An analytic solution for an uniaxial spherical resonator is presented using the method of Debye potentials. This serves as a starting point for the calculation of whispering gallery modes (WGM) in such a resonator. Suitable approximations for the radial functions are discussed in order to best characterize WGMs. The characteristic equation and its asymptotic expansion for the anisotropic case is also discussed, and an analytic formula with a precision of the order $O(\nu^{-1})$ is also given. Our careful treatment of both boundary conditions and asymptotic expansions makes the present work a particularly suitable platform for a quantum theory of whispering gallery resonators.

PACS numbers: 42.25.-p, 42.60.Da, 42.25.Lc

I. INTRODUCTION

London’s St. Paul’s Cathedral is famous for its rich history and architecture; one of the most unique aspects of this building is the whispering gallery that runs along the interior wall of its dome [1]. When sounds are uttered in low voice against the wall, sound waves generated circulate around the wall many times before fading away. As these waves propagate, they bring with them sounds that are audible on the opposite side of the dome. On the contrary, if the same sounds are uttered at higher volume, the frequencies of these sounds waves will not match and a lot of noise is created, making the message difficult to be heard at any part of the wall.

The physical explanation of this effect was firstly given more than a century ago in terms of reflection of acoustic rays from a surface near the dome apex. It was initially assumed that the rays that propagate along different large arcs of the dome in a form of a hemisphere should concentrate only at the point diametrically opposite to the source of the sound. Afterwards lord Rayleigh, in his Theory of sound [2], provided a different explanation of the effect that he named Whispering Gallery Waves: sound clutches to the wall surface and creeps along it without diverging as fast as during the free space propagation: these sound waves then propagate within a narrow layer adjacent to the wall surface. It was then discovered, at the beginning of the last century, that optical whispering gallery waves can exist even in dielectric spheres [3, 4]. An optical resonator that shows this particular wave structure was then called Whispering Gallery Resonator (WGR). In recent times whispering gallery waves have found new fame with the development of nano-optics, in particular with the ability to manufacture spherical and toroidal WGR with very high quality factors that ranges from $10^7$ to $10^{10}$ [6, 8]. This motivated a large theoretical and experimental work around these devices (see for example Ref. [5, 9, 11] and references therein). The ability to store light in microscopic spatial volumes for long periods of time (due to the high $Q$-factor) resulted in a significant enhancement of nonlinear interactions of various kinds like four-wave mixing [12, 13], Raman [14], parametric and Brillouin scattering [11, 15], microwave up-conversion [16], second and third order harmonic generation [17, 19]. Besides the field of nonlinear optics, WGRs were recently used even for cavity QED experiments [20, 21]. For an exhaustive review on the applications of WGRs see Ref. [22].

Since many of the applications of these resonators involve nonlinear optics, WGR are commonly fabricated using nonlinear materials or anisotropic crystals [11]. Despite the wide scientific production in the theory of anisotropic spherical resonators, that ranges from generalization of scattering methods [23, 25–28], potential method [24], dyadic Green function approach [25] and Fourier-based analysis [29], and even though an extensive study of isotropic WGRs was done in the past [33], detailed studies on anisotropic WGRs are still very few. To the knowledge of the authors anisotropic WGRs are mainly reported in literature as studied with FDTD models [30, 31], cavity loading [32] and direct solution of Maxwell’s equation with a surface nonlinear polarization as a forcing term [10].

In this work, we intend to develop a suitable analytic theory for WGRs, starting from a review of the solutions of Maxwell’s equations in an uniaxial spherical resonator, then presenting and discussing its mode structure in the limit of small anisotropy, and finally obtaining the spectrum of whispering gallery modes sustained by the resonator and their structure, discussing how anisotropy influence those modes. A detailed discussion on the application of boundary conditions to this resonator is also presented, pointing out how to apply correctly these conditions and discussing some of their basic features that, to the knowledge of the authors, is not present in earlier works. It is opinion of the authors that this discussion is important in order to better understand the physics behind this problem. This is the first main result of this work. Finally, we introduce a more accurate approximation for the field outside the resonator when the index of the Hankel function tends to infinity, as we noticed that the commonly used power expansion (as, for example,
the one presented in Ref. [33]) does not match the exact function completely, i.e. it has an additional phase factor respect to the real function. Such phase factor becomes relevant when field amplitude, as opposed to field intensity, turns to be fundamental. This happens, for example, when one wants to quantize the electromagnetic field inside the resonator, as required for a proper treatment of spontaneous emissions processes. Thus, the present work may serve as basis for a quantum theory of WGRs. This is the second main result of this work.

This paper is organized as follows: in section II the Debye method of potentials for solving Maxwell’s equations is briefly presented and then used in section III to develop the theory of an anisotropic spherical resonator for a dielectric uniaxial sphere. In section IV, Whispering Gallery Modes (WGMs) are obtained as limiting case of the normal modes of the dielectric sphere with high quantum numbers and their spectrum is discussed.

II. DEBYE METHOD OF POTENTIALS

A. Isotropic Solution

Before considering the problem of an uniaxial spherical resonator, it is pedagogical to briefly review the method of Debye potentials [33], largely used to solve Maxwell’s equations in integrable systems. Let us consider a monochromatic field with an harmonic time dependence (i.e., $\vec{E}(x,t) = \vec{E}(x) e^{-i\omega t}$) in an isotropic sourceless domain $\Omega$; Maxwell’s equations inside $\Omega$ can be written in the following compact form:

$$\nabla \times \vec{E} = -ik \vec{H}, \quad (1a)$$
$$\nabla \times \vec{H} = ik \vec{E}, \quad (1b)$$

where $k = \omega \sqrt{\epsilon}/c$ is the wavenumber in vacuum and $\epsilon$ is the dielectric constant inside $\Omega$. In order to fully determine the fields, it is necessary to specify their values on the domain boundary $\partial \Omega$. Electromagnetic boundaries are usually of two types: perfectly conducting walls (the field is zero on $\partial \Omega$), or open systems, where the field components inside $\Omega$ and the ones outside $\Omega$ must match on $\partial \Omega$. The boundary, together with symmetry considerations, gives a hint on which is the more suitable coordinate system to be used to solve the problem (e.g., spherical coordinates for spheres, cylindrical coordinates for wires etc.) [34].

Let us specify our problem by considering an open system constituted by a sphere of dielectric constant $\epsilon$ and radius $R$ surrounded by an isotropic medium (i.e. air) [34]. The set of Maxwell’s equations (1) in the spherical reference frame can be written in the following compact form:

$$\frac{\partial}{\partial \zeta}(L_m E_m) - \frac{\partial}{\partial \zeta}(L_n E_n) = -i k \epsilon \imath_{nm} L_n L_m H_l, \quad (2a)$$
$$\frac{\partial}{\partial \zeta}(L_m H_m) - \frac{\partial}{\partial \zeta}(L_n H_n) = i k \epsilon \imath_{nm} L_n L_m E_l, \quad (2b)$$

where $\{l, n, m\} \in \{1, 2, 3\}$, $\zeta_n$ are the spherical coordinates ($\zeta_1 = \varphi$, $\zeta_2 = \theta$ and $\zeta_3 = r$), and $L_m$ are the metric coefficients of the spherical reference frame ($L_1 = r \sin \theta$, $L_2 = r$, $L_3 = 1$). The Levi-Civita symbol $\epsilon_{nm}$ on the right-hand side of Eqs. (2) is equal to 1 if $\{l, n, m\}$ is equal to $\{1, 2, 3\}$ or any of its even permutation, is equal to $-1$ for any odd permutation of $\{1, 2, 3\}$ and equal to zero elsewhere.

In this reference frame, the fields can be decomposed in the so-called transverse electric (TE) and transverse magnetic (TM) waves: TE waves are characterized by having $E_r = 0$, i.e. the electric field is transversal with respect to the radial direction $r$. TM waves are instead characterized by having the magnetic field transversal with respect to the radial direction (i.e. $H_r = 0$) [34]. For the sake of simplicity, let us fix our attention on TM waves; the calculations for TE waves can be straightforward obtained by analogy. From Eqs. (2a) for $l = 3$, by substituting $H_\varphi = 0$ it is possible to introduce the $W$ function such that

$$r \sin \theta E_\varphi = \frac{\partial W}{\partial \varphi}, \quad (3a)$$
$$r E_\theta = \frac{\partial W}{\partial \theta}. \quad (3b)$$

By substituting relations (3) into Eqs. (2b) for $l = 1, 2$ and writing $W = \partial U/\partial r$, where $U$ is the TM Debye potential, from (2b) we obtain

$$H_\varphi = i k \frac{1}{r} \frac{\partial U}{\partial \varphi}, \quad (4a)$$
$$H_\theta = -i k \frac{1}{r \sin \theta} \frac{\partial U}{\partial \varphi}, \quad (4b)$$

and according to Eq (2a), the $r$-component ($l = 3$) of the electric field is given by

$$E_r = -\frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial \varphi} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \right] U. \quad (5)$$

Note that the differential operator that acts on the potential $U$ in this equation is the angular momentum operator $\hat{L} = \hat{r} \times \nabla$, that is the same operator that originates the centrifugal potential in the Hydrogen atom [49].

Therefore, all the components of the electric field are expressed in terms of the $U$ potential solely. In order to explicit them, it is necessary to find the equation which the $U$ potential satisfy. To do this, we can use one of the
last two equations left available from Eq. (2a), i.e. the ones with \( l = 1, 2 \). By using one of them it is possible to obtain the following wave equation that \( U \) must satisfy

\[
\frac{\partial^2 U}{\partial t^2} + \nabla_\perp^2 U + k^2 U = 0, \tag{6}
\]

where \( \nabla_\perp^2 \) is the angular part of the Laplace operator in spherical coordinates, i.e. the angular momentum operator \( \hat{L} \).

