Robust stability on averaging behaviour of linear time-varying uncertain systems

Ping-Min Hsua and Chun-Liang Linb

aAutomotive Research & Testing Center, Lukang, Changhua 50544, Taiwan; bDepartment of Electrical Engineering, National Chung Hsing University, Taichung 402, Taiwan

(Received 9 August 2014; accepted 9 February 2015)

This research proposes a new type of robust stability – mean robust asymptotic stability for linear time-varying (LTV) uncertain systems and its extension in robust asymptotic stability analysis. The system will feature this kind of stability if its state mean varies towards an equilibrium point. Robust input–output finite-time stability of the LTV uncertain system over a bounded time interval is first analysed by proposing a criterion to characterize the stability condition. The possible cases of steady-state error and state oscillation are then considered and tackled. Finally, some case studies are presented to demonstrate superiority of the proposed work.

Keywords: asymptotic stability; linear systems; robust stability; stability analysis; stability robustness

1. Introduction

Finite-time stability (FTS) was introduced based on the definition that a system with bounded initial states would be finite-time stable if the system states do not exceed a threshold over a bounded time interval. An extension of FTS, named input–output FTS (IO-FTS), has recently been investigated for the systems with norm-bounded input over a bounded time interval (Ma & Jia, 2011). It is said to be IO-FTS if its output does not exceed a predefined threshold over the time interval. Sufficient conditions of IO-FTS for stochastic Markovian jump systems and linear singular systems were proposed by Juan, June, Lin, and Yu (2011) and Ma and Jia (2011). In Ma and Jia (2011), the IO finite-time mean square stability of the stochastic Markovian jump system was studied. In Juan et al. (2011), the IO-FTS for the time-varying singular system was solved.

Amato, Carannante, De Tommasi, and Pironti (2012) summarized the sufficient and necessary conditions of IO-FTS for the linear systems; in particular, Theorem 3 in this work proposed a sufficient condition of stability for the linear time-varying (LTV) systems. Although it works well for the demonstrative case in Amato et al. (2012), this theorem would not be valid for systems having a singular system matrix at certain time instants or intervals. Moreover, robust stability analysis of the LTV system remains to be studied due to the existence of unknown noise input or modelling uncertainty. Robust IO-FTS was introduced in Amato, Cosentino, and De Tommasi (2011) for the linear system, which was treated as robust IO-FTS if it is IO-FTS for admissible uncertainties/disturbances. Nevertheless, the robust IO-FTS system may still suffer from state oscillation, which should not be ignored in practical applications. Robust asymptotic stability analysis (Bilman, 2004; Cheng, Su, & Chien, 2012; Corradini, Cristofaro, & Orlando, 2010; Kelly, 1996; Kerrigan & Maciejowski, 2004; Li, Gao, & Lu, 2011; Li, Wang, & Lu, 2012; Lu & Chen, 2009; Lu & Chen, 2010; Xu & Lam, 2005) is an appropriate method to remedy the above drawback. Although this analysis has been extensively discussed in the manner of linear matrix inequalities (LMI), quantized H∞ approach (Li et al., 2011), time-varying sliding surface (Corradini et al., 2010), new type of stage cost (Kerrigan & Maciejowski, 2004), and Lyapunov approach (Cheng et al., 2012; Li et al., 2012), the issue related to robust asymptotic stability study from the viewpoint of the state’s averaging behaviour is still absent. Especially, such discussions are useful for the network systems, since for that kind of systems, only its averaging behaviour can be technically characterized.

Motivated by the problems depicted above, we propose in this research a novel IO-FTS theorem focusing on the improvement of the result in Amato et al. (2012). For the IO-FTS LTV uncertain system, the issue of stability robustness is discussed, in the sense that it is robustly asymptotically stable with respect to its mean if the state mean values approach the equilibrium state. This contributes the benefit of analysing robust asymptotic stability by identifying the state oscillation and brings the launching point of the idea.

*Corresponding author. Email: chunlin@dragon.nchu.edu.tw

© 2015 The Author(s). Published by Taylor & Francis. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/Licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
2. Problem statement

In the remaining parts, $O$ represents a zero matrix with appropriate dimensions, $0$ denotes a zero vector, and $\lambda_i(M(t))$ denotes the $i$-th eigenvalue of the matrix $M(t)$ at $t$. Moreover, under the assumption that the ergodic hypothesis holds, the expected value of the random vector $d(t) = [v_{dij}(t)]_{m \times 1}$ is defined as $d_m(t) = E[d(t)] = \left[\int_0^t v_{dij}(\tau) d\tau/(t - t_0)\right]_{m \times 1}$.

2.1. System description

Consider the LTV uncertain system described by

$$\begin{align*}
\dot{x}(t) &= A(t)x(t) + B_1(t)u(t) + B_2(t)w(t), \\
y(t) &= C(t)x(t),
\end{align*}$$

(1)

where $x(t) \in \mathbb{R}^n$, the modelling uncertainty $w(x(t), u(t), d(t)) = [v_{dij}(t)]_{m \times 1}$, the control input $u(t) \in \mathbb{R}^u$, $B_1(t) \in \mathbb{R}^{n \times u}$, $B_2(t) = [v_{bij}(t)]_{n \times m}$, the system output $y(t) \in \mathbb{R}^q$, $A(t) = [v_{dij}(t)]_{n \times n}$, and $C(t) \in \mathbb{R}^{q \times n}$. Referring to Amato et al. (2012), we suppose that (i) $u(t) \in U_2(\Omega, R(\cdot)) \equiv \{u(\cdot) \in L_2(\Omega) : \|u\|_{2,R} \leq 1\}$ where $\Omega \triangleq [t_0, T]$, $L_2(\Omega) = \left\{l(\cdot) \left| \left(\int_\Omega \|l(t)\|^2 dt\right)^{1/2} < \infty \right. \right\}$, $\|l\|$ denotes the Euclidean norm of the vector $l$ with appropriate dimensions, $\|u\|_{2,R} \triangleq \left(\int_0^T u^T(t)R(t)u(t)dt\right)^{1/2}$, and $R(\cdot)$ is the symmetric norm-bounded continuous positive definite matrix-valued function; and (ii) $w(t) \in U_2(\Omega, R(\cdot))$. If $u(t) = K(t)x(t)$ where $K(\cdot)$ is the piecewise continuous matrix-valued function, the closed-loop system of (1) is

$$\begin{align*}
\dot{x}(t) &= A_{cl}(t)x(t) + B_2(t)w(t), \\
y(t) &= C(t)x(t),
\end{align*}$$

(2)

where $A_{cl}(t) = A(t) + B_1(t)K(t) = [v_{dij}(t)]_{n \times n}$.

