THE OPEN-CLOSED STRING MAP REVISITED

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Abstract. Let $E$ be a monotone convex symplectic manifold. We define and study the open-closed string map OC from Hochschild homology of the wrapped Fukaya category of $E$ to symplectic cohomology. We prove OC is a symplectic cohomology module map (proved independently by Ganatra in the exact setup), and we extend Abouzaid's generation criterion from the exact to the monotone setting. We furthermore prove OC commutes with the quilt maps defined by Lagrangian correspondences from closed manifolds. As sample applications, we show the wrapped category of $\mathcal{O}(-k) \to \mathbb{C}P^n$ is cohomologically finite for $k \leq m/2$, and we prove that lifts of split-generators for the compact Fukaya category under a finite free quotient are again split-generators.

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1. Introduction

1.1. Preamble. The wrapped Fukaya category $\mathcal{W}(E)$ of a symplectic manifold $E$ with convex contact boundary, introduced in \cite{19, 8}, is of increasing importance in both symplectic topology (topology of Stein manifolds \cite{21}, the nearby Lagrangian conjecture \cite{5}) and Homological Mirror Symmetry \cite{6, 33}. The wrapped category is a version of the Fukaya category which incorporates both closed Lagrangian submanifolds and non-compact Lagrangians modelled on a Legendrian cone at infinity, and which therefore takes into account the dynamics of the Reeb flow at the contact boundary. Incorporating non-compact Lagrangians typically leads to a substantial increase in complexity; the wrapped Floer cohomology of a cotangent fibre is isomorphic to the homology of the based loop space $\mathbb{L}$, in particular it is of infinite rank.
Despite its increasing prominence, there are surprisingly few cases in which the wrapped category has been computed. Essentially complete descriptions are known for cotangent bundles \([4]\) and for punctured spheres \([6]\), whilst we have partial information for certain Stein manifolds (given as Lefschetz fibrations \([21]\) or plumbings \([11]\)), and some general structural information concerning the behaviour of the category under Weinstein surgery \([13]\).

1.2. Foundational results. Following Abouzaid, we study the open-closed string map

\[
OC_E : HH_\ast(W(E)) \to SH^\ast(E)
\]
from Hochschild homology to symplectic cohomology. This is compatible with the decomposition of the wrapped category into subcategories indexed by the spectrum \(\{\lambda_j\} \subset \Lambda\) of quantum multiplication by the first Chern class,

\[
OC_E = \bigoplus_{\lambda} OC_{\lambda} : HH_\ast(W(E)_{\lambda}) \to SH^\ast(E)_{\lambda}
\]
(we work throughout with coefficients in the Novikov field \(\Lambda\) defined in \([8,9]\)). We prove two main results.

**Theorem 1.1.** Let \(\mathcal{C} \subset W(E)_{\lambda}\) be a full subcategory. If \(OC_{\lambda}|_{\mathcal{C}}\) hits an invertible element in \(SH^\ast(E)_{\lambda}\), then \(\mathcal{C}\) split-generates \(W(E)_{\lambda}\).

The Theorem subsumes two statements. First, we extend Abouzaid’s generation criterion \([2]\) from the exact to the monotone case: as discussed further below in Section 1.3, this is a fairly substantial undertaking. Second, we prove that \(OC\) is a symplectic cohomology module map, so hitting any invertible in a given eigensummand is as good as hitting the identity.

The module structure was observed independently, in the exact case, by Ganatra \([20]\), and the underlying \(QH^\ast\)-module structure is used in forthcoming work \([7]\) on generation in the closed case.

Our second main result studies a basic compatibility of \(OC\)-maps with the maps induced by functors of Fukaya categories arising from Lagrangian correspondences. Consider a monotone symplectic manifold \(B\), a convex monotone symplectic manifold \(E\), and a closed monotone Lagrangian correspondence \(\Gamma \subset \partial B \times \partial E\).

Composing monotone orientable Lagrangians \(L \subset B\) with the correspondence \(\Gamma\) gives rise to a set \(\mathcal{L} = L \circ \Gamma \subset E\). To ensure that \(\mathcal{L}\) is a monotone orientable Lagrangian, one needs to ensure that the composition \(L \circ \Gamma\) is transverse and embedded, and that \(\mathcal{L}\) is monotone. Call \(\Gamma\) very good if this is the case for all \(L\).

**Remark 1.2.** We will discuss the condition that \(\Gamma\) is “very good” in Section 9.9. To achieve this a priori condition for all possible \(L\) in such a general context, one would need to impose a strong condition on \(\Gamma\). For instance, it holds if \(\Gamma\) is the total space of a fibre bundle over \(B\), the projection \(\Gamma \to E\) is injective, and \(\pi_1(\Gamma) \to \pi_1(E)\) has torsion kernel.

However, in practice, for the purposes of proving a generation result (such as via Corollary \([7.4]\) below) it suffices that \(\Gamma\) is “very good” just with respect to a selected collection of Lagrangians \(L \subset B\) whose compositions with \(\Gamma\) one hopes will split-generate \(W(E)\) (or a summand thereof). Indeed, one applies our machinery to the subcategory generated by the selected \(L\) and the subcategory generated by their compositions \(L \circ \Gamma\). If one can deduce that \(OC\) restricted to the second subcategory hits an invertible element, then by Theorem \([7.4]\) those compositions \(L \circ \Gamma\) actually split-generate the whole category.

Let \(OC_B : HH_\ast(\mathcal{F}(B)) \to QH^\ast(B)\) be the open-closed map from the ordinary Fukaya category of \(B\) to small quantum cohomology.
Theorem 1.3. For monotone $B$, monotone convex $E$ and very good $\Gamma \subset B \times E$ as above, and for $\gamma \in HF^*(\Gamma, \Gamma)$, there is a commutative diagram

$$
\begin{array}{ccc}
\HH_*(\mathcal{F}(B)) & \longrightarrow & \HH_*(\mathcal{W}(E)) \\
OC_B & \downarrow & OC_E \\
\QH^*(B) & \longrightarrow & \SH^*(E).
\end{array}
$$

The vertical maps are $QH^*$ respectively $SH^*$-module maps. The horizontal arrows in the diagram are constructed using the quilt theory of Wehrheim-Woodward [37, 38, 39]. There are corresponding diagrams on the subcategories for particular $*_{C_1}$-eigenvalues.

Corollary 1.4. If for some full subcategory $B \subset \mathcal{F}(B)$ the map $q_{E, \gamma} \circ OC_B$ hits an $SH^*(E)$-invertible element, then the lifts of Lagrangians $L \in B$ to $E$ via $\Gamma$ split-generate $W(E)$, and for any monotone $L \subset E$ Legendrian at infinity, the wrapped Floer cohomology $HW^*(L, L)$ has finite total rank.

The possibility of incorporating the bulk deformation class $\gamma \in HF^*(\Gamma, \Gamma)$ is essential in applications. For instance, if $E \to B$ is the total space of a negative line bundle over $B$, the undeformed composite map $q_{E} \circ OC_B$ typically vanishes identically for degree reasons, whilst the deformed map can still be non-trivial. The diagram allows further deformations (for instance bulk deformations of $\mathcal{F}(B)$ and the corresponding $OC_B$-map), although for simplicity we do not state the results in the more general case.

En route to proving Theorem 1.3 we also construct for general monotone convex $E$ the commutative diagram

$$
\begin{array}{ccc}
\HH_*(\mathcal{F}(E)) & \longrightarrow & \HH_*(\mathcal{A}) \\
OC_E & \downarrow & OC_E \\
\QH^*(E) & \longrightarrow & \SH^*(E),
\end{array}
$$

which we call acceleration diagram, which relates the compact Fukaya category $\mathcal{F}(E)$ with the wrapped category. The analogue of Theorem 1.3 also holds for the compact category (namely: if $\mathcal{C} \subset \mathcal{F}(E)_\lambda$ is a full subcategory for which $OC_{\lambda|\mathcal{L}}$ hits an invertible element in $QH^*(E)_\lambda$, then $\mathcal{C}$ split-generates $\mathcal{F}(E)_\lambda$). The following is an illustrative application of the acceleration diagram and the generation theorem for the compact category.

Corollary 1.5. Let $E$ be a convex toric Fano variety. Let $z \in \text{Crit}(W)$ be a critical point of the superpotential with non-zero critical value $W(z) \neq 0$. If the generalized eigensummand $QH^*(E)_{W(z)}$ is 1-dimensional then the Lagrangian torus $L_z$ corresponding to $z$ split-generates $\mathcal{F}(E)_{W(z)}$ and $W(E)_{W(z)}$, in particular $W(E)_{W(z)}$ is cohomologically finite.

1.3. Linear versus quadratic Hamiltonians. There are two known constructions of the wrapped category of a Stein manifold, which take advantage of Hamiltonian functions which have either linear slope at infinity or which grow quadratically on the contact cone. In either set-up one must deal with the fact that products in Floer theory are most naturally defined as maps

$$
CF^*(L, \phi_{H_1}(L)) \otimes \cdots \otimes CF^*(L, \phi_{H_k}(L)) \longrightarrow CF^*(L, \phi_{H_1+\cdots+H_k}(L)) \quad (1.1)
$$

which involve (at least) multiples of a given Hamiltonian function (used to displace $L$ slightly to achieve self-transversality), and the fact that the Floer complex on the right of (1.1) may not be isomorphic, at chain level, to the factors on the left, even if all the $H_i$ coincide. In the setting of linear slope Hamiltonians, as in [8], one defines the morphism groups
via a “telescope” (homotopy direct limit) construction, $CW^*(L, L) = \oplus_k CF^*(L, \phi_k H(L))[q]$, involving arbitrarily large multiples of a fixed Hamiltonian of linear slope; whilst in the setting of quadratic Hamiltonians, one uses a rescaling trick (conjugating by a suitable multiple of the Liouville flow) to identify the output chain complex in \([11]\) with the input complexes appearing on the left.

Unfortunately, the rescaling trick is only available in the exact case, with no analogue for a general monotone convex symplectic manifold. For instance, for negative line bundles $E \to B$, the Liouville flow does not extend across the zero-section. On the other hand, Abouzaid proved the generation criterion using quadratic Hamiltonians, accordingly restricted to the exact case. Similar remarks apply to Ganatra’s discussion of the module structure of the open-closed string map.

The proof of each of the module structure of the OC-map and the generation criterion in the telescope model is surprisingly involved: the necessary modifications are numerous and non-trivial, and constitute one of the main contributions of this paper. At the cost of appealing to Kuranishi spaces or polyfolds, the arguments we give for Theorem 1.1 would adapt to hold for general convex symplectic manifolds, without any monotonicity assumptions.

1.4. Sample applications. Whilst this paper is largely conceived as foundational, we include two illustrative applications of Theorems 1.1 and 1.3.

1) The total space of the line bundle $O(-k) \to \mathbb{C}P^m$ is monotone whenever $k \leq m$. When $k \leq m/2$, $SH^*(O(-k))$ is known explicitly through work of the first author \([27]\). Making heavy use of the module structure and compatibility with eigenvalue splittings, we compute the open-closed string map explicitly, and prove that the wrapped category is cohomologically finite, split-generated by closed Lagrangian submanifolds. By contrast, in all previous situations in which the wrapped Fukaya category has been computed (punctured spheres, certain plumbings, cotangent bundles), it either vanishes or is cohomologically infinite. The special case of $O(-1) \to \mathbb{P}^1$ had been studied previously, compare to \([34, \text{Sec.4.4}]\).

2) The quilt diagram in Theorem 1.3 can be interpreted as a “transfer-of-generation” criterion, enabling one to show that lifts of Lagrangians under a very good correspondence define split-generators in terms of geometry on the domain of the correspondence. As a simple example, we show that if a finite group $G$ acts freely by symplectomorphisms on a closed monotone symplectic manifold $B$ with quotient $\pi : B \to B/G$, and if $OC : HH_*(\mathcal{F}(B/G)) \to QH^*(B/G)$ hits an invertible element when restricted to the subcategory generated by a collection of Lagrangians $\{L_1, \ldots, L_k\} \subset B/G$, then $L_1, \ldots, L_k$ are split-generators for $\mathcal{F}(B/G)$ and their preimages $\{\pi^{-1}(L_i)\}$ split-generate $\mathcal{F}(B)$.

It is not hard to envisage other applications. In a series of papers, Abouzaid \([3, 4, 5]\) proved that the cotangent fibre $T^*_qQ$ generates the wrapped category $W(T^*Q)$ of the cotangent bundle of a closed manifold equipped with its canonical exact symplectic structure $d\theta$. From a dynamical point of view, there is a great deal of interest in the convex non-exact manifolds defined by cotangent bundles equipped with magnetic forms, i.e. by $(T^*Q, d\theta + \pi^*\sigma)$ where $\sigma \in \Omega^2(Q)$ is some closed but not necessarily exact form pulled back from the base. It seems likely that our results can be used to show a cotangent fibre still generates the wrapped category after turning on the magnetic potential.

In another direction, Morse-Bott degenerations provide one class of examples to which Theorem 1.3 can be applied, cf. Subsection 11.7, either to give information on the OC$_E$ map, or conversely, when the OC$_E$ map is known, to constrain the behaviour of quilt maps on quantum cohomology, which are notoriously hard to compute in general.
Theorem 1.3 can also be considered a prototype for the more difficult case in which the correspondence $\Gamma$ is itself allowed to be non-compact. Whilst quilt theory is not manifestly sufficiently developed at present to allow such a generalisation, Ganatra’s recent work [20] takes many steps in the necessary direction. A commutative diagram of the shape presented in Theorem 1.3 but entwining open-closed maps on wrapped categories, would be of obvious interest in mirror symmetry.

1.5. Negative line bundles. Local Calabi-Yau’s (total spaces of canonical bundles over Fano varieties) are well-known testing grounds for many aspects of mirror symmetry, with rich symplectic topology [32, 14]. Monotone negative line bundles form another class of “local symplectic manifolds”, which arise naturally in symplectic birational geometry, cf. [34]. As remarked above, whilst in known computations in the Stein case the wrapped category $\mathcal{W}(E)$ either vanishes or is extremely complicated, the line bundles $\mathcal{O}(-k) \to \mathbb{P}^n$ show qualitatively different behaviour.

**Conjecture 1.6.** Let $E \to B$ be a monotone negative line bundle. If $\mathcal{O}_B$ hits an invertible element, then $\mathcal{W}(E)$ is cohomologically finite.

Conjecture 1.6 asserts an open-string analogue of a recent result of the first author [27], who gave an explicit description of the symplectic cohomology of a general negative line bundle $E$ in terms of Gromov-Witten theory on $B$. The symplectic cohomology is a quotient of quantum cohomology, so of finite rank.

The hypothesis that $\mathcal{O}_B$ hits an invertible is a version of a “resolution of the diagonal” statement for $B$. Using the cyclicity of the Fukaya category [17], and recent work of Abouzaid-Fukaya-Oh-Ohta-Ono [7], it is equivalent to the injectivity of the closed-open string map, a natural ring map

$$\text{CO}_B : QH^*(B) \to HH^*(\mathcal{F}(B)).$$

The condition that $\mathcal{O}_B$ hits $1 \in QH^*(B)$ is known to hold in several monotone examples: it holds for toric Fano varieties $B$ by [7], and it holds for products $B_1 \times \cdots \times B_r$ if it holds for the $B_i$ by [10]. There are also non-toric examples, for instance it holds for the Fano varieties given by a quadric and a complete intersection of two even-dimensional quadrics [34].

**Remark 1.7.** Monotonicity of the line bundle $E$ is a fairly strong requirement: for simply connected $B$, any such $E$ arises as the line bundle with first Chern class $ka$, where the first Chern class of $B$ is a non-trivial multiple $K\alpha$ of a primitive class $\alpha \in H^2(B; \mathbb{Z})$, and where the multiple $k$ satisfies $1 \leq k \leq k - 1$. Typical examples are the bundles $\mathcal{O}(-k) \to \mathbb{P}^n$ for $1 \leq k \leq m$, the square root of the canonical bundle for any Spin Fano variety, or appropriate bundles over the twistor space of a hyperbolic $2n$-manifold with $n \geq 4$ [16, Prop.33].

There is a canonical Lagrangian correspondence $\Gamma$ in $B \times E$, given by the coisotropic submanifold which is the circle bundle of $E$ of suitable radius, and the sequel [28] studies the diagram of Theorem 1.3 in this special case. For instance, if $B$ and $E$ are (monotone) toric, direct computations with the Landau-Ginzburg superpotential of $E$ imply that the unique toric fibre with non-trivial Floer cohomology is indeed the image, under $\Gamma$, of the corresponding unique monotone toric fibre in $B$, cf. [28].

Any precise categorical statement of the relationship between Floer theory in $B$ and in $E$, however, is necessarily rather involved. Because of monotonicity, the Fukaya categories $\mathcal{F}(B)$ and $\mathcal{W}(E)$ both split into collections of mutually orthogonal subcategories indexed by eigenvalues of the quantum product action of the first Chern class. Consideration of the behaviour of these eigenvalues under composition with a Lagrangian correspondence, cf. Section 1.3, shows that the image of $\mathcal{F}(B)$ in $\mathcal{W}(E)$ under a functor defined by $\Gamma$ will account for only particular idempotent summands, usually at most one. In the toric examples,
whilst the monotone Lagrangian tori of interest in $B$ and $E$ are superficially matched up by
the correspondence $\Gamma$, if one looks more closely at the local systems with which these tori
must be equipped for Floer cohomology to be non-vanishing, discrepancies start to appear.
To hope to obtain the whole of $\mathcal{W}(E)$ functorially from $\Gamma$, one should apparently incorpo-
rate bulk deformations of $B$, and deformations of the correspondence $\Gamma$ itself by non-trivial
self-Maurer-Cartan elements. The resulting moduli spaces contributing to OC are typically
obstructed, and explicit computations require virtual perturbations. We intend to address
some of these issues in a sequel to this paper.

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2. Hochschild homology of $A_\infty$-categories

In this Section we summarize the general algebraic constructions that we need to describe
the geometric constructions of Section 9. In particular, the reader might like to view the
somewhat complicated looking $A_\infty$-relations of this Section simply as a transcription of the
natural bubbling phenomena shown in the Figures of Section 9.

Remark 2.1. We restrict ourselves to non-curved $A_\infty$-categories. Later, we will view Fukaya
categories as direct sums of uncurved categories indexed by the eigenvalues of quantum product
by the first Chern class.

2.1. Grading conventions for $A_\infty$-categories. Recall that an $A_\infty$-category $B$ is: a collec-
tion of objects $\text{Ob}(B)$; graded vector spaces $\text{hom}_B(X_0, X_1)$ for all $X_0, X_1 \in \text{Ob}(B)$, working
over a chosen ground field; composition maps

$$\mu_B^r : B(X_r, \ldots, X_0) \to B(X_r, X_0),$$

where we abbreviate

$$B(X_r, X_{r-1}, \ldots, X_0) = \text{hom}_B(X_{r-1}, X_r) \otimes \text{hom}_B(X_{r-2}, X_{r-1}) \otimes \cdots \otimes \text{hom}_B(X_0, X_1)$$

for $r \geq 1$ (for $r = 0$ define it to equal the ground field; and we will typically denote generators
by $x_r \otimes \cdots \otimes x_1$ or $x_{r-1} \otimes \cdots \otimes x_0$); and finally the $\mu_B^r$ must satisfy the $A_\infty$-relations:

$$\sum (-1)^{\sigma_{j-i}^r - 1} \mu_B^r(x_r \otimes \cdots \otimes x_R \otimes \mu_B^{R-S+1}(x_R, \ldots, x_S), x_{S-1}, \ldots, x_1) = 0$$

summing over all obvious choices of $R, S$, and we now explain the sign and the gradings.

Let $|x|$ be the grading of a morphism in $B$, and define $\|x\| = |x| - 1$ the reduced grading.
Then grading signs are determined by the convention that the compositions $\mu_B^r$ are acting
from the right with degree 1. Abbreviate

$$\sigma_i^j = \sigma(x_i^j) = \sum_{i \leq \ell} \|x_\ell\|,$$

which is the reduced grading of $x_j \otimes x_{j-1} \otimes \cdots \otimes x_i$. The second expression emphasizes the
letter $x$ which is used to denote the variables indexed by $i, i + 1, \ldots, j$. 


2.2. Bimodules. A $B$-bimodule $M$ is a collection of graded vector spaces $M(X, Y)$ for any objects $X, Y$ of $B$, together with multi-linear maps

$$\mu^r_s: B(X_r, \ldots, X_0) \otimes M(X_0, Y_0) \otimes B(Y_0, \ldots, Y_s) \rightarrow M(X_r, Y_s)$$

for $r, s \geq 0$, and any objects $X_i, Y_j$ of $B$, satisfying the $A_\infty$-relations (Figure 3):

$$0 = \sum (-1)^s \mu^r_{s-R+S}[x_r, \ldots, x_{R+1}, \mu^r_{s-S}(x_r, \ldots, x_s), x_{S-1}, \ldots, x_1, \mu_{s-1} y_1, \ldots, y_s] +$$

$$+ \sum (-1)^s \mu^r_{s-R+S}[x_r, \ldots, x_{R+1}, \mu^r_{s-S}(x_r, \ldots, x_1, \mu_{s-1} y_1, \ldots, y_s), y_{S+1}, \ldots, y_s] +$$

$$+ \sum (-1)^s \mu^r_{s-R+S}[x_r, \ldots, x_1, \mu_{s-1} y_1, \ldots, y_{R-1}, \mu^r_{s-S}(y_{R-1}, \ldots, y_s), y_{S+1}, \ldots, y_s]$$

summing over all obvious choices of $R, S$, and where the signs are

$$\phi = \sigma(y)^{s+1}_{S+1}, \quad \ast = \sigma(y)^{s+1}_{1} + \deg(m) + \sigma(x)^{S-1}_{1}.$$

Note: generators of the domain of $\mu^r_s$ are written $x_r \otimes \cdots \otimes x_1 \otimes m \otimes y_1 \otimes \cdots \otimes y_s$, where we underline the bimodule element.

**Example.** $M = B$ is a $B$-bimodule, with $M(X_0, Y_0) = \text{hom}_B(Y_0, X_0) = B(X_0, Y_0)$ and composition maps $\mu^r_s = \mu^r_{r+1+s}$.

A morphism of $B$-bimodules $f: M \rightarrow N$ is a collection of multi-linear maps

$$f^r_s: B(X_r, \ldots, X_0) \otimes M(X_0, Y_0) \otimes B(Y_0, \ldots, Y_s) \rightarrow N(X_r, Y_s)$$

for $r, s \geq 0$, satisfying the $A_\infty$-relations (Figure 24):

$$0 = \sum (-1)^s f^r_{s-R+S}[x_r, \ldots, x_{R+1}, \mu^r_{s-S}(x_r, \ldots, x_s), x_{S-1}, \ldots, x_1, \mu_{s-1} y_1, \ldots, y_s] +$$

$$+ \sum (-1)^s f^r_{s-R+S}[x_r, \ldots, x_{R+1}, \mu^r_{s-S}(x_r, \ldots, x_1, \mu_{s-1} y_1, \ldots, y_s), y_{S+1}, \ldots, y_s] +$$

$$+ \sum (-1)^s f^r_{s-R+S}[x_r, \ldots, x_1, \mu_{s-1} y_1, \ldots, y_{R-1}, \mu^r_{s-S}(y_{R-1}, \ldots, y_s), y_{S+1}, \ldots, y_s] +$$

$$+ \sum (-1)^{\deg(f)} \mu^r_{s-R+S}(x_r, \ldots, x_{R+1}, \mu^r_{s-S}(y_{R-1}, \ldots, y_s), y_{S+1}, \ldots, y_s)$$

summing over the obvious choices of $R, S$, and where the signs are as before.

2.3. Cyclic bar complex and Hochschild homology. Let $M$ be a $B$-bimodule. Its cyclic bar complex in dimension $n \geq 0$ is the vector space

$$\text{CC}_n(B, M) = M(X_0, X_n) \otimes B(X_n, \ldots, X_0).$$

A typical generator will be denoted $m_n \otimes x_{n-1} \otimes \cdots \otimes x_0$. We underline the element belonging to the bimodule. The grading for $\text{CC}_n(B)$ is:

$$\|m_n \otimes x_{n-1} \otimes \cdots \otimes x_0\| = \deg(m_n) + \sigma^{n-1}_0.$$

The bar differential $b: \text{CC}_n(B, M) \rightarrow \text{CC}_{n-1}(B, M)$ on a generator $m_n \otimes x_{n-1} \otimes \cdots \otimes x_0$ is

$$\sum (-1)^{r-1} \mu^r_{s-1}(m_n \otimes x_{n-1} \otimes \cdots \otimes x_{r+1} \otimes \mu^r_{s-1}(x_{r+1}, \ldots, x_s) \otimes x_{s-1} \otimes \cdots \otimes x_0 +$$

$$+ \sum (-1)^{\deg(f)} \mu^r_{s-R}(x_{r+1}, \ldots, x_0, m_{n-1}, x_{n-1}, \ldots, x_r) \otimes x_{r-1} \otimes \cdots \otimes x_s$$

summing over all obvious choices of $r, s$, where $\dagger = \sigma^{n-1}_0(\deg(m_n)) + \sigma^{n-1}_0 + \sigma^{r-1}_s$.

The Hochschild homology is the resulting homology:

$$\text{HH}_n(B, M) = \text{H}_n(\text{CC}_n(B, M); b).$$

**Example.** $\text{HH}_n(B) = \text{H}_n(\text{CC}_n(B, B); b)$ is the Hochschild homology of $B$ (see above Example).
2.4. Change of rings. Let \( \Phi : \mathcal{B} \to \mathcal{E} \) be an \( A_\infty \)-functor, meaning a map \( \Phi : \text{Ob}(\mathcal{B}) \to \text{Ob}(\mathcal{E}) \) on objects together with a grading-preserving multi-linear map

\[
\Phi^n : \mathcal{B}(X_n, \ldots, X_0) \to \text{hom}_\mathcal{E}(\Phi X_0, \Phi X_n)
\]

for each \( n \geq 1 \) (using reduced gradings), satisfying the \( A_\infty \)-relations (Figure 20):

\[
0 = \sum \mu^{k_0}_{\mathcal{N}}(\Phi^k(x_n, \ldots, x_{n-k_s+1}), \ldots, \Phi^k_1(x_{k_1}, \ldots, x_1)) - \sum (-1)^{\sigma_r r} \Phi^{n-r+s}(x_{r}, \ldots, x_{r+1}, \mu^{r-s+1}_\mathcal{B} f \cdot \cdot \cdot f) \cdot \cdot \cdot (x_{s}, x_{s-1}, \ldots, x_1)
\]

summing over all partitions \( k_1 + \cdots + k_s = n \) and all \( \tilde{n} \).

**Lemma 2.2.** An \( A_\infty \)-functor \( \Phi : \mathcal{B} \to \mathcal{E} \) induces a change of rings which turns an \( \mathcal{E} \)-bimodule \( \mathcal{N} \) into a \( \mathcal{B} \)-bimodule \( \tilde{\mathcal{N}} \). Moreover, there is a tautological chain map

\[
\text{CC}_* (\mathcal{B}, \tilde{\mathcal{N}}) \to \text{CC}_* (\mathcal{E}, \mathcal{N}).
\]

**Proof.** For any \( X, Y \in \text{Ob}(\mathcal{B}) \), define

\[
\tilde{\mathcal{N}} (X, Y) = \mathcal{N}(\Phi X, \Phi Y).
\]

We extend the natural map \( \mu^{00}_\mathcal{N} = \text{id} : \tilde{\mathcal{N}} (X, Y) \to \mathcal{N}(\Phi X, \Phi Y) \) to define new \( \mu^{r,s}_\mathcal{N} \):

\[
\mu^{r,s}_\mathcal{N} = \sum \mu^{r,s}_\mathcal{N} \circ \left( \Phi^{k_r} \otimes \cdots \otimes \Phi^{k_1} \otimes \mu^{00}_\mathcal{N} \otimes \Phi^{\ell_1} \otimes \cdots \otimes \Phi^{\ell_s} \right)
\]

summing over all partitions \( k_1 + \cdots + k_r = r \) and \( \ell_1 + \cdots + \ell_s = s \), and all \( \tilde{r}, \tilde{s} \).

The tautological chain map \( \text{CC}_n (\mathcal{B}, \tilde{\mathcal{N}}) \to \text{CC}_n (\mathcal{E}, \mathcal{N}) \) is given by

\[
\sum \mu^{00}_\mathcal{N} \otimes \Phi^{\ell_1} \otimes \cdots \otimes \Phi^{\ell_s}
\]

summing over all partitions \( \ell_1 + \cdots + \ell_s = n \) and all \( \tilde{n} \). The fact that this is a chain map is a consequence of the \( A_\infty \)-relations satisfied by \( \Phi \).

\[ \square \]

2.5. Functorial properties of Hochschild homology.

**Lemma 2.3.** A morphism of \( \mathcal{B} \)-bimodules \( f : \mathcal{M}_1 \to \mathcal{M}_2 \) induces a chain map

\[
\text{CC}_* (f) : \text{CC}_* (\mathcal{B}, \mathcal{M}_1) \to \text{CC}_* (\mathcal{B}, \mathcal{M}_2).
\]

**Proof.** Define

\[
\text{CC}_n (f) : \mathcal{M}_1(X_n, X_n) \otimes \mathcal{B}(X_n, \ldots, X_0) \to \text{CC}_n (\mathcal{B}, \mathcal{M}_2)
\]

by sending \( m_n \otimes x_{n-1} \otimes \cdots \otimes x_0 \) to

\[
\sum (-1)^{\sigma_r r} f^{r,s}(x_{r-1} \otimes \cdots \otimes x_0 \otimes m_n \otimes x_{n-1} \otimes \cdots \otimes x_{n-s-1} \otimes \cdots \otimes x_r)
\]

where \( \sigma_0 r^{-1} (\deg(m_n) + \sigma_n n^{-1}) + \deg(f) r^{-s-1} \). This is a chain map.

\[ \square \]

Let \( \mathcal{M} \) be a \( \mathcal{B} \)-bimodule, and \( \mathcal{N} \) an \( \mathcal{E} \)-bimodule. Lemma 2.2 defines \( \tilde{\mathcal{N}} \).

**Corollary 2.4.** A map \( f : \mathcal{M} \to \tilde{\mathcal{N}} \) of \( \mathcal{B} \)-bimodules induces natural homomorphisms

\[
\text{HH}_* (\mathcal{B}, \mathcal{M}) \to \text{HH}_* (\mathcal{B}, \tilde{\mathcal{N}}) \to \text{HH}_* (\mathcal{E}, \mathcal{N}).
\]

**Proof.** At the chain level, the first map is \( \text{CC}_* (f) \), determined by Lemma 2.3 and the second map is the tautological map of Lemma 2.2.

\[ \square \]
2.6. Tensor products. This section is needed only in the proof of the generation criterion

Let \( A \) be an \( A_{\infty} \)-category. Suppose \( L, R \) are respectively a left and a right \( A \)-module.

This means to each object \( X \) in \( A \) we have associated objects \( L(X), R(X) \) of \( A \), and maps

\[
\mu^{r,s}_{L} : A(X_r, \ldots, X_0) \otimes L(X_0) \to L(X_r), \quad \mu^{s}_{R} : R(Y_0) \otimes A(Y_0, \ldots, Y_s) \to R(Y_s).
\]

We now define: a chain complex \( R \otimes_A L \), and an \( A \)-bimodule \( L \otimes R \), built so that

\[
HH_*(A, L \otimes R) \cong H^*(R \otimes_A L).
\]

The chain complex is given by

\[
R \otimes_A L = \bigoplus_{X \in \text{Ob}(A)} R(X_n) \otimes A(X_n, \ldots, X_0) \otimes L(X_0)
\]

graded by the sum of the degrees, using reduced degrees for \( A \), and the differential (of degree +1) on a generator \( x_n \otimes x_1 \otimes 1 \) is defined by

\[
\sum (-1)^{|x_n|+|x_1|} \mu^{n-1}_{L}(x_n, x_1) \otimes x_{n-1} \otimes \cdots \otimes x_1 \otimes 1 +
\]

\[
\sum (-1)^{|x_n|+|x_1|} \mu^{n+1}_{L}(x_n, x_1) \otimes x_{n+1} \otimes \cdots \otimes x_1 \otimes 1
\]

summing over the obvious \( R, S \). The \( A \)-bimodule is defined by

\[
(L \otimes R)(X, Y) = L(X) \otimes R(Y)
\]

with composition maps

\[
\mu^{r,s}_{L \otimes R} : A(X_r, \ldots, X_0) \otimes L(X_0) \otimes R(Y_0) \otimes A(Y_0, \ldots, Y_s) \to L(X_r) \otimes R(Y_s)
\]

defined on generators by

\[
\mu^{r,s}_{L \otimes R}(x_r, \ldots, x_1, 1) = 1 \otimes \mu^{s}_{R}(x_1, \ldots, y_s)
\]

and \( \mu^{r,s}_{L \otimes R} \) is zero whenever \( r, s \) are both non-zero. There is an obvious map from the cyclic bar complex for \( L \otimes R \) to the chain complex \( R \otimes A L \) which simply reorders the positions of \( 1 \) and \( \mathbb{1} \) (with the obvious sign change due to reordering dictated by the gradings). After the reordering, the bar differential becomes precisely the differential defined for \( R \otimes_A L \). Thus \( HH_*(A, L \otimes R) \cong H^*(R \otimes_A L) \).

2.7. A remark about gradings. For ease of reference, we summarize here all grading conventions and results arising in the paper, and we do not discuss these further (we refer to \[30][2][8] for a detailed discussion of gradings). On first reading this section should be skipped.

(1) The Novikov variable \( t \) for \( E \) has cohomological degree \( |t| = 2\lambda_E \);
(2) Unreduced gradings for Lagrangian Floer cohomology are as in Seidel \[30\];
(3) By convention, \( QH^*(B) \cong HF^*(H^B) \) has degree 0;
(4) and so \( c^* : QH^*(E) \to SH^*(E) \) has degree 0;
(5) Products preserve unreduced gradings, units lie in unreduced grading 0;
(6) We never reduce gradings of module inputs/outputs or of \( SH^* \);
(7) We reduce gradings of morphisms of \( A_{\infty} \)-categories, so all \( \mu^d \) have degree +1;
(8) \( A : HH_*(\mathcal{F}(E)) \to HH_*(\mathcal{W}(E)) \) has degree 0;
(9) \( CO_E : SH^*(E) \to HW^*(K, E) \) has degree 0 (unreduced gradings);
(10) \( OC_E : HH_*(\mathcal{W}(E)) \to SH^*(E) \) has degree \( \dim \mathbb{C}E \);
(11) \( OC_B : HH_*(\mathcal{F}(B)) \to QH^*(B) \) has degree \( \dim \mathbb{C}B \);
(12) \( \Delta^E_{L} : \mathcal{W}(L_r, \ldots, L_0) \otimes \mathcal{W}(L_0, L'_0) \otimes \mathcal{W}(L'_0, \ldots, L'_r) \to \mathcal{W}(L_r, K, L'_r) \) has degree \( \dim \mathbb{C}E \)
(the underlined terms are viewed as modules, so their degrees are not reduced).
(13) $H^*(q_{\Gamma, \gamma}) : QH^*(B) \to QH^*(E)$ has degree $\dim_{C}E - \dim_{C}B + k$ if $\gamma \in HF^k(\Gamma, \Gamma)$;
(14) $\HH_*(f_{\Gamma, \gamma}) : \HH_*(\mathcal{F}(B)) \to \HH_*(\mathcal{F}(E))$ has degree $k$ if $\gamma \in HF^k(\Gamma, \Gamma)$.

