Eductions of Edge Mode Effects

V.P. Nair

Physics Department, City College of the CUNY
New York, NY 10031

E-mail: vpnair@ccny.cuny.edu

Abstract

Edge modes in gauge theories, whose raison d'être is in the nature of the test functions used for imposing the Gauss law, have implications in many physical contexts. I discuss two such cases: 1) how edge modes are related to the interface term in the BFK formula and how they generate the so-called contact term for entanglement entropy in gauge theories, 2) how they describe the dynamics of particles in generalizing the Einstein-Infeld-Hoffmann approach to particle dynamics in theories of gravity.

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1 Introduction

For many students of high energy physics in the early 1980s in Syracuse, afternoons repeated a familiar pattern. Balachandran, or Bal as everyone referred to him, would appear at the door of the tiny office I shared with a fellow graduate student, in his signature attire, the greenish-grey sweater with elbow patches, coffee cup in hand, saying “Let us discuss”. We would walk down the hallway, collecting more students, sometimes visitors, to gather in Room 316, made famous by Fedele Lizzi, in his contribution to this volume. All matters great and small would come up for discussion. And some days we would then repair to his home where Indra’s tolerance and the excellent food would let us talk late into the evening. Bal was always fascinated by the mathematical side of things in physics. He, and his students and collaborators, had just been working on applying the coadjoint orbit actions to various problems when I arrived in Syracuse. This was also a time of an effervescence of many ideas in particle physics: the role of topology in physics was beginning to be appreciated, monopoles, solitons, instantons made frequent appearances in papers, anomalies were still intriguing, Bal talked of why he felt the effective action for anomalies constructed by Wess and Zumino should be important (prescient comments as it turned out) and the long dormant idea of Skyrmions was just about to be revived, Over time, all of us, his students and collaborators, have branched out in different ways, but a fascination and engagement with the mathematical side of things, from the Syracuse days with Bal, have remained the leitmotif of our work.

In appreciation, in this article dedicated to Bal’s 85th birthday, I would like to highlight a couple of effects related to one of his many favorite topics, namely, edge modes in gauge theories and gravity.

2 Entanglement in a gauge theory and the ‘contact term’

As is well known, in a gauge theory, physical states are annihilated by the Gauss law operator \( G(\theta) \) defined with test functions \( \theta(x) \) which vanish at the boundary of the space under consideration. In this case \( e^{iG(\theta)} \) generates gauge transformations \( e^{i\theta} \) which become the identity on the boundary. The same operator \( e^{iG(\bar{\theta})} \), with \( \bar{\theta} \) which do not vanish on the boundary, generate physical states; these are the edge modes. In the 1990s, Bal and collaborators explored the properties of such states for the Maxwell-
Chern-Simons theory, as well as in more general contexts [1, 2]. More recently, as interest in entanglement has increased, it has become clear that edge modes do make a contribution, the so-called contact term, to the entanglement entropy [3]. This is the facet of edge modes that I would first like to highlight.

We will consider the Maxwell theory in $2+1$ dimensions as this suffices to illustrate the main point. In a Hamiltonian framework, we can take $A_0 = 0$. The spatial components of the gauge potential $A_i$ and the electric field $E_i$, $(i = 1, 2)$ can be parametrized as

$$A_i = \partial_i \theta + \epsilon_{ij} \partial_j \phi, \quad E_i = \dot{A}_i = \partial_i \sigma + \epsilon_{ij} \partial_j \Pi \tag{1}$$

We consider the spatial manifold $M$ to be a square which is separated, by a straight line interface, into two rectangular regions which we will label as $L$ and $R$. The idea is to split the fields into three terms each, a field in $L$ which vanishes on the interface, a field in $R$ which also vanishes on the interface, and a field on the interface itself. Focusing on the region $L$ first, the fields can be split as

$$\chi_L(x) = \tilde{\chi}_L(x) + \int_{\partial L} \chi_0(y) n \cdot \partial_y G_L(y, x) \tag{2}$$

