Combinatorial analysis of the period mapping: the topology of 2D fibres

A. B. Bogatyrev

Abstract. We study the period mapping from the moduli space of real hyperelliptic curves to a Euclidean space. The mapping arises in the analysis of Chebyshev’s construction used in the constrained optimization of the uniform norm of polynomials and rational functions. The decomposition of the moduli space into polyhedra labelled by planar graphs allows us to investigate the global topology of low-dimensional fibres of the period mapping.

Bibliography: 23 titles.

Keywords: moduli space, real algebraic curve, Abelian integral, period mapping, foliations of a quadratic differential.

§ 1. Introduction

Period mappings are defined on the moduli spaces of curves with marked points on them and return the values of periods of differentials with singularities at the marked points which are uniquely associated to such curves. They appear in different settings (see [1], [18], [15] and [19]), mostly for the study of the geometry of the moduli spaces of curves. For instance in [18], the authors construct subvarieties of the moduli space as closed leaves of the foliation locally defined by the period mapping. The interrelation of the ‘integral’ leaves of certain foliations with the spectral curves of elliptic Calogero-Moser systems was shown in [16]. In [19] the authors used the period mapping to study the asymptotics of orthogonal polynomials. This particular research was motivated by uniform rational approximation problems.

A universal phenomenon known as the ‘Chebyshev Alternation Principle’ in the mathematical community or ‘Equiripple Property’ in the community of electrical engineers says that solutions to certain uniform norm optimization problems for polynomials look like waves of constant amplitude. The best-known function with this property is the (co)sine, so it is perhaps no wonder that the degree $n$ solutions $P_n(x)$ may be effectively evaluated by finding parameters in the following Chebyshev representation (Ansatz):

$$P_n(x) = \cos \left( n i \int_{(e,0)}^{(x,w)} d\eta \right)$$

This research was supported by the Russian Science Foundation under grant no. 16-11-10349.

AMS 2010 Mathematics Subject Classification. Primary 30F30; Secondary 32G15, 05E45.

© 2019 Russian Academy of Sciences (DoM) and London Mathematical Society
where $d\eta_M$ is a suitably normalized third-kind abelian differential on a hyperelliptic curve $M$. For the above formula to give a single-valued function on the Riemann sphere, all the periods of the abelian integral should be multiples of periods of the cosine, in other words lie in the lattice $2\pi n^{-1}\mathbb{Z}$. A deformation of the initial optimization problem brings us to a deformation of the curve $M$ and the induced deformation of the differential $d\eta_M$ such that all its periods are conserved since one cannot continuously jump from one point of the lattice to another. The above representation of polynomials in terms of Riemann surfaces also gives the solution to the Pell-Abel equation, which in its turn is related to the Poncelet problem, elliptic billiards [13] and the boundary-value problem for the string equation [10].

The content of this paper is briefly as follows. In §2 we introduce the moduli spaces of real hyperelliptic curves, a model of their universal covering—the labyrinth space—and define the period mapping from the labyrinth to a Euclidean space. The main object of our research is the global topology of the fibres of the latter map. In §3 we manufacture the main tool of our investigation, a description of curves in terms of weighted flat graphs composed of segments of two orthogonal foliations intrinsically associated to the curve. In §4 we study the decomposition of the moduli spaces into polyhedra spanned by the weights of the admissible graphs. Section 5 presents the period mapping as a linear map in terms of the local coordinates of each polyhedron. In §6 we construct a PL-model for fibres of the period mapping by attaching polyhedra obtained as sections of polyhedra from §4 by linear spaces, one to another. In particular, we conclude that the topology of 2D fibres is trivial.

Acknowledgements. The paper was started when the author held a visiting position at the University Bordeaux-I (LABRI), which provided excellent conditions for work and interdisciplinary communication. My special thanks are to Professor Alexandre Zvonkine, who reignited my interest in this topic after it was nearly extinguished and who has gently pushed me toward writing this text.

§ 2. Setting the problem

2.1. Moduli spaces. Let $\mathcal{H}$ be the moduli space of smooth real hyperelliptic curves $M$ with one marked point ‘$\infty$’ (in what follows we drop the quotes) on an oriented real oval. We assume that this marked point is not fixed by the hyperelliptic involution $J$ acting on each curve. Fixing the genus $g = 0, 1, \ldots$ of a curve and the number $k = 0, 1, \ldots, g + 1$ of its real ovals, we arrive at a decomposition of the moduli space into connected components with given values of the two topological invariants:

$$\mathcal{H} = \bigsqcup_{g,k} \mathcal{H}^{g,k}.$$  

Any element of $\mathcal{H}$ admits an affine model:

$$M = M(E) := \left\{ (x, w) \in \mathbb{C}^2 : w^2 = \prod_{s=1}^{2g+2} (x - e_s) \right\}, \quad e_s \in E, \quad (2.1)$$
with branching set $E = \overline{E}$ which is symmetric with respect to the real axis. We define the hyperelliptic and anticonformal involutions of the curve by

$$J(x, w) := (x, -w) \quad \text{and} \quad \overline{J}(x, w) := (\overline{x}, \overline{w}),$$

respectively. The marked point on the real oval is the point corresponding to $(x, w) = (+\infty, +\infty)$ in the natural two-point compactification of the curve (2.1).

An element of a component $\mathcal{H}_g^k$ of the total moduli space is represented by a set $E^+$ of $g-k+1$ distinct points in the open upper half-plane $\mathbb{H}$ and 2$k$ distinct points on the real axis $\mathbb{R}$ modulo translations and dilations of this set. In this paper we only consider the case $k > 0$, and the set $E^+$ may be normalized so that its two extreme real points are $\pm 1$. In this notation the branching set $E$ of the curve (2.1) is the union of the set $E^+$ and its complex conjugate.

**Lemma 1** (see [8] and [9], Ch. 3). The space $\mathcal{H}_g^k$ is a smooth real manifold of dimension $2g$, and its fundamental group is the group $\text{Br}_{g-k+1}$ of braids on $g-k+1$ strands.

### 2.2. The local period mapping.

On each curve $M$ from the moduli space $\mathcal{H}$ there is a unique third-kind abelian differential $d\eta_M$ with just two simple poles: the marked point $\infty$ and its involution $J \infty$, with residues $-1$ and $+1$, respectively, and purely imaginary periods (see [9], §2.1.3). For the algebraic model (2.1) of the curve $M$ this differential takes the form:

$$d\eta_M = (x^g + \cdots)w^{-1}dx \quad (2.2)$$

with dots standing for a polynomial of degree not greater than $g - 1$. One can check that the normalization conditions on the differential imply that the latter is real, that is, $\overline{J}d\eta_M = \overline{d\eta_M}$. In other words, the polynomial in (2.2) has real coefficients. An important consequence of this fact is this: half of the periods of this differential vanish.

Indeed, the anticonformal involution $\overline{J}$ acts naturally on the space of cycles $H_1(M, \mathbb{R})$ and splits it into a sum of two eigenspaces $H_1(M, \mathbb{R})^\pm$ corresponding to the eigenvalues $\pm 1$ of the operator. We call cycles that are invariant under the reflection $\overline{J}$ even and those that change sign odd. The periods of any real differential $d\eta$ over even and odd cycles are, respectively, real and purely imaginary:

$$\int_C d\eta = \pm \int_{\overline{J}C} d\eta = \pm \int_C \overline{J}d\eta = \pm \int_C \overline{d\eta} = \pm \int_{\overline{C}} d\eta.$$

In particular, we see that the integrals of the distinguished differential $d\eta_M$ over even cycles vanish as they should be real and imaginary simultaneously.

One can locally define the period mapping in a neighbourhood of a distinguished point $M_0$ of the moduli space by

$$\langle \Pi(M)|C \rangle := i\int_C d\eta_M, \quad C \in H_1^-(M_0, \mathbb{Z}); \quad (2.3)$$

here integration of the differential living on a curve $M$ over the cycles on the distinguished curve $M_0$ is possible due to the Gauss-Manin connection in the homological bundle over the moduli space (see [23] and [9], Ch. 5). Globally this mapping is not well defined due to the nontrivial holonomy of the connection: braids entangle the cycles on the curve. We study this effect later.
2.3. Universal covering of the moduli space. The usual way to define a multivalued mapping correctly is to lift it to the universal covering of the source space. The space \( \mathcal{H}_g^k \), the universal covering of \( \mathcal{H}_g^k \), has several models described in [7] and [9], Ch. 3. For instance, it may be represented as the Teichmüller space of a \((g - k + 1)\)-punctured disc with marked points on the boundary. In particular, the universal covering is homeomorphic to Euclidean space \( \mathbb{R}^{2g} \). Here we use another model of this space, the space of labyrinths.

A labyrinth \( \Lambda = (\Lambda_0, \ldots, \Lambda_g) \) attached to a point \( E^+ \) in the moduli space (see Figure 1) is a set of \( g + 1 \) disjoint arcs in the closed upper half-plane \( \overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{R} \cup \infty \). The first \( k \) of them are real intervals \( \Lambda_0, \ldots, \Lambda_{k-1} \) connecting real points of \( E^+ \) pairwise. The remaining \( g - k + 1 \) arcs \( \Lambda_k, \Lambda_{k+1}, \ldots, \Lambda_g \) have more freedom: they start at complex points of the branching set \( E^+ \) and meet the real axis to the right of the largest real point of it. All the arcs of a labyrinth are ordered by their intersections with the real axis. Two labyrinths \( \Lambda \) obtained one from the other by an isotopy of the upper half-plane which is fixed at the points of \( E^+ \) are declared to be equal.

![Figure 1](image)

Figure 1. A point \( E^+ \in \mathcal{H}_4^2 \) (a) is lifted to the universal covering by choosing the labyrinth that escorts it (b). A loop in the moduli space \( \mathcal{H}_3^1 \) is a braid on three strings (c).

**Definition 1.** The space of labyrinths \( \mathcal{L}_g^k \) is the set of equivalence classes of labyrinths \( \Lambda \) attached to the points \( E^+ \) running through the component \( \mathcal{H}_g^k \) of the moduli space. The total labyrinth space is the disjoint union of the connected components:

\[
\mathcal{L} := \bigsqcup_{g,k} \mathcal{L}_g^k;
\]

this is a useful notion if we do not want to specify the values of the topological invariants \( g \) and \( k \) of a real curve.

Speaking informally, the universal covering of a space is the history of motion from some marked point told for each point in this space. The variable part \( \Lambda_k, \ldots, \Lambda_g \) of a labyrinth has the meaning of the traces of the motion of branch points in the upper half-plane.

The braid group \( \text{Br}_{g-k+1} \) realized as the mapping class group [4] of the punctured half-plane \( \mathbb{H} \setminus E \) naturally acts on the space of labyrinths \( \mathcal{L}_g^k \) and corresponds
to covering transformations. Projection from the universal covering onto the (component of the) moduli space consists in wiping out the labyrinth, leaving the branch points only.

