An average of generalized Dedekind sums

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Classical Dedekind Sum

Generalized Dedekind Sum

A Different View

Bounds on the Second Moment
  Upper Bound
  Lower Bound

Conclusion
Classical Dedekind Sum
Definition

\[ B_1(x) = \begin{cases} 
0 & \text{if } x \in \mathbb{Z} \\
x - \lfloor x \rfloor - \frac{1}{2} & \text{otherwise.}
\end{cases} \]
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\[ s(a, c) = \sum_{j \mod c} B_1\left(\frac{j}{c}\right) B_1\left(\frac{aj}{c}\right) \]
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... one of its many guises:

\[ s(a, c) = \frac{1}{4c} \sum_{j \mod c} \cot\left(\frac{\pi j}{c}\right) \cot\left(\frac{\pi aj}{c}\right) \]
Dirichlet Characters

A Dirichlet character modulo $q$ is a function $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ that has

1. period $q$
2. $\chi(mn) = \chi(m)\chi(n)$
3. $\chi(n) = 0$ if and only if $\gcd(n, q) > 1$
4. $\chi(1) = 1$
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| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $\chi(n)$ | 0 | 1 | $-i$ | $i$ | $-1$ |
The function
defined by:
\[
\chi_{0,m}(n) = \begin{cases} 
1 & \text{if } \gcd(n, m) = 1 \\
0 & \text{otherwise.}
\end{cases}
\]
is the principal character modulo \( m \).
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Given \( \psi \) modulo \( q \), we can \textbf{induce} a character modulo \( mq \) by \( \psi \chi_{0,m} \).
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| \( n \) | 0 | 1 | 2 | 3 | 4 |
|---------|---|---|---|---|---|
| \( \psi(n) \) | 0 | 1 | \( i \) | \( -i \) | \(-1\) |

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------|---|---|---|---|---|---|---|---|---|---|
| \( \psi \chi_{0,2}(n) \) | 0 | 1 | 0 | \( -i \) | 0 | 0 | 0 | \( i \) | 0 | \(-1\) |
The function

\[ \chi_{0,m}(n) = \begin{cases} 
1 & \text{if } \gcd(n, m) = 1 \\
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| \( \psi \chi_{0,2}(n) \) | 0 | 1 | 0 | \( -i \) | 0 | 0 | 0 | \( i \) | 0 | \( -1 \) |

A **primitive** character is not induced by any other character.
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|
| $\psi(n)$ | 0 | 1 | 0 | 0 | 0 | -1 | 0 | 1 | 0 | 0 | 0 | -1 |
| $n$ | 0  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----|----|---|---|---|---|---|---|---|---|---|----|----|
| $\psi(n)$ | 0 | 1 | 0 | 0 | 0 | $-1$ | 0 | 1 | 0 | 0 | 0 | $-1$ |
| $n$ | $\psi(n)$ | $\psi^*(n)$ |
|-----|-----------|-------------|
| 0   | 0         | 0           |
| 1   | 1         | 1           |
| 2   | 0         | -1          |
| 3   | 0         | 0           |
| 4   | 0         | -1          |
| 5   | -1        | 0           |
| 6   | 0         | 1           |
| 7   | 1         | -1          |
| 8   | 0         | 0           |
| 9   | 0         | 0           |
| 10  | 0         | -1          |
| 11  | -1        | 0           |

| $n$ | $\psi^*(n)$ |
|-----|-------------|
| 0   | 0           |
| 1   | 1           |
| 2   | -1          |
The Dirichlet $L$-function associated with the character $\chi$ is

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$
The **Dirichlet L-function** associated with the character $\chi$ is

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Dirichlet used $L(1, \chi)$ to study primes in arithmetic progressions.
Walum’s Result

Walum evaluated

\[ \sum_{\chi \mod p} |L(1, \chi)|^2. \]
\[ \chi \mod p \]
\[ \chi(-1) = -1 \]

In principle, his technique works for all even powers.
Walum’s Result

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\[ \sum_{\chi \mod p \chi(-1)=-1} |L(1, \chi)|^2. \]

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**Theorem (Walum, 1982)**

\[ \sum_{\chi \mod p \chi(-1)=-1} |L(1, \chi)|^4 = \frac{\pi^4(p - 1)}{p^2} \sum_{a \mod p} |s(a, c)|^2. \]
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$$\sum_{\chi \mod p \atop \chi(-1)=-1} |L(1, \chi)|^4 = \frac{\pi^4(p - 1)}{p^2} \sum_{a \mod p} |s(a, c)|^2.$$

