HDG schemes for stationary convection-diffusion problems

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Abstract. For stationary linear convection-diffusion problems, we construct and study a hybridized scheme of the discontinuous Galerkin method on the basis of an extended mixed statement of the problem. Discrete schemes can be used for the solution of equations degenerating in the leading part and are stated via approximations to the solution of the problem, its gradient, the flow, and the restriction of the solution to the boundaries of elements. For the spaces of finite elements, we represent minimal conditions responsible for the solvability, stability and accuracy of the schemes.

1. Introduction

Recently, schemes of the discontinuous Galerkin method (DG or DGFEM-schemes) have been comprehensively studied and widely used for the solution of various boundary value problems for partial differential equations. Numerous papers deal with them, which explains a variety of approaches to their construction. Of these papers, we note [1-3].

DG-schemes are based on spaces of discontinuous finite elements (of arbitrary approximation order) and are locally and globally conservative. Just as in the finite-element method (FEM), there are two basic approaches to their construction. The main statement of the boundary value problem is used in the first of them, and a mixed statement is used in the other. In the first case, the scheme is stated via approximations to the solution of the problem \( u \), and in the other case, the scheme is stated via approximations to the solution of the problem \( u \) and to the flow vector \( q \). For linear problems, in the numerical implementation of schemes of the second type, the unknowns corresponding to the vector \( q \) can be eliminated element by element from the system of equations.

In comparison with the standard scheme of the FEM of the same accuracy order, a system of algebraic equations of the DG-scheme (for finding \( u \)) has a much larger dimension, which is a disadvantage of the method. To eliminate it from the scheme, we introduce the new unknown corresponding to the restriction of \( u \) to the boundaries of elements. This leads to hybridized schemes of the discontinuous Galerkin method (HDG-schemes; see [4-7]).

HDG-schemes constructed on the basis of the mixed statement of the problem are quite similar to hybridized mixed schemes of the FEM [8] and are also stated via approximations to \( u \), \( q \), and, say, \( \lambda \), where \( \lambda \) is the restriction of \( u \) to the boundaries of elements (see [9]). These schemes admit efficient implementation. For example, for the solution of linear problems by such methods, the unknowns corresponding to \( u \) and \( q \) are eliminated element by element from the scheme, and one obtains a system of algebraic equations for the unknown corresponding to \( \lambda \) alone. After finding \( \lambda \), the remaining unknowns are reconstructed element by element as well.
HDG-schemes for elliptic BVP of the second order were suggested in [10] on the basis of an extended mixed statement of the original problem and were studied from abstract viewpoints. Along with approximations to the unknowns $u$, $q$, and $\lambda$, they permit straightforwardly determining the approximation to the solution gradient ($\sigma = \nabla u$). The introduction of a new unknown vector variable into the scheme distinguishes these schemes from the known HDG-schemes. The introduction of a new unknown in a scheme insignificantly increases the amount of computations, because this unknown, like $u$ and $q$, can be eliminated element by element from the scheme.

In the present paper, we study two modifications of HDG-schemes in [10]. The first of them contains a stabilizing parameter and is designed for problems with dominating convection. The second scheme does not contain a stabilizing parameter and can be used for problems in which convection is not dominating. In the study of schemes, we obtain conditions on the spaces of finite elements ensuring the solvability, stability and accuracy of the schemes. The accuracy of schemes is estimated via the errors of $L_2$-orthoprojections of the solution onto the corresponding FE spaces.

