EMERGENCE IN CUCKER-SMALE DYNAMICAL SYSTEMS ON THE CIRCLE

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Abstract. This note studies large scale emergence in systems of collective behavior with short-range communication. We prove unconditional alignment for Cucker-Smale dynamics in the periodic one-dimensional environment. The result holds both for the discrete and hydrodynamic systems, for either smooth or singular communication kernels. Two new methods are presented – one based on a construction of a corrector to the energy balance which compensates for the missing long-range interactions, and another based on a dynamical approach.

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1. Introduction

We study long time behavior of Cucker-Smale type systems that govern evolution of a flock driven by laws of self-organization:

\[
\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} \phi(|x_i - x_j|)(v_j - v_i), \\
(x_i, v_i) &\in \Omega_d \times \mathbb{R}^d,
\end{align*}
\]

where $\Omega^d$ is an environment domain of the flock. The system was introduced by Cucker and Smale in [3] to demonstrate a basic mechanism of alignment in Galilean invariant settings, i.e. all $v_i \to \bar{v}$, where the limiting velocity is the total momentum $\bar{v} = \frac{1}{N} \sum_{i=1}^{N} v_i$. Such a result, accompanied by an exponential convergence rate, was achieved on $\mathbb{R}^d$ for sufficiently strong long-range communication: $\int_0^\infty \phi(r) \, dr = \infty$, see the original [3, 4], and more refined approach by Ha, Liu [7], and Tadmor, Tan [22]. Much of the current research, however, has been driven by demand to place the system into more realistic setting – local communication protocol,

\[
\phi(r) \geq \lambda I_{r<\rho_0},
\]

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for some possibly small \( r_0 > 0 \). Emergence of large scale patterns from such short-range interactions is a common phenomenon in many biological, social, and technological systems. We refer to these surveys and literature therein [23, 12, 17].

Obtaining alignment in short-range setting on the open environment \( \Omega^d = \mathbb{R}^d \) requires additional assumptions, such as graph-connectivity of the flock [9, 12] or strong local communication [10] (\( \lambda \gg 1 \)). Otherwise, it is easy to arrange for a configuration that never flocks: two separated groups of agents at a distance \( > r_0 \) launched in the opposite direction. Under more restrictive environmental settings, such as torus \( \Omega^d = \mathbb{T}^d \), the dynamics is recursive and so, heuristically, one would expect to obtain more generic alignment results for local kernels. This was recently demonstrated for a large class of so-called topological kernels in [18] in continuous no-vacuum settings. These are local symmetric non-convolution type kernels with adaptive diffusion that assist faster communication into regions with thinner density. In general, however, when no structural information about communication is given other than (2), and no connectivity is assumed, the question remains largely open. In multi-D case, again, solutions do not always align. This happens, for instance, when two agents move parallel to each other with different velocities, or two agents looping around the torus on perpendicular orbits with relatively rational velocities. All such examples represent measure-zero likelihood, and might be ruled out in various probabilistic settings. In this note, however, we will provide a definitive answer in one-dimensional case:

On \( \mathbb{T} \) under only local communication (2) the alignment occurs unconditionally.

The result will be established both for the discrete system (1) and its macroscopic counterpart, the Euler-alignment system, given by

\[
\begin{align*}
\begin{cases}
\partial_t \rho + (v \rho)_x &= 0 \\
\partial_t v + v \cdot \nabla v &= \int_\mathbb{T} \phi(x-y) (v(y) - v(x)) \rho(y) \, dy.
\end{cases}
\end{align*}
\]

See [8, 6, 11] for derivation in various settings, and [1, 5, 20, 21, 19, 22] for well-posedness results.

As to regularity assumption on the kernel, we consider two general classes: smooth kernels \( \phi \in C^1(\mathbb{R}^+) \), or singular kernels with non-integrable singularity:

\[
\int_0^1 \phi(r) \, dr = \infty,
\]

(limited information is known for lighter singularities, see [15, 13, 16]). For the smooth kernels the discrete system (1) is clearly well-posed, while (3) requires a threshold condition on the conserved quantity \( e = v_x + \phi \ast \rho \), namely, \( e_0 \geq 0 \). This avoids the typical Burgers shocks, see [1]. In the singular case, significance of condition (4) lies in the fact that such communication prevents collisions between agents and thus the discrete system (1) is well-posed even though the right hand side is not Lipschitz. This issue has received extensive global treatment in works of Peszek et al [14, 15, 2], and will be revisited in this note as an integral part of our approach. A more special class of singular kernels satisfying (4) and (2) is described by a power law communication:

\[
\int_{r_0}^{r_0} \frac{\lambda}{r^{\beta}} \leq \phi(r) \leq \frac{\Lambda}{r^{\beta}}, \quad 1 \leq \beta.
\]

Here the hydrodynamic system takes form of a fractional parabolic type, provided \( 1 < \beta < 3 \). An exhaustive global well-posedness theory in this case was developed in works [5, 20, 21, 19].

To summarize, in a variety of circumstances global smooth solutions to both systems, (1) and (3), exist. So, in what follows, we will not discuss existence results, rather focus on establishing alignment for a given classical solution.
To state our main results let us introduce the quadratic variation functional

$$\mathcal{V}_2(t) = \frac{1}{N^2} \sum_{i,j=1}^{N} |v_i(t) - v_j(t)|^2,$$

and its hydrodynamic analogue:

$$\mathcal{V}_2(t) = \int_{\mathbb{T}^2} |v(x,t) - v(y,t)|^2 \rho(x,t) \rho(y,t) \, dx \, dy. \quad (6)$$

In the discrete case we provide alignment result with \(N\)-dependent but faster-in-time rate.

**Theorem 1.1** (Alignment with \(N\)-dependent rate). Consider the system (1) with a local communication protocol (2) whose kernel is either smooth or satisfies (4). For any solution to such system we have

$$\mathcal{V}_2(t) \leq \frac{CN}{t}, \quad t \to \infty,$$

where \(C\) depends on the initial condition only.

Unfortunately, the dependence on \(N\) makes it unsuitable for the mean-field limit \(N \to \infty\). The next result establishes alignment with an \(N\)-independent bound but a slightly slower rate.

