Gravitational dynamics at $O(G^6)$: \[perturbative \text{ gravitational scattering meets experimental mathematics}\]

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A recently introduced approach to the gravitational dynamics of binary systems involves intricate
integrated gravitational wave signals emitted during the last orbits of coalescing black-hole binaries motivates the development of ever more accurate methods for describing the dynamics and radiation of gravitationally interacting binary systems.

The present work will demonstrate how progress in the theoretical description of binary systems can be reached by combining methods developed in General Relativity (GR) with ideas borrowed from computational methods in Quantum Field Theory (QFT). Specifically, we show how to complete a state-of-the-art approach to the classical dynamics of binary systems [2] by using advanced computing techniques developed for the evaluation of multi-loop Feynman integrals to obtain our results in two ways: high-precision arithmetic, yielding reconstructed analytic expressions, and direct integration via Harmonic Polylogarithms. The analytic expression of the tail contribution to the scattering involve transcendental constants up to weight four.

II. CLASSICAL PERTURBATIVE EXPANSION OF THE NONLOCAL-IN-TIME SCATTERING ANGLE

The approach of [2] (further developed in [11][13]) is based on a novel way of combining results from several theoretical formalisms, developed for studying the gravitational potential within classical GR: post-Newtonian (PN) expansion, post-Minkowskian (PM) expansion, multipolar-post-Minkowskian expansion, effective-field-theory, gravitational self-force approach, and effective one-body method. Within this framework, the Hamiltonian of binary systems is decomposed into three types of contributions,

\[H^{\text{tot}}(t) = H^{\text{loc},f}(t) + H^{\text{nonloc},h}(t) + H^{f-h}(t).\] (2.1)

Here, $H^{\text{loc},f}(t)$ is a local-in-time Hamiltonian, expressed, at some given PN accuracy, by an algebraic function of...
the instantaneous values, \(q(t), p(t)\), of position and momenta variables. By contrast, \(H_{\text{nonloc}}(t)\) is a nonlocal-in-time Hamiltonian, which involves integrals over (at least one) auxiliary time-shifted variable \(t' = t + \tau\):

\[
H_{\text{nonloc}}(t) = \frac{GM}{c^3} \operatorname{Pf}_{2r_{12}(t)/c} \int_{-\infty}^{+\infty} \frac{dt'}{|t - t'|} F^\text{split}(t, t') + \cdots
\]  

(2.2)

Here, \(M = \frac{GM}{c^3} \chi\) denotes the total conserved (center-of-mass) mass-energy of the binary system; \(\operatorname{Pf}_{2r_{12}(t)/c}\) denotes the partie-finie regularization, using the time scale \(\Delta t = 2r_{12}(t)/c\), of the logarithmically divergent integral at \(t' = t\); \(r_{12}^{\text{loc}}(t)\) denotes the harmonic-coordinate distance between the two bodies; and \(F^\text{split}(t, t')\) is a time-split version of the gravitational-wave energy flux absorbed and then emitted by the system\(^1\). The ellipsis in Eq. (2.2) denotes higher-order tail effects, containing higher powers of \(\frac{GM}{c^3}\), such as the second-order tail (\(\sim 2\chi^2\)) analytically derived in [13] up to the combined 6PM and 5.5PN accuracy. The local Hamiltonian \(H_{\text{loc}}(t)\) starts at Newtonian level, while \(H_{\text{nonloc}}(t)\) gets contributions from the 4PN order on, and its structure is known up to 6PN [11,13]. Finally, the last term \(H_{\text{loc}}(t)\) is a local-in-time contribution which involves the (unsplit) gravitational wave energy flux \(F_{\text{GW}}(t) = F^\text{split}(t, t')\), and a flexibility factor \(f(t) = 1 + O(\chi)\) that is a function of the instantaneous state of the system:

\[
H_{\text{loc}}(t) = +2\frac{GM}{c^3} F_{\text{GW}}(t) \ln (f(t))
\]  

(2.3)

The latter Hamiltonian contribution is local-in-time, but the determination of the flexibility factor \(f(t)\) depends on the explicit knowledge of the scattering angle induced by the local-in-time Hamiltonian \(H_{\text{loc}}(t)\).

