Points of low height on elliptic curves and surfaces
I: Elliptic surfaces over $\mathbb{P}^1$ with small $d$

Noam D. Elkies
Department of Mathematics, Harvard University, Cambridge, MA 02138 USA

Abstract. For each of $n = 1, 2, 3$ we find the minimal height $\hat{h}(P)$ of a nontorsion point $P$ of an elliptic curve $E$ over $\mathbb{C}(T)$ of discriminant degree $d = 12n$ (equivalently, of arithmetic genus $n$), and exhibit all $(E, P)$ attaining this minimum. The minimal $\hat{h}(P)$ was known to equal $1/30$ for $n = 1$ (Oguiso-Shioda) and $11/420$ for $n = 2$ (Nishiyama), but the formulas for the general $(E, P)$ were not known, nor was the fact that these are also the minima for an elliptic curve of discriminant degree $12n$ over a function field of any genus. For $n = 3$ both the minimal height $(23/840)$ and the explicit curves are new. These $(E, P)$ also have the property that that $mP$ is an integral point (a point of naive height zero) for each $m = 1, 2, \ldots, M$, where $M = 6, 8, 9$ for $n = 1, 2, 3$; this, too, is maximal in each of the three cases.

1. Introduction.

1.1 Statement of results. Let $K$ be a function field of a curve $C$ of genus $g$ over a field $k$ of characteristic zero, and $E$ a nonconstant elliptic curve over $K$. Let $d$ be the degree of the discriminant of $E$ (considered as a divisor on $C$), a natural measure of the complexity of $E$; and let $\hat{h} : E(K) \to \mathbb{Q}$ be the canonical height. Necessarily $12|d$; in fact it is known that $d = 12n$ where $n$ is the arithmetic genus of the elliptic surface $E$ associated with $E$. It is not hard to show that, given $d$, the set of numbers $H$ that can occur as the canonical height of a rational point on $E$ is discrete. In particular, for each $d = 12n$ there is a minimal positive height $\hat{h}_{\text{min}}(d)$, and also a minimal positive height $\hat{h}_{\text{min}}(g, d)$ for elliptic curves over function fields of genus $g$ (except for $g = d = 0$, when $E$ is a constant curve over $\mathbb{P}^1$ and thus has no points of positive height). It is thus a natural problem to compute or estimate these numbers $\hat{h}_{\text{min}}(d)$ and $\hat{h}_{\text{min}}(g, d)$. This paper is the first of a series concerned with different aspects of this problem.

In this paper we determine $\hat{h}_{\text{min}}(12n)$ for $n = 1, 2$ and $\hat{h}_{\text{min}}(0, 12n)$ for $n = 1, 2, 3$. Since we are working in characteristic zero, we may assume $k = \mathbb{C}$, when every genus-zero curve is isomorphic to $\mathbb{P}^1$ and its function field is isomorphic to $\mathbb{C}(T)$.

1 One can also usefully define the canonical height etc. in positive characteristic, but we need to use the ABC conjecture for $K$ and thus must assume that $K$ has characteristic zero.
Theorem 1.  i) (Oguiso-Shioda [7]) \( \hat{h}_{\min}(0, 12) = 1/30. \)
   ii) \( \hat{h}_{\min}(12) = 1/30. \) Moreover, let \( E \) be an elliptic curve with \( d = 12 \) over a complex function field \( K \), and \( P \in E(K) \). Then the following are equivalent:
   (a) \( \hat{h}(P) = 1/30; \) (b) Each of \( P, 2P, 3P, 4P, 5P, 6P \) is an integral point on \( E \);
   (c) \( K \cong \mathbb{C}(T) \), and \( (E, P) \) is equivalent to the curve
   \[
   E_1(q) : Y^2 + (s' - (q + 1)s)XY + qss'(s - s')Y = X^3 - qss'X^2
   \]
   over the \((s : s')\) line with the rational point \( P : (X, Y) = (0, 0) \), for some \( q \in \mathbb{C} \) other than 0 or 1.

Theorem 2.  i) (Nishiyama [2]) \( \hat{h}_{\min}(0, 24) = 11/420. \)
   ii) \( \hat{h}_{\min}(24) = 11/420. \) Moreover, let \( E \) be an elliptic curve with \( d = 24 \) over a complex function field \( K \), and \( P \in E(K) \). Then the following are equivalent:
   (a) \( \hat{h}(P) = 11/420; \) (b) \( mP \) is an integral point on \( E \) for each \( m = 1, 2, \ldots, 8 \);
   (c) \( K \cong \mathbb{C}(T) \), and \( (E, P) \) is equivalent to the curve
   \[
   E_2(u) : Y^2 + (r^2 - r'^2 + (u - 2)rr')XY
   - r^2rr'(r + r')(r + ur')(r + (u - 1)r')Y
   = X^3 - r^2rr'(r + r')(r + ur')X^2
   \]
   over the \((r : r')\) line with the rational point \( P : (X, Y) = (0, 0) \), for some \( u \in \mathbb{C} \) other than 0, 1.

Theorem 3.  i) \( \hat{h}_{\min}(0, 36) = 23/840. \)
   ii) Let \( E/\mathbb{C}(T) \) be an elliptic curve with \( d = 36 \), and \( P \) a rational point on \( E \). Then the following are equivalent: (a) \( \hat{h}(P) = 23/840; \) (b) \( mP \) is an integral point on \( E \) for each \( m = 1, 2, \ldots, 9 \);
   (c) \( (E, P) \) is equivalent to the curve
   \[
   E_3(A) : Y^2 + (At^3 + (1 - 2A)t^2t' - (A + 1)tt'^2 - t'^3)XY
   - t^3t'(t + t')(At + t')(At + (1 - A)t')(At^2 + tt' + t'^2)Y
   = X^3 - t^3t'(t + t')(At + t')(At^2 + tt' + t'^2)Y
   \]
   over the \((t : t')\) line with the rational point \( P : (X, Y) = (0, 0) \), for some \( A \in \mathbb{C} \) other than 0, 1.

The values of \( \hat{h}_{\min}(12) \) and \( \hat{h}_{\min}(24) \) are new. Note that we do not claim to determine \( \hat{h}_{\min}(36) \). As indicated, the values of \( \hat{h}_{\min}(0, 12) \) and \( \hat{h}_{\min}(0, 24) \) (the first parts of Theorems 1 and 2) were already known, but were obtained using techniques that are specific to the geometry of rational and K3 elliptic surfaces and do not readily generalize past \( n = 2 \). Our approach lets us treat all three cases uniformly, and in principle lets us determine \( \hat{h}_{\min}(0, 12n) \) for any \( n \), though the computations rapidly become infeasible as \( n \) grows beyond 3. The minimizing \((E, P)\) had not been previously exhibited, except for a single case of a rational
elliptic surface with a section of height 1/30 obtained by Shioda in a later paper [11], which we will identify with $E_1(4/5)$.

