ON ENDPOINT REGULARITY CRITERION OF THE 3D NAVIER-STOKES EQUATIONS

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Abstract. Let \( (u, \pi) \) with \( u = (u_1, u_2, u_3) \) be a suitable weak solution of the three dimensional Navier-Stokes equations in \( \mathbb{R}^3 \times [0, T] \). Denote by \( \dot{B}^{-1}_{\infty, \infty} \) the closure of \( C_0^\infty \) in \( B^{-1}_{\infty, \infty} \). We prove that if \( u \in L^\infty(0, T; \dot{B}^{-1}_{\infty, \infty}) \), \( u(x, T) \in \dot{B}^{-1}_{\infty, \infty} \), and \( u_3 \in L^\infty(0, T; L^3) \) or \( u_3 \in L^\infty(0, T; \dot{B}^{-1+3/p}_{p,q}) \) with \( 3 < p, q \leq \infty \), then \( u \) is smooth in \( \mathbb{R}^3 \times [0, T] \). Our result improves a previous result established by Wang and Zhang [Sci. China Math. 60, 637-650 (2017)].

1. Introduction

In this paper, we study the incompressible Navier-Stokes equations in \( \mathbb{R}^3 \times (0, T) \)

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla \pi &= 0, \\
\text{div } u &= 0, \\
u|_{t=0} &= u_0(x),
\end{align*}
\]

where \( u(x, t) = (u_1, u_2, u_3) \) denotes the velocity of the fluid, and \( \pi(x, t) \) represents the pressure.

In the pioneering works \([9, 13]\), Leray and Hopf proved the global existence of weak solutions with finite energy. However, the uniqueness and regularity of weak solutions still remains open. There exists many conditional regularity results of the three dimensional Navier-Stokes equations. The most well-known one is due to Serrin \([14]\) (see also Struwe \([17]\)), which states that if the weak solution \( u \) satisfies

\[
u \in L^q(0, T; L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad 3 < p \leq \infty,
\]

then \( u \) is regular. The limiting case \( u \in L^\infty(0, T; L^3(\mathbb{R}^3)) \) is solved by Escauriaza, Seregin and Šverák \([7]\) using blow-up analysis and backward uniqueness for heat equations.

In term of the chain of critical spaces

\[
L^3(\mathbb{R}^3) \hookrightarrow L^3,q(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)(3 < p, q < \infty) \hookrightarrow \text{BMO}^{-1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3),
\]

it is natural to extend the regularity condition \( u \in L^\infty(0, T; L^3(\mathbb{R}^3)) \) to \( u \in L^\infty(0, T; X) \) with \( X \) being one of the above spaces. Here, a Banach space \( X \) is called critical if we have \( \|\lambda u(\lambda x)\|_X = \|u\|_X \) for \( u \in X \). Many works have been
Phuc, Gallagher, Koch and Planchon and Wang and Zhang\\nSection 3 is devoted to the proof of main result.

notations of weak solutions and suitable weak solutions, as well as some useful lemmas.

C-Stokes scaling and that \( \dot{u} \) limit function, we take advantage of the property of involved function spaces –

property by combining the facts that \( \dot{u} \) vanishes at the last moment and has some spatial decay. We establish the first

of heat equations. We need to show two properties of the limit of scaled solutions:

Theorem 1.1 improves previous results due to Escauriaza, Seregin and Šverák [7],

Remark 1.2. Considering the inclusion relationship that for 3 < p, q < \( \infty \),

L^3(\mathbb{R}^3) \hookrightarrow L^{3,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3) \hookrightarrow VMO^{-1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3),

Theorem 1.1 improves previous results due to Escauriaza, Seregin and Šverák [7],

Remark 1.3. Denote by \( \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \) the closure of \( L^\infty(\mathbb{R}^3) \) in \( \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \). It holds

that \( \dot{B}^{-1}_{\infty,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \) for 1 ≤ q < \( \infty \). Tobias Barker points out that \( \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \) in Theorem 1.1 can be replaced with \( \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \). Namely, we have for every \( f \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \),

\[ \lim_{\lambda \to 0} \lambda f(\lambda \cdot) = 0 \quad \text{in} \quad \mathcal{D}' \]

