The generalized Racah algebra as a commutant

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Abstract. The Racah algebra $R(n)$ of rank $(n-2)$ is obtained as the commutant of the $\mathfrak{o}(2)^{\otimes n}$ subalgebra of $\mathfrak{o}(2n)$ in oscillator representations of the universal algebra of $\mathfrak{o}(2n)$. This result is shown to be related in a Howe duality context to the definition of $R(n)$ as the algebra of Casimir operators arising in recouplings of $n$ copies of $\mathfrak{su}(1,1)$. These observations provide a natural framework to carry out the derivation by dimensional reduction of the generic superintegrable model on the $(n-1)$ sphere which is invariant under $R(n)$.

1. Introduction

The Racah algebra $R(3)$ of rank 1 [1, 2] encodes the bispectrality properties of the Racah polynomials [3] and is the symmetry algebra of the generic superintegrable model on the 2-sphere with Hamiltonian $H$ given by [4]

$$H = \sum_{1 \leq i < j \leq 3} J_{ij}^2 + \sum_{i=1}^{3} \frac{a_i}{x_i^2}$$  \hspace{1cm} (1)

where

$$J_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad x_1^2 + x_2^2 + x_3^2 = 1$$  \hspace{1cm} (2)

and $a_1, a_2, a_3$ are parameters. For a review the reader is referred to [5]. Of particular relevance is the fact that $R(3)$ was seen to be the commutant in $\mathcal{U}(\mathfrak{su}(1,1)^{\otimes 3})$ of the embedding of $\mathfrak{su}(1,1)$.
in the three-fold tensor product of this algebra with itself, or in other words, that it is generated by the invariant operators arising in this Racah problem. This observation provided a way to generalize $R(3)$ to Racah algebras of arbitrary rank $(n - 2)$ [6] by extending the picture to $n$ factors and identifying the structure relations between the various Casimir operators arising in the possible recouplings. It follows that $R(n)$ thus defined is the symmetry algebra of the superintegrable model on the $(n - 1)$-sphere obtained by straightforwardly extending to $n$ variables the model on $S^2$ defined above.

We have found recently [7] that $R(3)$ can be realized as the commutant of the subalgebra $\mathfrak{o}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{o}(2) \subset \mathfrak{o}(6)$ in oscillator representations of the enveloping algebra of $\mathfrak{o}(6)$. We further observed that this description of $R(3)$ could be related to the one associated to the Racah problem for $\mathfrak{su}(1,1)$ through the Howe duality corresponding to the pair $(\mathfrak{o}(6), \mathfrak{su}(1,1))$. This provided a natural background for obtaining the superintegrable Hamiltonian [11] with $R(3)$ as symmetry algebra, under the dimensional reduction of a six-dimensional harmonic oscillator problem. We here wish to indicate how these results extend for $R(n)$, that is, for arbitrary ranks and dimensions.

The paper is structured as follows. In Section 2 we review how the Racah algebra $R(n)$ is defined as the algebra of the Casimir operators in the $n$-fold tensor product of $\mathfrak{su}(1,1)$ Lie algebras. The structure relations satisfied by these Casimirs are provided. In Section 3 we show that the generators of the commutant of the $\mathfrak{o}(2)^{n^2}$ subalgebra of $\mathfrak{o}(2n)$ satisfy the defining relations of $R(n)$. In Section 4 we invoke Howe duality to explain how the pairings between representations of $\mathfrak{o}(2n)$ and those of $\mathfrak{su}(1,1)$ underpin the connection between the tensorial and the commutant pictures of $R(n)$. How the $R(n)$-invariant superintegrable model on $S^{(n-1)}$ is obtained from an $n$-dimensional harmonic oscillator by modding out the action of the $n$-torus group is described in Section 5 and conclusions form Section 6.

