Cosmological Solutions in Nonlocal Models

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Abstract—A non-local modified gravity model with an analytic function of the d’Alembert operator that has been proposed as a possible way of resolving the singularities problems in cosmology is considered. We show that the anzats that is usually used to obtain exact solutions in this model provides a connection with $f(R)$ gravity models.

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1. INTRODUCTION

General relativity (GR) being a very efficient and simple theory of gravity featuring the second order equations of motion suffers from certain weaknesses. Among them the initial singularity problem which arises in the framework of the inflationary paradigm. This is the reflection of UV incompleteness of GR. One possibility to improve the ultraviolet behavior and even to get a renormalizable theory of quantum gravity is to add higher-derivative terms to the Einstein–Hilbert action. As one of the first papers we can mention \[1\] where curvature squared corrections were considered. Unfortunately, this model (and models with more than two but finite number of derivatives in equations of motion in general) has ghosts. An intriguing possibility to overcome this problem is to consider a non-local gravity with infinitely many derivatives.

The theoretical motivation behind an introduction of infinite derivative non-local corrections into a local theory is the string field theory (SFT) \[2\]. Such corrections naturally arise in the SFT and usually consist of exponential functions of the d’Alembertian operator acting on fields. The majority of non-local cosmological models motivated by such structures explicitly include an analytic or meromorphic function of the d’Alembertian operator \[3–9\].

Both GR and the modified gravity models yield a non-integrable system of equations of motion with only particular analytic solutions known. Having an analytic solution however is crucial in considering perturbations which are in turn the cornerstone of any cosmological model. Needless to say that finding analytic solutions in non-local nonlinear equations is an extremely hard task. Some studies of nonlocal modifications of GR resulted in analytic solutions can be found in \[5, 6, 9\].

The key to finding solutions in the non-local gravity models of interest is to employ an ansatz which relates finite powers of the d’Alembertian operator acting on the scalar curvature. The ansatz itself reduces the initial non-local model to an effective local model with more than two but finite number of derivatives. This does not mean that ghosts should appear as any ansatz is a relation in the background while perturbations enjoy all the new properties of the full non-local structure. Provided there is a background configuration satisfying the ansatz one simplifies the problem of solving the equations of motion considerably.

2. ACTION AND EQUATIONS OF MOTION FOR STRING-INSPIRED NON-LOCAL GRAVITY

The nonlocally modified gravity proposed in \[5\] is described by the following action:

\[ S = \int d^4 x \sqrt{-g} \left[ \frac{M_p^2}{2} R + \frac{1}{2} R \mathcal{F} \left( \frac{\Box}{M_\odot^2} \right) R - \Lambda + \mathcal{L}_M \right], \quad (1) \]

where $M_p$ is the Planck mass, $\Lambda$ is the cosmological constant, $M_\odot$ is the mass scale at which the higher derivative terms in the action become important, $\mathcal{L}_M$ is the matter Lagrangian. We use the convention where the metric $g$ has the signature $(-, +, +, +)$. An analytic function $\mathcal{F}(\Box/M_\odot^2) = \sum n \geq 0 f_n \Box^n$ is an ingredient inspired by the SFT. The operator $\Box$ is the covariant d’Alembertian. In the case of an infinite series we have a non-local action.

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Introducing dimensionless coordinates $\tilde{x}_\mu = M_\phi x_\mu$ and $\tilde{M}_\rho = M_\rho / M_\phi$ we get $\tilde{\mathcal{F}}(\tilde{x}/\tilde{M}_\phi) = \mathcal{F}(x)$, where $\Box$ is the d’Alembertian in terms of dimensionless coordinates. We shall use dimensionless coordinates only (omitting the bars).

A straightforward variation of action (1) yields the following system:

\[
\frac{1}{2} \left[ M_\rho^2 + 2 \tilde{\mathcal{F}}(\Box) R \right](2 R^\mu_\nu - \delta^\mu_\nu R) = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \left( \partial^\mu_\rho \partial^\nu_\sigma R \partial^\rho_\sigma \Box^{n-l-1} R + g^{\mu \rho} \partial^n_\rho \partial^{n-l-1} R - \delta^\mu_\nu (g^{\rho \sigma} \partial^n_\rho \partial^{n-l-1} R + \Box^{n-l-1} R) \right) + \frac{1}{2} R \tilde{\mathcal{F}}(\Box) R \delta^\mu_\nu - \Lambda \delta^\mu_\nu + T^\mu_\nu,
\]

where $\nabla_\mu$ is the covariant derivative, $T^\mu_\nu$ is the energy-momentum tensor of matter. The trace equation is

\[
\frac{1}{2} R \tilde{\mathcal{F}}(\Box) R \delta^\mu_\nu - \Lambda \delta^\mu_\nu + T^\mu_\nu = 0.
\]

