Integrability of classical strings dual for noncommutative gauge theories

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Abstract

We derive the gravity duals of noncommutative gauge theories from the Yang-Baxter sigma model description of the AdS$_5 \times$S$^5$ superstring with classical $r$-matrices. The corresponding classical $r$-matrices are 1) solutions of the classical Yang-Baxter equation (CYBE), 2) skew-symmetric, 3) nilpotent and 4) abelian. Hence these should be called abelian Jordanian deformations. As a result, the gravity duals are shown to be integrable deformations of AdS$_5 \times$S$^5$. Then, abelian twists of AdS$_5$ are also investigated. These results provide a support for the gravity/CYBE correspondence proposed in arXiv:1404.1838.

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1 Introduction

A particular class of gauge/gravity dualities can be seen as deformations of AdS/CFT [1]. With great progress, an integrable structure inhabiting AdS/CFT is well recognized now [2]. The Green-Schwarz string action on $\text{AdS}_5 \times \text{S}^5$ is constructed from a supercoset [3]

$$PSU(2, 2|4)/[SO(1, 4) \times SO(5)]$$

and the classical integrability follows from the $\mathbb{Z}_4$-grading [4]. Some deformations of the AdS/CFT correspondence may preserve the integrability and hence it would be interesting to consider a method to classify the integrable deformations.

A possible way is to employ the Yang-Baxter sigma model description [10, 11] (For $q$-deformed $su(2)$ and its affine extension, see [12] and [13], respectively). It has been applied to the $\text{AdS}_5 \times \text{S}^5$ superstring in [14]. According to this approach, integrable deformations of $\text{AdS}_5 \times \text{S}^5$ are given in terms of classical $r$-matrices satisfying modified classical Yang-Baxter equation (mCYBE). The case of [14] corresponds to the classical $r$-matrix of Drinfeld-Jimbo type [15–17]. The metric in the string frame and NS-NS two-form are obtained [18] and some generalizations to other cases are discussed in [19]. It is an intriguing issue to look for the complete gravitational solution. A mirror TBA is also proposed [20].

As a generalization of the Yang-Baxter sigma model description, one may consider classical Yang-Baxter equation (CYBE) rather than mCYBE. The classical action of the $\text{AdS}_5 \times \text{S}^5$ superstring has been constructed in [21]. The integrable deformations are basically regarded as Drinfeld-Reshetikhin twists [15] [16, 22] including Jordanian twists [23, 24] and abelian twists. Hence one can classify integrable deformations of this kind in terms of classical $r$-matrices. We will refer this picture as to the gravity/CYBE correspondence. The first example is presented in [25]. As another example, Lunin-Maldacena backgrounds [27, 28] have also been derived [29].

In this note, we derive the gravity duals of noncommutative (NC) gauge theories [33, 34] from the Yang-Baxter sigma model description of the $\text{AdS}_5 \times \text{S}^5$ superstring with classical $r$-matrices. The corresponding classical $r$-matrices are 1) solutions of CYBE, 2) skew-symmetric, 3) nilpotent and 4) abelian. Hence these should be called abelian Jordanian deformations. As a result, the gravity duals of NC gauge theories are shown to be integrable deformations of $\text{AdS}_5 \times \text{S}^5$. Then, abelian twists of $\text{AdS}_5$ are also investigated. These results provide a support for the gravity/CYBE correspondence proposed in [29].

1 For another formulation [4] of the $\text{AdS}_5 \times \text{S}^5$ superstring, the classical integrability is argued in [0]. For a classification of integrable supercosets, see [7, 8]. For an argument on non-symmetric cosets, see [9].

2 The solution is closely related to the one in Appendix C of [20].
This note is organized as follows. Section 2 gives a short summary of the Yang-Baxter sigma model description of the AdS$_5 \times $S$^5$ superstring with classical $r$-matrices satisfying CYBE. Then we introduce three classes of skew-symmetric solutions of CYBE. A new class of $r$-matrices induces abelian Jordanian deformations. Section 3 presents examples of abelian Jordanian type, which lead to the gravity duals of NC gauge theories. In section 4, we consider a deformation of AdS$_5$ with an abelian $r$-matrix concerned with a TsT transformation of AdS$_5$. Section 5 is devoted to conclusion and discussion. We argue some implications of this result and future directions in studies of the gravity/CYBE correspondence. In Appendix A our notation and convention are summarized. Appendix B presents the gravity duals of NC gauge theories with six deformation parameters. Appendix C describes the detailed computation of three-parameter abelian twists of AdS$_5$.

2 Integrable deformations of the AdS$_5 \times $S$^5$ superstring

We introduce here integrable deformations of the AdS$_5 \times $S$^5$ superstring based on the Yang-Baxter sigma model description with CYBE [21]. After giving a short review on the general form of deformed actions, we present three classes of classical $r$-matrices.

