Entangled light in transition through the generation threshold

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We investigate continuous variable entangling resources on the base of two-mode squeezing for all operational regimes of a nondegenerate optical parametric oscillator with allowance for quantum noise of arbitrary level. The results for the quadrature variances of a pair of generated modes are obtained by using the exact steady-state solution of Fokker-Planck equation for the complex P-quasiprobability distribution function. We find a simple expression for the squeezed variances in the near-threshold range and conclude that the maximal two-mode squeezing reaches 50\% relative to the level of vacuum fluctuations and is achieved at the pump field intensity close to the generation threshold. The distinction between the degree of two-mode squeezing for monostable and bistable operational regimes is cleared up.

PACS numbers: 03.67.Mn, 42.50.Dv, 42.50.-p

Recent years have witnessed an increasing interest in quantum information theory dealing with entanglement of continuous variable (CV) systems\textsuperscript{1}. CV entangled states of light hold the key for quantum communications at the high-intensity level, as CV analogues of various protocols developed originally in the framework of discrete quantum variables have been established (see, for example\textsuperscript{2,3}). An important example provides entangled Einstein-Podolski-Rosen (EPR) state of orthogonally polarized but frequency degenerate beams, efficiently generated via nonlinear optical process of parametric down-conversion. It has been pointed out in \textsuperscript{4} and has been demonstrated experimentally in \textsuperscript{5} for CV, employing sub-threshold nondegenerate optical parametric oscillator (NOPO). Then a CV entanglement source was built from two independent quadrature squeezed beams combined on a beam-splitter \textsuperscript{2}. The generation of CV EPR entanglement using an optical fibre interferometer is also demonstrated \textsuperscript{7}. It should be noted that so far most experimental realizations of the CV entanglement on NOPO’s have only been operated below threshold. A natural next step is the extension of these investigations to laser-like systems generating entangled bright-light states. Unfortunately, both theoretical and experimental studies of these problems are very complicated and only rare examples are known up to now. Many efforts have been devoted to the study of intensity correlated twin beams from NOPO above threshold \textsuperscript{8}. The conditional generation of sub-Poissonian light from bright twin beams in the CV regime is experimentally demonstrated in \textsuperscript{9}. Further experimental studies of bright two-mode entangled state from CW nondegenerate optical parametric amplifier have been made in Refs. \textsuperscript{10}. The generation of CV polarization entanglement by mixing a pair of polarization squeezed beams is realized in \textsuperscript{10}.

The purpose of this paper is to investigate physical properties and the presence of CV entanglement for NOPO in its transition through the generation threshold as well as in the regime of lasing. One of the principal problems in this study is the description of quantum fluctuations. In most theoretical works nonclassical effects and entanglement resources of nonlinear quantum systems are usually described within linear treatment of quantum noise. It is obvious that such approach does not describe the critical ranges (threshold, point of multistability, etc) where the level of quantum noise increases substantially. We use a more adequate approach within the framework of exact nonlinear treatment of quantum fluctuations via the solution of the Fokker-Planck equation for the quasiprobability distribution function. Deriving the quasiprobability functions so far has been performed only for a few simple models (see \textsuperscript{11} and references therein). For the NOPO the exact steady-state solution of the Fokker-Planck equation in the complex \( P \)-representation first was obtained in \textsuperscript{12}. We will use the generalized form of such solution \textsuperscript{13} which involves also detunings of the modes and hence admits bistability.

There are various questions that emerge in the study of these problems. Because NOPO displays both monostable and bistable regimes it is important to understand how the properties of entanglement depend on the operational regimes. What are peculiarities of entanglement in the vicinity of threshold? Will entanglement take place in the regime of lasing or how far it can be extended into the high intensity domain? Answering these questions is extremely important for a deeper understanding of quantum entanglement, and also from the perspective of creating an entangled light laser.

