A CHARACTERIZATION OF ERDŐS SPACE FACTORS

DAVID S. LIPHAM

Abstract. We prove that an almost zero-dimensional space $X$ is an Erdős space factor if and only if $X$ has a Sierpiński stratification of C-sets. This characterization shows that if $X$ is an almost zero-dimensional space which can be written as a countable union of complete C-sets, then $X$ is an Erdős space factor. We describe a strongly $\sigma$-complete almost zero-dimensional space which cannot be written as a countable union of complete C-sets. The space is a potential counterexample to a question by Dijkstra and van Mill regarding whether every almost zero-dimensional $F_{\sigma\delta}$-space is an Erdős space factor.

1. Introduction

All spaces under consideration are non-empty, separable and metrizable.

We say that a subset $A$ of a space $X$ is a $C$-set in $X$ if $A$ can be written as an intersection of clopen subsets of $X$. A space $X$ is almost zero-dimensional if every point $x \in X$ has a neighborhood basis consisting of $C$-sets of $X$. A (separable metric) topology $\Sigma$ on $X$ witnesses the almost zero-dimensionality of $X$ if $\Sigma$ is coarser than the given topology on $X$, $(X, \Sigma)$ is zero-dimensional, and every point of $X$ has a neighborhood basis consisting of sets that are closed in $(X, \Sigma)$. A space $X$ is almost zero-dimensional if and only if there is a topology on $X$ witnessing this fact [3, Remark 2.4].

The Erdős space

$$E := \{x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n < \omega\}$$

is almost zero-dimensional, as witnessed by the $F_{\sigma\delta}$ topology that $E$ inherits from $\mathbb{Q}^\omega$. We call a space $X$ an Erdős space factor if there is a space $Y$ such that $X \times Y$ is homeomorphic to $E$.

In [3], Dijkstra and van Mill proved:

Proposition 1 ([3, Theorem 9.2]). For any space $X$ the following are equivalent.

(a) $X$ is an Erdős space factor;
(b) $X \times E$ is homeomorphic to $E$;
(c) $X$ admits a closed embedding into $E$;
(d) there is an $F_{\sigma\delta}$ topology witnessing the almost zero-dimensionality of $X$.

Proposition 2 ([3, Corollary 9.3]). Every almost zero-dimensional $G_\delta$-space is an Erdős space factor.

They then asked:

Question 1 ([3, Question 9.7]). Is every almost zero-dimensional $F_{\sigma\delta}$-space an Erdős space factor?

2010 Mathematics Subject Classification. 54F45, 54H05, 30D05.

Key words and phrases. almost zero-dimensional, Erdős space factor, Cantor bouquet.
The goal of this paper is to provide an intrinsic characterization of Erdős space factors involving Sierpiński stratifications of C-sets (Theorem 1). Our characterization will imply that if $X$ is an almost zero-dimensional space which can be written as a countable union of C-set Erdős space factors, then $X$ is an Erdős space factor (Theorem 2). For example, we show that if $X$ is an Erdős space factor then so is the Vietoris hyperspace $\mathcal{F}(X)$ (Corollary 5).

Combining Theorem 2 with Proposition 2 will show that every almost zero-dimensional countable union of complete C-sets is an Erdős space factor (Corollary 4). We will show that this corollary applies to Dijkstra’s homogeneous almost zero-dimensional space $T(E,E')$ featured in [2]. On the other hand, we will present a potential counterexample to Question 1 in the form of a strongly $\sigma$-complete almost zero-dimensional space which cannot be written as a countable union of complete C-sets. The space emerges naturally in complex dynamics.

2. Main results

A tree $T$ on an alphabet $A$ is a subset of $A^{<\omega}$ that is closed under initial segments, i.e. if $\beta \in T$ and $\alpha < \beta$ then $\alpha \in T$. An element $\lambda \in A^\omega$ is an infinite branch of $T$ provided $\lambda \upharpoonright k \in T$ for every $k < \omega$. We let $[T]$ denote the set of all infinite branches of $T$. If $\alpha, \beta \in T$ are such that $\alpha < \beta$ and $\text{dom}(\beta) = \text{dom}(\alpha) + 1$, then we say that $\beta$ is an immediate successor of $\alpha$ and $\text{succ}(\alpha)$ denotes the set of immediate successors of $\alpha$ in $T$.

A system $(X_\alpha)_{\alpha \in T}$ is a Sierpiński stratification of a space $X$ if:
1. $T$ is a non-empty tree over a countable alphabet,
2. each $X_\alpha$ is a closed subset of $X$,
3. $X_\varnothing = X$ and $X_\alpha = \bigcup \{ X_\beta : \beta \in \text{succ}(\alpha) \}$ for each $\alpha \in T$, and
4. if $\lambda \in [T]$ then the sequence $X_{\lambda \upharpoonright 0}, X_{\lambda \upharpoonright 1}, \ldots$ converges to a point in $X$.

