[
p : \omega \tilde{\mathcal{R}}_{t,n}^\omega \omega^{-1} \to \omega \mathcal{R}_{t,n}^\omega \omega^{-1}.
\]
Since \(\omega \tilde{\mathcal{R}}_{t,n}^\omega \omega^{-1}\) and \(\omega \mathcal{R}_{t,n}^\omega \omega^{-1}\) are lower triangular \(p\) is well defined, bijective and has the inverse as claimed. □

Let
\[
A_n := \begin{pmatrix}
1 & f^{-1} & 0 & \cdots & 0 \\
0 & 1 & -f^{-1} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & -f^{-1} \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix} \in \text{GL}_n(F),
\]
\[
\tilde{A}_n := \begin{pmatrix}
f^{-1} & 0 & \cdots & \cdots & 0 \\
1 & -f^{-1} & \ddots & \vdots & \vdots \\
0 & 1 & -f^{-1} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & -f^{-1}
\end{pmatrix} \in \text{GL}_n(F),
\]
and
\[
B_n := f \cdot \tilde{A}_n \in I_n.
\]
Furthermore we let \( B_1 = C_1 = 1 \) and \( B_0 := 1_0 \). This guarantees that for all \( n \geq 0 \)

\[
\det(B_{n+1}C_{n+1}) = \det(B_n).
\]

Let \( w \) denote an Iwahori invariant \( \psi \)-Whittaker function on \( \text{GL}_n(F) \) and \( j : \text{GL}_n \to \text{GL}_{n+1} \) denote the canonical inclusion as before.

**Lemma 2.6.** For \( n \geq 0 \) and \( g \in \text{GL}_n(F) \) we have

\[
w(j(g)C_{n+1} \cdot D_{n+1}w_{n+1}) \cdot v(g) = \\
\psi(\lambda_n(gB_n)) \cdot w(j(gB_n \cdot D_nw_n)) \cdot v(gB_n).
\]

**Proof.** Since \( B_n \in I_n \) it suffices to show that

\[
\psi(\lambda_n(gB_n)) \cdot w(j(gB_n \cdot D_nw_n)) = w(j(g)C_{n+1} \cdot D_{n+1}w_{n+1}).
\]

To see this consider the matrix

\[
u = (u_{ij}) \in U_{n+1}(F),
\]

for \( j \leq n \) and \( 1 \leq i \leq n + 1 \) given by

\[
u_{ij} := \delta_{ij} \quad \text{(Kronecker delta)}
\]

and for \( 1 \leq i \leq n \) by

\[
u_{in+1} := -g_{i1} \cdot f^{-n}.
\]

Then, due to \( \tilde{A}_n = ((A_{n+1})_{ij+1})_{1 \leq i, j \leq n} \), we get

\[
u \cdot j(g)C_{n+1} \cdot A_{n+1} = \begin{pmatrix} 0 & \vdots & g \cdot \tilde{A}_n \\ \vdots \\ 0 & f^n & 0 & \ldots & 0 \end{pmatrix}.
\]

Finally we have

\[
u \cdot j(g)C_{n+1} \cdot A_{n+1} \cdot D_{n+1}w_{n+1} = \begin{pmatrix} 0 \\ \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}
\]

with

\[
\tilde{g} = g \cdot \tilde{A}_n f \cdot D_nw_n.
\]

We conclude that

\[
u \cdot j(g)C_{n+1} \cdot A_{n+1} \cdot D_{n+1}w_{n+1} = j(gB_n \cdot D_nw_n).
\]

Note that

\[
w_{n+1}D_{n+1}^{-1}A_{n+1} \cdot D_{n+1}w_{n+1} \in I_{n+1},
\]

which implies

\[
j(g)C_{n+1} \cdot A_{n+1} \cdot D_{n+1}w_{n+1} \in j(g)C_{n+1} \cdot D_{n+1}w_{n+1}I_{n+1}.
\]

For our Iwahori invariant \( \psi \)-Whittaker function this means that

\[
\psi(f^{-n} \cdot g_{n1}) \cdot w(j(gB_n \cdot D_nw_n)) = w(j(g)C_{n+1} \cdot D_{n+1}w_{n+1}).
\]

Thanks to

\[
B_n\phi_n = (f^{-n}, 0, \ldots, 0)^t
\]

the claim follows. \( \square \)
The next lemma is the key ingredient in the proof of Theorem 2.1.

**Lemma 2.7.** Let \( w \) and \( v \) be Iwahori invariant \( \psi - (\text{resp. } \psi^{-1}\text{-}) \) Whittaker functions on \( GL_n(F) \). For any \( n \geq 0, e \in \mathbb{Z}^n, \omega \in W_n \) and \( l \geq \max\{2n, n - e_1/\nu_p(f), \ldots, n - e_n/\nu_p(f)\} \) we have

\[
\sum_{g \in \mathfrak{g}^{\omega,\nu_p(f)}} \psi(\lambda_n(g)) \cdot w(g \cdot D_n w_n) \cdot v(g) \cdot (\det(g)) \cdot |\det(g)|^s =
\begin{cases}
\mathcal{M}(f)^{(l-2n)(n+1)} + \frac{1}{2} \sum_{n=1}^{\infty} 5^{-3/2} \cdot G(\chi)^{(n+1)} \cdot w(1_n) \cdot v(1_n), & \text{for } \omega = 1_n \text{ and } e = 0, \\
0, & \text{otherwise}.
\end{cases}
\]

**Proof.** We proceed by induction on \( n \). If \( n = 0 \), then \( W_0 = GL_0(\mathcal{O}_F) = \{1\} \), \( \mathfrak{R}_0^n = \{1\} \), \( Z^0 = \{0\} \). The case \( \omega \not= 1_0 \) or \( e \not= 0 \) actually never occurs. This concludes the case \( n = 0 \). Now let \( n \geq 1 \) and suppose that the claim is true for \( n - 1 \).

Remember that \( \overline{T}_{1,n} \) acts faithfully on the orbits of the action of \( T_n \) on \( \mathfrak{R}_l^\gamma \). Let \( S \subseteq T_n \) be any system of representatives for \( \overline{T}_{1,n} \). The decomposition of \( \mathfrak{R}_l^\gamma \) into orbits of this action naturally yields partial sums

\[
Z(r) := \sum_{\gamma \in S} \psi(\chi_n(\mathfrak{w}^\omega \gamma)) \cdot w(\mathfrak{w}^\omega \gamma r \cdot D_n w_n) \cdot v(\mathfrak{w}^\omega \gamma r) \cdot \chi(\mathfrak{w}^\omega \gamma r) \cdot |\det(\mathfrak{w}^\omega \gamma r)|^s,
\]

where \( r \in \mathfrak{R}_l^\gamma \) represents a \( T_n \)-orbit. Due to our hypothesis on \( l \) this sum is independent of the choice of \( S \) and furthermore \( Z(r) \) is by definition constant on the \( T_n \)-orbits. In particular \( Z(r) \) vanishes if and only if it vanishes on the whole orbit of \( r \).

Our strategy of proof is to see in which cases \( Z(r) \) vanishes. This will enable us to deduce equation (13). We will see that this is already enough to conclude the proof inductively by appealing to Lemma 2.6.

We have

\[
\mathfrak{w}^\omega \gamma r \cdot D_n w_n = \mathfrak{w}^\omega \gamma r \cdot \gamma 1_n \cdot D_n w_n = \mathfrak{w}^\omega \gamma r \cdot D_n w_n \cdot (w_n \gamma 1_n w_n) \in \mathfrak{w}^\omega \gamma r \cdot D_n w_n \cdot I_n,
\]

because \( w_n \gamma 1_n w_n \in I_n \). From this relation we get

\[
Z(r) = \sum_{\gamma \in S} \psi(\chi_n(\mathfrak{w}^\omega \gamma)) \cdot w(\mathfrak{w}^\omega \gamma r \cdot D_n w_n) \cdot v(\mathfrak{w}^\omega \gamma) \cdot \sum_{\gamma \in S} \chi(\gamma 1_n) \cdot \psi(\lambda_n(\mathfrak{w}^\omega \gamma r)).
\]

We have

\[
\psi(\lambda_n(\mathfrak{w}^\omega \gamma r)) = \prod_{\nu=1}^n \psi(\mathfrak{w}^\nu \gamma r \cdot \gamma_n),
\]
which yields

\[ \sum_{\gamma \in \mathcal{S}} \chi(\gamma \mathbf{1}_n) \cdot \psi(\lambda_n(\mathbb{G}^\omega \gamma r)) = \]

\[ \prod_{\nu = 1}^{n} \sum_{\nu_\sigma \in \mathcal{O}_{\mathbb{G}/11}}^n \chi(\gamma_{\nu_\sigma}) \cdot \psi \left( \mathbb{G}^\omega \mathcal{F}_{\nu_\sigma}^{r_{\sigma(\nu_\sigma)} \cdot \gamma_{\nu_\sigma}} \right). \]

Since \( r_{\sigma(\nu_\sigma)} \in \mathcal{O}_{\mathbb{G}/11}^\times \) and \( \nu_\mathcal{F}(f) \geq n \nu_\mathcal{F}(f) - e_n \) we conclude with equation (10) that we have an implication

\[ e_n \neq (n - \sigma(n)) \cdot \nu_\mathcal{F}(f) \Rightarrow Z(r) = 0. \]

Consequently let \( e_n = (n - \sigma(n)) \cdot \nu_\mathcal{F}(f) \). If \( \sigma(n) \neq n \), then \( e_n > 0 \) and therefore

\[ \| \mathbb{G}^\omega \mathcal{F}_{\nu_\sigma}^{r_{\sigma(n)} \cdot \gamma_n} \| < \| f^{-1} \| , \]

which implies again that

\[ Z(r) = 0. \]

Therefore we may furthermore assume that \( \sigma(n) = n \), from which we immediately get \( e_n = 0 \). Finally we have for any \( \nu < n \), because of (11), an implication

\[ (11) \quad \| r_{\nu \nu} \| \neq \| f^{n-\nu} \| \Rightarrow Z(r) = 0. \]

With (12) we can therefore assume that we have \( r_{\nu 1} = f^{n-1} \) and

\[ r_{\nu \nu} = -f^{n-\nu}, \quad 2 \leq \nu \leq n \]

(in the case \( n = 1 \) we have \( r_{\nu 1} = 1 \)), because \( Z(r) \) is constant on the \( T_n \)-orbits and in any orbit with \( Z(r) \neq 0 \) we may find a representative with the above property. Under these assumptions we get

\[ (12) \quad \sum_{\gamma \in \mathcal{S}} \chi(\gamma \mathbf{1}_n) \cdot \psi(\lambda_n(\mathbb{G}^\omega \gamma r)) = \chi(B_n) \cdot G(\chi)^n \cdot \mathfrak{M}(f)^{l-n-n}. \]