where $\chi = \theta, \phi, \sigma, \Pi$. $\tilde{\chi}_L$ vanishes on the boundary of $M$ as well as on the interface between $L$ and $R$. $\chi_0$ denotes the value of $\chi$ on the interface and $G_L(y, x)$ is the Green’s function for the Laplacian, again obeying Dirichlet (vanishing) conditions on the boundary of $M$ and on the interface $\partial L$. In (2), the value of the field on the interface, namely $\chi_0$, is continued into the interior of $L$ by Laplace’s equation and Green’s theorem, so that

$$\nabla^2 \int_{\partial L} \chi_0(y) n \cdot G_L(y, x) = 0 \tag{3}$$

This does not introduce any functional degrees of freedom in addition to $\chi_0$, the freedom of choosing arbitrary values of the field in $L$ is contained in $\tilde{\chi}_L$. Equation (2) gives a general parametrization of the fields in $L$.

Consider now the phase space path integral where, for the constraint $\nabla \cdot E \approx 0$, we choose the conjugate constraint $\nabla \cdot A \approx 0$ (Coulomb gauge). The path integral is given by

$$Z = \int \! d\mu \delta(\nabla \cdot E) \delta(\nabla \cdot A) \det(-\nabla^2) e^{iS}, \quad S = \int d^3x \left[ E_i \dot{A}_i - \mathcal{H} \right] \tag{4}$$
where $\mathcal{H}$ is the Hamiltonian density. The canonical two-form $\int \delta E_i \delta A_i$ serves to define the phase space measure $d\mu$ in (4). The canonical one-form $\mathcal{A} = \int E_i \delta A_i$ is given in terms of the parametrization (1) as

$$\mathcal{A} = \int_L \left[ (-\nabla^2 \tilde{\sigma}_L) \delta \tilde{\theta}_L + \tilde{\Pi}_L \delta B_L \right] + \int_{\partial L} \mathcal{E}_0 \delta \theta_0(x) + \int_{\partial L} Q_0 \delta \phi_0(x) \quad (5)$$

where $B = -\nabla^2 \phi$ is the magnetic field and

$$\mathcal{E}_0(x) = \int_y \sigma_0(y) M_L(y, x) + \partial_r \Pi_0(x)$$

$$Q_0(x) = \int_y \Pi_0(y) M_L(y, x) - \partial_r \sigma_0(x) \quad (6)$$

Here $\partial_r$ signifies the tangential derivative and $M_L(x, y) = n \cdot \partial_x n \cdot \partial_y G_L(x, y)$ is the Dirichlet-to-Neumann operator for the geometry we have. It is easily verified that this is essentially $(\sqrt{-\nabla^2})_{x,y}$. As a result, we can see that $\mathcal{E}_0$ and $Q_0$ are related by $C = \partial_y \int \mathcal{E}_1(x) M^{-1}(x, y) + Q_1(y) = 0$. This is another constraint in the problem, it is due to the freedom in defining $B$. Notice that $B = -\nabla^2 (\phi + f) = -\nabla^2 \phi$, if $\nabla^2 f = 0$. Nontrivial choices of such functions exist, for example, as

$$f(x) = \int_{\partial L} f_0(y) n \cdot \partial_y G_L(y, x) \quad (7)$$

The constraint $C$ encodes this additional “gauge freedom”, which is really an ambiguity of the parametrization (1). We can use the freedom of $f$ to set $\phi_0$ to zero, i.e., choose $\phi_0 \approx 0$ as the constraint conjugate to $C$. Eliminating $C$ and $\phi_0$ by standard symplectic reduction, we get

$$\mathcal{A} = \int_L \left[ (-\nabla^2 \tilde{\sigma}_L) \delta \tilde{\theta}_L + \tilde{\Pi}_L \delta B_L \right] + \int_{\partial L} \mathcal{E}_0 \delta \theta_0 \quad (8)$$

