Extensions of Motives and the Fundamental Group

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Abstract

In this paper we construct extensions of the mixed Hodge structure on the fundamental group of a smooth projective pointed curve which correspond to the regulators of certain motivic cohomology cycles on the self product of the curve constructed by Bloch and Beilinson. This leads to a new iterated integral expression for the regulator. This is a generalisation of a theorem of Colombo [Col02] where she constructed the extension corresponding to Collino’s cycles in the Jacobian of a hyperelliptic curve.

1 Introduction

A formula, usually called Beilinson’s formula — though independently due to Deligne as well — describes the motivic cohomology group of a smooth projective variety $X$ over a number field as the group of extensions in a conjectured category of mixed motives. If $M$ is the motive $h^i(X)(m)$ and $N = M^*(1) = h^i(X)(n)$ where $n = i + 1 - m$ then

$$\text{Ext}^1_{\mathcal{MM}}(M, \mathbb{Q}(1)) = \text{Ext}^1_{\mathcal{MM}}(\mathbb{Q}(0), N) = \begin{cases} CH^n_{\text{hom}}(X) \otimes \mathbb{Q} & \text{if } i + 1 = 2n \\ H^{i+1}_{\mathcal{MM}}(X, \mathbb{Q}(n)) & \text{if } i + 1 \neq 2n \end{cases}$$

Hence, if one had a way of constructing extensions in the category of mixed motives by some other method, it would provide a way of constructing motivic cycles.

One way of doing so is by considering the group ring of the fundamental group of the algebraic variety $\mathbb{Z}[\pi_1(X, P)]$. If $J_P$ is the augmentation ideal — the kernel of the map from $\mathbb{Z}[\pi_1(X, P)] \to \mathbb{Z}$ — then the graded pieces $J_a^p / J_b^p$ with $a < b$ are expected to have a motivic structure. These give rise to natural extensions of motives — so one could hope that these extensions could be used to construct natural motivic cycles.

Understanding the motivic structure on the fundamental group is complicated. However, the Hodge structure on the fundamental group is well understood [Hai87]. The regulator of motivic cohomology cycles can be thought of as the realisation of the extension of motives as an extension in the category of mixed Hodge structures. So while we may not be able to construct motivic cycles as extensions of motives coming from the fundamental group - we can hope to construct their regulators as extensions of mixed Hodge structures (MHS) coming from the fundamental group.

The aim of this paper is to describe this construction in the case of the motivic cohomology group of the self product of a curve. The first work in this direction was due to Harris and Pulte [Pul88, Hai87]. They showed that the Abel-Jacobi image of the modified diagonal cycle on the triple product of a pointed curve...
\((C, P)\), or alternatively the Ceresa cycle in the Jacobian \(J(C)\) of the curve, is the same as an extension class coming from \(J_P/J_P^3\), where \(J_P\) is the augmentation ideal in the group ring of the fundamental group of \(C\) based at \(P\).

In [Col02], Colombo extended this theorem to show that the regulator of a cycle in the motivic cohomology of a Jacobian of a hyperelliptic curve, discovered by Collino [Col97], can be realised as an extension class coming from \(J_P/J_P^\text{4}\), where \(J_P\) is the augmentation ideal of a related curve.

In this paper we extend Colombo’s result to more general curves. If \(C\) is a smooth projective curve with a function \(f\) with divisor \(\text{div}(f) = NQ - NR\) for some points \(Q\) and \(R\) and some integer \(N\) and such that \(f(P) = 1\) for some other point \(P\), there is a motivic cohomology cycle \(Z_{QR,P}\) in \(H^3_\text{M}(C \times C, \mathbb{Z}(2))\) discovered by Bloch [Blo00]. We show that the regulator of this cycle can be expressed in terms of an extensions coming from \(J/P\). When \(C\) is hyperelliptic and \(Q\) and \(R\) are ramification points of the canonical map to \(\mathbb{P}^1\), this is Colombo’s result as the image of \(Z_{QR,P}\) under the map from \(C \times C\) to \(J(C)\) takes this cycle to Collino’s cycle.

A crucial step in Colombo’s work is to use the fact that the modified diagonal cycle is torsion in the Chow group \(CH^2_{\text{hom}}(C^3)\) when \(C\) is a hyperelliptic curve. This means the extension coming from \(J_P/J_P^3\) splits and hence does not depend on the base point \(P\). This allows her to consider the extension for \(J_P/J_P^4\). In general, that is not true — in fact the known examples of non-torsion modified diagonal cycles come from the curves we consider. Our main contribution is to use an idea of Rabi [Rab01] to show that Colombo’s arguments can be extended to work in our case as well. As a result we have a more general situation — which has some arithmetical applications.

We have the following theorem (Theorem [5.6]):

**Theorem 1.1.** Let \(C\) be a smooth projective curve and \(Q\) and \(R\) be two points such that there is a function \(f_{QR}\) with divisor \(\text{div}(f_{QR}) = NQ - NR\) for some \(N\). Let \(P\) be a third point on \(C\). Normalize \(f_{QR}\) so that \(f_{QR}(P) = 1\).

Let \(Z_{QR} = Z_{QR,P}\) be the element of the motivic cohomology group \(H^3_\text{M}(C \times C, \mathbb{Z}(2))\) constructed by Bloch. There exists an extension \(\epsilon^4_{QR,P}\) in \(\text{Ext}_{MHS}(\mathbb{Z}(-2), \otimes^2 H^1(C))\) constructed from the mixed Hodge structure on the fundamental groups \(\pi_1(C \setminus Q, P)\) and \(\pi_1(C \setminus R, P)\) such that

\[
\epsilon^4_{QR,P} = (2g_C + 1) \text{reg}_Q(Z_{QR})
\]

in \(\text{Ext}_{MHS}(\mathbb{Z}(-2), \otimes^2 H^1(C))\).

In other words our theorem states that the regulator of a natural cycle in the motivic cohomology group of a product of curves, being thought of as an extension class is the same as that as a natural extension of MHS coming from the fundamental group of the curve. In fact, it is an extension of pure Hodge structures.

Our primary motivation are the conjectures relating regulators of the motivic cycles to special values of \(L\)-functions. One application we have is to the case of modular curves. Beilinson [BeÎ84] constructed a cycle in the group \(H^3_\text{M}(X_0(N) \times X_0(N), \mathbb{Q}(2))\) and showed that its regulator is related to a special value of the \(L\)-function. We construct the extension of \(MHS\) coming from the fundamental group which corresponds to the regulator of this cycle.
Since the mixed Hodge structure on the fundamental group is related to the iterated integrals we also get an expression for the regulator as an iterated integral. In a subsequent paper we apply this in the case of Fermat curves to get an explicit expression for the regulator in terms of hypergeometric functions — analogous to the works of Otsubo [Ots11], [Ots12].

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2 Iterated Integrals

Let $\alpha : [0, 1] \to X$ be a path and $\omega_1, \omega_2, \ldots, \omega_n$ be 1-forms on $X$. Suppose $\alpha^*(\omega_i) = f_i(t)dt$. The iterated integral of length $n$ is defined to be

$$\int_\alpha \omega_1 \omega_2 \ldots \omega_n := \int_{0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1} f_1(t_1)f_2(t_2)\cdots f_n(t_n)dt_n \ldots dt_2 dt_1.$$  

An iterated integral is said to be a homotopy functional if it only depends on the homotopy class of the path $\alpha$. A homotopy functional gives a functional on the group ring of the fundamental group or path space.

Iterated integrals can be thought of as integrals on simplices and satisfy the following basic properties — here we have only stated the results for length two iterated integrals, since that is the only type we will encounter in this paper —

**Lemma 2.1 (Basic Properties).** Let $\omega_1$ and $\omega_2$ be smooth 1-forms on $C$ and $\alpha$ and $\beta$ piecewise smooth paths on $C$ with $\alpha(1) = \beta(0)$. Then

1. $\int_{\alpha, \beta} \omega_1 \omega_2 = \int_\alpha \omega_1 \omega_2 + \int_\beta \omega_1 \omega_2 + \int_\alpha \int_\beta \omega_2$
2. $\int_\alpha \omega_1 \omega_2 + \int_\alpha \omega_2 \omega_1 = \int_\alpha \int_\alpha \omega_2$
3. $\int_\alpha df \omega_1 = \int_\alpha f \omega_1 - f(\alpha(0)) \int_\alpha \omega_1$
4. $\int_\alpha \omega_1 df = f(\alpha(1)) \int_\alpha \omega_1 - \int_\alpha f \omega_1$

**Proof.** This can be found in any article on iterated integrals, for instance Hain’s excellent article [Hai87]. □
3 Motivic Cohomology Cycles, Extensions and Regulators

3.1 Motivic Cohomology Cycles

Let $X$ be a smooth projective algebraic surface defined over a field $K$. An element of the motivic cohomology group $H^3_M(X, \mathbb{Z}(2)) = H^1(X, K_2) = CH^2(X, 1)$ has the following presentation. It consists of sums

$$Z = \sum_i (C_i, f_i)$$

where $C_i$ are curves on $X$ and $f_i : C_i \to \mathbb{P}^1$ are functions on them subject to the co-cycle condition

$$\sum_i \text{div}(f_i) = 0.$$ 

The relations in this group are given by the tame symbols of functions $\{f, g\}$ in $K_2(K(X))$.

$$\tau(\{f, g\}) = \sum_Z (-1)^{\text{ord}_Z(f) \text{ord}_Z(g)} \frac{f^{\text{ord}_Z(g)}}{g^{\text{ord}_Z(f)}}.$$ 

If $L/K$ is a finite extension, let $X_L = X \otimes_K L$. There is a norm map

$$Nm_{L/K} : H^3_M(X_L, \mathbb{Z}(2)) \to H^3_M(X, \mathbb{Z}(2)).$$ 

In the group $H^3_M(X, \mathbb{Z}(2))$ there are certain decomposable cycles coming from the product

$$H^3_M(X, \mathbb{Z}(2))_{\text{dec}} = \bigoplus_{L/K \text{ finite}} Nm_{L/K} \left( \text{Im} \left( H^1_M(X_L, \mathbb{Z}(1)) \otimes H^2_M(X_L, \mathbb{Z}(1)) \to H^3_M(X_L, \mathbb{Z}(2)) \right) \right).$$

The quotient is called the group of indecomposable cycles —

$$H^3_M(X, \mathbb{Z}(2))_{\text{ind}} = H^3_M(X, \mathbb{Z}(2))/H^3_M(X, \mathbb{Z}(2))_{\text{dec}}.$$ 

In general it is not easy to find non trivial elements in this group. One of the aims of this paper is to show that in certain cases the cycles we construct are indecomposable. One way to do that is by computing its regulator.

3.2 Regulators

Let $X$ be a smooth projective algebraic surface. The regulator map of Beilinson is a Chern class map from the motivic cohomology group to the Deligne cohomology group.

$$\text{reg}_Z : H^3_M(X, \mathbb{Z}(2)) \to H^3_D(X, \mathbb{Z}(2)) = \left( F^1H^2(X, \mathbb{C}) \right)^* \frac{H_2(X, \mathbb{Z}(1))}{H_2(X, \mathbb{Z}(1))}.$$ 

The Deligne cohomology group $H^3_D(X, \mathbb{Z}(2))$ is a generalised torus.

The map is defined as follows: Let $Z = \sum_i (C_i, f_i)$ be a cycle in $H^3_M(X, \mathbb{Z}(2))$, so $C_i$ are curves on $X$ and $f_i$ functions on them satisfying the cocycle condition. Let $\mu_i : \tilde{C}_i \to C_i$ be a resolution of singularities. We can
think of \( f_i \) as a function \( f_i : \tilde{C}_i \to \mathbb{P}^1 \). Let \([0, \infty]\) denote the positive real axis in \( \mathbb{P}^1 \) and \( \gamma_i = \mu_\ast(f_i^{-1}([0, \infty])) \).