3. The wrapped Fukaya category

We now define the wrapped Fukaya category $\mathcal{W}(E)$ of a monotone symplectic manifold $E$ which is conical at infinity. We will mimic the construction of Abouzaid-Seidel \cite{2} which dealt with exact $E$. The construction of Abouzaid \cite{2} for exact $E$ using quadratic Hamiltonians is somewhat simpler than \cite{3}, but relies heavily on having a globally defined Liouville flow (which is used to offset the problem that the $A_\infty$-composition maps increase the slope of the Hamiltonian). This construction cannot be applied to monotone $E$, because the Liouville flow is not globally defined: it is only defined outside of a compact subset of $E$.

3.1. Symplectic manifolds conical at infinity. $(E, \omega)$ will always denote a (non-compact) symplectic manifold which is conical at infinity, which means outside of a bounded domain $E^{\text{in}} \subset E$ there is a symplectomorphism

$$\psi : (E \setminus E^{\text{in}}, \omega|_{E \setminus E^{\text{in}}}) \cong (\Sigma \times [1, \infty), d(R\alpha)).$$

where $(\Sigma, \alpha)$ is a contact manifold, and $R$ is the coordinate on $[1, \infty)$.

The conical condition implies that outside of $E^{\text{in}}$ the symplectic form becomes exact:

$$\omega = d\Theta$$

where $\Theta = \psi^*(R\alpha)$. It also implies that the Liouville vector field $Z = \psi^*(R\partial_R)$ (defined by $\omega(Z, \cdot) = \Theta$) will point strictly outwards along $\partial E^{\text{in}}$. It also implies that $\psi$ is induced by the flow of $Z$ for time $\log R$, so we can write $\Sigma = \partial E^{\text{in}}$, $\alpha = \Theta|_{\Sigma}$ (pull-back).

We will call $E \setminus E^{\text{in}}$ the conical end of $E$.

By conical structure $J$ we mean an $\omega$-compatible almost complex structure on $E$ (so $\omega(\cdot, J\cdot)$ is a $J$-invariant metric) satisfying the contact type condition $J^*\Theta = dR$ for large $R$. On $\Sigma$ this implies $JZ = Y$ where $Y$ is the Reeb vector field for $(\Sigma, \alpha)$ defined by $\alpha(Y) = 1$, $d\alpha(Y, \cdot) = 0$.

By choosing $\alpha$ or $\Sigma$ generically, one ensures that $\alpha$ is sufficiently generic so that the periods of the closed orbits of $Y$ form a countable closed subset of $[0, \infty)$.

3.2. Monotonicity assumptions, Maslov index. We always assume $(E, \omega)$ is monotone:

$$c_1(TE)[u] = \lambda_E \omega[u] \text{ for all } [u] \in \pi_2(E)$$

for some monotonicity constant $\lambda_E > 0$ in $\mathbb{R}$, where the notation means we are integrating the cohomology class over the sphere $u$.

A submanifold $L \subset (E, \omega)$ is Lagrangian if $\dim_{\mathbb{R}}L = \dim_{\mathbb{C}}E$ and $\omega|_L = 0$. We will always assume that the Lagrangians are monotone, meaning:

$$\int_{D} u^*\omega = \mu([u]) \lambda_L \text{ for } [u] \in \pi_2(E, L)$$

for some constant $\lambda_L > 0$, where $\mu$ is the Maslov index.

Recall the Maslov index $\mu(u) \in \mathbb{Z}$ is associated to any continuous disc $u : (\mathbb{D}, \partial \mathbb{D}) \to (E, L)$, and $\mu(u)$ is even for orientable $L$. The Maslov index only depends on the $\pi_2(E, L)$ class of $u$. Moreover “gluing in a sphere” $v : S^2 \to E$ causes $\mu(v\#u) = 2c_1(TE)([v]) + \mu(u)$. This gluing relation forces $E$ to be monotone, and the monotonicity constants are related by:

$$\lambda_L = \frac{1}{2\lambda_E}$$
3.3. The Novikov ring. We will always work over the field
\[ \Lambda = \{ \sum_{i=0}^{\infty} a_i t^{n_i} : a_i \in \mathbb{K}, n_i \in \mathbb{R}, \lim n_i = \infty \}, \]
called the Novikov ring, where \( \mathbb{K} \) is some chosen ground field, and \( t \) is a formal variable. The choice of \( \mathbb{K} \) will not matter, except in the toric examples at the end of the paper, where we will specialize to \( \mathbb{K} = \mathbb{C} \).

The Novikov ring is graded by placing \( t \) in (real) degree \( |t| = 2\lambda_E \). In all future Floer constructions, solutions will be counted with Novikov weights. For example for the Floer differential, the Floer solutions \( u \) are counted with weight \( \pm t^{w[u]} \), which lies in degree \( 2\lambda_E \omega[u] = 2c_1(TE)[u] \). We will not discuss the orientation signs \( \pm \) in this paper (see [30, 8, 26]).

3.4. The objects \( \text{Ob}(\mathcal{W}(E)) \): the choice of Lagrangians. The objects of \( \mathcal{W}(E) \) are the closed monotone orientable Lagrangian submanifolds of \( E \), together with all non-compact monotone orientable Lagrangian submanifolds \( L \subset E \) which are conical at infinity, by which we mean:

1. \( L \) intersects \( \Sigma \) transversally;
2. the pull-back \( \Theta|_L = 0 \) near \( \Sigma \) and on the conical end.

The above conditions ensure that \( \ell = L \cap \Sigma \) is a Legendrian submanifold of the contact manifold \( (\Sigma, \Theta|_{\Sigma}) \), and that \( L \) has the form \( \ell \times [1 - \epsilon, \infty) \subset \Sigma \times [1 - \epsilon, \infty) \) near the conical end. Conversely, a monotone Lagrangian \( L \subset E^{\text{in}} \) with \( \Theta|_L = 0 \) near \( \Sigma \) can be extended to an \( L \) of the above form. In fact, any monotone Lagrangian \( L \subset E^{\text{in}} \) which intersects \( \Sigma \) in a Legendrian submanifold, can be Hamiltonianly isotoped in \( E^{\text{in}} \) relative to \( \partial E^{\text{in}} \) so that \( \Theta|_L \) will vanish near \( \Sigma \). This can be proved as in [8] Lemma 3.1 (it is a local argument, and the Liouville flow is defined near \( \Sigma \)).

Finally, as will be explained in detail in [8, 26], one must in fact restrict \( \text{Ob}(\mathcal{W}(E)) \) to Lagrangians having the same \( m_0 \)-value.

Remark 3.1 (Grading, characteristic). For \( \mathcal{W}(E) \) to be graded one would require \( 2c_1(E, L) = 0 \in H^2(E, L; \mathbb{Z}) \). For \( \mathcal{W}(E) \) to be defined over a field of characteristic \( \neq 2 \) one would require \( w_2(L) = 0 \in H^2(L; \mathbb{Z}/2) \) and then an \( L \) should come with a choice of Pin structure. For these details we refer to [30, 8]. We will not discuss gradings or orientation signs for sake of brevity.

3.5. Reeb chords and Hamiltonian chords. Fix a Hamiltonian \( H : E \to [0, \infty) \) which on the conical end has the form \( H(\sigma, R) = R^2 \) for \( (\sigma, R) \in \Sigma \times [1, \infty) \). Later, in 4.2 (see the Convention), we will make a global rescaling of \( H \), but the reader can ignore this for now.

Let \( X \) be the Hamiltonian vector field: \( \omega(\cdot, X) = dH \). The Reeb vector field on \( \Sigma \) is \( X|_{\Sigma} \).

Fix a countable collection \( L_i \in \text{Ob}(\mathcal{W}(E)) \), so they give rise to \( \ell_i \subset \Sigma \).

A Reeb chord of period \( w \) is a trajectory \( [0, 1] \to \Sigma \) of \( wX|_{\Sigma} \) with ends on \( \ell_i, \ell_j \) (for some, possibly non-distinct, indices \( i, j \)). It is an integer Reeb chord if \( w \) is an integer. By rescaling \( \omega \) (or by choosing \( \Sigma \) generically), there are no integer Reeb chords.

An integer X-chord of weight \( w \) is an orbit \( [0, 1] \to E \) of \( wX \) with ends on \( L_i, L_j \) for integer \( w \geq 1 \). These lie in \( E^{\text{in}} \) (otherwise, there would be one in the conical end projecting to an integer Reeb chord in \( \Sigma \)). For a generic choice of \( H \) (see [8]), integer X-chords are non-degenerate and any point of \( L_i \) cannot be both an endpoint and a starting point of two (possibly non-distinct) integer X-chords. So only finitely many integer X-chords have ends on given \( L_i, L_j \), and they lie in \( E^{\text{in}} \).

3.6. The Floer differential \( \delta \). Given \( L_i, L_j \in \text{Ob}(\mathcal{W}(E)) \), let \( CF^*(L_i, L_j; wH) \) denote the free \( \Lambda \)-module generated by the (finitely many) integer X-chords of weight \( w \) with ends on
Let $L_i, L_j$. Fix a conical structure $J$ on $E$ (we remark that the contact type condition in \cite[3.1]{AFOO}) on the conical end is equivalent to $JZ = X$. The chain differential
\[ \partial : CF^*(L_i, L_j; wH) \to CF^{*+1}(L_i, L_j; wH) \]
is defined as follows on a generator $y$. Let $u : \mathbb{R} \times [0, 1] \to E$ be an isolated (modulo $\mathbb{R}$-translation) solution of Floer’s equation $\partial_u + J(\partial_t u - wX) = 0$, satisfying: Lagrangian boundary conditions $u(\cdot, 0) \in L_i$ and $u(\cdot, 1) \in L_j$; and asymptotic conditions $u \to x, y$ as $s \to -\infty, +\infty$ respectively, where $x$ is an $X$-chord. Then $u$ contributes $\pm t^\omega[u] x$ to $\partial_y$, where $\omega|u] = \int u^*\omega \in \mathbb{R}$. We will always omit discussing orientation signs $\pm$, which is done carefully in \cite[3.7]{AFOO} by replacing each $\Lambda$ by an orientation line.

If $\partial^2 = 0$, then the Floer cohomology of $L_i, L_j$ using $wH$ is defined as
\[ HF^*(L_i, L_j; wH) = H^*(CF^*(L_i, L_j; wH); \partial) \]Since $E$ is monotone, $\partial^2 = 0$ precisely when $L_i, L_j$ have the same $m_0$-value (which we define and explain in \cite[3.7]{AFOO}).

3.7. The role of monotonicity. We keep the discussion quite general by supposing that we have a 1-dimensional family of Floer-type solutions (such as those arising in the maps we just mentioned) and we study the possible bubbleings.

First observe that sphere bubbling is ruled out for dimension reasons, since by monotonicity a non-constant holomorphic sphere will increase the index of the family by at least 2. So only disc bubbling can arise.

By disc bubble, we mean a non-constant holomorphic disc $u : D \to E$, bounding a single Lagrangian $u(\partial D) \subset L$ say, with one boundary marked point $z_0 \in \partial D$ landing at a prescribed generic point $p_L = u(z_0) \in L$ (so $z_0$ plays the role of the node when our family breaks). The index of families is additive, so only isolated discs will bubble off in our 1-dimensional family and so, by monotonicity and a dimension count, the disc must have Maslov index $\mu(u) = 2$.

Now consider the moduli space $M_1(\beta)$ of all such Maslov 2 discs in the class $\beta \in \pi_2(E, L)$ but do not impose a condition on the value at the marked point. Let $ev : M_1(\beta) \to L$ be the evaluation at the marked point. Then $ev_*[M_1(\beta)]$ determines a locally finite cycle in $L$ (it is a cycle and not just a chain, because $M_1(\beta)$ cannot have boundary degenerations, since by monotonicity 2 is the minimal Maslov index of non-constant disc bubbles). For dimension reasons this lf-cycle must be a scalar multiple of $[L]$ (viewed as a locally finite cycle since $L$ may be non-compact - we explain this in \cite[3.8]{AFOO}). Then define
\[ m_0(L) = \sum t^{\mu(\beta)} ev_*[M_1(\beta)] = m_0(L)[L] \in C^H_{\dim(L)}(L; \Lambda), \text{ where } m_0(L) \in \Lambda. \]

When there are no such $\mu = 2$ discs, then $m_0(L) = 0$. If this holds for all objects $L$ then the $A_\infty$-category is called unobstructed, and no disc bubbling discussion is necessary. In the applications we have in mind, we typically expect $\mu = 2$ discs. So we briefly explain why in the monotone case, thanks to the machinery of Fukaya-Oh-Ohta-Ono \cite{AFOO}, all bubbling contributions will in fact cancel out, so one can ignore all contributions coming from $m_0(L)$. In other words, monotonicity allows us to pretend that our $A_\infty$-categories $\mathcal{W}(E, \lambda)$ are all unobstructed.

Using monotonicity to deduce that the only non-constant holomorphic discs bounding $L$ have Maslov index at least 2, the machinery of Fukaya-Oh-Ohta-Ono \cite{AFOO} then yields a chain level model for $HF^*(L, L)$ using locally finite chains $C^H_\text{lf}(L)$ for which $HF^*(L, L)$ is a well-defined ring in which $[L]$ represents the unit. We briefly explain this. At the chain level the first $A_\infty$-equation for $L$ is
\[ \mu^1 \circ \mu^2(x) = -\mu^2(x, m_0(L)) + (-1)^{|x|} \mu^2(m_0(L), x), \]
for $x \in C^0_\mu(L)$. The fact that the right-hand side vanishes relies on the observation that these $\mu^2$ terms come from counting discs with three boundary marked points, but for which the incidence condition $[L]$ is automatic. So forgetting that unnecessary marked point shows that we are sweeping a chain contained inside $\mu^1(x)$, which has dimension one less than $x$ (unless the disc is constant). So only constant discs can contribute, and so it turns out that $\mu^2(x, [L]) = x$ and $\mu^2([L], x) = (-1)^{|x|}x$, so the above right-hand side becomes a multiple of $x - x = 0$. So $\mu^1$ really is a differential.

Similarly, feeding a copy of $[L]$ (or a copy of $m_0([L])$) into an entry of any of the higher $\mu^d$ composition maps (using only copies of $\mathcal{CF}^*(L, L)$), then again we get zero, and for dimension reasons, $\mu^1([L]) = 0$. So $[L]$ is a strict unit for the subcategory generated by $L$ [50 Sec.2a].

These arguments generalize to the case when one considers different Lagrangians: for $\mathcal{CF}^*(L, L')$ one gets $\mu^1 \circ \mu^1 = 0$ provided that $m_0(L) = m_0(L')$. So $HF^*(L, L')$ is only defined if $m_0(L) = m_0(L')$, and similarly the higher order compositions $\mu^n$ are only defined when all the Lagrangians have the same $m_0$ value. So the (wrapped) Fukaya category is actually a family of $A_\infty$-categories indexed by the possible $m_0$ values.

Since this is central to our treatment, we state it as a Lemma (the observation relating $m_0$ to the eigenvalues of $c_1$ is due to Kontsevich, Seidel and Auroux [12 Sec.6]):

**Lemma 3.2.** The category $\mathcal{W}(E)$ is a family of $A_\infty$-categories $\mathcal{W}(E, \lambda)$ indexed by $\lambda \in \Lambda$, where $\mathcal{W}(E, \lambda)$ is generated by the Lagrangians $L$ with $m_0(L) = \lambda$. Since $E$ is monotone, the non-triviality of $\mathcal{W}(E, \lambda)$ implies that $\lambda$ is an eigenvalue of quantum cup product by $c_1$:

$$c_1(TE) : \mathcal{QH}^*(E) \to \mathcal{QH}^*(E).$$

The analogous statement holds for the compact category $\mathcal{F}(E)$ or the Fukaya category $\mathcal{F}(B)$ of a closed (rather than convex) monotone manifold $B$.

**Convention.** From now on, we always treat $\mathcal{W}(E)$ as a non-curved $A_\infty$-category which splits into non-interacting pieces, not as a curved category. So it is always assumed that when we consider maps on $\mathcal{W}(E)$, we in fact mean some $\mathcal{W}(E, \lambda)$ so that all the Lagrangians involved have the same $m_0$ value. The same comment pertains to $\mathcal{F}(B)$ and $\mathcal{F}(E)$.

Our discussion shows that the $A_\infty$-equations for the wrapped category naturally involve also terms containing $m_0$’s, but that these terms in fact cancel out (since we assume that all Lagrangians involved have the same $m_0$-value). This implies that also, for instance, the bar differential for $CC_\ast(\mathcal{W}(E))$ will have terms involving $m_0$’s, and that these cancel.

3.8. The construction of $m_0$. Finally we make some technical remarks about why $m_0$ is defined in the non-compact setup (the machinery of Fukaya-Oh-Ohta-Ono [18] constructs $m_0$ in the closed monotone setup). We begin with an observation:

**Lemma 3.3.** Let $E$ be conical at infinity. Let $L \subset E$ be any Lagrangian submanifold conical at infinity [14]. Then all pseudo-holomorphic discs $u : (\mathbb{D}, \partial \mathbb{D}) \to (E, L)$ have $u(\partial \mathbb{D})$ lying either in $E_{\infty}$ or in a contact hypersurface $\Sigma \times \{R\}$ of the conical end.

**Proof.** For general reasons of convexity [8 Sec.7c], which essentially reduce to Green’s formula, any pseudo-holomorphic map $v : S \to E$ defined on a Riemann surface $S$ (which can have corners) with $v(S) \subset \Sigma \times \{R, \infty\}$ landing in the conical end of $E$, with $v(\partial S) \subset L \cup (\Sigma \times \{R\})$, must in fact lie entirely in $\Sigma \times \{R\}$. This maximum principle uses the fact that $\Sigma \times \{R\}$ is a contact hypersurface in $E$ on which the Liouville flow is outward-pointing. The claim follows immediately by taking $v$ to be the restriction of $u$ to $u^{-1}(\Sigma \times \{R, \infty\})$ (for generic $R$ so that $\Sigma \times \{R\}$ intersects the image of $u$ transversely).
By the Lemma, given a point $p_L$ of $L$, all holomorphic discs bounding $L$ satisfying $p_L \in u(\partial D)$ must lie entirely in a compact region of $E$ determined by $p_L$. Thus all the constructions relating to the moduli space of holomorphic discs and in particular the construction of $\mathfrak{m}_0(L)$ reduce to the constructions in the closed case, provided we replace cycles/chains by locally finite cycles and locally finite chains. For example, suppose $L, E$ are monotone. Then, using the analysis [13] known for the closed setup, $\mathfrak{m}_0$ is, up to a scalar multiple, a sum of chains which on each compact $C \subset E$ is building the fundamental cycle for the pair $(C \cap L, \partial C \cap L)$. Exhausting $E$ by a sequence of larger and larger compacts $C$ proves that $\mathfrak{m}_0(L)$ is, up to a scalar multiple, an infinite sum of chains which is precisely the locally finite cycle $[L]$.

3.9. The homotopy $\mathcal{R}$. Denote $\mathcal{R}$ the Floer continuation map

$$\mathcal{R} : CF^*(L_i, L_j; wH) \to CF^*(L_i, L_j; (w+1)H)$$

determined by a monotone homotopy from $(w+1)H$ to $wH$. This involves counting isolated solutions similar to those in the definition of $\mathcal{R}$, except now $J, H$ depend on the domain coordinate $s$ and there is no $\mathbb{R}$-reparametrization. Section 3.11 describes the auxiliary data carefully (for $\mathcal{R}$ one uses: $d = 1$, $F = \{1\}$, $w_0 = w + 1$, $w_1 = w$).

The maps $\mathcal{R}$ are the chain maps which at the level of cohomology are used to define the cohomology direct limit as $w \to \infty$:

$$HW^*(L_i, L_j) = \lim_{\to} HF^*(L_i, L_j; wH)$$

3.10. The telescope construction of the wrapped Floer complex. Because the $A_\infty$-operations act on the chain level and require changing the slope $w$ of $wH$, the wrapped Floer complex has to be built as a homotopy direct limit, in other words by a telescope construction (rather than taking the direct limit at the level of cohomology). Following [8, Sec.3g, Sec.5a],

**Definition 3.4.** The wrapped complex is

$$CW^*(L_i, L_j) = \bigoplus_{w=1}^\infty CF^*(L_i, L_j; wH)[q]$$

where $q$ is a formal variable of degree $-1$ satisfying $q^2 = 0$. This is a module over the exterior algebra $\mathbb{C}[\partial q]$, where the differentiation operator $\partial_q$ has degree $+1$. The chain differential is chosen to respect this module structure:

$$\mu^1(x + qy) = (-1)^{|x|} \partial x + (-1)^{|y|} (q \partial y + \mathcal{R} \partial y - y)$$

(using unreduced degrees)

Observe that for a $\partial$-cocycle $y$, the role of $qy$ is to identify $y$ and $\mathcal{R}y$ at the cohomology level, as expected in the cohomology direct limit; whereas the role of the subcomplex $(\partial_q = 0)$ is to yield representatives of $\oplus HF^* (L_i, L_j; wH)$. Indeed, recall [8 Sec.3g] the following about $C^b_a = \oplus_{w=a}^{b-1} CF^*(L_i, L_j; wH)[q] \oplus CF^*(L_i, L_j; bH)$:

1. $H^*(CW^*(L_i, L_j); \mu^1) \cong \lim \mu^1(C^b_a)$;
2. $C^b_a$ is quasi-isomorphic to $CW^*(L_i, L_j)$;
3. $C^b_a$ is quasi-isomorphic to $CW^*(L_i, L_j)$;
4. $C^b_a \subset C^{a-1}_b$ is a quasi-isomorphism for $a \leq b$, hence so is $C^b_b \subset C^b_0$;
5. $C^b_b \subset C^b_1$ is a quasi-isomorphism;
6. $\mathcal{R} : C^b_b \to C^{b+1}_b$ and $C^b_b \subset C^{b+1}_b$ commute up to [8], up to chain homotopy;
7. $H^*(CW^*(L_i, L_j); \mu^1) \cong HW^*(L_i, L_j)$ canonically;

[1] is because $C^b_0$ is an exhausting increasing filtration of $CW^*(L_i, L_j)$. The $C^b_{\infty}$, $C^b_0$ are decreasing filtrations for $CW^*(L_i, L_j)$, $C^1_b$ respectively; so [5]-[5] follow: successive quotients are cones on the identity, so acyclic. Combining yields [7].
3.11. The auxiliary data. By [8], there is a universal family $\mathcal{R}^{d+1,p} \to \mathcal{R}^{d+1}$ over the moduli space of stable discs, with elements $(S, F, p, \phi)$ where (Figure 1):

- $F$ is a finite set of indices $f$ (with $d \geq 1$, $d + |F| \geq 2$);
- A map $p : F \to \{1, \ldots, d\}$ defining labels $p_f$ (possibly not distinct);
- $S$ is a Riemann surface isomorphic to $\mathbb{D}$ with $d + 1$ distinct boundary punctures $z_0, z_1, \ldots, z_d$ (of which $z_0$ is distinguished, and the others are ordered according to the orientation of $\partial S$);
- $\phi = (\phi_f)_{f \in F}$ are holomorphic $\phi_f : S \to Z$ extending to smooth isomorphisms on the compactifications $\overline{S} \to \overline{Z}$, with $\phi_f(z_0) = -\infty$, $\phi_f(z_{p_f}) = +\infty$ (where $Z$ is the strip $\mathbb{R} \times [0, 1]$ viewed as a disc with 2 ends). This is the same as a choice of a preferred point on the hyperbolic geodesic joining $z_0, z_{p_f}$ (corresponding via $\phi_f$ to the point $(0, \frac{1}{2})$ and to the geodesic $\{t = \frac{1}{2}\} \subset Z$);

We will omit the discussion of finite symmetries [8, Sec.2c, 2f, Lemma 3.7], but we mention that all auxiliary data constructions are done so as to preserve the symmetries which exchange preferred points lying on the same geodesic.

The above family is compactified by boundary strata described by the natural bubbling configurations one expects when a subset of the preferred points of [4] converges towards a boundary puncture [8, Fig.2, Fig.3]. We omit discussing gluing [8, Sec.2d], but we mention one important technical aspect:

Remark 1. One actually allows the geodesics in [4] to be deformed (except near the ends). We explain why in an example. Suppose we want to glue the first two discs in Figure 1 at one puncture. Call the discs $S^-, S^+$ and we want to glue $z_1 \in S^-$ with $z_0 \in S^+$. Gluing gives a 6-punctured disc with a geodesic and 2 curves which are not geodesics (they overlap along a common curve carrying the preferred point $\phi_1$) but they can be deformed into two geodesics. The “glued” $F$ is $F = F^- \sqcup F^+ = \{1, 2, 7, 8, 9\}$. There are four choices for the “glued” $p$: $z_{p_1}$ can be any of the $z_1, z_2, z_3, z_4$ coming from $S^+$, and in the cases $z_2$ and $z_3$ we would need to draw a new geodesic before gluing, so that we obtain glued data as in [4] (up to deforming). Now observe the third disc in Figure 1. The dotted line is a “cut”: we consider a 1-parameter family of such discs in which the auxiliary data contained on one side of the cut converges towards a common boundary puncture. In the limit, the disc breaks into two, namely $S^-$ glued with $S^+$ with “glued” $p$ determined by $p(1) = 3$, and the new geodesic we draw on $S^+$ is remembering that $\phi_1$ came from the geodesic joining $z_0$ to $z_3$ before breaking.

Now assign further auxiliary data:

1. (Weight). $w = (w_0, \ldots, w_d)$ are positive integers with $w_0 = \sum_{k \geq 1} w_k + |F|$;
(2) (Strip-like ends). Holomorphic embeddings $\epsilon_k : Z_+ \to S$, for $k = 1, \ldots, d$, of $Z_+ = [0, \infty) \times [0, 1] \subset \mathbb{Z}$ mapping the boundaries $t = 0, 1$ to $\partial S$ and converging to $z_k$ as $s \to +\infty$. Similarly $\epsilon_0 : Z_- \to S$ for the negative strip $Z_- = (-\infty, 0] \times [0, 1]$;

(3) (Closed 1-forms). Closed 1-forms $\alpha_k$ on $S$ for $k = 1, \ldots, d$ pulling back to: 0 on $\partial S; dt$ via $\epsilon_k, \epsilon_0$ on $(|s| > 0) \subset Z_0; 0$ via the other $\epsilon_j$ on $(s > 0) \subset Z_+$;

(4) (Subclosed 1-forms). 1-forms $\beta_f$ on $S$ with $d\beta_f \leq 0$ (for the complex orientation of $S$), $d\beta_f = 0$ near $\partial S$, and such that $\beta_f$ pulls back to: 0 on $\partial S; dt$ via $\epsilon_0$ on $s < 0; 0$ via the other $\epsilon_j$ on $s > 0$. In practice, one always chooses $\beta_f$ so that $d\beta_f = 0$ except on a small neighbourhood of the preferred point $\phi_f$.

The above data determines $\gamma = \sum_{k \geq 1} w_k \alpha_k + \sum_{f \in F} \beta_f$, a 1-form on $S$ with $d\gamma \leq 0$, $d\gamma = 0$ near $\partial S$. Also $\gamma$ pulls-back to: 0 on $\partial S; w_j dt$ via $\epsilon_j$ on $(|s| > 0) \subset Z_0$. Observe that this is consistent with Stokes's theorem, $0 \leq -\int d\gamma = w_0 - \sum_{k \geq 1} w_k = |F|$.

The choices of $\alpha, \beta$ can be made to vary smoothly over the moduli space $[S, \text{Sec.2e,2f}]$ consistently with the boundary strata, by an inductive argument. We make one important technical remark about gluing:

**Remark 2.** When gluing a disc $S^-$ at $z^-_k$ with a disc $S^+$ at $z^+_0$, we require the weights to agree: $w^-_k = w^+_0$. The $\alpha$ forms glue naturally. As for the $\beta$ forms: they are indexed by $F$, not $p(F)$, so define $\beta_+ \equiv \alpha_k$ on $S^-$ and $\beta_- = 0$ on $S^+$, then the $\beta$'s glue in a natural way to yield $\beta_f$'s on the gluing indexed by $f \in F = F^- \sqcup F^+$.

The last piece of auxiliary data is the almost complex structure:

(1) (Almost complex structures). A family $(I_z)_{z \in S}$ of conical structures which pull-back via $\epsilon_j$ on $|s| > 0$ to almost complex structures $I^w_t$ depending only on $t$ (which are chosen before-hand for each integer $w$). This choice can be made smoothly over the moduli space, and compatibly with gluing;

(2) (Infinitesimal deformations). Certain deformation data $K$ for $I$ $[S, \text{Sec.3b}]$;

(3) (Regular almost complex structures) The $I, K$ give rise to a family of perturbed almost complex structures $J = I \exp(-IK)$, agreeing with $I$ asymptotically, and which vary smoothly over the moduli space.

3.12. Moduli space of pseudo-holomorphic maps. Let $x_k$ be $X$-chords of weight $w_k$ with ends on $(L_0, L_d), (L_0, L_1), (L_1, L_2), \ldots, (L_{d-1}, L_d)$, respectively, for $k = 0, \ldots, d$, and where $L_i \in \text{Ob}(\mathcal{W}(E))$. Denote by $R^{d+1,p}(x)$ the moduli space of solutions $u : S \to E$ of

$$(du - X \otimes \gamma)^{0,1} \equiv \frac{1}{2}[(du - X \otimes \gamma) + J \circ (du - X \otimes \gamma) \circ j] = 0$$

with Lagrangian boundary conditions $u(\partial S) \subset L_i$ where $\partial S$ is the component of $\partial S$ between $z_i$ and $z_{i+1}$; and asymptotic conditions $u \circ \epsilon_k \to x_k$ as $|s| \to \infty$. For a generic choice of $K$ the $R^{d+1,p}(x)$ are smooth manifolds of the expected dimension $[S, \text{Thm.3.5}]$.

In the exact setup of $[S]$, there is a natural compactification of $R^{d+1,p}(x)$ modeled on the compactification of $R^{d+1,p}$ $[S, \text{Sec.3e}]$. The compactness relied on two ingredients: (1) a maximum principle forcing solutions to lie in $E^n$ $[S, \text{Sec.7d}]$; (2) an a priori energy estimate:

$$E(u) = \int_S \frac{1}{2} |du - X \otimes \gamma|^2 = \int_S u^* \omega - u^* dH \wedge \gamma \leq \int_S u^* \omega - d(u^* H \wedge \gamma) \equiv E_{\text{top}}(u)$$

using $H \geq 0$, $d\gamma \leq 0$, and by Stokes's theorem the topological energy $E_{\text{top}}$ is bounded (in the exact setup) in terms of the asymptotics $x$ using action functionals $[S, \text{Sec.7b}]$.

In our non-exact setup, (1) still holds since the argument only relied on exactness of $\omega$ on the conical end. We no longer have an a priori estimate for $E_{\text{top}}$ since the action functionals are multi-valued. However, since we will count solutions with weight $f E_{\text{top}}(u)$ (notice this is a homotopy invariant of $u$ relative to the ends), we only need to ensure compactness of $R^{d+1,p}(x; C) \subset R^{d+1,p}(x)$ which is the union of the components which have $E_{\text{top}} \leq C$. But
this now follows as in the exact case (2) since we artificially imposed an a priori bound on $E(u)$ via $E(u) \leq E_{\text{top}}(u) \leq C$.

Remark. Observe that, because we work over the Novikov ring, we are not using the monotonicity assumptions here to obtain an energy-index relation which in turn would give a priori bounds on $E(u)$ in terms of the index (unlike \cite{55}, who use this approach since they are working over $\mathbb{Z}$). We only use the monotonicity assumptions in \cite{77} and in \cite{97} to control $m_0$.

3.13. The wrapped $A_\infty$-structure. We now define, for $d \geq 2$, the maps $\mu^{d,p,w}$:

$$\text{CF}^*(L_{d-1}, L_d; w_d H)[q] \otimes \cdots \otimes \text{CF}^*(L_0, L_1; w_1 H)[q] \to \text{CF}^*(L_0, L_d; w_0 H)[q]$$

First we determine the $q^0$ coefficient of the image. An isolated $u \in \mathcal{R}^{d+1,p}(x_0, x_1, \ldots, x_d)$ will contribute $\pm t^{E_{\text{top}}(u)} x_0$ to $\mu^{d,p,w}(q^d x_d \otimes \cdots \otimes q^1 x_1)$ where $i_k = 1$ for $k \in p(F)$, $i_k = 0$ for $k \notin p(F)$, and it will contribute 0 otherwise (for orientation signs, see \cite{55}, Sec.3i)). The reason for the mysterious-looking conditions on $p(F)$ is that when $i_k = 1$ the formal variable $q^1$ plays the algebraic role of remembering that there is a preferred point on the geodesic in $S$ connecting $z_0, z_k$. For symmetry reasons \cite{55} Lemma 3.7, there is a cancellation of the counts of pseudo-holomorphic maps involving auxiliary data where more than one preferred point lies on a geodesic, so we only need to keep track of whether there is or there isn’t a preferred point on each geodesic (encoded by $q^1$, $q^0$). Encoding this information in the algebra is necessary, so that the breaking analysis for the family $\mathcal{R}^{d+1,p} \to \mathcal{R}^{d+1}$ will indeed yield the $A_\infty$-equations for the $\mu^d$ that we define below (see Remarks 3 and 4 below, or see \cite{55} (3.29)).

The $q^1$ coefficient of the image is determined by the requirement that the map $\mu^{d,p,w}$ respects the $\partial_q$ operator as follows, for $d \geq 2$:

$$\partial_q \mu^{d,p,w}(c_d \otimes \cdots \otimes c_1) = \sum_{k=1}^{d} (-1)^{\sigma(c_k)} c_{k+1} \mu^{d,p,w}(c_d \otimes \cdots \otimes \partial_q c_k \otimes \cdots \otimes c_1),$$

where $\sigma(\cdot)$ is defined in \cite{21}. Observe that this equation determines all possible $\mu^{d,p,w}(q^d x_d \otimes \cdots \otimes q^1 x_1)$ inductively on the value $i_d + \cdots + i_1$ (for an example, see \cite{55} Ex.3.14). The geometrical meaning of $\partial_q$-equivariance is explained in Remark 3 below. Finally, recall that $q^2 = 0$, so we have determined the image.