Since furthermore \( Z(r) \) is constant on the orbits, we deduce from Proposition 2.4 that

\[ \sum_{g \in \mathbb{G}^\omega \mathfrak{M}_{\nu_\sigma}^{\gamma_n}} \psi(\lambda_n(g)) w(g \cdot D_n w_n) v(g) \chi(g) |\det(g)|^n = \]

\[ (13) \quad \mathfrak{M}(f)^{\frac{n(n-1)}{2}} \cdot \sum_{r \in \mathfrak{M}_{\nu_\sigma}^{\gamma_n}} Z(r). \]

In equation (14) we have already seen that

\[ \sum_{g \in \mathbb{G}^\omega \mathfrak{M}_{\nu_\sigma}^{\gamma_n}} \psi(\lambda_n(g)) w(g \cdot D_n w_n) v(g) \chi(g) |\det(g)|^n = 0 \]

for \( \sigma(n) \neq n \) or \( e_n \neq 0 \). Hence we may assume that \( \sigma(n) = n \) and \( e_n = 0 \). With a view to the induction step we define \( \tilde{e} := (e_\nu)_{1 \leq \nu \leq n-1}, \tilde{\nu} := \nu(\omega) \) and \( \tilde{\mathcal{F}} := \nu(\tilde{r}) \). Thanks to equations (9), (12), Proposition 2.4 and Lemma 2.6 we have

\[ \sum_{r \in \mathfrak{M}_{\nu_\sigma}^{\gamma_n}} Z(r) = \]

\[ \chi(B_n) \cdot G(\chi)^n \cdot \mathfrak{M}(f)^{l-n-n} \cdot \frac{\# \mathfrak{M}_{\nu_\sigma}^{\gamma_n}}{\# \mathfrak{M}_{\nu_\sigma}^{\gamma_n-1}}. \]
With the induction hypothesis this yields
\[
\sum_{\tilde{r} \in \mathcal{I}_{n-1}} \|\text{det}(\omega \tilde{r})\|^s \cdot \chi(\tilde{j}(\omega \tilde{r})C_n).
\]
\[
w(\tilde{j}(\omega \tilde{r} \cdot \tilde{r})C_n \cdot D_n w_n) \cdot v(\tilde{j}(\omega \tilde{r})) = G(\chi)^n \mathcal{H}(f)^{-n-n}.
\]
\[
\sum_{\tilde{r} \in \mathcal{I}_{n-1}} \|\text{det}(\omega \tilde{r})\|^s \cdot \chi(\omega \tilde{r} \cdot \tilde{r}B_{n-1}) \cdot \psi(\lambda_{n-1}(\omega \tilde{r} \cdot \tilde{r}B_{n-1})) \cdot w(\tilde{j}(\omega \tilde{r} \cdot \tilde{r}B_{n-1} \cdot D_{n-1} w_n)) \cdot v(\tilde{j}(\omega \tilde{r}))
\]

Multiplication with \(B_{n-1} \in I_{n-1}\) only permutes the double cosets, hence
\[
\sum_{r \in \mathcal{I}_{n}} Z(r) = \mathcal{H}(f)^{-n-n} \cdot G(\chi)^n.
\]
\[
\sum_{\tilde{r} \in \mathcal{I}_{n-1}} \|\text{det}(\omega \tilde{r})\|^s \cdot \chi(\omega \tilde{r} \cdot \psi(\lambda_{n-1}(\omega \tilde{r} \cdot \tilde{r}B_{n-1}))) \cdot w(\tilde{j}(\omega \tilde{r} \cdot \tilde{r}B_{n-1} \cdot D_{n-1} w_n)) \cdot v(\tilde{j}(\omega \tilde{r}))
\]

With the induction hypothesis this yields
\[
\sum_{r \in \mathcal{I}_{n}} Z(r) = 0,
\]
if \(e \neq 0\) or \(\omega \neq 1_n\). For \(e = 0\) and \(\omega = 1_n\) it follows that
\[
\mathcal{H}(f) \cdot \sum_{r \in \mathcal{I}_{n}} Z(r) = G(\chi)^n \cdot G(\chi)^{\frac{(n-1)n}{2}} \cdot \mathcal{H}(f)^{\frac{-n(n+1)}{2}} \cdot \mathcal{H}(f)^{\frac{(l-1)(n-1)n}{2}} \cdot \mathcal{H}(f) \cdot 5^{n-3} \cdot 5 \cdot w(1_n) \cdot v(1_n) = \mathcal{H}(f)^{\frac{(l-2)n(n+1)}{2}} \cdot 5^{n-3} \cdot 5 \cdot G(\chi)^{\frac{n(n+1)}{2}} \cdot w(1_n) \cdot v(1_n),
\]
concluding the proof of Lemma 2.7. \(\square\)

**Proof of Theorem 2.1** Propositions 2.2 and 2.3 show that the twisted local zeta integral of the theorem may be expressed as the finite sum of Lemma 2.7. We may choose \(l(0) = 2n\), so that the elementary formula
\[
\sum_{\nu=1}^{n} \nu(n+1-\nu) = n^3 + n^2 - \frac{1}{2} \sum_{\nu=1}^{n} (5\nu^2 - 3\nu)
\]
concludes the proof of the theorem. \(\square\)

For comparison with earlier work in the case \(m = n + 1\) and also for immediate applications we define
\[
h := \begin{pmatrix} \begin{array}{c} 1 \\ \vdots \\ w_n \\ \vdots \\ 0 \end{array} \end{pmatrix} \in \text{GL}_{n+1}(\mathbb{Z}),
\]
and for any \(f \in F^\times\)
\[
h(f) := t^{-1} \cdot h \cdot t,
\]
where
\[ t := \text{diag}(f^n, f^{n-1}, \ldots, f, 1) \in \text{GL}_{n+1}(F). \]

We note that for \( g = (g_{ij}) \in \text{GL}_{n+1}(F) \)
\[ t^{-1} \cdot g \cdot t = (f^{i-j} \cdot g_{ij})_{ij}. \]

**Corollary 2.8.** Let \( n \geq 0 \) and choose Iwahori invariant \( \psi \)- resp. \( \psi^{-1} \)-Whittaker functions \( w \) and \( v \) on \( \text{GL}_{n+1}(F) \) and \( \text{GL}_n(F) \) respectively. Then for any quasi-character \( \chi : F^\times \to \mathbb{C}^\times \) with non-trivial conductor \( \mathfrak{f} = f\mathcal{O}_F \) we have
\[
\int_{U_n(F) \backslash \text{GL}_n(F)} w \left( j(g) \cdot h(f) \right) \cdot v(g) \cdot \chi(\text{det}(g)) \cdot ||\text{det}(g)||^{s-\frac{1}{2}} dg = \
\prod_{\nu=1}^n (1 - \mathfrak{m}(p)^{-\nu})^{-1} \cdot \mathfrak{m}(f) - \sum_{k=1}^n k(n+1-k) \cdot G(\chi)^{\frac{-n+1}{2}} \cdot w(1_{n+1}) \cdot v(1_n).
\]

**Proof.** Remember the definition of the matrix \( B_n \) and define
\[
E_n := \begin{pmatrix}
0 & \ldots & \ldots & 0 & 1 \\
& \vdots & \ddots & \vdots & f \\
& & \ddots & \vdots & \vdots \\
0 & \ldots & f & \ldots & f^{n-2} \\
1 & -f & -f^2 & \ldots & -f^{n-1}
\end{pmatrix}.
\]

Then
\[
E_n^{-1} = \begin{pmatrix}
0 & \ldots & 0 & f & 1 \\
& \vdots & \ddots & -f & 0 \\
& & \ddots & \vdots & \vdots \\
-f & \ldots & \ddots & \vdots & \vdots \\
1 & 0 & \ldots & \ldots & 0
\end{pmatrix},
\]

and
\[
B_n^{-1} = \begin{pmatrix}
1 & 0 & \ldots & \ldots & 0 \\
f & -1 & \ddots & \vdots & \vdots \\
f^2 & -f & \ddots & \vdots & \vdots \\
& \vdots & \ddots & \ddots & \vdots \\
f^{n-1} & -f^{n-2} & \ldots & -f & -1
\end{pmatrix}.
\]

Obviously \( B_n \) and \( E_n^{-1} \) are related (let \( d_i \) denote the \( n \times n \) diagonal matrix which differs from the identity only in \((i, i)\) and displays a \(-1\) there; then \( d_1 B_n d_n E_n d_1 = -w_n \)). We are interested in these matrices because of the relation
\[
B_{n+1} h(f) E_{n+1} = \text{diag}(f^{-n}, f^{-(n-2)}, \ldots, f^{n-2}, f^n) = D_{n+1}.
\]

Note that conjugation with \( D_{n+1} \) equals conjugation with \( t^{-2} \). With this notation we have
\[
j(B_n) \cdot B_n^{-1} = C_{n+1}.
\]

Because of
\[
B_n \in I_n, E_{n+1} w_{n+1} \in I_{n+1}
\]
and due to the invariance of the local zeta integral of the corollary with respect to the substitution $g \mapsto g B_n$ we get

$$
\int_{U_n(F) \backslash GL_n(F)} w \left( j(g) \cdot k^{(f)} \right) \cdot v(g) \cdot \chi(g) \cdot |\det(g)|^{s - \frac{1}{2}} dg = \\
\chi(\det(B_n)).
$$

Another substitution $g \mapsto g B_n^{-1}$ yields

$$
\int_{U_n(F) \backslash GL_n(F)} \psi(\lambda_n(gB_n)) w(j(gB_n \cdot D_n w_n)) \cdot v(gB_n) \cdot \chi(gB_n) \cdot |\det(g)|^{s - \frac{1}{2}} dg.
$$

Lemma 2.6 shows that this equals

$$
\int_{U_n(F) \backslash GL_n(F)} \psi(\lambda_n(gB_n)) \cdot w(j(gB_n \cdot D_n w_n)) \cdot v(gB_n) \cdot \chi(gB_n) \cdot |\det(g)|^{s - \frac{1}{2}} dg.
$$

Another substitution $g \mapsto g B_n$ we get

$$
\int_{U_n(F) \backslash GL_n(F)} w(j(gB_n)) v(g) \chi(g) |\det(g)|^{s - \frac{1}{2}} dg,
$$

concluding the proof.

\[ \square \]

3. A General Global Birch Lemma

In this section $k$ denotes a global field, i.e. a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(T)$ and we fix $j : GL_n \to GL_m$ as before. Let $\pi$ and $\sigma$ be irreducible cuspidal automorphic representations of $GL_m(\mathbb{A}_k)$ and $GL_n(\mathbb{A}_k)$ respectively. By $S_\infty$ we denote the set of infinite places of $k$, being empty if $\text{char} \ k \neq 0$. Let $S$ denote the set of finite places where $\pi$ or $\sigma$ ramifies. Furthermore let $p \not\in S$ be a fixed finite place.

