Mass Generation in the large $N$-nonlinear $\sigma$-Model

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Abstract

Abstract: We study the infrared behaviour of the two-dimensional Euclidean $O(N)$ nonlinear $\sigma$-Model with a suitable ultraviolet cutoff. It is proven that for a sufficiently large (but finite!) number $N$ of field components the model is massive and thus has exponentially decaying correlation functions. We use a representation of the model with an interpolating bosonic field. This permits to analyse the infrared behaviour without any intermediate breaking of $O(N)$-symmetry. The proof is simpler than that of the corresponding result for the Gross-Neveu-Model [1].

1 Introduction

We want to study the infrared behaviour of the two-dimensional Euclidean nonlinear $\sigma$-model [2] which is formally given in terms of the Lagrangian

$$\mathcal{L} = \frac{N}{2\lambda} \{ (\partial \phi)^2 + \frac{K}{4}(\phi^2 - 1)^2 \}. \quad (1)$$

Here the constant $K$ is assumed to be of order 1, whereas we assume $N \gg 1$, for $\lambda$ see below. $\phi$ is a real-valued $N$-(flavour-)component bosonic field in the fundamental (vector) representation of $O(N)$. The minimum of $\mathcal{L}$ is thus situated at $\phi^2 = 1$, where the value 1 may be changed by rescaling the field variable. The ultraviolet (UV) cutoff as well as more precise statements on the lower bound for $N$ will be specified later. As regards $\lambda$, its value should not be much larger than 1, because otherwise the generated mass $m$ approaches the UV cutoff, see below (20). If it is much smaller than 1, on the other hand, the effective energy range of the UV cutoff model becomes large and therefore the bounds, which involve
factors of $\exp(4\pi/\lambda)$, deteriorate. The convergence proof then requires larger values of $N$. In the full renormalization group construction one would try to impose a condition $\lambda \sim 1$ by fixing the renormalized coupling $\lambda_0$ of the last renormalization group step to obey that condition since in the full construction $\lambda_0$ corresponds to our coupling $\lambda$.

The standard nonlinear $\sigma$-model has the constraint on the field variable (which we call $\phi$ instead of $\sigma$)

$$\phi^2 = 1.$$  \hspace{1cm} (2)

Condition (2) can be obtained from (1) by a suitable limit taking $K \rightarrow \infty$. Such a constraint however is immediately softened out when starting from the model with a large UV cutoff on integrating out high frequency modes, even after the first renormalization group step in a renormalization group construction. This can be seen from the renormalization group construction of the hierarchical model which has been performed by Gawedzki and Kupiainen[4] and later also by Pordt and Reiss[5]. It is rather obvious anyhow: Once you have (at least) two independent frequency modes, fluctuations of one may compensate those of the other such that the constraint (2) is restored for the sum. These fluctuations are not even highly improbable since neighbouring frequency modes may look similar in position space for frequencies close to the border line between the two. Thus we obtain for $K$ a value of order 1 after the first step. The much more difficult part of the ultraviolet analysis of the model - so far only performed in the hierarchical case for $N > 2$ and as long as the effective coupling stays small - is to show that the Lagrangian (1) is a good approximation to the full model. That implies in particular that the model has only one marginal direction which is well represented by the quartic term in (1). So our starting point is reasonable when giving credit to the evidence based on the hierarchical approximation. This hierarchical analysis in turn agrees with the seminal papers on the model based on perturbation theory by Brézin, Le Guillou and Zinn-Justin [6], and the analysis of Brézin and Zinn-Justin [6] also agrees with ours on the IR side in the limit $N \rightarrow \infty$. Furthermore the generally accepted view is confirmed by numerical simulations [7] and, which is of great importance in this respect, also by the Bethe Ansatz methods based on the exact $S$-matrix [3], which show in particular that the model has a mass gap. Nevertheless these results are not fully based on well proven assumptions and are rather self-consistent than rigorous. So we note that on the other hand that doubts against the general wisdom have been raised by Patrascioiu and Seiler [8].

We take an UV regularized version of (1) as our starting point. The scale is chosen such that the UV cutoff $\Lambda$ is situated at $\Lambda = 1$. The situation in constructive field theory is often complicated by the fact that the expansions around the situation where the degrees of

\footnote{For the analysis of the model in that limit a lattice regularization is probably most appropriate, see also [25, 26] and the comments below.}
freedom are to some extent decoupled start from regularized versions which tend to violate symmetries of the model in question. The symmetries on the other hand often greatly simplify the perturbative analysis if, as is often the case, an invariant regularization for perturbation theory is at hand. Fortunately this time we are on the easy side: Once we have introduced an interpolating field, which we now call $\sigma$, the whole analysis of the model can be performed without breaking the $O(N)$-symmetry, in complete agreement with the Mermin-Wagner-theorem [9],[10]. When the one-component scalar $\sigma$-field has been introduced we may integrate out the $\phi$-field thus obtaining a new interaction given by (the inverse of) a Fredholm determinant. For the UV cutoff model it is well-defined in finite volume. The infinite volume limit is taken in the end, once the cluster and Mayer expansions have been performed, which allow to divide out the divergent vacuum functional. The analysis of the Fredholm determinant proceeds similarly as that of the corresponding determinant in the case of the Gross-Neveu-Model [1]. It is simplified in the same way as the expansions are since we do not have to distinguish different zones characterized by the mean value of the $\sigma$-field - apart from the small field/large field splitting. The main new problem lies in the fact that for the inverted Fredholm determinant some of the estimates used to bound the determinant (together with antisymmetric tensor products generated by taking derivatives when cluster expanding, see [1],p.169 and more generally [11]) are no more valid. The problem is solved by deriving new bounds on inverted Fredholm determinants - in the last part of Ch.3, to show stability, by introducing a finer splitting of the large field configurations before cluster expanding to make sure that the cluster expansion derivatives always produce small terms, and by evaluating the expansion derivatives through Cauchy formulae.

The paper is organized as follows: Our specific choices for the regulators and the basic definitions are presented in Ch.2. They are dictated by technical simplicity. In Ch.3 we perform the small/large field splitting and develop the bounds on the various terms in the action ensuing from that splitting, as well as on the non-local operator kernels appearing. In particular we show that all the kernels appearing fall off exponentially in the small field region. In Ch.4 the cluster-expansion is performed which then allows to control the thermodynamic limit and to prove the exponential fall-off of the (two-point) correlation function(s).

After submitting this paper we learned about two important references on the subject. First the author was not aware of Kupiainen’s work [25]. Secondly, few weeks after submission there appeared a preprint by Ito and Tamura [26]. We close the introduction by shortly commenting on these papers. Kupiainen regards the $N$ component nonlinear $\sigma$-model on a unit width lattice for arbitrary dimensions $d$. He shows that the $1/N$-expansion is asymp-
totic above the spherical model critical temperature $T_S$, which is zero for $d = 2$. He also proves the existence of a mass gap for these temperatures and $N$ sufficiently large. Without attaching much importance to the numerical side we just say what 'N sufficiently large' means. We read from [25] (see equ.(19)) for the two-dimensional case that for given inverse temperature $\beta$ one needs

$$N > \text{cst} \, e^{50 \pi \beta}.$$ 

Since $\beta$ is to be identified with the inverse coupling $1/\lambda$ in our language this is basically the same as our bound: We require

$$N^{-1/6} < \text{cst} \, e^{-4\pi/\lambda}$$

since the small factor per cluster expansion step (see end of sect.4.4) has to beat the factor $O(m^{-2})$ from the spatial integration per link. Similarly the authors of [26] state their result in Theorem 24 for

$$N > \text{cst} \, e^{400 \pi \beta}$$

and $\beta$ large. They regard the same model as Kupiainen, the $N$ component nonlinear $\sigma$-model on a unit width lattice, for $d = 2$. Thus [25] and [26] analyse the lattice version of (1) where the limit $K \to \infty$ has been taken, i.e. the Heisenberg model. The result [26] only concerns the free energy or partition function which is shown to be analytic in $\beta$, given $N$ as above. Correlation functions have not yet been treated. It seems clear however that their method of proof which, as ours, is based on a small/large field cluster expansion is well adapted for that case too. We prove exponential fall-off of the two-point function, extension to any connected $n$-point function is straightforward using the Mayer expansion formulae for those, see e.g. [19]. The change in sect.4.5 would consist in singling out a connecting tree now for $n$ external points instead of two. Kupiainen’s result on the other hand is based on reflection positivity in the form of chess board estimates. It is not clear how the result on the exponential fall-off can be extended to general connected functions in this context, so strictly speaking (as he does) his result only holds for those correlation functions which have no nontrivial truncations.\footnote{In special cases he succeeds in performing truncations by a clever use of certain Ward identities.} An important point shared by [25] and [26] (in fact the authors of [26] could have referred themselves to [25] here) is that they both apply the Brydges-Federbush random walk representation to show and use exponential fall-off of the lattice kernels of $1/[p^2 + m^2 + i\sigma]$. In the continuum we only succeed in proving exponential fall-off for small fields $\sigma$. This is the main reason why we introduce a whole hierarchy of large field regions with larger and larger protection corridors (see (60)-(63)), and the fall-off over the corridors has to make up for the (possibly) absent fall-off in the large field domain. Apart from this [26] is technically closer to my paper than to [25]. It is more detailed on
some aspects of the expansions. A number of bounds take similar form here and in [26]. In [26] the building blocks of the cluster expansion are taken to be large also in the small field region. This has technical advantages, on the other hand treating many degrees of freedom as a whole generally tends to deteriorate the numerical bounds.

2 Presentation and Rigorous Definition of the Regularized Model

We want to show that the UV regularized large $N$ $\sigma$-model is massive, i.e. that the correlation functions decay exponentially. In our explicit representation we will restrict to the two-point function, generalizations to arbitrary $2N$-point functions being obvious. Thus formally we study the following object:

$$S_2(x, y) \sim \int D\phi \phi_i(x)\phi_i(y)e^{-\frac{N}{2\lambda}\int\{\partial\phi_i)^2 + \frac{K}{4}(\phi^2-1)^2\}}.$$  \hspace{1cm} (3)

Here $D\phi$ indicates the product of (ill-defined) Lebesgue measures $D\phi_1, \ldots, D\phi_N$. Before giving sense to this expression mathematically by imposing suitable regulators we want to introduce the interpolating field $\sigma$ as announced. We rewrite (3) as

$$S_2(x, y) \sim \int D\phi D\sigma \phi_i(x)\phi_i(y)e^{-\frac{N}{2\lambda}\int\{\partial\phi_i)^2 + i(\phi^2-1)\sigma + \frac{K}{4}\sigma^2\}}$$  \hspace{1cm} (4)

up to a global field-independent normalization factor. Now we can perform the Gaussian integrations over the $\phi$-fields to obtain

$$S_2(x, y) \sim \int D\sigma \left(\frac{1}{p^2 + i\sigma}\right)(x, y) \det^{-N/2}(p^2 + i\sigma) e^{-\frac{N}{2\lambda}\int \sigma^2 + \frac{K}{4}\int \sigma}$$  \hspace{1cm} (5)

again up to a global field-independent normalization factor and on rescaling $\phi^2 \rightarrow \phi'^2 = (N/2\lambda)\phi^2$. $(\frac{1}{p^2 + i\sigma})(x, y)$ denotes the position space kernel of the operator $\frac{1}{p^2 + i\sigma}$. Its existence in $L^2(\mathbb{R}^2)$ say, will be clear once the cutoffs and thus the support of the measure are specified below.\footnote{When studying higher order correlation functions it is preferable to work in the space $\bigoplus_1^N L^2(\mathbb{R}^2)$ and to suppress the exponent $N$ of $\det$ instead, because in this case the factor replacing $(\frac{1}{p^2 + i\sigma})(x, y)$ will depend on the flavour indices. We will adopt this convention only in the last part of the paper where it somewhat shortens the notation.} As regards \textbf{notation} we will generally use the same letters for position and momentum space objects. This lack in precision in our eyes is overcompensated by the gain in suggestive shortness. For the same reason and on the basis of the previous remarks

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on the size of the constants appearing we will abbreviate by \( O(1) \) sums of products of \( N \)-independent constants the largest of which appearing will actually be \( 1/m^2 \) (21). Without making this explicit we pay some attention not to collect astronomic numbers into \( O(1) \).

By performing a translation of the field variable \( \sigma \) according to

\[
\tau' = \sigma + i m^2
\]

we finally arrive at

\[
S_2(x, y) \sim \int D\tau \left( \frac{1}{p^2 + m^2 + ig\tau} \right)(x, y) \det^{-\frac{N}{2}} (1 + \frac{ig}{p^2 + m^2}\tau) e^{-\frac{1}{2} \int \tau^2 + i\sqrt{N}(\frac{m^2}{\sqrt{\lambda K}} + \sqrt{K} \lambda)} \int \tau. \quad (7)
\]

This time the change of normalization stems from three sources: from the translation, from a change of normalization of the Fredholm determinant and from a rescaling of the \( \tau' \)-field:

\( \tau' \to \tau = \sqrt{\frac{N}{\lambda K}} \tau' \). In (7) we introduced the coupling constant

\[
g = \sqrt{\frac{\lambda K}{N}}. \quad (8)
\]

The value of the translation parameter \( m \) is fixed below ((17)-(21)) by a gap equation. This eliminates the term in the interaction exponential which is linear in \( \tau \), and this in turn is a prerequisite in the \( 1/N \)-expansion, since that term has a coefficient \( \sim \sqrt{N} \). Before specifying the UV and IR regularizations we note that from the point of view of mathematical purity it would have been preferable to introduce them from the beginning. This however would have blown up the previous manipulations without a real gain since (3) and (7) are in fact to be viewed on equal footing as starting points: They both produce the same perturbation theory in \( 1/N \).

We now introduce the following regularizations:

**UV1**: We set the cutoff scale to be 1 and replace

\[
p^2 \to p_{\text{reg}}^2 = p^2 e^{p^2}
\]

in (7).