Summing up all $\mu^{d,p,w}$ as $p, w$ vary, we obtain the $A_\infty$-operations for $d \geq 2$:

$$\mu^d : CW^*(L_{d-1}, L_d) \otimes \cdots \otimes CW^*(L_0, L_1) \to CW^*(L_0, L_d).$$

We conclude with two technical remarks relating to gluing:

Remark 3. In Remark 1 of \cite{55} we described the gluings of the first two discs $S^-, S^+$ in Figure 7 for four choices of “glued” data $p$. Let us pretend $\phi_7$ is not present (otherwise $S^+$ will give cancelling contributions due to the symmetry which swaps $\phi_7$). The discs $S^-, S^+$ define two operators $\varphi^-, \varphi^+$, whose $q^0$-output is non-zero only if we input terms of the form $(q_{y_2}, q_{y_1})$, $(q_{x_4}, x_{x_3}, x_2, q_{x_1})$ respectively, and the $q^1$-outputs say $x_0, x_0^+$ are then a count of isolated solutions for the relevant data. The $\varphi^-, \varphi^+$ are in fact contributions to $\mu^2, \mu^3$. Gluing means composing operators, but $S^-$ requests a $q^1$-input at the gluing puncture (namely $q_{y_1}$). The $q^1$-output of $\varphi^+$ is determined by $\partial_q$-equivariance: for example $\varphi^+(q_{x_4}, q_{x_3}, x_2, q_{x_1}) = \pm x_0^+$ (ignore signs for simplicity). So in the composition, we take $q_{y_1} = \pm \varphi^+(q_{x_4}, q_{x_3}, x_2, q_{x_1}) = \pm \varphi^+(q_{x_4}, x_3, q_{x_2}, q_{x_1})$: those correspond to two of the four gluing choices (the other two involve $q^1 = 0$). For example, $\varphi^+(q_{x_4}, q_{x_3}, x_2, q_{x_1})$ corresponds to the choice $p(1) = 3$, and it arises from the breaking of Figure 7 (Right). In particular, at the level of composing operators, each term of the composition corresponds to precisely one type of breaking: so the ambiguity of the gluing choice in the auxiliary data disappears. This is crucial since distinct operator composites should correspond to distinct types of breakings.
Remark 4. We now explain the $A_\infty$-equations for the $\mu^d$ in an example. We ignore signs,

\begin{equation*}
\mu^1 \mu^2(q_{x_2}, x_1) + (-1)^{\|x_2\|} \mu^1(q_{x_2}, x_1) + \mu^2(q_{x_2}, \mu^1 x_1) = 0.
\end{equation*}

so we work modulo 2. Recall $\mu^1(x) = \partial x$ and $\mu^1(q_{x}) = (\partial q + \bar{R} \partial q)(q_{x}) + (\partial q \partial q)(q_{x})$. Consider the $A_\infty$-equation $\mu^1 \circ \mu^2 + \mu^2(\mu^1, \cdot) + \mu^2(\cdot, \mu^1) = 0$. Evaluate this equation on $q_{x_2}, x_1)$. First, the $q^1$-coefficient is: $\partial q \mu^2(x_2, x_1) + \mu^2(\partial q_{x_2}, x_1) + \mu^2(x, \partial x_1)$, which is the usual $A_\infty$-equation without $q$’s coming from the 3 possible cuts of a 3-punctured discs with no geodesics (in fact there are also $m_0$ terms, corresponding to cuts which don’t have any puncture on one side of the cut, but these cancel - see [7, 7]). Secondly, the $q^0$-coefficient: this has 7 terms of which two cancel, leaving $\partial q \mu^2(q_{x_2}, x_1) + \bar{R} \mu^2(x_1, x_1) + \mu^2(\bar{R} \partial q_{x_2}, x_1) + \mu^2((\partial q \partial q)(q_{x_2}, x_1) + \mu^2(q_{x_2}, \partial x_1)$. These are the breakings coming from the cuts shown in Figure 2. In general, $\partial q \partial q$ corresponds to the disc with 2 punctures, 1 geodesic, 1 preferred point; $\partial q \partial q$ arises as an incoming disc with 2 punctures, 1 geodesic, no preferred point (i.e. the $\partial$ disc with a new geodesic drawn).

4. Symplectic cohomology

4.1. Hamiltonian Floer cohomology. We refer to Salamon [25] for a precise definition of the Floer cohomology $HF^*(H^B)$ for a Hamiltonian $H^B: B \to \mathbb{R}$ defined on a closed monotone symplectic manifold $(B, \omega_B)$. In particular [25, Lecture 3] discusses the role of monotonicity. We recall that at the chain level, $CF^*(H^B)$ is generated, over the Novikov ring $\Lambda$, by all 1-periodic Hamiltonian orbits of $H^B$. The differential $\delta: CF^*(H^B) \to CF^{*+1}(H^B)$ is defined as follows on generators: there is a contribution $\pm e^{\omega_B[u, x_]} \delta(x_+)$ for each isolated (up to $\mathbb{R}$-reparametrisation) map $u: \mathbb{R} \times S^1 \to B$ solving Floer’s equation

$$\partial_s u + J^B(\partial_t u - X_{H^B}) = 0,$$

with asymptotic conditions $u(s, t) \to x_{\pm}(t)$ as $s \to \pm \infty$, where $J^B$ is a fixed almost complex structure on $B$ compatible with $\omega_B$. Up to isomorphism, the $HF^*(H^B)$ do not depend on the choice of $H^B, J^B$. Indeed, a homotopy $H^B, J^B$ which is constant for $|s| \gg 0$, induces a continuation map $HF^*(H^B_{+\infty}, J^B_{+\infty}) \to HF^*(H^B_{-\infty}, J^B_{-\infty})$ by counting solutions $u$ of Floer’s equation above where now $J^B_{+\infty}, H^B_{+\infty}$ depend on the coordinate $s \in \mathbb{R}$ (and we no longer have an $\mathbb{R}$-reparametrisation action). This continuation map is always an isomorphism (the inverse is defined using the “reversed homotopy” $H^B_{-\infty}, J^B_{-\infty}$). When $H^B$ is $C^2$-small and Morse, and $H^B, J^B$ are time-independent, then $CF^*(H^B)$ reduces to the Morse complex for $H^B$, in particular all 1-periodic Hamiltonian orbits are constant, and so by invariance we have an isomorphism

$$QHF^*(B) \cong HF^*(H^B),$$

which by PSS [24] (see 4.1) intertwines the quantum product and the pair-of-pants product.

4.2. Symplectic cohomology. For symplectic manifolds $E$ conical at infinity (see 3.1) $HF^*(H^E)$ heavily depends on the choice of $H^E: E \to \mathbb{R}$ because the larger $H^E$ is on the conical end the likelier it is that there are many 1-periodic Hamiltonian orbits there. The continuation maps are no longer isomorphisms, because they can no longer be reversed: only certain homotopies $H^E_s$ with $\partial_s H^E_s \leq 0$ are guaranteed to be well-defined. So to construct an
invariant one needs to choose a class of Hamiltonians which grow in a controlled way at infinity. We consider $H^E$ which at infinity are linear of positive slope in the radial coordinate $R$ (see \textsection 3.1). One then takes a direct limit over the continuation maps which increase the slope of $H^E$ at infinity: $HF^s(H^E) \to HF^s(H^E)$ where $\partial_s H^E_s \leq 0$. This defines the \emph{symplectic cohomology}

$$SH^*(E) = \lim_{\to} HF^*(H^E).$$

The symplectic cohomology in the non-exact setup was first constructed by Ritter \cite{ritter}, and was developed further in \cite{monotone}. The reader might like to consider first the original construction in the exact setup, due to Viterbo \cite{viterbo}, which is surveyed by Seidel \cite{seidel} and whose algebraic structures such as the pair-of-pants product are constructed in Ritter \cite{ritter}.

When $H^E$ is $C^2$-small on $E^{\text{in}}$, Morse, time-independent, and of small enough slope on the conical end, the chain complex $CF^*(H^E)$ reduces to the Morse complex for $E$, and since this is part of the direct limit construction there is always a canonical map (which can be non-injective and can be non-surjective):

$$c^*: QH^*(E) \to SH^*(E).$$

\textbf{Remark 4.1.} $c^*: QH^*(E) \to SH^*(E)$ splits into a sum $c^*_\lambda: QH^*(E)_\lambda \to SH^*(E)_\lambda$ indexed by the eigenvalues of the quantum product action of $c_1(TE)$ on $QH^*(E)$ (see Theorem 6.2).

In this paper we follow the conventions described in \cite{seidel}, except we use the simpler Novikov ring defined in \textsection 3.3 since we will always work in the monotone setting.

\textbf{Warning (Grading).} $SH^*(E)$ is typically not $\mathbb{Z}$-graded. Indeed, Floer cohomology is graded by the Conley-Zehnder index, which is a $\mathbb{Z}/2\mathbb{N}$-grading where $N\mathbb{Z} = c_1(TE)(\pi_2(E))$.

For the purpose of defining the open-closed string map in Section\textsection 6 it is actually convenient to view also $SH^*(E)$ as a homotopy direct limit (recall \textsection 3.11). Namely, define

$$SC^*(E) = \bigoplus_{w=1}^{\infty} CF^*(wH)[q]$$

with $\mu^1$ defined by the same formula as in Definition \textsection 3.3, where $\mathfrak{d}$ is now the differential defining Hamiltonian Floer cohomology and $\mathfrak{f}$ are the Floer continuation maps $CF^*(wH) \to CF^*((w+1)H)$ analogous to \textsection 3.9.

\textbf{Convention.} In \textsection 3.5 we chose $H$ to have slope 1 at infinity, but we could have rescaled $H$ by a constant $\epsilon > 0$ (once and for all). From now on, we assume $H$ is Morse and $C^2$-small on $E^{\text{in}}$, and linear on the conical end with slope $\epsilon > 0$ smaller than the minimal Reeb period of the closed Reeb orbits in $\Sigma$. This ensures that the only 1-periodic orbits of $H$ lie in $E^{\text{in}}$ and are constant (on the other hand $wH$ typically will have non-constant 1-periodic orbits on the conical end for large $w$).

\textbf{Lemma 4.2.} $SH^*(E) \cong H^*(SC^*(E); \mu^1)$ and the canonical map $c^*: HF^*(H) \to SH^*(E)$ arises as the inclusion of the subcomplex $CF^*(H) \to SC^*(E)$.

\textbf{Proof.} The first claim follows by the same arguments as in \textsection 3.10 the second claim follows by definition (and required the rescaling in the Convention). \hfill \Box

\textbf{4.3. Contractible vs Non-contractible orbits.} The Floer differential preserves the free homotopy class of the orbits, so one can restrict the chain complexes $CF^*(H^B)$ and $SC^*(E)$ to only contractible orbits. Denote their cohomologies by $HF^*_0(H^B)$ and $SH^*_0(E)$. By invariance, and considering small $H^B$ as mentioned in \textsection 4.1 we deduce $HF^*_0(H^B) = HF^*(H^B)$ for any $H^B$. Similarly when $H^E$ has small slope, such as the $H$ in \textsection 4.2 we have $HF^*(H^E) = HF^*_0(H^E)$, however this may not hold for general $H^E$. Moreover $SH^*_0(E) \subset SH^*(E)$ is a
subalgebra containing $c^*(QH^*(E))$ and the unit $1 = c^*(1)$, and there is a projection map $SH^*(E) \to SH_0^*(E)$. When $\pi_1(E) = 1$, $SH^*(E) = SH_0^*(E)$.

4.4. The PSS-isomorphisms. There are mutually inverse PSS-isomorphisms [24]
\[
\psi^+: QH^*(B) \to HF_0^*(H^B) \quad \psi^-: HF_0^*(H^B) \to QH^*(B)
\]
defined by counting spiked planes as follows. Parametrize $\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}$ by $z = e^{-2\pi(s+it)}$ for $(s,t) \in \mathbb{R} \times S^1$. Consider solutions $u: \mathbb{C}^* \to B$ of Floer’s equation
\[
\partial_s u + J^B(\partial_t u - c(s)X_{H^B}) = 0
\]
where $c: \mathbb{R} \to [0,1]$ is a decreasing cut-off function which equals 1 for $s \leq -2$ and equals 0 for $s \geq -1$. Notice $u$ is $J^B$-holomorphic near the puncture $z = 0$.

The energy of $u$ is defined as $\int_{\mathbb{C}^*} |\partial_s u|^2_{J^B} ds \wedge dt$. If the energy is finite, then $u$ converges to a Hamiltonian orbit of $H^B$ as $s \to -\infty$, and $u$ extends $J^B$-holomorphically over $z = 0$.

For a generic cycle $\alpha \in C_{2\dim B - s}(B)$, define $\psi^+(\alpha)$ by counting 0-dimensional moduli spaces of such finite energy solutions $u$ with $u(0) \in \alpha$.

The definition of $\psi^-$ is similar, except parametrize $\mathbb{C}^*$ by $z = e^{+2\pi(s+it)}$. In this case, solutions $u$ converge to a Hamiltonian orbit as $s \to +\infty$, and $u$ is $J^B$-holomorphic for $s \leq 0$ and extends $J^B$-holomorphically over the puncture $z = 0$.

In [27], it is proved that there are mutually inverse PSS-isomorphisms also for $E$,
\[
QH^*(E) \to HF_0^*(H^E) \quad HF_0^*(H^E) \to QH^*(E),
\]
where $H^E: E \to \mathbb{R}$ is radial at infinity and converges fast to a constant at infinity (so $H^E$ is globally bounded). Homotopying $H$ to $H^E$, where $H$ is as in [12], defines a continuation map $HF_0^*(H^E) \to HF^*(H)$ which is an isomorphism and the composite $QH^*(E) \to HF_0^*(H^E) \to HF^*(H) \to SH^*(E)$ equals the $c^*$ of [4,2] (for details, see [26]).

5. The (compact) Fukaya category and the acceleration functor

5.1. The Fukaya categories $\mathcal{B}$ and $\mathcal{E}$. Let $E$ be a monotone symplectic manifold conical at infinity, and let $B$ be a closed monotone symplectic manifold. Denote their Fukaya categories by $\mathcal{E} = \mathcal{F}(E)$, $\mathcal{B} = \mathcal{F}(B)$. Recall that their objects are constructed using only closed monotone Lagrangians as objects (compare: Ob$(\mathcal{W}(E))$ includes also non-compact Lagrangians conical at infinity). We briefly discuss $\mathcal{E}$, but much of what we say applies also to $\mathcal{B}$.

Suppose $L_0, \ldots, L_n$ are Lagrangians in $E$ (we continue with the convention in [37]). If the $L_i$ are pairwise transverse then no Hamiltonians arise in the definition of $\mu^B_\mathcal{E}: \mathcal{E}(L_n, \ldots, L_0) \to \mathcal{E}(L_n, L_0)$: one just counts suitable holomorphic polygons in $E$. Indeed, the analogues of all constructions for the compact category would correspond to allowing only weights $w = 0$ and only $q^0$ terms, in particular $CF^*(L_i, L_j)$ is generated by the intersections $L_i \cap L_j$.

When the Lagrangians don’t intersect transversely, one only uses $C^2$-small compactly supported Hamiltonians to flow the Lagrangians so that their intersections become transverse. For example, the ordinary Lagrangian Floer complexes are related to the wrapped versions via
\[
CF^*(L_i, L_j; H) \equiv CF^*(\varphi^1_H(L_i), L_j)
\]
where $\varphi^1_H$ is the time 1 flow of $X_H$, and where one must choose suitable almost complex structures (for example [26] explains this). When $L_i, L_j$ are not transverse, then $CF^*(L_i, L_j)$ is in fact defined as above using a $C^2$-small compactly supported $H$ so that $\varphi^1_H(L_i), L_j$ are transverse (how one sets up the $A_{\infty}$-category to keep track of the Hamiltonian perturbation data is described in detail in Seidel [30]).

So the construction of $\mathcal{E}$ is dramatically simpler than $\mathcal{W}(E)$: one no longer needs most of the auxiliary data in [3,11].
5.2. The bimodule structure for $B$. The composition maps for the $B$-bimodule $B$,

$$
\mu_B^{r|s} : B(L_r, L_s) \otimes B(L_0, L'_0) \otimes B(L_0', L'_1) \to B(L_r, L'_s),
$$

for $r, s \geq 0$ and any monotone Lagrangians $L_i, L'_i \subset B$, are defined by counting the 0-dimensional moduli spaces of maps described in Figure 3. The domain is a disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \}$ with punctures on the boundary: two distinguished fixed boundary punctures denoted by dashes at $z = \pm 1$ (+1 receives the module input from the $\mu_B^{r|s}$ map, and $-1$ receives the output), the small circular dots which lie on the two open arcs of $\partial \mathbb{D} \setminus \{ \pm 1 \}$ denote boundary punctures which are free to move (subject to being distinct and ordered on the respective arcs). In particular, $\mu_B^{0|0} = \text{id}$. The $A_\infty$-equations are a standard consequence of a bubbling analysis argument, by considering what happens when a subset of the free points converge together. The types of bubbles that appear are sketched in Figure 3.

5.3. The open-closed string map $OC_B : \text{HH}_*(B) \to \text{HF}^*(H_B)$. Define

$$
OC_B^n : B(L_0, L_n) \otimes B(L_0, L_n) \to CF^*(H_B)
$$

by counting maps as on the left in Figure 4. More precisely: the domain is the disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \}$, with one interior puncture and $n + 1$ boundary punctures of which one is distinguished. We fix the parametrization by fixing the boundary puncture at $z = 0$, we fix a cylindrical parametrization $(-\infty, 0] \times S^1$ where the $S^1$ is parametrized by the argument of $z$. Then we require that $u$ satisfies the Floer continuation equation

$$
\partial_s u + J(\partial_t u - c(s)X_H) = 0
$$

on this cylindrical parametrization, where $c : \mathbb{R} \to [0, 1]$ is a decreasing cut-off function which equals 1 for $s \leq -2$ and equals 0 for $s \geq -1$.

On the right in Figure 4 is an equivalent way of viewing the map $OC_B$: the domain is a half-infinite cylinder $(-\infty, 0] \times S^1$ with $n + 1$ punctures on the circle $\{0\} \times S^1$, one of which is a distinguished puncture at $(0, 0)$, and we require the maps to satisfy the above Floer equation.

The map $OC_B$ is a chain map. This can be seen by considering the possible degenerations of the 1-dimensional moduli spaces: (1) if marked points on the boundary move together, we obtain $\mu_B^1$ or $\mu_B^{1|s}$ contributions; and (2) if the Floer trajectory breaks (on the region
where $c(s) \equiv 1$, we obtain contributions described by the Floer differential $\partial : CF^*(H^B) \to CF^{*+1}(H^B)$. Thus we obtain a map on homology:

$$OC_B : \text{HH}^*(B) \to HF^*(H^B).$$

5.4. The bimodule structure for $E$ and $OC_E : \text{HH}^*(E) \to HF^*(HE)$. The constructions for $E$ are analogous to those for $B$. In particular, since all Lagrangians considered are closed, the maximum principle mentioned in 3.12 ensures that all solutions lie in a compact subset determined by the Lagrangians. So the non-compactness of $E$ plays no role (unlike the case of $W(E)$ where Lagrangians can be non-compact).

Composing $OC_E$ with the PSS isomorphism $HF^*(HE) \cong QH^*(E)$ of Section 4.4 corresponds to gluing a spiked disk onto the centre of the disk. By homotopying the Hamiltonian to zero for the glued configuration, one deduces that the composite map

$$OC_E : \text{HH}^*(E) \to QH^*(E)$$

is defined by the counts of holomorphic polygons which intersect a given test cycle in $E$ (we replaced the spike of the PSS-plane by a cycle intersection condition).

5.5. The acceleration functor $A : E \to W(E)$. We will now build an $A_\infty$-functor

$$A : E = \mathcal{F}(E) \to W(E)$$

from the compact category to the wrapped category. On objects, it is just the inclusion of the closed Lagrangians $L \subset E$ into the larger class of Lagrangians that define $\text{Ob}(W(E))$. To simplify the following discussion, we will in fact assume that we only consider a full subcategory of $E$ generated by a subcollection of Lagrangians which are pairwise transverse (something which can always be achieved for a countable collection of Lagrangians by flowing each one by a small generic compactly supported Hamiltonian flow). This is in order to avoid the tedious discussion of the auxiliary Hamiltonian perturbation data which is needed in the construction of the compact category, mentioned in Section 5.1.

If one identifies $CF^*(L_i, L_j; 0H) = CF^*(L_i, L_j)$, so the generators are the intersection points $L_i \cap L_j$ and the differential counts holomorphic strips, then one would like to define $A$ on morphisms as the inclusion

$$A : CF^*(L_i, L_j) \equiv CF^*(L_i, L_j; 0H) \subset CF^*(L_i, L_j; 0H)[q] \subset \bigoplus_{w=0}^{\infty} CF^*(L_i, L_j; wH)[q],$$

where one allows $w = 0$ in the definition of $CW^*(L_i, L_j)$ in Section 5.4. Unfortunately, in general one cannot allow weight $w = 0$ in the definition of $CW^*(L_i, L_j)$ when $L_i, L_j$ are both non-compact Lagrangians, since then they may intersect non-transversely at infinity, and it would no longer be transparent how to consistently define non-compactly supported
Hamiltonian perturbation data. However, allowing weight \( w = 0 \) is not problematic when \( L_i, L_j \) are compact Lagrangians, which is all we need.

**Definition 5.1.** Define the \( A_\infty \)-category \( W_0(E) \) to have the same objects \( \text{Ob}(W(E)) \) and the same morphism spaces \( CW^*(L_i, L_j) \) as \( W(E) \) except when \( L_i, L_j \) are both compact Lagrangians, in which case \( \hom_{W_i(E)}(L_i, L_j) = CF^*(L_i, L_j)[q] \oplus CW^*(L_i, L_j) \). The \( A_\infty \)-operations are defined by the same disc counts as in the construction of \( W(E) \), with the novelty that if one of the boundary punctures carries an input from \( CF^*(L_i, L_j)[q] \) then no closed 1-form \( \alpha \) is associated to that puncture (see Section 5.7), so the equation \( (du - X \otimes \gamma)^{0,1} = 0 \) turns into the \( J \)-holomorphic equation \( du = 0 \) near that puncture.

**Lemma 5.2.** The inclusion \( f : W(E) \to W_0(E) \) is a quasi-isomorphism.

**Proof.** The \( A_\infty \)-functor \( f \) is the identity on objects, the obvious inclusion on morphisms, and zero on higher order tensors. At the cohomology level \( f \) is an isomorphism since, as explained in Section 5.10, the wrapped cohomology is a direct limit over the weights \( w \to \infty \), so it does not matter if the weights start at \( w = 0 \) or \( w = 1 \).

**Definition 5.3.** The acceleration functor is the \( A_\infty \)-functor given by the composite

\[ A : E = F(E) \to W_0(E) \to W(E), \]

where the second map is a quasi-isomorphism which inverts the above \( f \) up to homotopy, and the first map is the inclusion on objects and morphisms, and is zero on higher order tensors.

**Remark 5.4.** As mentioned above, to simplify the discussion we assumed that in \( E \) we only considered a countable collection of compact Lagrangians which are pairwise transverse. So in the above discussion this restriction would also apply to \( W(E) \). However, the inclusion of this somewhat smaller \( W(E) \) into the full \( W(E) \) is a quasi-equivalence, since it is essentially surjective: a compact Lagrangian is isomorphic to a compactly supported Hamiltonian isotopy of itself. The analogous statement holds for \( F(E) \). So the acceleration functor \( F(E) \to W(E) \) can be defined by avoiding the discussion of Hamiltonian perturbations altogether.

5.6. The acceleration diagram. The acceleration functor determines an induced map

\[ HH_*(A) : HH_*(E) \to HH_*(W(E)). \]

In Section 6 we will extend \( OC_E : HH_*(E) \to HF^*(H^E) \) to a map \( OC_E : HH_*(W(E)) \to SH^*(E) \), taking \( H^E = H \).

**Theorem 5.5.** There is a commutative diagram

\[
\begin{array}{ccc}
HH_*(E) & \xrightarrow{HH_*(A)} & HH_*(W(E)) \\
OC_E \downarrow & & \downarrow OC_E \\
QH^*(E) & \xrightarrow{c^*} & SH^*(E)
\end{array}
\]

The square on the right tautologically commutes since \( f \) is the inclusion (Lemma 5.2). We remark that the lower horizontal map in that square at the chain level is the inclusion.
$SC^*(E) \to QC^*(E)[q] \oplus SC^*(E)$, where $SC^*(E) = \bigoplus_{w=1}^{\infty} CF^*(wH)[q]$. So $QC^*(E)$ plays the role of the “weight zero” summand, where we extend $\partial$ and $\mathfrak{R}$ to the new summand to be respectively the usual singular boundary operator on $\mathfrak{H}$-cycles and $\mathfrak{R} : QC^*(E) \to CF^*(H)$ to be the PSS map.

The map $e^*$ arises as an inclusion at the level of complexes, see Lemma $4.2$. So to prove that the left square commutes, we need to show that the extension $OC_E : \text{HH}_i(W(E)) \to \text{SH}^*(E)$ coincides with $OC_E : \text{HH}_i(E) \to HF^*(H_E)$ on the image of the inclusion $\text{HH}_i(\text{incl})$. To show that the two OC maps coincide it is enough to prove that no new solutions arise in the definition of OC restricted to compact Lagrangians, when one increases the slope of $H$ on the conical end. Since we are now working at the cohomology level, we just need to check that the OC maps agree on a combination of generators and we may assume that we increase the conical end. Since we are now working at the cohomology level, we just need to check that the OC maps agree on a combination of generators and we may assume that we increase the slope of $H$ only outside of a large compact set of $E$ which contains the compact Lagrangians involved in that combination of generators. Without loss of generality, we may assume that compact set is $E^{in}$. So we need to check that there are no solutions $u : S \to E$ of $(du - X \otimes \gamma)^{0,1} = 0$ bounding compact Lagrangians $L_i \in \text{Ob}(E)$ which converge to a closed Hamiltonian orbit outside $E^{in}$. Here $S$ is a disc with negative punctures on the boundary and one positive puncture at the centre of the disc, and $\gamma$ is a 1-form on $S$ satisfying $d\gamma \leq 0$ (for a detailed description see Section 6). It turns out that such solutions $u$ cannot enter $E \setminus E^{in}$: the proof of this is essentially [8, Lemma 7.2], except that in our case $S$ also has a negative puncture at the centre where $u$ converges to a closed Hamiltonian orbit in $E \setminus E^{in}$. But the Lemma of Abouzaid-Seidel can be adapted to this case (see for instance the Remarks in the “no escape Lemma” Section at the end of [26]), provided that the action of closed Hamiltonian orbits in $E \setminus E^{in}$ is negative (which can always be ensured, by choosing $H$ appropriately, see [26, Section 8]). We emphasize that the non-exactness of the symplectic form on $E^{in}$ is not an issue, since only the exact conical end $E \setminus E^{in}$ is used in the argument.

6. The open-closed string map on the wrapped Fukaya category

6.1. The open-closed string map. Recall the construction of $SC^*(E)$ in [4,2]. We now define the analogue of [4,2] for $W(E)$ at the chain level: $OC_E : CC_*(W(E)) \to SC^*(E)$. We first need to define a map

$$OC^{n,p,w}_E : CF^*(L_n, L_0; w_{n+1}H)[q] \otimes CF^*(L_{n-1}, L_n; w_nH)[q] \otimes \cdots$$
$$\cdots \otimes CF^*(L_0, L_1; w_1H)[q] \to CF^*(w_0H)[q].$$

Figure 5. Left: $n = 3$, $F = \{1, 2\}$, $p_1 = 3$, $p_2 = 2$. Right: two broken configurations caused because the preferred point $\phi_1$ converged to a puncture.
Observe Figure 3. Consider discs $S$ as in 3.11 for $d + 1 = n + 2$, except now $z_0$ is an interior puncture (this will receive the Hamiltonian orbit output coming from $SC^*(E)$). In addition, the last boundary puncture $z_{n+1} = z$ is distinguished and we denote by a dash in Figure 4 (this will receive the module input from $CC_n(W(E))$). We can fix the domain’s parametrization by taking $S$ to be a punctured disc with $z_0, z$ located at $0, +1$.

In 3.11 we modify $e_0$ to $e_0 : (-\infty, 0] \times S^1 \to S$ (asymptotic to $z_0$), and modify $\phi_f$ to $\phi_f : S \to \partial \setminus \{0, 1\}$, $\phi_f(z_0) = 0$, $\phi_f(z_{n+1}) = 1$. See the left of Figure 4.

Analogously to the $\mathcal{R}^{d+1,p}(x)$ defined in 3.12 except for the aforementioned modifications, we obtain a moduli space of maps solving $(du - X \otimes \gamma)^{0,1} = 0$ which we denote $OC^{n,p,w}(x)$. In particular, note that $(du - X \otimes \gamma)^{0,1} = 0$ turns into $\partial_t u + J(\partial_t u - w_0 X) = 0$ via $e_0$ near $z_0$.

To define $OC^{n,p,w}_E$ we mimic the definition of the $\mu^{n,p,w}$ in 3.13. First we determine the $q^0$ coefficient of the image. An isolated $u \in OC^{n,p}(x_0, x_1, \ldots, x_n, z)$ will contribute $\pm t^{E_{\mu}(a)} x_0$ to $OC^{n,p,w}(q^{i+1} x_0 \otimes q^{s} x_n \otimes \cdots \otimes q^1 x_1)$ where $i_k = 1$ for $k \in p(F)$, $i_k = 0$ for $k \notin p(F)$, and it will contribute $0$ otherwise (for signs mimic Sec.3h). The $q^1$ coefficient of the image is determined by the requirement that the map $\mu^{d,p,w}$ respects the $\partial_q$ operator.

**Example.** An isolated solution for the left configuration in Figure 5 contributes $\pm t^{\text{constant}} x_0$ to $OC^1_E(x \otimes q x_3 \otimes q x_2 \otimes x_1)$, using asymptotic data $x_0, x_1, x_2, x_3, z$ at $z_0, z_1, z_2, z_3, z$.

Finally, sum up the $OC^{n,p,w}_E$ as $p, w$ vary:

$$OC^{n,p,w}_E : W(E)(L_0, L_n) \otimes W(E)(L_n, \ldots, L_0) \to SC^*(E).$$

**Lemma 6.1.** $OC_E$ is a chain map, so it induces $OC_E : HH_*(W(E)) \to SH^*(E)$.

**Sketch of Proof.** The proof that $OC_E$ is a chain map is similar to the proof of the $A_\infty$-equations for $\mu^d_{W(E)}$ (see Remarks 3 and 4 in 3.13). We illustrate the argument in a concrete example. In Figure 5 consider the left configuration: if the preferred point $\phi_1$ moves towards $z_3$, the limit contributes to $OC^3_E(x \otimes (\delta \delta_q)(q x_3) \otimes q x_2 \otimes x_1)$ which is a part of $OC^3_E \circ (\id \otimes \mu^1_{W(E)} \otimes \id \otimes \id)$, and this corresponds to the right-most picture in Figure 5. If $\phi_1$ moves towards $z_0$, in the limit a 2-punctured sphere will bubble off with the preferred point $\phi_1$ lying on the geodesic joining the two punctures (see the broken configuration in the middle of Figure 5). By the construction of the parametrizations near the ends, this 2-punctured sphere can be identified with a cylinder $\mathbb{R} \times S^1$ with the preferred point $\phi_1$ lying at $(0, 0)$ on the geodesic line $t = 0$. So this breaking contributes to $\mu^1_{SC^*(E)}(\pm q OC^3_E(x \otimes q x_3 \otimes q x_2 \otimes x_1))$.

Consider the 1-dimensional components of the compactified moduli space $OC^{n,p}(x)$. The oriented sum of the possible boundary degenerations sums up to zero, and corresponds to the equation $OC_E \circ b = \mu^1_{SC^*(E)} \circ OC_E$ where $b$ is the bar differential. □

6.2. The $OC_E$ map respects $SH^*(E)$- and $QH^*(E)$-module actions. We warn the reader that $OC_E : HH_*(W(E)) \to SH^*(E)$ is not a ring homomorphism (and cannot be since $HH_*(W(E))$ does not admit a product, unlike $HH^*(W(E))$). However, we now prove that $OC_E : HH_*(W(E)) \to SH^*(E)$ respects a certain $SH^*(E)$-module structure on $HH_*(W(E))$, and for the compact category $OC_E : HH_*(F(E)) \to QH^*(E)$ respects a certain $QH^*(E)$-module structure, and that the two actions are compatible with the acceleration diagram of Theorem 5.5. The module structure was observed independently, in the exact case, by Ganatra [20], and the underlying $QH^*$-module structure is used in forthcoming work by Abouzaid-Fukaya-Oh-Ohta-Ono [7] in the closed case.

**Theorem 6.2.**

1. $HH_*(F(E))$ is a $QH^*(E)$-module; $QH^*(E)$ is a $QH^*(E)$-module via the quantum product;
OC_E : HH_*(\mathcal{F}(E)) \to QH^*(E) \text{ is a } QH^*(E)-\text{module homomorphism}

(2) \quad \text{HH}_*(\mathcal{W}(E)) \text{ is an } SH^*(E)-\text{module;}
SH^*(E) \text{ is an } SH^*(E)-\text{module via the pair-of-pants product;}
OC_E : HH_*(\mathcal{W}(E)) \to SH^*(E) \text{ is an } SH^*(E)-\text{module homomorphism}

(3) \quad \text{HH}_*(\mathcal{W}(E)), \text{ SH}^*(E) \text{ are } QH^*(E)-\text{modules via } c^* : QH^*(E) \to SH^*(E);
OC_E : HH_*(\mathcal{W}(E)) \to SH^*(E) \text{ is also a } QH^*(E)-\text{module homomorphism}
and the acceleration diagram of Theorem 5.6 respects the } QH^*(E)-\text{actions.}

(4) \quad \text{OC}_E \text{ splits up according to generalized eigensummands of the } c_1(TE)-\text{action,}
OC_E = \bigoplus_{\lambda} OC_{\lambda} : HH_*(\mathcal{F}(E)) \to QH^*(E)_\lambda
\quad \text{where } \lambda \text{ are the eigenvalues of the quantum product by } c_1(TE) \text{ on } QH^*(E), \text{ and } \mathcal{F}_\lambda(E)
\quad \text{is the Fukaya category generated by Lagrangians with } m_0(L) = \lambda \text{ (Lemma 5.2).}

(5) \quad \text{Similarly for the wrapped category,}
OC_E = \bigoplus_{\lambda} OC_{\lambda} : HH_*(\mathcal{W}(E)) \to SH^*(E)_\lambda

(6) \quad \text{The acceleration diagram therefore splits up according to the eigenvalues } \lambda \text{ of } c_1(TE):}
\begin{align*}
\text{OC}_E &= \mathcal{F}_\lambda(E) \\
\text{OC}_E &= \mathcal{F}_\lambda(E) \\
\text{\text{HH}_*(\mathcal{F}_\lambda(E))} & \xrightarrow{\text{HH}_*(A_{\lambda})} \text{HH}_*(\mathcal{W}_\lambda(E)) \\
\text{QH^*(E)_\lambda} & \xrightarrow{c_{\lambda}} \text{SH^*(E)_\lambda}
\end{align*}

\text{Remark: (1) and (4) also hold if we replace } E \text{ by a closed monotone symplectic manifold } B.

