Composition of Credal Sets via Polyhedral Geometry

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Abstract

Recently introduced composition operator for credal sets is an analogy of such operators in probability, possibility, evidence and valuation-based systems theories. It was designed to construct multidimensional models (in the framework of credal sets) from a system of low-dimensional credal sets. In this paper we study its potential from the computational point of view utilizing methods of polyhedral geometry.

1 Introduction

In the second half of 1990’s a new approach to efficient representation of multidimensional probability distributions was introduced with the aim to be an alternative to Graphical Markov Modeling. This approach is based on a simple idea: a multidimensional distribution is composed from a system of low-dimensional distributions by repetitive application of a special composition operator, which is also the reason why such models are called compositional models.

Later, these compositional models were introduced also in possibility theory [Vejnarová, 1998, 2007] (here the models are parametrized by a continuous t-norm) and ten years ago also in evidence theory [Jiroušek et al.,

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In all these frameworks the original idea is kept, but there exist some slight differences among these frameworks.

In [Vejnarová, 2013] we introduced a composition operator for credal sets, but due to the problem of discontinuity it needed a revision. After a thorough reconsideration in [Vejnarová, 2016] we presented a new definition avoiding this discontinuity. We showed that the revised composition operator keeps the basic properties of its counterparts in other frameworks, and therefore it enables us to introduce compositional models for multi-dimensional credal sets. Nevertheless, a problem how to make practical computations appeared and the need for effective computational procedures became urgent.

Credal sets are usually defined as convex sets of probability distributions. In finite case, a probability distribution can be represented as a point in a multidimensional space. Credal set — as a convex set of such points — can be interpreted as a convex polyhedron in respective space. This naturally leads to computational procedures based on methods used in polyhedral geometry [Grübaum et al., 1967].

This contributions is organized as follows. In Section 2 we summarise the basic concepts and notation. The definition of the operator of composition is presented in Section 3, which is devoted also to its basic properties. In Section 4 we describe proposed computational procedures, in Section 5 we illustrate their application on a few simple examples and Section 6 is devoted to implementation.

### 2 Basic Concepts and Notation

In this section we will briefly recall basic concepts and notation necessary for understanding the contribution.

#### 2.1 Variables and Distributions

For an index set \( N = \{1, 2, \ldots, n\} \) let \( \{X_i\}_{i \in N} \) be a system of variables, each \( X_i \) having its values in a finite set \( \mathcal{X}_i \) and \( \mathcal{X}_N = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n \) be the Cartesian product of these sets.

In this paper we will deal with groups of variables on its subspaces. Let \( X_K \) will denote a group of variables \( \{X_i\}_{i \in K} \) with values in \( \mathcal{X}_K = \mathcal{X}_{i \in K} \) throughout the paper.

Any group of variables \( X_K \) can be described by a probability distribution
(sometimes also called probability function)

\[ P : \mathcal{X}_K \rightarrow [0, 1], \]

such that

\[ \sum_{x_K \in \mathcal{X}_K} P(x_K) = 1. \]

Having two probability distributions \( P_1 \) and \( P_2 \) of \( X_K \) we say that \( P_1 \) is absolutely continuous with respect to \( P_2 \) (and denote \( P_1 \ll P_2 \)) if for any \( x_K \in \mathcal{X}_K \)

\[ P_2(x_K) = 0 \implies P_1(x_K) = 0. \]

This concept plays an important role in the definition of the composition operator.

### 2.2 Credal Sets

A credal set \( \mathcal{M}(X_K) \) describing a group of variables \( X_K \) is usually defined as a closed convex set of probability measures describing the values of these variables. In order to simplify the expression of operations with credal sets, it is often considered \cite{Moral2002} that a credal set is the set of probability distributions associated to the probability measures in it. Under such consideration a credal set can be expressed as a convex hull (denoted by CH) of its extreme distributions (ext)

\[ \mathcal{M}(X_K) = \text{CH}\{\text{ext}(\mathcal{M}(X_K))\}. \]

Consider a credal set \( \mathcal{M}(X_K) \). For each \( L \subset K \) its marginal credal set \( \mathcal{M}(X_L) \) is obtained by element-wise marginalization, i.e.

\[ \mathcal{M}(X_L) = \text{CH}\{P^{\downarrow L} : P \in \text{ext}(\mathcal{M}(X_K))\}, \tag{1} \]

where \( P^{\downarrow L} \) denotes the marginal distribution of \( P \) on \( \mathcal{X}_L \).

