ON THE RESTRICTION THEOREM FOR PARABOLOID
IN $\mathbb{R}^4$

CIPRIAN DEMETER

Abstract. We prove that the recent breaking [9] of the $\frac{3}{2}$ barrier in Wolff’s estimate on the Kakeya maximal operator in $\mathbb{R}^4$ leads to improving the $\frac{14}{5}$ threshold for the restriction problem for the paraboloid in $\mathbb{R}^4$. One of the ingredients is a slight refinement of a trilinear estimate from [5]. The proofs are deliberately presented in a nontechnical and concise format, so as to make the arguments more readable and focus attention on the key tools.

1. Kakeya and restriction estimates

A Kakeya set in $\mathbb{R}^n$ is a set containing a unit line segment in every direction. The following stands one of the most fascinating conjectures in geometric measure theory.

Conjecture 1.1 (Kakeya set conjecture). Each Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension $n$.

For quantities $A, B$ that depend on a scale parameter $P$ (typically the radius $R$ or eccentricity $\delta^{-1}$), we will write $A \lesssim B$ to denote the fact that $A \leq C \epsilon P^\epsilon B$ holds for all $\epsilon > 0$. An $(N_1, N_2)$-tube is a long cylinder with radius $N_1$ and length $N_2$. Its eccentricity is $N_2 N_1^{-1}$. The Kakeya set conjecture is known to be a consequence of the following conjecture.

Conjecture 1.2 (Kakeya maximal operator conjecture). Let $\Omega$ be a collection of tubes in $\mathbb{R}^n$ with eccentricity $\delta^{-1}$, equal sizes and $\delta$-separated directions (in particular, there is at most one tube in each of the $\sim \delta^{1-n}$ directions). Then for $\frac{n-1}{n-1} \leq r \leq \infty$

$$\| \sum_{T \in \Omega} 1_T \|_r \lesssim \left( \sum_{T \in \Omega} |T| \right)^{\frac{r}{n}} \delta^{-\frac{n}{n-1}(n-1)}. \quad (1)$$

This latter conjecture is in fact a theorem when $n = 2$ but is open in higher dimensions. When $n \geq 3$, it has been verified by Wolff [7] for $r \geq \frac{n+2}{n}$, and improvements in high dimensions have been obtained by Katz and Tao in [3]. Very recently, Zahl improved Wolff’s result to $r \geq \frac{88n}{57}$ when $n = 4$.

We point out that in general, an inequality of the form

$$\| \sum_{T \in \Omega} 1_T \|_r \lesssim \left( \sum_{T \in \Omega} |T| \right)^{\frac{r}{n}} \delta^{-\frac{n}{n-1}(n-1)}$$

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implies that the Hausdorff dimension of Kakeya sets is at least $n - sr'$, see [6].

Let
\[ \mathbb{P}^{n-1} = \{(\xi_1, \ldots, \xi_{n-1}, \xi_1^2 + \ldots + \xi_{n-1}^2) : |\xi_i| \leq 1\} \]
denote the truncated elliptic paraboloid in $\mathbb{R}^n$. For a cube $\tau \subset [-1, 1]^{n-1}$, $f_\tau$ will typically denote the restriction $f|_\tau$ of $f$ to $\tau$. Given $f : [-1, 1]^{n-1} \to \mathbb{C}$, denote by
\[
Ef(x_1, \ldots, x_n) = \int f(\xi_1, \ldots, \xi_{n-1}) e(\xi_1 x_1 + \ldots + \xi_{n-1} x_{n-1} + (\xi_1^2 + \ldots + \xi_{n-1}^2) x_n) d\xi_1 \ldots d\xi_{n-1}
\]
the extension operator.

Recall also the Restriction conjecture for the paraboloid.

