The law of large numbers for completely random behavior of market participants. Quantum economics

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Abstract

In this paper, we briefly discuss a mathematical concept that can be used in economics.

It has been known since Jacob Bernoulli that averaging in economics is nonlinear and does not obey the rules of ordinary arithmetics. For example, the average of winning or losing, say, $100,000 is not equivalent to customary life for a person who is not wealthy, since losing could mean their becoming homeless and even going to jail, whereas winning would not raise their living standards high enough to justify the mere risk of losing.

Accumulating interest. By pooling their capital, owners can become monopolists and raise prices. This also corroborates the nonarithmetic nature of addition. Further, one observes nonlinear addition when buying goods wholesale: the more you buy, the less you pay per item.

The mathematical problem of nonlinear averaging was attacked by the outstanding Russian mathematician Kolmogorov, who obtained a general formula for the average. I supplement his axioms by an additional axiom saying that if a (small) number is added to each term in the sum, then the average increases by same number.

This unambiguously results in the following rule for the average of two numbers $a$ and $b$: $(a \, + \, b)/2 = 1/\beta \log[(2^{\beta a} + 2^{\beta b})/2]$, where $\beta$ is an unknown parameter and base $2$ logarithm is used. For $\beta = 0$, one arrives at the conventional linear average. It is easily seen that $\beta$ is negative for purchase averaging and positive for sale averaging.

If the operation $a \, + \, b = 1/\beta \log[2^{\beta a} + 2^{\beta b}]$ is taken as addition and the conventional addition $a + b$ as multiplication, then the commutativity, associativity, and distributivity laws hold. This arithmetics is possibly used somewhere in the “Kingdom of Distorting Mirrors.”

In my opinion, this arithmetics is more adequate to a market economy than the conventional, classical arithmetics and permits one, using model examples, to explain stock price breakout, default, and a number of laws observed in statistical data processing.

In this note, I wish to present an economic effect that proved quite unexpected to myself. I speak of a statistical sensation that would be nicknamed “fool’s luck” by ordinary people. Since 2003, studies related to the analysis of operations at London stock exchange have been evoking quite a response. About five hundred papers that can be found in the Internet under the keywords zero intelligence discuss the problem of completely random behavior of market participants. The point is that traders buying and selling stock have to take into account so many various factors that they cannot make the right decision unambiguously, and so the behavior of market participants does not differ from a random behavior very much. And if it does, the trader usually loses. Here we assume that the sales volume, the number of participants, the number of nomenclatures of financial tools for each price, and the number of tools are sufficiently large.
Thus what is a random choice?

To make a random choice, one should first calculate all possible choices of purchases or sales and average them. The parameter $\beta$ can relatively easily be expressed via the budget restraint (BR) in the first case and via the required rate of return (RRR) in the second case.

Note that possible choices of purchases and sales obey the so-called Bose statistics. For example, the number of ways to buy nails and bread for 2 cents is as follows: 1) 2 cents for nails; 2) 2 cents for bread; 3) 1 cent for nails and 1 cent for bread. All in all, there are three ways, since bank notes or coins of the same value are indistinguishable. (The law that can be stated as “money does not smell”: nothing changes if one note is exchanged for another.)

How to make a random choice? It suffices to number all possible purchases whose price does not exceed BR and then choose a number randomly, as if throwing dice. Computer generates random (more precisely, “pseudorandom”) numbers. It turns out that the law of large numbers holds: given a BR, a vast majority of such random choices gives the same value for the number of purchases at a given price. This value can be determined in a very simple manner via the nonlinear average of all possible purchases by a small variation of the given price. (The number of tools purchased at a given price is equal to the partial derivative with respect to this price of the nonlinear average over all possible purchases.) A similar law holds for sellers.

Note that the parameter $\beta < 0$ for purchasers can also be determined by equating the derivative of the nonlinear average with respect to $1/\beta$ with the logarithm of the number of possible purchases whose price does not exceed BR. Accordingly, for sellers ($\beta > 0$) the derivative should be equated with the logarithm of the number of possible sales that give a profit not less than RRR.

Thus if the behavior of market participants is completely random, then we can find the number of goods purchased by a majority for a given price and even estimate the fraction constituted by this majority. Conversely, if we cannot do that, then the behavior of market participants cannot be viewed as random.

If the prices can be varied, then, by equating the amount of goods sold as a function of RRR with the amount of goods purchased as a function of BR, we can find equilibrium prices.

It follows from an analysis of papers in the “zero intelligence” series that traders who do not subtilize, i.e., who act at random rather than deliberately, mainly do not lose. It is not without reason that one speaks of fool’s luck or beginner’s luck: he who gambles for the first time (and hence has not yet been spoiled by calculations) does not lose as a rule. There is a large body of corroborating statistical evidence. Hence if we assume that a vast majority of traders have zero intelligence, then the law established by the author gives a right forecast of free market prices.

Let us discuss a similar situation for consumer goods. Suppose that a customer who has a certain amount BR of money for buying gifts in advance for a large number of people enters a gift shop (say, in Mexico) with $k$ shop floors each of which offers a large variety of souvenirs for the same price. The consumer does not know the tastes of all acquaintances whom the gifts are intended for and buys at random. Then the above-mentioned law applies, and one can rather accurately predict how much money the consumer will spend at each shop floor. More precisely, if there are many similar consumers, then a vast majority of them will spend exactly the amount predicted by this law at each shop floor.

