ON THE BOGOLUBOV-DE GENNES EQUATIONS

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Abstract. We consider the Bogolubov-de Gennes equations giving an equivalent formulation of the BCS theory of superconductivity. We are interested in the case when the magnetic field is present. We (a) discuss their general features, (b) isolate key physical classes of solutions (normal, vortex and vortex lattice states) and (c) prove existence of the normal, vortex and vortex lattice states and stability/instability of the normal states for large/small temperature or/and magnetic fields.

1. Introduction

The Bogolubov-de Gennes equations describe the remarkable quantum phenomenon of superconductivity. They present an equivalent formulation of the BCS theory and are among the latest additions to the family of important effective equations of mathematical physics. Together with the Hartree-Fock (Bogolubov), Ginzburg-Landau and Landau-Lifshitz equations, they are the quantum members of this illustrious family consisting of such luminaries as the heat, Euler, Navier-Stokes and Boltzmann equations.

There are still many fundamental questions about these equations which are completely open, namely

- Derivation;
- Well-posedness;
- Existence and stability of stationary magnetic solutions.

By the magnetic solutions we mean (physically interesting) solutions with non-zero magnetic fields. In this paper we address the third problem. The well-posedness (or existence) theory will be addressed elsewhere (cf. [2]).

The key special solutions of Bogoliubov-de Gennes (BdG) equations are normal, superconducting and mixed or intermediate states. The latter appear only for non-vanishing magnetic fields. For type II superconductors, they consist of the vortices and (magnetic) vortex lattices. In this paper, we prove the existence of the normal states for non-vanishing magnetic fields and of the vortex lattices and investigate the stability of the former.

There is a considerable physics literature devoted to the BdG equations, but, despite the role played by magnetic phenomena in superconductivity, it deals mainly with the zero magnetic field case, with only few disjoint remarks about the case when the magnetic fields are present, the main subject of this work.

As for rigorous work, it also deals exclusively with the case of zero magnetic field. The general (variational) set-up for the BdG equations is given in [3]. We use, like all subsequent papers, this set-up. The next seminal works on the subject are [5], where the...
authors prove the existence of superconducting states (the existence of the normal states under the assumptions of \[8\] is trivial), to which our work is closest, and \[6\], deriving the (macroscopic) Ginzburg-Landau equations. For an excellent, recent review of the subject, with extensive references and discussion see \[9\].

In the rest of this section we introduce the BdG equations, describe their properties and the main issues and present the main results of this paper. In the remaining sections we prove the these results, with technical derivations delegated to appendices. In the last appendix, following \[2\], we discuss a formal, but natural, derivation of the BdG equations.

1.1. Bogoliubov-de Gennes equations. In the Bogoliubov-de Gennes approach states of superconductors are described by the pair of operators $\gamma$ and $\alpha$, acting on the one-particle state space and satisfying (after peeling off the spin variables)

$$0 \leq \gamma = \gamma^* \leq 1 \quad \text{and} \quad \alpha^* = \alpha,$$

where $\overline{\gamma} := \mathcal{C}\gamma\mathcal{C}$, with $\mathcal{C}$, the operation of complex conjugation. $\gamma$ is a one-particle density operator, or diagonal correlation and $\alpha$ is a two-particle coherence operator, or off-diagonal correlation. $\gamma(x, x)$ is interpreted as the one-particle density, so that $\text{Tr}\gamma = \int \gamma(x, x) dx$ is the total number of particles.

The Bogoliubov-de Gennes equations form a system of self-consistent equations for $\gamma$ and $\alpha$. It is convenient to organize the operators $\gamma$ and $\alpha$ into the self-adjoint operator-matrix

$$\eta := \begin{pmatrix} \gamma & \alpha \\ \alpha^* & 1 - \overline{\gamma} \end{pmatrix}.$$ (1.2)

The relations (1.1) and the structure (1.2) of $\eta$ are equivalent to the following relations (cf. \[3\])

$$0 \leq \eta = \eta^* \leq 1 \quad \text{and} \quad J^* \eta J = 1 - \overline{\eta}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (1.3)$$

Since the BdG equations describe the phenomenon of superconductivity, they are naturally coupled to the electromagnetic field. We describe the latter by the vector and scalar potentials $a$ and $\phi$. Then the time-dependent BdG equations state (see e.g. \[5, 4, 12\])

$$i(\partial_t + i\phi)\eta_t = [\Lambda(\eta_t, a_t), \eta_t], \quad (1.4)$$

with

$$\Lambda(\eta, a) = \begin{pmatrix} h_{\gamma a} & v^*\alpha \\ v^2\alpha & -h_{\alpha a} \end{pmatrix},$$

where the operator $v^*$ is defined through the integral kernels as $(v^*\alpha)(x; y) := v(x, y)\alpha(x; y)$, $v(x, y)$ is the pair potential, and

$$h_{\gamma a} = h_a + v^*\gamma - v^*\gamma,$$

where $(v*\gamma)(x) = (v * d_{\gamma})(x) := \int v(x, y)d_{\gamma}(y)dy$, $d_{\gamma}(x) := \gamma(x, x)$. $v^*\gamma$ and $v^*\gamma$ are the direct and exchange self-interaction potentials, and $h_a = -\Delta a$. Eq (1.4) is coupled to the Maxwell equation (Ampère’s law)

$$\partial_t(\partial_t a + \nabla\phi) = -\text{curl}^*\text{curl}a + j(\gamma, a), \quad (1.6)$$
where \( j(\gamma, a)(x) \) := \( \frac{1}{2}[-i\nabla_a, \gamma](x, x) \) is the superconducting current. (Above, \( A(x, y) \) stands for the integral kernel of an operator \( A \).) In what follows, we assume that
\[
v(x, y) = v(y, x).
\]

**Remarks.**

1) In general, \( h_a \) might contain also an external potentials \( V(x) \) and \( A(x) \), due to the impurities, which, for simplicity of exposition, we do not consider.

2) For \( \alpha = 0 \), Eq (1.4) becomes the time-dependent von Neumann-Hartree-Fock equation for \( \gamma \) (and \( \alpha \)).

3) We may assume that the physical space is a finite box, \( \Omega \), in \( \mathbb{R}^d \) and \( \gamma \) and \( \alpha \) are trace class and Hilbert-Schmidt operators, respectively, see Subsection 1.6 for the precise formulation.

4) One can extend a formal derivation of (1.4) given in Appendix C to the coupled system (1.4)-(1.6) by starting with the hamiltonian (C.4) coupled to the quantized electromagnetic field.

**Connection with the BCS theory.** Eq (1.4) can be reformulated as an equation on the Fock space involving an effective quadratic hamiltonian (see [4, 5, 9] and [2], for the bosonic version). These are the effective BCS equations and the effective BCS hamiltonian.

1.2. **Symmetries and conservation laws.** The equations (1.4)-(1.6) are invariant under the *gauge transformations,*
\[
T^\text{gauge}_\chi : (\gamma, \alpha, a, \phi) \mapsto (e^{i\chi} \gamma e^{-i\chi}, e^{i\chi} \alpha e^{-i\chi}, a + \nabla \chi, \phi + \partial_t \phi),
\]
for any sufficiently regular function \( \chi : \mathbb{R}^d \to \mathbb{R} \), and, if the external potentials are zero and considering, for simplicity, the entire space \( \mathbb{R}^d \), then also under *translation* and *rotation* transformations,
\[
T^\text{trans}_h : (\gamma, \alpha, a) \mapsto (U_h \gamma U_h^{-1}, U_h \alpha U_h^{-1}, U_h a),
\]
\[
T^\text{rot}_\rho : (\gamma, \alpha, a) \mapsto (U_\rho \gamma U_\rho^{-1}, U_\rho \alpha U_\rho^{-1}, \rho U_\rho a),
\]
for any \( h \in \mathbb{R}^d \) and \( \rho \in O(d) \). Here \( T^\text{trans}_h \) and \( T^\text{rot}_\rho \) are the standard translation and rotation transforms \( U^\text{trans}_h : \phi(x) \mapsto \phi(x + h) \) and \( U^\text{rot}_\rho : \phi(x) \mapsto \phi(\rho^{-1} x) \). In terms of \( \eta \), say the gauge transformation, \( T^\text{gauge}_\chi \), could be written as
\[
T^\text{gauge}_\chi : \eta \to U_\chi \eta U_\chi^{-1}, \quad \text{where} \quad U_\chi = \begin{pmatrix} e^{i\chi} & 0 \\ 0 & e^{-i\chi} \end{pmatrix}
\]
(extended correspondingly to \( (\eta, a) \) by \( T^\text{gauge}_\chi(\eta, a) = (T^\text{gauge}_\chi(\eta), a + \nabla \chi) \)). Notice the difference in action of this transformation on the diagonal and off-diagonal elements of \( \eta \).

The invariance under the gauge transformations can be proven by using the relation
\[
\Lambda(T_\chi(\eta, a)) = T_\chi(\Lambda(\eta, a)),
\]
shown by using the operator calculus and the fact that \( U^\text{gauge}_\chi \) is unitary.

If the external fields are zero, as we assume in this paper, then the equations are translationally invariant (when considered in \( \mathbb{R}^2 \)). Because of the gauge invariance, it is natural to consider the simplest, gauge (magnetically) translationally invariant solutions, i.e. solutions invariant under the transformations
\[
T_\text{bs} : (\eta, a) \to (T^\text{gauge}_\chi)^{-1}T^\text{trans}_s(\eta, a),
\]
for any \( s \in \mathbb{R}^2 \), where \( \chi_s(x) := \frac{b}{2} (s \wedge x) \) (modulo \( \nabla f \)). (For a gauge-free expression, we give \( \chi_s(x) := x \cdot a_b(s) \), where \( a_b(x) \) is the vector potential with the constant magnetic field, \( \text{curl} a_b = b \).) We have

**Lemma 1.1.** The operators \( T_{bs} \) defined in (1.12) are unitary and satisfy

\[
T_{bs} T_{bt} = \hat{I}_{bst}, \quad T_{bt} = I_{bst} T_{bs},
\]

(1.13)

\[
\hat{I}_{bst} := I_{bst} - 1, \quad I_{bst} := \left( e^{i \frac{b}{2} (t \wedge s)} \right)
\]

(1.14)

**Proof.** The unitarity of \( T_{bs} \) is obvious. For the second statement, let \( U^\text{mt}_{bs} := (U^\text{gauge}_{bs})^{-1} U^\text{trans}_{bs} \). Then \( T_{bs} u = U^\text{mt}_{bs} u (U^\text{mt}_{bs})^{-1} \) and \( U^\text{mt}_{bs} U^\text{mt}_{bt} = I_{bst} U^\text{mt}_{bs+t} I_{bst} \) where we used that \( g_{st}(x) := \chi_s(x) + \chi_t(x + t) - \chi_{s+t}(x) = \frac{b}{2} (t \wedge s) \). Hence, the result follows. \( \square \)

**Particle-hole symmetry.** The evolution under the equations (1.4) - (1.6) preserves the relations in (1.3), i.e. if an initial condition has one of these properties, then so does the solution. This follows from the relation

\[
J^* \Lambda J = -\overline{\Lambda}.
\]

(1.15)

The second relation in (1.3) is called the particle-hole symmetry.

**Conservation laws.** Eqs (1.4) – (1.6) conserve the energy

\[
E(\eta, a, e) := E(\eta, a) + \int |e|^2,
\]

where \( e \) is the electric field and \( E(\eta, a) \) is given by

\[
E(\eta, a) = \text{Tr} \left( (-\Delta_a) \gamma \right) + \frac{1}{2} \text{Tr} \left( (v * d \gamma) \gamma \right) - \frac{1}{2} \text{Tr} \left( (v^2 \gamma) \gamma \right)
\]

\[
+ \frac{1}{2} \text{Tr} \left( \alpha^*(v^t \alpha) \right) + \int dx |\text{curl} a(x)|^2.
\]

(1.16)

The energy functional \( E(\eta, a) \) originates as \( E(\eta, a) := \varphi(H_a) \), where \( \varphi \) is a quasi-free state in question (see Appendix C) and \( H_a \) is the standard many-body given in (C.4), coupled to the vector potential \( a \).

Furthermore, the global gauge invariance implies the evolution conserves the number of particles, \( N := \text{Tr} \gamma \).

Finally, an important role in our analysis is played by the reflections symmetry. Let the reflection operator \( i^{\text{refl}} \) be given by conjugation by the reflections, \( u^{\text{refl}} \). We say that a state \( (\eta, a) \) is even (reflection symmetric) iff

\[
i^{\text{refl}} \gamma = \gamma \quad \text{and} \quad u^{\text{refl}} a = -a.
\]

(1.17)

The reflections symmetry of the BdG equations implies that if an initial condition is even then so is the solution every moment of time.

In what follows we always assume that solutions \( (\eta, a) \) are even.

**Remark.** We do not specify here the spaces and, consequently, the domain of integration, for \( \gamma, \alpha \) and \( a \). These are defined in Subsection 1.6.
Hamiltonian structure. The BdG equations (1.4) - (1.6) are hamiltonian, with the hamiltonian

$$\mathcal{H}(\kappa, \kappa^*, \alpha, \alpha^*, a, e) = \mathcal{E}(\kappa^* \kappa, \alpha, a) + \int |e|^2.$$ 

given in terms the canonically conjugate variables $\kappa, \kappa^*, \alpha, \bar{\alpha}^*$ and $a, -e$, where $e$ is the electric field, and the symplectic form (to be checked)

$$\text{Im}[\text{Tr}(\kappa^* \kappa') + \text{Tr}(\alpha^* \alpha')] + [e \cdot a' - e' \cdot a].$$

1.3. Stationary Bogoliubov-de Gennes equations. We consider stationary solutions to (1.4) of the form

$$\eta_t := T_{\chi}^{\text{gauge}} \eta = U_{\chi}^{\text{gauge}} \eta (U_{\chi}^{\text{gauge}})^{-1},$$

with $\eta$ and $\bar{\chi} \equiv \mu$ independent of $t$, $\chi$ independent of $x$, and $a$ independent of $t$ and $\phi = 0$. We have

**Proposition 1.2.** (1.18), with $\eta$ and $\bar{\chi} \equiv -\mu$ independent of $t$, is a solution to (1.4) iff $\eta$ solves the equation

$$[\Lambda_{\eta a}, \eta] = 0,$$

where $\Lambda_{\eta a} \equiv \Lambda_{\eta a} = \Lambda(\eta, a) - \mu S$, with $S := \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$, and is given explicitly

$$\Lambda_{\eta a} := \left( \begin{array}{cc} h_{\gamma a} & v^\sharp \alpha \\ v^\sharp \alpha^* & -\bar{h}_{\gamma a} \end{array} \right),$$

with $h_{\gamma a} := h_{\gamma a} - \mu$. 

