Matrix Riemann-Hilbert problems related to branched coverings of $\mathbb{CP}^1$

Dmitry Korotkin

In these notes we solve a class of Riemann-Hilbert (inverse monodromy) problems with an arbitrary quasi-permutation monodromy group. The solution is given in terms of Szegő kernel on the underlying Riemann surface. In particular, our construction provides a new class of solutions of the Schlesinger system. We present some results on explicit calculation of the corresponding tau-function, and describe divisor of zeros of the tau-function (so-called Malgrange divisor) in terms of the theta-divisor on the Jacobi manifold of the Riemann surface. We discuss the relationship of the tau-function to determinant of Laplacian operator on the Riemann surface.

1. Introduction

Apart from pure mathematical significance (see review of A.A.Bolibruch [1]), matrix Riemann-Hilbert (inverse monodromy) problems and related theory of isomonodromic deformations play an important role in mathematical physics. In particular, the RH problems are central in the theory of integrable systems (see for example [2, 3, 4]) and the theory of random matrices [5]. In applications the main object of interest is the so-called tau-function, which was first introduced by M.Jimbo, T.Miwa and their collaborators [8]; it was later shown by B.Malgrange [7] that the tau-function may be interpreted as determinant of certain Töplitz operator. The set of zeros of the tau-function in the space of singularities of the RH problem is called the Malgrange divisor ($\mathcal{D}$); it plays a crucial role in discussion of solvability of RH problem with given monodromy data.

For generic monodromy data neither the solution of a matrix RH problem nor the corresponding tau-function can be computed analytically in terms of known special functions [6]. However, there are exceptional cases, when the RH problem can be solved explicitly; surprisingly enough, these cases often appear in applications. For example, the solution of $2 \times 2$ RH problem with an arbitrary set of off-diagonal monodromy matrices was successfully applied to the problem of finding physically meaningful solutions of stationary axially symmetric Einstein equations [10, 11, 12] and to complete classification of $SU(2)$-invariant self-dual Einstein manifolds [4].
The solution of general $2 \times 2$ RH problem of this kind was given only in 1998 in the papers [14, 13] (however, some important ingredients of this solution were understood already three decades ago [16]). In [14] it was also calculated the tau-function corresponding to this RH problem, which turned out to coincide with an appropriately defined determinant of Cauchy-Riemann operator acting in certain spinor bundle on underlying hyperelliptic curve. In the framework of conformal field theory this determinant was first introduced by A.B.Zamolodchikov [17] (see also [18, 19, 20, 21, 22]).

From the mathematical point of view, determinants of Cauchy-Riemann operator appear in the context of holomorphic factorization of determinants of Laplacian operators naturally defined via corresponding zeta-function. For mathematical description of the determinant bundles over Riemann surfaces we refer the reader to works of D.Quillen, A.A.Beilinson, Yu.I.Manin, V.V.Schechtman, D.Freed and other authors (see [23] and references in the memoir by J.Fay [25]). In particular, the series of papers by L.A.Takhtajan and P.G.Zograf (see lecture notes [24] and references therein) contain the most elementary and simultaneously rigorous treatment of the problem of holomorphic factorization of Laplacian determinants in the framework of Teichmüller theory.

In the recent paper of the author [27] the solution of $2 \times 2$ case [14] was generalized to solve a class of essentially more non-trivial RH problems with quasi-permutation monodromies in any matrix dimension. It was also conjectured that the coincidence between corresponding tau-function and determinant of appropriately defined Cauchy-Riemann operator, observed in the $2 \times 2$ case, may be extended to an arbitrary $N \times N$ case.

Here we give further support to this conjecture, computing the tau-function up to a nowhere vanishing factor which depends only on the moduli of underlying $N$-fold covering of the Riemann sphere. Comparison with the works [18] and [21] suggests natural interpretation of this factor in the framework of the holomorphic factorization of determinants of Laplacians. We would like to notice also the paper [28] where the analogy between the tau-function of Kadomtsev-Petviashvili equation and Cauchy-Riemann determinants (although rather different from the determinants arising in our context) was observed.

One can hope that in these notes, as well as in the previous works [14, 27], we make a few steps towards complete solution of one of the problems formulated in lecture notes by V.G.Knizhnik [20] devoted to applications of geometry of the moduli spaces to perturbative string theory:

- To achieve a complete understanding of the links between isomonodromy deformations and determinants of Cauchy-Riemann operators on Riemann surfaces.

Let’s summarize some of the results presented below in more detail. Consider an arbitrary compact Riemann surface $\mathcal{L}$ realized as an $N$-sheeted branched covering of the Riemann sphere. Denote the coordinate on the Riemann sphere by $\lambda$ and projections of the branch points on the Riemann sphere by $\lambda_1, \ldots, \lambda_M$. 
Then the solution $\Psi(\lambda)$ of the inverse monodromy problem with a set of $N\times N$ quasi-permutation monodromy matrices, corresponding to the singular points $\lambda_1, \ldots, \lambda_M$, can be written in the following form (this formula is slightly generalized in the main text to allow an arbitrary choice of non-vanishing entries of the quasi-permutation monodromies):

$$
\Psi(\lambda)_{jk} = S(\lambda^{(j)}, \lambda_0^{(k)}) E_0(\lambda, \lambda_0), \quad j, k = 1, \ldots, N,
$$

where $\lambda^{(j)}$ denotes the point on the $j$th sheet of $\mathcal{L}$, having projection $\lambda$ on $\mathbb{CP}^1$. Here $S(P, Q)$ is the Szegő kernel (the reproducing kernel of the $\bar{\partial}$ operator acting in a spinor bundle over $\mathcal{L}$):

$$
S(P, Q) = \frac{1}{\Theta[p|q](0)} \frac{\Theta[p|q](U(P) - U(Q))}{E(P, Q)} ;
$$

$\Theta[p|q](z|\mathbf{B})$ is the theta-function on $\mathcal{L}$ ($\mathbf{B}$ is the matrix of $b$-periods on $\mathcal{L}$) with the argument $z \in \mathbb{C}^g$ and characteristics $p, q \in \mathbb{C}^g$; $E(P, Q)$ ($P, Q \in \mathcal{L}$) is the prime-form on $\mathcal{L}$ and $E_0(\lambda, \lambda_0) = (\lambda - \lambda_0)/\sqrt{\lambda d\lambda d\lambda_0}$ is the prime-form on $\mathbb{CP}^1$, appropriately lifted to $\mathcal{L}$. The constant vectors $p, q \in \mathbb{C}^g$ (where by $g$ we denote the genus of $\mathcal{L}$) are such that the combination $\mathbf{B} p + q$ does not belong to the theta-divisor ($\Theta$) on the Jacobi variety $J(\mathcal{L})$.

As follows from the Fay identity for the Szegő kernel [24], the function $\Psi(\lambda)$ has determinant 1 and is normalized at $\lambda = \lambda_0$ by the condition $\Psi(\lambda = \lambda_0) = I$. It solves the inverse monodromy problem with quasi-permutation monodromy matrices which can be expressed in terms of $p, q$ and intersection indexes of certain contours on $\mathcal{L}$. If parameter vectors $p$ and $q$ (and, therefore, also the monodromy matrices) don’t depend on $\{\lambda_j\}$ then the residues $A_m(\{\lambda_n\})$ of the function $\Psi \frac{\partial \Psi^{-1}}{\partial \lambda}$ at the singular points $\lambda_m$ satisfy the Schlesinger system.

The tau-function, corresponding to this solution of the Schlesinger system, has the following form:

$$
\tau(\{\lambda_m\}) = F(\{\lambda_m\}) \Theta[p|q](0|\mathbf{B}),
$$

where (holomorphic and non-vanishing outside of hyperplanes $\lambda_m = \lambda_n$) function $F$ depends only on the moduli of Riemann surface $\mathcal{L}$ (i.e. points $\{\lambda_m\}$) and does not depend on the elements of of monodromy matrices parametrized by vectors $p, q$. If all branch points of the Riemann surface $\mathcal{L}$ have multiplicity 1 (more general surfaces may be obtained from the surfaces of this class by simple limiting procedure), the function $F$ is a solution of the following compatible system of equations:

$$
\frac{\partial F}{\partial \lambda_m} = \frac{1}{24} R(\lambda_m),
$$

where $R$ is the projective connection of $\mathcal{L}$ corresponding to a natural choice of local coordinates on $\mathcal{L}$ in the neighbourhoods of the points $\lambda_m$. Therefore, $F$ is

---

3 A matrix is called matrix of quasi-permutation if each of its rows and each of its columns contain only one non-vanishing entry.
the generating function of the projective connection in our system of the local coordinates on \( \mathcal{L} \).

The compatibility of equations (1.2), which follows from the Schlesinger system, implies the following non-trivial equations for the values of projective connection at the branch points:

\[
\frac{\partial R_m}{\partial \lambda_n} = \frac{\partial R_n}{\partial \lambda_m},
\]

which were, probably, unknown before. The equations (1.3) are closely related to the analogous equations for the accessory parameters which arise in the problem of uniformization of punctured sphere (see [26]).

The function \( F \) turns out to be non-vanishing in the space of singularities outside of the hyperplanes \( \lambda_m = \lambda_n \); therefore, all the zeros of the tau-function (1.1) come from the zeros of the theta-function. This allows to establish the following simple link between the Malgrange divisor \( (\vartheta) \) in \( \{ \lambda_m \} \)-space and the theta-divisor \( (\Theta) \) in Jacobi variety \( J(\mathcal{L}) \) of the Riemann surface \( \mathcal{L} \):

\[\{ \lambda_m \} \in (\vartheta) \iff Bp + q \in (\Theta).\]

In the simplest case of \( N = 2 \) the factor \( F \) can also be calculated explicitly (see [14]) which leads to the following expression for the tau-function:

\[
\tau(\{ \lambda_m \}) = [\det A]^{-\frac{1}{2}} \prod_{m<n} (\lambda_m - \lambda_n)^{-\frac{1}{2}} \Theta \left[ \begin{bmatrix} p \\ q \end{bmatrix} \right] \left( 0 \right| B),
\]

where \( M = 2g + 2; \lambda_1, \ldots, \lambda_{2g+2} \) are branch points on the hyperelliptic curve \( \mathcal{L} \) defined by equation \( w^2 = \prod_{m=1}^{2g+2} (\lambda - \lambda_m) \); \( A_{\alpha\beta} = \oint a_\alpha \lambda^\beta, \lambda^{n-1} \) is a \( g \times g \) matrix of \( a \)-periods of non-normalized holomorphic differentials on this curve.

