One-Point Gradient-Free Methods for Composite Optimization with Applications to Distributed Optimization

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Abstract. This work is devoted to solving the composite optimization problem with the mixture oracle: for the smooth part of the problem, we have access to the gradient, and for the non-smooth part, only to the one-point zero-order oracle. We present a method based on the sliding algorithm. Our method allows to separate the oracle complexities and compute the gradient for one of the function as rarely as possible. The paper also examines the applicability of this method to the problems of distributed optimization and federated learning.

Keywords: zeroth order methods · one-point feedback · composite optimization · sliding · distributed optimization

1 Introduction

In this paper, we focus on the composite optimization problem:

$$\min_{x \in X} \Psi_0(x) := f(x) + g(x).$$  \hspace{1cm} (1)$$

This problem occurs in a fairly large number of applications. In particular, we can recall the problems of minimizing the objective function $f(x)$ with regularization $g(x)$, which can often be found in machine learning. Newer and more interesting applications of the composite problem arises in distributed optimization. In more detail, the goal of distributed optimization is to minimize the global objective function $f(x) = \sum_{m=1}^{M} f_m(x)$, where functions $f_1, \ldots, f_M$ are distributed among $M$ devices, and each device $m$ has access only to its local function $f_m$. Therefore, in order to solve this problem, you need to establish a communication process between devices. Two methods are distinguished: centralized and decentralized. In a centralized case, all devices can only communicate with the

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central server – transfer information about the $f_m$ function to it and receive responses. In a decentralized case, there is no longer a central server; all devices are connected into a network, which can be represented as an undirected graph, where vertices are devices, and edges are the presence of a connection between a pair of devices. Communication in a decentralized network is typically done with the gossip algorithm \cite{12, 4, 16}, which uses the so-called gossip matrix $W$.

This matrix is built on the basis of the properties of the communication graph.

It turns out that the centralized, that the decentralized distributed optimization problem can be written as a composite one \cite{15, 10, 3}:

\[
\min_{X \in \mathcal{X}^M} \sum_{m=1}^{M} f_m(x_m) + \frac{\lambda}{M} \sum_{m=1}^{M} \|x_m - \bar{x}\|^2_2
\]  

for the centralized case and

\[
\min_{X \in \mathcal{X}^M} \sum_{m=1}^{M} f_m(x_m) + \lambda \|\sqrt{W}X\|^2_2
\]  

for decentralized one. Here we introduce matrices $X = [x_1, \ldots, x_M]^T$, $Y = [y_1, \ldots, y_M]^T$, vector $\bar{x} = \frac{1}{M} \sum_{m=1}^{M} x_m$ and parameter of regularization $\lambda > 0$.

The essence of expressions (2) and (3) is very simple. On each device, we have local variables $x_m$, at the expense of the regularizer we penalize their deviations. In the centralized case, we penalize the deviation from the average across the entire network, and in the decentralized case, the difference between the connected devices (this is what the $W$ matrix is responsible for). In fact, in the decentralized case, we can also write the penalized problem in the form (2). But if in the case of a centralized architecture $\bar{x}$ is easy to calculate on the server, then in a decentralized network this is problematic (in particular, one of the devices have to be used as a server). Another important question is how to choose the $\lambda$ parameter. To get a close solution to the real solution of the distributed problem, one need to take $\lambda$ large enough. But more recently, problems (2) and (3) are considered from the point of view of personalized federated learning, in this case it makes sense to take small $\lambda$ as well.

Now let us go back to the original problem (1). As noted above, the function $g$ often plays the role of a regularizer, usually it is a simple function for which we can compute the gradient $\nabla g$. At the same time, the objective function $f$ can be quite complex. In this paper, we focus on the case when for the function $f$ only zero order oracle of is available to us (i.e., only the values of the function $f$, but not its gradient). In the literature, this concept is sometimes referred to as a black-box. It arises in optimization \cite{17}, adversarial training \cite{5}, RL \cite{8}. To make the problem statement even more practical we assume that we have access inexact values of function $f(x, \xi)$ with some random noise $\xi$. With the help of this oracle, it is possible to make some approximation of the gradient in terms...
of finite differences. Next we highlight two main approaches for such gradient estimates. The first approach is called a two-point feedback:

\[
\tilde{f}_r'(x, \xi) = \frac{n}{2r} (f(x + re, \xi) - f(x - re, \xi))e,
\]

where \(e\) is uniformly distributed on the unit euclidean sphere. For two-point feedback there are a lot of papers with theoretical analysis [6,19,17]. An important thing of this approach is that it is assumed that we were able to obtain the values of the function in points \(x + re\) and \(x - re\) with the same realization of the noise \(\xi\). But from a practical point of view, this is a very strong and idealistic assumption. Therefore, it is proposed to consider the concept of one-point feedback:

\[
\tilde{f}_r'(x, \xi^\pm) = \frac{n}{2r} (f(x + re, \xi^+) - f(x - re, \xi^-))e.
\]