**UV2**: We also introduce an UV cutoff for the \( \tau \)-field. When tracing this back to the original interaction (1) it amounts to smoothing out the pointlike quartic \( \phi^4 \)-interaction. To the expression \( \int D\tau e^{-\frac{1}{2} \int \tau^2} \) in (7) corresponds in rigorous notation integration with respect to the Gaussian measure \( d\mu_\delta(\tau) \) with mean zero and covariance \( C_\delta(x - y) = \delta(x - y) \), or in momentum space \( C_\delta(p) = 1(p) \). We replace the \( \delta \)-function by a regularized version

\[
1(p) \to \frac{1}{1 + \hat{f}(p)}, \quad \hat{f}(p) = \sqrt{1 + \pi(p) f(p) \sqrt{1 + \pi(p)}}, \quad (10)
\]
where \( \pi(p) \) is defined below, see (28). It is a smooth nonnegative function depending on \( p^2 \) only, bounded above by a constant of order \( 1/m^2 \) (see (29)). \( f(p) \) also is a smooth nonnegative function depending on \( p^2 \) only. It vanishes in the origin, grows monotonically with \( p^2 \) such that \( \alpha(p^2)^2 < f(p) < A(p^2)^2 \) with suitable \( 0 < \alpha < A < \infty \), and fulfills

\[
(\frac{1}{1+f})(x-y) = 0, \quad \text{if } |x-y| > 1. \tag{11}
\]

The last condition is the most important one. That all conditions are mutually compatible is rather credible. A proof is in the elementary Lemma 1 in [1]. There a suitable \( f \) (which in [1] is further restricted by demanding that it should vanish of high order in the origin) is constructed explicitly, basically by starting from the characteristic function of the unit ball in position space \( \mathbb{R}^2 \) and taking linear combinations of rescaled convolutions thereof. We should note that it is by no means crucial to have a cutoff with these particular properties. Only sufficient fall-off of \( 1/(1+f) \) in momentum and position space are required. So e.g. \( 1/(1+e^{p^2}) \) would do. The compact support property (11) is however helpful when fixing the final covariance of the model, taking into account large field constraints, see (67). It eliminates further small correction terms of similar nature as those appearing in \( \delta C_\gamma \) (70), cf. the remark after (79). In short the UV cutoff on the \( \tau \) field replaces the ultralocal covariance \( \delta(x-y) \) of this field by a smoothed compact support version of the \( \delta \)-function sandwiched between the two \( \sqrt{1+\pi} \)-factors. The growth properties of \( f(p) \) restrict the support properties of the corresponding Gaussian measure \( d\mu_f(\tau) \) to (real) continuous functions [12] and therefore we need not regularize expressions such as \( \tau^2 \) etc. In general we will view \( \tau \) as an element of the real Hilbert space \( \mathcal{L}^2(\mathbb{R}^2, \mathbb{R}) \).

**IR:** As an intermediate IR regularization to be taken away in the end we also introduce a finite volume cutoff. To be definite we choose a square

\[
\Lambda \subset \mathbb{R}^2 \tag{12}
\]

centered at the origin with volume

\[
|\Lambda| = 4n^2 >> 1, \quad n \in \mathbb{N}. \tag{13}
\]

We then restrict the support of the \( \tau \)-field to \( \Lambda \). But we do not restrict the Gaussian measure to \( \Lambda \) from the beginning, because this again would increase the number of correction terms later when we perform a configuration dependent change of covariance. We want to avoid

\[\text{In fact we did not prove monotonicity in [1]. This can however be achieved by a slight extension of the proof. We do not include it since monotonicity is not needed here, it might however be useful when performing a renormalization group construction on the same basis.}\]
this, but nevertheless want to suppress contributions in the measure supported outside $\Lambda$. We therefore introduce a term

$$\exp(-R \int_{\mathbb{R}^2-\Lambda} \tau^2), \quad R \gg 1$$

in the functional integral, and take the limit $R \to \infty$ later on. Note that absorbing this term in the measure and taking the limit right away would amount to restricting the covariance to $\Lambda$ from the beginning [13]. Again our particular choices for the IR cutoff are convenient, but not crucial.

With these preparations we now obtain the following rigorous expression for the regularized normalized two-point function:

$$S_2^\Lambda(x, y) = \frac{1}{Z^\Lambda} \int d\mu_f(\tau) \left( \frac{1}{p_{\text{reg}}^2 + m^2 + ig\chi_{\Lambda}} \right)(x, y)$$

$$\times \det^{-N/2}(1 + \frac{1}{p_{\text{reg}}^2 + m^2}ig\chi_{\Lambda}) e^{-R \int_{\mathbb{R}^2-\Lambda} \tau^2} e^{i\sqrt{N}(\frac{m^2}{\sqrt{\lambda K}} + \sqrt{\lambda K}) \int_{\Lambda} \tau}.$$  

The partition function $\hat{Z}^\Lambda$ is given by

$$\hat{Z}^\Lambda = \int d\mu_f(\tau) \det^{-N/2}(1 + \frac{1}{p_{\text{reg}}^2 + m^2}ig\chi_{\Lambda}) e^{-R \int_{\mathbb{R}^2-\Lambda} \tau^2} e^{i\sqrt{N}(\frac{m^2}{\sqrt{\lambda K}} + \sqrt{\lambda K}) \int_{\Lambda} \tau}.$$  

Here $\chi_{\Lambda}$ is the sharp characteristic function of the set $\Lambda$ in position space. Instead of $\chi_X$ we will mostly use $P_X$ to denote the orthogonal projector on the subspace of functions supported in $X$. From the bounds on the action is given in the next section it is clear that $\hat{Z}^\Lambda$ will not vanish in finite volume (see (118)). In the following we will mostly suppress explicit reference to the regulators by $\text{reg}$ and $\chi_{\Lambda}$ for shortness.

As announced the value of $m$ is fixed by imposing a gap equation eliminating the linear term in $\tau$ from the action, i.e. we demand:

$$i\sqrt{N}(\frac{m^2}{\sqrt{\lambda K}} + \sqrt{\lambda K}) \int_{\Lambda} \tau = N/2 Tr(\frac{1}{p_{\text{reg}}^2 + m^2}ig\chi_{\Lambda}).$$

When evaluating the $Tr$, the term $\int_{\Lambda} \tau$ factorizes an both sides of (17), and we obtain the relation

$$1/2 \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + m^2} = \frac{m^2}{\lambda K} + \frac{1}{2\lambda}.$$  

For a sharp cutoff at $p^2 = 1$ we would find from this

$$\frac{1}{4\pi} \ln\left(1 + \frac{m^2}{m^2} \right) = 1/\lambda + 2\frac{m^2}{\lambda K}.$$  

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with the solution

$$m^2 = e^{-\frac{4\pi}{\lambda K}} (1 + O(\frac{4\pi}{\lambda K} e^{-\frac{4\pi}{\lambda K}})).$$

(20)

For the case of an exponential cutoff the integral cannot be evaluated analytically, but it is easy to find suitable upper and lower bounds saying that

$$m^2 = c_m e^{-\frac{4\pi}{\lambda K}},$$

(21)

where the constant $c_m$ is close to one (lies between 0.9 and 1.1) for $\lambda \leq 1$. For definiteness we will assume from now on

$$\frac{2}{\pi} < \lambda < \pi$$

so that $e^{-10} < m < 1/6$.

(22)

Taking into account the constraint (17) we thus obtain for the two-point function

$$S_2^\Lambda(x, y) = \frac{1}{Z^\Lambda} \int d\mu_f(\tau) \left( \frac{1}{p^2 + m^2 + ig\tau} \right)(x, y) \det_2^{-N/2}(1 + \frac{1}{p^2 + m^2}ig\tau) e^{-Rf_{\mathbb{R}^2-\Lambda}\tau^2}$$

(23)

where we used the standard definition

$$\det_{n+1}(1 + K) = \det(1 + K) e^{-TrK + \frac{1}{2}TrK^2 + \cdots + (-1)^n\frac{1}{n!}TrK^n}$$

(24)

for any traceclass operator $K$ and $n \in \mathbb{N}$. In an expansion based on the parameter $1/N$ the canonical choice of covariance is such that it contains all terms of the action quadratic in the field $\tau$, possibly up to terms which are suppressed for $N \to \infty$. This is not yet the case for (23) since the term quadratic in $\tau$ from det is not suppressed for $N$ large: It contains $1/N$ from $g^2$ and $N$ from $\det^{-N/2}$ giving $N^0$ altogether. Thus the appropriate presentation of the two-point function is rather

$$S_2^\Lambda(x, y) = \frac{1}{Z^\Lambda} \int d\mu_C(\tau) \left( \frac{1}{p^2 + m^2 + ig\tau} \right)(x, y) \det_3^{-N/2}(1 + \frac{1}{p_{\text{reg}}^2 + m^2}ig\tau) e^{-Rf_{\mathbb{R}^2-\Lambda}\tau^2}.$$  

(25)

A corresponding change of definition has also been introduced when passing from $\hat{Z}^\Lambda$ to $Z^\Lambda$

$$Z^\Lambda = \int d\mu_C(\tau) \det_3^{-N/2}(1 + \frac{1}{p_{\text{reg}}^2 + m^2}ig\tau) \chi(\Lambda) e^{-Rf_{\mathbb{R}^2-\Lambda}\tau^2}.$$  

(26)

In (25), (26) $d\mu_C(\tau)$ represents the Gaussian measure with covariance

$$C = (1 + \hat{f} + P_\Lambda\pi P_\Lambda)^{-1}.$$  

(27)

$P_\Lambda$ is the orthogonal projector onto the subspace $\mathcal{L}^2(\Lambda)$ of $\mathcal{L}^2(\mathbb{R}^2)$, and $\pi$ is the quadratic part in $\tau$ from det. In momentum space it is given as

$$\pi(p) = \frac{\lambda K}{2} \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q^2 + m^2} \frac{1}{(p + q)^2 + m^2} > 0.$$  

(28)
Since the integral is UV convergent, it is largely independent of the cutoff functions $e^{p^2}$ which we did not write explicitly. We find in particular
\[ \pi(0) = \frac{C_\pi \lambda K}{8\pi m^2}, \quad (29) \]
where $C_\pi$ is again a constant close to 1. Furthermore one easily realizes that
\[ \pi \leq \pi(0) \text{ in the operator sense, or in momentum space } \pi(0) - \pi(p) \geq 0. \quad (30) \]
This can either be done by direct calculation or by noting that
\[ (\tau, \pi(0) \tau) = \frac{N}{2} Tr (V V^*) \geq \frac{N}{2} Tr V^2 = (\tau, \pi \tau), \quad (31) \]
for the operator
\[ V(\tau) = \frac{1}{p^2 + m^2} g \tau. \quad (32) \]

By $(\tau, \pi \tau)$ we denote the scalar product, which is given by $\int \tau(x) \pi(x - y) \tau(y)$ in position space. Note that $V$ has real expectation values in the real Hilbert space $L^2(\Lambda, \mathbb{R})$.

For later use we collect the following facts about the operator $\pi$ and (some functions of) the kernels of $\pi$ and of $1/(p^2 e^{p^2} + m^2)$ in position space.

**Lemma 1:**

a) The operator $\pi$ fulfills: $0 \leq \pi \leq \pi(0)$.

b) The kernel of $\pi$ in position space denoted as $\pi(x - y)$ (using translation invariance) satisfies:
   i) $|\pi(x - y)| \leq O(1) e^{-2m|x - y|}$,
   ii) furthermore $|\sqrt{1 + \pi} \pm 1 (x - y)| \leq O(1) e^{-2m|x - y|}$ for $x \neq y$.

Proof: The proof of a) was given previously. The statement b)i) follows from standard analyticity arguments: $\pi(p)$ is analytic in momentum space for $(Imp)^2 \leq 4m^2$ as is directly seen from the integrand in (28) by shifting the integration variable $q$ by $p/2$. The main reason to choose the analytic regulator function $e^{p^2}$ was that it does preserve (and even slightly enlarge) this analyticity domain so that b)i) follows. Coming now to the statement in b)ii) we first note that the condition $x \neq y$ eliminates the $\delta$-distribution contribution to the kernel so that we may regard in fact
\[ (\sqrt{1 + \pi})^{\pm 1} - 1 \quad (33) \]
As compared to b)i) we now also have to verify that the real part of $1 + \pi$ stays positive for $(Imp)^2 \leq 4m^2$ (so as to exclude a cut, i.e. a violation of analyticity due to the square root).
Again explicit calculation simply reveals this to be the case, where the regularization again slightly improves the situation.

c) This statement was proven in [1], Lemma 5. The lower bound follows from the representations

\[
\frac{1}{p^2 e^{p^2} + m^2} = e^{-p^2} \frac{1}{p^2 + 1} \sum_{n=0}^{\infty} \left( \frac{1}{p^2 + 1} - m^2 \frac{1}{p^2 + 1} e^{-p^2} \right)^n
\]

and

\[
\frac{1}{p^2 + 1} - m^2 \frac{1}{p^2 + 1} e^{-p^2} = \frac{1}{p^2 + 1} (1 - m^2 e) + m^2 e \int_0^1 ds e^{-s(p^2+1)}.
\]

Since \( m^2 e < 1 \) it becomes now obvious by explicit calculation of the Fourier transforms that the kernel of \( \frac{1}{p^2 e^{p^2} + m^2} \) is pointwise positive.

\[\square\]

### 3 Small/Large Field Decomposition and Bounds

The representation of the correlation functions according to (25), (26) is well-suited for an expansion in \( 1/N \), since the remnants of the action left in \( \text{det}_3 \) are all suppressed by factors of \( 1/\sqrt{N} \) or smaller. We then have to bound the contributions from \( \text{det}_3 \) for large values of the field variable \( \tau \) to show that it is integrable with respect to the Gaussian measure \( d\mu_C(\tau) \).

From our starting point we presume that this should be possible, since there the action was manifestly integrable. However, to obtain a convergent expansion of the correlation functions we have to perform a cluster expansion which makes visible the decoupling of the degrees of freedom with increasing separation in space. The cluster expansion interpolation formulae modify all nonlocal kernels of the theory, the modification being different for the measure and \( \text{det}_3 \). Therefore one global bound is not sufficient. What we rather need are local bounds per degree of freedom. The solution we adopt is similar as in [1], with simplifications due to the fact that we only have one phase, and complications due to the fact that the model is not fermionic in origin. The latter implies that certain sign cancellations due to the Pauli principle are absent in the outcome of the cluster expansion and necessitates finer distinctions on the size of the \( \tau \)-field than in [1].

We distinguish between small and (a series of) large field configurations depending on the size of \( \int_{\Delta} \tau^2 \), where \( \Delta \) is any (closed) unit square in \( \Lambda \) with integer valued lower left corner coordinates \((n_1, n_2)\). Then we sum over the possible choices for all squares. For a given configuration we take the union of large field squares, enlarge this region by adding all squares below some finite distance from those and divide (roughly speaking) the enlarged

\[\text{This fact will be useful later (see in particular (173)), but it is not crucial.}\]
region into its connected components. In the interior of any such component we do not introduce interpolation parameters, it is even reasonable not to absorb the quadratic part of \( \det_2 \) in the covariance there. Rather we use the large field criteria and certain bounds on inverted Fredholm determinants to show that these regions are suppressed in probability per large field square \( \Delta \) and according to the size of \( f_\Delta \tau^2 \). Then the expansion is largely restricted to the small field region, where the integrability of \( \det_3 \) is assured due to the small field criterion anyway. As usual such a cluster expansion with constraints goes in hand with a certain amount of combinatorics and technicalities coming from all sorts of correction terms. These are controlled by means of the large value of \( N \). We are now going to make this reasoning precise.

