ON ALGEBRAIC BI-LIPSCHITZ HOMEOMORPHISMS

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Abstract. Let \( X \subset \mathbb{C}^n \); \( Y \subset \mathbb{C}^m \) be closed affine varieties and let \( \phi : X \to Y \) be an algebraic bi-Lipschitz homeomorphism. Then \( \deg X = \deg Y \). Similarly, let \( (X, 0) \subset (\mathbb{C}^n, 0), (Y, 0) \subset (\mathbb{C}^m, 0) \) be germs of analytic sets and let \( f : (X, 0) \to (Y, 0) \) be a c-holomorphic and bi-Lipschitz homeomorphism. Then \( \text{mult}_0 X = \text{mult}_0 Y \). Finally we show that the normality is not a bi-Lipschitz invariant.

1. Introduction

In [4] Bobadilla, Fernandes and Sampaio study invariance of degree of complex affine varieties under bi-Lipschitz homeomorphisms. They proved that this problem is equivalent to the bi-Lipschitz version of the Zariski multiplicity conjecture. Moreover, they proved that the degree of curves and surfaces are such invariants. From other point of view in [3] Birbrair, Fernandes, Sampaio and Verbitsky give an example of two three-dimensional affine varieties, which are bi-Lipschitz equivalent but they have different degrees. In fact they showed that we have two different embeddings of \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) into \( \mathbb{CP}^5 \), say \( X \) and \( Y \), such that affine cones \( C(X), C(Y) \subset \mathbb{C}^9 \) are bi-Lipschitz equivalent, but they have different degrees. Hence in general the algebraic degree of an affine set is not a bi-Lipschitz invariant. However varieties \( X, Y \) of Birbrair, Fernandes, Sampaio and Verbitsky have codimension greater than one. Hence the problem of invariance of degree under bi-Lipschitz homeomorphisms is still open in the important case of affine hypersurfaces in \( \mathbb{C}^n \), where \( n > 3 \).

In this paper we introduce a new class of bi-Lipschitz homeomorphism- the algebraic bi-Lipschitz homeomorphisms. Assume that \( X, Y \) are affine varieties. We say that homeomorphism \( f : X \to Y \) is an algebraic bi-Lipschitz homeomorphism if it is a bi-Lipschitz and its graph is an algebraic variety. Inspired by our recent paper [1] (with L. Birbrair and A. Fernandes) we show that this class of mappings preserves the degree of affine varieties:

**Theorem 3.4** Let \( X \subset \mathbb{C}^n, Y \subset \mathbb{C}^m \) be affine algebraic varieties and let \( f : X \to Y \) be an algebraic bi-Lipschitz homeomorphism. Then \( \deg X = \deg Y \).

In a similar way we can prove:

**Theorem 4.1** Let \( (X, 0) \subset (\mathbb{C}^n, 0), (Y, 0) \subset (\mathbb{C}^m, 0) \) be germs of analytic sets and let \( f : (X, 0) \to (Y, 0) \) be a c-holomorphic and bi-Lipschitz homeomorphism. Then \( \text{mult}_0 X = \text{mult}_0 Y \).

Let us recall that a mapping \( f : X \to Y \) is c-holomorphic, if it is continuous and its graph is analytic in \( X \times Y \) (here \( X, Y \) are analytic sets).

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Corollary 4.2 Let $X \subset \mathbb{C}^n, Y \subset \mathbb{C}^m$ be affine algebraic varieties and let $f : X \to Y$ be an algebraic bi-Lipschitz homeomorphism. Then $\deg X = \deg Y$ and for every $x \in X$ we have $\text{mult}_x X = \text{mult}_{f(x)} Y$.

At the end of this paper we give an example of two algebraic affine varieties $X, Y$ such that $X$ is normal, $Y$ is not normal and there exists an algebraic bi-Lipschitz homeomorphism $f : X \to Y$. Hence the normality is not a bi-Lipschitz invariant.

