Clustering coefficient without degree correlations biases

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The clustering coefficient quantifies how well connected are the neighbors of a vertex in a graph. In real networks it decreases with the vertex degree, which has been taken as a signature of the network hierarchical structure. Here we show that this signature of hierarchical structure is a consequence of degree correlation biases in the clustering coefficient definition. We introduce a new definition in which the degree correlation biases are filtered out, and provide evidence that in real networks the clustering coefficient is constant or decays logarithmically with vertex degree.

The increasing availability of network data representing many real systems has motivated the development of new statistical measures to characterize large networks\(^1 23 4 5\). These measures has revealed that, as a difference with the classical Erdős Rényi\(^6\) random graph model, real networks are characterized by a power law distribution of vertex degrees\(^1 7 8\), a high clustering coefficient or transitivity\(^1 9\), and degree correlations between connected vertices\(^10 11 12\). Yet, it is important to characterize up to which extent the new measures provides new information about the studied networks. For instance, it has been shown that in some networks the degree correlations are a consequence of the existence of large degree vertices and, therefore, the sequence of vertex degrees is sufficient to characterize those networks\(^12 13 14\).

In this work we study the influence of degree correlations on the clustering coefficient. We show that most of the observed variations of the clustering coefficient with the vertex degrees\(^15\),\(^16\),\(^17\),\(^18\) are determined by the degree correlations among connected vertices. Based on this fact, we introduce a new definition of clustering coefficient, filtering out the effect of degree correlations. The similarities and differences between the two definitions are analyzed through the study of different real networks.

Consider undirected simple graphs on \(i = 1, \ldots, N\) vertices. Let \(k_i\) be the degree of a vertex and \(t_i\) the number of edges among its neighbors. The standard definition of local clustering coefficient is

\[
c_i = \frac{t_i}{\binom{k_i}{2}},
\]

where \(\binom{k_i}{2}\) is the number of pairs that can be made using \(k_i\) neighbors. Furthermore, to characterize the global clustering coefficient two different measures has been introduced. The first is just the average of \(c_i\) over all vertices with degree larger than one

\[
\langle c \rangle = \frac{\sum_{i|k_i>1} c_i}{\sum_{i|k_i>1} 1}.
\]

The second is obtained computing first the average of \(t_i\)

\[
\langle t \rangle = \frac{\sum_{i|k_i>1} t_i}{\sum_{i|k_i>1} 1},
\]

and \(\binom{k_i}{2}\) and then their ratio

\[
C = \frac{\sum_{i} t_i}{\sum_{i} \binom{k_i}{2}}.
\]

As noticed in\(^19\), the two definitions of global clustering coefficient may give different values. Consider, for instance, a double star of \(N\) vertices (Fig. 1). In this case \(\langle c \rangle \approx 1, C = O(1/N)\) and the two global clustering coefficients dramatically differ for \(N \gg 1\). The limitations in the clustering coefficient definition are not only related to the way averages are computed. The local clustering coefficient of any of the two central vertices of the double star, vertex 1 for instance, is \(c_1 = O(1/N)\), approaching zero for \(N \gg 1\). We cannot, however, increase the number of connections among the neighbors of vertex 1 without increasing the degree of its neighbors. In this sense, the neighbors of vertex 1 are as clustered as they can be, in contradiction with the small value of \(c_1\).

This example shows that the local clustering coefficient of a large degree vertex connected to vertices with much smaller degrees will be always small, no matter how its neighbors are interconnected. We would like instead a measure of clustering coefficient that allow us to quantify the connectivity among the neighbors of a vertex, independently of its degree and the degree of its neighbors. The clustering coefficient is a three vertex correlation measure and, as it is the general case in statistics, to define a three point correlation measure we should filter out two point correlations, represented here by the degree correlations between connected vertices. We tackle

FIG. 1: Double star with two vertices, 1 and 2, connected to \(N - 2\) other vertices. The neighbors of vertex 1 (or 2) are connected as most as their degrees allow. Yet, with the old definition of clustering coefficient we obtain \(c_1 = O(1/N)\), approaching zero in the limit \(N \gg 1\).
this problem defining the clustering coefficient relative to the maximum possible number of edges between the neighbors of a vertex, given their degree sequence. Let \( \omega_i \) be the maximum number of edges that can be drawn among the \( k_i \) neighbors of a vertex \( i \), given the degree sequence of its neighbors. A neighbor \( j \) can have at most \( \min(k_i - 1, k_j - 1) \) edges with the other neighbors, therefore

\[
\omega_i \leq \Omega_i = \left\lfloor \frac{1}{2} \sum_{\text{neighbors}} \min(k_i, k_j) - 1 \right\rfloor \leq \binom{k_i}{2}.
\] (4)