Our proof of Theorem 1.1 is based on the scheme developed by Escauriaza, Seregin and Šverák [7], which consists of blow up analysis and backward uniqueness of heat equations. We need to show two properties of the limit of scaled solutions: it vanishes at the last moment and has some spatial decay. We establish the first

property by combining the facts that \( \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \) is invariant under the Naiver-Stokes scaling and \( C_0^\infty(\mathbb{R}^3) \) is dense in \( \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \). To get spatial decay for the limit function, we take advantage of the property of involved function spaces – \( L^p \) is dense in \( \dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)(3 < p, q < \infty) \), and \( L^{3,\infty}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) + L^4(\mathbb{R}^3) \).

The rest of this paper is organized as follows. In Section 2, we recall the definitions of weak solutions and suitable weak solutions, as well as some useful lemmas. Section 3 is devoted to the proof of main result.

For more related results, see [2, 3, 4, 6, 11, 12, 15, 18] and references therein.

In this paper, we aim at improving Wang and Zhang’s results to the case \( u \in L^\infty(0, T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)) \). Our main results read as follows.

\textbf{Theorem 1.1.} Let \((u, \pi)\) be a suitable weak solution of in \( \mathbb{R}^3 \times [0, T] \). If \( u \in L^\infty(0, T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)) \), \( u(x, T) \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \), and \( u_3 \) satisfies

\[ u_3 \in L^\infty(0, T; \dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)) \quad \text{for} \quad 3 < p, q < \infty, \]

and \( u \in L^\infty(0, T; \text{BMO}^{-1}(\mathbb{R}^3)) \) with \( u(T) \in \text{VMO}^{-1}(\mathbb{R}^3) \).
2. Preliminaries

In this section we state definitions of Leray-Hopf weak solutions and suitable weak solutions, introduce some notations, and collect some useful lemmas.

First we recall the definitions of Leray-Hopf weak solutions \([9, 13]\) and suitable weak solutions \([5]\) to the Navier-Stokes equations.

**Definition 2.1.** A vector field \(u\) is called a Leray-Hopf weak solution of (1.1) in \(\mathbb{R}^3 \times (0, T)\) if
\[
(1) \ u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3));
(2) \ u \text{ satisfies (1.1) in } \mathbb{R}^3 \times (0, T) \text{ in the weak sense that for any } \psi \in C_c^\infty(\mathbb{R}^3 \times (0, T)) \text{ such that } \nabla \psi = 0,
\int_0^T \int_{\mathbb{R}^3} (-u \partial_t \psi + \nabla u \cdot \nabla \psi) \, dx \, dt = 0.
(3) \ u \text{ satisfies the energy inequality}
\int_{\mathbb{R}^3} |u(x, t)|^2 \, dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, ds \leq \int_{\mathbb{R}^3} |u_0(x)|^2 \, dx
\]
for all a.e. \(t \in [0, T]\).

**Definition 2.2.** Let \(\Omega\) be an open set in \(\mathbb{R}^3\) and \(T > 0\). A pair \((u, \pi)\) is called a suitable weak solution of (1.1) in \(\Omega \times (0, T)\) if
\[
(1) \ u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \ \pi \in L^2(0, T; H^1(\Omega));
(2) \ (u, \pi) \text{ satisfies (1.1) in } \Omega \times (0, T) \text{ in the sense of distribution};
(3) \ (u, \pi) \text{ satisfies the local energy inequality}
\int_\Omega |u(x, t)|^2 \varphi \, dx + 2 \int_0^t \int_\Omega |\nabla u|^2 \varphi \, dx \, ds
\leq \int_0^t \int_\Omega |u|^2 (\partial_x \varphi + \Delta \varphi) + u \cdot \nabla \varphi |u|^2 + 2\pi \, dx \, ds
\]
for all a.e. \(t \in [0, T]\) and all \(\varphi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R})\) such that \(\varphi \geq 0\) in \(\Omega \times (0, T)\).