\textbf{Proof. Step 1. The } QH^*(E)-\text{action on } HF^*(L,L').
\text{The action } \psi : QH^*(E) \otimes HF^*(L,L') \to HF^*(L,L') \text{ is described in more detail in [9,4] (and in great detail in Auroux [12, Sec.6.1]). Observe the first picture in Figure 6. Consider the disc } D \text{ with two fixed boundary punctures at } -1, +1 \text{ (respectively receiving the output and the input chain from } CF^*(L,L'), \text{ with an interior marked point } q_c = 0 \text{ (which receives as input the locally finite cycle } c \text{ from } QC^*(E)).}
\text{Remark. We fixed the } PSL(2,\mathbb{R})-\text{parametrization of the disc by fixing the punctures at } \pm 1 \text{ and by fixing at } q_c = 0 \text{ the marked point } q_c \text{ which is constrained to lie on the hyperbolic geodesic connecting the two punctures } \pm 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{}
\end{figure}

Then, for a generator } x_+ \text{ of } CF^*(L,L'), \text{ one defines } \psi(c,x_+) = \sum \pm t_{End(c)}(u)c_{-}, \text{ summing over generators } x_- \text{ of } CF^*(L,L'), \text{ where the discs } u : D \setminus \{\pm 1\} \to E \text{ are isolated among solutions which intersect } c \text{ at } u(q_c) \text{ and converge to } x_{\pm} \text{ at } \pm 1 \in D. \text{ It should be clear from}
the context what “solutions” we mean: here it means $u$ is $J$-holomorphic, it satisfies the obvious Lagrangian boundary conditions $L, L'$, and the asymptotic conditions $x_{\pm}$ at $\pm 1$, and $E_{\text{top}}(u) = \int u^* \omega$.

We now clarify more precisely the condition at $q_c$. Consider the moduli space $\mathcal{M}(A)$ of all such solutions lying in a fixed homotopy class $A \in \pi_2(E, L \cup L')$ but without imposing the condition $u(q_c) \in c$. Now fix a cohomology class $\text{PD}[c] \in H^*(E)$, and within it we pick a representative locally finite cycle $c$ which is transverse to the evaluation map $\text{ev} : \mathcal{M}(A) \to E$ at $q_c = 0 \in D$. Then $\text{ev}^{-1}(c) \subset \mathcal{M}(A)$ is the space of all solutions $u$ described above satisfying the condition $u(q_c) \in c$. Call $n_{A,c,x}$ the count of the isolated points (i.e. the zero-dimensional components) of $\text{ev}^{-1}(c) \subset \mathcal{M}(A)$ with appropriate orientation signs. Define $\psi(c,x) = \sum_{q_c} \sum_A n_{A,c,x} t^{E_{\text{top}}(A)}$, and extend $\psi$ so that it is $\Lambda$-linear in $CF^*(L,L')$.

**Technical Remark.** Two possible degenerations of a 1-family of such solutions involve a disc bubbling off at $-1$ or at $+1$, which respectively correspond to the Floer differential on the output and the input, and the count of these degenerations would show that the action of $c$ is a chain map (see the second and third picture in Figure 6). There is however also another degeneration, when a disc bubbles off at a boundary point $p$ different from $\pm 1$. This corresponds to a higher-order map which counts solutions as above but with an extra boundary marked point at $p$, whose input chain at $p$ is $m_0$. But our standing assumptions in 3.7 ensure that $m_0$ is a multiple of $[L]$ or $[L']$ (depending on which are of $\partial D \setminus \{\pm 1\}$ the point $p$ lies). So these solutions are never isolated since the condition $u(p) \in [L]$ or $[L']$ at $p$ is automatic since it is the Lagrangian boundary condition. For a more detailed discussion see Auroux [12] Lemma 6.5. Analogous remarks apply to the actions that we will define below, which we therefore omit. We will also omit the discussion that the module actions we define on cohomology only depend on the class $\text{PD}[c] \in QH^*(E)$ rather than on the chosen representative lf-cycle $c$, since this is a fairly standard argument.

We now prove that, on cohomology, $\psi$ is a module action. Instead of having just one interior marked point $q_c = 0$, consider now having two distinct interior marked points $q_c = 0$ and another $q_{c'} \in (0, 1)$ which is allowed to vary on the geodesic joining $q_c$ to $+1$. Consider a 1-parameter family of solutions $u$ which intersect lf-cycles $c, c'$ at the respective marked points (the asymptotic data $x_{\pm}$ being fixed). Apart from the usual disc bubbling on the boundary, there are two new degenerations that occur when $q_{c'}$ converges to $q_c = 0$ or to $+1$. In the first case, as sketched in the fourth picture in Figure 6 a sphere bubble forms at $0$ which carries the two marked points, corresponding to the quantum product $c \star c'$ (of which the lowest order $t$ term, namely when the sphere bubble is constant, corresponds to the classical intersection product of the lf-cycles $c, c'$). In the second case the bubbling corresponds to composing the actions of $c$ and $c'$. Thus the count of these boundary degenerations yields the equation $(c \star c') \cdot x = c \cdot (c' \cdot x)$ on cohomology, where $\cdot$ denotes the action.

**Step 2. The $QH^*(E)$-action on $HW^*(L,L')$.**

If we replace $CF^*(L,L')$ by $CW^*(L,L')$ in the above discussion (where $L,L'$ are now objects in the wrapped Fukaya category), then as in the construction of the wrapped category in 3.11 and 3.12 there will now be additional auxiliary data, in particular a Hamiltonian and preferred points, which determines a 1-form $\gamma$ on $D \setminus \{\pm 1\}$; and in this context a “solution” will mean that $u$ satisfies the Floer equation $(du - X \otimes \gamma)^{0,1} = 0$; and $E_{\text{top}}(u)$ is now defined as in 3.12.

There is however one crucial complication: in the proof that $\psi$ is a module action, when $q_{c'}$ converges to $q_c$ we want the degeneration to give rise to a holomorphic sphere bubble, so that standard gluing theorems for holomorphic maps apply. It is therefore necessary to ensure that the Floer equation $(du - X \otimes \gamma)^{0,1} = 0$ turns into the $J$-holomorphic equation $du^{0,1} = 0$
near \(q_e\) by constructing \(\gamma\) to be zero near \(q_e = 0 \in \mathbb{D}\), and similarly we need to ensure that 
\(\gamma = 0\) near both \(q_e, q_e^\prime\) when we prove that \(\psi\) is a module action.

So we need to define more precisely the auxiliary data for the domain \(S = \mathbb{D} \setminus \{ \pm 1 \}\) of 
the solutions counted by \(\psi(c, \cdot) : CW^\ast(L, L^\prime) \to CW^\ast(L, L^\prime)\) where, recall, 
\(CW^\ast(L, L^\prime) = \oplus_{n=1}^\infty CF^\ast(L, L^\prime; wH)\langle q \rangle\). Recall the auxiliary data described in 3.11. In this case, we have

1. a negative strip-like end \(c_0\) for the puncture \(z_0 = -1\);
2. a positive strip-like end \(c_1\) for the puncture \(z_1 = +1\);
3. only one closed 1-form \(\alpha_1\) on \(S\) associated with the end \(z_1 = +1\), which pulls back to 
\(dt\) near \(\pm 1\);
4. the labels \(p_f = 1\) for all \(f \in F\), so the preferred points \(\phi_f\) (if any are present) all lie 
on the geodesic joining \(z_0 = -1\) and \(z_1 = +1\), and one can ensure that the sub-closed 
forms \(\beta_f\) satisfy \(d\beta_f = 0\) except in a small neighbourhood \(N_f\) of \(\phi_f\).

To ensure \(\gamma = 0\) at \(q_e = 0\), we would need to impose \(\alpha_1 = 0\) and \(\beta_f = 0\) near \(q_e\), 
since \(\gamma = w_1 \alpha_1 + \sum_f \beta_f\) where the weight \(w_1\) is specified by the input at \(x_+\) coming from 
\(CF^\ast(L, L^\prime; w_1 H)\). If the \(N_f = \{ z \in \mathbb{D} : (d\beta_f)|_z \neq 0 \}\) do not contain \(q_e\), 
then one can always ensure \(\alpha_1 = 0, \beta_f = 0\) near \(q_e\) by adding suitable exact forms supported near \(q_e\) to \(\alpha_1, \beta_f\), 
because \(\alpha_1, \beta_f\) are closed near \(q_e\). The issue is that one cannot do this consistently in families, 
since the preferred point \(\phi_f\) may converge to \(q_e\). To fix this issue, observe the first picture in 
Figure 7 we simply deform the geodesic joining \(z_0, z_1\) in a fixed small neighbourhood of 
\(q_e = 0\), say a hyperbolic disc \(D_e = D(q_e; 1/2)\) of radius 1/2 centred at \(q_e\), so that the deformed 
geodesic avoids \(D_e\) by moving along the clockwise arc of \(\partial D_e\) from \(-1/2\) to \(1/2\). Moreover, 
we impose that the support of \(d\beta_f\) must be contained within a hyperbolic disc \(D(\phi_f; 1/4)\) of 
radius 1/4 centred at the preferred point \(\phi_f\). This ensures that \(d\beta_f = 0\) in \(D(q_e; 1/4)\), so we 
can always construct \(\alpha_1, \beta_f\) to be zero on \(D(q_e; 1/4)\).

![Figure 7](image)

As already mentioned in Remark 1 of 3.11 deforming the geodesics away from the ends is 
not an issue and is anyway a necessity in the gluing arguments. We could equivalently have 
said that we keep the geodesic undeformed, but we stipulate a canonical way of determining 
where \(N_f\) must lie with respect to the preferred point \(\phi_f\): this point of view is illustrated in 
the second picture of Figure 7. Namely, if \(\phi_f\) lies outside \(D(q_e; 1/2)\) then the centre \(c_f\) 
of the hyperbolic disc of radius 1/4 which contains \(N_f\) equals the preferred point: \(c_f = \phi_f\), 
otherwise if \(\phi_f \in D(q_e; 1/2)\) then define \(c_f = D_L \cap \gamma_f \cap \partial D_e\) where: \(D_L\) is the upper half-disc 
which bounds the geodesic \((-1, 1)\) and the arc of \(\partial S\) carrying the boundary condition 
\(L, \gamma_f\) is the hyperbolic geodesic through \(\phi_f\) orthogonal to the geodesic \((-1, 1)\). In this 
point of view, the preferred point does not play the role of the centre of the support disc 
\(D(c_f; 1/4)\), but instead plays the role of fixing the strip-like end parametrization as in 3.11 
which identifies \(S = \mathbb{D} \setminus \{ \pm 1 \} \cong Z\) with the strip \(Z = \mathbb{R} \times [0, 1]\) so that 
\(z_0 \mapsto -\infty, z_1 \mapsto +\infty\), (geodesic) \(\mapsto (t = \frac{1}{2}) \subset Z\) and (preferred point) \(\mapsto (0, \frac{1}{2})\). In particular, this identification
determines the image of $\mathbb{D}_c$ in $Z$, and the image of $\gamma_f$ is a vertical segment $s = \text{constant in } Z$. The 1-parameter family of automorphisms of $\mathbb{D} \setminus \{\pm 1\}$ which moves the preferred point along the geodesic $(-1, +1)$ corresponds to $\mathbb{R}$-translation on $Z$, so the images of $\mathbb{D}_c, \gamma_f$ in $Z$ will simply translate in $Z$ if we vary the preferred point in $S$. Therefore, by this point of view, it becomes clear that one can build the forms $\beta_f$ consistently in families, since the problem reduces to constructing families of model forms $\beta_f$ on $Z$.

The definition of $\psi(c, \cdot) : CF^*(L, L'; w_1H)[q] \to CW^*\text{-codim}(c)(L, L')$ now follows the usual routine: the $q^0$-coefficient of the image in $CF(L, L'; w_1H)$ counts the isolated Floer solutions which do not involve preferred points; the $q^0$-coefficient in $CF(L, L'; (w_1 + 1)H)$ counts the isolated solutions involving exactly one preferred point ($\beta_f$ bumps up the output weight by 1); as usual, by symmetry arguments [8, Lemma 3.7], the counts of Floer solutions which involve more than one preferred point on the geodesic cancel out; and finally we extend $\psi(c, \cdot)$ by requiring it to be $\partial_q$-equivariant.

Remark. The symmetry argument relies on showing that the transposition of two preferred points lying on the same geodesic switches the orientation sign with which the isolated discs are closed on $\mathbb{D}_c$ so that it avoids $\partial \mathbb{D}_c$ clockwise. Similarly, as shown in Figure 8, we now make a further deformation of the (deformed) geodesic so that it avoids $\mathbb{D}_{c'}$ by going around $\partial \mathbb{D}_{c'}$ clockwise (we emphasize that these are hyperbolic discs, so from the Euclidean point of view $\mathbb{D}_{c'}$ shrinks to a point if $q_{c'}$ converges to $+1$). Then, as before, we require that the preferred points $\phi_f$ lie on this new deformed geodesic, and that the supports $N_f$ of $d\beta_f$ lie inside $\mathbb{D}(\phi_f; 1/4)$. Thus, $\alpha_1, \beta_f$ are closed on $\mathbb{D}(q_c; 1/4), \mathbb{D}(q_{c'}; 1/4)$ so we can always construct $\alpha_1, \beta_f$ to be zero there (by adding exact forms supported there), so $\gamma = 0$ there.

![Figure 8.](image)

The proof that $\psi(c, \cdot) : HW^*(L, L') \to HW^*(L, L')$ is a module map now follows by the same argument as in Step 1 except we are using the auxiliary data described above.
Step 3. The $QH^*(E)$-action on $HH_*(F(E))$.
Consider the compact category $B = F(E)$ (so in the absence of Hamiltonians), and consider a generator of the $B$-bimodule $B$,

$$(x_r, \ldots, x_1, m, y_1, \ldots, y_s) \in B(L_r, \ldots, L_0) \otimes \mathcal{M}(L_0, L'_0) \otimes B(L'_0, \ldots, L'_s),$$

in the notation from 2.1.22, with $\mathcal{M}(L, K) = B(L, K) = CF^*(K, L)$ for closed Lagrangians $L, K \subset E$. For a locally finite cycle $c \in C^*_F(E)$, define $c \cdot (x_r, \ldots, x_1, m, y_1, \ldots, y_s)$ by:

$$\sum (-1)^{\deg(c) - \sigma(x)}c^{x_{R+1}}(x_r, \ldots, x_{R+1}, \psi_c(x_R, \ldots, x_1, m, y_1, \ldots, y_s), y_{S+1}, \ldots, y_s)$$

summing over all obvious choices of $R, S$, where we use the sign notation $\sigma$ defined in 2.1, where $\deg(c) = \text{codim}_E(c)$ is the cohomological degree, and where $\psi_c(x_R, \ldots, x_1, m, y_1, \ldots, y_s)$ is the count of discs $u : \mathbb{D} \rightarrow E$ illustrated in the first picture of Figure 9. The domain $S$ in that picture is a disc $\mathbb{D}$ with $R + 1 + S$ positive boundary punctures, of which the dash at $+1$ marks the input for the module element; one negative boundary puncture at $-1$ indicated by a dash; and one interior marked point at $q_c = 0$ (as in Step 1, the choices $-1, +1, 0$ fix the parametrization of the disc). The definition of $\psi_c$ is analogous to that of $\mu^*_\mathcal{M}$ except now there is additionally an interior marked point $q_c = 0$ and we impose that Floer solutions $u$ satisfy $u(q_c) \in c$ in the same sense as explained in Step 1. We emphasize that the boundary punctures which carry the inputs $x_1, \ldots, x_R$ and $y_S, \ldots, y_1$ are free to move in an order-preserving way along the two open arcs joining $-1, +1$.

![Figure 9](image-url)

The analogue here of the Technical Remark from Step 1 is illustrated in the second and third picture in Figure 9. These show two typical degenerations that can occur in a 1-family of such solutions $u$, when a subset of the boundary punctures converge towards $+1$ and $-1$ respectively. Observe that the bubbled discs which do not carry the marked point $q_c$ are precisely the discs counted by the bimodule homomorphisms $\mu^*_{\mathcal{M}}$. If a subset of the boundary punctures converge to a boundary point different from $\pm 1$, then a bubble appears which is counted by $\mu^*_{\mathcal{R}}$. These degenerations prove that the action of $c$ is a morphism of $B$-bimodules $c \cdot \cdot : \mathcal{M} \rightarrow \mathcal{M}$ (see the definition in 2.2). Therefore, $c \cdot \cdot$ induces a chain map on the Hochschild complex by Lemma 2.3. Thus, on homology, we obtain:

$$c \cdot \cdot : HH_*(F(E)) \rightarrow HH_*(F(E)).$$

We omit the standard argument that this map only depends on the cohomology class $PD[c] \in QH^*(E)$. To prove that this is a $QH^*(E)$-module action, we mimick Step 1 using two ordered interior marked points $q_c, q_{c'} \in (0, 1)$. If $q_{c'}$ converges to $q_c$ in a 1-family, then as usual a bubble counted by the quantum product appears as in the last picture in Figure 9. If $q_{c'}$ instead converges to $+1$, then a subset of the boundary punctures may converge together with $q_{c'}$ to $+1$, so the bubble that appears at $+1$ is of the same type as the first picture in Figure 9 with $q_{c'}$ in place of $q_c$. So these degenerations account for all the contributions of the composite of the actions of $c$ and $c'$. 


Step 4. The $QH^*(E)$-action on $HH_*(W(E))$.

For the wrapped category $\mathcal{B} = W(E)$ (so when Hamiltonians are present), we have the same complication as in Step 2: we need to ensure that the auxiliary data can be constructed so that $\gamma = 0$ near $q_c$ (and near $q_{\infty}$ in the proof of the module action), so that the Floer equation $(du - X \otimes \gamma)^{0,1} = 0$ turns into the $J$-holomorphic equation near $q_c$.

Observe Figure 10: this is the analogue of Figure 9, now decorated with a typical configuration of geodesics and preferred points $\phi_f$.

The auxiliary data is now be constructed as in 3.11, but we need to tweak the construction just like in Step 2 by deforming the geodesics so that they avoid the disc $\mathbb{D}_L$ (which bounds the geodesic $(-1,1)$ and the arc of $\partial S$) for the geodesics carrying the boundary conditions $L_0, \ldots, L_R$) we will deform the geodesics to avoid $\mathbb{D}_L$ by moving clockwise along $\partial \mathbb{D}_L$; whereas for the geodesics in the lower half-disc $\mathbb{D}_L = \mathbb{S} \setminus \mathbb{D}_L$ we will instead move anticlockwise along $\partial \mathbb{D}_c$. For the geodesic connecting $-1$ to $+1$ one needs to make a choice amongst these two possibilities: we will do the same as in Step 2, so we go around $\partial \mathbb{D}_c$ clockwise in this case. Although it seems that we are breaking the symmetry of the problem, this is only apparent because the module input at $+1$ has already divided the set of positive boundary punctures into two distinct subsets, depending on whether they lie in $\mathbb{D}_L$ or in $\mathbb{D}_R$, so we are only stipulating that $+1$ will belong to the former subset.

The upshot, just as in Step 2, is that the auxiliary forms $\alpha_i, \alpha'_j, \beta_f$ can all be consistently constructed so that they vanish on $\mathbb{D}(q_c; 1/4)$.

Now consider the wrapped category $\mathcal{B} = W(E)$ and a generator of the $\mathcal{B}$-bimodule $\mathcal{B}$,

$$(q_i x_1, \ldots, q^i x_0, q^i y_1, \ldots, q^{i'} y_s) \in \mathcal{B}(L_{r}, \ldots, L_0) \otimes \mathcal{M}(L_0, L'_0) \otimes \mathcal{B}(L'_0, \ldots, L'_s),$$

in the notation from 2.4.1,2.21 with $\mathcal{M}(L, K) = B(L, K) = CW^*(K, L)$ for (possibly non-compact) Lagrangians $L, K \subset E$, where $m \in CF^*(L'_0, L_0; w_0H)$, $x_1 \in CF^*(L_{i-1}, L_i; w_1 H)$, $y_i \in CF^*(L_{i}', L_{i-1}'; w_i H)$ for some weights $w_0, w_1, w_i$.

For an $s$-cycle $c \in C_s^*(E)$, define the action $c \bullet (q^i x_1, \ldots, q^{i} y_s) = \sum (-1)^{j} (q^i x_1, \ldots, q^{i} x_{R+1}, \psi_c(q^i x_1, \ldots, q^{i} x_1, \ldots, q^{i} y_1, \ldots, q^{i} y_s), q^{i} y_s, q^{R+1} y_{s+1}, \ldots, q^{i} y_s)$, where $\psi_c(q^i x_1, \ldots, q^{i} y_s) \in CF^*(L'_0, L_{R}; w_0 H)$ takes into account the $-1$ grading of $q_i$, and the $q^{0}$-coefficient of

$$\psi_c(q^i x_1, \ldots, q^{i} y_s) \in CF^*(L'_0, L_{R}; w_0 H)$$

counts the isolated solutions $u : \mathbb{D} \to E$ of Floer's equation $(du - X \otimes \gamma)^{0,1} = 0$ such as the first picture in Figure 10 (except the geodesics have been deformed as explained above, which ensured that the $\alpha_i, \alpha'_j, \beta_f$ are zero on $\mathbb{D}(q_{c}; 1/4)$ and so the same is true for $\gamma = w_0 o_0 + \sum w_i a_i + \sum w'_j a'_j + \sum \beta_f$).
where \(|F|\) is the number of preferred points. As usual, we extend \(\psi_c\) so that it is \(\partial_q\)-equivariant and \(\Lambda\)-multi-linear.

The proof that this action is a bimodule morphism is analogous to Step 3, and examples of the relevant degenerations of 1-families are illustrated in the second and third picture in Figure 10 so observe that the bubbled discs which do not carry the marked point \(q_c\) are precisely the discs counted by the bimodule homomorphisms \(\mu_{W_i(E)}^{r,s}\). As in Step 3, by Lemma 23 \(c \bullet\) induces a chain map on the Hochschild complex. The proof that this is a \(QH^*(E)\)-module action on homology is analogous to Step 3 (and an example of a relevant degeneration arising in the proof is illustrated in the last picture of Figure 10).

**Step 5. The \(QH^*(E)\)-action on \(SH^*(E)\).**

We will now construct the \(QH^*(E)\)-action \(\psi: QH^*(E) \otimes SH^*(E) \to SH^*(E)\). Recall \(SC^*(E) = \oplus CF^*(\mathfrak{wH})[q]\) from Section 4.2 which we already used in 6.1. In analogy with the wrapped category, by “solutions” \(u: \mathbb{R} \times S^1 \to E\) we now mean solutions to Floer’s equation \((du - X \otimes \gamma)^{0,1} = 0\) where \(\gamma\) depends on the auxiliary data. Let us recall this data in the case of the differential \(\mu^1\). The domain \(\mathbb{R} \times S^1\) can be viewed as the sphere \(\mathbb{C}P^1\) together with a negative puncture, a positive puncture, and a geodesic connecting the punctures corresponding to the line \(t = 0\). The auxiliary data is analogous to 3.11 except the ends are cylindrical rather than strip-like since we replace the time-interval \([0,1]\) by \(S^1\). In particular, we have a closed form \(\alpha\) which pulls back to \(dt\) in the cylindrical ends near the punctures; and a preferred point \(\phi_f\) (if present) on the geodesic \(t = 0\) corresponds to the translation \(\mathbb{R} \times S^1 \to \mathbb{R} \times S^1\), \(\phi_f \mapsto (0,0)\). These translations allow one to consistently build 1-forms \(\beta_f\) with \(d\beta_f \leq 0\), and \(d\beta_f = 0\) outside of a small neighbourhood of \(\phi_f\), analogously to the construction of the wrapped category.

To define the action \(\psi(c, \cdot): SC^*(E) \to SC^*(E)\) of an \(\text{lf}\)-cycle \(c \in C^\text{lf}_*(E)\) we mimick Step 1: on the domain we just described above, insert a new marked point \(q_c = (0,0) \in \mathbb{R} \times S^1\) on the geodesic \(t = 0\) (as usual, we chose the specific point \(q_c = (0,0)\) in order to fix the domain’s parametrization). Now require solutions \(u\) to satisfy \(u(q_c) \in c\) (in the sense explained in Step 1). Just as in Step 2, we need to ensure that \(\gamma = 0\) near \(q_c\): this can be achieved by deforming the geodesics as in Step 2 so that they avoid a hyperbolic disc \(D_c = D(q_c; 1/2) \subset \mathbb{C}P^1\) by moving clockwise along \(\partial D_c\), and one ensures \(\alpha = 0\) and \(\beta_f = 0\) on \(D(q_c; 1/4)\).

The proof that this is a chain map follows as usual: in a 1-family of solutions, \(q_c\) may converge to one of the two punctures which gives rise to a new twice-punctured sphere bubble carrying \(q_c\): whereas the twice-punctured sphere which does not carry \(q_c\) corresponds to the differential \(\mu^1\) (which involves either the Floer differential \(\mathfrak{w}\) if there are no preferred points, or the continuation map \(\mathfrak{R}\) if there is a preferred point, as defined in Section 4.2).

The proof that this is a module action on cohomology is analogous to the proof in Step 2, by placing two interior marked points \(q_c = (0,0), q_{c'} \in (0,\infty) \times \{0\}\) on the geodesic \(t = 0\). One deforms geodesics so that they avoid \(D_c, D_{c'} = D(q_c; 1/2)\) by going around their boundary clockwise, so that all auxiliary forms vanish on \(D(q_c; 1/4)\) and \(D(q_{c'}; 1/4)\). Then in a 1-family, if \(q_{c'}\) converges to \(q_c\), it will give rise to a holomorphic bubble (since \(\gamma = 0\) near \(q_c\)) counted by the quantum product; whereas if \(q_{c'}\) converges to \(q_c\), the twice-punctured sphere breaks up into two such spheres corresponding to the composite of the actions of \(c\) and \(c'\).

**Step 6. The \(SH^*(E)\)-action on \(SH^*(E)\).**

The pair-of-pants product \(\mu_{SC^*(E)}^2\) is constructed in analogy with \(\mu_{W_i(E)}^2\), except the domain is now a 3-punctured sphere (that is, a pair-of-pants), rather than a 3-punctured disc. Two of the punctures are positive: they receive inputs from \(CF^*(w_1H)[q], CF^*(w_2H)[q]\)
whereas the negative puncture receives the output from \( CF^*(\omega_H)[q] \). The count of solutions on these domains (with auxiliary data, such as preferred points) defines the contribution
\[
CF^*(\omega_H)[q] \otimes CF^*(\omega_H)[q] \to CF^*(\omega_H) \to \mu^2_{SC^*(E)}.
\]

As a comparison with the previous constructions, we could equivalently have said that we insert a new positive interior puncture \( q_e \) on the twice-punctured sphere. So rather than an interior marked point \( q_e \), as in Steps 1-5, we now have an \emph{interior puncture} together with a choice of positive cylindrical-end near \( q_e \); and rather than requiring that \( \gamma = 0 \) near \( q_e \), we now require that \( \gamma = w_2 dt \) in the cylindrical-end near \( q_e \) so that \((du - X \otimes \gamma)^{0,1} = 0\) turns into Floer’s equation for the Hamiltonian \( w_2H \) (rather than the \( J \)-holomorphic equation).

On cohomology, the pair-of-pants defines an action of \( SH^*(E) \) on itself by multiplication.

We need to check that the action of \( QH^*(E) \) on \( SH^*(E) \) defined in Step 5 agrees with this \( SH^*(E) \)-action via \( c^*: QH^*(E) \to SH^*(E) \). By Lemma \ref{lemma:action} \( c^* \) arises from a composition: the PSS map \( QC^*(E) \to CF^*(H) \) from \ref{thm:propagation} composed with the inclusion of the subcomplex \( CF^*(H) \to SC^*(E) \). At the level of cohomology, this composition corresponds to the gluing of a spiked plane (see \ref{thm:propagation}) onto the pair of pants. This gluing gives rise to a cylinder with a spike ending at an interior marked point of the cylinder. Observe that the spike is just an intersection condition with an \( \ell \)-cycle at the marked point, except it is phrased using Morse chains rather than singular chains. This concludes the proof that the two actions agree via \( c^* \). For more details on this gluing argument and on the pair-of-pants product, see \cite{25}.

**Step 7. The \( SH^*(E) \)-action on \( HH_*(W(E)) \).**

To upgrade the \( QH^*(E) \)-action on \( HH_*(W(E)) \) from Step 4 to an \( SH^*(E) \)-action, we make the following changes:

- the interior marked point \( q_e = 0 \in S \) in Step 4 gets replaced by a positive interior puncture \( q_e = 0 \notin S \), in analogy with the discussion in Step 6: so it comes with a choice of a cylindrical end, namely a holomorphic embedding \( \epsilon_e : (0, \infty) \times S^1 \to S \) converging to \( q_e \) as \( s \to \infty \);
- associated to \( q_e \) there is a weight \( w_e \) and a closed form \( \alpha_e \) which pulls back to \( dt \) in the strip-like end near \(-1\) and in the cylindrical end near \( q_e \). The asymptotic condition for solutions \( u : S \to E \) at \( q_e \) is now a generator \( c \in CF^*(\omega_H) \subset SC^*(E) \);
- a new type of geodesic may decorate the domain, as part of the auxiliary data: namely the geodesic \((-1,0)\) connecting the module output at \(-1\) with the puncture \( q_e = 0 \). A preferred point on this geodesic specifies a cylindrical parametrization, more precisely: it specifies a unique holomorphic map \( \phi_f : S \to Z' = \mathbb{R} \times S^1 \) which extends to a smooth isomorphism \( \overline{S} \to \overline{Z}' \) on the compactifications such that \( \phi_f(-1) = (-\infty,0) \), \( \phi_f(z_p) = (+\infty,0) \) (where \( \{\pm \infty\} \times S^1 \) are the circles compactifying the cylinder \( Z' \)), and such that \( \phi_f(\text{the geodesic}) = \mathbb{R} \times \{0\} \) and \( \phi_f(\text{preferred point}) = (0,0) \). Although, as usual, we will often just abusively call \( \phi_f \) the preferred point. Recall that these isomorphisms are used to consistently build the auxiliary sub-closed forms \( \beta_f \), with \( d\beta_f \) supported near the preferred point.
- just as discussed in Step 4, we need to deform all the geodesics so that the auxiliary forms (apart from those involving the puncture \( q_e \)) vanish on the hyperbolic disc \( \mathbb{D}(q_e;1/4) \subset S \cup \{q_e\} \). More precisely: the geodesics connecting \(-1\) to a boundary puncture in the upper half-disc \( D_L \) are deformed to avoid \( D_e = \mathbb{D}(q_e;1/2) \) by going around \( \partial D_e \) clockwise (this includes, by convention, the geodesic \((-1,1)\) connecting the module input/output); those connecting \(-1\) to a boundary puncture in the lower half-disc \( D_L \) go around \( \partial D_e \) anticlockwise. The geodesic described in the previous paragraph (if present) stays undeformed, since in this case by construction \( \beta_f = 0 \) near \( q_e \) and so \( \gamma = w_e \alpha_e \) near \( q_e \). (since all other auxiliary forms vanish on \( \mathbb{D}(q_e;1/4) \)).
The remainder of the construction of the \( W(E) \)-bimodule action of a cycle \( c \in SC^*(E) \) on \( W(E) \) is completely analogous to Step 4, so we obtain a chain map on the Hochschild complex, and thus an \( SH^*(E) \)-action on \( HH_* (W(E)) \).

To prove that it is a module action on cohomology, we now consider two positive interior punctures \( q_c = 0 \) and \( q_{c'} \in (0,1) \) (rather than interior marked points, as in Step 4). Just as in Step 4, all geodesics involving pairs of boundary punctures need to be deformed so that they avoid \( D(q_c; 1/2) \) and \( D(q_{c'}; 1/2) \) so that we can ensure that the relevant auxiliary forms vanish on \( D(q_c; 1/4) \) and \( D(q_{c'}; 1/4) \) (and as usual, the clockwise/anticlockwise convention for the deformation is as above). The novelty here is the possible presence of the geodesic \( G_{c'} = (-1,q_{c'} ) \) connecting \(-1\) to \( q_{c'} \). This geodesic \( G_{c'} \) does not need to be deformed in \( D(q_{c'}; 1/2) \) since the auxiliary forms for \( c' \) are constructed so that they pull-back correctly in the cylindrical-end near \( q_{c'} \) (and we just ensured that all other auxiliary forms vanish near \( q_{c'} \)). However, we need to deform \( G_{c'} \) near \( q_c \), so that the preferred point \( \phi_f \) (if present) on \( G_{c'} \) will not collide with \( q_c \) in a 1-family (unless also \( q_{c'} \) converges to \( q_c \)). We need to consistently deform \( G_{c'} \) without changing \( G_{c'} \) near the ends, so that it avoids a neighbourhood of \( q_c \); but to do this, we need to allow the diameter of the neighbourhood to depend on the relative distance between \( q_c, q_{c'} \).

Observe Figure 11. Define \( r = \min\{ \frac{1}{2}, \frac{1}{4}\text{dist}(q_c, q_{c'}) \} \) where \( \text{dist} \) refers to the hyperbolic distance. Then we deform \( G_{c'} \) so that it avoids \( D(q_c; r) \) by going clockwise around \( \partial D(q_c; r) \), and we require that \( N_f \) lies inside the disc \( D(\phi_f; r/2) \) so that \( d\beta_f = 0 \) on \( D(q_c; r/2) \), and thus we can always construct \( \beta_f \) so that \( \beta_f = 0 \) on \( D(q_c; r/2) \). In other words, for the geodesic \( G_{c'} \) instead of using \( D(q_{c'}; 1/2) \) we now use \( D(q_c; r) \) for a radius \( r \) which shrinks to zero as \( q_{c'} \) approaches \( q_c \), in particular we do not deform \( G_{c'} \) near the ends. Figure 11 compares how we deform \( G_{c'} \) (and we indicate a typical preferred point \( \phi_f \) and the support \( N_f \)) versus how we deform the geodesics which connect \(-1\) to a boundary puncture. Note that if \( q_{c'} \) lies outside \( D_c \) then \( D_c = D(q_c; r) \); so if \( q_{c'} \) converges to \(+1\) then the deformed geodesic \( G_{c'} \) converges to the usual deformation of the geodesic connecting \(-1\) to \(+1\).