2. The generalized Racah algebra and tensor products of $\mathfrak{su}(1,1)$

Let us recall how the generalized Racah algebra $R(n)$ is defined from the $n$-fold tensor product of $\mathfrak{su}(1,1)$. The $\mathfrak{su}(1,1)$ algebra has 3 generators, $J_0$, $J_\pm$ obeying the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0.$$  (3)

The Casimir element is given by

$$C = J_0^2 - J_+ J_- - J_0.$$  (4)

Let $[n] = \{1, 2, \ldots, n\}$ denote the set of the $n$ first integers and consider the tensor product $\mathfrak{su}(1,1)^{\otimes n}$. Coproduct embeddings of $\mathfrak{su}(1,1)$ in $\mathfrak{su}(1,1)^{\otimes n}$ are labelled by subsets $A \subset [n]$ with the generators mapped to

$$J^A = \sum_{i \in A} J^i,$$  (5)

and where the superindex denotes on which factor of $\mathfrak{su}(1,1)^{\otimes n}$ the operator $J^i$ is acting. Correspondingly, the Casimirs are sent to

$$C^A = (J_0^A)^2 - J_+^A J_-^A - J_0^A.$$  (6)

The generalized Racah algebra $R(n)$ is taken to be the algebra generated by all these intermediate Casimirs $C^A$ since this is the case for $R(3)$.

It is important to note that not all intermediate Casimirs $C^A$ are independent; indeed one has

$$C^A = \sum_{\{i,j\} \subset A} C^{ij} - (|A| - 2) \sum_{i \in A} C^i.$$  (7)
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where \(|A|\) stands for the cardinality of \(A\). In order to characterize \(R(n)\), given that the elements \(C^i\) are central, it therefore suffices to provide all the iterated commutators between the \(C^{ij}\)'s with \(i \neq j\) until closure is achieved. This has been carried out in [6]. It is convenient to introduce \(P^{ij}\) and \(F^{ijk}\):

\[
P^{ij} = C^{ij} - C^i - C^j, \quad F^{ijk} = \frac{1}{2}[P^{ij}, P^{jk}].
\] (8)

The defining relations of the Racah algebra \(R(n)\) then read

\[
[P^{ij}, P^{jk}] = 2F^{ijk}, \quad (9a)
\]

\[
[P^{jk}, F^{ijk}] = P^{ik}P^{jl} - P^{il}P^{jk} - 2P^{ik}C^j - 2P^{ij}C^k,
\] (9b)

\[
[F^{ijk}, F^{jkl}] = F^{ilm}P^{ik} - F^{ilm}P^{jk} - F^{iql}(P^{jl} + 2C^j) - F^{ijl}P^{jl},
\] (9d)

\[
[F^{ijk}, F^{klm}] = F^{ilm}P^{jk} - F^{ilm}P^{ik}.
\] (9e)

where \(i, j, k, l, m \in [n]\) are all different.

In the rank 1 case, (9c), (9d) and (9e) are redundant and the standard Racah algebra \(R(3)\) is fully described by (9a) and (9b). Note that the presentation that results from the specialization of these equations to \(n = 3\) is the equitable one. The relation between this presentation and the standard one used in [7] is given explicitly in [8]. The rank 2 Racah algebra (which has been studied in detail in [9]) only requires (9a)-(9c) to be characterized, while the relations (9d) and (9e) have to be added in order to define Racah algebras of rank 3 or higher.

