Abstract

Tangles of loops which approximate an aspect of the Kerr-Newman black hole metrics at large scales compared to the Planck length are constructed. The physical aspect the tangles approximate is discussed. This construction may be useful in the loop representation of canonical quantum gravity. Implications and applications of the tangles are remarked.

1 Introduction

It is known in the canonical formalism that the gauge field admits loop-wise excitations, if no external source is present, due to the gauge invariance requirement in contrast to non-gauge fields (i.e. scalar and Dirac fields), in which point-wise excitations are allowed. This means that a solution of the classical equation of the gauge field could be thought as a sum of loops along which the gauge potentials are “concentrated”. For the Maxwell theory in a flat space-time, for instance, the loop-wise excitations could be interpreted as lines of electric force in the absence of charged particles. A quantum description of the theory in terms of lines of force instead of the traditional description in terms of particles (photons) is possible.

General Relativity (GR) has been reformulated in the canonical formalism as a gauge field theory in terms of connection variables and hence suggesting the existence of a description in terms of loop-wise excitations. Then the loop representation has been applied to quantizing the theory. In this representation, the geometrical information is coded in loops; a quantum state is a functional of loops $\gamma : S^1 \to \Sigma$, where $\Sigma$ is a 3-space. An advantage of the use of loops in GR is that the diffeomorphism invariance, which is a key notion in GR, seems more tractable than with the use of metrics. In fact, diffeomorphism invariant quantum states have been realized as functions of knot and link classes of loops in $\Sigma$, namely “knot states.” In this respect relations between Quantum Gravity in physics and Knot theory in mathematics have been revealed. (For recent account, see.) A problem is the interpretation of such loops. If one interprets them as lines of gravitational force, then it is unclear how they describe classical space-time geometries, which are responsible to the gravitational force in GR.

Some quantum operators (i.e. the area operator), which have 3-metric information classically, have been constructed and shown to be diagonal in the loop representation. This fact could suggest that the loops more or less represent 3-metric geometries and some sets of loops representing a certain aspect of 3-metric geometry may exist. Tangles of loops, called “weaves,” which approximate known 3-metrics in the following sense have been constructed; the areas of a given surface measured on a weave state and its corresponding metric state
coincide up to an error of order of $l_p/L$, where $l_p$ is the Planck length and $L$ is the scale of the measurements, provided that the surface is slowly varying over $L$ with respect to the metric corresponding to the state. A hope could be that each of an infinite number of different weaves, a sector of the entire loop basis, might approximates one of an infinite number of different metrics and hence the loop basis might “generalize” the notion of metric geometry. Moreover, the loops might provide an aspect of “precise” picture of the Planck scale geometry, to which classical picture of geometry is insensitive, and induce 3-metrics at large scales compared to the Planck length.

However, in the constructions, the weaves were determined from classical metrics, hence they could not contain more information than the corresponding metrics. If the loop picture is correct, then ultimately any metric should be determined from a weave and hence the weaves should be more “precise” than the corresponding metrics. At present we know only small number of weaves determined from classical metrics: the weaves representing flat metrics [6], Schwarzschild metrics (outside the horizon) [7] and gravitational plane wave metrics [8].

Before exploring the “precise” structure of space-time in terms of loops, if it exists, one may need to understand at least a mechanism of determining a weave from a given metric, since one is familiar with only the notion of metric in classical GR. In order to understand the mechanism and how the loops represent metrics in a physically viable way, it could be useful to consider metrics with more physically meaningful parameters and metrics to which quantum effects are supposed to be important. In this paper, we construct weaves representing the Kerr-Newman black hole metrics, a three parameter family of solutions of the Einstein equation and known to be the only stationary black holes. Of course, our construction here by no mean exhausts the understanding of the mechanism. Rather, it should be understood as a step toward an unseen goal. In Sec. 2 we review the reference [6] about the idea of the weaves and a way of constructing weaves representing curved metrics from a flat weave. In Sec. 3 we construct weaves representing the Kerr-Newman black hole metrics and discuss physical information contained in the weaves. In Sec. 4 we give some remarks on implications and applications of the weaves constructed in this paper.

2 Flat and curved weaves

The bra states in the loop representation of quantum gravity labelled by countable loops are analogous to the countable energy-angular-momentum states of the hydrogen atom labelled by a set of integers $(n, l, m)$. However, our understandings of the two cases are different.