Besides marginalization we will also need the opposite operation, usually called extension. Vacuous extension of a credal set \( \mathcal{M}(X_L) \) describing \( X_L \) to a credal set \( \mathcal{M}(X_K) = \mathcal{M}(X_L)^{\uparrow K} \) \( (L \subset K) \) is the maximal credal set describing \( X_K \) such that \( \mathcal{M}(X_K)^{\downarrow L} = \mathcal{M}(X_L) \). A simple example of vacuous extension can be found in Section 5 (Example 5.3).

Having two credal sets \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) describing \( X_K \) and \( X_L \), respectively (assuming that \( K, L \subseteq N \)), we say that these credal sets are projective if their marginals describing common variables coincide, i.e. if

\[ \mathcal{M}_1(X_{K \cap L}) = \mathcal{M}_2(X_{K \cap L}). \]

Let us note that if \( K \) and \( L \) are disjoint, then \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are always projective, as \( \mathcal{M}_1(X_\emptyset) = \mathcal{M}_2(X_\emptyset) \equiv 1. \)
2.3 Strong Independence

Among the numerous definitions of independence for credal sets [Couso et al., 1999] we have chosen strong independence, as it seems to be the most appropriate for multidimensional models.

We say that (groups of) variables $X_K$ and $X_L$ ($K$ and $L$ disjoint) are strongly independent with respect to $M(X_{K∪L})$ iff (in terms of probability distributions)

$$M(X_{K∪L}) = \text{CH}\{P_1 \cdot P_2 : P_1 \in M(X_K), P_2 \in M(X_L)\}.$$ (2)

Again, there exist several generalizations of this notion to conditional independence, see e.g. [Moral and Cano, 2002], but as the following definition is suggested by the authors as the most appropriate for the marginal problem, it seems to be a suitable concept also in our case, since the operator of composition can also be used as a tool for solution of a marginal problem, as shown (in the framework of possibility theory) e.g. in [Vejnarová, 2007].

Given three groups of variables $X_K, X_L$ and $X_M$ ($K, L, M$ be mutually disjoint subsets of $N$, such that $K$ and $L$ are nonempty), we say that $X_K$ and $X_L$ are conditionally independent given $X_M$ under global set $M(X_{K∪L∪M})$ (to simplify the notation we will denote this relationship by $K \perp \perp L | M$) iff

$$M(X_{K∪L∪M}) = \text{CH}\{(P_1 \cdot P_2) / P_1^{im} : P_1 \in M(X_{K∪M}), P_2 \in M(X_{L∪M}), P_1^{im} = P_2^{im}\}.$$ (3)

This definition is a generalisation of stochastic conditional independence: if $M(X_{K∪L∪M})$ is a singleton, then $M(X_{K∪M})$ and $M(X_{L∪M})$ are also (projective) singletons and the definition reduces to the definition of stochastic conditional independence.

2.4 Polyhedral Geometry

A convex polytope may be defined in numerous ways, depending on what is more suitable for the problem at hand. Grünbaum’s definition [Grünbaum et al., 1967] is in terms of a convex set of points in space. Other important definitions are: as the intersection of half-spaces (H-representation) and as the convex hull of a set of vertices (V-representation). For a compact convex polytope, the minimal V-representation is unique and it is given by the set of the vertices of the polytope [Grünbaum et al., 1967]. In our experiments, we use both H and V-representations. However, for the purpose of representation of the polytope within the paper only V-representation is used.
As mentioned in the introduction, a credal set is a convex polytope (bounded polyhedron) in $|X_N|$-dimensional space. Each dimension corresponds to an element from $X_N$ (a combination of the variables $N$) - i.e. $x_N \in X_N$. Vertex $v$ of the space is nothing else than a probability distribution $P_v$ with $v[x_N] = P_v(x_N)$. Following the fact, that probability belongs to unit interval $[0, 1]$, we can restrict the space — it is, in fact, a hypercube.

The nature of the space imposes several restrictions to operations with convex polytopes. We do not consider projections into subspaces. Marginalization of a convex polytope representing a set of probability distributions of $X_N$ to $X_K$ ($K \subset N$) corresponds to a transformation of $|X_N|$-dimensional space into $|X_K|$-dimensional space. Naturally, for each vertex this is done by summing of all coordinates from $X_N$ with the projection to $X_K$. The similar holds for extensions.

3 Composition Operator

In this section we will recall the new definition of composition operator for credal sets introduced in Vejnarová, 2016. To enable the reader better understanding to the concept we will present it first in a precise probability framework.

