ROUGH METRICS ON MANIFOLDS AND QUADRATIC ESTIMATES

LASHI BANDARA

Abstract. We study the persistence of quadratic estimates related to the Kato square root problem across a change of metric on smooth manifolds by defining a class of Riemannian-like metrics that are permitted to be of low regularity and degenerate on sets of measure zero. We also demonstrate how to transmit quadratic estimates between manifolds which are homeomorphic and locally bi-Lipschitz. As a consequence, we demonstrate the invariance of the Kato square root problem under Lipschitz transformations of the space and obtain solutions to this problem on functions and forms on compact manifolds with a continuous metric. Furthermore, we show that a lower bound on the injectivity radius is not a necessary condition to solve the Kato square root problem.

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1. INTRODUCTION

Quadratic estimates are instrumental in the study of functions of bi-sectorial operators as these estimates are equivalent to the existence of a bounded holomorphic functional calculus. These operators capture an important class of partial differential operators and quadratic estimates provide a quantitative mechanism by which to analyse them. The survey papers [1] by Albrecht, Duong, and McIntosh, [2] by Auscher, Axelsson (Rosén), and McIntosh, and [16] by Hofmann and McIntosh are excellent sources which illustrate the virtues of quadratic estimates and their application to partial differential equations.

In this paper, we will be concerned with quadratic estimates associated to the Kato square root problem, which is the problem of determining the domains of square roots of uniformly elliptic second-order divergence-form differential operators. This problem on \(\mathbb{R}^n\) was solved in 2002 by Auscher, Hofmann, Lacey, McIntosh and

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Tchamitchian in [8]. This result was recaptured in a first-order framework in 2005 by Axelsson (Rosén), Keith, and McIntosh in [5]. In the same paper, the authors also solve the Kato square root problem for differential forms on $\mathbb{R}^n$, as well as similar problems on compact manifolds.

The AKM approach has been successfully adapted to solve a range of problems since its conception. In [4], the same authors tackle the Kato square root problem for domains with mixed boundary values, in [21] Morris adapts this framework to solve the Kato square root problem for Euclidean submanifolds with second fundamental form bounds, and in [8], McIntosh and the author solve Kato square root problems on manifolds under mild assumptions on the geometry. This AKM technology has also been applied to the study of square roots of sub-elliptic operators on Lie groups by ter Elst, McIntosh and the author in [9]. More recently, this framework has been adapted by Leopardi and Stern to study numerical problems in [19].

The central theme of this paper is to reveal an important connection between the study of quadratic estimates and the study of Kato square root problems on manifolds with non-smooth metrics. Our philosophy is to encode the lack of regularity of the "rough" metric into a rough coefficient operator on a nearby smooth metric. We emphasise that the first-order perspective of the AKM framework is of paramount importance in our work.

We remark that this philosophy is not necessarily our own insight and that we have borrowed it in large parts from the authors of [5]. They dedicate an entire section in their paper to deal with the holomorphic dependency of the functional calculus and show Lipschitz estimates for the calculus when the metric on a compact manifold is perturbed slightly. The main novelty here is that we allow our metrics to be of very low regularity and even degenerate on null measure sets.

We also derive motivation from [4], where the authors exploit the invariance of quadratic estimates under Lipschitz transforms to solve Kato square root problems on Lipschitz domains. However, since a metric on a manifold is the key global object which determines the underlying geometry, we concern ourselves with the study of non-smooth metrics, rather the special case of such transforms.

To increase the readability of this paper, we give an overview of our achievements section by section. In §2, we introduce our notation and provide an overview of the aspects of manifolds which are of importance to us, including their Lebesgue space theory. In particular, we outline the measure theoretic notions that will be of use to us in order to talk about degeneracy and low regularity in subsequent sections. For the benefit of the reader, we also present an overview of the AKM framework.

Following onto §3, we formulate a notion of a rough metric (Definition 3.1), which is a Riemannian-like metric that may not only be low in its regularity, but which may degenerate on large, non-trivial, but null measure sets. We also consider Lebesgue and Sobolev space theory for such metrics.
Then, in §4, we demonstrate how to reduce Kato square root problems on rough metrics to similar problems on a nearby smooth metric. A perspective that emerges from our developments in this section is that the Kato square root problems which concern us are dependent more on the measure than the induced distance metric. We also use the technology we build to show that the Kato square root problem on compact manifolds can always be solved in the affirmative when the metric is only assumed to be continuous (Theorem 4.9).

We dedicate §5 to the study of transmitting quadratic estimates between two manifolds. To make this study sufficiently general and non-trivial, we assume that the map between these two manifolds is a homeomorphism that is locally bi-Lipschitz. We show that if one manifold has a smooth metric and admits quadratic estimates, then so does the other in the induced, low regularity pullback metric (Theorem 5.15). As a consequence, we obtain that the Kato square root problem for functions is invariant under Lipschitz transformations of the geometry (Theorem 5.17).

Lastly, in §6, we consider the question posed by McIntosh as to whether a lower bound on the injectivity radius is necessary to solve Kato square root problems. This question is motivated by the fact that, in every setting where this problem is solved, lower bounds on injectivity radius are present. The most geometrically general such theorem (in [8]) reveals how this technicality assists in the proof. In this section, we answer this question in the negative by demonstrating the existence of smooth metrics on $\mathbb{R}^2$ with zero injectivity radius arbitrarily close to the Euclidean metric. We then apply the tools we have developed in §4 to perturb the Euclidean solution to a solution on the nearby metric (Theorem 6.7). More seriously, we also show that such solutions are abundant in all dimensions greater than two by constructing metrics with zero injectivity radius that are smooth everywhere but a point and which are arbitrarily close to any metric for which we can solve the Kato square root problem (Theorem 6.10).

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2. Preliminaries

2.1. Notation. Throughout this paper, we assume the Einstein summation convention. That is, whenever a raised index appears against a lowered index, we sum over that index, unless specified otherwise.

For a function or section \( f \) mapping into a topological vector space, we denote its support, by \( \text{spt} f = \{ x : f(x) \neq 0 \} \), the closure of the set of points that are non-zero under \( f \).

By \( L^p \) we denote the \( L^p \) spaces in the given context. Typically, we will write \( L^p(V, h) \) to denote the \( L^p \) space over the vector bundle \( V \) with metric \( h \). In writing \( C^{k,\alpha} \) we mean \( k \) times differentiable objects that are \( \alpha \)-Hölder continuous in the \( k \)-th derivative. When \( \alpha = 0 \), we simply write \( C^k \). The subspace of objects with compact support are denoted by \( C_c^{k,\alpha} \) and \( C_c^k \) respectively.

We reserve the symbol \( \mathcal{H} \) to denote a Hilbert space. By an operator \( \Gamma : \mathcal{H} \to \mathcal{H} \), we mean a linear map defined on a subset \( D(\Gamma) \subset \mathcal{H} \) which is the domain of the operator. The null space and range of this operator are denoted by \( \mathcal{N}(\Gamma) \) and \( \mathcal{R}(\Gamma) \) respectively. The algebra of bounded linear operators are then given by \( \mathcal{L}(\mathcal{H}) \).

In our analysis, we often write \( a \lesssim b \) to mean that \( a \leq Cb \), where \( C \) is some constant. The dependencies of \( C \) will either be explicitly specified or otherwise, clear from context. By \( a \asymp b \) we mean that \( a \lesssim b \) and \( b \lesssim a \).

Unless specified otherwise, we will reserve the symbol \( \delta \) to denote the standard inner product on \( \mathbb{R}^n \). The Lebesgue measure on \( \mathbb{R}^n \) will be denoted by \( \mathcal{L} \).

2.2. Manifolds and their Lebesgue spaces. The fundamental objects that lie at the heart of our study in this paper will consist of a pair \( (\mathcal{M}, g) \) where \( \mathcal{M} \) is manifold and \( g \) is a metric on \( \mathcal{M} \). By an \( n \)-manifold \( \mathcal{M} \), we will always mean a topological space that is second-countable, Hausdorff, and locally homeomorphic to open subsets of \( \mathbb{R}^n \). These homeomorphisms are coordinate charts on \( \mathcal{M} \) and their collection is called an atlas or differentiable structure. If the homeomorphisms are \( C^{k,\alpha} \)-diffeomorphisms, by which we mean that compositions of one chart with the inverse of another (as long as their domains have nonempty intersection) is a \( C^{k,\alpha} \)-diffeomorphism, then we call \( \mathcal{M} \) a \( C^{k,\alpha} \) manifold. A \( C^0 \) manifold is called a topological manifold, a manifold that is \( C^{0,1} \) is called Lipschitz, and \( C^\infty \) manifold is said to be smooth. The geometry of a manifold is dictated by the metric and it is to metrics that we shall devote significant attention throughout this paper.

Our wish is to study geometries that contain singularities. The word ‘singularity’ is used to mean many different things, but to us, it can mean one of two things: either
singularities arising from within the differentiable structure of $\mathcal{M}$, or singularities arising from a lack of regularity in $g$.

If we were to pursue the former notion, the most singular structure we can consider is $C^0$. It is difficult to envisage setting up analysis in this situation. However, due to a theorem of Sullivan (see [23]), it turns out that in all dimensions but 4, every topological manifold can be made into a Lipschitz manifold! This does not improve our situation much - simply because changes of coordinates for vectorfields locally involves derivatives of charts, and if the charts are at best bi-Lipschitz, then the vectorfields can be at most measurable in regularity. Consequently, analysis on Lipschitz manifolds is both difficult and limited, but remarkably, Gol’dshtein, Mitrea and Mitrea makes considerable progress setting up analysis on compact Lipschitz manifolds in [12].

Given what we have just said, and since we would like to measure varying degrees of regularity beyond simply Lipschitz or measurable in order to reduce low regularity problems to smooth ones, we are forced to confine our attention to higher regularity differentiable structures. It is reasonable to expect that we will have to consider the regularity classes $C^{k,\alpha}$ ($k \geq 1$, $\alpha \in [0, 1]$) separately. However, a classical result of Whitney (see [24]) states that every $C^1$ structure (and hence, every $C^{k,\alpha}$ ($k \geq 1$) manifold) can be smoothed. This exactly means that the manifold can be given a smooth atlas compatible with the initial $C^{k,\alpha}$ ($k \geq 1$) atlas. Let us remark that this is false for topological manifolds by a counterexample due to Kervaire in [17]. Thus, from this point onwards, without the loss of generality, we assume that $\mathcal{M}$ is smooth, and our efforts are concentrated on studying situations where the metric $g$ is typically of lower regularity.

Recall that for a $C^k$ ($k \geq 0$) metric $g$, the length functional of an absolutely continuous curve $\gamma : I \to \mathcal{M}$ is given by

$$\ell_g(\gamma) = \int_I |\dot{\gamma}(t)|_{g(\gamma(t))} \, dt,$$

where $|u|_{g(x)} = \sqrt{g_x(u, u)}$. The induced distance is given by

$$\rho_g(x, y) = \inf \{ \ell_g(\gamma) : \gamma(0) = x, \gamma(1) = y, \gamma \text{ abs. cts.} \}.$$

We remark that for $C^k$ metrics with $k \geq 0$, Burtscher demonstrates through Theorem 3.11 and Corollary 3.13 in her paper [10] that one can equivalently consider curves that are piecewise smooth.

A curve $\gamma : I \to \mathcal{M}$ is said to be a geodesic for $\rho_g$ if for each $t \in I$, there is a sub-interval $J_t \subset I$ with $t \in J_t$ such that for every $t_1, t_2 \in J_t$, $\rho_g(\gamma(t_1, t_2)) = |t_1 - t_2|$. This is a metric-space formulation of geodesy. A more geometric characterisation can be given to metrics $g$ of class $C^2$ or higher. Recall, that in this situation there exists the Levi-Cevita connection $\nabla$ with respect to $g$. Then, $\gamma$ is a geodesic if and only $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. The latter notion captures that the curve is lacking tangential acceleration and hence, is in free-fall. A curve $\gamma$ is said to be a minimising geodesic if $J_t = I$. It is easy to see that every geodesic is locally minimising, and that a minimising geodesic realises the shortest distance between the end points of the curve.
For a metric that is $C^2$ at a point $x \in \mathcal{M}$, the exponential map $\exp_x : V_x \subset T_x \mathcal{M} \to \mathcal{M}$ takes a velocity vector $v \in V_x$ as input and outputs the end point of the unique unit-length parametrised geodesic $\gamma_v$ of velocity $v$. For each such point $x$, there is an $r_x > 0$ such that $\exp_x : B_{r_x}(0) \subset T_x \mathcal{M} \to \mathcal{M}$ is a homeomorphism with its image. The largest such radius $r_x$ is then the injectivity radius $\text{inj}(\mathcal{M}, g, x)$ at the point $x \in \mathcal{M}$. The injectivity radius over the entire manifold is then given by $\text{inj}(\mathcal{M}, g) = \inf_{x \in \mathcal{M}} \text{inj}(\mathcal{M}, g, x)$.

A metric $g$ also induces a canonical volume measure on $\mathcal{M}$. It is constructed by pasting together the local expression

$$d\mu_g(x) = \sqrt{\det g(x)} \, dL(x),$$

via a smooth partition of unity subordinate to a covering of $\mathcal{M}$ by charts. This construction is readily checked to be well-defined.

In the analysis on manifolds with non-smooth metrics, it will be convenient for us to have certain measure-theoretic notions that can be formulated independently of a metric. In this spirit, we define the following.

**Definition 2.1** (Notions of measure). We say that:

(i) a set $A \subset \mathcal{M}$ is measurable if whenever $(U, \psi)$ is a chart satisfying $U \cap A \neq \emptyset$, then $\varphi(U \cap A) \subset \mathbb{R}^n$ is $\mathcal{L}$-measurable,

(ii) a function $f : \mathcal{M} \to \mathbb{C}$ is measurable if $f \circ \psi^{-1} : \psi(U) \to \mathbb{C}$ is $\mathcal{L}$-measurable for each chart $(U, \psi)$,

(iii) a tensor field $T : \mathcal{M} \to \mathcal{T}^{(r,s)} \mathcal{M}$ is measurable if the coefficients $T_{i_1, \ldots, i_r}^{j_1, \ldots, j_s}$ in each $(U, \psi)$ is measurable

(iv) a set $Z$ is a null set or set of null measure if requiring $\mathcal{L}(\varphi(U \cap Z)) = 0$ for each chart $(U, \psi)$,

(v) a property $P$ is valid almost-everywhere if it is valid $\mathcal{L}$-a.e. in each coordinate chart $(U, \psi)$.

In what is to follow, we always identify the $(r, s)$-tensors, the tensors of covariant rank $r$ and contravariant rank $s$, as a complex vector bundle after complexification of its real counterpart. The set of measurable sections for such tensors are then denoted by $\Gamma(\mathcal{T}^{(a,b)} \mathcal{M})$.

While we have formulated these notions independently of a metric, we can indeed recapture these definitions in terms of a background metric. This yields a global, geometric and coordinate independent perspective. But first, we require the following lemma concerning the preservation of measurability. We phrase this result in a far more general context than we need here because this generality will be of use to us later. Note that in the following proposition, we take the measure $\sigma$-algebra to be the maximal one. That is, a set $M \subset X$ is $\mu$-measurable if and only for every set $A \subset X$, $\mu(A) = \mu(A \setminus M) + \mu(A \cap M)$.

**Lemma 2.2.** Let $(X, \nu)$ be a measure-space and let $f : X \to [0, \infty]$ be a $\nu$-measurable function such that $0 < f(x) < \infty$ for $x$ $\nu$-a.e. Suppose that $d\mu(x) = f(x)d\nu(x)$ for
x \nu\text{-a.e. Then, a set } M \subset X \text{ is } \nu\text{-measurable if and only if it is } \mu\text{-measurable and } \mu \text{ and } \nu \text{ share the same sets of measure zero.}

Proof. Suppose that } M \subset X \text{ is } \nu\text{-measurable and } A \subset X \text{ is any other set. We show that } \mu(A) = \mu(A \cap M) + \mu(A \setminus M).

First, assume that } f \text{ is a simple function and let } f = \sum_{k=1}^{m} \chi_{F_k}. \text{ Then,}

\[ \mu(A) = \int_{A} \sum_{k=1}^{m} \chi_{F_k} \, d\nu(x) = \sum_{k=1}^{m} \nu(A \cap F_k). \]

Similarly, \( \mu(A \cap M) = \sum_{k=1}^{m} \nu(A \cap F_k \cap M) \) and since \( (A \setminus M) \cap F_k = (A \cap F_k) \setminus M, \mu(A \setminus M) = \sum_{k=1}^{m} \nu((A \cap F_k) \setminus M) \). By the \( \nu\)-measurability of \( M, \nu(A \cap F_k) = \nu(A \cap F_k \cap M) + \nu((A \cap F_k) \setminus M) \) which shows that \( \mu(A) = \mu(A \cap M) + \mu(A \setminus M). \)

Now, since } f \text{ is assumed to be non-negative and } \nu\text{-measurable, there exists a sequence of simple functions } f_n \text{ monotonically increasing to } f. \text{ Thus, by the monotone convergence theorem,}

\[ \mu(A) = \int_{A} f \, d\nu = \lim_{n \to \infty} \int_{A} f_n \, d\nu \]

and by what we have established previously, \( \int_{A} f_n \, d\nu = \int_{A \cap M} f_n \, d\nu + \int_{A \setminus M} f_n \, d\nu. \) Therefore, \( \mu(A) = \mu(A \cap M) + \mu(A \setminus M) \). Furthermore, it is easy to see that \( \nu(Z) = 0 \) implies \( \mu(Z) = 0. \)

Next, note that

\[ \left( \frac{1}{f} \right)^{-1}(\alpha, \infty) = \left\{ x \in X : \frac{1}{f(x)} > \alpha \right\} = \left\{ x \in X : f(x) < \alpha \right\} = f^{-1}[-\infty, \alpha]. \]

So \( 1/f \) is \( \nu\)-measurable which, by what we have just established, \( \mu\)-measurable. Also since \( 0 < f < \infty \text{-a.e.}, \) which implies \( \mu\text{-a.e.}, \) we can write \( d\nu(x) = 1/f(x) \, d\mu(x) \) for \( x \text{-a.e.} \) Thus, by repeating the previous argument with \( \nu \) and \( \mu \) interchanged, we obtain that a \( \mu\)-measurable set \( M \) is \( \nu\)-measurable and that whenever \( \mu(Z) = 0 \) implies \( \nu(Z) = 0. \)

This lemma then allows us to prove the following geometric rephrasing of our measure notions.

**Proposition 2.3.** Let \( \mathcal{M} \) be a smooth manifold, \( g \) a continuous metric, and \( \mu_g \) the induced volume measure. Then,

(i) a set \( M \subset \mathcal{M} \) is measurable if and only if \( M \) is \( \mu_g\)-measurable,
(ii) a function } f : \mathcal{M} \rightarrow \mathbb{C} \text{ is measurable if and only if it is } \mu_g\text{-measurable,
(iii) a set } Z \subset \mathcal{M} \text{ is a null set if and only if } \mu_g(Z) = 0,
(iv) a property } P \text{ holds a.e. in } \mathcal{M} \text{ if and only if it holds } \mu_g\text{-a.e.}
Proof. Take a covering of \( \mathcal{M} \) by coordinate charts, each of which have compact closure. Inside each chart, we can apply Lemma 2.2 with \( f = \sqrt{\det g} \). The conclusion then follows immediately.

Let us now fix a continuous metric \( g \) on our smooth manifold \( \mathcal{M} \). The \( L^p(\mathcal{T}^{(r,s)} \mathcal{M}, g) \) for \( 1 \leq p < \infty \) is defined as the space of sections \( \xi \in \Gamma(\mathcal{T}^{(r,s)} \mathcal{M}) \) such that

\[
\int_{\mathcal{M}} |\xi(x)|_{g(x)}^p \, d\mu(x) < \infty.
\]

The \( L^p \) norm is then simply the quantity

\[
\|\xi\|_p = \left( \int_{\mathcal{M}} |\xi(x)|_{g(x)}^p \, d\mu(x) \right)^{\frac{1}{p}}.
\]

Similarly, \( L^\infty(\mathcal{T}^{(r,s)} \mathcal{M}, g) \) consist of sections \( \xi \in \Gamma(\mathcal{T}^{(r,s)} \mathcal{M}) \) such that there exists \( C > 0 \) satisfying \( |\xi(x)|_{g(x)} \leq C \) for \( x \)-a.e. Then,

\[
\|\xi\|_\infty = \inf \{ C : |\xi(x)|_{g(x)} \leq C \text{ x-a.e.} \}.
\]

By \( W^{1,p}(\mathcal{M}, g) \), we denote the \( L^p \) Sobolev space over functions. That is, we let \( S_p = \{ u \in C^\infty \cap L^p : \nabla u \in L^p \} \) and define \( W^{1,p}(\mathcal{M}, g) \) as the closure of \( S_p \) with respect to the Sobolev norm

\[
\|u\|_{W^{1,p}} = \|u\|_p + \|\nabla u\|_p.
\]

The space \( W^{1,p}_0(\mathcal{M}, g) \) is defined by closing \( C^\infty_c(\mathcal{M}) \) under the same norm. Note that in general, \( W^{1,p}_0(\mathcal{M}, g) \neq W^{1,p}(\mathcal{M}, g) \). However, the following is true.