2.4. **A distinguished basis in the odd homology.** Any labyrinth $\Lambda$ gives us a distinguished basis in the lattice of odd cycles on the surface $M$. Take the closed upper half-plane $\mathbb{H}$ with removed labyrinth $\Lambda$. This set is simply connected and avoids the branch points of the two-sheeted covering $M \rightarrow M/J = \mathbb{C}P^1$. Therefore, this set can be uniquely lifted to the surface, the infinity point on the cut plane being mapped to the marked point $\infty$ of the surface. The union of $\mathbb{H} \setminus \Lambda$ embedded in the surface $M$ and its reflection $\overline{\mathbb{H}}(\mathbb{H} \setminus \Lambda)$ is a subsurface of $M$ with $g + 1$ boundary components which we endow with the standard boundary orientation\(^1\). We call these cycles $C_0, C_1, \ldots, C_g$; their enumeration is inherited from that of the components of the labyrinth.

**Lemma 2** (see [6], [8] and [9], Ch. 2). 1) The cycles $C_0, C_1, \ldots, C_g$ associated with a labyrinth make up a basis in the lattice of odd cycles on the surface $M(E)$ punctured at two points, $\infty$ and $J\infty$.

2) The same cycles make up horizontal sections (with respect to the Gauss-Manin connection) of the homological bundle over the universal covering of the moduli space.

\[ \beta_{s-k+1} : (C_s, C_{s+1}) \rightarrow (-C_{s+1}, 2C_{s+1} + C_s). \] 

\[^1\text{In contrast to the notations of [9], we have also changed the sign in the definition (2.3) of the period mapping.}\]
This matrix representation of the braid group is known as (a particular case of) the Burau representation [4].

2.5. The global period mapping.

Definition 2. The period mapping (2.3) evaluated at the distinguished basis of odd cycles gives a well-defined global mapping

$$\Pi(\Lambda): \mathcal{L}_g^k \to \mathbb{R}^{g+1}.$$  

The image of this map lies in a codimension-one Euclidean subspace: the sum of the distinguished cycles is homologous to the circle encircling the pole of the differential $d\eta_M$, and therefore the value of the sum of the components of the period mapping is always $2\pi$.

Theorem 1 (see [7] and [9], Ch. 5). 1) The period mapping $\Pi(\Lambda): \mathcal{L}_g^k \to \mathbb{R}^g$ has full rank everywhere.

2) The period mapping is equivariant with respect to the action of braids:

$$\beta \Pi = \Pi \beta, \quad \beta \in \text{Br}_{g-k+1}.$$  

Here braids act on the universal covering of the moduli space as covering transformations and on Euclidean space via the Burau representation (2.4).

Statement 1) in Theorem 1 was proved in [18] and [15] in a more general setting. The image of this period mapping was explicitly calculated in [8] and [9], §5.3.

2.6. The statement of the main result. In this paper we study the topology of fibres of the period mapping $\Pi$, that is, of the inverse images of points in the image of the mapping. It follows from Theorem 1 above that fibres are smooth $g$-dimensional submanifolds of $\mathcal{L}_g^k \cong \mathbb{R}^{2g}$. In §3 we introduce machinery that reduces the problem under investigation to certain combinatorial calculations. In §5 and §6 we perform these calculations for two-dimensional fibres and arrive at the main result.

Theorem 2. Any fibre of the period mapping defined in the space $\mathcal{L}_2^k$, $k = 1, 2, 3$, is a cell.

Remark 1. Similar calculations were performed by the author for 3D-fibres too and a new effect was discovered. Fibres may be disconnected, however each component of a fibre remains a cell, that is, a topological space homeomorphic to a Euclidean space.

This observation leads to the following.

Conjecture (see [5]). Components of fibres of the period mapping are always cells.

§3. Pictorial representation of curves

Here we work out the main tool of our investigation. We need a convenient description of curves will would allow us to effectively reconstruct the period mapping. The construction is briefly described below; more details may be found in [8] and [9], §4.1.
The idea of representing algebraic curves by (weighted) graphs is not new. The seminal input was possibly due to Grothendieck with his ‘Dessins d’Enfants’. Since appearing in mathematical physics, ribbon graphs today make up a flourishing industry established in works by Kontsevich, Penner, Chekhov and Fock (see [17], [20] and [11]) to name a few. Similar ideas were used by Bertola in his work on Boutroux curves (see [3], and also [2], [14] and [21]).

### 3.1. The global width function.

Suppose $M(E) \in \mathcal{H}$ and let $d\eta_M$ be the 3rd-kind differential associated with the curve $M$ as above. One immediately checks that the normalization conditions of $d\eta_M$ force the (width) function

$$W(x) := \left| \text{Re} \int_{(e,0)}^{(x,w)} d\eta_M \right|, \quad x \in \mathbb{C}, \ e \in E,$$

(3.1)

to obey the following properties:

- $W$ is single valued on the complex plane;
- $W$ is harmonic outside its zero set $\Gamma_1 := \{x \in \mathbb{C}: W(x) = 0\}$;
- $W$ has a logarithmic pole at infinity;
- $W$ vanishes at each branch point $e_s \in E$ (and hence the definition (3.1) is independent of the choice of the branch point $e$).

We only comment on the last property. Since $d\eta_M$ is odd with respect to the hyperelliptic involution of $M$, $W(e_s)$ is equal to one-half of the absolute value of the real part of some period of the differential. Normalization implies that all its periods are purely imaginary.

### 3.2. Constructing the graph $\Gamma(M)$.

To any curve $M \in \mathcal{H}$ we associate a weighted planar graph $\Gamma = \Gamma(M)$ composed of the finite number of segments of vertical and horizontal foliations [22] of the quadratic differential $(d\eta_M)^2$ pushed down to the Riemann sphere. The graph $\Gamma(M)$ is the union of the ‘vertical’ subgraph $\Gamma_1$ and the ‘horizontal’ subgraph $\Gamma_{\perp}$ (see Figure 3, a for an admissible graph). Sets of edges and vertices of a graph $\Gamma$ are denoted as $\text{Sk}_1(\Gamma)$ and $\text{Sk}_0(\Gamma)$, respectively.

**Definition 3.** Vertical edges $\text{Sk}_1(\Gamma_1)$ are arcs of the zero set of $W(x)$; they are segments of the vertical foliation $d\eta_M^2 < 0$ and are not oriented.

Horizontal edges $\text{Sk}_1(\Gamma_{\perp})$ include all segments of the horizontal foliation $(d\eta_M)^2 > 0$ (or the steepest descent lines for $W(x)$) which connect the finite critical points of the foliation to other such points or—as a rule—to the zero set of $W$. Horizontal edges are oriented with respect to the growth of $W(x)$.

Weights. Each edge is equipped with its length in the metric $ds = |d\eta_M|$ of the quadratic differential.

The vertices $\text{Sk}_0(\Gamma)$ of the graph $\Gamma$ comprise all finite points of the divisor of the quadratic differential $(d\eta_M)^2$ considered on the plane, as well as points in $\Gamma_1 \cap \Gamma_{\perp}$, the projections of the saddle points of $W$ onto its zero set along horizontal leaves.

Instead of assigning lengths to horizontal edges, it is more convenient to keep the values of the width function $W(x)$ at all vertices of the graph: thus the length of an oriented (that is, horizontal) edge is the increment of the width function along it.
Remark 2. The most important property of the description of curves in terms of their graphs $\Gamma$ is the following. The periods of $d\eta_M$ are integer linear combinations of the lengths of vertical edges.

Indeed, the weight of any edge is the absolute value of the abelian integral taken along this edge. Given a cycle on $M$, one just has to properly collapse it onto the graph $\Gamma$ to get the period of $d\eta_M$ along this cycle. We consider this in greater detail in §5.

Remark 3. The topological invariants $g$ and $k$ of the curve $M$ may be reconstructed from the combinatorics of the graph $\Gamma(M)$.

Indeed, one immediately checks that the multiplicity of a vertex $V$ of the graph in the divisor of the quadratic differential $(d\eta_M)^2$ is equal

$$\text{ord}(V) := d_{\text{vert}}(V) + 2d_{\text{hor}}(V) - 2,$$

where $d_{\text{vert}}$ is the degree of this vertex with respect to the vertical edges and $d_{\text{hor}}$ is the number of incoming horizontal edges. Branch points of the curve $M$ correspond to vertices with odd value of $\text{ord}(V)$ (or odd value of $d_{\text{vert}}(V)$, which is the same). The number $2k$ of branch points on the real axis determines the number of real ovals of the curve; the total number $2g + 2$ of branch points is related to the genus.

Definition 4. We call a vertex $V \in \text{Sk}_0(\Gamma)$ a branch point if the quantity $\text{ord}(V)$ is odd.

### 3.3. An axiomatic description of graphs.

We distinguish between the geometric graph $\Gamma(M)$ drawn on the plane, the class of weighted planar graphs $\{\Gamma\}$ modulo isotopies of the plane respecting complex conjugation, and the class of topological planar graphs $[\Gamma]$ without weights. We are going to describe all admissible types of graphs $\{\Gamma\}$ in an axiomatic way so that they could be listed by an automaton. It turns out that there are just three topological restrictions on the graph $[\Gamma]$ and two further normalization conditions on its weights. All of them are listed in the following.
**Lemma 3** (see [8] and [9], §4.1.4). (T1) $\Gamma$ is a tree with real symmetry axis.

(T2) Horizontal edges leaving the same vertex are separated by a vertical or an incoming edge, in particular, there are no hanging horizontal vertices like $\longleftrightarrow$.

(T3) If $\text{ord}(V) = 0$, then $V \in \Gamma_\infty \cap \Gamma_1$.

(W1) The width function increases along oriented edges, and $W(V) = 0$ if $V$ lies on the vertical part of the graph.

(W2) The weights of vertical edges are positive and their total sum is $\pi$.

**Sketch proof.** We say a few words about properties (T1), (T2) and (W2). The rest follow from the definition of $\Gamma$.

(T1) Suppose there is a cycle in the graph. Let us calculate the Dirichlet integral for the width function in the domain $\Omega$ bounded by this cycle by means of Green’s formula:

$$\int_{\Omega} |\text{grad} W(x)|^2 \, d\Omega = \int_{\partial\Omega} W(x) \frac{\partial W}{\partial n} \, ds.$$ 

The function $W$ vanishes on the vertical parts of the boundary while its normal derivative vanishes on the horizontal part of $\partial\Omega$, and therefore $W$ is constant. Now suppose the graph has several components. We sum up the quantities $\text{ord}(V)$ over all the vertices. Then we obtain the number $2\#\{\text{vertical edges}\} + 2\#\{\text{horizontal edges}\} - 2\#\{\text{vertices}\} = -2\#\{\text{trees in the forest}\}$. This is equal to the degree of the divisor of the quadratic differential $d\eta^2_M$ (that is, $-4$) plus the order of its pole at infinity (which is 2). Hence the graph $\Gamma$ is a tree.

