SINGULARITIES IN ARBITRARY CHARACTERISTIC
VIA JET SCHEMES

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Abstract. This paper summarizes recent results concerning singularities with respect to the Mather–Jacobian log discrepancies over an algebraically closed field of arbitrary characteristic. The basic point is that the inversion of adjunction with respect to Mather–Jacobian discrepancies holds under arbitrary characteristic. Using this fact, we will reduce several geometric properties of the singularities to jet scheme problems and try to avoid discussions that are distinctive to characteristic 0.

1. Introduction
Canonical and log-canonical singularities play important roles in birational geometry over the base field of characteristic 0 and are recognized as “good singularities,” which we admit on a minimal model. These singularities can be well described by jet schemes when they are locally a complete intersection.

On the other hand, [3] and [14] independently introduced singularities that can be described by jet schemes in general. These are based on Mather–Jacobian (MJ) discrepancies. We define MJ-canonical (MJ-log-canonical) singularities based on MJ discrepancy in a similar way to canonical (log-canonical) singularities. These MJ-canonical and MJ-log-canonical singularities have the following good properties: rationality (MJ-canonical), stability under small deformations (see [8]), and MJ-multiplier ideals (see [7]). These are all for a characteristic 0 case.

Now, for a positive characteristic case, one can also define canonical and log-canonical singularities as well as MJ-canonical and MJ-log-canonical singularities. However, canonical and log-canonical singularities of positive characteristic are difficult to treat, because the following results are unavailable:

1. resolution of singularities;
2. Bertini’s second theorem (generic smoothness); and
3. the Kodaira vanishing theorem,

all of which are used in discussions of singularities over characteristic 0.

For example, even the seemingly simple statement that “a general hyperplane section of a quasi-projective variety has at worst canonical singularities if the variety has such singularities” is not yet proved for positive characteristic case. This statement has been proved in the characteristic 0 case by resolution of the singularities and Bertini’s second theorem. If the variety is of dimension 3 over a base field of positive characteristic, then a resolution of the singularities exists [2], but this problem is not yet proved (although a result has been found under certain conditions [11]).

On the contrary, the MJ-version of this problem has the possibility to be solved, because MJ-singularities are well described by the jet schemes, which acts as an alternative to Bertini’s second theorem. As evidence, in Corollary 4.11, we prove this statement for three-dimensional MJ-canonical quasi-projective variety, thereby showing the possibility of MJ discrepancy for discussion of positive characteristics.
One of the aims of this paper is to summarize the results of MJ-singularities in the positive characteristic case and to clarify those which has been proved and those which has not yet been proved. Since such a review does not presently exist elsewhere, we think that it would be useful for us to summarize them here. The basic theorems showing that MJ-singularities are well described by jet schemes are Theorems 3.18 and 3.23. These theorems are proved in the same line as their proof in the characteristic 0-case, by carefully avoiding the use of resolutions of the singularities.

The other aim of this paper is to show that jet schemes are useful in the study of singularities over positive characteristic base fields (Corollary 4.11).

This paper is structured as follows: in Section 2, we give characteristic-free discussions of MJ discrepancies. In Section 3, we give preliminary information about jet schemes, define the codimension of a cylinder, and describe minimal log MJ discrepancies using jet schemes in the arbitrary-characteristic case. Note that these have been already established in the characteristic-0 case. In Section 4, we show some properties of MJ-canonical and MJ-log-canonical singularities of positive characteristic that have been proved and list some open problems for the positive-characteristic case.

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2. Mather–Jacobian discrepancy

Throughout this paper, a variety refers to a reduced pure-dimensional scheme of finite type over an algebraically closed field $k$ of arbitrary characteristic, unless otherwise stated.

Definition 2.1. Let $X$ be a variety of dimension $d$. Note that the projection
\[ \pi: \mathbb{P}(\wedge^n\Omega_X) \rightarrow X \]
is an isomorphism over the smooth locus $X_{\text{reg}} \subseteq X$. In particular, we have a section $\sigma: X_{\text{reg}} \rightarrow \mathbb{P}(\wedge^n\Omega_X)$.

The closure of the image of the section $\sigma$ is called the Nash blow-up of $X$, and is denoted by $\hat{X}$:
\[
\begin{array}{ccc}
\mathbb{P}(\wedge^n\Omega_X) & \supseteq & \hat{X} := \overline{\sigma(X_{\text{reg}})} \\
\downarrow \sigma & & \downarrow \pi \\
X_{\text{reg}} & \rightarrow & X.
\end{array}
\]

Definition 2.2. Let $X$ be a variety. We call a morphism $\varphi: Y \rightarrow X$ a partial resolution, if $\varphi$ is proper birational and $Y$ is normal. We also sometimes call $Y$ a partial resolution. A prime divisor over $X$ is a prime divisor that appears in a partial resolution of $X$. A prime divisor $E$ over $X$ is called an exceptional prime divisor if a partial resolution on which $E$ appears is not isomorphic at the generic point of $E$.

Definition 2.3. Let $E$ be a prime divisor over $X$, then the Mather discrepancy $\hat{k}_E \in \mathbb{Z}_{\geq 0}$ and the Jacobian discrepancy $j_E$ at $E$ are defined as follows:

Let $\varphi: Y \rightarrow X$ be a partial resolution of $X$ such that $E$ appears on $Y$, let $\hat{X} \rightarrow X$ be the Nash blow-up and let $Y \times_X \hat{X}$ be the fiber product. In the fiber product,
the union of the irreducible components each of which dominates an irreducible component of $X$ is called the main part.

Replacing $Y$ by the normalization of the main part of the fiber product, we may assume that $\varphi : Y \to X$ factors through the Nash blow-up $\hat{X} \to X$.

By the universality of the Nash blow-up, the image $Im$ of the following homomorphism is invertible:

$$\varphi^*(\wedge^d \Omega_X) \to \omega_Y,$$

where $d = \dim X$. Restricting the above homomorphism to the smooth locus $Y_{sm}$, we can describe

$$Im|_{Y_{sm}} = I\omega_{Y_{sm}}$$

with some invertible ideal sheaf $I$, as $\omega_{Y_{sm}}$ is invertible. As $Y$ is normal, the generic point $\eta$ of $E$ is in $Y_{sm}$. Let us define

$$\hat{k}_E = val_E(I)$$

and call it the Mather discrepancy of $X$ at $E$. We note that if $I = \mathcal{O}_Y(-\hat{K}_{Y/X})$ is expressed with a Cartier divisor $\hat{K}_{Y/X}$ on $Y_{sm}$, then $\hat{k}_E$ is the coefficient of $\hat{K}_{Y/X}$ at $E$.

For a coherent ideal sheaf $a \subset \mathcal{O}_X$, let us define

$$val_{\hat{E}}(a) = \min\{val_E(f) \mid f \in a\}.$$

Note that if $\varphi : Y \to X$ factors through the blow-up of $X$ by the ideal $a$, then $val_{\hat{E}}(a)$ is the coefficient at $E$ of the divisor $Z$ defined by $\mathcal{O}_Y(-Z) = a\mathcal{O}_Y$. In particular, when $a$ is the Jacobian ideal $J_X$ of $X$, we define

$$j_E = val_E(J_X)$$

and call it the Jacobian discrepancy of $X$ at $E$.

Then, the difference $\hat{k}_E - j_E$ is called the Mather–Jacobian (MJ) discrepancy of $X$ at $E$.

**Definition 2.4.** Let $X$ be a variety over $k$, let $a \subset \mathcal{O}_X$ be a non-zero coherent ideal sheaf, and let $n$ be a non-negative real number. A real number

$$a_{MJ}(E; X, a^n) = \hat{k}_E - j_E - n \cdot val_{\hat{E}}(a) + 1$$

is called the MJ-log discrepancy of the pair $(X, a^n)$ at $E$.

A pair $(X, a^n)$ is called MJ-canonical (resp. MJ-log-canonical) at a (not necessarily closed) point $x \in X$, if

$$a_{MJ}(E; X, a^n) \geq 1 \quad (\text{resp.} \geq 0)$$

for every exceptional prime divisor $E$ over $X$ whose center contains $x$.

A pair $(X, a^n)$ is called MJ-klt at a (not necessarily closed) point $x \in X$, if

$$a_{MJ}(E; X, a^n) > 0$$

for every prime divisor $E$ over $X$ whose center on $X$ contains $x$. Here, klt means “Kawamata log-terminal” which appears in Minimal Model Program and plays an important role (see, for example, [17]). Our “MJ-klt” is MJ-version of the usual klt.

We call a pair $(X, a^n)$ MJ-canonical (resp. MJ-log-canonical, MJ-klt) if it is MJ-canonical (resp. MJ-log-canonical, MJ-klt) at every point of $X$.

Here, MJ-klt is the MJ-version of the usual klt, i.e., Kawamata log terminal (see, for example, [17]).
Remark 2.5. (1) Note that the definitions of MJ-canonical and MJ-log-canonical require the condition only for “exceptional” prime divisors, while the definition of MJ-klt requires the condition for all prime divisors over $X$. However, regarding the MJ-log-canonical case, if $(X, a^n)$ is MJ-log-canonical at a point $x \in X$, then $m_{MJ}(E; X, a^n) \geq 0$ holds for every prime divisor over $X$ whose center contains $x$.

(2) If $X$ is normal and locally a complete intersection, the image of the canonical map $\wedge^d \Omega_X \to \omega_X$ is $\mathcal{J}_X \omega_X$ (see, for example, [6, Remark 9.6]). In this case $k_E - j_E = k_E$ for every prime divisor $E$ over $X$. Therefore, the MJ-canonical and MJ-log-canonical cases are equivalent to the usual canonical and log-canonical cases, respectively.

Definition 2.6. Let $X$ be a variety over $k$. The Mather–Jacobian minimal log discrepancies (MJ-mld) of the pair $(X, a^n)$ at a proper closed subset $W \subset X$ and at a point $\eta \in X$ are defined as follows:

$$m_{MJ}(W; X, a^n) = \inf \{ a_{MJ}(E; X, a^n) \mid E \text{ is a prime exceptional divisor over } X \text{ with the center in } W \}$$

$$m_{MJ}(\eta; X, a^n) = \inf \{ a_{MJ}(E; X, a^n) \mid E \text{ is a prime exceptional divisor over } X \text{ with the center } \{\eta\} \},$$

when $\dim X \geq 2$. When $\dim X = 1$ and the right-hand side is $\geq 0$, then we define $m_{MJ}$ by the right-hand side. Otherwise, we define $m_{MJ} = -\infty$.

Remark 2.7. (i) We strictly distinguish “center in $Z$” from “center $Z$.”

(ii) For a point $x \in X$, the pair $(X, a^n)$ is MJ-canonical (resp. MJ-log-canonical) if and only if $m_{MJ}(\eta; X, a^n) \geq 1$ (resp. $m_{MJ}(\eta; X, a^n) \geq 0$) for every $\eta \in X$ such that $x \in \{\eta\}$.

(iii) When $\dim X \geq 2$, we can prove that $m_{MJ}(W; X, a^n) = -\infty$, if $m_{MJ}(W; X, a^n) < 0$.

(iv) If $X$ is normal and of a complete intersection, then the MJ-log discrepancy coincides with the usual log discrepancy. Therefore, in this case, the MJ-canonical and MJ-log-canonical cases coincide with the usual canonical and log-canonical cases, respectively. In particular, if $X$ is non-singular $m_{MJ}(W; X, a^n) = \text{mld}(W; X, a^n)$, where the right-hand side is the usual minimal log discrepancy.

3. The arc space and jet schemes of a variety

Throughout this section, $X$ is always a $d$-dimensional variety over an algebraically closed field $k$ of an arbitrary characteristic. In this section, we prove the inversion of adjunction of minimal MJ-log discrepancies by means of discussions of arc spaces and jet schemes. This theorem has been independently proved in [3] and [14] based on the concept of [6], when the base field is of characteristic 0. Here, we will present a characteristic-free proof for this theorem.

The basic tool is the arc space. So first we introduce arcs and jets of a variety.