2.1 Deforming the AdS$_5 \times $S$^5$ superstring action with CYBE

A class of integrable deformations of the AdS$_5 \times $S$^5$ superstring can be described with classical $r$-matrices satisfying CYBE [21]. The deformed action is given by

$$S = -\frac{1}{4} (\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \int_{-\infty}^{\infty} d\tau \int_{0}^{2\pi} d\sigma \text{Str} \left( A_\alpha d \circ \frac{1}{1 - R_g \circ d} A_\beta \right),$$

where the left-invariant one-form $A_\alpha$ is defined as

$$A_\alpha \equiv g^{-1} \partial_\alpha g, \quad g \in SU(2,2|4).$$

Here $\gamma^{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ are the flat metric and the anti-symmetric tensor on the string world-sheet. The operator $R_g$ is defined as

$$R_g(X) \equiv g^{-1} R(gXg^{-1})g,$$

where a linear operator $R$ satisfies CYBE rather than mCYBE [14]. The R-operator is related to the tensorial representation of classical $r$-matrix through

$$R(X) = \text{Tr}_2 [r(1 \otimes X)] = \sum_i (a_i \text{Tr}(b_i X) - b_i \text{Tr}(a_i X))$$

with $r = \sum_i a_i \wedge b_i \equiv \sum_i (a_i \otimes b_i - b_i \otimes a_i).$
The operator $d$ is given by the following,

$$d = P_1 + 2P_2 - P_3,$$

(2.5)

where $P_i \ (i = 0, 1, 2, 3)$ are the projections to the $\mathbb{Z}_4$-graded components of $\mathfrak{su}(2, 2|4)$. $P_0, P_2$ and $P_1, P_3$ are the projectors to the bosonic and fermionic generators, respectively. In particular, $P_0(\mathfrak{su}(2, 2|4))$ is nothing but $\mathfrak{so}(1, 4) \oplus \mathfrak{so}(5)$.

For the action (2.1) with an R-operator satisfying CYBE, the Lax pair has been constructed [21] and the classical integrability is ensured in this sense. The $\kappa$-invariance has been proven as well [21].

### 2.2 A classification of classical $r$-matrices

According to the construction of the deformed string action, one may expect the correspondence between integrable deformations of $\text{AdS}_5 \times S^5$ and classical $r$-matrices, called the gravity/CYBE correspondence [29]. To study along this direction, it would be valuable to classify some typical class of skew-symmetric solutions of CYBE.

There are three types of classical $r$-matrices: i) Jordanian, ii) abelian, and iii) abelian Jordanian. In particular, the third class will play a crucial role in the next section.

In order to study deformations of $\text{AdS}_5$ later, let us consider the case of $\mathfrak{su}(2, 2)$.

**i) Jordanian $r$-matrix.**

The first class is classical $r$-matrices of Jordanian type,

$$r_{\text{Jor}} = E_{ij} \wedge (E_{ii} - E_{jj}) - 2 \sum_{i<k<j} E_{ik} \wedge E_{kj} \quad (1 \leq i < j \leq 4),$$

(2.6)

where $(E_{ij})_{kl} \equiv \delta_{ik}\delta_{jl}$ are the fundamental representation of $\mathfrak{su}(2, 2)$. The characteristic property of Jordanian type $r$-matrices is the nilpotency. Indeed, we could verify that the associated linear R-operator exhibits $(R_{\text{Jor}})^n = 0$ for $n \geq 3$.

Jordanian deformations of the $\text{AdS}_5 \times S^5$ superstring are considered in [21]. A simple example of the corresponding type IIB supergravity solution is presented in [25]. Only the $\text{AdS}_5$ part is deformed and it contains a three-dimensional Schrödinger spacetime as a subspace. Hence it may be regarded as a generalization of [30–32]. It seems likely that the resulting metric is closely related to a null Melvin twist [26].
ii) Abelian $r$-matrix.

The second class is abelian $r$-matrices composed of the Cartan generators as follows:

$$r_{Abe} = \sum_{1 \leq i < j \leq 3} \mu_{ij} (E_{ii} - E_{i+1,j+1}) \wedge (E_{jj} - E_{j+1,j+1}),$$

(2.7)

where $\mu_{ij} = -\mu_{ji}$ are arbitrary parameters. Since these commute with each other and hence satisfy CYBE obviously. The abelian $r$-matrix is a particular example of the Drinfeld-Reshetikhin twists [15,16,22]. Note that abelian $r$-matrices are intrinsic to higher rank cases (rank $\geq 2$).

It has been shown in [29] that abelian $r$-matrices lead to $\gamma$-deformed backgrounds [28], which include the Lunin-Maldacena background [27] as a particular case. In section 4, we will consider an abelian twist of AdS$_5$ with a single parameter. For multi-parameter cases, see Appendix C.

iii) Abelian Jordanian $r$-matrix.

The third class is composed of $r$-matrices which are nilpotent and abelian. These should be called abelian Jordanian $r$-matrices. A typical example takes the following form,

$$r_{AJ} = \sum_{i,k=1,2, j,l=3,4} \nu_{(ij),(kl)} E_{ij} \wedge E_{kl},$$

(2.8)

with arbitrary parameters $\nu_{(ij),(kl)} = -\nu_{(kl),(ij)}$. Because $E_{ij} (i = 1,2, j = 3,4)$ are the positive root generators and commute with each other, the square of the associated R-operator already vanishes like,

$$(R_{AJ})^2 = 0,$$

in comparison to Jordanian $r$-matrices (2.6) for which $(R_{Jor})^2 \neq 0$ and $(R_{Jor})^3 = 0$ in general.

In the next section, we will show that classical $r$-matrices of abelian Jordanian type correspond to the gravity duals of NC gauge theories [33,34].