The corresponding phase space volume element is

$$d\mu_1 = [d\tilde{\sigma} d\tilde{\theta}]_L [d\mathcal{E}_0 d\theta_0] [d\tilde{\Pi} dB]_L \det(-\nabla^2)_L \quad (9)$$

where the determinant is understood to be evaluated with Dirichlet conditions on its eigenfunctions. The action, the Hamiltonian and the constraints in (4) are given by

$$S_L = \int_L \left[ (-\nabla^2 \tilde{\sigma}_L) \dot{\tilde{\theta}}_L + \tilde{\Pi}_L \dot{B}_L \right] + \int_{\partial L} \mathcal{E}_0 \dot{\theta}_0(x) - \int dt \mathcal{H} \quad (10)$$

$$\mathcal{H} = \int_1^2 \left[ (\nabla \tilde{\sigma}_L)^2 + (\nabla \tilde{\Pi}_L)^2 + B^2_L \right] + \frac{1}{2} \int_{\partial L} \mathcal{E}_0(x) M^{-1}_L(x, y) \mathcal{E}_0(y)$$
\[ \delta(\nabla \cdot E) = (\det(-\nabla^2)_L)^{-1} \delta(\tilde{\sigma}_L) \quad \delta(\nabla \cdot A) = (\det(-\nabla^2)_L)^{-1} \delta(\tilde{\theta}_L) \]  

Using these results and carrying out integrations, including over \( B_L \), we find

\[ Z_L = \int [dE_0 d\theta_0] [d\Pi_L] \exp(i\tilde{S}) \]

\[ \tilde{S}_L = \int \frac{1}{2} \left[ \dot{\Pi}_L^2 - (\nabla \Pi_L)^2 \right] + \int_{\partial L} \left[ E_0 \dot{\theta}_0 - \frac{1}{2} E_0 (M_1 + M_{\Pi})^{-1} E_0 \right] \]

We see that the dynamics has been reduced to that of a scalar field \( \tilde{\Pi}_L \) (which obeys Dirichlet conditions on all of \( \partial L \)) and “edge modes” with dynamics given by the second term in \( \tilde{S}_L \).

We now carry out exactly the same analysis for the full space \( \mathcal{M} \). This will give an expression similar to (12), but for the whole space, and with no edge modes since we chose Dirichlet conditions on \( \partial \mathcal{M} \). However, we can also choose to split any field into \( \tilde{\chi}_L, \tilde{\chi}_R \) and \( \chi_0 \) with

\[ \chi = \begin{cases} 
\tilde{\chi}_L(x) + \int_{\partial L} \chi_0(y) n \cdot \partial G_L(y, x) & \text{in } L \text{ and on } \partial L \\
\tilde{\chi}_R(x) + \int_{\partial R} \chi_0(y) n \cdot \partial G_R(y, x) & \text{in } R \text{ and on } \partial R = \partial L 
\end{cases} \]

With this parametrization of the fields, the action takes the form

\[ S_{\text{split}} = \int_L \left[ (-\nabla^2 \tilde{\sigma}_L) \dot{\tilde{\theta}}_L - \frac{1}{2} (\nabla \tilde{\sigma}_L)^2 \right] + \int_R \left[ (-\nabla^2 \tilde{\sigma}_R) \dot{\tilde{\theta}}_R - \frac{1}{2} (\nabla \tilde{\sigma}_R)^2 \right] \\
+ \int \Pi \dot{B} - \frac{1}{2} \left[ (\nabla \Pi)^2 + B^2 \right] + \int_{\partial L} E_0 \dot{\theta}_0 - \frac{1}{2} E_0 (M_1 + M_{\Pi})^{-1} E_0 \]

From this, we can read off the canonical one-form

\[ A = \int_{L,R} \Pi \delta B + \int_L (-\nabla^2 \tilde{\sigma}_L) \delta \tilde{\theta}_L + \int_R (-\nabla^2 \tilde{\sigma}_R) \delta \tilde{\theta}_R + \int_{\partial L} E_0 \delta \theta_0 \]

The phase volume associated to this is

\[ d\mu_{\text{split}} = [d\tilde{\sigma} d\tilde{\theta}]_L [d\tilde{\sigma} d\tilde{\theta}]_R \det(-\nabla^2)_L \det(-\nabla^2)_R [dE_0 d\theta_0] [d\Pi dB] \]

