The influence of thermal fluctuations on uniform and nonuniform superconducting rings according to the Ginzburg–Landau and the Kramer–Watts-Tobin models

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Abstract

We evaluate the influence of thermal fluctuations on superconducting rings that enclose a magnetic flux, using the time-dependent Ginzburg–Landau model (TDGL) or the Kramer–Watts-Tobin model (KWT), while thermal fluctuations are accounted for by means of Langevin terms. This method is applicable in situations where previous methods are not, such as for nonuniform loops, rings with large width to radius ratio and loops with large coherence length to perimeter ratio. We evaluate persistent currents, the position and statistical behavior of flux-induced vortices, and the lifetime of metastable fluxoid states. The influence of nonuniformity on the persistent current does not depend strongly on the details of the cross section profile; it depends mainly on its first harmonic, but not only on it. As a consequence of nonuniformity the maximum of the persistent current shifts to smaller fluxes and the passage between fluxoid states remains non-hysteretic down to lower temperatures than in the case of a uniform sample. Our results obtained using TDGL agree remarkably well with recent measurements of the persistent current in superconducting rings and with measurements of the position of a vortex that mediates between fluxoid states in an asymmetric disc with a hole; they could also provide a plausible explanation for the unexpectedly short measured lifetimes of metastable states. Comparison of TDGL and KWT indicates that they lead to the same results for the persistent current, whereas KWT leads to larger lifetimes than TDGL.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Superconducting rings have attracted the interest of physicists for many decades, because they are an easily accessible system in which measurable quantities depend on the enclosed flux, rather than on local fields only. More precisely, these quantities depend on the magnetic flux modulo \( \Phi_0 = \hbar c / 2e \), as expected from quantum behavior. One of the milestones in this endeavor was the Little–Parks experiment [1], in which the transition temperature is an oscillatory function of the applied flux.

A closer look at the Little–Parks results reveals that, as expected from a one-dimensional system, there is actually no phase transition. The transition region is dominated by thermal fluctuations and what experimentalists call the ‘transition temperature’ is in most cases the temperature at which the resistance becomes some given fraction of the normal state resistance. Since the Little–Parks effect is most pronounced for small samples, fluctuations are especially important. Superconducting rings constitute a compelling system for studying fluctuations: theoretically, this is a nontrivial system which is not invariant under time reversal, and has a rich yet simple phase diagram, and also both Gaussian and non-Gaussian fluctuations may be involved; experimentally, the influence of fluctuations is quite directly controllable and detectable.
A theory for the evaluation of the fluctuation contribution to the conductivity above the critical temperature is due to Aslamazov and Larkin [2]. An exact method for evaluation of the average current around a uniform loop, as a function of the temperature and enclosed flux, was developed by von Oppen and Riedel (vOR) [3]. Although the vOR method is exact from the point of view of statistical mechanics (provided that the energy spectrum is described by the Ginzburg–Landau model), it is limited to a static situation and to rings that are perfectly uniform and one-dimensional; moreover, for small values of the parameter $\gamma$ (which will be defined in section 2.1) numeric implementation of this method becomes exponentially difficult. An independent theory which discusses the influence of the shape of the loop and uses a two-level approximation is difficult. An independent theory which discusses the influence of the shape of the loop and uses a two-level approximation is difficult. An independent theory which discusses the influence of the shape of the loop and uses a two-level approximation is difficult. An independent theory which discusses the influence of the shape of the loop and uses a two-level approximation is difficult. An independent theory which discusses the influence of the shape of the loop and uses a two-level approximation is difficult. An independent theory which discusses the influence of the shape of the loop and uses a two-level approximation is difficult. An independent theory which discusses the influence of the shape of the loop and uses a two-level approximation is difficult. An independent theory which discusses the influence of the shape of the loop and uses a two-level approximation is difficult. An independent theory which discusses the influence of the shape of the loop and uses a two-level approximation is difficult. An independent theory which discusses the influence of the shape of the loop and uses a two-level approximation is difficult. An independent theory which discusses the influence of the shape of the loop and uses a two-level approximation is difficult. An independent theory which discusses the influence of the shape of the loop and uses a two-level approximation is difficult.

The vOR theory was tested by Zhang and Price [6]. They measured the susceptibility and found values larger than predicted. Recently, Koshnick et al [7] repeated this experiment for several samples with widths of the order of 100 nm, using a scanning superconducting quantum interference device (SQUID), and found good agreement with the theory. A similar experiment with extremely thin and dirty samples, extending the fluctuation influence to lower temperatures, was performed by Bert et al [8]. Rosario et al [9] measured conductivity in ultrathin cylinders at half-integer number of fluxoids, and found exponential dependence on the temperature above the transition, in contrast with the power dependence predicted by the Aslamazov–Larkin approach.

In this paper we study the influence of the width and nonuniformity of the loop on the average current and on the transitions between fluxoid states. We focus on the Little–Parks temperature range, i.e. the range of temperatures (depending on the flux) for which the superconducting transition would occur in the absence of fluctuations. For higher temperatures, currents are small, and experimental and numeric values are noisy; for lower temperatures, there is hysteresis and the statistical average becomes irrelevant. Somewhat lower temperatures will also be considered, because fluctuations and nonuniformity lower the temperature at which hysteresis appears and, sporadically, we will enter the hysteretic range. Most of the article is devoted to evaluation of the current around the rings studied in [7] and comparison with their experimental values. For most samples, our results agree with the vOR theory (and therefore with the theoretical analysis in [7]); when they do not, we attribute the difference to the width and/or nonuniformity of the samples.

The most widespread theoretical tool in the study of dynamic properties of superconductors (see e.g. [2, 10, 11]) is the time-dependent Ginzburg–Landau model (TDGL) [12]. A recent review [13] criticizes the use of TDGL for gapped superconductors; nevertheless TDGL remains popular because of its simplicity and its ability to reproduce observed phenomena. Kramer and Watts-Tobin [14] generalized TDGL to a form that should be applicable to gapped superconductors as long as there is local equilibrium, while still retaining some of the simplifying features of the TDGL formalism.

The objectives of the present study are (i) to predict the influence of imperfections (i.e., nonuniformity and finite width) on fluctuation superconductivity in rings, (ii) to compare our predictions with experimental results and thus check the range of applicability of TDGL, at least as a phenomenological model, and (iii) to find how features of the fluctuation superconductivity (i.e., average current and lifetime of metastable states) are affected when TDGL is generalized to the Kramer–Watts-Tobin model (KWT).

This paper is organized as follows. In section 2 we describe our computational method, in section 3 we present general results that stem from our method and in section 4 we use our method to explain measured values. Besides making calculations that aim at fitting the currents measured in [7], we address experiments that measure the position of a vortex [15, 16] and lifetimes of metastable states [6]. In each of these sections we distinguish between thin rings, for which a one-dimensional treatment should suffice, and wide rings, for which corrections are necessary. In section 5 we extend our analysis to the Kramer–Watts-Tobin model and the central features of our results are briefly summarized in section 6.

Several technical details are made explicit in the appendix.

