Essentially Small Quasi-Dedekind modules and Anti-hopfian modules

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Abstract: In this paper we study the relationship between Essentially Small Quasi-Dedekind modules and anti-hopfian modules. Also, we give some examples which illustrate these relations.

Keywords: Essentially small Quasi-Dedekind modules, anti-hopfian modules.

1. Introduction

Let P be a ring with identity and let U be a unitary left module over P. In[1],we give the definition of Essentially Small Quasi-Dedekind and give some examples with basic properties. An P-module U is called Essentially Small Quasi-Dedekind (ESQD) if Hom(U/V,U) = 0 ∀ V ⊆ U. A ring P is ESQD if P is an ESQD P-module[1]. An P-module U is anti-hopfian if U is nonsimple and all nonzero factor modules of U are isomorphic to U; that is for all V ⊆ U, U/V ≅ U [2]. In this paper we study the relationship between ESQD with other modules such as anti-hopfian modules and continuous modules. Also, we give some examples which illustrate these relations.

Proposition 1 Assume U be an anti-hopfian P-module. Then S = Endp(U) is an ESQD ring.

Proof: Since U is an anti-hopfian P-module, thus, S = Endp(U) is an integral domain [2], so S = Endp(U) is an ESQD ring[1].

Remark 2 Every anti-hopfian module is not ESQD module.

Proof: Assume U be an anti-hopfian module, then for all V ⊆ U, U/V ≅ U. So Hom(U/V,U) ≅ Hom(U,U) ≠ 0. Hence every small submodule N of U is not small quasi-invertible. Therefore U is not ESQD module.

An P-module U has C1 property if, ∀ V ⊆ U , there exists K ≤ U with V ⊆ K. And an P-module U has C2 property if, for all V ≤ U and K ≤ U with V ≅ K , then K ≤ U . And an P-module U has C3 property if, for all V1, V2 ≤ U with V1 ∩ V2 = O , then V1 ⊕ V2 ≤ U . A module satisfying C1 property is called extending(CS), and a module satisfying C1 and C3 properties is called quasi-continuous(π-injective), and a module satisfying C1 and C3 is called continuous[3].

Proposition 3 If U is continuous, then Q/Δ is a (von Neumann) regular ring and Δ equal J, the Jacobson radical of Q.
Proof: Assume $\alpha \in \mathbb{Q}$ and assume $W$ be a complement of $K = \text{Ker} \alpha$. By $C_1$, $W \leq U_2$. Since $\alpha |_{W}$ is a monomorphism, $\alpha W \leq U_2$. Hence $(\alpha - \alpha \beta \alpha )(K \otimes W) = (\alpha - \alpha \beta \alpha )W = 0$, and so $K \otimes W \leq \text{Ker}(\alpha - \alpha \beta \alpha )$. Since $K \otimes W \leq \text{U}_{e} \alpha - \alpha \beta \alpha \in \Delta$. Subsequently $Q / \Delta$ is a regular ring. This also proves that $J \leq \Delta$.

Assume that $a \in \Delta$. Since $\text{Ker} a \cap \text{Ker}(1 - a) = 0$ and $\text{Ker} a \leq \text{U}_e \alpha$, $\text{Ker}(1 - a) = 0$. Hence $(1 - a)U \leq \text{U}_2$ by $C_2$. However $(1 - a)U \leq \text{U}_e$, since $\text{Ker} a \leq (1 - a)U$. Then $(1 - a)U = U$, and therefore $1 - a$ is a unit in $S$. It then follows that $a \in J$ and hence $\Delta \leq J$. Thus, $\Delta = J$.

Recall that an $P$-module $U$ is called small $K$-nonsingular if, for each $f \in \text{End}_P(U)$, $\text{Ker} f \leq \text{U}_e U$ implies $f = 0$.[1].

Proposition 4 If $U$ is a small $K$-nonsingular and continuous module, then $\text{End}_P(U)$ is regular and right continuous and $J(\text{End}_P(U)) = 0$.

