Thermalized Displaced and Squeezed Number States in the Coordinate Representation

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(April 1, 2022)

Within the framework of thermofield dynamics, the wavefunctions of the thermalized displaced number and squeezed number states are given in the coordinate representation. Furthermore, the time evolution of these wavefunctions is considered by introducing a thermal coordinate representation, and we also calculate the corresponding probability densities, average values and variances of position coordinate, which are consistent with results in the literature.

I. INTRODUCTION

Displaced number states and squeezed number states are generalization of coherent states and squeezed states of a harmonic oscillator, respectively [1]. The coherent state is constructed by displacing the ground state of the harmonic oscillator [2,3], and the squeezed state by first squeezing the ground state and then further displacing it (sometimes, by first displacing and then squeezing, or by only squeezing) [4]. In these constructions, number states (also called Fock states in quantum field theory) of the harmonic oscillator taking the place of the ground state will correspondingly produce the displaced number state and squeezed number state. Thermalizing the displaced and squeezed number states, one can get the thermalized displaced number state and squeezed number state, which will be discussed in the present paper. Evidently, the thermalized coherent and squeezed states are the special cases of the thermalized displaced number and squeezed number states. All the above states are interesting and important in physics.

As is well known, the coherent state can describe the coherent light, and its over-completeness gives rise to the coherent-state representation which is very useful in quantum optics, statistical physics, quantum field theory and particle physics, etc. [5,6]. This state mimics the motion of classical particles, and hence is also used for studying the Schrödinger cat states [7]. For the squeezed state, not only is it a minimum uncertainty state and similar to the classical motion, but also the quantum fluctuations in position can be suppressed at the expense of enhanced fluctuations in momentum [8], which is different from that of the coherent state. Therefore, the squeezed state has important technology applications in quantum computation and sensitive measurement [9]. So far the squeezed state has been receiving a great deal of investigations [10,11]. In the same way, a displaced number state follows the motion of a classical particle as well as keeps its shape in the course of the motion [12], and the squeezed number state can display the similar squeezed property to the squeezed state (Eqs.(38)—(41) in Ref. [13]) and hence promises hopeful applications in optical spectroscopy, communications, molecular and solid state physics [14] (This reference dealt with the squeezed displaced number states) [15]. Early in 1950's, the displaced number state and the squeezed number state were proposed and studied with the help of the coordinate representation [16]. Since then, these states received a few further investigations [11,14]. Recently, Ref. [17] reviewed these investigations, and gave the most general time-dependent wavefunctions and probability densities of them in the coordinate representation. Particularly, in view of the exponential realization of the optical and atomic squeezed (not displaced) states as well as the number states [15] (1996) [18], Nieto predicted that in the not too distant future, it is hopeful to observe the displaced and squeezed number states [19].

On the other hand, no thermal noises exist nowhere, and thus the influence of the noises on the above-mentioned states has to be studied. Such a investigation is often realized by using density matrices and master equation. However, within the framework of thermofield dynamics [19,10], a thermalizing operator acting on the states is also an important and useful way to introduce finite temperature effects [17,18]. Both the density-matrices and the thermofield-dynamics investigations give rise to a varieties of thermal partners of the above-mentioned states, such as the thermalized coherent, squeezed, displaced number, and squeezed number states, the displaced thermalized state, squeezed thermalized state, and so on. Many properties of various thermal coherent and squeezed states have been
studied by constructing directly a state vector \[|\tilde{\psi}\rangle\] and other methods, such as characteristic function, density operators, Glauber’s P-representation of density operator, etc. \[|\psi\rangle\] \[|\rho\rangle\] \[|\sigma\rangle\] ( Most of Refs. \[14,20,22\] were concerned with the coherent and squeezed thermalized states ). The connections between these thermal states have been revealed in Ref. \[18\], and Fearn and Collett also gave the physical interpretations of these states \[18\]. Besides, for the thermal coherent state, Barnett and Knight discussed the independence of the Glauber’s p-representation upon the order of displacing and thermalizing operators \[17\]. As for the thermalized displaced number and squeezed number states, there were few investigations of them, and just recently the thermalized squeezed number state ( not displaced, different from the state in the present paper ) was considered with its characteristic function for analysing the influence of thermal noise on higher-order squeezing properties of it \[24\].

This paper will address the wavefunctions and position probability densities of the thermalized displaced number and squeezed number states ( Hereafter, two of these states will also imply that the thermalized coherent and squeezed states are their special cases ). This problem hasn’t, to our knowledge, discussed in the literature, except for Ref. \[20\] (1993) and Ref. \[22\] (1965) in which the position probability density of coherent thermal state and squeezed thermal state ( not include number state ) was given by Glauber’s R-function and/or P-representation ). However, this problem is certainly interesting and meaningful. The wavefunctions of the coherent, squeezed, displaced number and squeezed number states contain all information about these states and hence describe completely these states. Therefore, the wavefunctions of the corresponding thermalized non-classical states will give the influence of finite temperature on the properties described by the zero-temperature wavefunctions, and can provide, at least, a quantum-mechanical intutional understanding for us. Moreover, the coordinate representation of their density operators can be obtained from the finite-temperature wavefunctions and consequently these wavefunctions can equip a coordinate-representation way for calculating the expectation values of all physical observables on the thermalized non-classical states, which is most usual way in quantum mechanics. Additionally, the position density probability can give the probability density of magnetic component of electromagnetic fields \[25\] Ref. \[22\] (1965).

Thermofield dynamics is unique formalism of finding the wavefunctions for the thermal non-classical states. In this paper, within the framework of thermofield dynamics, we shall give wavefunctions of the thermalized displaced number and squeezed number states in terms of the position coordinate, consider their time evolution, and calculate the position probability densities. In order to do so, we shall first derive the wavefunction of the thermal vacuum in the coordinate representation, which was almost given in Ref. \[3\], and introduce a thermal coordinate representation in the next section. Then the wavefunctions of the thermalized displaced number and squeezed number states will be given in terms of the position coordinate in Section III. Section IV will address the time evolution, the position probability densities, the position average values and variances of these states. We will conclude this paper at the end.

By the way, thermofield dynamics will be not introduced in this paper, and good expositions of them can be found in Ref. \[16\]. Besides, although this paper will discuss a harmonic oscillator with a mass and constant frequency, taking the mass as unit one can get the results which are usable for a one-mode electromagnetic field with the same frequency.

