Abstract.
This is a revised variant of my preprint:
Max-Plank Institut fuer Mathematik, Bonn, 2001, No 16.

Foreword
This is the first part of the detailed version of my work “On linear forms with
coefficients in \(\mathbb{N}\zeta(1 + \mathbb{N})\),” [11]. The goal of this work is to give a complete proof
of the following theorem.

Theorem. Let

\[
\phi_i(x_1, x_2) = \sum_{k=1}^{2} (2ik - i - k + 2) \zeta(i + k + 1) x_k, \quad i = 1, 2.
\]

For any \(d \in \mathbb{R}\), let \(\|d\|\) be the distance between \(d\) and \(\mathbb{Z}\); let \(\gamma = 43, 464412\). There
is a positive constant \(c\) such that

\[
\|\phi_1(x_1, x_2)\| + \|\phi_2(x_1, x_2)\| \geq c(|x_1| + |x_2|)^{-\gamma},
\]

where \(x_1 \in \mathbb{Z}\), \(x_2 \in \mathbb{Z}\), \(|x_1| + |x_2| > 0\).

Remark. The lattices \(L_1, L_2\) in \(\mathbb{R}^n\) are said to be incommensurable, if \(L_1 \cap L_2 = \{0\}\).
The qualitative part of the theorem asserts that lattice generated by the vectors
\(f_1 = (2\zeta(3), 3\zeta(4))\) and \(f_2 = (3\zeta(4), 6\zeta(5))\) and the lattice \(\mathbb{Z}^2\) (generated by the
vectors \((1, 0)\) and \((0, 1)\)) are incommensurable.
Corollary. If \( p \in \mathbb{Q}, q \in \mathbb{Q}, \) and \( p^2 + q^2 > 0, \) then
\[
\{2p \zeta(3) + 3q \zeta(4), 3p \zeta(4) + 6q \zeta(5)\} \not\in \mathbb{Q},
\]
and therefore
\[
\left\{ \frac{\zeta(3 + 2k)}{\zeta(4)}, \frac{12 \zeta(3) \zeta(5) - 9 \zeta(4)^2}{\zeta(4)} \right\} \not\in \mathbb{Q}, \quad k = 0, 1.
\]

In the first part, I give a short survey of the C. S. Mejer functions, and then define and compute my auxiliary functions. In what follows, given \( a \in \mathbb{C} \) and \( M \subseteq \mathbb{C}, M \neq \emptyset, \) let
\[
a + M = M + a = \{ w \in \mathbb{C} : w = a + z, \ z \in M \}.
\]

Acknowledgment. I am obliged to Professor A.G.Aleksandrov and to Professor B.Z.Moroz for their help and support.

§ 1. Short survey of the Mejer functions

In this work, I use the functions introduced and studied by C. S. Mejer in a long series of papers published during the decade 1936 – 1946. The functions of Mejer can be defined as follows ([9], ch. 5). Let
\[
G_{p,q}^{(m,n)} \left( \begin{array}{c} a_1, \ldots, a_p \\ a_1, \ldots, a_p \end{array} \right) = \frac{1}{2\pi i} \int_L \frac{z^s \prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} \ ds,
\]
where an empty product is, by definition, equal to 1, \( 0 \leq m \leq q, 0 \leq n \leq p; \) the parameters \( a_j \in \mathbb{C}, j = 1, \ldots, p, \) and \( b_k \in \mathbb{C}, k = 1, \ldots, q, \) are chosen in such a way that none of the poles of the function \( \Gamma(b_k - s), k = 1, \ldots, m, \) is equal to a pole of one of the functions \( \Gamma(1 - a_j + s), j = 1, \ldots, n. \) There are 3 possibilities to choose the curve \( L. \)

(A) First, the curve \( L = L_0 \) may be chosen to pass from \(-i\infty\) to \(+i\infty\) in such way that for any \( k = 1, \ldots, m \) all the poles of the function \( \Gamma(b_k - s) \) lie to the right of it and for any \( j = 1, \ldots, n \) all the poles of the function \( \Gamma(1 - a_j + s) \) lie to the left of it. The integral (1.1) is convergent in either of the following two cases:

(A1) \(|\arg(z)| < (m + n - \frac{p}{2} - \frac{q}{2})\pi;\)
(A2) \(|\arg(z)| \leq (m + n - \frac{p}{2} - \frac{q}{2})\pi\) and \(\frac{p-q}{2} + \text{Re} \; \Delta^* < -1\), where

\begin{equation}
\Delta^* = \sum_{k=1}^{q} b_k - \sum_{k=1}^{q} a_j.
\end{equation}

(B) Second, the curve \(L = L_1\) may be chosen to pass from \(+i\infty\) to \(+i\infty\), encircling each of the poles of the functions \(\Gamma(b_k - s), \; k = 1, \ldots, m\), in the negative direction, but not including any of the poles of the functions \(\Gamma(1 - a_j + s), \; j = 1, \ldots, n\). The integral (1.1) is convergent in each of the following three cases:

(B1) \(p < q\);
(B2) \(1 \leq p \leq q\) and \(|z| < 1\);
(B3) \(1 \leq p \leq q\), \(|z| \leq 1\) and \(\text{Re} \; \Delta^* < -1\).

(C) Third, the curve \(L = L_2\) may be chosen to pass from \(-\infty\) to \(-\infty\), encircling each of the poles of the functions \(\Gamma(1 - a_j + s), \; j = 1, \ldots, n\), in the positive direction, but not including any of the poles of the functions \(\Gamma(b_k - s), \; k = 1, \ldots, m\). The integral (1.1) is convergent in each of the following 3 cases:

(C1) \(q < p\);
(C2) \(1 \leq q \leq p\) and \(|z| > 1\);
(C3) \(1 \leq q \leq p\), \(|z| \geq 1\) and \(\text{Re} \; \Delta^* < -1\).

If both conditions (A) and (B) (respectively (A) and (C)) are satisfied, then the result does not depend on whether the curve \(L\) is defined as in (A) or as in (B) (respectively in (C)).

Let \(g\) denote the integrand of the integral (1.1), let \(G\) denote the integral (1.1) with \(L = L_k\), where \(k = 1, 2\), and let \(S_k\) be the set of all the unremovable singularities of \(g\) encircled by \(L_k\). If one of the conditions (B1) – (B3) (respectively (C1) – (C3)) holds for \(k = 1\) (respectively for \(k = 2\)), then

\begin{equation}
G = (-1)^k \sum_{s \in S_k} \text{Res} \; (g; s),
\end{equation}

where \(\text{Res} \; (g; s)\) stands for the residue of the function \(g\) at the point \(s\). Let us prove these assertions. Let \(\sigma_0 > 0\), \(\tau_0 > 0\), and let

\begin{equation}
\min(\sigma_0, \tau_0) \geq 1 + \sum_{j=0}^{p} 2(|a_j| + 1) + \sum_{k=0}^{q} 2(|b_k| + 1).
\end{equation}

The curve \(L_0\) may be chosen in such a way that, except for a compact piece, it coincides with a part of the imaginary axis \(|\text{Im} \; s| \geq \tau_0\); the curves \(L_1\) and \(L_2\)
may be chosen so that, except for a compact part, the curve \( L_1 \) coincides with the union of the two rays \( \sigma \pm i\tau_0, \sigma \geq \sigma_0 \), and the curve \( L_2 \), except for a compact part, coincides with the union of the two rays \( \sigma \pm i\tau_0, \sigma \leq -\sigma_0 \).

