The importance of being scrambled: supercharged Quasi Monte Carlo

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Abstract: In many financial applications Quasi Monte Carlo (QMC) based on Sobol’ low-discrepancy sequences (LDS) outperforms Monte Carlo showing faster and more stable convergence. However, unlike MC QMC lacks a practical error estimate. Randomized QMC (RQMC) method combines the best of two methods. Application of scrambled LDS allow to compute confidence intervals around the estimated value, providing a practical error bound. Randomization of Sobol’ LDS by two methods: Owen’s scrambling and digital shift are compared considering computation of Asian options and Greeks using hyperbolic local volatility model. RQMC demonstrated the superior performance over standard QMC showing increased convergence rates and providing practical error bounds around the estimated values. Efficiency of RQMC strongly depend on the scrambling methods. We recommend using Sobol’ LDS with Owen’s scrambling. Application of effective dimension reduction techniques such as the Brownian bridge or PCA is critical to dramatically improve the efficiency of QMC and RQMC methods based on Sobol’ LDS.

Keywords: Quasi Monte Carlo, Randomized Quasi Monte Carlo, Sobol sequences, Monte Carlo option pricing, Skew hyperbolic local volatility model

1 Introduction

Monte Carlo (MC) is a unique universal method widely used in valuation of complex financial instruments and risk management engines. Its convergence rate $O\left(\frac{1}{\sqrt{N}}\right)$ does not depend on the number of dimensions $d$ although it is
rather slow. Here $N$ is the number of sampled points or the number of path. It also provides practical error estimates through the computation of confidence intervals.

Unlike random numbers on which MC is based which are known to have bad uniformity properties, deterministic low-discrepancy sequences (LDS) are designed to fill multidimensional space as uniformly as possible. It results in the significantly improved convergence rate of the Quasi Monte Carlo (QMC) method based on LDS. Asymptotically, it is $O\left(\frac{1}{N}\right)$, which is much higher than the convergence rate of MC. However, this theoretical estimate depends on the dimensionality of the problem and in practice the number of paths $N$ required to achieve a given standard error may not be any lower than that of MC. For problems in quantitative finance, dimensions $d$ can reach many thousands. It has led to a misconception that QMC is not efficient in high dimensions. In practice, the effectiveness of QMC depends not on the nominal dimension $d$ but on the so-called effective dimensions. There are many problems for which the $d$-dimensional function $f$ is dominated by the first few variables (low effective dimension in the truncation sense) or can be well approximated by a sum of low-dimensional function in the ANOVA decomposition (low effective dimension in the superposition sense). Such functions are very efficiently integrated by QMC achieving the convergence rate close to $O\left(\frac{1}{N}\right)$. It has been shown that typically problems in finance have low effective dimensions or that effective dimensions can be reduced by applying special sampling schemes.

One of the main drawbacks of the QMC methods is that since LDS are deterministic there is no statistical method of computing the standard error of the estimate. It means that in particular there is no clear termination criterion for stopping simulation after reaching the required tolerance. There are techniques, known under the name of randomized QMC (RQMC), which introduce appropriate randomizations in the construction of LDS. It allows for measuring integration errors through a confidence interval similarly to MC while preserving and often improving the convergence rate of QMC. While the superior performance of the QMC methods based on Sobol LDS has been widely studied and used, there is only a handful of pure academic papers in which RQMC methods were applied to problems in finance. It was shown that RQMC offers both enhanced efficiency in comparison with pure QMC and ability to produce confidence intervals, however it is still not widely used in finance by practitioners. This work aims to bridge this gap. We consider two popular methods of LDS randomisation: random digital shift and Owen’s scrambling.