The connections with integral multiples of $P$ (see statement (b) of part (ii) of each Theorem) are also new. We do not expect them to persist past $n = 3$, and in fact find that for $n = 4$ the largest number of consecutive integral multiples occurs for $(E, P)$ with $\hat{h}(P) = 19/630$ or $13/360$, whereas $\hat{h}_{\text{min}}(0, 48) \leq 41/1540 < 19/630 < 13/360$. We shall say more about integrality later; for now we content ourselves with the following remarks. A point on an elliptic curve over a function field $k(C)$ is said to be integral if it is a nonzero point whose naive height vanishes. Geometrically, if we regard $E$ as an elliptic surface $E$ over $C$, and a rational point $P \in E(K)$ as a section $s_P$ of $E$, this means that $s_P$ is disjoint from the zero-section $s_0$ of $E$. Since $g = 0$ in our case, we can give an explicit algebraic characterization of integrality. Write $E$ in extended Weierstrass form as

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$$

where each $a_i$ is a homogeneous polynomial of degree $i \cdot n$ in two variables. Then a rational point $(X, Y)$ is integral if $X, Y$ are homogeneous polynomials of degrees $2n, 3n$ respectively. The equation (4) depends on the choice of coordinates $X, Y$ on $E$; replacing $X, Y$ by

$$\delta^2(X + \alpha_2), \quad \delta^3(Y + \alpha_1X + \alpha_3)$$

(some $\alpha_i$ and nonzero $\delta$) yields an isomorphic curve. If moreover $\delta \in C^*$ and each $\alpha_i$ is a homogeneous polynomial of degree $i \cdot n$ then the new equation for $E$ has the same discriminant degree and the same integral points.

1.2 Outline of this paper. For each $n = 1, 2, 3$ we prove Theorem $n$, except for the implications (a),(b)$\Rightarrow$(c) of part (ii), which require different methods that we defer to a later paper. Our proofs use the following ingredients:

- $\hat{h}(mP) = m^2 \hat{h}(P)$ for all $m \in \mathbb{Z}$.
- If $mP \neq 0$ then
  $$\hat{h}(mP) = h(mP) + \sum_v \lambda_v(mP),$$

where $h(\cdot)$ is the naive height and the sum extends over all places $v \in C(C)$ lying under singular fibers $E_v$ of $E$. (All places of $K$ are of degree 1 thanks to our use of the algebraically closed field $C$ for $k$.) The local corrections $\lambda_v(mP)$ are described further below.

- The naive height takes values in $\{0, 2, 4, 6, \ldots\}$, and satisfies $h(mP') \leq h(mP)$ for any integers $m, m'$ such that $m' \mid m$ and $mP \neq 0$.
- Each local correction $\lambda_v(mP)$ depends only on the Kodaira type of the fiber $E_v$ and on the component of $E_v$ meeting $P$. We shall call this component $c_v$. The values of $\lambda_v(\cdot)$ are known explicitly for all Kodaira types and each possible component, see for instance [13, Thm. 5.2].
Finally, the condition that $E$ have discriminant degree $d = 12n$ imposes two conditions on the Kodaira types of the singular fibers. The first condition is

$$d = \sum_v d_v,$$

where $d_v$ is the local discriminant degree of $E_v$. This allows only finitely many collections of fiber types. The second condition follows from an inequality due to Shioda [21 Cor. 2.7 (p.30)], and eliminates some of these collections that have too few fibers. According to this condition, if a nonconstant elliptic curve of discriminant degree $d$ over a function field $K = C(C)$ has a nontorsion point then the conductor degree of the curve strictly exceeds $(d/6) + \chi(C)$. Here $\chi(C) = 2 - 2g$ is the Euler characteristic of $C$. The conductor degree may be defined as the number of multiplicative fibers plus twice the number of additive fibers; thus it is also a sum of invariants of the singular fibers. When $(g, d) = (0, 12n)$ we have $\chi(C) = 2$ and $d/6 = 2n$, so the conductor degree is at least $2n + 3$.

We shall refer to these constraints as the “combinatorial conditions” on $h(P)$, $h(mP)$, and the collection of $(E_v, c_v)$ that arise for $(E, P)$. (For other uses of such conditions to obtain lower bounds on heights, see for instance $[31, 14]$ and work referenced in these sources.) In general the combinatorial conditions yield only a lower bound on $h_{\text{min}}(0, 12n)$, because they allow some possibilities that do not actually occur for any $(E, P)$. But for each of $n = 1, 2,$ and $3$ this lower bound turns out to be attained by some $(E, P)$ over $C(T)$, namely those exhibited in statement (c) of part (ii) of Theorem $n$. (Note that we do not yet need to derive the formulas for these $(E, P)$, nor to prove that they are the only ones possible.) Moreover, using $[9]$ we can check that $h(P) = h_{\text{min}}(0, 12n)$ if and only if the naive height $h(mP)$ vanishes for all $m$ up to $6, 8,$ or $9$ respectively.

Still, already at $n = 1$ we see some redundancy. The combinatorial conditions allow $h(P) = 1/30$ to be attained in any of five ways, four of which are realized by the curves $E_1(q)$ of Theorem $1$ for suitable choices of $q$. Shioda’s $E_1(4/5)$ has singular fibers of types $I_5$, $I_3$, $I_2$, and $II$. (We specify the components $c_v$ later in the paper.) The fibers of $E_1(-1)$ have types $I_5$, $IV$, $I_2$, and $I_1$, while those of $E_1(4)$ have types $I_5$, $I_3$, $III$, and $I_1$. In all other cases, the fibers of $E_1(q)$ have types $I_5$, $I_3$, $I_2$, $I_1$, $I_1$: the first three at $s = 0$, $s' = 0$, $s'' = s$, and the last two at the roots of the quadratic $(q + 1)^2s^2 = (11q^2 - 14q + 2)s + (q - 1)s^2$. When $q = 4/5$, these roots coincide and the two $I_1$ fibers merge to form a $II$; likewise at $q = -1$ or $q = 4$, one of the $I_1$ fibers merges with the $I_3$ or $I_2$ fiber to form a $IV$ or $III$ respectively. (The one merger that does not occur is $I_1 + I_1 \to I_2$.) But none of these degenerations changes $h(P)$, nor any $h(mP)$, nor the conductor degree $N$. In fact a fiber of type $II$, $III$, or $IV$ contributes as much to our formulas for $h(P), h(mP)$, $N$ as a pair of fibers of types $I_1$ and $I_\nu$ ($\nu = 1, 2,$ or $3$). Thus it is enough to minimize $h(P)$ under the further assumption that no fibers of type $II$, $III$, or $IV$ occur. We find similar replacements for all components of fibers of the remaining additive types $I^*_\nu$, $II^*$, $III^*$, $IV^*$. See Proposition 2. This simplifies
the computation of the combinatorial lower bound on \( \hat{h}_{\min}(0, 12n) \): instead of an exhaustive search over all combinations of \((E_v, c_v)\), we need only try those for which each \(E_v\) is multiplicative (of type \(I_\nu\) for \(\nu = d_v\)).

