Abstract. Let $(X, g)$ be a compact Riemannian manifold with quasi-positive Riemannian scalar curvature. If there exists a complex structure $J$ compatible with $g$, then the canonical bundle $K_X$ is not pseudo-effective and the Kodaira dimension $\kappa(X, J) = -\infty$. We also introduce the complex Yamabe number $\lambda_c(X)$ for compact complex manifold $X$, and show that if $\lambda_c(X) > 0$, then $\kappa(X) = -\infty$; moreover, if $X$ is also spin, then the Hirzebruch $\hat{A}$-hat genus $\hat{A}(X) = 0$.

1. Introduction

This is a continuation of our previous paper [37], and we investigate the geometry of Riemannian scalar curvature on compact complex manifolds.

The existences of various positive scalar curvatures are obstructed. For instance, it is well-known that, if a compact Hermitian manifold has quasi-positive Chern scalar curvature, then the Kodaira dimension is $-\infty$. On the other hand, a classical result of Lichnerowicz says that if a compact Riemannian spin manifold has quasi-positive Riemannian scalar curvature, then the $\hat{A}$-genus is zero. The first main result of our paper is

**Theorem 1.1.** Let $(X, g)$ be a compact Riemannian manifold with quasi-positive Riemannian scalar curvature. If there exists a complex structure $J$ compatible with $g$, then the canonical bundle $K_X$ is not pseudo-effective and $\kappa(X, J) = -\infty$.

Here quasi-positive means non-negative everywhere and strictly positive at some point. As it is well-known, the positivity of the Riemannian scalar curvature of $(X, J, g)$ can not imply that of the Chern scalar curvature. As a border line case, we obtain:
Theorem 1.2. Let \((X, g)\) be a compact Riemannian manifold with zero Riemannian scalar curvature. Suppose there exists a complex structure \(J\) compatible with \(g\). If \(\kappa(X, J) \geq 0\), then \((X, \omega_g)\) is a Kähler Calabi-Yau manifold and \(\text{Ric}(\omega_g) = 0\).

The proofs of Theorem 1.1 and Theorem 1.2 rely on several observations in our previous paper \cite{37} and a new scalar curvature relation in Theorem 3.8.

Note that, on Kähler Calabi-Yau surfaces (e.g. K3 surfaces, bi-elliptic surfaces), there is no Riemannian metrics with quasi-positive scalar curvature (e.g. \cite{18}, Theorem A). However, by Stolz’s solution (\cite{27}, Theorem A) to the Gromov-Lawson conjecture, on a simply connected Kähler Calabi-Yau manifold \(X\) with holonomy group \(SU(2m+1)\), there do exist Riemannian metrics with quasi-positive scalar curvature. On the contrary, as an application of Theorem 1.1 and Theorem 1.2, we show the Riemannian metrics with quasi-positive scalar curvature are not compatible with the Calabi-Yau complex structures, and more generally

Corollary 1.3. On a compact complex Calabi-Yau manifold \(X\) with torsion canonical bundle \(K_X\), there is no Hermitian metric with quasi-positive Riemannian scalar curvature. Moreover, if \(X\) is also non-Kähler, then there is no Hermitian metric with non-negative Riemannian scalar curvature.

It is well-known that all compact Kähler Calabi-Yau manifolds have torsion canonical bundle. On the other hand, many non-Kähler Calabi-Yau manifolds also have torsion canonical bundle. For instance, the connected sum \(\#_k(S^3 \times S^3)\) with \(k \geq 2\) (\cite{22}).

On a compact complex manifold \(X\) of complex dimension \(n \geq 2\), we introduce the complex Yamabe number \(\lambda_c(X)\):

\[
\lambda_c(X) = \sup_{g \text{ is Hermitian}} \inf_{\tilde{g} \text{ is conformal to } g} \left( \frac{\int_X s_{\tilde{g}} dV_{\tilde{g}}}{\left( \int_X dV_{\tilde{g}} \right)^{1-n}} \right),
\]

where \(s_{\tilde{g}}\) is the Riemannian scalar curvature of \(\tilde{g}\). Note that in (1.1), if the supremum is taken over all Riemannian metrics, then it is the classical Yamabe number \(\lambda(X)\) in conformal geometry. Our second main result is

Theorem 1.4. Let \(X\) be a compact complex manifold. If \(\lambda_c(X) > 0\), then \(\kappa(X) = -\infty\). Moreover, if \(X\) is also spin, then \(\hat{A}(X) = 0\).

According to the results of Gromov-Lawson \cite{13} and Stolz \cite{27}, on a simply connected Kähler Calabi-Yau manifold \(X\) with \(\text{dim}_\mathbb{C} X \geq 3\), one has \(\lambda(X) > 0\) and \(\hat{A}(X) = 0\). However, we have \(\lambda_c(X) \leq 0\) by Theorem 1.4.

As motivated by Theorem 1.1, Theorem 1.2, Theorem 1.4, various conjectures described in \cite{37}, Section 4] and classical works by Schoen-Yau [\cite{29, 30, 31}], Gromov-Lawson [\cite{13}], Stolz [\cite{27}] and LeBrun [\cite{18}] (see also Zhang [\cite{40}]), we propose

Conjecture 1.5. Let \(X\) be a compact Kähler manifold with \(\kappa(X) = -\infty\). If \(X\) has a spin structure, then \(\hat{A}(X) = 0\).

Note that Conjecture 1.5 holds when \(\text{dim}_\mathbb{C} X = 2\) or \(2m + 1\).

Finally, we would like to describe some applications of Theorem 1.1.
Proposition 1.6. Let \( X \) be a compact Kähler threefold. If there exists a Hermitian metric with quasi-positive Riemannian scalar curvature, then \( X \) is uniruled, i.e. \( X \) is covered by rational curves.

According to the uniruledness conjecture (e.g. [5, Conjecture]), Proposition 1.6 should be true on higher dimensional compact Kähler manifolds.

