Quasi-local Energy in 3D Gravity with Torsion

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Abstract

We show that for generic stationary spacetime and a series of Killing fields, Wald’s approach for quasi-local energy could always be generalized to the first order formalism straightforwardly without introducing the Lorentz-Lie derivative. Via this approach, we derive the general formula for the black hole entropy in the three dimensional torsional Mielke-Baekler gravity, reproducing correctly the total energy, the angular momentum as well as the black hole entropy for the BTZ-like solutions.

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1 Introduction

For gravitational field, there is no well defined local energy density because of the equivalence principle, since at a point one can always find a local frame of reference in which there is no gravitational field [1]. The best work one could do alternatively is to define the so-called quasi-local energy. There are many distinct approaches [2], for example, the Komar integral [3], the Brown-York approach which is based on the Hamilton-Jacobi method [4], Nester et al.’s approach in covariant Hamiltonian formalism [5,6], and the covariant phase space method developed by Wald et al [7–10]. It was shown via Wald’s approach that the black hole entropy is the Noether charge of diffeomorphism invariance evaluated at the horizon. This approach was generalized to diffeomorphism covariant gravitational theories with Chern-Simons term by Tachikawa [11].

It was realized that a straightforward generalization of the Wald’s approach to the first order formalism of gravitational theory leads to apparent puzzle [12]. This issue is attributed to the fact that the vielbeins are in general not invariant under the flow of the Killing field. By introducing the Lorentz-Lie derivative [13–16], Jacobson and Mohd reformulated Wald’s approach and concluded that the black hole entropy is the horizon Lorentz-diffeomorphism Noether charge [12]. Following this idea, generalizations to Chern-Simons-like theories of gravity [17] are developed in [18–22].

We notice, however, that for generic axially symmetric stationary spacetime, it is natural to choose an orthogonal frame that is invariant under the flow of a series of special Killing vectors, which are responsible for the definition of the total energy, the angular momentum as well as the black hole entropy. In this case, Wald’s approach could be applied straightforwardly in the first order formalism without introducing the Lorentz-Lie derivative, and the generalization to the Chern-Simons-like theories are much simpler. As pointed out in [12], such vielbeins are singular at the horizon and lead to divergent spin connection on the bifurcation surface, which is crucial for giving rise to non-vanishing black hole entropy via Wald’s formalism in the first order formalism. We generalize this treatment to gravitational theory with torsion in three dimension, obtaining a novel and consistent approach for deriving the total energy, the angular momentum as well as the black hole entropy in the first order formalism for three dimensional torsional gravity.

We apply this approach to a special gravitational theory, i.e., the Mielke-Baekler (MB) model [23,24]. This is a topological gravity with torsion in three dimension which could be formulated as a gauge theory. It had been realized that gravity could be formulated as a gauge theory of the Poincaré group (rather than the Lorentz group), in which torsion is the gauge field strength associated with translations [25]. Unlike its cousin in four dimension, i.e., the Einstein-Cartan theory in which fermion matter is necessary for non-vanishing torsion on shell, the MB model has non-trivial vacuum solutions with torsion, i.e., the BTZ-like black holes. We derive the general
entropy formula for stationary black holes in the MB model (see eq. (81)), and the total energy, the angular momentum as well as black hole entropy for the BTZ-like solutions, which are consistent with the first law of black hole thermodynamics as well as previous results in literature via different methods.

The rest of this paper is organized as follows. Section 2 reviews Wald’s approach for quasi-local energy and its generalizations; Section 3 introduces the Mielke-Baekler model and its solutions. In Section 4 we show that Wald’s approach could be applied straightforwardly in the first order formalism for stationary spacetime and specific Killing vectors, then calculate the total energy, the angular momentum as well as the black hole for BTZ-like solutions with torsion in MB model. We conclude in Section 5 with a brief discussion.