While \( \binom{k_i}{2} \) takes into account only the degree of the vertex, \( \Omega_i \) considers that occasionally, not all the \( k_i - 1 \) excess edges are available at the neighbors of \( i \). \( \omega_i \) considers, in addition, the possibility of the excess edges to actually form triangles. \( \omega_i \) can be computed using the following algorithm [20]: 1- Starting from the neighbor’s degree sequence \( \{k_1, \ldots, k_n\} \) \( (n = k_i) \), construct the list \( \{\min(k_1, k_i) - 1, \ldots, \min(k_i, k_n) - 1\} \), arranged in a decreasing order. 2- Draw an edge from the first element to as many as possible other elements in the list, always going from largest to smaller. Each time an edge is drawn, one is subtracted from the remaining degree of the connected vertices. 3 - Remove the first element and any zero from the list and sort the list in decreasing order. 4- Repeat the process and stop when the list is empty. The number of maximum possible connections \( \omega_i \) is the total number of edges drawn (see Fig. 2).

A proper definition of local clustering coefficient, removing the effects of degree correlations, is

\[
\tilde{c}_i = \frac{t_i}{\omega_i}.
\] (5)

and the two different measures of global clustering coefficient are the following. (i) If all the neighbors of a vertex has degree one (star) then its clustering coefficient is undefined. Indeed, the concept of clustering is meaningless for the central vertex of a star, as it is meaningless for degree one vertices. (ii) \( \tilde{c}_i \geq c_i \), as follows from (4). Therefore, when the clustering is one by the old definition it is one by the new definition. Notice that the opposite is not necessarily true (see Fig. 2(c)).

(iii) When all the \( k_i \) neighbors of a vertex \( i \) have degrees larger or equal to the degree of the vertex itself (a regular graph, for instance) \( \tilde{c}_i = c_i \).

The example in Fig. 2 shows how the old definition underestimates the clustering around a given vertex \( i \). In this case, the number of edges between neighbors is as large as it can be given their degree sequence. This picture is not captured by the clustering coefficient according to the old definition \( (c_i = 0.4) \), but it is correctly quantified using the new definition \( (\tilde{c}_i = 1) \). In the following we compare the old and new clustering coefficient definitions using the graph representation of four real systems. The degree of correlations present on these graphs is quantified by the assortativity coefficient \( r \) [11], taking values between -1 (highly disassortative) to 1 (highly assortative). The systems considered are, in increasing order of assortativity, 1- the autonomous system representation of the Internet, as for April 2001 [21], 2- the protein-protein interaction network of the yeast *Saccharomyces cerevisiae* [22], 3- the semantic web of English synonyms [17], and 4- the co-authorship network of mathematical publications between 1991 and 1999 [23].

In Table I we show the two global clustering coefficients as computed with the old and new definitions. For the two disassortative graphs \( (r < 0) \), there is an orders of magnitude difference between the global clustering coefficients \( c \) and \( \tilde{c} \) computed with the old definition. With the new definition, however, both global measures of clustering coefficient \( \tilde{c} \) gives values of the same order, independently of the degree correlations.

![Diagram](https://via.placeholder.com/150)

**FIG. 2:** Algorithm to compute \( \omega_i \). (a) A vertex \( i \) (open circle) is connected to five neighbors (filled circles) with degree sequence \( \{8, 7, 2, 2, 2\} \). (b) Since each neighbor can be connected at most with four other neighbors, we replace the neighbors degree sequence (lowest raw) by \( \{4, 4, 1, 1, 1\} \) (middle raw). It is easy to see that after connecting the first neighbor to all others, we get 4 triangles and 3 extra edges that can’t be used anymore (upper raw). Summarizing, for this example, \( \omega_i = 4, \Omega_i = 5 \) and \( \binom{5}{2} = 10 \). (c) Subgraph with maximum number of edges among the neighbors, with \( c_i = 0.4 \) and \( \tilde{c}_i = 1 \).

