Symmetry Breaking Dynamics in a Ring: Winding Number Statistics and Little-Parks Periodicities

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Abstract

Multiply connected geometry plays an important role in physics, such as the Aharonov-Bohm effect and the Little-Parks effect. We statistically realize the Little-Parks periodicities by simulating the dynamics of $U(1)$ symmetry breaking in a ring-shaped superconducting system from gauge-gravity duality. According to the Kibble-Zurek mechanism, quenching the system across the critical point to symmetry-breaking phase will result in topological defects – winding numbers – in a compact ring. In the final equilibrium state, the winding numbers are constrained in a normal distribution for a fixed magnetic flux threading the ring. The conserved current, critical temperatures, average condensates of the order parameter and free energies all perform periodic behaviors, also known as Little-Parks periodicities, versus the magnetic flux with periods equalling the flux quantum $\Phi_0$. The favorable solutions with distinct winding numbers will transit as the magnetic flux equals half-integers multiplying $\Phi_0$.

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I. INTRODUCTION

Little-Parks (LP) experiment [1] is a hallmark of demonstrating the pairing of electrons in the Bardeen-Cooper-Schrieffer (BCS) superconductors [2]. The experiment is composed of the hollow thin-walled superconducting cylinder threading by an axial magnetic field. The critical temperature turns out to be a periodic function of the flux quantum $\Phi_0 = 2\pi \hbar c/e^*$ enclosed by the cylinder, in which $e^*$ is the charge of electron pairs [3]. This periodicity arises from the existence of a single-valued complex wave function in the multiply connected geometry. Breaking of the ordinary $\Phi_0$ period was studied recently in a very small ring geometry [4, 5]. Extensions of LP effect to strongly coupled systems were carried out by virtue of the gauge-gravity duality in [6–8], in which the authors reproduced the periodicities of the critical transition points, the currents and the condensates of the order parameter. These work was done in static case and the winding numbers of order parameter were brought in a priori.

In this paper, we will generate the LP effect dynamically, in particular the winding numbers of the order parameter will turn out automatically and stochastically at the final equilibrium state. A natural way to simulate this dynamical process resorts to the Kibble-Zurek mechanism (KZM) [9–11], which states that quenching a system with higher symmetry across the critical point, the topological defects will arise in the symmetry-breaking phase. KZM has been widely tested in numerous systems [12–14] and was supported by various
numerical studies [15–17]. Please refer to the reviews [18, 19]. Previous holographic studies of KZM of spatial 1D and 2D systems were carried out in [20–26].

Here, we study the dynamical superconducting transition in a spatially 1D ring by decreasing the temperature across the critical point. The topological defects – winding numbers – will turn out due to the KZM [27–29]. In the final equilibrium state, the gradient of the phase of the order parameter is a constant, indicating a persistent superflow velocity along the ring. The winding number is a topologically invariant quantity due to the single valuedness of the order parameter around the ring. For a fixed value of the threading magnetic flux Φ, the statistics of winding numbers are constrained in a normal distribution with the mean equalling the ratio \( \Phi / \Phi_0 \). By varying the magnetic fluxes, the conserved currents, critical temperatures, average absolute values of the order parameter and the free energies all perform periodic relations with respect to the magnetic flux, with the identical periods equalling \( \Phi / \Phi_0 = 1 \). These are typical results of LP effect in a thin-walled cylinder. The favorable solutions with lower free energies constitute the scalloped shapes of curves, and transition from one winding number to another winding number occurs at the half-integers of \( \Phi / \Phi_0 \).

II. HOLOGRAPHIC MAPPING

The holographic superconducting system can be simulated by the Abelian-Higgs model in the bulk [30],

\[
L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |D\Psi|^2 - m^2 |\Psi|^2,
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the field strength of the U(1) gauge field \( A_\mu \), \( \Psi \) is the complex scalar field and \( D_\mu = \nabla_\mu - i A_\mu \) is the covariant derivative (we take the unit \( \hbar = c = e^* = 1 \)).