**Proof.** Plugging (1.18) into (1.4) and using (1.11) and that $\chi$ is independent of $x$, we see that

$$\mu[S, U_{\chi} \eta U_{\chi}^{-1}] = i \partial_t \eta_t = [\Lambda(U_{\chi} \eta U_{\chi}^{-1}, a), \eta]$$

$$= [\Lambda(U_{\chi} \eta U_{\chi}^{-1}, a + \nabla \chi), U_{\chi} \eta U_{\chi}^{-1}]$$

$$= U_{\chi} [\Lambda(\eta, a), \eta](U_{\chi})^{-1}. \quad (1.21)$$

Since $U$ and $U_{\chi}$ are diagonal, so the commute. It follows then $[\Lambda(\eta, a) - \mu S, \eta] = 0$, which is just the statement of the proposition. \qed

For any reasonable function $f$, solutions of the equation

$$\eta = f(\frac{1}{T} \Lambda_{\eta a}), \quad (1.24)$$

solve (1.19) and therefore give stationary solutions of (1.4). Under some conditions, the converse is also true. (The parameter $T > 0$, the temperature, is introduced here for the future references.)

The physical function $f$ is selected by either a thermodynamic limit (Gibbs states) or by a contact with a reservoir (or imposing the maximum entropy principle) It is given by the Fermi-Dirac distribution

$$f(h) = (1 + e^{2h/T})^{-1}.$$
Inverting the function $f$, one can rewrite (1.24) as $\Lambda_\eta a = T f^{-1}(\eta)$. Let $f^{-1} =: g'$. Then the stationary Bogoliubov-de Gennes equations can be written as

$$\Lambda_\eta a - T g'(\eta) = 0,$$

$$\text{curl}^* \text{curl} a = j(\eta, a).$$

(1.26)

(1.27)

Here $\Lambda_\eta a \equiv \Lambda(\eta, a)$ and $T \geq 0$ (temperature) and, as follows from the equations $g' = f^{-1}$ and (1.25), the function $g$ is given by

$$g(\lambda) = -\frac{1}{2}(\lambda \ln \lambda + (1 - \lambda) \ln(1 - \lambda)),$$

so that

$$g'(\lambda) = -\frac{1}{2} \ln \frac{\lambda}{1 - \lambda}.$$

(1.28)

(1.29)

Remarks. 1) One can express these equations in terms of eigenfunctions of the operator $\Lambda_\eta a$, which is the form appearing in physics literature (see [3, 2]).

2) For (1.24) to give $\eta$ of the form (1.2), the function $f(h)$ should satisfy the conditions

$$f(\bar{h}) = f(h)$$

and

$$f(-h) = 1 - f(h).$$

(1.30)

For $g(x)$ given in (1.28), the function $f(h)$ satisfies these conditions as can be checked from its explicit form (1.25). Indeed, the first condition in (1.30) means merely that $f$ is a real function, while the second condition in (1.30) is satisfied by functions $f(h) := (1 + e^{\tilde{g}(h)})^{-1}$, with $\tilde{g}(h)$, any odd function.

From now on, we assume $g(\lambda)$ either is given in (1.28) or satisfies the conditions (1.30), with $f(h) := (g')^{-1}(h)$, and

$$g'(1 - x) = -g'(x).$$

(1.31)

3) If we drop the direct and exchange self-interactions from $h_{\gamma a\mu}$, then $\Lambda_\eta a$ becomes independent of the diagonal part, $\gamma$, of $\eta$ and the equation (1.24) implies that (1.26) has always the solution

$$\eta_{\gamma a} = f\left(\frac{1}{T} \Lambda a\right),$$

where $\Lambda a := \Lambda_\eta a |_{\eta=0}.$

(1.32)

1.4. Free energy. The Bogoliubov-de Gennes equations arise as the Euler-Lagrange equations for the free energy functional

$$F_T(\eta, a) := E(\eta, a) - TS(\eta) - \mu N(\eta),$$

(1.33)

where $S(\eta) = -\text{Tr}(\eta \ln \eta)$ is the entropy (see Remark 2 after (1.35)), $N(\eta) := \text{Tr} \gamma$ is the number of particles, and $E(\eta, a)$ is the energy functional given in (1.16). With the spaces defined in Subsection 1.6 we have

Theorem 1.3. (a) The free energy functional $F_T$ is well defined on the space $D^1 \times (a_b + \delta_{\delta, \rho})$.

(b) $F_T$ is continuously (Gâteaux or Fréchet) differentiable at $(\eta, a) \in D^1 \times (a_b + \delta_{\delta, \rho})$, s.t. $\eta \ln \eta$ is trace class, with respect of perturbations $(\eta', a')$, with $\eta'$ satisfying

$$J^* J = -\eta', \; (\eta')^2 \lesssim [\eta(1 - \eta)]^2,$$

(1.34)

where $J$ is defined in (1.3).
(c) If $0 < \eta < 1$, strictly, $(\eta, a)$ is even in the sense of the definition (1.17) and if $v$ is even, then critical points of $F_T$ satisfy the BdG equations.

(d) Minimizers of $F_T$ over $D^1 \times (a_b + \hat{h}_{\delta, \rho})$ are its critical points.

This theorem is proven in Section 2. For the translation invariant case, it is proven in [8]. In general case, but with $a = 0$ (which is immaterial here), the fact that BdG is the Euler-Lagrange equation of BCS used in [6], but it seems with no proof provided.

As a result of Theorem 1.3, we write the Bogoliubov-de Gennes equations as

$$F'_T(\eta, a) = 0.$$ (1.35)

Remarks. 1) Usually in physics, Eq (1.35) appears in the minimization of $E(\eta, a)$, while keeping $S(\eta)$ and $N(\eta)$ fixed.

2) Due to the symmetry (1.3) of $\eta$, we see that

$$\text{Tr}(\eta \ln \eta) = \text{Tr}((1 - \eta) \ln(1 - \eta))$$ (1.36)

which implies that $S(\eta) = \text{Tr}g(\eta)$, with $g(\lambda)$ given in (1.28).

3) The map $F'_T(\eta, a)$ can be thought of a gradient map.

4) Condition (1.34) on perturbations are designed to handle a delicate problem of non-differentiability of $s(\lambda) := \lambda \ln \lambda$ at $\lambda = 0$, while allowing for sufficiently rich set to derive the BdG equations.

From now on, we consider only the cylindrical geometry, i.e. we assume the dimension is 2.

1.5. Special solutions of Bogoliubov-de Gennes equations. In general, equations (1.26)-(1.27) have the following special solutions:

1) Normal state: $(\eta, a)$, with $\alpha = 0$.

2) Superconducting state: $(\eta, a)$, with $\alpha \neq 0$ and $a = 0$.

3) Mixed state: $(\eta, a)$, with $\alpha \neq 0$ and $a \neq 0$.

We think of $v$ living on a microscopic scale (1), normal and superconducting states as living on a macroscopic scale ($\delta^{-1}$, the scale of the sample) and mixed states, on a mesoscopic one ($\delta^{-1}$), with the scales related as: micro $\ll$ meso $\ll$ macro or $1 \ll \delta^{-1} \ll \delta^{-1}$.

In what follows, we take $\delta' \to 0$, so we are left with $1 \ll \delta^{-1}$.

We discuss the above states in more detail.

Superconducting states. The existence of superconducting, translationally invariant solutions is proven in [8] (see this paper and [9] for the references to earlier results).

Normal states. For $b = 0$, we can choose $a = 0$. In this case, if we drop the direct and exchange self-interactions from $h_{\gamma a \mu}$, then, as was mentioned above, the normal state is given by (1.32), with $a = 0$. This result can be extended to the situation when the direct and exchange self-interactions are present (2).

These are normal translationally invariant states. For $b \neq 0$, the simplest normal states are the magnetically translation (mt-) invariant ones. The existence of the mt-invariant normal states for $b \neq 0$ is stated in
Theorem 1.4. Drop the exchange term $v^\sharp \gamma$ and let $|\int v|$ be small. Then the BdG equations have a mt-invariant normal solution, which is unique in among even, in the sense of the definition (1.17), solutions.

Moreover, this solution is of the form $(\eta = \eta_{T, b}, \ a = a_b)$, where (cf. (1.32))

$$\eta_{T, b} := \begin{pmatrix} \gamma_{T, b} & 0 \\ 0 & 1 - \gamma_{T, b} \end{pmatrix},$$  

(1.37)

with $\gamma_{T, b}$ a solution to the equation

$$\gamma = g^\sharp \left( \frac{1}{T} \gamma_{\gamma, a_b} \right),$$

where $g^\sharp := (g')^{-1}$, and $a_b(x)$ is the magnetic potential with the constant magnetic field $b$ ($\text{curl} \ a_b = b$). (For $g(x) = -(x \ln x + (1 - x) \ln(1 - x))$, we have $g^\sharp(h) = (e^{2h} + 1)^{-1}$ and therefore $\gamma_{T, b}$ solves the equation $\gamma = \left( e^{2\gamma_{\gamma, a_b}/T} + 1 \right)^{-1}$.)

The fact that $\gamma_{T, b}$ in (1.37) is diagonal should not come as a surprise as $a_b$ corresponds to a constant magnetic field $b$ throughout the sample and it corresponds to a normal state. It can be also seen from the following elementary statement

Proposition 1.5. If $\eta$ is mt-invariant, then $\alpha = 0$.

Proof. The mt-invariance implies that $\alpha = e^{-ibs \cdot a_b} \alpha$ for all $s, t \in \mathbb{R}^2$, which yields that $\alpha = 0$. □

Remark. Since $F_T$ is mt-invariant, it is infinite on mt-invariant states. Hence we can introduce the mt-invariant free energy density. As it turns out the latter reduces to the free energy density in the translation invariant case.

We address the question of the energetic stability of the mt-invariant states. We consider perturbations $\eta'$ satisfying the condition (1.34). Physically, the most important perturbations are of the form $\eta' = \phi(\alpha)$, where $\phi(\alpha)$ is the off-diagonal operator-matrix, defined by

$$\phi(\alpha) := \begin{pmatrix} 0 & \alpha \\ \alpha^* & 0 \end{pmatrix},$$  

(1.38)

and $\alpha$ is a Hilbert-Schmidt operator with appropriate smoothness (and equivariance conditions, e.g. from the space $L^2_{\delta, \rho}$, $\delta \geq 0$, of Subsection 1.6). Then $\eta' = \phi(\alpha)$ satisfies (1.34) iff $\alpha$ satisfies

$$\alpha \alpha^* \lesssim [\gamma_{T, b}(1 - \gamma_{T, b})]^2.$$  

(1.39)

Let $h^L$ and $h^R$ stand for the operators acting on other operators by multiplication from the left by the operator $h$ and from right by the operator $\bar{h}$, respectively, and recall $v^\sharp$ is defined after (1.4). We define the operator

$$L_{T, b} := K_{T, b} + v^\sharp,$$  

(1.40)

$$K_{T, b} := \frac{h^L_{T, b} + h^R_{T, b}}{\tanh(h^L_{T, b}/T) + \tanh(h^R_{T, b}/T)},$$  

(1.41)

where $h_{\gamma, a_b}$. Let $\langle \alpha, \alpha' \rangle := \text{Tr}(\alpha^* \alpha')$. We have the following result proven in Section 4, which generalizes that of [8] for $a = 0$:
Proposition 1.6. For perturbations \( \eta' = \phi(\alpha) \) satisfying (1.34), we have
\[
F_T(\eta_T b + \epsilon \eta', a_b) = F_T(\eta_T b, a_b) + \epsilon^2 \langle \alpha, L T b \alpha \rangle + O(\epsilon^3). \tag{1.42}
\]

Since \( h_T b := h_{a_b} + a \) uniformly bounded perturbation, and since \( h_{a_b} = \frac{1}{2} b - \mu \), we see that \( h_T b \geq \frac{1}{2} b - \text{const} \). Using this, it is not hard to show that \( \sigma_{\text{ess}}(L T b) \subset \{ 2 T \mu(b, T), \infty \} \), where \( \mu(b, T) \) is monotonically non-decreasing in \( b \) and \( T, \mu(0, T) = 1 \) and \( \mu(b, T) \to \infty \) as \( b \to \infty \). Hence, since \( v \) is independent of either \( T \) or \( b \), we have

Proposition 1.7. For either \( T \) or \( b \) sufficiently large, \( L T b > 0 \) and consequently the normal state \( (\eta_T b, a_b) \) is energetically stable under general \( \alpha \)-perturbations.

On the other hand, for \( T = 0 \), \( L T b = v^2 \), where \( v^2 \) is defined after (1.4) and therefore we can choose \( \alpha \) satisfying (1.39) and \( \langle \alpha, L T b \alpha \rangle \leq 0 \) given that \( v \leq 0 \), \( v \neq 0 \) (by taking first \( \alpha' \) satisfying (1.39) and then passing to \( \alpha := \chi^2 \alpha' \), where \( \chi \) is a cut-off function supported in the set \( \{ v \leq 0 \} \). Though \( L T b \) is not continuous at \( T = 0 \), we can extend this result to small \( T \) and \( b \).