In general \(\xi^+ \neq \xi^-\). In this paper we work with one-point concept.

The function \(f\) is “bad”, while the function \(g\) is “good”. The question arises how to minimize \(\Psi_0\) from (1). The easiest option is to add the gradient \(g\) and the ”gradient” \(f\) (from (4)) and make step along it. In this approach, there are no problems when \(g\) is just a Tikhonov regularizer, but if we look at problems (2) and (3), to compute the gradient of \(g\) we need to make communication, while to calculate the ”gradient” \(f\) we do not need it. But communications are the bottleneck of distributed algorithms, they require significantly more time than local computations. Therefore, one want to reduce the number communications and calculate the gradient \(g\) as rarely as possible.

This brings us to the goal of this paper: to come up with an algorithm that solves the composite optimization problem for one part of which we have a one-point zero order oracle, and for the other - a gradient. At the same time, we want to call the gradient as rarely as possible.

1.1 Our contribution

We present a new method based on the sliding algorithm for (1) with the mixture oracle. Our method solves the problems mentioned above in the introduction. It reduces the number of calls for the gradient \(\nabla g\) of the smooth part of the composite problem, while using one-point feedback for the non-smooth part \(f\).

We also discuss the applicability and relevance of this method for distributed and federated learning problems. It turns out that this method can be useful in terms of reducing the number of communications.

1.2 Related works

Let us note some works related to our paper.

Sliding. Our algorithm is based on the gradient sliding algorithm from [13,14]. Sliding technique allows us to separate the oracle complexities for the parts of the composite problem. This effect is achieved due to the fact that the
method consists of an outer and an inner loops. At the outer iterations, we calculate the gradient of the function $g$, while at the inner loop, only the function $f$ is used, with fixed information about the gradient $g$.

**Gradient-free methods.** Let us highlight the main works devoted to the zeroth-order methods: for two-point feedback [19,17,7,11], for one-point feedback [2,9,1,18].

**Mixture oracle.** The composite problem with a mixture oracle has already been studied in the literature. In particular, in the paper [3] the authors also assume that a gradient is available for one of the functions, and only a zero-order oracle – for the other. But unlike our work, this work explores two-point feedback. As noted earlier, this gradient-free concept is simpler and less practical. We use one-point feedback – this is the main difference.

## 2 Preliminaries

First, we define several notation. We denoted the inner product of $x, y \in \mathbb{R}^n$ as $\langle x, y \rangle \overset{\text{def}}{=} \sum_{i=1}^{n} x_i y_i$, where $x_i$ corresponds to the $i$-th component of $x$ in the standard basis in $\mathbb{R}^n$. Also, we denote $\ell^p_p$-norms as $\|x\|_p \overset{\text{def}}{=} (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ for $p \in (1, \infty)$ and for $p = \infty$ we use $\|x\|_\infty \overset{\text{def}}{=} \max_{1 \leq i \leq n} |x_i|$ And for the dual norm $\|\cdot\|_*$ for the norm $\| \cdot \|$ is defined in the following way: $\|y\|_* \overset{\text{def}}{=} \max \{ \langle x, y \rangle | \|x\| \leq 1 \}$. Operator $E[\cdot]$ denotes full mathematical expectation and operator $E_{\xi}[\cdot]$ express conditional mathematical expectation w.r.t. all randomness coming from random variable $\xi$.