Notice that \( d\mu \) involves \( \det(-\nabla^2) \) calculated separately for \( L \) and \( R \) with Dirichlet conditions. The determinant \( \det(-\nabla^2) \) for the full space appearing in the path integral (as in (4)) can also be displayed in the split form using the BFK gluing formula as

\[ \det(-\nabla^2) = \det(-\nabla^2)_L \det(-\nabla^2)_R \det(M_L + M_R) \]
As regards the constraints, we can write them as

\[
\begin{align*}
\int \partial_i f E_i &= \int_L \tilde{f}_L (-\nabla^2 \tilde{\sigma}_L) + \int_R \tilde{f}_R (-\nabla^2 \tilde{\sigma}_R) + \int f_0 \mathcal{E} \approx 0 \\
\int \partial_i h A_i &= \int_L \tilde{h}_L (-\nabla^2 \tilde{\theta}_L) + \int_R \tilde{h}_R (-\nabla^2 \tilde{\theta}_R) + \int h_0 (M_L + M_R) \theta_0 \approx 0
\end{align*}
\]

(18)

If we plan to integrate over the full space, the constraints eliminate \( \theta_0 \)-dependence everywhere, which is equivalent to writing

\[
\begin{align*}
\delta (\nabla \cdot E) \delta (\nabla \cdot A) &= \delta [-\nabla^2 \tilde{\sigma}_L] \delta [-\nabla^2 \tilde{\sigma}_R] \delta [-\nabla^2 \tilde{\theta}_L] \delta [-\nabla^2 \tilde{\theta}_R] \\
&\quad \times \delta [\mathcal{E}_0] \delta [(M_L + M_R) \theta_0] \\
&= \delta [\cdots] \det (-\nabla^2)^{-1}_L \det (-\nabla^2)^{-1}_R \det (M_L + M_R)^{-1}
\end{align*}
\]

(19)

where \( \delta [\cdots] \) indicates the product of \( \delta \)-functions for fields \( \tilde{\sigma}_L, \tilde{\sigma}_R, \tilde{\theta}_L, \tilde{\theta}_R, \mathcal{E}_0, \theta_0 \). Notice again the appearance of determinants separately for \( L \) and \( R \), but there is also one factor of \( \det (M_L + M_R)^{-1} \). This arises from the last term in the second of the constraints (18). It is then easy to check that the path integral for the full space is reproduced.

However, if we now consider integrating over fields in \( R \), the edge modes \( \mathcal{E}_0 \) and \( \theta_0 \) on the interface are physical degrees of freedom from the point of view of region \( R \). The test functions \( f_0, h_0 \) in (18) are to be taken to be zero, so we do not have the corresponding \( \delta \)-functions or the factor \( \det (M_L + M_R)^{-1} \) in (19). We then find, after integration over fields in \( R \),

\[
\begin{align*}
Z &= \det (M_L + M_R) \int [d\mathcal{E} d\theta_0] [d\Pi dB] e^{iS} \\
S &= \int \left[ \mathcal{E} \dot{\theta}_0 - \frac{1}{2} \mathcal{E} (M_L + M_R)^{-1} \mathcal{E} \right] + S_{\Pi, B}
\end{align*}
\]

(20)

The \( \Pi, B \) part, which we have not displayed in split-form or elaborated on, behaves as a scalar field and gives what is expected for a scalar field as regards entanglement. The new ingredient is that, while the result reduces to the expected one in region \( L \), there is an extra factor \( \det (M_L + M_R) \) from the phase volume, i.e., an extra degeneracy factor due to the edge modes. This contributes to the entanglement entropy, which is now of the form

\[
S_E = \log \det (M_L + M_R) + S_{E, \Pi}
\]

(21)

where \( S_{E, \Pi} \) is due to the \((\Pi, B)\)-sector. The extra contribution from the edge modes:

1. is the so-called contact term, calculated many years ago via the replica trick by Kabat [?].
2. is the interface term needed for the BFK gluing formula. This is one of the key effects of the edge modes we wanted to emphasize. For more details and extensions to the nonabelian, as well as Maxwell-Chern-Simons theories, see reference [3].