The Hamiltonian decomposition (2.4) yields a corresponding decomposition of the total scattering angle \(\chi_{\text{tot}}\), as displayed in Eq. (1.1). At the needed accuracy, the nonlocal contribution \(\chi_{\text{nonloc}}(E, J, \nu)\) can be written as [13],

\[
\chi_{\text{nonloc}}(E, J, \nu) = \frac{\partial W_{\text{nonloc}}(E, J, \nu)}{\partial J} \quad (2.4)
\]

\(W_{\text{nonloc}}(E, J, \nu) = \int_{-\infty}^{+\infty} dt \ H_{\text{nonloc}}(t)\) \quad (2.5)

is the integrated nonlocal action. Inserting Eq. (2.2) into Eq. (2.4), one sees that the knowledge of \(\chi_{\text{nonloc}}(E, J, \nu)\) depends on the evaluation of a (regularized) two-fold integral,

\[
W_{\text{nonloc}}(E, J, \nu) = \frac{GE}{c^5} \times \operatorname{Pf}_{2r_{12}(t)/c} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dt'}{|t - t'|} F^\text{split}(t, t') + \cdots
\]

(2.6)

The latter integral is to be evaluated along an hyperbolic-motion solution of the local-in-time Hamiltonian \(H_{\text{loc}}(t)\).

The method of [2] requires as crucial input the explicit knowledge of the double, PM and PN, perturbative expansion of \(W_{\text{nonloc}}(E, J, \nu)\), i.e., its combined expansion in powers of \(G\) (PM expansion), and of \(\frac{1}{J}\) (PN expansion). It is convenient to express the combined PM+PN expansion of \(W_{\text{nonloc}}(E, J, \nu)\) in terms of the dimensionless variables

\[
p_{\infty} = \sqrt{\gamma^2 - 1}, \quad \text{and} \quad j = \frac{cJ}{GM_{1}m_{2}},
\]

where the effective-one-body specific energy \(\gamma\) is defined as

\[
\gamma = \frac{E_{\text{eff}}}{\mu c^2} = \frac{E^2 - m_{1}^2 c^4 - m_{2}^2 c^4}{2m_{1}m_{2}c^2}.
\]

As \(j \approx \frac{cJ}{\mu G}\), the PM expansion of \(W_{\text{nonloc}}\) is equivalent to an expansion in inverse powers of \(j\), and reads (after setting aside the second-order tail contribution)

\[
\frac{cW_{\text{nonloc}}(\gamma, j; \nu)}{2Gm_{1}m_{2}} = -\nu p_{\infty}^4 \left( \frac{A_{h}^{(0)}(p_{\infty}, \nu)}{3j^3} + \frac{A_{h}^{(1)}(p_{\infty}, \nu)}{4p_{\infty}j^4} \right) + \frac{A_{h}^{(2)}(p_{\infty}, \nu)}{5p_{\infty}^2j^5} + O \left( \frac{1}{j^6} \right).
\]

(2.9)

Using Eq. (2.1), this corresponds to the following PM expansion of the corresponding nonlocal scattering angle

\[
\frac{1}{2} \chi_{\text{nonloc}}(\gamma, j; \nu) = +\nu p_{\infty}^4 \left( \frac{A_{h}^{(0)}(p_{\infty}, \nu)}{j^3} + \frac{A_{h}^{(1)}(p_{\infty}, \nu)}{p_{\infty}j^3} \right) + \frac{A_{h}^{(2)}(p_{\infty}, \nu)}{p_{\infty}^2j^2} + O \left( \frac{1}{j^3} \right)
\]

(2.10)

The dimensionless coefficients \(A_{h}^{(m)}(p_{\infty}, \nu), m = 0, 1, 2, \ldots\), then admit a PN expansion, i.e., an expansion in powers of \(p_{\infty} = O \left( \frac{cJ}{\mu G} \right)\), modulo logarithms of \(p_{\infty}\), say

\[
A_{m}^{(p_{\infty}, \nu)} = \sum_{n \geq 0} A_{mn}^{(n)} + A_{mn}^{(n)} \ln \left( \frac{p_{\infty}}{2} \right) n_{\infty}.
\]

(2.11)