This paper is organized as follows: Section 2 provides a brief review of MC, QMC and RQMC methods. In Section 3, we introduce the ANOVA decomposition and the concept of effective dimensions. The time-homogeneous hyperbolic local volatility model is presented in Section 4. Two different time discretization schemes are considered in the next Section. In Section 6, Monte Carlo simulation of option pricing and Greeks are discussed. Section 7 presents numerical results. Finally, the conclusions are given in the last Section.
2 MC, QMC and RQMC methods

The MC method solves a problem by simulating the underlying process and then calculating the average result of the process. It can be formulated as computation of the multidimensional integral

\[ I[f] = \int_{H^d} f(X) dX. \tag{1} \]

Here function \( f(X) \) is integrable in the \( d \)-dimensional unit hypercube \( H^d \). The MC quadrature formula is based on the probabilistic interpretation of an integral as an expectation. The standard MC estimator of the expectation is

\[ \hat{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} f(X_i), \tag{2} \]

where \( \{X_i\} \) is a sequence of random points of length \( N \) uniformly distributed in \( H^d \). The approximation \( \hat{\mu}_N \) converges to \( I[f] \) with probability 1. An integration error according to the Central Limit Theorem is \( \sigma(f) \sqrt{\frac{1}{N}} \), where \( \sigma^2(f) \) is the function variance. Although typically \( \sigma^2(f) \) is unknown, an unbiased estimate of it can be obtained as well as confidence intervals (Table 1). The convergence rate of MC does not depend on the number of variables \( d \) but it is rather slow. It is known that random number sampling is prone to clustering. As new points are added randomly, they do not necessarily fill the gaps between already sampled points.

In the classical Quasi-Monte Carlo (QMC) method independent random points \( \{X_i\} \) are replaced by a deterministic set of points such as LDS, which are designed to cover the unit hypercube more uniformly than random points. Successive LDS points “know” about the position of previously sampled points and “fill” the gaps between them. The QMC algorithm for the evaluation of the integral (1) has a form similar to (2) where instead of random points \( \{X_i\} \) LDS points \( \{Q_i\}, Q_i \in H^d, i = 1, \ldots, N \) are used. Sobol LDS also known as digital \((t, d)\) sequences in base 2 are the most known and widely used LDS in finance due to their efficiency (Glasserman, 2004).

The efficiency of a particular Sobol’ LDS generator depends on the so-called direction numbers. In this work we used BRODA’s SobolSeq generator (BRODA Ltd, 2022). Sobol’ sequences produced by BRODA’s SobolSeq satisfy additional uniformity properties: Property A for all dimensions (currently maximum dimension \( d = 131072 \)) and Property \( A' \) for adjacent dimensions. It has been shown in Sobol’ et al. (2011) on a number of different tests that BRODA’s SobolSeq generators generally outperform other considered in the paper LDS generators. These results were corroborated in other publications (Renzitti et al., 2020).

A major drawback of the QMC method is the lack of practical estimates of the integration error. A classical worst-case error bound for numerical integration by QMC is given by the Koksma-Hlawka inequality. Although this
Table 1: MC and RQMC sample standard deviations $\sigma$, RMSE errors $\varepsilon$ and confidence intervals. The total number of function evaluations $N = nK$.

| $\sigma_{MC}$ | $\sigma_{RQMC}$ |
|---------------|-----------------|
| $\frac{1}{N-1} \sum_{i=1}^{N} (f(X_i) - \hat{\mu}_N)^2$ | $\frac{1}{K-1} \sum_{k=1}^{K} (\hat{\mu}_n^k - \bar{\mu}_n)^2$ |
| $\varepsilon_{MC} = \frac{\sigma_{MC}}{\sqrt{N}}$ | $\varepsilon_{RQMC} = \frac{\sigma_{RQMC}}{\sqrt{K}}$ |
| $\hat{\mu}_N \pm z_\delta/2 \varepsilon_{MC}$ | $\bar{\mu}_n \pm z_\delta/2 \varepsilon_{RQMC}$ |

Randomized QMC (RQMC) method combines the accuracy of QMC with the MC-type error estimation. Consider a set of randomised replications $\{V_i\}$ of $\{Q_i\}$. In the RQMC method (a) for a fixed $i$ each point $V_i$ is uniformly distributed $V_i \sim U[0,1]^d$; (b) the point set $\{V_i\}, i = 1, \ldots, N$ is LDS with probability 1.