According to the general philosophy of holomorphic factorization [23, 20, 26], in general case the square of the module of function \( F \) should be equal to the determinant of Laplacian operator, up to the factor \( \det B \) and an appropriate Liouville action. The determinant of Laplacian operator with respect to the Poincare-Lobachevskii metric is in turn defined via zeta-function regularization.

The main technical tools used here are kernel functions on Riemann surfaces, Fay identities and deformation theory of Riemann surfaces. The systematic description of these objects may be found in Fay’s books [24, 25].

We expect present results to find an application to the problem of isolating the subclass of physically reasonable solutions of stationary axially symmetric Einstein-Maxwell system [10] in the spirit of works [11, 12], devoted to vacuum Einstein equations. For Einstein-Maxwell system the matrix dimension of RH problem is equal to three. Other potential areas of application are the theory of Frobenius manifolds [3] and random matrices [5].

Our results confirm an existence of deep internal connection between the algebro-geometric approach to integrable systems [6] and certain aspects of conformal field theory.
Let’s say a few words about organization of these notes. In section 2 we remind the formulation of general Riemann-Hilbert (inverse monodromy problem), associated isomonodromy deformation equations (Schlesinger system), and definition of Jimbo-Miwa tau-function. We further discuss quasi-permutation monodromy representations and their natural relationship to branched coverings of \( \mathbb{C}P^1 \).

In section 3 we review basic facts from the deformation theory of Riemann surfaces and adjust them to the situation when the Riemann surface is realized as a branched covering of the complex plane. Then the moduli space of the Riemann surfaces (more precisely, corresponding Hurwitz space) can be parametrized by the projections of branch points on \( \mathbb{C}P^1 \).

In section 4 we solve explicitly a class of RH problems corresponding to an arbitrary quasi-permutation monodromy representation such that the associated branched covering possesses the structure of compact Riemann surface.

In section 5 we prove formula (1.1) for the tau-function and show that the equations (1.2) for function \( F \) can be integrated for the simple case of \( 2 \times 2 \) monodromies to give (1.4). Here we also discuss general case.

2. Riemann-Hilbert problem with quasi-permutation monodromies and algebraic curves

2.1. Riemann-Hilbert problem, isomonodromy deformations and tau-function

Consider a set of \( M + 1 \) points \( \lambda_0, \lambda_1, \ldots, \lambda_M \in \mathbb{C} \), and a given \( GL(N) \) monodromy representation \( \mathcal{M} \) of \( \pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, \ldots, \lambda_M\}] \). Let us formulate the following Riemann-Hilbert problem:

Find function \( \Psi(\lambda) \in GL(N, \mathbb{C}) \), defined on universal cover of \( \mathbb{C}P^1 \setminus \{\lambda_1, \ldots, \lambda_M\} \), which satisfies the following conditions:

1. \( \Psi(\lambda) \) is normalized at point \( \lambda_0 \) on some sheet of the universal cover as follows:
   \[
   \Psi(\lambda_0) = I ;
   \]

2. \( \Psi(\lambda) \) has right holonomy \( \mathcal{M}_\gamma \) along contour \( \gamma \in \pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, \ldots, \lambda_M\}] \) for all \( \gamma \);

3. \( \Psi(\lambda) \) has regular singularities of the following form at the points \( \lambda_n \):
   \[
   \Psi(\lambda) = \{G_n + O(\lambda - \lambda_n)\}(\lambda - \lambda_n)^{T_n}C_n , \quad \lambda \sim \lambda_n ,
   \]
   where \( G_n, C_n \in GL(N, \mathbb{C}) \); \( T_n = \text{diag}(t_n^{(1)}, \ldots, t_n^{(N)}) \).

Consider the following set of standard generators \( \gamma_1, \ldots, \gamma_M \) of \( \pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, \ldots, \lambda_M\}] \). Choose \( \lambda_0 \) to be starting point and assume that the contour \( \gamma_n \) starts and ends
at \( \lambda_0 \) such that interior of \( \gamma_n \) contains only one marked point \( \lambda_n \) (our convention is that the point \( \lambda = \infty \) belongs to the exterior of any closed contour on \( \mathbb{CP}^1 \)). Moreover, we assume that these generators are ordered according to the following relation:
\[
\gamma_M \gamma_{M-1} \cdots \gamma_1 = 1 .
\]

The matrices \( \mathcal{M}_\gamma \equiv \mathcal{M}_m \) are called monodromy matrices; as a consequence of (2.7) we have:
\[
\mathcal{M}_M \mathcal{M}_{M-1} \cdots \mathcal{M}_1 = I .
\]

Monodromy matrices \( \mathcal{M}_n \) are related to coefficients of asymptotics (2.6) as follows:
\[
\mathcal{M}_n = C_n^{-1} e^{2\pi i T_n} C_n .
\]

The set of the matrices \( \{ \mathcal{M}_n , T_n , n = 1 , \ldots , M \} \) is called the set of monodromy data.

Solution \( \Psi(\lambda) \) of this RH problem satisfies the following matrix differential equation with meromorphic coefficients:
\[
d\frac{\Psi}{d\lambda} = \sum_{n=1}^{M} \frac{A_n}{\lambda - \lambda_n} \Psi ,
\]

where
\[
A_n = G_n T_n G_n^{-1} .
\]

Suppose now that all monodromy matrices don’t depend on positions of singularities \( \{ \lambda_n \} \) and that for any \( n \) none of the numbers \( t^{(j)}(\lambda_n) \) differ by integer. Then function \( \Psi(\lambda) \), in addition to (2.10), satisfies the equations with respect to positions of singularities \( \lambda_n \):
\[
\frac{d\Psi}{d\lambda_n} = \left( \frac{A_n}{\lambda_0 - \lambda_n} - \frac{A_n}{\lambda - \lambda_n} \right) \Psi .
\]

Compatibility conditions of equations (2.10) and (2.12) imply dependence of residues \( A_n \) on \( \{ \lambda_m \} \) described by the system of Schlesinger equations:
\[
\frac{\partial A_m}{\partial \lambda_n} = \frac{[A_n, A_m]}{\lambda_n - \lambda_m} - \frac{[A_n, A_m]}{\lambda - \lambda_m} , \quad m \neq n ;
\]

\[
\frac{\partial A_m}{\partial \lambda_m} = - \sum_{n \neq m} \left( \frac{[A_n, A_m]}{\lambda_n - \lambda_m} - \frac{[A_n, A_m]}{\lambda_m - \lambda_0} \right) .
\]

Once a solution of the Schlesinger system is given, one can define the locally holomorphic tau-function \( \tilde{\tau} \) by the system of equations
\[
\frac{\partial}{\partial \lambda_n} \ln \tau = H_n \equiv \frac{1}{2} \text{res}_{\lambda = \lambda_n} (\Psi \Psi^{-1})^2 ; \quad \frac{\partial \tau}{\partial \lambda_n} = 0 .
\]
The tau-function does not depend on normalization point $\lambda_0$. Namely, function $\Psi^*(\lambda)$, corresponding to the same monodromy data and normalized at a different point $\lambda^*_0$, has the form $\Psi^*(\lambda) = \Psi^{-1}(\lambda^*_0)\Psi(\lambda)$. Thus $\text{tr}(\Psi^*\Psi^{-1})^2 = \text{tr}(\Psi\Psi^{-1})^2$.

Another observation which we shall need below is that tau-functions corresponding to monodromy data $\{M_m, T_m\}$ and $\{\tilde{M}_m = DM_mD^{-1}, T_m\}$, where $D$ is an arbitrary non-degenerate matrix, independent of $\lambda$ and $\{\lambda_m\}$, coincide. Namely, the new set of monodromies corresponds to function $\tilde{\Psi} = \Psi(\lambda)D$, whose logarithmic derivative with respect to $\lambda$ coincides with the logarithmic derivative of $\Psi$.

According to Malgrange [7], the isomonodromic tau-function can be interpreted as determinant of certain Toeplitz operator. The important role in the theory of RH problems is played by the divisor of zeros of the tau-function in the universal covering of the space $\{\{\lambda_m\} \in \mathbb{C}^M \mid \lambda_m \neq \lambda_n \text{ if } m \neq n\}$. In analogy to the theta-divisor ($\Theta$) on a Jacobi variety, Malgrange denoted this divisor by ($\vartheta$). The importance of the Malgrange divisor ($\vartheta$) follows from the following fact: if $\{\lambda_n\} \in (\vartheta)$, the Riemann-Hilbert problem with the given set of monodromy matrices and eigenvalues $t^{(j)}_n$ does not have a solution. A close link between Malgrange divisor ($\vartheta$) and theta-divisor ($\Theta$) $\in J(L)$ for the class of quasi-permutation monodromy representations will be established in sect. 5.

2.2. Quasi-permutation monodromy representations and branched coverings

In this paper we shall consider two special kinds of $N \times N$ monodromy representations.

**Definition 2.1.** Representation $\mathcal{M}$ is called the permutations representation if matrix $\mathcal{M}_\gamma$ is a permutation matrix for each $\gamma \in \pi_1[\mathbb{CP}1 \setminus \{\lambda_1, \ldots, \lambda_M\}]$.

Remind that a matrix is called matrix of permutation if each raw and each column of this matrix contain exactly one non-vanishing entry and this entry equals to 1. Permutation matrices are in natural one-to-one correspondence with elements of permutation group $S_N$.

The definition (2.1) is self-consistent since the product of any two permutation matrices is again a permutation matrix.

Let us introduce now the notion of quasi-permutation monodromy representation:

**Definition 2.2.** Representation $\mathcal{M}$ is called the quasi-permutations representation if $\mathcal{M}_\gamma$ is a quasi-permutation matrix for each $\gamma \in \pi_1[\mathbb{CP}1 \setminus \{\lambda_1, \ldots, \lambda_M\}]$.

Again, this definition is natural since all quasi-permutation matrices form a subgroup in $GL(N, \mathbb{C})$. We repeat once more that a matrix is called the quasi-permutation matrix if each raw and each column of this matrix contain only one non-vanishing entry.
We shall call two quasi-permutation representations $\mathcal{M}$ and $\hat{\mathcal{M}}$ equivalent if there exists some diagonal matrix $D$ such that

$$\hat{\mathcal{M}}_\gamma = D\mathcal{M}_\gamma D^{-1}$$

for all $\gamma \in \pi_1[\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_M\}]$.

