Exact Energy and Momentum Conserving Algorithms for General Models in Nonlinear Elasticity

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Abstract

This paper develops implicit conserving time-integration schemes for initial boundary value problems in nonlinear elasticity which inherit the conservation laws of total energy, linear and angular momentum whenever they are present in the underlying problem. Schemes are constructed for general hyperelastic material models, both compressible and incompressible, and are implemented within the context of a finite element discretization in space. Numerical examples using Ogden-type material models are given to illustrate the performance of the proposed schemes. Within the context of the given examples the schemes are seen to exhibit excellent numerical stability properties without a compromise in accuracy relative to the standard implicit mid-point rule.

1. Introduction.

In this paper we develop conserving time-integration schemes for initial boundary value problems in nonlinear hyperelasticity. We consider both compressible and incompressible hyperelastic material models and employ the formalism presented in Gonzalez [1996a,b] to construct schemes which preserve the total energy of the system, along with the total linear and angular momentum whenever these conservation laws are present in the underlying problem. Within the context of incompressible hyperelasticity, this paper extends to the infinite-dimensional case the formalism for constrained systems presented in Gonzalez [1996b].

In Section 2 we present a displacement-traction initial boundary value problem for a general compressible hyperelastic continuum body, and review some of the Hamiltonian structure for this system. Using the methods of the aforementioned references we then present a conserving time-stepping scheme. As we will see, the resulting scheme is applicable to general models in hyperelasticity and inherits the conservation laws of total energy, linear and angular momentum whenever they are present in the underlying problem. Here we note that conserving schemes for general hyperelastic models in elastodynamics go back to Simo & Tarnow [1992a], who designed schemes based on mean value arguments which involved the solution of an extra nonlinear equation in each time step. This class of schemes were subsequently extended to nonlinear shells in Simo & Tarnow [1992b], and to nonlinear rods in Simo, Tarnow & Dobláré [1993]. As we will see, the class of schemes presented herein do not involve any extra variables or equations; in particular, the conservation properties are inherent to the time discretization.

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In Section 3 we develop a formulation of incompressible nonlinear hyperelasticity which fits within the framework for conserving integrators developed in Gonzalez (1996b); in particular, we will consider a differential-algebraic formulation of the dynamics (see e.g. Brenan, Campbell & Petzold [1989] for an explanation of this terminology). Such formulations follow naturally from the d’Alembert-Lagrange principle for constrained mechanical systems, and for conservative systems subject to holonomic constraints, it is well-known that such formulations may also be derived from a constrained variational principle (see e.g. Arnold [1989]). While many different differential-algebraic formulations can be used to describe the same system, our attention will be focused on a particular formulation that is “Hamiltonian” in an appropriate sense. For purposes of numerical implementation, we introduce a quasi-incompressible formulation in which the pointwise incompressibility condition is relaxed. In particular, the resulting quasi-incompressible formulation may be viewed as a generalization to elastodynamics of the work of Simo & Taylor [1991]. Using the general framework for conserving schemes mentioned above, we then construct a conserving time-stepping scheme for the quasi-incompressible formulation.

In Section 4 we discuss the finite element implementation of the time-stepping schemes presented herein. Since the finite element implementation for the compressible case is fairly standard, we concentrate on the quasi-incompressible case. In particular, we introduce a general mixed finite element formulation and reduce it to a two-field formulation. We then discuss a solution strategy which can be interpreted as the generalization to the dynamic case of the augmented multiplier methods used within the context of elastostatics, see e.g. Glowinski & Le Tallec [1981,1989] and Simo & Taylor [1991]. By using a nested iteration scheme, we will see that the two-field formulation may be effectively reduced to one field.

Finally, in Section 5, we present some numerical examples of both the compressible and incompressible formulations. We employ hyperelastic models of the Ogden type (see e.g. Ogden [1982, 1984]), and for the quasi-incompressible case we consider models in terms of deviatoric strain measures as employed in Simo & Taylor [1991].

2. General Compressible Hyperelasticity.

In this section we present a formulation of general compressible hyperelasticity. We first review some standard terminology for the subject and then state a displacement-traction initial boundary value problem for a hyperelastic continuum body. We next introduce a weak formulation, interpreted as a dynamic generalization of the principle of virtual work, which is suitable for numerical approximation. Given this weak formulation we discuss its Hamiltonian structure and then present a conserving time-stepping scheme. For more details on the physical and mathematical structure of the field equations presented in this section we refer the reader to the introductory text of Gurtin [1981], and the more advanced works of Marsden &
Hughes [1983], Ogden [1984], and Ciarlet [1988].

2.1. Preliminaries.

Let $\mathcal{B}$ denote a continuum body in 3-dimensional Euclidean space $\mathbb{R}^3$. We assume that $\mathcal{B}$, in its reference or undeformed state, occupies a closed subset $\tilde{\mathcal{B}}$ of $\mathbb{R}^3$ where $\mathcal{B}$ is a bounded path-connected open set with piecewise smooth boundary $\partial \mathcal{B}$. In particular, we identify material particles of $\mathcal{B}$ in its reference state with points $X = (X_1, X_2, X_3) \in \tilde{\mathcal{B}}$ and call $\tilde{\mathcal{B}}$ the reference configuration of $\mathcal{B}$.

A deformation of $\mathcal{B}$ in $\mathbb{R}^3$, relative to the reference configuration $\tilde{\mathcal{B}}$, is a mapping $\varphi: \tilde{\mathcal{B}} \to \mathbb{R}^3$ that is in some sense smooth, orientation-preserving, and injective in $\mathcal{B}$. If we denote by $Q$ the set of all deformations, then a motion of $\mathcal{B}$ in a time interval $[0, T]$ is a curve in $Q$ of the form $[0, T] \ni t \mapsto \varphi_t \in Q$. Hence, we regard a motion as a mapping $\varphi: \tilde{\mathcal{B}} \times [0, T] \to \mathbb{R}^3$, where for each $t \in [0, T]$ the mapping $\varphi_t = \varphi(\cdot, t): \mathcal{B} \to \mathbb{R}^3$ is a deformation of $\mathcal{B}$. Also, for any time $t \in [0, T]$ in a motion we define a material velocity field $V_t: \mathcal{B} \to \mathbb{R}^3$ by the expression

$$V_t = (\partial \varphi/\partial t)|_{t=0} = \dot{\varphi}_t.$$  

Let $M^3$ denote the vector space of all real $3 \times 3$ matrices equipped with the standard (Euclidean) inner product, defined for any $E, F \in M^3$ by the expression

$$E : F = \sum_{i,j=1}^{3} E^i_j F^i_j,$$

and consider the subsets $M^3_+, S^3$, and $S^3_{pd}$ of $M^3$ defined as

$$M^3_+ = \{ M \in M^3 \mid \det[M] > 0 \},$$

$$S^3 = \{ M \in M^3 \mid M^t = M \},$$

$$S^3_{pd} = \{ M \in M^3 \cap S^3 \mid c \cdot M c > 0 \quad \forall 0 \neq c \in \mathbb{R}^3 \},$$

where $\det: M^3 \to \mathbb{R}$ denotes the determinant function.

For any deformation $\varphi_t: \mathcal{B} \to \mathbb{R}$ in a motion of $\mathcal{B}$ we define a deformation gradient field $F_t: \tilde{\mathcal{B}} \to M^3_+$ by

$$F_t(X) = D\varphi_t(X), \quad \forall X \in \tilde{\mathcal{B}}$$

(2.5)

where $D\varphi_t(X)$ denotes the derivative of $\varphi_t$ evaluated at $X$, and we denote by $C_t: \mathcal{B} \to S^3_{pd}$ the Cauchy strain field defined by the expression

$$C_t = F_t^t F_t.$$  

(2.6)

For any $t \in [0, T]$ we denote by $S_t: \tilde{\mathcal{B}} \to S^3$ the second Piola-Kirchhoff stress field for the continuum body. Note that for a hyperelastic body $S_t$ is related to the deformation $\varphi_t$ locally via a (frame-invariant) constitutive relation of the form

$$S_t(X) = 2D_2 \tilde{\mathcal{W}}(X, \cdot)|_{\cdot = C_t(X)}, \quad \forall X \in \tilde{\mathcal{B}}$$

(2.7)

where $\tilde{\mathcal{W}}: \tilde{\mathcal{B}} \times S^3_{pd} \to \mathbb{R}$ is a hyperelastic stored energy function for the body relative to its reference state and $D_t$ denotes the derivative with respect to the $i^{th}$ argument.
2.2. Displacement-Traction Problem: Strong Form.

Let \( \Gamma_\varphi \) and \( \Gamma_\sigma \) be disjoint relatively open subsets of \( \partial B \) defined such that \( \partial B = \Gamma_\varphi \cup \Gamma_\sigma \). Then a displacement-traction initial boundary value problem for a motion of \( B \) may be stated as follows:

Find \( \varphi : B \times [0,T] \to \mathbb{R}^3 \) such that

\[
\begin{align*}
\text{Div}[F_t S_t] + b_t &= \rho \ddot{\varphi}_t \quad \text{in} \quad B \times (0,T) \\
F_t S_t N &= f_t \quad \text{in} \quad \Gamma_\sigma \times (0,T) \\
\varphi_t &= g_t \quad \text{in} \quad \Gamma_\varphi \times (0,T) \\
\varphi_0 &= \dot{\varphi}_0 \quad \text{in} \quad \bar{B} \\
\dot{\varphi}_0 &= \dot{V}_0 \quad \text{in} \quad \bar{B}
\end{align*}
\]

where superposed dots denote differentiation with respect to time and \( \text{Div} \) is the divergence operator in \( B \) relative to the cartesian coordinates \((X_1, X_2, X_3)\). For any field \( P : B \to \mathbb{R}^{3 \times 3} \) recall that \( \text{Div}[P](X) \in \mathbb{R}^3 \) for each \( X \) and has coordinates given by

\[
\text{Div}[P]^i_j(X) = \sum_{j=1}^{3} \partial_j P^{ij}(X), \quad (i = 1, 2, 3)
\]

where \( \partial_i \) denotes the partial derivative with respect to the coordinate \( X_i \).

In the above system \( b : B \times (0,T] \to \mathbb{R}^3 \) is a prescribed body force density per unit reference volume, \( \rho : B \to \mathbb{R}_+ \) is the mass density of the body in its reference configuration, \( N : \Gamma_\sigma \to \mathbb{R}^3 \) is the unit outward normal field on \( \Gamma_\sigma \subset \partial B \), \( f : \Gamma_\sigma \times (0,T] \to \mathbb{R}^3 \) is a prescribed surface force density per unit reference surface area, \( g : \Gamma_\varphi \times (0,T] \to \mathbb{R}^3 \) is a specified motion on \( \Gamma_\varphi \subset \partial B \), \( \varphi_0 : \bar{B} \to \mathbb{R}^3 \) is a prescribed deformation of \( B \), and \( \dot{V}_0 : \bar{B} \to \mathbb{R}^3 \) is a prescribed material velocity field. The partial differential equation \((2.8)_1\) expresses the local balance of linear momentum in \( B \) (whereas the local balance of angular momentum is implied by the symmetry of the stress field \( S_t \)), \((2.8)_2,3\) are traction and displacement boundary conditions, respectively, and \((2.8)_4,5\) are initial data for the motion.

Remark 2.1. In the above formulation we have excluded the general case in which the body force density \( b \), the surface force density \( f \) and the boundary condition \( g \), depend on the motion \( \varphi \). In addition, we can simplify matters further by making the assumption that \( b, f \) and \( g \) are independent of time, in which case we say that the body \( \bar{B} \) is subjected to dead loads. While this level of simplification is not necessary for the developments of this paper, we assume henceforth that all loads are dead unless mentioned otherwise.

Typically, we are interested in weak solutions of \((2.8)\); in particular, weak solutions satisfying an associated weak formulation interpreted as a dynamic generalization of the principle of virtual work, which we introduce next.
2.3. Displacement-Traction Problem: Weak Form.

Let $L^2(B, \mathbb{R}^3)$ denote the vector space of mappings from $B$ into $\mathbb{R}^3$ defined by

$$L^2(B, \mathbb{R}^3) = \{ \chi: B \to \mathbb{R}^3 \mid \int_B |\chi|^2 < \infty \}$$

(2.10)

where $|\cdot|$ is the standard Euclidean norm on $\mathbb{R}^3$, and denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $L^2(B, \mathbb{R}^3)$ defined by the expression

$$\langle v, w \rangle = \int_B v \cdot w$$

(2.11)

where the dot on the right-hand side denotes the standard inner product on $\mathbb{R}^3$. In analogy with $L^2(B, \mathbb{R}^3)$ we introduce the spaces $L^2(B, \mathbb{M}^3)$ and $L^2(\Gamma_\varphi, \mathbb{R}^3)$ with inner products, respectively, defined by the expressions

$$\langle F, G \rangle = \int_B F : G \quad \forall F, G \in L^2(B, \mathbb{M}^3)$$

(2.12)

$$\langle v, w \rangle_{\Gamma_\varphi} = \int_{\Gamma_\varphi} v \cdot w \quad \forall v, w \in L^2(\Gamma_\varphi, \mathbb{R}^3).$$

(2.13)

Taking $L^2(B, \mathbb{R}^3)$ as the setting for our weak formulation, we define the configuration space $Q$ for $B$ as a smooth (linear) manifold in $L^2(B, \mathbb{R}^3)$, namely

$$Q = \{ \psi \in L^2(B, \mathbb{R}^3) \mid D\psi \in L^2(B, \mathbb{M}^3), \quad D\psi(X) \in \mathbb{M}^3_+ \quad \forall X \in B \quad \text{and} \quad \psi(X) = g(X) \quad \forall X \in \Gamma_\varphi \}$$

(2.14)

where $g: \Gamma_\varphi \to \mathbb{R}^3$ is the specified boundary condition appearing in (2.8).3.

Remark 2.2. Following standard abuse of terminology, we call elements of $Q$ “configurations” of $B$ when really they are deformations of $B$ relative to the reference configuration $B$.

Given the above definition of $Q$, we now view a motion as a $C^1$ curve $[0, T] \ni t \mapsto \varphi_t \in Q$ with associated velocity $\dot{\varphi}_t$ viewed as a curve in $\mathcal{V}_Q$, i.e. $[0, T] \ni t \mapsto \dot{\varphi}_t \in \mathcal{V}_Q$, where the vector space $\mathcal{V}_Q$ is defined as

$$\mathcal{V}_Q = \{ \eta \in L^2(B, \mathbb{R}^3) \mid D\eta \in L^2(B, \mathbb{M}^3), \quad \eta(X) = 0 \quad \forall X \in \Gamma_\varphi \}. \quad (2.15)$$

With the above notation our weak formulation of (2.8) may be stated as follows:
Given $b : B \to \mathbf{R}^3$ and $f : \Gamma_\sigma \to \mathbf{R}^3$ find $\varphi : [0, T] \to Q$ such that
\[
\begin{align*}
\langle \rho \ddot{\varphi}_t, \eta \rangle + \langle F_t S_t, D \eta \rangle &= \langle b, \eta \rangle + \langle f, \eta \rangle |_{\Gamma_\sigma} \quad \text{in} \quad [0, T] \\
\langle \varphi_0, \eta \rangle &= \langle \varphi_0, \eta \rangle \\
\langle \dot{\varphi}_0, \eta \rangle &= \langle \dot{\varphi}_0, \eta \rangle
\end{align*}
\forall \eta \in \mathcal{V}_Q. \tag{2.16}
\]

Here we note that, if the deformation $\varphi_t \in Q$ is sufficiently smooth for each $t \in [0, T]$, then the weak formulation given above is equivalent to the initial boundary value problem given in (2.8).

