**1. Introduction**

Fluid dynamics plays a defining role in shaping our understanding of the vast majority of physical processes unfolding throughout the cosmos. Indeed, the dominant composition of the current Universe in the form stars, accretion discs, galaxies, galaxy clusters and the all-pervasive gaseous medium between such overdensities, is at heart a vast ensemble of inhomogeneous, magnetized, gravitating bodies of fluid. Astrophysical fluid dynamics provides the overarching structure needed to describe large-scale processes at work in our Universe. However, the physics of magnetized fluids is a messy business. The theorist must contend with distressingly long mean-free-paths, turbulence, plasma instabilities, magnetic stresses, radiation physics, and the list goes on. Often there is little choice but to resort to a highly phenomenological formalism and, when, as often happens, the phenomenology takes on a life of its own, the distinction between guesswork and fundamentals is blurred. At the end of complex simulation, what has actually been explained? The answer, more often than not, is unsatisfyingly ambiguous.
In this article, we will attempt to shine a bit of light through these murky depths, highlighting a series of problems in a broad range of settings, all of which contain an enlightening twist. We have tried to be careful to distinguish fundamental physics from any assumptions that need to be made. We hope that the article will serve the needs both of seasoned researchers looking for something new, and of those who may be embarking on their own research programmes in astrophysical fluid dynamics, who would welcome a broader introductory review.

The basic idea for this article stems from a series of monographs by Sir Rudolf Peierls on the theme of surprises in theoretical physics [56, 57] which inspired the 2011 Spitzer Lectures given at Princeton University given by one of us (SAB). Peierls discusses a series of problems, all of which contain some kind of enlightening twist: a contradiction that really isn’t, something that looks hard but turns out to be simple (or vice-versa), a technique that looks ideal for the problem at hand and then crashes, a technique that looks like it is being egregiously misapplied that works beautifully. With welcome rare exceptions, standard texts do not spend much time exploring seemingly promising approaches that actually fail. This is a pity, because it is all too easy to be fooled. Just how easy will perhaps be made apparent in this article, via several examples from a wide variety of different subject areas.

More importantly, we hope that this article will serve to illustrate how much remains to be understood of what would seem to be the most basic properties of astrophysical gases. The coupling between heating, cooling and buoyancy, the inertial quirks of rotationally-dominated dynamics, and especially the insidious effects of weak magnetic fields can all play havoc with one’s physical intuition. Theorists are still in the process of learning first-hand just how counterintuitively magnetized fluids can behave.

Thus motivated, the authors have selected a set of fundamental problems of general interest, each of which has a story to tell. In a few cases, the problem discussed has turned out to be important, but not in all cases. (At least not yet!) We chose these examples not for their immediate impact (though in a few cases the ultimate impact has been considerable), but because we have found them to be generally illuminating of important physical principles, or because there was an intellectual novelty to the problem not widely appreciated, or just because they were interesting.

This then is a review in the spirit of Peierls’ books, a collection of problems from the dynamics of astrophysical fluids that a practising expert might be able to read with enjoyment, but which might also serve as a technical introduction for a motivated beginner or outsider. We make no apologies for the idiosyncratic choice of topics; the unusual blend of problems is deliberate. We have tried to ensure that the astrophysical content of all problems is as clear and as self-contained as possible.

We begin by way of review with the fundamental set of governing equations relevant to astrophysical gas flows. This is followed by the first of our problems, that of thermal evaporation of cool interstellar clouds by a hot ambient medium, a problem first treated in detail for isolated clouds in the classic work of [24]. After establishing an interesting analogy with ordinary electrostatics, we then move on to examine the problem of thermal instability in a medium that is subject to bulk heating and cooling. Another classic problem of interstellar medium theory [27, 28], thermal instability theory in a stratified medium turns out to be rich in surprises with insights afforded by thermodynamic identities and very complex dynamics.

Continuing in this thermal vein, we revisit the thermal conduction problem, this time with a weak magnetic field included. For the very dilute plasmas of interest in astrophysics, a magnetic field makes a huge difference to the way in which heat is conducted. Because of the tiny electron Larmor radius, heat flows only along the magnetic lines of force, even for a very weak magnetic field. The surprises here are stunning: a host of new dynamical instabilities, when there is more-or-less any temperature gradient in a stratified medium.

For larger field strengths in a stratified medium the Newcomb–Parker [47, 54] instability is an issue, and is the topic of the next section. The surprise here is that the strong magnetic field makes no difference to the Schwarzschild instability criterion, when written as a constraint on the vertical density gradient in a gravitational field. The twist is a very simple demonstration of this. Of course the classic calculation does not include thermal conduction along field lines, so it would be amiss not to revisit Newcomb–Parker with this in mind. Surprise—or not, depending on your prior inclination—the Newcomb–Parker stability criterion is dramatically altered.

Next is a sort of local ‘theory-of-everything’, at least in the linear adiabatic regime. By this we mean a derivation of a very compact vector equation for small, three-dimensional Lagrangian displacements in an axisymmetric but otherwise arbitrary magnetized, differentially rotating and stratified background. The equation is so powerful that many well-known problems—including the endlessly surprising magnetorotational instability—can be almost read off from it directly. Our final two problems are (i) a discussion of the remarkable Papaloizou–Pringle instability, which keeps arising, often unrecognized, in different settings, and reminds us of the crucial role that boundary conditions may play in regulating stability behaviour; (ii) a very simple possible explanation of what seems to be a very complicated problem: the rotation pattern of the solar convection zone. Here the realization that convection tends to occur in cells of constant angular velocity and constant entropy (crucially, however, with the unstable driving radial gradient subtracted off!) allows a direct analytic solution for the shape of the isorotation contours. A welcome surprise: turbulent flow need not be totally unfathomable.

2. Fundamental equations

For ease of reference, to establish notation, to highlight a few points of interest, and with apologies to the well-initiated, let us remind ourselves of the fundamental equations of astrophysical fluid dynamics (e.g. [65]) and their immediate consequences. Throughout this review, we use the notation \((x, y, z)\)
for Cartesian coordinates, \((R, \phi, z)\) for (radial, azimuthal, axial) cylindrical coordinates, and \((r, \theta, \phi)\) for (radial, colatitude, azimuthal) spherical coordinates.

2.1. Mass conservation

If \(V\) is a fixed volume in space and \(S\) its bounding surface, then the mass \(M\) within \(V\) changes with time only if there is a net flux of mass integrated over \(S\). With \(\rho\) equal to the mass density and \(v\) the local velocity field, we have

\[
\frac{dM}{dt} = \int_V \frac{\partial \rho}{\partial t} \, dV = - \int_S \rho v \cdot dS.
\]

(1)

Since \(V\) is an arbitrary volume, the divergence theorem gives immediately

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0,
\]

(2)

the equation of mass conservation. It is also customary to use Cartesian index notation, with \(i, j, k\) used to represent the \(x, y, \) and \(z\) variables. As usual, repeated indices are summed over unless otherwise explicitly stated. Thus, mass conservation may also be written

\[
\partial_t \rho + \partial_i (\rho v_i) = 0,
\]

(3)

where the subscripted \(\partial\) symbol denotes partial differentiation with respect to Cartesian coordinate \(x_i\).

It will prove very useful to introduce the Langrangian time derivative following a fluid element, denoted \(D/Dt\):

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla = \partial_t + v_i \partial_i.
\]

(4)

In terms of the Lagrangian derivative, the mass conservation equation becomes

\[
\frac{D \ln \rho}{Dt} = -\nabla \cdot v.
\]

(5)

2.2. Entropy equation

Let \(S\) be the entropy per particle. Up to an irrelevant additive constant, for an ideal gas

\[
S = \frac{k_B}{\gamma - 1} \ln P \rho^{-\gamma}.
\]

(6)

Here, \(k_B\) is Boltzmann’s constant, \(\gamma\) is the ratio of specific heats at constant pressure to that of constant density, and \(P\) is the gas pressure. Typically, \(\gamma\) is 5/3 for a monatomic gas and 7/5 for a diatomic molecular gas. In the \(\gamma \rightarrow 1\) isothermal limit,

\[
S = -k_B \ln \rho.
\]

To change the entropy of a travelling fluid element, explicit heat sources (or sinks) are required: for a given Lagrangian fluid element, \(TDs = dQ\), where \(dQ\) is the change of heat per particle. It is more usual to work with the heat per unit volume. For example, if \(F\) is the heat flux in units of energy per area per time, then the volume specific heating rate is \(-\nabla \cdot F\). The volume specific change in entropy is simply \(n dS\), where \(n\) is the number density of particles. Thus, a typical astrophysical gasdynamic rendering of the entropy equation will take the form

\[
\frac{P}{\gamma - 1} \frac{D(\ln P \rho^{-\gamma})}{Dt} = -\nabla \cdot F + \text{sinks},
\]

(7)

where the ideal gas equation of state \(P = n k_B T\) has been used. Possible sources might include ohmic or viscous heating, or high energy external particles. A common sink term is losses from radiation (see section 2.6 below).

In contrast to the mass equation, whose form rarely changes, the entropy equation must be constructed anew for each environment. Happily, in many applications, the adiabatic limit of zero heat exchange is a sufficiently good approximation to use.

2.3. Dynamical equation of motion

Newton’s law of motion ‘\(F = ma\)’ takes the form

\[
\rho \frac{Dv}{Dt} = -\nabla P - \rho \nabla \Phi - \nabla \cdot \Pi + J \times B.
\]

(8)

where \(\Pi\) is the viscous stress tensor whose precise form need not concern us here (it is a complex subtopic unto itself), but the term may very often be ignored. Magnetic fields make an appearance in the final \(J \times B\) Lorentz force term. Here \(J\) is the current density and \(B\) the magnetic field vector. In the problems that we will discuss, Maxwell’s displacement current may be ignored (since the velocities are non-relativistic), so that

\[
\mu_0 J = \nabla \times B,
\]

(9)

where \(\mu_0\) is the usual vacuum permeability, \(4\pi \times 10^{-7}\) in SI units. The Lorentz force becomes, using a standard vector identity,

\[
\frac{1}{\mu_0} (\nabla \times B) \times B = -\nabla \left( \frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} (B \cdot \nabla) B.
\]

(10)

This may be interpreted as magnetic pressure force arising from the first term on the right, plus a restoring tension force from the second term. The tension arises from the distortion of the magnetic field’s line of force. Such distortions can propagate along a field line in a manner precisely analogous to waves propagating along a taut string under tension. This mode of response is peculiar to a magnetized gas, allowing it to host shear waves with no corresponding change in the density or pressure. These disturbances are called ‘Alfvén waves’, which play a key role in the behaviour of astrophysical gases in many environments. The equation of motion without the viscous term now reads:

\[
\rho \frac{Dv}{Dt} = -\nabla \left( P + \frac{B^2}{2\mu_0} \right) - \rho \nabla \Phi + \frac{1}{\mu_0} (B \cdot \nabla) B.
\]

(11)

2.4. Conservation of vorticity

In the absence of magnetic fields, the equation of motion simplifies to
\[
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = \frac{Dv}{Dt} = -\frac{1}{\rho} \nabla P - \nabla \Phi. \tag{12}
\]

Let us apply this equation to the following curious little problem. What is the Lagrangian derivative of a small line element \(dl\) as it moves embedded in the velocity field? Clearly, if one moves with one end of the moving line element and watches what happens to the other end, there is a change in \(dl\) only if this other end moves in this \(\nabla\)tive sense. In that case, the change in \(dl\) after a time \(\Delta t\) is given by
\[
\Delta(dl) = (dl \cdot \nabla)v \Delta t, \tag{13}
\]
which gives
\[
\frac{D(dl)}{Dt} = (dl \cdot \nabla)v. \tag{14}
\]

Now, if in equation (12), we restrict ourselves to the case in which \(P\) is a function only of \(\rho\) (a ‘barotropic’ fluid), or if \(\rho\) is a constant, then the right side of equation (12) is a pure gradient, say \(\nabla \Phi\). Then
\[
\frac{D(v \cdot dl)}{Dt} = \frac{Dv}{Dt} \cdot dl + v \cdot \frac{D(dl)}{Dt} = dl \cdot \nabla(\Phi + v^2/2), \tag{15}
\]
where (12) and (14) have been used. If we integrate \(v \cdot dl\) around a finite loop moving with the fluid, then the rate of change of this integral vanishes:
\[
\frac{D}{Dt} \oint v \cdot dl = \oint \frac{D}{Dt} (v \cdot dl) = \oint dl \cdot \nabla(\Phi + v^2/2) = 0 \tag{16}
\]
since the line integral of a pure potential vanishes around a closed loop. Notice that this holds even if the interior of the loop violates our potential flow restriction, only the nature of the flow along the boundary curve matters. The line integral of the velocity \(v\) taken round a closed loop is known as the circulation, and we have just demonstrated the conditions under which it is conserved.

Calculating the rate of change of area or volume elements embedded in the flow is just a matter of repeated application of equation (13) and retaining first order terms in \(\Delta t\). For example, applying (13) to each of \(dx, dy,\) and \(dz\) in the volume element expression
\[
dV = (dx \times dy) \cdot dz \tag{17}
\]
and keeping terms linear in \(\Delta t\) leads to
\[
\frac{D(dV)}{Dt} = dV \cdot \nabla \cdot v. \tag{18}
\]

The area element
\[
dS = dx \times dy \tag{19}
\]
may also be treated exactly the same way, but the result is not quite as slick. It is best to use index notation. In this case, we find:
\[
\frac{D(dS)}{Dt} = dS_\alpha \partial_v^\alpha - dS_\partial \partial_v^\alpha. \tag{20}
\]

Then, for an arbitrary vector \(\omega\),
\[
\omega \cdot \frac{D(dS)}{Dt} = (\omega \cdot dS) \nabla \cdot v - [(\omega \cdot \nabla) v] \cdot dS. \tag{21}
\]

The utility of this result becomes apparent when we consider the curl of equation (12). We define the vorticity as
\[
\omega = \nabla \times v. \quad \text{Noting}
\]
\[
(v \cdot \nabla)v = \nabla \frac{v^2}{2} + (\nabla \times v) \times v, \tag{22}
\]
the curl of equation (12) yields the Helmholtz vorticity equation:
\[
\frac{D\omega}{Dt} + \nabla \times (\omega \times v) = \frac{1}{\rho^2} \nabla \rho \times \nabla P. \tag{23}
\]

As before, we restrict ourselves to constant density or barotropic flow, so that the right side of the equation vanishes. Then, expanding the left side of the equation produces
\[
\frac{D\omega}{Dt} = -\omega \nabla \cdot v + (\omega \cdot \nabla)v, \tag{24}
\]
whence we find immediately
\[
\frac{D(\omega \cdot dS)}{Dt} = dS_\cdot \frac{D\omega}{Dt} + \omega \cdot \frac{DdS}{Dt} = 0. \tag{25}
\]

The content of equation (23) is that \(\omega \cdot dS\) is conserved as it moves with the flow: any change in the vorticity contribution is compensated by kinematic changes in the projected area element. This also follows from our previous result of the circulation \(\oint v \cdot dl\) remaining constant moving with the fluid and an application of Stokes’ theorem. Note as well that equation (24) with \(-D\ln \rho/Dt\) substituted for \(\nabla \cdot v\) yields
\[
\frac{D}{Dt} \frac{\omega}{\rho} = \left( \frac{\omega}{\rho} \cdot \nabla \right) v, \tag{26}
\]
i.e. \(\omega/\rho\) satisfies the same equation as the embedded line element \(dl\), equation (14). Thus vortex lines, normalized by the density, are in effect ‘frozen’ into the fluid.

A revealing application of the Helmholtz vorticity equation is the case of time-steady pure rotation, \(v = R \Omega e_\phi\). Equation (23) then has only a \(\phi\) component, and simplifies to
\[
-R \frac{\partial \Omega^2}{\partial z} = \frac{1}{\rho^2} (\nabla \rho \times \nabla P) \cdot e_\phi. \tag{27}
\]

A \(z\)-gradient of \(\Omega\) is special. If there are no other fluid motions, such a gradient can be supported only if isobaric (constant pressure) and isochoric (constant density) surfaces deviate from one another. If, by contrast, such surfaces coincide, then the external forces are derivable from a gradient and \(\Omega\) can only depend on \(R\)—it is constant on cylinders. In this case, one speaks of ‘barotropic flow’. Otherwise, if there is any \(z\) dependence of \(\Omega\) and isochores deviate from isobars, the flow is said to be ‘baroclinic’. In stars, a tiny offset between isochoric and isobaric surfaces can produce significant baroclinic differential rotation, as in the Sun (see section 9).

2.5. The induction equation

The addition of the magnetic field \(B\) into the problem requires an additional equation governing its evolution. This is provided by the Faraday induction equation
\[ \nabla \times E = -\frac{\partial B}{\partial t}, \]  
(28)

where \( E \) is the local electric field in the gas. Under many circumstances of interest, the gas is an excellent conductor (a very small ionization fraction will generally suffice), but we must not set \( E = 0 \) as a consequence. Any particular fluid element will be in motion, and it is only in the local rest frame of the element that the electric field may be expected to vanish\(^4\). This means

\[ E + \mathbf{v} \times \mathbf{B} = 0, \]  
(29)

where \( E \) and \( \mathbf{B} \) refer to the ‘observer frame’ in which the fluid element has velocity \( \mathbf{v} \). (It will not have escaped the reader’s notice that this statement is the same as requiring that the Lorentz force on individual charges must vanish.) Putting these last two equations together by eliminating \( E \) leads to the induction equation:

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \]  
(30)

The induction equation allows us to calculate the evolution of the magnetic field. As in our discussion of vorticity, this may be written in the alternative useful form

\[ \frac{\text{d}B_\phi}{\text{d}t} = -B \nabla \cdot \mathbf{v} + (\mathbf{B} \cdot \nabla)\mathbf{v}. \]  
(31)

The magnetic field follows the same equation as \( \omega \), so that once again \( \mathbf{B}/\rho \) are material lines, embedded in the moving fluid.

In the case of purely axisymmetric rotation, \( \mathbf{v} = R \Omega \mathbf{e}_\phi \), and the field evolves according to

\[ \frac{\text{d}B_\phi}{\text{d}t} = R(\mathbf{B} \cdot \nabla)\Omega, \]  
(32)

whence

\[ B_\phi(t) = B_\phi(0) + R(t \mathbf{B} \cdot \nabla)\Omega, \]  
(33)

an example of shear drawing out an azimuthal field from a radial component. Finally, since \( \mathbf{B} \) is derivable from a vector potential \( \mathbf{B} = \nabla \times \mathbf{A} \), we may ‘uncurl’ (30) to obtain:

\[ \frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{A}) + \nabla \Phi, \]  
(34)

where \( \Phi \) is an unspecified potential function, which may be chosen for convenience (e.g. to eliminate \( \nabla \cdot \mathbf{A} \)). Expanding the cross product and switching to index notation

\[ \frac{\text{D}A_k}{\text{D}t} = v_i \delta_{ik}A_i + \partial_k \Phi. \]  
(35)

Then, recalling (14),

\[ \frac{\text{D}(\text{D}A_i)}{\text{D}t} = \partial_k (\Phi A_i v_i). \]  
(36)

In other words, \( \text{D}(\mathbf{A} \cdot \text{D} \mathbf{v})/\text{D}t \) integrated around a closed loop moving with the flow vanishes since it amounts to integrating a potential function over a closed path. This is precisely analogous to our treatment of the conservation of the circulation integral. By Stokes’ theorem (a ‘uncurl’ \( \mathbf{B} \cdot \text{dS} \) must also be conserved, where \( \text{dS} \) is the area element bounded by the closed loop, in analogy to \( \omega \cdot \text{dS} \). Magnetic flux is conserved in exactly the same manner as vorticity is conserved.

Note the manner in which the Maxwell equations are satisfied. The vanishing of the divergence of \( \mathbf{B} \) is assured if \( \nabla \cdot \mathbf{B} = 0 \) is an initial condition. The Faraday equation and Biot–Savart law are already explicitly invoked in the fluid equation formulation. Finally, although the charge density computed from \( \nabla \cdot \mathbf{E} \) is ignored in the equation of motion, this turns out to be a second order relativistic term—the same order of smallness that is ignored by dropping the displacement current term and using the Biot–Savart magnetostatic limit. Thus, to first order in the velocity (normalized to the speed of light), the Maxwell equations are all satisfied.

### 2.6. Energy conservation

The equations (2), (7), (11) and (30) are a complete description of ideal magnetohydrodynamics (MHD) in the sense that no additional field equations are required. Individual physical processes can be added as separate terms to these equations as needed, but as long as we treat our system as a single fluid no additional evolutionary equation is needed. It is nevertheless valuable to have an equation expressing the conservation of energy. Such an equation is of course not independent from our earlier set, and can in fact be derived from this group of four by taking the dot product of equation (11) with \( \mathbf{v} \) and simplifying. In conservation form, it reads:

\[ \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathbf{F}_E = 0, \]  
(37)

where the energy density \( \mathcal{E} \) is

\[ \mathcal{E} = \frac{\rho v^2}{2} + \frac{P}{\gamma - 1} + \rho \phi + \frac{B^2}{2\mu_0} \]  
(38)

and the energy flux \( \mathbf{F}_E \)

\[ \mathbf{F}_E = \left( \frac{\rho v^2}{2} + \frac{\gamma P}{\gamma - 1} + \rho \phi \right) \mathbf{v} + \frac{1}{\mu_0} \mathbf{B} \times (\mathbf{v} \times \mathbf{B}) + \mathbf{F}. \]  
(39)

The magnetic field appears in \( \mathbf{F}_E \) as part of the electromagnetic Poynting flux. We have retained a thermal flux \( \mathbf{F} \), but have ignored viscous and resistive processes as well as external heating or cooling.