3 Examples - gravity duals of NC gauge theories

Let us consider examples of classical $r$-matrices of abelian Jordanian type. These lead to the gravity duals of NC gauge theories [33,34]. Hereafter we will concentrate on the AdS$_5$ part and $S^5$ is not deformed.

A possible example is given by

$$r_{AJ} = \mu p_2 \wedge p_3 + \nu p_0 \wedge p_1,$$

(3.1)
where $\mu, \nu$ are deformation parameters. Here $p_\mu$ ($\mu = 0, 1, 2, 3$) are the upper triangular matrices defined as

$$p_\mu \equiv \frac{1}{2} \gamma_\mu - m_{\mu 5}. \quad (3.2)$$

For our convention of $\gamma_\mu$ and the $\mathfrak{su}(2, 2)$ generators, see Appendix A. It should be emphasized that $p_\mu$'s are upper triangular and satisfy the following property:

$$p_\mu p_\nu = 0. \quad (3.3)$$

Thus the classical $r$-matrix (3.1) is of abelian Jordanian type and trivially satisfies CYBE.

To evaluate the Lagrangian (2.1), let us take the following coset parametrization [25]:

$$g_s = \exp \left[ p_0 x^0 + p_1 x^1 + p_2 x^2 + p_3 x^3 \right] \exp \left[ \frac{\gamma_5}{2} \log z \right] \in SU(2, 2)/SO(1, 4). \quad (3.4)$$

Then the AdS$_5$ part of (2.1) can be rewritten as

$$L = - \frac{1}{2} \left( \gamma^{\alpha \beta} - \epsilon^{\alpha \beta} \right) \text{Tr} [A_\alpha P_2 (J_\beta)]$$

with

$$J_\beta \equiv \frac{1}{1 - 2 \eta [R_{A\beta}] g} P_2 A_\beta. \quad (3.5)$$

Here $A_\alpha = g^{-1} \partial_\alpha g$ is restricted to $\mathfrak{su}(2, 2)$ and the associated R-operator $R_{A\beta}$ with (3.1) is determined by the relation (2.4).

It is convenient to divide the Lagrangian $L$ into two parts like $L = L_G + L_B$, where $L_G$ is the metric part and $L_B$ is the coupling to an NS-NS two-form, respectively:

$$L_G \equiv \frac{1}{2} \left[ \text{Tr} (A_\tau P_2 (J_\tau)) - \text{Tr} (A_\sigma P_2 (J_\sigma)) \right],$$

$$L_B \equiv \frac{1}{2} \left[ \text{Tr} (A_\tau P_2 (J_\sigma)) - \text{Tr} (A_\sigma P_2 (J_\tau)) \right]. \quad (3.7)$$

To derive the explicit form of $L$, it is sufficient to compute the projected current $P_2(\partial_\alpha)$ rather than $\partial_\alpha$ itself. Hence the computation is reduced to solving the following equation,

$$\left( 1 - 2 \eta P_2 \circ [R_{A\beta}] g \right) P_2 (\partial_\alpha) = P_2 (A_\alpha). \quad (3.8)$$

Note that $P_2 (A_\alpha)$ is expanded with $\gamma$ matrices as follows:

$$P_2 (A_\alpha) = \frac{\partial_\alpha x^0 \gamma_0 + \partial_\alpha x^1 \gamma_1 + \partial_\alpha x^2 \gamma_2 + \partial_\alpha x^3 \gamma_3 + \partial_\alpha z \gamma_5}{2z}. \quad (3.9)$$

Then, by combining (3.9) with (3.8), $P_2 (\partial_\alpha)$ can be obtained as

$$P_2 (\partial_\alpha) = \frac{z (\partial_\alpha x_0 + 2 \eta \nu \partial_\alpha x_1)}{2 (z^4 - 4 \eta^2 \nu^2)} \gamma_0 + \frac{z (\partial_\alpha x_1 + 2 \eta \mu \partial_\alpha x_0)}{2 (z^4 - 4 \eta^2 \nu^2)} \gamma_1$$

$$+ \frac{z (\partial_\alpha x_2 + 2 \eta \mu \partial_\alpha x_3)}{2 (z^4 + 4 \eta^2 \mu^2)} \gamma_2 + \frac{z (\partial_\alpha x_3 - 2 \eta \mu \partial_\alpha x_2)}{2 (z^4 + 4 \eta^2 \mu^2)} \gamma_3 + \frac{\partial_\alpha z}{2z} \gamma_5. \quad (3.10)$$
The resulting forms of $L_G$ and $L_B$ are given by, respectively,

$$L_G = \frac{-\gamma^{\alpha\beta}}{2} \left[ z^2 ( -\partial_\alpha x_0 \partial_\beta x_0 + \partial_\alpha x_1 \partial_\beta x_1 ) + \frac{z^2 ( \partial_\alpha x_2 \partial_\beta x_2 + \partial_\alpha x_3 \partial_\beta x_3 )}{z^4 + 4\eta^2 \mu^2} + \frac{\partial_\alpha z \partial_\beta z}{z^2} \right],$$  \hspace{1cm} (3.11)

$$L_B = \epsilon^{\alpha\beta} \left[ -\frac{2\eta \nu}{z^4 - 4\eta^2 \nu^2} \partial_\alpha x_0 \partial_\beta x_1 + \frac{2\eta \mu}{z^4 + 4\eta^2 \mu^2} \partial_\alpha x_2 \partial_\beta x_3 \right].$$  \hspace{1cm} (3.12)

Here two deformation parameters $\mu, \nu$ and one normalization factor $\eta$ are contained.

