Ising model $S = 1$, $3/2$ and $2$ on directed networks

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Abstract: On directed Barabási-Albert and Small-World networks the Ising model with spin $S = 1$, $3/2$ and $2$ is now studied through Monte Carlo simulations. In this model, the order-disorder phase transition of the order parameter is well defined on Small-World networks for Ising model with spin $S = 1$. We calculate the value of the critical temperature $T_c$ for several values of rewiring probability $p$ of the directed Small-World network. This model on directed Small-World networks we obtained a second-order phase transition for $p = 0.2$ and first-order phase transition for $p = 0.8$. The critical exponentes $\beta/\nu$, $\gamma/\nu$ and $1/\nu$ were calculated for $p = 0.2$. On directed Barabási-Albert we show that no there is phase transition for Ising model with spin $S = 1$, $3/2$ and $2$.

Keywords: Monte Carlo simulation, spins, networks, Ising.

Introduction

This paper deals with Ising spin on directed Barabási-Albert(BA) and Small-World(SW) networks. Sumour and Shabat [1, 2] investigated Ising models with spin $S = 1/2$ on directed BA networks [3] with the usual Glauber dynamics. No spontaneous magnetisation was found, in contrast to the case of undirected BA networks [4, 5, 6] where a spontaneous magnetisation was found below a critical temperature which increases logarithmically with system size. In S=1/2 systems on undirected, scale-free hierarchical-lattice SW networks [7], conventional and algebraic (Berezinskii-Kosterlitz-Thouless) ordering, with finite transition temperatures, have been found. Lima and Stauffer [8] simulated directed square, cubic and hypercubic lattices in two to five dimensions with heat bath dynamics in order to separate the network effects form the effects of directedness. They also compared different spin flip algorithms, including cluster flips [9], for Ising-BA networks. They found a freezing-in of the magnetisation similar to [1, 2], following an Arrhenius law at least in low dimensions. This lack of a spontaneous magnetisation (in the
usual sense) is consistent with the fact that if on a directed lattice a spin $S_j$ influences spin $S_i$, then spin $S_i$ in turn does not influence $S_j$, and there may be no well-defined total energy. Thus, they show that for the same scale-free networks, different algorithms give different results. The $q$-state Potts model has been studied in scale-free networks by Igloi and Turban [10] and depending on the value of $q$ and the degree-exponent $\gamma$ first- and second-order phase transitions are found, and also by Lima [11] on directed BA network, where only first-order phase transitions have been obtained independent of values of $q$ for values of connectivity $z = 2$ and $z = 7$ of the directed BA network. More recently, Lima [12] simulated the the Ising model for spin $S = 1$ on directed BA network and different from the Ising model for spin $S = 1/2$, the unusual order-disorder phase transition of order parameter was seen; this effect is re-evaluated in the light of the time dependence presented below. We study the Ising model for spin 1, 3/2 and 2 on directed BA network. The Ising model with spin 1, 3/2 and 2 was seen not to show a usual spontaneous magnetisation and this decay time for flipping of the magnetisation followed an Arrhenius law for HeatBath algorithms that agree with the results of the Ising model for spin $S = 1/2$ [1, 2] on directed BA network. Edina and Lima [13] calculate the exponentes $\beta/\nu$, $\gamma/\nu$, and $1/\nu$ exponents for majority-vote model on directed SW networks of Sánchez et al. [14], and on these networks the exponents are different from the Ising model a two-dimensional and independently on the values of rewiring probability $p$ of the directed SW networks. Here we also study the Ising model for spin $S = 1$ on directed SW, we obtained a second-order phase transition for $p = 0.2$ and a first-order phase transition for $p = 0.8$. We also calculate the critical exponents $\beta/\nu$, $\gamma/\nu$ and $1/\nu$ for $p = 0.2$.

Model and Simulation

Ising model on directed BA Networks:

We consider the spin 1, 3/2 and 2 Ising models on directed BA Networks, defined by a set of spins variables $S$ taking the values $\pm1$ and 0 for $S = 1$, $\pm3/2$ and 1/2 for $S = 3/2$, and $\pm1$, $\pm2$ and 0 for 2, respectivaly, located on every site of a directed BA Networks with $N$ sites.
Figure 1: Reciprocal logarithm of the relaxation times on directed BA networks for $S = 1$ to $S = 2$.

The probability for spin $S_i$ change its state in this *directed* networks is

$$p_i = 1/1 + \exp(2E_i/k_B T), \quad E_i = -J \sum_k S_i S_k$$  \hspace{1cm} (1)$$

where $k$-sum runs over all nearest neighbors of $S_i$. In this network, each new site added to the network selects with connectivity $z = 2$ already existing sites as neighbours influencing it; the newly added spin does not influence these neighbours.

To study the spin 1, 3/2 and 2 Ising models we start with all spins up, a number of spins equal to $N = 500000$, and Monte Carlo step (MCS) time up 2,000,000, in our simulations, one MCS is accomplished after all the $N$ spins are updated, here, with HeatBath Monte Carlo algorithm. Then we vary the temperature and study nine samples. The temperature is measured in units of critical temperature of the square-lattice Ising model. We determine the time $\tau$ after which the magnetisation has flipped its sign for the first time, and then take the median value of our nine samples. So we get different
values $\tau_1$ for different temperatures.