Since $\det D$ cancels out in (2.15), the action of matrix $D$ in (2.15) depends on $N-1$ constants. Therefore, taking (2.8) into account, we conclude that the $GL(N)$ quasi-permutation representations of $\pi_1[\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_M\}]$ form a $MN - 2N + 1$-parametric family.

Let us now discuss the correspondence between the quasi-permutation representations of $\pi_1[\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_M\}]$ and $N$-sheeted coverings of $\mathbb{CP}^1$.

Let $\mathcal{M}$ be a quasi-permutation representation of $\pi_1[\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_M\}]$. To every such representation we can naturally assign a permutation representation $\mathcal{M}'$ substituting 1 instead of all non-vanishing entries of all monodromy matrices.

Notice that if some monodromy matrix $\mathcal{M}_n$ is diagonal, the corresponding element $\mathcal{M}'_n$ of the permutation group is identical.

**Proposition 2.3.** There exists a one-to-one correspondence between permutation representations of $\pi_1[\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_M\}]$ and $N$-sheeted compact Riemann surfaces realized as ramified coverings of $\mathbb{CP}^1$ with projections of branch points on $\mathbb{CP}^1$ equal to $\lambda_1, \ldots, \lambda_M$.

**Proof.** Given a ramified covering $\mathcal{L}$ with projections $\lambda_1, \ldots, \lambda_M$ of branch points on $\mathbb{CP}^1$, we construct the corresponding permutation representation as follows. Denote the projection of $\mathcal{L}$ on $\mathbb{CP}^1$ by $\Pi$. Generators $\mathcal{M}'_n$ of permutation monodromy group are given by the following construction. Consider the pre-image $\Pi^{-1}(\gamma_n)$ of the generator $\gamma_n$. This pre-image is a union of $N$ (not necessary closed) disjoint contours on $\mathcal{L}$ which start and end at some of the points $\lambda_0^{(j)}$ (by $\lambda_0^{(j)}$ we denote the point of $j$th sheet of $\mathcal{L}$ which has projection $\lambda$ on $\mathbb{CP}^1$). Denote by $\gamma_n^{(j)}$ the component of $\lambda^{-1}(\gamma_n)$ which starts at the point $\lambda_0^{(j)}$; the endpoint of this contour is $\lambda_0^{(j_n)}$ for some $j_n \equiv j_n[j]$. If $\lambda_n^{(j)}$ is not a branch point, then $j_n[j] = j$, and contour $\gamma_n^{(j)}$ is closed; if $\lambda_n^{(j)}$ is a branch point, then $j_n[j] \neq j$ and contour $\gamma_n^{(j)}$ is non-closed. Then the monodromy matrix $\mathcal{M}'_n$ has the following form:

$$\mathcal{M}'_n[j,l] = \delta_{j_n[j],l}$$

and naturally corresponds to some element $s_n$ of the permutation group $S_N$. On the other hand, starting from some permutation monodromy representation we obviously can glue the sheets of the Riemann surface at the branch points $\{\lambda_n\}$ in such a way that it corresponds to the permutation monodromies (2.16). Moreover, this Riemann surface is obviously compact. \(\square\)

It is also clear that the branched covering $\mathcal{L}$ is connected iff the associated permutation representation $\mathcal{M}'$ is irreducible.
Remark 2.4. We notice that if some quasi-permutation monodromy matrix $M_n$ is diagonal, then corresponding matrix $M'_n$ is equal to $I$, and $\lambda_n$ is in fact not a projection of any branch point on $\mathbb{CP}$. However, in the sequel we shall treat such points in the same fashion as all other $\lambda_m$’s. Our formulas below explicitly contain multiplicities of all branch points; the formulas are written in such form that this does not lead to any inconveniences or inconsistencies.

In this paper we shall make two non-essential simplifying assumptions:

- First, we assume that different branch points $P_m$ have different projections $\lambda_m \equiv \Pi(P_m)$ on $\lambda$-plane, i.e. $\lambda_m \neq \lambda_n$ for $m \neq n$.
- Second, we assume that all the branch points $P_m$ are simple (i.e. have multiplicity 1 or, in other words, we assume that only two sheets coalesce at each $P_m$).

On the level of corresponding permutation representation these assumptions mean that the group element $s_m$ for each $m$ acts as elementary permutation of only two numbers of the set $(1, \ldots, N)$. An arbitrary RH problem with quasi-permutation representation, corresponding to non-singular curves, may be easily solved by degeneration of the construction presented below to a submanifold in the space of branch points where some of $\lambda_m$’s coincide. This follows, in particular, from possibility to represent any element of the permutation group $S_N$ as a product of the elementary permutations.

According to the Riemann-Hurwitz formula the genus of the Riemann surface $L$ is equal to

$$g = \frac{M}{2} - N + 1; \quad (2.17)$$

therefore, our assumptions about the structure of the covering $L$ imply, in particular, that the number $M$ is even.

3. Basic objects on Riemann surfaces. Variational formulas

3.1. Basic objects

Here we collect some useful facts from the theory of Riemann surfaces and their deformations. Consider a canonical basis of cycles $(a_\alpha, b_\alpha)$, $\alpha = 1, \ldots, g$ on $L$. Introduce the dual basis of holomorphic 1-forms $w_\alpha$ on $L$ normalized by $\oint_{a_\alpha} w_\beta = \delta_{\alpha\beta}$. The matrix of 6-periods $B$ and the Abel map $U(P)$, $P \in L$ are given by

$$B_{\alpha\beta} = \oint_{b_\alpha} w_\beta, \quad U_\alpha(P) = \int_{P_0}^P w_\alpha,$$

where $P_0$ is a basepoint. Consider theta-function with characteristics $\Theta_{[p]}^{[q]}(z|B)$, where $p, q \in \mathbb{C}^g$ are vectors of characteristics; $z \in \mathbb{C}^g$ is the argument. The
theta-function is holomorphic function of variable $z$ with the following periodicity properties:

$$
\Theta[p,q](z + e_\alpha) = \Theta[p,q](z)e^{2\pi i p \alpha};
$$

(3.19)

$$
\Theta[p,q](z + B e_\alpha) = \Theta[p,q](z)e^{-2\pi i q \alpha} e^{-2\pi i z_\alpha - \pi i B \alpha};
$$

where $e_\alpha \equiv (0, \ldots, 1, \ldots, 0)$ is the standard basis in $\mathbb{C}^g$. The theta-function satisfies the heat equation:

$$
\frac{\partial^2 \Theta[p,q]}{\partial z_\alpha \partial z_\beta} = 4\pi i \frac{\partial \Theta[p,q]}{\partial B_{\alpha\beta}}.
$$

(3.20)

Let us consider some non-singular half-integer characteristic $[p^*, q^*]$. The prime-form $E(P, Q)$ is the following skew-symmetric $(-1/2, -1/2)$-form on $\mathcal{L} \times \mathcal{L}$:

$$
E(P, Q) = \frac{\Theta[p^*, q^*](U(P) - U(Q))}{h(P)h(Q)},
$$

(3.21)

where the square of a section $h(P)$ of a spinor bundle over $\mathcal{L}$ is given by the following expression:

$$
h^2(P) = \sum_{\alpha=1}^g \partial z_\alpha \left\{ \Theta[p^*, q^*](0) \right\} w_\alpha(P).
$$

(3.22)

To completely define $h(P)$ we assume it to be a section of the spinor bundle corresponding to characteristic $[p^*, q^*]$. Then automorphy factors of the prime-form along all cycles $a_\alpha$ are trivial; the automorphy factor along each cycle $b_\alpha$ equals to $\exp\{-\pi iB_{\alpha\alpha} - 2\pi i(U_\alpha(P) - U_\alpha(Q))\}$. The prime-form has the following local behavior as $P \to Q$:

$$
E(P, Q) = \frac{x(P) - x(Q)}{\sqrt{dx(P)} \sqrt{dx(Q)}}(1 + o(1)),
$$

(3.23)

where $x(P)$ is a local parameter.

The meromorphic symmetric bidifferential on $\mathcal{L} \times \mathcal{L}$ with second order pole at $P = Q$ and biresidue 1, given by the formula

$$
\mathfrak{w}(P, Q) = d_P d_Q \ln E(P, Q),
$$

is called the Bergmann kernel. All $a$-periods of $\mathfrak{w}(P, Q)$ with respect to any of its two variables vanish. The period of Bergmann kernel along basic cycle $b_\alpha$ with respect to, say, variable $P$, is equal to $2\pi i w_\alpha(Q)$ and vice versa.

$^2$One can prove that all the zeros of the r.h.s. of (3.22) are of the second order; this allows to define consistently its square root.
The Bergmann kernel has double pole with the following local behavior on the diagonal $P \rightarrow Q$:

$$ w(P, Q) = \left\{ \frac{1}{(x(P) - x(Q))^2} + H(x(P), x(Q)) \right\} dx(P) dx(Q). $$  \hspace{1cm} (3.24)

where $H(x(P), x(Q))$ is a non-singular part of $w$ in each coordinate chart.

The restriction of function $H$ on the diagonal gives projective connection $R(x)$:

$$ R(x) = 6H(x(P), x(P)), $$ \hspace{1cm} (3.25)

which non-trivially depends on the chosen system of local coordinates on $L$. Namely, it is easy to verify that the projective connection transforms as follows with respect to change of local coordinate $x \rightarrow f(x)$:

$$ R(x) \rightarrow R(f(x))[f'(x)]^2 + \{f(x), x\} $$ \hspace{1cm} (3.26)

where

$$ \{f(x), x\} \equiv f'''f' - \frac{3}{2} \left(\frac{f''}{f'}\right)^2 $$

is the Schwarzian derivative.

Suppose that the Riemann surface $L$ is realized as branched covering of $\lambda$-plane, and the local coordinates are chosen in standard way, i.e. $x = \lambda - \Pi(P)$ for any point $P$ which does not coincide with branch points, and $x = (\lambda - \Pi(P_{m}))^{1/k_{m}}$ for any branch point of degree $k_{m}$. Projective connection corresponding to this choice of local coordinates will be denoted by $R^{H}(P)$.  

