Z' and the Appelquist-Carrazzone decoupling

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Abstract

We consider the electroweak theory with an additional neutral vector boson \( Z' \) at one loop. We propose a renormalization scheme which makes the decoupling of heavy \( Z' \) effects manifest. The proposed scheme justifies the usual procedure of performing fits to the electroweak data by combining the full SM loop corrections to observables with the tree level corrections due to the extended gauge structure. Using this scheme we discuss in the model with extra an \( U(1)' \) group factor 1-loop results for the \( \rho \) parameters defined in several different ways.
1 Introduction

For various reasons new physics is expected to show up at the TeV scale. One of the possibilities, not the least likely one, is that extra gauge boson with masses $\sim 1$ TeV should be discovered. They are predicted by various string inspired models as well as by some models aiming at solving the hierarchy problem of the SM. Here belong for example Little Higgs models [1] or models combining supersymmetry with the idea of the Higgs doublet as a pseudo-Goldston boson [2,3]. Before the advent of the LHC, the electroweak data are used to constrain parameter spaces of such models.

The standard methodology used in testing models of new physics against the electroweak data is that one combines the full one-loop (and also dominant two-loop) corrections to the relevant observables calculated within the SM with modifications stemming from new physics (new gauge bosons, new fermions, etc.) accounted at the tree level only. Given that the top quark mass is known fairly well, this allows to constrain other parameters of these models [4].

However, some doubts have been expressed in the literature [5–7] about the validity of this standard approach in models with extended gauge sector. In particular, it has been argued that this approach is not valid in theories in which at the tree level $\rho \neq 1$ since then the entire structure of loop correction is altered and the Appelquist-Carrazzone decoupling does not hold.

To investigate the problem in more detail we consider in this paper the simplest extension of the SM with additional $U(1)_E$ gauge group and study the one-loop renormalization of the model.\(^1\) We propose a renormalization scheme in which the Appelquist-Carrazzone decoupling is manifest. It combines the on-shell renormalization for the three input observables for which we conveniently choose $\alpha_{EM}$, $G_F$ and $M_W$ with the MS scheme for the additional parameters introduced by the extended gauge sector. The final expressions for measurable quantities are such that

- they coincide with the SM expression for $M_{Z'} \to \infty$,
- explicit renormalization scale dependence is only in the $M_{Z'}$ suppressed terms
- they are are scale independent when the RG running of the parameters is taken into account. Tadpoles play the crucial role here.

Our scheme can be contrasted with other renormalization schemes used in the literature in which the explicit decoupling of heavy particles ($Z'$) is lost because also the couplings related to the extended gauge sector (couplings of the $U(1)_E$ gauge boson) are expressed in terms of the additional to $\alpha_{EM}$, $G_F$ and $M_Z$ (or $M_W$) low energy observables like $\sin^2 \theta^\text{eff}_l$ or $\rho$. Our scheme can universally be used for $M_{Z'} \sim M_{Z^0}$ or $M_{Z'} \gg M_{Z^0}$ whereas the other ones are practical only for $M_{Z'} \sim M_{Z^0}$. Indeed, for $M_{Z'} \gg M_{Z^0}$, using e.g. $\sin^2 \theta^\text{eff}_l$ as an additional input parameter for fixing the coupling of $Z'$ leads, because of the lack in such a scheme of explicit Appelquist-Carrazzone decoupling, to uncertainties which become larger, the larger is the $Z'$

\(^1\)For earlier discussions of the renomalization of the $SU(2) \times U(1)_1 \times U(1)_2$ models see [8,9].
mass. The scheme proposed in this paper allows to directly constrain by the electroweak data the \( \overline{\text{MS}} \) running parameters of the extended model at a conveniently chosen renormalization scale \( \mu \), with \( \alpha_{\text{EM}} \), \( G_F \) and \( M_W \) chosen as input observables. Furthermore, for \( M_{Z'} \gg M_{Z^0} \) it lends justification to the standard approach to testing such a model against electroweak data and makes it rigorous by specifying what parameters are being constrained.

As an illustration of the use of our renormalization scheme and in order to demonstrate that it leads to explicit Appelquist-Carrazzone decoupling we clarify various aspects of the \( \rho \) parameter(s) in the \( SU(2) \times U(1)_1 \times U(1)_2 \) model. First of all, we discuss in detail various definitions of \( \rho \) and the corresponding tree level results. Interestingly enough, there exist a definition of \( \rho \) in terms of the low energy neutral to charged current ratio for neutrino processes which leads to \( \rho_{\text{low}} = 1 \) as in the SM. Next, we calculate loop corrections to these different \( \rho \) parameters and show that in the renormalization scheme with explicit Appelquist-Carrazzone decoupling the celebrated \( m_t^2/m_W^2 \) contribution is always present, The claimed in [5, 6] milder, logarithmic dependence on \( m_t \) is an artifact of a renormalization scheme in which there is no explicit Appelquist-Carrazzone decoupling.

We also elucidate some specific technical aspects of a theory with \( U(1)_1 \times U(1)_2 \) group factor related to the mixing of the two corresponding gauge bosons resulting in some peculiarities of the RG running of the \( U(1) \) gauge couplings.

The plan of the paper is as follows. In Section 2 we recall the general structure of a \( U(1)_1 \times U(1)_2 \) gauge theory and introduce effective charges which allow to cast the Lagrangian in a simple form. We express the renormalization group equations for the \( U(1) \) couplings in terms of these effective couplings. We also introduce the simplest extension of the SM by an extra \( U(1) \) group factor (with an \( SU(2) \) singlet scalar vacuum expectation value (VEV) breaking the extra \( U(1) \)) which will serve us as a laboratory to illustrate our main points concerning the loop corrections to electroweak observables. In Section 3 we define different \( \rho \) parameters, calculate them at tree level in the model introduced in Section 2 and show that the leading order contribution of \( Z' \) to these parameters can be also obtained in the approach using the Appelquist-Carrazzone decoupling. In Section 4 we define our renormalization scheme, and apply it in Section 5 to calculate the corrections to the low energy \( \rho \) parameter defined in terms of the neutrino processes. In Section 6 we illustrate the interplay of the proposed scheme with the renormalization group equations derived in Section 2 on the one-loop calculation of the \( Z^0 \) mass. Finally, in Section 7 we briefly discuss the calculation of the dominant top bottom contribution to the parameter \( \rho \) defined in terms of the \( Z^0 \), \( W^\pm \) gauge boson masses and \( \sin^2 \theta_{\text{eff}} \) parametrizing the coupling of on-shell \( Z^0 \) to leptons. Several appendices contain technical details necessary in the analyzes presented in the main text.
2 \textbf{\textit{U(1)}$_1$ × \textit{U(1)}$_2$ gauge theory: couplings and their RG equations}

The most general kinetic term for two \textit{U(1)} gauge fields has the form

$$\mathcal{L}^{\text{kin}} = -\frac{1}{4} f_{\mu\nu}^{1} f_{\mu\nu}^{1} - \frac{1}{4} f_{\mu\nu}^{2} f_{\mu\nu}^{2} - \frac{1}{2} \kappa f_{\mu\nu}^{1} f_{\mu\nu}^{2}.$$  \hspace{1cm} (1)

\(\kappa\) is a real constant constrained by the condition \(|\kappa| < 1\). The most general covariant derivative of a matter field \(\psi_k\) is

$$\mathcal{D}_\mu = \partial_\mu + i \sum_{a=1}^{2} \sum_{b=1}^{2} Y_k^a g_{ab} A_\mu^b,$$  \hspace{1cm} (2)

where the constants \(Y_k^a\) play the role of the \textit{U(1)} charges of \(\psi_k\) and \(g_{ab}\) are the coupling constants (running couplings in the \(\overline{\text{MS}}\) renormalization scheme). The gauge transformations then are

$$A_\mu^a \rightarrow A_\mu^a + \partial_\mu \theta^a,$$
$$\psi_k \rightarrow \exp \left( -i \sum_{a=1}^{2} \sum_{b=1}^{2} Y_k^a g_{ab} \theta^b \right) \psi_k.$$  \hspace{1cm} (3)

The existence of a whole matrix \(g_{ab}\) of couplings in place of only one gauge coupling per each \textit{U(1)} group factor is a peculiarity of the theory with multiple \textit{U(1)}’s [10,11]. Even if not introduced in the original Lagrangian, the last term in (1) and the matrix \(g_{ab}\) of couplings are generated in the effective action by radiative corrections.

To have simple forms of the tree level propagators, it is convenient to work in the basis in which the tree-level kinetic mixing is removed.\footnote{It is also possible to work with nondiagonal kinetic terms [11,12].} By expressing the original \(A_\mu^1, A_\mu^2\) fields in terms of the new fields denoted by \(A_\mu^Y\) and \(A_\mu^E\) (because they will play the roles of the weak hypercharge and extra \textit{U(1)} gauge bosons, respectively)

$$A_\mu^1 = \frac{1}{\sqrt{2(1 + \kappa)}} A_\mu^Y + \frac{1}{\sqrt{2(1 - \kappa)}} A_\mu^E, \quad A_\mu^2 = \frac{1}{\sqrt{2(1 + \kappa)}} A_\mu^Y - \frac{1}{\sqrt{2(1 - \kappa)}} A_\mu^E$$  \hspace{1cm} (4)

the kinetic cross term disappears (but there will be a counterterm \(- (1/2) \delta Z f_{\mu\nu} f_{Y\nu}^*\)) and the general form (2) of the covariant derivative does not change. Thus, for each matter field \(k\) there are charges \(Y_k^E\) and \(Y_k^Y\) and there are four couplings \(g_{YY}, g_{YE}, g_{YE}, g_{EE} \). Only three of them are independent [10]: the \textit{U(1)} gauge fields can be rotated: \(A^Y = \cos \vartheta \tilde{A}^Y - \sin \vartheta \tilde{A}^E, A^E = \sin \vartheta \tilde{A}^Y + \cos \vartheta \tilde{A}^E\), without reintroducing the kinetic cross term and such a rotation induces the corresponding rotations of couplings

$$\left( \tilde{g}_{YY} \tilde{g}_{YE} \right) = \left( \begin{array}{cc} \cos \vartheta & \sin \vartheta \\ - \sin \vartheta & \cos \vartheta \end{array} \right) \left( g_{YY} g_{YE} \right)$$
$$\left( \tilde{g}_{EE} \right) = \left( \begin{array}{cc} \cos \vartheta & \sin \vartheta \\ - \sin \vartheta & \cos \vartheta \end{array} \right) \left( g_{EE} \right)$$  \hspace{1cm} (5)
The angle $\vartheta$ can be chosen so that one of the four couplings vanishes. It is also easy to check that the following combinations
\begin{align}
g_{EE}g_{YY} - g_{EY}g_{YE} , & \quad g^2_{EE} + g^2_{EY} , \\
g_{YE}g_{EE} + g_{EY}g_{YV} , & \quad g^2_{YY} + g^2_{YE} ,
\end{align}
are the invariants of the rotations (5).

The renormalization group equations for the couplings $g_{ab}$ can be computed in the standard way [10, 11] with the result
\begin{equation}
\mu d g_{ba} = \frac{1}{16\pi^2} \sum_{c,d,e} g_{bc} \left[ \frac{2}{3} \sum_f (Y_f^d Y_f^e) + \frac{1}{3} \sum_s (Y_s^d Y_s^e) \right] g_{dc}g_{ea} \tag{7}
\end{equation}
where the first sum is over left-chiral fermions and the second one over complex scalars of the theory.

As an realistic extension of the SM we consider a theory with the $SU(2)_L \times U(1)_Y \times U(1)_E$ electroweak symmetry spontaneously broken down to $U(1)_{EM}$. The required symmetry breaking is ensured by vacuum expectation values of an $SU(2)$ doublet $H$ and of a singlet $S$. We assume that $S$ is charged under only one $U(1)$, that is $Y^S_H = 0$ (but $Y^Y_H \neq 0$ and $Y^E_H \neq 0$), so that $\langle S \rangle = v_S/\sqrt{2}$ leaves unbroken $SU(2)_L \times U(1)_Y$. It is then convenient to make the orthogonal field redefinition (which does not reintroduce the kinetic mixing term)
\begin{equation}
E_\mu = \frac{g_{EE} A^E_\mu + g_{EY} A^Y_\mu}{\sqrt{g^2_{EE} + g^2_{EY}}}, \quad B_\mu = \frac{-g_{EY} A^E_\mu + g_{EE} A^Y_\mu}{\sqrt{g^2_{EE} + g^2_{EY}}} \tag{8}
\end{equation}
where $E_\mu$ is the combination which becomes massive after $U(1)_E$ breaking by $v_S \neq 0$ and $B_\mu$ will play the role of the weak hypercharge gauge field. The couplings of a generic matter field $\psi_k$ to $E_\mu$ and $B_\mu$ are then given by
\begin{equation}
g_y Y_k B_\mu + \left( g_{EY} Y_k^E + g' Y_k^Y \right) E_\mu \tag{9}
\end{equation}
where
\begin{equation}
g_y \equiv \frac{g_{EE}g_{YY} - g_{EY}g_{YE}}{\sqrt{g^2_{EE} + g^2_{EY}}} , \quad g_E \equiv \sqrt{g^2_{EE} + g^2_{EY}} , \quad g' \equiv \frac{g_{YE}g_{EE} + g_{EY}g_{YY}}{\sqrt{g^2_{EE} + g^2_{EY}}} \tag{10}
\end{equation}
are invariants of the transformations (5). Because only three couplings are physical the last invariant, $g^2_{YY} + g^2_{YE}$ in (6), which does not enter the definitions of $g_y$, $g_E$, $g'$ can be expressed in terms of these
\begin{equation}
g^2_{YY} + g^2_{YE} = g^2_y + g'^2 . \tag{11}
\end{equation}

From (9) it follows that $Y_k^Y$ corresponds to the SM hypercharge. We assume therefore, that the factors $Y_k^Y$ are as in the SM, in particular, $Y_H^Y = \frac{1}{2}$. It will also
prove convenient to introduce effective charges $e_k$ and to rewrite the couplings of matter fields to the extra gauge boson $E_\mu$ in the form

$$g_E e_k \equiv g_E Y_k^E + g' Y_k^Y.$$  

(12)

With the factors $e_k$ the matter Lagrangian can be written in the naive form (frequently used in the literature [13, 14]) as if there was no mixing of the two $U(1)$ group factors. It is however important to remember that $e_k$ are just the way to compactly write the couplings. They are not quantum numbers (charges) - except for $e_S$ which is constant. They do run with the scale: their RG running can be determined from the running of $g_{EE}$, $g_{YY}$, $g_{EY}$, $g_{YE}$ and of $g_E$.