Proof: Since $U$ is continuous hence by Prop.3, $J(Q) = \{\varphi | \text{Ker} \varphi \leq \text{U}_e U\}$ and $Q / J(Q)$ is Von Neumann regular and right continuous. By small $K$-nonsingularity, $J(Q) = 0$.

Now, we give the following definition

An $P$-module $U$ is called small self-injective (small quasi-injective), if it is small $U$-injective.

Proposition 5 Let $U$ be a continuous $P$-module. If $U$ is an ESQD $P$-module, then $\text{End}_P(U)$ is regular and $J(\text{End}_P(U)) = 0$.

Proof: by Proposition 4.

Corollary 6 If $U$ is a small quasi-injective and ESQD $P$-module, then $\text{End}_P(U)$ is regular and $J(\text{End}_P(U)) = 0$.

Proof: Since $U$ is a small quasi-injective $P$-module, so $U$ is a continuous $P$-module. Hence the result is obtained by prop.5.

An $P$-homomorphism $f : U \to V$ is called small homomorphism if $\text{Ker} f \leq \text{U}_e U[4]$.

Assume $U$ and $V$ be $P$-modules, $V$ is called small $U$-injective, if for each $P$-small monomorphism $f : A \to U$ (where $A$ is $P$-module) and for each $P$-homomorphism $r : A \to V$, $\exists$ an $P$-homomorphism $k : U \to V$ such that $k \circ f = r$.

Theorem 7 Assume that $A$ be a small quasi-injective right $P$-module, and set $Q = \text{End}_P(A)$ then $J(Q) = \{f \in Q | \text{Ker} f \leq \text{U}_e A\}$.

Proof: Set $K = \{f \in Q | \text{Ker} f \leq \text{U}_e A\}$, consider any $f, g \in K$. Since $(\text{Ker} f) \cap (\text{Ker} g) \leq \text{Ker}(f - g)$, we infer that $\text{Ker}(f - g) \leq \text{U}_e A$, whence $f - g \in K$. Given any $h \in Q$, we have $\text{Ker}(fh) = h^{-1}(\text{Ker} f) \leq \text{U}_e A$, whence $fh \in K$. Also, since $\text{Ker} f \leq \text{Ker}(fh)$, we see that $fh \in K$ as well. Therefore $K$ is a two-sided ideal of $Q$. 

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Given any \( f \in K \), we have \( \ker f \ll_e A \) and \([\ker (1-f)] \cap [\ker f] = 0\); hence \( \ker (1-f) = 0\). Then \( 1-f \) provides an isomorphism of \( A \) onto \((1-f)A\), and the inverse isomorphism \((1-f) : A \to A \) extends to a map \( g \in Q \) such that \( g(1-f) = 1\). Thus \( f \) is a left quasi-regular element of \( Q \).

Now \( K \) is a left quasi-regular ideal of \( Q \), and so \( K \subseteq \mathcal{J}(Q) \). Before showing that \( K = \mathcal{J}(Q) \), we first prove that \( Q/K \) is regular ring, from which \( K = \mathcal{J}(Q) \) follows easily. Thus consider any \( f \in Q \), and let \( B \) be a relative complement for \( \ker f \) in \( A \). Noting that \( f \) restricts to isomorphism \( f_B : B \to B \) to some \( g \in Q \). Now \((gf)\mid_B \) is the identity on \( B \); hence \((fgf-f)B = 0\), and consequently \( B \oplus (\ker f) \leq \ker (fgf-f) \). In as much as \( B \oplus (\ker f) \ll_e A \), we thus obtain \( fgf-f \in K \), whence \( f \) in \( Q/K \) is regular ring.

Now \( K \) and \( \mathcal{J}(Q/K) \) have zero radical, therefore \( \mathcal{J}(Q/K) = 0 \). On the other hand, since \( K \subseteq \mathcal{J}(Q) \) we have \( \mathcal{J}(Q/K) = \mathcal{J}(Q)/K \), whence \( \mathcal{J}(Q) = K \).