Consider a set of $K$ randomised replication $\{V_i\} = V_i^k, k = 1, \ldots, K$. We denote by $\hat{\mu}_n^k$ the $k$-th RQMC estimator for (1):

$$\hat{\mu}_n^k = \frac{1}{n} \sum_{i=1}^{n} f(V_i^k),$$

and by $\bar{\mu}_n$ the sample mean

$$\bar{\mu}_n = \frac{1}{K} \sum_{k=1}^{K} \hat{\mu}_n^k.$$

We note that $\hat{\mu}_n^k$ are i.i.d., hence the sample standard deviation of this estimator and the corresponding root mean square error (RMSE) can be computed in the same way as for MC. Table 1 provides RMSE $\varepsilon_{MC}$ and $\varepsilon_{RQMC}$ for the MC and RQMC respectively. It also provides the expressions for confidence intervals. It is assumed that $K$ is large enough so that the sample mean $\bar{\mu}_n$ is normally distributed. $z_\delta$ denotes the $(1-\delta)$ quantile of the standard normal distribution with CDF $F$: $F(z_\delta) = 1 - \delta$. For a 95% confidence interval, $\delta = .05$ and $z_{.05/2} \approx 1.96$. In the case of small values of $K$ normal quantile should be replaced with the one from Student’s $t$ distribution on $K - 1$ degrees of freedom.

Assuming that $f$ is of bounded variation, the RMSE is of the order $O(1/(\sqrt{K}n^{(1-\alpha)}))$ with $\alpha > 0$. To obtain an accurate estimate of $\bar{\mu}_n$ one has to take large $n$ and small $K$ to keep the cost $nK$ at acceptable level.

Owen’s nested scrambling achieves maximum randomization of LDS while retaining multidimensional stratification (Owen, 1997). Consider a $b$-ary expansion of an LDS point in base $b$

$$Q_i^j = \sum_{p=1}^{m} q_{i,p}^j b^{-p}.$$ 

Here $Q_i^j$ is $j$-th dimensional component of $Q_i$, $j = 1, \ldots, d$, $i = 1, \ldots, N$, $N = b^m$, $b \geq 2$ and coefficients $q_{i,p}^j \in \{0, 1 \ldots, b-1\}$. Owen’s scrambled version $V_i^j$ of $Q_i^j$
is obtained by permuting the digits \( q^i_{j,p} \) in the following way: 
\[ v^j_{i,1} = \pi^j(q^i_{j,1}) , \]
\[ v^j_{i,2} = \pi^j(q^i_{j,2}) , \]
\[ v^j_{i,3} = \pi^j(q^i_{j,2}) , \]
and so on. All uniform random permutations \( \pi^j \) over the set of \( \{0, 1, \ldots, b-1\} \) are mutually independent but each of them depends on previous leading digits of \( Q^i_j \). Let \( M \) be the number of digits used in the binary number representation (\( M = 32 \) or \( M = 64 \)). Then the permutation tree in the \( d \)-dimensional case would consist of \( d(b^M-1)/(b-1) \) permutations. For Sobol’ LDS with \( b = 2 \), \( M = 32 \) and a low dimensional problem with \( d = 100 \) scrambling would require to store in memory \( \sim 4.3 \times 10^{11} \) permutations. One way to reduce computational costs would be to do permutations for the first \( k \) bits only and then generate the other bits randomly. In this work we use a modification of Owen’s scrambling with additional permutations [Atanassov and Kucherenko (2021)]. It has reduced memory and CPU requirements. Owen showed in Owen (1997) that for sufficiently smooth functions \( \varepsilon_{RQMC} \sim O(1/(n^{(3/2 - \alpha)})) \). It is \( \sqrt{n} \) times higher than the best achievable rate \( O(1/(n^{(1-\alpha)})) \) for the standard (non scrambled) nets. This reduction arises from random error cancellations.

The random digital shift (DS) method is simple to implement and it does not impose extra memory requirements as Owen’s scrambling. For simplicity we present it for the Sobol’ sequence. Consider a set \( d \)-dimensional Sobol’ points \( \{Q_i\} \) in base \( b = 2 \) Eq. (5). Generate a random vector \( U \sim U[0,1]^d \) and produce a randomised version \( V_i \) of \( Q_i \) with components \( v^j_{i,p} = (q^i_{j,p} \oplus u^p_j) \), \( i = 1, \ldots, N, j = 1, \ldots, d, p = 1, \ldots, m \). Here \( u^p_j \) is the \( p \)-th digit in the binary representation of \( U^j \). Symbol \( \oplus \) denotes the digital addition operation (a bitwise XOR operator). We note that \( K \) randomised replicas of \( \{Q_i\} \) are obtained with the same set of \( \{Q_i\} \) and different \( U^k \).