Ideally, before introducing an approximation scheme for (2.8) or (2.16) one would like to know whether or not these systems admit solutions, and if so, whether or not solutions are unique. That is, one would like to know if the systems are well-posed. In this paper we make no attempt to address these questions (the interested reader is referred to Marsden & Hughes [1983, Chapter 6] and Ciarlet [1988 Chapters 6, 7] for partial results). We proceed under the assumption that the above systems do admit solutions in some sense, and given this, we seek to construct numerical approximations to them. Before doing so, however, we first review some of the underlying Hamiltonian structure for the above systems.

2.4. Hamiltonian Structure and Conservation Laws.

In this section we review the Hamiltonian structure and conservation laws for (2.16). The treatment given here is brief and the interested reader is referred to Marsden & Hughes [1983] and Simo, Marsden & Krishnaprasad [1988] for more details. Since our goal is to eventually treat the incompressible case, we first review a variational characterization of motion and then use the associated Euler-Lagrange equations to formulate (2.16) as a Hamiltonian system.

To begin, we assume the body $B$ undergoes a motion in a time interval $[0, T]$ under the influence of conservative external forces with potential $V^{\text{ext}} : Q \to \mathbf{R}$. In particular, for dead loading with body force density $b : B \to \mathbf{R}^3$ and surface force density $f : \Gamma_\sigma \to \mathbf{R}^3$ we have
\[
V^{\text{ext}}(\varphi_t) = -\int_B b \cdot \varphi_t - \int_{\Gamma_\sigma} f \cdot \varphi_t. \tag{2.17}
\]

Let $V^{\text{int}} : Q \to \mathbf{R}$ denote the internal potential energy for the continuum body defined by the expression
\[
V^{\text{int}}(\varphi_t) = \int_B W(X, D\varphi_t(X)) \, dX \tag{2.18}
\]

where $W : B \times \mathcal{M}_+^3 \to \mathbf{R}$ is defined by
\[
W(X, F) = \tilde{W}(X, F^TF) \tag{2.19}
\]
and $\tilde{W}$ is the (frame-invariant) hyperelastic stored energy function in (2.7), and let $K : V_Q \to R$ denote the kinetic energy of the body defined as

$$K(\dot{\varphi}_t) = \int_B \frac{1}{2} \rho |\dot{\varphi}_t|^2. \quad (2.20)$$

With the above notation introduce a Lagrangian functional $\mathcal{L} : Q \times V_Q \to R$ for the body by the expression

$$\mathcal{L}(\varphi_t, \dot{\varphi}_t) = K(\dot{\varphi}_t) - V^{\text{int}}(\varphi_t) - V^{\text{ext}}(\varphi_t). \quad (2.21)$$

Then, under the assumption of conservative loading, the weak formulation presented in (2.16) is related to a variational principle which we briefly outline next.

Consider the set $\mathcal{C}$ of all possible motions in $[0, T]$ between two prescribed configurations $\hat{\varphi}_0, \hat{\varphi}_T \in Q$, i.e.

$$\mathcal{C} = \{ \varphi : [0, T] \to Q \mid \varphi \in C^1([0, T], Q), \quad \varphi_0 = \hat{\varphi}_0, \quad \text{and} \quad \varphi_T = \hat{\varphi}_T \}, \quad (2.22)$$

and note that, for any $\varphi \in \mathcal{C}$, the tangent space $T_\varphi \mathcal{C}$ is defined as

$$T_\varphi \mathcal{C} = \{ u : [0, T] \to V_Q \mid u \in C^1([0, T], V_Q) \quad \text{and} \quad u_0 = u_T = 0 \} \quad (2.23)$$

where $C^1(Y, Z)$ denotes the set of continuously differentiable mappings from $Y$ into $Z$. If we introduce the action functional $\mathcal{A} : \mathcal{C} \to R$ by

$$\mathcal{A}(\varphi) = \int_0^T \mathcal{L}(\varphi_t, \dot{\varphi}_t) \, dt, \quad (2.24)$$

then Hamilton’s principle states that the actual motion satisfies the following variational problem:

$$(V) \quad \text{Find } \varphi \in \mathcal{C} \text{ such that } D\mathcal{A}(\varphi) \cdot u = 0 \text{ for all } u \in T_\varphi \mathcal{C},$$

where $D\mathcal{A}(\varphi) \cdot u$ denotes the directional derivative of $\mathcal{A}$ at $\varphi \in \mathcal{C}$ in the direction $u \in T_\varphi \mathcal{C}$. Using standard arguments from the calculus of variations, we deduce that any solution of $(V)$ must satisfy the Euler-Lagrange equation

$$\frac{d}{dt} D_2 \mathcal{L}(\varphi_t, \dot{\varphi}_t) \cdot \eta = D_1 \mathcal{L}(\varphi_t, \dot{\varphi}_t) \cdot \eta \quad \text{in } (0, T), \quad \forall \eta \in V_Q. \quad (2.25)$$

Now, while the above variational principle (i.e. Hamilton’s principle) pertains to the boundary value problem of motion (i.e. motion between fixed configurations),
the Euler-Lagrange equation generated by the principle is suitable for the formulation of the initial value problem of motion in which \( \varphi_0 \) and \( \dot{\varphi}_0 \) are specified. In particular, one can verify that (2.25) may be written as

\[
\langle \rho \ddot{\varphi}_t, \eta \rangle + \langle F_t, S_t, D \eta \rangle = \langle b, \eta \rangle + \langle f, \eta \rangle \bigg|_{\eta = 0} \quad \text{in} \quad (0, T), \quad \forall \eta \in \mathcal{V}_Q,
\]

which is the weak equation (2.16) with dead loads \( b \) and \( f \). Given that (2.16) is formally equivalent to (2.25) we can now easily put the weak equations (2.16) in Hamiltonian form as follows.

For any fixed \( \varphi_t \in \mathcal{Q} \) consider the functional \( \mathcal{L}(\varphi_t, \cdot) : \mathcal{V}_Q \to \mathbb{R} \) and introduce the conjugate momenta \( \pi_t \in \mathcal{V}_Q \) by the weak expression

\[
\langle \pi_t, \eta \rangle = D_2 \mathcal{L}(\varphi_t, \dot{\varphi}_t) \cdot \eta \quad \forall \eta \in \mathcal{V}_Q.
\]

Next, construct a mapping \( \chi_{\varphi_t} : \mathcal{V}_Q \to \mathcal{V}_Q \) which solves (2.27) for \( \dot{\varphi}_t \) in the sense that

\[
\langle \dot{\varphi}_t, \eta \rangle = \langle \chi_{\varphi_t}(\pi_t), \eta \rangle = \langle \chi(\varphi_t, \pi_t), \eta \rangle \quad \forall \eta \in \mathcal{V}_Q.
\]

In particular, from (2.27) and the definition of the Lagrangian we have

\[
\chi(\varphi_t, \pi_t) = \rho^{-1} \pi_t.
\]

At this point, using (2.28) and (2.27), we can write the Euler-Lagrange equation (2.25) as

\[
\begin{align*}
\langle \dot{\varphi}_t, \eta \rangle & = \langle \chi(\varphi_t, \pi_t), \eta \rangle \\
\langle \dot{\pi}_t, \eta \rangle & = D_1 \mathcal{L}(\varphi_t, v) \cdot \eta \bigg|_{v = \chi(\varphi_t, \pi_t)} \quad \forall \eta \in \mathcal{V}_Q.
\end{align*}
\]

To complete the transition to Hamiltonian form we introduce a Hamiltonian functional \( \mathcal{H} : \mathcal{Q} \times \mathcal{V}_Q \to \mathbb{R} \) by

\[
\mathcal{H}(\varphi_t, \pi_t) = \langle \pi_t, \chi(\varphi_t, \pi_t) \rangle - \mathcal{L}(\varphi_t, \chi(\varphi_t, \pi_t)) \\
= \frac{1}{2} \langle \pi_t, \rho^{-1} \pi_t \rangle + V^{\text{int}}(\varphi_t) + V^{\text{ext}}(\varphi_t)
\]

and we note that

\[
D_1 \mathcal{H}(\varphi_t, \pi_t) \cdot \eta = -D_1 \mathcal{L}(\varphi_t, v) \cdot \eta \bigg|_{v = \chi(\varphi_t, \pi_t)} \quad \forall \eta \in \mathcal{V}_Q,
\]

and

\[
D_2 \mathcal{H}(\varphi_t, \pi_t) \cdot \eta = \langle \chi(\varphi_t, \pi_t), \eta \rangle \quad \forall \eta \in \mathcal{V}_Q.
\]

Using the above results in (2.30) we obtain

\[
\begin{align*}
\langle \dot{\varphi}_t, \eta \rangle & = D_2 \mathcal{H}(\varphi_t, \pi_t) \cdot \eta \\
\langle \dot{\pi}_t, \eta \rangle & = -D_1 \mathcal{H}(\varphi_t, \pi_t) \cdot \eta \quad \forall \eta \in \mathcal{V}_Q
\end{align*}
\]
which are just a weak version of Hamilton's canonical equations.

In view of the definition of $\mathcal{H}$ given in (2.31) our weak initial boundary value problem for the motion of $\mathcal{B}$ may be stated in Hamiltonian form as follows:

Given $b : B \to \mathbb{R}^3$ and $f : \Gamma_\sigma \to \mathbb{R}^3$ find $(\varphi, \pi) : [0, T] \to P = Q \times \mathcal{V}_Q$ such that

\[
\begin{align*}
\langle \dot{\varphi}_t, \eta \rangle &= \langle \rho^{-1} \pi_t, \eta \rangle \quad \text{in} \quad (0, T) \\
\langle \pi_t, \eta \rangle &= -(F_t S_t, D\eta) + \langle b, \eta \rangle + \langle f, \eta \rangle_{\Gamma_\sigma} \quad \text{in} \quad (0, T) \\
\langle \varphi_0, \eta \rangle &= \langle \dot{\varphi}_0, \eta \rangle \\
\langle \pi_0, \eta \rangle &= \langle \dot{\pi}_0, \eta \rangle
\end{align*}
\]  
\forall \eta \in \mathcal{V}_Q \tag{2.35}

where $(\dot{\varphi}_0, \dot{\pi}_0) \in P$ is given initial data.

Since our weak initial boundary value problem for a general hyperelastic body is Hamiltonian, we can use the general theory for Hamiltonian systems to establish the following conservation laws:

**Conservation of the Hamiltonian.** The Hamiltonian functional $\mathcal{H}$ is conserved along any solution $(\varphi, \pi) : [0, T_{\text{max}}] \to Q \times \mathcal{V}_Q$ in the sense that

\[
\frac{d}{dt} \mathcal{H}(\varphi_t, \pi_t) = 0 \quad \text{in} \quad (0, T_{\text{max}}).
\]  
\tag{2.36}

**Conservation of Linear and Angular Momentum.** Consider the case for which $\Gamma_\varphi = \emptyset$ and $\Gamma_\sigma = \partial B$, and suppose $b : B \to \mathbb{R}^3$ and $f : \Gamma_\sigma \to \mathbb{R}^3$ are the zero mappings. For this case we have that $\mathcal{H}$ is invariant under the following affine group actions:

1) $G^I = \mathbb{R}^3$ with action $\Phi^I : G^I \times P \to P$ defined as $\Phi^I(c, (\varphi_t, \pi_t)) = (\varphi_t + c, \pi_t)$ and associated momentum map $J^I : P \to \mathbb{R}^3$ defined by

\[
J^I(\varphi_t, \pi_t) = \int_B \pi_t
\]  
\tag{2.37}

called the linear momentum map.

2) $G^{II} = SO(3)$ with action $\Phi^{II} : G^{II} \times P \to P$ defined as $\Phi^{II}(A, (\varphi_t, \pi_t)) = (A \varphi_t, A \pi_t)$ and associated momentum map $J^{II} : P \to \mathbb{R}^3$ defined by

\[
J^{II}(\varphi_t, \pi_t) = \int_B \varphi_t \times \pi_t
\]  
\tag{2.38}

called the angular momentum map.
Given that $\mathcal{H}$ is invariant under the actions of $G^I$ and $G^{II}$, and that these actions possess momentum maps, we have two more conservation laws for our system. In particular, for the case we are considering, the linear and angular momentum maps are conserved along any solution $(\varphi, \pi) : [0, T_{\text{max}}) \rightarrow Q \times \mathcal{V}_Q$ in the sense that

$$
\frac{d}{dt} J^I(\varphi_t, \pi_t) = 0 \quad \text{and} \quad \frac{d}{dt} J^{II}(\varphi_t, \pi_t) = 0 \quad \text{in} \quad (0, T_{\text{max}}).
$$

We next construct a time-stepping scheme for (2.35) which inherits the above conservation laws.

2.5. Conserving Time-Stepping Scheme.

For concreteness consider the case for which $\Gamma_\varphi = \emptyset$ and $\Gamma_{\pi} = \partial B$, and suppose $b : B \rightarrow \mathbb{R}^3$ and $f : \Gamma_{\sigma} \rightarrow \mathbb{R}^3$ are the zero mappings. In this case the underlying system possesses three conservation laws which, ideally, we would like to design into a time-stepping scheme.