3. THE ANSatz FOR FINDING EXACT SOLUTIONS

It has been shown in [5, 6, 8, 9] that the following ansatz

\[
\Box R = r_1 R + r_2,
\]

with constants $r_1 \neq 0$ and $r_2$, is useful in finding exact solutions.

If the scalar curvature $R$ satisfies (4), then Eqs. (2) are

\[
\frac{1}{2} M_\rho^2 R - \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \left( \partial^\mu_\rho \partial^\nu_\sigma R \partial^\rho_\sigma \Box^{n-l-1} R + 2 \Box^l R \Box^{n-l} R \right) - 6 \Box \tilde{\mathcal{F}}(\Box) R = 4\Lambda - T^\mu_\nu.
\]

We proceed to consider a traceless radiation along with a cosmological constant. Under condition (4) the trace equation with $T^\mu_\nu = 0$ becomes especially simple:

\[
AR + \tilde{\mathcal{F}}(r_1)(2 r_2 R^2 + \partial_\rho R \partial^\rho R) + B = 0,
\]

where constants $A$ and $B$ are defined as follows:

\[
A = 4 \tilde{\mathcal{F}}(r_1) r_2 - M_\rho^2 - \frac{2 r_2}{r_1} (\tilde{\mathcal{F}}(r_1) - f_0) + 6 \tilde{\mathcal{F}}(r_1) r_1,
\]

\[
B = 4\Lambda - \frac{r_2 M_\rho^2 + r_2 A}{r_1}.
\]

The simplest way to get a solution to Eq. (6) is to impose $\tilde{\mathcal{F}}(r_1) = 0$ and to put $A = B = 0$. These relations fix values of $r_1$, $r_2$ and the cosmological constant:

\[
r_2 = - \frac{r_2 M_\rho^2 - 6 \tilde{\mathcal{F}}(r_1) r_1}{2 [\tilde{\mathcal{F}}(r_1) - f_0]},
\]

\[
\Lambda = -\frac{r_2 M_\rho^2 - 6 \tilde{\mathcal{F}}(r_1) r_1}{8 [\tilde{\mathcal{F}}(r_1) - f_0]}.
\]

Under conditions $A = B = 0$ the complete set of Eqs. (5) simplifies to

\[
2 \tilde{\mathcal{F}}(r_1)(R + 3 r_1) \delta^\mu_\nu = T^\mu_\nu + 2 \tilde{\mathcal{F}}(r_1) \times \left[ g^{\mu \rho} \nabla_\rho R - \frac{1}{4} \delta^\mu_\nu (R^2 + 4 r_1 R + r_2) \right].
\]

In general one is required to include radiative sources to get exact solutions of all equations [6]. Let us emphasize that equations are general and we do not take into account the properties of the metric.

4. RELATION BETWEEN THE ANSATS AND $R^2$ MODIFIED GRAVITY

It is well-known [10] that the $f(R)$ gravity model described by the action:

\[
S_f = \int d^4 x \sqrt{-g} \left[ M_\rho^2 f(R) + \mathcal{L}_M \right],
\]

has the following equations:

\[
M_\rho^2 f^{(1)}(R) R^\rho_\nu - \frac{1}{2} f(R) \delta^\rho_\nu
\]

\[
- (g^{\mu \rho} \nabla_\rho \partial_\nu \delta^\mu_\sigma f^{(1)}(R)) = T^\rho_\nu,
\]

where $f^{(1)}(R)$ is the first derivative of $f(R)$ with respect to $R$.