\section*{II. THERMAL VACUUM AND THERMAL COORDINATE REPRESENTATION}

In the fixed-time Schrödinger picture, for the quantum one-dimensional oscillator

\[ H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 = (a^\dagger a + \frac{1}{2})\hbar \omega , \]  

(1)

the ground state in the coordinate representation is the wavefunction

\[ \langle x|0\rangle = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} \exp\{-\frac{m\omega}{2\hbar}x^2\} , \]

(2)

where \( p = -i\hbar \frac{d}{dx} = -i\hbar \partial_x \), \( m \) is the mass, \( \omega \) the angular frequency, and

\[ a = \frac{1}{\sqrt{2m\hbar\omega}}(ip + m\omega x) , \quad a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-ip + m\omega x) \]

(3)

are the corresponding annihilation and creation operators, respectively. It is noticed that in Ref. \[1\], \( m, \omega \) and \( \hbar \) all are unit. In order to consider thermal effects, thermofield dynamics introduces a copy of the physical oscillator Eq.(1) ( called the tilde oscillator )

\[ \tilde{H} = \frac{1}{2m}\tilde{p}^2 + \frac{1}{2}m\omega^2 \tilde{x}^2 = (\tilde{a}^\dagger \tilde{a} + \frac{1}{2})\hbar \omega , \]

(4)
according to the tilde “conjugation”: \[ \tilde{CO} = C^* \hat{O} \] Here, \( C \) is any coefficient appeared in expressions of quantities for the physical system, \( O \) any operator, the superscript * means complex conjugation, and \( \hat{O} \) represents the corresponding operator for the tilde system. Exploiting the physical and tilde oscillators, one can have the thermal vacuum \[ |0, \beta > = T(\theta)|0, \hat{0} > \] (5) where, \(|0, \hat{0} > = |0 > |\hat{0} > \) is the product of ground states of the physical and tilde oscillators, \( \beta = \frac{1}{k_b T} \) with \( k_b \) the Boltzmann constant and \( T \) the temperature, and the unitary transformation \( T(\theta) \) (called thermal transformation) is

\[
T(\theta) = \exp\{-\theta(\beta)(a\hat{a} - a^\dagger\hat{a}^\dagger)\}
\]

with

\[
\tanh[\theta(\beta)] = e^{-\beta \hbar \omega / 2}.
\]

Notice that any physical operator commutes with any tilde operator. Consequently the thermal-vacuum average value agrees with canonical ensemble average in statistical mechanics.

It is evident that the thermal vacuum (5) is similar to the two-mode squeezed states discussed in Ref. [8] except for a minus difference between the exponents in Eq.(6) here and Eq.(37) there. Although the wavefunction of the two-mode squeezed state was given in the coordinate representation [8], here we still derive the position wavefunction of the thermal vacuum for the sake of both the completeness and the establishment of the thermal coordinate representation. Substituting Eq.(3) into Eq.(6), one can read

\[
T(\theta) = \exp\{i \frac{\theta}{\hbar} (\hat{x}\hat{p} - \hat{x}\hat{p})\}
\]

with \( \theta \equiv \theta(\beta) \). From Appendix B.4 in Ref. [26], the last formula can be unentangled as

\[
T(\theta) = \exp\{-\tanh(\theta)\hat{x} \partial_x\} \exp\{\ln[cosh(\theta)](x\partial_x - \hat{x}\partial_x)\} \exp\{-\tanh(\theta)x\partial_x\}.
\]

Using the following operator properties [27]

\[
e^{C_{\partial_y}} f(y) = f(y + C)
\]

and

\[
e^{C_y \partial_y} f(y) = f( ye^C),
\]

which are proved easily, we obtain the wavefunction of the thermal vacuum as

\[
< \hat{x}, x|0, \beta > = T(\theta)(\frac{m\omega}{\pi \hbar})^\frac{3}{2} \exp\{-\frac{m\omega}{2\hbar}(x^2 + \hat{x}^2)\}
\]

\[
= (\frac{m\omega}{\pi \hbar})^\frac{3}{2} \exp\{-\frac{m\omega}{2\hbar}(xcosh(\theta) - \hat{x}sinh(\theta))^2 + (\hat{x}cosh(\theta) - xsinh(\theta))^2\}.
\]

(11)

When \( \beta \to \infty \), \( < \hat{x}, x|0, \hat{0} > \) is reduced to \( < \hat{x}, x|0, \hat{0} > \). This expression Eq.(11) can be generalized to the Gaussian wavefunctional approach for equilibrium field theory in thermofield dynamics [28].

Such an expression of the thermal vacuum wavefunction Eq.(11) suggests the usefulness of introducing a thermal coordinate representation. In thermofield dynamics, for any operator of the physical or tilde oscillator \( Q \), its thermal counterpart is defined as \( Q_\beta \equiv T(\theta)Q T(\theta)^\dagger \) [10]. In particular, for the fundamental canonical conjugate pairs \( \{x, p = -i\hbar \partial_x\} \) and \( \{\hat{x}, \hat{p} = i\hbar \partial_x\} \), the corresponding thermal operators are

\[
x_\beta \equiv T(\theta)x T(\theta)^\dagger = xcosh(\theta) - \hat{x}sinh(\theta), \quad p_\beta \equiv T(\theta)p T(\theta)^\dagger = pcosh(\theta) - \hat{p}sinh(\theta)
\]

(12)

and

\[
\hat{x}_\beta \equiv T(\theta)\hat{x} T(\theta)^\dagger = \hat{x}cosh(\theta) - xsinh(\theta), \quad \hat{p}_\beta \equiv T(\theta)\hat{p} T(\theta)^\dagger = \hat{p}cosh(\theta) - psinh(\theta).
\]

(13)

Obviously, the thermal vacuum wavefunction Eq.(11) can be written as
\[ <\tilde{x}, x|0, \beta> = (\frac{m\omega}{\hbar})^\frac{3}{4} \exp\left\{-\frac{m\omega}{2\hbar} (x^2_\beta + \tilde{x}^2_\beta)\right\}, \]

which is the same form with the wavefunction \(<\tilde{x}, x|0, 0>\). Noticing that the commutators \([x_\beta, p_\beta] = i\hbar, [\tilde{x}_\beta, \tilde{p}_\beta] = -i\hbar\) and \([O_\beta, \tilde{O}_\beta] = 0\) hold, one can set \(p_\beta \equiv -i\hbar \frac{\partial}{\partial x_\beta}\) and \(\tilde{p}_\beta \equiv i\hbar \frac{\partial}{\partial \tilde{x}_\beta}\), and establish a representation for the thermal oscillator, in which any object (operators, wavefunctions) can be expressed in terms of \(x_\beta, \tilde{x}_\beta, \frac{\partial}{\partial x_\beta}\) and/or \(\frac{\partial}{\partial \tilde{x}_\beta}\). In this paper, we shall call it thermal coordinate representation. Evidently, this representation is reached through the unitary thermal transformation \(T(\theta)\) of the coordinate representation. When working in the representation, quantities will take similar forms to those in quantum mechanics, and hence it will simplify our derivation in the present paper.