We note that there are other types of LDS randomization which are less costly than nested Owen’s scrambling and more efficient than DS. A survey of these methods is given in L’Ecuyer (2018). They all satisfy properties a) and b) of RQMC above, but do not possess the increased rate of convergence of Owen’s scrambling.

3 ANOVA decomposition and effective dimension

ANOVA decomposition can be used to explain efficiency of QMC and RQMC methods in finance. Consider an integrable function \( f(x) \) defined in the unit hypercube \( H^d \). It can be decomposed as

\[
f(x) = f_0 + \sum_{i=1}^{d} f_i(X_i) + \sum_{i=1}^{d} \sum_{i<j}^{d} f_{ij}(X_i, X_j) + ... + f_{12...d}(X_1, X_2, ..., X_d). \quad (6)
\]

Each of the component \( f_{i_1,...,i_s}(X_{i_1}, ..., X_{i_s}) \) is a function of a unique subset of variables from \( x \). Components \( f_i(X_i) \) are called first order terms, \( f_{ij}(X_i, X_j)\)-
second order terms and so on. Under appropriate regularity conditions, the decomposition is unique if

$$
\int_0^1 f_{i_1,\ldots,i_s}(X_{i_1},\ldots,X_{i_s})dX_{i_k} = 0, \quad 1 \leq k \leq s.
$$

(7)

In this case terms are orthogonal with respect to integrations [Bianchetti et al. (2015)]. For square integrable functions, the total variance of $f$ decomposes as

$$
\sigma^2 = \sum_{i=1}^d \sigma_i^2 + \sum_{i=1}^d \sum_{i<j} \sigma^2_{ij} + \ldots + \sigma^2_{12\ldots d}.
$$

(8)

Here $\sigma^2_{i_1,\ldots,i_s} = \int_0^1 f_{i_1,\ldots,i_s}^2(X_{i_1},\ldots,X_{i_s})dX_{i_1}\ldots dX_{i_s}$ are called partial variances.

Let $|u|$ be a cardinality of a set of variables $u$. Define Sobol’ indices as

$$
S_u = \frac{\sigma^2_u}{\sigma^2}.
$$

The effective dimension of $f(x)$ in the superposition sense is the smallest integer $d_s$ s.t. $\sum_{0<|u|<d_s} S_u \geq 1 - \epsilon$ with small $\epsilon \geq 0$. If $d_s$ is close to 1, it means that $f$ is well approximated by a sum of $d_s$ (or less) dimensional functions. There are cases where the first few inputs are much more important than the others. If $\sum_{u\subseteq 1,\ldots,d_t} S_u \geq 1 - \epsilon$, then $f$ has an effective dimension $d_t$ in the truncation sense. Low effective dimension in the truncation sense can sometimes be achieved by redesigning the sampling scheme in such a way that the first few ANOVA components account for most of the variance in $f$ (see Section 5.2 for details).

4 Time-homogeneous hyperbolic local volatility model

It is well known that implied volatility (the volatility input to the Black-Scholes formula that generates the market European Call or Put price) in general depends on the strike $K$ and the maturity of the option $T$. When implied volatility is plotted against strike price, the resulting graph is typically downward sloping for equity markets, and the term "volatility skew" is often used. For other markets, such as FX options or equity index options, where the typical graph turns up at either end, the more familiar term "volatility smile" is used (for details see e.g. [Gatheral (2011)]). For our numerical analysis, we consider the time homogeneous hyperbolic local volatility model (HLV), which better captures the market skew. It corresponds to a parametric local volatility-type model in which the dynamic of the underlying under the risk neutral measure $Q$ is:

$$
dS(t) = rS(t)dt + \tilde{\sigma}(S(t))dW(t), \quad S_0 = 1,
$$

(9)

where $r$ is the risk free interest rate and

$$
\tilde{\sigma}(S) = \nu \left\{ \frac{(1-\beta + \beta^2)}{\beta} S + \frac{(\beta - 1)}{\beta} \left( \sqrt{S^2 + \beta^2(1-S)^2} - \beta \right) \right\}.
$$

(10)
Here \( \nu > 0 \) is the level of volatility, \( \beta \in (0,1] \) is the skew parameter and \( W \) is the standard Brownian motion. This model which was introduced in Jackel (2008) corresponds to the Black-Scholes model for \( \beta = 1 \) and exhibits a skew for the implied volatility surface when \( \beta \neq 1 \). We note that the skew increases significantly with decreasing value of \( \beta \). For example with \( \nu = 0.3 \), \( \beta = 0.2 \), the difference in volatility between strikes at 50% and at 100% is about 15%.