![Figure 11](image)

The remainder of the proof of the module action follows as in Step 4: in a 1-family of solutions, \( q_{c'} \) may converge to the puncture \( q_c \) or to \(+1\). In the first case, a three-punctured sphere carrying \( q_c,q_{c'} \) appears, corresponding to the pair-of-pants product \( \mu^2_{SC^*(E)} (c,c') \) (notice that preferred geodesics, if present, connect \( q_c,q_{c'} \) to the output puncture, as it should be). In the second case, a disc bubble appears carrying the puncture \( q_{c'} \), and the broken configuration corresponds to composing the actions of \( c \) and of \( c' \).

The fact that, on cohomology, the \( QH^*(E) \)-action on \( HH_* (W(E)) \) from Step 4 agrees with the \( SH^*(E) \)-action via \( c' : QH^*(E) \rightarrow SH^*(E) \) is proved as in Step 6. Namely, we glue a PSS-plane onto the puncture \( q_c \), thereby “filling it” and replacing it with an interior marked point \( q_c \) as in Step 4, the spike of the PSS-plane being the 1f-cycle intersection condition.
Step 8. The OC-maps are compatible with the actions.

We will prove that $OC : HH_*(W(E)) \rightarrow SH^*(E)$ preserves the $SH^*(E)$-action. The proof that the OC-map for the compact category preserves the $QH^*(E)$-action is analogous: just ignore all the decorations on the domains.

Consider the domain of the solutions which define OC: recall this is a disc with positive boundary punctures, of which one is distinguished (receiving the module input), and with one negative interior puncture (receiving the $SC^*(E)$-output). We fix the parametrization of the domain by placing the distinguished boundary puncture at 1 and the interior puncture at 0. Now introduce a new positive interior puncture $q_e$ which is free to move on the geodesic $(0, 1)$ connecting the punctures 0 and 1. A typical domain with decorations is shown in the first picture of Figure 13. The solutions $u$ of $(du - X \otimes \gamma)^{0,1} = 0$ will be required to satisfy Floer’s equation near $q_e$ just as in Step 6.

Recall that, as in the construction of the OC-map, the preferred geodesics connect the output interior puncture at 0 with a boundary puncture. The preferred point specifies a comment further on this below.

As in Step 4, we need to ensure that the auxiliary forms (except those related to $q_e$) vanish on $\mathbb{D}(q_e; 1/4)$, so as usual we need to deform the geodesics so as to avoid the hyperbolic disc $\mathbb{D}_e = \mathbb{D}(q_e; 1/2)$ by going around it (with the usual clockwise/anticlockwise convention depending on whether the geodesic lies in the upper or lower half-disc of $\mathbb{D} \setminus (-1, 1)$ – we comment further on this below)

The novelty here is:

- we emphasize that the interior puncture $q_e$ is now allowed to move freely on $(0, 1)$ (since we used the punctures $+1, 0$ to fix the parametrization of the disc);
- a new type of geodesic can arise, see the first picture in Figure 12: the geodesic $(0, q_e)$ connecting the output interior puncture 0 to the interior puncture $q_e$. This geodesic does not need to be deformed, since the relevant auxiliary forms for this geodesic are constructed so that they pull-back correctly in the cylindrical-ends near 0 and $q_e$;
- if $q_e$ is sufficiently close to 0, then $\mathbb{D}_e = \mathbb{D}(q_e; 1/2)$ contains 0. So in this case we cannot deform geodesics to avoid $\mathbb{D}_e$ since then they would never reach 0. So instead we apply a similar trick to Step 7 (Figure 11): instead of $\mathbb{D}_e = \mathbb{D}(q_e; 1/2)$ we always use the disc $\mathbb{D}(q_e; r)$ with radius $r = \min \left\{ \frac{1}{4}, \text{dist}(0, q_e)^2 \right\}$ when deforming geodesics (we will explain later why this time we square the hyperbolic distance between the two punctures).

Notice $\mathbb{D}(q_e; r)$ shrinks to zero if $q_e$ approaches the puncture 0, whereas it coincides with $\mathbb{D}_e$ whenever $q_e$ lies outside of $\mathbb{D}(0; 1/\sqrt{2})$ (and, since it is a hyperbolic disc, if $q_e$ approaches $+1$ it will shrink to a point when viewed as a Euclidean disc). Observe the second and third pictures in Figure 12: we deform (if necessary) any geodesic which connects 0 to a boundary puncture so that it avoids the disc $\mathbb{D}(q_e; r)$ by moving along $\partial \mathbb{D}(q_e; r)$ (with the usual clockwise/anticlockwise convention, in particular for the geodesic $(0, 1)$ from 0 to $+1$ we stipulate that the deformed geodesic goes around $\partial \mathbb{D}(q_e; r)$ clockwise: this does not break any symmetry in the boundary punctures since $+1$ already broke that symmetry by being the distinguished boundary puncture which receives the module input). The reason for making $r$ decrease quadratically in the hyperbolic distance between 0 and $q_e$, is to ensure that if we fix the slope of a straight line geodesic $G_x$ connecting 0 to a boundary point $x$ of the disc, then $G_x$ will not intersect $\mathbb{D}(q_e; r)$ when $q_e$ is sufficiently close to zero, and thus $G_x$ does not need to be deformed. Therefore, in a 1-family, if $q_e$ converges to 0, and $x$ converges to $p \neq 1$, then the deformations of $G_x$ will converge to the undeformed geodesic $G_p$. This also
holds in the borderline case $p = 1$ since $\mathbb{D}(q_c; r)$ shrinks to a point as $q_c \to 0$. This convergence result is necessary to ensure that when counting Floer solutions on these domains, and $q_c$ bubbles off to 0, then the resulting domain (without the bubble) is the domain used for the OC-map, namely where geodesics are undeformed straight line segments from 0 to a boundary puncture.

Figure 12.

Let us call $K_c$ the map which counts isolated solutions of $(du - X \otimes \gamma)^{0,1} = 0$ for the above domains with the indicated auxiliary data. Now consider a 1-family of such solutions. For example, the first picture in Figure 13 (for simplicity we draw the undeformed geodesics). The possible degenerations are:

1. $q_c$ converges to the output 0 (possibly together with a subset of preferred points). These are counted by the action of $c$ on the $SC^*(E)$-output of OC (see the second picture in Figure 13);
2. $q_c$ converges to the input 1: in this case a subset of the boundary punctures and preferred points can converge together with $q_c$ to 1. These degenerations are counted by the action of $c$ on the input of OC (see the third picture in Figure 13);
3. $q_c$ does not converge to one of the punctures, but a subset of the boundary punctures converges to a boundary point $p$ giving rise to a disc bubble. These degenerations are counted by the map $K_c$ composed with $\mu_{W(E)}^S$ when $p \neq 1$, or $K_c$ composed with $\mu_{W(E)}^R$ when $p = 1$. In other words, $K_c$ composed with the Hochschild differential $b_{CC,(W(E))}$ defined in 2.3;
4. $q_c$ does not converge to one of the punctures, but there is a twice-punctured sphere that bubbles off at 0. This degeneration is counted by $\mu_{SC^*(E)}^1$ composed with $K_c$.

Figure 13.

In conclusion, the counts of these degenerations correspond to the equation

$$(c \bullet) \circ OC - OC \circ (c \bullet) = K_c \circ b_{CC,(W(E))} + \mu_{SC^*(E)}^1 \circ K_c$$

which shows that $K_c$ is the chain homotopy required to prove that on cohomology OC preserves the $SC^*(E)$-action.
7. Wrapped Fukaya category with local systems

7.1. Novikov rings \( \Lambda_0, \Lambda_n^0, \Lambda_\pm \). Let \( \Lambda_0 \subset \Lambda \) be the subring of series having only non-negative powers of \( t \) (see \( \text{[33]} \)). Let \( \Lambda_n^0 \subset \Lambda_0 \) be the multiplicative group of units: the series having non-zero \( t^0 \)-coefficient. So \( \Lambda_0 \) is a local ring with maximal ideal \( \Lambda_\pm \) the kernel of the evaluation \( \Lambda_0 \to \mathbb{K} \) of the \( t^0 \)-coefficient, so the series with only strictly positive powers of \( t \).

7.2. Lagrangians with local systems. The original motivation for local systems of coefficients came from the SYZ-viewpoint on mirror symmetry \( \text{[33]} \), and is very natural in the context of toric varieties more generally.

Accordingly, we modify the \( A_\infty \)-category \( \mathcal{W}(E) \) to \( \mathcal{W}(E) \) where now the objects \( L \) come equipped with a local system of coefficients \( \Lambda^L \) with fibre \( \Lambda \) and structure group \( \Lambda_0 \). So:

1. \( \Lambda^L \equiv \Lambda \) over each \( x \in L \);
2. for any path \( \gamma \) in \( L \) from \( x \) to \( y \), there is an isomorphism \( P^L_\gamma : \Lambda^L \to \Lambda^L \) given by multiplication by an element in \( \Lambda_n^0 \) depending only on \( \gamma \) \( \in \pi_1(L,x,y) \);
3. concatenating paths yields composed \( P^L \).

For such \( L_0, L_1 \) and Hamiltonian \( wH \) the Floer complex is now

\[
\mathcal{CF}^*(L_0, L_1; wH) = \bigoplus \text{Hom}_\Lambda(\Lambda_{L_0}^L, \Lambda_{L_1}^L)
\]

summing over all integer \( x \)-chords \( x \) of weight \( w \), and a Floer solution \( u : (\mathbb{D} \setminus \{ \pm 1 \}, \partial \mathbb{D}) \to (E, L_0 \cup L_1) \) asymptotic to Hamiltonian chords \( x \), \( y \) at \( -1, +1 \), which contributed \( \pm t^{e[w]} x \) to the old differential \( \partial_y \), now contributes

\[
\text{Hom}_\Lambda(\Lambda_{L_0}^L, \Lambda_{L_1}^L) \ni \phi_y \mapsto \pm t^{e[w]} P_{\gamma_1} \circ \phi_y \circ P_{\gamma_0} \in \text{Hom}_\Lambda(\Lambda_{L_0}^L, \Lambda_{L_1}^L),
\]

where \( \gamma_0, \gamma_1 \) are the paths in \( L_0, L_1 \) swept by \( u|_{\partial \mathbb{D}} \) along the oriented arcs in \( \partial \mathbb{D} \) connecting \(-1 \) to \(+1 \), and \(+1 \) to \(-1 \). This defines the new differential \( \partial_x \).

Equivalently, one considers flat \( \Lambda_n^0 \)-connections on the trivial line bundle \( L \times \Lambda \) over \( L \). So the holonomy is a homomorphism \( \pi_1(L) \to \Lambda_n^0 \), which determines a class \([b_L] \in H^1(L; \Lambda_n^0)\), and the parallel transport \( P^L_\gamma : \{ x \} \times \Lambda \to \{ y \} \times \Lambda \) for a path \( \gamma \) from \( x \) to \( y \) is given by multiplication by \( b_L[\gamma] \in \Lambda_n^0 \) (evaluation of cocycles on chains). Independence of the homotopy class \( \gamma \) is recorded by the fact that \( b_L \) is a cocycle.

So we get the usual \( \Lambda \)-module \( \mathcal{CF}^*(L_0, L_1; wH) = \oplus \Lambda x \) generated by \( x \)-chords of weight \( w \), however the above \( u \) contributes \( \pm t^{e[w]} b_{L_0}[\gamma_0] b_{L_1}[\gamma_1] x \) to \( \partial_y x \). For zero \( b_{L_0}, b_{L_1} \), which is a trivial local system, this recovers the usual \( \mathcal{CF}^* \).

The key observation is that if we consider a 1-family of discs \( u \) as above with fixed asymptotics \( x, y \), then the homotopy class of the arcs \( \gamma_0, \gamma_1 \) is constant in the family, so \( t^{e[w]} b_{L_0}[\gamma_0] b_{L_1}[\gamma_1] \) is constant. Thus \( \partial^2 \) is 0 if and only if the old \( \partial^2 = 0 \).

Similarly, the analogue \( \text{of \text{[33]}} \) is defined as follows: \( \pm t^{e[w]} b_{L_0}[\gamma_0] b_{L_1}[\gamma_1] x \) is a contribution to \( \partial_x y \) precisely when \( \pm t^{e[u]} x \) is a contribution to \( \partial_u y \). Together with \( \partial_x \) this defines \( \mu^{(\Lambda)}(\mathcal{W}(E)) \).

Similarly, \( \mu^{(\Lambda)}(\mathcal{W}(E)) \) now counts solutions \( u \) with weight \( \pm t^{e[w]} b_{L_0}[\gamma_0] b_{L_1}[\gamma_1] \cdots b_{L_n}[\gamma_n] \) where \( \gamma_i \) are the images via \( u \) of the oriented arcs in \( \partial \mathbb{D} \) which land in \( L_i \). Analogously, one defines the map \( \text{OC}_{\mathcal{W}(E)} : H_!^*(\mathcal{W}(E)) \to SH^*(E) \).

For Fukaya categories since \( H \) is not present and assuming \( L_0 \cap L_1 \), the \( x \) are just intersections points in \( L_0 \cap L_1 \). In particular, in \( \text{[33]} \) for \( L = L_0 = L_1 \) we model \( \mathcal{CF}^*(L, L) \) by locally finite chains \( C^0_\Lambda(L; \text{End}_\Lambda(\Lambda_L^L)) \) with local coefficients.
Mimicking $\mathbf{3.7}$ the new $m_0$'s now depend on $[b_L] \in H^1(L; \Lambda^*_0)$:

$$m_0(L, b_L) = \sum \tau^*[\beta] b_L[\beta] \text{ev}_*[\mathcal{M}_1(\beta)] \in C^0_{\dim L}(L, \Lambda^*_0)$$

summing over the homotopy classes $\beta \in \pi_2(E, L)$ of Maslov index 2, where $\mathcal{M}_1(\beta) \subset \mathcal{M}_1$ corresponds to the disc bubbles $u$ in class $[u] = \beta$ (see $\mathbf{3.7}$). As argued in $\mathbf{3.7}$ $\mathcal{M}_1(\beta)$ is a $\dim(L)$-cycle so $\text{ev}_*[\mathcal{M}_1(\beta)]$ is a multiple of $[L]$, and finally we define $m_0(L, b_L) \in \Lambda^0$ by

$$m_0(L, b_L) = m_0(L, b_L) [L].$$

Just like in the absence of local systems, $W(E)$ splits up according to the $m_0$-values (see $\mathbf{3.7}$). Even though typically the $m_0$-values have changed by introducing local systems (i.e. $m_0(L) \neq m_0(L, b_L)$), the possible $m_0$-values are still constrained to lie in the spectrum of eigenvalues of $c_1(TE)$, because Lemma $\mathbf{3.2}$ holds also in the presence of local systems. The advantage of local systems is that a wider range of eigenvalues can now arise geometrically as $m_0$-values.

We will argue in $\mathbf{9.11}$ that all our results for $W(E)$ hold also for $\tilde{W}(E)$.

8. The Generation Criterion

8.1. Split-generation.

Definition 8.1 (Split-generating [30 Sec.4c.3j]). A full subcategory $\mathcal{S}$ split-generates an $A_\infty$-category $\mathcal{A}$ if for each $A \in \text{Ob}(\mathcal{A})$, there is a twisted complex built from $\text{Ob}(\mathcal{S})$ admitting $A$ as a summand. Concretely: $\text{Ob}(\mathcal{A})$ is obtained by starting from $\text{Ob}(\mathcal{S})$, forming mapping cones and shifts, repeating that process arbitrarily often, and then taking images under all idempotent endomorphisms of the resulting objects. If it is not necessary to take images under idempotents, then we say $\mathcal{S}$ generates $\mathcal{A}$. We sometimes say $\text{Ob}(\mathcal{S})$ split-generates or generates $\mathcal{A}$, respectively.

8.2. The generation criterion in the monotone setting.

Generation Criterion (Abouzaid [2]). Let $(M, d\theta)$ be an exact symplectic manifold $M$ conical at infinity, and $\mathcal{S}$ a full subcategory of $W(M)$. If $1$ lies in the image of

\[ HH_*(\mathcal{S}) \rightarrow HH_*(W(M)) \xrightarrow{\text{OC}_M} SH^*(M), \]

then $\mathcal{S}$ split-generates $W(M)$.

By Theorem $\mathbf{6.2}$ it in fact suffices that $\text{im}(\text{OC}_M)$ contains an $SH^*(M)$-invertible:

Lemma 8.2. If $s \in \text{im}(\text{OC}_M)$ is invertible in $SH^*(M)$ then $1 \in \text{im}(\text{OC}_M)$.

Proof. $\text{OC}_M(x) = s$ for some $x \in HH_*(W(M))$, so since $\text{OC}_M$ is an $SH^*(M)$-module map by Theorem $\mathbf{6.2}$ (which of course holds also in the exact setup), $\text{OC}_M(s^{-1} \cdot x) = s^{-1} \cdot s = 1$. \(\square\)

In this Section, we will prove that the generation criterion holds in the monotone setting.

Theorem 8.3. For any monotone symplectic manifold conical at infinity, the generation criterion holds for the wrapped Fukaya category defined as in $\mathbf{3.4}$.

Remark 8.4. Analogously, if an idempotent summand $\iota$ of $1$ is in the image of $\text{OC}_E$ on a full subcategory $\mathcal{S}$ of $W(E)$, then $\mathcal{S}$ split-generates the summand of $W(E)$ corresponding to $\iota$. We explain this further in Remark $\mathbf{8.5}$.

The reason the proof of Abouzaid does not immediately carry over to the monotone setting is that the $A_\infty$-categories have been constructed in significantly different ways. Abouzaid’s construction relies on a subtle use of the Liouville flow, which is globally defined in the exact case but not in the monotone case.
8.3. Outline of the proof of the generating criterion. Fix an object $K$ in $\mathcal{W}(M)$, where $M$ is a monotone symplectic manifold which is conical at infinity. Recall $\mathcal{S}$ is a full subcategory of $\mathcal{W}(M)$. Let $\mathcal{L}$, $\mathcal{R}$ denote the Yoneda $\mathcal{S}$-modules
\[
\mathcal{L}(X) = \text{hom}_\mathcal{S}(K, X) = \text{CW}^*(K, X), \quad \mathcal{R}(X) = \text{hom}_\mathcal{S}(X, K) = \text{CW}^*(X, K),
\]
where $X \in \text{Ob}(\mathcal{S})$. Recall $\mathcal{R}$ represents the object $K$ (see [30 Sec.(2g)]). Using these as the $\mathcal{L}$, $\mathcal{R}$ in [24] for the $A_\infty$-category $\mathcal{S}$, we deduce
\[
\text{HH}_*(\mathcal{S}, \mathcal{L} \otimes \mathcal{R}) \cong H^*(\mathcal{R} \otimes_\mathcal{S} \mathcal{L}).
\]
In [8.4] we will construct the coproduct $\Delta$, which is an $\mathcal{S}$-bimodule map
\[
\Delta : \mathcal{S} \to \mathcal{L} \otimes \mathcal{R},
\]
in particular $H^*(\Delta^0) : \text{HW}^*(X, X') \to \text{HW}^*(K, X) \otimes \text{HW}^*(X', K)$ is the usual coproduct. In Section [8.3] we will construct the closed-open string map $\text{CO} : \text{SH}^*(M) \to \text{HW}^*(K, K)$, constructed in analogy with the OC-maps.

This yields two interesting composite maps:
\[
\begin{array}{ccc}
\text{C}_1 : \text{HH}_*(\mathcal{S}, \mathcal{S}) & \xrightarrow{\text{OC}} & \text{SH}^*(M) & \xrightarrow{\text{CO}} & \text{HW}^*(K, K) \\
\text{C}_2 : \text{HH}_*(\mathcal{S}, \mathcal{S}) & \xrightarrow{\text{HH}_*(\Delta)} & \text{HH}_*(\mathcal{S}, \mathcal{L} \otimes \mathcal{R}) & \cong H^*(\mathcal{R} \otimes_\mathcal{S} \mathcal{L}) & \xrightarrow{\mu^\beta} \text{HW}^*(K, K) \\
\end{array}
\]
using the $\mu^\alpha_S$ to compose all morphisms of the chain complex $\mathcal{R} \otimes_\mathcal{S} \mathcal{L}$.

The key ingredient of the proof is to show that these composites are equal, which we will prove in [8.6]. The hypothesis that $\text{OC}$ hits $1 \in \text{SH}^*(M)$ implies $\text{C}_1$ hits the identity $e_K \in \text{HW}^*(K, K)$ since $\text{CO}$ is a ring homomorphism. Therefore $\text{C}_2$ hits $e_K$. For purely algebraic reasons [2 Appendix A] this implies that $\mathcal{R}$ (which represents $K$) is a summand of a twisted complex built from objects in $\mathcal{S}$. Therefore any object $K$ in $\mathcal{W}(M)$ is split-generated by objects in $\mathcal{S}$, as required.

We illustrate that purely algebraic reasoning in a very simple example: when $\mathcal{S}$ is generated by a single object $X$, and we assume that $\mu^2_S(\alpha, \beta) = e_K$ for some $\alpha \in \text{CW}^*(X, K)$, $\beta \in \text{CW}^*(K, X)$, let’s abbreviate that by $\alpha \cdot \beta = e_K$. Then $p = \beta \cdot \alpha \in \text{HW}^*(X, X)$ is an idempotent: $p \cdot p = \alpha \cdot (\beta \cdot \alpha) = \beta = p$. Thus
\[
\text{HW}^*(X, K) = H^*(\mathcal{R}(X)) \xrightarrow{\beta} \text{HW}^*(X, X) \xrightarrow{\alpha} \text{HW}^*(X, K) \quad \alpha \cdot \beta = e_K, \quad \beta \cdot \alpha = p
\]
displays $H^*(\mathcal{R}(X))$ as a summand of $\text{HW}^*(X, X)$: the image of the projection $p$.

Remark 8.5. Suppose $\mathcal{S}$ is a full subcategory of $\mathcal{W}(E)$ such that
\[
\text{HH}_*(\mathcal{S}) \to \text{SH}^*(E) \to \text{HW}^*(K, K)
\]
hits the identity $e_K$, and $K \in \text{Ob}(\mathcal{W}(E)_\lambda)$ lies in the summand $\mathcal{W}(E)_\lambda$ of $\mathcal{W}(E)$ corresponding to the eigenvalue $\lambda$ of $c_1(TE) \ast : \text{QH}^*(E) \to \text{QH}^*(E)$. Then in fact
\[
\text{HH}_*(\mathcal{S}_\lambda) \to \text{SH}^*(E)_\lambda \to \text{HW}^*(K, K)
\]
hits $e_K$, since the maps are $\text{QH}^*(E)$-module maps (by Theorem [6.3]). So it suffices to consider Lagrangians with a single $m_0$-value in the subsequent proof of the generation criterion. In particular, we never encounter holomorphic annuli with boundaries on Lagrangians with different $m_0$-values.

This outline is precisely the argument of Abouzaid [2], so the only difference for our setup is that we need to construct $\text{CO}$, $\Delta$ and prove the equality of $\text{C}_1$, $\text{C}_2$ in the monotone setting.
8.4. The closed-open string map. \( CO : SH^*(E) \to HW^*(K, K) \) arises at the chain level as the sum of certain maps \( CO^{p,w} : HF^*(w_*H)[q] \to CW^*(K, K; w_0H)[q] \). We now define \( CO^{p,w} \) by mimicking \( \mu_{\text{OC}} \). Consider

\[
S \cong \mathbb{D} \setminus \{0, -1\}
\]

(recall we used \( S \cong \mathbb{D} \setminus \{0, +1\} \) for \( OC^{0(0)} \)). Denote the interior puncture by \( z_* = 0 \) and the boundary puncture by \( z_0 = -1 \). Mimicking \( \mu_{\text{OC}} \) the auxiliary data is: an index set \( F \) with \( p \) (constant) map \( p : F \to \{0\} \), weights \( w = (w_1, w_0) \) with \( w_0 = w_* + |F| \), a closed form \( \alpha_0 \), and sub-closed forms \( \beta_f \) indexed by \( F \). The total 1-form on \( S \) is \( \gamma = w_0\alpha_0 + \sum \beta_f \), which pulls-back to \( w_* dt \) near \( z_* \) and \( w_0 dt \) near \( z_0 \). We can assume \( F \) is either \( \{0\} \) or empty, depending on whether we have a geodesic with one preferred point joining \( z_*, z_0 \) or not (the count of other configurations will cancel for symmetry reasons as in \([S]\) Lemma 3.7)).

Define the \( q^0x_* \) coefficient of \( CO^{p,w}(q^0x_*) \) as the count with weight \( \pm t_{E_{\text{typ}}(n)} \) of isolated solutions \( u : S \to E \) of \( (du - X \otimes \gamma)^{0,1} = 0 \) with Lagrangian boundary condition \( K \) and asymptotic conditions \( x_*, x_0 \) at \( z_* \) or \( z_0 \), using auxiliary data \( F = \{0\} \) if \( i = 1 \) and \( F = \emptyset \) if \( i = 0 \). The \( q^1 \) coefficient of \( CO^{p,w}(q^1x_*) \) is determined by requiring \( CO^{p,w} \) is \( R_q \)-equivariant.

An easy check shows \( CO \) is indeed a chain map. To prove that \( CO \) is a ring homomorphism on cohomology, we argue as in Theorem 6.2: consider the domain \( s \) as above except \( x_* \) is replaced by two interior punctures \( z_{1,1}, z_{1,2} \) lying on the geodesic joining \( -i, i \in \mathbb{D} \) symmetrically about \( 0 \). In a 1-family with fixed auxiliary data, if \( z_{1,1}, z_{1,2} \) converge together to \( 0 \), a 3-punctured bubble breaks off giving rise to the product on \( SH^* \); if \( z_{1,1}, z_{1,2} \) converge to the boundary then two disc bubbles form, contributing to the composite \( \mu_{\text{CW}^*(K,K)} \circ CO^{\otimes 2} \).

Remark 8.6. One can extend the above closed-open string map to

\[
CO_E : SH^*(E) \to HH^*(\mathcal{W}(E)),
\]

by having several outputs as opposed to just one output, and then introducing auxiliary data as we did for the open-closed string map \( OC_E \) of Section 8. In particular, this map is an acceleration of the analogous homomorphism \( CO_E : QH^*(E) \to HH^*(\mathcal{F}(E)) \) constructed using a small Hamiltonian \( H = H^E \) (compare \([7,0]\)).

The map \( CO_E \) is a module homomorphism over \( QH^*(E) \) and also over \( SH^*(E) \) (which is proved as for \( OC_E \) in Theorem 6.2), and it is a ring homomorphism (unlike \( OC_E \)).

8.5. The coproduct. For weights \( w = (w_r, \ldots, w_1) \), abbreviate

\[
\mathcal{W}(L_r, \ldots, L_0; w) = CF^*(L_{r-1}, L_r; w_*H)[q] \otimes \cdots \otimes CF^*(L_0, L_1; w_1H)[q].
\]

We need to define for \( r, s \geq 0 \), a map \( \Delta^r[s,p,w,p^w](\mathcal{W}, \mathcal{W}) \)

\[
\begin{align*}
\mathcal{W}(L_r, \ldots, L_0; w) & \otimes CF^*(L_0, L_0; w_0H)[q] \otimes \mathcal{W}(L'_r, \ldots, L'_{s}; w') \\
& \longrightarrow CF^*(K, L_r; w_0H)[q] \otimes CF^*(L_0, K; w_0H)[q],
\end{align*}
\]

Observe Figure 13. We are now considering a universal family \( \mathcal{R}^{r,s,p,p'} \to \mathcal{R}^{r+s+1} \) in which there are two distinguished boundary punctures \( z_0, z'_0 \) on the Riemann surface \( S \) which serve as the outputs for \( \Delta \), and one distinguished puncture \( z \) which serves as the input \( \Delta \) receives from the module \( CF^*(L_0, L_0; w_0H) \). These distinguished \( z_0, z'_0 \) are fixed to lie at \( 1, e^{2\pi i/3}, e^{4\pi i/3} \), after identifying \( S \) with the disc \( D \). This kills the reparametrization group of \( \mathcal{R}^{r+s+1} \) and the local coordinates for \( \mathcal{R}^{r+s+1} \) are given by the angular coordinates of the punctures \( z_1, \ldots, z_r \) lying on the arc \( z, z_0 \) and of the punctures \( z'_1, \ldots, z'_r \) lying on the arc \( z'_0, z' \). The punctures are ordered as shown in Figure 13.

We construct inductively two sets of auxiliary data, mimicking \( \mu_{\text{OC}} \) So \( w = (w_r, \ldots, w_1) \) and \( (w', w^w) = (w'_r, w'_1, \ldots, w'_s) \) are integer weights; \( F, F' \) are finite sets of indices \( f, f' \), coming with maps \( p : F \to \{r, \ldots, 1\}, p' : F' \to \{\infty, 1, \ldots, s\} \) defining labels \( p_f, p'_f \). (possibly not
distinct), where we use ∞ to denote \( z = z_{∞} \); we have sub-closed forms \( β_{f}, β_{f'} \); we have closed forms \( α, \ldots, α \) and \( α', α', \ldots, α' \). Finally this defines the total 1-form on \( S \):

\[
γ = \sum_{k=1}^{n} w_{k}α_{k} + \sum_{j=1}^{m} w'_{j}α'_{j} + \sum_{f \in F} β_{f} + \sum_{f' \in F'} β_{f'},
\]

with \( dγ ≤ 0, dγ = 0 \) near \( ∂S \). The pull-back of \( γ \) to \( ∂S \) is 0. Define \( w_{0} = \sum w_{k} + |F| \) and \( w'_{0} = \sum w'_{j} + |F'| \). Near the ends, in the relevant parametrization, \( γ \) has the form: \( w_{k} dt, w'_{j} dt, w_{0} dt, w'_{0} dt \) respectively near \( z_{k}, z_{j}', z_{0}, z_{0}' \). This is consistent with Stokes:

\[
0 ≤ -\int dγ = w_{0} - \sum w_{k} - \sum w'_{j} - w'_{0} = |F| + |F'|.
\]

By mimicking 3.12 we obtain a moduli space \( R^{r}x_{p}=p'(x, x') \) of solutions \( u : S → E \) of \( (du - X \otimes γ)^{0,1} = 0 \) satisfying the relevant asymptotic and boundary conditions. Arguing as in [8, Lemma 3.7], we can always assume that there is never more than one preferred point on a geodesic, since for symmetry reasons the contributions of isolated solutions will cancel when \( p \) or \( p' \) is not injective.

Let us abbreviate by \( ψ \) the map we need to define above. We will define it inductively on the total degree of \( q \) arising in the input (compare the definition of the OC maps in 6.1). We will later define the \( (q^{0}, q^{0}) \)-coefficient of the output of \( ψ \). Once this is done, by imposing \( \partial_{q} \)-equivariance of \( ψ \), the \( (q^{1}, q^{0}) \) - and the \( (q^{0}, q^{1}) \)-coefficients are determined. Finally the \( (q^{1}, q^{1}) \)-coefficient is determined by imposing \( \partial_{q} \)-equivariance of \( ψ \). An issue might arise at this stage: \( \partial_{q} q x \otimes q y = x \otimes q y + (-1)^{|x|+1} q x \otimes y = 0 \) (recall \( q \) has degree \(-1) \), so for consistency one needs in general that for a tensor \( w \) the output \( ψ(∂q w) = x \otimes q y \) and \( q x \otimes y \) arising with the same coefficient up to \((-1)^{|x|+1}\). But this is indeed the case by considering \( ψ(∂q∂q w) = 0 \) (using \( ∂q ∂q = 0 \)). Since \( q^{1} = 0 \), we have determined \( ψ \).

We now define the \( (q^{0}, q^{0}) \)-coefficient of the output of \( ψ \). Just as in 6.1 the only case when the output is non-zero is when the input \( q^{r} x_{1} \otimes \cdots \otimes q^{r} x_{1} \otimes q^{1} x_{1} \otimes q^{1} x_{1} ' \otimes q^{0} \) comes with the correct powers of \( q \), that is \( i_{k} = 1 \) iff \( k \in p(F) \) (respectively \( i_{j}' = 1 \) iff \( j \in p'(F') \)). In other words there is a \( q \) iff there is a geodesic (with one preferred point) joining the relevant puncture with \( z_{0} \) (respectively \( z_{0}' \)). If this is the case, then the \( x_{0} \otimes x_{0}' \) coefficient of the output is a count with weight \( ±e^{K_{x_{0},x_{0}'}(u)} \) of the isolated solutions \( u \in R^{r}x_{p}=p'(x, x', x') \).

That \( Δ \) satisfies the \( A_{∞} \)-equations required from an \( S \)-bimodule map \( S → L \otimes R \) is now an exercise in bookkeeping, compare Remark 4 in [8.13]. The key observation is that in codimension 1, either a subset of the punctures (and possibly some preferred points) converges to a common point on the boundary that is not \( z_{0}, z_{0}', z_{0} \) or a subset of the punctures and/or preferred points converges to one of \( z_{0}, z_{0}' \). In the first case it gives rise to a \( µ_{S} \)-term (which is part of the bar differential for the \( S \)-bimodule \( S \)). In the second case: if the common point

![Figure 14. Left: \( Δ^{0,0} : CF^*(L_{0}, L_{0}; w_{0} H) → CF^*(K, L_{0}; w_{0} H) \) \( \otimes CF^*(L_{0}, K; w_{0} H) \). Right: \( Δ^{0,0}(x, x, q x_{2}, x, q x_{1}, x_{2}, q x_{1}, x') \) using Hamiltonian chords \( x, x_{1}, x_{2}, x_{1}' \) as asymptotics at \( z_{0}, z_{0}' \).](attachment://figure.png)
is \( z_0 \) it gives rise to a \( R^{10} \)-term; if \( z'_0 \), it gives rise to a \( R^{01} \)-term; if \( z \), it gives rise to a \( R^{11} \)-term. In particular if no punctures but exactly one preferred point converges to a puncture then we get the \( R \)-contributions in \( R^{01} \), \( R^{10} \), \( R^{11} \).