## 2. TDGL with thermal fluctuations

The Ginzburg–Landau model describes the behavior of a superconductor by means of an order parameter field $\psi$ and an electromagnetic potential $A$. For a thin wire and close to the onset of superconductivity, the magnetic field in the wire can be approximated by the applied field. In this case and in cgs Gaussian units the model postulates the energy density $\alpha \vert \psi \vert^2 + \beta \vert \psi \vert^4/2 + (h^2/2m)\vert(\nabla \cdot - 2\pi A/\Phi_0)\psi \vert^2 + A \cdot j/c$, with $\alpha = 12\hbar k_B (T - T_c)/\pi m \epsilon_F k_B$ the Boltzmann constant, $T$ the temperature, $T_c$ the critical temperature, $m$ the mass of a Cooper pair, $\ell$ the mean free path, $\psi_f$ the Fermi velocity, $\beta = 8\pi (\epsilon_F/mc^2)$, $\kappa = 1.5 \times 10^{-6}/\ell$, $-e$ the electron charge and $j$ the total current density. The integral of the energy density over the volume of the sample will be denoted by $G$.

All the experiments [6, 7, 15] whose results we will compare with ours were performed on mesoscopic aluminum samples; we will therefore adopt $\psi_f = 1.5 \times 10^6$ cm s$^{-1}$, as appropriate for aluminum with a BCS coherence length $1.6 \mu$m; this value of $\psi_f$ is smaller than what would be obtained within the free-electron model. We will assume that the samples are in the dirty limit.

The characteristic temperature scale in our analysis will be $T_{LP} = \pi^2 \hbar \nu_f \ell/24k_B L^2$, where $L$ is the perimeter of the ring. $T_{LP}$ is the depression of the transition temperature that would be obtained in the absence of fluctuations for a uniform profile and half-integer magnetic flux (in units of $\Phi_0$). $T_{LP}$ equals the Thouless energy multiplied by $\pi/8k_B$. For the conductivity of normal electrons we have taken the aluminum value $\sigma = 1.22 \times 10^{11} \ell \Omega^{-1}$ cm$^{-2}$, but we have also checked other values.

All the rings that we will consider will be thin and narrow, on the scale of the coherence length $\xi = \hbar/\sqrt{2m|\alpha|}$ and the
magnetic penetration depth $\kappa \xi$. However, not all the rings will have a width which can be neglected when compared to its average radius; these rings will require a separate treatment.

2.1. One-dimensional loops

A numerical method for the implementation of TDGL with thermal fluctuations was presented in [17] for the case of thin wires and loops that do not enclose a magnetic field. In this section we review this reference and point out the required modification when magnetic flux is enclosed.

For computational purposes the loop is divided into $N$ segments of length $L/N$ and the fields and the energy are discretized. A can be taken tangential to the wire and we denote by $\psi_k$ and $A_k$ the values of $A$ and $A'$ in the $k$th segment. We also define $\hat{A}_k = 2\pi L A_k / N \Phi_0$. According to TDGL without fluctuations, both $\psi_k$ and $A_k$ obey evolution equations with the ‘canonical’ form $d\psi_k / dt = -\Gamma_\phi \partial G / \partial \psi_k$; invoking the fluctuation-dissipation theorem leads to the increments of $\psi_k$ and $A_k$ during a macroscopically short period of time $\tau$:

$$\Delta \text{Re}[\psi_k] = -\Gamma_\phi \partial \text{Re}[\psi] / \partial \psi_k \tau + \eta_k^{(\text{Re})},$$

with an analogous expression for $\Delta \text{Im}[\psi_k]$, and

$$\Delta A_k = -\Gamma_\phi \partial A / \partial A_k \tau + \eta_k^{(A)},$$

where $\Gamma_\psi = N m v_F e / 3 h^2 L w_k$ and $\Gamma_{A_k} = 4 e^2 L / N h^2 \sigma w_k$, with $w_k$ the cross section of segment $k$, and $\eta_k^{(\text{Re})}$ (resp. $\eta_k^{(A)}$) is a Langevin term with Gaussian distribution, zero average, and variance $2 \Gamma_{\psi,k} \tau$ (resp. $2 \Gamma_{A,k} \tau$).

The electromagnetic potential can be gauged out of the expression for the energy, and therefore from equation (1), by means of the transformation $\tilde{\psi}(s) = \mathcal{U}(s) \psi(s)$ with $\mathcal{U}(s) = \exp[(2\pi i / \Phi_0) \int_0^s A(s') ds']$. The price of this transformation is that the gauge-invariant order parameter $\tilde{\psi}$ does not obey a ‘canonical’ evolution equation.

Reference [17] adopts several normalizations that are appropriate for the fluctuation region $T \approx T_c$. The typical size of the order parameter is $\psi = (2 m k_B T_c^2 / \beta h^2 \sigma)^{1/3}$, where $\bar{w}$ is the average cross section, the typical coherence length is $\bar{\xi}_k = (\bar{w} \Phi_0^2 / 32 \pi^3 k_B T_c)^{1/3}$ and the time unit is $\bar{\tau} = 3 \xi_k^2 / \hbar v_F$. Under free-electron gas assumptions, the parameter $\gamma$ of [7] and [8] can be expressed as $\gamma = 0.5 (L / \xi_0)^3$.

Using the normalized quantities $\tilde{L} = L (2 m k_B T_c)^{1/2} / \hbar$, $\bar{R} = 6 e^2 L / \sigma m \hbar v_F$, $\bar{w}_k = \psi / \psi / \bar{w}$, $\psi_k = \psi / \psi / \bar{w}$ and $\alpha' = \alpha / \beta h^2$, equation (2) becomes

$$\Delta A_k = -(C_A / w_k) \{ \bar{h} / e k_B T_c \} \eta_k^{(A)} + \eta_k^{(A)},$$

where $\Delta_{\text{mac}} \dot{\psi}_k = -(\tau / \bar{\tau}) \left[ (\alpha' / |\psi_k|^2) \dot{\psi}_k + (N \bar{\xi}_k / \bar{L})^2 (2 \bar{w}_k)^{-1} \right]$

$$\times \left[ (w_k + w_{k+1}) (\bar{\psi}_k - \psi_k) + (w_k + w_{k-1}) (\bar{\psi}_k - \bar{\psi}_k) \right],$$

and $\psi_{k+1} = \psi_{k-1}$ for $k = 1, N$. Here $w_k$ is the width of the $k$th segment, $\bar{w}_k$ is the average radius; these rings will require a separate treatment.

In this study we assume that nonuniformity enters the problem through the cross section of the ring; we expect that chemical nonuniformity would have a similar effect.

Since the induced flux is negligible, if the applied flux enclosed by a ring remains constant, then $\sum A_k$ is also constant. This condition determines the current $I$; from equation (3) it leads to

$$I = e k_B T / h \sum \eta_k^{(A)} / \tau - (C_A \bar{\xi}_k / \bar{L}) \sum w_k^{-1} (\delta_k + \delta_{k+1}),$$

where the sums are over the $N$ segments of the ring. The first term in expression (5) is the Johnson noise and vanishes on average; the second term is the supercurrent.