We programmed the search over all partitions \(\{d_v\}\) of \(12n\) in \(\text{gp} [8]\) and ran it on a Sun Ultra 60. This took only a fraction of a second for \(n = 1\), five seconds for \(n = 2\), and five minutes for \(n = 3\). It took about an hour to carry out the same computation for \(n = 4\), and about 20 hours for \(n = 5\); but the resulting bounds are probably not attained: as we shall see in a later paper, the required \((E_v, c_v)\) data impose more conditions than the number of parameters needed to specify \((E, P)\). We do produce explicit \((E, P)\) that show \(\hat{h}_{\min}(0, 48) \leq 41/1540\) and \(\hat{h}_{\min}(0, 60) \leq 261/10010\), and conjecture that these are the correct values of \(\hat{h}_{\min}(0, 12n)\) for \(n = 4, 5\). We have not attempted to extend the computation past \(n = 5\).

1.3 Coming attractions. Happily, the computation of the surfaces \((1, 2, 3)\) not only completes the proofs of Theorems 1 through 3 but also points the way to further results and connections. We outline these here, and defer detailed treatment to a later paper in this series. In each step of the computation we in effect obtain a new birational model for the moduli space, call it \(X\), of pairs \((E, P)\) consisting of an elliptic curve and a point on it. Our new parametrizations of this rational surface \(X\) have several other applications. One is a geometric interpretation of Tate’s method for exhibiting the generic elliptic curve with an \(N\)-torsion point: we readily locate the modular curves \(X_1(N)\) \((N \leq 16)\) on \(X\), together with nonconstant rational functions of minimal degree that realize each \(X_1(N)\) as an algebraic curve of genus \(\leq 2\). Arithmetically, we can use our parametrizations of \(X\) to find \((E, P)\) over \(\mathbb{Q}\) (or over some other global field) such that \(P\) is a nontorsion point with small \(\hat{h}(P)\), and/or with many integral multiples in the minimal model of \(E\). For instance, we prove that there are infinitely many \((E, P)/\mathbb{Q}\) such that \(mP\) is integral for each \(m = 1, 2, \ldots, 11, 12\). Our numerical results for isolated curves \((E, P)\) over \(\mathbb{Q}\) may be found on the Web at \(\text{http://www.math.harvard.edu/~elkies/lowheight.html}\). They include new records for consecutive integral multiples and for the Lang ratio \(\hat{h}(P)/\log|\Delta_E|\).

We have \(mP\) integral for each \(m = 1, 2, \ldots, 13, 14\) for

\[ E : Y^2 + XY = X^3 - 139761580X + 158730304000, \]

an elliptic curve of conductor \(1029210 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 29\), and \(P\) the nontorsion point \((X, Y) = (11480, 1217300)\); and we find the curve

\[ Y^2 + XY = X^3 - 161020013035359930X + 24869250624742069048641252 \]

of conductor \(3476880330 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 31 \cdot 2111\) with the nontorsion point \((-296994156, 6818852697078)\) of canonical height\(^2\) \(\hat{h}(P) = .0190117 \ldots <

\(^2\) There are two standard normalizations, differing by a factor of 2, for the canonical height of a point on an elliptic curve over \(\mathbb{Q}\). We use the larger one, which is the one consistent with our formulas for function fields.
1.691732\cdot 10^{-4} \log |\Delta_E|$. The curves \([8,9]\) are the specializations of our formula \([3]\) with \((A,t/t') = (35/32, -8/15), (33/23, 115/77)\).

Our simplified formula for \(\hat{h}(mP)\) (Proposition 2) also bears on the asymptotic behavior of \(\hat{h}_{\text{min}}(g, 12n)\) for fixed \(g\) as \(n \to \infty\). Hindry and Silverman \([3]\) used the combinatorial conditions (except for the condition: \(h(m'P) \leq h(mP)\) if \(m'|m\)) to show that there exists \(C > 0\) such that

\[
\hat{h}(g, 12n) \geq Cn - O_g(1), \tag{10}
\]

This proved the function-field case of a conjecture of Lang \([4, \text{p.}92]\). The error terms \(O_g(1)\) are effectively computed, and can be omitted entirely if \(g \leq 1\). Hindry and Silverman also produce an explicit constant \(C\), but it is quite small: about \(7 \cdot 10^{-10}\). Their approach requires a point meeting every additive fiber in its identity component, which they achieved by working with \(12P\) instead of \(P\), at the cost of a factor of \(1/12^2\) in \(C\). Our results here let one apply the same methods directly to \(P\), thus saving a factor of \(12^2\) and raising \(C\) to about \(10^{-7}\). In a later paper we show how to gain another factor of approximately 5000, raising the lower bound on \(\liminf_n \hat{h}(g, 12n)/n\) to 1/2111. This is within an order of magnitude of the correct value: for all \(n \equiv 0 \mod 5\) we obtain \(\hat{h}_{\text{min}}(0, 12n) \leq 261n/50050\) via base change from our \(n = 5\) example.

2. The naïve and canonical heights.

We collect here the facts we shall use about elliptic curves \(E\) over function fields \(K\) in characteristic zero, the associated elliptic surface \(\mathcal{E}\), and the naïve and canonical height functions on \(E(K)\).

2.1 The naïve height. The naïve height \(h(P)\) of a nonzero \(P \in E(K)\) can be defined using intersection theory on the elliptic surface \(\mathcal{E}\) associated to some model of \(E\). Let \(s_0\) be the zero-section of the elliptic fibration \(\mathcal{E} \to C\), and \(s_P\) the section corresponding to \(P\). Then \(h(P) := 2s_P \cdot s_0\). Since we assumed that \(P \neq 0\), the sections \(s_0, s_P\) are distinct curves on \(\mathcal{E}\). Hence their intersection number \(s_P \cdot s_0\) is a nonnegative integer, and \(h(P)\) is a nonnegative even integer. Moreover \(h(P) = 0\) if and only if \(s_P\) is disjoint from \(s_0\), in which case we say that \(P\) is an integral point on \(E\).

When \(C = \mathbb{P}^1\), we can give an equivalent algebraic definition of \(h(P)\) in terms of a Weierstrass equation of \(E\). This definition emphasizes the analogy with the canonical height in the more familiar case of an elliptic curve over \(\mathbb{Q}\). Recall that each coefficient \(a_i\) in the Weierstrass equation \([4]\) is a homogeneous polynomial of degree \(i \cdot n\) in the projective coordinates on \(\mathbb{P}^1\). Then the coordinates \(x, y\) of a nonzero \(P \in E(K)\) are homogeneous rational functions of degrees \(2n, 3n\). If \(x, y\) are written as fractions “in lowest terms”, as quotients of coprime homogeneous polynomials, then the denominators are (up to scalar multiple) the square and cube of some polynomial \(\zeta\). The roots of \(\zeta\), with multiplicity, are the images on \(\mathbb{P}^1\) of the intersection points of \(s_0\) and \(s_P\). Hence \(s_P \cdot s_0 = \deg \zeta\). Therefore \(h(P)\) is the degree of the denominator \(\zeta^2\) of \(x\), which is also the number of poles.
of \( x \) counted with multiplicity. An integral point is one for which \( \zeta \) is a nonzero scalar and thus \( x, y \) are homogeneous polynomials of degrees \( 2n, 3n \).