It is easy to see the metric and the NS-NS two-form from (3.11) and (3.12). By introducing new parameter $a$ and $a'$ through the identification,

$$2\eta \mu = a^2, \hspace{1cm} 2\eta \nu = ia'^2,$$  \hspace{1cm} (3.13)

one can find that the resulting metric and two-form exactly agree with the ones of the gravity duals of NC gauge theories presented in \cite{33,34}, up to the coordinate change $z = 1/u$ and the Wick rotation $x_0 \rightarrow ix_0$. This result shows that the gravity duals of NC gauge theories \cite{33,34} are integrable deformation of AdS$_5$.

### 4 Abelian twists of AdS$_5$

As another kind of integrable deformation of AdS$_5$, we consider an abelian twist of AdS$_5$ with a single parameter$^3$. For a three-parameter generalization, see Appendix C.

Let us consider an abelian $r$-matrix,

$$r^{(\mu)}_{Abe} = \mu h_1 \wedge h_2,$$  \hspace{1cm} (4.1)

with a deformation parameter $\mu$. Here $h_i$ ($i = 1, 2$) are two of the Cartan generators of $su(2,2)$ and belong to the fundamental representation,

$$h_1 = \text{diag}(-1,1,-1,1), \hspace{1cm} h_2 = \text{diag}(-1,1,1,-1).$$  \hspace{1cm} (4.2)

Then, the AdS$_5$ part of the Lagrangian (2.1) is given by

$$L = L_G + L_B = -\frac{1}{2} (\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \text{Tr} [A_\alpha P_2 (J_\beta)] ,$$  \hspace{1cm} (4.3)

with $J_\beta \equiv \frac{1}{1 - 2\eta [R^{(\mu)}_{Abe}]_g \circ P_2} A_\beta,$

where the current $A_\alpha$ is $su(2,2)$-valued and the R-operator associated with (1.1) is defined by the rule (2.4).

\footnote{Abelian twists of S$^5$ have been studied in \cite{29}. The resulting geometries are three-parameter $\gamma$-deformed S$^5$ \cite{27,28}.}
The projected current $P_2(J_\alpha)$ is to be determined by solving the equation,

$$
\left( 1 - 2\eta P_2 \circ [R^g_{\text{Abel}}] \right) P_2(J_\alpha) = P_2(A_\alpha).
$$

By using the coset parameterization (4.5), $P_2(A_\alpha)$ is expanded with respect to $\gamma$ matrices,

$$
P_2(A_\alpha) = \frac{1}{2} \left[ -\partial_\alpha \rho \gamma_1 + i \cosh \rho \partial_\alpha \psi_3 \gamma_5 \\
- \sinh \rho \left( \cos \zeta \partial_\alpha \psi_1 \gamma_2 + \partial_\alpha \zeta \gamma_3 - i \sin \zeta \partial_\alpha \psi_2 \gamma_0 \right) \right].
$$

Then, by plugging (4.6) with (4.5), $P_2(J_\alpha)$ can be obtained as

$$
P_2(J_\alpha) = j_a^0 \gamma_0 + j_a^1 \gamma_1 + j_a^2 \gamma_2 + j_a^3 \gamma_3 + j_a^5 \gamma_5,
$$

with the coefficients

$$
j_a^0 = \frac{i}{2} \frac{\sin \zeta \sinh \rho}{1 + 16\eta^2 \mu^2 \sin^2 2\zeta \sinh^4 \rho} \left( \partial_\alpha \psi_2 + 8\eta \mu \cos^2 \zeta \sinh^2 \rho \partial_\alpha \psi_1 \right),
$$

$$
j_a^1 = -\frac{1}{2} \partial_\alpha \rho,
$$

$$
j_a^2 = \frac{1}{2} \frac{\cos \zeta \sinh \rho}{1 + 16\eta^2 \mu^2 \sin^2 2\zeta \sinh^4 \rho} \left( \partial_\alpha \psi_1 - 8\eta \mu \sin^2 \zeta \sinh^2 \rho \partial_\alpha \psi_2 \right),
$$

$$
j_a^3 = -\frac{1}{2} \sinh \rho \partial_\alpha \zeta,
$$

$$
j_a^5 = \frac{i}{2} \cosh \rho \partial_\alpha \psi_3.
$$

Finally, the resulting expressions of $L_G$ and $L_B$ are given by, respectively,

$$
L_G = -\frac{\hat{\gamma}^{\alpha\beta}}{2} \left[ \sinh^2 \rho \partial_\alpha \zeta \partial_\beta \zeta + \partial_\alpha \rho \partial_\beta \rho \right.
+ \frac{\sinh^2 \rho}{1 + \hat{\gamma}^2 \sin^2 \zeta \cos^2 \zeta \sinh^4 \rho} \left( \cos^2 \zeta \partial_\alpha \psi_1 \partial_\beta \psi_1 + \sin^2 \zeta \partial_\alpha \psi_2 \partial_\beta \psi_2 \right),
$$

$$
L_B = -\epsilon^{\alpha\beta} \frac{\hat{\gamma} \cos^2 \zeta \sin^2 \zeta \sinh^4 \rho}{1 + \hat{\gamma}^2 \cos^2 \zeta \sin^2 \zeta \sinh^4 \rho} \partial_\alpha \psi_1 \partial_\beta \psi_2.
$$