In the latter case, our understanding of the states is complete in the sense that we know the correct inner product and hence we have the corresponding ket states and these bra and ket states determine the quantum dynamics of the hydrogen atom. Rather than specifying the position of the electron as a function of time, a set of three integers $(n, l, m)$ is essential to specify the entire physical information of the hydrogen atom.

In the former case, our understanding of the states is incomplete in the sense that we do not know a correct inner product and hence we do not have ket states. We hope that rather than specifying the 3-metric $g_{ab}$ of space as a function of (coordinate) time, a countable number of loops is essential to specify at least an aspect of physical information of space
We do not expect that the loop basis determines the entire physical information of the dynamics of gravity. Rather we would like to use them as an ansatz to investigate the quantum dynamics of gravity in contrast to the case of the hydrogen atom, in which the energy-angular-momentum basis is a final result of quantum mechanics. Note that this ansatz is not an assumption but a result [6] in the kinematical level, namely before imposing the constraint conditions, by which the dynamics of gravity is induced.

For this strategy to be physically viable, it is desired to understand the physical meaning of the loops and their relation to the traditional description of space-time in terms of metrics. Since some quantum operators which have 3-metric information classically have been shown to be diagonal in the loop basis [3, 9], it is natural to expect that the loop states more or less represent 3-metric geometries.

An idea which relates loops to metrics is the construction of the weaves. A flat weave is a tangle of loops which approximates an aspect of a flat metric at large scales relevant in classical GR. The aspect is determined by a set of operators one is interested in. So far the area operator has been used and a resulting flat weave approximates the flat metric in the sense that the areas of a given surface measured on the weave state and its corresponding metric state coincide up to a error of order of $l_p/L$, where $L$ is the scale of the measurements. There are still an infinite number of weaves sharing this aspect of the metric. If one imposes more conditions by means of more operators, then one might determine a more restricted set of weaves reflecting a wider range of aspects of the metric. Within the restricted set of weaves, whether one particular weave or a linear combination of the weaves represents the classical metric has not been understood. The flat weave we make use of is a weave determined by the area operator only. Furthermore, it is a particular choice among an infinite number of the weaves sharing the same aspect determined by the area operator. The curved weaves constructed by deforming the flat weave also reflect the same aspect.

A way of constructing a flat weave is the following. Given a flat 3-metric $h_{ab}$, the flat weave $\Delta$ is a set of randomly oriented circles of radius $a$ whose centers are randomly distributed in the 3-space with an average number density $n = 1/a^3$; $a$ is of order of the Planck length $l_p$. Some of these circles may be linked to or may intersect with one another. We call each circle (or its deformation) a loop component. There is an infinite number of different flat weaves $\Delta$, and each one corresponds to a different loop state $\langle \Delta \rangle$, namely, a different point in the state space of loops; these states are denoted as flat weave states. If $S$ is a macroscopic surface whose extrinsic curvature varies slowly at the scale $L$, then the eigenvalue of the area operator acting on any of the flat weave states agrees with the area of $S$ with respect to the flat metric up to an error of order $l_p/L$, that is,

$$l_p^2 \int_S \left| \int ds \Delta^a(s) \delta^3(x, \Delta(s)) dS_a(x) \right| = \int_S \left( \tilde{z}^{ab} dS_a(x) dS_b(x) \right)^{1/2} + O(l_p/L),$$

(1)

up to a multiplicative constant in the left hand side, where $\tilde{z}^{ab}$ is the double densitized inverse of $h_{ab}$ and $dS_a(x)$ is an infinitesimal surface element at $x$. It has been shown by rigorously regularized calculations [3, 4] that the value is proportional to the number of times the flat weave $\Delta$ intersects the surface $S$. In this respect all the flat weave states approximate equally well the same flat metric for large enough $L$.