8.6. The key ingredient. We now prove that the composites \( C_1, C_2 \) from \( C_3 \) agree. This involves studying a moduli space of annuli, which in Abouzaid’s case is described in [2] Appendix C4,C5, and in our case will be endowed with additional decorations coming from geodesics and preferred points. Keeping track of this additional data may seem cumbersome at first, but it has the advantage that all gluings are natural: so in fact we will not need the first homotopy of Abouzaid [2 Sec.6.2] which changes the weights so that gluing becomes possible, and we will only need to construct the analogue of his second homotopy [2 Sec.6.3].

Consider the moduli space of all holomorphic annuli with two boundary punctures, one on each component of the boundary. For example the family of annuli

\[
\varepsilon_r = \{ z \in \mathbb{C} : 1 \leq |z| \leq r \} \setminus \{-1, +1\}.
\]

By doubling up an annulus, one obtains an elliptic curve \( \varepsilon \) with two marked points together with a real structure (an antiholomorphic involution \( \sigma \), whose fixed locus are the two boundary circles of the annulus, and it fixes the two marked points). The pairs \((\varepsilon, \sigma)\) form, via an étale forgetful map, a subset of the Deligne-Mumford space \( \mathcal{M}_{1,2} \) (recall \( \dim_{\mathbb{C}} \mathcal{M}_{1,2} = 2 \)). This is described in detail in Fukaya-Oh-Ohta-Ono [18 Chp.2].

The family \( \varepsilon_r \) determines a real 1-dimensional family in \( \mathcal{M}_{1,2} \) parametrized by the modular parameter \( \varepsilon' \in [e^1, \infty] \). A typical codim\(=1 \) degeneration of \( \mathcal{M}_{1,2} \) involves the shrinking of a hyperbolic geodesic in the punctured elliptic curve, which in the limit gives rise to a nodal curve with one node. However, since our family \( \varepsilon_r \) comes with real structures, the boundary strata will in fact carry two nodes (and \( \sigma \) either fixes or interchanges the two nodes), and so the elliptic curves in the family \( \varepsilon_r \) will degenerate into two spheres joined at two nodes.

When \( r \to \infty \), one of the two nodes is forming in the interior of our annulus, thus in the limit the annulus breaks into two discs joined at an interior point as in picture 1 of Figure 15 ignoring all decorations (this is the case when \( \sigma \) swaps the nodes). When \( r \to 1 \), the two nodes arise as a limit of two pairs of boundary points of the annulus connected by two shrinking geodesic arcs. In the limit this gives rise to two discs joined at two nodes as in picture 5 of Figure 15 ignoring all decorations (this is the case when \( \sigma \) fixes the nodes). Observe that in both cases the two original marked points will lie on opposite components of the nodal curve (in the second case this is because one disc would otherwise be unstable).

An analogous argument can be carried out when there are several marked points on the outer circle \( |z| = r \), which we view as a family lying over our family \( \varepsilon_r \) via a forgetful map. Thus the punctures at \(-1, +1\) are distinguished, which is why in Figure 15 we mark them with a dash, and we mark the other punctures with a cross.

The dotted lines in pictures 2 and 3 of Figure 15 are examples of hyperbolic geodesics which shrink in the limit to give respectively pictures 1 and 5. In picture 2, we work in the \((s, t)\) plane, where \( z = e^{s+it} \) parametrizes the annulus, so we’ve passed to the cover \( \mathbb{R} \to \mathbb{R}/\mathbb{Z} \) of the angular variable. The limit \( r \to \infty \) corresponds to stretching the strip 2 near the dotted line. Picture 4 is biholomorphic to 3, and shrinking the dotted lines in 3 corresponds to stretching the two handles in 4. The dotted lines play the same role as the “cuts” in Figures 4 and 2.

Observe our moduli space of annuli in picture 2 or 3 of Figure 15. The distinguished boundary punctures are \( z = -1 \) (which will be the output) and \( z' = z_\infty = +r \) (which will be the bimodule input), and the other boundary punctures \( z_1, \ldots, z_n \) ordered counterclockwise on the boundary \( s = r \). Mimicking [3,11], the auxiliary data is: a finite indexing set \( F \) with a map \( p : F \to \{\infty, 1, \ldots, n\} \) which keeps track of the presence of preferred points on geodesics...
joining $z$ with the other punctures (with the important convention that the geodesic joining $z, z'$, if present, is always the one which increases $t$); weights $w', w_1, \ldots, w_n$; closed forms $\omega, \omega_1, \ldots, \omega_n$; sub-closed forms $\beta_f$. These define a total $1$-form $\gamma = \sum w_k \omega_k + w' \omega + \sum \beta_f$, which pulls back to $w_k \ dt, w' \ dt$ near $z, z', z$ where $w = \sum w_k + w' + |F|$.

Consider a $1$-dimensional moduli space of solutions $u : S \to E$ of $(du - X \otimes \gamma)_0, 1 = 0$ defined on a fixed annulus $S$, and satisfying the relevant boundary and asymptotic conditions. Pictures 1 and 5 of Figure 15 are two examples of boundary points of this moduli space. Notice there are geodesics in pictures 1 and 5 which do not carry preferred points: this is a natural phenomenon that arises also in the breaking analysis for the $\mu_S$-maps, and Remarks 1 and 2 in 3.11 and Remarks 3 and 4 in 3.13 explain this: these geodesics encode how the breaking has occurred, and each breaking yields distinct composites of the relevant operators. If the cut in picture 2 had included more than one preferred point on the left, then picture 1 would have two or more preferred points on the same geodesic, and those broken configurations would cancel out at the level of operators (algebraically: $q^2 = 0$).

We conclude that (ignoring all $q$’s, to keep the discussion comprehensible) that picture 1 is a contribution to the coefficient of $x$ in

$$\text{CO} \circ \text{OC}(x', x_5, x_4, x_3, x_2, x_1),$$

whereas picture 5 is a contribution (ignoring all $q$’s) to the coefficient of $x$ in

$$\mu^3_S \circ \mathcal{T} \left( \Delta^{211}(x_2, x_1, x', x_5) \otimes x_4 \otimes x_3 \right),$$

where $\mathcal{T}$ is the reordering isomorphism from $\text{CC}^*$ which induces $\text{HH}_*(\mathcal{S}, \mathcal{L} \otimes \mathcal{R}) \cong H^*(\mathcal{R} \otimes_S \mathcal{L})$ (notice above that we are inputting into $\mathcal{T}$ a typical contribution to $\text{CC}^*(\mathcal{S})(x', x_5, x_4, x_3, x_2, x_1)$ as prescribed by Lemma 2.3).

This example now easily generalizes to show that the boundary points of such $1$-dimensional moduli spaces include the counts involved in the maps $C_1 = \text{CO} \circ \text{OC}$ and $C_2 = \mu^\text{max}_S \circ \mathcal{T} \circ \text{CC}^*(\Delta)$. There are however also two other contributions which we now describe.

Let $\varphi : \text{CC}^*(\mathcal{S}) \to \text{CW}^*(K, K)$ denote the weighted count of isolated solutions $u$ defined on an annulus as above. Now consider again a $1$-dimensional family of solutions on the annulus. Firstly, when a subset of the outer boundary punctures converges to a common point, we obtain bubbling configurations counted by $\varphi \circ b$ where $b$ is the bar differential for $\text{CC}^*(\mathcal{S})$. Secondly, when bubbling occurs on the inner boundary circle of the annulus, the broken configurations are counted by $\mu^{1}_{\text{CW}^*(K, K)} \circ \varphi$.
We remark on a particular case of the first type of bubbling: when the bubble carries none of the boundary punctures. These are the terms involving the \( m_0 \)'s that are supposed to arise in the bar differential and in the \( \mu_S \)-maps (see \[3.7\]). These terms will all cancel out since they cancel out for the \( A_\infty \)-composition maps.

Since the oriented count of boundary points of a compact 1-dimensional manifold is zero, we conclude that \( C_1 \) and \( C_2 \) differ at the chain level by \( \phi \circ b - \mu_{CW^*} (K,K) \circ \phi \), therefore they agree on cohomology. This concludes the proof of Theorem \[8.3\].

9. Geometric Constructions

9.1. Notation: \( B, E, \Gamma, B, E, W(E) \). From now on:

1. \( \Lambda \) is the Novikov ring (see \[5.3\]):
2. \( (B, \omega_B) \) is a closed monotone symplectic manifold;
3. \( B \) is the Fukaya category \( \mathcal{F}(B) \) of \( B \) (see \[5.1\]);
4. \( H^B : B \to \mathbb{R} \) is a Hamiltonian on \( B \) (see \[4.1\]);
5. \( (E, \omega) \) is a monotone symplectic manifold conical at infinity, see \[3.1\] (in Section \[11\] we will discuss the case when \( E \) is compact rather than non-compact);
6. \( E \) is the generalized Fukaya category of \( E \) (discussed in \[9.7\]);
7. \( W(E) \) is the wrapped Fukaya category of \( E \) (see \[3.4\]);
8. \( H^E : E \to \mathbb{R} \) is a Hamiltonian on \( E \) linear at infinity of small slope (see \[4.2\]);
9. \( B \times E \) denotes the product \( B \times E \) with symplectic form \( (\omega_B \otimes \omega_E) \) (the rescaling makes \( B \times E \) monotone with monotonicity constant \( \lambda_E \));
10. \( \Gamma \) is a closed Lagrangian correspondence from \( B \) to \( E \), that is: \( \Gamma \subset B \times E \) is a closed monotone oriented Lagrangian submanifold.

In this section, we use the quilt machinery of Wehrheim-Woodward \[37, 38, 39\]. This allows us to define maps by pictures of quilts and allows us to do gluing arguments pictorially.

9.2. The quilt morphism \( HF^*(H^B) \to HF^*(H^E) \). Define

\[
q_{\Gamma} : CF^*(H^B) \to CF^*+|\Gamma|(H^E)
\]

by counting isolated quilt maps as in Figure 16 (we use cohomological conventions), where \( H^E = H \) is small (see \[4.2\], \[4.4\]) and where \( |\Gamma| \) is the degree of the map which we will determine later. The quilt domain is an infinite cylinder \( \mathbb{R} \times S^1 \), with the seam condition \( \Gamma \) on the circle \( \{0\} \times S^1 \). After folding, the quilt maps are of the form

\[
u : (-\infty, 0] \times S^1 \to \overline{B} \times E
\]
satisfying the Floer equation

\[
\partial_s u + ((-J^B) \oplus J^E)[\partial_s u - c(s)X_{H^B} \oplus X_{H^E}] = 0,
\]

with boundary condition \( u|_{\{0\} \times S^1} \subset \Gamma \), where \( c : \mathbb{R} \to [0, 1] \) is a decreasing cut-off function which equals 1 for \( s \leq -2 \) and equals 0 for \( s \geq -1 \).

In general, such maps \( q_{\Gamma} \) have been investigated before in \[39\] Sec.6.1.

\[\text{Figure 16. } q_{\Gamma} : CF^*(H^B) \to CF^*+|\Gamma|(H^E).\]
9.3. The quilt morphism \( \text{QH}^*(B) \to \text{QH}^*(E) \). Abbreviate dimensions by
\[ |B| = \dim B, \quad |E| = \dim E. \]

**Theorem 9.1.** The composite (using the PSS maps from [4,4])
\[ Q_{\Gamma} : \text{QH}^*(B) \xrightarrow{\sim} \text{HF}^*(B) \xrightarrow{\phi} \text{HF}^+|\gamma|(|H^E|) \xrightarrow{\sim} Q \text{H}^+|\gamma|(|E|) \]
is identifiable with the homomorphism
\[ Q_{\Gamma} : \text{QH}_{|B|+|E|}(\overline{B} \times E) \to \Lambda, \quad (\alpha, \beta) \mapsto (\alpha, \beta) \bullet [\Gamma] \]
where \( \bullet \) is the intersection product \( H_*(\overline{B} \times E)^{\otimes 2} \to \mathbb{Z} \), and \( |\gamma| = |E| - |B| \).

**Proof.** By Poincaré duality, we identify
\[ PD_B : \text{QH}^*(B) \cong \text{QH}_{2|B|-*}(B) \quad PD_E : \text{QH}^*(E) \cong \text{QH}^*_2|E|-*(E) \]
For details about locally finite homology, we refer to [27].

Consider the intersection product
\[ \bullet : H_*(E) \otimes H^*_2|E|-*(E) \to \mathbb{K}. \]
In general, suppose we are given a locally finite pseudocycle \( ev : M \to E \) defined as an evaluation map at some marked point for some moduli space \( M \). Say \( ev_*[M] = \sum \lambda_i \gamma_i \), where \( \lambda_i \in \Lambda \) and \( \gamma_i \) is a basis of lf-cycles for \( E \). Then \( \lambda_i \) is the (Novikov-weighted) count of solutions in \( M \) which intersect the cycle \( \gamma_i \) at that marked point, where \( \gamma_i^* \) is the dual to \( \gamma_i \) with respect to \( \bullet \).

Using Poincaré duality, \( Q_{\Gamma} : \text{QH}_{2|B|-*}(B) \to \text{QH}^*_2|E|-*|\gamma|(|E|) \), then using \( \bullet \)-duality:
\[ Q_{\Gamma} : \text{QH}_{2|B|-*}(B) \otimes Q \text{H}^*_2|E|-*|\gamma|(|E|) \to \Lambda. \]
By a standard gluing argument, this counts certain quilt maps whose domain is a sphere \( S^2 \), with seam condition \( \Gamma \) along the equator. By a standard Floer homotopy argument we can assume the Hamiltonians have been homotoped to zero. After folding the sphere to obtain a disc with boundary condition \( \Gamma \), \( Q_{\Gamma}(\alpha, \beta) \) counts isolated \((-J^B) \oplus J^E\)-holomorphic discs
\[ u : \mathbb{D} \to \overline{B} \times E, \quad u|_{\partial \mathbb{D}} \subset \Gamma \]
intersecting \((\alpha, \beta)\) at \( z = 0 \) (here \( \mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \} \)). But no such non-constant \( u \) will be isolated: there is a rotational \( S^1 \)-symmetry of the domain (more precisely: the geometric image of the evaluation map from the fiber product of the moduli space of discs crossed with the domain, modulo reparametrization, evaluating into the manifold \( \overline{B} \times E \) will sweep a cycle that is too small to intersect the cycle condition imposed at the centre of the disc). So the only solutions being counted are the constants \( u \in (\alpha \times \beta) \cap \Gamma \) which yield the ordinary intersection product. Thus \( \dim \Gamma = 2|B| + |\gamma| \). Finally apply Kunneth’s theorem. \( \square \)

**Remark 9.2.** We remark that the map in the Theorem, which is part of a commutative diagram involving the OC map and the map \( HH_*(\mathcal{F}(B)) \to HH_*(\mathcal{F}(E)) \), can be constructed entirely in the holomorphic world (that is without appealing to \( HF^*(H) \) groups). The fact that it agrees with the above glued map boils down to the fact that the two PSS maps are indeed inverse to each other at the level of cohomology. By just working in the holomorphic world, one would bypass the “homotope \( H \) to zero” part of the argument. The way the argument is written above, one would actually need to explain why, after homotopying \( H \) to zero, there are no Gromov limits involving sphere/disc bubbles. The reason is that the principal component of the Gromov limit has to have full index otherwise the moduli space will not intersect the cycle condition, therefore the bubbles have to have index zero. But Chern number zero spheres impose a real codimension 4 condition so this bubbling does not occur for dimension reasons;
and Maslov index zero discs are constant by monotonicity so this is not actually a bubbling. This argument relies on the fact that the almost complex structure that we use when counting solutions must also be regular for the moduli spaces of spheres and of discs bounding the relevant Lagrangians, which of course is assumed and not problematic in the monotone setting.

9.4. The deformed quilt morphisms $Q_{\Gamma,\gamma} = 0$. If $\Gamma$ is null-homologous, then $Q_{\Gamma} = 0$. So to have a chance at making $Q_{\Gamma}$ non-zero, we would need to kill the $S^1$-freedom which arose in the previous proof by introducing an extra boundary marked point. We now show how this gives rise to a deformed map depending on a class in $\gamma \in HF^*(\Gamma, \Gamma)$.

Fix a puncture labeled $p_\gamma$ on the seam $\Gamma$ in Figure 16. Define $q_{\Gamma,\gamma}$ by counting maps as before, with the additional condition that $p_\gamma$ is an input for the Floer cycle $\gamma \in C_*(\Gamma; \Lambda) = CF^*(\Gamma, \Gamma)$. Observe that locally near $p_\gamma$ one can fold the solutions $u$ to give a local map $u = (u_{\overline{\mathbb{D}}}, u_E) : \mathbb{C} \cap \{ z \neq 0 : \text{Im}(z) \geq 0 \} \to \overline{B} \times E$ defined near $p_\gamma = (z = 0)$, with Lagrangian boundary condition $u(i\mathbb{R}) \subset \Gamma$, so the input $\gamma$ at $z = 0$ is well-defined. Moreover, observe that the following degenerations of 1-families of solutions can occur: breakings at the ends which define the Floer differentials for $H^B$ and $H^E$, and discs bubbling on $\Gamma$. The discs bubbling on $\Gamma$ must carry the boundary puncture $p_\gamma$, otherwise they would have too high index (recall $\Gamma$ is monotone and oriented). The disc bubbles carrying the boundary puncture $p_\gamma$ correspond to the Floer differential $\partial_{\text{Floer}}(\gamma)$, but this vanishes since $\gamma$ is a Floer cocyle. Therefore $q_{\Gamma,\gamma}$ is a chain map. Capping off the ends, using the PSS-maps, defines the map $Q_{\Gamma,\gamma}$.

**Remark 9.3.** It is not a priori known that $HF^*(\Gamma, \Gamma) \neq 0$. The undeformed map $Q_{\Gamma}$ corresponds to the case $\gamma = [\Gamma] \in HF^*(\Gamma, \Gamma)$.

**Theorem 9.4.** The deformed quilt homomorphisms

$$Q_{\Gamma,\gamma} : QH^*(B) \to QH^*(E)$$

are determined by the homomorphism

$$Q_{\Gamma,\gamma} : QH_*(\overline{B} \times E) \otimes HF^*(\Gamma, \Gamma) \to HF^*(\Gamma, \Gamma) \to \Lambda,$$

the first map is quantum intersection product, the second is the augmentation which projects to the coefficient of the generator $[\text{pt}]$ in $HF^*(\Gamma, \Gamma)$.

**Proof.** Arguing as for Theorem 9.1 we may identify $Q_{\Gamma,\gamma}$ with the map

$$QH_k(\overline{B} \times E) \times \{ [\gamma] \} \subset QH_k(\overline{B} \times E) \otimes H_*(\Gamma; \Lambda) \to \Lambda$$

which maps a generic cycle $(\alpha, \beta) \in QH_*(\overline{B} \times E)$ to the (Novikov-weighted) count of isolated $(-J^B) \oplus J^E$-holomorphic $u : (\overline{D}, \partial \overline{D}) \to (\overline{B} \times E, \Gamma)$, with $u(0) \in (\alpha, \beta)$ and $u(1) \in \gamma$ (where $p_\gamma = 1$ is the fixed marked point). For dimension reasons dictated by the quantum product, this forces $k = 2(|B| + |E|) - |\gamma|$ (using the homological grading for $\gamma$).

Next, we recall the quantum intersection product action in our setup, and we refer to Auroux [12 Sec.6.1] for details. Fix the three marked points $\pm 1, 0$ on the disc $D$ (alternatively, if one later quotients by domain reparametrizations, take two distinct freely moving boundary marked points and an interior marked point free to move along the hyperbolic geodesic joining the two boundary marked points).

Consider the moduli space $\mathcal{M}_{\Gamma, b}$ of $(-J^B) \oplus J^E$-holomorphic discs $u : (\overline{D}, \partial \overline{D}) \to (\overline{B} \times E, \Gamma)$, lying in class $[u] = b \in \pi_2(\overline{B} \times E, \Gamma)$. Let

$$\text{ev}_{b, \pm 1} : \mathcal{M}_{\Gamma, b} \to \Gamma \quad \text{ev}_{b, 0} : \mathcal{M}_{\Gamma, b} \to \overline{B} \times E$$

denote the evaluations at $\pm 1$ and 0 respectively. Given chains $(\alpha, \beta) \in C_*(\overline{B} \times E)$, $\gamma \in C_*(\Gamma)$ such that $\gamma \times (\alpha, \beta)$ is transverse to $\text{ev}_{b, 1} \times \text{ev}_{b, 0}$, define

$$(\alpha, \beta) \ast_b \gamma = (\text{ev}_{b, -1} \ast_b \text{ev}_{b, 1} \times \text{ev}_{b, 0})^{-1}(\gamma \times (\alpha, \beta)) \in C_*(\Gamma).$$
Define the weight \( \omega[b] = \int_D u^*\omega \) for any \( u \in \mathcal{M}_{\Gamma,b} \). The grading convention is

\[
|\tau^{\omega}[b]| = 2\lambda_E\omega[b] = 2\lambda_E\lambda_T\mu(b) = \mu(b).
\]

The quantum intersection product is by definition

\[
QH_*(\mathcal{B} \times E) \otimes HF^*(\Gamma, \Gamma) \to HF^*(\Gamma, \Gamma)
\]

\[
(\alpha, \beta) \ast \gamma = \sum_{b \in \tau^\omega(\mathcal{B} \times E, \Gamma)} \tau^{\omega}[b] ((\alpha, \beta) \ast_b \gamma) \in C_*(\Gamma; \Lambda)
\]

and it is degree-preserving if one uses cohomological gradings.

As explained more generally in the proof of Theorem 9.4, the requirement that \( \text{ev}_{b,-1} \) sweeps a multiple of the cycle \([pt]\) in \( \Gamma \) can be replaced by the condition that the discs \( u \) intersect the \( \bullet \)-dual cycle \([\Gamma]\) inside \( \Gamma \) at the marked point \( z = -1 \). But the latter condition is automatically satisfied, so composing the quantum intersection product with the augmentation gives precisely the count of holomorphic discs which define \( Q_{\Gamma,\gamma} \).

**Remark 9.5.** Observe that, since \( E \) is non-compact, one must take care to distinguish between \( QH_*(\mathcal{B} \times E) \) which is represented by cycles, and \( QH^*_B(\mathcal{B} \times E) \equiv QH^{2|B|+2|E|}(\mathcal{B} \times E) \) which is represented by locally finite cycles. The Theorem involves the former.

### 9.5. Dimension count

Recall that the moduli space \( \mathcal{M}_{\Gamma,b} \) defined above has

\[
\dim_k \mathcal{M}_{\Gamma,b} = \dim_k \Gamma + \mu(b),
\]

where \( \mu : \pi_2(\mathcal{B} \times E, \Gamma) \to \mathbb{Z} \) is the Maslov index. Therefore, in the definition of \( Q_{\Gamma,\gamma}(\alpha, \beta) \) the only \( b \) which contribute have Maslov index satisfying

\[
\dim_k \Gamma + \mu(b) - \text{codim}_{\mathcal{B} \times E}(\alpha, \beta) - \text{codim}_\Gamma(\gamma) = 0,
\]

so \( \mu(b) = \text{codim}(\alpha, \beta) - |\gamma| = 2(|B| + |E|) - |\alpha| - |\beta| - |\gamma| \) is the grading of \( Q_{\Gamma,\gamma}(\alpha, \beta) \). Thus

\[
Q_{\Gamma,\gamma} : QH^*(B) \to QH^{*+2|E|}[\gamma](E)
\]

\[
PD_B(\alpha) \mapsto \sum_\beta Q_{\Gamma,\gamma}(\alpha, \beta) \text{PD}_E(\beta^\vee).
\]

where the sum over \( \beta \) ranges over a basis of \( H_{*+2|E|}[\gamma](E) \), with \( \bullet \)-dual basis \( \beta^\vee \) in \( H_{*}[\gamma](E) \), and \( |\gamma| \) is the homological grading of \( \gamma \) (so \( 2|E| - |\gamma| = |E| - |B| + k \) if \( \gamma \in HF^k(\Gamma, \Gamma) \)).

### 9.6. Equivalent descriptions of \( Q_{\Gamma,\gamma} \)

**Lemma 9.6.** Consider the open-closed string map \( \text{OC} : CF^*(\Gamma, \Gamma) \to QC^*(\mathcal{B} \times E) \) for \( \Gamma \subset \mathcal{B} \times E \), and view \( \gamma \) as a Floer cocycle in \( HF^*(\Gamma, \Gamma) \). Then

\[
\text{OC}(\gamma) = \sum Q_{\Gamma,\gamma}(\alpha, \beta) [\alpha^\vee, \beta^\vee]
\]

where \( \alpha^\vee, \beta^\vee \) are respectively the cycle and the lf-cycle dual to the cycles \( \alpha, \beta \) via the ordinary intersection product, and we sum over a basis \([\alpha, \beta]\) of \( H_*(\mathcal{B} \times E) \).

**Proof.** This follows immediately upon comparing with the discs counted by \( Q_{\Gamma,\gamma} \), described in the proof of Theorem 9.4 and upon observing that the condition of intersecting \( (\alpha, \beta) \) at the centre of the disc is equivalent to requiring that the moduli space of such discs sweep \( (\alpha^\vee, \beta^\vee) \) under evaluation at the centre. \( \square \)
Lemma 9.7. For $\gamma \in HF^k(\Gamma, \Gamma)$, the map $Q_{\Gamma, \gamma}$ factors as:

$$
QH^*(B) \xrightarrow{\otimes 1} QH^*(\overline{B} \times E) \xrightarrow{\text{CO}} HF^*(\Gamma, \Gamma) \xrightarrow{\gamma} HF^{*+k}(\Gamma, \Gamma) \xrightarrow{\text{OC}} QH^{*+|\beta|+|E|+k}(\overline{B} \times E) \xrightarrow{|\text{pt}_B|} QH^{*+1|E|-|\beta|+k}(E)
$$

where $\cdot \gamma$ is the Floer triangle product by $\gamma$, and $/\text{pt}_B$ is slant product by the point class in $B$.

**Figure 17.** The coefficient of $\text{PD}_E(\varepsilon)$ in $Q_{\Gamma, \gamma}(\text{PD}_B(\beta))$.

**Proof.** Let $\varphi$ denote the map defined by the factorization in the claim. Observe Figure 17. On the left are the configurations which count the coefficient of the locally finite cycle $\varepsilon$ in $E$ in the output of $\varphi(\beta)$, where $\beta$ is a cycle in $B$, in particular the target space of the maps defined on those discs is the space $\overline{B} \times E$. Observe in the figure that we replaced the condition of sweeping $\text{pt}_B \times \varepsilon$ by the condition of intersecting $B \times \varepsilon$ at the left interior marked point, where $\varepsilon$ is the cycle in $E$ dual to $\varepsilon$ via the ordinary intersection product.

Now consider the figure on the right: discs with one boundary puncture at $-r, r$ where $0 < r < 1$. In a 1-family, as $r \to 1$ those configurations break to give the figure on the left. On the other hand, as $r \to 0$, a bubble appears at $z = 0$ carrying the two interior punctures: this is the quantum product of $(B, \varepsilon^\gamma)$ and $(\beta, E)$. Since $[B], [E]$ are the units for quantum product in $B, E$ respectively, we have $[B, \varepsilon^\gamma] \cdot [\beta, E] = [\beta, \varepsilon^\gamma]$. So, up to chain homotopy, $\varphi(\beta) = Q_{\Gamma, \gamma}(\beta)$. \qed

**Remark 9.8.** Even though OC, CO are $QH^*$-module maps, the map $Q_{\Gamma, \gamma} : QH^*(B) \to QH^*(E)$ is not in general a $QH^*$-module map (in any reasonable sense) due to the slant-product in the above factorization. Indeed, suppose we introduce a new interior marked point $p_{\beta'}$ on the ray $[0, \infty) \times \{0\}$ inside the quilt cylinder $\mathbb{R} \times S^1$ from Figure 16 and we impose the intersection condition at $p_{\beta'}$ with a cycle $\beta'$ in $B$. In a 1-family, $p_{\beta'}$ may diverge to infinity giving rise to the Floer product by $\beta'$. The other possible breaking is if $p_{\beta'}$ converges towards the marked point $p_{\varepsilon'}$ on the seam, giving rise to a disc bubble in $\overline{B} \times E$ bounding $\Gamma$, carrying $p_{\beta'}$ as an interior marked point and $p_{\gamma}$ as a boundary marked point. We can run a similar argument for a marked point $p_{\varepsilon'}$ on $(-\infty, 0) \times \{0\}$ and an lf-cycle $\varepsilon'$ in $E$.

Thus, abbreviating $Q(\gamma, \beta) = Q_{\Gamma, \gamma}(\beta)$, this proves that:

$$
Q(\gamma, \beta' \ast \beta) = Q(\beta' \times E) \ast Q(\gamma, \beta) \\
\varepsilon' \ast Q(\gamma, \beta) = Q(\overline{B} \times \varepsilon') \ast Q(\gamma, \beta)
$$

where $\ast \gamma$ denotes the quantum intersection product of $QH^*_*([\overline{B} \times E]_{\beta'})$ acting on $\gamma \in HF^*(\Gamma, \Gamma)$. So even if $\varepsilon'$ were obtained by a push-pull construction from $\beta'$, the difference $Q(\gamma, \beta' \ast \beta) - \varepsilon' \ast Q(\gamma, \beta)$ may be non-zero, depending on the quantum cohomology action on $HF^*(\Gamma, \Gamma)$. 

9.7. The generalized Fukaya category $\mathcal{E}$. For now we denote by $\mathcal{E} = \mathcal{F}_{\text{gen}}(E)$ the generalized Fukaya category of $E$ as defined by Wehrheim-Woodward [37] using generalized Lagrangian correspondences as objects. Generalized Lagrangian correspondence $L = (L_1, \ldots, L_k)$ will mean: $L_j \subset M_j \times M_{j+1}$ is a monotone closed oriented Lagrangian submanifold, $M_j$ are monotone symplectic manifolds, $M_1 = \text{pt}$, $M_{k+1} = E$ (with $M_j$ conical at infinity if $M_j$ is non-compact). The morphism spaces of $\mathcal{F}_{\text{gen}}(E)$ are the Floer cohomology groups $HF^*(L, L')$ defined in [37] (we later mention briefly what they are in our context).

For sufficiently nice $\Gamma$ one can in fact later switch back to the Fukaya category $E = \mathcal{F}(E)$ (see 9.9), but we will keep things general for now.

We emphasize: there is no wrapping for now, and all Lagrangians are closed.

9.8. The $A_\infty$-functor $\Phi : \mathcal{B} \to \mathcal{E}$. We aim to study the natural functor $\Phi = \Phi_\Gamma : \mathcal{B} \to \mathcal{E}$ which on objects is

$$\Phi(L) = (L, \Gamma)$$

where a monotone Lagrangian $L \subset B$ is viewed as a correspondence in $\text{pt} \times B$.

On morphisms, the map is given by “gluing in the identity $1_\Gamma$”, more precisely for monotone Lagrangians $L_0, L_1 \subset B$, the morphism

$$HF^*(L_0, L_1) \to HF^*((L_0, \Gamma), (L_1, \Gamma)) \equiv HF^*(L_0, \Gamma, \Gamma^T, L_1^T)$$

is defined by counting quilt maps schematically drawn in Figure 18.

![Figure 18](image.png)

Figure 18. Two equivalent pictorial descriptions of $\Phi^1$. The dash on the right is the input, the dash on the left is the output.

We extend $\Phi^1$ to an $A_\infty$-functor $\Phi : \mathcal{B} \to \mathcal{E}$:

$$\Phi^n : \mathcal{B}(L_n, \ldots, L_0) \to HF^*(\Phi(L_0), \Phi(L_n))$$

counts quilt maps as in Figure 19. In the second picture, the domain is a disc $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, there is a distinguished fixed puncture at $z = -1$ drawn as a dash (corresponding to the output of $\Phi$), and the remaining boundary punctures, drawn as small circular dots, are free to move on the arc $\partial \mathbb{D} \setminus \{-1\}$ (subject to being distinct and ordered anticlockwise).

The $A_\infty$-equations are a standard consequence of a bubbling analysis argument, by considering what happens when a subset of the free points converge together. The types of bubbles that appear are sketched in Figure 20.

9.9. $\Phi$ lands in the Fukaya category of $E$ for good $\Gamma$.

**Definition 9.9.** For a Lagrangian $L \subset B$, denote by the curly letter $\mathcal{L}$ the geometric composition with $\Gamma$, which is defined by the push-pull construction:

$$\mathcal{L} = L \circ \Gamma = \pi_E(\pi_B^{-1}(L) \cap \Gamma)$$

$$\equiv \pi_E([L \times \Gamma) \cap (\text{pt} \times B])$$

$$\equiv \{e \in E : \exists b \in B \text{ with } b \in L \text{ and } (b, e) \in \Gamma\}$$
where \( \pi_B, \pi_E \) are the projections. The composition is transverse if the intersection in the second line is transverse, and is embedded if \( \pi_E \) on the resulting intersection is injective.

**Definition 9.10.** Call \( \Gamma \) good if for all monotone orientable Lagrangians \( L \subset B \), the composite \( \mathcal{L} = L \circ \Gamma \) is transverse and embedded (so \( \mathcal{L} \) is an orientable Lagrangian). Call a good \( \Gamma \) very good if for all such \( L \subset B \), the composed Lagrangian \( L = L \circ \Gamma \) is monotone.

**Lemma 9.11.** If the projection \( \pi_B : \Gamma \to \overline{B} \) is a smooth fibre bundle and \( \pi_E : \Gamma \to E \) is injective, then \( \Gamma \) is good. If \( \pi_1(\Gamma) \to \pi_1(\overline{B} \times E) \) has torsion kernel, then \( \Gamma \) is monotone.

**Proof.** \( \pi_B \) being a smooth fiber bundle is equivalent to \( \pi_B \) being a surjective submersion. Surjectivity of \( d\pi_B : T\Gamma \to T\overline{B} \) implies transverseness. Injectivity of \( \pi_E \) on \( \Gamma \) implies embeddedness. For the final claim, consider the long exact sequence for the pair \( \Gamma \subset \overline{B} \times E \):

\[
\pi_2(\overline{B} \times E) \xrightarrow{\gamma} \pi_2(\overline{B} \times E, \Gamma) \xrightarrow{\alpha} \pi_1(\Gamma) \xrightarrow{\beta} \pi_1(\overline{B} \times E).
\]

By assumption \( \ker \beta \) is torsion, so also \( \text{coker} \gamma \cong \text{im} \alpha \cong \ker \beta \) is. So \( \gamma \) is surjective modulo torsion. Torsion does not affect Maslov indices as the Maslov index is additive on relative \( \pi_2 \).

So it suffices to verify monotonicity on spheres in \( \overline{B} \times E \) (discs mapping the boundary to a point in \( \Gamma \)). So monotonicity of \( \Gamma \) follows from the monotonicity of \( \overline{B} \times E \) (recall [31]).