Then \( \sum_i \text{div}(f_i) = 0 \) implies that the 1-chain \( \sum_i \gamma_i \) is closed and in fact exact. Hence

\[
\sum_i \gamma_i = \partial(D)
\]
for some 2-chain \( D \). For \( \omega \) a closed 2-form whose cohomology class lies in \( F^1H^2(X, \mathbb{C}) \),

\[
\langle \text{reg}_{Z}(Z), \omega \rangle := \sum_i \int_{C_i} \log(f_i) \omega + 2\pi i \int_D \omega
\]  

(1)

For a decomposable element \((C, a)\) the regulator is particularly simple:

\[
\langle \text{reg}_{Z}((C, a)), \omega \rangle = \int_C \log(a) \omega = \log(a) \int_C \omega
\]

### 3.3 Extensions

As stated in the introduction, conjecturally, there is a canonical description of the motivic cohomology group as an extension in the category of mixed motives. In our case, if one has a suitable category of mixed motives over \( \mathbb{Q} \), \( \mathcal{M}_M\mathbb{Q} \), one expects for a surface \( X \) \cite{Sch00},

\[
H^3_M(X, \mathbb{Z}(2)) \simeq \text{Ext}_{\mathcal{M}_M\mathbb{Q}}(\mathbb{Z}(-2), h^2(X)).
\]

One knows that the Deligne cohomology can be considered as an extension in the category of integral mixed Hodge structures,

\[
H^3_D(X, \mathbb{Z}(2)) \simeq \text{Ext}_{\text{MHS}}(\mathbb{Z}(-2), H^2(X))
\]

The regulator map above then has a canonical description as the map induced by the realisation map from the category of mixed motives to the category of mixed Hodge structures,

\[
\text{Ext}_{\mathcal{M}_M\mathbb{Q}}(\mathbb{Z}(-2), h^2(X)) \overset{\text{reg}}{\longrightarrow} \text{Ext}_{\text{MHS}}(\mathbb{Z}(-2), H^2(X)).
\]

One can take a further realisation in the category of Real mixed Hodge structures \( R - \text{MHS} \) to get the real regulator map to Deligne cohomology with \( R \)-coefficients

\[
\text{reg}_R : H^3_M(X, \mathbb{Z}(2)) \longrightarrow H^3_D(X, \mathbb{R}(2)) = \text{Ext}_{R - \text{MHS}}(\mathbb{R}(-2), H^2(X)) \simeq H^2_D(X, \mathbb{R}(1)) \cap H^{1,1}(X)
\]

hence the real regulator of a cycle can be viewed as a current on \((1, 1)\)-forms.

### 3.4 The regulator to Real Deligne cohomology

If \( X \) is defined over \( \mathbb{Q} \), one has the action of the \textit{Frobenius at infinity}, \( F_\infty \) on \( X(\mathbb{C}) \) induced by complex conjugation. Let \( X_\mathbb{C} = X \times_\mathbb{Q} \mathbb{C} \). For a subring \( A \subset \mathbb{R} \), \( F_\infty \) acts on the Deligne cohomology with coefficients in \( A \). We define the \textit{real} Deligne cohomology group as

\[
H^3_D(X/\mathbb{R}, \mathbb{R}(2)) = H^3_D(X_\mathbb{C}, \mathbb{R}(2))^{F_\infty = 1}
\]
This has a description as an extension —

\[ H^3_D(X/R, \mathbb{R}(2)) = \text{Ext}_{R-MHS}(R(-2), H^2(X_C, \mathbb{R}))^{F_\infty = 1} \simeq \frac{H^2_B(X_C, \mathbb{R}(1))^{F_\infty = 1}}{F^2 H^2_{DR}(X/R)} \]

and the realisation gives the regulator map

\[ \text{reg}_R : H^3_M(X, \mathbb{Q}(2)) \to H^3_D(X/R, \mathbb{R}(2)) = \text{Ext}_{R-MHS}(R(-2), H^2(X_C, \mathbb{R}))^{F_\infty} \]

The Real Deligne cohomology has a natural \( \mathbb{Q} \)-structure induced by the Betti and de Rham cohomologies. The determinant of that \( \mathbb{Q} \)-structure determines an element \( c_{BdR} \in \mathbb{R}^*/\mathbb{Q}^* \). The Beilinson conjectures then assert

- The image of \( \text{reg}_R \) gives another \( \mathbb{Q} \) structure. The determinant of that \( \mathbb{Q} \) structure is an element \( c_{\text{reg}} \in \mathbb{R}^*/\mathbb{Q}^* \).

- If \( L(H^2(X), s) \) is the \( L \)-function of \( H^2(X) \), then the first non-zero term in the Taylor expansion at \( s = 1 \), \( L^*(H^2(X), 1) \) satisfies

\[ L^*(H^2(X), 1) \cdot c_{BdR} \sim_{\mathbb{Q}^*} c_{\text{reg}} \]

where \( \sim_{\mathbb{Q}^*} \) means that this is up to a non-zero rational number.

Needless to say Beilinson conjectures are far more general — relating special values of motivic \( L \)-functions coming from motives or varieties of arbitrary dimension to certain higher regulators — but in the interest of brevity we have only stated things in the case at hand.

### 3.5 Extensions of Mixed Hodge Structures and Motives.

As stated above, the regulators of motivic cohomology cycles give extensions of mixed Hodge structures. The key point of this paper is that, in some cases, one can also obtain extensions of mixed Hodge structures in other ways. For instance, it was shown by Hain [Hai87] that the group ring of the fundamental group \( \mathbb{Z}[\pi_1(X, P)] \) of a pointed algebraic variety, as well as the graded quotients \( J^a_P/J^b_P \), with \( a \leq b \), where \( J_P \) is the augmentation ideal, carry mixed Hodge structures. Hence natural exact sequences involving them lead to extensions of mixed Hodge structures.

Our aim is to first construct some natural motivic cohomology cycles in the case when \( X = C \times C \). Their regulators will give rise to extensions of mixed Hodge structures. We will show that there are natural extensions of mixed Hodge structures coming from the Hodge structure on \( \mathbb{Z}[\pi_1(C, P)] \) for some suitable point \( P \) which give the same extensions. In particular, since the constructions can be carried out in at the level of mixed motives, if we had a good category of mixed motives the cycle itself would be an extension in the conjectured category of mixed motives coming from the fundamental group.

There are various candidates for the category of mixed motives [Lev05] — Voevodsky and Huber have candidates for the triangulated category of mixed motives and Nori and Deligne-Jannsen have candidates
for the Abelian category itself. Cushman [Cus00] showed that Nori’s motives can be used to get a motivic structure on the group ring of the fundamental group — so one expects that the same sequences we use would give extensions of mixed motives in Nori’s category.

An alternative to Cushman’s way of constructing Nori motives for the fundamental group was suggested to us by N. Fakhruddin. Nori’s category requires a realisation in terms of relative cohomology groups. In the case of the fundamental groups this is given in the paper of Deligne and Goncharov [DG05] Section 3, (Proposition 3.4).

If $X$ is a smooth variety and $x_0$ a distinguished point, they show that the Hodge structure on the graded pieces of the group ring of the fundamental group can be realised as the Hodge structure on the relative cohomology groups of pairs $(X^s, \cup_{i=0}^s X_i)$, where

- $X^s = X \times \cdots \times X$ $s$-times
- $X_0$ is the sub-variety given by $t_1 = x_0$ — namely $x_0 \times X^{s-1}$
- $X_i$ is the sub-variety given by $t_i = t_{i+1}$ for $0 < i < s$ — namely $X^{i-1} \times \Delta \times X^{s-(i+1)}$, where $\Delta$ is the diagonal in $X \times X$ in the $i$th and $(i + 1)^{st}$ places.
- $X_s$ is given by $t_s = x_0$ — namely $X^{s-1} \times x_0$.

We have

$$H^s(X^s/\cup_{i=0}^s X_i, \mathbb{C}) \simeq \text{Hom}(J/J^{s+1}, \mathbb{C}) = H^0(\bar{B}_s(X), x_0)$$

For example, when $s = 1$ we have

$$H^1(X/\{x_0\}, \mathbb{C}) \simeq H^1(X, \mathbb{C}) \simeq \text{Hom}(J/J^2, \mathbb{C}).$$

Hence the motive underlying the Hodge structure on the group ring of the fundamental group $\mathbb{Z}[\pi_1(X, x_0)]$ is the motive associated to the pair $(X^s, \cup_{i=0}^s X_i)$. Namely, to this object one can associate a de Rham, étale and Betti realization which are isomorphic when the field of coefficients is large enough.

In the special case when $C$ is a modular curve $X_0(N)$, we will show that certain natural elements of the motivic cohomology group constructed by Beilinson can be thought of as extensions coming from the fundamental group. In particular, we construct the extension in the Nori category of mixed motives which corresponds to the Beilinson elements. Kings [Kin97] showed this in the case of $H^2_{\text{mot}}(X_0(N), \mathbb{Z}(2))$ for Huber’s motives.

### 4 A Motivic Cohomology Cycle on $C \times C$.

In this section we construct a motivic cohomology cycle on $C \times C$ first introduced by Bloch in the case of $X_0(37)$. The cycle is similar, in fact, generalises, the cycle constructed by Collino [Col97]. This section generalises the work of Colombo [Col02] on constructing the extension corresponding to the Collino cycle and hence many of the arguments are adapted from her paper.
4.1 The cycle $Z_{QR}$

Let $C$ be a smooth projective curve defined over a number field $K$. Let $Q$ and $R$ be two points on $C$ such that there is a function $f = f_{QR}$ with divisor
\[ \text{div}(f_{QR}) = NQ - NR \]
for some $N \in \mathbb{N}$. To determine the function precisely, we choose a distinct third point $P$ and assume $f_{QR}(P) = 1$.

There exists notable examples of curves where such functions can easily be found. For instance, modular curves with $Q$ and $R$ being cusps, Fermat curves with the two points being among the ‘trivial’ solutions of Fermat’s Last Theorem, namely the points with one of the coordinates being 0, and hyperelliptic curves with the two points being Weierstrass points.

Consider the cycle in $C \times C$ given by
\[ Z_{QR,P} = (C \times Q, 1/f^Q) + (\Delta_C, f^\Delta) + (R \times C, 1/f^R) \]
where $\Delta_C$ is the diagonal, $f^Q = f_{QR} \times Q$, $f^R = R \times f_{QR}$ and $f^\Delta = f_{QR}$ being considered as a function on $\Delta_C$. Since
\[ \text{div}_{C \times Q}(1/f^Q) + \text{div}_{\Delta_C}(f^\Delta) + \text{div}_{R \times C}(1/f^R) = N(R, Q) - N(Q, Q) + N(Q, Q) - N(R, R) + N(R, R) - N(R, Q) = 0 \]
the cycle $Z_{QR,P}$ gives an element of $H^3_{\mathcal{M}}(C \times C, \mathbb{Z}(2))$. From now on we suppress the $P$ and simply write $Z_{QR}$.

This cycle was first described by Bloch [Blo00] in his celebrated Irvine lecture notes and later variants of this construction were used by Beilinson and others to verify the Beilinson conjectures in some special cases. If $C$ is a hyper-elliptic curve and $f$ is a function supported on the Weierstrass points, so $\text{div}(f) = 2Q - 2R$, then under the natural map:
\[ C \times C \to J(C) \]
\[ (x, y) \mapsto (x - y) \]
this cycle maps to the Collino cycle [C97].

4.2 The Regulator of $Z_{QR}$

Let $Z_{QR}$ be the motivic cohomology cycle in $H^3_{\mathcal{M}}(C \times C, \mathbb{Z}(2))$. We now obtain a formula for its regulator. The motivic cohomology group of $C \times C$ is the same as that of $h^2(C \times C)$. From the Künneth theorem the motive $h^2(C \times C)$ decomposes
\[ h^2(C \times C) = h^2(C) \otimes h^0(C) \oplus h^1(C) \otimes h^1(C) \oplus h^0(C) \otimes h^2(C). \]
We will essentially be concerned with $h^1(C) \otimes h^1(C)$. The regulator is a current on forms in $F^1(H^1(C) \otimes H^1(C))$ — that is, forms of the type $\phi \otimes \psi$ where $\phi$ and $\psi$ are 1-forms on $C$ and $\delta$ is of type $(1,0)$.