Let \( r_0 \geq \sigma_0 + \tau_0 \) be chosen big enough, so that the specified compact parts of \( L_0, L_1 \) and \( L_2 \) lie inside the disk \( K_0 = \{ s \in \mathbb{C} : |s| \leq r_0 \} \).

Let \( u \geq 0, \eta(u) = \int_0^u ([t] - t + \frac{1}{2}) dt; \) then \( 0 \leq \eta(u) \leq 1/8 \). Let

\[
\mathcal{D}_0 = \{ s \in \mathbb{C} : |s| \geq r_0 \} \backslash \{ s \in \mathbb{C} : \text{Re } s < 0, \ |\text{Im } s| < \tau_0 \}.
\]

If \( z = x + iy, \ |y| \geq \tau_0 \), then the integral in the complex Stirling formula ([10], ch. 4)

\[
\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log(2\pi) + \int_0^{\infty} \frac{\eta(u)}{(u + z)^2} du
\]

can be estimated as follows:

\[
\left| \int_0^{\infty} \frac{\eta(u)}{(u + z)^2} du \right| \leq \int_0^{\infty} \left| \frac{\eta(u)}{(u + z)^2} \right| du =
\]

\[
\int_0^{\infty} \frac{\eta(u)}{(u + x)^2 + y^2} du < \int_{-\infty}^{\infty} \frac{\eta(u)}{(u + x)^2 + y^2} du < \frac{\pi}{8\tau_0} = O(1). 
\]

If \( z = x + iy \) and \( x \geq \sigma_0 \), then

\[
\left| \int_0^{\infty} \frac{\eta(u)}{(u + z)^2} du \right| \leq \int_0^{\infty} \left| \frac{\eta(u)}{(u + z)^2} \right| du =
\]

\[
\int_0^{\infty} \frac{\eta(u)}{(u + x)^2 + y^2} du \leq \int_0^{\infty} \frac{\eta(u)}{(u + x)^2} du < \frac{1}{8\sigma_0} = O(1). 
\]

Therefore

(1.5) \[
\log \Gamma(s + c) = (s + c - \frac{1}{2})(\log s - 1) + O(1)
\]

in \( \mathcal{D}_0 \), for any constant \( c \) with \( 2|c| \leq \min\{\sigma_0, \tau_0\} \).

Let \( s = \sigma + i\tau = |s| e^{i\psi} \) with \( |\psi| \leq \pi, \ \phi = \text{arg } z \), then

(1.6) \[
\text{Re } \log \Gamma(s + c) = (\sigma + \text{Re } c - \frac{1}{2})(\log |s| - 1) - \tau \psi + O(1).
\]
One may write \( g(s) = g_1(s)g_2(s) \) with

\[
(1.7) \quad g_1(s) = z^s \prod_{j=1}^{p} \Gamma(1-a_j+s)/ \prod_{j=1}^{q} \Gamma(1-b_j+s),
\]

\[
(1.8) \quad g_2(s) = \prod_{j=n+1}^{p} \frac{\sin(\pi(a_j-s))}{\pi} \prod_{j=1}^{m} \frac{\pi}{\sin(\pi(b_j-s))}.
\]

Since \( \tau \psi \geq 0 \), it follows from (1.6) that

\[
(1.9) \quad g_1(s) = O(1) \cdot |z|^\sigma e^{\phi\|\tau\|}(\|s/e\|^{(p-q)(\sigma+\frac{1}{2})+\Re \Delta^*}) e^{-\tau(p-q)\psi} =
\]

\[
= O(1) \cdot |z|^\sigma (\|s/e\|^{(p-q)(\sigma+\frac{1}{2})+\Re \Delta^*}) e^{\tau(p-q)(\frac{1}{2}-|\psi|)+\tau|\phi-(p-q)\frac{1}{2}|}
\]

in \( \overline{D}_0 \). In view of our choice of \( \tau_0 \), it follows that

\[
|\sin(\pi(a_j-s))| = O(1) \cdot e^{\pi|\tau|}, \quad |\sin(\pi(b_k-s))|^{-1} = O(1) \cdot e^{-\pi|\tau|}
\]

for \( s \in \overline{D}_0, \ j = 1, \ldots, p \) and \( k = 1, \ldots, q \), and consequently

\[
(1.10) \quad g_2(s) = O(1) \cdot e^{\pi(p-m-n)|\tau|}.
\]

Therefore, according to (1.9) and (1.10) the following equality

\[
(1.11) \quad |g(s)| = O(1) \cdot |z|^\sigma e^{\phi\|\tau\|}(\|s/e\|^{(p-q)(\sigma+\frac{1}{2})+\Re \Delta^*}) e^{-|\tau|(p-q)\psi} e^{\pi(p-m-n)|\tau|} =
\]

\[
= O(1) \cdot |z|^\sigma (\|s/e\|^{(p-q)(\sigma+\frac{1}{2})+\Re \Delta^*}) e^{((p-q)(\frac{1}{2}-|\psi|)+|\phi|+\pi(\frac{p+q}{2}-m-n))|\tau|}
\]

holds in \( \overline{D}_0 \). According to (1.11), on the part of the curve \( L_0 \) lying on the rays \( \sigma = 0, |\tau| \geq \tau_0 \) we have the equality

\[
|g(s)| = O(1) \cdot (\|\tau/e\|^{\frac{p+q}{2}+\Re \Delta^*}) e^{\pi(\|\phi\|+\pi(\frac{p+q}{2}-m-n))},
\]

if the condition (A1) holds, then

\[
|g(s)| = |\tau|^{O(1)} e^{\pi(\|\phi\|+\pi(\frac{p+q}{2}-m-n))},
\]

where \( |\tau| \geq \tau_0 \), and the integral (1.1) is convergent; if the condition (A2) holds, then

\[
|g(s)| = O(1) \cdot (\|\tau/e\|^{\frac{p+q}{2}+\Re \Delta^*}),
\]
where $|\tau| \geq \tau_0$, and the integral (1.1) is convergent. Let the number $\sigma_0$ be chosen so that

$$\sigma_0 > 3 + \left| \frac{p-q}{2} + \text{Re} \Delta^* \right|. \quad (1.12)$$