2. Preliminaries

Definition 2.1. Let $X, Y$ be affine complex varieties and let $f : X \to Y$ be a continuous mapping. We say that $f$ is algebraic if the graph of $f$ is a complex algebraic set.

Proposition 2.2. Every algebraic mapping is rational, i.e., there exists a Zariski open subset $U \subset X$ such that the mapping $f|_U$ is a regular mapping.

Proof. Indeed, let $\Gamma$ be a graph of $f$ and $t : X \ni x \mapsto (x, f(x)) \in \Gamma$. Let $U \subset X$ be the set of smooth points of $X$. Then $U$ is a Zariski open dense subset of $X$. By the Zariski Main Theorem the projection $\pi : \Gamma \to X$ is an isomorphism over $U$. This means that $t$ is an isomorphism on $U$. But $f|_U = \pi \circ t|_U$, hence $f$ is regular on $U$. \hfill \Box

In other words an algebraic mapping is a mapping which additionally is rational. Here we are interested in these algebraic mappings, which are additionally bi-Lipschitz. Note that if a mapping $f : X \to Y$ is an algebraic bi-Lipschitz homeomorphism, then the mapping $f^{-1} : Y \to X$ is also an algebraic bi-Lipschitz homeomorphism. First of all let us recall the following statement, which follows from [1]:

Theorem 2.3. Let $X \subset \mathbb{C}^n$ be a $k$–dimensional affine variety. Then there is an algebraic bi-Lipschitz embedding $f : X \to \mathbb{C}^{2k+1}$.

We say that the mapping $f : X \to \mathbb{C}^n$ is an algebraic bi-Lipschitz embedding, if the mapping $f : X \to f(X)$ is algebraic and bi-Lipschitz. Since not every $k$ dimensional affine variety can be embedded into $\mathbb{C}^{2k+1}$ in a bi-regular way (see example below) we see that we have a lot of algebraic bi-Lipschitz homeomorphisms which are not biregular mappings.

Example 2.4. For $m > 3$ let $X \subset \mathbb{C}^m$ be a curve given by parametrization

$$X = \{x \in \mathbb{C}^m : x = (t^m, \ldots, t^{2m-1}); \ t \in \mathbb{C}\}.$$  

It is an easy observation that the Zariski tangent space $T_0X$ coincide with $\mathbb{C}^m$. Now consider a generic linear projection $\pi : X \to \mathbb{C}^3$. By [1] it is an algebraic bi-Lipschitz embedding. Since $T_0(\pi(X)) \subset \mathbb{C}^3$ and $T_0X = \mathbb{C}^m$ we see that the mapping $\pi$ is not bi-regular. In fact the algebraic bi-Lipschitz mapping $\pi^{-1} : \pi(X) \to X$ can not be extended to any $C^1$ mapping in a neighborhood of 0 in $\mathbb{C}^3$.

Example 2.5. Let $X = \{x \in \mathbb{C}^3 : x = (t, t^3 + t^2, t^5); t \in \mathbb{C}\}$ and $Y = \{x \in \mathbb{C}^3 : x = (t, t^3 + 2t^2, t^5); t \in \mathbb{C}\}$. Let $\phi : X \ni (t, t^3 + t^2, t^5) \mapsto (t, t^3 + 2t^2, t^5) \in Y$. Then $\phi$ is bi-regular and bi-Lipschitz but it is not a linear mapping.

It is bi-regular because $\phi(x, y, z) = (x, y + x^2, z)$. Now we show that $\phi$ is bi-Lipschitz outside some big ball. Let $a(t) = (t, t^3 + t^2, t^5), b(t) = (t, t^3 + 2t^2, t^5)$. Hence $\phi(a(t)) = b(t)$.

