Addendum to “Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees,”
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This addendum was submitted to the American Journal of Mathematics on June 6, 2000. On the same day a copy was sent to J. Noguchi who first drew our attention to a gap in our original paper. After three months of queries by e-mail, J. Noguchi informed the first author by e-mail on August 28, 2000 that he finally understood the arguments. This addendum is made available on the web at this time, because it has not yet appeared in a journal and is quoted in some recent papers of the first author. Moreover, from some public lectures a number of people had the wrong impression of the nature of the gap and did not know that it means only an easy modification of our original paper.

J. Noguchi, J. Winkelmann, and K. Yamanoi [NWY99] drew our attention to a difficulty in the proof of Lemma 2 of the paper in the title, which might arise when the projection of the space $J_k(D)$ of $k$-jets of the ample divisor $D$ of the $n$-dimensional abelian variety $A$ completely contains $W_k$, where $W_k \subset C^{nk}$ is the algebraic Zariski closure of the image of the differential of the holomorphic map $f : C \to A$. In this addendum we give an easy modification of our argument by using the uniqueness part of the fundamental theorem of ordinary differential equations.

The modification at the same time improves somewhat our result (see Theorem A). For a holomorphic map to an $n$-dimensional abelian variety $A$ whose image is not contained in a translate of an ample divisor $D$, the proximity function to $D$ plus the error obtained by truncating multiplicity in the counting function at order $k_n = k_n(D)$ is of logarithmic order of the characteristic function of the map, where $k_n$ is an explicit function of $n$ and the Chern number $D^n$.

The difficulty of [SY97] resulted from an attempt to use semi-continuity of cohomology groups in deformations to avoid employing Bloch’s technique from [B26] which involves the uniqueness part of the fundamental theorem.

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of ordinary differential equations. Here we simply go back to using Bloch’s technique to show that the projection of $J_k(D)$ cannot completely contain $W_k$ when $k$ is at least $k_n(D)$.

A sketch of the main idea in the modification given here is as follows. If the local function which defines $J_{k+1}(D)$ from $J_k(D)$ does not reduce the dimension or the multiplicity of the part of $J_k(D)$ over $W_k$ at a generic point, there it must locally belong to the ideal generated by local functions defining $J_k(D)$ and $W_k$. The multiplicity of the part of $J_k(D)$ over $W_k$ at a generic point admits an effective bound depending on $D_n$ and $k$. If the projection of $J_k(D)$ completely contains $W_k$, then the pullback of the defining function for a translate of $D$ by $f$ satisfies an ordinary differential equation in some open neighborhood of 0 in $C$ whose order admits an effective bound depending on $D_n$ and $k$ and whose coefficients for the highest-order derivative is 1. The uniqueness part of the fundamental theorem of ordinary differential equations yields the contradiction that some translate of $D$ must contain the image of $f$.

**Notations and Terminology.** Let $A$ be an abelian variety of complex dimension $n$. We denote by $\mathcal{O}_A$ the structure sheaf of $A$. We denote by $J_k(A) = A \times C^{nk}$ the space of all $k$-jets of $A$. We consider only subvarieties of $J_k(A) = A \times C^{nk}$ which are algebraic along the factor $C^{nk}$. In other words, they are defined by local functions on $J_k(A) = A \times C^{nk}$ which are holomorphic in the local coordinates of $A$ and are polynomials in the $nk$ global coordinates of $C^{nk}$. Here subvarieties of $J_k(A) = A \times C^{nk}$ are assumed without any further explicit mention to be algebraic along the factor $C^{nk}$. Zariski closures are defined using such subvarieties.

The projection $\pi_k : J_k(A) \to C^{nk}$ denotes the natural projection $J_k(A) = A \times C^{nk} \to C^{nk}$ onto the second factor. For $\ell > k$ let $p_{k,\ell} : C^{n\ell} \to C^{nk}$ denote the natural projection so that $\pi_k = p_{k,\ell} \circ \pi_\ell$.

