NON-ARCHIMEDEAN ENTIRE CURVES IN PROJECTIVE VARIETIES DOMINATING AN ELLIPTIC CURVE

JACKSON S. MORROW

ABSTRACT. Let $K$ be an algebraically closed, complete, non-Archimedean valued field of characteristic zero. We prove the non-Archimedean Green–Griffiths–Lang conjecture for projective surfaces of irregularity one. More precisely, we prove that if $X/K$ is a groupless, projective surface that admits a dominant morphism an elliptic curve, then $X$ is $K$-analytically Brody hyperbolic. The main ingredient in our proof is a theorem concerning the algebraic degeneracy of non-Archimedean entire curves in projective, pseudo-groupless varieties admitting a dominant morphism to an elliptic curve.

1. Introduction

The strong form of the Green–Griffiths–Lang conjecture predicts that a projective variety $X/C$ is of general type if and only if $X$ is pseudo-Brody hyperbolic (i.e., there is a proper closed subscheme $\Delta \subset X$ such every non-constant holomorphic map $C \to X(C)$ factors through $\Delta(C)$).

This conjecture is known when $X$ has irregularity $q(X) = h^0(X, \Omega^1_X) > \dim X$ by the celebrated theorems of Bloch–Ochiai–Kawamata [Blo26, Och77, Kaw80]. The works of Dethloff–Lu [DL07], Noguchi–Winkelmann–Yamanoi [NWY07, NWY13], and Winkelmann [Win11] proved a weak variant of Green-Griffiths–Lang conjecture when the irregularity is equal to the dimension. In particular, they show that for a smooth projective variety $X/C$ of general type with $q(X) = \dim X$, every complex entire curve $f: C \to X(C)$ is algebraically degenerate. To the author’s knowledge, there does not appear to be any literature on this conjecture when $q(X) < \dim X$.

It is natural to ask about non-Archimedean analogues of Brody hyperbolicity and of the Green–Griffiths–Lang conjectures, with the desideratum that all notions of hyperbolicity agree over an algebraically closed field of characteristic zero.

Let $K$ be an algebraically closed, complete, non-Archimedean valued field of characteristic zero, and for a variety $X/K$, let $X^\text{an}$ denote the $K$-analytic space (in the sense of Berkovich [Ber90]) associated to $X$. When investigating this analogy, one of the first questions encountered is: what is the “correct” notion of a non-Archimedean entire curve? The works of [Che94, Che96, CR04, ACW08, LW10, LW17, JV18, Mor21] have studied this question, and the results strongly suggest that, for the notions of hyperbolicity to agree, one should define a non-Archimedean entire curve to be an analytic morphism $G^\text{an}_{m,K} \to X^\text{an}$ (see Remark 2.10 for discussion).

Our first result concerns the algebraic degeneracy (Definition 2.11) of non-Archimedean entire curves in projective, pseudo-groupless (Definition 2.4) varieties admitting a dominant morphism to an elliptic curve. We remark that the notion of pseudo-groupless was introduced by Javanpeykar–Xie [JX20] and a consequence of the Green–Griffiths–Lang conjectures is that this notion is equivalent to being of general type [Jav20, Conjecture 12.1].

**Theorem A.** Let $K$ be an algebraically closed, complete, non-Archimedean valued field of characteristic zero. Let $X/K$ be a projective, pseudo-groupless variety admitting a dominant morphism to an elliptic curve. Then, every non-Archimedean entire curve $\varphi: G^\text{an}_{m,K} \to X^\text{an}$ is algebraically degenerate.
We will use Theorem A to prove a result on the non-Archimedean Green–Griffiths–Lang conjecture for projective varieties over $K$, which reads as follows.

**Conjecture 1.1** (Non-Archimedean Green–Griffiths–Lang, [JV18, Conjecture 1.1]). Let $K$ be an algebraically closed, complete, non-Archimedean valued field of characteristic zero, and let $X/K$ be a projective variety. Then the following are equivalent:

1. $X$ is groupless over $K$ (Definition 2.2);
2. $X$ is $K$-analytically Brody hyperbolic (Definition 2.6).

Using Theorem A, we prove Conjecture 1.1 for projective surfaces admitting a dominant morphism to an elliptic curve.

**Theorem B.** Let $K$ be an algebraically closed, complete, non-Archimedean valued field of characteristic zero, and let $S/K$ be a projective surface admitting a dominant morphism to an elliptic curve.

Then $S$ is groupless over $K$ if and only if $S$ is $K$-analytically Brody hyperbolic.