We consider a type-II phase matched NOPO with triply resonant cavity that supports the pump mode at the frequency \( \omega_3 \) and two orthogonally polarized modes at the same frequency \( \omega_1 = \omega_2 = \omega_3 / 2 \). The pump mode is driven by coherent field at the frequency \( \omega_L \simeq \omega_3 \). The
relevant interaction Hamiltonian is
\[ H = i\hbar \left( a_1^+ a_2^+ a_3 - a_1 a_2 a_3^+ \right) + \]
\[ + i\hbar \left( E e^{-i\omega_L t} a_3^+ - E^* e^{i\omega_L t} a_3 \right) + \sum_{i=1}^{3} \left( a_i \Gamma_i^+ + a_i^+ \Gamma_i \right) \]

where \( a_i \) (\( i = 1, 2, 3 \)) are the operators of the modes \( \omega_i, \kappa \) is the coupling constant, \( E = |E|e^{i\Phi} \) is the complex amplitude of the pump field. The last term in (1) describes mode damping in the cavity in terms of the reservoir operators \( \Gamma_i \) and \( \Gamma_i^+ \), which determine the damping factors \( \gamma_i \). We take into account the detunings of the cavity \( \Delta_1 = \omega_L - \omega_3 \), and \( \Delta_1 = \Delta_2 = \Delta = \omega_L/2 - \omega_1 \). We also assume perfect symmetry between the orthogonally polarized modes provided that they decay at the same rate: \( \gamma_1 = \gamma_2 = \gamma \).

It is well known that linear treatment of NOPO is not self-consistent due to the phase diffusion. According to this effect the difference between the phases of the signal and idler modes, as well as each of the phases can not be defined in the above threshold regime of generation. On the whole, the well-founded linearization procedure cannot be applied for this system. Nevertheless, the linearization procedure and analysis of quantum fluctuations for NOPO become possible due to the additional assumptions about temporal behavior of the difference between the phases of the generated modes [14]. Within this assumption the two-mode squeezing spectra are obtained for both below- and above-threshold regimes. These results are inapplicable in the near-threshold as well as in the bistable ranges of generation, where a more accurate treatment of quantum noise is necessary. Our analysis of NOPO is based on the steady-state exact solution of the Fokker-Planck equation. It is obvious, that in the framework of this approach we avoid both difficulties of the linear theory. The price one has to pay for this advantage is the impossibility to perform temporal description of quantum fluctuations and hence squeezing spectra cannot be calculated. Nevertheless, our analysis does not require spectral description of squeezing and therefore we do not encounter any principal difficulties.

We start with the Fokker-Planck equation of the system in the so called complex P-representation of the density matrix and in the case of high cavity losses for the pump mode (\( \gamma_3 \gg \gamma \)), however, in the operational regime the pump depletion effects are involved:

\[
\frac{\partial}{\partial t} P(\vec{\alpha}, t) = \left\{ -\frac{\partial}{\partial \alpha_1} \left[ -\frac{\pi_1 \alpha_1 + \kappa \left( \frac{E - \kappa \alpha_1 \alpha_2}{\pi_3} \right) \alpha_2^+}{\pi_3} \right] - \frac{\partial}{\partial \alpha_2} \left[ -\frac{\pi_2 \alpha_2 + \kappa \left( \frac{E - \kappa \alpha_1 \alpha_2}{\pi_3} \right) \alpha_1^+}{\pi_3} \right] + \frac{\kappa \partial^2}{\partial \alpha_1 \partial \alpha_2} \left[ \frac{E - \kappa \alpha_1 \alpha_2}{\pi_3} \right] + \frac{\kappa \partial^2}{\partial \alpha_1 \partial \alpha_2} \left[ \frac{E^* - \kappa \alpha_1 \alpha_2^+}{\pi_3} \right] \right\} P(\vec{\alpha}, t).
\]

Here \( \vec{\alpha} = (\alpha_1, \alpha_1^+, \alpha_2, \alpha_2^+, \alpha_3, \alpha_3^+) \), \( \alpha_i, \alpha_i^+ (i = 1, 2) \) are the independent complex variables corresponding to the operators \( a_i, a_i^+ \), and \( \pi_j = \gamma_j - i\Delta_j \) (\( \pi_1 = \pi_2 = \pi \)).

The normally-ordered moments of time-dependent operators are calculated through the P-quasiprobability distribution function as

\[
\langle a_1^+ (t)^k a_1 (t)^l a_2^+ (t)^m a_2 (t)^n \rangle = \int d\alpha_1^+ d\alpha_1 d\alpha_2^+ d\alpha_2 P(\vec{\alpha}, t) \alpha_1^+ \alpha_1^l \alpha_2^+ \alpha_2^m \alpha_1^2 \alpha_2^2.
\]