2 Wald’s approach and beyond

For a diffeomorphism invariant gravitational theory defined by Lagrangian $n$-form $L$ ($n$ is the dimension of the spacetime), denoting all the dynamical fields by $\phi$, the variation of the Lagrangian due to $\delta \phi$ is given by

$$\delta L = E \delta \phi + d \Theta(\phi, \delta \phi),$$

where $E$ defines the equations of motion by $E = 0$. The $(n-1)$-form $\Theta$, constructed locally out of $\phi$, $\delta \phi$ and linear in $\delta \phi$, is called the “symplectic potential”. Defining the “symplectic current” $(n-1)$-form by anti-symmetrizing the variation of $\Theta$:

$$\Omega(\phi, \delta_1\phi, \delta_2\phi) = \delta_1 \Theta(\phi, \delta_2\phi) - \delta_2 \Theta(\phi, \delta_1\phi),$$

the integration of $\Omega$ over a Cauchy surface $\Sigma$ then defines a (pre-)symplectic form in the space of field configuration, which is identified as the phase space of the theory.

Consider the variation due to an infinitesimal diffeomorphism generated by a vector field $\xi$:

$$\delta_\xi \phi = L_\xi \phi,$$

the diffeomorphism invariance of $L$ implies the variation of the Lagrangian is

$$\delta_\xi L = L_\xi L = d i_\xi L,$$

which is a total derivative, showing that $\xi$ generates a symmetry. For each $\xi$ there is an associated Noether current $(n-1)$-form:

$$j_\xi = \Theta(\phi, L_\xi \phi) - i_\xi L,$$

one could easily check that

$$d j_\xi = - E L_\xi \phi,$$
hence \( j_\xi \) is closed on-shell, which implies
\[
j_\xi \approx dQ_\xi ,
\]
where \( Q_\xi \) is the Noether charge \((n - 2)\)-form.

The Hamiltonian \( H_\xi \), which generates the phase space flow corresponding to the
diffeomophism generated by \( \xi \), is related to the symplectic form through Hamilton’s
equation
\[
\delta H_\xi = \int_\Sigma \Omega(\phi, \delta \phi, L_\xi \phi),
\]
which turns out to be a boundary integral on shell:
\[
\delta H_\xi = \oint_{\partial \Sigma} \delta Q_\xi - i_\xi \Theta .
\]
If one could find a \((n - 1)\)-form \( B \) such that
\[
\delta \int_{\partial \Sigma} i_\xi B = \int_{\partial \Sigma} i_\xi \Theta ,
\]
the Hamiltonian \( H_\xi \) exists and is given by integrating (9):
\[
H_\xi = \oint_{\partial \Sigma} Q_\xi - i_\xi B .
\]

In this case \( H_\xi \) gives a natural definition of the quasi-local “energy” for the region
\( \Sigma \) with respect to the vector \( \xi \).

If \( \xi \) generates a symmetry of the dynamical fields, i.e. \( L_\xi \phi = 0 \), it follows from
(8) that \( \delta H_\xi = 0 \), hence from (9)
\[
\oint_{\partial \Sigma} \delta Q_\xi - i_\xi \Theta = 0 .
\]

Consider a stationary black hole with bifurcate Killing horizon. Let \( \zeta \) be the Killing
vector field which generates the Killing horizon and vanishes on the bifurcation
\((n - 2)\)-surface \( B \),
\[
\zeta = \tau + \Omega H \psi ,
\]
in which \( \tau \) and \( \psi \) are the Killing fields generating the asymptotic time translation
and the asymptotic rotation, respectively; \( \Omega H \) is the angular velocity of the horizon.
Define the total energy \( E \) and the angular momentum \( J \) of the spacetime:

\[
E \equiv H_\tau = \oint_{\infty} Q_\tau - i_\tau B ,
\]
\[
J \equiv -H_\psi = -\oint_{\infty} Q_\psi - i_\psi B
\]
Q_\zeta \text{ depends on } \zeta \text{ only algebraically through } \zeta \text{ and } \nabla \zeta \text{ since } \zeta \text{ is a Killing vector. Note that } \zeta |_B = 0 \text{ and }
abla_\mu \zeta^\nu |_B = \kappa_H n^\nu_\mu ,
\text{ in which } \kappa_H \text{ is the constant surface gravity of the Killing horizont and } n^{\mu
u} \text{ is the binormal to } B \text{ (normalized to } -2). \text{ Defining }
S = 2\pi \oint_B \hat{Q}_\zeta ,
\text{ where } \hat{Q}_\zeta \text{ is obtained from } Q_\zeta \text{ by replacing } \nabla_\mu \zeta^\nu \text{ with } n^{\mu
u} , \text{ (16) then gives rise to the first-law of black hole thermodynamics }
T_H \delta S = \delta \mathcal{E} - \Omega_H \delta J ,
\text{ in which } T_H = \kappa_H / 2\pi \text{ is the Hawking temperature. It is concluded that the black hole entropy } S \text{ is proportional to the Noether charge associated with the horizon-generating Killing vector [8,9].}