5 Time discretization schemes

5.1 Euler discretization of the SDE

We consider the pricing of option on a single asset whose value \( S(t) \) is defined by SDE (9). To guarantee positive price in the simulation, the following transformation is used \( Y(t) = \ln(S(t)) \), then from (9) we obtain

\[
dY(t) = [r - \frac{1}{2} \sigma^2(Y(t))]dt + \sigma(Y(t))dW_t, \quad Y(0) = \log(S(0)),
\]

where \( \sigma(Y) = \frac{\tilde{\sigma}(e^Y)}{e^Y} \).

For a general MC pricing framework with SDE discretization, we use Euler-Maruyama scheme (Glasserman, 2004; Kloeden and Platen, 2013). In a discrete case of \( d \) equally distributed time steps, it has the following form:

\[
Y^d(t_{i+1}) = Y^d(t_i) + \left[r - \frac{1}{2} \sigma^2(Y^d(t_i))]\eta \frac{\Delta t}{d} \right](t_{i+1} - t_i) + \sigma(Y^d(t_i))\sqrt{t_{i+1} - t_i}(W(t_{i+1}) - W(t_i))
\]

with \( Y^d(0) = \log(S(0)), \Delta t = \frac{T}{d}, t_i = i \Delta t, i = 0, \ldots, d \).

In addition to the statistical noise, there is a discretisation error. Theorem 10.2.2 in Kloeden and Platen (2013) provides conditions for Euler-Maruyama scheme to have a strong error convergence of order \( \frac{1}{2} \). Under stronger conditions as in Kloeden and Platen (2013), theorem 14.5.2, the scheme reaches a weak error convergence of the order 1.

5.2 Discretization of the Wiener process

There are different algorithms for the discretization of the Brownian motion \( W \) in equation (11). The standard (incremental) discretization algorithm follows directly from the definition of \( W(t) \). It is defined by the relation:

\[
W(t_i) = W(t_{i-1}) + \sqrt{\Delta t}Z_i, \quad 1 \leq i \leq d,
\]
where \((Z_i)\) are independent standard normal variates obtained from random numbers or Sobol’ LDS using the inverse normal cumulative distribution function.

The Brownian bridge (BB) discretization is based on conditional distributions: the value of \(W(t_i)\) is generated from values of \(W(t_l), W(t_m), l \leq i \leq m\) at earlier and later time steps. This discretization first generates the Brownian motion at the terminal point

\[ W(T) = \sqrt{T}Z_1 \]

and then it fills other points using already found values of \(W(t_i)\). The generalised BB formula is given by

\[ W(t_i) = (1 - \gamma)W(t_l) + \gamma W(t_m) + \sqrt{\gamma(1 - \gamma)(m - l)}\Delta t Z_i, \quad (14) \]

where \(\gamma = \frac{i - l}{m - l}\). It can be seen from equation \((14)\) that the variance of the stochastic part of the BB formula \(\gamma(1 - \gamma)(m - l)\Delta t\) decreases rapidly at the successive levels of refinement and the first few points contain most of the variance. Moreover, the variance in the stochastic part of \((14)\) is smaller than that in \((13)\) for the same time steps.

For MC the BB scheme has the same efficiency as the standard one but it does affect the efficiency of QMC based on Sobol’ LDS. Sobol’ defined “Sobol’ sequence” as the \(LP\tau\) sequence. The \(\tau\)-value is a quality parameter which measures the uniformity of the point sets. The smaller the \(\tau\)-value is the more uniformly distributed the points are. This value is equal to 0 only for one and two dimensional Sobol’ sequences. In higher dimensions, as \(d\) increases, the smallest possible values of \(\tau\) increase as well. Hence, the initial coordinates of Sobol’ LDS are much better distributed than the later high dimensional coordinates.