3. The EIH method for Chern-Simons + Einstein gravity

We now turn to another effect of edge modes, namely, how they play a role in the context of the Einstein-Infeld-Hoffmann (EIH) method [6]. Recall that EIH argued that the Einstein field equations for gravity in the vacuum are sufficient to determine the dynamics of point-particles (i.e., matter dynamics) interacting gravitationally. They defined point-particles as singularities in the gravitational field, excised small spheres (or tubes when we include time) around the singularities to keep fields well-defined, and imposed the field equations on the resulting configurations. This led to a set of equations which are not only conceptually interesting, but actually can be applied in some astrophysical contexts. The question we pose here is: How does this work in more general theories of gravity, say, Chern-Simons (CS) gravity or in CS gravity with an Einstein-Hilbert term added [7]? To begin with let us consider the 2+1 dimensional CS action, with connections in the algebra of some Lie group $G$, given as

$$S = \frac{k}{4\pi} \int \text{Tr} (A dA + \frac{2}{3} A^3) + S_b(A, \psi)$$  \hspace{1cm} (22)

The spacetime manifold is taken as $M \times \mathbb{R}$, where $M$ has the topology of the disc. $S_b(A, \psi)$ is a boundary action (which may depend on some other fields $\psi$) which ensures the full gauge invariance of the action (22), including transformations on the boundary $\partial M$.

The bulk equation of motion is $F = 0$, i.e., $A$ is a pure gauge in the bulk. Consider now singular classical solutions on the disk $M$ of the form

$$A_i = a_i, \quad da + a^2 = \sum_{s=1}^{N} q_s \delta^{(2)}(x - x_s), \quad A_0 = a_0 = 0$$  \hspace{1cm} (23)

For simplicity, we take all $q_s$ to be in the Cartan subalgebra of $G$ so that $da + a^2 = da$. We still have $F = 0$ on $\tilde{M} = M - \{C_s\}$ where $C_s$ denote small disks around the singularities; thus $A$ is still a pure gauge on $\tilde{M}$. The general solution to the field equation is then a gauge transform of (23),

$$A_i = g^{-1} a_i g + g^{-1} \partial_i g, \quad A_0 = g^{-1} \partial_0 g$$  \hspace{1cm} (24)
The evaluation of the action on this configuration gives

\[ S = S[a] - \frac{k}{4\pi} \sum_s \oint_{\partial C_s} \left[ \text{Tr}(a \, dg \, g^{-1}) \right] \]  

(25)

We see that the dynamics is given in terms of the group elements \( g \) on \( \partial C_\alpha \); these represent the “edge modes”. Consider shrinking the size of the disks to almost zero radius, so that \( g \) can be taken to be \( g_s = g(\bar{x}_s) \equiv h_s^{-1} \). The \( g \)-dependent part of the action, after integrating over the spatial boundaries, is thus

\[ S = \frac{k}{4\pi} \sum_s \int dt \left[ \text{Tr}(q_s h_s^{-1} \partial_0 h_s) \right] \]  

(26)

This is a sum of co-adjoint orbit actions. We see that there is no real dynamical evolution, not surprising for the CS theory, but, in the usual manner of quantizing such actions, the states in the quantum theory will carry unitary representations of \( G \), of highest weights specified by the \( q_s \).

With this observation in mind, consider Einstein gravity in 2+1 dimensions (with a cosmological constant \( \sim l^{-2} \)) which may be described by the CS action\(^8\)

\[
S = \frac{k}{4\pi} \left[ \int \text{Tr} \left( A_L + \frac{2}{3} A^3 \right)_L - \int \text{Tr} \left( A_R + \frac{2}{3} A^3 \right)_R \right] 
= \frac{k}{4\pi} \int d^3 x \, \text{det} \left[ R - \frac{2}{l^2} \right] + \text{total derivative} \]  

(27)

where the gauge connections are given in terms of the spin connection \( \omega^{ab} \) and the frame field (dreibein in this case) \( e^a \) by

\[
A_L = (-iM_a) A^a_L = (-iM_a) \left( -\frac{1}{2} \eta^{ak} \epsilon_{kbc} \omega^{bc} + \hat{e}^a \right) \\
A_R = (-iN_a) A^a_R = (-iN_a) \left( -\frac{1}{2} \eta^{ak} \epsilon_{kbc} \omega^{bc} - \hat{e}^a \right) \]  

(28)

Here \( M_a, N_a \) are the generators of two independent \( SO(2,1) \) Lie algebras, with parity exchanging them. Newton’s constant for this problem can be identified as \( G = l/(4k) \).