Theorem 1.8. Suppose that \( v \leq 0 \), \( v \neq 0 \). For \( T \) and \( b \) sufficiently small, the operator \( L T b \) has a negative eigenvalue and consequently the normal state \( (\eta_T b, a_b) \) is energetically unstable under general \( \alpha \)-perturbations.

A proof of this proposition is given in Section 5.

Let \( T_c(b)/T'_c(b) \) be the largest/smallest temperature s.t. the normal solution is energetically unstable/stable under \( \alpha \)-perturbations, for \( (T < T_c(b))/(T > T_c(b))' \). Clearly, \( \infty \geq T'_c(b) \geq T_c(b) \geq 0 \). Proposition 1.7 and Theorem 1.8 imply

Corollary 1.9. Suppose that \( v \leq 0 \), \( v \neq 0 \). Then \( T_c(b) > 0 \) for \( b \) sufficiently small and \( T'_c(b) = 0 \) for \( b \) sufficiently large.

We conjecture that \( T_c(b) = T'_c(b) \).

The next corollary provides a convenient criterion for the determination of \( T_c(b) \) and \( T'_c(b) \).

Corollary 1.10. At \( T = T_c(b) \) and \( T = T'_c(b) \), zero is the lowest eigenvalue of the operator \( L T b \).

A proof of energetically stability under general perturbations, for either \( T \) or \( b \) sufficiently large, is more subtle. For it, one has to use the full linearized operator, \( dF_T'(\eta_T b, a_b) \), which is discussed in Appendix B. Our expressions in Appendix B suggest that \( 0 \) is the lowest eigenvalue of \( L T b \) iff \( 0 \) is the lowest eigenvalue of \( dF'_T(\eta_T b, a_b) \) and, consequently, \( T_c(b) \) and \( T'_c(b) \) apply also to the general perturbations.

The statement \( T_c = T_c(0) = T'_c(0) > 0 \) for \( a = 0 \) and therefore \( b = 0 \) and for a large class of potentials is proven, by the variational techniques, in §.

In conclusion of this paragraph, we mention that a simple computation shows

Proposition 1.11. The operator \( L T b \) commutes with the magnetic translations. The same is true for \( dF_T'(\eta_T b, a_b) \).

Remarks. 1) The question of when \( L T b \) has negative spectrum for a larger range of \( T \)’s is delicate one. For \( T \) close to \( T_c \), this depends, besides of the parameters \( T \) and \( b \), also on whether the superconductor is of Type I or II.

2) Since the components of magnetic translations (1.12) do not commute, the fiber decomposition of \( L T b \) is somewhat subtle (see §).
Mixed states. We consider \( d = 2 \), which means effectively the cylinder geometry, and let \( \mathcal{L}_\delta = \delta^{-1}(\mathbb{Z} + \tau \mathbb{Z}) \), where \( \tau \in \mathbb{C}, \text{Im} \tau > 0 \). Recall that the small parameter \( \delta > 0 \) defines the ratio of the microscopic and mesoscopic scales. For the mixed states, there are the following specific possibilities:

- Magnetic vortices: \( T^{\text{rot}}_\rho(\eta, a) = T^{\text{gauge}}_\rho(\eta, a) \) for every \( \rho \in O(2) \) and some functions \( g_\rho : O(2) \times \mathbb{R}^2 \to \mathbb{R} \);
- Vortex lattices: \( T^{\text{trans}}_s(\eta, a) = T^{\text{gauge}}_\chi(\eta, a) \), for every \( s \in \mathcal{L}_\delta \) and for some functions \( \chi_s : \mathcal{L}_\delta \times \mathbb{R}^2 \to \mathbb{R} \).

The maps \( g_\rho : O(2) \times \mathbb{R}^2 \to \mathbb{R} \) and \( \chi_s : \mathcal{L}_\delta \times \mathbb{R}^2 \to \mathbb{R} \) satisfy the co-cycle conditions, e.g.

\[
\chi_{s+t}(x) - \chi_s(x) - \chi_t(x) \in 2\pi \mathbb{Z}, \quad \forall s, t \in \mathcal{L}_\delta,
\]

and are called the summands of automorphy (see [13] for a relevant discussion). (The maps \( e^{ig} : O(2) \times \mathbb{R}^2 \to U(1) \) and \( e^{i\chi} : \mathcal{L}_\delta \times \mathbb{R}^2 \to U(1) \), where \( g(x, \rho) \equiv g_\rho(x) \) and \( \chi(x, s) \equiv \chi_s(x) \) are called the factors of automorphy.) One can take \( g_\rho \) to be independent of \( x \).

We think of \( \mathcal{L}_\delta \) as the (mesoscopic) vortex lattice of the solution.

**Magnetic flux quantization.** Denote by \( \Omega_\mathcal{L} \) a fundamental cell of \( \mathcal{L} \). One has the following results

- Magnetic vortices: \( \frac{1}{2\pi} \int_{\mathbb{R}^2} \text{curl} a = \deg g \in \mathbb{Z} \);
- Vortex lattices: \( \frac{1}{2\pi} \int_{\Omega_\mathcal{L}_\delta} \text{curl} a = c_1(\chi) \in \mathbb{Z} \).

Here \( \deg g \) is the degree (winding number) of the map \( e^{ig} : O(2) \to U(1) \) (which is map of a circle into itself, here we assume that \( g(\rho) \equiv g_\rho \) is independent of \( x \)) and \( c_1(\chi) \) is the first Chern number associated to the summand of automorphy \( \chi : \mathcal{L}_\delta \times \mathbb{R}^2 \to \mathbb{R} \) (see [13]).

**Existence of vortex lattice solutions.** With the spaces defined in Subsection 1.6, we have the following result on the existence of vortex lattices proven in Section 6.

**Theorem 1.12.** Drop the self-interaction terms \( 2v* d\gamma \) and \( -v^2 \gamma \) in (1.4), for simplicity. Assume that \( T \geq 0 \) and \( \|v\|_{\infty} \) is small. Then

- (i) for every Chern number \( c_1 \), there exists a (generalized) solution \( (\eta, a) \in \mathcal{D}_1 \times \mathcal{D}_b \) of the BdG equations (1.26) and (1.27); in particular, it satisfies \( T^{\text{trans}}_s(\eta, a) = T^{\text{gauge}}_\chi(\eta, a) \);
- (ii) this solution minimizes the free energy \( F_T \) (with the corresponding terms dropped, see (6.1)) for a given \( c_1 \);
- (iii) if the operator \( L_{Tb} \), given in (1.40) and defined on \( \mathcal{H}^2 \), has a negative eigenvalue, then \( (\eta, a) \) has \( \alpha \neq 0 \), i.e. this solution is a vortex lattice. In particular, the latter holds if \( T \) and \( b \) are sufficiently small, provided \( v \leq 0, v \neq 0 \).

The statement (iii) follows from (i) and Proposition 1.6.

**Remarks.** 1) The self-interaction terms \( 2v* d\gamma \) and \( v^2 \gamma \) in (1.5) are inessential for physics and are dropped for simplicity. We expect that these terms can be readily added back to the equation and will not drastically effect our proof.

2) The proof in Section 6 is a variational one and does not give much information about the solutions. The most interesting unanswered question here, which is related to the Ginzburg-Landau limit of the BdG equations, is the behaviour of the minimizer as \( \delta \to 0 \). (By the magnetic flux quantization \( \frac{1}{2\pi} \int_{\Omega_\mathcal{L}_\delta} \text{curl} a = n \in \mathbb{Z} \), where, recall, \( \Omega_\mathcal{L}_\delta \) a fundamental cell of \( \mathcal{L}_\delta \), and therefore \( \text{curl} a, b = O(\delta^2) \).
1.6. Spaces. In conclusion of this introduction, we define the spaces on which we work. To fix ideas, we assume in what follows, that $d = 2$.

We will define functions and operators on the entire space $\mathbb{R}^2$ and require that they are magnetic (gauge) periodic (or equivariant) w.r.to the lattice $\mathcal{L}_\delta$, with the norms defined on a fundamental cell $\Omega_\delta$. 

Let $\rho : \mathcal{L}_\delta \times \mathbb{R}^2 \rightarrow U(1)$ be an automorphy factor (it is related to $\chi_s$ in (1.11) and $\rho(s, x) \equiv e^{i\chi_s(x)}$). We begin with the one-particle state space

$$h_{\delta \rho} := \{ f \in L^2_{\text{loc}}(\mathbb{R}^2) : U_s f = f \text{ for all } s \in \mathcal{L}_\delta \}$$

where $U_s := U^\text{gauge}_s U^\text{trans}_s$, with the operators $U^\text{gauge}_s$ and $U^\text{trans}_s$ defined after (1.5), is the magnetic translation operator.

For an operator $A$ acting on $h_{\delta \rho}$, and consequently commuting with $U_s, \forall s \in \mathcal{L}_\delta$, the integral kernel, $A(x, y)$, if exists, restricted to the diagonal, is periodic w.r.to $\mathcal{L}_\delta$. The trace $\text{Tr}A$ and the trace norm $\|A\|_p$ for such operators are defined, as usual, as $\text{Tr}A := \sum_n \langle u_n, Au_n \rangle$, where $\{u_n\}$ is an orthonormal basis in $h_{\delta \rho}$, and $\|A\|_p := |\text{Tr}(AA^*)|^{p/2}/p$.

In terms of the integral kernels, we have $\text{Tr}A = \int_{\Omega_\delta} A(x, x) dx$.

Let $M_b := \sqrt{-\Delta_{\delta \rho}}$. We define Sobolev-type spaces for trace class operators by

$$I^{s, p} := \{ A : \mathfrak{h} \rightarrow \mathfrak{h} : [U_\gamma, A] = 0, \|A\|_{I^{s, p}} := \|M_b A M_b^*\|_p < \infty \},$$

$$\bar{I}^{s, p} := \{ A : \mathfrak{h}^* \rightarrow \mathfrak{h} : U_\gamma A = A U_\gamma^*, \|A\|_{\bar{I}^{p, s}} := \|AM_b\|_p < \infty \}.\tag{1.45}$$

Note that if $\gamma = \sigma \sigma^*$ where $\sigma \in \bar{I}^{s, 2}$, then $\gamma \in I^{s, 1}$ and $\|\gamma\|_{I^{s, 1}} = \|\sigma\|^2_{I^{s, 2}}$. We will use the notation $I^{0, p} = I^p$ and $\bar{I}^{0, p} = I^p$.

We denote by $I^{s, 1, 2}$ the Sobolev spaces of operators $\eta$, with $\gamma \in I^{s, 1}$ and $\alpha \in I^{s, 2}$, equipped with the norms

$$\|\eta\|_{s, \gamma} := \|\gamma\|_{I^{s, 1}} + \|\alpha\|_{I^{s, 2}}.\tag{1.47}$$

We consider the free energy functional, $F_T$, on the domain

$$D^s := D \cap I^{s, 1, 2},$$

with $s = 1$, where

$$D = \{ \eta \in \mathcal{L}(\mathfrak{h} \times \mathfrak{h}^*), \eta \text{ satisfies } (1.3), [\eta, U_s \oplus U^*_s] = 0 \forall s \in \mathcal{L}_\delta \}. \tag{1.49}$$

Similarly, we consider the Sobolev space of vector potentials

$$\tilde{b}_\delta^{\gamma} := \{ a \in H^1_{\text{loc}}(\mathbb{R}^2 ; \mathbb{R}^2) : T_s a = a \forall s \in \mathcal{L}_\delta, \text{div} a = 0, \langle a \rangle = 0 \}, \tag{1.50}$$

with the norm $\|a\|_{(r)} \equiv \|a\|_{H^r} \equiv \|a\|_{H^r(\Omega_\delta)}$, and the affine space

$$\tilde{b}_\delta^{\gamma} := a + \bar{b}_\delta^{\gamma}.\tag{1.51}$$

2. Energy and entropy functionals: Proof of Theorem 1.3

The proof of Theorem 1.3 consists of three parts: 1) differentiability of $F_T$, 2) identification of the BdG equations with the Euler-Lagrange equation of $F_T$, and 3) showing minimizers of $F_T$ among the set $D^1 \times \tilde{b}_1^1$ are critical points.

