A deterministic Bell model

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Abstract

Following ideas given by John Bell in a paper entitled Beables for quantum field theory, we show that it is possible to obtain a realistic and deterministic interpretation of any quantum field-theoretic model involving Fermi fields.

Key words: beables for quantum field theory, pilot-wave theory, fermion-number density
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1 Introduction

At the heart of the orthodox interpretation of the quantum theory is the measurement postulate. We recall the context in which this postulate has been introduced and why it is unsatisfactory\(^1\). It supposes that the wave-function is the complete description of a system of particles. If we assume that the wave-function evolves according to a linear equation, we face the problem of macroscopic superpositions (Schrödinger’s cat)\(^2\). Since those macroscopic superpositions are not observed, the measurement postulate is introduced, in order to stop the branching process somewhere, where systems encounter observers, who force them to collapse. By introducing the collapse, the world is split into two parts: a quantum world, made of systems, which can be in superpositions and are unable to perform measurements, and a classical world, made of observers, who on the contrary have the ability to cause a collapse and do not enter in superpositions. Systems are described by wave functions,

\(^1\) For more complete reviews of the measurement problem, see [2] and [5].
\(^2\) At this point, it should be stressed that superpositions have an absolute meaning, since any measurement is finally a measurement of positions.
whereas observers are described by positions. If one asks where the boundary between these two worlds is to be found or even why there is a boundary, the orthodox interpretation gives no precise answer. Nevertheless, such ill-defined concepts appear in the fundamental postulates of the orthodox interpretation. The second argument against the measurement is that the collapse postulate clashes with the reductionist project that has always guided the physicists: it has simply no meaning to speak of a wave-function of the universe in the Copenhagen interpretation. Would there be a theory that explains how the collapse appears for macroscopic objects, there would be nothing to worry about. There are interpretations following that way, but they imply modifications of the Schrödinger equation, for example the Ghirardi-Rimini-Weber theory. On the contrary of the orthodox interpretation, it has no need for a vague division of the world into quantum and classical parts. Another attitude towards superpositions is that of Everett (see [2], chapter 11,15 for a critical review).

Hidden-variables theories constitute the second category of interpretations: these are interpretations in which the wave-function is not the complete description with a system of particles. Now, the widespread claim is that those interpretations are ruled out by Bell’s inequality and the experiments that have been carried out later. In fact local ones are ruled out, but since non-locality is commonly claimed to be unacceptable (even weak non-locality, which does not lead to paradoxical situations), it is said that any hidden-variables theory is incompatible with the quantum theory\(^3\). That is the wrong way to present the theoretical situation. To present it correctly, we have to return to the EPR paradox, which in essence says that some quantum correlations cannot be explained in a local way, unless we say that the quantum theory is incomplete. Since local hidden-variables theories are ruled out by Bell’s inequality and experiments, the only way to explain quantum correlations is to revert to non-locality (which is in fact hidden in the collapse postulate). Then, to suppress the ill-defined measurement postulate, it is preferable to interpret the quantum world by a non-local hidden-variables theory. Such an interpretation exists for non-relativistic quantum mechanics: it is the de Broglie-Bohm pilot wave theory, whose John Bell, who is often credited of the refutation of hidden variables theories, has been one of the main advocates.

Pilot-wave theory is a realistic theory, in so far as the positions of the particles exist and are simply revealed by position measurements (the positions are beables, a term coined by Bell). If there are \(n\) particles, the universe is thus completely described by the couple \((\bar{X}(t), \Psi(t, \bar{X}))\), where \(\bar{X}\) is a point in a configuration space of dimension \(3n\). \(\Psi(t, \bar{X})\) evolves according to the

\(^3\) A theory is said to be non-local if it predicts that a cause and its effect are simultaneous in an inertial frame (even if it is impossible to distinguish the cause from the effect).
Schrödinger equation, whereas the equation of motion for $\vec{X}(t)$ is such that if we consider a set of universes with the same wave-function and initial configurations chosen according to the probability density $|\Psi(t_0, \vec{X})|^2$, then the final configurations will be distributed according to $|\Psi(t, \vec{X})|^2$ for any later time $t$ (see [3] for a survey of the non-relativistic pilot wave theory or [9] for a complete study).

The question that comes to mind is to know if the quantum field theory can also be interpreted as a non-local hidden-variable theory. At the time Bell wrote his paper [1], Bohm had already shown that it was possible to build a realistic interpretation of any bosonic quantum field theory [4]. To achieve that goal, Bohm took the field as the beable, however he was not able to do the same for fermions. The aim of Bell was then to show that it was also possible to build a realistic interpretation of any fermionic quantum field theory, along the pilot-wave ideas. Bell managed doing so but he took a different beable: the fermion number density. It is slightly different from the non-relativistic pilot-wave theory, whose beables are the positions of the particles. The model is also formulated on a spatial lattice (space is discrete but time remains continuous). His model is stochastic, but he suspected that the theory would become deterministic in the continuum limit.

This paper is organized in the following way. First, we will give a brief presentation of the lattice Bell model. Then we will ask ourselves what is the physical content of the Bell model, by studying the fermion-number density. Finally, we will show that the stochastic Bell model can be turned into a deterministic one.