The closed system of the RG equations for the three couplings (10) can be readily derived from the general formula (7). Note that these couplings are defined at any renormalization scale $\mu$ in the (rotating) basis in which the kinetic mixing term is absent. Using (11) one finds

$$
\begin{align*}
\frac{d}{dt}g_E & = A^{EE} g_E^3 + 2 A^{EY} g_E^2 g' + A^{YY} g_E g'^2, \\
\frac{d}{dt}g_y & = A^{YY} g_y^3, \\
\frac{d}{dt}g'_y & = A^{YY} (g'^2 + 2 g_y^2) + 2 A^{EY} g_E (g'^2 + g_y^2) + A^{EE} g_E g'^2, \\
\end{align*}
$$

(13)

where

$$A^{ab} = \frac{2}{3} \sum_f \left(Y^a_f Y^b_f\right) + \frac{1}{3} \sum_s \left(Y^a_s Y^b_s\right).$$

(14)

With the identification of $Y^Y_k$ as SM hypercharges, the running of $g_y$ is exactly as in the SM. This could be expected because of the $U(1)$ Ward identity which ensures the absence of threshold corrections to $g_y$ when the heavy massive $E_\mu$ field is decoupled.

In the calculations presented in the following sections we will need RG equations for the combinations $e_S^2 g_E^2$ and $e_H^2 g_E^2$ defined by (12). Using (13) and (14) these RG can be also expressed in terms of the effective couplings (12):

$$
\begin{align*}
\frac{d}{dt} e_S^2 g_E^2 & = 2 e_S^2 g_E^2 \left( \frac{2}{3} \sum_f (e_f g_E)^2 + \frac{1}{3} \sum_s (e_s g_E)^2 \right), \\
\frac{d}{dt} e_H^2 g_E^2 & = 2 e_H^2 g_E^2 \left( \frac{2}{3} \sum_f (e_f g_E)^2 + \frac{1}{3} \sum_s (e_s g_E)^2 \right) + 4 e_H g_E \left( \frac{2}{3} \sum_f e_f g_E Y^Y_f Y^Y_f + \frac{1}{3} \sum_s e_s g_E Y^Y_s Y^Y_s \right) g_y^2 \\
\end{align*}
$$

(15)

Finally, we recall the formulae derived in [13] for gauge boson masses appearing as a result of the electroweak breaking by $\langle S \rangle = v_S/\sqrt{2}$ and $\langle H^0 \rangle = v_H/\sqrt{2}$. The
$W^\pm$ boson mass is given as in the SM by $M_W^2 = \frac{1}{4}g_2^2v_H^2$, whereas the mass matrix of the neutral gauge bosons in the basis $(B_\mu, W^3_\mu, E_\mu)$ reads

$$
\mathcal{M}_{\text{neut}}^2 = \begin{pmatrix}
\frac{1}{4}g_2^2v_H^2 & -\frac{1}{4}g_yg_2v_H^2 & \frac{1}{2}g_yge Hv_H^2 \\
-\frac{1}{4}g_yg_2v_H^2 & \frac{1}{4}g_2^2v_H^2 & -\frac{1}{2}g_yge Hv_H^2 \\
\frac{1}{2}g_yge Hv_H^2 & -\frac{1}{2}g_yge Hv_H^2 & g_E^2(e_H^2v_H^2 + e_S^2v_S^2)
\end{pmatrix}
$$

(16)

It is diagonalized by two successive rotations so that the mass eigenstates are given by

$$
\begin{pmatrix}
B_\mu \\
W^3_\mu \\
E_\mu
\end{pmatrix} = \begin{pmatrix}
c & -sc' & ss' \\
s & cc' & -cs' \\
0 & s' & c'
\end{pmatrix}
\begin{pmatrix}
A_\mu \\
Z^0_\mu \\
Z'_\mu
\end{pmatrix},
$$

(17)

where $c \equiv \cos\theta_W$, $s \equiv \sin\theta_W$ are as in the SM: $s/c = g_y/g_2$, and $c' \equiv \cos \theta'$, $s' \equiv \sin \theta'$, where

$$
\tan 2\theta' = \frac{2(-\frac{1}{2}g_2^2 + g_2g_ey Hv_H^2)}{\frac{1}{4}(g_y^2 + g_2^2)v_H^2 - g_y^2(e_H^2v_H^2 + e_S^2v_S^2)}
$$

(18)

The masses of the two gauge bosons, $Z^0$ and $Z'$ are given by

$$
M_{Z^0}^2 = \frac{1}{2} \left( A + B - \sqrt{(A - B)^2 + 4D^2} \right),
$$

$$
M_{Z'}^2 = \frac{1}{2} \left( A + B + \sqrt{(A - B)^2 + 4D^2} \right),
$$

(19)

where $A = M_W^2/c^2$, $B = e_S^2g_2^2v_S^2 + e_H^2g_2^3v_H^2$ and $D = -(e/2sc)e_Hg_ey Hv_H^2$. The electric charge $e$ is given by the same formula as in the SM: $e = g_yc = g_2c$. In Appendix A we record some formulae which will prove indispensable in various manipulations.

The interactions of the matter fermions with $Z^0$ and $Z'$ bosons takes the form

$$
\mathcal{L}_{\text{int}} = -J_{Z^0\mu}Z^0_\mu - J_{Z'\mu}Z'_\mu
$$

where the currents are easily found to be

$$
J_{Z^0\mu} = \sum_{f = \nu, e, u, d} \left[ \frac{e}{sc} \left( T_f^3 - s^2Q_f \right) c' + e_f g_E s' \right] \tilde{\psi}_f \gamma^\mu \mathbf{P}_L \psi_f
$$

$$
+ \sum_{f = \nu, e, u, d} \left[ \frac{e}{sc} \left( -s^2Q_f \right) c' - e_f g_E s' \right] \tilde{\psi}_f \gamma^\mu \mathbf{P}_R \psi_f
$$

(20)

$$
J_{Z'\mu} = \sum_{f = \nu, e, u, d} \left[ -\frac{e}{sc} \left( T_f^3 - s^2Q_f \right) s' + e_f g_E c' \right] \tilde{\psi}_f \gamma^\mu \mathbf{P}_L \psi_f
$$

$$
+ \sum_{f = \nu, e, u, d} \left[ -\frac{e}{sc} \left( -s^2Q_f \right) s' - e_f g_E c' \right] \tilde{\psi}_f \gamma^\mu \mathbf{P}_R \psi_f
$$

(21)
where \( P_L = \frac{1}{2}(1 - \gamma^5) \), \( P_R = \frac{1}{2}(1 + \gamma^5) \). The factors in square brackets in (20) and (21) define the couplings \( c^Z_{fL,R} \) and \( c^{Z'}_{fL,R} \).

Gauge invariance of the Yukawa couplings of the matter fields

\[
L_{Yuk} = -y_e H_i^* l_i e^c - y_t \epsilon_{ij} H_i^* q_j u^c - y_d H_i^* q_i d^c
\]

imposes the conditions (see (3))

\[
Y_{ae}^a + Y_{l}^a - Y_{H}^a = 0 ,
Y_{ue}^a + Y_{q}^a + Y_{H}^a = 0 ,
Y_{de}^a + Y_{q}^a - Y_{H}^a = 0 ,
\]

where \( a = E, Y \). When combined with (12) they imply

\[
e_{ec} + e_l - e_H = 0 ,
e_{uc} + e_q + e_H = 0 ,
e_{dc} + e_q - e_H = 0 .
\]

3 \( \rho \) parameters in the \( SU(2)_L \times U(1)_Y \times U(1)_E \) model and the Appelquist-Carrazzone decoupling

In this section we define various measurable \( \rho \) parameters in the \( SU(2)_L \times U(1)_Y \times U(1)_E \) model and show that at the tree level the effects of heavy \( Z' \) decouple. We then identify the dimension six operators which, when added to the SM Lagrangian, reproduce at the tree level the leading (in inverse powers of \( v^2 \)) corrections to low energy observables due to \( Z' \).

3.1 \( \rho \) parameters

In the SM the measurable parameter \( \rho \) can be defined in several different ways. The simplest is the definition of \( \rho \) (call it \( \rho_{\text{low}} \)) as the ratio of the coefficients of the neutral and charged current terms in the effective low energy four-fermion Lagrangian. Another one is

\[
\rho = \frac{M^2_{W}}{M^2_{Z^0}(1 - \sin^2 \theta)} ,
\]

with \( \sin^2 \theta \) related to measurable quantities in various ways, e.g. as the parameter in the on-shell \( Z^0 \) couplings to fermions as in (24), or by the low energy neutral current Lagrangian for e.g. neutrino processes (i.e. as a parameter measuring the admixture of the vector-like electromagnetic current in the leptonic weak neutral current in the mentioned above low energy four-fermion Lagrangian). Finally, \( \rho \) (call it \( \rho_{Zf} \)) can be defined through the coupling of on-shell \( Z^0 \) to fermion-antifermion pairs expressed in terms of the Fermi constant measured in the muon decay:

\[
L^0_{\text{eff}} \quad \text{on shell} = - \left( \sqrt{2} G_F M^2_{Z^0} \rho_{Zf} \right)^{1/2} \bar{\psi}_f \gamma^\mu \left( T^3 - 2 Q_f \sin^2 \theta_{\text{eff}}^f - T^3_{f} \gamma^5 \right) \psi_f Z^0_{\mu} .
\]
Independently of the definition used, $\rho = 1$ at the tree level due to the custodial $SU(2)_V$ symmetry of the SM Higgs potential. Thus, in the SM $\rho = 1$ is the so-called natural relation, i.e. the prediction which does not depend on the values of the parameters of the model. Of course, quantum corrections to $\rho$ are numerically different for different definitions and do depend on the values of the SM parameters. The usefulness of $\rho$ stems from the fact that the dominant contributions (dependent on the top quark and Higgs boson masses) to it are universal, that is, the same for all definitions of $\rho$.

Although the different $\rho$ are observables (they are all defined in terms of measurable quantities) none of them can be used as an input observable in the procedure of renormalization of the SM, just because $\rho = 1$ is the natural relation.

In the $SU(2)_L \times U(1)_Y \times U(1)_E$ model custodial symmetry is broken at the tree level by the $Z^0-Z'$ mixing. It is then necessary to discuss the analogous $\rho$ parameters in some detail. Parameters $\rho$ and $\rho_{Zf}$ can be defined as in the SM, i.e. by the equations (23) and (24), respectively. The parameters $\rho_{\text{low}}$ is special, because it refers to the specific form of the low energy effective Lagrangian which needs not be the same as in the SM. In models in which the charged weak currents are unmodified with respect to the SM the effective Lagrangian for low energy weak interactions takes the general form

$$L_{\text{eff}} = -2\sqrt{2} G_F J^\mu J_{-\mu} \frac{1}{2} \sum_{f_1} \sum_{f_2} \left[ a^{f_1 f_2}_{LL} \left( \bar{\psi}_{f_1} \gamma^\mu P_L \psi_{f_1} \right) \left( \bar{\psi}_{f_2} \gamma^\mu P_L \psi_{f_2} \right) \right. \left. + a^{f_1 f_2}_{RR} \left( \bar{\psi}_{f_1} \gamma^\mu P_R \psi_{f_1} \right) \left( \bar{\psi}_{f_2} \gamma^\mu P_R \psi_{f_2} \right) \right]$$

where $J^\mu_{\pm}$ are the standard charged currents. In the SM the second part of (25) can be rewritten in the form of the product of two neutral currents

$$L_{\text{eff}}^{\text{NC}} = -2\sqrt{2} G_F J^\mu J_\mu ,$$

where

$$J^\mu = \sum_f \sqrt{\rho_f} \bar{\psi}_f \gamma^\mu \left( T^3_f P_L - \sin^2 \theta^\text{eff}_f Q_f \right) \psi_f$$

Moreover, if the fermion mass effects are neglected $\rho_f$ and $\sin^2 \theta^\text{eff}_f$ are universal, $\rho_f = \rho$, $\sin^2 \theta^\text{eff}_f = \sin^2 \theta^\text{eff}$. $\rho$ can be then factorized out of the neutral current $J^\mu$ and $\rho = 1$.

The necessary condition to define the low energy parameter $\rho_f$ (possibly dependent on the fermion type) in the $SU(2)_L \times U(1)_Y \times U(1)_E$ model is that the second part of (25) can be written in the current×current form (26). One would then have

$$\sqrt{\rho_{f_1} \rho_{f_2}} = \frac{a^{f_1 f_2}_{LL} + a^{f_1 f_2}_{RR} - a^{f_1 f_2}_{LR} - a^{f_1 f_2}_{RL}}{\sqrt{2} G_F 2T^3_{f_1,2} T^3_{f_2}}$$

8
Computing the diagrams with exchanges of $Z^0$ and $Z'$ between the two currents $J^\mu_{Z^0}$ (20) and two currents $J^\mu_{Z'}$ (21), respectively, and exploiting the relations (A.2) and (A.3) it is easy to find

$$a_{f_1f_2}^{\ell_f} + a_{f_1f_2}^{\ell_f} - a_{f_1f_2}^{\ell_f} - a_{f_1f_2}^{\ell_f} = -\frac{1}{v_H^2}2T_1^3 2T_2^3$$

$$- (2T_1^3 e_H g_E + e_f g_E + e_{f_1} g_E)(2T_2^3 e_H g_E + e_{f_2} g_E + e_{f_2} g_E)$$

$$\left(\frac{\epsilon_S^2 g_E^2 v_F^2}{\epsilon_S^2 g_E^2 v_F^2}\right)$$

Due to the relations (22), the second term vanishes and, since at the tree level $1/v_H^2 = \sqrt{2}G_F$, we find (to some surprise) that in the $SU(2)_L \times U(1)_Y \times U(1)_E$ model at the tree level

$$a_{f_1f_2}^{\ell_f} + a_{f_1f_2}^{\ell_f} - a_{f_1f_2}^{\ell_f} - a_{f_1f_2}^{\ell_f} = -2T_1^3 2T_2^3 \sqrt{2}G_F$$

as in the SM. However, writing the second part of (25) in the familiar current $\times$ current form is not always possible. It is only possible, if the following consistency condition holds

$$\left(a_{f_1f_2}^{\ell_f} - a_{f_1f_2}^{\ell_f}\right) \left(a_{f_1f_2}^{\ell_f} - a_{f_1f_2}^{\ell_f}\right) = -4 \sqrt{2}G_F a_{f_1f_2}^{\ell_f}$$

(it follows from the fact that the form (26) depends only on three unknown: $\sqrt{p_f} p_f$, $\sin^2 \theta_{e_h}$ and $\sin^2 \theta_{e_n}$, whereas the general form of the second term in (25) has four independent coefficients). It is straightforward to check that the condition (31) is not satisfied in general. It is satisfied only by that part of (25) which describes neutrino reactions. In this case $a_{f_1f_2}^{\ell_f} = a_{f_1f_2}^{\ell_f} = a_{f_1f_2}^{\ell_f} = 0$ and the condition (31) is trivially satisfied. Thus, for neutrino processes one can define the analog of the SM $\rho$ parameter as $\rho_{\text{low}} \equiv \sqrt{p_f} p_f$ and from (29) it follows that at the tree level $\rho_{\text{low}} = 1$ as in the SM.

In the general case in the $SU(2)_L \times U(1)_Y \times U(1)_E$ model even the generalized low energy parameters $\rho_f$ cannot be defined because the second part of the effective Lagrangian (25) cannot be written in the current $\times$ current form.