Below we consider only the stationary steady-state regime and drop the time dependence of operators when calculating one-time expectation values. Using the steady-state solution of the equation (2), obtained in [13], and the method of integration in the complex plane [12] we find:

\[
\langle a_1^+ a_1^l a_2^+ a_2^m \rangle = \frac{\epsilon^{l+k}}{N} \frac{\hbar}{(\Lambda + 1)_{l+k}} \sum_{j=0}^\infty \frac{(l+1)_{(k+1)j}}{(\Lambda + l + 1)_{j} (\Lambda^* + k + 1)_{j} (j!)^2} \frac{p^j}{(j!)^2},
\]

\[
\langle a_1^+ a_1^m a_2^+ a_2^m \rangle = \frac{p^m}{2^m + n N} \frac{m!}{(\Lambda + 1)m} \sum_{j=0}^\infty \frac{(m+1)_{(j+m)j}}{(\Lambda + m + 1)_{j} (j!)^2} \frac{p^j}{j!(j+m-n)!}, \quad \text{if } m \geq n,
\]
\[ \langle a_1^{+m}a_1^ma_2^{+n}a_2^n \rangle = \langle a_1^{+n}a_1^n a_2^{+m}a_2^m \rangle, \text{ if } n > m, \] (6)

where \( \varepsilon = E/\kappa, \rho = |2\varepsilon|^2 \) is the scaled pump intensity, \( \Lambda = 2\sqrt{\gamma_{13}/\kappa^2} \), \( \langle x \rangle := x(x+1)\ldots(x+j-1), \langle x \rangle_0 = 1 \), and

\[ N = \sum_{j=0}^{\infty} \frac{p^j}{(\Lambda + 1)^j} = \sum_{j=0}^{\infty} N_j. \] (7)

In addition to these expressions we note that the system under consideration has the following property, conditioned by its most general symmetries:

\[ \langle a_1^{+k}a_2^{+m}a_1^n a_2^p a_3^q \rangle = 0, \text{ if } k - l \neq m - n. \] (8)

This property is the consequence of the rotational symmetry of the Hamiltonian \( U(\theta) \) \( HU^{-1}(\theta) = H \) for any \( \theta \), where \( U(\theta) = \exp[\theta (a_1^2 - a_2^2)] \). Since the Lindblad part of the master equation is invariant with respect to such transformation too, then the steady state density operator of NOPO must commute with \( U(\theta) \): \( U(\theta) \rho U^{-1}(\theta) = \rho \), whereupon we arrive at the relation 8.

We will use only the photon number and the \( \langle a_1 a_2 \rangle \) moment in the further discussion, so we write their expressions explicitly:

\[ n = \langle a_1^+ a_1 \rangle = \langle a_2^+ a_2 \rangle = \frac{1}{2N} \sum_{j=1}^{\infty} \frac{jjp^j}{(\Lambda + 1)^j}, \] (9)

\[ \langle a_1 a_2 \rangle = \frac{e^{i\Phi_E}}{2N\sqrt{p}} \sum_{j=1}^{\infty} j(\Lambda^* + j)p^j. \] (10)

We note that the state generated in NOPO is non-Gaussian one, i.e. its Wigner function is non-Gaussian [15]. So far, the inseparability problem for bipartite non-Gaussian states is far from being understood. On the theoretical side, the necessary and sufficient conditions for the separability of bipartite CV systems have been fully developed only for Gaussian states [14]. To characterize CV entanglement we have chosen the inseparability criterion based on the total variance of a pair of EPR type operators. For a pair of optical beams generated in NOPO this criterion characterizes the entanglement in terms of quadrature operators

\[
X_k = X_k(\theta_k, t) = \frac{1}{\sqrt{2}} \left[ a_k^+(t)e^{-i\theta_k} + a_k(t)e^{i\theta_k} \right],
\]

\[
Y_k = Y_k(\theta_k, t) = X_k(\theta_k - \frac{\pi}{2}, t),
\] (11)

\( (k = 1, 2) \) and due to the mentioned symmetries is reduced to the following form:

\[ V(\theta_1, \theta_2) := V(X_1 - X_2) \equiv V(Y_1 + Y_2) < 1. \] (12)

Here \( V(x) \) is a convenient short-hand notation for the variance \( \langle x^2 \rangle - \langle x \rangle^2 \). Inequalities [13] require the variances of both conjugate variables \( X_1 - X_2 \) and \( Y_1 + Y_2 \) to drop below the level of vacuum fluctuations. Since the states of NOPO are non-Gaussian, the criterion (12) is only sufficient for inseparability.