If one of the condition (B), (C) holds, then

$$\sigma_0 > 3 + \left| p-q \right| + \left| \text{Re} \Delta^* \right|. \quad (1.12)$$

$$\frac{p-q}{2} - |\psi| = -|p-q| \left( \frac{\pi}{2} - |\psi| \right), \quad (1.14)$$

$$\log |z| = -|\sigma| |\log |z|| \quad \text{in } \overline{D_0},$$

and (1.9), (1.11) takes respectively the form

$$\left| g_1(s) \right| = O(1) \cdot e^{-|\sigma||\log |z||} \left( |s|/e \right)^{-|\sigma|} e^{-|p-q| + \frac{p-q}{2} + \text{Re} \Delta^*} \times \quad (1.16)$$

$$e^{-|p-q||\frac{\pi}{2} - |\psi|||\tau|} \left| e^{|\phi-(p-q)\frac{\pi}{2}|} \right|.$$
and the integral (1.1) is convergent. Finally, if one of condition (B3), (C3) holds, then, in view of (1.17),

\[(1.22)\]
\[|g(s)| = O(1) \cdot |\sigma|^\Re \Delta^*\]
on the rays \(\tau = \tau_0, \; |\sigma| > \sigma_0,\) and the integral (1.1) is convergent. For any \(r \geq r_0\) and \(k = 0, 1, 2\) let us denote by \(L_{r,k}\) the part of the curve \(L_k\) contained in the disk \(K_r = \{s \in \mathbb{C} : |s| \leq r\}\) and directed as the curve \(L_k\). Let further \(C_{r,k,j}\), where \(k = 1, 2, j = 0, 1,\) be the smallest counter-clockwise directed arc of the circle \(|s| = r\) connecting the point \(ir(-1)^j\) with the point \(i\tau_0(-1)^j - (-1)^k \sqrt{r^2 - \tau_0^2}\). Finally, let \(L^*_{r,k}\), where \(k = 1, 2,\) be the positively directed curve consisting of the curves \(L_{0,r}, L_{k,r}, C_{r,k,0}\) and \(C_{r,k,1}\). Clearly,

\[(1.23)\]
\[\int_{L_k} g(s) = \lim_{r \to \infty} \int_{L_{r,k}} g(s)ds\]
for \(k = 0, 1, 2\). The function \(g(s)\) has no singular points in the domain bounded by the curve \(L^*_{k}\). Therefore

\[(1.24)\]
\[\int_{L_{r,k}} g(s)ds - (-1)^k \int_{C_{r,k,0}} g(s)ds - \int_{L_{r,0}} g(s)ds - (-1)^k \int_{C_{r,k,1}} g(s)ds = \int_{L^*_{k}} g(s)ds = 0.\]

For \(k = 1, 2\) and \(j = 0, 1,\) let \(\hat{C}_{r,k,j}\) be the counter-clockwise directed part of the arc \(C_{r,k,j}\) lying in the strip \(|\sigma| \leq \sigma_0,\) and let \(\hat{C}_{r,k,j}\) be the counter-clockwise directed part of the arc \(C_{r,k,j}\) lying outside this strip. If \(p \neq q\) and one of conditions (B1), (C1) holds, then according to (1.19)

\[(1.25)\]
\[|g(s)| = O(1) \cdot (r/e)^{-3}\]
on the \(\hat{C}_{r,k,j}\) for \(k = 1, 2\) and \(j = 0, 1,\) and, because the length of the arc \(\hat{C}_{r,k,j}\) is not bigger than \(\pi r/2\), it follows that

\[(1.26)\]
\[\lim_{r \to \infty} \int_{\hat{C}_{r,k,j}} g(s)ds = 0.\]

Let \(\gamma = 4^{-1} \min\{|\phi| + \pi(p + q)/2 - m - n|, |\log|z||\}\). In view of (1.17), if \(p = q\) and one of conditions (B), (C) holds, then

\[(1.27)\]
\[|g(s)| = O(1) \cdot e^{-\gamma r}(r/e)^{\Re \Delta^*}\]
on the \( \hat{C}_{r,k,j} \) for \( k = 1, 2 \) and \( j = 0, 1 \); now, if the condition (A) is satisfied, then the equality \( \gamma = 0 \) implies the inequality \( \text{Re} \left( \Delta^* \right) < -1 \), and because the length of the arc \( \hat{C}_{r,k,j} \) is not bigger than \( \pi r/2 \), relation (1.26) still holds. Moreover,

\[ |\tau| = \sqrt{r^2 - \sigma_0^2} = r - O(1), \quad |\psi| = \frac{\pi}{2} - O(\arcsin(\sigma_0/r)) = \frac{\pi}{2} - O(1/r) \]

on the \( \hat{C}_{r,k,j} \) for \( k = 1, 2 \), \( j = 0, 1 \). Therefore if one of conditions (B), (C) holds, then according to (1.17)

\[(1.28) \quad |g(s)| = O(1) \cdot \left( \frac{r}{e} \right)^{\frac{p}{2} - q} + \text{Re} \Delta^* e^{(\phi + \pi(\frac{p+q}{2} - m - n))|\tau|} = O(1) \cdot \left( \frac{r}{e} \right)^{\frac{p}{2} - q} + \text{Re} \Delta^* e^{(\phi + \pi(\frac{p+q}{2} - m - n)r} \]

on the curve \( \hat{C}_{r,k,j} \). Consequently, if the condition (A1) and one of the condition (B), (C) hold, then

\[(1.29) \quad \lim_{r \to \infty} \int_{\hat{C}_{r,k,j}} g(s) ds = \lim_{r \to \infty} O(1) \cdot e^{(\phi + \pi(\frac{p+q}{2} - m - n)r} O(1) = 0 \]

because in this case \( |\phi| + \pi(\frac{p+q}{2} - m - n) < 0 \). If the condition (A2) holds, then

\[(1.30) \quad \lim_{r \to \infty} \int_{\hat{C}_{r,k,j}} g(s) ds = \lim_{r \to \infty} O(1) \cdot r^{\frac{p}{2} - q} + \text{Re} \Delta^* + 1 = 0. \]

We have thus proved that if both of the convergence conditions (A) and (B) hold, then

\[ \int_{\hat{L}_0} g(s) ds = \int_{\hat{L}_1} g(s) ds; \]

and if both of the convergence conditions (A) and (C) hold, then

\[ \int_{\hat{L}_0} g(s) ds = \int_{\hat{L}_2} g(s) ds. \]