For any finite place $q$ of $k$ we choose a fixed additive character $\psi_q : k_q \to \mathbb{C}^\times$ with conductor $\mathcal{O}_{k_q}$. Furthermore we choose non-trivial additive characters for the archimedean completions of $k$ such that $\psi := \bigotimes_{q \in M_k} \psi_q$ is a character of $k \backslash \mathbb{A}_k$.

We assume $m > n$ for simplicity. For any finite place $q$ with residue field cardinality $q$ we have for any pair

$$(w_q, v_q) \in \mathbb{W}^\prime(\pi_q, \psi_q) \times \mathbb{W}^\prime(\sigma_q, \psi_q^{-1})$$

of Whittaker functions the local zeta integral

$$
(14) \quad \Psi(w_q, v_q, s) := \int_{U_m(k_q) \backslash GL_m(k_q)} w_q(j(g)) v_q(g) |\det(g)|_q^{s - \frac{1}{2}} dg.
$$

This integral converges absolutely for $\text{Re}(s) \gg 0$, [22, 23, 25]. More precisely $\Psi(w_q, v_q, s)$ has a meromorphic continuation on $\mathbb{C}$ and is eventually a non-zero rational function in $q^{-s}$. Finally the collection of these integrals spans a fractional ideal in $\mathbb{C}[q^s]$ with respect to the subring $\mathbb{C}[q^{-s}, q^s]$. Any generator $T(s)$ of this ideal is of the form

$$
T(s) = P(q^{-s})^{-1}
$$

with a polynomial $P(X) \in \mathbb{C}[X]$. The local $L$-function $L(s, \pi_q \times \sigma_q)$ is defined as the unique generator for which $P(0) = 1$ holds [25]. In general $L(s, \pi_q \times \sigma_q)$ is not of the form $\Psi(w_q, v_q, s)$, but only a finite sum of local zeta integrals. In any case we find $L(s, \pi_q \times \sigma_q)$ in the image of the map

$$
\Psi : \mathbb{W}^\prime(\pi_q, \psi_q) \otimes \mathbb{W}^\prime(\sigma_q, \psi_q^{-1}) \to \mathbb{C}(q^s), \quad w_q \otimes v_q \mapsto \Psi(w_q, v_q, s).
$$
Therefore there exists always a good tensor $t^0_q \in \mathcal{W}(\pi_q, \psi_q) \otimes \mathcal{W}(\sigma_q, \psi_q^{-1})$ such that

$$L(s, \pi_q \times \sigma_q) = \Psi(t^0_q, s).$$

If $\pi_q$ and $\sigma_q$ are unramified, we may choose $t^0_q = w^0_q \otimes v^0_q$ with the corresponding new vectors $w^0_q$ and $v^0_q$. By Shintani’s explicit formula [50] (cf. [26, section 2]) we then have

$$L(s, \pi_q \times \sigma_q) = \det(1_{m_n} - \mathcal{M}(q)^{-s} A_{\pi_q} \otimes A_{\sigma_q})^{-1},$$

for any place $q \notin S \cup S_\infty$, where $A_{\pi_q}$ and $A_{\sigma_q}$ denote the corresponding Satake parameters ($\pi$ and $\sigma$ are always generic [43]).

Now let $(w, v) \in \mathcal{W}_0(\pi, \psi) \times \mathcal{W}_0(\sigma, \psi^{-1})$ be a pair of global Whittaker functions with factorizations $w = \otimes q w_q$, $v = \otimes q v_q$. Then by Fourier transform we have associated automorphic forms $\phi$ on $GL_m(A_k)$ and $\varphi$ on $GL_n(A_k)$ respectively (cf. [15] for example). Furthermore we have a projection $P^m_n$ from the space of cuspidal automorphic forms on $GL_m(A_k)$ to the space of cuspidal functions on $P_{n+1}(A_k)$, the standard mirabolic subgroup $P_{n+1} \subseteq GL_{n+1}$. For Re$(s) \gg 0$ the Euler product

$$\prod_q \Psi(w_q, v_q, s) = \int_{GL_n(k) \backslash GL_n(A_k)} P^m_n \phi \left( \begin{pmatrix} g & \psi \\ 1 & 1 \end{pmatrix} \right) \varphi(g) \|\text{det}(g)\|^{s-\frac{1}{2}} dg$$

converges absolutely and has an analytic continuation to $C$, because the global Rankin-Selberg integral on the right hand side is entire [14, Proposition 6.1], [27, Section 3.3]. As in the local setting the map

$$(\otimes w_q) \otimes (\otimes v_q) \mapsto [s \mapsto \prod_q \Psi(w_q, v_q, s)]$$

induces a $C$-linear map on the algebraic tensor product $\mathcal{W}_0(\pi, \psi) \otimes \mathcal{W}_0(\sigma, \psi^{-1})$. In the image of this map we find the global $L$-function

$$L(s, \pi \times \sigma) = \prod_q L(s, \pi_q \times \sigma_q),$$

modulo the Gamma factors (if $k$ is a number field). More precisely by [28] for any choice of $(w_q, v_q)$ for $q \in S_\infty$ there is an entire function $P$, such that

$$P(s) \cdot L(s, \pi \times \sigma) = \prod_{q \in S_\infty} \Psi(w_q, v_q, s) \cdot \prod_{q \notin S_\infty} \Psi(t^0_q, s),$$

where $t^0_q$ are good tensors as before. We know that $P(s)$ is a product of local integrals that depend on the choice of $(w_q, v_q)$ for archimedean $q \in S_\infty$. Furthermore for any $s_0$ there is a choice of $(w_q, v_q)$ for $q \in S_\infty$, such that so $P(s_0) \neq 0$ [14, Theorem 1.2]. In particular, at least in the case $m = n + 1$ the local $L$-functions at infinity are given by finite sums of Rankin-Selberg integrals as well [15, Theorem 1.3]. The question whether $P(\frac{1}{2}) \neq 0$ is intimately related to the problem if $\Omega \neq 0$. In the function field case there is no similar problem, we may assume $P \equiv 1$ in this case. This allows us to give a uniform treatment for all global field $k$.

Finally we have a representation

$$\otimes_{q \in S_\infty} (w_q \otimes v_q) \otimes \otimes_q t^0_q = \sum_i w_i \otimes v_i,$$
where any \( w_i \otimes v_i \) is a product of pure tensors. With the corresponding associated automorphic forms \((\phi_i, \varphi_i)\) we get the integral representation

\[
P(s) \cdot L(s, \pi \times \sigma) = 
\sum_{t} \int_{GL_n(k) \setminus GL_n(A_k)} P^m_n \phi_t \left( \left( \begin{array}{c} g \\ 1 \end{array} \right) \right) \varphi_t(g) \| \det(g) \|^{s-\frac{1}{2}} dg.
\]

In order to study the twisted \( L \)-function \( L(s, (\pi \otimes \chi) \times \sigma) \) for a quasi-character \( \chi \) with \( p \)-power conductor \( f \) and trivial at infinity. Then for any \( s \in \mathbb{C} \) and for any Iwahori invariant pair \((w_p, v_p)\) we have with the corresponding entire function \( P_s \),

\[
P(s) \delta(w_p, v_p) \chi(-1)^{n+1} G(\chi) \frac{n(n+1)}{2} \mathfrak{N}(f)^{-\sum_{k=1}^{n} k(n+1-k)} L(s, (\pi \otimes \chi) \times \sigma) = 
\sum_{t} \int_{GL_n(k) \setminus GL_n(A_k)} P^m_n \phi_t \left( \left( \begin{array}{c} g \\ 1 \end{array} \right) \right) \varphi_t(g) \chi(\det(g)) \| \det(g) \|^{s-\frac{1}{2}} dg,
\]

where

\[
\delta(w_p, v_p) := w_p(1_m) \cdot v_p(1_n) \cdot \prod_{\nu=1}^{n} (1 - \mathfrak{N}(p)^{-\nu})^{-1}.
\]

**Proof.** Any good tensor \( t^0_q \) for \((\pi_q, \sigma_q)\) gives rise to a good tensor \( \chi_q(\det) \cdot t^0_q \) for \((\pi_q \otimes \chi_q, \sigma_q)\). Furthermore we have

\[
L(s, (\pi_p \otimes \chi_p) \times \sigma_p) = 1,
\]

because \( \pi_p \) and \( \sigma_p \) are unramified at \( p \), but \( \chi_p \) is not. At \( p \) we define the Whittaker function

\[
g \mapsto \chi_p(\det(g)) \cdot w_p \left( g \cdot \hat{j}(h(f)) \right) =: w_p, \chi_p(g),
\]

where \( \hat{j} : GL_n+1 \to GL_n \) denotes the usual inclusion. Then corollary 2.8 yields

\[
\Psi(w_p, \chi_p, v_p, s) = 
\delta(w_p, v_p) \chi(-1)^{n+1} G(\chi) \frac{n(n+1)}{2} \mathfrak{N}(f)^{-\sum_{k=1}^{n} k(n+1-k)}.
\]

Composition of these local Whittaker functions to a global Whittaker function gives the formula of the global Birch lemma in a right half plane by Fourier transform. By analytic continuation we get the formula for any \( s \in \mathbb{C} \), concluding the proof. \( \square \)

An immediate consequence is

**Corollary 3.2.** In the case \( m = n+1 \) we have

\[
P(s) \cdot \delta(w_p, v_p) \cdot G(\chi) \frac{n(n+1)}{2} \mathfrak{N}(f)^{-\sum_{k=1}^{n} k(n+1-k)} L(s, (\pi \otimes \chi) \times \sigma) = 
\sum_{t} \int_{GL_n(k) \setminus GL_n(A_k)} \phi_t \left( j(g) \cdot h(f) \right) \varphi_t(g) \chi(\det(g)) \| \det(g) \|^{s-\frac{1}{2}} dg.
\]
Let $U_q := G_m(O_{F_q})$ for nonarchimedean $q$ and define $U_q := G_m(k_q)^0$ for $q \in S_\infty$. For an idèle $\alpha \in A_k^\times$ we let $C_{\alpha,f}$ denote the preimage of
\[ k^\times \backslash k^\times \cdot \alpha \cdot (1 + f) \cdot \prod_{q \in \mathfrak{M}} U_q \]
under the determinant map
\[ \det : \text{GL}_n(k) \backslash GL_n(A_k) \to k^\times \backslash A_k^\times. \]
Finally let $\varepsilon := \text{diag}(x, 1, \ldots, 1) \in \text{GL}_{n+1}(A_k)$ for $x \in A_k^\times$. As a consequence of the preceding corollary we get

**Corollary 3.3.** For any $\chi$ of finite order and conductor $\mathfrak{f}$ we have
\[ P\left( \frac{1}{2} \right) \cdot \delta(w_p, v_p) G(\chi) \sum_{k=1}^{\frac{n+1}{2}} \mathfrak{M}(f) \cdot \sum_{\mathfrak{f} = \mathfrak{f}} k^{(n+1-k)} L\left( \frac{1}{2}, (\pi \otimes \chi) \times \sigma \right) = \]
\[ \sum_{i, \alpha} \chi(\alpha) \cdot \sum_{x} \chi(x) \cdot \int_{C_{\alpha,\mathfrak{f}}} \phi_i \left( j(g) \cdot \varepsilon \cdot h(\mathfrak{f}) \right) \cdot \varphi_i(g) dg. \]

Here $\alpha$ runs through a system of representatives of the class group $k^\times \backslash A_k^\times / \prod_{q} U_q$ and $x \in O_{k,p}$ runs through a system of representatives for $(O_k/\mathfrak{f})^\times$.