To begin, let $P = Q \times \mathcal{V}_Q$ with points denoted by $z_t = (\varphi_t, \pi_t)$ and let $D\mathcal{H}(z_t)$ denote the derivative of $\mathcal{H}$ at $z_t$. In view the results presented in Gonzalez [1996a], the construction of a conserving time integration scheme for (2.35) depends on the construction of a $G^I, G^{II}$-equivariant discrete derivative for the function $\mathcal{H} : P \rightarrow \mathbb{R}$. This object (to be defined in Section 3) is a two-point approximation to the exact derivative of $\mathcal{H}$. For example, if we denote an equivariant discrete derivative for $\mathcal{H}$ by $D^\mathcal{H}(\cdot, \cdot)$, then

$$
D^\mathcal{H}(z_a, z_b) \approx D\mathcal{H}\left(\frac{z_a + z_b}{2}\right) \quad \text{(2.39)}
$$

for any two points $z_a, z_b \in P$. Given an equivariant discrete derivative $D^\mathcal{H}$ we define a set of weak Hamiltonian difference equations in $P = Q \times \mathcal{V}_Q$ analogous to (2.34), namely

\[
\begin{align*}
\langle \varphi_{n+1} - \varphi_n, \eta \rangle &= \Delta t D_2^\mathcal{H}(z_n, z_{n+1}) \cdot \eta \\
\langle \pi_{n+1} - \pi_n, \eta \rangle &= -\Delta t D_1^\mathcal{H}(z_n, z_{n+1}) \cdot \eta
\end{align*}
\quad \forall \eta \in \mathcal{V}_Q.
\]  

(2.40)

where $\Delta t > 0$ is the time step and the partial discrete derivatives $D_1^\mathcal{H}$ and $D_2^\mathcal{H}$ are defined by the relation

$$
D^\mathcal{H}(z_n, z_{n+1}) \cdot (v_1, v_2) = D_1^\mathcal{H}(z_n, z_{n+1}) \cdot v_1 + D_2^\mathcal{H}(z_n, z_{n+1}) \cdot v_2 \quad \forall (v_1, v_2) \in \mathcal{V}_Q \times \mathcal{V}_Q. \quad \text{(2.41)}
$$

Using techniques which are discussed in detail when we consider the incompressible formulation, we construct the following equivariant discrete derivative

$$
D^\mathcal{H}(z_n, z_{n+1}) \cdot (v_1, v_2) = \langle F_{n+\frac{1}{2}} S_{\text{algo}}, Dv_1 \rangle + \langle \rho^{-1} \pi_{n+\frac{1}{2}}, v_2 \rangle
\]  

(2.42)

where $S_{\text{algo}} : B \rightarrow S^3$, interpreted as an approximation to the second Piola-Kirchhoff stress field, is defined as follows.
For any $X \in B$ let $\tilde{W}_X = \tilde{W}(X, \cdot) : S^3_{\text{p}} \rightarrow R$ where $\tilde{W}$ is the (frame-invariant) hyperelastic stored energy function presented in (2.7) and define $D\tilde{W}_X : S^3_{\text{p}} \times S^3_{\text{p}} \rightarrow R$ by

\[
D\tilde{W}_X(A_n, A_{n+1}) = D\tilde{W}_X(A_{n+\frac{1}{2}})
+ \left[ \frac{\tilde{W}_X(A_{n+1}) - \tilde{W}_X(A_n) - D\tilde{W}_X(A_{n+\frac{1}{2}}) : \Delta A}{\Delta A : \Delta A} \right] \Delta A
\] (2.43)

where $\cdot_{n+\frac{1}{2}} = \frac{1}{2}(\cdot_n + \cdot_{n+1})$ and $\Delta A = A_{n+1} - A_n$. With the above notation the approximate stress field $S_{\text{algo}}$ is defined as

\[
S_{\text{algo}}(X) = 2D\tilde{W}_X(C_n(X), C_{n+1}(X))
\] (2.44)

and, in view of (2.40), (2.41) and (2.42), we obtain

\[
\begin{align*}
\langle \varphi_{n+1} - \varphi_n, \eta \rangle &= \Delta t \left\{ \langle \rho^{-1}\pi_{n+\frac{1}{2}}, \eta \rangle \right\} \\
\langle \pi_{n+1} - \pi_n, \eta \rangle &= -\Delta t \left\{ \langle F_{n+\frac{1}{2}} S_{\text{algo}}, D\eta \rangle \right\}
\end{align*}
\] \forall \eta \in V_Q. (2.45)

The above equations were derived for the case in which $\Gamma_\varphi = \emptyset$, $\Gamma_\sigma = \partial B$, and $b : B \rightarrow R^3$ and $f : \Gamma_\sigma \rightarrow R^3$ are the zero mappings. As such, they represent a discrete Hamiltonian system with symmetry in the sense of Gonzalez [1996a], and inherit the conservation laws discussed above. Extending the above results to the general displacement-traction problem with dead loads yields the following time-stepping scheme for (2.35):

Given $b : B \rightarrow R^3$, $f : \Gamma_\sigma \rightarrow R^3$, and $\Delta t > 0$ find a sequence $(\varphi_n, \pi_n)^{N}_{n=0}$ in $Q \times V_Q$ such that

\[
\begin{align*}
\langle \varphi_{n+1} - \varphi_n, \eta \rangle &= \Delta t \left\{ \rho^{-1}\pi_{n+\frac{1}{2}}, \eta \rangle \right\} \\
\langle \pi_{n+1} - \pi_n, \eta \rangle &= -\Delta t \left\{ \langle F_{n+\frac{1}{2}} S_{\text{algo}}, D\eta \rangle \right\} \\
\langle \varphi_0, \eta \rangle &= \langle \tilde{\varphi}_0, \eta \rangle \\
\langle \pi_0, \eta \rangle &= \langle \tilde{\pi}_0, \eta \rangle
\end{align*}
\] \forall \eta \in V_Q (2.46)

where $\tilde{\varphi}_0$ and $\tilde{\pi}_0$ are initial data.

As for the underlying system, the difference equations in (2.46) possess the following discrete conservation laws:
Conservation of the Hamiltonian. The Hamiltonian functional $\mathcal{H}$ is conserved along any solution sequence $(\varphi_n, \pi_n)_{n=0}^N$ in $Q \times \mathcal{V}_Q$ in the sense that
\[ \mathcal{H}(\varphi_n, \pi_n) = \mathcal{H}(\varphi_0, \pi_0) \quad \text{for} \quad 0 \leq n \leq N. \tag{2.47} \]
To see this, choose $\eta = \pi_{n+1} - \pi_n$ in (2.46) and choose $\eta = \varphi_{n+1} - \varphi_n$ in (2.46)$_2$. The result then follows by subtracting the two expressions.

Conservation of Linear and Angular Momentum. Consider the case for which $\Gamma_\varphi = \emptyset$, $\Gamma_\sigma = \partial B$, and $b : B \to \mathbb{R}^3$ and $f : \Gamma_\sigma \to \mathbb{R}^3$ are the zero mappings. For this case we have the following additional conservation laws:

1) The linear momentum map $J^I$ is conserved along any solution sequence $(\varphi_n, \pi_n)_{n=0}^N$ in $Q \times \mathcal{V}_Q$ in the sense that
\[ J^I(\varphi_n, \pi_n) = J^I(\varphi_0, \pi_0) \quad \text{for} \quad 0 \leq n \leq N. \tag{2.48} \]
To see this, consider any $\xi \in \mathbb{R}^3$ and choose $\eta(X) = \xi$ for all $X \in B$. The result then follows directly from (2.46)$_2$.

2) The angular momentum map $J^{II}$ is conserved along any solution sequence $(\varphi_n, \pi_n)_{n=0}^N$ in $Q \times \mathcal{V}_Q$ in the sense that
\[ J^{II}(\varphi_n, \pi_n) = J^{II}(\varphi_0, \pi_0) \quad \text{for} \quad 0 \leq n \leq N. \tag{2.49} \]
To see this, consider any $\xi \in \mathbb{R}^3$ and choose $\eta(X) = \xi \times \varphi_{n+\frac{1}{2}}(X)$ in (2.46)$_2$, and choose $\eta(X) = \xi \times \pi_{n+\frac{1}{2}}(X)$ in (2.46)$_1$. The result then follows using standard manipulations and the definition of the angular momentum map.

Remark 2.3. The time-stepping scheme in (2.46) was developed under the assumption that the loads $b$ and $f$ were dead. Note, however, that the general methods presented in Gonzalez [1996a] are not limited to this case. That is, conserving schemes can be constructed for general external loads $b$ and $f$ that are derivable from a potential. In this case we replace the term $\Delta t(b, \eta) + \Delta t(f, \eta)_{\Gamma_\sigma}$ in (2.46)$_2$ by $-\Delta t \mathcal{D} V^{\text{ext}}(\varphi_n, \varphi_{n+1}) \cdot \eta$, where $\mathcal{D} V^{\text{ext}}$ is a discrete derivative for the potential $V^{\text{ext}}$.

We next present a formulation of incompressible hyperelasticity and develop a conserving time-stepping scheme using the same ideas as above.

3. General Incompressible Hyperelasticity.

In this section we present a formulation of general incompressible hyperelasticity. Our goal is to develop a formulation for this system which falls within the framework of Gonzalez [1996b] and to use methods presented therein to construct conserving time-stepping schemes.
3.1. Hamiltonian Formulation.

Consider again the variational formulation of general compressible hyperelasticity given in Section 2.4. Let $Q$ denote the configuration space for $B$ considered earlier and let $C$ denote the space of possible motions of $B$ in a time interval $[0, T]$ between fixed endpoints. For the moment we assume $Q$ and $C$ contain only smooth mappings to justify the operations in this section.

Now, for the incompressible case, we require that any motion $\varphi \in C$ of our body $B$ satisfy a pointwise constraint of the form $G(D\varphi_t(X)) = 0$ in $B \times [0, T]$ for a smooth function $G : \mathbb{M}_+^3 \to \mathbb{R}$ given by

$$G(M) = \det[M] - 1. \quad (3.1)$$

In particular, the condition $G(D\varphi_t(X)) = 0$ in $B \times [0, T]$ says that, for any $t \in [0, T]$, the deformation $\varphi_t : B \to \mathbb{R}^3$ is volume-preserving at each point $X \in B$ (see e.g. Ogden [1984] or Marsden & Hughes [1983] for more details).

Given the above constraint, the basis for our formulation of incompressible hyperelasticity is the following constrained variational problem:

$$(CV) \quad \begin{cases}
\text{Find a stationary point } \varphi \in C \text{ of the functional } A : C \to \mathbb{R} \\
\text{subject to the pointwise condition } G(D\varphi_t(X)) = 0 \\
\text{for all } (X, t) \in B \times [0, T],
\end{cases}$$

where $A : C \to \mathbb{R}$ is the action functional defined in (2.24). Since constrained variational problems are often cumbersome to deal with directly we use the method of (Lagrange) multipliers to transform the constrained variational problem into an equivalent unconstrained one as follows.

Let $E$ denote the set of smooth functions $\phi : B \to \mathbb{R}$, and define a space $\mathcal{E}$ as the set of smooth mappings $\lambda : [0, T] \to E$. Next, consider an augmented action functional $\hat{A} : C \times \mathcal{E} \to \mathbb{R}$ defined by

$$\hat{A}(\varphi, \lambda) = A(\varphi) - \int_0^T \langle \lambda_t, G(D\varphi_t) \rangle \, dt$$

$$= \int_0^T \mathcal{L}(\varphi_t, \dot{\varphi}_t) - \langle \lambda_t, G(D\varphi_t) \rangle \, dt \quad (3.2)$$

where $\mathcal{L}$ denotes the Lagrangian functional defined in (2.21). Then, under suitable conditions on the constraint, the constrained problem (CV) is formally equivalent to the following unconstrained variational problem:

$$(UV) \quad \begin{cases}
\text{Find } (\varphi, \lambda) \in C \times \mathcal{E} \text{ such that } D\hat{A}(\varphi, \lambda) \cdot (u, v) = 0 \text{ for all } \\
(u, v) \in T_\varphi C \times T_\lambda \mathcal{E}.
\end{cases}$$
The Euler-Lagrange equations for this variational problem are

\[
\frac{d}{dt} D_2 \mathcal{L}(\varphi_t, \dot{\varphi}_t) \cdot \eta = D_1 \mathcal{L}(\varphi_t, \dot{\varphi}_t) \cdot \eta \\
- \langle \lambda_t D G(D \varphi_t), D\eta \rangle \quad \text{in} \quad (0, T), \quad \forall \eta \in \mathcal{V}_Q \quad \text{(3.3)}
\]

\[
\langle \phi, G(D \varphi_t) \rangle = 0 \quad \text{in} \quad (0, T), \quad \forall \phi \in E.
\]

To see that the weak equation (3.3) is a correct description of the motion we substitute the definition of \( \mathcal{L} \) and \( G \) into (3.3) to get

\[
\langle \rho \ddot{\varphi}_t, \eta \rangle + \langle F_t S_t + \text{det}[F_t] \lambda_t C_t^{-1}, D\eta \rangle = \langle b, \eta \rangle + \langle f, \eta \rangle_{\Gamma^*}
\]

which shows that the multiplier field \( \lambda_t : B \rightarrow \mathbb{R} \) may be interpreted as a pressure. In particular, the Euler-Lagrange equation (3.4) corresponds to the dynamic generalization of the equation of virtual work with an unknown pressure, which is determined by the incompressibility condition. Hence, from the point of view of continuum mechanics, the relation in (3.4) furnishes a correct description of the motion of \( B \) under the internal constraint of incompressibility (see e.g. Truesdell & Noll [1965]).

In view of (3.2) define an augmented Lagrangian \( \hat{\mathcal{L}} : Q \times \mathcal{V}_Q \times E \rightarrow \mathbb{R} \) by the expression

\[
\hat{\mathcal{L}}(\varphi_t, \dot{\varphi}_t, \lambda_t) = \mathcal{L}(\varphi_t, \dot{\varphi}_t) - \langle \lambda_t, G(D \varphi_t) \rangle.
\]

Then the Euler-Lagrange equations for (UV) become

\[
\frac{d}{dt} D_2 \hat{\mathcal{L}}(\varphi_t, \dot{\varphi}_t, \lambda_t) \cdot \eta = D_1 \hat{\mathcal{L}}(\varphi_t, \dot{\varphi}_t, \lambda_t) \cdot \eta \quad \text{in} \quad (0, T), \quad \forall \eta \in \mathcal{V}_Q
\]

\[
\langle \phi, G(D \varphi_t) \rangle = 0 \quad \text{in} \quad (0, T), \quad \forall \phi \in E.
\]

To pass to a formulation with some Hamiltonian structure we consider, for any fixed \( \varphi_t \in Q \) and \( \lambda_t \in E \), the functional \( \hat{\mathcal{L}}(\varphi_t, \cdot, \lambda_t) : \mathcal{V}_Q \rightarrow \mathbb{R} \) and introduce the conjugate momenta \( \pi_t \in \mathcal{V}_Q \) by the weak expression

\[
\langle \pi_t, \eta \rangle = D_2 \hat{\mathcal{L}}(\varphi_t, \dot{\varphi}_t, \lambda_t) \cdot \eta \quad \forall \eta \in \mathcal{V}_Q.
\]

Next, we construct a mapping \( \hat{\chi}_{\varphi_t, \lambda_t} : \mathcal{V}_Q \rightarrow \mathcal{V}_Q \) which solves (3.6) for \( \dot{\varphi}_t \) in the sense that

\[
\langle \dot{\varphi}_t, \eta \rangle = \langle \hat{\chi}_{\varphi_t, \lambda_t}(\pi_t), \eta \rangle = \langle \hat{\chi}(\varphi_t, \pi_t, \lambda_t), \eta \rangle \quad \forall \eta \in \mathcal{V}_Q.
\]

In particular, from (3.6) and the definition of the Lagrangian we have

\[
\dot{\varphi}_t = \hat{\chi}(\varphi_t, \pi_t, \lambda_t) = \rho^{-1} \pi_t.
\]
At this point, using (3.7) and (3.6), we can write the Euler-Lagrange equations (3.5) as
\[
\begin{align*}
\langle \dot{\varphi}_t, \eta \rangle &= \langle \dot{\chi} (\varphi_t, \pi_t, \lambda_t), \eta \rangle \\
\langle \dot{\pi}_t, \eta \rangle &= D_1 \mathcal{L} (\varphi_t, v, \lambda_t) \cdot \eta \bigg|_{v = \dot{\chi} (\varphi_t, \pi_t, \lambda_t)} \\
&\forall \eta \in \mathcal{V}_Q.
\end{align*}
\] (3.9)

Introducing a functional \( \mathcal{H} : Q \times \mathcal{V}_Q \times E \to \mathbb{R} \) by the relation
\[
\mathcal{H} (\varphi_t, \pi_t, \lambda_t) = \langle \pi_t, \dot{\chi} (\varphi_t, \pi_t, \lambda_t) \rangle - \dot{\mathcal{L}} (\varphi_t, \dot{\chi} (\varphi_t, \pi_t, \lambda_t), \lambda_t)
= \frac{1}{2} \langle \pi_t, \rho^{-1} \pi_t \rangle + V^{\text{int}} (\varphi_t) + V^{\text{ext}} (\varphi_t) + \langle \lambda_t, G (D \varphi_t) \rangle
\] (3.10)
we see that the Euler-Lagrange equations (3.5) lead to the following initial value problem:

Given \( b : B \to \mathbb{R}^3 \) and \( f : \Gamma_\sigma \to \mathbb{R}^3 \) find \( (\varphi, \pi, \lambda) : [0, T] \to Q \times \mathcal{V}_Q \times E \) such that
\[
\begin{align*}
\langle \dot{\varphi}_t, \eta \rangle &= D_2 \mathcal{H} (\varphi_t, \pi_t, \lambda_t) \cdot \eta \quad \text{in } (0, T), \quad \forall \eta \in \mathcal{V}_Q \\
\langle \dot{\pi}_t, \eta \rangle &= -D_1 \mathcal{H} (\varphi_t, \pi_t, \lambda_t) \cdot \eta \quad \text{in } (0, T), \quad \forall \eta \in \mathcal{V}_Q \\
\langle \phi, G (D \varphi_t) \rangle &= 0 \quad \text{in } (0, T), \quad \forall \phi \in E \\
\langle \dot{\varphi}_0, \eta \rangle &= \langle \phi_0, \eta \rangle \quad \forall \eta \in \mathcal{V}_Q \\
\langle \dot{\pi}_0, \eta \rangle &= \langle \pi_0, \eta \rangle \quad \forall \eta \in \mathcal{V}_Q.
\end{align*}
\] (3.11)

where \( \dot{\varphi}_0 \) and \( \dot{\pi}_0 \) are initial data.