We have to show that for some $K > 0$

$$\frac{1}{K}||a(t) - a(s)|| < ||b(t) - b(s)|| < K||a(t) - a(s)||.$$
Consider the fraction
\[
\frac{b_2(t) - b_2(s)}{a_2(t) - a_2(s)} = \frac{t^3 + 2t^2 - s^3 - 2s^2}{t^3 + t^2 - s^3 - s^2} = \frac{(1 - \epsilon_1 \alpha)(1 - \epsilon_2 \alpha) + 2/t(1 + \alpha)}{(1 - \epsilon_1 \alpha)(1 - \epsilon_2 \alpha) + 1/t(1 + \alpha)},
\]
where \(1, \epsilon_1, \epsilon_2\) are all roots of polynomial \(x^3 + 1\) and \(\alpha = s/t\). Note that we can always assume that \(|\alpha| \leq 1\). Denote \(g(\alpha) = (1 - \epsilon_1 \alpha)(1 - \epsilon_2 \alpha)\). This polynomial has roots in \(\epsilon_1, \epsilon_2\) only. Denote by \(U = \{\alpha : |\alpha - \epsilon_1| > r\} \cap \{\alpha : |\alpha - \epsilon_2| > r\}\). Since \(U\) does not contain roots of \(g\), we have \(|g(\alpha)| > \rho > 0\). We can assume that \(r\) is so small that in the set \(V := \{\alpha : |\alpha - \epsilon_1| \leq r\} \cup \{\alpha : |\alpha - \epsilon_2| \leq r\}\) there is no roots of polynomials \(x^3 + 1\). Now we estimate \((*)\). We have two possibilities:

a) \(\alpha \in U,\)

b) \(\alpha \in V.\)

In the case a) we have \(|g(\alpha)| > \rho > 0\) and in particular for \(|t| > R\) we have
\[
\frac{|b_2(t) - b_2(s)|}{|a_2(t) - a_2(s)|} < 2.
\]
Hence
\[
|b_2(t) - b_2(s)| < 2|a_2(t) - a_2(s)|.
\]
In the case b) we have \(|\phi(a(t)) - \phi(b(t))| = |b_3(t) - b_3(s)| = |a_3(t) - a_3(s)| = |a(t) - a(s)|\) for large \(|t|\) (we consider here the "sup" norm). Hence indeed the mapping \(\phi\) is Lipschitz outside a large ball. Similarly \(\phi^{-1}\) is Lipschitz outside a large ball. Hence \(\phi\) is bi-Lipschitz outside a large ball.

On the other hand \(\phi\) is bi-Lipschitz in any ball, because it is a smooth mapping. Combining this fact with the first step of our proof we see that we can reduce the general case to the case where \(|\alpha|\) is small and \(|t|\) is large. But this can be done in a similar way as in the first step (we left details to the reader). Hence finally the mapping \(\phi\) is bi-Lipschitz.

The mapping \(\phi\) is not a restriction of a linear mapping because otherwise \(t^3 + 2t^2 = at + b(t^3 + t^2) + cf^5 + d\), where \(a, b, c, d \in \mathbb{C}\). This is impossible.

3. Proof of the Theorem 3.4

**Definition 3.1.** Let \(L^s, H^{n-s-1}\) be two disjoint linear subspaces of \(\mathbb{C}P^n\). Let \(\pi_\infty\) be a hyperplane (a hyperplane at infinity) and assume that \(L^s \subset \pi_\infty\). By a projection \(\pi_L\) with center \(L^s\) we mean the mapping:
\[
\pi_L : \mathbb{C}^n = \mathbb{C}P^n \setminus \pi_\infty \ni x \mapsto <L^s, x> \cap H^{n-s-1} \in H^{n-s-1} \setminus \pi_\infty = \mathbb{C}^{n-s-1}.
\]
Here by \(<L, x>\) we mean a linear subspace spanned by \(L\) and \(\{x\}\).

**Lemma 3.2.** Let \(X\) be a closed subset of \(\mathbb{C}^n\). Denote by \(N \subset \pi_\infty\) the set of all secants of \(X\) and let \(\Sigma = \overline{N}\) where \(\pi_\infty\) is a hyperplane at infinity and we consider the projective closure. Let \(\pi_L : \mathbb{C}^n \rightarrow \mathbb{C}^l\) be a projection with center \(L\). Then \(\pi_L|X\) is a bi-Lipschitz embedding if and only if \(L \cap \Sigma = \emptyset\).