In addition, another significant result from the consideration of the weave state is that the
geometrical structure at the Planck scale is discrete (consisting of countable loops) due to the requirement that $a$ is a finite number of order $l_p$, because of which the weave approximates a classical continuum geometry at large scales \[8\]. The weave could be seen as a “classical”

One can construct a curved weave by deforming a flat weave. If one define a tensor $t^a_b$ such that a given curved metric $q_{ab}$ is expressed by $q_{ab} = h_{cd}t^c_a t^d_b$ and define a curved weave $\Delta_t$ such that the coordinates $\Delta^a_t$ of each point on the weave is $\Delta^a_t = t(t^{-1})_b^a \Delta^b$, where $\Delta$ is a flat weave and $t$ and $(t^{-1})_b^a$ are the determinant and the inverse of $t^a_b$ respectively, then

$$
\begin{align*}
L_p^2 \int_S \left| \int ds \Delta^a_t(s) \delta^3(x, \Delta_t(s)) dS_a(x) \right| &= L_p^2 \int_S \left| \int ds \Delta^b(s) \delta^3(x', \Delta(s)) t(t^{-1})_b^a dS_a(x) \right| \\
&= \int_S \left( \frac{\gamma^{cd}}{h} t(t^{-1})_c^a t(t^{-1})_d^b dS_a(x) dS_b(x) \right)^{1/2} + O(l_p/L) \\
&= \int_S \left( \frac{\gamma^{ab}}{q} dS_a(x) dS_b(x) \right)^{1/2} + O(l_p/L),
\end{align*}
$$

where $x'$ is the position transformed from $x$ by $t(t^{-1})_b^a$. This means that the eigenvalue of the area operator acting on the weave state $\langle \Delta^i \rangle$ agrees with the area of $S$ with respect to the

metric $q_{ab}$ ($\gamma^{ab}$ is its double densitized inverse). In other words, the area of $S$ is the number of times the weave $\Delta_t$ intersects the surface and its value agrees with the value of the area of $S$ measured with respect to the metric $q_{ab}$. In this sense the weave $\Delta_t$ represents the curved metric. This construction of a curved weave works where $q_{ab}$ is slowly varying with respect to $h_{ab}$. We use this technology to construct weaves representing the Kerr-Newman black hole metrics in the next section.

3 Weaving black holes

A Kerr-Newman metric with total mass $M$, electric charge $e$ and angular momentum per unit mass $b = |J|/M$ in geometrized units is \[10\]

$$
\begin{align*}
ds^2 &= -\frac{\Lambda}{\Sigma} \left( dt - b \sin^2 \theta d\phi \right)^2 + \frac{r^4 \sin^2 \theta}{\Sigma} \left( \frac{r^2 + b^2}{r^2} d\phi - \frac{b}{r^2} dt \right)^2 + \frac{\Sigma}{\Lambda} dr^2 + \Sigma d\theta^2,
\end{align*}
$$

where $\Sigma = r^2 + b^2 \cos^2 \theta$ and $\Lambda = r^2 + b^2 + e^2 - 2Mr$. It is known that, if $b^2 + e^2 \leq M^2$, this metric has an event horizon at $r_+ = M + \sqrt{M^2 - b^2 - e^2}$ and a complicated global structure with an infinite number of black and white holes and asymptotically flat regions, which can be explored by extending space-time through the coordinate singularities, where $\Lambda = 0$. However, here we are not interested in the entire structure of space-time but interested in an asymptotically flat region, where we are supposed to live, and a black (or white) hole region, which are supposed to be being discovered by our astronomers, if it exists in the universe. In order to extend space-time into a black (or white) hole region, we need an appropriate coordinate system. We choose namely “ingoing” (or alternatively “outgoing”) geodesic coordinates. The change of coordinates such that $dT = dt \mp \left( \frac{r^2 + b^2}{\Lambda} - 1 \right) dr$ and $d\Phi = d\phi \mp (b/\Lambda)dr$ \[11\] leads to

$$
\begin{align*}
ds^2 &= -dT^2 + dr^2 + \Sigma d\theta^2 + \left( r^2 + b^2 \right) \sin^2 \theta d\Phi^2 \pm 2b \sin^2 \theta dr d\Phi.
\end{align*}
$$
Furthermore, for the convenience of our construction of a weave, we adopt a “cartesian” coordinate system \([\mathbf{11}]\) defined by
\[
x = (r \cos \Phi \pm b \sin \Phi) \sin \theta, \quad y = (r \sin \Phi \mp b \cos \Phi) \sin \theta, \quad z = r \cos \theta.
\] (5)