Definition 3.1. Let $X$ be a scheme of finite type over $k$ and a $K \supset k$ a field extension. For $m \in \mathbb{Z}_{\geq 0}$ a $k$-morphism $\text{Spec}K[t]/(t^{m+1}) \to X$ is called an $m$-jet of $X$ and $k$-morphism $\text{Spec}K[[t]] \to X$ is called an arc of $X$. The unique point of $\text{Spec}K[t]/(t^{m+1})$ and the closed point of $\text{Spec}K[[t]]$ are both denoted by 0. and the image of 0 by the jet or the arc is called the center of the jet or the arc.
Theorem 3.2. Let $X$ be a scheme of finite type over $k$, $\text{Sch}/k$ the category of $k$-schemes and $\text{Set}$ the category of sets. A contravariant functor $F^X_m : \text{Sch}/k \rightarrow \text{Set}$ is defined as follows:

$$F^X_m(Z) = \text{Hom}_k(Z \times \text{Spec} k[t]/(t^{m+1}), X).$$

Then $F^X_m$ is representable by a scheme $X_m$ of finite type over $k$, therefore, there is a bijection as follows:

$$\text{Hom}_k(Z, X_m) \cong \text{Hom}_k(Z \times \text{Spec} k[t]/(t^{m+1}), X).$$

This scheme $X_m$ is called the space of $m$-jets or the jet scheme of $X$.

There exists the projective limit

$$X_\infty := \lim_{\rightarrow} X_m$$

and it is called the space of arcs or the arc space of $X$. There is also a bijection for a $k$-algebra $A$ as follows:

$$\text{Hom}_k(\text{Spec} A, X_\infty) \cong \text{Hom}_k(\text{Spec} A[[t]], X).$$

Definition 3.3. For a variety $X$ over $k$, let $X_m$ ($m \in \mathbb{N}$) and $X_\infty$ be the $m$-jet scheme and the arc space of $X$. Denote the canonical truncation morphisms by $\psi_m : X_\infty \rightarrow X_m$ and $\pi_m : X_m \rightarrow X$. In particular we denote the extremal morphism $\psi_0 = \pi_\infty : X_\infty \rightarrow X$ by $\pi$. We also denote the canonical truncation morphism $X_{m'} \rightarrow X_m$ ($m' > m$) by $\psi_{m', m}$. To specify the space $X$, we sometimes write $\psi_{m', m}$.

Definition 3.4. The pull-back $\psi^{-1}_m(S) \subset X_\infty$ of a constructible set $S \subset X_m$ ($m \in \mathbb{Z}_{\geq 0}$) is called a cylinder.

A subset $C \subset X_\infty$ is called a thin set if there is a closed subset $Z \subset X$ with dimension less than the dimension of $X$ such that $C \subset Z$.

A subset $C \subset \psi^{-1}_m(S)$ is called an irreducible component of the cylinder $\psi^{-1}_m(S)$ if it is a maximal irreducible closed subset of the cylinder. Here, we note that a maximal irreducible closed subset of a cylinder exists because the set consisting of irreducible closed subsets in the cylinder is an inductive set with respect to the inclusion order (the existence of a maximal irreducible component follows from Zorn’s lemma, which we always assume). If char $k = 0$ or dim $X \leq 3$, then every cylinder in $X_\infty$ has only a finite number of irreducible components (a resolution of singularities [2] is used to prove this result).

Definition 3.5. For an arc $\gamma \in X_\infty$, the order of an ideal $a \subset \mathcal{O}_X$ measured by $\gamma$ is defined as follows: let $\gamma^* : \mathcal{O}_{X, \gamma(0)} \rightarrow k[[t]]$ be the corresponding ring homomorphism of $\gamma$. Then, we define

$$\text{ord}_\gamma(a) = \sup\{r \in \mathbb{Z}_{\geq 0} \mid \gamma^*(a) \subset (t^r)\},$$

We define the subsets “contact loci” in the arc space as follows:

$$\text{Cont}^m(a) = \{\gamma \in X_\infty \mid \text{ord}_\gamma(a) = m\}$$

In a similar manner, we define

$$\text{Cont}^{\geq m}(a) = \{\gamma \in X_\infty \mid \text{ord}_\gamma(a) \geq m\}$$

By this definition, we can see that

$$\text{Cont}^{\geq m}(a) = \psi^{-1}_m(Z(a)_{m-1}),$$

Here, $Z(a)$ is the closed subscheme defined by the ideal $a$ in $X$. Therefore, the contact loci are cylinders in $X_\infty$. 


Definition 3.6. Let $E$ be a prime divisor over $X$ and $\varphi : Y \to X$ be a partial resolution of $X$ on which $E$ appears. Let $p \in E$ be the generic point. We define $C_X(\val_E) = \varphi_{\infty}((\pi_Y^*)^{-1}(p))$, also denoted by $N_E$ in the literature on the Nash problem. Furthermore, for $q \in \mathbb{N}$, let $p_{q-1} \in (\pi_Y^*)^{-1}((E \cap Y_{\reg})_{q-1})$ be the generic point. Then, we define

$$C_X(q \cdot \val_E) = \varphi_{\infty}(\psi_{q-1}^Y)(p_{q-1})$$

and call it the maximal divisorial set corresponding to the divisorial valuation $q \cdot \val_E$. This definition is the same as that in [4] and [13] in case $\text{char } k = 0$.

An irreducible closed subset $C \subset X_\infty$ is called a divisorial set, if there exist $q \in \mathbb{N}$ and a prime divisor $E$ over $X$ such that the divisorial valuation $q \cdot \val_E$ is given by $\text{ord}_p$, where $p$ is the generic point of $C$.

the generic point $\alpha \in C$ gives a divisorial valuation $q \cdot \val_E$ by $\text{ord}_\alpha$. A maximal divisorial set $C_X(q \cdot \val_E)$ is the maximal set among all divisorial sets corresponding to the valuation $q \cdot \val_E$.

Proposition 3.7 ([5, Lemma 4.1], [20, before lemma 3.2] or [21, 3.4], [6, Proposition 4.1]). For $e \in \mathbb{Z}_{\geq 0}$, the following subsets in $X_\infty$ and in $X_m$ are defined by using the Jacobian ideal $J_X$:

$$X^e_\infty := \{ \gamma \in X_\infty \mid \text{ord}_\gamma(J_X) = e \} \quad \text{and} \quad X^e_{m,\infty} := \psi_m(X^e_\infty).$$

Then, there exists $e \in \mathbb{N} \setminus \{0\}$ such that for an integer $m \geq e$ the canonical map $X^e_{m+1,\infty} \to X^e_{m,\infty}$ is a piecewise trivial fibration with fibers isomorphic to $\mathbb{A}^d$, where $d = \dim X$. Therefore, we also know that $X^e_{m+1,\infty} \to X^e_{m,\infty}$ is a piecewise trivial fibration with fibers isomorphic to $\mathbb{A}^d$, where we define

$$X^{\leq e}_\infty := \{ \gamma \in X_\infty \mid \text{ord}_\gamma(J_X) \leq e \} \quad \text{and} \quad X^{\leq e}_{m,\infty} := \psi_m(X^{\leq e}_\infty).$$

This proposition is stated in [5] under the condition that $\text{char } k = 0$; however in [20] and [6], its proof was also confirmed to apply to perfect fields in the positive-characteristic case.

Remark 3.8. Using Proposition 3.7, stable points of the space of arcs were defined as follows in [21]: let $\gamma$ be a point of $X_\infty$ (i.e., $\gamma$ is a prime ideal of $\mathcal{O}_{X_\infty}$) and let $Z(\gamma)$be the set of zeros of $\gamma$ on $X_\infty$; then, $\gamma$ is a stable point if there exists $m_0 \in \mathbb{N}$ and $G \in \mathcal{O}_{X_\infty}$, $G \notin \gamma$, $G \in \mathcal{O}_{X_{m_0}}$, such that, for $m \geq m_0$, the map $\psi_{m+1,m} : X_{m+1} \to X_m$ induces a trivial fibration $\psi_m(Z(\gamma)) \cap (X_{m+1})_G \to \psi_m(Z(\gamma)) \cap (X_m)_G$ with fiber $\mathbb{A}^d$, where $(X_m)_G$ is the open subset $X_m \setminus Z(G)$ of $X_m$.

The following is an easy consequence of the previous proposition when $\text{char } k = 0$, since in that case a cylinder has only finite number of irreducible components ([4, Proposition 3.6]), and therefore, an irreducible component of a cylinder is regarded generically as a cylinder. Herein, we give a proof that does not use the finiteness of the irreducible components of a cylinder; thus, this proof also works for a positive characteristic case.

Lemma 3.9. For an irreducible component $C$ of a cylinder in $X_\infty$ such that $C \not\subset \text{Sing}(X)_\infty$, its generic point $\gamma$ is a stable point of $X_\infty$. In particular, there exists $e$ such that

$$C^{\leq e}_m := \psi_m(C) \cap X^{\leq e}_{m,\infty}$$

is a nonempty open subset of $\psi_m(C)$ and the codimension of $C^{\leq e}_m$ inside $X^{\leq e}_{m,\infty}$ stabilizes for $m \gg e$. 
Proof. This lemma is proved in the same way as in the case of characteristic 0. Let $C$ be an irreducible component of a cylinder $\Gamma$ in $X_\infty$ such that $C \not\subset \text{Sing}(X)_\infty$. Let $\gamma \in C$ be the generic point. Then by $\gamma \notin \text{Sing}(X)_\infty$, we have $\text{ord}_\gamma(J_X) = e > 0$. Then, $C^{\leq e}_m$ is a nonempty open subset of $\psi_m(C)$. From this and Proposition 3.5, the result follows. \hfill \Box

Here, we show a result about a stable point, for the readers who are interested in this direction.

**Proposition 3.10.** If $\gamma$ is a stable point of $X_\infty$, then there exists an irreducible cylinder $C \subset X_\infty$ whose generic point is $\gamma$.

Proof. We may assume that $X$ is affine. Let $\gamma$ be a stable point of $X_\infty$. Then, by ([20, Theorem 4.1]) (see also the finiteness property of the stable points in [21, 3.10]), there exist $G \in \mathcal{O}_{X_\infty}$, $G \not\subset \gamma$ and $L_1, \ldots, L_r \in \mathcal{O}_{X_\infty}$ such that

\[
\gamma (\mathcal{O}_{X_\infty})_G = (L_1, \ldots, L_r) (\mathcal{O}_{X_\infty})_G.
\]

Since $\mathcal{O}_{X_\infty} = \lim_{\to} \mathcal{O}_{X_n}$, there exists $n_0 \in \mathbb{N}$ such that $G, L_1, \ldots, L_r \in \mathcal{O}_{X_{n_0}}$. Let us consider the constructible subset

\[ S := Z(L_1, \ldots, L_r) \cap (G \neq 0) \subset X_{n_0}. \]

Then $C = \psi^{-1}\gamma(S)$ is an irreducible cylinder whose generic point is $\gamma$ (see (1) and the definition of cylinder in [6, beginning of section 5]). \hfill \Box

**Definition 3.11.** For a variety $X$, let $C$ be an irreducible component of a cylinder in $X_\infty$ such that $C \not\subset \text{Sing}(X)_\infty$, then we define the codimension of $C$ in $X_\infty$ as follows:

\[ \text{codim}(C, X_\infty) := \text{codim}(C^{\leq e}_m, X^{\leq e}_m), \]

for $m \gg e$, where $e = \text{ord}_\gamma(J_X)$ for a generic point $\gamma \in C$.

Let $\Gamma \subset X_\infty$ be a cylinder not contained in $\text{Sing}(X)_\infty$, then we define the codimension of the cylinder $\Gamma$ in $X_\infty$ as the minimal value of the codimensions of the irreducible components not contained in $\text{Sing}(X)_\infty$.

The following lemma was proved in [5, Lemma 3.4] and in [6, Theorem 6.2, Lemma 6.3]. The former one was stated under the condition of characteristic 0, while the latter ones were stated under an arbitrary characteristic. Note that the statement of Lemma 6.2 in [6] assumed the properness of the morphism $f$, but the properness was not actually used in the proof.