The Szegö kernel $S(P, Q)$ is the $(1/2, 1/2)$-form on $L \times L$ defined by the formula

$$ S(P, Q) = \frac{1}{\Theta[p_q(0)]} \frac{\Theta[p_q]}{E(P, Q)} $$ \hspace{1cm} (3.27)

where $p, q \in \mathbb{C}^g$ are two vectors such that $\Theta[p_q(0)] \neq 0$. The Szegö kernel is the kernel of the integral operator $\bar{\partial}^{-1}$, where the operator $\bar{\partial}$ acts in the spinor bundle over $L$ with the holonomies $e^{2\pi ip_{\alpha}}$ and $e^{-2\pi i q_{\alpha}}$ along basic cycles. The Szegö kernel itself has holonomies $e^{2\pi ip_{\alpha}}$ and $e^{-2\pi i q_{\alpha}}$ along cycles $a_{\alpha}$ and $b_{\alpha}$, respectively, in its first argument and the inverse holonomies in its second argument.

The Szegö kernel is related to the Bergmann kernel as follows (24, p.26):

$$ S(P, Q)S(Q, P) = -w(P, Q) - \sum_{\alpha, \beta=1}^{g} \partial_{z_{\alpha}z_{\beta}} \ln \Theta[p_q(0)] w_{\alpha}(P) w_{\beta}(Q). $$ \hspace{1cm} (3.28)

For any two sets $P_{1}, \ldots, P_{N}$ and $Q_{1}, \ldots, Q_{N}$ of points on $L$ the following Fay identity takes place (see 24, p.33):

$$ \det\{S(P_{j}, Q_{k})\} $$

3Here “H” stands for “Hurwitz”
In particular, for $N = 2$ this formula is nothing but the famous Fay trisecant identity. The proof of (3.29) is quite simple: one can check this identity comparing analytical properties of the l.h.s. and the r.h.s. with respect to all variables $P_j$ and $Q_k$ using only the basic facts about holonomies and positions of zeros of the prime-form and theta-function.

Below we study dependence of these objects on the moduli of the Riemann surface. These facts will be required later for calculation of tau-function.

### 3.2. Variational formulas on a Riemann surface

If a Riemann surface is realized as a branched covering of $\mathbb{C}P^1$ then the positions of the branch points may be used as natural moduli parameters. The Riemann surfaces which can be obtained from a given Riemann surface by variation of positions of the branch points without changing their ramification type, span the so-called Hurwitz spaces (see [3, 29]). We shall start from the well-known variational formulas on an abstract Riemann surface and then show how these formulas look in the branched coverings realization. We shall mainly follow [25].

Consider a one-parametric family $\mathcal{L}\varepsilon$ of Riemann surfaces of genus $g$. It can be described as smooth deformation of the complex structure on a fixed Riemann surface $\mathcal{L}\varepsilon|_{\varepsilon=0} = \mathcal{L}$. If $x$ is a local coordinate on $\mathcal{L}$, the local coordinate $x\varepsilon$ on $\mathcal{L}\varepsilon$ is holomorphic in $\varepsilon$:

\begin{equation}
(3.30) \quad x\varepsilon = x + \varepsilon q(x, \overline{x}) + \ldots.
\end{equation}

Then the Beltrami differential $\mu$ (which is a $(-1)$-form with respect to $x$ and a 1-form with respect to $\overline{x}$), corresponding to the infinitesimal deformation of the curve $\mathcal{L}\varepsilon$ at $\varepsilon = 0$, is given by

\begin{equation}
(3.31) \quad \mu(x, \overline{x}) = \partial q(x, \overline{x}) d\overline{x} / dx .
\end{equation}

Let us introduce the following notations for the infinitesimal deformation defined by the Beltrami differential:

\begin{equation}
\delta\mu \equiv \frac{\partial}{\partial\varepsilon}|_{\varepsilon=0} , \quad \overline{\delta}\mu \equiv \frac{\partial}{\partial\varepsilon}|_{\varepsilon=0} .
\end{equation}

The infinitesimal variation of the basic holomorphic 1-forms and matrix of b-periods is given by the following Rauch formulas ([25], p.57):

\begin{equation}
(3.32) \quad \delta\mu w_\alpha(Q) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mu(P) w_\alpha(P) w(P, Q) , \quad \overline{\delta}\mu w_\alpha(Q) = 0 ;
\end{equation}

\begin{equation}
(3.33) \quad \delta\mu B_{\alpha\beta} = \int_{\mathcal{L}} \mu w_\alpha w_\beta , \quad \overline{\delta}\mu B_{\alpha\beta} = 0 .
\end{equation}
Taking into account that the integral of the Bergmann kernel \( w(P, Q) \) along cycle \( b_\beta \) with respect to variable \( Q \) is equal to \( 2\pi i w_\beta(P) \), the formulas (3.33) immediately follow from (3.32).

Let us apply these formulas to a Riemann surface \( L \) realized as a branched covering of \( \mathbb{CP}^1 \). We can consider projections \( \lambda_m \) of branch points on \( \mathbb{CP}^1 \) as coordinates on the Hurwitz space. Therefore, we will be interested in derivatives of basic holomorphic differentials and matrix of \( b \)-periods with respect to \( \lambda_m \).

**Theorem 3.5.** Basic holomorphic differentials and matrix of \( b \)-periods of an \( N \)-fold covering \( L \) of \( \mathbb{CP}^1 \) satisfy the following equations with respect to the positions of the branch points:

\[
\begin{align*}
\partial_{\lambda_m} \{ w_\alpha(P) \} &= \text{res}_{\lambda=\lambda_m} \left\{ \frac{-1}{(d\lambda)^2} \sum_j w_\alpha(\lambda^{(j)}) w(\lambda^{(j)}, P) \right\}, \\
\partial_{\lambda_m} \{ w_\alpha(P) \} &= 0; \\
\partial_{\lambda_m} \{ B_{\alpha\beta} \} &= \text{res}_{\lambda=\lambda_m} \left\{ \frac{-4\pi i}{(d\lambda)^2} \sum_{j<k} w_\alpha(\lambda^{(j)}) w_\beta(\lambda^{(k)}) \right\}, \\
\partial_{\lambda_m} B &= 0.
\end{align*}
\]

**Proof.** This theorem is valid for arbitrary multiplicities of the branch points. Here we check it under assumption that all branch points are simple and have different projections on \( \lambda \)-plane. In this case the local parameter in the neighborhood of a branch point \( \lambda_m \) is equal to \( x = \sqrt{\lambda - \lambda_m} \); this is the only local parameter which depends on the position of \( \lambda_m \). The Taylor series (3.30) looks as follows:

\[(\lambda - \lambda_m - \varepsilon)^{1/2} = (\lambda - \lambda_m)^{1/2} - \frac{\varepsilon}{2}(\lambda - \lambda_m)^{-1/2} + O(\varepsilon^2).\]

Therefore, the Beltrami differential

\[
\mu_m(P) = -\frac{1}{2} \left( \partial_{\frac{1}{x}} \left\{ \frac{1}{x} \right\} \right) \frac{dx}{d\tau} \equiv -\frac{\pi}{2} \delta(x) \frac{dx}{d\tau},
\]

where \( \delta(x) \) is two-dimensional delta-function, describes the infinitesimal deformation of complex structure under variation of position of the branch point \( \lambda_m \) [25]:

\[
\delta_{\mu_m} = \partial_{\lambda_m}.
\]

Substitution of Beltrami differential (3.36) into Rauch variational formulas (3.32), (3.33) gives (3.34) and the following formula for variation of matrix of \( B \)-periods:

\[
\frac{1}{2\pi i} \partial_{\lambda_m} \{ B_{\alpha\beta} \} = \text{res}_{\lambda=\lambda_m} \left\{ \frac{1}{(d\lambda)^2} \sum_{j=1}^N w_\alpha(\lambda^{(j)}) w_\beta(\lambda^{(j)}) \right\}.
\]
In turn, this formula implies (3.35) if we take into account the following lemma 3.6.

Lemma 3.6. An arbitrary holomorphic differential \( w(P) \) on a compact Riemann surface \( \mathcal{L} \), realized as \( N \)-fold covering of \( \mathbb{C}P^1 \), satisfies the following relation:

\[
\sum_{j=1}^{N} w(\lambda^{(j)}) = 0.
\]

Proof. This lemma is also valid without any restrictions on ramification type at the points \( \lambda_m \); it is sufficient to check that \( \sum_{j=1}^{N} w(\lambda^{(j)}) \) is a holomorphic differential on \( \mathbb{C}P^1 \). The only suspicious points are the branch points \( P_m \). Regularity of this sum at point \( P_m \) follows from analysis of its Laurent series in the neighborhood of \( \lambda_m \). For example, if the branch point \( P_m \) is simple, it is sufficient to observe that the local parameter \( x = \sqrt{\lambda - \lambda_m} \) has different signs on the sheets glued at the branch point \( \lambda_m \). □

4. Solution of Riemann-Hilbert problems with quasi-permutation monodromies and Szegö kernel

Here we are going to solve a class of Riemann-Hilbert problems for quasi-permutation monodromy representations \( \mathcal{M} \) which correspond to branch coverings with simple branch points. Solution of a RH problem corresponding to an arbitrary quasi-monodromy representation may be obtained as a limiting case of this construction. As before, denote projections of the branch points of this curve on \( \mathbb{C}P^1 \) by \( \lambda_1, \ldots, \lambda_M \); assume that all branch points are simple and have different projections on \( \mathbb{C}P^1 \). Genus \( g \) of the Riemann surface \( \mathcal{L} \) is equal to \( M/2 - N + 1 \); therefore, \( M \) should be always even.

In the sequel it will be convenient to assign degree to all of the points \( \lambda_m^{(j)} \) in the following way: \( k_m^{(j)} = 2 \) if \( \lambda_m^{(j)} \) is a (simple!) branch point, and \( k_m^{(j)} = 1 \) if \( \lambda_m^{(j)} \) is not a branch point.

Let us introduce on \( \mathcal{L} \) a contour \( S \), which connects certain initial point \( P_0 \) (it is convenient to assume that \( \Pi(P_0) = \lambda_0 \)) with all points \( \lambda_m^{(j)} \), including all branch points. Suppose that the point \( \lambda_0 \) does not belong to the set of projections of basic cycles \( (a_\alpha, b_\alpha) \) on \( \mathbb{C}P^1 \). Introduce the following objects:

- Intersection indexes of the contours \( l_m^{(j)} \) with all the basic cycles and the contour \( S \):

\[
I_m^{(j)} = l_m^{(j)} \circ a_\alpha, \quad J_m^{(j)} = l_m^{(j)} \circ b_\alpha, \quad K_m^{(j)} = l_m^{(j)} \circ S
\]

where \( m = 1, \ldots, M \); \( \alpha = 1, \ldots, g \); \( j = 1, \ldots, N \).
The contour $S$ can always be chosen in such a way that $K_m^{(j)} = 1$ if $\lambda_m^{(j)}$ is not a ramification point; if $\lambda_m^{(j)}$ is a branch point, then either $K_m^{(j)} = 1$ or $K_m^{(j)} = 0$.