(T2) Suppose that $W(V) > 0$ at the vertex $V$ of the graph. This is a saddle point of the width function, the meeting point of several alternating ‘ridges’ and ‘valleys’. A horizontal edge comes to $V$ from each valley. An outgoing edge (if any) goes along a ridge, so any two of them are separated. The same is true for $W(V) = 0$, with vertical edges coming from each ‘valley’.

(W2) The integral of $d\eta_M$ along the boundary of the plane cut along $\Gamma_1$ equals $2i$ times the sum of the weights of all vertical edges. The integration path may be contracted to the path encircling the pole at infinity, hence the integral is $2\pi i$.

It turns out that there are no further restrictions either on the topology or on the weights of graphs.

**Theorem 3** (see [8] and [9], §4.1.5). Each weighted graph $\{\Gamma\}$ satisfying the five properties in Lemma 3 stems from a unique curve $M = M\{\Gamma\} \in \mathcal{H}$.

**Sketch proof.** The Riemann surface $M$ may be glued from a finite number of strips in a way determined by the combinatorics and weights of the graph. Below we briefly describe the procedure.

Given a planar graph satisfying all the requirements of Lemma 3, we extend it by drawing $d_\text{out}(V) - d_\text{in}(V) + d_\text{out}(V) \geq 0$ outgoing horizontal arcs which connect each vertex $V$ to infinity and are disjoint except possibly at their endpoints. For each vertex of the extended graph $\text{Ext}\Gamma$ we require that all the outgoing edges, old and new, alternate with incident edges of other types: incoming or vertical. In particular, property (T2) in Lemma 3 is preserved for $\text{Ext}\Gamma$. Up to a (mirror-symmetric) isotopy of the plane, the extension of the graph is unique since the original graph is a tree (see Figure 3, b).
From the topological viewpoint all the components of the complement to the extended graph in the plane have the same structure (see Figure 3, b). They are 2-cells bounded by exactly one vertical edge and two chains of horizontal edges all pointing in the same direction and eventually meeting at infinity which are attached to the endpoints of the vertical edge. One easily checks that for the graph \( \Gamma(M) \) generated by an element of the moduli space, and its extension drawn by the horizontal trajectories, the (suitably chosen branch of) the abelian integral 
\[
\eta(x) = \int_x d\eta \text{ maps each of the above 2-cells to a horizontal half-strip of height equal to the weight } H \text{ of the vertical edge in the boundary of the cell:}
\]
\[
\Sigma(H) = \{ \eta \in \mathbb{C} : \Re \eta > 0; \ 0 < \Im \eta < H \}.
\]
This observation gives us the reconstruction rule for the curve \( M \): one has to glue 2\# \{vertical edges\} half-strips \( \Sigma(H_s) \) in the way dictated by the graph. The conformal mapping of the glued surface onto the sphere gives us a realization of the graph \( \Gamma \), its vertices with odd values of \( \text{ord}(V) \) make up the branching set \( \mathcal{E} \) of the curve \( M \).

3.4. The coordinate space of a graph. The weights of an admissible graph \{\( \Gamma \)\} have obvious linear restrictions. The dependencies arise from the mirror symmetry of the graph as well as conditions (W1) and (W2) in Lemma 3. Independent weights are given by the lengths \( H(R) \) of vertical edges \( R \) in the closed upper half-plane and by the nonzero values of the width function \( W(V) \) at vertices \( V \) in the closed upper half-plane. They fill out an open convex polyhedron\(^2\) implicitly described in conditions (W1) and (W2) in Lemma 3. This polyhedron is the product of the interior of the simplex \( \Delta[\Gamma] \) spanned by the positive variables \( H(R), R \in Sk_1(\Gamma) \cap \mathbb{H} \), with one normalization condition,
\[
\sum_{R \in Sk_1(\Gamma) \cap \mathbb{R}} H(R) + 2 \sum_{R \in Sk_1(\Gamma) \cap \mathbb{H}} H(R) = \pi,
\]
and the interior of the cone \( \mathcal{C}[\Gamma] \) spanned by the positive variables \( W(V), V \in (Sk_0(\Gamma) \setminus Sk_0(\Gamma_1)) \cap \mathbb{H} \), that respect the partial order of the nodes on the horizontal part of the graph that is given by the directions of arrows:
\[
0 < W(V_1) < W(V_2) \quad \text{if } V_1 < V_2.
\]

**Definition 5.** The coordinate space of the graph \( [\Gamma] \) is the open polyhedron \( \mathcal{A}[\Gamma] := \text{Int}(\Delta[\Gamma] \times \mathcal{C}[\Gamma]) \).

**Example 1.** For the graph \( \Gamma \) shown in Figure 3, a the genus \( g[\Gamma] = 2 \) and the number of real ovals \( k[\Gamma] = 1 \). The coordinate space
\[
\mathcal{A}[\Gamma] = \{H_1, H_2, H_3 > 0 : 2(H_1 + H_2) + H_3 = \pi\} \times \{0 < W_1 < W_2\}
\]
has full dimension: \( \dim \mathcal{A}[\Gamma] = \dim \mathcal{A}[\mathbb{H}] = 2g = 4 \).

\(^2\)Here and in what follows by the interior (\text{Int}) of a polyhedron we mean its relative interior, that is, its interior within its affine hull.
Theorem 4 (see [8] and [9], §§4.1.2 and 5.2). The mapping implied by Theorem 3 embeds the coordinate space $\mathcal{A}[\Gamma]$ real-analytically in the moduli space $\mathcal{H}$.

Lemma 4 (see [8] and [9], §4.2). The dimension of the cell $\mathcal{A}[\Gamma]$ is not greater than $2g[\Gamma]$ and it equals $2g[\Gamma]$ if and only if a neighbourhood of each vertex $V \in \text{Sk}_0(\Gamma)$ has one of the following forms

and neighbourhoods of vertices $V$ on the (real) symmetry axis may additionally be of the following types (up to central symmetry):

here the dotted line is the real axis.

Remark 4. A graph $\Gamma$ may be associated to an arbitrary hyperelliptic curve (not necessarily real) with two marked points interchanged by the involution $J$. The analogue of the above lemma for such curves $M$ says that the graph $\Gamma(M)$ of a typical curve is composed only of nodes with vicinities of the first three types in Lemma 4. The curve being mirror symmetric gives rise to additional stable nodes on the symmetry axis.

Remark 5. A combinatorial algorithm listing all stable admissible graphs with given topological invariants $g$ and $k$ is available and will be published elsewhere. For the low-dimensional components of the total moduli space all codimension-zero cells may be found manually just by trying all possible connections of the stable nodes listed above.

Example 2. The moduli spaces $\mathcal{H}_2^3$, $\mathcal{H}_2^2$ and $\mathcal{H}_2^1$ of dimension 4 will have 1, 5 and 9 cells of full dimension, respectively; their graphs $[\Gamma]$ are shown in Figures 4–6 up to central symmetry. The 6D-spaces $\mathcal{H}_3^1$, $\mathcal{H}_3^2$, $\mathcal{H}_3^3$ and $\mathcal{H}_3^4$ have 24, 20, 7 and 1 codimension-zero cells, respectively. The graphs encoding the 20 full-dimensional cells of $\mathcal{H}_3^2$ are listed in [9], §4.2.

Figure 4. The graph $\Gamma$ encoding the unique cell of the moduli space $\mathcal{H}_2^3$.

Figure 5. The graphs $[\Gamma_1]$, $[\Gamma_2]$ and $[\Gamma_3]$ and their central symmetric graphs $[\Gamma_1]$ and $[\Gamma_2]$ encode all full-dimensional cells of the moduli space $\mathcal{H}_2^2$.

Remark 6. Several enumeration problems may be put forward: find the number of admissible graphs $[\Gamma]$ with given topological invariants $g$ and $k$ and the number of full-dimensional admissible graphs (that is, $\dim \mathcal{A}[\Gamma] = 2g$). The same problems may be put forward for curves without mirror symmetry.
§ 4. Polyhedral model of the moduli space

In § 3 we built a decomposition of the total moduli space into smoothly embedded disjoint open polyhedra encoded by admissible (topological types of) graphs. We will show that this decomposition is natural in the following sense: polyhedra of lower dimensions lie in faces of higher-dimensional polyhedra. This allows us to effectively build a PL model for each component of the total moduli space, starting from the cells of codimension zero and gluing their faces in accordance with certain identifications.

4.1. Two types of faces of a coordinate space. Each coordinate space $\mathcal{H}[\Gamma]$ is defined by a finite set of strict linear inequalities for the independent graph weights $H(R)$ and $W(V)$ given in conditions (W1) and (W2) in Lemma 3. On a face of the polyhedron certain inequalities turn to equalities. In other words, the weights of certain edges of the tree $\Gamma$ vanish and we want to interpret such degenerate weighted trees as regular trees $\Gamma'$, but with lower dimension of the coordinate space.

![Figure 7](image)

Figure 7. Two typical mergers of branch points: $H_1 + H_2 \rightarrow 0$ (a) and $W \rightarrow 0$ (b).

We have to distinguish between two types of faces in each polyhedron $\mathcal{H}[\Gamma]$: outer or exterior faces correspond to a merger (that is, zero graph distance) of at least two branch points (see Figure 7) and all the rest of the faces, which we call inner or interior. Outer faces cannot be interpreted in terms of the same component
of the moduli space as the proposition below shows (see §4.3). Inner faces may be further subdivided and identified with lower-dimensional coordinate spaces of the same component of the moduli space. We call graphs \([Γ′]\) corresponding to inner faces of the coordinate space of the graph \([Γ]\) subordinate to the latter; they can be obtained by successive application of the two combinatorial procedures of contraction and zipping to the original graph which we describe below.

4.2. Eliminating zero-weight edges: definition. Weighted graphs \(\{Γ\}\) on the boundary of the coordinate space \(A[Γ]\) satisfy weak versions of axioms (W1) and (W2) in Lemma 3.

(W1*) The width function does not decrease along oriented edges, and \(W(V) = 0\) if \(V\) lies on the vertical part of the graph.

(W2*) The weights of vertical edges are nonnegative and their total sum is \(\pi\).

Remark 7. The modified axioms (W1*) and (W2*) allow zero-weight edges and, surprisingly, may affect the topology of the graph \(Γ\) too. For graphs satisfying the original version of the axioms the quantity \(d_{in}(V)\) is zero for vertices \(V\) of the vertical part of the graph.

Indeed, starting at \(V\) and moving against the arrows—which is possible due to condition (T2)—we eventually arrive at another component of \(Γ_{1}\). The value of the width function \(W\) at this point should be strictly negative on the one hand and zero on the other hand due to the same axiom (W1). The weak version of the axiom admits graphs with positive \(d_{in}(V)\) for \(V ∈ Sk_0(Γ_{1})\).

We will apply the two procedures, contraction and zipping, described below, to weighted graphs corresponding to inner faces of a coordinate space, that is, to faces with positive graph distance between any two branch points.

Contraction is applied to a zero-weight horizontal edge \(R\) of the graph \(Γ\); its action on the graph is shown in Figure 8.