It is suitable here to mention a mathematical property and the physical sense of the thermal transformation Eq.(6). It is shown easily that the action of \(T(\theta)\) on a function of the physical and tilde positions \(\{x, \tilde{x}\}\) amounts to just the thermal coordinate \(x_\beta, \tilde{x}_\beta\) taking the place of \(x, \tilde{x}\), that is,

\[ T(\theta)f(x, \tilde{x}) = f(x_\beta, \tilde{x}_\beta). \]

Thermal transformation is also called thermalizing operator \([18]\). It describes the effect of a thermal reservoir in which a quantum harmonic oscillator immerses. From Eq.(5), we can say loosely that a thermalizing operator heats the ground state of a zero-temperature harmonic oscillator into a thermal vacuum with a finite temperature. In quantum optics, the thermalizing operator describes the action of a source which excites one-mode electromagnetic field from its ground state to a chaotic state (thermalized radiation). Thus, in order to consider thermal noise, it is enough to perform the action of the thermalizing operator on the non-classical states mentioned in the last section. Next, we shall address them.

### III. THERMALIZED DISPLACED NUMBER AND SQUEEZED NUMBER STATE IN THE COORDINATE REPRESENTATION

Because both coherent and squeezed states are constructed with the displacing operator and squeezing operator acting on the ground state, there are three different states with squeezed effect: squeezed state (only the squeezing operator acting on the ground state), displaced squeezed state, and squeezed displaced state, which are all usually called squeezed state in the literature. In this paper, the terminology “squeezed state” means only the displaced squeezed state, for which the action of the displacing operator follows that of the squeezing operator. So does the squeezed number state. However, when introducing a finite temperature effect, one still faces more choices about the orders among displacing, squeezing and thermalizing. A different order will lead to a different thermal non-classical state \([18]\). Nevertheless, if using thermal creation and annihilation operators to work, i.e., doing as done in Ref. \([17, 19]\), one can escape the order problems with thermalizing operator. In this section, we shall introduce a finite temperature effect into the displaced number state and squeezed number state by using the thermal creation and annihilation operators with the vacuum \(|0, \beta >\)\([17, 18]\) and then give their expressions in the coordinate representation. This construction is utterly to thermalize the displaced number and squeezed number states, namely, it gives the thermalized displaced number and squeezed number states, as one shall be seen later.

The thermal annihilation and creation operators with the thermal vacuum Eq.(5) are \([10]\)

\[ a_\beta = T(\theta)aT^\dagger(\theta), a^\dagger_\beta = T(\theta)a^\dagger T^\dagger(\theta) \]

and

\[ \tilde{a}_\beta = T(\theta)\tilde{a}T^\dagger(\theta), \tilde{a}^\dagger_\beta = T(\theta)\tilde{a}^\dagger T^\dagger(\theta). \]

One can easily check that \(a_\beta|0, \beta >= 0, \tilde{a}_\beta|0, \beta >= 0\) and \([a_\beta, a^\dagger_\beta] = [\tilde{a}_\beta, \tilde{a}^\dagger_\beta] = 1\). With the aids of thermal creation operators \(a^\dagger_\beta\) and \(\tilde{a}^\dagger_\beta\), one can construct normalized thermal number states

\[ |n, m, \beta > = \frac{1}{\sqrt{n!m!}} a^\dagger_\beta n \tilde{a}^\dagger_\beta m |0, \beta > \]

with the closure relation

\[ \sum_{n,m} |n, m, \beta > |\beta, m, n > = 1 . \]
The so-called displaced number state $|\alpha, n> \equiv D(\alpha)|n>$ of the oscillator Eq.(1) is defined in Fock space as $|\alpha, n> \equiv D(\alpha)|n>$ and can have the following form in the coordinate representation:

\begin{equation}
<x|\alpha, n> = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2^n n!}} \exp\{-i\alpha_1a_2\} \cdot \exp\{-\frac{m\omega}{2\hbar}(x - \sqrt{\frac{2h}{m\omega}}\alpha_1)^2 + i\sqrt{\frac{2m\omega}{h}}\alpha_2x\}H_n[\sqrt{\frac{m\omega}{h}}x - \sqrt{2}\alpha_1] \tag{20}
\end{equation}

with $\alpha = (\alpha_1 + i\alpha_2)$ any complex number, $H_n[\cdot\cdot\cdot]$ the Hermite polynomials and the displacing operator

\begin{equation}
D(\alpha) = e^{\alpha a^+ - \alpha^* a}. \tag{21}
\end{equation}

In this definition, when the number state $|n>$ is replaced by the ground state $|0>$, the state $|\alpha, n>$ is reduced to the usual coherent state $|\alpha>$. Evidently, the state $|\alpha, n>$ is constructed just with the displacing operator $D(\alpha)$ acting on the number state $|n>$. Similarly, one can define the following state $|\alpha, n, \beta>$

\begin{equation}
|\alpha, n, \beta> = D_\beta(\alpha)\bar{D}_\beta(\alpha)|n, n, \beta> \tag{22}
\end{equation}

so as to introduce a finite temperature effect into the displaced number state. Here, the thermal displacing operators $D_\beta(\alpha)$ and $\bar{D}_\beta(\alpha)$ are

\begin{equation}
D_\beta(\alpha) = \exp\{\alpha a_\beta^+ - \alpha^* a_\beta\}, \tag{23}
\end{equation}

\begin{equation}
\bar{D}_\beta(\alpha) = \exp\{\alpha_\beta a^+ - \alpha^{*-\beta} a_\beta\}, \tag{24}
\end{equation}

respectively, which are generalization of the displacing operator $D(\alpha)$. Note that $\alpha = \alpha^*$ in the present paper (of course, one can take $\alpha$ as another parameter independent of $\beta$). When $n = 0$ the state $|\alpha, n, \beta>$ is just Eq.(11) with $\gamma = \alpha$ in Ref. [19] (the first paper) and Eq.(3.1) with $\varphi = \alpha$ in Ref. [17]. Employing the definitions (5), (16), (17), (18), we obtain

\begin{equation}
|\alpha, n, \beta> = T(\theta)|\alpha, n> \equiv |\tilde{\alpha}, n>, \tag{25}
\end{equation}