**Lemma 3.12 ([5], [6]).** Let $f : Y \to X$ be a birational morphism from a smooth variety $Y$ and $f_m : Y_m \to X_m$ the induced morphism. Let $\gamma \in Y_\infty$ be any arc such that $\tau := \text{ord}_\gamma(\hat{K}_{Y/X}) < \infty$. Then for $m \gg 0$, letting $\gamma_m = \psi^Y_m(\gamma)$, we have

\[ f_m^{-1}(f_m(\gamma_m)) \simeq \mathbb{A}^\tau. \]

Moreover, for every $\gamma'_m \in f_m^{-1}(f_m(\gamma_m))$, we have $\psi^Y_{m, -\tau}(\gamma_m) = \psi^Y_{m, -\tau}(\gamma'_m)$, where $\psi^Y_{m, -\tau} : Y_m \to Y_{m, -\tau}$ is the canonical truncation morphism.

**Theorem 3.13.** Let $E$ be a prime divisor over $X$ and $g : \hat{X} \to X$ be a partial resolution of $X$ such that $E$ appears on $\hat{X}$. Then $C_X(q \cdot \text{val}_E)$ is a cylinder of $X_\infty$ of codimension

\[ \text{codim}(C_X(q \cdot \text{val}_E), X_\infty) = q \cdot (\text{val}_E(\hat{K}_{Y/X}) + 1). \]

Proof. Let $f : Y \to X$ be the restriction of $g$ on an open subset $Y \subset \hat{X}$ such that $Y$ and $E \cap Y$ are smooth. Then, by using Lemma 3.12, we can prove the required equality in the same way as [4, Theorem 3.9]. Note that Theorem 3.9 in [4] was stated under the condition that $f : Y \to X$ is a resolution of the singularities of $X$,
but the resolution is not actually needed for the proof as the condition in Lemma 3.12 does not require the properness of $f$.

The following is a modified version of Lemma 3.12 for a blow-up $f$.

**Lemma 3.14.** Let $B \subset X$ be an irreducible reduced closed subset of dimension $s$ with the defining ideal $I_B$ in $X$ and let $\gamma \in X_{\infty}$ be an arc that is not contained in $\text{Sing}(X)_{\infty} \cup B_{\infty}$ and the center is smooth somewhere in $B$ with $e := \text{ord}_x(I_B) > 0$.

Let $f : Y \to X$ be the blow-up with the center $B$ and $\gamma' \in Y_{\infty}$ be the lifting of $\gamma$.

Let $f_m : Y_{m,\infty} \to X_{m,\infty}$ be the morphism induced by $f$. Then for $m \gg 0$, letting $\gamma'_m = \psi^Y_m(\gamma')$, we have

$$\dim f_m^{-1}(f_m(\gamma'_m)) \geq (d - s - 1)e.$$ 

Moreover, for every $\gamma''_m \in f_m^{-1}(f_m(\gamma'_m))$ we have $\psi^Y_{m,m-e}(\gamma''_m) = \psi^Y_{m,m-e}(\gamma'_m)$.

**Proof.** Since the problem is local, we may assume that $X$ is an affine scheme embedded in $A = k_N$, $B$ is defined by the ideal $(x_1, \ldots, x_{N-s})$ in $A = \text{Spec}k[x_1, \ldots, x_N]$ and the center of $\gamma$ is the origin $0 = (0, \ldots, 0) \in B \subset X \subset A$. Let $\gamma_m = \psi^X_m(\gamma)$, then by definition $f_m(\gamma'_m) = \gamma_m$. Then, we can express the ring homomorphism $\gamma^*_m : k[x_1, \ldots, x_N] \to k[t]/(t^{m+1})$ corresponding to $\gamma_m$ as follows:

$$\gamma^*_m(x_i) = \sum_{j=0}^{m} a_{i,j} t^j \quad (i = 1, \ldots, N-s)$$

$$\gamma^*_m(x_i) = \sum_{j=1}^{m} a_{i,j} t^j \quad (i = N-s + 1, \ldots, N),$$

where $a_{i,j} \in k$. Here, by this assumption, we may assume that $a_{1,e} \neq 0$, without loss of generality.

Let $f' : A' \to A$ be the blow-up with the center $B$. Then $A'$ is covered by $(N-s)$ affine spaces

$$U(i) = \text{Spec}k[x_i, x_{i+1}, \ldots, x_{N-s}, x_{N-s+1}, \ldots, x_N] \quad (i = 1, \ldots, N-s).$$

By this assumption, we may assume that $\gamma'_m \in U(1)_m$. We denote the restriction $f_m|_{U(1)_m} : U(1)_m \to A_m$ of $f'_m : A'_m \to A_m$ by the same symbol $f'_m$. Let $y_i = \frac{x_i}{x_e}$ for $i = 2, \ldots, N-s$, then $x_1, y_2, \ldots, y_N, x_N$ form a coordinate system of $U(1)$. We can express $\gamma'_m \in U(1)_m$ as follows:

$$\gamma^*_m(x_1) = \sum_{j=0}^{m} b_{1,j} t^j$$

$$\gamma^*_m(y_i) = \sum_{j=0}^{m} b_{i,j} t^j \quad (i = 2, \ldots, N-s)$$

$$\gamma^*_m(x_i) = \sum_{j=1}^{m} a_{i,j} t^j \quad (i = N-s + 1, \ldots, N),$$

where $b_{i,j}$’s satisfy $\gamma^*_m(y_i) \gamma^*_m(x_1) = \gamma^*_m(x_i) = \sum_{j=e}^{m} a_{i,j} t^j$ for $i = 2, \ldots, N-s$.

Then, we can see that a jet $\alpha \in (f'_m)^{-1}(\gamma_m)$ is expressed by

$$\alpha^*(x_1) = \sum_{j=e}^{m} a_{1,j} t^j$$

$$\alpha^*(y_i) = \sum_{j=0}^{m-e} b_{i,j} t^j + \sum_{j=m-e+1}^{m} y_{i,j} t^j \quad (i = 2, \ldots, N-s)$$
\[ \alpha^*(x_i) = \sum_{j=1}^{m} a_i^{(j)} t^j \quad (i = N - s + 1, \ldots, N), \]

where \( y_i^{(j)} \) \((i = 2, \ldots, N - s, j = m - e + 1, \ldots, m)\) can be an arbitrary element in \( k \). Therefore, we have that every \( \alpha \in (f_m')^{-1}(\gamma_m) \) is mapped to the same element \( \psi_{m,m-e}'(\gamma_m') \) by the truncation morphism \( \psi_{m,m-e}' \), in particular, which yields the second statement of the lemma.

Let \( \gamma'_{m-e} = \psi_{m,m-e}'(\gamma_m') \), then any element \( \beta \in (\psi_{m,m-e}' )^{-1}(\gamma_m') \) is expressed as

\[
\beta^*(x_i) = \sum_{j=1}^{m-e} a_i^{(j)} t^j + \sum_{j=m-e+1}^{m} x_i^{(j)} t^j \\
\beta^*(y_i) = \sum_{j=0}^{m-e} b_i^{(j)} t^j + \sum_{j=m-e+1}^{m} y_i^{(j)} t^j \quad (i = 2, \ldots, N - s) \\
\beta^*(x_i) = \sum_{j=1}^{m-e} a_i^{(j)} t^j + \sum_{j=m-e+1}^{m} x_i^{(j)} t^j \quad (i = N - s + 1, \ldots, N),
\]

where \( y_i^{(j)} \) and \( x_i^{(j)} \) are arbitrary elements of \( k \). Here, comparing the expressions of \( \alpha^* \) and \( \beta^* \) above, we obtain that

\[
(f_m')^{-1}(\gamma_m) = (f_m')^{-1}(\gamma_m) \cap (\psi_{m,m-e}' )^{-1}(\gamma_m') \subset (\psi_{m,m-e}' )^{-1}(\gamma_m')
\]

is defined by \((s + 1)e\) equations:

\[
x_i^{(j)} = a_i^{(j)}, \quad (j = m - e + 1, \ldots, m) \\
x_i^{(j)} = a_i^{(j)}, \quad (i = N - s + 1, \ldots, N, \quad j = m - e + 1, \ldots, m)
\]

in \((\psi_{m,m-e}' )^{-1}(\gamma_m')\).

Now, remember that \( Y \) is the strict transform of \( X \) in \( A' \), \( J_Y \) be the Jacobian ideal of \( Y \) and let \( r := \operatorname{ord}_{r}(J_Y) \). Take \( m \) such that it is sufficiently large with respect to \( r \) and \( e \). Let \( \rho_{m,m-1} : Y_{m,\infty}^{\leq r} \to Y_{m-1,\infty}^{\leq r} \) be the restriction of the truncation morphism \( \psi_{m,m-1}^1 : Y_m \to Y_{m-1} \). Then, \( \rho_{m,m-1} \) is a piecewise trivial morphism with the fiber \( A^d \). Therefore \( \rho_{m,m-e} : Y_{m,\infty}^{\leq r} \to Y_{m-e,\infty}^{\leq r} \) has exactly \( de \)-dimensional fibers. Here we note that \( \rho_{m,m-e}^{-1}(\gamma_m') \subset (\psi_{m,m-e}' )^{-1}(\gamma_m') \). By this and the above observation, we obtain that

\[
(f_m')^{-1}(\gamma_m) \cap \rho_{m,m-e}^{-1}(\gamma_m') \subset \rho_{m,m-e}^{-1}(\gamma_m')
\]

is defined by at most \((s + 1)e\) equations in \( de \)-dimensional variety \( \rho_{m,m-e}^{-1}(\gamma_m') \).

This implies that \( f_m : Y_m \to X_m \) has a fiber \( f_m^{-1}(\gamma_m) \) of dimension \( \geq (d - s - 1)e \).

\[\square\]

**Remark 3.15.** In the previous lemma, the center of the lifting \( \gamma' \in Y_\infty \) may be a singular point of \( X \), while the center of \( \gamma \in Y_\infty \) is non-singular in Lemma 3.12. This, the situation of Lemma 3.14 is different from that of Lemma 3.12.

As a corollary, we obtain the following. But the result has been proved in [21, Proposition 3.7 (vii)] using the concept of stable points in the space of arcs [21, Definition 3.6] (see Lemma 3.6).

This corollary was also proved in [4, Proposition 2.12] for the characteristic 0 case by using resolutions of singularities.
Corollary 3.16. Every irreducible component $C$ of a cylinder such that $C \not\subset \text{Sing}(X)_\infty$ gives a divisorial valuation. In particular, the closure of an irreducible component $C$ of finite intersection of contact loci $\text{Cont}^m(a_i)$ ($i = 1, \ldots, s$) is a maximal divisorial set.

Proof. Let $\alpha$ be the generic point of $C$. First, we prove that there exists a prime divisor $E$ over $X$ such that the lifting $\tilde{\alpha}$ of $\alpha$ onto $Y$ on which $E$ appears has the center $E$. Let $B \subset X$ be the center of $\alpha$. If $\dim B = d - 1$, then we have the required conclusion. Therefore, we assume that $\dim B = s < d - 1$. By replacing $X$ by a small affine neighborhood, we may assume that $X$ is embedded in the affine space $A = k^N$. Let $f^{(1)} : A^{(1)} \to A$ be the blow-up with the center $B$. Then, $f^{(1)}$ is isomorphism outside $B$ and the image of $\alpha$ is not contained in $B$, therefore $\alpha$ is lifted onto the proper transform $Y^{(1)}$ of $X$ on $A^{(1)}$ by the properness criteria applied to $f^{(1)}$. Then, the lifting $\alpha^{(1)}$ of $\alpha$ on $Y^{(1)}$ gives an irreducible component $C^{(1)}$ of a cylinder on $Y^{(1)}$. Indeed, if $C$ is an irreducible component of the cylinder $\Gamma = (\psi^X)^{-1}(S)$ for a constructible subset $S \subset X_m$, then $C^{(1)}$ is an irreducible component of the cylinder $(\psi^{X^{(1)}})^{-1}((f^{(1)}_n)^{-1}(S))$.