In the above system (3.11)$_{1,2}$ are interpreted as weak Hamiltonian evolution equations for \( (\varphi_t, \pi_t) \in Q \times \mathcal{V}_Q \), and expression (3.11)$_3$ is viewed as an equation which determines \( \lambda_t \); in particular, the system may be interpreted as a Hamiltonian differential-algebraic equation within the context of Gonzalez [1996b].

**Remark 3.1.** The above differential-algebraic formulation is not the only one which may be used to describe the dynamics of the incompressible system. For example, rather than append the constraint \( G (F_t (X)) = 0 \) to Hamilton's principle, one may append the time-differentiated constraint \( DG (F_t (X)) : \dot{F}_t (X) = 0 \) to obtain a different formulation. Such formulations, obtained by appending a time-differentiated configuration constraint to Hamilton's principle, are related to the so-called *vakonomic* approach to constrained systems. For more details on this approach see Kozlov [1983], Arnold [1988] and Lewis & Murray [1994] for the finite-dimensional case and Dichmann, Maddocks & Pego [1995], Maddocks & Pego [1995] and Dichmann, Li & Maddocks [1996] for variants of the vakonomic approach within an infinite-dimensional setting.

Here we note that, from the numerical point of view, the above formulation is susceptible to difficulties. The main problem lies in the fact that any smooth
solution of this system satisfies the pointwise condition \( \det[F_t(X)] = 1 \) in \( B \times [0, T] \). It is well-known that this class of deformations cannot be approximated well with standard (low-order) finite element spaces; in particular, one can expect difficulties such as the phenomena of locking (see e.g. Hughes [1987] for a summary account). We thus introduce another formulation to alleviate these expected difficulties.

3.2. A Quasi-Incompressible Formulation.

In this section we reformulate (3.11) to yield a system more suitable for numerical implementations. Following Simo, Taylor & Pister [1985] and Simo & Taylor [1991], the main idea is to introduce a field \( \Theta_t : B \rightarrow \mathbb{R}_+ \) which in some sense approximates the jacobian field \( J_t := \det[D\varphi_t] : B \rightarrow \mathbb{R}_+ \), and then enforce the condition \( \Theta_t(X) = 1 \) in \( B \times [0, T] \). Roughly speaking, since the incompressibility condition is enforced on \( \Theta_t \) rather than \( J_t \), the formulation is called quasi-incompressible.

To begin, let \( Q \) denote the configuration space for \( B \) defined in (2.14) and consider the spaces \( E = L^2(B, \mathbb{R}) \) and \( E_+ = L^2(B, \mathbb{R}_+) \). Now, for any \( \Theta_t \in E_+ \) and \( \varphi_t \in Q \) define a modified deformation gradient field \( \bar{F}_t : B \rightarrow M^3_+ \) by the expression

\[
\bar{F}_t(X) = \Theta_t(X) 1/3 F^\text{dev}_t(X),
\]

where \( F^\text{dev}_t : B \rightarrow M^3_+ \) is the deviatoric part of the actual deformation gradient field \( F_t \), i.e.

\[
F^\text{dev}_t(X) = J_t(X)^{-1/3} F_t(X),
\]

and define an associated modified Cauchy strain field \( \bar{C}_t : B \rightarrow S^3_{p_d} \) by

\[
\bar{C}_t(X) = \bar{F}_t(X) \bar{F}_t(X).
\]

Let \( W : B \times M^3_+ \rightarrow \mathbb{R} \) be the stored energy function defined in (2.19) and define a modified internal potential energy \( \bar{V}^{\text{int}} : Q \times E_+ \rightarrow \mathbb{R} \) by

\[
\bar{V}^{\text{int}}(\varphi_t, \Theta_t) = \int_B W(X, \bar{F}_t(X)).
\]

As before, denote by \( V^{\text{ext}} : Q \rightarrow \mathbb{R} \) the potential of the external loads.

Let \( \mathcal{H} : Q \times V_Q \times E_+ \times E \times E \rightarrow \mathbb{R} \) denote a modified Hamiltonian functional of the form

\[
\mathcal{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) = \frac{1}{2} \langle \pi_t, \rho^{-1} \pi_t \rangle + \bar{V}^{\text{int}}(\varphi_t, \Theta_t) + V^{\text{ext}}(\varphi_t) + \langle p_t, (J_t - \Theta_t) \rangle + \langle \lambda_t, \bar{G}(\Theta_t) \rangle,
\]

where \( \bar{G} : \mathbb{R}_+ \rightarrow \mathbb{R} \) is given by

\[
\bar{G}(\alpha) = \alpha - 1,
\]
and consider the following weak initial value problem:

Given \( b : B \rightarrow R^3 \) and \( f : \Gamma_\sigma \rightarrow R^3 \) find \((\varphi, \pi, \Theta, p, \lambda) : [0, T] \rightarrow Q \times V_Q \times E_+ \times E \times E\) such that

\[
\begin{align*}
\langle \dot{\varphi}_t, \eta \rangle &= D_2 \bar{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \eta \quad \text{in} \quad (0, T) \quad \forall \eta \in V_Q \\
\langle \dot{\pi}_t, \eta \rangle &= -D_1 \bar{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \eta \quad \text{in} \quad (0, T) \quad \forall \eta \in V_Q \\
D_3 \bar{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \phi &= 0 \quad \text{in} \quad [0, T] \quad \forall \phi \in E \\
D_4 \bar{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \phi &= 0 \quad \text{in} \quad [0, T] \quad \forall \phi \in E \\
D_5 \bar{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \phi &= 0 \quad \text{in} \quad [0, T] \quad \forall \phi \in E \\
\langle \varphi_0, \eta \rangle &= \langle \hat{\varphi}_0, \eta \rangle \quad \forall \eta \in V_Q \\
\langle \pi_0, \eta \rangle &= \langle \hat{\pi}_0, \eta \rangle \quad \forall \eta \in V_Q.
\end{align*}
\]

where \( \hat{\varphi}_0 \) and \( \hat{\pi}_0 \) are initial data.

The relation between (3.18) and (3.11) is established in the following

**Proposition 3.1.** If for each \( t \in [0, T] \) the fields \((\varphi_t, \Theta_t) \in Q \times E_+ \) are sufficiently smooth, then the formulation in (3.18) is equivalent to that given in (3.11).

**Proof.** Using (3.16) we arrive at the following directional derivatives of \( \bar{H} \):

\[
\begin{align*}
D_1 \bar{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \eta &= \left\langle D_2 W(\cdot, \bar{F}_t), \Theta_t^{-1/3} J_t^{-1/3} [D \eta - \frac{1}{3} F_t J_t^{-1} F_t^\top : D \eta] \right\rangle \\
&\quad + \left\langle p_t, J_t F_t^{-1} \right\rangle D \eta + DV^{\text{ext}}(\varphi_t) \cdot \eta \\
D_2 \bar{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \eta &= \left\langle \rho^{-1} \pi_t, \eta \right\rangle \\
D_3 \bar{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \phi &= \left\langle \frac{1}{3} \Theta_t^{-1} D_2 W(\cdot, \bar{F}_t) : \bar{F}_t + \lambda_t - p_t, \phi \right\rangle \\
D_4 \bar{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \phi &= \left\langle J_t - \Theta_t, \phi \right\rangle \\
D_5 \bar{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \phi &= \left\langle G(\Theta_t), \phi \right\rangle.
\end{align*}
\]

In view of (3.22) and (3.18)\textsubscript{4}, if \( J_t := \det[D \varphi_t] \) and \( \Theta_t \) are continuous, then we have \( J_t = \Theta_t \) in \( B \), which implies \( \bar{F}_t = F_t \) in \( B \). Using this result in (3.21) then gives

\[
\begin{align*}
\left\langle p_t, \phi \right\rangle &= \left\langle \lambda_t + \frac{1}{3} \Theta_t^{-1} D_2 W(\cdot, \bar{F}_t) : \bar{F}_t, \phi \right\rangle \\
&= \left\langle \lambda_t + \frac{1}{3} J_t^{-1} D_2 W(\cdot, F_t) : F_t, \phi \right\rangle,
\end{align*}
\]

which implies

\[
D_1 \bar{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \eta = \left\langle D_2 W(\cdot, F_t), D \eta \right\rangle + \left\langle \lambda_t, DG(\varphi_t) : D \eta \right\rangle + DV^{\text{ext}}(\varphi_t) \cdot \eta
\]

(3.25)
where we have used the fact that $DG(F_t) = J_t F_t^{-T}$. Also, we have

$$\tilde{G}(\Theta_t) = \tilde{G}(J_t) = J_t - 1 = G(D\varphi_t).$$

(3.26)

Hence, under the smoothness assumptions, the system in (3.18) may be written as

$$\begin{align*}
\langle \dot{\varphi}_t, \eta \rangle &= \langle \rho^{-1} \pi_t, \eta \rangle \quad \text{in} \quad (0, T] \quad \forall \eta \in \mathcal{V}_Q \\
\langle \dot{\pi}_t, \eta \rangle &= -\langle D_2 W(\cdot, F_t), D\eta \rangle - \langle \lambda_t, DG(D\varphi_t) : D\eta \rangle \\
&\quad - DV^{\text{ext}}(\varphi_t) \cdot \eta \quad \text{in} \quad (0, T] \quad \forall \eta \in \mathcal{V}_Q \\
\langle \phi, G(D\varphi_t) \rangle &= 0 \quad \text{in} \quad [0, T] \quad \forall \phi \in E \\
\langle \varphi_0, \eta \rangle &= \langle \dot{\varphi}_0, \eta \rangle \quad \forall \eta \in \mathcal{V}_Q \\
\langle \pi_0, \eta \rangle &= \langle \dot{\pi}_0, \eta \rangle \quad \forall \eta \in \mathcal{V}_Q
\end{align*}$$

(3.27)

which is precisely the system in (3.11).

From now on we will work exclusively with the quasi-incompressible formulation given in (3.18), which, as we will see later, is better suited for numerical implementations than its counterpart in (3.11). We next show that the quasi-incompressible formulation possesses conservation laws analogous to those of the general compressible formulation in (3.25).

3.3. Conservation Laws.

In this section we show that the quasi-incompressible formulation of incompressible elastodynamics given in (3.18) possesses conservation laws analogous to those of the general compressible system. We state these results in a sequence of propositions.

**Proposition 3.2.** The modified Hamiltonian functional $\tilde{H}$ is conserved along any solution $(\varphi, \pi, \Theta, p, \lambda) : [0, T_{\text{max}}] \to Q \times \mathcal{V}_Q \times E_+ \times E \times E$ of (3.18) in the sense that

$$\frac{d}{dt} \tilde{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) = 0 \quad \text{in} \quad (0, T_{\text{max}}).$$

(3.28)

**Proof.** The result follows from the Hamiltonian form of the evolution equations (3.18)$_{1,2}$, and the variational form of the equations (3.18)$_{3,4,5}$. In particular, for any $t \in (0, T_{\text{max}})$, choose $\eta = \dot{\pi}_t$ in (3.18)$_1$ and $\eta = \dot{\varphi}_t$ in (3.18)$_2$. Subtracting the resulting expressions yields

$$D_1 \tilde{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \dot{\varphi}_t + D_2 \tilde{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot \dot{\pi}_t = 0 \quad \text{in} \quad (0, T_{\text{max}}).$$

(3.29)

Next, choose $\phi = \dot{\Theta}_t$ in (3.18)$_3$, $\phi = \dot{p}_t$ in (3.18)$_4$, and $\phi = \dot{\lambda}_t$ in (3.18)$_5$. Summing the resulting expressions with (3.29) yields the desired result.  \[\square\]
Now consider the case for which \( \Gamma_\varphi = \emptyset \) and \( \Gamma_\sigma = \partial B \), and suppose \( V^{\text{ext}} \) is the zero function. In this case the modified Hamiltonian \( \tilde{H} \) is invariant under the following affine group actions:

1) \( G^I = \mathbb{R}^3 \) with action \( \Phi^I : G^I \times (Q \times V_Q) \to (Q \times V_Q) \) defined as \( \Phi^I(c, (\varphi, \pi_t)) = (\varphi + c, \pi_t) \) and associated momentum map \( J^I : (Q \times V_Q) \to \mathbb{R}^3 \) defined by

\[
J^I(\varphi, \pi_t) = \int_B \pi_t
\]  
(3.30)

called the linear momentum map.

2) \( G^{II} = SO(3) \) with action \( \Phi^{II} : G^{II} \times (Q \times V_Q) \to (Q \times V_Q) \) defined as \( \Phi^{II}(A, (\varphi, \pi_t)) = (A\varphi, A\pi_t) \) and associated momentum map \( J^{II} : (Q \times V_Q) \to \mathbb{R}^3 \) defined by

\[
J^{II}(\varphi, \pi_t) = \int_B \varphi_t \times \pi_t
\]  
(3.31)

called the angular momentum map.

Given that \( \tilde{H} \) is invariant under the actions of \( G^I \) and \( G^{II} \) in the sense that

\[
\tilde{H}(\Phi^I(c, (\varphi, \pi)), \Theta_t, p_t, \lambda_t) = \tilde{H}(\varphi, \pi, \Theta_t, p_t, \lambda_t) \quad \forall c \in G^I
\]
(3.32)
\[
\tilde{H}(\Phi^{II}(A, (\varphi, \pi)), \Theta_t, p_t, \lambda_t) = \tilde{H}(\varphi, \pi, \Theta_t, p_t, \lambda_t) \quad \forall A \in G^{II},
\]
(3.33)

we then have two more conservation laws for our system.