Here a new deformation parameter $\hat{\gamma}$ is defined as

$$
\hat{\gamma} \equiv 8\eta \mu.
$$

Now one can read off the metric and NS-NS two-form from (4.9) and (4.10). By performing the coordinate transformation,

$$
\rho_1 = \cos \zeta \sinh \rho, \quad \rho_2 = \sin \zeta \sinh \rho, \quad \rho_3 = i \cosh \rho,
$$

Here a new deformation parameter $\hat{\gamma}$ is defined as

$$
\hat{\gamma} \equiv 8\eta \mu.
$$

Now one can read off the metric and NS-NS two-form from (4.9) and (4.10). By performing the coordinate transformation,
the resulting metric and NS-NS two-form are given by

\[ ds^2 = d\rho_1^2 + d\rho_2^2 + d\rho_3^2 + \frac{\rho_1^2 d\psi_1^2 + \rho_2^2 d\psi_2^2}{1 + \gamma^2 \rho_1^2 \rho_2^2} + \rho_3^2 d\psi_3^2 + ds_{S^5}^2 , \]  

(4.13)

\[ B_2 = \frac{\gamma \rho_1^2 \rho_2^2}{1 + \gamma^2 \rho_1^2 \rho_2^2} d\psi_1 \wedge d\psi_2 . \]  

(4.14)

Here there is a constraint \( \sum_{i=1}^3 \rho_i^2 = -1 \).

These expressions are quite similar to a one-parameter \( \gamma \)-deformed \( S^5 \) \([27, 28]\) and thus the solution with the metric (4.13) and the NS-NS two-form (4.14) may be regarded as a single parameter \( \gamma \)-deformation of AdS\(_5\).

## 5 Conclusion and discussion

We have shown that the gravity duals of NC gauge theories \([33, 34]\) can be derived from the Yang-Baxter sigma model description of the AdS\(_5\)×\( S^5 \) superstring with classical \( r \)-matrices. The corresponding classical \( r \)-matrices are 1) solutions of CYBE, 2) skew-symmetric, 3) nilpotent and 4) abelian. These should be called abelian Jordanian deformations. As a result, the gravity duals are found to be integrable deformations of AdS\(_5\)×\( S^5 \). Then, abelian twists of AdS\(_5\) have also been investigated. These results provide a support for the gravity/CYBE correspondence proposed in \([29]\).

Our main result here is the integrability of \( N=4 \) super Yang-Mills (SYM) theory on noncommutative (NC) spaces. Now there are an enormous amount of arguments on the integrability for scattering amplitudes of \( N=4 \) SYM. Integrable deformations of it would be found on NC spaces. Our analysis has revealed a relation between classical \( r \)-matrices and deformations parameters of NC spaces. There may be a close connection to deformation quantization of Kontsevich \([35]\). Thus one may expect a deep mathematical structure behind the correspondence. We hope that our result could shed light on new fundamental aspects of integrable deformations.

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Appendix

A Notation and convention

We shall here summarize our notation and convention, which basically follow [36].

An element of $\mathfrak{su}(2, 2|4)$ is identified with an $8 \times 8$ supermatrix,

\[
M = \begin{bmatrix} m & \xi \\ \zeta & n \end{bmatrix}.
\] (A.1)

Here $m$ and $n$ are $4 \times 4$ matrices with Grassmann even elements, while $\xi$ and $\zeta$ are $4 \times 4$ matrices with Grassmann odd elements. These matrices satisfy a reality condition. Then $m$ and $n$ belong to $\mathfrak{su}(2, 2) = \mathfrak{so}(2, 4)$ and $\mathfrak{su}(4) = \mathfrak{so}(6)$, respectively.

We are concerned with deformations of AdS$_5$. An explicit basis of $\mathfrak{su}(2, 2)$ is the following. The $\gamma$ matrices are given by

\[
\gamma_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \quad \gamma_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},
\]

\[
\gamma_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},
\] (A.2)

and satisfy the Clifford algebra

\[
\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \quad \{\gamma_\mu, \gamma_5\} = 0, \quad (\gamma_5)^2 = 1.
\] (A.3)

The Lie algebra $\mathfrak{so}(1, 4)$ is formed by the generators

\[
m_{\mu\nu} = \frac{1}{4} [\gamma_\mu, \gamma_\nu], \quad m_{\mu 5} = \frac{1}{4} [\gamma_\mu, \gamma_5] \quad (\mu, \nu = 0, 1, 2, 3),
\] (A.4)

and then $\mathfrak{so}(2, 4) = \mathfrak{su}(2, 2)$ is spanned by the following set:

\[
m_{\mu\nu}, \quad m_{\mu 5}, \quad \gamma_\mu, \quad \gamma_5.
\] (A.5)

B Multi-parameter deformations of AdS$_5$

We present here multi-parameter deformations of AdS$_5$ by using the Yang-Baxter sigma model description with classical $r$-matrices. These may be regarded as a multi-parameter
generalization of the gravity duals of NC gauge theories discussed in \cite{33,34}. In the original construction \cite{33,34} based on twisted T-dualities, it would be intricate to perform T-dualities many times. A technical advantage of the Yang-Baxter sigma model description is that a single $r$-matrix gives the corresponding metric and NS-NS two-form in a more direct way.

