We develop statistical mechanics for a nonrelativistic ideal gas in the presence of Lorentz violating background fields. The analysis is performed using the Standard-Model Extension (SME). We derive the corresponding laws of thermodynamics and find that, to lowest order in Lorentz violation, the scalar thermodynamic variables are corrected by a rotationally invariant combination of the Lorentz terms which can be interpreted in terms of a (frame dependent) effective mass. We find that spin couplings can induce a temperature independent polarization in the gas that is not present in the conventional case.

1 Introduction

The Standard Model Extension (SME) provides a convenient framework for studying the effects of spontaneous Lorentz and CPT symmetry breaking within the context of conventional quantum field theory. In this report we develop a statistical mechanics formalism for calculations involving the SME. Our approach and our results are quite general and include a complete analysis of the effects of all Lorentz violating terms on a nonrelativistic ideal gas. Complete details of the results announced here will appear elsewhere.

2 Notation and Framework

We adopt the viewpoint of Jaynes. In this approach to statistical inference, one assumes a collection of states, \{ψᵢ\}_{i=1}^{∞}, and a finite collection of real valued functions on the collection of states, \{f_j\}_{j=1}^{l}. Given a distribution of states, \(q_i = q(ψ_i)\), we denote by brackets the corresponding expectations; \(⟨f_j⟩ = \sum_i f_j(ψ_i)q_i\). Given observations of the mean values \{⟨f_j⟩\}_{j=1}^{l}, Jaynes argues that the most likely distribution for the given data is obtained by maximizing the (information) entropy, \(S = −k \sum_i q_i \ln (q_i)\), subject to the constraints given by the observations (here \(k\) is a positive constant). A formal argument via
variational calculus then leads immediately to a solution of the form

$$q_i = e^{-\sum_{j=1}^i \lambda_j f_j(\psi_i)} Z,$$

where the Lagrange multipliers, $\lambda_j$, are real constants and $Z$ is the partition function, $Z(\lambda) = \sum e^{-\sum_{j=1}^i \lambda_j f_j(\psi_i)}$. The same formalism permits the observables $f_j$ to depend on a finite number of parameters.

When this formalism is applied to study the statistical mechanics of an ideal gas, one immediately identifies $k$ with the Boltzmann constant and the Lagrange multipliers with the usual thermodynamic quantities (i.e., scaled inverse temperature, chemical potential, etc). The central features of thermodynamics become consequences of formal computation. As was emphasized by Jaynes, the method provides accurate thermodynamic properties of a system assuming empirically accurate observations of the mean values and the correct laws of motion embedded in the Hamiltonian.

To use this framework to study ideal gases with Lorentz violation all that is required is the appropriate Hamiltonian. For nonrelativistic spin-$\frac{1}{2}$ fermions of mass $m$ this has been worked out by Kostelecky and Lane. The result to second order in $\frac{p^2}{2m}$, $p$ momentum, is

$$H = \frac{p^2}{2m} + A + B_j \sigma^j + C_j \frac{p_j}{m} + D_{jk} \frac{p_j p_k}{m} \sigma^k + F_{jk} \frac{p_j p_k}{2m} \sigma^l + G_{jkl} \frac{p_j p_k p_l}{2m} \sigma^m,$$

where $A$, $B_j$, $C_j$, $D_{jk}$, $F_{jk}$ and $G_{jkl}$ are real parameters which can be given explicitly in terms of the standard collection of parameters defining the Lagrangian of the minimal SME.

3 Single particle systems

We first consider a system consisting of a single free spin-$\frac{1}{2}$ particle governed by the Hamiltonian $H$ appearing in (2), constrained to a cube of side length $L$.

Denote by $\psi_{n,s}^{(0)}$ the standard unperturbed solutions for the Hamiltonian where $\mathbf{n} = (n_1, n_2, n_3)$ is a triple of positive integers and $s \in \{1, -1\}$ denotes a sign. Let $E_{n,s}^{(0)}$ denote the corresponding unperturbed energies. The first order correction to the energy levels due to the Lorentz-violating terms are found using standard degenerate perturbation theory as:

$$\langle \psi_{n,s} | H - \frac{p^2}{2m} | \psi_{n,s} \rangle = \frac{\pi^2 \hbar^2}{2mL^2} \left( An^2 + \sum_i F_{ii} n_i^2 + s|\mathcal{G}(\mathbf{n})| \right),$$

where the vector $\mathcal{G}(\mathbf{n})$ is defined with components

$$\langle \mathcal{G}(\mathbf{n}) \rangle_j = \frac{2mL^2}{\pi^2 \hbar^2} B_j + \sum_i G_{ij} n_i^2.$$