Recall that $f = f_{QR}$ is a function on $C$ with divisor $NQ - NR$ for some $N$. Let $\omega = \phi \otimes \psi$ and $\gamma = f^{-1}([0, \infty])$ be a path on $C$ from $Q$ to $R$. As $f$ is of degree $N$, $\gamma$ is the union of $N$ paths - each lying on a different sheet with only the points $Q$ and $R$ in common. We will denote by $\gamma_i$, $1 \leq i \leq N$. Each $\gamma_i$ is a path from $Q$ to $R$. Let $\gamma_Q, \gamma_R$ and $\gamma_\Delta$ denote the curve $\gamma$ on $C \times Q$, $R \times C$ and $\Delta_C$ respectively and similarly for the components $\gamma_i$. Then from the co-cycle condition one has

$$\gamma_\Delta \cdot \gamma_R^- \cdot \gamma_Q^- = \partial(D)$$

where $D$ is a 2-chain on $C \times C$. Here for a path $\alpha$, $\alpha^-$ is the inverse: $\alpha^-(t) = \alpha(1-t)$.

From equation (1) one has

$$\int_{\Delta_C} \log(f^Q) - \int_{R \times C} \log(f^R) + 2\pi i \int_D \omega = \int_{\Delta_C} \log(f^Q) - \int_{R \times C} \log(f^R) + 2\pi i \int_D \omega.$$ 

Our aim is to find a more explicit expression for $\text{reg}_Z$. For this we need an explicit description of $D$.

**Lemma 4.1.** Let

$$a(s,t) = t \quad \text{and} \quad b(s,t) = \frac{t(1-s)}{1-s(1-t)}$$

Define $F_i : [0,1] \times [0,1] \rightarrow C \times C$ by

$$F_i(s,t) = (\gamma_i(a(s,t)), \gamma_i(b(s,t)))$$

for $1 \leq i \leq N$ and let

$$D_i = \text{Im}(F_i).$$

Then

$$\partial(D_i) = \gamma_\Delta^- \cdot \gamma_R^+ \cdot \gamma_Q^+$$

In particular, if $D = \bigcup_{i=1}^N D_i$ then

$$\partial(D) = \gamma_\Delta^- \cdot \gamma_R^+ \cdot \gamma_Q^+$$

**Proof.** The oriented boundary of $D_i$ is

$$\partial(D_i) = F_i(0,t) \cup F_i(s,1) \cup F_i(1,1-t) \cup F_i(1-s,0).$$

Restricting $F_i$ to the boundary shows —

- $F_i(0,t) = (\gamma_i(t), \gamma_i(t)) = \gamma_\Delta^+ (t)$ - a path from $(Q,Q)$ to $(R,R)$.
- $F_i(s,1) = (\gamma_i(1)), \gamma_i(1-s)) = (R, \gamma_i(1-s)) = \gamma_R^+ (s)$ - a path from $(R,R)$ to $(R,Q)$.
- $F_i(1,1-t) = (\gamma_i(1-t), \gamma_i(0)) = (\gamma_i(1-t), Q) = \gamma_Q^+ (t)$ - a path from $(R,Q)$ to $(Q,Q)$.
\( F_i((1-s),0) = (\gamma^i(0),\gamma^i(0)) = (Q,Q) \) – the constant path at \((Q,Q)\).

Hence the boundary of \( D_i \) is \( \gamma_{\Delta}^i \cdot \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i} \). If \( D \) is the union of the \( D_i \) then its boundary is the union of the boundaries of the \( D_i \).

We can compute the second integral as an iterated integral as follows.

**Lemma 4.2.** Let \( \phi \) and \( \psi \) be closed 1-forms on \( C \) with \( \phi \) holomorphic and let \( D_i \) be a disc as in the above lemma. Then

\[
\int_{D_i} \phi \otimes \psi = \int_{\gamma_{\Delta}^i \cdot \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \phi \psi = \int_{\gamma_{\Delta}^i} \phi \psi.
\]

**Proof.** Any closed 1-form on a disc is exact, hence \( \phi|_{D_i} = d\rho \) for some function \( \rho \). From Stokes theorem one has

\[
\int_{D_i} \phi \otimes \psi = \int_{D_i} d(\rho \psi) = \int_{\partial(D_i)} \rho \psi = \int_{\gamma_{\Delta}^i \cdot \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \rho \psi.
\]

Using Lemma 2.1 (3) and choosing \( \rho \) such that \( \rho(\gamma_{\Delta}^i \cdot \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}(0)) = \rho((Q,Q)) = 0 \) we have

\[
\int_{\gamma_{\Delta}^i \cdot \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} d\rho \psi = \int_{\gamma_{\Delta}^i \cdot \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \rho \psi - \rho(\gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}(0)) \int_{\gamma_{\Delta}^i \cdot \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \psi = \int_{\gamma_{\Delta}^i \cdot \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \rho \psi.
\]

Hence

\[
\int_{D_i} \phi \otimes \psi = \int_{\gamma_{\Delta}^i \cdot \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} d\rho \psi = \int_{\gamma_{\Delta}^i} \phi \psi.
\]

Using Lemma 2.1 (1) with \( \alpha = \gamma_{\Delta}^i \) and \( \beta = \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i} \) we get

\[
\int_{\gamma_{\Delta}^i \cdot \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} d\phi \psi = \int_{\gamma_{\Delta}^i} \phi \psi + \int_{\gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \phi \psi + \int_{\gamma_{\Delta}^i} \phi \int_{\gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \psi.
\]

Since \( \gamma_{\Delta}^i \cdot \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i} = \partial(D_i) \) is exact, one has

\[
\int_{\gamma_{\Delta}^i \cdot \gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \phi = \int_{\gamma_{\Delta}^i} \phi + \int_{\gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \phi = 0.
\]

Using Lemma 2.1 (2) we have

\[
\int_{\gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \phi \psi = \int_{\gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \phi \psi - \int_{\gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \phi \int_{\gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \psi.
\]

Combining this with the above expression gives

\[
\int_{D_i} \phi \otimes \psi = \int_{\gamma_{\Delta}^i} \phi \psi - \int_{\gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \psi \phi.
\]

However, the integral

\[
\int_{\gamma_{R}^{-i} \cdot \gamma_{Q}^{-i}} \psi \phi = 0
\]
as $\gamma_Q^-$ and $\gamma_R^-$ are supported on $C \times Q$ and $R \times C$ hence one of $\phi$ or $\psi$ restricted to them will be 0. So we are left with

$$\int_{D_i} \phi \otimes \psi = \int_{\gamma^\Delta} \phi \psi.$$ 

Combining the above lemma with the earlier expression for the regulator we have the following theorem

**Theorem 4.3.** Let $Z_{QR}$ be the motivic cohomology cycle in $H^3_{\text{Mot}}(C \times C; \mathbb{Z}(2))$ and $\phi$ and $\psi$ two 1-forms on $C$ with $\phi$ holomorphic. Let $\omega = \phi \otimes \psi$. Then

$$\langle \text{reg}_{Z}(Z_{QR}), \omega \rangle = \int_{C \times Q} \log(f^Q) \omega + \int_{\Delta_C} \log(f^\Delta) \omega - \int_{R \times C} \log(f^R) \omega + N\pi i \int_{\gamma^\Delta} \phi \psi.$$ 

Here $N$ is the degree of the map $f_{QR}: C \to \mathbb{P}^1$. In fact, since $\omega|_{C \times Q} = \omega|_{R \times C} = 0$, the expression simplifies to

$$\langle \text{reg}_{Z}(Z_{QR}), \omega \rangle = \int_{\Delta_C} \log(f^\Delta) \omega + N\pi i \int_{\gamma^\Delta} \phi \psi.$$ 

**Proof.** This is immediate from the earlier lemmas. The only thing to be remarked is that the $N$ appears because there are $N$ components $D_i$ and all of them contribute the same integral. 

5 The Fundamental group and Mixed Hodge Structures

From this section onwards, $H^1(C)$ will denote the integral cohomology group $H^1(C, \mathbb{Z})$ and $H^1(C)_Q$ will denote $H^1(C, \mathbb{Q})$. Let $C$ be a smooth projective curve and $P$, $Q$ and $R$ be three points on $C$. Consider the open curve $C_Q = C \backslash \{Q\}$. Let $\mathbb{Z}[\pi_1(C_Q, P)]$ be the group ring of the fundamental group of $C_Q$ based at $P$. Let $J_{Q,P} := J_{C,Q,P}$ denote the augmentation ideal —

$$J_{Q,P} := J_{C,Q,P} = \text{Ker}\{\mathbb{Z}[\pi_1(C_Q, P)] \rightarrow \mathbb{Z}\}$$

Let $H^0(B_r(C_Q; P))$ denote the $F$-vector space, where $F$ is $\mathbb{R}$ or $\mathbb{C}$, of homotopy invariant iterated integrals of length $\leq r$. Chen \cite{Che77} showed that

$$H^0(B_r(C_Q; P)) \simeq \text{Hom}_\mathbb{Z}(\mathbb{Z}[\pi_1(C_Q, P)]/J_{C,Q,P}^{r+1}, F)$$

under the map

$$I \mapsto I(\gamma) = \int_{\gamma} I$$

Using this Hain \cite{Hai87} was able to put a natural mixed Hodge structure on the graded pieces $J_{Q,P}/J_{Q,P}^r$. 

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5.1 The extension \( E^3_{Q,P} \)

One can consider the extensions of mixed Hodge structures

\[
E^r_{Q,P} : 0 \to (J_{Q,P}/J_{Q,P}^{r-1})^* \to (J_{Q,P}/J_{Q,P}^r)^* \to (J_{Q,P}^r/J_{Q,P}^{r+1})^* \to 0
\]

The simplest non-trivial case is when \( r = 3 \). In this case \((J_{Q,P}/J_{Q,P}^3)^* \simeq H^1(C_Q) \simeq H^1(C)\) and \((J_{Q,P}^3/J_{Q,P}^4)^* \simeq \otimes^2 H^1(C)\) and the exact sequence becomes

\[
E^3_{Q,P} : 0 \to H^1(C) \to (J_{Q,P}/J_{Q,P}^3)^* \to \otimes^2 H^1(C) \to 0
\]

Hence \( E^3_{Q,P} \) gives an element in \( \text{Ext}_{MHS}(\otimes^2 H^1(C), H^1(C)) \). A similar construction with \( R \) in the place of \( Q \) gives us the extension \( E^3_{R,P} \), which also lies in the same Ext group.

There are a few natural morphisms of mixed Hodge structures which allow us to pull back and push forward the extensions

- There is a surjection \( \cup : \otimes^2 H^1(C) \to H^2(C)(-1) \) coming from the cup product composed with Poincaré Duality. Let \( K \) be the kernel of this map. The exact sequence of Hodge structures
  \[
  0 \to K \to \otimes^2 H^1(C) \xrightarrow{\cup} Z(-1) \to 0
  \]
  splits rationally to give a map \( \beta : 2g_C Z(-1) \to \otimes^2 H^1(C) \) and
  \[
  \otimes^2 H^1(C) = K \oplus 2g_C Z(-1)
  \]
- Let \( \Omega \) denote the polarisation on \( C \). There is an injection obtained by tensoring with \( \Omega \)
  \[
  J_\Omega = \otimes \Omega : H^1(C)(-1) \to \otimes^3 H^1(C).
  \]

From the Künneth theorem one has

\[
\text{Ext}(\otimes^2 H^1(C), H^1(C)) = \text{Ext}(K \oplus Z(-1), H^1(C)) = \text{Ext}(K, H^1(C)) \times \text{Ext}(Z(-1), H^1(C))
\]

It is well known that \( \text{Ext}(Z(-1), H^1(C)) \simeq \text{Pic}(C) \).

From the work of Hain [Hai87], Pulte [Pul88], Kaenders [Kae01] and Rabi [Rab01] one has the following theorem

**Theorem 5.1.** The class \( E^3_{Q,P} \) in \( \text{Ext}(\otimes^2 H^1(C), H^1(C)) \) is described as follows –

\[
E^3_{Q,P} = (m_P, k_{QP})
\]

where \( m_P \in \text{Ext}(K, H^1(C)) \) depends only on \( P \) and \( k_{QP} \) is given by

\[
2g_C Q - 2P - \kappa_C \in \text{Pic}(C)
\]

where \( \kappa_C \) is the canonical divisor of \( C \) and \( g_C \) is the genus of \( C \).
We now return to our situation where $P$, $Q$ and $R$ are three points such that there is a function $f_{QR}$ with $\text{div}(f_{QR}) = NQ - NR$. Recall that in the group $\text{Ext}$, addition is given by the Baer sum. We will denote this by $\oplus_B$ (or $\ominus_B$ if we are taking differences). Let $E^3_{Q,R,P}$ denote the Baer difference $E^3_{Q,P} \ominus_B E^3_{R,P}$.

