ARITHMETIC PROPERTIES OF $q$-FIBONACCI NUMBERS AND $q$-PELL NUMBERS

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ABSTRACT. We investigate some arithmetic properties of the $q$-Fibonacci numbers and the $q$-Pell numbers.

1. Introduction

The Fibonacci numbers $F_n$ are given by

$$F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad F_n = 2F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 2.$$  

For any odd prime $p$, it is well-known (cf. [7]) that

$$F_p \equiv \left( \frac{5}{p} \right) \quad (\text{mod} \ p), \quad (1.1)$$

$$F_{p+1} \equiv \frac{1}{2} (1 + \left( \frac{5}{p} \right)) \quad (\text{mod} \ p) \quad (1.2)$$

and

$$F_{p-1} \equiv \frac{1}{2} (1 - \left( \frac{5}{p} \right)) \quad (\text{mod} \ p), \quad (1.3)$$

where $\left( \frac{a}{p} \right)$ denotes the Legendre symbol. Indeed, we have

$$F_p = \frac{(1 + \sqrt{5})^p - (1 - \sqrt{5})^p}{2p\sqrt{5}} \equiv \frac{(1 + (\sqrt{5})^p) - (1 - (\sqrt{5})^p)}{2\sqrt{5}} = 5^{(p-1)/2} \quad (\text{mod} \ p).$$

For more results on the congruences involving the Fibonacci numbers, the readers may refer to [10], [11] and [12].

On the other hand, a sequence of polynomials $F_n(q)$ was firstly introduced by Schur (cf. [9]):

$$F_n(q) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ F_{n-1}(q) + q^{n-2}F_{n-2}(q) & \text{if } n \geq 2. \end{cases}$$
Also Schur considered another sequence $\hat{F}_n(q)$, which is given by

$$
\hat{F}_n(q) = \begin{cases} 
0 & \text{if } n = 0, \\
1 & \text{if } n = 1, \\
\hat{F}_{n-1}(q) + q^{n-1}\hat{F}_{n-2}(q) & \text{if } n \geq 2.
\end{cases}
$$

Obviously both $F_n(q)$ and $\hat{F}_n(q)$ are the $q$-analogues of the Fibonacci numbers. The sequences $F_n(q)$ and $\hat{F}_n(q)$ have been investigated in several papers (e.g., see [2], [3], [4], [5] and [6]). However, seemingly there are no simple expressions for $F_n(q)$ and $\hat{F}_n(q)$.

Now we can give the $q$-analogues of (1.1), (1.2) and (1.3). Suppose that $p \neq 5$ is an odd prime. Let $\alpha_p$ be the integer such that $1 \leq \alpha_p \leq 4$ and $\alpha_p p \equiv 1 \pmod{5}$. As usual we set

$$
[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}
$$

for any non-negative integer $n$.

**Theorem 1.1.**

$$
F_{p+1} \equiv \frac{1}{2}(1 + \left(\frac{5}{p}\right)) \pmod{[p]_q}
$$

and

$$
F_p \equiv \left(\frac{5}{p}\right) q^{(5-\alpha_p)p+1)/5} \pmod{[p]_q}.
$$

**Theorem 1.2.**

$$
\hat{F}_{p-1} \equiv \frac{1}{2}(1 - \left(\frac{5}{p}\right)) \pmod{[p]_q}
$$

and

$$
\hat{F}_p \equiv \left(\frac{5}{p}\right) q^{(\alpha_p p-1)/5} \pmod{[p]_q}.
$$

The Pell numbers $P_n$ are given by

$$
P_0 = 0, \ P_1 = 1 \ \text{and} \ \ P_n = 2P_{n-1} + P_{n-2} \ \text{for} \ n \geq 2.
$$