The BB discretization uses low well distributed coordinates from each \(d\)-dimensional LDS vector point to determine most of the structure of a path and reserves the later coordinates to fill in fine details. In other words, well distributed coordinates are used for important variables and higher not so well distributed coordinates are used for far less important variables. Thus the BB sampling reduces the effective dimension in the truncation sense which leads to the much higher convergence rate of the QMC algorithm for majority (but not all) of the payoffs (Bianchetti et al., 2015).

6 Monte Carlo simulation of option pricing and Greeks

6.0.1 Option pricing

We consider a geometric average Asian call option whose payoff function is given by

\[ P_A = \max(\bar{S} - K, 0), \quad (15) \]
where $\bar{S}$ is a geometric average at $d$ equally spaced time point:

$$\bar{S} = \left( \prod_{i=1}^{d} S_i \right)^{\frac{1}{d}}, \quad (16)$$

where $S_i$ is the asset price at time $t_i = \frac{iT}{d}, 1 \leq i \leq d$.

In a risk neutral setting, the value of a call option with maturity $T$ and strike $K$ is the discounted value of its payoff:

$$AC(T, K) = e^{-rT} \mathbb{E}^Q[P_A]. \quad (17)$$

There is no analytical formula for (17) in the HLV model and it is estimated by the MC method. Firstly, we approximate the asset price $S(t_i)$ with $S^d(t_i) = e^{Y^d(t_i)}$ by discretising the SDE (11) as described in Section 5.1. Secondly, the expectation of the Asian payoff (15) is computed with the MC estimator as an arithmetic average of payoffs taken over a finite number $N$ of simulated price path:

$$AC_N(T, K) = e^{-rT} \left[ \frac{1}{N} \sum_{i=1}^{N} \max(\bar{S}(i) - K, 0) \right], \quad (18)$$

where $\bar{S}(i)$ is an approximation of $\bar{S}$ using the simulated $i$-th price paths.

### 6.0.2 Sensitivity factors

Sensitivity factors or Greeks are derivatives of the price $AC(T, K)$ w.r.t specific parameters like spot price or volatility. They are computed for hedging and risk management purposes. In this work, we focus only on the Delta defined as $\Delta = \frac{\partial AC(T, K)}{\partial S_0}$, where $S_0$ is the current spot price. In the dynamic hedging, Delta corresponds to the number of assets one should hold for each option shorted to maintain a delta-neutral position. As there is no analytical formula for the value of $AC_N(T, K)$, Delta can be estimated by MC simulation and the finite difference method. In the case of the central difference scheme Delta is computed as

$$\Delta \approx \frac{AC_N(T, K, S_0 + \epsilon_s) - AC_N(T, K, S_0 - \epsilon_s)}{2\epsilon_s}, \quad (19)$$

where $\epsilon_s = hS_0$ is the increment, $h$ is a shift parameter.

Path recycling of both pseudo-random sequences and LDS is used to minimize the variance, as suggested e.g. in Glasserman (2004). We note that the error analysis for Greeks is more complex than that for prices, since the variance of the MC simulation is mixed with the bias due to the approximation of derivatives with finite differences. For the sensitivity factor estimation not to be entirely hidden by the MC noise, in our computations the shift is chosen to be large enough: $h = 0.01$ (see Glasserman (2004) for detailed discussions).
7 Numerical results

In this Section we present the results from simulations of prices and sensitivity factor $\Delta$ for Asian call options on a single underlying. The following parameters were used in simulations: $S_0 = 100$, $r = 3\%$, $T = 1$, $\nu = 30\%$, $\beta = 0.5$, number of discrete time steps $d = 256$. In the single underlying case, $d$ corresponds to the problem dimensionality. We consider in-the-money (ITM), at-the-money (ATM) and out-of-the-money (OTM) options with strike 80, 100, 120 respectively to investigate the effect of moneyness.

Numerical simulations using MC, QMC and RQMC methods were performed to compare convergence of each method. The standard (incremental) discretisation of Brownian motion was used in the MC method. The Brownian Bridge algorithm was used in QMC and RQMC methods (Section 5.2). The Mersenne Twister generator was used for MC simulations and RQMC with digital shift. BRODA’s Sobol’ sequence generator with additional uniformity properties described in Section 2 was used for QMC simulations BRODA Ltd (2022); Sobol’ et al. (2011). For QMC and RQMC to achieve an optimal uniformity sampling, the number of points $n$ was taken to be powers of two. The reference values of prices and Deltas were obtained by the MC method by averaging over $K = 10$ independent runs with each run using $n = 2^{18}$ paths.

