Multivariable Tangent and Secant
$q$-derivative Polynomials

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Abstract. The derivative polynomials introduced by Knuth and Buckholtz in their calculations of the tangent and secant numbers are extended to a multivariable $q$-environment. The $n$-th $q$-derivatives of the classical $q$-tangent and $q$-secant are each given two polynomial expressions. The first polynomial expression is indexed by triples of integers, the second by compositions of integers. The functional relation between those two classes is fully given by means of combinatorial techniques. Moreover, those polynomials are proved to be generating functions for so-called $t$-permutations by multivariable statistics. By giving special values to those polynomials we recover classical $q$-polynomials such as the Carlitz $q$-Eulerian polynomials and the $(t,q)$-tangent and -secant analogs recently introduced. They also provide $q$-analogs for the Springer numbers. Finally, the $t$-compositions used in this paper furnish a combinatorial interpretation to one of the Fibonacci triangles.

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1. Introduction

Back in 1967, Knuth and Buckholtz [KB67] devised a clever method for computing the tangent and secant numbers $T_{2n+1}$ and $E_{2n}$ for large values of the subscripts $n$. For that purpose they introduced two sequences of polynomials, referred to as derivative polynomials. A few years later, Hoffman [Ho95, Ho99] calculated the exponential generating functions for those polynomials, and found a combinatorial interpretation for their coefficients, in terms of so-called snakes. The goal of this paper is to obtain a multivariable $q$-analog of all those results. We first recall the contributions made by those authors, then, introduce the $q$-environment that makes it possible to derive a handy algebra for these new $q$-derivative polynomials.

1.1. The derivative polynomials. Recall that tangent and secant numbers $T_{2n+1}$ and $E_{2n}$ occur as coefficients in the Taylor expansions of $\tan u$ and $\sec u$:

$$\tan u = \sum_{n \geq 0} \frac{u^{2n+1}}{(2n+1)!} T_{2n+1}$$

$$= \frac{u}{1!} + \frac{u^3}{3!} + \frac{u^5}{5!} + \frac{u^7}{7!} + \frac{u^9}{9!} + \cdots$$

$$\sec u = \frac{1}{\cos u} = \sum_{n \geq 0} \frac{u^{2n}}{(2n)!} E_{2n}$$

$$= 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + \frac{u^8}{8!} + \frac{u^{10}}{10!} + \cdots$$

See, e.g., [Ni23, p. 177-178], [Co74, p. 258-259].

Let $(A_n(x))$ ($n \geq 1$) be the sequence of polynomials defined by

$$A_0(x) = x, \quad A_{n+1} = (1 + x^2) DA_n(x),$$

with $D$ being the differential operator. When writing $A_n(x) = \sum_{m \geq 0} a(n,m)x^m$, the coefficients $(a(n,m))$ satisfy the recurrence

$$a(0,m) = \delta_{1,m}, \quad a(n+1,m) = (m-1)a(n,m-1) + (m+1)a(n,m+1).$$

The $a(n,m)$’s form a triangle of integral numbers (see Table 1 in Section 11, where the numbers $a(n,m)$ are reproduced in boldface), now registered as the sequence A101343 in Sloane’s Encyclopedia of Integer Sequences [Sl06] with an abundant bibliography. Knuth and Buckholtz [KB67] showed that the $n$-th derivative $D^n \tan u$ was equal to the polynomial

$$D^n \tan u = \sum_{m \geq 0} a(n,m) \tan^m u.$$
Again, Knuth and Buckholtz [KB67] showed that the $n$-th derivative of $\sec u$ could be expressed as
\[ D^n \sec u := \sum_{m \geq 0} b(n, m) \tan^m u \sec u. \quad (1.5) \]

The triangle of numbers $(b(n, m))$ also appears in Sloane’s Encyclopedia [Sl06] with reference A008294. The first values of the numbers $b(n, m)$ are reproduced in Table 1 in plain type (not bold). From their very definitions the $a(n, m)$’s and $b(n, m)$’s can be imbricated in the same table, as done in Table 1. The meanings of the entries to the right of the table will be further explained.

The exponential generating functions for the polynomials
\[ A_n(x) := \sum_{m \geq 0} a(n, m) x^m \quad \text{and} \quad B_n(x) := \sum_{m \geq 0} b(n, m) x^m, \]
called derivative polynomials, have been derived by Hoffman [Ho95] in the form
\[ \sum_{n \geq 0} A_n(x) \frac{u^n}{n!} = \frac{x + \tan u}{1 - x \tan u}; \quad (1.6) \]
\[ \sum_{n \geq 0} B_n(x) \frac{u^n}{n!} = \frac{1}{\cos u - x \sin u}. \quad (1.7) \]

Those two exponential generating functions and recurrences for the $a(n, m)$’s and $b(n, m)$’s have also been obtained by other people in different contexts, in particular, by Carlitz and Scoville [CS72], Françon [Fr78].

By plugging $x = 1$ in (1.6) the right-hand side becomes $\tan 2u + \sec 2u$, so that the sum $\sum_m a(n, m)$ is equal to $2^n E_n$, if $n$ is even, and to $2^n T_n$, if $n$ is odd. Likewise, (1.7) yields $\sum_n B_n(1) u^n / n! = 1 / (\cos u - \sin u)$, which is the exponential generating function for the so-called Springer numbers $(1, 1, 3, 11, 57, 361, 2763, \ldots)$ (see [Sp71], [Du95]) originally considered by Glaisher [Gl98, Gl99, Gl14], as noted in Sloane’s Encyclopedia, under reference A001586. In Table 1 we have indicated the values of the row sums $\sum_m a(n, m)$ and $\sum_m b(n, m)$ to the right.

Finally, the combinatorial interpretations of the $a(n, m)$’s and $b(n, m)$’s are due to Hoffman in a later paper [Ho99]. An equivalent interpretation is also due to Josuat-Vergès [Jo11]. Both authors use the word snake of length $n$, a notion made popular by Arnold [Ar92, Ar92a] in the study of morsification of singularities, to designate each word $w = x_1 x_2 \cdots x_n$, whose letters are integers, positive or negative, with the further property that $x_1 > x_2$, $x_2 < x_3$, $x_3 > x_4$, \ldots in an alternating way and $|x_1|, |x_2|, \ldots, |x_n|$ is a permutation of $1, 2, \ldots, n$. Note that Josuat-Vergès, Novelli and Thibon [JNT12] have recently developed an algebraic combinatorics of snakes from the Hopf algebra point of view. In [Ho99] and [Jo11] Hoffman and Josuat-Vergès show that $A_n(x)$ is the generating polynomial for the set of all snakes of length $n$ by the number of sign changes; they also prove an analogous result for $B_n(x)$. 

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1.2. Towards a multivariable $q$-analog. In parallel with (1.4) and (1.5) the $q$-derivative operator $D_q$ (see [GR90, p. 22]), as well as the $q$-analogs of tangent and secant (see [St76], [AG78], [AF80], [Fo81], [St97, p. 148-149], [St10]) are to be introduced. The first problem is to see whether the $n$-th $q$-derivatives of those $q$-analogs can be expressed as polynomials in those functions, and they can! But, contrary to formulas (1.4) and (1.5), those $n$-th $q$-derivatives have several polynomial forms. As will be seen, two such polynomial forms are derived in this paper for each $q$-analog of tangent and secant. The second problem is to work out an appropriate algebra for the polynomials involved that must appear as natural multivariable $q$-analogs of the entries $a(n,m)$ and $b(n,m)$.

Before stating the first results of this paper we recall a few basic notions on $q$-Calculus. The $t$-ascending factorial in a variable $t$ is traditionally defined by

$$
(t; q)_n := \begin{cases} 
1, & \text{if } n = 0; \\
(1 - t)(1 - tq) \cdots (1 - tq^{n-1}), & \text{if } n \geq 1;
\end{cases}
$$

in its finite version and

$$
(t; q)_\infty := \lim_{n \to \infty} (t; q)_n = \prod_{n \geq 0} (1 - tq^n);
$$

in its infinite version. By $q$-series it is meant a series of the form $f(u) = \sum_{n \geq 0} f(n; q)u^n/(q; q)_n$, whose coefficients $f(n; q)$ belong to some ring [AAR00, chap. 10]. The $q$-derivative operator for fixed $q$ used below is defined by

$$
D_q f(u) := \frac{f(u) - f(qu)}{u},
$$

instead of the traditional $((f(u) - f(qu)))/(u(1 - q))$.

At this stage we just have to note (see Section 2) that there is only one $q$-tangent attached to the classical Jackson definitions [Ja04] of the $q$-sine and $q$-cosine, namely,

$$
\tan_q(u) := \frac{\sum_{n \geq 0} (-1)^n u^{2n+1}/(q; q)_{2n+1}}{\sum_{n \geq 0} (-1)^n u^{2n}/(q; q)_{2n}},
$$

but, as it has been rarely noticed, there are two $q$-secants

$$
\sec_q(u) := \frac{1}{\sum_{n \geq 0} (-1)^n u^{2n}/(q; q)_{2n}};
$$

$$
\Sec_q(u) := \frac{1}{\sum_{n \geq 0} (-1)^n q^n(2n-1)u^{2n}/(q; q)_{2n}}.
$$
1.3. The numerical and combinatorial background. Following Andrews [An76, chap. 4] a composition of a positive integer \( n \) is defined to be a sequence \( c = (c_0, c_1, \ldots, c_m) \) of nonnegative integers such that \( c_0 + c_1 + \cdots + c_m = n \), with the restriction that only \( c_0 \) and \( c_m \) can be zero; the \( c_i \)'s are the parts of the composition and \( m + 1 \) is the number of parts, denoted by \( \mu c + 1 \). A composition \( c = (c_0, c_1, \ldots, c_m) \) of \( n \) is said to be a \( t \)-composition, if the following two conditions hold:

(1) either \( m = 0 \), so that \( c = (c_0) \) and \( c_0 = n \) is an odd integer, or \( m \geq 1 \) and both \( c_0 \), \( c_m \) are even;

(2) if \( m \geq 2 \), then all the parts \( c_1, c_2, \ldots, c_{m-1} \) are odd.

For each \( n \geq 1 \) the set of all \( t \)-compositions of \( n \) is denoted by \( \Theta_n \). It is further assumed that \( \Theta_0 \) consists of a unique empty composition denoted by \((0,0)\).

The first \( t \)-compositions are the following:

\[
\begin{align*}
\Theta_0: & \quad (0,0) ; \\
\Theta_1: & \quad (1) , \quad (0,1,0) ; \\
\Theta_2: & \quad (0,2) , \quad (2,0) , \quad (0,1,1,0) ; \\
\Theta_3: & \quad (3) , \quad (0,1,2) , \quad (2,1,0) , \quad (0,3,0) , \quad (0,1,1,1,0) ; \\
\Theta_4: & \quad (0,4) , \quad (2,2) , \quad (4,0) , \quad (0,3,1,0) , \quad (0,1,3,0) , \quad (0,1,1,2) , \quad (2,1,1,0) , \\
& \quad (0,1,1,1,1,0).
\end{align*}
\]

A \( t \)-composition \( c = (c_0, c_1, \ldots, c_{m-1}, c_m) \) from \( \Theta_n \) such that \( c_m = 0 \) is called an \( s \)-composition. We write \( c^- = (c_0, c_1, \ldots, c_{m-1}) \). The subset of \( \Theta_n \) of all \( s \)-compositions is denoted by \( \Theta_n^- \).

For each composition \( c = (c_0, c_1, \ldots, c_m) \) of \( n \geq 1 \) let \( \tan_q(q^c u) := 1 \) if \( m = 0 \) and for \( m \geq 1 \), whatever \( n \geq 1 \), let

\[
\tan_q(q^c u) := \tan_q(q^{c_0} u) \tan_q(q^{c_0+c_1} u) \cdots \tan_q(q^{c_0+c_1+\cdots+c_{m-1}} u).
\]

Note that the rightmost part \( c_m \) does not occur in the previous expression. Also let \( \rho c := (c_m, \ldots, c_1, c_0) \) denote the mirror-image of \( c \).

A word \( w = y_1 y_2 \cdots y_m \), whose letters are positive integers, is said to be falling alternating, or simply alternating (resp. rising alternating), if \( y_1 > y_2 \), \( y_2 < y_3 \), \( y_3 > y_4 \), \ldots (resp. if \( y_1 < y_2 \), \( y_2 > y_3 \), \( y_3 < y_4 \), \ldots) in an alternating manner. The notion goes back to Désiré André [An79, An81], who showed that the number of falling (resp. rising) alternating permutations \( \sigma = \sigma(1) \sigma(2) \cdots \sigma(n) \) of \( 12 \ldots n \) is equal to \( T_n \) when \( n \) is odd, and to \( E_n \) when \( n \) is even. The length of each word \( w \) is denoted by \( \lambda w \) and the empty word is the word of length 0, denoted by \( \epsilon \).

Definition. A \( t \)-permutation of order \( n \) is defined to be a nonempty sequence \( w = (w_0, w_1, \ldots, w_m) \) of words having the properties:

(i) the juxtaposition product \( w_0 w_1 \cdots w_m \) is a permutation of \( 12 \cdots n \);

(ii) either \( m = 0 \) and \( w_0 \) is rising alternating of odd length, or \( m \geq 1 \) and \( w_0 \) is rising alternating of even length and \( w_m \) is (falling) alternating of even length;
(iii) if \( m \geq 2 \), then all the components \( w_1, w_2, \ldots, w_{m-1} \) are (falling) alternating of odd length.

For each \( n \geq 0 \) the set of all \( t \)-permutations of order \( n \) will be denoted by \( \mathcal{T}_n \).

The first \( t \)-permutations are the following:
\[
\mathcal{T}_0: \quad (\epsilon, \epsilon) ;
\mathcal{T}_1: \quad (1); \quad (\epsilon, 1, \epsilon);
\mathcal{T}_2: \quad (\epsilon, 21), \quad (12, \epsilon); \quad (\epsilon, 2, 1, \epsilon), \quad (\epsilon, 1, 2, \epsilon);
\mathcal{T}_3: \quad (132), \quad (231);
\quad (\epsilon, 321), \quad (\epsilon, 312, \epsilon), \quad (\epsilon, 213, \epsilon), \quad (12, 3, \epsilon), \quad (\epsilon, 1, 32), \quad (23, 1, \epsilon),
\quad (13, 2, \epsilon), \quad (\epsilon, 2, 31);
\quad (\epsilon, 1, 2, 3, \epsilon), \quad (\epsilon, 1, 3, 2, \epsilon), \quad (\epsilon, 2, 1, 3, \epsilon), \quad (\epsilon, 2, 3, 1, \epsilon),
\quad (\epsilon, 3, 1, 2, \epsilon), \quad (\epsilon, 3, 2, 1, \epsilon).
\]

1.4. The underlying statistics. The number of factors in each \( t \)-permutation \( w = (w_0, w_1, \ldots, w_m) \), minus one, is denoted by \( \mu w = m \). If \( w \) is of order \( n \), the sequence \( (\lambda w_0, \lambda w_1, \ldots, \lambda w_m) \) is a \( t \)-composition of \( n \), denoted by \( \Lambda w \). For each \( n \geq 0 \) the set of all \( t \)-permutations \( w \) from \( \mathcal{T}_n \), such that \( \mu w = m \), with \( 0 \leq m \leq n + 1 \) and \( m \equiv n + 1 \mod 2 \), will be denoted by \( \mathcal{T}_{n,m} \).