2 The Bell model

The Bell model is defined on a finite lattice, whose sites are labelled by an index

$$ l = 1, 2, \ldots, L , $$

where $L$ is very large. The lattice fermion-number operator is defined by

$$ \psi^\dagger(l) \psi(l) = \sum_{a=1}^{a=4} \psi_a^\dagger(l) \psi_a(l) , $$

(1)

where $\psi(l)$ is a four-component lattice Dirac field (we only consider a single species of fermions here). Since $[\psi^\dagger(l) \psi(l), \psi^\dagger(k) \psi(k)] = 0$, for $k \neq l$, it is possible to define eigenstates of the fermion-number density; those eigenstates are defined by $|nq\rangle$

$$ \psi^\dagger(l) \psi(l) |nq\rangle = F(l) |nq\rangle , $$
where \( q \) are eigenvalues of observables \( Q \) such that

\[
\{ \psi^\dagger(1)\psi(1), \ldots, \psi^\dagger(L)\psi(L), Q \}
\]

is a complete set of commuting observables, and \( n \) is a fermion-number density configuration \((n = (F(1), F(2) \ldots, F(L)))\). We will see that the \( F(l) \) belong to \( \{0, 1, 2, 3, 4\} \). The fermion-number operator is \( F = \sum_l \psi^\dagger(l)\psi(l) \).

Since it is impossible to build a no hidden-variables theorem concerning the fermion-number density (see [5] for a hint of the proof), the fermion-number density can be given the beable status. That means that any measurement of the fermion-number density at time \( t \) would simply reveal a pre-existing value

\[
n(t) = \{ n_1(t), n_2(t), \ldots, n_L(t) \}.
\]

This is the first element of reality in the description of the universe; the second element is the pilot-state \(|\Psi(t)\rangle\), which always evolves according to the Schrödinger equation

\[
 i \frac{\partial |\Psi(t)\rangle}{\partial t} = H |\Psi(t)\rangle,
\]

where \( H \) is the hamiltonian. Thus the complete description of the universe at time \( t \) would be given by the couple \((n(t), |\Psi(t)\rangle)\).

To complete the model, one must say how the real fermion-number density \( n(t) \) evolves in time. The equation of motion for \( n(t) \), called the velocity-law, must be such that the predictions of the orthodox quantum field theory are regained. Let us define \( P_n(t) \) as the probability for the universe to BE in configuration \( n \) at time \( t \). Since a measurement of the fermion-number density at time \( t \) reveals the pre-existing value \( n(t) \), the following relation

\[
P_n(t) = \sum_q |\langle n, q|\Psi(t)\rangle|^2
\]  

(2)

must be satisfied. Bell gives a stochastic equation of motion that reproduces eq. (2), provided that the initial fermion-number density configurations (at time \( t_0 \)) are distributed according to

\[
P_n(t_0) = \sum_q |\langle n, q|\Psi(t_0)\rangle|^2.
\]

This stochastic velocity-law is defined as follows. Let us take \( T_{nm}(t) \) to be the jump-rate, at time \( t \), to configuration \( n \), if the universe is in configuration \( m \)
at time $t$. If one takes

\[ T_{nm} = J_{nm} / D_m, \quad (3) \]

\[ J_{nm} = -2 \sum_{qp} \Re \{ \langle \Psi(t)|nq\rangle \langle nq| - iH|mp\rangle \langle mp|\Psi(t) \} , \]

\[ D_m = \sum_q |\langle mq|\Psi(t) \rangle|^2 , \]

if $J_{nm} > 0$, and $T_{nm} = 0$ otherwise, then it can be shown that the predictions of the orthodox quantum field theory are regained\(^4\). This constitutes the Bell model.

### 3 Further developments of the Bell model

In the continuum limit (lattice-spacing going to zero), the fermion-number density (eq. (1)) would become $\psi^\dagger(\vec{x})\psi(\vec{x})$, so the fermion-number density is directly related to the charge-density operator, except that no normal-ordering is implied in the definition of the fermion-number density (otherwise its eigenvalues would not be positive). It is worth mentioning that it is impossible to build eigenstates of the fermion-number density which are also eigenstates of the particle-number (number of electrons plus number of positrons). The proof (that it is possible to find test functions $f$ such that

\[ \left[ \int d^3\bar{x} f(\bar{x})\psi^\dagger(\bar{x})\psi(\bar{x}), N \right] \neq 0 , \]

where $N$ is the particle number) is given in the appendix A of [7]. Then one may ask why this operator is called the fermion-number density; the answer will be given in the next section.