Now I shall prove the equality (1.3). Let us choose \( \sigma_1 \) so that it is bigger than \( r_0 \) and lies outside of the countable set

\[ \bigcup_{\mu=0}^{1} \bigcup_{j=1}^{p} ((-1)^{\mu} \text{Re} (a_j) + \mathbb{Z}) \bigcup_{\nu=0}^{1} \bigcup_{k=1}^{q} ((-1)^{\nu} \text{Re} (b_k) + \mathbb{Z}). \]
For $k = 1, 2$ and $n \in \mathbb{N} - 1$, let $r(n) = \sqrt{\tau_0^2 + (\sigma_1 + n)^2}$, let $c_{n,k}$ denote the segment 

$$[-(-1)^k(\sigma_1 + n) - i\tau_0, -(-1)^k(\sigma_1 + n) + i\tau_0],$$

and let $\tilde{L}_{n,k}$ be the counter-clockwise directed curve consisting of the curve $L_{r(n),k}$ and the segment $c_{n,k}$; finally, let $\tilde{S}_{n,k}$ denote the set of all the unremovable singularities encircled by the curve $\tilde{L}_{n,k}$. It is clear that

$$\frac{1}{2\pi i} \int_{\tilde{S}_{n,k}} g(s)ds = \sum_{s \in \tilde{S}_{n,k}} \text{Res}(g;s),$$

where $k = 1, 2$. The equality (1.3) will be proved if we establish that for each $k = 1, 2$ the equality

$$\lim_{n \to \infty} \int_{c_{n,k}} g(s)ds = 0$$

holds. First, consider the case $k = 1$. In view of the choice of $\sigma_1$, the function $g_2(s)$ is continuous on the segment $c_{0,k}$, and therefore there is a constant $M_0$ with $|g_2(s)| \leq M_0$ for $s \in c_{0,k}$, where $k = 1, 2$. Since the period of the function $|g_2(s)|$ is equal to one, we have

$$\lim_{n \to \infty} \int_{c_{n,k}} |g_2(s)|ds = 0$$

for $s \in c_{n,k}$, $k = 1, 2$, and $n \in \mathbb{N}$. Let us first consider the case $k = 1$. If condition (B) holds, then (1.31) follows from (1.16). Suppose now that condition (C) is satisfied. Write

$$g_1(s) = g_3(s)g_4(s)$$

with

$$g_3(s) = z^s \prod_{j=1}^{q} \frac{\Gamma(b_j - s)}{\Gamma(a_j - s)},$$

and

$$g_4(s) = \left( \prod_{j=1}^{p} \frac{\sin(\pi(a_j - s))}{\pi} \prod_{j=1}^{q} \frac{\pi}{\sin(\pi(b_j - s))} \right)^{-1}.$$
Since the period of the function $|g_4(s)|$ is equal to one, there is a constant $M_2$ such that

\begin{equation}
|g_4(s)| \leq M_2
\end{equation}

for $s \in c_{n,2}$, $n \in \mathbb{N} - 1$, in view of the choice of $\sigma_1$. Because $|\tau| \leq \tau_0$ if $s \in c_{n,2}$ and $-c_{n,2}$ lies in $\overline{D_0}$, $n \in \mathbb{N} - 1$, it follows from (1.6) that

\begin{align*}
g_1(s) &= O(1) \cdot |z|^{\sigma} \left( |s|/e \right)^{\sum_{j=1}^N \text{Re}(b_j - \sigma - \frac{1}{2}) - \sum_{j=1}^N \text{Re}(b_j - \sigma - \frac{1}{2})} \\
&= O(1) \cdot |z|^{\sigma} \left( |s|/e \right)^{(p-q)(\sigma + \frac{1}{2}) + \text{Re} \Delta^*}
\end{align*}

with $|s| \geq -\sigma = \sigma_1 + n$. Therefore the equality (1.31) holds also for $k = 2$. This proves (1.3).

§ 2. My auxiliary functions

I shall work with the set $\Omega^*$ consisting of all the points $z \in \mathbb{C}$ for which

\begin{equation}
|z| \geq 1 \quad \text{and} \quad -\frac{3\pi}{2} < \arg(z) \leq \frac{\pi}{2}.
\end{equation}

Thus $\log(-z) = \log z - i\pi$ for $\text{Re}(z) > 0$. Let $\Delta \in \mathbb{N}$, $1 < \Delta$, $\delta_0 = 1/\Delta$,

\begin{equation}
\gamma_1 = (1 - \delta_0)/(1 + \delta_0), \quad d_l = \Delta + (-1)^l, \quad l = 1, 2.
\end{equation}

To introduce the first of my auxiliary function $f_1(z, \nu)$, I use the auxiliary set

\begin{equation}
\Omega^{(h)} = \{z \in \mathbb{C} : |z| \leq 1, -\frac{3\pi}{2} < \phi = \arg(z) \leq \frac{\pi}{2}\}.
\end{equation}

I shall prove that, for each $\nu \in \mathbb{N}$, the function $f_1(z, \nu)$ belongs to $\mathbb{Q}[z]$; therefore using the principle of analytic continuation we may regard it as being defined in $\mathbb{C}$, and consequently, in $\Omega^*$. For $\nu \in \mathbb{N}$, let

\begin{equation}
f_1(z, \nu) = \left(-1\right)^{\nu(\Delta+1)} G_{6,6}^{(1,3)} \left( \begin{array}{cccccc}
-\nu d_1, & -\nu d_1, & -\nu d_1, & 1 + \nu d_2, & 1 + \nu d_2, & 1 + \nu d_2 \\
0, & 0, & 0, & \nu, & \nu, & \nu \\
\end{array} \right) = \left(-1\right)^{\nu(\Delta+1)} \frac{1}{2\pi i} \int_{L_1} g_{6,6}^{(1,3)}(s) ds,
\end{equation}
where

\[ g_{6,6}^{(1,3)}(s) = z^s \Gamma(-s) \Gamma(1+s)^{-2} \left( \Gamma(1+\nu d_1 + s) / (\Gamma(1 - \nu + s) \Gamma(1 + \nu d_2 - s)) \right)^3, \]

and the curve \( L_1 \) passes from \( +\infty \) to \(+\infty\) encircling the set \( \mathbb{N} - 1 \) in the negative direction, but not including any point of the set \( -\mathbb{N} \). Here \( p = q = 6 \), \( m = 1 \), \( n = 3 \), \( a_1 = a_2 = a_3 = -\nu d_1 \), \( a_4 = a_5 = a_6 = 1 + \nu d_2 \), \( b_1 = b_2 = b_3 = 0 \), \( b_4 = b_5 = b_6 = \nu \), \( \Delta^* = -3\nu - 3 \) and, since we take \( |z| \leq 1 \), convergence conditions (B2) and (B3) hold. To compute the function \( f_1(z, \nu) \), we use formula (1.3) and the well-known formula

\[ (2.3) \quad \Gamma(s) = \Gamma(s + l) \prod_{k=1}^{l} (s + l - k)^{-1} \]

with \( l \in \mathbb{N} \). The set of unremovable singular points of the function \( g_{6,6}^{(1,3)}(s) \), which are encircled by the curve \( L_1 \), consists of the points \( s = \nu, \ldots, \nu d_2 \), all these points are poles of the first order, and, for \( k = 0, \ldots, \nu \Delta \), the following equality holds:

\[
\Res(g_{6,6}^{(1,3)}; \nu + k) = -(-z)^{\nu + k}((\nu + k)!)^{-3}((\nu \Delta + k)!)^{-3}((\nu \Delta - k)!)^{-3} = (-z)^{\nu + k}((\nu d_1)!/((\nu \Delta)!))^3 \left( \frac{\nu \Delta}{k} \right)^3 \left( \frac{\nu \Delta + k}{\nu d_1} \right)^3.
\]

The function \( f_1(z, \nu) \) is equal to a finite sum

\[ (2.4) \quad f_1(z, \nu) = ((\nu d_1)!/((\nu \Delta)!))^3 z^\nu (-1)^{\nu \Delta} \sum_{k=0}^{\nu \Delta} (-z)^k \left( \frac{\nu \Delta}{k} \right)^3 \left( \frac{\nu \Delta + k}{\nu d_1} \right)^3. \]

Therefore, as it has been already remarked, using the principle of analytic continuation we may regard it as being defined in \( \mathbb{C} \) and consequently in \( \Omega^* \).