**Conjecture 1.3 (Restriction conjecture).** For each $p > \frac{2n}{n-1}$ and each $f : [-1, 1]^{n-1} \to \mathbb{C}$
\[
\|Ef\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_p. \tag{2}
\]

A standard randomization argument shows that the validity of (2) for some $p$ implies the following weaker form of (1)
\[
\|\sum_{T \in \Omega} 1_T\|_r \lesssim (\sum_{T \in \Omega} |T|)^\frac{1}{r} \delta^{\frac{2n}{r} - 2(n-1)} \tag{3}
\]
with $r = \frac{p}{2}$. As observed earlier, this in turn implies that the Hausdorff dimension of Kakeya sets in $\mathbb{R}^n$ is at least
\[
d_{p,n} := \frac{2p - n(p - 2)}{p - 2}.
\]

In particular, since $d_{\frac{2n}{n-1},n} = n$, the Restriction conjecture is stronger than the Kakeya set conjecture. In fact (3) shows that it is also stronger than the Kakeya maximal operator conjecture.

The Restriction conjecture is also known when $n = 2$ and open in all other dimensions. In dimensions three and higher than four, the best known restriction estimates are weaker than the best known Kakeya estimates. This means that the Hausdorff dimension of Kakeya sets in $\mathbb{R}^n$ is known to be strictly larger than $d_{p,n}$, where $p$ is the smallest value for which (2) is known to hold. Interestingly, when $n = 4$, recent advances due to Guth have allowed for the restriction theory to catch up with Wolff’s result for the Kakeya set conjecture. Indeed, it is proved in [5] that (2) holds with $p = \frac{14}{5}$ when $n = 4$ and note that $3 = d_{\frac{14}{5},4}$. The main goal of this note is to show that any improvement over Wolff’s exponent $r = \frac{3}{2}$ in (1) leads to improvements over the restriction index $\frac{14}{5}$, too.

More precisely, we will prove the following result.
Theorem 1.4. Let $n = 4$. If (1) holds for some $r < \frac{3}{2}$ then
\[ \|Ef\|_{L^p(B_R)} \lesssim \|f\|_{\infty} \]
holds for some $p < \frac{14}{5}$ and each ball $B_R$ with radius $R$.

The dependence of $p$ on $r$ can be extracted from the argument. Using known arguments, $\|f\|_{\infty}$ may be replaced with $\|f\|_p$ and $B_R$ may be replaced with $\mathbb{R}^n$. In particular, combining Theorem 1.4 with the new result [9] on the Kakeya maximal function leads to a slight improvement of the restriction index, $p = \frac{14}{5} - \frac{2}{416515}$.

The proof of the theorem will be presented in sections 3 and 4 and will involve a slight reshuffling of the techniques from [4], [5] and [1]. Our hypothesis on $r < \frac{3}{2}$ will be used twice in the argument. First, a corollary of this (inequality (4)) is used in Section 3 to get a new trilinear restriction estimate. Second, the full strength of the hypothesis is used in Section 4 to bridge the gap between the trilinear and the desired linear restriction estimate.

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2. TANGENT TUBES

Let $Z$ be an $m$-dimensional variety in $\mathbb{R}^n$. The polynomial method developed in [4] and [5] introduces the concept of a tube tangent to the variety $Z$. For all practical purposes we may think of this as being an $(R^\frac{3}{2}, R)$-tube that is contained in the $CR^\frac{1}{2}$-neighborhood of $Z$.\(^1\) We will call this the wall, and will denote it by $W_{Z,R}$.

It seems very intuitive to conjecture the following, see Conjecture 11.1 from [5].

Conjecture 2.1. Let $\Omega$ be a collection of $(R^\frac{3}{2}, R)$-tubes in $B_R \subset \mathbb{R}^n$ with $R^{-\frac{3}{2}}$-separated directions. Assume these tubes are tangent to some $n - 1$ dimensional variety $Z$ of degree at most $D$. Then the number of tubes in $\Omega$ satisfies
\[ \#\Omega \lesssim D^C R^\frac{2}{2}, \]
for some $C$ independent of $D, R$.

To put things into perspective, we show the connection between this and the Kakeya maximal operator conjecture.