It is easy to check that

$$
P_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}}.
$$

Hence for odd prime $p$, we have

$$
P_p = \left(\frac{1 + \sqrt{2}}{2\sqrt{2}}\right)^p - \left(\frac{1 - \sqrt{2}}{2\sqrt{2}}\right)^p \equiv \frac{2(\sqrt{2})^p}{2\sqrt{2}} = 2^{(p-1)/2} \equiv \left(\frac{2}{p}\right) \pmod{p}.
$$
Define the $q$-Pell numbers $P_n(q)$ and $\hat{P}_n(q)$ by

\[
P_n(q) = \begin{cases} 
0 & \text{if } n = 0, \\
1 & \text{if } n = 1, \\
(1 + q^{n-1})P_{n-1}(q) + q^{n-2}P_{n-2}(q) & \text{if } n \geq 2,
\end{cases}
\]

and

\[
\hat{P}_n(q) = \begin{cases} 
0 & \text{if } n = 0, \\
1 & \text{if } n = 1, \\
(1 + q^{n-1})\hat{P}_{n-1}(q) + q^{n-1}\hat{P}_{n-2}(q) & \text{if } n \geq 2.
\end{cases}
\]

**Theorem 1.3.** Let $p$ be an odd prime. Then

\[
q^{(p^2-1)/8}P_p(q) \equiv \left(\frac{2}{p}\right) \pmod{[p]_q} \quad (1.9)
\]

and

\[
\hat{P}_p(q) \equiv \left(\frac{2}{p}\right) q^{(p^2-1)/8} \pmod{[p]_q}. \quad (1.10)
\]

Furthermore, we have

\[
P_{p+1}(q) - P_p(q) \equiv \hat{P}_{p+1}(q) - \hat{P}_p(q) \equiv 1 \pmod{[p]_q}. \quad (1.11)
\]

Since $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$, (1.9) and (1.10) can be respectively rewritten as

\[
(-q)^{(p^2-1)/8}P_p(q) \equiv 1 \pmod{[p]_q}
\]

and

\[
\hat{P}_p(q) \equiv (-q)^{(p^2-1)/8} \pmod{[p]_q}.
\]

The proofs of Theorems 1.1, 1.2 and 1.3 will be given in Sections 2 and 3.

2. Proofs of Theorems 1.1 and 1.2

For any $n, m \in \mathbb{Z}$, the $q$-binomial coefficient $\left[n\atop m\right]_q$ is given by

\[
\left[n\atop m\right]_q = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-m+1})}{(1 - q^m)(1 - q^{m-1}) \cdots (1 - q)}
\]

when $m \geq 0$, and let $\left[n\atop m\right]_q = 0$ if $m < 0$. Obviously $\left[n\atop m\right]_q$ is a polynomial in the variable $q$ with integral coefficients since the $q$-binomial coefficients satisfy the recurrence relation

\[
\left[n+1\atop m\right]_q = q^m \left[n\atop m\right]_q + \left[n\atop m-1\right]_q.
\]
Let \( |x| \) denotes the greatest integer not exceeding \( x \). Then for any non-negative integer \( n \), we have

\[
\mathcal{F}_{n+1}(q) = \sum_{0 \leq 2j \leq n} q^{j^2} \left[ \frac{n-j}{j} \right]_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{(5j+1)/2} \left[ \left[ \frac{n}{5j} \right] + \frac{1}{2} \right]_q \tag{2.1}
\]

and

\[
\hat{\mathcal{F}}_{n+1}(q) = \sum_{0 \leq 2j \leq n} q^{j^2+j} \left[ \frac{n-j}{j} \right]_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{(5j-3)/2} \left[ \left[ \frac{n+1}{5j} \right] + 1 \right]_q \tag{2.2}
\]

We mention that (2.1) and (2.2) can be considered as the finite forms of the first and the second of the Rogers-Ramanujan identities respectively (the full proofs of (2.1) and (2.2) can be found in [1]).

In this section we assume that \( p \neq 5 \) is an odd prime.