Rearranging, we have an average of Dedekind sums:

$$\sum_{a \mod p} |s(a, p)|^2 = \frac{p^2}{\pi^4(p - 1)} \sum_{\chi \mod p \atop \chi(-1)=-1} |L(1, \chi)|^4.$$
Generalized Dedekind Sum
Let $\chi_1 \mod q_1$ and $\chi_2 \mod q_2$ be non-trivial primitive Dirichlet characters. The **generalized Dedekind sum** is

$$S_{\chi_1, \chi_2}(a, c) = \sum_{j \mod c} \sum_{n \mod q_1} \overline{\chi_2}(j) \overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right)$$
Definition

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... one of its many guises:

$$S_{\chi_1,\chi_2}(a, c) = K \sum_{s \mod c} \sum_{r \mod q_2} \chi_1(s) \chi_2(r) \cot\left(\pi \left(\frac{r}{q_2} - \frac{as}{c}\right)\right) \cot\left(\frac{\pi S}{c}\right)$$
The Second Moment

**Theorem (D. and G., 2019)**

Let $\chi_1$ and $\chi_2$ be nontrivial primitive characters such that $\chi_1 \chi_2(-1) = 1$, and let $q_1 q_2 \mid c$. Then

$$\sum_{a \mod c \atop (a,c)=1} |S_{\chi_1, \chi_2}(a, c)|^2 = \frac{\varphi(c)}{\pi^4} \sum_{\psi \mod c \atop \psi \chi_1(-1) = -1} |L(1, \overline{\psi}^* \chi_1)|^2 |L(1, (\psi \chi_2)^*)|^2 |g_{\chi_1, \chi_2}(\psi; c)|^2.$$
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$$g_{\chi_1,\chi_2}(\psi; c) = K(\psi) \sum_{d | c \atop d \equiv 0 \mod q(\psi)} \frac{\overline{\chi_2}(c/d)}{\varphi(d)} ((\overline{\psi} \chi_2)^* \mu * 1)(d) (\chi_1 * \mu \psi^*) \left( \frac{d}{q(\psi)} \right)$$
Theorem (D. and G., 2019)

Let $\chi_1$ and $\chi_2$ be nontrivial primitive characters modulo $q_1$ and $q_2$, respectively, such that $\chi_1\chi_2(-1) = 1$, and let $q_1q_2 \mid c$. For every $\varepsilon > 0$, there exist positive $A_\varepsilon$ and $B_\varepsilon$ such that

$$A_\varepsilon c^{2-\varepsilon} \leq \sum_{a \mod c \atop (a,c)=1} |S_{\chi_1,\chi_2}(a, c)|^2 \leq B_\varepsilon c^{2+\varepsilon}.$$ 

Corollary

For all $c > 0$, $S_{\chi_1,\chi_2}(a, c)$ does not vanish.
A Different View
**Definition**

The special linear group $\text{SL}_2(\mathbb{Z})$ is the set of $2 \times 2$ matrices \[
\begin{pmatrix}
 a & b \\
 c & d
\end{pmatrix}
\] such that $ad - bc = 1$. 

The Dedekind sum is a map from $\mathcal{O}(q_1; q_2)$ to $\mathbb{C}$ by $S(q_1; q_2)(a; c)$. 

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An average of generalized Dedekind sums
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Definition

For $N \in \mathbb{N}^+$, the subgroup of $\text{SL}_2(\mathbb{Z})$ such that $N$ divides $c$ is denoted $\Gamma_0(N)$. 

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### Definition

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### Definition

For $N \in \mathbb{N}^+$, the subgroup of $\text{SL}_2(\mathbb{Z})$ such that $N$ divides $c$ is denoted $\Gamma_0(N)$.

The Dedekind sum is a map from $\Gamma_0(q_1q_2)$ to $\mathbb{C}$ by

$$S_{\chi_1, \chi_2}(\gamma) = S_{\chi_1, \chi_2}(a, c).$$
Let $\chi(\gamma) = \chi(d)$. Then

$$S_{\chi_1, \chi_2}(\gamma_1 \gamma_2) = S_{\chi_1, \chi_2}(\gamma_1) + \chi_1 \overline{\chi_2}(\gamma_1) S_{\chi_1, \chi_2}(\gamma_2).$$

If $\chi_1 = \chi_2$, then $\chi_1 \overline{\chi_2}(\gamma_1) = 1$, so $S_{\chi_1, \chi_2}(\gamma)$ is a homomorphism.
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If $\chi_1 = \chi_2$, then $\chi_1 \overline{\chi_2}(\gamma_1) = 1$, so $S_{\chi_1, \chi_2}(\gamma)$ is a homomorphism.

