STRICT GENERATORS OF THE SUBREGULAR $\mathcal{W}$-ALGEBRA $\mathcal{W}^{K-N}(\mathfrak{sl}_N, f_{\text{sub}})$ AND COMBINATORIAL DESCRIPTION AT CRITICAL LEVEL

NAOKI GENRA AND TOSHIRO KUWABARA

Abstract. We construct explicitly strong generators of the affine $\mathcal{W}$-algebra $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{\text{sub}})$ of subregular type $A$. Moreover, we are able to describe the OPEs between them at critical level. We also give a description the affine $\mathcal{W}$-algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{\text{sub}})$ in terms of certain fermionic fields, which was conjectured by Adamović.

1. Introduction

For a reductive Lie algebra $\mathfrak{g}$, a nilpotent element $f \in \mathfrak{g}$ and $k \in \mathbb{C}$, the affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is defined as a vertex algebra constructed by the generalized quantum Drinfeld-Sokolov reduction; see [FF90], [KRW03], [KW04]. In this paper, we discuss the affine $\mathcal{W}$-algebra $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{\text{sub}})$ associated with $\mathfrak{sl}_N$ and a subregular nilpotent element $f_{\text{sub}} \in \mathfrak{sl}_N$ with level $K_0 - N$, which we call the subregular $\mathcal{W}$-algebra. Recently, in [Gen17, Section 6], the first author described the subregular $\mathcal{W}$-algebra by using certain screening operators, and showed that the subregular $\mathcal{W}$-algebra is isomorphic to a vertex algebra $\mathcal{W}_N^{(2)}$ introduced by Feigin and Semikhatov in [FS04].

For a principal nilpotent element $f_{\text{pr}} \in \mathfrak{sl}_N$, the corresponding affine $\mathcal{W}$-algebra $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{\text{pr}})$ is a vertex algebra such that, at critical level $K_0 = 0$, $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{\text{pr}})$ coincides with the center of affine vertex algebra $V^{-N}(\mathfrak{sl}_N)$, called the Feigin-Frenkel center. In [AM17, Section 2], Arakawa and Molev explicitly constructed strong generators of the vertex algebra $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{\text{pr}})$. Their images through the Miura map are described by a certain noncommutative analog of the elementary symmetric polynomials, which recovers a result of Fateev and Lukyanov in [FL90].

In Section 3 of this paper, we discuss construction of certain strong generators for the subregular $\mathcal{W}$-algebra $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{\text{sub}})$. Our construction is based on the Feigin-Semikhatov description of $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{\text{sub}})$, which describes $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{\text{sub}})$ as intersection of the kernels of screening operators on a certain lattice vertex algebra. We construct elements $W_m$ ($m = 2, \ldots, N$) of the lattice vertex algebra by using the noncommutative elementary symmetric polynomials of the elements of the Heisenberg part of the lattice vertex algebra, and show that they lie in the intersection of the kernels of the screening operators (Definition 3.8 and Proposition 3.5). We also show that these elements are algebraically independent in the Zhu’s $C_2$ Poisson algebra of the vertex algebra $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{\text{sub}})$, and it implies that the elements $W_2, \ldots, W_{N-1}$, together with the generators $E$, $H$, $F$ of Feigin-Semikhatov’s, strongly generate $\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{\text{sub}})$ (Theorem 3.13).

In Section 4, we discuss the subregular $\mathcal{W}$-algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{\text{sub}})$ at critical level, $K_0 = 0$. At critical level, the vertex algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{\text{sub}})$ has a nontrivial center as a vertex algebra, and the center is naturally isomorphic to the Feigin-Frenkel center of the affine vertex algebra $V^{-N}(\mathfrak{sl}_N)$. We show that our elements $W_2, \ldots, W_N$ strongly generate the center (Proposition 4.1). Moreover, we give an explicit form of OPEs between the strong generators (Theorem 4.4).
In [Ada15], Adamović conjectured that the subregular $W$-algebra $W^{−N}(\mathfrak{sl}_N, f_{\text{sub}})$ is isomorphic to a vertex algebra generated by certain fields, consisting of certain fermionic fields and the generators of the Feigin-Frenkel center. We prove his conjecture by using our strong generators (Theorem 4.5).

**Acknowledgments.** One of the main results of this paper, Theorem 4.5, was first conjectured by Dražen Adamović. The authors would like to express their gratitude to him for showing us his private notes [Ada15]. The authors are deeply grateful to Tomoyuki Arakawa for valuable comments. The authors also thank Boris Feigin and Alexei Semikhatov for fruitful discussion on their construction of the vertex algebra $W_N^{(2)}$. The second author thanks Yoshihiro Takeyama for discussion on the proof of Lemma 3.6.

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2. **Subregular $W$-algebras and Feigin-Semikhatov screenings**

The subregular $W$-algebra $W^{K_0} = W^{K_0−N}(\mathfrak{sl}_N, f_{\text{sub}})$ at level $K_0−N$ is a vertex algebra defined by the generalized quantum Drinfeld-Sokolov reduction associated with $\mathfrak{sl}_N, f_{\text{sub}}$ and $K_0 \in \mathbb{C}$ [KRW03], where $f_{\text{sub}} = e_{−a_2} + \cdots + e_{−a_{N−1}} \in \mathfrak{sl}_N$ is a subregular nilpotent element in $\mathfrak{sl}_N$. We introduce a free field realization of $W^{K_0}$ following [FS04] and [Gen17].

We follow [FBZ04, Kac98] for definitions of vertex algebras, and denote by $A(z) = Y(A, z) = \sum_{n \in \mathbb{Z}} A_n(a)$ a field on $V$ for an element $A$ in a vertex algebra $V$. Let $K$ be an indeterminate and $W^{K}_C[K]$ the subregular $W$-algebra associated with $\mathfrak{sl}_N, f_{\text{sub}}, K$ over $\mathbb{C}[K]$. By definition, we have $W^{K}_C[K] \otimes \mathbb{C}_{K_0} = W^{K_0}$, where $\mathbb{C}_{K_0} = \mathbb{C}[K]/(K − K_0) \simeq \mathbb{C}$ is a one-dimensional $\mathbb{C}[K]$-module on which $K$ acts by $K_0$. See e.g. [ACL18]. Let $V = \mathbb{C}[K] A_{N−1} \oplus \cdots \oplus \mathbb{C}[K] A_1 \oplus \mathbb{C}[K] Q \oplus \mathbb{C}[K] Y$ be a free $\mathbb{C}[K]$-module of rank $N + 1$. We define a symmetric bilinear form on $V$ given by the Gram matrix

\[
\begin{pmatrix}
2K & −K & 0 & 0 & \cdots & \cdots & 0 \\
−K & 2K & −K & 0 & \cdots & \cdots & 0 \\
0 & −K & 2K & −K & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \ \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & −K & 2K & −K & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & −K & 1 & 1 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0
\end{pmatrix}.
\]

Consider a Heisenberg vertex algebra $\mathcal{H}^K$ over $\mathbb{C}[K]$ associated with the bilinear form on $V$. It is a vertex algebra generated by the elements $A_i, Q$ and $Y (i = 1, \ldots, N−1)$ subject to the OPE $a(z)b(w) ∼ (a, b)/(z−w)^2$ where $a, b = A_i, Q$ or $Y$, and $(\cdot, \cdot)$ is the bilinear form on $V$. For a vector $w \in V$, let $\mathcal{H}^K_w$ be the $\mathcal{H}^K$-module of highest weight $w \in V$. We denote the anti-derivation of $w$ by $\partial w$ and the highest weight vector of $\mathcal{H}^K_w$ by $e^w$. The direct sum $V^K_{\mathbb{C}[K]} = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}^K_{mw}$ is equipped with a vertex algebra structure. Indeed, the vertex operator $e^m Y(z) = \sum_n e^m Y_n(z) z^{−n−1}$ corresponding to the highest weight vector $e^m Y$ is a field with the following OPEs

\[
e^m Y(z) e^n Y(w) ∼ 0, \quad a(z) e^m Y(w) ∼ \langle a, mY \rangle \frac{e^m Y}{z−w}
\]