**Remark 1.2.** While we only state our results for Berkovich $K$-analytic spaces, our results hold for adic spaces (in the sense of Huber [Hub94]), under the added assumption that $K$ is non-trivially valued. This follows because the category of strict, Hausdorff Berkovich $K$-analytic spaces is equivalent to the category of taut, locally of finite type adic spaces over $\text{Spa}(K,K^\circ)$ by [Hub96, Proposition 6.3.7].

**Related Results.** The previous literature on the complex and non-Archimedean Green–Griffiths–Lang conjecture focused on the setting of varieties with irregularity greater than or equal to their dimension. Above, we mentioned several results in the complex analytic setting. In the non-Archimedean analytic setting, Cherry [Che94, Theorem 3.6] proved Conjecture 1.1 for closed subvarieties of an abelian variety, and hence for proper surfaces of irregularity greater than two.

We stress that our results are complementary to these as we focus on varieties with irregularity strictly less than the dimension. Also, combining Cherry’s results with Theorem B, the remaining cases of Conjecture 1.1 for irregular surfaces (i.e., surfaces with positive irregularity) are those with irregularity two.

**Outline of paper.** In Section 2, we recall definitions and results on hyperbolicity in the algebraic and non-Archimedean analytic setting, the topological notion of semi-coverings, and topological properties of Berkovich spaces. In Section 3, we prove Theorem A, and in Section 4, we prove our Theorem B. Finally, in Section 5, we provide an example of a non-pseudo-groupless surface that dominates an elliptic curve and admits a non-Archimedean entire curve which is not algebraically degenerate.

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**Notation.** We establish the following notations throughout our work.

**Fields and algebraic geometry.** Let $K$ be an algebraically closed, complete, non-Archimedean valued field of characteristic zero. By a variety $X$ over $K$, we mean an integral separated scheme of finite type over $K$ with positive dimension. For a variety $X$, let $q(X) = h^0(X,\Omega_X^1) = h^1(X,\mathcal{O}_X)$ denote the irregularity of $X$. Since we work in the characteristic zero setting, the irregularity of $X$ corresponds to the dimension of the Albanese variety $\text{Alb}(X)$ associated to $X$. 

Berkovich spaces. We will use script letters \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) to denote Berkovich \( K \)-analytic spaces (in the sense of [Ber90] or good \( K \)-analytic spaces in the sense of [Ber93]). We let \(|\mathcal{X}|\) denote the underlying topological space of \( \mathcal{X} \). In this work, our \( K \)-analytic spaces will be good. For a variety \( X/K \), we will use \( X^{an} \) to denote the \( K \)-analytic space associated to \( X \). By [Ber90, Theorem 3.2.1] and [Tem15, Fact 4.3.1.1], the \( K \)-analytic space \( X^{an} \) is good, path-connected, locally compact, locally path-connected, and Hausdorff. When \( X \) is a quasi-projective variety, an important result of Hrushovski–Loeser [HL16, Theorem 14.4.1] states that the \( K \)-analytic space \( X^{an} \) admits a topological universal cover, which we denote by \( \tilde{X} \).

2. Preliminaries

In this preliminary section, we recall definitions and results on hyperbolicity in the algebraic and analytic setting, define the notion of non-Archimedean entire curves and algebraic degeneracy, and discuss the topological notion of semi-coverings and topological properties of Berkovich spaces.

2.1. Hyperbolicity in the algebraic and non-Archimedean analytic setting. To begin, we recall definitions and results concerning notions of hyperbolicity in the algebraic setting. We will focus on the setting when our variety is proper, and for further discussion on the non-proper setting, we refer the reader to [JK20, JX20].

Definition 2.2 ([JK20, Definition 2.1, Lemma 2.4, & Lemma 2.5]). A proper variety \( X \) over \( K \) is groupless (over \( K \)) if, for every abelian variety \( A/K \), every morphism \( A \to X \) is constant.

Definition 2.3 ([JX20, Definition 3.1 & Corollary 3.17]). Let \( X/K \) be a proper variety and let \( \Delta \subset X \) be a closed subscheme. \( X \) is groupless modulo \( \Delta \) (over \( K \)) if, for every abelian variety \( A/K \) and every dense open subscheme \( U \subset A \) with \( \text{codim}(A \setminus U) \geq 2 \), every non-constant morphism of varieties \( U \to X \) factors over \( \Delta \).

Definition 2.4 ([JX20, Definition 3.2]). We say that a proper variety \( X \) is pseudo-groupless (over \( K \)) if there exists a proper closed subscheme \( \Delta \subset X \) such that \( X \) is groupless modulo \( \Delta \).

Later, we will need the important property that being pseudo-groupless is invariant under finite étale morphisms.