Now, take $m$ to be sufficiently large and a general arc $\gamma \in C$ whose center is a closed point in $B$. Let $e = \text{ord}_\gamma I_B$. Then, we obtain that the morphism $f^{(1)} : C^{(1)}_m \to C_m$ has relative dimension $\geq (d - s - 1)e \geq 1$. Therefore, $\dim C^{(1)}_m \geq \dim C_m$, which yields $\text{codim}(C^{(1)}, Y^{(1)}_m) < \text{codim}(C, X_m)$. If the center of the generic point of $C^{(1)}$ is not of dimension $d - 1$, we blow-up at the center and obtain the irreducible component whose codimension is less than the previous one. In this way, we obtain the successive blow-ups:

$$A^{(n)} \to \cdots \to A^{(2)} \to A^{(1)} \to A$$

with the sequence of irreducible components

$$C^{(n)}, \ldots, C^{(2)}, C^{(1)}, C$$

of cylinders in $Y^{(n)}, \ldots, Y^{(2)}, Y^{(1)}, X$ such that

$$\text{codim} C^{(n)} < \cdots < \text{codim} C^{(2)} < \text{codim} C^{(1)} < \text{codim} C.$$

Because the codimension is finite, this procedure should terminate, i.e., there exists a number $n$ such that the dimension of the center of the generic point of $C^{(n)}$ has dimension $d - 1$. This is the conclusion to our first claim.

Then, we prove that the lifting $\alpha^{(n)}$ of $\alpha$ gives a divisorial valuation $q \cdot \text{val}_E$. Replacing $Y^{(n)}$ by its normalization if necessary, we may assume that $Y^{(n)}$ is normal and the center of the lifting $\alpha^{(n)}$ of $\alpha$ on $Y^{(n)}$ is a prime divisor, say $E$. The arc $\alpha^{(n)}$ gives a ring homomorphism of local rings $\alpha^{(n)*} : \mathcal{O}_{Y^{(n)}, \alpha^{(n)}(0)} \to K[[t]]$, where $\alpha^{(n)}(0)$ is the center of $\alpha^{(n)}$ on $Y^{(n)}$ and $K$ is the residue field of $\alpha^{(n)} \in Y^{(n)}_m$. Here, we note that the center $\alpha^{(n)}(0)$ is the generic point of $E$. Then, we can see that $\alpha^{(n)*}$ factors through

$$\beta : \hat{\mathcal{O}}_{Y^{(n)}, \alpha^{(n)}(0)} = k(E)[[\tau]] \to K[[t]],$$

where $k(E)$ is the rational function field of $E$ and $\tau$ is the generator of the maximal ideal of the local ring $\mathcal{O}_{Y^{(n)}, \alpha^{(n)}(0)}$. If we denote $\beta(\tau) = t^q$, then it implies that $\text{ord}_{\alpha^{(n)}} = q \cdot \text{val}_E$.\hspace{1cm} \Box

Finally, we assume that $C$ is an irreducible component of the intersection of $\text{Cont}^m(a_i)$'s. The generic point $\alpha$ of $C$ gives a divisorial valuation $q \cdot \text{val}_E$ for a prime divisor over $X$ by the above discussions. By definition of $\text{Cont}^m(a_i)$, it contains also the generic point of the maximal divisorial set $C(q \cdot \text{val}_E)$ for every $i$. This complete the proof of the last assertion of the corollary.
For the interpretation of MJ-minimal log discrepancies with center at a (not necessarily closed) point in terms of jet schemes, we need the following definition:

**Definition 3.17.** Let $X$ be a variety and $\eta \in X$ a (not necessarily closed) point. For a cylinder $S \subset X_\infty$ we define the codimension of $S \cap \pi^{-1}(\eta)$ as follows:

$$\text{codim}(S \cap \pi^{-1}(\eta), X_\infty)$$

$$:= \inf \left\{ \text{codim} C \mid C \text{ is an irreducible component of } S \cap \pi^{-1}\{\eta\} \text{ dominating } \{\eta\} \text{ and not contained in } (\text{Sing}X)_\infty \right\}.$$

By Corollary 3.16, we obtain the following interpretation of the MJ minimal log discrepancy by the arc space. This is proved [14] for the characteristic zero case.

**Theorem 3.18.** Let $(X,a)$ be a pair consisting of an arbitrary variety $X$ and a non-zero coherent ideal sheaf $a \subset \mathcal{O}_X$. Let $n \geq 0$ be a real number, let $W$ be a proper closed subset of $X$ and let $I_W$ be the (reduced) ideal of $W$. Then,

$$\text{mld}_{MJ}(W; X, a^n)$$

$$= \inf_{m_i \in \mathbb{N}} \{ \text{codim}(\text{Cont}^{m_1}(a) \cap \text{Cont}^{m_2}(\mathcal{J}_X) \cap \text{Cont}^\geq 1(I_W)) - m_1 n - m_2 \}$$

(2)

$$\text{mld}_{MJ}(W; X, a^n)$$

(3)

For $a$ (not necessarily closed) point $\eta \in X$, we also have

$$\text{mld}_{MJ}(\eta; X, a^n)$$

(4)

$$= \inf_{m_i \in \mathbb{N}} \{ \text{codim}(\text{Cont}^{m_1}(a) \cap \text{Cont}^{m_2}(\mathcal{J}_X) \cap \pi^{-1}(\eta)) - m_1 n - m_2 \}.$$

(5)

*Proof.* For char $k = 0$, we proved these in [14, Proposition 3.8 and Remark 3.8], using [4, Theorem 3.9]. The proof for positive characteristic case follows just in the same way by using Theorem 3.13 and Corollary 3.16. However, for the reader’s convenience, we write down the proof here. For the proof of the equality (2), we will show first the following inequality:

$$\text{mld}_{MJ}(\eta; X, a^n)$$

$$\geq \inf_{m_i \in \mathbb{N}} \{ \text{codim}(\text{Cont}^{m_1}(a) \cap \text{Cont}^{m_2}(\mathcal{J}_X) \cap \pi^{-1}(\eta)) - m_1 n - m_2 \}.$$

Let $E$ be any prime divisor over $X$ with the center in $W$. Let $m_1 = \text{val}_E(a)$, $m_2 = \text{val}_E(\mathcal{J}_X)$ and $v = \text{val}_E$. Then, there is a non-empty open subset $C$ of the maximal divisorial set $C_X(v)$ such that $C \subset \text{Cont}^{m_1}(a) \cap \text{Cont}^{m_2}(\mathcal{J}_X) \cap \text{Cont}^\geq 1(I_W)$.

Hence,

$$\hat{k}_E - \text{val}_E(\mathcal{J}_X) - n \cdot \text{val}_E(a) + 1 = \text{codim}(C_X(v)) - m_1 n - m_2$$

$$\geq \text{codim}(\text{Cont}^{m_1}(a) \cap \text{Cont}^{m_2}(\mathcal{J}_X) \cap \text{Cont}^\geq 1(I_W)) - m_1 n - m_2,$$

which yields the required inequality unless $\dim X = 1$ and $\text{mld}_{MJ}(X, a^n) = -\infty$.

When $\dim X = 1$ and $\text{mld}_{MJ}(X, a^n) = -\infty$, there is a prime divisor $E$ over $X$ with the center in $W$ such that $k_E - \text{val}_E(\mathcal{J}_X) - n \cdot \text{val}_E(a) + 1 < 0$. Let $m_1 = \text{val}_E(a)$ and $m_2 = \text{val}_E(\mathcal{J}_X)$, then $\text{codim}_{C_X(\text{val}_E) - m_1 n - m_2} < 0$. Here, for every $q \in \mathbb{N}$, by Proposition 3.13,

$$\text{codim}_{C_X(q \cdot \text{val}_E) - q m_1 n - q m_2} = q(\text{codim}_{C_X(\text{val}_E) - m_1 n - m_2}) < 0.$$
As a non-empty open subset of $C_X(q \cdot \text{val}_E)$ is contained in \(\text{Cont}^{m_1}(a) \cap \text{Cont}^{m_2}(J_X) \cap \text{Cont}^{\geq 1}(I_W)\), we have
\[
\text{codim}(\text{Cont}^{m_1}(a) \cap \text{Cont}^{m_2}(J_X) \cap \text{Cont}^{\geq 1}(I_W)) - q m_1 n - q m_2 \\
\leq \text{codim}(C_X(q \cdot \text{val}_E)) - q m_1 n - q m_2 \\
= q(\text{codim}(C_X(\text{val}_E)) - m_1 n - m_2) < 0.
\]
Here, if \(q \to \infty\), then, we have
\[
\text{codim}(\text{Cont}^{m_1}(a) \cap \text{Cont}^{m_2}(J_X) \cap \text{Cont}^{\geq 1}(I_W)) - q m_1 n - q m_2 \to -\infty,
\]
which implies that the right-hand side of (2) in the theorem is \(-\infty\).

For the proof of the converse inequality,
\[
\text{mld}_{MJ}(q; X, a\nu) \\
\leq \inf_{m \in \mathbb{N}}\{\text{codim}(\text{Cont}^{m_1}(a) \cap \text{Cont}^{m_2}(J_X) \cap \pi^{-1}(\eta)) - m_1 n - m_2\},
\]
we may assume that \(\widehat{k_E} - \text{val}_E(J_X) - n \cdot \text{val}_E(a) + 1 \geq 0\) for every prime divisor \(E\) over \(X\) with the center in \(W\). Indeed if there is a prime divisor \(E\) with the center in \(W\) and \(\widehat{k_E} - \text{val}_E(J_X) - n \cdot \text{val}_E(a) + 1 < 0\), then \(\text{mld}_{MJ}(W; X, a\nu) = -\infty\) by Remark 2.7, (iii), and therefore, the required inequality is trivial.

For \(m_i \in \mathbb{N}\), let \(C \subset \text{Cont}^{m_1}(a) \cap \text{Cont}^{m_2}(J_X) \cap \text{Cont}^{\geq 1}(I_W)\) be an irreducible component that gives the codimension of the cylinder. Then, the closure \(\overline{C}\) is \(C_X(v)\) for some divisorial valuation \(v\) by Corollary 3.16. Let \(v = q \cdot \text{val}_E\). Then \(m_1 = v(a)\), \(m_2 = v(J_X)\) and \(E\) is a prime divisor over \(X\) with the center in \(W\) and
\[
\text{codim}(\text{Cont}^{m_1}(a) \cap \text{Cont}^{m_2}(J_X) \cap \text{Cont}^{\geq 1}(I_W)) - m_1 n - m_2 \\
= \text{codim}(C_X(v)) - m_1 n - m_2 \\
= q(\widehat{k_E} + 1) - q n \cdot \text{val}_E(a) - q \cdot \text{val}_E(J_X) \geq \widehat{k_E} + 1 - n \cdot \text{val}_E(a),
\]
which yields the required inequality.

For the proof of (3) of the theorem, let
\[
a_m = \text{codim}(\text{Cont}^{m_1}(a) \cap \text{Cont}^{m_2}(J_X) \cap \text{Cont}^{\geq 1}(I_W)) - m_1 n - m_2, \\
b_m = \text{codim}(\text{Cont}^{\geq m_1}(a) \cap \text{Cont}^{\geq m_2}(J_X) \cap \text{Cont}^{\geq 1}(I_W)) - m_1 n - m_2.
\]
As \(\text{Cont}^{m}(a) \subset \text{Cont}^{\geq m}(a)\), we have \(a_m \geq b_m\). Therefore, it follows
\[
\inf_{m \in \mathbb{N}}\{a_m\} \geq \inf_{m \in \mathbb{N}}\{b_m\}.
\]

Next, we prove the converse inequality. For every \(m \in \mathbb{N}\), let \(C_X(v)\) be the irreducible component of \(\text{codim}(\text{Cont}^{2m_1}(a) \cap \text{Cont}^{2m_2}(J_X) \cap \text{Cont}^{\geq 1}(I_W))\) that gives the codimension. Then, for \(m_1' := v(a) \geq m_1\) and \(m_2' := v(J_X) \geq m_2\) we have
\[
\text{codim}(\text{Cont}^{2m_1}(a) \cap \text{Cont}^{2m_2}(J_X) \cap \text{Cont}^{\geq 1}(I_W)) \\
= \text{codim}(\text{Cont}^{m_1'}(a) \cap \text{Cont}^{m_2'}(J_X) \cap \text{Cont}^{\geq 1}(I_W)).
\]
Hence, \(b_m \geq a_{m'}\), which yields \(\inf_{m \in \mathbb{N}}\{b_m\} \geq \inf_{m \in \mathbb{N}}\{a_m\}\).