Recall that the ligne of route of a permutation \( \sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \) of \( 12 \cdots n \), denoted by Ligne \( \sigma \), is defined to be the set of all \( i \) such that \( 1 \leq i \leq n - 1 \) and \( \sigma(i) > \sigma(i + 1) \); also, the inverse ligne of route, Iligne \( \sigma \), to be the set of all \( \sigma(i) \) such that \( \sigma(j) = \sigma(i) + 1 \) for some \( j \leq i - 1 \). In an equivalent manner, Iligne \( \sigma = \text{Ligne} \sigma^{-1} \). Next, let
\[
\text{ides} \sigma = \# \text{Iligne} \sigma;
\text{imaj} \sigma = \sum_i \sigma(i) \quad (\sigma(i) \in \text{Iligne} \sigma);
\]
and let \( \text{inv} \) be the number of inversions of \( \sigma \), as being the number of pairs \((i, j)\) such that \( 1 \leq i < j \leq n \) and \( \sigma(i) > \sigma(j) \).

The inverse ligne of route, Iligne \( w \), of a \( t \)-permutation \( w = (w_0, w_1, \ldots, w_m) \) is then defined by
\[
\text{Iligne} w := \text{Iligne}(w_0 w_1 \cdots w_m);
\]
and the number of inversions, \( \text{inv} w \), by
\[
\text{inv} w := \text{inv}(w_0 w_1 \cdots w_m).
\]
This makes sense, as the latter juxtaposition product is a permutation, say, \( \sigma \) of \( 12 \cdots n \) if \( w \in \mathcal{T}_n \). Finally, let \( \min w := a \) if 1 is a letter in \( w_a \).

For example, with the \( t \)-permutation \( w = (45, 1113, 1079, 6, 82) \), the elements of \( \text{Iligne} w \) being reproduced in boldface, we have: \( \text{ides} w = 6 \), \( \text{imaj} w = 3 + 10 + 9 + 6 + 8 + 2 = 38 \), \( \text{inv} w = 27 \) and \( \min w = 1 \).

1.5. The main results. For each triple \((k, a, b)\) let \( \mathcal{T}_{n,k,a,b} \) denote the set of all \( t \)-permutations \( w \) from \( \mathcal{T}_n \) such that \( \text{ides} w = k \), \( \min w = a \) and \( a + b = \mu w \).
Theorem 1.1 (Multivariable \(q\)-analog of (1.4)). Let

\[(1.8) \quad A_{n,k,a,b}(q) = \sum_{w \in T_{n,k,a,b}} q^{\text{imaj} w} \]

be the generating polynomial for the set \(T_{n,k,a,b}\) by the statistic “imaj.” Then

\[(1.9) \quad D_n^q \tan_q(u) = \sum_{k,a,b} A_{n,k,a,b}(q) (\tan_q(q^{k+1}u))^b (\tan_q(q^k u))^a; \]

where \(0 \leq k \leq n - 1\) and \(0 \leq a + b \leq n + 1\).

In the same manner, for each \((k,a,b)\) let \(T_{n,k,a,b+1}^-\) denote the set of all \(t\)-permutations \(w = (w_0, w_1, \ldots, w_m, w_{m+1})\) in \(T_{n,k,a,b+1}\) such that \(w_{m+1} = \epsilon\).

Theorem 1.2 (Multivariable \(q\)-analog of (1.5)). Let

\[(1.10) \quad B_{n,k,a,b}(q) = \sum_{w \in T_{n,k,a,b+1}^-} q^{\text{imaj} w} \]

be the generating polynomial for the set \(T_{n,k,a,b+1}^-\) by “imaj.” Then

\[(1.11) \quad D_n^q \sec_q(u) = \sum_{k,a,b} B_{n,k,a,b}(q) (\tan_q(q^{k+1}u))^b \sec_q(q^{k+1}u) (\tan_q(q^k u))^a; \]

\[(1.12) \quad D_n^q \Sec_q(u) = \sum_{k,a,b} q^{n(n-1)/2} B_{n,n-1-k,b,a}(q^{-1}) \times (\tan_q(q^{k+1}u))^b \Sec_q(q^{k+1}u) (\tan_q(q^k u))^a. \]

where \(0 \leq k \leq n - 1\) and \(0 \leq a + b \leq n\).

Theorem 1.3 (Composition \(q\)-analogs of (1.4) and (1.5)). For each \(n \geq 1\) and each \(t\)-composition \(c\) of \(n\) let \(A_{n,c}(q)\) be the polynomial

\[(1.13) \quad A_{n,c}(q) = \sum_{w \in T_n, \Lambda w = c} q^{\text{inv} w}. \]

Then,

\[(1.14) \quad D_n^q \tan_q(u) = \sum_{c \in \Theta_n} A_{n,c}(q) \tan_q(q^c u); \]

\[(1.15) \quad D_n^q \sec_q(u) = \sum_{c \in \Theta_n} A_{n,c}(q) \tan_q(q^{c^-} u) \sec_q(q^n u); \]

\[(1.16) \quad D_n^q \Sec_q(u) = \sum_{c \in \Theta_n^-} q^{n(n-1)/2} A_{n,c}(q^{-1}) \tan_q(q^{c^-} u) \Sec_q(u). \]

Remark. The polynomials \(A_{n,k,a,b}(q)\) are not uniquely defined by identity (1.9). On the contrary, both (1.11) and (1.12) uniquely define the polynomials \(B_{n,k,a,b}(q)\) and \(A_{n,c}(q)\).

Polynomials indexed by triples \((k,a,b)\) and those by compositions \(c\) are related to each other, as indicated in the next theorem.
Theorem 1.4. We have

$$\sum_{k \geq 0, a+b=m} A_{n,k,a,b}(q) = \sum_{c \in \Theta_{n,m}} A_{n,c}(q);$$

$$\sum_{k \geq 0, a+b=m} B_{n,k,a,b}(q) = \sum_{c \in \Theta_{n,m+1}} A_{n,c}(q).$$

Now, form the generating polynomials

$$A_n(x, q) := \sum_{m \geq 0} x^m \sum_{k \geq 0, a+b=m} A_{n,k,a,b}(q) = \sum_{m \geq 0} x^m \sum_{c \in \Theta_{n,m}} A_{n,c}(q);$$

$$B_n(x, q) := \sum_{m \geq 0} x^m \sum_{k \geq 0, a+b=m} B_{n,k,a,b}(q) = \sum_{m \geq 0} x^m \sum_{c \in \Theta_{n,m+1}} A_{n,c}(q).$$

Theorem 1.5. The factorial generating functions for the polynomials $A_n(x, q)$ and $B_n(x, q)$ are given by:

$$\sum_{n \geq 0} A_n(x, q) \frac{u^n}{(q; q)_n} = \tan_q(u) + \sec_q(u)(1 - x \tan_q(u))^{-1} x \sec_q(u);$$

$$\sum_{n \geq 0} B_n(x, q) \frac{u^n}{(q; q)_n} = \sec_q(u)(1 - x \tan_q(u))^{-1}.$$

Those two formulas, derived in Section 9, are true $q$-analogs of Hoffman’s identities (1.6) and (1.7), as the latter ones can be rewritten as:

$$\sum_{n \geq 0} A_n(x) \frac{u^n}{n!} = \tan(u) + \sec(u)(1 - x \tan(u))^{-1} x \sec(u),$$

$$\sum_{n \geq 0} B_n(x) \frac{u^n}{n!} = \sec(u)(1 - x \tan(u))^{-1}.$$

The four-variable polynomials $A_n(t, x, y, q) = \sum_{k,a,b} A_{n,k,a,b}(q)t^k x^a y^b$ and $B_n(t, x, y, q) := \sum_{k,a,b} B_{n,k,a,b}(q)t^k x^a y^b$, that may be considered as multivariable $q$-analogs of the entries $a(n, m)$ and $b(n, m)$, have several interesting specializations studied in Section 10. First, $tA_n(t, 0, 0, q)$ and $tB_n(t, 0, 0, q)$ are shown to be the $(t, q)$-analogs $T_n(t, q)$ and $E_n(t, q)$ of tangent and secant numbers, only defined so far [FH11] by their factorial generating functions $\sum_n T_{2n+1}(t, q)u^{2n+1}/(t; q)_{2n+2}$ and $\sum_n E_{2n}(t, q)u^{2n}/(t; q)_{2n+1}$. The recurrences of the polynomials $A_{n,k,a,b}(q)$ and $B_{n,k,a,b}(q)$ provide a handy method for calculating them. We also prove that the polynomial $A_n(t, x, q) := \sum_{k,a} A_{n,k,a,n+1-a}(q)t^k x^a$ is a refinement of the Carlitz $q$-analog [Ca54] $A_n(t, q)$.
of the Eulerian polynomial, as $A_n(t, 1, q) = A_n(t, q)$, with an explicit combinatorial interpretation. Referring to Tables 2 and 3 of the polynomials $A_{n,k,a,b}(q)$ and $B_{n,k,a,b}(q)$ displayed at the end of the paper, it is shown that the sum of the polynomials occurring in each box along the two top diagonals can be explicitly evaluated. Furthermore, the Springer numbers are given two $q$-analogues. Finally, generating functions and recurrence relations are provided for both $t$- and $s$-compositions.

2. A detour to the theory of $q$-trigonometric functions

By means of the $q$-binomial theorem ([GR90, § 1.3]; [AAR00, § 10.2]) we can express the first $q$-exponential $e_q(u)$ and the second $q$-exponential $E_q(u)$, either as an infinite $q$-series, or an infinite product:

\begin{align}
e_q(u) &= \sum_{n \geq 0} \frac{u^n}{(q; q)_n} = \frac{1}{(u; q)_\infty}; \\
E_q(u) &= \sum_{n \geq 0} \frac{u^{n(n-1)/2}}{(q; q)_n} = (-u; q)_\infty,
\end{align}

two results that go back to Euler [Eu48]. As already done by Jackson [Ja04] (also see [GR90, p. 23]), they both serve to define the $q$-trigonometric functions $q$-sine and $q$-cosine:

$$
\sin_q(u) := \frac{e_q(iu) - e_q(-iu)}{2i} = \sum_{n \geq 0} (-1)^n \frac{u^{2n+1}}{(q; q)_{2n+1}};
$$

$$
\cos_q(u) := \frac{e_q(iu) + e_q(-iu)}{2} = \sum_{n \geq 0} (-1)^n \frac{u^{2n}}{(q; q)_{2n}};
$$

$$
\Sin_q(u) := \frac{E_q(iu) - E_q(-iu)}{2i} \quad \text{and} \quad \Cos_q(u) := \frac{E_q(iu) + E_q(-iu)}{2}.
$$

We can also define: $\tan_q(u) := \sin_q(u) / \cos_q(u)$ and $\Tan_q(u) := \Sin_q(u) / \Cos_q(u)$; but, as $\sin_q(u) \Cos_q(u) - \Sin_q(u) \cos_q(u) = 0$, we have:

$$
\tan_q(u) = \frac{\sin_q(u)}{\cos_q(u)} = \frac{\Sin_q(u)}{\Cos_q(u)} = \Tan_q(u),
$$

so that there is only one $q$-tangent. However, there are two $q$-secants:

$$
\sec_q(u) := \frac{1}{\cos_q(u)} = \sum_{n \geq 0} \frac{(-1)^n u^{2n}}{(q; q)_{2n}};
$$

$$
\Sec_q(u) := \frac{1}{\Cos_q(u)} = \sum_{n \geq 0} (-1)^n q^{n(2n-1)} u^{2n} / (q; q)_{2n}.
$$
**Theorem 2.1.** The $q$-derivatives of the series $\tan_q(u)$, $\sec_q(u)$, $\Sec_q(u)$ can be evaluated as follows:

\begin{align*}
(2.3) \quad D_q \tan_q(u) &= 1 + \tan_q(u) \tan_q(qu); \\
(2.4) \quad D_q \sec_q(u) &= \sec_q(qu) \tan_q(u); \\
(2.5) \quad D_q \Sec_q(u) &= \Sec_q(u) \tan_q(qu).
\end{align*}

**Proof.** These three identities can be proved, either by working with $e_q(u)$ and $E_q(u)$, when expressed as infinite $q$-series, or by using the infinite products appearing in (2.1) and (2.2). We choose the former way, because of its closeness to the traditional trigonometric calculus.

First,

\[
e_q(\alpha u) - e_q(\alpha qu) = \sum_{n \geq 1} \frac{(\alpha u)^n(1 - q^n)}{(q; q)_n} = \alpha u \sum_{n \geq 1} \frac{(\alpha u)^n}{(q; q)_{n-1}} = \alpha u e_q(\alpha u);
\]

\[
E_q(\alpha u) - E_q(\alpha qu) = \sum_{n \geq 1} \frac{(\alpha u)^n q^{n(n-1)/2}(1 - q^n)}{(q; q)_n} = \alpha u \sum_{n \geq 1} \frac{(\alpha u q)^n}{(q; q)_{n-1}} = \alpha u E_q(\alpha qu).
\]

Hence,

\[
D_q e_q(\alpha u) = \alpha e_q(\alpha u); \quad D_q E_q(\alpha u) = \alpha E_q(\alpha qu).
\]

Next, by applying $D_q$ to the familiar identities: $e_q(iu) = \cos_q(u) + i \sin_q(u)$ and $E_q(iu) = \Cos_q(u) + i \Sin_q(u)$, we get

\[
D_q \cos_q(u) = -\sin_q(u); \quad D_q \sin_q(u) = \cos_q(u); \\
D_q \Cos_q(u) = -\Sin_q(u); \quad D_q \Sin_q(u) = \Cos_q(u).
\]

Finally, we take advantage of the next formula that yields an expression for the $q$-derivative of a ratio $f(u)/g(u)$ of two $q$-series

\[
D_q \frac{f(u)}{g(u)} = \frac{g(qu) D_q f(u) - f(qu) D_q g(u)}{g(u)g(qu)},
\]

and use it for the ratios $\sin_q(u)/\cos_q(u)$, $1/\cos_q(u)$, $1/\Cos_q(u)$ to get:

\[
D_q \tan_q(u) = \frac{\cos_q(qu) \cos_q(u) - \sin_q(qu)(-\sin_q(u))}{\cos_q(u) \cos_q(qu)} = 1 + \tan_q(u) \tan_q(qu);
\]

\[
D_q \sec_q(u) = \frac{\sin_q(u)}{\cos_q(u) \cos_q(qu)} = \sec_q(qu) \tan_q(u);
\]

\[
D_q \Sec_q(u) = \frac{\Sin_q(qu)}{\Cos_q(u) \Cos_q(qu)} = \Sec_q(u) \tan_q(qu).
\]
Identities (2.4) and (2.5) above show a certain duality between the \( q \)-derivatives of \( \sec u \) and \( \Sec u \), which is to be explored by using the base \( q^{-1} \) instead of \( q \). First, for each \( q \)-series \( f(u) \) let \( Qf(u) := f(qu) \) and \( Uf(u) := (1/u)f(u) \). We have the identities:

\[
UQ = qQU, \quad Q^{-1}U = qUQ^{-1}.
\]

Next, the \( q \)-difference operators \( D_q \) and \( D_{q^{-1}} \), the latter being defined by

\[
D_{q^{-1}}f(u) := (1/u)(f(u) - f(q^{-1}u)),
\]

also read

\[
D_q = U(I - Q); \quad D_{q^{-1}} = U(I - Q^{-1}).
\]

Hence,

\[
D_q = -D_{q^{-1}}Q.
\]

Next, we have the relations:

\[
(2.8) \quad e_{q^{-1}}(u) = QE_q(-u), \quad E_{q^{-1}}(u) = Qe_q(-u).
\]

\[
(2.9) \quad \sin_{q^{-1}}(u) = -Q\Sin_q(u); \quad \cos_{q^{-1}}(u) = Q\Cos_q(u);
\]

\[
(2.10) \quad \tan_{q^{-1}}(u) = -Q\tan_q(u); \quad \sec_{q^{-1}}(u) = Q\Sec_q(u).
\]

We end this Section with three technical lemmas that will be used in Section 5 (resp. Section 6) to show how identity (1.12) (resp. (1.16)) in Theorem 1.2 (resp. in Theorem 1.3) can be obtained from (1.11) (resp. from (1.15)).