\(2\)}
Using the perturbed energy expression and standard approximations, the partition function is

\[ Z^{(1)} \simeq 2e^{-\beta A_{nQ}V} \left( 1 - \frac{1}{2} \text{Tr}(F) \right), \tag{5} \]

where \( V \) is the volume of the box, \( n_Q = (m/2\beta \pi \hbar^2)^{3/2} \) is the quantum concentration, and \( \text{Tr}(F) = \sum F_{ii} \). It follows that only the \( A \) term corrects the energy and corresponds to a constant shift in all of the energy levels.

The correction to the partition function due to the \( F \) term can be incorporated into an effective mass for the fermion

\[ m^* = \left( 1 - \frac{1}{3} \text{Tr}(F) \right) m. \tag{6} \]

The expectation value of the spin can be calculated similarly:

\[ \langle s^{(1)} \rangle \simeq -\beta B - \frac{1}{2} \text{Tr}(G), \tag{7} \]

where the vector \( \text{Tr}(G) \) is defined by \( (\text{Tr}(G))_k \equiv \sum_i G_{ik} \).

## 4 Classical gas

The grand partition function for the classical gas system can be written in terms of the single-particle partition function and a chemical potential. This gives expressions for the expected particle number and energy. From these expressions it follows that there is no change in the ideal gas law.

It is possible to solve for the chemical potential \( \mu^{(C)} \)

\[ \mu^{(C)} = -kT \left( \ln \left( \frac{2n_Q}{n^{(C)}} \right) - \frac{1}{2} \text{Tr}(F) \right), \tag{8} \]

where \( n_Q \) is the quantum concentration and \( n^{(C)} \equiv \langle N^{(C)} \rangle/V \) is the concentration of the classical gas. One can also solve for the entropy, and there is a modification of the Sackur-Tetrode equation:

\[ S^{(C)} = \langle N^{(C)} \rangle k \left[ \frac{5}{2} - \frac{1}{2} \text{Tr}(F) + \ln \left( \frac{2n_Q}{n^{(C)}} \right) \right]. \tag{9} \]

Finally, the expectation of the spin is

\[ \langle s^{(C)} \rangle = -\langle N^{(C)} \rangle \left[ \beta B + \frac{1}{2} \text{Tr}(G) \right]. \tag{10} \]
5 Quantum Gas - Fermions

With notation from previous sections, and using zero subscripts to represent unperturbed quantities, the partition function for the grand canonical ensemble associated to a Fermi gas is given by

\[
\ln \left( Z_0^{(Q)}(\alpha_0) \right) = \frac{2}{\lambda^3} V f_\frac{3}{2}(e^{-\alpha_0}) ,
\]

where \( \lambda = \frac{\hbar}{2\pi mkT} \) is the thermal wavelength and \( f_\nu(e^{-\alpha}) \) is the appropriate Fermi-Dirac integral. As in the classical case, first order corrections to the partition function occur only for Lorentz violating terms of type \( F \). The corresponding partition function can be written as

\[
\ln \left( Z^{(Q)}(\alpha) \right) \simeq \left( 1 - \frac{1}{2} \text{Tr}(F) \right) \ln \left( Z_0^{(Q)}(\alpha) \right) .
\]

As above, the perturbation can be absorbed as an effective mass. Calculations then give

\[
\langle N^{(Q)}(\alpha) \rangle = \left( 1 - \frac{1}{2} \text{Tr}(F) \right) \frac{2}{\lambda^3} V f_\frac{3}{2}(e^{-\alpha}) , \]

\[
\langle E^{(Q)}(\alpha) \rangle = \frac{3}{2} \langle N^{(Q)}(\alpha) \rangle kT \frac{f_\frac{3}{2}(e^{-\alpha})}{f_\frac{3}{2}(e^{-\alpha})} ,
\]

as well as the ideal gas law

\[
\frac{P}{n^{(Q)}kT} = \frac{f_\frac{3}{2}(e^{-\alpha})}{f_\frac{3}{2}(e^{-\alpha})} ,
\]

where \( n^{(Q)} \) is the concentration of the quantum gas.