3.1 Composition Operator of Probability Distributions

First let us recall the definition of composition of two probability distributions Jiroušek, 1997. Consider two index sets $K, L \subset N$. We do not put any restrictions on $K$ and $L$; they may be but need not be disjoint, and one may be a subset of the other. Let $P_1$ and $P_2$ be two probability distributions of (groups of) variables $X_K$ and $X_L$; then

$$ (P_1 \triangleright P_2)(X_{K \cup L}) = \frac{P_1(X_K) \cdot P_2(X_L)}{P_2(X_{K \cap L})}, $$

whenever $P_1(X_{K \cap L}) \ll P_2(X_{K \cap L})$; otherwise, it remains undefined.

It is specific property of composition operator for probability distributions — in other settings the operator is always defined Vejnarová, 2007, Jiroušek et al. 2007.

3.2 Definition

Let $M_1$ and $M_2$ be credal sets describing $X_K$ and $X_L$, respectively. Our goal is to define a new credal set, denoted by $M_1 \triangleright M_2$, which will describe
\(X_{K \cup L}\) and will contain all of the information contained in \(\mathcal{M}_1\) and, as much as possible, from \(\mathcal{M}_2\). In other words, we want to find a common extension of \(\mathcal{M}_1\) and \(\mathcal{M}_2\) (if it is possible).

The required properties were already met by Definition 1 in [Vejnarová, 2013]. However, that definition exhibits a kind of discontinuity and was thoroughly reconsidered. In [Vejnarová, 2016] we proposed the following one.

**Definition 3.1** For two credal sets \(\mathcal{M}_1\) and \(\mathcal{M}_2\) describing \(X_K\) and \(X_L\), their composition \(\mathcal{M}_1 \triangleright \mathcal{M}_2\) is defined as a convex hull of probability distributions \(P\) obtained as follows. For each couple of distributions \(P_1 \in \mathcal{M}_1(X_K)\) and \(P_2 \in \mathcal{M}_2(X_L)\) such that \(P_2^{K \cap L} \in \arg\min\{Q_2 \in \mathcal{M}_2(X_{K \cap L}) : d(Q_2, P_1^{K \cap L})\}\), distribution \(P\) is obtained by one of the following rules:

[a] if \(P_1^{K \cap L} \ll P_2^{K \cap L}\)

\[
P(X_{K \cup L}) = \frac{P_1(X_K) \cdot P_2(X_L)}{P_2^{K \cap L}(X_{K \cap L})}.
\]

[b] otherwise

\[
P(X_{K \cup L}) \in \text{ext}\{P_1^{K \cup L}(X_K)\}.
\]

Function \(d\) used in the definition is a suitable distance function. In this paper we use Euclidean distance, as it is natural choice in polyhedral geometry.

Let us note, that this definition of composition operator does not differ from the original one [Vejnarová, 2013] in case of projective credal sets, as in this case the only distributions in \(\mathcal{M}_1 \triangleright \mathcal{M}_2\) are those satisfying \(P = (P_1 \cdot P_2)/P_2^{K \cap L}\), where \(P_1^{K \cap L} = P_2^{K \cap L}\) (and those belonging to their convex hull). However, it differs in the remaining cases. It will be illustrated in Section 5 using our computational procedures.

In the next subsection we will summarize the most important basic properties of the composition operator.

**3.3 Basic Properties**

The following lemma proven in [Vejnarová, 2016] suggests that the above-defined composition operator possesses basic properties required at the beginning of this section. Its last item characterizes condition under which common extension of \(\mathcal{M}_1\) and \(\mathcal{M}_2\) can be obtained.

\(^1\)Let us note that the definition is based on Moral’s concept of conditional independence with relaxing convexity.
Lemma 3.2 For two credal sets $M_1$ and $M_2$ describing $X_K$ and $X_L$, respectively, the following properties hold true:

1. $M_1 \triangleright M_2$ is a credal set describing $X_{K \cup L}$.
2. $(M_1 \triangleright M_2)(X_K) = M_1(X_K)$.
3. $M_1 \triangleright M_2 = M_2 \triangleright M_1$ iff $M_1(X_{K \cap L}) = M_2(X_{K \cap L})$.

This lemma, together with the following theorem, proven in [Vejnárová, 2013], expressing the relationship between strong independence and the operator of composition, are the most important assertions enabling us to introduce compositional models.

Theorem 3.3 Let $M$ be a credal set describing $X_{K \cup L}$ with marginals $M(X_K)$ and $M(X_L)$. Then

$$M(X_{K \cup L}) = (M_{\|K} \triangleright M_{\|L})(X_{K \cup L})$$

iff

$$(K \setminus L) \perp \perp (L \setminus K | (K \cap L)).$$

This theorem remains valid also for the revised definition of the composition operator, as $M(X_K)$ and $M(X_L)$ are marginals of $M(X_{K \cup L})$, and therefore only [a] (for projective distributions) is applicable.