Let us return to the lattice case. We assume that the lattice fermionic fields satisfy the following canonical anti-commutation relations

\[ \{\psi_a(k), \psi_b(l)\} = 0 \quad \{\psi_a(k), \psi^\dagger_b(l)\} = \delta_{ab}\delta_{lk} , \quad (4) \]

with $l, k \in \{1,2,\ldots,L\}$, $a,b \in \{1,2,3,4\}$, and we want to construct the eigenstates of the fermion-number density. For that purpose, we start from a state $|\Phi\rangle$ such that

\[ \psi^\dagger(l)\psi(l)|\Phi\rangle = F(l)|\Phi\rangle \]

and we apply an operator $\psi^\dagger_b(k)$ on it. From the relations (4), it can be seen that

\[ \sum_a \psi^\dagger_a(l)\psi_a(l)\psi^\dagger_b(k)|\Phi\rangle = (\delta_{kl} + F(l))\psi^\dagger_b(k)|\Phi\rangle , \]

\(^4\) The idea is to take the time-derivative of eq. (2) and to show that it is indeed verified (see [1]).
so that any of the four operators $\psi^\dagger_b(k)$ ($k$ fixed) creates a quantum of the fermion-number at site $k$. In the same way, the operators $\psi_b(k)$ are annihilators of the fermion-number at site $k$. To obtain the eigenstates of the fermion-number density, we thus have to start from a state $|\psi\rangle$ such that

$$\psi_a(l)|\psi\rangle = 0 \quad \forall \, a \in \{1, 2, 3, 4\} \quad \forall \, l \in \{1, 2, \ldots, L\}$$

and apply various creators on it. Since $\psi_a(l)$ contains operators that destroy electrons and operators that create positrons, the state $|\psi\rangle$ is simply the positronic sea $|\psi\rangle$ (all the positron states occupied):

$$\psi_a(l)|\psi\rangle = \psi_a(l)|\psi\rangle = \psi_a(l)|\psi\rangle \prod_{\vec{p},s} d^\dagger_s(\vec{p})|01\rangle = 0 \quad \forall \, a, l ,$$

where $|01\rangle$ is the usual vacuum (no electron, no positron), $s$ are helicity symbols and $d^\dagger_s(\vec{p})$ is the operator that creates a positron of momentum $\vec{p}$, energy $\sqrt{|\vec{p}|^2 + m^2}$ and helicity $(-1)^{s+1}$.

Let us consider an example: an eigenstate of the fermion-number density with fermion-number equal to 2, with one quantum localized at site $l_1$ and another at site $l_2$. There are 16 eigenstates of the fermion-number density with that eigenvalue ($F(l) = \delta_{l_1} + \delta_{l_2}$), namely

$$\psi^\dagger_{a_1}(l_1)\psi^\dagger_{a_2}(l_2)|\psi\rangle \quad a_1, a_2 \in \{1, 2, 3, 4\} .$$

It should be stressed again that these eigenstates are not eigenstates of the particle number and that it is generally impossible to take particular combinations of them in order to obtain eigenstates of the particle number, the two exceptions being the positronic sea and the electronic sea.

Since $[H, F] = 0$, the fermion-number is conserved and in accordance with the well-known super-selection rule that forbids superpositions of states with different eigenvalues of the fermion-number, we have that

$$F|\Psi(t)\rangle = \omega|\Psi(t)\rangle ,$$

where $\omega$ is a positive integer belonging to $\{0, 1, \ldots, 4L\}$. Thus $|\Psi(t)\rangle$ can always be decomposed along the eigenstates of the fermion-number density with fermion-number equal to $\omega$. Those eigenstates are

$$\psi^\dagger_{a_1}(l_1)\ldots\psi^\dagger_{a_\omega}(l_{\omega})|\psi\rangle \quad a_1, \ldots, a_\omega \in \{1, 2, 3, 4\}, l_1, \ldots, l_{\omega} \in \{1, 2, \ldots, L\} .$$

Then

$$|\Psi(t)\rangle = \frac{1}{\omega!} \sum_{a_1 l_1} \ldots \sum_{a_\omega l_\omega} |\psi^\dagger_{a_1}(l_1)\ldots\psi^\dagger_{a_\omega}(l_{\omega})\rangle \psi^\dagger_{a_1}(l_1) \ldots \psi^\dagger_{a_\omega}(l_{\omega})|\psi\rangle ,$$

where the lattice wave-function $\Psi_{a_1 \ldots a_\omega}(t, l_1, \ldots, l_{\omega})$ is antisymmetric under any transposition of two labels belonging to $\{1, \ldots, \omega\}$. 
Physically, $\omega$ is the number of negative charges that one has to add to the charge contained in the positronic sea ($2eL$) in order to obtain the charge contained in $|\Psi(t)\rangle$. In the Bell ontology, these $\omega$ negative charges jump from site to site according to the stochastic velocity-law (eq. (3)) and a measurement of the fermion-number density simply reveals their positions (or more exactly the sites that they occupy). If the fermion-number density is sufficient to describe the outputs of the measuring devices, then we have a realistic interpretation of any lattice quantum field theory involving Fermi fields. Due to the simple relation between fermion-number density and charge density, it is also strictly equivalent to say that the charge density is the beable.

Now we ask the question whether it is possible that the Bell model becomes deterministic in the continuum limit (lattice spacing going to zero) or whether it is possible to turn the stochastic Bell model into a deterministic one. There are two strong arguments in favor of that conjecture; the deterministic character of the Schrödinger equation and the conservation of the fermion-number.