The coefficient \(A_{h}^{(n)}(\nu)\) parametrizes a term of order \(p_{\infty}^{4-n-m} \sim \frac{G^{4-m}}{1+m} \) (with \(m \geq 0, n \geq 0\)) in the combined PM+PN expansion of the nonlocal scattering angle. The leading-order contribution to the nonlocal dynamics is at the combined 4PM and 4PN level, i.e.,

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\(^1\) We consider the conservative dynamics of a binary interacting in a time-symmetric way.
\( \propto G^4/c^8 \) [13]. The corresponding nonlocal scattering coefficient, coming from \( m = 0 \) and \( n = 0 \), is \( A_0^0(p_{\infty}, \nu) = \pi \left[ -\frac{3}{24} \ln \left( \frac{\nu}{\nu_c} \right) - \frac{\nu_c}{\nu} \right] + O(\nu^2) \) [14]. The higher-order logarithmic coefficients \( A_{mn}^\nu(\nu) \) were analytically determined [11–13] so that we shall henceforth focus on the non-logarithmic coefficients \( A_{mn}^\nu(\nu) \). The classical GR perturbative approach of [2, 11–13] yields explicit integral expressions for the non-logarithmic coefficient \( A_{mn}^\nu(\nu) \) with integrands that are polynomials in the symmetric mass ratio \( \nu \). Writing \( A_{mn}(\nu) = \sum_k A_{mnk}\nu^k \), with \( k = 0, 1, 2, \ldots \), this finally yields explicit, parameter-free double-integral expressions for the (numerical) coefficient \( A_{mnk} \) of \( \nu^k \) in the polynomial \( A_{mn}(\nu) \), say

\[
A_{mnk} = \int_{-1}^{+1} \int_{-1}^{+1} \frac{dTdT'}{|T - T'|} a_{mnk}(T, T').
\]  

The structure of the integrands \( a_{mnk}(T, T') \) reads

\[
a_{mnk}(T, T') = R_{0}^{mnk}(T, T')
+ R_{1}^{mnk}(T, T') (\arctanh(T) - \arctanh(T'))
+ R_{2}^{mnk}(T, T') (\arctanh(T) - \arctanh(T'))^2
+ R_{3}^{mnk}(T, T') (\arctanh(T) - \arctanh(T')) ,
\]  

where the coefficients \( R_{N}^{mnk}(T, T') \) are rational functions of \( T \) and \( T' \). The integration variables are related via \( T = \tanh \frac{t}{2} \) and \( T' = \tanh \frac{t'}{2} \) to the “hyperbolic eccentric anomalies” \( v \) and \( v' \) that parametrize the original time variables \( t \) and \( t' \) via the relativistic generalization [16, 17] of the Keplerian representation of hyperbolic motion. The latter notably involves a relativistic version of the hyperbolic Kepler equation: \( n(t - t_0) = e_i \sinh v - v + O(1) \).

It was possible to analytically compute the numerical coefficients \( A_{mnk} \) appearing at the 4PM (\( G^4 \)) and 5PM (\( G^5 \)) levels (i.e., for \( m = 0, 1 \)) up to the 6PN, i.e., \( 1/\pi \) accuracy. By contrast, the integrands of Eq. (2.12) become so involved at the 6PM order (corresponding to 5-loop classical scattering diagrams), that the use of standard GR integration methods failed to give the analytical values of the 6PM scattering coefficients \( A_{220}, A_{240}, A_{241}, A_{242} \). Even the numerical evaluation of the latter coefficients in [13] met with difficulties and only produced 8-digit-accurate results.

The lack of analytical determination of the 6PM coefficients \( A_{220}, A_{240}, A_{241}, A_{242} \) is an imperfection that limits the application of the method of [2] at the 6PN level. In particular, the combination

\[
D = \frac{1}{\pi} \left( \frac{5}{2} A_{221} + \frac{15}{8} A_{200} + A_{242} \right),
\]  

crucially enters the definition of the flexibility factor \( f(t) \), and thereby the analytical definition of the third contribution, \( H^{3-b}(t) \), Eq. (2.3), to the total Hamiltonian. The coefficient \( D \), Eq. (2.14), is of direct physical significance for the dynamics of coalescing binary systems because it enters the elliptic-motion observables (such as periastron precession).