Let $D$ be an ample divisor in $A$. We denote by $\theta_D$ the theta function on $C^n$ whose divisor is $D$. The space of $k$-jets of $D$ is denoted by $J_k(D)$.

For a holomorphic map $f$ from $C$ to $A$, let $W_{k,f}$ be the (algebraic) Zariski closure of $\pi_k \left( \operatorname{Im} (d^k f) \right)$ in $C^{nk}$, where $d^k f : J_k(C) \to J_k(A)$ is the map induced by $f$. Clearly, $p_{k,\ell}(W_{\ell,f}) = W_{k,f}$ for $\ell > k$. When $Z_k$ is a proper subvariety of $W_{k,f}$, $p_{k,\ell}^{-1}(Z_k)$ is a proper subvariety of $W_{\ell,f}$ for $\ell > k$. Let $m(r, f, D)$ denote the proximity function for the map $f$ and the divisor $D$.
at the radius $r$. Let $N_k(r, f, D)$ be the counting function for the divisor $D$ which truncates multiplicity at $k$, i.e., with multiplicity replaced by $k$ if it is greater than $k$. Let $N(r, f, D)$ denote the usual counting function for the divisor $D$ without any truncation of multiplicity.

**Theorem A.** Let $A$ be an abelian variety of complex dimension $n$ and $D$ be an ample divisor of $A$. Inductively let $k_0 = 0$ and $k_1 = 1$ and

$$k_{\ell+1} = k_{\ell} + 3^{n-\ell-1} (4(k + 1))^\ell D^n$$

for $1 \leq \ell < n$. Then for any holomorphic map $f : \mathbb{C} \to A$ whose image is not contained in any translate of $D$,

$$m(r, f, D) + (N(r, f, D) - N_{k_n}(r, f, D)) = O(\log T(r, f, D) + \log r) \quad ||$$

when $||$ means that the equation holds for $r$ outside some set whose measure with respect to $\frac{dr}{r}$ is finite.

The rest of this note is devoted to the proof of Theorem A. Our first step is to verify the following claim.

**Claim 1.** $\pi_{k_n}(J_{k_n}(D))$ does not contain $W_{k_n,f}$.

Assume that Claim 1 is false and we are going to derive a contradiction.

In order to have an effective bound on the multiplicities of the highest-dimensional branches of

$$\pi_{k}^{-1}(\gamma) \cap J_k(D) = \pi_{k}^{-1}(\gamma) \cap \{\theta_D = d\theta_D = \cdots = d^k\theta_D\},$$

we introduce a meromorphic connection for the line bundle $D$ so that the covariant differentials of $\theta_D$ with respect to the meromorphic connection are line-bundle-valued jet differentials on $A$, making it possible to bound the multiplicities by Chern numbers. By Lefschetz’s theorem, $mD$ is a very ample line bundle over $A$ for $m \geq 3$. Choose $\tau_j \in \Gamma(A, jD)$ for $j = 3, 4$ with the property that for $1 \leq k \leq k_n$ there exists a proper subvariety $Z_k$ of $W_{k,f}$ such that for $\gamma_k \in W_{k,f} - Z_k$ the set $\{\tau_3\tau_4 = 0\} \cap J_k(D) \cap \pi_k^{-1}(\gamma_k)$ is a nowhere dense subvariety of $J_k(D) \cap \pi_k^{-1}(\gamma_k)$.