**Proposition 2.4.** The operator \( \nabla_p : C^\infty \cap L^p \rightarrow C^\infty \cap L^p \) is closable in \( L^p \), as is the operator \( \nabla_c : C^\infty_c(\mathcal{M}) \rightarrow C^\infty_c(\mathcal{M}) \). Moreover, \( W^{1,p}(\mathcal{M}, g) = D(\nabla_p) \) and \( W^{1,p}_0(\mathcal{M}, g) = D(\nabla_c) \).

**Proof.** Since \( \nabla_c \subset \nabla_p \), it is enough to show that \( \nabla_p \) is closable. For this, all we need to show is that whenever \( u_n \in S_p \) with \( u_n \to 0 \) and \( \nabla_p u_n \to v \), then \( v = 0 \).

Fix a chart \( (U, \psi) \) such that \( \overline{U} \) is compact and note that \( \nabla_p u_n \to v \) implies that \( \nabla_p u_n \to v \) in \( L^p(\overline{U}, g) \). Then,

\[
\int_{\overline{U}} |\nabla u_n - v|^p_{g} \, d\mu_g = \int_{\psi(U)} |\psi^{-1*} \nabla u_n - \psi^{-1*} v|^p \sqrt{\det g} \, d\mathcal{L} \\
\geq \left( \text{essinf}_{x \in \psi(U)} \sqrt{\det g(x)} \right) \int_{\psi(U)} |\nabla \psi^{-1*} u_n - \psi^{-1*} v| \, d\mathcal{L}.
\]

The essinf_{x \in \psi(U)} \sqrt{\det g(x)} > 0 since \( \overline{\psi(U)} \) is compact and therefore, \( \nabla_p u_n \to v \) in \( L^p(\overline{U}, g) \) implies that \( \nabla_p \psi^{-1*} u_n \to \psi^{-1*} u \) in \( L^p(\psi(U), \mathcal{L}) \). But \( \nabla \) here is the exterior derivative which commutes with pullbacks, and hence, since \( d \) is closable in \( \mathbb{R}^n \), we have that \( \psi^{-1*} v = 0 \) in \( \varphi(U) \). Then, \( v = 0 \) in \( U \) and since we can cover the manifold \( \mathcal{M} \) by countably many such charts \( (U, \psi) \), we obtain that \( v = 0 \). That the Sobolev spaces can be equated with the domains of the closure of the relevant operator is then immediate. \( \square \)
In the more specific case of a complete metric, we obtain that the two Sobolev spaces are equal.

**Proposition 2.5.** Suppose that $g$ is a continuous metric that is complete. Then, $W_{0}^{1,p}(\mathcal{M}, g) = W^{1,p}(\mathcal{M}, g)$.

**Proof.** Fix $\varepsilon > 0$ and note that since $g$ is a continuous and complete metric, there exists $R_{\varepsilon} > 0$ such that whenever $r \geq R_{\varepsilon}$, there exists a $\rho_{g}$-Lipschitz cutoff function $\varphi_{r} : \mathcal{M} \to [0, 1]$ such that $\varphi_{r} = 1$ on $B_{r}$, spt $\varphi_{r}$ is compact and $|\nabla \varphi_{r}|^{p} = (\text{Lip } \varphi_{r})^{p} < \varepsilon$ almost-everywhere.

Now, let $u \in W^{1,p}(\mathcal{M}, g)$ and note that

$$
\hat{\int}_{\mathcal{M}} (|u|^{p} + |\nabla u|^{p}) \, d\mu_{g} = \lim_{r \to \infty} \int_{B_{r}} (|u|^{p} + |\nabla u|^{p}) \, d\mu_{g}.
$$

As a consequence, note that we can find another $R'_{\varepsilon} > 0$ large enough so that whenever $r \geq R'_{\varepsilon}$,

$$
\int_{\mathcal{M} \setminus B_{r}} |u|^{p} \, d\mu_{g} < \varepsilon \quad \text{and} \quad \int_{\mathcal{M} \setminus B_{r}} |\nabla u|^{p} \, d\mu_{g} < \varepsilon.
$$

Also note that $W^{1,p}(\mathcal{M}, g)$ remains invariant under multiplication by bounded Lip-}
schitz functions. Therefore, upon taking $R$ to be the larger of $R_{\varepsilon}$ and $R'_{\varepsilon}$, we note that for almost-every $x \in \mathcal{M}$

$$
|\nabla u - \nabla (\varphi_{r} u)|^{p} \lesssim |\nabla u - \varphi_{r} \nabla u|^{p} + |u \nabla \varphi_{r}|^{p},
$$

and that

$$
\int_{B_{r}} |\nabla u - \varphi_{r} \nabla u|^{p} \, d\mu_{g} = 0.
$$

Furthermore,

$$
\int_{\mathcal{M} \setminus B_{r}} (1 - \varphi_{r}) |\nabla u|^{p} \, d\mu_{g} \leq \int_{\mathcal{M} \setminus B_{r}} |\nabla u|^{p} \, d\mu_{g} < \varepsilon,
$$

and

$$
\int_{\mathcal{M}} |u \nabla \varphi_{r}|^{p} \, d\mu_{g} = \int_{\mathcal{M} \setminus B_{r}} |u \nabla \varphi_{r}|^{p} \, d\mu_{g} < \varepsilon \int_{\mathcal{M} \setminus B_{r}} |u|^{p} \, d\mu_{g} < \varepsilon^{2}.
$$

Therefore,

$$
\int_{\mathcal{M}} |\nabla u - \nabla (\varphi_{r} u)|^{p} \, d\mu_{g} < \varepsilon + \varepsilon^{2}.
$$

Thus, we can approximate $u \in W^{1,p}(\mathcal{M}, g)$ by a sequence $u_{n} \in W^{1,p}(\mathcal{M}, g)$ such that spt $u_{n}$ is compact.

By what we have just proved, to establish our claim, it suffices to restrict our attention to $u \in W^{1,2}(\mathcal{M}, g)$ with spt $u$ compact. Let us take a partition of unity $\{\xi_{i}\}$ with respect to the compact charts $(U_{i}, \psi_{i})$ covering spt $u$. Then, $u = \sum_{i=1}^{N} \xi_{i} u$ and on denoting the standard symmetric mollifier in $\mathbb{R}^{n}$ by $\eta^{\varepsilon}$, define

$$
u_{n} = \sum_{i=1}^{N} \psi_{i}^{*} (\eta^{\varepsilon} * \psi_{i}^{-1} \xi_{i} u),$$

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for \( \varepsilon > 0 \) small enough so that \( \text{spt} (\eta^\varepsilon \ast \psi_i^{-1} \xi_i u) \subset \xi_i(U_i) \). Now, \( u_\varepsilon \in C^\infty_c(M) \) and

\[
\int_M |u - u_\varepsilon|^p \, d\mu_g \leq \sum_{i=1}^N \int_{U_i} |\psi_i^*(\eta^\varepsilon \ast \psi_i^{-1} \xi_i u) - \xi_i u|^p \, d\mu_g = \sum_{i=1}^N \int_{\psi_i(U_i)} |\eta^\varepsilon \ast \psi_i^{-1} \xi_i u - \psi_i^{-1} \xi_i u|^p \sqrt{\det g} \, d\mathcal{L}.
\]

But \( g \) is continuous and \( \overline{\psi_i(U_i)} \) is compact, and therefore, there exist \( C_1, C_2 > 0 \) such that \( C_1 \leq \sqrt{\det g} \leq C_2 \). Therefore, \( \eta^\varepsilon \ast \psi_i^{-1} \xi_i u \to \psi_i^{-1} \xi_i u \) in \( L^p(\mathbb{R}^n) \) and hence, \( u_\varepsilon \to u \) in \( L^p(M) \). Since \( \nabla_p = \nabla = d \) on functions, and since the exterior derivative commutes with pullbacks, we obtain by a similar argument that \( \nabla_p u_\varepsilon \to \nabla_p u \). \( \square \)

Let \( \Omega^k(M) \) denote the bundle of \( k \)-forms, and \( \Omega(M) \) the bundle of forms. Recall that the exterior derivative is then the differential operator \( d : C^\infty(\Omega^k(M)) \to C^\infty(\Omega^{k+1}(M)) \). For \( k = 1 \), this is just \( \nabla \). Let

\[
S^d_p = \{ u \in C^\infty \cap L^p(\Omega(M)) : du \in L^p(\Omega(M)) \}
\]

and define the norm \( \|u\|_{d,p} = \|u\|_p + \|du\|_p \). We define the Sobolev spaces of \( d \) as follows. Let the space \( W^{d,p}(M, g) \) denote the closure of \( S^d_p \) under \( \|\cdot\|_{d,p} \), and \( W^{0,p}(M, g) \) as the closure of \( C^\infty_c(\Omega(M)) \) under the same norm. We refrain from proving a result similar to Proposition 2.4 here since we will prove it later with the aid of better tools.

While we have defined these function spaces for general \( p \), we shall only concentrate here on the case of \( p = 2 \). Note that \( L^2(T^{(r,s)}M, g) \) is a Hilbert space, with the induced inner product given by

\[
\langle u, v \rangle = \int_M \langle u(x), v(x) \rangle_{g(x)} \, d\mu_g(x).
\]

It is a standard fact that \( C^\infty(T^{(r,s)}M) \) is a dense subset of \( L^p(T^{(r,s)}M, g) \) for \( 1 \leq p < \infty \), so this is in particular true for \( L^2(M, g) \). Therefore, operator theory guarantees the existence of closed, densely-defined adjoints to \( \nabla_2 \) and \( \nabla_c \) which we identify as the divergence operators \( \text{div}_g = -\nabla_2 \) and \( \text{div}_{0,g} = -\nabla_c \).

### 2.3. Axelsson-Keith-McIntosh framework in Hilbert spaces

At the core of our achievements in this paper lies the Axelsson-Keith-McIntosh framework. First formulated in 2005 in their paper \([5]\), it provides a first-order framework unifying the resolution of many problems in modern harmonic analysis, including the boundedness of the Cauchy integral operator on Lipschitz curves, the Kato square root problem for systems on \( \mathbb{R}^n \), and the Kato square root problem on compact manifolds.

As aforementioned in §1, this framework has been used and adapted many times to solve a range of different problems. The framework at the level of Hilbert spaces as presented in \([5]\) is a Dirac-type setup consisting of three hypotheses (H1)-(H3) as follows. We emphasise here that \( \mathcal{H} \) is a Hilbert space and \( \Gamma \) an unbounded operator on \( \mathcal{H} \).
Let us now define $\Pi_B$ conditions (H1)-(H3) guarantee that $\Pi_B$ satisfies quadratic estimates (Q)

to say that $\Pi_B$ satisfies quadratic estimates means that

\[(Q) \quad \int_0^\infty \|Q_t u\|^2 \frac{dt}{t} \simeq \|u\|^2,\]

for all $u \in \mathcal{R}(\Pi_B)$. Such estimates are the fundamental objects of study, for they provide access to the tools of harmonic analysis in proving Kato square root theorems. The following is the main consequence. Note that here, the function $\chi_+$ takes value 1 on the right half of the complex plane and $\chi_-$ takes 1 on the left half.

**Theorem 2.6** (Kato square root type theorem). Suppose that $\Pi_B$ satisfies the quadratic estimates (Q). Then,

(i) there is a spectral decomposition:

\[\mathcal{H} = \mathcal{N}(\Pi_B) \oplus E_+ \oplus E_-\]

where $E_\pm = \chi_\pm(\Pi_B)$ (the sum is, in general, non-orthogonal),

(ii) $\mathcal{D} = \mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(B_1\Gamma^*B_2) = \mathcal{D}(I + \Pi_B^2)^{-1}$ with

\[\|\Pi_B u\| \simeq \|\Gamma u\| + \|\Gamma^*B_2 u\| \simeq \sqrt{\|\Pi_B^2 u\|}\]

for all $u \in \mathcal{D}$.

This theorem and a description of its consequences (in particular the stability of the perturbation $\hat{B} \mapsto \Pi_{B+\hat{B}}$ as Lipschitz estimates) can be found in [5]. A slightly more general exposition of similar results are contained in §1.8 of [7].

Let us conclude this section by demonstrating the connection between this setup and the aforementioned theorem to the Kato square root problem (for functions). Let $(\mathcal{M}, g)$ be a manifold and $\mathcal{H} = L^2(\mathcal{M}) \oplus L^2(\mathcal{M}) \oplus L^2(\mathcal{T}^*\mathcal{M})$. Let $S = (I, \nabla) : L^2(\mathcal{M}) \to L^2(\mathcal{M}) \oplus L^2(\mathcal{M})$, and suppose $A \in L^\infty(\mathcal{L}(L^2(\mathcal{M}) \oplus L^2(\mathcal{T}^*\mathcal{M})))$ and $a \in L^\infty(\mathcal{L}(L^2(\mathcal{M})))$ satisfying $\text{Re} \langle au, u \rangle \geq \kappa_1 \|u\|^2$ and $\text{Re} \langle ASv, v \rangle \geq \kappa_2 \|v\|^2_{W^{1,2}}$ for all $u \in L^2(\mathcal{M})$ and $v \in W^{1,2}(\mathcal{M})$. Then, set

\[\Gamma = \begin{pmatrix} 0 & 0 \\ S & \end{pmatrix}, \quad B_1 = \begin{pmatrix} a & 0 \\ 0 & \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.\]
If the quadratic estimates (Q) hold, then the conclusions of Theorem 2.6 are true, and in particular an easy calculation will reveal that the following Kato square root estimate holds:

$$D(\sqrt{aS^*AS}) = W^{1,2}(\mathcal{M}) \text{ and } \|\sqrt{aS^*ASu}\| \simeq \|u\|_{W^1}$$

for all $u \in W^1(\mathcal{M})$.

3. Rough metrics and their properties

In this section, we formulate a notion of a rough metric as a Riemann-like metric which we permit to be of low regularity and degenerate. We will see in subsequent parts of this paper that such metrics provide a sufficiently large class of geometries on which the quadratic estimates we consider are stable. We establish some of their key properties, their associated Lebesgue spaces, and consider implications for rough geometries when there are regular geometries nearby.

3.1. Rough metrics. We begin our discussion with the following definition, noting that $\delta$ denotes the usual Euclidean inner product.

**Definition 3.1 (Rough metric).** Suppose that $g \in \Gamma(\mathcal{T}^{(2,0)}\mathcal{M})$ is symmetric and satisfies the following local comparability condition: for every $x \in \mathcal{M}$, there exists a chart $(U, \psi)$ near $x$ and constant $C \geq 1$ such that

$$C^{-1}|u|_{\psi^*\delta(y)} \leq |u|_{g(x)} \leq C|u|_{\psi^*\delta(y)}$$

for $u \in T_xU$ and for almost-every $y \in U$. Then we say that $g$ is a rough metric.

By covering $\mathcal{M}$ by a countable collection of charts satisfying the local comparability condition, it is easy to see that $0 < |u|_{g(x)} < \infty$ for $0 \neq u \in T_x\mathcal{M}$ for almost-every $x \in \mathcal{M}$. As a consequence, we define the singular set of $g$ as

$$\text{Sing}(g) = \{x \in \mathcal{M} : |u|_{g(x)} = \infty \text{ or } |v|_{g(x)} = 0, \ v \neq 0\}.$$

It is easy to see that this set is of null measure. We define the regular set of $g$ by $\text{Reg}(g) = \mathcal{M} \setminus \text{Sing}(g)$.

The measurability of $g$, which if we recall is exactly the measurability of its coefficients inside coordinate charts against the $\mathcal{L}$-measure, provides us with a means to define a volume measure $\mu_g$. First, we note the following.

**Proposition 3.2.** Let $(U, \psi)$ and $(V, \varphi)$ be two coordinate charts satisfying the local comparability condition such that $W = U \cap V \neq \emptyset$. Then, for any measurable set $A \subset W$,

$$\int_{\psi(A)} \sqrt{\det g(x)} \ d\mathcal{L}(x) = \int_{\varphi(A)} \sqrt{\det g(y)} \ d\mathcal{L}(y).$$

**Proof.** Let $\{x^i\}$ and $\{y^j\}$ be the the coordinates in $(U, \psi)$ and $(V, \varphi)$ respectively. Then, let us denote $g$ in $U$ and $V$ respectively as

$g(x) = g_{ij}(\psi(x)) \ dx^i \otimes dx^j$ and $g(y) = \tilde{g}_{kl}(\varphi(y)) \ dy^k \otimes dy^l$. 

for almost all \( x \in U \) and almost all \( y \in V \). Then, on the intersection, the transformation formula is similar to the case of a continuous metric but holding for almost every \( x \in \mathcal{M} \),

\[
g_{ij}(\psi(x)) = \tilde{g}_{ij}(\varphi \circ \psi^{-1}(x)) \partial_x y^k \partial_x y^l,
\]

where \( (\partial_x y^i) = D(\varphi \circ \psi^{-1}) \). Letting \( G_x = (g_{ij}) \) and \( \tilde{G}_y = (\tilde{g}_{ij}) \), we note that

\[
G_x = D(\varphi \circ \psi^{-1}) \tilde{G}_y D(\varphi \circ \psi^{-1})^t,
\]

and hence \( \det G_x = (\det \tilde{G}_y)(\det D(\varphi \circ \psi^{-1}))^2 \).

Recall that for an integrable function \( \xi : \psi(W) \to \mathbb{R} \), the change of variable formula is given by

\[
\int_{\psi(A)} \xi(x) \, d\mathcal{L}(x) = \int_{\varphi(A)} (\xi \circ \psi \circ \varphi^{-1})(y) |\det D(\psi \circ \varphi^{-1})(y)| \, d\mathcal{L}(y).
\]

On noting that \( (\psi \circ \varphi^{-1})^{-1} = \varphi \circ \psi^{-1} \), and combining this with our previous calculation of \( \det G_x \), the conclusion follows.

As for the case of a continuous metric, we define the induced measure \( \mu_g \) by pasting together the local expression

\[
d\mu_g(x) = \sqrt{\det g(x)} \, d\mathcal{L}(x)
\]

via a smooth partition of unity subordinate to a countable covering of \( \mathcal{M} \) by charts satisfying the local comparability condition. The previous proposition guarantees that this object is well-defined under a change of coordinates. We observe the following properties of this measure.

**Proposition 3.3** (Compatibility of the measure). Let \( g \) be a rough metric. Then,

(i) a set \( A \subset \mathcal{M} \) is measurable if and only if it is \( \mu_g \)-measurable,

(ii) a function \( f : \mathcal{M} \to \mathbb{C} \) is measurable if and only if it is \( \mu_g \)-measurable,

(iii) \( Z \) is a set of null measure if and only if \( \mu_g(Z) = 0 \),

(iv) a property \( P \) holds a.e. in \( \mathcal{M} \) if and only if it holds \( \mu_g \)-a.e.,

**Proof.** Let \( (U, \psi) \) be a chart satisfying the local comparability condition. Then, inside each such chart, we can apply Lemma 2.2 since \( 0 < |u|_{g(x)} < \infty \) for almost-every \( x \in \mathcal{M} \) and \( 0 \neq u \in T_x \mathcal{M} \). Thus, we are able to conclude the properties hold inside \( U \) and again by the observation that the manifold can be covered by countably many such charts, the conclusion follows.

We remark that if we had simply asked for \( \text{Sing}(g) \) to be a null measure set in place of the local comparability condition, the conclusions of this proposition would still be valid. In what is to follow, we will see that the local comparability condition becomes important to establish regularity properties of the measure, as well as some desirable properties of Lebesgue and Sobolev spaces for such metrics. First, let us note the following lemma.