Before closing this section, let us present one more result proven in [Vejnárová, 2017] concerning the relationship between the original composition operator for precise probabilities and that studied in this contribution.

Lemma 3.4 Let $M_1(X_K)$ and $M_2(X_L)$ be two singleton credal sets describing $X_K$ and $X_L$, respectively, where $M_1(X_{K \cap L})$ is absolutely continuous with respect to $M_2(X_{K \cap L})$. Then $(M_1 \triangleright M_2)(X_{K \cup L})$ is also a singleton.

The reader should however realize that the definition of the operator of composition for singleton credal sets is not completely equivalent to the definition of composition for probabilistic distributions. They equal each other only in case that the probabilistic version is defined. This is ensured in Lemma 3.3 by assuming the absolute continuity. In case it does not hold, the probabilistic operator is not defined while its credal version introduced in this paper is always defined. However, in this case, the result is not a singleton credal set, as can be seen from Example 5.3 in Section 5.
4 Computational Procedures for Composition Operator

The experiments were performed in R environment using RStudio. To implement polyhedral geometry, we have used the `rcdd` package based on GMP GNU library. For quadratic programming methods — i.e. in case of finding a projection of a given vertex on a certain polytope (we considered Euclidean distance in this paper) — we have used methods of quadratic programming implemented in the `quadprog` package.

To implement various extensions to higher dimensional space — as described in Definition 3.1 — we have used the advantage of equivalence of H- and V-representations of convex polytopes. To move from $|X_N|$-dimensional space to $|X_K|$-dimensional space where $K \subset N$ we can easily create a 0-1 transformation matrix that allows us to convert an arbitrary vertex from one space to the other one. With the help of such a transformation matrix, a set of related (in)equalities can be transformed as well. This is extremely handy when we need to extend a vertex or a convex polytope from a space to the higher-dimensional one. First, we convert it to its H-representation and then transform it using a transformation matrix to the higher dimensional space.

To find a part of a polytope with given projection, it is enough to extend H-projection of the projection to the original space and combine both H-representations (to get the intersection of the polytopes).

5 Examples

In this Section we will demonstrate the application of Definition 1 via our computational procedures. Let us start with the case of projective credal sets.

Example 5.1 Let $\mathcal{M}_1(X_1X_2)$ and $\mathcal{M}_2(X_2X_3)$ be credal sets about variables $X_1X_2$ and $X_2X_3$, respectively, with extreme vertices as listed in Table 1. These two credal sets are projective, as

$$\mathcal{M}_1(X_2) = \text{CH}\{[0.2, 0.8], [0.5, 0.5]\} = \mathcal{M}_2(X_2)$$
Following Definition 1, $M_1 \supset M_2$ can be expressed as a convex polytope with V-representation defined in Table 2.

Table 1: V-representations of credal sets $M_1$ and $M_2$ from Example 5.1

| $x_1x_2$ | $x_1\bar{x}_2$ | $\bar{x}_1x_2$ | $\bar{x}_1\bar{x}_2$ |
|---------|---------------|----------------|-----------------|
| 1       | 0.2           | 0.2            | 0.6             |
| 2       | 0.1           | 0.4            | 0.1             |
| 3       | 0.25          | 0.25           | 0.25            |
| 4       | 0.2           | 0.3            | 0.2             |

| $x_2x_3$ | $x_2\bar{x}_3$ | $\bar{x}_2x_3$ | $\bar{x}_2\bar{x}_3$ |
|---------|---------------|----------------|-----------------|
| 1       | 0.2           | 0.3            | 0.3             |
| 2       | 0             | 0.2            | 0.8             |
| 3       | 0.5           | 0              | 0.5             |
| 4       | 0.2           | 0.3            | 0.2             |

$\mathcal{M}_1(X_1X_2)$ $\mathcal{M}_2(X_2X_3)$

It can easily be checked that both $(M_1 \supset M_2)(X_1X_2) = M_1(X_1X_2)$ and $(M_1 \supset M_2)(X_2X_3) = M_2(X_2X_3)$. ◊

Here for any distribution $P_1(X_1X_2)$ in $M_1$ exists (at least one) distribution $P_2(X_2X_3)$ in $M_2$ such that $P_1(X_2) = P_2(X_2)$, therefore all the extreme points are obtained as a simple conditional product. In this case, furthermore, $M_1 \supset M_2 = M_2 \supset M_1$, as corresponds to Lemma 3.2 (and can be easily checked).

