Lagrangian extensions of multi-dimensional integrable equations. I. The five-dimensional Martínez Alonso–Shabat equation

I. S. Krasil’shchik1 · O. I. Morozov1

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Abstract
We study a Lagrangian extension of the 5d Martínez Alonso–Shabat equation $\mathcal{E}$

$$u_{yz} = u_{tx} + u_y u_{xs} - u_x u_{ys},$$

that coincides with the cotangent equation $\mathcal{J}^* \mathcal{E}$ to the latter. We describe the Lie algebra structure of its symmetries (which happens to be quite nontrivial and is described in terms of deformations) and construct two families of recursion operators for symmetries. Each family depends on two parameters. We prove that all the operators from the first family are hereditary, but not compatible in the sense of the Nijenhuis bracket. We also construct two new parametric Lax pairs that depend on higher-order derivatives of the unknown functions.

Keywords 5D Martínez Alonso–Shabat equation · Cotangent covering · Symmetries · Algebras of Kac–Moody type · Recursion operators

Mathematics Subject Classification 58H05 · 58J70 · 35A30 · 37K05 · 37K10

1 Introduction

In what follows, we consider the two-component system

$$u_{yz} = u_{tx} + u_y u_{xs} - u_x u_{ys},$$
$$v_{yz} = v_{tx} + u_y v_{xs} - u_x v_{ys} + 2 u_{ys} v_x - 2 u_{xs} v_y,$$

(1)

1 Trapeznikov Institute of Control Sciences, 65 Profsoyuznaya Street, Moscow 117997, Russia
which is nothing else but the cotangent equation (see, e.g., [24] and references therein) to the 5-dimensional Martínez Alonso–Shabat equation

\[ u_{yz} = u_{tx} + u_y u_{xs} - u_x u_{ys}, \]  
\[ (2) \]

see [4, 30, 31, 33]. We describe the Lie algebra structure of its symmetries and construct two families of recursion operators for symmetries. Each family depends on two parameters. It is proved that all the operators from the first family are hereditary, but not compatible in the sense of the Nijenhuis bracket. We also construct two new parametric Lax pairs that depend on higher-order derivatives of the unknown functions.

Equation (2) admits a number of symmetry reductions. The 4-dimensional reductions include the reduced quasi-classical self-dual Yang–Mills equation [12]

\[ u_{yz} = u_{tx} + u_y u_{xx} - u_x u_{xy}, \]  
\[ (3) \]

the four-dimensional Martínez Alonso–Shabat equation [33]

\[ u_{ty} = u_z u_{xy} - u_y u_{xz}, \]  
\[ (4) \]

and the 4D universal hierarchy equation [9]

\[ u_{zz} = u_{tx} + u_z u_{xy} - u_x u_{yz}. \]  
\[ (5) \]

In their turn, these equations admit 3D symmetry reductions: the hyper-CR equation for Einstein–Weyl structures [11, 28, 37, 40]

\[ u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}, \]  
\[ (6) \]

the 3D rdDym equation [7]

\[ u_{ty} = u_x u_{xy} - u_y u_{xx}, \]  
\[ (7) \]

the 3D universal hierarchy equation [32, 33]

\[ u_{yy} = u_y u_{tx} - u_x u_{ty}, \]  
\[ (8) \]

and the modified Veronese web equation [1, 13]

\[ u_{ty} = u_t u_{xy} - u_y u_{tx}. \]  
\[ (9) \]

The Lax representation (21) as well as the recursion operator (23), (24) survive in these reductions when \( \kappa = \mu = 0 \) and thus provide Lax representations and recursion operators for the cotangent extensions of the above reduced equations. The questions of whether these cotangent extensions possess parametric families of Lax representations and recursion operators, and whether the recursion operators are hereditary, are more subtle. The affirmative answer to the second question for Eqs. (7) and (6) was obtained
The Lax representation of Eq. (4) with two non-removable parameters [38] was used to construct new recursion operators for this equation in a recent preprint [44].

The other reductions of Eq. (13) as well as some other multi-dimensional integrable PDE will be considered in the forthcoming parts of the work.

In Sect. 2, we very briefly discuss the approach adopted in the study. A more detailed information can be found in the monographs [8, 24] and in [20]. Section 3 contains the main results. Namely, Subsection 3.1 deals with a full description of the Lie algebra of symmetries of Eq. (1), which consists not only of point symmetries (a typical situation for multi-dimensional systems), but contains one higher (of order 3) one. In Subsection 3.2, we find a two-parameter Lax pair for System (1) which is used to construct two families of recursion operators. We show that the operators from the first family are hereditary, but pair-wise incompatible in the sense of the Nijenhuis bracket. The action of the operators from the first family on symmetries is described in Subsection 3.4. In Subsection 3.5, using specific properties of the above-mentioned higher symmetry, we construct two families of Lax pairs that depend on variables of order 5 and 4. Finally, in Sect. 4, we discuss some the perspectives.

2 Preliminaries and notation

Our goal is to study various invariants (symmetry algebras, Lax pairs, recursion operators, etc.) of canonical Lagrangian extensions associated to multi-dimensional systems. The approach adopted here is based on the geometrical theory of PDEs (see [8] for the main definitions and standard notation and [26] for the details of the nonlocal theory).

Namely, any PDE (its infinite prolongation, to be more precise) is treated as a submanifold $E \subset J^\infty(\pi)$, where $J^\infty(\pi)$ is the space of infinite jets of some locally trivial vector bundle $\pi: E \to M$, $\dim M = n$, rank $\pi = m$. There exists a natural projection $\pi_\infty: E \to M$ and $E$ is always endowed with an integrable $n$-dimensional $\pi_\infty$-horizontal distribution $\mathcal{C} \subset T E$ (the Cartan distribution)\footnote{Since $\mathcal{C}$ is horizontal and $n$-dimensional, a flat connection in $\pi_\infty$ is associated to this distribution (the Cartan connection).}. Locally, $\mathcal{C}$ annihilates all the Cartan forms $\omega^j_\sigma = du^j_\sigma - \sum_i u^j_{\sigma i} dx^i$.

A $\pi_\infty$-vertical vector field $Z$ on $E$ is a symmetry if $[Z, \mathcal{C}] \subset \mathcal{C}$. Any such a $Z$ has the form of an evolutionary derivation $E_\varphi$, where $\varphi$ is the generating section of $Z$; we do not distinguish between $Z$ and $\varphi$. The space of symmetries $\text{sym} \ E$ carries the structure of a Lie algebra with respect to the commutator. The corresponding bracket on generating sections is denoted by $\{ \cdot, \cdot \}$ and is called the Jacobi bracket. Let $\mathcal{E}$ be given by the system $\{ F = 0, \ F = (F^1, \ldots, F^r) \}$. To find symmetries, one needs to solve the linear system

$$\ell_\mathcal{E}(\varphi) = 0,$$

where $\ell_\mathcal{E}$ is the restriction of the linearization operator $\ell_F$ to $\mathcal{E}$.