**Proof.** a) Assume that \(L \cap \Sigma = \emptyset\). We will proceed by induction. Since a linear affine mapping is a bi-Lipschitz homeomorphism, we can assume that \(\pi_L\) coincide with the projection \(\pi : \mathbb{C}^n \ni (x_1, ..., x_n) \mapsto (x_1, ..., x_k, 0, ..., 0) \in \mathbb{C}^k \times \{0, ..., 0\}\). We can decompose \(\pi\) into two projections: \(\pi = \pi_2 \circ \pi_1\), where \(\pi_1 : \mathbb{C}^n \ni (x_1, ..., x_n) \mapsto (x_1, ..., x_{n-1}, 0) = \mathbb{C}^{n-1} \times 0\) is a projection with a center \(P_1 = (0 : 0 : ... : 1)\) and \(\pi_2 : \mathbb{C}^{n-1} \ni (x_1, ..., x_{n-1}, 0) \mapsto (x_1, ..., x_k, 0, ..., 0) \in \mathbb{C}^k \times \{0, ..., 0\}\) is a projection with a center \(L' = \{x_0 = 0, ..., x_k = 0\}\). Since \(P_1 \in L\) and consequently \(P_1 \notin \Sigma\), we prove that \(\pi_1\) is an algebraic bi-Lipschitz
homeomorphism. Indeed, let $P_1 \in \mathbb{C}P^{n-1} \setminus \Sigma$ and let $H \subset \mathbb{C}^n$ be a hyperplane, such that $P \not\in H$. Since a complex linear isomorphism is a bi-Lipschitz homeomorphism, we can assume that $P_1 = (0 : 0 : \ldots : 1)$ and $H = \{x_n = 0\}$. We show that the projection $p : X \to H$ with center at $P_1$ is an algebraic bi-Lipschitz homeomorphism. Of course $||p(x) - p(y)|| \leq ||x - y||$. Assume that $p$ is not bi-Lipschitz, i.e., there is a sequence of points $x_j, y_j \in X$ such that

$$\frac{||p(x_j) - p(y_j)||}{||x_j - y_j||} \to 0,$$

as $n \to \infty$. Let $x_j - y_j = (a_1(j), \ldots, a_{n-1}(j), b(j))$ and denote by $P_j$ the corresponding point $(a_1(j) : \ldots : a_{n-1}(j) : b(j))$ in $\mathbb{C}P^{n-1}$. Hence

$$P_j = \frac{(a_1(j) : \ldots : a_{n-1}(j) : b(j))}{||x_j - y_j||}.$$

Since $(a_1(j) \ldots a_{n-1}(j)) = \frac{p(x_j) - p(y_j)}{||x_j - y_j||} \to 0$, we get that $P_j \to P$. It is a contradiction. Notice that if $\pi_1(X) = X'$, then $\Sigma' = \pi_1(\Sigma)$. Moreover $L' = L \cap \{x_n = 0\}$ and $< L', P_1 >= L$. This means that $\Sigma' \cap L' = 0$. Now we can finish by induction.

b) Assume that $\pi_{L|X}$ is a bi-Lipschitz map and $\Sigma \cap L \neq \emptyset$. As before we can change a system of coordinates in such a way that $\pi : \mathbb{C}^n \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0) \in \mathbb{C}^k \times \{0, \ldots, 0\}$. Moreover, we can assume that $P_1 = (0 : 0 : \ldots : 1) \in \Sigma$. Actually $\pi_1$ is not bi-Lipschitz. Indeed there is a sequence of secants $l_n = (x_n, y_n)$ of $X$ whose directions tend to $P_1$. Let $x_j - y_j = (a_1(j), \ldots, a_{n-1}(j), b(j))$ and denote by $P_j$ the corresponding point $(a_1(j) : \ldots : a_{n-1}(j) : b(j))$ in $\mathbb{C}P^{n-1}$. Hence

$$P_j = \frac{(a_1(j) : \ldots : a_{n-1}(j) : b(j))}{||x_j - y_j||}.$$

Since $P_j \to P$ we have $(a_1(j) \ldots a_{n-1}(j)) = \frac{p(x_j) - p(y_j)}{||x_j - y_j||} \to 0$. Hence the mapping $\pi_1$ is not bi-Lipschitz.