The above approach is constructed for diffeomorphism invariant theory. For gravity theory with Chern-Simons term, the action is diffeomorphism invariant only up to a surface term, i.e.,
\delta_\xi L = \mathcal{L}_\xi L + d\Xi_\xi .
\text{ In this case Wald’s approach was generalized by Tachikawa [11]. Both the Noether charge } Q_\xi \text{ and the Hamiltonian } H_\xi \text{ receive additional contribution coming from the surface term. By identifying the black hole entropy with the the horizon Noether charge plus additional modification terms, the first law is still satisfied.}

On the other hand, Wald’s approach is constructed in the second order formalism of gravitational theory, in which the appearance of the } \nabla \xi \text{ dependence in } Q_\xi \text{ is essential for nonzero entropy. If one tries to generalize Wald’s approach straightforwardly to the first order formalism with vielbeins and spin connections, it seems that the black hole entropy would be vanishing, since there will be no } \nabla \xi \text{ dependence in } Q_\xi . \text{ This puzzle was clarified by Jacobson and Mohd [12]. The crucial point is that for vielbeins satisfying}
\mathcal{L}_\xi e^a = 0 ,
\text{ the spin-connection } \omega \text{ would become divergent at the bifurcation surface } B , \text{ giving rise to finite value of } i_\xi \omega \text{ at } B , \text{ which appears in } Q_\xi \text{ and leads to nonzero black
hole entropy. In general, however, the Lie derivative of the vielbein with respect to a Killing vector $\xi$ is non-zero, since the vielbein might undergo a Lorentz transformation under the flow generated by $\xi$. In this case, one need to introduce the Lorentz-Lie (LL) derivative \[13{-16}\] instead which contains a compensating local Lorentz transformation. The LL derivative of $e^a$ with respect to a Killing vector is always vanishing, and one could reformulate Wald’s approach by replacing the Lie derivative with the LL derivative, with the conclusion that the black hole entropy is the horizon Noether charge for a combination of diffeomorphism and local Lorentz symmetry \[12\].

For Chern-Simons-like theories of gravity \[17\] in the first order formalism, the Lagrangian is invariant up to a surface term under the local Lorentz transformation, for which an approach incorporating the Lorentz-Lie derivative as well as the surface contribution was developed in \[18{-21}\], giving rise to a general formula for black hole entropy in such theories (see \[22\] for a comprehensive review).

3 Torsional gravity in three dimension

Before exploring Wald’s approach in the first order formalism for gravitational theories with torsion, we review briefly the torsional gravity in three dimension in this section.

In the first-order formalism of gravity, the independent fields are the vielbeins $e^a$ and the spin connections $\omega^{ab}$. The torsion 2-form and the curvature 2-form are defined by

\[ T^a = de^a + \omega^a_b \wedge e^b, \]
\[ R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b. \] (22)

In three dimension, it is convenient to introduce the dual spin connection $\omega^a$ and the dual curvature $R^a$ :

\[ \omega^a = \frac{1}{2} \epsilon^{abc} \omega^bc, \quad R^a = \frac{1}{2} \epsilon^{abc} R^{bc}. \] (24)

