ABSTRACT. We study the discrete spectrum of the two-particle Schrödinger operator $\hat{H}_{\mu\lambda}(K)$, $K \in \mathbb{T}^2$, associated to the Bose-Hubbard Hamiltonian $\hat{H}_{\mu\lambda}$ of a system of two identical bosons interacting on site and nearest-neighbor sites in the two dimensional lattice $\mathbb{Z}^2$ with interaction magnitudes $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, respectively. We completely describe the spectrum of $\hat{H}_{\mu\lambda}(0)$ and establish the optimal lower bound for the number of eigenvalues of $\hat{H}_{\mu\lambda}(K)$ outside its essential spectrum for all values of $K \in \mathbb{T}^2$. Namely, we partition the $(\mu, \lambda)$-plane such that in each connected component of the partition the number of bound states of $\hat{H}_{\mu\lambda}(K)$ below or above its essential spectrum cannot be less than the corresponding number of bound states of $\hat{H}_{\mu\lambda}(0)$ below or above its essential spectrum.

1. INTRODUCTION

In this paper we consider the family $\hat{H}_{\mu\lambda}(K)$ Schrödinger operators, associated to the Bose-Hubbard Hamiltonian of a system of two identical bosons on the two dimensional lattice $\mathbb{Z}^2$ with on-site interaction $\mu \in \mathbb{R}$ and nearest-neighbor interaction $\lambda \in \mathbb{R}$.

Lattice Bose-Hubbard models have become popular in recent years since they represent a minimal, natural Hamiltonian in ultracold atoms in optical lattices, systems with highly controllable parameters such as lattice geometry and dimensionality, particle masses, tunneling, two-body potentials, temperature etc. (see e.g., [2, 6, 7, 17] and references therein). Unlike the traditional condensed matter systems, where stable composite objects are usually formed by attractive forces and repulsive forces separate particles in free space, the controllability of collision properties of ultracold atoms has enabled to experimentally observe a stable repulsive bound pair of ultracold atoms in the optical lattice $\mathbb{Z}^3$, see e.g., [4, 20, 23, 24]. In all these observations Bose-Hubbard Hamiltonians became a link between the theoretical basis and experimental results.

The main difficulty in solving Bose-Hubbard Hamiltonian even with the on-site interaction is due to the tunneling, i.e., the kinetic energy necessary for a boson to hop from site to site, since it is highly nonlocal. Moreover, unlike its continuous counterpart, the lattice Hamiltonian, corresponding to short-range interacting systems of particle pairs, is not rotationally invariant, hence, the separation of lattice Hamiltonian related to the center of mass-motion is not possible. However, the translation-invariance of the Hamiltonian (in $d$-dimensional lattice $\mathbb{Z}^d$) allows to use the Floquet-Bloch decomposition: the underlying Hilbert space $\ell^2(\mathbb{Z}^d)^2$ and the total two-particle Hamiltonian $\hat{H}$ are represented as the von Neumann direct integral

$$\ell^2((\mathbb{Z}^d)^2) \simeq \int_{K \in \mathbb{T}^d} \oplus\ell^2(\mathbb{Z}^d) dK, \quad \hat{H} \simeq \int_{K \in \mathbb{T}^d} \oplus\hat{H}(K) dK,$$
associated to the representation of the discrete group $\mathbb{Z}^d$ by the shift operators, here $\mathbb{T}^d$ is the $d$-dimensional torus, a Pontryagin dual group to $\mathbb{Z}^d$. The fiber Hamiltonians $\hat{H}_\mu(K)$ nontrivially depend on the so-called quasi-momentum $K \in \mathbb{T}^d$ (see e.g., [1, 3, 18, 19]).

In what follows any nonzero solution $\tilde{\psi}_K$ to the Schrödinger equation

$$\hat{H}(K)\tilde{\psi}_K = e_K\tilde{\psi}_K, \quad \tilde{\psi}_K \in \ell^2(\mathbb{Z}^d),$$

will be called bound state with energy $e_K$.

In the current paper, we study the family

$$\hat{H}_{\mu}(K) := \hat{H}_0(K) + \hat{V}_\mu, \quad K \in \mathbb{T}^2,$$

discrete Schrödinger operators associated to the Hamiltonian $\hat{H}_{\mu}(\lambda)$ of a system of two identical bosons on the two dimensional lattice $\mathbb{Z}^2$ with zero-range on-site interaction $\mu \neq 0$ and nearest-neighbor interaction $\lambda \neq 0$ (see (2.5) in Subsection 2.3). To the best of our knowledge, this is a new, exactly solvable model, for which the exact number of eigenvalues and their locations as well as exact lower and upper bounds for all values of the pair interactions $\mu, \lambda \in \mathbb{R}$ can be found.

First we observe that the essential spectrum of $\hat{H}_{\mu}(K)$ consists of a segment $[\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)]$, where

$$\mathcal{E}_{\min}(K) := 2\sum_{i=1}^{2} \left(1 - \cos \frac{K_{2i}}{2}\right), \quad \mathcal{E}_{\max}(K) := 2\sum_{i=1}^{2} \left(1 + \cos \frac{K_{2i}}{2}\right)$$

(see Subsection 2.4).

Now we study the discrete spectrum of $\hat{H}_{\mu}(K)$. The eigenvalue problem for $\hat{H}_{\mu}(K)$ with general $K$ is not an easy problem; note that $\hat{H}_{\mu}(K)$ becomes a small perturbation of $\hat{V}_\mu$ if $K$ is close to $\vec{\pi} := (-\pi, \pi)$, and $\hat{V}_\mu$ is a small perturbation of $\hat{H}_0$ provided that $\mu, \lambda$ is small. However, it turns out that for any $\mu, \lambda \in \mathbb{R}$, if $e_n(K; \mu, \lambda)$ and $E_n(K; \mu, \lambda)$ are the $n$-th eigenvalues of $\hat{H}_{\mu}(K)$ below and above the essential spectrum, respectively, then the functions $\mathcal{E}_{\min}(K) - e_n(K; \mu, \lambda)$ and $E_n(K; \mu, \lambda) - \mathcal{E}_{\max}(K)$ as a function of $K$ will have a minimum at $K = 0$ (see Lemma 4.9). Subsequently, the number of eigenvalues $n_+ (\hat{H}_{\mu}(K))$ (resp. $n_-(\hat{H}_{\mu}(K))$) of $\hat{H}_{\mu}(K)$ above (resp. below) the essential spectrum is not less than that of $\hat{H}_{\mu}(0)$ (Theorem 3.1). This result is a generalization of [1, Theorems 1 and 2] which contain the assertion related only to the ground state of $\hat{H}_{\mu}(K)$.

To study the discrete spectrum of $\hat{H}_{\mu}(0)$ we introduce the Fredholm determinants $\Delta^{(s)}(z)$ and $\Delta^{(a)}(z)$, associated to the restriction of $\hat{H}_{\mu}(0)$ onto symmetric and antisymmetric functions in $\ell^2,\ell^2(\mathbb{Z} \times \mathbb{Z})$ w.r.t. coordinate permutations, respectively. It is well-known that eigenvalues of $H_{\mu}(0)$ are zeros of these determinants, and vice versa (see also Lemma 4.5). Moreover, the number of zeros of $\Delta^{(s)}(z)$ and $\Delta^{(a)}(z)$ can change if and only if their asymptotics as $z$ approaches the thresholds of the essential spectrum vanish.