**Theorem 1.2** (Alignment with \(N\)-independent rate). Consider the system (1) whose kernel is either smooth or satisfies (4). The following alignment holds for any solution:

(i) For smooth or sub-quadratic kernels satisfying

$$\lambda I_{r < r_0} \leq \phi(r) \leq \frac{\Lambda}{r^2} \quad (7)$$

one has

$$\mathcal{V}_2(t) \leq C \frac{\ln t}{t}, \quad t \to \infty,$$

where \(C\) depends only on the initial condition.

(ii) If the kernel satisfies more singular assumption (5) for \(\beta > 2\), then

$$\mathcal{V}_2(t) \to 0, \quad t \to \infty,$$

with a rate independent of \(N\).

On the way of proving Theorem 1.2 we revisit the non-collision result from [2] and first, generalize it to the singularity condition (4), and second improve exponential lower bound on minimal distance to an algebraic one for the case \(\beta > 2\), namely, \(|x_i(t) - x_j(t)| \gtrsim t^{-\beta/2}\), see Section 2.1.

**Remark 1.3.** It is important to note that the lower bound on the kernel (2) in the assumptions of Theorem 1.1, Theorem 1.2 (i) can be replaced with a lower bound over any subinterval of the torus:

$$\phi(r) \geq \lambda I_I, \quad (8)$$

where \(I \subset (-\pi, \pi)\). In the case of continuous kernel this simply means that the kernel is not completely trivial, \(\phi(x_0) > 0\) for some \(x_0 \in \mathbb{T}\).

Kernels of type (8) define systems with heterophilious communication studied extensively in [12]. Numerics presented in that study shows that stronger communication in the region \(r_0 < r < r_1\) enhances alignment in first order swarming models. Our analysis does not distinguish between this situation and the local one (2), however it is conceivable that a more refined study may reveal such distinction.
With rates being independent of $N$ we can prove a direct analogue of Theorem 1.2 (and Remark 1.3) for the hydrodynamic system (3) in terms of continuous $L^2$-variation (6), see Theorem 3.1. The alignment, however, is expected to hold in the uniform metric as well since it is natural to the maximum principle. Our last result claims such $L^\infty$-based alignment in the case of smooth kernels. To state it let us denote by $M = \int_T \rho(x,t) \, dx$ the total conserved mass of the flock and note that the momentum $\int_T v \rho \, dx$ is also conserved. This identifies the projected limiting velocity to be the average

$$\bar{v} = \frac{1}{M} \int_T v \rho \, dx.$$ 

**Theorem 1.4.** Consider the system (3) on $\mathbb{T}$ with smooth kernel $\phi \geq 0$ which is non-trivial at least at one point $\phi(x_0) > 0$. Then any global classical solution aligns:

$$\sup_x |v(x,t) - \bar{v}| \leq C \left( \frac{\ln t}{t} \right)^{\frac{1}{5}}.$$ 

Let us note that technically the theorem works even if $\phi = 0$, although in this case it is trivial – the only global smooth solution to Burgers on $\mathbb{T}$ is a constant one. Positivity of $\phi$ makes it possible to have global solutions with $e_0 \geq 0$ as discussed earlier, although in the course of the proof we are not using the $e$-quantity directly. Connectivity of the flock is not assumed either, i.e. the solution may have vacuum initially.

The approaches to prove $L^2$-based results versus $L^\infty$-based ones are quite different. In the $L^2$ settings we construct a corrector which complements the usual energy balance law

$$\frac{d}{dt} \mathcal{V}_2 = - \frac{1}{N^2} \sum_{i,j=1}^N \phi(x_i - x_j) |v_i(t) - v_j(t)|^2,$$

to cover the missing long-range interactions on the right hand side: note that if $\phi > \lambda$ held everywhere, then the conclusion would have been trivial and the alignment would hold exponentially fast. Such construction is novel to the theory. In the $L^\infty$ settings we provide a different dynamical argument based on finding colliding characteristics in misaligned solutions.

### 2. Discrete Systems

In what follows we use the following abbreviations:

$$v_{ij} = v_i - v_j, \quad x_{ij} = x_i - x_j, \quad \phi_{ij} = \phi(x_{ij}), \quad \text{etc.}$$

Let us define the following variation functionals

$$\mathcal{V}_p' = \frac{1}{N^2} \sum_{i,j=1}^N |v_{ij}|^p, \quad p \geq 1.$$ 

We observe that all these functionals are non-increasing. Indeed, (for $p = 1$ in distributional sense) we have

$$\frac{d}{dt} \mathcal{V}_p' = \frac{p}{N^3} \sum_{i,j=1}^N |v_{ij}|^{p-1} \text{sgn}(v_{ij}) \sum_{k=1}^N (\phi_{ki}v_{ki} - \phi_{kj}v_{kj}) = \frac{2p}{N^3} \sum_{i,j,k=1}^N |v_{ij}|^{p-1} \text{sgn}(v_{ij}) \phi_{ki}v_{ki} = \frac{p}{N^3} \sum_{i,j,k=1}^N (|v_{ij}|^{p-1} \text{sgn}(v_{ij}) - |v_{kj}|^{p-1} \text{sgn}(v_{kj})) \phi_{ki}v_{ki}.$$
Since the function $t \to |t|^{p-1}\text{sgn}(t)$ is non-decreasing, and $v_{ki} = v_{kj} - v_{ij}$, the sum above is negative. More quantitatively, let us denote

$$\mathcal{J}_p = \frac{p}{N^2} \sum_{i,j=1}^{N} |v_{ij}|^p \phi_{ij}.$$ 

Then for $p = 2$ we have the exact identity (energy law):

$$\frac{d}{dt} \mathcal{J}_2 = -\mathcal{J}_2.$$ 

For $p > 2$, using elementary inequality $(|t''|^{p-1}\text{sgn}(t'') - |t'|^{p-1}\text{sgn}(t'))(t'' - t') \geq |t'' - t'|^p$ we obtain

$$\frac{d}{dt} \mathcal{J}_p \leq -\mathcal{J}_p.$$ 

No similar quantitative dissipation bound can be seen in the case $1 < p < 2$. However, for $p = 1$ all $\text{sgn}(v_{ij}) - \text{sgn}(v_{kj})$ either have opposite sign to $v_{ki}$, or 0, and at least

$$\left| \sum_{j=1}^{N} \text{sgn}(v_{ij}) - \text{sgn}(v_{kj}) \right| \geq 2$$

due to $j = i, k$. So, we obtain

$$\frac{d}{dt} \mathcal{J}_1 \leq -\frac{2}{N} \mathcal{J}_1.$$ 

Note that the bound depends on $N$ now, which is not sustainable under the large crowd limit $N \to \infty$.