The equalities (4) and (hmldptgeq) follow in the same way, indeed one has only to be careful to replace “center in \(W\)” by “center \(\{\eta\}\).”

\[\square\]

**Remark 3.19.** Our formula can be easily extended for the combination of ideals \(a_1, a_2, \cdots, a_r\), instead of one ideal \(a\). I.e., we have
\[
\text{mld}_{MJ}(W; X, a_1^{e_1} a_2^{e_2} \cdots a_r^{e_r}) = \\
\inf_{m_i \in \mathbb{N}} \left\{ \text{codim} \left( \left( \bigcap_{i=1}^{r} \text{Cont}^{m_i}(a_i) \right) \cap \text{Cont}^{m_1+1}(J_X) \cap \text{Cont}^{\geq 1}(I_W) \right) - \sum_{i=1}^{r} m_i e_i - m_{r+1} \right\},
\]
where \(e_i\)'s are positive real numbers. Here, any of \(\text{Cont}^{m_i}(a_i)\)'s can be replaced by \(\text{Cont}^{\geq m_i}(a_i)\). For simplicity of the notation and the proofs, we keep formulating
the forthcoming formulas for one ideal only. But note that the formulas in this section are also valid under this combination form.

The description of log-canonical threshold on a non-singular variety by jet schemes in positive characteristic case is obtained by Zhu [22].

Here, we will show the inversion of adjunction Formula for MJ-minimal log discrepancies for the base field of arbitrary characteristic. The proof basically follows the idea in the case of characteristic 0. For the proof, we prepare some lemmas. The first one was proved in [6] for an arbitrary characteristic.

Lemma 3.20 ([6, Lemma 8.4]). Let $A$ be a non-singular variety and $M = H_1 \cap \cdots \cap H_c$ a codimension c complete intersection in $A$. If $C$ is an irreducible locally closed cylinder in $A_\infty$ such that
\[
C \subset \bigcap_{i=1}^c \text{Cont}^{\geq d_i}(H_i),
\]
and if there is an arc $\gamma \in C \cap M_\infty$ with $\text{ord}_x(\mathcal{J}_M) = e$, then
\[
\text{codim}(C \cap M_\infty, M_\infty) \leq \text{codim}(C, A_\infty) + e - \sum_{i=1}^c d_i.
\]

3.21. Consider $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} = \text{Spec}k$ as a multiplicative group scheme. For $m \in \mathbb{N} \cup \{\infty\}$, the morphism $k[[t]] \to k[s, s^{-1}][[t]]$ defined by $t \mapsto s \cdot t$ gives an action
\[
\phi : \mathbb{G}_m \times \text{Spec}k[[t]] \to \text{Spec}k[[t]]
\]
of $\mathbb{G}_m$ on $\text{Spec}k[[t]]$. Therefore, it gives an action
\[
\Phi : \mathbb{G}_m \times \text{Spec}k X_\infty \to X_\infty
\]
of $\mathbb{G}_m$ on $X_\infty$. As $\phi$ is extended to a morphism: $\overline{\phi} : \mathbb{A}^1 \times \text{Spec}k[[t]] \to k[[t]]$, we obtain the extension
\[
\overline{\Phi} : \mathbb{A}^1 \times \text{Spec}k X_\infty \to X_\infty
\]
of $\Phi$.

In a similar way, we have an action of $\mathbb{G}_m$ on $X_r$ for $r \in \mathbb{N}$ and note that the truncation morphisms are $\mathbb{G}_m$-equivariant.

The following lemma is proved in Lemma 8.3 in [6] for characteristic 0 case by using a resolution of singularities. To give a characteristic free proof, we have only to check that the proof in [6] works under the normalized blow-up instead of a resolution. Let $\overline{\Phi} : \mathbb{A}^1 \times X_\infty \to X_\infty$ be the morphism given as above. Note that if an arc $\gamma$ has the center $x \in X$, then $\Phi(0, \gamma)$ is the constant arc over $x$.

Lemma 3.22 ([6]). Let $C$ be a non-empty cylinder on $X_\infty$ for a variety $X$. If $\overline{\Phi}(\mathbb{A}^1 \times C) \subset C$, then $C \not\subset (\text{Sing} X)_\infty$.

Proof. Take an arc $\gamma \in C$ and denote $C = (\psi^X_m)^{-1}(S)$ for some constructible set $S \subset X_m$. Let $x \in X$ be the center of $\gamma$, then by $\mathbb{A}^1$-invariance of $C$, the trivial $m$-jet $0^e_m$ at $x$ belongs to $S$. Let $f : X' \to X$ be the normalized blow-up at the point $x$ and take a non-singular point $x'$ of $X'$ inside $f^{-1}(x)$. Let $C'$ be the cylinder $(\psi^{X'}_m)^{-1}(0^e_m)$, then $f_\infty(C') \subset C$. As $C'$ is not a thin set in $X'_\infty$, $C$ is not thin in $X_\infty$ either, which gives that $C \not\subset (\text{Sing} X)_\infty$.

The following is the inversion of adjunction formula for MJ-minimal log discrepancies which was proved for a characteristic 0 base field in [3] and [14] independently. We give here a proof for the arbitrary characteristic case. We prepared necessary lemmas for the proof in arbitrary characteristic case that were used in the proof.
in characteristic 0 case. So the outline of the proof can be the same as that of the characteristic 0 case (presented in [3] and [14] which are based on [6]). Here, we remind the reader that if \( \dim X = 1 \) and there is a prime divisor \( E \) over \( X \) such that \( a_{MJ}(E; X, a^n) < 0 \), then we define \( \mld_{MJ}(x; X, a^n) = -\infty \) (see, Definition 2.6).

**Theorem 3.23** (inversion of adjunction). Let \( X \) be a variety over an algebraically closed field \( k \) of an arbitrary characteristic and \( A \) a smooth variety containing \( X \) as a closed subscheme of codimension \( c \). Let \( \tilde{\alpha} \subset \mathcal{O}_A \) be a coherent ideal sheaf such that its image \( \alpha := \tilde{\alpha}\mathcal{O}_X \subset \mathcal{O}_X \) is non-zero. Denote the ideal of \( X \) in \( A \) by \( I_X \). Then, for a proper closed subset \( W \) of \( X \), we have

\[
\mld_{MJ}(W; X, a^n) = \mld(W; A, \tilde{\alpha}^n I_X).
\]

For a point \( \eta \in X \), we have

\[
\mld_{MJ}(\eta; X, a^n) = \mld(\eta; A, \tilde{\alpha}^n I_X).
\]

**Proof.** First, we prove the inequality \( \geq \) in (6). We assume the contrary and will induce a contradiction. By the assumption, there exist \( \epsilon, m \in \mathbb{N} \) and an irreducible component \( C \subset \text{Cont}^{2m}(\alpha) \cap \text{Cont}^e(J_X) \cap \text{Cont}^e(I_W) \) such that \( \text{codim} C - mn < \mld(W; A, \tilde{\alpha}^n I_X) \). Then, for a sufficiently large \( s \in \mathbb{N} \), we obtain

\[
(s + 1)d - \dim \psi_s(C) - mn = \text{codim} C - mn < \mld(W; A, \tilde{\alpha}^n I_X),
\]

where \( I_W \) is the defining ideal of \( W \) in \( X \). As \( \psi_s(C) \subset \text{Cont}^{2m}(\alpha) \cap \text{Cont}^e(I_W) \), we have

\[
C \subset (\psi_s(C))^{-1}(\text{Cont}^{2m}(\tilde{\alpha}) \cap \text{Cont}^e(I_W)) \cap \text{Cont}^e(I_X) =: S,
\]

where \( \psi_s: A_{\infty} \to A_s \) is the truncation morphism.

Now we obtain

\[
(d + c)(s + 1) - \dim \psi_s(C) = \text{codim}(\psi_s(C), A_s) = \text{codim}((\psi_s^{-1})^{-1}(\psi_s(C), A_{\infty}) \geq \text{codim}(S, A_{\infty}) \geq \mld(W; A, \tilde{\alpha}^n I_X) + mn + c(s + 1),
\]

which is a contradiction to (8).

For the converse inequality in (6), we have only to show the following claim

**Claim 3.24.** For a prime divisor \( F \) over \( A \) with center in \( W \), there is a prime divisor \( E \) over \( X \) with center in \( W \) and an integer \( q \geq 1 \) such that

\[
q \cdot a_{MJ}(E; X, a^n) \leq a_{MJ}(F, A, \tilde{\alpha}^n I_X).
\]

We can prove that this induces the inequality

\[
\mld_{MJ}(W; X, a^n) \leq \mld(W; A, \tilde{\alpha}^n I_X)
\]

as follows: If \( \mld_{MJ}(W; X, a^n) = -\infty \), then the required inequality is trivial. If \( \mld_{MJ}(W; X, a^n) \geq 0 \) then \( a_{MJ}(E; X, a^n) \geq 0 \) in (9), which yields \( a_{MJ}(F, A, \tilde{\alpha}^n I_X) \geq \frac{1}{q} a_{MJ}(F, A, \tilde{\alpha}^n I_X) \geq a_{MJ}(E; X, a^n) \geq \mld_{MJ}(W; X, a^n) \) for every prime divisor \( F \) over \( A \) with center in \( W \).

In the following, we prove Claim 3.24. Consider the maximal divisorial set

\[
V = C_A(\text{val}_F) \subset A_{\infty};
\]

then we have

\[
\text{codim}(V, A_{\infty}) = k_F + 1.
\]

The intersection \( V \cap X_\infty \subset X_\infty \) is a non-empty cylinder in \( X_\infty \) and is not contained in \( (\text{Sing} X)_\infty \) by Lemma 3.22. Let \( C \) be an irreducible component of \( V \cap X_\infty \) that is not contained in the arc space of the singular locus of \( X \), and the generic point \( \gamma \in C \) gives the minimal value \( \text{ord}_x(J_X) = e \) among the points in \( V \cap X_\infty \). We
can also assume that $C$ has the minimal codimension among the components with
\[ \text{ord}_c(J_X) = e. \]
Then, by Corollary 3.16, there exists a prime divisor $E$ over $X$ with
the center in $W$, such that the generic point $\gamma$ of $C$ gives the divisorial valuation
$q \cdot \text{val}_E$ for some $q \in \mathbb{N}$. Therefore, $C \subset C_X(q \cdot \text{val}_E)$ and we have the following inequalities:

\[ q \cdot (k_E + 1) = \text{codim}(C_X(q \cdot \text{val}_E), X_\infty) \leq \text{codim}(C, X_\infty) \]  

Now, by an appropriate choice of $e$ generators of $I_X$, we can take a complete
intersection scheme $M$ of dimension $d$ containing $X$ such that $\text{ord}_e(J_M) = e$. Then,
by [6, Corollary 9.2], we have
\[ J_M \cdot O_X \subset ((I_M : I_X) + I_X)/I_X. \]
It follows that $\gamma$ belongs to the open cylinder
\[ V_0 := V \cap \text{Cont}^e(J_M) \cap \text{Cont}^e(I_M : I_X). \]
Here, we note that
\[ V_0 \cap X_\infty = V_0 \cap M_\infty. \]
Indeed, by the definition of $V_0$ an arc $\alpha \in V_0 \cap M_\infty$ has finite order along $(I_M : I_X)$
which is a defining ideal of the union $X'$ of the components of $M$ other than $X$.
This means $\alpha \not\in X'$.  
Here, for every $\beta \in M_\infty$, we note that $\text{ord}_\beta(J_X) \leq \text{ord}_\beta(J_M)$; therefore, by the
definition of $C$, we obtain

\[ \text{codim}(C, X_\infty) = \text{codim}(V_0 \cap X_\infty, X_\infty) = \text{codim}(V_0 \cap M_\infty, M_\infty). \]

Apply Lemma 3.20 to $V_0$ and we obtain
\[ \text{codim}(V_0 \cap M_\infty, M_\infty) \leq \text{codim}(V_0, A_\infty) + e - \sum_{i=1}^c d_i, \]
where $d_i$'s satisfy $V_0 \subset \bigcap_{i=1}^c \text{Cont}^e(M_i)$. Then, as $V_0$ is an open subset of $V = C_A(\text{val}_F)$, the term $\sum d_i$ in the above inequality satisfies
\[ \sum d_i \geq c \cdot \text{val}_F(I_X) \]
and the equality $\text{codim}(V_0, A_\infty) = \text{codim}(V, A_\infty)$ holds. On the other hand, $e = \text{ord}_\gamma(J_X) = q \cdot \text{val}_E(J_X) = q \cdot j_E$. Therefore, we have the following inequality

\[ \text{codim}(V_0 \cap M_\infty, M_\infty) \leq \text{codim}(V_0, A_\infty) + q \cdot \text{val}_E(J_X) - c \cdot \text{val}_F(I_X). \]

Now combining (10), (11), and (12), we obtain
\[ q \cdot (k_E - j_E + 1) \leq \text{codim}(V, A_\infty) - c \cdot \text{val}_F(I_X). \]

Note that for any proper coherent ideal sheaf $b \subset O_M$ not vanishing on any component of $X$, we have $\text{val}_C(b | X) \geq \text{val}_V(b)$ by the inclusion $C \subset V$. In particular, this implies that
\[ q \cdot \text{val}_E(b | X) \geq \text{val}_F(b), \]
which yields the inequality in Claim 3.24.