In diffuse astrophysical gases, nonadiabatic heating and cooling processes can operate on time scales of practical interest. We therefore introduce the net loss function \( \mathcal{L} \), with dimensions of energy per second per unit mass; \( \rho \mathcal{L} \) is then the cooling per unit volume. \( \mathcal{L} \) is generally a function of two thermodynamic variables, \( \rho \) and \( T \) say. It is ordinarily not a function of spatial gradients, which instead appear explicitly elsewhere in the equations (for example, \( \mathbf{F} \) above). On the other hand, \( \mathcal{L} \) could include bulk heating, for example, in the form of impulsive collisions of the thermal medium by high energy particles.
energy particles. The physics of the cooling process might be electron–ion thermal bremsstrahlung or inelastic electron–ion collisions. A typical form for $\mathcal{L}$ is given by

$$\rho \mathcal{L} = n^2 \Lambda(T) - n \Gamma,$$

(40)

where $\Lambda$ depends only on the temperature of gas and details of the atomic processes involved, and $\Gamma$ is a heating rate. In classical applications, the heating is often due to cosmic rays, in which case it is not sensitive to the properties of the thermal gas. The number density $n$ could be chosen to be the electron density in an ionized gas, or the dominant species in a predominantly neutral gas. The fact that $\Lambda$ as written depends only upon $T$ is a reflection of the assumption of thermodynamic equilibrium between the ions and electrons, a very important restriction indeed. It is often, but not always, valid in cases of interest. We will not treat the much more complicated case of out-of-equilibrium gases here.

Amending the entropy and energy equations to include bulk losses, we find

$$\frac{P}{\gamma - 1} \frac{D(\ln P \rho^{-\gamma})}{Dt} + \nabla \cdot \mathbf{F} = -\rho \mathcal{L},$$

(41)

and

$$\frac{\partial \xi}{\partial t} + \nabla \cdot \mathbf{F}_E = -\rho \mathcal{L}.$$

(42)

2.7 The Boussinesq and anelastic approximations

The Boussinesq limit, or Boussinesq approximation, is a powerful simplification which allows progress to be made by filtering out extraneous compressive modes, generally acoustic waves, from the system under study. It often causes confusion, however, as the name is used in at least two different ways. A small digression is appropriate.

The limit in which the Boussinesq approximation applies is that of constant density everywhere, except in the buoyancy term of the equation of motion. In this case, the equation of mass conservation reads

$$\nabla \cdot \mathbf{v} = 0 \quad \text{(Boussinesq).}$$

(43)

The reason for confusion is that the Boussinesq approximation is used, quite correctly, in situations where the density is not constant.

To set the stage, we write the general mass conservation equation in the form

$$\varepsilon \frac{D \ln \rho}{Dt} + \nabla \cdot \mathbf{v} = 0,$$

(44)

where $\varepsilon$ has been inserted as a small parameter to remind us that the density changes are, in a sense we need to make more precise, small. The density changes in this equation might be small in the relative sense that the spatial gradients of the velocity are very large, or in an absolute sense: the change in the relative density of a liquid is likely to be ignorable even when the velocity gradients involve the longest global length scales. In either case, the leading order Boussinesq approximation is that the velocity divergence vanishes.

In its classical implementation, the Boussinesq approximation is really a Lagrangian statement: when displaced, a fluid element maintains its density. For this statement to have any content, the background density gradient cannot be exactly zero. We can, however, impose the condition exactly that a fluid element’s density is constant. A vertical salinity gradient in seawater, for example, will give rise to a very small density gradient, say $\rho'(z)$. Then, a strictly vertical displacement $\xi$ of a fluid element will produce a density change relative to the element’s new surroundings of $\delta \rho = -\xi \rho'$. The displacement creates a buoyancy force

$$\frac{\delta \rho}{\rho^2} P'(z),$$

where $P'$ is the background pressure gradient due to the presence of a gravitational acceleration $g$, $P' = -\rho g$. (Here and above $\rho$ is treated as an approximate constant). Thus, the equation of motion for $\xi$ is

$$\dot{\xi} = \frac{g \rho'}{\rho} \xi.$$ 

(45)

With $\rho' < 0$, this gives rise to buoyant oscillations with an oscillation frequency $\omega^2 = -g \rho'/\rho$ [41]. When heated from below, a fluid may locally invert the sign of $\rho'$, in which case $\omega^2 < 0$ and unstable thermal convection ensues.

There is an appealing internal consistency in this example: $\mathbf{D} \rho/\mathbf{D}t$ is set to zero, which is ‘as it should be’ if $\nabla \cdot \mathbf{v} = 0$. In astrophysical settings, gas dynamical considerations prevail, not liquid water, and $\mathbf{D} \rho/\mathbf{D}t$ is clearly not a vanishing quantity. Instead it is the entropy $S$ that is modelled as strictly conserved with moving fluid elements in the rigorous adiabatic limit $\mathbf{D} s/\mathbf{D}t = 0$, where $s = \ln P \rho^{-\gamma}$. How then, may we justify the constraint $\nabla \cdot \mathbf{v} = 0$?

The answer is more WKB (named for physicists Wentzel, Kramers and Brillouin) than Boussinesq in spirit, though the vanishing velocity divergence condition is often informally referred to in the astrophysical literature as ‘Boussinesq’. The often-made WKB approximation consists of assuming that over the scale of a perturbation, the change in the background quantities is negligible. This is also referred to as the local approximation. The point is that the combination of a very small perturbation length scale and a long time scale leads to a vanishing velocity divergence, but only to leading order. $\mathbf{D} \ln \rho/\mathbf{D}t$ should then be interpreted as the error incurred.

Moreover, displaced fluid elements remain very nearly in pressure equilibrium, differing only to the extent that the (very short) sound crossing time over a wavelength is comparable to other times of interest. Relative pressure perturbations may be thus be ignored compared with relative density perturbations, and then the adiabatic constraint becomes $\gamma \delta \rho = \xi \sigma'$. The frequency of oscillation emerges as $\omega^2 = -g \sigma'/\gamma$, a quantity known as the Brunt–Väisälä frequency. We recover the classical Boussinesq limit by letting $\gamma \rightarrow \infty$.

What might be done to eliminate acoustic oscillations if the background equilibrium density is very strongly stratified in position, as is often the case in stars? One approach, known as
the anelastic approximation is equivalent to the mass conservation asymptotic regime

\[ \epsilon \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \]  

(46)

Note the difference with equation (44): the Lagrangian change was regarded as small, here it is the explicit Eulerian time dependence that is small. The anelastic approximation amounts to writing

\[ \rho(x, t) = \rho_0(x) + \epsilon \rho(x, t) \]  

(47)

and regarding the leading order mass conservation equation as

\[ \nabla \cdot (\rho_0 \mathbf{v}) = 0. \]  

(48)

The spatial dependence of \( \rho_0 \) is of course retained. The anelastic approximation is used extensively in simulations of the convective zone in the Sun [70] and other stars.

The foundations of the anelastic approximation are less firm than those of the Boussinesq limit, however. When is the velocity large enough to justify dropping the lead time derivative? The velocity field may itself be linear in \( \rho_0 \partial \rho_0 \). Moreover, it has recently been pointed out that the standard anelastic approximation, lacking the density time derivative, is not consistent with an energy conservation formulation. Efforts continue as of this writing to include modifications to render the anelastic approximation compatible with this requirement (e.g. [17]).

Throughout this paper we will work, when needed, in the Boussinesq \( \nabla \cdot \mathbf{v} = 0 \) limit, appropriate to large wavenumber disturbances in a stratified background.

### 3. Thermal conduction–electrostatic analogy

#### 3.1. Preliminaries

A common astrophysical environment involves high density gas concentrations embodied within a more diffuse medium. (Interstellar gas is a good example of this sort of ‘cloudy medium’). If the high density clouds are in approximate pressure equilibrium with their environment, they must be cool relative to the warmer confining gas. Heat will diffuse towards the cloud surface by thermal conduction from the hot surroundings, and if the clouds are small, the concentrated flux will evaporate the surface layers. The outer layer will heat faster than it is able to cool. It must then expand, and flow outward in a wind. (See figure 1.) An analogous process has been studied in tokamaks: pellets containing a deuterium-tritium ice are ablated by laser heating, and the evaporative outflow in surrounding, and if the clouds are small, the concentrated flux will evaporate the surface layers. The outer layer will heat faster than it is able to cool. It must then expand, and flow outward in a wind. (See figure 1.) An analogous process has been studied in tokamaks: pellets containing a deuterium-tritium ice are ablated by laser heating, and the evaporative outflow in

We begin with the calculation of the mass loss rate \( \dot{m} \) from an isolated spherical cloud of radius \( R \) immersed in a hot gas at a given ambient temperature [24]. In its simplest form, the heat flux \( \mathbf{F} \) is given by [67]

\[ \mathbf{F} = -\kappa \nabla T_e, \]  

(49)

where \( T_e \) is the electron temperature and \( \kappa \) is the thermal conductivity coefficient, itself highly temperature dependent:

\[ \kappa \approx 6 \times 10^{-7} T_e^{5/2}. \]  

(50)

The units of \( \kappa \) are erg cm\(^{-1}\) s\(^{-1}\) K\(^{-1}\). (Note, because of the specialized nature of the material, in this section only we will use cgs and esu units).

The evaporation rate from a spherical cloud in an ambient gas at temperature \( T_h \) was first worked out by [24]. Gravity is negligible in this problem. If the heat flux is given by this simple diffusion approximation, the evaporation rate is

\[ \dot{m} = \frac{16\pi \kappa B R}{25B}, \]  

(51)

where \( \kappa_b = \kappa(T_b) \) is the value of the thermal conductivity coefficient in the ambient gas, \( \mu \) is the mean mass per particle, and \( B \) is the Boltzmann constant. Note the interesting result that the rate is not proportional to the surface area presented by the cloud, as one might naively expect, but to the radius. There is a deep and interesting reason for this on which we shall have much to say in the next section.

The precise numerical prefactor in equation (51) depends upon the assumption of a spherical cloud (which is of course an idealization) and the fact that the thermal conductivity due to Coulomb collisions is proportional to the temperature \( T \) to the 5/2 power. The origin of this particular value is not difficult to understand. The characteristic length at which the kinetic and potential energies of two electrons are equal is \( r_c \sim e^2/k_B T \), where \( e \) is the electron charge. The Coulomb cross section \( \sim \pi r_c^2 \) will then have a 1/T\(^2 \) dependence, and the mean free path \( \lambda_{ei} \) should then scale as \( T^2/n_e \), where \( n_e \) is the number density of electrons. The diffusion of the heat flux in the small mean free path limit is proportional to

\[ (m,n_e)C_e \lambda_{ei} \nabla T_e^2 \sim \frac{T_e^{5/2}}{\sqrt{m_e} \nabla T}, \]  

(52)

where \( c_e^2 = k_B T_e/n_e \) is the square of the electron thermal velocity. This shows the origin of the 5/2 power and justifies the assumption that the electrons are the heat carrying population—at least when the electron and ion temperatures are equal.

The precise calculation of the thermal conductivity coefficient \( \kappa \) in an ionized gas is by no means simple, involving a detailed kinetic treatment of the electron distribution function and delicate handling of a logarithmically-divergent integral (the ‘Coulomb logarithm’) in the course of calculating the scattering cross section. The problem was worked out in complete detail in the 1950s by Lyman Spitzer and his colleagues [67]. For our purposes, it will suffice to use equation (50). (This number actually has a very weak temperature and density dependence stemming from a formal ‘Coulomb-logarithm’ term. The value selected in equation (50) is appropriate for most dilute astrophysical plasmas.)

#### 3.2. Mass loss as capacitance

Even with a nonlinear temperature dependence, the thermal evaporation problem can be cast in a form in which it is entirely analogous to classical electrostatics. This is our first surprise, and it is a pleasant one, for we may then bring to bear on evaporation problems a powerful mathematical formalism and an intuition honed from electrostatic potential theory.
A unique solution is obtained by specifying the temperature in the ambient gas (i.e. at ‘infinity’) and on each cloud. These are classical Dirichlet boundary conditions. Clearly, $T^{5/2}$ must be proportional to the electrostatic potential $\Phi$ in this analogy. The constant of proportionality is set by demanding that the electric field analogue $-\nabla \Phi$ be our mass flux:

$$\rho \mathbf{v} = \frac{4\mu \chi}{25k_B} \nabla T^{5/2} = -\nabla \Phi,$$

where $k_B$ is the Boltzmann constant and $m$ the mean mass per particle. Hence,

$$\Phi = -\frac{4\mu \chi}{25k_B} T^{5/2}.$$

If we now use the analogue of Gauss’ Law (‘the integral of $E \cdot dS$ over a closed surface is $4\pi$ times the enclosed charge’), then

$$m = \int \rho \mathbf{v} \cdot dS = 4\pi C \Delta \Phi,$$

where we have represented the enclosed charge as the product of the capacitance $C$ and the potential difference $\Delta \Phi$ between the common ground of the cloud surfaces and infinity. (The clouds are all supposed to be cold, or at zero common potential, relative to the hot surroundings.) Thus, with $T_h$ the temperature of the hot gas far from the clouds,

$$\Delta \Phi = \frac{4\mu \chi}{25k_B} T_h^{5/2},$$

hence [4],

$$m = \frac{16\pi \mu \chi C}{25k_B} T_h^{5/2}.$$

As for $C$, this must be the electrostatic capacitance of ... exactly what?

By way of an answer, note that in contrast to our potential and electric field, $C$ is not some mathematical analogue quantity, it really is the actual capacitance in esu units, a length. We are, in effect, solving the standard Dirichlet problem of a system of conductors with specified surface potentials (all zero in this case) and given potential at infinity. If we were calculating the evaporation of a lone spherical cloud, for example, then $C = R$, the cloud radius [24]. Then the result (61) for $m$ is exactly the expression (51). But why should we limit ourselves to a single evaporating spherical cloud? What about a hemispherical shell of radius $R$? Then, $C = R(1/2 + 1/\pi)$. Two spherical clouds of radius $R$ in contact? $C = 2R \ln 2$.

While one can look up the esu capacitance for any particular shape in a compiled table and thereby determine how rapidly the corresponding gas cloud would evaporate (e.g. [66]), the real value of equation (61) is not this handy convenience. It is the more general insight that $T^{5/2}$ and $m$ behave respectively as a potential and a capacitance in complex systems percolating with clouds.

### 3.3. Evaporation in a cloudy medium

One consequence of the electrostatic analogy is the remarkable (and generally unappreciated) Faraday cage behaviour
of a cloudy medium. Just as a Faraday cage can shield its interior from external fields when the cage wires cover only a tiny fraction of the effective surface, a very low filling factor of clouds in a two-phase medium will profoundly affect the system’s thermal behaviour. The easiest way to compute this effect is to imagine the mathematically equivalent but reverse problem of hot spherical clouds (radius $R$, temperature $T_h$) embedded in a cold $T = 0$ medium. An isolated cloud would exhibit the ‘monopole’ temperature profile

$$T_{5/2}(r) = \frac{T_h^{5/2}R}{r}. \quad (62)$$

When does the assumption of isolation break down? It breaks down when, at a typical cloud’s surface, the superposed contribution from all other clouds results in a net augmentation of $T_{5/2}$ that is comparable to $T_h^{5/2}$.

Consider a spherical medium of radius $X$ which is actually comprised of individual embedded spherical clouds, each of radius $R$. The volume fraction of the medium filled by clouds, hereafter the ‘filling factor’, is $f$. Then, the number of clouds per unit volume is $3f/4\pi R^3$. The condition for a cloud near the center of the system to be marginally isolated from the other clouds may then be defined as

$$\int_0^X 4\pi r^2 (3f/4\pi R^3) (R/r) \, dr = 1 \quad (63)$$

or

$$f = 2R^2/3X^2 \quad (sphere). \quad (64)$$

The same calculation for a gross slab of vertical thickness $h$ and radius $X$ gives

$$f = 2R^2/3Xh \quad (slab). \quad (65)$$

These are much more stringent criteria than the naive guess $f \ll 1$! Indeed, in most astrophysical applications they are far from satisfied. We are facing a sort of ‘Olbers paradox’, in which the superposed small effects from numerous distant sources overwhelms the local contribution.

What are the consequences of this? The primary physical consequence is that if cool clouds are evaporating into a high medium at temperature $T_h$, the process unfolds in two stages. The first is evaporation from the clouds into a local intercloud medium with a temperature at some value intermediate between the cool cloud temperature (effectively zero) and $T_h$. Let us call this temperature $T_i$. The potential function $T_i^{5/2}$ in our electrostatic analogy cannot obtain an extremal value in the interior of its domain since it satisfies the Laplace equation. Accordingly, we expect little variation in $T_i$ within this intercloud region. The second stage of the evaporation is that this intercloud medium at temperature $T_i$ is heated by an extended hot phase at temperature $T_h$, driving a large scale evaporative outflow. (See figure 2.)

The total change in the temperature potential $T_i^{5/2}$ is the change going from the clouds $T_i^{5/2} = 0$ to $T_i^{5/2} = T_h^{5/2}$ in the immediate intercloud medium, and thence to $T_h^{5/2}$. In terms of the system capacitance $C_s$ of the ensemble of clouds and the gross capacitance $C_g$ of the large scale shape assumed by the ensemble,

$$\frac{m}{C_s} = \frac{m_c}{R} + \frac{m}{C_g}, \quad (66)$$

where $m_c$ is the evaporation rate of an individual cloud. But if $N$ is the total number of clouds, then $Nm_c = \dot{m}$ since mass is conserved. Hence

$$\frac{1}{C_s} = \frac{1}{NR} + \frac{1}{C_g}. \quad (67)$$

As though they were components in a laboratory circuit, the individual cloud capacitances first add in parallel, then in series with a capacitance $C_g$ (Note: this is unlike resistors!) Expressed in terms of the filling factor $f$ and gross system volume $V$, $C_s$ is thus given by

$$C_s = \frac{C_g}{1 + 4\pi R^2 C_g/(3V)}. \quad (68)$$
For the specific case of the gross slab geometry we introduced above,

\[ C_s = \frac{2\pi f X}{1 + 8R^2/(3\pi f X h)}. \]  

(69)

As a practical matter, an astrophysical cloudy medium in (say) the disc of a galaxy, would evaporate in a very hot gas as though there were no clouds at all, as if the entire disc were a continuum of gas \((C_s = C_g).\) This is the first surprise. The second surprise is that radiative losses, which are often unimportant when considering the thermal evaporation of a single cloud, make a great impact on the cloudy medium problem—one that is still not well understood.

The intermediate temperature \(T_I\) may be determined by mass conservation: the total mass evaporation rate from all the clouds into the gas at \(T = T_I\) must equal the evaporation rate from \(T_I\) into the gas at \(T = T_h.\) If there are total of \(N\) spherical clouds in our system, then

\[ NRT_I^{5/2} = C_s T_h^{5/2}. \]

(70)

Once again, consider a slab galaxy with a cloudy medium. Using \(C_s = 2X/\pi,\) the intermediate temperature gas has a temperature of

\[ T_I = \left[ \frac{8R^2}{3\pi f X h} \right]^{2/5} T_h. \]

(71)

Typical values might be 1 pc for \(R, 2 \times 10^4\) pc for \(X, 100\) pc for \(h,\) and \(10^{-2}\) for \(f.\) With a hot gas temperature of \(T_h = 10^8\) K expected for the intracluster medium of a rich x-ray cluster, \(T_I\) is a few million degrees. While this value fits nicely into the intermediate asymptotic regime which is both very large compared with the cloud temperature and very small compared with \(T_h,\) it is not a temperature regime that can maintain itself for long: radiative losses generally cannot be offset by conductive heating.

We return once again to our canonical evaporation problem of a slab of clouds immersed in a very hot gas at \(T_h = 10^8\) K, a disc galaxy in a hot intracluster medium. The clouds start to evaporate as per above, rapidly driving the hot gas out of immediate contact with the clouds, replacing it with gas at \(T = T_I \sim 10^8\) K. Thus far, all is well.

But now the \(10^8\) degree gas starts to cool (by thermal bremsstrahlung and atomic collisions) more rapidly than it evaporates. Our solution is not quite self-consistent: the \(T_I\) gas loses pressure support as it cools, and the ambient \(T_h\) gas re-enters the galaxy! This now creates fresh \(T_I\) gas, and our problem simply repeats.

The problem seems to be intrinsically time-dependent, and it is yet to be solved in any quantitative sense. The most likely scenario is one in which after an initial transient (assuming the clouds do not all evaporate), the gas settles into an evolving outwardly-moving, mass-loaded (because of ongoing evaporation), cooling conduction front. How such a system would manifest itself observationally is an interesting astrophysical question.

The surprises here are several. A seemingly complicated problem (the thermal evaporation of many clouds at once) turns out at first to be simple (ordinary electrostatics) but in the end complicated again (because of radiation effects). But the key insight that flows from clouds have important long range interactions even at low filling factors is one that is very likely to survive beyond the idealizations we have adopted here.