We achieve here the important goal of analytically determining all the 6PM scattering coefficients, \( A_{2nk} \) (and thereby also the coefficient \( D \), Eq. (2.14)), by applying to the integral representations, Eqs. (2.12), (2.13), some of the high-precision numerical techniques and analytic methods that have been developed for evaluating QFT observables, expressed in terms multi-loop Feynman integrals.

### III. \( A_{2nk} \) AND COMPANION COEFFICIENTS

In the following we use the notation of Ref. [13] and parametrize the (non-logarithmic) scattering coefficients \( A_{2nk} \) in terms of the equivalent set of coefficients denoted \( d_{nk} \), and related to them via

\[
\begin{align*}
\pi^{-1} A_{200} &= d_{00}, \\
\pi^{-1} A_{220} &= d_{20} + 3d_{00}, \\
\pi^{-1} A_{221} &= d_{21} - 2d_{00}, \\
\pi^{-1} A_{240} &= d_{20} + d_{40} + \frac{3}{2} d_{00}, \\
\pi^{-1} A_{241} &= d_{21} - \frac{11}{2} d_{00} + d_{41} - 2d_{20}, \\
\pi^{-1} A_{242} &= d_{42} - 2d_{21} + 3d_{00}.
\end{align*}
\]  

The coefficients \( d_{00} \) and \( d_{21} \) (and therefore \( A_{200} \) and \( A_{221} \)) were computed analytically [13],

\[
\begin{align*}
d_{00} &= -\frac{99}{4} - \frac{2079}{8} \zeta(3), \\
d_{21} &= \frac{1541}{8} + 306 \zeta(3).
\end{align*}
\]  

In addition, some parts of the integrals giving \( d_{20}, d_{40}, d_{41} \) and \( d_{42} \) could be analytically evaluated, leaving as remaining unknown coefficients the quantities \( Q_{20}, Q_{40}, Q_{41} \) and \( Q_{42} \) related to \( d_{20}, d_{40}, d_{41} \) and \( d_{42} \) (and thereby

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2 For simplicity, we shall not use here the coefficients \( c_{nk} \) that are related to the \( d_{nk} \)’s through Eq. (4.14) of Ref. [13].
to $A_{220}$, $A_{240}$, $A_{241}$, and $A_{242}$) via

$$
\pi d_{20} = \left( \frac{32813}{192} + \frac{66999}{224} \right) \ln(2) \pi + \frac{5}{2} Q = \frac{25883}{720} - \frac{22333}{56} K, \quad 
\pi d_{40} = \left( \frac{293499}{512} + \frac{44237}{2688} \right) \ln(2) \pi + \frac{5}{2} Q = \frac{750674317}{762048} - \frac{442237}{2016} K, \\
\pi d_{41} = \left( \frac{4431841}{48384} - \frac{28735}{64} \right) \ln(2) \pi + \frac{5}{2} Q = \frac{703435949}{635040} + \frac{28735}{48} K, \\
\pi d_{42} = \left( \frac{1105777}{6048} + \frac{4497}{32} \right) \ln(2) \pi + \frac{5}{2} Q = \frac{59610947}{317520} + \frac{1499}{8} K. 
$$

(3.3)

Here $K$ denotes Catalan’s constant, defined as $K = \beta(2) = \sum_{n=1}^{\infty} (-1)^n/(2n + 1)^2 = 0.915966 \ldots$ (with $\beta$ being the Dirichlet function). The coefficients $Q_{20}$, $Q_{40}$, $Q_{41}$ and $Q_{42}$ are all expressed as two-fold integrals of the type indicated in Eq. (2.12). The explicit forms of the integrands $q_{nk}(T, T')$, for $(nk) = (20), (40), (41)$, yielding the corresponding coefficients $Q_{nk}$ after integration on the square $(T, T') \in [-1, 1] \times [-1, 1]$, are given in the Supplemental Material of the present paper. See below for the integrand $q_{42}(T, T')$ of $Q_{42}$.