**Remark 2.3.** If \(u\) is a Leray-Hopf weak solution and \(u \in L^\infty(0, T; \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3))\), by Lemma 2.8 then \(u \in L^3(\mathbb{R}^3 \times (0, T))\), which means that \(u\) satisfies the local energy inequality and is a suitable weak solution.

We now fix some notations. Let \((u, \pi)\) be a solution to the Navier-Stokes equations (1.1). Then for \(\lambda \in \mathbb{R}\) \((u_\lambda(x, t), \pi_\lambda(x, t))\)
\[
u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad \pi_\lambda(x, t) = \lambda^2 \pi(\lambda x, \lambda^2 t),
\]
also solves the Navier-Stokes equations (1.1). For \(z_0 = (x_0, t_0)\), the following quantities are invariant under the above scaling:
\[
A(u; r, z_0) := \sup_{r^2 + t_0 \leq r \leq t} \int_{B_r(z_0)} |u(y, t)|^2 \, dy, \quad C(u; r, z_0) := r^{-2} \int_{Q_r(z_0)} |u(y, s)|^3 \, dy \, ds,
E(u; r, z_0) := r^{-1} \int_{Q_r(z_0)} |\nabla u(y, s)|^2 \, dy \, ds, \quad D(u; r, z_0) := r^{-2} \int_{Q_r(z_0)} |\pi(y, s)|^2 \, dy \, ds.
\]
We denote
\[
B(x_0, r) := \{ x \in \mathbb{R}^3 : |x - x_0| < R \}, \quad B_r := B(0, r);
\]
Definition 2.4 (Littlewood-Paley decomposition). Let \( \phi \) be a smooth function with values in \([0,1]\) such that \( \phi \in C_0^\infty(\mathbb{R}^n) \) in the annulus \( \mathcal{C} := \{ \xi \in \mathbb{R}^n : \frac{2}{3} \leq |\xi| \leq \frac{5}{3} \} \) satisfying
\[
\sum_{j \in \mathbb{Z}} \phi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.
\]
For every \( u \in \mathcal{S}' \), we define the homogeneous dyadic blocks \( \hat{\Delta}_j \) and the homogeneous low-frequency cutoff operators \( S_j \) for all \( j \in \mathbb{Z} \) as
\[
\hat{\Delta}_j f = \phi(2^{-j}D)f = \mathcal{F}^{-1}(\phi(2^{-j}\xi)\hat{f}(\xi)).
\]
Definition 2.5. Let \( P \) be the set of polynomials. The Besov space \( \dot{B}_{p,q}^s(\mathbb{R}^3) \) with \( s \in \mathbb{R} \), \( 1 \leq p, q \leq \infty \), consists of \( f \in \mathcal{S}'(\mathbb{R}^3)/P \) satisfying
\[
\|f\|_{\dot{B}_{p,q}^s} := \left\|2^{js}\|\Delta f\|_{L^p(\mathbb{R}^3)}\right\|_{\ell^q} < \infty.
\]
Lemma 2.6. Let \( 1 \leq p_1 \leq p_2 \leq \infty \), and \( 1 \leq r_1 \leq r_2 \leq \infty \), \( s \in \mathbb{R} \). Then, we have
\[
\dot{B}_{p_1,r_1}^s \hookrightarrow \dot{B}_{p_2,r_2}^{s+\delta/p_2-\delta/p_1}.
\]
Remark 2.7. Lemma (2.6) can be found in Chapter 2 in Bahouri et.al. [1].

The following improved Galirado-Nirenberg inequality (Ledoux [10]) and its local version (Seregin and the second author [16]) will be used in the proof of main theorem.