4. Thermal instability

4.1. Preliminaries

A fascinating and important property of a gas subject to bulk heating and cooling is the tendency for its thermodynamic behaviour to be unstable. The underlying cause is easy to understand. Imagine a slightly overdense region embedded in pressure equilibrium in a surrounding gas. The increased density leads to a higher collision rate among the constituent gas particles, which in turn leads to a higher rate of thermal energy loss. If this enhanced loss prevails over any corresponding enhanced heating (as it generally does: heating depends less sensitively on the density), the overdense region becomes yet more dense as it is compressed by the surrounding medium. The losses become yet more rapid and the process runs away.

This notion was first put on a quantitative footing by [27].

In terms of the radiative loss function \(L,\) Field showed that the thermal instability in the manner described above occurs when

\[ \left( \frac{\partial L}{\partial T} \right)_P < 0. \]

(72)

In retrospect, this is a rather obvious mathematical rephrasing of the physical description that precedes the equation. Note the crucial point that the temperature derivative must be taken with the pressure \(P\) held constant; the loss function is more naturally given as a function of density \(\rho\) and temperature \(T.\) Fixing \(P\) in the temperature derivative reflects the constant pressure conditions imposed by the ambient medium.

But are matters really so simple? After all, astrophysical gases do not, in general, pervade the universe homogeneously. Rather, they are trapped and held in gravitational potential wells. Under such conditions, an overdensity will not stay put as it grows, it will fall down, like all heavy objects, in the direction the gravitational field is pointing! As it falls, the element of fluid does not keep its pressure fixed, the pressure grows to match that of the surrounding fluid at lower depths. Indeed, the growing pressure can squeeze the element and adiabatically heat it to the point at which the element becomes warmer and thus less dense than the nonadiabatic surroundings. We no longer have an overdensity, we have an underdensity that will rise back up in the opposite direction of the gravitational field. In other words, the thermal instability is trying to play out in a fluid element that is actually undergoing buoyant oscillations in the gas. These oscillations will generally be more rapid than the timescale associated with the cooling, so to leading order it is the entropy \(S\) of the fluid element that remains constant as the fluid element evolves, not the pressure \(P!\) Should our criterion be

\[ \left( \frac{\partial L}{\partial T} \right)_S < 0 \]

(73)
for instability? This is a very different, and in practice much more difficult, inequality to satisfy. The surprises begin.

Suppose, for example that the cooling function had the canonical form

\[ L = A \rho T^d - B, \quad (74) \]

where \( A \) and \( B \) are constants and \( d \) is a positive real number less than but of order unity. Then

\[ \left( \frac{\partial L}{\partial T} \right)_p = (d - 1)A \rho T^{d-1} < 0. \quad (75) \]

But with the entropy \( S \) held constant,

\[ \left( \frac{\partial L}{\partial T} \right)_S = \left( d + \frac{1}{\gamma} - 1 \right)A \rho T^{d-1} > 0. \quad (76) \]

The two results are clearly inconsistent with one another.

To confuse the issue thoroughly, we offer the following plausible argument [26], which suggests that we had the criterion correct the first time, in equation (72). Consider a convectively stable oscillating fluid element at the peak of its upward (against the sense of the gravitational field) displacement. As before, it is in pressure equilibrium with its surroundings. Assume that the buoyant oscillations are growing in amplitude with time, due to thermal instability. Then, cooling at the maximum upward displacement will be enhanced relative to the equilibrium \( L = 0 \) ambient surroundings. But the element’s temperature \( T \) must be less than the surroundings, consistent with the sense of the downward buoyant force on this phase of the oscillation. Imagine now crossing from the ambient medium into the fluid element along an isobar (constant \( P \)). The change in \( L \) will be positive, the change in \( T \) negative. In other words, equation (72) must be satisfied for thermal instability, even when the instability is actually an oscillating over stability.

So what is going on here? The problem is the tacit assumption of the existence of convectively stable buoyant oscillations, independently of whether Field-style thermal instability is present or not. The surprise is that this assumption is incorrect.

4.2. Eulerian and Lagrangian perturbations

4.2.1. A useful digression. To probe more deeply, we need to sharpen the distinction between perturbations affecting a particular fluid element, and perturbations referring to a particular spatial location. The former are Lagrangian perturbations, the latter are Eulerian perturbations [42, 64]. Let \( \xi(t) \) be the vector field representing the displacement length of a fluid element located at \( r \) in equilibrium, i.e. the displaced fluid element is located at \( r + \xi \). The displacement is considered to be small compared with all scale heights of interest. To be definite, we work here with the cooling function \( L \), but the results apply to any flow quantity of interest, including vector field components. The equilibrium cooling function will be denoted simply as \( L \), the perturbed cooling function as \( L' \). The Eulerian change in \( L \), denoted \( \delta L \), is

\[ L'(r, t) - L(r) = \delta L. \quad (77) \]

The Lagrangian change in the \( L \), denoted \( \Delta L \), is then, to leading order in \( \xi \):

\[ L'(r + \xi, t) - L(r) = \Delta L = L'(r, t) - L(r) + (\xi \cdot \nabla) L. \quad (78) \]

where in the final term we have used the equilibrium \( L \), the distinction with \( L' \) introducing only higher order corrections. Abstracting the key equality from the above,

\[ \Delta L = \delta L + (\xi \cdot \nabla) L, \quad (79) \]

our fundamental relation between Lagrangian and Eulerian perturbations. We will use this definition for both scalar and vector functions alike, taking care to include any unit vectors in the \( \xi \cdot \nabla \) derivative.

What is the relationship between the Eulerian radial velocity disturbance \( \delta v \) and the Lagrangian change \( \Delta v \)? Here some care is needed. The Lagrangian change of the velocity field \( \delta v \) is by definition the difference between the time derivative of the displacement of a perturbed element and its unperturbed counterpart, i.e. \( \Delta v = D\xi / Dt \).

Hence,

\[ \frac{D\xi}{Dt} \equiv \frac{\delta L}{\partial t} + (v \cdot \nabla) \xi = \delta v + (\xi \cdot \nabla) v, \quad (80) \]

the latter equality following from (79). In other words,

\[ \delta v = \frac{\partial \xi}{\partial t} + (v \cdot \nabla) \xi - (\xi \cdot \nabla) v, \quad (81) \]

an important result.

To illustrate the power of the Lagrangian approach as well as to derive some powerful equations that we shall use throughout this work, consider the equations of mass conservation, (adiabatic) entropy conservation, and the MHD induction equation of a perfect conductor. These are all of the form

\[ \frac{\partial Q}{\partial t} = L(v), \quad (82) \]

where \( Q \) is a non-velocity flow attribute, and the operator \( L \) is linear in the velocity \( v \), and independent of time derivatives. In equilibrium, which of course need not be static, \( L(v) = 0 \). Both \( Q \) and \( L \) may be either scalar or vector quantities.

Imagine continuously deforming an equilibrium solution, altering the structural variables (i.e. pressure, density, magnetic field) by adding a finite velocity field \( w \) (not necessarily small compared with \( v \)) to the unperturbed velocity \( v \), then tabulating the brief evolution over an infinitesimal time interval \( \delta t \). This will normally cause a small change in \( Q \), as per the governing equation (82). Since the partial derivative \( \partial Q / \partial t \) is Eulerian, taken at a fixed spatial location, it may equally well be regarded as the Eulerian change \( \delta Q \) induced by the deformation, divided by the interval \( \delta t \). Using the fact that \( L(v) = 0 \), and that \( L \) is linear in the velocity and free of time derivatives,

\[ \delta Q = [L(v + w)] \delta t = [L(v) + L(w)] \delta t = 0 + L(w) \delta t = L(w \delta t) = L(\xi), \quad (83) \]

where \( \xi \equiv w \delta t \) is the infinitesimal displacement of a fluid element associated with the continuous deformation. From this...
general result follows three equations of constraint, in the sense that they display no explicit time evolution, but instead embody the Lagrangian conservation of mass, entropy, and magnetic flux. The relations allow a direct calculation of the Eulerian changes associated with fluid displacements:

\[ \delta \rho = - \nabla \cdot (\rho \xi) \]  

(84)

\[ \delta \ln P \rho^{-\gamma} \equiv \left( \frac{\delta P}{P} - \frac{\gamma}{\gamma - 1} \frac{\delta \rho}{\rho} \right) \equiv - \xi \cdot \nabla \ln P \rho^{-\gamma}, \]  

(85)

\[ \delta B = \nabla \times (\xi \times B). \]  

(86)

These may also be derived directly from the fundamental equations and the definitions of Eulerian and Lagrangian perturbations, but with considerably more effort.

In this work, we will be working in the WKB limit of local plane wave behaviour \( \exp(ik \cdot r) \), with \( |k| \) large compared with reciprocal equilibrium scale heights. We will also exclude acoustic disturbances by working in the Boussinesq limit, meaning here that the terms comprising \( \nabla \cdot \xi \) divergence are large compared with \( \xi \cdot \nabla \rho \) in (84). This also means that disturbances are very close to pressure equilibrium with their surroundings, so that \( \delta \ln P \) may be neglected in comparison with \( \delta \ln \rho \). The pressure balance argument should include a contribution from the magnetic field as well, but our interest will be in those cases in which the magnetic field is sufficiently weak that its pressure contribution may be ignored—but not its tension. (An explicit exception to this restriction is the Newcomb–Parker problem, section 6.) To leading order, our three equations then simplify to

\[ \nabla \cdot \xi = 0, \]  

(87)

\[ \delta \rho = \frac{1}{\gamma} \xi \cdot \nabla \ln P \rho^{-\gamma}, \]  

(88)

\[ \delta B = i(k \cdot B) \xi. \]  

(89)

These three equations of constraint are extremely useful across a wide variety of fluid problems.

4.3. When does thermal instability mean convective instability?  

We return to the problem at hand. For a medium whose equilibrium cooling is described by \( \mathcal{L} = 0 \), the distinction between \( \delta \mathcal{L} \) and \( \Delta \mathcal{L} \) is lost, and it is an interesting fact that the earlier considerations presented above may now be distilled to a single apparently trivial equation:

\[ (\Delta \mathcal{L})_S = (\delta \mathcal{L})_p. \]  

(90)

What this says is the following. Follow a local fluid element (a ‘blob’), initially in an \( \mathcal{L} = 0 \) state, remaining in pressure equilibrium with its surroundings, on its adiabatic upward displacement. (As before, ‘upward’ means against gravity. Downward displacements work too.) Once displaced, the blob’s local thermodynamic variables no longer satisfy \( \mathcal{L} = 0 \); there is some \( \Delta \mathcal{L} \). But \( \Delta \mathcal{L} \) due to this adiabatic displacement must be the same change in \( \mathcal{L} \) that would be found by crossing from the immediate surrounding undisturbed \( \mathcal{L} = 0 \) medium (at the same pressure) into the displaced fluid element. By either the Lagrangian adiabatic or Eulerian isobaric route, we start at \( \mathcal{L} = 0 \) and end up in the same blob with the same new value of \( \mathcal{L} \). From this obvious little equation follows everything.

Equation (90) may be written

\[ \Delta T \left( \frac{\partial \mathcal{L}}{\partial T} \right)_S = \delta T \left( \frac{\partial \mathcal{L}}{\partial T} \right)_p. \]  

(91)

For an adiabatic displacement,

\[ \frac{\Delta T}{T} = \frac{1}{\gamma} (\gamma - 1) \frac{\Delta P}{P} = \frac{1}{\gamma} (\gamma - 1) \left( \frac{\delta P}{P} + \xi \frac{\delta \ln P}{\delta r} \right), \]  

(92)

where \( \xi \) is the radial component of the displacement. (We are assuming that the background is spherically symmetric in radius \( r \).) Thus, since \( \delta P = 0 \),

\[ \frac{\Delta T}{T} = \frac{\xi}{\gamma} (\gamma - 1) \frac{\partial \ln P}{\partial r}. \]  

(93)

On the other hand,

\[ \frac{\delta T}{T} = \frac{\Delta T}{T} - \xi \frac{\partial \ln T}{\partial r}. \]  

(94)

Putting the last two equations together with \( \delta \ln T = \delta \ln P - \delta \ln \rho \),

\[ \frac{\delta T}{T} = \frac{\xi}{\gamma} \left( \frac{\partial \ln P}{\partial r} + \frac{\partial \ln \rho}{\partial r} \right). \]  

(95)

When used in (91), equations (93) and (95) lead to

\[ - (\gamma - 1) \frac{\partial \ln P}{\partial r} \left( \frac{\partial \mathcal{L}}{\partial T} \right)_S = \frac{\partial \ln \rho}{\partial r} \left( \frac{\partial \mathcal{L}}{\partial T} \right)_p. \]  

(96)

Finally, assuming the medium is in hydrostatic equilibrium with a gravitational field \( g \) satisfying \( -\rho g = \partial \mathcal{P}/\partial r \), our equation becomes

\[ (\gamma - 1) \frac{g}{c^2_s} \left( \frac{\partial \mathcal{L}}{\partial T} \right)_S = \frac{\partial \ln \rho}{\partial r} \left( \frac{\partial \mathcal{L}}{\partial T} \right)_p. \]  

(97)

where \( c^2_s = \mathcal{P}/\rho \) is the square of the isothermal sound speed. This remarkable equation states that if the adiabatic temperature gradient of \( \mathcal{L} \) with respect to \( T \) is positive, as is almost always the case for any astrophysical loss function, the corresponding thermodynamic isobaric temperature gradient must have the same sign as the spatial gradient of the entropy! In particular the Field criterion, which requires a negative isobaric temperature gradient for thermal instability, will always be accompanied by a negative spatial gradient of the specific entropy. The latter is a prescription for convective instability, normally a much more rapidly growing disruption than a simple thermal stability.

Now here is a nice surprise: were we to try to construct an atmosphere in both hydrostatic and thermal equilibrium subject to bulk heating and cooling, then for realistic thermal loss functions, classic thermal instability would not occur without
a simultaneous, much more disruptive, convective instability. In this case, thermal runaway is an afterthought to adiabatic heating and cooling. Rising convective plumes are warm relative to their surroundings and are simply made warmer by unbalanced external heating; the same holds in reverse for descending cool parcels.

This resolves the immediate puzzle: we should not have tried to analyze a simple thermal instability in a gas in both hydrostatic and thermal equilibrium under the assumption that there were simple buoyant oscillations present. But the surprises, as we shall presently see, are only just beginning.

4.4. Cooling in a cooling medium

Although the problem we analyzed in the previous section was well-posed and physically sensible, in realistic astrophysical settings there are seemingly slight differences from our idealization that in fact profoundly affect the thermal behaviour of the gas. What is the fate, for example, of a gas that is cooling without a significant heat source present? This is an interesting question because a typical astrophysical environment will be characterized by a thermal cooling time much longer than a dynamical sound crossing times: as it cools, the gas passes from one state of hydrostatic equilibrium to another. This is similar to the notion of individual fluid elements passing from one state of collisional thermal equilibrium to another in flow.) At one time (e.g. [23]), this was a simple standard model for the very hot x-ray emitting gas one finds cradled within the potential of a megaparsec scale cluster of galaxies, though current models of the cluster tend to be far more complex. (This almost certainly means there is something important still missing in our understanding of the dynamics.) The question then arises of the stability of the bulk cooling process. Even using what we’ve just learned, this is a surprisingly tricky problem, as the cooling instability time scale and the background flow time scale are comparable. This is analogous to the classical cosmology problem of linear perturbations growing by gravitational instability in a critical Friedmann universe.

4.4.1. Equilibrium cooling. The equations governing the unperturbed flow, taken to be spherical, are (i) mass conservation:

\[ \dot{m} = 4\pi r^2 \dot{v} = \text{constant}, \]

where the equilibrium velocity is \( v = v_e \); note that both \( \dot{m} \) and \( v \) will be negative for inflow. Next, we impose (ii) hydrostatic equilibrium:

\[ \frac{\dot{P}}{\dot{r}} = -\rho \dot{g}, \]

where \( -\dot{g} \) is the radial gravitational field. Finally, we require (iii) the entropy loss equation without a heating term in \( \mathcal{L} \):

\[ P_v \frac{\mathrm{d} \ln P}{\mathrm{d} r} = -\rho \mathcal{L}. \]

In fact, these equations do not represent a rigorous global equilibrium (a finite amount of material eventually collapses to the origin \( r = 0 \)), but rather an extended quasi-static phase of a slowly cooling atmosphere.

4.4.2. Cooling disturbances. To understand the growth of local disturbances in this inflowing gas, we will follow the cosmologists and work in a comoving, Lagrangian coordinate frame. The transformation between a small physical radial separation \( \delta r \), and the separation represented in local comoving coordinates, \( \delta r' \), in which radial stretching caused by the inflow does not appear, is given by:

\[ \delta r = a \delta r', \]

where we take \( a \) to be a function of \( r' \) and \( t \). This is equivalent to defining \( a \) by

\[ a = \left( \frac{\partial r}{\partial r'} \right)_t. \]

The radial scale of the perturbations is taken to be much smaller than the global scales of the problem, so that while \( a \) in general would depend on \( t \) and \( r' \), in a small Lagrangian neighbourhood \( r' + \epsilon \) around our fluid element, with no restrictions on \( t \), \( a \) may sensibly be written as \( a(t) \). Equation (14) then tells us

\[ \frac{D(\delta r)}{Dt} = \delta r \frac{dv}{dr} \]

or using (102)

\[ \frac{dv}{dr} = \frac{1}{a} \frac{Da}{Dt}. \]

The idea is that while the original equations are written in terms of the usual \( r \) spatial coordinates, it is only in small neighbourhoods of these embedded comoving \( r' \) coordinates that the displacement perturbations exhibit a simple WKB plane wave form. A ‘local’ calculation is truly local only in these coordinates. If (say) the radial displacement \( \xi \) has a spatial dependence \( \sim \exp(ik'r') \), with \( k' \) independent of time in these Lagrangian coordinates, and \( k'r' \gg 1 \), then

\[ \frac{\partial}{\partial r} = \frac{\partial}{\partial r'} \frac{\partial}{\partial r'} = \frac{ik'}{a}. \]

The wavenumber is, in effect, frozen into the flow expansion. Note as well that (104) implies

\[ \frac{Dv_r}{Dr} = \nu_r \frac{dv_r}{dr} = \nu_r \frac{Da}{Dr}, \]

so that \( a \) is linearly proportional to the velocity \( v_r \).

4.4.3. Linear dynamical response. The perturbed dynamical equation of motion is

\[ \frac{D\delta v}{Dt} + (\delta v \cdot \nabla) v = -\frac{1}{\rho} \nabla \delta P + \frac{\delta p}{\rho^2} \nabla P. \]

Now, the \( r \) component of the left side of this equation is

\[ \frac{D\delta v_r}{Dr} + \delta v_r \frac{dv_r}{dr} = \frac{1}{a} Da \delta v_r, \]

where we have used (104). A similar expression emerges for the angular component of \( \delta v \), denoted \( \delta v_{\text{ang}} \), but with \( r \) replacing \( a \). Now the left side of the angular part of the equation of motion becomes
where the final equality is aided by the angular component of equation (80). We note that only if there is a change in the ratio of the angular displacement relative to \( r \) is there a true change in the angular velocity at a particular location, a sensible result.

To make further progress, we need to specify our perturbations more precisely. We will use spherical perturbations of the form

\[
\delta v = \delta v_{\text{in}} + \mathcal{R}(r) \nabla_{\text{ang}} Y_{\text{in}},
\]

where \( Y_{\text{in}} \) is the usual spherical harmonic function of colatitude \( \theta \) and azimuth \( \phi \), and \( \nabla_{\text{ang}} \) is a vector operator whose components are the angular components of the standard \( \nabla \) gradient. The quantities \( \delta v \) and \( \mathcal{R} \) are \( r \)-dependent amplitudes of the \( Y_{\text{in}} \) and their angular gradients. These should not be confused with \( \delta v_{\text{av}} \) and \( \nabla_{\text{ang}} \mathcal{R}_{\text{av}} \), the full vector components depending on \( r \), \( \theta \), and \( \phi \). The use of spherical harmonics here is useful to display the \( 1/r \) dependence explicitly in the angular structure, because fluid elements may traverse an extended distance in radius over the course of the calculation.

Note that

\[
\nabla \cdot \delta v = \left[ \frac{1}{r^2} \frac{\partial r^2 \delta v}{\partial r} - \frac{l(l+1)}{r^2} \mathcal{R} \right] Y_{\text{in}}.
\]

Upon integration over angles, the angular equation of motion takes the compact form

\[
\frac{D}{Dt}(Y_{\text{in}} \mathcal{R}) = -\delta P/\rho.
\]

Finally, we use the standard Boussinesq approximation \( \nabla \cdot \delta v \approx 0 \) to eliminate sound waves. The previous two equations lead immediately to

\[
\frac{D}{Dt} \left( \frac{Y_{\text{in}}}{l(l+1)} \frac{\partial r^2 \delta v}{\partial r} \right) = -\delta P/\rho.
\]

Using this in the radial equation of motion (108) (to eliminate \( \delta P \)), together with the large wavenumber approximation, we find that upon grouping terms,

\[
1 - a \frac{D}{Dt} a \delta v \left( 1 + \frac{k^2 r^2}{a^2 l(l+1)} \right) \equiv 1 - a \frac{D}{Dt} \left( \frac{\delta v}{\beta^2} \right) = -\frac{\delta P}{\rho} - \frac{\delta v}{\gamma},
\]

which defines the geometrical parameter \( \beta \leq 1 \).