**Proposition 3.3.** The linear momentum map \( J^I : Q \times V_Q \to \mathbb{R}^3 \) is conserved along any solution \( (\varphi, \pi, \Theta, p, \lambda) : [0, T_{\text{max}}) \to Q \times V_Q \times E_+ \times E \times E \) of (3.18) in the sense that

\[
\frac{d}{dt} J^I(\varphi_t, \pi_t) = 0 \quad \text{in} \quad (0, T_{\text{max}}).
\]  
(3.34)

**Proof.** Let \( \xi \in \mathbb{R}^3 \) be arbitrary and choose in (3.18) the variation \( \eta(X) = \xi \) for all \( X \in B \). We then have

\[
\langle \hat{\pi}_t, \xi \rangle = \frac{d}{dt} J^I(\varphi_t, \pi_t) \cdot \xi = 0 \quad \forall \xi \in \mathbb{R}^3,
\]

which establishes the result. \( \blacksquare \)

**Proposition 3.4.** The angular momentum map \( J^{II} : Q \times V_Q \to \mathbb{R}^3 \) is conserved along any solution \( (\varphi, \pi, \Theta, p, \lambda) : [0, T_{\text{max}}) \to Q \times V_Q \times E_+ \times E \times E \) of (3.18) in the sense that

\[
\frac{d}{dt} J^{II}(\varphi_t, \pi_t) = 0 \quad \text{in} \quad (0, T_{\text{max}}).
\]  
(3.35)
\textbf{Proof.} Let \( \xi \in \mathbb{R}^3 \) be arbitrary and choose, for any \( t \in (0, T_{\text{max}}) \), the variation \( \eta = \xi \times \pi_t \) in (3.18)1 and \( \eta = \xi \times \varphi_t \) in (3.18)2. Then, since

\[
\begin{align*}
D_1 \mathcal{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot (\xi \times \varphi_t) &= 0 \\
D_2 \mathcal{H}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \cdot (\xi \times \pi_t) &= 0
\end{align*}
\]

we have

\[
\langle \dot{\varphi}_t, \xi \times \pi_t \rangle + \langle \dot{\pi}_t, \xi \times \varphi_t \rangle = \frac{d}{dt} J^{11}(\varphi_t, \pi_t) \cdot \xi = 0 \quad \forall \xi \in \mathbb{R}^3,
\]

which establishes the result. \( \square \)

We next construct a time-stepping scheme for the quasi-incompressible formulation that inherits the above conservation laws.

\subsection*{3.4. Conserving Time-Stepping Scheme.}

For concreteness consider the case for which \( \Gamma_\varphi = \emptyset \) and \( \Gamma_\sigma = \partial B \), and suppose \( b : B \to \mathbb{R}^3 \) and \( f : \Gamma_\sigma \to \mathbb{R}^3 \) are the zero mappings. In this case the underlying system possesses three conservation laws which, ideally, we would like to preserve under time discretization. Since the quasi-incompressible formulation outlined in (3.18) defines a (weak) Hamiltonian differential-algebraic system in the sense of Gonzalez [1996b], we use the results of that paper to construct a conserving time-stepping scheme.

To begin, let \( P = Q \times V_Q \times E_+ \times E \times E \times E \) with points denoted by \( \chi_t = (\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) \) and \( V_P = V_Q \times V_Q \times E \times E \times E \). The construction of a conserving time-stepping scheme for (3.18) depends on the construction of a \( G^{1,11} \)-equivariant discrete derivative for the modified Hamiltonian functional \( \tilde{\mathcal{H}} : P \to \mathbb{R} \).

In the most basic sense, a discrete derivative for \( \tilde{\mathcal{H}} \) is a two-point approximation to the exact derivative of \( \tilde{\mathcal{H}} \). For example, if we denote an equivariant discrete derivative for \( \tilde{\mathcal{H}} \) by \( D^\sigma \tilde{\mathcal{H}}(\cdot, \cdot) \), then

\[
D^\sigma \tilde{\mathcal{H}}(\chi_a, \chi_b) \approx D \tilde{\mathcal{H}} \left( \frac{\chi_a + \chi_b}{2} \right)
\]

for any two points \( \chi_a, \chi_b \in P \). Following the developments of Gonzalez [1996b] we require \( D^\sigma \tilde{\mathcal{H}}(\cdot, \cdot) \) to satisfy the following properties:

1) Directionality \( D^\sigma \tilde{\mathcal{H}}(\chi_a, \chi_b) \cdot v_{ab} = \tilde{\mathcal{H}}(\chi_b) - \tilde{\mathcal{H}}(\chi_a) \) for any \( \chi_a, \chi_b \in P \) where \( v_{ab} = \chi_b - \chi_a \).

2) Consistency \( D^\sigma \tilde{\mathcal{H}}(\chi_a, \chi_b) \cdot v = D \tilde{\mathcal{H}}(\chi_c) \cdot v + \mathcal{O}(\|\chi_b - \chi_a\|)(v) \) for all \( v \in V_P \) where \( \chi_c = \frac{1}{2}(\chi_a + \chi_b) \). (Here \( \| \cdot \| \) denotes a norm for the vector space \( V_P \).)
3) Orthogonality Conditions

i) \( D^c \mathcal{H}(x_a, x_b) \cdot (\xi, 0, 0, 0) = 0, \quad \forall \xi \in T_e G^I \cong \mathbb{R}^3 \)

ii) \( D^c \mathcal{H}(x_a, x_b) \cdot (\xi \times \varphi_c, \xi \times \pi_c, 0, 0) = 0, \quad \forall \xi \in T_e G^{II} \cong \mathbb{R}^3 \)

for any \( x_a, x_b \in P \) where \( \langle \cdot, \cdot \rangle_c = \frac{1}{2} [\langle \cdot, \cdot \rangle_a + \langle \cdot, \cdot \rangle_b] \).

Given a \( G^{I, II} \)-equivariant discrete derivative for \( \mathcal{H} \) let \( D^c_1 \mathcal{H}, \ldots, D^c_5 \mathcal{H} \) denote partial discrete derivatives defined by the relation

\[
D^c \mathcal{H}(x_a, x_b) \cdot (v_1, \ldots, v_5) = \sum_{i=1}^{5} D^c_i \mathcal{H}(x_a, x_b) \cdot v_i, \quad \forall v = (v_1, \ldots, v_5) \in \mathcal{V}_P. \tag{3.38}
\]

Then a conserving time-stepping scheme for (3.18) may be stated as follows:

Given \( b : B \to \mathbb{R}^3, f : \Gamma_\sigma \to \mathbb{R}^3 \), and \( \Delta t > 0 \) find a sequence \( (x_n)_{n=0}^N \) in \( P \) such that

\[
\begin{align*}
\langle \varphi_{n+1} - \varphi_n, \eta \rangle &= \Delta t D^c_3 \mathcal{H}(x_n, x_{n+1}) \cdot \eta \quad \forall \eta \in \mathcal{V}_Q \\
\langle \pi_{n+1} - \pi_n, \eta \rangle &= -\Delta t D^c_4 \mathcal{H}(x_n, x_{n+1}) \cdot \eta \quad \forall \eta \in \mathcal{V}_Q \\
D^c_3 \mathcal{H}(x_n, x_{n+1}) \cdot \phi &= 0 \quad \forall \phi \in E \\
D^c_4 \mathcal{H}(x_n, x_{n+1}) \cdot \phi &= 0 \quad \forall \phi \in E \\
D^c_5 \mathcal{H}(x_n, x_{n+1}) \cdot \phi &= 0 \quad \forall \phi \in E \\
\langle \varphi_0, \eta \rangle &= \langle \hat{\varphi}_0, \eta \rangle \quad \forall \eta \in \mathcal{V}_Q \\
\langle \pi_0, \eta \rangle &= \langle \hat{\pi}_0, \eta \rangle \quad \forall \eta \in \mathcal{V}_Q 
\end{align*}
\tag{3.39}
\]

where \( \hat{\varphi}_0 \) and \( \hat{\pi}_0 \) are initial data.

The conservation properties of (3.39) follow by the general results presented in Gonzalez [1996b]. For completeness, however, we state these properties in the following sequence of propositions.

**Proposition 3.5.** The modified Hamiltonian functional \( \mathcal{H} \) is conserved along any solution sequence \( (x_n)_{n=0}^N \) of (3.18) in the sense that

\[
\mathcal{H}(x_n) = \mathcal{H}(x_0) \quad \text{for} \quad 0 \leq n \leq N. \tag{3.40}
\]

**Proof.** The result follows from the directionality property of the discrete derivative, the discrete Hamiltonian form of the equations (3.39)\(_1,2\), and the discrete variational form of the equations (3.39)\(_3,4,5\). In particular, for any \( 0 \leq n < N \), choose \( \eta = \)
\[ \pi_{n+1} - \pi_n \text{ in } (3.39)_1 \text{ and } \eta = \varphi_{n+1} - \varphi_n \text{ in } (3.39)_2. \]

Subtracting the resulting expressions yields

\[
D_1^c \mathcal{H}(\chi_n, \chi_{n+1}) \cdot (\varphi_{n+1} - \varphi_n) + D_2^c \mathcal{H}(\chi_n, \chi_{n+1}) \cdot (\pi_{n+1} - \pi_n) = 0 \quad \text{for } 0 \leq n < N. \quad (3.41)
\]

Next, choose \[ \phi = \Theta_{n+1} - \Theta_n \text{ in } (3.39)_3, \phi = p_{n+1} - p_n \text{ in } (3.39)_4, \text{ and } \phi = \lambda_{n+1} - \lambda_n \text{ in } (3.39)_5. \]

Summing the resulting expressions with (3.41) yields

\[
D_3^c \mathcal{H}(\chi_n, \chi_{n+1}) \cdot (\chi_{n+1} - \chi_n) = 0 \quad \text{for } 0 \leq n < N. \quad (3.42)
\]

The result then follows from the directionality property of the discrete derivative and induction.

For the problem we are considering the modified Hamiltonian \( \mathcal{H} \) is invariant under the actions of \( G^I \) and \( G^{II} \) on \( Q \times \mathcal{V}_Q \) and the underlying system preserves the momentum maps \( J^I \) and \( J^{II} \). The following results show that these conservation laws are preserved by (3.39).

**Proposition 3.6.** The linear momentum map \( J^I : Q \times \mathcal{V}_Q \to \mathbb{R}^3 \) is conserved along any solution sequence \( (\chi_n)_{n=0}^N \) of (3.39) in the sense that

\[
J^I(\varphi_n, \pi_n) = J^I(\varphi_0, \pi_0) \quad \text{for } 0 \leq n \leq N. \quad (3.43)
\]

**Proof.** Let \( \xi \in \mathbb{R}^3 \) be arbitrary and choose in (3.39)_2 the variation \( \eta(X) = \xi \) for all \( X \in B \). We then have

\[
\langle \pi_{n+1} - \pi_n, \xi \rangle = -\Delta t D_1^c \mathcal{H}(\chi_n, \chi_{n+1}) \cdot \xi
= -\Delta t D_3^c \mathcal{H}(\chi_n, \chi_{n+1}) \cdot (\xi, 0, 0, 0)
= 0 \quad \forall \xi \in \mathbb{R}^3, \quad (3.44)
\]

by the orthogonality condition on the discrete derivative. The result follows by noting that

\[
\langle \pi_{n+1} - \pi_n, \xi \rangle = [J^I(\varphi_{n+1}, \pi_{n+1}) - J^I(\varphi_n, \pi_n)] \cdot \xi.
\]

**Proposition 3.7.** The angular momentum map \( J^{II} : Q \times \mathcal{V}_Q \to \mathbb{R}^3 \) is conserved along any solution sequence \( (\chi_n)_{n=0}^N \) of (3.39) in the sense that

\[
J^{II}(\varphi_n, \pi_n) = J^{II}(\varphi_0, \pi_0) \quad \text{for } 0 \leq n \leq N. \quad (3.45)
\]
Proof. Let $\xi \in \mathbb{R}^3$ be arbitrary and choose, for any $0 \leq n < N$, the variation
\[ \eta = \xi \times \pi_{n+\frac{1}{2}} \] in (3.39)1 and $\eta = \xi \times \varphi_{n+\frac{1}{2}}$ in (3.39)2, where as usual $(\cdot)_{n+\frac{1}{2}} = \frac{1}{2}[(\cdot)_n + (\cdot)_{n+1}]$. Subtracting the resulting expressions gives
\[
\begin{align*}
\langle \varphi_{n+1} - \varphi_n, \xi \times \pi_{n+\frac{1}{2}} \rangle - \langle \pi_{n+1} - \pi_n, \xi \times \varphi_{n+\frac{1}{2}} \rangle \\
= \Delta t D^w_2 \mathcal{H}(\chi_n, \chi_{n+1}) \cdot (\xi \times \varphi_{n+\frac{1}{2}}) + \Delta t D^w_2 \mathcal{H}(\chi_n, \chi_{n+1}) \cdot (\xi \times \pi_{n+\frac{1}{2}}) \\
= \Delta t D^w_2 \mathcal{H}(\chi_n, \chi_{n+1}) \cdot (\xi \times \varphi_{n+\frac{1}{2}}, \xi \times \pi_{n+\frac{1}{2}}, 0, 0, 0) \\
= 0 \quad \forall \xi \in \mathbb{R}^3,
\end{align*}
\] (3.46)
by the orthogonality condition on the discrete derivative. The result follows by noting that
\[
\begin{align*}
\langle \varphi_{n+1} - \varphi_n, \xi \times \pi_{n+\frac{1}{2}} \rangle - \langle \pi_{n+1} - \pi_n, \xi \times \varphi_{n+\frac{1}{2}} \rangle \\
= [J^{II}(\varphi_{n+1}, \pi_{n+1}) - J^{II}(\varphi_n, \pi_n)] \cdot \xi.
\end{align*}
\]
\[\blacksquare\]

3.4.1. An Equivariant Discrete Derivative.

To complete the time-stepping scheme given in (3.39) all we need to do is construct a discrete derivative for the modified Hamiltonian functional $\mathcal{H}: \mathbb{P} \rightarrow \mathbb{R}$. From Gonzalez [1996a] note that an equivariant discrete derivative can be constructed for the functional $\bar{\mathcal{H}}$ if the functional can be expressed in terms of a density $\mathcal{H}$, and if an equivariant discrete derivative can be constructed for $\mathcal{H}$.