Lemma 2.8. (1) If \( f \in \dot{B}_{2,1}^{-1}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \), then we have
\[
\|f\|_{L^4(\mathbb{R}^3)} \leq c\|f\|_{\dot{B}_{2,1}^{-1}(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}.
\]

(2) If \( f \in \dot{B}_{2,1}^{-1}(\mathbb{R}^3) \cap H^1(B_{2r}(x_0)) \), then we have
\[
\|f\|_{L^4(B_2r(x_0))} \leq c\|f\|_{\dot{B}_{2,1}^{-1}(\mathbb{R}^3)}\left(\frac{1}{r}\|f\|_{L^2(B_{2r}(x_0))} + \|\nabla f\|_{L^2(B_{2r}(x_0))}\right)^{\frac{1}{2}}.
\]
We recall the definition of weak Lebesgue space, as well as decomposition of functions in weak Lesbegue spaces (see [4]).

Definition 2.9. The weak Lesbegue space \( L^{3,\infty}(\mathbb{R}^3) \) consists of local integrable functions \( f \) satisfying
\[
\|f\|_{L^{3,\infty}(\mathbb{R}^3)} = \sup_{\lambda > 0} \lambda \int_{\mathbb{R}^3} |f(x)|^\frac{3}{\lambda} \, dx < \infty.
\]

Lemma 2.10. Let \( 1 < t < r < s \leq \infty \), and \( f \in L^{r,\infty}(\mathbb{R}^3) \). Then we have \( f = f_1 + f_2 \) for some \( f_1 \in L^r(\mathbb{R}^3) \) and \( f_2 \in L^s(\mathbb{R}^3) \), which satisfy
\[
\|f_1\|_{L^r(\mathbb{R}^3)} \leq c(s,r)\|f\|_{L^{r,\infty}(\mathbb{R}^3)},
\]
and
\[
\|f_2\|_{L^s(\mathbb{R}^3)} \leq c(r,t)\|f\|_{L^{r,\infty}(\mathbb{R}^3)}.
\]
We also need a bound for scaled energy of Naiver-Stokes equations by Seregin and the second author \[16\].

**Lemma 2.11.** Let \((u, \pi)\) be a suitable weak solution of \((1.1)\) in \(\mathbb{R}^3\times(0, T)\). Moreover, it is supposed that 
\[
u \in L^{\infty}(0, T; \dot{B}_{-1, \infty}^{-1}(\mathbb{R}^3)).
\]
Then, for any \(z_0 \in \mathbb{R}^3 \times (0, T)\), we have estimate
\[
\sup_{0 < r < r_0} \left\{ A(u; z_0, r) + C(u; z_0, r) + D(u; z_0, r) + E(u; z_0, r) \right\} \\
\leq c \left[ \frac{1}{r_0^2} \int_{Q_{r_0}} |u|_3^3 dx \right] + c \left[ \frac{1}{r_0^2} \int_{Q_{r_0}} |\pi|_-^{3/2} dx \right],
\]
where \(r_0 \leq \frac{1}{2} \min\{1, t_0\}\) and \(c\) depends on \(C(z_0, 1)\) and \(D(z_0, 1)\) only.

We conclude this section by recalling the small energy regularity results due to Wang and Zhang \[19\].

**Lemma 2.12.** Let \((u, \pi)\) be a suitable weak solution of \((1.1)\) in \(Q_1\). If \(u\) satisfies
\[
\sup_{0 < r < 1} \left\{ A(u; r) + E(u; r) \right\} \leq M,
\]
where \(M > 0\), then there exists a positive constant \(\varepsilon\) depending on \(M\) such that if
\[
\frac{1}{r_*^2} \int_{Q_{r_*}} |u|_3^3 dx \leq \varepsilon,
\]
for some \(r_*\) with \(0 < r_* < \min\{\frac{1}{2}, (C(u; 1) + D(\pi; 1))^{-2}\}\), then \((0, 0)\) is a regular point.