It is interesting to contrast $\rho_{\text{low}}$ discussed above, for which $\rho_{\text{low}} = 1$ at the tree level is a natural relation, with e.g. $\rho = M_{W}^2/M_{Z^0}^2(1 - \sin^2 \theta)$, with $\sin^2 \theta$ identified with $\sin^2 \theta_{\text{eff}}$ in (24). We find

$$\sin^2 \theta_{\text{eff}} = s^2 \left(1 - \frac{e_f e_{f_1} g_E s'}{1 - 2s c e_H e_f e_{f_1}} \right) \approx s^2 + s^2 \left(2 s c e_H - \frac{c}{s} e_f \right) \frac{g_E s'}{e f} + \ldots$$

$$= s^2 + \left(s^2 e_H - \frac{1}{2} e_f \right) \frac{e_H v_H^2}{e_S v_S^2} + \ldots$$

(32)
where we have used (A.1). Using (19) we get then
\[
\rho \approx \left(1 + \frac{e_H^2 g_E^2 v_H^2}{e_S^2 g_E^2 v_S^2} + \ldots\right) \left[1 + \left(\frac{s^2}{e^2} e_H - \frac{1}{2 c^2} e_H e_c\right) \frac{e_H v_H^2}{e_S^2 v_S^2} + \ldots\right] = 1 + O\left(\frac{v_H^2}{v_S^2}\right). \tag{33}
\]

The important difference between \(\rho_{\text{low}}\) and \(\rho\) in the \(SU(2)_L \times U(1)_Y \times U(1)_E\) model is that the latter does depend on some combination of the Lagrangian parameters. From the above results it is clear that the Appelquist-Carrazzone decoupling holds at the tree level in the \(SU(2)_L \times U(1)_Y \times U(1)_E\) model. It is also easy to show that it can be easily masked by choosing a low energy observable like \(\sin^2 \theta\) (in addition \(M_{Z'}\)) to fix e.g. the coupling \(g_E\). To simplify the argument, let us assume that \(e_\ell c = 0\) (at the renormalization scale we are working). Then \(e_H^2 v_H^2 / e_S^2 v_S^2\) in (33) can be directly expressed in terms of \(\sin^2 \theta_{\text{eff}}\) from (32) so that
\[
\rho \approx \left(\sin^2 \theta_{\text{eff}} + \ldots\right) \left[1 + \frac{\sin^2 \theta_{\text{eff}} - s^2}{e^2} + \ldots\right] \tag{34}
\]
and the decoupling is lost!

In the next subsection show the dimension six operators completing the SM Lagrangian, which reproduce leading terms of the corrections to electroweak observables found at the tree level.

### 3.2 Decoupling at the tree level

At the tree level the subgroup \(U(1)_E\) can be broken independently of the breaking of \(SU(2)_L \times U(1)_Y\). In this case the gauge field \(E_\mu\) becomes \(Z'\) with a mass \(M_{Z'}^2 = e_S^2 g_E^2 v_S^2\). For \(v_S\) much higher than the Fermi scale, the electroweak observables can be calculated in the \(SU(2)_L \times U(1)_Y\) effective theory (which is just the SM) supplemented with higher dimensional operators generated by decoupling of heavy \(Z'\). This approach yields corrections to the electroweak observables due to \(Z'\) effects in the form of power series in \(1/v_S\). Below we display the dimension six operators which reproduce the corrections to different \(\rho\) and \(\sin^2 \theta\) from the preceding subsection up to \(O(1/v_S^4)\).

Exchanges of \(Z'\) between fermion lines are taken into account by adding to the SM Lagrangian the four-fermion nonrenormalizable operators of the type
\[
\Delta L_{\text{SM}} = -\frac{1}{e_S^2 g_E^2 v_S^2} e_H^2 g_E^2 [\bar{\psi}_{lA} \gamma^\mu P_L \psi_{lA}] [\bar{\psi}_{lB} \gamma^\mu P_L \psi_{lB}]
\]

\[3\text{Defining } \sin^2 \theta \text{ in terms of the structure of the current (27) for neutrino processes we would get}
\]
\[
\sin^2 \theta = s^2 + (e_H + e_\ell)(s^2 e_H - \frac{1}{2} e_H e_c) \frac{v_H^2}{e_S^2 v_S^2}.
\]

\[4\text{The fact that at the tree level } \rho_{\text{low}} = 1 \text{ as in the SM makes this observable useless for constraining the } SU(2)_L \times U(1)_Y \times U(1)_E \text{ model as the effects of new physics will be always much larger in observables which are modified already at the tree level.}
\]
The kinetic term of the electroweak Higgs doublet $H$ gives rise, through the first diagram of figure 1, to a nonrenormalizable term of the form

$$\Delta L_{\text{SM}} = -\frac{1}{2} (2 e H g E)^2 \frac{1}{e^2 g E v_S^2} \left[ H^\dagger \left( g_2 W^a T^a + \frac{1}{2} g_B B \right) H \right]^2 .$$  

(36)

Finally, the second diagram shown in figure 1 gives rise to the interaction:

$$\Delta L_{\text{SM}} = \sum_f 2 e_f e H g E \frac{1}{e^2 g E v_S^2} \left[ H^\dagger \left( g_2 W^a T^a + \frac{1}{2} g_B B \right) H \right] \bar{\bar{f}} \bar{\bar{f}} \bar{\bar{f}} .$$  

(37)

After the electroweak symmetry breaking the operator (36) gives correction to the $Z^0$ mass squared $\Delta M_{Z^0}^2 = - (M_{Z^0}^2)_{\text{SM}} (e^2 H v_S^2 / e^2 g E v_S^2)$ whereas the operator (37) modify the $Z^0$ couplings to SM fermions:

$$\Delta L_{\text{SM}} = - \sum_f \frac{e_f e H}{2 s c} \frac{e^2 g E v_H^2}{v_S^2} Z_\mu \left[ \bar{\bar{f}} \bar{\bar{f}} \bar{\bar{f}} \bar{\bar{f}} \right] \approx - \sum_f e_f g E s' Z_\mu \left[ \bar{\bar{f}} \bar{\bar{f}} \bar{\bar{f}} \bar{\bar{f}} \right]$$

they just correspond to terms $e_f g E s'$ expanded to order $1/v_S^2$ in the $Z^0$ couplings (20).

Figure 1: Generating four-fermion operators by the heavy $Z'$.  

At the tree level the three operators (35), (36) and (37) reproduce to order $1/M_Z^2 \sim 1/v_S^2$ all corrections to the low energy (compared to $v_S$) observables due to the extended gauge structure of the model. This is equivalent to the statement that the Appelquist-Carrazone decoupling works for $Z'$ (at least) at the tree level.

We can illustrate this approach by calculating the corrections due to the higher dimensional operators (35), (36) and (37) to the parameter $\rho_{\text{low}}$. To this end it is sufficient to find the difference $a_{\mu}^{\mu} - a_{\mu}^{\mu}$ of the coefficients in the effective Lagrangian (25). In the SM $a_{\mu}^{\mu} - a_{\mu}^{\mu} = (e^2 / 4 s^2 c^2 M_Z^2) = 1/v_H^2$ and since at the tree level $1/v_H^2 = \sqrt{2} G_F$, $\rho_{\text{low}} = 1$. The corrections due to the extended gauge structure read

$$\left( \Delta a_{LL}^{\mu} \right)_{Z'} = - \frac{1}{g_E^2 e^2 g E v_S^2} \epsilon_1 e_1 g E^2 , \quad \left( \Delta a_{RL}^{\mu} \right)_{Z'} = \frac{1}{g_E^2 e^2 g E v_S^2} \epsilon_1 e_1 g E^2$$  

(38)
from the operator (35),
\[ (\Delta a_{L \nu}^{\ell \nu})_{Z} = \frac{e^2}{4s^2 c^2} \frac{1}{M_{Z_0}^2} \frac{e_H v_H^2}{e_S v_S^2}, \quad (\Delta a_{R \nu}^{\ell \nu})_{Z} = \frac{e^2}{4s^2 c^2} \frac{(-2s^2)}{M_{Z_0}^2} \frac{e_H v_H^2}{e_S v_S^2}, \tag{39} \]
from the correction to the Z⁰ mass produced by the operator (36) and
\[ (\Delta a_{L \nu}^{\ell \nu})_{Z} = -\frac{e^2}{2c^2 M_{Z_0}^2} \frac{e_i e_H v_H^2}{e_S^2 v_S^2}, \quad (\Delta a_{R \nu}^{\ell \nu})_{Z} = -\frac{e^2}{4s^2 c^2} \frac{1}{M_{Z_0}^2} (2s^2 e_i - e_c) \frac{e_H v_H^2}{e_S^2 v_S^2}, \tag{40} \]
from the correction to the Z⁰ couplings produced by the operator (37). Combining these three corrections we find, using the relations (22) that \( \Delta (a_{L \nu}^{\ell \nu} - a_{R \nu}^{\ell \nu}) = 0. \)

Other observables can be checked similarly. Subleading in \( 1/v_S \) corrections can be also reproduced upon inclusion in the SM Lagrangian operators of dimension higher than six.

The equivalence of the two approaches (full calculation versus higher dimensional operators) checked above shows that the Appelquist-Carrazzone decoupling holds at the tree level. The expectation that it should hold in the \( SU(2)_L \times U(1)_Y \times U(1)_E \) model to all orders is based on the observation that \( U(1)_E \) can be broken independently of the breaking of \( SU(2)_L \times U(1)_Y \). We will propose the scheme which makes it explicit at one loop and thus show that in particular it is not spoiled by the mixing of the gauge fields corresponding to the two \( U(1) \) groups.

### 4 Renormalization scheme

Before we define our renormalization scheme for the \( SU(2)_L \times U(1)_Y \times U(1)_E \) extension of the SM, it is instructive to recall the simplest possible approach to calculating loop corrections to the electroweak observables within the SM [15, 16].

Basic (running) parameters of the SM are:\(^5\) \( \hat{g}_y, \hat{g}_2 \) and \( \hat{v}_H \) (or any three other functions of these parameters, e.g. \( \hat{\alpha}, \hat{M}_Z \) and \( \hat{s}^2 \)). In the renormalization procedure they are expressed in terms of the values of the three experimentally measured observables. Traditionally one chooses for this purpose \( G_F, \alpha_{EM} \) and \( M_Z \). These quantities are computed in perturbation calculus using for example the dimensional regularization and the \( \overline{MS} \) subtraction:

\[
\alpha_{EM} = \frac{\hat{g}_y^2 \hat{g}_2^2}{4\pi(\hat{g}_y^2 + \hat{g}_2^2)} + \delta\alpha_{EM} = \frac{\hat{e}^2}{4\pi} + \delta\alpha_{EM} = \hat{\alpha} + \delta\alpha_{EM}
\]

\[
M_Z^2 = \frac{1}{4} (\hat{g}_y^2 + \hat{g}_2^2) \hat{v}^2 = \frac{1}{4} \frac{\hat{e}^2}{s^2 c^2} \hat{v}^2 + \delta M_Z^2 = \hat{M}_Z^2 + \delta M_Z^2 \tag{41}
\]

\[
G_F = \frac{1}{\sqrt{2} \hat{v}^2} + \delta G_F = \frac{\hat{e}^2}{\sqrt{2} s^2 c^2 \hat{M}_Z^2} + \delta G_F = \hat{G}_F + \delta G_F
\]

\(^5\)We denote running parameters which are traded for observables by a hat.
As the corrections $\delta \alpha_{\text{EM}}, \delta M_Z^2, \delta G_F$ are calculated in terms of the parameters $\hat{\alpha}, \hat{M}_Z^2,$ $s^2$ the above relations have to be inverted recursively. At the one loop order this is particularly simple:

\begin{align*}
\hat{\alpha} &= \alpha_{\text{EM}} - \delta \alpha_{\text{EM}} \\
\hat{M}_Z^2 &= M_Z^2 - \delta M_Z^2 \\
\hat{G}_F &= G_F - \delta G_F 
\end{align*}

(42)

where in $\delta \alpha_{\text{EM}}, \delta M_Z^2, \delta G_F$ one replaces the parameters $\hat{\alpha}, \hat{M}_Z^2, s^2$ by $\alpha_{\text{EM}}, M_Z$ and $G_F$ using the tree level relations. For any other measurable quantity $A$ we then have

\begin{align*}
A &= A^{(0)}(\hat{\alpha}, \hat{M}_Z^2, \hat{G}_F) + \delta A(\hat{\alpha}, \hat{M}_Z^2, \hat{G}_F) + \ldots 
\end{align*}

(43)

where $\delta A$ is the one loop contribution to the quantity $A$. This is next written as

\begin{align*}
A &= A^{(0)}(\alpha_{\text{EM}}, M_Z^2, G_F) + \delta A(\alpha_{\text{EM}}, M_Z^2, G_F) \\
&- \frac{\partial A^{(0)}}{\partial \alpha_{\text{EM}}} \delta \alpha_{\text{EM}} - \frac{\partial A^{(0)}}{\partial M_Z^2} \delta M_Z^2 - \frac{\partial A^{(0)}}{\partial G_F} \delta G_F .
\end{align*}

(44)

The expression (44) is finite and independent of the renormalization scale $\mu$.

The free running parameters of the $SU(2)_L \times U(1)_Y \times U(1)_E$ extension of the SM are $g_2, v_H, v_S$ and the couplings $g_{EE}, g_{EY}, g_{YY}, g_{YE}$ (in fact only three of them). One way of organizing higher loop calculations in such a model is to follow the recipe sketched above and to chose the appropriate number of input observables, in terms of which one would express all the running parameters.

Clearly, for $M_{Z'} \gg M_{Z'}^0$ the parameters of the model form two sets: $g_2, g_y$ and $v_H$ describe the SM electroweak sector and $v_S$, and the remaining gauge couplings describe the $Z'$ sector. However, since the $Z'$ boson has not yet been discovered and its mass is unknown (assuming it exists), the best way to organize loop calculations is such that the Appelquist-Carrazzone decoupling (in the case $Z'$ is heavy) would be manifest. This condition is not satisfied by schemes in which additional parameters related to the heavy particle sector are expressed in terms of *low energy observables*. Decoupling would be manifest if all additional parameters were related to measurable characteristics of the heavy particles. Independently of the question of decoupling, renormalization schemes using the number of observables equal to the number of free parameters may be difficult to implement in practice as one has to solve for the running parameters a larger set of equations than in the SM and the resulting analytical formulae may be very complicated and unmanageable.

In the fits to the electroweak data, breakdown of explicit Appelquist-Carrazzone decoupling in a scheme chosen to compute the observables may even incorrectly produce *upper bounds* on the additional heavy particles (gauge bosons, Higgs scalars).