This equation indicates that the state $|\alpha, n, \beta>$ is just the thermalized displaced number state. In the last equation, $|\tilde{\alpha}, n>$ is the tilde displaced number state and can be obtained from Eq.(20) according to the tilde rules. When $n = 0$, the state $|\alpha, 0, \beta>$ is the thermalized coherent state, being similar to Eq.(3.3) in Ref. [17]. Employing Eqs.(15),(12) and (13), one can have

\begin{equation}
<x, \alpha, n, \beta> = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2^n n!}} \exp\{-\frac{m\omega}{2\hbar}(x - \sqrt{\frac{2h}{m\omega}}\alpha_1)^2 + (\tilde{x}_\beta - \sqrt{\frac{2h}{m\omega}}\alpha_1)^2\]
\begin{align*}
+ i\sqrt{\frac{2m\omega}{h}}\alpha_2(\tilde{x}_\beta - \sqrt{2}\alpha_1)H_n[\sqrt{\frac{m\omega}{h}}x - \sqrt{2}\alpha_1]H_n[\sqrt{\frac{m\omega}{h}}\tilde{x}_\beta - \sqrt{2}\alpha_1] \tag{26}
\end{align*}
\end{equation}

\begin{equation}
= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2^n n!}} \exp\{-\frac{m\omega}{2\hbar}(x\cosh(\theta) - \tilde{x}_\theta\sinh(\theta) - \sqrt{\frac{2h}{m\omega}}\alpha_1)^2\]
\begin{align*}
+ (\tilde{x}_\theta\cosh(\theta) - \sqrt{\frac{2h}{m\omega}}\alpha_1)] + i\sqrt{\frac{2m\omega}{h}}\alpha_2(\cosh(\theta) + \sinh(\theta))(x - \tilde{x}) \tag{27}
\end{align*}
\end{equation}

\begin{align*}
\cdot H_n[\sqrt{\frac{m\omega}{h}}(x\cosh(\theta) - \tilde{x}_\theta\sinh(\theta)) - \sqrt{2}\alpha_1]H_n[\sqrt{\frac{m\omega}{h}}(\tilde{x}_\theta\cosh(\theta) - x\sinh(\theta)) - \sqrt{2}\alpha_1].
\end{align*}

The r.h.s of Eq.(26) is the wavefunction of the thermalized displaced number state in the thermal coordinate representation, and Eq.(27) is just the wavefunction in the coordinate representation. When $\beta \rightarrow \infty$, Eq.(27) is reduced to a product of $x$-function and $\tilde{x}$-function, each factor resembling Eq.(15) in Ref. [1].

Now we are at the position to discuss the thermalized squeezed number state. The squeezed number state $|\alpha, z, n> \equiv S(z|\alpha, n>$ of the oscillator Eq.(1) is constructed by using the squeezing operator $S(z)$

\begin{equation}
S(z) = \exp\{-\frac{1}{2}(z^* a a - za^+ a^+)\} \tag{28}
\end{equation}
In analogy with the definition of the squeezed number state, we introduce the thermal squeezing operator
\[
|\alpha, z, n > \equiv D(\alpha)S(z)|n > .
\]

Here, \(z\) is any complex constant. When \(n = 0\), \(|\alpha, z, 0 >\) is the usual squeezed state. From Ref. \[1\], the wavefunction of \(|\alpha, z, n >\) in the coordinate representation is (here in terms of our notations)
\[
<x|\alpha, z, n > = (\frac{m \omega}{\pi \hbar})^{\frac{1}{2}} \frac{(|F_3|)^n}{\sqrt{2^n n!}} \exp\{-ia_1 a_2\}
\cdot \exp\{-\frac{m \omega}{2 \hbar} F_2[x - \sqrt{\frac{2 \hbar}{m \omega}} a_1]^2 + i \sqrt{\frac{2m \omega}{h}} a_2 x] H_n[\sqrt{\frac{m \omega}{h}} (F_4)^{-1} (x - \sqrt{\frac{2 \hbar}{m \omega}} a_1)]
\]
(30)

with \(z = z_1 + iz_2 = re^{i \phi}, S = \cosh(r) + z_1 \sinh(r)/r\) and \(\kappa = z_2 \sinh(r)/(2rS)\). In Eq.(30), for the convenience of comparison later, we adopted the notations \(F\)’s in Ref. \[1\], that is,
\[
F_1 = S(1 + i2\kappa), \quad F_2 = \frac{1}{S^2(1 + i2\kappa)} - i2\kappa,
\]
\[
F_\beta = \frac{1 - i2\kappa}{1 + i2\kappa}, \quad F_4 = S(1 + 4\kappa^2)^{\frac{1}{2}}.
\]
(31)