**Lemma 2.13.** Let \((u, \pi)\) be a suitable weak solution of \((1.1)\) in \(Q_r\). If \((u, \pi)\) satisfies
\[
\frac{1}{r^2} \int_{Q_r} |u|^3 + |\pi|_-^{3/2} dx dt \leq M,
\]
where \(M > 0\), then there exists a positive constant \(\varepsilon\) depending on \(M\) such that if
\[
\frac{1}{r^2} \int_{Q_r} |u|_3^3 dx \leq \varepsilon,
\]
then \((0, 0)\) is a regular point.

### 3. Proof of Theorem 1.1

Since the Naiver-Stokes equations are translation and scaling invariant, thus it is sufficient to prove our main results in the domain \(\mathbb{R}^3 \times [-1, 0)\). The proof is based on blow-up analysis and backward uniqueness of parabolic equations developed in Escauriaza, Seregin and Šverák \[7\]. We argue by contradiction. Without loss of generality, we assume that \((0, 0)\) is a singular point of \(u\).

**Proof.** **Step 1: Blow-up analysis**

By the assumption, we have
\[
\|u\|_{L^{\infty}(-1, 0; \dot{B}_{-1, \infty}^{-1}(\mathbb{R}^3))} \leq c.
\]
From Lemma 2.11 we get, for \(z_0 \in B_{1/2} \times (-1/4, 0)\) and \(0 < r < \frac{1}{2}\),
\[
A(u; z_0, r) + C(u; z_0, r) + D(u; z_0, r) + E(u; z_0, r) \leq c(C(u; 1), D(u; 1)).
\]
Since \( (0,0) \) is a singular point, Lemma 2.12 ensures that there exists a sequence \( R_k \) such that \( R_k \to 0 \) as \( k \to +\infty \)

\[
R_k^{-2} \int_{Q_{R_k}} |u_3(x,t)|^3 \, dx \, dt \geq \varepsilon.
\]

Define

\[
u^k(y, s) = R_k u(R_k y, R_k^2 s), \quad \pi^k(y, s) = R_k^2 \pi(R_k y, R_k^2 s),
\]

where \((y, s) \in \mathbb{R}^3 \times (-\frac{1}{R_k^2}, 0)\). Then the pair \((u^k, \pi^k)\) is still a suitable weak solution to \((1.1)\).

Since \( A(u; r), C(u; r), D(\pi; r) \) and \( E(u; r) \) are invariant under the Navier-Stokes scaling, we obtain that, for any \( a > 0 \) and \( z_0 = (x_0, t_0) \in \mathbb{R}^3 \times (-\infty, 0] \),

\[
A(u^k; z_0, a) + C(u^k; z_0, a) + D(\pi^k; z_0, a) + E(u^k; z_0, a)
= A(u^k; z_0^k, r_k a) + C(u^k; z_0^k, r_k a) + D(\pi^k; z_0^k, r_k a) + E(u^k; z_0^k, r_k a),
\]

where \( z_0^k = (r_k x_0, r_k^2 t_0) \). From (3.1), we get that for enough large \( k \),

\[
A(u^k; z_0, a) + C(u^k; z_0, a) + D(\pi^k; z_0, a) + E(u^k; z_0, a) \leq c.
\]

We also have

\[
\int_{Q_1} |u_3^k(x,t)|^3 \, dx \, dt = R_k^{-2} \int_{Q_{R_k}} |u_3(x,t)|^3 \, dx \, dt \geq \varepsilon
\]

for all \( k \in \mathbb{N} \).