3. The generalized Racah algebra and \(\mathfrak{o}(2n)\)

Let us now indicate how the relations (9) given above are satisfied by the generators in \(U(\mathfrak{o}(2n))\) of the commutant of \(n\) copies of \(\mathfrak{o}(2)\) sitting in \(\mathfrak{o}(2n)\). The algebra \(\mathfrak{o}(2n)\) has \(n(2n-1)\) generators \(L_{\mu\nu} = -L_{\nu\mu}, \ \mu, \nu = 1, \ldots, 2n\) obeying

\[
[L_{\mu\nu}, L_{\rho\sigma}] = \delta_{\nu\rho}L_{\mu\sigma} - \delta_{\nu\sigma}L_{\mu\rho} - \delta_{\mu\rho}L_{\nu\sigma} + \delta_{\mu\sigma}L_{\nu\rho},
\] (10)

and possesses the following quadratic Casimir:

\[
C = \sum_{1 \leq \mu < \nu \leq n} L_{\mu\nu}^2.
\] (11)

We will use the realization

\[
L_{\mu\nu} = \xi_\mu \frac{\partial}{\partial \xi_\nu} - \xi_\nu \frac{\partial}{\partial \xi_\mu}, \quad \mu \neq \nu, \quad \mu, \nu = 1, \ldots, 2n.
\] (12)

Pick the \(\mathfrak{o}(2)^{\oplus n}\) subalgebra of \(\mathfrak{o}(2n)\) generated by the commutative set \(\{L_{12}, L_{34}, \ldots, L_{2n-1,2n}\}\). We want to focus on the commutant in \(U(\mathfrak{o}(2n))\) of this Abelian algebra.

It is easy to see that the set of invariants \(\{G^i, K^{ij}\}_{1 \leq i < j \leq n}\)

\[
G^i = L_{2i-1,2i}^2, \quad K^{ij} = L_{2i-1,2j}^2 + L_{2i-1,2j-1}^2 + L_{2i-2,2j-1}^2 + L_{2i-2,2j}^2 + L_{2i-2,2j-1}^2 + L_{2j-1,2j}^2 + L_{2j-1,2j-1}^2,
\] (13)

(14)
is sufficient to generate this commutant and it happens to be the generalized Racah algebra. Indeed, with the following redefinitions

\[ C^i = -\frac{1}{4} G^i + \frac{1}{4}, \]

\[ C^{ij} = -\frac{1}{4} K^{ij}, \]

\[ P^{ij} = -\frac{1}{4} K^{ij} + \frac{1}{4} (G^i + G^j) + \frac{1}{2}, \]

\[ F^{ijk} = \frac{1}{32} [K^{ij}, K^{jk}], \]

a long but straightforward calculation in the realization (12) shows that the defining relations (9) of the algebra \( R(n) \) are obeyed.

4. The \( \mathfrak{su}(1,1) \) and \( \mathfrak{o}(2n) \) descriptions of \( R(n) \) and Howe duality

In the last two sections we indicated that the generalized Racah algebra \( R(n) \) is the commutant of \( \mathfrak{su}(1,1) \) in \( \mathcal{U}(\mathfrak{su}(1,1)^{\otimes n}) \) and of \( \mathfrak{o}(2)^{\otimes n} \) in oscillator representations of \( \mathcal{U}(\mathfrak{o}(2n)) \). The connection between these two descriptions is rooted in Howe duality.

It is known [10, 11, 12, 13] that \( \mathfrak{o}(2n) \) and \( \mathfrak{sp}(2) \) form a dual pair in \( \mathfrak{sp}(4n) \), with these two subalgebras being their mutual commutants. This implies that \( \mathfrak{o}(2n) \) and \( \mathfrak{sp}(2) \simeq \mathfrak{su}(1,1) \) have dual actions on the Hilbert space of \( 2n \) oscillator states. That means that their irreducible representations can be paired and this can be done through the Casimirs in the following way.