**Theorem 8** Let \( U \) be a small quasi-injective \( P \)-module such that \( \mathcal{J}(\text{End}_P(U)) = 0 \). If \( D \) is an essential small \( P \)-submodule of \( U \) then \( D \) is a small quasi-invertible \( P \)-submodule of \( U \).

**Proof:** Suppose that \( D \) is an essential small \( P \)-submodule of \( U \) and \( f \in \text{Hom}(U/D, U) \), \( f \neq 0 \). Define \( g = f \circ \pi \), where \( \pi : U \to U/D \) is the natural homomorphism. It is clear that \( g \in \text{End}_P(U) \), \( g \neq 0 \) and \( D \subseteq \ker g \). Since \( D \) is essential small, then \( \ker g \) is essential small by Th. 7, \( g \in \mathcal{J}(\text{End}(U)) \) and hence \( g = 0 \). Then \( f = 0 \). This is a contradiction. Therefore \( \text{Hom}(U/D, U) = 0 \) and hence \( D \) is a small quasi-invertible \( P \)-submodule.

**Proposition 9** Let \( U \) be a small quasi-injective \( P \)-module with \( \mathcal{J}(\text{End}_P(U)) = 0 \). Then \( U \) is an ESQD \( P \)-module.

**Proof:** Let \( A \ll_e U \). So by Th. 8, \( A \) is a small quasi-invertible submodule of \( U \). Thus \( U \) is an ESQD \( P \)-module.

**Corollary 10** If \( U \) is a small quasi-injective \( P \)-module. Then \( U \) is an ESQD \( P \)-module if and only if \( \mathcal{J}(\text{End}_P(U)) = 0 \).

**Proof:** By Prop. 9 and Coro 6, we get the result.

**Remark 11** Note that the condition \( \mathcal{J}(\text{End}_P(U)) = 0 \) is necessarily in prop. 9, for example: \( Z_4 \) as \( Z \)-module is small quasi-injective, but it is not ESQD. In fact \( \mathcal{J}(\text{End}_P(Z_4)) \cong Z_2 \neq 0 \).

**Proposition 12** If \( U \) is a \( P \)-module such that \( \text{End}_P(U) \) is a regular ring, then \( U \) is an ESQD \( P \)-module.

**Proof:** Suppose that \( f \in \text{End}_P(U) \), \( f \neq 0 \). To prove that \( \ker f \ll_e U \). Since \( \text{End}_P(U) \) is a regular ring, implies that, there exists \( O \neq g \in \text{End}_P(U) \) such that \( f = f \circ g \circ f \) and so that \( g = (g \circ f)^2 \), and \( g \circ f \neq 0 \). Hence \( g \circ f \) is an idempotent element in \( \text{End}_P(U) \), then \( U = \ker(g \circ f) \oplus \text{Im}(g \circ f) \), that is \( \ker(g \circ f) \leq \mathcal{J}(U) \), so \( \ker(g \circ f) \ll_e M \), because \( \text{Im}(g \circ f) \neq 0 \) and \( \ker(g \circ f) \cap \text{Im}(g \circ f) = 0 \). But \( \ker g \subseteq \ker(g \circ f) \), implies \( \ker f \ll_e U \). Thus \( U \) is an ESQD \( P \)-module.

**Corollary 13** Assume that \( U \) be a small quasi-injective \( P \)-module. Thus \( U \) is an ESQD \( P \)-module if and only if \( \text{End}_P(U) \) is a regular ring.
Proof: It follows by Coro 6 and Prop 12.

Corollary 14 Let $U$ be a multiplication $P$-module. If each cyclic submodule of $U$ is injective, then $U$ is an ESQD $P$-module.

Proof: Since $U$ is a multiplication $P$-module with any cyclic submodule of $U$ is injective, then by[5], $\text{End}_{P}(U)$ is a regular ring. Thus by Prop 12, $U$ is an ESQD $P$-module.

The converse of Prop.12 is not true in general, consider the following example.

Example 15 It is well-known that $Z$ as a $Z$-module is ESQD, but $\text{End}_Z(Z) \cong Z$ which is not a regular ring.