For the problem we are considering the functional $\bar{\mathcal{H}}: \mathbb{P} \rightarrow \mathbb{R}$ is given by
\[
\bar{\mathcal{H}}(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t) = \int_B \frac{1}{2} \rho^{-1} |\pi_t|^2 + W(X, \bar{F}_t) + p_t(J_t - \Theta_t) + \lambda_t \bar{G}(\Theta_t).
\] (3.47)

To express $\mathcal{H}$ in terms of a density $\mathcal{H}$ we introduce the space $\mathcal{U} = \mathbb{M}^3_+ \times \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ and we denote points in this space by $\mathcal{E} = (M, w, r, \alpha, \beta)$. For each $\chi \in \mathbb{P}$ we define a mapping $\zeta(\chi): B \rightarrow \mathcal{U}$ by the relation
\[
\zeta(\varphi_t, \pi_t, \Theta_t, p_t, \lambda_t)(X) = (D \varphi_t(X), \pi_t(X), \Theta_t(X), p_t(X), \lambda_t(X)).
\] (3.48)

Then (3.47) may be written as
\[
\bar{\mathcal{H}}(\chi_t) = \int_B \bar{\mathcal{H}} \circ \zeta(\chi_t)
\] (3.49)
where, omitting the explicit dependence on $X \in B$, the density $\bar{\mathcal{H}}: \mathcal{U} \rightarrow \mathbb{R}$ is defined by
\[
\bar{\mathcal{H}}(M, w, r, \alpha, \beta) = \frac{1}{2} \rho^{-1} |w|^2 + W(r^{1/3} \det[M]^{-1/3}M) + \alpha(\det[M] - r) + \beta \bar{G}(r).
\] (3.50)
To construct an equivariant discrete derivative for the density $\bar{H}$ we need to know the actions in $\mathcal{U}$ induced by those on $Q \times \mathcal{V}_Q$. Given the mapping $\zeta(\chi): B \to \mathcal{U}$ we note that the density $\bar{H}$ is invariant under the following induced actions in $\mathcal{U}$:

1) $\Pr \Phi^I : G^I \times \mathcal{U} \to \mathcal{U}$ defined by

$$\Pr \Phi^I (c, (M, w, r, \alpha, \beta)) = (M, w, r, \alpha, \beta).$$  \hspace{1cm} (3.51)

2) $\Pr \Phi^{II} : G^{II} \times \mathcal{U} \to \mathcal{U}$ defined by

$$\Pr \Phi^{II} (\Lambda, (M, w, r, \alpha, \beta)) = (\Lambda M, \Lambda w, r, \alpha, \beta).$$  \hspace{1cm} (3.52)

According to the methods of Gonzalez [1996a], our first task is to find $G^{I,II}$-invariant maps $\Pi^i : \mathcal{U} \to \tilde{\mathcal{U}}^i$ ($i = 1, \ldots, k$) for some $k \geq 1$, and a function $\bar{H} : \tilde{\mathcal{U}}^1 \times \cdots \times \tilde{\mathcal{U}}^k \to \mathbb{R}$ such that

$$\bar{H}(\Xi) = \bar{H}(\Pi^1(\Xi), \ldots, \Pi^k(\Xi)) = (\bar{H} \circ \Pi)(\Xi)$$  \hspace{1cm} (3.53)

where $\Pi = (\Pi^1, \ldots, \Pi^k)$. If the maps $\Pi^i(\Xi)$ are all at most degree two, then, for any two points $\Xi_a, \Xi_b \in \mathcal{U}$, a $G^{I,II}$-equivariant discrete derivative for $\bar{H}$ is

$$D^c \bar{H}(\Xi_a, \Xi_b) = \sum_{i=1}^k \frac{1}{2} \left( D\bar{H}_{ab}^i(\Pi^1_a, \Pi^1_b) \cdot D\Pi^i(\Xi_c) + D\bar{H}_{ba}^i(\Pi^1_a, \Pi^1_b) \cdot D\Pi^i(\Xi_c) \right),$$  \hspace{1cm} (3.54)

where $\bar{H}_{ab}^i, \bar{H}_{ba}^i : \tilde{\mathcal{U}}^i \to \mathbb{R}$ are defined by the relations

$$\bar{H}_{ab}^i(v) = \bar{H}(\Pi^1_a, \ldots, \Pi^i_a - 1, v, \Pi^i_a + 1, \ldots, \Pi^k_b),$$  \hspace{1cm} (3.55)

$$\bar{H}_{ba}^i(v) = \bar{H}(\Pi^1_b, \ldots, \Pi^i_b - 1, v, \Pi^i_a + 1, \ldots, \Pi^k_a),$$  \hspace{1cm} (3.56)

$\Pi^i_a = \Pi^i(\Xi_a)$ and $\Pi^i_b = \Pi^i(\Xi_b)$ ($i = 1, \ldots, k$), and $\Xi_c = \frac{1}{2}(\Xi_a + \Xi_b)$.

The operator “$\mathcal{D}$” used in the above constructions is defined for any differentiable mapping $f : (\mathcal{U}, \langle \cdot, \cdot \rangle_{\mathcal{U}}) \to \mathbb{R}$ by the expression

$$\mathcal{D} f(x, y) = D f(z) + \frac{f(y) - f(x) - \langle D f(z), v_{xy} \rangle_{\mathcal{U}}}{\langle v_{xy}, v_{xy} \rangle_{\mathcal{U}}} v_{xy}$$  \hspace{1cm} (3.57)

for all $x, y \in \mathcal{U}$ where $z = (x + y)/2$ and $v_{xy} = y - x$. 
Now, for our example we find a mapping \( \Pi = (\Pi^1, \ldots, \Pi^5) : \mathcal{U} \rightarrow \mathcal{U} \), where 
\[ \mathcal{U} = S^3_{p_d} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \]
defined by the components
\[
\begin{align*}
\Pi^1(M, w, r, \alpha, \beta) &= M^T M \\
\Pi^2(M, w, r, \alpha, \beta) &= |w|^2 \\
\Pi^3(M, w, r, \alpha, \beta) &= r \\
\Pi^4(M, w, r, \alpha, \beta) &= \alpha \\
\Pi^5(M, w, r, \alpha, \beta) &= \beta,
\end{align*}
\]
and we find a function \( \bar{H} : \mathcal{U} \rightarrow \mathbb{R} \) of the form
\[
\bar{H}(A, \gamma, r, \alpha, \beta) = \frac{1}{2} \rho^{-1} \gamma + \tilde{W}(r^{2/3} \det[A]^{-1/3} A) + \alpha (\det[A]^{1/2} - r) + \beta \tilde{G}(r)
\]
(3.59)
such that \( \bar{H} = \bar{H} \circ \Pi \). Here \( \tilde{W} \) is the frame-invariant stored energy function introduced earlier (with the explicit dependence on \( X \in B \) now omitted for brevity).

To simplify matters, introduce a modified stored energy function \( \bar{W} : S^3_{p_d} \times \mathbb{R}_+ \rightarrow \mathbb{R} \) defined by the relation
\[
\bar{W}(A, r) = \tilde{W}(r^{2/3} \det[A]^{-1/3} A).
\]
(3.60)
For any \( r \in \mathbb{R}_+ \) let \( \bar{W}_r = \tilde{W}(\cdot, r) : S^3_{p_d} \rightarrow \mathbb{R} \), and for any \( A \in S^3_{p_d} \) let \( \bar{W}_A = \tilde{W}(A, \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R} \). Then, using the notation \( A = M^T M \), an equivariant discrete derivative for \( \bar{H} \) is defined by the relation
\[
D^G\bar{H}(\Xi_n, \Xi_{n+1}) \cdot (v_1, \ldots, v_5) = \sum_{i=1}^{5} D^G_i \bar{H}(\Xi_n, \Xi_{n+1}) \cdot v_i
\]
(3.61)
where
\[
\begin{align*}
D^G_1 \bar{H}(\Xi_n, \Xi_{n+1}) &= M_n + \frac{1}{2} \left( D\bar{W}_r_n (A_n, A_{n+1}) + D\bar{W}_r_{n+1} (A_n, A_{n+1}) \right) \\
&\quad + 2 \alpha_n + \frac{1}{2} M_n + \frac{1}{2} D \left[ \det[A]^{1/2} \right] (A_n, A_{n+1}) \\
D^G_2 \bar{H}(\Xi_n, \Xi_{n+1}) &= M_n + \frac{1}{2} D \bar{W}_r_n (A_n, A_{n+1}) + D\bar{W}_r_{n+1} (A_n, A_{n+1}) \\
&\quad - \alpha_n + \frac{1}{2} \beta_n + \frac{1}{2} \bar{G}(r_n, r_{n+1}) \\
D^G_3 \bar{H}(\Xi_n, \Xi_{n+1}) &= \frac{1}{2} \left( D\bar{W}_r_n (r_n, r_{n+1}) + D\bar{W}_r_{n+1} (r_n, r_{n+1}) \right) \\
&\quad + \alpha_n + \frac{1}{2} \beta_n + \frac{1}{2} \bar{G}(r_n, r_{n+1}) \\
D^G_4 \bar{H}(\Xi_n, \Xi_{n+1}) &= \frac{1}{2} \left( \det[M_n] + \det[M_{n+1}] \right) - r_n + \frac{1}{2} \beta_n + \frac{1}{2} \bar{G}(r_n, r_{n+1}) \\
D^G_5 \bar{H}(\Xi_n, \Xi_{n+1}) &= \frac{1}{2} \left( \beta_n + \frac{1}{2} \bar{G}(r_n) + \bar{G}(r_{n+1}) \right)
\end{align*}
\]
Using the above result for the density $\bar{\rho}$, an equivariant discrete derivative for the functional $\bar{\mathcal{H}} : P \to \mathbb{R}$ then follows via the relation

$$
D^c \bar{\mathcal{H}}(x_n, x_{n+1}) \cdot u = \int_B D^c \bar{\mathcal{H}}(\zeta(x_n), \zeta(x_{n+1})) \cdot \zeta(u).
$$

(3.63)

In terms of partial discrete derivatives we have

$$
D^c \bar{\mathcal{H}}(x_n, x_{n+1}) \cdot (u_1, \ldots, u_5) = \sum_{i=1}^{5} D^c_i \bar{\mathcal{H}}(x_n, x_{n+1}) \cdot u_i
$$

(3.64)

where

$$
\begin{align*}
D^c_1 \bar{\mathcal{H}}(x_n, x_{n+1}) \cdot \eta &= \left\langle F_{n+\frac{3}{2}} \left( DW_{\theta_n}(C_n, C_{n+1}) + DW_{\theta_{n+1}}(C_n, C_{n+1}) \right) \rightangle \\
&\quad + 2p_{n+\frac{1}{2}} F_{n+\frac{1}{2}} D \left[ \det[C]^{1/2} \right] (C_n, C_{n+1}), D\eta \\
D^c_2 \bar{\mathcal{H}}(x_n, x_{n+1}) \cdot \eta &= \left\langle \rho^{-1} \pi_{n+\frac{1}{2}}, \eta \right\rangle \\
D^c_3 \bar{\mathcal{H}}(x_n, x_{n+1}) \cdot \phi &= \left\langle \frac{1}{2} \left( DW_{\theta_n}(\theta_n, \theta_{n+1}) + DW_{\theta_{n+1}}(\theta_n, \theta_{n+1}) \right) \rightangle \\
&\quad - p_{n+\frac{1}{2}} + \lambda_{n+\frac{1}{2}} DG(\theta_n, \theta_{n+1}), \phi \\
D^c_4 \bar{\mathcal{H}}(x_n, x_{n+1}) \cdot \phi &= \frac{1}{2} \left\langle (J_{n+1} - \theta_{n+1}), \phi \right\rangle + \frac{1}{2} \left\langle (J_n - \theta_n), \phi \right\rangle \\
D^c_5 \bar{\mathcal{H}}(x_n, x_{n+1}) \cdot \phi &= \frac{1}{2} \left\langle \bar{G}(\theta_{n+1}), \phi \right\rangle + \frac{1}{2} \left\langle \bar{G}(\theta_n), \phi \right\rangle. 
\end{align*}
$$

(3.65)

Substituting the above expressions into (3.39) yields a conserving time integration scheme for our formulation of quasi-incompressible elastodynamics. Recall, however, that the above expressions were derived for the case in which $\Gamma_\varphi = \emptyset$, $\Gamma_\sigma = \partial B$ and $V^{\text{ext}} = 0$ (i.e. no external loads). In this case, the scheme inherits all the conservation laws discussed in the previous section. As we discussed in Section 2.5 for the compressible case, we can extend the above results to the general displacement- traction problem with conservative loading by including a term of the form $-\Delta t D V^{\text{ext}}(\varphi_n, \varphi_{n+1}) \cdot \eta$ in the discrete momentum balance equation (3.39)\_2. The particular case of dead loads is summarized in the next section.

### 3.4.2. Summary.

In view of the scheme outlined in (3.39), and the expressions in (3.65), we now summarize a conserving scheme for our formulation of quasi-incompressible elastodynamics. In particular, for a displacement- traction problem with dead loads we have the following:

Let $b : B \to \mathbb{R}^3$, $f : \Gamma_\sigma \to \mathbb{R}^3$, and $\Delta t > 0$ be given, and consider initial data $(\varphi_0, \pi_0, \theta_0, p_0, \lambda_0)$ satisfying

$$
\left\langle (J_0 - \theta_0), \phi \right\rangle = 0 \quad \text{and} \quad \left\langle \bar{G}(\theta_0), \phi \right\rangle = 0 \quad \forall \phi \in E.
$$

(3.66)
Then, the algorithmic problem is to find a sequence \((\varphi_n, \pi_n, \Theta_n, p_n, \lambda_n)_{n=0}^{N}\) in \(P\) such that

\[
\begin{align*}
\langle \varphi_{n+1} - \varphi_n, \eta \rangle &= \Delta t \left\langle \rho^{-1} \pi_{n+\frac{1}{2}}, \eta \right\rangle \quad \forall \eta \in \mathcal{V}_Q \\
\langle \pi_{n+1} - \pi_n, \eta \rangle &= -\Delta t \left\langle F_{n+\frac{1}{2}} \left( D\vec{W}_{\Theta_n}(C_n, C_{n+1}) + D\vec{W}_{\Theta_{n+1}}(C_n, C_{n+1}) \right) \\
&\quad + 2p_{n+\frac{1}{2}} F_{n+\frac{1}{2}} D \left[ \det [C]^{1/2} \right] (C_n, C_{n+1}), D\eta \right\rangle \\
&\quad + \Delta t \langle b, \eta \rangle + \Delta t \langle f, \eta \rangle_{\Gamma_n} \quad \forall \eta \in \mathcal{V}_Q \\
\langle \frac{1}{2} (D\vec{W}_C(\Theta_n, \Theta_{n+1}) + D\vec{W}_{C_{n+1}}(\Theta_n, \Theta_{n+1})) \\
&- p_{n+\frac{1}{2}} + \lambda_{n+\frac{1}{2}} D\vec{G}(\Theta_n, \Theta_{n+1}), \phi \rangle &= 0 \quad \forall \phi \in E \\
\langle (J_{n+1} - \Theta_{n+1}), \phi \rangle &= 0 \quad \forall \phi \in E \\
\langle \vec{G}(\Theta_{n+1}), \phi \rangle &= 0 \quad \forall \phi \in E.
\end{align*}
\]

(3.67)

In the above system, \((3.67)_{1,2}\) are interpreted as discrete evolution equations for \((\varphi_n, \pi_n) \in Q \times \mathcal{V}_Q\) with parameters \(\Theta_n, \Theta_{n+1}, p_{n+\frac{1}{2}}\) and \(\lambda_{n+\frac{1}{2}}\), expression \((3.67)_3\) is viewed as an equation which determines \(p_{n+\frac{1}{2}}\), expression \((3.67)_4\) is viewed as an equation which determines \(\Theta_{n+1}\), and expression \((3.67)_5\) is viewed as an equation which determines the multiplier \(\lambda_{n+\frac{1}{2}}\). The operator “\(D\)” appearing in the above formulation is defined in \((3.57)\).