\(^1\)There is a small lie here, the analysis in [4] and [5] introduces a small parameter $\delta$ and works with $R^\delta$ enlargements of both the tubes and the wall. This complication is of entirely technical nature and will be ignored here.
Proposition 2.2. Let $\Omega$ be a collection of tubes as in Conjecture 2.1. Inequality (1) for some $r$ implies the bound

$$\# \Omega \lesssim D^C R^{n-1-\frac{r}{r'}}. \quad (4)$$

In particular, the Kakeya maximal operator conjecture implies Conjecture 2.1.

Proof The proof is an immediate application of Hölder’s inequality and Wongkew’s inequality [8], which states that the volume of $W_{Z,R} \cap B_R$ is $\lesssim D^C R^{n-\frac{r}{r'}}$. Indeed

$$\# \Omega R^{\frac{n+1}{r}} = \sum_{T \in \Omega} |T| = \| \sum_{T \in \Omega} 1_T \|_1 \leq \| \sum_{T \in \Omega} 1_T \|_{W_{Z,R} \cap B_R}^{\frac{1}{r'}} \lesssim D^C \left( \sum_{T \in \Omega} |T| \right)^{\frac{1}{r'}} R^{n-\frac{n}{r} - \frac{n-\frac{n}{r}}{r'}},$$

and this is easily seen to imply (4).

Conjecture 2.1 has been verified by Guth [4] when $n = 3$. When $n = 4$, Zahl [9] proved a slightly weaker version of the conjecture with $D^C$ replaced by an unspecified constant $C_D$. The validity of the conjecture in higher dimensions is unknown.

3. An improved trilinear restriction theorem in $\mathbb{R}^4$

In the following, we will restrict attention to $n = 4$. For $\xi = (\xi_1, \xi_2, \xi_3) \in [-1,1]^3$, define the normal vector to $\mathbb{P}^3$

$$n(\xi) = (-2\xi_1, -2\xi_2, -2\xi_3, 1).$$

Fix three cubes $\tau_1, \tau_2, \tau_3 \subset [-1,1]^3$ with side length $\sim 1$. We will assume the transversality condition

$$\inf_{\xi \in \tau_i} |n(\xi^1) \wedge n(\xi^2) \wedge n(\xi^3)| \gtrsim 1.$$

A very close version of the following result is proved in [5].

Theorem 3.1. For each $f : \bigcup \tau_i \to \mathbb{C}$ and $R \geq 1$ we have for $p \geq \frac{14}{5}$

$$\left\| \min_{i=1}^3 |Ef_{\tau_i}| \right\|_{L^p(B_R)} \lesssim \|f\|_2.$$  

\[\text{2The slight lie here is that in [5] the minimum is taken over a larger number of contributions than just three, by still maintaining a trilinear profile. More precisely, the term } \| \min_{i=1}^3 |Ef_{\tau_i}| \|_{L^p(B_R)} \text{ here is a substitute for the quantity } \|Ef\|_{BL^p_{n=4}(B_R)} \text{ from [5] that we do not bother to define. The distinction between three and the higher number considered in [5] is irrelevant for our analysis.}\]
It is conjectured that the minimum can be replaced with the average 
$$\left(\prod_{i=1}^{3} |Ef_{\tau_i}|\right)^{\frac{1}{3}},$$
but this stronger result would not help improve the argument presented here. \footnote{It would be of independent interest to determine whether the polynomial method can be used to make progress on this trilinear restriction conjecture regarding geometric averages. In its current formulation, the polynomial method does not control well the interactions between tangent and transverse tubes that are inherent to geometric averages. The choice of a substitute norm in [5] is precisely made to avoid such interactions.}

The exponent $\frac{14}{5}$ is sharp, if the $L^2$ norm of $f$ is used on the right hand side. We will show how to lower the exponent $\frac{14}{5}$ by replacing the $L^2$ norm with the $L^\infty$ norm.