Proposition 16 Let $U$ an $P$-module, and let $S = \text{End}_{P}(U)$. If $U$ is an ESQD and quasi-continuous module, then $S_{5}$ satisfies $C_{3}$. Conversely, if $S_{5}$ has $C_{3}$, then $U$ has $C_{3}$, for an arbitrary module $U$.

Proof: Since $U$ is ESQD, so $\Delta = \{ \phi \in S : \text{Ker} \phi \ll_{e} U \} = 0$, where $S = \text{End}_{P}(U)$, and by[3], $S_{5}$ has $C_{3}$.

To prove the converse. Assume $U_{1}, U_{2} \leq \oplus U$ with $U_{1} \cap U_{2} = 0$, implies $U = U_{1} \oplus K_{1}$, $U = U_{2} \oplus K_{2}$ for some $K_{1}, K_{2} \leq U$. Consider the following: $U \xrightarrow{\alpha} U_{1} \xrightarrow{\beta} U$ and $U \xrightarrow{i_{1}} U_{2} \xrightarrow{i_{2}} U$, where $\rho_{1}, \rho_{2}$ are the natural projection mappings, and $i_{1}, i_{2}$ are the inclusion mappings. Then $i_{1} \rho_{1} \in S = \text{End}_{P}(U)$, $i_{2} \rho_{2} \in S = \text{End}_{P}(U)$. So $i_{1} \rho_{1} = (i_{1} \rho_{1})^{2}$, then $i_{1} \rho_{1}$ is an idempotent element in $S_{5}$, thus $\rho \subseteq \rho \leq \rho S, \rho \subseteq \rho, \rho \subseteq \rho S, \rho \subseteq \rho$. But $(i_{1} \rho_{1})S \cap (i_{2} \rho_{2})S = 0$, because if, there exists $g \in (i_{1} \rho_{1})S$, $g \in (i_{2} \rho_{2})S$ such $g = (i_{1} \rho_{1})f_{1}$ and $g = (i_{2} \rho_{2})f_{2}$ for some $f_{1}, f_{2} \subseteq S$.

Hence $g(u) = (i_{1} \rho_{1})f_{1}(u) = (i_{1} \rho_{1})(u_{1} + k_{1}) = u_{1}$ and $g(u) = (i_{2} \rho_{2})f_{2}(u) = (i_{2} \rho_{2})(u_{2} + k_{2}) = u_{2}$, then $g(u) = u_{1} = u_{2}$, so $g(u) = U_{1} \cap U_{2} = 0$, thus $g = 0$. But $S_{5}$ satisfies $C_{3}$, we get $(i_{1} \rho_{1})S \oplus (i_{2} \rho_{2})S \leq \rho S$. Thus $[(i_{1} \rho_{1})S \oplus (i_{2} \rho_{2})S] \oplus B = S$, for some $B \leq S$. Since $I \subseteq S$ (where $I = \text{identity map on } U$), therefore $I = [(i_{1} \rho_{1})g_{1} + (i_{2} \rho_{2})g_{2}] + \psi$, where $\psi \in B$, $g_{1}, g_{2} \subseteq S$.

Thus: $U = I(U) \equiv [(i_{1} \rho_{1})g_{1}(U) + (i_{2} \rho_{2})g_{2}(U)] + \psi(U) \equiv [(i_{1} \rho_{1})(U) + (i_{2} \rho_{2})(U)] + \psi(U) = (U \oplus U_{2}) \oplus \psi(U)$. Hence $U \oplus U_{2} \leq \oplus U$. Therefore $U$ has $C_{3}$.

An $P$-module $U$ is called to have (strong) summand intersection property (SSIP), if for an (infinite) finite index set $I$ and $\forall (U_{i})_{i \in I}$ with $U_{i} \leq U$, $i \in I$, then $\bigcap_{i \in I} U_{i} \leq \oplus U$.