In the study the critical behavior this Ising model (with spins 1, 3/2 and 2) we define the variable $m = \sum_{i=1}^{N} S_i / N$ as normalized magnetisation.

Our BA simulations, using the HeatBath algorithm, indicate that the spins $S = 1, 3/2$ and 2 Ising model do not display a phase transition and the plot of the time $1/\ln \tau$ versus temperature in Fig. 1 shows that our BA results for all spins agree with the modified Arrhenius law for relaxation time, defined as the first time when the sign of the magnetisation flips: $1/\ln(\tau) \propto T + \ldots$. This result agrees with the results of Sumour et al. for spin $S = 1/2$ on 7 million of sites, see Fig. 2 (figure retired of reference [2]) and our results are more reliable than both Lima [12] and Sumour et al. [21] because that this are longer simulation times.
Figure 3: Magnetisation as a function of the temperature $T$, for $N = 16384$ sites. A second-order phase transition for values of $p = 0.2$ and a first-order phase transition for $p = 0.8$.

**Ising model on directed SW Networks:**

We consider the Ising model with spin $S = 1$, on directed SW Networks, defined by a set of spins variables $S$ taking the values $-1$, 0 and $+1$, situated on every site of a directed SW Networks with $N$ sites.

The probability for spin $S_i$ to change its state in these directed network is given eq. (1) where the sum runs over all nearest neighbors of $S_i$. In this network, created for Sánchez et al. [14], we start from a two-dimensional square lattice consisting of sites linked to their four nearest neighbors by both outgoing and incoming links. Then, with probability $p$, we reconnect nearest-neighbor outgoing links to a different site chosen at random. After repeating this process for every link, we are left with a network with a density $p$ of SW directed links. Therefore, with this procedure every site will have exactly four outgoing links and varying (random) number of incoming links. To study the critical behavior of the model we use the HeatBath algorithm.
and define the variable \( e = E/N \), where \( E = 2 \sum E_i \) and \( m = \sum_{i=1}^{N} S_i / N \). From the fluctuations of the \( e \) measurements we can compute: the average \( e \), the specific heat \( C \) and the fourth-order cumulant of \( e \),

\[
    u(K) = \langle E \rangle / N, \\
    C(K) = K^2 N \langle e^2 \rangle - \langle e \rangle^2, \\
    B_i(K) = [1 - \frac{\langle e^4 \rangle}{3 \langle e^2 \rangle^2}]_{av},
\]

where \( K = J/k_B T \), with \( J = 1 \), and \( k_B \) is the Boltzmann constant. Similarly, we can derive from the magnetization measurements the average magnetization, the susceptibility, and the magnetic cumulants,

\[
    m(K) = \langle |m| \rangle_{av}, \\
    \chi(K) = KN \langle m^2 \rangle - \langle |m| \rangle^2, \\
    U_4(K) = [1 - \frac{\langle m^4 \rangle}{3 \langle |m| \rangle^2}]_{av},
\]

where \( \langle ... \rangle \) stands for a thermodynamic average and \( [...]_{av} \), square brackets for a averages over the 20 realizations.

In order to verify the order of the transition this model, we apply finite-size scaling (FSS) [15]. Initially we search for the minima of \( e \) fourth-order cumulant.
parameter of eq. (4). This quantity gives a qualitative as well as a quantitative description of the order of the transition \([16]\). It is known \([17]\) that this parameter takes a minimum value \(B_{i,\text{min}}\) at effective transition temperature \(T_c(N)\). One can show \([18]\) that for a second-order transition \(\lim_{N \to \infty} (2/3 - B_{i,\text{min}}) = 0\), even at \(T_c\), while at a first-order transition the same limit measuring the same quantity is small and \((2/3 - B_{i,\text{min}}) \neq 0\).

A more quantitative analysis can be carried out through the FSS of the fluctuation \(C_{\text{max}}\), the susceptibility maxima \(\chi_{\text{max}}\) and the minima of the Binder parameter \(B_{i,\text{min}}\). If the hypothesis of a first-order phase transition is correct, we should then expect, for large systems sizes, an asymptotics FSS behavior of the form \([19, 20]\).

\[
\begin{align*}
C_{\text{max}} &= a_C + b_C N + \ldots \quad (8) \\
\chi_{\text{max}} &= a_\chi + b_\chi N + \ldots \quad (9) \\
B_{i,\text{min}} &= a_{B_i} + b_{B_i} N + \ldots \quad (10)
\end{align*}
\]

Therefore, if the hypothesis of a second-order phase transition is correct, we should then expect, for large systems sizes, an asymptotics FSS behavior.
Figure 6: Binder’s fourth-order cumulant of the magnetisation versus temperature for $p = 0.2$.

