The Existence of Pseudoharmonic Maps For Small Horizontal Energy *

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Abstract: In this paper, we consider the pseudoharmonic heat flow with small initial horizontal energy and give the existence of pseudoharmonic maps from closed pseudo-Hermitian manifolds to closed Riemannian manifolds.

1 Introduction

The existence of harmonic mappings is an important issue which has been widely studied in Geometric Analysis. Eells and Sampson [4] proved the existence of harmonic maps from closed Riemannian manifolds into nonpositively curved closed Riemannian manifolds through the heat flow method. As for general target manifolds, we cannot get an Eell-Sampson’s type theorem in general. In [5], Mitteau discussed the harmonic heat flow with small initial energy and established an existence theorem of harmonic maps when target manifold $N$ is not assumed to be nonpositively curved.

For pseudoharmonic case, Barletta et al. in [10] introduced the notion of pseudoharmonic maps in CR geometry, which is a generalization of harmonic maps. Chang and Chang [6] studied the solution of the pseudoharmonic heat flow and proved the existence of pseudoharmonic maps from closed pseudo-Hermitian manifolds into a nonpositively curved closed Riemannian manifolds under the commutation condition $[\Delta_b, \xi] = 0$, where $\Delta_b$ is the sub-Laplacian and $\xi$ is the Reeb vector field. In [3], Ren and Yang obtained the Eells-Sampson’s type theorem without the commutation condition in [6].

In the present paper, following the idea of Mitteau, we obtain the global existence of the pseudoharmonic heat flow and the existence of pseudoharmonic maps if the initial horizontal energy is small enough. The main result is the following theorem.

Theorem 1.1 Let $(M^{2m+1}, HM, J, \theta)$ be a closed pseudo-Hermitian manifold with $m \geq 2$. Let $(N^n, g_N)$ be a closed Riemannian manifold and $\kappa$ be an upper bound for the sectional curvature of $N$. Suppose $h : M \to N$ is a smooth map with $e(h) \leq D$ for some constant $D \geq 0$. Then there exists a constant $\epsilon \geq 0$

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depending on $M, N, D$ such that if the horizontal energy $E_b(h) \leq \epsilon$, then the pseudoharmonic heat flow

$$\frac{\partial u}{\partial t} = \tau(u), u(x, 0) = h(x)$$

has a solution on $M \times [0, \infty)$. Moreover, there exists a sequence $t_i \to \infty$ such that $u(\cdot, t_i) \to u_\infty$ uniformly to a pseudoharmonic map $u_\infty$.

Assume that $u$ is a solution of the pseudoharmonic heat flow (1), then the key point now is to estimate the supremum of the total energy density of $u$, which will be denoted by $\rho$ in the following passage. The proof is mainly divided into two parts. First we point out that $\rho$ has an upper bound in a finite time interval $[0, t_0]$ through a control function. Next for $t \geq t_0$, we show that if the initial horizontal energy is sufficiently small, $\rho$ has an upper bound by the continuity of $\rho$ of $t$.

2 Pseudo-Hermitian Geometry

In this section we introduce some basic notations in pseudo-Hermitian geometry (cf. [9]). A smooth manifold $M^{2m+1}$ is a CR manifold if there is a smooth rank $m$ complex subbundle $T^{1,0}M \subseteq TM \otimes \mathbb{C}$ such that

$$T^{1,0}M \cap T^{0,1}M = \{0\}$$

$$[T^{1,0}M, T^{1,0}M] \subseteq T^{1,0}M$$

where $T^{0,1}M = \overline{T^{1,0}M}$. Equivalently, the CR structure may also be described by the real bundle $HM = \text{Re}\{T^{1,0}M \oplus T^{0,1}M\}$ and an almost complex structure $J$ on $HM$, $J(X + \bar{X}) = \sqrt{-1}(X - \bar{X}), \forall X \in T^{1,0}M$. Set

$$E = \{\omega \in T^*M|\omega(HM) = 0\}.$$ 

If $M$ is oriented, then $E$ admits global nowhere vanishing sections. A section $\theta \in \Gamma(E\setminus\{0\})$ is called a pseudo-Hermitian structure. The Levi form is defined by

$$L_\theta(X, Y) = d\theta(X, JY), \forall X, Y \in HM.$$ 

A pseudo-Hermitian manifold is the quadruple $(M, HM, J, \theta)$ with $L_\theta$ is positive definite.