In the function field case the sum over $\alpha$ is countably infinite but absolutely convergent.

Note that in the number field case $k^\times \backslash A_k^\times / \prod_{q} U_q$ not only has the classical ideal class group as a factor group, but also contains the group
\[ \left( O_k^\times / O_{k,+}^\times \right) \backslash \left( G_m(kR)/ G_m(kR)^0 \right) \cong \left( O_k^\times / O_{k,+}^\times \right) \backslash \bigcap_{\mathfrak{r} \text{ real}} \left( R^\times / (R^\times)^0 \right), \]
which is the kernel of the canonical map on the ideal class group. Here $O_{k,+}^\times = O_k^\times \cap G_m(kR)^0$ denotes the subgroup of totally positive elements in $O_k^\times$.

### 4. Hecke relations and distributions

For the classical theory of Hecke operators we refer to [16, chapter 3] and [37, chapter 2, §7]. Let $G$ be a group. A Hecke pair $(R, S)$ (in $G$) consists of a subgroup $R \subseteq G$ and of a sub half group $S \subseteq G$ with $RS = SR = S$ and the additional property that for any $s \in S$ the double coset $RsR$ is a finite union of right (or left) cosets modulo $R$.

The condition $RS = SR$ always holds in the case $S = G$. The second condition is satisfied if $G$ is a locally compact topological group and $R$ is a compact open subgroup.

For any Hecke pair $(R, S)$ we have a natural embedding of the free $\mathbb{Z}$-module $H_Z(R, S)$ over the set of all double cosets $RsR$ into the free $\mathbb{Z}$-module $\mathcal{H}_Z(R, S)$ over the set of the right cosets $sR$, $s \in S$, which is induced by
\[ RsR = \bigsqcup_{i} s_i R \mapsto \sum_{i} s_i R. \]

We may identify $H_Z(R, S)$ with its image under this embedding. Then $H_Z(R, S)$ becomes the $\mathbb{Z}$-module of $R$-invariants under the action
\[ R \times \mathcal{H}_Z(R, S) \to \mathcal{H}_Z(R, S), \quad (r, sR) \mapsto rsR. \]
Finally $H_Z(R, S)$ admits a structure of an associative $\mathbb{Z}$-algebra with the multiplication
\[
\left(\sum_i s_i R\right) \cdot \left(\sum_j t_j R\right) := \sum_{i,j} s_i t_j R.
\]
This algebra is unitary if and only if $R \cap S \neq \emptyset$. For any commutative ring $A$ we let
\[
H_A(R, S) := H_Z(R, S) \otimes \mathbb{Z} A.
\]
$H_A(R, S)$ is an associative algebra over $A$. We define $H(R, S) := H_C(R, S)$ and call it the Hecke algebra of the pair $(R, S)$.

Now let $G$ be a locally compact topological group and fix a compact open subgroup $K \leq G$. In this case $H_Z(K, G)$ may be interpreted as the $\mathbb{Z}$-module of locally constant right $K$-invariant mappings $f : G \to \mathbb{Z}$ with compact support and $H_Z(K, G)$ is just the submodule of left $K$-invariant mappings. The multiplication is given by convolution
\[
\alpha * \beta : x \mapsto \int_G \alpha(g) \beta(xg^{-1}) dg,
\]
where $dg$ is the right invariant Haar measure on $G$ which gives $K$ measure 1. This interpretation generalizes to any Hecke algebra $H_A(R, S)$ over any subring $A \subseteq \mathbb{C}$.

All Hecke algebras we consider arise in this topological context. We have the elementary

**Proposition 4.1.** Let $G$ denote a locally compact group, $H \leq G$ a closed subgroup and let $K \leq G$ be a compact open subgroup such that $L = H \cap K$ and $HK = G$. Then the restriction
\[
\alpha \mapsto \alpha|_H
\]
defines a monomorphism $H_A(K, G) \to H_A(L, H)$ of $A$-algebras.

The proof is elementary and may be found in [1, Proposition 3.1.6].

Fix a global field $k$ and a finite place $p$ of $k$. The Hecke algebra for the pair $(K, G)$ given by $K = \text{GL}_n(\mathcal{O}_k)$ and $G = \text{GL}_n(k_p)$ is commonly referred to as the standard Hecke algebra at $p$. Following Tamagawa [51] (see [42, Theorem 6] as well) we have an isomorphism (the so-called Satake map)
\[
S : H(K, G) \to \mathbb{C}[X_{1}^{\pm 1}, \ldots, X_{n}^{\pm 1}]^{S_n},
\]
\[
T_\nu \mapsto \mathfrak{h}(p)^{\nu(n+1)/2} \cdot \sigma_\nu(X_1, \ldots, X_n), \quad (0 \leq \nu \leq n)
\]
where $S_n$ is the symmetric group, operation by permutation on the $X_i$, and
\[
T_\nu := K \begin{pmatrix} 1_{n-\nu} & 0 \\ 0 & \sigma_\nu(1_{\nu}) \end{pmatrix}
\]
is independent of the choice of a prime $\varpi$. Furthermore $\sigma_\nu$ is the elementary symmetric polynomial of degree $\nu$ in $X_1, \ldots, X_n$.

As before $B_n(k_p)$ denotes the standard Borel subgroup of $\text{GL}_n(k_p)$. We define $K_{B_p} := B_n(k_p) \cap K = B_n(\mathcal{O}_k)$ and the parabolic Hecke algebra as $H_{B_p} := H(K_{B_p}, B_n(k_p))$. Then Iwasawa decomposition [21, Proposition 2.33], [22, Section 8.2] guarantees that the hypothesis of Proposition 4.1 is fulfilled and we see that
\( \mathcal{H}_{B_p} \) is a ring extension of \( \mathcal{H}_p := \mathcal{H}(K, G) \), with respect to the explicit embedding \\
\( \epsilon : \mathcal{H}_p \to \mathcal{H}_{B_p} \) given by \\
\( \sum_i a_i \cdot g_i K \mapsto \sum_i a_i \cdot g_i K_{B_p} \), \\
where we may assume that \( g_i \in B_n(k_p) \) thanks to the Iwasawa decomposition.

In \( \mathcal{H}_{B_p} \) we have the Hecke operators \\
\( U_i := K_{B_p} \begin{pmatrix} 1_{n-i} & 0 & 0 \\ 0 & \varpi & 0 \\ 0 & 0 & 1_{n-i} \end{pmatrix} K_{B_p} \), \\
which commute \([19, \text{Lemma 2}]\). Gritsenko \([19, \text{Theorem 2}]\) showed that over \( \mathcal{H}_{B_p} \) we have a decomposition of the Hecke polynomial \\
\( H_p(X) := \sum_{\nu=0}^n (-1)^\nu \mathfrak{m}(p)^{\frac{(\nu-1)n}{2}} T_\nu X^{n-\nu} \in \mathcal{H}_p(X) \)
into linear factors \\
\( H_p(X) = \prod_{i=1}^n (X - U_i) \).

Following \([30, \text{section 4}]\) we define for \( 1 \leq \nu \leq n \) the operators \\
\( V_{p,\nu} := \mathfrak{m}(p)^{\frac{(\nu-1)n}{2}} \cdot U_1 U_2 \cdots U_\nu \in \mathcal{H}_{B_p} \), \\
and \\
\( t(p) := \text{diag}(\varpi^{n-1}, \varpi^{n-2}, \ldots, 1) \).

In complete analogy with \([30, \text{Lemma 4.1}]\) we then have

**Lemma 4.2.** We have \\
\( V_{p,\nu} = K_{B_p} \begin{pmatrix} \varpi \cdot 1_\nu & 0 \\ 0 & 1_{n-\nu} \end{pmatrix} K_{B_p} = \bigcup_A \begin{pmatrix} \varpi \cdot 1_\nu & A \\ 0 & 1_{n-\nu} \end{pmatrix} K_{B_p} \), \\
where \( A \in \mathcal{O}_{k_p}^{n \times n-\nu} \) runs through is a system of representatives modulo \( p \). Furthermore the Hecke operators \( V_{p,\nu} \) commute and \\
\( K_{B_p} t(p) K_{B_p} = \prod_{\nu=1}^{n-1} V_{p,\nu} = \bigcup u t(p) K_{B_p} \), \\
where \( u \) runs through a system of representatives of \( U_n(\mathcal{O}_{k_p})/t(p) U_n(\mathcal{O}_{k_p}) t^{-1}(p) \).

Denote by \( \mathcal{M}_p \) the \( \mathbb{C} \)-vector space of \( \mathbb{C} \)-valued right \( K_{B_p} \)-invariant mappings on \( \text{GL}_n(k_p) \). The Hecke algebra \( \mathcal{H}_{B_p} \) operates from the left on \( \mathcal{M}_p \) by the rule \\
\( \mathcal{H}_{B_p} \times \mathcal{M}_p \to \mathcal{M}_p \) \\
\( \left( \sum_i a_i \cdot g_i K_{B_p}, \psi \right) \mapsto \sum_i a_i \cdot [g \mapsto \psi(g g_i)] \).