We subdivide the volume \( \Lambda \) into the \( 4n^2 \) unit squares \( \Delta \) specified above and regard some given \( \tau \in L^2(\Lambda) \). We say that \( \Delta \in \Lambda \) is a large field square w.r.t. \( \tau \) if

\[
\lambda K \int_\Delta \tau^2 \geq N^{1/6},
\]

and \( \Delta \in \Lambda \) is a small field square w.r.t. \( \tau \) if

\[
\lambda K \int_\Delta \tau^2 < N^{1/6}.
\]

We introduce a smoothed monotonic step function \( \theta \in C^\infty(\mathbb{R}) \) fulfilling

\[
\theta(x) = \begin{cases} 
0, & \text{for } x \leq -1/4 \\
1, & \text{for } x \geq 1/4
\end{cases}
\]

Then we also introduce \( 4n^2 \) factors of 1 into the functional integral according to

\[
1_\Delta = \theta\left(\frac{\lambda K \Vert \tau_\Delta \Vert_2^2}{N^{1/6}} - 1\right) + \left(1 - \theta\left(\frac{\lambda K \Vert \tau_\Delta \Vert_2^2}{N^{1/6}} - 1\right)\right) \tag{38}
\]

In (39) we set as usual

\[
\Vert \tau_\Delta \Vert_2^2 = \int_\Delta \tau^2.
\]

Now the first factor is decomposed further writing

\[
\theta\left(\frac{\lambda K \Vert \tau_\Delta \Vert_2^2}{N^{1/6}} - 1\right) = \sum_{n=1}^{\infty} \left[\theta\left(\frac{\lambda K \Vert \tau_\Delta \Vert_2^2}{N^{n/6}} - 1\right) - \theta\left(\frac{\lambda K \Vert \tau_\Delta \Vert_2^2}{N^{(n+1)/6}} - 1\right)\right] =: \sum_{n=1}^{\infty} \theta_n(\Vert \tau_\Delta \Vert_2^2). \tag{40}
\]

We then may rewrite (25), (26) as a sum of \( 2^{4n^2} \) terms each carrying for any square \( \Delta \) a factor which is either the first or the second summand in (39). For a square carrying the first factor the functional integral is then split up further according to (41). To fix the language
we say

**Definition:** A square carrying the factor

\[
\theta^s_\Delta(\tau) := 1 - \theta\left(\frac{\lambda K||\tau_{\Delta}||^2}{N^{1/6}} - 1\right)
\]

is called a small field or s-square. A square \(\Delta\) carrying the factor

\[
\theta^l_\Delta(\tau) := \theta\left(\frac{\lambda K||\tau_{\Delta}||^2}{N^{1/6}} - 1\right)
\]

is called a large field or l-square. More specifically we call it an \(l^n\)-square if it carries a factor \(\theta_n(||\tau_{\Delta}||^2)\) resulting from the splitting (41).

An \(l\)-square then only contributes to the functional integral if

\[
\lambda K||\tau_{\Delta}||^2 \geq \frac{3}{4} N^{1/6},
\]

an \(l^n\)-square only if

\[
\frac{5}{4} N^{(n+1)/6} > \lambda K||\tau_{\Delta}||^2 \geq \frac{3}{4} N^{n/6},
\]

and an s-square only contributes, if

\[
\lambda K||\tau_{\Delta}||^2 < \frac{5}{4} N^{1/6},
\]

so that *we will always assume the respective inequality to hold once a square has been specified to be l, \(l^n\) or s*, since in this paper we are only bounding contributions to the functional integral. As regards notation we will write \(P_s\), \(P_l\), \(P_{l^n}\) and \(P_{\Delta}\) for the orthogonal projectors onto functions with support in \(\Lambda_s\), \(\Lambda_l\), \(\Lambda_{l^n}\) and \(\Delta\) respectively. Here we denote by \(\Lambda_s \subset \Lambda\) resp. \(\Lambda_l \subset \Lambda\) resp. \(\Lambda_{l^n} \subset \Lambda\) the set of small field resp. large field resp. \(l^n\) - squares in \(\Lambda\).

Note \(\Lambda_s \cup \Lambda_l = \Lambda\), \(\bigcup_{n \in \mathbb{N}} \Lambda_{l^n} = \Lambda_l\). Before proceeding further with the \(1/s\) decomposition we want to show that the small field condition is sufficient to obtain a small upper bound in norm on the operator appearing in det:

**Proposition 2:** For \(\tau \in \mathcal{L}^2(\Lambda)\) let \(\Lambda_s \subset \Lambda\) be a collection of unit squares such that for \(\Delta \in \Lambda_s\) we have

\[
\lambda K||\tau_{\Delta}||^2 < \frac{5}{4} N^{1/6}.
\]

Then the operator norm of \(A_s : \mathcal{L}^2(\Lambda) \to \mathcal{L}^2(\Lambda)\) satisfies:

\[
||A_s|| \leq O(1)N^{-5/12} \leq N^{-2/5}.
\]

Here \(A\) is the operator \(P_{\Lambda} \frac{1}{\beta_{reg+m^2}} g \tau P_{\Lambda}\), and \(A_s\) is defined to be \(P_s A P_s\).
Proof: We first regard $A_\Delta = P_\Delta A P_\Delta$ for $\Delta \in \Lambda_s$. For $\varphi \in \mathcal{L}^2(\Delta)$ and $||\varphi||_2 = 1$ we find:

$$|(A_\Delta \varphi, A_\Delta \varphi)| \leq g^2 \int_{x,y,z} |\varphi(x)\tau(x)F(x-y)\chi_\Delta(y)F(y-z)\tau(z)\varphi(z)|$$  \hspace{1cm} (49)

$$\leq g^2 F^2(0) \int_{x,y} |\varphi(x)\tau(x)\tau(y)\varphi(y)| \leq g^2 F^2(0) \int_{x \in \Delta} \tau^2(x) < F^2(0) \frac{5}{4} N^{-5/6}.$$  

Here $F(x-y)$ is the pointwise positive kernel (see Lemma 1)

$$F(x-y) = \int \frac{d^2q}{(2\pi)^2} \frac{e^{iq(x-y)}}{q^2 e^q + m^2},$$  \hspace{1cm} (50)

which is obviously bounded by its value at 0, which in turn is bounded by $O(1/\lambda)$, which we absorb in $O(1)$, which we bound by $N^{-2/5+5/6}$. This proves the assertion for a single square $\Delta$. To go from here to the general case one has to exploit the exponential fall-off of the kernel $F(x-y)$ (Lemma 1), which deteriorates the bound by a factor of $O(1/m^2)$, which we absorb in $O(1)$ and bound it again by $N^{-2/5+5/6}$. So now let $\varphi \in \mathcal{L}^2(\Lambda)$ with $||\varphi||_2 = 1$.

$$|(A_s \varphi, A_s \varphi)| \leq \sum_{\Delta, \Delta', \Delta'' \in \Lambda_s} |(A_\Delta \varphi, A_\Delta' \varphi)|$$  \hspace{1cm} (51)

$$\leq O(1) \sum_{\Delta, \Delta', \Delta'' \in \Lambda_s} \exp\left\{-m(dist(\Delta, \Delta') + dist(\Delta', \Delta''))\right\} \int_{x,y} |\tau(x)\chi_\Delta(x)\varphi(x)\tau(y)\chi_{\Delta''}(y)\varphi(y)|.$$  

By performing first the sum over $\Delta''$ and then over $\Delta'$ and using the bound on $\tau$, the Schwarz inequality and the fact that $\varphi$ is normalized, we obtain the bound

$$O(1) N^{-1} N^{1/12} \sum_{\Delta \in \Lambda_s} \left(\int_{\Delta} \varphi^2\right) \left(\int_{\Delta} \tau^2\right)^{1/2} \leq O(1) N^{-5/6}. $$  \hspace{1cm} (52)

This ends the proof.

As announced we want - for given l/s-regions - enlarge the l-regions by security belts of sufficient width such that the fall-off of the kernels from Lemma 1 will produce a small factor if the kernels have to bridge these belts. This procedure generally will merge together some of the different connected components of the l-region. Let $\Lambda_t^1, \ldots, \Lambda_t^r$ be the connected components of $\Lambda_t$. We say there is a connectivity link between $\Lambda_t^i$ and $\Lambda_t^j$, $1 \leq i, j \leq r$, $i \neq j$, if there exists some $\Delta_i \in \Lambda_t^i$ and some $\Delta_j \in \Lambda_t^j$ such that there exists $\Delta \in \Lambda$ with

$$dist(\Delta_i, \Delta) + dist(\Delta_j, \Delta) \leq 2M,$$  \hspace{1cm} (53)

where we choose for definiteness

$$M = \frac{2}{m} \ln N.$$  \hspace{1cm} (54)
Then we call $l_1, \ldots, l_s$ the maximal subsets of $\Lambda_l$ connected by connectivity links and call them *connectivity components*. Obviously $s \leq r$. Now we set

$$\Gamma = \Gamma(l) = \{ \Delta \subset \Lambda | \text{dist}(\Delta, \Lambda_l) \leq M \}$$

(55)

and

$$\Gamma_i = \Gamma(l_i) = \{ \Delta \subset \Lambda | \Lambda_i^k \subset l_i, \text{dist}(\Delta, \Lambda_i^k) \leq M \} .$$

(56)

Thus there is a one-to-one relation between the $\Gamma_i$ and the $l_i$, and the $\Gamma_i$ are connected \footnote{It requires some (elementary) work to really give an explicit proof of that fact, which amounts basically to transferring the square $\Delta$ constituting the connectivity link between $\Delta_i$ and $\Delta_j$ to the centre of a line of minimal length connecting $\Delta_i$ and $\Delta_j$ and showing that then either this transferred square or two of its neighbours touching each other connect together $\Delta_i$ and $\Delta_j$ within some $\Gamma_k$. We skip the proof since it is not crucial for us that the $\Gamma_i$ are connected.} (in the standard sense), and we have

$$\Gamma_i \cap \Gamma_j = 0 \text{ for } i \neq j , \text{ and } \bigcup_1^n \Gamma_i = \Gamma .$$

(57)

In set-theoretic relations we always denote by 0 a set of (standard) Lebesgue measure 0. We also introduce the sets $\gamma_i$ which (roughly speaking) lie between $l_i$ and $\Gamma_i$ :

$$\gamma_i = \{ \Delta \subset \mathbb{R}^2 | \Lambda_i^k \subset l_i , \text{dist}(\Delta, \Lambda_i^k) \leq M/2 \} , \quad \gamma = \bigcup_1^n \gamma_i$$

(58)

so that

$$\text{dist}(\gamma, \Lambda - \Gamma) \geq M/2 - \sqrt{2} .$$

(59)

Note that for technical reasons we have defined $\gamma$ as a subset of $\mathbb{R}^2$, not necessarily of $\Lambda$. We do so because this definition of $\gamma$ is useful when fixing the covariance in the presence of large field configurations such that it has good positivity and fall-off properties (see (67)-(70) and Lemma 3).

The previous definitions now are extended to the situation where we split up further the $\Lambda_l$-region into the components $\Lambda_{l^n}$. If the size of the field is very large we also need very large security belts to protect our large field regions - such that the decay of the kernel across this belt again assures a small contribution. We start again from the connected components $\Lambda_1^l, \ldots, \Lambda_r^l$ of $\Lambda_l$ and say that there is an *$e$-connectivity link* (or extended connectivity link) between $\Lambda_i^l$ and $\Lambda_j^l$, $1 \leq i, j \leq r, i \neq j$, if there exists some $\Delta_i \in \Lambda_i^l \cap \Lambda_{l^n}$ and some $\Delta_j \in \Lambda_j^l \cap \Lambda_{l^n}$ such that there exists $\Delta \in \Lambda$ with

$$\text{dist}(\Delta_i, \Delta) + \text{dist}(\Delta_j, \Delta) \leq (n' + n'')M .$$

(60)
The *e*-connectivity components are then the maximal subsets of $\Lambda$ connected by *e*-connectivity links. We call them $\ell_i^e$, $1 \leq i \leq s'$, and obviously $s' \leq s \leq r$. Now we set

$$\Gamma^e = \Gamma^e(l) = \bigcup_n \{ \Delta \subset \Lambda | \text{dist}(\Delta, \Lambda_{\ell n}) \leq n M \} \quad (61)$$

and

$$\Gamma^e_i = \Gamma^e(l^e_i) = \bigcup_n \{ \Delta \subset \Lambda | \Lambda_{\ell n}^e := \Lambda_{\ell n} \cap \Lambda_{\ell i}^e \subset \ell^e_i , \text{dist}(\Delta, \Lambda_{\ell n}^e) \leq n M \} . \quad (62)$$

Again there is a one-to-one relation between the $\Gamma^e_i$ and the $\ell_i^e$, and as before

$$\Gamma^e_i \cap \Gamma^e_j = 0 \quad \text{for} \quad i \neq j , \quad \text{and} \quad \bigcup_i \Gamma^e_i = \Gamma^e . \quad (63)$$

Starting from the l/s- decomposition of the volume $\Lambda$ we now decompose the Fredholm determinant, define the $s$-dependent final covariance and bound the large field action. With the definition of the operator $A$ (Proposition 2) we can write the Fredholm determinant as $\det(1 + iA)$. We first separate $A_s$ from the rest of $A$ via the standard relation

$$\det^{-1}(1 + iA) = \det^{-1}(1 + iA_s) \det^{-1}(1 + \frac{1}{1+iA_s}iA'') \quad (64)$$

with

$$A'' := A - A_s = A' + A_l , \quad A' := P_s A P_l + P_l A P_s . \quad (65)$$

Since $A$ has real spectrum, the operator $1/(1 + iA)$ is well-defined. For $A_s$ we now proceed as indicated before (see (25)), i.e. we absorb the quadratic part in $\tau$ into the covariance. When doing so we obtain the following (transitory) expression for the inverse propagator $C_{ls}^{-1}$:

$$C_{ls}^{-1} = P_s \pi P_s + 1 + \hat{f} . \quad (66)$$

We express (66) in terms of $C^{-1}_\gamma$ (67), the basic reason for this being the fact that we are not able to deduce suitable fall-off properties in position space for the inverse of (66). Our final choice for the configuration dependent covariance will rather be

$$C^{-1}_\gamma = \sqrt{1 + \pi(1 - P_\gamma + \varepsilon P_\gamma + f)}\sqrt{1 + \pi} . \quad (67)$$