For an arbitrary base curve \( C \), the coefficients \( a_i \) are global sections of \( \mathcal{L} \otimes i \) for some line bundle \( \mathcal{L} \) on \( C \), and \( x, y \) are meromorphic sections of \( \mathcal{L} \otimes 2, \mathcal{L} \otimes 3 \). The pole divisors of \( x, y \) are \( 2Z, 3Z \) for some effective divisor \( Z \) on \( C \), whose degree is \( s_P \cdot s_0 \); thus again \( h(P) \) is the degree of the pole divisor \( 2Z \) of \( x \), and \( P \) is integral iff \( Z = 0 \) iff \( x, y \) are global sections of \( \mathcal{L} \otimes 2, \mathcal{L} \otimes 3 \). A linear change of coordinates according to (5) yields the same notion of integrality if and only if \( \delta \in \mathbb{C}^* \) and \( \alpha_i \in \Gamma(\mathcal{L} \otimes i) \) for each \( i \).

We shall need one more property of the naïve height beyond its relation with the canonical height and the fact that \( h(mP) \in \{0, 2, 4, 6, \ldots\} \) (\( mP \neq 0 \)).

**Lemma 1.** Let \( P \) be a point on an elliptic curve over \( k(C) \), and let \( m, m' \) be any integers such that \( m' \mid m \) and \( mP \neq 0 \). Then \( h(m'P) \leq h(mP) \).

**Proof:** Each point of \( s_{m'P} \cap s_0 \) is also a point of intersection of \( s_{mP} \) with \( s_0 \), to at least the same multiplicity. Hence \( s_{m'P} \cdot s_0 \leq s_{mP} \cdot s_0 \), so

\[
h(m'P) = 2s_{m'P} \cdot s_0 \leq 2s_{mP} \cdot s_0 = h(mP)
\]

as claimed. \( \square \)

**Remarks:**

1. We could also state the result as: The naïve height of a point is less than or equal to the naïve height of any of its multiples that is not the zero point. This is a more natural formulation (the first point does not have to be written as \( m'P \)), but less convenient for our purposes.

2. In the proof, “at least the same multiplicity” can be strengthened to “exactly the same multiplicity” in our characteristic-zero setting. In general \( h(mP) \) may strictly exceed \( h(m'P) \) because \( s_{mP} \cap s_0 \) may also contain points where \( m'P \) reduces to a nontrivial \((m/m')\)-torsion point.

The naïve height satisfies further inequalities along the lines of Lemma 1 for instance

\[
h(6P) + h(P) \geq h(2P) + h(3P).
\]

(11)

Lemma 1 suffices for the proofs of Theorems 1–3 in the genus-zero case; but inequalities such as (11) are sometimes needed to exclude possible configurations with positive \( g \), as we shall see for \( d = 24 \). The strongest such inequality we found is:

**Lemma 2.** Let \( P \) be a point on an elliptic curve over \( k(C) \), and let \( m \) be any integer such that \( mP \neq 0 \). Then

\[
\sum_{m' \mid m} \mu(m/m') h(m'P) \geq 0.
\]

(12)
Proof: The left-hand side can be interpreted as twice the number of points of $C$, counted with multiplicity, at which $mP = 0$ but $m'/P \neq 0$ for each proper factor of $m$. 

Inequality (11) is the special case $m = 6$ of this Lemma. The sum in (12) may be considered as an analogue of the formula \( \prod_{m'} |m/(m')^\mu | \) for the $m$-th cyclotomic polynomial. We recover Lemma 1 by summing the inequality (12) over all factors of $m$, including $m$ itself but not 1, to obtain $h(mP) \geq h(P)$, which is equivalent to Lemma 1 by the first Remark above.

2.2 Local invariants, and Shioda’s inequality. To go from the naive to the canonical height we must use the minimal model of $E$ for the elliptic surface $\mathcal{E}$. We next describe this model, collect some known facts on the singular fibers of $\mathcal{E}$, and give Shioda’s lower bound on the conductor degree.

Whereas a naive height could be defined for any model of $E$,\(^3\) the canonical height requires the Néron minimal model. It is known that there exists a minimal line bundle $\mathcal{L}$ on $C$ with the following property: let $D$ be a divisor on $C$ such that $O(D) \cong \mathcal{L}$; then $E$ is isomorphic to a curve with an extended Weierstrass equation \( \mathcal{E} \) whose coefficients $a_i$ are global sections of $iD$. In characteristic zero we can easily obtain $D$ and $\mathcal{L}$ by putting $E$ in narrow Weierstrass form $Y^2 = X^3 + a_4X + a_6$. Then $D$ is the smallest divisor such that $(a_4) + 4D \geq 0$ and $(a_6) + 6D \geq 0$. In other words, we can regard $a_4, a_6$ as global sections of $\mathcal{L}^{\otimes 4}, \mathcal{L}^{\otimes 6}$ such that there is no point of $C$ where $a_4$ and $a_6$ vanish to order at least 4 and 6 respectively. Once we have $a_i \in \Gamma(\mathcal{L}^{\otimes i})$, we can regard the Weierstrass equation \( \mathcal{E} \) as a surface in the plane bundle $\mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}$ over $C$. If all the roots of the discriminant $\Delta \in \Gamma(\mathcal{L}^{\otimes 12})$ are distinct then this surface is smooth and is the minimal model of $E$. Otherwise it has isolated singularities, which we blow up as many times as needed (we may follow Tate’s algorithm \([15]\) to obtain the minimal model $E$. This is a smooth algebraic surface of arithmetic genus $n = \deg \mathcal{L}$, equipped with a map to $C$ with generic fiber $E$ and $\omega_{E/C} \cong \mathcal{L}$.

See for instance \([1] \text{pp.149ff.}\].