\textbf{Theorem 3.2.} Assume (4) holds for some $r < \frac{3}{2}$. Then there is $q < \frac{14}{5}$ such that for each $f : \cup \tau_i \to \mathbb{C}$ and $R \geq 1$ we have
\begin{equation}
\| \min_i |Ef_{\tau_i}| \|_{L^q(B_R)} \lesssim \|f\|_{\infty}. \tag{5}
\end{equation}

\textbf{Proof} We will prove the following slightly stronger result.

Assuming that $f$ satisfies
\begin{equation}
\int_{\theta} |f|^2 \lesssim |\theta|, \text{ for each } R^{-1/2} - \text{cube } \theta \subset [-1,1]^3, \tag{6}
\end{equation}
we will show that
\begin{equation}
\| \min_i |Ef_{\tau_i}| \|^q_{L^q(B_R)} \lesssim \|f\|^2_{L^2}. \tag{7}
\end{equation}

It is clear that this implies (5).

The proof of (7) follows very closely the approach in [4], with the input (8) from [5]. We briefly sketch it and refer the reader to [4] for details.

There is a double induction on $R$ and $\|f\|_2$. Use a polynomial $P$ of appropriate degree $D$ to create $\sim D^4$ cells. The degree $D$ is chosen to depend on $R$, but can be thought of as $\lesssim 1$. Call $Z$ the zero set of $P$, and let $W_{Z,R}$ be the corresponding wall.

One needs to estimate the cellular contribution and the contribution from the wall. The cellular contribution is controlled via the induction on $\|f\|_2$.

Roughly speaking, on the wall one has a decomposition of the form
\[ Ef_{\tau_i} = Ef_{\text{tang},\tau_i} + Ef_{\text{trans},\tau_i}, \]
with $Ef_{\text{tang},\tau_i}$ supported on $(R^{1/2},R)$-tubes tangent to $Z$, and $Ef_{\text{tang},\tau_i}$ supported on $(R^{1/2},R)$-tubes that intersect the variety in a transverse (non-tangential) way. The transverse contribution for the wall is controlled via the induction on $R$. To address the tangent term contribution to the wall, it will suffice to prove
\[ \| \min_i |Ef_{\text{tang},\tau_i}| \|_{L^2(B_R)} \lesssim \|f\|^\frac{8}{5}_{L^2}. \]

This is the only new estimate, and here is how it follows. By Proposition 8.1 from [5] ($n = 4, m = k = 3$), we have for $2 \leq q \leq \frac{14}{5}$
\[
\| \min_i |E_{f_{\text{tan},\tau_i}}| \|_{L^q(B_R)} \lesssim R^{\frac{1}{2} - \frac{7}{2q} \left( \frac{1}{2} - \frac{1}{q} \right)} \|f\|_2.
\]  \quad (8)

Using (4) and (6) we get
\[
\|f\|_2 \lesssim (R^{\frac{3}{2} - \frac{r'}{2}} - \frac{1}{2})^\frac{1}{2} = R^{-t},
\]
with \( t > 0 \). This is the place where we use the fact that the dependence in (4) is polynomial in \( D \), as \( D \lesssim 1 \) guarantees \( D^C \lesssim 1 \). Thus, for \( \frac{8}{3} < q < \frac{14}{5} \), (8) can be dominated by
\[
R^{\frac{1}{2} - \frac{7}{2} \left( \frac{1}{2} - \frac{1}{q} \right)} R^{-t(1 - \frac{8}{3q})} \|f\|_2^{\frac{8}{3q}}.
\]
It suffices to choose \( q \) sufficiently close to \( \frac{14}{5} \) so that the exponent of \( R \) is \( \leq 0 \).

The argument above shows that we may take \( q = \frac{2(9+4r')}{3(r'+2)} \). In particular, using \( r = \frac{85}{57} \) as in [9], gives \( q = \frac{8}{3} \times \frac{148}{141} \). If we assume (4) holds for \( r = \frac{4}{3} \), then the corresponding value is \( q = \frac{25}{9} \).