- Two vectors $\mathbf{p}, \mathbf{q} \in \mathbb{C}^g$.
- Constants $r_m^{(j)} \in \mathbb{C}$ assigned to each point $\lambda_m^{(j)}$; we assume that the constants $r_m^{(j)} = r_m^{(j)'}$ coincide if $\lambda_m^{(j)} = \lambda_m^{(j)'}$, i.e. if $\lambda_m^{(j)}$ is a branch point. We require that

$$\sum_{m=1}^M \sum_{j=1}^N r_m^{(j)} = 0 \quad (4.41)$$

Therefore, among constants $r_m^{(j)}$ we have only $MN - 2g - 2N + 1$ independent parameters naturally assigned to non-coinciding points among $\lambda_m^{(j)}$.

Hence, altogether we introduced $MN - 2N + 1$ independent constants $\mathbf{p}, \mathbf{q}$ and $r_m^{(j)}$; as we saw above, this number exactly equals the number of non-trivial parameters carried by the non-vanishing entries of the quasi-permutation monodromy matrices of our RH problem.

Now we are in position to define $N \times N$ matrix-valued function $\Psi(\lambda)$ which will later turn out to solve a Riemann-Hilbert problem. We define the germ of function $\Psi(\lambda)$ in a small neighborhood of the normalization point $\lambda_0$ by the following formula:

$$\Psi(\lambda)_{kj} = \tilde{S}(\lambda^{(j)}, \lambda^{(k)}_0) E_0(\lambda, \lambda_0). \quad (4.42)$$

Here $\tilde{S}(P, Q)$ is a section of certain spinor bundle on $\mathcal{L} \times \mathcal{L}$, given by the following formula inside of the fundamental polygon of Riemann surface $\mathcal{L}$:

$$\tilde{S}(P, Q) = \frac{\Theta[p^0]}{\Theta[p^1]} (U(P) - U(Q) + \Omega) \prod_{m=1}^M \prod_{l=1}^N \left[ \frac{E(P, \lambda_m^{(l)})}{E(Q, \lambda_m^{(l)'})} \right] r_m^{(l)} \quad (4.43)$$

By $E_0$ we denote the prime-form on $\mathbb{C}P^1$

$$E_0(\lambda, \lambda_0) = \frac{\lambda - \lambda_0}{\sqrt{d(\lambda - \lambda_0)^2}}, \quad (4.44)$$

naturally lifted to $\mathcal{L}$ (the precise way to lift $E_0$ from $\mathbb{C}P^1$ to $\mathcal{L}$ we shall discuss below);

$$\Omega \equiv \sum_{m=1}^M \sum_{j=1}^N r_m^{(j)} U(\lambda_m^{(j)}). \quad (4.45)$$

The vector $\Omega$ does not depend on the choice of initial point of the Abel map due to assumption $(4.41)$. The formula $(4.42)$ makes sense if $\Theta[p^0] (\Omega) \neq 0$. 

To define the function $\Psi$ completely we need to specify how to lift the spinor $\sqrt{d\lambda}$ from $\mathbb{CP}^1$ to $L$. Being lifted on $L$, the 1-form $d\lambda$ has simple zeros at all the branch points $P_m$. Therefore, $\sqrt{d\lambda}$ is not a holomorphic section of a spinor bundle on $L$. However, we can define it in such a way that the ratio $h(P)/\sqrt{d\lambda}$ (where $\lambda = \Pi(P)$; $h(P)$ is the spinor used in definition of the prime-form) has trivial automorphy factors along all basic cycles. This function has poles of order $1/2$ at the branch points $P_m$ and holonomies $-1$ along small cycles encircling the branch points $P_m$.

Consider now the ratio of two prime-forms

$$f(P, Q) \equiv \frac{E_0(\lambda, \mu)}{E(P, Q)},$$

where $\lambda = \Pi(P)$, $\mu = \Pi(Q)$. Consider holonomies of $f(P, Q)$ along cycles $a_\alpha$ and $b_\alpha$ with respect to, say, variable $P$. From the previous discussion we conclude that these holonomies are equal to $e^{\pi i p_\alpha}$ and $e^{-\pi i q_\alpha - 2\pi i (U_\alpha(P) - U_\alpha(Q))}$, respectively. Notice, that these holonomies do depend on the choice of the odd half-integer $[p_\ast q_\ast]$, in contrast to the holonomies of the prime-form $E(P, Q)$ itself!

In addition, $f(P, Q)$ has holonomies $e^{2\pi i (k_m - 1)} = \pm 1$ along small cycles encircling branch points $P_m$.

The following theorem gives a solution to a class of RH problems with quasi-permutation monodromies. This is the main result of present section:

**Theorem 4.7.** Suppose that $\Theta([p_\ast q_\ast] (\Omega)) \neq 0$. Let us analytically continue function $\Psi(\lambda)$ from the neighborhood of the normalization point $\lambda_0$ to the universal covering $\hat{T}$ of $\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_M\}$. Then the function $\Psi(\lambda)$ is non-singular and non-degenerate on $\hat{T}$. It has regular singularities at the points $\lambda = \lambda_m$, satisfies the normalization condition $\Psi(\lambda = \lambda_0) = I$ and solves the Riemann-Hilbert problem with the following quasi-permutation monodromies:

$$(4.47) \quad (M_{n})_{k,l} = \exp \left\{ 2\pi i \{k_n^{(k)} r_n^{(k)} + 1/2 \} - 1/2 \right\} K_n^{(k)}$$

$$+ \sum_{\alpha=1}^{g} \left\{ j_{n}^{(k)}(p_\alpha + p_\alpha^*) - I_{n}^{(k)}(q_\alpha + q_\alpha^*) \right\} \delta_{jm,[k,l]}$$

where all constants $p, q$ and $r_n^{(k)}$ were introduced above; $j_{n}^{(k)}$ stands for the number of sheet where the contour $i_{m}^{(k)}$ ends.

**Proof.** Choose in the Fay identity (3.29) $P_j \equiv \lambda^{(j)}$ and $Q_k \equiv \lambda^{(k)}_0$. Then, taking into account the holonomy properties of the prime-form and asymptotics (3.23), we conclude that

$$\det \Psi = \prod_{m=1}^{M} \prod_{j,k=1}^{N} \left[ \frac{E(\lambda^{(j)}, \lambda^{(k)}_m)}{E(\lambda^{(j)}_0, \lambda^{(k)}_m)} \right] r_n^{(k)}$$
which, being considered as function of $\lambda$, does not vanish outside of the points $\lambda^{(j)}_k$; thus $\Psi \in GL(N)$ if $\lambda$ does not coincide with any of $\lambda_m$. The normalization condition $\Psi_{jk}(\lambda_0) = \delta_{jk}$ is an immediate corollary of the asymptotic expansion of the prime form (3.23).

Expressions (4.47) for the monodromy matrices of function $\Psi$ follow from the simple consideration of the components of function $\Psi$. Suppose for a moment that the function $\hat{S}(P, \lambda^{(k)}_0)E_0(\lambda, \lambda_0)$, defined by (4.43), would be a single-valued function on $L$ (as function of $P \in L$). Then all monodromy matrices would be matrices of permutation: the analytical continuation of the matrix element $\hat{S}(\lambda^{(j)}_k, \lambda^{(k)}_0)E_0(\lambda, \lambda_0)$ along contour $l_jm$ would simply give the matrix element $\hat{S}(\lambda^{(j)}_k, \tilde{\lambda}^{(m)}_0)E_0(\lambda, \lambda_0)$. However, since in fact the function $\hat{S}(P, \lambda^{(k)}_0)E_0(\lambda, \lambda_0)$ gains some non-trivial multipliers from crossing the basic cycles $a_\alpha, b_\alpha$ and contour $S$, we get in (4.47) an additional exponential factor. Its explicit form is a corollary of the definition of intersection indexes which enter this expression, and periodicity properties of the theta-function and the prime-form.

Remark 4.8. If we assume that all constants $r^{(j)}_m$ vanish, the formula (4.42) may be nicely rewritten in terms of the Szegő kernel (3.27) as follows:

\begin{equation}
\Psi(\lambda)_{kj} = S(\lambda^{(j)}_k, \lambda^{(k)}_0)E_0(\lambda, \lambda_0)
\end{equation}

where $E_0(\lambda, \lambda_0) = (\lambda - \lambda_0)/\sqrt{d\lambda, d\lambda_0}$ is the prime-form on $\mathbb{C}P1$.

If we now assume that vectors $p, q$ and constants $r^{(j)}_m$ don’t depend on $\{\lambda_m\}$ then the monodromy matrices $M_j$ also don’t carry any $\{\lambda_m\}$-dependence and the isomonodromy deformation equations are satisfied.

Theorem 4.9. Assume that vectors $p$ and $q$ and constants $r^{(j)}_m$ don’t depend on $\{\lambda_m\}$. Then the functions

\begin{equation}
A_n(\{\lambda_m\}) = \text{res}|_{\lambda=\lambda_n} \{\Psi \Psi^{-1}\},
\end{equation}

where $\Psi(\lambda)$ is defined in (4.43), satisfy the Schlesinger system (2.13) outside of the hyperplanes $\lambda_n = \lambda_m$ and a submanifold of codimension one defined by the condition

\begin{equation}
Bp + q + \Omega \in (\Theta),
\end{equation}

where $(\Theta)$ denotes the theta-divisor on Jacobian $J(L)$.

Remark 4.10. The formula (4.42) remains valid for solution of RH problem with an arbitrary quasi-permutation monodromy representation corresponding to a non-singular branched covering. In other words, our assumption of simplicity of all branch points is non-essential. The expressions for monodromy matrices (4.47) also remain valid if we assume that the degree $k^{(j)}_m$ stands for the number of sheets glued at the point $\lambda^{(j)}_m$. 


5. Isomonodromic tau-function and Cauchy-Riemann determinants

5.1. Tau-function and projective connection

According to the definition of the tau-function (2.14), let us start with calculation of expression $\text{tr} \left( \Psi(\lambda) \Psi^{-1} \right)^2$. Notice that this object is independent of the choice of normalization point $\lambda_0$ [substitution of $\lambda_0$ by another point $\tilde{\lambda}_0$ corresponds to the $\lambda$-independent “gauge” transformation $\Psi(\lambda) \rightarrow \tilde{\Psi}(\lambda) = \Psi^{-1}(\tilde{\lambda}_0)\Psi(\lambda)$].