In analogy with the definition of the squeezed number state, we introduce the thermal squeezing operator
\[
S_\beta(z) = \exp\{-\frac{1}{2} (z^* a_\beta a_\beta - za_\beta^* a_\beta^*)\}, \quad \tilde{S}_\beta(z) = \exp\{-\frac{1}{2} (\tilde{z}^* \tilde{a}_\beta \tilde{a}_\beta - \tilde{z} \tilde{a}_\beta^* \tilde{a}_\beta^*)\}
\]
(32)

and define the following state \(|\alpha, z, n, \beta >\)
\[
|\alpha, z, n, \beta > \equiv D_\beta(\alpha)\tilde{D}_\beta(\alpha)S_\beta(z)|n, n, \beta > .
\]
(33)

to introduce a finite temperature effect. Note that \(\tilde{z} = z^*\) in the present paper. (Of course, one can take \(\tilde{z}\) as another parameter independent of \(z\).) It is easily shown that
\[
|\alpha, z, n, \beta > = T(\theta)|\alpha, z, n > |\tilde{\alpha}, \tilde{z}, n >
\]
(34)

with \(|\tilde{\alpha}, \tilde{z}, n >\) the tilde version of \(|\alpha, z, n >\). Evidently, the last equation indicates that the state \(|\alpha, z, n, \beta >\) is just the thermalized displaced number state. When \(n = 0\), \(|\alpha, z, 0, \beta >\) is just the thermalized squeezed state, being similar to Eq.(11) in Ref. \[2\] (the second paper). Employing Eqs.(15),(30),(12) and (13), we obtain
\[
<x|\tilde{\alpha}, x|\alpha, z, n, \beta > = (\frac{m \omega}{\pi \hbar})^{\frac{1}{2}} \frac{|F_3|^n}{|F_1|^2 n!} \exp\{-\frac{m \omega}{2 \hbar} [F_2(x_\beta - \sqrt{\frac{2 \hbar}{m \omega}} a_1)^2 + F_2(\tilde{x}_\beta - \sqrt{\frac{2 \hbar}{m \omega}} a_1^2)] + i \sqrt{\frac{2m \omega}{h}} a_2 (x_\beta - \tilde{x}_\beta)] \}
\cdot H_n[\sqrt{\frac{m \omega}{h}} (F_4)^{-1} (x_\beta - \sqrt{\frac{2 \hbar}{m \omega}} a_1)] H_n[\sqrt{\frac{m \omega}{h}} (F_4)^{-1} (\tilde{x}_\beta - \sqrt{\frac{2 \hbar}{m \omega}} a_1)]
\]
(35)
\[
= (\frac{m \omega}{\pi \hbar})^{\frac{1}{2}} \frac{|F_3|^n}{|F_1|^2 n!} \exp\{-\frac{m \omega}{2 \hbar} [F_2(xcosh(\theta) - \tilde{x}sinh(\theta) - \sqrt{\frac{2 \hbar}{m \omega}} a_1)^2]
\cdot H_n[\sqrt{\frac{m \omega}{h}} (F_4)^{-1} (xcosh(\theta) - \tilde{x}sinh(\theta) - \sqrt{\frac{2 \hbar}{m \omega}} a_1)]
\cdot H_n[\sqrt{\frac{m \omega}{h}} (F_4)^{-1} (\tilde{x}cosh(\theta) - xsinh(\theta) - \sqrt{\frac{2 \hbar}{m \omega}} a_1)].
\]
(36)
The expression Eq.(36) is just the wavefunction of the thermalized squeezed number state in the coordinate representation. When $\beta \to \infty$, the x-part of Eq.(36) is consistent with Eq.(20) in Ref. [1].

In this section, we have constructed the thermalized displaced number and squeezed number states with the thermal creation and annihilation operators, and given their wavefunctions. These states are physically meaningful. For a displaced thermalized squeezed state, Fearn and Collett gave a physical interpretation that it corresponds to the output from a linear photon amplifier whose input is a squeezed state if the amplifier’s added noise is regarded as thermal photons [8]. Thus, according to this interpretation, it is not difficult to give physical interpretations for the states in the present paper: thermalized displaced number and squeezed number states. We shall discuss the more practical situations in a separate paper. Next, we shall discuss the time evolution of the two thermalized non-classical states.

**IV. TIME-EVOLUTION OF THE THERMALIZED NON-CLASSICAL STATES**

In this section, we first consider the time-evolution of the thermalized displaced number and squeezed number states and then calculate position probability densities of them.

In thermofield dynamics, Hamiltonian $\hat{H}$ of the combined system of the physical and tilde oscillators is [16]

$$\hat{H} = H - \hat{H} = (a^\dagger a - \tilde{a}^\dagger \tilde{a}) h \omega = (a^\dagger \beta a_\beta - \tilde{a}^\dagger \beta \tilde{a}_\beta) h \omega. \quad (37)$$

Hence the time-evolution operator of the combined system is [16]

$$U(t) = \exp\left\{-\frac{i}{\hbar} \hat{H} t\right\} = \exp\left\{-i \omega t a^\dagger_\beta a_\beta\right\} \exp\left(i \omega t \tilde{a}^\dagger_\beta \tilde{a}_\beta\right) \quad (38)$$

with $t$ the time. From Eqs.(3), (16) and (17), we have

$$a_\beta = \frac{1}{\sqrt{2m\hbar \omega}}(ip_\beta + m \omega x_\beta), \quad a^\dagger_\beta = \frac{1}{\sqrt{2m\hbar \omega}}(-ip_\beta + m \omega x_\beta) \quad (39)$$

and

$$\tilde{a}_\beta = \frac{1}{\sqrt{2m\hbar \omega}}(-ip_\beta + m \omega \tilde{x}_\beta), \quad \tilde{a}^\dagger_\beta = \frac{1}{\sqrt{2m\hbar \omega}}(ip_\beta + m \omega \tilde{x}_\beta). \quad (40)$$

Thus, in the thermal coordinate representation, the time-evolution operator can be unentangled as

$$U(t) = \frac{1}{\cos(\omega t)} \exp\left(-i \frac{m \omega}{2\hbar} \tan(\omega t) x_\beta^2\right) \exp\left(-\log(\cos(\omega t)) x_\beta \frac{\partial}{\partial x_\beta}\right) \exp\left(i \frac{\hbar}{2m \omega} \tan(\omega t) \frac{\partial^2}{\partial x_\beta^2}\right) \cdot \exp\left(i \frac{m \omega}{2\hbar} \tan(\omega t) \tilde{x}_\beta^2\right) \exp\left(-\log(\cos(\omega t)) \tilde{x}_\beta \frac{\partial}{\partial \tilde{x}_\beta}\right) \exp\left(-i \frac{\hbar}{2m \omega} \tan(\omega t) \frac{\partial^2}{\partial \tilde{x}_\beta^2}\right). \quad (41)$$