The proof of the equality (7) follows in the same way. Indeed, we have only to
be careful to replace “with the center in $W$” by “with the center $\overline{\eta}$” in the proof above.

\[ \square \]

**Remark 3.25.** The formula in the theorem can be easily extended for the combination
of ideals $a_1, a_2, \ldots, a_r$ instead of one ideal $a$. I.e., we have
\[ \text{mld}(W; X, a_1^{e_1} a_2^{e_2} \cdots a_r^{e_r}) = \text{mld}(W; A, \hat{a}_1^{e_1} \hat{a}_2^{e_2} \cdots \hat{a}_r^{e_r} I_X), \]
where $e_i$'s are non-negative real numbers and $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_r$ are coherent ideal sheaves of $O_A$ such that the extensions $a_i = \hat{a}_i O_X$'s are not zero.
Corollary 3.26. Let $\eta \in X$ be a point of a variety $X$ of dimension $d$. For each $m \in \mathbb{N}$ we denote $r_m = \dim \pi_m^{-1}(x)$ for a general closed point $x \in \{\eta\}$. Then, we have the equality:

$$
\text{mld}_{M, J}(\eta; X) := \text{mld}_{M, J}(x; X, \mathcal{O}_X) = \inf_m \left\{ (m + 1)d - \dim \pi_m^{-1}(\eta) \right\}
= \inf_m \left\{ (m + 1)d - (\dim \{\eta\} + r_m) \right\}.
$$

Proof. Since the statement is local, we can assume that $X$ is a closed subvariety of codimension $c$ of a non-singular variety $A$. By Theorem 3.23, we have

$$
\text{mld}_{M, J}(\eta; X) = \text{mld}(\eta; A, I^\infty_X)
$$

and by Theorem 3.18, we have

$$
\text{mld}(\eta; A, I^\infty_X) = \inf_{m \geq 0} \left\{ \text{codim}(\text{Cont}^{m+1}(I_X) \cap (\pi^A)^{-1}(\eta), A_\infty) - (m + 1)c \right\},
$$

where, for our convenience later, we shift $m$ to $m + 1$ on the right-hand side. By Definition 3.17, $\text{codim}(\text{Cont}^{m+1}(I_X) \cap (\pi^A)^{-1}(\eta), A_\infty)$ is the minimal codimension of irreducible components $C \subset (\psi^A_m)^{-1}(X_m) \cap \pi^{-1}(\{\eta\})$ in $A_\infty$ that dominate $\{\eta\}$. Therefore,

$$
\text{codim}(\text{Cont}^{m+1}(I_X) \cap (\pi^A)^{-1}(\eta), A_\infty) = \text{codim}(\text{Cont}^{m+1}(\pi_m^{-1}(\eta), A_m)).
$$

As $\text{codim}(\pi_m^{-1}(\eta), A_m) = (m + 1)(d + c) - \dim (\pi_m^{-1}(\eta)$, we have the first equality in the corollary. For the second equality, we need to know that

$$
\dim \pi_m^{-1}(\eta) = \dim \{\eta\} + r_m.
$$

□

Corollary 3.27. Let $X$ be a variety of dimension $d$ and $a$ a coherent ideal sheaf of $\mathcal{O}_X$. Let $V \subset W$ be two irreducible proper closed subsets of $X$ and $\eta_V$ and $\eta_W$ are the generic points of $V$ and $W$, respectively. Then,

(i) We have the following inequality:

$$
\text{mld}_{M, J}(\eta_V; X, a) \leq \text{codim}(V, X),
$$

where the equality holds if and only if $\eta_V$ is a regular point and $a = \mathcal{O}_X$ around $\eta_V$.

In particular, if $x \in X$ is a closed point, then

$$
\text{mld}_{M, J}(x; X, a) \leq d
$$

and the equality holds if and only if $X$ is smooth at $x$ and $a = \mathcal{O}_X$ around $x$. (This statement for the usual mld is Shokurov’s conjecture and is not yet proved.)

(ii) $\text{mld}_{M, J}(\eta_V; X, a) \leq \text{mld}_{M, J}(\eta_W; X, a) + \text{codim}(V, W)$,

where the equality holds for very general $V$ in $W$; i.e., $\eta_V$ is in the complement of a countable number of closed subsets in $W$. (If $k$ is an uncountable field, then this subset is non-empty.)

(iii) If $\text{char } k = 0$, then the equality in (ii) holds for general $V$ in $W$.

Proof. First note that if $a \neq \mathcal{O}_X$ around $\eta_V$, then

$$
\text{mld}_{M, J}(\eta_V; X, a) < \text{mld}_{M, J}(\eta_V; X, \mathcal{O}_X)
$$

by the definition of MJ-log discrepancy. Here, by Corollary 3.26, we have

$$
mld_{M, J}(\eta_V; X, \mathcal{O}_X) \leq (m + 1)d - r_m - \dim V
$$

and by Theorem 3.23, we have

$$
mld_{M, J}(\eta_V; X, \mathcal{O}_X) = \inf_{m \geq 0} \left\{ \text{codim}(\text{Cont}^{m+1}(I_X) \cap (\pi^A)^{-1}(\eta_V), A_\infty) - (m + 1)c \right\},
$$

where $\text{codim}(\text{Cont}^{m+1}(I_X) \cap (\pi^A)^{-1}(\eta_V), A_\infty)$ is the minimal codimension of irreducible components $C \subset (\psi^A_m)^{-1}(X_m) \cap \pi^{-1}(\{\eta\})$ in $A_\infty$ that dominate $\{\eta\}$. Therefore,

$$
\text{codim}(\text{Cont}^{m+1}(I_X) \cap (\pi^A)^{-1}(\eta_V), A_\infty) = \text{codim}(\text{Cont}^{m+1}(\pi_m^{-1}(\eta), A_m)).
$$

As $\text{codim}(\pi_m^{-1}(\eta), A_m) = (m + 1)(d + c) - \dim (\pi_m^{-1}(\eta)$, we have the first equality in the corollary. For the second equality, we need to know that

$$
\dim \pi_m^{-1}(\eta) = \dim \{\eta\} + r_m.
$$

□
for every $m \geq 0$. Therefore, in particular for $m = 0$, we obtain
\[
\mld_{MJ}(\eta_V; X, \mathcal{O}_X) \leq \text{codim}(V, X).
\]

Now assume the equality in (i): $\mld_{MJ}(\eta_V; X, \mathcal{O}_X) = d - \dim V$. Then, by the first comment in this proof, we have $a = \mathcal{O}_X$. Consider the inequality (14) for $m = 1$, we obtain
\[
d - \dim V \leq (d - \dim V) + d - r_1.
\]
Here, we note that $r_1$ is the dimension of the tangent space of $X$ at a general closed point $x \in V$. Then, the inequality (15) gives $r_1 = d$, which means that general points of $V$ are non-singular in $X$.

For the proof of (ii), let
\[
s_V(m, n) := \text{codim} \left( \text{Cont}^{\geq m}(I_X) \cap \text{Cont}^{\geq n}(a) \cap (\psi^1)^{-1}(\eta_V), A_\infty \right) - mc - n.
\]
\[
s_W(m, n) := \text{codim} \left( \text{Cont}^{\geq m}(I_X) \cap \text{Cont}^{\geq n}(a) \cap (\psi^1)^{-1}(\eta_W), A_\infty \right) - mc - n.
\]
Then, by Theorem 3.18 and Theorem 3.23, we have
\[
\mld_{MJ}(\eta_V; X, a) = \inf_{m, n} s_V(m, n) \quad \text{and} \quad \mld_{MJ}(\eta_W; X, a) = \inf_{m, n} s_W(m, n).
\]
Remember the action of $\mathbb{G}_m$ on $A_r$ (3.21). Then, for each $m, n$, by an appropriate $r = r(m, n) \in \mathbb{N}$ and a $\mathbb{G}_m$-invariant closed subset
\[
S_{m, n} = \psi_r \left( (\text{Cont}^{\geq m}(I_X) \cap \text{Cont}^{\geq n}(a)) \right) \subset A_r,
\]
we can express
\[
s_V(m, n) - s_W(m, n) = \dim S_{m, n} \cap \pi_r^{-1}(\eta_W) - \dim S_{m, n} \cap \pi_r^{-1}(\eta_V).
\]
\[
= (\dim W + \delta_W) - (\dim V + \delta_V),
\]
where $\delta_V$ and $\delta_W$ are the dimensions of general fibers of $\pi_r|_{S_{m, n} \cap \pi_r^{-1}(V)}$ and $\pi_r|_{S_{m, n} \cap \pi_r^{-1}(W)}$, respectively. As $S_{m, n}$ is $\mathbb{G}_m$-invariant, the restricted morphism $\pi_r|_{S_{m, n} \setminus \sigma(A)} : S_{m, n} \setminus \sigma(A) \to A$ factors through the projective morphism $\pi'_r : (S_{m, n} \setminus \sigma(A))/\mathbb{G}_m \to A$. Here, $\sigma(A) \subset A_r$ is the subset consisting of the trivial $r$-jets on $A$. Therefore, dimension of fibers of $\pi'_r$ and also $\pi_r|_{S_{m, n}}$ are upper-semi-continuous, which implies the inequality $\delta_W \leq \delta_V$. This yields
\[
s_V(m, n) - s_W(m, n) \leq \text{codim}(V, W)
\]
for every $m, n$, which yields the inequality in (ii).

For a fixed $m, n$, there exists a closed subset $F_{m, n} \subset W$ such that for every point $\eta_V \in W$ which is not contained in $F_{m, n}$ satisfies $\delta_V = \delta_W$. Then, for these $V$ we obtain
\[
s_V(m, n) - s_W(m, n) = \text{codim}(V, W).
\]
Therefore, if $\eta_V$ is not contained in $F = \bigcup_{m, n} F_{m, n}$, then, we obtain the equality
\[
\mld_{MJ}(\eta_V; X, a) = \mld_{MJ}(\eta_W; X, a) + \text{codim}(V, W).
\]
For the proof of (iii), assume that $\text{char } k = 0$. Let $f : A' \to A$ be an embedded log resolution of $(X, a)$ with at least one exceptional divisor with the center $\{\eta_W\}$. Then, there is an exceptional prime divisor $E \subset A'$ computing the $\mld_{MJ}(\eta_W; X, a)$. If $\eta_V \in W$ is not in the union of the lower dimensional centers of the exceptional divisors of $f$, then, the divisor obtained by the blow-up with the center $f^{-1}(V) \cap E$ computes
\[
\mld_{MJ}(\eta_V; X, a) = \mld_{MJ}(\eta_W; X, a) + \text{codim}(V, W).
\]
\[\square\]
Remark 3.28. As is seen in the proof of (iii), one can see that for a positive characteristic case if (iii) in Corollary 3.27 does not hold, it shows a counter example of the existence of resolution of singularities.

4. MJ-CANONICAL AND MJ-LOG-CANONICAL SINGULARITIES

We say that $X$ has MJ-canonical (resp. MJ-log-canonical) singularities if the pair $(X, \mathcal{O}_X)$ has MJ-canonical (resp. MJ-log-canonical) singularities. We denote $\text{mld}_{MJ}(x; X, \mathcal{O}_X)$ by $\text{mld}_{MJ}(x; X)$. In this section, we will study the nature of MJ-canonical singularities and MJ-log-canonical singularities.