**Lemma 2.1.** Let

\[
L(j) = \frac{j(5j+1)}{2} - \left( \left[ \frac{(p-1-5j)/2}{2} \right] + 1 \right),
\]

and let

\[
\hat{L}(j) = \frac{j(5j-3)}{2} - \left( \left[ \frac{(p-1-5j)/2}{2} \right] + 2 \right).
\]

Then

\[
L(2j) - L(2j-1) = \hat{L}(2j) - \hat{L}(2j-1) = p.
\]

**Lemma 2.2.** Let

\[
S_p = \{ j \in \mathbb{Z} : 0 \leq \left[ \frac{(p-1-5j)/2}{2} \right] + 1 \leq p-1 \}
\]

and

\[
\hat{S}_p = \{ j \in \mathbb{Z} : 0 \leq \left[ \frac{(p-1-5j)/2}{2} \right] + 1 \leq p-1 \}.
\]

We have

\[
S_p = \{ j \in \mathbb{Z} : -\lfloor p/5 \rfloor \leq j \leq \lfloor p/5 \rfloor \},
\]

and

\[
\hat{S}_p = \begin{cases} 
\{ j \in \mathbb{Z} : \lfloor p/5 \rfloor + 1 \leq j \leq \lfloor p/5 \rfloor \} & \text{if } p \equiv 1 \mod 5, \\
\{ j \in \mathbb{Z} : -\lfloor p/5 \rfloor \leq j \leq \lfloor p/5 \rfloor \} & \text{if } p \equiv 2, 3 \mod 5, \\
\{ j \in \mathbb{Z} : -\lfloor p/5 \rfloor \leq j \leq \lfloor p/5 \rfloor + 1 \} & \text{if } p \equiv 4 \mod 5.
\end{cases}
\]
These two lemmas above can be verified directly, so we omit the details here.

Proof of Theorem 1.1. By (2.1), we have

$$F_{p+1}(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \left[ \left\lfloor \frac{p}{(p-5j)/2} \right\rfloor_q \right].$$

Observe that

$$\left[ \frac{p}{k} \right]_q \equiv \begin{cases} 1 \pmod{[p]_q} & \text{if } k = 1 \text{ or } p, \\ 0 \pmod{[p]_q} & \text{if } 1 \leq k \leq p - 1. \end{cases}$$

Then it follows that

$$F_{p+1}(q) \equiv \sum_{\left\lfloor (p-5j)/2 \right\rfloor = 0 \text{ or } p} (-1)^j q^{j(5j+1)/2} \pmod{[p]_q}.$$ 

It is easy to check that

$$\{ j : \left\lfloor (p-5j)/2 \right\rfloor = 0 \text{ or } p \} = \begin{cases} \{(p-1)/5\} & \text{if } p \equiv 1 \pmod{5}, \\ \{-p+1)/5\} & \text{if } p \equiv 4 \pmod{5}, \\ \emptyset & \text{if } p \equiv 2, 3 \pmod{5}. \end{cases}$$

Thus

$$F_{p+1}(q) \equiv \sum_{\left\lfloor (p-5j)/2 \right\rfloor = 0 \text{ or } p} (-1)^j q^{j(5j+1)/2} \equiv \begin{cases} 1 \pmod{[p]_q} & \text{if } p \equiv 1, 4 \pmod{5}, \\ 0 \pmod{[p]_q} & \text{if } p \equiv 2, 3 \pmod{5}. \end{cases}$$

This concludes the proof of (1.4).

Also, applying (2.1) and Lemma 2.2, we deduce that

$$F_p(q) = \sum_{0 \leq \left\lfloor (p-1-5j)/2 \right\rfloor \leq p-1} (-1)^j q^{j(5j+1)/2} \left[ \left\lfloor \frac{p-1}{(p-1-5j)/2} \right\rfloor_q \right].$$

Suppose that $p \equiv 1 \pmod{5}$. Noting that $(p-1)/5$ is even and

$$(2j-1) + \left\lfloor (p-1 - 5(2j-1))/2 \right\rfloor = 2j + \left\lfloor (p-1 - 5 \cdot 2j)/2 \right\rfloor + 1 = (p-1)/2 - 3j + 1,$$
we have
\[ \mathcal{F}_p(q) \equiv (-1)^{-(p-1)/5}q^{L(-(p-1)/5)} + \sum_{j=-(p-6)/5}^{(p-1)/5} (-1)^{j+[(p-1-5j)/2]}q^{L(j)} \]
\[ = q^{(p-1)(p-2)/10-p(p-1)/2} + \sum_{j=-(p-11)/10}^{(p-1)/10} (-1)^{(p-1)/2-3j}(q^{L(2j)} - q^{L(2j-1)}) \]
\[ \equiv q^{(4p+1)/5} \pmod{[p]_q}. \]