Let us rewrite once more the formula (4.42) for $\Psi_{jk}$:

\[ \Psi_{kj}(\lambda, \lambda_0) = \hat{S}(\lambda^{(j)}, \lambda_0^{(k)}) \frac{\lambda - \lambda_0}{\sqrt{d\lambda} \sqrt{d\lambda_0}} \]  

(5.51)

where $\hat{S}(P, Q)$ is given by expression (4.43). Consider the limit $\lambda_0 \rightarrow \lambda$. In this limit the matrix elements of the function $\Psi$ behave as follows:

\[ \Psi_{kj}(\lambda, \lambda_0) = \lambda_0 - \lambda \frac{\hat{S}(\lambda^{(j)}, \lambda^{(k)})}{\sqrt{d\lambda} \sqrt{d\lambda_0}} + O\left\{ (\lambda_0 - \lambda)^2 \right\}, \quad k \neq j \]  

(5.52)

\[ \Psi_{jj}(\lambda, \lambda_0) = 1 + \lambda_0 - \lambda \frac{W_1(\lambda^{(j)}) - W_2(\lambda^{(j)})}{\sqrt{d\lambda} \sqrt{d\lambda_0}} \]  

(5.53)

where $W_1(P)$ is a linear combination of the basic holomorphic 1-forms on $L$:

\[ W_1(P) = \frac{1}{\Theta [P] (\Omega)} \sum_{\alpha=1}^{g} \partial z_{\alpha} \{ \Theta [P] (\Omega) \} w_{\alpha}(P), \]  

(5.54)

and $W_2(P)$ is the following meromorphic 1-form with simple poles at the points $\lambda_m^{(j)}$ and the residues $r_m^{(j)}$:

\[ W_2(P) = \sum_{m=1}^{M} \sum_{j=1}^{N} r_m^{(j)} dP \ln E(P, \lambda_m^{(j)}). \]  

(5.55)

Taking into account independence of the expression $\text{tr} \left( \Psi(\lambda) \Psi^{-1} \right)^2$ on position of the normalization point $\lambda_0$, we have

\[ \text{tr} \left( \Psi(\lambda) \Psi^{-1} \right)^2 = 2 \sum_{j<k} \hat{S}(\lambda^{(j)}, \lambda^{(k)}) \hat{S}(\lambda^{(k)}, \lambda^{(j)}) + \sum_{j=1}^{N} \left( W_1(\lambda^{(j)}) - W_2(\lambda^{(j)}) \right)^2. \]

To transform this expression we first notice that (2.34), p.26

\[ \hat{S}(P, Q)\hat{S}(Q, P) = -w(P, Q) - \sum_{\alpha, \beta=1}^{g} \partial^2_{z_{\alpha} \bar{z}_{\beta}} \{ \ln \Theta [P] (\Omega) \} w_{\alpha}(P)w_{\beta}(Q). \]
Furthermore, since $W_1(P)$ is a holomorphic 1-form on $L$, the expression $\sum_{j=1}^N W_1(\lambda^{(j)})$ vanishes identically according to Lemma 3.6; hence

$$
\sum_{j=1}^N \{W_1(\lambda^{(j)})\}^2 = -2 \sum_{j=1}^N \sum_{\alpha_1, \beta_1=1}^g \partial_z^{(\alpha_1)} \{\partial_2 \{\theta_2 \{p_3\} (\Omega)\} \partial_z^{(\beta_1)} \{\ln \theta_2 \{p_3\} (\Omega)\} w_\alpha(\lambda^{(j)}) w_\beta(\lambda^{(k)})\}. 
$$

Similarly, we can conclude that $\sum_{j=1}^N \{W_2(\lambda^{(j)})\}^2$ is a meromorphic 2-form on $CP^1$ which has poles only at the points $\lambda_m$; calculation of its residues gives

$$(5.56) \quad \sum_{j=1}^N \{W_2(\lambda^{(j)})\}^2 = \sum_{m,n=1}^M \frac{r_{mn}(d\lambda)^2}{(\lambda - \lambda_n)(\lambda - \lambda_m)},$$

where

$$(5.57) \quad r_{mn} = \sum_{j=1}^N r_m^{(j)} r_n^{(j)}.$$

Therefore, as the first step of our calculation, we get the following expression:

$$(5.58) \quad \frac{1}{2} \text{tr} \left( \Psi \lambda \Psi^{-1} \right)^2 (d\lambda)^2 = -\sum_{j<k} w(\lambda^{(j)}, \lambda^{(k)})$$

$$- \frac{1}{\theta_2 \{p_3\} (\Omega)} \sum_{j<k} \sum_{\alpha_1, \beta_1=1}^g \partial_z^{(\alpha_1)} \{\theta_2 \{p_3\} (\Omega)\} \partial_z^{(\beta_1)} \{\ln \theta_2 \{p_3\} (\Omega)\} w_\alpha(\lambda^{(j)}) w_\beta(\lambda^{(k)}) + \frac{1}{2} \sum_{m,n} \frac{r_{mn}(d\lambda)^2}{(\lambda - \lambda_n)(\lambda - \lambda_m)}$$

$$- \frac{1}{\theta_2 \{p_3\} (\Omega)} \sum_{j<k} \sum_{\alpha_1, \beta_1=1}^g \partial_z^{(\alpha_1)} \{\theta_2 \{p_3\} (\Omega)\} \partial_z^{(\beta_1)} \{\ln \theta_2 \{p_3\} (\Omega)\} w_\alpha(\lambda^{(j)}) d_\lambda \ln E(P, \lambda_m^{(j)}).$$

Let us now analyze the Hamiltonians

$$H_m \equiv \frac{1}{2} \text{res}_{\lambda = \lambda_m} \left\{ \text{tr} \left( \Psi \lambda \Psi^{-1} \right)^2 \right\}.$$ 

Using the heat equation for theta-function (3.20), we can represent $H_m$ in the following form:

$$
(5.59) \quad H_m = -\text{res}_{\lambda = \lambda_m} \left\{ \sum_{j<k} w(\lambda^{(j)}, \lambda^{(k)}) \right\} + \frac{1}{2} \sum_{n \neq m} \frac{r_{mn}}{\lambda_m - \lambda_n} 
$$

$$+ \frac{1}{\theta_2 \{p_3\} (\Omega)} \sum_{\alpha_1, \beta_1=1}^g \partial_\alpha \{\theta_2 \{p_3\} (\Omega)\} \partial_\alpha \{B_{\alpha \beta}\} + \frac{1}{\theta_2 \{p_3\} (\Omega)} \sum_\alpha \partial_z^{(\alpha)} \{\theta_2 \{p_3\} (\Omega)\} \partial_\alpha \{\Omega_{\alpha}\},$$

or, equivalently,

$$H_m = -\text{res}_{\lambda = \lambda_m} \left\{ \frac{1}{(d\lambda)^2} \sum_{j<k} w(\lambda^{(j)}, \lambda^{(k)}) \right\} + \partial_{\lambda_m} \ln \left\{ \prod_{l<n} (\lambda_l - \lambda_n)^{r_{lm} \theta_2 \{p_3\} (\Omega)} \right\}.$$
Therefore, we come to the following

**Theorem 5.11.** The tau-function corresponding to solution (4.49) of Schlesinger system, is given by

\[
\tau(\lambda_n) = F(\{\lambda_n\}) \prod_{m,n=1}^M (\lambda_m - \lambda_n)^{r_{mn}} \Theta(p_q) (\Omega|B),
\]

where function \( F(\{\lambda_n\}) \) does not depend on constants \( p, q \) and \( r_{n}^{(j)} \), and satisfies the following system of compatible equations

\[
\partial_{\lambda_m} \ln F = -\text{res}_{\lambda=\lambda_m} \left\{ \frac{1}{(d\lambda)^2} \sum_{j \neq k} \frac{w(\lambda^{(j)}, \lambda^{(k)})}{(d\lambda)^2} \right\}.
\]

One can check that only the non-singular part of the Bergmann kernel contributes to the residue in expression (5.61); therefore, we can further express \( \partial_{\lambda_m} \ln F \) in terms of the projective connection \( R^H \) on the Riemann surface \( L \) corresponding to the natural choice of local coordinates on \( L \).

**Lemma 5.12.** Function \( F(\{\lambda_n\}) \), defined by (5.61), satisfies the following system of compatible equations:

\[
\partial_{\lambda_m} \ln F = \frac{1}{24} R(\lambda_m) \equiv -\frac{1}{12\pi} \int_L \mu_m R^H(dx)^2,
\]

where

\[
\mu_m = -\frac{\pi}{2} \delta(x) \frac{dx}{d\sigma} \quad \text{with} \quad x = (\lambda - \lambda_m)^{1/2}
\]

is the Beltrami differential \( \delta(x) \) corresponding to variation of the branch point \( \lambda_m \); \( R^H(P) \) is the projective connection corresponding to our choice of local parameters on \( L \): \( x = (\lambda - \lambda_m)^{1/2} \) in the neighborhood of a branch point \( P_m \) (all branch points are simple according to our assumption); \( \delta(x) \) is two-dimensional delta-function.

**Proof.** Formula (5.61) can be rewritten in terms of the non-singular part of the Bergmann kernel (3.24), which immediately leads to (5.62) taking into account the definition of projective connection \( R \). \( \square \)

According to this lemma, the function \( F \) plays the role of generating function of the projective connection corresponding to the natural choice of coordinate system on the branched covering \( L \).
**Remark 5.13.** In the case of higher multiplicity of branch points the formula (5.62) suffers only minor modification: instead of value of the projective connection at the point $\lambda_m$ this formula contains an appropriate derivative of $R$ at this point.

**Remark 5.14.** Integrability of equations (5.62) for the function $F$ follows from integrability of the equations (2.14) for the isomonodromic tau-function. We would like to notice that it is rather non-trivial fact from the point of view of the theory of Riemann surfaces that equations

\[ \frac{\partial R_H(\lambda_m)}{\partial \lambda_n} = \frac{\partial R_H(\lambda_n)}{\partial \lambda_m} \]  

are always satisfied if the Riemann surface $L$ has only simple branch points (for higher multiplicities the values of the projective connection in (5.63) should be substituted by an appropriate derivatives). This fact looks analogous to similar equations for accessory parameters which appear in the uniformization problem of punctured sphere [26].