2. Boundary value problem
Let $\Omega$ be a polygon in $R^d$ with boundary $\Gamma$, $d = 2, 3$. Consider the convection-diffusion equation

$$-\nabla \cdot (\varepsilon A \nabla u - bu) + au = f, \quad x \in \Omega. \quad (1)$$

We assume that the original data satisfy the conditions

$$A \in [L_{\infty}(\Omega)]^{d \times d}, \quad b \in [L_{\infty}(\Omega)]^d, \quad \nabla \cdot b, a \in L_{\infty}(\Omega), \quad f \in L_2(\Omega), \quad (2)$$

$$m_A |\xi|^2 \leq A(x) \xi \cdot \xi \leq M_A |\xi|^2, \quad \forall \xi \in R^d, \quad a(x) + 0.5 \nabla \cdot b(x) \geq m_a, \quad \forall x \in \Omega. \quad (3)$$

Here $\varepsilon$ is a positive parameter, $m_A > 0$, and $m_a \geq 0$. In what follows, to study the scheme stability and accuracy, we impose additional conditions on the problem data. We supplement Eq. (1) with the boundary conditions

$$u = g_D, \quad x \in \Gamma_D,$$

$$(\varepsilon A \nabla u - bu) \cdot n = g_N, \quad x \in \Gamma_R,$$

$$A \nabla u \cdot n = g_N, \quad x \in \Gamma_N.$$

Here $n$ is the field of unit outward normals on $\Gamma$, $g_D$ is the trace of some function $u_D \in H^1(\Omega)$, and $g_N$ belongs to $L_2(\Gamma_R \cup \Gamma_N)$; the sets $\Gamma_D$, $\Gamma_R$, and $\Gamma_N$ define a partition of the boundary $\Gamma$ into disjoint parts; moreover, $\text{meas}(\Gamma_D) > 0$, and

$$\Gamma_R \subseteq \Gamma^\text{in} = \{x \in \Gamma : b(x) \cdot n(x) < 0\}, \quad \Gamma_N \subseteq \Gamma^\text{out} = \{x \in \Gamma : b(x) \cdot n(x) \geq 0\}.$$

Note that either $\Gamma_R$ or $\Gamma_N$ can be the empty set. To construct discrete schemes, we rewrite Eq. (1) in the form of the system of first-order equations

$$\sigma = \nabla u, \quad q = \varepsilon A \sigma, \quad -\nabla \cdot q + \nabla \cdot (bu) + au = f.$$

3. Spaces of discontinuous finite elements

3.1. Triangulation of the domain
Let $h$ be a positive parameter; let $\hat{K}$ be a polyhedron in $R^d$ that is a basic finite element (a unit triangle or square for $d = 2$; a tetrahedron, cube, or prism for $d = 3$); let $T_h$ be a triangulation of $\Omega$ into finite elements of the maximum diameter $h$, which are images of $\hat{K}$ under an affine transformation.
We assume that $T_h$ is a conformal, regular and coordinated with a boundary partition. In what follows, $K$ stands for an arbitrary element of $T_h$, and the faces of $K$ are treated as its $(d-1)$-dimensional faces. The set of all faces is denoted by $F_h$; by $F_h^D$, $F_h^R$, and $F_h^N$ we denote the sets of faces lying on $\Gamma_D$, $\Gamma_R$, and $\Gamma_N$, respectively, and $F_h^{RN} = F_h^R \cup F_h^N$.

3.2. Function spaces

We use the standard notation $H^s(D)$ for the scalar Sobolev space of order $s$. The norm and seminorm on it are denoted by $\| \cdot \|_{s,D}$ and $| \cdot |_{k,D}$. For the space of vector functions $\tau = (\tau_1, \ldots, \tau_d)$ with components in $H^s(D)$, we use the notation $[H^s(D)]^d$. The norm and seminorm on it are denoted by analogy with the scalar case. The notation $(\cdot, \cdot)_D$ is used for the inner product both in the scalar case of $L_2(D)$ and in the vector case of $[L_2(D)]^d$.

Both for scalar functions and for vector functions, we set

$$\langle w, \eta \rangle_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \langle w, \eta \rangle_K, \quad |w|_{\mathcal{T}_h} = \left( \sum_{K \in \mathcal{T}_h} |w|_{L^2(K)}^2 \right)^{1/2}, \quad \| w \|_{\mathcal{T}_h} = \left( \sum_{K \in \mathcal{T}_h} \| w \|_{L^2(K)}^2 \right)^{1/2}.$$

On the basis of the triangulation $T_h$ we define the space $H^s(T_h)$ as the subspace of functions in $L_2(\Omega)$ whose restrictions to each element $K \in T_h$ belong to the same $H^s(K)$ with the norm $\| \cdot \|_{k,\mathcal{T}_h}$.