where Lemma 2.1 is applied in the last step. Similarly we obtain that
\[ \mathcal{F}_p \equiv \begin{cases} 
(-1)^{(p-2)/5}q^{L(-(p-2)/5)} \equiv -q^{(2p+1)/5} \pmod{[p]_q} & \text{if } p \equiv 2 \pmod{5}, \\
(-1)^{(p-3)/5}q^{L(-(p-3)/5)} \equiv -q^{(3p+1)/5} \pmod{[p]_q} & \text{if } p \equiv 3 \pmod{5}, \\
(-1)^{(p-4)/5}q^{L(-(p-4)/5)} \equiv q^{(p+1)/5} \pmod{[p]_q} & \text{if } p \equiv 4 \pmod{5}. 
\end{cases} \]

\[ \hat{\mathcal{F}}_p(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j-3)/2} \left[ \left\lfloor \frac{p}{5j} \right\rfloor + 1 \right]_q. \]

And we have
\[ \{ j : \left\lfloor (p-5j)/2 \right\rfloor + 1 = 0 \text{ or } p \} = \begin{cases} \{ -(p-1)/5 \} & \text{if } p \equiv 1 \pmod{5}, \\
\{ -(p-2)/5 \} & \text{if } p \equiv 2 \pmod{5}, \\
\{ (p+2)/5 \} & \text{if } p \equiv 3 \pmod{5}, \\
\{ (p+1)/5 \} & \text{if } p \equiv 4 \pmod{5}. 
\end{cases} \]

Therefore
\[ \hat{\mathcal{F}}_p(q) \equiv \begin{cases} 
(-1)^{-(p-1)/5}q^{(p+2)(p-1)/10} \equiv q^{(p-1)/5} \pmod{[p]_q} & \text{if } p \equiv 1 \pmod{5}, \\
(-1)^{(p-2)/5}q^{(p+1)(p-2)/10} \equiv -q^{(3p-1)/5} \pmod{[p]_q} & \text{if } p \equiv 2 \pmod{5}, \\
(-1)^{(p+2)/5}q^{(p+2)(p-1)/10} \equiv -q^{(2p-1)/5} \pmod{[p]_q} & \text{if } p \equiv 3 \pmod{5}, \\
(-1)^{(p+1)/5}q^{(p+1)(p-2)/10} \equiv q^{(4p-1)/5} \pmod{[p]_q} & \text{if } p \equiv 4 \pmod{5}. 
\end{cases} \]

Now let us turn to the proof of (1.7). From (2.2), it follows that
\[ \hat{\mathcal{F}}_{p-1}(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j-3)/2} \left[ \left\lfloor \frac{p-1}{5j} \right\rfloor + 1 \right]_q \]
\[ = \sum_{0 \leq \left\lfloor (p-1-5j)/2 \right\rfloor + 1 \leq p-1} (-1)^j q^{j(5j-3)/2} \left[ \left\lfloor \frac{p-1}{5j} \right\rfloor + 1 \right]_q. \]
If \( p \equiv 1 \mod 5 \), then by Lemmas 2.1 and 2.2, we have
\[
\hat{F}_{p-1}(q) = \sum_{j=-\lfloor(p-1)/5\rfloor}^{\lfloor(p-1)/5\rfloor} (-1)^j q^{j(5j-3)/2} \left[ \frac{p-1}{(p-1-5j)/2} + 1 \right]_q
\]
\[
\equiv \sum_{j=-\lfloor(p-6)/5\rfloor}^{\lfloor(p-6)/5\rfloor} (-1)^j + [(p-1-5j)/2] + 1 q \hat{L}(j)
\]
\[
= \sum_{j=-\lfloor(p-11)/10\rfloor}^{\lfloor(p-11)/10\rfloor} (-1)^{(p-1)/2-3j+1}(q \hat{L}(2j) - q \hat{L}(2j-1))
\]
\[\equiv 0 \pmod{[p]_q}.\]