Lemma 4.1 ([8, Proposition 3.3]). Let $x \in X$ be a closed point. If $X$ is MJ-canonical at $x$, then the embedding dimension satisfies

$$\text{emb}(X, x) \leq 2d - 1.$$ 

If $X$ is MJ-log-canonical at $x$, then the embedding dimension satisfies

$$\text{emb}(X, x) \leq 2d.$$

Proof. This is proved in [8, Proposition 3.3] in case char $k = 0$. However, the main point of the proof is the formula in Corollary 3.26 and we have seen in the proof of the corollary that the formula holds true for an arbitrary characteristic. Therefore, the same proof works for the lemma in the positive characteristic case too. □

Definition 4.2. We say that a variety $X$ has a pseudo rational singularity at $x \in X$ if

1. $X$ is normal around $x$;
2. For every partial resolution $f : Y \to X$, which means a proper birational morphism with normal $Y$, the equality

$$f_*\omega_Y = \omega_X$$

holds around $x$;
3. $X$ is Cohen–Macaulay around $x$.

Note that if char $k = 0$ or if dim $X = 2$, this definition is equivalent to the following:

1. $X$ is normal around $x$;
2'. For every resolution $f : Y \to X$ the vanishing

$$R^if_*\mathcal{O}_Y = 0$$

holds for $i > 0$ around $x$.

The singularity $(X, x)$ satisfying (1) and (2') is called a rational singularity.

Proposition 4.3. Let a variety $X$ have at worst MJ-canonical singularities. Then $X$ has normal hypersurface singularities in codimension 2.

If char $k = 0$, then $X$ is normal; furthermore, the singularities on $X$ are rational.

Proof. The second statement is proved in [3, Theorem 7.7] and [7, Corollary 3.7] independently. Regarding the first statement, since the problem is local, we may assume that $X$ is a closed subvariety in the affine space $A = \mathbb{A}^N$. Then, as $X$ is MJ-canonical, it follows that $\text{mld}_{MJ}(\eta; X) = \text{mld}(\eta, A, I_X^c) \geq 0$, where $I_X$ is the defining ideal of $X$ in $A$ and $c = \text{codim}(X, A)$.

The first statement is proved by blow-up $A$ at an irreducible component of the singular locus of $X$ and checking the discrepancy of $(A, I_X^c)$ as in the proof of Proposition 4.7. But, here, we present a proof using jet schemes discussions.

Let $\eta \in X$ be the generic point of an irreducible closed subset of codimension one. Then, as $X$ is MJ-canonical, it follows that $\text{mld}_{MJ}(\eta; X) \geq 1$. On the other
hand, we have \( \text{mld}_{\text{MJ}}(\eta; X) \leq d - \dim \{ \eta \} = 1 \) by Corollary 3.27. Then, we obtain that the equality in (i) in Corollary 3.27 holds, which yields that \( X \) is regular at \( \eta \). Now, \( \dim \text{Sing}(X) \leq d - 2 \). Let \( \zeta \in X \) be the generic point of an irreducible component of \( \text{Sing}(X) \) of codimension 2. Then, by inequality (14), it follows

\[
1 \leq \text{mld}_{\text{MJ}}(\zeta; X) \leq (m + 1)d - r_m - \dim \{ \zeta \},
\]

where \( r_m \) is the dimension of a general fiber of \( \pi_m \). Considering the case \( m = 1 \), we obtain

\[
1 \leq \text{mld}_{\text{MJ}}(\zeta; X) \leq 2d - r_1 - (d - 2) \leq 1.
\]

Here, the last inequality is shown as follows: note that \( r_1 \) is the dimension of Zariski tangent space of \( X \) at a general point; therefore, \( r_1 \geq d + 1 \), as the point is a singular point. Therefore, all inequalities in (16) become equalities, in particular \( r_1 = d + 1 \), which means that a general point is a hypersurface singularity. Hence, \( X \) is a Gorenstein variety in codimension 2 and satisfies \( R_1 \), which yields that \( X \) is normal in codimension 2 by Serre’s criteria.

**Corollary 4.4.** A two-dimensional singularity \((X, x)\) is MJ-canonical if and only if it is a rational double point in arbitrary characteristic.

**Proof.** In case \( \text{char} \ k = 0 \), this statement is proved in [8]. The following is a characteristic free proof. As a rational double point of dimension two is a normal hypersurface singularity, the canonicity in the usual sense is equivalent to MJ-canonicity. Therefore, a rational double point of dimension two is MJ-canonical.

Conversely, if \((X, x)\) is MJ-canonical, then, by Proposition 4.3, it is a normal hypersurface singularity. Therefore it is canonical in the usual sense, which yields that it is rational double.

**Corollary 4.5.** If an MJ-canonical variety \( X \) is locally a complete intersection, then \( X \) is normal and has pseudo rational singularities. In particular, for \( \text{char} \ k = 0 \), \( X \) has rational singularities.

**Proof.** As \( X \) is locally a complete intersection, it is Gorenstein. By Proposition 4.3, \( X \) satisfies \( R_1 \), therefore by Serre’s criteria, it is normal. Since \( X \) is locally a complete intersection, we have that \( \hat{k}_E - j_E = k_E \) (Remark 2.5) for every prime divisor \( E \) over \( X \). On the other hand, when we take a partial resolution \( f : \hat{Y} \to X \), for every prime divisor \( E \) on \( \hat{Y} \), we obtain \( \text{val}_E(K_{\hat{Y}} - f^*K_X) = k_E = \hat{k}_E - j_E \geq 0 \) by the assumption that \( X \) is MJ-canonical. Therefore, on \( \hat{Y} \), we have the inequality \( K_{\hat{Y}} \geq f^*K_X \) which yields

\[
f_*\omega_{\hat{Y}} \supset f_*f^*\omega_X = \omega_X.
\]

Now, we obtain \( f_*\omega_{\hat{Y}} = \omega_X \), since the opposite inclusion of (17) is trivial.

**Definition 4.6.** Let \( x \in X \) be a closed point of a variety \( X \). We say that \( x \) is a normal crossing double point in \( X \) if \( \hat{\sigma}_{x, X} = k[[x_1, \ldots, x_N]]/(x_1 \cdot x_2) \).

**Proposition 4.7.** Let a variety \( X \) have at worst MJ-log-canonical singularities. Then a general point of the singular locus of codimension one is normal crossing double.

**Proof.** Since the problem is local, we may assume that \( X \) is a closed subvariety in a smooth affine variety \( A \). Then, as \( X \) is MJ-log-canonical, it follows that \( \text{mld}_{\text{MJ}}(\eta; X) = \text{mld}(\eta, A, I_X^c) \geq 0 \), for every (not necessarily closed) point \( \eta \in X \), where \( I_X^c \) is the defining ideal of \( X \) in \( A \) and \( c = \text{codim}(X, A) \). It seems to be well known that if \((A, I_X^c)\) is log-canonical, then \( X \) is a hypersurface with at worst normal crossing double singularities in codimension 1 when the base field is of
characteristic 0. We will write the proof that works for an arbitrary characteristic case.

Let $S \subset X$ be an irreducible component of codimension 1 in the singular locus and let $\eta \in S$ be the generic point. Take $A$ as a minimal dimensional smooth ambient space around $\eta$. Let $f : A_1 \to A$ be the blow-up of $A$ with the center $S$ and let $E_1$ be the exceptional divisor dominating $S$. Then, we obtain

$$a(E_1; A, I_X^\eta) = k_{E_1} - c \cdot \text{val}_{E_1}(I_X) + 1 \geq 0.$$  

Here, we note that $k_{E_1} = c$. By the minimality of $\dim A$, every element of $I_X$ has multiplicity $\geq 2$ at $\eta$, which yields $\text{val}_{E}(I_X) \geq 2$. Then, we obtain

$$0 \leq k_{E_1} - c \cdot \text{val}_{E_1}(I_X) + 1 \leq c - 2c + 1 = -c + 1 \leq 0.$$  

Therefore all equalities should hold, in particular, $a(E_1; A, I_X^\eta) = 0$, $c = 1$ and $\text{val}_{E_1}(I_X) = 2$. These mean that $X$ has hypersurface double points in codimension 1.

If a general point of $S$ is not a normal crossing double point, then $E_1$ and the proper transform $X_1$ contact with the order $\geq 2$ along a closed subset $S_1$ dominating $S$. Let $A_2 \to A_1$ be the blow-up with the center $S_1$ and $E_2$ the exceptional divisor dominating $S_1$. Let $E'_1 \subset A_2$ and $X_2 \subset A_2$ be the proper transforms of $E_1$ and $X_1$, respectively.

Then, three divisors $E_2, E'_1$ and $X_2$ in $A_2$ still have an intersection at a closed subset $S_2$ of dimension $d - 1$. Now, blow up $A_3 \to A_2$ with the center $S_2$ and let $E_3$ be the exceptional divisor dominating $S_2$. Then, the log discrepancy at $E_3$ is

$$a(E_3; A, I_X) = k_{E_3} - \text{val}_{E_3}I_X + 1 \leq 3 - 5 + 1 = -1,$$  

a contradiction. Therefore, $E_1$ and $X_1$ have the reduced intersection at $S_1$ over general points of $S$, which implies that $X$ is normal crossing double at a general point of $S$.

\[ \square \]

**Theorem 4.8.** Let $k$ be an algebraically closed field. Let $p \in X$ be a closed point of an arbitrary variety $X$ over $k$. A pair $(X,B)$ consisting of $X$ and an effective $\mathbb{R}$-Cartier divisor $B$ on $X$ satisfies

\[ \dim X - 1 \leq \text{mld}(p; X, B) \]

if and only if either

(i) $\dim X \geq 2$, $B = 0$ and $(X,p)$ is a compound Du Val singularity,

(ii) $B = 0$, and $(X,p) \subset (\mathbb{A}^3, 0)$ is given by:

\[
\begin{align*}
xy &= 0, \\
z^2 + xy^2 &= 0 \text{ if char } k \neq 2, \text{ or} \\
z^2 + xy^2 + yzg(x,y) &= 0 \text{ if char } k = 2, \text{ where mult}_g \geq 1 \text{ and either } g = 0 \text{ or } g(x,0) \neq 0.
\end{align*}
\]

(iii) $(X,p)$ is non-singular and $0 \leq \text{mult}_p B \leq 1$.

In cases (i) and (ii), we have $\text{mld}(p; X, J_X) = \dim X - 1$ and in case (iii) we have $\text{mld}(p; X, B) = \text{mld}(p; X, B) = \dim X - \text{mult}_p B$ and the minimal log discrepancy is computed by the exceptional divisor of the first blow-up at $p$.

**Proof.** As in [15, Theorem 4.1], it suffices to show that equality in (18) holds if and only if (i) or (ii) are satisfied. Germs of varieties $(X, p)$, where $p$ is a closed point, satisfying $\text{mld}(p; X, B) = \dim X - 1$ are called top singularities in [15].

Recall the invariant $\tau(X, p)$ introduced in [10]. In the case of a germ of an hypersurface $(X, p)$ in $(\mathbb{A}^{d+1}, 0)$ defined by $f \in k[x_1, \ldots, x_{d+1}]$, $\tau$ is the smallest possible dimension of a linear subspace $V_0$ of $V = kx_1 + kx_2 + \cdots + kx_{d+1}$ such that the initial term $\text{in}_f$ of $f$ lies in the subalgebra $k[V_0]$ of $k[x_1, \ldots, x_{d+1}]$. It is known to
be an invariant of \((X, p)\).

The following results, proved in [15] if \(\text{char } k = 0\), are based on equality (13), hence, remain true if \(\text{char } k > 0\):

(R0) [15, Lemma 3.5] Let \(X\) be a \(d\)-dimensional variety and let \(X' \subset X\) be a \((d - c)\)-dimensional subvariety, which is defined as the zero locus of \(c\) elements of \(O_X\). Let \(p\) be a closed point in \(X'\). If \((X', p)\) is a top singularity, then \((X, p)\) is a top singularity.

(R1) [15, Lemma 3.6] If \(X\) has a top singularity at \(p\), then \(X\) has a hypersurface singularity of multiplicity 2 at \(p\).