4.4.4. Linear thermal response. Our final step begins with the linearly perturbed entropy equation,

\[
\frac{D}{Dt} \left( \frac{\delta P}{P} - \frac{\delta \rho}{\rho} \right) + \delta v \frac{d \ln P \rho^{-\gamma}}{dr} = - \left( \gamma - 1 \right) \frac{\delta T}{T} \Theta_{T,P} + \frac{\delta P}{P} \Theta_{P,T}.
\]

where \( T \) is the gas temperature. We have introduced the following notational scheme:

\[
\frac{\partial (\rho \mathcal{L} / P)}{\partial X}_Y = \left( \frac{\partial \Theta}{\partial X}\right)_Y \equiv \Theta_{X|T}.
\]

Notice that the stability discriminant is no longer the gradients of \( L \), as in the static problem of section 4.3, but the gradients of \( \Theta = \rho \mathcal{L} / P \). This is a destabilizing influence, increasing the effect of the cooling as the temperature drops.

It will be noted that we have chosen to regard the cooling parameter \( \Theta \) as a function of \( P \) and \( T \). The disturbances of interest are very nearly isobaric so that the relative change in pressure is much smaller than that of either the density or temperature. (The quasi-isobaricity stems from \( \delta P \sim 1/k r \) scaling.) With the \( \delta P \) term dropped, the entropy equation becomes

\[
\frac{D}{Dt} + \left( \frac{\gamma - 1}{\gamma} \right) T \Theta_{T,P} \frac{\delta P}{\gamma} \frac{\delta v}{\rho} \frac{\rho \ln P}{\gamma} \frac{dr}{dr} = 0.
\]

Finally, substituting from \( \delta P / \rho \) from equation (114), we arrive at the governing equation for \( \delta v \):

\[
\frac{D}{Dt} + \left( \frac{\gamma - 1}{\gamma} \right) T \Theta_{T,P} \frac{1}{\gamma} \frac{D}{ag} \frac{a \delta v}{\beta^2} + \frac{N^2}{\epsilon^2 g} \delta v = 0.
\]

where

\[
N^2 \equiv \frac{g}{\gamma} \frac{\partial \ln P \rho^{-\gamma}}{\partial r}.
\]

is the Brunt–Väisälä (buoyancy) frequency we first encountered in section 2. Free oscillations in an adiabatic gas respond at this characteristic frequency and more generally propagate as internal gravity waves. Note that we have inserted a small parameter ‘tag’ \( \epsilon \) in order to set the scale for the final term as being large, and thus to formalize a perturbation treatment. This will allow for a (surprisingly) revealing WKB solution of this differential equation.

We seek a solution to equation (118) of the form (e.g. [15]):

\[
y \equiv \left( \frac{a \delta v}{\beta^2} \right) = A \exp \left[ \frac{S_0}{\epsilon} + S_1 \right],
\]

where \( A \) is a fiducial constant whose value is immaterial and the \( S_i \) are functions of time to be determined. The differential equation to be solved is

\[
\frac{D}{Dt} + \left( \frac{\gamma - 1}{\gamma} \right) T \Theta_{T,P} \frac{1}{\gamma} \frac{D \delta v}{ag} + \frac{\beta^2 N^2}{\epsilon^2 g a} = 0.
\]

Substituting (120) into (121), and sorting out terms of order \( \Omega \epsilon^2 \) and \( \Omega \epsilon \) leads to the two equations:

\[
(S_0)'' + \beta^2 N^2 = 0,
\]

\[
2S_1 + \frac{S_0''}{S_0} + \frac{\gamma - 1}{\gamma} T \Theta_{T,P} = 0.
\]

where we have used the primed ‘ notation to represent \( \mathrm{D}/\mathrm{D}t \). This pair of first order differential equations decouples, is solved by elementary methods, and \( \delta v \) may then be obtained via (120):
\[
\delta v_i = \beta \left( \frac{g}{a_0 n^2} \right)^{1/2} \exp \int^t \left( \pm i \beta N - \frac{\gamma - 1}{2 \gamma} T \Theta_{\tau_P} \right) dt. \tag{124}
\]

4.5. Eliciting the instability

At first glance, it might seem that the stability of the Eulerian velocity perturbation is regulated by the thermodynamic derivative \( \Theta_{\tau_P} \); if this is negative, there is exponential growth. This is just the classical [25] result. But of course this assessment is too crude. Despite appearances, the amplitude modulation terms are on the same footing as the exponential cooling. More surprises.

It is a bit more revealing to work with the Lagrangian displacement \( \xi \), which to leading WKB order is just \( b_\nu l(\beta N) \). This implies

\[
\frac{\xi}{\beta} = \left( \frac{1}{\beta N} \right)^{1/2} \left( \frac{g}{a_0 n^2} \right)^{1/2} \exp \int^t \left( \pm i \beta N - \frac{\gamma - 1}{2 \gamma} T \Theta_{\tau_P} \right) dt, \tag{125}
\]

Now,

\[
\left( \frac{a_0 n^2}{g} \right) \propto v \frac{d \ln P_{\rho^{-\gamma}}}{dr} \propto \Theta, \tag{126}
\]

where the last relation follows from the entropy equation. Continuing our equation juggling,

\[
\frac{1}{\Theta^{1/2}} = \exp \left( -\frac{1}{2} \int \frac{d \Theta}{\Theta} \right) \exp \left( \frac{1}{2} \int \frac{(d \Theta/dr)}{\Theta/\nu} \right) dr
\]

\[
= \exp \left( \frac{\gamma - 1}{2} \int \frac{(d \Theta/dr)}{d \ln P_{\rho^{-\gamma}}/dr} \right), \tag{127}
\]

where in the final equality we have once again used the entropy equation. We have thus found

\[
\frac{\xi}{\beta} = \left( \frac{1}{\beta N} \right)^{1/2} \exp \int^t \left( \pm i \beta N - \frac{\gamma - 1}{2 \gamma} \right) \left[ T \Theta_{\tau_P} - \frac{\gamma (d \Theta/dr)}{d \ln P_{\rho^{-\gamma}}/dr} \right] \right) \tag{128}
\]

Our last trick is to bring a thermodynamic identity into play [14]:

\[
\frac{d \Theta}{dr} = T \Theta_{\tau_P} \frac{d \ln P_{\rho^{-\gamma}}}{dr} + (\gamma - 1) T \Theta_{\tau_S} \frac{d \ln P_{\rho^{-\gamma}}}{dr}, \tag{129}
\]

which, it will be noted, reduces to the remarkable equation (97) in the limit that \( \Theta = 0 \). The \( \Theta \) terms in our expression for \( \xi/\beta \) collapse, and upon using hydrostatic equilibrium to substitute for \( d \ln P/dr \), we are left with

\[
\frac{\xi}{\beta} = \left( \frac{1}{\beta N} \right)^{1/2} \exp \int^t \left( \pm i \beta N - \frac{\gamma - 1}{2 \gamma} \right) \left[ T \Theta_{\tau_P} - \frac{\gamma (d \Theta/dr)}{c_s^2 d \ln P_{\rho^{-\gamma}}/dr} \right] \right) \tag{130}
\]

where, as before, \( c_s^2 \) is the square of the isothermal sound speed, \( P/\rho \). Thus, the behaviour of \( \xi/\beta \) ultimately consists of a buoyant oscillation, modified by a thermal loss term whose stability is regulated by an adiabatic thermal gradient. (Knowledgable readers will recognise the action-conserving amplitude, \( [\beta N]^{-1/2} \) ) Whether there is growth or not depends not just on the sign of this gradient, but on how the thermal behaviour competes with the evolution of \( \beta \). The classical 1965 Field stability criterion \( \Theta_{\tau_P} > 0 \) is nowhere to be found.

This in itself is surprising and very often misunderstood; perhaps more surprising still is the sheer complexity of the problem. The reader who has navigated through this somewhat harrowing section from beginning to end will have developed a healthy respect for the subtlety of understanding thermal instability in moving backgrounds, with the full interplay between the dynamics of the background and the developing perturbation.

5. Magnetothermal and heat-flux buoyancy instabilities

5.1. Magnetized heat flux

Consider an ionized plasma whose thermal physics is dominated by heat conduction. In astrophysical environments this will typically mean a hot diffuse plasma such as a galactic halo or an intracluster medium. The kinetics of such a gas will be dictated by any magnetic field that might be present, even a very weak one. This is because in what we shall call a ‘dilute’ gas, both the ion and electron gyroradii are much smaller than the respective particle’s mean free path. Under these circumstances, the standard form of the collisional heat flux (equation (49)) is no longer valid. Instead, it must be modified to take into account (i) that only the component of the gradient parallel to the local magnetic field lines contributes significantly to the flow of heat, and (ii) that the resultant heat flux flows parallel to the same field lines. In other words, we must replace the scalar thermal conductivity \( \kappa \) with a tensor conductivity \( \kappa b b T \) where \( b_i \) and \( b_j \) are components of the unit vector parallel to the magnetic field \( B \). (For the scalar conductivity case, \( b_i b_j \) in effect reverts to the Kronecker delta \( \delta_{ij} \).) The magnetic heat flux then takes the form [18]:

\[
F_i = -\kappa b_i b_j \partial_j T, \tag{131}
\]

where \( \partial_j \) is the partial derivative with respect to the Cartesian variable \( x_j \), while \( x_i \) held constant. What makes this interesting is that the \( b_j \) are more than just labels for a field-based coordinate system, they are dynamical variables in their own right.

The change of the form of the conductivity has particularly important consequences for the stability of hot plasmas. Given a time steady equilibrium, the Eulerian change in the heat flux \( \delta F_i \) in the WKB limit of rapidly spatially varying perturbations is given by [6, 7]

\[
\delta F_i = -\kappa (\delta b_i b_j T) \partial_j + b_i \delta b_j (T \delta T) + b_j \delta b_i (T \delta T) \tag{132}
\]

The first term redirects a pre-existing heat flux, the second alters the directional temperature gradient being tapped as a heat source, and the final takes into account the new temperature gradient along a pre-existing field line. The complexity
of equation (132) should be compared with $-\kappa \partial_t \delta T$, the sole term that would be present in $\delta F_\ell$ for a scalar conductivity.

### 5.2. Magnetothermal instability

To see this process in its clearest manifestation, consider a slab of hot gas, hot on the bottom and cooler on top. The temperature stratification is entirely along the $z$ axis. There is a very weak magnetic field present, of no dynamical significance whatsoever. The field’s kinetic significance is that it limits the electron gyroradius to be much less than a Coulomb mean free path, so that equation (131) describes the conductive heat flux. In equilibrium, the field is taken to be uniform along the $x$ axis, $B = B_0 e_x$, which ensures that there is no heat flux unless the system is disturbed. The gravitational field points in the $-z$ direction. We assume equal electron and ion temperatures, and that the specific entropy $S$ satisfies $\delta S/\delta z > 0$, so that in the absence of heat conduction, any disturbances would correspond to stable buoyant oscillations.

This static equilibrium is disturbed by displacements of the form

$$\xi = \xi \exp(ikx + \sigma t) e_z.$$  

Equation (87) is then trivially satisfied, equation (88) does not apply because the presence of heat flow renders the dynamics nonadiabatic, and equation (89) has only one component:

$$\delta B_x = i k B_0 \xi.$$  

The change in the unit magnetic field vector $b$ is simply

$$\delta b_z = \frac{\delta B_z}{B} = i k \xi,$$  

implying a perturbed heat flux of

$$\delta F_\ell = -\kappa \left[ \frac{\delta B_z}{B} \partial_z T + i k \delta T \right] = -i k \kappa [\xi \partial_z T + \delta T] \equiv -i k \kappa \Delta T.$$  

The heat flux is proportional to the Lagrangian change in the temperature $\Delta T$. This should be contrasted with the zero field result, $\delta F_\ell = -i k \kappa \delta T$, where only the Eulerian change is important. The difference is profound: Eulerian and Lagrangian temperature changes may well have opposite signs.

The perturbed dynamical equation of motion in the $x$ direction reduces to $i k \delta P = 0$, so that the pressure perturbation is identically zero, implying $\delta \ln \rho = \delta \ln T$. In the $z$ direction, with $\delta v \equiv \sigma \xi$,

$$\sigma^2 \xi = \frac{\delta \rho}{\rho} \frac{\partial P}{\partial z} = -\frac{\delta \rho}{\rho} g,$$  

(136)

where $-\rho g = \partial P/\partial z$ defines the gravitational field strength $g$. Finally, the perturbed entropy equation reads

$$-\sigma \gamma \frac{\delta \rho}{\rho} + \sigma \xi \frac{\partial \ln P}{\partial z} = -\gamma \frac{1}{P} \nabla \cdot \delta F_\ell.$$  

(137)

Replacing $\delta \ln \rho$ with $-\delta \ln T$ and combining (135) and (137) we obtain

$$\left[ \sigma \gamma + \frac{k^2 \kappa T (\gamma - 1)}{P} \right] \frac{\delta \rho}{\rho} = \left[ \frac{\partial \ln P}{\partial z} \gamma - \frac{k^2 \kappa T (\gamma - 1)}{P} \right] \frac{\delta \ln T}{\partial z}. $$  

(138)

Finally, substituting for $\delta \rho$ from equation (136), the dispersion relation emerges:

$$\sigma^3 + \sigma^2 \gamma - \frac{1}{\gamma} k^2 \kappa T \gamma \sigma N^2 + \gamma - 1 \frac{k^2 \kappa T}{\gamma} \frac{\partial \ln T}{\partial z} = 0,$$  

(139)

where we have used a standard notation for the Brunt-Väisälä oscillation frequency,

$$N^2 \equiv -\frac{g}{\gamma} \frac{\partial \ln P}{\partial z} \gamma.$$  

(140)

which is a positive quantity for stable adiabatic perturbations. Equation (139) may be compared with the dispersion relation for nonmagnetized thermal conduction

$$\sigma^2 + \sigma \gamma - \frac{1}{\gamma} k^2 \kappa T \gamma \sigma N^2 = 0,$$  

(141)

here the thermal term serves only to damp the adiabatic Brunt–Väisälä oscillations. On the other hand, when even a tiny

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**Figure 3.** Numerical MHD simulation of the magnetothermal instability [45]. Field lines are shown in black, and colour indicates temperature (with red hotter, blue colder). Small perturbations in the initial horizontal field are convectively unstable. This instability eventually leads to the complete disruption of the initial configuration and the onset of sustained non-linear turbulence.
magnetic field is included, one branch of long wavelength disturbances are characterized by a balance between the third and fourth terms in equation (139) and the growth/damping rate is

$$\sigma = -(\gamma - 1) \frac{k^2 \sigma T}{P} \frac{d \ln T}{dz} / d \ln T \rho^{-2} / dz$$  \hspace{1cm} \text{(long wavelengths), (142)}$$
the other two solutions correspond to buoyant oscillations, a dominant balance between the first and third terms. For short wavelengths, the interesting solution is a dominant balance between the second and fourth terms:

$$\sigma^2 = -g \frac{d \ln T}{dz}$$  \hspace{1cm} \text{(short wavelengths). (143)}$$
Evidently, if \(dT/\Delta T < 0\), the equilibrium is unstable. This is quite surprising: in the presence of dissipative thermal conduction, short wavelength disturbances might normally be thought to be prone to strong damping, not a robust instability growing on a free-fall time! Figure 3 shows a simulation of the full nonlinear development of the MTI.

Perhaps the greatest surprise here is that the magnetized dispersion relation makes no reference to all at a magnetic field, yet produces results that are completely different from a nonmagnetized gas. The key point is that the magnetic field affects the dynamics when the Lorentz force becomes comparable to either the pressure or rotational forces. On the other hand, the magnetic field affects the kinetics when the electron or ion gyroradius (depending upon whether thermal conduction or viscosity is involved) is small compared with a Coulomb mean free path, a very different requirement. For the dilute astrophysical plasma in which Coulomb thermal conduction is important, an astonishingly minute magnetic field will suffice—10^{-21} T is enough to make an electron gyroradius smaller than the Coulomb mean free path for a 10^6 K plasma with an electron density of 10^4 m^{-3}. Interstellar magnetic fields are more than ten orders of magnitude in excess of this. The effect of the magnetic field is masked in our use of equation (131) for the heat flux, serving only to set the cross-field diffusion equal to zero. For a sufficiently weak magnetic field cross-field diffusion would have to be included, in which case the transition from magnetic to nonmagnetic plasma would be quite smooth. Even when our interest is purely at the level of dynamics, it is often not a straightforward matter to decide when a seemingly weak magnetic field might be important. The question always is, ‘weak’ compared to what? We will have surprises in store for us along these lines in sections 7 and 8.

To summarize: the most rapidly growing modes are at short wavelengths, with buoyantly unstable behaviour reminiscent of violating the Schwarzschild criterion for the onset of convection—except that it is not the entropy gradient that enters, it is the temperature gradient.

5.3. Physics of the magnetothermal instability

Recall the reason for classical convective instability in a gas: an adiabatically displaced fluid element will cool at constant entropy on an upward displacement, in pressure balance at all times with its surroundings. If this constant entropy element is nevertheless still warmer than the equilibrium background at the displaced location then (i) the background entropy must have been decreasing upward, and (ii) the element will continue to rise by buoyant forces. Thus, the criterion for instability is that the equilibrium entropy profile must be decreasing upwards. The presence of thermal conduction would only lessen the growth rate, by lessening the temperature difference (and thus the density difference) between the element and its immediate surroundings.

When a weak magnetic field alters the thermal conduction, heat flows most efficiently along the field line, and at large wavenumbers the flux is so efficient that nearly isothermal conditions are maintained along a field line (i.e. \(\Delta T \approx 0\)) and it is the component of \(\nabla T\) across the field line and parallel to gravity that is important. Therefore, if the background temperature decreases with height, an upwardly displaced element is always warm relative to the surroundings, and vice-versa for a downwardly displaced element. This must be convectively unstable. At small wavenumbers (long wavelengths), the growth rate is small. To first order in \(\sigma\), equation (136) suggests ignoring changes in \(\delta \rho\), hence also in \(\Delta T\) since \(\delta \rho = 0\). The instability is only in the perturbed velocity. Indeed, at first sight it appears not to be a buoyant instability at all: the growth rate (142) is independent of the gravitational field. The dominant balance is thermal, between the heat deposition along the perturbed field line and the corresponding rise in entropy of an upwardly moving fluid element. But the role of gravity is hidden: in this regime, \(g\) is in effect a large parameter. Indeed, taking \(g \rightarrow \infty\) in equation (139) produces the same growth rate. This ensures that an element moves ‘instantly’ to the proper \(\delta \rho = 0\) position as the gas is heated by conduction along the displaced field line. Equation (136) shows that the precise \(\delta \rho\) is small but negative, i.e. the element really is driven by buoyant forces.

The final surprise is the astrophysical significance of the instability: a weak to moderate magnetic field cannot thermally isolate a hot dilute plasma in a gravitational field. In fact, such configurations appear to be candidates for a dynamo amplification of the magnetic field [45, 50].

A problem of widespread interest is one in which cool gas is stratified at the bottom of a gravitational potential well surrounded by hotter exterior plasma. This is the disposition of the intracluster medium in rich clusters of galaxies. The question is whether thermal conduction from the hot gas will evaporate the cool gas or whether the cool gas reservoir will grow by radiative losses. It came as a great surprise when it emerged that the act of heating cool gas from above by thermal conduction is actually unstable. When a weak magnetic field is present, that is.

5.4. A buoyant heat flux instability

Remarkably, as first shown by [58], the magnetothermal instability is only half the story. Let us reconsider the effects of a heat flux in the background equilibrium. As before, gravity points in the downward \(z\) direction. The simplest manifestation of the heat flux instability involves a purely vertical
magnetic field, parallel to the gravitational field. We must also allow for both vertical and horizontal displacements \((\xi_x, \xi_z)\), and the same for the wavenumber components \((k_x, k_z)\). As before, the disturbances have the leading order WKB form \(\exp(\alpha t - i k \cdot \mathbf{r})\); the background field is presumed nearly constant on the scale of the perturbation. Mass conservation (87) then implies

\[
k_x \xi_x + k_z \xi_z = 0. \quad (144)
\]

There is now an equilibrium heat flux present, assumed to be divergence free:

\[
F = -\kappa \mathbf{b} \frac{\partial T}{\partial z} = -e_z k \frac{\partial T}{\partial z}. \quad (145)
\]

The perturbed heat flux is given once again by equation (132):

\[
\delta F = -\kappa (\xi_x \mathbf{b}_x + \xi_z \mathbf{b}_z T_z + \mathbf{b}_z \mathbf{b}_z T_z + b_x b_x T_x + b_x b_y T_y). \quad (146)
\]

but now the first term in this expression, which was not present in our earlier magnetothermal calculation, serves to redirect the equilibrium flux. The perturbed magnetic field unit vector,

\[
\delta \mathbf{b} = \delta \left( \frac{\mathbf{B}}{B} \right) = \mathbf{b} \times \left( \frac{\delta \mathbf{B}}{B} \times \mathbf{b} \right). \quad (147)
\]

must always be orthogonal to the unperturbed \(\mathbf{b}\), and has only an \(x\) component:

\[
\delta b_x = i (k \cdot \mathbf{b}) \xi_x, \quad (148)
\]

where (89) has been used.