Then, our final form of the Kerr-Newman metric, which has a Kerr-Shaid form \([\mathbf{11}]\), is
\[
ds^2 = -dT^2 + dx^2 + dy^2 + dz^2 + \frac{2Mr - e^2}{r^2 + b^2} \left( \pm dT + \frac{1}{r^2 + b^2} \left[ r(x dx + y dy) \pm b(x dy - y dx) \right] + \frac{1}{r} zdz \right)^2,
\] (6)
where \(r\) is not the radial coordinate corresponding to \(\sqrt{x^2 + y^2 + z^2}\) but a function of \(x, y\) and \(z\) satisfying
\[
x^2 + y^2 + \frac{z^2}{r^2} = 1.
\] (7)

The metric is asymptotically flat and its induced 3-metric \(q_{ab}\) at a constant \(T\) is positive definite except at the true singularity where \(\Sigma = 0\) and a small region where \(r \leq -M + \sqrt{M^2 + e^2 - b^2} \cos \theta\). In this paper, we do not consider these places. The lower (upper) signs in the right hand side of Eq \([\mathbf{11}]\) or \([\mathbf{11}]\) lead to the black (white) hole interpretation.

Note that for a fixed \(r\) Eq \([\mathbf{7}]\) specifies an elliptical surface and the surface with \(r = r_+\) corresponds to the event horizon.

Let us now focus on the 3-metric \(q_{ab}\). Define a tensor \(t^a_b\) such that \(q_{ab} = h_{cd} t^c_a t^d_b\), where \(h_{ab}\) is a flat 3-metric and the indices take the value \(x, y\) or \(z\) (\(h_{ab} \equiv \delta_{ab}\) in our coordinate system). The components of \(t^a_b\) are
\[
\begin{bmatrix}
t^x_x & t^x_y & t^x_z \\
t^y_x & t^y_y & t^y_z \\
t^z_x & t^z_y & t^z_z \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \alpha
\end{bmatrix} \begin{bmatrix}
-XZ/\rho & -YZ/\rho & \rho \\
Y/\rho & -X/\rho & 0 \\
X & Y & Z
\end{bmatrix},
\] (8)
with
\[
X = \frac{r x \mp by}{r^2 + b^2}, \quad Y = \frac{ry \pm bx}{r^2 + b^2}, \quad Z = \frac{z}{r},
\]
\[
\rho = (X^2 + Y^2)^{1/2} = \frac{z}{r} \sqrt{X^2 + Y^2}, \quad \alpha = \left(1 + \frac{2Mr - e^2}{r^2 + b^2} \right)^{1/2}.
\] (9)

Note that the condition \([\mathbf{11}]\) implies \(X^2 + Y^2 + Z^2 = 1\).

Define a weave \(\Delta_t\) such that the coordinates \(\Delta_t^a\) of each point on the weave is \(\Delta_t^a = t(t^{-1})_b^a \Delta^b\), where a dot means the derivative with respect to a parameter of the loop and \((t^{-1})_b^a\) and \(t\) are the inverse and the determinant of \(t^a_b\) respectively. We assume that \(t^a_b\) (thus \((t^{-1})_b^a\)) is uniform over a single loop component of the weave since each loop component is of a size of order \(l_p\); in other words, \(t^a_b\) constructed from a classical metric is considered to be insensitive to the Planck scale structure of space-time. Thus the weave is determined by
\[
\Delta_t^a = t(t^{-1})_b^a \Delta^b,
\] (10)
up to a loop-wisely undetermined additive constant. We assume here that each of $\Delta_t^a$ and $\Delta^a_t$ is one of the coordinates of each point on a loop component relative to the center of the loop component, the positions of the centers of all the loop components are unchanged so that the undetermined constants are zero, and that $(t^{-1})^a_b$ is defined at the center of each loop component. The resulting weave, representing the Kerr-Newman metric, is such that

$$\begin{bmatrix}
\Delta_t^x \\
\Delta_t^y \\
\Delta_t^z
\end{bmatrix} =
\begin{bmatrix}
-XZ/\rho & Y/\rho & X \\
-YZ/\rho & -X/\rho & Y \\
\rho & 0 & Z
\end{bmatrix}
\begin{bmatrix}
\alpha & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\Delta^x \\
\Delta^y \\
\Delta^z
\end{bmatrix}.$$  (11)

This transformation consists of two parts, uniform expansions (or contractions) of loop components in the direction perpendicular to the $z$-direction and rotations of loop components so that the non-expanded (or non-contracted) direction coincides with $(X, Y, Z)$ at the center of each loop component $(x, y, z)$, (See Fig.1). The magnitude of the changes from the flat weave depends on $\sqrt{x^2 + y^2}$ and $z$. This construction is well defined everywhere except where $\Sigma = 0$ or $r \leq -M + \sqrt{M^2 + e^2 - b^2 \cos^2 \theta}$ as mentioned above.