We let \\
\( V_{p,0} := K_{B_p} 1_n K_{B_p} \) 

denote the unit element of \( \mathcal{H}_{B_p} \). We have as in \([30, \text{Proposition 4.2}]\)
Proposition 4.3. Let \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{C}^{n-1} \) and \( \psi \in \mathcal{M}_p \) such that 
\[
\forall \nu = 1, 2, \ldots, n-1 : \quad H_p(\lambda_\nu) \cdot \psi = 0.
\]
Then 
\[
\psi_\lambda := \prod_{i=1}^{n-1} \prod_{j=1}^{n} (\lambda_i \mathfrak{m}(p)^{1-j} V_{p,j-1} - V_{p,j}) \cdot \psi
\]
is a simultaneous eigenfunction of \( V_{p,1}, \ldots, V_{p,n-1} \). More precisely with 
\[
\eta_\nu := \mathfrak{m}(p)^{-\nu(n-1)} \prod_{i=1}^{\nu} \lambda_i
\]
for \( 1 \leq \nu \leq n-1 \) we have the relation 
\[
V_{p,\nu} \cdot \psi_\lambda = \eta_\nu \cdot \psi_\lambda.
\]

Now let \( \pi \) and \( \sigma \) denote automorphic representations of \( \text{GL}_n \) and \( \text{GL}_{n-1} \) respectively, unramified at \( p \). Let 
\[
\lambda_{p,1}, \ldots, \lambda_{p,n} \in \overline{\mathbb{Q}}
\]
and 
\[
\alpha_{p,1}, \ldots, \alpha_{p,n-1} \in \overline{\mathbb{Q}}
\]
denote the roots of the corresponding Hecke polynomials \( H_p \) of \( \pi_p \) and \( \sigma_p \) respectively. If \( \pi \) and \( \sigma \) are cohomological, then \( \pi_p \) and \( \sigma_p \) are defined over a number field \([13] \) Théorème 3.13 resp. Proposition 3.16] and consequently the Hecke roots are algebraic in this case. We say that \( \pi \) (resp. \( \sigma \)) are ordinary at \( p \), if (with a suitable numbering) for \( 1 \leq i \leq n-1 \)
\[
\|\lambda_{p,i}\|_p = \|\mathfrak{m}(p)\|_p^{i-1}
\]
(resp. for \( 1 \leq j \leq n-2 \) \( \|\alpha_{p,j}\|_p = \|\mathfrak{m}(p)\|_p^{j-1} \)). We write 
\[
\Lambda(p) := (\lambda_{p,1}, \ldots, \lambda_{p,n-1}) \in \overline{\mathbb{Q}}^{n-1},
\]
\[
\alpha(p) := (\alpha_{p,1}, \ldots, \alpha_{p,n-2}) \in \overline{\mathbb{Q}}^{n-2},
\]
and furthermore 
\[
\kappa_\Lambda(p) := \prod_{\nu=1}^{n-1} \lambda_{p,\nu}^{-\nu},
\]
\[
\kappa_\alpha(p) := \prod_{\nu=1}^{n-2} \alpha_{p,\nu}^{-1-\nu},
\]
\[
\hat{\kappa}_\Lambda(p) := \mathfrak{m}(p)^{-\frac{(n-1)(n-2)}{6}} \cdot \kappa_\Lambda(p),
\]
\[
\hat{\kappa}_\alpha(p) := \mathfrak{m}(p)^{-\frac{(n-1)(n-2)(n-3)}{6}} \cdot \kappa_\alpha(p).
\]
Under the ordinarity assumption \( \hat{\kappa}_\Lambda(p) \) and \( \hat{\kappa}_\alpha(p) \) are \( p \)-adic units.

We may assume that the \( p \)-factor of the Fourier transform of the automorphic forms \( \phi_i \) resp. \( \varphi_i \) is class-1. Then \( \phi_i \) and \( \varphi_i \) are normalized eigenvectors of the corresponding Hecke algebras \( \mathcal{H}_p \) at \( p \). By Proposition \([13] \) we get modified \( p \)-Iwahori invariant automorphic forms \( \tilde{\phi}_i \) resp. \( \tilde{\varphi}_i \), which are eigenvectors of the corresponding operators 
\[
V_p := V_{p,1} \cdots V_{p,n-1}
\]
with eigenvalues \( \hat{\kappa}_\Lambda(p) \) resp. \( \hat{\kappa}_\alpha(p) \).
For any nontrivial \( p \)-power \( f \) let
\[
\kappa(f) := \frac{\mathcal{N}(f) \prod_{q \mid p} (\kappa_{\Delta(p)} \cdot \kappa_{\Omega(p)})^{\nu_p(f)}}{(\kappa_{\Delta(p)} \cdot \kappa_{\Omega(p)})^{\nu_p(f)}},
\]
and for \( \alpha \in \mathbb{A}_k^\times \) and \( x \in \mathcal{O}_k^\times \)
\[
\mu_\alpha(x + f) := \kappa(f) \cdot \sum_{\iota} P_{\alpha,\iota}(\varepsilon_x \cdot h(f), f),
\]
where
\[
P_{\alpha,\iota}(u, f) := \int_{C_{\alpha,1}} \tilde{\phi}_{\iota}(j(g) \cdot u) \cdot \tilde{\varphi}_{\iota}(g) dg.
\]
Here \( h(f) \) is an element of \( \GL_n(k_p) \). Furthermore let
\[
\Theta := k^\times \backslash \mathbb{A}_k^\times / \prod_{q \mid p} U_q \cong \lim_{f \to 1} k^\times \backslash \mathbb{A}_k^\times / (1 + f) \prod_{q \mid p} U_q,
\]
and
\[
\Theta(\alpha) := k^\times \backslash k^\times \cdot \alpha \cdot \prod_{q \mid p} U_q / \prod_{q \mid p} U_q \cong \lim_{f \to 1} k^\times \backslash k^\times \cdot \alpha \cdot \prod_{q \mid p} U_q / (1 + f) \prod_{q \mid p} U_q.
\]
Then \( \Theta \) is a disjoint union of the compact open sets \( \Theta(\alpha_1), \ldots, \Theta(\alpha_h) \). We may assume that \( \alpha_1, \ldots, \alpha_h \) are trivial at \( p \).

**Theorem 4.4.** If \( p^{\frac{n(n-1)}{2}} \) is principal, then \( \mu_{\alpha_1}, \ldots, \mu_{\alpha_h} \) are distributions on \( \Theta(\alpha_1) \) which give rise to a \( \mathbb{C} \)-valued distribution \( \mu \) on \( \Theta \).

For any character \( \chi : k^\times \backslash \mathbb{A}_k^\times \to \mathbb{C}^\times \) of finite order with nontrivial \( p \)-power conductor \( f \), trivial at \( \infty \), we have
\[
\int_{\Theta} \chi d\mu = P(\frac{1}{2}) \cdot \delta(\pi, \sigma) \cdot \kappa(f) \cdot G(\chi) \frac{n(n-1)}{2} \cdot L(\frac{1}{2}, \tau_1 \otimes \chi). \quad (1 + \mathcal{N}(\chi) - 1).
\]
Here \( \kappa(f) \) and \( \delta(\pi, \sigma) \) are given explicitly by
\[
\kappa(f) := \mathcal{N}(f)^{\frac{n(n-1)(n-2)}{6}} \cdot (\kappa_{\Delta(p)} \cdot \kappa_{\Omega(p)})^{-\nu_p(f)},
\]
and
\[
\delta(\pi, \sigma) := \tilde{w}_p(1_n) \cdot \tilde{v}_p(1_{n-1}) \cdot \prod_{\nu=1}^{n-1} (1 - \mathcal{N}(p)^{\nu})^{-1}.
\]
\( \tilde{w}_p \) and \( \tilde{v}_p \) denote the local Whittaker functions at \( p \), corresponding to the \( p \)-factor of the Fourier transform of \( \tilde{\phi}_\epsilon \) and \( \tilde{\varphi}_\iota \).

We could renormalize our Whittaker functions in such a way that \( \delta(\pi, \sigma) = 1 \). The precise value of \( \delta(\pi, \sigma) \) is given in [30] Proposition 4.12 and lies in the field generated by the Hecke roots.

**Proof.** Our proof follows the proofs of [30] Proposition 4.9] and [44] Theorem 3.1]. Introduce the notation
\[
U_n := U_n(\mathcal{O}_{k_p}),
\]
and
\[
U_n(\mathcal{O}) := t_p U_n t_p^{-1}.
\]
Proposition 4.3 and Lemma 4.2 give
\[
\forall g \in \text{GL}_n(A_k) : \sum_{u(U_n) \in U_n/U_n} \tilde{\phi}_i(gut(p)) = \tilde{\kappa}_{\Delta(p)} \cdot \tilde{\phi}_i(g)
\]
and
\[
\forall g \in \text{GL}_{n-1}(A_k) : \sum_{u(U_n) \in U_{n-1}/U_{n-1}} \tilde{\phi}_i(gut(p)) = \tilde{\kappa}_{\Delta(p)} \cdot \tilde{\phi}_i(g).
\]
We conclude that
\[
\sum_{u(U_n) \in U_n/U_n} \tilde{\phi}_i \left( j(g) \cdot \varepsilon \cdot h(f) \cdot ut(p) \right) \cdot \tilde{\phi}_i(gut(p)) = 
\sum_{u(U_n) \in U_n/U_n} \int_{C_{\alpha,i}} \tilde{\phi}_i \left( j(g) \cdot \varepsilon \cdot h(f) \cdot ut(p) \right) \cdot \tilde{\phi}_i(gut(p)) 
\]
since \( \det(t_p) \) globally represents a principal ideal. This shows that
\[
P_{\alpha,i}(\varepsilon \cdot h(f), \chi) = (15) \tilde{\kappa}_{\Delta(p)} \cdot \tilde{\kappa}_{\Delta(p)} \cdot \sum_{u(U_n) \in U_n/U_n} \int_{C_{\alpha,i}} \tilde{\phi}_i \left( j(g) \cdot \varepsilon \cdot t_{-1}^{-1}j(w)^{-1}h(f) \cdot ut(p) \right) \cdot \tilde{\phi}_i(gut(p)) \]
Now [33] Lemma 3.2] easily generalizes to any local field, which means that we have
\[
tj(w)^{-1}h(f)^{-1}ut^{-1} = tj(w)^{-1}t^{-1}h(f)^{-1}ut^{-1} = h(1) \pmod{f},
\]
since
\[
tj(w)^{-1}t^{-1} = tut^{-1} = 1_n \pmod{f}.
\]
Finally we have the formula
\[
\left( U_n : U_n^{(f)} \right) = \mathfrak{N}(p) \frac{(n+1)n(n-1)}{6}
\]
cf. [33] p. 110], hence
\[
P_{\alpha,i}(\varepsilon \cdot h(f), \chi) = \tilde{\kappa}_{\Delta(p)} \cdot \tilde{\kappa}_{\Delta(p)} \cdot \mathfrak{N}(p) \frac{(n+1)n(n-1)(n-2)}{6} \cdot P_{\alpha,i}(\varepsilon \cdot h(f), \chi) \cdot \mathfrak{N}(p) \frac{(n+1)n(n-1)(n-2)}{6}.
\]
Due to the relation
\[
P_{\alpha,i}(\varepsilon \cdot h(f), \chi) = \sum_{a \pmod{p}} P_{\alpha,i}(\varepsilon \cdot h(f), \chi)
\]
we conclude that
\[
\mu_\alpha(x + f) = \sum_{a \pmod{p}} \mu_\alpha(x + f)
\]
proving the distribution relation. The interpolation formula follows from corollary 3.3. This proves the theorem. 