2.2. Introduction of robust IO-FTS

The idea of robust IO-FTS (2) is defined below:

DEFINITION 1 (Amato et al., 2011): Given $U_2$, $\Omega$, and a continuous positive definite matrix-valued function $Q(\cdot)$, the system (2) is said to be robust IO-FTS corresponding to $(U_2, Q(\cdot), \Omega)$ if $y^T(t)Q(t)y(t) < 1$, $\forall t \in \Omega$ for any admissible $w(t)$.

Theorem 3 in Amato et al. (2012) can be applied to guarantee robust IO-FTS of (2) based on Definition 1. It is stated as

THEOREM 1 (Amato et al., 2012): Consider the system (2). Given $U_2(\Omega, R(\cdot)), Q(\cdot), \text{and} \Omega$. The system is robust IO-FTS with respect to $(U_2, Q(\cdot), \Omega)$ if there is $(K(\cdot), P(\cdot) > 0)$ satisfying

$$\begin{align*}
\begin{bmatrix}
P(t) + A_{cl}(t)P(t) + P(t)A_{cl}(t) \\
B_2^T(t)P(t)
\end{bmatrix} < 0,
\end{align*}$$

(3)

$$
P(t) > C^T(t)Q(t)C(t),$$

(4)

over $\Omega$.

In Amato et al. (2012), the LMI (3) has been equalized to

$$\begin{align*}
\dot{P}(t) + P(t)A_{cl}(t) + A_{cl}(t)\dot{P}(t) + P(t)B_2(t)R^{-1}(t)B_2^T(t)P(t) + Q_p(t) = 0,
\end{align*}$$

(5)

by using Schur complements, where $Q_p(t) > 0, \forall t \in \Omega$.

2.3. Problem statement

Theorem 1, requiring $P(t)$ satisfying (4) and (5) over $\Omega$, is invalid for (2) because of violation of (5) over $\Omega$ if it features the singular $A_{cl}(t)$ at some $t \in \Omega$ (such as a linearized system of variable stiffness joint dynamics (Choi, Hong, Lee, Kang, & Kim, 2011) around its equilibrium point). More precisely, (5) is a Lyapunov matrix differential equation if $B_2(t) = 0$; it does not uniquely admit

$$P(t) = \Phi_{cl}^{-T}(t, t_0)P(t_0)\Phi_{cl}^{-1}(t, t_0)$$

$$- \int_0^t \Phi_{cl}^{-T}(t, \tau)Q_p(\tau)\Phi_{cl}(t, \tau) d\tau,$$

(6)

proposed by Abou-Kandil, Freiling, Ionesco, and Jank (2003) and Davis, Gravagne, Marks, and Ramos (2010) at these instances with the state transition matrix $\Phi_{cl}(t, t_0)$ of (2) since $\Phi_{cl}^{-1}(t, t_0)$ does not exist at the same times.

To illustrate, without loss of generality, let $A_{cl}(t) \equiv A_a(t) - A_{\beta}(t)$ where $A_a(t)$ is non-singular at $t \in U_t \equiv \{t \in \Omega \mid A_{cl}(t) \text{ is singular at } t\}$ and $A_a(t) \neq A_{\beta}(t), \forall t \in U_t$. By contradiction, we suppose that $\Phi_{cl}^{-1}(t, t_0)$ exists at all $t \in U_t$. Furthermore,

$$\det(\Phi_{cl}(t, t_0)) = \det(A_{cl}(t)) \det(\Phi_{cl}(t, t_0)) = 0, \forall t \in U_t,$$

(7)

where $\det(A_{cl}(t)) = 0$ over $U_t$. Substituting $A_{cl}(t) = A_a(t) - A_{\beta}(t)$ into (7) gives

$$\det(A_a(t)\Phi_{cl}(t, t_0) - A_{\beta}(t)\Phi_{cl}(t, t_0))$$

$$= \det(A_a(t)) \det(\Phi_{cl}(t, t_0)) \det(I - A_a^{-1}(t)A_{\beta}(t)) = 0,$$

(8)

$\forall t \in U_t$. Hence, $\Phi_{cl}^{-1}(t, t_0)$ does not exist at all $t \in U_t$. Nevertheless, the considered system may still be IO-FTS with respect to $(U_2, Q(\cdot), \Omega)$.
3. Main results

This section deals with three issues: (i) robust IO-FTS by improving Theorem 1; (ii) robust asymptotic stability with respect to its mean; and (iii) robust asymptotic stability. The first issue requires a redefinition of $U_2$ with $U_{2b}(T) \equiv \{u(\cdot) \in L_2(\Omega) : \|u\|_{2,\Omega} \leq \varepsilon_{b0}\}$ where $\varepsilon_{b0} \equiv (\int_0^T u^T(t)u(t)dt)^{1/2}$. $U_2(\Omega, R(\cdot)) \subseteq U_{2b}(T)$ under $\varepsilon_{b0} \equiv 1/\gamma_{0}^{1/2}$ since $\|u\|_{2,\Omega} \leq \|u\|_{2,R} \leq 1$ where $u^T(t)R(t)u(t) \equiv \gamma(t)u^T(t)u(t)$, $0 < \gamma(t) < \infty$, $\forall t \in \Omega$, and $\gamma_0 \equiv \min_{t \in \Omega} \gamma(t)$. The next discussion is fulfilled by assuming $w(t) \in U_{2b}(T)$.

### 3.1. IO-FTS analysis of the LTV system

**Theorem 2** Consider the system (2). Suppose that $Q_1(t) \geq Q_2(t)$, and $Q_3(t) > 0$, $\forall t \in \Omega$. The system is robust IO-FTS with respect to $(U_{2b}(T), \hat{Q}(\cdot), \Omega)$ where $\hat{Q}(t) = Q_2(t)/[\varepsilon_{b0}^2 + x^T(t_0)\hat{P}(t_0)x(t_0)]$ if there is $(K(\cdot), A_1(\cdot), A_2(\cdot), \hat{P}(\cdot))$ ensuring

$$
A(t) + B_2(\cdot)K(t) = A_1(t) - A_2(t),
$$

$$
A_1^T(\cdot)\hat{P}(\cdot) + \hat{P}(\cdot)A_1(\cdot) = -Q_1(\cdot), \quad \tau \in [t_0, t],
$$

$$
\hat{P}(\cdot) = A_1^T(\cdot)\hat{P}(\cdot) + \hat{P}(\cdot)A_2(\cdot) - \hat{P}(\cdot)
$$

$$
B_2(\cdot)B_2^T(\cdot)\hat{P}(\cdot) + Q_2(\cdot), \quad \tau \in [t_0, t],
$$

$$
\hat{P}(\cdot) \geq C^T(\cdot)Q_3(\cdot)C(\cdot),
$$

where $\hat{P}(\cdot) > 0$, $\forall t \in \Omega$.