This is a very simple model, which can be generalized. For example, consider the case in which the goods are divided into $i$ clusters with close prices within each cluster. Then
one determines the (nonlinear) average profit over all goods if \( N_i \) goods are sold for these prices and then the average price for the \( i \)th cluster is found. The above-mentioned law remains valid in this case and determines the amount spent by a customer for a given cluster of goods.

In this paper, we briefly discuss a mathematical concept that can be used in economics.

1 Nonlinear averaging in the sense of Kolmogorov

A sequence of functions \( M_n \) determines the regular type of the average if the following (Kolmogorov) conditions are satisfied:

I. \( M(x_1, x_2, \ldots, x_n) \) is a continuous and monotone function in each variable. To be definite, we assume that \( M \) increases in each variable.

II. \( M(x_1, x_2, \ldots, x_n) \) is a symmetric function\(^1\).

III. The average of identical numbers is equal to their common value: \( M(x, x, \ldots, x) = x \).

IV. A group of values can be replaced by their average so that the common average does not change:

\[
M(x_1, \ldots, x_m, y_1, \ldots, y_n) = M_{n+m}(x_1, x_2, \ldots, y_n),
\]

where \( x = M(x_1, \ldots, x_n) \).

**Theorem 1 (Kolmogorov)** Under conditions I–IV, the average \( M(x_1, x_2, \ldots, x_n) \) takes the form

\[
M(x_1, x_2, \ldots, x_n) = \frac{\psi(\varphi(x_1)) + \varphi(x_2) + \ldots + \varphi(x_n)}{n},
\]

where \( \varphi \) is a continuous strictly monotone function and \( \psi \) is its inverse.

For the proof of the theorem, see [1].

2 The main averaging axiom

It is rather obvious for a stable system that the following axiom must hold.

If the same value \( \omega \) is added to \( x_k \), then their average increases by this value \( \omega \).

Obviously, the nonlinear averaging of \( x_i \) under normal conditions must also increase by this value. We take this fact as **Axiom 5**.

This axiom leads to a unique solution in the nonlinear case, i.e., the linear case (the arithmetic mean) naturally satisfies this axiom, as well as a unique (up to the same constant by which we can multiply all the incomes \( x_i \)) nonlinear function.

In fact, the incomes \( x_i \) are calculated in some currency and, in general, must be multiplied by a quantity \( \beta \), which is responsible for the purchasing power of this currency, so that this constant (the parameter \( \beta \)) must *a priori* be contained in the definition of the income. Hence we can state that there exists a unique nonlinear function that satisfies Axiom 5.

The function \( f(x) \) has the form

\[
f(x) = C \exp(Dx) + B,
\]

where \( C, D \neq 0 \) and \( B \) are numbers independent of \( x \).

\( ^1 \)In our case, the symmetry follows from the Bose statistics for bank notes.
3 Semiring, an example of self-adjoint linear operators

We consider the semiring generated by nonlinear averaging and the space $L_2$ ranging in this semiring.

First, we consider a heat equation of the form
\[
\frac{\partial u}{\partial t} = \frac{h}{2} \frac{\partial^2 u}{\partial x^2}.
\] (3)

Here $h$ is a small parameter, but we do not use its smallness now.

Equation (3) is a linear equation. As is known, this means that if $u_1$ and $u_2$ are its solutions, then the linear combination
\[
u = \lambda_1 u_1 + \lambda_2 u_2
\] (4)
is also its solution. Here $\lambda_1$ and $\lambda_2$ are constants.

Now we perform the following change. We set
\[
u = e^{-\frac{w}{h}}.
\] (5)

Then we obtain the following nonlinear equation for the unknown function $w(x,t)$:
\[
\frac{\partial w}{\partial t} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \frac{h}{2} \frac{\partial^2 w}{\partial x^2} = 0.
\] (6)

This well-known equation is sometimes called the Bürgers equation\(^2\).

The solution $u_1$ of Eq. (3) is associated with the solution $w_1 = -h \ln u_1$ of Eq. (6), and the solution $u_2$ of Eq. (3) is associated with the solution $w_2 = -h \ln u_2$ of Eq. (6).

The solution of Eq. (3) is associated with the solution $w(\mu_i) = -h \ln(e^{-\frac{w_1}{h}} + e^{-\frac{w_2}{h}})$, where $\mu_i = -h \ln \lambda_i$, $i = 1, 2$.

This implies that Eq. (6) is a linear equation, but it is linear in a function space, where the following operations were introduced:
- the operation of taking the sum $a \oplus b = -h \ln(e^{\frac{a}{h}} + e^{\frac{b}{h}})$;
- and the operation of multiplication $a \odot \lambda = a + \lambda$.

In this case, the change $w = -h \ln u$ takes zero to infinity and the unity to zero. Thus, $\infty$ is a generalized zero in this new space: $\emptyset = \infty$, and the usual zero is a generalized unity: $0 = 0$. The function space, where the operations $\oplus$ and $\odot$ are introduced, with the associated zero $\emptyset$ and the unity $0$ is isomorphic to the usual function space with the usual multiplication and addition.