It is a long-standing open problem to determine whether the six-sphere \( S^6 \) admits a complex structure or not\(^1\). Now assuming \( X := S^6 \) has a complex structure \( J \). As pointed out in [16, p. 122], it is not at all clear whether \( \kappa(X, J) = -\infty \), and proving this would seem to be as complicated as to show that there are no divisors on \( X \) at all. It is obvious that \( c_1(X) = 0 \in H^2(X, \mathbb{Z}) \) and it is also proved in [34] that \( c_1^{BC}(X, J) \neq 0 \) and in particular, \( K_X \) is not holomorphically torsion. For more related discussions, we refer to [1]. Let \( \mathcal{S} \) be the space of Riemannian metrics with non-negative scalar curvature. We have

Theorem 1.7. If there exists a complex structure \( J \) which is compatible with some \( g \in \mathcal{S} \), then \( K_X \) is not pseudo-effective and

\[ \kappa(X, J) = -\infty. \]

It is known that there is no complex structure compatible with metrics in a small neighborhood of the round metric on \( S^6 \) (e.g. [20, 24, 32, 6]).

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2. Preliminaries

2.1. Ricci curvature on complex manifolds. Let \( (X, \omega_g) \) be a compact Hermitian manifold. Locally, we write

\[ \omega_g = \sqrt{-1} g_{ij} dz^i \wedge d\overline{z^j} \]

The (first Chern-)Ricci form \( \text{Ric}(\omega_g) \) of \( (X, \omega_g) \) has components

\[ R_{i\overline{j}} = \frac{\partial^2 \log \det(g_{ik})}{\partial z^i \partial \overline{z^j}} \]

which also represents the first Chern class \( c_1(X) \) of the complex manifold \( X \) (up to a constant).

The Chern scalar curvature \( s_C \) of \( (X, \omega_g) \) is given by

\[ s_C = \text{tr}_{\omega_g} \text{Ric}(\omega_g) = g^{ij} R_{i\overline{j}}. \]

The total Chern scalar curvature of \( \omega_g \) is

\[ \int_X s_C \cdot \omega_g^n = n \int \text{Ric}(\omega_g) \wedge \omega_g^{n-1}. \]

\(^1\)Recently, it was announced in [4] by Sir Michael Atiyah that there is no complex structure on \( S^6 \).
where $n$ is the complex dimension of $X$.

### 2.2. The Bott-Chern classes

The Bott-Chern cohomology on a compact complex manifold $X$ is given by

$$H^{p,q}_{BC}(X) := \frac{\text{Ker} \cap \Omega^{p,q}(X)}{\text{Im} d \cap \Omega^{p,q}(X)}.$$ 

Let $\text{Pic}(X)$ be the set of holomorphic line bundles over $X$. As similar as the first Chern class map $c_1 : \text{Pic}(X) \to H^{1,1}_{BC}(X)$, there is a 

**first Bott-Chern class** map

$$c^1_{BC} : \text{Pic}(X) \to H^{1,1}_{BC}(X).$$

Given any holomorphic line bundle $L \to X$ and any Hermitian metric $h$ on $L$, its curvature form $\Theta_h$ is locally given by $-\sqrt{-1}\partial\bar\partial \log h$. We define $c^1_{BC}(L)$ to be the class of $\Theta_{h^{-1}}$ in $H^1_{BC}(X)$. For a complex manifold $X$, $c^1_{BC}(X)$ is defined to be $c^1_{BC}(K_X^{-1})$ where $K_X^{-1}$ is the anti-canonical line bundle.

### 2.3. Special manifolds

Let $X$ be a compact complex manifold.

1. A Hermitian metric $\omega_g$ is called a Gauduchon metric if $\partial\bar\partial \omega_g^{n-1} = 0$. It is proved by Gauduchon ([12]) that, in the conformal class of each Hermitian metric, there exists a unique Gauduchon metric (up to scaling).
2. A Hermitian metric $\omega_g$ is called a Kähler metric if $d\omega_g = 0$.
3. $X$ is called a Calabi-Yau manifold if $c_1(X) = 0 \in H^2(X, \mathbb{Z})$.

### 2.4. Kodaira dimension of compact complex manifolds

The Kodaira dimension $\kappa(L)$ of a line bundle $L$ over a compact complex manifold $X$ is defined to be

$$\kappa(L) := \limsup_{m \to +\infty} \frac{\log \text{dim}_\mathbb{C} H^0(X, L^m)}{\log m}$$

and the **Kodaira dimension** $\kappa(X)$ of $X$ is defined as $\kappa(X) := \kappa(K_X)$ where the logarithm of zero is defined to be $-\infty$. In particular, if

$$\text{dim}_\mathbb{C} H^0(X, K_X^{\otimes m}) = 0$$

for every $m \geq 1$, then $\kappa(X) = -\infty$.

### 2.5. Spin manifold and $\hat{A}$-genus

Let $X$ be a compact oriented Riemannian manifold. It is called a spin manifold, if it admits a spin structure, i.e. its second Stiefel-Whitney class $w_2(X) = 0$. It is well-known that all compact Calabi-Yau manifolds are spin.