Let \( \Omega_0^* = \{ z \in \Omega^* : \Re(z) > 0 \} \). If \( z \in \Omega_0^* \), then

\[-\frac{3\pi}{2} < \phi = \arg(-z) = \arg(z) - \pi < -\frac{\pi}{2},\]

and therefore \(-z \in \Omega^* \). Now, let me introduce my second auxiliary function defined for \( z \in \Omega_0^* \).

Let

\[ f_2(z, \nu) = -(-1)^{\nu \Delta} \frac{1}{2\pi i} \int_{L_2} g_{6,6}^{(4,3)}(s) ds = \]
\[ (-1)^{\nu \Delta} G_{6,6}^{(4,3)} \left( -z \middle| \begin{array}{ccccccc}
-\nu d_1, & -\nu d_1, & -\nu d_1, & 1 + \nu d_2, & 1 + \nu d_2, & 1 + \nu d_2 \\
0, & 0, & \nu, & n u, & \nu \end{array} \right), \]

where \( z \in \Omega_{0*}, \nu \in \mathbb{N}, \) and

\[ g_{6,6}^{(4,3)}(s) = (-z)^s \Gamma(-s)^3 \Gamma(-s + \nu) \Gamma(1 - \nu + s)^{-2} \Gamma(1 + \nu d_1 + s)^3 \Gamma(1 + \nu d_2 - s)^{-3}; \]

and the curve \( L_2 \) passes from \(-\infty\) to \(-\infty\), encircling the set \(-\mathbb{N}\) in the positive direction but no point in the set \( \mathbb{N} - 1 \). Here \( p = q = 6, n = 4, m = 3, a_1 = a_2 = a_3 = -\nu d_1, a_4 = a_5 = a_6 = 1 + \nu d_2, b_1 = b_2 = b_3 = 0, b_4 = b_5 = b_6 = \nu, \Delta^* = -3\nu - 3; \) since now \(| -z | \geq 1\), convergence conditions (C2) and (C3) are satisfied. To compute the function \( f_2(z, \nu) \), we use formula (1.3). The set of all the unremovable singular points of the function \( g_{6,6}^{(4,3)}(s) \), encircled by the curve \( L_2 \), consists of the points \( s = -1 - \nu d_1 - k \) with \( k \in \mathbb{N} - 1 \); each of these points is a pole of the first order. Therefore making use of (2.3) one obtains

\[ \text{Res} \left( g_{6,6}^{(4,3)}; -1 - \nu d_1 - k \right) = \]

\[ = (-z)^{-1 - \nu d_1 - k} ((\nu d_1 + k)!)^3 ((\nu \Delta + k)!) \Gamma(1 + 2\nu \Delta + k)!)^{-3} = \]

\[ = (-1)^{1 + \nu d_1} z^{-(1 + \nu d_1 + k)} \left( \frac{\nu \Delta - \nu}{\prod_{j=0}^\nu \Delta} \frac{1 + \nu \Delta - \nu + k - j}{\prod_{j=0}^1 (1 + \nu \Delta + k + j)} \right)^3, \]

(2.5) \[ f_2(z, \nu) = \sum_{k=0}^{+\infty} z^{-(1 + \nu d_1 + k)} \left( \frac{\nu \Delta - \nu}{\prod_{j=0}^\nu \Delta} \frac{1 + \nu \Delta - \nu + k - j}{\prod_{j=0}^1 (1 + \nu \Delta + k + j)} \right)^3. \]

Let \( a \in \mathbb{N} - 1, b \in \mathbb{N} + a, \) and

(2.6) \[ R(a; b; t) = \frac{b!}{(b - a)!} \prod_{\kappa=a+1}^b (t - \kappa) \prod_{\kappa=0}^b \frac{1}{(t + \kappa)}, \quad R_0(t; \nu) = R(\nu; \nu \Delta; t). \]

Let \( t = 1 + \nu \Delta + k \) with \( k \in \mathbb{N} - 1 \); in view of (2.5), it follows that

(2.7) \[ f_2(z, \nu) \left( (\nu \Delta)! / (\nu d_1)! \right)^3 = (-1)^\nu \sum_{t = \nu \Delta + 1}^{+\infty} R_0(t; \nu)^3 z^{t + \nu}. \]
Since $R_0(t; \nu) = 0$ for $t = \nu + 1, \ldots, \nu \Delta$, we have

$$\tag{2.8} f_2(z, \nu)\left((\nu \Delta)!/(\nu d_1)\right)^3 = (-1)^\nu \sum_{t=\nu+1}^{\infty} R_0(t; \nu)^3 z^{-t+\nu}.$$ 

Let

$$f_3 = (-1)^{\nu(\Delta+1)} \frac{1}{2\pi i} \int_{L_2} g_{6,6}^{(5,3)}(s) ds =$$

$$(\nu d_1, -\nu d_1, -\nu d_1, 1 + \nu d_2, 1 + \nu d_2, 1 + \nu d_2),$$

where $z \in \Omega_0^*$, $\nu \in \mathbb{N}$, and

$$g_{6,6}^{(5,3)} = g_{6,6}^{(5,3)}(s) = z^s \Gamma(-s)^3 \Gamma(\nu-s)^2 \Gamma(1-\nu+s)^{-1} \Gamma(1+\nu d_1+s)^3 \Gamma(1+\nu d_2-s)^{-3}.$$ 

Here $p = q = 6$, $m = 5$, $n = 3$, $a_1 = a_2 = a_3 = -\nu d_1$, $a_4 = a_5 = a_6 = 1 + \nu d_2$, $b_1 = b_2 = b_3 = 0$, $b_4 = b_5 = b_6 = \nu$, $\Delta^* = -3\nu - 3$; convergence conditions (C2) and (C3) are satisfied since now $|z| \geq 1$. The set of all the unremovable singular points of the function $g_{6,6}^{(5,3)}(s)$, encircled by the curve $L_2$, consists of the points $s = -1 - \nu d_1 - k$ with $k \in \mathbb{N} - 1$; each of these points is a pole of the second order. Therefore

$$\text{Res} \left(g_{6,6}^{(5,3)}; -\nu d_1 - 1 - k\right) = \lim_{s \to -\nu d_1 - 1 - k} \frac{\partial}{\partial s} \left( (s + \nu d_1 + 1 + k)^2 g_{6,6}^{(5,3)} \right),$$

where $k \in \mathbb{N} - 1$.