We work with the Eddington-Finkelstein coordinates in AdS\(_4\) black brane to inspect the temporal dynamics of the holographic superconducting system [31],

\[
ds^2 = \frac{1}{z^2} \left( -f(z)dt^2 - 2dtdz + dx^2 + dy^2 \right),
\]

where \( f(z) = 1 - (z/z_h)^3 \) and \( z_h \) represents the horizon location (we have set the AdS radius to be \( l = 1 \)). The AdS boundary is at \( z = 0 \) and the horizon can be scaled as \( z_h = 1 \). Therefore, the Hawking temperature is given by \( T = 3/(4\pi) \). In the probe limit, the equations of motions read,

\[
D_\mu D^\mu \Psi - m^2 \Psi = 0, \quad \nabla_\mu F^{\mu\nu} = i (\Psi^* D^\nu \Psi - \Psi (D^\nu \Psi)^*).
\]

Holographically, we propose a toy model in the boundary field theory to mimic the LP experiment, which consists of a thin-walled cylinder threading by an axial magnetic field. That is we set the periodic boundary conditions of all the fields along \( x \)-direction to represent the compact ring, and impose the homogeneous dependence of fields on the \( y \)-direction indicating the axial direction of the cylinder. Therefore, the ansatz for the fields are \( \Psi = \Psi(t, z, x), A_t = A_t(t, z, x), A_x = A_x(t, z, x) \) and \( A_z = A_y = 0 \). Without loss of generality, we set the scalar mass square as \( m^2 = -2 \). The asymptotic behaviors of fields near \( z \rightarrow 0 \) are
FIG. 1: Time evolutions of the phases and the condensate of the order parameter in the quenched dynamics. (a) Temporal evolutions of the phase $\theta$ from the initial time $(t = -8.4)$ to the final equilibrium state $(t = 1000)$, and the corresponding evolution of the average condensate of the order parameter. The phase evolves from initial random distributions to the final equilibrium state with constant gradient $\nabla \theta$ along the ring. The dashed lines indicate spurious jumps of the phase at the edges $\theta = \pm \pi$. The winding number in the final equilibrium state is $W = +2$. The four specific instants are denoted in the right plot for the evolution of the order parameter condensate. (b) A stereographic view of the windings of the phase along the ring at time $t = 1000$. The spurious jumps of the phase in panel (a) disappear on this $(\theta, x)$ torus, and the phase wraps the torus clock-wisely twice along the $x$-direction. In this figure, we have set $a_x = 0$ and the quench rate $\tau_Q = 20$.

$A_\mu \sim a_\mu + b_\mu z + \mathcal{O}(z^2), \Psi = z (\Psi_0 + \Psi_1 z + \mathcal{O}(z^2))$. From the dictionary of gauge-gravity duality, $a_t, a_x$ and $\Psi_0$ are interpreted as the chemical potential, potentials of the spatial component of gauge fields and source of scalar operators on the boundary, respectively. Correspondingly, $b_t, b_x$ and $\Psi_1$ are related to the charge density $\rho$, the conserved current $J_x$ and condensate of the order parameter $\langle O \rangle$. At the horizon $z_h$, we set $A_t(z_h) = 0$ and the regular finite boundary conditions for other fields. At the boundary $z \to 0$, we impose $\Psi_0 = 0$ in the standard quantization [30] and set the Dirichlet boundary conditions for $A_x$.

III. RESULTS

A. Symmetry breaking dynamics & formation of winding numbers

According to KZM, quenching the system across the critical point into the symmetry-breaking phase will result in the topological defects dynamically and statistically [9–11]. In our model, this topological defect is the winding number of phase of the order parameter along the ring [20, 27]. (Details of the quench profile and numerical schemes can be referred to in the Methods of Appendix.) We denote the compact ring as $\mathcal{C}$ with the circumference
\( L = 50 \), i.e., \( x \in [0,50] \). Therefore, the winding number of the phase \( \theta \) of the order parameter \( \Psi_1 = |\Psi_1|e^{i\theta} \) can be defined as

\[
W = \frac{1}{2\pi} \oint_C d\theta.
\] (4)