Now it is enough to note that $||\pi_2(x) - \pi_2(y)|| \leq ||x - y||$, hence $||\pi(x_n) - \pi(y_n)|| = ||\pi_2(\pi_1(x_n)) - \pi_2(\pi_1(y_n))|| \leq ||\pi_1(x_n) - \pi_1(y_n)||$. Thus

$$\frac{||x_n - y_n||}{||\pi(x_n) - \pi(y_n)||} \geq \frac{||x_n - y_n||}{||\pi_1(x_n) - \pi_1(y_n)||} \to \infty.$$

This contradiction finishes the proof. \hfill $\square$

**Lemma 3.3.** Let $X \subset \mathbb{C}^n$ be a closed set and let $f : X \to \mathbb{C}^m$ be an algebraic Lipschitz homeomorphism. Let $Y := \text{graph}(f) \subset \mathbb{C}^n \times \mathbb{C}^m$. Then the mapping $\phi : X \ni x \mapsto (x, f(x)) \in Y$ is an algebraic bi-Lipschitz homeomorphism.

**Proof.** The mapping $\phi$ is algebraic by Proposition 2.2. Since $f$ is Lipschitz, there is a constant $C$ such that

$$||f(x) - f(y)|| < C||x - y||.$$

We have

$$||\phi(x) - \phi(y)|| = ||(x - y, f(x) - f(y))|| \leq ||x - y|| + ||f(x) - f(y)|| \leq ||x - y|| + C||x - y|| \leq (1 + C)||x - y||.$$

Moreover

$$||x - y|| \leq ||\phi(x) - \phi(y)||.$$

Hence

$$||x - y|| \leq ||\phi(x) - \phi(y)|| \leq (1 + C)||x - y||.$$
Theorem 3.4. Let $X \subset \mathbb{C}^n, Y \subset \mathbb{C}^m$ be affine algebraic varieties and let $f : X \to Y$ be an algebraic bi-Lipschitz homeomorphism. Then $\deg X = \deg Y$.

Proof. Let $X \subset \mathbb{C}^n$ and let $f : X \to Y$ be an algebraic bi-Lipschitz homeomorphism. Denote by $\Gamma \subset \mathbb{C}^n \times \mathbb{C}^n$ the graph of $f$. By Lemma 3.3 projections $\pi_X : \Gamma \to X$ and $\pi_Y : \Gamma \to Y$ are algebraic bi-Lipschitz homeomorphism. Let $\pi_1 : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$ and $\pi_2 : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$ and denote by $S_1, S_2 \subset \pi_\infty = \mathbb{C}^{pn+n-1}$ centers of these projections. Denote by $\Lambda \subset \pi_\infty$ the set of directions of all secants of $\Gamma$ and let $\Sigma = cl(\Lambda)$. Since $\pi_1|_{\Gamma} = \pi_X$ and $\pi_2|_{\Gamma} = \pi_Y$ we have by Lemma 3.2 that $\Sigma \cap S_1 = \Sigma \cap S_2 = \emptyset$. But $\overline{\Gamma} \cap \Gamma \subset \Sigma$ and consequently $\overline{\Gamma} \cap S_i = \emptyset$ for $i = 1, 2$. Now let $L \subset \mathbb{C}^n$ be a generic linear subspace of dimension $k = \text{codim} X$. Hence $\#(L \cap X) = \deg X$ and $L$ has not common points with $X$ at infinity. Since $\overline{\Gamma} \cap S_1 = \emptyset$ we see that $\#(<S_1, L > \cap \Gamma) = \deg \Gamma$ where by $<S_1, L>$ we mean a linear (projective) subspace spanned by $L$ and $S_1$. However the mapping $\pi_X$ is a bijection, hence $\#(<S_1, L > \cap \Gamma) = \deg X$. In particular $\deg \Gamma = \deg X$. In the same way $\deg \Gamma = \deg Y$. Hence $\deg X = \deg Y$. □