The following example is more complicated, as it deals with non-projective credal sets.

Example 5.2 Let $M_1(X_1X_2)$ and $M_2(X_2X_3)$ be two credal sets describing binary variables $X_1X_2$ and $X_2X_3$, respectively, defined as a convex hull of vertices in Table 3.

These two credal sets are not projective, as $M_1(X_2) = \text{CH}\{[0.2, 0.8], [0.6, 0.4]\}$, while $M_2(X_2) = \text{CH}\{[0.3, 0.7], [0.5, 0.5]\}$. Therefore $M_2(X_2) \subset M_1(X_2)$.
Table 3: V-representations of credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ from Example 5.2

| $x_1x_2$ | $x_1\bar{x}_2$ | $\bar{x}_1x_2$ | $\bar{x}_1\bar{x}_2$ | $\bar{x}_2x_3$ | $x_2\bar{x}_3$ | $\bar{x}_2x_3$ | $\bar{x}_2\bar{x}_3$ |
|-----------|----------------|-----------------|----------------------|---------------|---------------|---------------|----------------|
| 1 0.2     | 0.8            | 0               | 0                    | 1             | 0.3           | 0             | 0.7            |
| 2 0.1     | 0.4            | 0.1             | 0.4                  | 2             | 0.2           | 0.1           | 0.4            |
| 3 0.3     | 0.2            | 0.3             | 0.2                  | 3             | 0.25          | 0.25          | 0.25           |
| 4 0.6     | 0.4            | 0               | 0.6                  | 4             | 0.5           | 0             | 0.5            |

$\mathcal{M}_1(X_1X_2)$ $\mathcal{M}_2(X_2X_3)$

Table 3: V-representations of credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ from Example 5.2

Definition 1 in this case leads to a credal set $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_1X_2X_3)$ with 23 extreme points listed in Table 4. On the other hand $(\mathcal{M}_2 \triangleright \mathcal{M}_1)(X_1X_2X_3)$

| $x_1x_2x_3$ | $x_1\bar{x}_2\bar{x}_3$ | $\bar{x}_1x_2\bar{x}_3$ | $\bar{x}_1\bar{x}_2x_3$ | $\bar{x}_1\bar{x}_2\bar{x}_3$ | $\bar{x}_2\bar{x}_3x_3$ | $\bar{x}_2\bar{x}_3\bar{x}_3$ | $\bar{x}_2x_3\bar{x}_3$ | $\bar{x}_2x_3\bar{x}_3$ |
|--------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 1 0.15       | 0                        | 0                        | 0.35                     | 0                        | 0.15                     | 0                        | 0.35                     | 0                        |
| 2 0.075      | 0.3                      | 0                        | 0                        | 0.225                    | 0                        | 0.4                      | 0                        | 0.4                      |
| 3 0.05       | 0.171                    | 0.129                    | 0.15                     | 0.075                    | 0.229                    | 0.171                    | 0                        | 0.171                    |
| 4 0.15       | 0                        | 0.6                      | 0                        | 0.15                     | 0                        | 0.1                      | 0                        | 0.1                      |
| 5 0.1        | 0.343                    | 0.257                    | 0.1                      | 0.05                     | 0.057                    | 0.043                    | 0                        | 0.043                    |
| 6 0.225      | 0                        | 0.65                     | 0                        | 0.075                    | 0                        | 0.05                     | 0                        | 0.05                     |
| 7 0.15       | 0.371                    | 0.279                    | 0.05                     | 0.025                    | 0.029                    | 0.021                    | 0                        | 0.021                    |
| 8 0.125      | 0.125                    | 0.125                    | 0.125                    | 0.125                    | 0.125                    | 0.125                    | 0.125                    |
| 9 0.25       | 0                        | 0.25                     | 0                        | 0.25                     | 0                        | 0                        | 0                        | 0                        |
| 1 0.012      | 0.012                    | 0.05                     | 0.05                     | 0.238                    | 0.238                    | 0.2                      | 0                        | 0.2                      |
| 11 0.025     | 0.1                      | 0                        | 0.475                    | 0                        | 0.4                      | 0                        | 0                        | 0                        |
| 12 0.025     | 0.1                      | 0.225                    | 0                        | 0.225                    | 0.15                     | 0.15                     | 0                        | 0.15                     |
| 13 0.05      | 0                        | 0.45                     | 0                        | 0.45                     | 0                        | 0.3                      | 0                        | 0                        |
| 14 0.138     | 0.138                    | 0.175                    | 0.175                    | 0.113                    | 0.113                    | 0.075                    | 0.075                    |
| 15 0.275     | 0                        | 0.35                     | 0                        | 0.225                    | 0                        | 0.15                     | 0                        | 0                        |
| 16 0.2       | 0                        | 0.8                      | 0                        | 0                        | 0                        | 0                        | 0                        | 0                        |
| 17 0.133     | 0.067                    | 0.457                    | 0.343                    | 0                        | 0                        | 0                        | 0                        | 0                        |
| 18 0.1       | 0                        | 0.4                      | 0                        | 0.1                      | 0                        | 0.4                      | 0                        | 0.4                      |
| 19 0.067     | 0.033                    | 0.229                    | 0.171                    | 0.067                    | 0.033                    | 0.229                    | 0.171                    |
| 2 0.3       | 0                        | 0.3                      | 0                        | 0.3                      | 0                        | 0.2                      | 0                        | 0.2                      |
| 21 0.15     | 0.1                      | 0.15                     | 0.1                      | 0.15                     | 0.15                     | 0.1                      | 0.1                      |
| 22 0        | 0                        | 0                        | 0.6                      | 0                        | 0.4                      | 0                        | 0                        | 0                        |
| 23 0        | 0                        | 0                        | 0.3                      | 0.3                      | 0.2                      | 0                        | 0.2                      |