One of the questions addressed in [17] is that of the relative importance of $\eta^{(A)}$ and $\eta^{(A)}$. In typical numerical studies (e.g. [18]), $\eta^{(A)}$ is usually ignored. According to an early study [19], fluctuations of the electromagnetic field are most important, and determine the nature of the normal–superconducting transition; on the other hand, in the case of an isotropic 3D superconductor in [20], it was found that electromagnetic fluctuations become relevant only for extremely large coherence lengths. The results of [17] show that, in the regime in which the coherence length and the perimeter of a ring are of the same order of magnitude, ignoring $\eta^{(A)}$ leads to functions $\bar{\psi}(s)$ that are significantly flatter than predicted by statistical mechanics. Unlike in the 3D case, where the mechanism by which $\eta^{(A)}$ affects $\bar{\psi}(s)$ involves the induced magnetic fields [21], in the 1D case the influence of $\eta^{(A)}$ on $\bar{\psi}(s)$ arises directly from electroneutrality.

The numerical implementation of our method and averaging procedure is outlined in Appendix A.1. The only modification that is required in order to apply the procedure of [17] is in the boundary condition for $\psi$, which in the case where flux $\Phi$ is enclosed becomes $\bar{\psi}(L) = \exp(2 \pi i \Phi / \Phi_0) \bar{\psi}(0)$.

2.2. Wide rings

Although theory is ‘cleaner’ for 1D loops, in real life this idealization might not be justified. Moreover, wide rings carry larger currents and may provide us with reliable measurements in fluctuation regions of the $\Phi - T$ plane, where thin rings have a low signal to noise ratio. It will therefore prove useful to develop a simple model for rings that are not ideally 1D.

We consider superconducting rings with a width that is very small compared with the lengths over which the superconducting variables change significantly (coherence length, effective magnetic penetration depth), but is not sufficiently small compared with the radius. As a consequence,
the fluxes enclosed by the inner and by the outer boundaries will not be the same. Nevertheless, we will build a quasi-1D model in which the superconducting variables will be represented by quantities that do not depend on the radial coordinate.

Let us denote by \( R \) the average radius and the width by \( D(\theta) \); the thickness is \( w(\theta)/D(\theta) \). The ring occupies the region \( R - D/2 < r < R + D/2 \) and is immersed in a uniform perpendicular magnetic field \( B \). At this stage we neglect the induced magnetic field and \( B \) will be taken as constant.

With the purpose of averaging over \( r \), we separate the electromagnetic vector potential into a contribution of the applied magnetic field and a contribution \( a \) due to thermal fluctuations, i.e. we write \( A = (Br + a)\hat{\theta} \). In order to have a 1D model we assume that \( \psi \) and the voltage are functions of \( \theta \) and time only, which implies

\[
a = a_R \frac{R}{r}, \quad \nabla \psi = \psi_R \frac{R}{r} \hat{\theta},
\]

where \( a_R \) is the value of \( a \) at \( r = R \) and \( \psi_R \) is the derivative of \( \psi \) with respect to the arc length \( s = R\theta \), taken along the circle \( r = R \). We will also approximate the current density by the form \( j = j_R(r/R)\hat{\theta} \), with \( j_R \) independent of \( r \) and proportional to \( 1/w(\theta) \).

We now integrate the energy density over \( R - D/2 \leq r \leq R + D/2 \) and neglect terms of order higher than \((D/R)^2\). In this approximation the volume averages are given by equation (8). Hence, the effective energy becomes

\[
G = \int w(\theta)\{(2\psi^2 + \beta|\psi|^2)/2 + \left(\frac{h^2}{2m}\right)\left((2\pi/\Phi_0)^2|\psi|^2(B^2(4R^2 + D^2))/16 + a_RBR + a_R^2f_D) + |\psi|^2(f_D + (2\pi i/\Phi_0)(BR/2 + a_Rf_D)(\psi^*\psi_R - \psi\psi_R^*)) + (BR/2 + f_Da_R)\hat{\theta}/c\}
\]

where \( f_D = 1 + D^2/12R^2 + O((D/R)^4) \).

We can now introduce a new variable, \( \hat{\Lambda} = (2\psi/\Phi_0)(BR/2f_D + a_R) \), and rewrite \( G \) in the form

\[
G = \int w(\theta)\{\left(\frac{eBD^2}{(6mc^2)}\right)|\psi|^2 + \beta|\psi|^2/2 + f_D\hbar^2/2m\}(|\psi|^2 + \hat{\Lambda}^2)|\psi|^2 + i\hat{\Lambda}(\psi\psi_R^* - \psi^*\psi_R)
\]

where \( \hat{\Lambda} = f_D\Phi_0\hat{A}/(2\pi c) \).

Discrete variables can now be defined as in [17]: \( \psi_k \) as the average of \( \psi \) in cell \( k \) and \( \hat{\Lambda}_k \) as the integral of \( \hat{\Lambda} \) along the length \( L/N \) of this cell. It can be checked that these variables behave as ‘canonical’ in the sense of [17].

Integrating over \( r \) the TDGL expression for \( d\psi/dt \), under natural averaging assumptions we recover equation (1), with \( G \) given by equation (8). However, using \( (\Phi_0/2\pi)\int d\Lambda/\partial t = (\partial \Lambda/\partial t) = -c(jn/\sigma) \), where \( jn \) is the normal current density, we obtain that the coefficient \( \Gamma_{AA} \) is smaller by a factor \( f_D \) than the expression obtained for a 1D ring. The same simplifications obtained in [17] for 1D loops can be achieved here by defining the gauge-invariant order parameter \( \psi(s) = U(s)\psi(s) \) with \( U(s) = \exp(i\int_0^s \hat{A}(s) \, ds) \).

By detailed comparison we conclude that a ring with non-negligible width can be treated by means of equations (3) and (4) and the fluctuations described under them, provided that we include the following corrections: (i) \( \alpha \) has to be replaced with \( \alpha + (eBD^2)/(6mc^2) \), which means adding (1/3)(\( D\partial\psi/\partial \phi_0R^2 \)) to \( \alpha' \); (ii) the term proportional to \( \beta^2 \) in equation (4) has to be multiplied by \( f_D \) and (iii) the variance of \( \Delta\Lambda_k \) has to be divided by a factor \( f_D \).

As long as we neglect the self-inductance, the integral of the electric field (and hence of \( da_R/dt \)) around the ring has to vanish; in this case the instantaneous current is given by equation (5). If there is no-negligible influence of the ring self-inductance, which we denote by \( L_e \), the total flux through the ring will not be the applied flux \( \Phi_i \), but rather \( \Phi = \Phi_i + cL_e I \). This has two consequences. The first is that the average \( A \cdot ds \) varies with \( I \) and equation (5) generalizes to

\[
I = \left\{ \frac{2\psi L_i}{h\tau}I_{prev} + \sum \eta_k^{(A)}/\tau - (C\Lambda\beta/L) \right\} 
\]

\[
\times \left( \sum u_k^{-1}(\delta_k + \delta_{k+1}) \right) 
\]

\[
\times \left\{ \left( \frac{2\psi L_i}{h\tau} + (h\Lambda/ck_T) \sum u_k^{-1} \right) \right\}^{-1},
\]

where \( I_{prev}(t) = I(t - \tau) \). For the range of parameters that we considered, this refinement turned out to have no noticeable effect. The second consequence is that when comparing our results with experiments, we should present the results as functions of \( \Phi_e = \Phi - cL_e I \), since usually this is the controlled quantity. For the parameters we considered, \( |\Phi_e - \Phi| \) is small but noticeable.