**Lemma 3.4.** Let \( B \) be a symmetric, positive matrix on \( \mathbb{R}^n \) such that there exists \( C \geq 1 \) satisfying \( C^{-1} |u| \leq u^t B u \leq C |u| \). Then, \( C^{-n} \leq \det B \leq C^n \).
Proof. Let \( \langle \cdot, \cdot \rangle \) be the standard Euclidean inner product. Then, \( u^* B u = \langle u, B u \rangle = \langle B u, u \rangle \) by the symmetry of \( B \). Choosing \( |u| = 1 \), we have that the numerical range \( \text{nr}(B) \subset [C^{-1}, C] \). But \( \sigma(B) \subset \text{nr}(B) \subset [C^{-1}, C] \). Letting \( \sigma(B) = \{ b_1, \ldots, b_n \} \), we have that \( C^{-n} \leq \det B = \prod_{i=1}^n b_i \leq C^n \). □

Using this lemma, we are first able to prove the following regularity result for the measure \( \mu_g \).

**Proposition 3.5.** The measure \( \mu_g \) is Borel and finite on compact sets.

**Proof.** That \( \mu_g \) is Borel is a simple consequence of (i) in Proposition 3.3. We show that it attains finite measure on compact sets. Let \( K \subset M \) be compact and let \( \{ U_i \}_{i=1}^N \) be a finite cover of \( K \) by charts \( (U_i, \psi_i) \) each of which satisfy the local comparability condition with constants \( C_i \). Inside \( U_i \), we have that

\[
\mu_g(U_i \cap K) = \int_{\psi_i(U_i \cap K)} \sqrt{\det g(x)} \, d\mathcal{L}(x) \leq C_i^2 \int_{\psi_i(U_i \cap K)} d\mathcal{L}(x) < \infty.
\]

Thus, \( \mu_g(K) \leq \sum_{i=1}^N \mu_g(U_i \cap K) < \infty \). □

**Remark 3.6.** We do not know whether such a measure is also Borel-regular.

We remark that we do not attempt to explore any distance metric properties that rough metrics may induce, simply because such metrics may not even provide us with a sufficiently finite length functional. Although additional conditions can remedy such an effect, we are primarily concerned with constructing Lebesgue and Sobolev spaces with certain desirable properties, and we will see in later parts that a possible lack of a metric structure will not hinder our efforts.

It is easy to see that every continuous Riemannian metric is a rough metric. Thus, from this point onwards, we shall simply refer to rough metrics as metrics. In order to illustrate that our definition is non-trivial, we provide the following two examples of degenerate rough metrics.

**Example 3.7.** Let \( M = \mathbb{R}^2 \) and define \( g(u, v) = \langle u, v \rangle_{\mathbb{R}^2} \) on \( \mathbb{R}^2 \setminus (\mathbb{R} \times 0) \) and \( g(u, v) = 0 \) on \( \mathbb{R} \times 0 \). Then, \( g \) is a rough metric with \( \mu_g = \mathcal{L} \). This space can be seen as \( \mathbb{R}^2 \) with the line \( \mathbb{R} \times 0 \) collapsed to a single point.

**Example 3.8.** Let \( (M, h) \) be a smooth manifold with a continuous metric and let \( Z \subset M \) be a null set. Then, set \( g = h \) on \( M \setminus Z \) and \( g = 0 \) on \( Z \) and we have that \( \mu_g = \mu_h \). We remark that \( Z \) can even be a dense subset.

### 3.2. Lebesgue and Sobolev spaces.

In this section, we define Lebesgue and Sobolev spaces for rough metrics. Since we are interested in their relationship to differential operators, we prove that compactly supported functions are dense in the \( L^p \) theory, which, as in the continuous metric case, allows us to obtain the Sobolev spaces as domains of operators.
First, for a rough metric $g$, we define the space $L^p(T^{(r,s)}M, g)$ precisely as we did for a continuous metric. Note that such spaces can be defined for a measure space alone. The Sobolev spaces $W^{1,2}(M, g)$ and $W^{1,2}_0(M, g)$ are also defined similarly.

While for continuous metrics it is clear that $C^\infty_c(T^{(r,s)}M)$ is dense in $L^p(T^{(r,s)}M)$, this is not immediate for rough metrics. We dedicate some energy to verify this is indeed the case. To aid us, we first demonstrate the following simple lemma.

**Lemma 3.9.** For every smooth manifold $M$, there exist a sequence of open sets $U_i$ such that $U_i$ is compact, $U_i \subset U_{i+1}$, and $M = \bigcup_i U_i = \lim_{i \to \infty} U_i$.

**Proof.** Fix $x \in M$ and let $(U, \psi)$ be a chart near $x$. Then, let $B(x, r) \subset \psi(U)$ be a Euclidean ball, and let $V_x = \psi^{-1}(B(x, 1/2r))$. It is easy to see that $V_x$ is open and $\overline{V_x}$ is compact. Since $M$ is second-countable, we are able to extract a countable subcollection $\{V_i\}$. Now, set $U_i = \cup_{j=1}^i V_i$ which finishes the proof. □

An immediate consequence is the following approximation result.

**Lemma 3.10.** For $p \in [1, \infty)$ and for every $u \in L^p(T^{(r,s)}M, g)$, there exists a sequence $u_n \in L^p(T^{(r,s)}M, g)$ such that $\text{spt } u_n$ is compact and $u_n \to u$ in $L^p$.

**Proof.** Let $u \in L^p(T^{(r,s)}M, g)$ and fix $\varepsilon > 0$. Let $U_i$ be the collection of sets guaranteed by the previous lemma. We claim that there exists an $N$ such that

$$\int_{M \setminus U_N} |u| \, d\mu_g < \varepsilon.$$ 

To argue by contradiction, suppose not. That is, for every $i$,

$$\int_{M \setminus U_i} |u|^p \, d\mu_g \geq \varepsilon.$$ 

But we have that

$$\int_M |u|^p \, d\mu_g = \int_{U_i} |u|^p \, d\mu_g + \int_{M \setminus U_i} |u|^p \, d\mu_g \geq \int_{U_i} |u|^p \, d\mu_g + \varepsilon.$$ 

Also, $\lim_i U_i = M$ and therefore, we have that

$$\int_M |u|^p \, d\mu_g \geq \int_M |u|^p \, d\mu_g + \varepsilon,$$

which means that $\varepsilon \leq 0$, which is a contradiction. Since each $U_i$ is compact, the sequence $u_n$ can be obtained simply by setting $\varepsilon = 1/n$ to extract sets $U'_i$ and setting $u_n = \chi_{U'_i} u$. □

Note that we did not use any properties of the metric $g$ in proving this lemma. However, in the proof of the following proposition, the locally comparability condition becomes of crucial importance.

**Proposition 3.11.** Whenever $g$ is a rough metric and $p \in [1, \infty)$, the space $C^\infty_c(T^{(r,s)}M)$ is dense in $L^p(T^{(r,s)}M, g)$. 


Proof. Fix \( u \in L^p(\mathcal{T}^{(r,s)}\mathcal{M}, g) \). By Lemma 3.10, we can assume that \( \text{spt} \ u \) is compact, and even further, without the loss of generality, let us assume that \( \text{spt} \ u \subseteq U \), where \( U \) is a chart where the local comparability condition is valid.

Noting that \( u = u^I_I \, dx^I \otimes \partial_{x^J} \) (where \( I \) and \( J \) are multi-indices), write \( u_\varepsilon = \psi^{-1} (\eta^\varepsilon * u^I_I \, dx^I \otimes \partial_{x^J}) \). Then, for some \( \kappa > 0 \), we have that \( \text{spt} \ u_\varepsilon \subseteq U \) for all \( \varepsilon < \kappa \). It is easy to see that \( u_\varepsilon \in C_\infty^c(\mathcal{T}^{(r,s)}\mathcal{M}) \). By invoking Lemma 3.4

\[
\int_U |u_\varepsilon - u|^p_g \, d\mu \leq \int_U |u_\varepsilon - u|_{\psi^*\delta}^p \, d\mu.
\]

But, inside \((U, \psi)\), \( d\mu(x) = \sqrt{\det g(x)} \, d\mathcal{L}(x) \) and by Lemma 3.4, on writing \( g(u, v) = u^{tr} G v \) at regular points, we obtain that \( C^{-\frac{n}{2}} \leq \sqrt{\det g} \leq C^{\frac{n}{2}} \). Thus,

\[
\int_U |u_\varepsilon - u|^p_g \, d\mu \leq \int_{\psi(U)} |\psi^* u_\varepsilon - \psi^* u|^p_{\delta} \, d\mathcal{L} \to 0
\]

by the standard results on mollification in Euclidean space. \( \square \)

By using this proposition, we are able to assert the following important properties of \( \nabla_p \) and obtain the Sobolev spaces associated to \( g \) as domains of the closure of this operator.

**Proposition 3.12.** For a rough metric \( g \), \( \nabla_p : C^\infty \cap L^p{\mathcal{M}} \to C^\infty \cap L^p(\mathcal{T}^*\mathcal{M}) \) and \( \nabla_c : C_\infty^c(\mathcal{M}) \to C_\infty^c(\mathcal{T}^*\mathcal{M}) \) are closable, densely-defined operators. Furthermore, \( W^{1,p}(\mathcal{M}) = \mathcal{D}(\nabla_p) \) and \( W_0^{1,p} = \mathcal{D}(\nabla_c) \).

**Proof.** Since \( \nabla_c \subseteq \nabla_p \), it suffices to prove the statement for \( \nabla_p \). The fact that \( \nabla_p \) and \( \nabla_c \) are densely-defined is immediate from Proposition 3.11.

To show that \( \nabla_p \) is closable, let \( u_n \in C^\infty \cap L^p(\mathcal{M}) \) such that \( \nabla_p u_n \in L^p(\mathcal{M}) \) and \( u_n \to 0 \) and \( \nabla_p u_n \to v \). It suffices to show that \( v = 0 \) a.e. in a countable collection of charts \( U_i \) satisfying the local comparability condition. For this, we can replicate the proof of Proposition 2.4 since we only require that the quantity \( \text{essinf}_{x \in U} \sqrt{\det g(x)} > 0 \), which we have as a consequence of the local comparability condition coupled with Lemma 3.4. Obtaining the Sobolev spaces \( W^{1,p}(\mathcal{M}) \) and \( W_0^{1,p}(\mathcal{M}) \) as \( \mathcal{D}(\nabla_p) \) and \( \mathcal{D}(\nabla_c) \) is then immediate. \( \square \)

As a consequence of Proposition 3.12, in the \( L^2 \) theory, we define the divergence operators to be \( \text{div}_g = -\nabla_2^* \) and \( \text{div}_{0,g} = -\nabla_0^* \). Note that since \( \nabla_2 \) and \( \nabla_0 \) are closed and densely-defined, so are \( \text{div}_g \) and \( \text{div}_{0,g} \) by the reflexivity of \( L^2 \).

### 3.3. Uniformly close metrics and their properties

Since we are ultimately interested in demonstrating how to pass quadratic estimates between two geometries that are close, we define an appropriate notion of closeness. We also demonstrate how certain desirable properties of one metric are preserved for the other nearby geometry. Our starting point is the following definition.
Definition 3.13 (Uniformly close metrics). Let $g$ and $\tilde{g}$ be two rough metrics and suppose there exists $C \geq 1$ such that
\[
C^{-1}|u|_g(x) \leq |u|_{\tilde{g}}(x) \leq C|u|_{\tilde{g}}(x),
\]
for $u \in T_xM$ and almost-every $x$ in $M$. Then, we say that $g$ and $\tilde{g}$ are uniformly close or $C$-close. If the inequality holds everywhere, then we say that the two metrics are $C$-close everywhere.

By Proposition 3.3, this particularly means that the two uniformly close metrics $g$ and $\tilde{g}$ satisfies the inequality in the definition $\mu_g$-a.e as well as $\mu_{\tilde{g}}$-a.e. Also, note that this inequality is invariant under interchanging $g$ and $\tilde{g}$. Heuristically speaking, this condition captures that these two geometries look almost-everywhere uniformly close as viewed from either metric, and that one geometry is almost-everywhere trapped by a uniform scaling of the other.

A first result we prove is that for the continuous case, the notion of $C$-close and $C$-close everywhere are equivalent. For the purpose of convenience, from this point onwards, let us denote the largest set on which the $C$-close inequality holds by $R$.

Proposition 3.14. Two continuous metrics $g$ and $\tilde{g}$ are $C$-close if and only if they are $C$-close everywhere.

Proof. The direction $C$-close everywhere implies $C$-close is easy. So we shall concentrate on the mildly harder opposite direction.

If we prove that $M = R$, then we are done. To draw a contradiction, suppose not and fix $x \in M \setminus R$. First, we claim that there exists a sequence $x_n \in R$ such that $x_n \to x$. We argue this by contraction, so suppose this is not true. Then, that means that there exists some open set $U_x$ near $x$ such that $U_x \cap R = \emptyset$. The only way this could happen is if $U_x \subset M \setminus R$. Thus, $U_x$ must be a null measure set. However, in some coordinate chart $(V, \psi)$ near $x$, there exists some $r > 0$ such that $U'_x = \psi^{-1}(B_r(\psi(x))) \subset U_x$. This implies that $U'_x$ is a set of measure zero. However, $\psi(U'_x) = B_r(\psi(x))$ which does not attain zero Lebesgue measure, and hence, we arrive at a contradiction.

Now, consider a chart $(U, \psi)$ near $x$, and for $n \geq N$, we have that $x_n \in U$. Furthermore, via the chart, we can define a smooth $u' : U \to TM$ such that $u'(x) = u$ and $C^{-1}|u'(x_n)|_{\tilde{g}(x_n)} \leq |u'(x_n)|_{g(x_n)} \leq C|u'(x_n)|_{\tilde{g}(x_n)}$. Each quantity is in this inequality is continuous and therefore, we can interchange the function and the limit. Thus, we find that $C^{-1}|u|_{\tilde{g}}(x) \leq |u|_g(x) \leq C|u|_{\tilde{g}}(x)$. Therefore, $x \in R$ which is a contradiction.

In order to explore the deeper properties of $C$-close metrics, it is convenient to be able to write one metric in terms of the other. To aid us in this direction, we first prove the following lemma.
Lemma 3.15. Let $V$ be a vector space of dimension $n$ and let $h_1$ and $h_2$ be two inner products on $V$. Then, there exists a bounded, symmetric, positive operator $B : V \to V$ such that $h_1(Bu, v) = h_2(u, v)$.

Proof. Since $V \cong \mathbb{R}^n$, we assume without loss of generality that $V = \mathbb{R}^n$. Then, we can write $h_i(u, v) = u^\text{tr}H_iv$ with $H_i$ being a positive, symmetric matrix. Indeed, such a matrix is diagonalisable, and so let us write $H_i = P_i^\text{tr}D_iP_i$ where $P_i$ is the matrix of eigenvectors and $D_i$ is the corresponding diagonal. Again, by the properties of $P_i$ and $D_i$, $H_i = P_i^\text{tr}\sqrt{D_i}P_i = (\sqrt{D_i})^\text{tr}(\sqrt{D_i})P_i$.

First, we show there exists a matrix $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $h_1(Au, Av) = h_2(u, v)$. This is equivalent to asking $(Au)^\text{tr}(\sqrt{D_1}P_1)^\text{tr}(\sqrt{D_1}P_1)Av = u^\text{tr}(\sqrt{D_2}P_2)^\text{tr}(\sqrt{D_2}P_2)v$. Thus, it suffices to solve for $\sqrt{D_2}P_2A = \sqrt{D_2}P_2$. But $D_i$ are invertible by their positivity and therefore, $A = A^*A = P_2^\text{tr}\sqrt{D_1}P_2$. Then it is easy to see that $B = A^*A = P_2^\text{tr}D_1^{-1}D_2P_2$. \hfill \blacksquare

An immediate consequence of this is the following, which allows us to capture the difference between two metrics as a $(1,1)$-tensor field.

Proposition 3.16. Let $g$ and $\tilde{g}$ be two rough metrics that are $C$-close. Then, there exists $B \in \mathbf{Γ}(T^*M \otimes TM)$ such that it is symmetric, almost-everywhere positive and invertible, and

$$\tilde{g}_x(B(x)u, v) = g_x(u, v)$$

for almost-every $x \in M$. Furthermore, for almost-every $x \in M$,

$$C^{-2}|u|_{\tilde{g}(x)} \leq |B(x)u|_{g(x)} \leq C^2|u|_{\tilde{g}(x)},$$

and the same inequality with $\tilde{g}$ and $g$ interchanged. If $\tilde{g} \in C^k$ and $g \in C^l$ (with $k, l \geq 0$), then the properties of $B$ are valid for all $x \in M$ and $B \in C^{\min(k,l)}(T^*M \otimes TM)$.

Proof. As before, let $\mathcal{R}$ be the largest set on which the $C$-close inequality holds. Then, for $x \in \mathcal{R}$, we invoke Lemma 3.15 to obtain a $B_x$ such that $\tilde{g}_x(B_xu, v) = g_x(u, v)$ for $u, v \in TM$. Then, set $B(x) = B_x$. Near $x$, the coefficients of $B$ consist of the coefficients of $\tilde{g}$, $g$ and their inverses, we have that $B \in \mathbf{Γ}(T^*M \otimes TM)$. That it is almost-everywhere positive, symmetric and invertible comes from the fact that $M \setminus \mathcal{R}$ is a null measure set.

By noting that for such a point $x \in \mathcal{R}$, $\tilde{g}_x(B_xu, v) = \tilde{g}_x(\sqrt{B_x}u, \sqrt{B_x}v)$, we have by the $C$-close condition that

$$C^{-1}|u|_{\tilde{g}(x)} \leq |u|_{g(x)} \leq C|u|_{\tilde{g}(x)},$$

from which the inequality in the conclusion of the theorem follows. The fact that this remains unchanged under the interchange of $\tilde{g}$ and $g$ is obvious.

If $\tilde{g} \in C^k$ and $g \in C^l$, then $\mathcal{R} = M$ by Proposition 3.14 and so the conclusions we have obtained so far are valid everywhere. As aforementioned, $B$ consists of the coefficients of $\tilde{g}$, $g$ and their inverses near $x$, and so it follows that $B \in C^{\min(k,l)}(T^*M \otimes TM)$. \hfill \blacksquare
Remark 3.17. By denoting the canonical extensions of these metrics to $T^{(r,s)}M$ by the same symbols, we note that we can prove there exists $B \in \Gamma(T^{(r,s)}M \otimes T^{(r,s)}M)$ satisfying the same properties as in the proposition, except that the inequality in the conclusion becomes

$$C^{-2(r+s)}|u|_g \leq |u|_{\tilde{g}} \leq C^{2(r+s)}|u|_{\tilde{g}}.$$ 

Using this operator $B$, we express the induced volume measures with respect to each other.

**Proposition 3.18.** The measure $d\mu_{\tilde{g}}(x) = \sqrt{\det B(x)} d\mu_g(x)$ for $x$-a.e. and $C^{-\frac{n}{2}} \mu_g \leq \mu_{\tilde{g}} \leq C^\frac{n}{2} \mu_g$.

**Proof.** Let $(U, \psi)$ be a chart near $x$. Then, for $y \in R$,

$$\psi^{-1} d\mu_g(y) = \sqrt{\det g(\psi(y))} d\mathcal{L}(y),$$

and letting $G$ denote the matrix of $\tilde{g}$ in these coordinates and $\tilde{G}$ the coordinates of $g$, we have that $\tilde{G} = BG$. Thus, $\det G = \det B \det g$. Therefore, it follows that

$$\psi^{-1} d\mu_g(y) = \sqrt{\det B(\psi(y))} \sqrt{\det \tilde{g}(\psi(y))} d\mathcal{L}(y) = \sqrt{\det B(\psi(y))} \psi^{-1} d\mu_{\tilde{g}}(y).$$

For the estimate, at a regular point $x \in \mathcal{R}$, we can apply Lemma 3.4 to conclude that $C^{-n} \leq \det B(x) \leq C^n$. 

With the aid of these two observations, we can now demonstrate how the $L^p$ spaces and Sobolev spaces of two uniformly close rough metrics compare. For the purposes of readability, we write $\theta(x) = \sqrt{\det B(x)}$ from here on.