In addition, we introduce the notation

$$\langle w \cdot n, \eta \rangle_{\mathcal{F}_h} = \sum_{K \in \mathcal{F}_h} \langle w \cdot n_K, \eta \rangle_{\partial K}, \quad \langle w, \eta \rangle_{F_h^e} = \sum_{F \in F_h^e} \langle w, \eta \rangle_F,$$

where $n_K$ is the field of unit outward normal’s on $\partial K$, and $S \in \{R, N, RN\}$.

3.3. Spaces of finite elements

Let $P_m(D)$ stand for the set of polynomials of degree $\leq m$ defined on $D$. On $\hat{K}$, we introduce the finite-dimensional spaces $V(\hat{K})$ and $W(\hat{K})$ such that

$$P_0(\hat{K}) \subseteq V(\hat{K}) \subseteq H^1(\hat{K}), \quad [P_0(\hat{K})]^d \subseteq W(\hat{K}) \subseteq [H^1(\hat{K})]^d.$$

To each face $F \subset \partial \hat{K}$ we assign the finite-dimensional space $\Lambda(F)$, $\quad P_0(\hat{F}) \subseteq \Lambda(\hat{F}) \subseteq L_2(\hat{F})$.

By using these spaces, we introduce the spaces $V(K)$, $W(K)$, and $\Lambda_K(F)$, $F \subset \partial K$. Let $x = F_K(\hat{x}) = B_K \hat{x} + c_K$ be an invertible affine transformation $\hat{K} \to K$. Set

$$V(K) = \{ v = \hat{v} \circ F_K^{-1}, \hat{v} \in V(\hat{K}) \}, \quad \Lambda_K(F) = \{ \lambda = \hat{\lambda} \circ F_K^{-1}, \hat{\lambda} \in \Lambda(\hat{F}) \},$$

$$W(K) = \{ \tau = |B_K|^{-1} B_K (\hat{\tau} \circ F_K^{-1}), \hat{\tau} \in W(\hat{K}) \},$$

where $|B_K| = \det(B_K)$. To each face $F \in F_h$ we assign a finite-dimensional space $\Lambda(F)$, by assuming that if $F$ is a common face of two elements $K$ and $L$, then

$$\Lambda(F) = \Lambda_K(F) = \Lambda_L(F),$$

and $\Lambda(F) = \Lambda_K(F)$, if $K$ is a boundary element and its face is $F \subset \Gamma$. Let $1_D \in P_0(D)$ be the function equal to the identical unity, and let

$$\Lambda(\partial K) = \{ \hat{\lambda} \in L_2(\partial \hat{K}) : \lambda \mid_{\partial K} \in \Lambda(F), F \subset \partial K \},$$

$$\mathbf{V}(K) = V(K) \times \Lambda(\partial K), \quad \mathbf{V}_0(K) = \{ (v, \lambda) \in \mathbf{V}(K) : (v, \lambda) = c(1_K, 1_{\partial K}), c \in R \}.$$
The main constraints for the above-introduced spaces is given by the following conditions.

\((H_1)\) For \((v, \lambda) \in V(K) \times \Lambda(\partial K)\), the relation

\[-(v, \nabla \cdot \tau) + (\lambda, \tau \cdot n_K)_{\partial K} = 0, \quad \tau \in W(K),\]

implies that \((v, \lambda) \in V_h(K)\).

\((H_2)\) \(\nabla \cdot W(K) = \{\nabla \cdot w : w \in W(K)\} \subseteq V(K)\).

\((H_3)\) \(\Lambda(F) = \{w \cdot n_K | F, w \in W(K)\}\) for every face \(F \subset \partial K\).