![Figure 8](image)

Figure 8. a) The initial tree with zero-weight horizontal edge \(R = [V_a, V_b]\) and the subtrees \(A\) and \(B\) rooted in \(V_a\) and \(V_b\), respectively; b) the edge \(R\) is contracted.

Elimination of a zero-weight vertical edge of the graph is more complicated as it leads to the collapse of a whole strip in the decomposition of the plane considered in the sketch proof of Theorem 3 and therefore to a deeper modification of the graph.

Zipping caused by a zero-weight vertical edge \(R\) of the graph is defined as follows. This edge is a vertical side of exactly two half-strips in the complement of the extended graph \(Ext Γ\). Each half-strip is collapsed to a ray so that points on the opposite sides with equal value of the width function \(W\) are identified (see Figure 9). The horizontal edges of the modified extended graph that go to infinity should be removed afterwards.

---

3Perhaps a more fruitful viewpoint is to consider points in outer faces as nodal curves living on the boundary of the component of the moduli space. Further desingularization allows us to identify them with lower-genus curves.
Informally speaking, the opposite sides of the strip which pass through the zero-weight vertical edge \( R \) are treated as the two pieces of a zipper fastener which are attached one to the other in a natural way. The topology of the modified graph depends on the relations between the values of the width function \( W \) at vertices on opposite sides of the strip. It is exactly at this step that faces of the coordinate space of the initial graph may be subdivided into smaller polyhedra.

**Remark 8.** As a byproduct of zipping, zero-weight horizontal edges on the sides of the collapsed strip are contracted. Their further contraction should be considered as a trivial action on the graph.

**Remark 9.** The procedure inverse to the elimination of zero-weight edges leads to graphs with higher-dimensional coordinate space. It consists in the decay of unstable vertices \( V \) into more stable ones. The measure \( \text{codim}(V) \) of the stability of a vertex \( V \) was introduced in [8] and [9], §4.1.3.

### 4.3. Elimination of edges: properties

Using the above two procedures leads—in a finite number of steps—to an admissible graph \( \Gamma' \) as shown by the following.

**Lemma 5.** Contraction and zipping operations respect topological restrictions (T1)–(T3) and weak restrictions on the weights (W1*) and (W2*) of the graph. The number of zero-weight edges decreases after each operation.

**Proof.** (T1) Both operations obviously preserve the dendritic nature of the graph: graphs in the right-hand sides of Figures 8, b and 9, b remain trees if \( A, B, \ldots \) are trees. The mirror symmetry of the weighted tree may be violated at some step, but eventually the symmetry will be repaired as two symmetric edges are collapsed or not collapsed simultaneously (see Lemma 6 on permutability of operations).

(T2) The horizontal edges leaving a vertex of the extended graph \( \text{Ext} \Gamma' \) alternate with incident edges of other types—incoming and vertical. Merging two vertices...
during a contraction of an edge or zipping preserves this property (see Figure 10).
To return to the usual graph, we delete all edges leaving for infinity, so that all the
remaining outgoing edges are separated by edges of other types.

(T3) Assume that a contraction or zipping leads to an appearance of a vertex
as a merger of several vertices $V_a, V_b, \ldots$. We show that
$$\text{ord}(V) = \text{ord}(V_a) + \text{ord}(V_b) + \cdots.$$ (4.1)

For the contraction shown in Figure 8 we have a chain of equalities:
$$\text{ord}(V_{ab}) := d_A^1(V_{ab}) + d_B^1(V_{ab}) + 2d_A^2(V_{ab}) + 2d_B^2(V_{ab}) - 2$$
$$= (d_A^1(V_a) + 2d_A^2(V_a) - 2) + (d_B^1(V_b) + 2(d_B^2(V_b) + 1) - 2)$$
$$= \text{ord}(V_a) + \text{ord}(V_b),$$

where $d_A^1(V)$ is the degree of the vertex $V$ with respect to the vertical part of the
subtree $A$, $d_A^2(V)$ is the number of the edges incoming from the subtree $A$ and so on.

Zipping a strip may lead to the merging of more than two nodes. A preliminary
contraction of zero-weight edges on the horizontal sides of the strip reduces the
proof of the addition formula (4.1) for the orders of vertices to just a merger of two
nodes.

If $W(V_a) = W(V_b) = 0$ as in Figure 9, a, then
$$\text{ord}(V_{ab}) := d_A^1(V_{ab}) + d_B^1(V_{ab}) + 2d_A^2(V_{ab}) + 2d_B^2(V_{ab}) - 2$$
$$= ((d_A^1(V_a) + 1) + 2d_A^2(V_a) - 2) + ((d_B^1(V_b) + 1) + 2d_B^2(V_b) - 2)$$
$$= \text{ord}(V_a) + \text{ord}(V_b).$$
Otherwise if \( W(V_{a2}) = W(V_{b2}) > 0 \) as in Figure 9, a, then

\[
\text{ord}(V_{a2b2}) := 2d_{in}^A(V_{a2b2}) + 2d_{in}^B(V_{a2b2}) \\
= 2(d_{in}^A(V_{a2}) + 1) - 2 + 2(d_{in}^B(V_{b2}) + 1) - 2 = \text{ord}(V_{a2}) + \text{ord}(V_{b2}).
\]

Suppose that the order of a newborn node is zero. We forbid mergers of branch points and hence all orders in the above sum \((4.1)\) are even. Moreover, property (T2) implies that \( \text{ord}(V) > -2 \), and therefore all the parent vertices of the newborn one have zero order. According to property (T3) of the initial graph, each parent vertex is incident to two vertical and one or two outgoing edges only. This may only happen in the case of an Ext \( \Gamma \) fragment shown in Figure 11, which after zipping gives a graph with property (T3).

![Figure 11. Zipping a zero-height strip leading to a merger of two vertices of order zero.](image)

Contracting and zipping change the topology of the graph, not the weights. Therefore, properties (W1*) and (W2*) remain intact. Contraction removes exactly one zero-weight horizontal edge, zipping eliminates one vertical edge of zero weight and all zero-weight horizontal edges on the sides of the strip supported by this vertical edge.

Lemma 5 is proved.

**Lemma 6.** Any two procedures eliminating zero-weight edges — contractions or zippings — commute.

**Proof.** Essentially, this fact follows from the local nature of both operations: nothing happens outside the edge we eliminate in the case of contraction or outside the strip in the case of zipping. It follows from property (T1) that two strips are either disjoint or have one edge of the extended graph in common. (See also Remark 8.)

**Lemma 7.** Eliminating zero-weight edges does not merge branch points together.

**Proof.** We apply the procedures of contraction and zipping only to graphs with positive graph distance between the branch points.

The above three statements show that the outcome of the full chain of contractions and zippings applied to a weighted graph \( \{\Gamma\} \) from a face of the coordinate space does not depend on the order of the operations and corresponds to an admissible weighted graph \( \{\Gamma'\} \) from the same component of the moduli space. We say that the topological type of \( \Gamma' \) is subordinate to that of \( \Gamma \) and write \( [\Gamma'] < [\Gamma] \) (an example is shown in Figure 12).

**Lemma 8.** If \( [\Gamma'] < [\Gamma] \), then the coordinate space \( \mathcal{A}[\Gamma'] \) is embedded (possibly more than once; see Remark 10) in a face of the polyhedron \( \mathcal{A}[\Gamma] \).
Proof. Subordination of the topological type of $\Gamma'$ to that of $\Gamma$ means that there is a chain of contractions and zippings transforming the latter into the former. On each step the weights may be naturally and uniquely lifted to the larger graph. Eventually, we obtain weights on the graph $\Gamma$, but some of those will be zeros. This corresponds to a point on the boundary of the polyhedron $A[\Gamma]$.

The lemma is proved.

Definition 6. By dressing a coordinate space $A[\Gamma]$ we mean attaching all its interior faces:

$$\hat{A}[\Gamma] := \bigcup_{[\Gamma'] \leq [\Gamma]} A[\Gamma'];$$

this set is equipped with the ambient Euclidean space topology.

The basis for the incidence relations of the coordinate spaces of graphs which we have described here is the following.

Proposition. The natural embedding of any dressed coordinate space $\hat{A}[\Gamma]$ in the moduli space $\mathcal{H}$ is continuous and has no continuous extension to the exterior faces of $\hat{A}[\Gamma]$.

This statement is intuitively clear, however its rigorous proof requires special analytical techniques, which are beyond the scope of this article. It will be given in a separate publication.

4.4. Building moduli spaces and labyrinth spaces. A piecewise-linear model of any component of the total moduli space $\mathcal{H}$ may be assembled in three steps:

- list all admissible graphs $[\Gamma]$ with full-dimensional coordinate space $A[\Gamma]$ (there are only finitely many of them once the genus is fixed);
- dress each zero-codimension coordinate space $A[\Gamma]$;
- glue the dressed polyhedra $\hat{A}[\Gamma]$ along their interior faces.

Remark 10. Note that at the last stage a full-dimensional polyhedron may be glued to itself. For instance, if the graph $\Gamma$ contains the fragment shown in Figure 7,a, then two of its hyperfaces, $\{H_1 = 0\}$ and $\{H_2 = 0\}$, correspond to the same subordinate graph $\Gamma'$ and should be glued one to the other.

Gluing a labyrinth space from the dressed full-dimensional polyhedra $\hat{A}[\Gamma]$ is somewhat more tricky. Lifting a point from the moduli space to the labyrinth space means attaching a labyrinth $\Lambda$ to the branching divisor $E^+$ in the upper half-plane.
Definition 7. We call a labyrinth $\Lambda$ escorting the branching divisor $E^+$ canonical if and only if it does not intersect the graph $\Gamma$ of the curve $M(E)$ in the open upper half-plane except in points in the branching divisor.

For nonexceptional graphs $[\Gamma]$ the canonical labyrinth is unique up to isotopy. Here nonexceptional means that all branch points $V$ in the open upper half-plane are hanging vertices of the graph (which are stable: with degree $d(V) = 1$). In particular, all full-dimensional graphs $[\Gamma]$ are nonexceptional. Exceptional graphs $[\Gamma]$ admit several canonical labyrinths $\Lambda, \Lambda'$... which are interchanged by the action of braids: $\beta \cdot \Lambda := \Lambda'$, $\beta \in \text{Br}$ (see Figure 13). The algorithm for reconstructing the braid from a couple of labyrinths escorting the same sets of branch points can be found in [12].

Figure 13. The upper half of an exceptional graph $[\Gamma]$ with $g = 3$ and $k = 1$ which admits three canonical labyrinths transformed by the standard generators $\beta_1$ and $\beta_2$ of $\text{Br}_3$.