In this paper we propose to organize loop calculations into a hybrid scheme in which the parameters $\hat{g}_2, \hat{g}_y$ and $\hat{v}_H$ are expressed in terms of $\alpha_{\text{EM}}, G_F$ and $M_{Z'}^0$ (or $M_W$) as in the SM and the remaining parameters are kept in the calculations as the $\overline{\text{MS}}$ scheme running parameters. The renormalization scale $\mu$ for them can be chosen arbitrarily.
As we will show by explicit calculations in the $SU(2)_L \times U(1)_Y \times U(1)_E$ model, the advantage of such a hybrid scheme\footnote{In fact, such a hybrid scheme is adopted for the usual treatment of the strong interaction corrections to the electroweak observables: $\hat{\alpha}_s(\mu)$ is not traded for any observable quantity; instead one relies on the fact that the explicit $\mu$ dependence of the two-loop contributions should cancel against the $\mu$ dependence of $\hat{\alpha}_s(\mu)$ in one-loop terms.} is twofold: the Appelquist-Carrazzone decoupling of heavy particle effects is made manifest - for heavy particle masses taken to infinity the expressions for the observables measured at energies of the order of the electroweak scale (or lower) coincide with the SM expression due to the presence of explicit suppression by a large mass scale (in the $SU(2)_L \times U(1)_Y \times U(1)_E$ model by factors of $1/v_2^2$). Moreover, explicit renormalization scale dependence remains only in the terms suppressed by the large mass scale(s). The expressions for observables are in fact scale independent when the RG running of the parameters is taken into account. Tadpoles play the crucial role here [17]. Last but not least, our scheme does not require solving for running parameters complicated set of equations; in this respect it is as practical in use as the usual schemes in the SM.

Extensions of the SM are constrained by precision electroweak observables. In our scheme observables are calculated in terms of $\alpha_{EM}$, $G_F$ and $M_Z$ or $M_W$ (because in the $SU(2)_L \times U(1)_Y \times U(1)_E$ model the tree level formula (19) for the $Z^0$ mass is complicated it is much more convenient to take as the three input observables $\alpha_{EM}$, $G_F$ and $M^2_W$ and compute instead $M^2_{Z^0}$ in terms of these) and the additional parameters of the model at a conveniently chosen renormalization scale $\mu$. Fits to the data can then give constraints on these running parameters. Moreover, in theories in which the Appelquist-Carrazzone decoupling holds, because the loop corrections reduce to their SM form as the heavy mass scale is sent to infinity, fairly accurate estimate of the limits imposed by the precision data on the additional parameters of the model is possible by combining the SM loop corrections with the tree level corrections due to “new physics”.

The one loop expressions for the chosen basic input observables read (see Appendix B for details):

\begin{align}
\hat{\alpha} &= \frac{\alpha_{EM}}{1 + \hat{\Pi}_\gamma(0) - (\hat{\alpha}/\pi) \ln \frac{M^2_W}{\mu^2}} \approx \alpha_{EM} \left(1 - \hat{\Pi}_\gamma(0) + \frac{\alpha_{EM}}{\pi} \ln \frac{M^2_W}{\mu^2}\right) \\
\hat{M}^2_W &= M^2_W \left(1 - \hat{\Pi}_{WW}(\frac{M^2_Z}{M^2_W})\right) \\
\hat{v}^2_H &= \frac{1}{\sqrt{2}G_F}(1 + \Delta_G) \\
\hat{v}^2_H &= \frac{\pi\alpha_{EM}}{\sqrt{2}G_FM^2_W}(1 + \Delta) \equiv s^2(0) + s^2(0)\Delta \\
\hat{c}^2 &= \frac{\sqrt{2}G_FM^2_W - \pi\alpha_{EM}(1 + \Delta)}{\sqrt{2}G_FM^2_W} \equiv c^2(0) - s^2(0)\Delta
\end{align}
where

$$\Delta = -\hat{\Pi}_\gamma(0) + \frac{\hat{\alpha}}{\pi} \ln \frac{M_W^2}{\mu^2} + \frac{\hat{\Pi}_{WW}(M_W^2)}{M_W^2} + \Delta_G$$

(as usually $\hat{\Pi}_\gamma(q^2)$ is defined by $\hat{\Pi}_{\gamma\gamma}(q^2) = q^2 \hat{\Pi}_\gamma(q^2)$, i.e. it is the residue of the photon propagator).

Using this scheme we will explicitly demonstrate that in the $SU(2)_L \times U(1)_Y \times U(1)_E$ extension of the SM the Appelquist-Carrazzone decoupling does hold. To this end we will compute in our scheme the two different $\rho$ parameters defined as in Section 3 in terms of the observables: $\rho_{\text{low}}$ defined by the effective Lagrangian for $\nu_\mu e^- \rightarrow \nu_\mu e^-$ elastic scattering and $\rho \equiv M_W^2/M_Z^2(1 - \sin^2\theta^\text{eff}_\ell)$ where $\sin^2\theta^\text{eff}_\ell$ parametrizes the effective coupling of on-shell $Z^0$ to $l^+l^-$ pair. In particular we will demonstrate that the celebrated $m_t^2/M_W^2$ term is present in both cases.

5 Decoupling of $Z'$ effects in $\rho_{\text{low}}$ at 1-loop

As an exercise, in order to demonstrate the working of our renormalization scheme we will compute one loop corrections to the low energy parameter $\rho_{\text{low}}$ defined by the $\nu_\mu e^- \rightarrow \nu_\mu e^-$ elastic scattering. Since $\rho_{\text{low}} = 1$ at the tree level is a natural relation in the $SU(2) \times U(1)_Y \times U(1)_E$ model, the one loop corrections to $\rho_{\text{low}}$ should be finite when $1/v_H^2$ in (29) is expressed in terms of $G_F$ with one loop accuracy.

At one loop the direct generation number dependent fermion contribution comes through the “oblique” corrections to $a_{LL}^{\nu \nu} - a_{RL}^{\nu \nu}$:

$$(a_{LL}^{\nu \nu} - a_{RL}^{\nu \nu})_{1\text{-loop}} = c_{\nu L}^{Z_0} a_{\nu}^{Z_0} \frac{1}{M_W^2} \frac{\Pi_0^{Z_0Z_0}(0)}{M_Z^2} + c_{\nu L}^{Z'} a_{\nu}^{Z'} \frac{1}{M_W^2} \frac{\Pi_0^{Z_0Z_0}(0)}{M_Z^2} + c_{\nu L}^{Z_0} a_{\nu}^{Z_0} \frac{1}{M_W^2} \frac{\Pi_0^{Z_0Z_0}(0)}{M_Z^2} + c_{\nu L}^{Z'} a_{\nu}^{Z'} \frac{1}{M_W^2} \frac{\Pi_0^{Z_0Z_0}(0)}{M_Z^2},$$

where $Z_i$ denotes $Z^0$ or $Z'$, $a_i^{Z_0} = c_{jL}^{Z_0} - c_{jR}^{Z_0}$, and the couplings $c_{jL,R}^{Z_0}$, $c_{jL,R}^{Z'}$ of $Z^0$ ($Z'$) to left and right-chiral leptons are defined by (20), (21). The self energies $\Pi_{Z_iZ_j}$ contain in principle also tadpole contributions. Another generation number dependent contribution to $\rho$ arises from $\hat{\Pi}_{WW}(0)/\hat{M}_W^2$ after expressing $1/v_H^2$ in the tree level term (29) with one loop accuracy

$$(a_{LL}^{\nu \nu} - a_{RL}^{\nu \nu})_{\text{tree}} = \frac{1}{v_H^2} = \sqrt{2}G_F(1 - \Delta_G)$$

with $\Delta_G$ given by (15,4).

Fermionic contribution to $\rho_{\text{low}}$

The top-bottom quark contribution to 1-particle irreducible part of $\hat{\Pi}_{WW}$ is the same as in the SM

$$\hat{\Pi}_{WW}(0) = \frac{\hat{e}^2}{s^2} N_c \left[ 2\hat{A}(0, m_t, m_b) - \frac{1}{2}(m_t^2 + m_b^2)\frac{\partial}{\partial(t,b)}(0, m_t, m_b) \right],$$

(50)
where $N_c = 3$. The 1-particle irreducible part of $\hat{\Pi}_{Z_iZ_j}(0)$ can be simplified to

$$\hat{\Pi}_{Z_iZ_j}(0) = -2a_t^2 a_t Z^0_i N_c m_t^2 b_0(0,m_t,m_t) - 2a_b^2 a_b Z^0_i N_c m_b^2 b_0(0,m_b,m_b)$$

Contributions of the other fermion fermions can be written analogously. When inserted into (48) the fermion $f$ contribution to $\hat{\Pi}_{Z_iZ_j}(0)$ factorizes

$$(a_{LL}^{e\nu} - a_{RL}^{e\nu})_{1_{loop}} = -\left(\frac{g_\varepsilon^0 Z^0_i}{M_{Z^0}^2} + \frac{g_\varepsilon^0 Z_i}{M_{Z_i}^2} + \frac{g_\varepsilon^0 Z_i^0}{M_{Z_i^0}^2} \right) 2m_f^2 N_c \ b_0(0,m_f,m_f)$$

and computing the factors in brackets using (20), (21) and the formulae (A.2), (A.3) one finds (omitting $1/16\pi^2$)

$$(a_{LL}^{e\nu} - a_{RL}^{e\nu})_{1_{loop}} = \frac{2}{v_H^4} m_t^2 N_c \ln \frac{\mu^2}{\mu^2} \times \left[ 1 - \frac{v_H^2 (e_l + e_e - e_H)(e_q + e_w + e_H)}{v_S^2 e_S^2} \right] \times \left[ 1 + \frac{v_H^2 (e_l + e_e - e_H)(e_q + e_w + e_H)}{v_S^2 e_S^2} \right]$$

$$+ \frac{2}{v_H^4} m_b^2 N_c \ln \frac{\mu^2}{\mu^2} \times \left[ 1 + \frac{v_H^2 (e_l + e_e - e_H)(e_q + e_w - e_H)}{v_S^2 e_S^2} \right] \times \left[ 1 - \frac{v_H^2 (e_l + e_e - e_H)(e_q + e_w - e_H)}{v_S^2 e_S^2} \right]$$

The first terms in square brackets reproduce the SM contribution. The other terms are simply zero due to the relations (22). Combining this with the top bottom contribution to $\hat{\Pi}_{WW}(0)$ in (49) one finds that the fermionic "oblique" contribution to $\rho_{low}$ is finite and exactly reproduces the one-loop SM result

$$\Delta \rho_{low} = \frac{N_c}{16\pi^2} \sqrt{2} G_F g(m_t,m_b) + \ldots = \frac{N_c}{16\pi^2} \sqrt{2} G_F m_t^2 + \ldots$$

(51)

(the function $g(m_1,m_2)$ is defined in Appendix F). Thus, we explicitly demonstrate that in the $SU(2)_L \times U(1)_Y \times U(1)_E$ model the celebrated $\propto m_t^2$ contribution is present in the $\rho$ parameter defined in terms of low energy neutrino processes.

**Bosonic contribution $\rho_{low}$**

The circumstance simplifying calculation of the the vertex and self energy corrections to external lines to the $\nu_\mu e^- \rightarrow \nu_\mu e^-$ amplitude is that (due to the corresponding $U(1)$ Ward identities) the corrections to the vertices due to virtual $Z^0$ and $Z'$ are exactly canceled by the virtual $Z^0$ and $Z'$ contributions to the self energies. For the corrections due to virtual $W$ one finds

$$16\pi^2 (a_{LL}^{e\nu})_{1_{loop}} = \left[ c_{eL}^{Z_0} \frac{1}{M_{Z_0}^2} e^3 \frac{c}{s^3} + c_{eL}^{Z'} \frac{1}{M_{Z'}^2} \left( -e^3 \frac{c}{s^3} s' \right) \right] \left[ \eta_{\text{div}} + \ln \frac{\hat{M}_W^2}{\mu^2} \right]$$

(52)
The divergences of the $Z_a$ contributions to a tadpole contributions to the vector boson self energies cancel out in the difference the other “oblique” bosonic contributions are finite. It is also easy to check that the corresponding contributions to $\alpha$ and similarly for $\alpha_a$ are such that they can be written in the form

$$\Pi_{Z_i Z_j}(q^2) = \alpha_{Z_i Z_j}^{(k)}(q^2)$$

which when used in the $e\nu \rightarrow e\nu$ amplitude leads to the factorization observed already for the fermionic contribution:

$$a_{LL} = \left( \frac{c_{\nu L} \alpha_{Z_0}^{(k)}}{M_{Z_0}^2} + \frac{c_{\nu L} \alpha_{Z_0}^{(k)}}{M_{Z_0}^2} \right) \left( \frac{c_{\nu L} \alpha_{Z_0}^{(k)}}{M_{Z_0}^2} + \frac{c_{\nu L} \alpha_{Z_0}^{(k)}}{M_{Z_0}^2} \right) \Pi^{(k)}(q^2)$$

and similarly for $a_{RL}$. This allows to easily calculate the divergent part of the corresponding contributions to $a_{LL} - a_{RL}$ (of these only $W^+ W^-$ and $G^\pm W^\mp$ are divergent). Using the tricks (A.2), (A.3) and (22) it is

$$\frac{1}{\hat{v}_H^2} 2e^2 \hat{c}^4 \left[ 1 + \frac{v_H^2}{v_S^2} \frac{e_H(e_H + e_l)}{e_s^2} \right] - \hat{c}^2 \left( 2s^2 - 2e^2 \frac{v_H^2}{v_S^2} \frac{e_H(e_H + e_l)}{e_s^2} \right) \eta_{\text{div}}$$

The divergences of the $Z^0 h^0$ and $Z^0 h^0$ loop contributions to $\Pi_{Z_i Z_j}$ can be combined to yield

$$\left[ \Pi_{Z_i Z_j} \right]_{\text{div}} = \alpha_{Z_i Z_j} \left( \frac{\hat{c}^2}{s^2 c^2} + 4e_H^2 g_E^2 \right) \frac{\hat{v}_L^2}{4} \eta_{\text{div}}$$

with $\alpha_{Z_0} = -\frac{\hat{c}^2}{s^2 c^2} + 2e_H g_E s'$ and $\alpha_{Z^0} = \frac{\hat{c}^2}{s^2 c^2} + 2e_H g_E s'$. The corresponding divergent contributions to $a_{LL} - a_{RL}$ is then

$$- \frac{1}{\hat{v}_H^2} \left( \frac{\hat{c}^2}{s^2 c^2} + 4e_H^2 g_E^2 \right) \eta_{\text{div}}$$

The other “oblique” bosonic contributions are finite. It is also easy to check that the tadpole contributions to the vector boson self energies cancel out in the difference $a_{LL} - a_{RL}$.
Finally we record for completeness the finite contributions of the box diagrams to the coefficients $a_{\nu e}^{\nu e}_{LL}$ and $a_{\nu e}^{\nu e}_{LR}$ of the low energy Lagrangian \((25)\). We find

\[ \begin{align*}
16\pi^2 a_{\nu e}^{\nu e}_{LL} &= \frac{1}{M_{Z_0}^2} 3(c_{\nu L}^0)^2(c_{\nu L}^0)^2 + \frac{1}{M_{Z}^2} \left( \frac{M_{Z}^2}{M_{Z_0}^2} \right) 6 c_{\nu L}^0 c_{\nu L} c_{\nu R}^0 c_{\nu L}^0 + \frac{\hat{e}^4}{\hat{s}^4 M_W^2} \\
16\pi^2 a_{\nu e}^{\nu e}_{LR} &= \frac{1}{M_{Z_0}^2} 3(c_{\nu L}^0)^2(c_{\nu R}^0)^2 - \frac{1}{M_{Z}^2} \left( \frac{M_{Z}^2}{M_{Z_0}^2} \right) 6 c_{\nu L}^0 c_{\nu L} c_{\nu R}^0 c_{\nu R}^0 - \frac{1}{M_{Z}^2 - M_{Z_0}^2} \ln \left( \frac{M_{Z}^2}{M_{Z_0}^2} \right) 6 c_{\nu L}^0 c_{\nu L} c_{\nu R}^0 c_{\nu R}^0.
\end{align*} \] (59)

From these formulae the box contribution to $\rho_{\text{low}}$ can be easily obtained.