Lemma 2.5 ([Jav20, Lemma 6.5]). Let \( f: X \to Y \) be a finite étale morphism of proper varieties over \( K \). Then \( X \) is pseudo-groupless over \( K \) if and only if \( Y \) is pseudo-groupless over \( K \).

We now turn to the non-Archimedean notion of hyperbolicity following [JV18].

Definition 2.6 ([JV18, Definition 2.3, Lemma 2.14, Lemma 2.15]). A variety \( X \) over \( K \) is \( K \)-analytically Brody hyperbolic if every analytic morphism \( \mathbb{G}_{m,K}^{an} \to X^{an} \) is constant, and for every abelian variety \( A \) over \( K \), every algebraic morphism \( A \to X \) is constant.

We recall that Conjecture 1.1 is true for projective curves.

Theorem 2.7 ([Che94, Theorem 3.6]). Let \( C/K \) be a connected, projective curve. Then, \( C \) is groupless if and only if \( C \) is \( K \)-analytically Brody hyperbolic.

2.8. Non-Archimedean entire curves and algebraic degeneracy. We now introduce the notions of non-Archimedean entire curves and algebraic degeneracy.

Definition 2.9. For a \( K \)-analytic space \( \mathcal{X} \), an analytic morphism \( \varphi: \mathbb{G}_{m,K}^{an} \to \mathcal{X} \) is called non-Archimedean entire curve in \( \mathcal{X} \).
Remark 2.10. In the complex analytic setting, the exponential and logarithm maps provide an isomorphism between \( \mathbb{C} \) and \( \mathbb{C}^\times \), and so Brody hyperbolicity could equivalently be defined by the non-existence of non-constant morphisms from \( \mathbb{C}^\times \) into a complex analytic manifold. However, in the non-Archimedean setting, the exponential map is not convergent everywhere, and so we do not have such an isomorphism; in fact, by [Che94, Proposition 3.3], every analytic map from \( \mathbb{A}^{1,\text{an}}_K \rightarrow \mathbb{C}^{\text{an}}_{m,K} \) is constant. As a result, testing hyperbolicity on analytic morphisms from \( \mathbb{A}^{1,\text{an}}_K \) or \( \mathbb{C}^{\text{an}}_{m,K} \) can yield different results. For example, a result of Cherry (loc. cit. Theorem 3.5) states that for an abelian variety \( A/K \), every analytic map \( \mathbb{A}^{1,\text{an}}_K \rightarrow A^{\text{an}} \) is constant.

However, there can exist non-constant analytic morphisms \( \mathbb{C}^{\text{an}}_{m,K} \rightarrow A^{\text{an}} \) if \( A \) does not have good reduction over \( \mathcal{O}_K \). The reason for this is that analytic tori appear in the non-Archimedean uniformization of abelian varieties [BL84, Theorem 8.8]. Moreover, Definition 2.9 appears to be the “correct” one as it aligns with our desideratum from Section 1 and gives insight into the K-analytic Brody hyperbolicity of a K-analytic space.

Definition 2.11. For a variety \( X/K \), a non-Archimedean entire curve \( \varphi : \mathbb{C}^{\text{an}}_{m,K} \rightarrow X^{\text{an}} \) is algebraically degenerate if there exists a proper closed subscheme \( Y_\varphi \subset X \), which depends on \( \varphi \), such that \( \varphi(\mathbb{C}^{\text{an}}_{m,K}) \subset Y_\varphi \).

Remark 2.12. When \( X/K \) is proper, the condition that every non-Archimedean entire curve is algebraically degenerate is equivalent to the non-existence of a Zariski dense non-Archimedean entire curve. Indeed, if a non-Archimedean entire curve is not Zariski dense, then its Zariski closure is a proper closed subscheme of \( X \) by Berkovich analytic GAGA [Ber90, Corollary 3.4.13].

2.13. Semi-coverings and topological properties of Berkovich spaces. To conclude this preliminary section, we recall the notion of semi-covering from work of Brazas [Bra12] and discuss some topological properties of Berkovich spaces.

Definition 2.14 ([Bra12, Definition 3.1]). A semi-covering is a local homeomorphism with continuous lifting of paths and homotopies.

By [Bra12, Remark 3.6], one can check that a covering is in fact a semi-covering, so this notion generalizes the notion of covering. Unlike coverings, path-connected semi-coverings satisfy a very useful “two out of three” property.

Proposition 2.15 ([Bra12, Lemma 3.4 & Corollary 3.5]). Let \( p : X \rightarrow Y \), \( q : Y \rightarrow Z \) and \( r = q \circ p \) be maps of path-connected spaces. If two of \( p \), \( q \), \( r \) are semi-coverings, then so is the third.