Let us consider the following classical $r$-matrix,

$$ r_{AI} = \mu_1 p_2 \wedge p_3 + \mu_2 p_3 \wedge p_1 + \mu_3 p_1 \wedge p_2 $$

$$ + \nu_1 p_0 \wedge p_1 + \nu_2 p_0 \wedge p_2 + \nu_3 p_0 \wedge p_3, $$

where $\mu_1, \mu_2, \mu_3$ and $\nu_1, \nu_2, \nu_3$ are six deformation parameters, and $p_\mu$ are defined in \cite{[B.2]}. By following the analysis in section 3, it is straightforward to get the deformed string action. For simplicity, we shall write down only the resulting metric and NS-NS two-form,

$$ ds^2 = \frac{dz^2}{z^2} + z^2 G \left[ -(z^4 + 4\eta^2(\mu_1^2 + \mu_2^2 + \mu_3^2))dx_0^2 + (z^4 + 4\eta^2(\mu_1^2 - \nu_2^2 - \nu_3^2))dx_1^2 
 + (z^4 + 4\eta^2(\mu_2^2 - \nu_1^2 - \nu_3^2))dx_2^2 + (z^4 + 4\eta^2(\mu_3^2 - \nu_1^2 - \nu_2^2))dx_3^2 
 - 8\eta^2[(\mu_2\nu_3 - \mu_3\nu_2)dx_0dx_1 + (\mu_3\nu_1 - \mu_1\nu_3)dx_0dx_2 + (\mu_1\nu_2 - \mu_2\nu_1)dx_0dx_3 
 - (\mu_1\mu_2 + \nu_1\nu_2)dx_1dx_2 - (\mu_2\mu_3 + \nu_2\nu_3)dx_2dx_3 - (\mu_1\mu_3 + \nu_1\nu_3)dx_1dx_3] \right], $$

$$ B_2 = 2\eta G \left[ (z^4\mu_1 - \eta^2\nu_1 K)dx_2 \wedge dx_3 - (z^4\nu_1 + \eta^2\mu_1 K)dx_0 \wedge dx_1 
 + (z^4\mu_2 - \eta^2\nu_2 K)dx_3 \wedge dx_1 - (z^4\nu_2 + \eta^2\mu_2 K)dx_0 \wedge dx_2 
 + (z^4\mu_3 - \eta^2\nu_3 K)dx_1 \wedge dx_2 - (z^4\nu_3 + \eta^2\mu_3 K)dx_0 \wedge dx_3 \right]. $$

Here a scalar function $G$ and a constant parameter $K$ are defined as

$$ G^{-1} \equiv z^8 + 4\eta^2 z^4(\mu_1^2 + \mu_2^2 + \mu_3^2 - \nu_1^2 - \nu_2^2 - \nu_3^2) - \eta^4 K^2, $$

$$ K \equiv 4(\mu_1\nu_1 + \mu_2\nu_2 + \mu_3\nu_3). $$

By taking the following identification of the parameters

$$ 2\eta \mu_1 = a^2, \quad 2\eta \nu_1 = ia^2, \quad \mu_2 = \mu_3 = \nu_2 = \nu_3 = 0, $$

and performing a Wick rotation $x_0 \to ix_0$, one can reproduce the metric and NS-NS two-form of the two-parameter case \cite{[33,34]}.

Note that the metric \cite{[B.2]} and NS-NS two-form \cite{[B.3]} are complemented with the other field components and gives a complete solution of type IIB supergravity. It gives a consistent string background because it is basically obtained by performing a chain of (twisted) T-dualities for AdS$_5$. 
C Three-parameter abelian twists of AdS$_5$

Let us consider here a three-parameter generalization of the abelian deformation of AdS$_5$ discussed in section 4.

We will consider the following classical $r$-matrix,
\[
  r^{(\mu_1,\mu_2,\mu_3)}_{\text{Abe}} = \mu_3 h_1 \wedge h_2 + \mu_1 h_2 \wedge h_3 + \mu_2 h_3 \wedge h_1 ,
\]
(C.1)
with deformation parameters $\mu_i$. Here $h_i$ are the three Cartan generators of $\mathfrak{su}(2,2)$ and belong to the fundamental representation,
\[
  h_1 = \text{diag}(-1,1,-1,1), \quad h_2 = \text{diag}(-1,1,1,-1), \quad h_3 = \text{diag}(1,1,-1,-1) .
\]
(C.2)

By using the $r$-matrix (C.1), the AdS$_5$ part of (2.1) can be rewritten as
\[
  L = L_G + L_B = -\frac{1}{2} (\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \text{Tr} \left[ A_\alpha P_2 (J_\beta) \right] ,
\]
(C.3)
with $J_\beta \equiv \frac{1}{1 - 2 \eta [R^{(\mu_1,\mu_2,\mu_3)}_{\text{Abe}}]_g \circ P_2} A_\beta$
(C.4)
where $A_\alpha = g^{-1} \partial_\alpha g$ is restricted to $\mathfrak{su}(2,2)$ and the R-operator associated with (C.1) is determined by the rule (2.3).