It is possible to prove that the function $F$ does not vanish as long as the Riemann surface $L$ remains compact. Therefore, in particular, it does not vanish outside of the hyperplanes $\lambda_m = \lambda_n$. This allows to claim that the zeros of the tau-function (5.60) coincide with the zeros of the theta-function $\Theta[p] (\Omega | B)$. Therefore, we come to the following relationship between the Malgrange divisor $(\vartheta)$ in the space of singularities (corresponding to quasi-permutation monodromy groups considered here) and the theta-divisor in Jacobi manifold of the Riemann surface $L$:

**Theorem 5.15.** The set of singularities $\{\lambda_m\}$ belongs to the Malgrange divisor $(\vartheta)$ iff the vector $Bp + q + \Omega$ belongs to the theta-divisor $(\Theta)$ in the Jacobi manifold $J(L)$ of the Riemann surface $L$.

We remind that in the expression $Bp + q + \Omega$ the $\{\lambda_m\}$-dependence is hidden inside the matrix of $b$-periods and the vector $\Omega$.

### 5.2. Function $F$ and holomorphic factorization of determinant of Laplacian operator

Let us make a few comments concerning the link of the function $F$ with the determinant of Laplacian operator in spirit of previous works [20, 26]. Consider, for example, the case $g > 1$. Let us denote by $z = f(P)$ the fuchsian uniformization map of the curve $L$ to the fundamental domain $H/\Gamma$ of a fuchsian group $\Gamma$ (by $z$ we denote the complex coordinate on the upper half-plane $H$). Then we can write down the Poincare metric of gaussian curvature $-1$ on $L$:

\[ ds^2 = \frac{dzd\overline{z}}{(3z)^2} \equiv e^{\phi(x)} dx d\overline{x}, \]

\footnote{For $g = 0$ and $g = 1$ function $z(x)$ maps $L$ to the Riemann sphere or fundamental parallelogramm, respectively.}
where
\[ \phi(x) = \ln \left| \frac{z'(x)}{3z(x)} \right|^2 \]
is a real function in each coordinate chart (notice that \( \phi(x) \) does transform under coordinate change i.e. it is not a scalar). Function \( \phi(x) \) satisfies in each chart the Liouville equation
\[ \phi_{xx} = \frac{1}{2} e^\phi, \]
which formally provides the extremals of the the Liouville action
\[ S = \int \left( |\phi_x|^2 + e^\phi \right) dx d\tau. \]

However, since the function \( \phi \), defined by (5.64), does not behave like a scalar with respect to the coordinate change, this expression has to be accurately defined in each case taking into account the terms coming from the boundaries of the coordinate charts [26]. This can, for example, be explicitly done if the local coordinate \( x \) corresponds to the Schottky uniformization of the Riemann surface \( \mathcal{L} \) [30]. It is also easy to write down these boundary terms in system of local parameters associated to the ramified covering realization, but we will not discuss them here.

The Laplace operator on \( \mathcal{L} \)
\[ \Delta = (z - \overline{z})^2 \frac{\partial^2}{\partial z \partial \overline{z}} \]
is invariant with respect to the Möbius group. Therefore, it can be naturally defined on \( \mathcal{L} \), where it is self-adjoint, non-negative, and has discrete spectrum in the Hilbert space of functions on \( \mathcal{L} \). The Hilbert space is equipped with the natural inner product provided by the Poincare metric. The zeta-function of the Laplacian \( \Delta \) is defined in terms of its eigenvalues \( \kappa_j \) as follows: \( \zeta(s) = \sum_j \kappa_j^{-s} \).

In turn, the determinant of the Laplacian \( \Delta \) is defined via analytical continuation of the zeta-function to zero:
\[ \det \Delta = \exp \{ \zeta'(0) \} . \]

An infinitesimal variation of the moduli of the curve \( \mathcal{L} \) by a Beltrami differential \( \mu \) leads to the following variation of the determinant of Laplacian operator [22],
\[ \delta_{\mu} \ln \frac{\det \Delta}{\det \mathfrak{B}} = \frac{1}{6\pi} \int_{H/\Gamma} \mu(z)R^F(z)(dz)^2 , \]
where \( R^F(z), z \in H/\Gamma \) is the projective connection corresponding to Fuchsian uniformization of \( \mathcal{L} \).

On the other hand, the action of the Beltrami differential \( \mu_m \) on the logarithm of the real function \(|F|^2\) follows from (5.62):
\[ \partial_{\lambda_m} \ln |F|^2 = -\frac{1}{12\pi} \int_{\mathcal{L}} \mu_m(x)R^H(x)(dx)^2 , \]
where $R^H(x)$ denotes the projective connection with respect to the natural system of local parameters on $\mathcal{L}$ arising from the realization of $\mathcal{L}$ as branched covering of $\mathbb{C}P^1$. Therefore, there must exist a real-valued function $S^H(\{\lambda_m\})$ such that

$$|F|^2 = \left( \frac{\det \Delta}{\det \mathcal{B}} \exp \left\{ \frac{1}{12} S^H \right\} \right)^{-1/2}.$$  

(5.68)

According to (5.66) and (5.67), the function $S^H$ satisfies the following compatible system of equations:

$$\frac{\partial S^H}{\partial \lambda_m} = 2 \int_{\mathcal{L}} \left[ R^H(x) - R^F(z(x)) \frac{dz}{dx} \right]^2 \mu_m(dx)^2 .$$  

(5.69)

Taking into account that the Beltrami differential $\mu_m$ is, up to the factor $-\pi/2$, nothing but the delta-function with support at the branch point $\lambda_m$, and that the projective connection transforms according to (3.26) under a change of the local coordinate, we get the following equations for $S^H$ in terms of Schwartzian derivative of the local parameters:

$$\frac{\partial S^H}{\partial \lambda_m} = -\left\{ z(x_m), x_m \right\} \bigg|_{x_m=0} ,$$  

(5.70)

where $x_m = \sqrt{\lambda - \lambda_m}$ is the local parameter near the branch point. According to the previous experience (see [26], where the relationship between Fuchsian and Shottky uniformizations is discussed in detail, and more recent paper [33]), the solution of equations (5.70) should coincide with an appropriately defined Liouville action (5.65). In addition to the bulk term (5.65), the Liouville action contains suitable boundary terms, which we don’t write down explicitly.

According to the general philosophy of holomorphic factorization [23], the formula (5.68) allows to identify $F$ with $[\det \mathcal{O}_0]^{-1/2}$. The operator $\mathcal{O}_0$ acts on sections of the trivial bundle over $\mathcal{L}$; this operator should be understood as differentiation with respect to our standard system of local parameters on the branched covering.

We would like to refer also to work [33], where the generating function for projective connection was computed with respect to the Fuchsian uniformization; this gives further support to the hypothesis of close relationship between function $F$ and appropriate version of Liouville action in our parametrization.

5.3. Hyperelliptic curves and $2 \times 2$ Riemann-Hilbert problems with off-diagonal monodromies

Here we consider the simplest case of $N = 2$, when any matrix of quasi-permutation is either diagonal or off-diagonal. We shall consider monodromy groups containing only off-diagonal monodromies; the insertion of additional diagonal monodromies according to the general scheme is straightforward. In this case the branched covering $\mathcal{L}$ corresponds to hyperelliptic algebraic curve with branch points $\lambda_1, \ldots, \lambda_M$. 


and function $F$ may be calculated explicitly [14]. We have $M = 2g + 2$, where $g$ is the genus of the branched covering $L$; this branched covering is the Riemann surface of the algebraic curve

$$w^2 = \prod_{m=1}^{2g+2} (\lambda - \lambda_m)$$

(5.71)

It is convenient to put all $r_j^{(j)} = 0$; in this case the formula (4.48) gives the solution $\Psi(\lambda) \in SL(2)$ of the RH problem with arbitrary off-diagonal monodromies having unit determinant:

$$M_m = \begin{pmatrix} 0 & d_m \\ -d_m^{-1} & 0 \end{pmatrix},$$

where constants $d_m$ may be expressed in terms of the elements of vectors $p, q$. Let us count the number of essential parameters in the monodromy matrices and in the construction of function $\Psi$. The matrices $M_m$ contain altogether $2g + 2$ constants; however, there is one relation (product of all monodromies gives $I$). One more parameter is non-essential due to possibility of simultaneous conjugation of all monodromies with an arbitrary diagonal constant matrix. Therefore, the set of monodromy matrices contains $2g$ non-trivial constants in accordance with number of non-trivial constants contained in vectors $p$ and $q$.

To integrate the remaining equations

$$\partial_{\lambda_m} \ln F = \frac{1}{24} R^H(\lambda_m)$$

(5.72)

on hyperelliptic curve (5.71) we use the following formula (24, p.20) for the projective connection at arbitrary point of $P$ of the hyperelliptic curve $L$ (where $x$ is the local parameter in the neighborhood of the point $P$, $\lambda = \Pi(P)$ is the projection of $P$ on $\lambda$-plane):

$$R^H(P) = \{\lambda(x), x\}(P) + \frac{3}{8} \left( \frac{d}{dx} \ln \prod_{\lambda_m \in T} (\lambda - \lambda_m) \right)^2 (P)$$

$$- \frac{6}{\Theta \left[ p^T \right] (0)} \sum_{\alpha, \beta = 1}^{g} \partial^2_{z_\alpha z_\beta} \left( \Theta \left[ p^T q^T \right] (0) \right) \frac{\partial a_\alpha}{dx} (P) \frac{\partial b_\beta}{dx} (P).$$

(5.73)

Here $\{\lambda, x\}$ is the Schwarzian derivative of $\lambda$ with respect to $x$; $T$ is an arbitrary divisor consisting of $g + 1$ branch points, which satisfies certain non-degeneracy condition. Characteristic $\left[ p^T \right]$ is the even half-integer characteristic corresponding to the divisor $T$ according to the following equation:

$$Bp^T + q^T = \sum_{\lambda_m \in T} U(\lambda_m) - K,$$

(5.74)
where $K$ is the vector of Riemann constants; the initial point of the Abel map is chosen to be, say, $\lambda_1$. In this case the r.h.s. of (5.74) is a linear combination, with integer or half-integer coefficients, of the vectors $e\alpha$ and $Be\alpha$. These coefficients are composed in vectors $p\alpha \mathbf{T}$ and $q\alpha \mathbf{T}$. The non-degeneracy requirement imposed on the divisor $T$ gives rise to the condition that the vector $Bp\alpha + q\alpha \mathbf{T}$ does not belong to the theta-divisor on $J(L)$, i.e. $\Theta [p\alpha \mathbf{T}, q\alpha \mathbf{T}] (0) \neq 0$.