4. The proof of Theorem 1.4

There are two types of mechanisms introduced in [1] that allow to convert multilinear estimates into linear ones. The reader can check that the more basic one does not suffice for our purposes, as the treatment of the planar contribution\(^4\) turns out to be too costly\(^5\). The more elaborate mechanism minimizes the cost for the planar term by using Kakeya type estimates. The proof in this section follows very closely this more elaborate approach.

We will use the following version of inequality (3.4)-(3.5) from [1] (see also Lemma 4.3.1 from [2]), valid for \( x \in B_R \)
\[
|Ef(x)| \lesssim \sum_{R^{-1/2} \lesssim \delta \lesssim 1} \max \left[ \sum_{\tau \in \mathcal{E}_\delta} (\phi_\tau(x) \min_i |Ef_{\tau_i}(x)|)^2 \right]^{1/2} \quad (9)
\]
\[
+ \max \left[ \sum_{\tau \in \mathcal{E}_{R^{-1/2}}} (\phi_\tau(x)|Ef_{\tau}(x)|)^2 \right]^{1/2} \quad (10)
\]
where

(A1) \( \mathcal{E}_\delta \) is an arbitrary collection consisting of \( O(\delta^{-1}) \) \( \delta \) – cubes \( \tau \)

(A2) for each \( \mathcal{E}_\delta \) as above, \( \tilde{\mathcal{E}}_\delta \) is any collection of the form
\[
\tilde{\mathcal{E}}_\delta = \{ \tilde{\tau} := (\tau, \tau_1, \tau_2, \tau_3) : \tau \in \mathcal{E}_\delta \}
\]

\(^4\)The terms \( Ef_\tau \) with \( \tau \) intersecting a line in \( \mathbb{R}^3 \)

\(^5\)In short, while Theorem 4 gives a favorable estimate below \( \frac{14}{5} \) for the trilinear term, there is no obvious way to duplicate this estimate for the bilinear term
where \( \tau_1, \tau_2, \tau_3 \subset \tau \) are arbitrary \( \frac{\delta}{K} \)-cubes satisfying the non collinearity assumption

\[
\inf_{\xi \in \tau} |n(\xi^1) \land n(\xi^2) \land n(\xi^3)| \gtrsim \delta^2.
\]

\((A3)\) \( \phi_\tau \geq 0 \) and \( \frac{1}{|B|} \int_B \phi_\tau \lesssim 1 \), for each \((\delta^{-1}, \delta^{-2})\)-tube \( B \) dual to \( \tau \).

Here \( K \) is a large enough parameter satisfying \( K \lesssim 1 \). The idea behind such a decomposition is to iterate the following dichotomy. Either there are three transverse cubes that contribute significantly, or all such cubes cluster near a line in \( \mathbb{R}^3 \), in which case one uses the standard \( L^4 \) Cordoba type estimate.

To prove Theorem 1.4 we may assume that \( \|f\|_\infty = 1 \). It suffices to show that there exists \( p < \frac{14}{5} \) such that \( \|(9)\|_{L^p(B_R)} \lesssim 1 \) and \( \|(10)\|_{L^p(B_R)} \lesssim 1 \). We will show this for the term (9), the analysis for the other term is entirely similar.

Let \( q \) be the number from Theorem 3.2. Parabolic rescaling shows that for each \( \tau_i \) as in \((A2)\) and each \( s \geq q \)

\[
\| \min_i |E f_{\tau_i}| \|_{L^s(B_R)} \lesssim \delta^{3s-5}.
\]

\((11)\)

We will get three estimates for (9) that we will then interpolate using H"older. To describe these estimates, let

\[
f_1(z) = \frac{3}{2} - 4z, \quad f_2(z) = \frac{5}{2} - 7z.
\]