for $m, n \in \mathbb{Z}$, and $a \in V$ and the derivative of $e^m Y(z)$ is given by $\partial e^m Y(z) = e^m Y(z) e^m Y(z)$.
Similarly to $e^Y(z)$, we also have the vertex operator $e^Q(z)$ (resp. $e^{A_i}(z)$) associated with $Q \in \mathcal{V}$ (resp. $A_i \in \mathcal{V}$ for $i = 1, \ldots, N - 1$). The OPEs between these vertex operators and fields of $\mathcal{W} \otimes \mathbb{C}[K]$ are given as follows:

$$a(z)e^b(w) \sim \frac{(a, b)}{z - w} e^b(w), \quad e^{A_i}(z)e^mY(w) \sim 0,$$

$$e^Q(z)e^mY(w) \sim (z - w)^m e^{mY+Q}(w)$$

where $a, b \in \mathcal{V}$, $i = 1, \ldots, N - 1$ and $m \in \mathbb{Z}$. The residue of $e^Q(z)$ (resp. $e^{A_i}(z)$) gives an operator on $\mathcal{V} \otimes \mathbb{C}[K]$ such that $e^Q_{(0)} : \mathcal{H}_{mY} \rightarrow \mathcal{H}_{mY+Q}$ (resp. $e^{A_i}_{(0)} : \mathcal{H}_{mY} \rightarrow \mathcal{H}_{mY+A_i}$) for $m \in \mathbb{Z}$ and $i = 1, \ldots, N - 1$. The operators $e^Q_{(0)}$, $e^{A_i}_{(0)}$ are called screening operators. The vertex algebra given as intersection of the kernels of these screening operators were introduced by Feigin and Semikhatov in [FS04]. Recently, the first author showed that their vertex algebra is isomorphic to the subregular $\mathcal{W}$-algebra.

**Proposition 2.1** ([Gen17], Theorem 6.9). *As a vertex algebra over the ring $\mathbb{C}[K]$, we have an isomorphism*

$$\mu^K : \mathcal{W} \otimes \mathbb{C}[K] \cong \mathcal{V} = \prod_{i=1}^{N-1} \mathcal{V} \otimes \mathcal{W} \otimes \mathbb{C}[K],$$

***Since $\mathcal{W} \otimes \mathbb{C}[K] \cong \mathcal{W} \otimes \mathcal{K}_0$, we have an embedding $\mathcal{W} \otimes \mathbb{C}[K] \hookrightarrow \mathcal{W} \otimes \mathbb{C}[K]$.***

We remark that the embedding is obtained as the composition of three maps $\mu, \mu_\beta, \mu_{\beta \gamma}$, defined as follows. Applying the specialization functor $\otimes \mathbb{C}[K_0]$ to embeddings

$$\text{Ker} e^Q_{(0)} \cap \text{Ker} e^{A_i}_{(0)} \rightarrow \text{Ker} e^Q_{(0)} \cap \text{Ker} e^{A_i}_{(0)} \rightarrow \text{Ker} e^Q_{(0)} \rightarrow \mathcal{V} \otimes \mathbb{C}[K],$$

we have vertex algebra homomorphisms

$$\mathcal{W} \otimes \mathbb{C}[K] \xrightarrow{\mu} \mathcal{V} \otimes \mathbb{C}[K] \cong \mathcal{V} \otimes \mathbb{C}[K],$$

where $\mathcal{V} \otimes \mathbb{C}[K]$, is the affine vertex algebra associated with the Lie subalgebra $\mathfrak{g}_0$ and the bilinear form $\tau_{K_0}$ on $\mathfrak{g}_0$ defined in [Gen17, (2.2)], and $\mathcal{D}^{\mathfrak{h}}(\mathbb{C})$ is the vertex algebra of $\beta\gamma$-system of rank one. It then follows that $\mu, \mu_\beta, \mu_{\beta \gamma}$ are injective maps, called the Miura map for $\mathcal{W} \otimes \mathbb{C}[K]$ [KW04, Gen17], Wakimoto realization for $\mathcal{W} \otimes \mathbb{C}[K]$ [Wak86, Fre05] and Friedan-Martinec-Shenker bosonization [FMS86] respectively.

3. STRONG GENERATORS OF THE SUBREGULAR $\mathcal{W}$-ALGEBRA

In this section, we explicitly construct elements $W_1, \ldots, W_N$ of the subregular $\mathcal{W}$-algebra, and show that $E, F, H, W_1, \ldots, W_{N-1}$ strongly generate the vertex algebra $\mathcal{W} \otimes \mathbb{C}[K]$ for any $K_0 \in \mathbb{C} \setminus \{1\}$. In the rest of this paper, we extend the vertex algebra $\mathcal{W} \otimes \mathbb{C}[K]$ by the ring $R := \mathbb{C}[K, (K - 1)^{-1}]$, and write $\mathcal{W} \otimes \mathbb{C}[K] \otimes \mathbb{C}[K]$ for short.

Set $t_N(K) = K(N - 1)/N - 1$. Define three elements in $\mathcal{W}$

\[ H = t_N(K)Y + Q + \frac{N - 1}{N}A_1 + \cdots + \frac{2}{N}A_{N-2} + \frac{1}{N}A_{N-1}, \]

\[ E = e^Y, \quad F = -\rho_N \cdots \rho_2 \rho_1 e^{-Y} \]
where \( \rho_i = (K-1)(\partial + Y_{(-1)}) + Q_{(-1)} + \sum_{j=1}^{i-1} A_{j(-1)} \) for \( i = 1, \ldots, N \). Note that
\[
\rho_1 e^{jY} = Q_{(-1)} e^{jY} \text{ because } 0 e^{jY} = -Y_{(-1)} e^{jY}.
\]
In [FS04], they showed that these three elements generate the vertex algebra \( \mathcal{W}_R^K \). The first goal of the present paper is to construct a set of strong generators of \( \mathcal{W}_R^K \), including these three elements \( H, E \) and \( F \).

Define \( N \) elements in the Heisenberg part \( \mathcal{H}^K \) of the vertex algebra \( \mathcal{V}_R^K \)
\[
X_i = -\frac{K}{N} Y - \sum_{j=1}^{i-1} \frac{j}{N} A_j + \sum_{j=i}^{N-1} \frac{N-j}{N} A_j \in \mathcal{H}^K
\]
for \( i = 1, \ldots, N \). Then, we have \( \rho_i = (K-1)\partial + H_{(-1)} - X_{i(-1)} \) for \( i = 1, \ldots, N \). Also, we define
\[
X_0 = -\frac{K}{N} Y + Q + \frac{N-1}{N} A_1 + \cdots + \frac{1}{N} A_{N-1} \in \mathcal{H}^K \subset \mathcal{V}_R^K
\]
and \( \rho_0 = (K-1)\partial + H_{(-1)} - X_{0(-1)} = (K-1)(\partial + Y_{(-1)}) \).

Recall the definition of the noncommutative elementary symmetric polynomials (cf. [Mol18, (12.48)]). Let \( \xi_1, \ldots, \xi_N \) be mutually noncommutative \( N \) operators on a certain vector space. Define the \( m \)-th noncommutative elementary symmetric polynomial in \( \xi_1, \ldots, \xi_N \),
\[
e_m(\xi_1, \ldots, \xi_n) = \sum_{i_1 > i_2 > \cdots > i_m} \xi_{i_1}\xi_{i_2}\cdots \xi_{i_m}.
\]
Note that we arrange the operators reverse-lexicographically.

Let \( \sigma_i = (K-1)\partial - X_{i(-1)} \) be an operator on \( \mathcal{V}_R^K \) for \( i = 0, 1, \ldots, N \). For \( m = 1, \ldots, N \), we define an element of \( \mathcal{V}_R^K \)
\[
(1) \quad W'_m = \sum_{k=0}^{N} (-1)^k \prod_{j=1}^{k} \frac{j(K-1)+1}{j(K-1)} \sum_{1 \leq t_1 < \cdots < t_{N-k} \leq N} e_m(\sigma_0, \ldots, \sigma_{t_1}, \ldots, \sigma_{t_{N-k}}).
\]

We also introduce the generating function of these elements \( W'_m \) \( (m = 1, \ldots, N) \). Let \( u \) be an indeterminate which commutes with all other elements. Note that, for operators \( \xi_1, \ldots, \xi_N \), we have
\[
(u + \xi_N) \cdots (u + \xi_1) = \sum_{m=0}^{N} e_m(\xi_1, \ldots, \xi_N) u^{N-m}
\]
by the definition of \( e_m \). Setting \( \sigma_i(u) = u + \sigma_i \), we have
\[
(2) \quad W^{(N)}(u) := \sum_{m=0}^{N} W'_m u^{N-m} = \sum_{k=0}^{N} (-1)^k \prod_{j=1}^{k} \frac{j(K-1)+1}{j(K-1)} \sum_{N \geq t_1 > \cdots > t_{N-k} \geq 1} \sigma_{t_1}(u) \cdots \sigma_{t_{N-k}}(u) \sigma_0(u)^k 1
\]
where \( W^{(N)}_0 \) is equal to \( 1 \) up to multiplication by a certain constant.