Lemma 2.2. We have \( D_{q^{-1}}Q = qQD_{q^{-1}} \), so that for each \( n \geq 1 \)

\[
(2.11) \quad (D_{q^{-1}}Q)^n = q^{(n+1)n/2}Q^nD_{q^{-1}}^n.
\]

Proof. By (2.6) we have: \( D_{q^{-1}}Q = U(I - Q^{-1})Q = UQ - U = qQU - qQUQ = qQU(I - Q^{-1}) = qQD_{q^{-1}} \). Identity (2.11) is then a simple consequence, as there are \((n+1)n/2\) transpositions \( QD_{q^{-1}} \leftrightarrow D_{q^{-1}}Q \) to be made to go from \((QD_{q^{-1}})^n\) to \(Q^nD_{q^{-1}}^n\). \( \square \)

Lemma 2.3. For each composition \( c = (c_0, c_1, \ldots, c_m) \) of an integer \( n \geq 1 \), we have

\[
(2.12) \quad Q^n\tan_q((q^{-1})^{\rho c}u) = \tan_q(q^c u).
\]

Proof. A simple verification:

\[
\tan_q((q^{-1})^{\rho c}u) = \tan_q((q^{-1})^{c_0}u)\tan_q((q^{-1})^{c_m+c_{m-1}}u)\cdots\tan_q((q^{-1})^{c_m+\cdots+c_1}u).
\]
Hence, the left-hand side of (2.12) is equal to
\[
Q^n \tan_q((q^{-1})^p u)
\]
\[
= \tan_q(q^{n-c_m}u) \tan_q(q^{n-c_m-c_{m+1}}u) \times \cdots \times \tan_q(q^{n-c_m-\cdots-c_1}u)
\]
\[
= \tan_q(q^{c_0+\cdots+c_{m-2}}u) \tan_q(q^{c_0+\cdots+c_{m-2}}u) \times \cdots \times \tan_q(q^{c_0}u)
\]
\[
= \tan_q(q^c u).
\]

**Lemma 2.4.** We have:
\[
D_q^n \Sec_q(u) = (-1)^n q^{n(n-1)/2} Q^{n-1} D_{q^{-1}} \Sec_{q^{-1}}(u).
\]

**Proof.** Just write
\[
D_q^n \Sec_q(u) = (-D_{q^{-1}} Q)^n Q^{-1} \Sec_{q^{-1}}(u) \quad \text{[by (2.7) and (2.10)]}
\]
\[
= (-1)^n (D_{q^{-1}} Q)^n D_{q^{-1}} \Sec_{q^{-1}}(u)
\]
\[
= (-1)^n q^{n(n-1)/2} Q^{n-1} D_{q^{-1}} \Sec_{q^{-1}}(u). \quad \square \quad \text{[by (2.11)]}
\]

3. **Transformations on t-permutations**

Say that a t-permutation \( w = (w_0, w_1, \ldots, w_m) \) is of the first (resp. second) kind, if 1 appears (resp. does not appear) as a one-letter factor among the \( w_i \)'s. Each set \( T_{n,m} \) can be partitioned into two subsets \( T_{n,m}^* \) and \( T_{n,m}^\ast \), the former one consisting of all permutations from \( T_{n,m} \) of the first kind, the latter one of those of the second kind. Let \( [m] := \{1, 2, \ldots, m\} \) and for each integer \( y \) let \( y^+ := y + 1 \) and \( v^+ := y_1^+ y_2^+ \cdots y_m^+ \) for each word \( v = y_1 y_2 \cdots y_m \), whose letters are integers.

For each pair \((m,n)\) such that \( 0 \leq m \leq n+1 \) and \( m \equiv n+1 \mod 2 \), we construct two bijections

\[
\Delta^* : [m] \times T_{n,m} \to T_{n+1,m+1}^*; \quad \Delta : [m] \times T_{n,m} \to T_{n+1,m+1}^\ast;
\]

Let \( w = (w_0, w_1, w_2, \ldots, w_m) \) belong to \( T_{n,m} \) and \( 1 \leq i \leq m \). The sequence

\[
(w_0^+ \cdots, w_i^+ \cdots, w_{i-2}^+, w_{i-1}^+, 1, w_i^+, w_{i+1}^+, \ldots, w_m^+)
\]

is obviously from \( T_{n+1,m+1}^* \); denote it by \( \Delta^*(i,w) \). Next, the sequence

\[
(w_0^+ \cdots, w_i^+ \cdots, w_{i-2}^+, (w_{i-1}^+ 1 w_i^+), w_{i+1}^+, \ldots, w_m^+)
\]

is then from \( T_{n+1,m+1}^\ast \), and will be denoted by \( \Delta(i,w) \); its \( i \)-th factor appears as \( w_{i-1}^+ 1 w_i^+ \), as indicated in the inner parentheses in (3.3). By the above two bijections, recurrence (1.3) holds with the interpretation \( a(n,m) = \#T_{n,m} \).

The next step is to study the actions of the transformations \( \ast \Delta \) and \( \Delta^* \) on the statistics “ides,” “imaj,” “min,” “inv” introduced in Subsection 1.4. Taking again the example used in that Subsection, namely the t-permutation \( w = (45, 1113, 1079, 6, 82) \), whose underlying statistics are ides \( w = 6 \), imaj \( w = 38 \), inv \( w = 27 \) and min \( w = 1 \), we get:
Theorem 4.1. Let \( w \) be a \( t \)-permutation and let \( \text{Iligne } w = \{ j_1 < j_2 < \cdots < j_r \} \). Furthermore, let \( \Delta^*(i, w) \) and \( \Delta^*(i, w) \) be defined as in (3.1) - (3.3). Then,

\[
\text{Iligne } \Delta^*(i, w) = \begin{cases} 
\text{Iligne } \Delta(i, w) & \text{if } i \leq \min w; \\
\{ 1, (j_1 + 1), (j_2 + 1), \ldots, (j_r + 1) \} & \text{if } \min w < i.
\end{cases}
\]

Hence,

\[
\text{ides } \Delta^*(i, w) = \text{ides } \Delta(i, w) = \begin{cases} 
\text{ides } w & \text{if } i \leq \min w; \\
1 + \text{ides } w & \text{if } \min w < i.
\end{cases}
\]

\[
\text{imaj } \Delta^*(i, w) = \text{imaj } \Delta(i, w) = \begin{cases} 
\text{imaj } w & \text{if } i \leq \min w; \\
1 + \text{imaj } w & \text{if } \min w < i;
\end{cases}
\]

Furthermore,

\[
\text{inv } \Delta^*(i, w) = \text{inv } \Delta(i, w) = \text{inv } w + \lambda(w_0 w_1 \cdots w_{i-1}), \\
\min(\Delta^*(i, w)) = i, \quad \min(\Delta(i, w)) = i - 1, \quad \text{for } i \geq 1.
\]

4. Proof of Theorem 1.1

Two steps are needed to achieve the proof: first, the derivation of a recurrence relation for the polynomials \( A_{n,k,a,b}(q) \), presented in the next theorem, then, an explicit algorithm for calculating them, described in Lemma 4.2.

Theorem 4.1 (Recurrence for the polynomials \( A_{n,k,a,b}(q) \)). With \( m := a + b \) and \( m' := a' + b' \) we have:

\[
A_{n+1,k',a',b'}(q) = q^k \left( \sum_{0 \leq a' \leq a' - 1} A_{n,k'-1,a,m'-1-a}(q) + \sum_{1 \leq a' \leq m'-1} A_{n,k',a,m'-1-a}(q) \right.
\]

\[
+ \sum_{0 \leq a \leq a'} A_{n,k'-1,a,m'+1-a}(q) + \sum_{a'+1 \leq a \leq m'+1} A_{n,k',a,m'+1-a}(q) \bigg),
\]

valid for \( n \geq 0 \) with the initial condition: \( A_{0,k,a,b}(q) = \delta_{k,0}\delta_{a,1}\delta_{b,0} \).
Lemma 4.2. Let \( w' = (w'_0, w'_1, \ldots, w'_{m'}) \in \mathcal{T}_{n+1, k'} \). When \( w' \) is of the first kind, the inverse \( \Delta^{-1}(w') \) is obtained by deleting the unique \( w'_{a'} \) equal to 1 and subtracting 1 from all the other letters of the components \( w'_j \) \((j = 0, \ldots, a' - 1, a' + 1, \ldots, m')\). When \( w' \) is of the second kind, the inverse \( \Delta^{-1}(w') \) is obtained by deleting 1 from the component \( w'_{a'} = u'_a v'_a \) and subtracting 1 from all the letters of the components \( w'_0, \ldots, w'_{a'-1}, w'_{a'}, w'_{a'+1}, \ldots, w'_{m'} \). The \( t \)-permutations \( w' \) from \( \mathcal{T}_{n+1, k', a', b'} \) fall into four categories:

1. \( w' \) is of the first kind and 2 is to the left of 1 in the product \( w'_0 w'_1 \cdots w'_{m'} \); then \( 1 \leq a' \leq m' - 1 \), because, when \( n + 1 \geq 2 \), the components \( w'_0 \) and \( w'_m \), are, either empty, or their lengths are at least equal to 2. Hence, \( \Delta^{-1}(w') = (a, w) \) with \( w \in \mathcal{T}_{n, k'-1, a, m'-1} - a \) and \( 0 \leq a \leq a' - 1 \leq m' - 2 \), as \( \text{ides } w = \text{ides } w' - 1 \). Also, \( \text{imaj } w' = 1 + \text{ides } w + \text{imaj } w = k' + \text{imaj } w \). [By Theorem 3.1]

2. \( w' \) is of the first kind and 2 is to the right of 1 in \( w'_0 w'_1 \cdots w'_{m'} \); for an analogous reason as in case (1) we have: \( \Delta^{-1}(w') = (a, w) \) with \( w \in \mathcal{T}_{n, k', a, m'-1} - a \) and \( a' \leq a \leq m' - 1 \). In this case \( \text{imaj } w' = \text{ides } w + \text{imaj } w = k' + \text{imaj } w \).

3. \( w' \) is of the second kind and 2 is to the left of 1 in \( w'_0 w'_1 \cdots w'_{m'} \), so that 2 can still belong to \( w'_{a'} \), or to any one of the components \( w'_0, \ldots, w'_{a'-1} \). Hence, \( \Delta^{-1}(w') = (a, w) \) with \( w \in \mathcal{T}_{n, k'-1, a, m'+1} - a \) and \( 0 \leq a \leq a' \), for the number of components has increased by 1. Again, \( \text{imaj } w' = k' + \text{imaj } w \).

4. \( w' \) is of the second kind and 2 is to the right of 1 in \( w'_0 w'_1 \cdots w'_{m'} \), so that 2 can still belong to \( w'_{a'} \), or to any one of the components \( w'_{a'+1}, \ldots, w'_{m'} \). Hence, \( \Delta^{-1}(w') = (a, w) \) with \( w \in \mathcal{T}_{n, k', a, m'+1} - a \) and \( a' + 1 \leq a \leq m' + 1 \). Also, \( \text{imaj } w' = k' + \text{imaj } w \).

Thus, identity (4.1) holds.

A consequence of the combinatorial interpretation is the symmetry property

\[
q^{n(n-1)/2} A_{n, n-1-k, b, a}(q^{-1}) = A_{n, k, a, b}(q),
\]

whose proof is easy and will be omitted.

Lemma 4.2. Let \([k, a, b] := \tan_q(q^{k+1}u) b \tan_q(q^{k}u)a \). Then,

\[
D_q[k, a, b] = q^k \sum_{0 \leq i < a - 1} [k, i, a + b - 1 - i] + q^{k+1} \sum_{0 \leq i < b - 1} [k + 1, a + i, b - 1 - i] + q^k \sum_{1 \leq i < a} [k, i, a + b + 1 - i] + q^{k+1} \sum_{1 \leq i < b} [k + 1, a + i, b + 1 - i].
\]

Proof. For taking the \( q \)-derivative of a product \( f_1(u) f_2(u) \cdots f_n(u) \) of \( q \)-series we use the formula:

\[
(4.2) \quad D_q \prod_{1 \leq i \leq n} f_i(u) = \sum_{1 \leq i \leq n} f_1(u) \cdots f_{i-1}(u) (D_q f_i(u)) f_{i+1}(qu) \cdots f_n(qu).
\]
TANGENT AND SECANT $q$-DERIVATIVE POLYNOMIALS

In particular,

$$D_q (\tan_q (q^{k+1} u)) \tan_q (q^k u)^a$$

$$= \sum_{0 \leq i \leq a-1} (\tan_q (q^{k+1} u))^b (\tan_q (q^k u))^i (D_q \tan_q (q^k u)) (\tan_q (q^{k+1} u))^{a-1-i}$$

$$+ \sum_{0 \leq j \leq b-1} (\tan_q (q^{k+1} u))^j (D_q \tan_q (q^{k+1} u)) (\tan_q (q^{k+2} u))^{b-1-j} (\tan_q (q^{k+1} u))^a$$

$$= q^k \sum_{0 \leq i \leq a-1} (\tan_q (q^{k+1} u))^{a+b-1-i} (\tan_q (q^k u))^i$$

$$+ q^k \sum_{0 \leq i \leq a-1} (\tan_q (q^{k+1} u))^{a+b-i} (\tan_q (q^k u))^{i+1}$$

$$+ q^{k+1} \sum_{0 \leq j \leq b-1} (\tan_q (q^{k+2} u))^{b-1-j} (\tan_q (q^{k+1} u))^{a+j}$$

$$+ q^{k+1} \sum_{0 \leq j \leq b-1} (\tan_q (q^{k+2} u))^{b-j} (\tan_q (q^{k+1} u))^{a+j+1} \quad \Box$$

With the notation $[k, a, b]$ we have $[0, 1, 0] = \tan_q (u)$, and identity (1.9) can be rewritten

$$D_q^n \tan_q (u) = \sum_{k, a, b} A_{n, k, a, b} (q) [k, a, b],$$

so that Lemma 4.2 can be used to calculate the polynomials $A_{n, k, a, b} (q)$ by iteration:

$$D_q [0, 1, 0] \quad (= D_q \tan_q (u)) = [0, 0, 0] + [0, 1, 1],$$

so that $A_{1, 0, 0, 0} (q) = A_{1, 0, 1, 1} (q) = 1$;

$$D_q^n [0, 1, 0] \quad (= D_q^n \tan_q (u)) = D_q [0, 1, 1] = [0, 0, 1] + [q, 1, 1, 0] + [0, 1, 2] + q[1, 2, 1],$$

so that $A_{2, 0, 0, 1} (q) = 1, A_{2, 1, 1, 0} (q) = q, A_{2, 0, 1, 2} (q) = 1, A_{2, 1, 2, 1} (q) = q$;

$$D_q^n [0, 1, 0] \quad (= D_q^n \tan_q (u)) = (q + q^2)[1, 0, 0]$$

$$+ [0, 0, 2] + q^2[1, 0, 2] + (2q + 2q^2)[1, 1, 1] + q[1, 2, 0] + q^3[2, 2, 0]$$

$$+ [0, 1, 3] + q^2[1, 1, 3] + (q + q^3)[1, 2, 2] + q[1, 3, 1] + q^3[2, 3, 1],$$

so that $A_{3, 1, 0, 0} (q) = q + q^2, A_{3, 0, 0, 2} (q) = 1, \text{ etc.}$

The polynomials $A_{n, k, a, b} (q)$ in Table 2 (Section 11) have been obtained using the previous calculation.