7.1 Price and delta convergence

Firstly, we analyze convergence plots, namely values of price (Figs. 1) and Delta (Figs. 2) versus the number of paths. One trial ($K = 1$) is used for MC and RQMC runs. For simulated prices RQMC with Owen’s scrambling outperforms all other methods for the ITM and ATM cases converging quicker to the reference levels. Its efficiency is followed by QMC and RQMC with DS methods, which outperform MC for the ITM and ATM cases. These two methods show similar performance between themselves. Similar but less pronounced trends are present in the case of OTM.

For simulated Deltas RQMC with Owen’s scrambling marginally outperform all other methods for the ITM case. It shows a similar performance to QMC for the ATM case. Both RQMC and QMC methods slightly outperform MC for the case of OTM.

We note, that the step-like behaviour of the convergence patterns of in some QMC, RQMC plots is likely to be a result of the inherent design of Sobol’ sequence generators as can be seen from Fig. 2, p. 70 in Sobol’ et al. (2011).

We can conclude that all considered cases (ITM, ATM and OTM), RQMC method with Owen’s scrambling shows a faster convergence than other methods. RQMC with DS shows similar to QMC performance.

7.2 Confidence intervals

Tables 2, 3 show RMSE (Table 1) for MC and RQMC methods of prices and Deltas estimations, respectively. Ratios of MC to RQMC with Owen’s
Figure 1: Asian call price (a) ITM (b) ATM (c) OTM w.r.t number of paths $N = n$, $K = 1$ (in $\log_2$ scale).
Figure 2: Asian call Delta (a) ITM (b) ATM (c) OTM w.r.t number of paths $N = n$, $K = 1$ (in log2 scale).
Table 2: $\varepsilon_{MC}$ and $\varepsilon_{RQMC}$ at $K=10$, $n=2^{12}$ ($N=40960$) of price estimations.

|          | ITM       | ATM       | OTM       |
|----------|-----------|-----------|-----------|
| $\varepsilon_{MC}$ | $3.26 \times 10^{-2}$ | $2.2 \times 10^{-2}$ | $1.24 \times 10^{-3}$ |
| $\varepsilon_{RQMC}$ (Owen) | $3.67 \times 10^{-4}$ | $6.09 \times 10^{-4}$ | $4.04 \times 10^{-4}$ |
| $\varepsilon_{RQMC}$ (DS)   | $7.12 \times 10^{-4}$ | $1.03 \times 10^{-3}$ | $4.59 \times 10^{-4}$ |
| $\varepsilon_{MC}/\varepsilon_{RQMC}$ (Owen) | 89          | 36          | 3          |

Table 3: $\varepsilon_{MC}$ and $\varepsilon_{RQMC}$ at $K=10$, $n=2^{12}$ ($N=40960$) of Deltas estimations.

|          | ITM       | ATM       | OTM       |
|----------|-----------|-----------|-----------|
| $\varepsilon_{MC}$ | $3.29 \times 10^{-4}$ | $2.43 \times 10^{-3}$ | $4.09 \times 10^{-4}$ |
| $\varepsilon_{RQMC}$ (Owen) | $3.23 \times 10^{-5}$ | $5.98 \times 10^{-4}$ | $1.41 \times 10^{-4}$ |
| $\varepsilon_{RQMC}$ (DS)   | $4.04 \times 10^{-5}$ | $4.34 \times 10^{-4}$ | $1.46 \times 10^{-4}$ |
| $\varepsilon_{MC}/\varepsilon_{RQMC}$ (Owen) | 10          | 4          | 3          |

scrambling error estimates show a dramatic improvement with using RQMC with the largest improvement ratio for ITM. RQMC with Owen’s scrambling method on average is better than RQMC with DS producing smaller $\varepsilon_{RQMC}$.