We will show that the elements \( W'_m \) \( (m = 1, \ldots, N) \) belong to the subregular \( \mathcal{W}_R^K \) by calculating the action of the screening operators \( e_{(0)}^{j\mathcal{A}_i}, e_{(0)}^{j\mathcal{A}_i} \) \( (i = 1, \ldots, N-1) \).

Lemma 3.1. (1) For \( i = 1, \ldots, N-1, j = 0, \ldots, N \) and \( m \in \mathbb{Z} \), we have
\[
[e_{(m)}^{j\mathcal{A}_i}, \sigma_j(u)] = \begin{cases} 
(m(K-1) + K) e_{(m-1)}^{j\mathcal{A}_i} & (j = i) \\
(m(K-1) - K) e_{(m-1)}^{j\mathcal{A}_i} & (j = i+1) \\
m(K-1) e_{(m-1)}^{j\mathcal{A}_i} & (j \neq i, i+1)
\end{cases}
\]
in $\bigoplus_{m \in \mathbb{Z}} \text{Hom}_{\mathbb{C}[K]}(H_{mY}^K, \mathcal{H}_{mY+Q})[u]$.

(2) For $j = 0, \ldots, N$ and $m \in \mathbb{Z}$, we have

$$[e_f^{(Q)}, \sigma_j(u)] = \begin{cases} 
(m + 1)(K - 1)e_f^{(m-1)} & (j = 0) \\
(m(K - 1) - K)e_f^{(m-1)} & (j = 1) \\
m(K - 1)e_f^{(m-1)} & (j = 2, \ldots, N)
\end{cases}$$

in $\bigoplus_{m \in \mathbb{Z}} \text{Hom}_{\mathbb{C}[K]}(H_{mY}^K, \mathcal{H}_{mY+Q})[u]$.

Proof. Note that $(A_i, X_i) = K, (A_i, X_{i+1}) = -K, (A_i, X_j) = 0$ for $j \neq i, i+1$, and $(Q, X_0) = -K+1, (Q, X_1) = -K, (Q, X_j) = 0$ for $j = 2, \ldots, N$. Then, both (1) and (2) can be checked by direct computation. \hfill $\square$

**Proposition 3.2.** For $i = 1, \ldots, N - 1$, we have $e_f^{(0)} \tilde{W}^{(N)}(u) = 0$.

Proof. First, note that we have

$$e_f^{(A_i)} \sigma_N(u) \ldots \sigma_{i+2}(u) \sigma_{i+1}(u) \ldots 1 = \sigma_N(u) \ldots \sigma_{i+2}(u) e_f^{(A_i)} \sigma_{i+1}(u) \ldots 1$$

since the screening operator $e_f^{(A_i)}$ commutes with $\sigma_j(u)$ for $j \neq i, i+1$ by Lemma 3.1.

In $\tilde{W}^{(N)}(u)$, there exists three kinds of terms; i) $\ldots \sigma_{i+1}(u) \sigma_i(u) \ldots 1$, terms with both factors $\sigma_{i+1}(u)$ and $\sigma_i(u)$, ii) $\ldots \sigma_{i+1}(u) \ldots 1$ or $\ldots \sigma_i(u) \ldots 1$, terms with either $\sigma_{i+1}(u)$ or $\sigma_i(u)$, iii) $(\sigma_{i+1}(u))^\lambda (\sigma_i(u))^\lambda \ldots 1$, terms without $\sigma_{i+1}(u)$ nor $\sigma_i(u)$. We consider the action of $e_f^{(A_i)}$ in these three cases individually.

i) By Lemma 3.1 (1), we have

$$e_f^{(A_i)} \ldots \sigma_{i+1}(u) \sigma_i(u) \ldots 1$$

$$= \ldots [e_f^{(A_i)}, \sigma_{i+1}(u)] \sigma_i(u) \ldots 1 + \ldots \sigma_{i+1}(u)[e_f^{(A_i)}, \sigma_i(u)] \ldots 1$$

$$= K \cdot (\sigma_{i+1}(u) - \sigma_i(u)) e_f^{(A_i)} \ldots 1$$

$$- K \cdot ((-1)(K - 1) + K)e_f^{(A_i)} \ldots 1$$

$$= K \cdot \{ A_i(-1) e_f^{(A_i)} - e_f^{(A_i)} \} \ldots 1 = 0.$$

Here we used $[e_f^{(A_i)}, X_{j(i)}] = 0$ for $j \neq i, i+1$ with $m, n \in \mathbb{Z}$, and $e_f^{(A_i)} = (A_i(-1) e_f^{(A_i)})(-1)$. 

ii) Note that the generating function $\tilde{W}^{(N)}(u)$ is symmetric by permutation between factors $\sigma_1(u), \ldots, \sigma_N(u)$. This implies that, for a term $(\ldots) \sigma_{i+1}(u) (\ldots \sigma_1(u) (\ldots 1$ of type ii), we also have a term $(\ldots) \sigma_i(u) (\ldots \sigma_1(u) (\ldots 1$ of exactly the same form except for replacing $\sigma_{i+1}(u)$ by $\sigma_i(u)$. Here the factors different from $\sigma_{i+1}(u)$ and $\sigma_i(u)$ are denoted by $(\ldots)_j (j = 1, 2)$. Then, by Lemma 3.1 (1), we have

$$e_f^{(A_i)} \{ (\ldots) \sigma_{i+1}(u) (\ldots \sigma_1(u) (\ldots \sigma_1(u) (\ldots 1$$

$$= (\ldots)(-K e_f^{(A_i)}(\ldots)_1 \sigma_1(u)(\ldots)_1 + (\ldots)(+K e_f^{(A_i)}(\ldots)_1 \sigma_1(u)(\ldots)_1 = 0.$$

iii) A term without $\sigma_{i+1}(u)$ and $\sigma_i(u)$ trivially vanishes by the action of the screening operator $e_f^{(A_i)}$ by Lemma 3.1 (1).

As a consequence, we have $e_f^{(A_i)} \tilde{W}^{(N)}(u) = 0$. \hfill $\square$

The action of another screening operator $e_f^{(Q)}$ is more complicated. Previous to the calculation of $e_f^{(Q)} \tilde{W}^{(N)}(u)$ we prepare the following lemma.
Lemma 3.3. For \( m \geq 0 \), we have

\[
e^{\int_Q (\sigma_1(u)\sigma_0(u)^m - \frac{(m+1)(K-1)+1}{(m+1)(K-1)} \sigma_0(u)^{m+1})} = 0
\]

Proof. First, note that

\[
[e^{\int_Q}, \sigma_0(u)^l] = \sum_{i=1}^{l} (-1)^i (K-1)^i \frac{l!}{(l-i)!} \sigma_0(u)^{l-i} e^{\int_Q},
\]

\[
[e^{\int_Q}, \sigma_0(u)^{l-1}] = \sum_{i=2}^{l+1} (-1)^{i-1}(K-1)^{i-1} \frac{l!}{(l-i+1)!} \sigma_0(u)^{l-i+1} e^{\int_Q},
\]

by Lemma 3.1 (2).