The operator $U(t)$ acting on a wavefunction will yield the time-evolution of the wavefunction. With the help of Eqs.(9) and (10) and the following operator property [27]

$$\exp(C \partial_y^2) f(y) = \frac{1}{\sqrt{4\pi C}} \int_{-\infty}^{\infty} \exp\left(-\frac{(w-y)^2}{4C}\right) f(w) dw, \quad (42)$$

which can be shown by using the identity

$$e^{CQ^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\sigma^2/2} e^{\sqrt{2\sigma}Q} d\sigma,$$

we can perform the action of $U(t)$ on various thermalized non-classical states here.
The density is easy to calculate and the result is

\[
|\langle \tilde{x}, x | \alpha, n, \beta, t >= \rangle|^2 = \frac{m\omega}{\pi h} \frac{1}{2n!} \exp\left\{ \frac{\alpha^2}{A} + \frac{\alpha^*}{A^*} \cos(\omega t) \right\}
\]

with \( A = \cos(\omega t) + i\sin(\omega t) \). In finishing the relevant integral with Hermite polynomial, we used the formula 7.374 (8) in Ref. [29]. Eq.(43) indicates that the wavefunction of the thermalized squeezed number state is dependent upon time, which is different from the thermal vacuum wavefunction.

Letting \( U(t) \) act on the wavefunction Eq.(14), one can find that the wavefunction Eq.(14) is invariant and hence the wavefunction of the thermal vacuum is independent of time, i.e., \( \langle \tilde{x}, x | \alpha, n, \beta, t >= \rangle = \langle \tilde{x}, x | \alpha, n, \beta > \). This is understandable because the average value of any physical observable on the thermal vacuum is equal to its ensemble average value, which does not vary with time.

\( U(t) \) Eq.(41) acting on Eq.(26) gives the time-dependent wavefunction of the thermalized displaced number state \( \langle \tilde{x}, x | \alpha, z, n, \beta, t >= \rangle \) as

\[
\langle \tilde{x}, x | \alpha, z, n, \beta, t >= \rangle = \frac{m\omega}{\pi h} \frac{1}{2n!} \exp\left\{ \frac{\alpha^2}{A} + \frac{\alpha^*}{A^*} \cos(\omega t) \right\}
\]

\[
\cdot \exp\left\{- \frac{m\omega}{2\hbar} \left[ (x_\beta - \sqrt{\frac{2h}{m\omega}} \alpha) + (\bar{x}_\beta - \sqrt{\frac{2h}{m\omega}} \alpha^*) \right]^2 \right\}
\]

\[
\cdot H_n\left[ \frac{m\omega}{\hbar} (x_\beta - \sqrt{2}(\alpha_1 \cos(\omega t) + \alpha_2 \sin(\omega t))) \right] H_n\left[ \frac{m\omega}{\hbar} (\bar{x}_\beta - \sqrt{2}(\alpha_1 \cos(\omega t) + \alpha_2 \sin(\omega t))) \right]
\]

Eq.(43) indicates that the wavefunction of the thermalized displaced number state is dependent upon time.

Similarly, \( U(t) \) Eq.(41) acting on Eq.(35) yields the time-dependent wavefunction of the thermalized squeezed number state \( \langle \tilde{x}, x | \alpha, z, n, \beta, t >= \rangle \) as

\[
\langle \tilde{x}, x | \alpha, z, n, \beta, t >= \rangle = \frac{m\omega}{\pi h} \frac{1}{2n!} \left| \frac{F_3}{|F_3|^2} \right| B^* \cdot \exp\left\{ \frac{\alpha^2}{A} + \frac{\alpha^*}{A^*} \cos(\omega t) \right\}
\]

\[
\cdot \exp\left\{- \frac{m\omega}{2\hbar} \left[ (x_\beta - \sqrt{\frac{2h}{m\omega}} \alpha_1 + i\alpha_2) + (\bar{x}_\beta - \sqrt{\frac{2h}{m\omega}} \alpha_1 - i\alpha_2) \right]^2 \right\}
\]

\[
\cdot \exp\left\{- \frac{m\omega}{2\hbar} \left[ (x_\beta - \sqrt{\frac{2h}{m\omega}} \alpha_1 + i\alpha_2) - (\bar{x}_\beta - \sqrt{\frac{2h}{m\omega}} \alpha_1 - i\alpha_2) \right]^2 \right\}
\]

\[
\cdot H_n\left[ \frac{m\omega}{\hbar} (F_3 |B|)^{-1} (x_\beta - \sqrt{\frac{2h}{m\omega}} (\cos(\omega t) \alpha_1 + \sin(\omega t) \alpha_2)) \right]
\]

\[
\cdot H_n\left[ \frac{m\omega}{\hbar} (F_3 |B|)^{-1} (\bar{x}_\beta - \sqrt{\frac{2h}{m\omega}} (\cos(\omega t) \alpha_1 + \sin(\omega t) \alpha_2)) \right].
\]

Eq.(44) indicates that the wavefunctions of both the thermalized displaced number state and the thermalized squeezed number state are dependent upon time, which is different from the thermal vacuum wavefunction. This point is because both the thermalized displaced number state and the thermalized squeezed number state are not eigenstates of the Hamiltonian \( \hat{H} \), while the thermal vacuum is an eigenstate of the Hamiltonian \( \hat{H} \) with zero eigenvalue.

Now we can calculate the position probability densities. The probability density is the modulus square of the position-coordinate wavefunction with the tilde coordinate integrated. At the first, we consider the thermal vacuum. The density is easy to calculate and the result is

\[
\rho_\omega(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} <x, t|\tilde{x}, x|\alpha, \beta, t> d\tilde{x} = \frac{m\omega}{\pi h} \tanh \frac{\beta h\omega}{2} \exp\left\{ - \frac{m\omega}{h} \tanh \left( \frac{\beta h\omega}{2} \right) x^2 \right\}.
\]
With the help of the formula on page 225 of Ref. [30]
this is just the familiar result in statistical mechanics.