Of course, the projective connection $R$, as well as the function $F$, are independent of the choice of the divisor $T$, which plays only intermediate role. If in (5.73) we choose $P = \lambda_m$, the local parameter is $x = \sqrt{\lambda - \lambda_m}$. Then all terms in $R(\lambda_m)$ which don’t contain theta-function can be integrated explicitly; the terms containing theta-function can be represented as logarithmic derivative with respect to $\lambda_m$ by making use of the heat equation for theta-function (3.20) and Rauh formula (3.35). These terms are equal to

$$-6 \frac{\partial}{\partial \lambda_m} \ln \Theta [p\alpha \mathbf{T}, q\alpha \mathbf{T}] (0) .$$

This expression may be rewritten using the Thomae formula [34]

$$\Theta [p\alpha \mathbf{T}, q\alpha \mathbf{T}] (0) = \pm (\text{det} \mathcal{A})^2 \prod_{\lambda_m, \lambda_n \in T} (\lambda_m - \lambda_n) \prod_{\lambda_m, \lambda_n \not\in T} (\lambda_m - \lambda_n),$$

where $\mathcal{A}_{\alpha\beta} = \oint \lambda_{\alpha\beta}^{-1} w$ is the $g \times g$ matrix of $\alpha$-periods of non-normalized holomorphic differentials on $L$.

Collecting together all the explicit factors arising from the Thomae formula and expression (5.73), we get the following answer for the function $F$:

$$(5.75) \quad F = (\text{det} \mathcal{A})^{-\frac{1}{2}} \prod_{m < n} (\lambda_m - \lambda_n)^{-\frac{1}{2}}$$

which coincides with the expression for the determinant $\{\text{det} \partial_0\}^{-1/2}$ of Cauchy-Riemann operator acting in trivial bundle over $L$ with respect to our system of local parameters [17, 20]. For the tau-function itself we get the following expression

$$\tau(\{\lambda_m\}) = (\text{det} \mathcal{A})^{-\frac{1}{2}} \prod_{m < n} (\lambda_m - \lambda_n)^{-\frac{1}{2}} \Theta [p\alpha \mathbf{T}] (0) |B| ,$$

which, according to the same papers, coincides with naturally defined determinant of the Cauchy-Riemann operator acting on spinors which have holonomies $e^{2\pi i \nu_\alpha}$ and $e^{-2\pi i \nu_\alpha}$ along basic cycles of $L$.

As we saw above, for general curves the interpretation of the factor $F$ as $\{\text{det} \partial_0\}^{-1/2}$ can, probably, be preserved. Interpretation of the whole tau-function as $\text{det} \partial_1^{1/2}$ in a twisted spinor bundle remains valid for arbitrary curves, if all constants $r_m^{(j)}$ vanish.
Remark 5.16. In the article [35] it was argued that the tau-function for isomonodromy deformations with arbitrary (not only quasi-permutation) monodromy matrices can be interpreted as determinant of Cauchy-Riemann operator in appropriate spinor vector bundle over punctured sphere with cuts. We don’t know at the moment how to establish an explicit link between the framework of [35] and our present scheme, where the spinor line bundles over compact Riemann surfaces appear.

Acknowledgements
I would like to thank A.Bobenko, J.Harnad, J.Hurtubise, A.Kokotov, V.B.Matveev, A.Orlov and A.N.Tyurin for comments and discussions. This work was supported by NSERC and FCAR grants, and Laboratoire Gevrey de Mathématique Physique, Université de Bourgogne. I thank V.B.Matveev for hospitality at Université de Bourgogne, where this work was completed.

References
[1] Bolibruch, A., The Riemann-Hilbert problem, *Russ. Math. Surveys* **45** (1990), 1-58.
[2] Zakharov, V.E., Manakov, S.V., Novikov, S.P., Pitaevskii, L.P., *Theory of Solitons. The inverse scattering method*, Consultants Bureau, New York, 1984.
[3] Dubrovin, B., Geometry of 2D topological field theories, in: *Integrable systems and quantum groups* 120-348, Lecture Notes in Math., v.1620, Springer, Berlin, 1996.
[4] Hitchin, N., Twistor spaces, Einstein metrics and isomonodromic deformations, *J. Diff. Geom.* **42** (1995), 30-112.
[5] Deift, P., Its, A., Zhou, X., A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics, *Annals of Math.* **146** (1997), 149-235.
[6] Belokolos, E., Bobenko, A., Enolski, V., Its, A., Matveev, V., *Algebro-geometrical integration of non-linear differential equations*, Springer Verlag Berlin Heidelberg New York 1992.
[7] Malgrange, B., Sur les Déformation Isomonodromiques, in *Mathématique et Physique (E.N.S. Séminaire 1979-1982)*, p.401-426, Birkhäuser, Boston, 1983.
[8] Jimbo, M., Miwa, T., Ueno, K., Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I, *Physica* **2D** (1981), 306-352.
[9] Okamoto, K., Studies on the Painlevé Equations. I. Sixth Painlevé Equation $P_{VI}$, *Annali Mat. Pura Appl.*, **146** (1987), 337-381.
[10] Korotkin, D., Finite-gap solutions of stationary axially symmetric Einstein equations in vacuum, *Theor. Math. Phys.* **77** (1989), 1018-1031.
[11] Neugebauer, G., Meinel, R., General relativistic gravitational field of the rigidly rotating disk of dust: Solution in terms of ultraelliptic functions, *Phys.Rev.Lett.* **75** (1995), 3046-3048.
[12] Klein, C., Richter, O., Explicit solutions of Riemann-Hilbert problem for the Ernst equation, *Phys.Rev.D* 57 (1998), 857-862.

[13] Korotkin, D., Babich, M., Self-dual SU(2)-invariant Einstein manifolds and modular dependence of theta-functions, *Lett.Math.Phys.* 46 (1998), 323-337.

[14] Kitaev, A., Korotkin, D., On solutions of Schlesinger equations in terms of theta-functions, *Intern.Math.Res.Notices* 17 (1998), 877-905.

[15] Deift, P., Its, A., Kapaev, A., Zhou, X., On the algebro-geometric integration of the Schlesinger equations. *Commun. Math. Phys.* 203 (1999), 613–633.

[16] Zverovich, E.I., Boundary value problems in the theory of analytic functions in Hölder classes on Riemann surfaces, *Russ. Math. Surveys* 26 (1971), 117-192.

[17] Zamolodchikov, A.B., Conformal scalar field on the hyperelliptic curve and critical Ashkin-Teller multipoint correlation functions, *Nucl.Phys.* B285 (1986), 481-503.

[18] Belavin, A.A., Knizhnik, V.G., Algebraic geometry and the geometry of quantum strings, *Phys.Lett.* 168B (1986), 201-206.

[19] Knizhnik, V.G., Analytic fields on Riemann surfaces II, *Commun.Math.Phys.* 112 (1987), 567-590.

[20] Knizhnik, V.G., Multi-loop amplitudes in the theory of quantum strings and complex geometry, *Sov.Phys.Uspekhi.* 32 (1989), 945-971.

[21] Bershadsky, M., Radul, A., Fermionic fields on $\mathbb{Z}_N$-curves, *Commun.Math.Phys.* 116, (1988), 689-700.

[22] Alvarez-Gaume, L., Moore, G., Vafa, C., Theta-functions, Modular Invariance and Strings, *Commun.Math.Phys.* 106, 1-40 (1986)

[23] Quillen, D., Determinants of Cauchy-Riemann operators over Riemann surface, *Funct.Anal.Appl.* 19 No.1 (1984), 37-41.

[24] Fay, J., *Theta Functions on Riemann Surfaces*, Lect.Notes in Math., 352, Springer, Berlin, 1973

[25] Fay, J., *Kernel functions, Analytic torsion and Moduli spaces*, Memoirs of the American Mathematical Society, 96 No.464 (1992), 1-123.

[26] L.Takhtajan, Semi-classical Liouville theory, complex geometry of moduli spaces, and uniformization of Riemann surfaces, in *New Symmetry Principles in Quantum Field Theory*, ed. by Frölich, J., et al, Plenum Press, New York, 1992.

[27] Korotkin, D., Isomonodromic deformations and Hurwitz spaces, math-ph/0103023, to appear in *Isomonodromic deformations and applications*, ed. by Harnad, J. and Its, A., American Mathematical Society, 2001.

[28] Grinevich P., Orlov A., Flag Spaces in KP theory and Virasoro action on $\det D_j$ and Segal-Wilson $\tau$-function, in *Research reports in physics. Problems of modern quantum field theory*, p. 86-106, ed. by Belavin, A.A., Klimuk, A.U., Zamolodchikov, A.B., Springer Berlin, Heidelberg, 1989.

[29] Natanzon, S., Turaev, V., A compactification of the Hurwitz space, *Topology* 38 (1999), 889-914.

[30] P.Zograf, L.Takhtajan, On uniformization of Riemann surfaces and the Weil-Petersson metric on Teichmüller and Schottky spaces, *Mat.Sbornik.* 132 No.3 (1987), 304-327.
[31] P.Zograf, The Liouville action on moduli spaces and uniformization of degenerating Riemann surfaces, *Algebra i Analiz* 1 No.4 (1989), 136-160.

[32] P.Zograf, L.Takhtajan, A potential for the Weil-Petersson metric on Torelli space, *Zap.Nauch.Sem. LOMI* 160 (1987), 110-120.

[33] E.Androvandi, L.Takhtajan, Generating functional in CFT and effective action for two-dimensional quantum gravity on higher-genus Riemann surfaces, *Commun.Math.Phys.* 188 (1997), 29-67.

[34] D.Mumford, *Tata Lectures on Theta, I,II*, Progress in Mathematics 28,43 Birkhauser, Boston 1983,84

[35] J.Palmer, Determinants of Cauchy-Riemann operators as $\tau$-functions, *Acta Appl. Mathematicae* 18 (1990), 199-223.

Department of Mathematics and Statistics
Concordia University
7141 Sherbrook West, Montreal
H3B 1R6 Quebec
Canada

1991 Mathematics Subject Classification. Primary 35Q15; Secondary 30F60, 32G81.

Received