Consider an overdetermined PDE $W$ whose compatibility conditions coincide with $E$. Then the surjection $\tau: W \to E$ is a covering. Coordinates in the fiber of $\tau$ are
called nonlocal variables. If the defining equations of $W$ are linear in nonlocal variable, this covering is a Lax pair for $E$. An $\mathbb{R}$-linear derivation $S: \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(W)$ is a nonlocal $\tau$-shadow if it preserves Cartan distributions (commutes with the action of Cartan connections). The defining equation for a shadow $S = \tilde{E}_\varphi$ is

$$\tilde{\ell}_E(\varphi) = 0,$$

where $\tilde{\ell}_E$ is the lift of $\ell_E$ from $E$ to $W$, while $\varphi$ is the $m$-component generating section of $S$ that lives on $W$.

Let $\tau_\lambda : W_\lambda \to E$ be a $\lambda$-parametrized family of coverings, $\lambda \in \mathbb{R}$. The parameter $\lambda$ is said to be non-removable if the coverings $\tau_\lambda$ are pair-wise non-equivalent. There exists a regular way to insert a parameter in a covering with the unliftable symmetry, see [26]. Namely, let $Z$ be an integrable symmetry of $E$, i.e., such that it possesses local trajectories. Assume that $Z$ cannot be lifted to $W$, i.e., there exists no symmetry $\tilde{Z}$ of $W$ such that $\tau_\lambda(\tilde{Z}) = Z$. Then $\exp(\lambda Z)$ generates the desired family of coverings.

The following construction underlies our approach to recursion operators for symmetries of $E$. Introduce the bundle $t : TE = T E / \mathcal{C} \to \mathcal{C}$ which is called the tangent covering of $E$. In coordinates, if $E$ is given by $\{F = 0\}$, then $TE$ is defined by the system

$$F(u) = 0, \quad \ell_E(q) = 0,$$

where $q = (q^1, \ldots, q^m)$ is a new unknown which is assumed to be odd (of parity 1).

The algebra of super-functions on $T E$ is identified with the Grassmann algebra $\Lambda^*_i(\mathcal{E})$ of Cartan forms on $E$. The Cartan differential

$$d_C : \Lambda^i(\mathcal{E}) \to \Lambda^{i+1}(\mathcal{E}), \quad d_C(f) = \sum \frac{\partial f}{\partial u^\sigma} \omega^\sigma_j, \quad d_C(\omega^\sigma_j) = 0,$$

defines a canonical nilpotent vector field $X$ on $TE$ of parity 1. Sections of $t$ that preserve the Cartan distributions coincide with symmetries of $E$.

Let $\tau : W \to TE$ be a covering with fibers of parity 1. Let also $S$ be a $t \circ \tau$-shadow linear in the nonlocal variables of $\tau$. Then it defines another covering $\tau_S : W \to TE$ and we obtain the diagram

$$\mathcal{R} : \quad \begin{array}{ccc} \mathcal{T}E & \xrightarrow{\tau_S} & W & \xrightarrow{\tau} & \mathcal{T}E, \end{array}$$

i.e., a recursion operator (a Bäcklund auto-transformation of $T E$), cf. [34]. We say that $\mathcal{R}$ is a regular operator if there exists a symmetry $\tilde{S} = E_\varphi \in \text{sym} W$ that extends $S$, i.e., such that the restriction of $\tilde{S}$ to $\mathcal{C}^\infty(E)$ coincides with $S$.

Consider two regular operators of the form (10) and let $E_{\varphi_1}, E_{\varphi_2}$ be the corresponding symmetries. Then the generating section $[[\varphi_1, \varphi_2]]$ of their super-commutator

$$[E_{\varphi_1}, E_{\varphi_2}] = E_{\varphi_1} \circ E_{\varphi_2} + E_{\varphi_2} \circ E_{\varphi_1}$$
(a symmetry of parity 2) is called their Nijenhuis bracket. An operator is hereditary if \( \| \varphi, \varphi \| = 0 \), two operators are compatible if \( \| \varphi_1, \varphi_2 \| = 0 \).

Regularity is established in two steps: lift to \( TE \) and subsequent lift to \( W \). First, using the interpretation of functions on \( TE \) as Cartan forms, we set

\[
S(\omega^j_a) = S(X(u^j_a)) = -X(S(u^j_a)).
\]

The action of \( X \) on local variables is known. To compute its values at nonlocal ones, we take the defining equations of the covering \( \tau \)

\[
w^j_{x^i} = W^j_i, \quad i = 1, \ldots, n, \quad j = 1, \ldots, \text{rank } \tau,
\]

and apply \( X \) to them:

\[
D_{x^i}(X(w^j)) = X(W^j_i), \quad (11)
\]

where \( D_{x^i} \) are the total derivatives. Solving Eq. (11) with respect to \( X(w^j) \) (if possible) gives the desired result. In a similar way, to find the action of \( S \) on \( w^j \), we solve the system

\[
D_{x^i}(S(w^j)) = S(W^j_i)
\]

with respect to \( S(w^j) \).

To finish this introductory part, let us say a few words about the extensions we work with below. Let \( E \) be an equation given by \( \{ F = 0 \} \) in local coordinates. Then the system

\[
\mathcal{T}^*E: \quad \ell^*_E(p) = 0, \quad F(u) = 0, \quad (12)
\]

where \( \ell^*_E \) is the adjoint operator, is called the cotangent equation to \( \mathcal{E} \). System (12) is always a Lagrangian one with the Lagrangian density \( \mathcal{L} = \langle p, F \rangle dx^1 \wedge \cdots \wedge dx^n \).

Though \( \mathcal{T}^*E \) is defined in coordinates, it can be shown that when \( E \) is a two-line equation (see [43]), the object is well defined, see [24] for the proof.

### 3 The 5D Martínez Alonso–Shabat equation

We consider the five-dimensional Martínez Alonso–Shabat equation \( E \)

\[
uyz = utx + uy uxs - ux uys.
\]

The linearization of this equation reads

\[
qyz = qtz + uy qzs + uxs qy - ux qys - uys qx.
\]
The cotangent covering \( \tilde{\mathcal{E}} = T^* \mathcal{E} \to \mathcal{E} \) (see, e.g., [24]) is obtained by appending the adjoint linearization

\[
v_{yx} = v_{tx} + u_y v_{xx} - u_x v_{yx} + 2 u_{yx} v_x - 2 u_{xs} v_y
\]

(15)
to (13). 