We shall need much information about the singular fibers that can arise for the elliptic fibration $\mathcal{E} \rightarrow C$. We extract from Tate’s table \([16] \text{p.46}\) the following local data for each possible Kodaira type of a singular fiber $E_v$: the discriminant degree $d_v$, the conductor degree $N_v$, and the structure of the group $E_v/(E_v)_0$ of multiplicity-1 components. We also list in each case the root lattice $L_v$ that $E_v$ contributes to the Néron-Severi lattice $\text{NS}(\mathcal{E})$ of $\mathcal{E}$. In each case, $L_v$ has rank $d_v - N_v$, and $E_v/(E_v)_0 \cong L_v^* / L_v$ where $L_v^* \subset L_v \otimes \mathbb{Q}$ is the dual lattice. The lattice “$A_0$” that appears for Kodaira types I$_1$ and II is the trivial lattice of rank zero. For Kodaira type I$_\nu^*$, the group $E_v/(E_v)_0$ always has order 4, and has exponent 2 or 4 according as $\nu$ is even or odd. For positive $\nu$ of either parity, a fiber of type I$_\nu^*$ has a distinguished multiplicity-1 component of order 2 in $E_v/(E_v)_0$, namely

\(^3\) Two models may yield different heights $h, h'$, but $h' = h + O(1)$ holds for any pair of naive heights on the same curve. It also follows that the property $\hat{h} = h + O(1)$ of the canonical height does not depend on the choice of naive height $h$.\n
the one closest to the identity component. In the \( L_v \) picture, the distinguished component corresponds to the nontrivial coset of \( D_4 \) in \( \mathbb{Z}^{4+\nu} \). When \( \nu = 0 \) there is no distinguished component: all three non-identity components of multiplicity 1 are equivalent, as are all three nontrivial cosets due to the triality of \( D_4 \).

| Kodaira type | I\(_\nu\)(\( \nu > 0 \)) | II | III | IV | IV* | III* | II* |
|--------------|--------------------------|----|-----|----|-----|------|-----|
| \( d_v \)    | \( \nu \) | 2  | 3   | 4  | 6 + \( \nu \) | 8   | 9   | 10  |
| \( N_v \)    | 1  | 2  | 2   | 2  | 2   | 2    | 2   |
| \( E_v/(E_v)_0 \) | \( \mathbb{Z}/\nu\mathbb{Z} \) | \{0\} | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/3\mathbb{Z} \) | \( D_4^{+\nu}/D_4^{+\nu} \) | \( \mathbb{Z}/3\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \{0\} |
| root lattice | \( A_{\nu-1} \) | \( A_0 \) | \( A_1 \) | \( A_2 \) | \( D_4^{+\nu} \) | \( E_6 \) | \( E_7 \) | \( E_8 \) |

The discriminant and conductor degrees \( d, N \) of \( \mathcal{E} \) are sums of the discriminant and conductor degrees of the singular fibers:

\[
d = 12n = \sum_v d_v, \quad N = \sum_v N_v. \tag{13}\]

Hence \( d - N = \sum_v (d_v - N_v) = \sum_v \text{rk} L_v \) is the rank of the subgroup \( \oplus_v L_v \) of \( \text{NS}(\mathcal{E}) \) due to the singular fibers. Shioda used this to prove [9, Cor. 2.7 (p.30)]:

**Proposition 1.** Let \( E \) be a nonconstant elliptic curve over a function field \( K = k(C) \) of genus \( g \), with discriminant and conductor degrees \( d = 12n \) and \( N \). Then

\[
N \geq 2n + (2 - 2g) + r, \tag{14}\]

where \( r \) is the rank of the Mordell-Weil group \( E(K) \).

**Proof:** Let \( T \subseteq \text{NS}(\mathcal{E}) \) be the subgroup spanned by \( s_0 \), the generic fiber, and \( \oplus_v L_v \). Then we have a short exact sequence (see for instance [10, Thm. 1.3]):

\[
0 \to T \to \text{NS}(\mathcal{E}) \to E(K) \to 0, \tag{15}\]

where the map \( \text{NS}(\mathcal{E}) \to E(K) \) is the sum on the generic fiber. Taking ranks, we find

\[
\text{rk} \text{NS}(\mathcal{E}) = \text{rk} T + \text{rk} E(K) = 2 + (d - N) + r. \tag{16}\]

But \( \text{NS}(\mathcal{E}) \) embeds into \( H^{1,1}(\mathcal{E}, \mathbb{Z}) \), a group of rank \( h^{1,1}(\mathcal{E}) = 10n + 2g \). Hence

\[
\text{rk} \text{NS}(\mathcal{E}) \leq 10n + 2g. \text{ Therefore}
\]

\[
N \geq (d + 2 + r) - (10n + 2g) = 2n + (2 - 2g) + r,
\]

as claimed. \( \blacksquare \)

**Remarks:**

1. Since \( r \geq 0 \) it follows that

\[
N \geq 2n + (2 - 2g) = (d/6) + \chi \tag{17}\]
for any nonconstant elliptic surface. This weaker inequality is sufficient for most of our purposes, even though we are interested in curves with a non-torsion point, for which the strict inequality \(N > (d/6) + \chi\) holds because \(r > 0\).

2. The inequality (17) is now usually known as the “Szpiro inequality”, but Shioda’s paper [9] predates Szpiro’s [15] by almost two decades (see also [12, p.114]). It is by now well-known that (17) can be proved by elementary means via Mason’s theorem [5] (the ABC inequality for function fields).

Can one also give an elementary proof of Shioda’s inequality, or even of its consequence that \(r = 0\) if \(N = (d/6) + \chi\)?

3. The requirement that \(E\) not be a constant curve is essential. There is an analogous statement for constant curves but many details must change. Suppose \(E\) is such a curve, that is, \(E = C \times E_0\) for some elliptic curve \(E_0/k\). Then \(E(K)/E_0(k)\) is finitely generated, and identified with the group \(\text{NS}(E)/T\).

Again we call the rank of this group \(r\). Since \(n = d = N = 0\) in this setting, we obtain the inequality \(r + 2 \leq h_{1,1}(C \times E_0)\). But for a constant curve, \(h_{1,1}(C \times E_0) = 2g + 2\), instead of the \(2g\) that one would expect from the \(10n + 2g\) formula. Hence \(r \leq 2g\). This can also be proved using the identification of \(E(K)/E_0(k)\) with \(\text{End}(\text{Jac}(C), E_0)\), an approach that also yields the equality condition: clearly \(r = 2g\) if \(g = 0\); if \(g > 0\) then \(r = 2g\) if and only if \(E_0\) has complex multiplication and \(\text{Jac}(C)\) is isogenous with \(E_0^g\). See for instance [2].

4. The hypothesis of characteristic zero, too, is essential here. In positive characteristic, one cannot decompose the second Betti number \(b_2(E)\) as \(h^{2,0} + h^{1,1} + h^{0,2}\), so one has only the weaker upper bound \(b_2(E)\) on \(\text{rk}(\text{NS}(E))\). This upper bound exceeds the characteristic-zero bound by \(2g\) for a constant curve and \(2(n + g - 1)\) for a nonconstant one. For instance, a constant curve \(C \times E_0\) has \(r \leq 4g\), with equality if and only if either \(g = 0\) or \(E_0\) and \(\text{Jac}(C)\) are both supersingular. In general \(E\) is said to be “supersingular” if \(\text{NS}(E) \cong \mathbb{Z}^{2g(E)}\); such surfaces were studied and used in [10,2].

2.3 Local height corrections. We next list the local height corrections \(\lambda_v(mP)\) for each of the Kodaira types. For convenience we abuse notation by using \(mP\) to refer also to the section \(s_{mP}\).