The Reeb vector filed is the unique vector field $\xi$ on $M$ such that

$$\theta(\xi) = 1, \ d\theta(\xi, \cdot) = 0.$$ 

Then there is a decomposition of the tangent bundle $TM, TM = HM \oplus \mathbb{R}\xi$, which induces the projection $\pi_0 : TM \to HM$. We can extend $L_\theta$ to get a Riemannian metric $g_\theta$, called Webster metric, by

$$g_\theta = L_\theta + \theta \otimes \theta.$$ 

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We also consider the vertical energy density \( e \) if the tensor field
\[
dV
\]
where
\[
\{ \}
\]
can define the horizontal energy density \( \theta \)
\[
T_N = 2d\theta(X, Y)\xi, \ T(\xi, JX) + JT(\xi, X) = 0, \forall X, Y \in HM.
\]
The pseudo-Hermitian torsion \( A \) is defined by \( A(X, Y) = g_0(T(\xi, X), Y) \) for any \( X, Y \in TM \).

Let \((M, HM, J, \theta)\) be a pseudo-Hermitian manifold of dimension \( 2m + 1 \). Assume that \( \{\eta_1, \ldots, \eta_m\} \) be a local frame of \( T^{1,0}M \) on an open set \( U \subseteq M \) and \( \{\theta^1, \ldots, \theta^m\} \) its dual coframe. Then we have the structure equations for the Tanaka-Webster connection.

\[
d\theta = 2\sqrt{-1}\theta^\alpha \wedge \theta^\beta,
\]

\[
d\theta^\alpha = \theta^\beta \wedge \theta^\beta + A^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma, \quad \theta^\beta = \theta^\beta + \theta^\beta = 0,
\]

\[
d\theta^\alpha = \theta^\beta \wedge \theta^\beta + \Pi^\alpha_{\beta}
\]

where \( \{\theta^\alpha\} \) are the Tanaka-Webster connection 1-forms with respect to \( \{\eta_\alpha\} \).

\[
W^\alpha_{\beta\mu} = A^\alpha_{\mu,\beta}, \quad W^\alpha_{\beta\mu} = A^\beta_{\mu,\alpha}, \quad R^\alpha_{\beta\mu\lambda} \text{ is the Webster curvature.}
\]

For a Riemannian manifold \((N, g_N)\) with the Levi-Civita connection \( \nabla^N \).

Let \( \{\sigma^1\} \) be a local orthonormal coframe of \( T^*N \) and \( \{E_i\} \) its dual frame of \( TN \). For a smooth map \( f : M \to N \), there are two induced connections on \( f^*TN \) and \( TM \otimes f^*TN \), also denoted by \( \nabla \). Now we give some basic notations of energy and energy density. Let us recall the total energy density \( e(f) \) and the total energy \( E(f) \) for a \( C^2 \) map \( f : M \to N \)

\[
e(f) = \frac{1}{2}|df|^2, \quad E(f) = \frac{1}{2} \int_M e(f) dV
\]

where \( dV = \theta \wedge (d\theta)^m \) is the volume form. For pseudo-Hermitian geometry, we can define the horizontal energy density \( e_h(f) \) and the horizontal energy \( E_h(f) \) by

\[
e_h(f) = \frac{1}{2}|d_hf|^2 = \frac{1}{2}|df \circ \pi_h|^2, \quad E_h(f) = \frac{1}{2} \int_M e_h(f) dV.
\]

We also consider the vertical energy density \( e_v(f) \) and the vertical energy \( E_v(f) \)

\[
e_v(f) = e(f) - e_h(f) = \frac{1}{2}|f_0|^2, \quad E_v(f) = \frac{1}{2} \int_M e_v(f) dV.
\]