First we deal with the first term of the equality of the lemma. Using the fact

\[
\sigma_1(u) = \sigma_0(u) + Q_{(-1)}
\]

and the identity

\[
Q_{(-1)} \sigma_0(u)^{m-i} = \sum_{j=0}^{m-i} (-1)^j \frac{(m-i)!}{(m-i-j)!} \sigma_0(u)^{m-i-j} Q_{(-1)},
\]

we obtain

\[
(3) \quad e^{\int_Q} \sigma_1(u)\sigma_0(u)^m 1 = \sigma_1(u) e^{\int_Q} \sigma_0(u)^m 1 - Ke^{\int_Q} \sigma_0(u)^m 1
\]

\[
= -K \sigma_0(u)^m e^{\int_Q}_{(-1)} 1
\]

\[
+ \sum_{i=2}^{m+1} (-1)^i (K-1)^i \frac{l!}{(m-i+1)!} \sigma_0(u)^{m-i+1} e^{\int_Q}_{(-1)} 1
\]

\[
+ \sum_{i=1}^{m} (-1)^i (K-1)^i \frac{l!}{(m-i)!} \sigma_1(u)\sigma_0(u)^{m-i} e^{\int_Q}_{(-1)} 1
\]

\[
= \sum_{i=1}^{m+1} (-1)^i (K-1)^i \frac{l!}{(m-i+1)!} \sigma_0(u)^{m-i+1} e^{\int_Q}_{(-1)} 1
\]

\[
+ \sum_{i=1}^{m} (-1)^i (K-1)^i \frac{l!}{(m-i)!} \sigma_0(u)^{m-i} \sigma_0(u)^{m-i} e^{\int_Q}_{(-1)} 1
\]

\[
+ \sum_{i=1}^{m} (-1)^i (K-1)^i \frac{l!}{(m-i)!} Q_{(-1)} \sigma_0(u)^{m-i} e^{\int_Q}_{(-1)} 1
\]

\[
= \sum_{i=1}^{m+1} (-1)^i (K-1)^i \frac{l!}{(m-i+1)!} \sigma_0(u)^{m-i+1} e^{\int_Q}_{(-1)} 1
\]

\[
+ \sum_{i=1}^{m} (-1)^i (K-1)^i \frac{l!}{(m-i)!} \sigma_0(u)^{m-i} \sigma_0(u)^{m-i} e^{\int_Q}_{(-1)} 1
\]

\[
+ \sum_{i=1}^{m} \sum_{j=0}^{m-i} (-1)^{i+j}(K-1)^{i+j} \frac{l!}{(m-i-j)!} \sigma_0(u)^{m-i-j} Q_{(-1)} e^{\int_Q}_{(-1)} 1.
\]
On the other hand, for the second term of the equality of the lemma, we have

\[
\begin{align*}
(4) \quad e^{(Q) m+1} \frac{(m+1)(K-1)+1}{(m+1)(K-1)} \sigma_0(u)^{m+1} 1 \\
= \sum_{i=1}^{m+1} (-1)^i(K-1)^i \frac{(m+1)!}{(m-i+1)!} \sigma_0(u)^{m+1-i} e^{(Q) i} 1 \\
+ \sum_{i=1}^{m+1} (-1)^i(K-1)^{i-1} \frac{m!}{(m-i+1)!} \sigma_0(u)^{m+1-i} e^{(Q) i} 1
\end{align*}
\]

The second term in (3) and the first term in (4) are canceled out. Since \( \partial e^{(Q)} = Q_{(-1)} e^{(Q)} \), we have

\[
\sum_{j=0}^{k} Q_{(-j-1)} e^{(Q) (-k+j)} 1 = k e^{(Q) (-k-1)} 1
\]

for \( k \geq 1 \). Therefore, we obtain

\[
\begin{align*}
e^{(Q) (1)} \left( \sigma_1(u) \sigma_0(u)^m \cdot \frac{(m+1)(K-1)+1}{(m+1)(K-1)} \sigma_0(u)^{m+1} 1 \right) \\
= \sum_{i=2}^{m+1} (-1)^i(K-1)^i \frac{m!}{(m-i+1)!} (i-1) \sigma_0(u)^{m-i+1} e^{(Q) i} 1 \\
+ \sum_{k=1}^{m} \sum_{j=0}^{k} (-1)^k(K-1)^k \frac{m!}{(m-k)!} \sigma_0(u)^{m-k} Q_{(-j-1)} e^{(Q) (-k+j)} 1 \\
= \sum_{i=2}^{m+1} (-1)^i(K-1)^i \frac{m!}{(m-i+1)!} (i-1) \sigma_0(u)^{m-i+1} e^{(Q) i} 1 \\
+ \sum_{k=1}^{m} (-1)^k(K-1)^k \frac{m!}{(m-k)!} \sigma_0(u)^{m-k} k e^{(Q) (-k-1)} 1 = 0.
\end{align*}
\]

\[\square\]

**Proposition 3.4.** We have \( e^{(Q) \tilde{W}^{(N)}(u)} = 0 \).

**Proof.** We split terms in the definition of \( \tilde{W}^{(N)}(u) \) into two parts; terms with the factor \( \sigma_1(u) \) and terms without \( \sigma_1(u) \). Note that the screening operator \( e^{(Q)} \) commutes with \( \sigma_1(u) \) for all \( i \neq 0, 1 \). Then we have

\[
\begin{align*}
e^{(Q) \tilde{W}^{(N)}(u)} = e^{(Q) m} \sum_{m=0}^{N} \frac{1}{m} \prod_{j=1}^{m} j(K-1) + 1 \sum_{N \geq \ell_1 > \cdots > \ell_{N-m} \geq 1} \sigma_i(u) \cdots \sigma_{iN-m} \sigma_0(u)^{m} 1 \\
= e^{(Q) m} \left\{ \sum_{m=0}^{N} \frac{1}{m} \prod_{j=1}^{m} j(K-1) + 1 \sum_{N \geq \ell_1 > \cdots > \ell_{N-m-1} \geq 2} \sigma_i(u) \cdots \sigma_{iN-m-1} \sigma_1(u) \sigma_0(u)^{m} 1 \\
+ \sum_{m=0}^{N} \frac{1}{m+1} \prod_{j=1}^{m+1} j(K-1) + 1 \sum_{N \geq \ell_1 > \cdots > \ell_{N-m-1} \geq 2} \sigma_i(u) \cdots \sigma_{iN-m-1} \sigma_0(u)^{m+1} 1 \right\} = 0
\end{align*}
\]

by Lemma 3.3. \[\square\]
By Proposition 3.2 and Proposition 3.4, we have \( W'_m \in \ker \epsilon^{f_Q}_{(0)} \cap \bigcap_{k=1}^{N-1} \ker \epsilon^{f_{A_k}}_{(0)} \) for \( m = 1, \ldots, N \). Thus, we have the following proposition by a consequence of Proposition 2.1

**Proposition 3.5.** For \( m = 1, \ldots, N \), we have \( W'_m \in W^K_R \).

To construct elements which give strong generators of the vertex algebra \( W^K_R \), we need to normalize the elements \( W'_m \) for \( m = 2, \ldots, N \).

**Lemma 3.6.** Let \( q \) be an indeterminate. For \( l, m \geq k \), we have the following identity:

\[
\sum_{l=k}^{N} (-1)^{l-k} \binom{N-m+k}{l} \frac{l}{k} \prod_{j=k+1}^{l} \left( 1 + \frac{q}{j} \right) = \frac{(-1)^{N-m}}{(N-m)!} \prod_{j=0}^{N-m-1} (q-j).
\]

**Proof.** By direct calculation, we have

\[
\sum_{l=k}^{N} (-1)^{l-k} \binom{N-m+k}{l} \frac{l}{k} \prod_{j=k+1}^{l} \left( 1 + \frac{q}{j} \right)
= \sum_{l=0}^{N-m} (-1)^l \binom{N-m+k}{l+k} \frac{l}{l+k} \prod_{j=1}^{l} (q+k+j)
= \frac{(N-m+k)!}{(N-m)! l!} \sum_{l=0}^{N-m} \frac{(-N+m)_l (q+k+l) \cdot 1!}{(k+1)_l} l!
\]

where \((x)_l = \prod_{j=0}^{l-1} x+j\). The RHS can be described by the hypergeometric function \( _2F_1(a, b; c; z) \), and thus by the \( \Gamma \)-function \( \Gamma(z) \) with applying Gauss’s hypergeometric theorem. Then we have

\[
\sum_{l=k}^{N} (-1)^{l-k} \binom{N-m+k}{l} \frac{l}{k} \prod_{j=k+1}^{l} \left( 1 + \frac{q}{j} \right)
= \frac{(N-m+k)!}{(N-m)! l!} \cdot _2F_1(-N+m, q+k+1, k+1; 1) \cdot \frac{(N-m)! \Gamma(k+1) \Gamma(N-m-q)}{(N-m)! \Gamma(N-m+k+1) \Gamma(-q)}
= \frac{(N-m+k)!}{(N-m)! l!} \cdot \prod_{j=0}^{N-m-1} (-q+j) = \frac{(-1)^{N-m}}{(N-m)!} \prod_{j=0}^{N-m-1} (q-j).
\]