From (10) for a traceless matter, we get the following trace equation:

\[
f^{(1)}(R) R - 2 f(R) + 3 \Box f^{(1)}(R) = 0.
\]
One can see that at
\[ f(R) = \frac{\bar{F}(r_1)}{M_p^2} [R^2 + 6r_1R + 3r_1^2]. \]  
(12)

Eq. (11) is coincide with the ansatz (4). Moreover, Eqs. (8) are equivalent to (10). Note that condition (7) gives the following connection:
\[ M_p^2 = \frac{2}{r_1^2} [3\bar{F}(r_1)r_1^2 - (\bar{F}(r_1) - f_0)r_1]. \]

We proved that any solution of the modified gravity model (9) with \( f(R) \) given by (12) and traceless matter is a solution of the initial system of Eqs. (2) on condition that \( A = B = \bar{F}(r_1) = 0 \) and the anzats (4) is satisfied.

CONCLUSIONS

We have shown that any solution of the \( R^2 \) modified gravity model (9) either without matter, or with the radiation is a solution of the corresponding SFT inspired non-local gravity models (1). We do not assume some special form of the metric to prove this result. Note that the initial nonlocal model is not totally equivalent to an \( R^2 \) model. Full consideration of the model includes both the background solution, and perturbations, which may not satisfy the ansatz (4) in general. The analysis of the cosmological perturbations in the considered non-local models is given in [8]. Note that, the \( R^2 \) modified gravity is very well studied [10] and looks as a realistic modification of gravity. In particular, the modern cosmological and astrophysical data [11] confirm the predictions of the Starobinsky inflationary model [12].