This can be interpreted in the following way: somewhere on another planet, the people are used to deal with precisely these operations $\oplus$ and $\odot$, and then Eq. (3) is a linear equation from their viewpoint.

Everything written here is, of course, trivial, and the people on our planet need not study new arithmetic operations, because, using a change of the function, one can pass from Eq. (3) to Eq. (6), which is linear in the usual sense. But it turns out that the “Kingdom of distorting mirrors” given by this semiring is related to the “capitalistic” economics.

\(^2\)The usual Bürgers equation is obtained from this equation by differentiating with respect to $x$ and applying the change $v = \frac{\partial w}{\partial x}$. 
In the function space ranging in the ring \( a \oplus b = -h \ln(e^{-\frac{W}{h}} + e^{-\frac{W}{h}}), \) \( \lambda \odot b = \lambda + b, \) we introduce the inner product

\[
(w_1, w_2) = -h \ln \int e^{-\frac{\lambda w_1 + \lambda w_2}{h}} dx.
\]

We show that the product in this space has the following bilinear properties: \( (a \oplus b, c) = (a, c) \oplus (b, c) \) and \( (\lambda \odot a, c) = \lambda \odot (a, c) \). Indeed,

\[
(a \oplus b, c) = -h \ln \left( \int \exp \left( \frac{-h \ln(e^{-\frac{a}{h}} + e^{-\frac{b}{h}}) + c}{h} \right) dx \right) =
= -h \ln \left( \int (e^{-\frac{a}{h}} + e^{-\frac{b}{h}}) e^{-\frac{c}{h}} dx \right) = -h \ln \left( \int e^{-\frac{a+c}{h}} dx + \int e^{-\frac{b+c}{h}} dx \right) = (a, c) \oplus (b, c).
\]

We consider an example of self-adjoint operators in this space, namely, the operator

\[
L : W \rightarrow W \odot (-h \ln \left( \frac{(W')^2}{h^2} - \frac{W''}{h} \right)).
\]

And now we verify whether it is self-adjoint:

\[
(W_1, LW_2) = -h \ln \int e^{-\frac{\lambda W_1 + \lambda W_2}{h}} dx =
= -h \ln \int \exp \left[ - \left( W_1 + W_2 - h \ln \left( \frac{W_1'}{h^2} - \frac{W_2'}{h} \right) \right) / h \right] dx =
= -h \ln \int e^{-\frac{W_1}{h}} e^{-\frac{W_2}{h}} \left( \frac{(W_1')^2}{h^2} - \frac{W_2''}{h} \right) dx = -h \ln \int e^{-\frac{W_1}{h}} \frac{d^2}{dx^2} e^{-\frac{W_2}{h}} dx =
= -h \ln \int e^{-\frac{W_1}{h}} e^{-\frac{W_2}{h}} dx = -h \ln \int e^{-\frac{W_1}{h}} \left( \frac{(W_1')^2}{h^2} - \frac{W_2'}{h} \right) e^{-\frac{W_2}{h}} dx =
= -h \ln \int \exp \left[ - \left( W_1 - h \ln \left( \frac{(W_1')^2}{h^2} - \frac{W_2'}{h} \right) \right) / h \right] dx =
= -h \ln \int e^{-\frac{LW_1 + LW_2}{h}} dx = (LW_1, W_2).
\]

Its linearity can also be verified easily.

We construct the resolvent operator of the Burgers equation: \( L : W_0 \rightarrow W, \) where \( W \) is a solution of Eq. \( (8) \) satisfying the initial condition \( W|_{t=0} = 0. \)

The solution of Eq. \( (3) \) satisfying the condition \( u|_{t=0} = u_0 \) has the form

\[
u = \frac{1}{\sqrt{2\pi h}} \int e^{-\frac{(\xi - \tau)^2}{2h}} u_0(\xi) d\xi.
\]

Taking into account that \( u = e^{-\frac{W}{h}} \) and \( W = -h \ln u, \) we obtain the resolvent \( L_t \) of the Burgers equation

\[
L_t W_0 = -\frac{h}{\sqrt{2\pi h}} \ln \int e^{-\frac{(\xi - \tau)^2}{2h} + \frac{w(\xi)}{h}} d\xi.
\]

The operator \( L_t \) is self-adjoint in the new inner product.
4 Entropy for the producer and the consumer. Condition for the producer income and the consumer expenditure. Production and consumption. Equilibrium prices

We consider a large group of producers manufacturing goods of \( M \) types. The corresponding production structure is characterized by the vector \( \omega^N = (K_1, \ldots, K_M) \), where \( K_i \) is the number of goods of the \( i \)th type, \( i = 1, \ldots, M \), and \( \sum_{i=1}^{M} K_i = N \). The consumption structure is treated similarly. Suppose that \( \epsilon_i \) is the price of goods of the \( i \)th type. The income obtained by selling \( N \) units of goods is equal to \( E = \sum_{i=1}^{M} \epsilon_i K_i \).

The concepts introduced below are based on the notion of nonlinear averaging of the incomes obtained by realizing \( N \) units of goods over all possible versions \( \omega^N \) of the production structure for a given positive value of the parameter \( \beta \):

\[
M_N = \frac{1}{\beta} \log \left( \frac{1}{L} \sum_{K_1+\ldots+K_M=N} 2^{\beta \sum_{i=1}^{M} \epsilon_i K_i} \right). \tag{10}
\]

The nonlinear averaging is discussed in detail in [2].