We introduce the spaces of finite elements

\[V_h = \{v \in L_2(\Omega) : v|_K \in V(K), K \in T_h\},\]

\[W_h = \{w \in [L_2(\Omega)]^d : w|_K \in W(K), K \in T_h\},\]

\[\Lambda_0^h = \{\lambda \in L_2(F_h) : \lambda|_F \in \Lambda(F), F \in F_h; \lambda|_{F_{x \in F}^d} = P_L u_d |_{F_{x \in F}^d}\},\]

and the set of functions

\[\Lambda_h = \{\lambda \in L_2(F_h) : \lambda|_F \in \Lambda(F), F \in F_h; \lambda|_{F_{x \in F}^d} = P_L u_d |_{F_{x \in F}^d}\}.\]

Here \(P_L\) is the \(L_2\)-projection \(L_2(F_h) \rightarrow \Lambda_h\). Set \(V_h = V_h \times \Lambda_h\) and \(W_h^0 = V_h \times \Lambda_h^0\).

In what follows, we use the space \(V_h\) for the approximation to the problem solution \(u\), \(W_h\) for \(\sigma\) and \(q\), and \(\Lambda_h\) for the approximation to the trace of \(u\) on element faces.

### 3.4. Discrete gradient

Let us introduce the operator \(\nabla_K : (V(K) \rightarrow W(K)\) (a local discrete gradient \([15, 16]\)) acting by the rule

\[
(\nabla_K v, w)_K = -(v, \nabla \cdot w)_K + (\mu, w \cdot n_K)_{\partial K} \quad \forall w \in W(K),
\]

where \(v = (v, \mu)\). Note that condition \((H_1)\) implies that the kernel \(\nabla_K\) is formed only by "constant" functions; more precisely, \(\text{ker}(\nabla_K) = V_0(K)\). We define the discrete gradient \(\nabla_h : V_h \rightarrow W_h\) by the relations \((\nabla_h v)|_K = \nabla_K (v|_K)\), i.e., \((\nabla_h v, w)_h = -(v, \nabla \cdot w)_h + (\mu, w \cdot n)|_{\partial T_h} \quad \forall w \in W_h\).

### 4. Families of HDG-schemes

Let us consider the following discrete problem referred to as scheme \(P_h\): find \((\sigma_h, q_h, u_h, \lambda_h) \in W_h \times W_h \times V_h \times \Lambda_h\) such that the identities

\[
(\sigma_h, w)_K + (u_h, \nabla \cdot w)_K - (\lambda_h, w \cdot n_K)_{\partial K} = 0,
\]

\[
(q_h - \varepsilon A \sigma_h, w)_K = 0,
\]

\[
(-\nabla \cdot q_h + au_h, v)|_{T_h} - (b u_h, \nabla v)|_{T_h} + (b \cdot n \lambda_h, v)|_{\partial T_h} = (f, v)|_{T_h}, \tag{4}
\]

\[
(q_h, n, \mu)|_{\partial T_h} - (b \cdot n \lambda_h, \mu)|_{\partial T_h} + (b \cdot n \lambda_h, \mu)|_{\partial T_h} = (g_N, \mu)|_{\partial T_h}, \tag{5}
\]

hold for arbitrary \(w \in W_h\), \(v \in V_h\), \(\mu \in \Lambda_h^0\) and \(K \in T_h\).

As is shown below, this scheme is stable if the convective term (or \(h\)) is sufficiently small. By introducing stabilizing terms into Eqs. (4) and (5), we obtain a scheme stable for arbitrary \(\varepsilon\) and \(h\).

We refer to it as scheme \(P_h^\varepsilon\). It differs from \(P_h\) only by two terms and has the following form: find \((\sigma_h, q_h, u_h, \lambda_h) \in W_h \times W_h \times V_h \times \Lambda_h\) such that

\[
(\sigma_h, w)_K + (u_h, \nabla \cdot w)_K - (\lambda_h, w \cdot n_K)_{\partial K} = 0,
\]

\[
(q_h - \varepsilon A \sigma_h, w)_K = 0,
\]
Basic problem for scheme satisfying condition (8). In this case, the functions are smooth and 