The coordinate space of a nonexceptional (full-dimensional, for instance) graph $[\Gamma]$ has a canonical lift to the labyrinth space, which we denote by $\mathcal{LA}[\Gamma]$. The dressed polyhedron $\mathcal{A}[\Gamma] = \bigcup_{\Gamma' \leq \Gamma} \mathcal{A}[\Gamma']$ also has a canonical lift designated as $\mathcal{L}[\Gamma]$, since all the attached low-dimensional faces inherit a labyrinth from $\Gamma$ (which may no longer be canonical). All other lifts differ by an action of covering transformations of the universal covering, which are represented by braids. Therefore, to build a PL model of the labyrinth space $\mathcal{L}_g^k$ we take all possible (that is, finitely many) dressed full-dimensional polyhedra $\mathcal{A}[\Gamma]$ corresponding to the given topological invariants $g$ and $k$ and label them by braids $\beta \in \text{Br}_{g-k+1}$ (of which there are infinitely many if $g > k$). Now we identify the common face $\mathcal{A}[\Gamma_{12}]$ of two polyhedra $\beta^1 \cdot \mathcal{L}\mathcal{A}[\Gamma_1]$ and $\beta^2 \cdot \mathcal{L}\mathcal{A}[\Gamma_2]$, $[\Gamma_{12}] < [\Gamma_1], [\Gamma_2]$, if and only if $\beta^2 = \beta^1 \beta^{12}$, where the braid $\beta^{12} \in \text{Br}_{g-k+1}$ maps the labyrinth inherited by $\Gamma_{12}$ from $\Gamma_2$ to the labyrinth inherited from $\Gamma_1$.

More formally, the total labyrinth space may be represented as follows:

$$\mathcal{L} := \bigcup_{g,k} \mathcal{L}_g^k = \bigcup_{\text{codim } \mathcal{A}[\Gamma] = 0, \beta \in \text{Br}} \beta \cdot \mathcal{L}\mathcal{A}[\Gamma] / \sim \quad (4.2)$$

with the following equivalence relation $\sim$ on the boundaries of the dressed coordinate spaces. If $[\Gamma']$ is subordinated to both $[\Gamma_1]$ and $[\Gamma_2]$, then the coordinate space $\mathcal{A}[\Gamma']$ has two natural inclusions in the labyrinth space, as a face

---

\(^4\)Namely $\prod_V d(V)$, where the product is taken over the branch points in the open upper half-plane.
of $\mathcal{L} \mathcal{A} [\Gamma_s]$, $s = 1, 2$. We identify $\mathcal{L} \mathcal{A} [\Gamma'_1]$ with $\beta^{12} \cdot \mathcal{L} \mathcal{A} [\Gamma'_2]$ where the braid $\beta^{12}$ maps the labyrinth of $\Gamma'$ inherited from $[\Gamma_2]$ to the one inherited from $[\Gamma_1]$. The case $\Gamma_1 = \Gamma_2$ is not excluded here: there may be more than one inclusion of the same coordinate space $\mathcal{A} [\Gamma']$ in a dressed coordinate space $\mathcal{A} [\Gamma]$. 

§ 5. A PL model of the period mapping

A point $M(\mathcal{E}) \in \mathcal{M}$ in the total moduli space is presented by a (normalized) mirror-symmetric branching divisor $\mathcal{E}$. Points over $M$ in the covering labyrinth space are distinguished by the choice of the labyrinth $\Lambda$ that escorts the set $\mathcal{E}^+$. A labyrinth determines a basis in the lattice of odd integer cycles on the (twice punctured at infinity) curve $M$ which is transported by the Gauss-Manin connection (see Lemma 2). The period mapping from each component of the labyrinth space to a suitable Euclidean space is defined by (2.3) and it is evaluated at the distinguished basis of odd cycles on the surface. Given a point $\mathcal{E}^+ \in \mathcal{M}$, we start with evaluation of the period mapping for the canonical labyrinth $\Lambda$ (that is, the one not intersecting the graph of $M(\mathcal{E})$ in the upper half-plane) which is unique for nonexceptional curves $M$.

5.1. The period mapping restricted to a coordinate space. To calculate the period mapping we need some preliminary considerations. One can single out a branch of the distinguished differential $d\eta_M$ in any simply connected domain in the plane that avoids the branch points of the curve $M$. We take the complements to the labyrinth and to the graph in the open upper half-plane as such sets. In particular, we introduce the harmonic function

$$H'(x) = \text{Im} \int_x^\infty d\eta_M$$

in the complement $\mathbb{H} \setminus \Lambda$ to the canonical labyrinth, with normalization $H'(x) = 0$ for large real argument. The branch of the differential here is singled out by the following condition: its residue at infinity equals $-1$. A harmonic function $H(x)$ conjugate to $W(x)$ is defined by the same formula in the complement $\mathbb{H} \setminus \Gamma$ to the graph. The differentials $dH$ and $dH'$ coincide up to sign in the components of the set $\mathbb{H} \setminus \{\Gamma \cup \Lambda\}$. This sign changes when we cross either the labyrinth or the vertical edge of the graph $\Gamma$ and remains the same when we cross a horizontal edge. In particular, the sign $dH'/dH$ equals $(-1)^{g+1}$ in the component of the complement $\mathbb{H} \setminus \{\Gamma \cup \Lambda\}$ bounded by the arcs $\Lambda_l$ and $\Lambda_{l+1}$ of the labyrinth, $l = k, \ldots, g - 1$. We call a component of the complement positive or negative depending on the sign of $dH'/dH$ in it.

The boundary of the half-plane cut along the graph $\Gamma$ has a natural orientation and is divided into pieces labelled by vertical and horizontal edges of the graph. The Cauchy-Riemann equations show that the boundary value $H(v)$ is locally constant on the pieces corresponding to horizontal edges and strictly decreases on the banks of vertical edges. Therefore, the value of $H$ at a vertex $v$ of the boundary can be found as follows: we crawl along the boundary from $v$ toward $+\infty$ and sum up the weights $H(R)$ of all vertical edges we encounter (some of these may appear twice).
Example 3. The boundary values of the function $H(v)$ for the curve $M(\Gamma)$ with the graph shown in Figure 14 are as follows:

- $H(v_2) = H_1 + H_2,$
- $H(v_3) = H_1 + 2H_2 + H_3,$
- $H(v_4) = H_1 + 2H_2 + 2H_3 + H_4 + H_5 + H_6 + H_7,$
- $H(v_5) = H_1 + 2H_2 + 2H_3 + H_4 + H_5 + H_6 + 2H_7 + H_8,$
- $H(v_6) = H_1 + 2H_2 + 2H_3 + H_4 + H_5 + H_6 + 2H_7 + 2H_8 + H_9.$

Figure 14. The upper half of a graph $\Gamma$, $M(\Gamma) \in \mathcal{H}_6^2$; the canonical labyrinth $\Lambda = (\Lambda_0, \ldots, \Lambda_6)$ (dotted lines in the upper half-plane); the weights of vertical edges $H_1, \ldots, H_{10}$; the auxiliary points $v_s$.

Lemma 9 (see [8] and [9], §5.3). The period mapping

$$\Pi_s := \langle \Pi|C_s \rangle := i \int_{C_s} d\eta_M,$$

evaluated at the odd cycles $C_s$ determined by the canonical labyrinth $\Lambda$ is given by

$$\Pi_s(\Lambda) = \begin{cases} 
2 \sum_{R \subset \Lambda_s} \epsilon(R) H(R), & s = 0, \ldots, k - 1, \\
4(-1)^s + g H(v_s), & s = k, \ldots, g, 
\end{cases} \quad (5.1)$$

where the sum in the formula for $s = 0, \ldots, k - 1$ is taken over all real vertical edges $R$ of the graph $\Gamma$ which make up the segment $\Lambda_s$ and $\epsilon(R)$ is the sign of the component of $\mathbb{H} \setminus (\Lambda \cup \Gamma)$ bordered by $R$; $v_s$ in the other formulat, for $s = k, \ldots, g$, is the meeting point of the graph and the $s$th arc of the labyrinth in the upper half-plane, it lies on the boundary of $\mathbb{H} \setminus \Gamma$.

Proof. Each of the basic cycles $C_s$ changes its orientation after complex conjugation, hence we may integrate along the upper half of this cycle alone:

$$\Pi_s := - \text{Im} \int_{C_s} d\eta_M = -2 \text{Im} \int_{C_s \cap \mathbb{H}} d\eta_M = -2 \int_{C_s \cap \mathbb{H}} dH'.$$

We consider two cases separately:
a) $s = 0, \ldots, k - 1$, when the integration path is the upper bank of the real segment $\Lambda_s$;

b) $s = k, \ldots, g$, when we integrate along both banks of the cut $\Lambda_s$ in the upper half-plane.

a) We integrate $dH'$ along the upper bank of the real segment $\Lambda_s$:
\[
\frac{\Pi_s}{2} = -\int_{\Lambda_s} dH' = -\sum_{R \subset \Lambda_s} \int_R dH' = -\sum_{R \subset \Lambda_s} \pm \int_R dH = \sum_{R \subset \Lambda_s} \epsilon(R) H(R),
\]
where $\epsilon(R)$ in the last formula is the sign of the component of the complement to the graph and the labyrinth whose boundary contains the edge $R$.

b) We integrate $dH'$ along both banks of the cut $\Lambda_s$. Instead, we can integrate along the left-hand bank and double the result. This integration path lies in the component of the complement to both the labyrinth and the graph $\Gamma$ that is bounded by the arcs $\Lambda_s$ and $\Lambda_{s-1}$. We integrate $(-1)^{s+g+1} dH$ instead of $dH'$:
\[
(-1)^{s+g} \frac{\Pi_s}{4} = H(v_s) = \sum_{R > v_s} H(R),
\]
where $v_s$ is the point where the arc $\Lambda_s$ touches the graph; the sum is taken over all vertical edges of the boundary of the upper half-plane cut along the graph that lie between $v_s$ and $+\infty$.

The lemma is proved.

Remark 11. We see that the period mapping depends only on the coordinates $H$ of the simplicial factor $\Delta[\Gamma]$ of the space $\mathcal{M}[\Gamma]$. If this does not lead to confusion (for example, in the case of a nonexceptional $[\Gamma]$) we consider the period mapping (5.1) directly from the simplex $\Delta[\Gamma]$ to a Euclidean space and denote it by the same letter $\Pi$.