Since there is a nonzero cosmological constant, it is clear from (27) that the solution for the vacuum state, i.e., the solution of the bulk equation of motion, is the anti-de Sitter (AdS) space in 2+1 dimensions. As with (22), we can now add a boundary term (on \( \partial M \)) to obtain full gauge invariance for (27) and then introduce “point-particles” via ansätze of the form (23), (24). The action on these configurations becomes

\[ S = -\frac{k}{4\pi} \sum_s \int dt \left[ \text{Tr}(q_s h_s^{-1} \partial_0 h_s)_L - \text{Tr}(q_s h_s^{-1} \partial_0 h_s)_R \right] \]  

(29)

\(^1\)another of Bal’s favorite topics
We get two sets of co-adjoint orbit actions, leading to unitary representations of $SO(2,1) \times SO(2,1)$ upon quantization. The full isometry group for the AdS space is $SO(2,1) \times SO(2,1)$, with the diagonal $SO(2,1)$ as the Lorentz group, while the coset directions correspond to translations. Thus a point-particle in AdS space must be defined in the quantum theory as a unitary irreducible representation of $SO(2,1) \times SO(2,1)$, and this is exactly what is obtained, realizing the EH strategy. The mass and spin, read off from the identification of translations and Lorentz transformations, are

$$m = \frac{q_R + q_L}{32\pi G}, \quad s = \frac{l}{16G}(q_L - q_R)$$  \hspace{1cm} (30)

In 2+1 dimensional gravity, curvatures are localized at the positions of the particles, so that the result (29) with no interactions between particles is indeed what we expect. So it is interesting to consider a similar analysis in 4+1 dimensions, where the Einstein-Hilbert action is distinct from a combination of CS terms [9] and can be included as an additional term in the action. We can construct an $SO(4,2)$ algebra using $4 \times 4$ Dirac $\gamma$-matrices, $\gamma_\alpha, \Sigma_{ab} = (i/4)[\gamma_a, \gamma_b], a, b = 0, 1, 2, 3, 5$. We then define the connections

$$A_L = -\frac{i}{2}(\omega^{ab}\Sigma_{ab} + e^a \gamma_a), \quad A_R = -\frac{i}{2}(\omega^{ab}\Sigma_{ab} - e^a \gamma_a)$$  \hspace{1cm} (31)

As for the action, we will consider the parity-invariant combination

$$S = CS(A_L) - CS(A_R) + S_b(A, \psi) + S_{EH}$$

$$CS(A) = -\frac{ik}{24\pi^2} \int \text{Tr} \left[ AdAdA + \frac{3}{2} A^3dA + \frac{3}{5} A^5 \right]$$  \hspace{1cm} (32)

$S_b(A, \psi)$ is, as before, included to cancel any boundary terms from gauge transformations, and $S_{EH}$ is the Einstein-Hilbert action with a cosmological constant $\sim l^{-2}$. In terms of the curvature $R^{ab} = d\omega^{ab} + (\omega \omega)^{ab}$ and torsion $T^a = de^a + \omega^{ab}e^b$, we find

$$F_{L,R} = (-i\Sigma_{ab}/2)(R^{ab} + e^a e^b) \pm (-i\gamma_a)T^a$$  \hspace{1cm} (33)

The bulk equation of motion for the $CS$-part in (32) is thus satisfied by $R^{ab} = -e^a e^b$ and $T^a = 0$. With the scaling, $e^a \rightarrow \sqrt{12} e^a/l$, we see that this is AdS spacetime. For simplicity, we take $S_{EH}$ to have the same value for the cosmological constant so that the term from $S_{EH}$ in the bulk equation of motion is also zero for the same AdS spacetime.