The definition of the $\hat{A}$-genus of a compact oriented Riemannian manifold $X$ is as follows. Let $\hat{A}(p_1, \ldots, p_i)$ be the multiplicative sequence of polynomials in the Pontryagin classes $p_i$ of $X$ belonging to the power series

$$\frac{\sqrt{z}}{\sinh \left( \frac{\sqrt{z}}{2} \right)} = 1 - \frac{1}{24} z^2 + \frac{7}{2^7 \cdot 3^2 \cdot 5} z^4 + \cdots .$$

The first few terms are

$$\hat{A}_1(p_1) = -\frac{1}{24} p_1, \quad \hat{A}_2(p_1, p_2) = \frac{1}{2^7 \cdot 3^2 \cdot 5} (-4p_2^2 + 7p_1^2).$$

The $\hat{A}$-genus, $\hat{A}(X)$ is by definition the real number $(\sum_i \hat{A}(p_1, \ldots, p_i))[X]$, where $[X]$ means evaluation of the cohomology class on the fundamental cycle of $X$. Since $p_i \in H^{4i}(X, \mathbb{Z})$,
\( \hat{A}(X) \) is zero unless \( \dim \mathbb{R} X \equiv 0 \pmod{4} \). Moreover, if \( X \) is a spin manifold, \( \hat{A}(X) \) is an integer. The following result is well-known (for more historical explanations, we refer to [17, 35] and the reference therein):

**Lemma 2.1.** On a compact spin manifold \( X \), if it admits a Riemannian metric with quasi-positive scalar curvature, then \( \hat{A}(X) = 0 \).

For more necessary background materials, we refer to [17, 23, 24, 36, 37, 25] and the references therein.

3. The Riemannian scalar curvature and Kodaira dimension

Let \((X, \omega)\) be a compact Hermitian manifold. We first give several computational results.

**Lemma 3.1.** For any smooth real valued function \( f \in C^\infty(X, \mathbb{R}) \), we have

\[
\bar{\partial}^* (f \omega) = f \partial^* \omega + \sqrt{-1} \partial f.
\]

**Proof.** For any smooth \((1,0)\)-form \( \eta \in \Gamma(X, T^{*1,0}X) \), we have the global inner product

\[
\left( \bar{\partial}^* (f \omega), \eta \right) = (f \omega, \partial \eta) = (\omega, f \partial \eta) = (\omega, \bar{\partial}(f \eta)) - (\omega, \partial f \wedge \eta) = (\bar{\partial} \omega, \eta) - (\bar{\partial} f \wedge \eta) = (\bar{\partial} \omega, \eta) + \sqrt{-1} (\partial f, \eta)
\]

where the last identity follows from the fact that \( f \) is real valued. \( \square \)

**Lemma 3.2.** For any \((1,0)\) form \( \eta \) and real valued function \( f \in C^\infty(X, \mathbb{R}) \), we have

\[
\bar{\partial}^* (f \eta) = f \partial^* \eta - (\eta, \partial f).
\]

**Proof.** For any smooth function \( \phi \in C^\infty(X) \), we have

\[
(\bar{\partial}^* (f \eta), \phi) = (f \eta, \partial \phi) = (\eta, f \partial \phi) = (\eta, \partial (f \phi) - \partial f \cdot \phi) = (f \partial^* \eta, \phi) - (\eta, \partial f, \phi)
\]

and we obtain (3.2). \( \square \)

Let \( \omega_f = e^f \omega \) for some \( f \in C^\infty(X, \mathbb{R}) \). We denote by \( \overline{\partial}_f, \partial_f \) and \( \partial^*, \bar{\partial}^* \) the adjoint operators taking with respect to \( \omega_f \) and \( \omega \) respectively. The local and global inner products with respect to \( \omega \) and \( \omega_f \) are indicated by \( (\cdot, \cdot) \), \( (\cdot, \cdot) \) and \( (\cdot, \cdot)_f \), \( (\cdot, \cdot)_f \) respectively.

**Lemma 3.3.** For any \((1,0)\) form \( \eta \) and real valued function \( f \in C^\infty(X, \mathbb{R}) \), we have

\[
\partial_f^* \eta = e^{-f} \left[ \partial^* \eta - (n-1) (\eta, \partial f) \right].
\]

**Proof.** For any \( \varphi \in C^\infty(X) \), we have

\[
(\partial_f^* \eta, \varphi)_f = (\eta, \partial \varphi)_f = (e^{(n-1)f} \eta, \partial \varphi) = \left( \partial^* \left( e^{(n-1)f} \eta \right), \varphi \right)
\]
where the second identity holds since $\eta$ is a $(1, 0)$-form. By (3.2), we obtain
$$\left(\partial_f^* \eta, \varphi \right)_f = \left(e^{(n-1)f} \partial^* \eta, \varphi \right) - \left(\eta, \partial e^{(n-1)f}, \varphi \right).$$

Hence,
$$\left(\partial_f^* \eta, \varphi \right)_f = \left(e^{-f} \partial^* \eta, \varphi \right)_f - \left((n - 1)e^{-f} \langle \eta, \partial f \rangle, \varphi \right)_f,$$
which verifies (3.3).

\begin{lemma}
We have
\begin{equation}
\omega_f = \partial^* \omega + (n - 1) \sqrt{-1} \partial f.
\end{equation}
\end{lemma}

\begin{proof}
For any $\eta \in \Gamma(X, T^{1, 0}X)$, we have
$$\left(\partial_f^* \omega, \eta \right)_f = \left(e^{(n-1)f} \cdot \omega, \eta \right).$$

Now by (3.1), we have
$$\left(\partial_f^* \omega, \eta \right)_f = \left(e^{(n-1)f} \left[ \partial^* \omega + (n - 1) \sqrt{-1} \partial f \right], \eta \right).$$

since $\eta$ is a $(1, 0)$ form. Therefore, we obtain (3.4).
\end{proof}

\begin{lemma}
We have
\begin{equation}
\sqrt{-1} \partial_f^* \partial^* \omega_f = e^{-f} \left( \sqrt{-1} \partial^* \sqrt{-1} \partial f - (n - 1) \left\{ \Delta_d f + \text{tr} \omega \sqrt{-1} \partial f \right\} + (n - 1)^2 |\partial f|^2 \right).
\end{equation}
\end{lemma}

\begin{proof}
By formulas (3.2) and (3.4), we have
$$\sqrt{-1} \partial_f^* \partial^* \omega_f = \sqrt{-1} \partial_f^* \left( \partial^* \omega + (n - 1) \sqrt{-1} \partial f \right)$$
$$= e^{-f} \left( \sqrt{-1} \partial^* \sqrt{-1} \partial f \right)$$
$$\left( \partial^* \omega, \partial f \right) - \left( \partial^* \partial f, \partial f \right) - \left( (n - 1) \partial \sqrt{-1 \partial f}, \partial f \right).$$