Proof of Theorem 1.1. By induction. Assume that (1.9) is true for $n$.

$$D_q^{n+1} [0, 1, 0] = D_q D_q^n [0, 1, 0]$$

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For the second sum we have
\[ A \] with the convention that
\[ k \]
\[ D_q \sum_{k,a,b} A_{n,k,a,b}(q)[k, a, b] = \sum_{k,a,b} A_{n,k,a,b}(q)D_q[k, a, b] \]
\[ + q^{k+1} \sum_{0 \leq i \leq b-1} [k+1, a+i, b-1-i] \]
\[ + q^k \sum_{1 \leq i \leq a} [k, i, a+b+1-i] + q^{k+1} \sum_{1 \leq i \leq b} [k+1, a+i, b+1-i] \).

We calculate the contribution of each sum to the triple \([k', a', b']\). For the first sum we have \([k, i, a+b-1-i] = [k', a', b'] (0 \leq i \leq a-1)\) if and only if \(k = k', i = a', a+b-1 = a'+b', a'+1 \leq a \leq a'+b'+1\), so that the contribution is
\[ q^{k'} \sum_{a'+1 \leq a \leq a'+b'+1} A_{n,k',a,a'+b'+1-a}(q). \]

For the second sum we have \([k+1, a+i, b-1-i] = [k', a', b'] (0 \leq i \leq b-1)\) if and only if \(k = k'-1, a+i = a', b-1-i = b', 0 \leq a'-a \leq b-1\), that is, \(k = k'-1, i = a'-a, a+b = a'+b'+1, 0 \leq a \leq a'\), so that the contribution is
\[ q^{k'} \sum_{0 \leq a \leq a'} A_{n,k'-1,a,a'+b'+1-a}(q). \]

with the convention that \(A_{n-1, a,a'+b+1-a}(q) = 0\). For the third sum we have \([k, i, a+b+1-i] = [k', a', b'] (1 \leq i \leq a)\) if and only if \(k = k', i = a', a+b+1 = a'+b', 1 \leq a' \leq a \leq a+b = a'+b'-1\), so that the contribution is
\[ q^{k'} \sum_{1 \leq a' \leq a \leq a'+b'-1} A_{n,k',a,a'+b'-1-a}(q). \]

For the fourth sum we have \([k+1, a+i, b+1-i] = [k', a', b'] (1 \leq i \leq b)\) if and only if \(k = k'-1, a+i = a', b+1-i = b', 1 \leq i \leq b\), that is, \(k = k'-1, i = a'-a, a+b+1 = a'+b', 0 \leq a \leq a'-1 = a+i-1 \leq a+b-1 \leq a'+b'-2\).

The contribution is then
\[ q^{k'} \sum_{0 \leq a \leq a'-1 \leq a'+b'-2} A_{n,k'-1,a,a'+b'-1-a}(q). \]

By Theorem 4.1 we have
\[ D_q^{k'+1}[0, 1, 0] = \sum_{k',a',b'} A_{n+1,k',a',b'}[k', a', b']. \]

5. Proof of Theorem 1.2

As for Theorem 1.1, two steps are used to complete the proof of Theorem 1.2: first, the derivation of a recurrence relation, then, the construction of an explicit algorithm.
Theorem 5.1 (Recurrence relation for the polynomials $B_{n,k,a,b}(q)$). With $m := a + b$ and $m' := a' + b'$ we have:

\[(5.1)\] \[B_{n+1,k',a',b'}(q) = q^{k'} \left( \sum_{0 \leq a \leq a'-1} B_{n,k'-1,a,m'-1-a}(q) + \sum_{1 \leq a' \leq m'-1} B_{n,k',a,m'-1-a}(q) \right) + \sum_{0 \leq a \leq a'} B_{n,k'-1,a,m'+1-a}(q) + \sum_{a'+1 \leq a \leq m'+1} B_{n,k',a,m'+1-a}(q),\]

valid for $n \geq 0$ with the initial condition: $B_{0,k,a,b}(q) = \delta_{k,-1} \delta_{a,0} \delta_{b,0}$.

Proof. Let $w' = (w'_0, w'_1, \ldots, w'_{m'}, \epsilon) \in T'_{n+1,k',a',b'+1}$. What has been said for recurrence (4.1) can be reproduced, except for the first sum, when $w'$ is of the first kind and 2 is to the left of 1 in the product $w'_0 w'_1 \cdots w'_{m'}$; now, $1 \leq a' \leq m'$ and not $m' - 1$. Hence, $\Delta^{-1}(w') = (a, w)$ with $w \in T'_{n,k'-1,a,m'-a}$ and $0 \leq a \leq a' - 1 \leq m' - 1$. Thus, identity (5.1) holds.

Lemma 5.2. Let $\langle k, a, b \rangle := (\tan_q(q^{k+1}u))^b \sec_q(q^{k+1}u)(\tan_q(q^{k}u))^a$. Then

\[(5.2)\] \[D_q \langle k, a, b \rangle := q^k \sum_{0 \leq i \leq a-1} \langle k, i, a - 1 - i \rangle + q^k \sum_{1 \leq i \leq a} \langle k, i, a + b - 1 - i \rangle + q^{k+1} \sum_{0 \leq i \leq b-1} \langle k + 1, a + i, b - 1 - i \rangle + q^{k+1} \sum_{1 \leq i \leq b+1} \langle k + 1, a + i, b + 1 - i \rangle.\]

Proof. By using (4.2) we derive

\[
D_q(\tan_q(q^{k+1}u))^b \sec_q(q^{k+1}u)(\tan_q(q^{k}u))^a
= \sum_{0 \leq i \leq a-1} (\tan_q(q^{k+1}u))^{a+b-1-i} \sec_q(q^{k+1}u)(\tan_q(q^{k}u))^i (D_q \tan_q(q^{k}u))
+ \sum_{0 \leq j \leq b-1} (\tan_q(q^{k+1}u))^{a+j} (D_q \tan_q(q^{k+1}u))(\tan_q(q^{k+2}u))^{b-1-j} \sec_q(q^{k+2}u)
+ (\tan_q(q^{k+1}u))^b D_q(\sec_q(q^{k+1}u)) (\tan_q(q^{k}u))^a
= q^k \sum_{0 \leq i \leq a-1} (\tan_q(q^{k+1}u))^{a+b-1-i} \sec_q(q^{k+1}u)(\tan_q(q^{k}u))^i
+ q^k \sum_{0 \leq i \leq a-1} (\tan_q(q^{k+1}u))^{a+b-i} \sec_q(q^{k+1}u)(\tan_q(q^{k}u))^{i+1}
+ q^{k+1} \sum_{0 \leq j \leq b-1} (\tan_q(q^{k+2}u))^{b-1-j} \sec_q(q^{k+2}u)(\tan_q(q^{k+1}u))^{a+j}
+ q^{k+1} \sum_{0 \leq j \leq b-1} (\tan_q(q^{k+2}u))^{b-j} \sec_q(q^{k+2}u)(\tan_q(q^{k+1}u))^{a+j+1}
+ q^{k+1}(\tan_q(q^{k+1}u))^{a+b+1} \sec_q(q^{k+2}u). \]

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Lemma 5.2 provides a way to calculate the polynomials \(B_{n,k,a,b}(q)\). As \((-1, 0, 0) = \sec_q(u)\), we get
\[
D_q(-1, 0, 0) = (0, 1, 0), \text{ so that } B_{1,0,1,0}(q) = 1;
\]
\[
D_q^2(-1, 0, 0) = D_q(0, 1, 0) = (0, 0, 0) + (0, 1, 1) + q(1, 2, 0),
\]
so that \(B_{2,0,0,0}(q) = B_{2,0,1,1}(q) = 1, B_{2,1,2,0}(q) = q;\)
\[
D_q^3(-1, 0, 0) = D_q((0, 0, 0) + (0, 1, 1) + q(1, 2, 0))
\]
\[
= q(1, 1, 0) + (0, 1, 2) + q(1, 1, 0) + q(1, 2, 1) + q(1, 3, 0)
\]
\[
+ (q^2(1, 1, 0) + q^2(1, 1, 0) + q^2(1, 2, 1) + q^2(1, 2, 1) + q^2(2, 3, 0))
\]
\[
= 0, 0, 1 + q^2(1, 0, 1) + (2q + q^2)(1, 1, 0)
\]
\[
+ (0, 1, 2) + q^2(1, 1, 2) + (q + q^2)(1, 2, 1) + q(1, 3, 0) + q^3(2, 3, 0);
\]
so that \(B_{3,0,0,1}(q) = 1, B_{3,1,0,1}(q) = q^2, B_{3,1,1,0}(q) = 2q + q^2, B_{3,0,1,2}(q) = 1, B_{3,1,1,2}(q) = q^2, B_{3,1,2,1}(q) = q + q^2, B_{3,1,3,0}(q) = q, B_{3,2,3,0}(q) = q^3.\)

The polynomials \(B_{n,k,a,b}(q)\) in Table 2 have been calculated by means of the previous algorithm.

**Proof of (1.11).** By induction. Assume that (1.11) is true for \(n\). Then,
\[
D_q^{n+1}(-1, 0, 0) = D_q D_q^n(-1, 0, 0)
\]
\[
= D_q \sum_{k,a,b} B_{n,k,a,b}(q) (k, a, b) = \sum_{k,a,b} B_{n,k,a,b}(q) D_q (k, a, b)
\]
\[
= \sum_{k,a,b} B_{n,k,a,b}(q) \left( q^k \sum_{0 \leq i \leq a-1} \langle k, i, a+b-1-i \rangle 
\]
\[
+ q^{k+1} \sum_{0 \leq i \leq b-1} \langle k+1, a+i, b-1-i \rangle
\]
\[
+ q^k \sum_{1 \leq i \leq a} \langle k, i, a+b+1-i \rangle + q^{k+1} \sum_{1 \leq i \leq b+1} \langle k+1, a+i, b+1-i \rangle \right).
\]

We calculate the contribution of each sum to the triple \(\langle k', a', b' \rangle\). For the first three sums we can simply reproduce the arguments developed in the proof of Theorem 1.1. Only the fourth sum is to be checked. This time, the double inequality \(1 \leq i \leq b+1\) prevails, instead of \(1 \leq i \leq b\). This leads to the sequence \(0 \leq a \leq a' - 1 = a + i - 1 \leq a + (b + 1) - 1 \leq a + b \leq a' + b' - 1\). Hence, the contribution is
\[
q^{k'} \sum_{0 \leq a \leq a' - 1 \leq a'+b'-1} B_{n,k'-1,a,a'+b'-1-a}(q).
\]

By Theorem 5.1 we have
Hence, as \(0 \leq k \leq n\),

\[
Q^{n-1}D_q^{n-1}\sec_q(u) = \sum_{k,a,b} B_{n-1,k,a,b}(q^-1)(\tan_q(q^{-k-1}u))^b \sec_q(q^{-k-1}u)(\tan_q(q^{-k}u))^a
\]

Hence, as \(0 \leq k \leq n-1\),

\[
Q^{n-1}D_q^{n-1}\sec_q(u)
= \sum_{k,a,b} B_{n-1,k,a,b}(q^-1)(-1)^{a+b}(\tan_q(q^{n-1-k}u))^b \sec_q(q^{n-1-k}u)(\tan_q(q^{n-k}u))^a
= \sum_{k,a,b} B_{n-1,n-1-k,a,b}(q^-1)(-1)^{a+b}(\tan_q(q^k u))^b \sec_q(q^k u)(\tan_q(q^{k+1}u))^a.
\]

Finally, as \(n\) and \(a + b\) are of the same parity, we have by Lemma 2.4

\[
D_q^n \sec_q(u) = (-1)^n q^{n(n-1)/2} Q^{n-1}D_q^{n-1}\sec_q(q^{-1} u),
= \sum_{k,a,b} q^{n(n-1)/2} B_{n-1,n-1-k,a,b}(q^-1)(\tan_q(q^{k}u))^b \sec_q(q^k u)(\tan_q(q^{k+1}u))^a.
\]

This establishes identity (1.12). □

6. Proof of Theorem 1.3

Let \(c' = (c'_0, c'_1, \ldots, c'_{m'}) \in \Theta_{n+1}\) and \(0 \leq i \leq m'\). If \(c'_0 \geq 2\), and if \(1 \leq 2j + 1 \leq c'_0\), let

\[
c'[0, 2j + 1] := (2j, c'_0 - (2j + 1), c'_1, \ldots, c'_{m'}).
\]

If \(1 \leq i \leq m'\) and \(2 \leq 2j \leq c'_i\), let

\[
c'[i, 2j] := (c'_0, \ldots, c'_{i-1}, 2j - 1, c'_i - 2j, c'_{i+1}, \ldots, c'_{m'}).
\]

Finally, if \(1 \leq i \leq m' - 1\) and \(c'_i = 1\), let

\[
c'[i, 1] := (c'_0, \ldots, c'_{i-1}, c'_{i+1}, \ldots, c'_{m'}).
\]
Theorem 6.1 (Recurrence relation for the polynomial $A_{n,c}(q)$). With $c' = (c'_0, c'_1, \ldots, c'_{m'})$ we have:

\begin{align*}
A_{n+1,c'}(q) &= \sum_{0 \leq 2j+1 \leq c'_0} q^{2j} A_{n,c'[0,2j+1]}(q) \\
&+ \sum_{1 \leq i \leq m'} q^{c'_0 + \cdots + c'_{i-1}} \sum_{2 \leq 2j \leq c'_i} q^{2j-1} A_{n,c'[i,2j]}(q) \\
&+ \sum_{1 \leq i \leq m'-1} \chi(c'_i = 1) q^{c'_0 + \cdots + c'_{i-1}} A_{n,c'[i,1]}(q),
\end{align*}

where $\chi(c'_i = 1)$ is equal to 1 if $c'_i = 1$ holds and 0 otherwise.

Proof. Let $w' = (w'_0, w'_1, \ldots, w'_{m'})$ be a $t$-permutation from $T_{n+1}$ such that $Aw' = c' = (c'_0, c'_1, \ldots, c'_{m'})$. Two cases are to consider: (i) 1 belongs to the component $w_i$ ($0 \leq i \leq m'$) having at least two letters; write $w \in T_{n+1,c';2}$; (ii) the component $w_i$ is equal to the one-letter 1: write $w \in T_{n+1,c';1}$. For each word $v$, whose letters are integers, let $v^-$ designate the word obtained from $v$ by subtracting 1 from each of its letters. By convention, $\epsilon^- := \epsilon$.