\(\square\)

**Lemma 3.7.** Let \( \xi_0, \xi_1, \ldots, \xi_N \) be operators. For \( m = 1, \ldots, N \), we have

\[
\sum_{k=0}^{N} (-1)^k \prod_{j=1}^{k} \frac{j(K-1)+1}{j(K-1)} \sum_{1 \leq i_1 < \cdots < i_{N-k} \leq N}^{k\text{-times}} e_m(\xi_0, \xi_{i_1}, \ldots, \xi_{i_{N-k}})
= \prod_{j=1}^{N} \frac{j(K-1)-K}{j(K-1)} \sum_{k=0}^{m} (-1)^k \left( \prod_{j=1}^{k} \frac{j(K-1)+1}{j(K-1)} \right) e_{m-k}(\xi_1, \ldots, \xi_N) \xi_0^k.
\]
Proof. Below we write \( e_{m-k}(\xi_1, \ldots, \xi_N) \) by \( e_{m-k} \) for short.

\[
\sum_{k=0}^{N} (-1)^k \prod_{j=1}^{k} \frac{j(K-1)+1}{j(K-1)} \sum_{1 \leq i_1 < \cdots < i_{N-k} \leq N}^{k\text{-times}} e_{m}(\xi_0, \ldots, \xi_{i_1}, \ldots, \xi_{i_{N-k}}) = \sum_{l=0}^{N} (-1)^l \prod_{j=1}^{l} \frac{j(K-1)+1}{j(K-1)} \sum_{\min(l,m)}^{m\text{-times}} e_{m}(\xi_{i_1}, \ldots, \xi_{i_{l-1}}, \ldots, \xi_{i_{l-m-k}}) = \sum_{k=0}^{m} (-1)^k \left( \prod_{j=1}^{k} \frac{j(K-1)+1}{j(K-1)} \right) \cdot \sum_{l=k}^{N} (-1)^{l-k} \left( \frac{N-m+k}{l} \right) \prod_{j=k+1}^{l} \left( 1 + \frac{1}{j(K-1)} \right) e_{m-k} \xi_0^k
\]

Applying Lemma 3.6 for \( q = 1/(K-1) \), we obtain the identity of the lemma. \( \square \)

Applying the above lemma for \( \xi_i = \sigma_i \), we have identities for the elements \( W'_m (m = 1, \ldots, N) \).

\[
W'_m = \sum_{k=0}^{N} (-1)^k \left( \prod_{j=1}^{k} \frac{j(K-1)+1}{j(K-1)} \right) \cdot \sum_{1 \leq i_1 < \cdots < i_{N-k} \leq N}^{k\text{-times}} e_{m}(\sigma_{i_0}, \ldots, \sigma_{i_1}, \ldots, \sigma_{i_{N-k}}) 1
\]

\[
\sum_{l=k}^{N} (-1)^{l-k} \left( \frac{N-m+k}{l} \right) \prod_{j=k+1}^{l} \left( 1 + \frac{1}{j(K-1)} \right) e_{m-k}(\sigma_1, \ldots, \sigma_N) \sigma_0^k 1
\]

Definition 3.8. For \( m = 1, \ldots, N \), set

\[
(5) \quad W'_m = (-1)^m \prod_{j=1}^{N-m} \frac{j(K-1)}{j(K-1)-K} W'_m
\]

\[
\sum_{k=0}^{m} (-1)^{m+k} \left( \prod_{j=1}^{k} \frac{j(K-1)+1}{j(K-1)} \right) \cdot e_{m-k}(\sigma_1, \ldots, \sigma_N) \sigma_0^k 1 \in W^K_R
\]

Then, we define elements \( W_m \in W^K_R \) for \( m = 1, \ldots, N \) inductively as follows:

\[
(6) \quad W_m = W'_m - \sum_{k=0}^{m-1} (-1)^k \left( \prod_{j=1}^{k} \frac{j(K-1)+1}{j(K-1)} \right) \cdot W_{m-k(-1)}(H(-1))^k 1,
\]

and \( W_1 = 0 \).

In [FS04, Lemma 2.3.5], a conformal vector \( \omega \) of the vertex algebra \( W^K_{\mathbb{C}[K]} \otimes \mathbb{C}[K, K^{-1}] \) is explicitly given over \( \mathbb{C}[K, K^{-1}] \). By direct calculation, we have the following relations between \( W_1 \), \( W_2 \), and \( H, \omega \).

Proposition 3.9. We have the following identities between the elements \( W_2 \) and \( H, \omega \):

\[
\omega = -\frac{1}{K} W_2 + \frac{N-1}{2} \frac{2K-1}{(K-1)^2} H(-1)H + \frac{N(2N-3)K - (2N-2)}{2} \frac{1}{K-1} \partial H
\]

over \( \mathbb{C}[K, K^{-1}, (K-1)^{-1}] \).
In the rest of this section, we show that the \( N+1 \) elements \( E, F, H \) and \( W_m \) for \( m = 2, \ldots, N-1 \) strongly generate the subregular \( W \)-algebra \( W^K_0 \otimes \mathbb{C}K_0 \) for \( K_0 \in \mathbb{C} \setminus \{1\} \).

For a vertex algebra \( V \), let \( \overline{A}(V) = V/C_2(V) \) be Zhu’s \( C_2 \) Poisson algebra of \( V \), where \( C_2(V) = V(-2)V \). For an element \( a \in V \), we denote its image in \( \overline{A}(V) \) by \( \overline{a} \).

**Lemma 3.10.** For arbitrary \( K_0 \in \mathbb{C} \setminus \{1\} \), we have \( \overline{W}_m = e_m(\overline{X}_1, \ldots, \overline{X}_N) \) in \( \overline{A}(V^K_0) \) for \( m = 1, \ldots, N \).

**Proof.** Since \( Y = e(Y)e^{-jY} \equiv 0 \) modulo \( C_2(V^K_0) \), we have \( \overline{H} = \overline{X}_0 \). Thus, by (5), we have

\[
\overline{W}_m' = \sum_{k=0}^{m} (-1)^k \left( \prod_{j=1}^{k} \frac{j(K-1)+1}{j(K-1)} \right) \cdot e_{m-k}(\overline{X}_1, \ldots, \overline{X}_N) \overline{H}^k.
\]

Then, the claim of the lemma follows from (7) and (6) by induction on \( m \).

**Lemma 3.11.** For arbitrary \( K_0 \in \mathbb{C} \setminus \{1\} \), the elements \( \overline{H}, \overline{W}_2, \ldots, \overline{W}_N \in \overline{A}(V^K_0) \) are algebraically independent over the field \( \mathbb{C} \).

**Proof.** First, note that it is enough to see that \( \overline{H}, \overline{W}_2, \ldots, \overline{W}_N \) are algebraically independent in \( \overline{A}(V^K_0) \), since we have \( C_2(V^K_0) \subset C_2(V^K_0) \). Below we write \( e_m = e_m(\overline{X}_1, \ldots, \overline{X}_N) \) for short. Note that we only have elements of form \( e^{m(Y)} \) for \( m \in \mathbb{Z} \) in the lattice part of \( V^K_0 \), and thus \( \overline{Q}, \overline{A}_1, \ldots, \overline{A}_{N-1} \) are linearly independent in \( \overline{A}(V^K_0) \), while \( Y = e(Y)e^{-jY} \equiv 0 \) modulo \( C_2(V^K_0) \). Since \( \overline{X}_i \in \bigoplus_{j=1}^{N-1} \mathbb{C} \overline{A}_j \), for all \( i = 1, \ldots, N \) and \( \overline{H} = \overline{X}_0 \notin \bigoplus_{j=1}^{N-1} \mathbb{C} \overline{A}_j \), \( \overline{H} \) is algebraically independent of \( \overline{W}_2 = \overline{X}_2, \ldots, \overline{W}_N = \overline{X}_N \).