We also observe that
\begin{equation}
\sqrt{-1} \langle \partial^* \omega, \partial f \rangle = \partial^* \partial f + \text{tr} \omega \sqrt{-1} \partial f.
\end{equation}

Indeed, for any function $\varphi \in \mathcal{C}^\infty(X)$, we have
$$\left( \sqrt{-1} \langle \partial^* \omega, \partial f \rangle, \varphi \right) = \sqrt{-1} \left( \partial^* \omega, \varphi \partial f \right)$$
$$= \sqrt{-1} \left( \omega, \varphi \partial f \right)$$
$$= \omega, \varphi \partial f + \varphi \partial f$$
$$= \left( \partial f, \varphi \right) + \left( \partial f, \varphi \right)$$
$$= \left( \partial f, \varphi \right) + \left( \text{tr} \omega \sqrt{-1} \partial f, \varphi \right)$$
which gives (3.6). Since $\Delta_d f = d^* df = \partial^* \partial f + \partial^* \partial f$, we obtain (3.5).
\end{proof}

The following observation is one of the key ingredients in the curvature computations.
Lemma 3.6. Let \((X, \omega)\) be a compact Hermitian manifold. Then
\[
\langle \partial \bar{\partial} \omega, \omega \rangle = |\partial^* \omega|^2 - \sqrt{-1} \partial^* \partial \omega.
\]
In particular, if \(\omega\) is a Gauduchon metric, we have
\[
\langle \partial \bar{\partial} \omega, \omega \rangle = |\partial^* \omega|^2.
\]

Proof. For any smooth real valued function \(\varphi \in C^\infty(X, \mathbb{R})\), we have
\[
\left( \langle \partial \bar{\partial} \omega, \omega \rangle, \varphi \right) = \left( \langle \partial \bar{\partial} \omega, \varphi \rangle, \omega \right) = \left( \langle \partial^* \omega, \varphi \rangle + \sqrt{-1} \partial \varphi \right)
\]
where we use formula (3.1) in the second identity. Since \(\varphi\) is an arbitrary smooth real function, we obtain (3.7). If \(\omega\) is Gauduchon, i.e. \(\partial \bar{\partial} \omega^n = 0\), we have \(\partial^* \partial \omega = 0\), and so (3.8) follows from (3.7). \(\square\)

Corollary 3.7. On a compact Hermitian manifold \((X, \omega)\), the Riemannian scalar curvature \(s\) and the Chern scalar curvature \(s_{\text{C}}\) are related by
\[
s = 2s_{\text{C}} - 2\sqrt{-1} \partial^* \partial \omega - \frac{1}{2} |T|^2.
\]
where \(T\) is the torsion tensor of the Hermitian metric \(\omega\).

Proof. By Lemma 6.2 in the Appendix, we have
\[
s = 2s_{\text{C}} + \left( \langle \partial \bar{\partial}^* \omega + \partial^* \partial \omega, \omega \rangle - 2|\partial^* \omega|^2 \right) - \frac{1}{2} |T|^2.
\]
Hence, by formula (3.7) we obtain (3.9). \(\square\)

Let \(\omega_f = e^f \omega\) be a smooth Gauduchon metric (i.e. \(\partial \bar{\partial} \omega^n_f = 0\)) in the conformal class of \(\omega\) for some smooth function \(f \in C^\infty(X, \mathbb{R})\).

Theorem 3.8. The total Chern scalar curvature of the Gauduchon metric \(\omega_f\) is
\[
n \int_X \text{Ric}(\omega_f) \wedge \omega_f^{n-1} = \int_X e^{(n-1)f} \left( \frac{s}{2} + \frac{|T|^2}{4} \right) \omega^n + (n - 1)^2 ||\partial f||^2_{\omega_f}.
\]

Proof. Indeed, since \(\omega_f\) satisfies \(\partial \bar{\partial} \omega^n_f = 0\), we have
\[
n \int_X \text{Ric}(\omega_f) \wedge \omega_f^{n-1} = n \int_X \text{Ric}(\omega) \wedge \omega_f^{n-1}
\]
\[
= n \int_X e^{(n-1)f} \cdot \text{Ric}(\omega) \wedge \omega^{n-1} = \int_X e^{(n-1)f} \cdot s_{\text{C}} \cdot \omega^n
\]
\[
= \int_X e^{(n-1)f} \left( \frac{s}{2} + \frac{|T|^2}{4} \right) \omega^n + \int_X e^{(n-1)f} \cdot \sqrt{-1} \partial^* \partial \omega \cdot \omega^n,
\]
where we use the scalar curvature relation (3.9) in the third identity. Since \(\omega_f\) is Gauduchon, and we have \(\partial^* \partial \omega = 0\). By formula (3.5), we have
\[
\sqrt{-1} \partial^* \partial \omega = (n - 1) \left( \Delta_{df} + \text{tr}_{\omega} \sqrt{-1} \partial \bar{\partial} f \right) - (n - 1)^2 |\partial f|^2.
\]
It is easy to show that
\[ \int_X e^{(n-1)f} \sqrt{-1} \partial \overline{\partial} f \cdot \omega^n = n \int_X \sqrt{-1} \partial \overline{\partial} f \wedge \omega_f^{n-1} = 0 \]
and
\[ \int_X e^{(n-1)f} |\partial f|^2 \omega^n = \|\partial f\|_{\omega_f}^2. \]
Moreover,
\[ \int_X e^{(n-1)f} \Delta df \omega^n = \left( d^* df, e^{(n-1)f} \right) 
= (n-1) \left( df, e^{(n-1)f} df \right) 
= (n-1) (df, df)_f \]
since \( df \) is a 1-form. Finally, we obtain
\[ \int_X e^{(n-1)f} \cdot \sqrt{-1} \partial \overline{\partial} \cdot \omega^n = (n-1)^2 \|\partial f\|_{\omega_f}^2 - (n-1)^2 \|\partial f\|_{\omega_f}^2 = (n-1)^2 \|\partial f\|_{\omega_f}^2. \]
Putting all together, we get (3.10). \( \square \)