3. Representative results

3.1. One-dimensional loops

Our formalism enables us to study samples with cross section given by any periodic function \( w(\theta) \) of the polar coordinate \( \theta \). Therefore, our first question is that of what nonuniformity profiles it would be interesting to study. Since any periodic function can be written as a Fourier series \( w(\theta) = \psi(1 + \sum \beta_j \cos j\theta + \sum \gamma_j \sin j\theta) \), another way to pose the question is to ask what Fourier coefficients \( \beta_j \) and \( \gamma_j \) it would be interesting to consider.

We will restrict ourselves to rings with mirror symmetry \( w(-\theta) = w(\theta) \), so that \( \gamma_j = 0 \), and we will have to decide on the choice of the \( \beta_j \)'s. We may be guided by our results for rings at \( T \approx T_e \) and small deviations from uniformity, in the absence of fluctuations [16, 22, 23]. In those studies, which were corroborated by other groups [24, 25], it was found that nonuniformity can qualitatively modify the Little–Parks phase diagram and there is a range of temperatures for which there is a continuous passage between consecutive fluxoid states. At the lower end of this range, there is a critical point \( P_2 \) such that below \( P_2 \) the passage between fluxoid states is hysteretic and such that at \( P_2 \) the derivative of the current with respect to the flux diverges. The important aspect of our previous results for the question at hand is that the salient features of the phase diagram, such as the position of \( P_2 \), depend only on \( \beta_1 \), whereas
higher harmonics have no influence at the leading order. It is therefore natural to consider a shape in which only the first harmonic is present, i.e. \( u(\theta) = \hat{u}(1 + \beta_1 \cos \theta) \). This profile will be called 'sinusoidal'.

In order to gain some understanding of how other profiles behave, we also consider cross sections with discontinuous functions \( w(\theta) \), since in this case the coefficients \( \beta_j \) have the slowest decrease with \( j \), providing a situation complementary to the sinusoidal profile one. The simplest discontinuous profile is that of a ring with a constriction. This situation might also be of experimental interest, since the constriction might represent a Josephson junction or a defect. Finally, in order to have a case opposite to the sinusoidal profile one, we consider the case of rings with an additional mirror symmetry \( w(\pi - \theta) = w(\theta) \), since in this case \( \beta_1 = 0 \). A profile like this will be called 'symmetric'.

Figure 1 shows the current \( I \) as a function of the magnetic flux \( \Phi \) for fixed temperatures. The choice of geometric and material parameters was inspired by those of the samples for which data are available [7]. The upper panel is for \( T = T_c - 1.64 T_{LP} \) and all the curves represent sinusoidal cross section profiles. We see that nonuniformity has two qualitative effects: one of them is an overall decrease of the current and the second is a shift of the maximum current to lower fluxes, so the decay of \( I \) as \( \Phi \rightarrow \Phi_0/2 \) becomes more gradual. In contrast to the well known behavior in the absence of thermal fluctuations, even for uniform cross sections and \( T < T_c - T_{LP} \), the average current vanishes at \( \Phi = \Phi_0/2 \) and the passage between consecutive winding numbers is smooth.

The lower panel in figure 1 is for \( T = T_c - 2.67 T_{LP} \). For a uniform sample at this temperature, the average current does not go to zero for \( \Phi \rightarrow \Phi_0/2 \). According to a blind application of statistical mechanics, this average should vanish, since the energies for clockwise or counter-clockwise currents are equal; however, there is an energy barrier between the two fluxoid states, so in real experiments or simulations only one of them is probed. For all the nonuniform samples shown in the graph, the current does go to zero as \( \Phi \rightarrow \Phi_0/2 \), meaning that either there is no energy barrier, as is the case for temperature above that of \( P_2 \), or the barrier is not large in comparison to thermal fluctuations.

The profiles with a constriction in figure 1 had a segment of length \( 0.8L \) with large cross section and a constriction of length \( 0.2L \). We see that the \( I(\Phi) \) curves for these profiles practically coincide with those of appropriate sinusoidal profiles. (Only one line was drawn in these cases for both sets of symbols.) From here we may adopt the working assumption that for any reasonable profile there will be an equivalent sinusoidal profile, so that by studying sinusoidal profiles we may expect to obtain most of the interesting information. The values of \( \beta_1 \) of the samples with a constriction are smaller than those of sinusoidal samples with similar \( I(\Phi) \) curves, indicating that the \( I(\Phi) \) curve is influenced most significantly by \( \beta_1 \), but not only by it. As an extreme example, we considered a symmetric sample, for which \( \beta_1 = 0 \). This sample consisted of two wide segments, each of length 0.2L, connected by constrictions of length 0.3L. In spite of a very large ratio between the cross sections, which manifests itself in a strong inhibition of the average current, the position of the maximum remains quite close to \( \Phi = \Phi_0/2 \). As a general observation we might say that the influence of nonuniformity on \( I(\Phi) \) is stronger for lower temperatures and close to \( \Phi_0/2 \).

Figure 2 shows the ratio of the current at a given fixed flux to the current at flux \( \Phi = 0.1 \Phi_0 \), as a function of temperature. Again, we see that the influence of nonuniformity increases as \( \Phi \) approaches \( \Phi_0/2 \) and, for the parameters used in this graph, is noticeable for \( T \lesssim T_c - 1.1 T_{LP} \).

3.2. Wide rings

We consider first the case in which the cross section is uniform along the ring. We denote by \( I_D \) the current around the ring for non-negligible width, and by \( I_0 \) the current in the case \( D \rightarrow 0 \), \( w/D \rightarrow \infty \) (the cross sections are the same in the two cases). In all the cases we considered, we found \( 0 < I_D/I_0 < 1 \), i.e., the finite width inhibits the current, but does not wipe it out. The inset in figure 3 compares \( I_D \) and \( I_0 \) for a typical situation.

In order to make more quantitative statements, we classify our results into those outside and within the fluctuation regions. In the absence of fluctuations, our quasi-1D model leads to vanishing order parameter when

\[
(1/3)(D\Phi/R\Phi_0)^2 + f_D(\Phi/\Phi_0 - n)^2 = (T_c - T)/4T_{LP}, \quad (10)
\]
by straight lines. For visibility, the curves for symbol meanings are the same as in figure 1. The circles are for a sample with a constriction of length 0.2L and cross section equal to 1/2.14 of that of the rest of the sample; they are shown for $I(0.4\Phi_0)/I(0.1\Phi_0)$ only. The calculated symbols have been joined by straight lines. For visibility, the curves for $I(0.4\Phi_0)/I(0.1\Phi_0)$ (red online) have been raised by 1.5 units and those for $I(0.4\Phi_0)/I(0.1\Phi_0)$ (black) have been raised by 3 units.

$I_D - I_0$ is proportional to $D^2$ and its value is independent of $T_c = T$.

Figure 4 shows the current within fluctuation regions in the vicinity of the region border, $\Phi_{n+}$, for two fixed temperatures. For visibility, the values for $T = T_c - 0.33T_L$ have been shifted 0.02 units to the right. Symbols: $\Theta, \Phi_{n+} = \Phi_{0+}, D/R = 1/3; \times, \Phi_{n+} = \Phi_{0+}, D/R = 2/3; +, \Phi_{n+} = \Phi_{1+}, D/R = 1/3; \ast, \Phi_{n+} = \Phi_{1+}, D/R = 1/5; D, D = 0$, arbitrary $\Phi_{n+}$. The other parameters are as for figure 3.