Example 4. The period mapping for the curve $M\{\Gamma\}$ with the graph $[\Gamma]$ shown in Figure 14 equals
\[
\begin{align*}
\Pi_0 &= 2(H_{10} - H_5), \\
\Pi_1 &= -2(H_4 + H_1), \\
\Pi_2 &= 4(H_1 + H_2), \\
\Pi_3 &= -4(H_1 + 2H_2 + H_3), \\
\Pi_4 &= 4(H_1 + 2H_2 + 2H_3 + H_4 + H_5 + H_6 + H_7), \\
\Pi_5 &= -4(H_1 + 2H_2 + 2H_3 + H_4 + H_5 + H_6 + 2H_7 + H_8), \\
\Pi_6 &= 4(H_1 + 2H_2 + 2H_3 + H_4 + H_5 + H_6 + 2H_7 + 2H_8 + H_9).
\end{align*}
\]

One can check that
\[
\sum_{s=0}^{6} \Pi_s = 2(H_1 + H_4 + H_5 + H_{10}) + 4(H_2 + H_3 + H_6 + H_7 + H_8 + H_9) = 2\pi
\]
due to the normalization of the weights of vertical edges.
The equality \( \sum_{s=0}^{g} \Pi_s = 0 \) follows from the fact that the sum of all basic odd cycles is homological to a small circle enclosing the pole of the distinguished differential \( d\eta_M \). This means that the image of the period mapping lies in the hyperplane of \( \mathbb{R}^{g+1} \). We check the equality by a straightforward calculation:

\[
\sum_{s=k}^{g} \Pi_s = 4(H(v_g) - H(v_{g-1})) + 4(H(v_{g-2}) - H(v_{g-3})) + \cdots + (-1)^{g+k}4H(v_k)
\]

the last sum is taken over the vertical edges in the boundaries of all ‘negative’ domains in the decomposition of \( \mathbb{H} \) by the graph and the labyrinth. Adding \( \sum_{s=0}^{k-1} \Pi_s \) to the latter sum, we get the quadruple sum of the weights \( H(R) \) of all vertical edges in the open upper half-plane and the double sum of the weights of the vertical edges of the graph lying on the real axis. The total value equals \( 2\pi \) due to the weight normalization condition (W2).

5.2. The image of the coordinate space and local fibres of the period mapping. Lemma 9 allows us to calculate the image of the period mapping. We know that any labyrinth space is tiled by translations of copies of the coordinate spaces \( \mathcal{A}[\Gamma] \). A coordinate space is embedded in a component of the moduli space and then lifted to its covering space by drawing a (usually unique) canonical labyrinth. Lemma 9 computes the period mapping on the latter cell; the mapping of translated cells is related to the one computed by means of the Burau representation (2.4); see Theorem 1. We know that any coordinate space is the product of the interior of the simplex spanned by the vertical variables \( H(R) \) and the interior of the cone spanned by the horizontal variables \( W(V) \). The period mapping depends on the weights of the vertical edges only, so the image of a coordinate space under the period mapping can be obtained as follows. One evaluates the period mapping at the vertices of the simplicial factor of the coordinate space and take the relative interior of the convex hull of the points thus obtained.

Following this procedure, the range of the period mapping was calculated for all values of the topological invariants \( g \) and \( k \) in \cite{8} and \cite{9}, §5.3. Here we reproduce the result for genus \( g = 2 \) and \( k = 1, 2, 3 \) (see Figure 15). For the purposes of §6 we also calculate the fibres of the period mapping in each full-dimensional cell of the moduli space corresponding to \( g = 2 \).

![Figure 15](image-url)

Figure 15. The images \( \Pi(\mathcal{L}_2^k) \) for \( k = 3, 2, 1 \) (only part of the image for \( k = 1 \)).

5.2.1. Three real ovals. The image of the unique coordinate space in \( \mathcal{L}_2^3 = \mathcal{H}_2^3 \), projected onto the plane of the variables \( \Pi_0 \) and \( \Pi_1 \) is the open triangle

\[
\Pi_0 > 0, \quad \Pi_1 > 0, \quad \Pi_0 + \Pi_1 < 2\pi.
\]
Fibres of this map are the quadrants $\mathbb{R}^+_2$ spanned by the weights $W_1$ and $W_2$ of the two vertices of the horizontal subgraph (see Figure 4).

### 5.2.2. Two real ovals.

The space $\mathcal{L}_2^2 = \mathcal{H}_2^2$ contains five full-dimensional polyhedra corresponding to the topological graphs $[\Gamma_1]$, $[\Gamma_2]$ and $[\Gamma_3]$ in Figure 5 and their central symmetric graphs $[-\Gamma_1]$ and $[-\Gamma_2]$. The simplicial factor $\Delta[\Gamma]$ of the coordinatespace $\mathcal{A}[\Gamma]$ is 3-dimensional for $[\Gamma] = \pm [\Gamma_1]$ and 2-dimensional in the remaining three cases. Hence a fibre of the restriction of the period mapping to the coordinate spaces $\mathcal{A}[\pm [\Gamma_1]]$ is a half-strip, the product of an interval (the section of a tetrahedron by a line) and a ray (the conical factor of the coordinate space). In the three other coordinate spaces corresponding to $[\Gamma] = \pm [\Gamma_2]$, $[\Gamma_3]$, fibres are their conical factors, the open quadrants spanned by the two positive weights of the vertices in the horizontal subgraphs. The projections of the images of full-dimensional coordinate spaces onto the plane of coordinates $\Pi_0$, $\Pi_1$ as well as local fibres of the period mapping in those polyhedra are given in Table 1 below.

| $\Gamma$ | The image $\Pi(\mathcal{L}A[\Gamma])$ | The fibre $\Pi^{-1}(\Pi^*)$ for $\Pi^*$ belonging to the image |
|----------|----------------------------------------|-------------------------------------------------------------|
| $\Gamma_1$ | Int$(a_+ \cup b)$ | half-strip $(H,W): 0 < 4H < 2\pi - |\Pi_6^*| + \Pi_1^*, 0 < W$ |
| $-\Gamma_1$ | Int$(a_- \cup b)$ | half-strip $(H,W): 0 < 4H < 2\pi - \Pi_6^* - |\Pi_1^*|, 0 < W$ |
| $\pm \Gamma_2$ | Int$(a_{\pm})$ | quadrant $(W_1, W_2): 0 < W_1, W_2$ |
| $\Gamma_3$ | Int$(b)$ | quadrant $(W_1, W_2): 0 < W_1, W_2$ |

Here $a_{\pm}, b$ are the three closed triangles shown in Figure 16; $H$ is the weight of the unique vertical edge in the upper half-plane; $W_{1,2}$ are the weights of the vertices in the horizontal subgraph.

![Figure 16](image-url)  
**Figure 16.** The range of the period mapping in the coordinate spaces of $\mathcal{H}_2^2$. Black dots are the images of vertices of the simplexes $\Delta[\Gamma]$.  

### 5.2.3. One real oval.

The moduli space $\mathcal{H}_2^3$ contains nine full-dimensional polyhedra corresponding to the topological graphs $[\Gamma_1], \ldots, [\Gamma_6]$ in Figure 6 and also their central symmetric graphs $[-\Gamma_4]$, $[-\Gamma_5]$ and $[-\Gamma_6]$. The dimension of the simplicial factor $\Delta[\Gamma]$ of the coordinatespace $\mathcal{A}[\Gamma]$ is 4 for $[\Gamma] = [\Gamma_1], 3$ for $[\Gamma] = [\Gamma_2], [\pm \Gamma_5]$
and 2 in the remaining five cases. A fibre of the period mapping restricted to a coordinate space \( L\mathcal{A}[\Gamma_1] \), which is a section of an open 4-simplex by a 2-plane, may be either a pentagon, a quadrilateral, or a triangle. A fibre of the period mapping in the coordinate space for \( [\Gamma] = [\Gamma_2], [\pm \Gamma_5] \) is a half-strip, the product of an interval (the section of a tetrahedron by a line) and a ray (the conical factor of the coordinate space). In the other five coordinate spaces corresponding to \( [\Gamma] = [\Gamma_3], [\pm \Gamma_4], [\pm \Gamma_6] \) fibres are their conical factors spanned by the two positive weights of the vertices in the horizontal subgraphs. The projections of the images of full-dimensional coordinate spaces onto the plane of the coordinates \( \Pi_1, \Pi_2 \) as well as fibres of the period mapping in these polyhedra are given in Table 2.

| \( \Gamma \) | The image \( \Pi(L\mathcal{A}[\Gamma]) \) | The fibre \( \Pi^{-1}(\Pi^*) \) for \( \Pi^* \) belonging to the image |
|-----|----------------------------------|-------------------------------------------------|
| \( \Gamma_1 \) | \( \text{Int}(a \cup b \cup c_+ \cup c_- \cup d) \) | rectangle, \( \Pi^* \in \text{Int}(a) \) |
| | | pentagon, \( \Pi^* \in \text{Int}(b) \) |
| | | trapezoid, \( \Pi^* \in \text{Int}(c_+ \cup c_-) \) |
| | | triangle, \( \Pi^* \in \text{Int}(d) \) |
| \( \Gamma_2 \) | \( \text{Int}(b \cup c_+ \cup c_- \cup d) \) | half-strip \((H, W): \min(-\Pi_1^*, \Pi_1^* + \Pi_2^*) < 4H < \max(0, 2\Pi_2^* + \Pi_1^* - 4\pi), \ W > 0 \) |
| \( \Gamma_3 \) | \( \text{Int}(a) \) | quadrant \((W_1, W_2) \in \mathbb{R}_+^2 \) |
| \( \pm \Gamma_4 \) | \( \text{Int}(b \cup c_{\pm}) \) | cone \((W_1, W_2): 0 < W_1 < W_2 \) |
| \( \pm \Gamma_5 \) | \( \text{Int}(a \cup b \cup c_{\pm}) \) | half-strip |
| \( \pm \Gamma_6 \) | \( \text{Int}(b \cup c_{\pm}) \) | quadrant \((W_1, W_2) \in \mathbb{R}_+^2 \) |

Here \( a, b, c_{\pm} \) and \( d \) are the closed triangles shown in Figure 17, a; \( H \) is the weight of the vertical edge in the upper half-plane; \( W_{1,2} \) are the weights of the vertices in the horizontal subgraph (see Figure 6).

**Example 5.** Consider a section of the 4-simplex \( \mathcal{A}[\Gamma_1] \) in greater detail. We parametrize a fibre \( \Pi^{-1}(\Pi^*) \) of the period mapping by the weights \( H_1 \) and \( H_2 \) of the two vertical edges of the graph in the upper half-plane (see \([\Gamma_6]\) in Figure 6). As the weights of the three vertical edges of \([\Gamma_1]\) on the real (symmetry) axis are positive, we have

\[
0 < 4H_1 < -\Pi_1^*, \quad 4(H_1 + H_2) < \Pi_1^* + \Pi_2^*, \quad 0 < 4H_2 < 4\pi - \Pi_2^*.
\]

Depending on the relations between components of \( \Pi^* \) we obtain either a pentagon a quadrilateral, or a triangle (see Figure 17, b).

The braid group \( \text{Br}_2 = \mathbb{Z} \) acts by covering transformations on the labyrinth space \( L\mathcal{A}_2^1 \). The images of noncanonical lifts of coordinate spaces are related to the images we have just calculated by the action of the Burau representation generated by the matrix

\[
B := \begin{pmatrix}
0 & -1 \\
1 & 2
\end{pmatrix}.
\]
5.3. Fibres on the boundary of a coordinate space. Typically, a fibre of the period mapping intersects transversally the boundaries of the full-dimensional cells which tile a labyrinth space. However, some exceptional fibres may (locally) lie on the boundary. Here we specify conditions for this relatively rare phenomenon.

The period mapping of a dressed coordinate space $\hat{A}[\Gamma]$ lifted to the labyrinth space depends only on the coordinates $H(\cdot)$ in the simplex $\Delta[\Gamma]$.