When the source space of a semi-covering is locally arcwise connected and the base space is locally simply connected, the semi-covering is actually a covering map.

Proposition 2.16 ([DC16, Section 5–6A, Proposition 6]). Let \( p : X \rightarrow Y \) be a semi-covering. If \( Y \) is locally simply connected and \( X \) is locally arcwise connected, then \( p \) is a covering map.

We now recall that topological covers of Berkovich spaces (see [dJ95, Definition 2.1]) correspond to covering of the underlying topological space of the Berkovich space.

Lemma 2.17 ([dJ95, Lemma 2.6]). Let \( \mathcal{X}/K \) be a K-analytic Berkovich space. The category of topological covering spaces of \( \mathcal{X} \) is equivalent to the category of covering spaces of \([\mathcal{X}]\).

To conclude, we summarize the above results into a useful proposition.

Proposition 2.18. Let \( \mathcal{X}/K \) be a path-connected, locally simply connected K-analytic space, and let \( Y/K \) be a path-connected, locally path-connected topological space in the K-analytic topology. Any semi-covering \( Y \rightarrow [\mathcal{X}] \) is in fact a covering map.

Furthermore, \( Y \) can be endowed with the structure of a K-analytic space, which is unique up to isomorphism, and if \( \mathcal{X} \) is a K-analytic group, then \( Y \) can also be given the structure of a K-analytic group, again unique up to isomorphism.
Proof. This follows from Proposition 2.16, Lemma 2.17, and the well-known fact that a covering space of a path-connected, locally path-connected topological group is a topological group. \[\square\]

3. Proof of Theorem A

In this section, we will prove Theorem A. Let $X/K$ be a projective, pseudo-groupless variety, let $E/K$ be an elliptic curve, let $a: X \to E$ be a dominant morphism, and let $\varphi: G^a_{m,K} \to X^a$ be a non-Archimedean entire curve. By Remark 2.12, it suffices to prove that there does not exist a non-Archimedean entire curve, which is Zariski dense. We also note that we can and do assume that $X$ has dimension $\geq 2$, as Theorem A is true when $X$ is a curve by Theorem 2.7. Indeed, the notions of groupless and pseudo-groupless coincide for curves (c.f. [JI20, Remark 3.4]).

If $E$ has good reduction over $O_K$, then our result follows immediately from a result of Cherry.

Lemma 3.1. With the notation as above, suppose that $E$ has good reduction over $O_K$. Then $\varphi$ cannot be Zariski dense.

Proof. The composed morphism $\alpha^a \circ \varphi: G^a_{m,K} \to E^a$ is constant by [Che94, Theorem 3.2], and hence the image of $\varphi$ is contained in a fiber $F$ of $a$, which has dimension $\leq 1$. Furthermore, the Zariski closure of $F$ will be of dimension $\leq 1$, and hence our claim follows. \[\square\]

For the remainder of the section, we will assume that $E$ has multiplicative reduction over $O_K$ and suppose to the contrary that $\varphi: G^a_{m,K} \to X^a$ is Zariski dense.

The condition that $E/K$ has multiplicative reduction allows us to import techniques from algebraic topology. In particular, $E^a$ is topologically uniformized by $G^a_{m,K}$, and hence, as a $K$-analytic space, $E^a$ is isomorphic to $G^a_{m,K}/qZ$ for some $q \in K$ with $0 < |q| < 1$. Since $G^a_{m,K}$ is simply connected [Ber90, Section 6.3], the morphisms $\varphi$ and $\alpha^a \circ \varphi$ uniquely lift to the topological universal cover of $X^a$ and of $E^a$. To summarize the situation, we have the following diagram:

\[
\begin{array}{cccc}
G^a_{m,K} & \xrightarrow{\tilde{\varphi}} & \tilde{X} & \xrightarrow{\tilde{a}} & G^a_{m,K} \\
\varphi & & \downarrow{\pi_X} & & \downarrow{\pi_E} \\
X^a & \xrightarrow{\alpha^a} & E^a,
\end{array}
\]

where $\pi_X: \tilde{X} \to X^a$ and $\pi_E: G^a_{m,K} \to E^a$ are the universal covering morphisms.

A result of Cherry [Che94, Proposition 3.4] tells us that the morphism $\tilde{a} \circ \tilde{\varphi}: G^a_{m,K} \to \tilde{X} \to G^a_{m,K}$ is \textit{algebraic} (i.e., $\tilde{a} \circ \tilde{\varphi}: z \mapsto cz^d$ for some $c \in K$, $d \in \mathbb{Z}$). As $a^a$ is dominant [Ber90, Proposition 3.4.7] and $\varphi$ is assumed to be Zariski dense, we have that $\alpha^a \circ \varphi$ is Zariski dense, and hence we know that $c \neq 0$ and $d \neq 0$. Moreover, after translation and post-composition with the automorphism $z \mapsto z^{-1}$, we may and do assume that $\tilde{a} \circ \tilde{\varphi}: z \mapsto z^d$ where $d$ is a positive integer.