We first add these \( 2n \) representations by coupling them pairwise

\[ J^{(\mu;\nu)} = J^{(\mu)} + J^{(\nu)}. \]

In what follows, we will always assume that the pairs denoted \( (\mu;\nu) \) are such that \( (\mu;\nu) = (2i-1;2i), \ i = 1, \ldots, n \). Now take \( A \subset [n] \) to be any subset that is the union of \( N \) such pairs:

\[ A = \bigcup_{i=1}^{N} \{j_{i};v_{i}\}, \]

with \( |A| = 2N \) and \( 1 \leq N \leq n \). The \( \mathfrak{su}(1,1) \) realization associated to such a subset \( A \) reads

\[ J^A_+ = \frac{1}{2} \sum_{\mu \in A} \xi^2_{\mu}, \quad J^A_- = \frac{1}{2} \sum_{\mu \in A} \frac{\partial^2}{\partial \xi^2_{\mu}}, \quad J^A_0 = \frac{1}{2} \left( \frac{|A|}{2} + \sum_{\mu \in A} \xi_{\mu} \frac{\partial}{\partial \xi_{\mu}} \right). \]

It is then straightforward to show that the Casimir for an embedding labelled by the subset \( A \) is given by

\[ C^A = (J^A_0)^2 - J^A_+ J^A_- - J^A_0 = \frac{|A|(|A| - 4)}{16} - \sum_{\mu < \nu \atop \mu, \nu \in A} \frac{(L_{\mu\nu})^2}{4}. \]
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As already noted, not all \( C^A \)'s are independent. The translation of (24) shows that all \( C^A \)'s can be rewritten as

\[
C^A = \sum_{(\mu;\nu),(\rho;\sigma) \in A} C^{(\mu;\nu),(\rho;\sigma)} - \frac{|A| - 4}{2} \sum_{(\mu;\nu) \in A} C^{(\mu;\nu)}
\]  
(24)

with

\[
C^{(\mu;\nu),(\rho;\sigma)} = -\frac{1}{4} (L_{\mu\nu}^2 + 1), \quad C^{(\mu;\nu),(\rho;\sigma)} = -\frac{1}{4} (L_{\mu\nu}^2 + L_{\mu\rho}^2 + L_{\nu\rho}^2 + L_{\nu\sigma}^2 + L_{\rho\sigma}^2).
\]  
(25)

This shows that all higher order Casimirs can be reexpressed in terms of those of lowest orders.

We thus observe that the intermediate \( \mathfrak{sp}(2) \) Casimirs correspond (up to an affine transformation) to the generators of the commutant of \( \{L_{1,2}, \ldots, L_{2n-1,2n}\} \) in \( \mathcal{U}(\mathfrak{o}(2n)) \). We know from Section 3 that the intermediate \( \mathfrak{sp}(2) \) Casimirs realize the commutation relations of the generalized Racah algebra. This will hence be the case also for the commutant generators and we have here our duality connection.

5. The generalized Racah algebra and the generic superintegrable model on \( S^{n-1} \)

We can now complete the picture by performing the dimensional reduction from \( \mathbb{R}^{2n} \) to \( \mathbb{R}^+ \times S^{n-1} \) to obtain the generic superintegrable model with Hamiltonian \( H \) (introduced in Section 1) and to recover its symmetries. Starting from the oscillator representation (19), make the following change of variables:

\[
\xi_{2i-1} = x_i \cos \theta_i, \quad \xi_{2i} = x_i \sin \theta_i, \quad L_{2i-1,2i} = \xi_{2i-1} \frac{\partial}{\partial \xi_{2i-1}} - \xi_{2i} \frac{\partial}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}, \quad i = 1, \ldots, n.
\]  
(26)

Eliminate the ignorable \( \theta_i \)'s by separating these variables and setting \( L_{2i-1,2i}^2 \sim k_i^2 \). After performing the gauge transformation \( O \mapsto \tilde{O} = x_1^{1/2} O x_i^{-1/2} \) one obtains the reduced system

\[
\tilde{J}_+^{(2i-1,2i)} = \frac{1}{2} x_i^2, \quad \tilde{J}_-^{(2i-1,2i)} = \frac{1}{2} \left( \frac{\partial^2}{\partial x_i^2} + \frac{a_i}{x_i^2} \right), \quad \tilde{J}_0^{(2i-1,2i)} = \frac{1}{2} \left( x_i \frac{\partial}{\partial x_i} + \frac{1}{2} \right),
\]  
(27)