**Proof** Set $V(t) = x^T(\cdot)\hat{P}(\cdot)x(\cdot)$ then

$$
\dot{V}(t) = x^T(\cdot)[A_1^T(\cdot)\hat{P}(\cdot) + \hat{P}(\cdot)A_1(\cdot) + \hat{P}(\cdot)]
$$

$$
\cdot x(\cdot) + w^T(\cdot)
$$

$$
B_2^T(\cdot)\hat{P}(\cdot)x(\cdot) + x^T(\cdot)\hat{P}(\cdot)B_2(\cdot)w(\cdot),
$$

where $A_1^T(\cdot) = A_1(\cdot) - A_2(\cdot)$ due to (10). From (11)–(12), it follows

$$
\dot{V}(t) = -x^T(\cdot)[Q_1(\cdot) - Q_2(\cdot)]x(\cdot) + w^T(\cdot)w(\cdot)
$$

$$
- a^T(\cdot)a(\cdot) < w^T(\cdot)w(\cdot),
$$

over $t \in [t_0, t]$ where $Q_1(\cdot) - Q_2(\cdot) \geq 0$ and $a(\cdot) = w(\cdot) - B_2^T(\cdot)\hat{P}(\cdot)x(\cdot)$. Integrating (15) over $[t_0, t]$ yields

$$
x^T(t)\hat{P}(t)x(t) - x^T(t_0)\hat{P}(t_0)x(t_0) \leq \int_{t_0}^{t} w^T(\cdot)w(\cdot)d\tau
$$

$$
\leq \int_{\Omega} w^T(\cdot)w(\cdot)d\tau \leq \varepsilon_{b0}^2,
$$

since $w^T(\cdot)w(\cdot) \geq 0$ and $w(\cdot) \in U_{2b}(T)$. That is, $x^T(t)\hat{P}(t)x(t) < \varepsilon_{b0}^2 + x^T(t_0)\hat{P}(t_0)x(t_0)$. From (13), $y^T(t)Q_3(\cdot)y(t) < \varepsilon_{b0}^2 + x^T(t_0)\hat{P}(t_0)x(t_0)$. That is $y^T(t)\hat{Q}(\cdot)y(t) < 1$ where $\hat{Q}(t) = Q_3(t)/[\varepsilon_{b0}^2 + x^T(t_0)\hat{P}(t_0)x(t_0)]$. According to Definition 1, the system (2) is thus possessing robust IO-FTS with respect to $(U_{2b}(T), \hat{Q}(\cdot), \Omega)$.

**Remark 1** The pair $(K(\cdot), A_1(\cdot), A_2(\cdot), \hat{P}(\cdot))$ can be decided via five steps: (i) determine $A_1(\cdot)$ as a stable upper triangular matrix over $\Omega$ so that (11) must admit $\hat{P}(\cdot)$; (ii) evaluate $A_2(\cdot)$ by substituting $\hat{P}(\cdot)$ into (12); (iii) compute $K(\cdot)$ via (10); (iv) verify (13) with the obtained $\hat{P}(\cdot)$; and (v) repeat Step (i) by setting another $A_1(\cdot)$ such that $\hat{P}(\cdot)$ has larger Re $[\lambda_i(\hat{P}(\cdot))]$ for all $i$ if (13) does not work.
Comparing Theorem 1 with Theorem 2 when \( R(t) = I \), the matrix inequality (3) can be derived by combining (11) and (12) while (13) remains invariant as (4). To illustrate, combining (11) and (12) yields
\[
\dot{\bar{P}}(t) + A_{d2}^T(t)\dot{\bar{P}}(t) + \bar{P}(t)A_{d2}(t) + \dot{\bar{P}}(t)B_{2}(t)B_{2}^T(t)\bar{P}(t) = -[Q_1(t) - Q_2(t)],
\]
(17)
where \( Q_1(t) - Q_2(t) > O \) is determined for the system (2) with the non-singular \( A_{d2}(t) \) over \( \Omega \). Equation (17) is consistent with (5); as a result, the sufficient condition in Theorem 1 becomes a special case to Theorem 2 when \( R(t) = I \).

### 3.2. Robust asymptotic stability of IO-FTS systems with respect to its mean

This section focuses on the steady-state error rejection of (2) with \( \dot{\bar{P}}_{2}(t) = \left[ \int_{t_0}^{t} \nu_{bij}(\tau) d\tau / (t - t_0) \right]_{m \times n} \neq O \) and \( w(t) \in \tilde{U}_2(T) = \{ u(t) : \| u(t) \|_2 \leq \varepsilon_0, t \in [t_0, T] \} \).

**Definition 2** The system (2) is said to be robustly asymptotically stable with respect to its mean if \( \lim_{t \to \infty} E[x(t)] = 0 \).

The major difference between robustly asymptotic stability and that in Definition 2 is that the former investigates actual state response while the proposed stability specified the averaging state behaviour. The definition is useful for stability characterization of the local network system (Misra, Gong, & Towsley, 2000), whose averaging dynamic behaviour can be illustrated by the following fluid-flow model (Holllot, Misra, & Towsley, 2002):
\[
\dot{\bar{q}}(t) = \bar{w}(t) - \bar{q}(t)/R_{a}(t) - C_{p},
\]
where \( R_{a}(t) = a + \bar{q}/C_{p} \) denotes the averaging round-trip time, \( a \) refers to the propagation delay, \( \bar{q}/\bar{w} \) represents the averaging queue length/window size, \( C_{p} \) is the capacity, \( nf \) is the number of Transmission Control Protocol flows, and \( p \) is the dropping probability. Permuting linearization with respect to (18) around its equilibrium point \((w_0, q_0, p_0)\) yields the linearized model of the following form:
\[
\Delta \dot{w} = -\frac{nf}{R_{a}^{2}C_{p}}\Delta \bar{w} - \frac{nf}{R_{a}^{2}C_{p}}\Delta \bar{w}(t - R_{a}) - \frac{R_{0}C_{p}^{2}}{2nf^{2}}\Delta p(t - R_{a}) - \frac{\Delta \bar{q}}{R_{a}^{2}C_{p}} + \Delta \bar{q}(t - R_{a}),
\]
\[
\Delta \dot{\bar{q}} = \frac{nf}{R_{0}}\Delta \bar{w} - \frac{\Delta \bar{q}}{R_{0}},
\]
(19)
where \( R_0 = a + q_0/C_{p}, \Delta \bar{w} = \bar{w} - w_0, \Delta q = \bar{q} - q_0, \) and \( \Delta p = p - p_0 \). The linearized system describing the behaviour of \((q, w)\) around \((w_0, q_0, p_0)\) is focused here with \( q(t)/w(t) \) being the real queue length/window size. It can be deduced by applying (19) and extended network disturbances (END) (Hsu & Lin, 2012). Using \( \Delta p(t) = K(t)x(t) \), the closed-loop system can be formulated in the form of (2) where \( A_{d2}(t) = A_{e} + B_{e}K(t), B_{2}(t) = I_{2}, C(t) = I_{2}, A_{e} = \left[ \begin{array}{cc} -2nf/R_{0}^{2}C_{p} & 0 \\ nf/R_{0} & -1/R_{0} \end{array} \right], B_{e} = \left[ \begin{array}{cc} -R_{0}C_{p}^{2}/2nf^{2} \\ 0 \end{array} \right] \).