**Lemma 5.2.** Under the hypothesis that there is a function with divisor $\text{div}(f_{QR}) = NQ - NR$ the extension $N_{2gC}E^3_{Q,R,P}$ splits. That is

$$N_{2gC}E^3_{Q,R,P} \cong H^1(C) \oplus 2H^1(C)$$

**Proof.** This follows quite easily from Theorem 5.1. Since the class of $E^3_{Q,P} = (m_P, k_{QP})$ where $m_P$ does not depend on $Q$, the class of the difference is $E^3_{Q,R,P} = (m_P, k_{QP}) - (m_P, k_{RP}) = (0, k_{QP} - k_{RP})$.

The class

$$k_{QP} - k_{RP} = (2gCQ - 2P - \kappa C) - (2gCR - 2P - \kappa C) = 2gC(Q - R)$$

in $\text{Pic}(C)$. By hypothesis, this is torsion. Hence the extension whose class is given by the Baer difference splits when multiplied by $N_{2gC}$. 

**Remark 5.3.** This extension represents the class $Q - R$, at least up to a integral multiple, and is hence the first example of the theme of this paper - namely the Abel-Jacobi image of a null-homologous cycle is described in terms of extensions coming from the fundamental group.

A consequence of this is that there is a morphism of integral mixed Hodge structures

$$r_3 : \frac{N}{2gC}E^3_{Q,R,P} \longrightarrow H^1(C)$$

given by the projection.

### 5.2 The extensions $E^4_{Q,P}$ and $E^4_{R,P}$

From the work of Hain, Pulte, Harris and others one knows that the class $m_P$ in $\text{Ext}(K, H^1(C))$ corresponds to the extension of mixed Hodge structures determined by the Ceresa cycle in $J(C)$, or alternately, the modified diagonal cycle in $C^3$. We would like to construct a similar class corresponding to the motivic cohomology cycle $Z_{QR}$. To that end, we now consider, with $C, P, Q$ and $R$ as before, the extension corresponding to $r = 4$

$$E^4_{Q,P} : 0 \longrightarrow (J_{Q,P}/J^3_{Q,P})* \longrightarrow (J_{Q,P}/J^4_{Q,RP})* \longrightarrow (J^3_{Q,P}/J^4_{Q,RP})* \longrightarrow 0$$

We have that $(J^3_{Q,P}/J^4_{Q,P})* \cong \otimes^3 H^1(C)$ and this does not depend on $P, Q$ or $R$. However, from Theorem 5.1 $(J_{Q,P}/J^3_{Q,P})*$ depends on $Q$ and $P$ and similarly $(J_{R,P}/J^3_{R,P})*$ depends on $R$ and $P$.

When $C$ is hyperelliptic $E^3_{Q,P} \otimes Q$ and $E^3_{R,P} \otimes Q$ are split. Hence one gets two classes

$$E^4_{Q,P}, E^4_{R,P} \in \text{Ext}(\otimes^3 H^1(C)_Q, \otimes^2 H^1(C)_Q \oplus H^1(C)_Q)$$

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and one can project to get two classes \( e^4_{Q,P} \) and \( e^4_{R,P} \) in \( \text{Ext}(\otimes^3 H^1(C)_Q, H^1(C)_Q) \). Colombo \cite{Col02} shows that the class 

\[
e^4_{QR,P} = e^4_{Q,P} \oplus_B e^4_{R,P} \in \text{Ext}(\otimes^3 H^1(C)_Q, H^1(C)_Q)
\]

corresponds to the extension determined by the cycle \( Z_{QR} \) - after pulling back and pushing forward with some standard maps.

Unfortunately, in general the extensions \( E^3_{Q,P} \) and \( E^3_{R,P} \) are not split rationally. They correspond to the instances where the Ceresa cycle is non-torsion. The instances where this is known are precisely the cases we have in mind - modular curves and Fermat curves\cite{Har83}. Hence we cannot use this argument immediately. However, since we know from Lemma 5.2 that their difference \( E^3_{QR,P} \) is split rationally, we would like to get an extension of the form

\[
0 \rightarrow E^3_{QR,P} \otimes \mathbb{Q} \cong E^4_{QR,P} \cong \otimes^3 H^1(C)_Q \rightarrow 0
\]

where “\( E^4_{QR,P} \)” is a sort of generalised Baer difference of the two extensions \( E^4_{Q,P} \) and \( E^4_{R,P} \). We cannot simply consider \( E^4_{QR,P} = E^4_{Q,P} \oplus_B E^4_{R,P} \) as the two extensions lie in different Ext-groups. So we have to consider a generalisation of Baer sums to not necessarily exact sequences which we came across in a paper of Rabi \cite{Rab01}.

### 5.3 The Baer sum and a generalisation

Recall that if we have two exact sequences of modules

\[
E_j : 0 \rightarrow A \rightarrow B_j \rightarrow C \rightarrow 0
\]

for \( j \in \{1, 2\} \), the Baer difference \( E_1 \oplus_B E_2 \) is constructed as follows. We have

\[
0 \rightarrow A \oplus A \rightarrow B_1 \oplus B_2 \rightarrow C \oplus C \rightarrow 0
\]

Let \( \psi : B_1 \oplus B_2 \rightarrow C \) be the map

\[
\psi(b_1, b_2) = p_1(b_1) - p_2(b_2)
\]

and let \( H = \text{Ker}(\psi) = \{(b_1, b_2) | p_1(b_1) = p_2(b_2)\} \). Let \( D \) be the image of \( \tilde{f} : A \rightarrow A \oplus A \rightarrow H \)

\[
\tilde{f}(a) = (f_1(a), f_2(a))
\]

Let \( B = H/D \). The map \( f : A \oplus A \rightarrow B \) given by

\[
f(a_1, a_2) = (f_1(a_1), f_2(a_2))
\]

factors through \( (A \oplus A)/A \cong A \) and so one has a map \( \tilde{f} : A \rightarrow B \),

\[
a \rightarrow (f_1(a), 0) = (0, -f_2(a))
\]
and a sequence

\[ 0 \longrightarrow A \xrightarrow{\mathbf{j}} B \xrightarrow{p_1 (\text{or } p_2)} C \longrightarrow 0 \]

The class of \( B \) in \( \text{Ext}(C, A) \) is the Baer difference \( E_1 \ominus E_2 \). The Baer sum \( E_1 \oplus E_2 \) is the sequence obtained when one of the maps \( f_2 \) or \( p_2 \) is replaced by its negative. If the modules and morphisms have additional structure — for instance, if we are working in the category of mixed Hodge structures — then the Baer sum is also in the same category. The Baer sum is essentially the push-out over \( A \) in the category of modules.

5.3.1 A generalisation

Now suppose we have two exact sequences, for \( j = \{1, 2\} \),

\[ E_j : 0 \longrightarrow B_1^j \xrightarrow{f_j} B_2^j \xrightarrow{p_j} B_3 \longrightarrow 0 \]

and a diagram of the following type:

\[
\begin{array}{ccccccc}
0 & \downarrow & A_1 & \xrightarrow{i_j} & 0 \\
& & \downarrow & & \downarrow \\
0 & \xrightarrow{f_j} & B_2^j & \xrightarrow{p_j} & B_3 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \\
& & C_1 & \longrightarrow & 0 \\
\end{array}
\]

where the vertical and horizontal sequences are exact for all values of \( j \). We would like to take the Baer difference of the \( E_j \) — but since they do not lie in the same Ext group we cannot quite do that. However, using the vertical exact sequence, we can still salvage something out of it.

We have \( B_1^j \in \text{Ext}(C_1, A_1) \) hence we can form their Baer difference. Let \( B_1 = B_1^1 \ominus_B B_1^2 \). Define \( B_2 \) as follows: Let \( H_2 = \text{Ker}(\psi) \), where \( \psi \) is the ‘difference’ map

\[ \psi : B_2^1 \oplus B_2^2 \longrightarrow B_3 \]

\[ \psi((b_1^1, b_2^2)) = (p_1(b_1^1) - p_2(b_2^2)) \]

Let \( D_2 \) be the image of the map

\[ A_1 \longrightarrow B_1^1 \oplus B_1^2 \longrightarrow H_2 \]

\[ a \longrightarrow (f_1(i_1(a)), f_2(i_2(a))) \]
Define $B_2 = H_2/D_2$. We call this the *generalised* Baer difference of $B_2^1$ and $B_2^2$. Observe that this is almost the Baer difference of $B_2^1$ and $B_2^2$ in the sense that if $B_1 = B_1^1 = B_1^2$, then we could take the difference in $\text{Ext}(B_3, B_1)$. Since that is not the case, we do the best we can - we take the difference of the *inexact* sequences

$$0 \rightarrow A_1 \rightarrow B_2^j \rightarrow B_3 \rightarrow 0.$$  

As a result of this one has a sequence

$$0 \rightarrow B_1 \xrightarrow{f_1 \oplus f_2} B_2 \xrightarrow{p_1 (\text{or } p_2)} B_3 \rightarrow 0.$$  

However, this sequence is *not* exact — $\text{Ker}(p_1)$ is larger than $(f_1 \oplus f_2)(B_1)$. The next lemma describes this difference.

**Lemma 5.4** (Rabi [Rab01]). Let $F = B_2/B_1$. Then one has the following diagram, in which the horizontal and vertical sequences are exact.

$$\begin{array}{ccccccc}
0 & \rightarrow & B_1 & \xrightarrow{f} & B_2 & \xrightarrow{\eta} & F & \rightarrow & 0 \\
& & \downarrow \phi & & \downarrow \bar{\psi} & & \\
& & C_1 & & B_3 & & \\
& \downarrow \phi & & \downarrow & & \downarrow & & 0 \\
\end{array}$$

**Proof.** The horizontal sequence is exact by construction. To show the vertical sequence is exact we have to first describe the map $\phi$. It is defined as follows. One has maps $\pi_j : B_2^j \rightarrow C_1$. Consider the map

$$\phi : C_1 \oplus C_1 \xrightarrow{(\pi_1^{-1}, \pi_2^{-1})} B_2^1 \oplus B_2^2 \xrightarrow{(f_1, f_2)} H_2$$

$$\phi(c_1, c_2) = (f_1(\pi_1^{-1}(c_1)), f_2(\pi_2^{-1}(c_2)))$$

This is well defined modulo the image of $A_1$, the kernel of $\pi_j$ — which is $D_2$. Also, it maps to the kernel of $\psi$. Hence gives a map to $B_2 = H_2/D_2$. Further, the image of the diagonal $\Delta C_1 = \{(c, -c) | c \in C_1\}$ is 0, so this map factors through $(C_1 \oplus C_1)/\Delta C_1 \simeq C_1$. The pre-image of $\Delta C_1$ is the Baer difference $B_1$ — hence the map $\phi$ maps to $F$.  

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As a result we get a sequence

\[ 0 \to C_1 \xrightarrow{\phi} F \xrightarrow{\bar{p}} B_3 \to 0 \]

This map is exact as if \( b = (b_1^1, b_2^2) \) is in \( F \) and \( \bar{p}(b) = 0 \), then \( p_1(b_1^1) = p_2(b_2^2) = 0 \). So \( b_1^1 \) and \( b_2^2 \) lie in the image of \( B_1^1 \oplus B_1^2 \) — say \( b_1^1 = f_1(b_1^1) \) and \( b_2^2 = f_2(b_2^2) \). Let \( c_1 = \pi_1(b_1^1) \) and \( c_2 = \pi_2(b_2^2) \). Then

\[ b = \phi(c_1, c_2) \]

so it lies in the image of \( \phi : C_1 = (C_1 \oplus C_1) / \Delta C_1 \).

As a result of this lemma, we get an extension class \( f_{12} \in \text{Ext}(B_3, C_1) \) corresponding to \( F \). This measures, in a sense, the obstruction to having a exact sequence involving \( B_1, B_2 \) and \( B_3 \).