**Proposition 3.19.** Let $g$ and $\tilde{g}$ be two $C$-close rough metrics. Then,

(i) whenever $p \in [1, \infty)$, $L^p(T^{(r,s)}M, g) = L^p(T^{(r,s)}M, \tilde{g})$ with

$$C^{-(r+s+\frac{n}{2p})} \|u\|_{p, g} \leq \|u\|_{p, \tilde{g}} \leq C^{r+s+\frac{n}{2p}} \|u\|_{p, \tilde{g}},$$

(ii) for $p = \infty$, $L^\infty(T^{(r,s)}M, g) = L^\infty(T^{(r,s)}M, \tilde{g})$ with

$$C^{-(r+s)} \|u\|_{\infty, g} \leq \|u\|_{\infty, \tilde{g}} \leq C^{r+s} \|u\|_{\infty, \tilde{g}},$$

(iii) the Sobolev spaces $W^{1,p}(M, g) = W^{1,p}(M, \tilde{g})$ and $W^{1,p}_0(M, g) = W^{1,p}_0(M, \tilde{g})$ with

$$C^{-(1+\frac{n}{2p})} \|u\|_{W^{1,p}, g} \leq \|u\|_{W^{1,p}, \tilde{g}} \leq C^{1+\frac{n}{2p}} \|u\|_{W^{1,p}, \tilde{g}},$$

(iv) the Sobolev spaces $W^{d,p}(M, g) = W^{d,p}(M, \tilde{g})$ and $W^{d,p}_0(M, g) = W^{d,p}_0(M, \tilde{g})$ with

$$C^{-(n+\frac{n}{2p})} \|u\|_{W^{d,p}, g} \leq \|u\|_{W^{d,p}, \tilde{g}} \leq C^{n+\frac{n}{2p}} \|u\|_{W^{d,p}, \tilde{g}},$$

(v) the divergence operators satisfy $\text{div}_g = \theta^{-1} \text{div}_{\tilde{g}} B$ and $\text{div}_{0,g} = \theta^{-1} \text{div}_{0,\tilde{g}} B$.

**Proof.** To prove (i), by the density of $C^\infty_c(T^{(r,s)}M)$ in $L^p(T^{(r,s)}M, g)$ and $L^p(T^{(r,s)}M, \tilde{g})$, it suffices to prove the inequality for $u \in C^\infty_c(T^{(r,s)}M)$. The $C$-close condition implies that

$$C^{-p(r+s)} |u|_g^p \leq |u|_{\tilde{g}}^p \leq C^{p(r+s)} |u|_{\tilde{g}}^p,$$
and Proposition 3.18 gives

\[ C^{-\frac{2}{n}} \int_{\mathcal{M}} |u|^p \, d\mu_g \leq \int_{\mathcal{M}} |u|^p \, d\mu_{\tilde{g}} \leq C^2 \int_{\mathcal{M}} |u|^p \, d\mu_{\tilde{g}}. \]

Combining the two proves (i).

To prove (ii), note that if \( C' > 0 \) such that \( |u(x)|_{\tilde{g}(x)} < C' \) x-a.e., then \( |u(x)|_{\tilde{g}(x)} \leq C'C^{r+} \) x-a.e. Therefore, \( \|u\|_{M,g} \leq C^{r+}\|u\|_{\infty,\tilde{g}} \). It is easy to see that the same is true with \( g \) and \( \tilde{g} \) interchanged.

Fix \( u \in C^\infty \cap L^p(\mathcal{M}) \) such that \( \nabla u \in C^\infty \cap L^p(\mathcal{T}^*\mathcal{M}) \) (we omit the \( g \) and \( \tilde{g} \) dependence of \( L^p \) since by (i), the \( L^p \) spaces are the same under both \( g \) and \( \tilde{g} \)). Then,

\[ \|u\|_{W^{1,p,g}} = \|u\|_{p,g} + \|\nabla u\|_{p,g} \leq C \|u\|_{p,\tilde{g}} + C^{\frac{1}{n}} \|\nabla u\|_{p,\tilde{g}} \leq C^{\frac{2}{n}} \|u\|_{W^{1,p,\tilde{g}}}. \]

Similarly, we can interchange \( g \) and \( \tilde{g} \) to obtain the lower inequality which proves (iii).

The claim in (iv) is proven similarly, on noting that \( \Omega(\mathcal{M}) \subset \oplus_{s=0}^{n-1}\mathcal{T}^{(s,0)}(\mathcal{M}) \) and thus the largest constant that can appear is \( C^n \).

To prove the last claim, let \( u \in \mathcal{D}(\nabla_2) = W^{1,2}(\mathcal{M}) \) and \( v \in \mathcal{D}(\text{div}_g) \). By construction of \( \text{div}_g \), we have that \( \langle \nabla_2 u, v \rangle_g = \langle u, -\text{div}_g v \rangle_g \). By what we have already established, we have that \( \langle \nabla_2 u, \theta \rangle_g = \langle \nabla_2 u, \theta \rangle_{\tilde{g}} \) and \( \langle u, \theta \text{div}_g v \rangle_g = \langle u, \theta \text{div}_g v \rangle_{\tilde{g}} \). Combining these two calculations, we obtain that \( \langle \nabla_2 u, \theta \rangle_{\tilde{g}} = \langle u, -\theta \text{div}_g v \rangle_{\tilde{g}} \), and therefore, \( \theta \text{div}_g v \in \mathcal{D}(\text{div}_g) \). For the reverse inclusion, let \( v \in \mathcal{D}(\text{div}_g) \) and then,

\[ \langle u, -\theta^{-1} \text{div}_g \theta Bv \rangle_g = \langle \nabla_2 u, \theta Bv \rangle_g = \langle \nabla_2 u, v \rangle_g. \]

Hence \( v \in \mathcal{D}(\text{div}_g) \) with \( \text{div}_g v = \theta^{-1} \text{div}_g \theta Bv \).

By replacing \( \nabla_2 \) by \( \nabla_0 \) and hence \( \text{div}_g \) and by \( \text{div}_{0,g} \) and \( \text{div}_{\tilde{g}} \) by \( \text{div}_{0,\tilde{g}} \) proves \( \text{div}_{0,g} = \theta^{-1} \text{div}_{0,\tilde{g}} \theta B \).

\[ \square \]

**Remark 3.20.** (i) Note that in (iii) and (iv), we only consider Sobolev spaces over functions or the exterior derivative. The exterior derivative depends only on the topology of \( \mathcal{M} \) and it is independent of the metric. An attempt to prove similar results for tensors is a futile effort (at least using these methods) for the simple fact that the Levi-Civita connection depends on the metric. In fact, we do not even know what we mean by a Levi-Cevita connection if the metric is of regularity less than \( C^1 \).

(ii) Defining \( \text{div}_g \) abstractly as the negative of the adjoint of \( \nabla_2 \) prevents us from knowing whether \( \text{div}_g \) is even a differential operator. Indeed, for a smooth metric \( h \) and a corresponding compatible connection \( \nabla^h \), we have \( \langle \nabla^h u, v \rangle = \langle u, -\text{tr} \nabla^h v \rangle \) whenever \( u \in C_c^\infty(\mathcal{M}) \) and \( v \in C_c^\infty(\mathcal{T}^*\mathcal{M}) \), so that \( \text{div}_h \supset -\text{tr} \nabla^h \). It is a lack of such a compatibility formula for rough metrics which prevents us from knowing the differential properties of \( \text{div}_g \). However, when \( \tilde{g} \) is a smooth metric, (v) illustrates that \( \text{div}_g \) is indeed the differential operator \( \text{div}_g \) but with measurable coefficients.
3.4. **There are smooth geometries near continuous ones.** In this final section, we consider the special case of continuous metrics as rough metrics. For such geometries, we establish that there are always $C$-close smooth geometries for any choice of $C > 1$. As a consequence, we show that the operators $d_2$ (and hence, $d_0$) are closable operators for every continuous geometry, a result we promised we would prove with the aid of better tools in §2.2.

**Lemma 3.21.** Let $g$ be a continuous metric. Then, for each $x \in M$, there exists an $r_x > 0$ such that $B(x, r_x)$ is a smooth coordinate system and at $x$, the metric in the coordinate directions $\{\partial_i\}$ satisfy $g_{ij}(x) = g_x(\partial_i, \partial_j) = \delta_{ij}$.

**Proof.** Fix $x \in M$ and let $\{e_j\}$ be an orthonormal basis with respect to $g(x)$ for $T_xM$. This exists via a Gram-Schmidt process since $g$ is non-degenerate.

Next, we note that the manifold $M$ admits a connection $\nabla$ because it admits a smooth metric $\tilde{g}$ and we can take $\nabla$ to be the Levi-Cevita connection of this metric.

We consider the exponential map, $\exp_x : T_xM \to M$, with respect to $\nabla$. Now, given the basis $\{e_j\}$ at $x$, we are able to obtain a neighbourhood $U \subset T_xM$ for which $\exp_x : U \to M$ is a smooth injection since it is the exponential map of the smooth metric $\tilde{g}$. Thus, by identifying $T_xM$ with $\mathbb{R}^n$, we obtain a coordinate system $U'$ near $x$ such that $\partial_j(x) = e_j$. Now we pick $r_x$ such that $B(x, r_x) \subset U'$. □

Using this Lemma, we can prove the following result that asserts that there are smooth geometries arbitrarily close to continuous ones.

**Proposition 3.22.** Let $g$ be a continuous metric. Given any $C > 1$, there exists a smooth metric $\tilde{g}$ which is $C$-close to $g$.

**Proof.** Fix $x \in M$. As a consequence of Lemma 3.21, there exists an $r_x$ and a coordinate system $(B(x, r_x), \psi_x)$ such that at $x$, $|u|_g = |u|_{\psi_x^*}\delta$ for all $u \in T_xM$.

Let $v \in C^\infty(TB(x, r'_x))$, we can find $r'_x \leq r_x$ such that

$$\frac{1}{C}|v(y)|_{\psi_x^*}\delta \leq |v(y)|_{\tilde{g}(y)} \leq C|v(y)|_{\psi_x^*}\delta$$

for all $y \in B(x, r'_x)$. Since we only have $n$ independent directions, we can obtain $r'_x$ independent of $v$.

Now, let $B_i$ be a countable subcover of $\{B(x, r'_x)\}$, and let $\{\varphi_i\}$ be a smooth partition of unity subordinate to $\{B_i\}$. Then, for $x \in M$ define

$$\tilde{g}(x) = \sum_{i=1}^{\infty} \varphi_i(x)(\psi_i^*\delta)(x).$$

Indeed, each of our coordinate systems $(B_i, \psi_i)$ are smooth, and hence, $\tilde{g}$ is smooth. By construction, we have that $\tilde{g}$ is $C$-close to $g$. □

As promised, we illustrate the following immediate consequence.
Corollary 3.23. Let \( g \) be a continuous metric. Then the operators \( d_p : C^\infty \cap L^p(\Omega(\mathcal{M})) \to C^\infty \cap L^p(\Omega(\mathcal{M})) \) and \( d_0 : C^\infty(\Omega(\mathcal{M})) \to C^\infty(\Omega(\mathcal{M})) \) are closable, densely-defined operators. Moreover, \( W^{d_p}(\mathcal{M}) = \mathcal{D}(\overline{d_p}) \) and \( W^{d_0}(\mathcal{M}) = \mathcal{D}(\overline{d_0}) \).

Proof. By the previous proposition, there exists a smooth metric \( \tilde{g} \) that is 2-close everywhere to \( g \). The conclusion then follows from Proposition 3.19. \( \square \)

For the \( p = 2 \) case, this asserts that the adjoints \( \delta_p = d_p^* \) and \( \delta_0 = d_0^* \) both exist as densely-defined, closed operators. Furthermore, we immediately obtain the following as a corollary.

Corollary 3.24. Let \( g \) be a continuous, complete metric. Then \( d_p = d_0 \) and \( \delta_0 = \delta_p \).

Proof. Again, we obtain a smooth 2-close everywhere metric \( \tilde{g} \), which is guaranteed to be complete since \( g \) is complete. That the conclusion holds for a smooth metric is well known fact in the folklore. \( \square \)

4. Reducing low regularity problems to smooth ones

The goal of this section is to demonstrate the reduction of low regularity Kato square root problems to smooth ones via quadratic estimates. We first establish a general framework at the level of Hilbert spaces. We then apply this technology, along with the results we have previously obtained, to illustrate how to pass quadratic estimates for the Kato square root problem on functions between two uniformly close metrics, as well as a similar problem for inhomogeneous Hodge-Dirac operators as considered by the author in §6 of [7].

4.1. Reduction at the level of the AKM framework. We begin by describing a general framework that encapsulates the reduction we consider later. The primary feature of our viewpoint, borrowed from [5], is that one part of the operator remains fixed while the other part changes under a change of inner product.

Recall from §2.3 that \( \Gamma : \mathcal{D}(\Gamma) \subset \mathcal{H} \to \mathcal{H} \) is a closed, densely-defined and nilpotent operator. Let \( \langle \cdot , \cdot \rangle_1 \) and \( \langle \cdot , \cdot \rangle_2 \) be two inner products on \( \mathcal{H} \) and suppose there exists a bounded, self-adjoint, invertible operator \( \Phi \in \mathcal{L}(\mathcal{H}) \) such that \( \langle u, v \rangle_1 = \langle \Phi u, v \rangle_2 \) for all \( u, v \in \mathcal{H} \).

Let \( \Gamma^*_1 \) denote the adjoint of \( \Gamma \) with respect to \( \langle \cdot , \cdot \rangle_1 \) and similarly \( \Gamma^*_2 \) denote the adjoint of \( \Gamma \) with respect to \( \langle \cdot , \cdot \rangle_2 \). We first obtain the following transformation rule whose proof is similar to the proof of case (v) of Proposition 3.19.

Proposition 4.1. The operator \( \Gamma^*_1 = \Phi^{-1}\Gamma^*_2\Phi \).

Let \( B_i \in \mathcal{L}(\mathcal{H}) \) satisfying (H2) and (H3) of the AKM framework from §2.3 with respect to \( \langle \cdot , \cdot \rangle_1 \). By the previous proposition, we write

\[
\Pi_{B_1} = \Gamma + B_1\Gamma^*_1B_2 = \Gamma + B_1\Phi^{-1}\Gamma^*_2\Phi B_2.
\]
Thus, we define $\Pi_{B,2} = \Gamma + \tilde{B}_1 \Gamma_2 \tilde{B}_2$ where $\tilde{B}_1 = B_1 \Phi^{-1}$ and $\tilde{B}_2 \Phi$ so that $\Pi_{B,1} = \Pi_{B,2}$. We prove later (under a very mild additional assumption) that $\Pi_{B,1}$ satisfies quadratic estimates if and only if $\Pi_{B,2}$ satisfies quadratic estimates.

Recall that in the AKM framework, the $B_1$ satisfies an accretivity assumption with respect to $\mathcal{R}(\Gamma^*)$. In our situation, we need to tweak this assumption to reflect the fact that we have two adjoint operators arising from the two different inner products. We present this modification as (H2') below.

(H2') Suppose there exists $\mathcal{H} \subset \mathcal{H}$ such that $\Phi \mathcal{H}^* = \mathcal{H}^*$, $\mathcal{R}(\Gamma_1^*) \cup \mathcal{R}(\Gamma_2^*) \subset \mathcal{H}^*$, and

$$\Re\langle B_1 u, u \rangle_1 \geq \kappa_1 \| u \|_1^2,$$

and

$$\Re\langle B_2 v, v \rangle_1 \geq \kappa_2 \| v \|_1^2,$$

for all $u \in \mathcal{H}^*$ and $v \in \mathcal{R}(\Gamma)$.

We remark that in practise, the space $\mathcal{H}^*$ arises naturally.

It is unreasonable to expect that changing from the operator $\Pi_{B,1}$ to $\Pi_{B,2}$ can be done for free. The following proposition calculates the cost we must pay in accretivity.

**Proposition 4.2 (Cost in accretivity for change of operator).** Suppose that

$$\| \Phi^{-\frac{1}{2}} u \|_2^2 = \langle \Phi^{-1} u, u \rangle_2 \geq \eta_1 \| u \|_2^2,$$

$$\| \Phi^{\frac{1}{2}} v \|_2^2 = \langle \Phi v, v \rangle_2 \geq \eta_2 \| v \|_2^2,$$

for $u \in \mathcal{H}^*$ and $v \in \mathcal{R}(\Gamma)$. Then, assuming that (H1), (H2') and (H3) are satisfied for $\Gamma$ and $B_i$ in $\langle \cdot, \cdot \rangle_1$ with constants $\kappa_i$, then the same assumptions are satisfied for $\tilde{B}_i$ in $\langle \cdot, \cdot \rangle_2$. The constants in (H2') are then $\kappa_1 \eta_1$ and $\kappa_2 \eta_2$.

**Proof.** It is an easy observation that (H1) is satisfied in $\langle \cdot, \cdot \rangle_2$.

Let us first consider (H3). We note that $\tilde{B}_1 \tilde{B}_2 = B_1 \Phi^{-1} \Phi B_2 = B_1 B_2$. Hence, it is trivial that $\tilde{B}_1 \tilde{B}_2 \mathcal{R}(\Gamma) = B_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$. Next, note that $\tilde{B}_2 B_1 = \Phi B_2 B_1 \Phi^{-1}$. Also, as a consequence of Proposition 4.1, we have that $\Gamma_2' = \Phi \Gamma_1^* \Phi^{-1}$. Thus,

$$\tilde{B}_2 \tilde{B}_1 \Gamma_2^* = \Phi B_2 B_1 \Phi^{-1} \Phi \Gamma_1^* \Phi = \Phi B_2 B_1 \Gamma_1^* \Phi = 0.$$

Thus, $\tilde{B}_2 \tilde{B}_1 \mathcal{R}(\Gamma_2^*) \subset \mathcal{N}(\Gamma_2^*)$.

Now, we show that (H2') is satisfied for $\Gamma$ and $\tilde{B}_i$ in $\langle \cdot, \cdot \rangle_2$. Let us fix $u \in \mathcal{H}^*$ and note that $\langle \tilde{B}_1 u, u \rangle_2 = \langle B_1 \Phi^{-1} u, u \rangle_2$. Let $u' = \Phi^{-1} u$ and $u' \in \mathcal{H}^*$ by assumption. Then,

$$\langle \tilde{B}_1 u, u \rangle_2 = \langle B_1 u', \Phi u' \rangle_2 = \langle B_1 u', u' \rangle_1 \geq \kappa_1 \| u' \|_1^2 = \geq \kappa_1 \| \Phi^{-1} u \|_1^2.$$

But we have by assumption on $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ that $\| u \|_2^2 = \langle u, u \rangle_1 = \langle \Phi u, u \rangle_2 = \langle \Phi^2 u \rangle_2$. Thus, $\| \Phi^{-1} u \|_1^2 = \| \Phi^{\frac{1}{2}} u \|_2 \geq \eta_1 \| u \|_2$. This proves that $\langle \tilde{B}_1 u, u \rangle_2 \geq \kappa_1 \eta_1 \| u \|_2^2$. 
Next, let \( v \in \mathcal{R}(\Gamma) \). Then,

\[
\langle \hat{B}_2 v, v \rangle_2 = \langle \Phi B_2 v, v \rangle_2 = \langle B_2 v, v \rangle_1 \geq \kappa_2 \| v \|_1^2 = \kappa_2 \| \Phi v \|_2^2 \geq \kappa_2 \eta_2 \| v \|_2^2,
\]

which finishes the proof. \( \square \)

The main tool that we shall require in later sections is the following.

**Proposition 4.3.** The quadratic estimate

\[
\int_0^\infty \| t \Pi_{B,1} (1 + t^2 \Pi_{B,1}^2)^{-1} u \|_1^2 \frac{dt}{t} \simeq \| u \|_1^2
\]

is satisfied for all \( u \in \overline{\mathcal{R}(\Pi_{B,1})} \) if and only if

\[
\int_0^\infty \| t \Pi_{B,2} (1 + t^2 \Pi_{B,2}^2)^{-1} v \|_2^2 \frac{dt}{t} \simeq \| v \|_2^2
\]

is satisfied for all \( v \in \overline{\mathcal{R}(\Pi_{B,2})} \).

**Proof.** It suffices to note that \( \mathcal{R}(\Pi_{B,2}) = \mathcal{R}(\Pi_{B,1}) \) and that \( \| \cdot \|_1 \simeq \| \cdot \|_2 \). \( \square \)

### 4.2. The Kato square root problem for functions.

With the aid of this framework, we first consider the Kato square root problem for functions. Let \( g \) and \( \tilde{g} \) be two \( C \)-close metrics, and suppose that \( \tilde{g} \) is at least continuous and complete. Let \( \mathcal{H} = L^2(\mathcal{M}) \oplus L^2(\mathcal{M}) \oplus L^2(\text{T}^*\mathcal{M}) \) and let \( \langle \cdot, \cdot \rangle_1 = \langle \cdot, \cdot \rangle_g \) and \( \langle \cdot, \cdot \rangle_2 = \langle \cdot, \cdot \rangle_{\tilde{g}} \). Define \( \Phi : \mathcal{H} \to \mathcal{H} \) by \( \Phi(u, v, w) = (0u, 0v, 0w) \). It is easy to see that \( \Phi \in L(\mathcal{H}) \), symmetric, positive, invertible and that \( \langle u, v \rangle_g = \langle \Phi u, v \rangle_{\tilde{g}} \).