The model (2) in (C1) would be robustly asymptotically stable with respect to its mean if \( \lim_{t \to \infty} E[B_{2}(t)w(t)] = 0 \).

**Lemma 1** For \( w(t) = E[w(t)] \) with bounded \( \| w(t_0) \|_2 \), \( \lim_{t \to \infty} w(t_0) = 0, \text{ if and only if, } w(t) \in \tilde{U}_2(\infty) \).

**Proof** For the sufficient condition, one has \( w(t) = \left[ \int_{t_0}^{t} \nu_{wij}(\tau) d\tau / (t - t_0) \right]_{m \times 1} \in \tilde{U}_2(\infty) \text{ since } \| w(t) \|_2 \leq \int_{t_0}^{t} \| w(\tau) \|_2 d\tau / (t - t_0) \leq \varepsilon_0, \forall t \in [t_0, \infty) \) where \( \| w(t) \|_2 \leq \varepsilon_0 \) over \( \Omega \) and \( w(t) \in \tilde{U}_2(\infty) \). Since \( \dot{\bar{w}} = [w(t) - w_0] / C_p \) and \( \Delta \bar{w} = \bar{w} - w_0 \), the proof is completed.
Lemma 1. Proof

\[
\lim_{t \to \infty} \|\dot{w}_m(t)\|_2 \leq \frac{\lim_{t \to \infty} 2\varepsilon_0}{t - t_0} = 0,
\]

(20)

where \( \|w(\infty)\|_2 \leq \varepsilon_0 \) and \( \|w_m(\infty)\|_2 \leq \varepsilon_0 \) because \( w(t), w_m(t) \in \tilde{U}_2(\infty) \). Clearly, (20) reveals \( \lim_{t \to \infty} \|\dot{w}_m(t)\|_2 = 0 \). As a result, \( \lim_{t \to \infty} \|\dot{w}_m(t)\|_2 = 0 \). For the necessary part with \( \lim_{t \to \infty} \|\dot{w}_m(t)\|_2 = 0 \), this reveals that \( \|\dot{w}_m(t)\|_2 \) is bounded and inversely proportional to \( t \). Hence, without loss of generality, \( \|\dot{w}_m(t)\|_2 = \varepsilon_1(t)(t - t_0) \) and \( (\varepsilon_1(t), \dot{\varepsilon}_1(t_0)) = (0, 0) \). Substituting it into (20) gives \( \varepsilon_1(t) = \|w(t) - w_m(t)\|_2 \) that implies

\[
\varepsilon_1(t) \geq \|w(t)\|_2 - \|w_m(t)\|_2,
\]

(21)

where \( \|w_m(t)\|_2 \) is bounded over \([t_0, \infty)\) due to the invariant \( \lim_{t \to \infty} \|w_m(t)\|_2 \leq \infty \). That is, \( \|w_m(t)\|_2 = \varepsilon_2(t) < \infty \), \( \forall t \in [t_0, \infty) \) and \( \varepsilon_2(\infty) = 0 \). Substituting \( \|w_m(t)\|_2 = \varepsilon_2(t) \) into (21) gives \( \|w(t)\|_2 \leq \varepsilon_1(t) + \varepsilon_2(t) < \infty \) where there is a finite constant \( \varepsilon_{30} \in \mathbb{R}^+ \) such that \( \|w(t)\|_2 \leq \varepsilon_1(t) + \varepsilon_2(t) < \varepsilon_{30} \), \( \forall t \in [t_0, \infty) \). Hence \( w(t) \in \tilde{U}_2(\infty) \).

In addition to \( \tilde{U}_2(\infty) \), Lemma 1 can be extended to the case of

\[
\tilde{U}(T, F, \eta) = \{F(\cdot)\|F(\cdot)\|_2 \leq \varepsilon_{30}, \forall t \in [T, T] \}
\]

(22)

where \( \varepsilon_{30} < \infty \), \( F(t) = [v_{ij}(t)]_{n_1 \times n_2} \), the induced norm \( \|F(t)\|_2 = \sup_{t \in [T, T]}\|F(t)\|_2 \), \( \varepsilon_1(t) \in \mathbb{R}^n \).

**Lemma 2**

Given \( \tilde{U} \) and \( \tilde{F}(t) \)

\[
= \left[ \int_{t_0}^t v_{ij}(\tau) d\tau / (t - t_0) \right]_{n_1 \times n_2}
\]

with bounded \( \|\tilde{F}(t)\|_2 \), \( \lim_{t \to \infty} \tilde{F}(t) = O, \) if and only if \( \dot{F}(t) \in \dot{U}(\infty, F, \eta) \).

**Proof** Follow the similar process of the proof of Lemma 1. ■

We now present the following result with \( \tilde{F}_1(t) = \left[ \int_{t_0}^t v_{ij}(\tau) d\tau / (t - t_0) \right]_{n_1 \times n_2}, F_1(t) = [v_{ij}(t)]_{n_1 \times n_2}, w_{1m}(t) = E[w_1(t)] = [v_{ij}(t)]_{n_1 \times 1} \), and the random vector \( w_1(t) \in \mathbb{R}^{n_2 \times 1} \).

**Lemma 3**

Given \( \lim_{t \to \infty} E[F_1(t)(w_1(t) - w_{1m}(t))] = \alpha_0 \), \( \lim_{t \to \infty} E[F_1(t)w_1(t)] = \tilde{F}_1(\infty)w_{1m}(\infty) + \alpha_0 \), if \( F_1(t) \in \tilde{U}(\infty, F_1, w_1) \) and \( w_1(t) \in \tilde{U}_2(\infty) \).