Our next goal is to show that we may reduce to the case where $d = 1$. A result of Tate [Tat95, p. 325] states that the endomorphism $z \mapsto z^d$ on $G^a_{m,K}$ uniquely induces a morphism of smooth, proper, connected, commutative, 1-dimensional $K$-analytic groups $\psi: G^a_{m,K}/(q^{1/d})Z \to G^a_{m,K}/qZ$, which is in fact an isogeny [Tat95, p. 325, Theorem], whence a finite étale morphism. By Berkovich analytic GAGA [Ber90, Corollary 3.4.13], we have that $G^a_{m,K}/(q^{1/d})Z$ is algebraic (i.e., there exists an elliptic curve $E'/K$ such that $E^a \cong G^a_{m,K}/(q^{1/d})Z$). Moreover, we can enhance the above
Consider the following fibered product $\mathcal{X}$, which exists as a $K$-analytic space (cf. [Ber93, p. 34]).

![Diagram](image)

We claim that the $K$-analytic space $\mathcal{X}$ is algebraic. As finite étale morphisms are stable under base change [And03, Remark 1.2.4(iv)], we have that the morphism $\psi' : \mathcal{X} \to X^{\text{an}}$ is finite étale, and by Berkovich analytic GAGA [Ber90, Corollary 3.4.13], $\mathcal{X}$ is algebraic, so there exists a proper separated scheme of finite type $X'$ such that $\mathcal{X} \cong X^{\text{an}}$. Using this, we identify $\mathcal{X}$ with $X'$.

By the definition of fibered product, we have that there exists a unique morphism $\varphi' : G_{m,K}^{an} \to X'$, and note that the morphism $\bar{\varphi} : G_{m,K}^{an} \to \bar{X}' \to G_{m,K}^{an}$ is the identity map. Moreover, the morphism $\bar{\varphi}'$ is injective and $\bar{\varphi}'$ is surjective as $\bar{\varphi}' \circ \bar{\varphi}'$ is bijective. To summarize, we have reduced the proof of Theorem A to the setting in below.

![Diagram](image)

**Lemma 3.2.** The morphism $\varphi' : G_{m,K}^{an} \to \varphi'(G_{m,K}^{an})$ is a covering map. Furthermore, $\varphi'(G_{m,K}^{an})$ is path-connected and locally path-connected.

**Proof.** To begin, we claim that $\bar{\varphi}'(G_{m,K}^{an})$ is homeomorphic to $G_{m,K}^{an}$. Since $G_{m,K}^{an}$ is locally compact and $\bar{X}'$ is Hausdorff (as $X'$ is Hausdorff), the injection $\bar{\varphi}'$ is a local homeomorphism, and hence $\bar{\varphi}' : G_{m,K}^{an} \to \bar{\varphi}'(G_{m,K}^{an})$ is a bijective, local homeomorphism, which is a homeomorphism.

Note that $\pi_{X'} : \bar{\varphi}'(G_{m,K}^{an}) \to \varphi'(G_{m,K}^{an})$ is a covering map which fits into the following commutative diagram:

$$\begin{array}{ccc}
G_{m,K}^{an} & \xrightarrow{\bar{\varphi}'} & \bar{X}' \\
\downarrow \pi_{X'} & \downarrow & \downarrow \pi_{X'} \\
\varphi'(G_{m,K}^{an}) & \xrightarrow{\varphi'} & E'^{an}
\end{array}$$

where the top arrow is a homeomorphism. Therefore, we have that $\varphi' : G_{m,K}^{an} \to \varphi'(G_{m,K}^{an})$ is the composition of a homeomorphism with a covering map, and hence is a covering map.

To conclude, we have that $\varphi'(G_{m,K}^{an})$ is path-connected because the continuous image of a path-connected space is path-connected, and since $\varphi'$ is a local homeomorphism, we have that $\varphi'(G_{m,K}^{an})$ is locally path-connected. □
Lemma 3.3. The morphism $\alpha^\prime_{an}|_{\varphi'(G^an_{m,K})} : \varphi'(G^an_{m,K}) \to E^{'an}$ is a covering map. Moreover, $\varphi'(G^an_{m,K})$ has the structure of a $K$-analytic group.