The solution can be easily found with the method of separations of variables; writing the potential as \( U(r, \theta, \varphi) = \Psi(r)\Theta(\theta)\Phi(\varphi) \) and substituting this into Eq. (6), we obtain the following equations for the functions \( \Psi(r) \), \( \Theta(\theta) \) and \( \Phi(\varphi) \) [34]:

\[
\begin{align*}
\frac{d^2 \Psi}{dr^2} + \left( k^2 - \frac{c_3}{r^2} \right) \Psi &= 0, \tag{7a} \\
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \left( c_1 - \frac{c_2}{\sin^2 \theta} \right) \Theta &= 0, \tag{7b} \\
\frac{d^2 \Phi}{d\varphi^2} + c_2 \Phi &= 0, \tag{7c}
\end{align*}
\]

where \( c_1, c_2 \) and \( c_3 \) are the separation constants appearing in the equations by separating the variables, whose values must be \( c_1 = n(n+1) \) and \( c_2 = n^2 \) in order to the solution to these equations to be unique, i.e. physically meaningful; \( n \) and \( m \) are integers, including zero. With these values the solution is straightforward. Equations (7b) and (7c) give rise to the so-called spherical harmonics \( Y_{nm}(\theta, \varphi) = \mathcal{N} P^m_n(\cos \theta) e^{im\varphi} \), i.e. the eigensolutions of the angular momentum operator [34], where \( P^m_n(\cos \theta) \) are the associated Legendre functions of the first kind that are solution of Eq. (7b), while the complex exponential is a solution of Eq. (7c). \( \mathcal{N} \) is a normalization constant that guarantees that the integral over the solid angle is unitary. The radial equation (7a) can be transformed into a Bessel equation by the substitution \( \Psi(r) = \sqrt{kr} Z(kr) \) that brings to:

\[
\frac{d^2 Z}{dx^2} + \frac{1}{x} \frac{dZ}{dx} + \left( 1 - \frac{\nu^2}{x^2} \right) Z = 0, \tag{8}
\]

where \( x = kr \) and \( \nu = n + 1/2 \); the solutions to this equation are the four Bessel functions \( J_{\nu}(x) \), \( N_{\nu}(x) \), \( H^{(1)}_{\nu}(x) = J_{\nu}(x) + i N_{\nu}(x) \) and \( H^{(2)}_{\nu}(x) = J_{\nu}(x) - i N_{\nu}(x) \). Physically, the solution inside the sphere must be finite at the origin, and the only plausible solution is \( J_{\nu}(x) \) because \( N_{\nu}(x) \) has a divergence at the origin. Outside the sphere, instead, the solution should have the form of a runaway wave with the Sommerfeld condition at the infinity (i.e. , the solution must drop at infinity as the inverse square of the distance). For this reason the correct solution in this domain is the Hankel function of the first kind \( H^{(1)}_{\nu}(x) \) because its asymptotic form decreases to zero as the inverse square of the distance for \( x \to \infty \). Putting everything together, the TM Debye potential reads as follows:

\[
U_{nm}^{\text{int/ext}}(r, \theta, \varphi) = C^{\text{int/ext}} \sqrt{kr} Z_{\nu}(kr) Y_{nm}(\theta, \varphi), \tag{9}
\]

where \( n, m \) are the angular quantum numbers that address the single mode of the resonator, \( Z_{\nu}(kr) \) is the radial Bessel-type function that is equal to the Bessel function \( J_{\nu}(kr) \) inside the dielectric sphere, and is equal to the Hankel function of the first kind \( H^{(1)}_{\nu}(k_0 r) \) outside the dielectric sphere. The constants \( C^{\text{int/ext}} \) are to be determined by applying suitable boundary conditions. Note that the argument of the Bessel function inside the sphere contains the sphere dielectric constant \( \varepsilon \) via the wavevector \( k = \omega \sqrt{\varepsilon} / c \) while the argument of the Hankel function outside the sphere contains only the vacuum wavevector \( k_0 = \omega / c \) because \( \varepsilon_{\text{air}} = 1 \).

The components of the electric and magnetic fields for TM waves can then be written as a function of \( U \) as follows:

\[
\begin{align*}
E_r &= \left( \frac{\partial^2}{\partial t^2} + k^2 \right) U, \tag{10a} \\
E_\theta &= \frac{1}{r} \frac{\partial^2 U}{\partial \theta \partial t}, \tag{10b} \\
E_\varphi &= \frac{1}{r \sin \theta} \frac{\partial^2 U}{\partial r \partial \varphi}, \tag{10c} \\
H_r &= 0, \tag{10d} \\
H_\theta &= -i k \frac{1}{r} \frac{\partial U}{\partial \varphi}, \tag{10e} \\
H_\varphi &= i k \frac{1}{r} \frac{\partial U}{\partial \theta}. \tag{10f}
\end{align*}
\]

Note that in obtaining the expression of \( E_r \) we have combined Eqs. (5) and (6).

If we proceed in a similar manner for TE waves, we obtain:

\[
\begin{align*}
E_r &= 0, \tag{11a} \\
E_\theta &= -i k \frac{1}{r} \frac{\partial V}{\partial \varphi}, \tag{11b} \\
E_\varphi &= i k \frac{1}{r} \frac{\partial V}{\partial \theta}, \tag{11c} \\
H_r &= \left( \frac{\partial^2}{\partial r^2} + k^2 \right) V, \tag{11d} \\
H_\theta &= \frac{1}{r} \frac{\partial V}{\partial r}, \tag{11e} \\
H_\varphi &= \frac{1}{r \sin \theta} \frac{\partial V}{\partial \varphi}, \tag{11f}
\end{align*}
\]

where \( V \) is the TE field potential obtained by Eqs. (2) by substituting the TE ansatz \( E_r = 0 \).
B. Boundary Conditions

Prior to investigate the structure of the modes for the anisotropic resonator, it is important to discuss the boundary conditions that have to be applied to this problem. At the resonator surface \( r = R \), the wave vector \( k \) inside the dielectric sphere has to match the wave vector \( k_0 = \omega / c \) outside the sphere and the constants \( C_{\text{int}} \) and \( C_{\text{ext}} \) should be chosen properly. There is not a unique way to fulfill boundary conditions: in fact, one could apply "pure" or "mixed" conditions: the former consists in applying the boundary conditions to all the components of only electric or magnetic field, while the latter applies the boundary to certain components of one field and certain other components of the other field. Obviously, these two different paths bring to the same physical solutions [34]. Among these possibilities, in this work we chose to apply "pure" boundary condition, i.e. we impose that the tangential electric (magnetic) field components for TM (TE) waves has to be continuous at the resonator surface \( r = R \), while the radial component of the displacement vector \( \vec{D} = \varepsilon \vec{E} \) is continuous across the resonator surface. For TE fields, the radial condition is automatically fulfilled, since the resonator is non-magnetic (i.e. \( \mu = 1 \)).

The condition for the radial component of the displacement vector \( (\varepsilon_{\text{int}} E_r^{\text{int}} = \varepsilon_{\text{ext}} E_r^{\text{ext}}) \) across the resonator surface gives the ratio between the inner and outer coefficients

\[
\frac{C_{\text{ext}}}{C_{\text{int}}} = \varepsilon^{1/4} \frac{j_\nu(k_0 \sqrt{\varepsilon} R)}{h_\nu^{(1)}(k_0 R)}, \tag{12}
\]

while the continuity of the tangential component \( E_{\phi,\varphi}^{\text{int}} = E_{\phi,\varphi}^{\text{ext}} \) of the field gives rise to the so called characteristic equation, that allows to determine the allowed values for the wave vector \( k \) (i.e. to find the spectrum of the allowed modes) inside the resonator, and it turns out to be

\[
\frac{[j_\nu(kR)]'}{j_\nu(kR)} = \sqrt{\varepsilon} \frac{[h_\nu^{(1)}(k_0 R)]'}{h_\nu^{(1)}(k_0 R)}, \tag{13}
\]

for TM waves and

\[
\frac{[j_\nu(kR)]'}{j_\nu(kR)} = \frac{1}{\sqrt{\varepsilon}} \frac{[h_\nu^{(1)}(k_0 R)]'}{h_\nu^{(1)}(k_0 R)}, \tag{14}
\]

for TE waves. In these equations \( j_\nu(x) = \sqrt{x} J_\nu(x) \) and \( h_\nu^{(1)}(x) = \sqrt{x} H_\nu^{(1)}(x) \) are the Riccati-Bessel functions, and the prime indicates the total derivative with respect to the argument on which the functions depend, i.e. over \( kR \) or \( k_0 R \).

Although formally corrected, as they are these boundary conditions do not provide a unique solution to the determination of the mode patterns in the resonator. In order to better understand this not-uniqueness of the solution, let us consider the general structure of eqs. [13] and [14]. Let \( T(x) \) be a piecewise function defined across an interface, placed at \( x = 1 \), between two region of space, such that \( T(x) = C^{(i)} f(x) \) for \( x < 1 \) and \( T(x) = C^{(e)} g(x) \) for \( x > 1 \), with \( f(x) \) and \( g(x) \) two arbitrary real valued and regular functions. The constants \( C^{(i,e)} \) are to be determined by the boundary conditions and they must be chosen in such a way that the following characteristic equation is satisfied:

\[
f'(x) f(x) = \alpha' g'(x) g(x), \tag{15}
\]

where the apex stands for the derivative of the two functions with respect to their arguments. Since this is a generalization of the characteristic equations [13] and [14], this equation must hold at the interface between the two region of space considered, i.e. its validity is limited to \( x = 1 \). From eq. [15] it is clear that if we admit that the derivatives \( f'(x) \) and \( g'(x) \) of the functions are equal at the separation interface \( x = 1 \), then the functions themselves will be discontinuous with a jump that has the value of \( 1/\alpha \). On the other hand if we now admit that the the functions \( f(x) \) and \( g(x) \) are equal at the separation interface \( x = 1 \), then their derivatives must be discontinuous, and the magnitude of the discontinuity is precisely \( \alpha \).