By interpolation between \( A(u^k; a) \) and \( E(u^k; a) \), we get \( u^k \in L_t^6 L_x^{18/7}(Q_a) \). Then we have by Hölder’s inequality that \( u^k \cdot \nabla u^k \in L_t^2 L_x^\frac{9}{2}(Q_a) \). Appealing to the linear Stokes estimate, we deduce that

\[
|\partial_t u^k| + |\nabla u^k| + |\nabla \pi^k| \in L_t^2 L_x^\frac{9}{2}(Q_a).
\]

Applying the Aubin-Lions lemma, we can extract a subsequence, still denoted by \((u^k, \pi^k)\), such that \((u^k, \pi^k)\) converges weakly to some limit functions \((v, \pi')\), for any \( a > 0 \),

\[
u^k \rightharpoonup v \quad \text{in} \quad L^\infty(-a^2, 0; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)),
\]

\[
u^k \rightharpoonup v \quad \text{in} \quad L^\infty(-a^2, 0; L^2(B_a)),
\]

\[
u^k \rightharpoonup v \quad \text{strongly in} \quad C([-a^2, 0]; L^\frac{9}{2}(B_a)),
\]

\[
s \nabla u^k \rightharpoonup \nabla v \quad \text{in} \quad L^2(Q_a),
\]

\[
\pi^k \rightharpoonup \pi' \quad \text{in} \quad L^\frac{9}{2}(Q_a).
\]

Since \( u^k \) is uniformly bounded in \( L^{10/3}(Q_a) \) by interpolation between \( A(u^k; a) \) and \( E(u^k; a) \), we get by Hölder’s inequality,

\[
u^k \rightharpoonup v \quad \text{strongly in} \quad L^3(Q_a).
\]

Furthermore, we have in case \( u_3 \in L^\infty(-1, 0; L^{3,\infty}(\mathbb{R}^3)) \) that

\[
u_3^k \rightharpoonup v_3 \quad \text{in} \quad L^\infty(-a^2, 0; L^{3,\infty}(\mathbb{R}^3)),
\]

or in case \( u_3 \in L^\infty(-1, 0; \dot{B}_{p,q}^{-1+3/p}(\mathbb{R}^3)) \) that

\[
u_3^k \rightharpoonup v_3 \quad \text{in} \quad L^\infty(-a^2, 0; \dot{B}_{p,q}^{-1+3/p}(\mathbb{R}^3)).
\]
The above convergence implies that \((v, \pi')\) satisfy the Navier-Stokes equations in \(\mathbb{R}^3 \times (-\infty, 0)\). Moreover, due to lower semi-continuity of norm, it holds that for any \(z_0 \in \mathbb{R}^3 \times (-\infty, 0)\) and \(a > 0\),

\[
A(v; z_0, a) + E(v; z_0, a) + C(v; z_0, a) + D(\pi'; z_0, a) \leq c, \tag{3.5}
\]

\[
\|v_3\|_{L^\infty((-a^2,0); B^{-1+3/p}_{p,q}(\mathbb{R}^3))} \leq c, \quad \text{if } u_3 \in L^\infty((-a^2,0); B^{-1+3/p}_{p,q}(\mathbb{R}^3)); \tag{3.6}
\]

\[
\|v_3\|_{L^\infty(-a^2,0; L^{3,\infty}(\mathbb{R}^3))} \leq c, \quad \text{if } u_3 \in L^\infty((-a^2,0; L^{3,\infty}(\mathbb{R}^3)). \tag{3.7}
\]

Thanks to (3.4), we find

\[
\|v_3\|_{L^\infty(-a^2,0; L^{3,\infty}(\mathbb{R}^3))} \leq c, \tag{3.8}
\]

\[
\int_{Q_1} |v_3(x,t)|^3 \, dxdt \geq \varepsilon.
\]

**Step 2:** Prove that the limit function \(v(x,0) = 0\ in \mathbb{R}^3\)

Since \(u(x,0) \in \mathring{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)\), where \(\mathring{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)\) is the closure of \(C_c^\infty(\mathbb{R}^3)\) in \(\mathring{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)\), thus for any \(\varepsilon > 0\), there exists a function \(U^*(x,0) \in C_0^\infty(\mathbb{R}^3)\) such that \(\|u(x,0) - U^*\|_{\mathring{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)} < \varepsilon\). Then for any \(\varphi \in C_c(\mathbb{R}^3)\) with \(a > 0\), we have

\[
|\int_{B_a} v(x,0)\varphi(x) \, dx| \leq |\int_{B_a} (v(x,0) - u^k(x,0))\varphi(x) \, dx| + |\int_{B_a} u^k(x,0)\varphi(x) \, dx| = I_1 + I_2.
\]