4 The continuum Dirac quantum field theory

In the continuum limit, the finite lattice is replaced by a cubic box of volume $V$ and the fermion-number density operator becomes

$$\psi^\dagger(\vec{x})\psi(\vec{x}) = \sum_{a=1}^{a=4} \psi^\dagger_a(\vec{x})\psi_a(\vec{x}) ,$$

The components of the Dirac fields satisfy the following canonical anti-commutation relations

$$\{\psi^\dagger_a(\vec{x}), \psi_b(\vec{y})\} = \delta_{ab}\delta^3(\vec{x} - \vec{y}) \quad \{\psi_a(\vec{x}), \psi_b(\vec{y})\} = 0 . \quad (5)$$

Integrating the fermion-number density over the cubic box of volume $V$, we get the fermion-number

$$F = \int_V d^3\vec{x}\psi^\dagger(\vec{x})\psi(\vec{x}) . \quad (6)$$

The free hamiltonian is defined by

$$H_0 = \int_V d^3\vec{x}\psi^\dagger(\vec{x})[-i\vec{\alpha} \cdot \nabla + m\beta]\psi(\vec{x}) . \quad (7)$$

Let us take $u_s(\vec{p})e^{-iE_{\vec{p}}t}e^{i\vec{p} \cdot \vec{x}}/\sqrt{V}$ to be a free solution of the classical Dirac equation with momentum $\vec{p}$, energy $E_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$ and helicity $(-1)^{s+1}$, and $v_s(\vec{p})e^{-iE_{\vec{p}}t}e^{-i\vec{p} \cdot \vec{x}}/\sqrt{V}$ to be a free solution of the classical Dirac equation.
with momentum \(-\vec{p}\), energy \(-E_{\vec{p}}\) and helicity \((-1)^s\). For the free solutions to be orthogonal and normalized in a covariant way, the following relations

\[ u^\dagger_s(\vec{p})u_s(\vec{p}) = \delta_{rs} \frac{E_{\vec{p}}}{m}, \quad u^\dagger_s(\vec{p})u_s(-\vec{p}) = 0, \quad v^\dagger_s(\vec{p})v_s(\vec{p}) = \delta_{rs} \frac{E_{\vec{p}}}{m} \]  

must be satisfied. Since the quantum field \(\psi\) is a solution of the Dirac equation, it is a superposition of free solutions with operators as coefficients; taking

\[ \psi(t, \vec{x}) = \sqrt{\frac{1}{V}} \sum_s \sum_{\vec{p}} \sqrt{\frac{m}{E_{\vec{p}}}} \left[ c_s(\vec{p})u_s(\vec{p})e^{-iE_{\vec{p}}t}e^{i\vec{p} \cdot \vec{x}} + d_s(\vec{p})v_s(\vec{p})e^{iE_{\vec{p}}t}e^{-i\vec{p} \cdot \vec{x}} \right], \]  

with \(\{c_s(\vec{p}), d^\dagger_s(\vec{q})\} = \delta_s^r \delta_{\vec{p} \vec{q}}, \{d_s(\vec{p}), d^\dagger_s(\vec{q})\} = \delta^r_s \delta_{\vec{p} \vec{q}}\) and all other anti-commutators vanishing, the anti-commutation relations (5) are regained.

The vacuum is still called \(|0_1\rangle\). From the expressions of the various observables in the momentum space, it appears that the operator \(c^\dagger_s(\vec{p})\) \((d^\dagger_s(\vec{p}))\) creates an electron (a positron) of momentum \(\vec{p}\), energy \(E_{\vec{p}}\) and helicity \((-1)^{s+1}\).

With the help of eq. (8) and eq. (9), we find that the expression of the fermion-number (eq. (6)) in the momentum space is

\[ F = \sum_{s=1}^{s=2} \sum_{\vec{p}} [c^\dagger_s(\vec{p})c_s(\vec{p}) + d^\dagger_s(\vec{p})d_s(\vec{p})]. \]

Hence, there is only one eigenstate of the fermion-number with the lowest eigenvalue (zero), it is the positronic sea \(|ps\rangle\). The eigenstates of the fermion-number density are obtained from the positron sea by applying creators \(\psi^\dagger_a(\vec{x})\) on it. For example, there are 16 eigenstates of the fermion-number density with eigenvalue \(f(\vec{x}) = \delta^3(\vec{x} - \vec{x}_1) + \delta^3(\vec{x} - \vec{x}_2)\); those eigenstates are

\[ \psi^\dagger_{a_1}(\vec{x}_1)\psi^\dagger_{a_2}(\vec{x}_2)|ps\rangle, \quad a_1, a_2 \in \{1, 2, 3, 4\}. \]

Instead of working with positrons, we can keep the states of negative energy. This can be accomplished by making the substitutions

\[ d^\dagger_s(\vec{p}) \rightarrow \zeta^\dagger_s(-\vec{p}), \quad d_s(\vec{p}) \rightarrow \zeta_s(-\vec{p}), \]

where \(\zeta^\dagger_s(\vec{p})\) is the operator that creates an electron of momentum \(\vec{p}\), energy \(-E_{\vec{p}}\) and helicity \((-1)^{s+1}\). This is Dirac’s prescription (a hole in the Dirac sea is equivalent to a positron). Then, if that interpretation is used, another vacuum has to be defined; we call it \(|0_2\rangle\)

\[ c_s(\vec{p})|0_2\rangle = 0, \quad \zeta^\dagger_s(\vec{p})|0_2\rangle = 0, \quad \forall s, \vec{p}. \]