(R2) [15, Lemma 3.20] Let \((X, p) \subset (\mathbb{A}_k^{d+1}, 0)\) be a germ of a hypersurface singularity of multiplicity 2 at \(p\) and \(\tau(X, p) = 1\). Hence, the equation of \((X, p) \subset (\mathbb{A}_k^{d+1}, 0)\) is expressed as

\[
 f = x_1^2 - g(x_2, \ldots, x_{d+1}) + x_1 g'(x_1, \ldots, x_{d+1}) = 0.
\]

Here, we note that \(x_1 g' = x_1^2 - g(x_2, \ldots, x_{d+1}) \in x_1(x_1, \ldots, x_{d+1})^2\). Then, if \((X, p)\) is a top singularity, we have \(d \geq 2\) and \(\text{mult } g = 3\).

(R3) [15, Proposition 3.23] In the conditions of (R2), suppose also that the initial form of \(g\) has only one factor. Hence, the equation of \((X, p) \subset (\mathbb{A}_k^{d+1}, 0)\) gives

\[
 x_1^2 + x_3^2 + g_3(x_3, \ldots, x_{d+1}) \cdot x_2 + g_4(x_3, \ldots, x_{d+1}) \\
 \in x_1(x_3, \ldots, x_{d+1})^2 + x_1 x_2(x_3, \ldots, x_{d+1}) + (x_1 x_2^2) + x_2^2(x_3, \ldots, x_{d+1})^2
\]

where \(\text{mult } g_i \geq i\), for \(i = 3, 4\). Then, if \((X, p)\) is a top singularity, we have either \(\text{mult } g_3 = 3\) or \(4 \leq \text{mult } g_4 \leq 5\).

On the other hand, in pages 256–268 of [18], a characterization of pseudo rational double points over any field is given.

Restricting to an algebraically closed field \(k\), the following is proved: Let us assume the hypothesis H1:

H1) \(X \subset \mathbb{A}_k^3\) be a surface of multiplicity 2. Then \(\tau \leq 3\) and we have:

Case I: \(\tau = 3\) if and only if \((X, p) \subset (\mathbb{A}_k^3, 0)\) has \(A_1\)-singularity.

Case II (II a) in [18]): \(\tau = 2\) is equivalent to \((X, p) \subset (\mathbb{A}_k^3, 0)\) having \(A_n\)-singularity \((n \geq 2)\) if we assume that \((X, p)\) is pseudo rational.

If \(\tau = 1\), the equation of \(X\) gives \(z^2 - G(x, y) \in zM^2\) with \(\text{mult } G \geq 3\), where \(M\) is the maximal ideal at the origin. Besides, the hypothesis \(X\) pseudo rational implies that

H2) \(\text{mult } G = 3\).

Let \(\overline{G}\) be the initial form of \(G\). Then one of the following cases occurs:

Case III (III c in [18]): \(\tau = 1\) and \(\overline{G}\) has 3 factors. This is equivalent to \(X\) being a \(D_3\)-singularity

Case IV: \(\tau = 1\) and \(\overline{G}\) is the product of a linear factor and the square of another factor. This is equivalent to \((X, p) \subset (\mathbb{A}_k^3, 0)\) having a \(D_n\)-singularity \((n \geq 5)\) if we assume that \((X, p)\) is pseudo rational.

Case V: \(\tau = 1\) and \(\overline{G}\) has only 1 factor. Then, either \((X, p)\) is an \(E_6\)-singularity or it can be expressed as

\[
 z^2 + y^3 + \rho x^3 y + \sigma x^5 \in (z x y, z y^2, x^2 y^2, x^3 z).
\]

Again \((X, p)\) pseudo rational implies:

H3) either \(\rho\) is a unit or \(\sigma\) is a unit.
Finally, we have

**Case V c**: $\rho$ being a unit is equivalent to $(X, p)$ being an $E_7$-singularity.

**Case V d**: $\rho$ not a unit and $\sigma$ a unit, is equivalent to $(X, p)$ being an $E_8$-singularity.

Note that rational double points are top singularities (apply the proof for char $k = 0$ given in [15], example 3.12). From this, applying (R0), it follows that (i) in the theorem implies equality in (16). The fact that (ii) implies equality in (16) can also be checked. In fact, for the last case in (ii), note that the surface given by $z^2 + xy^2 + yzg(x, y) = 0$ (char $k = 2$), where $\text{mult}g \geq 1$ and either $g = 0$ or $g(x, 0) = 0$, can be desingularized by the blow-up of $(y, z)$; if $E$ is the exceptional curve appearing then $\hat{k}_E = 1$ and $\text{ord}_E(J_X) = 1$.

Finally, let us prove that if $(X, p)$ is a top singularity, then either (i) or (ii) hold. In fact, let $(X', p)$ be the surface obtained by a general cut of $(X, p)$ with $d - 2$ hyperplanes. Since (R1) holds for $(X, p)$, we have that (H1) holds for $(X', p)$. Analogously, since (R2) holds for $(X, p)$, it follows that for $(X', p)$ we have that if $\tau = 1$ then (H2) holds. Finally, (R3) for $(X, p)$ implies that if $(X', p)$ satisfies the hypothesis of Case V, either $(X', p)$ is an $E_6$-singularity or (H3) holds.

We thus conclude that if we assume that $(X', p)$ is pseudo rational in Cases II and IV, then $(X, p)$ being a top singularity would imply that $(X', p)$ is a rational double point; hence, $(X, p)$ is a compound Du Val singularity.

Finally, for cases II and IV without the hypothesis of pseudo rational, we obtain

1. If $(X', p)$ is in case II and not pseudo rational, then it can be expressed as $xy = 0$.
2. If $(X', p)$ is in case IV and not pseudo rational, then it can be expressed as:

   \[
   z^2 + xy^2 = 0 \text{ if char } k \neq 2, \text{ or } \\
   z^2 + xy^2 + yzg(x, y) = 0 \text{ if char } k = 2,
   \]
   where $\text{mult}g \geq 1$ and either $g = 0$ or $g(x, 0) \neq 0$.

Thus, we conclude the result.

\[\square\]

**Remark 4.9.** For a closed point $x \in X$ in a variety $X$. Define

\[
s_m(x) := (m + 1)d - \dim \pi_m^{-1}(x).
\]

We know that $\text{mld}_{MJ}(x; X) = \inf_{m \in \mathbb{N}} s_m(x)$ by the formula (13) in Corollary 3.26. The proof of the previous theorem shows that $s_m \geq d - 1$ ($m \leq 5$) yields that $\text{mld}_{MJ}(x; X) \geq d - 1$.

**Proposition 4.10.** Let $X$ be a variety over an algebraically closed field $k$. Assume that $X$ has at worst MJ-canonical singularities. Then, $X$ has at worst cDV singularities in codimension 2.

**Proof.** In Proposition 4.3, it is proved that $X$ has normal in codimension 2. Let $\eta \in X$ be the generic point of an irreducible component of the singular locus of codimension 2. As $X$ is MJ-canonical, it follows $\text{mld}_{MJ}(\eta; X) \geq 1$; therefore, by the Corollary 3.26

\[
1 \leq \text{mld}_{MJ}(\eta; X) = \inf_m \left\{ (m + 1)d - (\dim \overline{\{\eta\}} + r_m) \right\},
\]

where $r_m = \dim \pi_m^{-1}(x)$ ($m \in \mathbb{N}$) for a general closed point $x \in \overline{\{\eta\}}$. Considering the cases for $m = 1, \ldots, 5$, we obtain

\[
s_m(x) := (m + 1)d - r_m \geq d - 1
\]
Corollary 4.11. Let $X$ be an MJ-canonical quasi-projective variety of dimension 3 over an algebraically closed field $k$. Then, a general hyperplane section $H$ of $X$ has at worst Du Val singularities.

Proof. Let $X$ be a locally closed subvariety of $\mathbb{P}^N$. As the linear system of hyperplane in $\mathbb{P}^N$ is very ample, we can apply the original Bertini's theorem (see, for example, [9, Theorem 8.18, II]) that works for arbitrary characteristic. Then, it follows that a general hyperplane section $H$ is non-singular away from the singular locus of $X$. Therefore, if $\dim \text{Sing}X = 0$, then $H$ is non-singular. So we assume that $\dim \text{Sing}X = 1$. By Proposition 4.10, a general hyperplane section intersects the singular locus at finite number of cDV points. We have only to show that the intersection is general at each point $x \in H \cap \text{Sing}X$. Let $|O_{\mathbb{P}^N}(1)|$ be the complete linear system of hyperplanes on $\mathbb{P}^N$. For a point $x \in \text{Sing}X$, we define a subset $D_x \subset |O_{\mathbb{P}^N}(1)|$ as

$$D_x = \{ \mathcal{H} \in |O_{\mathbb{P}^N}(1)| \mid X \subset \mathcal{H} \text{ or } (\mathcal{H} \cap X, x) \text{ not rational double} \}.$$ 

As $O_{\mathbb{P}^N}(1)$ is very ample, the canonical $k$-linear map

$$\varphi_x : \Gamma(\mathbb{P}^N, O_{\mathbb{P}^N}(1)) \to O_{\mathbb{P}^N}(1) \otimes O_X / m_x^2 \simeq O_X / m_x^2$$

is surjective. Let $\tilde{D}_x \subset \Gamma(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ be the subset corresponding to $D_x$ for a cDV point $x \in \text{Sing}X$, then $\tilde{D}_x$ is the pull-back by $\varphi_x$ of the proper closed subset of $k^4 = m_x^2 / m_x^2 \subset O_X / m_x^2$. Then,

$$\dim \tilde{D}_x \leq N + 1 - 5 + 3 = N - 1;$$

therefore

$$\dim D_x \leq N - 2.$$ 

On the other hand, if $x \in \text{Sing}X$ is not a cDV point, then

$$\dim D_x \leq N - 1.$$ 

Let $D \subset \text{Sing}X \times |O_{\mathbb{P}^N}(1)|$ be the set $\{(x, \mathcal{H}) \mid \mathcal{H} \in D_x \}$. Then, as $\text{Sing}X$ is of 1-dimensional and non cDV singularities are isolated, we have

$$\dim D \leq N - 1.$$ 

Hence, the image $p(D) \subset |O_{\mathbb{P}^N}(1)|$ of $D$ by the projection

$$p : \text{Sing}X \times |O_{\mathbb{P}^N}(1)| \to |O_{\mathbb{P}^N}(1)|$$

has dimension $N - 1 \leq N = \dim |O_{\mathbb{P}^N}(1)|$, which yields that general elements of $|O_{\mathbb{P}^N}(1)|$ is not in $p(D)$. This completes the proof. \qed

The usual canonical version of this statement is proved in [11] under certain conditions.

Here, we list open problems for the positive characteristic case, which are all proved for characteristic 0 (see [7] for (1) and see [8] for (2),(3),(4)):

**Open problems for positive characteristic case:**

1. Is an MJ-canonical singularity normal? Cohen–Macaulay?
2. Is the map $X \to Z, x \mapsto \text{mld}_{MJ}(x, X)$ lower semi-continuous?
3. Is an MJ-canonical (MJ-log-canonical) singularity open condition?
4. Is a small deformation of MJ-canonical singularity again MJ-canonical?
Here, we note that if there exist resolutions of singularities, then we would have the affirmative answer to (2), (3), and (4). On the other hand, without resolutions we can prove these if the following natural conjecture holds (this will be discussed in a forthcoming paper by one of the authors):

**Conjecture 4.12.** Let $0 \leq \delta \leq d$, there is a number $N_{\delta,d}$ depending only on $\delta$ and $d$ such that if

$$s_m(x) \geq \delta \geq 0,$$

for all $m \leq N_{\delta,d}$

then $\text{mld}_{M^1}(x,X) \geq \delta$.

We observe that this holds true for $\delta = d - 1$. Indeed, we can take $N_{d-1,d} = 5$ as is seen in the proof of Remark 4.9.

This conjecture is equivalent to the following conjecture (this will be showed also in the forthcoming paper):

**Conjecture 4.13.** There exists a number $N_d$ depending only on $d$ such that

$$\min\{s_m(x) \mid m \leq N_d\} = \text{mld}_{M^1}(x,X),$$

if $\text{mld}_{M^1}(x,X) \geq 0$,

and

$$s_m(x) < 0,$$

for some $m \leq N_d$, if $\text{mld}_{M^1}(x,X) = -\infty$.

**Remark 4.14.** We stated the results in this section under the condition that $k$ is algebraically closed. However, we can weaken this condition such that $k$ is perfect in all results except for Theorem 4.8 and Corollary 4.11.

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