**The proof of Theorem 1.1.** Let \( \omega \) be the Hermitian metric of \((g, J)\). Let \( \omega_f = e^f \omega \) be a smooth Gauduchon metric in the conformal class of \( \omega \). If the Riemannian scalar curvature \( s \) of \( \omega \) is quasi-positive, then by formula (3.10), the total Chern scalar curvature of the Gauduchon metric \( \omega_f \) is strictly positive, i.e.
\[ n \int_X \text{Ric}(\omega_f) \wedge \omega_f^{n-1} > 0. \]
By [37, Theorem 1.1] and [37, Corollary 3.2], \( K_X \) is not pseudo-effective and \( \kappa(X, J) = -\infty \). \( \square \)

**The proof of Theorem 1.2.** Suppose \( \omega \) is not a Kähler metric, i.e. the torsion \(|T|^2\) is not identically zero, then by formula (3.10), there exists a Gauduchon metric with positive total Chern scalar curvature. Hence, by [37, Corollary 3.2] we have \( \kappa(X) = -\infty \) which is a contradiction. Therefore, \( \omega \) is Kähler and so in formula (3.10), \( f \) is a constant and \( T = 0 \). That means, \( \omega \) is a Kähler metric with zero scalar curvature. By [37, Corollary 1.6], \( X \) is a Calabi-Yau manifold since \( \kappa(X) \geq 0 \). By the Calabi-Yau theorem ([39]), there exists a Kähler Ricci-flat metric \( \omega_{CY} \), i.e. \( \text{Ric}(\omega_{CY}) = 0 \). Hence,
\[ \text{Ric}(\omega) = \text{Ric}(\omega) - \text{Ric}(\omega_{CY}) = \sqrt{-1} \partial \overline{\partial} F \]
where \( F = \log \left( \frac{\omega_{CY}^{n}}{\omega}\right) \). Since \( \omega \) has zero scalar curvature, we have
\[ \Delta F = \text{tr} \omega \sqrt{-1} \partial \overline{\partial} F = 0, \]
which implies \( F = \text{const} \) and \( \text{Ric}(\omega) = 0 \). \( \square \)

**Corollary 3.9.** Let \((X, g)\) be a compact Riemannian manifold with nonnegative Riemannian scalar curvature. If there exists a complex structure \( J \) which is compatible with \( g \), then either
\( (1) \ \kappa(X, J) = -\infty; \) or
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(2) $\kappa(X, J) = 0$ and $(X, J, g)$ is a Kähler Calabi-Yau.

Proof. If the Riemannian scalar curvature is not identically zero, or $s$ is identically zero but the metric is not Kähler, then the total Chern scalar curvature in formula (3.10) is strictly positive. Hence $\kappa(X, J) = -\infty$. The last possibility is that $(g, J)$ is a Kähler metric with zero scalar curvature. In this case, it follows from [37, Corollary 1.6] that either $\kappa(X) = -\infty$, or $\kappa(X, J) = 0$ and $(X, J, g)$ is a Kähler Calabi-Yau. □

Proposition 3.10. Suppose $X$ is a compact complex manifold with $c_{BC}^1(X) \leq 0$. Then

(1) there exists a Hermitian metric with non-positive Riemannian scalar curvature;
(2) there is no Hermitian metric with quasi-positive Riemannian scalar curvature.

Moreover, $X$ admits a Hermitian metric $g$ with zero Riemannian scalar curvature if and only if $(X, g)$ is a Kähler Calabi-Yau.

Proof. Note that by definition there exists a $d$-closed non-positive $(1,1)$ form $\eta$ which represents $c_{BC}^1(X)$. By [28, Theorem 1.3], there exists a non-Kähler Gauduchon metric $\omega_G$ such that $\text{Ric}(\omega_G) = \eta \leq 0$.

Hence, for any Gauduchon metric $\omega$,

\begin{equation}
\int_X \text{Ric}(\omega) \wedge \omega^{n-1} = \int_X \text{Ric}(\omega_G) \wedge \omega^{n-1} \leq 0.
\end{equation}

(1). Since $\omega_G$ is Gauduchon, by formula (3.9), we have

$$s = 2s_C - \frac{1}{2}|T|^2 = 2\text{tr}_{\omega_G}\text{Ric}(\omega_G) - \frac{1}{2}|T|^2 \leq 0.$$

(2). If there exists a Hermitian metric with quasi-positive Riemannian scalar curvature, then it induces a Gauduchon metric with positive total Chern scalar curvature which is a contradiction.

Suppose $X$ admits a Hermitian metric $g$ with zero Riemannian scalar curvature, then by formulas (3.10) and (3.12), we have $T = 0$ and $f = 0$, i.e. $(X, \omega_g)$ is a Kähler manifold with zero scalar curvature. Since $c_{BC}^1(X) = c_1(X) \leq 0$, we have $\text{Ric}(\omega_g) = 0$. □

The proof of Corollary 1.3. If $K_X$ is a torsion, i.e. $K_X^{\otimes m} = O_X$ for some $m \geq 1$, then $\kappa(X) = 0$. The first part of Corollary 1.3 follows from Theorem 1.1, and the second part follows from Theorem 1.2. □

The proof of Proposition 1.6. By Theorem 1.1, $K_X$ is not pseudo-effective and $\kappa(X) = -\infty$. Hence by [7, Corollary 1.2] or [15, Corollary 1.4], we conclude $X$ is uniruled. □