This vacuum \(|0_2\rangle\) is not the usual vacuum \(|0_1\rangle\); in fact, \(|0_1\rangle\) is equivalent to the Dirac sea, and can be obtained from \(|0_2\rangle\) by filling all the negative-energy
states. The expression of the fermion-number in the momentum space becomes

\[ F = \sum_{s=1}^{s=2} \sum_{\vec{p}} \left[ c_s^\dagger(\vec{p}) c_s(\vec{p}) + \zeta_s^\dagger(\vec{p}) \zeta_s(\vec{p}) \right] , \]

so that the fermion-number is the number of electrons, but of positive and negative energy. Following that interpretation, the fermion-number is really what its name implies and the beables of the Bell model would be the positions of the electrons (but of positive and negative energy). The state with lowest fermion-number, destroyed by any annihilator \( \psi_a(\vec{x}) \), is the vacuum \( |0_2\rangle \). The signification of the operator \( \psi_a^\dagger(\vec{x}) \) is that it creates an electron localized at point \( \vec{x} \). But the state \( \zeta_s^\dagger(\vec{p})|0_2\rangle \) has no direct interpretation, showing that the negative-energy electrons are not appropriated to the study of properties related to the momentum space. The point we want to make is that these negative-energy states are well suited to the study of localized properties. Let us show it. We start from the Schrödinger equation

\[ i\hbar \frac{\partial |\Psi(t)\rangle}{\partial t} = H_0 |\Psi(t)\rangle . \]

Since \([H_0, F] = 0\), and in accordance with the super-selection rule, we know that \(|\Psi(t)\rangle\) is an eigenstate of the fermion-number. Let us consider the case where there is only one quantum of the fermion-number:

\[ F|\Psi(t)\rangle = |\Psi(t)\rangle . \]

Then \(|\Psi(t)\rangle\) can be decomposed along the eigenstates of the fermion-number density with fermion-number equal to 1, which are

\[ \psi_a^\dagger(\vec{x})|0_2\rangle \quad (\vec{x} \in \mathbb{R}^3, \ a \in \{1, 2, 3, 4\}) , \]

since \(|ps\rangle = |0_2\rangle\). Thus, in our case,

\[ |\Psi(t)\rangle = \sum_a \int d^3\vec{x} \Psi_a(t, \vec{x}) \psi_a^\dagger(\vec{x})|0_2\rangle . \]

Inserting the previous equation in the Schrödinger equation, using the eq. (5), and the definition of the hamiltonian (eq. (7)), one finds that

\[ i \frac{\partial \Psi(t, \vec{x})}{\partial t} = -i\vec{\alpha} \cdot \vec{\nabla} \Psi(t, \vec{x}) + m\beta \Psi(t, \vec{x}) , \]

which is the Dirac equation. So the link is made between the first and the second quantization. And that shows that the beables of the Bell model are the same hidden variables that are used in the Bohm theory for Dirac particles (first quantization). This is self-consistent since the same observable (fermion-number density) is used in both theories.
All the occurrences of the positronic sea $|ps⟩$ will now be replaced by the vacuum $|0_2⟩$, since this point of view seems more fundamental.

From now on, the hamiltonian is not restricted to the free case ($H_0 → H = H_0 + H_I$ and the interaction terms are made of the Dirac field, for instance $H_I = g(\bar{\psi}\psi)^2$, where $g$ is a coupling constant). Since $[H, F] = 0$, the pilot-state is an eigenstate of the fermion-number (super-selection rule) and the corresponding eigenvalue is still denoted by $\omega$:

$$\int d^3\vec{x}\bar{\psi}(\vec{x})\psi(\vec{x})|Ψ(t)⟩ = \omega|Ψ(t)⟩ .$$  

A problem is that in the continuum limit, $\omega$ becomes infinite for any state containing a finite number of electrons and positrons. But let us consider the case $\omega$ finite for the moment (it will be of interest for us in the next section).

Thus $|Ψ(t)⟩$ can be decomposed along the eigenstates of the fermion-number density with fermion-number equal to $\omega$:

$$|Ψ(t)⟩ = \frac{1}{\omega!} \sum_{a_1=1}^{a_1=4} \cdots \sum_{a_\omega=1}^{a_\omega=4} \int d^3\vec{x}_1 \cdots d^3\vec{x}_\omega \Psi_{a_1 \cdots a_\omega}(t, \vec{x}_1, \cdots, \vec{x}_\omega) \psi_{a_1}^\dagger(\vec{x}_1) \cdots \psi_{a_\omega}^\dagger(\vec{x}_\omega)|0_2⟩ ,$$

where the wave function $Ψ_{a_1 \cdots a_\omega}(t, \vec{x}_1, \cdots, \vec{x}_\omega)$ is antisymmetric, under any transposition of two of the labels $1$ to $\omega$.