The first situation corresponds to require that the derivative of the function \( T(x) \) is continuous at the separation interface (i.e. \( T'(1)^+ = T'(1)^- \)), where the plus or minus superscript stands for the expression of \( T(x) \) for \( x > 1 \) and \( x < 1 \) respectively). This implies that \( C^{(e)} / C^{(i)} = f'(1)/g'(1) \) and the function \( T(x) \) can be written as:

\[
T(x) = \begin{cases} f(x) & x < 1, \\ \left[ \frac{f'(1)}{g'(1)} \right] g(x) & x \geq 1. \end{cases}
\]

It is then clear that taking \( T'(x) \) to be continuous at the interface results in a discontinuity in the behavior of \( T(x) \) while passing through \( x = 1 \), whose magnitude is \( f'(1)/g'(1) \), as is depicted in Fig[1].

Conversely, the second condition on the functions \( f(x) \) and \( g(x) \) implies that the function \( T(x) \) to be continuous at the separation interface (i.e. \( T(1)^+ = T(1)^- \)), we have \( C^{(e)} / C^{(i)} = f(1)/g(1) \) and the function \( T(x) \) has the following form:
FIG. 1. (color online) The figure shows the behavior of the function \( T(x) \) and its derivative \( T'(x) \) when the condition of continuous derivative at the separation interface \( x = 1 \) is considered. As can be noted, in this case the function shows a discontinuity while its derivative is (obviously) continuous. For this example we have used \( f(x) = \text{Ai}(x) \) and \( g(x) = e^{-x} \), and the magnitude of the discontinuity in the function \( T(x) \) at the separation interface is \( f'(1)/g'(1) \approx 0.5457 \).

\[
T(x) = \begin{cases} 
  f(x) & x < 1, \\
  \left[ \frac{f(1)}{g(1)} \right] g(x) & x \geq 1.
\end{cases}
\]

In this case, taking \( T(x) \) to be continuous at the interface results in a discontinuity in its derivative, whose magnitude is \( f(1)/g(1) \), as Fig. 2 underlines.

In both cases, however, it is not possible to make the functions and their derivatives both continuous at the same time. This fact makes only possible to obtain the ratio between the two constants \( C_{\text{int}}, C_{\text{ext}} \) and not their explicit values: in order to do that, another condition must be applied to the problem. This condition depends on the particular problem we are dealing on; in scattering problems, for example, the incoming field is known, and determines the field pattern on the resonator surface. In this case \( C_{\text{ext}} \) is known and the ambiguity is removed. Another situation in which the ambiguity is overcome is by embedding the whole system (resonator plus surrounding medium) in an ideal perfectly reflective sphere of big, but finite radius \( R_0 \), in such a way that the boundary conditions at the metallic surface will completely determine the fields: this second approach is very useful if we are dealing with the quantization of the field in such a system.

We want to end this discussion by pointing out that the first situation (derivative continuous at the interface) corresponds to the boundary condition for the electric field across a dielectric surface: the normal component with respect to the separation surface is discontinuous by a factor equal to the ratio of the two dielectric constants of the two regions, while the tangential components (i.e., the derivative of the radial field in our spherical case) is continuous at the interface. The second situation, instead, corresponds to put continuous the normal component of the displacement vector across the separation surface, resulting in a discontinuity of the tangential component of the displacement vector at the interface. While the former correspond to the usual way of imposing boundary conditions in an electromagnetic problem, the latter is never used, but still valid.

In this work, however, we are neither interested on scattering problems nor on field quantization, and so in the rest of the paper this ambiguity will not be removed. This does not create too much problems because we are only interested on the mode structure of the resonator. We leave this problem of not-uniqueness to future works.

III. NORMAL MODES OF AN UNIAXIAL SPHERICAL RESONATOR

Let us consider the same dielectric spherical resonator of radius \( R \) of the previous section, but with an uniaxial anisotropy along the \( z \)-axis described by the following dielectric tensor:
\[
\hat{\varepsilon} = \begin{pmatrix}
\varepsilon_{xx} & 0 & 0 \\
0 & \varepsilon_{xx} & 0 \\
0 & 0 & \varepsilon_{zz}
\end{pmatrix} = \varepsilon_{xx}(\hat{\mathbf{x}} \hat{\mathbf{x}} + \hat{\mathbf{y}} \hat{\mathbf{y}}) + \varepsilon_{zz}\hat{\mathbf{z}} \hat{\mathbf{z}}.
\] (16)

In order to use this dielectric tensor in Eqs. (2), it should be converted in spherical coordinates; this operation is simply done by converting the cartesian dyadics \(\hat{\mathbf{x}} \hat{\mathbf{x}}, \hat{\mathbf{y}} \hat{\mathbf{y}}\) and \(\hat{\mathbf{z}} \hat{\mathbf{z}}\) into the spherical dyadics \(\hat{\mathbf{r}} \hat{\mathbf{r}}, \hat{\mathbf{\theta}} \hat{\mathbf{\theta}}\) and \(\hat{\varphi} \hat{\varphi}\) using the standard cartesian-to-spherical transformation relations [34]. By performing this transformation, the dielectric tensor in spherical coordinates reads

\[
\hat{\varepsilon} = \begin{pmatrix}
\varepsilon_{rr} & -\varepsilon_{r\theta} & 0 \\
-\varepsilon_{r\theta} & \varepsilon_{\theta\theta} & 0 \\
0 & 0 & \varepsilon_{\perp}
\end{pmatrix},
\] (17)

with \(\varepsilon_{\perp} = (\varepsilon_{zz} + \varepsilon_{xx})/2\) and \(\varepsilon_{\perp} = \varepsilon_{xx}\). We have then defined \(\varepsilon_{rr} = \varepsilon_+ + \varepsilon_- \cos(2\theta), \varepsilon_{\theta\theta} = \varepsilon_+ - \varepsilon_- \cos(2\theta)\) and \(\varepsilon_{r\theta} = \varepsilon_- \sin(2\theta)\). The fact that the tensor components depend on the polar coordinate \(\theta\) makes the problem to find the eigenmodes of the spherical resonator much more difficult. Moreover, in an anisotropic system it is in general no longer possible to divide the electric and magnetic fields in their TM and TE components. In order to overcome the latter problem, we will focus our attention on the case of small anisotropy, i.e. \(\lambda = \varepsilon_- / \varepsilon_+ \ll 1\) (this approximation is very good if we consider, for example, a dielectric sphere made of Lithium Niobate (LiNbO3) for which we have \(\varepsilon_{xx} = 5.3, \varepsilon_{zz} = 6.47\) and therefore \(\lambda \simeq 0.01\)). In such a way the fields can be decomposed in quasi-TE and quasi-TM oscillations, allowing us to solve the problem using the method of Debye potentials illustrated above. To this aim, and for the sake of clearness, let us rewrite the set of Eqs. (2) for the anisotropic case as follows:

\[
\frac{\partial}{\partial r}(r \sin \theta E_{\varphi}) - \frac{\partial}{\partial \varphi}(E_r) = ik_0 \sin \theta (r H_\varphi), \tag{18a}
\]

\[
\frac{\partial}{\partial \theta}(E_r) - \frac{\partial}{\partial r}(r E_\theta) = ik_0 (r H_\varphi), \tag{18b}
\]

\[
\frac{\partial}{\partial \varphi}(r E_\theta) - \frac{\partial}{\partial \theta}(r \sin \theta E_{\varphi}) = ik_0 r \sin \theta (r H_r), \tag{18c}
\]

and

\[
\frac{\partial}{\partial r}(r \sin \theta H_\varphi) - \frac{\partial}{\partial \varphi}(H_r) =
\]

\[
= ik_0 \sin \theta [\varepsilon_{r\theta} (r E_r) - \varepsilon_{\theta\theta} (r E_\theta)], \tag{19a}
\]

\[
\frac{\partial}{\partial \theta}(H_r) - \frac{\partial}{\partial r}(r H_\theta) = -ik_0 \varepsilon_{\perp} (r E_{\varphi}), \tag{19b}
\]

\[
\frac{\partial}{\partial \varphi} (r H_\varphi) - \frac{\partial}{\partial \theta} (r H_\theta) = \frac{1}{\sin \theta} [i k_0 \sin \theta \varepsilon_{r\theta} (r E_r) + \varepsilon_{\theta\theta} (r E_\theta)], \tag{19c}
\]

As in the previous section, we solve the problem for the quasi-TM component of the field (i.e. \(H_r = 0\)); the quasi-TE solution is again obtained using similar arguments. We follow the solving procedure described in Ref.[37]. Equation (18c) defines the \(W\) function as in (3). Let us combine (19a) and the derivative with respect to \(r\) of (18a):

\[
\frac{\partial}{\partial r}(r H_\theta) = \frac{1}{\sin \theta} \varepsilon_{r\theta} \frac{\partial W}{\partial \varphi}, \tag{19a}
\]

\[
\frac{\partial^2}{\partial r^2} (W) - \frac{\partial^2}{\partial \varphi^2} (E_r) = ik_0 \sin \theta \frac{\partial}{\partial r} (r H_\theta). \tag{19b}
\]

If we substitute the expression of \(\frac{\partial}{\partial r} (r H_\theta)\) obtained from the first equation into the second one and if we define the differential operator \(\hat{t}_x = \partial^2 / \partial r^2 + k_0^2 \varepsilon_{\perp}\) we obtain:

\[
\frac{\partial E_r}{\partial r} = \hat{t}_x W. \tag{20}
\]