And a module $U$ has the summand sum property (SSP), if $\forall V, K \leq \oplus U$, then $V + K \leq \oplus U$ [6].
The left annihilator of $V \subseteq U$ in $S = \operatorname{End}_{P}(U)$ (i.e. all elements $\phi \in S$ such that $\phi V = 0$) is denoted by $L_{0}(V)$, the right annihilator of $T \subseteq S$ in $U$ (i.e. all elements $u \in U$ such that $Tu = 0$) is denoted by $r_{0}(T)$.[7]

Recall that an $P$-module $U$ is Baer if, for all $V \subseteq U$, $L_{0}(V) = \{e\}$, with $e^{2} = e \in S = \operatorname{End}_{P}(U)$. Equivalently, $U$ is Baer if, for all ideal $I \subset S$, $r_{0}(I) = eU$ with $e^{2} = e \in S = \operatorname{End}_{P}(U)$.[7]

Proposition 17 An ESQD and quasi-continuous $P$-module $U$ has SSIP and SSP.

Proof: Since $U$ has $C_1$ property; that is $U$ is an extending $P$-module. But $U$ be an essentially small quasi-Dedekind extending $P$-module, implies $U$ is a Baer $P$-module, by[7]. So by[6], $U$ has SSIP. Now, Assume $U_{1}, U_{2} \leq_{P} U$. Thus by SSIP, $U_{1} \cap U_{2} \leq_{P} U$. Let $H = U_{1} \cap U_{2}$, implies there exists $K \subseteq U$ such that $H \oplus K = U$. But $U_{1} \leq_{P} U$, implies $U_{1} \oplus A = U$, for some $A \subseteq U$. So, $H \oplus K = U = U_{1} \oplus A$. Similarly, $U_{2} \leq_{P} U$, so $U_{2} \oplus H = U$, for some $H \subseteq U$. We claim that $H \oplus (K \cap U_{1}) = U_{1}$ ...(1), $H \oplus (K \cap U_{2}) = U_{2}$ ...(2). To prove (1). Since $H \oplus (K \cap U_{1}) = (U_{1} \cap U_{2}) \oplus (K \cap U_{1}) \subseteq U_{1}$. Conversely, let $x \in U_{1}$, therefore $x = x + 0 \in U_{1} \oplus A = U = H \oplus K$, then $x = a + b$, where $a \in H = (U_{1} \cap U_{2})$ and $b \in K$, implies $b = x - a \in K \cap U_{1}$, thus $x = a + b \in H \oplus (K \cap U_{1})$, similarly we can get(2). Assume that $W_{1} = K \cap U_{1}$, $W_{2} = K \cap U_{2}$, $U_{1} + U_{2} = (H \oplus W_{1}) + (H \oplus W_{2}) = (U \oplus W_{1}) \oplus W_{2}$, thus $U_{1} + U_{2} = (H \oplus W_{1}) \oplus W_{2}$, but $H \oplus W_{1} = U_{1} \leq_{P} U$, also $W_{2} \leq_{P} U$, (since $H \oplus W_{2} = U_{2}$ and $U_{2} \oplus B = U$, therefore $H \oplus W_{2} \oplus B = U_{2} \oplus B = U$, then $W_{2} \leq_{P} U$). Now, to prove $(H \oplus W_{1}) \cap W_{2} = 0$. Let $x \in (H \oplus W_{1})$ and $x \in W_{2} = K \cap U_{2}$, hence $x = h + w_{1}$, $x \in K$, $x \in U_{2}$, therefore $w_{1} \in W_{1} = K \cap U_{1}$, so $w_{1} \in U_{1}$, also $h \in H = U_{1} \cap U_{2}$, so $h \in U_{1}$ and $h \in U_{2}$, thus $x = h + w_{1} \in U_{1}$, but $x \in U_{2}$, thus $x \in U_{1} \cap U_{2} = H$, but $x \in K$, thus $x \in K \cap H = 0$. Therefore by property $C_{3}$, $(H \oplus W_{1}) \oplus W_{2} \leq_{P} U$ and therefore $U_{1} + U_{2} \leq_{P} U$. So $U$ has SSP.

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