Proof. By Lemma 3.2 and Proposition 2.15, we have that $\alpha^\prime_{an}|_{\varphi'(G^an_{m,K})}$ is a semi-covering. Since $E^{'an}$ is locally simply connected [Ber99, Corollary 9.5] and path-connected, Proposition 2.18 gives us the desired result.

Proposition 3.4. The image of $\varphi'(G^an_{m,K})$ is isomorphic, as a $K$-analytic group, to the analytification of an elliptic curve $E''$, which is isogenous to $E'$.

Proof. By Lemma 3.2 and Lemma 3.3, we have the following commutative diagram of covering maps:

$$
\begin{array}{ccc}
\mathcal{G}^an_{m,K} & \xrightarrow{\varphi'} & E^{'an} \\
\downarrow \varphi' & \nearrow \pi_{E'} \\
\varphi'(G^an_{m,K}) & \xrightarrow{\alpha^\prime_{an}|_{\varphi'(G^an_{m,K})}} & E^{'an}.
\end{array}
$$

Since topological fundamental group of $E^{'an}$ is isomorphic to $\mathbb{Z}$, the above diagram and the theory of covering maps tells us that the deck transformation group $\text{Aut}(\varphi')$ is isomorphic to $m\mathbb{Z}$ where $m$ is some positive integer and that $\text{Aut}(\alpha^\prime_{an}|_{\varphi'(G^an_{m,K})})$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$.

We have three cases to consider.

1. If $m = 0$, then $\varphi'$ is a homeomorphism and so the covering map $\alpha^\prime_{an}|_{\varphi'(G^an_{m,K})} : \varphi'(G^an_{m,K}) \to E^{'an}$ is the universal covering map. However, this cannot happen as $\alpha^\prime_{an}$ is algebraic and the universal covering map of a Tate curve is analytic.

2. If $m$ is a non-zero, square, then $\alpha^\prime_{an}|_{\varphi'(G^an_{m,K})} : \varphi'(G^an_{m,K}) \to E^{'an}$ is a finite topological (hence finite étale) cover of $E^{'an}$. By Proposition 2.18 and Berkovich analytic GAGA [Ber90, Corollary 3.4.13], we have that $\varphi'(G^an_{m,K})$ is the analytification of an algebraic group $E''$ which admits a finite étale cover to $E'$ of degree $m$. Since $m$ is a square, we know that such a cover exists, and moreover, $\varphi'(G^an_{m,K})$ is isomorphic to $E''^an$ as $K$-analytic spaces.

3. Finally, consider the case where $m$ is not a square. The above argument again implies that $\varphi'(G^an_{m,K})$ is the analytification of an algebraic group $E''$ which admits a finite étale cover to $E'$ of degree $m$, but since $m$ is not a square, no such cover cannot exist. Indeed, this follows because the algebraic fundamental group of $E'$ is isomorphic to $\hat{\mathbb{Z}}^2$.

To conclude, we have shown that the only possibility is that the above $m$ is a non-zero, square, and hence $\varphi'(G^an_{m,K})$ is the analytification of an algebraic group $E''$, where $E''$ is isogenous to $E'$.

Proof of Theorem A. By Remark 2.12, it suffices to prove that there does not exist a Zariski dense, non-Archimedean entire curve in $X^an$. By Lemma 3.1, we may assume that $E/K$ has multiplicative reduction over $O_K$.

Suppose that $\varphi : G^an_{m,K} \to X^an$ is a Zariski dense, non-Archimedean entire curve in $X^an$. In the setting of multiplicative reduction, we have shown above that exists a finite étale cover $X' \to X$ such that $\varphi$ lifts to a morphism $\varphi' : G^an_{m,K} \to X^{'an}$. We claim that the morphism $\varphi' : G^an_{m,K} \to X^{'an}$ is Zariski dense. Suppose to the contrary, and let $Z$ denote the open $K$-analytic space corresponding to the complement of the Zariski closure of $\varphi'(G^an_{m,K})$ in $X^{'an}$. Since $X^{'an} \to X^an$ is étale, the image of $Z$ in $X^an$ is open and not contained in the image of $\varphi$, which contradicts that $\varphi$ is Zariski dense.

Since $X' \to X$ is finite étale, Lemma 2.5 tells us that $X'$ is pseudo-groupless, and Proposition 3.4 says that the image of $\varphi'$ is the analytification of an algebraic group. As $X'$ is pseudo-groupless
and $\varphi'$ factors through the analytification of an algebraic group, we have that $\varphi'$ cannot Zariski dense, and so we have reached a contradiction to our assumption on the existence of a Zariski dense, non-Archimedean entire curve in $X^{an}$. Therefore, we can conclude that any entire non-Archimedean curve in $X^{an}$ is algebraically degenerate, as desired. \hfill \Box

4. Proof of Theorem B

In this section, we will prove that for $S/K$ a projective surface admitting a dominant morphism to an elliptic curve, $S$ being groupless is equivalent to $S$ being $K$-analytically Brody hyperbolic.