We now define covariant differentials of $\theta_D$ by using usual differentials of meromorphic functions after meromorphically trivialize the line bundle $D$
by means of \( \tau_3, \tau_4 \). The above condition in the choice of \( \tau_3, \tau_4 \) is to make sure that the additional pole-sets and zero-sets introduced because of the meromorphicity of the connection are irrelevant to our purpose. Let

\[
D^k\theta_D = (\tau_4)^{k+1} d^k \left( \frac{\theta_D \tau_3}{\tau_4} \right)
\]

for \( 1 \leq k \leq k_n \). Then \( D^k\theta_D \) is a \((4(k+1)D)\)-valued holomorphic \( k \)-jet differential on \( A \) of weight \( k \). For \( \ell \geq k \) and \( \gamma \in C^{n\ell} \) we denote by \( \langle D^k\theta_D, \gamma \rangle \) the element of \( \Gamma(A, 4(k+1)D) \) which is obtained by evaluating \( D^k\theta_D \) at the element of \( J_k(D) \) whose image under \( \pi_k \) is equal to \( p_{k,\ell}(\gamma) \).

We use the convention that a set of complex dimension \(-1\) means the empty set.

For \( 1 \leq \ell \leq n \) we introduce the following statement \( (A)_\ell \) and use the convention that \( (A)_0 \) always holds. Let \( 0 \leq \ell_0 \leq n \) be the largest integer inclusively between 0 and \( n \) such that the following statement \( (A)_\ell \) holds with \( \ell = \ell_0 \).

\( (A)_\ell \): There exists a proper subvariety \( E_\ell \) of \( W_{k,\ell} \) such that the complex dimension of the common zero-set of

\[
\theta_D, \langle D\theta_D, \gamma \rangle, \langle D^2\theta_D, \gamma \rangle, \cdots, \langle D^{k_\ell}\theta_D, \gamma \rangle
\]

in \( A - \{\tau_3\tau_4 = 0\} \) is no more than \( n - 1 - \ell \) for \( \gamma \in W_{k,\ell} - E_\ell \).

We can choose

(i) \( \sigma_0 \in \Gamma(A, 4(k_{\ell_0} + 1)D) \) and \( \sigma_j \in \Gamma(A, 4(k_{\ell_0} - j)D) \) for \( 1 \leq j < k_{\ell_0} \),

(ii) complex numbers \( c_{i,j} \) for \( 1 \leq i \leq \ell_0 \) and \( 1 \leq j \leq k_{\ell_0} \), and

(iii) a proper subvariety \( \tilde{E}_{\ell_0} \) of \( W_{k_{\ell_0},f} \) containing \( E_{\ell_0} \),

such that, when we set \( \sigma_{k_{\ell_0}} \equiv 1 \) and

\[
\rho_{i,\gamma} = \sum_{j=0}^{k_{\ell_0}} c_{i,j} \sigma_j \langle D^j\theta_D, \gamma \rangle
\]

for \( 1 \leq i \leq \ell_0 \) and \( \gamma \in W_{k_{\ell_0},f} - \tilde{E}_{\ell_0} \), the common zero-set of

\[
\rho_{1,\gamma}, \cdots, \rho_{\ell_0,\gamma} \in \Gamma(A, 4(k_{\ell_0} + 1)D)
\]
in $D - \{\tau_3 \tau_4 = 0\}$ is a subvariety of pure complex dimension $n - 1 - \ell_0$ in $D$.

Because of the assumption that $\pi_{k_n}(J_{k_n}(D))$ contains $W_{k_n,f}$, we know that $\ell_0 < n$. Let $\kappa_n = \theta_D$. We can choose

$$\kappa_1, \cdots, \kappa_{n-\ell_0-1} \in \Gamma(A, 3D)$$

such that the intersection of the common zero-set of $\kappa_1, \cdots, \kappa_{n-\ell_0-1}$ and every $\ell_0$-dimensional branch of the common zero-set of $\rho_{1,\gamma}, \cdots, \rho_{\ell_0,\gamma}$ in $D - \{\tau_3 \tau_4 = 0\}$ is zero-dimensional. Then the sum, over all points $Q \in A - \{\tau_3 \tau_4 = 0\}$, of the dimension over $C$ of the (vector-space) stalk at $Q$ of the sheaf