In local coordinates $x^\mu$, we have $e^a = e_\mu dx^\mu$, $\omega^a = \omega^a_\mu dx^\mu$ and the torsion 2-form $T^a = e_\lambda T^a_\mu dx^\mu \wedge dx^\nu$, where $T^a_\mu := \Gamma^a_\mu\nu$ is the torsion tensor. The affine connection $\Gamma^a_\mu\nu$ could be decomposed into

\[ \Gamma^a_\mu\nu = \tilde{\Gamma}^a_\mu\nu + K^a_\mu\nu, \] (25)

where $\tilde{\Gamma}^a_\mu\nu$ is the Levi-Civita connection and $K^a_\mu\nu$ is the contorsion tensor

\[ K^a_\mu\nu = T^a_\mu\nu + T^\lambda_\mu\nu + T^\lambda_\nu\mu. \] (26)
Defining the contorsion 1-form \( k^a_b = K^a_{\mu b} dx^\mu \) and the dual contorsion 1-form \( k^a = \frac{1}{2} \epsilon^a_{bc} k^{bc} \), one could verify the following identity:

\[
\omega^a = \tilde{\omega}^a + k^a , \tag{27}
\]

where \( \tilde{\omega}^a \) is the dual Riemannian spin connection defined by the torsion-free condition \( de^a + \epsilon^a_{bc} \tilde{\omega}^b \wedge e^c = 0 \). From (27) one also has

\[
T^a = \epsilon^a_{bc} k^b \wedge e^c . \tag{28}
\]

The Lagrangian of the Mielke-Baekler (MB) model [23,24] contains the Einstein-Cartan term, the cosmological constant term, the Chern-Simons term for the curvature, as well as a translational Chern-Simons term linear in torsion and veilbein:

\[
L = L_{EC} + L_\Lambda + L_{CS} + L_T + L_M , \tag{29}
\]

where

\[
L_{EC} = \frac{1}{\pi} e^a \wedge R_a ,
\]

\[
L_\Lambda = -\frac{\Lambda}{6\pi} \epsilon_{abc} e^a \wedge e^b \wedge e^c ,
\]

\[
L_{CS} = -\theta_L \left( \omega^a \wedge d\omega_a + \frac{1}{3} \epsilon_{abc} \omega^a \wedge \omega^b \wedge \omega^c \right) ,
\]

\[
L_T = \frac{\theta_T}{2\pi^2} e^a \wedge T_a , \tag{30}
\]

in which \( \Lambda \) is the cosmological constant and \( \theta_L, \theta_T \) are coupling constants, and \( L_M \) is the Lagrangian of the matter. From now on we will focus on the case with vanishing source.

Variation of the action (29)(30) with respect to \( e^a \) and \( \omega^a \) gives rise to the field equations:

\[
2\pi R_a + 2\theta_T T_a - \pi \Lambda \epsilon_{abc} e^b \wedge e^c = 0 , \tag{31}
\]

\[
2\pi T_a - 4\pi^2 \theta_L R_a + \theta_T \epsilon_{abc} e^b \wedge e^c = 0 . \tag{32}
\]

Assuming that \( 1 + 2\theta_T \theta_L \neq 0 \), the field equations are solved by

\[
T^a = \frac{T}{\pi} \epsilon^a_{bc} e^b \wedge e^c , \tag{33}
\]

\[
R^a = -\frac{\mathcal{R}}{2\pi^2} \epsilon^a_{bc} e^b \wedge e^c , \tag{34}
\]

in which

\[
T \equiv -\theta_T + 2\pi^2 \Lambda \theta_L , \quad \mathcal{R} \equiv -\frac{\theta_T^2 + \pi^2 \Lambda}{1 + 2\theta_T \theta_L} . \tag{35}
\]
Noting (28), eq. (33) is equivalent to
\[ k^a = \frac{T}{\pi} e^a. \] (36)

The equations (33) (34) have a family of BTZ-like solutions [26]. The vielbeins take the “diagonal” form of the BTZ black holes [27]:
\[ e^0 = N \, dt, \quad e^1 = \frac{dr}{N}, \quad e^2 = r \left( d\phi + N^\phi dt \right), \] (37)
in which\(^1\)
\[ N^2(r) = -M - \Lambda_{\text{eff}} r^2 + \frac{J^2}{4r^2}, \quad N^\phi(r) = -\frac{J}{2r^2} \] (38)
with
\[ \Lambda_{\text{eff}} \equiv -\frac{T^2 + \mathcal{R}}{\pi^2}. \] (39)

The dual spin connections, however, contain both the Riemannian part and the contorsion part, according to eq. (36):
\[ \omega^a = \tilde{\omega}^a + \frac{T}{\pi} e^a, \] (40)
where the Riemannian dual spin connections are determined totally by vielbeins (37) through the torsion-free condition:
\[ \tilde{\omega}^0 = N d\phi, \quad \tilde{\omega}^1 = -\frac{N^\phi}{N} dr, \quad \tilde{\omega}^2 = -\Lambda_{\text{eff}} r dt + r N^\phi d\phi. \] (41)

If \( \theta_L = \theta_T = 0 \), we have \( T = 0 \) and \( \Lambda_{\text{eff}} = \Lambda \), and the above solutions are reduced to the standard BTZ black holes.