We identify the vector space \( \bigoplus_{j=1}^{N-1} \mathbb{C} \overline{A}_j \) with the Cartan subalgebra

\[
\mathfrak{h} = \left\{ \sum_{i=1}^{N} c_i \varepsilon_i \in \bigoplus_{i=1}^{N} \mathbb{C} \varepsilon_i \mid c_1 + \cdots + c_N = 0 \right\} \simeq \mathbb{C}^{N-1}
\]

by the standard way; \( \overline{A}_j = \varepsilon_j - \varepsilon_{j+1} \) (\( j = 1, \ldots, N-1 \)). Under this identification, we have \( \overline{X}_i = \varepsilon_i - (1/N) \sum_{j=1}^{N} \varepsilon_j \) for \( i = 1, \ldots, N \). Then, the symmetric polynomials \( \overline{\tau}_2, \ldots, \overline{\tau}_N \) are algebraically independent and we have \( \mathbb{C}[\varepsilon_0 \varepsilon_N] = \mathbb{C}[\overline{\tau}_2, \ldots, \overline{\tau}_N] \), while \( \overline{\tau}_1 \) is zero by the classical fact on the Weyl-group-invariant subalgebra \( \mathbb{C}[\varepsilon_0 \varepsilon_N] \). Thus, we have the assertion of the lemma.

**Proposition 3.12.** The Poisson center of \( C_2 \) Poisson algebra \( \overline{A}(V^K_0) \) is generated by \( \overline{W}_2, \ldots, \overline{W}_N \).

**Proof.** Since \( (X_i, Y) = 0 \) for \( i = 1, \ldots, N \), it is easy to check that \( \{ \overline{X}_i, \overline{F} \} = \{ \overline{X}_i, P \} = 0 \). Thus, \( \overline{\tau}_2, \ldots, \overline{\tau}_N \) are Poisson central in \( \overline{A}(V^K_0) \). By [Gen18, Lemma 6.12], \( \mu_{\beta_\gamma} \circ \mu_{\omega} \circ \mu : V(K_0)^\circ \rightarrow V(K_0)^\circ \) induces an embedding \( \overline{A}(V^K_0) \hookrightarrow \overline{A}(V^K_0) \), and thus \( \overline{\tau}_2, \ldots, \overline{\tau}_N \) are Poisson central also in \( \overline{A}(V^K_0) \). By a consequence of [DSK06], the Poisson center of \( \overline{A}(V^K_0) \) is isomorphic to \( \mathbb{C}[\varepsilon_0 \varepsilon_N] \). Therefore, the Poisson center of \( \overline{A}(V^K_0) \) is \( \mathbb{C}[\overline{W}_2, \ldots, \overline{W}_N] \simeq \mathbb{C}[\varepsilon_0 \varepsilon_N] \).

It is easy to check that the elements \( E, H, W_2, W_3, \ldots, W_{N-1}, F \) have conformal weights \( 1, 2, 3, \ldots, N-1, N-1 \) respectively. Note that, for arbitrary \( K_0 \in \mathbb{C} \), the vertex algebra \( W^K_0 \otimes \mathbb{C}K_0 \) is of type \( \mathbb{W}(1, 1, 2, 3, \ldots, N-1, N-1) \), i.e. \( W^K_0 \otimes \mathbb{C}K_0 \) has \( N+1 \) strong generators of conformal weight \( 1, 2, 3, \ldots, N-1, N-1 \).
Theorem 3.13. For arbitrary $K_0 \in \mathbb{C} \setminus \{1\}$, the elements $E, H, W_2, \ldots, W_{N-1}$, $F$ strongly generate the vertex algebra $\mathcal{W}_R^K \otimes \mathcal{C}_{K_0} = \mathcal{W}_R^K$.

Proof. The vertex algebra $V = \mathcal{W}_R^K \otimes \mathcal{C}_{K_0}$ is decomposed as $V = \bigoplus_{d \geq 0} V_d$ where $V_d$ is the subspace of conformal weight $d$. Since $V$ is of type $\mathcal{W}(1, 1, 2, \ldots, N-1, N-1)$, $V_1$ is two-dimensional. On the other hand, there exist two linearly independent elements $E$ and $H$, and thus we have $V_1 = \mathcal{C}E \oplus \mathcal{C}H$.

By induction on $d$, we show that

$$V_d \subseteq \text{Span}\{a_{i(-n_1)} \cdots a_{k(-n_k)} | a_i = E, H, W_2, \ldots, W_d, n_i \geq 1\}$$

for $d = 2, \ldots, N-1$. Assume that (8) holds for $V_{d'}$ with $d' \leq d - 1$. Set

$$U_d = V_d \cap \text{Span}\{a_{i(-n_1)} \cdots a_{k(-n_k)} | a_i = E, H, W_2, \ldots, W_{d-1}, n_i \geq 1\}.$$

Then, $U_d$ is codimension one in $V_d$ since $V$ has exactly one strong generator of conformal weight $d$. We show that $W_d \notin U_d$. Indeed, assume that we have an identity

$$W_d = \sum_p c^{(p)} a^{(p)}_{1(-n_1)} \cdots a^{(p)}_{k(-n_k)} 1$$

where $c^{(p)} \in \mathbb{C}$, $a^{(p)}_i = E, H, W_2, \ldots, W_{d-1}$ and $n_i \geq 1$. Terms containing $E_i(-n)$ ($n \geq 1$) have positive $H_{(0)}$-eigenvalues while $H_{(0)} W_m = 0$ for all $m$ and $H_{(0)} H = 0$. By using decomposition into $H_{(0)}$-eigenspaces, we may assume that the identity (9) holds for $a^{(p)}_1 = H, W_2, \ldots, W_{d-1}$. Taking modulo $\mathcal{C}_2(V)$, we have an identity

$$\overline{W_d} = \sum_p \overline{c^{(p)}} \overline{a^{(p)}_{1(-n_1)} \cdots a^{(p)}_{k(-n_k)}} 1$$

in the algebra $\overline{\mathcal{A}}(V)$. It contradicts Lemma 3.11, and hence we have $W_d \notin U_d$. Therefore, we have $V_d = U_d \oplus \mathbb{C}W_d$ and the induction completes.

Similarly to the above, setting $U_{N-1} = V_{N-1} \cap \text{Span}\{a_{i(-n_1)} \cdots a_{k(-n_k)} 1 | a_i = E, H, W_2, \ldots, W_{N-1}, n_i \geq 1\}$, we have $\dim V_{N-1}/U_{N-1} = 2$. Since the element $\overline{W_{N-1}}$ is algebraically independent of $\overline{W_1}, \overline{W_2}, \ldots, \overline{W_{N-2}}$ in $\overline{\mathcal{A}}(V)$ by Lemma 3.11, we have $V_{N-1} \notin U_{N-1}$. Note that the element $F$ has weight $-1$ with respect to the action of $H_{(0)}$, and thus we have $F \notin U_{N-1} \oplus \mathbb{C}W_{N-1}$. Therefore we have $V_{N-1} = U_{N-1} \oplus \mathbb{C}W_{N-1} \oplus \mathbb{C}F$.

Since $V$ has no strong generator with conformal weight bigger than $N-1$, we have

$$V = \text{Span}\{a_{i(-n_1)} \cdots a_{k(-n_k)} 1 | a_i = E, H, W_2, \ldots, W_{N-1}, F, n_i \geq 1\}.$$

It implies that $E, H, W_2, \ldots, W_{N-1}, F$ strongly generates $V$. \hfill \Box

Remark 3.14. Consider another set of elements

$$U_m = \sum_{k=0}^{m} (-1)^{m+k} \left( \prod_{j=1}^{k} \frac{j(K-1)+1}{j(K-1)} \right) \epsilon_{m-k}(\rho_1, \ldots, \rho_N) \rho_0^k 1$$

of the vertex algebra $\mathcal{W}_{\mathbb{C}[K]}^K$ instead of $W_m$ for $m = 1, \ldots, N$, where $\rho_i = \sigma_i + H_{(-1)}$ for $i = 0, 1, \ldots, N$. Since $\rho_0 = (K-1)(\partial + Y_{(-1)})$, $U_m$ is well-defined even over the ring $\mathbb{C}[K]$ for $m = 1, \ldots, N$. One can easily check that we have $c^{(0)}_0 U_m = c^{(0)}_0 U_m = 0$ for all $i = 1, \ldots, N-1$ and $m = 1, \ldots, N$ merely by replacing $\sigma_i$ by $\rho_i$ for $i = 0, \ldots, N$ in the proofs of lemmas and propositions in this section, since $H$ lies in the kernels of the screening operators. Thus, $U_1, \ldots, U_N$ are elements of $\mathcal{W}_{\mathbb{C}[K]}^K$.