3.1 Symmetries

Symmetries of the system at hand are described by the following

Proposition 1 The Lie algebra \( \mathfrak{s} \) of the local infinitesimal symmetries of order \( \leq 3 \) for System (13), (15) is generated by the functions

\[
\begin{align*}
\psi_1 &= (x u_x + y u_y - u_x v_x + y v_y), \\
\psi_2 &= (-t u_t - x u_x - 2 z u_z - u_t v_t - x v_x - 2 z v_z), \\
\psi_3 &= (-t u_t + x u_x - y u_y + z u_z - t v_t + x v_x - y v_y + z v_z), \\
\psi_4 &= (-t z v_t - x z u_x - t x u_y - z^2 u_z - z u, \\
&\quad \quad - z v - x z u_x - t z v_t - t x v_y - z^2 v_z), \\
\psi_5 &= (-y u_x - t u_z, -y v_x - t v_z), \\
\psi_6 &= (t^2 u_t + y z u_x + t y u_y + t z u_z + t u, \\
&\quad \quad t^2 v_t + y z v_x + t y v_y + t z v_z + t v), \\
\psi_7 &= (u_z, v_z), \\
\psi_8 &= (-z u_t - x u_y, -z v_t - x v_y), \\
\psi_9 &= (-u_t, -v_t), \\
\psi_{10} &= (-z u_x - t u_y, -z v_x - t v_y), \\
\psi_{11} &= (u_x, v_x), \\
\psi_{12} &= (-u_y, -v_y), \\
\psi_{13} &= (0, -u_{sss}), \\
\psi_{14} &= (0, v), \\
\phi_{0,0}(A) &= (-A u_s + A_s u + A z x + A_t y, -A v_s - 2 A_s v), \\
\phi_{0,1}(A) &= (A, 0), \\
\phi_{1,0}(A) &= (0, 2 A u_s + A_s u + A z x + A_t y), \\
\phi_{1,1}(A) &= (0, A),
\end{align*}
\]

where \( A = A(t, z, s) \) and \( B = B(t, z, s) \) below are arbitrary smooth functions of their arguments. Besides, the map \( (t, x, y, z) \mapsto (z, y, x, t) \) is a discrete (finite) symmetry
of this system. There hold
\[
\{\varphi_{i,m}(A), \varphi_{j,n}(B)\} = \begin{cases}
\varphi_{i+j,m+n}(AB_s - (1 - 3j)BA_s), & i + j \leq 1, \\
0, & m + n \leq 1,
\end{cases}
\]
for \(i \leq j\) and
\[
\{\psi_i, \varphi_{i,m}(A)\} = (i - m) \varphi_{i,m}(A),
\]
\[
\{\psi_2, \varphi_{i,m}(A)\} = \varphi_{i,m}(tA_t + 2\varepsilon A_z + (i - m) A),
\]
\[
\{\psi_3, \varphi_{i,m}(A)\} = \varphi_{i,m}(tA_t - \varepsilon A_z),
\]
\[
\{\psi_4, \varphi_{i,m}(A)\} = \varphi_{i,m}(t^2 A_t + \varepsilon^2 A_z + m A),
\]
\[
\{\psi_5, \varphi_{i,m}(A)\} = \varphi_{i,m}(tA_t),
\]
\[
\{\psi_6, \varphi_{i,m}(A)\} = \varphi_{i,m}(-t^2 A_t - t\varepsilon A_z - m A),
\]
\[
\{\psi_7, \varphi_{i,m}(A)\} = \varphi_{i,m}(-\varepsilon A_z),
\]
\[
\{\psi_8, \varphi_{i,m}(A)\} = \varphi_{i,m}(zA_t),
\]
\[
\{\psi_9, \varphi_{i,m}(A)\} = \varphi_{i,m}(A_t),
\]
\[
\{\psi_{10}, \varphi_{i,m}(A)\} = \begin{cases}
\varphi_{i,1}(tA_t + \varepsilon A_z), & m = 0, \\
0, & \text{otherwise},
\end{cases}
\]
\[
\{\psi_{11}, \varphi_{i,m}(A)\} = \begin{cases}
\varphi_{i,1}(-\varepsilon A_z), & m = 0, \\
0, & \text{otherwise},
\end{cases}
\]
\[
\{\psi_{12}, \varphi_{i,m}(A)\} = \begin{cases}
\varphi_{i,1}(A_t), & m = 0, \\
0, & \text{otherwise},
\end{cases}
\]
\[
\{\psi_{13}, \varphi_{i,m}(A)\} = \begin{cases}
\varphi_{1,m}(A_{x,x}), & i = 0, \\
0, & \text{otherwise},
\end{cases}
\]
\[
\{\psi_{14}, \varphi_{i,m}(A)\} = -i \varphi_{i,m}(A),
\]
\[
\{\psi_i, \psi_{14}\} = 0, \quad i \in \{1, ..., 12\},
\]
\[
\{\psi_i, \psi_{13}\} = \begin{cases}
-\psi_{13}, & i \in \{1, 2, 14\}, \\
0, & \text{otherwise}.
\end{cases}
\]
The subalgebra \(\langle \psi_1, ..., \psi_{12} \rangle \subseteq s\) is isomorphic to the Lie algebra \(\mathfrak{gl}(3, \mathbb{R}) \times \mathbb{R}^3\) with the isomorphism given by the map
\[
(\psi_1, ..., \psi_{12}) \mapsto (E_{11} + E_{22} + E_{33}, E_{11} - E_{22}, E_{22} - E_{33}, E_{12}, E_{23}, \\
E_{13}, E_{21}, E_{32}, E_{31}, e_1, e_2, e_3),
\]
where \(E_{ij} \in \mathfrak{gl}(3, \mathbb{R})\) is the matrix with the only non-zero \((i, j)\)-entry 1 and for the vectors \(e_k \in \mathbb{R}^3\) there hold \(E_{ij} e_k = \delta_{jk} e_i\). \(\square\)
The Lie algebra $\mathfrak{s}_0$ of the contact symmetries of Eq. (13) is generated by the first components of the symmetries $\psi_1, \ldots, \psi_{12}, \varphi_0, 0 (A)$, and $\varphi_0, 1 (A)$.