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REFERENCES

1. K. S. Stelle, “Renormalization of higher derivative quantum gravity,” Phys. Rev., Ser. D 16, 953 (1977).
2. I. Ya. Aref’eva, D. M. Belov, A. A. Giryavets, et al., “Noncommutative field theories and (super)string field theories,” hep-th/0111208.
3. S. Jhingan, S. Nohji, S. D. Odintsov, et al., “Phantom and non-phantom dark energy: the cosmological relevance of non–locally corrected gravity,” Phys. Lett., Ser. B 663, 424 (2008); arXiv:0803.2613; T. S. Koivisto, “Dynamics of nonlocal cosmology,” Phys. Rev., Ser. D 77, 123513 (2008); arXiv:0803.3399; S. Capozziello, E. Elizalde, S. Nofrj, and S. D. Odintsov, “Accelerating cosmologies from nonlocal higher-derivative gravity,” Phys. Lett., Ser. B 671, 193 (2009); arXiv:0809.1535; L. Modesto, “Super-renormalizable quantum gravity,” Phys. Rev., Ser. D 86, 044005 (2012), arXiv:1107.2403; E. Elizalde, E. O. Pozdeeva, and S. Yu. Vernov, “Reconstruction procedure in nonlocal cosmological models,” Class. Quantum Grav. 30, 035002 (2013), arXiv:1209.5957; A. S. Koshelev, “Stable analytic bounce in non-local Einstein-Gauss-Bonnet cosmology,” Class. Quantum Grav. 30, 155001 (2013), arXiv:1302.2140; E. Elizalde, E. O. Pozdeeva, S. Yu. Vernov, and Y.-I. Zhang, “Cosmological solutions of a nonlocal model with a perfect fluid,” J. Cosmol. Astropart. Phys. 1307, 034 (2013), arXiv:1302.4330; S. Deser and R. P. Woodard, “Observational viability and stability of nonlocal cosmology,” J. Cosmol. Astropart. Phys. 1311, 036 (2013), arXiv:1307.6639; R. P. Woodard. “Nonlocal models of cosmic acceleration,” Found. Phys. 44, 213 (2014), arXiv:1401.0254.
4. N. Barnaby, T. Biswas, and J. M. Cline, “p–adic inflation,” J. High Energy Phys. 0704, 056 (2007), hep-th/0612230; A. S. Koshelev, “Non-local SFT tachyon and cosmology,” J. High Energy Phys. 0704, 029 (2007), arXiv:hep-th/0701103; I. Ya. Aref’eva, L. V. Joukovskaya, and S. Yu. Vernov, “Bouncing and accelerating solutions in nonlocal stringy models,” J. High Energy Phys. 0707, 087 (2007), arXiv:hep-th/0701184; D. J. Mulryne and N. J. Nunes, “Diffusing non-local inflation: solving the field equations as an initial value problem,” Phys. Rev., Ser. D 78, 063519 (2008), arXiv:0805.0449; I. Ya. Aref’eva and A. S. Koshelev, “Cosmological signature of tachyon condensation,” J. High Energy Phys. 0809, 068 (2008), arXiv:0804.3570; A. S. Koshelev and S. Yu. Vernov, “Cosmological perturbations in SFT inspired non-local scalar field models,” Eur. Phys. J., Ser. C 72, 2198 (2012), arXiv:0903.5176; G. Calcagni and G. Nardelli, “Nonlocal gravity and the diffusion equation,” Phys. Rev., Ser. D 82, 123518 (2010), arXiv:1004.5144; A. S. Koshelev and S. Yu. Vernov, “Analysis of scalar perturbations in cosmological models with a non-local scalar field,” Class. Quant. Grav. 28, 085019 (2011), arXiv:1009.0746; A. S. Koshelev, “Modified non-local gravity,” Rom. J. Phys. 57, 894 (2012), arXiv:1112.6410.
5. T. Biswas, A. Mazumdar, and W. Siegel, “Bouncing universes in string-inspired gravity,” J. Cosmol. Astropart. Phys. 0603, 009 (2006), arXiv:hep-th/0508194.
6. T. Biswas, T. Koivisto, and A. Mazumdar, “Towards a resolution of the cosmological singularity in non-local higher derivative theories of gravity,” J. Cosmol. Astropart. Phys. 1011, 008 (2010), arXiv:1005.0590.
7. T. Biswas, E. Gerwick, T. Koivisto, and A. Mazumdar, “Towards singularity and ghost free theories of gravity,” Phys. Rev. Lett. 108, 031101 (2012), arXiv:1110.5249.
8. T. Biswas, A. S. Koshelev, A. Mazumdar, and S. Yu. Vernov, “Stable bounce and inflation in non-local higher derivative cosmology,” J. Cosmol. Astropart. Phys. 1208, 024 (2012), arXiv:1206.6374.
9. A. S. Koshelev and S. Yu. Vernov, “On bouncing solutions in non-local gravity,” Phys. Part. Nucl. 43, 666 (2012), arXiv:1202.1289; I. Dimitrijevic, I. Dragovich, J. Gruic, and Z. Rakic, “New cosmological solutions in nonlocal modified gravity,” arXiv:1302.2794.
10. T. Ruzmaikina and A. Ruzmaikin, “Quadratic corrections to the Lagrangian density of the gravitational field and the singularity,” Sov. Phys. JETP 30, 372 (1970); V. Ts. Gurovich and A. A. Starobinsky, “Quantum effects and regular cosmological models,” Sov. Phys. JETP 50, 844 (1979); K. I. Maeda, “Towards the Einstein–Hilbert action via conformal transformation,” Phys. Rev., Ser. D 39, 3159 (1989); S. Carloni, P. K. S. Dunsby, S. Capozziello, and A. Troisi, “Cosmological dynamics of $R^n$ gravity,” Class. Quant. Grav. 22, 4839 (2005), arXiv:gr-qc/0410046; J. D. Barrow and S. Hervik, “Evolution of universes in quadratic theories of gravity,” Phys. Rev., Ser. D 74, 124017 (2006), arXiv:gr-qc/0610013; M. M. Ivanov and A. V. Toporensky, “Stable super-inflating cosmological solutions in $f(R)$-Gravity,” Int. J. Mod. Phys., Ser. D 21, 1250051 (2012), arXiv:1112.4194; S. Domazet, V. Radovanovic, M. Simonovic, and H. Stefancic, “On analytical solutions of $f(R)$ modified gravity theories in FLRW cosmologies,” Int. J. Mod. Phys., Ser. D 22, 1350006 (2013), arXiv:1203.5220; S. Capozziello and V. Faraoni, “Beyond Einstein gravity: a survey of gravitational theories for cosmology and astrophysics,” in Fund. Theor. Phys. (Springer, N.Y., 2011), Vol. 170.

11. P. A. R. Ade et al. (Planck Collab.), Planck 2013 Results, XXII: Constraints on Inflation, arXiv:1303.5082.

12. A. A. Starobinsky, “Relict gravitation radiation spectrum and initial state of the universe,” JETP Lett. 30, 682 (1979); A. A. Starobinsky, “A new type of isotropic cosmological models without singularity,” Phys. Lett., Ser. B 91, 99 (1980).

13. A. Vilenkin, “Classical and Quantum Cosmology of the Starobinsky Inflationary Model,” Phys. Rev. D 32, 2511 (1985).