Of the form

$$C = C_{reg} + L^{\alpha/\nu} f_C(x)[1 + ...],$$

$$[<|m|>|]_{av} = L^{-\beta/\nu} f_m(x)[1 + ...],$$

$$\chi = L^{\gamma/\nu} f_\chi(x)[1 + ...],$$

$$\frac{dU_4}{dT} = L^{1/\nu} f_{U}(x)[1 + ...],$$

where $C_{reg}$ is a regular background term, $\nu$, $\alpha$, $\beta$, and $\gamma$, are the usual critical exponents, and $f_i(x)$ are FSS functions with

$$x = (K - K_c)L^{1/\nu}$$

being the scaling variable, and the brackets $[1 + ...]$ indicate corrections-to-scaling terms. Therefore, from the size dependence of $M$ and $\chi$ we obtained the exponents $\beta/\nu$ and $\gamma/\nu$, respectively. The maximum value of susceptibility also scales as $L^{\gamma/\nu}$. Moreover, the value of $T$ for which $\chi$ has a maximum,
\( T_c^{\chi_{\text{max}}} = T_c(L), \) is expected to scale with the system size as
\[
T_c(L) = T_c + b L^{-1/\nu},
\]
were the constant \( b \) is close to unity. Therefore, the relations (14) and (16) are used to determine the exponent \( 1/\nu \).

We have performed Monte Carlo simulation on directed SW network with various values of probability \( p \). For a given \( p \), we used systems of size \( L = 8, 16, 32, 64, \) and 128. We waited 50,000 Monte Carlo steps (MCS) to make the system reach the steady state, and the time averages were estimated from the next 50,000 MCS. In our simulations, one MCS is accomplished after all the \( N \) spins are updated. For all sets of parameters, we have generated 20 distinct networks, and have simulated 20 independent runs for each distinct network.

**Results and Discussion**
In Fig. 3 we show the dependence of the magnetisation $M$ on the temperature, obtained from simulations on directed SW network with $L = 128 \times 128$ sites and two values of probability $p$: $p = 0.2$ for second-order transition and $p = 0.8$ for first-order transition. In Fig. 4 we plot Binder’s fourth-order cumulant of $e$ for different values of $L$ and two different values of $p$: $p = 0.2$ in part (a) and $p = 0.8$ in part (b). In Fig. 5 the difference $2/3 - B_{i,\text{min}}$ is shown as a function of parameter $1/N$ for probability $p = 0.2$ (circles) and $p = 0.8$ (squares) obtained from the data of Fig. 4. In Fig. 6 we show Binder’s fourth-order cumulant of the magnetisation versus temperature for $p = 0.2$. The temperature obtained is $T_c = 1.890(6)$. Figs. 7 we plot the dependence of the magnetisation at $T = T_c$ versus the system size $L$. The slopes of curves correspond to the exponent ratio $\beta/\nu$ of according to Eq. (12). The results show that the exponent ratio $\beta/\nu$ at $T_c$ is 0.379(21).

In Fig. 8 we display the scalings for susceptibility at $T = T_c(L)\chi(T_c(L))$ (circles), and for its maximum amplitude, $\chi_{L,\text{max}}^\text{max}$ (squares), and the scalings for susceptibility at $T = T_c$ obtained from Binder’s cumulant, $\chi(T_c)$ versus $L$. 

Figure 8: Plot of $\ln \chi_{L,\text{max}}^\text{max}$ (square) and $\ln \chi(T_c)$ (circle) versus $\ln L$ for $p = 0.2$. 
for probability $p = 0.2$. The exponents ratio $\gamma/\nu$ are obtained from the slopes of the straight lines. The exponents $\gamma/\nu$ of the two estimates agree (within errors). The values obtained are $\gamma/\nu = 1.252(13)$ (circles) and $1.264(34)$ (squares), respectively. Therefore we can use the Eq. (16), for $p = 0.2$, obtain the critical exponent $1/\nu$, that is equal to $1.209(165)$ and $1.248(196)$ obtained of according to Eq. (14). To improve our results obtained above we start with all spins up, a number of spins equal to $N = 640000$, and time up $2,000,000$ (in units of Monte Carlo steps per spins). Then we vary temperature $T$ and at each $T$ study the time dependence for 9 samples. We determine the time $\tau$ after which the magnetisation has flipped its sign for first time, and then take the median values of our nine samples. So we get different values $\tau_1$ for different temperatures $T$. In Fig. 9 show that the decay time goes to infinity at some positive $T$ value. This behavior ensures that there is a phase transition for Ising model spin $S = 1$ on directed SW network.
Conclusion

In conclusion, we have presented the Ising model spin spins $S = 1, 3/2$ and 2 on directed BA and directed SW network ($S = 1$). The Ising model does not display a phase transition on directed BA for spins $S = 1/2, 1, 3/2$ and 2. In the directed SW network [11] this model presents a first- and second-order phase transition which occurs with probability $p = 0.8$ and $0.2$, respectively. The exponents obtained for $p = 0.2$ are different from the exponents the Ising model on square lattice, that it suggests that these exponents belong to one another class of universality.

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