2.1. **Collisions under singular communication.** The calculations above are valid in the smooth case or when no collisions are present. In this section we will collect all necessary facts about the collision issue. We start with two short examples to illuminate the basic mechanism (see also [14]).

**Example 2.1.** First, in the smooth case let us assume $\phi = 1$ in a neighborhood of 0 for simplicity. Let us arrange two agents $x = x_1 = -x_2$ with $0 < x(0) = \varepsilon \ll 1$. And let $v_1 = -v_2 < 0$ be very large. Clearly $x(t)$ will remain in the same neighborhood of 0 as where it has started, and so the system reads

$$\frac{d}{dt} x = v, \quad \frac{d}{dt} v = -2v.$$ 

Solving it explicitly we can see that the two agents will collide at the origin. In the smooth kernel case such hard interactions are common but present no difficulty for well-posedness simply because the right hand side of (1) remains smooth.

**Example 2.2.** Singularity, obviously present a problem if agents collide. However if it is sufficiently strong the agents align faster than a collision has a chance to happen. Just how much singularity is necessary can be seen from the following example. Let the kernel be given by $\phi(r) = \frac{1}{r^\beta}$, and let us consider the same setup as previously. Then we obtain the system

$$\frac{d}{dt} x = v, \quad \frac{d}{dt} v = -2\frac{v}{x^\beta}.$$ 

This system has a conservation law provided $\beta < 1$: $v + \frac{2^{1-\beta}}{1-\beta} = C_0$. So, if initially $C_0 \ll 0$, then $v < C_0 \ll 0$ as well. This means that $x$ will reach the origin in finite time.

**Theorem 2.3.** Under the strong singularity condition (4) the flock experiences no collisions between agents for any non-collisional initial datum. Consequently, any non-collisional initial datum gives rise to a unique global solution.
Proof. We start as in [2]. Let us assume that for a given non-collisional initial condition \((x_i, v_i)_i\) a collision occurs at time \(T^*\) for the first time. Let \(\Omega^* \subset \Omega = \{1, ..., N\} \) be one set of indexes of the agents that collided at one point (note that other groups of agents may collide as well at other points of space). Hence, there exists a \(\delta > 0\) such that \(|x_{ik}(t)| \geq \delta\) for all \(i \in \Omega^*\) and \(k \in \Omega \setminus \Omega^*\). Denote
\[
\mathcal{D}^*(t) = \max_{i,j \in \Omega^*} |x_{ij}(t)|, \quad \mathcal{V}^*(t) = \sum_{i,j \in \Omega^*} |v_{ij}(t)|^2.
\]
Directly from the characteristic equation we obtain \(\dot{\mathcal{D}}^* \leq \sqrt{\mathcal{V}^*}\), and hence
\[
(13) \quad -\dot{\mathcal{D}}^* \leq \sqrt{\mathcal{V}^*}.
\]
From the momentum equation we obtain
\[
\frac{d}{dt} \mathcal{V}^* = 2 \sum_{k \in \Omega, i,j \in \Omega^*} \phi_{ki} v_{ki} v_{ij} - \phi_{kj} v_{kj} v_{ij}.
\]
Switching \(i, j\) in the second sum results in the same as first sum. So, we obtain
\[
\frac{d}{dt} \mathcal{V}^* \leq \frac{4}{N} \sum_{k \in \Omega, i,j \in \Omega^*} \phi_{ki} v_{ki} v_{ij} \leq \frac{4}{N} \sum_{k \in \Omega \setminus \Omega^*, i,j \in \Omega^*} \phi_{ki} v_{ki} v_{ij} \leq C_1 \sqrt{\mathcal{V}^*} - C_2 \phi(\mathcal{D}^*) \mathcal{V}^*.
\]
We thus obtain a system
\[
\begin{cases}
\frac{d}{dt} \mathcal{V}^* \leq -C_2 \phi(\mathcal{D}^*) \sqrt{\mathcal{V}^*} + C_1 \\
-\frac{d}{dt} \mathcal{D}^* \leq \sqrt{\mathcal{V}^*}.
\end{cases}
\]
Let us denote the energy functional
\[
E(t) = \sqrt{\mathcal{V}^*(t)} + C_2 \int_{\mathcal{D}^*(t)} \phi(r) \, dr.
\]
We readily find that \(\frac{d}{dt} E \leq C_1\), hence \(E\) remains bounded up to the critical time. This means that \(\mathcal{D}^*(t)\) cannot approach zero value.

The global existence part is now a routine application of the Picard iteration and the standard continuation argument. □

A quantitative version on the minimal distance between agents was obtained in [2] in the case of power kernels (5) with \(\beta \geq 2\), namely \(|x_{ij}(t)| \geq ce^{-Ct}\). We can considerably improve this bound in the course of the computation below that will be needed in the proof of Theorem 1.2.

So, let us consider the local version of the collision functional introduced in [2] for \(\beta \geq 2\):
\[
\mathcal{C} = \begin{cases}
\frac{1}{N^2} \sum_{i,j=1}^N \frac{1}{(|x_{ij}| \land r_0)^{\beta - 2}}, & \beta > 2 \\
\frac{1}{N^2} \sum_{i,j=1}^N \ln(|x_{ij}| \land r_0), & \beta = 2.
\end{cases}
\]
Note that as long as $b < 3$ this will stay finite in the limit $N \to \infty$ for a smooth distribution. Let us now look at the time-derivative of $\mathcal{C}$ for $\beta > 2$:

\[
\frac{d\mathcal{C}}{dt} = \frac{(2 - \beta)}{N^2} \sum_{i,j=1}^{N} \frac{d}{dt} \left( \frac{\left| x_{ij} \right| \wedge r_0}{\left| x_{ij} \right| \wedge r_0} \right)^{\beta - 1} \leq \frac{|\beta - 2|}{N^2} \sum_{i,j=1}^{N} \frac{1}{\left| x_{ij} \right| \wedge r_0} \int_{x_{ij} < r_0} \left| v_{ij} \right| I_{|x_{ij}| < r_0} \\
\leq |\beta - 2| \left( \frac{1}{N^2} \sum_{i,j=1}^{N} v_{ij}^2 \frac{1}{\left| x_{ij} \right| \wedge r_0} \right)^{1/2} \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \left| x_{ij} \right|^{2-\beta} I_{|x_{ij}| < r_0} \right)^{1/2} \\
\leq C \sqrt{\mathcal{F}_2} \sqrt{\mathcal{C}}.
\]