Fig.1(a) shows the temporal evolutions of the phases and average condensate of the order parameter from the initial time (at temperature \( T = 1.4T_c \)) to the final equilibrium state \( (T = 0.8T_c) \). At the initial time \( t = -8.4 \), the system is in the normal state with vanishing order parameter (refer to the right plot of Fig.1(a)). Tiny random seeds of the scalar field are thrown into the system at the initial time in order to induce spatial inhomogeneities during the quenching process. Thus, the phase is randomly distributed in space at \( t = -8.4 \). Decreasing the temperature across the critical point \( T_c \), spontaneous U(1) symmetry breaking will take place and the winding numbers will turn out in the ring.

At the time \( t = 50 \) the system is already in a superconducting state, but still in a far-from-equilibrium state which can be seen from the right plot in Fig.1(a). The phase \( \theta \) at this stage has some roughly constant ‘plateaus’, which is a direct result of KZM’s prediction that the symmetry will spontaneously break and the phase will randomly choose some constant values in different spatial regions. Since the system is still in the far-from-equilibrium state, the winding numbers at this stage may be destroyed by the non-equilibrium dynamics for some reason. For instance, at \( t = 50 \) the winding number is \( W = +3 \), however at the later time \( t = 67 \) the winding number changes to be \( W = +2 \).\(^1\)

The instant \( t = 67 \) is at the early stage when the condensate of order parameter arrives at the equilibrium value. From \( t = 67 \) to \( t = 1000 \) the absolute value of the condensate of order parameter will keep invariant as well as the winding numbers. However, the phase of the order parameter will still undergo dynamical processes until its gradient is a constant. For instance, at the final equilibrium state as \( t = 1000 \) the phase becomes ‘piecewise’ straight lines which is different from that at \( t = 67 \). This is from the fact that in the final equilibrium state, the superflow has a constant velocity \( \nabla \theta \) along the ring. The phase has a range \( \theta \in [-\pi,\pi] \), therefore, the dashed lines in Fig.1(a) are just the spurious jumps of the phases at the edges \( \theta = \pm \pi \). At the equilibrium state \( \nabla \theta \) is a constant, thus from Eq.(4) it is easy to deduce that \( \nabla \theta = 2\pi W/L \). In fact the phase is smooth at these jumps if one views the phase in a 2-torus in the \((\theta,x)\)-coordinates, refer to Fig.1(b). The phase smoothly wraps the torus clock-wisely twice if we go along the \( x \)-direction. Therefore, the phase has winding number \( W = +2 \) as we defined.

\(^1\) We define the winding number to be \( W = +n \) (\( n \geq 0 \) and \( n \in \mathbb{Z} \)) if the phase goes from \(-\pi\) to \(+\pi\) and wrap it \( n \) times along the \( x \)-direction. Negative winding number can be defined correspondingly.
FIG. 2: Distributions of the gauge invariant velocities and winding numbers. (a) Probability densities of the gauge invariant velocity $u$. The dashed line is the best fit to the numerical results, which satisfies the normal distribution with the mean $\langle u \rangle \approx 0$ and the standard deviation $\sigma(u) \approx 0.1033$. (b) Distributions of winding numbers for various magnetic fluxes. Dashed lines are the best fits to the numerical results. The dashed lines satisfy the normal distributions, with their means $\langle W \rangle \approx \Phi/\Phi_0$ in each plot as the dotted dashed lines indicate, and the standard deviations are identical to $\sigma \approx 0.7927$. We have made 7000 times of independent simulations in this figure.