Similarly when \( p \equiv 4 \mod 5 \),
\[
\hat{F}_{p-1}(q) = \sum_{j=-\lfloor(p-4)/5\rfloor}^{\lfloor(p-4)/5\rfloor} (-1)^j + [(p-1-5j)/2] + 1 q \hat{L}(j) \equiv 0 \pmod{[p]_q}.
\]

Finally, suppose that \( p \equiv 2 \) or \( 3 \mod 5 \). Then
\[
\hat{F}_{p-1}(q)
\]
\[
\equiv \sum_{j=-\lfloor(p/5\rfloor)}^{\lfloor(p/5\rfloor)} (-1)^j + [(p-1-5j)/2] + 1 q \hat{L}(j)
\]
\[
\equiv \begin{cases} 
-1)^{(p-2)/5+1} q \hat{L}((-p-2)/5) = q^{p(p-7)/10} \equiv 1 \pmod{[p]_q} & \text{if } p \equiv 2 \mod 5, \\
-1)^{(p-3)/5+(p-1)} q \hat{L}(-(p-3)/5) = q^{p(1-2p)/5} \equiv 1 \pmod{[p]_q} & \text{if } p \equiv 3 \mod 5.
\end{cases}
\]
All are done. \( \square \)

3. \( q \)-PELL NUMBER

To prove Theorem 1.3, we need the similar identities as (2.1) and (2.2) for \( \mathcal{P}_n(q) \) and \( \hat{\mathcal{P}}_n(q) \) respectively. Fortunately, such identities have been established by Santos and Sills in [8]. Let
\[
T_1(n, m, q) = \sum_{j=0}^{n} (-q)^j \frac{n}{j} \frac{2n-2j}{q^2} \frac{n-m-j}{q}.
\]

**Lemma 3.1** (Santos and Sills, [8]). Let \( n \) be a non-negative integer. Then
\[
\mathcal{P}_{n+1}(q) = \sum_{j=0}^{n} \sum_{k=0}^{j} q^{(j^2+j+k^2-k)/2} \frac{j}{k} \frac{n-k}{q} \frac{n}{j} \frac{n-m-j}{q}
\]
\[
= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2} T_1(n+1, 4j+1, \sqrt{q}), \quad (3.1)
\]
and
\[
\hat{P}_{n+1}(q) = \sum_{j=0}^{n} \sum_{k=0}^{j} q^{(j^2 + j^2 + 2k + n)/2} \binom{j}{k}_q \binom{n-k}{j}_q
\]
\[
= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2 + j} T_1(n + 1, 4j + 1, \sqrt{q}). \tag{3.2}
\]

**Proof of Theorem 1.3.** For arbitrary two polynomials \(f(q)\) and \(g(q)\), it is easy to see that \(f(q^2) \mid g(q^2)\) implies \(f(q) \mid g(q)\). Indeed, if
\[
g(q^2)/f(q^2) = \sum_{k=0}^{n} a_k q^k,
\]
then
\[
\sum_{k=0}^{n} a_k (-q)^k = g((-q)^2)/f((-q)^2) = g(q^2)/f(q^2) = \sum_{k=0}^{n} a_k q^k,
\]
whence \(a_k = 0\) for each odd \(k\). Thus
\[
g(q)/f(q) = \sum_{0 \leq k \leq n/2} a_{2k} q^k.
\]

Now it suffices to show that
\[
q^{(p^2-1)/4} \mathcal{P}_p(q^2) \equiv \left(\frac{2}{p}\right) \pmod{[p]_{q^2}}
\]
and
\[
\hat{P}_p(q^2) \equiv \left(\frac{2}{p}\right) q^{(p^2-1)/4} \pmod{[p]_{q^2}}.
\]
By (3.1), we have
\[
\mathcal{P}_p(q^2) = \sum_{j=-\infty}^{\infty} (-1)^j q^{4j^2} T_1(n + 1, 4j + 1, q)
\]
\[
= \sum_{j=-\infty}^{\infty} (-1)^j q^{4j^2} \sum_{k=0}^{p} (-q)^k \left[ p \atop k \right]_{q^2} \left[ 2p - 2k \atop p - 4j - k - 1 \right]_q
\]
\[
\equiv \sum_{j=-\infty}^{\infty} (-1)^j q^{4j^2} \left[ 2p \atop p - 4j - 1 \right]_q \pmod{[p]_{q^2}}.
\]
Since
\[
\left[ 2p \atop k \right]_q \equiv 0 \pmod{[p]_{q^2}}
\]
when \( p \nmid k \), we have