**Definition 2.1** (cf. [10]) A smooth map \( f : M \to N \) is called pseudoharmonic if the tensor field

\[
\tau(f) = 0,
\]
the proof of Theorem 1.1. In this paper, we employ the index conven tions small initial horizontal energy. Now we give some results which will be u sed in this paper with the long time existence of the pseudoharmonic heat flow with small initial horizontal energy. Therefore we are only concerned in this paper with the long time existence of the pseudoharmonic heat flow with small initial horizontal energy. Now we give some results which will be used in the proof of Theorem 1.1. In this paper, we employ the index conventions
\[ A, B, C = 0, 1, \ldots, m, 1, \ldots, m, \]
\[ \alpha, \beta, \gamma = 1, 2, \ldots, m, \]
\[ i, j, k = 1, 2, \ldots, n, \]
and use the Einstein summation convention. First let us recall the CR Bochner formulas.

\textbf{Lemma 2.1 (cf. [2])} We choose a local coframe \( \{ \theta, \theta^\alpha, \theta^\beta \} \) on \( M \), a local frame \( \{ E_j \} \) on \( N \) and denote the pseudo-Hermitian Ricci curvature by \( R^M_{\alpha \beta} \). For any smooth map \( f : M \rightarrow N \), we denote the components of \( df \), \( \nabla df \) by \( f_\alpha \), \( f_{\alpha \beta} \).

Then we have
\begin{align*}
\Delta_b |d_b f|^2 &= 2 |\nabla_b d_b f|^2 + 2 \langle \nabla_b \tau (f), d_b f \rangle + 8 \sqrt{-1} \left( f^i_\alpha f^i_\beta - f^i_\beta f^i_\alpha \right) \\
&\quad + 4 R^M_{\alpha \beta} f^i_\alpha f^i_\beta - 4 \sqrt{-1} (m - 2) \left( f^i_\alpha f^i_\beta A_{\alpha \beta} - f^i_\beta f^i_\alpha A_{\alpha \beta} \right) \\
&\quad + 4 \left( f^i_\alpha f^i_\beta R^N_{ijkl} + f^i_\alpha f^i_\beta f^i_\gamma f^i_\delta R^N_{ijkl} \right) \quad (2) \\
\Delta_b |f_0|^2 &= 2 |\nabla_b f_0|^2 + 2 \langle \nabla_\xi \tau (f), f_0 \rangle + 4 f^i_\alpha f^i_\beta f^i_\gamma f^i_\delta R^N_{ijkl} \\
&\quad + 4 \left( f^i_\alpha f^i_\beta A_{\beta \alpha, \alpha} + f^i_\alpha f^i_\beta A_{\beta \alpha, \beta} + f^i_\alpha f^i_\beta A_{\beta \alpha, \beta} + f^i_\alpha f^i_\beta A_{\beta \alpha, \alpha} \right) \quad (3)
\end{align*}

\textbf{Corollary 2.1 (cf. [3])} There exists constants \( C_1, C_2 \), where \( C_1 \) only depending on the pseudo-Hermitian Ricci curvature, the pseudo-Hermitian torsion and its divergence and \( C_2 \) only depending on the sectional curvature of \( N \), such that for a solution \( u \) of (1), we have
\begin{equation}
(\Delta_b - \partial_t) e(u) \geq -C_1 e(u) - C_2 e^2(u). \tag{4}
\end{equation}

In order to estimate the vertical energy, we needs some formulas in [1].

\textbf{Lemma 2.2 (cf. [1])} Let \( (M^{2m+1}, H M, J, \theta) \) be a closed pseudo-Hermitian manifold with \( m \geq 2 \) and \( N^n \) be a closed Riemannian manifold. Suppose \( f : M \rightarrow N \) is a smooth map. Then
\[ \sqrt{-1} \left( f^i_\alpha f^i_\beta - f^i_\beta f^i_\alpha \right) \left( (Pf + \overline{Pf}, d_b f) - \frac{1}{2m} (d_b f, \nabla_b \tau (f)) \right) \]
\[
\int_M (Pf + Pf\cdot df) dV \leq \frac{2m}{m-1} \int_M f^{\alpha}_{\bar{\alpha}} f^{\beta}_{\bar{\beta}} f^{\alpha}_{\bar{\alpha}} f^{\beta}_{\bar{\beta}} R^N_{jikl} dV
\]  

(5)

where \( Pf = (P^i_{\beta} f) \theta^i \otimes E_j \), \( P^i_{\beta} f = f^i_{\bar{\alpha}} \alpha \beta + 2\sqrt{-1} m A_{\beta \alpha} f^i_{\bar{\alpha}} \).