The first inequality will be

\[
\|(9)\|_{L^q(B_R)} \lesssim \delta^{f_2(\frac{1}{q})}
\]

and will follow from the new trilinear estimate in Theorem 3.2. The advantage of this inequality is that it holds at \( q < \frac{14}{5} \), while its deficit comes from the fact that \( f_2(\frac{1}{q}) < 0 \). We will compensate this deficit by proving an estimate of the form\(^6\)

\[
\|(9)\|_{L^{2r}(B_R)} \lesssim \delta^{f_1(\frac{1}{2r})}
\]

with \( \frac{14}{5} < 2r < 3 \) as in Theorem 1.4. The strength of this estimate comes from the fact that \( f_1(\frac{1}{2r}) > f_2(\frac{1}{2r}) > 0 \). These inequalities combined with the fact that \( f_2(z) > 0 \) for \( z < \frac{5}{14} \) will be enough to prove Theorem 1.4.

Here is how to get the first estimate. Let \( q \leq s \leq 4 \). Write first using H"older

\[
\max_{\tilde{E}_3} \left[ \sum_{\tau \in \tilde{E}_3} (\phi_\tau(x) \min_i |E f_{\tau_i}(x)|)^2 \right]^{1/2} \lesssim \delta^{\frac{1}{2} - \frac{1}{s}} \left[ \sum_{\tau} (\phi_\tau(x) \min_i |E f_{\tau_i}(x)|)^s \right]^{1/s}.
\]

\((12)\)

Note that the sum on the right is over all cubes \( \tau \) in a partition of \([-1, 1]^3 \). Consider a finitely overlapping cover of \( B_R \) with \((\delta^{-1}, \delta^{-2})\)-tubes \( B \) dual to

\(^6\)There will be certain losses involving truncation parameters \( \lambda \) and \( \mu \), but these will be balanced with a third inequality.
τ. Since \( \min_i |Ef_{\tau_i}(x)| \) is essentially constant on each tube \( B \), we get using (A3), (11) and the fact that \( s \leq 4 \)

\[
\int_{B_R} (\phi \tau \min_i |Ef_{\tau_i}|)^s \approx \sum_B \left( \int_B (\min_i |Ef_{\tau_i}|)^s \frac{1}{|B|} \right) \int_B \phi \tau^s \\
\lesssim \int_{B_R} \min_i |Ef_{\tau_i}|^s \lesssim \delta^{3s-5}.
\]

Combining this with (12) leads to the following estimate for \( q \leq s \leq 4 \)

\[
\| \max_{E_\delta} \left[ \sum_{\tau \in E_\delta} (\phi \tau \min_i |Ef_{\tau_i}|)^{2s} \right]^{1/2} \|_{L^4(B_R)} \lesssim \delta^{\frac{2-s}{2}}.
\] (13)

We will later use this with \( s = q \).

Here is how to refine this estimate. For dyadic parameters \( 0 < \lambda \leq 1 \) and \( \mu \geq 1 \) write for each \( \tilde{\tau} := (\tau, \tau_1, \tau_2, \tau_3) \in \tilde{E}_\delta \)

\[
g_{\tilde{\tau},\lambda} = \min_i |Ef_{\tau_i}| 1_{\{\min_i |Ef_{\tau_i}| \sim \lambda \delta^3\}}
\]

\[
\phi_{\tau,\mu} = \phi_{\tau} 1_{\phi_{\tau} \sim \mu}, \quad \mu > 1
\]

\[
\phi_{\tau,1} = \phi_{\tau} 1_{\phi_{\tau} \leq 1}.
\]

Note that

\[
\min_i |Ef_{\tau_i}| = \sum_\lambda g_{\tilde{\tau},\lambda}
\]

\[
\phi_{\tau} = \sum_\mu \phi_{\tau,\mu}.
\]

Because of the triangle inequality, it suffices to focus on fixed values of \( \lambda, \mu \).