Case (i): we may write: $w'_1 = w'_i1v'_i$, where at least one of the factors $u'_i$, $v'_i$ is nonempty. To $w'$ there corresponds a unique triple $(w, i, \lambda(u'_i1))$ where $w := (w'_0, \ldots, w'_i, v'_i, w'_{i+1}, w'_{m'})$. As $w'_0$ is rising alternating (when nonempty) and the other components falling alternating, the length $\lambda(u'_i1)$ of the word $u'_i1$ is odd, say, $2j+1$, when $i = 0$ and even, say, $2j$, when $i \geq 1$.

Thus, the mapping $w \mapsto (w, i, \lambda(u'_i1))$ is a bijection of $T_{n+1,c';2}$ to the set of triples $(w, i, j)$, where $0 \leq i \leq m'$; $w \in T_n$, $Aw$ is equal to $c'[0,2j+1]$ when $i = 0$ and to $c'[i,2j]$ when $1 \leq i \leq m'$; $0 \leq 2j+1 \leq c'_0$ when $i = 0$ and $2 \leq 2j \leq c'_i$ when $1 \leq i \leq m'$. Moreover,

\[
\text{inv } w' = \begin{cases} 
2j + \text{inv } w, & \text{when } i = 0; \\
c'_0 + \cdots + c'_{i-1} + 2j - 1 + \text{inv } w, & \text{when } 1 \leq i \leq m'.
\end{cases}
\]

Case (ii): the mapping $w' \mapsto (w, i)$ is a bijection of $T_{n+1,c';1}$ to the set of the pairs $(w, i)$, where $w := (w'_0, \ldots, w'_{i-1}, w'_{i+1}, w'_{m'})$ and $1 \leq i \leq m'-1$. Thus, $Aw = c'[i,1]$. Moreover, $\text{inv } w' = c'_0 + \cdots + c'_{i-1} + \text{inv } w$.

Accordingly, (6.4) holds. \[
\square
\]
Lemma 6.2. Let \( c = (c_0, c_1, \ldots, c_m) \in \Theta_n \). Then,

\[
D_q \tan_q(q^c u) = \sum_{1 \leq i \leq m} q^{c_0 + \cdots + c_{i-1}} (\tan_q(q^{(i)} c u) + \tan_q(q^{(i)} u)).
\]

Proof.

\[
D_q \tan_q(q^c u) = D_q \prod_{1 \leq i \leq m} \tan_q(q^{c_0 + \cdots + c_{i-1}} u)
\]

\[
= \sum_{1 \leq i \leq m} \tan_q(q^{c_0} u) \cdots \tan_q(q^{c_0 + \cdots + c_{i-2}} u)
\]

\[
\times q^{c_0 + \cdots + c_{i-1}} (1 + \tan_q(q^{c_0 + \cdots + c_{i-1}} u) \tan_q(q^{c_0 + \cdots + c_{i-1}+1} u))
\]

\[
= \sum_{1 \leq i \leq m} q^{c_0 + \cdots + c_{i-1}} (\tan_q(q^{(i)} c u) + \tan_q(q^{(i)} u)). \quad \square
\]

In the next example, let \( c := \tan_q(q^c u) \) by convention, so that

\[
D_q c = \sum_{1 \leq i \leq m} q^{c_0 + \cdots + c_{i-1}} ((i) c + c^{(i)}).
\]

We get

\[
D_q((0,0)) := (1) + (0,1,0)
D_q^2((0,0)) := D_q((1) + (0,1,0))
\]

\[
= 0 + ((2,0) + q(0,2) + (0,1,1,0) + q(0,1,1,0))
\]

\[
= (2,0) + q(0,2) + (1 + q)(0,1,1,0);
\]

\[
D_q^3((0,0)) := D_q((2,0) + q(0,2) + (1 + q)(0,1,1,0))
\]

\[
= (q^2(3) + q^2(2,1,0)) + q((3) + (0,1,2))
\]

\[
+ (1 + q)((2,1,0) + q(0,3,0) + q^2(0,1,2)
\]

\[
+ (0,1,1,1,0) + q(0,1,1,1,1,0) + q^2(0,1,1,1,0))
\]

\[
= (q + q^2)(3) + (1 + q + q^2)(2,1,0) + (q + q^2 + q^3)(0,1,2)
\]

\[
+ (q + q^2)(0,3,0) + (1 + q)(1 + q + q^2)(0,1,1,1,0).
\]

We can then calculate the polynomials (see also Table 3):

\[
A_{1,(1)}(q) = A_{1,(0,1,0)}(q) = 1;
\]

\[
A_{2,(0,2)}(q) = 1, A_{2,(0,2)}(q) = q, A_{2,(0,1,1,0)}(q) = 1, A_{2,(0,1,1,0)}(q) = q;
\]

\[
A_{3,(0,3)}(q) = q + q^2, A_{3,(2,1,0)}(q) = 1 + q + q^2, A_{3,(0,1,2)}(q) = q + q^2 + q^3,
\]

\[
A_{3,(0,1,1,0)}(q) = q + q^2, A_{3,(0,1,1,0)}(q) = (1 + q)(1 + q + q^2).
\]
Proof of (1.14). Let $c' = (c'_0, c'_1, \ldots, c'_m)$. By (6.1) and (6.7) we have \(c = c'\) when \(c'_0 \geq 2\), if and only if \(c = c'[0, 2j + 1]\) for some \(j\) such that \(1 \leq 2j + 1 \leq c'_0\). By (6.2) and (6.8) the relation \((c) = c'\) holds for \(i\) such that \(1 \leq i \leq m'\) and \(c'_i \geq 2\), if and only if \(c = c'[i, 2j]\) for some \(j\) such that \(2 \leq 2j \leq c'_i\). Finally, \((c)^{(i)} = c'\) holds for \(i\) such that \(1 \leq i \leq m' - 1\) and \(c'_i = 1\), if and only if \(c = c'[1, i]\).

Finally, (1.14) is proved by induction. Assume it is true for \(n\). Then,

\[
D_q^{n+1} \tan_q(u) = D_q D_q^n \tan_q(u) = D_q \left( \sum_{c \in \Theta_n} A_{n,c}(q) \tan_q(q^c u) \right)
\]

\[
= \sum_{c \in \Theta_n} A_{n,c}(q) D_q \tan_q(q^c u)
\]

\[
= \sum_{c \in \Theta_n} \left( \sum_{1 \leq i \leq m' \leq c'_0} q^{c_0+\cdots+c_i-1} \left( \tan_q((c)_i u) + \tan_q(q^{(i)}_c u) \right) \right)
\]

\[
= \sum_{c' \in \Theta_{n+1}} \left( \sum_{0 \leq 2j+1 \leq c'_0} q^{2j} A_{n,c'}[0, 2j+1](q) \right)
\]

\[
+ \sum_{1 \leq i \leq m'} q^{c'_0+\cdots+c_i-1} \sum_{2 \leq 2j \leq c'_i} q^{2j-1} A_{n,c'}[i, 2j](q)
\]

\[
+ \sum_{1 \leq i \leq m'-1} \chi(c'_i = 1) q^{c'_0+\cdots+c_i-1} A_{n,c'}[i, 1](q) \tan_q(q^{c'} u)
\]

\[
= \sum_{c' \in \Theta_{n+1}} \left( A_{n+1,c'}(q) \right) \tan_q(q^{c'} u). \quad \text{[By Theorem 6.1]}
\]

For each \(c = (c_0, c_1, \ldots, c_m, 0) \in \Theta_n\), let \(\Phi(c) := \tan_q(q^{-c} u) \sec_q(q^n u)\).

Lemma 6.3. Let \(c = (c_0, c_1, \ldots, c_m, 0) \in \Theta_n\). Then,

\[
D_q \Phi(c) = \sum_{1 \leq i \leq m} \left( q^{c_0+\cdots+c_i-1} (\Phi^{(i)} c) + \Phi^{(i)}(c) \right) + q^n \Phi c^{(m+1)}.
\]

Proof.

\[
D_q \Phi(c) = D_q \tan_q(q^{-c} u) \sec_q(q^n u)
\]

\[
= \sum_{1 \leq i \leq m} \tan_q(q^{c_0 u} \cdots \tan_q(q^{c_0+\cdots+c_{i-1} u} u)
\]

\[
\times q^{c_0+\cdots+c_i-1} (1 + \tan_q(q^{c_0+\cdots+c_i-1 u}) \tan_q(q^{c_0+\cdots+c_i-1+1 u}))
\]

\[
\times \tan_q(q^{c_0+\cdots+c_{i-1}+1+c_i u}) \cdots \tan_q(q^{c_0+\cdots+c_i-1+1+c_i+\cdots+c_{m-1} u})
\]

\[
\times \sec_q(q^{n+1} u)
\]

\[
+ q^n \tan_q(q^{-c} u) \tan_q(q^n u) \sec_q(q^{n+1} u)
\]

\[
= \sum_{1 \leq i \leq m} \left( q^{c_0+\cdots+c_i-1} (\Phi^{(i)} c) + \Phi^{(i)}(c) \right) + q^n \Phi c^{(m+1)}. \quad \square
\]
Proof of (1.15). Again (1.15) is proved by induction. Assume that (1.15) holds for $n$. Then,

$$D_q^{n+1} \sec_q(u) = D_q D_q^n \sec_q(u) = D_q \left( \sum_{c \in \Theta_n^-} A_{n,c}(q) \Phi(c) \right)$$

$$= \sum_{c \in \Theta_n^-} A_{n,c}(q) D_q \Phi(c)$$

$$= \sum_{c \in \Theta_n^-} A_{n,c}(q) \left( \sum_{1 \leq i \leq m} (q^{c_0+\cdots+c_{i-1}}(\Phi(i)c) + \Phi(c(i))) + q^n \Phi(c^{(m+1)}) \right)$$

$$= \sum_{c' \in \Theta_{n+1}^-} \left( \sum_{0 \leq 2j+1 \leq c_0'} q^{2j} A_{n,c'[0,2j+1]}(q) \right)$$

$$+ \sum_{1 \leq i \leq m'} \sum_{1 \leq i \leq m' - 1} \sum_{2 \leq 2j \leq c_i'} \sum_{1 \leq i \leq m' - 1} \chi(c'_{i-1} = 1)q^{c_0' + \cdots + c_{i-1}} A_{n,c'[i,2j]}(q) \Phi(c')$$

$$= \sum_{c' \in \Theta_{n+1}^-} A_{n+1,c'}(q) \Phi(c'). \text{ [By Theorem 6.1]}$$

Proof of (1.16). In our proof of (1.16) given next we again make a full use of the duality derived in Section 2 between the $q$-series Sec$_q(u)$, tan$_q(u)$ and their analogs sec$_{q^{-1}}(u)$, tan$_{q^{-1}}(u)$. Since tan$_{q^{-1}}((q^{-1})^{c-} u) = (-1)^{\mu c} \tan_q((q^{-1})^{c-} qu)$, identity (1.15) can be rewritten

$$D_q^n \sec_{q^{-1}}(u) = \sum_{c \in \Theta_n^-} A_{n,c}(q^{-1}) \tan_{q^{-1}}((q^{-1})^{c-} u) \sec_{q^{-1}}((q^{-1})^n u)$$

$$= (-1)^{\mu c} \sum_{c \in \Theta_n^-} A_{n,c}(q^{-1}) \tan_q((q^{-1})^{c-} qu) \Sec_q((q^{-1})^n qu).$$

By Lemma 2.3,

$$Q^{n-1} D_q^n \sec_{q^{-1}}(u) = (-1)^{\mu c} \sum_{c \in \Theta_n^-} A_{n,c}(q^{-1}) \tan_q(q^{\mu c-} u) \Sec_q(u).$$

By Lemma 2.4, and since $n$ and $\mu c$ are of the same parity, we get

$$D_q^n \Sec_q(u) = (-1)^{n} q^{n(n-1)/2} Q^{n-1} D_q^n \sec_{q^{-1}}(u)$$

$$= \sum_{c \in \Theta_n^-} q^{n(n-1)/2} A_{n,c}(q^{-1}) \tan_q(q^{\mu c-} u) \Sec_q(u).$$

This proves identity (1.16).
7. More on $q$-trigonometric functions

Some parts of this section are of semi-expository nature, although, to our knowledge, the combinatorial properties of “$\text{Sec}_q(u)$” have not been explicitly written down. Let

\begin{align*}
\tan_q(u) &= \sum_{n \geq 0} A_{2n+1}(q) \frac{u^{2n+1}}{(q; q)_{2n+1}}; \\
\sec_q(u) &= \sum_{n \geq 0} A_{2n}(q) \frac{u^{2n}}{(q; q)_{2n}}; \\
\text{Sec}_q(u) &= \sum_{n \geq 0} A_{2n}^{\text{Sec}}(q) \frac{u^{2n}}{(q; q)_{2n}};
\end{align*}

be the $q$-expansions of the three series $\tan_q(u)$, $\sec_q(u)$, $\text{Sec}_q(u)$, respectively.

Let $[N]_q := (q; q)_N/((q; q)_M (q; q)_{N-M})$ ($0 \leq M \leq N$) be the Gaussian polynomial. Identities (2.3)-(2.5) yield

\begin{align*}
A_{2n+1}(q) &= \sum_{0 \leq k \leq n-1} \left[ \begin{array}{c} 2n \\ 2k + 1 \end{array} \right] q^{2k+1} A_{2k+1}(q) A_{2n-2k-1}(q), \\
A_{2n}(q) &= \sum_{0 \leq k \leq n-1} \left[ \begin{array}{c} 2n-1 \\ 2k \end{array} \right] q^{2k} A_{2k}(q) A_{2n-2k-1}(q), \\
A_{2n}^{\text{Sec}}(q) &= \sum_{0 \leq k \leq n-1} \left[ \begin{array}{c} 2n-1 \\ 2k \end{array} \right] q^{2k} A_{2k}^{\text{Sec}}(q) q^{2n-2k-1} A_{2n-2k-1}(q),
\end{align*}

for $n \geq 1$ with the initial conditions $A_1(q) = 1$, $A_0(q) = 1$ and $A_0^{\text{Sec}}(q) = 1$. Hence, the coefficients $A_n(q), A_{2n}^{\text{Sec}}(q)$ ($n \geq 0$) occurring in the $q$-expansions of $\tan_q(u), \sec_q(u), \text{Sec}_q(u)$ in (7.1)-(7.2) are polynomials with positive integral coefficients.