- If \(mP\) is on the identity component of \(E_v\) then
  \[
  \lambda_v(mP) = d_v/6. \quad (18)
  \]

In particular this covers fibers of type II or II*.

- If \(E_v\) is of type I, and \(P\) passes through component \(a \in \mathbb{Z}/\nu\mathbb{Z}\), let \(x = \bar{a}/\nu\) for any lift \(\bar{a}\) of \(a\) to \(\mathbb{Z}\); then
  \[
  \lambda_v(mP) = \nu B(mx), \quad (19)
  \]

where \(B(\cdot)\) is the second Bernoulli function \(B(z) := \sum_{n=1}^{\infty} \cos(2\pi n)/(\pi n)^2\).

Since \(B\) is \(\mathbb{Z}\)-periodic, the choice of \(\bar{a}\) does not matter. Likewise, since
\[ B(z) = B(-z) \] it does not matter that \( a \) cannot be canonically distinguished from \(-a\). We have
\[ B(z) = z^2 - z + \frac{1}{6} \]
for all \( z \in [0, 1] \), so in particular \( B(0) = 1/6 \). Hence \( \lambda_v(mP) = \nu/6 \) if \( mP \) passes through the identity component of \( E_v \), as also asserted by Lemma 4 in that case.

- If \( E_v \) is of type III, IV, I_0^*, III^*, or IV^*, and \( mP \) passes through a non-identity component of \( E_v \), then \( \lambda_v(mP) = 0 \).
- Finally, suppose \( E_v \) is of type I_1^* (\( \nu > 0 \)) and that \( mP \) passes through a non-identity component. If that component is the distinguished one of order 2 then \( \lambda_v(mP) = \nu/6 \). Otherwise \( \lambda_v(mP) = -\nu/12 \). (We could have also allowed \( \nu = 0 \), when there is no distinction among the three non-identity components, but \( \lambda_v(mP) = \nu/6 = -\nu/12 = 0 \) for all of them.)

We record two applications of these formulas for future use:

**Lemma 3.** Let \( E \) be an elliptic curve of discriminant degree \( 12n \) over a function field \( K \), and \( P \) any nonzero point of \( E(K) \). Then
\[ -n \leq \hat{h}(P) - h(P) \leq 2n. \]  

**Proof:** For each \( v \) we have \(-d_v/12 \leq \lambda_v \leq d_v/6\). Summing over \( v \) yields (21).

**Lemma 4.** Let \( E \) be an elliptic curve of discriminant degree \( 12n \) over a function field \( K \), and \( P \) any point of \( E(K) \). If for some integer \( m \) the multiple \( mP \) is a nonzero integral point then \( \hat{h}(mP) \leq 2n/m^2 \).

**Proof:** By our formulas for \( \lambda_v \) we have \( \lambda_v(mP) \leq d_v/6 \) for all \( v \). Hence
\[ m^2 \hat{h}(P) = \hat{h}(mP) = h(mP) + \sum_v \lambda_v(mP) \leq h(mP) + \sum_v d_v/6. \]  
But \( h(mP) = 0 \) since \( mP \) is integral, and \( \sum_v d_v/6 = d/6 = 2n \). Hence \( m^2 \hat{h}(P) \leq 2n \), and the Lemma follows.

**2.4 Reduction to the semistable case.** Recall that an elliptic curve is said to be semistable if all its singular fibers are of type I_v for some \( v \). Suppose \( E/K \) is semistable and \( P \) is a nontorsion point in \( E(K) \). We associate to \( (E, P) \) an element \( \gamma \) of the abelian group \( G \) of formal \( \mathbb{Z} \)-linear combinations of orbits of \( \mathbb{Q} \) under the infinite dihedral group \( D_\infty \) generated by \( z \mapsto z + 1 \) and \( z \mapsto 1 - z \).

We denote by \([z]\) the generator of \( G \) corresponding to the orbit of \( z \). Then \( \gamma \) is defined as a sum of local contributions \( \gamma_v \in G \) that record the types \( \nu(v) \) of the singular fibers \( E_v \) and the component \( c_v = a(v) \in \mathbb{Z}/(\nu(v))\mathbb{Z} \) of each fiber that contains \( P \), as follows:
\[ \gamma_v := \sum_v \gcd(a(v), \nu(v)) \cdot \left[ \frac{a(v)}{\nu(v)} \right]. \]
Then each of the height corrections $\hat{h}(mP) - h(mP)$, as well as the discriminant degree, are images of $\gamma$ under homomorphisms $\lambda_m, d$ from $G$ to $\mathbb{Q}$ or $\mathbb{Z}$, and the conductor is bounded above by the image of a homomorphism $N : G \to \mathbb{Z}$.

We define these homomorphisms on the generators of $G$ and extend by linearity. Suppose $\mathbb{Q} \ni z = a/b$ with $b > 0$ and $\gcd(a, b) = 1$. Note that $b$ is an invariant of the action of $D_\infty$. Then we set

$$\lambda_m([z]) := b B_2(mz), \quad d([z]) := b, \quad N([z]) := 1.$$  

(24)

Then our formulas (19, 13) yield the identities

$$\hat{h}(mP) = h(mP) + \lambda_m(\gamma) \quad (m = 1, 2, 3, \ldots), \quad 12n = d = d(\gamma)$$  

(25)

and the estimate

$$N \leq N(\gamma).$$  

(26)

(This last is an upper bound rather than an identity because each $v$ contributes 1 to $N$ and $\gcd(a(v), \nu(v)) \geq 1$ to $N(\gamma)$.) It follows that

$$N(\gamma) \geq N \geq (d/6) + (2 - 2g) + r \geq \frac{1}{6} d(\gamma) + 3 - 2g.$$  

(27)

The second step is Shioda’s inequality (Prop. 1), and the third step uses the positivity of $r$, which follows from our hypothesis that $P$ is nontorsion.

To generalize these formulas to curves that may not be semistable, it might seem that we would have to extend $G$ with generators that correspond to Kodaira types other than $I_\nu$. But we can associate to any additive fiber $E_v$ an element $\gamma_v \in G$, depending on $(E_v, c_v)$ as follows:

1. If $E_v$ is multiplicative, $\gamma_v$ is defined by (23).
2. If $c_v$ is the identity component then $\gamma_v := d_v [0]$.
3. If $c_v$ is a non-identity component of a fiber $E_v$ of type III, IV, IV*, or III* then $\gamma_v$ is respectively

$$[1/2] + [0], \quad [1/3] + [0], \quad 2 \cdot [1/2] + 2 \cdot [0], \quad 3 \cdot [1/3] + 3 \cdot [0].$$

4. If $c_v$ is a distinguished component of a fiber $E_v$ of type $I_\nu$, then

$$\gamma_v := 2 [1/2] + (\nu + 2) [0].$$

Proposition 2. Let $E$ be an elliptic curve over a function field $K$ of genus $g$, and $P \in E(K)$ a nontorsion point. Define for each singular fiber $E_v$ a positive $\gamma_v \in G$, depending on $(E_v, c_v)$ as follows:

- If $E_v$ is multiplicative, $\gamma_v$ is defined by (23).
- If $c_v$ is the identity component then $\gamma_v := d_v [0]$.
- If $c_v$ is a non-identity component of a fiber $E_v$ of type III, IV, IV*, or III* then $\gamma_v$ is respectively

$$[1/2] + [0], \quad [1/3] + [0], \quad 2 \cdot [1/2] + 2 \cdot [0], \quad 3 \cdot [1/3] + 3 \cdot [0].$$

- If $c_v$ is a distinguished component of a fiber $E_v$ of type $I_\nu$, then

$$\gamma_v := 2 [1/2] + (\nu + 2) [0].$$
If \( c_v \) is a non-distinguished, non-identity component of a fiber \( E_v \) of type \( \Gamma_v \), then
\[
\gamma_v := (\mu + 2) [1/2] + 2 [0]
\]
if \( \nu = 2\mu \), and
\[
\gamma_v := [1/4] + (\mu + 1) [1/2] + [0]
\]
if \( \nu = 2\mu + 1 \) for some integer \( \mu \).

Then:

i) \( \lambda_v(mP) = \lambda_m(\gamma_v) \) for each \( m = 1, 2, 3, \ldots \);

ii) \( d_v = d(\gamma_v) \); and

iii) \( N_v \leq N(\gamma_v) \).

Thus \((22, 26, 27)\) hold for \( \gamma := \sum_v \gamma_v \). Equality in (iii) holds if and only if \( E_v \) is either a multiplicative fiber with \( \gcd(a, \nu) = 1 \), a fiber of type III or IV with \( c_v \) a non-identity component, or a fiber of type II.

[Note that, as was true for the \( \lambda_v \) formulas, the first two formulas in Prop. 2 overlap in the case of a multiplicative fiber with \( a(\nu) = 0 \), but give the same answer in this case. Here both prescriptions yield \( \gamma_v = \nu(\nu) \cdot [0] \) for such \( \nu \).]

Proof: The multiplicative case was seen already. For each of the other Kodaira types, it is straightforward to verify that \( \lambda_v(mP) = \lambda_m(\gamma_v) \) for each nonnegative \( m \) less than the exponent of the finite group \( E_v / (E_v)_0 \) (which is at most 4), and to check that \( d_v = d(\gamma_v) \), and that \( N_v \leq N(\gamma_v) \), with strict inequality except in the three cases listed. We recover \((22, 26, 27)\) by summing over \( \nu \).

3. The values of \( \hat{h}_{\min}(0, 12n) \) for \( n = 1, 2, 3 \), and consecutive integral multiples.

For each \( n \) we can use the formulas and results above to obtain a lower bound on \( \hat{h}_{\min}(g, 12n) \). When \( g = 0 \) and \( n = 1, 2, 3 \) we also show that this bound is attained if and only if \( mP \) is integral for \( m \leq M = 6, 8, 9 \), and verify that the \( (E, P) \) exhibited in Theorem \( n \) satisfy those conditions.

Suppose \( E \) is an elliptic curve over \( C(T) \) with discriminant degree 12n. Let \( P \) be a nontorsion rational point on \( E \), and \( \gamma \) the associated element of \( G \). From \( \gamma \) and \( \hat{h}(P) \) we can recover all the naïve heights \( h(mP) \) from the first formula in \((22)\): \( h(mP) = m^2 \hat{h}(P) - \lambda_m(\gamma) \). Given \( n \) and an upper bound \( H \) on \( \hat{h}(P) \), there are only finitely many candidates for the pair \( (\gamma, \hat{h}(P)) \): there are finitely many \( \gamma > 0 \) with \( d(\gamma) = 12n \), and for each one there are only finitely many possible choices for \( \hat{h}(P) \) consistent with \( \hat{h}(P) + \lambda_1(\gamma) = \hat{h}(P) \in (0, H] \). For each candidate \( (\gamma, \hat{h}(P)) \) we can check the condition \( m' | m \Rightarrow h(mP) \geq h(m'P) \geq 0 \).

Only finitely many \( m \) need be checked for each \( (\gamma, \hat{h}(P)) \); by Lemma \((4)\) we know that \( h(mP) \geq 0 \) once \( m^2 \hat{h}(P) \geq n \), and \( h(mP) \geq h(m'P) \) for each \( m' | m \) once \( m^2 \hat{h}(P) \geq 4n \). The minimal \( h(P) \) among the \( (\gamma, \hat{h}(P)) \) that pass these tests is then our lower bound on \( \hat{h}_{\min}(g, 12n) \). [We could also test the more complicated
inequality of Lemma 2, which may further improve the bound; instead we checked that inequality after the fact when necessary.

We wrote a gp program to compute this bound by exhaustive search, and ran it with \( H = 2n/M^2 \) for \( n = 1, 2, 3 \). We chose this upper bound \( H \) to ensure that, by Lemma 3, we would also find all feasible \( (\gamma, \hat{h}(P)) \) such that \( h(mP) = 0 \) for each \( m = 1, 2, 3, \ldots, M \). For \( n = 1 \), we found that the minimum occurs for

\[
\gamma = \left[\frac{1}{5}\right] + \left[\frac{1}{3}\right] + \left[\frac{1}{2}\right] + 2\left[0\right], \quad \hat{h}(P) = 1/30, \tag{28}
\]

and is the unique \( (\gamma, \hat{h}(P)) \) such that \( h(mP) = 0 \) for each \( m \leq 6 \). For \( n = 2 \), we found that the minimum occurs for

\[
\gamma = \left[\frac{1}{11}\right] + 2\left[\frac{2}{5}\right] + \left[\frac{1}{3}\right], \quad \hat{h}(P) = 4/165; \tag{29}
\]

but this is not feasible because \( h(mP) = 0, 2, 2, 2 \) for \( m = 2, 4, 6, 12 \), so inequality \( 11 \) is violated when \( m = 2 \). Our lower bound on \( \hat{h}_{\text{min}}(g, 24) \) is thus the next-smallest value, which occurs for

\[
\gamma = \left[\frac{1}{7}\right] + \left[\frac{2}{5}\right] + \left[\frac{1}{4}\right] + \left[\frac{1}{3}\right] + \left[\frac{1}{2}\right] + 3\left[0\right], \quad \hat{h}(P) = 11/420, \tag{30}
\]

and is the unique \( (\gamma, \hat{h}(P)) \) such that \( h(mP) = 0 \) for each \( m \leq 8 \).

On the other hand, the \( (\gamma, \hat{h}(P)) \) pairs of \( 28 \) are also those associated with the curves and points \( E, P \) exhibited in \( 11 \). Hence those \( E, P \) attain our lower bounds \( 1/30, 11/420 \) on \( \hat{h}_{\text{min}}(12), \hat{h}_{\text{min}}(24) \), as well as the upper bounds \( 6 \) and \( 8 \) on the number of consecutive integral multiples for \( n = 1 \) and \( n = 2 \). This proves all of Theorems 1 and 2 except for the claims that every \( (E, P) \) attaining those bounds is isomorphic with some \( E_1(q) \) or \( E_2(u) \).