4.1 Psychology of an ordinary depositor. Psychological law of status quo preservation

We note that losing $100,000 is much heavier in its psychological “cost” than winning the same sum. This means that an ordinary person prefers to preserve status quo, i.e., not to take risk of losing $100,000. Therefore, if a person deposits a certain sum \( N \) in several banks at high interest, he must calculate what sum is sufficient for him to live like a rentier, to preserve status quo, and, accordingly, how to spread the money over several banks so as not to take any risk and not to lose his status.

This purely psychological fact is the base of our mathematical calculations. We note that an ordinary depositor can rather easily calculate the sum of income, but it is difficult for him to evaluate the reliability of a bank proposing high interest rates. Therefore, in the formulas given below, the free parameter \( \beta \) can be determined in terms of the sum of income introduced above, and then we obtain graphs showing how the bank deposits \( n_i \) depend on the \textit{a priori} given income.

The problem of calculating the “surviving probability,” which is close to the “status quo preservation law,” has been discussed by specialists in mathematical economics.

If the required rate of return (RRR) \( \sum n_i \lambda_i = E \) is assumed to be an independent variable, then the nonlinear averaging based on the Kolmogorov axioms and the additional axiom proposed by the author is unique.

We assume that production runs normally if the producer’s income \( \sum_1^n \epsilon_i N_i \) is larger than or at least equal to a quantity \( E_1 \). The consumer cannot spend more than \( E_2 = \sum_1^n \epsilon_i N_2 \) (the budget restraint – BR). We consider the “entropies” of the producer and the consumer as follows: \( H_1(\epsilon_1, \ldots, \epsilon_n, E_1) \) is the base 2 logarithm of the number of sale versions for a sum no less than \( E_1 \); \( H_2(\epsilon'_1, \ldots, \epsilon'_n, E_2) \) is the base 2 logarithm of the number of purchase versions for a sum not exceeding \( E_2 \).

Let \( \tilde{M} \) be the sum (10) for \( \beta < 0 \), and let \( \theta = \frac{1}{\beta} \).
The equation
\[ \frac{\partial M}{\partial \theta} = H_1 \]
allows us to obtain \( \theta = \theta_1(H_1) \), and the equation
\[ \frac{\partial \tilde{M}}{\partial \theta} = H_2 \]
allows us to obtain \( \theta = \theta_2(H_1) \).

We choose \( k \) possible purchase versions at random.

It turns out that, for the majority of these versions, the sum of money spent for buying goods at price \( \varepsilon_i \), is “almost” equal to
\[ \frac{\partial M}{\partial \varepsilon_i}|_{\theta = \theta_1} \]
(this is an analog of the law of large numbers; the exact estimates are the same as in the usual law of large numbers).

A similar statement holds for the seller. The equilibrium prices follows from the relation
\[ \frac{\partial M}{\partial \varepsilon_i}|_{\theta = \theta_1(\varepsilon)} = \frac{\partial \tilde{M}}{\partial \varepsilon_i}|_{\theta = \theta_2(\varepsilon)}, \quad i = 1, \ldots, n. \]
Thus, the “resources–producer–consumer–etc.” vertical line is divided into pairs each of which contains two new numbers, RRR and BR. The entropy (the Kolmogorov complexity) and the degree of risk are “hidden” in the intermediate calculations. This is how the usual model of economical equilibrium (the general equilibrium) varies.

The models of dynamical equilibrium (the intertemporal general equilibrium) vary similarly according to formula (41) given in [3], p. 276.

The price equilibrium condition is determined by the relations
\[ \frac{\partial M}{\partial \varepsilon_i}|_{\theta = \theta_1(\varepsilon, E_1)} = \frac{\partial \tilde{M}}{\partial \varepsilon_i}|_{\theta = \theta_2(\varepsilon, E_2)}. \] (11)

It follows from the “pair” law that the derivative of the average with respect to the “temperature” \( \theta \) is the entropy and the derivative with respect to the price is the quantity of goods. Relation (11) is well known in the linear case. Here we generalize it to the non-linear case. This generalization allows us to take the Kolmogorov complexity into account, and hence the entropy, which is one of the most important notions in economics. Its conjugate, the temperature, determines the degree of risk, and sometimes, the volatility. But the variables in the final formulas contain only the incomes RRR and the expenditures BR, which can be calculated easily. Thus, \( H \) and \( \theta \) are “hidden parameters” here.

5 Tunnel canonical operator in economics

The equilibrium prices are determined by the condition that the demand and offer must be the same for each item of goods and resources. The following pairs can be determined similarly: flows of goods and services – prices; flows of trade of different kinds – salary rates; flows of raw material resources – rents; interest – loan volume.