**Remark 2.2** (6) is contained in the proof of Theorem 4.1 of [1].

**Lemma 2.3** (cf. [3]) If \( \phi \in C^\infty(M \times (0, \delta)) \) is nonnegative and satisfies

\[
(\Delta_b - \partial_t) \phi \geq 0,
\]

then for any \( \epsilon \in (0, \delta) \), \( t \in [\epsilon, \delta] \), we find that

\[
\phi(x, t) \leq C_\epsilon \int_{t-\epsilon}^t \int_M \phi(y, s) dV ds,
\]

where \( C_\epsilon \) only depends on \( \epsilon \).

### 3 Long Time Existence for Small Horizontal Energy

Let \( u = u(x, t) \) be a solution of (1) and \([0, T_0] \) be the maximal existence time interval of \( u \), where \( 0 \leq T_0 \leq +\infty \). Then we have the following lemma.

**Lemma 3.1** If \( e(h) \leq D \) in the pseudoharmonic heat flow (1), then for the maximal time \( T_0 \) of (1) and the total energy density \( e(u) \), we have the following estimate

\[
T_0 \geq \frac{1}{C_1} \log(1 + \frac{C_1}{DC_2})
\]  

(7)

and

\[
e(u) \leq \frac{C_1 De^{C_1t}}{C_1 + C_2 D - C_2 De^{C_1t}}
\]  

(8)

on \([0, \frac{1}{C_1} \log(1 + \frac{C_1}{DC_2})] \), where \( C_1, C_2 \) are given by Corollary 2.1.

**Proof.** From Corollary 2.1, we have

\[
(\Delta_b - \partial_t) e(u) \geq -C_1 e(u) - C_2 e^2(u).
\]

We define a function \( g \) by

\[
g(t) = \frac{C_1}{C_2} \left( \frac{1}{1 - C_2 e^{B-C_1t}} - 1 \right) = \frac{C_1}{C_2} \frac{C_2 e^{B+C_1t}}{1 - C_2 e^{B+C_1t}}
\]
and $g$ satisfies
\[
\frac{\partial g}{\partial t} = \frac{C_1 e^{B+C_1 t}}{1 - C_2 e^{B+C_1 t}} = \frac{C_1}{1 - C_2 e^{B+C_1 t}} = g(C_1 + C_2 g) > 0, \quad (9)
\]
where $B = \log \frac{D+\delta}{C_1+C_2(D+\delta)}$, $\delta > 0$. Also
\[
g(0) = \frac{C_1 e^{B}}{1 - C_2 e^{B}} = \frac{C_1}{1 - C_2 e^{B}} = \frac{C_1(D+\delta)}{C_1} = D + \delta > e(h).
\]

Now we claim that:
\[
e(u)(x, t) < g(t), \quad \forall (x, t) \in M \times [0, -(B + \log C_2) / C_1]. \quad (10)
\]
Otherwise, there exits a $T \in (0, -(B + \log C_2) / C_1)$ is the first time such that $\inf_{M \times [0, T]} (g - e(u)) = 0$. Note that
\[
(\triangle_b - \partial_t)(g - e(u)) \leq -C_1 g - C_2 g^2 + C_1 e(u) + C_2 e(u) = -C_1(g - e(u)) - C_2(g - e(u))(g + e(u)). \quad (11)
\]
From (9), we have
\[
g(t) + e(u) > g(0) = D + \delta \quad (12)
\]
on $(0, T)$. (11)+(12) yield that
\[
(\triangle_b - \partial_t)(g - e(u)) \leq -(C_1 + C_2(D + \delta))(g - e(u))
\]
and
\[
(\triangle_b - \partial_t)e^{-(C_1+C_2(D+\delta)t)}(g - e(u)) \leq 0.
\]
By the maximum principle, we have
\[
e^{-(C_1+C_2(D+\delta)t)}(g - e(u)) \geq g(0) - e(h) \geq \delta, \quad \forall x \in M, \ t \in [0, T]. \text{ Let } t = T, \text{ we find that}
\]
\[
(g - e(u))(x, T) \geq \delta e^{(C_1+C_2(D+\delta)T} > 0, \quad \forall x \in M.
\]
But this leads to a contradiction with \( \inf_{M \times [0,T]} (g - e(u)) = 0 \), which proves the claim. We can complete the proof by taking \( \delta \to 0 \) in (3.4).