Let us now consider nonuniform cross sections. We have to deal separately with the case in which nonuniformity is due to the thickness, so that $D$ is constant, and the case in which nonuniformity is due to the width, so that $D$ is a function of $\theta$. For the parameters we considered, there was no significant difference; the results we report in the following are for uniform $D$. Figure 5 compares the shapes of the current–flux curve for a uniform and a nonuniform sample. As in the case of 1D loops, nonuniformity brings about a more gradual decay at $\Phi \sim 0.5\Phi_0$.

We would finally like to know whether the influence of $\beta_1$ is enhanced or weakened by the width of the ring. Denoting by $I_D(\beta_1, \phi)$ the current for a ring of width $D$ with deviation from uniformity $\beta_1$ at flux $\phi\Phi_0$, we may regard the ratio $I_D(\beta_1, 0.45)/I_D(\beta_1, 0.1)$ as a representative index for the shape of the function $I_D(\beta_1, \phi)$, and the difference $I_D(\beta_0, 0.7, 0.45)/I_D(\beta_0, 0.7, 0.1) - I_D(\beta_0, 0.45)/I_D(\beta_0, 0.1)$ as a representative index of the influence of nonuniformity on the shape of $I_D(\beta_1, \phi)$. We see from the inset in figure 5 that for wider samples the influence of nonuniformity is smaller.

In our calculations we used several values for the conductivity, with a ratio $\sim 3 \times 10^5$ between the largest and the smallest. The value of $\sigma$ had no appreciable influence on the average current. This can be understood, since the average current is an equilibrium quantity and $\sigma$ only affects the dynamics.
4. Comparison with experiments

4.1. One-dimensional loops

We have used our method to analyze the 15 samples studied in [7]. Samples for which $D < R/4$ were treated as ‘one-dimensional’ ones. We modeled each sample by a ring with sinusoidal cross section profile. For most of the samples, the best fit was obtained when a uniform cross section was assumed. However, there were a few samples for which a better fit was obtained for $\beta_1 \neq 0$, and two of them are presented in figure 6, each for a temperature above or close to $T_c - T_{LP}$ and another temperature well below $T_c - T_{LP}$. The temperature of the critical point $P_2$ is $\sim T_c - (1 + 2\beta_1)T_{LP}$ [22, 26], so we may expect nonuniformity to extend the range over which there is no hysteresis at $\Phi = \Phi_0/2$ by an amount of $\sim 2\beta_1 T_{LP}$. In figure 6, the lines marked with a temperature are the experimental representative curves for $I(\Phi)$, the points are calculated values using our method, and the lowest line is an interpolation for values calculated assuming a uniform cross section. Sample 13 is the one with largest deviation from uniformity and, indeed, scanning electron microscopy inspection revealed that this ring has imperfections. In general our calculated points are in good agreement with the experimental lines. For the lower temperatures shown in figure 6, the theoretical curves are more rounded than the experimental curves; this is probably due to our model assumption that the cross section profile is sinusoidal.

4.2. Wide rings

4.2.1. Current–flux curves. In this section we deal with the samples of [7] that have $D > R/4$. As discussed in section 2.2,

1 The entire set of currents as functions of sample, temperature and flux was sent to us by the authors; the properties of each sample are tabulated in the supporting online material for [7].

the contribution of the induced magnetic field is not quite negligible. Since the field generated by self-induction is not uniform, and since additional magnetic induction mechanisms may be present besides self-induction, we initially attempted to regard $L_s$ as an adjustable parameter. The results that we obtained were of the order of the tabulated values [27], but systematically smaller; in some cases we obtained $L_s < 0$. In view of this behavior, and in order to reduce the number of parameters, we decided to neglect self-induction altogether. This omission may influence the effective value of $\beta_1$.

Figure 7 shows our results for sample 5. This sample has a width/radius ratio $D/R = 0.39$. Note that for this sample the vOR method could not be applied [7]. We chose the value of $\ell$ from the data for $T = T_c - 0.17 T_{LP}$, the value of $\beta_1$ was adjusted to reproduce the experimental ratio $I(0.507\Phi_0)/I(0.1\Phi_0)$ at $T = T_c - 1.56 T_{LP}$ and the mutual inductance was adjusted at $T = T_c - 1.17 T_{LP}$. We also used the data at $T = T_c - 1.56 T_{LP}$ to refine the calibration of the applied flux. Our calculated results in the hysteresis region (inset) are in surprising agreement with the experiment. However, the calculated decays of metastable states are much sharper than the observed decays; the lack of sharpness in the observed decay is probably due to the fact that the experiment was not perfectly static: the range of flux involved in the passage between fluxoid states was swept during a lapse of time of the order of $10^{-4}$ s, which is comparable with the filtering time.
The larger currents are for $T = T_c - 1.17T_{cP}$ and the smaller currents (blue online) for $T = T_c - 0.67T_{cP}$. The parameters taken from [7] are $T = 2.2 \mu m$, $\bar{w} = 7.83 \times 10^{-11} \text{cm}$, $D = 1.35 \times 10^{-5} \text{cm}$, $T_c = 1.24 K$; the parameters fitted here are $\ell = 30.2 \text{nm}$, $\beta = 0.07$ and the mutual inductance $M_{L,R} = 0.035 \mu \Phi_0 \text{nA}^{-1}$. Inset: hysteresis region; the curves (actually, a dense set of points, red online) are experimental data for $T = T_c - 1.56T_{cP}$ and the black lines are calculated.

**Figure 8.** Current as a function of the magnetic flux for another wide sample. The larger currents are for $T = T_c - 1.42T_{cP}$ and the smaller currents (blue online) for $T = T_c - 0.42T_{cP}$. For visibility, the currents in the case $T = T_c - 0.42T_{cP}$ have been multiplied by a factor of 3. The parameters taken from [7] are $L = 3.14 \mu m$, $\bar{w} = 1.01 \times 10^{-10} \text{cm}^2$, $D = 1.75 \times 10^{-5} \text{cm}$, $T_c = 1.244 K$; the parameters fitted here are $\ell = 30.9 \text{nm}$, $\beta = 0$ and the mutual inductance $M_{L,R} = 0.082 \mu \Phi_0 \text{nA}^{-1}$.

Our fitted values of $\ell$ vary from sample to sample. The largest value equals almost twice the smallest value, but they are all (including those for thin and wide samples) within 15% of the empirical expression $1/\ell = 1.5 \times 10^5 \text{cm}^{-1} + 2/D + 14.5/L$. Our values of $\ell$ are larger than those of [7]; the difference is mainly due to the value of $v_F$ adopted.

The fluctuation region around $\Phi = 0.5\Phi_0$ for sample 12 is shown enlarged in figure 9. In the absence of fluctuations, the current would vanish for $0.32 \leq \Phi/\Phi_0 \leq 0.71$. The lines that show the current that would be present without fluctuations were obtained with the same code as all the other results, but the Langevin terms were divided by a factor of 100. Note that all the adjustable parameters of our model were fixed outside this region.

Our fitted values of $\ell$ vary from sample to sample. The largest value equals almost twice the smallest value, but they are all (including those for thin and wide samples) within 15% of the empirical expression $1/\ell = 1.5 \times 10^5 \text{cm}^{-1} + 2/D + 14.5/L$. Our values of $\ell$ are larger than those of [7]; the difference is mainly due to the value of $v_F$ adopted.