Moreover, the proofs of Lemma 3.11 and Theorem 3.13 work also analogously for $U_2, \ldots, U_{N-1}$. Therefore, we have another set of strong generators $E, H, U_2,$
..., \ U_{N-1}, \ F \ of \ the \ vertex \ algebra \ \mathcal{W}_K^{K_0} \otimes \mathbb{C}_{K_0} = \mathcal{W}_0 \ for \ arbitrary \ K_0 \in \mathbb{C}.

While this set of elements gives strong generators even for \( K_0 = 0 \), we prefer the elements \( W_m (m = 2, \ldots, N) \) because \( W_2, \ldots, W_N \) strongly generate the center of the vertex algebra \( \mathcal{W}_K^{K_0} \otimes \mathbb{C}_0 = \mathcal{W}_0 = \mathcal{W}^{-N}(\mathfrak{sl}_N, f_{\text{sub}}) \) at critical level as we discuss in the following section.

4. Structure of the subregular \( \mathcal{W} \)-algebra at critical level

In this section, we consider the strong generators \( E, H, W_2, \ldots, W_{N-1}, \ F \) of the subregular \( \mathcal{W} \)-algebra at critical level \( (K = 0) \), and study the OPEs between these generators. Throughout this section, we specialize \( K \) to 0, and consider the elements \( E, F, H, W_2, \ldots, W_N \) in the vertex algebra \( \mathcal{W}^{-N}(\mathfrak{sl}_N, f_{\text{sub}}) = \mathcal{W}_{K_0}^H \otimes \mathbb{C}_0 \).

First, by Definition 3.8 and (1), we have
\[
W_m = (-1)^m W_m' = e_m(\partial + X_1(-1), \ldots, \partial + X_{N}(-1)) \mathbf{1}
\]
for \( m = 2, \ldots, N \). Since \( X_j \in \bigoplus_{j=1}^{N-1} \mathbb{C}A_j \) and \( (A_j, \mathbf{1}) = 0 \) for \( i = 1, \ldots, N \) and \( j = 1, \ldots, N-1 \), the element \( W_m \) is central for \( m = 2, \ldots, N \).

**Proposition 4.1.** The elements \( W_2, \ldots, W_N \) strongly generate the center of the vertex algebra \( \mathcal{W}^{-N}(\mathfrak{sl}_N, f_{\text{sub}}) \).

**Proof.** By [Ara12, Theorem 1.1], the center of the vertex algebra \( \mathcal{W}^{-N}(\mathfrak{sl}_N, f_{\text{sub}}) \) coincides with the center of the universal affine vertex algebra \( V^{-N}(\mathfrak{sl}_N) \), and it is \( \mathbb{Z}_{\geq 0} \)-graded. Hence, it is the vertex algebra of type \( W(2,3,\ldots,N) \). Note that the elements \( \overline{W}_2, \ldots, \overline{W}_N \) are algebraically independent in the \( C_2 \) Poisson algebra by Lemma 3.11. Thus, applying the same argument in the proof of Theorem 3.13 to the elements \( W_2, \ldots, W_N \) of the center, the assertion of the proposition holds. \( \square \)

Note that we know the OPEs
\[
H(z)E(w) \sim \frac{1}{z-w}E(w), \quad H(z)F(w) \sim \frac{-1}{z-w}F(w).
\]
To describe complete structure of the vertex algebra \( \mathcal{W}^{-N}(\mathfrak{sl}_N, f_{\text{sub}}) \) algebraically, we discuss the OPE between \( E \) and \( F \). First, by direct computation, we have the following lemma.

**Lemma 4.2.** We have the commutation relation \( e^{(m)}_i \rho_i = (-m-1)e^{(m)}_{i-1} \) for \( i = 1, \ldots, N \), where \( \rho_i = -\partial + H(-1) - X_i(-1) \) is the operator defined in Section 3.

In the following, we set \( W_0 = 1 \) and \( W_1 = 0 \). The following lemma is essentially due to Molev. See [Mol18, Proposition 12.4.4].

**Lemma 4.3.** For \( m = 1, \ldots, N \), we have the following identity
\[
e_m(\partial - H(-1) + X_1(-1), \ldots, \partial - H(-1) + X_{N}(-1))
\]
\[
= \sum_{k=0}^{m} \binom{N-k}{m-k} W_k(-1)(\partial - H(-1))^{m-k}.
\]

**Proof.** Define \( \zeta_m \in \mathbb{C}[\mathfrak{g}_{X_i(-n)} | i = 1, \ldots, N, n \geq 1] \) by
\[
\sum_{m=0}^{N} \zeta_m \mathbf{1} = (\partial + X_{N}(-1)) \cdots (\partial + X_1(-1)).
\]
Proof. For (11) Applying Lemma 4.3. Here we used \( e_{(i)}^Y e^{-f Y} = 0 \) for \( n \geq 0 \) and \( e_{(i)}^Y \) commutes with \( \rho_i \) \((i = 1, \ldots, N)\). Applying Lemma 4.3, we obtain (11)
\[
E_{(N-m)} F = (-1)^{N+1}(N - m + 1)! \sum_{k=0}^{m-1} \binom{N-k}{m-k-1} W_{k(1)} (\partial - H_{(-1)})^{m-k-1} \mathbf{1},
\]
and the OPE of the theorem is an immediate consequence of it. \( \square \)

Now we describe the OPE between \( \mathcal{E} \) and \( \mathcal{F} \) in terms of our strong generators.

**Theorem 4.4.** We have the following OPE:
\[
E(z) F(w) \sim (-1)^{N+1} \sum_{n=1}^{N-1} \frac{n!}{(z - w)^n} \sum_{m=0}^{N-n} \binom{N-m}{n} (W_{m(1)} (\partial - H_{(-1)})^{N-n-m} \mathbf{1})(w)
\]

Proof. For \( m \leq N + 1 \), we have by using Lemma 4.2 repeatedly
\[
E_{(N-m)} F = -e_{(N-m)}^Y \rho_N \cdots \rho_1 e^{-f Y}
\]
\[
= -\rho_N \cdots \rho_1 e_{(N-m)}^Y e^{-f Y} - \cdots - \left( \prod_{j=1}^{N-m} (-j - 1) \right) \sum_{\iota_1 > \cdots > \iota_m} \rho_{\iota_1} \cdots \rho_{\iota_m} e_{(0)}^Y e^{-f Y}
\]
\[
+ \left( \prod_{j=1}^{N-m} (-j - 1) \right) \sum_{\iota_1 > \cdots > \iota_m} \rho_{\iota_1} \cdots \rho_{\iota_m-1} e_{(-1)}^Y e^{-f Y}
\]
\[
= (-1)^{N+1}(N - m + 1)! e_{m-1}(\partial - H_{(-1)}) + X_{N-1}, \ldots, \partial - H_{(-1)} + X_{N-1}) \mathbf{1}.
\]
Here we used \( e_{(n)}^Y e^{-f Y} = 0 \) for \( n \geq 0 \) and \( e_{(i)}^Y \) commutes with \( \rho_i \) \((i = 1, \ldots, N)\). Using the strong generators \( \mathcal{E}, \mathcal{F}, \mathcal{H} \) and \( W_m \) \((m = 2, \ldots, N - 1)\), we can determine the structure of the Zhu algebra of \( W^{-N}(\mathfrak{sl}_N, f_{\text{sub}}) \) explicitly.
Let $V = \bigoplus_{\Delta \geq 0} V\Delta$ be a $\mathbb{Z}_{\geq 0}$-graded vertex algebra, and we denote the degree of a homogeneous element $a \in V$ by $\Delta(a)$. For $a \in V\Delta$ and $b \in V$, we define

$$a \circ b = \sum_{j=0}^{\Delta} \left( \begin{array}{c} \Delta \\ j \end{array} \right) a(j-2)b, \quad a * b = \sum_{j=0}^{\Delta} \left( \begin{array}{c} \Delta \\ j \end{array} \right) a(j-1)b.$$ 

Then, the vector space $A(V) = V/(V \circ V)$ has a structure of an associative algebra by the multiplication induced by $*$, called the Zhu algebra of $V$ [Zhu96, FZ92, DSK06]. For $V = \mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{sub})$, the Zhu algebra $A(\mathcal{W}^{K_0-N}(\mathfrak{sl}_N, f_{sub})) = U(\mathfrak{sl}_N, f_{sub})$ is known as the finite $\mathcal{W}$-algebra associated with $\mathfrak{sl}_N$ and $f_{sub}$ by the result of De Sole and Kac in [DSK06], and in particular it does not depend on the level $K_0-N$. See also [Ara17]. Moreover, in [Pre02], Premet showed that the finite $\mathcal{W}$-algebra $U(\mathfrak{sl}_N, f_{sub})$ is isomorphic to Smith’s algebra introduced by Smith in [Smi90]. Below we describe the structure of $A(\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub}))$.