The locations of the horizons are determined by \( N^2(r) = 0 \):
\[ r_+^2 = \frac{1}{2\Lambda_{\text{eff}}} \left( -M + \sqrt{M^2 + \Lambda_{\text{eff}} J^2} \right) \] (42)
(note that \( \Lambda_{\text{eff}} < 0 \) for asymptotic AdS solutions), one could verify that
\[ r_+ r_- = \frac{J}{2\sqrt{-\Lambda_{\text{eff}}}}. \] (43)

The angular velocity of the outer horizon is given by
\[ \Omega_H = -N^\phi(r_+) = \frac{J}{2r_+^2}. \] (44)

Vanishing of the conical singularity of the Euclidean BTZ metric gives the black hole temperature
\[ T_H = \frac{\Lambda_{\text{eff}} \left( r_+^2 - r_-^2 \right)}{2\pi r_+}, \] (45)
and the surface gravity is \( \kappa_H = 2\pi T_H \).

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\(^1\)The convention in [26] is related to ours by the following replacement: \( \chi \to 1 \), the gravitational constant \( \ell \to \pi \), the effective cosmological constant \( \Lambda_{\text{eff}} \to -\Lambda_{\text{eff}} \).
4 Quasi-local energy in 3D torsional gravity

In this section we investigate the quasi-local energy in the MB model, beginning with revisiting Wald’s approach in the first order formalism.

4.1 Wald’s approach in first order formalism revisited

As reviewed in Section 2, in general the Lie derivative of the vielbein with respect to a Killing vector field is non-zero, hence the Lorentz-Lie derivative is needed to reformulate Wald’s approach. We notice, however, that for a large amount of geometries, it is not only possible but also natural to choose vielbeins which are invariant under the flow of some special Killing vectors. Consider generic axially symmetric stationary spacetime in three dimension:

$$\begin{align*}
ds^2 &= -A(r)^2N(r)^2dt^2 + \frac{dr^2}{B(r)^2N(r)^2} + C(r)^2 \left(d\phi + N^\phi(r)dt\right)^2,
\end{align*}$$

(46)

it is natural to choose the “diagonal” vielbeins:

$$\begin{align*}
e^0 &= A(r)N(r)dt, & e^1 &= \frac{dr}{B(r)N(r)}, & e^2 &= C(r)\left(d\phi + N^\phi(r)dt\right),
\end{align*}$$

(47)

Note that $e^a = e^a(r)$. If one consider the Killing vector

$$\xi = a\partial_t + b\partial_\phi,$$

(48)

with $a$, $b$ constants, it follows immediately that

$$\mathcal{L}_\xi e^a = 0$$

(49)

since $e^a$ do not depend on $t, \phi$. The Riemannian dual spin connections $\tilde{\omega}^a$ determined completely by the vielbein do not depend on $t, \phi$, either. If one assume further that

$$k^a = k^a(r),$$

(50)

which is natural due to the spacetime symmetry, the Lie-derivative of the dual spin connection $\omega^a$ with respect to $\xi$ also become vanishing

$$\mathcal{L}_\xi \omega^a = 0,$$

(51)

hence Wald’s original approach would still be valid in the first order formalism for diffeomorphism invariant gravity theories; there’s no need to introduce Lorentz-Lie derivative in such cases.