**Proof**

Let \( F_1(t)w_{1m}(t) = [v_{ij}(t)]_{n_1 \times 1} \),

\[
\lim_{t \to \infty} \left[ \int_{t_0}^t v_{ij}(\tau) d\tau / (t - t_0) \right]_{n_1 \times 1} - \tilde{F}_1(t)w_{1m}(t)
\]

(23)

where

\[
\tilde{F}_1(t)w_{1m}(t) = \sum_{k=1}^{n_2} \int_{t_0}^t v_{ijk}(\tau)[v_{w_{1k}(\tau)}(t) - v_{w_{1k}(\infty)}] d\tau / (t - t_0) \]

and \( w_{1m}(\infty) = [v_{ij}(\infty)]_{n_2 \times 1} \) is constant from Lemma 1 due to \( w_1(t) \in \tilde{U}_2(\infty) \). This reveals \( v_{ij}(\tau)[v_{w_{1k}(t)}(t) - v_{w_{1k}(\infty)}] \to 0 \) in (23) as \( t \to \infty \) for \( i = 1 \sim n_2 \) since \( |v_{ij}(\tau)| < \infty \) over \([t_0, t] \), upon \( F_1(t) \in \tilde{U}(\infty, F_1, w_1) \). In other words, \( \lim_{t \to \infty} \tilde{F}_1(t)w_{1m}(t) = 0 \) because \( \lim_{t \to \infty} \int_{t_0}^t v_{ij}(\tau)[v_{w_{1k}(t)}(t) - v_{w_{1k}(\infty)}] d\tau / (t - t_0) = 0 \) if \( \int_{t_0}^t v_{ij}(\tau)[v_{w_{1k}(t)}(t) - v_{w_{1k}(\infty)}] d\tau < \infty \); otherwise,

\[
\lim_{t \to \infty} \int_{t_0}^t v_{ij}(\tau)[v_{w_{1k}(t)}(t) - v_{w_{1k}(\infty)}] d\tau / (t - t_0)
\]

(24)

Therefore, \( \lim_{t \to \infty} \int_{t_0}^t v_{ij}(\tau) d\tau / (t - t_0) \)

= \( F_1(\infty)w_{1m}(\infty) \) from (23). As a result,

\[
\lim_{t \to \infty} E[F_1(t)w_1(t)] = \lim_{t \to \infty} \left[ \int_{t_0}^t v_{ij}(\tau) d\tau / (t - t_0) \right]_{n_1 \times 1}
\]

and \( \lim_{t \to \infty} E[F_1(t)w_1(t)] = \tilde{F}_1(\infty)w_{1m}(\infty) + \alpha_0 \),

(25)

where from Lemma 1, \( \alpha_0 \) is constant since \( F_1(t)[w_1(t) - w_{1m}(t)] \in \tilde{U}_2(\infty) \), in which \( w_{1m}(t) \in \tilde{U}_2(\infty) \). When \( w_1(t) \in \tilde{U}_2(\infty) \),

Lemma 3 shows that \( E[F_1(t)w_1(t)] = \tilde{F}_1(t)w_{1m}(t) - \alpha_0 \) is proportional to \( \tilde{F}_1(t) \) and \( \dot{w}_{1m}(t) \); consequently, let

\[
\frac{dE[F_1(t)w_1(t)]}{dt} = \frac{E[F_1(t)w_1(t)] - E[F_1(t)w_1(t)]}{t + \alpha_0}
\]

(26)

gives

\[
\lim_{t \to \infty} \frac{dE[F_1(t)w_1(t)]}{dt}
\]

\[
F_1(t)w_1(t) - \tilde{F}_1(t)w_{1m}(t) - \tilde{F}_1(t)\dot{w}_{1m}(t) - \tilde{F}_1(t)\dot{w}_{1m}(t) \]

\[
= \lim_{t \to \infty} \left[ \frac{\dot{F}_1(t) - \frac{\sum_{k=1}^{n_2} \dot{w}_{1k}(t)}{t - t_0}}{t - t_0} \right]_{n_1 \times 1}
\]

(27)

where \( \dot{F}_1(t) = \tilde{F}_1(t)(t - t_0) + \tilde{F}_1(t) \) and \( w_1(t) = \dot{w}_{1m}(t)(t - t_0) + w_{1m}(t) \).
Theorem 3 Consider the system \((2)\) which satisfies (C2). Suppose \(\alpha _1 > 0, B_2(t) \in \hat{U}(\infty , B_2, w), A_{cl}(t) = \int _{t_0}^{\infty } T_v (\tau ) d\tau / (t - t_0)\), \(\tilde{A}_T(t) = \left[ \begin{array}{c} O \\ I \\ A_{cl}(t) \end{array} \right] \), and \(Q_{m1}(t) = Q_{m2}(t) \geq \Omega _{m2}(t) = Q_{m2}(t)\), \(\forall t \in [t_1, \infty )\). This system is robustly asymptotically stable with respect to its mean, if there are \(K(\cdot ), \tilde{A}_d(\cdot ), \tilde{A}_b(\cdot ), \) and \(\tilde{P}_1(\cdot ) = \tilde{P}_1(\cdot ) > 0\) satisfying
\[
\tilde{A}_T(K(t)) = \tilde{A}_d(t) - \tilde{A}_b(t),
\]
\[
\tilde{A}_T^T(t)\tilde{P}_1(t) + \tilde{P}_1(t)\tilde{A}_T(t) = -Q_{m1}(t),
\]
\[
\tilde{P}_1(t) = \tilde{A}_T^T(t)\tilde{P}_1(t) + \tilde{P}_1(t)\tilde{A}_T(t) - \alpha _1\tilde{P}_1(t) + Q_{m2}(t),
\]
\(\forall t \in [t_1, \infty )\) and \(\tilde{A}_T(\infty )\) is full rank.

Proof Consider the associated model
\[
E[\dot{x}(t)] = E[A_{cl}(t)x(t) + B_2(t)w(t)].
\]

It is desirable to introduce a transformation to convert it into the form without uncertainties by Lemmas 1–3. Expanding (31) at \(x(t_0)\) gives
\[
E[\dot{x}(t)] = \frac{[x(t) - x_m(t)]}{t - t_0} + \frac{[x_m(t) - x(t_0)]}{t - t_0}
\]
\[
= \dot{x}_m(t) + \frac{[x_m(t) - x(t_0)]}{t - t_0}
\]
\[
= E[A_{cl}(t)x(t)] + E[B_2(t)w(t)].
\]

Then,
\[
\dot{x}_m(t) = E[A_{cl}(t)x(t)] + E[B_2(t)w(t)] - \frac{x_m(t)}{t - t_0} + \frac{x(t_0)}{t - t_0}.
\]

The system would approach to a limit system (Lee, Liaw, & Chen, 2001) as \(t \to \infty\). First, since \(B_2(t) \in \hat{U}(\infty , B_2, w)\) and \(w(t) \in \hat{U}_2(\infty )\), expanding \(\lim _{t \to \infty } E[B_2(t)w_m(t)]\) in (32) and using Lemma 3 gives
\[
\lim _{t \to \infty } E[B_2(t)w_m(t)] = \hat{B}_2(\infty )w_m(\infty ) + \phi _0,
\]
where \(\phi _0\) is a constant. Substituting (33) into (32) and performing differentiation gives
\[
\lim _{t \to \infty } \dot{x}_m(t) = \lim _{t \to \infty } \left\{ \frac{dE[A_{cl}(t)x(t)]}{dt} + \hat{B}_2(t)w_m(t) \right\} + \lim _{t \to \infty } \left\{ \dot{x}_m(t) - \frac{x_m(t)}{t - t_0} \right\} + \frac{x_m(t)}{(t - t_0)^2}
\]

where \(\dot{w}_m(\infty ) = 0\) and \(\hat{B}_2(\infty ) = 0\) (from Lemmas 1 and 2), because \(w(t) \in \hat{U}_2(\infty )\) and \(B_2(t) \in \hat{U}(\infty , B_2, w)\). Second, \(dE[A_{cl}(t)x(t)]/dt\) in (34) is expanded as (27) where (28) gives \(\lim _{t \to \infty } \tilde{x}_m(t) = \lim _{t \to \infty } \tilde{A}_T(t)x_m(t)\) where \(\tilde{A}_T(t) = \left[ \begin{array}{c} O \\ I \\ A_{cl}(t) + I/(t - t_0)^2 \end{array} \right] \), as \(t \to \infty\) \(\tilde{x}_m(t) = \tilde{x}_m(t)\) and \(\tilde{A}_T(\infty )\) is full rank. This reveals that (31) reaches a limit which is defined by
\[
\dot{x}_m(t) = \tilde{A}_T(t)x_m(t),
\]

which has the equilibrium state \(\tilde{x}_m = 0\).