We now introduce solutions with point-like singularities on the spatial manifold. These will be taken as the point-like limit of 4d-instantons in $SO(4) \in SO(4,2)$. We
excise small balls around each point, and on \( M - \{ C_s \} \), we take the solution to be of the form

\[
a = t_1 U^{-1} dU, \quad t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[ U = \phi^0 + i \sigma_i \phi^i, \quad \phi^0 \phi^0 + \phi^i \phi^i = 1 \quad (34) \]

\( t_1 \) is a projector to either of the two \( SU(2) \)'s in \( SO(4) \sim SU(2) \times SU(2) \). The instanton number is the integral of \((1/12\pi^2) \epsilon_{\mu \nu \alpha \beta} \phi^\mu d\phi^\nu d\phi^\alpha d\phi^\beta \). We then consider the more general configurations with the same singularity structure:

\[
A_L = g^{-1}(a_L + \delta a_L) g + g^{-1} d g, \quad A_R = g^{-1}(a_R + \delta a_R) d g + g^{-1} d g \quad (35)
\]

\( a_L \) and \( a_R \) are terms of the form (34), and are related by parity. \( \delta a_L \) and \( \delta a_R \) correspond to perturbations of the metric as \( g_{\mu \nu} \rightarrow g_{\mu \nu} + h_{\mu \nu} \), i.e.,

\[
\delta a_{L,R} = (-i/2) \left[ \pm \gamma_a e^{-1/\alpha} h_{\mu \nu} - \Sigma_{ab} e^{-1/\beta} e^{-1/\gamma} \nabla_\alpha h_{\beta \mu} \right] d x^\mu \quad (36)
\]

Also \( g \) is an element of \( SO(4,2) \) parametrized as

\[
g = S^{-1} A V, \quad V = \frac{1}{\sqrt{z}} \begin{pmatrix} z & i X \\ 0 & 1 \end{pmatrix}, \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (37)
\]

Here \( X = x^0 - i \sigma_i x^i \), and \( z = x^5 \) is the radial coordinate of AdS and \( \Lambda \in SO(4,1) \) denotes a Lorentz transformation. Evaluating the action (32) on the configurations (35), we find

\[
S = \frac{k}{2} \sum_s \int (Q_s^1 - Q_s^2) \eta_{ab}(\Lambda_s) \left[ e^b_\mu + e^{-1/\alpha} h_{\mu \nu} \right] d x^\mu + \frac{1}{2} \int h_{\mu \nu} \nabla_{\mu} \phi_{\nu} h_{\alpha \beta} \quad (38)
\]

where \( Q_s^1 \) and \( Q_s^2 \) are the instanton numbers for the two \( SU(2) \)'s in \( SO(4) \) and \( \nabla_{\mu \alpha \beta} \) is the Lichnerowicz operator (i.e., the quadratic fluctuation operator) for \( S_{\text{EH}} \). Without the \( h_{\mu \nu} \)-terms, this is a sum of co-adjoint orbit actions. The mass may be identified as \( m_s = -\frac{1}{2} k(Q_s^1 - Q_s^2) \). (Spin can be included by considering different orientations of the \( SU(2) \)'s in \( SO(4) \) for different instantons.) Further, if we write \( \Lambda_s^0 = e^0_\mu dx^\mu \), which is consistent with its properties, the first term in \( S \) becomes \(-m \int ds \), corresponding to free particle motion. Solving for \( h_{\mu \nu} \) from its equation of motion à la (38), and using it back again in (38), we get interactions between the particles. In the nonrelativistic limit, the result is

\[
S = - \sum_s m_s \int ds + \frac{1}{6} \sum_{s \neq s'} \int d x_s^0 m_s m_{s'} \quad (39)
\]

\[ x_s^0 = x_{s'}^0 + |\vec{x}_s - \vec{x}_{s'}| \]
showing, correctly, the 4d Coulomb-like potential, if we neglect retardation effects. In principle, one can include higher corrections systematically, but this suffices to prove our main point: The edge modes of the CS+Einstein gravity allow us to realize the Einstein-Infeld-Hoffmann method of defining point-particles as singularities of the gravitational field and then obtaining their equations of motion from the field equations.

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References

[1] Entanglement is a topic with an enormous literature by now, it is impossible to make even a vaguely comprehensive list in the short space available. The papers cited below can be used to trace most of the relevant articles.

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