By the assumption of continuity and completeness on \( \tilde{g} \), we conclude from Proposition 2.4 that \( \nabla_0 = \nabla_2 \) and \( \text{div}_{0,\tilde{g}} = \text{div}_g \). Write \( S = (1, \nabla_2) \) and \( A \in L^\infty(\mathcal{L}(L^2(\mathcal{M}) \oplus L^2(\text{T}^*\mathcal{M}))) \) such that the following ellipticity assumption holds: there exists \( \kappa_1, \kappa_2 > 0 \) such that

\[
(E_g) \quad \text{Re}(au, u)_g \geq \kappa_1 \| u \|_g^2 \quad \text{and} \quad \text{Re}(ASv, Sv)_g \geq \kappa_2 \| v \|_{W^{1,2}_g}^2
\]

for \( u \in L^2(\mathcal{M}, g) \) and \( v \in W^{1,2}(\mathcal{M}, g) \).

We recall that the Kato square root problem for functions is then to determine whether the following holds:

\[
(K_{S,g}) \quad \mathcal{D}(\sqrt{aS^*AS}) = W^{1,2}(\mathcal{M}, g) \text{ with } \| \sqrt{aS^*AS} u \|_g \simeq \| u \|_{W^{1,2}_g}
\]

for \( u \in W^{1,2}(\mathcal{M}, g) \).

As outlined in \( \S 2.3 \), we let

\[
\Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} a & 0 \\ 0 & \tilde{g} \end{pmatrix}, \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix},
\]

and \( \mathcal{R} = L^2(\mathcal{M}) \oplus 0 \oplus 0 \). Then, we note that \((E_g)\) is equivalent to

\[
\text{Re}(B_1 u, u)_g \geq \kappa_1 \| u \|_g^2 \quad \text{and} \quad \text{Re}(B_2 v, v)_g \geq \kappa_2 \| v \|_g^2
\]
whenever \( u \in \mathcal{R}^* \) and \( v \in \mathcal{R}(\Gamma) \). It is also easy to see that \( \Phi \mathcal{R}^* = \mathcal{R}^* \), \( \mathcal{R}(\Gamma^*_g) \cup \mathcal{R}(\Gamma^*_\tilde{g}) \subset \mathcal{R}^* \) and that

\[
\tilde{B}_1 = \begin{pmatrix} a\theta^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \tilde{B}_2 = \begin{pmatrix} 0 & 0 \\ 0 & TA \end{pmatrix},
\]

where \( T : L^2(\mathcal{M}) \oplus L^2(\mathcal{T}^*\mathcal{M}) \to L^2(\mathcal{M}) \oplus L^2(\mathcal{T}^*\mathcal{M}) \) via \( T(u, v) = (\theta u, \theta Bv) \).

As an immediate consequence to Proposition 4.2, we obtain the following.

**Proposition 4.4.** The operators \( \tilde{B}_i \in L(\mathcal{H}) \) satisfy

\[
\Re \langle \tilde{B}_1 u, u \rangle _{\tilde{g}} \geq \frac{\kappa_1}{C^2} \| u \| _{\tilde{g}}^2,
\]

\[
\Re \langle \tilde{B}_2 v, v \rangle _{\tilde{g}} \geq \frac{\kappa_2}{C^{1+\frac{1}{2}}} \| v \| _{\tilde{g}}^2
\]

whenever \( u \in \mathcal{R}^* \) and \( v \in \mathcal{R}(\Gamma) \).

**Proof.** By Proposition 4.2, it suffices to simply compute lower bounds for \( \| \Phi^{-\frac{1}{2}} u \| _{\tilde{g}}^2 \) and \( \| \Phi^\frac{1}{2} v \| _{\tilde{g}}^2 \) for appropriate \( u \) and \( v \).

First, fix \( u \in \mathcal{R}^* \). Then,

\[
\| \Phi^{-\frac{1}{2}} u \| _{\tilde{g}}^2 = \| \theta^{-\frac{1}{2}} u_1 \| _{\tilde{g}}^2 \geq C^{-\frac{1}{2}} \| u_1 \| _{\tilde{g}}^2 = C^{-\frac{1}{2}} \| u \| _{\tilde{g}}^2.
\]

Next, let \( v \in 0 \oplus L^2(\mathcal{M}) \oplus L^2(\mathcal{T}^*\mathcal{M}) \supset \mathcal{R}(\Gamma) \). Then,

\[
\| \Phi^\frac{1}{2} v \| _{\tilde{g}}^2 = \| \theta^\frac{1}{2} v_2 \| _{\tilde{g}}^2 + \| \theta Bv_3 \| _{\tilde{g}}^2 \geq C^{\frac{1}{2}} \| v_2 \| _{\tilde{g}}^2 + C^{-(1+\frac{1}{2})} \| v_3 \| _{\tilde{g}}^2 \geq C^{-(1+\frac{1}{2})} \| v \| _{\tilde{g}}^2
\]

which finishes the proof. \( \square \)

Combining these results, we obtain the following main theorem of this section as a consequence of Proposition 4.3.

**Theorem 4.5.** Let \( g \) be a rough metric and \( \tilde{g} \) continuous, complete and suppose that they are uniformly close. Further suppose that

\[
\int_0^\infty \| t\Pi_{B,\tilde{g}}(1 + t^2\Pi_{B,\tilde{g}})^{-1} u \| _{\tilde{g}}^2 \frac{dt}{t} \sim \| u \| _{\tilde{g}}^2
\]

for all \( u \in \mathcal{R}(\Pi_{B,\tilde{g}}) \). Then,

\[
\int_0^\infty \| t\Pi_{B,g}(1 + t^2\Pi_{B,g})^{-1} u \| _{\tilde{g}}^2 \frac{dt}{t} \sim \| u \| _{\tilde{g}}^2
\]

for all \( u \in \mathcal{R}(\Pi_{B,\tilde{g}}) \).

By combining this with Theorem 1 of McIntosh and the author in [6], we obtain the following important corollary.

**Corollary 4.6.** Let \( \tilde{g} \) be a smooth, complete metric and suppose that there exists \( \kappa > 0 \) and \( \eta > 0 \) such that \( \text{inj}(\mathcal{M}, \tilde{g}) \geq \kappa \) and \( \text{Ric}(\tilde{g}) \leq \eta \). Then, for any rough metric \( g \) that is uniformly close, quadratic estimates are satisfied for \( \Pi_{B,\tilde{g}} \). In particular \( (K_{S,\tilde{g}}) \) holds under the assumption \( (E_g) \).
4.3. The Kato square root problem for differential forms. In his thesis [7], the author considers versions of the Kato square root problem for perturbations of inhomogeneous Hodge-Dirac operators under a natural and mild curvature assumption on the bundle of forms.

Let $g$ be a rough metric and $A \in L^\infty(\mathcal{L}(L^2(\Omega(M))) \oplus L^2(\Omega(M))))$. We assume that $A$ satisfies the following ellipticity condition with respect to $g$: there exists $\kappa_2 > 0$ such that

$$\text{Re}\langle Au, u \rangle_g \geq \kappa_2 \|u\|^2_g$$

for every $u \in L^2(\Omega(M)))$. Indeed, we immediately obtain that there exists $\kappa_1 > 0$ such that $\text{Re}\langle A^{-1}u, u \rangle_g \geq \kappa_1 \|u\|^2_g$.

Let $D_{A,g} = d + A^{-1}\delta_g A$. Given some $0 \neq \beta \in \mathbb{C}$, the Kato square root problem for forms as outlined in §6.4 in [7] is then to determine that $D(\sqrt{D_{A,g}} + |\beta|) = D(A_{,g})$

for $u \in D(A_{,g})$.

Now, let $\tilde{g}$ be at least continuous and complete, and assume that it is $C$-close to $g$. We denote the induced canonical metrics on $\Omega(M)$ by the same symbols. As we have noted previously, for almost-every $x \in M$ and every $u \in \Omega_x(M)$, we have the inequality

$$C^{-n}\|u\|_{\tilde{g}(x)} \leq \|u\|_{\hat{g}(x)} \leq C^n\|u\|_{\tilde{g}(x)}.$$

An argument along the lines of the proof of Proposition 3.16 guarantees an operator $E : \Gamma(\Omega^*(M) \otimes \Omega(M))$ such that $g_x(u, v) = \tilde{g}_x(E(x)u, v)$ satisfying the inequality

$$C^{-2n}\|u\|_{\hat{g}(x)} \leq \|E(x)u\|_{\tilde{g}(x)} \leq C^{-2n}\|u\|_{\hat{g}(x)}$$

for almost-every $x \in M$.

Let $\mathcal{H} = L^2(\Omega(M)) \oplus L^2(\Omega(M))$ and note that $g$ and $\tilde{g}$ induces $\langle \cdot, \cdot \rangle_g$ and $\langle \cdot, \cdot \rangle_{\hat{g}}$ respectively. On setting $\Phi(w, z) = (\theta E w, \theta E z)$, we can see that $\langle u, v \rangle_{\tilde{g}} = \langle \Phi u, v \rangle_{\hat{g}}$.

To encode the problem into a Dirac-type operator, fix $\beta \in \mathbb{C}$ with $\beta \neq 0$ and let

$$d_{,\beta} = \begin{pmatrix} d & 0 \\ \beta & -d \end{pmatrix}.$$

The operator $d$ here is the operator $d_0$ or $\overline{d}_2$, which are equal in both metrics $g$ and $\tilde{g}$ as a consequence of Corollary 3.24, the continuity and completeness of $\tilde{g}$, and the $C$-closeness of the two metrics. The adjoint of $d_{,\beta}$ with respect to $g$ and $\tilde{g}$ are denoted by $\delta_{\beta,g}$ and $\delta_{\beta,\tilde{g}}$ respectively. It is easy to see that these are given by the operator matrices

$$\delta_{\beta,g} = \begin{pmatrix} \delta_g & \beta \\ 0 & \delta_g \end{pmatrix}$$

and

$$\delta_{\beta,\tilde{g}} = \begin{pmatrix} \delta_{\tilde{g}} & \beta \\ 0 & \delta_{\tilde{g}} \end{pmatrix}.$$
By repeating the argument proving (v) of Proposition 3.19, we obtain that \( \delta_g = (E\theta)^{-1} \delta_{\tilde{g}} (E\theta) \) and also that \( \delta_{\beta,g} = \Phi^{-1} \delta_{\beta,\tilde{g}} \Phi \).

Next, define \( B_1, B_2 \in L(H) \) by
\[
B_1 = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.
\]

On setting \( R^* = H \), by the ellipticity assumption on \( A \), we obtain that
\[
\Re \langle B_1 u, u \rangle_g \geq \kappa_1 \|u\|_g^2 \quad \text{and} \quad \Re \langle B_2 u, u \rangle_g \geq \kappa_2 \|u\|_g^2
\]
for all \( u \in H \). Recall the operator \( \Pi_{B,g} = \Gamma + B_1 \Gamma^* B_2 \) from the AKM framework and note that
\[
\Pi_{B,g}^2 = \begin{pmatrix} D_{A,g}^2 + |\beta| & 0 \\ 0 & D_{A,g}^2 + |\beta| \end{pmatrix}.
\]

It is for the operator \( \Pi_{B,g} \) for which we consider quadratic estimates to ultimately yield a solution to \( (K_{D,g}) \).

As in §4.2, we reduce the non-smooth problem to a smooth one. It is easy to see that \( \Pi_{B,g} = \Pi_{\tilde{B},\tilde{g}} \) where \( \tilde{B}_1 = B_1 \Phi^{-1} \) and \( \tilde{B}_2 = \Phi B_2 \). By applying a similar argument as in the proof of Proposition 4.4, we obtain the following change of accretivity in moving from \( \Pi_{B,g} \) to \( \Pi_{\tilde{B},\tilde{g}} \).

**Proposition 4.7.** The operators \( \tilde{B}_1 \) and \( \tilde{B}_2 \) satisfy
\[
\Re \langle \tilde{B}_i u, u \rangle_{\tilde{g}} \geq \frac{\kappa_i}{C_3^2} \|u\|_{\tilde{g}}^2
\]
for \( u \in L^2(\Omega_x(M)) \) and \( i = 1, 2 \).

**Proof.** As a consequence of Proposition 4.2, it suffices to compute a lower bound for \( \|\Phi^2 u\|_{\tilde{g}} \) for \( u \in L^2(\Omega_x(M)) \). But it is easy to observe that \( \|\Phi^2 u\|_{\tilde{g}} \geq C_3^{(1+\frac{1}{2})} \|u\|_{\tilde{g}} \).

Recall that for a smooth, complete metric \( \tilde{g} \), the curvature endomorphism \( R : \Omega_x(M) \to \Omega_x(M) \) is given by
\[
R\omega = -Rm_{ijkl}(x) \theta^i \wedge (\theta^j \wedge (\theta^k \wedge (\theta^l \wedge \omega))),
\]
where \( \{\theta^i\} \) are an orthonormal frame at \( x \) and \( \omega \in \Omega_x(M) \). In Theorem 6.4.3 of [7], the author shows that the quadratic estimates for \( \Pi_{\tilde{B},\tilde{g}} \) are satisfied under appropriate bounds on the geometry of \( \tilde{g} \) and on \( R \). Coupling this result with Proposition 4.3, we have the following main theorem of this section.

**Theorem 4.8.** Let \( g \) be a rough metric uniformly close to \( \tilde{g} \), a smooth, complete metric, and suppose that:

(i) there exists \( \kappa > 0 \) such that \( \text{inj}(\mathcal{M}, \tilde{g}) \geq \kappa \),
(ii) there exists \( \eta > 0 \) such that \( |Ric(\tilde{g})| \leq \eta \), and
(iii) there exists \( \zeta \in \mathbb{R} \) such that \( \tilde{g}(R\omega, \omega) \geq \zeta \|\omega\|_{\tilde{g}}^2 \).

Then, whenever \( A \in L^\infty(\mathcal{L}(L^2(\Omega_x(M)))) \) satisfies \( (E_{D,\tilde{g}}) \), we obtain \( (K_{D,\tilde{g}}) \).
4.4. Applications to compact manifolds. In this short section, we present the following theorem which is a culmination of results we have obtained so far. It demonstrates that the aforementioned Kato square root problems can always be solved on compact manifolds for every rough metric.

**Theorem 4.9.** Let $\mathcal{M}$ be a smooth, compact manifold and $g$ a rough metric. Whenever $A$ satisfies $(E_g)$ then $(K_{S,g})$ holds. Similarly, if $A$ satisfies $(E_{D,g})$ then $(K_{D,g})$ holds.

*Proof.* By the compactness of $\mathcal{M}$, we have a finite number of charts $(U_i, \psi_i)$ covering $\mathcal{M}$ satisfying the local comparability condition with constants $C_i$. On letting $\varphi_i$ be a smooth partition of unity subordinate to $\{U_i\}$, write

$$\tilde{g}(x) = \sum_i \varphi_i \psi_i^* \delta(x).$$

It is easy to see that $\tilde{g}$ is a smooth metric. On setting $C = \max_i \{C_i\}$, we obtain that $g$ and $\tilde{g}$ are $C$-close.

Since $|\text{Ric}_{\tilde{g}}|_{\tilde{g}} : \mathcal{M} \to \mathbb{R}$ is smooth and in particular continuous, by the compactness of $\mathcal{M}$, there is an $\eta > 0$ and $\zeta \in \mathbb{R}$ such that $|\text{Ric}_{\tilde{g}}|_{\tilde{g}} \leq \eta$ and $\tilde{g}(\text{R} \omega, \omega) \geq \zeta \|\omega\|$ for $\omega \in \Omega(\mathcal{M})$. That there exists $\kappa > 0$ such that $\text{inj}(\mathcal{M}, \tilde{g}) \geq \kappa$ also follows from compactness of $\mathcal{M}$ and smoothness of $\tilde{g}$. See Theorem III.2.1 and the discussion prior to Theorem III.2.3 in [11]. The conclusion is then obtained by invoking Corollary 4.6 and Theorem 4.8. □

**Remark 4.10.** If the metric $g$ was continuous, then we can choose any $C > 1$ and by invoking Proposition 3.22, we can find $\tilde{g}$ to be smooth and $C$-uniformly everywhere close to $g$.

### 5. Quadratic estimates and isometries

In our achievements so far, we have always considered the situation of fixing a manifold and studying the persistence of quadratic estimates under suitable changes of the metric. Another important situation to consider is the transmission of quadratic estimates between manifolds which are isometric. Our ability to do this will depend on the regularity of the isometry. We first consider a general description of this problem at the level of the AKM framework.

#### 5.1. Isometries between Hilbert spaces.

The first results we obtain are concerned with pushing and pulling forward Dirac-type operators on Hilbert spaces. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively. We assume that $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is an isometric isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$, by which we mean that $\Phi$ is a vector space isomorphism satisfying $\langle \Phi u, \Phi v \rangle_2 = \langle u, v \rangle_1$ for all $u, v \in \mathcal{H}_1$. On letting $\Gamma_1 : \mathcal{H}_1 \to \mathcal{H}_1$ be closed, densely-defined operators on $\mathcal{H}_1$ related via $\Phi$, we obtain the following transformation rule for their adjoints.

**Lemma 5.1.** Suppose that $\Gamma_2 = \Phi \Gamma_1 \Phi^{-1}$, by which we mean that $\mathcal{D}(\Gamma_2) = \Phi \mathcal{D}(\Gamma_1)$ and $\Gamma_2 u = \Phi \Gamma_1 \Phi^{-1} u$ for all $u \in \mathcal{D}(\Gamma_2)$. Then, $\Gamma_2^* = \Phi \Gamma_1^* \Phi^{-1}$ and $\overline{\mathcal{R}}(\Gamma_2) = \Phi \overline{\mathcal{R}}(\Gamma_1)$. 

Proof. First, we prove that $D(\Gamma_2^*) = \Phi D(\Gamma_1^*)$. Fix, $u \in D(\Gamma_2^*)$. Then, whenever $v \in D(\Gamma_2)$,

$$\langle \Gamma_2^* u, v \rangle_2 = \langle u, \Gamma_2 v \rangle_2 = \langle \Phi^{-1} u, \Phi^{-1} \Gamma_2 v \rangle_1 = \langle \Phi^{-1} u, \Gamma_1 \Phi^{-1} v \rangle_1.$$ 

But since $D(\Gamma_2) = \Phi D(\Gamma_1)$, we can write $v' = \Phi^{-1} v \in D(\Gamma_1)$ and furthermore,

$$\langle \Gamma_2^* u, v' \rangle_2 = \langle \Phi^{-1} \Gamma_2^* u, \Phi^{-1} v' \rangle_1$$

and thus, $\langle \Phi^{-1} \Gamma_2^* u, \tilde{v} \rangle_1 = \langle \Phi^{-1} u, \Gamma_1 \tilde{v} \rangle_1$ for all $\tilde{v} \in D(\Gamma_1)$. Therefore, $\Phi^{-1} u \in D(\Gamma_1^*)$.

For the opposite direction, let $u \in D(\Gamma_1^*)$ and let $v \in D(\Gamma_2^*)$. Then, a similar calculation holds:

$$\langle \Gamma_1^* u, v \rangle_1 = \langle u, \Gamma_1 v \rangle_1 = \langle \Phi u, \Phi \Gamma_1 v \rangle_2 = \langle \Phi u, \Phi \Gamma_1 \Phi^{-1} (\Phi v) \rangle_2 = \langle \Phi u, \Gamma_2 (\Phi v) \rangle_2.$$ 

So, $\langle \Phi \Gamma_1^* u, \tilde{v} \rangle_2 = \langle \Phi u, \Gamma_2 \tilde{v} \rangle_2$ for all $\tilde{v} \in D(\Gamma_2)$, and thus, we conclude that $\Phi u \in D(\Gamma_2^*)$.

Note that this implies that $D(\Gamma_2^*) \subset \Phi D(\Gamma_1^*)$, for if not, then there would be a $u \in D(\Gamma_2)$ such that $u \notin \Phi v$ for all $v \in D(\Gamma_1^*)$, but setting $v = \Phi^{-1} u \in D(\Gamma_1^*)$ would produce a contradiction by what we have just proved. On combining these two calculations, we obtain that $D(\Gamma_2^*) = \Phi D(\Gamma_1^*)$.