$$\mathcal{O}_A / \left(\sum_{j=1}^{n-\ell_0} \mathcal{O}_A \kappa_j + \sum_{i=1}^{\ell_0} \mathcal{O}_A \rho_{i,\gamma}\right)$$

is no more than

$$3^{n-\ell_0-1} (4(k_{\ell_0} + 1))^{\ell_0} D^n.$$ 

Since the above estimates imply that the complex dimension $d_s$ of the stalk at $Q \in D - \{\tau_3 \tau_4 = 0\}$ of

$$\mathcal{O}_A / \left(\sum_{j=1}^{n-\ell_0} \mathcal{O}_A \kappa_j + \sum_{i=1}^{s} \mathcal{O}_A \langle D^s \theta_D, \gamma \rangle\right)$$

is no more than $k_{\ell_0+1} - k_{\ell_0}$ for $\gamma \in (p_{k_{\ell_0}, k_{\ell_0+1}})^{-1} \left(W_{k_{\ell_0}, f} - \tilde{E}_{\ell_0}\right)$ and $k_{\ell_0} \leq s \leq k_{\ell_0+1}$, it follows that there exists some $k_{\ell_0} < q \leq k_{\ell_0+1}$ such that $d_{q-1} = d_q$. The $q$ depends on the choice of $Q, \gamma, \kappa_1, \cdots, \kappa_{n-\ell_0}$, but we can make $q$ independent of the choice for $Q$ in some Zariski open subset of $D - \{\tau_3 \tau_4 = 0\}$, for $(\kappa_1, \cdots, \kappa_{n-\ell_0})$ in some Zariski open subset of the product of $n - \ell_0 - 1$ copies of $\Gamma(A, 3D)$, and for $\gamma$ in some Zariski open subset of $(p_{k_{\ell_0}, k_{\ell_0+1}})^{-1} \left(W_{k_{\ell_0}, f} - \tilde{E}_{\ell_0}\right)$.

Since $(A)_{\ell_0+1}$ does not hold, there exists a proper subvariety $F_{\ell_0+1}$ of $W_{\ell_0+1,f}$ containing $p_{\ell_0, \ell_0+1}^{-1} \left(E_{\ell_0}\right)$ such that that the complex dimension of the common zero-set of

$$\theta_D, \langle D \theta_D, \gamma \rangle, \langle D^2 \theta_D, \gamma \rangle, \cdots, \langle D^{k_{\ell_0+1}} \theta_D, \gamma \rangle$$
in $A - \{\tau_3 \tau_4 = 0\}$ is $n - 1 - \ell_0$ for $\gamma \in W_{k_{\ell_0+1}} - F_{\ell_0+1}$. Let $V_{\ell_0+1}$ be the minimum subvariety in $J_q(D) \cap \pi_q^{-1}(W_{q,f})$ which contains all the branches of dimension $< n - 1 - \ell_0$ of $J_q(D) \cap \pi_q^{-1}(\gamma)$ for all $\gamma \in W_{q,f}$. Then $V_{\ell_0+1}$ is a proper subvariety of $J_q(D) \cap \pi_q^{-1}(W_{q,f})$. After replacing $V_{\ell_0+1}$ by a larger proper subvariety of $J_q(D) \cap \pi_q^{-1}(W_{q,f})$ to take care of the requirement of $Q, \gamma, \kappa_1, \cdots, \kappa_{n-\ell_0-1}$ being in some appropriate Zariski open subsets in the choice of $q$ made above, we conclude that the following statement $(B)_\ell$ holds with $\ell = \ell_0 + 1$.