A space is absolute $F_{\sigma \delta}$ if and only if it has a Sierpiński stratification [5, Théorème].

**Theorem 1.** An almost zero-dimensional space $X$ is an Erdős space factor if and only if $X$ has a Sierpiński stratification of C-sets.

**Proof.** Suppose that $X$ is an Erdős space factor. Then $X$ is homeomorphic to a closed subset of $\mathcal{E}$. By [3, Proposition 9.1] we have $\mathcal{E} \simeq Q^\omega \times C$, where
\[ C := \{ x \in \ell^2 : x_n \in \mathbb{P} \text{ for all } n < \omega \} \]
is the complete Erdős space. Thus we may assume that $X$ is a closed subset of $Q^\omega \times C$. Let $(A_\alpha)_{\alpha \in S}$ be the obvious Sierpiński stratification of $Q^\omega$ in which $S$ is a tree over $Q$. Let $d$ be a complete metric for $C$. For each $n < \omega$ let $\{ B^n_i : i < \omega \}$ be a C-set covering of $C$ such that $\text{diam}(B^n_i) < 1/n$ in the metric $d$. Let $B_\varnothing = C$. For each non-empty $\beta \in \omega^{<\omega}$ define $B_\beta = \bigcap \{ B^n_\beta(n) : n < \text{dom}(\beta) \}$. Then $T := \{ \beta \in \omega^{<\omega} : B_\beta \neq \varnothing \}$ is a tree over $\omega$, and by completeness of $(C,d)$ we have that $(B_\beta)_{\beta \in T}$ is a Sierpiński stratification of $C$. If $\alpha \in Q^{<\omega}$, $\beta \in \omega^{<\omega}$, and $n = \text{dom}(\alpha) = \text{dom}(\beta)$, then we define
\[ \alpha * \beta = (\langle \alpha(0), \beta(0) \rangle, \ldots, (\alpha(n-1), \beta(n-1))) \].

Note that $S * T := \{ \alpha * \beta : \alpha \in S, \beta \in T, \text{ and dom}(\alpha) = \text{dom}(\beta) \}$ is a tree over $Q \times \omega$. $(A_\alpha \times B_\beta)_{\alpha \beta \in S \times T}$ is a Sierpiński stratification of $Q^\omega \times C$, and each $A_\alpha \times B_\beta$ is a C-set in $Q^\omega \times C$. Then $((A_\alpha \times B_\beta) \cap X)_{\alpha \beta \in S \times T}$ is a Sierpiński stratification of $X$ consisting of C-sets in $X$. 

...
Now suppose that \((A_\alpha)_{\alpha \in T}\) is a Sierpiński stratification of \(X\) where every \(A_\alpha\) is a C-set in \(X\). For each \(\alpha \in T\) write \(A_\alpha = \bigcap \{C^\alpha_n : n < \omega\}\) where each \(C^\alpha_n\) is clopen in \(X\). Let \(\{B_i : i < \omega\}\) be a neighborhood basis of C-sets for \(X\), and for each \(i < \omega\) write \(B_i = \bigcap \{C^i_{ij} : j < \omega\}\) where each \(C^i_{ij}\) is clopen in \(X\). The topology \(\mathcal{T}\) that is generated by the sub-basis
\[
\{C^\alpha_n, X \setminus C^\alpha_n : (\alpha, n) \in T \times \omega\} \cup \{C_{ij}, X \setminus C_{ij} : i, j < \omega\}
\]
is easily seen to be a second countable, regular, zero-dimensional topology on \(X\). It witnesses the almost zero-dimensionality of \(X\) because \(\mathcal{T}\) is coarser than the original topology of \(X\) and every \(B_i\) is \(\mathcal{T}\)-closed. Further, every \(A_\alpha\) is \(\mathcal{T}\)-closed and \((A_\alpha)_{\alpha \in T}\) is a Sierpiński stratification of \((X, \mathcal{T})\). Therefore \((X, \mathcal{T})\) is absolute \(F_{\sigma\delta}\).

By Proposition 1 \(X\) is an Erdős space factor. □

**Theorem 2.** If \(X\) is an almost zero-dimensional space which is the union of countably many C-set Erdős space factors, then \(X\) is an Erdős space factor.