| Network                | \( r \)  | \( < c > \) | \( C \)  | \( < \tilde{c} > \) | \( \tilde{C} \) |
|------------------------|---------|-------------|--------|---------------------|-------------|
| Internet               | -0.19   | 0.45        | 0.0090 | 0.49                | 0.45        |
| protein interaction    | -0.13   | 0.12        | 0.055  | 0.16                | 0.19        |
| semantic               | 0.085   | 0.75        | 0.31   | 0.83                | 0.59        |
| co-authorship          | 0.67    | 0.65        | 0.56   | 0.78                | 0.85        |

\[
\langle \tilde{c} \rangle = \frac{\sum_{i} t_i \omega_i > 0}{\sum_i \omega_i} \quad \text{and} \quad \tilde{C} = \frac{\sum_i t_i}{\sum_i \omega_i}.
\] (6)

Some general properties of the new definition of clustering coefficient are the following. (i) If all the neighbors of a vertex has degree one (star) then its clustering coefficient is undefined. Indeed, the concept of clustering is meaningless for the central vertex of a star, as it is meaningless for degree one vertices. (ii) \( \tilde{c}_i \geq c_i \), as follows from (4). Therefore, when the clustering is one by the old definition it is one by the new definition. Notice that the opposite is not necessarily true (see Fig. 2(c)). (iii) When all the \( k_i \) neighbors of a vertex \( i \) have degrees larger or equal to the degree of the vertex itself (a regular graph, for instance) \( \tilde{c}_i = c_i \).
Another characteristic feature of the old definition of clustering coefficient is that, when the average is restricted to vertices with the same degree \( \langle c \rangle_k \), it decays as \( \langle c \rangle_k \sim k^{-\alpha} \) with vertex degree \( k \). This decay can be observed in Fig. 3 for the four graphs considered here, being more pronounced for the two disassortative graphs in Fig. 3(a) and (b), and almost absent for the highly assortative co-authorship graph in Fig. 3(d). In contrast, when computed with the new definition (2), \( \langle \tilde{c} \rangle_k \) does not exhibit a strong variation with increasing vertex degree (see Fig. 3).

In particular, the decreasing trend is completely absent for the Internet (Fig. 3(a)), and the variations between the smallest and largest new clustering coefficient are no more than a factor of two, indicating variations previously observed with the standard definition are reflecting degree correlations. The large variations of \( \langle c \rangle_k \) with the vertex degree \( k \) have been interpreted as the existence of a hierarchical structure, with high degree vertices interconnecting highly connected subgraphs made of smaller degree vertices, but with no or few connections among vertices in different subgraphs. The existence of this hierarchical structure, however, was already predicted from the analysis of the degree correlations. The present work make the bridge between these two differences approaches to quantify the hierarchical structure of the Internet, showing that the variations in the clustering coefficient with the vertex degrees, as measured with the old definition, are just reflecting the existence of degree correlations. These conclusions are also applicable for the protein-protein interaction graph, with a degree of disassortative close to that of the Internet graph.

In the case of the Internet we can also follow changes in the clustering coefficient as the network evolves, with around 3000 vertices in 1997 to 10000 vertices in 2001. \( \langle \tilde{c} \rangle_k \) remains essentially stationary within this period (data not shown), as does \( \langle c \rangle_k \). In contrast, in random graphs with fixed degree distribution and degree correlations the local clustering coefficient approaches zero with increasing graph size, independently of the vertex degree \( k \). Therefore, the Internet is more clustered than expected from the degree distribution and degree correlations alone.

In the case of the semantic web (Fig. 3(c)), although the clustering coefficient variations are reduced after filtering out the degree correlations, there is still a logarithmic decrease with increasing the vertex degree (see inset of Fig. 3(c)). Using a deterministic growing graph model introduced in (27), we show that this logarithmic decay may be the general case for graphs where \( \langle c \rangle_k \sim 1/k \). In the deterministic model, we start with one edge at time \( t = -1 \). At each time step we create a new triangle on each existing edge by connecting its two endpoints to a new vertex. At time \( t = 0 \) we get one triangle and at time \( t = 1 \), we will have the triangle from the previous step and three new ones, each is using one edge from the existing old triangle and two new edges with a new vertex. At each time step we create a new triangle on each existing edge by connecting its two endpoints to a new vertex. At time \( t = 0 \) we get one triangle and at time \( t = 1 \), we will have the triangle from the previous step and three new ones, each is using one edge from the existing old triangle and two new edges with a new vertex.