B. Statistics of gauge invariant velocities & winding numbers

Near the boundary $z \to 0$ we impose the Dirichlet boundary conditions for $A_x$ by fixing $a_x$. Integration of $a_x$ along $C$ is the magnetic flux $\Phi$ threading the ring,

$$\Phi = \int B dS = \int (\nabla \times a_x) dS = \oint_C a_x dx = a_xL. \quad (5)$$

The last equality holds because of the homogeneity of $a_x$ along $x$-direction as we imposed. The gauge invariant velocity of the superflow can be defined as [3]

$$u = \nabla \theta - a_x. \quad (6)$$

For a fixed value of $a_x$ (equivalently to fix $\Phi/\Phi_0$ where $\Phi_0 = 2\pi$ is the flux quantum), it is possible to get various winding numbers according to the statistical properties of KZM [20, 24]. At final equilibrium $u = \nabla \theta - a_x = \frac{2\pi}{L}(W - \Phi/\Phi_0)$ from the Eqs.(4) and (5), thus, as one varies $a_x$ and makes multiple simulations one can arrive at a distribution of $u$. Due to the ‘central limit theorem’ [32], large number of independent simulations (7000 times in this paper) of $u$ should satisfy the normal distribution. Fig.2(a) shows the probability density for the distributions of $u$. The dashed line is the normal distribution function with the mean $\langle u \rangle \approx 0$ and standard deviation $\sigma(u) \approx 0.1033$. The numerical results matches the normal distribution very well. We further check that the higher cumulants of $u$ are
\( \kappa_3(u) \approx 4.9064 \times 10^{-6} \) and \( \kappa_4(u) \approx -1.6690 \times 10^{-5} \), which are very tiny and numerically verifies that \( P(u) \) is a normal distribution [33]. It should be noted that the winding number is integer, therefore, its distribution can never be normal distribution. However, one can first assume \( W \) as a continuous variable which satisfy the normal distribution as the dashed lines in Fig.2(b) shows. Then, the probabilities for each integer winding number are in fact the probabilities of the normal distribution at those integers. The dashed lines in Fig.2(b) have the means \( \langle W \rangle \approx \Phi/\Phi_0 \) for each plot as the dotted dashed lines indicate. Their standard deviations are identical as \( \sigma(W) \approx 0.7927 \).

At the final equilibrium, \( \langle W \rangle \approx \Phi/\Phi_0 \) is understandable since \( \langle u \rangle \approx 0 \) and \( u = \frac{2\pi}{L}(W - \Phi/\Phi_0) \). From the normal distribution, one can readily get that the standard deviation of \( W \) has a relationship to \( u \) as \( \sigma(W) = \sigma(u)L/(2\pi) \). Theoretically, \( \sigma(W) = 0.1033 \times 50/(2\pi) \approx 0.8220 \) which has 3.5% difference to the numerical results \( \sigma(W) \approx 0.7927 \). This small difference supports our conclusion above that the distribution of \( W \) was constrained as probabilities at integers in the normal distribution.

From the analysis in [3], there is a square term \( (W - \Phi/\Phi_0)^2 \) in the free energy, thus the requirement of the minimal free energy imposes \( W = \Phi/\Phi_0 \) statically. Therefore, the above requirement of minimal free energy in static case is consistent with the statistical results \( \langle W \rangle = \Phi/\Phi_0 \). This also in turn implies \( \langle u \rangle = 0 \), consistent with the numerical results.

C. Little-Parks periodicities of currents, condensates and free energies

From the AdS/CFT dictionary, the conserved current \( J_x \) in the boundary field theory is \( J_x = -b_x \). The inset plot in Fig.3(a) shows a linear relation between \( J_x \) and the gauge invariant velocity \( J_x \propto u \). Since \( u = \frac{2\pi}{L}(W - \Phi/\Phi_0) \) at the equilibrium, we expect that \( J_x \) has a periodic linear relation with respect to \( \Phi/\Phi_0 \) for each winding number, and the period is \( \Phi/\Phi_0 \). This relation is shown in Fig.3(a) with the open circles being the numerical data and the linear lines being the linear fitting \( J_x \propto (W - \Phi/\Phi_0) \). Different colors stand for different winding numbers. The periodicity in Fig.3(a) is the result from the LP effect in a compact geometry [3]. It should be noted that the solid thick lines represent the parts which correspond to the minimal values of free energies (refer to Fig.3(c)). Fig.3(b) shows the periodic relations between \( (J_x)^2 \) and magnetic flux \( \Phi/\Phi_0 \). The colored parabolas have the relation \( (J_x)^2 \propto (W - \Phi/\Phi_0)^2 \). The solid thick lines have lower free energies than the dashed lines (see Fig.3(c)), therefore, a favorable solution between \( (J_x)^2 \) to the \( \Phi/\Phi_0 \) is reflected by the periodic solid lines, which is similar to the profiles in LP oscillations in the weakly coupled condensed matter physics [3].