Combining now (19a) and the derivative with respect to \(r\) of (18b) gives:

\[
\frac{\partial}{\partial r} (r \sin \theta H_\varphi) = ik_0 r \sin \theta \left[ \varepsilon_{r\theta} E_r - \frac{1}{r} \varepsilon_{\theta\theta} \frac{\partial W}{\partial \theta} \right], \tag{20a}
\]

\[
\frac{\partial^2}{\partial r^2} (E_r) - \frac{\partial^2}{\partial \varphi^2} (r E_\theta) = ik_0 \frac{\partial}{\partial r} (r H_\varphi). \tag{20b}
\]

Again, by substituting the expression for \(\frac{\partial}{\partial r} (r H_\varphi)\) obtained from the first equation into the second one, and by defining the differential operator \(\hat{t}_\theta = \partial^2 / \partial r^2 + k_0^2 \varepsilon_{\theta\theta}\) we obtain

\[
\left[ k_0^2 \varepsilon_{\perp} r + \frac{\partial^2}{\partial r \partial \theta} \right] E_r = \hat{t}_\theta W. \tag{21}
\]

As in the isotropic case, we want to define \(W\) as a function of the quasi-TM potential \(U\), in order to fully determine the components of the fields as a function of the quasi-TM potential \(U\) solely. In order to do this, let us compare Eqs. (20) and (21). By noting that the differential operator \(\hat{t}_\theta\) commutes with the operator \((k_0^2 \varepsilon_{\theta\theta} + \partial^2 / \partial r \partial \theta)\) that appears in (21), it is possible to define, after some simple algebra, the function \(W\) as a function of the quasi-TM potential \(U\) as follows:

\[
W = \frac{\partial}{\partial r} \left( \hat{t}_\theta U \right). \tag{22}
\]
This allows us to write the components of the TM electric and magnetic field in terms of the quasi-TM potential $U$ as follows \[36\]:

\[
\begin{align*}
E_{r}^{TM} & = \hat{l}_x \hat{i}_0 U, \\
E_{\theta}^{TM} & = \frac{1}{\sin \theta} \frac{\partial^2}{\partial \theta \partial \varphi} (\hat{i}_0 U), \\
E_{\varphi}^{TM} & = (\varepsilon_r \epsilon_k \hat{k}_r^2 + \frac{\partial^2}{\partial \theta \partial \varphi}) \hat{l}_x U, \\
H_{r}^{TM} & = 0, \\
H_{\theta}^{TM} & = \frac{i k_0 \varepsilon_\perp}{\sin \theta} \frac{\partial}{\partial \varphi} (\hat{i}_0 U), \\
H_{\varphi}^{TM} & = \frac{i k_0}{\sin \theta} \left[ \varepsilon_{\theta \theta} \frac{\partial}{\partial \theta} + \varepsilon_{r \theta} \left( 1 - r \frac{\partial}{\partial r} \right) \right] \hat{l}_x U.
\end{align*}
\]

Again, if we proceed in a similar manner for the quasi-TE waves we obtain:

\[
\begin{align*}
H_{r}^{TE} & = \hat{l}_x \hat{i}_0 V, \\
H_{\theta}^{TE} & = \frac{\partial^2}{\partial \varphi^2} (\hat{i}_0 V), \\
H_{\varphi}^{TE} & = \frac{1}{\sin \theta} \frac{\partial^2}{\partial \theta \partial \varphi} (\hat{i}_0 V), \\
E_{r}^{TE} & = 0, \\
E_{\theta}^{TE} & = \frac{i k_0}{\sin \theta} \frac{\partial}{\partial \varphi} (\hat{i}_0 V), \\
E_{\varphi}^{TE} & = -\frac{i k_0}{\sin \theta} \left[ \varepsilon_{\theta \theta} \frac{\partial}{\partial \theta} + \varepsilon_{r \theta} \left( 1 - r \frac{\partial}{\partial r} \right) \right] \hat{l}_x V,
\end{align*}
\]

where $V$ represents the quasi-TE potential.

Equations \[23\] and \[24\] represent the quasi-TM and quasi-TE components of electric and magnetic field inside an anisotropic spherical resonator in terms of the TM and TE quasi-potentials $U$ and $V$.

The next step consists in constructing an equation that the quasi-potentials $U$ and $V$ satisfy, whose solutions give the mode fields of the resonator. To do that, let us consider a general electric and magnetic field, whose components are written as the superposition of the quasi-TE and quasi-TM oscillations, i.e. $E_i = E_i^{TM} + E_i^{TE}$ and $H_i = H_i^{TM} + H_i^{TE}$. Substituting this ansatz in Eqs. \[19\], after some algebra we arrive at a set of two coupled equations for the quasi-potentials $U$ and $V$ that reads \[37\] where we have defined:

\[
\begin{align*}
\hat{L}_H & = (\nabla_\perp^2 + r^2 \hat{l}_x) \hat{i}_0 - 2 \varepsilon_\perp k_0 \frac{\partial^2}{\partial \varphi^2}, \\
\hat{L}_E & = \hat{\xi}_0 + \lambda^2 \hat{\xi}_x, \\
\hat{\xi}_\perp & = \left[ \nabla_\perp^2 + r^2 \frac{\partial^2}{\partial r^2} + \varepsilon_\perp (1 - \lambda^2) k_0^2 \right] \hat{l}_x + 2 \varepsilon_\perp \frac{\partial^2}{\partial \varphi^2}, \\
\hat{\xi} & = \cos(2\theta) \left[ r^2 \frac{\partial^2}{\partial r^2} - \nabla_\perp^2 + 1 - 3 (1 - r \frac{\partial}{\partial r}) \right] + \left( 3 - 2r \frac{\partial}{\partial r} \right) \sin(2\theta) \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \varphi^2},
\end{align*}
\]

and $\lambda = \varepsilon_- / \varepsilon_\perp$ is the anisotropy parameter.

From Eqs. \[25\] it is evident that the anisotropy gives rise to a coupling between the two quasi-potentials $U$ and $V$: this coupling is absent in the isotropic case in which the two potentials are independent one each other. These equations, in fact, contain the isotropic solution in the limit of $\lambda = 0$. Making this substitution in equation \[25a\] and using the definition of the operator $\hat{L}_H$, we obtain:

\[
\langle \nabla_\perp^2 + r^2 \hat{l}_x \rangle \hat{i}_0 U = 0.
\]

Because we set $\lambda = 0$, the two differential operators $\hat{l}_x$ and $\hat{i}_0$ are equal, since $\varepsilon_{\theta \theta} = \varepsilon_\perp - \varepsilon_\perp \cos 2\theta = \varepsilon_\perp = \varepsilon$ and $\varepsilon_\perp = \varepsilon$, i.e. no anisotropy is present anymore.

We now define $\hat{i}_0 U = A$ as the isotropic potential, and we assume that this potential can be written in a separable way, i.e. $A(r, \theta, \varphi) = \Psi(r) Y_{nm}(\theta, \varphi)$, where $Y_{nm}(\theta, \varphi)$ are the eigensolutions of the angular momentum operator, whose eigenvalues are $-n(n+1)$ (i.e. , $\nabla_\perp^2 Y_{nm} = -n(n+1) Y_{nm}$). Substitution of this ansatz into the previous equation and consequent simplification of the angular part then brings to the following radial equation:

\[
[n(n+1) - r^2 \hat{l}_x \Psi(r) = 0,
\]

that is precisely the radial equation \[7a\] for the isotropic potential, whose solutions are the Bessel functions given in the previous section. The same procedure applied to the quasi-potential $V$ brings to its isotropic counterpart. Since we stated that the anisotropy is small (i.e. , $\lambda \ll 1$), then we can use the method of separations of variable
to solve the coupled equations (25). We then write the quasi-potentials as follows:

\[ U(r, \theta, \varphi) = \sum_p U_p(r, \theta, \varphi) = \sum_p u_p(r)Y_{nm}(\theta, \varphi), \] (29a)

\[ V(r, \theta, \varphi) = \sum_p V_p(r, \theta, \varphi) = \sum_p v_p(r)Y_{nm}(\theta, \varphi), \] (29b)

where the subscript \( p \) stands for the three indexes \( n, m \) and \( q \) on which the quasi-potential depends; the polar index \( n \) determines the number of field nodes along the polar coordinate \( \theta \), the azimuthal number \( m \) characterizes the nodes in the \( \varphi \) direction and, finally, the radial index \( q \) gives the number of field oscillations along the radial direction \( r \) that is related with the solution of the characteristic equation. Substituting \( u_p \) into \( \sum_p u_p \) and equating the terms with equal angular part \( Y_{nm}(\theta, \varphi) \), we obtain the following set of differential equations for the radial components \( u_p(r) \) and \( v_p(r) \) of the quasi-potentials [35]:

\[ a_{1,n}^{TM} u_n - a_{2,n}^{TM} u_{n-2} - \frac{a_{1,n}^{TM}}{2} u_{n+2} = - \frac{2\varepsilon - mk_0}[b_{1,n-1}v_{n-1} - b_{2,n+1}v_{n+1}], \] (30a)

\[ a_{1,n}^{TE} v_n - \lambda(a_{2,n}^{TE} v_{n-2} + a_{3,n}^{TE} v_{n+2}) = - \frac{2amk_0}[b_{1,n-1}v_{n-1} - b_{2,n+1}v_{n+1}]. \] (30b)

For the sake of clarity, the expression of the coefficients \( a_{1,n}^{TM/TE} \) and \( b_i \) are reported in Appendix A.