Next, consider a Lyapunov function candidate given by
\[
\dot{V}_m(t) = \tilde{x}_m^T(t)\tilde{P}_1(t)\tilde{x}_m(t).
\]

Expanding (31) at \(x(t_0)\) gives
\[
\dot{V}_m(t) = \tilde{x}_m^T(t)\tilde{P}_1(t)\tilde{x}_m(t) - \alpha _1\tilde{x}_m(t)\tilde{P}_1(t)\tilde{x}_m(t)
\]

for \(t \in [t_1, \infty )\). As a result, \(2\) converges to (35). That is, the system is robustly asymptotically stable with respect to its mean.

The quaternion set \((K(\cdot ), \tilde{A}_d(\cdot ), \tilde{A}_b(\cdot ), \tilde{P}_1(\cdot ))\) can be designed by following the steps depicted in Remark 1 where \(\tilde{A}_d(\cdot ), \tilde{A}_b(\cdot ), \) and \(\tilde{P}_1(\cdot )\) correspond, respectively, to \(A_d(\cdot ), A_b(\cdot ),\) and \(\tilde{P}_1(\cdot )\) (10)–(12) are substituted by (28)–(30), respectively.

Theorem 3 presents a function for identifying the existence of the steady-state error. It is suitable for theoretical analysis of the results in Example 2 (Hollot et al., 2002). This work did not provide explanations why the system was not suffering from steady-state error under the treatment \((\hat{w}, \hat{q}) = (w, q)\) for realization of the controller in terms of \((\hat{w}, \hat{q})\). Theorem 3 can be used to enhance the required analysis. The network work system (2), equipped with the congestion controller (Hollot et al., 2002) in terms of \((w, q)\) instead, in Section 3.2 will not reveal steady-state error if it satisfies Theorem 3. This reveals advantage of Theorem 3.
3.3. Robust asymptotic stability analysis via behaviour identification

This section deals with oscillation behaviour of the system (2) where the system has been robustly asymptotically stabilized with respect to its mean by determining \( K(\cdot) \) according to Theorem 3. Without loss of generality, its solution is denoted by \( x(t) \equiv r(t) + x_m(t) \) where \( x_m(\infty) = 0 \) and \( E[r(t)] = 0 \) over \([t_0, \infty)\).

**Theorem 4** Consider the robustly asymptotically stabilized system (2) with respect to its mean. Suppose \( \alpha_2 > 0 \), \( \dot{A}_r(t) = \begin{bmatrix} 0 & 1 \\ \dot{A}_{cl}(t) + A_{cl}(t) & A_{cl}(t) \end{bmatrix} \) with \( \dot{A}_r(\infty) \) being full rank, \( \dot{A}_{cl}(t) \in \dot{U}(\infty, \dot{A}_{cl}, x_m) \), \( \dot{A}_{cl}(t) \in \dot{U}(\infty, \dot{A}_{cl}, x_m) \), and \( Q_{r1}(t) = Q_{r1}^T(t) \geq Q_{r2}(t) = Q_{r2}^T(t) \) for all \( t \in [t_1, \infty) \). This system is robustly asymptotically stable, if there are \( \dot{A}_r(t) \) and \( P_2(t) = P_2(t) > 0 \) satisfying

\[ [\dot{A}_r(t) + \dot{A}_{cl}(t)]^T P_2(t) + P_2(t) [\dot{A}_r(t) + \dot{A}_{cl}(t)] = -Q_{r1}(t), \quad \forall t \in [t_1, \infty), \]

(38)

\[ \dot{P}_2(t) = \dot{A}_r(t)^T \dot{P}_2(t) + \dot{P}_2(t) \dot{A}_r(t) - \alpha_2 \dot{P}_2(t) + Q_{r2}(t), \quad \forall t \in [t_1, \infty). \]

(39)

**Proof** Substituting \( x(t) = r(t) + x_m(t) \) into (2) yields

\[ \dot{r}(t) = A_{cl}(t) r(t) + A_{cl}(t) x_m(t) + \dot{x}(t) - A_{cl}(t) \dot{x}(t) = A_{cl}(t)(r(t) - A_{cl}(t)x_m(t)(t - t_0) + \dot{x}(t)(t - t_0) + \dot{x}_m(t), \]

(40)

where \( x_m(t) = x(t) - \dot{x}_m(t)(t - t_0) \). Then

\[ \lim_{t \to \infty} \dot{r}(t) = \lim_{t \to \infty} \{ A_{cl}(t) r(t) - A_{cl}(t) x_m(t)(t - t_0) + \dot{x}(t)(t - t_0) + \dot{x}_m(t) \} \]

(41)

where \( \lim_{t \to \infty} \dot{x}_m(t) = \dot{A}_{cl}(\infty) x_m(\infty) + A_{cl}(\infty) \dot{x}_m(\infty) \) from (35), \( \dot{x}_m(\infty) = 0 \), and \( A_{cl}(t) = [A_{cl}(t) - \dot{A}_{cl}(t)]/(t - t_0) \). In other words,

\[ \lim_{t \to \infty} \dot{r}(t) = \lim_{t \to \infty} \{ A_{cl}(t) r(t) + A_{cl}(t) \dot{r}(t) + [A_{cl}(t) - \dot{A}_{cl}(t)] x_m(t) + [A_{cl}(t) - \dot{A}_{cl}(t)] \dot{x}_m(t) \}

(42)

\[ = \lim_{t \to \infty} \{ A_{cl}(t) r(t) + A_{cl}(t) \dot{r}(t) + \dot{A}_{cl}(t) r(t) \}, \]

where \( \dot{x}_m(t) = r(t)/(t - t_0) \), \( \dot{A}_{cl}(t) = [A_{cl}(t) - \dot{A}_{cl}(t)]/(t - t_0) \), and \( [A_{cl}(\infty) - \dot{A}_{cl}(\infty)] x_m(\infty) = 0 \) since \( x_m(\infty) = 0 \) and \( A_{cl}(t) - \dot{A}_{cl}(t) \in \dot{U}(\infty, A_{cl} - \dot{A}_{cl}, x_m) \) with \( A_{cl}(t) \in \dot{U}(\infty, A_{cl}, x_m) \) and \( \dot{A}_{cl}(t) \in \dot{U}(\infty, \dot{A}_{cl}, x_m) \). Since \( A_{r}(\infty) \) is full rank, the limit system \( \dot{x}_r(t) = \dot{A}_r(t) x_r(t) \) has an equilibrium point \( x_r = 0 \) where \( x_r(t) = [r(t) - r(t)]^T \).