\[
\mathcal{P}_p(q^2) \equiv \begin{cases} 
(-1)^{(p-1)/4} q^{(p-1)^2/4} & \text{if } p \equiv 1 \mod 4 \\
(-1)^{-(p+1)/4} q^{(p+1)^2/4} & \text{if } p \equiv 3 \mod 4 
\end{cases}
\]

\[
\equiv \left( \frac{2}{p} \right) q^{-(p^2-1)/4} \quad \text{(mod } [p]_{q^2}).
\]

The proof of (1.10) is very similar. From (3.2), we deduce that

\[
\hat{P}_p(q^2) = \sum_{j=-\infty}^{\infty} (-1)^j q^{4j^2+2j} \sum_{k=0}^{p} (-q)^k \left[ \begin{array}{c} p \\ k \end{array} \right] q^2 \left[ \begin{array}{c} 2p-2k \\ p-4j-k-1 \end{array} \right]_q
\]

\[
\equiv \sum_{j=-\infty}^{\infty} (-1)^j q^{4j^2+2j} \left[ \begin{array}{c} 2p \\ p-4j-1 \end{array} \right]_q
\]

\[
\equiv \begin{cases} 
(-1)^{(p-1)/4} q^{(p-1)^2/4+(p-1)/2} & \text{if } p \equiv 1 \mod 4 \\
(-1)^{-(p+1)/4} q^{(p+1)^2/4-(p+1)/2} & \text{if } p \equiv 3 \mod 4 
\end{cases}
\]

\[
= \left( \frac{2}{p} \right) q^{(p^2-1)/4} \quad \text{(mod } [p]_{q^2}).
\]

Thus the proofs of (1.9) and (1.10) are completed.

We need the following lemma in the proof of (1.11).

**Lemma 3.2.**

\[
\left[ \begin{array}{c} 2p+2 \\ p \end{array} \right]_q \equiv 1 + q^p \quad \text{(mod } [p]_{q^2})
\]

for any odd prime \( p \).

**Proof.** Write

\[
\left[ \begin{array}{c} 2p+2 \\ p \end{array} \right]_q = \frac{(1-q^{2p+2})(1-q^{2p+1})(1-q^{2p}) \prod_{j=1}^{p-1} (1-q^{2p-j})}{(1-q^{p+2})(1-q^{p+1})(1-q^p) \prod_{j=1}^{p-1} (1-q^j)}.
\]

Since

\[
\frac{1-q^{2p-j}}{1-q^j} = \frac{q^j-q^{2p}}{q^j(1-q^j)} = \frac{(1-q^{2p}) - (1-q^j)}{q^j(1-q^j)} \equiv -q^{-j} \quad \text{(mod } [p]_{q^2}),
\]

we have

\[
\left[ \begin{array}{c} 2p+2 \\ p \end{array} \right]_q \equiv (-1)^{p-1} q^{-\binom{q}{2}} \frac{(1-q^{2p+2})(1-q^{2p+1})(1-q^{2p})}{(1-q^{p+2})(1-q^{p+1})(1-q^p)} \quad \text{(mod } [p]_{q^2}).
\]
And noting that
\[
q^{-\left(\frac{p}{2}\right)} \frac{(1 - q^{2p+2})(1 - q^{2p+1})}{(1 - q^{p+2})(1 - q^{p+1})} \equiv 1 \pmod{[p]_q},
\]
it follows that
\[
q^{-\left(\frac{p}{2}\right)} \frac{(1 - q^{2p+2})(1 - q^{2p+1})(1 - q^{2p})}{(1 - q^{p+2})(1 - q^{p+1})(1 - q^{p})} \equiv 1 + q^p \pmod{(1 + q^p)[p]_q = (1 + q)[p]_{q^2}).
\]