The first values of the polynomials $A_n(q)$ and $A_{2n}^{\text{Sec}}(q)$ ($n \geq 0$) can be calculated by means of (7.4)—(7.6):

\begin{align*}
A_1(q) &= 1; \quad A_2(q) = q + q^2; \quad A_3(q) = q^2 + 2q^3 + 3q^4 + 4q^5 + 3q^6 + 2q^7 + q^8; \\
A_0(q) &= A_2(q) = 1; \quad A_4(q) = q + 2q^2 + q^3 + q^4; \\
A_6(q) &= q^2 + 3q^3 + 5q^4 + 8q^5 + 10q^6 + 10q^7 + 9q^8 + 7q^9 + 5q^{10} + 2q^{11} + q^{12}; \\
A_0^{\text{Sec}}(q) &= A_2^{\text{Sec}}(q) = q; \quad A_4^{\text{Sec}}(q) = q^2 + q^3 + 2q^4 + q^5; \\
A_6^{\text{Sec}}(q) &= q^3 + 2q^4 + 5q^5 + 7q^6 + 9q^7 + 10q^8 + 10q^9 + 8q^{10} + 5q^{11} + 3q^{12} + q^{13};
\end{align*}

We note that the two identities (7.5) and (7.6) can be combined into the following single formula, valid for $n \geq 1$ with $A_0(q) = 1$:

$$A_n(q) = \sum_{0 \leq k \leq \lfloor n/2 \rfloor - 1} \left[ \begin{array}{c} n-1 \\ 2k \end{array} \right] q^{n-2k-2} A_{2k+1}(q) A_{n-2k-2}(q).$$
The polynomials $A_{2n+1}(q)$ (resp. $A_{2n}(q)$) defined by (7.1) and (7.2) are usually called the \textit{$q$-tangent numbers} and \textit{$q$-secant numbers}, respectively. No traditional name exists for the polynomials $A_{2n}^\text{Sec}(q)$, as they are intimately related to the $A_{2n+1}(q)$'s by identity (7.12).

For each $n \geq 0$ the set of all rising (resp. falling) alternating permutations $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $12\cdots n$ is denoted by $\mathcal{R}_n$ (resp. $\mathcal{F}_n$). As already mentioned in Section 7, the old result by Désiré Andrée [An81] asserts that both $\#\mathcal{R}_n$ and $\#\mathcal{F}_n$ (resp. $\#\mathcal{R}_n^\text{Sec}$ and $\#\mathcal{F}_n^\text{Sec}$) are equal to the tangent number $T_{2n+1}$ (resp. secant number $E_{2n}$) occurring in (1.1) and (1.2).

For the $q$-analog of this result we keep the same combinatorial set-up, namely, $\mathcal{R}_n$ and $\mathcal{F}_n$ ($n \geq 0$), but, as there are two different $q$-secants, $\text{sec}_q(u)$ and $\text{Sec}_q(u)$, the coefficients in their $q$-expansions will have different combinatorial interpretations. For each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $12\cdots n$ (not necessarily an alternating permutation), let $\text{inv } \sigma$ denote the traditional number of inversions of $\sigma$.

**Theorem 7.1.** Let $A_{2n+1}(q)$ (resp. $A_{2n}(q)$, resp. $A_{2n}^\text{Sec}(q)$) be the coefficients in the $q$-expansion of $\tan_q(u)$ (resp. of $\sec_q(u)$, resp. of $\text{Sec}_q(u)$), as defined in (7.1)–(7.2). Then,

\begin{align*}
A_{2n+1}(q) &= \sum_{\sigma \in \mathcal{R}_n} q^{\text{inv } \sigma} = \sum_{\sigma \in \mathcal{F}_n} q^{\text{inv } \sigma}; \quad (7.7) \\
A_{2n}(q) &= \sum_{\sigma \in \mathcal{R}_n} q^{\text{inv } \sigma}; \quad (7.8) \\
A_{2n}^\text{Sec}(q) &= \sum_{\sigma \in \mathcal{F}_n} q^{\text{inv } \sigma}. \quad (7.9)
\end{align*}

The proofs of (7.7) – (7.9) are not reproduced here. It suffices to $q$-mimick Desiré Andrée’s [An81] classical proof.

The statistics “Ligne” and “imaj” have been defined in Section 8. By means of the so-called “second fundamental transformation” (see, e.g., [Lo83, §10.6], [Fo68], [FS78]) we can construct a bijection $\Phi$ of the group of all permutations onto itself with the property that

\begin{align*}
\text{Ligne } \sigma &= \text{Ligne } \Phi(\sigma) \quad \text{and} \quad \text{inv } \sigma = \text{imaj } \Phi(\sigma). \quad (7.10)
\end{align*}

Saying that a permutation $\sigma$ is falling (resp. rising) alternating is equivalent to saying that $\text{Ligne } \sigma = \{1, 3, 5, \ldots\}$ (resp. $= \{2, 4, 6, \ldots\}$). Accordingly, we also have

\begin{align*}
A_{2n+1}(q) &= \sum_{\sigma \in \mathcal{R}_n} q^{\text{imaj } \sigma} = \sum_{\sigma \in \mathcal{F}_n} q^{\text{imaj } \sigma}; \quad (7.11)
\end{align*}

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For each permutation \( \sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \) let \( \rho \) (the mirror-image) and \( \gamma \) (the complement) be defined by
\[
\gamma \sigma(i) := n + 1 - \sigma(i); \quad \rho \sigma(i) := \sigma(n + 1 - i) \quad (1 \leq i \leq n).
\]
The transformation \( \rho \gamma \) is a bijection of \( \mathcal{RA}_{2n+1} \) onto \( \mathcal{FA}_{2n+1} \) preserving the number of inversions. This makes up a combinatorial proof of the second identity in (7.7).

**Proposition 10.1.** We have
\[
\begin{align*}
q^{(2n+1)(2n)/2} A_{2n+1}(q^{-1}) &= A_{2n+1}(q); \\
q^{(2n)(2n-1)/2} A_{2n}(q^{-1}) &= A_{2n}^{\text{Sec}}(q).
\end{align*}
\]

**Proof.** The transformation \( \rho \) is a bijection of \( \mathcal{RA}_{2n+1} \) onto \( \mathcal{RA}_{2n+1} \) with the property that: \( \text{inv} \sigma + \text{inv} \rho \sigma = (2n)(2n + 1)/2 \). By (7.7) we have
\[
A_{2n+1}(q) = \sum_{\sigma \in \mathcal{FA}_{2n+1}} q^{\text{inv} \sigma} = \sum_{\sigma \in \mathcal{FA}_{2n+1}} q^{\text{inv} \rho \sigma} = q^{2n(2n+1)/2} \sum_{\sigma \in \mathcal{FA}_{2n+1}} q^{-\text{inv} \sigma} = q^{2n(2n+1)/2} A_{2n+1}(q^{-1}).
\]
In the same manner, the transformation \( \rho \) is a bijection of \( \mathcal{RA}_{2n} \) onto \( \mathcal{FA}_{2n} \) with the property that: \( \text{inv} \sigma + \text{inv} \rho \sigma = (2n)(2n - 1)/2 \). Hence,
\[
A_{2n}^{\text{Sec}}(q) = \sum_{\sigma \in \mathcal{FA}_{2n}} q^{\text{inv} \sigma} = \sum_{\sigma \in \mathcal{RA}_{2n}} q^{\text{inv} \rho \sigma} = q^{2n(2n-1)/2} \sum_{\sigma \in \mathcal{RA}_{2n}} q^{-\text{inv} \sigma} = q^{2n(2n-1)/2} A_{2n}(q^{-1}).
\]

8. **Proof of Theorem 1.4**

By means of the second fundamental transformation \( \Phi \), already mentioned in (7.10), and the bijection \( i \) that maps each permutation \( \sigma \) from the symmetric group \( \mathcal{S}_n \) onto its inverse \( \sigma^{-1} \), we can form \( \psi := i \Phi i \). The latter bijection has the following properties:

\[
\text{Ligne } \psi(\sigma) = \text{Ligne } \sigma; \quad \text{inv } \psi(\sigma) = \text{imaj } \sigma.
\]

If \( c = (c_0, c_1, \ldots, c_m) \) is a composition of \( n \) and \( \sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \) a permutation from \( \mathcal{S}_n \), let
\[
(\text{Ligne } \backslash c)\sigma := \text{Ligne } \sigma \setminus \{c_0, c_0 + c_1, \ldots, c_0 + c_1 + \cdots + c_{m-1}\},
\]
so that for every composition \( c \) of \( n \)
\[
(\text{Ligne } \backslash c)\psi(\sigma) = (\text{Ligne } \backslash c)\sigma; \quad \text{inv } \psi(\sigma) = \text{imaj } \sigma.
\]
TANGENT AND SECANT $q$-DERIVATIVE POLYNOMIALS

Now, consider a $t$-permutation $w = (w_0, w_1, \ldots, w_m)$ of order $n$ such that $\Lambda w = c = (c_0, c_1, \ldots, c_m)$. The transformation $\psi$ maps the permutation $\sigma := w_0 w_1 \cdots w_m$ to another permutation $\psi(\sigma)$. Let $(w'_0, w'_1, \ldots, w'_m)$ be the factorization of $\psi(\sigma)$, written as a word of $n$ letters, defined by: $\lambda w'_0 = c_0$, $\lambda w'_1 = c_1$, $\lambda w'_m = c_m$. Property (8.1) implies that the mapping
\[
\psi : w = (w_0, w_1, \ldots, w_m) \mapsto w' := (w'_0, w'_1, \ldots, w'_m)
\]
is a bijection of $T_n$ onto itself having the properties:
\[
(8.2) \quad \Lambda w' = \Lambda w = c, \quad \operatorname{inv} w' = \operatorname{imaj} w.
\]

For instance, (see [FS78, p. 147] where the same numerical example is here reproduced) we have: $\Phi(749261583) = 472619583$. Hence, $w = 649275183 \xrightarrow{1} 749261583 \xrightarrow{\Phi} 472619583 \xrightarrow{1} 539174286 = w'$. The $t$-permutation $w = (\epsilon, 6, 4, 927, 5183)$ is then mapped under $\psi$ onto $w' = (\epsilon, 5, 3, 917, 4286)$. Moreover, $\operatorname{imaj} w = 1+3+5+8 = 17 = 1+3+1+3+5+1+3 = \operatorname{inv} w'$.

From Theorem 1.3 and (8.2) it then follows that
\[
A_{n,c}(q) = \sum_{w \in T_n, \Lambda w = c} q^{\operatorname{inv} w} = \sum_{w \in T_n, \Lambda w = c} q^{\operatorname{imaj} w}.
\]

Now, sum the previous identity over all $t$-compositions of $n$. We get:
\[
\sum_{\mu \in m} A_{n,\mu}(q) = \sum_{\mu \in m} \sum_{w \in T_n, \Lambda w = c} q^{\operatorname{inv} w} = \sum_{\mu \in m} \sum_{w \in T_n, \Lambda w = c} q^{\operatorname{imaj} w} = \sum_{k \geq 0, a+b=m} q^{\operatorname{imaj} w} = \sum_{k \geq 0, a+b=m} A_{n,k,a,b}(q).
\]
This proves (1.17).

For instance, from the tables 2 and 3 we can verify that (1.17) holds for $n = 3$ and for $m = 2$: $A_{3,0,0,2}(q) + A_{3,1,0,2}(q) + A_{3,1,1,1}(q) + A_{3,1,2,0}(q) + A_{3,2,2,0}(q) = 1 + q^2 + (2q + 2q^2) + q + q^3 = (q + q^2) + (1 + q + q^2) + (q + q^2 + q^3) = A_{3,3}(q) + A_{3,2,1}(q) + A_{3,0,1,2}(q)$.

The same technique of proof can be used for the $s$-permutations. We get:
\[
\sum_{c \in \Theta_{n,m+1}} A_{n,c}(q) = \sum_{c \in \Theta_{n,m+1}} \sum_{w \in T_n, \Lambda w = c} q^{\operatorname{inv} w} = \sum_{c \in \Theta_{n,m+1}} \sum_{w \in T_n, \Lambda w = c} q^{\operatorname{imaj} w} = \sum_{k \geq 0, a+b=m} q^{\operatorname{imaj} w} = \sum_{k \geq 0, a+b=m} B_{n,k,a,b}(q).
\]
This proves (1.18).
9. Proof of Theorem 1.5

If $J = (J_0, J_1, \ldots, J_{m-1}, J_m)$ is a sequence of disjoint subsets of the interval $[n] := \{1, 2, \ldots, n\}$ of union $[n]$ with $m \geq 1$, then $(\#J_0, \#J_1, \ldots, \#J_m)$ is a composition $c = (c_0, c_1, \ldots, c_m)$ of $n$. We then write $\#J := c$. Also, let $\text{inv} J$ denote the number of ordered pairs $(x, y)$ where $x \in J_k$, $y \in J_l$, $k < l$ and $x > y$. A classical result that goes back to MacMahon (see, e.g. [An 76, §3.4]) makes it possible to write for each composition $c$ of $n$

$$\sum_{J, \#J = c} q^{\text{inv} J} = \left[ \begin{array}{c} n \\ c_0, c_1, \ldots, c_m \end{array} \right]_q,$$

where the right-hand side is the $q$-multinomial coefficient equal to

$$\frac{(q; q)_n}{(q; q)_{c_0}(q; q)_{c_1} \cdots (q; q)_{c_m}}.$$

**Theorem 9.1.** For each $t$-composition $c = (c_0, c_1, \ldots, c_{m-1}, c_m)$ of $n$ ($n \geq 1$) we have:

$$A_{n,c}(q) = \left[ \begin{array}{c} n \\ c_0, c_1, \ldots, c_{m-1}, c_m \end{array} \right]_q A_{c_0}(q)A_{c_1}(q) \cdots A_{c_{m-1}}(q)A_{c_m}^{\text{Sec}}(q).$$

**Proof.** Each $t$-permutation $w$ from $T_n$ such that $\Lambda w = c$ and $\mu c = m \geq 1$ is completely characterized by a sequence

$$((I_0, \sigma_0), (I_1, \sigma_1), \ldots, (I_m, \sigma_m)),$$

having the following properties:

(i) the sequence $I(w) := (I_0, I_1, \ldots, I_m)$ consists of disjoint subsets of the interval $[n] := \{1, 2, \ldots, n\}$ of union $[n]$; moreover, $\#I(w) := c$;

(ii) $\sigma_0 \in RA_{c_0}$, $\sigma_1 \in FA_{c_1}$, $\ldots$, $\sigma_m \in FA_{c_m}$.

If $I(w) = J$, then $\text{inv} w = \text{inv} J + \text{inv} \sigma_0 + \text{inv} \sigma_1 + \cdots + \text{inv} \sigma_{m-1} + \text{inv} \sigma_m$.

Hence,

$$A_{n,c}(q) = \sum_{w \in T_n, \Lambda w = c} q^{\text{inv} w} = \sum_{w \in T_n, \#I(w) = c} q^{\text{inv} w}$$

$$= \sum_{J, \#J = c} \sum_{w, I(w) = J} q^{\text{inv} w}$$

$$= \sum_{J, \#J = c} \sum_{\sigma_0 \in RA_{c_0}} q^{\text{inv} J} \sum_{\sigma_1 \in FA_{c_1}} q^{\text{inv} \sigma_0} \sum_{\sigma_2 \in FA_{c_2}} q^{\text{inv} \sigma_1} \cdots \sum_{\sigma_m \in FA_{c_m}} q^{\text{inv} \sigma_m}$$

$$= \left[ \begin{array}{c} n \\ c_0, c_1, \ldots, c_m \end{array} \right]_q A_{c_0}(q)A_{c_1}(q) \cdots A_{c_m}^{\text{Sec}}(q).$$

$\square$
The factorial generating functions for the polynomials

\[ A_n(x, q) := \sum_{c \in \Theta_n} x^c A_{n,c}(q), \quad B_n(x, q) := \sum_{c \in \Theta_n} x^{c-1} B_{n,c}(q), \]

can be derived from Theorem 9.1.