These equations require, in general, a numerical approach to be solved. However, in the limit of small anisotropy, i.e. \( \lambda \ll 1 \), a solution to Eqs. (30) can be searched in terms of power series in the factor \( \lambda \). The zeroth order solution brings (as shown before) to the solution of the isotropic spherical resonator in terms of the Riccati-Bessel functions [see Eq. (28)]. The first order solution, i.e. the anisotropic correction we are searching for, is obtained by neglecting the terms that are proportional to \( \lambda^2 \) in Eqs. (30): however, the resulting equations contain in the right-hand side a term that is not zero (like in the zeroth order solution) but depends on the quasi-potentials \( u_{n \pm 1} \) and \( v_{n \pm 1} \). This coupling among neighbor radial modes is a signature of the anisotropy, that on one hand breaks the azimuthal degeneracy [the azimuthal quantum number appears in the definition of the coefficients of Eqs. (30)] and on the other hand results in a coupling between radial modes. Although this coupling results in an impossibility of an analytic solution, it can be demonstrated [37] that these terms are of the order \( \lambda^2 \) and at the first order they can be neglected. With this argument, Eqs. (30) at the leading order \( \lambda \) read

\[ r^2 \left( \frac{d^2}{dr^2} + \frac{\gamma_1^2}{r^2} \right) - n(n+1) \right] l_+ u_n(r) = 0 \] (31a)

\[ \left[ r^2 \left( \frac{d^2}{dr^2} + \frac{\gamma_1^2}{r^2} \right) - n(n+1) \right] l_+ v_n(r) = 0 \] (31b)

where \( \gamma_1 \) and \( \gamma_2 \) are the TM and TE (respectively) anisotropic factor given by

\[ \gamma_1^2 = k_0^2 \varepsilon_+ \left( 1 - \lambda \left[ 1 - \frac{2m^2}{n(n+1)} \right] \right) \]

\[ \gamma_2^2 = k_0^2 \varepsilon_+ \left( 1 - \lambda \left[ 1 - \frac{4m^2}{4n(n+1) - 3} + \frac{2m^2}{n(n+1)} \right] \right) \]

Eqs. (31) have the same structure of the radial equation for the isotropic case \( 34 \). The only difference is the presence of the \( \gamma_1 \) terms that modify the arguments of the Riccati-Bessel functions and the quasi-potentials can be written as

\[ U_{nm}(r, \theta, \varphi) = C_{int/ext} z_\nu(x) Y_{nm}(\theta, \varphi) \] (33a)

\[ V_{nm}(r, \theta, \varphi) = C_{int/ext} z_\nu(x) Y_{nm}(\theta, \varphi) \] (33b)

where \( z_\nu(x) \) corresponds to \( j_\nu(x) \) inside the sphere and to \( h_\nu^{(1)}(x) \) outside the sphere. Note also that inside the sphere, where the anisotropy exists, \( x = \gamma_1 r \), while outside the sphere \( x = k_0 r \) (the surrounding medium is still isotropic). Substituting these expressions in Eqs. (23) and (24) we obtain all the components of the electric and magnetic fields in an uniaxial anisotropic spherical resonator

\[ E_r = \frac{n(n+1)}{r^2} \left[ \sqrt{\frac{\pi}{2}} z_\nu(\gamma_1 x) Y_{nm}(\theta, \varphi) \right] \] (34a)

\[ r E_\theta = - \frac{\partial^2}{\partial r \partial \theta} \left[ \sqrt{\frac{\pi}{2}} z_\nu(\gamma_1 x) Y_{nm}(\theta, \varphi) \right] \] (34b)

\[ r E_\varphi = \frac{1}{\sin \theta} \frac{\partial^2}{\partial r \partial \varphi} \left[ \sqrt{\frac{\pi}{2}} z_\nu(\gamma_1 x) Y_{nm}(\theta, \varphi) \right] \] (34c)

\[ H_r = 0 \] (34d)

\[ \frac{r k_0}{} \frac{\partial}{\sin \theta} \frac{\partial}{\partial \varphi} \left[ \sqrt{\frac{\pi}{2}} z_\nu(\gamma_1 x) Y_{nm}(\theta, \varphi) \right] \] (34e)

\[ r H_\varphi = i \frac{k_0}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \sqrt{\frac{\pi}{2}} z_\nu(\gamma_1 x) Y_{nm}(\theta, \varphi) \right] \] (34f)

for quasi-TM fields. Similar expressions can be written for the quasi-TE fields by replacing \( \gamma_1 \) with \( \gamma_2 \), exchanging the role of the electric and magnetic field and setting \( \varepsilon_\perp = 1 \).

The characteristic equation can be found by applying the boundary conditions and it turns out to be

\[ \tilde{\gamma}_1 = \frac{\gamma_1}{k_0 \sqrt{\varepsilon_\perp}} \]

\[ \tilde{\gamma}_1 \left[ j_\nu(\gamma_1 kR) \right] = \frac{\varepsilon_\perp \left| h_\nu^{(1)}(k_0 R) \right|}{\sqrt{\varepsilon_\perp} \left| h_\nu^{(1)}(k_0 R) \right|} \] (35)
out-of-plane (sphere surface is of the order of its index, i.e. $\nu/x$ argument of the Bessel function for a WGM near the problem) point of view. The appropriate approximation, to work with easier functions that suit better onto the analytical (where the approximation gives the possibility functions of large index is highly time-consuming) or either from the numerical (where computing Bessel functions of large index is highly time-consuming) or

as can be seen from the previous equations, in the small anisotropy regime, the only effect of the anisotropy is a rescaling of the radial coordinate; this is in accordance with the fact that an uniaxial crystal shows two different refractive indexes: one in-plane ($\sqrt{\varepsilon_{xx}}$) and the other out-of-plane ($\sqrt{\varepsilon_{zz}}$). Different refractive indexes correspond to different optical paths, and this is exactly reflected in the rescaling effect of the anisotropy onto the radial part of the modes of the resonator. Note also that at this level of analysis, the anisotropy doesn’t affect the angular structure ($\theta$ and $\varphi$) of the modes. Another difference respect to the isotropic case is the value of the coefficients on the right-hand side of the characteristic equations: while the coefficient for the quasi-TE wave is analogous to its isotropic counterpart (if we substitute the isotropic dielectric constant $\varepsilon$ with the anisotropy-averaged dielectric constant $\varepsilon_+$), the coefficient for the quasi-TM wave reveals the presence of the anisotropy, since it is a ratio between the in-plane dielectric constant and the anisotropy-averaged one. This is not so surprising because for a dielectric uniaxial crystal, only the TM component suffer directly anisotropy, while the TE component does not, because the crystal is magnetic isotropic.

IV. WHISPERING GALLERY MODES

A. Radial functions with large indices

With the term whispering gallery mode (WGM) are commonly addressed the set of modes with a large index $n$; strictly speaking, the real WGMs are only those for which it results that $n = m$ and the radial wavefunction shows no roots inside the resonator. However, modes with indices $n \neq m$ and with $q > 1$, but close to unity, have properties that are close to those of WGMs: this means that there is no great difference between a “pure” WGM and other modes with nearest indices.

To study such modes, the first thing we have to do is to find a suitable approximation of Riccati-Bessel functions for large index. This approximation is useful either from the numerical (where computing Bessel functions of large index is highly time-consuming) or analytical (where the approximation gives the possibility to work with easier functions that suit better onto the problem) point of view. The appropriate approximation, however, should be searched bearing in mind that the argument of the Bessel function for a WGM near the sphere surface is of the order of its index, i.e. $\nu/x \simeq 1$.

for the quasi-TE waves.

\[
\gamma^2 \frac{j_\nu(\gamma_2 k_R)}{j_\nu(\gamma_1 k_R)} = \frac{1}{\sqrt{\varepsilon_+}} \frac{h_\nu^{(1)}(k_0 R)}{h_\nu^{(1)}(k_0 R)}
\]

for the quasi-TE waves.

As can be seen from the previous equations, in the small anisotropy regime, the only effect of the anisotropy is a rescaling of the radial coordinate; this is in accordance with the fact that an uniaxial crystal shows two different refractive indexes: one in-plane ($\sqrt{\varepsilon_{xx}}$) and the other out-of-plane ($\sqrt{\varepsilon_{zz}}$). Different refractive indexes correspond to different optical paths, and this is exactly reflected in the rescaling effect of the anisotropy onto the radial part of the modes of the resonator. Note also that at this level of analysis, the anisotropy doesn’t affect the angular structure ($\theta$ and $\varphi$) of the modes. Another difference respect to the isotropic case is the value of the coefficients on the right-hand side of the characteristic equations: while the coefficient for the quasi-TE wave is analogous to its isotropic counterpart (if we substitute the isotropic dielectric constant $\varepsilon$ with the anisotropy-averaged dielectric constant $\varepsilon_+$), the coefficient for the quasi-TM wave reveals the presence of the anisotropy, since it is a ratio between the in-plane dielectric constant and the anisotropy-averaged one. This is not so surprising because for a dielectric uniaxial crystal, only the TM component suffer directly anisotropy, while the TE component does not, because the crystal is magnetic isotropic.

\[
\gamma^2 \frac{j_\nu(\gamma_2 k_R)}{j_\nu(\gamma_1 k_R)} = \frac{1}{\sqrt{\varepsilon_+}} \frac{h_\nu^{(1)}(k_0 R)}{h_\nu^{(1)}(k_0 R)}
\]

for the quasi-TE waves.