Let $\mathfrak{w} = \text{Der}(\mathcal{C}^\infty(\mathbb{R})) = \{ f(s) \partial_s \mid f \in \mathcal{C}^\infty(\mathbb{R}) \}$ be the Lie algebra of smooth vector fields on $\mathbb{R}$ (or, equivalently, $\mathbb{R}$-linear derivations of $\mathcal{C}^\infty(\mathbb{R})$). In other words, this Lie algebra is the vector space $\mathcal{C}^\infty(\mathbb{R})$ endowed with the bracket $[f, g]_0 = f' g - g' f$. For $n \in \mathbb{N}$, consider the commutative unital algebra of the truncated polynomials in the (formal) variable $h$ of degree $n$: $\mathbb{R}_n[h] = \mathbb{R}[h]/(h^{n+1} = 0)$. Then

$$\mathfrak{s}_0 \cong (\mathfrak{gl}(3, \mathbb{R}) \times \mathbb{R}^3) \times (\mathcal{C}^\infty(\mathbb{R}^2) \otimes \mathbb{R}_1[h] \otimes \mathfrak{w}).$$

We need the following construction to describe the structure of the Lie algebra $\mathfrak{s}$.

The Lie algebra $\mathfrak{q}_{n, 0} = \mathbb{R}_n[\tau] \otimes \mathfrak{w}$ admits the deformation\(^2\) generated by the cocycle $\Psi \in H^2(\mathfrak{q}_{n, 0}, \mathfrak{q}_{n, 0})$,

$$\Psi(\tau^k \otimes f, \tau^m \otimes g) = \begin{cases} \tau^{k+m} \otimes (k f g_s - m g f_s), & k + m \leq n, \\ 0, & \text{otherwise}. \end{cases}$$

For each $\varepsilon \neq 0$, this cocycle defines a new bracket $[\cdot, \cdot]_\varepsilon = [\cdot, \cdot] + \varepsilon \Psi(\cdot, \cdot)$ on the linear space $\langle \tau^m \otimes f \mid m \leq n, f \in \mathcal{C}^\infty(\mathbb{R}) \rangle$. We denote the resulting Lie algebra by $\mathfrak{q}_{n, \varepsilon}$. In other words, $\mathfrak{q}_{n, \varepsilon}$ is isomorphic to the linear space of functions $F(s, \tau) = f_0(s) + \tau f_1(s) + \cdots + \tau^n f_n(s), f_k \in \mathcal{C}^\infty(\mathbb{R})$, equipped with the bracket

$$[F, G]_\varepsilon = F G_\varepsilon - G F_\varepsilon + \varepsilon \tau (F_\tau G_\varepsilon - G_\tau F_\varepsilon)$$

such that for $k > n$ there holds $\tau^k = 0$. Arguments similar to [10] show that the subalgebra of $\mathfrak{q}_{n, \varepsilon}$ obtained by replacing $\text{Der}(\mathcal{C}^\infty(\mathbb{R}))$ to $\text{Der}(\mathbb{R}[s, s^{-1}])$ in the definition of $\mathfrak{w}$ is a proper subalgebra of the affine Kac–Moody Lie algebra $\mathfrak{g}(A_{m}^{(1)})$ with the generalized Cartan matrix $A_{m}^{(1)}$ [17] for some $m \geq n$. Therefore the algebras $\mathfrak{q}_{n, \varepsilon}$ are referred to as the Lie algebras of Kac–Moody type.

The map $D_0 = \tau \partial_\tau : \mathfrak{q}_{n, \varepsilon} \to \mathfrak{q}_{n, \varepsilon}, D_0 : \tau^k \otimes f \mapsto k \tau^k \otimes f$, is an outer derivation of the Lie algebra $\mathfrak{q}_{n, \varepsilon}$ for all $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$. For some special values of the parameters $n$ and $\varepsilon$ this algebra has other outer derivations.

**Proposition 2** The Lie algebra $\mathfrak{q}_{1, -3}$ admits the outer derivation

$$D_1(\tau^k \otimes f) = \begin{cases} \tau \otimes f_{sss}, & k = 0, \\ 0, & k = 1. \end{cases}$$

**Proof** For $f, g \in \mathcal{C}^\infty(\mathbb{R})$ and $D_1 : \mathfrak{q}_{1, \varepsilon} \to \mathfrak{q}_{1, \varepsilon}$, there hold

\[ D_1([f, g]_\varepsilon) - [D_1(f), g]_\varepsilon - [f, D_1(g)]_\varepsilon = (\varepsilon + 3) \tau (f_s g_{sss} - g_s f_{sss}), \]
\[ D_1([f, \tau g]_\varepsilon) - [D_1(f), \tau g]_\varepsilon - [f, D_1(\tau g)]_\varepsilon = \tau^2 ((\varepsilon + 3) f_s g_{sss} - (4\varepsilon + 3) g_s f_{sss} - 3\varepsilon f_{ss} g_{ss}) = 0, \]

\(^2\) For a full description of the deformations of the subalgebra $\mathbb{R}_n[\tau] \otimes \text{Der}(\mathbb{R}[s]) \subset \mathfrak{q}_{n, 0}$ see [45].
and
\[
D_1([\tau f, \tau g]) - [D_1(\tau f), \tau g] - [\tau f, D_1(\tau g)] = (4 \varepsilon + 3) \tau^3 (f_s g_{sss} - g_s f_{sss}) = 0,
\]
therefore \(D_1 \in \text{Der}(q_{1, \varepsilon})\) if and only if \(\varepsilon = -3\). Furthermore,
\[
D_1(f) - [g_1 + \tau g_2, f] = f g_{1,s} - g_1 f_s + \tau (f_{sss} - (1 + \varepsilon) g_2 f_s + f g_{2,s})
\]
for all \(f, g_1, g_2 \in C^\infty(\mathbb{R})\), and there is no choice of functions \(g_1, g_2\) that would eliminate the term \(\tau f_{sss}\) in the right-hand side of the last equation. Thus \(D_1 \in \text{Der}_{out}(q_{1, -3})\).

Denote by \(Q\) the two-dimensional ‘right’ extension, \([14, \S 1.4.4], \langle D_0, D_1 \rangle \ltimes q_{1, -3}\) of the Lie algebra \(q_{1, -3}\) associated to the derivations \(D_0 \) and \(D_1\). As a vector space \(Q = \langle w_0, w_1 \rangle \oplus q_{1, -3}\), and the bracket on \(q_{1, -3}\) is extended to the new basis elements \(w_0\) and \(w_1\) by the formulas
\[
[w_0, f + \tau g] = D_0(f + \tau g) = \tau g, [w_1, f + \tau g] = D_1(f + \tau g) = \tau f_{sss},\]
and \([w_0, w_1] = w_1\). Then we have
\[
s \cong (\mathfrak{gl}(3, \mathbb{R}) \ltimes \mathbb{R}^3) \ltimes (C^\infty(\mathbb{R}^2) \otimes \mathbb{R}[h] \otimes Q),
\]
where the derivatives \(D_0 \) and \(D_1\) correspond to the symmetries \(\psi_{14}\) and \(\psi_{13}\).