**Remark 3.2.** As mentioned above, the algorithmic problem is completed by specification of initial data \((\varphi_0, \pi_0, \Theta_0, p_0, \lambda_0)\) in \(P\). Given initial data \(\varphi_0\) and \(\Theta_0\) satisfying \((3.66)\), the question arises as to what freedom one has in specifying the remaining data \(\pi_0, p_0\) and \(\lambda_0\). In general, the full initial data set must be consistent with the constraints. That is, the full set of initial data should satisfy \((3.66)\) along with as many time derivatives of \((3.66)\) as possible.

### 4. Numerical Implementation.

In this section we discuss the finite element implementation of the time-stepping scheme \((3.67)\) for quasi-incompressible elastodynamics. We first discuss a mixed finite element formulation for our scheme and then reduce it to a two-field formulation for efficient implementation. We then discuss a solution strategy which can be interpreted as the generalization to the dynamic case of the augmented multiplier methods used in Simo & Taylor [1991]. For more details on these methods we refer the reader to Bertsekas [1982], and to Glowinski & Le Tallec [1981, 1989] for applications to finite elasticity.
4.1. Mixed Finite Element Formulation.

Consider a partition of the domain \( B \subset \mathbb{R}^3 \) into \( m_{\text{elem}} \geq 1 \) nonoverlapping subdomains \( B^e \ (e = 1, \ldots, m_{\text{elem}}) \), each defined by a set of \( m_{\text{en}} \geq 1 \) element nodes denoted by \( X_a^e \in B^e \ (a = 1, \ldots, m_{\text{en}}) \). We assume the elements \( B^e \) are defined such that \( B = \bigcup_e B^e \), and we introduce, for each \( e = 1, \ldots, m_{\text{elem}} \), the sets

\[
\nu^e = \{ a \mid a = 1, \ldots, m_{\text{en}} \}, \\
\nu^e_\partial = \{ a \in \nu^e \mid X_a^e \in \Gamma_e \}.
\tag{4.1}
\]

In accordance with a standard isoparametric discretization, we parametrize each subdomain \( B^e \) with a mapping \( \Psi^e : \Omega \rightarrow B^e \) of the form

\[
\Psi^e(\xi) = \sum_{a=1}^{m_{\text{en}}} N^a(\xi) X_a^e,
\tag{4.2}
\]

where \( \Omega \subset \mathbb{R}^3 \) is typically the bi-unit cube and \( N^a : \Omega \rightarrow \mathbb{R} \ (a = 1, \ldots, m_{\text{en}}) \) are standard isoparametric shape functions. In particular, the shape functions satisfy the condition \( N^a(\xi_b) = \delta^a_b \), where \( \xi_b \in \bar{\Omega} \ (b = 1, \ldots, m_{\text{en}}) \) are element nodes for the parent domain \( \Omega \).

Given a discretization of \( B \) as described above, we introduce finite-dimensional spaces \( Q^h \) and \( V^h_Q \) (interpreted as finite-dimensional approximations of \( Q \) and \( V_Q \), respectively) defined by

\[
Q^h = \{ \varphi^h : \bar{B} \rightarrow \mathbb{R}^3 \mid \varphi^h \in C^0(\bar{B}, \mathbb{R}^3), \varphi^h|_{B^e} \circ \Psi^e = \sum_{a \in \nu^e_\partial} N^a d_a^e + \sum_{a \in \nu^e} N^a g(X_a^e), \ d_a^e \in \mathbb{R}^3 \},
\tag{4.3}
\]

and

\[
V^h_Q = \{ \eta^h : \bar{B} \rightarrow \mathbb{R}^3 \mid \eta^h \in C^0(\bar{B}, \mathbb{R}^3), \eta^h|_{B^e} \circ \Psi^e = \sum_{a \in \nu^e_\partial} N^a c_a^e, \ c_a^e \in \mathbb{R}^3 \}.
\tag{4.4}
\]

In a similar manner, we construct finite-dimensional approximations to \( E \) and \( E_+ \). In particular, we consider approximations by spaces of discontinuous functions of the form

\[
E^h = \{ p^h : \bar{B} \rightarrow \mathbb{R} \mid p^h|_{B^e} \circ \Psi^e = \sum_{i=1}^{m_{\text{dis}}} \hat{N}^i \alpha_i^e, \ \alpha_i^e \in \mathbb{R} \},
\tag{4.5}
\]

\[
E_+^h = \{ \Theta^h : \bar{B} \rightarrow \mathbb{R}_+ \mid \Theta^h|_{B^e} \circ \Psi^e = \sum_{i=1}^{m_{\text{dis}}} \hat{N}^i \tau_i^e, \ \tau_i^e \in \mathbb{R}_+ \}.
\tag{4.6}
\]
where $m_{\text{dis}} \geq 1$ and $\tilde{N}^i : \Omega \to \mathbb{R}$ ($i = 1, \ldots, m_{\text{dis}}$) are smooth interpolation functions. Note that the functions in $E^h$ and $E^h_+$ are smooth within element domains, but are allowed to be discontinuous across element boundaries.

A mixed finite element formulation of our time-stepping scheme is now constructed by simply restating (3.67) with the finite-dimensional spaces $Q^h$, $V^h_Q$, $E^h$ and $E^h_+$. We then use standard arguments to reduce the formulation to a system of nonlinear algebraic equations. (See e.g. Hughes [1987] for an exposition of these standard ideas.)

**Remark 4.1.** Since the finite element formulation of (3.67) has the same structure as (3.67) itself, it is easy to see that the finite element formulation possesses conservation laws analogous to those of (3.67).

### 4.2. Reduction to a Two-Field Formulation.

In this section we eliminate the fields $\pi^h_{n+1}$, $\Theta^h_{n+1}$ and $\rho^h_{n+1}$ from the finite element formulation of (3.67). The result will be a two-field finite element formulation in which we need only solve for $\varphi^h_{n+1}$ and $\lambda^h_{n+1}$ at each step of the algorithm.

To begin, we assume for simplicity that the density $\rho : B \to \mathbb{R}$ is a constant function. Then, in view of (3.67)$_1$, we can eliminate $\pi^h_{n+1}$ from the finite element formulation using the expression

$$\pi^h_{n+1} = \frac{2}{\Delta t} \rho(\varphi^h_{n+1} - \varphi^h_n) - \pi^h_n. \quad (4.7)$$

Now assume the interpolation functions $\tilde{N}^i : \Omega \to \mathbb{R}$ ($i = 1, \ldots, m_{\text{dis}}$) are non-trivial and introduce, for each $e = 1, \ldots, m_{\text{elem}}$, an $m_{\text{dis}} \times m_{\text{dis}}$ symmetric positive-definite matrix $H_e$ by the component expressions

$$(H_e)^{ij} = \int_D \tilde{N}^i \tilde{N}^j \det[D\Psi^e], \quad (4.8)$$

and let $H_e^{-1}$, an $m_{\text{dis}} \times m_{\text{dis}}$ matrix with components $(H_e^{-1})_{ij}$, be the inverse of $H_e$. In view of the fact that the spaces $E^h$ and $E^h_+$ are spanned by functions allowed to be discontinuous across element boundaries, we can use standard arguments and reduce the finite element version of (3.67)$_4$ to

$$\langle \Theta^h_{n+1}|_{B_e}, \phi^h|_{B_e} \rangle_{B_e} = \langle J^h_{n+1}|_{B_e}, \phi^h|_{B_e} \rangle_{B_e} \quad \forall \phi^h|_{B_e} \quad (e = 1, \ldots, m_{\text{elem}}) \quad (4.9)$$

where $J^h_{n+1} := \det[D\varphi^h_{n+1}]$. For any $e = 1, \ldots, m_{\text{elem}}$ we then arrive at the equation

$$\langle \Theta^h_{n+1}|_{B_e} \circ \Psi^e, \tilde{N}^i \det[D\Psi^e] \rangle_{\Omega} = \langle J^h_{n+1}|_{B_e} \circ \Psi^e, \tilde{N}^i \det[D\Psi^e] \rangle_{\Omega} \quad (4.10)$$
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which must holds for \( i = 1, \ldots, m_{\text{dis}} \). To solve this equation for \( \Theta_{n+1}^h |_{B^e} \) we consider an interpolation of the form

\[
\Theta_{n+1}^h |_{B^e} \circ \Phi^e = \sum_{j=1}^{m_{\text{dis}}} \hat{N}^j \Theta_{j,n+1}^e
\]

which, in view of (4.10), leads us to the relation

\[
\sum_{j=1}^{m_{\text{dis}}} (H_e)^{ij} \Theta_{j,n+1}^e = \left\langle J_{n+1}^h |_{B^e} \circ \Phi^e, \hat{N}^i \det[D\Phi^e] \right\rangle \Omega \quad (i = 1, \ldots, m_{\text{dis}}).
\] (4.12)

Solving (4.12) for \( \Theta_{j,n+1}^e \) \( (j = 1, \ldots, m_{\text{dis}}) \) and substituting the result into (4.11) then yields

\[
\Theta_{n+1}^h |_{B^e} \circ \Phi^e = \sum_{i,j=1}^{m_{\text{dis}}} \hat{N}^j (H_e^{-1})^{ji} \left\langle \left[ \frac{1}{2} \left( D\mathcal{W}_{n}^h (\Theta_n^h, \Theta_{n+1}^h) + D\mathcal{W}_{n+1}^h (\Theta_n^h, \Theta_{n+1}^h) \right) \right] + \lambda_{n+\frac{1}{2}} \mathcal{D} \left( \Theta_n^h, \Theta_{n+1}^h \right) \right\rangle_{B^e} \circ \Phi^e, \hat{N}^i \det[D\Phi^e] \right\rangle \Omega
\]

which expresses \( \Theta_{n+1}^h |_{B^e} \) as a function of \( \varphi_{n+1}^h |_{B^e} \).

Similarly, we can develop an expression for the restriction of the field \( p_{n+1}^h \) to any element \( B^e \). In particular, we obtain

\[
p_{n+1}^h |_{B^e} \circ \Phi^e = 2 \sum_{i,j=1}^{m_{\text{dis}}} \hat{N}^j (H_e^{-1})^{ji} \left\langle \left[ \frac{1}{2} \left( D\mathcal{W}_{n}^h (\Theta_n^h, \Theta_{n+1}^h) + D\mathcal{W}_{n+1}^h (\Theta_n^h, \Theta_{n+1}^h) \right) \right] + \lambda_{n+\frac{1}{2}} \mathcal{D} \left( \Theta_n^h, \Theta_{n+1}^h \right) \right\rangle_{B^e} \circ \Phi^e, \hat{N}^i \det[D\Phi^e] \right\rangle \Omega
\] (4.14)

Let \( P^h = Q^h \times \mathcal{V}_Q^h \times E_4^h \times E^h \times E^h \). Then our two-field finite element formulation of the conserving time-stepping scheme in (3.67) may be stated in algorithmic form as follows:

Given \( (\varphi_n^h, \pi_n^h, \Theta_n^h, p_n^h, \lambda_n^h) \in P^h \) find \( (\varphi_{n+1}^h, \lambda_{n+1}^h) \in Q^h \times E^h \) such that

\[
\begin{aligned}
\left\langle \rho(\varphi_{n+1}^h - \varphi_n^h), \eta^h \right\rangle - \Delta t \left\langle \pi_n^h, \eta^h \right\rangle \\
+ \frac{1}{2} \Delta t^2 \left\langle F_n^h + F_{n+\frac{1}{2}}^h \left( D\mathcal{W}_{n}^h (C_n^h, C_{n+1}^h) + D\mathcal{W}_{n+1}^h (C_n^h, C_{n+1}^h) \right) \right\rangle \\
+ 2\eta_{n+\frac{1}{2}}^h \left[ \det[C_n^h]^{1/2} \right] (C_n^h, C_{n+1}^h), D\eta^h \right\rangle \\
- \frac{1}{2} \Delta t^2 \left\langle b, \eta^h \right\rangle - \frac{1}{2} \Delta t^2 \left\langle f, \eta^h \right\rangle_{\Gamma_e} = 0 \quad \forall \eta^h \in \mathcal{V}_Q^h
\end{aligned}
\] (4.15)

\[
\langle \mathcal{G} (\Theta_{n+1}^h) |_{B^e}, \phi^h |_{B^e} \rangle_{B^e} = 0 \quad \forall \phi^h \in E^h \quad (e = 1, \ldots, m_{\text{elem}})
\]
where $\Theta_{n+1}^h$ and $p_{n+1}^h$ are considered functions of $\varphi_{n+1}^h$ and $\lambda_{n+1}^h$ via expressions (4.13) and (4.14), respectively, and $\pi_{n+1}^h$ is computed according to (4.7).

**Remark 4.2.** If we choose $m_{\text{dis}} = 1$ and consider the interpolation function $\tilde{N}(\xi) = 1$ for all $\xi \in \Omega$, then we obtain

$$\Theta_{n+1}^h|_{B^e} = \frac{1}{\text{vol}(B^e)} \int_{B^e} J_{n+1}^h.$$  \hfill (4.16)

That is, the restriction of $\Theta_{n+1}^h$ to any element is constant and equal to the average jacobian value over $B^e$. In this case, the incompressibility condition we are enforcing is $\Theta_{n+1}^h|_{B^e} = 1$ ($e = 1, \ldots, m_{\text{elem}}$). Hence, the incompressibility condition is being enforced on an averaged jacobian, and not the point-wise jacobian $J_{n+1}^h$, which helps circumvent numerical difficulties such as locking. (See e.g. Hughes [1987].)

### 4.3. Solution Strategy

Following Simo, Taylor & Pister [1985] consider frame-invariant stored energy functions $\tilde{W} : B \times S^p_{\text{pd}} \to R$ of the form

$$\tilde{W}(X, A) = \tilde{W}^\text{dev}(X, A^\text{dev}) + \tilde{U}(X, \det[A])$$  \hfill (4.17)

where $\tilde{W}^\text{dev} : B \times S^3_{\text{pd}} \to R$ is interpreted as a *deviatoric* stored energy function, $\tilde{U}(X, r^2) = U(X, r)$ for some function $U : B \times R_+ \to R$ which is convex in $r$ for each $X \in B$, and $A^\text{dev} \in S^3_{\text{pd}}$ is the deviatoric part of $A$ defined by $A^\text{dev} = \det[A]^{-1/3} A$. In view of (3.60) and (4.17), the modified stored energy function $\tilde{W}$ appearing in our formulation becomes

$$\tilde{W}(X, A, r) = \tilde{W}^\text{dev}(X, A^\text{dev}) + U(X, r).$$  \hfill (4.18)

For our quasi-incompressible formulation we can assume, without loss of generality, that the function $U$ is independent of $X \in B$ and of the form

$$U(r) = K \gamma(r)$$  \hfill (4.19)

where $K > 0$ is a constant and $\gamma : R_+ \to R$ is convex, non-negative and satisfies $\gamma(r) = 0$ if and only if $r = 1$.

In accordance with standard penalty formulations of constrained variational problems, one expects that any motion of a hyperelastic body with stored energy function as described above satisfies $\Theta_t(X) \to 1$ in $B \times [0, T]$ as $K \to \infty$. In fact, if we drop the quasi-incompressibility condition (4.15) and set $\lambda_{n}^h = 0$ for all $n$, then the resulting finite element formulation (with stored energy function as described above) could be interpreted as a standard penalty formulation of quasi-incompressible
elastodynamics. These types of formulations, however, are typically ill-conditioned (see e.g. Luenberger [1984] or Bertsekas [1982]).