**Remark 4.5.** The condition on \( p \) may be weakened by a modified construction, if we forget the finer structure provided by the \( \mu_\alpha \) and consider
\[
\mu' = \sum_{i=1}^h \mu_\alpha.
\]
This sum is invariant under translations, which gives the distribution relation in the general case along the same lines. But at the same time this limits integration.
to characters constant on $\alpha_1, \ldots, \alpha_h$. If for any character $\chi$ under consideration we may find an unramified character $\chi_{nr}$, such that $\chi_{nr}\chi$ becomes trivial on $\alpha_1, \ldots, \alpha_h$, then the transition from $\pi$ to the twist $\pi \otimes \chi_{nr}^{-1}$ gives us distributions $\mu'_{\chi_{nr}}$, which give rise to a distribution $\mu$ on $\Theta$. In more algebraic terms, the class group $C_k$ of $k$ is in this case a direct factor of $\Theta = \Theta' \times C_k$ and the Iwasawa algebra

$$C[[\Theta]] = \lim_{\leftarrow} C[k^\times \backslash A_k^\times / (1 + f) \prod U_q]_{q|f}$$

of $C$-valued distributions on $\Theta$ decomposes canonically into a tensor product $C[C_k] \otimes C[C[[\Theta']]]$ (cf. [36, section (7.3)]). To be more precise, the linear independence of the unramified characters shows that there are distributions $\mu_{\chi_{nr}}$ on $C_k$ such that

$$\int_{C_k} \chi_i^{-1} d\mu_{\chi_i} = \delta_{ij} \quad \text{(Kronecker delta).}$$

The distribution

$$\mu := \sum_{i=1}^h \mu_{\chi_i} \otimes \mu'_{\chi_i} \in C[[\Theta]]$$

then has the interpolation property of Theorem 4.4 for all idèle class characters with $p$-power conductor.

**Remark 4.6.** Note that the short exact sequence

$$(16) \quad 1 \to \mathcal{O}_k^\times \to \mathcal{O}_p^\times \times \pi_0(k_\mathfrak{R}^\times) \to \Theta \to C_k \to 1$$

does not split in general. A simple counter example is given by $k = Q(\sqrt{-15})$. Here we have $C_k \cong Z/2Z$ and the ray class group $C^\theta$ is cyclic of order 4 if $p \mid 5$ and cyclic of order 16 if $p \mid 17$. In the latter case we have $\lim_{\leftarrow} C^\theta = Z_{17}$. For almost all prime places the group $\mathcal{O}_p^\times$ still contains an open pro-$p$-subgroup $H$, where $p$ denotes the residue field characteristic of $p$. Consequently the condition $(p, \#C_k) = 1$ guarantees that any character of $H$ as a continuation on $\Theta$ which may be assumed trivial on $\alpha_1, \ldots, \alpha_h$. Due to the finiteness of $\mathcal{O}_p^\times / H$ we still get the interpolation property for a subgroup of all characters of $p$-power conductor of finite index.

**Theorem 4.7.** If $p$ does not divide the order of the class group of $k$, there is a distribution $\mu$ on $\theta$, such that the interpolation formula of theorem 4.4 holds for all nontrivial characters $\chi$ in a subgroup of finite index of all finite order characters with $p$-power conductor.

**Remark 4.8.** To illustrate an extreme situation in the case $p \mid \#C_k$ let $k = Q(\sqrt{-23})$. This imaginary quadratic field has the class group $C_k \cong Z/3Z$ and for the corresponding ray class groups are given by $C^\theta = Z/3Z$ for any $r \geq 1$ if $p \mid 3$. Hence we have $\lim_{\leftarrow} C^\theta = Z_3$, which means that $\Theta / \pi_0(k_\mathfrak{R}^\times)$ is pro-cyclic. Thus it may eventually happen that no nontrivial character of $\mathcal{O}_p^\times$ has a lift to $\Theta$ trivial on $\alpha_1, \ldots, \alpha_h$.

Note that Ash and Ginzburg overlooked this phenomenon in [3]. In section 2.2 of loc. cit. they implicitly assume that any character of $\mathcal{O}_k^\times$ lifts to an idèle class group character trivial on $\alpha_1, \ldots, \alpha_h$.
Remark 4.9. All known constructions suffer this restriction. Already Manin, constructing $p$-adic $L$-functions for Hilbert modular forms in [34] by basically the same method, noted that the nontriviality of the class group introduces some new phenomena. Manin argues that the effect of restricting an idèle class group character $\chi$ on $\mathcal{O}_{K}^\times$ gives rise to a controllable modification in the corresponding twisted $L$-function. Consequently he would consider the distribution $\mu$ that we construct ($\mu$ always exists) as the $p$-adic analogue of the complex $L$-function $L(s, \pi \times \sigma)$.

Another analogy between our construction and Manins construction is the following. Instead of restricting to $\mu$-functions Manin considers general conductors $\mathfrak{f}$ with prime divisors in a fixed finite set $S$. Our general global Birch lemma easily generalizes to this more general situation and analogously gives rise to a distribution $\mu$ interpolating twists with characters that are $S$-complete in the sense of Manin. This means that all places in $S$ are divisors of the conductors $\mathfrak{f}$. We restricted to the case $S = \{\mathfrak{p}\}$ to give a less technical treatment.

5. Algebraicity and boundedness of the distribution

Fix a number field $k/\mathbb{Q}$ with $r_1$ real and $r_2$ complex places. We write $S_\infty$ for its set of archimedean places. Let $G_n := \text{Res}_{k/\mathbb{Q}} \text{GL}_n$, where by abuse of notation $\det : G_n \to G_1$ is the restriction of scalars of $\det : \text{GL}_n \to \text{GL}_1$. Then $\mathcal{C}(G_n)^0 = \text{Res}_{k/\mathbb{Q}} \mathcal{C}(\text{GL}_n)^0$ and (cf. [39] section 1.4)

$$\text{rank}_\mathbb{Q}(\mathcal{C}(G_n)) = 1.$$ 

On the other hand

$$\text{rank}_\mathbb{Q}(\mathcal{C}(G_n)) = [k : \mathbb{Q}],$$

and it is easily seen that

$$\text{rank}_\mathbb{R} \mathcal{C}(G_n) = r_1 + r_2.$$

We may assume that $\text{GL}_n(\mathcal{O}) = G_n(\mathbb{Z})$, $\text{GL}_n(k) = G_n(\mathbb{Q})$, $\text{GL}_n(k_{\mathbb{R}}) = G_n(\mathbb{R})$, and $\text{GL}_n(A_k) = G_n(A_{\mathbb{Q}})$.

Now let $\pi, \sigma$ be irreducible cohomological cuspidal representations of $\text{GL}_n(A_k)$ and $\text{GL}_{n-1}(A_k)$ with trivial central character. Choose compact open subgroups $K \leq \text{GL}_n(A_k)$ and $K' \leq \text{GL}_{n-1}(A_k)$ such that

$$\det(K) = \det(K') = \widehat{\mathcal{O}}_k^\times.$$

Therefore we may assume that there is a $K$- (resp. $K'$-) right invariant new vector $w_{\ell}$ (resp. $v_{\ell}$) of the finite component $\pi_{\ell}$ (resp. $\sigma_{\ell}$). Furthermore we may assume (cf. [24] section (4.1), Théorème) that $K$ contains the image of $K'$ under the embedding

$$j : \text{GL}_{n-1} \to \text{GL}_n, \quad g \mapsto \begin{pmatrix} g & \varepsilon \ell \nu \phi \Gamma \end{pmatrix}.$$

Finally we assume that the modified automorphic forms $\tilde{\phi}_\ell$ and $\tilde{\phi}_t$ are right-$K$- and right-$K'$-invariant respectively and that locally $K_p = I_n(\mathcal{O}_{k_p})$ resp. $K'_p = I_{n-1}(\mathcal{O}_{k_p})$. Let $K$ and $K'$ be small enough such that all arithmetic subgroups in the sequel are torsion free.

For any $x \in \mathcal{O}_{k_p}^\times$, an idèle $a$ and a generator $f \in \mathcal{O}_{k_p}$ of a nontrivial $p$-power $\mathfrak{f}$ our aim is to give a cohomological interpretation of the integral $P_{n,i}(\varepsilon x \mathcal{h}^{(f)}, \mathfrak{f})$. Thanks
to the $\pi_0(\GL_n(k_R)) \times \GL_n(A_k^\epsilon)$- (resp. $\pi_0(\GL_n-1(k_R)) \times \GL_n-1(A_k^\epsilon)$)-action on the space of automorphic forms we get

$$P_{\alpha,i}(\varepsilon_x h^{(f)}, f) = \int_{C_{1,1}} \tilde{\phi}_i^{\alpha}(j(g)\varepsilon_x h^{(f)}) \cdot \tilde{\varphi}_i^{\alpha}(g) \, dg,$$

where $\tilde{\phi}_i^{\alpha}$ (resp. $\tilde{\varphi}_i$) denotes the image of $\tilde{\phi}_i$ under the action of $\varepsilon_{\alpha}$ (resp. $\varphi_i$) as an element of $\pi_0(G_n(R)) \times \GL_n(A_k^\alpha)$. We write

$$K_{\alpha} := \varepsilon_{\alpha} K \varepsilon_{\alpha}^{-1}$$

and we have the corresponding arithmetic subgroup

$$\Gamma_{\alpha} := \{ \gamma \in \GL_n^+(k) \mid \gamma \in K_{\alpha} \},$$

with $\GL_n(k) := \GL_n(k) \cap \GL_n(k_R)^0$. Strong approximation for $\SL_n$ yields the decomposition

$$\GL_n(A_k) = \bigcup_i \GL_n(k) \cdot \varepsilon_{\alpha_i} \cdot (\GL_n(k_R)^0 \times K_{\alpha})$$

corresponding to the fibers of the determinant map. We find $\gamma_{x,f} \in \GL_n(k)$, $\gamma_{x,f,\infty} \in \GL_n(k_R)^0$, $\gamma_{f} \in K_{\alpha}^0$ and $1 = \alpha(f) \in \{ \alpha_1, \ldots, \alpha_k \}$ with

$$\varepsilon_x \cdot h^{(f)} = \gamma_{x,f}^{-1} \cdot (\gamma_{x,f,\infty} \cdot \varepsilon_{\alpha(f)} \cdot \gamma_{f}) = \gamma_{x,f}^{-1} \cdot (\gamma_{x,f,\infty}, \gamma_{f}).$$