Given the equilibrium point, the Lyapunov function candidate \( \dot{V}_r(t) = x_r^T(t) \dot{P}_2(t) x_r(t) \) is defined where \( \dot{P}_2(t) > 0 \) for all \( t \in [t_1, \infty) \). From which one has

\[ \dot{V}_r(t) = x_r^T(t) [\dot{A}_r(t) \dot{P}_2(t) + \dot{P}_2(t) \dot{A}_r(t) + \dot{P}_2(t)] x_r(t) \]

(43)

\[ = -x_r^T(t)[Q_{r1}(t) - Q_{r2}(t)] x_r(t) - \alpha_2 x_r^T(t) \dot{P}_2(t) x_r(t) \]

because of (38) and (39) where \( Q_{r1}(t) \geq Q_{r2}(t) \), \( \forall t \in [t_1, \infty) \).

The dynamic equation (2) thus admits the solution \( x(t) = r(t) + x_m(t) \) where \( x_m(\infty) = 0 \) and \( r(\infty) = 0 \) imply robust asymptotic stability.

As mentioned in Section 3.2, although (19) can be asymptotically stabilized with \( \Delta P = K[\Delta \dot{w} \Delta q]^T \) (Hollop et al., 2002), this does not imply that (2) possesses asymptotic stability. Asymptotic stability of (2) requires suppression of the steady-state error and oscillation. The former can be fulfilled using Theorem 3 while the latter could be ensured by Theorem 4, which is capable of identifying existence of the oscillation nature.

4. Case study

4.1. Verification of IO-FTS LTV system

As mentioned in Section 2.3, Theorem 1 is invalid for (9). Theorem 2 was used here to study its robust IO-FTS over \( \Omega = [0, 50] \) with \( A(t) = A_1(t) - A_2(t) \), \( Q_1(t) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \), and \( Q_2(t) = \begin{bmatrix} Q_{21} & Q_{22} \\ Q_{22} & Q_{22} \end{bmatrix} \) as shown in Figure 2(a) and 2(b), respectively. Based on the setting, solving (10)–(13) yields

\[ A_1(t) = \begin{bmatrix} -3 & \sin(t) \\ 0 & -2 \end{bmatrix}, \]

\[ A_2(t) = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}. \]

Figure 2. Responses of (a) \( Q_1(t) \) and (b) \( Q_2(t) \).
Figure 3. Response of $\tilde{P}(t)$.

$$\begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, \quad \text{and } \tilde{P}(t) \text{ approaches } \begin{bmatrix} 0.3333 & 0.1667 \\ 0.1667 & 0.5 \end{bmatrix} \quad \text{(see Figure 3)} \quad \text{as } t \to 50.$$ Consequently, from Theorem 2, the system (9) is IO-FTS with respect to $(\dot{U}_2(T), \dot{Q}(\cdot), \Omega)$.

4.2. Case study of robust stability

Consider the system (1) with

$$A(t) = \begin{bmatrix} t & 2t - 1 \\ 3t + 2 & -t + 1 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = I_2,$$

$$w(t) = r_w(t) = 10 \sin (t) + 10.$$

To demonstrate the advantage of Theorems 3 and 4, Theorem 2 was first applied to guarantee its robust IO-FTS by considering $u(t) = K(t)x(t)$ with $Q_2 = I$ (see Figure 4(a)), and $Q_1$ (see Figure 4(b)). Note that $Q_1(2,1)$ approaches $Q_1(1,2)$ gradually. In spite of determining the diverging $Q_1(t)$ in Theorem 2 to ensure $Q_1(t) \geq Q_2$ over $\Omega \in [0,30]$, conducting the steps depicted in Remark 1 yields $K(t) = [3t - 2 \quad t - 3]$, $A_1(t) = \begin{bmatrix} 3t - 3 & t - 1 \\ 0 & -3 \end{bmatrix}$, $A_2(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and $\tilde{P}(t) > 0$ over $\Omega \in [0,30]$ (see Figure 5(a)) with $\tilde{P}(30) = \begin{bmatrix} 0.433 & -0.067 \\ -0.067 & 0.433 \end{bmatrix}$. The system (2) is thus possessing robust IO-FTS with respect to $(\dot{U}_2(T), \dot{Q}(\cdot), \Omega)$. However, using the control input in Figure 4(c), the result of Figure 5(b) exhibits steady-state error caused by the nonzero $w_m(t)$ and state oscillation induced by $w(t)$. This should thus be improved by redesigning $K(t)$.

To remedy, Theorem 3 was adopted to design the control law $u(t) = K(t)x(t)$. Under the settings of $Q_{m2} = 2I$, $\alpha_1 = 0.1$, $t_0 = 1$, and $Q_{m1}(t) = \begin{bmatrix} Q_d(t) & Q_{b}(t) \\ Q_{c}(t) & Q_d(t) \end{bmatrix}$ with $Q_d = [Q_d(i,j)]_{2 \times 2} \to \begin{bmatrix} 1.999 & 0 \\ 0 & 1.999 \end{bmatrix}$, $Q_c = [Q_c(i,j)]_{2 \times 2} \to \begin{bmatrix} 0 & 0 \\ 0 & -0.05 \end{bmatrix}$, and $Q_{b} = [Q_{b}(i,j)]_{2 \times 2}$ with $Q_{b}(1,2) = Q_{b}(2,1) \to -0.2999$ as $t \to 40$ in Figure 6, solving (28)–(30) upon Remark 1 gives $A_b(t) = -10$, $A_d(t) = \begin{bmatrix} -10 & 0 & 1 \\ 0 & -10 & 0 \\ -1 & 0 & -2t - 10 \end{bmatrix}$, and $\tilde{K}(t) = \begin{bmatrix} 3t - 3 & t - 1 \end{bmatrix}$, $\tilde{P}_1(t) = \begin{bmatrix} \tilde{P}_a(t) & \tilde{P}_b(t) \\ \tilde{P}_c(t) & \tilde{P}_d(t) \end{bmatrix}$, and $\tilde{P}_1(40) = 0.1I$.