Part 1: differentiability. We consider first the variation $\eta + \epsilon \eta'$ for $\epsilon > 0$ small and perturbations satisfying (1.3a). Note that such $\eta'$ satisfies, for $\epsilon$ small enough,

$$0 \leq \eta + \epsilon \eta' \leq 1.\tag{2.1}$$
Let $d_{\epsilon}F_T(\eta, a)\eta' := \partial_{\epsilon}F_T(\eta + \epsilon\eta', a) \big|_{\epsilon=0}$, if the r.h.s. exists. From (1.16) and (1.33), we see that
\[
d_{\epsilon}F_T(\eta, a)\eta' = \text{Tr}(A(\eta, a)\eta') - TdS(\eta)\eta',
\]
provided $dS(\eta)\eta' := \partial_{\epsilon}S(\eta + \epsilon\eta') \big|_{\epsilon=0}$ exists. Hence the differentiability of $F_T$ follows from (2.2) and

**Proposition 2.1.** Let $\eta \in D^1$ be such that $s(\eta) := \eta\ln\eta$ is trace class and $\eta'$ satisfy (1.34). Then $S$ is $C^1$ and its derivative is given by
\[
dS(\eta)\eta' = \text{Tr}(s'(\eta)\eta').
\]

**Proof.** Denote $\eta'' := \eta + \epsilon\eta'$. We write
\[
S(\eta'') - S(\eta) = \text{Tr}(\eta'\ln\eta'') + \epsilon\eta'(\ln\eta'' - \ln\eta + \epsilon\eta'\ln\eta)
\]
\[
= A + B + \epsilon\text{Tr}(\eta'\ln\eta).
\]
Using the formula $\ln a - \ln b = \int_0^\infty [(b + t)^{-1} - (a + t)^{-1}]dt$ and the second resolvent equation, we compute
\[
A := \text{Tr}(\eta(\eta'' - \ln\eta))
\]
\[
= \int_0^\infty \text{Tr}\{\eta[(\eta + t)^{-1} - (\eta'' + t)^{-1}]\}dt
\]
\[
= \int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1}\epsilon\eta'(\eta'' + t)^{-1}\}dt
\]
\[
= \int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1}\epsilon\eta'(\eta + t)^{-1}\}dt
\]
\[
- \int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1}\epsilon\eta'(\eta'')\eta''(\eta + t)^{-1}\}dt.
\]
Similarly, we have
\[
B := \text{Tr}(\epsilon\eta'(\eta'' - \ln\eta))
\]
\[
= \int_0^\infty \text{Tr}\{\epsilon\eta'[(\eta + t)^{-1} - (\eta'' + t)^{-1}]\}dt
\]
\[
= \int_0^\infty \text{Tr}\{\epsilon\eta'(\eta + t)^{-1}\epsilon\eta'(\eta'' + t)^{-1}\}dt.
\]
Combining the last two relations with (2.5), we find
\[
S(\eta + \epsilon\eta') - S(\eta) = \epsilon S_1 + \epsilon^2 R_2,
\]
\[
S_1 := \text{Tr}\eta'\ln\eta + \int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1}\eta'(\eta + t)^{-1}\}dt,
\]
\[
R_2 := - \int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1}\eta'(\eta + t)^{-1}\eta''(\eta'' + t)^{-1}
\]
\[
- \eta'(\eta + t)^{-1}\eta'(\eta'' + t)^{-1}\}dt.
\]
The estimates below show that the integrals on the r.h.s. converge. Computing the integral $\int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1}\eta'(\eta + t)^{-1}\}dt = \int_0^\infty \text{Tr}\{\eta(\eta + t)^{-2}\eta'\}dt = \text{Tr}\eta'$ in the expression...
for \(S_1\) and transforming the expression for \(R_2\), we obtain
\[
S_1 := \text{Tr}\{\eta' \ln \eta + \eta'\},
\]
\[
R_2 := \int_0^\infty \text{Tr}\{t(\eta + t)^{-1} \eta'((\eta + t)^{-1} \eta'' + t)^{-1}\}dt.
\]

To demonstrate the convergence in (2.9) and (2.12), we estimate the integrand on the r.h.s. of (2.12). We have
\[
|\text{Tr}\{\eta'((\eta + t)^{-1} \eta'' + t)^{-1}\}| \leq \|\eta'((\eta + t)^{-1} \eta'' + t)^{-1}\|_2^2 
\]
and therefore
\[
\|\eta'((\eta + t)^{-1} \eta'' + t)^{-1}\|_2 \leq \|\eta((1 - \eta')(\eta + t)^{-1}\|_2^2 
\]
where \(\eta'\) is either \(\eta\) or \(\eta''\) and \(\xi := \eta'(1 - \eta'').\) Let \(\mu_n\) be the eigenvalues of the operator \(\xi := \eta'(1 - \eta'').\) Then we have
\[
\|\xi((\xi + t)^{-1}\|_2^2 = \sum_n \mu_n^2(\mu_n + t)^{-2},
\]
and therefore
\[
\int_0^\infty \|\xi((\xi + t)^{-1}\|_2^2 dt = \int_0^\infty \sum_n \mu_n^2(\mu_n + t)^{-2} dt 
\]
\[
= \sum_n \mu_n = \text{Tr}\xi.
\]

Since \(\eta(1 - \eta)\) and \(\eta''(1 - \eta'')\) are trace class operators, this proves the claim and, with it, the convergence of the integral on the l.h.s. Similarly, one shows the convergence of the other integrals.

To sum up, we proved the expansion (2.8) with \(S_1\) given by (2.11), which is the same as (2.3), and \(R_2\) bounded as \(|R_2| \lesssim 1\). In particular, this implies that \(S\) is \(C^1\) and its derivative is given by (2.3).

**Part 2: Euler-Lagrange equation.** Next, we show that if \(d_\eta F_T(\eta,a) \eta' = 0\) and \(d_\eta F_T(\eta,a) a' = 0\) for all \(a' \in \mathfrak{h}^1\) and \(\eta'\) satisfying (1.34), and \(0 < \eta' < 1\), then \((\eta,a)\) satisfies the BdG equation. We start with \(d_\eta F_T \eta' = 0\) for all \(\eta'\) such that \((\eta')^2 \lesssim [\eta(1 - \eta)]^2\). Recall from equation (2.2), (2.3) and (1.36) that
\[
d_\eta F_T(\eta,a) \eta' = \text{Tr} \left[ (A(\eta,a) - T g' (\eta)) \eta' \right].
\]

We consider \(\eta' = \eta'(x):= P_x - P_{ J\bar{x}}\) for some \(x \in \mathfrak{h} \times \mathfrak{h}^*\) of unit norm, where \(P_x\) is the orthogonal projection on to \(x\). Note that \(J \bar{x}\) and \(x\) are orthogonal so that \(P_x\) and \(P_{ J\bar{x}}\) commute. It is easy to show that \(\eta'\) satisfies (1.34) if for all \(y \in \mathfrak{h} \times \mathfrak{h}^*\), we have that
\[
|\langle x,y \rangle| \leq C\|\eta(1 - \eta)y\|
\]
for some $C > 0$, with possibly a different constant $C$. This shows that $(\eta')^2 \leq C[\eta(1-\eta)]^2$ if $x$ is in the range of $\eta(1-\eta)$. Moreover, for any $x$ in the range of $\eta(1-\eta)$, equation $d_\eta F_T(\eta,a)\eta' = 0$, with (2.16) and $\eta' = \eta'(x) := P_x - P_{Jx}$ becomes

$$0 = \langle x, Hx \rangle - \langle Jx, HJx \rangle = 2\langle x, Hx \rangle$$

(2.18)

where $J$ is given in (1.3), $H := \frac{1}{2}\Lambda(\eta,A) - Tg'(\eta)$, and the last equality follows from the fact that $H$ satisfies the relation

$$J^*HJ = -\bar{H}.$$  

(2.19)

The assumption $0 < \eta < 1$ shows that $\eta$ has dense range and therefore we conclude that $\frac{1}{2}\Lambda(\eta,A) - Tg'(\eta) = 0$.

Now, we consider the equation $d_\eta F_T(\eta,a)a' = 0$. As was mentioned above, one can easily show that

$$d_\eta F_T(\eta,a)a' = 2\int dx[\text{curl}^*\text{curl} a - j(\gamma,a)] \cdot a',$$

(2.20)

where the perturbation $a' \in \mathfrak{h}$ is divergence free and mean zero. Hence, to conclude that

$$\text{curl}^*\text{curl} a - j(\gamma,a) = 0,$$

(2.21)

we have to show that the l.h.s of this equation is divergence free and mean zero. Clearly, the term $\text{curl}^*\text{curl} a$ is divergence free and mean zero.

So we show that $J(\gamma,a)$ is divergence free and mean zero. To this end, we use the fact that our free energy functional is invariant under gauge transformation. In fact, it suffices to use the gauge invariance of the first line in (1.16), $E_1(\eta,a) := \text{Tr}((-\Delta a)\gamma) + \text{Tr}((v * \rho_\gamma)\gamma) - \frac{1}{2}\text{Tr}((v^\gamma\gamma)\gamma)$. It gives

$$0 = \partial_t|_{t=0} E_1(T_{\gamma}^{\text{gauge}}(\eta,a))$$

(2.22)

for all $\chi \in H^1_{\text{loc}}$ which are $\mathcal{L}$-periodic. Using the cyclicity of trace, we compute this explicitly

$$0 = \text{Tr}(\text{Re}(2i\nabla a) \cdot \nabla \chi) + \text{Tr}((\gamma, h_{\gamma, a}) \chi).$$

(2.23)

Since $(\eta,a)$ solves the BdG equation, we see that $[\Lambda(\eta,a), \eta] = 0$. Taking the upper left component of this operator valued matrix equation, we see that

$$[h_{\gamma,a}, \gamma] + (v^\gamma\alpha)\alpha - \alpha(v^\gamma\alpha) = 0$$

(2.24)

Since $v(x) = v(-x)$, we conclude that the integral kernel of $(v^\gamma\alpha)\alpha - \alpha(v^\gamma\alpha)$,

$$\int (v(x-z) - v(-z-y))\alpha(x,z)\alpha(z,y)dz,$$

(2.25)

is zero on the diagonal. Thus, the same conclusion holds for $[\gamma, h_{\gamma,a}]$. Consequently, $\text{Tr}((\gamma, h_{\gamma,a}) \chi) = 0$ and we conclude, by (2.23), that

$$0 = -\text{Tr}(\text{Re}(2i\nabla a) \cdot \nabla \chi) = \int_{\Omega_S} j(\gamma,a) \cdot \nabla \chi = 0,$$

(2.26)

Since this is true for every $\chi \in H^1_{\text{loc}}$ which are $\mathcal{L}$-periodic, it follows that $\text{div} j(\gamma,a) = 0$.

Furthermore, since for our class of solutions $\gamma$ is even and $a$, odd, we conclude that $j(\gamma,a)$ is odd under reflections and therefore $\langle j(\gamma,a) \rangle = 0$. Hence (2.21) holds.
Since $\text{div} \, a = 0$, we may replace $\text{curl}^* \, \text{curl}$ by $-\Delta$. Hence, the elliptic regularity theory shows that $a \in H^1_{\gamma}$. This completes the proof. □

**Part 3: minimizers are critical points.** For a minimizer $(\eta, a)$, we have that $d_{\eta} F_T(\eta, a) \eta' \neq 0$, $d_{a} F_T(\eta, a) a' \neq 0$. Since $H^1_{\gamma}$ is linear, $a' = H^1_{\gamma}$ if and only if $-a' = H^1_{\gamma}$. So $d_{a} F_T(\eta, a) a' = 0$ for all $a \in H^1_{\gamma}$. Similarly, we note that $\eta'$ satisfies the assumption (1.34) if and only if $-\eta'$ satisfies the same requirement. Hence we conclude that $0 = dF_T(\eta, a) \eta'$, completes the proof. □

3. The normal states with non-vanishing magnetic fields: Proof of Theorem 1.4

When $\alpha = 0$, the BdG equations reduces to the following equations for $\gamma$ and $a$,

$$\gamma = g^2 \left( \frac{1}{T} h_{\gamma,a} \right),$$

$$\text{curl}^* \, \text{curl} \, a = j(\gamma, a)$$

where, recall, $j(\gamma, a)(x) := \frac{1}{2} [i \nabla_{\gamma} - \Delta ] (x, x)$. We show first that the second equation is automatically satisfied for $a = a_0$ and $\gamma$ is a magnetically translation invariant operator, which even in the sense of (1.17).

We define $t_{\alpha}^{\text{int}} := t_{\alpha}^{\text{gauge}} \eta^\text{trans}$, where $t_{\alpha}^{\text{gauge}} : \gamma \mapsto e^{i \chi \alpha - i x}$, for any sufficiently regular function $\chi : \mathbb{R}^d \to \mathbb{R}$, and $t_{\alpha}^{\text{trans}} : \gamma \mapsto U_{h} \gamma U_{h}^{-1}$, for any $h \in \mathbb{R}^d$, and $g_{x}(x) := \frac{1}{2} s \cdot J x$, where $J$ is the usual complex structure as in (1.3). Recall that the reflection operator $t^{\text{refl}}$ is given by conjugation by the reflections.

**Proposition 3.1.** If a trace class operator $\tilde{\gamma}$ satisfies $t_{\alpha}^{\text{int}} \tilde{\gamma} = \tilde{\gamma}$, then $\tilde{\gamma}(x, x) = \tilde{\gamma}(0, 0)$ for all $x$. If, in addition, $t_{\alpha}^{\text{refl}} \tilde{\gamma} = -\tilde{\gamma}$, then $\tilde{\gamma}(x, x) = 0$.

**Proof.** The equation $t_{\alpha}^{\text{int}} \tilde{\gamma} = \tilde{\gamma}$ implies that the integral kernel of $\tilde{\gamma}$ obeys $e^{i q_{a}(x)} \tilde{\gamma}(x + h, y + h) e^{-i q_{a}(y)} = \tilde{\gamma}(x, y)$. Taking $y = x$ this gives $\tilde{\gamma}(x + h, x + h) = \tilde{\gamma}(x, x)$, which implies $\tilde{\gamma}(x, x) = \tilde{\gamma}(0, 0)$.

Next, the equation $t_{\alpha}^{\text{refl}} \tilde{\gamma} = -\tilde{\gamma}$ implies that $\tilde{\gamma}(-x, -y) = -\tilde{\gamma}(x, y)$, which, together with the previous relation, gives $\tilde{\gamma}(x, x) = \tilde{\gamma}(0, 0) = 0$, as claimed. □

If $\gamma$ is magnetically translationally invariant and even, then $\tilde{\gamma} = -i \nabla_{\alpha} \gamma$ is a magnetically translationally invariant and odd. Applying this proposition to $\tilde{\gamma} = -i \nabla_{\alpha} \gamma$, where $\gamma$ is a magnetically translationally invariant and even trace class operator, gives (3.2).