The asymptotics of \( M \) and \( \tilde{M} \) is given by the tunnel canonical operator in the phase space of pairs.
We consider the phase space $R^{2n}$, where the intensive variables play the role of coordinates and the extensive variables play the role of momenta. In economics, the role of values of a random variable $\lambda_i$ can be played by the prices of the corresponding goods, and $N_i$ can, for example, be the number of sold goods, i.e., the number of people who bought goods of this particular type, or the interest paid by the $i$th bank, etc. Obviously, the price depends on the demand, i.e., $\lambda_i(N_i)$ is a curve in the two-dimensional phase space. In the two-dimensional phase space, to each point (vector) $\lambda_i, i = 1, \ldots, n$, there corresponds a vector $N_i(\lambda_1, \ldots, \lambda_n), i = 1, \ldots, n$. In a more general case, this is an $N$-dimensional manifold (surface), where the “coordinates” and “momenta” locally depend on $n$ parameters, and the following condition is satisfied: the Lagrange brackets of the “coordinates” and “momenta” are zero with respect to these parameters. Hence such a manifold was called a Lagrangian manifold by the author. In other words, the form $\sum N_i d\lambda_i$ is closed (see the Afterword in [4] and [5]). This means that $\int N_i d\lambda_i$ is independent of the path and is called an action like $\int pdq$ ($p$ is the momentum and $q$ is the coordinate) in mechanics.

The producer acquires resources and transforms the resources expenditure vector into the vector of material wealth production. Then the consumer acquires this material wealth. Thus, the equilibrium prices of resources and of the consumer material wealth are determined according to the above relations.

In addition to such equilibrium prices, there can also be vertical pairs for some types of material wealth and seller–buyer pairs (i.e., permanent seller – permanent buyer pairs), and the prices related to these pairs are formed. This is an analog of the Cooper pairs in quantum statistics.

This construction requires the use of the ultra-secondary quantization method in abstract algebraic form, which could be applied in economics. The “vertical” clusters are also formed in this theory.

Thus, we have determined the parameter $\beta$, and the problem is solved completely.

Example. In what follows, a person who buys stocks will be called a player. Suppose that there are only two types of stocks, the first are conditionally called “cheap” stocks and the stocks of the second type are said to be “expensive.” We assume that a player buys a packet of $N$ stocks in which the number of cheap stocks is $N_1$ and the number of expensive stocks is $N_2 = N - N_1$, respectively. A player spends money to buy stocks, and the number of purchased stocks affects the price of stocks of both types. In particular, the larger is the number of expensive stocks bought by a player, the less their price will be. Consequently, in what follows, we assume that the player’s expenditures for a packet of stocks depend nonlinearly on the number of purchased cheap and expensive stocks. For example, it depends quadratically as follows:

$$E(N_1) = \lambda_1 N_1 + \lambda_2 N_2 - \frac{\gamma N_2^2}{2N} - \frac{\gamma N_1^2}{2N} + (\lambda_1 - \lambda_2 + \gamma)N_1 - \frac{\gamma N_1^2}{N},$$

where the numbers $\lambda_1, \lambda_2,$ and $\gamma$ satisfy the conditions

$$\frac{\gamma}{2} < \lambda_1 < \lambda_2, \quad \lambda_2 - \lambda_1 < \gamma < 2(\lambda_2 - \lambda_1).$$

It follows from conditions (13) that the function (12) with $N_1 = 0, 1, \ldots, N$ has the global minimum for $N_1 = N$ and a local minimum for $N_1 = 0$.

If, at the initial moment, the player buys $N_1$ cheap stocks and $N_2$ expensive stocks so that $N_1 < \frac{\lambda_1 - \lambda_2 + \gamma}{\gamma}N$, then selling one cheap and one expensive stock so as to decrease $E(N_2)$, he will come to a local minimum at $N_1 = 0$, i.e., he will buy all expensive stocks.
But if $N_1 > \frac{\lambda_1 - \lambda_2 + \gamma}{\gamma} N$, then, as a result of a monotone process, the player will buy all cheap stocks.

Now we consider local financial averagings of the player income. We assume that $G_1$ dealers sell cheap stocks and $G_2$ dealers sell expensive stocks. In this case, the number of different ways in which the player can buy a packet of stocks is

$$\Gamma(N_1) = \frac{(N_1 + G_1 - 1)! (N - N_1 + G_2 - 1)!}{(G_1 - 1)! N_1! (G_2 - 1)!(N - N_1)!}. \quad (14)$$

**Remark.** Instead of introducing different dealers, we can assume that the cheapest and the most expensive stocks are, respectively, of $G_1$ and $G_2$ different types, but of the same price.

We assume that, for $\beta = \infty$, the player is at the point of local minimum for $N_1 = 0$. Since he tries to change the stocks pairwise and gradually (monotonically) so that not to increase his expenditures $^3$, we can consider the averaging only in a neighborhood of the point of local minimum (the local financial averaging). If $\beta$ varies slowly and $N \to \infty$, then the asymptotics of $M_\beta$ as $N \to \infty$ again corresponds to the local minimum

$$\mathcal{E}(N_1) = \beta (\lambda_1 N_1 + \lambda_2 N_2 - \frac{\gamma N_1^2}{2N} - \frac{\gamma N_2^2}{2N}) + \ln \frac{(N_1 + G_1 - 1)! (N - N_1 + G_2 - 1)!}{(G_1 - 1)! N_1! (G_2 - 1)!(N - N_1)!}. \quad (15)$$

Figure 3 shows how the entropy depends on the temperature and the local and global minima.