Let $s = -\nu d_1 - 1 - k + u$ and

$$H_1(u) = g_{6,6}^{(5,3)}(-\nu d_1 - 1 - k + u) = z^{-\nu d_1 - 1 - k + u} \Gamma(\nu d_1 + 1 + k - u)^3 \times$$

$$\Gamma(\nu \Delta + 1 + k - u)^2 \Gamma(-\nu \Delta - k + u)^{-1} \Gamma(-k + u)^3 \Gamma(2\nu \Delta + 2 + k - u)^{-3} =$$

$$z^{-\nu d_1 - 1 - k + u} \Gamma(\nu d_1 + 1 + k - u)^3 \Gamma(\nu \Delta + 1 + k - u)^{-3} \times$$

$$\Gamma(-\nu \Delta - k + u)^{-1} \Gamma(1 + k - u)^3 \Gamma(-k + u)^3 \Gamma(1 + k - u)^{-3} \Gamma(2 + 2\nu \Delta + k - u)^{-3} =$$

$$(\pi/ \sin(\pi u))^2 (1)^{\nu \Delta} z^{-\nu d_1 - 1 - k + u} \Gamma(\nu d_1 + 1 + k - u)^3 \times$$

$$\Gamma(\nu \Delta + 1 + k - u)^3 \Gamma(1 + k - u)^{-3} \Gamma(2 + 2\nu \Delta + k - u)^{-3} = \left(\pi/ \sin(\pi u)\right)^2 H^*(u),$$

where

$$H^*(u) = (-1)^{\nu \Delta} z^{-\nu d_1 - 1 - k + u} \times$$
\[
\Gamma(\nu d_1 + 1 + k - u)^3 \Gamma(1 + k - u)^{-3} \Gamma(1 + \nu \Delta + k - u)^3 \Gamma(2 + 2\nu \Delta + k - u)^{-3} = \\
= (-1)^{\nu\Delta} z^{-T+\nu} \Gamma(T)^3 \Gamma(T + 1 + \nu \Delta)^{-3} \Gamma(1 + \nu \Delta)^{-3} \Gamma(T - \nu \Delta)^{-3} \Gamma(T - \nu)^3 = \\
= (-1)^{\nu\Delta} z^{-T+\nu} \left( \prod_{\kappa=\nu+1}^{\nu \Delta} (T - \kappa) \right)^3 \left( \prod_{\kappa=0}^{k+\nu \Delta} (T + \kappa)^{-1} \right)^3 = \\
= (-1)^{\nu\Delta} z^{-T+\nu} R_0(T; \nu)^3 ((\nu \Delta)! / (\nu d_1)!)^{-3}
\]

and \( T = \nu \Delta + 1 + k - u \). Therefore

\[
(-1)^{\nu d_2} ((\nu \Delta)! / (\nu d_1)!)^3 \text{Res} (g_{6,6}^{(5,3)}; -1 - \nu d_1 - k) = \\
= (-1)^{\nu} z^{-T+\nu} \left( R_0(T; \nu)^3 \log z - \frac{\partial}{\partial T} R_0(T; \nu)^3 \right) \bigg|_{T=1+\nu \Delta+k}
\]

because \((\pi u / (\sin(\pi u))^2\) is an even function. Thus

\[
f_3(z, \nu) = f_2(z, \nu) \log z - (-1)^{\nu} ((\nu \Delta)! / (\nu d_1)!)^{-3} \sum_{t=\nu \Delta+1}^{\infty} z^{-t+\nu} \frac{\partial}{\partial t} R_0(t; \nu)^3;
\]

since \( R_0(t; \nu)^3 \) has zeros of the third order in the points \( t = 1, \ldots, \nu \Delta \), it follows that

\[
(2.9) \quad f_3(z, \nu) = f_2(z, \nu) \log z - (-1)^{\nu} ((\nu \Delta)! / (\nu d_1)!)^{-3} \sum_{t=1+\nu}^{\infty} z^{-t+\nu} \frac{\partial}{\partial t} R_0(t; \nu)^3.
\]

Let

\[
(2.10) \quad f_4(z, \nu) = -((\nu \Delta)! / (\nu d_1!))^{-3} (-z)^{\nu} \sum_{t=\nu \Delta+1}^{\infty} z^{-t} \frac{\partial}{\partial t} R_0(t; \nu)^3;
\]

then

\[
(2.11) \quad f_3(z, \nu) = f_2(z, \nu) \log z + f_4(z, \nu).
\]

Let

\[
f_5'(z, \nu) = -\frac{(-1)^{\nu \Delta}}{2\pi i} \int_{L_2} g_{6,6}^{(6,3)}(s) ds = \\
= -(-1)^{\nu \Delta} G_{6,6}^{(6,3)} \left( -z \begin{vmatrix} -\nu d_1, & -\nu d_1, & -\nu d_1, & 1 + \nu d_2, & 1 + \nu d_2, & 1 + \nu d_2 \\ 0, & 0, & 0, & \nu, & \nu, & \nu \end{vmatrix} \right),
\]
where \( z \in \Omega^*_0 \), \( \nu \in \mathbb{N} \), and

\[
g^{(6,3)}_{6,6} = g^{(6,3)}_{6,6}(s) = (-z)^s \Gamma(-s)^3 \Gamma(\nu - s)^3 \Gamma(1 + \nu d_1 + s)^3 \Gamma(1 + \nu d_2 - s)^{-3}.
\]

Here \( p = q = 6 \), \( m = 6 \), \( n = 3 \), \( a_1 = a_2 = a_3 = -\nu d_1 \), \( a_4 = a_5 = a_6 = 1 + \nu d_2 \), \( b_1 = b_2 = b_3 = 0 \), \( b_4 = b_5 = b_6 = \nu \), \( \Delta^* = -3\nu - 3 \); since \( -z \in \Omega^* \), each of the convergence conditions (C2) and (C3) is satisfied. The set of all the unremovable singular points of the function \( g^{(6,3)}_{6,6}(s) \), encircled by the curve \( L_2 \), consists of the points \( s = -1 - \nu d_1 - k \) with \( k \in \mathbb{N} - 1 \); each of these points is a pole of the third order. Therefore

\[
\text{Res} \left( g^{(6,3)}_{6,6}; -\nu d_1 - 1 - k \right) = \lim_{s \rightarrow -\nu d_1 - 1 - k} \frac{1}{2} \left( \frac{\partial}{\partial s} \right)^2 ((s + \nu d_1 + 1 + k)^2 g^{(6,3)}_{6,6}),
\]

where \( k \in \mathbb{N} - 1 \).

Let \( s = -\nu d_1 - 1 - k + u \), and

\[
H_2(u) = g^{(6,3)}_{6,6}(-\nu d_1 - 1 - k + u) = (-z)^{-\nu d_1 - 1 - k + u} \Gamma(-k + u)^3 \Gamma(\nu d_1 + 1 + k - u)^3 \Gamma(2 + 2\nu \Delta + k - u)^{-3}
\]

\[
= (-1)^k (-z)^{-\nu d_1 - 1 - k + u} \Gamma(1 + \nu \Delta + k - u)^3 \Gamma(2 + 2\nu \Delta + k - u)^{-3} \times \Gamma(\nu d_1 + 1 + k - u)^3 \Gamma(1 + k - u)^{-3} (\pi / \sin(\pi u))^3 = (\pi / \sin(\pi u))^3 H^*_2(u),
\]

where

\[
H^*_2(u) = (-1)^k (-z)^{-T + \nu} R_0(T; \nu)^3 ((\nu \Delta)! / (\nu d_1)!)^{-3}
\]

and \( T = \nu \Delta + 1 + k - u \), \( t = \nu \Delta + 1 + k \), as before.