$(B)_\ell$. There exist $k_{\ell-1} < q \leq k_\ell$, a subvariety $V_\ell$ of $J_q(D) \cap \pi_q^{-1}(W_{q,f})$ containing $J_q(D) \cap \{\tau_3 \tau_4 = 0\}$, and a proper subvariety $F_\ell$ of $W_{q,f}$, such that

(i) $(J_q(D) - V_\ell) \cap \pi_q^{-1}(\gamma_q)$ is a nonempty $(n - \ell)$-dimensional subvariety of $J_q(A) - V_\ell$ for every $\gamma_q \in W_{q,f} - F_\ell$, and

(ii) on $J_q(A) - \pi_q^{-1}(F_\ell) - V_\ell$ the function $D^q \theta_D$ locally belongs to the ideal sheaf which is locally generated by the functions

$$\theta_D, D\theta_D, D^2 \theta_D, \cdots, D^{q-1} \theta_D$$

and by the pullbacks to $J_q(A)$ of the local holomorphic functions on $C^{uq}$ vanishing on $W_{q,f}$.

The numbers $k_0 = 0$ and $k_1 = 1$ are chosen so that $(B)_\ell$ holds with $\ell = 1$ when $(A)_\ell$ fails for $\ell = 1$.

There exists some $\zeta_0 \in C$ such that

$$\pi_q((d^q f)(\zeta_0)) \in W_{q,f} - F_{\ell_0+1},$$

where $(d^q f)(\zeta_0)$ denotes the element of $J_q(A)$ defined by $f$ at $\zeta_0$. By Condition (i) of $(B)_{\ell_0+1}$ there exists some point $Q_0 \in D$ such that when we define

$$g(\zeta) = (Q_0 - f(\zeta_0)) + f(\zeta) \in A$$

for $\zeta \in C$, we have

$$\frac{d^q g(\zeta_0)}{d\zeta} \in J_q(D) - V_{\ell_0+1} - \pi_q^{-1}(F_{\ell_0+1}).$$

Since $\pi_q((d^q g)(\zeta)) = \pi_q((d^q f)(\zeta)) \in W_{q,f}$ for $\zeta \in C$, from Condition (ii) of $(B)_{\ell_0+1}$ it follows that $\theta_D \circ g$ satisfies a differential equation

$$\frac{d^q}{d\zeta} (\theta_D \circ g)(\zeta) = \sum_{j=0}^{q-1} h_j(\zeta) \frac{d^j}{d\zeta^j} (\theta_D \circ g)(\zeta)$$
on some open neighborhood \( U \) of \( \zeta_0 \) in \( \mathbb{C} \) for some holomorphic functions \( h_0(\zeta), \ldots, h_{q-1}(\zeta) \) on \( U \). By (1) and (2) and the uniqueness part of the fundamental theorem of ordinary differential equations, \( (\theta_D \circ g)(\zeta) \) is identically zero for \( \zeta \in \mathbb{C} \). Hence the image of \( f \) is contained in the translate of \( D \) by \( Q_0 - f(\zeta_0) \). This contradicts the assumption that the image of \( f \) is not contained in any translate of \( D \) and therefore finishes the verification of Claim 1.

Claim 1 yields the following lemma.

**Lemma 1.** There exists a polynomial

\[
P = P\left(\{d^\mu z_\nu\}_{1 \leq \mu \leq k_n, 1 \leq \nu \leq n}\right)
\]

in the variables \( d^\mu z_\nu \) \((1 \leq \mu \leq k_n, 1 \leq \nu \leq n)\) with coefficients in \( \mathbb{C} \) such that \( P \) is not identically zero on \( W_{k_n,f} \) and \( \pi^* k_n P \) is identically zero on \( J_{k_n}(D) \).

Let \( h_{\alpha,\bar{\beta}} \) be the positive definite Hermitian matrix such that

\[
\sum_{\alpha,\beta=1}^{n} h_{\alpha,\beta}dz_\alpha d\bar{z}_\beta
\]

represents the Chern class of the ample line bundle \( D \). Denote by \( \nabla \) the covariant differentiation for sections of \( D \) with respect to the metric of \( D \) whose curvature form is \( \sum_{\alpha,\beta=1}^{n} h_{\alpha,\beta}dz_\alpha d\bar{z}_\beta \). Then for a local section \( s \) of the ample line bundle \( D \),

\[
\nabla s = ds + \left( \sum_{\alpha,\beta=1}^{n} h_{\alpha,\beta}z_\beta d\bar{z}_\alpha \right) s.
\]

The following lemma is an immediate consequence of Lemma 1.