\( \lambda, \mu \in \mathcal{A}_h \) and 

\[
\lambda_h = \lambda_h(\chi, \mu), \quad \lambda = \lambda(x)
\]

in relation (7). For arbitrary \( w \in W_h \), \( v \in V_h \), \( \mu \in \mathcal{A}_h \) and \( K \in T_h \). Here \( \mathbf{u}_h = (u_h, \lambda_h) \), and 

\[
b_h(\mathbf{u}_h)_{|K} = 0.5b(u_h + \lambda_h)_{|K} + \alpha(u_h - \lambda_h)_{|K} n_K,
\]

where \( \alpha = \alpha(x) \) is some nonnegative stabilization parameter that is a scalar function on \( F_h \).

One can readily see that the scheme \( P_h \) is obtained from \( P_h^\alpha \) for \( \alpha_{|K} = -0.5b \cdot n_K \) in relation (7). If we set \( \alpha_{|K} = 0.5 |b \cdot n_K| \), then we obtain 

\[
b_h(\mathbf{u}_h)_{|K} = ((b \cdot n_K)^+ u_h - (b \cdot n_K)^- \lambda_h)_{|K},
\]

where \( v^\pm = \max\{0, \pm v\} \) are the positive and negative parts of \( v \). In this case, relation (6) is an upwind approximation to the convection-diffusion equation [6].

In what follows, we assume that 

\[
\alpha_0 \leq \alpha(x) \leq C \quad \forall x \in \partial K, \quad K \in T_h,
\]

where \( \alpha_0 \) and \( C \) are positive constants independent of \( h \) and \( \varepsilon \).

For the numerical implementation of the scheme, as the stabilizing function \( \alpha \) one can choose a piecewise constant function on \( F_h \) satisfying condition (8). In this case, the function \( b_h(\mathbf{u}_h)_{|F} \) has the same smoothness as \( b \cdot n_K \) if \( \lambda_h \) if \( \lambda_h \) are smooth functions. This is important from the viewpoint of an approximate computation of the corresponding integrals for the implementation of the scheme.

4.1. Basic statement of the schemes

Let us present other statements of the above-constructed schemes, which are referred below to as basic statements and which are convenient both for the study and for the numerical implementation. They are obtained by the elimination of the unknowns \( \sigma_h \) and \( q_h \) elementwise from the schemes and are stated for the pair of unknowns \( \mathbf{u}_h = (u_h, \lambda_h) \). Basic problem for scheme \( P_h \) has the following form:

\[
\mathbf{u}_h \in V_h : \quad \mathbf{A}_h(\mathbf{u}_h, v_h) = \mathbf{f}_h(v_h) \quad \forall \mathbf{v}_h \in V_h,
\]

where

\[
\mathbf{A}_h(\mathbf{u}_h, v_h) = \mathbf{a}_h(\mathbf{u}_h, v_h) + \mathbf{b}_h(\mathbf{u}_h, v_h), \quad \mathbf{f}_h(v_h) = (f, v_h)_T + (g_N, \mu_h)_{\Gamma_h^\varepsilon},
\]

\[
\mathbf{a}_h(\mathbf{u}_h, v_h) = (\varepsilon A\nabla_{\mathbf{u}_h}, \nabla v_h)_T + (\alpha u_h, v_h)_T,
\]

\[
\mathbf{b}_h(\mathbf{u}_h, v_h) = - (b u_h, \nabla v_h)_T + (b \cdot n \lambda_h, v_h - \mu_h)_{\Gamma_h^\varepsilon} + (b \cdot n \lambda_h, \mu_h)_{\Gamma_h^\varepsilon}.
\]

After the solution of Eq. (9), the unknowns \( \sigma_h \) and \( q_h \) are determined elementwise in accordance with relations (\( P_h \) is the \( L_2 \) projection of \( [L_2(\Omega)]^d \) onto \( W_h \))

\[
\sigma_h = \nabla_h^\perp \mathbf{u}_h, \quad q_h = P_h^\varepsilon \varepsilon A\sigma_h.
\]