Now, we solve the first equation (3.1) on magnetic translation invariant $\gamma$’s. If we drop both the direct and exchange self-interactions from $h_{\gamma a_{\mu}}$, then the latter equation becomes the definition of $\gamma_{TB}$: $\gamma_{TB} = g^2 (\frac{1}{T} h_{a_{\mu}})$, where, recall, $h_{a_{\mu}} := -\Delta_{a_{\mu}} - \mu$. Otherwise, we have to treat this equation as a fixed point problem. This problem simplifies considerably if we drop the exchange term, as in this case it reduces to a fixed point problem for a real number $\xi$:

$$\xi = v \cdot d_{g^2((h_{a_{\mu}} + \xi)/T)}.$$  (3.3)

Then $\xi$ is a real since $-\Delta_{a_{\mu}}$ is self-adjoint and is a multiple of the identity map by magnetic translation invariance due to Proposition 3.1. Suppose that a real $\xi$ solves (3.3), then define
\[ \gamma := g^\sharp((h_{ab} + \xi)/T). \] Since \( g^\sharp > 0 \) and \( h_{ab} + \xi \) is self-adjoint (for real \( \xi \)), we see that \( \gamma \geq 0 \). Then, we see that

\[ \gamma := g^\sharp((h_{ab} + \xi)/T) \] \( = g^\sharp((h_{ab} + v*\rho^\gamma_{ab} + \xi)/T)/T \) \( = g^\sharp((h_{ab} + v*\rho)/T). \) (3.4)

Hence \( \gamma \) satisfies (3.1). Conversely, if \( \gamma \) solves the BdG equation (3.1), then \( \xi = v*d_\gamma \) satisfies the equation

\[ \xi = v*d_\gamma = v*d_{\gamma^2}((h_{ab} + v*\rho_{ab} + \xi)/T) \] \( = v*d_{\gamma^2}((h_{ab} + \xi)/T). \) (3.5)

So this \( \xi \) is solution to (3.3). We have therefore shown the following correspondence:

**Lemma 3.2.** There exists a solution \( (\gamma_b, a = 0, a = a_b) \) with \( \gamma_b \geq 0 \) and \( \gamma_b \) is a function of \( -\Delta a_b \) if and only if the fixed point problem (3.3) has a solution. Moreover, this solution is unique.

We show in Appendix A that the fixed point problem (3.3) has a unique solution if \( |f| \) is small. Thus, we obtain an unique magnetic translation invariant solution. So we prove uniqueness among the class of \( \gamma \)'s such that \( \gamma \) is a function of \( -\Delta a \).

Assume that \( \gamma_1, \gamma_2 \) are two solutions to (3.1). Then we may form their corresponding \( \xi_i = v*d_{\gamma_i} \) for \( i = 1, 2 \). Uniqueness of solution of equation (3.3) dictates that \( \xi_1 = \xi_2 \). Therefore,

\[ \gamma_1 = g^\sharp((h_{ab} + v*d_{\gamma_1})/T) \] \( = g^\sharp((h_{ab} + \xi_1)/T) \) \( = g^\sharp((h_{ab} + \xi_2)/T) \) \( = g^\sharp((h_{ab} + v*d_{\gamma_2})/T) \) \( = \gamma_2. \) (3.9)

What remains to be done is to show that the solution is unique among solutions \( (\gamma, a) \) such that \( \gamma \) is magnetic translation invariant. It suffices to show that \( a = a_b \); then equation (3.1) shows that \( \gamma \) is a function of \( -\Delta a_b \) and we can conclude uniqueness by Lemma 3.2. We decompose \( a = a_b + a' \), where \( a' \) is defined by this expression. Using equation (3.2), we see that

\[ \text{curl}^* \text{curl} a' = -a'(x)\gamma(0,0), \] (3.14)

since, by Proposition 3.1, \( \gamma(x,x) = \gamma(0,0) \), the term \( \text{Re}(-i\nabla_{ab}\gamma)(x,x) \) vanishes, and \( \text{curl}^* \text{curl} a_b = 0 \). Multiplying both sides by \( a' \) and integrate, we see

\[ \int |\text{curl} a'|^2 + \gamma(0,0) \int |a'(x)|^2 = 0 \] (3.15)

Since \( h_{ab} + \xi \) is bounded below and \( g^\sharp \) is strictly positive and increase, we see that \( \gamma = g^\sharp((h_{ab} + \xi)/T) \geq c > 0 \). Thus \( \gamma(0,0) > 0 \). It follows that \( a' = 0 \) and the proof is complete.

Finally, we can show that \( \xi \) is smooth in \( T \).
Lemma 3.3. Assume that \( \int v \leq 0 \) and \( | \int v | \) is small. Then \( \xi \) is negative for all \( T \geq 0 \) and is bounded for all \( T \) small. It is smooth and has the expansion, for \( t \) small,

\[
\xi(T) = \frac{B \mu}{1 + B} + \frac{TB}{2} e^{(T+2\mu)^2} + O(Te^{-\frac{2(\xi-\mu)}{T}} + \delta^2)
\]  

(3.16)

The proof is given in Appendix A. \( \square \)

4. Expansion of the free energy: Proof of Proposition 1.6

We begin with the following

Proposition 4.1. \( S \) is \( C^3 \) at \( \eta_{Tb} \) w.r.to perturbations \( \eta' \) satisfying \( (1.34) \). Moreover, we have

\[
S(\eta_{Tb} + \epsilon \eta') = S(\eta_{Tb}) + \epsilon \text{Tr}(g'(\eta_{Tb})\eta') + \epsilon^2 S''(\eta', \eta') + O(\epsilon^3)
\]  

(4.1)

where \( g(x) = -\frac{1}{2}(x \ln x + (1 - x) \ln(1 - x)) \), \( S''(\eta', \eta') \) is a quadratic form,

\[
S''(\eta', \eta') = \frac{1}{2} \int_0^\infty \text{Tr}\{(\eta + t)^{-1} \eta'(\eta + t)^{-1} \eta'} dt \quad (4.2)
\]

\[
= \frac{1}{4} \int_0^\infty \text{Tr}\{((\eta + t)^{-1} \eta'(\eta + t)^{-1} + (1 - \eta + t)^{-1} \eta'(1 - \eta + t)^{-1})\eta'} dt, \quad (4.3)
\]

and the error term is uniform in \( \eta' \) and is bounded by \( \epsilon^3 \text{Tr}(\eta_{Tb}(1 - \eta_{Tb})) \). For \( \eta' = \phi(\alpha) \), the quadratic term becomes

\[
S''(\eta', \eta') = - \text{Tr}(\bar{\alpha}_K_{Tb} \alpha), \quad K_{Tb} := \frac{h_{L}^{Tb} + h_{R}^{Tb}}{T \tanh(h_{L}^{Tb}/T) + \tanh(h_{R}^{Tb}/T)},
\]  

(4.4)

where \( h_{Tb} := h_{\gamma_{Tb}, a_{Tb}} \).

Proof. For the duration of the proof we omit the subindex \( Tb \) in \( \eta_{Tb} \). Recall \( (2.5) \) - \( (2.7) \) and continuing computing \( A \) and \( B \) in \( (2.6) \) - \( (2.7) \) in the same fashion as in the derivation of these equations, we find

\[
A = \int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1} \eta'(\eta + t)^{-1}\} dt
\]

\[
- \int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1} \eta'(\eta + t)^{-1} \eta'(\eta + t)^{-1}\} dt
\]

\[
+ \int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1} \eta'(\eta + t)^{-1} \eta'(\eta + t)^{-1} \eta'(\eta'' + t)^{-1}\} dt,
\]

(4.5)

and

\[
B = \int_0^\infty \text{Tr}\{\eta'(\eta + t)^{-1} \eta'(\eta + t)^{-1}\} dt
\]

\[
- \int_0^\infty \text{Tr}\{\eta'(\eta + t)^{-1} \eta'(\eta + t)^{-1} \eta'(\eta'' + t)^{-1}\} dt.
\]

(4.6)
Combining the last two relations with (4.10) and recalling the computation of $S_1$, we find
\[
S(\eta + \epsilon \eta') - S(\eta) = \epsilon S_1 + \epsilon^2 S_2 + \epsilon^3 R_3 \tag{4.7}
\]

\[
S_2 := -\int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1}\eta(\eta + t)^{-1}\eta'(\eta + t)^{-1} \eta'(\eta + t)^{-1}\} dt,
\]

\[
R_3 := \int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1}\eta(\eta + t)^{-1}\eta'(\eta + t)^{-1} \eta'(\eta + t)^{-1}\} dt. \tag{4.8}
\]

Transforming the expressions for $S_2$ and $R_3$, we obtain
\[
S_2 = \int_0^\infty \text{Tr}\{t(\eta + t)^{-1}\eta(\eta + t)^{-1}\eta'(\eta + t)^{-1}\} dt, \tag{4.9}
\]

\[
R_3 = -\int_0^\infty \text{Tr}\{t(\eta + t)^{-1}\eta(\eta + t)^{-1}\eta'(\eta + t)^{-1}\} dt. \tag{4.10}
\]

Estimates similar to those done after (2.12) show that the integrals on the r.h.s. converge. This proves the expansion (4.1) with $S_1$ and $S_2$ given by (2.5), which is the same as (2.3), and (4.8) and $R_3$ bounded as $|R_3| \leq 1$. Identifying the quadratic form $S_2$ with $S''(\eta', \eta')$, we arrive at the expansion (4.1).

Before computing $S_2 \equiv S''(\eta', \eta')$, we find a simpler representation for it. Integrating the r.h.s. of (4.9) by parts, we find
\[
S'' = \int_0^\infty \text{Tr}\{t(\eta + t)^{-2}\eta(\eta + t)^{-1}\eta'\} dt
\]

\[
= \int_0^\infty \text{Tr}\{(\eta + t)^{-1}\eta'(\eta + t)^{-1}\eta'\} dt
\]

\[- \int_0^\infty \text{Tr}\{t(\eta + t)^{-1}\eta'(\eta + t)^{-2}\eta'\} dt. \tag{4.11}
\]

But by the cyclicity of the trace the last integral is equal to the first one and therefore we have (4.2). Eq (1.36) gives (4.3).

Now, we use (4.13) to compute to $S''$ for $\eta' = \phi(\alpha)$. First, we recall that $\eta = \eta_Tb$ and observe that for $\eta' = \phi(\alpha)$,
\[
\text{Tr}((\eta_Tb + t)^{-1}\eta'(\eta_Tb + t)^{-1}\eta') = 2\text{Tr}((\gamma_Tb + t)^{-1}\alpha(1 - \gamma_Tb + t)^{-1}\alpha)
\]

\[
= 2\text{Tr}((x + t)^{-1}(y + t)^{-1}\alpha) \tag{4.12}
\]

\[
= 2\text{Tr}((x + t)^{-1}(y + t)^{-1}\alpha) \tag{4.13}
\]

where $x$ and $y$ are regarded as operators acting on $\alpha$ from the left by multiplying by $\gamma_Tb$ and from the right, by $1 - \gamma_Tb$. Putting this together with a similar expression for the second term on the r.h.s. of (4.8) and performing the integral in $t$, we obtain
\[
S''(\eta', \eta') = -\text{Tr} [\tilde{\alpha} K(\alpha)], \tag{4.14}
\]

\[
K := \frac{\log(x) - \log(y)}{x - y} + \frac{\log(1 - x) - \log(1 - y)}{(1 - x) - (1 - y)}, \tag{4.15}
\]

with $x$ acting on the left and $y$ acting on the right. (4.15) can be written as
\[
K = \frac{\log(x^{-1} - 1) - \log(y^{-1} - 1)}{x - y}. \tag{4.16}
\]
Recalling that \( \gamma_{Tb} = g^s(h_{Tb}/T) = (1 + e^{2h_{Tb}/T})^{-1} \), where \( h_{Tb} := h_{\gamma_{Tb}a_b} \), and therefore \( x^{-1} - 1 = e^{2h_{Tb}/T} \) and \( y^{-1} - 1 = e^{-2h_{Tb}/T} \), we see that

\[
K = \frac{1}{T} \frac{h_{Tb}^L + h_{Tb}^R}{(1 + e^{2h_{Tb}/T})^{-1} + (1 + e^{2h_{Tb}/T})^{-1}},
\]

which together the hyperbolic functions identities, \( (1 + e^h)^{-1} = \frac{1}{2}(1 - \tanh h) \) and \( (1 + e^{-h})^{-1} = \frac{1}{2}(1 + \tanh h) \), gives (4.17).

Now, proceed to the proof of Proposition 1.6. Using the definition (1.33), with (1.16), and the expansion (4.1), we find, for perturbations \( \eta' = \phi(\alpha) \) obeying the condition (1.34),

\[
F_T(\eta_{Tb} + \epsilon \eta', a_b) = F_T(\eta_{Tb}, a_b) + e^2 \text{Tr}(\alpha v^s \alpha) + e^2 S''(\eta', \eta') + O(\epsilon^3).
\]

The absence of the linear term is due to the fact that \( (\eta_{Tb}, 0) \) is a critical point of the energy function. This together with (4.4) gives (4.12).

5. Instability of the normal states for small \( T \) and \( b \): Proof of Theorem 1.8

Set \( w = -v \geq 0 \). We note that \( L_{Tb} = K_{Tb} - w \) acts on \( \mathfrak{h} \otimes \mathfrak{h} \) (see Section 6). We use Birman-Schwinger principle to show that \( L_{Tb} \) has negative eigenvalue. \( K_{Tb} - w \) has a negative eigenvalue \(-E\) if and only if \( w^{1/2}(K_{Tb} + E)^{-1}w^{1/2} \) is bounded below by 1 for some \( E > 0 \). Let \( \alpha(x, y) \in \mathfrak{h} \otimes \mathfrak{h} \) and \( \phi(x, y) := (w^{1/2}\alpha)(x, y) \). Expanding

\[
\phi = \sum_{nn'} P_n \otimes P_{n'} \phi,
\]

where \( \{P_n\} \) is a basis of eigenprojections for \(-\Delta_{a_b} \) with the eigenvalues \( \lambda_n = b(2n + 1) \), we see that

\[
\langle \phi, (K_{Tb} + E)^{-1} \phi \rangle = \sum_{nn'} \left( \frac{\lambda_{n'} - \mu + \lambda_n - \mu}{\tanh((\lambda_{n'} - \mu)/T) + \tanh((\lambda_n - \mu)/T)} + E \right)^{-1} ||P_n \otimes P_{n'} \phi||^2. \tag{5.1}
\]

Now, if we take \( b \) to be small, then we can find \( N \) such that \( |\lambda_N - \mu| \ll 1 \) and \( ||P_N \otimes P_N \phi|| \neq 0 \). (Since \( w \geq 0, w \neq 0 \), we can always find such \( \alpha \) so that \( ||P_N \otimes P_N \phi|| \neq 0 \).) Hence, since the function \( \frac{x}{\tanh(x/T)} \) has the minimum \( T \) at \( x = 0 \), we have the estimate

\[
0 \leq \frac{\lambda_N - \mu}{\tanh((\lambda_N - \mu)/T)} + E \leq 2T + E \tag{5.2}
\]

for \( \delta \) sufficiently small. Since the function \( \frac{x+y}{\tanh(x/T) + \tanh(y/T)} \) is monotonically increasing, the function \( \frac{x+y}{\tanh(x/T) + \tanh(y/T)} \) is positive (in fact, it has the minimum \( T \) at \( x = y = 0 \)). This together with the relations (5.1) and (5.2) gives

\[
\langle \phi, (K_{Tb} + E)^{-1} \phi \rangle \geq \frac{||P_N \otimes P_N \phi||^2}{2T + E}. \tag{5.3}
\]

Since \( ||P_N \otimes P_N \phi|| \neq 0 \), the r.h.s. can be made arbitrarily large if \( T \) and \( E \) are small. This proves Theorem 1.8. \( \Box \)
6. The existence of the vortex lattices

In this section, we prove Theorem 1.12 on existence of the vortex lattice solutions to the BdG equations with an arbitrary vortex (the first Chern) number \( n \). Recall that to simplify the exposition and to illustrate the main techniques, we drop self-interaction terms \( v^*d \gamma \) and \( v^\#\gamma \) so that the free energy functional \( F_T(\eta, a) \) in (1.33) becomes

\[
F(\eta, a) = \text{Tr}(h_a\gamma) + \frac{1}{2}\text{Tr}(\alpha^*v^\#\alpha) + \int dx |\text{curl} a(x)|^2 - \mu\text{Tr}(\gamma) - TS(\eta). \tag{6.1}
\]

We expect that these terms can be readily added back to the equation and will not drastically effect the proof below.