The equations of motion are

\[
\sigma^2 \xi_x = -i k_x \frac{\delta P}{\rho} \quad (149)
\]

\[
\sigma^2 \xi_z = -\frac{\delta P}{\rho} g - i k_z \frac{\delta P}{\rho} \quad (150)
\]

and with \(k \cdot \xi = 0\), we may eliminate \(\delta P\) from the above to find

\[
\frac{\delta P}{\rho} = -\frac{\sigma^2 k_x^2 + k_z^2}{g k_x^2} \xi_x. \quad (151)
\]

where \(k^2 = k_x^2 + k_z^2\). The perturbed heat flux is

\[
\delta F = -\kappa \left( \delta b_x \frac{\partial T}{\partial z} + i (k \cdot \mathbf{b}) \delta T \right) \quad (152)
\]

and

\[
\nabla \cdot \delta F = -\kappa \left( ik \cdot \delta \mathbf{b} \frac{\partial T}{\partial z} - (k \cdot \mathbf{b})^2 \delta T \right) = -k^2 \kappa (\xi_x \frac{\partial}{\partial z} T - \delta T). \quad (153)
\]

With the zero pressure condition \(\delta \ln T = -\delta \ln \rho\) and equation (151), this becomes

\[
\nabla \cdot \delta F = \kappa T k^2 \xi_x \left( \frac{\sigma^2 k^2}{g k_x^2} \right. \left. - \delta \ln T. \right) \quad (154)
\]

Even before we arrive at a dispersion relation, it is clear that efficient heat transport will lead to a buoyant instability in this problem. As \(k_z \to \infty\), there must be a near cancellation of the two terms in the flux divergence of the right side of equation (154). The first is the ordinary thermal diffusion term arising from temperature fluctuations, the second represents heat flow along redirected field lines. We will presently see that the precise growth rate is slightly less than that given by a balance by the terms on the right, so \(\xi_x > 0\) corresponds to \(\nabla \cdot \delta F < 0\). In other words, the redirected field lines converge when \(\xi_x > 0\), and diverge when \(\xi_x < 0\). This is a prescription for buoyant instability, since there is conductive heating (cooling) for an upward (downward) displacement. The dispersion relation becomes:

\[
\sigma^3 + \frac{\gamma - 1}{\gamma} \frac{\kappa T}{P} k^2 \sigma^2 + \frac{k_z^2 g}{k_x^2} \frac{\partial \ln P}{\partial \ln \rho} \sigma \quad (155)
\]

Just as suggested by equation (154), there is now instability if \(\partial T / \partial z > 0\), exactly the opposite of the magnetothermal instability configuration of the previous section. In other words, even if the cold gas is on the bottom, the configuration is unstable! Now that really is a surprise.

The problem is not so much the cold gas on the bottom, it is the heat flux flowing into this gas, trying to disrupt this happy configuration, that is the root of the difficulties. This heat-flux buoyancy instability, or HBI as it is known, achieves its most important astrophysical application in x-ray clusters.
of galaxies. The space between the galaxies in such a cluster is filled with a hot, diffuse plasma, cooler toward the central region at the base of the cluster well. We have just seen that any flux into this cool region will be unstable, a fact not known until Quataert’s important 2008 paper. The nature of the response of the gas to the HBI is to try to draw out a tangential field, thermally shielding the interior (see the simulation of figure 4), but in real clusters conditions may be too disturbed.

As an epilogue to this story, there is a Braginskii viscosity as well as a Braginskii conductivity [18, 35], which channels the momentum flux in an ionized plasma along field lines in a manner similar to the heat flux. Recent studies [37–39] point to the importance of Braginskii viscosity in limiting the thermal insulation properties of the HBI in the nonlinear regime. (The magnetothermal instability is much less affected.) The most basic properties of the thermal behaviour of the gas in x-ray clusters is still not at all well-understood. How the various Braginskii diffusion coefficients and their associated instabilities play themselves out promises to be full of surprises, ripe for future elucidation.

6. The Newcomb–Parker problem

6.1. Introduction

Often just called the ‘Parker instability’ in the astrophysical literature, the Newcomb–Parker problem addresses the behaviour of a gas in a gravitational field with substantial magnetic support. Before the astrophysicists took it up [54], the problem had been thoroughly studied in the 1950s and early 1960s by the magnetic confinement community of plasma physicists. This is perhaps not too surprising: if one’s goal is to confine a thermonuclear plasma in the laboratory, it behooves one to understand any instabilities that might be present.

The astrophysical side of the Parker instability arose from attempts to understand hydrostatic equilibrium in a vertical gravitational field, in practice the disc of the Milky Way Galaxy. (More recent applications include accretion disc theory.) In the Galactic problem, in addition to the presence of significantly magnetized interstellar gas, cosmic rays are also thought to comprise an important dynamic component. In some discussions of this problem, the cosmic rays are central to an explanation of the instability itself, with these energetic particles moving upwards along magnetic field lines anchored in place by sinking cool interstellar gas. In the 1970s, the Parker instability was studied as a means for initiating star formation [16, 46].

Discussion of the Parker instability in the astrophysical literature often tends to be somewhat confusing; Parker’s 1966 paper is itself far from an easy read. The surprise here is that the essence of the instability is really simplicity itself. The linearly perturbed equation of motion (here, hydrostatic balance) is:

\[ 0 = -\frac{1}{\rho} \nabla \left( \delta P + \delta P_\text{cr} + \frac{B \cdot \delta B}{\mu_0} \right) + \frac{\delta \rho}{\rho} g + \frac{B_0 \delta B}{\mu_0} + e_x \cdot \nabla B, \]  

(159)

(The subscripted notation \( \partial \) denotes partial differentiation with respect to coordinate \( i \).) The induction equation for the magnetic field is

\[ \delta B = \nabla \times (\xi \times B) = \nabla \times (\xi B_e), \]  

(160)

or

\[ \delta B_x = B_0 \partial_i \xi, \quad \delta B_0 = -\xi \partial_0 B. \]  

(161)

Notice that we retain background gradient terms that would be dropped in a WKB treatment. Using the above relations, we obtain

\[ \delta B_0 \partial_i B = (B_0 \partial_i \xi) \partial_0 B = B_0 \partial_i (\xi \partial_0 B) = -B_0 \partial_i \delta B_x \]  

(162)

so that the final two terms in the \( x \) equation of motion exactly cancel. This leaves

\[ 0 = -\frac{1}{\rho} \partial_i \left( \delta P + \delta P_\text{cr} + \frac{B \cdot \delta B}{\mu_0} \right) \equiv -\frac{1}{\rho} \partial_i (\delta P_\text{mb}). \]  

(163)

6.2. Equilibrium state

Consider a slab of interstellar gas lying in the \( xy \) plane with vertical coordinate \( z \). The gravitational field \( g = -ge_x \) points downward in the \( -z \) direction. In equilibrium, the gas contains a magnetic field pointing in the horizontal \( e_x \) direction, depending only upon \( z \):

\[ B = B(z)e_x. \]  

(156)

The gas also contains cosmic rays, with pressure \( P_\text{cr}(z) \). The cosmic ray pressure is taken to remain constant along field lines; i.e. any gradient is immediately eliminated by rapid particle streaming:

\[ B \cdot \nabla P_\text{cr} = 0. \]  

(157)

This is our effective equation of state for the cosmic rays. The equation for hydrostatic equilibrium is

\[ \frac{d}{dz} \left[ P + P_\text{cr} + \frac{B^2}{2\mu_0} \right] = -\rho g. \]  

(158)

The field \( g \) may be any suitable function of \( z \).

6.3. Departures from equilibrium

Next, consider displacement perturbations in the vertical direction, \( \xi = \xi e_x \), the most unstable modes. We are free to assume an \( x \) dependence of \( \exp(ikx) \) since there is no \( x \) dependence in the equilibrium state. The \( z \) dependence is left unspecified. We will work at the point of marginal stability, which will allow an exact treatment of the problem. Accordingly, there is no time dependence in either the equilibrium or perturbed states.

The linearly perturbed equation of motion (here, hydrostatic balance) is:

\[ 0 = -\frac{1}{\rho} \nabla \left( \delta P + \delta P_\text{cr} + \frac{B \cdot \delta B}{\mu_0} \right) + \frac{\delta \rho}{\rho} g + B_0 \delta B + e_x \cdot \nabla B, \]  

(159)

The induction equation for the magnetic field is

\[ \delta B = \nabla \times (\xi \times B) = \nabla \times (\xi B_e), \]  

(160)

or

\[ \delta B_x = B_0 \partial_i \xi, \quad \delta B_0 = -\xi \partial_0 B. \]  

(161)

Notice that we retain background gradient terms that would be dropped in a WKB treatment. Using the above relations, we obtain

\[ \delta B_0 \partial_i B = (B_0 \partial_i \xi) \partial_0 B = B_0 \partial_i (\xi \partial_0 B) = -B_0 \partial_i \delta B_x \]  

(162)

so that the final two terms in the \( x \) equation of motion exactly cancel. This leaves

\[ 0 = -\frac{1}{\rho} \partial_i \left( \delta P + \delta P_\text{cr} + \frac{B \cdot \delta B}{\mu_0} \right) \equiv -\frac{1}{\rho} \partial_i (\delta P_\text{mb}). \]  

(163)
Since the operator $\partial_\xi$ amounts to multiplication by $ik$, the total perturbed pressure $\delta P_{\text{tot}}$ vanishes for these most unstable modes. The $z$ force balance is then very simple:

$$0 = \frac{\delta \rho}{\rho} g + (kv)^2 \xi$$

(164)

where the Alfvén velocity is given by

$$v_A = \frac{B}{(\rho \mu_0)^{1/2}}.$$  

(165)

with $\mu_0$ being the vacuum permeability.

6.4. Stability criterion

The most unstable modes are clearly those with $k \to 0$, so the marginal stability condition is just neutral buoyancy:

$$\frac{\delta \rho}{\rho} = 0.$$  

(166)

No magnetic field in sight. Neither Karl Schwarzschild nor Baron Rayleigh would have been surprised at this simple and intuitive result. Astrophysical complexity should not be allowed to obscure this basic physical point.

6.4.1. Adiabatic disturbances. For adiabatic conditions, equation (88) gives

$$\frac{\delta \rho}{\rho} = 1 \left( \frac{\delta P}{P} + \frac{\partial \ln P}{\partial \rho} \xi \right).$$

(167)

In the problem without the magnetic field or cosmic rays, the $\delta P$ term vanishes, and we recover the classical Schwarzschild criterion that the entropy should increase upward:

$$\frac{\partial \ln P}{\partial \rho} > 0 \quad \text{(Schwarzschild stability criterion),}$$

(168)

for stability. (The sign is determined by requiring $\delta \rho > 0$ for $\xi > 0$). Since $\partial P/\partial \rho = -\rho g$, this may also be written

$$\frac{\partial \ln P}{\partial \rho} + \frac{g}{a^2} < 0 \quad \text{(Stability),}$$

(169)

where $a^2 = \gamma P/\rho$ is the adiabatic sound speed. In this form, the result is in fact completely general, even with cosmic rays and magnetic fields, for an adiabatic gas.

To see this, note the following cosmic ray manipulations:

$$0 = \delta (B \cdot \nabla P_{\text{cr}}) = \delta B \cdot \nabla P_{\text{cr}} + B \cdot \nabla \delta P_{\text{cr}}$$

(170)

$$= \delta B \cdot \partial_\xi P_{\text{cr}} + ikB \delta P_{\text{cr}}$$

(171)

$$= (B \cdot \partial_\xi) \partial_\xi P_{\text{cr}} + ikB \delta P_{\text{cr}}$$

(172)

$$= ikB (\delta P_{\text{cr}} + \xi \cdot \nabla P_{\text{cr}}) = ikB \Delta P_{\text{cr}}.$$  

(173)

In other words, the Lagrangian pressure disturbance of the cosmic rays vanishes, $\delta P_{\text{cr}} = -\xi \partial_\xi P_{\text{cr}}$.

Similarly, for the magnetic pressure,

$$\mu_0 \delta P_{\text{mag}} = B \cdot \delta B = B \partial_\xi B = B(-\xi \partial_\xi B) = -\xi \partial_\xi (B^2/2).$$

(174)

and since

$$0 = \delta P_{\text{tot}} = \delta P + \delta P_{\text{cr}} + \delta P_{\text{mag}},$$

(175)

we find that

$$\delta P = \xi \frac{\partial}{\partial \xi} \left( P_{\text{cr}} + \frac{B^2}{2\mu_0} \right).$$

(176)

Inserting this result into (167), using equation (158), and demanding $\delta \rho > 0$ for stability, leads us directly to the condition (169) once again.

The destabilizing role of the cosmic rays and magnetic field becomes more apparent if the stability diagnostic is the vertical temperature gradient. Then, combining equations (169) and (158) yields the stability criterion

$$\frac{\partial \ln a^2}{\partial \rho} > \left[ \frac{g}{a^2} + \frac{1}{P} \frac{\partial}{\partial \rho} \left( \frac{B^2}{2\mu_0} + P_{\text{cr}} \right) \right] \xi + \frac{\delta \rho}{P},$$

(Stability).

(177)

In other words, a less negatively steep temperature gradient will destabilize.

There are three surprises here. The first is that once the gravitational field is specified, the presence of magnetic fields and cosmic rays makes no difference to the upper limit of the inverse density scale height for buoyant stability. It is always $g/a^2$. (The critical temperature scale is, however, affected.)

The second surprise is that the derivation has been simple and general, at both the conceptual and technical levels. We are dealing with elementary buoyancy forces and nothing more. Students of astrophysical gasdynamics would do well to examine the more extravagant claims on behalf of the Parker instability with a discerning eye.

The third surprise is that the criterion is entirely incorrect for a dilute plasma.

6.4.2. Parker–Newcomb–magnetothermal instability. To include the effects of thermal conduction along the field lines, for the calculation of $\delta \rho/\rho$ from the entropy equation we return to equation (138). This goes through just as before, only now we must retain the $\delta P/P$ term because of cosmic ray pressure and a dynamically important magnetic field:

$$\left[ \sigma^2 \gamma + \frac{k^2 T(\gamma - 1)}{P} \right] \frac{\delta \rho}{\rho} = \left[ \sigma \frac{\partial}{\partial \rho} \right] \frac{\partial \ln P}{\partial \rho}$$

$$+ \frac{k^2 T(\gamma - 1)}{P} \frac{\partial T}{\partial \rho} \left( \frac{\xi}{\sigma} + \frac{\delta P}{P} \right).$$

(178)

Equation (176) is unchanged by the inclusion of thermal conduction, so (178) becomes

$$\left[ \sigma^2 \gamma + \frac{k^2 T(\gamma - 1)}{P} \right] \frac{\delta \rho}{\rho} = \left[ -\sigma \frac{g}{a^2} + \frac{\partial}{\partial \rho} \right] \frac{\partial \ln P}{\partial \rho}$$

$$+ \frac{k^2 T(\gamma - 1)}{P} \frac{\partial T}{\partial \rho} \frac{\partial \ln T}{\partial \rho}.$$  

(179)

In the absence of thermal conduction ($\kappa = 0$), we recover the classic result of the previous section. But when thermal conduction is present, the $\sigma \to 0$ limit is a bit more delicate. In the limit of small temperature gradients, the two terms in square
brackets on the right side of the equation form the dominant balance. Thus, we obtain not only a stability criterion, but a leading order growth rate! When we are stable by the classic Newcomb–Parker criterion, there is instability when the temperature gradient decreases upwards. This is precisely the magnetothermal instability criterion (and leading order growth rate), recovered just as the classic Schwarzschild criterion is recovered in the conduction free case: without reference to the magnetic field at all.

The classic Newcomb–Parker stability criterion is simply not applicable to dilute astrophysical plasmas. Surprise.

7. Local, 3D, weakly-magnetized, adiabatic perturbations in 2D rotating, stratified backgrounds

We begin rather formally, deriving and examining a kind of master equation for the evolution of three-dimensional Lagrangian displacements in axisymmetric, two-dimensional, magnetized, shearing, stratified backgrounds. The magnetic field is considered to be weak in that it does not affect the equilibrium state, only the perturbations. We are then in the regime of weak fields and large wavenumbers.

This is a very general equation. Moreover, it has a sort of elegant transparency: the emergent force balance is pleasingly intuitive. We will use various limits of the equation to (i) derive directly a very general form of the magnetorotational dynamical equations; (ii) derive some new results on linear convection theory in a background shear similar to that of the Sun.

The surprises here are many, and they will be highlighted in the development to come. One of the most important will be obvious in the early stages of the development, and that is the deceptive ease with which magnetism can be included in the analysis, yet its effects are often profound. We have already seen evidence of this in our discussions of various thermal conduction driven instabilities due to kinetic heat flow along field lines—a tiny field goes a long way. That was all kinetic transport. We will see here that the direct dynamical influence of a supposedly weak magnetic field can be no less subtle.

7.1. Governing equation

7.1.1. Equilibrium. Our background equilibrium state might be a star, disc, or model galaxy. The gas is axisymmetric about a rotation axis, denoted by $z$. It will be convenient to use either spherical $(r, \theta, \phi)$ or cylindrical $(R, \phi, z)$ coordinates depending upon the problem at hand. The density $\rho$, pressure $P$, and angular velocity $\Omega$ are then regarded as functions either of $r$ and $\theta$, or $R$ and $z$. The same holds for any other variables constructed from these quantities. As usual, the velocity vector is denoted by $v$.

The partial derivatives of entropy and angular momentum cannot be chosen entirely arbitrarily, even if our interest is a simple local calculation in which these quantities are viewed as given background-defined constants. Rather, because the magnetic field is negligible in the equilibrium state, the derivatives are linked by the $\phi$ component of the vorticity equation (27), which reads:

$$R \frac{\partial \Omega}{\partial z} = \frac{1}{\rho^2} \left( \frac{\partial \rho}{\partial R} \frac{\partial P}{\partial z} - \frac{\partial \rho}{\partial z} \frac{\partial P}{\partial R} \right) \quad (180)$$

For future reference, it is also convenient to have an alternative form of this equation in terms of the entropy-like variable $\sigma = \ln P \rho^{-\gamma}$ rather than the density $\rho$.

$$R \frac{\partial \Omega^2}{\partial z} = \frac{1}{\gamma \rho} \left( \frac{\partial \sigma}{\partial z} \frac{\partial P}{\partial R} - \frac{\partial \sigma}{\partial R} \frac{\partial P}{\partial z} \right) \quad (181)$$

7.1.2. Inertial terms from Lagrangian derivatives. We are interested in linear departures from the fundamental equation given by (11). We shall work in the local WKB limit, so that for the weak magnetic field $B$ only terms involving the gradients of $\delta B$ are retained, as these are boosted by the assumed large wavenumber. If we divide equation (11) by $\rho$ and take Eulerian perturbations, there results

$$\frac{D}{Dt} (\delta v + (\delta \nu \cdot \nabla) w) = \frac{\delta \rho}{\rho^2} \nabla P - \frac{1}{\rho} \nabla \left( \delta P + \frac{B \cdot \delta B}{\mu_0} \right)$$

$$+ \frac{1}{\rho \mu_0} (B \cdot \nabla) \delta B, \quad (182)$$

where now

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}. \quad (183)$$

It is easiest to begin with a representation in cylindrical coordinates. The projected components of the left-side operator of equation (182) are:

$$e_R \left[ \frac{D}{Dt} (\delta v + (\delta \nu \cdot \nabla) w) \right] = \frac{D \delta v_R}{Dt} - 2\Omega \delta v_\phi \quad (184)$$

$$e_\phi \left[ \frac{D}{Dt} (\delta v + (\delta \nu \cdot \nabla) w) \right] = \frac{D \delta v_\phi}{Dt} + 2\Omega \delta v_R + R (\delta \nu \cdot \nabla) \Omega \quad (185)$$

$$e_z \left[ \frac{D}{Dt} (\delta v + (\delta \nu \cdot \nabla) w) \right] = \frac{D \delta v_z}{Dt}. \quad (186)$$

Let $\xi$ be the Lagrangian displacement field. From (78),

$$\frac{D}{Dt} \xi \equiv \Delta \xi = \delta v + \xi \cdot \nabla (R \xi) e_R = \delta v + e_R \xi \cdot \nabla (R \xi) - e_R \xi \Omega, \quad (187)$$

whence

$$\delta v_R = \frac{D \xi_R}{Dt}, \quad \delta v_\phi = \frac{D \xi_\phi}{Dt} - R \xi \cdot \nabla \Omega, \quad \delta v_z = \frac{D \xi_z}{Dt}. \quad (188)$$

Using (188) in (184)–(186), we arrive at

$$\frac{D}{Dt} \delta v + (\delta \nu \cdot \nabla) w = \xi + 2 \Omega \times \xi + e_R (R \xi \cdot \nabla) \Omega^2. \quad (189)$$

where $\Omega = \Omega e_R$. The overhead ‘dot’ notation indicates a time derivative of the vector components but not the unit vectors. In other words, the coordinates are now treated as locally Cartesian, with $e_R$ replacing $e_R$, $e_\phi$ replacing $e_\phi$, and $e_z$ remaining as such. The physical meaning of the additional two inertial forces on the right side of (189) is readily grasped: the first
is obviously the Coriolis force, and the second is difference between centrifugal and rotational forces, a tidal force that is in balance only at $\xi = 0$.