Table 4: V-representation $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_1X_2X_3)$ from Example 5.2
has 16 extreme points. They are listed in Table 5.

|   | \(x_1\) | \(x_2\) | \(x_3\) | \(x_4\) | \(x_5\) | \(x_6\) | \(x_7\) | \(x_8\) | \(x_9\) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.15 | 0 | 0.35 | 0 | 0.15 | 0 | 0.35 |
| 2 | 0 | 0.075 | 0 | 0.3 | 0 | 0.225 | 0 | 0.4 |
| 3 | 0 | 0.15 | 0 | 0.6 | 0 | 0.15 | 0 | 0.1 |
| 4 | 0 | 0.225 | 0 | 0.65 | 0 | 0.075 | 0 | 0.05 |
| 5 | 0.1 | 0.05 | 0.2 | 0.15 | 0.1 | 0.05 | 0.2 | 0.15 |
| 6 | 0.05 | 0.025 | 0.171 | 0.129 | 0.15 | 0.075 | 0.229 | 0.171 |
| 7 | 0.1 | 0.05 | 0.343 | 0.257 | 0.1 | 0.05 | 0.057 | 0.043 |
| 8 | 0.15 | 0.075 | 0.371 | 0.279 | 0.05 | 0.025 | 0.029 | 0.021 |
| 9 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 |
| 1 | 0.012 | 0.012 | 0.05 | 0.05 | 0.238 | 0.238 | 0.2 | 0.2 |
| 11 | 0.025 | 0.025 | 0.1 | 0.1 | 0.225 | 0.225 | 0.15 | 0.15 |
| 12 | 0.138 | 0.138 | 0.175 | 0.175 | 0.113 | 0.113 | 0.075 | 0.075 |
| 13 | 0.25 | 0.25 | 0 | 0.25 | 0 | 0.25 | 0 | 0 |
| 14 | 0.025 | 0 | 0.1 | 0 | 0.475 | 0 | 0.4 | 0 |
| 15 | 0.05 | 0 | 0.2 | 0 | 0.45 | 0 | 0.3 | 0 |
| 16 | 0.275 | 0 | 0.35 | 0 | 0.225 | 0 | 0.15 | 0 |

Table 5: V-representation of \((M_2 \triangleright M_1)(X_1X_2X_3)\) from Example 5.2.

This difference deserves a more detailed explanation. Both \(M_1 \triangleright M_2\) and \(M_2 \triangleright M_1\) keep the first marginal, but not the second one (which corresponds to Lemma 3.2).

The extreme vertices of marginal of the composition \(M_2 \triangleright M_1\) of \(X_1X_2\) are listed in Table 6. It is not obvious just from the table itself, but the convex polytope corresponding to \((M_2 \triangleright M_1)(X_1X_2)\) is smaller than the one of the original credal set \(M_1(X_1X_2)\). Actually, any extreme vertex of \((M_2 \triangleright M_1)(X_1X_2)\) is an inner point of \(M_1(X_1X_2)\).

The opposite inclusion holds for the marginal of the composition \(M_1 \triangleright M_2\) of \(X_2X_3\). The V-representation of respective credal set is given in Table 7. This credal set is bigger than \(M_2 \triangleright M_1\) \((X_2X_3)\), i.e., any extreme vertex of \(M_2(X_2X_3)\) is contained in \((M_1 \triangleright M_2)(X_2X_3)\).