A repeat of the earlier argument using now

\[
\frac{1}{|B|} \int_B \phi_{\tau,\mu}^s \approx \mu^{s-4}
\]

and

\[
\int_{B_R} g_{\tilde{\tau},\lambda}^s \lesssim (\lambda \delta^3)^{s-q} \int_{B_R} \min_i |Ef_{\tau_i}|^q \lesssim \lambda^{s-q} \delta^{3s-5}
\] (14)

leads to the estimate for \( q \leq s \leq 4 \)

\[
\| \max_{E_\delta} \left[ \sum_{\tau \in E_\delta} (\phi_{\tau,\mu} g_{\tilde{\tau},\lambda})^2 \right]^{1/2} \|_{L^4(B_R)} \lesssim \lambda^{1-\frac{q}{4}} \mu^{1-\frac{q}{4}} \delta^{\frac{3s-5}{2}}.
\]

To simplify computations, we will later use the above with \( s = \frac{14}{5} \)

\[
\| \max_{E_\delta} \left[ \sum_{\tau \in E_\delta} (\phi_{\tau,\mu} g_{\tilde{\tau},\lambda})^2 \right]^{1/2} \|_{L^{14/5}(B_R)} \lesssim \lambda^{1-\frac{14}{40}} \mu^{1-\frac{14}{40}} \delta^{\frac{3s-5}{2}}.
\] (15)

Let us now get the third estimate. Using our hypothesis, for each collection \( \Omega \) consisting of \((\delta^{-1}, \delta^{-2})\)-tubes with \( \delta \)-separated directions we have

\[
\| \sum_{T \in \Omega} 1_T \|_r \lesssim \delta^{-3-\frac{4}{r}}.
\] (16)
A standard consequence via convexity is the estimate
\[ \| \sum_{T \in \Omega} \sum_{k \in \Lambda_T} c_{T,k} 1_{T+k} \|_r \lesssim \delta^{-3-\frac{q}{2}}, \] (17)
whenever \( c_{T,k} \geq 0 \) and \( \max_{T \in \Omega} \sum_{k \in \Lambda_T} c_{T,k} \lesssim 1 \), with \((T+k)_{k \in \Lambda_T}\) a tiling of \( \mathbb{R}^4 \).

For each \( \tau \) let \( T_\tau \) be the tube dual to \( \tau \) passing through the origin. We can think of each \(|E_{f_{T_\tau}}|\), and thus also of \( g_{\bar{\tau},\lambda}^2 \) as being essentially constant on each \( k + T_\tau \), \( k \in \Lambda_{T_\tau} \). More precisely
\[ (g_{\bar{\tau},\lambda}(x))^2 \lesssim \delta^5 \int (g_{\bar{\tau},\lambda}(z))^2 1_{T_\tau}(x-z)dz. \] (18)

A computation similar to (14) shows that
\[ \int_{B_R} g_{\bar{\tau},\lambda}^2 \lesssim (\lambda \delta^3)^{2-q} \int_{B_R} \min_{i} |E_{f_{T_\tau}}|^q \lesssim \lambda^{2-q} \delta. \] (19)

We can rewrite (18) and (19) as
\[ (g_{\bar{\tau},\lambda}(x))^2 \lesssim \delta^6 \lambda^{2-q} \int 1_{T_\tau}(x-z)c_{\tau,\lambda}(z)dz \]
with \( c_{\tau,\lambda} \) essentially constant on each \( k + T_\tau \) and satisfying
\[ \int c_{\tau,\lambda} \lesssim 1. \]

Note that for each \( \tau \) there can be \( \lesssim 1 \) many \( \bar{\tau} \) with first entry \( \tau \). Thus
\[ \max_{\tilde{E}_\delta} \left[ \sum_{\tau \in \tilde{E}_\delta} (\phi_{\tau,\mu}g_{\bar{\tau},\lambda}(x))^2 \right]^{1/2} \lesssim \mu \left[ \sum_{\bar{\tau}} (g_{\bar{\tau},\lambda}(x))^2 \right]^{1/2} \lesssim \delta^3 \lambda^{1-\frac{q}{4}} \left[ \int \sum_{\tau} 1_{T_\tau}(x-z)c_{\tau,\lambda}(z)dz \right]^{1/2}. \]