Now let us introduce a few definitions

**Definition 1 (L-smoothness).** Function $g$ is called $L$-smooth in $X \subseteq \mathbb{R}^n$ with $L > 0$ w.r.t. norm $\| \cdot \|$ when it is differentiable and its gradient is $L$-Lipschitz continuous in $X$, i.e.

$$\| \nabla g(x) - \nabla g(y) \|_* \leq L \| x - y \|, \quad \forall x, y \in X.$$  

One can show that $L$-smoothness implies

$$g(x) \leq g(y) + \langle \nabla g(y), x - y \rangle + \frac{L}{2} \| x - y \|^2, \quad \forall x, y \in X. \quad (5)$$

**Definition 2 (Bregman divergence).** Assume that function $\nu(x)$ is 1-strongly convex w.r.t. $\| \cdot \|$-norm and differentiable on $X$ function. Then for any two points $x, y \in X$ we define Bregman divergence $V(x, y)$ associated with $\nu(x)$ as follows:

$$V(x, y) \overset{\text{def}}{=} \nu(y) - \nu(x) - \langle \nabla \nu(x), y - x \rangle.$$  

Then we denote the Bregman-diameter of the set $X$ w.r.t. $V(x, y)$ as $D_{X,V} := \max \{ \sqrt{2V(x, y)} | x, y \in X \}$.

**Definition 3 (convex function).** Continuously differentiable function $g(x)$ is called convex in $\mathbb{R}^n$ if the inequality holds for $x, y \in \mathbb{R}^n$ any

$$g(y) \geq g(x) + \langle \nabla g(x), y - x \rangle. \quad (6)$$
3 Main part

Recall that we consider the composite optimization problem (1). To take into account the "geometry" of the problem, we work in a certain (not necessarily Euclidean) norm \( \| \cdot \| \) (with dual norm \( \| \cdot \|_* = \| \cdot \|_q \)), and also measure the distance using the Bregman divergence, \( V \). Assume that \( X \subseteq \mathbb{R}^n \) is a compact and convex set with diameter \( D_{X,V} \), function \( g \) is convex and \( L \)-smooth w.r.t. \( \| \cdot \| \) norm on \( X \), function \( f \) is convex differentiable function on \( X \). Assume we can use the first-order oracle for \( g(x) \) and zeroth-order oracle with unbiased stochastic noise for \( f(x) \), i.e. we have access to

\[
\tilde{f}(x, \xi) \overset{\text{def}}{=} f(x) + \xi
\]

where \( \xi \) is generated randomly regardless of the point \( x \). Additionally, we assume that the noise is bounded:

\[
\mathbb{E}\xi = 0, \quad \mathbb{E}[\xi^2] \leq \sigma^2.
\]

Also assume that for all \( x \in X \)

\[
\|\nabla f(x)\|_2 \leq G.
\]

Note that the boundedness of the gradient \( f \) is needed only for the theoretical analysis; in practice, the method uses only the oracle with the values of the function.

3.1 Algorithm

As mentioned above, our algorithm is based on the sliding algorithm. The idea is to use an internal prox-sliding procedure, which provides "sliding" along one of the functions. Our algorithm is a modification of the first order sliding with a zero-order oracle.

The following theorem gives an estimate for the convergence of this method:

**Theorem 1.** Suppose that \( p_t = \frac{t}{2}, \quad \theta_t = \frac{2(t+1)}{(t+3)}, \quad \beta_k = \frac{2L}{k}, \quad \gamma_k = \frac{2}{k+1}, \quad T_k = \frac{4N(9n^2(p^2_0(G^2+\sigma^2)))}{3D_{X,V}^2} \) for \( t \geq 1, k \geq 1 \) and fixed number of iteration \( N \). Then for all \( N \) it holds that

\[
\mathbb{E}[\psi_0(x_N) - \psi_0(x^*)] \leq 2rG + \frac{12LD_{X,V}^2}{N(N+1)}.
\]

Additionally, if we put \( r = \Theta \left( \frac{\varepsilon}{M} \right) \), then the number of calling for \( \nabla g \) and \( f \) required by Algorithm 1 to find an \( \varepsilon \)-solution \( x_N \) of (1) that \( \mathbb{E}[\psi_0(x_N)] - \psi(x^*) \leq \varepsilon \), is bounded by

\[
O \left( \sqrt{\frac{LD_{X,V}^2}{\varepsilon}} \right)
\]
**Algorithm 1** One-Point Zeroth-Order Sliding Algorithm (OPZOSA)