Secondly, we calculate the position probability density of the thermalized displaced number state

\[ \rho_c(x, t) \equiv \int_{-\infty}^{\infty} <t, x, \tilde{x} | x > ^* <x, \tilde{x}, t | x, \tilde{x}, t> d\tilde{x}. \]  \hspace{1cm} (46)

Substituting Eqs. (12), (13) and (43) into Eq. (46) and reducing it, we have

\[ \rho_c(x, t) = \frac{m\omega}{\pi \hbar} \left( \frac{1}{2^{n!}n!} \right)^2 \int_{-\infty}^{\infty} \exp\left\{ -(a_1\tilde{x} + a_2)^2 - (b_1\tilde{x} + b_2)^2 \right\} \left( H_n[a_1\tilde{x} + a_2]^2 \right) \left( H_n[-b_1\tilde{x} - b_2]^2 \right) d\tilde{x} \]  \hspace{1cm} (47)

where,

\[ a_1 = \sqrt{\frac{m\omega}{\hbar}} \cosh(\theta), \quad a_2 = -\sqrt{\frac{m\omega}{\hbar}} \sinh(\theta) - \sqrt{2}(\alpha_1 \cos(\omega t) + \alpha_2 \sin(\omega t)) \]
\[ b_1 = \sqrt{\frac{m\omega}{\hbar}} \sinh(\theta), \quad b_2 = -\sqrt{\frac{m\omega}{\hbar}} \cosh(\theta) + \sqrt{2}(\alpha_1 \cos(\omega t) + \alpha_2 \sin(\omega t)). \]

With the help of the formula on page 225 of Ref. [30]

\[ H_m[x]H_n[x] = \sum_{r=0}^{\min(m,n)} 2^r r! C_m^r C_n^r H_{m+n-2r}[x] \]  \hspace{1cm} (48)

with \( C_n^r \) and \( C_m^r \) being combinations, Eq. (47) can be written as

\[ \rho_c(x, t) = \frac{m\omega}{\pi \hbar a_1} \left( \frac{1}{2^{n!}n!} \right)^2 \sum_{j,k=0}^{n} 2^{j+k+j!k!} (C_j^j C_k^k)^2 \int_{-\infty}^{\infty} \exp\left\{ -y^2 - (ay + b)^2 \right\} H_{2n-2j}[y] H_{2n-2k}[ay + b] dy \]  \hspace{1cm} (49)

with \( a = \frac{\beta}{a_1} \) and \( b = -aa_2 + b_2 \). Using repeatedly the formula

\[ e^{-x^2} H_n[x] = -\frac{d}{dx} \{ e^{-x^2} H_{n-1}[x] \} \]

and then the formula 7.374(8) in Ref. [27], one can obtain

\[ \rho_c(x, t) = \frac{m\omega}{\pi \hbar a_1} \left( \frac{1}{2^{n!}n!} \right)^2 \sqrt{\frac{\pi}{a^2 + 1}} \exp\left\{ -\frac{b^2}{a^2 + 1} \right\} \]
\[ \cdot \sum_{j,k=0}^{n} 2^{j+k+j!k!} (C_j^j C_k^k)^2 a^{2(n-j)}(a^2 + 1)^{j+k-2n} H_{2(2n-j-k)} \left( \frac{b}{\sqrt{a^2 + 1}} \right) \]  \hspace{1cm} (50)

\[ = \sqrt{\frac{m\omega}{\pi \hbar}} \left( \frac{1}{2^{n!}n!} \right)^2 \tan h^2 \left( \frac{\beta \hbar \omega}{2} \right) \]
\[ \cdot \exp\left\{ -\left[ \sqrt{\frac{m\omega}{\hbar}} \tan h \left( \frac{\beta \hbar \omega}{2} \right) x - \sqrt{2}(\alpha_1 \cos(\omega t) + \alpha_2 \sin(\omega t)) \right] \sqrt{1 + \text{sech}^2 \left( \frac{\beta \hbar \omega}{2} \right)} \right\} \]
\[ \cdot \sum_{j,k=0}^{n} 2^{j+k-2n} j! k! (C_j^j C_k^k)^2 \left( \frac{1}{2} (j-k) \hbar \omega \beta \right) \left( \cosh \left( \frac{\beta \hbar \omega}{2} \right) \right)^{j+k-2n} H_{2(2n-j-k)} \left[ \sqrt{\frac{m\omega}{\hbar}} \tan h \left( \frac{\beta \hbar \omega}{2} \right) x - \sqrt{2}(\alpha_1 \cos(\omega t) + \alpha_2 \sin(\omega t)) \right] \sqrt{1 + \text{sech}^2 \left( \frac{\beta \hbar \omega}{2} \right)} \]  \hspace{1cm} (51)

As for the thermalized squeezed number state, the calculation of the position probability density is completely similar to that of the thermalized displaced number state. Finishing calculations similar to the above, one can have the position probability density of the thermalized squeezed number state Eq. (44).
\[ \rho_s(x, t) \equiv \int_{-\infty}^{\infty} (|\alpha, z, n, \beta, t\rangle) \langle \alpha, z, n, \beta, t| \, d\bar{x} \]

\( = \frac{m\omega}{\pi \hbar a_1} (\frac{1}{2^{n!}})^2 \sqrt{\frac{\pi}{a^2 + 1}} \bar{F}_4 B^* \exp\{ - \frac{b^2}{(a^2 + 1) \bar{F}_4 B^*} \} \times \sum_{j, k=0}^{n} 2^{j+k} j! k!(C_n^j C_n^k)^2 a^{2(n-j)} (a^2 + 1)^{j+k-2n} H_{2(2n-j-k)} \left( \frac{b}{\sqrt{a^2 + 1} \bar{F}_4} \right) \]  

(53)

When \( \beta \to \infty \), the existence of the factor \( a^{2(n-j)} \) enforces the summation index \( j \) to have a unique value \( n \) because \( a = 0 \). Thus, employing Eq.(48), one has

\[ \sum_{j, k=0}^{n} 2^{j+k} j! k!(C_n^j C_n^k)^2 a^{2(n-j)} (a^2 + 1)^{j+k-2n} H_{2(2n-j-k)} \left( \frac{b}{\sqrt{a^2 + 1} \bar{F}_4} \right) = 2^n n! \left( \frac{b}{\sqrt{a^2 + 1} \bar{F}_4} \right)^2, \]  

and hence Eq.(54) with \( \beta \to \infty \) can give Eq.(50) in Ref. [1]. Meanwhile, the probability density Eq.(51) with \( \beta \to \infty \) can also lead to Eq.(50) with \( z = 0 \) in Ref. [1].