**Example 9.12** (Coisotropic Submanifolds). Let \( C \subset E \) be a closed coisotropic submanifold. Then \( TC^\omega \) is an integrable distribution by \cite[Lemma 5.33]{22}, so \( C \) is foliated by isotropic leaves. Assume \( C \) is regular: the leaves are connected compact submanifolds of \( C \). Denote...
by $Q$ the quotient $\pi_Q : C \to Q = C/\sim$, identifying points lying on the same leaf. Then $T_{[c]}Q = T_c C/T_c C$ and $Q$ inherits a symplectic form $\omega_Q$ which is natural in the sense that $\pi_Q^*\omega_Q = \omega_C$. \cite{22} Lemma 5.35. Also $C$ naturally gives rise to a Lagrangian submanifold

$$\Gamma = \{(\pi_Q(c), c) : c \in C\} \subset \bar{Q} \times E,$$

where $\bar{Q}$ is $Q$ with the symplectic form $-\omega_Q$. By construction $\Gamma \subset \bar{Q} \times E$ is good, and $Q \ni L \mapsto \bar{L} \subset E$ is the map which takes the union of leaves corresponding to points in $L$.

**Lemma 9.13.** Let $C \subset E$ be a regular coisotropic with reduction $Q$ as above. Suppose $(Q, \omega_Q)$ is monotone. By rescaling, assume $\lambda_Q = \lambda_E$, so also $\bar{Q} \times E$ is monotone with $\lambda_{\overline{Q} \times E} = \lambda_E$. Assume further that $\pi_1(C) \to \pi_1(E)$ has torsion kernel. Then $\Gamma$ is monotone and very good.

**Proof.** Since $\Gamma \cong C$, $(\pi_Q(c), c) \mapsto c$, the map $\pi_1(C) \to \pi_1(E)$ can be identified with $\pi_1(\Gamma) \to \pi_1(\overline{B} \times E)$. So monotonicity of $\Gamma$ follows by Lemma 9.11.

Suppose $L \subset Q$ is a monotone Lagrangian. We now verify monotonicity for a disc $u : (\overline{D}, \partial \overline{D}) \to (E, \Lambda)$. Consider a neighbourhood $\nu$ of $C \subset E$. We can assume $\nu$ retracts onto $C$ (so $L \subset \overline{D}$ is fixed) and $\partial \nu$ intersects $u$ transversely along some circles. The concatenation of these circles is trivial in $\pi_1(E)$ since it is homotopic to $u(\partial \overline{D})$, which has a null-homotopy $u(D) \in E$. Since $\pi_1(C) \to \pi_1(E)$ has torsion kernel, we can assume the concatenation is null-homotopic in $\nu$ (it suffices to verify monotonicity on a multiple of the class $[u] \in \pi_2(\overline{E}, \Lambda)$).

By adding and subtracting the null-homotopy, we can assume $u$ is a gluing of a disc $(\overline{D}, \partial \overline{D}) \to (\nu, \Lambda)$ and some spheres in $E$. For spheres monotonicity follows from monotonicity of $E$, so we can now assume $u(\overline{D}) \subset \nu$. By retracting $\nu$ we can assume $u : (\overline{D}, \partial \overline{D}) \to (C, \Lambda)$.

So $u$ induces $(\pi_Q \circ \overline{u}, u) : (\overline{D}, \partial \overline{D}) \to (\bar{Q} \times E, \Gamma)$ where $\overline{u}$ is $u$ composed with an orientation reversing isomorphism of $\overline{D}$. By additivity for Maslov indices, $\mu(\pi_Q \circ \overline{u}, u) = \mu(\pi_Q \circ \overline{u}) + \mu(u)$, and for areas: $\int (\pi_Q \circ \overline{u}, u)^* (\omega_{\overline{Q}} \oplus \omega) = \int (\pi_Q \circ \overline{u})^* \omega_Q + \int u^* \omega$. But $\Gamma, L$ are monotone with $\lambda_{\overline{\Gamma}} = 1/(2\lambda_E)$ (see 3.2) so monotonicity of $\Lambda$ with $\lambda_{\overline{\Lambda}} = 1/(2\lambda_E)$ follows. \hfill $\Box$

**Lemma 9.14.** If $\pi_{\overline{\Gamma}} : \Gamma \to \overline{B}$ is a smooth fibre bundle and $\pi_E : \Gamma \to E$ is injective, then $\Gamma$ arises by reduction from the coisotropic submanifold $C = \pi_E(\Gamma) \subset E$.

**Proof.** Define $\pi_C : C \to \overline{B}$ so that $\pi_C \circ \pi_E = \pi_{\overline{\Gamma}}$. So $\Gamma = \{(\pi_C(c), c) : c \in C\}$. Via the diffeomorphism $\pi_E : \Gamma \to C$ any $\tilde{c}_1 \in TC$ gives rise to a unique $\tilde{b}_1 = d\pi_C(\tilde{c}_1) \in TB$ with $(\tilde{b}_1, \tilde{c}_1) \in T\Gamma$. Since $\Gamma$ is Lagrangian, $\omega_E(\tilde{c}_1, \tilde{c}_2) = \omega_B(\tilde{b}_1, \tilde{b}_2)$. To prove that $C$ is coisotropic, it remains to show that $TC \cong \ker d\pi_C$, and since the inclusion $TC \subset \ker d\pi_C$ is clear it suffices to compare their real dimensions. Via the diffeomorphism, $\ker d\pi_C \cong \ker d\pi_{\overline{\Gamma}}$. Since $\pi_{\overline{\Gamma}}|\Gamma$ is a fiber bundle, dim ker $d\pi_{\overline{\Gamma}}|\Gamma$ = dim $\Gamma$ − dim $B$. Finally: rank $TC \cong$ rank $TE$ = rank $TC$ = dim $E$ − dim $\Gamma$. Thus the dimensions agree since dim $\Gamma = \frac{1}{2} \dim E + \dim B$. \hfill $\Box$

**Remark 9.15 (Monotonicity assumptions).** We discuss in detail in \cite{9.17} that the next Lemma makes sense when $m_0(L_0) = m_0(L_1)$ because then also $m_0(L_0) = m_0(L_1)$. In particular, we ensure that $B, E$ have the same monotonicity constant $\lambda_E$ by rescaling $\omega_B$ to $\frac{1}{\lambda_E} \omega_B$.

**Lemma 9.16.** For good $\Gamma$, $HF^*(\Phi(L_0), \Phi(L_1)) \cong HF^*(\bar{L}_0, \bar{L}_1)$.

**Sketch Proof.** This is a standard consequence of the “strip-shrinking” theorem of Wehrheim-Woodward \cite{38}, but we include a sketch of the key ideas of the proof for the convenience of the reader. Consider the first quilt in Figure 21 Recall that the generators of the Floer chain group for $HF^*(\Phi(L_0), \Phi(L_1))$ are constant such quilts, and the differential is defined by counting the 1-dimensional moduli spaces (modulo $\mathbb{R}$-translation) of such quilts. Consider shrinking the width of the strips mapping into $B$ (so we are deforming the domain of the quilt maps). A standard parametrized moduli space argument shows that the Floer chain
Figure 21. Strip-shrinking identifies \( \text{hom}_\mathcal{E}(\Phi(L_0), \Phi(L_1)) \equiv \text{hom}_\mathcal{E}(\mathcal{L}_0, \mathcal{L}_1) \).

groups change by a quasi-isomorphism if we change the width. Wehrheim-Woodward show that when the width is very small, the quilt maps become approximately constant on the thin strips. An implicit function theorem argument then shows that there is a bijection between the quilt maps for small width and the second type of quilt maps in Figure 21 where the pair of seams \( L_i \) and \( \Gamma \) are replaced by the composite seam \( L_i \circ \Gamma = \mathcal{L}_i \). Therefore, the generators can be canonically identified, and under this identification the Floer differentials agree. So the Floer cohomologies are isomorphic.

The Lemma is a consequence of the strip-shrinking theorem of Wehrheim-Woodward \[38, \text{Rmk. 5.4.2(b)} \].

Lemma 9.17. \( \Phi \) lands in a full subcategory of \( \mathcal{F}_\text{gen}(E) \) which is quasi-equivalent to \( \mathcal{F}(E) \).

Proof. By shrinking one strip at a time in the proof of the previous Lemma, defines

\[
HF^*(\mathcal{L}, \mathcal{L}) \cong HF^*((L, \Gamma), \mathcal{L}) \cong HF^*((L, \Gamma), (L, \Gamma))
\]

where \( e \) are identity elements. Similarly, one obtains \( [f^-] \in HF^*(\mathcal{L}, (L, \Gamma)) \). A gluing argument for quilts and a strip-shrinking argument shows that \( f^- \circ f^- = e_{\mathcal{L}} \circ e_{\mathcal{L}} = e_{\mathcal{L}} \) on cohomology, and similarly \( f^- \circ f^- \cong e_{(L, \Gamma)} \). So \( \mathcal{L} \) and \( (L, \Gamma) \) are quasi-isomorphic in \( \mathcal{F}_\text{gen}(E) \).

Consider the two full subcategories \( \mathcal{F}(E), \mathcal{E}' \) of \( \mathcal{F}_\text{gen}(E) \) generated respectively by all objects \( \mathcal{L} = L \circ \Gamma \) and \( (L, \Gamma) \). Since each \( \mathcal{L} \) is quasi-isomorphic to \( (L, \Gamma) \), the inclusions \( \mathcal{F}(E) \to \mathcal{F}_\text{gen}(E) \) and \( \mathcal{E}' \to \mathcal{F}_\text{gen}(E) \) are quasi-equivalences of \( A_\infty \)-categories (Seidel [30, Sec. 10a]). Therefore \( \mathcal{F}(E), \mathcal{E}' \) are quasi-equivalent, and the claim follows.

9.10. Additivity of the \( m_0 \)-values. We will assume \( B, E \) have the same monotonicity constant \( \lambda_E \) by rescaling \( \omega_B \) to \( \frac{\lambda_E}{\lambda_E} \omega_B \). For good \( \Gamma \), we always abbreviate \( \mathcal{L} = L \circ \Gamma \) (see 9.10).

Lemma 9.18. For very good \( \Gamma \), if \( L_0, \ldots, L_n \subset B \) are monotone oriented with equal \( m_0 \)-values then the \( \mathcal{L}_0, \ldots, \mathcal{L}_n \subset E \) are monotone oriented with equal \( m_0 \)-values, indeed:

\[
m_0(L \circ \Gamma) = m_0(L) + m_0(\Gamma).
\]

Sketch proof. The Lemma is a consequence of the strip-shrinking theorem of Wehrheim-Woodward [38, Rmk. 5.4.2(b)]. We sketch the argument. Consider the Floer chain complex \( CF^*(L_1, \Gamma, \Gamma^T, L_2^T) \) of the cyclic correspondence \( (L_1, \Gamma, \Gamma^T, L_2^T) : pt \to B \to E \to B \to pt \). Note the Lagrangians are monotone and orientable, and the symplectic manifolds \( B, E \) are monotone with the same monotonicity constant \( \lambda_E \). Consider the Floer differential \( \partial \): the obstruction to \( \partial^2 = 0 \) are disc bubbles of Maslov index 2 (see 5.7), giving rise to:

\[
\partial^2 = (m_0(L_1) + m_0(\Gamma) - m_0(L_2)) \text{Id} = (m_0(L_1) - m_0(L_2)) \text{Id}.
\]

If \( L_1, L_2 \) have the same \( m_0 \)-values then \( \partial^2 = 0 \) so \( HF^*(L_1, \Gamma, \Gamma^T, L_2^T) \equiv HF^*((L_1, \Gamma), (L_2, \Gamma)) \) is well-defined. By strip-shrinking the pair \((L_1, \Gamma)\) (see the proof of Lemma 9.10), one deduces
that $\partial^2 = 0$ also for the cyclic correspondence $(L_1, \Gamma^T, \mathbb{L}^T) : pt \to E \to B \to pt$, and strip-shrinking $(L_2, \Gamma)$ we obtain $\partial^2 = 0$ for $(L_1, \mathbb{L}^T) : pt \to E \to E \to pt$. Thus:

$$m_0(L_1) - m_0(\Gamma) - m_0(L_2) = 0, \quad m_0(L_1) - m_0(L_2) = 0.$$

Remark. Our $m_0(L)$ is defined over the Novikov ring, whereas in [38] Thm. 5.4.1] the notation $W(L)$ refers to the same Maslov 2 disc count over the integers. The two are related by: $m_0(L) = w(L) t^{2\lambda_L} = w(L) t^{1/\lambda_E}$ since this lies in grading 2 (see [32]).

Corollary 9.19. For very good $\Gamma$, the functor $\Phi = \Phi_\Gamma : F(B) \to F(E)$ respects $m_0$-values:

$$\Phi_\Gamma : F(B)_\lambda \to F(E)_{\lambda + m_0(\Gamma)}.$$

So $\Phi_\Gamma$ can only be non-zero on summands with $\lambda \in \text{Spec}(C_1(TB)) \ast \cdots : QH^*(B) \to QH^*(B))$ and $\lambda + m_0(\Gamma) \in \text{Spec}(C_1(TE)) \ast \cdots : QH^*(E) \to QH^*(E))$ (compare Lemma 3.3).

9.11. Using local systems. Recall from section 7 the construction of the (wrapped) Fukaya category with local systems. We now briefly explain why our results for $W(E)$ will hold also for $\mathbb{W}(E)$.

Inserting local systems of coefficients does not affect the moduli spaces of solutions that we want to count, it only changes the weight with which we count solutions. So it suffices to show that the counts are consistent with breaking. Just as for $B$, the key observation is that once the asymptotic data is chosen and we consider any family of solutions with that asymptotic data, then the total weight associated with solutions in the family is constant, even for broken solutions arising in the boundary strata of the family. This is because by (2) in [7.2] the weight is constant along a 1-family of paths with fixed ends, and by (3) this holds even if the path breaks (the weight of a concatenation of paths equals the product of the weights for the components of the broken path).

Finally, consider the new functor $\Phi : B \to \mathcal{E}$ where for now $\mathcal{E}$ is the generalized Fukaya category for $E$ with local systems. For simplicity, we will not put a local system on $\Gamma$ (one can easily do this as well). The Lagrangian boundary conditions along the outermost seams of the quilt domains involved in $\Phi$ are always objects $L_i \in \text{Ob}(B)$, so it suffices to keep track of the local systems on the $L_i$. So we only need to comment on [9.9] when we compose $\mathcal{L} = L \circ \Gamma$.

We now assume $\Gamma$ is good. Then a local system on $L$ induces one on $\Gamma$ by pulling back via the projection $\pi_{\mathbb{L}} : \Gamma \to B$. Since $\pi_E : \pi_{\mathbb{L}}^{-1}(L) \cap \Gamma \to \mathcal{L}$ is injective, we can uniquely lift to $\Gamma$ any paths and homotopies in $\mathcal{L}$, so the local system on $\Gamma$ induces one on $\mathcal{L}$ (equivalently, we are taking the direct image of sheaves). We need to show that this canonical choice is compatible with strip-shrinking. Suppose a quilt map $u$ has a seam condition $(u_{\mathbb{L}}(\gamma), u_E(\gamma)) \in \mathcal{E}$, where $u_{\mathbb{L}}(\gamma)$ is a path on the quilt, then in the limit of strip-shrinking one obtains $(u_E(\gamma), u_E(\gamma)) \in (L_i, L_i) \cap \Gamma$ for some Lagrangian $L_i$ (after subdividing the arc $\gamma$ according to the Lagrangian targets). By construction, $b_{\mathbb{L}}[u_E(\gamma)] = b_{\mathbb{L}}[(u_{\mathbb{L}}(\gamma), u_E(\gamma))] = b_{\mathbb{L}}[u_E(\gamma)]$. So the parallel transports equal in $\mathbb{A}_{\mathbb{L}}$. Thus, after strip-shrinking, we obtain $\Phi : B \to \mathcal{F}(E)$, which on objects is given by

$$(L_i, L_i) \to (\mathcal{L}, (\pi_E)_*(\pi_{\mathbb{L}})^*L_i) \equiv (\mathcal{L}, \mathcal{L}_i)$$

and Lemma 9.17 becomes:

$$HF^\ast((L_0, \mathbb{L}_0), \mathbb{\Gamma}), ((L_1, \mathbb{L}_1), \mathbb{\Gamma})) \cong HF^\ast((L_0, \mathbb{A}_0), (L_1, \mathbb{A}_1)).$$

9.12. Bimodule structure for $\mathcal{E}$ over $\mathcal{E}$. We now continue with $\mathcal{E} = \mathcal{F}_{\text{gen}}(E)$, without local systems, and we do not assume that $\Gamma$ is good.

The bimodule maps $\mu^r_{\mathcal{E}}$ for $\mathcal{E}$ are analogous to those for $B$: in Figure 8 replace $B$ by $E$, and replace $L_i, L_j$ by Lagrangian correspondences $L_i, L_j$ involving any monotone symplectic manifolds $M_0 \to pt, M_1, \ldots, E$. The $A_{\infty}$-relations for $\mu^r_{\mathcal{E}}$ are proved just as for $B$ (in Figure 8 make the analogous replacements).
9.13. Bimodule structure for $\tilde{E}$ over $B$. By Lemma 2.2 the functor $\Phi : B \to E$ induces a change of rings which turns $E$ into a $B$-bimodule $\tilde{E}$. Recall that the new composition maps $\mu_{\tilde{E}}^{r|s}$ involve composing the $\mu_{E}^{r|s}$ maps with several $\Phi^d$ maps, see Figures 22 and 23.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure22.png}
\caption{A contribution $\mu_{E}^{2|1} \circ (\Phi^{k_2} \otimes \Phi^{k_1} \otimes \text{id} \otimes \Phi^{\ell_1})$ to $\mu_{\tilde{E}}^{r|s}$ (before gluing).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure23.png}
\caption{On top: two equivalent ways of viewing the new compositions $\mu_{\tilde{E}}^{r|s}$ (after gluing). On bottom: the $A_\infty$-relations for $\mu_{\tilde{E}}^{r|s}$.}
\end{figure}

9.14. Bimodule morphism $f : B \to \tilde{E}$. Define

$$f^{r|s} : B(L_r, \ldots, L_0) \otimes B(L_0, L_0') \otimes B(L_0', \ldots, L_s') \to \tilde{E}(L_r, L_s') \equiv E(\Phi(L_r), \Phi(L_s'))$$

by counting quilt maps as shown in Figure 24.
9.15. **The induced homomorphism** \( \text{HH}^*_c(\mathcal{B}) \to \text{HH}^*_c(\mathcal{E}) \). By Corollary 2.4 the morphism \( f : \mathcal{B} \to \tilde{\mathcal{E}} \) of \( \mathcal{B} \)-bimodules together with the functor \( \Phi : \mathcal{B} \to \mathcal{E} \) induce a homomorphism \( \text{HH}^*_c(f) : \text{HH}^*_c(\mathcal{B}) \to \text{HH}^*_c(\mathcal{B}, \tilde{\mathcal{E}}) \to \text{HH}^*_c(\mathcal{E}) \).

9.16. **The open-closed string map** \( \text{OC}_E \). By mimicking 5.3, Figure 4, define \( \text{OC}_E : \text{HH}^*_c(\mathcal{E}) \to HF^*(\mathcal{E}) \).

As in 9.12 one replaces the \( L_i \) in Figure 4 by Lagrangian correspondences \( \tilde{L}_i \). The fact that \( c(s) \) is decreasing ensures a maximum principle (3.12) prohibiting solutions from escaping to infinity (the radial coordinate is bounded by the maximum that the radial coordinate attains on the (compact!) Lagrangian correspondences or on the Hamiltonian orbit at \( s = -\infty \)).

9.17. **The deformed morphism** \( \text{HH}^*_c(\mathcal{f}_\Gamma, \gamma) : \text{HH}^*_c(\mathcal{B}) \to \text{HH}^*_c(\mathcal{E}) \). So far, we have defined an \( A_\infty \)-functor \( \Phi_\Gamma : \mathcal{B} \to \mathcal{E} \), and a morphism \( \text{HH}^*_c(\mathcal{f}_\Gamma) : \text{HH}^*_c(\mathcal{B}) \to \text{HH}^*_c(\mathcal{E}) \) where we emphasize the dependence on \( \Gamma \). We now deform \( \mathcal{f}_\Gamma \) to \( \mathcal{f}_{\Gamma, \gamma} \), depending on a class \( \gamma \) in \( HF^*(\Gamma, \Gamma) \) in analogy with Subsection 9.4.

Fix a cycle \( \gamma \) representing a Poincaré dual class in \( H^*(\Gamma) \). Introduce a new fixed marked point \( p_\gamma \) on the seam \( \Gamma \) in Figure 24. We impose the new condition that the quilt maps \( u \) must intersect \( \gamma \) at the new marked point (after folding \( u \) near \( p_\gamma \) to obtain a local map into \( \mathcal{B} \times \mathcal{E} \)). More precisely, we have an evaluation map \( ev_{p_\gamma} \) defined on each moduli space of such quilt maps (having imposed boundary conditions, and having fixed a homology class of such maps). We pick \( \gamma \) generically so that \( ev_{p_\gamma} \) is transverse to it, and finally we consider \( (ev_{p_\gamma})^{-1}(\gamma) \). The map

\[
\mathcal{f}^*_{\Gamma, \gamma} : \mathcal{B}(L_r, \ldots, L_0) \otimes \mathcal{B}(L_0, L'_0, \ldots, L'_s) \to \tilde{\mathcal{E}}(L_r, L'_s) \equiv \mathcal{E}(\Phi(L_r), \Phi(L'_s))
\]

counts the resulting zero dimensional components.

We emphasize that the marked point \( p_\gamma \) is fixed, so the \( A_\infty \)-relations and the compatibility with the (undeformed) bimodule structure continue to hold and are proven in exactly the same manner. The original map \( f \) is given by \( \mathcal{f}_{\Gamma, [\gamma]} \).
9.18. The first commutative diagram.

**Theorem 9.20.** There is a commutative diagram

\[
\begin{array}{c}
\text{HH}_*(B) \xrightarrow{HH_*(f_{\Gamma, \gamma})} \text{HH}_*(B, \tilde{E}) \\
\text{OC}_B \xrightarrow{OC_E} \text{HH}_*(B, \tilde{E}) \xrightarrow{\text{tautological}} \text{HH}_*(E)
\end{array}
\]

\[
\begin{array}{c}
\text{HF}^*(H^B) \xrightarrow{HF^*(q_{\Gamma, \gamma})} \text{HF}^*(H^E) \xrightarrow{\text{identity}} \text{HF}^*(H^E)
\end{array}
\]

using the tautological map from Lemma 2.2 and the map \(\tilde{OC}_E^n : CC_n(B, \tilde{E}) \to CF^*(H^E)\) defined by counting quilts shown on the left in Figure 26.

**Proof.** Consider the square on the left in the diagram. The composite \(q_{\Gamma, \gamma} \circ OC^n_B\) is shown on the left in Figure 25 (after gluing). Observe this is equivalent to the quilt on the right.

![Figure 25. Two equivalent descriptions of \(q_{\Gamma, \gamma} \circ OC^n_B\), \(OC_E \circ f_{\Gamma, \gamma}\) after homotopy.](image)

Now consider the disc on the right in Figure 25 except we make a hyperbolic translation of the seam-circle: we assume it is a hyperbolic circle, whose hyperbolic centre is the interior puncture \(*\) which we allow to freely move on the geodesic joining \(-1, 0\), and we fix the boundary point \(p_{\gamma}\) to lie at \(z = 1/2\). If \(*\) is at 0 we get precisely the quilt on the right in Figure 25. In a 1-family with fixed auxiliary data, if \(*\) converges to \(-1\) then we obtain the broken configuration on the right in Figure 26.

![Figure 26. Left: \(\tilde{OC}_E^n\). Right: a contribution to \(\tilde{OC}_E \circ f_{\Gamma, \gamma}\) (before gluing).](image)

There are other possible degenerations in the above two cases. Let \(\varphi\) be the count of isolated configurations shown in Figure 25. In a 1-family, when a subset of the boundary punctures converge together, we get contributions to \(\varphi \circ b\) where \(b\) is the bar differential for \(CC_*(B)\), and when a Floer cylinder breaks off at the puncture \(*\) we get contributions to \(\partial_{CF^*(H^E)} \circ \varphi\). So the difference at the chain level between the two composites in the left square is counted by \(\varphi \circ b - \partial_{CF^*(H^E)} \circ \varphi\), so the composites agree on homology, and the square commutes.

Now consider the right square in the diagram. The composite of the tautological map and the \(OC_E\) map is precisely the count of quilt maps shown in Figure 22 except we replace the
output dash at $-1$ by an interior puncture $*$ at $z = 0$. So the proof that the square on the right commutes is analogous to the proof of the $A_\infty$-equations for the functor $\Phi$ shown in Figure 20 (except that Figure is rotated by $180^\circ$ so that the output dash at $-1$ becomes our input dash at $+1$ and we have additionally an interior puncture $*$ at $z = 1/2$ or $0$). □

9.19. The second commutative diagram.

**Theorem 9.21.** For very good $\Gamma \subset \overline{B} \times E$ (Definition 9.10), there is a commutative diagram

\[
\begin{array}{ccccccc}
\HH_\ast(B) & \xrightarrow{\HH_\ast(f_{\Gamma, \gamma})} & \HH_\ast(E) & \xrightarrow{\HH_\ast(A)} & \HH_\ast(W(E)) \\
OC_B & \downarrow & OC_E & \downarrow & OC_E \\
HF^\ast(H^B) & \xrightarrow{H^\ast(\Gamma, \gamma)} & HF^\ast(H^E) & \xrightarrow{c^\ast} & SH^\ast(E)
\end{array}
\]

In particular, $c^\ast$ is a ring homomorphism; $\HH_\ast(A), OC_B, OC_E$ are $QH^\ast$-module maps; and the second $OC_E$ is an $SH^\ast$-module map.

**Proof.** Consider the diagram from Section 9.18. We can replace $E = \mathcal{F}_{\text{gen}}(E)$ by $E = \mathcal{F}(E)$ by Lemma 9.17 since $\Gamma$ is very good. In particular, we use that the $\mathcal{F}(E), \mathcal{E}'$ of the proof of Lemma 9.17 have isomorphic $\HH_\ast$ since they are quasi-equivalent $A_\infty$-categories (the $A_\infty$-analogue of Morita equivalence). Moreover, a strip-shrinking argument as in 9.16 shows that the $OC_E$ maps on cohomology commute with the isomorphism $HF^\ast(\mathcal{F}(E)) \cong HH_\ast(\mathcal{E}')$. This yields the left square and proves that it commutes. The square on the right commutes by Theorem 5.5. □

10. Generation and Functoriality for the Compact Category

Our previous discussion always assumed that $E$ was a (non-compact) monotone symplectic manifold conical at infinity. However, several of the results hold also if $E$ is a closed monotone symplectic manifold – in fact they become easier to prove since they do not require accelerating the Hamiltonian, so much of the auxiliary data which decorates the disc counts can be omitted. We state the two main results, and in the following sections we will show applications thereof.

10.1. Generation for compact categories.

**Theorem 10.1.** Let $B$ be a closed monotone symplectic manifold, and let $S$ a full subcategory of $\mathcal{F}(B)$. If the composite

\[
\HH_\ast(S) \xrightarrow{OC_B} \HH_\ast(\mathcal{F}(B)) \xrightarrow{QH^\ast(B)} QH^\ast(B),
\]

hits an invertible element, then $S$ split-generates $\mathcal{F}(B)$. The analogous statement holds if one restricts to generalized eigensummands by the Remark in Theorem 6.2 and Remark 8.4.

10.2. The second commutative diagram for compact categories.

**Theorem 10.2.** Let $B_1, B_2$ be closed monotone symplectic manifolds. For very good $\Gamma \subset B_1 \times B_2$ (Definition 9.10), and any $\gamma \in HF^\ast(\Gamma, \Gamma)$, there is a commutative diagram

\[
\begin{array}{ccccccc}
\HH_\ast(\mathcal{F}(B_1)) & \xrightarrow{\HH_\ast(f_{\Gamma, \gamma})} & \HH_\ast(\mathcal{F}(B_2)) \\
OC_{B_1} & \downarrow & QH^\ast(B_1) & \xrightarrow{Q_{\Gamma, \gamma}} & QH^\ast(B_2) \\
OC_{B_2} & \downarrow & OC_{B_2} \\
QH^\ast(B_1) & \xrightarrow{Q_{\Gamma, \gamma}} & QH^\ast(B_2)
\end{array}
\]

In particular, $OC_{B_1}, OC_{B_2}$ are $QH^\ast$-module maps.
11. Applications

11.1. Quotients by the free action of a finite group. Let \((M, \omega_M)\) be a closed monotone symplectic manifold. Let \(\pi : \tilde{M} \to M\) be a cover with finite deck group \(G\), so

\[M = \tilde{M}/G.\]

We use the symplectic form \(\omega_{\tilde{M}} = \pi^*\omega_M\) on the cover \(\tilde{M}\).

We briefly recall some basic facts from algebraic topology. Firstly, the pull-back \(\pi^* : H^*(M) \to H^*(\tilde{M})\) is injective, since \(\pi \circ \ell = |G| \cdot \text{id} : C_*(M) \to C_*(\tilde{M})\) where \(\ell : C_*(M) \to C_*(\tilde{M})\) sends a chain \(\sigma\) to the sum of its lifts \(\sum_{g \in G} g \cdot \tilde{\sigma}\).

Secondly, by naturality of the first Chern class, \(c_1(\tilde{M}) = \pi^*c_1(M)\). In particular \(M, \tilde{M}\) have the same monotonicity constant (note that holomorphic spheres lift, since \(\pi_1(S^2) = 1\)). One can verify directly that the pull-back \(\pi^* : QH^*(M) \to QH^*(\tilde{M})\) preserves the quantum product, since spheres always lift since \(\pi_1(S^2) = 1\).

**Corollary 11.1.** The natural pull-back map \(QH^*(M; \omega_M) \to QH^*(\tilde{M}; \pi^*\omega_M)\) preserves the quantum product.

The corollary can easily be reformulated at the Floer level (recall that \(HF^* \cong QH^*\) for closed monotone symplectic manifolds). The pull-back map at the Floer level is given by

\[\pi^* : CF^*(M; H, J) \to CF^*(\tilde{M}; \pi^*H, \pi^*J), \quad x \mapsto \sum g \cdot \tilde{x}.\]

At the Floer level, one can see that the pair-of-pants product is indeed preserved since there is a bijection between the pair-of-pants moduli spaces:

\[\mathcal{M}_{\text{POP}}^M(x; y_1, y_2; H, J) \cong \bigcup_{g_0 \in G, g_2 \in G} \mathcal{M}_{\text{POP}}^{\tilde{M}}(g_0 \cdot \tilde{x}; g_1 \cdot \tilde{y}_1, g_2 \cdot \tilde{y}_2; \pi^*H, \pi^*J)\]

(once you fix the lift \(\tilde{y}_1\) of an asymptote of a pair-of-pants map \(u : P \to M\), there is a unique lift of \(u\) to \(\tilde{M}\)). The fact that \(\pi^*\) preserves the POP-product then follows since in \(\tilde{M}\) the POP-moduli spaces have a free \(G\)-action given by applying the deck transformations to the POP-maps \(u : P \to \tilde{M}\).

**Lemma 11.2.** Let \(\Gamma\) be the transpose of the graph of \(\pi : \tilde{M} \to M\),

\[\Gamma = \{(\pi(m), \tilde{m}) : \tilde{m} \in \tilde{M}\} \subset \overline{M} \times \tilde{M},\]

where \(\overline{M}\) means \(M\) with the opposite symplectic form and opposite almost complex structure. Then \(\Gamma\) is very good (Definition [9.17]).

**Proof.** It suffices to verify that Lemma [9.11] applies. The projection \(\Gamma \to \overline{M}\) is a smooth fibre bundle since it is a covering. The projection \(\Gamma \to \tilde{M}\) is injective. Finally, \(\pi_1(\Gamma) \to \pi_1(\overline{M} \times \tilde{M})\) has torsion kernel since \(\pi_1(\tilde{M}) \cong \pi_1(\Gamma)\) includes into \(\pi_1(\overline{M} \times \tilde{M})\). \(\square\)

**Corollary 11.3.** The diagram from Theorem [10.2], with no marked point on the correspondence \(\Gamma\) (or equivalently, taking \(\gamma = [\Gamma]\)), becomes

\[
\begin{array}{ccc}
\HH_* (\mathcal{F}(M)) & \xrightarrow{\HH_* (fr)} & \HH_* (\mathcal{F}(\tilde{M})) \\
QC_M & & QC_{\tilde{M}} \\
QH^* (M) & \xrightarrow{Qr} & QH^* (\tilde{M})
\end{array}
\]
Lemma 11.4. In the above diagram, $Q_\Gamma(1) = 1$.

Proof. By Theorem 11.4, the lower horizontal map $QH^*(M) \to QH^*(\tilde{M})$ is determined by the intersection product with $[\Gamma]$:

$$Q_\Gamma(\alpha) = \sum((\alpha, \beta) \cdot [\Gamma])\beta^\vee$$

where $\alpha \in C_*(\tilde{M})$, $\beta \in C_*(\tilde{M})$, $\beta^\vee \in C_*(\tilde{M})$ is the intersection dual to $\beta$, and the sum is over a basis of cycles representing $H_*(\tilde{M})$. For dimension reasons we need $|\alpha| + |\beta| = \dim_R(\Gamma) = \dim_R(M)$. Thus

$$Q_\Gamma(1) = Q_\Gamma([M]) = (([\tilde{M}], [pt]) \cdot [\Gamma])[pt]^\vee = [\tilde{M}] = 1 \in QH^*(\tilde{M})$$

since they intersect once transversely at $(\pi(pt), pt)$. \hfill \Box

The functor $\Phi_\Gamma : \mathcal{F}(M) \to \mathcal{F}(\tilde{M})$ sends a Lagrangian $L \subset M$ to the (possibly disconnected) preimage Lagrangian $\mathcal{L} = \pi^{-1}(L)$, since

$$L \circ \Gamma = \pi_{\tilde{M}}((L \times \Gamma) \cap (pt \times \Delta_M \times \tilde{M})) = \pi^{-1}(L).$$

Corollary 11.5. If $\mathcal{S}$ is a full subcategory of $\mathcal{F}(M)$ such that $OC_M : HH_*(\mathcal{S}) \to QH^*(M)$ hits an invertible, then $\mathcal{S}$ split-generates $\mathcal{F}(M)$ and $\Phi_\Gamma(\mathcal{S})$ split-generates $\mathcal{F}(\tilde{M})$.

In particular, if this $\mathcal{S}$ is generated by Lagrangians $L_i$ (possibly with holonomies) then the preimage Lagrangians $\mathcal{L}_i = \pi^{-1}(L_i)$ (with pulled-back holonomies) split-generate $\mathcal{F}(\tilde{M})$.