One can also get extension classes \( e_1 \) and \( e_2 \) in \( \text{Ext}(B_3, C_1) \) by pushing forward the extensions \( E_j \) under the maps \( \pi_j \). From the construction of the map \( \phi \), one has the following corollary of the above lemma.

**Corollary 5.5.** Let \( e_1, e_2 \) and \( f_{12} \) be the extensions in \( \text{Ext}(B_3, C_1) \) described above. Then

\[ f_{12} = e_1 \otimes_B e_2 \]

In the next section we apply these constructions in our particular case to get the extension class we want.

### 5.4 The extension \( e_{QR,P}^4 \)

In this section we construct an extension \( e_{QR,P}^4 \) in \( \text{Ext}(\otimes^2 H^1(C), H^1(C)) \) which generalises the element \( e_{QR,P}^4 \otimes_B e_{R,P}^4 \) constructed by Colombo. Recall that we have an exact sequence

\[ E_{Q,P}^3 : 0 \to H^1(C) \to (J_{Q,P} / J_{Q,P}^3)^* \to \otimes^2 H^1(C) \to 0 \]

and a similar one for \( E_{R,P}^3 \). Also, we have the sequence

\[ E_{Q,P}^2 : (J_{Q,P} / J_{Q,P}^2)^* \to (J_{Q,P} / J_{Q,RP}^4)^* \to (J_{Q,P} / J_{Q,P}^3)^* \to 0 \]
and a similar one for $E^4_{R,P}$. This gives us a diagram as in Lemma 5.4 with $B^1_1 = (J_{Q,P}/J^3_{Q,P})^*$, $B^2_1 = (J_{R,P}/J^3_{R,P})^*$, $B^1_2 = (J_{Q,P}/J^4_{Q,P})^*$ and $B^2_2 = (J_{R,P}/J^4_{R,P})^*$.

\[
\begin{array}{cccccccc}
0 & \rightarrow & (J_{Q,P}/J^3_{Q,P})^* & \rightarrow & (J_{Q,P}/J^4_{Q,P})^* & \rightarrow & \otimes^3 H^1(C) & \rightarrow & 0 \\
& \downarrow & f_j & \downarrow & p_j & \downarrow & i_j & \downarrow & 0 \\
& H^1(C) \rightarrow & \otimes^2 H^1(C) \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\end{array}
\]

Let $E^4_{Q,R,P}$ denote the generalised Baer difference as in Lemma 5.4 and $F_{QR} = E^4_{Q,R,P}/E^3_{Q,R,P}$. We get an exact sequence

\[
\begin{array}{cccccccc}
0 & \rightarrow & E^3_{Q,R,P} & \rightarrow & E^4_{Q,R,P} & \rightarrow & F_{QR} & \rightarrow & 0 \\
& \downarrow & f & \downarrow & \eta & \downarrow & \phi & \downarrow & \phi \\
& \otimes^2 H^1(C) \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\end{array}
\]

We know from Lemma 5.2 that the extension $\frac{N}{2g_C}E^3_{Q,R,P}$ splits – so

\[
\frac{N}{2g_C}E^3_{Q,R,P} \simeq \otimes^3 H^1(C) \oplus H^1(C)
\]

Let $e^{23}_{Q,P}$ and $e^{23}_{R,P}$ denote the extensions in $\text{Ext}(\otimes^3 H^1(C), \otimes^2 H^1(C))$ determined by pushing forward the extensions $E^4_{Q,P}$ and $E^4_{R,P}$ respectively. From Corollary 5.3 we have that

\[
F_{QR} = e^{23}_{Q,P} \oplus_B e^{23}_{R,P}
\]

From Rabi [Rab01], Corollary 3.3, the extension $e^{23}_{Q,P}$ is

\[
e^{23}_{Q,P} = E^3_{Q,P} \otimes H^1(C) \oplus_B H^1(C) \otimes E^3_{Q,P}
\]

and similarly for $e^{23}_{R,P}$. Hence their difference $F_{QR}$ splits rationally as well! Precisely,

\[
\frac{N}{2g_C}F_{QR} \simeq \otimes^3 H^1(C) \oplus \otimes^2 H^1(C)
\]
Hence from Lemma [5.4] we get
\[
\frac{N}{2g_C} E_{QR,P}^4 \in \text{Ext}(\otimes^3 H^1(C) \oplus \otimes^2 H^1(C), \otimes^3 H^1(C) \oplus H^1(C))
\]

From the Künneth theorem,
\[
\text{Ext}(\otimes^3 H^1(C) \oplus \otimes^2 H^1(C), \otimes^3 H^1(C) \oplus H^1(C)) = \prod_{i \in \{2,3\}, j \in \{1,3\}} \text{Ext}(\otimes^i H^1(C), \otimes^j H^1(C))
\]

Define
\[
e^4_{QR,P} \in \text{Ext}(\otimes^3 H^1(C), H^1(C))
\]
to be the projection on to that component. Note that if \( C \) is hyperelliptic, this class \( e^4_{QR,P} \) is precisely the class \( e^4_{QR,P} = e^4_{Q,P} \otimes_B e^4_{R,P} \) constructed by Colombo.

### 5.5 Statement of the main theorem

Armed with the class \( e^4_{QR,P} \in \text{Ext}(\otimes^3 H^1(C), H^1(C)) \) we can proceed as in Colombo. We first pull back the class using the map \( J_\Omega \) to get a class in
\[
J_\Omega^*(e^4_{QR,P}) \in \text{Ext}(H^1(C)(-1), H^1(C)).
\]

Tensoring with \( H^1(C) \) we get a class in
\[
J_\Omega^*(e^4_{QR,P}) \otimes H^1(C) \in \text{Ext}(\otimes^2 H^1(C)(-1), \otimes^2 H^1(C)).
\]

Once again pulling back using the map \( \beta : 2g_C \mathbb{Z}(-1) \to \otimes^2 H^1(C) \) gives us a class in
\[
e^4_{QR,P} \in \text{Ext}(\mathbb{Z}(-2), \otimes^2 H^1(C)) \subset \text{Ext}(\mathbb{Z}(-2), H^2(C \times C))
\]

Our main theorem is

**Theorem 5.6.** Let \( C \) be a smooth projective curve and \( Q \) and \( R \) be two points such that there is a function \( f_{QR} \) with \( \text{div}(f_{QR}) = NQ - NR \) for some \( N \). Let \( P \) be a third point on \( C \). Normalize \( f_{QR} \) so that \( f_{QR}(P) = 1 \). Let \( Z_{QR} = Z_{QR,P} \) be the element of the motivic cohomology group \( H^3_M(C \times C, \mathbb{Z}(2)) \) constructed above. Let \( e^4_{QR,P} \) be the extension in \( \text{Ext}_{MHS}(\mathbb{Z}(-2), \otimes^2 H^1(C)) \) constructed above. Then
\[
e^4_{QR,P} = (2g_C + 1) \text{reg}_Q(Z_{QR})
\]
in \( \text{Ext}_{MHS}(\mathbb{Z}(-2), \otimes^2 H^1(C)) \).

In other words our theorem states that the regulator of a natural cycle in the motivic cohomology group of a product of curves, being thought of as an extension class is the same as that as a natural extension of MHS coming from the fundamental group of the curve. In fact, it is an extension of pure Hodge structures.

**Remark 5.7.** The dependence on \( P \) is not so serious. If we do not normalise \( f_{QR} \) with the condition that \( f_{QR}(P) = 1 \) then one has to add an expression of the form \( \log(f_{QR}(P)) f_C \cdot \) to the term — and this corresponds to adding a decomposable element of the form \( (\Delta_C, \log(f_{QR}(P))) \) to our element \( Z_{QR} \).
5.6 Carlson’s representatives

The proof of the above theorem will follow by showing that they induce the same current. For that we have to understand the how the extension class induces a current. This comes from understanding the Carlson representative. In the section we once again follow Colombo [Col02] and adapt her arguments to our situation.

If $V$ is a MHS all of whose weights are negative, then the Intermediate Jacobian of $V$ is defined to be

$$J(V) = \frac{V_C}{T_0 V_C \oplus V_Z}$$

This is a generalised torus - namely a group of the form $C^a/\mathbb{Z}^b \simeq (\mathbb{C}^*)^b \times (\mathbb{C})^{a-b}$ for some $a$ and $b$.

An extension of mixed Hodge structures

$$0 \rightarrow A \xrightarrow{i} H \xrightarrow{\pi} B \rightarrow 0$$

is called separated if the lowest non-zero weight of $B$ is greater than the largest non-zero weight of $A$. This implies that $\text{Hom}(B, A)$ has negative weights. Carlson [Car80] showed that

$$\text{Ext}_{\text{MHS}}(B, A) \simeq J(\text{Hom}(B_C, A_C))$$

This is defined as follows. As as extension of Abelian groups, the extension splits. So one has a map $r_Z : H \rightarrow A$ which is a retraction — namely $r_Z \circ i = id$. Let $s_F$ be a section in $\text{Hom}(B_C, H_C)$ preserving the Hodge filtration. Then the Carlson representative of an extension is defined to be the class of

$$r_Z \circ s_F \in J(\text{Hom}(B_C, A_C))$$

5.7 The Carlson representative of $\epsilon^4_{QR}$

We now describe the explicitly the Carlson representative of the extension $\epsilon^4_{QR}$ constructed in the previous section. This is done in three steps, first for $\epsilon^4_{QR,P}$ and then for its various pullbacks and push forwards to obtain that for $\epsilon^4_{QR}$. We first describe the Carlson representative of the extension

$$\epsilon^4_{QR,P} \in \text{Ext}_{\text{MHS}}(\otimes^3 H^1(C), H^1(C)).$$

Let $P, Q, R$ be as above. Fix a set of loops $\alpha_1, \alpha_2, \ldots, \alpha_{2g}$ based at $P$ in $C_{Q,R} = C \setminus \{Q, R\}$ such that they give a symplectic basis for $H^1(C)$ - so the intersection matrix is of the form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Let $dx_i$ be the dual basis of this basis. We may assume that the 1-forms $dx_i$ are harmonic. Let

$$c(i) = \begin{cases} 1 & \text{if } i \leq g_C \\ -1 & \text{if } i > g_C \end{cases}$$

and let

$$\sigma(i) = i + c(i)g_C$$

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so one has
\[ \int_{\alpha} dx_j = c(i) \delta_j \sigma(i) \]
where \( \delta_{ij} \) is the Konecker \( \delta \) function.

From the above description, we have that the Carlson representative is given by
\[ s_F \circ r_Z \circ p_1 \]
where \( p_1 \) is the projection of \( \frac{N}{2g_C} E^3_{QR,P} \rightarrow H^1(C) \).

To describe \( s_F \) we need a little more. Let \( \ominus \hat{\mathcal{B}} \) be the generalised Baer difference. Let
\[ s_F : \otimes^3 H^1(C) \rightarrow \frac{N}{2g_C} E^4_{QR,P} \simeq \frac{N}{2g_C} ((J_{Q,P}/J_{Q,P}^4)^* \ominus \hat{\mathcal{B}} (J_{R,P}/J_{R,P}^4)^*) \]
be the section preserving the Hodge filtration given by
\[ s_F(dx_i \otimes dx_j \otimes dx_k) = \left( I^{ij}_{Q}, I^{ij}_{R} \right) \]
Here \( I^{ij}_{\bullet} \in (J_{\bullet,P}/J_{\bullet,P}^4)^* \) for \( \bullet \in \{Q, R\} \) are two iterated integrals with
\[ I^{ij}_{\bullet} = \frac{N}{2g_C} \left( \int dx_i dx_j dx_k + dx_i \mu_{jk,\bullet} + \mu_{ij,\bullet} dx_k + \mu_{ijk,\bullet} \right) \quad (3) \]
where \( \mu_{ij,\bullet}, \mu_{jk,\bullet} \) and \( \mu_{ijk,\bullet} \) are smooth, logarithmic \((1, 0)\) forms on \( C_\bullet \) such that

- \( d\mu_{jk,\bullet} + dx_j \wedge dx_k = 0 \)
- \( d\mu_{ij,\bullet} + dx_i \wedge dx_j = 0 \)
- \( dx_i \wedge \mu_{jk,\bullet} - \mu_{ij,\bullet} \wedge dx_k + d\mu_{ijk,\bullet} = 0 \).