Fig.3(c) exhibits the periodic parabolas of the reduced free energy \( \Delta F/V_y \) with respect to \( \Phi/\Phi_0 \). Here, \( \Delta F = F - F_{W=0,\Phi/\Phi_0=0} \) and \( V_y = \int dy \) is the volume along the \( y \)-direction. It is clear that the solid bold lines stand for the minimal values of the free energies. The period is still \( \Phi/\Phi_0 = 1 \) and the lower free energies will change from one winding number to another winding number at half-integers of \( \Phi/\Phi_0 \), i.e., \( \Phi/\Phi_0 = Z \pm 1/2 \). For a specific value of \( \Phi/\Phi_0 \), a favorable solution is that with \( W = [\Phi/\Phi_0] \), in which \([\Phi/\Phi_0]\) represents the integers closest to \( \Phi/\Phi_0 \). For instance, the closest integer to \( \Phi/\Phi_0 = 0.75 \) is 1, thus the favorable solution has winding number \( W = 1 \) which is indicated by the solid thick green line. It reminds us of
FIG. 3: Little-Parks periodicities with respect to $\Phi/\Phi_0$. (a), (b), (c) and (d) show the periodic dependences of the conserved current $J_x$, the square of the current $(J_x)^2$, the reduced free energies $\Delta F/V_y$ and the average condensate of the order parameter $\langle |O| \rangle$ versus the magnetic flux $\Phi/\Phi_0$, respectively. All the periods are $\Phi/\Phi_0 = 1$. Numerical results are indicated by the open circles. Each colored line is the best fit to the numerical results, and corresponds to specific winding number. The solid thick lines represent the favorable solutions corresponding to lower free energies. This also means dashed lines have higher free energies and are unfavorable. The inset plot in panel (a) shows a linear relation between the current $J_x$ and the gauge invariant velocity $u$.

the distributions of winding numbers in Fig.2(b) that for a fixed $\Phi/\Phi_0 = 0.75$ the frequently appeared winding number is $W = 1$ as well. In addition, if $\Phi/\Phi_0 = 0.5$, the closest integer to 0.5 is either 0 or 1, thus the favorable solution has two choices with either $W = 0$ or $W = 1$. This is represented by the intersecting point between $W = 0$ and $W = 1$ solid lines. It also was reflected in Fig.2(b) that for a fixed $\Phi/\Phi_0 = 0.5$, the solutions $W = 0$ and
$W = 1$ almost have the same probabilities. Thus, the free energy analysis is consistent with the statistical distribution of winding numbers in the preceding subsection.

Fig.3(d) shows the periodic parabolas of the average condensate of the order parameter versus $\Phi/\Phi_0$. It also has the period $\Phi/\Phi_0 = 1$ and the solid thick lines correspond to the lower free energy parts in Fig.3(c). The transit of the favorable solutions between different winding numbers take places at half-integers of $\Phi/\Phi_0$ as well. $^2$

IV. CONCLUSIONS

Utilizing the KZM, we dynamically realized the winding numbers of the order parameter in a compact ring. At the final equilibrium state, the distributions of the gauge invariant velocity satisfies the normal distribution, while the distributions of integer winding numbers are constrained by this normal distribution that the probability for a specific winding number equals the probability density in the normal distribution at that winding number. Little-Parks periodicities turned out statistically as we varied the magnetic flux and simulated it multiple times. In particular, the conserved currents, the free energy and the average condensate of the order parameters exhibit the periodic dependences with respect to the magnetic flux. They all have the periods as $\Phi/\Phi_0 = 1$. Considering the minimum of free energies, the favorable static solutions correspond to the scalloped shape of the corresponding curves, and the scallops changes from one winding number to another winding number at the half-integers of $\Phi/\Phi_0$. Moreover, the analysis of the free energies were consistent with the statistics of winding numbers, which indicated that frequently appeared solutions had lower free energies.