The curve breaks at the point $T \approx 40$. At this point, the derivative $\frac{\partial S}{\partial T}$ becomes infinite, and the modified Laplace method, which could be used for the asymptotics of

$^3$The least risk principle in economics.
Figure 2: Graph of $E(N_1)$ for $T = 5$, $G_2 = 30$, and $\gamma = 1.5$
Figure 3: The heavy line corresponds to the local minimum, the global minimum lies below. At the point $T = 40$, the local minimum and the global maximum coincide and do not exist for $T > 40$. 
the local $M_\beta$ at the other points, cannot be used near this point. The local minimum obtained by computer shows unstable "spreading." It turns out that the asymptotics near this point can be expressed in terms of the Airy function of an imaginary argument, and this fact removes all the problems listed above.

For $T > 40$, no equilibrium can exist when small changes in purchasing and selling occur. Therefore, to return to the equilibrium point formed as a result of changes in the local averaging near another local minimum, the player must change a large amount of stocks at once (see Fig. 3).

A similar situation occurs when $\kappa = 1$ and the player wants to win as much as possible. Then the points of minima are replaced by the points of maxima between which, in the case of a quadratic dependence on $N_1$ and $N_2$, there is a minimum. If $T$ varies from zero to some $T_0$ at which the local maximum disappears, then a jump occurs, which can be treated as a stock price break-down. In our case, if the player simultaneously and very fast sells a large amount of stocks of one type and buys a large amount of stocks of a different type, then he can again get to another equilibrium point.

We have considered only the simplest model. In a more complicated economical situation concerning the interests of a great mass of the population, the people cannot change their behavior very fast, passing, for example, from the usual consumer basket to a different basket and thus changing their mode of life. Then there is no equilibrium (balance) point in general, and a sharp disbalance leads to general default.

**Generalization.** In our example, we obtained a one-dimensional curve corresponding to a local minimum of the entropy dependence on the “temperature” in the two-dimensional $S,T$-space and considered its projection on the $T$-axis at the point $T_0$. This projection is not “good,” and we said that the asymptotics of $M_\beta$ in a neighborhood of this point must be replaced by the Airy function.

What picture appears in the general case where we have two “conjugate” pairs: the entropy–temperature pair and, for example, the number of people $N$ corresponding to some average salary $\varepsilon_k$.

In this case, we consider a four-dimensional (phase) space, where $T, \varepsilon$ are the “coordinates” and $N, S$ are the momenta. The surface corresponding to our curve is two-dimensional and can (locally) be written in parametric form as

$$T = T(\alpha_1, \alpha_2), \quad \varepsilon = \varepsilon(\alpha_1, \alpha_2), \quad N = N(\alpha_1, \alpha_2), \quad S = S(\alpha_1, \alpha_2),$$

where $\alpha_1, \alpha_2$ are parameters.

Because this surface must be obtained (at least, at the simple points of projection on the “coordinate” plane) from the asymptotics of sums of the form

$$M_\beta = \frac{1}{\kappa \beta} \ln \left( \frac{e^{\kappa \beta a} + e^{\kappa \beta b}}{2} \right), \quad \kappa = \pm 1, \quad \beta > 0,$$

it must be a Lagrangian manifold (this notion was introduced by the author in [5]). At the points of “bad projection” on the $T, \varepsilon$-plane, the asymptotics is given by the tunnel canonical operator [5], and its simplification, depending on the form of the surface near the point of “bad projection,” can be obtained using [6].

These general considerations can help constructing the corresponding model of a given economical situation if the appropriate statistical data are available.
6 The law of large numbers

We consider the sellers who sell goods of \( M \) types. The corresponding structure of sold goods is characterized by the vector \( \omega^N = \langle K_1, \ldots, K_M \rangle \), where \( K_i \) is the number of goods of the \( i \)th type, \( i = 1, \ldots, M \), and \( \sum_{i=1}^{M} K_i \leq N \). The structure of purchased goods is considered similarly. Suppose that \( \epsilon_i \) is the price of goods of the \( i \)th type. The income obtained by realizing \( K_1 + \cdots + K_n \) units of goods is \( E(\omega^N) = \sum_{i=1}^{M} \epsilon_i K_i \).

The concepts introduced below are based on the notion of nonlinear averaging of the incomes obtained by realizing \( \leq N \) units of goods over all possible versions \( \omega^N \) of the structure of sold goods for a given positive value of the parameter \( \beta \):

\[
M_N(\beta) = \frac{1}{\beta} \log \left( \frac{1}{L} \sum_{\omega^N} 2^{\beta E(\omega^N)} \right) = \frac{1}{\beta} \log \left( \frac{1}{L} \sum_{K_1+\cdots+K_M \leq N} 2^{\beta \sum_{i=1}^{M} \epsilon_i K_i} \right),
\]

where \( L \) is the number of terms in the sum. The nonlinear averaging is discussed in detail in [2]. In the present paper, we show that, under certain natural assumptions, the average can be calculated only in two ways: as the usual linear average and as the nonlinear average of the form (16).

We divide the goods of all types into groups with different prices. Suppose that the total number of groups of such goods is \( n \), \( n \leq M \), the \( i \)th group contains \( G_i \) types of goods with the common price \( \lambda_i \) and \( \lambda_1 < \lambda_2 < \cdots < \lambda_n \).