Therefore, we partition $(\mu, \lambda)$-plane of interactions into connected components by means of hyperbolas $\lambda \mu + 4 \lambda + 2 \mu = 0$ and $\lambda \mu - 4 \lambda - 2 \mu = 0$ and straight lines $\lambda = \pm \frac{4}{\pi - \pi}$ (see Figure 1), where $\lambda \mu \pm 4 \lambda \pm 2 \mu$ and $\lambda = \frac{4}{\pi - \pi}$ appear as constant in front of the main term in the asymptotics of the Fredholm determinants $\Delta^{(s)}(z)$ and $\Delta^{(a)}(z)$, respectively, as $z$ converges to the edges of the essential spectrum (Proposition 4.4). As we noticed above, the number of

\footnotesize
\begin{enumerate}
\item Note that the terms 'symmetric' and 'antisymmetric' are not related to the particle symmetry, but rather to the invariant subspaces of $\hat{H}_{\mu}(0)$ related to the permutation of coordinates in $\mathbb{Z}^2 := \mathbb{Z} \times \mathbb{Z}$ of one-particle.
\end{enumerate}
eigenvalues of \( H_{\mu \lambda} (0) \) changes if and only if one of the 'constants' \( \lambda \mu \pm 4 \lambda \pm 2 \mu \) and \( \lambda \pm \frac{\pi}{4 - \pi} \) changes its sign (see also Lemmas 4.6-4.8).

Hence, while \( (\mu, \lambda) \) runs in \( \mathbb{R}^2 \) and does not cross any of these curves, no qualitative or quantitative changes occur in the discrete spectrum of \( \widehat{H}_{\mu \lambda} (0) \); however, as soon as \( (\mu, \lambda) \) crosses any of those hyperbolas resp. straight lines, the essential spectrum of \( \widehat{H}_{\mu \lambda} (0) \) either 'gives birth' or 'absorb' a bound state of \( \widehat{H}_{\mu \lambda} (0) \) which is symmetric resp. antisymmetric w.r.t. permutation of coordinates (Theorem 3.4).

This fact is tightly connected to coupling constant threshold phenomenon [10, 14, 15]: if \( H(t), t \geq 0 \), is a one-parameter family of Schrödinger operators (in continuum or lattice) and 0 is a lower edge of the essential spectrum, then \( t_0 \geq 0 \) is a coupling constant threshold if and only if there exists a negative eigenvalue \( E(t) \) of \( H(t) \) for \( t > t_0 \) such that \( E(t) \nearrow 0 \) as \( t \searrow t_0 \), i.e., as \( t \searrow t_0 \) an eigenvalue is absorbed at the threshold of continuum, and conversely, as \( t \nearrow t_0 + \varepsilon \) an eigenvalue is released from the continuum. Indeed, considering \( \widehat{H}(t) := \widehat{H}_{t \mu, t \lambda} \) for \( t \geq 0 \), from Figure 1 below we immediately obtain that the only coupling constant thresholds are \( t_0 = 0, t_0 = \left| \frac{4 \lambda + 2 \mu}{\lambda \mu} \right| \) (when \( \lambda \mu \neq 0 \)) and \( t_0 = \frac{\pi}{4 - \pi} \).

Surprisingly, the maximum number of isolated eigenvalues is achieved only in four connected components in which both \( \mu \) and \( \lambda \) run on infinite intervals.

In general, some results such as existence of an eigenvalue and also the finiteness of the number of eigenvalues can be obtained for wide classes of operators (see e.g., [8, 9, 10, 22]). However, Figure 1 shows that the study of a qualitative change in the number of eigenvalues of \( \widehat{H}_{\mu \lambda}(K) \), even for \( K = 0 \), is very delicate: there are discs in the \( (\mu, \lambda) \)-plane with arbitrarily small radius in which the number of eigenvalues has jump (see Theorem 3.4).

Recall that in [8] authors extend the results of [9, 22] to the lattice case. Namely, the spectral properties of one-particle discrete Schrödinger operator

\[
\tilde{H}_t = \tilde{H}_0 + t \tilde{v}, \quad t \geq 0,
\]

in \( \mathbb{Z}^1 \) and \( \mathbb{Z}^2 \) have been studied, here \( \tilde{H}_0 \) is a self-adjoint Laurent-Toeplitz-type operator generated by a dispersion relation \( \tilde{E} : \mathbb{Z}^d \to \mathbb{C} \) of the particle and the potential \( \tilde{v} \) is the multiplication operator by \( \tilde{v} : \mathbb{Z}^d \to \mathbb{R} \). Under certain regularity assumption on \( \tilde{E} \) and a decay assumption on \( \tilde{v} \), authors show that if \( \sum \tilde{v}(x) \) is nonnegative resp. nonpositive, then the discrete spectrum

\[ \text{FIGURE 1. A partition of the } (\mu, \lambda) \text{-plane using the hyperbolas } \lambda \mu \pm 4 \lambda \pm 2 \mu = 0 \text{ and straight lines } \lambda = \pm \frac{\pi}{4 - \pi}. \]
of \( \hat{H}_t \) above resp. below its essential spectrum is non-empty for any \( t > 0 \). Moreover, authors prove the existence of \( t_0 > 0 \) (depending only on \( \hat{v} \) and \( \hat{E} \)) such that

(a) if \( \sum \hat{v}(x) > 0 \), then for any \( t \in (0, t_0) \) the operator \( \hat{H}_t \) has no eigenvalues below the essential spectrum and has a unique eigenvalue above the essential spectrum;
(b) if \( \sum \hat{v}(x) < 0 \), then for any \( t \in (0, t_0) \) the operator \( \hat{H}_t \) has no eigenvalues above the essential spectrum and has a unique eigenvalue below the essential spectrum;
(c) if \( \sum \hat{v}(x) = 0 \), then for any \( t \in (0, t_0) \) the operator \( \hat{H}_t \) has unique eigenvalues in both below and above the essential spectrum.

Note that for our model \( \sum \hat{v}(x) = \mu + 2\lambda \). Hence, our results for the discrete spectrum of \( \hat{H}_{\mu\lambda}(0) \) show the (exact!) dependence of \( t_0 \) on \( \mu \) and \( \lambda \): namely, \( t_0 \) will the intersection of the half-line \( \{(t\mu, t\lambda) : t \geq 0\} \) with the branches of hyperbolas \( \lambda\mu \pm 4\lambda \pm 2\mu = 0 \) passing through the origin (see Figure 2).

Moreover, from Figure 2 we observe that the unique eigenvalue of [8] appears from the symmetric space and eigenvalues in the antisymmetric space come out later (see Theorem 4.2).

In general, it is well-known that in the case of \( d \geq 3 \) or in the case of fermions with \( d \geq 1 \) the bound states appear from the essential spectrum either as a threshold bound state or as a threshold resonance [11, 12]. However, our results show that in the case \( d = 2 \), even though antisymmetric bound states come out from the threshold eigenvalues, all symmetric bound states of \( \hat{H}_{\mu\lambda}(0) \) appear (only!) from the singularity of the associated Fredholm determinant at the thresholds, namely, associated Fredholm determinant cannot be analytically (even continuously) extended to the edges of the essential spectrum (see Proposition 4.4). Such a result holds also in \( d = 1 \) (see [16]). This is a strict mathematical explanation in the difference of the appearance of eigenvalues for \( d = 1, 2 \) and \( d \geq 3 \), or for bosons and fermions.

As far as we know for continuous two-body Schrödinger operators \( \mathbb{R}^2 \) there are no analogous examples and results. In the discrete case similar results for the number of eigenvalues of one-particle Schrödinger operators in \( \mathbb{Z}^d \) with zero-range on-site and nearest-neighbor interactions have been obtained, for instance, in [12] for \( d = 3 \) with attractive potential field, in [16] for \( d = 1 \), and in [5] for all \( d \geq 1 \) considering only negative eigenvalues.

The paper is organized as follows: In Section 2 we introduce the Hamiltonian \( \hat{H}_{\mu\lambda} \) of a system of two bosons in the position and momentum space and also the Schrödinger operator \( \hat{H}_{\mu\lambda}(K) \) associated to \( \hat{H}_{\mu\lambda} \). Main results of the paper are stated in Section 3 and their proofs are contained in Section 4. We conclude the paper with an appendix containing the proof of Proposition 4.4.