Remark 3.5. In fact a more general statement is true. We can say after [4] that the mapping $f : X \to Y$ is bi-Lipschitz at infinity if there exist compact sets $K, K'$ such that the mapping $f' : X \setminus K \ni x \mapsto f(x) \in Y \setminus K'$ is bi-Lipschitz. It is easy to see that our proof works if $f$ is an algebraic homeomorphism, which is bi-Lipschitz at infinity. Indeed under this assumption we still have $\overline{\Gamma} \cap \Gamma \subset \Sigma$, where $\Gamma = \text{graph}(f)$ and $\Sigma$ is the set of directions of secants of $\text{graph}(f')$. Moreover, there are sufficiently general linear subspaces which omit $K$ or $K'$. Hence in fact we can state:

Theorem 3.6. Let $X \subset \mathbb{C}^n, Y \subset \mathbb{C}^m$ be affine algebraic varieties and let $f : X \to Y$ be a birational correspondence. Assume that there exist compact sets $K, K'$ such that $f : X \setminus K \to Y \setminus K'$ is defined (as continuous mapping) and bi-Lipschitz. Then $\deg X = \deg Y$.

Remark 3.7. It is worth to say, that affine cones $C(X), C(Y)$ mentioned in the introduction are birationally and bi-Lipschitz equivalent, but they have different degrees.

4. Proof of the Theorem 4.1

In a similar way we can prove:

Theorem 4.1. Let $(X, 0) \subset (\mathbb{C}^n, 0), (Y, 0) \subset (\mathbb{C}^m, 0)$ be germs of analytic sets and let $f : (X, 0) \to (Y, 0)$ be a $c$-holomorphic and bi-Lipschitz homeomorphism. Then $\text{mult}_0 X = \text{mult}_0 Y$.

Proof. Let $U, V$ be small neighborhoods of $0$ in $\mathbb{C}^n$ and $\mathbb{C}^m$ such that the mapping $f : U \cap X = X' \to V \cap Y = Y'$ is defined and it is by-Lipschitz. Denote by $\Gamma \subset U \times V$ the graph of $f$. By Lemma 3.3 projections $\pi_{X'} : \Gamma \to X'$ and $\pi_{Y'} : \Gamma \to Y'$ are bi-Lipschitz homeomorphism. Let $\pi_1 : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$ and $\pi_2 : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$ and denote by $S_1, S_2 \subset \pi_\infty = \mathbb{C}^{pn+n-1}$ centers of these projections. Denote by $\Lambda \subset \pi_\infty$ the set of directions of all secants of $\Gamma$ and let $\Sigma = cl(\Lambda)$. Since $\pi_1|_{\Gamma} = \pi_X$ and $\pi_2|_{\Gamma} = \pi_Y$ we have by Lemma 3.2 that $\Sigma \cap S_1 = \Sigma \cap S_2 = \emptyset$. But $\overline{C(0, \Gamma)} \setminus C(0, \Gamma) \subset \Sigma$ and consequently $\overline{C(0, \Gamma)} \cap S_i = \emptyset$ for $i = 1, 2$. Now let $L \subset \mathbb{C}^n$ be a generic linear subspace of dimension $k = \text{codim} X$. Hence $\#(L \cap X') = \text{mult}_0 X$ and $L$ has not common points with $C(0, X)$ at infinity (we can shrink $U, V$ if necessary!). Since $\overline{C(0, \Gamma)} \cap S_1 = \emptyset$ we see that
\#(< S_1, L > \cap \Gamma) = \text{mult}_0 \Gamma \text{ where by } < S_1, L > \text{ we mean a linear (projective) subspace spanned by } L \text{ and } S_1. \text{ However the mapping } \pi_{X'} \text{ is a bijection, hence } \#(< S_1, L > \cap \Gamma) = \text{mult}_0 X. \text{ In particular mult}_0 \Gamma = \text{mult}_0 X. \text{ In the same way mult}_0 \Gamma = \text{mult}_0 Y. \text{ Hence mult}_0 X = \text{mult}_0 Y. \square