This implies

\[
\sqrt{\mathcal{C}(t)} \leq \sqrt{\mathcal{C}(0)} + C \int_{0}^{t} \sqrt{\mathcal{F}_2(s)} \, ds,
\]

and recalling that $\mathcal{F}_2$ is integrable on $\mathbb{R}^+$ we conclude

\[
\mathcal{C}(t) \lesssim t.
\]

For $\beta = 2$, exact same computation gives $\frac{d}{dt} \mathcal{C} \leq C \sqrt{\mathcal{F}_2}$, hence $\mathcal{C}(t) \lesssim \sqrt{t}$. We thus arrive at the following bounds

\[
|x_{ij}(t)| \geq \begin{cases} \frac{c}{t^{\frac{\beta-2}{2}}} & \beta > 2, \\ ce^{-C\sqrt{t}} & \beta = 2. \end{cases}
\]

**Remark 2.4.** All the results of this section carry over to the multidimensional settings ad verbatim.

**2.2. Unconditional alignment on the 1D torus.** In this section we provide proofs of our discrete system results.

**Proof of Theorem 1.1.** First, we present a quick dynamical argument that sets the intuition behind the result.

By the Galilean invariance we can assume that the momentum vanishes, $\sum_{i=1}^{N} v_i = 0$. Since $\int_{0}^{\infty} \mathcal{J}_1(s) \, ds < \infty$ for a fixed $\varepsilon > 0$ there is a time $T$ such that

\[
\int_{T}^{\infty} \mathcal{J}_1(s) \, ds < \varepsilon.
\]

This immediately implies that velocity variations from that time on remain small:

\[
|v_i(t + T) - v_i(T)| \leq \varepsilon.
\]

Let $U$ be the maximal velocity at time $T$, and assume it is large: $U > 2\varepsilon$. Let $v_i(T) = U$ for some $i$. Noting that $v_i(t + T) > U/2$ for all $t > 0$ we conclude

\[
x_i(t + T) > x_i(T) + tU/2 = x_i(T) + 4\pi,
\]

for $t = 8\pi/U$. At the same time, since the total momentum is zero, we can find $j$ with $v_j(T) < 0$, hence

\[
x_j(t + T) < x_j(T) + t\varepsilon < x_j(T) + \varepsilon8\pi/U < x_j(T) + \pi,
\]

provided $U > 8\pi\varepsilon$. So, unless $U \leq 8\pi\varepsilon$, during this time the two agents will collide, which is a contradiction in the singular case. In the smooth case, we argue that the agents will remain $r_0$-close on a time span $t'' - t' = r_0/U$. However, in this case

\[
\lambda \int_{t'}^{t''} |v_i(t) - v_j(t)| \, dt \leq \int_{t'}^{t''} \mathcal{J}_1(t) \, dt < \varepsilon.
\]
This implies that at some time $t \in [t', t'']$, $|v_i(t) - v_j(t)| \lesssim \varepsilon$, which brings us to $U \lesssim \varepsilon$. In either case we arrive at the same conclusion $U \lesssim \varepsilon$.

This argument will not give a good quantitative estimate on the rate of decay. Instead we go back to (10)–(12) and construct a corrector functional $G$ which serves as a compensator for the missing non-local interactions.

First we define the directed distance function:

$$d_{ij}(t) = (x_i - x_j) \text{sgn}(v_j - v_i) \mod 2\pi,$$

where $x_i, x_j \in [0, 2\pi)$ are viewed on the same coordinate chart. In other words, the distance picks up the length of the arch between $x_i$ and $x_j$ which dynamically contracts at time $t$ under the evolution of the two agents. The distance clearly undergoes jumps at $x_i = x_j$ and $v_i = v_j$. Otherwise, we have

$$\frac{d}{dt}d_{ij} = -|v_{ij}|.$$  \hspace{1cm} (19)

Next we define an auxiliary communication kernel $\psi \geq 0$ as follows

$$\psi(x) = \begin{cases} -x + r_0, & -r_0 \leq x \leq r_0 \\ r_0 - \pi - r_0, & r_0 < x < 2\pi - r_0 \end{cases} \quad \text{and extended periodically on the line. With this we define}

$$G(t) = \frac{1}{N^2} \sum_{i,j=1}^{N} |v_{ij}| \psi(d_{ij}).$$

Let us observe the formula for the derivative of $G$:

$$\frac{d}{dt}G = -\frac{1}{N^2} \sum_{i,j=1}^{N} |v_{ij}|^2 \psi'(d_{ij}) + \frac{2}{N^3} \sum_{i,j=1}^{N} \psi(d_{ij}) \text{sgn}(v_{ij}) \sum_{k=1}^{N} v_{ki} \phi_{ki}.$$ \hspace{1cm} 

Indeed, at those times when there is not jump, i.e. $x_i \neq x_j$ and $v_i \neq v_j$ it follows directly from (19). When two agents pass each other $x_i = x_j$ we use periodicity of $\psi$, and when $v_{ij} = 0$, the pre-factor $|v_{ij}|$ vanishes.

From this we continue

$$\frac{d}{dt}G(t) = \frac{1}{N^2} \sum_{i,j=1}^{N} |v_{ij}|^2 \mathbb{1}_{|x_{ij}| \leq r_0} - \frac{r_0}{\pi - r_0} \frac{1}{N^2} \sum_{i,j=1}^{N} |v_{ij}|^2 \mathbb{1}_{|x_{ij}| > r_0} + R,$$

where

$$R = \frac{2}{N^3} \sum_{i,j,k=1}^{N} \psi(d_{ij}) \text{sgn}(v_{ij}) v_{ki} \phi_{ki} \leq cF_1.$$ \hspace{1cm} (20)

So, we obtain

$$\frac{d}{dt}G(t) \leq \frac{\pi}{c_0(\pi - r_0)} F_2 - \frac{r_0}{\pi - r_0} F_2 + cF_1 = aF_2 - bF_2 + cF_1.$$

With this we can form another Lyapunov functional:

$$L = G + \frac{cN}{2}V_1 + btV_2 + aV_2,$$

so that $\frac{d}{dt}L \leq 0$. This implies the conclusion of the theorem immediately.