Combining the results \((54), (57), (58)\) with the divergent part of $\Delta G$ in \((49)\) given by \((B.5)\) and \((D.7)\) one easily finds that the total one loop contribution to the $\rho_{\text{low}}$ parameter defined in terms of the $\nu e \to \nu e$ scattering amplitude is finite and, since the coefficient of $\ln(1/\mu^2)$ is the same as that of $\eta_{\text{div}}$, independent of the renormalization scale. Moreover, it is easy to see, that in the limit $\upsilon_S \to \infty$ one recovers the SM result i.e. the Appelquist-Carrazzone decoupling is manifest.

If $\sin^2 \theta_{\text{eff}}^\ell$ is used as an additional observable, the explicit decoupling is lost. This is because one has then to express $g_E$ and $\upsilon_S$ in the one-loop contribution through $M_{Z'}$ and $\sin^2 \theta_{\text{eff}}^\ell$ (to zeroth order accuracy) with the effect already described: the explicit suppression factor $\propto 1/\upsilon_S^2$ is then replaced by the difference of $\sin^2 \theta_{\text{eff}}^\ell - s_{\text{(0)}}^2$ which is finite and does not vanish as $\upsilon_S \to \infty$.

6 \hspace{1cm} One loop calculation of $M_{Z^0}^2$

In this section we compute in our scheme $M_{Z^0}^2$. Unlike the previous example of $\rho_{\text{low}}$, the tree level formula for $M_{Z^0}^2$ does depend on the parameters of the extended gauge sector. Therefore, in the one loop result for $M_{Z^0}^2$ in our scheme explicit dependence on the renormalization scale $\mu$ will remain. We will however show that the conditions for the heavy $Z'$ effects to decouple are satisfied: the part of the result which does not vanish as $\upsilon_S \to \infty$ is independent of $\mu$ and takes the SM form. Furthermore, we will show that the whole result for $M_{Z^0}^2$ is independent of the renormalization scale if the dependence on $\mu$ of the parameters in the zeroth-order expression is taken into account. This constitutes a nontrivial check of the renormalization group equations \((13)-(15)\) and of our renormalization scheme.

We calculate now the one loop corrections to the $Z^0$ boson mass. It is given by the formula\(^7\)

\[ M_{Z^0}^2 = \hat{M}_{Z^0}^2 + \Pi_{Z^0 Z^0}(M_{Z^0}^2) \]

\(^7\)Mixing of $Z^0$ with $Z'$ is formally a two-loop effect.
where the tree level term $\tilde{M}_{Z^0}^2$ is given by (19). The running parameters $\hat{e}$, $\hat{s}$, $\hat{c}$, $\hat{\upsilon}_H$ in $\tilde{M}_{Z^0}^2$ have to be expressed in terms of the input observables $G_F$, $M_W^2$ and $\alpha_{EM}$ with one loop accuracy by using the relations (145), (146). This gives

$$A_0 + \delta A = \frac{M_W^2}{c_0^2} \left\{ 1 - \tilde{\Pi}_{WW}(M_W^2) + \frac{s_0^2}{c_0^2} \Delta \right\}$$

$$B_0 + \delta B = g_E^2 \hat{e}^2 \hat{s}^2 + \frac{g_E^2 \hat{e}^2}{\sqrt{2} G_F} (1 + \Delta_G)$$

$$D_0 + \delta D = -\frac{1}{2} e_H g_E \frac{e_0}{s_0 c_0 \sqrt{2} G_F} \left\{ 1 + \frac{s_0^2}{c_0^2} \Delta - \frac{1}{2} \tilde{\Pi}_{WW}(M_W^2) + \frac{1}{2} \Delta_G \right\}$$

where $e_0 = \sqrt{4\pi \alpha_{EM}}$ and $\Delta$ and $\Delta_G$ are given by (47) and (B.4) respectively. In agreement with the prescription (44) we then have $2M_Z^2 = 2(M_{Z^0}^2)_{(0)} + 2\delta M_{Z^0}^2$ where $(M_{Z^0}^2)_{(0)}$ is given by (19) with $A$, $B$, $D$ replaced by $A_0$, $B_0$, $D_0$, respectively and

$$2\delta M_{Z^0}^2 = \delta A + \delta B - \frac{(A_0 - B_0)(\delta A - \delta B) + 4 D_0 \delta D}{\sqrt{(A_0 - B_0)^2 + 4 D_0^2}} + 2 \hat{\Pi}_{Z^0 Z^0}(M_Z^2)$$

$$= \frac{M_W^2}{c_0^2} \left[ -\tilde{\Pi}_{WW}(M_W^2) + \frac{s_0^2}{c_0^2} \right] + 2 \hat{\Pi}_{Z^0 Z^0}(M_Z^2)$$

$$+ \frac{g_E^2 \hat{e}^2 \hat{s}^2 + \frac{g_E^2 \hat{e}^2}{\sqrt{2} G_F} - \frac{M_W^2}{c_0^2}}{\sqrt{\cdots}} \left\{ M_W^2 \left[ -\tilde{\Pi}_{WW}(M_W^2) + \frac{s_0^2}{c_0^2} \right] - \frac{g_E^2 \hat{e}^2}{\sqrt{2} G_F} \Delta_G \right\}$$

$$+ \frac{g_E^2 \hat{e}^2 \Delta_G - \frac{g_E^2 \hat{e}^2}{\sqrt{2} G_F} - \frac{M_W^2}{c_0^2}}{\sqrt{\cdots}} \left[ \frac{1 - \frac{s_0^2}{c_0^2}}{2} \Delta - \frac{1}{2} \tilde{\Pi}_{WW}(M_W^2) + \frac{1}{2} \Delta_G \right]$$

where the self energies $\hat{\Pi}_{WW}$ and $\hat{\Pi}_{Z^0 Z^0}$ include the tadpole contributions. We would like now to demonstrate that $i)$ in the limit $v_S \to \infty$ the SM result is recovered, and $ii)$ that the above result is independent of the renormalization scale $\mu$.

### 6.1 SM limit - decoupling of the heavy $Z'$ effects

For $v_S \to \infty$ the tree level term $(M_{Z^0}^2)_{(0)}$ obviously gives the SM result $M_W^2/c_0^2$. Moreover, the prefactor in the third line of (61) is then $1 + O(1/v_S^4)$ and the prefactor of the last term is also suppressed by $1/v_S^4$. Thus in the limit one recovers superficially the SM formula.

$$2\delta M_{Z^0}^2 \to 2 \frac{M_W^2}{c_0^2} \left[ -\tilde{\Pi}_{WW}(M_W^2) + \frac{s_0^2}{c_0^2} \right] + 2 \hat{\Pi}_{Z^0 Z^0}(M_Z^2).$$

However, one still has to check that the appropriate combinations of $\hat{\Pi}_{WW}$, $\hat{\Pi}_{Z^0 Z^0}$, $\Delta$ do not contain terms which would grow too fast as $v_S \to \infty$ invalidating the argument.
In order to show that they do not, we first note that the the $S^0$ tadpole $T_{S^0}$ which contributes only to $\hat{T}_{Z^0Z^0}$ is suppressed (as we show below, the $h^0$ tadpoles cancel out exactly in the full formula (61) similarly as in the SM). Indeed, the $S^0$ coupling to $Z^0Z^0$ is proportional to $s^2v_S \sim 1/v_S^2$; the $S^0$ propagator is $\sim 1/v_S^2$; the $S^0$ coupling to $Z'Z'$ and $S^0S^0$ pairs is proportional to $v_S$ so that these particles circulating in the tadpole loop give to $T_{S^0}$ contributions $\sim v_S^3$. Hence, the $S^0$ tadpole contribution to $\hat{T}_{Z^0Z^0}$ goes as $\sim (1/v_S^3)(1/v_S^3)(v_S^3) \sim 1/v_S^3$.

Furthermore, $\Delta$ approaches in this limit its SM form due to cancellation of the leading for $v_S \to \infty$ terms between $\Lambda$ and $\Sigma_{\nu L} + \Sigma_{e L}$ and between $\hat{T}_{WW}(M_W^2)$ and $\hat{T}_{WW}(0)$. Moreover, $\Delta_G + \hat{T}_{WW}(M_W^2)/M_W^2$ grows only as $\ln(v_S^2)$, so the contribution of the last bracket in (61) vanishes for $v_S \to \infty$. Thus, in this limit one indeed gets (62) and it remains to check that the difference of the $Z^0$ and $W^\pm$ self-energies approaches the SM form.

For the fermionic contribution to (62) this is clear: for $\hat{T}_{WW}(M_W^2)$ it is exactly as in the SM and that to $\hat{T}_{Z^0Z^0}(M_Z^2)$ is different, but the difference is only due to $Z^0$ couplings which, as is follows from (20) and (A.1) approach as $v_S \to \infty$ their SM form. In particular this means that in the $SU(2)_L \times U(1)_Y \times U(1)_E$ model the celebrated contribution $\propto m_t^2/M_W^2$ is present in the $M_W^2 \leftrightarrow M_Z^2$ relation.

Bosonic contributions to $\hat{T}_{WW}(M_W^2)$ and $\hat{T}_{Z^0Z^0}(M_Z^2)$ individually contain terms which grow as $v_S \to \infty$ (the last term in the third line of (2.6) and the $Z'h^0$ contribution to $\hat{T}_{Z^0Z^0}$) but it is easy to check that they cancel out in (62) and the difference $\hat{T}_{WW}(M_W^2)/M_W^2 - \hat{T}_{Z^0Z^0}(M_Z^2)/M_Z^2$ approaches its SM form too.

Thus, we have demonstrated that in the limit $v_S \to \infty$ the finite SM expression for $M_{Z^0}$ is recovered.

### 6.2 Renormalization scale $\mu$ independence of $M_{Z^0}$ at one-loop

#### $h^0$ tadpoles cancelation

As a first step we show that the $h^0$ tadpoles $T_{h^0}$ drop out of the formula (61). The contribution of $T_{h^0}$ to $2\hat{T}_{Z^0Z^0}$ is

$$2\hat{T}_{Z^0Z^0}^{h^0\text{ tad}} = 2 \left[ \hat{e}^2 \frac{c^2}{4s^2\hat{c}^2} - \hat{e} \hat{c} H g_E e c' s' - \left( \frac{\hat{e}^2}{4s^2\hat{c}^2} - \frac{e_H g_E^2}{s^2 \hat{c}^2} \right) s'^2 \right] \left( 2\hat{v}_H \frac{T_{h^0}}{M_{h^0}^2} \right). \tag{63}$$

With one loop accuracy and using the formulae (A.1) this can be rewritten as

$$\left[ \frac{M_{h^0}^2}{c_0^2} - \left( \frac{M_W^2}{c_0^2} - \frac{g_E^2 c_H^2}{\sqrt{2} G_F} \right) \frac{A_0 - B_0}{\sqrt{\cdots}} + \frac{g_H^2 c_H^2}{\sqrt{2} G_F} - \frac{e_{(0)}^2 - e_{(0)}^2}{s_0^2 c_0^2 \sqrt{2} G_F} \right] \left( 2 \frac{T_{h^0}}{\hat{v}^2_H M_{h^0}^2} \right). \tag{64}$$

It is then clear that each term finds in (61) its counterpart with $-\hat{T}_{WW}^{h^0 \text{ tad}}/M_W^2 = (2/\hat{v}_H)(T_{h^0}/M_{h^0}^2)$ and exactly the same coefficient.

**Contribution proportional to fermion masses squared**

20
Next we consider contributions to $M_{Z^0}^2$ proportional to the fermion masses squared. These are hidden in $\hat{\Pi}_{Z^0Z^0}$, $\hat{\Pi}_{WW}(M_W^2)$ and in $\hat{\Pi}_{WW}(0)$. As usual, they can be isolated by approximating the first two self energies by $\hat{\Pi}_{Z^0Z^0}(0)$ and $\hat{\Pi}_{WW}(0)$, respectively. From the the formula (D.8) we get

$$\Pi_{Z^0Z^0}^{\text{ferm}}(q^2) = -2 \sum_f N_c^{(f)} (c_{fL}^{Z^0} - c_{fR}^{Z^0})^2 m_f^2 b_0(0, m_f, m_f)$$  \hspace{1cm} (65)

Using the couplings (20) and the relations (22), (A.1) we can write

$$(c_{fL}^{Z^0} - c_{fR}^{Z^0})^2 = \frac{1}{4\hat{s}^2\hat{c}^2} + \left( \frac{\hat{e}^2}{4\hat{s}^2\hat{c}^2} - \hat{g}_E^2\hat{c}_H^2 \right) \frac{B-A}{\sqrt{\cdots}} + \hat{c}_H^2\hat{g}_E^2 - \frac{\hat{e}^2}{\sqrt{\cdots}} \frac{g_E^2\hat{c}_H^2}{\hat{c}_H^2}$$  \hspace{1cm} (66)

This makes clear that to each term in $2\left[\hat{\Pi}_{Z^0Z^0}^{\text{ferm}}\right]_{\text{mass}}$ there is a corresponding term with $\hat{\Pi}_{WW}$ in the formula (61), so that the divergences proportional to fermion masses squared properly cancel out. Hence, the terms quadratic in fermion masses arising from “oblique” corrections are finite (and, hence, $\mu$-independent) just as they are in the SM. For the one-loop top-bottom contribution using (50) we get

$$M_{Z^0}^2 = \frac{1}{2} \left( A_0 + B_0 - \sqrt{(A_0 - B_0)^2 + 4D_0^2} \right)$$

$$- \left( c_{fL}^{Z^0} - c_{fR}^{Z^0} \right)^2 \frac{N_c}{16\pi^2} g(m_t, m_b) + \text{other contributions}$$  \hspace{1cm} (67)

And in the limit $v_S \to \infty$ one recovers the SM relation (computed using as input observables $M_W$, $G_F$ and $\alpha_{EM}$).