We define the free energy functional \( F(\eta, a) \) in (6.1) on \( D^1 \times \vec{h}^1_{\delta,b} \), if \( S(\eta) < \infty \). Otherwise we set \( F(\eta, a) = \infty \).

We derive the existence of the solutions from the existence of the minimizers of free energy (6.1).

Remark. 1) The trace of \( 2 \times 2 \) matrix-operator valued matrices is defined as the \( \mathfrak{h} \otimes \mathbb{C}^2 \) trace where we trace the diagonal entries individually in \( \mathfrak{h} \). If the matrix-operator in question is trace class, then this trace coincides with the standard one.

2) If \( A \geq 0 \), then the two forms of trace agrees for the operator \( A \). More precisely, \( A \) is trace class if and only if its \( \mathfrak{h} \otimes \mathbb{C}^2 \) trace is finite. In particular, \( S(\eta) < \infty \) if and only if \( g(\eta) \) is trace class.

6.1. The main theorem. Theorem 1.12 follows from Theorem 1.3 and the following

**Theorem 6.1.** Assume that \( T > 0 \) and \( \|v\|_{\infty} \) is small. There exists a finite energy minimizer \((\eta_*, a_*) \in D^1 \times \vec{h}^1_{\delta,b} \) of the functional \( F(\eta, a) \) on the set \( D^1 \times \vec{h}^1_{\delta,b} \). This minimizer satisfies \( 0 < \eta_* < 1 \) and \( g(\eta_*) \) is trace class.

The last statement, \( g(\eta) \) is trace class, follows from the fact that if \( A \geq 0 \), then the two form of traces (the usual trace and the \( \mathfrak{h} \otimes \mathbb{C}^2 \) trace) are the same.

Since the equivariance property, and therefore the magnetic flux, are preserved under passing to the limit in a minimizing sequence, the minimizer is, in fact, a vortex lattice with a given magnetic flux through the fundamental cell.

By combining this result with Theorem 1.3 we obtain Theorem 1.12.

One can likewise perform minimization among diagonal \( \eta \)'s. This way, we obtain a variational proof of Theorem 1.4 on the existence of normal states.

6.2. Proof of Theorem 6.1. We will use standard minimization techniques proving that \( F(\eta, a) \) is coercive and weakly lower semi-continuous, and \( D^1 \times \vec{h}^1_{\delta,b} \) weakly closed.

**Part 1: coercivity.** The main result of this step is the following proposition:

**Proposition 6.2.** For \( T > 0 \) and \( \|v\|_{\infty} \) small enough, we have

\[
F_T(\eta, a_b + a') \geq C_1(\|\eta\|_{(1)} + \|a\|_{(1)}) - C_2 \tag{6.3}
\]

for suitable \( C_1, C_2 > 0 \).
We estimate $F_T(\eta, a)$ from below by reducing the full functional to tractable simpler ones. One first major step is to be able to estimate the entropy term $S(\eta)$. So we give an estimate before we dive into the proof of the proposition.

We define and recall the diagonal and off-diagonal operator-matrix $\eta_0$ and $\phi$ as

$$\eta_0 := \begin{pmatrix} \gamma & 0 \\ 0 & 1 - \gamma \end{pmatrix}, \quad \phi(\beta) := \begin{pmatrix} 0 & \beta \\ \beta^* & 0 \end{pmatrix}.$$  

(6.4)

Recall that $S(\eta) = \text{Tr}(s(\eta))$ for $s(x) = -x \ln x$ and define the relative entropy

$$S(A|B) = \text{Tr}(s(A|B)), \quad s(A|B) := A(\ln A - \ln B) \geq A - B,$$  

(6.5)

**Proposition 6.3.** We have for $\eta = \eta_0 + \phi(\alpha)$,

$$S(\eta) = S(\eta_0) - S(\eta|\eta_0) \leq S(\eta_0).$$  

(6.6)

(The last inequality follows from Klein’s inequality and the fact that $\text{Tr}\eta = \text{Tr}\eta_0$.)

**Proof.** We note that for $\eta := \eta_0 + \phi(\alpha)$,

$$\eta \ln \eta - \eta_0 \ln \eta_0 = \eta \ln \eta - \eta \ln \eta_0 + \eta \ln \eta_0 - \eta_0 \ln \eta_0$$

(6.7)

$$= s(\eta, \eta_0) + (\eta - \eta_0) \ln \eta_0$$

(6.8)

$$= s(\eta, \eta_0) + \phi(\alpha) \ln \eta_0.$$  

(6.9)

the last term $\phi(\alpha) \ln[\eta_0(1 - \eta_0)^{-1}]$ has zero trace since it is off-diagonal, we have the first equation in (6.6). □

**Proof of Proposition 6.2.** We estimate $\text{Tr}(h_a \gamma)$. Writing $a = a_b + a'$ and using $\text{div} a' = 0$, we write

$$\text{Tr}((-\Delta a) \gamma) = \text{Tr}((-\Delta a_b) \gamma) + 2i \text{Tr}(a' \cdot \nabla a_b \gamma) + \text{Tr}(|a'|^2 \gamma)$$

(6.11)

$$\geq (1 - \epsilon) \text{Tr}((-\Delta a_b) \gamma) + (1 - \epsilon^{-1}) \text{Tr}(|a'|^2 \gamma),$$  

(6.12)

for any $\epsilon > 0$. Since $\langle a' \rangle = 0$, the Poincaré’s inequality shows that

$$\text{Tr}(|a'|^2 \gamma) \leq \int_{\Omega^\delta} |a'|^2 \leq C \int_{\Omega^\delta} |Da'|^2,$$  

(6.13)

where $|Da'|^2$ is the sum of all squares of derivatives of $a$, for some constant $C$ (dependent of $\delta$). The last two inequalities give

$$\text{Tr}((-\Delta a) \gamma) \geq (1 - \epsilon) \text{Tr}((-\Delta a_b) \gamma) + (1 - \epsilon^{-1}) C \int_{\Omega^\delta} |Da'|^2,$$  

(6.14)

assuming $\epsilon < 1$. Furthermore, using $a = a_b + a'$, $\langle a' \rangle = 0$ and $\text{div} a' = 0$, we find

$$\int_{\Omega^\delta} |\text{curl} a|^2 = \int_{\Omega^\delta} |Da'|^2 + b|\Omega^\delta|.$$  

(6.15)
By considering the 2,2-entry, we have that
\[ \alpha \leq 0 \]

Proof.

From (1.2), we see that the 1,1-entry of
\[ M_{\alpha \alpha} \]

This shows that
\[ \| | \alpha \| | \leq 0 \]

Proof. Since 0 \leq \gamma \leq 1, we see that 0 \leq \gamma(x, x) \leq 1. Definition (6.1), the inequalities above, Proposition 6.3, and the definition \( h_a = -\Delta_a - \mu \) give, for any \( \delta, \epsilon > 0, \)
\[
\mathcal{F}(\eta, a) \geq \delta \text{Tr}((-\Delta_a)\gamma) - \mu \text{Tr} \gamma + \int_{\Omega_\delta} |\text{curl} a|^2
\]
\[
+ \frac{1}{2} \text{Tr}(\alpha^* v^* \alpha) - TS(\gamma)
\]
\[
\geq \delta(1 - \epsilon)\text{Tr}((-\Delta_a)\gamma) + \frac{1}{2} \text{Tr}(\alpha^* v^* \alpha) - TS(\gamma)
\]
\[
+ \delta(1 - \epsilon - 1)C \int_{\Omega_\delta} |D a|^2 + \int_{\Omega_\delta} |D a'|^2 + b|\Omega_\delta|,
\]
\[
(6.16)
\]
\[
(6.17)
\]
\[
(6.18)
\]
\[
(6.19)
\]
Now, take \( \delta = 1/(2C(\epsilon^{-1} - 1)) \) and let \( \delta' := \delta(1 - \epsilon) \). We can first minimize over \( \gamma \) to obtain
\[
\frac{1}{2} \delta' \text{Tr}((-\Delta_a)\gamma) - TS(\gamma) \geq -C_\epsilon \text{ for some positive constant } C_\epsilon.
\]
This, together with the previous estimate, this gives
\[
\mathcal{F}(\Gamma, a) \geq \frac{1}{2} \delta' \text{Tr}((-\Delta_a)\gamma) + \frac{1}{2} \text{Tr}(\alpha^* v^* \alpha)
\]
\[
+ \frac{1}{2} \int_{\Omega_\delta} |D a|^2 + b|\Omega_\delta| - C_\epsilon,
\]
\[
(6.20)
\]
To estimate the second term on the r.h.s. of (6.20), we bound \( \alpha \) by \( \gamma \) via the constraint
\[ 0 \leq \eta \leq 1: \]
Lemma 6.4. The constraint \( 0 \leq \eta \leq 1 \) shows that
\[ \begin{align*}
(1) & \quad 0 \leq \eta(1 - \eta) \leq 1. \\
(2) & \quad \alpha^* \alpha \leq \gamma(1 - \gamma) \text{ and } \alpha \alpha^* \leq \gamma(1 - \gamma). \\
(3) & \quad \text{Tr}(M \alpha \alpha^* M^*) \leq \|M \gamma^{} M\|_1 \text{ for any operator } M.
\end{align*} \]

Proof. Since \( 0 \leq \eta \leq 1 \), then \( 0 \leq 1 - \eta \leq 1 \) as well and therefore \( \eta(1 - \eta) \leq 1 \), proving the first claim. From (1.2), we see that the 1,1-entry of \( \eta(1 - \eta) \) is \( 0 \leq \gamma(1 - \gamma) - \alpha \alpha^* \). By considering the 2,2-entry, we have that \( \alpha^* \alpha \leq \gamma(1 - \gamma) \). Finally, since \( 1 - \gamma \leq 1 \), we see that \( M \alpha \alpha^* M^* \leq M \gamma M^* \) completes the proof.

Since \( v \) is bounded, this lemma gives
\[
|\text{Tr}(\alpha^* v^* \alpha)| \leq \|v\|_\infty \text{Tr}(\alpha^* \alpha) \leq \|v\|_\infty \text{Tr}(\gamma).
\]
\[
(6.21)
\]
This shows that
\[
\frac{1}{2} \delta' \text{Tr}((-\Delta_a)\gamma) + \frac{1}{2} \text{Tr}(\alpha^* v^* \alpha) \geq (1 - \epsilon)\text{Tr}(h_a \gamma) - \frac{1}{2} \|v\|_\infty \text{Tr}(\gamma).
\]
This together with inequality (6.20) implies the bound, claimed in Proposition 6.2 provided \( \|v\|_\infty \) is small.

Part 2: weak lower semi-continuity.

Lemma 6.5. The functional \( \mathcal{F}_T \) is weakly lower semi-continuous in \( D^1 \times (a_b + \delta^1) \).

Proof. We study the functional \( \mathcal{F}_T(\eta, a) \) term by term. For the first term on the r.h.s. of (6.23), we use \( a = a_b + a' \) to write
\[
\text{Tr}(h_a \gamma) = \text{Tr}(h_a \gamma) + 2i\text{Tr}(a' \cdot \nabla_{a_b} \gamma) + \text{Tr}(|a'|^2 \gamma).
\]
\[
(6.22)
\]
Since the first term of (6.22) satisfies \( \text{Tr}(h_{a_{\beta}^2}) = \|\gamma\|_{I,1} - \mu \text{Tr} \gamma \) and is linear, it is \( \|\cdot\|_{I,1} \)-weakly lower semi-continuous.