Here $\varepsilon$ is introduced so that $C_\gamma$ is bounded also in the large field region. We fix it as

$$\varepsilon = N^{-\frac{2}{5}} . \quad (68)$$

Choosing (67) we have to control the difference between (66) and (67), since it is (66) which is isolated from the action. Writing

$$C^{-1}_\gamma - C_{ls}^{-1} = \delta C_\gamma - P_l$$

(69)
we obtain for $\delta C_\gamma$ the sum of terms:

$$
\delta C_\gamma = \sum_{i=1}^{i=4} \delta C_i(\gamma),
$$

$$
\delta C_1(\gamma) = -P_s(\sqrt{1 + \pi})P_\gamma(\sqrt{1 + \pi})P_s,
$$

$$
\delta C_2(\gamma) = P_l(\sqrt{1 + \pi})(1 - P_\gamma)(\sqrt{1 + \pi})P_\gamma + P_\gamma(\sqrt{1 + \pi})(1 - P_\gamma)(\sqrt{1 + \pi})P_l
$$

$$
\delta C_3(\gamma) = (1 - P_\Lambda)(\sqrt{1 + \pi})(1 - P_\gamma)(\sqrt{1 + \pi})P_\Lambda + P_\Lambda(\sqrt{1 + \pi})(1 - P_\gamma)(\sqrt{1 + \pi})(1 - P_\Lambda) + (1 - P_\Lambda)(\sqrt{1 + \pi})(1 - P_\gamma)(\sqrt{1 + \pi})(1 - P_\Lambda),
$$

$$
\delta C_4(\gamma) = \sqrt{1 + \pi} \varepsilon P_\gamma \sqrt{1 + \pi}.
$$

Having introduced the final covariance we may now rewrite the expression for the two-point function based on the Gaussian measure $d\mu_\gamma$ with covariance $C_\gamma$ normalized such that

$$
\int d\mu_\gamma(\tau) = 1.
$$

Since our covariance is configuration dependent there will be a change of normalization of the functional integral when changing the l/s-assignment. Relative to the situation where $\gamma = \emptyset$ this normalization factor is given by [13]

$$
Z_\gamma = \det^{1/2}(C_\gamma/C_0),
$$

where $C_0$ is given below (76). Taking into account this factor we may rewrite (25) as

$$
S_2^\Lambda(x, y) = \frac{\sum_{l,s} Z_\gamma \int d\mu_\gamma(\tau) \left( \frac{1}{\sqrt{\tau^2 + iA_s^2 + i\pi^2}} \right) G_\gamma(x, y) G_\gamma(x, y) \sum_{l,s} Z_\gamma \int d\mu_\gamma(\tau) G_\gamma(\tau)}{\sum_{l,s} Z_\gamma \int d\mu_\gamma(\tau) G_\gamma(\tau)} , \quad \gamma = \gamma(l).
$$

For the action $G_\gamma$ we find collecting the results of the previous manipulations:

$$
G_\gamma = \Pi_{\Delta \in \Lambda} \theta^l_\Delta(\tau) \Pi_{\Delta \in \Lambda} \theta^s_\Delta(\tau) e^{-1/2 \int_{\Lambda} \tau^2 \det_3^{-N/2}(1 + iA_s) \det_2^{-N/2}(1 + \frac{1}{1 + iA_s} iA''')} (74)
$$

$$
\times e^{-R \int_{R_{2-L}} \tau^2} e^{1/2(\tau, \delta C_\gamma, \tau)}.
$$

Note that we get indeed $\det_3^{-N/2}(1 + iA_s) \det_2^{-N/2}(1 + \frac{1}{1 + iA_s} iA'')$ after using the gap equation and absorbing the quadratic part of $\det^{-N/2}(1 + iA_s)$ since

$$
TrA = TrA_s + TrA_l, \quad TrA_l = Tr\left( \frac{1}{1 + iA_s} A_l \right),
$$

$$
17
$$
on using $Tr A' = 0$ and $A_k A_l = 0$.

We first analyse the covariance $C_\gamma$. Then we bound the normalization factors $Z_\gamma$ and the correction terms $\delta C_\gamma$. Finally we bound the large field determinant. Calling $C_0$ the covariance $C_\gamma$ for the case that $\gamma = \emptyset$ which means

$$C_0 = \frac{1}{\sqrt{1 + \pi}} \frac{1}{1 + f} \frac{1}{\sqrt{1 + \pi}}$$

we may write the inverse of (67) as

$$C_\gamma = C_0 + C_0(C_0^{-1} - C_\gamma^{-1})C_\gamma = C_0 \sum_{r=0}^{\infty}[(C_0^{-1} - C_\gamma^{-1})C_0]^r$$

(77)

$$= C_0 + \frac{1}{\sqrt{1 + \pi}} \frac{1}{1 + f} \sum_{r=0}^{\infty} \frac{1}{\sqrt{1 + f}} P_\gamma (1 - \varepsilon) \frac{1}{\sqrt{1 + f}} \frac{1}{\sqrt{1 + \pi}}$$

$$= C_0 + \frac{1}{\sqrt{1 + \pi}} \frac{1}{1 + f} \sum_{r=0}^{\infty} P_\gamma (1 - \varepsilon) \frac{1}{1 + f} P_\gamma (1 - \varepsilon) \frac{1}{\sqrt{1 + \pi}} \frac{1}{\sqrt{1 + \pi}}.$$  

The sums are obviously norm-convergent. At this stage the support properties of $1/(1 + f)$ (11) become very helpful. They imply that in position space $C_\gamma$ may be written in terms of a simple sum over disconnected pieces with support restricted to (a neighbourhood of) $\gamma_i$.

We obtain

$$C_\gamma = C_0 + C^\gamma, \quad C^\gamma := \sum_{i=1}^{\infty} C^{\gamma_i} ,$$

$$C^{\gamma_i} := \frac{1}{\sqrt{1 + \pi}} \frac{1}{1 + f} P_{\gamma_i} (1 - \varepsilon) \sum_{r=0}^{\infty} \frac{1}{1 + f} P_{\gamma_i} (1 - \varepsilon) \frac{1}{1 + f} \frac{1}{\sqrt{1 + \pi}} .$$

(79)

If we had only imposed exponential fall-off for $1/(1 + f)$, arbitrarily many terms coupling the various $\gamma_i$ would have appeared. They could be shown to be small using the distance between the various $\gamma_i$ of size $\sim M$ and the fall-off of $1/(1 + f)$ and of $\frac{1}{\sqrt{1 + \pi}}$, but still they would be a nuisance. The fall-off properties of $C_0$ have been analysed in Lemma 1. The complications stemming from nonempty $\gamma$ are controlled easily in

**Lemma 3:** The kernel $C^\gamma$ for $\gamma \neq \emptyset$ satisfies the following estimates:

$$|C^\gamma(x, y)| \leq O(1) N^{2/5} \exp\{-2m(\text{dist}(x, \gamma) + \text{dist}(y, \gamma))\} .$$

(80)

For $x, y \in \Lambda - \Gamma$ or $x \in \Gamma_i$, $y \in \Gamma_j$ with $i \neq j$ we find:

$$|C^\gamma(x, y)| \leq O(1) \frac{1}{N^{18/5}}$$

(81)

and for $x \in \Gamma$, $y \in \Lambda - \Gamma$

$$|C^\gamma(x, y)| \leq O(1) \frac{1}{N^{8/5}} .$$

(82)
Finally we have
\[ |C_0(x, y)| \leq O(1) \exp\{-2m|x-y|\}. \tag{83} \]

**Proof:** We have to control the contribution of the infinite sum over \( r \) in (79). We abbreviate
\[ O = P_{\gamma_i} (1-\varepsilon) \sum_{r=0}^{\infty} \left[ \frac{1}{1+f} \right]^{r_i} P_{\gamma_i} (1-\varepsilon)^{r_i} \]
so that
\[ C_{\gamma_i} = BOB^* \tag{85} \]
Obviously \( ||O|| \leq N^{2/5} \) and \( ||B|| \leq 1 \). Furthermore the kernel of \( B \) is continuous and pointwise bounded by \( O(1) \). By inserting characteristic functions of squares \( \Delta \) between \( B \) and \( O \) and between \( O \) and \( B^* \), summing over the squares, using the fall-off properties of the kernels and bounding
\[ ||(\chi_{\Delta}, O \chi_{\Delta'})|| \leq N^{2/5} \tag{86} \]
we then arrive at the bounds stated in Lemma 3. For the required properties of the kernels see Lemma 1 and (11). The minimal distances of points fulfilling the conditions specified in Lemma 3 follow from the definitions (54)-(59).

We remark that the bounds in Lemma 3 could be somewhat improved on by using methods similar to those employed in the proof of Lemma 4. We do not do so because this improvement would not strengthen our final bounds anyway. Note in particular that the cluster expansion will be performed such that only \( C^\gamma \)-terms bridging the gap between \( \gamma \) and \( \Lambda - \Gamma \) will be produced. Now we are going to bound the factors \( Z_{\gamma} \).

**Lemma 4:** Let \( |\gamma| \) denote the volume of \( \gamma \). Then
\[ 1 \leq Z_{\gamma} \leq e^{O(1)|\gamma|}. \tag{87} \]

**Proof:** Using (76),(78),(79) we have
\[ Z_{\gamma} = \det^{1/2}(C_{\gamma}/C_0) = \det^{1/2}(1 + \sum_i C_{\gamma_i}/C_0) = \det^{1/2}(1 + \sum_{i, r_i \geq 1} [(1-\varepsilon) P_{\gamma_i} \frac{1}{1+f}]^{r_i}) \tag{88} \]
\[ = \Pi_i \det^{1/2}(1 + \sum_{r_i \geq 1} [(1-\varepsilon) P_{\gamma_i} \frac{1}{1+f}]^{r_i}) = \Pi_i \det^{1/2}(C_{\gamma_i}/C_0) = \Pi_i Z_{\gamma_i}. \]
Again we used the support properties of \( 1/(1+f) \) to factorize the determinant. For \( Z_{\gamma_i} \) we now find
\[ Z_{\gamma_i} = \det^{1/2}(1 + \frac{(1-\varepsilon) P_{\gamma_i} \frac{1}{1+f}}{1 - (1-\varepsilon) P_{\gamma_i} \frac{1}{1+f}}) = \det^{-1/2}(1 - (1-\varepsilon) P_{\gamma_i} \frac{1}{1+f}) \tag{89} \]
\[ = \exp \text{Tr}\{(-1/2) \ln(1 - (1 - \varepsilon) P_{\gamma} \frac{1}{1+f})\} = \exp \text{Tr}\{1/2 \sum_{r \geq 1} \frac{1}{r}(1 - \varepsilon)^r [P_{\gamma} \frac{1}{1+f}]^r\} . \]

This expression implies \( Z_{\gamma} \geq 1 \). On the other hand we may use Lemma 3' from [1], which says that for an Hermitian trace class operator \( A \) and orthogonal Projector \( P \) we have the inequality :

\[ \text{Tr}(PA^r) \leq \text{Tr}P A^r P. \] (90)

Applying this to \( \frac{1}{1+f} \) and \( P_{\gamma} \) we may bound

\[ \text{Tr}\left( \frac{1}{1+f} P_{\gamma}\right)^r = \text{Tr}(P_{\gamma} \frac{1}{1+f} P_{\gamma})^r \leq \text{Tr}(P_{\gamma} (\frac{1}{1+f})^r P_{\gamma}) \leq O(1) |\gamma_i| . \]

using the fact that

\[ \int d^2 p \left( \frac{1}{1+f(p)} \right)^r \leq O(1) \int d^2 p \left( \frac{1}{1+(p^2)^2} \right)^r \leq O(1)r^{-1/2} . \] (92)

Using this we obtain

\[ Z_{\gamma} \leq \exp\{O(1)|\gamma| \sum_{r \geq 1} \frac{1}{r^{3/2}} (1 - \varepsilon)^r \} \leq \exp(O(1)|\gamma|) . \] (93)

This proves Lemma 3.

We now come to the bounds on the correction terms \( \delta C_i(\gamma) \) from (70).

**Lemma 5:**

i) \( \delta C_1(\gamma) \leq 0 \) (as an operator),

ii) \( \delta C_3(\gamma) \leq 1 + \pi(0) , \delta C_4(\gamma) \leq O(1) N^{-2/5} \) (as operators),

iii) \( ||\delta C_2(\gamma)|| \leq O(1) N^{-2} , |\delta C_2(\gamma)(x - y)| \leq O(1) \inf\{e^{-2m|x-y|}, N^{-2}\} . \)

**Proof:** i) is immediately obvious from the positivity of \( \pi \).

ii) The first statement is obvious since

\[ \delta C_3(\gamma) = (\sqrt{1+\pi})(1 - P_\gamma)(\sqrt{1+\pi}) - P_\Lambda(\sqrt{1+\pi})(1 - P_\gamma)(\sqrt{1+\pi})P_\Lambda . \] (94)

Note that \( \delta C_3(\gamma) \) only enters through interactions with field configurations of support outside \( \Lambda \), which will be suppressed anyway when taking \( R \to \infty \), (Prop.8,(114)). The bound on \( \delta C_4(\gamma) \) follows from the definition of \( \varepsilon \) in (68).

\footnote{Unfortunately the factor of \( r^{-1/2} \) appearing in (92) is falsely written as \( 2^{-r} \) in [1]. This mistake fortunately is of no consequence however.}
iii) The first statement in iii) follows from the exponential fall-off of \( \sqrt{1 + \pi} \) (Lemma 1) and the fact that \( \text{dist}(\Lambda_l, (\mathbb{R}^2 - \gamma)) \geq \frac{m \kappa}{m} \) (see (58)). This implies a bound on \( \delta C_2(\gamma) \) of the form in iii), closer inspection shows that \( O(1) \) is basically given by \( m^{-3} \), two powers coming from the integration over the kernel bridging the distance gap and one coming from a norm bound on the second \( \sqrt{1 + \pi} \). The second statement in iii) also follows from the definition of \( \gamma \) and from the fall-off of the kernel of \( \sqrt{1 + \pi} \).

Now we come to the bound on the nondiagonal term \( \det_{-\frac{N}{2}}^{-1/2}(1 + \frac{1}{1+iA_s}iA'') \) in the action (74). We need to get a suitable bound for this term which is sufficiently stable under the modifications caused by the cluster expansion parameters. We (temporarily) introduce the operator \( B \) through

\[
B = \frac{1}{1 + iA_s}iA'' = \frac{1}{(1 + iA_s)(1 - iA_s^*)}(i + A_s^*)A''.
\] (95)

Using the facts that the \( A \)-operators have real expectation values in real Hilbert space, that \( TrA^nA'' = 0 \) and cyclicity we find

\[
|\det_{-\frac{N}{2}}^{-1}(1 + B)| = |\det_{-\frac{N}{2}}^{-1}(1 + B)| = |\det_{-\frac{N}{2}}^{-1}(1 + B^*)| = \det_{-\frac{N}{2}}^{-1/2}(1 + D),
\] (96)

where

\[
D = B + B^* + B^* B.
\] (97)

Now we may apply the norm bound on \( A_s \) from Proposition 2 to realize that \( B \) coincides with \( iA'' \) up to small corrections, more precisely:

**Lemma 6:** For \( \varphi \in \mathcal{L}^2(\Lambda) \) we find

\[
B\varphi = iA''\varphi + \delta A''\varphi, \quad B^*\varphi = -iA''^*\varphi + A''^*\delta^*\varphi,
\] (98)

where the operator \( \delta \) is bounded in norm as

\[
||\delta|| \leq (1 + \alpha)||A_s|| \leq (1 + \alpha)N^{-2/5} \ll 1
\] (99)

with suitable \( 0 < \alpha \ll 1 \).