2. Discrete Schrödinger operators on lattices

2.1. The two-particle Hamiltonian: the position-space representation. Let \( \mathbb{Z}^2 \) be the two-dimensional lattice and let \( \ell^2(\mathbb{Z}^2 \times \mathbb{Z}^2) \) be the Hilbert space of square-summable symmetric functions on \( \mathbb{Z}^2 \times \mathbb{Z}^2 \).
In the position-space representation the two particle Hamiltonian $\hat{H}_{\mu\lambda}$, associated to a system of two bosons interacting via zero-range and nearest-neighbor potential $\hat{v}_{\mu\lambda}$ is a bounded self-adjoint operator acting in $L^{2,s}(\mathbb{Z}^2 \times \mathbb{Z}^2)$ as

$$\hat{H}_{\mu\lambda} = \hat{H}_0 + \hat{v}_{\mu\lambda}, \quad \mu, \lambda \in \mathbb{R}.$$  

Here the free Hamiltonian $\hat{H}_0$ of a system of two identical particles (bosons) is a bounded self–adjoint operator acting in $L^{2,s}(\mathbb{Z}^2 \times \mathbb{Z}^2)$ as

$$\hat{H}_0 \hat{f}(x, y) = \sum_{n \in \mathbb{Z}^2} \tilde{\varepsilon}(x - n) \hat{f}(n, y) + \sum_{n \in \mathbb{Z}^2} \tilde{\varepsilon}(y - n) \hat{f}(x, n),$$

where

$$\tilde{\varepsilon}(s) = \begin{cases} 2 & \text{if } |s| = 0, \\
-\frac{1}{2} & \text{if } |s| = 1, \\
0 & \text{if } |s| > 1, \end{cases} \quad (2.1)$$

and $|s| = |s_1| + |s_2|$ for $s = (s_1, s_2) \in \mathbb{Z}^2$.

The interaction $\hat{v}_{\mu\lambda}$ is the multiplication operator

$$\hat{v}_{\mu\lambda} \hat{f}(x, y) = \hat{v}_{\mu\lambda}(x - y) \hat{f}(x, y),$$

given by the function

$$\hat{v}_{\mu\lambda}(s) = \begin{cases} \mu & \text{if } |s| = 0, \\
\frac{\lambda}{2} & \text{if } |s| = 1, \\
0 & \text{if } |s| > 1. \end{cases} \quad (2.2)$$

2.2. The two-particle Hamiltonian: the momentum-space representation. Let $\mathbb{T}^2 = (\mathbb{R}/2\pi \mathbb{Z})^2 \equiv [-\pi, \pi)^2$ be the two dimensional torus, the Pontryagin dual group of $\mathbb{Z}^2$, equipped with the Haar measure $dp$, and let $L^{2,s}(\mathbb{T}^2 \times \mathbb{T}^2)$ be the Hilbert space of square-integrable symmetric functions on $\mathbb{T}^2 \times \mathbb{T}^2$. Let $\mathcal{F} : L^2(\mathbb{Z}^2) \rightarrow L^2(\mathbb{T}^2)$ be the standard Fourier transform

$$\mathcal{F} \hat{f}(p) = \frac{1}{2\pi} \sum_{x \in \mathbb{Z}^2} \hat{f}(x) e^{ip \cdot x},$$

where $p \cdot x := p_1 x_1 + p_2 x_2$ for $p = (p_1, p_2) \in \mathbb{T}^2$ and $x = (x_1, x_2) \in \mathbb{Z}^2$.

In the momentum-space via $\mathcal{F} \otimes \mathcal{F}$ the two-particle Hamiltonian is represented in $L^{2,s}(\mathbb{T}^2 \times \mathbb{T}^2)$ as

$$\mathbb{H}_{\mu\lambda} := (\mathcal{F} \otimes \mathcal{F}) \hat{H}_{\mu\lambda}(\mathcal{F} \otimes \mathcal{F})^* := \mathbb{H}_0 + \nabla_{\mu\lambda}.$$  

Here the free Hamiltonian $\mathbb{H}_0 = (\mathcal{F} \otimes \mathcal{F}) \hat{H}_0(\mathcal{F} \otimes \mathcal{F})^*$ is the multiplication operator:

$$\mathbb{H}_0 f(p, q) = [\varepsilon(p) + \varepsilon(q)] f(p, q),$$

where

$$\varepsilon(p) := \sum_{i=1}^2 (1 - \cos p_i), \quad p = (p_1, p_2) \in \mathbb{T}^2,$$

is the dispersion relation of a single boson. The interaction $\nabla_{\mu\lambda} = (\mathcal{F} \otimes \mathcal{F}) \hat{v}_{\mu\lambda}(\mathcal{F} \otimes \mathcal{F})^*$ is the (partial) integral operator

$$\nabla_{\mu\lambda} f(p, q) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} v_{\mu\lambda}(p - u) f(u, p + q - u) du,$$
where
\[ v_{\mu\lambda}(p) = \mu + \lambda \sum_{i=1}^{2} \cos p_i, \quad p = (p_1, p_2) \in \mathbb{T}^2. \]

2.3. The Floquet-Bloch decomposition of $H_{\mu\lambda}$ and discrete Schrödinger operators. Since $\hat{H}_{\mu\lambda}$ commutes with the representation of the discrete group $\mathbb{Z}^2$ by shift operators on the lattice, we can decompose the space $L^{2,e}(\mathbb{T}^2 \times \mathbb{T}^2)$ and $\mathbb{H}_{\mu\lambda}$ into the von Neumann direct integral as
\[ L^{2,e}(\mathbb{T}^2 \times \mathbb{T}^2) \simeq \int_{K \in \mathbb{T}^2} \oplus L^{2,e}(\mathbb{T}^2) \, dK \]  
and
\[ \mathbb{H}_{\mu\lambda} \simeq \int_{K \in \mathbb{T}^2} \oplus H_{\mu\lambda}(K) \, dK, \]
where $L^{2,e}(\mathbb{T}^2)$ is the Hilbert space of square-integrable even functions on $\mathbb{T}^2$ (see, e.g., [1]).

The fiber operator $H_{\mu\lambda}(K), K \in \mathbb{T}^2$, is a self-adjoint operator defined in $L^{2,e}(\mathbb{T}^2)$ as
\[ H_{\mu\lambda}(K) := H_0(K) + V_{\mu\lambda}, \]
where the unperturbed operator $H_0(K)$ is the multiplication operator by the function
\[ \mathcal{E}_K(p) := 2 \sum_{i=1}^{2} \left( 1 - \cos p_i \cos p_i \right), \]
and the perturbation $V_{\mu\lambda}$ is defined as
\[ V_{\mu\lambda} f(p) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \left( \mu + \lambda \sum_{i=1}^{2} \cos p_i \cos q_i \right) f(q) \, dq. \]

In the literature the parameter $K \in \mathbb{T}^2$ is called the two-particle quasi-momentum and the fiber $H_{\mu\lambda}(K)$ is called the discrete Schrödinger operator associated to the two-particle Hamiltonian $\mathbb{H}_{\mu\lambda}$.

Using the Fourier transform (2.3) and (2.4) can be represented as
\[ \ell^{2,e}(\mathbb{Z}^2 \times \mathbb{Z}^2) \simeq \int_{K \in \mathbb{T}^2} \oplus \ell^{2,e}(\mathbb{Z}^2) \, dK \]
and
\[ \mathbb{H}_{\mu\lambda} \simeq \int_{K \in \mathbb{T}^2} \oplus \tilde{H}_{\mu\lambda}(K) \, dK, \]
where $\ell^{2,e}(\mathbb{Z}^2)$ is the Hilbert space of square-summable even functions on $\mathbb{Z}^2$ and
\[ \tilde{H}_{\mu\lambda}(K) := \mathcal{F}^* H_{\mu\lambda}(K) \mathcal{F}, \]
where
\[ \tilde{H}_{\mu\lambda}(K) = \tilde{H}_0(K) + \tilde{V}_{\mu\lambda}(K), \]  
and
\[ \tilde{H}_0(K) f(x) = \sum_{s \in \mathbb{Z}^2} \tilde{\mathcal{E}}_K(x - s) \tilde{f}(s), \quad f \in \ell^{2,e}(\mathbb{Z}^2) \]
with
\[ \tilde{\mathcal{E}}_K(x) = 2\mathcal{E}(x) \cos \frac{K \cdot x}{2} \]
and the operator $\tilde{V}_{\mu\lambda}$ acts in $l^2(\mathbb{Z}^2)$ as
\[
\tilde{V}_{\mu\lambda}f(x) = \tilde{v}_{\mu\lambda}(x)f(x),
\]
and the functions $\tilde{\varepsilon}$ and $\tilde{v}_{\mu\lambda}$ are given by (2.1) and (2.2), respectively.