**Corollary 4.2.** Let \( X \subset \mathbb{C}^n, Y \subset \mathbb{C}^m \) be affine algebraic varieties and let \( f : X \rightarrow Y \) be an algebraic bi-Lipschitz homeomorphism. Then \( \deg X = \deg Y \) and for every \( x \in X \) we have \( \text{mult}_x X = \text{mult}_{f(x)} Y. \)

**5. Normality is not a bi-Lipschitz invariant**

In \([2]\) (see also \([6]\)) the authors proved that the smoothness of an analytic space is a bi-Lipschitz invariant. The natural question is whether the normality is also a bi-Lipschitz invariant. Directly from \([2]\) and \([6]\) we have:

**Theorem 5.1.** Let \((X, 0), (Y, 0)\) be analytic hypersurfaces in \((\mathbb{C}^n, 0)\). If they are bi-Lipschitz equivalent and \( X \) is normal, then also \( Y \) is normal.

**Proof.** Indeed, such a hypersurface is normal if it is smooth in codimension one. Hence our result follows directly from \([2]\) and \([6]\). \square

In this section we show that in general Theorem 5.1 does not hold. We start with:

**Lemma 5.2.** Let \( X \mathbb{P}^n \) be a projective variety, which is not contained in any hyperplane. Let \( C(X) \subset \mathbb{C}^{n+1} \) be a cone over \( X \). Then \( T_0(C(X)) = n + 1 \).

**Proof.** Note that \( T_0(C(X)) \) is the dimension of a minimal smooth germ \( Y_0 \) which contains the germ \( C(X)_0 \). However since \( C(X)_0 \subset Y_0 \) we have that \( C(X) \subset T_0 Y \). Denote by \( \pi \cong \mathbb{P}^n \) the hyperplane at infinity. By the assumption we have \( T_0 Y \cap \pi = \pi \). Hence \( T_0 Y = \mathbb{C}^{n+1} \). \square

**Theorem 5.3.** For every \( r > 1 \) there is a normal affine algebraic variety \( X^r \) of dimension \( r \) and an algebraic bi-Lipschitz homeomorphism \( \phi : X^r \rightarrow Y^r \) such that \( Y^r \) is not normal.

**Proof.** Let us take \( d \)-tuple embedding \( \phi_d \) of \( \mathbb{P}^r \) to \( \mathbb{P}^N(d) \). Denote \( A_d = \phi_d(\mathbb{P}^r) \). Let us note that the space \( A_d \subset \mathbb{P}^N(d) \) is projectively normal i.e., the mapping

\[
\Gamma(\mathbb{P}^N(d), O_{\mathbb{P}^N(d)}(k)) \rightarrow \Gamma(A_d, O_X(k))
\]

is surjective for every \( k \). This means that the cone \( X := C(A_d) \) is a normal space (for details see \([Har]\), p.126, 5.14). Moreover it is easy to check that \( A_d \) is not contained in any linear subspace of \( \mathbb{P}^N(d) \). Hence by Lemma 5.2 we have \( T_0 X = N(d) + 1 \). We can take \( d \) so large that \( N := N(d) > 2r + 1 \). Now consider a generic projection \( \pi : \mathbb{C}^N \rightarrow \mathbb{C}^{2r+1} \). We know by \([1]\) that \( \pi \) restricted to \( X_d \) is by-Lipschitz embedding. Let \( Y = \pi(X) \). The variety \( Y \) is not normal. Indeed, otherwise by the Zariski Main Theorem the mapping \( \pi|X : X \rightarrow Y \) has to be an isomorphism, in particular \( T_0 X = \text{dim} T_0 Y \). Since \( \text{dim} T_0 Y \leq 2r + 1 < N \) it is a contradiction. \square
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