*Proof.* Suppose that \(X\) is almost zero-dimensional and \(X = \bigcup \{A_n : n < \omega\}\) where each \(A_n\) is both an Erdős space factor and a C-set in \(X\). By Theorem 1, for each \(n < \omega\) there is a Sierpiński stratification \((B^\alpha_n)_{\alpha \in T}\) of \(A_n\) such that each \(B^\alpha_n\) is a C-set in \(A_n\). Define a tree
\[
T = \bigcup_{n<\omega} \{(n)^\sim \alpha : \alpha \in T^n\}.
\]
Put \(X_\emptyset = X\) and \(X_{(n)^\sim \alpha} = B^\alpha_n\) for each \(n < \omega\) and \(\alpha \in T^n\). By [3, Corollary 4.20] each \(B^\alpha_n\) is a C-set in \(X\). Thus \((X_\alpha)_{\alpha \in T}\) is a Sierpiński stratification of \(X\) consisting of C-sets in \(X\). By Theorem 1 \(X\) is an Erdős space factor. □

By Proposition 1 and Theorem 2 we have:

**Corollary 3.** Let \(X\) be an almost zero-dimensional space. If \(X\) can be written as the union of countably many C-sets which admit closed embeddings into \(\mathcal{E}\), then \(X\) has a closed embedding in \(\mathcal{E}\).

By Theorem 2 and Proposition 2 we have:

**Corollary 4.** If \(X\) is almost zero-dimensional space that is a countable union of complete C-sets, then \(X\) is an Erdős space factor.

**Application to** \(\mathcal{F}(X)\). For any space \(X\) and \(n < \omega\) we let \(\mathcal{F}_n(X)\) denote the set of all non-empty subsets of \(X\) of cardinality \(\leq n\). The set of all non-empty finite subsets of \(X\) is denoted
\[
\mathcal{F}(X) = \bigcup_{n<\omega} \mathcal{F}_n(X).
\]
The Vietoris topology on \(\mathcal{F}(X)\) has a basis of open sets
\[
\langle U_0, \ldots, U_{k-1} \rangle = \{F \in \mathcal{F}(X) : F \subset \bigcup_{i<k} U_i \text{ and } F \cap U_i \neq \emptyset \text{ for each } i < k\},
\]
where \(k < \omega\) and \(U_0, \ldots, U_{k-1}\) are non-empty open subsets of \(X\).

**Corollary 5.** If \(X\) is an Erdős space factor then so is \(\mathcal{F}(X)\).

*Proof.* Suppose that \(X\) is an Erdős space factor. Then \(\mathcal{F}(X)\) is almost zero-dimensional by [7, Proposition 2.2]. By [7, Corollary 5.2] each \(\mathcal{F}_n(X)\) is an Erdős space factor. And each \(\mathcal{F}_n(X)\) is a C-set in \(\mathcal{F}(X)\). Indeed, if \(F \in \mathcal{F}(X) \setminus \mathcal{F}_n(X)\) then we can partition \(X\) into \(m = |F|\) pairwise disjoint non-empty clopen sets
prove this, for each \( n < \omega \) such that \( \langle C_0, \ldots, C_m \rangle \) is clopen in \( \mathcal{F}(X) \) and \( \langle C_0, \ldots, C_m \rangle \subset \mathcal{F}(X) \setminus \mathcal{F}_n(X) \), this shows that \( \mathcal{F}_n(X) \) is a C-set in \( \mathcal{F}(X) \).

In conclusion, \( \mathcal{F}(X) \) is an almost zero-dimensional countable union of C-set Erdős space factors. By Theorem 2 \( \mathcal{F}(X) \) is an Erdős space factor.

3. Strongly \( \sigma \)-complete examples

Recall that a space \( X \) is strongly \( \sigma \)-complete if \( X \) can be written as a countable union of closed complete subspaces. Examples include spaces which are countable unions of complete C-sets. Strongly \( \sigma \)-complete spaces are known to be absolute \( F_{\sigma \delta} \), and are thus relevant to Question 1.

**Example 1.** Dijkstra’s homogeneous almost zero-dimensional space is a countable union of complete C-sets and is therefore an Erdős space factor.

If \( X \) is an almost zero-dimensional space and \( A \) is a C-set in \( X \), then define

\[
T(X, A) = \bigcup_{n<\omega} (X \setminus A)^n \times A.
\]

Let \( \pi : T(X, A) \to X \) be defined by \( \pi(x_0, \ldots, x_n) = x_0 \). Consider the collection \( \mathcal{B} \) of subsets of \( T(X, A) \) that consists of all sets of the form \( O_0 \times \ldots \times O_{n-1} \times \pi^{-1}(O_n) \), where \( n < \omega \), each \( O_i \) is an open subset of \( X \), and \( O_i \subset X \setminus A \) for each \( i < n \). By [2, Claim 7], \( \mathcal{B} \) forms a basis for an almost zero-dimensional topology on \( T(X, A) \).