As can be seen from the previous equations, in the small anisotropy regime, the only effect of the anisotropy is a rescaling of the radial coordinate; this is in accordance with the fact that an uniaxial crystal shows two different refractive indexes: one in-plane ($\sqrt{\varepsilon_{xx}}$) and the other out-of-plane ($\sqrt{\varepsilon_{zz}}$). Different refractive indexes correspond to different optical paths, and this is exactly reflected in the rescaling effect of the anisotropy onto the radial part of the modes of the resonator. Note also that at this level of analysis, the anisotropy doesn’t affect the angular structure ($\theta$ and $\varphi$) of the modes. Another difference respect to the isotropic case is the value of the coefficients on the right-hand side of the characteristic equations: while the coefficient for the quasi-TE wave is analogous to its isotropic counterpart (if we substitute the isotropic dielectric constant $\varepsilon$ with the anisotropy-averaged dielectric constant $\varepsilon_+$), the coefficient for the quasi-TM wave reveals the presence of the anisotropy, since it is a ratio between the in-plane dielectric constant and the anisotropy-averaged one. This is not so surprising because for a dielectric uniaxial crystal, only the TM component suffer directly anisotropy, while the TE component does not, because the crystal is magnetic isotropic.

\[
\gamma^2 \frac{j_\nu(\gamma_2 k_R)}{j_\nu(\gamma_1 k_R)} = \frac{1}{\sqrt{\varepsilon_+}} \frac{h_\nu^{(1)}(k_0 R)}{h_\nu^{(1)}(k_0 R)}
\]

for the quasi-TE waves.

As can be seen from the previous equations, in the small anisotropy regime, the only effect of the anisotropy is a rescaling of the radial coordinate; this is in accordance with the fact that an uniaxial crystal shows two different refractive indexes: one in-plane ($\sqrt{\varepsilon_{xx}}$) and the other out-of-plane ($\sqrt{\varepsilon_{zz}}$). Different refractive indexes correspond to different optical paths, and this is exactly reflected in the rescaling effect of the anisotropy onto the radial part of the modes of the resonator. Note also that at this level of analysis, the anisotropy doesn’t affect the angular structure ($\theta$ and $\varphi$) of the modes. Another difference respect to the isotropic case is the value of the coefficients on the right-hand side of the characteristic equations: while the coefficient for the quasi-TE wave is analogous to its isotropic counterpart (if we substitute the isotropic dielectric constant $\varepsilon$ with the anisotropy-averaged dielectric constant $\varepsilon_+$), the coefficient for the quasi-TM wave reveals the presence of the anisotropy, since it is a ratio between the in-plane dielectric constant and the anisotropy-averaged one. This is not so surprising because for a dielectric uniaxial crystal, only the TM component suffer directly anisotropy, while the TE component does not, because the crystal is magnetic isotropic.

\[
\gamma^2 \frac{j_\nu(\gamma_2 k_R)}{j_\nu(\gamma_1 k_R)} = \frac{1}{\sqrt{\varepsilon_+}} \frac{h_\nu^{(1)}(k_0 R)}{h_\nu^{(1)}(k_0 R)}
\]

for the quasi-TE waves.

As can be seen from the previous equations, in the small anisotropy regime, the only effect of the anisotropy is a rescaling of the radial coordinate; this is in accordance with the fact that an uniaxial crystal shows two different refractive indexes: one in-plane ($\sqrt{\varepsilon_{xx}}$) and the other out-of-plane ($\sqrt{\varepsilon_{zz}}$). Different refractive indexes correspond to different optical paths, and this is exactly reflected in the rescaling effect of the anisotropy onto the radial part of the modes of the resonator. Note also that at this level of analysis, the anisotropy doesn’t affect the angular structure ($\theta$ and $\varphi$) of the modes. Another difference respect to the isotropic case is the value of the coefficients on the right-hand side of the characteristic equations: while the coefficient for the quasi-TE wave is analogous to its isotropic counterpart (if we substitute the isotropic dielectric constant $\varepsilon$ with the anisotropy-averaged dielectric constant $\varepsilon_+$), the coefficient for the quasi-TM wave reveals the presence of the anisotropy, since it is a ratio between the in-plane dielectric constant and the anisotropy-averaged one. This is not so surprising because for a dielectric uniaxial crystal, only the TM component suffer directly anisotropy, while the TE component does not, because the crystal is magnetic isotropic.

\[
\gamma^2 \frac{j_\nu(\gamma_2 k_R)}{j_\nu(\gamma_1 k_R)} = \frac{1}{\sqrt{\varepsilon_+}} \frac{h_\nu^{(1)}(k_0 R)}{h_\nu^{(1)}(k_0 R)}
\]

for the quasi-TE waves.
FIG. 4. (color online) The figure shows the accuracy $\sigma$ as a function of the Bessel index $\nu$; the accuracy is defined as the ratio between the difference of the true function $j_\nu(\nu)$ and its Airy approximation $Ai(\zeta^*)$ and their sum, i.e. $\sigma = \left[ j_\nu(\nu) - Ai(\zeta^*) \right] \left[ j_\nu(\nu) + Ai(\zeta^*) \right]$, where $\zeta^*$ is $\zeta$ evaluated for $x = \nu$. As can be seen, as $\nu$ grows, the approximation becomes more precise.

the former is very good when the argument of the Hankel function is greater than the index (that is precisely the case of the outer functions), the latter is not suitable in this region, either for being out of phase with respect to Hankel function (as shown in Figs. 5 and 6) or to not approximate in the correct way the original function (Fig. 7). Moreover, Fig. 7 shows that the field outside the resonator has all the characteristics of an evanescent wave, i.e. it decays exponentially as the distance from the resonator surface grows.

In order to justify this evanescent behavior outside the resonator, one can directly solve Eq.(8) in the limit $r > R$ (but still close to the resonator surface), where the terms $x = k_0 r$ outside the derivation symbol can be substituted with $k_0 R$, leading to the following equation:

$$\frac{d^2 Z}{dx^2} + \frac{1}{k_0 R} \frac{dZ}{dx} + \left[ 1 - \frac{\nu^2}{(k_0 R)^2} \right] Z = 0, \quad (39)$$

whose solution is:

$$Z_\nu(x) = C_0 e^{-\delta x}, \quad (40)$$

where

$$\delta = \left[ \sqrt{\left( \frac{\nu}{k_0 R} \right)^2 - 1} + 1 - \frac{1}{4(k_0 R)^2} - \frac{1}{2k_0 R} \right].$$

This is, how we are expecting, the expression of an exponentially decreasing field, that is in perfect agreement with the hypothesis that the field outside the resonator is evanescent due to total internal reflection.

FIG. 5. (color online) Comparison between the real part of $H^1_\nu(x)$ (black solid line), Eq. (38) (dashed red line) and Eq. (31) of Ref.[33] (dashed blue line) for $\nu = 1000.5$. Our approximation works very well in the region in which the argument is greater than the index (i.e. $x > 1010$), while the approximation presented in Ref.[33] is out of phase respect to the Hankel function.

FIG. 6. (color online) Same as Fig. 5 but the comparison is made for the imaginary part.

It can be moreover noted that the oscillatory behavior of the field components outside the resonator (as depicted in Figs. 5 and 6 for the radial component of the electric field) is not in contrast with this hypothesis, since it only represent the behavior of the Hankel function as $r \to \infty$, i.e. it behaves like a runaway wave whose intensity is decreasing as $1/r^2$. In the case of WGM, however, no radiation will run away towards infinity since the external field is evanescent, i.e. the radiation is trapped inside the WGM and rapidly decreases toward zero when the field goes outside the resonator.
B. Angular functions with large indices

For large indices \( n \), the WGM field is concentrated in a narrow interval of angles \( \theta \) near \( \theta_0 = \pi/2 \); this makes possible to approximate the associated Legendre functions (i.e. the \( \theta \)-part of the scalar spherical harmonics) with large indices, with Hermite polynomials with small indices as follows:

\[
Y_{nm}(\theta, \varphi) \simeq \frac{\sqrt{m}}{2^n n! \sqrt{\pi} \pi^n} H_n(\sqrt{m} x) e^{-\frac{1}{2} x^2} e^{i m \varphi}.
\]

(41)

Detailed calculations for obtaining this result are shown in Appendix B.

C. Roots of characteristic equations

The approximations exploited in the previous section are very useful in finding an analytical solution to the characteristic equation for the eigenfrequencies of the resonator; however, due to the anisotropy, some changes in the definition of the variables used above must be done. First of all, the \( x \) appearing in Eqs. (37) and (38) has to be different for the inner and outer functions, due to the fact that the anisotropy is confined only inside the resonator; we can then define \( x = k_0 R \) as the outer variable and by consequence the inner variable results to be \( y = \gamma_i \sqrt{\varepsilon} x \). Then, the definition of \( \zeta \) must be changed into \( \zeta = (2/\nu)^{1/3}(\nu - y) \). After that, by substituting Eqs. (37) and (38) into Eq. (35), the characteristic equation for quasi-TE field gives

\[
\gamma_i \frac{1}{\text{Ai}(\zeta)} \frac{d\text{Ai}(\zeta)}{d\zeta} = \frac{\varepsilon_\perp}{\sqrt{\varepsilon_\parallel}} \left( \frac{\nu}{2} \right)^{1/3} \times \left[ \frac{1}{4} \left( \frac{2x}{x^2 - \nu^2} \right) - i \sqrt{1 - \frac{\nu^2}{x^2}} \right],
\]

(42)

the equation for the quasi-TE field can be deduced by this one upon changing \( \gamma_i \) with \( \gamma_2 \) and putting \( \varepsilon_\perp = 1 \). In order to find an approximate formula for the solutions of this equation, let us firstly analyze the limiting case in which \( \nu \to \infty \); in this right-hand side of the equation goes to infinity and the only possible solution is that \( \text{Ai}(\zeta) = 0 \), whose solutions are the zeros of the Airy function \( \zeta_q \). Let us denote with \( \Delta \zeta_q \) the first order correction to these roots; expanding both left-hand and right-hand size of Eq. (42) in power series with a first order accuracy to terms \( \Delta \zeta_q \) we can obtain the first order correction to the roots \( \zeta_q \), whose expression is

\[
\Delta \zeta_q = \frac{\gamma_i \sqrt{\varepsilon_\perp}}{\varepsilon_\parallel \alpha} \left( \frac{\nu}{2} \right)^{1/3},
\]

(43)

where

\[
\alpha = \frac{x_q}{2(x^2_q - \nu^2)} - i \sqrt{1 - \frac{\nu^2}{x^2_q}},
\]

(44)

and \( x_q \) is obtained by substituting the value of the first zero of the Airy function (\( \zeta_q = -2.33811 \)) into the definition of \( \zeta \) and inverting that relation with respect to \( x \).