The conical factor $C[\Gamma]$ of the coordinate space will always be a factor of a local fibre. Hence we have to study the conditions when the restriction of the linear map $\Pi$ to a simplex $\Delta$ has a fibre $f$ of maximum dimension which lies on the boundary of the simplex.

The fibre $f$ belongs to a translated kernel of $\Pi$, hence the inequality

$$\dim \Delta - \dim \Pi \Delta \geq \dim f,$$

which becomes an equality, for instance, when $f$ intersects the (relative) interior of the simplex.

**Lemma 10.** The fibre $f$ has the (maximum possible) dimension equal to the left-hand side of (5.3) if and only if there exists a boundary simplex $\Delta' \subseteq \Delta$ such that $f$ intersects the interior$^5$ of $\Delta'$ and the codimension of this simplex (in $\Delta$) equals the codimension of its image $\Pi \Delta'$ (in $\Pi \Delta$).

**Proof.** 1. Suppose we have found a boundary simplex $\Delta' \subseteq \Delta$ with the required property:

$$\dim \Delta - \dim \Delta' = \dim \Pi \Delta - \dim \Pi \Delta',$$

$^5$By the interior of a 0-simplex we mean the point itself.
and $f$ intersects the relative interior of $\Delta'$. Consider $f' := f \cap \Delta'$, a fibre of $\Pi$ in a smaller simplex. The inequality (5.3) now becomes

$$\dim \Delta' - \dim \Pi \Delta' = \dim f'. \tag{5.4}$$

The two equalities together with the obvious inequality $\dim f' \leq \dim f$ give the inequality reverse to (5.3).

2. Suppose now that we have equality in (5.3). Let $\Delta' \leq \Delta$ be the largest simplex in the boundary such that $f$ intersects its interior (take the convex hull of all simplexes with this property). This simplex contains the whole of $f$. Indeed, a point in $f \setminus \Delta'$ belongs to the interior of a unique simplex $\Delta'' \leq \Delta$ and either the convex hull of $\Delta'$ and $\Delta''$ is larger than $\Delta'$ or $\Delta'' \leq \Delta'$. Subtracting equality (5.3) from (5.4) with $f = f'$ we get the required equality of codimensions.

The lemma is proved.

Now we apply this result to our computations of exceptional 2D fibres:

5.3.1. Cells of the space $\mathcal{H}_2^2$. For $[\Gamma] = [\pm \Gamma_2], [\Gamma_3]$ the simplicial factor $\Delta[\Gamma]$ of the coordinate space is a triangle. The period mapping sends these triangles to the plane $(\Pi_0, \Pi_1)$ without degeneracy so there are fibres of maximum dimension on the boundary. However the boundary of the triangles is the projection of the exterior faces of the appropriate coordinate spaces. The simplex $\Delta[\pm \Gamma_1]$ is three-dimensional and exactly one of its faces satisfies the condition in Lemma 10. However, this face lifts to an outer face of $\mathcal{A}[\pm \Gamma_1]$. Summarizing, all fibres of the period mapping intersect transversally the boundaries of full-dimensional cells of the labyrinth space $\mathcal{L}_2^2$.

5.3.2. Cells of the space $\mathcal{H}_2^1$. For $[\Gamma] = [\Gamma_1], [\Gamma_2]$ the simplicial factor $\Delta[\Gamma]$ of the coordinate space has no boundary simplexes that meet the condition in Lemma 10 (see Figure 17,a). In all the remaining cases such boundary simplexes exist but they correspond to the outer boundary of the coordinate space $\mathcal{A}[\Gamma]$, with two exceptions: $[\Gamma] = [\pm \Gamma_4]$. In each of the latter cases the two codimension-one boundary simplexes satisfy the condition in Lemma 10 and they lift to inner faces of the coordinate space. In the case when $[\Gamma] = [\Gamma_4]$ the exceptional values of the period mapping belong to the interiors of the two sides of the triangle $\Pi \Delta[\Gamma_4]$ incident to $\Pi^* = (0, 0)$. In the case when $[\Gamma] = [-\Gamma_4]$ the interior points in the sides of the triangle $\Pi \Delta[-\Gamma_4]$ incident to $\Pi^* = (-4\pi, 4\pi)$ also lift to 2D-fibres.

§ 6. Polyhedral model of fibres of the period mapping

We have seen that an investigation of fibres of the period mapping requires different instruments at different zooms. At the microscopic level we use differential geometry to show that fibres are smooth. At the mesoscale, linear algebra and convex analysis say that within blocks commensurate with the size of the moduli space, fibres remain cells. At the cosmological scale—in the labyrinth space—we have to use combinatorics to analyse the global structure of fibres, which are composed of glued-together polyhedra.

A decomposition of the labyrinth space $\mathcal{L}$ into polyhedra so that $\Pi(\cdot)$ is a linear map in each of these gives the following natural recipe for the construction of the fibre of the global period mapping above some value $\Pi^*$. 
6.1. The algorithm. Step 1. Given a value $\Pi^*$ and the topological invariants $g$ and $k$ of a real curve, list all braids $\beta \in \text{Br}_{g-k+1}$ satisfying the inclusions\(^6\)

$$\beta \cdot \Pi^* \in \bigcup_{\Gamma} \Pi(\mathcal{L} \mathcal{A}[\Gamma]), \quad \text{codim}[\Gamma] = 0.$$ 

It is technically convenient to consider the closure of the right-hand side of the inclusion, the images of the simplexes $\Delta[\Gamma]$, and then sift out the braids $\beta$ that send the value $\Pi^*$ to the image of the exterior faces of the coordinate space.

Step 2. For all $\beta$ and $\Gamma$ from the above step we find the section of the closed coordinate space $\hat{\mathcal{A}}[\Gamma]$ by a $g$-plane of equal values of the period mapping:

$$\mathcal{A}[\Gamma, \beta \cdot \Pi^*] = \{(H, W) \in \mathcal{A}[\Gamma] : \Pi(H) = \beta \cdot \Pi^*\}.$$ 

For points $\beta \cdot \Pi^*$ lying in the interior of the image of a full-dimensional dressed coordinate space we obtain $g$-dimensional polyhedra; for points $\beta \cdot \Pi^*$ lying in the boundary of such an image we select local fibres of maximum dimension with the help of Lemma 10.

Step 3. Glue the arising $g$-dimensional polyhedra using the same equivalence relations we used in (4.2) to construct the labyrinth space:

$$\Pi^{-1}(\Pi^*) = \bigsqcup_{\beta, [\Gamma]} \beta^{-1} \cdot \mathcal{L} \mathcal{A}[\Gamma, \beta \cdot \Pi^*] \simeq, \quad \text{codim}[\Gamma] = 0.$$ 

6.2. The first step for genus-two curves. For the number of ovals $k = 2, 3$, no braids appear and this step of the algorithm is trivial. For the space $\mathcal{L}_2^1$ we suppose without loss of generality that the value $\Pi^*$ belongs to the image $\Delta := a \cup b \cup c_\pm \cup d$ of the canonical lifts of all full-dimensional cells. Here $a, \ldots, d$ are the closed triangles shown in Figure 17,a.

Lemma 11. The orbit of a point $\Pi^* \in \Delta$ under the action of $\text{Br}_2 = \mathbb{Z}$ has more than one point in $\Delta$ if and only if $\Pi^* \in c_\pm \cup d$. In this case the orbit has exactly one point in each triangle $c_\pm \setminus d$.

Proof. The action of the braid generator on the plane consists in translating the point $\Pi = (\Pi_1, \Pi_2)$ parallel to the line $\{\Pi_1 + \Pi_2 = 0\}$ by twice the Euclidean distance to this line:

$$B(\Pi_1, \Pi_2)^t = (\Pi_1 - (\Pi_1 + \Pi_2), \Pi_2 + (\Pi_1 + \Pi_2))^t.$$ 

Therefore, the intersection of the triangle $\Delta$ with a full orbit of the Burau action of braids on a point $\Pi^*$ in our case has the form

$$\Pi, B\Pi, B^2\Pi, \ldots, B^s\Pi, \quad \Pi \in \Delta, \quad s \geq 0.$$ 

If $s > 0$, then for the first point in this chain we have $B\Pi \in \Delta \not\subset B^{-1}\Pi$, which can be written as

$$\Pi \in B^{-1}\Delta \setminus B\Delta = c_+ \setminus d.$$ 

\(^6\)In the case $g = 2$ that we consider in detail below the number of braids satisfying the inclusion is finite. However, already for $g = 3$ and $k = 1$ the number of suitable braids may be infinite, so one has to glue countably many polyhedra to obtain the model of a fibre.
The last point in the chain satisfies the inclusion
\[ \beta^s \Pi \in B \Delta \setminus B^{-1} \Delta = c_+ \setminus d, \]
and all the other points of the orbit lie in
\[ B^{-1} \Delta \cap B \Delta = d. \]

The lemma is proved.

6.3. The second step: cutting out the patches.

6.3.1. The space \( L^2 \). For a point \( \Pi^* \) in the interior of any of the three triangles \( a_\pm \) and \( b \) in Figure 16 we reconstruct the fibre of the period mapping over it from Table 1. In the interior of the intersections \( a_\pm \cap b \) fibres can be only half-strips lying in \( \mathcal{A}[\pm \Gamma_1] \) as follows from §5.3.1.

6.3.2. The space \( L^1 \). For points \( \beta \cdot \Pi^* \) in the interior of each of the five triangles \( a, b, c_\pm \) and \( d \) in Figure 17 the fibres of the period mapping over them are listed in Table 2. The analysis given in §5.3.2 shows that at the interfaces between the triangles the local fibres of the period mapping are listed in Table 3.

| Range of \( \beta \cdot \Pi^* \) | Polyhedron \( \mathcal{A}[\Gamma] \) | Fibre inside the polyhedron |
|---------------------------------|---------------------------------|-----------------------------|
| \( \text{Int}(a \cap b) \)     | \( \mathcal{A}[\Gamma_1] \)    | rectangle                   |
|                                 | \( \mathcal{A}[\pm \Gamma_5] \) | half-strips                 |
| \( \text{Int}(b \cap c_\pm) \) | \( \mathcal{A}[\Gamma_1] \)    | trapezoid                   |
|                                 | \( \mathcal{A}[\pm \Gamma_5] \) | half-strip                  |
|                                 | \( \mathcal{A}[\pm \Gamma_4] \), \( \mathcal{A}[\pm \Gamma_5] \), \( \partial_{\text{int}} \mathcal{A}[\pm \Gamma_4] \) | sectors                     |
| \( \text{Int}(c_\pm \cap d) \) | \( \mathcal{A}[\Gamma_1] \)    | triangle                    |
|                                 | \( \mathcal{A}[\Gamma_2] \)    | half-strip                  |
|                                 | \( \partial_{\text{int}} \mathcal{A}[\pm \Gamma_4] \) | sectors                     |
| \( b \cap d \)                 | \( \mathcal{A}[\Gamma_1] \)    | triangle                    |
|                                 | \( \mathcal{A}[\Gamma_2] \), \( \mathcal{A}[-\Gamma_2] \) | half-strip                  |
|                                 | \( \partial_{\text{int}} \mathcal{A}[\Gamma_4] \), \( \partial_{\text{int}} \mathcal{A}[-\Gamma_4] \) | sectors                     |

6.4. The third step: the patchwork. Having prepared the polygons \( \mathcal{A}[\Gamma, \beta \cdot \Pi^*] \) labelled by a graph and a braid each, we assemble them together using the identification rule from (4.2). For convenience, we first glue the polygons labelled by the same braid \( \beta \). Typically, we identify sides of different patches if and only if these sides are labelled by the same graph \( [\Gamma] \) of positive codimension. For the space \( L^3 \), which consists of a unique coordinate space, the job was already done in §5.2.1: the fibre is always an open quadrant.
6.4.1. Assembling fibres of the space $L^2_2$. Since the braids are trivial in the case of genus-two curves with two real ovals, the final answer is shown in Figure 18 for the point $\Pi^*$ lying in the interior of the triangles $a_\pm$ and $b$. On interfaces between these triangles a global fibre of the period mapping reduces to an open half-strip.