**4.2.2. Flux-induced vortices.** A recent experiment on a doubly connected asymmetric disc [15, 16] found that, for the sample examined, the order parameter practically vanishes in the narrow part of the sample when passing between the fluxoid state 0 and the fluxoid state 1; when passing between 2 and 3 the order parameter practically vanishes in the wide part, and no local vanishing is observed in the passage between 1 and 2.

We consider now a sample with geometric and material parameters similar to those reported in the experiment and check whether our theoretical method reproduces the experimental behavior. Our results are shown in table 1. We denote by $\psi_0$ the order parameter at angle $\theta$, where $\theta = 0$ is the angle at which the sample is widest; $\langle \cdots \rangle$ is the average over time; for each temperature and flux we use the normalization constant $C = 0.5((|\psi_0|^2) + (|\psi_0|^2))$; $\text{stdev}_{\psi}$ stands for $((|\psi_0|^2) - (|\psi_0|^2))^2)^{1/2}$. We do indeed see that for $\Phi = 0.5\Phi_0$ and for a temperature within the appropriate experimental range, $|\psi|^2$ is particularly small for $\theta = \pi$, whereas for $\Phi = 2.5\Phi_0$ the small value is obtained for $\theta = 0$. In the table we show the first two moments only; higher moments indicate that in these two situations $|\psi|^2$ has exponential distribution.
Appropriate temperatures were taken from [15].

\[ D(\theta) = 168(1 + 0.625 \cos \theta) \text{ nm}, \quad \dot{w} = 5.04 \times 10^{-11} \text{ cm}^2, \]
\[ T_c = 1.36 \text{ K and } \ell = 15.6 \text{ nm.} \]
\[ C = 0.5((\psi_{\pi/2})^2 + (\psi_{-\pi/2})^2)). \]

\[ \Phi/\Phi_0 \quad T (\text{K}) \quad \langle |\psi_0|^2 \rangle / C \quad \text{stdev}_0 / C \quad \langle |\psi_1|^2 \rangle / C \quad \text{stdev}_1 / C \]

| \( \Phi/\Phi_0 \) | 0.5 | 1.23 | 0.08 | 0.016 | 0.017 |
|-----|-----|-----|-----|-----|-----|
| 1.5 | 1.20 | 0.64 | 0.61 | 1.23 | 1.19 |
| 2.5 | 1.05 | 0.012 | 0.012 | 4.45 | 0.79 |

Moreover, for \( \Phi = 1.5\Phi_0 \), \( \langle |\psi|^2 \rangle \) is not small for any \( \theta \), in agreement with the experimental observation. Figure 10 shows that for this flux and temperature, \( \langle |\psi|^2 \rangle \) does not depend very strongly on \( \theta \). However, we note that \( \text{stdev}_0 \) (respectively \( \text{stdev}_1 \)) is almost as large as \( \langle |\psi_0|^2 \rangle \) (\( \langle |\psi_1|^2 \rangle \)), suggesting that there are frequent transitions in both directions between the fluxoid states 1 and 2. Since a continuous change in winding number must involve a place where the order parameter vanishes, we may anticipate that small averages of \( |\psi|^2 \) will be obtained if we average over transition steps only.

For this purpose we need a criterion for deciding what is meant by a ‘transition step’. We start by noting that in the fluxoid state 2 a positive current flows around the sample for \( \Phi \approx 1.5\Phi_0 \), whereas in the state 1 the current is negative, so in a transition between the two states the current should change sign. However, there may be many small steps in which the sign changes back and forth; we will not count each of these changes as a transition, but only the first one after the sample has been in a ‘typical’ state. Let us denote by \( I_+ \) (\( I_- \)) the average of the current over those steps in which it is positive (negative). We found that \( |I_-| \approx I_+ \). We regarded a step as a ‘transition up’ (‘transition down’) if it is the first one for which the current is positive (negative) after having been smaller than \( -I_+ / 8 \) (larger than \( I_+ / 8 \)). When a transition step was detected, we identified the cell in which \( |\psi|^2 \) has its minimum value as the place where a phase slip occurs.

**4.2.3. Lifetime of metastable states.** Zhang and Price [6] performed direct measurements of the lifetime of metastable states, as a function of the temperature and the flux. They prepared states with winding number 0 that enclosed flux larger than \( 0.5\Phi_0 \) and waited until a change in the magnetic susceptibility was detected. The elapsed time was measured and the same procedure was repeated several times, until a significant average was obtained.

There are many reasons not to expect our method to be able to reproduce these experimental values of the lifetime. First, TDGL is not expected to be valid at the temperatures
Table 2. Flux for which the lifetime of the metastable state is 10^{-4} s. The values of \( \phi_{-4} \) in the second column are experimental and those in the third and fourth columns were calculated using TDGL; the last two columns are for the Kramer–Watts-Tobin model (section 5).

| \( T (K) \) | \( \phi_{-4} \) (exp.) | \( \phi_{-4}^{TDGL} (\beta_1 = 0) \) | \( \phi_{-4}^{TDGL} (\beta_1 = 0.25) \) | \( \phi_{-4}^{KWT} (\beta_1 = 0) \) | \( \phi_{-4}^{KWT} (\beta_1 = 0.25) \) |
|----------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 1.192    | 0.63              | 0.72              | 0.62              | 0.79              | 0.62              |
| 1.137    | 0.79              | 0.88              | 0.73              | 0.88              | 0.73              |

considered; second, the width/radius ratio for their sample was 0.54, so the quasi-1D description seems inappropriate; third, the direct measurements could be performed for lifetimes larger than 1 s, whereas the lifetimes that we could practically study were smaller than 10^{-3} s; fourth, since the fluctuations above \( T_c \) in [6] were much larger than predicted by theory, we may suspect that some non-thermal perturbation was present in the experiment and that this perturbation (rather than thermal activation) could induce the decays. In spite of all these difficulties, having an estimate of the influence of nonuniformity on the lifetime of a metastable state is at least of academic interest.

Let us denote by \( \phi_{-4} \) the value of \( \Phi_\psi/\Phi_0 \) for which the lifetime of the state with winding number 0 is 10^{-4} s. The values of \( \phi_{-4} \), for temperatures at which measurements were performed and are not too far from \( T_c \), are shown in table 2. The experimental value of \( \phi_{-4} \) was determined by extrapolating the exponential dependence found in [6]. For comparison, \( \phi_{-4} \) was also calculated with our method, using the physical parameters reported as the best fit in [6] and two cross section profiles; in one case the cross section was taken as uniform and in the other case we assumed uniform width, whereas the thickness had sinusoidal dependence, with \( \beta_1 = 0.25 \), which is not far from the value 0.15 used for the same sample in [22] in order to mimic the shape of the phase diagram. The initial values of the order parameter were those that minimize the free energy for winding number 0; evolution followed, governed by TDGL with fluctuations, and we decided that the initial state had decayed when a positive current was reached; after a decay, the process was repeated, during a total of 3 \times 10^3 time units.

Although in section 4.2.1 we ignored self-inductance, in the present case the flux sensitivity is very large and we therefore decided to take self-inductance into account. Self-inductance was taken into account also in [6], but the influence that they found had the opposite sign to what we find here: in our (resp. their) case, self-inductance increases (resp. reduces) metastability. In our case the main effect of self-inductance is due to the difference between \( \Phi_\psi \) and \( \Phi_\psi \); in their case the self-inductance is used in a model for estimating the height of the potential barrier.