The only possible problem, as pointed out in [12] for Einstein gravity, is that the spin connection might become divergent at the the horizon, hence need to be
treated carefully. Below we give an argument generalizing that in \cite{12} to incorporate torsion. From (49) we have

\[ 0 = \mathcal{L}_\xi e^a = i_\xi de^a + di_\xi e^a = i_\xi De^a + D(i_\xi e^a) - (i_\xi \omega^a_b) \land e^b, \tag{52} \]

in which D is the Lorentz covariant exterior derivative when acting on p-forms:

\[ De^a = de^a + \omega^a_b \land e^b = T^a. \tag{53} \]

Specifying the action of D on tensor indices to be the covariant derivative \( \nabla \) (defined in terms of connection with torsion), we interpret D to be the full derivative. The tetrad postulate states that

\[ D_{\mu}e^a_{\nu} = \partial_{\mu}e^a_{\nu} - \Gamma^\lambda_{\mu\nu}e^a_{\lambda} + \omega^a_{b\mu}e^b_{\nu} = 0. \tag{54} \]

Substituting (53) (54) into (52), we have

\[ i_\xi \omega^a_b = e^a_{\mu} (i_\xi T^a)_\mu + e^a_{\mu} e^b_{\nu} \nabla_{\mu} \xi^\nu. \tag{55} \]

Noticing that

\[ \frac{1}{2} \epsilon^{\mu\nu}_{bc} e_{\mu} e_{\nu} \nabla_{\mu} \xi^\nu = \frac{1}{2} \epsilon^{\mu\nu}_{bc} e_{\mu} e_{\nu} \tilde{\nabla}_{\mu} \xi^\nu - \frac{1}{2} e_{\mu} e_{\nu} (i_\xi T^b)_\mu + i_\xi k^a, \tag{56} \]

in which \( \tilde{\nabla} \) is defined in terms of the Levi-Civita connection, we finnaly obtain

\[ i_\xi \omega^a = \frac{1}{2} \epsilon^{\mu\nu}_{bc} e_{\mu} e_{\nu} \tilde{\nabla}_{\mu} \xi^\nu + i_\xi k^a. \tag{57} \]

If we consider a Killing field \( \zeta \) which generates a Killing horizon with surface gravity \( \kappa_H \), noticing (17) we obtain

\[ i_\zeta \omega^a \vert_{B} = -\frac{1}{2} \kappa_H \epsilon^{\mu\nu}_{bc} n^b_{\mu} + i_\zeta k^a \vert_{B}. \tag{58} \]

Assuming that \( i_\zeta k^a \) is also finite on \( B \), the limit of \( i_\zeta \omega^a \) turns out to have finite value on the bifurcation surface, indicating the divergence of the spin connection on \( B \).

To summarize, one could still adopt Wald’s original approach in the first order formalism for diffeomorphism invariant gravitational theory, as long as the vielbeins are of the form (47) and the Killing fields are of the form (48), just paying attention to the relation (58).

In particular, the above argument is valid for the total energy, the angular momentum as well as the black hole entropy of the BTZ-like black holes (37)-(41) in the MB model. In the next subsection, we derive the quasi-local energy for the BTZ-like solutions in the first-order formalism by applying Wald’s original approach straightforwardly.
4.2 Quasi-local energy

The Lagrangian of the MB model (29)(30) is manifestly invariant under the diffeomorphism, while is invariant only up to a surface term under the local Lorentz transformation due to the presence of the Chern-Simons term. However, as explained in the previous subsection, if one just considers the diffeomorphism generated by Killing vectors $\xi$ of the form (48), the natural vielbeins (47) of generic stationary spacetime are always invariant under the flow of $\xi$, hence undergoes no Lorentz transformation. As a result, there’s no need to deal with the surface contribution, and the original formulation of Wald’s approach applies straightforwardly.

The dynamical fields in the Lagrangian (29)(30) are the vielbeins $e^a$ and the dual spin connections $\omega^a$, variation of the Lagrangian with respect to these fields is given by

$$\delta L = \delta e^a \wedge E_a^{(e)} + \delta \omega^a \wedge E_a^{(\omega)} + d\Theta(\phi, \delta \phi),$$

where

$$\Theta(\phi, \delta \phi) = \frac{1}{\pi} \delta \omega^a \wedge e_a + \frac{\theta_T}{2\pi^2} \delta e^a \wedge e_a - \theta_L \delta \omega^a \wedge \omega_a,$$

$$E_a^{(e)} = \frac{1}{\pi} \left( R_a + \frac{\theta_T}{\pi} T_a - \frac{\Lambda}{2} \epsilon_{abc} e^b \wedge e^c \right),$$

$$E_a^{(\omega)} = \frac{1}{\pi} \left( T_a - 2\pi \theta_L R_a + \frac{\theta_T}{2\pi} \epsilon_{abc} e^b \wedge e^c \right).$$