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—Appendix—

I. METHODS

Holographic renormalization & holographic free energy: From holographic renormalization [34], the holographic free energy can be computed from the on-shell action and

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$^2$ In the Section II of Appendix, we use the Sturm-Liouville eigenvalue problems to analytically get the periodic relations between the critical phase transition points $\rho_c$ with respect to $\Phi/\Phi_0$. 

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the corresponding counter terms. The generic on-shell action of the Lagrangian (1) is \[ S_{\text{on-shell}} = -\frac{1}{2} \int d^4x \partial_\mu \left[ \sqrt{-g} \left( A_\mu F^{\mu\nu} + \Psi^* \partial^\mu \Psi + \Psi \partial^\mu \Psi^* \right) \right] \] (S1)\[ + i \frac{1}{2} \int d^4x \sqrt{-g} A_\mu (\Psi^* D^\mu \Psi - \Psi (D^\mu \Psi)^*) \].

The counter term in this case is \[ S_{\text{counter}} = \int d^3x \sqrt{-h} (\Psi^* \Psi), \] where \( h \) is the reduced metric on the \( z \to 0 \) boundary. The finite renormalized on-shell action \( S_{\text{renormalized}} \) can be obtained by \( S_{\text{renormalized}} = S_{\text{on-shell}} + S_{\text{counter}} \). The free energy is \( F = -TS_{\text{renormalized}} \) where \( T \) is the temperature of the black hole. Replacing the fields and the corresponding boundary conditions into the free energy, we finally arrive at

\[ F = -T S_{\text{renormalized}} = -\frac{1}{2} V_y \int dx \left( A_t \partial_x A_t - A_x \partial_x A_x \right) \bigg|_{z=0} \] (S2)\[ - V_y \int dz dx \left( A_t \text{Im} (\psi \partial_z \psi^*) - A_x \text{Im} (\psi \partial_x \psi^*) - A_x^2 |\psi|^2 \right). \]

where \( V_y = \int dy \) is the volume along \( y \)-direction and \( \psi = \Psi / z \).

**Numerical schemes:** From the dimensional analysis, the temperature of the black hole \( T \) has mass dimension one, while the mass dimension of the charge density \( \rho \) on the boundary is two. Therefore, \( T/\sqrt{\rho} \) is a dimensionless quantity. From the holographic superconductor [30], reducing the temperature is equivalent to increasing the charge density. Thus, in order to linearly quench the temperature as \( T(t)/T_c = 1 - t/\tau_Q \) from KZM [9–11] (\( \tau_Q \) is the so-called quench rate), one can indeed quench the charge density \( \rho \) as

\[ \rho(t) = \frac{\rho_c}{(1 - t/\tau_Q)^2}, \] (S3)

where \( \rho_c \) is the critical charge density in the homogeneous and static holographic superconducting system. Its value is explored in the next Section II in this Appendix. We evolve the system by using the 4th-order Runge-Kutta method with time step \( \Delta t = 0.1 \). In the AdS radial direction \( z \), we adopt the Chebyshev pseudo-spectral methods with 21 grid points. Since all the fields are periodic in the \( x \)-directions, we use the Fourier decomposition along \( x \)-directions with 201 grid points.