For \(I_1\), the convergence in (3.3) shows that

\[
I_1 \leq \int_{B_a} |v(x,0) - u^k(x,0)| \, dx \to 0 \quad \text{as } k \to +\infty.
\]

For \(I_2\), we obtain

\[
I_2 = |\int_{B_a} u^k(x,0)\varphi(x) \, dx| = R_k |\int_{B_a} u(R_k x,0)\varphi(x) \, dx|
\]

\[
\leq R_k |\int_{B_a} (u(R_k x,0) - U^*(R_k x))\varphi(x) \, dx| + R_k \int_{B_a} U^*(R_k x)\varphi(x) \, dx| \leq cR_k \|u(R_k x,0) - U^*(R_k x)\|_{\mathring{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)} + R_k \int_{B_a} |U^*(R_k x)\varphi(x)| \, dx
\]

\[
\leq c\|U(x) - U^*(x)\|_{\mathring{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)} + R_k \int_{B_a} |U^*(R_k x)\varphi(x)| \, dx \leq c\varepsilon,
\]

as \(k \to +\infty\), where we used the fact that \(U^*\) is continuous at 0.

Thus we conclude that \(v(x,0) = 0\).

**Step 3: Spatial Decay**

1. **Case** \(u_3 \in L^\infty(0,T; L^{3,\infty}(\mathbb{R}^3))\)

Using Lemma 2.10, we decompose \(v_3(\cdot, t) \in L^{3,\infty}(\mathbb{R}^3)\) as

\[
v_3(\cdot, t) = v_3(\cdot, t) + \tilde{v}_3(\cdot, t),
\]

where \(v_3(\cdot, t) \in L^2(\mathbb{R}^3)\), \(\tilde{v}_3(\cdot, t) \in L^4(\mathbb{R}^3)\), and

\[
\|v_3(\cdot, t)\|_{L^2} \leq c\|v_3(\cdot, t)\|_{L^{3,\infty}(\mathbb{R}^3)}, \quad \|\tilde{v}_3(\cdot, t)\|_{L^4} \leq c\|v_3(\cdot, t)\|_{L^{3,\infty}(\mathbb{R}^3)}.
\]
Then we have $\bar{v}_3(x,t) \in L^\infty(-a^2,0; L^2(\mathbb{R}^3))$ and $\bar{v}_3(x,t) \in L^\infty(-a^2,0; L^4(\mathbb{R}^3))$.  

Then it follows that  

\begin{equation}
\int_{Q_1(z_0)} |v_3|^2 \, dx \, dt \leq \int_{Q_1(z_0)} |v_3|^2 \, dx \, dt + \int_{Q_1(z_0)} |\bar{v}_3|^2 \, dx \, dt
\quad \to 0 \quad \text{as} \quad |z_0| \to \infty.
\end{equation}

(2) **Case** $u_3 \in L^\infty(0,T; \dot{B}_p^{-1+3/p}(\mathbb{R}^3))$, $3 < p < \infty$

Let $v_3^N = \sum_{i=-N}^{N} \Delta_i v_3$. We first show a functional property of Besov space: for any $\beta \in [1, \infty)$ and $T > 0$, it holds that

\begin{equation}
\lim_{N \to +\infty} \|v_3(t) - v_3^N(t)\|_{\dot{B}_p^{-1+3/p}(\mathbb{R}^3)} = 0.
\end{equation}

In fact, from the definition of Besov space, we have for any $t \in (-T,0]$,

$\lim_{N \to +\infty} \|v_3(t) - v_3^N(t)\|_{\dot{B}_p^{-1+3/p}(\mathbb{R}^3)} = 0.
$

Since

$\|v_3^N(t)\|_{\dot{B}_p^{-1+3/p}(\mathbb{R}^3)} \leq c \|v_3(t)\|_{\dot{B}_p^{-1+3/p}(\mathbb{R}^3)},$

then (3.10) is a consequence of Lebesgue’s dominated convergence theorem.