$E_a^{(e)} = 0$ and $E_a^{(\omega)} = 0$ give rise to the equations of motion (31)(32). The diffeomorphism Noether current 3-form (5) turns out to be

$$j_\xi = dQ_\xi + C_\xi,$$

in which

$$Q_\xi = \frac{1}{\pi} (i \xi \omega^a) \wedge e_a + \frac{\theta_T}{2\pi^2} (i \xi e^a) \wedge e_a - \theta_L (i \xi \omega^a) \wedge \omega_a,$$

$$C_\xi = - (i \xi e^a) \wedge E_a^{(e)} - (i \xi \omega^a) \wedge E_a^{(\omega)}.$$  

Clearly the Noether current $j_\xi$ is exact on shell. The variation of quasi-local energy, according to (9), is given by

$$\delta H_\xi = \oint_{\partial \Sigma} \left\{ \frac{1}{\pi} \left[ (i \xi e^a) \wedge \delta \omega_a + (i \xi \omega^a) \wedge \delta e_a \right] + \frac{\theta_T}{\pi^2} (i \xi e^a) \wedge \delta e_a - 2\theta_L (i \xi \omega^a) \wedge \delta \omega_a \right\}.$$  

The integrability of (66) will be shown for specific cases below.
4.2.1 Energy and angular momentum

The variation of the total energy and the angular momentum of the BTZ-like solution (37)-(41) are obtained by taking $\xi$ to be $\partial_t$ and $\partial_\phi$ in the expression (66) respectively and integrating over the spatial infinity. By perturbing the parameters

$$M \to M + \delta M, \quad J \to J + \delta J,$$

the variations turn out to be

$$\delta \mathcal{E} \equiv \delta H_{\partial_t}|_\infty = \delta M - 2\theta_L (T \delta M + \pi \Lambda_{\text{eff}} \delta J),$$

$$\delta \mathcal{J} \equiv -\delta H_{\partial_\phi}|_\infty = \delta J + 2\theta_L (\pi \delta M - T \delta J),$$

Assuming that the pure AdS$_3$ is the ground state, i.e., $\mathcal{E} = \mathcal{J} = 0$ for $M = J = 0$, the total energy and the angular momentum are obtained by integrating (68)(69):

$$\mathcal{E} = M - 2\theta_L (T M + \pi \Lambda_{\text{eff}} J),$$

$$\mathcal{J} = J + 2\theta_L (\pi M - T J),$$

which is consistent with the previous results in the literature obtained via different approaches [26, 28].

4.2.2 Black hole entropy

To derive the black hole entropy, we take the Killing vector to be

$$\zeta = \partial_t + \Omega_H \partial_\phi$$

in (66) and integrate over the bifurcation surface $\mathcal{B}$ at $r = r_+$. Note that for a generic axially symmetric stationary black hole of the form (47), the location of the horizon $r_+$ as well as the angular velocity of the horizon $\Omega_H$ are determined by

$$N(r_+) = 0, \quad \Omega_H = -N^\phi(r_+)$$

separately, from which it is straightforward to check that

$$i_\zeta e^a|_\mathcal{B} = 0.$$

Since the dual contorsion 1-form $k^a$ is proportional to $e^a$ according to the equation of motion (36), one also has

$$i_\zeta k^a|_\mathcal{B} = 0.$$

As analyzed in Subsection 4.1, the vanishing of the Lie derivative of $e^a$ with respect to Killing vector $\zeta$ indicates (58); noticing (75) we have