Therefore

\[
-(1)^{\nu \Delta} ((\nu \Delta)! / (\nu d_1)!)^3 \text{Res} \left( g^{(6,3)}_{6,6}; -1 - \nu d_1 - k \right) = -(1)^{\nu \Delta + k} \left[ \frac{1}{2} (\log(-z))^2 (-z)^{-t + \nu} R_0(t; \nu)^3 - (-z)^{-t + \nu} \log(-z) \frac{\partial}{\partial T} R_0(t; \nu)^3 + \right.
\]

\[
\left. + \frac{1}{2} (-z)^{-t + \nu} \left( \frac{\partial}{\partial T} \right)^2 R_0(t; \nu)^3 + \pi^2 (-z)^{-t + \nu} R_0(t; \nu)^3 \right)
\]

because \( (\pi u / \sin(\pi u))^3 = 1 + \frac{1}{2}(\pi u)^2 + \ldots \).

Consequently,

\[
f^{\nu}_5(z, \nu) = \frac{1}{2} f_2(z, \nu) ((\log(-z))^2 + \pi^2) + f_4(z, \nu) \log(-z) +
\]
\[
\begin{align*}
&+ \frac{1}{2}(-z)^\nu((\nu\Delta)!/(\nu d_1)!)^{-3} \sum_{t=\nu\Delta+1}^{\infty} z^{-t} \left( \frac{\partial}{\partial T} \right)^2 R_0(t; \nu)^3 = \\
&= \frac{1}{2} f_2(z, \nu)(\log z)^2 + f_4(z, \nu) \log z + \\
&+ \frac{1}{2}(-z)^\nu((\nu\Delta)!/(\nu d_1)!)^{-3} \sum_{t=\nu\Delta+1}^{\infty} z^{-t} \left( \frac{\partial}{\partial T} \right)^2 R_0(t; \nu)^3 - \pi if_3(z, \nu),
\end{align*}
\]

where \( z \in \Omega_0^* \).

Since \( R_0(t; \nu)^3 \) has zeros of the third order at the points \( t = 1 + \nu, \ldots, \nu\Delta \), it follows that

\[
\begin{align*}
&f_5^\nu(z, \nu) = \frac{1}{2} f_2(z, \nu)(\log z)^2 + f_4(z, \nu) \log z + \\
&+ \frac{1}{2}(-z)^\nu((\nu\Delta)!/(\nu d_1)!)^{-3} \sum_{t=\nu\Delta+1}^{\infty} z^{-t} \left( \frac{\partial}{\partial T} \right)^2 R_0(t; \nu)^3 - \pi if_3(z, \nu).
\end{align*}
\]

Let

\[
(2.13) \quad f_6(z, \nu) = \frac{1}{2}(-z)^\nu((\nu\Delta)!/(\nu d_1)!)^{-3} \sum_{t=1}^{\infty} z^{-t} \left( \frac{\partial}{\partial T} \right)^2 R_0(t; \nu)^3,
\]

\[
\begin{align*}
&f_5(z, \nu) = f_6(z, \nu) + \frac{1}{2} f_2(z, \nu)(\log z)^2 + f_4(z, \nu) \log z,
\end{align*}
\]

where \( z \in \Omega_0^* \). Then

\[
(2.14) \quad f_5(z, \nu) = f_5^\nu(z, \nu) + i \pi f_3(z, \nu),
\]

where \( z \in \Omega_0^* \). Let further

\[
(2.15) \quad f_j^\nu(z, \nu) = ((\nu\Delta)!/(\nu d_1)!)^{3} f_j(z, \nu), \quad j = 1, \ldots, 6.
\]

Expanding function \( R_0(t; \nu)^3 \) into partial fractions, we obtain

\[
R_0(t; \nu)^3 = \sum_{k=0}^{\nu\Delta} \alpha_{\nu,k}^* (t + k)^{-3} + \sum_{k=0}^{\nu\Delta} \beta_{\nu,k}^* (t + k)^{-2} + \sum_{k=0}^{\nu\Delta} \gamma_{\nu,k}^* (t + k)^{-3}
\]

with

\[
(2.16) \quad \alpha_{\nu,k}^* = (-1)^{\nu+\nu\Delta+k} \left( \begin{array}{c} \nu\Delta \k \nu\Delta+k \end{array} \right)^3 \left( \begin{array}{c} \nu\Delta \k \nu\Delta-k \end{array} \right),
\]

\[ \beta_{\nu,k} = \lim_{t \to -k} \frac{\partial}{\partial t} (R_0(t; \nu)^3(t + k)^3) = \alpha_{\nu,k}^* \left( 3 \left( - \sum_{\kappa = \nu + k + 1}^{\nu \Delta + k} \kappa^{-1} - \sum_{\kappa = 1}^{\nu \Delta - k} \kappa^{-1} + \sum_{\kappa = 1}^{k} \kappa^{-1} \right) \right), \]

\[ 2 \gamma_{\nu,k}^* = \alpha_{\nu,k}^* \left( 3 \left( - \sum_{\kappa = \nu + k + 1}^{\nu \Delta + k} \kappa^{-1} - \sum_{\kappa = 1}^{\nu \Delta - k} \kappa^{-1} + \sum_{\kappa = 1}^{k} \kappa^{-1} \right) \right)^2 - \alpha_{\nu,k}^* \left( 3 \left( \sum_{\kappa = \nu + k + 1}^{\nu \Delta + k} \kappa^{-2} - \sum_{\kappa = 1}^{\nu \Delta - k} \kappa^{-2} - \sum_{\kappa = 1}^{k} \kappa^{-2} \right) \right), \]

where \( k = 0, \ldots, \nu \Delta \). It follows from (2.4), (2.8), (2.10), (2.13) and (2.15) - (2.18) that

\[ f_2^* (z, \nu) = (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} R_0(t; \nu)^3 = \]

\[ = (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \sum_{k=0}^{\nu \Delta} \alpha_{\nu,k}^* (t + k)^{-3} + (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \sum_{k=0}^{\nu \Delta} \beta_{\nu,k}^* (t + k)^{-2} + \]

\[ + (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \sum_{k=0}^{\nu \Delta} \gamma_{\nu,k}^* (t + k)^{-1} = (-z)^\nu \sum_{k=0}^{\nu \Delta} \alpha_{\nu,k}^* z^k \sum_{t=1+\nu+k}^{+\infty} z^{-t} t^{-3} + \]

\[ + (-z)^\nu \sum_{k=0}^{\nu \Delta} \beta_{\nu,k}^* z^k \sum_{t=1+\nu+k}^{+\infty} z^{-t} t^{-2} + (-z)^\nu \sum_{k=0}^{\nu \Delta} \gamma_{\nu,k}^* z^k \sum_{t=1+\nu+k}^{+\infty} z^{-t} t^{-1} = \]

\[ = \alpha^* (z; \nu)L_3(z^{-1}) + \beta^* (z; \nu)L_2(z^{-1}) + \gamma^* (z; \nu)(- \log(1 - 1/z)) - \phi^* (z; \nu) \]

with

\[ L_n(z) = \sum_{t=1}^{+\infty} z^t t^{-n}, \]

\[ \alpha^* (z; \nu) = (-z)^\nu \sum_{k=0}^{\nu \Delta} \alpha_{\nu,k}^* z^k = f_1^* (z; \nu), \]

\[ \beta^* (z; \nu) = (-z)^\nu \sum_{k=0}^{\nu \Delta} \beta_{\nu,k}^* z^k \]
(2.23) \[ \gamma^*(z; \nu) = (-z)^\nu \sum_{k=0}^{\nu \Delta} \gamma_{\nu, k}^* z^k, \]