At this point we must note the difference between the focus of our paper and most of the preceding work devoted to the study of two-mode squeezing. It is an established standard to describe squeezing with the spectra of quantum fluctuations of the considered variables. The recent experiment on spectral investigation of criteria for CV entanglement was presented in the paper [17]. Unlike that, we aim to analyze the separability properties of the system solely with the help of the criterion (12), which involves a single integral characteristic of the NOPO quantum state. For the completeness of the results it is useful to remind how intracavity variances are related to the spectra of squeezing of output fields. For this, let us consider the special scheme of generation, when the couplings of in- and out-fields occur at only one of the ring-cavity mirrors. For the case when only the fundamental mode is coherently driven by the pump field, while subharmonic modes are initially in the vacuum state, we have for the output fields

\[ a_{m}(t) = \sqrt{2\gamma} a_1(t), (t = 1, 2). \]

The spectra of two-mode quadrature amplitude squeezing is

\[ S(\omega, \theta_1, \theta_2) = \int_{-\infty}^{\infty} e^{i\omega t}(X_{out}^{\omega}(\theta_1, \theta_2, t)X_{out}^{\omega}(\theta_1, \theta_2, t + \tau))d\tau. \] (13)

Here \( X_{out}^{\omega}(\theta_1, \theta_2, t) = \sqrt{2\gamma} [X_1(\theta_1, t) - X_2(\theta_2, t)] \), and \( t \) is sufficiently large \( (t \gg \gamma^{-1}) \) to ensure steady state, so that \( S(\omega, \theta_1, \theta_2) \) does not depend on \( t \).

The variance \( V(\theta_1, \theta_2) \) is expressed with the above spectra in the following way:

\[ 2\gamma V(\theta_1, \theta_2) = 2\gamma \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega, \theta_1, \theta_2)d\omega. \right. \] (14)

Let us turn to the integral variances. With an appropriate selection of the phases \( \theta_k \) of the quadrature amplitudes, namely

\[ \theta_1 + \theta_2 = \arg\langle a_1 a_2 \rangle, \] (15)

we arrive at the following minimum value of \( V(\theta_1, \theta_2) \)

\[ V_{min} = 1 + 2(n - |\langle a_1 a_2 \rangle|). \] (16)

Detailed physical analysis of the variance [18] on the base of the exact expressions [14], [15] seems to be a rather complicated task. To proceed further we transform the formula (14) to a more appropriate form

\[ \langle a_1 a_2 \rangle = e^{i\Phi_E} \frac{n}{\sqrt{p}} \left( \Lambda^* + 2n + \frac{n}{\partial n} \frac{\partial n}{\partial p} \right). \] (17)
What is important is that $\langle a_1 a_2 \rangle$ is expressed with $n$ in a simple enough form, much easier to understand than its expansion in powers of $p$. The advantage of that form is due to the intuitiveness of the behavior of $n$ in various feasible operational regimes. Moreover, semiclassical solution for the mean photon number, where applicable, can be employed to obtain a semiclassical solution for $\langle a_1 a_2 \rangle$. To this end, we remind relevant results of the semiclassical approximation. In the regime below threshold $p < p^{th}$ the excitations of modes are at the level of spontaneous noise, and above threshold they have the form

$$n_{cl} = \frac{1}{2} \left( -\Re \Lambda + \sqrt{p - (\Im \Lambda)^2} \right).$$

The case $\Re \Lambda > 0$ corresponds to monostable dynamics with threshold $p^{th}_m = |\Lambda|^2$. (The corresponding threshold value of the pump field $E^{th} = |(\gamma - i\Delta)(\gamma - i\Delta_3)|/\kappa$.) The opposite case $\Re \Lambda < 0$ is bistable with threshold $p^{th}_b = (\Im \Lambda)^2$ and the stability region of the zeroth solution extending up to $p = |\Lambda|^2$. For completeness, we present in Fig. 1 quantum-mechanical results for mean photon number of the modes for three values of detunings corresponding to monostable, bistable and interjacent ($\Re \Lambda = 0$) regimes. For the bistable dynamics the critical region, i.e. the range of the pump intensity, where the system passes onto the nontrivial classical branch, lies between $p^{th}_b$ and $|\Lambda|^2$. The quantum critical region of monostable dynamics is in the vicinity of the generation threshold $p^{th}_m = |\Lambda|^2$.

We are now in a position to study the entanglement effects and will state the main results of the paper. What is important is that we find the quantities (9)-(10) through the exact steady-state solution of the Fokker-Planck equation and carry out an exact quantum statistical analysis.