**Proof of Theorem 1.1** By a similar argument of the short-time existence in [3], the long-time existence for (1) will hold if \( e(u) \) is uniformly bounded inside any finite time. Due to Lemma 3.1 and (9), there exists a \( t_0 \) such that

\[
e(u)(x,t) \leq g(t) \leq g(t_0) = 2D
\]

on \([0,t_0]\).

For a given \( T \geq t_0 \), we denote

\[
\rho = \sup_{M \times [0,T]} e(u)(x,t).
\]

Then Corollary 2.1 yields that

\[(\triangle b - \partial_t)e(u) \geq -(C_1 + C_2 \rho)e(u),\]

which implies

\[(\triangle b - \partial_t)e^{-(C_1 + C_2 \rho)t}e(u) \geq 0\]

on \( M \times [0,T] \). Then Lemma 2.3 yields that

\[
e^{-(C_1 + C_2 \rho)t}e(u)(x,t)
\]

\[
\leq C_s \int_{t-s}^t \int_M e^{-(C_1 + C_2 \rho)r} e(u)(y,r)dV_y dr
\]

\[
\leq C_s e^{-(C_1 + C_2 \rho)(t-s)} \int_{t-s}^t \int_M [e_0(u) + e_0(u)](y,r)dV_y dr,
\]

\((14)\)

\( \forall x \in M, \, 0 < s \leq t \leq T \), where \( C_s \) only depends on \( s \). Integrate (5), we conclude that

\[
\int_{t-s}^t \int_M \sqrt{-1}(u_\alpha^i u_\beta^i - u_\beta^j u_\alpha^j)(y,r)dV_y dr
\]

\[
= \frac{1}{m} \int_{t-s}^t \int_M \langle Pu + \nabla u, du \rangle(y,r)dV_y dr
\]

\[
+ 2 \int_{t-s}^t \int_M \sqrt{-1}(u_\alpha^i u_\beta^j A_\alpha^j - u_\beta^j u_\alpha^i A_j^\alpha)(y,r)dV_y dr
\]

\[
- \frac{1}{2m} \int_{t-s}^t \int_M (du, \nabla \tau(u))(y,r)dV_y dr.
\]

\((15)\)

The left hand side of (15) is

\[
\int_{t-s}^t \int_M \sqrt{-1}(u_\alpha^i u_\beta^i - u_\beta^j u_\alpha^j)
\]
\[
\int_{t-s}^{t} \int_{M} \sqrt{-1}w_{0}(u_{\alpha\alpha}^{i} - u_{\alpha\alpha}^{i})(y, r) dV_y d\tau
= 2m \int_{t-s}^{t} \int_{M} |u_{0}^{i}|^2(y, r) dV_y d\tau.
\]  