**Remaining fermion contribution - the use of RG equations**

The remaining divergent fermionic contribution (D.8) to $\Pi_{Z^0Z^0}$ is proportional to $q^2$

$$2 \left[\Pi_{Z^0Z^0}^{\text{ferm}}(q^2)\right]_{\text{part}}^2 = \frac{4}{3} q^2 \sum_f N_c^{(f)} [(c_{fL}^{Z^0})^2 + (c_{fR}^{Z^0})^2] \eta_{\text{div}}$$

Using the couplings (20) and the relations (22), (A.1) the right hand side takes the form

$$\frac{2}{3} M_{Z^0}^2 \left\{ \left( 1 - \frac{A-B}{\sqrt{\cdots}} \right) \frac{\hat{e}^2}{4\hat{s}^2\hat{c}^2} 2 - 4\hat{s}^2 + 8\hat{s}^4 + N_c(2 - 4\hat{s}^2 + \frac{40}{9}\hat{s}^4) \right\}$$

$$+ \frac{\hat{e}^2}{\hat{c}^2} \frac{g_E^2\hat{c}_H^2}{\sqrt{\cdots}} \left[ 2\hat{e}_l - 2\hat{e}_s + N_c(-\frac{2}{3}\hat{e}_q + \frac{4}{3}\hat{e}_u - \frac{2}{3}\hat{e}_d) \right]$$

$$+ \left( 1 + \frac{A-B}{\sqrt{\cdots}} \right) g_E^2 \left[ 2\hat{c}_l^2 + \hat{e}_q^2 + N_c(2\hat{c}_q^2 + \hat{e}_u^2 + \hat{e}_d^2) \right] \eta_{\text{div}}$$  \hspace{1cm} (68)

With one loop accuracy the prefactor of the first line can be transformed into

$$M_{Z^0}^2 \left( 1 - \frac{A-B}{\sqrt{\cdots}} \right) = M_W^2 \frac{c_0}{c_0} + \frac{B_0 - A_0 M_W^2}{\sqrt{\cdots}} - \frac{1}{2} \frac{\hat{g}_E^2}{\hat{c}_H} \frac{e_0^2}{2\sqrt{\cdots}} \frac{1}{2G_F^2}$$
after which different terms arising from the first line of (68) combine with the appropriate fermionic contributions to

\[
- \frac{M_W^2}{c_W^2} \left[ - \frac{\Pi_{WW}(M_W^2)\eta^{\text{part}}}{M_W^2} + \frac{s_W^2 c_W^2}{c_W^2} \right]_{\text{div}}
\]

in (61) canceling their divergences and the \( \mu \) dependence exactly as in the SM.

In our renormalization scheme (outlined in Section 4) the two other divergent terms in (68) are cut off by the \( \overline{\text{MS}} \) procedure. In order to see that \( M_{Z^0}^2 \) computed at one loop is nevertheless renormalizations scale \( \mu \) independent we have to consider the dependence on \( \mu \) of \( 2(M_{Z^0}^2)_{(0)} \)

\[
(2M_{Z^0}^2)_{(0)} = A_0 + B_0(\mu) - \sqrt{[A_0 - B_0(\mu)]^2 + 4D_0^2(\mu)} \quad (69)
\]

The superscripts 0 on \( A, B \) and \( D \) mean that the parameters \( \tilde{e}^2, \tilde{s}^2, \tilde{c}^2, \tilde{v}_H \) have been expressed in terms of the basic observables \( \alpha_{\text{EM}}, M_W \) and \( G_F \) to zeroth order accuracy. The \( \mu \) dependence is due to the parameters \( e_H g_E, e_E g_E, v_s \) which are still the running parameters of the full theory. Using the renormalization group equations (15) and (C.5) for an infinitesimal change of scale \( \mu \) we have:

\[
B_0(\mu) = B_0(\mu') + \delta B_1 + \delta B_2 + \delta B_v
\]

\[
4D_0^2(\mu) = 4D_0^2(\mu') + 4\delta D_1^2 + 4\delta D_2^2
\]

where

\[
\delta B_1 = \frac{1}{\sqrt{2}G_F} 2 \left( \sum_{f} e_H g_E e_f g_E Y_H Y_f + \frac{1}{3} \frac{e_H g_E Y_H Y_f}{2} \right) g_y \ln \frac{\mu^2}{\mu'^2}
\]

\[
\delta B_2 = \left( \frac{1}{\sqrt{2}G_F} e_H g_E + \frac{s_H g_E}{2} \right) \left( \frac{2}{3} \sum_{f} e_H g_E Y_H Y_f + \frac{1}{3} e_H g_E Y_H Y_f \right) \ln \frac{\mu^2}{\mu'^2}
\]

\[
\delta B_v = \frac{\tilde{e}_s g_E}{s_2} \left( -\frac{3}{2} \frac{\tilde{g}_E}{s_2} + \frac{3\tilde{e}_s g_E v_S^2 - 12 \frac{\tilde{g}_E}{s_2} v_S^2}{\frac{\tilde{g}_E}{s_2}} \right) \ln \frac{\mu^2}{\mu'^2}
\]

\[
4\delta D_1^2 = \frac{e_H^2}{2 G_F^2} \left( \frac{2}{3} \sum_{f} e_H g_E e_f g_E Y_H Y_f + \frac{1}{3} e_H g_E Y_H Y_f \right) g_y \ln \frac{\mu^2}{\mu'^2}
\]

\[
4\delta D_2^2 = \frac{e_H^2}{2 G_F^2} \left( \frac{2}{3} \sum_{f} e_H g_E e_f g_E Y_H Y_f + \frac{1}{3} e_H g_E Y_H Y_f \right) \ln \frac{\mu^2}{\mu'^2}
\]

The formula (69) then takes the form

\[
(2M_{Z^0}^2)_{(0)} \approx A_0 + B_0(\mu') - \sqrt{[A_0 - B_0(\mu')]^2 + 4D_0^2(\mu')} + (\delta B_1 + \delta B_2 + \delta B_v) \left( 1 + \frac{A_0 - B_0}{\sqrt{\ldots}} \right) - \frac{1}{2\sqrt{\ldots}} (4\delta D_1^2 + 4\delta D_2^2)
\]

(71)
It is then a matter of some simple algebra to check that the fermion generation number dependent terms in \((71)\) precisely match the \(\ln(1/\mu^2)\) proportional terms associated with the two last lines of \((68)\) changing in these terms \(\mu\) into \(\mu'\). Hence, up to one loop accuracy the entire fermionic contribution to \(M_{Z^0}^2\) is renormalization scale independent.

**Renormalizations scale independence of the bosonic contribution to \(M_{Z^0}^2\)**

The scale independence of the remaining one-loop contribution can be checked in a similar way (using judiciously the relations collected in \text{Appendix A}): part of the divergences with the associated \(\mu\) dependence cancels out explicitly in the formula \((61)\) as a result of expressing \(\dot{c}, \dot{s}, \dot{c}', \dot{v}_H\) in terms of the basic observables \(\alpha_{EM}, M_W\) and \(G_F\) with one loop accuracy. Other divergences are cut-off by the prescription and the explicit renormalization scale dependence is compensated by the change with \(\mu\) dictated by the RG of the parameters \(e_k, g_E, v_S\) in the zeroth order term \((M_{Z^0}^2)_{(0)}\) \((63)\). Here we only would like to show that the \(S^0\) tadpole contribution to \(2\hat{\Pi}_{Z^0Z^0}^{tad}\) plays a crucial role in the working of the scheme \([17]\).

The couplings of \(S^0\) to \(S^0S^0\) and to \(G'G', G^0G^0\) can be easily computed.\(^8\) For the \(S^0\) tadpole we then get

\[
T_{S^0} = \frac{3}{4} \lambda_S v_S a(M_{S^0}) + \frac{1}{4} \lambda_S v_S c^2 a(M_{G'}) + \frac{1}{4} \lambda_S v_S s^2 a(M_{G'}) + 3g_E^2 c^2 v_S \left[ c^2 M_{Z^0}^2 \left( \ln \frac{M_{Z^0}^2}{\mu^2} - \frac{1}{3} \right) + s^2 M_{Z^0}^2 \left( \ln \frac{M_{Z^0}^2}{\mu^2} - \frac{1}{3} \right) \right]
\]

where \(c'\) and \(s'\) are the mixing angles of \(G^0\) and \(G'\). \(c'\) and \(s'\) are different than \(c\) and \(s\) but still one has the usual relations \(M_{G^0}^2 = \xi M_{Z^0}^2\) and \(M_{G'}^2 = \xi M_{Z^0}^2\). The \(S^0\) mass is \(M_{S^0}^2 = \frac{1}{2} \lambda_S v_S^2\). As usually we work in the Feynman gauge \(\xi = 1\).

The \(S^0\) tadpole gives

\[
2\hat{\Pi}_{Z^0Z^0}^{tad} = 2 \cdot 2g_E^2 c^2 v_S s^2 \left( -\frac{T_{S^0}}{M_{S^0}^2} \right) = -4g_E^2 c^2 v_S \left( 1 + \frac{A - B}{\sqrt{\cdots}} \right) \frac{1}{\lambda_S v_S^2} \ln \frac{1}{\mu^2} + \ldots
\]

where we have used the relations \(s^2 M_{Z^0}^2 + c^2 M_{Z^0}^2 = g_E^2 c^2 v_S^2 + g_E^2 s^2 v_H^2\) and \(s^2 M_{Z^0}^2 + c^2 M_{Z^0}^2 = c^2 g_E^2 v_S^2\).

From \((2M_{Z^0}^2)_{\text{tree}}\) \((69)\), we have instead:

\[
(2M_{Z^0}^2)_{\text{tree}} \supset \left( 1 + \frac{A_0 - B_0}{\sqrt{\cdots}} \right) \delta B_v .
\]

\(^8\)As explained in \text{Appendix C} in order to simplify the formulae we assume that at the scale we are working the scalar potential is the sum \(V = V_H(H) + V_S(S)\). The physical Higgs scalars \(S^0\) and \(h^0\) are then pure real parts of the singlet \(S\) and of the neutral component of the doublet \(H\). The \(S^0\) does not couple then to \(h^0h^0\).
This explicitly shows that in the $S^0$ tadpole contribution the scale $\mu$ is properly replaced by $\mu'$ in the terms $\propto \lambda_S$ and $\propto (1/\lambda_S)$ (as we have checked, the $\lambda_S$ independent terms in $T_{S^0}$ combine with the bosonic contribution $\hat{\Pi}_{Z^0Z^0}$).

We have shown, that in the one loop expression for $M_{Z^0}^2$, consistently with the Appelquist-Carrazzone decoupling the explicit renormalization scale dependence is only in terms suppressed by inverse powers of $\nu_S$. Moreover, the whole expression is in fact renormalization scale independent, it one takes into account the $\mu$ dependence of the RG running of the parameters in the tree level term.

7 On-shell $Z^0$ couplings to fermions

In this section we briefly consider the parameter $\rho$ defined in terms of physical $Z^0$ and $W^\pm$ masses and the Weinberg angle:

$$\rho = \frac{M_W^2}{M_{Z^0}^2(1 - \sin^2\theta_{\text{eff}}^\ell)}.$$  \hspace{1cm} (72)

where $\sin^2\theta_{\text{eff}}^\ell$ is defined by the form (23) of the effective coupling of on-shell $Z^0$ to fermions (we take leptons for definitness)

$$\mathcal{L}^{Z^0\text{eff \ on-shell}} = \bar{\psi}_L \gamma^\mu (F_L P_L + F_R P_R) \psi_R Z^0_\mu.$$  \hspace{1cm} (73)

Comparison of (73) with (24) gives $\sin^2\theta_{\text{eff}}^\ell = F_R/2(F_R - F_L)$. For the formfactors $F_{L,R}$ we have the formulae

$$F_{L,R} = -c_{\ell,L,R}^Z - \frac{1}{2} \hat{\Pi}_{Z^0Z^0}(M_{Z^0}^2) c_{\ell,L,R}^Z + \frac{\hat{\Pi}_{Z^0Z'}(M_{Z^0}^2)}{M_{Z^0}^2 - M_{Z'}^2} c_{\ell,L,R}^{Z'} + \ldots.$$  \hspace{1cm} (74)

Since we are interested only in the dominant universal top bottom contribution, we have not written down neither the genuine vertex corrections nor the final fermion self energies.

Expressing the running coupling constants in $c_{\ell,L,R}^Z$ in terms of $M_W^2$, $G_F$ and $\alpha_{\text{EM}}$ with one loop accuracy we find

$$c_{\ell,R}^Z = e_0 \frac{s_0}{c_0} \left\{ 1 - \frac{1}{2} \hat{\Pi}_\gamma(0) - \frac{\alpha_{\text{EM}}}{2\pi} \ln \frac{M_W^2}{\mu^2} + \frac{1}{2c_0(0)} \Delta \right\} c'(0)$$
$$- e_\ell e_E s'_0 + e_0 \frac{s_0}{c_0} \delta c'$$

$$c_{\ell,L}^Z = -\frac{e_0}{2s_0c_0} \left\{ 1 - \frac{1}{2} \hat{\Pi}_\gamma(0) - \frac{\alpha_{\text{EM}}}{2\pi} \ln \frac{M_W^2}{\mu^2} + \frac{s_0(0) - c_0(0)}{2c_0(0)} \Delta \right\} c'(0)$$
$$+ e_0 \frac{s_0}{c_0} \left\{ 1 - \frac{1}{2} \hat{\Pi}_\gamma(0) - \frac{\alpha_{\text{EM}}}{2\pi} \ln \frac{M_W^2}{\mu^2} + \frac{1}{2c_0(0)} \Delta \right\} c'(0)$$
$$+ e_\ell e_E s'_0 - \frac{e_0}{2s_0c_0} (1 - 2s_0(0)) \delta c' + e_\ell e_E \delta s',$$  \hspace{1cm} (75)

24
where $\Delta$ is given in [17]. We have also introduced $\delta c'$ and $\delta s'$ because original $c'$ and $s'$ depend on $\hat{e}$, $\hat{s}$, $\hat{c}$ and $\hat{v}\hat{H}$. The quantities $c'_0(0)$ and $s'_0(0)$ are then given by the same expressions as $c'$ and $s'$ but with $\hat{e}$, $\hat{s}$, $\hat{c}$ and $\hat{v}\hat{H}$ replaced by $e_0(0)$, $s_0(0)$, $c_0(0)$ and $1/\sqrt{2}G_F$, respectively.

In our renormalization scheme the formfactors $F_{L,R}$ given by (74) and (75) are finite if the $\overline{\text{MS}}$ scheme is employed. Moreover their nonvanishing as $v_S \to \infty$ parts are renormalization scale independent (i.e. they are just finite) and the explicit $\mu$ dependence of the one-loop terms is compensated by the change of the running parameters $e_H g_E$, $\ell g_E$, $e_{\ell} g_E$ and $v_S$ entering the zeroth order contributions.