In our statement, each \( \lambda_i \) is equal to one of \( \epsilon_j \). We denote \( \bar{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \). By definition, we have \( \sum_{i=1}^{n} G_i = M \). Let \( K_{i,j} \) be the total number of units of goods of the \( j \)th type in the \( i \)th group of goods with the common price \( \lambda_i \). Thus, the structure of sold goods is given by the vector

\[
\omega^N = \langle K_{1,1}, \ldots, K_{1,G_1}, \ldots, K_{i,1}, \ldots, K_{i,G_i}, \ldots, K_{n,1}, \ldots, K_{n,G_n} \rangle.
\]

Let \( N_i = \sum_{j=1}^{G_i} K_{i,j} \) be the number of all units of goods with the price \( \lambda_i \), and let \( \nu_i = N_i/N \) be the part of all such goods in their total amount, where \( i = 1, \ldots, n \). The price structure of sold goods is characterized by the vector \( \langle N_1, \ldots, N_n \rangle \) or, in fractions, by the vector

\[
\langle \nu_1, \ldots, \nu_n \rangle.
\]

Then the total cost of all \( N \) units of goods is

\[
E(\omega^N) = \sum_{i=1}^{n} \lambda_i N_i = N \sum_{i=1}^{n} \lambda_i \nu_i.
\]

By \( \Xi^{N_1, \ldots, N_n}_N \) we denote the set of all versions of the structure of sold goods with a prescribed price structure \( \langle N_1, \ldots, N_n \rangle \). The number of elements in this set is

\[
|\Xi^{N_1, \ldots, N_n}_N| = \left( \begin{array}{c} N_1 + G_1 - 1 \\ G_1 - 1 \end{array} \right) \cdots \left( \begin{array}{c} N_n + G_n - 1 \\ G_n - 1 \end{array} \right).
\]

\(^4|A|\) denotes the number of elements in the set \( A \).
If the goods are separated in groups, then the sum (16) becomes

\[ M_N(\beta, \lambda) = \frac{1}{\beta} \log \left( \frac{1}{L} \sum_{N_1, \ldots, N_n \leq N} |\Xi_N^{N_1, \ldots, N_n}| \right), \]

(21)

where \( L \) is the total number of all possible versions of the structure of sold goods, and the second sum in (16) is taken over all the price structures \((N_1, \ldots, N_n)\).

We assume that the sellers have sold at most \( N \) units of goods of different types. We consider the problem of predicting the “most expected” structure of goods realized in the free market, which is necessary for the average income to be at least \( E_1 \). Thus, the following inequality must hold:

\[ \sum_{i=1}^{n} \lambda_i N_i \geq E_1. \]

(22)

In addition, the following natural inequality must hold:

\[ \sum_{i=1}^{n} N_i \leq N. \]

(23)

We introduce the main definitions. The base 2 logarithm of the number of possible versions of sale of \( N \) units of goods for which the income is no less than \( E_1 \), i.e., (22) is satisfied, will be called the entropy \( H(E_1, \lambda, N) \).

By \( \theta(E_1, \lambda, N) \) we denote the solution (with respect to \( \theta \)) of the equation

\[ \frac{\partial M_N(\theta^{-1}, \lambda)}{\partial \theta} = H(E_1, \lambda, N). \]

(24)

The solution of this equation exists and is unique. This follows from the monotonicity. We set

\[ \tilde{\nu}_i(E_1, \lambda, N) = \frac{1}{N} \frac{\partial M_N}{\partial \lambda_i} |_{\theta = \theta(E_1, \lambda)} \]

(25)

for \( i = 1, \ldots, n \).

The notions introduced above are of general character and can be used for all possible versions of the goods distribution. In what follows, we consider a specific case for which we find the asymptotics of the introduced variables and prove a certain simplest version of the “law of large numbers” under the following conditions.

We introduce the notation \( \nu_i = N_i / N \) and \( p_i(N) = G_i / N \) for \( i = 1, \ldots, n \). We assume that the variables \( p_i(N) \) have the limit \( p_i \) as \( N \to \infty \), and this limit is greater than zero for all \( i \). We also assume that \( E_1 = e_1 N \). Let \( \rho = N/M \) be a constant (one can also assume that \( N/M \to \rho \) as \( N \to \infty \)).

In what follows, we also need the restriction \( e_1 \leq \rho \sum_{i=1}^{n} \lambda_i p_i \) on the value of the average income per unit sold goods.

**Theorem 2** Let \( n \geq 3 \). Then, for an arbitrary \( \epsilon > 0 \), the part of all versions of the structure of sold goods \((16)\) for which the average income per unit goods is no less than \( e_1 \) \(^5\) and \(|\nu_i - \tilde{\nu}_i| \geq \epsilon \) at least for a single \( i \), \( 1 \leq i \leq n \), does not exceed \( 2^{-c\epsilon^2 N} \), where \( c \) is a constant.

\(^5\)i.e., under restrictions (22) and (23).
The conditions of the theorem contain the assumption that the number of nomenclatures of goods with the same price is sufficiently large. But one can unite a group of goods with different prices and, calculating the nonlinear average of these prices, replace this group of prices in the statement of the theorem by this nonlinear average price.

We shall write the corresponding formula.