In their original forms, the integrands $q_{nk}(T, T')$ (including the factor $\frac{1}{T - T'}$) pulled out in Eq. (2.12) are finite but discontinuous across the first diagonal $T = T'$ of the square. One, however, gets a continuous, and simpler, integrand by working (as it is allowed) with a symmetrized integrand $q_{nk}^{\text{sym}}(T, T') = \frac{1}{2}(q_{nk}(T, T') + q_{nk}(T', T))$. It is then found that $q_{nk}^{\text{sym}}(T, T')$ has a four-fold symmetry. Namely, it is symmetric under reflection through the two diagonals of the square $(T, T') \in [-1, 1] \times [-1, 1]$. In other words, $q_{nk}^{\text{sym}}(T, T') = + q_{nk}^{\text{sym}}(T', T) = q_{nk}^{\text{sym}}(-T, T) = q_{nk}^{\text{sym}}(-T', -T)$. In addition, $q_{nk}^{\text{sym}}(T, T')$ is continuous (though not differentiable) across the first diagonal $T = T'$. The value of the integral $Q_{nk}$ can then be obtained from integrating $q_{nk}^{\text{sym}}(T, T')$ on the triangle $-1 < T < 1, -1 < T' < T$ (or even on the subtriangle obtained by modding out the symmetry under the second diagonal).

For concreteness, let us discuss in detail the structure of the integrand $q_{42}^{\text{sym}}(T, T')$. We have

$$
|T' - T'q_{42}^{\text{sym}}(T, T') = r_{42}(T, T') + \bar{c}_1(T, T') (A - A') \frac{1}{T - T'} - \frac{1}{2} \frac{dA}{dT} - \frac{1}{2} \frac{dA'}{dT'} \\
+ \bar{c}_2(T, T') \left[ \left( \frac{A - A'}{T - T'} \right)^2 - \frac{1}{2} \left( \frac{dA}{dT} \right)^2 - \frac{1}{2} \left( \frac{dA'}{dT'} \right)^2 \right],
$$

(3.4)

where $r_{42}(T, T')$, $\bar{c}_1(T, T')$ and $\bar{c}_2(T, T')$ are rational functions, and when we used the shorthand notation $A \equiv \text{arctanh}(T)$, and $A' \equiv \text{arctanh}(T')$. The integral corresponding to $r_{42}(T, T')$ can be explicitly performed, namely

$$
Q_{42} = \frac{+1 + \frac{1}{T - T'} r_{42}(T, T')}{\pi} = \frac{3463}{30} K + \frac{33256213}{340200} \\
- \pi \left( \frac{20}{8306153213} + \frac{19776073}{7257600} \right) \ln(2), 
$$

(3.5)

while the rational coefficients $\bar{c}_1(T, T')$ and $\bar{c}_2(T, T')$ can be factorized as

$$
\bar{c}_1(T, T') = -16(1 - T^2)^2(1 - T'^2)^2(1 - T T')^3 P_1(T, T'), \\
\bar{c}_2(T, T') = -16(1 - T^2)^3(1 - T'^2)^3 (1 - T T')^3 P_2(T, T'),
$$

(3.6)

(3.7)

where $P_1(T, T')$ and $P_2(T, T')$ are (symmetric) polynomials in $T$ and $T'$ which are given in the Supplemental Material. The total degree in $T$ and $T'$ of $P_1(T, T')$ is 32, while that of $P_2(T, T')$ is 28. Note that the integrands corresponding to $\bar{c}_1(T, T')$ and $\bar{c}_2(T, T')$ vanish proportionally to $(T - T')^2$ across the first diagonal.

IV. ANALYTIC EVALUATION OF THE SCATTERING INTEGRALS $Q_{nk}$

To determine the analytic expressions of the scattering integrals $Q_{nk}$ (equivalent to the $A_{nk}$'s), we adopt a two-step strategy:

1. Experimental Mathematics and Analytic Recognition;

2. Analytic Integration and Harmonic Polylogarithms.

Such a strategy is often used in the realm of multi-loop Feynman calculus, when a direct analytic integration seems prohibitive, See e.g. Refs. 13, 14, 15, 22. Previous uses of experimental mathematics and high-precision
arithmetics within studies of binary systems include Refs. [25, 27]. We note in particular that one of the integrals (in momentum space) contributing to the 4PN-static term of the two-body potential, used in [27] and originally obtained by analytic recognition [28], was later analytically confirmed by direct integration (in position space) [29].