A modification of the proof under more general lower bound (8) is only required in the definition of $\psi$, where we place the negative linear slope exactly over $I$, and the rest reconnects with positive slope. \hspace{1cm} $\square$
Proof of Theorem 1.2. The previous proof is repeated up to the remainder estimate (20) which we will replace using the collision potential (15) in order to avoid the use of $N$-dependent $\mathcal{F}_1$. Symmetrizing over $i, k$, we find

$$\mathcal{R} = \frac{1}{N^3} \sum_{i,j,k=1}^{N} (\psi(d_{ij}) \text{sgn}(v_{ij}) - \psi(d_{kj}) \text{sgn}(v_{kj})) v_{ki} \phi_{ki}.$$ 

If $v_i \geq v_j \geq v_k$ or $v_i \leq v_j \leq v_k$, then the summand is negative, and so we can drop it. Hence,

$$\mathcal{R} \leq \frac{1}{N^3} \sum_{i,j,k=1}^{N} (\psi(d_{ij}) \text{sgn}(v_{ij}) - \psi(d_{kj}) \text{sgn}(v_{kj})) v_{ki} \phi_{ki} \mathbb{1}_{v_j > \max(v_i, v_k)}$$

$$+ \frac{1}{N^3} \sum_{i,j,k=1}^{N} (\psi(d_{ij}) \text{sgn}(v_{ij}) - \psi(d_{kj}) \text{sgn}(v_{kj})) v_{ki} \phi_{ki} \mathbb{1}_{v_j < \min(v_i, v_k)}$$

$$= \frac{1}{N^3} \sum_{i,j,k=1}^{N} (\psi(d_{kj}) - \psi(d_{ij})) v_{ki} \phi_{ki} \mathbb{1}_{v_j > \max(v_i, v_k)}$$

$$+ \frac{1}{N^3} \sum_{i,j,k=1}^{N} (\psi(d_{ij}) - \psi(d_{kj})) v_{ki} \phi_{ki} \mathbb{1}_{v_j < \min(v_i, v_k)}.$$ 

In the considered cases $v_j > \max(v_i, v_k)$ and $v_j < \min(v_i, v_k)$, we see that $v_i - v_j$ and $v_k - v_j$ have the same sign so that $d_{ij}$ and $d_{kj}$ are computed in the same direction. Hence we find by the Lipschitz continuity of $\psi$ and by the triangle inequality that $|\psi(d_{ij}) - \psi(d_{kj})| \leq C|x_i - x_k|$. Therefore,

$$\mathcal{R} \leq \frac{C}{N^3} \sum_{i,j,k=1}^{N} |x_{ik}| v_{ik} \phi_{ki} = \frac{C}{N^2} \sum_{i,k=1}^{N} |x_{ik}| v_{ik} \phi_{ki} \leq \frac{1}{t} \frac{C}{bN^2} \sum_{i,k=1}^{N} |x_{ik}|^2 \phi_{ki} + \frac{b}{N^2} \sum_{i,k=1}^{N} v_{ik}^2 \phi_{ki}.$$ 

Let us proceed now under the assumptions of (i). Here we obtain

$$\mathcal{R} \leq \frac{c}{t} + bt\mathcal{F}_2.$$ 

Then the corrector equation becomes

$$\frac{d}{dt} \mathcal{G}(t) \leq a\mathcal{F}_2 + b(t - \mathcal{F}_2) + \frac{c}{t}.$$ 

With this we can form another functional:

$$\mathcal{L} = \mathcal{G} + bt\mathcal{F}_2 + a\mathcal{F}_2,$$

satisfying $\frac{d}{dt} \mathcal{L} \leq \frac{c}{t}$. Hence, $\mathcal{L}(t) \leq \ln t$, and the resulting bound is as claimed in (i).

For part (ii) we use the collision potential:

$$\mathcal{R} \leq C \left( \frac{1}{N^2} \sum_{i,k=1}^{N} |x_{ik}|^{2-\beta} |x_{ik}|^{\beta} \phi_{ki} \right)^{1/2} \sqrt{\mathcal{F}_2} \lesssim \sqrt{\mathcal{F}_2} \sqrt{\mathcal{G}}.$$ 

Recalling (16),

$$\mathcal{R} \leq c_1 \sqrt{\mathcal{F}_2(t)} + c_2 \sqrt{\mathcal{F}_2(t)} \int_0^t \sqrt{\mathcal{F}_2(s)} \, ds.$$ 

We can replace as before

$$c_1 \sqrt{\mathcal{F}_2(t)} \leq \frac{c_3}{t} + bt\mathcal{F}_2(t),$$
obtaining
\[
\frac{d}{dt} \mathcal{I}(t) \leq a \mathcal{I}_2 + b(t, \mathcal{I}_2 - \mathcal{I}_2') + \frac{c_3}{t} + c_2 \sqrt{\mathcal{I}_2(t)} \int_0^t \sqrt{\mathcal{I}_2(s)} \, ds.
\]
With \( \mathcal{L} \) defined as before,
\[
\frac{d}{dt} \mathcal{L} \lesssim \frac{c_3}{t} + c_2 \sqrt{\mathcal{I}_2(t)} \int_0^t \sqrt{\mathcal{I}_2(s)} \, ds.
\]
Integrating,
\[
\mathcal{L}(T) \lesssim \mathcal{L}(0) + \ln T + \left( \int_0^T \sqrt{\mathcal{I}_2(s)} \, ds \right)^2.
\]
Hence,
\[
\mathcal{I}_2(T) \leq \frac{1}{T} \mathcal{L}(T) \lesssim \frac{\ln T}{T} + \frac{1}{T} \left( \int_0^T \sqrt{\mathcal{I}_2(s)} \, ds \right)^2.
\]
The right hand side obviously tends to zero which can be seen by splitting the integral into \((0, T')\) and \((T', T)\), where \( T' \) is large. \( \square \)

3. HYDRODYNAMIC SYSTEMS

We now turn our attention to the hydrodynamic case (3). In what follows we assume that a classical solution \((v, \rho)\) is given. Existence of such solutions has been surveyed in the Introduction for various cases.