Dijkstra’s main example \( T(\hat{E}, E') \) in [2] is generated by a particular complete almost zero-dimensional space \( \hat{E} \simeq \mathcal{E}_c \) and a C-set \( E' \) in \( \hat{E} \). We claim that for \( X = \hat{E} \) and \( A = E' \), the space \( T(X, A) \) is a countable union of complete C-sets. To prove this, for each \( n < \omega \) put

\[
T_n(X, A) = \bigcup_{k \leq n} (X \setminus A)^k \times A.
\]

Write \( A \) as an intersection of clopen subsets of \( X \); \( A = \bigcap \{ C_i : i < \omega \} \). Note that

\[
T_n(X, A) = \bigcap_{i < \omega} \bigcup_{k \leq n} (X \setminus A)^k \times \pi^{-1}(C_i).
\]

For each \( i < \omega \) note that \( \bigcup_{k \leq n} (X \setminus A)^k \times \pi^{-1}(C_i) \) is clopen in \( T(X, A) \) because it is a union of basic open sets and its complement is the basic open set \( (X \setminus A)^n \times \pi^{-1}(X \setminus C_i) \). Therefore \( T_n(X, A) \) is a C-set in \( T(X, A) \). And each \( T_n(X, A) \) is complete by [2, Claim 6].

**Example 2.** There is a strongly \( \sigma \)-complete almost zero-dimensional space which cannot be written as a countable union of complete C-sets.

The Julia set \( J(f) \) of the entire function \( f(z) = \exp(z) - 1 \) is known to be a Cantor bouquet of rays in the complex plane and thus has a natural endpoint set \( E(f) \). Consider the set

\[
\hat{E}(f) := \{ z \in E(f) : \{ f^n(z) : n \in \mathbb{N} \} \neq J(f) \}
\]

consisting of all endpoints whose (forward) orbits are not dense in \( J(f) \). By [1, Lemma 1] \( \hat{E}(f) \) is a first category \( F_{\sigma} \)-subset of the completely metrizable almost zero-dimensional space \( E(f) \). In particular, \( \hat{E}(f) \) is strongly \( \sigma \)-complete and almost zero-dimensional. The escaping endpoint set

\[
\bar{E}(f) := \{ z \in E(f) : f^n(z) \to \infty \}
\]
is densely contained in $\dot{E}(f)$, and in [4] we proved that $\dot{E}(f)$ cannot be written as a countable union of nowhere dense C-sets. Therefore $\dot{E}(f)$ cannot be written as a countable union of nowhere dense C-sets. By Baire’s theorem every completely metrizable subset of $\dot{E}(f)$ is nowhere dense, this proves that $\dot{E}(f)$ cannot be written as a countable union complete C-sets.

**Remark 1.** van Mill [6] proved that $\mathbb{Q} \times \mathbb{P}$ is the unique zero-dimensional space which is strongly $\sigma$-complete, nowhere $\sigma$-compact, and nowhere complete. There is no such classification of almost zero-dimensional spaces, as $\dot{E}(f)$ is strongly $\sigma$-complete, nowhere $\sigma$-compact, nowhere complete, and is not a $\mathbb{Q}$-product.

**Question 2.** Is $\dot{E}(f)$ an Erdős space factor?

**References**

[1] I. N. Baker and P. Domínguez, Residual Julia Sets, Journal of Analysis, Vol. 8, 2000, pp. 121–137.
[2] J. J. Dijkstra, A homogeneous space that is one-dimensional but not cohesive, Houston J. Math. 32 (2006), no. 4, 1093–1099.
[3] J. J. Dijkstra, J. van Mill, Erdős space and homeomorphism groups of manifolds, Mem. Amer. Math. Soc. 208 (2010), no. 979.
[4] D. S. Lipham, Distinguishing endpoint sets from Erdős space, arXiv preprint (2020); https://arxiv.org/pdf/2006.04783.pdf.
[5] W. Sierpiński, Sur une définition topologique des ensembles $F_{\sigma\delta}$, Fund. Math. 6 (1924), 24–29.
[6] J. van Mill, Characterization of some zero-dimensional separable metric spaces, Trans. Amer. Math. Soc. 264 (1981), 205–215.
[7] A. Zaragoza, Symmetric products of Erdős space and complete Erdős space, Topology Appl. 284 (2020), 1–10.

**Department of Mathematics, Auburn University at Montgomery, Montgomery AL 36117, United States of America**

**E-mail address:** ds100032@auburn.edu