**Proof of Theorem 1.5.** For \( m \geq 1 \) we have:

\[
\sum_{n \geq 0} \frac{u^n}{(q;q)_n} \sum_{\mu \in \Theta_n, \ c = \mu} A_{n,c}(q) = \sum_{n \geq 0} \frac{u^n}{(q;q)_n} \sum_{c_0 + c_1 + \cdots + c_m = n} [c_0, c_1, \ldots, c_m]q A_{c_0}(q) A_{c_1}(q) \cdots A_{c_m}(q).
\]

When \( m \geq 1 \), the integers \( c_0 \) and \( c_m \) are even; if, furthermore, \( m \geq 2 \), then \( c_1, \ldots, c_{m-1} \) are odd. Hence, the previous identity may be rewritten as

\[
\sum_{n \geq 0} \frac{u^n}{(q;q)_n} \sum_{\mu \in \Theta_n, \ c = \mu} A_{n,c}(q) = \sum_{j_0 \geq 0} A_{2j_0}(q) \frac{u^{2j_0}}{(q;q)_{2j_0}} \left( \sum_{j \geq 0} A_{2j+1}(q) \frac{u^{2j+1}}{(q;q)_{2j+1}} \right)^{m-1} \sum_{j_m \geq 0} A_{2j_m}(q) \frac{u^{2j_m}}{(q;q)_{2j_m}} = \sec_q(u) (\tan_q(u))^{m-1} \sec_q(u).
\]

When \( m = 0 \), then \( n \) is odd. Hence, \( A_{n,c}(q) = A_n(q) \) (\( n \) odd) and

\[
\sum_{n \geq 0} \frac{u^n}{(q;q)_n} \sum_{\mu \in \Theta_n, \ c = \mu} A_{n,c}(q) = \tan_q(u).
\]

Thus,

\[
\sum_{n \geq 0} A_n(x, q) \frac{u^n}{(q;q)_n} = \sum_{n \geq 0} \frac{u^n}{(q;q)_n} \sum_{m \geq 0} x^m \sum_{\mu \in \Theta_n, \ c = \mu} A_{n,c}(q)
\]

\[
= \sum_{n \geq 0} \frac{u^n}{(q;q)_n} \sum_{\mu \in \Theta_n, \ c = \mu} A_{n,c}(q) + \sum_{m \geq 1} x^m \sum_{n \geq 0} \frac{u^n}{(q;q)_n} \sum_{\mu \in \Theta_n, \ c = \mu} A_{n,c}(q)
\]

\[
= \tan_q(u) + \sum_{m \geq 1} \sec_q(u) (x \tan_q(u))^{m-1} x \sec_q(u)
\]

\[
= \tan_q(u) + \sec_q(u) (1 - x \tan_q(u))^{-1} x \sec_q(u),
\]

which proves (1.19).
The proof of (1.20) is quite similar. The only difference is the fact that \( c = (c_0, c_1, \ldots, c_m, 0) \) is now an \( s \)-composition, so that, \( c_0 \) is even and if \( m \geq 1 \) all the other \( c_i \)'s are odd. We then get:

\[
\sum_{n \geq 0} \frac{u^n}{(q; q)_n} \sum_{w \in \Theta_n^{-} \atop \mu \, c = m+1} A_{n,c}(q) = \sum_{j_0 \geq 0} A_{2j_0}(q) \frac{u^{2j_0}}{(q; q)_{2j_0}} \left( \sum_{j \geq 0} A_{2j+1}(q) \frac{u^{2j+1}}{(q; q)_{2j+1}} \right)^m
\]

\[= \sec(u) (\tan(u))^m.\]

Then

\[
\sum_{n \geq 0} B_n(x,q) \frac{u^n}{(q; q)_n} = \sum_{m \geq 0} x^m \sum_{n \geq 0} \frac{u^n}{(q; q)_n} \sum_{w \in \Theta_n^{-} \atop \mu \, c = m+1} A_{n,c}(q)
\]

\[= \sum_{m \geq 0} \sec(u) (x \tan(u))^m
\]

\[= \sec(u)(1 - x \tan(u))^{-1},\]

which proves (1.20).

10. Specializations

Our three families of polynomials \((A_{n,k,a,b}(q))\), \((B_{n,k,a,b}(q))\), and \((A_{n,c}(q))\) involve specializations that relate to other classes of generating polynomials or classical numbers that have been studied in previous works. Those polynomials are displayed in Tables 2–4 at the end of the paper. Each table appears as a matrix, whose \((n,m)\)-cell contains several polynomials. In the \((n,m)\)-cell of Table 2 (resp. Table 3) are reproduced all the polynomials \(A_{n,k,a,b}(q)\) (or \(B_{n,k,a,b}(q)\)) such that \(a + b = m\) (resp. \(A_{n,c}(q)\) (or \(B_{n,c}(q)\)) such that \(\mu \, c = m\).

The specializations we deal with refer to rows, columns or diagonals of those tables. Others are obtained by summing the above polynomials with respect to certain subscripts.

To this end we use the following notations:

\[
(10.1) \quad A_{n,k,a+b=m}(q) := \sum_{a+b=m} \quad A_{n,k,a,b}(q);
\]

\[
(10.2) \quad A_{n,a+b=m}(q) := \sum_{k \geq 0 \atop a+b=m} \quad A_{n,k,a,b}(q);
\]

By Theorem 1.4 we also have:

\[
A_{n,a+b=m}(q) := \sum_{\mu \, c = m} \quad A_{n,c}(q).
\]

Analogous definitions are made for the polynomials \(B_{n,k,a,b}(q)\).
10.1. The first column of \((A_{n,k,a,b}(q)), \ (B_{n,k,a,b}(q))\). The \((t,q)\)-analogs of tangent and secant have been introduced in our previous paper [FH11]. For each \(r \geq 0\) form the \(q\)-series:

\[
\begin{align*}
\sin_q^{(r)}(u) &:= \sum_{n \geq 0} (-1)^n \frac{(q^r;q)_{2n+1} u^{2n+1}}{(q;q)_{2n+1}}, \\
\cos_q^{(r)}(u) &:= \sum_{n \geq 0} (-1)^n \frac{(q^r;q)_{2n} u^{2n}}{(q;q)_{2n}}, \\
\tan_q^{(r)}(u) &:= \frac{\sin_q^{(r)}(u)}{\cos_q^{(r)}(u)}, \\
\sec_q^{(r)}(u) &:= \frac{1}{\cos_q^{(r)}(u)}.
\end{align*}
\]

The \((t,q)\)-analogs of the tangent and secant numbers have been defined as the coefficients \(T_{2n+1}(t,q)\) and \(E_{2n}(t,q)\), respectively, in the following two series:

\[
\begin{align*}
\sum_{r \geq 0} t^r \tan_q^{(r)}(u) &= \sum_{n \geq 0} \frac{u^{2n+1}}{(t;q)_{2n+2}} T_{2n+1}(t,q); \\
\sum_{r \geq 0} t^r \sec_q^{(r)}(u) &= \sum_{n \geq 0} \frac{u^{2n}}{(t;q)_{2n+1}} E_{2n}(t,q).
\end{align*}
\]

It was then proved that \(T_{2n+1}(t,q)\) and \(E_{2n}(t,q)\) have the following combinatorial interpretations:

\[
\begin{align*}
T_{2n+1}(t,q) &= \sum_{\sigma \in \mathcal{RA}_{2n+1}} t^{1+\text{ides}} \sigma q^{\text{imaj} \sigma}, \\
E_{2n}(t,q) &= \sum_{\sigma \in \mathcal{RA}_{2n}} t^{1+\text{ides}} \sigma q^{\text{imaj} \sigma}.
\end{align*}
\]

Now, the set \(\mathcal{T}_{2n+1,k,0,0}\) is the set of all \(t\)-permutations \(w = (w_0)\) of order \((2n+1)\), where \(w_0\) is simply an element of \(\mathcal{RA}_{2n+1}\), that is, a rising alternating permutation of order \((2n+1)\). Hence,

\[
T_{2n+1}(t,q) = \sum_{k \geq 1} t^{k+1} A_{2n+1,k,0,0}(q).
\]

Accordingly, Theorem 4.1 provides a method for calculating the polynomials \(T_{2n+1}(t,q)\), only defined so far by their generating function (10.3). In an
In the same way, we get:

\[ \sum_{r \geq 0} t^r \tan_q^{(r)}(u) = \sum_{n \geq 0} \frac{u^{2n+1}}{(t; q)_{2n+2}} \sum_{k \geq 1} t^{k+1} A_{2n+1,k,0,0}(q). \]

In the same way, we get:

\[ E_{2n}(t, q) = \sum_{k \geq 1} t^{k+1} B_{2n,k,0,0}(q), \]

which also provides, either a way of calculating the \((t, q)\)-analogs \(E_{2n}(t, q)\) of the secant numbers, or writing the factorial generating function for the first column of the matrix \((B_{n,k,a,b}(q))\).

10.2. The super-diagonal \((n, n + 1)\) of the matrix \((A_{n,k,a,b}(q))\). As will be shown, the polynomials \(A_{n,k,a,b}(q)\) \((a + b = n + 1)\) of that super-diagonal provide a refinement of the Carlitz \(q\)-analogs of the Eulerian polynomials [Ca54]. Twenty-one years later [Ca75] Carlitz also showed that they were generating polynomials for the symmetric groups by the pair “des” (number of descents) and “maj” (major index). Let \((A_n(t, q))\) be the sequence of those polynomials, written as \(A_n(t, q) = \sum_{j \geq 0} A_{n,j}(q)\) \((n \geq 0)\). The recurrence

\[ A_{n,j}(q) = (1 + q + \cdots + q^j) A_{n-1,j}(q) + (q^j + q^{j+1} + \cdots + q^{n-1}) A_{n-1,j-1}(q), \]

with the initial conditions \(A_{0,j}(q) = A_{1,j}(q) = \delta_{0,j}\), provides a method for calculating them.

Their first values are reproduced in the following table:

\[
\begin{align*}
A_0(t, q) &= A_1(t, q) = 1; \quad A_2(t, q) = 1 + t q; \quad A_3(t, q) = 1 + 2 t q (q + 1) + t^2 q^3; \\
A_4(t, q) &= 1 + t q(3 q^2 + 5 q + 3) + t^2 q^3(3 q^2 + 5 q + 3) + t^3 q^6; \\
A_5(t, q) &= 1 + t q(4 q^3 + 9 q^2 + 9 q + 4) + t^2 q^3(6 q^4 + 16 q^3 + 22 q^2 + 16 q + 6) + t^3 q^6(4 q^3 + 9 q^2 + 9 q + 4) + t^4 q^{10}.
\end{align*}
\]

Now, go back to the recurrence for the polynomials \(A_{n,k,a,b}(q)\) shown in (4.1) and rewrite it when \(a' + b' = m' = n + 2\). The coefficients \(A_{n,k'-1,a,m'+1-a}(q) = A_{n,k'-1,a,n+3-a}(q)\) and \(A_{n,k',a,m'+1-a}(q) = A_{n,k',a,n+3-a}(q)\) vanish, because \(A_{n,k,a,b}(q) = 0\) when \(a + b \geq n + 2\). Hence,

\[
A_{n+1,k',a',n+2-a'}(q) = q^{k'} \left( \sum_{0 \leq a \leq a'-1 \leq n} A_{n,k'-1,a,n+1-a}(q) + \sum_{1 \leq a' \leq a \leq n+1} A_{n,k',a,n+1-a}(q) \right),
\]

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valid for \( n \geq 0 \) with the initial condition: \( A_{0,k,a,b}(q) = \delta_{k,0} \delta_{a,1} \delta_{b,0} \). For \( n \geq 0 \) let \( A_{n,k,a}(q) := A_{n,k,a,n+1−a}(q) \), so that the previous recurrence can be written in the form

\[
A_{n+1,k',a'}(q) = q^{k'} \left( \sum_{0 \leq a \leq a'−1 \leq n} A_{n,k'-1,a}(q) + \sum_{1 \leq a' \leq a \leq n+1} A_{n,k',a}(q) \right),
\]

with \( 0 \leq a' \leq n+2 \) and makes it possible the calculations of those polynomials:

\[
A_{1,0,1}(q) = 1;
A_{2,0,1}(q) = q;
A_{3,0,1}(q) = 1, A_{3,1,2}(q) = q^2, A_{3,1,3}(q) = q + q^2, A_{3,2,3}(q) = q^3;
A_{4,0,1}(q) = 1, A_{4,1,1}(q) = 2q^2 + 3q^3, A_{4,1,2}(q) = q + 2q^2 + q^3, A_{4,1,3}(q) = q + q^2,
A_{4,1,4}(q) = q, A_{4,2,1}(q) = q^5, A_{4,2,2}(q) = q^4 + q^5, A_{4,2,3}(q) = q^3 + 2q^4 + q^5,
A_{4,2,4}(q) = 2q^3 + 2q^4, A_{4,3,4}(q) = q^6.
\]

Theorem 10.1. For each \( n \geq 0 \) and each \( j \geq 0 \) the coefficient \( A_{n,j}(q) \) of \( t^j \) in the Carlitz \( q \)-Eulerian polynomial \( A_n(t, q) \) admits the following refinement:

\[
A_{n,j}(q) = \sum_{a \geq 0} A_{n,j,a}(q).
\]

For instance, \( A_{4,1}(q) = 3q + 5q^2 + 3q^3 = (2q^2 + 2q^3) + (q + 2q^2 + q^3) + (q + q^2) + q = A_{4,1,1}(q) + A_{4,1,2}(q) + A_{4,1,3}(q) + A_{4,1,4}(q) \).

Proof. Each polynomial \( A_n(t, q) \) is the generating polynomial for \( S_n \) by the pair (“number of descents”, “major index”), as established by Carlitz [Ca75], or, in an equivalent manner, by the pair (ides, imaj). This can also be expressed by

\[
A_{n,j}(q) = \sum_{\sigma \in S_n, \text{ides } \sigma = j} q^{\text{imaj } \sigma},
\]

By Theorem 9.1 \( A_{n,j,a}(q) = A_{n,j,a,n+1−a}(q) \) is the generating polynomial for the set of all \( t \)-permutations \( w \) of order \( n \) such that \( \mu w = n + 1 \), \( \text{ides } w = j \), and \( \min w = a \) by “imaj.” Such \( t \)-permutations are of the form \( w = (w_0, w_1, w_2, \ldots, w_{n+1}) \), so that necessarily, \( w_0 = w_{n+1} = \epsilon \), and the other components \( w_i \) are one-letter words. Accordingly, \( A_{n,j,a}(q) \) is the generating polynomial for all (ordinary) permutations \( \sigma = w_1 w_2 \cdots w_n \) of \( 12 \cdots n \) such that \( \text{ides } \sigma = j \) and \( w_a = 1 \), that is,

\[
A_{n,j,a}(q) = \sum_{\sigma \in S_n, \text{ides } \sigma = j, \text{min } \sigma = a} q^{\text{imaj } \sigma}.
\]

Thus (10.5) is a consequence of (10.6) and (10.7). □
For each integer $n \geq 1$ let $[n]_q := (q; q)_n/(1 - q)^n = 1 + q + \cdots + q^{n-1}$. By summing the $A_{n,j,a}$’s over the pair $(j, a)$ we get the polynomial $A_{n,a+b=n+1}(q)$ defined in (10.2), which is the generating polynomial for $\mathcal{G}_n$ by “inv,” well-known to be equal to

$$A_{n,a+b=n+1}(q) = [1]_q [2]_q \cdots [n]_q,$$

also equal (using the same combinatorial interpretation) to $B_{n,a+b=n}(q) = \sum_{k \geq 0, a+b=n} B_{n,k,a,b}(q)$.