Let us introduce some notation. We consider the density measure
\[
dm_t = \rho(x, t) \, dx.
\]
In view of the transport nature of the continuity equation, this measure is transported along the flow of \( v \). Namely, if
\[
\frac{d}{dt} x(\alpha, t) = v(x(\alpha, t), t), \quad t > 0
\]
\[
x(\alpha, 0) = \alpha,
\]
then \( dm_t \) is a push-forward of \( dm_0 \) under \( x(\cdot, t) \): \( dm_t = x(\cdot, t) \# dm_0 \). In other words, for any \( f \),
\[
\int_T f(x(\alpha, t)) \, dm_0(\alpha) = \int_T f(x) \, dm_t(x).
\]
(21)
We also define the 2-variation and its dissipation as before
\[
\mathcal{I}_2(t) = \int_{T^2} |v(x, t) - v(y, t)|^2 \rho(x, t) \rho(y, t) \, dx \, dy
\]
\[
\mathcal{J}_2(t) = 2 \int_{T^2} |v(x, t) - v(y, t)|^2 \rho(x, t) \rho(y, t) \phi(x - y) \, dx \, dy
\]
\[
\frac{d}{dt} \mathcal{I}_2 = -\mathcal{J}_2.
\]
We will work in Lagrangian coordinates in what follows and use the short notation
\[
u(\alpha, t) = v(x(\alpha, t), t), \quad u(\alpha, t) - u(\beta, t) = u_{\alpha\beta}, \quad x(\alpha, t) - x(\beta, t) = x_{\alpha\beta}, \quad \phi(x(\alpha, t) - x(\beta, t)) = \phi_{\alpha\beta},
\]
and \( dm(\alpha, \beta) \) to denote product measures. Thus, the quantities above can be rewritten as
\[
\mathcal{I}_2(t) = \int_{T^2} |u_{\alpha\beta}|^2 \, dm_0(\alpha, \beta)
\]
\[
\mathcal{J}_2(t) = 2 \int_{T^2} |u_{\alpha\beta}|^2 \phi_{\alpha\beta} \, dm_0(\alpha, \beta).
\]

**Theorem 3.1.** For a given smooth solution \((v, \rho)\) to (3) the following alignment holds:
(i) For smooth or sub-quadratic kernels satisfying (7) one has

\[ T_2(t) \leq C \frac{\ln t}{t}, \quad t \to \infty, \]

where \( C \) depends only on the initial condition.

(ii) For power law communication (5) with \( 2 < \beta < 3 \), one has

\[ T_2(t) \to 0, \quad t \to \infty. \]

**Proof.** Proceeding as in the discrete case we define the directed distance

\[ d_{\alpha\beta}(t) = (x(\alpha, t) - x(\beta, t)) \text{sgn}(u(\beta, t) - u(\alpha, t)) \mod 2\pi, \]

and the corrector with \( \psi \) as above:

\[ G = \int_{T^2} |u_{\alpha\beta}| \psi(d_{\alpha\beta}) \, dm_0(\alpha, \beta). \]

With the same justification as above in case of jumps, we calculate the derivative of \( G \):

\[
\frac{d}{dt} G = -\int_{T^2} |u_{\alpha\beta}|^2 \psi'(d_{\alpha\beta}) \, dm_0(\alpha, \beta) + \int_{T^2} \text{sgn}(u_{\alpha\beta}) \psi(d_{\alpha\beta}) \phi_{\alpha\gamma} u_{\gamma\alpha} \, dm_0(\alpha, \beta, \gamma) \\
\leq a \mathcal{I}_2 - b T_2 + \mathcal{R}.
\]

Here,

\[
\mathcal{R} = \int_{T^3} \text{sgn}(u_{\alpha\beta}) \psi(d_{\alpha\beta}) \phi_{\alpha\gamma} u_{\gamma\alpha} \, dm_0(\alpha, \beta, \gamma) \\
= \frac{1}{2} \int_{T^3} [\text{sgn}(u_{\alpha\beta}) \psi(d_{\alpha\beta}) - \text{sgn}(u_{\gamma\beta}) \psi(d_{\gamma\beta})] \phi_{\alpha\gamma} u_{\gamma\alpha} \, dm_0(\alpha, \beta, \gamma) \\
\leq \int_{T^3} (\psi(d_{\alpha\beta}) - \psi(d_{\gamma\beta})) \phi_{\alpha\gamma} u_{\gamma\alpha} \mathbb{1}_{u(\beta) < \min\{u(\alpha), u(\gamma)\}} \, dm_0(\alpha, \beta, \gamma) \\
+ \int_{T^3} (\psi(d_{\gamma\beta}) - \psi(d_{\alpha\beta})) \phi_{\alpha\gamma} u_{\gamma\alpha} \mathbb{1}_{u(\beta) > \max\{u(\alpha), u(\gamma)\}} \, dm_0(\alpha, \beta, \gamma) \\
\leq \int_{T^2} |x_{\alpha\gamma}| |u_{\gamma\alpha}| \phi_{\alpha\gamma} \, dm_0(\alpha, \gamma).
\]

In the case (i) we then obtain

\[ \mathcal{R} \leq \frac{c}{t} + bt \mathcal{I}_2, \]

and the proof ends as in the discrete settings. In the case (ii) we consider the collision potential

\[ \mathcal{C} = \int_{T^2} \frac{\, dm_0(\alpha, \beta)}{|x_{\alpha\beta} \land r_0|^{\beta - 2}}. \]

Note that it is well-posed for \( \beta < 3 \) (in view also of the fact that the density is bounded for regular solutions). Computation similar to the discrete proves (16), and from this point on the proof is exactly the same. \( \square \)

Let us note that the content of Remark 1.3 still applies to the continuous settings of Theorem 3.1.