**Corollary**

*The crossed homomorphism $S_{\chi_1, \chi_2}$ is nontrivial. In fact, for each $c > 0$, there exists some $a \in \mathbb{Z}$ so that $S_{\chi_1, \chi_2}(a, c) \neq 0$.***
Questions?
Bounds on the Second Moment
Recall that:

\[ A_\varepsilon c^{2-\varepsilon} \leq \sum_{\substack{a \mod c \quad (a,c)=1}} |S_{\chi_1,\chi_2}(a, c)|^2 \leq B_\varepsilon c^{2+\varepsilon} \]
Sketchy Outline: Upper bound

\[ \sum_{a \mod c \atop (a,c)=1} |S_{\chi_1,\chi_2}(a,c)|^2 = \frac{\varphi(c)}{\pi^4} \sum_{\psi \mod c \atop \psi\chi_1(-1)=-1} |L(1, \overline{\psi}\chi_1)|^2 |L(1, (\psi\chi_2)^*)|^2 |g_{\chi_1,\chi_2}(\psi; c)|^2 \]
Bound the $L$-functions:

- For $\chi$ modulo $q$, there exists $K > 0$ so that $|L(1, \chi)| \leq K \log q$
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Bound $g$:

- Use the triangle inequality
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Bound the \( L \)-functions:

\begin{itemize}
  \item For \( \chi \) modulo \( q \), there exists \( K > 0 \) so that \( |L(1, \chi)| \leq K \log q \)
\end{itemize}

Bound \( g \):

\begin{itemize}
  \item Use the triangle inequality
  \item Terms inside sum become 1
\end{itemize}
Sketchy Outline: Upper bound

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\sum_{a \mod c, (a, c)=1} |S_{\chi_1, \chi_2}(a, c)|^2 = \frac{\varphi(c)}{\pi^4} \sum_{\psi \mod c} |L(1, \overline{\psi} \chi_1)|^2 |L(1, (\psi \chi_2)^*)|^2 |g_{\chi_1, \chi_2}(\psi; c)|^2
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Bound the \(L\)-functions:

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Bound \(g\):

- Use the triangle inequality
- Terms inside sum become 1
- Bound by divisor function
**Definition**

\[ d(n) \] is the number of positive divisors of \( n \).

**Example:** The divisors of 12 are \( \{1, 2, 3, 4, 6, 12\} \), so \( d(12) = 6 \).
Divisor Function

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Claim

For all \(\varepsilon > 0\) there exists \(K_\varepsilon > 0\) such that \(d(n) \leq K_\varepsilon n^\varepsilon\).
**Definition**

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**Claim**

*For all \( \varepsilon > 0 \) there exists \( K_\varepsilon > 0 \) such that \( d(n) \leq K_\varepsilon n^\varepsilon \).*

**Property**

If \( \gcd(m, n) = 1 \), then \( d(mn) = d(m)d(n) \).

So look at \( d(p^k) \) for primes \( p \).
Want to show that $d(p^k) \leq K_\varepsilon p^{k\varepsilon}$, so consider

$$\frac{d(p^k)}{p^{k\varepsilon}}.$$
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Calculate: $d(p^k) = k + 1$. 

$$\frac{k + 1}{(p^\varepsilon)^k}.$$
Want to show that \( d(p^k) \leq K_\varepsilon p^{k\varepsilon} \), so consider

\[
\frac{d(p^k)}{p^{k\varepsilon}}.
\]

Calculate: \( d(p^k) = k + 1 \).

\[
\frac{k + 1}{(p^\varepsilon)^k} \leq K_\varepsilon
\]

Therefore \( d(n) \leq K_\varepsilon n^\varepsilon \).
Sketchy Outline: Lower bound

\[ \sum_{a \mod c \ (a,c)=1} |S_{\chi_1,\chi_2}(a, c)|^2 \geq A_\varepsilon c^{2-\varepsilon} \]
Sketchy Outline: Lower bound

\[
\sum_{\substack{a \mod c \ (a,c)=1}} |S_{\chi_1, \chi_2}(a, c)|^2 = \frac{\varphi(c)}{\pi^4} \sum_{\substack{\psi \mod c \ \psi \chi_1(-1) = -1}} |L(1, \overline{\psi} \chi_1)|^2 |L(1, (\psi \chi_2)^*)|^2 |g_{\chi_1, \chi_2}(\psi; c)|^2
\]

Bound the \(L\)-functions:

- For \(\chi\) modulo \(q\), there exists \(K_\varepsilon > 0\) so that
  \[|L(1, \chi)| \geq K_\varepsilon q^{-\varepsilon}\]
\[
\sum_{\substack{a \mod c \\
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Bound the \(L\)-functions:

- For \(\chi\) modulo \(q\), there exists \(K_{\varepsilon} > 0\) so that
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Bound \(g\):

- Restrict the sum
Bound the $L$-functions:

- For $\chi$ modulo $q$, there exists $K_\varepsilon > 0$ so that
  \[ |L(1, \chi)| \geq K_\varepsilon q^{-\varepsilon} \]