**Proof:** Since we have

\[
\delta = i\left(\frac{1}{1+iA_s} - 1\right)
\] (100)

the statements of the Lemma follow directly from Proposition 2, where \( \alpha \) may be chosen to obey an upper bound of size \( \sim ||A_s|| \).
Now we can also bound the operator $D$. For $\varphi \in \mathcal{L}^2(\Lambda)$ normalized to one we obtain
\[
(\varphi, D\varphi) = i(\varphi, A''\varphi) - i(A''\varphi, \varphi) + (\varphi, \delta A''\varphi) + (\delta A''\varphi, \varphi) + (A''\varphi, \varphi) + (A''\varphi, \frac{1}{1-iA_s^*}(1+iA_s) - 1)A''\varphi.
\]
Since the first two terms drop out, this entails
\[
(\varphi, D\varphi) \geq (1 - \eta)||A''\varphi||_2^2 - \eta||A''\varphi||_2 \geq -\frac{4\eta^2}{25(1-\eta)}
\]
with the choice
\[
\eta = 2||\delta|| << 1.
\]
Splitting the selfadjoint operator $D$ into its negative part $D_-$ and its nonnegative part $D_+$
\[
D = D_+ - D_-
\]
we thus have obtained that
\[
0 \leq D_- \leq \frac{4\eta^2}{25(1-\eta)} \leq N^{-4/5}.
\]
Using this we may now proceed to a bound on $|\det_2^{-1}(1 + B)| = \det^{-1/2}(1 + D)$. We find
\[
det^{-1/2}(1 + D) = e^{-1/2TrD} det_2^{-1/2}(1 + D)
\]
\[
= e^{-1/2TrB^*B} det_2^{-1/2}(1 + D_+) det_2^{-1/2}(1 - D_-) \leq det_2^{-1/2}(1 - D_-).
\]
Here we used again the fact that $TrB = iTrA_l = TrB^*$. Evaluating the trace of $D_-$ in an eigenbasis of $D_-$ one may easily establish the bound
\[
\det_2^{-1/2}(1 - D_-) \leq \exp\left(\frac{1}{4 - 4||D_-||} TrD_-\right).
\]
To verify the first inequality one observes that for $x > -\varepsilon > -1$ we have $x - \ln(1+x) \leq \frac{x^2}{2-\varepsilon + x}$. Now it remains to bound $TrD_-^2$. We call $\{\varphi_-\}$ a suitable set of normalized eigenfunctions of $D_-$ and find
\[
TrD_-^2 = \sum(\varphi_-, D^2\varphi_-) = \sum[(\varphi_-, D\varphi_-)]^2
\]
\[
\leq \sum(\frac{4}{5}\eta||A''\varphi_-||)^2 \leq \frac{16}{25}\eta^2 Tr(A''A'') \leq \frac{16}{25}\eta^2 Tr(A^*A)
\]
\[
\leq \frac{16}{25}\eta^2 g^2(\int_{\Lambda} \tau^2)(\int_{\Lambda} \frac{d^2p}{(2\pi)^2} \frac{1}{p_{reg}^2 + m^2})^2 \leq O(1)N^{-4/5}g^2(\int_{\Lambda_s} \tau^2 + \int_{\Lambda_l} \tau^2).
\]
In the first inequality in (108) we made use of (102). It is admittedly pedantic to insist on factors as \((4/5)^2\) in our context. To pass from \(Tr(A''^*A'')\) to the expression in the last line in (108) it is sufficient to take away the projectors \(P_l\) or \(P_s\) and thus to bound \(Tr(A''^*A'')\) in terms of \(Tr(A^*A)\) which is given by the double integral. We have obtained

\[
|\det_2^{-N/2} (1 + \frac{1}{1+iA_s})| \leq e^{O(1)N^{-4/5} \int_{\Lambda} \tau^2}. \tag{109}
\]

Finally we may also bound somewhat further the terms \(\langle \tau, \delta C_3(\gamma) \tau \rangle\) and \(\langle \tau, \delta C_4(\gamma) \tau \rangle\) appearing in \(G_\gamma\). Writing \(\hat{\tau} := \tau \chi_{\mathbb{R}^2 - \Lambda}\) we have

\[
\frac{1}{2} \langle \tau, \delta C_3(\gamma) \tau \rangle = (\hat{\tau}, \delta C_3(\gamma) \hat{\tau}) + 1/2 \langle \hat{\tau}, \delta C_3(\gamma) \hat{\tau} \rangle. \tag{110}
\]

We may then bound

\[
(\hat{\tau}, \delta C_3(\gamma) \hat{\tau}) + 1/2 \langle \hat{\tau}, \delta C_3(\gamma) \hat{\tau} \rangle \leq R/2 (\hat{\tau}, \hat{\tau}) + \frac{O(1)}{R} (\tau \chi_{\Lambda}, \tau \chi_{\Lambda}) \tag{111}
\]

for \(R\) large enough, using Lemma 5. As for \(\delta C_4(\gamma)\) we find

\[
\langle \tau, \delta C_4(\gamma) \tau \rangle \leq O(1) N^{-2/5} [(\tau \chi_{\Lambda}, \tau \chi_{\Lambda}) + (\hat{\tau}, \hat{\tau})]. \tag{112}
\]

Now we dispose of a complete control of the action \(G_\gamma\) from (74) and may collect our in findings in

**Proposition 8:** For \(R\) large enough we have

\[
Z_\gamma G_\gamma(\tau) \leq e^{-\frac{49}{100} \int_{\Lambda_l} \tau^2} e^{O(1) N^{-2/5} \int_{\Lambda_s} \tau^2} e^{-R/2 \int_{\mathbb{R}^2 - \Lambda} \tau^2} \tag{113}
\]

Thus we now take the limit \(R \to \infty\) and absorb the term \(e^{-R/2 \int_{\mathbb{R}^2 - \Lambda} \tau^2}\) in the covariance, which implies that we may replace

\[
C_\gamma \to P_\Lambda C_\gamma P_\Lambda \tag{114}
\]

and restrict the action to configurations \(\tau(x)\) with \(\text{supp } \tau \subset \Lambda\).

**Proof:** The proof concerning (114) is to be found e.g. in [13], so we have only to gather the pieces for the proof of (113). In the first term \(e^{-\frac{49}{100} \int_{\Lambda_l} \tau^2}\) we collected together the contribution \(e^{-1/2 \int_{\Lambda_l} \tau^2}\) from (74) the \(\Lambda_l\)-contribution from (109), the term from the bound on \(Z_\gamma\) in Lemma 4, where we used

\[
O(1) |\gamma| \leq O(1) \left(\frac{\ln N}{m}\right)^2 |\Lambda_l| \leq O(1) \left(\frac{\ln N}{m}\right)^2 N^{-1/6} \int_{\Lambda_l} \tau^2, \tag{115}
\]

and the contributions from (111) and (112) in \(\Lambda_l\). Finally we also absorbed in this term a contribution coming from \(\delta C_2(\gamma)\). In the term \(e^{O(1) N^{-2/5} \int_{\Lambda_s} \tau^2}\) we have absorbed the
contribution in $\Lambda_s$ from (109),(111),(112) and again a contribution coming from $\delta C_2(\gamma)$.
Finally we absorbed the one from the bound on $|\det_3^{-N/2}(1 + iA_s)|$ to be derived now: $|\det_3^{-N/2}(1 + iA_s)|$ can be bounded using the inequality

$$|TrA^n| \leq ||A^{n-2}||Tr(A^*A)$$  \hspace{1cm} (116)$$
valid for any traceclass operator $A$ and $n \geq 2$. To bound $||A^{n-2}||$ we use Proposition 2.
We may restrict to $n = 3$, the subsequent terms in the expansion of $\det_3^{-N/2}(1 + iA_s)$ being much smaller

$$|TrA_3^2| \leq ||A_s|| Tr(A_s^*A_s) \leq N^{-2/5}N^{-1}O(1)(\tau\chi_s, \pi(0) \tau\chi_s) \leq O(1)N^{-7/5} \int_{\Lambda_s} \tau^2 .$$  \hspace{1cm} (117)$$
With the help of the previous remarks and this relation we can verify the bound $|\det_3^{-N/2}(1 + iA_s)| \leq e^{O(1)N^{-2/5} \int_{\Lambda_s} \tau^2}$.
The following result now is immediate.

**Corollary:** Reducing the volume $\Lambda$ to a single square $\Delta$ equipped with a small field condition (42) we find

$$Z^\Delta = \int d\mu^\Delta(\tau) G^\Delta = 1 + o(N^{-1/5}) ,$$  \hspace{1cm} (118)$$
where $G^\Delta$ is the integrand from (26) restricted to the single small field square volume, and $d\mu^\Delta(\tau)$ is the normalized measure with covariance $\chi_\Delta C_0 \chi_\Delta$.
The statement follows from the bound (113) restricted to one small field square in $\Lambda$.

It will be useful later on to bound the large field contribution in (113), r.h.s. by a product of suppression factors in probability per square $\Delta \in \Gamma$. If (43) holds we may write

$$\int_{\Delta \in \Lambda_t} \tau^2 \geq 1/2 \int_{\Delta \in \Lambda_t} \tau^2 + \frac{3\lambda K}{8} N^{1/6} \quad \text{and} \quad \int_{\Delta \in \Lambda_m} \tau^2 \geq 1/2 \int_{\Delta \in \Lambda_m} \tau^2 + \frac{3\lambda K}{8} N^{n/6} .$$  \hspace{1cm} (119)$$
Therefore we obtain

**Lemma 9:**

$$e^{-\frac{49}{100} \int_{\Lambda_t} \tau^2} \leq e^{-1/4 \int_{\Lambda_t} \tau^2} e^{-N^{1/8}|\tau^2|} \prod_{\Delta \in \Lambda_n} e^{-N^{n-1} \frac{8}{8}} .$$  \hspace{1cm} (120)$$
**Proof:** It suffices to observe that for $N$ sufficiently large we have (using (42),(45))

$$O(1)(\frac{n \ln N}{m})^{-2} N^{n/6} \geq N^{\frac{n}{8}} .$$

Now we have sufficient control of the action to start with the expansions.
4 The Expansions, Proof of Mass Generation

4.1 The General form of the Expansions

The cluster expansion allows to control the spatial correlations of the model. When combined with a subsequent Mayer expansion, which frees the clusters from their hard core constraints, it allows to take the thermodynamic limit and to bound the decay of the correlation functions.

We proceed similarly as in [1] and use the general formalism for cluster expansions presented in [18], which in turn is an elaboration on a theme which has been the subject of several seminal papers by Brydges and collaborators over more than a decade. We apply in particular the Brydges-Kennedy formulae [15]. For general references on cluster expansions see also [13], where the presentation is close to the original way of introducing cluster expansions in constructive field theory, and [14], [19], which are close to our way of presentation.

The cluster expansion is a technique to select explicit connections between different spatial regions. The best formulas for the clusters involve trees, which are the minimal way to connect abstract objects together. We call the subsequent formulae forest formulae, the forests generally consisting of several disconnected trees. The basic building blocks of our expansion are the large field blocks $\Gamma_i^e$ (62) composed of (generally many) large field squares and their security belts, and the individual small field squares $\Delta$ from

$$S := \Lambda - \Gamma^e.$$  \hspace{1cm} (121)

From the point of view of the presentation it seems advantageous to connect together these large field blocks $\Gamma_i^e$ already by a first cluster expansion, and then to proceed to a second one, the building blocks of which are given by the outcome of the first. Then the expansion really connects together unit size squares which allows to somewhat unify the language as regards convergence criteria etc. In view of the existence of the excellent presentations to be found in [14]-[19] and since we stick very closely to [18] we hardly give indications on the proofs of cluster expansion formulae here. The general forest formula we are going to use will be given now. We introduce the following notation:

Let $I$ be a finite index set (in our context the set $\Lambda$ of the squares $\Delta \in \Lambda$) and $P(I)$ the set of all unordered pairs $(i, j) \in I \times I$, $i \neq j$. A (unordered) forest $\mathcal{F}$ on $I$ is a subset of $P(I)$ which does not contain loops $(i_1, i_2) \ldots (i_n, i_1)$. Any such forest splits as a single union of disjoint trees, and it gives also a decomposition of $I$ into $|I| - |\mathcal{F}|$ clusters (some of them possibly singletons). The non-trivial clusters are connected by the (non-empty) trees of the forest. Let $H$ be a function of variables $x_{ij}$, $ij \in P$. Then the following forest formula due to Brydges is proven in [18]:

$$25$$
\[ H(1, ..., 1) = \sum_{\mathcal{F}} \left( \prod_{l \in \mathcal{F}} \int_0^1 dh_l \right) \left( \prod_{l \in \mathcal{F}} \frac{d}{dx_l} H \right) (h^F_{ij}(h)) , \quad (122) \]

where
\[ h^F_{ij}(h) = \inf \{ h_l, l \in L_F(i, j) \} \quad (123) \]

and \( L_F(i, j) \) is the unique path in the forest \( \mathcal{F} \) connecting \( i \) to \( j \). If no such path exists, by convention \( h^F_{ij}(h) = 0 \).

This interpolation formula will subsequently be applied to our expression for the two-point function, more precisely to the summands in the numerator and denominator of (73) with given \( l/s \)-assignments. As mentioned we proceed in two steps. The first rather trivial one is to connect together the squares in the components \( \Gamma^e_\alpha \) of \( \Gamma^e \). Let \( P_{\Gamma^e} \) be the set of all pairs \((i, j)\) of distinct squares in \( \Gamma^e \). We define \( \varepsilon_{ij} = 0 \), if \( \Delta_i \cap \Delta_j = \emptyset \) or if \( \Delta_i \) and \( \Delta_j \) belong to different components \( \Gamma^e_\alpha(i) \neq \Gamma^e_\alpha(j) \) of \( \Gamma^e \), \( \varepsilon_{ij} = 1 \) otherwise, and \( \eta_{ij} = 1 - \varepsilon_{ij} \). Our first forest formula is simply
\[ 1 = \sum_{\mathcal{F}_1} \prod_{l \in \mathcal{F}_1} \left( \varepsilon_l \int_0^1 dh_l \right) \prod_{l \notin \mathcal{F}_1} \left( \eta_l + \varepsilon_l h^F_{li}(h) \right) . \quad (124) \]

This follows directly from the application the forest formula to \( H(\{x_{ij}\}) = \prod_{ij \in P_{\Gamma^e}} (x_{ij} \varepsilon_{ij} + \eta_{ij}) \) using that here \( H(1, ..., 1) = 1 \).