Next, we prove the \( L^\infty \)-based result, Theorem 1.4. Heuristically, the alignment still holds in spite of all the lacking assumptions because in regions where the density is non-negligible the alignment term works faster than the transport to avoid characteristic collisions. At the same time in regions where the density is thin, the equation acts as Burgers. So, to avoid a blowup it must have low velocity variations in those regions as well.
Proof of Theorem 1.4. By Galilean invariance we can assume throughout that \( \bar{v} = 0 \). As a consequence of the energy equality and (23) we obtain
\[
\int_T^\infty \int_\Omega \phi_{\alpha\beta} |u(\alpha, t) - u(\beta, t)|^2 \, dm_0(\alpha, \beta) \, dt \leq C \frac{\ln T}{T} := \varepsilon.
\]

Denote
\[
F(\alpha, T) = \int_T^\infty \int_\Omega \phi_{\alpha\beta} |u(\alpha, t) - u(\beta, t)|^2 \, dm_0(\beta) \, dt.
\]

So, we have
\[
\int_T^T F(\alpha, T) \, dm_0(\alpha) \leq \varepsilon.
\]

Let us fix another small parameter \( \delta > 0 \) and define the “good set”:
\[
G_\delta(T) = \{ \alpha : F(\alpha, T) \leq \delta \}.
\]

We denote by \( G_\delta^c \) the complement of \( G_\delta \), so that \( m_0(G_\delta^c) = M - m_0(G_\delta) \) (recall that \( M \) is the total mass of the flock). By the Chebychev inequality,
\[
(24)\quad m_0(G_\delta^c) < \frac{\varepsilon}{\delta}.
\]

Thus, the good set occupies almost all of the domain provided \( \varepsilon \ll \delta \). We now proceed by proving that alignment occurs first on the good set identified above, and then on the rest of the torus later in time within a controlled time scale.

Lemma 3.2 (Alignment on the good set). We have
\[
\sup_{\alpha_1, \alpha_2 \in G_\delta(T), t \geq T} |u(\alpha_1, t) - u(\alpha_2, t)| \lesssim \delta^{2/3}.
\]

Proof. Note that it suffices to establish alignment at time \( T \) only. This simply follows from monotonicity of the our \( F \)-function:
\[
F(\alpha, t) \leq F(\alpha, T), \quad t > T,
\]

which implies that the good sets are increasing in time, \( G_\delta(T) \subset G_\delta \).

Integrating the equation
\[
\frac{d}{dt} u(\alpha, t) = \int_\Omega \phi_{\alpha\beta} u_{\alpha\beta} \, dm_0(\beta)
\]
over \([T, t]\) for any \( \alpha \in G_\delta \) we obtain
\[
(25)\quad |u(\alpha, t) - u(\alpha, T)| \leq \int_T^t \int_\Omega \phi_{\alpha\beta} |u_{\alpha\beta}| \, dm_0(\beta) \lesssim \delta \sqrt{t - T}.
\]

Suppose that for some \( \alpha_1, \alpha_2 \in G_\delta \) we have
\[
u(\alpha_1, T) - u(\alpha_2, T) > U,
\]

where \( U \) to be determined later. Then in view of (25),
\[
u(\alpha_1, t) - u(\alpha_2, t) > \frac{U}{2},
\]

as long as
\[
(26)\quad t - T \lesssim \frac{U^2}{\delta^2}.
\]

During this time the corresponding characteristics will undergo a significant relative displacement
\[
x(\alpha_1, t) - x(\alpha_2, t) \geq x(\alpha_1, T) - x(\alpha_2, T) + \frac{1}{2} U (t - T) \mod 2\pi,
\]
where $\frac{1}{U}(t - T) > 4\pi$ as long as $t - T \geq \frac{1}{U}$. If this is allowed to happen, then the characteristics will find themselves at the separation distance equal $2\pi = 0$, so they collapse. We necessarily obtain

$$
\frac{1}{U} \geq \frac{U^2}{\delta^2},
$$

which gives $U \lesssim \delta^{2/3}$ as claimed. \hfill \Box

Next step is to show that the solution aligns completely at a not too distant later time $t > T$.

**Lemma 3.3** (Alignment outside the good set). For all $t \geq T + \frac{1}{\delta^{1/3} + (\varepsilon/\delta)^{1/2}}$ we have

$$
\sup_{\alpha \in \mathcal{Y}, \gamma \in G_\delta(T)} |u(\alpha, t) - u(\gamma, t)| \lesssim \delta^{1/3} + (\varepsilon/\delta)^{1/2}.
$$

**Proof.** Let $\alpha \in \mathcal{Y}$ and $\gamma \in G_\delta(T)$. Let us write the momentum equation in Lagrangian coordinates as follows

$$
\frac{d}{dt} u(\alpha, t) = \int_{\mathcal{Y}} u_{\beta\alpha} \phi_{\alpha\beta} \, dm_0(\beta) = \int_{\mathcal{Y}} (u_{\beta\gamma} + u_{\gamma\alpha}) \phi_{\alpha\beta} \, dm_0(\beta)
$$

$$
= \left( \phi \ast \rho \right)(x(\alpha, t), tu_{\alpha}) + \int_{\mathcal{Y}} u_{\beta\gamma} \phi_{\alpha\beta} \, dm_0(\beta).
$$

The integral term will remain small for all $t \geq T$, in view Lemma 3.2 and (24). Indeed,

$$
\left| \int_{\mathcal{Y}} u_{\beta\gamma}(t) \phi_{\alpha\beta} \, dm_0(\beta) \right| = \left| \int_{G_\delta(T)} u_{\beta\gamma}(t) \phi_{\alpha\beta} \, dm_0(\beta) \right| + \left| \int_{G_\delta(T)} u_{\beta\gamma}(t) \phi_{\alpha\beta} \, dm_0(\beta) \right|
$$

$$
\lesssim \delta^{2/3} + \frac{\varepsilon}{\delta}.
$$

So,

$$
(\phi \ast \rho)u_{\alpha\gamma} - \delta^{2/3} - \frac{\varepsilon}{\delta} \leq \frac{d}{dt} u(\alpha, t) \leq (\phi \ast \rho)u_{\alpha\gamma} + \delta^{2/3} + \frac{\varepsilon}{\delta}.
$$

Let us fix a time $t \geq T + \frac{1}{\delta^{1/3} + (\varepsilon/\delta)^{1/2}}$, and assume that $u_{\alpha\gamma}(t) = U > 0$, where $U$ is to be determined later. Let us drive the dynamics backwards in time from the moment $t$. For a time period $[s, t]$, where $T < s < t$, the difference will remain positive $u_{\alpha\gamma}(s) > 0$. On that time period, the right hand side of (28) implies

$$
\frac{d}{dt} u \leq \delta^{2/3} + \frac{\varepsilon}{\delta}
$$

and hence,

$$
u(\alpha, t) - (\delta^{2/3} + \frac{\varepsilon}{\delta})(t - s) \leq u(\alpha, s).
$$