On the other hand, keeping the formulation in (4.15) as it is (i.e. with a multiplier field $\lambda_n^h$ enforcing the quasi-incompressibility condition), and employing a stored energy function as described above, we can interpret our system as an augmented lagrangian formulation, in the sense of optimization theory. That is, we enforce the quasi-incompressibility condition (4.15) by means of a multiplier field and a penalty parameter $K$. Given this interpretation of our formulation, we have at our disposal various efficient solution strategies (see e.g. Luenberger [1984], Bertsekas [1982] and Fortin & Fortin [1985].) One example is the classical Uzawa algorithm, which, using $k$ as an iteration index, may be summarized as follows:

1) Let $(\varphi_n^h, \pi_n^h, \Theta_n^h, p_n^h, \lambda_n^h) \in P^h$ be given and set $\lambda_{n+1}^{h,k}|_{k=0} = \lambda_n^h$.

2) For fixed $\lambda_{n+1}^{h,k}$, solve (4.15) for the field $\varphi_{n+1}^h$. Call this solution $\varphi_{n+1}^{h,k}$.

3) If (4.15) is not satisfied to a specified tolerance, then update the multiplier field using the expression

$$\left\langle \lambda_{n+1}^{h,k+1}, \phi^h \right\rangle = \left\langle \lambda_{n+1}^{h,k}, \phi^h \right\rangle + K \left\langle \tilde{G}(\Theta_{n+1}^{h,k}), \phi^h \right\rangle \quad \forall \phi^h \in E^h,$$

and repeat Step 2.

**Remarks 4.3.**

1) An example of an admissible penalty function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ is

$$\gamma(r) = \frac{1}{2}(r^2 - 1) - \ln(r).$$  \hfill (4.20)

2) Note that the above algorithm effectively reduces the two-field formulation to one field. Hence, Step 2 can be carried out using standard finite element codes suitable for nonlinear problems.

### 5. Numerical Examples.

In this section we illustrate the performance of the time-stepping schemes presented herein by means of three example problems: the tumbling of an elastic block, the stretching of a rectangular plate, and the inversion of a spherical cap. In all cases we employ hyperelastic models of the Ogden type, namely

$$\tilde{W}(C) = \tilde{w}(\lambda_1, \lambda_2, \lambda_3) = \sum_{A=1}^{3} w(\lambda_A)$$  \hfill (5.1)
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where

\begin{equation}
    w(\lambda_A) = \sum_{m=1}^{k} \left[ \frac{\mu_m}{\alpha_m} (\lambda_A^{\alpha_m} - 1) - \mu_m \ln(\lambda_A) \right].
\end{equation}

Here $C$ is the right Cauchy strain tensor, $\lambda_A^2$ ($A = 1, 2, 3$) are the eigenvalues of $C$, $\mu_m$ and $\alpha_m$ ($m = 1, \ldots, k$) are parameters, and $k$ is the number of terms in the Ogden model. As discussed in Ogden [1982,1984] and Valanis & Landel [1967] models of the form presented above are particularly useful for rubber-like materials.

For our compressible calculations we employ the above model with $k = 3$ and the following parameters:

\begin{equation}
    \begin{aligned}
        \mu_1 &= 0.690 \times 10^6 N/m^2, \quad \alpha_1 = 1.3, \\
        \mu_2 &= 0.010 \times 10^6 N/m^2, \quad \alpha_2 = 4.0, \\
        \mu_3 &= -0.012 \times 10^6 N/m^2, \quad \alpha_3 = -2.0,
    \end{aligned}
\end{equation}

which are taken from Ogden [1984, p.498].

An extension of the above model to incompressible materials is given by

\begin{equation}
    \tilde{W}(C) = \tilde{W}^{\text{dev}}(C^{\text{dev}}) = \sum_{A=1}^{3} w^{\text{dev}}(\lambda_A^{\text{dev}})
\end{equation}

together with the constraint $J = \lambda_1 \lambda_2 \lambda_3 = 1$, where $w^{\text{dev}}$ is of the form

\begin{equation}
    w^{\text{dev}}(\lambda_A^{\text{dev}}) = \sum_{m=1}^{k} \frac{\mu_m}{\alpha_m} [(\lambda_A^{\alpha_m})^{\alpha_m} - 1].
\end{equation}

Here $C^{\text{dev}}$ is the deviatoric part of $C$ and $(\lambda_A^{\text{dev}})^2$ ($A = 1, 2, 3$) are the eigenvalues of $C^{\text{dev}}$. Within our quasi-incompressible framework we introduced a \textit{modified} stored energy function $\tilde{W}(C, \Theta)$. In terms of the function in (5.4) we have

\begin{equation}
    \tilde{W}(C, \Theta) = \tilde{W}^{\text{dev}}(C^{\text{dev}}) + U(\Theta)
\end{equation}

where $U$ is interpreted as a penalty function, which we take as

\begin{equation}
    U(\Theta) = K \gamma(\Theta)
\end{equation}

where $K > 0$ is a penalty parameter and $\gamma : \mathbb{R}_+ \to \mathbb{R}$ is given in (4.20). For our quasi-incompressible calculations we use the same constants as in (5.3).

For the mixed finite element formulation we construct the spaces $Q^h$ and $V_Q^h$ using standard trilinear shape functions. For the spaces $E^h$ and $E_+^h$ we take $m_{\text{dis}} = 1$ and choose constant interpolation functions over the elements.
Example 5.1. **Tumbling Elastic Block.** In this simple example we consider a cube of dimension $l = 0.02m$ composed of a homogeneous elastic material of Ogden type with parameters as given in (5.3) and density $\rho = 1000 kg/m^3$. In the simulation, the cube is initially at rest and is subjected to tractions $F_1$ and $F_2$ for a short period of time. The forces are then removed and the cube is allowed to tumble freely in space. Figure 5.1 shows a schematic of the cube in its reference configuration along with the mentioned forces. Specifically, the traction $F_1$ is a spatially uniform force distribution on the face $X_3 = -l/2$ defined by $F_1 = p(t)(0, -f/4, -f/8)$ and the traction $F_2$ is a spatially uniform force distribution on the face $X_3 = l/2$ defined by $F_2 = p(t)(0, f, f/2)$ where $f = 32 N/cm^2$ and $p(t)$ is given by

$$p(t) = \begin{cases} t, & 0 \leq t \leq 0.005s \\ 0, & t > 0.005s. \end{cases}$$

![Figure 5.1. Schematic of elastic cube showing applied forces.](image)

For this example problem we compute solutions for both the compressible and quasi-incompressible formulations using a uniform spatial discretization of the cube into 27 elements. For the compressible formulation we consider two time-stepping schemes: the standard mid-point rule and the exact energy-momentum preserving scheme in (2.46). For the quasi-incompressible formulation we use the conserving scheme presented in (3.67). For this case, we employ the nested iteration strategy outlined in the previous section and iterate until $\max_c |\tilde{G}(\Theta_n^e)| < 5 \times 10^{-10}$ for each time step.

Figure 5.2 shows snapshots, at time increments of $0.005s$, in the motion of the cube during the time interval $[0, 0.1s]$. The plot was computed using the compressible formulation, however, the incompressible motion is indistinguishable for the given resolution. For clarity, we show the snapshots in the $X_2$-$X_3$ plane only.

Figures 5.3 and 5.4 show total energy and angular momentum time histories, computed using various time steps, for the mid-point rule and conserving schemes, respectively, applied to the compressible formulation. Note that the mid-point rule
experiences numerical blow-up for the time step $\Delta t = 0.005\,s$ in the time interval of the computation.

Figure 5.3 shows how the conserving scheme compares with the mid-point rule with respect to accuracy for the compressible formulation. For this comparison we considered the above motion in the time interval $[0.0100\,s, 0.0125\,s]$. For each scheme we supplied the same initial conditions at $t_0 = 0.0100\,s$ and computed the solution out to a time $t_1 = 0.0125\,s$. Denoting the solution at time $t_1$ for a given scheme and time step by $\varphi_{t_1,\Delta t}$, we define the $L_2$-error in displacements for that scheme and time step by

$$L_2\text{ Error } = \left( \int_B |\varphi_{t_1,\Delta t} - \varphi_{t_1,\text{conv}}|^2 \right)^{1/2}$$
where $\varphi_{t_1,\text{conv}}$ is the solution at $t_1$ computed using a time step of $\Delta t = 2.5 \times 10^{-7}$ s. As shown in Figure 5.5 the conserving scheme and the mid-point rule have nearly identical accuracy properties, at least for the model problem considered.

Figure 5.6 shows total energy and angular momentum time histories, computed using a time step size of $\Delta t = 0.005 s$, for the conserving scheme applied to the quasi-incompressible formulation. As mentioned above, we employed the nested iteration strategy outlined in the previous section and iterated until $\max_e |\hat{G}(\Theta_n^e)| < 5 \times 10^{-10}$ for each time step.

We next consider another example problem using the quasi-incompressible formulation. In contrast to the last example, the deformations in this next problem are rather severe.
Example 5.2. Stretched Elastic Plate. In this example we consider a thick rectangular plate of thickness $w = 0.002m$, length $l = 0.01m$ and height $h = 0.01m$ as shown in Figure 5.7. The plate is composed of a homogeneous elastic material of Ogden type with parameters as given in (5.3) and density $\rho = 1000kg/m^3$. In the simulation, the plate is initially at rest and is subjected to a traction $F$ for a short period of time. The force is then removed and the plate is allowed to vibrate freely subject to a zero displacement boundary condition along the shaded region shown in the figure. Specifically, the traction $F$ is applied to the face $X3 = -h/2$, is uniform across the thickness, and varies in the $X2$-direction according to

$$F(X2,t) = \begin{cases} p(t)(f/2,0,-f), & -\frac{l}{2} \leq X2 < 0 \\ p(t)(0,0,-f), & 0 \leq X2 \leq \frac{l}{2} \end{cases}$$

(5.8)

where $f = 5kN/cm^2$ and $p(t)$ is given by

$$p(t) = \begin{cases} t, & 0 \leq t \leq 0.004s \\ 0, & t > 0.004s. \end{cases}$$

(5.9)

The zero displacement boundary condition is imposed along the surface defined by $\{X3 = h/2\} \cap \{-l/8 \leq X2 \leq l/2\}$.

In the spatial discretization of the plate we use 2 elements through the thickness and a uniform discretization in the $X2$-$X3$ plane consisting of 25 elements, for a total of 50 elements. Figure 5.8 shows snapshots at various times in the motion of the plate computed using the conserving scheme for the quasi-incompressible formulation with a time step of $\Delta t = 0.0001s$. We employed the nested iteration strategy outlined in the previous section and iterated until $\max_e |\tilde{G}(\Theta_n^e)| < 5 \times 10^{-10}$ for each time step.

Figure 5.9 shows total energy and angular momentum time histories for the computed motion of the elastic plate. Note that total energy is conserved after the loads are removed at $t = 0.0040s$, but the angular momentum is not conserved due to the displacement boundary condition.

FIGURE 5.6. Time history of total energy and angular momentum for the conserving scheme applied to the quasi-incompressible formulation with step size of $\Delta t = 0.005s$. 

FIGURE 5.7. Stretching history of the plate. 

FIGURE 5.8. Snapshots at various times in the motion of the plate computed using the conserving scheme for the quasi-incompressible formulation with a time step of $\Delta t = 0.0001s$. 

FIGURE 5.9. Total energy and angular momentum time histories for the computed motion of the elastic plate. Note that total energy is conserved after the loads are removed at $t = 0.0040s$, but the angular momentum is not conserved due to the displacement boundary condition.
We next consider another example problem using the compressible formulation. In contrast to the elastic cube example, the deformations will be more severe.

**Example 5.3. Inversion of a Spherical Cap.** In this example we consider a thick spherical cap with dimensions $r = 0.1m$, $\beta = 0.008m$ and $\alpha = \pi/4$ as shown in Figure 5.10. The cap is composed of a homogeneous elastic material of Ogden type with parameters as given in (5.3) and density $\rho = 100kg/m^3$. In the simulation, the cap is initially at rest and is subjected to a system of equilibrated force distributions $F_1$ and $F_2$ for a short period of time. The forces are then removed and the cap is allowed to vibrate freely in space. Specifically, the force distribution $F_1$ is applied to the convex face of the cap, acts in the negative $X_1$ direction and is uniform in a disk centered about the $X_1$ axis. The force $F_1$ has a resultant given by $p(t)(-f, 0, 0)$ where $f = 160kN$ and

$$p(t) = \begin{cases} t, & 0 \leq t \leq 0.001s \\ 0, & t > 0.001s \end{cases}$$

The force distribution $F_2$ is applied along the edge of the cap as shown in the figure, acts in the positive $X_1$ direction and has a resultant given by $p(t)(f, 0, 0)$.

In the spatial discretization of the cap we use 4 elements through the thickness and a somewhat uniform discretization of each surface of constant radius into 128 elements, for a total of 512 elements. Figure 5.11 show snapshots at time increments of 0.0005s in the motion of the cap computed using the conserving scheme.

Figures 5.12 and 5.13 show total energy and angular momentum time histories, computed using a time step of $\Delta t = 0.0005s$, for the conserving scheme and midpoint rule, respectively, applied to the spherical cap problem. For the conserving scheme we see that the total energy is conserved after the loads are removed at $t = 0.0010s$. Also, since the force distributions were equilibrated and symmetric with
FIGURE 5.8. Snapshots at various times in the motion of the stretched elastic plate as computed using the conserving scheme for the quasi-incompressible formulation.
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FIGURE 5.9. Time history of total energy and angular momentum for the conserving scheme applied to the quasi-incompressible elastic plate with step size of $\Delta t = 0.0001\ s$.

FIGURE 5.10. Schematic of elastic spherical cap showing geometry and applied force distributions.

respect to the $X1$ axis we see that the total angular momentum of the cap remains zero throughout the simulation. Note that the mid-point rule experiences numerical blow-up for the given time step size.
FIGURE 5.11. Snapshots from the back view at various times in the motion of the elastic spherical cap as computed using the conserving scheme for the compressible formulation.
6. Closing Remarks.

In this paper we have developed time-integration schemes for initial boundary value problems in nonlinear elasticity; in particular, conserving time-stepping schemes for general hyperelastic models, both compressible and incompressible. The approach taken herein was based on Gonzalez [1996a,b] where a general framework for conserving integrators for Hamiltonian systems is established. Using standard finite element discretizations in space for the compressible formulation, and a mixed finite element discretization for the incompressible case, we applied our results to three example problems. We saw that the conserving schemes perform relatively well as compared to a standard (symplectic) scheme: the implicit mid-point rule. In particular, we saw that the mid-point rule is prone to numerical blow-up at time steps
not too large when compared with the scales of the overall or average motion. In addition, for relatively small time steps, we saw that the conserving schemes have accuracy properties similar to that of the mid-point rule.

While strategies for the actual solution of the fully discrete system were not addressed in this paper, we note in closing that all the examples in this paper were performed using a standard Newton iteration scheme with a consistent linearization. Within this framework note that the conserving schemes presented herein yield unsymmetric linearizations whereas the mid-point rule yields a symmetric linearization. Hence, in the approach we have taken, the conserving schemes were considerably more computationally intensive. However, we note that it is possible to lessen the computational costs of conserving schemes by employing more efficient solution strategies.

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