### 3.2 Lax representations

Equation (13) admits the Lax representation
\[
w_t = \Lambda w_y - u_y w_s, \quad w_z = \Lambda w_x - u_x w_s,
\]
(16)
where
\[
\Lambda = \frac{\lambda - \kappa y - \mu x}{1 + \kappa t + \mu z}
\]
with the parameters \(\kappa, \mu, \lambda \in \mathbb{R}\) (see [4]). When \(\kappa = \mu = 0\), this Lax representation coincides with the Lax representation found in \([30]\) and used intensively in \([31]\). The parameters \(\kappa, \mu, \) and \(\lambda\) are non-removable, that is, the differential coverings defined by system (16) with the different constant values of \(\kappa, \mu, \) and \(\lambda\) are not equivalent. To ensure this, we note that the first components of the symmetries \(\psi_4, \psi_6, \) and \(\psi_{10}\) from Proposition 1 do not admit lifts to symmetries of the system
\[ w_t = -u_y w_s, \]
\[ w_z = -u_x w_s. \]  
(18)

Then we take the vector field associated with the first component of the linear combination \(-\mu \psi_4 + \kappa \psi_6 + \lambda \psi_{10}\). The flow of its first prolongation maps system (18) to system (16). In accordance with [26, §§ 3.2, 3.6], [15, 16, 19, 35] this proves the claim.

In what follows, we will use another Lax representation for Eq. (13). We observe that the function
\[ \theta = \frac{1}{(\lambda - \kappa y - \mu x) w_s} \]  
(19)
is a shadow of a nonlocal symmetry for Eq. (13) in the covering (16). We express \(w_s\) from (19), differentiate the result with respect to \(t\) and \(z\), and substitute to Eqs. (16). This yields a new covering
\[ \theta_t = \Lambda \theta_y - u_y \theta_s + \left( u_{ys} - \frac{\kappa}{\kappa t + \mu z + 1} \right) \theta, \]
\[ \theta_z = \Lambda \theta_x - u_x \theta_s + \left( u_{xs} - \frac{\mu}{\kappa t + \mu z + 1} \right) \theta \]  
(20)
over Eq. (13). This covering admits an extension to a covering over \(\mathcal{M}^*\):

**Proposition 3** Systems (20) and
\[ \omega_t = \Lambda \omega_y - u_y \omega_s - \left( 2 u_{ys} + \frac{\kappa}{\kappa t + \mu z + 1} \right) \omega + 2 v_y \theta_s + v_{ys} \theta, \]
\[ \omega_z = \Lambda \omega_x - u_x \omega_s - \left( 2 u_{xs} + \frac{\mu}{\kappa t + \mu z + 1} \right) \omega + 2 v_x \theta_s + v_{xs} \theta \]  
(21)
define a Lax representation for System (13), (15).

Moreover, we have

**Proposition 4** A solution \((\theta, \omega)\) of System (20), (21) is a shadow of a nonlocal symmetry for System (13), (15).

Another covering over \(\mathcal{M}^*\) is defined by a lift of System (17):

**Proposition 5** Systems (17) and
\[ W_t = \lambda W_y - u_y W_s - 3 u_{ys} W + v_y w_s, \]
\[ W_z = \lambda W_x - u_x W_s - 3 u_{xs} W + v_x w_s. \]  
(22)
provide a Lax representation for System (13), (15).
In Sect. 3.4 below we will use nonlocal variables of the so-called negative and positive coverings, see [4], associated to covering (17), (22). To construct the negative covering, we substitute the formal expansions \( w = \sum_{n \geq 0} \lambda^{-n} w_{-n}, \) \( W = \sum_{n \geq 0} \lambda^{-n} W_{-n} \) into (17), (22) and obtain the infinite tower of Abelian two-component coverings (nonlocal conservation laws) given by equations

\[
\begin{align*}
\frac{\partial w_0}{\partial x} &= 0, \\
\frac{\partial w_0}{\partial y} &= 0, \\
\frac{\partial w_{-n-1}}{\partial x} &= w_{-n,z} + u_x w_{-n,s}, \\
\frac{\partial w_{-n-1}}{\partial y} &= w_{-n,t} + u_y w_{-n,s},
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial W_0}{\partial x} &= 0, \\
\frac{\partial W_0}{\partial y} &= 0, \\
\frac{\partial W_{-n-1}}{\partial x} &= W_{-n,z} + u_x W_{-n,s} + 3 u_{xx} W_{-n} - v_x w_{-n,s}, \\
\frac{\partial W_{-n-1}}{\partial y} &= W_{-n,t} + u_y W_{-n,s} + 3 u_{ys} W_{-n} - v_y w_{-n,s}.
\end{align*}
\]

In particular, if we put \( w_0 = s, \) \( w_{-1} = u, \) \( W_0 = 0, \) \( W_{-1} = -v \) and denote \( p = w_{-2}, \) \( r = -W_{-2} + 3 u_s v, \) we obtain the local conservation law

\[
\begin{align*}
p_x &= u_x + u_x u_s, \\
p_y &= u_t + u_y u_s, \\
r_x &= v_x + u_x v_x - 2 u_s v_x, \\
r_y &= v_t + u_y v_s - 2 u_s v_y
\end{align*}
\]

for \( \mathcal{J}^* \mathcal{E}. \)

Likewise, the positive covering is generated by the expansions \( w = \sum_{n \geq 0} \lambda^n w_n, \) \( W = \sum_{n \geq 0} \lambda^n W_n \) that produce systems

\[
\begin{align*}
\frac{\partial w_0}{\partial t} &= -u_y w_{0,s}, \\
\frac{\partial w_0}{\partial z} &= -u_x w_{0,s}, \\
\frac{\partial w_{n+1}}{\partial t} &= -u_y w_{n+1,s} + w_{n,y}, \\
\frac{\partial w_{n+1}}{\partial z} &= -u_x w_{n+1,s} + w_{n,x}
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial W_0}{\partial t} &= -u_y W_{0,s} + v_y w_{0,s} - 3 u_{ys} W_0, \\
\frac{\partial W_0}{\partial z} &= -u_x W_{0,s} + v_x w_{0,s} - 3 u_{xs} W_0, \\
\frac{\partial W_{n+1}}{\partial t} &= -u_y W_{n+1,s} + v_y w_{n+1,s} - 3 u_{ys} W_{n+1} + W_{n,y}, \\
\frac{\partial W_{n+1}}{\partial z} &= -u_x W_{n+1,s} + v_x w_{n+1,s} - 3 u_{xs} W_{n+1} + W_{n,x}.
\end{align*}
\]

### 3.3 Recursion operators

In view of Propositions 4, we can use Systems (20), (21) for deriving recursion operators for symmetries of Eqs. (13), (15) by the method of [41], see also [21, 22, 27, 29,
36, 39]. Since these equations are independent on the parameters $\kappa, \mu, \lambda$, we expand $\theta, \omega$ in the formal Laurent series with respect to one of the parameters, substitute this series into systems (20), (21), and collect terms at the same powers of the parameter.