![FIG. 3: Average clustering as a function of the vertex degree, as computed using the old definition (circles), the new definition approximating \( \omega_i \) by \( \Omega_i \) (squares) and the new definition using \( \omega_i \) (triangles). The graphs are shown in increasing order of their assortativity, with the most disassortative graph in the top, and the more assortative graph on the bottom.](image)
to the vertex degree. This observation is in agreement with the se-
hibit a logarithmic decrease with increasing the vertex degree. These results will eventually force us to reevaluate the clustering based analysis of complex networks, and other approaches \cite{18,26,27,28} based on this magnitude.

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\begin{thebibliography}{99}
  \bibitem{1} R. Albert and A.-L. Barabási, Rev. Mod. Phys. \textbf{74}, 47 (2001).
  \bibitem{2} S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. \textbf{51}, 1079 (2002).
  \bibitem{3} M. E. Newman, SIAM Rev. \textbf{45}, 167 (2003).
  \bibitem{4} S. Bornholdt and H. G. Schuster, eds., \textit{Handbook of Graphs and Networks: From the Genome to the Internet} (Wiley-VCH, Weinheim, 2003).
  \bibitem{5} R. Pastor-Satorras and A. Vespignani, \textit{Evolution and structure of the Internet: A Statistical Physics approach} (Cambridge University Press, Cambridge, 2004).
  \bibitem{6} P. Erdős and A. Rényi, Publications Mathematicaa \textbf{6}, 290 (1959).
  \bibitem{7} M. Faloutsos, P. Faloutsos, and C. Faloutsos, Comput. Commun. Rev. \textbf{29}, 251 (1999).
  \bibitem{8} A.-L. Barabási, R. Albert, H. Jeong, and G. Bianconi, Science \textbf{287}, 2115a (2000).
  \bibitem{9} D. J. Watts and S. H. Strogatz, Nature \textbf{393}, 440 (1998).
  \bibitem{10} R. Pastor-Satorras, A. Vázquez, and A. Vespignani, Phys. Rev. Lett. \textbf{87}, 258701 (2001).
  \bibitem{11} M. E. J. Newman, Phys. Rev. Lett. \textbf{89}, 208701 (2002).
  \bibitem{12} S. Maslov and K. Sneppen, Science \textbf{296}, 910 (2002).
  \bibitem{13} J. Park and M. E. J. Newman, Phys. Rev. E \textbf{68}, 026112 (2003).
  \bibitem{14} M. Catanzaro, M. B. na, and R. Pastor-Satorras, \texttt{arXiv:cond-mat/0408110}.
  \bibitem{15} A. Vázquez, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. E \textbf{65}, 066130 (2002).
  \bibitem{16} E. Ravasz, A. L. Somera, D. A. Mongru, Z. N. Oltvai, and A.-L. Barabási, Science \textbf{297}, 1515 (2002).
  \bibitem{17} E. Ravasz and A.-L. Barabási, Phys. Rev. E p. 026112 (2003).
  \bibitem{18} A. Vázquez, Phys. Rev. E \textbf{67}, 056104 (2003).
  \bibitem{19} B. Bollobás and O. M. Riordan, in \textit{Handbook of Graphs and Networks: From the Genome to the Internet}, edited by S. Bornholdt and H. G. Schuster (Wiley-VCH, Weinheim, 2003), pp. 1–34.
  \bibitem{20} S. N. Soffer and A. Vázquez, unpublished.
  \bibitem{21} \textit{The national laboratory for applied network research (NLARR), national science foundation, http://moat.nlanr.net/}.
  \bibitem{22} \textit{The Database of Interacting Proteins (DIP), http://dip.doe-mbi.ucla.edu/}
  \bibitem{23} A.-L. Barabási, H. Jeong, Z. Néda, E. Ravasz, A. Schubert, and T. Vicsek, Physica A \textbf{311}, 590 (2002).
  \bibitem{24} S. N. Dorogovtsev, arXiv:cond-mat/0308444.
  \bibitem{25} S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. E \textbf{65}, 066122 (2002).
  \bibitem{26} M. E. J. Newman, Phys. Rev. E \textbf{68}, 026121 (2003).
  \bibitem{27} A. Barrat, M. Barthélemy, R. Pastor-Satorras, and A. Vespignani, Proc. Natl. Acad. Sci. USA \textbf{101}, 3747 (2004).
  \bibitem{28} A. Vázquez, R. Dobrin, D. Sergi, J.-P. Eckmann, Z. Oltvai, and A.-L. Barabási, arXiv:cond-mat/0408431.
\end{thebibliography}