To compute the element of \( \text{Hom}(\otimes^3 H^1(C)_\bullet, H^1(C)_\bullet) \) obtained as the projection under \( p_1 \), we describe it as an element of \( H_1(C)_\bullet \simeq \text{Hom}(H_1(C)_Q, C) \). The map from
\[ H^1(C) \rightarrow (H^1(C) \oplus H^1(C))/\Delta_{H^1(C)} \]
is given by
\[ x \rightarrow (x, -x) \]
Further, if \( \alpha \) is a smooth loop based at \( P \), the class in \( H_1(C) \) corresponding to it is \( \alpha - 1 \). So one has \( s_F \circ r_Z \circ p_1 \in \text{Hom}(\otimes^3 H^1(C)_\bullet, H^1(C)_\bullet) \)
\[ s_F \circ r_Z \circ p_1(dx_i \otimes dx_j \otimes dx_k)(\alpha) = \int_{\alpha-1} I_Q - \int_{\alpha-1} I_R \]
\[ = \frac{N}{2g_C} \left( \int_{\alpha-1} dx_i (\mu_{jk,Q} - \mu_{jk,R}) + (\mu_{ij,Q} - \mu_{ij,R}) dx_k + (\mu_{ijk,Q} - \mu_{ijk,R}) \right) \]

**Remark 5.8.** We can choose the logarithmic forms \( \mu_{ij,\bullet} \) and \( \mu_{ijk,\bullet} \), for \( \bullet \in \{Q, R\} \), satisfying the following
From (3) one has

\[ \mu_{ij,\bullet} = -\mu_{ji,\bullet}. \]

For \( |i - j| \neq g_C \), \( \mu_{ij,\bullet} \) is smooth on \( C \), as \( dx_i \wedge dx_j = 0 \). As \( H^2(C_{Q,R},\mathbb{Z}) = 0 \) and \( \mu_{ij,\bullet} \) is smooth, it is orthogonal to all closed forms, that is, \( \mu_{ij,\bullet} \wedge dx_k = 0 \). If \( dx_i \) is harmonic, then \( \mu_{ij,\bullet} \wedge dx_k = 0 \).

\[ \mu_{i\sigma(i),\bullet} \] has a logarithmic singularity at \( \bullet \) with residue \( c(i) \).

\[ \mu_{ij,Q} - \mu_{ij,R} = 0 \] if \( |i - j| \neq g_C \).

\[ \mu_{i\sigma(i),Q} - \mu_{i\sigma(i),R} = \frac{c(i)}{N} d\log(f), \] where \( f = f_{QR} \) is a function such that \( \text{div}(f) = NQ - NR \). We can normalise \( f_{QR} \) once again by requiring that \( f_{QR}(P) = 1 \).

In terms of the basis of harmonic forms of \( H^1(C), \Omega \in \otimes^2 H^1(C) \) is expressed as

\[ \Omega = \sum_{i=1}^{2g_C} dx_i \otimes dx_{(i+g_C)} - dx_{(i+g_C)} \otimes dx_i = \sum_{i=1}^{2g_C} c(i) dx_i \otimes dx_{\sigma(i)} \]

and under the map

\[ \otimes^2 H^1(C) \xrightarrow{\mu} H^2(C) \cong \mathbb{Z}(-1) \]

\[ \sum_{i=1}^{2g_C} c(i) dx_i \cup dx_{\sigma(i)} = 2g_C \cdot \omega_C \]

where \( \omega_C \) is the generator of \( H^2(C) \cong \mathbb{Z}(-1) \). With the choices of \( \mu_{ij,\bullet} \) and \( \mu_{ijk,\bullet} \) as above, we have the following theorem

**Theorem 5.9.** Let \( G_{QR,P} \in \text{Hom}(H^1(C)(-1)_C, H^1(C)_C) \) be the Carlson representative corresponding to the extension class \( J^*_\Omega(c^4_{QR,P}) \). It is given by

\[ G_{QR,P}(dx_k \otimes \Omega)(\alpha_j) = \int_{a_j} \frac{(2g_C + 1)}{2g_C} \log(f) dx_k + \frac{N}{2g_C} \int_{a_j} W(dx_k) \]

in \( J(\text{Hom}(H^1(C)_Q(-1), H^1(C)_Q)) \), where

\[ W(dx_k) = \sum_{i=1}^{2g_C} c(i)(\mu_{k\sigma(i),Q} - \mu_{k\sigma(i),R}) \]

**Proof.** Let \( S_F \) denote the map \( S_F = s_F \circ J_\Omega : H^1(C)(-1) \rightarrow c^4_{QR,P} \). This is given by

\[ S_F(dx_k \otimes \Omega) = \sum_{i=1}^{2g_C} c(i) s_F(dx_k \otimes dx_i \otimes dx_{\sigma(i)}) \]

From this one has

\[ S_F(dx_k \otimes \Omega) = \left( \sum_{i=1}^{2g_C} c(i) \int_{Q} F^{\kappa_{\sigma(i)}}_{Q} \right) \]

\[ \sum_{i=1}^{2g_C} c(i) \int_{R} F^{\kappa_{\sigma(i)}}_{R} \]

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Evaluating on a path \( \alpha_j \) based at \( P \) using the maps described above, this is

\[
\sum_{i=1}^{2g_C} \int_{\alpha_j} c(i) \left( I_Q^{k_i \sigma(i)} - I_R^{k_i \sigma(i)} \right)
\]

From Remark 5.8, the leading terms and several of the lower order terms cancel out and

\[
\mu_{k_i,Q} - \mu_{k_i,R} = -c(i) \delta_{k_i \sigma(i)} d \log(f)/N
\]

and finally

\[
\mu_{i \sigma(i),Q} - \mu_{i \sigma(i),R} = c(i) d \log(f)/N
\]

so what remains is

\[
\frac{N}{2g_C} \left( \sum_{i=1}^{2g_C} \int_{\alpha_j} \frac{dx_k}{N} \frac{df}{f} - \int_{\alpha_j} \frac{df}{f} \int_{\alpha_j} c(i) \int_{\alpha_j} \left( \mu_{k_i \sigma(i),Q} - \mu_{k_i \sigma(i),R} \right) \right)
\]

Let

\[
W(dx_k) = \sum_{i=1}^{2g_C} c(i) \left( \mu_{k_i \sigma(i),Q} - \mu_{k_i \sigma(i),R} \right)
\]

Recall that \( \gamma = f^{-1}([0, \infty]) \). On \( C \setminus \gamma \), \( d \log(f) \) is exact. So if \( \alpha_j \cap \gamma = \emptyset \) then we can evaluate the integral using Lemma 2.1(3). If \( \alpha_j \cap \gamma \neq \emptyset \), one has to do the computation on a path lifting of \( \alpha_j \) on a covering of \( C \) where \( d \log(f) \) is exact. The difference in the two integrals is given by a multiple of \( 2\pi i \int_{\alpha_j} dx_k \) – hence is in \( \text{Hom}_\mathbb{Z}(H^1(C)(-1), H^1(C)) \) – which is \( 0 \) in the intermediate Jacobian.

Hence we have, using Lemma 2.1(3) and the fact that we have chosen \( f \) with \( f(P) = 1 \),

\[
\int_{\alpha_j} \frac{dx_k}{N} \frac{df}{f} = \frac{1}{N} \int_{\alpha_j} \log(f) dx_k
\]

and

\[
\int_{\alpha_j} \frac{df}{f} \int_{\alpha_j} c(i) \int_{\alpha_j} \left( \mu_{k_i \sigma(i),Q} - \mu_{k_i \sigma(i),R} \right)
\]

So the integral is

\[
\frac{2g_C + 1}{N} \int_{\alpha_j} \log(f) dx_k + \int_{\alpha_j} W(dx_k)
\]

Multiplying this by the factor \( \frac{N}{2g_C} \) gives us the final result.

\[\square\]

**Remark 5.10.** It is convenient to have the iterated integral expression for the Carlson representative as well so we note it here

\[
G_{QR,P}(dx_k \otimes \Omega)(\alpha_j) = \frac{2g_C + 1}{2g_C} \int_{\alpha_j} \frac{df}{f} dx_k + \frac{N}{2g_C} \int_{\alpha_j} W(dx_k)
\]

We have computed the Carlson representative of our class in \( \text{Ext}(H^1(C)(-1), H^1(C)) \). We now tensor with \( H^1(C) \) and pull back using the map \( \beta : 2g_C \mathbb{Z}(-1) \rightarrow \otimes^2 H^1(C) \). This gives us an element of \( \text{Ext}(2g_C \mathbb{Z}(-2), \otimes^2 H^1(C)) \).
Lemma 5.11. The Carlson representative of the class in $\text{Ext}(\mathbb{Z}(-2), \otimes^2 H^1(C))$ is given by

$$F_{QR,P} = (G_{QR,P} \times \text{Id}) \circ \beta$$

in $(\otimes^2 H^1(C))^*$. On an element $\alpha_j \otimes \alpha_k$, since $\Omega = \beta(1)$, it is given by

$$F_{QR,P}(\Omega)(\alpha_j \otimes \alpha_k) = \int_{\alpha_j} \frac{2gc + 1}{N} \log(f)dx_k + \int_{\alpha_j} W(dx_k)$$

Proof. Recall that

$$\Omega = \sum_i c(i)dx_i \otimes dx_{\sigma(i)}$$

From above we have

$$(G_{QR,P} \times \text{Id})(\Omega)(\alpha_j \otimes \alpha_k) = \sum_i c(i)G_{QR,P}(dx_i \otimes \Omega)(\alpha_j) \cdot \text{Id}(dx_{\sigma(i)})(\alpha_k)$$

From the choice of $\alpha_k$ one has

$$\text{Id}(dx_{\sigma(i)})(\alpha_k) = \int_{\alpha_k} dx_{\sigma(i)} = c(k)\delta_{ki}$$

Hence, in the sum above, precisely one term survives – when $i = k$, and we have

$$(G_{QR,P} \times \text{Id})(\Omega)(\alpha_j \otimes \alpha_k) = c(k)^2G_{QR,P}(dx_k \otimes \Omega)(\alpha_j)$$

Since $c(k)^2 = 1$, we get

$$F_{QR,P}(\Omega)(\alpha_j \otimes \alpha_k) = G_{QR,P}(dx_k \otimes \Omega)(\alpha_j) =$$

$$= \int_{\alpha_j} \frac{2gc + 1}{2gc} \log(f)dx_k + \frac{N}{2gc} \int_{\alpha_j} W(dx_k)$$

\[ \square \]

We now recall a lemma due to Colombo which relates integrals over the curve $C - \gamma$ with integrals over paths. This is crucial in relating the two expressions for the regulator.

Lemma 5.12 (Colombo, Prop. 3.3). Let $\gamma$ be the curve $f^{-1}([0, \infty])$. Let $\alpha$ be a smooth, simple loop on $C$ transverse to $\gamma$. Let $\phi$, $\psi$ and $\omega$ be three smooth 1-forms on $C$ such that $\phi$, $\psi$ and $\Theta = (\log(f)\psi + \omega)$ are closed and $\phi$ is the Poincaré dual of the class of $\alpha$. Then

$$\int_{\alpha} \Theta = \int_{C - \gamma} \phi \wedge \Theta + 2\pi i \int_{\gamma} \phi \psi$$

Proof. We first recall the following explicit construction of a differential form $\eta = \eta_{\alpha}$ which is the Poincaré dual of $\alpha$ as in [FK91, II, Section 3.3]. Let $\Omega$ be a tubular neighbourhood of $\alpha$ obtained by covering $\alpha$ by a finite number of co-ordinate discs. We can assume $\Omega$ is an annulus and $\Omega - \alpha$ is the union of two annuli $\Omega^+ \cup \Omega^-$. Assume $\alpha$ is oriented so that $\Omega^-$ is to the left.
Let $\Omega_0$ and correspondingly $\Omega_0^+$ and $\Omega_0^-$ be sub-annuli of the $\Omega$. We can find a $C^\infty$-function $F$ on $C - \alpha$ such that

$$F(z) = \begin{cases} 1 & \text{if } z \in \Omega_0^- \\ 0 & \text{if } z \in C - \Omega^- \end{cases}$$

Let $\eta = \eta_\alpha$ be defined as follows

$$\eta = \begin{cases} dF & \text{on } \Omega - \alpha \\ 0 & \text{on } (C - \Omega) \cup \alpha \end{cases}$$

The $\eta$ is a smooth, closed, differential form with compact support in $\Omega$ which is the Poincaré dual of $\alpha$.