**Proposition 4.1.** Let $S/K$ be a groupless, projective surface. If an analytic morphism $\varphi: G^{an}_{m,K} \to S^{an}$ is not Zariski dense, then $\varphi$ is constant.

**Proof.** Suppose that $\varphi$ is non-constant. Since $\varphi: G^{an}_{m,K} \to S^{an}$ is not Zariski dense, the Zariski closure $\overline{\varphi(G^{an}_{m,K})}$ of the image of $\varphi$ is a connected, closed, 1-dimensional $K$-analytic subvariety of $S^{an}$, and by Berkovich analytic GAGA [Ber90, Corollary 3.4.13], we have that $\overline{\varphi(G^{an}_{m,K})}$ is the analytification of a connected, projective curve (i.e., $\overline{\varphi(G^{an}_{m,K})}$ is isomorphic to $Z^{an}$ for a connected, projective curve $Z \subset S$). Since $S$ is groupless, we have that $Z$ is groupless, and hence Theorem 2.1 states that the map from $G^{an}_{m,K} \to Z^{an}$ must be constant, which contradicts our initial assumption. Therefore, we can conclude that $\varphi$ must be constant. \hfill \Box

We are now in a position to prove Theorem B.

**Proof of Theorem B.** We first note that if $S$ is $K$-analytically Brody hyperbolic, then it is clearly groupless, so it suffices to prove the reverse direction. Suppose that $S$ is groupless. By Definition 2.6, we need to show that every analytic morphism $\varphi: G^{an}_{m,K} \to S^{an}$ is constant. We note that $S$ is groupless modulo $\emptyset$, and hence Theorem A implies that $\varphi$ cannot be Zariski dense. The result now follows as Proposition 4.1 says that if $\varphi$ is not Zariski dense, then it must be constant. \hfill \Box

**Remark 4.2.** In a previous version of this manuscript, we claimed that for $S/K$ a projective surface admitting a dominant morphism to an elliptic curve, $S$ being pseudo-groupless is equivalent to $S$ being pseudo-$K$-analytically Brody hyperbolic. However due to recent improvements to the notion of pseudo-$K$-analytically Brody hyperbolic, we are currently unable to prove this claim.

The notion of pseudo-$K$-analytically Brody hyperbolic was first defined in [Mor21, Definition 2.3] and was equivalent to being pseudo-groupless and having every non-constant analytic map from $G^{an}_{m,K}$ to the analytification of the variety factoring through the analytification of a closed subset. In recent work [MR21], the author and Giovanni Rosso offer a new, more natural definition of pseudo-$K$-analytically Brody hyperbolic (loc. cit. Definition 4.1) and prove (loc. cit. Theorem 4.4) that one may test this notion on analytic maps from $G^{an}_{m,K}$ and from big analytic opens of analytifications of abelian varieties. While this new definition is stronger, it does seem to be the “correct” way to define pseudo-$K$-analytically Brody hyperbolic (c.f. loc. cit. Theorem A). The obstacle in proving the above result for this new definition is the analysis of analytic maps from big analytic opens of analytifications of abelian varieties.

5. A non-pseudo-groupless surface dominating an elliptic curve with Zariski dense, non-Archimedean entire curve

In this final section, we describe an example, communicated to us by Marco Maculan, of a non-pseudo-groupless surface that dominates an elliptic curve and admits a Zariski dense, non-Archimedean entire curve. This example illustrates that the assumption of pseudo-groupless cannot be removed from Theorem A.

Let $K = \mathbb{C}_p$. Let $E/K$ be an elliptic curve with multiplicative reduction over $\mathcal{O}_K$. Let $\mathcal{L}$ be a line bundle over $E$. Let $U = V(\mathcal{L})$ be the total space of $\mathcal{L}$, and let $X = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_E)$ be the projective bundle associated to $\mathcal{L} \oplus \mathcal{O}_E$. The variety $X$ is a ruled surface and $U$ is an open subset in $X$. 


Lemma 5.1. A ruled surface $S/K$ is not pseudo-groupless over $K$.