Since the map \( \alpha \to f_\frac{3}{2}(e^{-\alpha}) \) is invertible, we have a formal expression for the chemical potential in terms of the inverse \( F \)

\[
\mu^{(Q)} \simeq -kT \text{Tr} \left( \frac{\lambda^3(m^*)^2 n^{(Q)}}{2} \right) .
\]

Equation (16) can be used to obtain expressions for relevant thermodynamic quantities; for example, the Fermi energy and associated perturbations of the chemical potential at low temperature, low temperature expressions for specific heat and entropy, and low temperature perturbations of the ideal gas law.

The expectation value for the spin can be calculated in the quantum regime using the fractional occupancies:

\[
\langle s^{(Q)} \rangle \simeq -\langle N^{(Q)} \rangle \left[ 2 \frac{\beta}{\lambda^3} f_\frac{3}{2}(e^{-\alpha}) B + \frac{1}{2} \text{Tr}(G) \right] .
\]

At low temperatures, the contribution from the \( B \) term can be written as

\[
\langle s_B^{(Q)} \rangle \simeq -\langle N^{(Q)} \rangle \frac{3}{2} B \frac{\beta}{\lambda^3} \left[ 1 - \frac{\pi^2}{12} \left( \frac{kT}{\mathcal{E}_F} \right)^2 \right] ,
\]

where \( \mathcal{E}_F \) is the Fermi energy.
6 Quantum gas - Bosons

It is possible to generate a model for a free spin-0 boson gas by combining two fermions into a singlet representation of the spin group. The resulting Hamiltonian is given by

\[ H = \frac{p^2}{2m} + A + C_j \frac{p_i}{m} + F_{jk} \frac{p_ip_k}{2m} . \]  

Choosing the ground state energy to be zero, employing the notation of the previous sections and making the standard approximation, the associated grand partition function for the unperturbed case can be written as

\[ \ln(Z^{(QB)}(\alpha_0)) = \frac{1}{\lambda^3} V g_\frac{3}{2}(e^{-\alpha_0}) - \ln(1 - e^{-\alpha_0}) , \]  

where \( \lambda \) is the thermal wavelength, \( g_\frac{3}{2}(e^{-\alpha}) \) is the appropriate Bose-Einstein integral, and the ground state has been separated out. The only nontrivial leading order perturbation in (19) arises from the \( F \) term. A calculation which follows that done for the case of fermions gives

\[ \ln\left( Z^{(QB)}(\alpha) \right) \simeq \left( 1 - \frac{1}{2} \text{Tr}(F) \right) \frac{1}{\lambda^3} V g_\frac{3}{2}(e^{-\alpha}) - \ln(1 - e^{-\alpha}) . \]  

It follows that for the perturbed case we have

\[ \langle N^{(QB)}(\alpha) \rangle - \langle N_{G0} \rangle = \left( 1 - \frac{1}{2} \text{Tr}(F) \right) \frac{1}{\lambda^3} V g_\frac{3}{2}(e^{-\alpha}) , \]

\[ \langle E^{(QB)}(\alpha) \rangle = \frac{3}{2} kT \frac{V}{\lambda^3} g_\frac{3}{2}(e^{-\alpha}) . \]  

As in the Fermi case, the chemical potential can be expressed as a function of the number of particles in excited states. Because only Tr(\( F \)) enters into the grand partition function, it is possible to use the concept of effective mass to absorb the effect as before. Standard results of Bose-Einstein condensation therefore hold in a given laboratory frame.

7 Conclusion

Using Jaynesian formalism, we have developed a framework for statistical mechanics in the presence of symmetry violation which parallels the conventional case. We find that the laws of thermodynamics are the same as in the conventional case, with specific expectation values of thermodynamic quantities modified by the Lorentz-violating terms.

For an ideal gas in the absence of any external applied fields, expectation values for scalar thermodynamic quantities such as energy and particle number
were unaltered except for an overall scaling factor $\text{Tr}(F)$. This correction can be incorporated into theory as an effective mass $m^* = (1 - \frac{1}{3} \text{Tr}(F))m$ in the Hamiltonian, although the effective mass defined in this way depends on the observer’s Lorentz frame.

Focussing on spin, we find nontrivial changes in the net spin expectation value arising from the terms that couple to the spin. The pure-spin coupling $B_j$ mimics a constant background magnetic field and induces a corresponding magnetic moment per unit volume in the gas. The derivative-spin coupling $G_{ijk}$ generates a fundamentally new type of effect that induces a temperature-independent polarization in the classical gas that is proportional to $\text{Tr}(G)$.

References

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