We are done. □

From the definition of \(T_1\), we have
\[
T_1(p + 1, 4j + 1, q) + qT_1(p, 4j + 1, q)
\]
\[
= \sum_{k=0}^{p+1} (-q)^k \left[ \frac{p + 1}{k} \right] \left[ \frac{2p + 2 - 2k}{p - 4j - k} \right]_q - \sum_{k=0}^{p} (-q)^k \left[ \frac{p}{k} \right]_q \left[ \frac{2p - 2k}{p - 4j - k - 1} \right]_q
\]
\[
= \left[ \frac{2p + 2}{p - 4j} \right]_q + \sum_{k=0}^{p} (-q)^k \left[ \frac{p + 1}{k + 1} \right]_q - \left[ \frac{p}{k} \right]_q \left[ \frac{2p - 2k}{p - 4j - k - 1} \right]_q
\]
\[
= \left[ \frac{2p + 2}{p - 4j} \right]_q + (-q)^p \left[ \frac{2}{-4j} \right]_q \pmod{[p]_{q^2}).
\]

Thus
\[
P_{p+1}(q^2) + qP_p(q^2) = \sum_{j=-\infty}^{\infty} (-1)^j q^{4j^2} (T_1(p + 1, 4j + 1, q) - qT_1(p, 4j + 1, q))
\]
\[
\equiv \sum_{j=-\infty}^{\infty} (-1)^j q^{4j^2} \left[ \frac{2p + 2}{p - 4j} \right]_q + \sum_{j=-\infty}^{\infty} (-1)^{p-j} q^{4j^2+p} \left[ \frac{2}{-4j} \right]_q
\]
\[
\equiv \sum_{j=-\infty}^{\infty} (-1)^j q^{4j^2} \left[ \frac{2p + 2}{p - 4j} \right]_q - q^p \pmod{[p]_{q^2}).
\]

Note that
\[
\left[ \frac{2p + 2}{k} \right]_q \not\equiv 0 \pmod{[p]_{q^2}}
\]
only if \( k \in \{0, 1, 2, p, p + 1, p + 2, 2p, 2p + 1, 2p + 2\} \). Hence we have

\[
\sum_{j=-\infty}^{\infty} (-1)^j q^{4j^2} \left[ \frac{2p + 2}{p - 4j} \right] q \\
\equiv \left[ \frac{2p + 2}{p} \right] q + \begin{cases} 
(-1)^{(p-1)/4} q^{(p-1)^2/4} \left[ \frac{2p + 2}{1} \right] q & \text{if } p \equiv 1 \mod 4 \\
(-1)^{-(p+1)/4} q^{(p+1)^2/4} \left[ \frac{2p + 2}{2p + 1} \right] q & \text{if } p \equiv 3 \mod 4
\end{cases}
\]

\[
\equiv 1 + q^p + \left( \frac{2}{p} \right) q^{-(p^2-1)/4} (1 + q) \\
\equiv 1 + q^p + (1 + q) P_p(q^2) \pmod{[p]q^2}.
\]

This concludes that

\[
P_{p+1}(q^2) \equiv 1 + P_p(q^2) \pmod{[p]q^2}.
\]

Similarly, we have

\[
\hat{P}_{p+1}(q^2) + q \hat{P}_p(q^2) = \sum_{j=-\infty}^{\infty} (-1)^j q^{4j^2+2j} (T_1(p + 1, 4j + 1, q) - qT_1(p, 4j + 1, q)) \\
\equiv \sum_{j=-\infty}^{\infty} (-1)^j q^{4j^2+2j} \left[ \frac{2p + 2}{p - 4j} \right] - q^p \\
\equiv \left[ \frac{2p + 2}{p} \right] q + \left( \frac{2}{p} \right) q^{(p^2-1)/4} (1 + q) - q^p \\
\equiv 1 + q^p + (1 + q) \hat{P}_p(q^2) - q^p \pmod{[p]q^2},
\]

whence

\[
\hat{P}_{p+1}(q^2) \equiv 1 + \hat{P}_p(q^2) \pmod{[p]q^2}.
\]

\[\square\]

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