The fact that \((M_2 \triangleright M_1)(X_1X_2)\) is smaller (more precise) than \((M_2 \triangleright M_1)\)(\(X_1X_2)\) corresponds to the idea that we want \(M_2 \triangleright M_1\) to keep all the information contained in \(M_2\). Therefore, we do not consider those distributions from \(M_1\) not corresponding to any from \(M_2\) — although these distributions are taken into account when composing \(M_1 \triangleright M_2\).
Table 6: \((M_2 \triangleright M_1)(X_1X_2)\)

| \(x_1x_2\) | \(x_1\bar{x}_2\) | \(\bar{x}_1x_2\) | \(\bar{x}_1\bar{x}_2\) |
|------------|----------------|----------------|----------------|
| 1          | 0.15           | 0.35           | 0.15           | 0.35           |
| 2          | 0.075          | 0.3            | 0.225          | 0.4            |
| 3          | 0.15           | 0.6            | 0.15           | 0.1            |
| 4          | 0.225          | 0.65           | 0.075          | 0.05           |
| 5          | 0.25           | 0.25           | 0.25           | 0.25           |
| 6          | 0.025          | 0.1            | 0.475          | 0.4            |
| 7          | 0.05           | 0.2            | 0.45           | 0.3            |
| 8          | 0.275          | 0.35           | 0.225          | 0.15           |

Table 7: \((M_1 \triangleright M_2)(X_2X_3)\)

| \(x_2x_3\) | \(x_2\bar{x}_3\) | \(\bar{x}_2x_3\) | \(\bar{x}_2\bar{x}_3\) |
|------------|----------------|----------------|----------------|
| 1          | 0              | 0.3            | 0              | 0.7            |
| 2          | 0.25           | 0.25           | 0.25           | 0.25           |
| 3          | 0.5            | 0              | 0.5            | 0              |
| 4          | 0              | 0.2            | 0              | 0.8            |
| 5          | 0.133          | 0.067          | 0.457          | 0.343          |
| 6          | 0.6            | 0              | 0.4            | 0              |
| 7          | 0.3            | 0.3            | 0.2            | 0.2            |

V-representation of two marginals of composition of \(M_1\) and \(M_2\) from Example 5.2

The last example demonstrates the case, when \([b]\) of Definition 1 is applied. Simultaneously, it is the case, when probabilistic composition operator remains undefined, but composition of two precise probabilities taken as singleton credal sets is defined (and the result is, naturally, a credal set).

Table 8: Two singleton credal sets \(M_1\) and \(M_2\) from Example 5.3

| \(x_1x_2\) | \(x_1\bar{x}_2\) | \(\bar{x}_1x_2\) | \(\bar{x}_1\bar{x}_2\) | \(x_2x_3\) | \(x_2\bar{x}_3\) | \(\bar{x}_2x_3\) | \(\bar{x}_2\bar{x}_3\) |
|------------|----------------|----------------|----------------|------------|----------------|----------------|----------------|
| 1          | 0.25           | 0.25           | 0.25           | 0.25       | 1              | 0.5            | 0              | 0            |

\(M_1(X_1X_2)\) \hspace{1cm} \(M_2(X_2X_3)\)

\textbf{Example 5.3} Let \(M_1(X_1X_2)\) and \(M_2(X_2X_3)\) be two singleton credal sets describing variables \(X_1X_2\) and \(X_2X_3\), respectively. They are defined in Table 8. Let us compute \(M_1 \triangleright M_2\). As \(M_1(X_2) = \{[0.5, 0.5]\}\), while \((M_2(X_2) = \{[1, 0]\}\), it is evident, that \(M_1\) is not absolutely continuous with respect to \(M_2\). Therefore, using part \([b]\) of Definition 1, we get the full extension of \(M_1(X_2) = \{[0.5, 0.5]\}\), to \(X_{1,2,3}\)-dimensional space of probability distributions (i.e. to a hypercube). The V-representation of such a polytope is listed in Table 9.