For a given time, for example, \( t = 0 \), one can get the position probability densities \( \rho_c(x) \) and \( \rho_s(x) \) without considering the time evolution from Eqs.(51) and (54). Making a contrast between \( \rho_c(x) \) and \( \rho_s(x, t) \) as well as \( \rho_s(x) \) and \( \rho_s(x, t) \), one can find that just the displacement and squeeze parameters in the densities experience changes with the evolution of time. That is, for the thermalized displaced number state, the real part of the displacement parameter \( \alpha \) becomes \( \alpha_1 \cos(\omega t) + i\alpha_2 \sin(\omega t) \), which is similar to that in Ref. [22] (1965), and for the thermialized squeezed number state, besides the same change of \( \alpha_1 \), the parameter \( \bar{F}_4 \) becomes \( F_4[B] \).

The expressions of both \( \rho_c(x, t) \) and \( \rho_s(x, t) \) are complicated, but one can easily prove that each of them is normalized, because only the term with \( j = k = n \) is not zero when \( \rho_c(x, t) \) or \( \rho_s(x, t) \) is integrated with respect to \( x \). Furthermore, one can calculate the average value of the position coordinate \( x \) on the thermalized squeezed number state as

\[ < x > \equiv \int_{-\infty}^{\infty} x \rho_s(x, t) \sqrt{\coth(\frac{\beta\hbar \omega}{4})}\left( \frac{2h}{m\omega} \right) (\alpha_1 \cos(\omega t) + i\alpha_2 \sin(\omega t)) . \]  

(55)

Evidently, \( < x > \) is independent of \( n \) and the squeeze parameter \( z \), which is similar to the result of the squeezed number state [1]. Eq.(55) indicates that the average value of \( x \) on any thermalized squeezed number state follows the motion of a classical harmonic oscillator, and the amplitude of the oscillation increases with the increase of the temperature. When \( \beta \to \infty \), Eq.(55) is consistent with Eq.(36) in Ref. [1], and contrasting to Eq.(36) in Ref. [1], Eq.(55) has just an additional temperature factor \( \sqrt{\coth(\frac{\beta\hbar \omega}{4})} \).

Besides, one also easily obtain the variance of \( x \) on a thermalized squeezed number state as

\[ (\Delta_n x)^2 \equiv \int_{-\infty}^{\infty} (x-< x >)^2 \rho_s(x, t) = \coth(\frac{\beta\hbar \omega}{2})(2n+1) \frac{hF_4[B]^2}{2m\omega} . \]  

(56)

When \( \beta \to \infty \), \( (\Delta_n x)^2 \) is consistent with Eq.(38) in Ref. [1]. When \( n = 0 \), \( (\Delta_0 x)^2 \) is consistent with Eq.(15a) in Ref. [19] ( the second paper ). A comparison of Eq.(56) with Eq.(38) in Ref. [1] tells us that \( (\Delta_n x)^2 \) just has an additional temperature factor \( \coth(\frac{\beta\hbar \omega}{2}) \).

Finally, we give two examples of \( \rho_s(x, t) \) to end this section. For \( n = 0 \), we have

\[ \rho_s(x, t) = \sqrt{\frac{1}{2\pi(\Delta_0 x)^2}} \exp\{ - \frac{(x-< x >)^2}{2(\Delta_0 x)^2} \} . \]  

(57)
Using the relation between the thermalized squeezed state and squeezed thermalized state in Ref. [18](1991), one can find that the last equation with $t = 0$ is consistent with Eq.(6.6a) in Ref. [20](1993). For $n = 1$, we have

$$\rho_s(x, t) = \sqrt{\frac{2}{\pi (\Delta_0 x)^2}} \exp\left\{-\frac{(x-<x>)^2}{2(\Delta_0 x)^2}\right\} \left\{ \frac{1}{2} \text{sech}^2\left(\frac{\beta \hbar \omega}{2}\right) \left[ (x-<x>)^2 - \frac{3}{2} \right] + \frac{3}{2} \right\} . \quad (58)$$

From the last equation, we see that by introducing a finite temperature effect, $x$-polynomial factor in the expression of $\rho_s(x, t)$ is not just the quadratic term of $(x-<x>)$ as Eq.(52) in Ref. [1], but have additional quadruplicate term of $(x-<x>)$. The appearance of the quadruplicate term is understandable because thermalizing a nonclassical state amounts to doubling the freedom number of the systems within the framework of thermofield dynamics [16].

From Eq.(55) to the last equation, we give some explicit results only about the thermalized squeezed number state. As for the thermalized displaced number state, taking $z = 0$ in Eqs.(55)—(58), one can obtain the corresponding results about them, which are also consistent with those in the literature.

V. CONCLUSION

In this paper, we have given the wavefunctions of the thermalized displaced number and squeezed number states in the coordinate representation. Furthermore, with the help of the thermal coordinate representation, we obtain the time-dependent expressions of these wavefunctions. Although the thermal vacuum wavefunction is time-independent, but either the thermal displaced number state or the thermalized squeezed number state varies with time. We also give the probability densities, average values and variances of the position coordinate on these states. Each of the wavefunctions, the time-dependent wavefunctions, the probability densities, average values or variances here is consistent with that in the literature when the temperature tends to zero. Setting $n = 0$ in the expressions of this paper, one can obtain results of the usual thermalized coherent and squeezed states. Additionally, setting $t = 0$, one can obtain the probability densities without considering the time evolution of these states.

In the thermal coordinate representation, the forms of the wavefunctions Eqs.(14),(26),(35), (43) and (44) resemble their own zero-temperature limits. Of course, this resemblance does not exist in the coordinate representation at all, and the probability densities are different from those at the zero-temperature case. From section IV, one has seen that the thermal coordinate representation simplified greatly the calculations there. Perhaps, the thermal coordinate representation would simplify other calculations related to the thermal non-classical states.

Finally, although the thermalized displaced number and squeezed number states are discussed, the above results of them can give easily the corresponding ones of other similar states, such as displaced thermalized number state, squeezed thermalized number state, etc., with a simple parameter transformation [18](1991) [20](1993). Additionally, no matter how complicate the expressions (51) and (54) are, it is not difficult to compute them numerically for a given number $n$. We believe that once the displaced number state and squeezed number state is prepared in laboratories some day, the results in the present paper will be useful.

ACKNOWLEDGMENTS

This project was supported jointly by the President Foundation of Shanghai Jiao Tong University and the National Natural Science Foundation of China with grant No. 19875034.

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