To show that the second term on the r.h.s of (6.22) is continuous in the norm \( \|\cdot\|_{(s)} \), s < 1, we write, omitting the prime at \( a' \),

\[
\text{Tr}(a_n \cdot \nabla a_{\beta} \gamma_n) - \text{Tr}(a_s \cdot \nabla a_{\beta} \gamma_s) = \text{Tr}((a_n - a_s) \cdot \nabla a_{\beta} \gamma_n) - \text{Tr}(a_s \cdot \nabla a_{\beta} (\gamma_n - \gamma_s)).
\]

(6.23)

This shows that we have to estimate terms of the form \( \text{Tr}(a \cdot \nabla a_{\beta} \gamma) \). We write \( \gamma = \kappa^2 \), denote by \( \kappa'(y, x) \) the integral kernel of \( \kappa \). Then

\[
|\text{Tr}(a \cdot \nabla a_{\beta} \gamma)| = \left| \int \int a(x)(\nabla a_{\beta} \kappa'(x, y))\kappa'(y, x)dx\,dy \right|
\leq \left( \int \int |\nabla a_{\beta} \kappa'|^2 \right)^{1/2} \left( \int \int |a\kappa'|^2 \right)^{1/2}.
\]

(6.24)

Using the definition of the spaces \( I^{s,1} \) and \( I^{s,2} \) and a Sobolev embedding theorem, we find furthermore,

\[
\left( \int \int |\nabla a_{\beta} \kappa'|^2 \right)^{1/2} \leq \|\kappa\|_{I^{2,1}},
\]

(6.25)

\[
\int \int |a\kappa'|^2 \leq (\int |a|^4)^{1/2} \int dy(\int |\kappa'|^4 dx)^{1/2}
\leq \|a\|^2 \int dy \|\kappa'|_{H^2_s}^2 \leq \|a\|^2 \|\kappa\|^2_{L^{s,2} \rightarrow 2},
\]

(6.26)

with \( s > \frac{4}{3} \). The last three inequalities and the relation \( \|\kappa\|^2_{L^{s,2} \rightarrow 2} = \|\gamma\|^2_{I^{s,1}} \), give finally

\[
|\text{Tr}(a \cdot \nabla a_{\beta} \gamma)| \leq \|a\|_{H^s} \|\kappa\|_{I^{s,2}} \|\kappa\|_{I^{2,1}} = \|a\|_{H^s} (\|\gamma\|_{I^{s,1}} \|\gamma\|_{I^{s,1}})^{1/2}.
\]

(6.27)

Applying this inequalities to the terms on the r.h.s. of (6.23), we find, for \( 3/4 < s < 1 \),

\[
|\text{Tr}(a_n \cdot \nabla a_{\beta} \gamma_n) - \text{Tr}(a_s \cdot \nabla a_{\beta} \gamma_s)|
\leq \|a_n - a_s\|_{H^s} \|\gamma_n\|_{I^{1,1}}
\]

\[
+ \|a_s\|_{H^s} (\|\gamma_n - \gamma_s\|_{I^{1,1}} \|\gamma_n - \gamma_s\|_{I^{s,1}})^{1/2}.
\]

(6.28)

Now, by a standard result \( \tilde{H}^s \) is compactly embedded in \( \tilde{H}^{s'} \), for any \( s' > s \). Similarly, passing, as above, from trace class operators to their square roots, and then to the integral kernels of the latter, we see that \( I^{s,1} \) is compactly embedded in \( I^{s',1} \), for any \( s' > s \). Moreover, \( \|\gamma_n\|_{I^{s,1}} \) are uniformly bounded. Hence, we conclude that the r.h.s. of (6.28) converges to 0 as \( n \to \infty \).

Now, we consider the final term \( \text{Tr}(|a'|^2 \gamma) \) in (6.22). Again, omitting the prime at \( a' \), we estimate

\[
|\text{Tr}(|a_n|^2 \gamma_n) - \text{Tr}(|a_s|^2 \gamma_s)| \leq |\text{Tr}(|a_n|^2 (\gamma_n - \gamma_s))| + |\text{Tr}(|a_s|^2 (\gamma_n - \gamma_s))| \leq \|\gamma_n - \gamma_s\|_{\infty} \|a_n\|^2_{L^2} + \|\gamma_s\|_{\infty} \|a_s\|^2_{L^2} - \|a_s\|^2_{L^2}.
\]

(6.29)

(6.30)

and therefore, as above the r.h.s. converges to 0.

The second term on the r.h.s. of (6.1) is quadratic in \( \alpha \) and therefore it is continuous in \( D \times \tilde{H}^1 \) since \( v \in L^\infty \). It follows that it is weakly lower semi-continuous in \( D^1 \times \tilde{H}^1 \).
The third term on the r.h.s. of (6.1), $\int_{\Omega_0} |\text{curl} a|^2$, is clearly convex. So its norm lower semi-continuity is equivalent to weak semi-continuity. Since it is clearly $\tilde{h}^1$ norm continuous, it is $\tilde{h}^1$ weakly lower semi-continuous.

Finally, we study the term $-TS(\eta)$. We use an idea from [11] which allows to reduce the problem to a finite-dimensional one. To the latter end, we use (6.6), to pass from $-S(\eta)$ to the relative entropy, $S(\eta|\eta_0)$, defined in (6.5), with $\eta_0$ of the form (6.4), with $\text{Tr}\gamma_0 < \infty$ and s.t. $S(\eta_0) < \infty$. By Klein’s inequality and the relation $\text{Tr}(\eta_0 - \eta) = \text{Tr}(\gamma_0 - \gamma) - \text{Tr}(\gamma_0 - \bar{\gamma}) = 0$, it is a non-negative functional. Moreover,

$$S(\eta) = S(\eta_0) - S(\eta|\eta_0) - \text{Tr}[(\eta - \eta_0) \ln \eta_0].$$

(6.31)

We choose $\eta_0$ so that to have the w.s.c. of the term $\text{Tr}[(\eta - \eta_0) \ln \eta_0]$ above,

$$\eta_0 = g^\ast(H_0/T), \quad H_0 := \text{diag}(\sqrt{-\Delta_{a_0}}, -\sqrt{-\Delta_{a_0}}).$$

(6.32)

Indeed, since $g^\ast(h) = (e^{2h} + 1)^{-1}$, we see that that

$$0 \leq -\ln \eta_0 = \ln(1 + e^{2H_0/T}) \leq 1 + 2H_0/T.$$  

(6.33)

This estimate and (6.32) shows that $\text{Tr}[(\eta - \eta_0) \ln \eta_0]$ is $D^1$-norm continuous. Furthermore, since this term is affine in $\eta$, it is convex. Thus it is w.s.c.

The positivity of $S(\eta|\eta_0)$ allows us to represent it as $S(\eta|\eta_0) = \sup_P \text{Tr}(Ps(\eta|\eta_0))$, where the supremum is taken over all finite dimensional projections $P$ of the form $P_1 \oplus P_2$ (We note that the two notions of trace agree for positive operators, so we may, in fact, take supremum over all finite dimensional projections).

Since $\eta_n \to \eta_*$ in $\| \cdot \|_{(0)}$, hence in operator norm, and $-x \ln x$ is continuous on $[0, 1]$, we see that

$$s(\eta_n|\eta_0) \to s(\eta_*|\eta_0)$$

(6.34)

in operator norm, where we recall that

$$s(A|B) = A(\ln A - \ln B).$$

(6.35)

In particular, for any finite dimensional projection $P$,

$$\text{Tr}(Ps(\eta_n|\eta_0)) \to \text{Tr}(Ps(\eta_*|\eta_0)).$$

(6.36)

Since $\text{Tr}(Ps(\eta_*|\eta_0))$ is continuous, then, by a standard argument, $\text{Tr}_s(\eta|\eta_0) = \sup_P \text{Tr}(Ps(\eta|\eta_0))$ is lower semi-continuous. Indeed, $\liminf_n \sup_P (\text{Tr}Ps(\eta_n|\eta_0)) \geq \liminf_n \text{Tr}(P'Ps(\eta_n|\eta_0))$ for every $P'$, and therefore

$$\liminf_n \sup_P \text{Tr}P(s(\eta_n|\eta_0)) \geq \sup \liminf_n \text{Tr}P(s(\eta_n|\eta_0)) = \text{Tr}_s(\eta_*|\eta_0).$$

(6.37)

(6.38)

Since $\eta_n - \eta_0$ is trace class on the diagonal, we conclude that

$$\text{Tr}_s(\eta_*|\eta_0) \leq \liminf_n \text{Tr}_s(\eta_n|\eta_0).$$

(6.39)

Hence all the terms on the r.h.s. of the expression (6.1) for $\mathcal{F}(\eta, a)$ are lower semi-continuous under the convergence indicated and therefore so is $\mathcal{F}(\eta, a)$, which completes the proof. \qed
Proof of Theorem 6.1 With the results above, the proof is standard. Let \((\eta_n, a_n) \in D_1 \times \vec{h}_1^\delta, b)\) be a minimizing sequence for \(F\). Proposition 6.2 shows that \(F\) is coercive. Hence \(\|\eta_n\|_{(1)} + \|a_n\|_{(1)}\) is bounded uniformly in \(n\). By Sobolev-type embedding theorems, \((\eta_n, a_n)\) converges strongly in \(D^s \times \vec{h}^s, \delta, b)\) for any \(s < 1\). Moreover, together with the Banach-Alaoglu theorem, the latter implies that \((\eta_n, a_n)\) converges weakly in \(D_1 \times \vec{h}_1^\delta, b)\). Hence, denoting the limit by \((\eta^*, a^*)\), we see that, by Lemma 6.5, \(F\) is lower semi-continuous: 

\[
\liminf_{n \to \infty} F(\eta_n, a_n) \geq F(\eta^*, a^*) \tag{6.40}
\]

Hence, \((\eta^*, a^*)\) is indeed a minimizer.

Next, we show that any minimizer, \((\eta^*, a^*)\), satisfies \(0 < \eta^* < 1\). Assume not, then \(\eta^*\) has non trivial kernel. Let \(P_0\) be the orthogonal projection onto any one dimensional subspace of \(\text{Null} \eta\). Then we see that \(JJ_0^JP^*\) is a projection into the 1-eigenspace of \(\eta\). Furthermore, \(P = P_0 - JJ_0^JP^*\) is of the form \((1.2)\). Consider the perturbation \(\eta^* + \frac{1}{2} \epsilon P\). Then, expanding in the eigenfunctions of \(\eta\) we find 

\[
S(\eta^* + \frac{1}{2} \epsilon P) = S(\eta^*) + \epsilon \ln \epsilon + O(\epsilon) \tag{6.41}
\]

For \(0 < \epsilon\) sufficiently small, we see that \(|\epsilon \ln \epsilon| \ll O(\epsilon)\). Since \(\epsilon \ln \epsilon < 0\), it follows that 

\[
F(\eta^* + \epsilon P, a^*) < F(\eta^*, a^*) \tag{6.42}
\]

This is a contradiction since \((\eta^*, a^*)\) is a minimizer.

Finally, since a minimizing sequence converges to \(a^*\) strongly in \(\vec{h}_1^\delta, b)\) for any \(s < 1\), we have by the magnetic flux quantization \(\frac{1}{2\pi} \int_{\Omega_{L^\delta}} \text{curl} a^* = c_1(\rho) \in \mathbb{Z}\), where, recall, \(\Omega_{L^\delta}\) a fundamental cell of \(L^\delta\). □

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Appendix A. Proof of the existence of solution to (3.3)

We define \(f_T : \mathbb{R} \to \mathbb{R}\) by

\[
f_T(\xi) := v \ast g^2((\Delta u - \mu + \xi)/T) \tag{A.1}
\]

We can derive a more explicit formula for \(f_T\).

Lemma A.1. For each \(T > 0\),

\[
|f_T(\xi) - TB \int_{(\xi - \mu)/T}^\infty g^2(y)dy| \leq C \delta^2 \tag{A.2}
\]

where \(B = \frac{\bar{v}(0)(\text{Im} v)^2}{4\pi T}\) and \(C\) is independent of \(T\).

Proof. We have already seen that 

\[
f_T(\xi) = \bar{v}(0) h(x, x) \quad \text{any} \ x \in \mathbb{R}^2 \tag{A.3}
\]
where \( h(x, y) \) is the integral kernel of \( g^\sharp((-\Delta_{a_b} - \mu + \xi)/T) \). Using the eigenbasis \( \psi_{m,j} \) of 
\(-\Delta_{a_b}, \) with the eigenvalues \( b(2m + 1), \ j \in \{1, \ldots, n\} \), we have

\[
h(x, x) = \sum_{m \geq 0, j=0,\ldots,n-1} g^\sharp((b(2m + 1) - \mu + \xi)/T) |\psi_{m,j}|^2(x),
\]

which gives

\[
f_T(\xi) = \int f_T(\xi) \delta^2 \text{Im} \tau \, dx = \delta^2 \hat{v}(0) \text{Im} \tau \int h(x, x) \, dx
\]

\[
= n \delta^2 \hat{v}(0) \text{Im} \tau \sum_{m \geq 0} g^\sharp((b(2m + 1) - \mu + \xi)/T).
\]

We note that \( g^\sharp \) is monotonically decreasing. Therefore,

\[
\sum_{m \geq -1} g^\sharp((b(2m + 1) - \mu + \xi)/T) \delta^2 \geq \int_0^\infty g^\sharp((b_0 x - \mu + \xi)/T)
\]

\[
\geq \sum_{m \geq 1} g^\sharp((b(2m + 1) - \mu + \xi)/T) \delta^2,
\]

where \( b_0 = \frac{4\pi n}{|\text{Im} \tau|} \). Since \( 0 \leq g^\sharp \leq 1 \), the difference between upper and lower sum is \( O(\delta^2) \).

Moreover

\[
\lim_{\delta \to 0} n \delta^2 \hat{v}(0) \text{Im} \tau \sum_{m \geq 0} g^\sharp((b(2m + 1) - \mu + \xi)/T)
\]

\[
= n \hat{v}(0) \text{Im} \tau \int_0^\infty g^\sharp((b_0 x - \mu + \xi)/T) \, dx
\]

\[
= \frac{T n \hat{v}(0)}{b_0} \text{Im} \tau \int_{(\xi-\mu)/T}^\infty g^\sharp(y) \, dy,
\]

by the change of variable \( y = (b_0 x - \mu + \xi)/T \). \( \square \)

Now we use fixed the point theorem to show existence of solution. To this end, we need an estimate on \( f_T(a) - f_T(b) \) so that we can use the Banach contraction mapping principle.

**Lemma A.2.** Assume that \( \delta^2 \ll T \). If \( \delta \) is small enough, then

\[
|f_T(a) - f_T(b)| < C |\hat{v}(0)| |a - b|
\]

and \( C \) is independent of \( T \) and \( \delta \).