Proof. By the Enriques–Kodaira classification, $S$ is birational to the trivial $\mathbb{P}^1$-bundle over some curve $C$; in particular, it is birational to $\mathbb{P}^1 \times C$. Since Kodaira dimension is a birational invariant and additive with respect to products, we have that the Kodaira dimension of $S$ is negative, and so $S$ is not of general type. The result now follows from the contrapositive of [JX20, Lemma 3.23]. □

Let $\pi: \mathcal{G}^{an}_{m,K} \to \mathcal{E}^{an}$ be the uniformization map, so the kernel of $\pi$ is of the form $qZ$ for some $q \in K$ such that $0 < |q| < 1$. Let $\varepsilon: Z \to \overline{Q}$ be a bijection. Since $qZ$ is discrete and $\mathcal{G}^{an}_{m,K}$ is Stein (see [MP18]), there is a $K$-analytic function $f: \mathcal{G}^{an}_{m,K} \to \mathcal{A}^{1,an}_K$ such that $f(q^n) = \varepsilon(n)$ for all $n \in \mathbb{Z}$. The topological universal cover of $\mathcal{U}^{an}$ is isomorphic to $\mathcal{G}^{an}_{m,K} \times \mathcal{A}^{1,an}_K$ because $\pi^*\mathcal{L}$ is the trivial line bundle [BL91, Lemma 2.2]. Let $\tilde{\pi}: \mathcal{G}^{an}_{m,K} \times \mathcal{A}^{1,an}_K \to \mathcal{U}^{an}$ be the uniformization map.

Consider the composite map

$$\varphi: \mathcal{G}^{an}_{m,K} \xrightarrow{(id,f)} \mathcal{G}^{an}_{m,K} \times \mathcal{A}^{1,an}_K \xrightarrow{\tilde{\pi}} \mathcal{U}^{an} \xrightarrow{\pi} \mathcal{X}^{an}$$

where the last arrow is the open immersion of $\mathcal{U}^{an}$ into $\mathcal{X}^{an}$.

Proposition 5.2. The image of $\varphi$ is Zariski dense in $\mathcal{X}^{an}$.

Proof. Since $\mathcal{X}^{an}$ is projective, the Zariski closure of the image of $\varphi$ is the analytification of a closed subscheme $Z$ of $\mathcal{X}$. Furthermore, the Zariski closure of the closed analytic subspace defined by the kernel of the homomorphism $\mathcal{O}^{an}_X \to \mathcal{O}^{an}_{\mathcal{G}^{an}_{m,K}}$, and therefore the induced map $\mathcal{O}^{an}_Z \to \mathcal{O}^{an}_{\mathcal{G}^{an}_{m,K}}$ is injective. This shows that $Z$ is irreducible because $\mathcal{G}^{an}_{m,K}$ is.

If we can prove that $Z$ has dimension 2, then we are done. Since $\varphi$ is non-constant, the dimension of $Z$ is positive. By definition of $f$, $Z$ contains the fiber at 0 of the projection $g: \mathcal{X} \to \mathcal{E}$ because $\overline{Q}$ is dense in $\mathbb{P}^{1,an}_K$. The image of $g|_Z: Z \to \mathcal{E}$ is either one point or all of $\mathcal{E}$. If $Z$ had dimension 1, then the image would only be the point 0 as $Z$ contains $g^{-1}(0)$. However, this is not possible because $g \circ \varphi: \mathcal{G}^{an}_{m,K} \to \mathcal{E}^{an}$ is the uniformization map. Indeed, the map $(id,f): \mathcal{G}^{an}_{m,K} \to \mathcal{G}^{an}_{m,K} \times \mathcal{A}^{1,an}_K$ when composed with the first projection is by definition the identity. The first projection $\mathcal{G}^{an}_{m,K} \times \mathcal{A}^{1,an}_K \to \mathcal{G}^{an}_{m,K}$ is moreover equivariant with respect to the action of the topological fundamental group of $\mathcal{E}^{an}$ (cf. [BL91, Lemma 2.3(a)]), and it descends to the natural projection $\mathcal{G}^{an}_{m,K} \times \mathcal{A}^{1,an}_K \to \mathcal{E}^{an}$. Therefore the composite map $\mathcal{G}^{an}_{m,K} \to \mathcal{G}^{an}_{m,K} \times \mathcal{A}^{1,an}_K \to \mathcal{U}^{an} \xrightarrow{\pi} \mathcal{X}^{an}$ is the uniformizing map. □

Combining Lemma 5.1 and Proposition 5.2, we see that the ruled surface $\mathcal{X} = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_E)$ is a non-pseudo-groupless, projective surface that dominates an elliptic curve and admits a Zariski dense morphism from a non-Archimedean entire curve, as desired.

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EEBEC H3T 1J4, CAN

Email address: jmorrow4692@gmail.com

URL: https://sites.google.com/site/jacksontsalvatoremorrow/