We define the group

$$K'_{x,f} := j^{-1} \left( j \left( (K')_{\alpha}^{\infty} \cap \varepsilon_x h^{(f)} K_{\alpha}^{\infty} h^{(f)^{-1}} \varepsilon_x^{-1} \right) \right).$$

Then $K'_{x,f}$ operates on $C_{\alpha,f}$ via right translation (cf. [44, Prop. 3.4]) and we have

$$\Gamma'_{\alpha} := \{ \gamma \in \Gamma_{\alpha}^{+} \mid \gamma \in (K')_{\alpha}^{\infty} \},$$

$$\Gamma_{\alpha,x,f} := \{ \gamma \in \Gamma_{\alpha}^{+} \mid \gamma \in K'_{x,f} \}.$$ Then

$$\Gamma_{\alpha,x,f}^{\infty} = \{ \gamma \in \Gamma_{\alpha}^{+} \mid j(\gamma) f \in \varepsilon_x h^{(f)} K_{\alpha}^{\infty} h^{(f)^{-1}} \varepsilon_x^{-1} \} \subseteq \Gamma'_{\alpha},$$

and we get the diffeomorphism

$$i_{\alpha,x,f} : \Gamma_{\alpha,x,f}^{\infty} \to \Gamma_{\alpha,x,f}^{\infty} / K_{\alpha,x,f},$$

$$\Gamma_{\alpha,x,f}^{\infty} \cdot g_{\infty} \mapsto \Gamma_{\alpha,x,f}^{\infty} \cdot (\varepsilon_x, h^{(f)}, K_{\alpha,x,f}).$$

Since

$$\GL_n(k) \cdot (\varepsilon_x, h^{(f)}) \cdot K_{\alpha} = \GL_n(k) \cdot (\gamma_{x,f,\infty} g_{\infty}, 1) \cdot K_{\alpha},$$

we conclude that

$$\vol \left( (K_{\alpha,x,f})^{-1} \cdot P_{\alpha,i}(\varepsilon_x h^{(f)}, f) \right) =$$

$$\int_{\Gamma_{\alpha,x,f}^{\infty} \setminus G_{n-1}(R)} \tilde{\phi}_i^{\alpha}(\gamma_{x,f,\infty}, j(g_{\infty})) \cdot \tilde{\varphi}_i^{\alpha}(g_{\infty}) \, dg_{\infty}.$$ Note that

$$\Gamma'_{\alpha,x,f} = \{ \gamma \in \Gamma'_{\alpha} \mid j(\gamma) f \in (\gamma_{x,f,\infty})^{-1} \gamma_{x,f} f \} =: \Gamma_{\alpha,x,f}.$$ We give a direct cohomological interpretation of this integral. Let $\mathcal{X}_n$ be the symmetric space of $G_n(R)$ with respect to the standard maximal compact subgroup $K_{\infty}$, hence canonically

$$\mathcal{X}_n = \prod_{q \in S_{\infty} \text{ real}} \GL_n(R) / O(n) \times \prod_{q \in S_{\infty} \text{ complex}} \GL_n(C) / U(n).$$
Let \( \mathcal{X}_n^1 \) be the symmetric space to the standard maximal compact subgroup of \( G_n^\text{ad}(\mathbb{R}) \). By means of the canonical isogeny \( G_n^{\text{der}} \to G_n^\text{ad} \) we may consider \( \mathcal{X}_n^1 \) as a symmetric space of \( G_n^{\text{der}}(\mathbb{R}) \) as well.

We write
\[
\begin{align*}
   b_n &:= \frac{n^2 - n + 2 \left\lfloor \frac{n}{2} \right\rfloor}{4}, \\
   \tilde{b}_n &:= \frac{n(n-1)}{2}, \\
   c_n &:= \dim(\mathfrak{g}_n) - \dim(\mathfrak{o}_n) = \frac{n^2 + n}{2}, \\
   \tilde{c}_n &:= \dim_{\mathbb{R}}(\mathfrak{g}_n \otimes_{\mathbb{R}} \mathbb{C}) - \dim_{\mathbb{R}}(U(n)) = 2n^2 - n^2 = n^2,
\end{align*}
\]
Then \( b_n + b_{n-1} = c_{n-1}, \tilde{b}_n + \tilde{b}_{n-1} = \tilde{c}_{n-1} \), and
\[
   r_1c_n + r_2\tilde{c}_n = \dim \mathcal{X}_n = \dim \mathcal{X}_n^1 + r_1 + r_2.
\]

Denote by \( \tilde{\mathfrak{g}}_n \) the Lie subalgebra of \( \tilde{\mathfrak{g}}_n := \text{Lie}(G_n(\mathbb{R})) = \bigoplus_{q \in S_\infty} \mathfrak{g}_n \oplus \bigoplus_{q \in S_\infty \text{ complex}} \mathfrak{g}_n \otimes_{\mathbb{R}} \mathbb{C} \) given by matrices with componentwise totally imaginary trace. We write \( \tilde{\mathfrak{sl}}_n \) for the Lie subalgebra of \( \mathfrak{g}_n \otimes_{\mathbb{R}} \mathbb{C} \) given by matrices with totally imaginary trace.

We may assume that \( \pi \) and \( \sigma \) occur in dimension \((r_1b_n + r_2\tilde{b}_n)\) resp. \((r_1\tilde{b}_{n-1} + r_2b_{n-1})\) of the cohomology of the corresponding Lie algebras. Then \( \pi \) and \( \sigma \) occur with multiplicity one. More precisely
\[
   H^{r_1b_n + r_2\tilde{b}_n}(\tilde{\mathfrak{g}}_n, K_\infty; H_{\pi_\infty}^{(K_\infty)}) \cong \mathbb{C},
\]
where
\[
   \pi_\infty = \bigotimes_{q \in S_\infty} \pi_q
\]
and \( H_{\pi_\infty}^{(K_\infty)} \) is the space of \( K_\infty \)-finite elements of the representation space \( H_{\pi_\infty} \) of \( \pi_\infty \). We implicitly used that \( \pi_\infty \) is uniquely determined by its (irreducible) restriction to
\[
   G^\pm_n := \{ g \in \text{GL}_n(k_{\mathbb{R}}) \mid \forall q \in S_\infty : \| \det(g_q) \|_q = 1 \}
\]
The Lie algebra of this group is just \( \tilde{\mathfrak{g}}_n \). The claimed multiplicity one follows from \cite{[13] Lemme 3.14} via Künneth formalism \cite{[11]} chap. I, \S 1.3 and \S 5, (4)], which in our case reads
\[
   H^{r_1b_n + r_2\tilde{b}_n}(\tilde{\mathfrak{g}}, K_\infty; H_{\pi_\infty}^{(K_\infty)}) \cong
\]
\[
   \bigotimes_{v \text{ real}} H^{b_n}(\mathfrak{sl}_n, O(n), H_{\pi_q}^{(O(n))}) \otimes \bigotimes_{v \text{ complex}} H^{\tilde{b}_n}(\mathfrak{sl}_n, U(n), H_{\pi_q}^{(U(n))}),
\]
\[(17)\]
\[
   H^{r}(\mathfrak{sl}_n, O(n), H_{\pi_q}^{(O(n))}) = 0,
\]
resp. for \( s < \tilde{b}_n \)
\[
   H^{s}(\mathfrak{sl}_n, U(n), H_{\pi_q}^{(U(n))}) = 0.
\]
We have the \((\mathfrak{g}_n, K_\infty)\)-module
\[
\mathcal{W}_0(\pi_\infty, \psi_\infty) := \bigotimes_{q|\infty} \mathcal{W}_0(\pi_q, \psi_q).
\]

By \([11\text{, chap. II, \S3, Corollary 3.2}]\) we have
\[
H_{b_1 b_n + r_2 b_n} \left( \tilde{\mathfrak{g}}_n, K_\infty, H(\pi_\infty) \right) \cong \left( \bigwedge^{r_1 b_n + r_2 b_n} \tilde{h}_n^* \otimes \mathcal{W}_0(\pi_\infty, \psi_\infty) \right)^{K_\infty},
\]
where \(\tilde{h}_n\) denotes the \((-1)\)-eigenspace of the Cartan involution in \(\tilde{\mathfrak{g}}\).