**Proof.** Using the same method as Lemma A.1. We see that

\[
f_T(a) - f_T(b) = n \delta^2 \hat{v}(0) \text{Im} \tau \sum_{m \geq 0} g^\sharp((b(2m + 1) - \mu + a)/T)
\]

\[
- g^\sharp((b(2m + 1) - \mu + b)/T),
\]

which, by the mean value theorem and the fact that the resulting expression is a Riemann sum of a \( L^1 \) function, gives (A.12). \( \square \)
Thus, we see that $f_T$ is a contraction on $\mathbb{R}$. Hence, we conclude that it has an unique fixed point on $\mathbb{R}$.

Now we carry out the proof of Lemma $\ref{lem:fixed-point}$. To see the first claim we only need to note that $\text{sgn}B = \text{sgn}\hat{\phi}(0) < 0$. But the summand in $\ref{eq:summand}$ is positive. Hence $f_T(\xi) < 0$ for all $\xi$. So a fixed point of $f_T$ must also be negative.

We use Lemma $\ref{lem:fixed-point}$ We see that
\[
|\xi| \leq T|B| \int_{\frac{B}{1+B}}^{\infty} g^\sharp(y) dy + O(\delta^2)
\]
\[
\leq T|B| \int_{\frac{B}{1+B}}^{0} g^\sharp(y) dy + T|B| \int_{0}^{\infty} g^\sharp(y) dy + O(\delta^2)
\]
\[
\leq T|B| |\mu - \xi| + O(T + \delta^2)
\]
\[
\leq |B| (|\mu - \xi| + O(T + \delta^2))
\]
Since $|B| < 1$, we see that $\xi$ is bounded.

Now, we explicitly integrate the expression in $\ref{eq:integrate}$. We see that
\[
\int_{\frac{B}{1+B}}^{\infty} g^\sharp(y) dy = \frac{1}{2} \left( \log(e^{\frac{2(\xi - \mu)}{T}}) + 1 - \frac{2(\xi - \mu)}{T} \right)
\]
Expanding the above expression for $T$ small, we see that
\[
\xi = -B(\xi - \mu) + \frac{TB}{2} e^{\frac{2(\xi - \mu)}{T}} + O(\text{Te}^{\frac{4(\xi - \mu)}{T}} + \delta^2)
\]
\[
= -\frac{B\mu}{1+B} + \frac{TB}{2} e^{\frac{2(\xi - \mu)}{T}} + O(\text{Te}^{\frac{4(\xi - \mu)}{T}} + \delta^2)
\]
\[
= -\frac{B\mu}{1+B} + \frac{TB}{2} e^{\frac{-2\mu}{(1+B)T}} + O(\text{Te}^{\frac{4(\xi - \mu)}{T}} + \delta^2)
\]
Finally, we show that $\xi(T)$ is smooth. This can be seen by noticing that $f_T(\xi)$ is smooth in $\xi$ and $T$ for $T > 0$ and apply the implicit function theorem. The requirement $\partial_\xi f_T(\xi) \neq 0$ (for $T$ small) can be seen by Lemma $\ref{lem:fixed-point}$ and the fact $g^\sharp > 0$ always.

**Appendix B. The operator $dF'_T(u_{Tb})$**

The general structure of the operator $dF'_T(u_{Tb})$ is given by

**Proposition B.1.**

\[
dF'_T(u_{Tb}) = \begin{pmatrix}
\partial^2_{\gamma\gamma} F_T(u_{Tb}) & 0 & \partial^2_{a\gamma} F_T(u_{Tb}) \\
0 & \partial^2_{ao} F_T(u_{Tb}) & 0 \\
\partial^2_{a\alpha} F_T(u_{Tb}) & 0 & \partial^2_{ao} F_T(u_{Tb})
\end{pmatrix}.
\]

**Sketch of the proof.** We can decompose Eq $\ref{eq:decomposition}$ into the diagonal and off-diagonal parts which give the equations for $\gamma$ and $\alpha$, respectively. The corresponding decomposition of $\Lambda_{oa}$ is straightforward. To decompose $g'(\eta)$, we write $\eta = \eta_d(\gamma) + \phi(\alpha)$, where
\[
\eta_d(\gamma) := \begin{pmatrix}
\gamma \\
0
\end{pmatrix}
\quad\text{and}\quad
\phi(\alpha) := \begin{pmatrix}
0 & \alpha \\
\alpha^* & 0
\end{pmatrix},
\]
and expand $g'(\eta_d(\gamma) + \phi(\alpha))$ in $\phi(\alpha)$. Then the diagonal part, $g'_{\text{diag}}$, of $g'(\eta)$ is even in $\phi(\alpha)$ while the off-diagonal one, $g'_{\text{off-diag}}$, odd. Hence, the derivatives of $g'_{\text{diag}}$ and $g'_{\text{off-diag}}$
with respect to $\alpha$ and $\gamma$, respectively, vanish at $\alpha = 0$. Moreover, the equation for $\alpha$ (the off-diagonal part of (1.20)) is independent of $a$ and Eq (1.27) is independent of $\alpha$ and therefore their derivative w.r.t. $a$ and $\alpha$, respectively, vanish as well. \hfill \square

Furthermore, a direct computation gives
\[
\bar{J}dF_T(u_{Tb})\bar{J} = d\bar{F}_T(u_{Tb}), \quad \bar{J} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
where
\[
d\bar{F}_T(u_{Tb}) := \begin{pmatrix} \partial^2 a a F_T(u_{Tb}) & 0 \\ 0 & \partial^2 t a F_T(u_{Tb}) \end{pmatrix},
\]
with
\[
d^2 t a F_T(u_{Tb}) = \begin{pmatrix} \partial^2 \gamma a F_T(u_{Tb}) & \partial^2 \gamma t F_T(u_{Tb}) \\ \partial^2 \gamma a F_T(u_{Tb}) & \partial^2 \gamma t F_T(u_{Tb}) \end{pmatrix}.
\]
We expect that for either $T$ or $b$ sufficiently large, $d^2 t a F_T(u_{Tb})$ is positive definite, provided the gauge is fixed appropriately. Using the definition of $F_T$ in (1.33), the non-zero entries in (B.1) are computed as $\partial^2 F_T(u_{Tb}) = T \partial_{\gamma} S(\gamma_{TB})$, where $\partial_{\gamma} S(\gamma_{TB})$ is defined by
\[
\langle \gamma', \partial_{\gamma} S(\gamma_{TB}) \gamma' \rangle := \partial^2 h S(\eta_{TB} + \epsilon d(\gamma'))|_{\epsilon = 0}
\]
and is given in Proposition B.2 below, and $d^2 t a F_T(u_{Tb}) = 2 \nabla a$. Now, we compute $\partial_{\gamma} S(\gamma_{TB})$.

**Proposition B.2.** The hessian operator $\partial_{\gamma} S(\gamma_{TB})$ is given by
\[
\partial_{\gamma} S(\gamma_{TB}) = \frac{1}{T \tanh(h_{TB}^L/T) - \tanh(h_{TB}^R/T)},
\]
where, recall, $h_{TB} := h_{\gamma_{TB}, a}$.

**Proof.** Our starting point in the formula (1.2) and hence we begin with the computation of the term $\int_0^\infty \text{Tr}[(\eta + t)^{-1} \eta'(\eta + t)^{-1} \eta']dt$, with $\eta' = d(\gamma')$, where $d(\gamma')$ denotes the perturbation in $\gamma$ given by
\[
d(\gamma') := \begin{pmatrix} \gamma' & 0 \\ 0 & \gamma' \end{pmatrix}.
\]

First, we recall that $\eta = \eta_{TB}$ and observe that for $\eta' = d(\gamma')$,
\[
\text{Tr}((\eta_{TB} + t)^{-1} \eta'(\eta_{TB} + t)^{-1} \eta') = \text{Tr}((\gamma_{TB} + t)^{-1} \gamma'(\gamma_{TB} + t)^{-1} \gamma') + (1 - \gamma_{TB} + t)^{-1} \gamma'(1 - \gamma_{TB} + t)^{-1} \gamma')
\]
where the last follows from $\text{Tr}(A) = \text{Tr}(\bar{A})$ for self-adjoint operators, and $x$ and $x'$ are regarded as operators acting on $\gamma'$ from the left by multiplying by $\gamma_{TB}$ and from the right, by $\gamma_{TB}$. Performing the integral in $t$, we obtain $S''(\eta', \eta') = -\text{Tr} [\gamma' K'(\gamma')]$, where the operator $K'$ is given by
\[
K' := \frac{\log(x) - \log(x')}{x - x'} + \frac{\log(1 - x) - \log(1 - x')}{(1 - x) - (1 - x')},
\]
with $x$ acting on the left and $x'$ acting on the right. Clearly, $K'$ is identified with $\partial_{\gamma'}S(\gamma Tb)$. Rewrite the operator $K'$ as

$$K' = -\log(x^{-1} - 1) - \log(x'^{-1} - 1)$$

(B.12)

Recalling that $\gamma Tb = g^s(h Tb/T) = (1 + e^{2h Tb/T})^{-1}$, where $h Tb := h_{\gamma Tb, \alpha b}$, we see that

$$K = \frac{1}{T} \left( \frac{h^L_{Tb} - h^R_{TB}}{(1 + e^{h^L_{Tb}/T})^{-1} - (1 + e^{h^R_{TB}/T})^{-1}} \right)$$

(B.13)

which, together with (B.12) and the hyperbolic functions identities, $(1 + e^h)^{-1} = \frac{1}{2}(1 - \tanh h)$ and $(1 + e^{-h})^{-1} = \frac{1}{2}(1 + \tanh h)$, gives (B.6).

One can conduct the proof also in two steps: first consider the off-diagonal perturbations and use the previous results and then the diagonal ones.

**Appendix C. Quasifree reduction**

In general, a many body evolution can be defined on states (i.e. positive linear (‘expectation’) functionals) on the CAR or Weyl CCR algebra $\mathfrak{W}$ over, say, Schwartz space $S(\mathbb{R}^d)$. Elements of this algebra are operators acting on the fermionic/bosonic Fock space $\mathcal{F}$, with annihilation and creation operators $\psi(x)$ and $\psi^*(x)$. Given a quantum Hamiltonian $H$ on $\mathcal{F}$, the evolution of states is given by the von Neumann-Landau equation

$$i\partial_t \omega_t(A) = \omega_t([A, H]), \quad \forall A \in \mathfrak{W}.$$  

(C.1)

(We leave out technical questions such as a definition of $\omega_t([A, H])$ as $[A, H]$ is not in $\mathfrak{W}$.)

Let $N := \int dx \psi^*(x)\psi(x)$ be the particle number operator. We distinguish between

(a) **confined** systems with $\omega(N) < \infty$, as in the case of BEC experiments in traps, and

(b) **thermodynamic** systems with $\omega(N) = \infty$. In the former case the states are given by density operators on $\mathcal{F}$, i.e. $\omega(A) = \text{Tr}(AD)$, where $D$ is a positive, trace-class operator with unit trace on $\mathcal{F}$ (see e.g. [3], Lemma 2.4).

As the evolution (C.1) is practically intractable, one is interested in manageable approximations. The natural and most commonly used ones are one-body ones, which trade the number of degrees of freedom for the nonlinearity.

The most general one-body approximation is given in terms of quasifree states. A quasifree state $\varphi$ determines and is determined by the truncated expectations to the second order:

$$\begin{cases}
\phi(x) := \varphi(\psi(x)), \\
\gamma(x, y) := \varphi[\psi^*(y)\psi(x)] - \varphi[\psi^*(y)]\varphi[\psi(x)], \\
\alpha(x, y) := \varphi[\psi(x)\psi(y)] - \varphi[\psi(x)]\varphi[\psi(y)].
\end{cases}$$

(C.2)

Let $\gamma$ and $\alpha$ denote the operators with the integral kernels $\gamma(x, y)$ and $\alpha(x, y)$. After stripping off the spin components, this definition implies (1.1).

However, the property of being quasifree is not preserved by the dynamics (C.1) and the main question here is how to project the true quantum evolution onto the class of quasifree states.

---

3For a more detailed informal description, see [2].
Following \([2]\), we define self-consistent approximation as the restriction of the many-body dynamics to quasifree states. More precisely, we map the solution \(\omega_t\) of (C.1), with an initial state \(\omega_0\), to the family \(\varphi_t\) of quasifree states satisfying
\[
i\partial_t \varphi_t(A) = \varphi_t([A,H])
\]
for all observables \(A\), which are at most quadratic in the creation and annihilation operators, with an initial state \(\varphi_0\), which is the quasifree projection of \(\omega_0\). We call this map the quasifree reduction of equation (C.1).

Of course, we cannot expect \(\varphi_t\) to be a good approximation of \(\omega_t\), if \(\omega_0\) is far from the manifold of quasifree states.

Evaluating (C.3) for monomials \(A \in \{\psi(x)\psi^*(y), \psi(x)\psi(y), \psi^*(x)\psi^*(y)\psi(x)\psi(y)\}\), yields a system of coupled nonlinear PDE's for \((\phi_t, \gamma_t, \alpha_t)\). For the standard any-body Hamiltonian,
\[
H = \int dx \psi^*(x)h\psi(x) + \frac{1}{2} \int dxdy v(x,y)\psi^*(x)\psi^*(y)\psi(x)\psi(y),
\]
with \(h := -\Delta + V(x)\) acting on the variable \(x\) and \(v\) a pair potential of the particle interaction, defined on Fock space, \(\mathcal{F}\), these give the (time-dependent) Bogolubov-de Gennes (BdG) or the (time-dependent) Hartree-Fock-Bogoliubov (HFB) equations, depending on whether we deal with fermions or bosons. In the former case, the starting Hamiltonian is given by the BCS theory and one takes \(\phi_t(x) = 0\) and \(v(x,y)\) is non-local, in the latter case, \(v(x, y) \equiv v(x - y)\) is a local operator.

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