Bound $g$:

- Restrict the sum
- All the terms are 1!

\[ \sum_{\substack{a \mod c \\ (a,c)=1}} |S_{\chi_1, \chi_2}(a, c)|^2 = \frac{\varphi(c)}{\pi^4} \sum_{\substack{\psi \mod c \\ \psi \chi_1(-1)=-1}} |L(1, \psi^* \chi_1)|^2 |L(1, (\psi \chi_2)^*)|^2 |g_{\chi_1, \chi_2}(\psi; c)|^2 \]
\[ \sum_{a \mod c, (a,c)=1} |S_{\chi_1, \chi_2}(a, c)|^2 = \frac{\varphi(c)}{\pi^4} \sum_{\psi \mod c, \psi\chi_1(-1)=-1} |L(1, \overline{\psi} \chi_1)|^2 |L(1, (\psi \chi_2)^*)|^2 |g_{\chi_1, \chi_2}(\psi; c)|^2 \]

Bound the $L$-functions:

- For $\chi$ modulo $q$, there exists $K_\varepsilon > 0$ so that
  \[ |L(1, \chi)| \geq K_\varepsilon q^{-\varepsilon} \]

Bound $g$:

- Restrict the sum
- All the terms are 1!
- Clever counting
A lemma that counts I

**Question**

How many primitive characters modulo $q$ are there?

Recall that a primitive character is **not** induced by a character of lower modulus.

Let $\varphi^*(q)$ be the number of primitive characters modulo $q$. 
Look at characters modulo $p^n$.

**Idea:** count the opposite.
Pick a prime . . .

Look at characters modulo $p^n$.

**Idea:** count the opposite.

A character is **not** primitive if it is induced by a character modulo $p^{n-1}$.

So we just need to find the number of characters modulo $p^{n-1}$. 
A Dirichlet Digression

Definition

Let $n \in \mathbb{N}^+$. The set

$$\left( \mathbb{Z}/n\mathbb{Z} \right)^* := \{ a \in \mathbb{Z}/n\mathbb{Z} : \gcd(a, n) = 1 \}$$

is a group under multiplication.
A Dirichlet Digression

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(\mathbb{Z}/n\mathbb{Z})^* \:= \{ a \in \mathbb{Z}/n\mathbb{Z} : \gcd(a, n) = 1 \}
\]

is a group under multiplication.

We can also define a Dirichlet character \( \chi \mod q \) as a homomorphism \((\mathbb{Z}/q\mathbb{Z})^* \rightarrow \mathbb{C}^*\). (This means that \( \chi(1) = 1 \) and \( \chi(mn) = \chi(m)\chi(n) \).)
A Dirichlet Digression

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We can also define a Dirichlet character $\chi \mod q$ as a homomorphism $(\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$. (This means that $\chi(1) = 1$ and $\chi(mn) = \chi(m)\chi(n)$.)

Then extend $\chi$ to $\mathbb{Z}$ by setting

$$\chi(n) = \begin{cases} 
\chi(n \mod q) & \text{if } \gcd(n, q) = 1 \\
0 & \text{otherwise.}
\end{cases}$$
Fact
The number of characters modulo $q$ is equal to the number of elements of $(\mathbb{Z}/q\mathbb{Z})^*$.

Definition
The number of positive integers less than $q$ that are relatively prime to $q$ is denoted $\varphi(q)$.

So there are $\varphi(p^{n-1})$ characters modulo $p^{n-1}$. 

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Modulo $p^n$, there are

1. $\varphi(p^n)$ characters
2. $\varphi(p^{n-1})$ imprimitive characters
3. $\varphi(p^n) - \varphi(p^{n-1})$ primitive characters.
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Claim: $\varphi(p^n) = p^n - p^{n-1}$. 
A lemma that counts II

Modulo $p^n$, there are

1. $\varphi(p^n)$ characters
2. $\varphi(p^{n-1})$ imprimitive characters
3. $\varphi(p^n) - \varphi(p^{n-1})$ primitive characters.

**Claim:** $\varphi(p^n) = p^n - p^{n-1}$.

**Proposition**

$$\varphi^*(p^n) = p^n - 2p^{n-1} + p^{n-2}.$$
Conclusion
Conclusion, being the Place in which we Recapitulate the High Points previously stated to you Fine Folk, and including a Small Sampling of the Exceedingly Excellent Problems related thereto

- $S_{X_1,X_2}$ is a generalization of Dedekind sum
- $S_{X_1,X_2} : \Gamma_0(q_1q_2) \to \mathbb{C}$
- Exact formula and bounds for second moment
- Proved that $S_{X_1,X_2}$ is always a nontrivial map into $\mathbb{C}$.

**Future work**

Find formula for or asymptotics of higher moments
Thank You!

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