In the current work, in order to perform the first step, we started by numerically computing the definite integrals of \( q_{nk}(T, T') \) on the triangle \(-1 < T < 1, -1 < T' < T \) to a very high precision (with a few hundreds of digits), using a double-exponential change of variables [30]. Indeed, the latter method is well-adapted to our integrals which are mildly singular on the boundaries of the triangle \(-1 < T < 1, -1 < T' < T \). The 200-digit accuracy level that we used was amply sufficient for reconstructing the analytic expressions of the \( Q_{nk} \)'s by using the PSLQ algorithm [31] and a basis of transcendental constants indicated both by the structure of the integrands, and the analytical results (3.2), (3.3). Our numerical results are given in Table I, while Table II gives the reconstructed analytic expressions of the \( Q_{nk} \)'s, and the related \( d_{nk} \)'s.

We have \textit{a posteriori} checked that the so-reconstructed exact values of the \( \det q_{nk} \)'s agree (within our estimated error \( \pm 1 \times 10^{-8} \)) with the values given in Table VI of [13].

Having the semi-analytic expressions given in Table II in hands, we proceed to the (purely analytical) second step of our strategy.

We first perform the integration over \( T' \), beginning with an integration by-parts of the terms that contain, in the denominator, polynomials in \( T' \) with integer exponents bigger than one. We are then left with integrals containing powers of \( \arctan(T') \), or \( \arctanh(T') \), in the numerator, and powers of \( (T' \pm 1) \), \( (T' \pm i) \), \( (T' - T) \) and \( (T' + 1/T) \), in the denominator. These integrals are carried out by differentiating with respect to \( T' \), repeatedly if needed, until the integration in \( T' \) is straightforward. Thereby, the original integral is obtained as a repeated quadrature in \( T \), whose first layer reads as,

\[
\begin{align*}
\int_{T_0}^{T} dT' g(T, T') &= f(T_0) + \int_{T_0}^{T} dT' \frac{\partial}{\partial T} \int_{-1}^{T} dT'' g(T, T'') \\
&= f(T_0) + \int_{T_0}^{T} dT \left( g(T, T) + \int_{-1}^{T} dT'' \frac{\partial g(T, T'')}{\partial T} \right).
\end{align*}
\]

The integrand \( f(T) \) obtained after the integration over \( T' \) contains Nielsen polylogarithms [32] (up to weight 3). For convenience, we fold the integral over the interval \( T \in [0, 1] \): \( \int_{-1}^{1} dT f(T) = \int_{0}^{1} dT [f(T) + f(-T)] \).

The final integration over \( T \in [0, 1] \) is performed in three steps. First, we integrate by-parts the factors \( (T \pm i)^{-n} \) or \( (T \pm 1)^{-n} \) with \( n > 1 \), until \( n \) is reduced to 1. Second, we map the resulting integrals containing \( T^{-1} \) and \( (T \pm 1)^{-1} \) (but not \( (T \pm i)^{-1} \)) to Harmonic Polylogarithms (HPLs) [8]. The HPLs are defined as recursive integrals,

\[
H_{i_1 i_2 \ldots i_n}(x) = \int_{0}^{x} dt_1 f_{i_1}(t_1) H_{i_2 \ldots i_n}(x),
\]

with \( f_{\pm 1}(x) = (1 \mp x)^{-1} \), \( f_0(x) = 1/x \), and \( H_{\pm 1}(x) = \ln(1 \mp x) \), \( H_0(x) \equiv \ln(x) \).

For a given HPL, \( H_{i_1 i_2 \ldots i_n}(x) \), the number \( n \) of indices is called its \textit{weight}, and corresponds to the number of iterations appearing in its nested integral representation. \( H \)-functions obey integration-by-parts relations and shuffle algebra relations which can be used to identify, weight-by-weight, a minimal (albeit not unique) subset of them to be considered as independent. For instance, at weights \( w = 2, 3 \), and 4 the minimal subsets are formed by 3, 8, and 18 elements, respectively (see [33] for a Mathematica implementation).