In the standard interpretation, the probability density to observe the universe in a configuration $(\vec{x}_1, \cdots, \vec{x}_\omega)$ is

$$\rho(t, \vec{x}_1, \cdots, \vec{x}_\omega) = \sum_{a_1=1}^{a_1=4} \cdots \sum_{a_\omega=1}^{a_\omega=4} |⟨Ψ(t)|\psi_{a_1}^\dagger(\vec{x}_1) \cdots \psi_{a_\omega}^\dagger(\vec{x}_\omega)|0_2⟩|^2 ,$$  \hspace{1cm} (10)

and we have the relation

$$\int d^3\vec{x}_1 \cdots d^3\vec{x}_\omega \rho(t, \vec{x}_1, \cdots, \vec{x}_\omega) = 1 .$$  \hspace{1cm} (11)

Its time-derivative could be deduced from the relation

$$\frac{\partial ρ(t, \vec{X})}{\partial t} + \vec{∇} \cdot \vec{J}(t, \vec{X}) = 0 ,$$  \hspace{1cm} (12)

where $\vec{X} = (\vec{x}_1, \cdots, \vec{x}_\omega)$ and $\vec{J}$ is a probability density current in the configuration space of dimension $3\omega$, provided that the fields go to zero fast enough as $|\vec{X}| → 0$. Let us consider the free case for the moment ($H$ is the free Dirac
Hamiltonian $H_0$). The time derivative of eq. (11) gives the relation:

$$\frac{d}{dt} \int d^3 \vec{x}_1 \ldots d^3 \vec{x}_\omega \rho(t, \vec{x}_1, \ldots, \vec{x}_\omega) = \sum_{a_1=1}^{a_\omega=4} \ldots \sum_{a_\omega=1}^{a_\omega=4} \int d^3 \vec{x}_1 \ldots d^3 \vec{x}_\omega$$

$$\frac{d}{dt} \left[ \langle \Psi(t_0) | \psi_{a_1}^\dagger(\vec{x}_1, t) \ldots \psi_{a_\omega}^\dagger(\vec{x}_\omega, t) | 0_2 \rangle \langle 0_2 | \psi_{a_\omega}(\vec{x}_\omega, t) \ldots \psi_{a_1}(\vec{x}_1, t) | \Psi(t_0) \rangle \right] = 0,$$

where we have switched to the Heisenberg picture. It can be simplified, knowing that

$$i \frac{\partial \psi(t, \vec{x})}{\partial t} = -i \vec{\alpha} \cdot \vec{\nabla} \psi(t, \vec{x}) + m \beta \psi(t, \vec{x}).$$

Terms containing $\beta$ cannot contribute (it amounts to take the real part of an imaginary number). Thus we obtain the following current for the $i$-th coordinate ($\vec{J}(t, \vec{X}) = (\vec{J}_1(t, \vec{X}), \ldots, \vec{J}_\omega(t, \vec{X}))$):

$$\vec{J}_i(\vec{x}_1, \ldots, \vec{x}_\omega, t) = \sum_{s_i, a_1, \ldots, a_\omega} \Psi^*_{a_1 \ldots a_\omega}(t, \vec{x}_1, \ldots, \vec{x}_\omega) \vec{\alpha}_{a_i s_i} \Psi_{a_1 \ldots a_\omega}(t, \vec{x}_1, \ldots, \vec{x}_\omega),$$

where

$$\Psi_{a_1 \ldots a_\omega}(t, \vec{x}_1, \ldots, \vec{x}_\omega) = \langle 0_2 | \psi_{a_\omega}(\vec{x}_\omega) \ldots \psi_{a_1}(\vec{x}_1) | \Psi(t) \rangle.$$

That particular form of the current is unchanged when there is an interaction term $g(\bar{\psi} \psi)^2$.

### 5 A deterministic Bell model

We have already mentioned the problem with the continuum limit: the fermion-number contained in any physical state (any state having a finite number of electrons and positrons) becomes infinite. So the fermion-number density is infinite too. In order to keep everything finite, the ultra-violet cut-off must be maintained. Nevertheless, the positions of the quanta of the fermion-number must belong to $\mathbb{R}^3$, otherwise the theory cannot be deterministic. So, let us see if we can obtain a deterministic Bell model by pursuing these two lines of thought.

The idea is the following. Take the universe to be a finite cubic box of volume $V$, made of $L$ smaller cubic boxes of volume $\lambda^3$ (where $\lambda^3$ is very small). The boxes are indexed by a label

$$l = 1, 2, \ldots, L.$$

There can be up to four quanta of the fermion-number in each box. A particular configuration of the fermion-number density is still defined by

$$n = (n_1, n_2, \ldots, n_L).$$
There are four operators that create a quantum of the fermion-number in box $l$; these are denoted by $\psi_a^\dagger(l)$ ($a \in \{1, 2, 3, 4\}$). And there can be up to $4L$ quanta of the fermion-number in the universe. Again, we take the pilot-state to be an eigenstate of the fermion-number, according to the super-selection rule, with eigenvalue $\omega$. So the solution of the Schrödinger equation amounts to give the wave-function $\Psi_{a_1...a_\omega}(t, l_1, \ldots, l_\omega)$.