2.4. The essential spectrum of discrete Schrödinger operators. Since $V_{\mu\lambda}$ has rank at most three, by Weyl’s Theorem for any $K \in \mathbb{T}^2$ the essential spectrum $\sigma_{\text{ess}}(H_{\mu\lambda}(K))$ coincides with the spectrum of $H_0(K)$, i.e.,
\[
\sigma_{\text{ess}}(H_{\mu\lambda}(K)) = \sigma(H_0(K)) = [E_{\text{min}}(K), E_{\text{max}}(K)],
\]
where
\[
E_{\text{min}}(K) := \min_{p \in \mathbb{T}^2} E_K(p) = 2 \sum_{i=1}^2 \left(1 - \cos \frac{K_i}{2}\right) \geq 0 = E_{\text{min}}(0),
\]
\[
E_{\text{max}}(K) := \max_{p \in \mathbb{T}^2} E_K(p) = 2 \sum_{i=1}^2 \left(1 + \cos \frac{K_i}{2}\right) \leq 8 = E_{\text{max}}(0).
\]

3. MAIN RESULTS

Our first main result is the following generalization of [1, Theorems 1 and 2].

**Theorem 3.1.** Suppose that $H_{\mu\lambda}(0)$ has $n$ eigenvalues below resp. above the essential spectrum for some $\mu, \lambda \in \mathbb{R}$. Then for every $K \in \mathbb{T}^2$ the operator $H_{\mu\lambda}(K)$ has at least $n$ eigenvalues below resp. above its essential spectrum.

Next we find exact lower bound for the number of eigenvalues of $H_{\mu\lambda}(K)$ depending only on $\mu$ and $\lambda$.

In the $(\mu, \lambda)$-plane let us define the following nine sets:

\begin{align*}
S_{01} & := \{(\mu, \lambda) \in \mathbb{R}^2 : \lambda > \frac{\pi}{4 - \pi}\}, \\
S_{00} & := \{(\mu, \lambda) \in \mathbb{R}^2 : |\lambda| < \frac{\pi}{4 - \pi}\}, \\
S_{10} & := \{(\mu, \lambda) \in \mathbb{R}^2 : \lambda < \frac{\pi}{\pi - 4}\}, \\
C_0^+ & := \{(\mu, \lambda) \in \mathbb{R}^2 : \lambda \mu - 4\lambda - 2\mu > 0, \lambda < 2\}, \\
C_1^+ & := \{(\mu, \lambda) \in \mathbb{R}^2 : \lambda \mu - 4\lambda - 2\mu < 0\}, \\
C_2^+ & := \{(\mu, \lambda) \in \mathbb{R}^2 : \lambda \mu - 4\lambda - 2\mu > 0, \lambda > 2\}, \\
C_0^- & := \{(\mu, \lambda) \in \mathbb{R}^2 : \lambda \mu + 4\lambda + 2\mu > 0, \lambda > -2\}, \\
C_1^- & := \{(\mu, \lambda) \in \mathbb{R}^2 : \lambda \mu + 4\lambda + 2\mu < 0\}, \\
C_2^- & := \{(\mu, \lambda) \in \mathbb{R}^2 : \lambda \mu + 4\lambda + 2\mu > 0, \lambda < -2\}
\end{align*}

(see Figures 3a-3d below).

Let $n_+ (H_{\mu\lambda}(K))$ resp. $n_- (H_{\mu\lambda}(K))$ be the number of $H_{\mu\lambda}(K)$ above resp. below its essential spectrum.
Theorem 3.2. Let $K \in \mathbb{T}^2$ and $(\mu, \lambda) \in \mathbb{R}^2$. Then

\begin{align*}
(\mu, \lambda) \in C_2^+ \cap S_{01} & \implies n_+(H_{\mu \lambda}(K)) = 3, \\
(\mu, \lambda) \in C_2^+ \Delta S_{01} & \implies n_+(H_{\mu \lambda}(K)) \geq 2, \\
(\mu, \lambda) \in C_1^+ \setminus S_{01} & \implies n_+(H_{\mu \lambda}(K)) \geq 1, \\
(\mu, \lambda) \in \overline{C_0^+} & \implies n_+(H_{\mu \lambda}(K)) = 0,
\end{align*}

and

\begin{align*}
(\mu, \lambda) \in C_2^- \cap S_{10} & \implies n_-(H_{\mu \lambda}(K)) = 3, \\
(\mu, \lambda) \in C_2^- \Delta S_{10} & \implies n_-(H_{\mu \lambda}(K)) \geq 2, \\
(\mu, \lambda) \in C_1^- \setminus S_{10} & \implies n_-(H_{\mu \lambda}(K)) \geq 1, \\
(\mu, \lambda) \in \overline{C_0^-} & \implies n_-(H_{\mu \lambda}(K)) = 0,
\end{align*}

where $A \Delta B := (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of sets and $\overline{A}$ is the closure of a set $A$. 

FIGURE 3. Schematic locations of sets $S_{ij}$ and $C_i^\pm$ in (3.1).
Theorem 3.2 provides a lower bound for the number of eigenvalues for both sides of the essential spectrum of $H_{\mu\lambda}(K)$. Recall that by the min-max principle $H_{\mu\lambda}(K)$ can have at most three bound states outside its essential spectrum.

**Remark 3.3.** Theorem 3.2 implies that in some subsets of the $(\mu, \lambda)$-plane (for example in $C_1^+ \cap C_1^-$) the eigenvalues of $H_{\mu\lambda}(K)$ can appear simultaneously on both sides of the essential spectrum.

The following result shows that the estimates for $n_\pm(H_{\mu\lambda}(K))$ given by Theorem 3.2 are sharp.

**Theorem 3.4.** Let $K = 0$. Then all inequalities in (3.2) and (3.3) are in fact equalities. Moreover, the spaces

$$L^{2,e,s}(\mathbb{T}^2) := \left\{ f \in L^{2,e}(\mathbb{T}^2) : f(p_1, p_2) = f(p_2, p_1), p_1, p_2 \in \mathbb{T} \right\}$$

and

$$L^{2,e,a}(\mathbb{T}^2) := \left\{ f \in L^{2,e}(\mathbb{T}^2) : f(p_1, p_2) = -f(p_2, p_1), p_1, p_2 \in \mathbb{T} \right\}$$

of symmetric and antisymmetric even functions are invariant with respect to $H_{\mu\lambda}(0)$ and:

1. if $(\mu, \lambda) \in \overline{S_{00}}$, then $H_{\mu\lambda}(0)$ has no antisymmetric bound states outside the essential spectrum;
2. if $(\mu, \lambda) \in S_{01}$ resp. $(\mu, \lambda) \in S_{10}$, then $H_{\mu\lambda}(0)$ has a unique antisymmetric bound state above resp. below the essential spectrum;
3. if $(\mu, \lambda) \in C^+_0$ resp. $(\mu, \lambda) \in C^-_0$, then $H_{\mu\lambda}(0)$ has a exactly two symmetric bound states above resp. below the essential spectrum;
4. if $(\mu, \lambda) \in C_1^+ \cup \partial C_2^+$ resp. $(\mu, \lambda) \in C_1^- \cup \partial C_2^-$, then $H_{\mu\lambda}(0)$ has a unique symmetric bound state above resp. below the essential spectrum, where $\partial A$ is the topological boundary of a set $A$;
5. if $(\mu, \lambda) \in C_0^+$ resp. $(\mu, \lambda) \in C_0^-$, then $H_{\mu\lambda}(0)$ has no symmetric bound states above resp. below the essential spectrum.