For $\delta c'$ and $\delta s'$ we find

$$
\delta s' = - \frac{c'_0(0)}{4 s'_0(0)} \delta c' = \frac{1}{4 s'_0(0)} \frac{1}{(\sqrt{\cdots})^3} \left[ 4D_0^2(\delta A - \delta B) - (A_0 - B_0)4D_0\delta D \right]
$$

$$
= \frac{c'_0(0)}{(\sqrt{\cdots})^2} \frac{e_0(0)}{2 s'_0(0)} \frac{e_H e_S^2 g_E^2 v_S^2}{\sqrt{2}G_F} \left[ - \hat{\Pi}_{WW}(0) \right] + \ldots \quad (76)
$$

where in the second line, in order to isolate the dominant top-bottom contributions to the formfactors $F_L$ and $F_R$, we have isolated only the term with $\hat{\Pi}_{WW}(0)$. Combining this with

$$
\hat{\Pi}_{Z^0}(M_{Z^0}^2) \approx \hat{\Pi}_{Z^0}(0) = - \sum_f (c_{fL}^{\nu} - c_{fR}^{\nu})(c_{fL}^{\nu'} - c_{fR}^{\nu'}) \bar{2}N_c(f) m_f^2 b_0(0, m_f, m_f)
$$

$$
= - \frac{1}{(\sqrt{\cdots})} \hat{\Pi}_{Z^0}(0) \approx - \bar{2}N_c(f) m_f^2 b_0(0, m_f, m_f) \quad (77)
$$

(where again we have used the results (20), (21) and (A.2)) and using the fact that $M_{Z^0}^2 - M_{Z^0}^2 = -\sqrt{\cdots}$ we find

$$
F_{L,R}^{t,b} \approx - \frac{1}{(\sqrt{\cdots})^2} \frac{\hat{\Pi}_{WW}(0)}{2 \hat{s}_c} \frac{e_H e_S^2 g_E^2 v_S^2}{\sqrt{2}G_F} \bar{2}N_c(f) \left( m_t, m_b \right) \quad (78)
$$

Since $(\sqrt{\cdots})^2 \equiv (A_0 - B_0)^2 + 4D_0^2 \sim v_S^4$ as $v_S \to \infty$, this contribution is explicitly suppressed in this limit. It is easy to see that the expressions for $F_L$ and $F_R$ (74), (75) do not involve any other contributions proportional to $m_t^2$ and $m_b^2$ and, therefore, no contribution $\propto m_t^2/M_W^2$ enter $\sin^2\theta_{\text{eff}}^\ell$ at one loop.\footnote{In the SM the formfactors $F_L$ and $F_R$ do not receive any such contribution if the scheme based on $M_W$, $G_F$ and $\alpha_{\text{EM}}$ as input observables is employed.} Since we have already shown that for $v_S \to \infty$ one recovers also the SM expression for $M_{Z^0}$, we conclude, that in the $U(1)_Y \times U(1)_E$ model

$$
\rho = \frac{M_W^2}{M_{Z^0}^2(1 - \sin^2\theta_{\text{eff}}^\ell)} = 1 + \frac{N_c}{16\pi^2} 2G_F g(m_t, m_b) + O(m_t^2/v_S^2) + \ldots \quad (79)
$$

where dots stand for other SM contribution as well as for other terms suppressed in the limit $v_S \to \infty$ (also those arising from the tree level contribution contributon to...
\( \rho \) (see eq. (33)). Similar result can be proven also for \( \rho_{Zf} \) defined by the effective Lagrangian (24).

It should be stressed that unlike \( \rho_{\text{low}} \) to which one loop corrections have been computed in section 5, the parameter \( \rho \) defined in (72) is not equal to unity at the tree level. Therefore the one loop result for \( \rho \) does depend on the renormalization scheme and in particular on the chosen set of input observables. This observation is helpful in understanding the apparent discrepancy of our results with the claim of refs. [5–7] that in models like the one considered here the contribution to \( \rho \) proportional to \( m_t^2/M_W^2 \) is absent. Refs. [5–7] use \( \sin^2 \theta_{\text{eff}} \) as one of the input observables and then, as we have, commented earlier, explicit Appelquist-Carrazzone decoupling is lost. However, our point is that the renormalization scheme can be chosen in such a way that new physics effects can be treated as corrections to the well established SM results.

8 Conclusions

In this paper we have discussed some technical aspects related to the \( U(1)_E \) extension of the standard electroweak theory. We have elucidated the correct treatment of the additional coupling constants and presented the one loop renormalization group equations in the form adapted to practical calculations. Furthermore we have proposed a renormalization scheme employing as in the SM only three input observables (for technical convenience we have chosen to work with \( M_W, G_F \) and \( \alpha_{\text{EM}} \) instead of the customary set \( M_Z, G_F \) and \( \alpha_{\text{EM}} \)) which has the virtue of making the decoupling of heavy \( Z' \) effects manifest. To demonstrate this we have computed the parameter \( \rho \) defined either in terms of the low energy neutrino scattering processes or in terms of physical \( M_W^2, M_Z^2 \) and \( \sin^2 \theta_{\text{eff}} \) as measured in \( Z^0 \to \ell^+\ell^- \). In addition, in both cases we have shown explicitly in a renormalization scheme in which the Appelquist-Carrazzone decoupling is manifest the \( \propto G_F m_t^2 \) contribution to the \( \rho \) parameters is present and up to terms vanishing as \( M_{Z'} \to \infty \) take the form as in the SM. Our calculation supports therefore similar observation made in [9] long time ago.

Our choice of \( M_W, G_F \) and \( \alpha_{\text{EM}} \) as input observables instead of the commonly used set \( M_Z, G_F \) and \( \alpha_{\text{EM}} \) was dictated by the desire of demonstrating crucial aspects of our renormalization scheme (in particular the role of the renormalization group equations in proving scale independence of computed observables) analytically. We have checked however, that the explicit decoupling of heavy \( Z' \) effects (that the expressions for the electroweak observables approach their SM form for \( v_S \propto M_{Z'} \to \infty \), do not depend on whether one uses \( M_W \) or \( M_Z \).

The Appelquist-Carrazzone decoupling offers a possibility of a systematic inclusion of all large logarithmic \( \sim \ln(M_{Z'}/M_Z^0) \) corrections by taking into account the RG running of the Wilson coefficients of nonrenormalizable operators generated by decoupling of the heavy \( Z' \) sector.
Acknowledgments

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Appendix A Useful formulae

The mass matrix of $Z^0$ and $Z'$ which arises as a $2 \times 2$ submatrix after rotating (16) by the angle $\theta_W$ reads

$$
\begin{pmatrix}
\frac{1}{2}(g^2_y + g^2_2)v_H^2 & -\frac{1}{2}\sqrt{g^2_y + g^2_2g_Ee_Hv_H^2} \\
-\frac{1}{2}\sqrt{g^2_y + g^2_2g_Ee_Hv_H^2} & e^2_Hg^2_Ev_H^2 + e^2_Sg^2_Ev_S^2
\end{pmatrix} = \begin{pmatrix} A & D \\ D & B \end{pmatrix}
$$

It is diagonalized by the rotation by the angle $\theta'$ determined from (18). For $s'^2$, $c'^2$ and $s'c'$ one derives the following useful expressions

$$
s'^2 = \frac{1}{2} \left( 1 + \frac{A - B}{\sqrt{(A - B)^2 + 4D^2}} \right) = \frac{1}{4} \frac{g^2_y + g^2_2 e^4_Hv^4_H}{g^2_Ee_Sv_S^4} + \ldots ,
$$

$$
c'^2 = \frac{1}{2} \left( 1 - \frac{A - B}{\sqrt{(A - B)^2 + 4D^2}} \right) = 1 - \frac{1}{8} \frac{g^2_y + g^2_2 e^4_Hv^4_H}{g^2_Ee_s^4v_S^4} + \ldots , \quad (A.1)
$$

$$
s'c' = \frac{-D}{\sqrt{(A - B)^2 + 4D^2}} = \frac{1}{2} \frac{g^2_y + g^2_2 e^2_Hv^2_H}{g^2_Ee_Sv_S^2} + \ldots
$$

Other useful expressions are

$$
\frac{s'^2}{M^2_{Z^0}} + \frac{c'^2}{M^2_{Z'}} = \frac{1}{M^2_{Z^0}M^2_{Z'}} \left( \frac{e^2}{4s^2c^2}e^4_Hv^4_H \right)
$$

$$
\frac{c'^2}{M^2_{Z^0}} + \frac{s'^2}{M^2_{Z'}} = \frac{1}{M^2_{Z^0}M^2_{Z'}} \left( g^2_Ee^2_Hv^2_H + g^2_2e^2_Sv^2_S \right) \quad (A.2)
$$

$$
s'c' \left( \frac{1}{M^2_{Z^0}} - \frac{1}{M^2_{Z'}} \right) = \frac{1}{M^2_{Z^0}M^2_{Z'}} \left( \frac{e}{2s}e^2_Hv^2_H \right)
$$

and

$$
M^2_{Z^0}M^2_{Z'} = \frac{e^2}{4s^2c^2}g^2_Ee^2_Sv^2_Hv^2_S \quad (A.3)
$$

Other useful relations are

$$
c'^2M^2_{Z^0} = Ac'^2 + Ds'c'
$$

$$
s'^2M^2_{Z^0} = Bs'^2 + Ds'c'
$$

$$
c'^2M^2_{Z'} = Bc'^2 - Ds'c'
$$

$$
s'^2M^2_{Z'} = As'^2 - Ds'c' \quad (A.4)
$$
Appendix B Calculation of the input observables $\alpha_{\text{EM}}, \ G_F$ and $M_W$

Here we outline the calculation in the $SU(2)_L \times U(1)_Y \times U(1)_E$ model of the basic input observables $\alpha_{\text{EM}}, \ G_F$ and $M_W$. The formula for $M_W$ is simple

\[
M_W^2 = \frac{\hat{e}^2}{4s^2}\hat{v}_H^2 + \hat{\Pi}_{WW}(M_W^2) \tag{B.1}
\]

where $\hat{\Pi}_{WW}(M_W^2)$ includes in principle also the tadpole contribution. Expressions for $\alpha_{\text{EM}}$ and $G_F$ are derived below.

Appendix B.1 Calculation of $\delta\alpha_{\text{EM}}$

This is most easily computed using the effective Lagrangian technique [16]. Below the electroweak scale (the renormalizable part of) the effective Lagrangian for electromagnetic interactions has the form

\[
\mathcal{L} = -\frac{1}{4}(1 + \delta z_L) f_{\mu\nu} f^{\mu\nu} \\
+ (1 + \delta z_L^L) \bar{\psi}_e i \not\partial \psi_e - \left( e + \delta e + \hat{e} \delta z_2^L + \frac{1}{2} \hat{e} \delta z_2 \right) \bar{\psi}_e q_e \not A_L \psi_e \tag{B.2} \\
+ (1 + \delta z_R^R) \bar{\psi}_e i \not\partial \psi_e - \left( \hat{e} \delta z_2^R + \frac{1}{2} \hat{e} \delta z_2 \right) \bar{\psi}_e q_e \not A_R \psi_e \\
+ \text{counterterms}.
\]

$\hat{e} + \delta e$ is the electromagnetic coupling of QED at the scale just below the Fermi scale threshold; it can be easily related to $\alpha_{\text{EM}}$ via the RG running.

The factors $\delta z_L^L$ and $\delta z_R^R$ are such that they reproduce at the tree level contributions of virtual $W$, $Z^0$ and $Z'$ to the electron self-energies (computed at zero momentum). Similarly,

\[
\delta z_\gamma = -\left[\hat{\Pi}_\gamma(0)\right]_{W,G^+,f} \tag{B.3}
\]

reproduces at the tree level the vacuum polarization due to decoupled heavy particles $W^\pm$ and top quark.

The vertex corrections determining the combinations $\delta e + \hat{e} \delta z_2^L + \frac{1}{2} \hat{e} \delta z_2$ are shown in figure 2. Owing to the $U(1)_Y$ and $U(1)_E$ Ward identities the $Z^0$ and $Z'$ contributions to $\delta e$ are exactly canceled by the $Z^0$ and $Z'$ contributions to $\delta z_2^L$ and $\delta z_2^R$, respectively. The second diagram in figure 2 is exactly as in the SM and combines with the $W$ contribution to $\delta z_2^L$. As a result from the photon coupling to left-chiral electrons one gets

\[
\delta e = \frac{1}{2} \hat{e} \hat{\Pi}_\gamma(0) + c_{eL}^Z \hat{\Pi}_{\gamma Z^0}(0) M_{Z^0}^2 + c_{eL}^{Z'} \hat{\Pi}_{\gamma Z'}(0) M_{Z'}^2 - \frac{\hat{e}^3}{16\pi^2s^2} \left( \eta_{\text{div}} + \ln \frac{M_W^2}{\mu^2} \right)
\]
Figure 2: Corrections to the photon-electron vertex in a model with extra $U(1)$. The external line momenta can be off-shell but must be $\ll M_Z$.

The self energies $\tilde{\Pi}_{\gamma Z^0}(0)$ and $\tilde{\Pi}_{\gamma Z'}(0)$ receive contributions only from the virtual $W^+W^-$ and $W^\pm G^\mp$ pairs. We get

$$\delta e = \frac{1}{2} \hat{c} \tilde{\Pi}_{\gamma}(0) - \frac{1}{16\pi^2 s^2} \left( \eta_{\text{div}} + \ln \frac{\hat{M}_W^2}{\mu^2} \right)$$

$$+ \frac{1}{16\pi^2} 2\hat{c}_{eL} \left[ -\hat{c}^3 \frac{\hat{e}'}{s} - \hat{c} \left( \frac{\hat{s}}{\hat{c}} \hat{c}' - 2\hat{e} g_E s' \right) \right] \frac{\hat{M}_W^2}{\hat{M}_{Z'}^2} \ln \frac{\hat{M}_W^2}{\mu^2}$$

$$+ \frac{1}{16\pi^2} 2\hat{c}_{eL} \left[ \hat{e}^2 \frac{\hat{c}}{s} + \hat{e} \left( \frac{\hat{s}}{\hat{c}} s' + 2\hat{e} g_E c' \right) \right] \frac{\hat{M}_W^2}{\hat{M}_{Z'}^2} \ln \frac{\hat{M}_W^2}{\mu^2}$$

By using the relation (A.2), (A.3) this can be reduced to

$$\delta e = \frac{1}{2} \hat{c} \tilde{\Pi}_{\gamma}(0) - \frac{\hat{c}^3}{8\pi^2} \left( \eta_{\text{div}} + \ln \frac{\hat{M}_W^2}{\mu^2} \right)$$

which (as could be expected) is the same as in the SM. The same result is obtained by considering the photon coupling to right-chiral electron.

**Appendix B.2 Calculation of $\delta G_F$**

Calculation of $\delta G_F$ proceeds as in the SM. The only modification is that there are additional box diagrams with $Z'$ and in addition the $W$ boson self energy $\Pi_{WW}(q^2)$ as well as the self energies of external line fermions are modified by the presence of $Z'$ (there are contributions from virtual $Z'$ and the couplings of $Z^0$ are modified).

Still the formula takes the form

$$G_F = \frac{1}{\sqrt{2} \hat{v}_H^2} (1 + \Delta_G) = \frac{\hat{e}^2}{4\sqrt{2} s^2 \hat{M}_W^2} (1 + \Delta_G)$$

with $\Delta_G$ given by (B.4)

$$\Delta_G = -\frac{\tilde{\Pi}_{WW}(0)}{\hat{M}_W^2} + B_{W\gamma} + B_{WZ^0} + B_{WZ'} + 2\hat{A} + \hat{S}_{eL} + \hat{S}_{\nu L} . \quad (B.4)$$
Here $B_{W\gamma}$ is the contribution (in units of the tree level $W$ exchange) of the $W\gamma$ box with subtracted contribution of the photonic vertex correction to the tree level diagram in the low energy effective four-Fermi theory of $\mu^-$ decay:

$$B_{W\gamma} = \frac{\hat{e}^2}{16\pi^2} \left( \eta_{\text{div}} + \frac{1}{2} + \ln \frac{M_W^2}{\mu^2} \right)$$

(this contribution is the same as in the SM) and $B_{WZ^0}$ and $B_{WZ'}$ denote the contributions of the box diagrams with $WZ^0$ and $WZ'$, respectively:

$$B_{WZ^0} = \frac{1}{16\pi^2} \left[ \left(c_{eL}^{Z^0}\right)^2 + \left(c_{\nu L}^{Z^0}\right)^2 - 8 c_{eL}^{Z^0} c_{\nu L}^{Z^0} \right] \frac{M_W^2}{M_W^2 - M_{Z^0}^2} \ln \frac{M_W^2}{M_{Z^0}^2}$$

and $B_{WZ'}$ is given by a similar expression with $c_{eL}^{Z^0} \rightarrow c_{eL}^{Z'}$ and $M_{Z^0}^2 \rightarrow M_{Z'}^2$.