Taking into account the definition of \( \zeta \), the eigenvalues of the wave numbers for the anisotropic resonator can be represented in the following explicit form

\[
k_{0q} = \nu - \left( \frac{\nu}{2} \right)^{1/3} (\zeta_q + \Delta \zeta_q) \frac{1}{\gamma_i \sqrt{\varepsilon} R}.
\]

(45)

Note that because the quantity \( \Delta \zeta_q \) is complex, the eigenvalues of the wave number are also complex. The real part of the wave number then determines the eigenfrequencies of the mode. Complex eigenfrequencies are fully compatible with the open cavity. As can be seen from Eq. (43), this approximation has an accuracy of \( \nu^{-1/3} \) more accurate asymptotic expressions that allow the calculation of the positions of resonances of the modes in an isotropic dielectric spherical resonator have been largely studied in literature (see for example Ref. [42–47] and references therein) and they were given with various accuracy with respect to the index \( \nu \); in Ref. [47] analytic calculations are carried out to the order \( \nu^{-1/2} \), in Ref. [43] the eigenfrequencies are calculated with an accuracy of \( O(\nu^{-2/3}) \), while in Ref. [46] the
authors give an expression up to the order $O[\nu^{-8/3}]$. Here we report the anisotropic correction of the formula found in Ref. [42] that gives the eigenfrequencies with a precision of the order of $O[\nu^{-1}]$

$$
\gamma_i(\nu) = \left\{ \nu - \left( \frac{\nu}{2} \right)^{1/3} \right\} \left\{ \frac{1}{\nu}\right\}^{1/3} \zeta_q - \sqrt{\frac{\epsilon_+}{\epsilon_+ - 1}} P + \frac{3}{10} \left( \frac{1}{4\nu} \right)^{1/3} \zeta_q - \left( \frac{1}{2
u^2} \right)^{1/3} \left( \frac{\epsilon_+}{\epsilon_+ - 1} \right)^{3/2} P
\times P\left( \frac{2}{3} \right)^2 \left( \frac{\epsilon_+}{\epsilon_+ - 1} \right) + O[\nu^{-1}],
$$

where $P = 1/(\gamma_1 \epsilon_+)$ for quasi-TM modes and $P = \epsilon_+ / (\gamma_2 \epsilon_+)$ for quasi-TE modes.

D. Whispering Gallery Modes

We now have all the elements for writing the explicit expressions for the radial, polar and azimuthal components of the quasi-TM and quasi-TE WGMs. Taking the approximations [37], [38] and [41], the equations for the components of the quasi-TM fields defined in Eqs. [34] become

$$
E_r = \frac{n(n+1)\sqrt{m}}{r^2 2^w \sqrt{\pi w!}} \left( \frac{\nu}{2} \right)^{1/6} \text{Ai}(\zeta) \times \times H_w(\sqrt{m}\alpha)e^{-\frac{\pi}{4\alpha^2}e^{im\varphi}}, \quad (47a)
$$

$$
\times \frac{\partial}{\partial \theta} \left[ H_w(\sqrt{m}\alpha)e^{-\frac{\pi}{4\alpha^2}e^{im\varphi}} \right], \quad (47b)
$$

$$
E_r = \frac{\partial}{\partial \varphi} \left[ H_w(\sqrt{m}\alpha)e^{-\frac{\pi}{4\alpha^2}e^{im\varphi}} \right], \quad (47c)
$$

$$
H_r = 0, \quad (47d)
$$

$$
H_\theta = \frac{k_0\epsilon_1}{\sin \theta^{2w_1}} \left( \frac{\nu}{2} \right)^{1/6} \text{Ai}(\zeta) \times \times H_w(\sqrt{m}\alpha)e^{-\frac{\pi}{4\alpha^2}e^{im\varphi}}, \quad (47e)
$$

$$
H_\varphi = \frac{i k_0 \epsilon_1}{\sin \theta^{2w_1}} \left( \frac{\nu}{2} \right)^{1/6} \text{Ai}(\zeta) \times \times \frac{\partial}{\partial \theta} \left[ H_w(\sqrt{m}\alpha)e^{-\frac{\pi}{4\alpha^2}e^{im\varphi}} \right], \quad (47f)
$$

for the field inside the resonator, while for the field outside the resonator the expressions are the following:

$$
E_r = \frac{n(n+1)\sqrt{m}}{r^2 2^w \sqrt{\pi w!}} \left( \frac{\nu}{2} \right)^{1/6} \text{Ai}(\zeta) \times \times H_w(\sqrt{m}\alpha)e^{-\frac{\pi}{4\alpha^2}e^{im\varphi}}, \quad (48a)
$$

$$
\times \frac{\partial}{\partial \theta} \left[ H_w(\sqrt{m}\alpha)e^{-\frac{\pi}{4\alpha^2}e^{im\varphi}} \right], \quad (48b)
$$

$$
E_r = \frac{\partial}{\partial \varphi} \left[ H_w(\sqrt{m}\alpha)e^{-\frac{\pi}{4\alpha^2}e^{im\varphi}} \right], \quad (48c)
$$

$$
H_r = 0, \quad (48d)
$$

$$
H_\theta = \frac{k_0\epsilon_1}{\sin \theta^{2w_1}} \left( \frac{\nu}{2} \right)^{1/6} \text{Ai}(\zeta) \times \times H_w(\sqrt{m}\alpha)e^{-\frac{\pi}{4\alpha^2}e^{im\varphi}}, \quad (48e)
$$

$$
H_\varphi = \frac{i k_0 \epsilon_1}{\sin \theta^{2w_1}} \left( \frac{\nu}{2} \right)^{1/6} \text{Ai}(\zeta) \times \times \frac{\partial}{\partial \theta} \left[ H_w(\sqrt{m}\alpha)e^{-\frac{\pi}{4\alpha^2}e^{im\varphi}} \right], \quad (48f)
$$

Similar expressions can be found for quasi-TE WGMs by interchanging the roles of the electric and magnetic field in the previous expressions, changing $\gamma_1$ with $\gamma_2$ and $i$ with $-i$.

Figures [8] and [9] show the behavior of the fundamental quasi-TM radial (no nodes in radial direction, i.e. $q = 1$) and polar (i.e. $n = m$) WGM component $r^2E_r$ and the “first excited” radial ($q = 2$) and polar ($m = n + 1$) mode for the same component of the quasi-TM field; the physical parameters have been set to be $\epsilon_{xx} = 53.0, \epsilon_{zz} = 6.47$ (LiNbO$_3$) and $\lambda = 1064$ nm. Note that the radial component has its maximum very close to the sphere surface (dashed vertical line in Fig. [8]), and its position shifts on the left, i.e. on the inner part of the resonator as the radial number $q$ increases. The polar part, instead, is localized around $\theta = \pi/2$ in its fundamental state and as $m$ becomes smaller than $n$, the maxima of the polar component tend to repel each other from $\theta = \pi/2$. In figures [10] to [13] the intensity distribution of the total electric field of a quasi-TM (i.e. $E_{TM} = E_r \hat{r} + E_\theta \hat{\theta} + E_\varphi \hat{\varphi}$) is shown, where the components $E_k$ ($k = r, \theta, \varphi$) are given by Eqs. [17] for the field inside the resonator and Eqs. [18] for the field outside the resonator; $\hat{r}, \hat{\theta}$ and $\hat{\varphi}$ represent the unit vectors of the spherical basis ($r, \theta, \varphi$). In order to obtain the intensity distribution of such a field, one has to sum the square modulus of each component of the electric field; however, in this particular case, the contribution of $E_\theta$ and $E_\varphi$ is very small and localized at the resonator surface, and the total field is, with a good level of approximation, fully determined by its radial component $E_r$. The intensity distribution for the
FIG. 8. (color online) Radial part of the fundamental (black line) and first excited (red line) WGM for the anisotropic resonator; the vertical dashed line indicates the position of the resonator surface. The fundamental mode has indices \( n = m = 1000 \) and \( q = 1 \), while the first excited mode has the same \( n \) and \( m \) indices but \( q = 2 \).

FIG. 9. (color online) Polar part of the fundamental (black line) and first polar-excited (red line) WGM for the anisotropic resonator. The fundamental mode has indices \( n = m = 1000 \) and \( q = 1 \), while the first polar-excited mode has \( n = 1000, m = n - 1 \) and \( q = 1 \).

FIG. 10. (color online) Intensity distribution of the electric field of the fundamental quasi-TM WGM. The WGM quantum numbers are \( n = m = 1000,q = 1 \).

CONCLUSIONS

In this work, we have developed a classical-optics theory for an uniaxial spherical whispering gallery resonator. We have presented and discussed the mode structure in the limit of small anisotropy for such resonator, and obtained its spectrum. Moreover, we have furnished a thorough discussion on the boundary conditions and asymptotic expressions for the electromagnetic field in WGRs. Our results may be easily generalized to achieve a quantum theory of WGRs.

ACKNOWLEDGEMENTS

The authors want to thank Josef Fürst, Christoph Marquardt and Dmitry Strekalov for fruitful discussions.

APPENDIX A: COEFFICIENTS OF EQS. (22)

Here are reported the explicit expressions of the coefficients that appear on Eqs. (47) and (48) or by nothing that the \( H_\theta \) component of the magnetic field has the same intensity distribution as the radial electric field component \( E_r \) and the \( H_\phi \) component, because of the presence of the derivative with respect to \( \theta \), has the same intensity distribution as the one depicted in Fig. 12.