4. **Proofs of the Main Results**

4.1. **Discrete spectrum of $H_{\mu\lambda}(0)$.** Unlike the case $K \neq 0$ in the case $K = 0$ the Fredholm determinant $\Delta_{\mu\lambda}(0, z)$ is easier to study. Notice that the operator $H_0(0)$ is the multiplication operator by the symmetric function $\epsilon_0(p) = 2\epsilon(p)$ in $L^{2,e}(\mathbb{T}^2)$. Hence, both $L^{2,e,s}(\mathbb{T}^2)$ and $L^{2,e,a}(\mathbb{T}^2)$ are invariant with respect to $H_0(0)$. Moreover, since

$$2 \cos p_1 \cos q_1 + 2 \cos p_2 \cos q_2$$

$$= (\cos p_1 + \cos p_2)(\cos q_1 + \cos q_2) + (\cos p_1 - \cos p_2)(\cos q_1 - \cos q_2),$$

the spaces of symmetric and antisymmetric even functions in $\mathbb{T}^2$ are invariant also with respect $V_{\mu\lambda}$. Thus,

**Lemma 4.1.**

$$\sigma(H_{\mu\lambda}(0)) = \sigma(H_{\mu\lambda}^s) \cup \sigma(H_{\mu\lambda}^a),$$

where

$$H_{\mu\lambda}^s := H_0(0) + V_{\mu\lambda}^s \quad \text{and} \quad H_{\mu\lambda}^a := H_0(0) + V_{\mu\lambda}^a,$$

with

$$V_{\mu\lambda}^s f(p) = \frac{\mu}{4\pi^2} \int_{\mathbb{T}^2} f(q) \, dq + \frac{\lambda}{8\pi^2} (\cos p_1 + \cos p_2) \int_{\mathbb{T}^2} (\cos q_1 + \cos q_2) f(q) \, dq$$

and

$$V_{\mu\lambda}^a f(p) = \frac{\mu}{4\pi^2} \int_{\mathbb{T}^2} f(q) \, dq - \frac{\lambda}{8\pi^2} (\cos p_1 + \cos p_2) \int_{\mathbb{T}^2} (\cos q_1 + \cos q_2) f(q) \, dq.$$
Proposition 4.4. There exist \( \operatorname{are established in the following proposition.} \)

\[ \begin{align*}
\operatorname{Proposition 4.4.} & \quad \text{There exists } V^\alpha \lambda f(p) = \frac{\lambda}{8\pi^2} (\cos p_1 - \cos p_2) \int_{\mathbb{T}^2} (\cos q_1 - \cos q_2) f(q) \, dq, \\
& \quad \text{are the restrictions of } H_{\mu\lambda}(0) \text{ onto } L^{2,e,s}(\mathbb{T}^2) \text{ and } L^{2,e,a}(\mathbb{T}^2). \\
\end{align*} \]

Theorem 4.2. Fix \((\mu, \lambda) \in \mathbb{R}^2.\)

(a) If \(2\lambda + \mu \geq 0,\) then \(H^{s}_{\mu\lambda}\) has at least one eigenvalue greater than 8.

(b) If \(2\lambda + \mu \leq 0,\) then \(H^{a}_{\mu\lambda}\) has at least one negative eigenvalue.

Proof. Note that the trace of \(V^\alpha \lambda\) is exactly \(2\lambda + \mu.\) Hence, the proof can be done along the essentially same lines of \([8, \text{Theorem 1.3}]\) introducing the non-symmetric Birman-Schwinger operator in \(L^{2,e,s}(\mathbb{T}^2)\) in place of \(L^2(\mathbb{T}^2)\) and rewriting it as a small perturbation of rank-one projection (which is not identically zero in \(L^{2,e,s}(\mathbb{T}^2)\)). \(\square\)

Remark 4.3. A statement in \(L^{2,e,a}(\mathbb{T}^2)\) analogous to Theorem 4.2 cannot be proven using the arguments of \([8, \text{Theorem 1.3}].\) Since in this case the rank-one projection (which is norm-close to the Birman-Schwinger operator) is identically zero in \(L^{2,e,a}(\mathbb{T}^2).\)

Note that the essential spectrum of all Hamiltonians \(H_{\mu\lambda}(0), H^{s}_{\mu\lambda}\) and \(H^{a}_{\mu\lambda}\) coincide with the segment \([0, 8].\) Let us find an (implicit) equation for the discrete eigenvalues of \(H_{\mu\lambda}(0).\) By Lemma 4.1 it is enough to solve

\[ \begin{align*}
H^{s}_{\mu\lambda}f &= zf \quad \text{resp. } H^{a}_{\mu\lambda}f = zf \\
\end{align*} \]

in \(z \in \mathbb{C} \setminus [0, 8]\) and nonzero \(f.\) Here we apply Fredholm’s determinants theory (see, e.g., \([21]).\)

For shortness, writing

\[ \begin{align*}
a(z) := \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \frac{dp}{\mathcal{E}_0(p) - z}, & \quad b(z) := \frac{1}{8\pi^2} \int_{\mathbb{T}^2} \frac{(\cos p_1 + \cos p_2)^2 \, dp}{\mathcal{E}_0(p) - z}, \\
c(z) := \frac{1}{8\pi^2} \int_{\mathbb{T}^2} \frac{(\cos p_1 + \cos p_2) \, dp}{\mathcal{E}_0(p) - z}, & \quad d(z) := \frac{1}{8\pi^2} \int_{\mathbb{T}^2} \frac{(\cos p_1 - \cos p_2)^2 \, dp}{\mathcal{E}_0(p) - z},
\end{align*} \]

define the Fredholm determinants associated to \(H^{s}_{\mu\lambda}\) and \(H^{a}_{\lambda},\) respectively, as

\[ \begin{align*}
\Delta^{(s)}_{\mu\lambda}(z) := \Delta_{\mu 0}(z) & \Delta_{0\lambda}(z) - 2:\mu\lambda c(z)^2, \\
\Delta^{(a)}_{\lambda}(z) := 1 + \lambda d(z),
\end{align*} \]

and

\[ \begin{align*}
\Delta_{\mu 0}(z) := 1 + \mu a(z), & \quad \Delta_{0\lambda}(z) := 1 + \lambda b(z).
\end{align*} \]

Note that the functions \(a(\cdot), b(\cdot)\) and \(d(\cdot)\) are analytic in \(\mathbb{C} \setminus [0, 8],\) strictly increasing in \(\mathbb{R} \setminus [0, 8],\) negative in \((8, +\infty)\) and positive in \((-\infty, 0).\) Their behaviour near \(z = 0\) and \(z = 8\) are established in the following proposition.

Proposition 4.4. There exists \(\delta > 0\) such that for every \(\omega \in \{a, b, c, d\}\) there exist functions \(P_{\omega}^0,\) and \(Q_{\omega}^0,\) analytic in the disc \(\{z \in \mathbb{C} : |z| < \delta\},\) and \(P_{\omega}^1,\) and \(Q_{\omega}^1,\) analytic in the disc \(\{z \in \mathbb{C} : |z - 8| < \delta\},\) such that

\[ \begin{align*}
& \text{for every } z \in (-\delta, 0) \\
& \quad \omega(z) = P_{\omega}^0(z) \ln(-z) + Q_{\omega}^0(z), \\
& \quad \omega(z) = P_{\omega}^1(z) \ln(z) + Q_{\omega}^1(z),
\end{align*} \]

where

\[ P_{a}^0(0) = -\frac{1}{4\pi}, \quad P_{b}^0(0) = -\frac{1}{2\pi}, \quad P_{c}^0(0) = -\frac{1}{4\pi}, \quad P_{d}^0(0) = 0 \]
Lemma 4.6. Let \( \lambda \in \mathbb{R} \).