Now let
\[
0 \neq \eta_\infty \in \left( \bigwedge^{r_1 b_n + r_2 b_n} \tilde{h}_n^* \otimes \mathcal{W}_0(\pi_\infty, \psi_\infty) \right)^{K_\infty}.
\]
We write
\[
\eta_\infty = \sum_{#I = r_1 b_n + r_2 b_n} \omega_I \otimes w_{I, \infty}
\]
with \(w_{I, \infty} \in \mathcal{W}_0(\pi_\infty, \psi_\infty)\), similarly for \(\sigma\). We get a form
\[
\eta'_\infty = \sum_{#I' = r_1 b_{n-1} + r_2 b_{n-1}} \omega'_{I'} \otimes v_{I', \infty}
\]
of degree \(r_1 b_{n-1} + r_2 b_{n-1}\) with \(v_{I', \infty} \in \mathcal{W}_0(\pi_{\infty}, \psi_\infty)\). For the pair \((w_{s,t}, v_{s,t})\) of finite Whittaker functions corresponding to our forms \((\phi_s, \tilde{\phi}_s)\) this gives
\[
\eta_{s,0} := w_{s,t} \otimes \eta_\infty \in \left( \bigwedge^{r_1 b_n + r_2 b_n} \tilde{h}_n^* \otimes \mathcal{W}_0(\pi, \psi) \right)^{K_\infty},
\]
and
\[
\eta'_{s,0} := v_{s,t} \otimes \eta'_\infty \in \left( \bigwedge^{r_1 b_{n-1} + r_2 b_{n-1}} \tilde{h}_{n-1}^* \otimes \mathcal{W}_0(\pi, \psi) \right)^{K'_{\infty}}.
\]
Coefficientwise Fourier transform yields
\[
\eta_s \in \left( \bigwedge^{r_1 b_n + r_2 b_n} \tilde{h}_n^* \otimes \mathcal{W}_0(\pi_{\infty}, \psi_{\infty}) / (\mathcal{W}_0(\pi_{\infty}, \psi_{\infty}) / (\mathcal{W}_0(\pi_{\infty}, \psi_{\infty})\mathcal{W}_0(\pi_{\infty}, \psi_{\infty})^0 K_\infty K_{\infty}) \right)^{K_\infty},
\]
and analogously an \(\eta'_s\). Hence we get cohomology classes
\[
[\eta_{s,t}] \in H_{\text{cusp}}^{r_1 b_n + r_2 b_n}(\Gamma_{\alpha} \setminus \mathfrak{g}_n, \mathcal{C}) \subseteq H_{c}^{r_1 b_n + r_2 b_n}(\Gamma_{\alpha} \setminus \mathfrak{g}_n, \mathcal{C}),
\]
\[
[\eta'_{s,t}] \in H_{\text{cusp}}^{r_1 b_{n-1} + r_2 b_{n-1}}(\Gamma'_{\alpha} \setminus \mathfrak{g}_{n-1}, \mathcal{C}) \subseteq H_{c}^{r_1 b_{n-1} + r_2 b_{n-1}}(\Gamma'_{\alpha} \setminus \mathfrak{g}_{n-1}, \mathcal{C}),
\]
where the arithmetic groups operate via the adjoint action. Let \(\phi_{s,t}\) be the Fourier transform of \(w_{s,t} \otimes w_{s,t}\), and we fix the notation
\[
\eta_s = \sum_{#I = r_1 b_n + r_2 b_n} \omega_I \otimes \phi_{s,t}
\]
and
\[
\eta'_s = \sum_{#I' = r_1 b_{n-1} + r_2 b_{n-1}} \omega'_{I'} \otimes \varphi_{s,t}.\]
respectively. By Lemma 1.6 ad induces a proper projection
\[ p : 0 \mathcal{G}_{n-1}^* \to \mathcal{G}_{n-1}^*. \]
Furthermore we have the central morphism \(\text{ad} \circ j : \text{GL}_{n-1} \to \text{PGL}_n\) and its restriction of scalars
\[ s := \text{Res}_{k/q} \text{ad} \circ j = \text{ad} \circ \text{Res}_{k/q} j : G_{n-1} \to G_n^\text{ad}. \]
Finally we have the cohomological formula
\[ \text{vol} \left( K'_{\alpha,x,f} \right)^{-1} \cdot P_{\alpha,x}(\varepsilon_x h(f), f) = \int_{\Gamma_{n-1} \backslash \mathcal{G}_{n-1}} s_{\alpha,x,f} \cdot \left( \eta_{\alpha,s} \right) \wedge p^*(\eta'_{\alpha,s}) = \mathcal{G}_{n-1}^*(\left[ \eta_{\alpha,r} \right], [\eta'_{\alpha,s}]) . \]

**Theorem 5.1.** Let \(\pi\) and \(\sigma\) be ordinary at \(p\). By suitable cohomological choice of the Whittaker functions at \(\infty\) Theorems 4.4 and 4.7 give distributions with values in the ring of integers \(\mathcal{O}_E\) of a field \(E = \mathbb{Q}(\pi,\sigma)\), i.e. they are \(p\)-adically bounded.

**Proof.** Due to multiplicity one we may assume that one, and hence all, classes lie in the cuspidal cohomology
\[ [\eta_{\alpha,s}] \in H^r_{\text{cusp}}(\mathcal{G}_{n-1}, \mathcal{O}_E) , \]
\[ [\eta'_{\alpha,s}] \in H^r_{\text{cusp}}(\mathcal{G}_{n-1}, \mathcal{O}_E) \]
with entire coefficients. This immediately implies that
\[ \text{vol} \left( K'_{\alpha,x,f} \right)^{-1} \cdot P_{\alpha,x}(\varepsilon_x h(f), f) \in \mathcal{O}_E(\pi,\sigma) . \]

To see the boundedness of the distribution, we must show that
\[ \left| \kappa(f) \cdot \text{vol} \left( K'_{\alpha,x,f} \right) \right|_p = \left| \eta(f)(n+1)n(n-1)/(n+2) \right|_p \cdot \text{vol} \left( K'_{\alpha,x,f} \right)_p \]
is bounded for any nontrivial \(p\)-power \(f\). Since \(\alpha\) and \(x\) commute, \(\varepsilon_x\) normalizes the group \((K')^\alpha\). Therefore
\[ K'_{\alpha,x,f} = \varepsilon_x \cdot K'_{\alpha,1,f} \cdot \varepsilon_x^{-1} . \]

Conjugation with \(h(f)\) affects only the components at \(p\), hence for \(q \nmid f\) we have
\[ (K'_{\alpha,x,f})_q = j(K'_{q,f})^\alpha . \]
Therefore we are left to consider
\[ (K'_{\alpha,x,f})_p = j(I_{n-1})^\alpha \cap h(f) P_{n} h(f)^{-1} . \]

We may assume that \(\alpha\) has a trivial \(p\)-component. Then our claim follows from the natural generalization of [44] Lemmas 3.6 and 3.7. \(\square\)

**Remark 5.2.** The proof of the theorem shows that we get an estimate of the order of the distribution along the same lines if we drop the assumption of ordinarity.

Now assume \(k\) to be totally real. Then cohomologicality of \(\pi\) means that for any \(q \in S_\infty\)
\[ H^*\left( \mathfrak{gl}_n, SO(n) \mathcal{G}(GL_n)(R)^0 ; H^0_{\mathcal{G}(n)q} \right) \neq 0 \]
and we have a similar statement for \(\sigma\). The question wether \(P(1/2) \neq 0\) is a local problem, because the Künneth formalism yields factorizations
\[ \eta_\infty = \otimes q |q| \eta_q . \]
\[ \eta'_\infty = \otimes_{q|\infty} \eta'_q, \]
where the local components satisfy
\[ \eta_q \in \left( \bigwedge^{b_n} \mathfrak{p}_n^* \otimes \mathcal{W}_0(\pi_q, \psi_q) \right)^{K_q} \]
and
\[ \eta'_q \in \left( \bigwedge^{b_{n-1}} \mathfrak{p}_{n-1}^* \otimes \mathcal{W}_0(\sigma_q, \psi'_q) \right)^{K'_q}. \]
The index set \( I \) decomposes into local sets \( I_q \) and we have
\[ P(s) = \sum_{# I = r_1 b_n + r_2 b_n} \varepsilon_I P_I(s), \]
where
\[ \Psi(w_I, v_I, 1/2) = \prod_{q|\infty} \frac{\Psi(w_{I_q}, v_{I_q}, 1/2)}{L(1/2, \pi_q \times \sigma_q)}. \]
Equation (18) now reads
\[ P_I(s) = \prod_{q|\infty} \frac{\Psi(w_{I_q}, v_{I_q}, s)}{L(s, \pi_q \times \sigma_q)}. \]
The question whether
\[ P_q(1/2) := \frac{\Psi(w_{I_q}, v_{I_q}, 1/2)}{L(1/2, \pi_q \times \sigma_q)} \neq 0 \]
is classical for \( n = 2 \) and the case \( n = 3 \) is treated in [43, Theorem 3.8]. This proves

**Proposition 5.3.** If \( k \) is totally real, \( n \in \{2, 3\} \), and if \( s = 1/2 \) is critical for \( L(s, \pi \times \sigma) \), then \( P(1/2) \neq 0 \).}

6. **The \( p \)-adic symmetric cube of elliptic curves**

Fix a totally real number field \( k \) and assume that \( E_1 \) and \( E_2 \) are modular elliptic curves over \( k \) (without CM). This is the case for example if \( k/\mathbb{Q} \) is solvable and \( E_1 \) and \( E_2 \) are defined over \( \mathbb{Q} \) [52, 54, 12, 32, 2]. We assume that \( E_1 \) and \( E_2 \) have good ordinary reduction at a finite place \( p \) (assumed to be principal for simplicity). Then \( \text{Sym}^2 E_1 \) is modular as well [18] and Theorems 4.4 and 5.1 yield the existence of a \( p \)-adic measure \( \mu \) such that for any abelian Artin representation \( \chi \) with \( p \)-power conductor
\[ \int_{\Gamma} \chi d\mu = \Omega \cdot \hat{\kappa}(f) \cdot G(\chi) \frac{n(n-1)}{2} \cdot L(2, \text{Sym}^2 E_1 \otimes E_2 \otimes \chi), \]
where \( \Gamma \) denotes the Galois group of the maximal abelian extension of \( k \), unramified outside \( p \), and \( \Omega \in \mathbb{C}^\times \) by Proposition 5.3 \( \mu \) naturally corresponds to an abelian \( p \)-adic \( L \)-function.

To show that \( \Omega \) corresponds to Deligne’s periods in the sense of [53], we would like to appeal to [17, Theorem 6.2]. Unfortunately the parallel weight 2 case is excluded in this theorem (cf. 3.4.8 and Remark 4.6.3 of loc. cit.; note that [48, Theorem 4.3] still holds in our case due to [41]).
Now if \( E_1 = E_2 =: E \) we have \( \text{Sym}^2 E \otimes E = \text{Sym}^3 E \oplus E(-1) \). Manin [34] showed the existence of the \( p \)-adic \( L \)-function \( L_p(E, s) \) for \( E \). If \( E \) is already defined over \( \mathbb{Q} \) and \( k/\mathbb{Q} \) is abelian and if \( p \) lies above a non-split rational prime \( p \), it is easily seen that [40] implies that \( L_p(E, s) \) does not vanish identically. Therefore, with
\[
L_p(\text{Sym}^3 E, s) := \frac{L_p(\text{Sym}^2 E \otimes E, s)}{L_p(E, s)},
\]
we get the \( p \)-adic \( L \)-function for \( \text{Sym}^3 E \) in this case.

Note that by [31] \( \text{Sym}^3 E \) is modular as well so that the results of [3] are applicable. On the other hand Ash and Ginzburg must suppose the existence of a so-called \( H \)-model to construct the \( p \)-adic \( L \)-function for \( \text{Sym}^3 E \). The existence of the latter is equivalent to the condition that our \( p \)-adic \( L \)-function \( L_p(\text{Sym}^3 E, s) \) does not vanish identically (section 5 of loc. cit.). This remains an open problem, even in the case \( k = \mathbb{Q} \).

The pool of explicit examples for our theory is limited by the yet small number of proved instances of Langland’s functoriality conjectures. We are optimistic that this situation will change in the future.

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