The only non-zero terms in this formula are those for which the clusters associated to the forest \( \mathcal{F}_1 \) are exactly the set of connected components \( \Gamma^e_\alpha \) of the large field region. Indeed they cannot be larger because of the factor \( \prod_{l \notin \mathcal{F}_1} \varepsilon_l \), nor can they be smaller because of the factor \( \prod_{l \notin \mathcal{F}_1} (\eta_l + \varepsilon_l h^F_{li}(h)) \), which is zero if there are some neighbours belonging to the same component (for which \( \eta_{ij} = 0 \)) belonging to different clusters (for which \( h^F_{ij}(h) = 0 \)). Therefore this formula simply associates connecting trees of “neighbour links” to each such connected component, but in a symmetric way without arbitrary choices. We remark finally that in (124) the interpolated factors \( \prod_{l \notin \mathcal{F}_1} (\eta_l + \varepsilon_l h^F_{li}(h)) \) after giving the necessary constraints on the clusters can be bounded simply by 1.

The second cluster expansion links together the previous clusters by interpolating all the non-local kernels in the theory. It gives a forest formula which is an extension of the first one. We consider all non-local kernels in our theory, that is
\[ \frac{1}{\sqrt{1 + \pi}} , \quad \frac{1}{\sqrt{1 + \pi}^2} , \quad \frac{1}{p^2 + m^2} . \quad (125) \]

Note that due to the support property (11) of \( 1/(1 + f) \) and our choice of treating each \( \Gamma^e_\alpha \) as one connected block of the second expansion, we need not interpolate \( C_\gamma = C_0 + C^\gamma \) as a whole: When all kernels appearing in (125) are interpolated such that they do not connect
any more different clusters of the second expansion, then $C_\gamma$ - with these interpolated kernels replacing the noninterpolated ones in the expression for $C_\gamma$ - does not connect different clusters either. The three kernels from (125) will be generically called $K$. Now the second expansion takes into account the connections built by the first, i.e. it interpolates only the links

$$K_l(x, y) = K_{ij}(x, y) = \Delta_i(x) K(x, y) \Delta_j(y)$$

(126)

for squares which belong to different clusters of the first forest. Let $Z(K, \Gamma^e, \Lambda)$ be a generic name for the quantities we want to compute, namely the numerator and denominator in (73). Then the second forest formula gives:

$$Z(K, \Gamma^e, \Lambda) = \sum_{F_1} \prod_{l \in F_1} (\varepsilon_l \int_0^1 dh_l \prod_{l \notin F_1} (\eta_l + \varepsilon_l h_l^F(h)) \times$$

$$\times \sum_{F_2 \supset F_1} \prod_{l \in F_2 - F_1} (\int_0^1 dh_l \prod_{l \in F_2 - F_1} (d/dx_l)Z(K(h_{F_2 - F_1}), \Lambda),$$

where $Z(K(h_{F_2 - F_1}), \Lambda)$ is a functional integral with interpolated kernels $K(h_{F_2 - F_1})$. These interpolated kernels are defined by $K(h_{F_2 - F_1}) = h_{F_1} F_1, F_2(h) K_l(x, y)$, where $h_{F_1} F_1, F_2(h)$ is the inf of the $h$ parameters of the lines of $F_2 - F_1$ on the unique path in $F_2$ joining $\Delta_i$ to $\Delta_j$ (if $l = (i, j)$). Again if no such path exists, by convention $h_{F_1} F_1, F_2(h) = 0$. In other words the path is computed with the full forest, but only the parameters of the forest $F_2 - F_1$ are taken into account for the interpolated non-local kernels.

The product $\prod_{l \in F_2 - F_1} (d/dx_l)$ is a short notation for an operator which derives with respect to a parameter $x_l$ multiplying $K_l$ where $K$ is any of the non-local kernels, and then takes $x_l$ to 1. Therefore the action of $\prod_{l \in F_2 - F_1} (d/dx_l)$ creates the product $\prod_{l \in F_2 - F_1} K_l$ (with summation over the finite set of possible $K$’s), multiplied either by functional derivatives hooked to both ends (for the case where the derivatives apply to the measure and are evaluated by partial integration) or by other terms descended from action exponential, if the derivatives apply directly to the action. In section IV.4 we give the list of the corresponding derived “vertices” produced by these derivatives. The important fact to be shown is that because these derivatives act on terms which carry a factor $N^{-x}$, $x > 0$, in fact to each such vertex, hence to each link of this second expansion, is associated a factor which tends to zero as $N \to \infty$.

It is an important property of the forest formulas of this type that they preserve positivity properties [18], so that if $K$ is a positive operator, $K(h_{F_2 - F_1})$ is also positive. This is not obvious at first sight form the infimum rule of (123), but it is true because for any ordering of the $h$ parameters (say $h_1 \leq ... \leq h_n$) there is a way (which varies with the ordering) to
rewrite the interpolated $K(h)$ as an explicit sum of positive operators [18]:

$$K(h) = \sum_p (h_p - h_{p-1}) \sum_{q=1}^p \chi_{p,q} K \chi_{p,q}$$

(128)

The functions $\chi_{p,q}$ are the characteristic functions of the clusters built with the part of the forest made of lines $p, p+1, \ldots, n$. For us (as for anyone interpolating Gaussian measures) this preservation of positivity is crucial when the covariance $C_\gamma$ is interpolated.

### 4.2 The Cluster amplitudes. Factorization

From (127) we realize that the quantities $Z(K, \Gamma^e, \Lambda)$ factorize over contributions, the mutually disjoint supports of which - to be called polymers - are the blocks connected together by the links of the disjoint trees in the forest $\mathcal{F}_2$. So they take the form

$$A(K, \Gamma^e, Y) = \sum_{\text{trees } \{\mathcal{T}_{1a}\} =: \mathcal{T}_1} \prod_{l \in \mathcal{T}_1} (\varepsilon_l \int_0^1 dh_l) \prod_{l \notin \mathcal{T}_1} (\eta_l + \varepsilon_l h_{l}^{\mathcal{T}_1}(h)) \times$$

$$\times \sum_{\text{trees } \mathcal{T}_2 \text{ on } Y, \mathcal{T}_2 \supset \mathcal{T}_1} \prod_{l \in \mathcal{T}_2 - \mathcal{T}_1} \int_0^1 dh_l \prod_{l \notin \mathcal{T}_2 - \mathcal{T}_1} \left(\frac{d}{dx_l}\right) A(K(\{h_{\mathcal{T}_2 - \mathcal{T}_1}\}), Y),$$

(129)

The trees $\mathcal{T}_{1a}$ join together the connected subsets of $Y \cap \Gamma^e_a$, their union, called $\mathcal{T}_1$, (which in fact is a forest) becomes a subset of a single tree when adding the links from $\mathcal{T}_2 - \mathcal{T}_1$. The trees $\mathcal{T}_2$ connect together all of the polymer $Y$, so they have $|Y| - 1$ elements. Then (similarly as above (122)) $A(K(\{h_{\mathcal{T}_2 - \mathcal{T}_1}\}), Y)$ is a functional integral with interpolated kernels $K(\{h_{\mathcal{T}_2 - \mathcal{T}_1}\})$. These kernels are defined by $K(\{h_{\mathcal{T}_2 - \mathcal{T}_1}\}) = h_{l}^{\mathcal{T}_1, \mathcal{T}_2}(h) K_l(x, y)$, where $h_{l}^{\mathcal{T}_1, \mathcal{T}_2}(h)$ is the inf of the $h$ parameters of the lines of $\mathcal{T}_2 - \mathcal{T}_1$ on the unique path in $\mathcal{T}_2$ joining $\Delta_i$ to $\Delta_j$ for $l = (i, j)$.

Now regarding more explicitly the two-point function (73) we get the following formula as result of the cluster expansion:

$$S^\Lambda_2(x, y) = \frac{\sum_l \prod_{a} Z_{\gamma_a} \sum_{q,Y_l} A^l(Y_1, x, y)(1/(q - 1)!) \prod_{l=2}^q A^l(Y_l)}{\sum_l \prod_{a} Z_{\gamma_a} \sum_{q,Y_l} (1/q!) \prod_{l=1}^q A^l(Y_l)}$$

(130)

with the following explanations:

1) The amplitudes for the polymers depend on the choice $l$ of the large field region. By shorthand notation $l$ stands for the infinite series of possible choices $s, l^1, l^2, \ldots$. Correspondingly the sum $\sum_l$ stands for the infinite sum over those choices. We note already that
there is no convergence problem associated with this infinite sum due to the suppression factors (120).

2) The difference between the numerator and the denominator in (130) is that in the numerator there is one external polymer depending on the source points $x$ and $y$. Note that there is no nonzero contribution in which the points $x$ and $y$ lie in two distinct polymers. This would necessitate to cut the factor $(\frac{1}{p^2 + m^2 + igτx_λ})(x, y)$ into a product of two pieces of disjoint support, one containing $x$ and the other $y$. Such a contribution obviously vanishes. The absence of such a contribution can be traced back to the symmetry $φ → −φ$ of the action (1).

Since by the rule of our cluster expansion, each component $γ_a$ of the large field region is contained in exactly one polymer $Y$, we may absorb each normalization factor $Z_{γ_a}$ into its cluster, defining

$$\tilde{A}(Y) := A(Y) \prod_{a/γ_a \subset Y} Z_{γ_a}.$$ (131)

The simplest cluster is a single small field square $Δ \subset S = Λ − Γ^e$. Due to (118) we find in this case

$$A_0(Δ) = 1 + o(N^{-1/5}).$$ (132)

Therefore it is convenient to cancel out the background of trivial single square small field clusters, hence to introduce for a polymer $Y$ the normalized amplitude

$$a(Y) = \frac{\tilde{A}(Y)}{\prod_{Δ \subset Y} A_0(Δ)}.$$ (133)

Then we obtain the usual dilute polymer representation:

$$S_2(x, y) = \frac{\sum_l \sum_{q, Y_l} a_l(Y_1, x, y) \frac{1}{(q-1)!} \prod_{i=2}^l a_l(Y_i)}{\sum_l \sum_{q, Y_l} (1/q!) \prod_{i=1}^l a_l(Y_i)}.$$ (134)

To get factorization we must analyze how the choice of $l$ affects the cluster amplitudes. The choice of the large field regions $Λ_l^n$ for fixed $n$ is a local one, which means that the constraints implied by the choice are of finite range. The sum over these choices therefore

\[ \frac{1}{p^2 + m^2 + igτ} = \frac{1}{1 + \frac{1}{p^2 + m^2} igτ} \frac{1}{p^2 + m^2} \]

and interpolate the kernel $1/(p^2 + m^2)$, see also the proof of the Theorem below.
can be absorbed into the value of (redefined) factorized amplitudes. Indeed we can replace the global sums over \(s, l^1, l^2, \ldots\) by local ones:

\[
\sum_l \sum_{q,Y_l} a_l(Y_1, x, y) \frac{1}{(q-1)!} \prod_{i=2}^q a_l(Y_i) = \sum_{q,Y_l} b(Y_1, x, y) \frac{1}{(q-1)!} \prod_{i=2}^q b(Y_i),
\]

with the explanations:
(i) The right sum is over all sets \(\{Y_1, ..., Y_q\}\) where the \(Y_i\) are sets of \(\Delta\)'s, a single \(\Delta\) being excluded, (except if it is an external square containing one of the source points \(x\) and \(y\)). One has the disjointness or hard core constraints \(Y_i \cap Y_j = 0\) for \(i \neq j\).

(ii) \(b(Y)\) is computed from \(a(Y)\) through

\[
b(Y) = \sum' a_l(Y),
\]

where the sum is over all assignments of large field regions \textit{included} in \(Y\). This sum \(\sum'\) is submitted to constraints (as indicated): We define \(\Lambda_l(Y) := \Lambda_l \cap Y = \bigcup_n \Lambda_n(Y)\), \(\Lambda_n(Y) := \Lambda_n \cap Y\) and sum over the \(s, l^n\)-assignments within \(Y\) with the following restriction: For given \(Y\) any assignment for which there exists some \(\Delta \in \Lambda_n(Y)\) with

\[
\text{dist} (\Delta, (\partial Y - \partial \Lambda)) \leq n M
\]

is forbidden, because otherwise our polymer would not contain the whole of the large field block \(\Gamma^e_a\) containing \(\Delta\) and associated with \(\Lambda_l(Y)\). It is also evident that it does contain this block if (138) does not hold for any square from \(\Lambda_n(Y)\). With this definition of the amplitudes \(b(Y)\) we now obtain factorization:

\[
S_2(x, y) = \frac{\sum_{q,Y_l} b(Y_1, x, y) \frac{1}{(q-1)!} \prod_{i=2}^q b(Y_i)}{\sum_{q,Y_l} (1/q!) \prod_{i=1}^q b(Y_i)}
\]

4.3 The Mayer Expansion and the Convergence Criterion

(139) has now the form required for the application of the Mayer expansion in a standard way. The hard core interaction between two clusters or polymers \(X, Y\) is \(V(X, Y) = 0\) if \(X \cap Y = 0\), and \(V(X, Y) = +\infty\) if \(X \cap Y \neq 0\), and the disjointness constraint for the
polymers can be replaced by the inclusion of an interaction $e^{-V(Y_i, Y_j)}$ between each pair of polymers. A configuration $M$ is an ordered sequence of polymers. We define $b^T(M)$ by

$$b^T(M) = T(M)(\frac{1}{q!} \prod_{i=1}^{q} b(Y_i)),$$

where the connectivity factor $T(M)$ is defined using connected Graphs $G$ on $M$, by

$$T(M) := \sum_{G \text{ connected on } M} \prod_{ij \in G} (e^{-V(X_i, X_j)} - 1).$$

Then we can divide by the vacuum functional to obtain

$$S_2(x, y) = \sum_{M \text{ (x,y)-configuration}} b^T(M),$$

where $M$ is a sequence of overlapping polymers $Y_1, ..., Y_q$, the first of which contains the squares containing $x$ and $y$ and thus includes the factor $(\frac{1}{p^2 + m^2 + i\tau \chi \Lambda})(x, y)$ from (73). The sufficient condition for the convergence of (142) in the thermodynamic limit is well known: It is a particular bound on the sum over all clusters, containing a fixed square or point to break translation invariance [14, 18, 19]. We state it as

**Proposition 10:**

$$| \sum_{Y \in Y} b(Y) e^{|Y|} | \leq 1/2$$

for $N$ sufficiently large, uniformly in $\Lambda$, $|Y|$ being the number of squares in $Y$.