(a) If \( \lambda \in [\frac{\pi}{4}, \frac{3\pi}{4}] \), then \( \Delta^{(a)}(\cdot) > 0 \) in \( \mathbb{R} \setminus [0, 8] \).

(b) If \( \lambda < \frac{\pi}{4-\pi} \), then \( \Delta^{(a)}(\cdot) \) has a unique zero in \( (-\infty, 0) \) and \( \Delta^{(a)}(\cdot) > 1 \) in \( (8, +\infty) \).

(c) If \( \lambda > \frac{\pi}{4-\pi} \), then \( \Delta^{(a)}(\cdot) \) has a unique zero in \( (8, \infty) \) and \( \Delta^{(a)}(\cdot) > 1 \) in \( (-\infty, 0) \).

Proof. Note that the map \( z \mapsto d(z) \) strictly increases in both connected components of \( \mathbb{R} \setminus [0, 8] \). Hence, the equation \( \Delta^{(a)}(z) = 0 \) has at most one zero in \( \mathbb{R} \setminus [0, 8] \). Moreover,

\[
\lim_{z \to \pm\infty} \Delta^{(a)}(z) = 1
\]  

and by Proposition 4.4

\[
\lim_{z \to 0^+} \Delta^{(a)}(z) = 1 + \frac{(4 - \pi)\lambda}{\pi}, \quad \lim_{z \to 8^-} \Delta^{(a)}(z) = 1 + \frac{(\pi - 4)\lambda}{\pi}.
\]  

Now the assertions of the lemma follow from the strict monotonicity and continuity of \( z \mapsto \Delta^{(a)}(z) \), and (4.3)-(4.4).

The following lemma provides the dependence of the number of zeros of \( \Delta^{(s)} \) in \( (8, +\infty) \) on \( \mu \) and \( \lambda \).

Lemma 4.7. Let \( (\mu, \lambda) \in \mathbb{R}^2 \).

\[
Q^0_a(0) = \frac{5\ln 2}{4\pi}, \quad Q^0_b(0) = \frac{5\ln 2 - \pi}{2\pi}, \quad Q^0_c(0) = \frac{5\ln 2 - \pi}{4\pi}, \quad Q^0_d(0) = \frac{4 - \pi}{\pi};
\]

• for every \( z \in (8, 8 + \delta) \)

\[
\omega(z) = P^1_a(z) \ln(z-8) + Q^1_a(z),
\]  

where

\[
P^1_a(8) = \frac{1}{4\pi}, \quad P^1_b(8) = \frac{1}{2\pi}, \quad P^1_c(8) = \frac{1}{4\pi}, \quad P^1_d(8) = 0
\]

and

\[
Q^1_a(8) = -\frac{5\ln 2}{4\pi}, \quad Q^1_b(8) = -\frac{5\ln 2 - \pi}{2\pi}, \quad Q^1_c(8) = -\frac{5\ln 2 - \pi}{4\pi}, \quad Q^1_d(8) = \frac{\pi - 4}{\pi}.
\]  

The proof of Proposition 4.4 is postponed to the Appendix A.

Lemma 4.5. A number \( z \in \mathbb{C} \setminus [0, 8] \) is an eigenvalue of \( H_{\mu\lambda}^s \) resp. \( H_{\mu\lambda}^\mu \) of multiplicity \( m \geq 1 \) if and only if it is a zero of \( \Delta_{\mu\lambda}^{(s)}(\cdot) \) resp. \( \Delta_{\mu\lambda}^{(a)}(\cdot) \) of multiplicity \( m \). Moreover, in \( \mathbb{R} \setminus [0, 8] \) the function \( \Delta_{\mu\lambda}^{(s)}(\cdot) \) has at least one and at most two zeros and the function \( \Delta_{\mu\lambda}^{(a)}(\cdot) \) has at most one zero.

Proof. The first assertion follows from the Fredholm determinants theory. By Theorem 4.2 \( H_{\mu\lambda}^s \) has at least one eigenvalue outside the essential spectrum. Moreover, since \( H_{\mu\lambda}^s \) is of rank two resp. rank one, by the min-max principle, it has at most two eigenvalues outside the essential spectrum. Hence, by the first part of the proposition \( \Delta_{\mu\lambda}^{(s)}(\cdot) \) has at least one and at most two zeros in \( \mathbb{R} \setminus [0, 8] \).

The last assertion follows from the rank-one property of \( H_{\mu\lambda}^{(a)} \) and the first part of the proposition. \( \square \)

Next we study zeros of \( \Delta_{\mu\lambda}^{(a)} \).
Lemma 4.8. Let $\lambda, \mu \in \mathbb{R}^2$. 

(a) If $\lambda \mu - 4 \lambda - 2 \mu \geq 0$ and $\lambda < 2$, then $\Delta_{\mu\lambda}^{(s)}$ has no zeros in $(8, +\infty)$.

(b) If $\lambda \mu - 4 \lambda - 2 \mu < 0$ or $\lambda \mu - 4 \lambda - 2 \mu = 0$ with $\lambda > 2$, then $\Delta_{\mu\lambda}^{(s)}$ has a unique zero in $(8, +\infty)$.

(c) If $\lambda \mu - 4 \lambda - 2 \mu > 0$ and $\lambda > 2$, then $\Delta_{\mu\lambda}^{(s)}$ has two zeros in $(8, +\infty)$.

Proof. Note that

$$
\lim_{z \to +\infty} \Delta_{\mu\lambda}^{(s)}(z) = 1
$$

and by Proposition 4.4

$$
\Delta_{\mu\lambda}^{(s)}(z) = -\frac{\lambda \mu - 4 \lambda - 2 \mu}{8\pi} \ln(z - 8) + \left(1 - \frac{5 \ln 2}{4\pi} \mu - \frac{\ln 2 - \pi}{2\pi} \lambda + \frac{\ln 2 - \pi}{8\pi} \lambda \mu\right) + o(1)
$$

as $z \searrow 8$ so that

$$
\lim_{z \searrow 8} \Delta_{\mu\lambda}^{(s)}(z) = \begin{cases} 
+\infty & \text{if } \lambda \mu - 4 \lambda - 2 \mu > 0, \\
-\infty & \text{if } \lambda \mu - 4 \lambda - 2 \mu < 0, \\
1 - \mu/4 & \text{if } \lambda \mu - 4 \lambda - 2 \mu = 0.
\end{cases}
$$

Notice that $\lambda \mu - 4 \lambda - 2 \mu = 0$ and $\lambda > 2$, then $1 - \mu/4 < 0$.

(a) We observe that if $\lambda \mu - 4 \lambda - 2 \mu \geq 0$ and $\lambda < 2$, then either $\lambda \leq 0$ or $\mu \leq 0$. Hence, by the minmax principle, $H_{\mu\lambda}^{s}$ can have at most one eigenvalue above the essential spectrum. By Lemma 4.5 $\Delta_{\mu\lambda}^{(s)}$ has at most one zero in $(8, +\infty)$. Then (4.5) and (4.6) imply that $\Delta_{\mu\lambda}^{(s)}(z) > 0$ in $(8, +\infty)$, otherwise $\Delta_{\mu\lambda}^{(s)}(\cdot)$ would cross $(8, +\infty)$ at least two times.

(b) If $\lambda \mu - 4 \lambda - 2 \mu < 0$ or $\lambda \mu - 4 \lambda - 2 \mu = 0$ with $\lambda > 2$, then by (4.5) and (4.6) the continuous function $\Delta_{\mu\lambda}^{(s)}(\cdot)$ changes sign in $(8, +\infty)$ so that the equation $\Delta_{\mu\lambda}^{(s)}(z) = 0$ has at least one zero in $(8, +\infty)$. If this equation had at least two zeros, then $\Delta_{\mu\lambda}^{(s)}(\cdot)$ should have at least three zeros since $\Delta_{\mu\lambda}^{(s)}(\cdot)$ has different signs at the endpoints of $(8, +\infty)$. This contradicts to Lemma 4.5.