Next, we fix $u \in D(\Gamma_2^*)$ and compute:

$$\langle \Gamma_2^* u, v \rangle_2 = \langle \Phi^{-1} u, \Gamma_1 \Phi^{-1} v \rangle_1 = \langle \Gamma_1 \Phi^{-1} u, \Phi^{-1} v \rangle_1 = \langle \Phi \Gamma_1^* \Phi^{-1} u, v \rangle_2$$

for all $v \in D(\Gamma_2)$, and so $\Gamma_2^* u = \Phi \Gamma_1^* \Phi^{-1} u$ by the density of $D(\Gamma_2)$ in $\mathcal{H}$.

Now, let us show that $\overline{R}(\Gamma_2) = \Phi \overline{R}(\Gamma_1)$. Fix $u \in D(\Gamma_2)$. Then, write $v = \Gamma_2 u = \Phi \Gamma_1 \Phi^{-1} u$. That is $\Phi^{-1} v = \Gamma_2 \Phi^{-1} u$ which shows that $\Phi^{-1} \overline{R}(\Gamma_2) \subset \overline{R}(\Gamma_1)$. For the other direction, let $u \in D(\Gamma_1)$ and so $v = \Gamma_1 u = \Phi^{-1} \Gamma_2 \Phi u$. Then, we can conclude by similar reasoning that $\Phi \overline{R}(\Gamma_1) \subset \overline{R}(\Gamma_2)$ and hence $\overline{R}(\Gamma_2) = \Phi \overline{R}(\Gamma_1)$. The proof is completed on observing that $\|v\|_1 = \|\Phi v\|_2$. $\Box$

Let us now assume that the operator $\Gamma_1$ and $B_i \in \mathcal{L}(\mathcal{H}_1)$ satisfy the hypotheses (H1)-(H3) of the AKM framework as outlined in §2.3. By virtue of the previous lemma, the definition of $B_i$, and as a consequence of the fact that $\Phi$ is an isometry, the operators $\tilde{B}_i$ satisfy the same coercivity estimate in (H2) with the exact same constants as $B_i$. In fact, it is easy to see that the operators $\Gamma_2$ and $\tilde{B}_i$ satisfy the entire set of hypotheses (H1)-(H3). Define $\Pi_B = \Gamma_1 + B_1 \Gamma_1 B_2$ and $\tilde{\Pi}_B = \Gamma_2 + \tilde{B}_1 \Gamma_2 \tilde{B}_2$. We note the following.

**Lemma 5.2.** $(1 + t^2 \tilde{\Pi}_B^2)^{-1} u = \Phi (1 + t^2 \Pi_B^2)^{-1} \Phi^{-1} u$ for all $u \in \mathcal{H}_2$.

**Proof.** First, it is an easy fact that $\tilde{\Pi}_B = \Phi \Pi_B \Phi^{-1}$. So, now, fix $v = \Phi (1 + t^2 \Pi_B^2)^{-1} \Phi^{-1} u$ so that $\Phi^{-1} v = (1 + t^2 \Pi_B^2)^{-1} \Phi^{-1} u$. Then, $\Phi^{-1} u = (1 + t^2 \Pi_B^2) \Phi^{-1} v = \Phi^{-1} v + t^2 \Pi_B^2 \Phi^{-1} v$ and multiplying both sides by $\Phi$ then yields $u = (1 + t^2 \Pi_B^2) \Phi v$. Thus, $v = (1 + t^2 \Pi_B^2)^{-1} u$. $\Box$

On combining these two lemmas, we obtain the following result pertaining to the transmission of quadratic estimates across isometries.
Proposition 5.3. The quadratic estimate
\[
\int_0^\infty \|t\Pi_B(1 + t^2\Pi_B^2)^{-1}u\|_2^2 \frac{dt}{t} \simeq \|u\|_2^2
\]
for \(u \in \overline{\mathcal{R}(\Pi_B)}\) is satisfied if and only if
\[
\int_0^\infty \|t\Pi_B(1 + t^2\Pi_B^2)^{-1}v\|_1^2 \frac{dt}{t} \simeq \|v\|_1^2
\]
for all \(v \in \overline{\mathcal{R}(\Pi_B)}\).

Proof. First, by identifying \(\Pi_B\) with \(\Gamma_1\) in Lemma 5.1, we find that \(\overline{\mathcal{R}(\Pi_B)} = \Phi\overline{\mathcal{R}(\Pi_B)}\). Thus, for \(u \in \overline{\mathcal{R}(\Pi_B)}\) we have that \(\|u\|_2 = \|\Phi^{-1}u\|_1\). For the same \(u\),
\[
\|t\Pi_B(1 + t^2\Pi_B^2)^{-1}u\|_2 = \|t\Phi^{-1}\Pi_B\Phi^{-1}\Phi(1 + t^2\Pi_B^2)^{-1}\Phi^{-1}u\|_2 = \|t\Pi_B(1 + t^2\Pi_B^2)^{-1}\Phi^{-1}u\|_1.
\]
Setting \(v = \Phi^{-1}u\) finishes the proof. \(\square\)

5.2. Local Lipeomorphisms between manifolds. Let \(\mathcal{M}\) and \(\mathcal{N}\) be smooth manifolds and \(h\) a \(C^k\) \((k \geq 1)\) metric on \(\mathcal{N}\). We would like to consider maps \(F : \mathcal{M} \to \mathcal{N}\) with which we can pull the metric \(h\) across to \(\mathcal{M}\) in a way that this new geometry reflects the regularity of \(F\). If \(F\) is a \(C^k\) \((k \geq 1)\) diffeomorphism, then we are able to simply consider the pullback metric of \(h\) of regularity \(C^{k-1}\).

The key point to notice here is that we require the first derivatives of \(F\) to exist on a suitably large set of points in \(\mathcal{M}\). At a glance, it seems that it would suffice to ask \(F\) to be a Lipeomorphism, which we define as an invertible Lipschitz map with a Lipschitz inverse between two metric spaces. However, without specifying a metric a priori on \(\mathcal{M}\), we are unable to make sense of this terminology. The existence of derivatives is a local problem and therefore the notion of a local Lipeomorphism can be formulated between two manifolds by resorting to their locally Euclidean structure. We begin our discussion by formalising this notion. We note that the definition given below is independent of the metric \(h\) on \(\mathcal{N}\).

Definition 5.4 (Local Lipeomorphism). Let \(\mathcal{M}\) and \(\mathcal{N}\) be two smooth manifolds. Then, we say that \(F : \mathcal{M} \to \mathcal{N}\) is local Lipeomorphism if

(i) \(f\) is a homeomorphism, and
(ii) for all \(x \in \mathcal{M}\), there exists a chart \((U, \psi)\) near \(x\) and \((V, \varphi)\) near \(F(x)\) and a constant \(C \geq 1\) such that
\[
C^{-1}|x' - y'| \leq |(\varphi \circ F \circ \psi^{-1})(x') - (\varphi \circ F \circ \psi^{-1})(y')| \leq C|x' - y'|.
\]
For the sake of nomenclature, we call the charts \((U, \psi)\) and \((V, \varphi)\) the Lipeo-admissible charts.
As a consequence of this definition, the differential of \( \tilde{F} = (\varphi \circ F \circ \psi^{-1}) \) exists almost-everywhere on Lip
eo-admissible charts and hence, we are able to define the pushforward \( F_* \) almost-everywhere in \( \mathcal{M} \).

From this point onwards, let us assume \( h \) to be a complete \( C^k \) \( (k \geq 0) \) metric on \( \mathcal{N} \). Define \( g = F^* h \) and note that it is a rough metric. We will prove in this section that \( g \) has considerably better properties than an arbitrary rough metric.

5.2.1. The distance metric \( \rho_g \) and geodesy. Recall that the length of an absolutely continuous curve \( \gamma : I \to \mathcal{M} \) is given by:

\[
\ell_g(\gamma) = \int_I |\gamma(t)|_{g(\gamma(t))} \, dt = \int_I |F \circ \gamma(t)|_{h(F \circ \gamma(t))} \, dt < \infty.
\]

As for continuous metrics, by taking an infimum over the lengths \( \ell_g(\gamma) \) for such curves \( \gamma \) between points \( x, y \in \mathcal{M} \), we obtain a distance metric \( \rho_g \). The following proposition then gives a regularity criteria for geodesics of \( (\mathcal{M}, g) \).

**Proposition 5.5.** For every \( x, y \in \mathcal{M} \), \( \rho_g(x, y) = \rho_h(F(x), F(y)) \). Furthermore, if \( x', y' \in \mathcal{N} \) with a \( C^k \)-minimising geodesic (for \( k \geq 1 \)) between them, then there exists a Lipschitz curve on \( \mathcal{M} \) that is a minimising geodesic between \( F^{-1}(x') \) and \( F^{-1}(y') \).

**Proof.** Fix \( x, y \in \mathcal{M} \) and let \( \gamma : [0, 1] \to \mathcal{N} \) be an absolutely continuous curve between \( F(x) \) and \( F(y) \). Then, \( \rho_g(x, y) \leq \ell_g(F^{-1} \circ \gamma) = \ell_h(\gamma) \). Taking an infimum over such curves then gives that \( \rho_g(x, y) \leq \rho_h(F(x), F(y)) \). Conversely, if \( \sigma : [0, 1] \to \mathcal{M} \) is an absolutely continuous curve between \( x \) and \( y \), then \( \rho_h(F(x), F(y)) \leq \ell_h(F \circ \sigma) = \ell_g(\sigma) \) and so \( \rho_h(F(x), F(y)) \leq \rho_g(x, y) \).

Suppose now that \( \gamma : [0, 1] \to \mathcal{N} \) is a minimising geodesic between \( x', y' \in \mathcal{N} \) that is of class \( C^k \) for \( k \geq 1 \). Then, for \( t_1, t_2 \in I \), \( \rho_h(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2| \). On letting \( \sigma = F \circ \gamma \), we note that \( \rho_g(\sigma(t_1), \sigma(t_2)) = \rho_h(\gamma(t_1), \gamma(t_2)) \), which shows that \( \sigma \) is a minimising geodesic in \( g \). It is easy to see that \( \sigma \) is Lipschitz. \( \square \)

**Remark 5.6.**

1. Burtscher points out (in a private communication) that for \( C^1 \) metrics there exist “geodesics” in the sense of curves satisfying the Euler-Lagrange equations that are nowhere minimising due to a result of Hartman-Wintner in [15]. More seriously such curves may not be unique (see [14]). This forces us to only consider minimising geodesics.

2. When a metric is \( C^{0,1} \), it turns out that the regularity of its minimising geodesics (when they exist) are \( C^{1,1} \). When it is \( C^{0,\alpha} \) for \( \alpha \in (0, 1) \), then its minimising geodesics are \( C^{1,\alpha/2} \). Indeed, minimising geodesics of a \( C^k \) metric \( (k \geq 1) \) are \( C^{k+1} \). In each case, we see that the minimising geodesics have one integer exponent of regularity higher than the metric. While this may tempt us to embrace this observation as a maxim, it is not true for purely continuous metrics! Pettersson in his remarkable paper [22] gives a continuous metric on \( \mathbb{R}^2 \) for which a minimising geodesic passing through the origin is given by \( t(\sin(a(t), \cos(a(t)) \) where \( a(r) = \log(-\log|t|) \). It is easy to see that this curve is continuous but it is not differentiable at the origin.
5.2.2. The induced measure $\mu_g$. Recall the induced measure for rough metrics from §3.1. Here, we establish a relationship between the measures $\mu_h$ and $\mu_g$. This is necessary in order for us to relate the Lebesgue and Sobolev space theory of the two geometries. First, we present the following lemma which provides us with a formula expressing $\mu_g$ with respect to $\mu_h$ inside charts.

**Lemma 5.7.** Let $(U, \psi)$ be a chart on $\mathcal{M}$ near $x \in \mathcal{M}$ and $(V, \varphi)$ a chart near $F(x) \in \mathcal{N}$. Then,

$$d\mu_g(y) = |\det D\tilde{F}(y)| \sqrt{\det h(\tilde{F}(y))} \, d\mathcal{L}(y),$$

where $\tilde{F} = \varphi \circ F \circ \psi^{-1}$ and for $\mathcal{L}$-almost every $y \in \psi(U)$.

**Proof.** Let $\{x^i\}$ denote coordinates in $U$ and $\{y^j\}$ coordinates in $V$. It suffices to show that $\det g(y) = |\det D\tilde{F}(y)|^2 \det h(\tilde{F}(y))$ for $\mathcal{L}$-a.e. $y \in U$.

Inside the chart $V$, we can write $h(w) = h_{kl}(w) \, dy^k \otimes dy^l$ and hence, by the linearity of the pullback, $F^*h(y) = h_{kl}(F(y)) \, F^*dy^k \otimes F^*dy^l$. A straightforward calculation gives that $F^*dy^k = \partial_x \tilde{F} dx^i$ and hence, $F^*h(y) = h_{kl}(\partial_x \tilde{F}) dx^i \otimes dx^j$. Therefore, $g_{ij}(y) = h_{kl}(F(y)) \partial_x \tilde{F}^k \partial_x \tilde{F}^l$ and so $(g_{ij}) = (h_{kl}) D\tilde{F} \cdot D\tilde{F}$. Thus, $\det g(y) = (\det h(F(y)))(\det D\tilde{F})^2$. \hfill $\Box$

On combining this result with Lemma 2.2, we are able to compare the measure algebras of $\mu_g$ and $\mu_h$ via $F$.

**Proposition 5.8.** A function $\xi : \mathcal{M} \to \mathbb{C}$ is $\mu_g$-measurable if and only if $F^*\xi = \xi \circ F^{-1} : \mathcal{N} \to \mathbb{C}$ is $\mu_h$-measurable.

**Proof.** Let $(U_j, \psi_j)$ and correspondingly $(V_j = F(U_j), \varphi_j)$ be Lipeo-admissible charts and fix $\alpha \in \mathbb{R}$. Define

$$X_j = U_j \cap \xi^{-1}(\alpha, \infty) = \{x \in U_j : \xi(x) > \alpha\},$$

$$Y_j = V_j \cap (\xi \circ F^{-1})^{-1}(\alpha, \infty) = \{y \in V_j : \xi \circ F^{-1}(x) > \alpha\}.$$

Then, it is easy to see that $x \in X_j$ if and only if $F(x) \in V_j = F(U_j)$ and $\xi(x) > \alpha$ if and only if $(\xi \circ F^{-1})(F(x)) > \alpha$. Thus, $Y_j = F(X_j)$.

By Lemma 5.7, we have inside $U_j$ that $d\mu_g(x) = |\det D\tilde{F}(x)| \sqrt{\det h(\tilde{F}(x))} \, d\mathcal{L}(x)$ and therefore, on setting $f = |\det D\tilde{F}(x)| \sqrt{\det h(F(x))}$, we can conclude from Lemma 2.2 that $X_j$ is $\mu_g$-measurable if and only if $Y_j$ is $\mu_h$-measurable. Thus, $\xi^{-1}(\alpha, \infty) = \cup_j X_j$ is $\mu_g$ measurable if and only if $(\xi \circ F^{-1})^{-1}(\alpha, \infty) = \cup_j Y_j$ is $\mu_h$-measurable. \hfill $\Box$

As we would expect, we obtain that $\mu_g$ is indeed the pullback measure of $\mu_h$ under $F$. 

Proposition 5.9. The measure $\mu_g = F^* \mu_h$. That is if $\xi : \mathcal{M} \to \mathbb{C}$ $\mu_g$-integrable, then,

$$\int_{\mathcal{M}} \xi(x) \, d\mu_g(x) = \int_{\mathcal{N}} \xi \circ F^{-1}(y) \, d\mu_h(y).$$

Proof. Fix $(U, \psi)$ and $(V, \varphi)$ Lipschitz-admissible charts with $V = F(U)$. Then, first consider $\xi : \mathcal{M} \to \mathbb{C}$ with spt $\xi \subset U$ which is $\mu_g$ integrable. Then,

$$\int_{\mathcal{M}} \xi(x) \, d\mu_g(x) = \int_{\psi(U)} \xi \circ \psi^{-1}(x) \sqrt{\det g(x)} \, d\mathcal{L}(x).$$

Also, $\tilde{\xi} = \xi \circ F^{-1}$ is $\mu_h$ measurable with spt $\tilde{\xi} \subset V$ and

$$\int_{\mathcal{N}} \tilde{\xi}(y) \, d\mu_h(y) = \int_{\psi(V)} \tilde{\xi} \circ F^{-1} \circ \varphi^{-1}(y) \sqrt{\det h(y)} \, d\mathcal{L}(y).$$

Now, by our choice of $(U, \psi)$ and $(V, \varphi)$, we have that the map $\tilde{F} = \varphi \circ F \circ \psi^{-1} : \psi(U) \to \varphi(V)$ is bi-Lipschitz and therefore we have an integration by substitution formula:

$$\int_{\psi(U)} \xi \circ \psi^{-1}(x) \sqrt{\det g(x)} \, d\mathcal{L}(x) = \int_{\tilde{F}(\psi(U))} \xi \circ \psi^{-1} \circ \tilde{F}^{-1}(x) \sqrt{\det g(\tilde{F}^{-1}(x))} \det D\tilde{F}^{-1}(x) \, d\mathcal{L}(x).$$

From Lemma 5.7, $\sqrt{\det h(x)} = \sqrt{\det g(\tilde{F}^{-1}(x))} \det D\tilde{F}^{-1}(x)$ and $\xi \circ \psi^{-1} \circ \tilde{F}^{-1}(x) = \tilde{\xi} \circ \varphi^{-1}(x)$. Therefore, $\int_{U} \xi \, d\mu_g = \int_{V} \xi \circ F^{-1} \, d\mu_h$. For a general $\xi$, we can cover $\mathcal{M}$ by $(U_i, \psi_i)$ and corresponding $(V_i, \varphi_j)$ and the patch the integral together through a partition of unity.

As a direct consequence, we point out the following regularity result.

Proposition 5.10. $\mu_g$ is a Radon measure.

Proof. Since $g$ is a rough metric, by Proposition 3.5, we obtain that $\mu_g$ is Borel and finite on compact sets. Hence, we only need to prove Borel-regularity. For that, let $A \subset \mathcal{M}$. Then, there exists $\bar{B} \subset \mathcal{N}$ such that $F(A) \subset \bar{B}$ and $\mu_h(F(A)) = \mu_h(\bar{B})$. Set $B = F^{-1}(\bar{B})$ and note that $A \subset B$ and $\mu_g(A) = \mu_h(F(A)) = \mu_h(\bar{B}) = \mu_g(B)$. □

5.2.3. Lebesgue and Sobolev spaces. In our previous analysis where the manifold $\mathcal{M}$ was fixed and we dealt with two uniformly close metrics $g$ and $\tilde{g}$, we were able to relate the Lebesgue and Sobolev spaces of the two metrics to each other in a more or less straight forward manner. The situation we now face is different - we need to relate these spaces via $F$. The primary difficulty is that the pullback of $F$ does not preserve smoothness. Rather, it sends smooth functions to Lipschitz ones, and smooth tensors to tensors with only measurable coefficients. As a consequence we we dispense with our attempts to setup this analysis on differential forms and only...
concentrate on functions. Furthermore, we demonstrate the somewhat unsurprising fact that Lipschitz functions play a sufficient role in the Lebesgue and Sobolev theory so that we may use them in our analysis instead of smooth objects.

We begin our efforts by presenting the following easy but important observation which is immediate from Proposition 5.9.

**Proposition 5.11.** The map $F$ induces an isometry between $L^p(T^{(a,b)}\mathcal{M}, g)$ and $L^p(T^{(a,b)}\mathcal{N}, h)$.

Let us now fix some notation. Fix $\mathcal{P}$ to be a smooth manifold and let us denote the exterior derivative on this manifold by $\nabla^\mathcal{P}$. Let $\text{Lip}_\text{loc}(\mathcal{P})$ be the space of locally Lipschitz functions, defined by appealing to the local Euclidean structure of $\mathcal{P}$ so this space can be formulated independently of a metric on $\mathcal{P}$. Define the subspace of such functions with compact support by $\text{Lip}_c(\mathcal{P})$. The idea is to substitute $\text{Lip}_\text{loc}(\mathcal{P})$ for $C^k(\mathcal{P})$. The following lemma gives credence to our efforts.