11.2. Brief survey on toric varieties and the Landau-Ginzburg superpotential. Let $(X, \omega_X)$ be a closed real $2n$-dimensional symplectic manifold together with an effective Hamiltonian action of the $n$-torus $U(1)^n$. This action determines a moment map $\mu_X : X \to \mathbb{R}^n$, which is determined up to an additive constant (we tacitly identify $\mathbb{R}^n$ with the dual of the Lie algebra of $U(1)^n$).

The image $\Delta = \mu_X(X) \subset \mathbb{R}^n$ is a convex polytope, called the moment polytope. By Delzant’s theorem $\Delta$ determines, up to isomorphism, $(X, \omega_X)$ together with the action. The moment polytope has the form

$$\Delta = \{ y \in \mathbb{R}^n : (y, e_i) \geq \lambda_i \text{ for } i = 1, \ldots, r \},$$

where $\lambda_i \in \mathbb{R}$ are parameters and $e_i \in \mathbb{Z}^n$ are the primitive inward-pointing normal vectors to the codimension 1 faces of $\Delta$. For Delzant’s theorem to hold, we always assume that at each vertex $p$ of $\Delta$ there are exactly $n$ edges of $\Delta$ meeting at $p$; that the edges are rational, that is they are of the form $p + n \mathbb{Z} v_i$ for $v_i \in \mathbb{Z}^n$; and that these $v_1, \ldots, v_n$ are a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.

Example: $X = \mathbb{P}^m$. The natural rotation action of $U(1)^m$ on the entries of $\mathbb{C}^m$ extends to $\mathbb{P}^m$. Choosing $\lambda_1 = \cdots = \lambda_m = 0$ and $\lambda_{m+1} = -1$, the associated moment map is

$$\mu_{\mathbb{P}^m}(x) = \frac{(|x_1|^2, \ldots, |x_m|^2)}{|x_1|^2 + \cdots + |x_{m+1}|^2}$$

and the moment polytope is

$$\Delta_{\mathbb{P}^m} = \{ y \in \mathbb{R}^m : y_1 \geq 0, \ldots, y_m \geq 0, \sum y_j \leq 1 \}.$$

To the moment polytope one associates the Landau-Ginzburg superpotential $W : \Lambda^n \to \Lambda$, where $\Lambda$ is the Novikov ring defined over $K = \mathbb{C}$, given by

$$W(z_1, \ldots, z_n) = \sum_{j=1}^r t^{-\lambda_i} z_i e_i.$$

Example: $X = \mathbb{F}^m$. Using the above choices, $W = z_1 + \cdots + z_m + tz_1^{-1} \cdots z_m^{-1}$. 

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Consider the Lagrangian tori $L_r = \mu_r^{-1}(r)$ arising as the fibres of the moment map.

Let \( \text{val}_t \) denote the valuation for the \( t \)-filtration, whose value on a Laurent series is the lowest exponent of \( t \) arising in the series. The condition \( \text{val}_t(t^{-\lambda_i}z^{c_i}) > 0 \) is equivalent to the equations of the moment polytope. So restricting to those \( z \in \Lambda^n \) which satisfy that positivity condition, each \( z \) determines a toric Lagrangian \( L_r \) by taking \( r = \text{val}_t(z) \in \text{interior}(\Delta) \subset \mathbb{R}^n \), together with a choice of holonomy around each generating circle of the torus \( L_r \) given by \( i^{-\text{val}_t(z)}z \in H^1(L, \Lambda^n_0) \) (recall \( \Lambda^n_0 \) from Section 7.1).

Abbreviate by \( L_z \) the Lagrangian \( L_r \) with the holonomy determined by \( z \). From now on we work with local systems in the sense of Section 7.2. We will only consider cases where \( X \) is monotone, so in particular \( L_z \) does not bound any non-constant holomorphic discs of Maslov index strictly less than 2. As a consequence of the machinery of Fukaya-Oh-Ohta-Ono [18] defined. Moreover, \( HF \) is a multiple of the fundamental class (see 3.7 and 7.2) and therefore \( HF \) as a module over \( QH \) satisfies

\[
\langle x, y \rangle = \langle [x], [y] \rangle = m_0([L_z]) \langle [L_z], [L_z] \rangle.
\]

The superpotential \( W \) in fact expresses the same count of Maslov 2 discs as \( m_0(L_z) \) (see Auroux [12] Definition 3.3), so

\[
m_0(L_z) = W(z).
\]

The key relationship between critical points of \( W \) and the Floer cohomology of the \( L_z \) is the following result of Cho-Oh (see also the discussion thereof in [12] Proposition 6.9).

**Theorem 11.6** (Cho-Oh [13]). \( HF^*(L_z, L_z) \neq 0 \) if and only if \( z \in \text{Crit}(W) \), indeed this holds if and only if the point class \([pt] \in C_*(L_z)\) is a Floer cycle in \( HF^*(L_z, L_z) \). In particular, if this holds, then \( L_z \) is not displaceable by a Hamiltonian isotopy.

11.3. **Generation results for toric negative line bundles.** The original application which we had in mind, which spurred the general results of this foundational paper, was to apply our machinery in the toric setting to negative line bundles in order to prove generation statements for the wrapped category of the total space in terms of generation statements for the Fukaya category of the base. A general treatment is given in the sequel [28], but here we indicate why the situation is more delicate than one might at first suspect.

Recall that a complex line bundle \( \pi : E \to B \) over a closed symplectic manifold \((B, \omega_B)\) is called a **negative line bundle** if

\[
c_1(E) = -k[\omega_B] \text{ for some real } k > 0.
\]

By convention, \( c_1(E) \) will always mean the first Chern class of the line bundle \( E \), whereas \( c_1(TE) \) denotes the first Chern class of the total space \( E \).

We remark that a symplectic manifold \((B, \omega_B)\) admits a Hermitian line bundle \( E \to B \) with a connection with curvature form \(-k\omega_B\) if and only if \(-k[\omega_B] \) is in the image of \( H^2(X; \mathbb{R}) \to H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{R}) \) (the map induced by the exact sequence \( 0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0 \)).

As explained for example in [27] (taking \( \varepsilon = \pi \) in [27] Sec.7.2), the negativity condition ensures that there is a symplectic form \( \omega \) on \( E \) such that

\[
\omega|_{\text{fibre} \equiv \mathbb{C}} = \omega_C \quad \omega|_{\text{base} = B} = k\omega_B,
\]
so \([\omega] = k[\omega_B]\) via \(H^2(E) \cong H^2(B)\). It is shown carefully in [27] that \(E\) is a symplectic manifold conical at infinity (see [51]), and the contact hypersurface \(\Sigma\) can be chosen to be any sphere subbundle of \(E\). The Reeb flow is given by the \(S^1\)-rotation in the fibres.

We always assume that we are in the monotone setting:

\[
c_1(TE)(A) = \lambda_E[\omega](A), \quad \forall A \in \pi_2(E) \quad \text{some } \lambda_E > 0.
\]

This is equivalent, via \(\lambda_B = k(1 + \lambda_E)\), to the condition that \(B\) is monotone with

\[
c_1(TB)(A) = \lambda_B[\omega_B](A), \quad \forall A \in \pi_2(B), \quad \text{where } 0 < k < \lambda_B.
\]

Our motivation for studying this particular class of non-compact manifolds, is because it is the only family of examples outside of exact cotangent bundles where the symplectic cohomology has been computed:

**Theorem 11.7 (Ritter [27]).** There is an isomorphism of algebras

\[
SH^*(E) \cong QH^*(E)/\ker r^n
\]

where \(r : QH^*(E) \to QH^{*+2}(E)\) is quantum product by \(\pi^*c_1(E)\), and \(n \geq \text{rank } H^*(B)\).

Moreover, the action of \(r\) descends to an automorphism

\[
\mathcal{R} = \text{quantum product by } \pi^*c_1(E) : SH^*(E) \to SH^{*+2}(E).
\]

**Remark.** Since \(E\) is monotone and \([\omega] = k\pi^*[\omega_B]\), all the classes \([\omega]\), \(c_1(TE), \pi^*c_1(E), \pi^*c_1(TB), \pi^*[\omega_B]\) are non-zero real multiples of each other.

**Remark.** Typically one cannot compute \(SH^*\) unless the relevant moduli spaces of solutions to the Floer equation are empty, because one can rarely solve the Floer equation explicitly for a generic almost complex structure. In the case of negative line bundles the relevant counts can be rephrased in terms of Gromov-Witten invariants involving explicit obstruction classes, which can be computed by algebro-geometric techniques.

We now define the correspondence \(\Gamma\) to which we would like to apply our machinery. Recall \(\overline{B} \times E\) means \(B \times E\) with symplectic form \(-\frac{\lambda_B}{\lambda_E} \omega_B \oplus \omega\), so \(\overline{B} \times E\) is monotone.

**Lemma 11.8.** Let \(\Gamma \subset \overline{B} \times E\) be the \(S^1\)-bundle over the diagonal \(\Delta_B \subset \overline{B} \times B\) which fibrewise is the circle of radius \(1/\sqrt{\pi \lambda_E}\). Then \(\Gamma\) is monotone and very good (Definition 9.10).

**Proof.** Apply Example 9.12 to the coisotropic submanifold \(C = SE \subset E\) given by the sphere subbundle of radius \(r_C\). Then \(\Gamma \subset \overline{B} \times E\) is the \(S^1\)-bundle over \(\Delta_B\) which fibrewise is the circle of radius \(r_C\). Also \(\omega_C = (1 + \pi^2 r_C^2)k\pi^*\omega_B\), and via the projection \(\pi\) we may identify \((Q, \omega_Q) \cong (B, (1 + \pi^2 r_C^2)k\omega_B)\). To make \(\overline{Q} \times E\) monotone, we want \(\lambda_B/\lambda_E = (1 + \pi^2 r_C^2)k\), so choose \(r_C\) so that \(\pi r_C^2 = \frac{1}{\lambda_E}\). Finally, we check \(\pi_1(SE) \to \pi_1(E)\) has torsion kernel, so the claim will follow by Lemma 9.13. The \(S^1\)-fibration \(\pi : SE \to B\) yields

\[
0 \to \pi_2(SE) \to \pi_2(B) \overset{c_*}{\to} \pi_1(S^1) \overset{\pi_1}{\to} \pi_1(SE) \overset{\pi_1}{\to} \pi_1(B).
\]

The map \(c_*\) is in fact evaluation by the Euler class of \(E\) after identifying \(\pi_1(S^1) \cong \mathbb{Z}\) (this can be checked by comparing the Gysin sequence for \(f^*SE\) and the long exact sequence for the pair \((f^*E \setminus 0) \subset f^*E\), for any sphere \(f : S^2 \to B\). In our case, \(e(E) = c_1(E) = -k[\omega_B] \neq 0 \in H^2(B)\), so \(\text{coker } c_* \cong \text{im } j_* = \ker \pi_*\) is torsion. Finally, \(\pi_1(SE) \to \pi_1(E)\) has torsion kernel since the composite \(\pi_1(SE) \to \pi_1(E) \to \pi_1(B)\) has torsion kernel \(\ker \pi_*\).

For our \(\Gamma\), in Theorem 9.1 we obtain \(Q\Gamma = 0\) since \([\Gamma] = 0\) (a homology is given by the disc bundle over \(\Delta_B\)). So instead we wish to use the \(Q\Gamma\) map of 9.4 for the class

\[
\gamma = \text{PD}_{\text{Floer}}(1).
\]
using the Poincaré duality \( HF^0(\Gamma, \Gamma) \cong HF^{\dim \Gamma}(\Gamma, \Gamma) \).

It turns out that the lowest-order \( t \) term in \( Q_{\Gamma, \gamma}(1) \) counts the standard Maslov 2 disc in a reference fibre \( C \) of \( E \) bounding the circle \( S^1 \) in the sphere bundle \( SE \) of radius \( 1/\sqrt{\pi \lambda_E} \).

Thus:

\[
Q_{\Gamma, \gamma}(1) = 2\lambda c \omega_Q^{\dim B} + \text{(higher order \( t \) terms)} \neq 0 \in QH^*(E)
\]

where \( c \neq 0 \) is a normalization constant: \( PD_B[pt] = c_\omega_B^{\dim B} \).

Suppose that the base \( B \) has the property that \( OC_B : HH_*(S) \to QH^*(B) \) hits an invertible for some full subcategory \( S \subset \mathcal{F}(B) \). Since \( OC_B \) is a \( QH^*(B) \)-module map, it also hits 1. Thus if we could verify that \( Q_{\Gamma, \gamma}(1) \) is invertible, then the diagram of Theorem 11.9 would imply that \( OC_B : HH_*(S) \to SH^*(E) \) hits an invertible. This in turn implies by Theorem 11.8 that \( \Phi(S) \) split-generates \( W(E) \).

Unfortunately, without performing further deformations of the correspondence \( \Gamma \) (i.e. allowing several marked points with cycle conditions in \( HF^*(\Gamma, \Gamma) \), rather than just one marked point) it appears that \( Q_{\Gamma, \gamma}(1) \) vanishes in \( SH^*(E) \), at least in the case \( E = \mathcal{O}(-k) \to \mathbb{P}^m \).

For the sake of space, we will not discuss this phenomenon in detail here, nor will we pursue the aspect of these further deformations – we will relegate such discussions to later work \[28\].

We will however state the following general result about negative line bundles, which we prove in \[28\] by a careful analysis of the superpotential, which shows that the wrapped category for a large class of negative line bundles is non-trivial.

**Theorem 11.9.** \[28\] Let \( E \to B \) be a monotone toric negative line bundle over a closed symplectic manifold \( B \) for which \( c_1(TB) \in QH^*(B) \) is non-nilpotent.

Then \( c_1(E) \in QH^*(E) \) is non-nilpotent and \( SH^*(E) \neq 0 \). Moreover, there is a non-displaceable monotone Lagrangian torus \( \mathcal{L}_z \subset E \) with \( HF^*(\mathcal{L}_z, \mathcal{L}_z) \neq 0 \) and \( HW^*(\mathcal{L}_z, \mathcal{L}_z) \neq 0 \). If \( L \subset B \) is the Lagrangian torus corresponding to the barycentre of the moment polytope of \( B \), then geometrically \( \mathcal{L}_z = L \circ \Gamma \) is the torus in the sphere bundle \( SE \) lying over \( L \subset B \).

**Conjecture 11.10.** For closed monotone toric manifolds \( B \), \( c_1(TB) \in QH^*(B) \) is always non-nilpotent.

We will not discuss this result since it relies heavily on computations with the superpotential, which are beyond the purpose of this paper. Rather, for the purposes of this paper, we wish to give a quick and elegant concrete application of the OC-map and of the generation criterion in the monotone setting.

1. There is a sign \((-1)^{k-1} \) missing in version 1 of \[27\], which will be corrected in revision 2. The sign arises in Theorem 63: \( A_{\alpha} = (-1)^{n-k} n^2 \tau_{\alpha, \gamma} \). This is because, within the proof, \( A\alpha_i + B\alpha_j \) should be \((-A\alpha_i + B\alpha_j)\) because the duality sign on the \( \mathcal{H}^0 \) reverses the sign of the weights.

2. The \( t^k \) used in this paper corresponds to the \( t \) used in \[27\]. In \[27\], \( t = [pt] \in H_2(\mathbb{P}^m, \mathbb{Z}) \). But in the conventions of this paper, this variable comes with a power of the area \( \int_{\mathcal{F}[pt]} \omega_E = k \).
Recall that $QH^\ast(\mathbb{P}^m) \cong \Lambda[w]/(w^{1+m} - T)$, $\omega_{\mathbb{P}^m} \mapsto w$, where $T$ is the Novikov variable for $B = \mathbb{P}^m$. So the eigenvalues of $c_1(TB) \ast \cdot$ on $QH^\ast(B)$ are $(1 + m)$-th roots of unity (up to a fixed constant factor). On the other hand, for $E = \mathcal{O}_{\mathbb{P}^m}(-k)$ the above calculation shows that the eigenvalues of $c_1(TE) \ast \cdot$ on $QH^\ast(E)$ are $(1 + m - k)$-th roots of unity (up to a constant factor).

**Remark 11.12.** The eigenvalues in this example highlight the difficulty mentioned in Section [11.2] regarding the necessity of further deformations of $\Gamma$. Indeed, by the additivity formula in Lemma [9.18] and Corollary [9.19], the functor $\Phi_t$ translates $m_0$ values by $m_0(\Gamma)$. In general, therefore, one would expect $\Phi_t$ to be non-trivial on at most one summand $F(B)_{\lambda}$; in many cases one sees that $\Gamma$ induces the zero functor.

As a toric variety, $E = \mathcal{O}(-k) \to \mathbb{P}^m$ can be described by the fan whose edges in $\mathbb{Z}^{m+1}$ are given by $(b_1,0), \ldots, (b_m,0), (b_{m+1},k)$ where $b_i$ are the edges for the fan for $\mathbb{P}^m$: $b_1 = (1,0,\ldots,0), \ldots, b_m = (0,0,\ldots,1), b_{m+1} = (-1,\ldots,-1)$. The moment polytope for $\mathbb{P}^m$ is given by $\{y \in \mathbb{R}^m : y_1 \geq 0, \ldots, y_m \geq 0, \sum y_j \leq 1\}$, and the polytope for $E$ is

$$\{y \in \mathbb{R}^{m+1} : y_1 \geq 0, \ldots, y_m \geq 0, \sum_{j=1}^m y_j - ky_{m+1} \leq k\}.$$ 

The Landau-Ginzburg superpotential associated to this polytope is

$$W_E = z_1 + \cdots + z_m + t^k z_1^{-1} \cdots z_m^{-1} z_{m+1}. $$

The critical points of $W_E$ are the following: for each solution $w$ of $w^{1+m-k} = (-k)^k t^k$, there is one critical point:

$$z_{\text{crit}} = (w, \ldots, w, -kw) \quad W_E(z_{\text{crit}}) = (1 + m - k) w.$$ 

The Jacobian ring $\text{Jac}(W_E) \equiv \Lambda[z_1^{\pm 1}, \ldots, z_{m+1}^{\pm 1}]/(\partial_z W_E, \ldots, \partial_{z_{m+1}} W_E)$ has rank $1 + m - k$:

$$\text{Jac}(W_E) \equiv \Lambda[w]/(w^{1+m-k} - (-k)^k t^k).$$

**Corollary 11.13.** $SH^\ast(E) \cong \text{Jac}(W_E)$ for $1 \leq k \leq m/2$, via $W_E = (1 + m - k)w \mapsto c_1(TE)$ (viewing $SH^\ast(E)$ as a quotient of $QH^\ast(E)$ by Theorem [11.7]).

This Corollary is noteworthy since, for closed toric Fano varieties, the Jacobian ring coincides with quantum cohomology.

Now $F(B)$ is generated by the Lagrangian toric fibre $L_{\text{crit}}$ over the barycentre of the polytope for $B = \mathbb{P}^m$, taken with finitely many appropriate local systems, with one local system for each eigenvalue of $c_1(TB)$ (corresponding to the $m_0$ value). These local systems on the monotone torus $L_{\text{crit}}$ arise precisely as the critical points of the superpotential for $B$, which is $W_B = z_1 + \cdots + z_m + T z_1^{-1} \cdots z_m^{-1}$. A calculation shows that the lift $L_{\text{crit}}$ to the sphere bundle $SE$ in $E = \mathcal{O}(k)$ is a monotone Lagrangian toric fibre of $B$ and that the critical points of $W_E$ are precisely $L_{\text{crit}}$ with finitely many local systems. The $m_0$-values for $L_{\text{crit}}$ with these local systems are precisely the critical values $W_E(z_{\text{crit}})$ above, so they exhaust all non-zero eigenvalues of $c_1(TE) \ast \cdot$.

**Remark 11.14.** Again, this highlights the necessity of further deformations of $\Gamma$, since the naive lift of $L_{\text{crit}}$ with its local systems to the sphere bundle would be $L_{\text{crit}}$ with local systems which necessarily are trivial in the last factor. So the holonomies are not being lifted correctly by the functor $\Phi_t$ as written, and this is consistent with the issue of the additivity of the $m_0$-values mentioned in the previous remark.

We are now in the position to prove our application.

**Theorem 11.15.** Let $E = \mathcal{O}_{\mathbb{P}^m}(-k)$, for $1 \leq k \leq m/2$. Then
(1) The wrapped category $\mathcal{W}(E)$ is compactly generated, indeed split-generated by one
Lagrangian torus $\mathcal{L} = \mathcal{L}_{\text{crit}}$ with $1 + m - k$ choices of holonomy;
(2) $\mathcal{W}(E)$ is cohomologically finite (so $\dim HF^*(L_1, L_2) < \infty$ for any objects $L_1, L_2$);
(3) $HF^*(\mathcal{L}, \mathcal{L}) \neq 0$, so $\mathcal{L}$ is not Hamiltonianly displaceable.

Remark. $\mathcal{L}_{\text{crit}}$ is the lift of the Clifford torus in the base $\mathbb{P}^m$ to the sphere subbundle $SE$.

Proof. By Theorem 11.11 for non-zero eigenvalues $\lambda$, the eigensummands of $c_1(E)$ acting on
$QH^*(E)$ are 1-dimensional over the Novikov field $\Lambda = \mathbb{C}(t)$ of formal Laurent series in $t$. Therefore
$QH^*(E)$ splits as a direct sum of the generalized 0-eigenspace and 1-dimensional field summands corresponding to the non-zero eigenvalues. The map $c^* : QH^*(E) \to SH^*(E)$
is the quotient by the generalized 0-eigenspace, by Theorem 11.7.

Let $\mathcal{L}$ be a Lagrangian torus in $E$ with appropriate holonomy arising as a critical point of
the superpotential $W_E$.

Call $\lambda_\mathcal{L} = m_0(\mathcal{L})$ the eigenvalue of $c_1(TE)$ corresponding to $\mathcal{L}$.
Then $OC : HF^*(\mathcal{L}, \mathcal{L}) \to QH^*(E)$ satisfies

$$OC([pt]) = -kw \cdot \pi^* \omega^m_{\mathbb{P}^m} + (\text{higher order } t \text{ terms}) \neq 0 \in QH^*(E),$$

where the first term arises from the regular Maslov 2 disc in the fibre (the standard disc filling $S^1 \subset \mathbb{C}$), and $-kw \neq 0 \in \Lambda$ is the holonomy for the $S^1$-fibre direction of $\mathcal{L}$. Observe that
there is no order $t^0$ term because the constant disc would sweep the $I$-cycle $[pt] \in H^0_I(E) \cong H^{2m+2}(E) = 0$. The $\pi^* \omega^m_{\mathbb{P}^m}$ factor arises because the fibre disc intersects once transversely
the cycle intersection condition $[\mathbb{P}^m] \in C_*(E)$, and the intersection dual of this is the $I$-cycle
$fibre] \in H^2_I(E)$ which corresponds to $\pi^* \omega^m_{\mathbb{P}^m} \in H^{2m}(E)$ (the fibre is the preimage of a point
in the base, and taking preimages is Poincaré dual to pull-back on cohomology). It is a general
feature of the toric setting that the Maslov index of a disc bounding a toric Lagrangian is
twice the number of intersections of the disc with the anticanonical divisor (see Auroux [12]
Lemma 3.1]). In our case, the anticanonical divisor viewed as an $I$-cycle, consists of the base
$\mathbb{P}^m$ together with the preimage under projection of the canonical divisor in the base $\mathbb{P}^m$. So
the fibre disc is the only Maslov two disc that can intersect the cycle $[\mathbb{P}^m]$. By monotonicity,
all other discs with Maslov index greater than 2 will contribute higher order $t$ terms.

This proves that $OC([pt]) \neq 0$ (in particular, this uses that $[pt]$ is a cycle by Theorem 11.5). The key observation is to consider the quantum module structure of the OC map and the
eigensummands $QH^*(E)_{\lambda_\mathcal{L}}$ of $QH^*(E)$:

By Theorem 6.2, $OC([pt])$ can only be non-zero in the eigensummand $QH^*(E)_{\lambda_\mathcal{L}}$.
But that eigensummand is a field and $OC([pt])$ is non-zero.
So $OC([pt])$ is invertible in that eigensummand.
By Theorem 8.3, $\mathcal{L}$ split-generates $\mathcal{W}(E)_{\lambda_\mathcal{L}}$.

Since the critical points of the superpotential show that there is such a Lagrangian $\mathcal{L}$ for
each eigensummand of $\mathcal{W}(E)$, this proves the claim. The finite-dimensionality claim follows
because the split-generators are closed Lagrangians. In particular, $HF^*(\mathcal{L}, \mathcal{L}) \cong HW^*(\mathcal{L}, \mathcal{L})$
is generated at the chain level by the finitely many intersections of $\mathcal{L}$ with a Hamiltonianly
deformed copy of $\mathcal{L}$. \hfill \Box

Remark 11.16. We emphasise that although the wrapped category is cohomologically finite,
it does contain objects which are non-compact Lagrangians – such as real line bundles over
suitable Lagrangians in the base or Lefschetz thimbles for appropriate Lefschetz fibrations –
whose wrapped Floer cochain complexes are infinitely generated.

Remark 11.17. The module structure of the OC-map played a crucial role: without this
module structure one would need to compute all the terms in the expansion of $OC([pt])$. It
turns out that in addition to the discs which can be detected in the usual trivializations of \(\mathcal{O}_{\mathbb{P}^m}(-k)\) (which are easily shown to be regular) there are also “hidden discs” that do not entirely lie in one given trivialization and they are not regular for the standard complex structure (in fact, they are broken configurations consisting of a disc bounding \(L\) with a holomorphic sphere bubble in \(\mathbb{P}^m\) attached to the disc). These hidden discs arise for Maslov indices \(2 + 2(1 + m - k)\) and above, and they perfectly cancel the terms involving \(1, \omega, \ldots, \omega^{k-1}\) contributed by the “visible discs”. This is forced to happen by the module structure, since \(\text{OC}([\text{pt}])\) must be divisible by \(\omega^{k}\) since it must vanish in the 0-eigensummand of \(QH^*(E)\). It would be difficult to justify this computation without the module structure, as it would involve discussing gluing results to turn the irregular hidden discs into genuine pseudo-holomorphic discs for a perturbed almost complex structure.

11.5. Generation results for closed toric Fano varieties. Another short and elegant application of the quantum module structure of the OC map and of the generation theorem in the monotone setting applies to closed toric varieties which are Fano (that is, monotone). The following result is a special case of a theorem of Abouzaid, Fukaya, Oh, Ohta and Ono [7]. We include a proof since it gives a nice illustration of the usefulness of the module structure of the OC map.

**Theorem 11.18.** Let \(B\) be a closed toric Fano variety, such that the eigenvalues of \(c_1(TB)\) acting on \(QH^*(B)\) are all distinct and have multiplicity one. Then \(\mathcal{F}(B)\) is split-generated by the Lagrangian tori with suitable holonomies arising as the critical points of the superpotential for \(B\).

**Proof.** The assumption on the eigenvalues of \(c_1(TB)\) ensures that \(QH^*(B)\) splits into 1-dimensional eigensummands. A classical fact about closed Fano varieties is that quantum cohomology is isomorphic to the Jacobian ring of the superpotential. Therefore, to each 1-dimensional eigensummand there corresponds a critical point \(z\) of the superpotential, and therefore a toric Lagrangian \(L_z\) with suitable holonomy (as discussed in Section 11.2).

But now observe that \(\text{OC} : HF^*(L_z, L_z) \to QH^*(E)\) satisfies
\[
\text{OC}([\text{pt}]) = \text{vol}_B + (\text{higher order t terms}) \neq 0 \in QH^*(B),
\]
where the first term corresponds to the constant disc, and we used the fact that the Poincaré dual of \([\text{pt}] \in H_0(B)\) is the cohomology class of the volume form.

But by the \(QH^*(B)\)-module structure of \(\text{OC}\), the element \(\text{OC}([\text{pt}])\) can only be non-zero in the eigensummand \(QH^*(B)_{\lambda_{L_z}}\) corresponding to the relevant eigenvalue \(\lambda_{L_z} = m_0(L_z)\) of \(c_1(TB)\). Being non-zero in a field implies invertibility, so \(\text{OC}\) hits an invertible element in the relevant eigensummand. So by the generation Theorem 11.1 the Lagrangians \(L_z\) split-generate \(\mathcal{F}(B)\).

Abouzaid, Fukaya, Oh, Ohta and Ono work in the more general setting of not-necessarily-Fano closed toric varieties, with no assumption on the simplicity of the spectrum of quantum product by the first Chern class, and prove that the Fukaya category and its bulk-deformations are still split-generated by toric fibres.

11.6. Generation results for convex symplectic manifolds. Let \((E, \omega)\) be a symplectic manifold conical at infinity, satisfying the monotonicity assumption
\[
c_1(TE) = \lambda E[\omega] \quad (\text{some } \lambda_E > 0)
\]
(this is equivalent to the monotonicity definition in Section 5.2 when \(E\) is simply connected).

Let \(L \in \text{Ob}(\mathcal{F}(E))\), so \(L \subset E\) is a monotone closed orientable Lagrangian with \(m_0(L) = \lambda \in \text{Spec}(c_1(TE)^*)\).
Theorem 11.19. If $\lambda \neq 0$ and $\text{[pt]}$ is a Floer cycle in $HF^*(L,L)$, then

1. $\text{OC}([\text{pt}]) \neq 0 \in QH^*(E);$ 
2. if the generalized eigensummand $QH^*(E)_\lambda$ is 1-dimensional, then $L$ split-generates $\mathcal{F}(E)_\lambda$ and $\mathcal{W}(E)_\lambda$, in particular $\mathcal{W}(E)_\lambda$ is cohomologically finite.

Proof. By Kontsevich, Seidel and Auroux [12, Sec.6], the image under $\text{CO} : QH^*(E) \to HF^*(L,L)$ of $c_1(TE)$ is $m_0(L)[L]$ (this relies on $L$, $E$ being monotone). The proof involves showing that the count of holomorphic discs $D \to E$ bounding an If-cycle representative of $PD(c_1(TE)) \in H^L_0(E)$ at $0 \in D$ and passing through a generic $[\text{pt}] \in C_*(L)$ at $1 \in \partial D$ is precisely $m_0(L)$.

Now $c_1(TE)$ is a multiple of $[\omega]$, and $\omega$ is exact at infinity. Therefore $c_1(TE)$ can be represented by a closed de Rham form which is compactly supported, which in turn is Poincaré dual to a cycle $C \in H_*(E)$. Thus we can represent $c_1(TE)$ by a compact If-cycle $C$. Since the disc count above does not depend on the choice of representative, we can use $C$. But then that disc count is precisely the coefficient of $PD(C^\vee)$ in $\text{OC}([\text{pt}])$, where $C^\vee \in H^L_0(E)$ is the intersection dual of $C \in H_*(E)$.

The other terms in the expansion of $\text{OC}([\text{pt}])$ are obtained by extending $PD(C^\vee)$ to a basis of $H^*(E)$ and counting discs which hit the Poincaré-dual basis representatives in $H_*(E)$ at the centre. In particular, the other terms cannot cancel the term contributed by $PD(C^\vee)$ because of linear independence over $H^*(E)$, thus:

$$\text{OC}([\text{pt}]) = m_0(L) \cdot PD(C^\vee) + \text{linearly independent terms} \in QH^*(E).$$

This proves the first claim since $m_0(L) = \lambda \neq 0$ by assumption.

Example: To clarify the argument, consider the example $E = \mathcal{O}_{\mathbb{P}^m}(-k)$. Since $[\mathbb{P}^m]$ as an If-cycle represents $\pi^*c_1(\mathbb{E}) = -k[\pi^*\omega_{\mathbb{P}^m}]$, and $c_1(TE) = \pi^*c_1(\mathbb{P}^m) + \pi^*c_1(\mathbb{E}) = (1 + m - k)\pi^*\omega_{\mathbb{P}^m}$, we can take $C = \frac{1 + m - k}{-k}[\mathbb{P}^m]$. Then $C^\vee = \frac{k}{1 + m - k}[\text{fibre}]$. Finally $PD(C^\vee) = \frac{-k}{1 + m - k} \pi^*\omega_{\mathbb{P}^m}^{m}$ (pull-back is Poincaré dual to taking preimages, and $[\text{pt}]$ is PD to $\omega_{\mathbb{P}^m}^{m}$ in $\mathbb{P}^m$). By the calculations in Section 11.13, the non-zero eigenvalues of $c_1(TE)$ are $\lambda = (1 + m - k)\omega$ for solutions of $\omega^{1 + m - k} = (-k)^k k^k$. Thus the term we are getting in $\text{OC}([\text{pt}])$ is $m_0(L) \cdot PD(C^\vee) = m_0(L) \frac{-k}{1 + m - k} \pi^*\omega_{\mathbb{P}^m}^{m} = -k\omega \cdot \pi^*\omega_{\mathbb{P}^m}^{m}$, which agrees with the contribution of the fibre disc computed in the proof of Theorem 11.13.

The second claim follows by the same argument as in the proof of Theorem 11.15 and by the acceleration diagram (Theorem 11.14).

Corollary 11.20. Let $E$ be any (non-compact) toric Fano variety convex at infinity. Let $z \in \text{Crit}(W)$ be a critical point of the superpotential $W$ with non-zero critical value $W(z) \neq 0$. If the generalized eigensummand $QH^*(E)_{W(z)}$ is 1-dimensional then the Lagrangian torus $L_z$ split-generates $\mathcal{F}(E)_{W(z)}$ and $\mathcal{W}(E)_{W(z)}$, in particular $\mathcal{W}(E)_{W(z)}$ is cohomologically finite.

Proof. The fact that $[\text{pt}]$ is a Floer cycle in $HF^*(L_z,L_z)$ follows by Theorem 11.16. The Lagrangian $L_z$ (with holonomy) associated to $z$ has eigenvalue $\lambda = m_0(L_z) = W(z) \neq 0$. So the claim follows by the previous Theorem.

11.7. Morse-Bott degenerations. Finally, we point out another interesting class of examples where the diagram of Theorem 11.14 applies. These examples arise when a quasi-projective variety $P$ is the total space of a Morse-Bott degeneration to $E_0$ such that $B = \text{Sing}(E_0)$ is closed and monotone. Recall this means that there is a Lefschetz fibration $p : P \to \mathbb{D}$ with singular fibre $E_0$ such that the Hessian of $\pi$ is non-degenerate on the normal bundle of $B = \text{Sing}(E_0) \subset P$. This ensures that there is a Lagrangian correspondence in $\mathbb{B} \times E$ where $E$ is a generic smooth fibre of $p : P \to \mathbb{D}$, since the vanishing cycle of the degeneration is
an $S^k$-bundle over $B$, which embeds as a coisotropic submanifold in $E$. These spaces were studied by Perutz [23, Def.2.1].

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