At the level of the beables, the quanta of the fermion-number have well defined positions $\vec{x}_1(t), \ldots, \vec{x}_\omega(t)$ and we assume that when a quantum of the fermion-number enters the box $l$, it will certainly be detected in box $l$ but its exact position remains unknown. So we have an $\vec{X}(t) = (\vec{x}_1(t), \ldots, \vec{x}_\omega(t))$ which determines univocally the configuration $n(t)$ that will be observed, if a measurement of the fermion-number density is performed at time $t$. The determination is made with the formula

$$n_l(t) = \sum_{j=1}^{j=\omega} (1 \text{ if } \vec{x}_j(t) \cap \text{box } l \neq \emptyset) .$$

The next step is to extend the domain of definition of the wave-function $\Psi_{a_1...a_\omega}(t, l_1, \ldots, l_\omega)$, in order to obtain a wave-function $\Psi_{a_1...a_\omega}(t, \vec{x}_1, \ldots, \vec{x}_\omega)$ defined on $\mathbb{R}^{3\omega}$ that can guide the beables $\vec{x}_1(t), \ldots, \vec{x}_\omega(t)$ in the configuration space $\mathbb{R}^{3\omega}$. To do that, we replace the fields $\psi_a^\dagger(l)$ by the operators

$$\frac{1}{\lambda^3} \int_{\text{box } l} \psi_a^\dagger(\vec{x}) d^3\vec{x} ,$$

where $\psi_a^\dagger(\vec{x})$ is defined at eq. (9). For example, the eigenstate of the fermion-number density $\psi_1^\dagger(l)|0_2\rangle$ would become

$$\frac{1}{\lambda^3} \int_{\text{box } l} d^3\vec{x} \psi_1^\dagger(\vec{x})|0_2\rangle .$$

Then the complete description of the universe at time $t$ would be given by the couple $(n(t), |\Psi(t)\rangle)$, which is equivalent to $(\vec{X}(t), |\Psi(t)\rangle)$, since $n(t)$ is fixed by $\vec{X}(t)$. The next step is to give the equations of motion for these quantities. For the pilot-state, the Schrödinger equation is retained. For the vector $\vec{X}(t)$, we must choose a law that ensures that the predictions of the orthodox quantum field theory are regained. Let us consider a set of universes, labelled by an index $j$, with the same pilot-state. We can define a probability density for the universe to be in configuration $\vec{X}$ at time $t$: we call it $r(t, \vec{X})$. In order to regain the predictions of the orthodox theory, it is sufficient to say that the following condition

$$r(t, \vec{X}) = \rho(t, \vec{X})$$

must be satisfied ($\rho(t, \vec{X})$ is defined at eq. (10)). That ensures that the condition (2), which is

$$P_n(t) = \sum_q |\langle n, q|\Psi(t)\rangle|^2 ,$$

12
is also satisfied. Let us assume that the initial configurations $\vec{X}_j(t_0)$ are chosen according to the probability density

$$r(t_0, \vec{X}) = \rho(t_0, \vec{X}) .$$

Then the condition (13) is equivalent to

$$\frac{\partial r(t, \vec{X})}{\partial t} = \frac{\partial \rho(t, \vec{X})}{\partial t} .$$

(14)

We suppose that the universe moves in a deterministic way; then its velocity $\vec{V}(t)$ must be obtained from the quantities $|\Psi(t)\rangle$ and $\vec{X}(t)$. We also have the continuity equation

$$\frac{\partial r(t, \vec{X})}{\partial t} + \vec{\nabla} \cdot (r(t, \vec{X}) \vec{V}(t, \vec{X})) = 0 .$$

(15)

With the help of eq. (12) and eq. (15), eq. (14) becomes

$$\vec{\nabla} \cdot (r(t, \vec{X}) \vec{V}(t, \vec{X})) = \vec{\nabla} \cdot \vec{J}(t, \vec{X}) .$$

Thus if take the velocity

$$\vec{V}(t) = \frac{\vec{j}(t, \vec{X})}{\rho(t, \vec{X})} \bigg|_{\vec{X} = \vec{X}(t)} ,$$

all the predictions of the orthodox theory are regained.

The Bell model is non-local, but this is a necessary property of any realistic interpretation of the quantum field theory, following the EPR paradox, Bell’s inequality and related experiments. To show it explicitly, one can consider the case of two electrons in a 1 + 1 space-time. These electrons are described by the beables $x_1(t)$ and $x_2(t)$ and they move according to the velocity-law

$$v_1(t) = \frac{j_1(t, x_1, x_2)}{\rho(t, x_1, x_2)} , \quad v_2(t) = \frac{j_2(t, x_1, x_2)}{\rho(t, x_1, x_2)} .$$

Due to the exchange interaction, required by the Pauli principle, it can be shown that the current $j_1(t, x_1, x_2)$ cannot be factorized (it is impossible to find two functions $j_A$ and $j_B$ such that $j_1(t, x_1, x_2) = j_A(t, x_1)j_B(t, x_2)$), so the Bell model is non-local; the velocity of an electron, at time $t$, depends on the position of the other electron, at the same time. Since the velocity-law has the same expression as that given by Bohm for Dirac particles (first quantization), the velocity-law is not covariant (see [8] for a discussion of the Lorentz covariance of the velocity-law).