We can prove that the scheme \( P_h \) can be reduced to system (9), (10). By reversing the argument, one can readily show that the solution of system (9), (10) can be reduced to the solution of the original scheme \( P_h \). The basic statement of the scheme \( P_h^\alpha \) is obtained by analogy. It has the form

\[
\mathbf{u}_h \in V_h : \quad \mathbf{A}_h^\alpha(\mathbf{u}_h, v_h) = \mathbf{a}_h^\alpha(\mathbf{u}_h, v_h) + \mathbf{b}_h^\alpha(\mathbf{u}_h, v_h) = \mathbf{f}_h(v_h) \quad \forall \mathbf{v}_h \in V_h,
\]

\[
\mathbf{b}_h^\alpha(\mathbf{u}_h, v_h) = - (b u_h, \nabla v_h)_T + (b \cdot a(\mathbf{u}_h) \cdot n, v_h - \mu_h)_{\Gamma_h^\varepsilon} + (b \cdot n \lambda_h, \mu_h)_{\Gamma_h^\varepsilon},
\]
where $a_h$ and $f_h$ are the same forms as above. After the solution of Eq. (11), the other unknowns of the scheme are found elementwise in accordance with (10).

4.2. Stability of the schemes

On the space $V_h$, we introduce the seminorm $| \cdot |_{0,h}$ and the norm $\| \cdot \|_{l,h}$ by the relations

$$\| v \|_{0,h}^2 = m_u \| v \|^2_{0,T_h} + (\alpha (\mu - v)^2)_{\partial T_h} + 0.5 (| b \cdot n |, \mu^2)_{\partial T_h}, \quad \| v \|_{l,h}^2 = \| v \|^2_{0,h} + \epsilon \| \nabla_h v \|^2_{0,\Omega}.$$

**Lemma 1.** Let conditions (2), (3) be satisfied, and let $c = \min\{m_A, 1\}$. Then

$$A_h^*(v_h, v_h) \geq c \| v_h \|^2_{0,h} \quad \forall v_h \in V_h^0.$$

As consequence, the scheme $P_h^*$ is uniquely solvable.

Let $C_h$ be defined such that $\alpha + 0.5 | b | \leq C_h$. In the study of the scheme $P_h$, we assume that

$$\epsilon = 1, \quad m_u \geq 0, \quad m_A - C_h \Omega \| b \|_{L^2(\Omega)} \geq \tilde{m}_A > 0. \quad (12)$$

**Lemma 2.** Let conditions (2), (3) and (12) be satisfied. Then

$$A_h(v_h, v_h) \geq \tilde{m}_A \| \nabla_h v_h \|^2_{0,\Omega}.$$

As consequence, the scheme $P_h$ is uniquely solvable.

5. Estimate for the schemes accuracy

We estimate the accuracy of the schemes $P_h^*$ and $P_h$ under extra constraints:

$$(H_d) \quad \nabla V(K) = \{ \nabla v : v \in V(K) \} \subset [V(K)]^d.$$

We estimate the accuracy of schemes under the following conditions additional to (2) and (3):

$$b \in [W^1_0(T_h)]^d, \quad q \in [H^1(T_h)]^d. \quad (13)$$

**Theorem 1.** Let conditions (2), (3) with $m_u > 0$, (12), (13), and $(H_{d-4})$ be satisfied, and $(q_h, \sigma_h, u_h, \lambda_h)$ be a solution of the scheme $P_h^*$. Then

$$\| u - u_h \|_{0,\Omega} + \epsilon^{1/2} \| \sigma - \sigma_h \|_{0,\Omega} \leq c E_h^*(u), \quad \| q - q_h \|_{0,\Omega} \leq c \epsilon^{1/2} E_h^*(u) + \| q - P_w q \|_{0,\Omega},$$

$$\| u - \lambda_h \|_{H^{1/2}} \leq c (h^{-1/2} + \epsilon^{-1/2}) E_h^*(u),$$

where

$$E_h^*(u) = \| u - P_h u \|_{0,\partial T_h} + \| u - P_h u \|_{0,\partial T_h} + \| u - P_h u \|_{0,\partial T_h} + \| u - P_h u \|_{0,\partial T_h} + \| \nabla u - P_w (\nabla u) \|_{H^{1/2}} + \epsilon^{-1/2} \| q - P_w q \|_{0,\Omega}.$$