Invoking (17) we get our third main estimate
\[ \| \max_{\tilde{E}_\delta} \left[ \sum_{\tau \in \tilde{E}_\delta} (\phi_{\tau,\mu}g_{\bar{\tau},\lambda}(x))^2 \right]^{1/2} \|_{L^{2r}(B_R)} \lesssim \mu^{1-\frac{q}{4}} \delta^{\frac{1}{4}} \lambda^{-\frac{q}{4}}. \] (20)

Interpolate (15) and (20) as follows. Let \( \theta \in (0,1) \) be such that
\[ \theta(1 - \frac{q}{2}) + (1 - \theta)(1 - \frac{5q}{14}) = 0, \]
so \( \theta = \frac{14 - 5q}{2q} \). We may assume\(^7\) \( 2r > \frac{14}{5} \). Define \( p_1 \in (\frac{14}{5}, 2r) \) via
\[ \frac{1}{p_1} = \frac{\theta}{2r} + \frac{5(1-\theta)}{14}. \]

Then (15) and (20) give
\[ \| \max_{\tilde{E}_\delta} \left[ \sum_{\tau \in \tilde{E}_\delta} (\phi_{\tau,\mu}g_{\bar{\tau},\lambda}(x))^2 \right]^{1/2} \|_{L^{p_1}(B_R)} \lesssim \mu^{1-\frac{q}{4}} \delta^{\frac{1}{4}} \lambda^{-\frac{q}{4}}. \]

\(^7\)Otherwise we actually have a stronger estimate and things get easier
The choice of $\theta$ was made in order to make the exponent of $\lambda$ zero. Recall that $\mu \geq 1$. The key facts are that $f_1(z) > f_2(z)$ for $z > \frac{1}{3}$ and that $10\theta - 3 < 0$. These together with the fact that $f_2$ is affine allows us to rewrite the above inequality as follows

$$\| \max_{E_\delta} \left[ \sum_{\tau \in E_\delta} (\phi_\tau |Ef_1|)^{1/2} \right] \|_{L^{p_1}(B_R)} \lesssim \delta f_2(\frac{1}{p_1})^{1/2},$$

for some $\Delta > 0$ whose exact value is not important. We may in fact choose a slightly larger $\theta$, so that we have a saving in $\lambda$ that allows to sum over both $\mu \geq 1$ and $\lambda \leq 1$. We conclude that

$$\| \max_{E_\delta} \left[ \sum_{\tau \in E_\delta} (\phi_\tau \min_i |Ef_\tau|)^{1/2} \right] \|_{L^{p_1}(B_R)} \lesssim \delta f_2(\frac{1}{p_1})^{1/2},$$

for some $p_1 > \frac{4}{5}$. Interpolate this with (13) ($s = q$) which we rewrite as follows

$$\| \max_{E_\delta} \left[ \sum_{\tau \in E_\delta} (\phi_\tau \min_i |Ef_\tau|)^{1/2} \right] \|_{L^{q}(B_R)} \lesssim \delta f_2(\frac{1}{q}).$$

Since $f_2(\frac{5}{14}) = 0$ and $f_2$ is affine, there is $\alpha \in (0, 1)$ so that

$$\alpha f_2(\frac{1}{p_1}) + (1 - \alpha) f_2(\frac{1}{q}) = 0$$

and so that $p$ defined via

$$\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1 - \alpha}{q}$$

satisfies $p < \frac{44}{5}$. With this choice, Hölder leads to

$$\| \max_{E_\delta} \left[ \sum_{\tau \in E_\delta} (\phi_\tau \min_i |Ef_\tau|)^{1/2} \right] \|_{L^{p}(B_R)} \lesssim 1.$$  

The desired inequality $\| (9) \|_{L^{p}(B_R)} \lesssim 1$ is now immediate since there are $\lesssim 1$ many scales $\delta$.

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