Third, we consider the integrals containing \( (T \pm i)^{-1} \); these cannot be directly cast in HPL format. Therefore, we modify the integrands by a suitable insertion of a parameter \( x \), to be later eliminated, in order to obtain the original integral back. The integral, now function of \( x \), will be subsequently reconstructed by repeated differentiations with respect to \( x \) and quadratures (as explained earlier, in the case of the \( T' \) integration). Let us show an example of this technique: all the \( Q_{nk} \) contain the same combination of integrals with \( w = 4 \),

\[
J = \int_{0}^{1} dT \frac{16 \arctan^3(T) - 3 \text{Li}_3 \left[ \frac{1}{1 + T^2} \right]}{1 + T^2}.
\]

Notice that \( \arctan(T) = -(1/2)\ln((1 - T)/(1 + T)) \). We modify the integral (4.2), to let it acquire a dependence on the variable \( x \), i.e. \( J \rightarrow J(x) \), in the following way:

\[
J(x) \equiv \int_{0}^{1} dT \left( 1 - x^2 \right) \times \\
\frac{16 \arctan^3(T) - 3 \text{Li}_3 \left[ \left( \frac{1-T}{1+T} \right)^2 \right]}{2i(T + x)(T + 1/x)}.
\]

Then, the original integral is recovered at \( x = i \), that is \( J = J(i) \). By differentiating and re-integrating over \( x \), \( J(x) \) can be conveniently written in terms of HPL's at
TABLE I: Numerical values of the $Q_{nk}$ integrals with 200-digit accuracy.

| $Q_{20}$ | $Q_{40}$ | $Q_{41}$ | $Q_{42}$ |
|----------|----------|----------|----------|
| 524.76729218021258434273595570310175847614199995573900173972871123849883893009711209390731581 | 9606803170623899925025677050206794678749466475413470110101455883184170170829347212071124106113165 | 863485679 | 544.493991570170677225845815842157013358458333264830495036760834156682498158607542856530298622 |
| 870842521519234233393649815247226338070650337694321196917178743743144282667704148469493992691447 | 2804761699 | $-1029.52887537403849684626249062859153113439810449676867454201338394155153339406867109916000880$ | $-802.8850570507866427558862950690344599707368650584366549641788985002426423211047940727300850018$ |
| 742678716203784351105139654474676789525511182468350940535176176197645060875802781537593191860287 | 843381464 | $843381464$ | $843381464$ |

$w = 4$, as,

$$ i J(x) = \frac{23}{240} \pi^4 - 21 \ln 2 \zeta(3) + \pi^2 \ln^2 2 - \ln^4 2 - 24 a_4 $$

Using $a_4 = L_1(1/2)$, and the values of the HPLs at $x = i$, listed in Table [11] (see the Appendix), one finds the following value for (4.2):

$$ J(i) = -\frac{1}{2} \pi^2 K + \frac{9}{2} \pi \zeta(3) = J. \quad (4.5) $$

Using a similar strategy, the expressions of all the coefficients $Q_{nk}(x)$ can be obtained analytically. The size of the occurring intermediate expressions, similar to [4.3], is too large to be presented here. Anyway, the crucial results concern the final analytic expressions for the so-obtained $Q_{nk} \equiv Q_{nk}(x = i)$. They are found to be drastically simpler than the intermediate results, and, as expected, to be in perfect agreement with the semi-analytical expressions discussed earlier, and given in Table [11].

V. SCATTERING ANGLE AND PERIASTRON PRECESSION AT 6PM, $O(G^6)$

The 6PM-accurate ($O(G^6)$) scattering coefficient $A_2^s(p_\infty, \nu)$ associated with the integrated nonlocal action $W_{\text{nonloc}}$, Eq. (2.5), when PN-expanded in powers of $p_\infty$, reads,

$$ A_2^s(p_\infty; \nu) = A_2^\text{tail.h,N} + A_2^\text{tail.h,1PN} + A_2^\text{tail2.h,1.5PN} + A_2^\text{tail.h,2PN} + O(p_\infty^6). \quad (5.1) $$