For the contributions $\hat{\Lambda}^{(i)}$ of individual diagrams to the vertex corrections $\hat{\Lambda} = (1/16\pi^2) \sum_i \hat{\Lambda}^{(i)}$ one finds:

$$\hat{\Lambda}^{Z^0eL} = -c_{eL}^{Z^0, Z^0} \left( \eta_{\text{div}} + \frac{1}{2} + \ln \frac{M_{Z^0}^2}{\mu^2} \right)$$

$$\hat{\Lambda}^{Z'eL} = -c_{eL}^{Z', Z'} \left( \eta_{\text{div}} + \frac{1}{2} + \ln \frac{M_{Z'}^2}{\mu^2} \right)$$

$$\hat{\Lambda}^{WZ^0} = -3c_{\nu L}^{Z^0} \left( \hat{e}^2 - c_s^2 \right) \left( \eta_{\text{div}} - \frac{5}{6} + \ln \frac{M_W^2}{\mu^2} + \frac{M_{Z^0}^2}{M_{Z^0}^2 - M_W^2} \ln \frac{M_{Z^0}^2}{M_W^2} \right)$$

$$\hat{\Lambda}^{WZ'} = -3c_{\nu L}^{Z'} \left( \hat{e}^2 - c_s^2 \right) \left( \eta_{\text{div}} - \frac{5}{6} + \ln \frac{M_W^2}{\mu^2} + \frac{M_{Z'}^2}{M_{Z'}^2 - M_W^2} \ln \frac{M_{Z'}^2}{M_W^2} \right)$$

$$\hat{\Lambda}^{\epsilon Z^0W} = 3c_{eL}^{Z^0} \left( \hat{e}^2 - c_s^2 \right) \left( \eta_{\text{div}} - \frac{5}{6} + \ln \frac{M_W^2}{\mu^2} + \frac{M_{Z^0}^2}{M_{Z^0}^2 - M_W^2} \ln \frac{M_{Z^0}^2}{M_W^2} \right)$$

$$\hat{\Lambda}^{\epsilon Z'^W} = 3c_{eL}^{Z'} \left( \hat{e}^2 - c_s^2 \right) \left( \eta_{\text{div}} - \frac{5}{6} + \ln \frac{M_W^2}{\mu^2} + \frac{M_{Z'}^2}{M_{Z'}^2 - M_W^2} \ln \frac{M_{Z'}^2}{M_W^2} \right)$$

$$\hat{\Lambda}^{\epsilon W} = -3\hat{e}^2 \left( \eta_{\text{div}} - \frac{5}{6} + \ln \frac{M_W^2}{\mu^2} \right)$$

so that the divergent part of $\hat{\Lambda}$ is

$$\hat{\Lambda}_{\text{div}} = \frac{1}{16\pi^2} \left( \hat{e}^2 \frac{1 - 2s^2 - 12c^2}{4s^2\hat{e}^2} - \hat{e}^2 g_E^2 \right) \eta_{\text{div}}$$

Finally, for the self-energies $\hat{\Sigma}_{\nu L}$ and $\hat{\Sigma}_{e L}$ of the left-chiral electron and neutrino, respectively one gets

$$16\pi^2 \hat{\Sigma}_{\nu L} = \frac{\hat{e}^2}{2s^2} \left( \eta_{\text{div}} + \frac{1}{2} + \ln \frac{M_W^2}{\mu^2} \right) + \left(c_{\nu L}^{Z^0}\right)^2 \left( \eta_{\text{div}} + \frac{1}{2} + \ln \frac{M_{Z^0}^2}{\mu^2} \right)$$
\[
16\pi^2 \hat{\Sigma}_{eL} = \frac{\hat{e}^2}{2s^2} \left( \eta_{\text{div}} + \frac{1}{2} + \ln \frac{\hat{M}_W^2}{\mu^2} \right) + \left( \hat{e}_{\hat{c}L}^Z \right)^2 \left( \eta_{\text{div}} + \frac{1}{2} + \ln \frac{\hat{M}_{Z_0}^2}{\mu^2} \right)
\]

with the divergent part

\[
(\hat{\Sigma}_{\nu L} + \hat{\Sigma}_{e L})_{\text{div}} = \frac{1}{16\pi^2} \left( \frac{\hat{e}^2}{s^2} + \frac{\hat{e}^2}{4s^2c^2} \left[ 1 + (1 - 2s^2)^2 \right] + 2\hat{e}^2g_E^2 \right) \eta_{\text{div}}
\]

Collecting all divergent parts together we get for boxes, vertex and self energy corrections exactly the same divergent part as in the SM

\[
(B_{\text{boxes}} + 2\hat{\Lambda} + \hat{\Sigma}_{e L} + \hat{\Sigma}_{\nu L})_{\text{div}} = -\frac{\hat{e}^2}{16\pi^2 s^2} \eta_{\text{div}} \tag{B.5}
\]

**Appendix C  RG equation for \( v_S \)**

The most general scalar field potential in the model considered in this paper is

\[
V = m_S^2 S^* S + \frac{\lambda_S}{4} (S^* S)^2 + m_H^2 H^\dagger H + \frac{\lambda_H}{4} (H^\dagger H)^2 + \kappa (S^* S)(H^\dagger H)
\]

In order to simplify the formulae we have assumed that at one particular renormalization scale \( \mu \), at which we chose to work, \( \kappa(\mu) = 0 \). However, to derive the renormalization group equation for \( v_S \) one has to keep \( \kappa \). With

\[
S = \frac{1}{\sqrt{2}} (v_S + S^0 + iG_S) \quad H = \frac{1}{\sqrt{2}} \left( \sqrt{2}G^+ \right) \quad \left( v_H + h^0 + iG_H \right)
\]

(\[C.1])

(where \( h^0 \) and \( S^0 \) are the physical Higgs scalars and \( G_H \) and \( G_S \) are the fields whose appropriate linear combinations \( G^0 \) and \( G' \) become the longitudinal components of the massive \( Z_0 \) and \( Z' \)), the formulae determining \( v_S^2 \) and \( v_H^2 \) read

\[
m_H^2 + \frac{1}{4} \lambda_H v_H^2 + \frac{1}{2} \kappa v_S^2 = 0
\]

\[
m_S^2 + \frac{1}{4} \lambda_S v_S^2 + \frac{1}{2} \kappa v_H^2 = 0 \tag{C.2}
\]

Differentiating the second one with respect to \( \mu \) we get at \( \kappa = 0 \):

\[
\mu \frac{dv_S^2}{d\mu} = -4 \frac{1}{\lambda_S} \left( \mu \frac{dm_S^2}{d\mu} + \frac{1}{4} v_S^2 \mu \frac{d\lambda_S}{d\mu} + \frac{1}{2} v_H^2 \mu \frac{d\kappa}{d\mu} \right) \tag{C.3}
\]
Thus, to find the derivative of $v_S^2$ at the scale $\mu$ such, that $\kappa(\mu) = 0$ we need to get also $d\kappa/dt$. Calculating derivatives appering in the right hand side of (C.3) is standard

$$
\frac{d}{d\mu} \lambda_S = 2\epsilon \lambda_S + 5\lambda_S^2 - 12\lambda_S g_E^2 \epsilon_S^2 + 24 g_E^4 \epsilon_S^4
$$

$$
\frac{d}{d\mu} m_S^2 = m_S^2 (2\lambda_S - 6g_E^2 \epsilon_S^2)
$$

$$
\frac{d}{d\mu} \kappa = 12g_E^4 \epsilon_S^2 \epsilon_H^2
$$

Using these results and (C.3) it is easy to derive

$$
\frac{d}{d\mu} v_S^2 = v_S^2 \left( -3\lambda_S + 6g_E^2 \epsilon_S^2 \right) - \frac{24 g_E^4 \epsilon_S^2 v_S^2 + g_E^4 \epsilon_S^2 \epsilon_H v_H^2}{\lambda_S}
$$

(C.5)

**Appendix D  Vector boson self energies**

The fermionic one loop contribution to $\Pi_{WW}(q^2)$ in the $SU(2) \times U(1)_E \times U(1)_Y$ is as in the SM. For the bosonic part of $\Pi_{WW}(q^2)$ we have

$$
-\frac{\epsilon^2}{s^2} \tilde{A}(q^2, \hat{M}_W, \hat{M}_h^0) - \frac{\epsilon^2}{s^2} \tilde{A}(q^2, \hat{M}_W, \hat{M}_Z^0) + \frac{\epsilon^2}{s^2} \hat{M}_W^2 b_0(q^2, \hat{M}_W, \hat{M}_h^0) + \epsilon^2 \hat{M}_W^2 b_0(q^2, \hat{M}_W, 0)
$$

$$
+ \left( -\frac{\epsilon S}{c} + 2\epsilon H g_E s' \right)^2 \hat{M}_W^2 b_0(q^2, \hat{M}_W, \hat{M}_Z^0) + \left( \frac{\epsilon S}{c} s' + 2\epsilon H g_E c' \right)^2 \hat{M}_W^2 b_0(q^2, \hat{M}_W, \hat{M}_{Z'})
$$

$$
-\epsilon^2 \frac{c^2}{s^2} \left[ 8 \tilde{A}(q^2, \hat{M}_W, \hat{M}_{Z'}) + (4q^2 + \hat{M}_W^2 + \hat{M}_{Z'}^2) b_0(q^2, \hat{M}_W, \hat{M}_{Z'}) - \frac{2}{3} \frac{q^2}{16\pi^2} \right]
$$

$$
-\epsilon^2 \frac{s^2}{s^2} \left[ 8 \tilde{A}(q^2, \hat{M}_W, 0) + (4q^2 + \hat{M}_W^2) b_0(q^2, \hat{M}_W, 0) - \frac{2}{3} \frac{q^2}{16\pi^2} \right]
$$

(D.6)

The divergent part of this contribution taken at $q^2 = 0$ is

$$
16\pi^2 [\Pi_{WW}(0)]^{\text{bos}}_{\text{div}} = \left( \epsilon^2 \frac{s^2 - c^2}{s^2 c^2} \hat{M}_W^2 + 4\epsilon^2 H g_E \hat{M}_W^2 \right) \eta_{\text{div}}
$$

(D.7)

(we have used $c^2 \hat{M}_{Z'}^2 + s^2 \hat{M}_{Z'}^2 = \hat{M}_W^2/\epsilon^2$). It differs from the SM only by the last term.

Below we list all bosonic contributions to $\Pi_{Z_1Z_2}(q^2)$ for $Z_1Z_2 = Z^0Z^0$, $Z'Z'$, $Z'Z'$

**$W^+W^-$**

$$
W^+W^- : -\frac{\epsilon^2 c^2}{s^2} \left[ 8 \tilde{A}(q^2, \hat{M}_W, \hat{M}_W) + (4q^2 + 2\hat{M}_W^2) b_0(q^2, \hat{M}_W, \hat{M}_W) - \frac{2}{3} \frac{q^2}{16\pi^2} \right] \times \left( \begin{array}{c} c^2 \\ s^2 \\ -c's' \\
\end{array} \right)
$$

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To simplify the calculations we have assumed here that the scalar fields \( H \) and \( S \) do not mix in the potential, so that the Higgs boson \( h^0 \) comes only from the doublet \( H \) and \( S^0 \) only from the singlet \( S^0 \).

The fermion contribution to \( \Pi_{Z^0,Z^0}^{\text{form}}(q^2) \) reads

\[
\Pi_{Z^0,Z^0}^{\text{form}}(q^2) = \sum_f N_c^{(f)} \left\{ 2(c_{fL}^Z c_{fR}^Z + c_{fR}^Z c_{fL}^Z) m_f^2 b_0(q^2, m_f, m_f) \right. \\
\left. + (c_{fL}^Z c_{fL}^Z + c_{fR}^Z c_{fR}^Z) \left[ 4\tilde{A}(q^2, m_f, m_f) + (q^2 - 2m_f^2) b_0(q^2, m_f, m_f) \right] \right\}
\]

where \( N_c \) is the colour factor and the couplings \( c_{fL}^Z, c_{fR}^Z \) can be read off from (20) and (21).
Appendix E  Loop functions

Here we define some loop functions to make the calculations presented in the text complete.

\[ 16\pi^2 a(m) = m^2 \left( \eta_{\text{div}} - 1 + m^2 + \ln \frac{m^2}{\mu^2} \right) \]  
(E.1)

\[ 16\pi^2 b_0(q^2, m_1, m_2) = \eta_{\text{div}} + \int_0^1 dx \ln \frac{q^2 x(x-1) + x m_1^2 + (1-x) m_2^2}{\mu^2} \]  
(E.2)

\[ 16\pi^2 b_0(0, m_1, m_2) = \eta_{\text{div}} - 1 + \frac{m_1^2}{m_1^2 - m_2^2} \ln \frac{m_1^2}{\mu^2} + \frac{m_2^2}{m_2^2 - m_1^2} \ln \frac{m_2^2}{\mu^2} \]  
(E.3)

\[ \tilde{A}(q^2, m_1, m_2) = -\frac{1}{6} a(m_1) - \frac{1}{6} a(m_2) + \frac{1}{6} \left( m_1^2 + m_2^2 - \frac{q^2}{2} \right) b_0(q^2, m_1, m_2) \]
\[ + \frac{m_1^2 - m_2^2}{12q^2} \left[ a(m_1) - a(m_2) - (m_1^2 - m_2^2) b_0(q^2, m_1, m_2) \right] \]
\[ - \frac{1}{16\pi^2} \left( \frac{1}{6} \left( m_1^2 + m_2^2 - \frac{q^2}{3} \right) \right) \]  
(E.4)

The divergent part of \( \tilde{A}(q^2, m_1, m_2) \) is

\[ 16\pi^2 \left[ \frac{\tilde{A}(q^2, m_1, m_2)}{\text{div}} \right] = -\frac{1}{12} q^2 \eta_{\text{div}} , \]  
(E.5)

\( \tilde{A}(0, m_1, m_2) \) is finite and reads

\[ 16\pi^2 \tilde{A}(0, m_1, m_2) = -\frac{1}{8} \left[ m_1^2 + m_2^2 - \frac{2m_1^2 m_2^2}{m_1^2 - m_2^2} \log \frac{m_1^2}{m_2^2} \right] = -\frac{1}{8} g(m_1, m_2) \]  
(E.6)

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