$$i_\zeta \omega^a|_\mathcal{B} = -\frac{1}{2} \kappa_H e^a_{bc} n^{bc}.$$
For simplicity we introduce
\[ N^a \equiv \frac{1}{2} \varepsilon^{a}_{bc} n^{bc}, \tag{77} \]
hence
\[ i \zeta \omega^a |_B = - \kappa_H N^a. \tag{78} \]
Noticing (74)(78), the variation of the quasi-local energy (66) with respect to \( \zeta \) turns out to be
\[ \delta H_\zeta = - \kappa_H \oint_B N^a \left( \frac{1}{\pi} \delta e_a - 2 \theta_L \delta \omega_a \right). \tag{79} \]
As \( N^a \) is just a constant vector on \( B \), the above expression is obviously integrable and gives rise to
\[ H_\zeta = - \kappa_H \oint_B N^a \left( \frac{1}{\pi} e_a - 2 \theta_L \omega_a \right), \tag{80} \]
Since the only non-zero components of the binormal to the bifurcation surface \( B \) is \( n_{01} = - n_{10} = 1 \), the only non-zero component of \( N^a \) is just \( N^2 = -1 \).
Motivated by (18), we define the black hole entropy by replacing \( H_\zeta \) with \( H_\zeta / \kappa_H \) and multiplying a factor \( 2\pi \), obtaining a general formula for the entropy of rotating black holes in the MB model:
\[ S = \frac{2\pi}{\kappa_H} H_\zeta = 2\pi \oint_B \left( \frac{1}{\pi} e^2 - 2 \theta_L \omega^2 \right), \tag{81} \]
which contains explicit contribution coming from the Chern-Simons action. The first law of black hole thermodynamics
\[ T_\mu \delta S = \delta M - \Omega_\mu \delta J \tag{82} \]
is satisfied by construction.

By submitting explicitly the solution (37)-(41) into our entropy formula (81) and noticing (43), we obtain the black hole entropy for the BTZ-like solutions in the MB model:
\[ S = 4\pi r_+ - 8\pi \theta_L \left( T r_+ - \frac{\pi}{\ell_{\text{eff}}^{-2}} r_- \right), \tag{83} \]
in which we have introduced an effective AdS radius \( \ell_{\text{eff}}^{-2} \equiv - \Lambda_{\text{eff}} \). This is consistent with previous result in [28] via the Euclidean partition function approach \(^2\), as well as that in [29] via a direct integration from the first law. Our entropy formula (81) is also consistent with the generic expression for black hole entropy in Chern-Simons-like theories in Section XVI of [22] via a different approach employing the Lorentz-Lie derivative.

Before conclusion, we give a brief discussion on the torsional effect. The torsion of the BTZ-like solution (37)-(41) is characterized by the parameter \( T \), which originates from the non-vanishing coupling \( \theta_T, \theta_L \) according to (35). It is explicit that
\(^2\)The convention in [28] is related to ours by replacing \( G \to 1/8 \), \( \alpha_3 \to - \theta_L \), \( p \to 2T / \pi \) and \( \ell \to \ell_{\text{eff}} \).
the total energy (70), the angular momentum (71) as well as the black hole entropy (83) contains direct contribution coming from the torsion; on the other hand, for solutions with vanishing torsion, i.e., $\mathcal{T} = 0$, the conserved charges as well as the black hole entropy still differ from those in Einstein gravity, due to the Chern-Simons coupling $\theta_L$. In general, the quasi-local energy (66) receives corrections from both the $\theta_T$-term and the $\theta_L$-term; however, it turns out that only the $\theta_L$-term gives non-vanishing contribution to the total energy, the angular momentum as well as the black hole entropy.

5 Conclusion and discussion

In this paper, we showed that for axially symmetric stationary spacetime and specific Killing vectors, Wald’s approach could be applied straightforwardly in the first order formalism. In this method we studied the quasi-local energy in three dimensional torsional MB model, derived the general entropy formula for stationary black holes, reproduced correctly the total energy, the angular momentum as well as the black hole entropy for the BTZ-like solutions.

It should be pointed out, however, that if one tries to consider more general Killing fields other than (48), the Lie derivative of the vielbeins might not be vanishing, and one needs to introduce the Lorentz-Lie derivative in such cases. One example is the Killing vector that vanishes on the boundary of the entanglement wedge in AdS spacetime, it has been demonstrated in [30,31] that its corresponding quasi-local energy in Einstein gravity is equivalent to the relative entropy in the dual boundary CFT. This argument was generalized to Einstein-Cartan gravity in [32,33]. It would be interesting to investigate if there exists similar relationship for three dimensional torsional MB model [34].

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