Now $\phi$ is also a closed form dual to $\alpha$. Hence

$$\phi = \eta + dg$$

with $dg$ exact. So we can break the integral in to two parts –

$$\int_{C - \gamma} \phi \wedge \Theta = \int_{C - \gamma} (\eta + dg) \wedge \Theta = \int_{C - \gamma} \eta \wedge \Theta + \int_{C - \gamma} dg \wedge \Theta.$$

We first tackle the second term. One has

$$dg \wedge \Theta = dg \wedge (\log(f) \psi + \omega) = d(g \log(f) \psi + \omega)$$

which is exact. So one can evaluate that integral by Stokes theorem applied to the Riemann surface with boundary, $C - \gamma$. The boundary $\partial(C - \gamma)$ consists of two copies of $\gamma$, $\gamma_1$ and $\gamma_2$ on which the log differs by $2\pi i$, so one has

$$\log(\gamma_1(t)) - \log(\gamma_2(t)) = 2\pi i$$

Conventionally, one orients the boundary of $\partial(P^1 - [0, \infty])$, which consists of two copies of the line $[0, \infty]$ in such a manner that the plane is always to the right. With that orientation and its induced orientation on $\gamma$ one has

$$\partial(C - \gamma) = -\gamma_1 \cup \gamma_2$$

Hence, by Stokes Theorem,

$$\int_{C - \gamma} dg \wedge (\log(f) \psi + \omega) = \int_{-\gamma_1} g(\log(f) \psi + \omega) + \int_{\gamma_2} g(\log(f) \psi + \omega) = -2\pi i \int_{\gamma} g \psi$$

(4)

Let $P$ be the base point of $\alpha$. We can choose $g$ such that $g(P) = 0$ hence, using Lemma 2.4, the above expression can be rewritten as

$$\int_{\gamma} g \psi = \int_{\gamma} (g - g(P)) \psi = \int_{\gamma} dg \psi$$

We now deal with the other part, namely $\int_{C - \gamma} \eta \wedge \Theta$. Recall that $\eta$ is the Poincaré dual of $\alpha$. First suppose $\alpha \cap \gamma = \emptyset$. Then we can choose $\Omega$ above such that $\Omega \cap \gamma = \emptyset$. So $\Theta$ is then a closed form well defined one the support of $\eta$. Then by Poincaré duality one has

$$\int_{C - \gamma} \eta \wedge \Theta = \int_{\alpha} \Theta = \int_{\alpha} (\log(f) \psi + \omega)$$

(5)
Further, since $\phi = \eta + dg$ and $\eta|_\gamma = 0$ one has $\phi|_\gamma = dg|_\gamma$. Hence adding equations $\textcircled{4}$ and $\textcircled{5}$ we get the result.

If $\alpha \cap \gamma \neq \emptyset$ then $\log(f)$ is no longer well defined on $\Omega$. Hence we have to compute the integral on the disjoint union of regions which make up $\Omega' = \Omega - \gamma$. Since $\eta$ is supported on $\Omega$ one has

$$\int_{C-\gamma} \eta \wedge \Theta = \int_{\Omega'} \eta \wedge \Theta$$

Since $\eta = dF$ on $\Omega - \alpha$ and 0 elsewhere, we have

$$\partial(\Omega' - \alpha) = \alpha \cup (-\gamma_1 \cap \Omega^-) \cup (\gamma_2 \cap \Omega^-)$$

So from Stokes theorem, the integral becomes

$$\int_{\Omega' - \alpha} dF \wedge \Theta = \int_{\alpha} \Theta + \int_{\gamma \cap \Omega^-} \Theta.$$  

As in Equation $\textcircled{4}$ above,

$$\int_{\gamma \cap \Omega^-} \Theta = \int_{\gamma \cap \Omega^-} (\log(f) \psi + \omega) = -2\pi i \int_{\gamma \cap \Omega^-} dF \psi$$

Since $\eta = 0$ outside $\Omega^-$ and $\eta|_\gamma = dF|_\gamma$ and $\phi|_\gamma = \eta|_\gamma + dg|_\gamma$ one gets

$$\int_{\alpha} \Theta = \int_{C-\gamma} \phi \wedge \Theta + 2\pi i \int_{\gamma} \phi \psi$$

We now apply this in the case of interest to us.

**Corollary 5.13.** Choose $\alpha_j$ to be simple closed loops transverse to $\gamma$. Then we have

$$F_{QR,P}(\Omega)(\alpha_j \otimes \alpha_k) = (2gC + 1)c(j))(\int_{C-\gamma} dx_{\sigma(j)} \wedge \left( \log(f) dx_k - \frac{N}{(2gC + 1)} W(dx_k) \right))$$

$$+ 2\pi i \int_{\gamma} dx_{\sigma(j)} dx_k$$

**Proof.** One has $c(j) dx_{\sigma(j)}$ is dual to $\alpha_j$. Hence we can apply the above lemma with

- $\phi = c(j) dx_{\sigma(j)}$
- $\psi = dx_k$
- $\Theta = \log(f) dx_k - \frac{N}{(2gC + 1)} W(dx_k)$

Note that $\Theta$ is closed because

$$d\Theta = d\log(f) \wedge dx_k - \frac{N}{(2gC + 1)} dW(dx_k)$$
and
\[ dW(dx_k) = \sum_{i=1}^{2g_C} c(i) d \left( \mu_{ki\sigma(i),Q} - \mu_{ki\sigma(i),R} \right) \]
Recall that
\[ d_{\mu_{ij},Q} = \mu_{ij,Q} \wedge dx_k - dx_i \wedge \mu_{jk,Q} \]
So the sum becomes
\[ dW(dx_k) = \sum_{i=1}^{2g_C} c(i) \left( (\mu_{ki,Q} \wedge dx_{\sigma(i)} - dx_k \wedge \mu_{i\sigma(i),Q}) - (\mu_{ki,R} \wedge dx_{\sigma(i)} - dx_k \wedge \mu_{i\sigma(i),R}) \right) \]
\[ = \sum_{i=1}^{2g_C} c(i) \left( (\mu_{ki,Q} - \mu_{ki,R}) \wedge dx_{\sigma(i)} - dx_k \wedge (\mu_{i\sigma(i),Q} - \mu_{i\sigma(i),R}) \right) \]
\[ = c(\sigma(k))(\mu_{k\sigma(k),Q} - \mu_{k\sigma(k),R}) \wedge dx_k - \sum_{i=1}^{2g_C} c(i) \left( \frac{c(i)}{N} dx_k \wedge d \left( \log(f) \right) \right) \]
\[ = c(\sigma(k)) \left( \frac{c(k)}{N} d \log(f) \right) \wedge dx_k - \sum_{i=1}^{2g_C} c(i) \left( \frac{c(i)}{N} d \log(f) \right) \]
\[ = \frac{(2g_C + 1)}{N} d \log(f) \wedge dx_k \]
Hence it cancels out and we have \( d\Theta = 0 \). Applying the proposition we have
\[ \int_{\alpha_j} \log(f) dx_k - \frac{N}{2g_C + 1} W(dx_k) = \]
\[ = \int_{\gamma} c(j) dx_{\sigma(j)} \wedge \left( \log(f) dx_k - \frac{N}{2g_C + 1} W(dx_k) \right) + 2\pi i \int_{\gamma} c(j) dx_{\sigma(j)} dx_k \]
Hence
\[ F_{QR,P}(\Omega)(\alpha_j \otimes \alpha_k) = (2g_C + 1)c(j) \left( \int_{\gamma} dx_{\sigma(j)} \wedge \left( \log(f) dx_k - \frac{N}{2g_C + 1} W(dx_k) \right) + 2\pi i \int_{\gamma} dx_{\sigma(j)} dx_k \right) \]
\[ \square \]
\( F_{QR,P}(\Omega) \) determines an element of the intermediate Jacobian of \((\otimes^2 H^1(C))^*\)
\[ J(\otimes^2 H^1(C))^* \approx \frac{F^1(\otimes^2 H^1(C,\mathbb{C}))^*}{(\otimes^2 H^1(C,\mathbb{Z}))^*} \]
so to determine \( F_{QR,P}(\Omega) \) it suffices to evaluate it on elements of \( F^1(\otimes^2 H^1(C,\mathbb{C}))^* \), namely linear combinations of forms of the type \( \zeta_i \otimes \alpha_j \) and \( \alpha_i \otimes \zeta_j \), where \( \{\zeta_i\}_{i=1}^g \) is a basis for the Poincare duals of the holomorphic 1-forms, \( H^{1,0}(C) \). Let \( dz_j \) denote the dual of \( \zeta_j \). We can choose the basis \( \{\zeta_i\} \) such that it satisfies
\[ \int_{\alpha_i} dz_j = \delta_{ij} \quad 1 \leq i \leq g \]
where \( \{\alpha_i\} \) is the symplectic basis. Since \( c(j) dx_{\sigma(j)} \) is dual to \( \alpha_j \),
\[ dz_j = dx_{j+g} + \sum_{i=1}^g A_{ji} dx_i \quad \text{where } A_{ji} = \int_{\alpha_i} dz_j \]
Proposition 5.14 (Colombo, [Col02], Prop 3.4). The map $F$ evaluated on elements of the form $\zeta_i \otimes \alpha_j$ is

$$ F_{QR,P}(\Omega)(\zeta_i \otimes \alpha_j) = (2g_C + 1) \left( \int_{C-\gamma} \log(f)dz_i \wedge dx_j + 2\pi i \int_{\gamma} dz_i dx_j \right) $$

In other words

$$ dz_i \wedge W(dx_j) = 0. $$

Proof. $dz_i$ and $W(dx_j)$ are both $(1, 0)$ forms. Hence their wedge product is a $(2, 0)$ form and is therefore 0. 

In fact, the theorem holds for the other term as well.

Proposition 5.15. For a suitable choice of $\mu_{ijk,Q}$ and $\mu_{ijk,R}$ one has

$$ W(dz_i) := W(dx_{j(g)}) + \sum_{i=1}^{g} A_{ji} W(dx_i) = 0 $$

Proof. This is essentially the same as Colombo [Col02] Lemma 3.1.

Hence we have

Theorem 5.16.

$$ F_{QR,P}(\Omega)(\alpha_j \otimes \zeta_i) = (2g_C + 1) \left( \int_{C-\gamma} \log(f)dx_j \wedge dz_i + 2\pi i \int_{\gamma} dx_j dz_i \right) $$

Comparing this with the regulator term in Theorem 4.3 we get

Theorem 5.17. Let $Z_{QR}$ be the motivic cohomology cycle constructed above and $\epsilon_{QRP}^1$ the extension in $\text{Ext}_{MHS}(\mathbb{Q}(2), \otimes^2 H^1(C))$. We used $\epsilon_{QRP}^1$ to denote its Carlson representative as well. Then one has

$$ \langle \epsilon_{QRP}^1, \omega \rangle = (2g_C + 1)(\text{reg}_\mathbb{Q}(Z_{QR}), \omega) $$

Proof. The Carlson representative is given by $F_{QR,P}$ and the result follows from comparing the two expressions. 

Recall that we have assumed in both cases that $f_{QR}(P) = 1$. If we do not do that, then one has a term corresponding to the decomposable element $(\Delta_C, \log(f_{QR}(P)))$ that one has to account for. However, if we work modulo the decomposable cycles we can ignore that term.

We can also consider the regulator to the Real Deligne cohomology $H^3_D(C \times C/R, \mathbb{R}(2))$ which is the same as the group $\text{Ext}_{\mathbb{R}-\text{MHS}}(\mathbb{R}(2), \otimes^2 H^1(C))$ in the category of $\mathbb{R}$-mixed Hodge structures with the action of the infinite Frobenius. We can take the realisation of the extension $\epsilon_{QRP}^1$ in that extension group and one has

Theorem 5.18. The real regulator is

$$ \text{reg}_\mathbb{R}(Z_{QR}) = (2g_C + 1)\epsilon_{QRP}^1 $$

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and on a (1, 1) form $dz_i \otimes d\bar{z}_j$ it is

$$\langle \text{reg}_R(Z_{QR}), dz_i \otimes d\bar{z}_j \rangle = \int_{\bar{\zeta}_j} \log |f^\Delta| dz_i$$

where $\bar{\zeta}$ is the Poincaré dual of $d\bar{z}_j$.