The values of the first, $A_2^\text{tail.h,N}$, and third, $A_2^\text{tail2.h,1.5PN}$, contributions (respectively contributing to the 4PN and 5.5PN orders) were obtained in Ref. [13]. New with the present work is the complete analytical determination of the two other contributions to Eq. (5.1), namely, $A_2^\text{tail.h,1PN}$, and $A_2^\text{tail.h,2PN}$. The latter contributions are...
both at the 6PM ($O(G^6)$) order, and they respectively belong to the 5PN ($O(c^{-10})$) and 6PN ($O(c^{-12})$) levels. Recalling also the 4PN contribution to $A^2_{\text{h}}(p_\infty; \nu)$, we have now the complete, 6PN-accurate analytical results for $A^2_{\text{tail,h}}$:

\begin{align*}
A^2_{\text{tail,h},N} &= \pi \left[ -\frac{2079}{8} \zeta(3) - \frac{99}{4} - 122 \ln \left( \frac{p_\infty}{2} \right) \right], \\
A^2_{\text{tail,h},1PN} &= \pi \left[ -\frac{13831}{56} + \frac{811}{2} \nu \ln \left( \frac{p_\infty}{2} \right) - \frac{41297}{112} \right] + \frac{1937}{8} \nu + \left( \frac{49941}{64} + \frac{3303}{4} \nu^2 \right) \zeta(3) - \frac{9216}{7} \ln(2) \right] p_\infty^2, \\
A^2_{\text{tail,h},2PN} &= \pi \left[ \frac{75595}{168} \nu + \frac{64579}{1008} - 785 \nu^2 \ln \left( \frac{p_\infty}{2} \right) + \frac{1033549}{4536} + \frac{8008171}{8064} \nu - \frac{583751}{864} \nu^2 \right] .
\end{align*}

As a consequence of our results, we can also now compute the analytical value of the (minimal value of the) flexibility coefficient, $f(t)$, and thereby the effect of $H^{l-h}(t)$, Eq. (2.3), on the near-zone gravitational physics, such as periastron precession. They both depend on the crucial combination $D$, Eq. (2.14). Though all the building blocks entering $D$ (which can equivalently be written as $D = \frac{1}{2} d_{21} + d_{42} - \frac{1}{2} d_{00}$) contain $\zeta(3)$, it is remarkably found that $D$ turns out to be equal to the rational number

\begin{equation}
D = -\frac{12607}{108},
\end{equation}

which is compatible with the previous numerical estimate $D_{\text{num}} = -116.73148147(1)$.

The value of $D$ then determines the minimal value of the flexibility coefficient $D_3^{\text{min}}$ (see Eq. (7.28) in [13]), namely

\begin{equation}
D_3^{\text{min}} = -\frac{68108}{945} \nu,
\end{equation}

as well as the $f$-related, 6PN-level contribution to the periastron precession (see Eq. (8.30) in [13]):

\begin{equation}
K^{l-h, \text{circ}, \text{min}}(j) = \frac{68108}{945} \nu^3 j^{12}.
\end{equation}

VI. CONCLUSIONS

By using advanced computing techniques developed for the evaluation of multi-loop Feynman integrals, we have completed the analytical knowledge of classical gravitational scattering (and periastron precession) at the sixth order in $G$, and at the sixth post-Newtonian accuracy. We think that the present work exemplifies a new type of synergy between classical GR and QFT techniques that can be developed in many directions, and can significantly help to improve the theoretical description of gravitationally interacting binary systems.

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Appendix

We collect in Table [II] the values of the independent bases of HPLs, $H_{i_1 \ldots i_w}(x)$ at the point $x = i$, up to weight $w = 4$, required for the analytic evaluations described in Sec. [I].
TABLE III: Independent sets of HPLs, at the point $x = i$, up to weight four.

| $H_{-1}(i)$ | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
|--------------|----------------------------------|
| $H_0(i)$    | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_1(i)$    | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |

| $H_{0,-1}(i)$ | $\frac{\ln 2}{2} + iK$ |
|---------------|------------------------|
| $H_{0,1}(i)$  | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,-1}(i)$ | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1}(i)$  | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |

| $H_{0,-1,-1}(i)$ | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
|------------------|----------------------------------|
| $H_{0,-1,0}(i)$  | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{0,1,-1}(i)$  | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |

| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
|------------------|----------------------------------|
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |
| $H_{1,1,0}(i)$   | $\frac{\ln 2}{2} + \frac{i\pi}{4}$ |

| $L_{4}(1/2)$ | $a_4$ |
|--------------|-------|
| $\text{Im} L_{4}(i)$ | $K$ |
| $\text{Im} L_{4}(i)$ | $\beta(4)$ |
| $\text{Im} H_{0,1,1,1}(i)$ | $Q_3$ |
| $\text{Im} H_{0,1,1,1}(i)$ | $Q_4$ |
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