There is another deterministic interpretation of the fermionic quantum field theories, due to Holland ([9], section 10.6.2), where the fermionic field is
treated as a collection of rotators. For the simplest case of a spin 0 field quantized according to the Fermi-Dirac statistics, there is a rotator for each normal mode \( \vec{k} = \frac{2\pi}{V^{1/3}}(n_1, n_2, n_3) \), with \( n_1, n_2, n_3 \in \mathbb{R}^3 \). For each rotator, there are two independent states, \( u_{\vec{k}-}(\vec{\alpha}_{\vec{k}}) \) and \( u_{\vec{k}+}(\vec{\alpha}_{\vec{k}}) \), respectively of spin down and spin up, \( \vec{\alpha}_{\vec{k}} \) being the Euler angles of that rotator. If the rotator \( \vec{k} \) is in the spin down state, there is no particle of momentum \( \vec{k} \) in the universe and conversely if the rotator is in the spin up state, there is a particle of momentum \( \vec{k} \) in the universe. In Holland’s model, the hidden variables are the Euler angles \( \vec{\alpha}_{\vec{k}}(t) \), \( \vec{\alpha}_{\vec{k}}(t) \) determining univocally if there is a particle of momentum \( \vec{k} \) in the universe at time \( t \). Each \( \vec{\alpha}_{\vec{k}}(t) \) evolves according to a deterministic law; \( \dot{\vec{\alpha}}_{\vec{k}}(t) \) depends on the other Euler angles (at the same time) and on the state \( |\Psi(t)\rangle \).

The extension to a Dirac field seems direct; the number of hidden variables would be multiplied by four (positron, electron and their two states of spin). How does this model relate to the Bell model? In Holland’s model, which relies on a particle ontology, the specification of the Euler angles determine the number of particles present in the universe and their velocities. Clearly, Holland’s model has the virtue to show that it is possible to give an objective and deterministic interpretation of any fermionic quantum field-theoretic model based on a particle ontology, but the price to pay is that the hidden variables belong to the momentum space. But it seems to be a necessary price; once we look at localized properties, we have to expect that these properties will not commute with the particle number. Nevertheless, since any measurement is finally a measurement of positions, the Euler angles are not revealed. In the Bell model, a measurement of the charge density reveals a pre-existing value, but the link with particles is lost. Both models rely on a quantum-based ontology (quanta of the particle number in Holland’s model, quanta of the fermion number in Bell’s model), something which seems justified by the pilot-wave theory of non-relativistic quantum mechanics.

Later, another realistic (and still deterministic) interpretation of the fermionic quantum field theory, due to Valentini [11], has been advanced. It is the logical completion of Bohm’s program for the quantum field theory (fields as beables). Is there some relation between the Bell model and Valentini’s model? In [11], a Van der Waerden field \( \phi(t, \vec{x}) \) is used, instead of the usual Dirac field. It is a two-component complex field satisfying the equation

\[
\frac{\partial^2 \phi(t, \vec{x})}{\partial t^2} - (\vec{\sigma} \cdot \vec{\nabla})^2 \phi(t, \vec{x}) + m^2 \phi(t, \vec{x}) = 0.
\]

The Van der Waerden quantum field theory is supposed to be equivalent to the Dirac quantum field theory. In the Valentini model, the fields \( \phi_a(t, \vec{x}) \) and \( \phi^*_a(t, \vec{x}) \) (\( a = 1, 2 \)) are the beables. The point we want to make here is that this model and the Bell model are not equivalent. For example, in the one-quantum case, a field cannot mimic the beable \( \vec{x}(t) \). Even if the field is localized around \( \vec{x}(t_0) \) at the initial time \( t_0 \), there are solutions of the pilot-state that will make
the field spread. This is shown in [11], when the non-relativistic limit is studied and when the most probable field configurations are deduced, but also in [6], from the velocity-law.

The Bell model has also been studied in [10], where it is shown that for a non-relativistic Hamiltonian

\[
H_{n-r} = \sum_s \int d^3 \vec{x} \ C_s^\dagger(\vec{x}) \left[ -\frac{\Delta^2}{2m} + V(\vec{x}) \right] C_s(\vec{x}),
\]

where \( C_s(\vec{x}) = \int d^3 \vec{p} \ c_s(\vec{p}) e^{i \vec{p} \cdot \vec{x}} \), and for the case where there is only one electron, the Bell model is equivalent to the non-relativistic de Broglie-Bohm model. Since it is assumed that there are only electrons of positive energy, the fermion-number density commutes with the particle-number, but this only valid for that non-relativistic model. Intuitively, when there are only low-energy electrons (positive-energy electrons), it seems indeed reasonable that they cannot excite the electrons of the Dirac sea, so that the positive-energy electrons decouple from the negative-energy electrons, but a complete study should also take account of the potential energy (virtual particles).

6 Conclusion

We have shown that the Bell model is not a model concerned about the trajectories of the particles; the real beable is the charge density. And it does not commute with the particle number. That stems from general physical arguments: to measure localized properties with high precision, one has to use high energy and that leads to pairs creation.

Can one build a similar interpretation of the Klein-Gordon theory? It seems that the answer is no, for it is impossible to define a state destroyed by a charge annihilator in the Klein-Gordon theory. But this is not even necessary since all measuring devices are made of fermions. Other observables could be given the beable status, provided that they are not forbidden by no hidden-variables theorems, but it seems that only conserved quantities must be chosen for the model to remain deterministic.

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