The fixed point is chosen to be 0 without restriction. For $N$ large enough, (143) in fact holds if one replaces the number $e$ in (143) by any other constant. To deduce convergence of (142) under condition (143) requires to reorganize the connectivity factor $T(M)$ according to a tree formula. We can use again the basic forest formula (122) to obtain a symmetric sum over all trees. We define

$$v_{ij} = (e^{-V(X_i, X_j)} - 1) \text{ for } i \neq j.$$

We call $P$ the set of pairs $1 \leq i < j \leq n$. Expanding $\prod_{(ij) \in P}(1 + v_{ij})$ with (122) we get another forest formula, on which we can read the connectivity factor

$$T(M) = \sum_{T} \prod_{t \in T} \left(v_{t_{\text{ini}}} \int_{0}^{1} dh_t \right) \prod_{(ij) \notin T} (1 + h_T(i, j)v_{ij}),$$

where $h_T(i, j)$ is the inf of all parameters in the unique path in the tree $T$ joining $i$ to $j$. This formula is then used e.g. like in [14, 18, 19] to derive the convergence of (142). Remark that again every tree coefficient forces the necessary overlaps and is bounded by 1.
It remains to prove Proposition 10. We do not give a first principles proof here, but we do show how to sufficiently control those contributions to the polymer amplitudes, which do not appear in analogous form in e.g. UV-regularized massive $\varphi^4$-theory, since the latter is clearly exposed in many reviews and textbooks, e.g. [14,19,22]. Cluster expansion techniques are nowadays applied to much more complicated situations than this, recently also with accent on a clear and systematic presentation [20,21]. The aspects not to be encountered in a $\varphi^4$-treatment are analyzed in section 4.4. Here we reduce the proof to certain bounds on functional derivatives generated by the links of the second tree $T_2 - T_1$ in (129). Because the amplitude $b(Y)$ is given by a tree formula we will sum over all squares in $Y$ by following the natural ordering of the tree, from the leaves towards the root, i.e. the particular square containing 0. The factorial of the Cayley theorem counting the number of (unordered) trees is compensated in the usual way by the symmetry factor $1/|Y|!$ that one naturally gets when summing over all positions of labeled squares [14,19]. Then the only requirements to complete the proof of (143) are

(i) summable decay of the factor associated to each tree link. This is obvious for the $\varepsilon_{ij}$ links of $T_1$, because these extend only over neighbours, so have bounded range. For the tree links of $T_2 - T_1$, it follows from the decay of the corresponding kernels (125), see Lemmas 1,3.

(ii) A small factor for each tree link, or equivalently for each square of $Y$. This will compensate in particular for the combinatorial factors to choose which term of the action to act on by the derivatives etc. For tree links of $T_1$ this small factor comes from the one associated to each of the large field squares, hence from Lemma 9. Once a square is chosen large field we still have the choices $l^1, \ldots, l^{n_0}$, the value of $n_0$ depending on the distance of the square from the boundary of $Y$. The sum over the $n$-values converges (rapidly) due to (120). For the tree links of $T_2 - T_1$ the small factor comes from the negative powers of $N$ generated at the ends of these links ("vertices"). These small factors are described in more detail in the next section. Remark that all types of small factors tend to zero as $N \to \infty$.

We note that the small factor per square should be there on taking into account the bound on the action as a net effect. (113) was derived before performing the cluster expansion. Does it still hold once the interpolation parameters and support restrictions are introduced? It does indeed, because support restrictions do not cause any harm in the reasoning of Ch.3, because all interpolated kernels are bounded in modulus by the modulus of their noninterpolated versions (see (128)), and because the interpolated versions of the operator $A$ still have real spectrum. Then one easily realizes that all statements go through as before, in particular the proof of Proposition 2 and of Lemma 7. A slightly more serious modification of the action is caused by the use of the Cauchy formula below, it will be controlled by Lemmas 12 and 13.
4.4 The Outcome of the Derivatives

With the tools previously developed we now want to show the existence of the correlation functions in the thermodynamic limit. We have at our disposal exponentially decaying kernels, a suitable stability bound on the action (Proposition 8), and we have arranged things such that derivatives will produce a small factor corresponding to the small coupling. As compared to a treatment of UV-regularized $\varphi^4$. The main new features to be analysed are the following:

a) The action is nonlocal, and the covariance is interpolated twice.
b) There is a small/large field split, and thus small factors per derivative appear in various different forms.
c) The action is nonpolynomial, which implies in particular that terms descended from the action by derivation may be redervied arbitrarily often.

The amplitudes of the polymers $Y$ are given as sums over trees (129) which are the factorized contributions coming from the forest formula (122). When performing the $h$-derivatives those may either apply to $d\mu_{\gamma}(Y)$ or to

$$\left(\frac{1}{p_{\text{reg}}^2 + m^2 + ig\tau\chi_Y}\right)(x, y) \det^{-N/2}_3(1 + iA_s) \det^{-N/2}_2(1 + \frac{1}{1 + iA_s}iA'') e^{1/2}(\tau, \deltaC) .$$

(146)

Here we went back to (73) (remembering that the term $e^{-\int_{\mathbb{R}^2}A^2/\Lambda^2}$ is now absent, cf. Proposition 8). In (146) the kernels from (125), which appear in $C_\gamma$ and the action, are to be replaced by their $h$-dependent versions. We write shortly $K(h)$ for $K(\{h_{T_2-T_1}\})$ and have (see (128),(129)...) $K(h)(x, y) = \chi_Y(x) h_{T_1-T_2}(h) K(x, y) \chi_Y(y) .

(147)

Application of derivatives with respect to $d\mu_{\gamma}$ is evaluated by partial integration ([13], Chap. 9):

$$\partial_{h_i} \int d\mu_{\gamma}(h, \tau) \ldots = \int d\mu_{\gamma}(h, \tau) \int_{x,y} \frac{\delta}{\delta\tau(x)} (\partial_{h_i} C_{\gamma}(h))(x-y) \frac{\delta}{\delta\tau(y)} \ldots$$

(148)

In $C_\gamma$ the kernels $S = \frac{1}{\sqrt{1+\pi}}$ are interpolated. Thus $\partial_{h_i} C_{\gamma}(h)$ is of the form

$$\partial_{h_i} C_{\gamma}(h) = (\partial_{h_i} S(h)) \hat{C}_\gamma S(h) + S(h) \hat{C}_\gamma \partial_{h_i} S(h) .$$

(149)

The supports of the derived kernels, i.e. $\partial_{h_i} S(h)$, are by construction restricted to the two squares linked by the $h_i$ derivation [18], which adds a link to the previous tree. Therefore the $\tau$ functional derivatives are either directly localized in these squares - in the case where $\partial_{h_i}$ applies to the first (second) kernel $S(h)$ in $C_\gamma$, and we consider the $\frac{\delta}{\delta\tau}$ derivative on the
left (right), or they are only essentially localized - when e.g. \( \partial_h \) applies to the first (second) kernel \( S(h) \) in \( C_\gamma \), and we consider the \( \frac{\delta}{\delta \tau} \) derivative on the right (left). In the last case this means that the \( \frac{\delta}{\delta \tau} \) functional derivative is linked to its localization square via the second (underived) kernel \( S(h) \), which is supported over the polymer in question, see (128),(147). It has exponential decay, so the links to squares distant from the localization square rapidly decrease with distance. Summing over them gives an additional factor \( \sim 1/m^2 \). Since this tolerable deterioration of the bound per derivative is the only effect of essential localization, we may forget about this difference from now on.

The \( (T_2 - T_1) \)-\( h \)-derivatives can apply also to the terms in (107). To roughly keep track of the combinatorial factors involved we note that any \( h \)-derivative may apply to any kernel in (146) \( (\sim 10 \text{ terms}) \). If it applies to the measure there appear two terms with two functional derivatives which again may apply to the action \( (\sim 40 \text{ terms}) \). Still one should note that the effect of these combinatorics is not very important since going through the terms in detail (which we shall not do too explicitly) reveals that most of them give much smaller \((\text{in } N)\) contributions than the dominating ones. This is also true for the sum over the \( l \)-assignments: Large field contributions, in particular for \( n > 1 \), are tiny corrections due to (120). Therefore e.g. all the contributions coming from the terms in \( \delta C_\gamma \) are unimportant: They are only present when \( \Lambda_l \subset \gamma \) is not empty. There is one more source of combinatoric increase of the number of terms, namely due to the fact that the derivatives may also act on terms produced by previous derivatives. For the polynomial part of the action this may only happen a few times. But it needs special discussion when regarding the determinants. So we will now go through the various contributions and comment how the derivatives act on them. We can be short about

\( \delta C_\gamma \): In all terms we have the kernels \( \sqrt{1 + \pi} \), which fall off as \( \exp(-2m|x - y|) \). The contributions are listed in (70). When applying an \( h \)-derivative to \( \delta C_1(\gamma) \) the small factor in \( N \) comes from \( \text{dist}(\gamma, \Lambda - \Gamma) \geq \ln N/m \). Due to the fall-off this gives a factor \( \sim N^{-2} \). We may then e.g. write in the bound for the kernel

\[
\exp(-2m|x - y|) = \exp(-\frac{5m}{4}|x - y|) \exp(-\frac{3m}{4}|x - y|)
\]

(150)

and keep the first factor as a kernel with exponential fall-off and bound the second by \( N^{-3/4} \) using the support restrictions. This is then the small factor per derivative. Note that we could also do without extracting this factor from (150), extracting it as a part of (120) instead. The same splitting (150) can be applied when the \( h \)-derivatives act on \( \delta C_2(\gamma) \). For \( \delta C_4(\gamma) \) we may invoke support restrictions to extract \( N^{-3/4} \) as above, additionally we get a factor of \( \varepsilon \sim N^{-2/5} \). The term \( \delta C_3(\gamma) \) does no more contribute due to the limit \( R \to \infty \). The same mechanism produces the small factors also, when we apply the functional derivatives
\( \delta/\delta \tau \) instead of \( h \)-derivatives. Remember the above remarks concerning essential localization. By the derivatives we also produce \( \tau \)-fields (essentially) localized in some square \( \Delta \) (two fields per \( h \)-derivative, one per \( \delta/\delta \tau \)-derivative). If the square \( \Delta \) is in \( \Lambda_s \), we have the choice to perform Gaussian integration or to bound the contribution directly using (46)

\[
| \ldots \int_{\Delta} K_1(z - x) \tau(x) K_2(x - y) \ldots | \leq O(1) N^{1/12} \quad \sup_{x \in \Delta} |K_1(z - x) K_2(x - y)| \quad \ldots |
\]

This is maybe the simplest way of doing. Note that in this case we still can keep aside a factor of \( N^{-3/4 + 2/12} < N^{-1/2} \) per \( h \)-derivative. If the square is in \( \Lambda_{l_n} \), the bound is achieved using (44),(45) and (120). The above-mentioned rederivation of derived terms allows to apply (at most) two \( \delta/\delta \tau \) on an \( h \)-derived term so that the factor has to be distributed over three derivatives leaving in this worst case \( N^{-1/6} \) per derivative (without invoking large field suppression).

Maybe we should also mention shortly the wellknown and well-solved local factorial problem. There is the possibility that a large number of \( \tau \)-fields accumulate in a single square \( \Delta \), even when regarding only the polynomial part of the action, namely if the tree in question has a large coordination number \( d \) at that square: There are \( d \) links of the type \( l_{i,j_\nu} \), \( \nu = 1, \ldots d \) in the tree, \( i \) referring to \( \Delta \). Then bounding the at most \( 2d \) \( \tau \)-fields in \( \Delta \subset \Lambda_l \) using (120) (and the Schwarz inequality) gives

\[
\left[ \int_{\Delta} \tau^2 \right]^d e^{-1/4 \int_{\Delta} \tau^2} \leq 4^d d!
\]

This is not tolerable as a bound when aiming to prove (143), but the solution is in the fact that most of the \( d \) squares associated to the links \( l_{i,j_\nu} \) have to be at a large distance from \( \Delta \) for large \( d \). Extracting a small fraction \( \eta \) of the kernel decay we can isolate a factor associated to \( d >> 1 \), which is much smaller than \( \frac{1}{d!} \). For a more thorough discussion of the point see [14,19] or also [1].

Now we regard the Fredholm determinants. As compared to [1] we have to regard an inverted determinant. This is related to the fact that we regard a bosonic model, and it means that the sign cancellations appearing as a consequence of the Pauli principle which sometimes improve the convergence properties are absent. The inverted determinants are raised to the power \( N/2 \). For shortness we will change the notation for the rest of this section and suppress this power assuming instead the operators \( A_s, \ldots \) to act in \( \bigoplus_{k=1}^{N/2} L^2(\Lambda) \). We assume \( N \) to be even, otherwise we still would have to carry around a power \( 1/2 \) (without consequence). This change entails that we absorb a factor of \( N/2 \) in \( \text{Tr} \) as well.

\[ \text{It is of order } e^{-\delta d^{3/2}}. \]
We rewrite the product of the two Fredholm determinants appearing in terms of a single one. This is possible, since the interpolation acts equally on all $A$-operators. We have
\[
\det_3^{-1}(1 + iA_s) \det_2^{-1}(1 + \frac{1}{1 + iA_s} iA'') = \det^{-1}(1 + iA) e^{\operatorname{Tr}(iA_s - 1/2(iA_s)^2 + \frac{1}{1 + iA_s} iA'')} .
\] (153)

Since the $\operatorname{Tr}$ of $A''$ multiplied by any power of $A_s$ vanishes, whereas $\operatorname{Tr}(A_s + A'') = \operatorname{Tr}A$, we may rewrite (153) as
\[
\det_2^{-1}(1 + iA) \exp \operatorname{Tr}\{-1/2(iA_s)^2\} .
\] (154)

The cluster derivatives acting on (154) will be evaluated as Cauchy integrals over suitable (large) contours. Similar reasoning has been used by Iagolnitzer and Magnen [23] in a renormalization group analysis of the Edwards model and earlier by Spencer in the analysis of the decay of Bethe-Salpeter kernels [24]. To obtain useful bounds using this method requires that the derivatives $\partial_{h_l}A$ are always small in norm. At this stage we therefore really need the whole cascade of large field splittings from the previous chapter. We have

**Lemma 11:** Let $l \in \mathcal{T}_2 - \mathcal{T}_1$ be a link of the cluster expansion joining two squares $\Delta, \Delta'$ such that $\partial_{h_l}A = P_{\Delta'} A P_{\Delta}$. Then we have
\[
||\partial_{h_l}A|| \leq O(1) N^{-5/12} \exp\{-m d_l\} ,
\] (155)

if $\Delta$ is a small field square. Here we set $d_l = \operatorname{dist}(\Delta, \Delta')$. If $\Delta$ is a large field square in $\Lambda_{\ln}$, we find
\[
||\partial_{h_l}A|| \leq O(1) N^{-1/2}(\int_{\Delta} \tau^2) \exp\{-m d_l\} \leq O(1) N^{5/12} N^{-1/2} - \frac{1}{2} - 2n .
\] (156)

**Proof:** The result is obtained in the same way as when proving Proposition 2, if $\Delta$ is a small field square. If $\Delta$ is in $\Lambda_{\ln}$, the distance between the squares is by our expansion rules larger than $\frac{2n \ln N}{m}$ which assures (156) through the decay of $\frac{1}{p^2 + m^2}$ (remember in particular (44),(45),(62)).