(2.24) \[ \phi^*(z; \nu) = (-z)^\nu \sum_{k=0}^{\nu \Delta} \alpha_{\nu, k}^* z^k \sum_{t=1}^{k+\nu} z^{-t} t^{-3} + \]
\[ + (-z)^\nu \sum_{k=0}^{\nu \Delta} \beta_{\nu, k}^* z^k \sum_{t=1}^{k+\nu} z^{-t} t^{-2} + (-z)^\nu \sum_{k=0}^{\nu \Delta} \gamma_{\nu, k}^* z^k \sum_{t=1}^{k+\nu} z^{-t} t^{-1} = \]
\[ = (-z)^\nu \sum_{t=1}^{\infty} z^{-t} \left( t^{-3} \alpha^*(z; \nu) + t^{-2} \beta^*(z; \nu) + t^{-1} \gamma^*(z; \nu) \right) + \]
\[ + (-z)^\nu \sum_{k=0}^{\nu \Delta} \gamma_{\nu, k}^* z^k \sum_{t=1+\nu}^{k+\nu} z^{-t} t^{-1}; \]

(2.25) \[ f^*_4(z, \nu) = -(-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \frac{\partial}{\partial t} R_0(t; \nu)^3 = \]
\[ = (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \sum_{k=0}^{\nu \Delta} 3 \alpha_{\nu, k}^* (t + k)^{-4} + (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \sum_{k=0}^{\nu \Delta} 2 \beta_{\nu, k}^* (t + k)^{-3} + \]
\[ + (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \sum_{k=0}^{\nu \Delta} \gamma_{\nu, k}^* (t + k)^{-2} = (-z)^\nu \sum_{k=0}^{\nu \Delta} 3 \alpha_{\nu, k}^* z^k \sum_{t=1+\nu+k}^{+\infty} z^{-t} t^{-4} + \]
\[ + (-z)^\nu \sum_{k=0}^{\nu \Delta} 2 \beta_{\nu, k}^* z^k \sum_{t=1+\nu+k}^{+\infty} z^{-t} t^{-3} + (-z)^\nu \sum_{k=0}^{\nu \Delta} \gamma_{\nu, k}^* z^k \sum_{t=1+\nu+k}^{+\infty} z^{-t} t^{-2} = \]
\[ = 3 \alpha^*(z; \nu) L_4(z^{-1}) + 2 \beta^*(z; \nu) L_3(z^{-1}) + \gamma^*(z; \nu) L_2(z)^{-1} - \psi^*(z; \nu), \]

where

(2.26) \[ \psi^*(z; \nu) = (-z)^\nu \sum_{k=0}^{\nu \Delta} 3 \alpha_{\nu, k}^* z^k \sum_{t=1}^{k+\nu} z^{-t} t^{-4} + \]
\[ (+z)^\nu \sum_{k=0}^{\nu\Delta} 2\beta^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t} t^{-3} + (+z)^\nu \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t} t^{-2} = \]

\[ = (-z)^\nu \sum_{t=1}^{\nu} z^{-t} \Big( t^{-4} 3\alpha^*(z;\nu) + t^{-3} 2\beta^*(z;\nu) + t^{-2} \gamma^*(z;\nu) \Big) + \]

\[ + (-z)^\nu \sum_{k=0}^{\nu\Delta} 3\alpha^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t} t^{-4} + (-z)^\nu \sum_{k=0}^{\nu\Delta} 2\beta^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t} t^{-3} + \]

\[ + (-z)^\nu \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t} t^{-2}; \]

\[ (2.27) \]

\[ f^*_0(z,\nu) = \frac{1}{2} (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \left( \frac{\partial}{\partial t} \right)^2 R_0(t;\nu)^3 = \]

\[ = (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \sum_{k=0}^{\nu\Delta} 6\alpha^*_{\nu,k} (t+k)^{-5} + (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \sum_{k=0}^{\nu\Delta} 3\beta^*_{\nu,k} (t+k)^{-4} + \]

\[ + (-z)^\nu \sum_{t=1+\nu}^{+\infty} z^{-t} \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k} (t+k)^{-3} = (-z)^\nu \sum_{k=0}^{+\infty} 6\alpha^*_{\nu,k} z^k \sum_{t=1+\nu+k}^{+\infty} z^{-t} t^{-5} + \]

\[ + (-z)^\nu \sum_{k=0}^{\nu\Delta} 3\beta^*_{\nu,k} z^k \sum_{t=1+\nu+k}^{+\infty} z^{-t} t^{-4} + (-z)^\nu \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k} z^k \sum_{t=1+\nu+k}^{+\infty} z^{-t} t^{-3} = \]

\[ = 6\alpha^*(z;\nu) L_5(z^{-1}) + 3\beta^*(z;\nu) L_4(z^{-1}) + \gamma^*(z;\nu) L_3(z^{-1}) - \xi^*(z;\nu), \]

and

\[ (2.28) \]

\[ \xi^*(z;\nu) = (-z)^\nu \sum_{k=0}^{\nu\Delta} 6\alpha^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t} t^{-5} + \]

\[ + (-z)^\nu \sum_{k=0}^{\nu\Delta} 3\beta^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t} t^{-4} + (-z)^\nu \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t} t^{-3} = \]

\[ = (-z)^\nu \sum_{t=1}^{\nu} z^{-t} \Big( 6t^{-5} \alpha^*(z;\nu) + 3t^{-4} \beta^*(z;\nu) + t^{-3} \gamma^*(z;\nu) \Big) + \]
\[ + (-z)^\nu \sum_{k=0}^{\nu\Delta} 6\alpha_{\nu,k}^* z^k \sum_{t=1+\nu}^{k+\nu} z^{-t} t^{-5} + (-z)^\nu \sum_{k=0}^{\nu\Delta} 3\beta_{\nu,k}^* z^k \sum_{t=1+\nu}^{k+\nu} z^{-t} t^{-4} + \]
\[ + (-z)^\nu \sum_{k=0}^{\nu\Delta} \gamma_{\nu,k}^* z^k \sum_{t=1+\nu}^{k+\nu} z^{-t} t^{-3}. \]

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