Prices and Deltas with confidence intervals versus the number of simulation paths $n$ for the ITM call are shown in Figures 3. Visually the results for the ATM and OTM cases are similar and they are not shown. As expected the confidence intervals are tightening with increasing the number of paths. RQMC with Owen’s scrambling offers the most accurate results by reducing significantly the bounds of confidence intervals.

7.3 Performance analysis

We also analyze the relative performance of considered methods in terms of convergence rates. For all considered sampling schemes the following power law for the integration error is observed empirically in numerical tests:

$$\varepsilon_n \sim \frac{C}{n^\alpha}.$$  (20)

For the MC method $\alpha = 0.5$. For applications of the QMC and RQMC methods to financial problems quite commonly $\alpha > 0.5$. Its value can be very close to 1 irrespective of the nominal dimension when the effective dimensions are low.

It has been discussed in Section 2 that there are no statistical measures like variances associated with LDS because they are deterministic. Hence, the constant $C$ in (20) is not a variance and (20) does not have a probabilistic interpretation. In practice, the root mean square error (RMSE) for both MC, RQMC and QMC methods for any fixed $n$ can be estimated by computing the
Figure 3: Asian Call Delta with 95% confidence intervals computed at $K = 10$ for ITM call for RQMC (a) Owen (b) DS methods w.r.t number of paths $n$ (in $\log_2$ scale).
Table 4: Extracted α in QMC and RQMC (Owen’s scrambling) methods.

|            | ITM | ATM | OTM |
|------------|-----|-----|-----|
| QMC (Price)| 1.0 | 0.95| 0.82|
| RQMC (Price)| 0.7 | 0.79| 0.79|
| QMC (Delta)| 0.65| 0.64| 0.71|
| RQMC (Delta)| 0.66| 0.62| 0.62|

following error averaged over $K$ independent runs:

$$\epsilon_n = \sqrt{\frac{1}{K} \sum_{k=1}^{K} \left( V - V_n^{(k)} \right)^2},$$

(21)

where $V$ is the exact, or estimated value of the integral (option price or Delta in our case) obtained at a very large $n \rightarrow \infty$, $V_n^{(k)}$ is the simulated value for the $k$-th run, performed using $n$ paths.

We note some difference between definitions of $\epsilon_{RQMC}$ given in Table 1 (it is computed with a reference to $\bar{\mu}_n$) and $\epsilon_n$ in (21) (it is computed with a reference to $V$).

For MC and RQMC, runs based on different seed points are statistically independent. In the case of QMC, different runs are obtained using non overlapping sections of the LDS. In our computations $K = 10$.

Figures 4, 5 show the RMSE versus the number of paths $n$ for MC, QMC and RQMC methods in log2-log2 scale. We fitted the regression lines that follow the power law (20) to extract convergence rates $\alpha$: they are the slopes of the regression lines (Table 4). We also extracted the intercepts of regression lines ($\log_2(C)$, (20)) (not presented here). They provide useful information about the efficiency of the QMC, RQMC and MC methods: lower intercepts mean that the simulated value starts closer to the exact value.

As expected, for MC $\alpha = 0.5$ for all cases, these results are not presented in the Table 4. For QMC and RQMC $\alpha > 0.5$ for all cases. It is higher for price than for Delta. Although it is marginally higher for QMC but the intercepts $C$ of regression lines are always lower for RQMC than for other methods for considered ranges of $n$. It makes RQMC the most efficient method (although efficiencies of RQMC and QMC are similar for the OTM case).

8 Conclusions

We present and discuss the results of an application of MC, QMC and RQMC methods for derivative pricing and risk analysis based on the hyperbolic local volatility model. The results presented for the Asian option show the superior performance of the QMC and RQMC methods. RQMC not only increases the rate of convergence of QMC but also allows to compute confidence intervals around the estimated value. Efficiency of RQMC strongly depends on the
Figure 4: RMSE for ITM Asian call Prices (a) ITM (b) ATM (c) OTM w.r.t number of paths n, K = 10. Owen’s scrambling is used in RQMC.
Figure 5: RMSE for ITM Asian call Deltas (a) ITM (b) ATM (c) OTM w.r.t number of paths $n$, $K = 10$. Owen’s scrambling is used in RQMC.
scrambling methods. We advise to use Sobol’ LDS with Owen’s scrambling as the most efficient method.
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