10.3. The subdiagonal $(n, n - 1)$ of the matrix $(A_{n,k,a,b}(q))$. Our purpose is to evaluate the polynomial $A_{n,a+b=n-1}(q)$ for each $n \geq 1$, which is the generating function for all $t$-permutations of order $n$ such that $\mu w = n - 1$ by “inv.” Such $t$-permutations $w$ have one of the three forms:

1. $w = (x_1 x_2, x_3, x_4, \ldots, x_n, \epsilon)$ with $x_1 < x_2$;
2. $w = (\epsilon, x_1, x_2, \ldots, x_{n-2}, x_{n-1} x_n)$ with $x_{n-1} > x_n$;
3. $(\epsilon, x_1, \ldots, x_{i-1}, x_i x_{i+1} x_{i+2}, x_{i+3}, \ldots, x_n, \epsilon)$ with $1 \leq i \leq n-2, x_i > x_{i+1}, x_{i+1} < x_{i+2}$.

The g.f. of the $t$-permutations of form (1) or (2) by “inv” is equal to

$$(1 + q) \left[ \frac{n}{2, 1^{n-2}} \right]_q = [2]_q [3]_q [4]_q \cdots [n]_q.$$

The g.f. of the $t$-permutations of form (3) by “inv” is equal to

$$\sum_{1 \leq i \leq n-2} \left[ \frac{n}{1^{i-1}, 3, 1^{n-i-2}} \right]_q \times q(q + 1) = (n - 2)q(q + 1) \left[ \frac{n}{3, 1^{n-3}} \right]_q$$

so that the total g.f. is equal to

$$[2]_q [4]_q \cdots [n]_q ([3]_q + (n - 2)q)$$

$$= [2]_q [4]_q \cdots [n]_q (1 + q + q^2 + (n - 2)q)$$

$$= [1]_q [2]_q (1 + (n - 1)q + q^2) [4]_q \cdots [n]_q.$$

Thus, $A_{n,a+b=n-1}(q) = [1]_q [2]_q (1 + (n - 1)q + q^2) [4]_q \cdots [n]_q$.

10.4. The subdiagonal $(n, n - 2)$ of the matrix $(B_{n,k,a,b}(q))$. Using the same combinatorial technique as in 10.3, but this time operating with the $s$-permutations we get the following evaluation for each $n \geq 2$:

$$B_{n,a+b=n-2}(q) = (1 + (n - 1)q + (n - 1)q^2) [4]_q \cdots [n]_q.$$
10.5. Two $q$-analog of the Springer numbers. It was recalled in the Introduction (1.7 and 1.20) that $\sec(u)(1 - x \tan(u))^{-1}$ for $x = 1$ was the exponential generating function for the Springer numbers. Referring to (1.21) we then see that $\sec_q(u)(1 - \tan_q(u))^{-1}$ is the factorial generating function for the $q$-analog of the Springer numbers, which are simply the generating polynomials for the $s$-permutations by “imaj” or “inv.”

Note that $\sec_q(u)(1 - \tan_q(u))^{-1}$ is also the factorial generating for such $q$-analog, but this time for the $S$-permutations by “imaj” or “inv.”

10.6. $t$-compositions and Fibonacci triangle. Let $\alpha(n,m) := \#\Theta_{n,m}$ (resp. $\beta(n,m) := \#\Theta_{n,m+1}$) be the number of $t$-compositions (resp. $s$-compositions). From the previous lists of the $\Theta_i$’s made in Section 1 we have the next table, where the $\alpha(n,m)$’s (resp. $\beta(n,m)$’s) have been reproduced in bold face (resp. plain type). To the right are displayed the row sums of those entries, which will be proven to be the classical Fibonacci numbers.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| 0   | 1 |   |   |   |   |   |   |   |
| 1   | 1 | 1 |   |   |   |   |   |   |
| 2   | 1 | 2 | 1 |   |   |   |   |   |
| 3   | 1 | 2 | 3 | 1 |   |   |   |   |
| 4   | 1 | 3 | 3 | 4 | 1 |   |   |   |
| 5   | 1 | 3 | 6 | 4 | 5 | 1 |   |   |
| 6   | 1 | 4 | 6 | 10| 5 | 6 | 1 | 1 |

Fig. 10.1. The coefficients $\alpha(n,m)$ and $\beta(n,m)$

The mapping $(c_0, c_1, \ldots, c_m, 0) \mapsto (c_0, c_1, \ldots, c_m - 1)$ is a bijection of $\Theta_{n,m+1}$ onto $\Theta_{n-1,m}$, because $c_m$ is odd. Hence $\beta(n,m) = \alpha(n - 1, m)$ $(n \geq 1)$. We now only study the numbers $\alpha(n,m)$.

**Proposition 10.2.** With the initial values $\alpha(0,m) = \delta_{1,m}$, $\alpha(1,m) = \delta_{0,m} + \delta_{2,m}$, the entries $\alpha(n,m)$ are inductively given by

$$\alpha(n,m) = \alpha(n - 1, m - 1) + \alpha(n - 2, m) \quad (n \geq 2).$$

**Proof.** Let $c = (c_0, c_1, \ldots, c_{m-1}, c_m) \in \Theta_{n,m}$. If $c_m = 0$ we define $\phi(c) = (c_0, c_1, \ldots, c_{m-1} - 1) \in \Theta_{n-1,m-1}$. If $c_m \geq 1$, then $c_m \geq 2$ because $c_m$ is even. We define $\phi(c) = (c_0, c_1, \ldots, c_{m-1}, c_m - 2) \in \Theta_{n-2,m}$. We verify that $\phi$ is a bijection between $\Theta_{n,m}$ and $\Theta_{n-1,m-1} + \Theta_{n-2,m}$. \[ \]

For each $n \geq 0$ let $A_n(x) = \sum_{m \geq 0} \alpha(n,m)x^m$ be the generating polynomials of the coefficients $\alpha(n,m)$. From Proposition 10.23 it follows that $A_0(x) = x$, $A_1(x) = 1 + x^2$ and the recurrence formula

$$A_{n+1}(x) = xA_n(x) + A_{n-1}(x) \quad (n \geq 1).$$
Let $A(x; u) := \sum_{n \geq 0} A_n(x) u^n$. Then, $A(x; u) - x - u(1 + x^2) = xu(A(x; u) - x) + u^2A(x; u)$, so that $A(x; u)(1 - xu - u^2) = x + u$ and

$$A(x; u) = \sum_{n \geq 0} A_n(x) u^n = \sum_{c \in \Theta} x^\mu c u^{|c|} = \frac{x + u}{1 - u(x + u)}.$$ 

Let $x = 1$ we obtain the generating function for the row sums $A_n(1) = \sum_{m \geq 0} a(n, m)$ ($m \geq 0$), which is equal to $(1 + u)/(1 - u - u^2)$. Thus, the row sums are the classical Fibonacci numbers.

The polynomials $A_n(x)$ are related to the polynomials $F_n(x)$ already introduced in Sloane’s Integer Encyclopedia [Sl06] under reference A102426 by $A_n(x) = x^{n+1}F_{n+1}(x^{-2})$ for $n \geq 0$. Accordingly, the $t$-compositions provide a natural combinatorial interpretation for their coefficients.

10.7. Further comment. Dominique Dumont [Du12] has drawn our attention to the two papers by Carlitz-Scoville [CS72] and Françon [Fr78]. Instead of $t$-permutations or snakes, Carlitz and Scoville have dealt with “up-down sequences of length $n + m$ with $m$ infinite elements.” Such a sequence is a rising alternating permutation $x_1x_2 \cdots x_{n+m}$ containing all the integers $1, 2, \ldots, n$ and $m$ letters equal to $-\infty$. Note that replacing all the commas in each $t$-permutation $w = (w_0, w_1, \ldots, w_m)$ by $-\infty$ makes up a bijection of the set of all $t$-permutations onto the set of all Carlitz-Scoville sequences. In Françon [Fr78] can be found an unexpected combinatorial interpretation of the entries $b(n, m)$ in terms of computer file histories.

11. Tables

Four tables are being displayed, the first one containing the values of $a(n, m)$ and $b(n, m)$ for $0 \leq n \leq 6$, the second one containing the values of the polynomials $A_{n,k,a,b}(q)$ and $B_{n,k,a,b}(q)$ for $0 \leq n \leq 4$, the third one for the polynomials $A_{n,c}(q)$ for $0 \leq n \leq 4$. The last one contains the values of the polynomials $A_{n,a+b=n}(q)$ and $B_{n,a+b=n}(q)$, whose definitions are given in (10.1)–(10.2).

| $m =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------|---|---|---|---|---|---|---|---|
| $n = 0$ | 1 | 1 |   |   |   |   | 1.2^0 | 1 |
| 1     | 1 | 1 | 1 |   |   |   | 1.2^1 | 1 |
| 2     | 1 | 2 | 2 | 2 |   |   | 1.2^2 | 3 |
| 3     | 2 | 5 | 8 | 6 | 6 |   | 2.2^3 | 11 |
| 4     | 5 | 16 | 28 | 40 | 24 | 24 | 5.2^4 | 57 |
| 5     | 16 | 61 | 136 | 180 | 240 | 120 | 120 | 16.2^5 | 361 |
| 6     | 61 | 272 | 662 | 1232 | 1320 | 1680 | 720 | 720 | 61.2^6 | 2763 |

Table 1. The coefficients $a(n, m)$ and $b(n, m)$
### Table 2. Polynomials $A_{n,k,a,b}(q)$ and $B_{n,k,a,b}(q)$ for $0 \leq n \leq 4$

$A_{n,k,a,b}(q)$ and $B_{n,k,a,b}(q)$

| $m=0$ | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| $n=0$ | $B_{0,-1,0,0}=1$ | $A_{0,0,1,0}=1$ | | |
| 1     | $A_{1,0,0,0}=1$ | $B_{1,0,1,0}=1$ | $A_{1,0,1,1}=1$ | |
| 2     | $B_{2,0,0,0}=1$ | $A_{2,0,0,1}=1$ | $B_{2,0,1,1}=1$ | $A_{2,0,1,2}=1$ |
| 3     | $A_{3,1,0,0}=q+q^2$ | $B_{3,0,0,1}=1$ | $A_{3,0,0,2}=1$ | $B_{3,0,1,2}=1$ |
|       | $B_{3,1,0,1}=q^2$ | $B_{3,1,1,0}=2q^2+q^3$ | $A_{3,1,0,2}=q^2$ | $B_{3,1,1,2}=q^2$ |
|       | $A_{4,2,1,0}=2q^3+3q^4+q^5$ | $A_{3,1,1,1}=2q+2q^2$ | $A_{3,1,1,1}=2q+2q^2$ | $B_{3,1,2,1}=q+q^2$ |
|       | $A_{4,2,1,0}=2q^3+3q^4+q^5$ | $A_{3,1,2,0}=q$ | $A_{3,1,2,0}=q$ | $B_{3,1,3,0}=q$ |
|       | $A_{4,2,2,0}=q^3$ | $A_{3,2,0,0}=q^3$ | $A_{3,2,0,0}=q^3$ | $B_{3,2,3,0}=q^3$ |

| $m=4$ | 4 | 5 |
|-------|---|---|
| $n=3$ | $A_{3,0,1,3}=1$ | $A_{4,0,1,4}=1$ |
|       | $A_{3,1,1,3}=q^2$ | $A_{4,1,1,4}=2q^2+2q^3$ |
|       | $A_{3,1,2,2}=q+q^2$ | $A_{4,1,2,3}=q+2q^2+q^3$ |
|       | $A_{3,1,3,1}=q$ | $A_{4,1,3,2}=q+q^2$ |
|       | $A_{3,2,3,1}=q^3$ | $A_{4,1,4,1}=q$ |
| 4     | $B_{4,0,1,3}=1$ | $A_{4,0,1,4}=1$ |
|       | $B_{4,1,1,3}=2q^2+2q^3$ | $A_{4,1,1,4}=2q^2+2q^3$ |
|       | $B_{4,1,2,2}=q+2q^2+q^3$ | $A_{4,1,2,3}=q+2q^2+q^3$ |
|       | $B_{4,1,3,1}=q+q^2$ | $A_{4,1,3,2}=q+q^2$ |
|       | $B_{4,1,4,0}=q$ | $A_{4,1,4,1}=q$ |
|       | $B_{4,2,1,3}=q^5$ | $A_{4,2,1,4}=q^5$ |
|       | $B_{4,2,2,2}=q^4+q^5$ | $A_{4,2,2,3}=q^4+q^5$ |
|       | $B_{4,2,3,1}=q^3+2q^4+q^5$ | $A_{4,2,3,2}=q^3+2q^4+q^5$ |
|       | $B_{4,2,4,0}=2q^3+q^4$ | $A_{4,2,4,1}=2q^3+2q^4$ |
|       | $B_{4,3,4,0}=q^6$ | $A_{4,3,4,1}=q^6$ |
| $m = 0$ | $n = 0$ | $A_{0(00)} = 1$ | $2$ | $A_{2(01)} = 1$ | $3$ | $A_{3(02)} = 1$ |
|-------|-------|----------------|-----|----------------|-----|----------------|
| $1$   | $A_{1(10)} = 1$ | $A_{1(01)} = 1$ |     | $A_{2(10)} = 1 + q$ |     |                 |
| $2$   | $A_{2(20)} = 1$ | $A_{2(02)} = q$ |     | $A_{3(10)} = 1 + q + q^2$ |     |                 |
| $3$   | $A_{3(30)} = 1 + q + q^2$ | $A_{3(03)} = q + q^2$ |     | $A_{4(01)} = (1 + q + q^2)(1 + q + q^2 + q^3)$ |
|       | $A_{4(04)} = q^2 + 2q^3 + 4q^4$ | $A_{4(22)} = (q + q^3)(1 + q + q^2)$ | $A_{4(30)} = (q + q^2)^2$ | $A_{4(0130)} = (q + q^2)(1 + q + q^2 + q^3)$ |
|       | $A_{4(40)} = q + 2q^2 + q^3 + q^4$ | $A_{4(0120)} = (q + q^2 + q^3)$ |     | $A_{4(0310)} = (q + q^2 + q^3)$ |
|       | $A_{4(21)} = (1 + q + q^2)(1 + q + q^2 + q^3)$ |     |     | $A_{4(2110)} = (1 + q + q^2)(1 + q + q^2 + q^3)$ |

**Table 3. Polynomials $A_{n,c}(q)$ for $0 \leq n \leq 4$, $(A_{n(c_0,...,c_m)} := A_{n,(c_0,...,c_m)}(q))$**

| $m = 4$ | $n = 3$ | $A_{3(01110)} = (1 + q)(1 + q + q^2)$ |     | $A_{4(011110)} = (1 + q)(1 + q + q^2)(1 + q + q^2 + q^3)$ |
|---------|--------|-----------------------------------|-----|-----------------------------------|
| $5$     |        |                                   |     |                                   |

**Table 4. Polynomials $A_{n,a+b=m}(q)$ ($n \equiv m + 1 \pmod{2}$) and $B_{n,a+b=m}(q)$ ($n \equiv m \pmod{2}$) for $0 \leq n \leq 4$**
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