### 5.8 A generalization

We will apply this calculation to compute the regulator in a slightly more general situation, which is particularly relevant to our applications to modular curves and Fermat curves. Let $C$ be a curve and $S \subset C$ a finite subset of points on $C$ such that any divisor of degree 0 supported on $S$ is torsion. Examples of such sets include the set of points $\{Q, R\}$ above, cusps on a modular curve and the $3N$ ‘trivial solutions’ on the Fermat curve $F_N : X^N + Y^N = Z^N$.

Suppose $f$ is a function whose divisor is supported on $S$,

$$\text{div}(f) = \sum_{P \in S} a_P P$$

then we can construct a motivic cohomology cycle $Z_f$ as follows. Define a simple function to be a function $f_{QR}$ with

$$\text{div}(f_{QR}) = NQ - NR$$

with $Q$ and $R$ in $S$. Using the fact that any divisor of degree 0 on $S$ is torsion, one can decompose $f$ into a product

$$f^k = \prod f_{QR}$$

for some simple functions $f_{QR}$ and natural numbers $k$. This is far from unique. Then one can easily check that the cycle

$$Z_f = k(\Delta_C, f) - \left( \sum (Q \times C, f_{QR} \times C) + (C \times R, C \times f_{QR}) \right)$$

where $\Delta_C$ is the diagonal, is an element of $H^3_{\text{MHS}}(C \times C, \mathbb{Q}(2))$. One can see that

$$Z_f - \sum_{Q,R} Z_{QR} = 0 \in H^3_{\text{MHS}}(C \times C, \mathbb{Q}(2))$$

Hence one has

**Theorem 5.19.** Let $Z_f$ be the motivic cohomology class corresponding to a function $f$ as above. Then there is an extension class

$$\epsilon^4_f \in \text{Ext}_{\text{MHS}}(\mathbb{Q}(-2), \otimes^2 H^1(C))$$

which corresponds to the regulator of $Z_f$. This class is given by

$$\epsilon^4_f = \sum \epsilon^4_{QR}$$

While this theorem is immediate from the earlier considerations, it will be useful in the next section on modular curves.
Modular curves and the Beilinson conjectures

One of the few cases where the Beilinson conjectures are known is that of $H^2_{\text{mot}}(X, \mathbb{Q}(2))$ where $X$ is a modular curve. To prove it he first decomposed the motive of the modular curve into motives of modular forms of weight 2. The $L$-function also decomposes. He then constructed an element of $H^2_{\text{mot}}(X, \mathbb{Q}(2))$ and showed that the regulator of the projection on to the various modular form components of this element was related to the special value of the $L$-function. In [Kin97], Kings constructs an extension in the category of mixed motives defined by Huber which corresponds to the element constructed by Beilinson, in the sense that the realisation of Kings’ extension, which is an extension in the category of R-mixed Hodge structures with an $F_\infty$-action, is the same as the regulator to the real Deligne cohomology of Beilinson’s element.

The other case which Beilinson proved in his seminal paper [Be˘ı84] was that of $H^3_{\text{mot}}(X \times X, \mathbb{Q}(2))$. In this section we show that here too one can construct an extension in the category of mixed Hodge structures coming from the fundamental group which corresponds to Beilinson’s element. As remarked earlier, this can likely be made to work at the level of Nori’s mixed motives itself using the description of Deligne-Goncharov.

6.1 An explicit description of Beilinson’s element

We give an explicit description of Beilinson’s element. This is done in [BS04]. Recall that a modular unit is a function on a modular curve with its divisor supported on the cusps. For $N$ a square free integer, let $X_0(N)$ denote the modular curve with level $N$ structure. Since $N$ is square free, the set of cusps is represented by points $P_d = [d^{-1}]$ for $d|N$.

Let $\Delta(z)$ denote the Ramanujan $\Delta$-function, the classical modular form of weight 12 for $SL_2(\mathbb{Z})$. For $N$ a square free integer let

$$
\Delta_N(z) := \prod_{d|N} \Delta \left( \frac{Nz}{d} \right)^{\mu(d)}
$$

Since $\sum_{d|N} \mu(d) = 0$, this is a modular function, in fact, it is a modular unit. One has [BS04],

$$
div(\Delta_N) = \prod_{p|N} (p - 1) \cdot \sum_{d|N} \mu(N/d) P_d
$$

Define a simple unit to be a modular unit $f_{PQ}$ such that

$$
div(f_{PQ}) = kP - kQ
$$

for some cusps $P$ and $Q$. One has, by the Manin-Drinfeld Theorem, that $\Delta_N^\kappa$ is a product of simple units. More precisely, we have [BS04], Theorem 3.1,

**Theorem 6.1.** Let

$$
N = \prod_{i=0}^r p_i \quad \kappa = \prod_{i=1}^r (p_i + 1)
$$

Then one has

$$
\Delta_N^\kappa = \prod_{d|N/p_0} F_d
$$
where \( F_d \) is the simple unit with divisor

\[
\text{div}(F_d) = \Lambda_d(P_d - P_{dp_0})
\]

where

\[
\Lambda_d = (p_0 - 1)\mu(N/d) \prod_{i=1}^{r} (p_i^2 - 1)
\]

From the above discussion we have an element \( Z_{\Delta_N} \) in \( H^3_{\text{et}}(X_0(N) \times X_0(N), \mathbb{Q}(2)) \) and the extension class is given by

\[
\epsilon^4_{\Delta_N} = \sum_{d|N/p_0} \epsilon^4_{F_d}
\]

### 6.2 The Real Regulator

Beilinson’s conjecture, or theorem, in this case, relates the element \( Z_{\Delta_N} \) with the special value of the \( L \)-function of \( H^2(X_0(N) \times X_0(N)) \) at \( s = 1 \). The theorem shows that the regulator evaluated on the \((1, 1)\) form \( \omega_{f,g} = f(z)\overline{g}(z)dzd\overline{z} \) is related to the \( L \)-function of the motive \( L(M_f \otimes M_g, 1) \).

\[
\langle \text{reg}_{\mathbb{R}}(Z_{\Delta_N}), \omega_{f,g} \rangle \sim_{\mathbb{Q}} L(M_f \otimes M_g, 1)
\]

Beilinson proves the theorem by relating the two sides using the integral formula for the Rankin-Selberg convolution and the Kronecker Limit formula. We use the above expression to get an iterated integral formula for the regulator.

From the Carlson representative for the extension class we have that

\[
\langle \text{reg}_{\mathbb{R}}(Z_{\Delta_N}), \omega_{f,g} \rangle = F_{\Delta_N, \rho}(\Omega)(\omega^*_{f,g})
\]

Combining this with the earlier calculations we have the following iterated integral representation of the regulator.

\[
\langle \text{reg}_{\mathbb{R}}(Z_{\Delta_N}), \omega_{f,g} \rangle = \int_{\gamma_{\overline{g}}} \log |\Delta_N(z)|f(z)dz = \int_{\gamma_g} f(z_1)E_N(z_2)dz_1dz_2
\]

where \( \gamma_{\overline{g}} \) is the Poincaré dual of \( \omega_{\overline{g}} \). This follows from the fact that

\[
d\log |\Delta_0(N)| = E_N(z)dz
\]

where \( E_N(z) \) is a holomorphic Eisenstein Series of weight 2 for \( \Gamma_0(N) \).

### 6.3 Remarks on Degenerations

Collino shows that his cycle can be viewed as a ‘degeneration’ of the Ceresa cycle. We expect that the Bloch-Beilinson cycle too can be viewed as a suitable degeneration of the modified diagonal cycle. In a recent preprint [IM14], Iyer and Müller-Stach have worked out a special case of this. In a subsequent paper, we hope to show this in general and derive some additional consequences.
A curious special case is the case of modular curves. Here, if one looks at the regulators - the regulator of the modified diagonal cycle can be expressed as an iterated integral of two cusp forms over the dual of a third. As you degenerate, the regulator of the Beilinson cycle is an iterated integral of a cusp form and an Eisenstein series over the dual of a cusp form. Degenerating further, one has that the regulator of some elements of $K_2$ of a modular curve can be expressed as the iterated integral of two Eisenstein series over the dual of a cusp form. Finally, one expects that there should be an expression for the special value of the $\zeta$-function of the field of definition of a cusp corresponding to $K_3$ as an integral of two Eisenstein series over the dual of a third Eisenstein series.

References

[Bei84] A. A. Beilinson. Higher regulators and values of $L$-functions. In *Current problems in mathematics, Vol. 24*, Itogi Nauki i Tekhniki, pages 181–238. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.

[Blo00] Spencer J. Bloch. *Higher regulators, algebraic $K$-theory, and zeta functions of elliptic curves*, volume 11 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2000.

[BS04] Srinath Baba and Ramesh Sreekantan. An analogue of circular units for products of elliptic curves. *Proc. Edinb. Math. Soc. (2)*, 47(1):35–51, 2004.

[Car80] James A. Carlson. Extensions of mixed Hodge structures. In *Journées de Géométrie Algébrique d’Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 107–127. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[Che77] Kuo Tsai Chen. Iterated path integrals. *Bull. Amer. Math. Soc.*, 83(5):831–879, 1977.

[Col97] A. Collino. Griffiths’ infinitesimal invariant and higher $K$-theory on hyperelliptic Jacobians. *J. Algebraic Geom.*, 6(3):393–415, 1997.

[Col02] Elisabetta Colombo. The mixed Hodge structure on the fundamental group of hyperelliptic curves and higher cycles. *J. Algebraic Geom.*, 11(4):761–790, 2002.

[Cus00] Matthew Wayne Cushman. *The motivic fundamental group*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–The University of Chicago.

[DG05] Pierre Deligne and Alexander B. Goncharov. Groupes fondamentaux motiviques de Tate mixte. *Ann. Sci. École Norm. Sup. (4)*, 38(1):1–56, 2005.

[FK91] Gerhard Frey and Ernst Kani. Curves of genus 2 covering elliptic curves and an arithmetical application. In *Arithmetic algebraic geometry (Texel, 1989)*, volume 89 of *Progr. Math.*, pages 153–176. Birkhäuser Boston, Boston, MA, 1991.
[Hai87] Richard M. Hain. The geometry of the mixed Hodge structure on the fundamental group. In Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), volume 46 of Proc. Sympos. Pure Math., pages 247–282. Amer. Math. Soc., Providence, RI, 1987.

[Har83] Bruno Harris. Homological versus algebraic equivalence in a Jacobian. Proc. Nat. Acad. Sci. U.S.A., 80(4 i.):1157–1158, 1983.

[IM14] J. N. Iyer and S. Müller-Stach. Degeneration of the modified diagonal cycle. ArXiv e-prints, January 2014.

[Kae01] Rainer H. Kaenders. The mixed Hodge structure on the fundamental group of a punctured Riemann surface. Proc. Amer. Math. Soc., 129(5):1271–1281, 2001.

[Kin97] Guido Kings. Extensions of motives of modular forms. Math. Ann., 309(3):375–399, 1997.

[Lev05] Marc Levine. Mixed motives. In Handbook of K-theory. Vol. 1, 2, pages 429–521. Springer, Berlin, 2005.

[Ots11] Noriyuki Otsubo. On the regulator of Fermat motives and generalized hypergeometric functions. J. Reine Angew. Math., 660:27–82, 2011.

[Ots12] Noriyuki Otsubo. On the Abel-Jacobi maps of Fermat Jacobians. Math. Z., 270(1-2):423–444, 2012.

[Pul88] Michael J. Pulte. The fundamental group of a Riemann surface: mixed Hodge structures and algebraic cycles. Duke Math. J., 57(3):721–760, 1988.

[Rab01] Reuben Rabi. Some variants of the logarithm. Manuscripta Math., 105(4):425–469, 2001.

[Sch00] Anthony J. Scholl. Integral elements in K-theory and products of modular curves. In The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), volume 548 of NATO Sci. Ser. C Math. Phys. Sci., pages 467–489. Kluwer Acad. Publ., Dordrecht, 2000.