Since the time derivatives are taken following fluid elements, the appropriate fixed spatial coordinates when a time derivative is taken are those following the unperturbed fluid elements. Denoting these coordinates by primed notation $R'$, $\phi'$, and $z'$, we have

$$R' = R, \quad \phi' = \phi - t \Omega (R, z), \quad z' = z. \quad (190)$$

The usual chain rule then gives

$$\frac{\partial}{\partial R} = \frac{\partial}{\partial R'} - \frac{\partial \Omega}{\partial R} \frac{\partial}{\partial \phi'}, \quad \frac{\partial}{\partial \phi'} = \frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}. \quad (191)$$

and more generally

$$\nabla = \nabla' - t (\nabla \Omega) \frac{\partial}{\partial \phi'}. \quad (192)$$

In the WKB limit, the embedded perturbations of interest will take the plane wave form $\exp(i R k' + z' k_z + m \phi')$. As $m$ is the same in both frames, there is no primed superscript. Equation (192) then implies

$$\nabla = ik' - i m t (\nabla \Omega) \equiv ik(t), \quad (193)$$

when operating upon perturbed $\delta$-variables. The poloidal components of $k$ thus tend to align with $-\nabla$ as time goes on. A particularly useful result follows with some help from equation (33):

$$k(t) \cdot B(t) = k' \cdot B(0). \quad (194)$$

That is, even though the (Eulerian) wavenumber $k$ and toroidal field component $B_\phi$ are both time dependent, the dot product $k \cdot B$ is a local constant.

7.1.3. External forces. There remains the right side of (182). The constraint equations (87) and (88) now yield

$$\frac{\delta P}{\rho} \nabla P = \frac{\nabla P}{\gamma \rho} \xi \cdot \nabla \ln \rho \rho^{-\gamma} \quad (195)$$

$$\frac{1}{\rho \mu_0} (B \cdot \nabla) \delta B = \frac{-(k \cdot v_A)^2}{\rho \mu_0} \xi \equiv -(k \cdot v_A)^2 \xi, \quad (196)$$

where as earlier we use the Alfvén velocity,

$$v_A = \frac{B}{\sqrt{\rho \mu_0}}. \quad (197)$$

Equation (196) implies a very simple ‘Hooke’s Law’ behaviour for magnetic tension: the force is restorative and proportional to the displacement. Surprisingly, ‘restorative’ forces in rotating systems can have precisely the opposite effect, as we shall soon see.

Putting together equations (182), (189), (192), (195), (196) yields the set of equations

$$\frac{\delta P}{\rho} \nabla P \xi + 2 \Omega \times \xi + e_R (R \xi \cdot \nabla) \Omega^2 - \nabla P \xi \cdot \nabla \ln \rho \rho^{-\gamma} + \frac{ik(t) \delta P_{\text{tot}}}{\rho} = 0 \quad (198)$$

$$\delta P_{\text{tot}} = \delta P + \frac{B \cdot \delta B}{\mu_0} \quad (199)$$

$$k(t) \cdot \xi = 0, \quad (200)$$

$$k(t) = k' - mt (\nabla \Omega), \quad (201)$$

It is useful to eliminate the final total pressure term $\delta P_{\text{tot}}$, a process effected by taking the dot product of (198) with $k$ to isolate the final term of equation (198). Then, equation (198) is replaced with

$$\left( I - \frac{kk'}{k^2} \right) \cdot \mathcal{L}(\xi) = 0, \quad (202)$$

where $I$ is the identity matrix with entries $\delta_{ij}$ (Kronecker delta), and $\mathcal{L}(\xi)$ is the linear operator

$$\mathcal{L}(\xi) = \frac{\delta P}{\gamma \rho} \xi \cdot \nabla \ln \rho \rho^{-\gamma}. \quad (203)$$

Each of the individual terms in $\mathcal{L}(\xi)$ is readily identifiable. From left to right we have acceleration, magnetic tension, Coriolis, tides, and finally buoyancy. Note as well the similar $\xi \cdot \nabla$ couplings that the tidal and buoyant forces both invoke.

We shall argue in section 9 below that this entropy-angular momentum pairing is important to an understanding of the Sun’s rotation pattern.

The lead factor in (202) is a projection operator, ensuring that only $\mathcal{L}(\xi)$ forces perpendicular to the wavevector $k$ enter into dynamical consideration. Finally, given the generality of our assumptions, the magnetic force is surprisingly simple, appearing only as the spring-like term $(k \cdot v_A)^2 \xi$.

7.2. The magnetorotational instability

7.2.1. Reduced system. To begin our exploration of the remarkable equation (202), consider first a disc in which the local equilibrium pressure gradient is negligible. (Such conditions typically prevail in the midplane of a rotationally supported disc.)

For axisymmetric ($m = 0$) disturbances, the wavenumber is independent of time, and we will begin here. With disturbances proportional to $\exp(ik_z z)$, only displacements in the plane of the disc enter, and the reduced equations are:

$$\xi_R - 2\Omega \xi_o + \frac{\partial \Omega^2}{\partial \ln R} \xi_R + (k \cdot v_A)^2 \xi_o = 0, \quad (204)$$

$$\xi_o + 2\Omega \xi_R + (k \cdot v_A)^2 \xi_o = 0. \quad (205)$$

7.2.2. Hydrodynamic limit. When the magnetic field vanishes, the two equations are yet more simple,

$$\xi_R - 2\Omega \xi_o + \frac{\partial \Omega^2}{\partial \ln R} \xi_R = 0, \quad (206)$$

$$\xi_o + 2\Omega \xi_R = 0. \quad (207)$$
The local normal modes with time dependence \( \exp(-i \omega t) \) satisfy the dispersion relation
\[
\omega^2 = 4 \Omega^2 + \frac{\partial \Omega^2}{\partial \ln R} = \frac{1}{R^4} \frac{\partial (R^4 \Omega^2)}{\partial R} \equiv \kappa^2, \quad (208)
\]
a quantity known as the epicyclic frequency. Displaced fluid elements, viewed from the point-of-view of an observer at the undisturbed circular orbit location, execute a retrograde ellipse. It is retrograde in the sense that the element moves round the ellipse clockwise (counterclockwise) when the main circular orbit is counterclockwise (clockwise). The center of the ellipse is the undisturbed location of the element’s circular orbit.

This is a simple consequence of angular momentum conservation. The element is moving on the ellipse—the ‘epicycle’—around the central point while maintaining the angular momentum of the undisturbed circular orbit. Thus, when the element is on its outward excursion at a greater radial distance, it must rotate a little more slowly. It lags behind relative to a point on the undisturbed circular orbit. When the element is on its inward excursion, it must rotate a little more rapidly. This results in retrograde circulation. In astrophysical discs, generally \( \kappa < 2 \Omega \). From equation (207), the epicycle is elongated along the circular orbit, with major-to-minor axis ratio of \( 2 \Omega/\kappa \).

Equation (208) indicates that if the specific angular momentum \( R^2 \Omega \) increases outward, then \( \kappa^2 > 0 \) and our description of elliptical epicycles is entirely self-consistent. But if the specific angular momentum decreases with increasing \( R \), \( \kappa^2 < 0 \) and the displacements do not form a bounded ellipse, but instead grow exponentially. In this case, the flow is unstable to infinitesimal axisymmetric disturbances. The constraint that the specific angular momentum increase outward for stable flow is known as the Rayleigh criterion, after its discoverer Lord [59]. The Rayleigh criterion is satisfied for Keplerian flow \( R^2 \Omega \propto R^{1/2} \), galactic flat rotation curves \( \propto R \), and essentially all other astrophysical environments.

The Rayleigh criterion clearly is not a guarantee of stability to nonaxisymmetric disturbances. A sharp outwardly increasing discontinuity in \( \Omega \) in a disc would be liable to Kelvin–Helmholtz instabilities; less dramatically, a simple inflection point in the flow (where \( d^2 \Omega/d R^2 = 0 \)) may be enough for destabilization [1]. But these are nonlocal flows in the sense that the rotational profile exhibits a marked deviation from the undisturbed location of the element.

The onset of turbulence is a deeply contentious subject, one that is difficult to study analytically, numerically, or in the laboratory. Laboratory experiments designed to measure the rotation velocity field by Doppler techniques find no nonlinear shear instabilities in Keplerian-like flows [36, 62]. This is in accord with the findings of early numerical work [11] and a very detailed follow-up [40]. A recent claim for turbulence and enhanced angular momentum transport in Keplerian shear flow [49] is now understood to be a boundary layer effect stemming from presence of the axial endcaps [3, 48]. It was precisely to mitigate these effects that the Princeton experiment, described in [62], split the endcaps of their Couette apparatus into differentially rotating rings. Even with such precautions, end effects can still induce turbulent flow. When viscous effects are sufficiently small, however, the turbulence is spatially confined, and as of this writing experiments and simulations seem to be in good accord (Avila, private communication). Whatever turbulence is present in laboratory Couette flow is due to viscous boundary layers, not local shear flow.

2.3. Magnetic fields. The inclusion of the apparently restorative magnetic acceleration term \((k \cdot \mathbf{v}_k)^2 \xi \) in equations (204)–(205) produces a surprisingly expanded dispersion relation. If we seek solutions of the form \( \exp(\pm \kappa z + i \omega t) \), the resulting equation is
\[
\omega^4 - \omega^2 [\kappa^2 + 2(k \cdot \mathbf{v}_k)^2] + (k \cdot \mathbf{v}_k)^2 \left[(k \cdot \mathbf{v}_k)^2 + \frac{\partial \Omega^2}{\partial \ln R}\right] = 0. \quad (209)
\]
This is a quadratic equation in \( \omega^2 \), and it is easily seen that if the final constant term is negative, solutions exist with \( \omega^2 < 0 \), i.e. local instabilities. But it is easy for this constant term to be negative because \( \partial \Omega^2/\partial R < 0 \) quite generally in astrophysical discs. Thus, provided the magnetic field is not too strong, \( k \cdot \mathbf{v}_k \) can always be chosen sufficiently small at long wavelengths so that instability is present. This is the magnetorotational instability, or MRI, and it is thought to be the underlying cause of turbulence in even moderately ionized astrophysical discs [9, 10]. Figure 5 shows an early 2D nonlinear simulation of the MRI, starting with a tube of magnetic flux.

The MRI is endlessly surprising. The reader may enjoy the algebraic exercise of showing that the maximum growth rate of the MRI is
\[
|\omega_{\max}| = \frac{1}{2} \left| \frac{d \Omega}{d \ln R} \right|, \quad (210)
\]
which is achieved at wavenumbers corresponding to
\[
(k \cdot \mathbf{v}_k)_{\max}^2 = \Omega^2 \left[ 1 - \left( \frac{\kappa}{2 \Omega} \right)^2 \right], \quad (211)
\]
for a displacement eigenvector corresponding to
\[
\xi_R = -\xi_\phi. \quad (212)
\]
Though we have not shown it here, this applies to the case of any magnetic field geometry, including a purely toroidal field [10]. In other words:

Instability occurs only in the presence of a magnetic field, but neither the instability criterion, nor the maximum growth rate, nor the most unstable displacement eigenvector depend upon any properties of the magnetic field, including its geometry.
Now all of this is one whopping surprise, and the fact that none of this was even remotely anticipated undoubtedly contributed to preliminary MRI work languishing uninvestigated for more than thirty years, despite some interesting leads on global instabilities of axial magnetic fields in Couette experiments [21, 71]. Chandrasekhar’s text, for example, noted in passing that the Rayleigh criterion was not recovered in the limit $B \to 0$, speculated that field-freezing ought to be involved somehow, and left it at that.

These seemingly remarkable properties of the MRI can be understood with the help of a simple physical model. We have noted that the acceleration brought on by the magnetic field line tension is identical in form to a simple spring-like coupling in which the force is proportional to (minus) the displacement. Imagine then two nearby point masses, connected by a weak spring, which are in orbit about a central mass. The relative displacement of the two masses would satisfy a system of equations mathematically identical to (204) and (205). We will refer to the spring constant (per unit mass) as $K$. It corresponds to $(k \cdot r_1)^2$ in the MHD system.

Let $m_i$ refer to the mass that is orbiting slightly closer to the centre and $m_o$ to the mass slightly farther out. The spring pulls

**Figure 5.** The original 2D numerical MHD simulation by [33] showing the fastest growing mode of the magnetorotational instability with an initial net magnetic field in the $z$-direction. The perturbations grow until they form channeling solutions which disrupt the initial configuration. In 3D, the non-linear saturation of the MRI leads to sustained turbulence and angular momentum transport.
max 0.3, this may be written as a quadratic equation in max. Thus, in the MHD 22. The dispersion relation resulting from 40, i.e. only Alfvén and + = Ω ω ⋅ .

The magnetic force between two orbiting fluid elements may be represented as a spring-like force connecting two masses. Here, the spring is a simple blue line, the masses are mi and m₀, and the figure shows the tethered configuration at three subsequent times, proceeding anticlockwise round the orbit. See text for an explanation.

back on m₀ because it is tethered to the more slowly orbiting mass mia, and forward on mₐ for the oppositely analogous reason. But the backward-pulling torque on mₐ causes it to lose angular momentum, so the mass to drop to an orbit closer in, where it actually speeds up. Conversely, the forward-pulling torque on m₀ causes it to acquire angular momentum, so the mass moves outward to a more slowly rotating orbit. The net torque between the two masses increases, mₐ spirals inward more rapidly, m₀ spirals out more rapidly, and the process runs away (see figure 6). The very act of transporting angular momentum from one mass to another by a reactive, spring-like force is intrinsically unstable.

This description tacitly assumes that the spring constant K is not so large that there are many oscillations over a time scale rapid compared with an orbital period. Were this the case, the orbital dynamics would become irrelevant, and only the spring-induced oscillations would matter. Indeed, when k ⋅ vₐ is large, the MHD dispersion relation reduces to ω² = (k ⋅ vₐ)², i.e. only Alfvén waves are present.

Moreover, in the limit of very small K, the instability is still present, but the rate of growth becomes arbitrarily long. Therefore, there must be a well-tuned value of K that maximizes the angular momentum transfer rate between the masses without causing a coupling so strong that it isolates the mass pair from the orbital dynamics, thereby stabilizing the interaction. This value of K, call it Kmax, is in fact the right side of equation (211). Since Kmax depends only on the properties of Ω, so will the maximum growth rate |ωmax|. Thus, in the MHD problem, the magnetic field strength sets the absolute scale for the fastest growing wavenumber kmax, but plays no role in determining |ωmax|.

For the astrophysically important case of Keplerian flow, κ = Ω and

\[(k \cdot vₐ)^2 = 15Ω^2/16, \quad |ωmax| = 3Ω/4 \quad (Kepler). \quad (213)\]
The most rapidly growing wavenumber increases a factor in amplitude of 111 each orbit!

There are a host of surprises associated with the detailed behaviour of the MRI, but perhaps the biggest of all is the fact that the calculation is worth doing at all: the MRI is formally present and vigorous in a rotating disc when the magnetic field is, it would seem, ‘negligibly small’. The perceived complexities of treating a significantly magnetized disc in which an embedded magnetic field was generating yet more field by radial shear was the principal reason for avoiding a detailed MHD analysis of accretion discs. Throughout the 1970s and 80s, the topic remained anathema to most disc theorists. The understanding that the most salient features of MHD disc theory remain accessible in the limit B → 0 with k ⋅ vₐ finite, a regime that marginalizes the importance of the equilibrium magnetic field behaviour, was the key realization that started our modern understanding of accretion disc turbulence.

72.4. General axisymmetric disturbances. Using equation (202), the full axisymmetric problem presents no particular difficulties. Equation (205) for ξᵦ remains valid. The normal modes have the plane wave form

ξ ∝ exp(ik₃R + ik₉z − iωt) \quad (214)

with k₃k₉ + k₉ζ = 0. The dispersion relation resulting from the z component of (202) is:

\[\frac{k^2}{k_z^2} \omega^4 + \omega^2 \left[ \frac{1}{ρ^2} (D^2 D) Dσ + \frac{1}{R^2} D σ^2 \right] - 4Ω^2 (k \cdot vₐ)^2 = 0, \quad (215)\]

where

\[ω^2 = \omega^2 = (k \cdot vₐ)², \quad D ≡ \left[ k₉ \frac{∂}{∂z} - \frac{∂}{∂R} \right], \quad \sigma = ln Pρ⁻γ. \quad (216)\]

(The R component produces only a longer route to the same equation). It is telling to compare the marginal stability condition of equation (215) with and without the magnetic field. In the absence of any magnetic field, the marginal stability condition is simply

\[\frac{1}{ω^2} (D^2 D) Dσ + \frac{1}{R^2} D σ^2 = 0. \quad (217)\]

With x ≡ k₃/k₉, this may be written as a quadratic equation in x:

\[x^2 N_x^2 + \frac{x}{ω^2} \left[ \frac{∂P}{∂z} \frac{∂σ}{∂z} + \frac{∂P}{∂R} \frac{∂σ}{∂R} - k₉ ω^2 \frac{∂σ}{∂t} \right] + N_σ^2 + \frac{1}{ρ^2} \frac{∂^2 σ}{∂R^2} = 0, \quad (218)\]

where we have introduced the Brunt–Väisälä frequencies

---

6Technically, the waves include an Alfvénic and slow mode branch.
\[ N_z^2 = -\frac{1}{\gamma\rho} \frac{\partial P}{\partial z} \frac{\partial \sigma}{\partial z}, \quad N_R^2 = -\frac{1}{\gamma\rho} \frac{\partial P}{\partial R} \frac{\partial \sigma}{\partial R}. \]  

(219)

To ensure stability, the polynomial on the left of (218) should be positive somewhere and have no zeros for real \( x \). The most economical way of doing this is to insist that the sum of the \( x^2 \) coefficient and constant term be positive (then at least one of the terms must itself be positive), and that the quadratic discriminant \((b^2 - 4ac)\) is everywhere negative. The first of these requirements is easily written down:

\[ N^2 + \frac{1}{R^3} \frac{\partial^2 P}{\partial R^2} > 0, \quad N^2 = -\frac{1}{\gamma\rho} (\nabla P) \cdot \nabla \sigma. \]  

(220)

The second is more of an algebraic challenge and will not be repeated here. The result of the calculation is the condition [5, 69]

\[ -\frac{\partial}{\partial z} \left( \frac{\partial^2 \sigma}{\partial R^2} \frac{\partial \sigma}{\partial z} - \frac{\partial^2 \sigma}{\partial R \partial z} \right) > 0. \]  

(221)

The motivated reader who may wish to verify this directly should note that equation (180) is needed on more than one occasion in the course of the derivation.

Normally, \(-\partial P / \partial z > 0\), so that equation (221) states that the \( \phi \) component of \( \nabla \sigma \times \nabla P \) should be positive for stability. An example of such a stable configuration is \( \ell \) increasing from the rotation axis, stratified on constant \( R \) cylinders, with a spherically symmetric entropy profile increasing outward. Should the \( \sigma \) gradient acquire a negative \( z \) component with a positive \( R \) component, the same angular momentum distribution would be unstable.

The surprise comes when we add a weak but otherwise arbitrary magnetic field, with at least some poloidal component. Repeating our calculation in exactly the same way, but now including the \((k \cdot v_A)^2\) terms in equation (215), leads to the following stability criteria [5]; see also [30, 53, 2]:

\[ N^2 + \frac{\partial}{\partial \ln R} \frac{\partial \Omega^2}{\partial \ln R} > 0, \]  

(222)

\[ -\frac{\partial P}{\partial z} \left( \frac{\partial^2 \sigma}{\partial R^2} \frac{\partial \sigma}{\partial z} - \frac{\partial^2 \sigma}{\partial R \partial z} \right) > 0. \]  

(223)

The sole difference between (220)–(221) and (222)–(223) is that the \( \ell \) gradients have been replaced by \( \Omega \) gradients! This is remarkable considering the general disposition of the magnetic field, but there is a certain logical consistency to all this: by tethering fluid elements, the presence of the magnetic field creates a pathway tapping into the true free energy source in this problem, which is the shear itself. We encountered a similar phenomenon in our study of magnetothermal behaviour, in which the stability went from being regulated by a conserved quantity gradient (the entropy) to a free energy gradient (the temperature).

Free energy gradients are special. Not only do they make life possible, they are palpable in a way that entropy or angular momentum gradients are not: they hurt. The extreme unpleasantness of a punch in the nose is due to the free energy source provided by the relative shear present between one person’s head and another’s fist. A burn from a hot stove is a thermal counterpart. It is not the large entropy difference that gives us pain, it is \( dT \). We don’t particularly care about how many microstates gives rise to the stove’s macrostate, the salient feature is that it is hot. It is interesting, therefore, that while free energy gradients normally make their presence known in fluids only through rather ghostly diffusive effects (thermal conduction and viscosity), a magnetic field is a catalyst for turning them into agents of active dynamical destabilization.

Equations (220) and (221) are known as the Høiland criteria for the stability [69]; so (222) and (223) may be thought of as magnetic Høiland criteria. They pertain to axisymmetric perturbations, and this is one case where we must closely heed to this restriction. Our next surprising example will show why.