**Theorem 2.** Let conditions (2), (3), (13) and $(H_{d-4})$ be satisfied, and $(q_h, \sigma_h, u_h, \lambda_h)$ be a solution of the scheme $P_h$. Then

$$\| u - u_h \|_{0,\Omega} + \| \sigma - \sigma_h \|_{0,\Omega} \leq c E_h(u), \quad \| q - q_h \|_{0,\Omega} \leq c E_h(u) + \| q - P_w q \|_{0,\Omega},$$

$$\| u - \lambda_h \|_{H^{1/2}} \leq c h^{-1/2} E_h(u),$$

where

$$E_h(u) = h^{1/2} \| u - P_h u \|_{0,\partial T_h} + \| u - P_h u \|_{0,\partial T_h} + \| \nabla u - P_w (\nabla u) \|_{0,\partial T_h} + \| q - P_w q \|_{0,\Omega}.$$

6. Estimation of the accuracy of a particular family of schemes

For the practical use of the studied schemes $P_h^*$ and $P_h$, one should define finite-dimensional spaces $V(K)$, $W(K)$, and $\Lambda(F)$ satisfying conditions $(H_{d-4})$. It was shown in [10, 11] that the spaces widely used in mixed FEM satisfy these conditions. They can be used for the definition of particular schemes $P_h^*$ and $P_h$ whose accuracy estimate was mentioned above. As an example, we study only one family of similar schemes. Let $K$ be a $d$-dimensional simplex, and let
\[ W(K) = [P_k^d(K)]^d + xP_k(K), \quad V(K) = P_k^e(K), \quad \Lambda(F) = P_k^e(F), \quad k \geq 0. \]  

(14) 

Here \( W(K) \) is the space of Raviart-Thomas polynomials [12].

Theorem 3. Let \( u \in H^{r+1}(T_h) \), \( q = \varepsilon AVu \in H^s(T_h) \), and \( (q_h, \sigma_h, u_h, \lambda_h) \) be a solution of the scheme \( P_h^e \) on the basis of the finite elements (14). Then for \( 1 \leq s \leq k+1 \)

\[
\| u - u_h \|_{0, \Omega} + \varepsilon^{1/2} \| \sigma - \sigma_h \|_{0, \Omega} + \varepsilon^{-1/2} \| q - q_h \|_{0, \Omega} \leq c E_h^e(u),
\]

\[
\| u - \lambda_h \|_{0, \partial \Omega_h} \leq c (h^{-1/2} + \varepsilon^{-1/2} h^{1/2}) E_h^e(u),
\]

under conditions (2), (3) and (13). Here \( E_h^e(u) = h^{-1} (h^{1/2} \| u \|_{s, T_h} + \varepsilon^{1/2} h (\| u \|_{s+1, T_h} + |AVu|_{s, T_h})) \).

Theorem 4. Let \( u \in H^{r+1}(T_h) \), \( q = \varepsilon AVu \in H^s(T_h) \), and \( (q_h, \sigma_h, u_h, \lambda_h) \) be a solution of the scheme \( P_h^e \) on the basis of the finite elements (3). Then for \( 1 \leq s \leq k+1 \) the estimate

\[
\| u - u_h \|_{0, \Omega} + \| \sigma - \sigma_h \|_{0, \Omega} + \| q - q_h \|_{0, \Omega} + h^{1/2} \| u - \lambda_h \|_{0, \partial \Omega_h} \leq c h^s [u]_{s+1, T_h}
\]

hold under conditions (2), (3), (12) and (13). Here \([u]_{s+1, T_h} = |u|_{s, T_h} + |u|_{s+1, T_h} + |AVu|_{s, T_h} \).

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