**Lemma 5.12.** For every $f \in \text{Lip}_\text{loc}(\mathcal{N})$, $\nabla^\mathcal{N} f$ exists for $\mu_h$-a.e. Furthermore, if $\xi \in C^k(\mathcal{M})$ for $k \geq 1$, then $F^{-1}\star \xi \in \text{Lip}_\text{loc}(\mathcal{N})$ and $\nabla^\mathcal{M} \xi = F^\ast \nabla^\mathcal{N} F^{-1} \star \xi \mu_g$-a.e. Similarly, if $\eta \in C^k(\mathcal{N})$ for $k \geq 1$, then, $F \star \eta \in \text{Lip}_\text{loc}(\mathcal{M})$ and $\nabla^\mathcal{M} \eta \mu_g \ast F^\ast \nabla^\mathcal{N} F^{-1} \star \eta \mu_h$-a.e.

**Proof.** The conclusions follows immediately from the fact that $\nabla^\mathcal{P}$ is the exterior derivative on $\mathcal{P}$, $\psi^\ast \nabla^\mathcal{P} = \nabla^{\mathbb{R}^n} \psi^\ast$, and since $\varphi \circ F \circ \psi^{-1}$ is Lipschitz and a.e. differentiable.

The following proposition then yields that $\text{Lip}_c(\mathcal{N})$ is a suitable substitute for $C^k_c(\mathcal{N})$, $k \geq 1$.

**Proposition 5.13.** $\text{Lip}_c(\mathcal{N}) \subset W^{1,2}(\mathcal{N})$ and moreover, $\text{Lip}_c(\mathcal{N})$ is dense in $W^{1,2}(\mathcal{N})$.

**Proof.** First, consider $f \in \text{Lip}_c(\mathcal{N})$ with $\text{spt } f \subset V$, where $(V, \varphi)$ is a compact chart on $\mathcal{N}$. Then, $f \circ \varphi^{-1} : \varphi(V) \to \mathbb{C}$ is Lipschitz, and furthermore, $\text{spt } (f \circ \varphi^{-1}) \subset \varphi(V)$. Then, there exists $\delta > 0$ such that for all $\varepsilon < \delta$, $\text{spt } (\eta \ast f \circ \varphi^{-1}) \subset \varphi(V)$, where $\eta$ is the standard symmetric mollifier. Writing

$$\tilde{f}_\varepsilon = \begin{cases} \eta \ast f \circ \varphi^{-1} \circ \varphi(x) & x \in V \\ 0 & x \notin V, \end{cases}$$

defines $\tilde{f}_\varepsilon \in C^\infty_c(\mathcal{N})$. Then,

$$\int_{\mathcal{N}} |\tilde{f}_\varepsilon - f|^2 \ d\mu_h = \int_V |\tilde{f}_\varepsilon - f|^2 \ d\mu_h$$

$$= \int_{\varphi(V)} |\eta \ast f \circ \varphi^{-1} - f \circ \varphi^{-1}|^2 \sqrt{\det h} \ d\mathcal{L}$$

$$\leq C \int_{\varphi(V)} |\eta \ast f \circ \varphi^{-1} - f \circ \varphi^{-1}|^2 \ d\mathcal{L} \to 0$$
as \( \varepsilon \to 0 \) and where \( C \) depends on \( h \) and \( V \) by the continuity of \( h \) and compactness of \( V \).

Also,

\[
\int_{\mathcal{N}} |d\tilde{f}_\varepsilon - df_\varepsilon|_h^2 = \int_{\varphi(V)} |\varphi^{-1}\ast d\tilde{f}_\varepsilon - \varphi^{-1}\ast df_\varepsilon|_h^2 \sqrt{\det h} \ d\mathcal{L} \\
\leq C \int_{\varphi(V)} |\varphi^{-1}\ast d\tilde{f}_\varepsilon - \varphi^{-1}\ast df_\varepsilon|_h^2 \ d\mathcal{L}
\]

where \( C \) depends on \( h \) and \( V \) as before. Also,

\[
\partial_i \eta \ast f \circ \varphi^{-1} = \eta \ast \partial_i (f \circ \varphi^{-1})
\]

and therefore,

\[
\int_{\varphi(V)} |\varphi^{-1}\ast d\tilde{f}_\varepsilon - \varphi^{-1}\ast df_\varepsilon|_h^2 \ d\mathcal{L} \to 0
\]

as \( \varepsilon, \varepsilon' \to 0 \), which implies that there exists \( v \in L^2(\mathcal{N}, h) \) such that \( d\tilde{f}_\varepsilon = \nabla^\mathcal{N} \tilde{f}_\varepsilon \to v \).

Combining this with the fact that \( \tilde{f}_\varepsilon \to f \) and because \( \nabla^\mathcal{N} \) is a closable operator, we have that \( f \in W^{1,2}(\mathcal{N}) \) and \( v = \nabla^\mathcal{N} f \). Through a partition of unity argument, the requirement that \( V \) is compact and \( \text{spt } f \subset V \) can be dropped and makes the result valid for every \( f \in \text{Lip}_\varepsilon(\mathcal{N}) \).

Recall that, since \( g \) is a rough metric, the two operators \( \nabla^\mathcal{M} \) and \( \nabla^\mathcal{N}_2 \) are both automatically densely-defined are closable as a consequence of Proposition 3.12. Since \( \nabla^\mathcal{N}_c = \nabla^\mathcal{N}_2 \) due the completeness of \( h \), we obtain the following similar result on \( \mathcal{M} \).

**Proposition 5.14.** \( W^{1,2}_0(\mathcal{M}) = W^{1,2}(\mathcal{M}) \) and \( F^* W^{1,2}(\mathcal{N}) = W^{1,2}(\mathcal{M}) \) with \( \nabla^\mathcal{M} = F^\ast \nabla^\mathcal{N}_c F^{-1}\ast \).

**Proof.** First, we show that \( F^* W^{1,2}(\mathcal{N}) = W^{1,2}(\mathcal{M}) \). For this, fix \( f \in W^{1,2}(\mathcal{M}) \). Then, there exists \( f_j \in C^\infty \cap L^2(\mathcal{M}) \to L^2(T^*\mathcal{M}) \) such that \( f_j \to f \) and \( \nabla^\mathcal{M} f_j \to \nabla^\mathcal{M} f \). But \( F^{-1}\ast f_j \to F^{-1}\ast f \) and \( \nabla^\mathcal{N} F^{-1}\ast \to v \). By the closedness of \( \nabla^\mathcal{N}_c \), we have that \( F^{-1}\ast f_j \in W^{1,2}(\mathcal{N}) \) and \( v = \nabla^\mathcal{N}_c F^{-1}\ast f \). This shows that \( F^{-1}\ast W^{1,2}(\mathcal{M}) \subset W^{1,2}(\mathcal{N}) \). A similar argument then establishes that \( F^* W^{1,2}(\mathcal{N}) \subset W^{1,2}(\mathcal{M}) \). The formula \( \nabla^\mathcal{M} = F^\ast \nabla^\mathcal{N}_c F^{-1}\ast \) follows immediately.

Next we prove \( W^{1,2}_0(\mathcal{M}) = W^{1,2}(\mathcal{M}) \). For this, note that since \( F^* W^{1,2}(\mathcal{N}) = W^{1,2}(\mathcal{M}) \) and \( F^* \) is an \( L^2 \) isometry implies that \( F^* \text{Lip}_c(\mathcal{N}) \) is a dense subset of \( W^{1,2}(\mathcal{M}) \). It is enough to prove that \( F^* \text{Lip}_c(\mathcal{N}) \subset W^{1,2}_0(\mathcal{M}) \). In fact, we prove that \( \text{Lip}_c(\mathcal{M}) \subset W^{1,2}_0(\mathcal{M}) \) is a dense subset, and it is an easy observation that \( F^* \text{Lip}_c(\mathcal{N}) \subset \text{Lip}_c(\mathcal{M}) \).

Let \( f \in \text{Lip}_c(\mathcal{M}) \) such that \( \text{spt } f \subset U \), where \((U, \psi)\) and \((F(U) \subset V, \varphi)\) is a Lipeo-admissible compact chart. Then, we can arrange \( \varepsilon > 0 \) small such that \( f_\varepsilon = \)}
\(\psi^* \circ (\eta^* \circ f \circ \psi^{-1}) \in C^\infty_c(U)\), and extend it to 0 outside of \(U\). Then, \(f_\varepsilon \to f\) in \(L^2(\mathcal{M})\) and
\[
\int_{\mathcal{M}} |\nabla_c^\mathcal{M} f_\varepsilon - \nabla_c^\mathcal{M} f_{\varepsilon'}| \, d\mu_g \\
= \int_{\psi(U)} |d(\eta^* \circ f \circ \psi^{-1}) - d(\eta^* \circ f \circ \psi^{-1})| \, | \det D\tilde{F}| \sqrt{\det h(\tilde{F})} \, d\mathcal{L} \\
\leq C \int_{\psi(U)} |d(\eta^* \circ f \circ \psi^{-1}) - d(\eta^* \circ f \circ \psi^{-1})| \, d\mathcal{L} \to 0
\]
where \(\tilde{F} = \varphi \circ F \circ \psi^{-1}\) and the \(C\) depends on the lower bound of \(| \det D\varphi \circ F \circ \psi^{-1}|\), which exists since \(U\) and \(V\) are Lipeo-admissible and by the continuity of \(\sqrt{\det h}\) in \((\varphi \circ F)(U)\) which is guaranteed to have compact closure. Therefore, \(\nabla_c^\mathcal{M} f_\varepsilon \to v\) as \(\varepsilon \to 0\) and by the closedness of \(\overline{\nabla_c^\mathcal{M}}\), we conclude \(f \in W^{1,2}_0(\mathcal{M})\).

### 5.2.4. Transmitting quadratic estimates via \(F\)

We now demonstrate how to transmit quadratic estimates via \(F\) between \(L^2(\mathcal{M})\) and \(L^2(\mathcal{N})\). As we have previously mentioned, we only concentrate on the case of functions.

As before, let \(S_\mathcal{M} = (I, \overline{\nabla_c^\mathcal{M}})\) with domain \(D(S_\mathcal{M}) = W^{1,2}(\mathcal{M})\) and similarly define \(S_\mathcal{N}\). Then, write \(\mathcal{H}_1 = L^2(\mathcal{M}) \oplus L^2(T^*\mathcal{M}) \oplus L^2(T^*\mathcal{M})\) and define \(\Gamma_\mathcal{M} : \mathcal{H}_1 \to \mathcal{H}_1\) by
\[\Gamma_\mathcal{M} = \begin{pmatrix} 0 & 0 \\ S_\mathcal{M} & 0 \\ 0 & 0 \end{pmatrix}.
\]
Similarly, define \(\Gamma_\mathcal{N}\) upon replacing \(S_\mathcal{M}\) by \(S_\mathcal{N}\). Since \(\mathcal{H}_1\) is a Hilbert space and \(\Gamma_\mathcal{M}\) is densely-defined and closed, it follows that \(\Gamma_\mathcal{M}^*\) exists and it is also densely-defined and closed.

Write \(\mathcal{H}_2 = L^2(\mathcal{N}) \oplus L^2(T^*\mathcal{N}) \oplus L^2(T^*\mathcal{N})\) and define \(\Phi : \mathcal{H}_1 \to \mathcal{H}_2\) by
\[\Phi(u, v, w) = (F^{-1*}u, F^{-1*}v, F^{-1*}w),
\]
which is readily checked to be an isometry between \(\mathcal{H}_1\) and \(\mathcal{H}_2\). Letting \(\text{div}_g\) denote the divergence with respect to \(\nabla^\mathcal{M}\) and metric \(g\), we note that \(\Gamma_\mathcal{M}^* = \Phi^{-1}\Gamma_\mathcal{N}\Phi\) as a consequence of Lemma 5.1. That is precisely \(\text{div}_g = F^*\text{div}_h F^{-1*}\).

Let \(A \in L^\infty(\mathcal{L}(\mathcal{M} \oplus T^*\mathcal{M})), a \in L^\infty(\mathcal{L}(\mathcal{M}))\) and suppose they satisfy
\[\text{Re} \langle AS_Mv, v \rangle_1 \geq \kappa_1 \|v\|_{W^{1,2}(\mathcal{M})} \quad \text{and} \quad \text{Re} \langle au, u \rangle_1 \geq \kappa_2 \|u\|_1
\]
for \(v \in W^{1,2}(\mathcal{M})\) and \(u \in L^2(\mathcal{M})\). Set
\[B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.
\]
Write \(\tilde{B}_1 = F^{-1}B_1\Phi\), so that
\[\tilde{B}_1 = \begin{pmatrix} F^*aF^{-1*} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{B}_2 = \begin{pmatrix} 0 & 0 \\ 0 & F^*AF^{-1*} \end{pmatrix}.
\]
Then, on letting \(\Pi_{\mathcal{M},\tilde{B}} = \Gamma_\mathcal{M} + B_1\Gamma_\mathcal{M}B_2\) and \(\Pi_{\mathcal{N},\tilde{B}} = \Gamma_\mathcal{N} + \tilde{B}_1\Gamma_\mathcal{N}\tilde{B}_2\), we obtain the following main theorem of this section.
Theorem 5.15. The quadratic estimate
\[ \int_0^\infty \| t \Pi_{\mathcal{M},B}(I + t^2 \Pi_{\mathcal{M},B}^2)^{-1} u \|_g^2 \frac{dt}{t} \simeq \| u \|_g^2 \]
for all \( u \in \overline{\mathcal{R}(\Pi_{\mathcal{M},B})} \) if and only if
\[ \int_0^\infty \| t \Pi_{\mathcal{N},B}(I + t^2 \Pi_{\mathcal{N},B}^2)^{-1} v \|_h^2 \frac{dt}{t} \simeq \| v \|_h^2 \]
for all \( v \in \overline{\mathcal{R}(\Pi_{\mathcal{N},B})} \).

5.3. Lipschitz transformations. Let \((\mathcal{M}, g)\) and \((\mathcal{N}, h)\) now be two smooth manifold with continuous metrics. Suppose that the map \( F : (\mathcal{M}, g) \rightarrow (\mathcal{N}, h) \) is now a \textit{Lipseomorphism}, by which we mean that there exists \( C \geq 1 \) satisfying \( C^{-1} \rho_h(x, y) \leq \rho_h(F(x), F(y)) \leq C \rho_h(x, y) \). Our first observation is that such a map is a local Lipseomorphism as we have previous defined.

Proposition 5.16. \( F : \mathcal{M} \rightarrow \mathcal{N} \) is a local Lipseomorphism.

Proof. Since \( F \) is an invertible Lipschitz map with a Lipschitz inverse and since the Lipschitz property implies continuity, we have that \( F \) is a homeomorphism in the topologies induced on \( \mathcal{M} \) and \( \mathcal{N} \) by \( g \) and \( h \) respectively. But, by the continuity of \( g \) and \( h \), the induced topologies agrees with the natural topologies of the manifolds.

Now, let us show that it is locally Lipschitz. Choose charts \((U, \psi)\) near \( x \) and \((V, \varphi)\) near \( F(x) \) such that \( \overline{U}, \overline{V} \) are compact and that \( V = F(U) \). Let \( \tilde{F} = \varphi \circ F \circ \psi^{-1} \). Then, for any two points \( x', y' \in \varphi(U) \) and any absolutely continuous curve \( \gamma \) connecting these points. Then,
\[ |\dot{\gamma}(t)|^2 = h(\dot{\gamma}(t), \dot{\gamma}(t)) = h_{ij} \dot{\gamma}^i(t) \dot{\gamma}^j(t) \leq \left( \sup_{x \in U} \max_{ij} h_{ij}(x) \right) |\dot{\gamma}|_h(t)^2, \]
where continuity of \( h \) and compactness of \( U \) guarantees the finiteness of the supremum. Therefore, \( \rho_g(x', y') \lesssim |x' - y'| \) and therefore, \( \rho_h(\tilde{F}(x'), \tilde{F}(y')) \lesssim |x' - y'| \). But by considering an absolutely continuous curve \( \sigma \) connecting the points \( \tilde{F}(x') \) and \( \tilde{F}(y') \) in \( \varphi(V) \), through a similar argument we obtain that \(|\tilde{F}(x') - \tilde{F}(y')| \lesssim \rho_h(\tilde{F}(x'), \tilde{F}(y'))| \), with the constant depending on \( h \) and \( V \). In fact, a repetition of argument yields \( \rho_h(\tilde{F}(x'), \tilde{F}(y')) \lesssim |\tilde{F}(x') - \tilde{F}(y')| \) and since \( \rho_g(x', y') \lesssim \rho_h(\tilde{F}(x'), \tilde{F}(y')) \), by again repeating this argument, we get that \(|x' - y'| \lesssim \rho_g(x', y') \). This shows that \( \tilde{F} \) is a bi-Lipschitz mapping between \( \psi(U) \) and \( \varphi(V) \), with the constant depending on \( U, V, h \) and the global Lipschitz constant \( C \). \( \square \)

Let us now apply this result to the case \((\mathcal{N}, h) = (\mathcal{M}, g)\). Consider the pullback geometry \( \tilde{g} = F^*g \). This geometry is a \textit{Lipschitz transformation} of \((\mathcal{M}, g)\). As a consequence of Theorem 5.15, we are able to conclude that if quadratic estimates are satisfied for \((\mathcal{M}, g)\), then they are also satisfied for \((\mathcal{M}, \tilde{g})\).
Theorem 5.17. Let \((\mathcal{M}, g)\) be a smooth manifold with continuous metric. Then the quadratic estimate

\[
\int_0^\infty \| t\Pi_{\mathcal{M},B}(1 + t^2\Pi_{\mathcal{M},B}^2)^{-1}u \| ^2 \frac{dt}{t} \simeq \| u \|^2
\]

for all \(u \in \mathcal{R}(\Pi_{\mathcal{M},B})\) remains invariant under Lipschitz transformations of \((\mathcal{M}, g)\).

6. Metrics without lower bounds on injectivity radius

In each of the papers [5], [4], [21], [9] and [8] which establish solutions to the Kato square root problem, the underlying geometry possesses lower bounds on injectivity radius. Certainly, this is not an exhaustive list of references, however, we do not know of a solution to this problem where the underlying geometry fails to satisfy such bounds. In [8] harmonic coordinates are used to obtain uniform control at a small scale on the manifold, making it explicit how the assumption of such bounds assist in the proof.

As a consequence, McIntosh has asked whether this assumption is a necessary condition. In this section, we demonstrate that it is not. We first construct very simple 2-dimensional geometries which fail these bounds as suggested by Anton Petrunin and Sergei V. Ivanov. We demonstrate by application of previous results that the Kato square root problem can be solved on these geometries. Furthermore, we demonstrate that such examples are abundant in all dimensions above 2.

6.1. Cones as Lipschitz graphs. Cones and their smooth counterparts will be instrumental to the results we obtain in this section. We begin by examining \(n\)-cones and establishing some of their properties that will be of use to us.

Let \(h, r > 0\) and consider the \(n\)-cone in \(\mathbb{R}^{n+1}\) given by

\[
C_{r,h}^n = \{(x, t) \in \mathbb{R}^{n+1} : |x| = \frac{r}{h}(h - t), \ t \in [0, h]\}.
\]

This is a cone of height \(h\) and radius \(r\). Letting \(H : B_r(0) \to \mathbb{R}\) be the height function given by

\[
H_{r,h}(x) = h \left(1 - \frac{|x|}{r}\right),
\]

we note that the cone can be realised as the image of the graph function \(F_{r,h}(x) = \text{graph} H(x) = (x, H_{r,h}(x))\).

Let \(U\) be an open set in \(\mathbb{R}^n\) such that \(B_r(0) \subset U\). Then, define \(G_{r,h} : U \to \mathbb{R}^{n+1}\) as the map \(F_{r,h}\) whenever \(x \in B_r(0)\) and \((x, 0)\) otherwise. First, we prove the following.

Proposition 6.1. The map \(G_{r,h}\) satisfies

\[
|x - y| \leq |G_{r,h}(x) - G_{r,h}(y)| \leq \sqrt{1 + \frac{h^2}{r^2}} |x - y|.
\]
Proof. Suppose \( x, y \notin B_r(0) \). Then, it is easy to see that \( |G_{r,h}(x) - G_{r,h}(y)| = |x - y| \).

When \( x, y \in B_r(0) \), then \( G_{r,h}(x) - G_{r,h}(y) = (x - y, h/r^2(|y| - |x|)) \) and by the reverse triangle inequality \( ||x| - |y|| \leq |x - y| \) we obtain the inequality in the conclusion.