The proof of Theorem 1.7. Since $H^2(X, \mathbb{R}) = 0$, the Hermitian metric $(g, J)$ is not Kähler. Then Theorem 1.7 follows from Theorem 1.1 and Theorem 1.2. □
4. The Yamabe number, $\hat{A}$-genus and Kodaira dimension

Let $(X, g_0)$ be a compact Riemannian manifold with real dimension $2n$. The \textit{Yamabe invariant} $\lambda(X, g_0)$ of the conformal class $[g_0]$ is defined as

\begin{equation}
\lambda(X, g_0) = \inf_{g = e^f g_0, f \in C^\infty(X, \mathbb{R})} \frac{\int_X s_g dV_g}{(\int_X dV_g)^{1-\frac{n}{4}}}
\end{equation}

where $s_g$ is the Riemannian scalar curvature of $g$. Moreover, one can define the Yamabe number

\begin{equation}
\lambda(X) = \sup_{\text{all Riemannian metric } g} \lambda(X, g).
\end{equation}

As analogous to (4.2), on a compact complex manifold $X$, one can define the complex version

\begin{equation}
\lambda_c(X) = \sup_{\text{all Hermitian metric } g} \lambda(X, g).
\end{equation}

**Theorem 4.1.** Let $X$ be a compact complex manifold. If $\lambda_c(X) > 0$, then $\kappa(X) = -\infty$. Moreover, if $X$ is also spin, then $\hat{A}(X) = 0$.

\textit{Proof.} Suppose $\lambda_c(X) > 0$, then there exists a Hermitian metric $g_0$ such that

\[ \lambda(X, g_0) = \inf_{g \in [g_0]} \frac{\int_X s_g dV_g}{(\int_X dV_g)^{1-\frac{n}{4}}} > 0. \]

Let $\omega_f = e^f \omega_{g_0}$ be a Gauduchon metric in the conformal class of $\omega_{g_0}$. Hence, $\omega_f$ has positive total Riemannian scalar curvature

\[ \int_X s_f \cdot \omega_f^n > 0. \]

Moreover, by formula (3.9), the total Chern scalar curvature of $\omega_f$ is

\begin{equation}
\int_X (s_C)_f \cdot \omega_f^n = \int_X \frac{s_f}{2} \cdot \omega_f^n + \frac{1}{4} \int_X |T_f|^2 \cdot \omega_f^n > 0,
\end{equation}

where we use the fact that $\omega_f$ is Gauduchon, i.e. $\partial_f \bar{\partial}_f \omega_f = 0$. Therefore, the Gauduchon metric $\omega_f$ has positive total Chern scalar curvature, and by [37, Corollary 3.2], $\kappa(X) = -\infty$. We also have $\lambda(X) \geq \lambda_c(X) > 0$. It is well-known that $\lambda(X) > 0$ if and only if there exists a Riemannian metric with positive Riemannian scalar curvature. If $X$ is spin, then $\hat{A}(X) = 0$ by Lichnerowicz’s result. \hfill $\square$

Note that on a simply connected Kähler Calabi-Yau manifold $X$ with $\dim \mathbb{C} X = 2m + 1$, one has $\lambda(X) > 0$ and $\hat{A}(X) = 0$. However, $\lambda_c(X) \leq 0$.

**Question 4.2.** On a compact Kähler (or complex) manifold $X$, find sufficient and necessary conditions such that $\lambda(X)$ and $\lambda_c(X)$ have the same sign, or $\lambda(X) = \lambda_c(X)$.

A result along this line is

**Corollary 4.3.** Let $X$ be a simply compact complex manifold with $\dim \mathbb{C} X \geq 3$. If $\lambda_c(X)$ has the same sign as $\lambda(X)$, then $\kappa(X) = -\infty$.

\textit{Proof.} By Gromov-Lawson [13] and Stolz [27], if $X$ is a simply connected complex manifold with $\dim \mathbb{C} X \geq 3$, then $X$ has a Riemannian metric with positive scalar curvature, hence $\lambda(X) > 0$ and so $\lambda_c(X) > 0$. By Theorem 4.1, we obtain $\kappa(X) = -\infty$. \hfill $\square$
Finally, we want to present a nice result of LeBrun, which answers Conjecture 1.5 affirmatively when X is a compact spin Kähler surface (for related works, see also [14] and [20]):

**Theorem 4.4.** [18, Theorem A] Let X be a compact Kähler surface, then

\[
\begin{align*}
\lambda(X) > 0 & \text{ if and only if } \kappa(X) = -\infty; \\
\lambda(X) = 0 & \text{ if and only if } \kappa(X) = 0 \text{ or } 1; \\
\lambda(X) < 0 & \text{ if and only if } \kappa(X) = 2.
\end{align*}
\]

According to Theorem 1.1, Theorem 1.2, Theorem 1.4 and [37, Theorem 1.1], there should be some analogous results for \(\lambda_c(X)\) on compact Kähler manifolds, which will be addressed in future studies. For some related settings, we refer to [3, 8, 2, 21] and the references therein.

5. Examples on compact non-Kähler Calabi-Yau surfaces

In this section, we discuss two special Calabi-Yau surfaces of class VII. One is the diagonal Hopf surface \(S^1 \times S^3\) and the other one is the Inoue surface. It is well-known, they are non-Kähler Calabi-Yau surfaces with Kodaira dimension \(-\infty\), \(b_1(X) = 1\) and \(b_2(X) = 0\). A straightforward computation shows that on \(S^1 \times S^3\), there are Hermitian metrics with positive (resp. negative, zero) Riemannian scalar curvature ([24, Section 6]). We show by the following example that the converses of Theorem 1.1 and Theorem 1.4 are not valid in general:

**Proposition 5.1.** On an Inoue surface X, it has \(\kappa(X) = -\infty\) and \(\hat{A}(X) = 0\). However, it cannot support a Hermitian metric with non-negative Riemannian scalar curvature. In particular, \(\lambda_c(X) \leq 0\).