(c) If $\lambda \mu - 4 \lambda - 2 \mu > 0$ and $\lambda > 2$, then $\mu + 2 \lambda > 0$, and hence, by Theorem 4.2 $H_{\mu\lambda}^{s}$ has at least one eigenvalue above the essential spectrum. Then by Lemma 4.5 $\Delta_{\mu\lambda}^{(s)}(\cdot)$ has at least one zero in $(8, +\infty)$. On the other hand, by (4.5) and (4.6) $\Delta_{\mu\lambda}^{(s)}(\cdot)$ has the same signs at the endpoints of $(8, +\infty)$, hence, by continuity the equation $\Delta_{\mu\lambda}^{(s)}(z) = 0$ has at least two solutions. Now Lemma 4.5 implies that $\Delta_{\mu\lambda}^{(s)}(\cdot)$ has two zeros in $(8, +\infty)$.

The zeros of $\Delta_{\mu\lambda}^{(s)}$ in $(-\infty, 0)$ are studied in the following lemma whose proof can be done along the lines of Lemma 4.7.

Lemma 4.8. Let $(\mu, \lambda) \in \mathbb{R}^2$.

(a) If $\lambda \mu + 4 \lambda + 2 \mu \geq 0$ and $\lambda > -2$, then $\Delta_{\mu\lambda}^{(s)}$ has no zeros in $(-\infty, 0)$.

(b) If $\lambda \mu + 4 \lambda + 2 \mu < 0$ or $\lambda \mu + 4 \lambda + 2 \mu = 0$ with $\lambda < -2$, then $\Delta_{\mu\lambda}^{(s)}$ has a unique zero in $(-\infty, 0)$.

(c) If $\lambda \mu + 4 \lambda + 2 \mu > 0$ and $\lambda < -2$, then $\Delta_{\mu\lambda}^{(s)}$ has two zeros in $(-\infty, 0)$.

Proof of Theorem 3.4. The assertions of the theorem follow from Lemmas 4.1, 4.5, 4.6, 4.7 and 4.8.
4.2. The discrete spectrum of $H_{\mu \lambda}(K)$. For every $n \geq 1$ define
\[
e_n(K; \mu, \lambda) := \sup_{\phi_1, \ldots, \phi_{n-1} \in L^2(eT^2)} \inf_{\psi \in [\phi_1, \ldots, \phi_{n-1}]^\perp, \|\psi\|=1} (H_{\mu \lambda}(K)\psi, \psi)\]
and
\[
E_n(K; \mu, \lambda) := \inf_{\phi_1, \ldots, \phi_{n-1} \in L^2(eT^2)} \sup_{\psi \in [\phi_1, \ldots, \phi_{n-1}]^\perp, \|\psi\|=1} (H_{\mu \lambda}(K)\psi, \psi).
\]
By the minmax principle, $e_n(K; \mu, \lambda) \leq \mathcal{E}_{\min}(K)$ and $E_n(K; \mu, \lambda) \geq \mathcal{E}_{\max}(K)$. Moreover, choosing $\phi_1 \equiv 1$, $\phi_2(p) = \cos p_1$ and $\phi_3(p) = \cos p_2$ we immediately see that $e_n(K; \mu, \lambda) = \mathcal{E}_{\min}(K)$ and $E_n(K; \mu, \lambda) = \mathcal{E}_{\max}(K)$ for all $n \geq 4$.

**Lemma 4.9.** Let $n \geq 1$ and $i \in \{1, 2\}$. Then the map
\[
K_i \in \mathbb{T} \mapsto \mathcal{E}_{\min}(K) - e_n(K; \mu, \lambda)
\]
is non-increasing in $(-\pi, 0]$ and non-decreasing in $[0, \pi]$. Similarly, the map
\[
K_i \in \mathbb{T} \mapsto E_n(K; \mu, \lambda) - \mathcal{E}_{\max}(K)
\]
is non-increasing in $(-\pi, 0]$ and non-decreasing in $[0, \pi]$.

**Proof.** Without loss of generality we assume that $i = 1$. Given $\psi \in L^2(\mathbb{T}^2)$ consider
\[
((H_0(K) - \mathcal{E}_{\min}(K))\psi, \psi) = \int_{\mathbb{T}^2} \sum_{i=1}^2 \cos \frac{K_i}{2} (1 - \cos q_i) |\psi(q)|^2 \, dq.
\]
Thus, the map $K_1 \in \mathbb{T} \mapsto ((H_0(K) - \mathcal{E}_{\min}(K))\psi, \psi)$ is non-decreasing in $(-\pi, 0]$ and is non-increasing in $[0, \pi]$. Since $V_{\mu \lambda}$ is independent of $K$, from the definition of $e_n(K; \mu, \lambda)$ the map $K_1 \in \mathbb{T} \mapsto e_n(K; \mu, \lambda) - \mathcal{E}_{\min}(K)$ is non-decreasing in $(-\pi, 0]$ and is non-increasing in $[0, \pi]$.

The case of $K_i \mapsto E_n(K; \mu, \lambda) - \mathcal{E}_{\max}(K)$ is similar. □

**Proof of Theorem 3.1.** From Lemma 4.9 for any $K \in \mathbb{T}^2$ and $m \geq 1$
\[
0 \leq \mathcal{E}_{\min}(0) - e_m(0; \mu, \lambda) \leq \mathcal{E}_{\min}(K) - e_m(K; \mu, \lambda),
\]
and
\[
E_m(K; \mu, \lambda) - \mathcal{E}_{\max}(K) \geq E_m(0; \mu, \lambda) - \mathcal{E}_{\max}(0) \geq 0.
\]
By assumption, $e_n(0; \mu, \lambda)$ is a discrete eigenvalue of $H_{\mu \lambda}(0)$ for some $\mu, \lambda \in \mathbb{R}$. Thus, $\mathcal{E}_{\min}(0) - e_n(0; \mu, \lambda) > 0$, and hence, by (4.7) and (2.6) $e_n(K; \mu, \lambda)$ is a discrete eigenvalue of $H_{\mu \lambda}(K)$ for any $K \in \mathbb{T}^2$. Since $e_1(K; \mu, \lambda) \leq \ldots \leq e_n(K; \mu, \lambda) < \mathcal{E}_{\min}(K)$, it follows that $H_{\mu \lambda}(K)$ has at least $n$ eigenvalues below the essential spectrum.

The case of $E_n(K; \mu, \lambda)$ is similar. □

**Proof of Theorem 3.2.** Just combine Theorem 3.1 with Theorem 3.4. □

**Appendix A. Proof of Proposition 4.4**

The general form of expansions (4.1) and (4.2) of $a, b, c, d$ can be established following, for example, to [13]. The first two terms of these expansions, i.e., the pairs $(P^1_\omega(0), Q^1_\omega(0))$ and $(P^1_\omega(8), Q^1_\omega(8))$ are given by the following lemma.
Lemma A.1.