For \( n = 3 \), we find that there is a unique \( (\gamma, \hat{h}(P)) \) such that \( h(mP) = 0 \) for each \( m \leq 9 \), namely

\[
\gamma = \left[\frac{1}{8}\right] + \left[\frac{3}{7}\right] + \left[\frac{1}{5}\right] + \left[\frac{1}{4}\right] + 2\left[\frac{1}{3}\right] + \left[\frac{1}{2}\right] + 4\left[0\right], \quad \hat{h}(P) = 23/840. \tag{31}
\]

Again these are the \( \gamma \) and \( \hat{h}(P) \) for the \( (E, P) \) exhibited in the Introduction (formula 15). But we do not claim that \( \hat{h}_{\text{min}}(36) = 23/840 \); Lemma 2 eliminates the second-smallest pair

\[
(\gamma, \hat{h}(P)) = \left(\left[\frac{1}{13}\right] + \left[\frac{3}{8}\right] + \left[\frac{3}{7}\right] + \left[\frac{1}{5}\right] + \left[\frac{1}{4}\right] + \left[\frac{1}{3}\right], \quad 229/10920\right)
\]

(which violates the inequality 11 in the same way that 29 did), but not several other possibilities with \( \hat{h}(P) < 23/840 \). We next list all these possibilities, in order of increasing \( \hat{h}(P) \):

\begin{center}
\begin{tabular}{c|c|c}
\hline
\( \gamma \) & \( \hat{h}(P) \) & \\
\hline
\[\left[\frac{1}{13}\right] + \left[\frac{3}{11}\right] + \left[\frac{3}{8}\right] + 2\left[\frac{1}{2}\right] \] & \( 23/1144 \approx .02010 \) & \\
\[\left[\frac{1}{13}\right] + \left[\frac{3}{8}\right] + \left[\frac{2}{7}\right] + \left[\frac{1}{4}\right] + 2\left[\frac{1}{2}\right] \] & \( 17/728 \approx .02335 \) & \\
\[\left[\frac{1}{11}\right] + \left[\frac{4}{9}\right] + \left[\frac{2}{7}\right] + \left[\frac{1}{4}\right] + \left[\frac{1}{3}\right] + 2\left[0\right] \] & \( 65/2772 \approx .02345 \) & \\
\[\left[\frac{1}{12}\right] + \left[\frac{3}{11}\right] + \left[\frac{3}{8}\right] + 2\left[\frac{1}{2}\right] + \left[0\right] \] & \( 7/264 \approx .02652 \) & \\
\[\left[\frac{1}{11}\right] + \left[\frac{3}{7}\right] + 2\left[\frac{1}{5}\right] + \left[\frac{1}{4}\right] + 2\left[\frac{1}{2}\right] \] & \( 41/1540 \approx .02662 \) & \\
\hline
\end{tabular}
\end{center}
Points of low height on elliptic curves and surfaces I

(For comparison, \(229/10920 \approx 0.02097\) and \(23/840 \approx 0.02738\).) We have \(d(\gamma) \leq 7\) for each entry in the table \(\text{[22]}\); therefore by Prop. \(\text{[1]}\) none of them can occur for an elliptic curve over \(\mathbb{P}^1\). (Even the weaker inequality \(\text{[17]}\) would suffice here; either of those inequalities also excludes \(\text{[29]}\) for \(n = 2\), and would thus be enough to obtain \(\hat{h}_{\min}(0, 24)\), but the determination of \(\hat{h}_{\min}(24)\) required a further argument.) Thus \(\hat{h}_{\min}(0, 36) = 23/840\), proving Theorem 3 except for the claim that every \((E, P)\) satisfying conditions (a) and (b) is of the form \(E_3(A)\) for some \(A\).

Acknowledgements. I thank J. Silverman and T. Shioda for helpful correspondence concerning their papers and related issues, and M. Watkins for carefully reading a draft of this paper. This work was made possible in part by funding from the Packard Foundation and the National Science Foundation.

References

1. Barth, W., Peters, C., Van de Ven, A.: Compact Complex Surfaces. Berlin: Springer, 1984.
2. Elkies, N.D.: Mordell-Weil lattices in characteristic 2, I: Construction and first properties. International Math. Research Notices 1994 #8, 343–361.
3. Hindry, M., Silverman, J.H.: The canonical height and integral points on elliptic curves, Invent. Math. 93 (1988), 419–450.
4. Lang, S.: Elliptic Curves: Diophantine Analysis. Berlin: Springer, 1978.
5. Mason, R.C.: Diophantine Equations over Function Fields, London Math. Soc. Lect. Note Ser. 96, Cambridge Univ. Press 1984. See also pp.149–157 in Springer LNM 1068 (1984) [=proceedings of Journées Arithmétiques 1983 (Noordwijk, H. Jager, ed.).
6. Nishiyama, K.-i.: The minimal height of Jacobian fibrations on K3 surfaces, Tohoku Math. J. (2) 48 (1996), 501–517.
7. Oguiso, K., Shioda, T.: The Mordell-Weil lattice of a rational elliptic surface, Comment. Math. Univ. St. Pauli 40 (1991), 83–99.
8. PARI/GP, versions 2.1.1–4, Bordeaux, 2000–4, http://pari.math.u-bordeaux.fr.
9. Shioda, T.: Elliptic Modular Surfaces, J. Math. Soc. Japan 24 (1972), 20–59.
10. Shioda, T.: On the Mordell-Weil lattices. Comment. Math. Univ. St. Pauli 39 (1990), 211–240.
11. Shioda, T.: Existence of a Rational Elliptic Surface with a Given Mordell-Weil Lattice, Proc. Japan Acad. (Ser. A) 68 (1992), 251–255.
12. Shioda, T.: Some remarks on elliptic curves over function fields, Astérisque 209 (1992) [=proceedings of Journées Arithmétiques 1991 (Genève), D.F. Coray and Y.-F. S. Pétermann, eds.], 99–114.
13. Silverman, J.H.: Computing Heights on Elliptic Curves, Math. of Computation 51 #183 (July 1988), 339–358.
14. Silverman, J.H.: A lower bound for the canonical height on elliptic curves over abelian extensions, J. Number Theory 104 (2005), 353–372.
15. Szpiro, L.: Discriminant et conducteur des courbes elliptiques. Astérisque 183 (1990) [=Séminaire sur les Pinceaux de Courbes Elliptiques, Paris 1988], 7–18.
16. Tate, J.: Algorithm for Determining the Type of a Singular Fiber in an Elliptic Pencil. Pages 33–52 in Modular Functions of One Variable IV (Lect. Notes in Math. 476 (1975); Birch, B.J., Kuyk, W., eds.).