**Proof.** On each Inoue surface, there exists a smooth Gauduchon metric with non-positive Ricci curvature. Indeed, let \((w, z) \in \mathbb{H} \times \mathbb{C}\) be the holomorphic coordinates, then by the precise definition of each Inoue surface ([9, 33, 10, 25]), we know the form

\[
\sigma = \frac{dw \wedge dz}{\text{Im}(w)}
\]

descends to a smooth nowhere vanishing \((2,0)\) form on X, i.e. \(\sigma \in \Gamma(X, K_X)\). Then it induces a smooth Hermitian metric \(h\) on \(K_X\) given by \(h(\sigma, \sigma) = 1\). In the holomorphic frame \(e = dw \wedge dz\) of \(K_X\), we have

\[
h = h(e, e) = [\text{Im}(w)]^2.
\]

It also induces a Hermitian metric \(h^{-1}\) on \(K_X^{-1}\), and the curvature of \(h^{-1}\) is

\[
-\sqrt{-1} \partial\bar{\partial} \log h^{-1} = \sqrt{-1} \partial\bar{\partial} \log [\text{Im}(w)]^2 = -\frac{\sqrt{-1}}{2} \frac{dw \wedge d\bar{w}}{[\text{Im}(w)]^2},
\]

which also represents \(c_1^{BC}(X)\). By Theorem [28, Theorem 1.3], there exists a Gauduchon metric \(\omega_G\) with non-positive Ricci curvature

\[
\text{Ric}(\omega_G) = -\frac{\sqrt{-1}}{2} \frac{dw \wedge d\bar{w}}{[\text{Im}(w)]^2} \leq 0.
\]
(Note also that the Riemannian scalar curvature of $\omega_G$ is strictly negative according to (3.9).) Hence, for any Gauduchon metric $\omega$, the total Chern scalar curvature

$$2 \int_X \text{Ric}(\omega) \wedge \omega = 2 \int_X \text{Ric}(\omega_G) \wedge \omega < 0.$$ 

If $X$ admits a Hermitian metric $\omega$ with non-negative Riemannian scalar curvature, then by formula (3.10), there exists a Gauduchon metric with positive total Chern scalar curvature. This is a contradiction. □

6. Appendix: The scalar curvature relation on compact complex manifolds

Let’s recall some elementary settings (e.g. [24, Section 2]). Let $(M, g, \nabla)$ be a $2n$-dimensional Riemannian manifold with the Levi-Civita connection $\nabla$. The tangent bundle of $M$ is also denoted by $T_R M$. The Riemannian curvature tensor of $(M, g, \nabla)$ is

$$R(X, Y, Z, W) = g \left( \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \right)$$

for tangent vectors $X, Y, Z, W \in T_R M$. Let $T_C M = T_R M \otimes \mathbb{C}$ be the complexification. We can extend the metric $g$ and the Levi-Civita connection $\nabla$ to $T_C M$ in the $\mathbb{C}$-linear way. Hence for any $a, b, c, d \in \mathbb{C}$ and $X, Y, Z, W \in T_C M$, we have

$$R(aX, bY, cZ, dW) = abcd \cdot R(X, Y, Z, W).$$

Let $(M, g, J)$ be an almost Hermitian manifold, i.e., $J : T_R M \to T_R M$ with $J^2 = -1$, and for any $X, Y \in T_R M$, $g(JX, JY) = g(X, Y)$. The Nijenhuis tensor $N_J : \Gamma(M, T_R M) \times \Gamma(M, T_R M) \to \Gamma(M, T_R M)$ is defined as

$$N_J(X, Y) = [X, Y] + J[X, Y] + J[X, JY] - [JX, JY].$$

The almost complex structure $J$ is called integrable if $N_J \equiv 0$ and then we call $(M, g, J)$ a Hermitian manifold. We can also extend $J$ to $T_C M$ in the $\mathbb{C}$-linear way. Hence for any $X, Y \in T_C M$, we still have $g(JX, JY) = g(X, Y)$. By Newlander-Nirenberg’s theorem, there exists a real coordinate system $\{x^i, x^J\}$ such that $z^i = x^i + \sqrt{-1}x^J$ are local holomorphic coordinates on $M$. Let’s define a Hermitian form $h : T_C M \times T_C M \to \mathbb{C}$ by

$$h(X, Y) := g(X, Y), \quad X, Y \in T_C M.$$ 

By $J$-invariant property of $g$,

$$h_{ij} := h \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right) = 0, \quad \text{and} \quad h_{\overline{ij}} := h \left( \frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^j} \right) = 0$$

and

$$h_{\overline{i}j} := h \left( \frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial z^j} \right) = \frac{1}{2} \left( g_{ij} + \sqrt{-1}g_{iJ} \right).$$

It is obvious that $(h_{\overline{i}j})$ is a positive Hermitian matrix. Let $\omega$ be the fundamental 2-form associated to the $J$-invariant metric $g$:

$$\omega(X, Y) = g(JX, Y).$$
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In local complex coordinates,

\[(6.5) \quad \omega = \sqrt{-1} h_{i\bar{j}} dz^i \wedge d\bar{z}^j.\]

In the local holomorphic coordinates \(\{z^1, \ldots, z^n\}\) on \(M\), the complexified Christoffel symbols are given by

\[(6.6) \quad \Gamma^C_{AB} = \sum_E \frac{1}{2} g^{CE} \left( \frac{\partial g_{AE}}{\partial z^B} + \frac{\partial g_{BE}}{\partial z^A} - \frac{\partial g_{AB}}{\partial z^E} \right) = \sum_E \frac{1}{2} h^{CE} \left( \frac{\partial h_{AE}}{\partial z^B} + \frac{\partial h_{BE}}{\partial z^A} - \frac{\partial h_{AB}}{\partial z^E} \right),\]

where \(A, B, C, E \in \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}\) and \(z^A = z^i\) if \(A = i\), \(z^A = \bar{z}^i\) if \(A = \bar{i}\). For example

\[(6.7) \quad \Gamma^k_{ij} = \frac{1}{2} h^{k\ell} \left( \frac{\partial h_{j\ell}}{\partial z^i} + \frac{\partial h_{i\ell}}{\partial z^j} \right), \Gamma^k_{ij} = \frac{1}{2} h^{k\ell} \left( \frac{\partial h_{j\ell}}{\partial z^i} - \frac{\partial h_{i\ell}}{\partial z^j} \right).\]