To evaluate the Lagrangian (C.3), let us adopt the following coset parametrization [18] :
\[
  g = \Lambda(\psi_1, \psi_2, \psi_3) \Xi(\zeta) \tilde{g}_\rho(\rho) \in SU(2,2)/SO(1,4).
\]
(C.5)

Here the matrices $\Lambda$, $\Xi$ and $\tilde{g}_\rho$ are defined as
\[
  \Lambda(\psi_1, \psi_2, \psi_3) \equiv \exp \left[ i \frac{1}{2} (\psi_1 h_1 + \psi_2 h_2 + \psi_3 h_3) \right] ,
\]

\[
  \Xi(\zeta) \equiv \begin{pmatrix}
    \cos \frac{\zeta}{2} & \sin \frac{\zeta}{2} & 0 & 0 \\
    -\sin \frac{\zeta}{2} & \cos \frac{\zeta}{2} & 0 & 0 \\
    0 & 0 & \cos \frac{\zeta}{2} - \sin \frac{\zeta}{2} & 0 \\
    0 & 0 & \sin \frac{\zeta}{2} & \cos \frac{\zeta}{2}
  \end{pmatrix} ,
\]

and $\tilde{g}_\rho(\rho) \equiv \begin{pmatrix}
    \cosh \frac{\rho}{2} & 0 & 0 & \sinh \frac{\rho}{2} \\
    0 & \cosh \frac{\rho}{2} - \sinh \frac{\rho}{2} & 0 & 0 \\
    0 & -\sinh \frac{\rho}{2} & \cosh \frac{\rho}{2} & 0 \\
    \sinh \frac{\rho}{2} & 0 & 0 & \cosh \frac{\rho}{2}
  \end{pmatrix} .
\]

To find the projected current $P_2(J_\alpha)$, it is necessary to solve the following equation,
\[
  \left( 1 - 2 \eta P_2 \circ [R^{(\mu_1,\mu_2,\mu_3)}_{\text{Abe}}]_g \right) P_2(J_\alpha) = P_2(A_\alpha) .
\]
(C.6)

Note that $P_2(A_\alpha)$ is expanded with respect to the $\gamma$ matrices,
\[
  P_2(A_\alpha) = \frac{1}{2} \left[ -\partial_\alpha \rho \gamma_1 + i \cosh \rho \partial_\alpha \psi_3 \gamma_5 \\
    - \sinh \rho (\cos \zeta \partial_\alpha \psi_1 \gamma_2 + \partial_\alpha \zeta \gamma_3 - i \sin \zeta \partial_\alpha \psi_2 \gamma_0) \right] .
\]
(C.7)
Then, by combining (C.7) with (C.6), \( P_2(J_a) \) can be obtained as

\[
P_2(J_a) = j^0_a \gamma_0 + j^1_a \gamma_1 + j^2_a \gamma_2 + j^3_a \gamma_3 + j^5_a \gamma_5 ,
\]
with the coefficients

\[
j^0_a = \frac{-i}{2} \frac{\sin \zeta \sinh \rho}{1 - 16 \eta^2 [(\mu_1^2 \sin^2 \zeta + \mu_2^2 \cos^2 \zeta) \sinh^2 2 \rho - \mu_3^2 \sin^2 2 \zeta \sinh^4 \rho]} \times \left[ (-1 + 16 \eta^2 \mu_1^2 \sin^2 \zeta \sinh^2 2 \rho) \partial_\alpha \psi_2 \
- 8 \eta (\mu_1 - 8 \eta \mu_2 \mu_3 \cos^2 \zeta \sinh^2 \rho) \cosh^2 \rho \partial_\alpha \psi_3 \
- 8 \eta (\mu_3 - 8 \eta \mu_1 \mu_2 \cosh^2 \rho) \cos^2 \zeta \sinh^2 \rho \partial_\alpha \psi_1 \right] ,
\]

\[
j^1_a = -\frac{1}{2} \partial_\rho \psi , \]

\[
j^2_a = \frac{1}{2} \frac{\cos \zeta \sinh \rho}{1 - 16 \eta^2 [(\mu_1^2 \sin^2 \zeta + \mu_2^2 \cos^2 \zeta) \sinh^2 2 \rho - \mu_3^2 \sin^2 2 \zeta \sinh^4 \rho]} \times \left[ (-1 + 16 \eta^2 \mu_1^2 \sin^2 \zeta \sinh^2 2 \rho) \partial_\alpha \psi_1 \
+ 8 \eta (\mu_3 + 8 \eta \mu_1 \mu_2 \cosh^2 \rho) \sin^2 \zeta \sinh^2 \rho \partial_\alpha \psi_2 \
+ 8 \eta (\mu_2 + 8 \eta \mu_1 \mu_3 \sin^2 \zeta \sinh^2 \rho) \cosh^2 \rho \partial_\alpha \psi_3 \right] ,
\]

\[
j^3_a = -\frac{1}{2} \sinh \rho \partial_\alpha \zeta ,
\]