![Diagram](image1)

**Figure 18.** a) The intersection of the fibre $\Pi^{-1}(\Pi^*)$ with a full-dimensional cell of the space $L^2_2$ when $\Pi^* \in \text{Int}(a_+), \text{Int}(b), \text{Int}(a_-)$; b) the graphs on the interface between the full-dimensional cells $[\pm \Gamma_{12}] < [\pm \Gamma_1, \Gamma_2]$ and $[\pm \Gamma_{13}] < [\pm \Gamma_1, \Gamma_3]$.

6.4.2. Assembling fibres of the space $L^1_2$. First we assemble the patches $\mathcal{A}[\Gamma, \beta \cdot \Pi^*]$ with the same value of $\beta \cdot \Pi^*$ which lies in the interior of the triangles $a$, $b$, $c_\pm$ or $d$ shown in Figure 17 (see Figures 19 and 20).

Fibres on the interfaces between the triangles are as follows:

- $\Pi^* \in \text{Int}(a \cap b)$: the same picture as in the left-hand panel of Figure 19, a but without the patch corresponding to $\Gamma_3$; the whole boundary is exterior.
- $\Pi^* \in \text{Int}(b \cap c_\pm)$: the same picture as in the right-hand panel of Figure 19, a but without the two patches corresponding to $[\mp \Gamma_5]$ and $[\mp \Gamma_6]$ and with a trapezoid block instead of the pentagonal one corresponding to $[\Gamma_1]$. The patch corresponding to $[\mp \Gamma_4]$ lies on the boundary of the coordinate space and one side of this patch is the outer boundary.
- $\Pi^* \in \text{Int}(c_\pm \cap d)$: from a fibre over $\text{Int}(c_\pm)$ (Figure 20, the left-hand and central panels) we remove the two patches corresponding to $[\pm \Gamma_5]$ and $[\pm \Gamma_6]$ and take a triangular block instead of the trapezoid one corresponding to $[\Gamma_1]$. The patch corresponding to $[\pm \Gamma_4]$ lies on the boundary of the coordinate space and one side of this patch is an outer boundary.
- $\Pi^* \in b \cap d$: from a fibre over $\text{Int}(b)$ (Figure 19, a, the right-hand panel) we remove the four removed patches corresponding to $[\pm \Gamma_5]$, $[\pm \Gamma_6]$ and take a triangular block instead of the pentagonal one corresponding to $[\Gamma_1]$. The whole of the boundary of this patchwork is outer.

For the value $\Pi^*$ of the period mapping in $a \cup b$ the global fibre is ready. For the remaining cases $\Pi^* \in c_\pm \cup d$ one has to glue together pieces with different braid labels. The result of this operation is shown in Figure 21.

Summarizing, we see that any global 2D fibre of the period mapping is a cell, as we stated in the main theorem, Theorem 2.
Figure 19. a) The intersection of the fibre $\Pi^{-1}(\Pi^*)$ with full-dimensional cells of the space $\mathcal{L}_2$ when $\Pi^* \in \text{Int } a$, $\text{Int } b$; b) a meeting point of several patches, $[\Gamma_{13\pm 5}] < [\Gamma_1], [\Gamma_3], [\Gamma_{\pm 5}]$ and $[\Gamma_{12456}] < [\Gamma_1], [\Gamma_2], \ldots, [\Gamma_6]$.

Figure 20. The intersection of the fibre $\Pi^{-1}(\Pi^*)$ with the canonical lifts of full-dimensional cells of the space $\mathcal{L}_2$ when $\Pi^* \in \text{Int } c_+, \text{Int } c_-, \text{Int } d$ (from left to right).

Figure 21. The large patchwork for $\Pi^* \in c_+ \cup c_- \cup d$. 
§ 7. Conclusion

We have developed the effective combinatorial approach to the study of the period mapping proposed in [8] and [9], Chs. 4 and 5. We have proved that the global 2D fibres of the period mapping in 4D labyrinth spaces have the simplest topology. Basically, the same technique can be applied to higher-dimensional moduli spaces. The broad use of combinatorial techniques is vital already for the investigation of 3D fibres. Indeed, the number of full-dimensional cells in the decomposition of the 6D moduli space $\mathcal{H}_3^1$ is too large to consider fibres in each cell and then glue them together. One has to use larger ‘building blocks’, say clusters of coordinate spaces with nonexceptional graphs $\Gamma$. As we have mentioned already, a new effect appears in the 3D world: fibres may become disconnected, however each component remains a cell. We are going to describe the 3D calculations in a forthcoming publication.

Bibliography

[1] V.I. Arnol’d, S.M. Gusein-Zade and A.N. Varchenko, *Singularities of differentiable maps*, vol. 2: *Monodromy and asymptotics of integrals*, Nauka, Moscow 1984, 336 pp.; English transl., Reprint of the 1988 transl., Mod. Birkhäuser Class., Birkhäuser/Springer, New York 2012, x+492 pp.

[2] Yu. Baryshnikov, “Bifurcation diagrams of quadratic differentials”, *C. R. Acad. Sci. Paris Sér. I Math.* 325:1 (1997), 71–76.

[3] M. Bertola, “Boutroux curves with external field: equilibrium measures without a variational problem”, *Anal. Math. Phys.* 1:2–3 (2011), 167–211.

[4] J.S. Birman, *Braids, links and mapping class groups*, Ann. of Math. Stud., vol. 82, Princeton Univ. Press, Princeton, NJ; Univ. of Tokyo Press, Tokyo 1975, ix+228 pp.

[5] A. Bogatyrev, “Fibers of periods map are cells?”, *J. Comp. Appl. Math.* 153:1–2 (2003), 547–548.

[6] A.B. Bogatyrev, “Effective approach to least deviation problems”, *Mat. Sb.* 193:12 (2002), 21–40; English transl. in *Sb. Math.* 193:12 (2002), 1749–1769.

[7] A.B. Bogatyrev, “Representation of moduli spaces of curves and calculation of extremal polynomials”, *Mat. Sb.* 194:4 (2003), 3–28; English transl. in *Sb. Math.* 194:4 (2003), 469–494.

[8] A.B. Bogatyrev, “Combinatorial description of a moduli space of curves and of extremal polynomials”, *Mat. Sb.* 194:10 (2003), 27–48; English transl. in *Sb. Math.* 194:10 (2003), 1451–1473; “Errata” 194:12 (2003), 1899.

[9] A. Bogatyrev, *Extremal polynomials and Riemann surfaces*, Moscow Center for Continuous Mathematical Education, Moscow 2005, 173 pp.; English transl., Springer Monogr. Math., Springer, Heidelberg 2012, xxvi+150 pp.

[10] V.P. Burskii and A.S. Zhedanov, “On Dirichlet, Poncelet and Abel problems”, *Commun. Pure Appl. Anal.* 12:4 (2013), 1587–1633.

[11] V.V. Fock and L.O. Chekhov, “A quantum Teichmüller space”, *Teor. Mat. Fiz.* 120:3 (1999), 511–528; English transl. in *Theor. Math. Phys.* 120:3 (1999), 1245–1259.

[12] P. Dehornoy, I. Dynnikov, D. Rolfsen and B. Wiest, *Ordering braids*, Math. Surveys Monogr., vol. 148, Amer. Math. Soc., Providence, RI 2008, x+323 pp.

[13] V. Dragović and M. Radnović, *Poncelet porisms and beyond. Integrable billiards, hyperelliptic Jacobians and pencils of quadrics*, Front. Math., Birkhäuser/Springer Basel AG, Basel 2011, viii+293 pp.
[14] A. Frolova and A. Vasil’ev, “Combinatorial description of jumps in spectral networks defined by quadratic differentials”, Proc. Amer. Math. Soc. (to appear); arXiv:1509.00674.

[15] S. Grushevsky and I. Krichever, “The universal Whitham hierarchy and the geometry of the moduli space of pointed Riemann surfaces”, Geometry of Riemann surfaces and their moduli spaces, Surv. Differ. Geom., vol. 14, Int. Press, Somerville, MA 2009, pp. 111–129.

[16] S. Grushevsky and I. Krichever, Foliations on the moduli space of curves, vanishing in cohomology, and Calogero-Moser curves, arXiv:1108.4211v1.

[17] M. L. Kontsevich, “Intersection theory on the moduli space of curves”, Funktsional. Anal. i Prilozhen. 25:2 (1991), 50–57; English transl. in Funct. Anal. Appl. 25:2 (1991), 123–129.

[18] I. M. Krichever and D. H. Phong, “On the integrable geometry of soliton equations and \( N = 2 \) supersymmetric gauge theories”, J. Differential Geom. 45:2 (1997), 349–389; arXiv:hep-th/9604199.

[19] A. B. J. Kuijlaars and Man Yue Mo, “The global parametrix in the Riemann-Hilbert steepest descent analysis for orthogonal polynomials”, Comput. Methods Funct. Theory 11:1 (2011), 161–178.

[20] R. C. Penner, “The decorated Teichmüller space of punctured surfaces”, Comm. Math. Phys. 113:2 (1987), 299–339.

[21] A. Yu. Solynin, “Quadratic differentials and weighted graphs on compact surfaces”, Analysis and mathematical physics, Trends Math., Birkhäuser, Basel 2009, pp. 473–505.

[22] K. Strebel, Quadratic differentials, Ergeb. Math. Grenzgeb. (3), vol. 5, Springer-Verlag, Berlin 1984, xii+184 pp.

[23] V. A. Vassiliev, Ramified integrals, Moscow Center for Continuous Mathematical Education, Moscow 2000, 432 pp.; Extended English transl., V. A. Vassiliev, Ramified integrals, singularities and lacunas, Math. Appl., vol. 315, Kluwer Acad. Publ., Dordrecht 1995, xviii+289 pp.