Using (6), the terms in the right hand side of (15) can be estimated as

\[
\frac{1}{m} \int_{t-s}^{t} \int_{M} \langle Pu + Pu, d_{b}u \rangle (y, r) dV_y d\tau 
\leq 2m \int_{t-s}^{t} \int_{M} |d_{b}u|^2(y, r) dV_y d\tau;
\]

\[
\frac{1}{2m} \int_{t-s}^{t} \int_{M} \langle d_{b}u, \nabla_b \tau (u) \rangle (y, r) dV_y d\tau
\leq \frac{1}{4m} \int_{t-s}^{t} \int_{M} \partial_{\gamma} \langle d_{b}u, d_{b}u \rangle (y, r) dV_y d\tau
\leq \frac{1}{2m} (E_{b}(u)(t-s) - E_{b}(u)(t)),
\]

\[
2 \int_{t-s}^{t} \int_{M} \sqrt{-1}(u_{\alpha\alpha}^{i}u_{\alpha\beta}^{i}A_{\alpha\beta} - u_{\alpha\alpha}^{i}u_{\beta\alpha}^{i}A_{\alpha\beta})(y, r) dV_y d\tau
\leq 2m' \int_{t-s}^{t} \int_{M} |d_{b}u|^2(y, r) dV_y d\tau,
\]

where \( K^N \leq \kappa, \ |A| \leq \kappa' \). Since

\[
\frac{d}{dr} E_{b}(u)(r) = \int_{M} \langle \nabla_{\alpha} d_{b}u, d_{b}u \rangle (y, r) dV_y = \int_{M} \langle \nabla_{b} \partial_{\gamma} u, d_{b}u \rangle (y, r) dV_y
= - \int_{M} |\partial_{\gamma} u|^2(y, r) dV_y \leq 0,
\]

we have

\[
E_{b}(u) \leq E_{b}(h) \leq \epsilon.
\]

Then (15)-(21) yields that

\[
\int_{t-s}^{t} \int_{M} e_{0}(u)(y, r) dV_y d\tau \leq C_3(1 + \rho)E_{b}(h) \leq C_3(1 + \rho)\epsilon,
\]

where \( C_3 \) is a constant depends on \( M, N, s \). Hence (14)-(22) asserts that

\[
e^{-(C_1 + C_2)/t}e(u)(x, t) \leq C_4(1 + \rho)e^{-(C_1 + C_2)(t-s)} \epsilon,
\]
implying
\[ e(u)(x, t) \leq C_4(1 + \rho)e^{(C_1 + C_2 \rho)s} \epsilon, \]  
(23)

\( \forall x \in M, \; 0 < s \leq t \leq T, \) where \( C_4 \) is a constant depends on \( M, N, s. \) We now are ready to estimate the total energy density \( e(u) \). Let us choose \( s \) sufficiently small such that
\[ s < \frac{1}{D(4D + 2)C_2}. \]  
(24)

We now claim that
\[ \rho < -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{C_2 s}}, \]
if \( \epsilon \) is small enough. We see this by examining two cases.

(i) If \( \rho = \sup \limits_{M} e(u)(x, t) \) for some \( t \in [0, t_0] \), then (13) and (24) deduce that
\[ \frac{1}{4} + \frac{1}{C_2 s} > \frac{1}{4} + (4D + 2)D > (2D + \frac{1}{2})^2, \]
implying
\[ \rho \leq 2D < -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{C_2 s}}. \]

(ii) If \( \rho = \sup \limits_{M \times [t_0, T]} e(u)(x, t) \), then (23) yields that
\[ e^{-C_2 \rho s}e(u) \leq C_4(1 + \rho)e^{C_1 s} \epsilon, \]
\( \forall (x, t) \in M \times [t_0, T] \). Take the supremum on the left, we have
\[ \frac{e^{-C_2 \rho s}}{1 + \rho} \leq C_4 e^{C_1 s} \epsilon. \]

Define the function \( \phi \) by
\[ \phi(x) = \frac{e^{-C_2 s x}}{1 + x}. \]

Compute that
\[ \phi'(x) = \frac{(-C_2 s e^{-C_2 s x} + e^{-C_2 s x})(1 + x) - e^{-C_2 s x}}{(1 + x)^2} \]
\[ = \frac{e^{-C_2 s x}(-C_2 s x^2 - C_2 s x + 1)}{(1 + x)^2}, \]
then $\phi$ attach its maximal value at maximal point $x_0 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{C_2s}}$ with value $\phi(x_0)$. If $\epsilon$ is small enough such that 