with \( a_i = k_i^2 + \frac{1}{4} \). Defining \( \tilde{J}_i \equiv \tilde{J}^{(2i-1,2i)} \), the reduced Casimirs

\[
\tilde{C}_i = (\tilde{J}_0)^2 - \tilde{J}_+ \tilde{J}_- - \tilde{J}_0^2,
\]  
(28)

\[
\tilde{C}_{ij} = (\tilde{J}_0 + \tilde{J}_j)^2 - (\tilde{J}_+ + \tilde{J}_-)(\tilde{J}_+ + \tilde{J}_-) - (\tilde{J}_0 + \tilde{J}_0^2),
\]  
(29)

are easily computed and have the following expressions:

\[
\tilde{C}_i = -\frac{1}{4} \left( a_i + \frac{3}{4} \right), \quad \tilde{C}_{ij} = -\frac{1}{4} \left[ J_{ij}^2 + a_i x_i^2 + a_j x_j^2 + a_i + a_j + 1 \right], \quad J_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad i < j.
\]  
(30)

Using the fact that \( \tilde{J}^{[n]} = \sum_{i=1}^n \tilde{J}_i \), the total Casimir \( \tilde{C}^{[n]} \) is obtained:

\[
\tilde{C}^{[n]} = -\frac{1}{4} \sum_{1 \leq i < j \leq n} J_{ij}^2 - \frac{1}{4} \left( \sum_{i=1}^n x_i^2 \right) \sum_{j=1}^n a_j + \frac{n(n-4)}{16},
\]  
(31)
and assuming $\sum_{i=1}^{n} x_i^2 = 1$, one thereby obtains the Hamiltonian of the generic model on $S^{n-1}$ (up to an affine transformation). The basic intermediate Casimirs are essentially the conserved quantities:

$$Q_{ij} = J_{ij}^2 + a_i x_i^2 x_j + a_j x_j^2 x_i, \quad 1 \leq i < j \leq n$$

and they generate $R(n)$ which is hence the symmetry algebra of the superintegrable model on the $(n-1)$ sphere. (Note that the $Q_{ij}$’s are affinely related to the $P_{ij}$’s in the relations (32).)

6. Conclusion

Summing up, we have shown that the generalized Racah algebra $R(n)$ can be defined as the commutant of the $\mathfrak{o}(2) \oplus n$ subalgebra of $\mathfrak{o}(2n)$ in oscillator representations of $\mathcal{U}(\mathfrak{o}(2n))$. This offers an alternative to the definition of $R(n)$ as the algebra of the intermediate Casimirs associated to the $\mathfrak{su}(1, 1)$ embeddings in $\mathfrak{su}(1, 1) \otimes n$. We have related these two pictures in the context of Howe duality and obtained the generic $R(n)$-invariant superintegrable model on $S^{n-1}$ through the dimensional reduction scheme stemming from the analysis. This has provided a generalization to arbitrary ranks and dimensions of the study carried in [7] for the standard Racah algebra.

We wish to remark that since $\mathfrak{o}(nd)$ and $\mathfrak{sp}(2)$ form a dual pair in $\mathfrak{sp}(2nd)$, it is also possible to realize the generalized Racah algebra as the commutant of the $\mathfrak{o}(d) \oplus n$ subalgebra of $\mathfrak{o}(nd)$. We have concentrated on the case $d = 2$ because it offers the simplest situation that allows to obtain the superintegrable system on $S^{n-1}$ by dimensional reduction.