Calculated lifetimes divided by their averages have typically Poissonic distributions.

While bearing in mind the reservations raised at the beginning of this section, table 2 suggests that nonuniformity could be a reason for the low lifetimes found in [6].

5. Beyond TDGL

5.1. The Kramer–Watts-Tobin model

Kramer and Watts-Tobin [14] extended TDGL so as to render it applicable to gapped superconductors and valid as long as there is local equilibrium. KWT has been successfully used to describe the current–voltage characteristic of thin wires [29–31]. The energy functional is the same as in the Ginzburg–Landau model; in a gauge such that the electrochemical potential is uniform, equation (1) generalizes to

\[
\frac{1}{\sqrt{1 + K|\psi|^2}} \left[ \frac{d}{dt} + \frac{K}{2} \frac{d|\psi|^2}{dt} \right] \text{Re}[\psi] = -\Gamma_{\phi,1} \frac{\partial G}{\partial \text{Re}[\psi]} \tag{11}
\]

and similarly for \( \text{Im}[\psi] \). Here \( K = 247k^2e^2k_BT_c \ell v_F^2/\hbar^2c^2 \), where \( \tau_{ph} \) is the electron–phonon inelastic scattering time. In the limit \( K \to 0 \), TDGL is recovered. Aluminum is challenging since it has a particularly large \( \tau_{ph} \), i.e., \( \tau_{ph} = 10^{-8} \) s. As a consequence, the expected correction to TDGL should be large. Also, the range of validity of KWT should be small, \( T_c - T \lesssim \hbar/k_BT_{ph} \approx 1 \) mK.

Equation (11) can be brought into canonical form by writing \( \psi_k = |\psi_k| \exp(i\chi_k) \). We obtain

\[
d|\psi_k|/dt = -h_{|\psi|}\langle\psi_k|\Gamma_{\psi,k}\partial G/\partial|\psi_k|, \tag{12}
\]

\[
d\chi_k/\partial t = -h_{\phi}\langle\psi_k|\Gamma_{\phi,k}\partial G/\partial \chi_k, \tag{13}
\]

with \( h_{|\psi|}(\psi_k) = (1 + K|\psi_k|^2)^{-1/2} \) and \( h_{\phi}(\psi_k) = 1/(h_{|\psi|}|\psi_k|^2) \).

5.2. Appropriate Langevin terms

Following the reasoning of [17], for a period of time \( \tau \) over which \( |\psi_k| \) does not change significantly, the fluctuating parts of the changes of \( |\psi_k| \) and of \( \chi_k \) would be expected to have variances \( \langle \eta_{|\psi|}^2 \rangle = 2|\psi_k|^2\Gamma_{\psi,k}k_BT \) and \( \langle \eta_{\chi}^2 \rangle = 2\Gamma_{\phi,k}k_BT \tau/h_{|\psi|} \langle |\psi_k|^2 \rangle \). However, we have pointed out elsewhere [28] that in the case of equation (12) this is not the full story. Due to the dependence on \( |\psi_k| \) of \( h_{|\psi|} \) and of the Jacobian \( W = \partial \text{Re}[\psi_k]/\partial |\psi_k|/\partial \text{Im}[\psi_k]/\partial \chi_k \), fluctuations of \( |\psi_k| \) do not have zero average but rather \( \langle \eta_{|\psi|} \rangle = \partial \ln(h_{|\psi|}W)/\partial |\psi_k|h_{|\psi|}\Gamma_{\psi,k}k_BT\tau \). Using the expressions for \( h_{|\psi|} \) and \( W \) we obtain

\[
|\psi_k|(t + \tau) - \langle |\psi_k| \rangle(t) = h_{|\psi|}\Gamma_{\psi,k}\left[ \frac{h_{|\psi|}k_BT}{|\psi_k|^2} \frac{\partial G}{\partial |\psi_k|} \right] \tau + \tilde{\eta}_{|\psi|}, \tag{14}
\]

where \( \tilde{\eta}_{|\psi|} \) has Gaussian distribution, \( \langle \tilde{\eta}_{|\psi|} \rangle = 0 \) and \( \langle \tilde{\eta}_{|\psi|}^2 \rangle = 2h_{|\psi|}^2\Gamma_{\psi,k}k_BT\tau \).

5.3. Results and comparison with experiments

We have limited our study to two experiments. The first is the case of persistent current in a wide ring in the fluctuation region, shown in figure 9. The oblique crosses are for \( \tau_{ph} = 10^{-8} \) s and the upright crosses for \( \tau_{ph} = 10^{-8} \) s; the other parameters are as for TDGL. We see that even for large
values of $\tau_{ph}$ the currents obtained coincide within statistical uncertainty with those obtained with TDGL.

The second experiment that we studied was that investigating the lifetime of metastable states, considered in section 4.2.3. Figure 12 shows our results for the flux at which the lifetime is $10^{-5}$ s. The results are reasonably well fitted by expressions of the form $\phi_{-5}(\tau_{ph}) = \phi_{-5}(0) + a[(b^2 + \tau_{ph}^2)^{1/4} - \sqrt{b}]$, where $a$ and $b$ are adjustable constants. Using a fit of this form for $\phi_{-4}$ and extrapolating to $\tau_{ph} = 10^{-8}$ s leads to the results shown in table 2 (the increment of $\phi_{-4}$ due to KWT for $\beta_1 = 0.25$ is less than 0.01).

6. Discussion

We have evaluated the influence of thermal fluctuations on persistent currents (i.e., average current), the position of the flux-induced vortices, and the lifetime of metastable fluxoid states for aluminum rings that enclose magnetic flux, in a range down to $\sim 10^2$ mK below the critical temperature, using TDGL with the addition of Langevin terms. In the cases that we considered to be most interesting, the evaluation was also performed using KWT. The rings considered were not ‘ideal’, in the sense that they were either wide, and so the ‘enclosed’ flux was not sharply defined, or nonuniform, and so expansion of the order parameter into a Fourier series does not simplify the problem. In all cases we obtained at least qualitative agreement between TDGL and the experimental results, and in many cases we obtained quantitative agreement, in spite of the fact that we considered temperatures far beyond the range where TDGL is justified by microscopic theory.

Nonuniformity leads to smaller persistent currents (in comparison with a uniform ring with the same average parameters), to smaller slopes of the current as a function of the flux when passing between fluxoid states, and to a larger temperature range for which the passage between fluxoid states is non-hysteretic. It also leads to a narrower range of metastability. Finite width leads to smaller persistent currents (in comparison with 1D rings) and to lesser sensitivity to nonuniformity.

For several experiments in which the rings were intended to be uniform, better agreement with theory can be obtained by assuming that the rings actually had some unintentional nonuniformity; for those rings that were intentionally nonuniform, good agreement with theory was found using the reported values for the shape of the ring.

Several years ago it was predicted that the passage between fluxoid states may be mediated by a flux-induced vortex [32] and this prediction has been experimentally confirmed [15, 16]. Our present study confirms the existence and position of this vortex. Moreover, it describes a situation in which the intermediate state between two fluxoid states is actually dynamic, with sporadic migrations between both states and involving a place where the order parameter vanishes.