**Lemma 2.** The pullback \( \tilde{P} \) to \( \mathbb{C}^n \) of \( P \) can be written in the form

\[
(3) \quad \tilde{P} = \sum_{\mu=0}^{k_n} \tilde{\rho}_\mu (\nabla^\mu \theta_D),
\]

where \( \tilde{\rho}_\mu \) is a polynomial in \( d^\mu z_\nu \) \((1 \leq \mu \leq k_n, 1 \leq \nu \leq n)\) whose coefficients are bounded smooth functions on \( \mathbb{C}^n \).
Let $A_r(\cdot)$ denote the average over the circle of radius $r$ centered at 0. Let $T(r, f, D)$ be the Nevanlinna characteristic function for the function $f$ and the ample line bundle $D$ of $A$ at the radius $r$. Let $\tilde{f} : \mathbb{C} \to \mathbb{C}^n$ be the lifting of $f : \mathbb{C} \to A$. For a meromorphic function $F$ on $\mathbb{C}$ we denote by $N(r, F, \infty)$ the counting function of the map $\mathbb{C} \to \mathbb{P}_1$ defined by $F$ and denote by $T(r, F)$ its Nevanlinna characteristic function (with respect to the line bundle of $\mathbb{P}_1$ of degree 1).

**Final Step in the Proof of Theorem A.** From $d\mu z^\nu = d\mu \log(e^z)$ and the logarithmic derivative lemma it follows that

$$A_r \left( \log^+ \left| \tilde{f}^* (d\mu z^\nu) \right| \right) = O(\log T(r, f, D) + \log r) \ ||$$

and

$$A_r \left( \log^+ \left| \tilde{f}^* \tilde{P} \right| \right) = O(\log T(r, f, D) + \log r) \ ||.$$  \hspace{1cm} (4)

Here we interpret $\tilde{f}^* d\mu z^\nu$ as a function of $\zeta$ which is the coefficient of the power of $d\zeta$. The notation $\tilde{f}^* \tilde{P}$ above and $\tilde{f}^* \left( \frac{\tilde{P}}{\theta_D} \right)$ carry similar meanings and are functions of $\zeta$. By (3) and the logarithmic derivative lemma it follows that

$$A_r \left( \log^+ \left| \tilde{f}^* \left( \frac{\tilde{P}}{\theta_D} \right) \right| \right) = O(\log T(r, f, D) + \log r) \ ||.$$  \hspace{1cm} (5)

From (3) we conclude that

$$N \left( r, \tilde{f}^* \left( \frac{\tilde{P}}{\theta_D} \right), \infty \right) \leq N_{k_n}(r, f, D).$$  \hspace{1cm} (6)

It follows from the First Main Theorem of Nevanlinna theory that

$$m(r, f, D) + N(r, f, D)$$

$$= T \left( r, \tilde{f}^* \left( \frac{1}{\theta_D} \right) \right) + O(1)$$

$$= T \left( r, \tilde{f}^* \left( \frac{\tilde{P}}{\theta_D} \right) \right) + O(1)$$

$$\leq T \left( r, \tilde{f}^* \left( \frac{\tilde{P}}{\theta_D} \right) \right) + T \left( r, \frac{1}{\tilde{P}} \right) + O(1).$$
Using (4), (5), and (6), we obtain

\[ m(r,f,D) + N(r,f,D) \leq N_{ku}(r,f,D) + O(\log T(r,f,D) + \log r) \quad ||. \]

Q.E.D.

**Remark.** The same arguments work for semi-abelian varieties with straightforward modifications.

**References.**

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