By Hölder’s inequality and Lemma 2.18 noting that $\|v_3\|_{\dot{B}_p^{-1+3/p}(\mathbb{R}^3)} \leq c \|v_3\|_{\dot{B}_p^{-1+3/p}(\mathbb{R}^3)},$ we obtain

\begin{equation}
\int_{Q_1(z_0)} |v_3|^2 \, dx \, dt \leq \int_{Q_1(z_0)} |v_3 - v_3^N|^2 \, dx \, dt + \int_{Q_1(z_0)} |v_3^N|^2 \, dx \, dt
\quad \leq c \|v_3 - v_3^N\|_{L^2(-T;\dot{B}_p^{-1+3/p}(\mathbb{R}^3))} \|v_3 - v_3^N\|_{L^p(Q_1(z_0))} + c \|v_3^N\|_{L^p(Q_1(z_0))}
\quad \leq c \|v_3 - v_3^N\|_{L^2(-T;\dot{B}_p^{-1+3/p}(\mathbb{R}^3))} \left(\|v_3\|_{L^2(H^1_0(Q_2(z_0)))} + c \|v_3^N\|_{L^2(H^1_0(Q_2(z_0)))}\right)
\quad + c \|v_3^N\|_{L^p(Q_1(z_0))}.
\end{equation}

In (3.11), by (3.10) and the fact that $\|v_3^N\|_{W^{1,p}(\mathbb{R}^3)} \leq c(N)\|v_3\|_{\dot{B}_p^{-1+3/p}(\mathbb{R}^3)},$ sending first $z_0$ to $\infty$ and then $N$ to $\infty,$ we arrive at

\begin{equation}
\int_{Q_1(z_0)} |v_3| \, dx \, dt \to 0 \quad \text{as} \quad |z_0| \to \infty.
\end{equation}

**Step 4: Backward uniqueness and unique continuation**

Spatial decay results in Step 3 and small regularity criterion in Lemma 2.13 ensure that there exists a constant $R > 0$ such that

\begin{equation}
|v(x,t)| + |\nabla v(x,t)| \leq c,
\end{equation}

for $(x,t) \in (\mathbb{R}^3 \setminus B_R) \times (-T,0)$.

Let $\omega = \nabla \times v.$ Since $v(x,0) = 0$, we get $w(x,0) = 0$. Moreover, as $\omega$ satisfies

$\partial_t \omega - \Delta \omega = -v \cdot \nabla \omega + \omega \cdot \nabla v,$

we find that by (3.13)

$|\partial_t \omega - \Delta \omega| \leq c(|\omega| + |\nabla \omega|) \quad \text{in} \quad (\mathbb{R}^3 \setminus B_R) \times (-T,0).$

Applying backward uniqueness of parabolic operator [7] yields that

$\omega(x,t) = 0, \quad (x,t) \in (\mathbb{R}^3 \setminus B_R) \times (-T,0).$

By unique continuation argument as in [7], we see that

\begin{equation}
\omega(x,t) = 0 \quad \text{in} \quad \mathbb{R}^3 \times (-T,0).
\end{equation}
Combining the incompressible condition $\text{div} v = 0$ and (3.13), we find that $\Delta v(\cdot, t) = 0$ in $\mathbb{R}^3$. Then we get from the energy bound (3.5) that $v \in L^\infty(-T, 0; L^\infty(\mathbb{R}^3))$. The Liouville Theorem for harmonic functions implies that $v(\cdot, t)$ is constant. Due to the spatial decay bound in Step 3, it follows that $v_3(\cdot, t) = 0$ for any $t \in (-T, 0)$, which contradicts with (3.8). This completes the proof of Theorem 1.1. □

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