As the reader can see from these figures, the field is nonzero even after the resonator surface \( (x = 1) \); this is not surprising because in this region the total field is evanescent due to the fact that it has been total internal reflected by the resonator, i.e. the field is confined in the resonator WGM.
FIG. 11. (color online) Same as Fig. 10 but for $q = 2$; this mode represents the first radially excited WGM.

$$g_{\pm} = k_0^2 \xi_{\pm},$$

$$f_n = \frac{1}{2n+2},$$

$$T_n = r^2 \tilde{l}_x - n(n+1),$$

$$\tilde{l}_+ = \frac{\partial^2}{\partial r^2} + k_0^2 \xi_+.$$

With these parameters defined, the $a$ and $b$ coefficients of Eq. (30) become:

$$a_{1,n}^{TM} = T_n \left[ l_+ - \frac{1 - 4m^2}{4n(n+1) - 3} g_- \right] + 2g_- m^2,$$

$$a_{2,n}^{TM} = 2g_- (n - m + 1)(n - m + 2) f_n f_{n+1} T_n,$$

$$a_{3,n}^{TM} = 2g_- (n + m)(n + m - 1) f_n f_{n-1} T_n,$$

$$b_{1,n} = f_n (n - m + 1) \left[ r \tilde{l}_x - (n + 2) \frac{d}{dr} \right],$$

$$b_{2,n} = f_n (n + m) \left[ r \tilde{l}_x + (n - 1) \frac{d}{dr} \right].$$

APPENDIX B: APPROXIMATION OF SCALAR SPHERICAL HARMONICS FOR LARGE INDICES

The equation for the $\theta$-part of spherical harmonics is the following

$$\frac{1}{\sin \theta} \frac{d}{d \theta} \left( \sin \theta \frac{df}{d \theta} \right) + \left[ n(n+1) - \frac{m^2}{\sin^2 \theta} \right] f = 0,$$

whose solutions are the associated Legendre functions $f(\theta) = P_n^m (\cos \theta)$. Since WGMs are located near the equator of the resonator, the correspondent functions $f(\theta)$ will be peaked near the angle $\theta_0 = \pi/2$; in order to find an approximate expression for the polar part of the spherical harmonics, let us introduce the new variable
\[ \alpha = \frac{\pi}{2} - \theta: \text{substituting into equation above gives} \]
\[ \frac{d^2 f}{d\alpha^2} - \tan \alpha \frac{df}{d\alpha} + \left[ n(n+1) - \frac{m^2}{\cos^2 \alpha} \right] f = 0. \]

over the real axis is one. This equation gives the approximated form of the associated Legendre functions for WGMs; substituting it into the definition of the scalar spherical harmonics gives exactly Eq (41).

 FIG. 13. (color online) Same as Fig. 10 but for \( n - m = 1 \) and \( q = 2 \); this mode represents the first radially and polar excited WGM.

We note that, since the modes are localized near the equator, \( \alpha \ll 1 \) and this allow us to expand in power series the trigonometric functions that appear in the previous equation, i.e. \( \tan \alpha \approx \alpha \) and \( 1/\cos^2 \alpha \approx 1 + \alpha^2 \). Substituting in the previous equation, writing \( f(\alpha) = G(\alpha)e^{\alpha^2/4} \) and performing the change of variables \( \alpha = x/\left[(m^2+1/4)^{1/4}\right] = \xi x \) we obtain

\[ \frac{d^2G}{dx^2} + \left\{ \xi^2 \left[ n(n+1) - m^2 \right] - x^2 \right\} G = 0, \]

Introducing the quantity \( w = n - m \) and remembering that WGMs are characterized by high values of the indices, i.e. \( n, m \gg 1 \), the first term that appears inside the curly brackets can be simplified as \( 2w + 1 \). With this substitution the last equation is precisely the Hermite-Gauss equation, whose solutions have the form \( G(x) \approx H_w(x)e^{-x^2/2} \). Function \( f(\alpha) \) then becomes

\[ f(\alpha) = P^m_n(\cos \theta) \approx NH_w(\sqrt{m} \alpha)e^{-\frac{\alpha^2}{2}}, \]

where \( N \) is a normalization factor whose expression could be found by requiring that the norm of \( f(\alpha) \) integrated

*marco.ornigotti@mpe.mpg.de

[1] http://en.wikipedia.org/wiki/St_Paul’s_Cathedral

[2] Baron John William Strutt Rayleigh, *The Theory of Sound: Volume II*, Dover Publication (1945)

[3] G. Mie, Ann. Physik. 25, 377 (1908)

[4] P. Debye, Ann. Physik. 30, 57 (1909)

[5] V.S. Ilchenko et al., Phys. Rev. Lett. 92, 049303(4)(2004)

[6] M.L. Gorodetsky et al., Opt. Lett. 21, 453(1996)

[7] I.S. Grudinin et al., Phys. Rev. A 74, 063806(2006)

[8] A.A. Savchenkov et al., Phys. Rev. A 70, 051804(R)(2004)

[9] V.S. Ilchenko et al., J. Opt. Soc. Am. B 20, 1304(2003)

[10] G. Kozyreff et al., Phys. Rev. A 77, 043817(2008)

[11] A.A. Savchenkov et al., Opt. Lett. 32, 157(2007)

[12] A.A. Savchenkov et al., Phys. Rev. Lett. 93, 243905(2004)

[13] P. Del’Haye et al., Nature (London) 450, 1214(2007)

[14] A.A. Savchenkov et al., Phys. Rev. Lett. 101, 093902(2008)

[15] I.S. Grudinin, A.B. Matsko, and L. Maleki, Phys. Rev. Lett. 102, 043902(2009)

[16] D.V. Strekalov et al., Opt. Lett. 34, 713(2009)

[17] V.S. Ilchenko et al., Phys. Rev. Lett. 92, 043903(2004)

[18] T. Carmon and K.J. Vahala, Nature Phys. 3, 430(2007)

[19] J.U. Fürt et al., Phys. Rev. Lett. 104, 153901(2010)

[20] D.W. Vernooy et al., Phys. Rev. A 57, R2293(1998)

[21] J.R. Buck and H.J. Kimble, Phys. Rev. A 67, 033806(2003)

[22] A.B. Matsko et al., IPN Progress Report 42, 162(2005)

[23] G.W. Ford and S.A. Werner, Phys. Rev. B 18, 6752(1978)

[24] C.W. Qu et al., Phys. Rev. E 75, 026609(2007)

[25] R.E. Colin, Electromagnetics 6, 183(2010)

[26] S.N. Papadakis et al., J. Opt. Soc. Am. A 7, 991(1990)

[27] H. Chen et al., J. Phys.: Condens. Matter 16, 165(2004)

[28] Y.L. Geng et al., Phys. Rev. E 70, 056609(2004)

[29] W. Ren, Phys. Rev. E 47, 664(1993)

[30] N. Okada and J.B. Cole, J. Opt. Soc. Am. B 27, 631(2010)

[31] D.V. Strekalov et al., Phys. Rev. A 80, 033810(2009)

[32] J.M. le Floch et al., Phys. Lett. A 359, 1(2007)

[33] A.N. Oraevsky, Quantum Electronics 32, 377(2002)

[34] J.D. Jackson, *Classical Electrodynamics*, Wiley (Third Edition)

[35] In this paper we consider a nonmagnetic resonator, i.e. \( \mu = 1 \) surrounded by air (\( \varepsilon_{\infty} = 1 \)).

[36] In obtaining the azimuthal component of the quasi-TM magnetic field following relation was used: \[ \frac{\partial}{\partial \theta} \frac{\varepsilon_{\theta}}{\varepsilon_{\infty}} - \varepsilon_{r \theta} \frac{\partial}{\partial \theta} \frac{\varepsilon_{\infty}}{\varepsilon_{\infty}} \]

[37] Y.V. Proponenko et al., Technical Physics 49, 459(2004)

[38] Eq. (5) of Ref. [57] contains an error: the expression for \( \cos(2\theta)Y_{n,m}(\theta, \phi) \) is incorrect. Here we report the correct relation, in which the argument of the spherical harmonics is omitted for the sake of simplicity:
\[
\cos(2\theta)Y_{n,m} = A_{n,m}Y_{n+2,m} + B_{n,m}Y_{n-2,m} + C_{n,m}Y_{n,m}
\]
with \(A_{n,m} = 2(n-m+1)(n-m+2)f_nf_{n+1}, B_{n,m} = 2(n+m)(n+m+1)f_nf_{n-1}\) and \(C_{n,m} = (1-4m^2)/(4n(n+1)-3)\)

[39] M.A. Abramowitz and I. Stegun (Editors), *Handbook of Mathematical Functions: with Formulas, Graphs and Mathematical Tables*, Dover (1965)

[40] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academy Press (2007)

[41] To write this equation we have used the expression in Eq (38) stopped at the order \(O[\nu^{-1}]\)

[42] S. Schiller and R.L. Byer, Opt. Lett 16, 1138(1991)

[43] V.S. Ilchenko et. al., J. Opt. Soc. Am. A 20, 157(2003)

[44] M. Gadtone *et al.*, IEEE Trans. Microwave Theory and Techniques MTT-15, 694(1997)

[45] B.R. Johnson, J. Opt. Soc. Am. A 10, 343(1993)

[46] S. Schiller, Appl. Opt 32, 2181(1993)

[47] C.C. Lam *et al.*, J. Opt. Soc. Am. B 9, 1585(1992)

[48] As can be seen from Eqs. (10), the radial component of the TM field has a dependence on the radial wavefunction itself, via the second derivative of the potential. The tangential component, instead, depends on the derivative (with respect to the radial variable) of the wavefunction.

[49] A. Messiah, *Quantum Mechanics*, Dover (1999)