$$C_4 e^{C_1s} \epsilon < \phi(x_0),$$

we can find that

$$\rho < -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{C_2s}}$$

or

$$\rho > -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{C_2s}}.$$ 

Since for $T = t_0$

$$\rho \leq 2D < -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{C_2s}}$$

and $\rho$ is a continuous function of $T$, we have $\rho < -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{C_2s}}$. Finally we conclude that

$$\rho < -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{C_2s}}, \quad (25)$$

for all $t$. Then there exists a sequence $t_i$, such that

$$u(\cdot, t_i) \to u_\infty, \quad (26)$$

where $u_\infty$ is a Lipschitz map. By the regularity of the solution of the subelliptic parabolic system (cf. [3]), we only need to show that $|u_{t_i}| \to 0$ as $t_i \to \infty$, $\forall x \in M$.

First note that, from (20) and (26) we have

$$\int_{t_i-1}^{t_i} \int_M |u_{t_i}|^2(x, r)dV dr = E_b(t_i - 1) - E_b(t_i) \to 0 \quad (27)$$

as $t_i \to \infty$. Since

$$(\triangle_b - \partial_t)|u_t|^2 = 2(u_{tt})^2 + 2u_{tii}u_t - 2u_{tt}u_t \
\geq -2\kappa e_b(u)|u_t|^2 \geq -2\kappa \rho |u_t|^2,$$

where $\rho = \sup_{M \times [0, \infty)} e(u)$ and $K^N \leq \kappa$. Then we find that

$$(\triangle_b - \partial_t)e^{-2\kappa \rho t}|u_t|^2 \geq 0,$$

hence applying Lemma 2.3, we conclude that

$$e^{-2\kappa \rho t_i}|u_t|^2(x, t_i) \leq C \int_{t_i-1}^{t_i} \int_M e^{-2\kappa \rho r}|u_r|^2(x, r)dV dr$$

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\[
\leq C \epsilon^{-2\kappa \rho (t_i^{-1})} \int_{t_i^{-1}}^{t_i} \int_M |u_r|^2(x, r) dV dr.
\]
Together with (27), this implies that
\[
|u_i|^2(x, t_i) \leq C \epsilon^{2\kappa \rho} \int_{t_i^{-1}}^{t_i} \int_M |u_r|^2(x, r) dV dr
\]
\[
\to 0,
\]
as \(t_i \to \infty\). This completes the proof.

\[\square\]

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References

[1] T. Chong, Y.X. Dong, Y.B. Ren, G.L. Yang, On harmonic and pseudoharmonic maps from pseudo-Hermitian manifolds, Nagoya Math. J. 234 (2019) 170–210.

[2] Y.B. Ren, G.L. Yang, T. Chong, Liouville theorem for pseudoharmonic maps from Sasakian manifolds, J. Geom. Phys. 81 (2014) 47–61.

[3] Y.B. Ren, G.L. Yang, Pseudo-harmonic maps from closed pseudo-Hermitian manifolds to Riemannian manifolds with nonpositive sectional curvature, Calc. Var. Partial Differential Equations 57 (5) (2018) Paper No. 128, 27.

[4] J. Eells, J. H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964) 109–160.

[5] J.-C. Mitteau, Sur les applications harmoniques, J. Differential Geometry 9 (1974) 41–54.

[6] S.-C. Chang, T.-H. Chang, On the existence of pseudoharmonic maps from pseudohermitian manifolds into Riemannian manifolds with nonpositive sectional curvature, Asian J. Math. 17 (1) (2013) 1–16.

[7] S. Dragomir, G. Tomassini, Differential geometry and analysis on CR manifolds, Vol. 246 of Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 2006.

[8] S. M. Webster, Pseudo-Hermitian structures on a real hypersurface, J. Differential Geometry 13 (1) (1978) 25–41.
[9] N. Tanaka, A differential geometric study on strongly pseudo-convex manifolds, Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 9, Kinokuniya Book-Store Co., Ltd., Tokyo, 1975.

[10] E. Barletta, S. Dragomir, H. Urakawa, Pseudoharmonic maps from nondegenerate CR manifolds to Riemannian manifolds, Indiana Univ. Math. J. 50 (2) (2001) 719–746.