The KWT was applied to two situations, mainly in order to estimate the expected discrepancy incurred when using TDGL. The first case was that of the persistent current in a fluctuation range, i.e., a range for which the current would vanish in the absence of thermal fluctuations. Remarkably, even for the large value of $\tau_{ph}$ in the case of aluminum, the currents obtained for KWT agree with those obtained for TDGL, and they both agree with the measured values. This agreement may be understood if we bear in mind that the persistent current is an equilibrium quantity and therefore depends only on the energy spectrum and not on the dynamics; since the energy functional for KWT is the same as for TDGL, they lead to the same currents. In more general terms, we can say that even if some features of a model, such as TDGL, are not an accurate description of reality, the model can accurately describe situations in which these features are irrelevant.

The second case in which we used KWT was the study of the lifetime of metastable states. We find that the ring can remain in the metastable state for longer times than predicted by TDGL. Since, according to equations (12) and (13), KWT brings about a slower evolution of the absolute value of the order parameter and a faster evolution of its phase, we can conclude that the evolution rate of the absolute value is dominant in determining metastability. This can be understood, since escape from the fluxoid state implies change in the winding number of the order parameter, which cannot occur as long as the absolute value of the order parameter is positive everywhere; on the other hand, when the order parameter does vanish at some point, the fast evolution of the phase could lead to multiple transitions, as discussed in [33].

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Appendix. Numerical procedures

A.1. Evolution of the order parameter

We start from an arbitrary set of values for the order parameter; the initial values of \( \hat{A} \) are not required, since only the increments of \( \hat{A} \) enter our equations. \( \tilde{\psi}_{\pm k}(t + \tau) \) is then obtained through explicit Euler–Mayurama steps, i.e., by adding to \( \tilde{\psi}_{\pm k}(t) \) the term \( \Delta_{\text{max}}\tilde{\psi}_{\pm k} \) according to (4), a Langevin term with variance \( N \tilde{w} \tilde{E}_{B}/\tilde{r} w_{k} L \), and then modifying its phase according to (3). The initial steps are intended for relaxation to some typical state, and subsequent steps are used for statistical averaging.

The Langevin terms are obtained by multiplying pseudorandom normally distributed numbers by the required standard deviation. These numbers were obtained by two methods, leading to the same results. One method was the Box–Muller algorithm. In the second method, we had an initial list \( R_{0}, R_{1}, R_{2}, \ldots, R_{1000} \) of pseudorandom normally distributed numbers, with average exactly set to 0 and variance exactly set to 1. Then our pseudorandom numbers were evaluated by using the expression \( (R_{i} + R_{j} + R_{k} + R_{l})/4 \), with \( i, j, k, l \) random integers in the range \( 1 \leq i, j, k, l \leq 1000. \)

Averages of interest, such as \( \langle I \rangle \), were obtained as averages over steps. Additional technical details were described in [17].

A.2. Fits to experimental data for \( I(\Phi) \)

For a given sample and temperature, our first task was the generation of a representative curve \( I(\Phi) \) for the experimental data. In the case of 1D rings, we subtracted the linear background as the first step. In some cases, we assumed that for a 1D sample the maximum value of \( |I| \) in the range \( 0 < |\Phi| < 0.5\Phi_{0} \) has to be the same as for \( 0.5\Phi_{0} < |\Phi| < \Phi_{0} \). We then shifted the origin of the applied field to enforce \( \int_{-\Phi_{0}/2}^{\Phi_{0}/2} I(\Phi) \cos(2\pi \Phi/\Phi_{0}) d\Phi = 0 \), and finally fitted the data in the range \(-0.5\Phi_{0} \leq \Phi \leq 0.5\Phi_{0} \) to a high order odd polynomial that vanishes at \( \Phi = \pm 0.5\Phi_{0} \). In the case of wide rings the overall slope of the data was not subtracted, the shift of the origin to render \( I(\Phi) \) an odd function was fixed ‘by eye’, and different polynomials were used to fit the data in different ranges.

The second task was the choice of parameters \( T_{c}, \ell, \beta_{1} \) and the mutual inductance between the sample and the scanning SQUID. \( T_{c} \) was taken from [7]. \( \ell \) was chosen by assuming a uniform ring, considering the highest temperature for which measurements are reasonably reproducible and periodic, and then requiring that our calculated values reproduce the experimental smoothed ratio \( I(\Phi_{1})/I(\Phi_{2}) \) where, typically, \( \Phi_{1} \approx 0.1\Phi_{0} \) and \( \Phi_{2} \approx 0.35\Phi_{0}. \) In order to fix \( \beta_{1} \), we again require agreement with the experimental ratio \( I(\Phi_{1})/I(\Phi_{2}) \), but this time we consider the lowest temperature for which there is no hysteresis and \( \Phi_{2} \approx 0.47\Phi_{0}. \) After \( \beta_{1} \) was fixed, we re-evaluated \( \ell \), using the value of \( \beta_{1} \) obtained rather than \( \beta_{1} = 0 \); we found that this second iteration did not modify \( \ell \) significantly. Finally, the mutual inductance was fixed by fitting the calculated points to the experimental \( I(\Phi) \) for the lowest temperature and in the range \( 0 < \Phi \lesssim 0.4\Phi_{0}. \) The mutual inductances that we found are typically \((10 \pm 5)\% \) smaller than those reported in [7]. This small discrepancy can be attributed to the experimental uncertainty in the sample–SQUID distance.

A.3. Numerical aspects of KWT

In the case of TDGL, we may regard the evolution of the order parameter as isotropic in \( \psi \)-space. On the other hand, equations (12) and (13) show that according to KWT, variations of \( \psi_{\pm k} \) in the ‘radial’ direction are inhibited by a factor \( h_{0} \), whereas in the ‘angular’ direction they are enhanced by a factor \( 1/h_{0}. \) For situations in which \( K |\psi_{\pm k}|^{2} > 1, \) this factor can be important, and this forces us to divide steps in the angular direction into several steps, i.e., for every step lasting time \( \tau \) in the radial direction there are \( n_{\sigma \text{ steps}} \) steps in the angular direction, each lasting time \( \tau/n_{\sigma} \) \( (n_{\sigma} \sim h_{0}^{-2}). \)

In cases in which we evaluate an equilibrium property, it might be unnecessary to perform all the steps in the angular direction, since, due to the fast variation of \( \chi_{\sigma} \) while \( |\psi_{\pm k}| \) remains frozen, we may assume that after a relatively small number of steps \( n_{\text{prob}} \ll n_{\sigma} \), we have already obtained the equilibrium value of the property evaluated for the given value of \( |\psi_{\pm k}| \), and we may go on and probe new values of \( |\psi_{\pm k}|. \) In this case it is important to note that the statistical weight of this partial average is \( \tau \) and not \( n_{\text{prob}}/n_{\sigma}. \)

Equation (14) was implemented through the assignment \( \psi_{k}(t + \tau) = \psi_{k}(t) + \Delta|\bar{\psi}_{k}|/|\psi_{k}| \). Some caution is required due to the presence of a factor \( |\psi_{k}|^{2} \) in the denominator, which might by chance be very small. We replaced it with \( |\psi_{k}|^{2} + \tau \tilde{\psi}^{2}/\tilde{r} \), unless \( |\psi_{k}| \) is exceptionally small, this replacement gives rise to a negligible \( O(\tau) \) contribution, while it guarantees that \( \psi_{k} \) remains conveniently bounded in these exceptional cases.

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