\[
\int_{\mathbb{T}^2} \frac{dq_1 dq_2}{2 - \cos q_1 - \cos q_2 - z} = \begin{cases} 
-2\pi \ln(-z) + 8\pi \ln 2 + o(1) & \text{as } z \to 0, \\
2\pi \ln(z - 4) - 8\pi \ln 2 + o(1) & \text{as } z \searrow 4,
\end{cases} \tag{A.1}
\]

\[
\int_{\mathbb{T}^2} \frac{\cos q_1 \cos q_2 dq_1 dq_2}{2 - \cos q_1 - \cos q_2 - z} = \begin{cases} 
-2\pi \ln(-z) + 8\pi(\ln 2 - 1) + o(1) & \text{as } z \to 0, \\
2\pi \ln(z - 4) - 8\pi(\ln 2 - 1) + o(1) & \text{as } z \searrow 4.
\end{cases} \tag{A.2}
\]

\[
\int_{\mathbb{T}^2} \frac{(\cos q_1 + \cos q_2) dq_1 dq_2}{2 - \cos q_1 - \cos q_2 - z} = \begin{cases} 
-4\pi \ln(-z) + 16\pi \ln 2 - 4\pi^2 + o(1) & \text{as } z \to 0, \\
4\pi \ln(z - 4) - 16\pi \ln 2 + 4\pi^2 + o(1) & \text{as } z \searrow 4,
\end{cases} \tag{A.3}
\]

\[
\int_{\mathbb{T}^2} \frac{(\cos q_1 + \cos q_2)^2 dq_1 dq_2}{2 - \cos q_1 - \cos q_2 - z} = \begin{cases} 
-8\pi \ln(-z) + 32\pi \ln 2 - 8\pi^2 + o(1) & \text{as } z \to 0, \\
8\pi \ln(z - 4) - 32\pi \ln 2 + 8\pi^2 + o(1) & \text{as } z \searrow 4,
\end{cases} \tag{A.4}
\]

\[
\int_{\mathbb{T}^2} \frac{(\cos q_1 - \cos q_2)^2 dq_1 dq_2}{2 - \cos q_1 - \cos q_2 - z} = \begin{cases} 
32\pi - 8\pi^2 + o(1) & \text{as } z \to 0, \\
-32\pi + 8\pi^2 + o(1) & \text{as } z \searrow 4.
\end{cases} \tag{A.5}
\]

Proof. Since the idea is the same, we prove (A.1)-(A.5) only for \( z < 0 \). By simmetry, it suffices to prove only (A.1) and (A.2). First we observe that if \(|u| > 1\), then

\[
\int_{T} \frac{dt}{u - \cos t} = \begin{cases} 
\frac{-2\pi}{\sqrt{u^2 - 1}} & \text{if } u < -1, \\
\frac{-2\pi}{\sqrt{u^2 - 1}} & \text{if } u > 1.
\end{cases} \tag{A.6}
\]

Let us prove (A.1). Since \( 2 - z - \cos q_2 > 1 \) for any \( z \in \mathbb{R} \setminus [0, 4] \) and \( q_2 \in \mathbb{T} \), by (A.6)

\[
\int_{\mathbb{T}^2} \frac{dq_1 dq_2}{2 - \cos q_1 - \cos q_2 - z} = \int_{\mathbb{T}} \frac{dq_2}{\sqrt{(2 - z - \cos q_2)^2 - 1}}.
\]

Then using the change of variables \( q_2 := 2 \arctan \left( \frac{z - 2}{z - 4} \right)^{1/2} \) we get

\[
\int_{\mathbb{T}^2} \frac{dq_1 dq_2}{2 - \cos q_1 - \cos q_2 - z} = \frac{8\pi}{|2 - z|} \int_{0}^{\infty} \frac{dv}{\sqrt{(\frac{z^2 - 4z}{z^2 - 2})^2 + v^2}}(1 + v^2) = J_1(z) + J_2(z) + J_3(z),
\]
where

\[
J_1(z) := \frac{8\pi}{|2-z|} \int_0^1 \frac{dv}{\sqrt{\frac{z^2-4v}{z-2} + v^2}},
\]

\[
J_2(z) := \frac{8\pi}{|2-z|} \int_0^1 \frac{(1+v^2)^{-1/2} - 1}{\sqrt{\frac{z^2-4v}{z-2} + v^2}} dv,
\]

\[
J_3(z) := \frac{8\pi}{|2-z|} \int_1^\infty \frac{dv}{\sqrt{\left(\frac{z^2-4v}{z-2}\right) + v^2} (1+v^2)}.
\]

Note that

\[
J_1(z) = \frac{8\pi}{|2-z|} \left( \ln \left[1 + \sqrt{1 + \frac{z^2-4v}{(z-2)^2}}\right] - \ln \frac{\sqrt{z^2-4v}}{|z-2|} \right) = -2\pi \ln(-z) + 4\pi \ln 2 + o(1)
\]

as \( z \to 0 \). Moreover, \( J_2 \) and \( J_3 \) are continuous at \( z = 0 \) and \( J_2(0) = 4\pi \ln \frac{2}{1+\sqrt{2}} \) and \( J_3(0) = 4\pi \ln(1 + \sqrt{2}) \). Thus, (A.1) follows.

Now we prove (A.2). Using \( \int_T \cos q_2 dq_2 = 0 \) and (A.6)

\[
\int_{T^2} \frac{\cos q_1 \cos q_2 dq_1 dq_2}{2 - \cos q_1 - \cos q_2 - z} = \int_T \frac{(2 - z - \cos q_2) \cos q_2 dq_2}{\sqrt{(2 - z - \cos q_2)^2 - 1}}.
\]

Thus changing variables as \( q_2 = 2 \arctan u \) we get

\[
\int_{T^2} \frac{\cos q_1 \cos q_2 dq_1 dq_2}{2 - \cos q_1 - \cos q_2 - z} = I_1(z) + I_2(z),
\]

where

\[
I_1(z) := 8\pi (1 - z) \int_0^\infty \frac{du}{(1+u^2)^2 \sqrt{-z + (2-z)u^2} \sqrt{2-z + (4-z)u^2}},
\]

\[
I_2(z) := 8\pi \int_0^\infty \frac{(2u^2 - (3-z)u^4) du}{(1+u^2)^2 \sqrt{-z + (2-z)u^2} \sqrt{2-z + (4-z)u^2}}.
\]

Obviously, \( I_2 \) is continuous at \( z = 0 \) and \( I_2(0) = 2\pi(\pi - 5) \). We represent \( I_1(z) \) as

\[
I_1(z) = I_{11}(z) + I_{12}(z) + I_{13}(z),
\]

where

\[
I_{11}(z) := \frac{8\pi (1 - z)}{\sqrt{2-z}} \int_0^1 \frac{du}{\sqrt{-z + (2-z)u^2}}
\]

\[
I_{12}(z) := 8\pi (1 - z) \int_0^1 \frac{[(1+u^2)^{-2}(2-z + (4-z)u^2)^{-1/2} - (2-z)^{-1/2}]}{\sqrt{-z + (2-z)u^2}} du
\]

\[
I_{13}(z) := 8\pi (1 - z) \int_1^\infty \frac{du}{(1+u^2)^2 \sqrt{-z + (2-z)u^2} \sqrt{2-z + (4-z)u^2}}.
\]

Then

\[
I_{11}(z) = \frac{8\pi (1 - z)}{2-z} \left[ \ln \left(1 + \sqrt{1 + \frac{z}{z-2}}\right) - \ln \sqrt{\frac{z}{z-2}} \right] = -2\pi \ln(-z) + 6\pi \ln 2 + o(1).
\]
as $z \to 0$. Note that the functions $I_{12}(z)$ and $I_{13}(z)$ are continuous at $z = 0$ and

$$I_{12}(0) = 4\pi \left[ \ln \frac{2}{1 + \sqrt{3}} - \frac{\pi}{6} + \frac{2 - \sqrt{3}}{4} \right]$$

and

$$I_{13}(0) = 4\pi \left[ \ln \frac{\sqrt{3} + 1}{\sqrt{2}} - \frac{\pi}{3} + \frac{\sqrt{3}}{4} \right].$$

Hence

$$I_1(z) = -2\pi \ln(-z) + 8\pi \ln 2 - 2\pi^2 + 2\pi + o(1)$$

and (A.2) follows.

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(S. Lakaev) SAMARKAND STATE UNIVERSITY, UNIVERSITY BOULEVARD, 15, 140104 SAMARKAND, UZBEKISTAN
Email address: slakaev@mail.ru

(Sh. Kholmatov) UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, 1090 VIENNA, AUSTRIA
Email address: shokhrukh.kholmatov@univie.ac.at

(Sh. Khamidov) SAMARKAND STATE UNIVERSITY, UNIVERSITY BOULEVARD, 15, 140104 SAMARKAND, UZBEKISTAN
Email address: shoh.hamidov1990@mail.ru