\diamond
Algorithm 1 Implementation of $\mathcal{M}_1 \triangleright \mathcal{M}_2$

```
procedure $V(\mathcal{M})$
    return $V$-representation of $\mathcal{M}$ - set of extreme vertices
end procedure

procedure COMPOSE($\mathcal{M}_1(X_K), \mathcal{M}_2(X_L)$)
    $\mathcal{M}_{KL} \leftarrow \mathcal{M}_1^{KL} \cap \mathcal{M}_2^{KL}$
    $\mathcal{M}_1^{\text{projective}} \leftarrow \mathcal{M}_1^{KL} \cap \mathcal{M}_1$
    $\mathcal{M}_2^{\text{projective}} \leftarrow \mathcal{M}_2^{KL} \cap \mathcal{M}_2$
    result $\leftarrow \emptyset$
    for $P_1 \in V(\mathcal{M}_1^{\text{projective}})$ do
        for $P_2 \in V(\mathcal{M}_2^{\text{projective}})$ do
            if $P_1^{KL} = P_2^{KL}$ then
                add $P_1 \triangleright P_2$ to the result
            end if
        end for
    end for
    for $P_1 \in V(\mathcal{M}_1)$ do
        $Q_2 \leftarrow$ find a projection of $P_1^{KL}$ on $\mathcal{M}_2^{KL}$
        (comment: i.e. minimize distance $\|P_1^{KL} - p\|$ subject to $p \in \mathcal{M}_2^{KL}$)
        (comment: in case of Euclidean distance we can use methods of quadratic programming)
        if $P_1^{KL} \ll Q_2$ then
            for $P_2 \in V(\mathcal{M}_2 \cap Q_2^{KL})$ do
                add $P_1 \triangleright P_2$ to the result
            end for
        end if
        if $P_1^{KL} \not\ll Q_2$ then
            add $V(P_1^{KL})$ to the result
        end if
    end for
    return convex hall of the result
end procedure
```
|    | $x_1x_2x_3$ | $x_1x_2x_3$ | $x_1x_2x_3$ | $x_1x_2x_3$ | $x_1x_2x_3$ | $x_1x_2x_3$ | $x_1x_2x_3$ |
|----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1  | 0.25        | 0           | 0           | 0.25        | 0           | 0.25        | 0.25        |
| 2  | 0.25        | 0           | 0           | 0.25        | 0           | 0.25        | 0           |
| 3  | 0.25        | 0           | 0           | 0.25        | 0.25        | 0           | 0.25        |
| 4  | 0.25        | 0           | 0           | 0.25        | 0.25        | 0           | 0.25        |
| 5  | 0.25        | 0           | 0.25        | 0           | 0.25        | 0           | 0           |
| 6  | 0.25        | 0.25        | 0           | 0.25        | 0           | 0.25        | 0           |
| 7  | 0.25        | 0           | 0.25        | 0           | 0           | 0.25        | 0           |
| 8  | 0.25        | 0           | 0.25        | 0           | 0           | 0.25        | 0           |
| 9  | 0           | 0.25        | 0.25        | 0           | 0           | 0.25        | 0           |
| 10 | 0           | 0.25        | 0.25        | 0           | 0.25        | 0           | 0           |
| 11 | 0           | 0.25        | 0.25        | 0           | 0           | 0.25        | 0           |
| 12 | 0           | 0.25        | 0.25        | 0           | 0           | 0.25        | 0           |
| 13 | 0           | 0.25        | 0          | 0.25        | 0.25        | 0           | 0.25        |
| 14 | 0           | 0.25        | 0          | 0.25        | 0.25        | 0           | 0           |
| 15 | 0           | 0.25        | 0          | 0.25        | 0.25        | 0           | 0           |
| 16 | 0           | 0.25        | 0          | 0.25        | 0.25        | 0           | 0.25        |

Table 9: V-representation of $\mathcal{M}_1(X_2) \uparrow X_1X_2X_3$

6 Implementation

The implementation of $\mathcal{M}_1 \triangleright \mathcal{M}_2$ via Definition 1 is based on a conjecture that it is sufficient to deal with two finite groups of vertices only. The first group coincides with the set of extreme vertices of each polytope (credal set). The other group corresponds to projective parts of the credal sets - i.e. the parts whose marginals describing common variables coincide. For better explanation see the pseudo-code of the implementation, as described in Algorithm 1.

7 Conclusions and Future Work

We have presented computational procedures for composition of credal sets. We utilized the fact, that a credal set can be viewed as a special case of a convex polyhedron and that the methods in polyhedral geometry are developed for a long period. It seems to be useful, as the composition can hardly be performed without this computational support (with the exception of the simplest examples).

Nevertheless, it is only the first step in the construction of multidimen-
sional models. The repetitive application of the composition operator is theoretically solved for so-called perfect sequences of credal sets, but the computational issues have not been tackled yet. And the problems connected with other generating sequences could be another research direction. The shift of the distance from the Euclidean on to some divergence of probability distributions as e.g., Kullback-Leibler divergence, total variation or some other f-divergence [Vajda, 1989].

Last but not least, a number of theoretical issues concerning relationship between credal compositional models and other kinds of multidimensional models in the framework of credal sets or compositional models in other frameworks are to be solved.

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