73. Convection and rotation

We return once again to the master equation (198 et sequentia) to consider nonaxisymmetric perturbations in rotating but nonshearing systems. The space-time dependence of our displacements is now \( \exp(i k_R R + m\phi + k_z z - \omega t) \). Choosing any two components of this equation together with the mass conservation condition (200) leads to a dispersion relation quite analogous to the general axisymmetric condition (215), but with the addition of one more remarkable term proportional to \( m^2 \):

\[ \omega^4 + \omega^2 \left[ k_z^2 \left( \frac{1}{\gamma\rho} (\nabla P)^2 - 4\Omega^2 \right) + \frac{m^2}{\rho R^2 k^2} (\nabla P) \cdot (\nabla \sigma) \right] - \frac{4k_z^2}{k^2} \Omega^2 (k \cdot v_A)^2 = 0, \]  

(224)

where now \( k^2 = k_R^2 + k_z^2 + m^2/R^2 \). This is the magnetized version [13] of a classical result due to [25], but this is one classic whose lessons have not been fully absorbed. What we should learn from (224) is that moderate rotation cannot stabilize an unfavourable alignment of the pressure and entropy gradients (entropy and pressure increasing in the same direction). This is because purely azimuthal wavenumbers with \( k_z = 0 \) are apparently immune to Coriolis forces. This is all the more surprising because when \( k_R = 0 \) as well, the growth rate is the maximum possible. In the case of a slowly rotating star like the Sun, in which the background pressure gradient is (spherically) radial to a part in \( 10^3 \), the fluid displacements associated with this maximum growth rate are also radial to the same high level of accuracy. Coriolis deflections surely ought to produce distortions much larger than this! So what is happening?

What is happening is that a condition well-known to geophysicists is being set-up: geostrophic balance. Latitudinal and longitudinal motions on the Earth’s surface are subject to the Coriolis force due to the planet’s rotation. A time steady balance can be achieved between this velocity dependent force and the pressure gradient. This gives rise, in the northern hemisphere, to clockwise circulating high pressure regions and counterclockwise circulating low pressure regions. (The pressure gradient force is balanced by a velocity flow along...
isobars.) Here, ‘heliostrophic balance’ is achieved by balancing the azimuthal Coriolis term $2 \Omega \zeta$, with $(1/\rho)(dP/d\phi)$, and there is no $\phi$ deflection. The radial equation of motion therefore has no Coriolis term, and is instead a simple balance between the radial acceleration and the unstable buoyancy force. As far as fluid element displacements are concerned, it is as though the Coriolis force were absent.

That ought to be a surprise, and more than a bit unsettling. As we shall see in the next section, there is evidence that the Sun is a few degrees warmer on average at the poles compared with the equator. This is often said to be a consequence of the Coriolis force: after all, it must be easier for a hot, outwardly rising gas parcel to move parallel to the rotation axis at high latitudes toward the pole then for it to move orthogonal to the rotation axis at lower latitudes toward the equator. If so, it is a more subtle process, for we have just seen that there is nothing in the simplest version of buoyant instability to suggest that Coriolis forces have any effect on the most rapidly moving convective blobs.

Remember, these considerations apply only to uniform rotation. Does background shear at a level present in the Sun make a difference? Indeed it does. That however brings its own plate of surprises, which we will discuss in section 9.

8. The Papaloizou–Pringle instability

The Coriolis force exerts a powerful stabilizing influence on the linear response of Keplerian or near-Keplerian discs. A constant density disc, for example, responds to perturbations with propagating inertial waves driven by the angular momentum gradient, and there are no linear instabilities.

Matters are no longer so straightforward when a boundary is present, which will bring in its own dynamical behaviour. One such remarkable instability was discovered by [51]. It caused a great stir at the time, because it demonstrated that a then-popular (non-Keplerian) disc model of launching jets was in fact quite unstable. The Papaloizou–Pringle instability is surprisingly subtle, yet, as shown by [31], occurs in very simple systems. Our inventory of gas dynamical surprises would be incomplete without at least a cursory visit.

8.1. Setting the stage

Start with a standard disc. For simplicity, we will take the density to be constant. We ignore vertical structure, so the disc is really an axial cylinder. The radial extent of the disc is assumed to be very narrow. The mid-radius will be denoted as $R_0$, and at a radius $R$ in the disc we define a local Cartesian coordinate by $R = R_0 + x$ with $x \ll R_0$. The central rotation rate $\Omega_0$ is given by

$$\Omega_0^2 = \frac{GM}{R_0^3},$$

where $M$ is the central mass. Expanding to linear order around $R_0$,

$$\Omega = \Omega_0 (1 - qx/R_0).$$

We introduce the enthalpy function, $d\mathcal{H} = dP/\rho$, where as usual $P$ is pressure and $\rho$ density. Then the equation of motion for the unperturbed flow may be written

$$R\Omega^2 = \frac{d\mathcal{H}}{dR} + \frac{GM}{R^3},$$

or, expanding to leading order in $x$:

$$\frac{d\mathcal{H}}{dx} = (3 - 2q)\Omega_0^2 x.$$

This is our formal background equilibrium condition.

Consider Eulerian velocity perturbations $\delta u$ with radial ($x$) and azimuthal ($\phi$) components. The perturbed enthalpy is denoted $\delta \mathcal{H}$. A perturbed variable $\delta Q$, is assumed to have a space-time dependence of

$$\delta Q(x, y, t) = \delta Q(x) \exp[i(ky - \omega_0 t - \omega t)] = \delta Q(x) \exp[iky - i(\omega + kR\Omega_0)t].$$

We work in an inertial frame, so the effective frequency has a rotational kinematic boost $kR\Omega_0$ included. Note the distinction between $\Omega_0$ and $\Omega$, and the use of $R$, as opposed to $R_0$.

The equation of mass conservation is

$$\frac{d(\delta u_x)}{dx} + i k \delta u_x = 0.$$  

Introducing the notation

$$\tilde{\omega} = \omega + q\Omega kx,$$

the equations of motion may be taken from (184) and (185):

$$-i \tilde{\omega} \delta u_x - 2\Omega \delta u_y = - \frac{d(\delta \mathcal{H})}{dx},$$

$$-i \tilde{\omega} \delta u_y + \frac{k^2}{2\Omega} \delta u_x = - i k \delta \mathcal{H},$$

where $k^2$ is given by (208), and we have dropped the ‘0’ subscript on $\Omega$, now a constant.

Eliminating $\delta \mathcal{H}$ and $\delta u_y$ from (230)–(233) yields a Laplace equation for $\delta u_x$:

$$\frac{d^2(\delta u_x)}{dx^2} - k^2 \delta u_x = 0.$$

Surprisingly, since the equations depend upon $x$ through $\tilde{\omega}$, this is independent of everything except the constant wave-number $k$. The general solution is a superposition of sinh and cosh functions:

$$\delta u_x = A \cosh(kx) + B \sinh(kx).$$

A similar result is found for surface water waves [41], a consequence of divergence- and curl-free flow. As with water waves, the crucial dynamics here lies in the free surface boundary condition: the Lagrangian change in the pressure (or enthalpy here) must vanish. In other words, at the free surface $x = \pm s$,

$$\delta \mathcal{H} = -\xi \frac{d\mathcal{H}}{dx},$$

where $\xi$ is the radial displacement,
\[ \xi = \frac{i\delta u_x}{\tilde{\omega}} \]  

and \( dH/dx \) is the equilibrium enthalpy profile (228). Expressing (236) in terms of \( \delta u_x \) yields

\[ \omega^2 \frac{d(\delta u_x)}{dx} + \left( k\omega - \frac{\kappa^2}{2\Omega} + k^2 \frac{dH}{dx} \right) \delta u_x = 0, \]  

applied at \( x = \pm s \), the thin disc boundaries. The emergent dispersion relation, in the limit \( ks \to 0 \), is

\[ \omega^2 - \Omega^2 \omega^2 + 3(3 - q^2)k^2\Omega^4 = 0. \]  

The condition for unstable modes to be present is simply \( q > \sqrt{3} = 1.732 \) [52]. This contrasts with the local Rayleigh instability criterion of \( q > 2 \). It is easier to destabilize the flow if the disc boundaries are free. (An example of Papaloizou–Pringle instabilities in protostellar discs is discussed by [43], in which the instability is mediated by Rossby-like modes.)

Goldreich et al [31] provided a detailed physical explanation for how this arises. It depends on the fact that a wave propagating through a moving fluid can either increase the local energy density on average, in which case it is a positive energy wave, or it can decrease the energy density, in which case it is a negative energy wave. The onset of instability for \( q > \sqrt{3} \) corresponds to the appearance of regions of both positive and negative perturbation energy densities, lying on either side of a so-called corotation radius. Energy flows from the region of negative energy, and the loss causes an increase in amplitude, i.e. it is yet more negative. The energy flows into the region of positive energy density, also increasing its amplitude. In other words, the flow of energy from the negative to the positive energy region is intrinsically unstable!

The enlightening surprise here is that although one’s intuition can be shaped by local flow behaviour, under conditions in which the edge of the system is well-defined, entirely new dynamics can appear. In the particular example we have analysed, it might be said that conditions were artificial and certainly not directly applicable to any known astrophysical environment. But the physical content of the Goldreich et al explanation suggests that this casual dismissal misses the point. The key notion is one of trapped waves, the existence of finite regions in waves of positive or negative energy density are confined. In the problem we chose this was set up in a simple way by the use of edge dynamics; more complex examples might involve forming such regions by appropriate background gradients in the equilibrium flow. The elegant physics responsible for the destabilization is just the same.

### 8.2. Another example

There is more to learn and more interesting surprises to be found with our simple example. Consider the same problem, this time with a hard wall in place at \( x = 0 \), so that the local relevant boundary condition is \( \delta u_x = 0 \). Then \( \delta u_x \) is proportional to \( \sinh(kx) \). At \( x = s \), the free surface condition becomes

\[ \tilde{\omega}^2 + \left[ k\omega - \frac{\kappa^2}{2\Omega} + ks\Omega^2 (3 - 2q) \right] \tanh ks = 0, \]  

where \( \tilde{\omega} \) is understood to be evaluated at \( x = s \). This is analogous to the dispersion relation that emerges for Rayleigh’s surface water waves in a sea of depth \( s \) [41]:

\[ \omega^2 = g k \tanh(k s), \]  

where \( g \) is the (uniform) gravitational acceleration. Indeed, if we set \( q = 2 \) so that \( \kappa^2 = 0 \) and epicyclic oscillations are eliminated, our dispersion relation becomes

\[ \tilde{\omega}^2 = (s\Omega^2) k \tanh(k s), \]  

the equivalent of water waves (Doppler boosted in frequency by \( gks\Omega \) with \( g = \Omega^2 s \)).

The surprise comes when we put back the epicyclic waves, creating an interplay between the two types of response. With \( \kappa^2 = 2\Omega^2 (2 - q) \), the boundary condition at \( x = s \) becomes the dispersion relation

\[ \tilde{\omega}^2 + \tilde{\omega} \Omega (2 - q) \tanh(k s) + ks \tanh(k s) \Omega^2 (3 - 2q) = 0. \]  

It suits our present purposes to leave this in terms of \( \tilde{\omega} \) instead of \( \omega \).

Instability corresponds to \( \omega \), and thus \( \tilde{\omega} \), acquiring a positive imaginary component. This, in turn, necessitates the condition

\[ \Omega^2 (2 - q)^2 - 4ks \Omega^2 \tanh(k s) < 0 \]  

or

\[ \frac{\tanh ks}{ks} < \frac{(3 - 2q)}{(2 - q)^2}. \]  

The left side function has a maximum of 1 at small \( ks \), falling to a minimum of zero at large \( ks \). There is instability (in fact, overstability), if \( q < 1.5 \). Keplerian flow is once again stable, but uniform rotation (\( q = 0 \)) is not! The key is the sign of the pressure gradient: in the presence of a hard wall at the ‘bottom’, all unstable modes are characterized by an increasing outward pressure. The edge modes reinforce the epicyclic oscillations under these conditions—an increasing outward pressure makes the restoring gravity effectively more powerful—whereas no such reinforcement is present when the pressure decreases outward.

Even in this very simple system, we have discovered two very different types of instability depending upon which boundary condition is used. One depends upon extracting the free energy of differential rotation by the joint presence of positive and negative energy waves, the other taps into an adverse pressure gradient to mutually reinforce surface gravity and epicyclic oscillations. The surprise here is that all this occurs quite apart from local shear instability, the classic focus of the rotational destabilization. Boundary conditions, which can be difficult to pin down for accretion discs, cause qualitative changes in behaviour.

### 9. Convection and rotation in the Sun

#### 9.1. Helioseismology results

The elucidation of the dynamical state of the Sun’s interior, including a detailed rotation profile, is one of the most impressive achievements of 20th century astronomy. Not only do we
have more information for the velocity field of the Sun than for any other astrophysical fluid, we have far more information about how the interior of the Sun is rotating than we have for a typical laboratory fluid experiment! The physics behind this remarkable observational feat is the ability to extract thousands of global eigenmode frequencies in the observed acoustic oscillation spectrum of the Sun [68]. Differences between the frequencies corresponding to different azimuthal wavenumbers then allow the precise angular rotation rate $\Omega$ to be determined as a function of spherical radius $r$ and colatitude $\theta$. By analogy with terrestrial earthquakes, which allow the Earth’s interior to be probed, the techniques that allow the precision determination of the Sun’s interior state is known as helioseismology.

Figure 8 shows the results of helioseismology analysis. A meridional slice of the Sun is depicted. The outermost circular arc is the solar surface. The black interior curves are contours of constant angular velocity. (Ignore the white curves for the moment.) Reckoned in units of nano-Herz ($10^{-9}$ rotations per second, abbreviation nHz), the uppermost polar contour is about 320, equatorial rotation is 460, and the intermediate contours are equally spaced intervals. An average rotation rate of 400 nHz corresponds to $2.5 \times 10^{-6} \text{s}^{-1}$, about a one month period. Latitudinal variations amount to some 15%. The question is why do the contours look like this? In particular, why are the rotation contours so insensitive to depth in the bulk of the outer layers? This will be our focus here.

The interior of the Sun is distinguished by three principal zones. A small inner core, comprising some 10% of the Sun’s mass is the region of nuclear energy production. From this core out to $0.72R_\odot$, the energy diffuses outwards at a rate proportional to the temperature gradient. (The radius of the Sun is denoted $R_\odot$.) This region is known as the radiative zone. The outer 28% of the Sun, the third region, is in a state of turbulence in which the thermal energy is transported by convection: buoyant hot gas rising, together with cooler, relatively heavy, gas sinking. This may be thought of as a sort of boiling due to the intense heating from below. The convective motions are typically very slow, measured in 10s to 100s of meters per second, because of the efficiency of this mechanical transport process. A tiny velocity can move a great deal of bulk thermal energy compared with radiative diffusion. Very near the surface, however, convection velocities can become transonic.

Regions of different energy transport—diffusion, slow convection, rapid convection—leave their distinctive imprint on the profile of the Sun’s differential rotation. At the base of the convective zone and into the radiative zone the rotation is only weakly dependent on $\theta$. In the bulk of the convective zone, by contrast, where the turbulent velocity is low, the rotation is only weakly dependent on $r$. In the near surface layers there is a strong dependence upon both $r$ and $\theta$. The shear is very marked here.

The question we pose here concerns the simplest portion of the solar rotation problem. In the bulk of the convection zone, the velocities are very small and the stratification is adiabatic to a remarkable accuracy: probably about 1 part in $10^5$. This means that the pressure should be very well described by a barotropic equation of state, $P = P(\rho)$. Indeed, the near
spherical symmetry by itself should be enough to ensure a barotropic
equation of state? Why then is the rotation not stratified
on cylinders?

Start with equation (27) in a dimensionless form:

$$R \frac{\partial \ln \Omega^2}{\partial z} = \frac{1}{\rho^2 \Omega^2} (\nabla \rho \times \nabla P) \cdot e_\phi. \quad (246)$$

From the helioseismology data, the left side of this equation is
a number of order 10%. The right side, if we simply go by
magnitudes and do not worry about the alignment of the gra-
dients, is a number of order 10^2 (ratio of gravity to centrifugal
forces). The point is that if there is the slightest misalign-
ment of pressure and density isosurfaces—or, equivalently,
of temperature and density isosurfaces—this will be reflected
in greatly magnified, easily observed, baroclinic rotational
velocity gradients.

Barotropic rotation (which of course includes uniform
rotation) distorts the temperature and density surfaces from
spherical, while still permitting the equilibrium forces to be
derived from a potential function. This means that isobaric
and isothermal surfaces coincide with one another (and with
equipotentials). This nonspherical distortion, however, is gen-
erally inconsistent with radiative equilibrium (e.g. [22, 63]).

In a star rotating on cylinders, the fact that pressure and
density surfaces coincide when $\Omega = \Omega(R)$ may be read off
from equation (27); that these surfaces coincide with those of
the effective potential $\Phi$ follows directly from inspection of
the hydrostatic equilibrium equation

$$\frac{1}{\rho} \nabla P = -\nabla \Phi, \quad (247)$$

where $\Phi$ is the effective potential

$$\Phi = \Phi_\pi - \int^R R' \Omega^2 \, dR', \quad (248)$$

and $\Phi_\pi$ the gravitational potential. The radiative flux $F$ may be
written in the form

$$F = -\chi \nabla T = -\chi \frac{dT}{d\Phi} \nabla \Phi, \quad (249)$$

where the diffusivity $\chi$ depends only on $\rho$ and $T$—and there-
fore only on $\Phi$. Then,

$$\nabla \cdot F = -\frac{d}{d\Phi} \left( \chi \frac{dT}{d\Phi} \right) \nabla \Phi^2 - \chi \frac{dT}{d\Phi} \nabla^2 \Phi. \quad (250)$$

Since

$$\nabla^2 \Phi = 4\pi G \rho - \frac{1}{R} \frac{d}{dR} (R^2 \Omega^2), \quad (251)$$

when $\Omega$ is constant, everything on the right side of (250) is
constant on equipotential surfaces, except for $|\nabla \Phi|^2$. This
latter term can’t possibly be constant, since the equipoten-
tials are compressed at the poles and distended at the equa-
tor. Thus the right side cannot be identically zero. In the case of
$\Omega = \Omega(R)$, the same surfaces coincide and the functional
constraints imposed by setting $\nabla \cdot F = 0$ are still too restric-
tive to be compatible with hydrostatic equilibrium (e.g. [68]).
If $\Omega = \Omega(R, z)$ the restrictions break down completely.

Radiative equilibrium is generally thought to be main-
tained by tiny velocities (less than terrestrial plate tectonics)
of meridional, or Eddington–Sweet, circulation. The classical
argument, as presented by [63] and others, is based on tapping
the star’s thermal energy gradient source to move matter in
bulk to offset radiative imbalance. However, it is not energy
per se that is needed, it is in particular heat—i.e. entropy. It is
much more difficult to satisfy the radiative entropy equation.
The focus here is on the region where the entropy gradient
would vanish in a star with barotropic rotation, so any bulk
motion is locally ineffective. What then? One solution is that
the rotation profile could become baroclinic. Curiously, this is
just what the solar data show, in just the right location.

Let us return to, with the right side written in spherical
coordinates:

$$R \frac{\partial \Omega^2}{\partial z} = \frac{1}{\gamma \rho r} \left( \frac{\partial \sigma}{\partial r} \frac{\partial P}{\partial \sigma} - \frac{\partial \sigma}{\partial \theta} \frac{\partial P}{\partial \theta} \right), \quad (252)$$

The advantage of using the entropy variable $\sigma = \ln \rho P^{\gamma - 1}$ is that
in the convective zone down to the radiative boundary the sper-
chiral rotation would produce an energy flow huge in
comparison with the Sun’s. On the right side of the above equa-
tion we need only retain the term involving $\partial P/\partial r$. This gives,

$$R \frac{\partial \Omega^2}{\partial z} = \frac{g}{\gamma \rho r}, \quad (253)$$

where $g = -(1/\rho)(\partial P/\partial r)$ is the dominant radial gravitational
field magnitude. Equation (253) is known as the thermal wind
equation, widely used in geophysics [55]. The efficiency of
convexion is invoked explicitly by ignoring the term in $\partial P/\partial \theta$,
and tacitly by ignoring convective velocities in the equation of
motion, which assumes hydrostatic equilibrium. For present
purposes, we note that an axial gradient in the angular velo-
city is intimately associated with latitudinal gradients in the
entropy. This has important consequences for understanding
radiative equilibrium at convective-radiative boundary.

Near the radiative boundary, the Sun’s rotation is strongly
baroclinic. As such, there is no reason why a strict radiative
equilibrium could not be enforced. The argument against
this—that the heat flux divergence cannot vanish everywhere
on a common equipotential/isothermal surface [61]—breaks
down for baroclinic flow, because equipotentials are not iso-
thermal surfaces. Indeed, one could turn the argument on its
head. Why is the Sun’s rotation baroclinic at all? One answer
might be that since the entropy gradient vanishes at the radi-
active/convective boundary in a barotropic model of solar rota-
tion, it is ineffective for circulation to offset a finite heat flux
divergence. The rotation profile $\Omega(r, \theta)$ must then alter itself
non-baroclinically until the heat flux divergences vanishes.
In this view, altered rotation, not the appearance of circulation,
is the key to the energetics, and the region remains in strict
radiative balance.