So, if we choose the parameter $\lambda$, we substitute $\theta = \sum_{n \in \mathbb{Z}} \theta_n \lambda^n, \omega = \sum_{n \in \mathbb{Z}} \omega_n \lambda^n$, then collect the terms at $\lambda^n$ for a fixed arbitrary $n \in \mathbb{Z}$, and finally rename $\theta_{n-1} \mapsto \tilde{\theta}$, $\theta_n \mapsto \theta$, $\omega_{n-1} \mapsto \tilde{\omega}$, $\omega_n \mapsto \omega$. This gives two systems

\[
\tilde{\theta}_x = -(\kappa t + \mu z + 1) (\theta_z + u_x \theta_s - u_{xs} \theta) - (\kappa y + \mu x) \theta_x - \mu \theta,
\]

\[
\tilde{\theta}_y = -(\kappa t + \mu z + 1) (\theta_t + u_y \theta_s - u_{ys} \theta) - (\kappa y + \mu x) \theta_y - \kappa \theta
\]

and

\[
\tilde{\omega}_x = -(\kappa t + \mu z + 1) (\omega_z + u_x \omega_s + 2 u_{xs} \omega - 2 v_x \theta_s - v_{xs} \theta) - (\kappa y + \mu x) \omega_x - \mu \omega,
\]

\[
\tilde{\omega}_y = -(\kappa t + \mu z + 1) (\omega_t + u_y \omega_s + 2 u_{ys} \omega - 2 v_y \theta_s - v_{ys} \theta) - (\kappa y + \mu x) \omega_y - \kappa \omega.
\]

Likewise, expanding $\theta$ and $\omega$ into the Laurent series with respect to $\mu$ gives the systems

\[
\tilde{\theta}_x = -\frac{1}{x} \left( z (\tilde{\theta}_z + u_x \tilde{\theta}_s - u_{xs} \tilde{\theta}) + \tilde{\theta} + (\kappa y - \lambda) \theta_x + (\kappa t + 1) (\theta_z + u_x \theta_s - u_{xs} \theta) \right),
\]

\[
\tilde{\theta}_y = -\frac{1}{x} \left( z (\tilde{\theta}_t + u_y \tilde{\theta}_s - u_{ys} \tilde{\theta}) + \kappa \theta + (\kappa y - \lambda) \theta_y + (\kappa t + 1) (\theta_t + u_y \theta_s - u_{ys} \theta) \right)
\]

and

\[
\tilde{\omega}_x = -\frac{1}{x} \left( z (\tilde{\omega}_z + u_x \tilde{\omega}_s + 2 u_{xs} \tilde{\omega} - 2 v_x \tilde{\theta}_s - v_{xs} \tilde{\theta}) + \tilde{\omega} + (\kappa y - \lambda) \theta_x + (\kappa t + 1) (\omega_z + u_x \omega_s + 2 u_{xs} \omega - 2 v_x \theta_s - v_{xs} \theta) \right),
\]

\[
\tilde{\omega}_y = -\frac{1}{x} \left( z (\tilde{\omega}_t + u_y \tilde{\omega}_s + 2 u_{ys} \tilde{\omega} - 2 v_y \tilde{\theta}_s - v_{ys} \tilde{\theta}) + \kappa \tilde{\omega} + (\kappa y - \lambda) \theta_y + (\kappa t + 1) (\omega_t + u_y \omega_s + 2 u_{ys} \omega - 2 v_y \theta_s - v_{ys} \theta) \right).
\]

The correspondence $\theta \mapsto \tilde{\theta}$ defined by System (23) will be denoted by $\tilde{\theta} = R_{\kappa,\mu}(\theta)$, and its lift $(\theta, \omega) \mapsto (\tilde{\theta}, \tilde{\omega})$ defined by Systems (23), (24) will be denoted by $(\tilde{\theta}, \tilde{\omega}) = \hat{R}_{\kappa,\mu}(\theta, \omega)$. Likewise, the correspondences defines by System (25) and by Systems (25), (26) will be denoted by $\hat{\theta} = S_{\kappa,\lambda}(\theta)$ and $(\hat{\theta}, \hat{\omega}) = \hat{S}_{\kappa,\lambda}(\theta, \omega)$, respectively. Since each component of the above Laurent series is a shadow of a symmetry, we obtain the following assertion.
Proposition 6 Systems (23) and (25) define two-parametric families of recursion operators $\mathcal{R}_{k,\mu}$ and $\hat{\mathcal{S}}_{k,\lambda}$ for symmetries of Eq. (13). Likewise, Systems (23), (24) and (25), (26) define two-parametric families of recursion operators $\hat{\mathcal{R}}_{k,\mu}$ and $\hat{\mathcal{S}}_{k,\lambda}$ for symmetries of the cotangent extension of Eq. (13). □

Remark 1 From Systems (23) and (24) we have $\mathcal{R}_{k,\mu} = (\kappa t + \mu z + 1) \mathcal{R}_{0,0} + (\kappa y + \mu x) \mathcal{I}$ and $\hat{\mathcal{R}}_{k,\mu} = (\kappa t + \mu z + 1) \hat{\mathcal{R}}_{0,0} + (\kappa y + \mu x) \mathcal{I}$, where $\mathcal{I}$ is the identical map on the spaces of shadows of Eq. (13) and its cotangent extension. ◀

Remark 2 The recursion operators produced by expanding System (20), (21) with respect to the parameter $\kappa$ are the compositions of the recursion operators $\mathcal{S}_{\mu,\lambda}, \hat{\mathcal{S}}_{\mu,\lambda}$, respectively, with the finite symmetry $(t, x, y, z) \mapsto (z, y, x, t)$ of Eq. (13). ◀

Unlike Systems (25) and (26), systems (23) and (24) define Abelian coverings over Eqs. (13) and (15). This allows one to use the results of [20, 25] to prove the following

Proposition 7 The recursion operator $\mathcal{R}_{k,\mu}$ is hereditary for each choice of the parameters $k, \mu$. Two such operators with different values of the parameters $k, \mu$ are compatible, i.e., their Nijenhuis bracket vanishes.