We also have \(\Gamma^k_{ij} = \Gamma^k_{ji} = 0\) by the Hermitian property \(h_{pq} = h_{q\bar{p}} = 0\). The complexified curvature components are

\[(6.8) \quad R^{D}_{ABC} = \sum_E R^{D}_{ABCE} h^{ED} = -\left( \frac{\partial R^D_{AC}}{\partial z^B} - \frac{\partial R^D_{BC}}{\partial z^A} + \Gamma^F_{AC} \Gamma^D_{FB} - \Gamma^F_{BC} \Gamma^D_{AF} \right).\]

By the Hermitian property again, we have

\[(6.9) \quad R^g_{ijk} = -\left( \frac{\partial R^g_{ik}}{\partial z^j} - \frac{\partial R^g_{jk}}{\partial z^i} + \Gamma^s_{ik} \Gamma^g_{js} - \Gamma^s_{jk} \Gamma^g_{is} - \Gamma^g_{ik} \Gamma^g_{js} \right).\]

It is computed in [24, Lemma 7.1] that

**Lemma 6.1.** On the Hermitian manifold \((M, h)\), the Riemannian Ricci curvature of the Riemannian manifold \((M, g)\) satisfies

\[(6.10) \quad \text{Ric}(X, Y) = h^7 \left[ R \left( \frac{\partial}{\partial z^i}, X, \frac{\partial}{\partial \bar{z}^j} \right) + R \left( \frac{\partial}{\partial \bar{z}^i}, Y, \frac{\partial}{\partial z^j} \right) \right]\]

for any \(X, Y \in T_R M\). The Riemannian scalar curvature is

\[(6.11) \quad s = 2 h^7 \left( 2 R^g_{ik} - R^g_{ij} \right).\]

The following result is established in [24, Corollary 4.2] (see also some different versions in [11]). For readers’ convenience we include a straightforward proof without using “normal coordinates”.

**Lemma 6.2.** On a compact Hermitian manifold \((M, \omega)\), the Riemannian scalar curvature \(s\) and the Chern scalar curvature \(s_C\) are related by

\[(6.12) \quad s = 2 s_C + \left( (\partial \bar{\partial} \omega + \bar{\partial} \partial \omega) - 2 |\partial \bar{\partial} \omega|^2 \right) - \frac{1}{2} |T|^2,\]

where \(T\) is the torsion tensor with

\[(T^k_{ij}) = h^{k\ell} \left( \frac{\partial h_{j\ell}}{\partial z^i} - \frac{\partial h_{i\ell}}{\partial z^j} \right).\]
Proof. For simplicity, we denote by 
\[ s_R = h_{i}^{j} h_{k}^{\ell} R_{ij\ell k} \quad \text{and} \quad s_H = h_{i}^{j} h_{k}^{\ell} R_{ij\ell k}. \]
Then, by formula (6.11), we have \( s = 4s_R - 2s_H. \) In the following, we shall show
\[ s_H = s_C - \frac{1}{2} \langle \partial \partial^{*} \omega + \partial \bar{\partial}^{*} \omega, \omega \rangle - \frac{1}{4} |T|^2 \tag{6.13} \]
and
\[ s_R = s_C - \frac{1}{2} |\partial^{*} \omega|^2 - \frac{1}{4} |T|^2. \tag{6.14} \]

It is easy to show that
\[ \bar{\partial}^{*} \omega = 2 \sqrt{-1} \Gamma_{ik}^j dz^i \tag{6.15} \]
and so
\[ - \frac{\partial \partial^{*} \omega + \partial \bar{\partial}^{*} \omega}{2} = \sqrt{-1} \left( \frac{\partial \Gamma_{ik}^j}{\partial z^j} + \frac{\partial \Gamma_{ik}^j}{\partial \bar{\sigma}^j} \right) dz^i \wedge d\bar{\sigma}^j. \tag{6.16} \]

On the other hand, by formula (6.9), we have
\[ R_{jk}^k = - \frac{\partial \Gamma_{ik}^j}{\partial \bar{\sigma}^j} + \frac{\partial \Gamma_{ik}^j}{\partial z^j} + \Gamma_{jk}^i \Gamma_{ik}. \tag{6.17} \]
A straightforward calculation shows
\[ h_{i}^{j} h_{k}^{\ell} \Gamma_{ik}^j \Gamma_{ik}^\ell = - \frac{1}{4} |T|^2. \]
Moreover, we have
\[ - \frac{\partial \Gamma_{ik}^j}{\partial \bar{\sigma}^j} + \frac{\partial \Gamma_{ik}^j}{\partial z^j} \quad \text{and} \quad - \frac{\partial \Gamma_{ik}^j}{\partial \bar{\sigma}^j} - \frac{\partial \Gamma_{ik}^j}{\partial z^j} = - \partial^2 \log \det(g) \frac{\partial \log \det(g)}{\partial z^i \partial \bar{\sigma}^j} \tag{6.18} \]
where the last identity follows from (6.7). Indeed, we have
\[ \Gamma_{ik}^j + \Gamma_{jk}^i = k_{i}^{k} \partial \frac{\partial \log \det(g)}{\partial z^i}. \]
Hence, we obtain
\[ s_H + \frac{1}{2} \langle \partial \partial^{*} \omega + \partial \bar{\partial}^{*} \omega, \omega \rangle = s_C - \frac{1}{4} |T|^2 \]
which proves (6.13). Similarly, one can show (6.14). \( \square \)

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