1: Initial point $x_0 \in X$ and iteration limit $N$. Let $\beta_k \in \mathbb{R}_{++}$, $\gamma_k \in \mathbb{R}_{+}$, and $T_k \in \mathbb{N}$, $k = 1, 2, ..., $ be given and set $x_0 = x_0$.
2: for $k = 1, ..., N$ do
3: \hspace{1em} Set $x_k = (1 - \gamma_k)x_{k-1} + \gamma_kx_{k-1}$ and let $h_k(y) = g(x_k) + \langle \nabla g(x_k), y - x_k \rangle$.
4: \hspace{1em} Set $(x_k, \tilde{x}_k) = PS(h_k, x_{k-1}, \beta_k, T_k)$
5: \hspace{1em} Set $x_k = (1 - \gamma_k)x_{k-1} + \gamma_k\tilde{x}_k$
6: end for
7: Result: $\tilde{x}_N$

The $PS$ procedure: $(x_k, \tilde{x}_k) = \text{PS}(h_k, x_{k-1}, \beta_k, T_k)$

1: Let the parameters $p_t \in \mathbb{R}_{++}$ and $\theta_t \in [0, 1]$, $t = 1, ..., $ be given. Set $u_0 = \tilde{u}_0 = x$.
2: for $t = 1, ..., T$ do
3: \hspace{1em} $u_t = \text{argmin}_{u \in X} \left\{ h(u) + \left\langle f_t(u_{t-1}), u \right\rangle + \beta V(x, u) + \beta p_t V(u_{t-1}, u) \right\}$.
4: \hspace{1em} $\tilde{u}_t = (1 - \theta_t)\tilde{u}_{t-1} + \theta_t u_t$
5: end for
6: Set $x^+ = u_T$ and $\tilde{x}^+ = \tilde{u}_T$.

$$O\left( \sqrt{\frac{LD_{X,V}^2}{\varepsilon}} + \frac{n \cdot p^2(n)G^2D_{X,V}^2}{\varepsilon^2} + \frac{n^2 \cdot p^2(n)G^2\sigma^2D_{X,V}^2}{\varepsilon^4} \right),$$

where $p(n) = \min\{2q - 1, 32 \ln n - 8\}n^{\frac{2}{q} - 1}$, for $n \geq 3$.

Note that the second estimate depends on the ”geometry” of the problem. In particular, in the Euclidean case $q = 2$ and then we have.

$$O\left( \sqrt{\frac{LD_{X,V}^2}{\varepsilon}} + \frac{nG^2D_{X,V}^2}{\varepsilon^2} + \frac{n^2G^2\sigma^2D_{X,V}^2}{\varepsilon^4} \right).$$

A more interesting case is the case when we work on a probability simplex, then $\| \cdot \| = \| \cdot \|_1$ and $q = \infty$ and $D_{X,V}^2 = 2\ln n$, in this case the estimate is transformed into

$$O\left( \sqrt{\frac{L \ln n}{\varepsilon}} + \frac{\ln n \cdot G^2}{\varepsilon^2} + \frac{n \ln n \cdot G^2\sigma^2}{\varepsilon^4} \right).$$

### 3.2 Applications to distributed optimization

An interesting question is how to obtain estimates for problems (2) and (3) from the general results of the previous section. We consider the Euclidean case. Let assume that all functions $f_m$ have bounded gradient with constant $G$. Then $f(X)$ has also bounded gradient. One can note that $g$ is $\lambda$-smooth in (2) and $\lambda\lambda_{\max}(W)$-smooth in (3). Recall that the number of the computations of $\nabla g(X)$ corresponds to the number of the communication rounds, and the calls $f(X)$ – to the local gradient-free calculations. Then the following estimates are valid for the number of communications and local iterations:
\[ O\left(\sqrt{\frac{\lambda D^2}{\varepsilon}}\right) \] communication rounds and

\[ O\left(\sqrt{\frac{\lambda D^2}{\varepsilon}} + \frac{nG^2D^2}{\varepsilon^2} + \frac{n^2G^2\sigma^2D^2}{\varepsilon^4}\right) \] local computations

\[ O\left(\sqrt{\frac{\lambda\lambda_{\max}(W)D^2}{\varepsilon}}\right) \] communication rounds and

\[ O\left(\sqrt{\frac{\lambda\lambda_{\max}(W)D^2}{\varepsilon}} + \frac{nG^2D^2}{\varepsilon^2} + \frac{n^2G^2\sigma^2D^2}{\varepsilon^4}\right) \] local computations

Here \( D^2 \) – the Euclidean diameter of the set.

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