In the near future, we plan on exploring similarly the Askey-Wilson (AW) and the Bannai-Ito (BI) algebras which share features with the Racah algebra since both encode the bispectrality properties of the eponym polynomials and appear through tensor products of $\mathcal{U}_q(\mathfrak{sl}(2))$ [14] and $\mathfrak{osp}(1|2) \simeq \mathfrak{sl}_{-1}(2)$ [15] respectively. Moreover, the BI algebra is the symmetry algebra of a superintegrable model on the sphere involving reflection operators [16] as well as a Dirac-Dunkl equation in $\mathbb{R}^3$ [6]. It would be of interest to build on the work of the present paper to obtain a Howe duality setting for the interpretation of the AW and BI algebras as commutants; moreover extensions along the lines of this paper would shed interesting light on the higher rank versions of these algebras.

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References

[1] Granovskii Y A and Zhedanov A S 1988 J. Exp. Theor. Phys. 94 49-54 URL http://www.jetp.ac.ru/cgi-bin/dn/e_067_10_1982.pdf
[2] Genest V X, Vinet L and Zhedanov A 2014 Lett. Math. Phys. 104 931-952 URL http://link.springer.com/10.1007/s11005-014-0697-γ
[3] Koekoek R, Lesky P A and Swarttouw R F 2010 Hypergeometric Orthogonal Polynomials and Their q-Analogues Springer Monographs in Mathematics (Springer Berlin Heidelberg) ISBN 978-3-642-05013-8 URL http://link.springer.com/10.1007/978-3-642-05014-5
[4] Kalnins E G, Miller W and Post S 2007 J. Phys. A Math. Theor. 40 11525–11538 URL http://iopscience.iop.org/article/10.1088/1751-8113/40/38/005
[5] Genest V X, Vinet L and Zhedanov A 2014 J. Phys. Conf. Ser. 512 012011 ISSN 1742-6588 URL http://iopscience.iop.org/article/10.1088/1742-6596/512/1/012011
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[6] De Bie H, Genest V X, van de Vijver W and Vinet L 2017 J. Phys. A Math. Theor. 51 025203 URL http://iopscience.iop.org/article/10.1088/1751-8121/aa9756

[7] Gaboriaud J, Vinet L, Vinet S and Zhedanov A 2018 URL https://arxiv.org/abs/1808.05261

[8] Genest V X, Vinet L and Zhedanov A 2013 J. Phys. A Math. Theor. 45 025203 URL http://iopscience.iop.org/article/10.1088/1751-8113/47/2/025203

[9] Post S 2015 SIGMA. Symmetry, Integr. Geom. Methods Appl. 11 057 URL http://www.emis.de/journals/SIGMA/2015/057/

[10] Howe R 1987 Proceedings, Appl. Gr. Theory Phys. Math. Phys. ed Flato M, Sally P and Zuckerman G (Chicago: AMS) chap 6, pp 179–206

[11] Howe R 1989 Trans. Am. Math. Soc. 313 539–570 ISSN 00029947 URL http://www.ams.org/journals/tran/1989-313-02/S0002-9947-1989-0986027-X

[12] Howe R 1989 J. Am. Math. Soc. 2 535 URL http://www.ams.org/journals/jams/1989-02-03/S0894-0347-1989-0985172-6

[13] Rowe D J, Carvalho M J and Repka J 2012 Rev. Mod. Phys. 84 711–757 URL https://link.aps.org/doi/10.1103/RevModPhys.84.711

[14] Granovskii Y I and Zhedanov A S 1993 Hidden Symmetry of the Racah and Clebsch-Gordan Problems for the Quantum Algebra $sl_q(2)$ Tech. rep. UdeM-LPN-TH Montreal URL https://arxiv.org/pdf/hep-th/9304158.pdf

[15] Genest V X, Vinet L and Zhedanov A 2014 Proc. Am. Math. Soc. 142 11970–11978 URL www.ams.org/journals/proc/2014-142-05/S0002-9939-2014-11970-8

[16] Genest V X, Vinet L and Zhedanov A 2014 J. Phys. A Math. Theor. 47 205202 (Preprint 1401.1525) URL http://iopscience.iop.org/article/10.1088/1751-8113/47/20/205202