That there are plausible grounds for expecting steady
non-circulating rotation patterns was put forth long ago (e.g.
[60, 61, 68]), but they seem to have faded with time owing
to problems of stability. These profiles see at face value to be vulnerable to the Goldreich–Schubert–Fricke (GSF) instability [29, 32], which afflicts rotational flow not stratified on cylinders, i.e., baroclinic flow. More recently, the question of the stability of baroclinic rotation in the upper radiative zone was raised by [20], who calculated the explicit form of rotation profiles using static radiative equilibrium as a requirement. Within the uncertainties of the helioseismology observations, it is not difficult to reproduce the observed rotation profile by imposing diffusive radiative equilibrium. Moreover, their baroclinic form, the [20] profiles seem to be locally GSF stable in the upper radiative zone—with or without a magnetic field [19]. The more complex question of global stability is not yet resolved. There is no consensus model of this radiative/convective transition zone, known as the tachocline, at the time of this writing. It is a very lively field of investigation.

9.2. Solution of the thermal wind equation

Surprisingly, the thermal wind equation (253) can be solved in a very useful way for the bulk of the solar convection zone. Let us begin by asking what convection actually does, both thermally and dynamically.

Thermally, convection redistributes entropy (S) from a higher S to a lower S region. Doing so tends to flatten the radial entropy gradient, but does not eliminate it. Some slightly negative (unstable) gradient is required to maintain convection, but this gradient is very tightly regulated. It must allow precisely one solar luminosity worth of thermal energy to pass through the convective zone via convective transport. A small change in the gradient would produce a large change in the outward heat flux.

What effect does rotation have? One might guess that Coriolis forces act more adversely against motion perpendicular to the axis of rotation than motion along the axis. In that case, the poles of the Sun should perhaps be slightly warmer than the equator. The data suggest, in fact, that they are.

The surprise here has been previewed in section 7.3, namely there is nothing in the dynamics of the linear theory of convection to suggest that this is what actually happens. For a uniformly rotating model of a star, [25] made a point of noting this, now almost 65 years ago! The most rapidly growing mode associated with the dispersion relation (224) corresponds to radial displacements and purely azimuthal (e^{\text{azm}}) wavenumbers. [13] showed the same state of affairs prevails even in a star undergoing barotropic shear, Ω = Ω(R). (More precisely, they showed that there was no effect of the shear to linear order in ∂Ω/∂r.) The Coriolis force is annulled by the offsetting azimuthal pressure gradient in the φ equation of motion, a state of geostrophic balance, common in planetary atmospheres and oceanography. But without a φ velocity, there is no Coriolis term in the radial force equation to deflect hot parcels poleward! So why are the poles warmer in the Sun? Shall we simply put it down to nonlinear complications?

As neither uniform rotation nor barotropic differential rotation affects the behaviour of the most rapidly growing convective displacements, the last possibility is baroclinic differential rotation. Does a z gradient cause any departures from radial motion to leading linear order?

Indeed it does. Balbus and Schaan [13] carried out the analysis and found that the components of the displacement vector $\xi$ satisfy the equations

$$\dot{\xi}_r = -\frac{1}{\rho \gamma} \left( \frac{\partial P}{\partial r} \right) \xi_r$$

$$\dot{\xi}_\theta = R \frac{\partial \Omega^2}{\partial r} \xi_r$$

so that there is a slight (northern) poleward drift of high entropy elements when $\partial \Omega^2/\partial z < 0$, as it is throughout most of the (northern) convective zone. This appears to show a nicely self-consistent feature, namely that the convergence of high entropy fluid parcels at the poles could serve to raise the local temperature by a few degrees and maintain the small latitudinal entropy gradient needed for baroclinic rotation in the first place.

To understand why the z gradient appears whereas the R gradient does not, recall that we are dealing with embedded perturbations in a shearing medium. Under these circumstances, the wavenumber of a perturbation is sheared with the medium itself (see equation (193)). The relevant perturbations are dominated by the azimuthal m wavenumber, so that the poloidal components $k_p$ and $k_c$ are directly proportional to $\nabla \Omega$. The coupling to the differential rotation disappears when $k_c$ does, so it is this purely kinematic relationship between $k_c$ and $\partial \Omega/\partial z$ that brings baroclinic shear effects into our problem.

Let us return now to the helioseismology data of figure 8. What is striking is that in the bulk of the convective zone the isorotation curves seem to resemble convective cells: predominantly radial, but with a slight poleward bias. In fact, [13] show both that the perturbed Eulerian azimuthal velocity $\delta \xi_\theta$ is very small and that it is proportional to $\xi \cdot \nabla \Omega$. In other words, the fluid element displacement velocities tend to lie in constant Ω surfaces This of course means that the elements do not conserve their individual angular momenta. There is no mystery here; it is due to the presence of strong azimuthal pressure gradients. This also means that the $\xi \cdot \nabla \Omega^2$ term in equation (203) is very small: the excess centrifugal term that brings baroclinic shear effects into our problem.

What a convective fluid parcel is trying to do is to eliminate the entropy gradient, as we have discussed. Both because the radial entropy gradient is tightly regulated and because there is a latitudinal entropy gradient, we expect that within a (nearly radial) convective cell the radial behaviour of the entropy profile will be given by some function, $\sigma(r)$ say, with a (nearly) constant offset, i.e.

$$\sigma = \sigma_0 + \text{constant}. \quad (256)$$

The ‘constant’ need not be the same constant at each latitude of course—indeed it cannot be, if $\partial \sigma/\partial \theta$ is present. Moreover, the convective mixing process itself has no effect whatsoever
on this entropy constant; it is the entropy gradient along the gravitational field that is affected. The latitudinal entropy gradient may need to be present to drive baroclinic flow, thereby ensuring radiative equilibrium in the presence of ineffective meridional circulation in the upper radiative zone. What emerges in the convective zone is a picture in which fluid elements move in surfaces of constant \( \Omega \) and at the same time in surfaces of constant \( \sigma \). We shall refer to this difference as ‘residual entropy’, \( \delta \sigma \). Since only the \( \theta \) gradient appears in the thermal wind equation (253), we may replace \( \sigma \) with \( \delta \sigma \) on the right side. Our reasoning then suggests that we investigate solutions of (253) with its right side a function \( f \) of \( \Omega^2 \) only:

\[
\frac{\partial \sigma}{\partial \theta} = \frac{\partial \delta \sigma}{\partial \theta} = \frac{\partial f(\Omega^2)}{\partial \theta} = \left( \frac{df}{d\Omega^2} \right) \frac{\partial \Omega^2}{\partial \theta}.
\]  

(257)

At the poles and the equator, this ansatz must be true just by the symmetry of our problem. The real content applies to the bulk of the convective zone, away from these symmetry regions.

Combining equations (253) and (257) and switching to fully spherical coordinates,

\[
\frac{\partial \Omega^2}{\partial r} - \left( \frac{\tan \theta}{r} + \frac{gf'}{r^2 \sin \theta \cos \theta} \right) \frac{\partial \Omega^2}{\partial \theta} = 0,
\]  

(258)

where we have written \( f' \) for \( df/d\Omega^2 \). Equation (258) is an equation for precisely what we would like to know: the isorotation contours of \( \Omega^2 \). In particular, \( \Omega^2 \) must be constant on the characteristic curve given by

\[
\frac{d\theta}{dr} = -\left( \frac{\tan \theta}{r} + \frac{gf'}{r^2 \sin \theta \cos \theta} \right).
\]  

(259)

Now \( f' \) is not known, but as it depends only upon \( \Omega^2 \), we may be sure that along the curve of interest, \( f' \) is a constant. So equation (259) is a perfectly well-posed ordinary differential equation. Moreover, despite its awkward appearance, it folds up nicely:

\[
\frac{d(r^2 \sin^2 \theta)}{dr} = -\frac{2f'}{\gamma} g = -\frac{2f'}{\gamma} \frac{d\Phi}{dr},
\]  

(260)

where we have assumed that \( g \), the gravitational field strength, is a function of \( r \); \( \Phi(r) \) is the potential. Our equation integrates immediately to

\[
r^2 \sin^2 \theta \equiv R^2 = A + B\Phi(r),
\]  

(261)

where \( A \) is an integration constant and

\[
B = -2f'/\gamma > 0
\]  

(262)

is in practice a parameter to be fit from the data (in principle known from turbulence theory) and \( B \) is positive since \( f' \) is negative.

The surprise here is how simple equation (261) is. After a lengthy and somewhat laboured discussion of the nature of convection and rotation (motivated by linear theory) we come to the conclusion that the isorotation contours are curves on which \( R^2 \) is a linear function of the potential! For the Sun, 98% of the mass is below the convective zone, so to an
excellent degree of approximation, \( \Phi = -GM_\odot r \), where \( M_\odot \) is the mass of the Sun. If \( \Omega^2 \) is specified as a function of \( \theta \) on some surface \( r = r_0 \), as it must be for a proper formulation of the solution of this class of partial differential equation, then the characteristic isorotation curves may be written

\[
r^2 \sin^2 \theta \equiv R^2 = r_0^2 \sin^2 \theta_0 + \beta r_0^2 \left( 1 - \frac{r_0}{r} \right),
\]

(263)

where

\[
\beta = -\frac{2f' GM}{\gamma r_0^3}
\]

(264)
is a dimensionless number of order unity and \( \theta_0 \) is the value of \( \theta \) at the start of the contour. The \( \beta \) parameter must be constant along a contour, but can vary from one contour to another with \( f' (\Omega^2) \).

Even without any detailed calculations, it is clear that equation (263) has the right kind of properties to account for the general appearance of the solar isorotation contours. As \( r \) increases, the dominant balance is between \( R^2 \) and the constants on the right, i.e. the flow tends toward constant on cylinders. As \( R \) becomes small near the rotation axis, the dominant balance is between the \( 1/r \) and constant terms on the right, i.e. the flow tends toward constant rotation on spherical surfaces.

Using only one free parameter to fit to the entire Sun, \( \beta = 0.55 \), the match is already quite striking (e.g. [12]). If we allow ourselves the indulgence of a three parameter polynomial fit for the same region:

\[
\beta = A + B \sin \theta + C \sin^2 \theta
\]

(265)

the result is nearly perfect (the white curves in figure 8), apart from the boundary layers. This level of agreement suggests that the basic assumptions of the theory—the validity of the thermal wind equation and the sharing of angular velocity and residual entropy surfaces—is basically sound. Thus, this is a potentially useful approach for analysing the rotation profiles arising in many other problems involving convection and rotation. The surprise here has been that the dynamics of a complicated turbulent system has allowed itself to be encapsulated in such a simple mathematical prescription: \( f' = f(\Omega^2) \) plus thermal wind balance is enough to understand the shapes of the Sun’s isorotation contours. In displaying its isorotational contours, the Sun is acting like a great analogue computer, graphically presenting the solution characteristics of the thermal wind equation.

10. Concluding remarks

The dedicated reader will be aware of the recurring lessons running through many of these problems. Among the most significant is the remarkable dichotomy between problems in which a simple, naive treatment works wonders because of the insensitivity of the problem to anything but the dominant dynamical forces (i.e. the Newcomb–Parker instability); and, in sharp contrast, problems in which the inclusion of a subtle dynamical or microphysical effect discretely and profoundly changes the behaviour of a system (i.e. weak magnetic fields). One of the reasons that, regardless of one’s experience, the subject of astrophysical gasdynamics is so rich, counterintuitive and surprising, is that it is rarely obvious a priori which of these two regimes a particular problem will fall into. Even an intuition honed by experience can be led astray by the underlying complex or delicate interactions amongst different dynamical and thermal processes, which may be on very distinct timescales. Starting afresh with a more careful and rigorous approach is then necessary.

The potent ability of weak magnetic fields to destabilize otherwise stable configurations is particularly striking. That magnetic fields wield such a profound influence seems bizarre, how can it make any difference when the magnetic energy density is such a tiny fraction of the thermal energy density? The answer lies not in strength, but in the new degrees of freedom that a field imparts to a fluid. A magnetic field needn’t compete with pressure. Instead it propagates disturbances through the gas strictly on its own terms, in the form of an Alfvén wave. These are shear modes, without any hydrodynamic counterpart to compete with. Masses on linked springs are standard models for solids and magnetic media, both of which are in turn venues for propagating elastic waves. Hydrodynamic fluids are not.

Another important lesson to bear in mind is the surprising ability of gravity to reverse the effects of usually straightforward interactions. Remember that the world of orbital dynamics is one in which pulling on a body in the direction of its motion slows it down and hindering its motion speeds it up. In a gravitationally stratified medium adding heat to a body can lower its temperature and extracting heat can raise it. Forces that are attractive in a static environment become repulsive in a rotating one, and ordinary thermal conduction can enhance temperature differences in the presence of gravitational fields. These already counterintuitive systems are further complicated with added magnetic degrees of freedom. It is, in retrospect, perhaps not surprising that so many new instabilities appear in the weak magnetic field regime. It is all but certain that there will be more to follow, and that we don’t yet fully understand the consequences of the ones we already supposedly know about.

Beware of instabilities that appear simple but are in fact complex. A slowly evolving process may be hypersensitive to the background equilibrium state and especially to its microphysics. Many of the most knotty current problems of theoretical astrophysics, for example, involve the thermal behaviour of astrophysical plasmas. On the other hand, physics is full of examples in which complexity is only apparent. Once established, it is usually the case that fundamental explanations are rarely unduly complicated. On the contrary, they generally seem embarrassingly obvious. What ever was the problem in the first place? How could so many clever people have overlooked that? For shame! How unfortunate that our current problems are ever so much more complicated. Let naivety be a guide and maintain a playful spirit. There are patterns and constraints to be discerned. Linear theory can be surprisingly helpful, so make sure it is understood before taking on a full analysis. Unless we have been the victim of coincidences, the example of solar rotation offers hope that the days of pencil and paper analysis remain viable even in domains normally viewed as the province of large scale numerical simulation.

The current trend in our discipline has been to ever larger numerical simulations, and ever more strained rationalizations
attempting to justify additions and alterations to the governing equations so that the desired answers appear. Some of this is perhaps inevitable as our ambitions continue to mount; the course of time will tell us whether this is a healthy trend. In the meanwhile, one would do well to distinguish between a subtle explanation and a complicated one. Genuine solutions to profound puzzles are usually the former, and only occasionally the latter. When they are the latter, they will, as a rule, involve the former as well. Finally, we should never forget the wisdom of the ancients. There is still much to learn from the investigations that have attained classic status. The challenge may simply be to recognize the same underlying physics in what may be a very different or somewhat more general setting.

Astrophysical fluid dynamics is teeming with tractable but unsolved problems. Regardless of our methods, there will always be a need for powerful—and often simple—insights. We may all look forward to the surprises to come.

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References

[1] Acheson D J 1990 Elementary Fluid Dynamics (Oxford: Clarendon)
[2] Acheson D J and Hide R 1973 Rep. Prog. Phys. 36 159
[3] Avila M 2013 Phys. Rev. Lett. 108 124501
[4] Balbus S A 1985 Astrophys. J. 291 518
[5] Balbus S A 1995 Astrophys. J. 453 380
[6] Balbus S A 2000 Astrophys. J. 534 420
[7] Balbus S A 2001 Astrophys. J. 562 909
[8] Balbus S A, Bonart J, Latter H and Weiss N O 2009 Mon. Not. R. Astron. Soc. 400 176
[9] Balbus S A and Hawley J F 1991 Astrophys. J. 376 214
[10] Balbus S A and Hawley J F 1998 Rev. Mod. Phys. 70 1
[11] Balbus S A, Hawley J F and Stone J M 1996 Astrophys. J. 467 76
[12] Balbus S A, Latter H and Weiss N 2012 Mon. Not. R. Astron. Soc. 420 2457
[13] Balbus S A and Schaan E 2012 Mon. Not. R. Astron. Soc. 426 1546
[14] Balbus S A and Soker N 1989 Astrophys. J. 341 611
[15] Bender C M and Orszag S A 1978 Advanced Mathematical Methods for Scientists and Engineers (New York: McGraw-Hill)
[16] Blitz L and Shu F H 1980 Astrophys. J. 238 148
[17] Brown B P, Vasil G M and Zweibel E 2012 Astrophys. J. 756 109
[18] Braginskii S I 1965 Phys. Fluids 8 22
[19] Caleb A and Balbus S A 2016 Mon. Not. R. Astron. Soc. 457 1711
[20] Caleb A, Balbus S A and Potter W J 2015 Mon. Not. R. Astron. Soc. 448 2077
[21] Chandrasekhar S 1961 Hydrodynamic and Hydromagnetic Stability (New York: Clarendon)
[22] Clayton D D 1983 Principles of Stellar Evolution and Nucleosynthesis (Chicago: University of Chicago)
[23] Cowie L L and Binney J 1977 Astrophys. J. 215 723
[24] Cowie L L and McKee C F 1977 Astrophys. J. 211 135
[25] Cowling T G 1951 Astrophys. J. 114 272
[26] Defouw R J 1970 Astrophys. J. 160 659
[27] Field G B 1965 Astrophys. J. 142 531
[28] Field G B, Goldsmith D W and Habing H J 1969 Astrophys. J. 155 149
[29] Fricke K 1968 Z. Astrophys. 68 317
[30] Fricke K 1969 Astron. Astroph. 1 388
[31] Goldreich P, Goodman J and Narayan R 1986 Mon. Not. R. Astron. Soc. 221 339
[32] Goldreich P and Schubert G 1967 Astrophys. J. 150 571
[33] Hawley J F and Balbus S A 1991 Astrophys. J. 376 223
[34] Hawley J F 1987 Mon. Not. R. Astron. Soc. 225 677
[35] Islam T and Balbus S 2005 Astrophys. J. 633 328
[36] Ji H, Burin M, Schartman E and Goodman J 2006 Nature 444 343
[37] Kunz M W 2011 Mon. Not. R. Astron. Soc. 417 602
[38] Kunz M W, Bogdanovic T, Reynolds C S and Stone J M 2012 Astrophys. J. 754 122
[39] Latter H N and Kunz M W 2012 Mon. Not. R. Astron. Soc. 423 1964
[40] Lesur G and Longaretti P-Y 2005 Astron. Astroph. 444 25
[41] Lighthill J 1978 Waves in Fluids (Cambridge: Cambridge University Press)
[42] Lynden-Bell D and Ostriker J P 1967 Mon. Not. R. Astron. Soc. 136 293
[43] Lyra W and MacLow M-M 2012 Astrophys. J. 756 62
[44] Mayer F J 1982 Phys. Rev. Lett. 48 1400
[45] McCourt M, Parrish J I, Sharma P and Quataert E 2011 Mon. Not. R. Astron. Soc. 413 1295
[46] Mouschovias T Ch 1974 Mon. Not. R. Astron. Soc. 192 426
[47] Mouschovias T Ch 1974 Mon. Not. R. Astron. Soc. 192 37
[48] Newcomb W A 1961 Phys. Fluids 4 391
[49] Nordsieck F, Huisman S G, van de Veen R C A, Sun C, Lohse D and Lathrop D P 2015 arXiv:1408.1059
[50] Paoletti M S and Lathrop D P 2011 Phys. Rev. Lett. 106 024501
[51] Parrish I J and Stone J M 2009 Astrophys. J. 664 135
[52] Papaloizou J C B and Pringle J E 1984 Mon. Not. R. Astron. Soc. 208 721
[53] Papaloizou J C B and Pringle J E 1985 Mon. Not. R. Astron. Soc. 213 799
[54] Papaloizou J C B and Szuszkiewicz E 1992 Geophys. Astrophys. Fluid. Dynam. 66 233
[55] Parker E N 1966 Astrophys. J. 145 811
[56] Pedlosky J 1987 Geophysical Fluid Dynamics 2nd edn (New York: Springer)
[57] Peierls R 1979 Surprises in Theoretical Physics (Princeton: Princeton University Press)
[58] Peierls R 1991 More Surprises in Theoretical Physics (Princeton: Princeton University Press)
[59] Quataert E 2008 Astrophys. J. 673 758
[60] Rayleigh J W S 1916 Proc. R. Soc. A 93 148
[61] Roxburgh I W 1964 Mon. Not. R. Astron. Soc. 128 157
[62] Roxburgh I W 1966 Mon. Not. R. Astron. Soc. 132 201
[63] Schartman E, Ji H, Burin M J and Goodman J 2012 Astron. Astroph. 543 94
[64] Schwartzchild M 1958 Principles of Stellar Evolution (New York: Dover)
[65] Shapiro S A and Teukolsky S L 1983 Black Holes, White Dwarfs, and Neutron Stars: the Physics of Compact Objects (New York: Wiley)
[66] Shu F H 1991 The Physics of Astrophysics, Volume II, Gas Dynamics (Sausalito, CA: University Science)
[67] Smythe W and Yeh C 1972 American Institute of Physics Handbook ed D E Gray (New York: McGraw-Hill) pp 5–12
[68] Spitzer L 1962 Physics of Fully Ionized Gases 2nd edn (New York: Wiley)
[69] Roxburgh I W and Strittmatter P A 1966 Mon. Not. R. Astron. Soc. 122 345
[70] Tassoul J-L 1978 Theory of Rotating Stars (Princeton: Princeton University Press)
[71] Thompson M J, Christensen-Dalsgaard J, Miesch M S and Toomre J 2003 Astron. Astroph. 41 599
[72] Velikhov E P 1959 Sov.—JETP 36 995