We consider a set of prices \( \lambda_i \), where \( i = 1, \ldots, n \), and a set of numbers \( g_i \) equal, for example, to the number of goods of different types but with the same price \( \lambda_i \). The numbers \( g_i \) are quantities of the order of 1. By \( N_i \) we denote the number of goods purchased at the price \( \lambda_i \). Taking into account the fact that \( g_i \) different goods can be purchased at the price \( \lambda_i \), the number of different ways for buying \( N_i \) goods at the price \( \lambda_i \) is given by the formula

\[
\gamma_i(N_i) = \frac{(N_i + g_i - 1)!}{N_i!(g_i - 1)!},
\]

and the number of different ways for buying the set of goods \( \{ N \} = N_1, \ldots, N_n \) is, respectively, equal to

\[
\Gamma(\{ N \}) = \prod_{i=1}^{n} \gamma_i(N_i) = \prod_{i=1}^{n} \frac{(N_i + g_i - 1)!}{N_i!(g_i - 1)!}.
\]

We assume that the goods are divided in \( m \leq n \) groups as follows. Suppose that there are two sequences \( i_\alpha \) and \( j_\alpha \), where \( \alpha = 1, \ldots, m \), such that

\[
i_\alpha \leq j_\alpha, \quad i_{\alpha+1} = j_\alpha + 1, \quad \alpha = 1, \ldots, m, \quad i_1 = 1, \quad j_m = n.
\]

In this case, we say that the goods belong to the group of goods with number \( \alpha \) if their price \( \lambda_i \) satisfies the condition \( i_\alpha \leq i \leq j_\alpha \). By \( \tilde{N}_\alpha \) we denote the number of purchased goods from the group with number \( \alpha \). This number is given by the formula

\[
\tilde{N}_\alpha = \sum_{i=i_\alpha}^{j_\alpha} N_i.
\]

We introduce \( w_\alpha \), which is the nonlinearly averaged price of goods in group \( \alpha \),

\[
w_\alpha = \frac{1}{\beta \tilde{N}_\alpha} \left( \log \left( \sum_{\{ N \}} \prod_{i=i_\alpha}^{j_\alpha} \gamma_i(N_i) 2^{-\beta \lambda_i N_i} \right) - \log \left( \tilde{\gamma}_\alpha(\tilde{N}_\alpha) \right) \right),
\]

where \( \beta \) is an economic parameter, for example, the volatility. We also use the notation

\[
\sum_{\{ N \}} \gamma_i(N_i)
\]

which means that the summation is performed over all the sets of nonnegative integers \( N_{i_\alpha}, \ldots, N_{j_\alpha} \) such that

\[
\sum_{i=i_\alpha}^{j_\alpha} N_i = \tilde{N}_\alpha.
\]

Moreover, \( \tilde{\gamma}_\alpha(\tilde{N}_\alpha) \) is given by the formula

\[
\tilde{\gamma}_\alpha(\tilde{N}_\alpha) = \sum_{\{ N \}} \prod_{i=i_\alpha}^{j_\alpha} \gamma_i(N_i).
\]
We can show that (33) satisfies the formula
\[ \tilde{\gamma}_\alpha(\tilde{N}_\alpha) = \frac{(\tilde{N}_\alpha + g_\alpha - 1)!}{\tilde{N}_\alpha!(\tilde{g}_\alpha - 1)!}, \] (34)

where
\[ \tilde{g}_\alpha = \sum_{i=i_\alpha}^{j_\alpha} g_i. \] (35)

We note that, in general, the nonlinearly averaged price of goods in group \( \alpha \) depends on \( \beta \) and \( \tilde{N}_\alpha \).

Now we consider the nonlinear expectation of the income
\[ M(\beta, \lambda, N) = -\frac{1}{\beta} \log \left( \sum_{\{N\}} ' \prod_{i=1}^{n} \gamma_i(N_i)2^{-\beta \lambda_i N_i} \right), \] (36)

where
\[ \sum_{\{N\}} ' \] (37)
denotes the sum over all sets of nonnegative integers \( \{N\} = N_1, \ldots, N_n \) such that
\[ \sum_{i=1}^{n} N_i = N. \] (38)

**Proposition 1** The nonlinear expectation (36) satisfies the relation
\[ M(\beta, \lambda, N) = -\frac{1}{\beta} \log \left( \sum_{\{\tilde{N}\}} ' \prod_{\alpha=1}^{m} \tilde{\gamma}_\alpha(\tilde{N}_\alpha)2^{-\beta \omega_\alpha \tilde{N}_\alpha} \right), \] (39)

where
\[ \sum_{\{\tilde{N}\}} ' \] (40)
denotes the sum over all sets of nonnegative integers \( \{\tilde{N}\} = \tilde{N}_1, \ldots, \tilde{N}_m \) such that
\[ \sum_{\alpha=1}^{m} \tilde{N}_\alpha = N. \] (41)

This assertion readily follows from the definitions of nonlinearly averaged prices (34) and nonlinear expectation of income (36).

Now we have
\[ \tilde{\nu}_\alpha = \frac{\partial M}{\partial \omega_\lambda}, \]
and the law of large numbers theorem remains the same, where \( \tilde{\nu}_\alpha \) is the number of goods purchased at the prices \( \lambda_i \) for \( i_\alpha \leq i \leq j_\alpha \).
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