II. **STURM-LIOUVILLE EIGENVALUE PROBLEMS & LITTLE-PARKS PERIODICITIES OF STATIC TRANSITION POINTS**

This part will analytically study the homogeneous and static holographic superconducting phase transition point in the main text by using the variational method for the Sturm-Liouville eigenvalue problems [36–38]. It is convenient to work in the AdS-Schwarzschild black brane,

\[ ds^2 = \frac{1}{z^2} \left( -f(z) dt^2 + \frac{dz^2}{f(z)} + dx^2 + dy^2 \right), \quad \text{with} \quad f(z) = 1 - \left( \frac{z}{z_h} \right)^3. \] (S4)
In the static case, the fields are homogeneous in the \( x \)-direction except the phase of the order parameter will depend on \( x \) along the compact ring \([6–8]\). Thus, we make ansatz as \( \Psi = \Psi(z)e^{i\frac{2\pi W}{L}x} \), \( A_t = A_t(z) \) and \( A_x = A_x(z) \), where \( W \) is the winding number and \( L \) is the circumference of the ring. Therefore, the equations of motions read,

\[
\left( \frac{1}{z^2} - z \right) \Psi'' - \left( 1 + \frac{2}{z^3} \right) \Psi' - \left[ \frac{m^2}{z^4} + \frac{A_t^2}{z^2(z^3 - 1)} + \frac{1}{z^2} \left( A_x - \frac{2\pi W}{L} \right)^2 \right] \Psi = 0, \quad (S5)
\]

\[
A_t'' + \frac{2\Psi^2 e^{i\frac{2\pi W}{L}x}}{z^2(z^3 - 1)} A_t = 0, \quad (S6)
\]

\[
A_x'' + \frac{3z^2}{z^3 - 1} A_x' + \frac{2\Psi^2 e^{i\frac{2\pi W}{L}x}}{z^2(z^3 - 1)} A_x = 0, \quad (S7)
\]

where \( ' \) denotes the derivative with respect to \( z \). Near the boundary \((z \to 0)\), we have the expansions,

\[
\Psi = \Psi_0 z^{\Delta_-} + \Psi_1 z^{\Delta_+}, \quad \text{where} \quad \Delta_{\pm} = \frac{3}{2} \pm \sqrt{\frac{9}{4} + m^2} \quad (S8)
\]

\[
A_t = \mu - \rho z, \quad (S9)
\]

\[
A_x = a_x + b_x z. \quad (S10)
\]

In the main text, we take \( m^2 = -2 \) and focus on the standard quantization to set \( \Psi_0 = 0 \). Thus, the expectation values of the order parameter is \( \langle O \rangle = \Psi_1 \). At the transition point \( T_c \) (or equivalently \( \rho_c \)), the scalar field is nearly vanishing \( \Psi \approx 0 \). Therefore, the equation of motion Eq.(S6) reduces to \( A_t'' \sim 0 \), with the solution \( A_t \sim c_1 + c_2 z \) where \( c_{1,2} \) are constants. Imposing the vanishing boundary conditions of \( A_t \) at the horizon \( z = z_h = 1 \), one finally gets \( A_t(z) = \rho(1 - z) \). Similarly, one can get \( A_x = a_x \) which is a constant from the Eq.(S7) by imposing \( \Psi \sim 0 \).

Therefore, as \( T \to T_c \) the equation Eq.(S5) for the scalar field \( \Psi \) becomes

\[
-\Psi'' + \frac{2 + z^3}{z(1 - z^3)} \Psi' + \left( \frac{-2 + z^2 \left( a_x - \frac{2\pi W}{L} \right)^2}{z^2(1 - z^3)} - \frac{\rho^2}{(1 + z + z^2)^2} \right) \Psi = 0. \quad (S11)
\]

To solve this equation, we introduce a trial function \( F(z) \) for \( \Psi \) near \( z = 0 \) as \([36–38]\)

\[
\Psi|_{z \to 0} \approx \langle O \rangle z^2 F(z) \quad (S12)
\]

The boundary condition for \( F(z) \) is \( F(0) = 1 \) and \( F'(0) = 0 \). Hence, the equation of motion for \( F(z) \) is,

\[
-F'' + \frac{1}{z} \left( \frac{2 + z^3}{1 - z^3} - 4 \right) F' + \left( \frac{4z^2 + \left( a_x - \frac{2\pi W}{L} \right)^2}{1 - z^3} - \frac{\rho^2}{(1 + z + z^2)^2} \right) F = 0. \quad (S13)
\]