Let us turn to discuss the meaning of the transformation. We are interested in physical information contained in the transformation and hence in the weave produced. In particular, we examine the areas of the event horizons for different values of $M$, $e$ and $b$. The area of the horizon computed from the metric Eq (3) or (4) is $4\pi (r_+^2 + b^2)$. Remember that the area of a surface in the loop representation is proportional to the number of times loops intersect the surface. By expanding or contracting the flat weave, the number of intersections with the surface changes. If the surface is closed, then an expansion or contraction in the direction transverse to the normal of the surface does not alter the total number of intersections but one in the normal direction does. If it is expanded (contracted) by a factor, the number of intersections increases (decreases) by the same factor since the weave consists of microscopic loops with a size of order $l_p$ whose centers are distributed randomly with an average number density of order $l_p^{-3}$.

First, if $M = 0$ and $e = 0$, that is, $\alpha = 1$, then $\Delta_t^a$ would be another flat weave obtained from $\Delta$ by just changing the orientations of the loop components regardless of the value of $b$ but fixed. The corresponding geometry is still flat up to the Planck scale structure, which is undetermined from classical metrics. This means that if we fix a surface, then the net change of the number of times the loops intersect it is zero in macroscopical sense, that is, the area of the surface does not change. Therefore, the transformations with $\alpha = 1$ keep the classical flat geometry unchanged although they change the Planck scale structure, to which the classical geometry is insensitive.

Next, consider cases of static black holes, $b = 0$ but $M \neq 0$ (the Schwarzschild metric for $e = 0$ and the Reissner-Nordstrom metric for $e \neq 0$). In these cases, due to the presence of non-unit $\alpha > 1$, the number of times the loops intersect a surface does change. However, the direction of expansion of each loop component is rotated to be transverse to the radial direction at the center of the loop component. In other words, $(X, Y, Z)$ agrees with $(x/r, y/r, z/r)$. Therefore, the total number of times the loop intersect a spherical surface does not change. Since $b = 0$ (static black holes), the horizon is a spherical surface at $r = r_+$; hence, the area of the horizon does not change from the value in the flat geometry, which is $4\pi r_+^2$. 

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The cases of rotating black holes, \( b \neq 0 \) and \( M \neq 0 \) (the Kerr metric for \( e = 0 \) and the Kerr-Newman metric for \( e \neq 0 \)), are different. The direction of the expansion of each loop component is rotated not to be transverse to the radial direction at the center of the loop component. In other words, \((X, Y, Z)\) does not agree with \((x/r, y/r, z/r)\). Furthermore, the horizon is not a spherical surface but an elliptical one in this coordinate system. Therefore, the number of times the loops intersect the horizon may change; hence, its area may change from the value at the same position in the flat geometry, which is

\[
4\pi(r_+^2 + b^2)[\frac{1}{2} + \frac{\sinh^{-1}(b/r_+)}{b/r_+}]\]

More precise computation of the area of the horizon is as follows. The area of an infinitesimal surface \( dS_a(x) \) at \( x \) by means of the black hole weave is

\[
dA = \left| l^2 \oint du \Delta^a_t(u) \delta^a(x, \Delta(u)) dS_a(x) \right| \tag{12}
\]

up to a multiplicative constant. This expression means that the area is the number of times the loops in the weave intersect the surface \( dS_a \). Then, by inserting Eq (10), rewrite it in terms of the flat weave

\[
dA = \left| l^2_p \oint du \Delta^b_t(u) \delta^3(x', \Delta(u)) t(t^{-1})^a_b dS_a(x) \right| , \tag{13}
\]

where \( x' \) is the position transformed from \( x \) by \( t(t^{-1})^a_b \). From the definition of the flat weave \( \Delta \), this is simply the area of the surface \( t(t^{-1})^a_b dS_a \) at \( x' \) in the flat space, namely