The system is thus robustly asymptotically stable with respect to its mean in spite of setting the diverging $Q_{m1}(t)$ in Theorem 3 to meet $Q_{m1}(t) \geq Q_{m2}$ for $t \geq 10$.
Third, robust asymptotic stability of the system (1) with \( u(t) = K(t)x(t) = [-3t \quad -1]x(t) \) was examined using Theorem 4, in which \( \hat{A}_1(t) = -50I \) and \( \hat{P}_2(t) = \begin{bmatrix} P_a(t) & P_b(t) \\ P_c(t) & P_d(t) \end{bmatrix} > 0 \) for \( t \in [10, \infty) \) with \( \hat{P}_2(40) = 0.1I \). Response of \( \hat{P}_2(t) \) is shown in Figure 8, when the matrix was evaluated for \( Q_{r2}(t) = 10I \) and \( \alpha_2 = 0.1 \), and \( Q_{r1}(t) = \begin{bmatrix} Q_a(t) & Q_b(t) \\ Q_c(t) & Q_d(t) \end{bmatrix} \) was selected to guarantee \( Q_{r1}(t) \geq Q_{r2} \) for \( t \geq 10 \) where \( Q_a \to \begin{bmatrix} 9.999 & 0 \\ 0 & 9.999 \end{bmatrix} \), \( Q_b \to \begin{bmatrix} 0.2 & 0 \\ 0 & 0.05 \end{bmatrix} \), \( Q_c \to \begin{bmatrix} 0.2 & 0 \\ 0 & 0.05 \end{bmatrix} \), and \( Q_d(1,2) \to -0.3 \) as \( t \to 40 \) (see Figure 9). By Theorem 4, the system (2) is thus robustly asymptotically stable.
Finally, the result of Figure 10(a) illustrates that the states response converge to zero as $t \to \infty$. Figure 10(b) displays the control command.

5. Conclusions
This paper has proposed a novel type of stability robustness–mean robust asymptotic stability, and applies it to a class of LTV uncertain systems. The results are proposed to enhance the insufficiency of robust IO-FTS for certain LTV uncertain system – in particular, when the dynamic equation exists, a singular system matrix at some time over a bounded time interval. For the situation of the systems exhibit steady-state errors or state oscillation, a refinement is further proposed to by identifying the oscillating behaviour. Our main results guarantee robust stability for a class of LTV systems with modelling uncertainty. All the propositions have been demonstrated via case studies.

Disclosure statement
No potential conflict of interest was reported by the authors.

Funding
This research was sponsored by the Ministry of Science and Technology, Taiwan, R.O.C. [grant number NSC 101-2221-E-005-015-MY3], [grant number MOST 103-2218-E-005-005-MY2].

References
Abou-Kandil, H., Freiling, G., Ionesco, V., & Jank, G. (2003). *Matrix Riccati equations in control and systems theory* (pp. 1–20). Birkhäuser verlag: Basel.
Amato, F., Caramanite, G., De Tommasi, G., & Pironti, A. (2012). Input-output finite-time stability of linear systems: Necessary and sufficient conditions. *IEEE Transactions on Automatic Control*, 57(12), 3051–3063.
Amato, F., Cosentino, C., De Tommasi, G. (2011, September). Sufficient conditions for robust input-output finite-time stability of linear systems in presence of uncertainties. Proceedings of IFAC World Congress, Milano, 7643–7647.
Bilman, P. A. (2004). A convex approach to robust stability for linear systems with uncertain scalar parameters. *SIAM Journal on Control and Optimization*, 42(6), 2016–2042.
Cheng, C. C., Su, G. L., & Chien, C. W. (2012). Block backstepping controllers design for a class of perturbed non-linear systems with r blocks. *IET Control Theory and Applications*, 6(13), 2021–2030.
Choi, J., Hong, S., Lee, W., Kang, S., & Kim, M. (2011). A robot joint with variable stiffness using leaf springs. *IEEE Transactions on Robotics*, 27(2), 229–238.
Corradini, M. L., Cristofaro, A., & Orlando, G. (2010). Robust stabilization of multi input plants with saturating actuators. *IEEE Transactions on Automatic Control*, 55(2), 419–425.
Davis, J. M., Gravagne, I. A., Marks II, R. J., & Ramos, A. A. (2010). *Algebraic and dynamic Lyapunov equations on time scales*. Proceedings of South Eastern Symposium on System Theory, Tyler, Texas, 329–334.
Holot, C. V., Misra, V., & Towsley, D. (2002). Analysis and design of controllers for AQM routers supporting TCP flows. *IEEE Transactions on Automatic Control*, 47(6), 945–959.
Hsu, P. M., & Lin, C. L. (2012, July). Time-delay compensation for time-varying delayed systems using extended network disturbance. Proceedings of IEEE Industrial Electronics and Applications, Singapore, 388–393.
Jain, Y., June, F., Lin, S., & Yu, Z. (2011, July). Input-output finite-time stability of time-varying linear singular systems. Proceedings of Chinese Control Conference, Yantai, 62–66.
Kelly, R. (1996). Robust asymptotically stable visual servo of planar robots. *IEEE Transactions on Robotics and Automation*, 12(5), 759–766.
Kerrigan, E. C., & Maciejowski, J. M. (2004). Feedback min-max model predictive control using a single linear program: Robust stability and the explicit solution. *International Journal of Robust and Nonlinear Control*, 14, 395–413.
Lee, T. C., Liaw, D. C., & Chen, B. S. (2001). A general invariance principle for nonlinear time-varying systems and its applications. *IEEE Transactions on Automatic Control*, 46(12), 1989–1993.
Li, C., Wang, J., & Lu, J. (2012). Observer-based robust stabilization of a class of non-linear fractional-order uncertain systems: A linear matrix inequality approach. *IET Control Theory & Applications*, 6(18), 2757–2764.
Li, H., Gao, H., & Lu, H. (2011). Robust quantised control for active suspension systems. *IET Control Theory & Applications*, 5, 1955–1969.
Lu, J. G., & Chen, G. (2009). Robust stability and stabilization of fractional-order interval systems: An LMI approach. *IEEE Transactions on Automatic Control*, 54(6), 1294–1299.
Lu, J. G., & Chen, Y. Q. (2010). Robust stability and stabilization of fractional-order interval systems with the fractional order $\alpha$: The $0 < \alpha < 1$ case. *IEEE Transactions on Automatic Control*, 55(1), 152–158.
Ma, H., & Jia, Y. (2011, December). Input-output finite-time stability and stabilization of stochastic Markovian jump systems. Proceedings of IEEE Decision and Control and European Control Conference, Orlando, 8027–8031.
Misra, V., Gong, W. B., & Towsley, D. (2000, August). Fluid-based analysis of a network of AQM routers supporting TCP flows with an application to RED. Proceedings of ACM SIGCOMM, Stockholm, 151–160.
Xu, S., & Lam, J. (2005). Improved delay-dependent stability criteria for time-delay systems. *IEEE Transactions on Automatic Control*, 50(3), 384–387.