\[
j^5_a = \frac{i}{2} \frac{\cosh \rho}{1 - 16 \eta^2 [(\mu_1^2 \sin^2 \zeta + \mu_2^2 \cos^2 \zeta) \sinh^2 2 \rho - \mu_3^2 \sin^2 2 \zeta \sinh^4 \rho]} \times \left[ (1 + 16 \eta^2 \mu_3^2 \sin^2 2 \zeta \sinh^4 \rho) \partial_\alpha \psi_3 \
- 8 \eta (\mu_2 - 8 \eta \mu_1 \mu_3 \sin^2 \zeta \sinh^2 \rho) \cos^2 \zeta \sinh^2 \rho \partial_\alpha \psi_1 \
+ 8 \eta (\mu_1 + 8 \eta \mu_2 \mu_3 \cos^2 \zeta \sinh^2 \rho) \sin^2 \zeta \sinh^2 \rho \partial_\alpha \psi_2 \right] .
\]

Finally, \( L_G \) and \( L_B \) are given by, respectively,

\[
L_G = -\gamma^{\alpha \beta} \frac{1}{2} \left[ - \sinh^2 \rho \partial_\alpha \partial_\beta \rho \
+ (\sin \zeta \sinh \rho \partial_\alpha \zeta - \cos \zeta \cosh \rho \partial_\alpha \rho) (\sin \zeta \sinh \rho \partial_\beta \zeta - \cos \zeta \cosh \rho \partial_\beta \rho) \
+ (\cos \zeta \sinh \rho \partial_\alpha \zeta + \sin \zeta \cosh \rho \partial_\alpha \rho) (\cos \zeta \sinh \rho \partial_\beta \zeta + \sin \zeta \cosh \rho \partial_\beta \rho) \
+ \hat{G} [\sinh^2 \rho (\cos^2 \zeta \partial_\alpha \psi_1 \partial_\beta \psi_1 + \sin^2 \zeta \partial_\alpha \psi_2 \partial_\beta \psi_2) - \cosh^2 \rho \partial_\alpha \psi_3 \partial_\beta \psi_3 \
- \cos^2 \zeta \sin^2 \zeta \cosh^2 \rho \sinh^2 \rho (\sum \hat{\gamma}_i \partial_\alpha \psi_i) (\sum \hat{\gamma}_j \partial_\beta \psi_j)] \right],
\]

\[
L_B = -\epsilon^{\alpha \beta} \hat{G} \left[ \hat{\gamma}_3 \cos^2 \zeta \sin^2 \zeta \sinh^4 \rho \partial_\alpha \psi_1 \partial_\beta \psi_2 \
- \sinh^2 \rho \cosh^2 \rho (\hat{\gamma}_2 \cos^2 \zeta \partial_\alpha \psi_3 \partial_\beta \psi_1 + \hat{\gamma}_1 \sin^2 \zeta \partial_\alpha \psi_2 \partial_\beta \psi_3) \right].
\]

Here a scalar function \( \hat{G} \) is defined as

\[
\hat{G}^{-1} \equiv 1 - (\hat{\gamma}_1^2 \sin^2 \zeta + \hat{\gamma}_2^2 \cos^2 \zeta) \cosh^2 \rho \sinh^2 \rho + \hat{\gamma}_3^2 \cos^2 \zeta \sin^2 \zeta \sinh^4 \rho ,
\]
and new deformation parameters $\hat{\gamma}_i$ are

$$\hat{\gamma}_i \equiv 8\eta \mu_i \quad (i = 1, 2, 3). \quad (C.13)$$

By performing the coordinate transformation (4.12), the metric and NS-NS two-form associated with (C.10) and (C.11) are written into a compact forms,

$$ds^2 = \sum_{i=1}^{3} (d\rho_i^2 + \hat{G}_i \rho_i^2 d\psi_i^2) + \hat{G}_1 \rho_1^2 \rho_2 \rho_3 \left( \sum_{i=1}^{3} \hat{\gamma}_i d\psi_i \right)^2 + ds_{S^5}^2, \quad (C.14)$$

$$B_2 = \hat{G} \left( \hat{\gamma}_3 \rho_1^2 \rho_2^2 d\psi_1 \wedge d\psi_2 + \hat{\gamma}_1 \rho_2^2 \rho_3^2 d\psi_2 \wedge d\psi_3 + \hat{\gamma}_2 \rho_3^2 \rho_1^2 d\psi_3 \wedge d\psi_1 \right). \quad (C.15)$$

Here there is a constraint $\sum_{i=1}^{3} \rho_i^2 = -1$ and $\hat{G}$ turns out to be

$$\hat{G}^{-1} = 1 + \hat{\gamma}_3 \rho_1^2 \rho_2^2 + \hat{\gamma}_1 \rho_2^2 \rho_3^2 + \hat{\gamma}_2 \rho_3^2 \rho_1^2. \quad (C.16)$$

These are quite similar to $\gamma$-deformed $S^5$ [27, 28] and hence the metric (C.14) and NS-NS two-form (C.15) may be regarded as $\gamma$-deformed AdS$_5$.

The one-parameter result in section 4 is reproduced by setting the parameters as

$$\hat{\gamma}_1 = \hat{\gamma}_2 = 0, \quad \hat{\gamma}_3 = \hat{\gamma}. \quad (C.17)$$

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