The vertex algebra $\mathcal{V}^0$ is $\mathbb{Z}$-graded by $\Delta(A_i) = 1$ for $i = 1, \ldots, N-1$, $\Delta(Q) = 1$, $\Delta(Y) = 1$ and $\Delta(\pm \mathcal{J}Y) = \pm 1$. This grading induces $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub}) = \mathcal{V}^0 \subset \mathcal{V}$; i.e. we have $\Delta(E) = 1$, $\Delta(F) = N - 1$, $\Delta(H) = 1$ and $\Delta(W_m) = m$ for $m = 2, \ldots, N$. Note that the grading coincides with the conformal weight in spite of $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ is not a vertex operator algebra.

First, we discuss the $C_2$ Poisson algebra $\mathcal{A} := \mathcal{A}(\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub}))$. The elements $\mathcal{W}_2, \ldots, \mathcal{W}_N$ are Poisson central and algebraically independent by Proposition 4.1 and Lemma 3.11, and thus $\mathcal{A}$ is a Poisson algebra over $\mathbb{C}[\mathcal{W}_2, \ldots, \mathcal{W}_{N-1}]$. By (11) for $m = N-1$, we have

$$CE = \sum_{k=1}^{N} (-1)^{k+1} \mathcal{W}_k \mathcal{H}^{N-k}.$$ 

By the result of [DSK06] or [Ara17], the Poisson algebra $\mathcal{A}$ is the graded algebra of $U(\mathfrak{sl}_N, f_{sub})$, and thus is the coordinate ring of the Slodowy slice $\mathcal{S}$ in $\mathfrak{sl}_N$ associated with $f_{sub}$, which is known as the simultaneous deformation of the Kleinian singularity of type $A_{N-1}$. These fact gives an isomorphism of commutative algebras

$$\mathcal{A} = \mathbb{C}[\mathcal{H}, \mathcal{F}, \mathcal{F}, \mathcal{W}_m | m = 2, \ldots, N-1] / (CE - \sum_{k=1}^{N} (-1)^{k+1} \mathcal{W}_k \mathcal{H}^{N-k}),$$ 

and the subalgebra $\mathbb{C}[\mathcal{W}_2, \ldots, \mathcal{W}_N]$ is the Poisson center of $\mathcal{A}$. Note that the Poisson brackets between these elements are given by $\{ \mathcal{H}, \mathcal{F} \} = \mathcal{E}, \{ \mathcal{F}, \mathcal{F} \} = -\mathcal{F},$ and $\{ \mathcal{E}, \mathcal{F} \} = \sum_{k=1}^{N} (-1)^{k+1} (k+1) \mathcal{W}_k \mathcal{H}^{N-k}$ by (11).

In [Smi90], Smith studied an associative algebra $R$ generated by the elements $x, y$ and $h$ subject to the relation $[h, x] = x$, $[h, y] = -y$, $[x, y] = f(h)$ where $f$ is a polynomial of degree $N - 1$. The Zhu algebra $A := A(\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub}))$ is generated by the image of $E, F, H$ and $W_m$ ($m = 2, \ldots, N-1$) by Theorem 3.13. Note that, by Proposition 4.1, the center of $A$ is $\mathbb{C}[\mathcal{W}_2, \ldots, \mathcal{W}_N]$. As an associative algebra over $\mathbb{C}[\mathcal{W}_2, \ldots, \mathcal{W}_{N-1}]$, $A$ is generated by $E, F$ and $H$. It is easy to see that $[H, E] = E$ and $[H, F] = -F$ by (10). Using (11) and the skew-symmetry $F_{(n)}E = \sum_{j=1}^{n} (-1)^{n-j-1} j! (E_{(n+j)}F)/j!$, $[E, F]$ is a polynomial in $H$ of degree $N - 1$ with coefficients in $\mathbb{C}[\mathcal{W}_2, \ldots, \mathcal{W}_{N-1}]$. Thus, the Zhu algebra $A$ is Smith’s algebra over $\mathbb{C}[\mathcal{W}_2, \ldots, \mathcal{W}_{N-1}]$. The isomorphism between the finite $\mathcal{W}$-algebra $U(\mathfrak{sl}_N, f_{sub})$ and Smith’s algebra is a well-known result by Premet [Pre02, Theorem 7.10].

The vertex algebra $\mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub})$ can be realized also by using fermionic fields. The realization was conjectured by Adamović in [Ada15]. Below we discuss such a realization.
Set \( \alpha = Q + \frac{N-1}{N}A_1 + \cdots + \frac{1}{N}A_{N-1}, \quad \beta = Y - \alpha, \)

elements of the vertex algebra \( \mathcal{V}^0. \) Note that we have \( (\alpha, \alpha) = 1, \ (\beta, \beta) = -1, \)
\( (\alpha, \beta) = 0. \) We consider the fermionic vertex operators \( \Psi^\pm(z) := e^{\pm \int \alpha(z)} \) and \( e^{\pm f \beta(z)}. \) Note that \( H = -\beta \) and \( e^{-fY} = \Psi_{(-1)}^{-} e^{-f \beta} \) by definition, and hence we have \( (\partial - H_{(-1)})^n e^{-fY} = (\partial^n \Psi^{-}_{(-1)}) e^{-\beta} \) for \( n \geq 0. \) Then, we have
\[
F = -\rho_N \cdots \rho_1 e^{-fY} = (-1)^{N+1}(\partial - H_{(-1)} + X_{N(-1)}) \cdots (\partial - H_{(-1)} + X_{1(-1)}) e^{-fY}
\]
\[
= (-1)^{N+1} \sum_{m=0}^{N} W_{m(-1)}(\partial - H_{(-1)})^{N-m} e^{-fY}
\]
\[
= (-1)^{N+1} \sum_{m=0}^{N} W_{m(-1)}(\partial^{N-m} \Psi^{-}_{(-1)}) e^{-\beta}
\]
since \( H_{(-1)} \) commutes with \( X_{i(-1)} \) for \( i = 1, \ldots, N. \) Therefore, we obtain the following realization of the vertex algebra \( \mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub}). \)

**Theorem 4.5** (Adamović’s conjecture [Ada15]). For \( N \geq 2, \) the subregular \( \mathcal{W}-\)algebra \( \mathcal{W}^{-N}(\mathfrak{sl}_N, f_{sub}) \) of type \( A_{N-1} \) is isomorphic to the vertex algebra strongly generated by the following fields:
\[
\circ e^{f \beta(z)} \Psi^+(z) \circ, \quad H(z), \quad W_m(z) \ (m = 2, \ldots, N - 1), \quad \text{and} \quad \sum_{m=0}^{N} \circ W_m(z) e^{-f \beta(z)} \partial^{N-m} \Psi^{-}(z) \circ.
\]

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N.G.: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 JAPAN, E-mail address: gnr@kurims.kyoto-u.ac.jp

T.K.: Department of Mathematics, Faculty of Pure and Applied Sciences, University of Tsukuba, Tsukuba, Ibaraki 305-8571, JAPAN, E-mail address: kuwabara@math.tsukuba.ac.jp