At the same time, by (25) applied to $\gamma \in G_\delta$, we have

$$
|u(\gamma, t) - u(\gamma, s)| \leq \delta(t - s)^{1/2}.
$$

So, combined with the previous,

$$
U - (\delta^{2/3} + \frac{\varepsilon}{\delta})(t - s) - \delta(t - s)^{1/2} = u_{\alpha\gamma}(t) - (\delta^{2/3} + \frac{\varepsilon}{\delta})(t - s) - \delta(t - s)^{1/2} \leq u_{\alpha\gamma}(s).
$$

We will have

$$
u_{\alpha\gamma}(s) \geq \frac{U}{2},
$$

as long as $(t - s) \lesssim \frac{U^{2/3} + \frac{1}{\delta}}{\delta^{2/3}}$ and $(t - s) \lesssim \frac{t^{1/2}}{\delta^2}$. The former is more restrictive, unless $U \lesssim \delta^{4/3}$, in which case we have achieved our objective. Arguing as in the proof of Lemma 3.2 we obtain collision backwards in time, provided $(t - s) \sim 1/U$. This is possible when $U \gtrsim \delta^{1/3} + (\varepsilon/\delta)^{1/2}$ on the time interval $t - T \gtrsim 1/U$, which is true under the assumption.

Arguing from the opposite end, $u_{\alpha\gamma}(t) = -U < 0$, we obtain the bound from below as well. \hfill \Box
Lemma 3.3 implies the global alignment: for all \( t \gtrsim T + \frac{1}{\delta^{1/4} (\varepsilon/\delta)^{1/2}} \)

\[
\sup_{\alpha, \gamma \in T} |u(\alpha, t) - u(\gamma, t)| \lesssim \delta^{1/3} + (\varepsilon/\delta)^{1/2}.
\]

Optimizing over \( \delta \), we pick \( \delta = \varepsilon^{3/5} \), and recalling that \( \varepsilon = \ln T / T \), we obtain

\[
\sup_{\alpha, \gamma \in T} |u(\alpha, t) - u(\gamma, t)| \lesssim \left( \frac{\ln T}{T} \right)^{1/5},
\]

for \( t \sim T + \left( \frac{T}{\ln T} \right)^{1/5} \sim T \). This proves the result.

\[\square\]

**References**

[1] José A. Carrillo, Choi, Young-Pil, Eitan Tadmor, and Changhui Tan. Critical thresholds in 1D Euler equations with non-local forces. *Math. Models Methods Appl. Sci.*, 26(1):185–206, 2016.

[2] José A. Carrillo, Young-Pil Choi, Piotr B. Mucha, and Jan Peszek. Sharp conditions to avoid collisions in singular Cucker-Smale interactions. *Nonlinear Anal. Real World Appl.*, 37:317–328, 2017.

[3] Felipe Cucker and Steve Smale. Emergent behavior in flocks. *IEEE Trans. Automat. Control*, 52(5):852–862, 2007.

[4] Felipe Cucker and Steve Smale. On the mathematics of emergent behavior. *Jpn. J. Math.*, 2(1):197–227, 2007.

[5] Tam Do, Alexander Kiselev, Lenya Ryzhik, and Changhui Tan. Global Regularity for the Fractional Euler Alignment System. *Arch. Ration. Mech. Anal.*, 228(1):1–37, 2018.

[6] Alessio Figalli and Moon-Jin Kang. A rigorous derivation from the kinetic Cucker-Smale model to the pressureless Euler system with nonlocal alignment. *Anal. PDE*, 12(3):843–866, 2019.

[7] Seung-Yeal Ha and Jian-Guo Liu. A simple proof of the Cucker-Smale flocking dynamics and mean-field limit. *Commun. Math. Sci.*, 7(2):297–325, 2009.

[8] Seung-Yeal Ha and Eitan Tadmor. From particle to kinetic and hydrodynamic descriptions of flocking. *Kinet. Relat. Models*, 1(3):415–435, 2008.

[9] A. Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions Automatic Control*, 48:988–1001, 2003.

[10] E. Tadmor, J. Morales, and J. Peszek. Flocking with short-range interactions. https://arxiv.org/abs/1812.03567.

[11] Trygve Karper, Antoine Mellet, and Konstantina Trivisa. Hydrodynamic limit of the kinetic cucker-smale flocking model. *Mathematical Models and Methods in Applied Sciences*, 25(1):131–163, 2015.

[12] Sebastien Motsch and Eitan Tadmor. Heterophilious dynamics enhances consensus. *SIAM Rev.*, 56(4):577–621, 2014.

[13] Piotr B. Mucha and Jan Peszek. The cuckersmale equation: singular communication weight, measure-valued solutions and weak-atomic uniqueness. *Arch. Rational Mech. Anal.*, 227:273–308, 2018.

[14] J. Peszek. Existence of piecewise weak solutions of a discrete cucker–smale’s flocking model with a singular communication weight. *J. Differential Equations*, 257:2900–2925, 2014.

[15] Jan Peszek. Discrete Cucker-Smale flocking model with a weakly singular weight. *SIAM J. Math. Anal.*, 47(5):3671–3686, 2015.

[16] David Poyato and Juan Soler. Euler-type equations and commutators in singular and hyperbolic limits of kinetic Cucker-Smale models. *Math. Models Methods Appl. Sci.*, 27(6):1089–1152, 2017.

[17] Kevin W. Rio, Gregory C. Dachner, and William H. Warren. Local interactions underlying collective motion in human crowds. *Proc. R. Soc. B*, 285:20180611, 2018.

[18] R. Shvydkoy and E. Tadmor. Topological models for emergent dynamics with short-range interactions. *ArXiv e-prints*, June 2018.

[19] Roman Shvydkoy and Eitan Tadmor. Eulerian dynamics with a commutator forcing III: Fractional diffusion of order \( 0 < \alpha < 1 \). to appear in Physica D.

[20] Roman Shvydkoy and Eitan Tadmor. Eulerian dynamics with a commutator forcing. *Transactions of Mathematics and Its Applications*, 1(1):tnx001, 2017.

[21] Roman Shvydkoy and Eitan Tadmor. Eulerian dynamics with a commutator forcing II: Flocking. *Discrete Contin. Dyn. Syst.*, 37(11):5503–5520, 2017.

[22] Eitan Tadmor and Changhui Tan. Critical thresholds in flocking hydrodynamics with non-local alignment. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 372(2018):20130401, 22, 2014.

[23] T. Vicsek and A. Zefeiris. Collective motion. *Physics Reprints*, 517:71–140, 2012.