The recursion operator $\hat{\mathcal{R}}_{k,\mu}$ is hereditary for each choice of the parameters $k, \mu$. The operators $\hat{\mathcal{R}}_{k_1,\mu_2}$ and $\hat{\mathcal{R}}_{k_2,\mu_2}$ are compatible if and only if $k_1 = k_2$ and $\mu_1 = \mu_2$.

Proof We introduce two functions $U = (\kappa t + \mu z + 1) u_s \theta - \eta$ and $V = (\kappa t + \mu z + 1) (v_s \theta - 2 u_s \omega) - \zeta$, where $\eta$ and $\zeta$ are the nonlocal variables defined by the systems

$$
\eta_x = (\kappa t + \mu z + 1) (\theta_z + u_s \theta_x + u_x \theta_s) + (\kappa y + \mu x) \theta_x + \mu \theta_x,
$$

$$
\eta_y = (\kappa t + \mu z + 1) (\theta_t + u_s \theta_y + u_y \theta_s) + (\kappa y + \mu x) \theta_y + \kappa \theta_y.
$$

and

$$
\zeta_x = (\kappa t + \mu z + 1) (\omega_z + u_x \omega_y - 2 u_s \omega_x + v_s \theta_x - 2 v_x \theta_s)
+ (\kappa y + \mu x) \omega_x + \mu \omega_x,
$$

$$
\zeta_y = (\kappa t + \mu z + 1) (\omega_t + u_y \omega_x - 2 u_s \omega_y + v_s \theta_y - 2 v_y \theta_s)
+ (\kappa y + \mu x) \omega_y + \kappa \omega_y.
$$

Here $\omega, \theta, \eta$, and $\zeta$ are understood as odd variables in the fibers of the tangent covering $T\tilde{E} \rightarrow \tilde{E}$ and the symbol $\wedge$ below denotes their anti-commutative multiplication.

Then $U$ is a shadow of a symmetry of Eq. (13), while the pair $(U, V)$ is a shadow of a symmetry for System (13), (15). By the technique used in the proof of Proposition 3 from [20] we obtain □

Lemma 1 The shadow $U$ admits the lift $\Phi_{k,\mu}$ to a symmetry of System (23). The shadow $(U, V)$ has the lift $\Phi_{k,\mu}$ to a symmetry of System (23), (24). For the associated
evolutionary vector fields $E_{\Phi,\mu}$ and $\hat{E}_{\Phi,\mu}$ we have

\[
\Theta = E_{\Phi,\mu}(\theta) = (\kappa t + \mu z + 1) \theta \wedge \theta,
\]
\[
\Omega = E_{\Phi,\mu}(\omega) = (\kappa t + \mu z + 1) (\theta \wedge \omega + 2\theta_s \wedge \omega),
\]
\[
H = E_{\Phi,\mu}(\eta) = (\kappa t + \mu z + 1) \theta \wedge \eta,
\]
\[
Z = \hat{E}_{\Phi,\mu}(\zeta) = (\kappa t + \mu z + 1) (\theta \wedge \zeta + 2 \eta_s \wedge \omega) + 6 (\kappa t + \mu z + 1)^2 u_s \omega \wedge \theta.
\]

(29)

Proof of Lemma  Straightforward computations. \[\square\]

To finish the prove of Proposition 7 we use formulas (29) to compute

\[
[[E_{\Phi,\mu_1}, E_{\Phi,\mu_2}]] = 0
\]

and

\[
[[\hat{E}_{\Phi,\mu_1}, \hat{E}_{\Phi,\mu_2}]] = \Psi
\]

with

\[
\Psi = \begin{pmatrix}
0 \\
-6 ((\kappa_1 - \kappa_2) t + (\mu_1 - \mu_2) z)^2 u_s \omega \wedge \theta_s \\
0 \\
0
\end{pmatrix}.
\]

The last equation shows that $[[\hat{E}_{\Phi,\mu_1}, \hat{E}_{\Phi,\mu_2}]] = 0$ if and only if $\kappa_1 = \kappa_2$ and $\mu_1 = \mu_2$. \[\square\]

3.4 Actions

In this Subsection we study the actions of the recursion operators from Proposition 6 in the simplest cases. Namely, we consider the actions of the operators $\hat{R}_{\kappa,\mu}$ and $R_{-1,0}$.

The results of [5, 6, 23, 44] show that the adequate setting for studying the actions of the operators $R_{-1,0}, \hat{R}_{-1,0}, S_{k,\lambda}, \hat{S}_{-1,0}, S_{-1,\lambda}, \hat{S}_{-1,\lambda}$, and $\hat{S}_{-1,\lambda}$ should include consideration of nonlocal symmetries in various coverings. We intent to deal with this issue in the forthcoming research.

As usual, the image of zero under the action of a recursion operator is non-trivial. Below the actions of the recursion operators are computed modulo images of zero.

According to Remark 1 one can readily express the action of the operators $\hat{R}_{\kappa,\mu}$ in terms of the action of $\hat{S}_{0,0}$. The direct computations show that the map defined the last recursion operator is given by the following formulas, where the nonlocal variables $p$
and \( r \) were defined in Subsection 3.2:

\[
\begin{align*}
\psi_1 & \mapsto (2p - yu_t - uu_s - xu_z, r - yv_t - uv_s - xv_z), \\
\psi_2 & \mapsto (tp_t + 2zp_z + 2p - uu_s (turt + 2zu_z + u) + xu_z, \\
& \quad tr_t + 2r z + r + 2uu_s (tv_t + 2zu_z) \\
& \quad - uv_s (turt + 2zu_z + u) + xv_z), \\
\psi_3 & \mapsto (tp_t - zp_z - xu_z + yu_t - uu_s (tu_t - zu_z), \\
& \quad tr_t - rz + r - yv_t + 2uu_s (tv_t - zv_z) - uv_s (tu_t - zu_z)), \\
\psi_4 & \mapsto (z^2 p_z + tz p_t + 2z p + (x - zu_z) (tu_t uu_s + zu_z + u), \\
& \quad z^2 z + rz + r + (x + 2zu_z) (tv_t + xv_z) \\
& \quad - z uv_s (zu_t + zu_z + u)), \\
\psi_5 & \mapsto (zp_t - tu u_s + yu_z, tr_z + yv_z - tv u_s + 2tu u_s z), \\
\psi_6 & \mapsto (-t^2 p_t - zp_z - 2tp + (tu_s - y) (tu_t + zu_z + u), \\
& \quad - tv_t (tu_t + zu_z + u) + tv_s (tu_t + zu_z + u)), \\
\psi_7 & \mapsto (-p_z + uu_z - r - 2uu_s z + uu_z), \\
\psi_8 & \mapsto (zp_t - uu_t uu_s + xu_t, zr_t + 2uu_s vt + xv_t - zu_t), \\
\psi_9 & \mapsto (p_t - uu_t uu_s, rt_t + 2uu_s vt - uu_t), \\
\psi_{10} & \mapsto (tu_t + uu_z + uu_t, tv_t + vu_z + v), \\
\psi_{11} & \mapsto (-uu_z, -v_z) = -\psi_7, \\
\psi_{12} & \mapsto (uu_t, v_t) = -\psi_9, \\
\psi_{13} & \mapsto (0, ps_{sss} - uu_s uu_{sss} - \frac{3}{2} u_{sss}^2), \\
\psi_{14} & \mapsto (0, -t - 2uu_s v), \\
\varphi_{0,0}(A) & \mapsto (Ap_s - A p + uu_s (A_s u + At y + Az x - A u_s) \\
& \quad - \frac{1}{2} (u A_{ss} u + 2Azx + A_{ts} y) + Azz x^2 + 2A_{tz} x y + A_{tt} y^2), \\
& \quad Ar_s + 2As r + vs (A u_s + As u + At y + Az x v_s + A u_s v_s) \\
& \quad + 2v^2 (2A u_s u + As s u + At y + Azs x), \\
\varphi_{0,1}(A) & \mapsto (A u_s - A u_t y - Az x, A v_s + 2As v), \\
\varphi_{1,0}(A) & \mapsto (0, -2Ap_s - As (p + 2uu_s) - uu_s (A u_s + 2At y + 2Az x) \\
& \quad - \frac{1}{2} (u A_{ss} u + 2A_{ts} y + 2Azs x + Azz x^2 + 2At y y + A_{tt} y^2)), \\
\varphi_{1,1}(A) & \mapsto (0, -2A u_s - As u - At y - Az x).
\end{align*}
\]

The action of the recursion operator \( \mathcal{R}_{0,0} \) is given by the first components of the above formulas.

The action of the operator \( \mathcal{R}_{0,0}^{-1} \circ \text{pr}_u \), where \( \text{pr}_u \) ‘forgets’ the \( v \)-component of shadows, is presented in terms of the nonlocal variables \( w_0 \) and \( w_1 \) as follows\( ^3 \):

\[ \text{Most of these formulas were obtained in [4].} \]
3.5 Higher order Lax representations

The symmetry $\psi_{13} = (0, -u_{sss})$ has no lift to the Lax representation (21). We can use this fact to produce new Lax representations for Eq. (13) and its cotangent covering.

Indeed, the flow of the prolongation of the vector field $u_{sss} \partial_v$ transforms System (21) to the system

$$
\begin{align*}
\omega_t &= \Lambda \omega_y - u_y \omega_s - \left( 2u_{ys} + \frac{\kappa}{\kappa t + \mu z + 1} \right) \omega \\
&\quad + 2 (v_y + u_{ysss}) \theta_s + (v_{ys} + u_{yssss}) \theta, \\
\omega_z &= \Lambda \omega_x - u_x \omega_s - \left( 2u_{xs} + \frac{\mu}{\kappa t + \mu z + 1} \right) \omega \\
&\quad + 2 (v_x + u_{xsss}) \theta_s + (v_{xs} + u_{xssss}) \theta.
\end{align*}
$$

(30)

System (20), (30) defines a two-parametric family of higher order Lax representations for System (13), (15). If we put $v = 0$ in (30), we obtain the system

$$
\begin{align*}
\omega_t &= \Lambda \omega_y - u_y \omega_s - \left( 2u_{ys} + \frac{\kappa}{\kappa t + \mu z + 1} \right) \omega \\
&\quad + 2u_{ysss} \theta_s + u_{yssss} \theta, \\
\omega_z &= \Lambda \omega_x - u_x \omega_s - \left( 2u_{xs} + \frac{\mu}{\kappa t + \mu z + 1} \right) \omega \\
&\quad + 2u_{xsss} \theta_s + u_{xssss} \theta.
\end{align*}
$$

(31)

System (20), (31) is compatible by virtue of Eq. (13) alone and defines a two-parametric family of higher order Lax representations for this equation.
Likewise, the action of the flow of $u_{xxx} \partial_v$ on System (22) produces the systems

$$
W_t = \lambda \ W_y - u_y \ W_x - 3 \ u_{ys} \ W + (v_y + u_{ysss}) \ w_s,
$$

$$
W_z = \lambda \ W_x - u_x \ W_s - 3 \ u_{xs} \ W + (v_x + u_{xsss}) \ w_s
$$

(32)

and

$$
W_t = \lambda \ W_y - u_y \ W_x - 3 \ u_{xs} \ W + u_{ysss} \ w_s,
$$

$$
W_z = \lambda \ W_x - u_x \ W_s - 3 \ u_{ys} \ W + u_{xsss} \ w_s
$$

(33)

These systems together with (17) give one-parametric families of higher order Lax representations for System (13), (15) and Eq. (13), respectively.

Remark 3 Generally speaking, higher symmetries, contrary to classical (contact) ones, cannot be used to insert a parameter to Lax pairs (coverings), since they do not possess trajectories. But in the case of $\psi_{13}$ the result of [26] remains valid, because the first component of the symmetry at hand vanished, while the second one is independent of $u$. Of course, this is true for all symmetries of such a type.

4 Conclusions

Let us conclude our exposition with the following remarks.

(1) Since the cotangent equation $\tilde{E} = T^\ast E$ is a Lagrangian one, one has $\ell_{\tilde{E}} = \ell_{\tilde{E}}^\ast$ and thus the spaces of symmetries and cosymmetries for $\tilde{E}$ coincide. Therefore, the recursion operators found above are good candidates for Hamiltonian structures.

(2) Contrary to many other examples, the 5D Martínez Alonso–Shabat equation admits a rich family of recursion operators. It would interesting to describe the group structure of this family.

(3) It is generally accepted that nonlinear multi-dimensional equations do not possess higher symmetries. Our experience shows that cotangent equations deliver a ‘regular’ counter-example to this statement. What is the reason of this phenomenon and what is the role of higher symmetries in geometry of multi-dimensional systems?

(4) As it was noticed in Sect. 1, a number of Lax integrable systems are obtained as symmetry reduction of our equation or are related to it by Bäcklund transformations. What is the behavior of various invariants (symmetry algebras, recursion operators, Lax pairs, etc.) under these reduction and/or relations?

(5) Finally, it is interesting to compare the obtained results with the invariants of other multi-dimensional equations.

These and other problems are subjects of future research.

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Declarations

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