For shortness of notation we introduce
\[
\det^{-1}(1 + Q) := \det^{-1}(1 + iA)
\] (157)
and first describe how the derivatives act on (157) instead of (154). Namely we write
\[
\partial_{h_1} \ldots \partial_{h_n} \det^{-1}(1 + Q) = \left[\partial_{\alpha_1} \ldots \partial_{\alpha_n} \det^{-1}(1 + Q + \alpha_1 \partial_{h_1} Q + \ldots + \alpha_n \partial_{h_n} Q)\right]_{\alpha_1, \ldots, \alpha_n = 0}
\] (158)
We evaluate (158) by means of a Cauchy formula for the \( n \) independent complex variables \( \alpha_i \). The idea is to regain the small factor per derivative and the distance decay by choosing the \( \alpha \)-parameters sufficiently large. We note first that \( \det^{-1}(1 + Q + \alpha_1 \partial_{h_1} Q + \ldots + \alpha_n \partial_{h_n} Q) \) is analytic in the \( \alpha \)-parameters, see Simon [11], as long as \( 1 + Q + \alpha_1 \partial_{h_1} Q + \ldots + \alpha_n \partial_{h_n} Q \) has no 0 eigenvalues. This restricts the maximal size of the \( |\alpha_i| \). We choose the size of the \( \alpha_l \)-parameter corresponding to the link \( l \) as follows:

\[
R_l := |\alpha_l| = N^{\frac{2}{5}} e^{\frac{m_1}{10} d_l} .
\] (159)

We now find

**Lemma 12:** If the \( \alpha_l \) are chosen according to (159) then

\[
||\sum_l \alpha_l \partial_{h_l} A|| \leq O(1) N^{-1/4}
\] (160)

**Proof:** For the individual entries in the sum the bound follows on inspection. If the supports of the links (i.e. the pairs \( \Delta, \Delta' \)) are mutually disjoint it stays true, since then the \( \partial_{h_l} A \) are mutually orthogonal. If they are not, we again employ the argument (see above (152)) that in this case the links corresponding to a large coordination number \( d \) in the tree have to grow longer and longer. In this case the sum may be performed using the remnant decay \( e^{-\frac{m_1}{10} d_l} \).

**Remark:** When proving the exponential decay of the two-point function in the end of the paper we would like have exponential decay with mass \( m \) up to corrections small with \( N \) (without invoking the analyticity improvement due to the UV cutoff). It may then be necessary to use the **full** decay for **at most two** links \(^{13}\) among those appearing at a branch point of the respective tree (see below, proof of Theorem). Obviously this does not change the norm bound (160) at all, since we may bound the sum in the same way keeping aside a fraction of the decay for \( d - 2 \) links only.

So we now evaluate (158) through

\[
|\partial_{h_1} \ldots \partial_{h_n} \det^{-1}(1 + Q)| = \left| \frac{1}{2\pi i} \int_{R_1 \ldots R_n} \frac{1}{\alpha_1^2 \ldots \alpha_n^2} \det^{-1}(1 + Q + \sum_l \alpha_l \partial_{h_l} Q) \right| \leq \left( \frac{1}{2\pi} \right)^n \frac{1}{R_1, \ldots, R_n} \sup_{\alpha} |\det^{-1}(1 + Q + \sum_l \alpha_l \partial_{h_l} Q)| ,
\] (161)

\(^{13}\) if these links are indispensable to join via the tree the squares containing the points \( x \) and \( y \) in the external polymer \( A(Y, x, y) \).
where the sup is to be taken over the \( \alpha_l \)-parameters on the circles \( R_l \). Thus we obtain indeed per derivative a factor
\[
\frac{1}{2\pi} N^{-1/6} e^{-\frac{9m}{10} dt}.
\]
(162)

Before ending the discussion of how to evaluate derivatives acting on \( \det \) we mention how we treat the \( \delta/\delta \tau_m \)-derivatives. In this case we choose (in modification of (159))
\[
R_m(\tau) = N^{1/4}.
\]
(163)

We thus collect a smaller factor in \( N \) from the \( \tau \)-derivative because \( \delta/\delta \tau \) annihilates a possibly large \( \tau \)-factor, on the other hand we do not get a distance decay factor and need not do so, because it is already present in the term \( \partial h_i S(h) \) which accompanies \( \delta/\delta \tau \) (see (149)).

Of course it remains to give suitable bounds on the Fredholm determinants modified by the \( \alpha \)-parameters. We have to remember that our true object of interest is not \( \det^{-1}(1 + Q) \) but rather the subtracted determinant (154). First we note that we may still evaluate the derivatives acting on (154) by introducing \( \alpha \)-parameters, on replacing as before for a given choice of of \( h \)- and \( \tau \)-derivatives
\[
A_\alpha := A \to A + \sum_l \alpha_l \partial h_i A + \sum_m \alpha_m \delta \tau_m A \text{ and similarly for } A_s, A''.
\]
(164)

So after bounding the Cauchy integrals we have to bound
\[
\sup_\alpha |\det^{-1}(1 + i A_\alpha) \exp\left\{\frac{1}{2} Tr(A_{\alpha s})^2\right\}|.
\]
(165)

The task is to reproduce the bounds on the action from Ch.3 on replacing \( A \to A_\alpha \). Inspection shows that the proofs of Propositions 2, Lemmas 6,7 and part of Proposition 8 (as far as (117) is concerned) have to be redone with this modification on \( A \). We collect our findings in

**Lemma 13:** We assume that the kernels \( A \) are restricted to a given polymer \( Y \subset \Lambda \) of the cluster expansion. Then we have
a) \( |\|A_{\alpha s}\| | \leq O(1)N^{-1/4} \) (replacing (48))
b) \( \det^{-1}(1 + \frac{1}{1+iA_\alpha s} iA''_\alpha) \leq \exp\{O(1)N^{-1/4} f_Y(\tau^2 + 1)\} \) (replacing (109))
c) \( |Tr(A_{\alpha s}^3)| \leq N^{-1/4} f_{Y \cap \Lambda_s} \tau^2 \) (replacing (117))

**Remark:** Note again that due to our change of notation a factor of \( N/2 \) has been absorbed in \( Tr \) together with a corresponding change in \( \det \).

**Proof:** The proof of a) is trivial from Proposition 2 and Lemma 12. As for b) we have to go again through the considerations leading from (95) to (109). Since the reasoning is
analogous, we will be rather short. Introducing the quantities $B_\alpha$, $D_\alpha$ as we did for $A_\alpha$ we find that (95) to (97) stay true also for complex $\alpha$. The essential modification occurs in (101),(102): since the $\alpha_i$ are complex we find

$$i(\varphi, A''_\alpha \varphi) - i(\varphi, A''_\alpha \varphi) = -2 \sum_{\alpha_i} \text{Im} \alpha_i (\varphi, \partial_{h_i} A''_\alpha \varphi) - 2 \sum_{\alpha_m} \text{Im} \alpha_m (\varphi, \delta_{\tau_m} A''_\alpha \varphi)$$

(166)

instead of 0 for $\alpha \equiv 0$. Correspondingly we have to modify (102). The norm bound (160) then still implies

$$D_\alpha \leq O(1) N^{-1/4}.$$  

(167)

which is weaker than (105) but sufficient for us. In evaluating $Tr D^2_{\alpha-}$ we take into account the additional contribution too. Since now

$$0 \leq D_{\alpha-} \leq (2 + \eta) |\delta A''_\alpha| + \sum_l 2 R_l |\partial_{h_l} A''_\alpha| + \sum_m 2 R_m (\tau) |\delta_{\tau_m} A''_\alpha|, \quad \eta << 1$$

(168)

it is straightforward to realize that we may bound $Tr D^2_{\alpha-}$ by

$$Tr D^2_{\alpha-} \leq O(1) N^{-1/4}(\int_{\Lambda \cap Y} \tau^2 + 1).$$

(169)

The first contribution is obtained similarly as in (108), it is quadratic in $\tau$, but we can keep aside a small factor. The additional contribution is proportional to the number of squares touched by $\delta_\tau$-derivatives ($\leq |Y|$), thus it is independent of the size of $\tau$. This ends the proof of b).

c) The proof is as in (116), (117).

From Lemma 13 we now find that (113) (on restriction to $Y \subset \Lambda$ and on using interpolated kernels) is to be replaced by

$$Z^Y_{\gamma} G^Y_{\gamma,\alpha} (\tau) \leq e^{-\frac{1}{20} \int_{\Lambda \cap Y} \tau^2} e^{O(1) N^{-1/4} \int_{\Lambda \cap Y} (\tau^2 + 1)}.$$  

(170)

So the large field suppression stays unaltered and in the bound on the polymer amplitudes there is at most a factor of $\sim 1 + O(N^{-1/12})$ per small field square from the action to beat (we could tolerate $O(1)$).

Here we may end our discussion on the outcome of the derivatives. We have shown that we have a small factor $\sim N^{-1/6}$ per derivative and factor of $e^{-N^{1/8}}$ per large field square. All links are through kernels decaying exponentially with mass $> m$. This is sufficient to beat the factors $O(1)$ per square from the combinatoric choices and from the action. We pointed out that this is sufficient for the proof of Proposition 10.

\footnote{where we mentioned already that just taking the maximal value gives a crude bound since most terms are much smaller than the leading ones}
4.5 Exponential Decay of the Correlation Functions

Now we have proven the existence of $S_2(x,y)$ in the TD limit we want to proceed to the announced result on its exponential decay.

**Theorem:** For $N \gg 1$ sufficiently large the infinite volume two-point function decays exponentially

$$|S_2(x,y)| \leq O(1) e^{-m'|x-y|} \quad (171)$$

with

$$m' = m(1 + o(N^{1/10})) \quad (172)$$

**Remarks:** $O(1)$ is an $N$-independent positive number. The estimate on the exponent of $N$ in (172) is of course not optimal. The proof goes through without much change also for any $2n$-point function. Using the effects of the UV-cutoff we could replace $m'$ by $m$.

**Proof:** The reasoning is very similar to that of [1] though somewhat simpler. The point is now to realize that the convergence proof still works when we put aside the decay factor appearing in (171). We may assume $x$ and $y$ far apart. They both have to be contained in the same polymer $A(Y,x,y)$, and we have to extract the decay factor when calculating its amplitude. More specifically we shall extract it from the sum over trees $T_2$ in (129), where we first only deal with those trees $T$ for which $T_1$ is empty, namely we first assume that $Y$ does not contain large field squares, which is the dominant contribution. Obviously the decay is associated with the factor

$$\left[\frac{1}{p^2 + m^2 + ig\tau} (x,y) = \left[\frac{1}{1 + \frac{1}{p^2 + m^2} ig\tau} \frac{1}{p^2 + m^2}\right] (x,y) \quad (173)$$

which appears in the external polymer. The kernel $\frac{1}{p^2 + m^2}$ is interpolated and thus in particular restricted in support to $Y$. Let $\Delta_1$ and $\Delta_2$ be the squares in $Y$ containing $x$ and $y$. For given tree $T$ there is a unique path in $T$ connecting $\Delta_1$ and $\Delta_2$. We call it $T'$, noting that $T'$ is a tree with coordination numbers $d_i = 2$, apart from the ends, where they equal 1. Its complement in $T$ will be called $T''$. It has several connected components in general. Each of these connected components may be viewed as being rooted at some square (attached to links) from $T'$. Keeping these squares fixed for the moment and summing over the positions of the other squares in the various connected components of $T''$ then provides us for these connected components with the usual polymer bound (Proposition 10) sufficient for convergence. It remains to sum over the positions of the squares in $T'$ apart from $\Delta_1$ and $\Delta_2$, which are sitting on the ends. For given positions of those squares we may isolate a factor of

$$\varepsilon^{|T'|} \prod_{\nu \in T'} K''(x\nu, y\nu) \quad (174)$$
Here $\varepsilon \sim o(N^{-1/10})$ is part of the small factor per small field derivative, the other being used to beat the combinatoric constants etc., see above. The kernels $K_{l'}(x_{l'}, y_{l'})$ are those generated by the derivatives of the expansion. They all fall off exponentially with at least the rate of $\frac{1}{p^2 + m^2}$, so they all may be bounded by the modulus of $\left[\frac{1}{p^2 + m^2}\right](x_{l'}, y_{l'})$ up to a constant $\sim O(1)$, which we absorb in $\varepsilon$. The coordinates $(x_{l'}, y_{l'})$ are situated in the two squares linked by $l' \in T'$ and are to be integrated over those squares. In Lemma 1 we showed that the kernel of $\frac{1}{p^2 + m^2}$ is pointwise positive. From this we then obtain easily that (174), when integrated over the intermediate squares and summed over their positions is bounded by

$$\varepsilon^{|T'|} \left[\left(\frac{1}{p^2 + m^2}\right)^{|T'|}\right](x, y)$$

(175)

(up to a constant $\sim O(1)$, which we absorb in $\varepsilon$.) Note that having split up the tree $T$ does not change the way in which the sum over the trees is performed. We succeeded in extracting the factor (175) due to the fact that two squares in the external polymer are fixed instead of only one as in Proposition 10. When summing over all possible values of $|Y|$ and using the polymer bound (143) we now obtain a bound of the form

$$|S_2(x, y)| \leq O(1) \left[\frac{1}{p^2 + m^2} \right](1 + \sum_{|T'|} \varepsilon^{|T'|} \left(\frac{1}{p^2 + m^2}\right)^{|T'|})(x, y) .$$

(176)

Here the first term is the contribution for $|Y| = 2$ and where the single $h$-derivative applies to the second factor in (173). This is the only case where it does not produce a factor $\leq \varepsilon$. Performing the geometric series in (176) now proves (171) on using

$$\left(\frac{1}{p^2 + m^2 - \varepsilon}\right)(x, y) \leq O(1) \exp\{-m - \varepsilon/m\}|x - y|$$

(177)

Finally we have to make sure that large field contributions do not spoil our estimate. For this it suffices to note that in the large field region we have at our disposal a factor of $\leq \exp(-N^{1/8})$ per square of $\Gamma^e$, half of which may be put aside per each square of $\Gamma_t^e$, on which ends some $l' \in T'$. Then we only have to note that this factor is much smaller than the factor of $\varepsilon$ which we loose instead, and that the links within $\Gamma_t^e$ are of short range.

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15 apart from $x_{l'} = x$ and $y_{|T'|} = y$ which are fixed
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