\[
dA = \left( \hat{h}^{cd}(x') t(t^{-1})^a_c dS_a(x) t(t^{-1})^b_d dS_b(x) \right)^{1/2} \tag{14}
\]

and hence it becomes

\[
dA = \left( \hat{q}^{ab}(x) dS_a(x) dS_b(x) \right)^{1/2} \tag{15}
\]

by definition of the transformation \( t^a_b \). Therefore, by integrating over the surface of the event horizon at \( r = r_+ \), we can find the area of it, namely \( A = 4\pi(r_+^2 + b^2) \). Here, we have recovered the area of the event horizon from the language of loops.

Note that in any case, \( \Delta_t \) coincides with the reoriented flat weave with \( b = 0 \) as \( \sqrt{x^2 + y^2 + z^2} \to \infty \). This fact implies that the black hole weaves are “asymptotically flat.”

4 Remarks

In this section we remark implications and applications of the black hole weaves constructed in this paper.

(i) The coordinate system we chosen is just a choice for convenience. Another appropriate choice of coordinates may give a different weave, which is supposed to be related to ours by a diffeomorphism up to Planck scale structures. Physical information invariant under diffeomorphisms must be extracted in an appropriate way in the loop representation quantum gravity.
(ii) The weaves themselves cannot determine causal structures and cannot distinguish black and white holes (the change of the alternative signs in $X$ and $Y$ is merely equivalent to a rotation of coordinate system about $x$- or $y$-axis by the amount of $\pi$ radians), which are determined by canonical conjugate variables. In the loop representation, the canonical conjugate information is coded in topological operations of loops [3]; moreover, in the quantum theory, the conjugate information is uncertain when the 3-metric is certain. Therefore, we have to understand the physical meaning of our construction of black holes in the language of the loop representation quantum gravity.

(iii) The construction of our weave is appropriate only where $q_{ab}$ is slowly varying with respect to $h_{ab}$, that is, where $t \equiv \alpha$ is close to the unity. Therefore, at small $r$ our weave may fail to approximate the corresponding 3-metric. In particular, at the singularity and the region $r \leq -M + \sqrt{M^2 + e^2 - b^2 \cos^2 \theta}$ we do not consider its validity.

(iv) Our construction of the black hole weaves is independent from the choice of a flat weave. That is, given a flat weave, one can determine the corresponding black hole weave with the three parameters $M$, $e$ and $b$. Therefore, the aspect of the metric the black hole weave approximates is the same aspect of the flat metric the flat weave approximates. By choosing a flat weave with a wider range of aspects of the metric, one can construct the corresponding black hole weave with the wider range of aspects of the black hole metric. This could be done by imposing more conditions on the flat weave by means of more operators one is interested in (i.e. the volume operator [3]).

(v) The weaves are constructed from classical metrics. Therefore, they do not have information about the Planck scale geometry at this stage. However, they may have a potential to have further structure at the Planck scale. In this sense, we expect that the weave might provide an aspect of “precise” picture of the Planck scale geometry, to which the classical picture of geometry is insensitive.

(vi) A graviton theory on the flat weave, which describes graviton physics in the flat metric space, has been constructed by means of the non-perturbative loop representation quantum gravity [12, 13]. It may be interesting to construct a graviton theory on the black hole weave constructed here, which describes a graviton physics in the corresponding black hole background metric, and study the Hawking effect, as suggested by Ashtekar [14].

(vii) Another possible application might be to study the black hole thermodynamics. Since the weave might provide an aspect of “precise” picture of the Planck scale geometry, to which classical picture of geometry is insensitive, if differential geometry of GR describes the black hole “thermodynamics,” then “loop geometry” of the loop representation might describe a black hole “statistical mechanics.”

The author thanks Carlo Rovelli for helpful comments and discussions.

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Figure 1: A loop component at \((x, y, z)\). For each \((x, y, z)\), there exist \(r\) and \((X, Y, Z)\) such that \((x, y, z) = r(X, Y, Z) + b(Y, -X, 0), \frac{x^2 + y^2}{R^2} + \frac{z^2}{r^2} = 1\) and \(X^2 + Y^2 + Z^2 = 1\), where \(R = (r^2 + b^2)^{1/2}\).