Multiplying

\[
K(z) = z^2(1 - z^3) \quad (S14)
\]
to both sides of the above equation, the equation of motion for $F(z)$ reduces to
\[
\frac{d}{dz} \left( K(z) F' \right) + P(z) F + Q(z) \rho^2 F = 0 \quad (S15)
\]
where
\[
P(z) = 4z^4 + z^2 \left( a_x - \frac{2\pi W}{L} \right)^2, \quad Q(z) = -\frac{z^2(1-z)}{1+z+z^2} \quad (S16)
\]
From Sturm-Liouville eigenvalue problem, the minimal eigenvalues of $\rho^2$ can be obtained by taking variations with the following functional,
\[
\mu^2 = \frac{\int_0^1 dz (KF'^2 + PF^2)}{\int_0^1 dz QF^2} \quad (S17)
\]
The trial function $F(z)$ can be assumed to be $F(z) \equiv 1 - \alpha z^2$ with $\alpha$ a constant. Thus, we obtain
\[
\rho^2(\alpha) = 2 \times \frac{1 + \frac{1}{3}(a_x - \frac{2\pi W}{L})^2 - \left[ \frac{4}{3} + \frac{2}{5}(a_x - \frac{2\pi W}{L})^2 \right] \alpha + \left[ \frac{4}{5} + \frac{1}{7}(a_x - \frac{2\pi W}{L})^2 \right] \alpha^2}{3 - \ln 3 + \frac{\pi}{\sqrt{3}} + \left( \frac{13}{3} - 4 \ln 3 \right) \alpha + \left( \frac{\pi}{\sqrt{3}} - \frac{7}{10} + \ln 3 \right) \alpha^2} \quad (S18)
\]

**FIG. S1:** Little-Parks periodic relations between $\rho_c$ and $\Phi/\Phi_0$. Each colored parabola corresponds to each winding number. The period is $\Phi/\Phi_0 = 1$. The solid thick curves represent lower free energy solutions, while the dashed lines correspond to unfavorable solutions.

In the main text, we set $L = 50$. In order to get the minimal value of $\rho$ (equivalently $\rho_c$), one needs to fix the values of $W$ and $a_x$ with $W \in \mathbb{Z}$. Thus one can find the critical value of $\rho_c$ from some specific values of $\alpha$ from the Eq.($S18$). Then, one may sweep the values of $a_x$ and $W$ to find a series of values of $\rho_c$’s. This will provide a relation between the critical values of $\rho_c$ and $(a_x - \frac{2\pi W}{L})$. As we mentioned in the main text, in the final equilibrium state
(or at the static case), the magnetic flux $\Phi = a_x L$, therefore, $a_x - \frac{2\pi W}{L} = \frac{2\pi}{L} (\Phi/\Phi_0 - W)$. The relations between the critical points $\rho_c$ and $\Phi/\Phi_0$ are given in Fig.S1, with the period $\Phi/\Phi_0 = 1$. Different colored lines correspond to different winding numbers, while the solid thick lines correspond to lower free energies in the main text. The dashed lines correspond to unfavorable solutions. The scalloped shapes of the lower curves remind us of the Little-Parks effect [3]. The transition of the lower curves from one winding number to another winding number happens at half-integers of $\Phi/\Phi_0$. We find that the quantity $(a_x - \frac{2\pi W}{L})$ appear as a whole in Eq.(S18). Therefore, it is easy to deduce that the relation between the critical points $\rho_c$ should be a function of $(a_x - \frac{2\pi W}{L})$. By fitting the data in Fig.S1 we find that it is a parabolic relation as

$$\rho_c = a_1 \left( \frac{L}{2\pi} a_x - W \right)^2 + \rho_{\min} = a_1 \left( \frac{\Phi}{\Phi_0} - W \right)^2 + \rho_{\min}, \quad (S19)$$

with $a_1 \approx 0.0094$ and $\rho_{\min} \approx 4.16$.

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