For the remaining case, let \( x \in B_r(0) \) and \( y \notin B_r(0) \). Then, \( G_{r,h}(x) - G_{r,h}(y) = (x - y, h(1 - |x|/r^2)) \). But by choice of \( x \) and \( y \), \( r \leq |y| \leq |x - y| + |x| \) which implies that

\[
1 - \frac{|x|}{r} \leq \frac{|x - y|}{r}
\]

and the desired estimate follows by direct calculation. \( \square \)

This proposition tells us that, in particular, the map \( G_{r,h} \) is almost-everywhere differentiable. Therefore, we define the pullback metric \( g = G_{r,h}^*(\cdot, \cdot)_{\mathbb{R}^{n+1}} \) on \( U \).

The following proposition is immediate.

**Proposition 6.2.** Let \( \gamma : I \to U \) be a smooth curve such that \( \gamma(0) \notin \{0\} \cup \partial B_r(0) \). Then,

\[
|\gamma'(0)| \leq |(G_{r,h} \circ \gamma)'(0)| \leq \sqrt{1 + \frac{h^2}{r^2}}|\gamma'(0)|.
\]

Moreover, for \( u \in T_xU, x \notin \{0\} \cup \partial B_r(0) \) (and in particular for almost-every \( x \)),

\[
|u|_{\delta} \leq |u|_g \leq \sqrt{1 + \frac{h^2}{r^2}}|u|_{\delta},
\]

where \( \delta \) is the usual inner product on \( U \) induced by \( \mathbb{R}^n \).

A particular consequence of this observation is that the metrics \( g \) and \( \delta \) are \( \sqrt{1 + (hr^{-1})^2} \)-close on \( U \).

6.2. Two dimensional examples. In this section, we construct 2-dimensional examples to illustrate that the Kato square root problem can be solved for metrics with zero injectivity radius. First, we establish some basics facts about 2-cones. We first realise the 2-cone as a quotient space since this makes it easier for us to study its geodesics.

Let \( T_{r,h} \) denote the region enclosed by the sector of angle \( \theta = \frac{2\pi r}{\sqrt{h^2 + r^2}} \) inside the ball \( B_{\sqrt{h^2 + r^2}}(0, S) \) where \( S = \sqrt{h^2 + r^2} \cos \left( \frac{\pi r}{\sqrt{h^2 + r^2}} \right) \) containing the triangular region

\[
\left\{ (x, t) \in \mathbb{R}^2 : |x| \leq \frac{R}{S}(S - t), \ t \in [0, S] \right\},
\]

of base length \( 2R \) and height \( S \), with \( R = \sqrt{h^2 + r^2} \sin \left( \frac{\pi r}{\sqrt{h^2 + r^2}} \right) \) as illustrated in Figure 1.
Define the map \( q_{r,h} : \mathcal{T}_{r,h} \to \mathbb{R}^2 \) as the identity map in \( \mathcal{T}_{r,h} \), the interior of \( \mathcal{T}_{r,h} \), and by identifying the points \((R(1-tS^{-1}), t)\) with \((-R(1-tS^{-1}), t)\).

**Lemma 6.3.** The quotient space \( \mathcal{T}_{r,h}/q_{r,h} \) is isometric to \( \mathcal{C}^2_{r,h} \).

**Proof.** Intersect the cone \( \mathcal{C}^2_{r,h} \) with a 2-plane containing the \( z \)-axis in \( \mathbb{R}^3 \). This produces a triangle whose height is \( h \) and radius \( r \) with hypotenuse \( \sqrt{h^2 + r^2} \). This along with the fact that the base of the cone has circumference \( 2\pi r \) allows us to compute \( S, R \) and \( \theta \) as in our definition above. The identification is then obvious. \( \square \)

Given this, we observe the following about geodesics of \( \mathcal{C}^2_{r,h} \).

**Lemma 6.4.** The straight lines inside \( \mathcal{T}_{r,h} \) between points map to geodesics in \( \mathcal{C}^2_{r,h} \) under \( q_{r,h} \).

**Proof.** The map \( q_{r,h} \) bends \( \mathcal{T}_{r,h} \) to the cone \( \mathcal{C}^2_{r,h} \). That is, it is an isometry. \( \square \)

The next is a very important lemma which establishes the existence of non-unique geodesics of decreasing length.

**Lemma 6.5.** Given \( \varepsilon > 0 \), there exists two points \( x, x' \) and distinct minimising smooth geodesics \( \gamma_{1,\varepsilon} \) and \( \gamma_{2,\varepsilon} \) between \( x \) and \( x' \) of length \( \varepsilon \). Furthermore, there are two constants \( C_{1,r,h,\varepsilon}, C_{2,r,h,\varepsilon} > 0 \) depending on \( h, r \) and \( \varepsilon \) such that the geodesics \( \gamma_{1,\varepsilon} \) and \( \gamma_{2,\varepsilon} \) are contained in \( G_{r,h}(A_{\varepsilon}) \) where \( A_{\varepsilon} \) is the Euclidean annulus

\[
\{ x \in B_r(0) : C_{1,r,h,\varepsilon} < |x| < C_{2,r,h,\varepsilon} \}.
\]

**Proof.** As a consequence of the previous lemma, it suffices to realise \( \gamma_{1,\varepsilon} \) and \( \gamma_{2,\varepsilon} \) as straight lines of length \( \varepsilon \) in \( \mathcal{T}_{r,h} \). This is easy. If we solve for \( t \) in \( R(-tS^{-1}) = \varepsilon \), we find that \( t = (1 - \frac{\varepsilon}{R}) \). Thus, fix the point \( y = (0, S(1 - \frac{\varepsilon}{R})) \). This can be identified with the point \( x \). Now consider \( y_\pm = (\pm \varepsilon, S(1 - \frac{\varepsilon}{R})) \). These two points are identified under the quotient map \( q_{r,h} \), so identify this with \( x' \). Then, we can take the straight line segment \( l_+ \) from \( y \) to \( y_+ \) and identify this with \( \gamma_{1,\varepsilon} \). Then \( \gamma_{2,\varepsilon} \) can be obtained
by the straight line $l_\perp$ from $y$ to $y_-$. Since $q_{r,h}$ is an isometry, $\gamma_{1,\varepsilon}$ and $\gamma_{2,\varepsilon}$ each have length $\varepsilon$.

Next, consider the ball of radius $\tau$ centred $(0,h)$, the apex of the region region $\mathcal{T}_{r,h}$. It is easy to verify that the intersection $B_\tau(0,h) \cap \mathcal{T}_{r,h}$ correspond to balls $B_\tau(0) \subset \mathbb{R}^2$ under quotiening and via the inverse of $G_{r,h}$ which is the projection map. It is easy to see that the lines $l_\pm$ are contained in $B_{C_1, r, h, \varepsilon}(0,h) \cap B_{C_2, r, h, \varepsilon}(0,h)$ where $C_i$ are constants dependent on the identification, $r$, $h$ and $\varepsilon$. Thus, it follows that $\gamma_{1,\varepsilon}, \gamma_{2,\varepsilon} : I \to A_\varepsilon$. 

We first demonstrate that the Kato square root problem can be solved for a non-smooth metric with zero injectivity radius.

**Theorem 6.6.** For any $C > 1$, there exists a metric $g$ which is $C$-close to the Euclidean metric $\delta$ obtained via a Lipschitz pullback for which $\text{inj}(\mathbb{R}^2, g) = 0$. Furthermore, the Kato square root problem $(K_{S,g})$ can be solved for $(\mathbb{R}^2, g)$ under the ellipticity assumptions $(E_g)$.

**Proof.** Set $r = 1$ and choose $h > 0$ such that $\sqrt{1 + h^2} = C$ and apply Proposition 6.2 with $U = \mathbb{R}^2$. As a consequence of Lemma 6.5, there exist distinct minimising-geodesics $\gamma_{1,\varepsilon}, \gamma_{2,\varepsilon}$ between two points $x_\varepsilon, x'_\varepsilon$ for each $\varepsilon > 0$. Therefore, $\text{inj}(\mathbb{R}^2, g, x_\varepsilon) \leq \varepsilon$ and hence $\text{inj}(\mathbb{R}^2, g) = 0$. The fact that the Kato square root problem $(K_{S,g})$ has solutions is immediate from Corollary 4.6. 

Using a similar construction as we did in the previous example, we produce a smooth metric with zero injectivity radius solving the aforementioned problem.

**Theorem 6.7.** For any $C > 1$, there exists a smooth metric which is $C$-close to $\delta$ on $\mathbb{R}^2$ for which $\text{inj}(\mathbb{R}^2, g) = 0$. The Kato square root problem $(K_{S,g})$ then has a solution on $(\mathbb{R}^2, g)$ under $(E_g)$.

**Proof.** As in the proof of the previous theorem, set $r = 1$ and choose $h$ such that $\sqrt{1 + h^2} = C$. Consider the 2-cone of radius 1 contained in $B_2(0)$. Then, for any $k > 0$, we can find an annulus $A_{\frac{1}{k}}$ containing two points $x_k$ and $x'_k$ and non-unique minimising geodesics $\gamma_{1,k}$ and $\gamma_{2,k}$ between them of length $\frac{1}{k}$ by Lemma 6.5. As a consequence, we can smooth the cone at the apex above $A_{\frac{1}{k}}$ and below at the base to obtain a smooth map $G^\infty_{1,h,k} : B_2(0) \to \mathbb{R}^{n+1}$. These geodesics are still geodesics in this new space as $A_{\frac{1}{k}}$ is an open set and hence, it is totally geodesic. Now, choose a set of points $y_k = (0, 3k) \in \mathbb{R}^2$ for $k \geq 0$ and consider the map:

$$G(x) = \begin{cases} x & x \notin B_2(y_k) \\ G^\infty_{1,h,k}(x - y_k) & x \in B_2(y_k). \end{cases}$$

It is easy to see that $G$ is a smooth map and pullback metric $g = G^*(\cdot, \cdot)_{\mathbb{R}^{n+1}}$ is smooth and $C$-close to the Euclidean metric. By construction, $\text{inj}(\mathbb{R}^2, g, x_k) \leq \frac{1}{k}$ and hence, $\text{inj}(\mathbb{R}^2, g) = 0$. As before, the fact that the Kato square root problem $(K_{S,g})$ has solutions is immediate from Corollary 4.6. 

6.3. Higher dimensions. In this section, we demonstrate that metrics with zero injectivity radius are abundant in the space of rough metrics. We use this to show that the Kato square root problem can be solved for a wide class of metrics and also demonstrate that there exist low regularity metrics on compact manifolds for which injectivity radius bounds fail. Throughout this section, we assume that the dimension is at least 2.

First, as in the previous section, we define the annulus \( A^n_\varepsilon \) in our more general situation as

\[
A^n_\varepsilon = \{ x \in B_r(0) : C_{1,r,h,\varepsilon} < |x| < C_{2,r,h,\varepsilon} \},
\]

where the constants \( C_{1,r,h,\varepsilon} \) and \( C_{2,r,h,\varepsilon} \) are the ones guaranteed by Lemma 6.5. We then obtain the following natural generalisation of this lemma.

**Lemma 6.8.** For every \( \varepsilon > 0 \), the set \( G_{r,h}(A^n_\varepsilon) \) contains two points \( x_\varepsilon \) and \( x'_\varepsilon \) and two distinct minimising geodesics \( \gamma_{1,\varepsilon}, \gamma_{2,\varepsilon} \) between these two points such that their length is \( \varepsilon \).

**Proof.** Let \( x = (x_1, \ldots, x_n) \in S^{n-1} \subset \mathbb{R}^n \) be points on the \((n-1)\)-sphere. Then, let \( s(x) = (x_1, x_2, -x_3, \ldots, -x_n) \). Note that the fixed point set of this map \( s \) precisely the circle \( S^1 \).

Now, for points \((x,t) \in C^n_{r,h}\), define \( \Phi(x,t) = (s(x),t) \). The map \( \Phi : C^n_{r,h} \to C^n_{r,h} \) is an isometry. Moreover, the restricted map \( \Phi_\varepsilon(x,t) = \Phi|_{A^n_\varepsilon} : A^n_\varepsilon \to A^n_\varepsilon \) is also an isometry. It is easy to see that \( A^2_\varepsilon = \{ (x,t) : \Phi_\varepsilon(x,t) = (x,t) \} \). Since \( A^n_\varepsilon \) stays away from the base and apex of \( C^n_{r,h} \), it is a smooth manifold submanifold of \( \mathbb{R}^{n+1} \) and by the fact that \( \Phi_\varepsilon \) is an isometry, Theorem 1.10.15 in [18] guarantees us that \( A^2_\varepsilon \) is a totally geodesic submanifold. Invoking Lemma 6.5 completes the proof. \( \square \)

The following final lemma is instrumental in proving the main theorem of this section. Note that we write \( B^*_\varepsilon(x) = \psi^{-1}(B_r(\psi(x))) \) inside a coordinate chart \((U, \psi)\).

**Lemma 6.9.** Let \( \mathcal{M} \) be a smooth manifold with a continuous metric \( g \). Then, for any \( x_0 \) and \( C > 1 \), there exists an at most countable collection \( \mathcal{C} \) of charts covering \( \mathcal{M} \) and an \( r_0 > 0 \) such that

(i) for all \( y \in U_i \) and \( u \in T_y U_i \),

\[
C^{-1}|u|_{\psi_i',\delta} \leq |u|_g \leq C|u|_{\psi_i',\delta},
\]

(ii) \( B^*_r(x_0) \subset U_0 \), and

(iii) \( B^*_r(x_0) \cap U_i = \emptyset \) for \( i > 0 \).

**Proof.** For \( x \in \mathcal{M} \), we can find a chart \((V_x, \psi_x)\) satisfying the inequality \( C^{-1}|u|_{\psi_x',\delta} \leq |u|_g \leq C|u|_{\psi_x',\delta} \) for all \( u \in T_x V_x \) by the continuity of \( g \).

Let \( \{V_i\} \) be a countable subcover, choosing the index so that \( x_0 \in V_0 \). We can choose \( r_0 > 0 \) small such that \( B_{r_0}(\psi_0(x_0)) \subset \psi_0(V_0) \). Define a new set of charts by
restricting the sets \( V_i \) on setting \( \tilde{U}_0 = V_0 \) and \( \tilde{U}_i = V_i \setminus B_{r_0}^r(x_0) \) for \( i > 0 \). Then, define \( \mathcal{C} = \{ \tilde{U}_i : \tilde{U}_i \neq \emptyset \} \).

It is easy to see that \( \mathcal{C} \) covers \( \mathcal{M} \). Since the \( \tilde{U}_i \) are obtained as restrictions of the \( V_i \), it is easy to see that the desire inequality in (i) still holds. This shows (i). That (ii) is true is immediate, and (iii) is true by the construction of the \( \tilde{U}_i \).

Whit the aid of this tool, we prove the following main theorem of this section.

**Theorem 6.10.** Let \( \mathcal{M} \) be a smooth manifold of dimension at least 2 and \( g \) a continuous metric. Given \( C > 1 \), and a point \( x_0 \in \mathcal{M} \), there exists a rough metric \( h \) such that:

(i) it preserves the topology of \( \mathcal{M} \),

(ii) it is smooth everywhere but at \( x_0 \),

(iii) the geodesics through \( x_0 \) are Lipschitz,

(iv) it is \( C \)-close to \( g \),

(v) \( \text{inj}(\mathcal{M} \setminus \{x_0\}, h) = 0 \).

**Proof.** For a given \( x_0 \in \mathcal{M} \) take the cover given by Lemma 6.9 with constant \( C_1 > 1 \) to be chosen later. Let \( \{\varphi_i\} \) be a smooth partition of unity subordinate to \( \mathcal{C} = \{U_0, U_1, \ldots \} \). Note that since \( \overline{B_{r_0}^r(x_0)} \cap U_i = \emptyset \) for \( i > 0 \), we have that \( \varphi_0 \equiv 0 \) on \( \overline{B_{r_0}^r(x_0)} \).

Define a new metric \( \tilde{g} = \sum_i \varphi_i \psi_i^* \delta \). It is easy to see that this metric is smooth and a direct calculation at an arbitrary point \( x \) will show that it is \( C_1 \)-close to \( g \).

Set \( r = \frac{r_0}{10} \), and choose \( h > 0 \) such that \( \sqrt{1 + \frac{h^2}{r^2}} \leq C_2 \), where \( C_2 > 0 \) to be chosen later. Inside \( \varphi_0(U_0) \), remove the Euclidean ball \( B_r(y_0) \subset \mathbb{R}^n \) where \( y_0 = \psi_0(x_0) \) and affix an \( n \)-cone of radius \( r \) and height \( h \) via the map \( G_{r,h} : B_{r_0}(\varphi(x_0)) \to \mathbb{R}^n+1 \) on setting \( U = B_{r_0}(\varphi(x_0)) \) in our construction in \( \S 6.1 \). Note that \( G_{r,h}^* (\cdot, \cdot)_{\mathbb{R}^n+1} = (\cdot, \cdot)_{\mathbb{R}^n} \) in the annulus \( B_{r_0}(y_0) \setminus \overline{B_r(y_0)} \). We can smooth the base of the cone to produce a map \( \tilde{G}_{r,h} : B_r(y_0) \to \mathbb{R}^n+1 \) such that there exists \( \tau \in (0, r) \) with \( \tilde{G}_{r,h}^* (\cdot, \cdot)_{\mathbb{R}^n+1} = (\cdot, \cdot)_{\mathbb{R}^n} \) for \( B_r(y_0) \setminus \overline{B_{r-\tau}(y_0)} \), a \( c \in (0, 1) \) with \( \tilde{G}_{r,h} = G_{r,h} \) in \( B_{cr}(y_0) \), and so that the Lipschitz constant \( C_2 \) is unaltered.

Now, consider the pullback metric \( \tilde{g}_0 = \tilde{G}_{r,h}^* \) on \( B_{r_0}(y_0) \). It is easy to see that this metric preserves the topology in \( B_{r_0}(y_0) \). Thus, define

\[
h(x) = \begin{cases} 
\psi_0^* \tilde{g}_0(x) & x \in B_{r_0}^r(x_0), \\
\tilde{g}(x) & \text{otherwise}.
\end{cases}
\]

This is a smooth metric away from \( x_0 \), because by what we have said before, \( \varphi_0 \equiv 1 \) on the ball \( B_{r_0}^r(x_0) \), \( \tilde{g}_0 = \delta \) in \( B_{r_0}(y_0) \setminus \overline{B_{r-\tau}(y_0)} \), and because \( \tilde{g}_0 \) fails to be smooth only at \( y_0 \). As we have already mentioned, \( \tilde{G}_{r,h} = G_{r,h} \) in a neighbourhood of the
apex, and hence, it is a Lipschitz map there. As a consequence, geodesics through $x_0$ are Lipschitz.

So far, we have shown (i) to (iii). To show (iv), note that $h$ is $C_2$-close to $\tilde{g}$ and $\tilde{g}$ is $C_1$ close to $g$, we have that $h$ is $C_1 C_2$-close to $g$. On setting $C_1 = C_2 = \sqrt{C}$, we obtain that $g$ is $C$-close to $h$.

To show that $\text{inj}(\mathcal{M}, h) = 0$, note that there is some $\varepsilon_0$ so that whenever $\varepsilon \in (0, \varepsilon_0)$, $\tilde{G}_{r,h}(A_\varepsilon) = G_{r,h}(A_\varepsilon)$. The set $A_\varepsilon$ is an open set in $\mathbb{R}^n$ and hence, $G_{r,h}(A_\varepsilon)$ it is an open set in $\text{Img} \tilde{G}_{r,h}$. Open sets are totally geodesic submanifolds, and by Lemma 6.8, $G_{r,h}(A_\varepsilon)$ contains a two distinct minimising geodesics of length $\varepsilon$ between points $y_\varepsilon$ and $y'_\varepsilon$. Thus, on setting $x_\varepsilon = \psi_0^{-1}(G_{r,h}(x_\varepsilon))$, we have that $\text{inj}(\mathcal{M}, h, x_\varepsilon) \leq \varepsilon$. Therefore, $\text{inj}(\mathcal{M}, h) = 0$. □

As aforementioned, we are currently able to prove the Kato square root problem when $g$ is complete, smooth, $|\text{Ric}| \leq \eta$ and $\text{inj}(\mathcal{M}, g) \geq \kappa > 0$. However, by this theorem, we are able to find arbitrarily close metrics $h$ to $g$ for which the injectivity radius bounds fail, and by Corollary 4.6, we can solve the Kato square root problem for such metrics. This leads us to believe that the lower bounds on injectivity radius in the proofs of the Kato square root problem is a technical assumption.

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lashi Bandara, Centre for Mathematics and its Applications, Australian National University, Canberra, ACT, 0200, Australia

URL: http://maths.anu.edu.au/~bandara

E-mail address: lashi.bandara@anu.edu.au