Fig. 2 illustrates the dependence of $V_{\text{min}}$ on the pump amplitude. One can see that for monostable dynamics (curve 1) entanglement is realized in the entire range of pump intensities. In all cases maximal degree of two-mode squeezing $V_{\text{min}} = 0.5$ is achieved within the critical region. Above the critical point mean photon numbers of the modes increase considerably and variance $V_{\text{min}}$ starts to increase too. For the bistable case (curve 3) growth of $V_{\text{min}}$ is much faster and larger so that the sufficient criterion for inseparability is not fulfilled. Note that $V_{\text{min}}$ has a sharp peak in the critical range of bistability. Anyway in the far-above threshold region the CV entanglement of the system is guaranteed. More exactly, the asymptotic value of $V_{\text{min}}$ for all detunings is equal to 0.75.

Attentive reader of this article must be somewhat confused by the unnatural value of $\kappa/\gamma$ ratio selected to plot

**Fig. 1:** Scaled mean photon number versus dimensionless amplitude of the pump field $E_s = (2\kappa/\gamma_3)|E|$ for monostable (curve 1, $\Delta/\gamma = 1$), interjacent (curve 2, $\Delta/\gamma = 3$) and bistable (curve 3, $\Delta/\gamma = 7$) regimes ($\kappa/\gamma = 0.5$, $\gamma_3/\gamma = 18$). 

**Fig. 2:** Minimized variance $V(X_1 - X_2)$ versus dimensionless amplitude of the pump field $E_s$ for the same parameters as for Fig. 1.

**Fig. 3:** Illustration of the critical region squeezing for small values of $\kappa/\gamma$ ratio. $\kappa/\gamma = 0.5$ for the solid line, and $\kappa/\gamma = 10^{-6}$ for the dot line.
the figures. We emphasize that the choice of parameters is conditioned merely by illustrative purposes. For smaller, experimentally available values of the parameter $\kappa/\gamma \sim 10^{-6} \div 10^{-8}$, the behavior of $V_{\text{min}}$ is not changed qualitatively in comparison with the results of Fig. 4.

To illustrate this we plot in Fig. 3 the behavior of $V_{\text{min}}$ in the interjacent regime and for two values of $\kappa/\gamma$. One can see that for the more realistic parameter value $\kappa/\gamma = 10^{-6}$ behavior in the vicinity of the critical point is much more abrupt, which is just a result of critical region being much narrower. For the bistable dynamics this narrowing of the critical region makes the peak of $V_{\text{min}}$ much sharper. For the bistable dynamics this narrowing of the critical region makes the peak of $V_{\text{min}}$ much sharper. For the bistable dynamics this narrowing of the critical region makes the peak of $V_{\text{min}}$ much sharper.

It is remarkable that equations (16), (17) allow to explain qualitatively and even quantitatively the numerical results obtained.

1. In the far below-threshold regime mean photon number can be represented linear in $p$: $n = p/2(\Lambda + 1)^2 \ll 1$. Using this formula it is easy to check that the condition $V_{\text{min}} < 1$ is fulfilled for any $\Lambda$ within the entire domain where the weak pump approximation is valid.

2. In the far above-threshold range ($p \gg |\Lambda|^2$) we can make use of the semiclassical expression (18) together with (19) to study the variance $V_{\text{min}}$. To this end we write $n = n_d + \delta n$, and neglect $\delta n$ only where that doesn’t lead to loss of accuracy in the formula (16). After some algebra (see Appendix A) we arrive at the following expression:

$$\lim_{p \to \infty} V_{\text{min}} = 0.5 - 2\delta n. \quad (19)$$

Straightforward, but complicated analytical calculations on the formula (16) show that $\delta n \rightarrow -0.125$ in the limit $p \to \infty$, which leads to the asymptotic value $V_{\text{min}} = 0.75 < 1$ (find in Appendix B several hints regarding our analysis of $\delta n$). Therefore, as our analysis shows with allowance for quantum fluctuations of arbitrary level, CV entanglement is always achieved in NOPO above threshold. What is remarkable, is that a very small quantum correction $\delta n$ to the semiclassical intracavity photon number plays an essential role to the production of entanglement in high-intensity level. Obtained result seems interesting as provides the example of preserving CV entanglement in the high-intensity domain.

3. In the near-threshold and bistability regimes analytical study is too complicated. We have carried out strict analysis only for the monostable case ($\Delta \Delta_3 < \gamma\gamma_3$), assuming $\kappa/\gamma \ll 1$. We perform expansion of $V_{\text{min}}$ around the minimum point $p_{\text{min}}$ in powers of parameter $s := |\Lambda|^{-1/2} (s \sim \kappa/\gamma)$ keeping only the significant terms (technics similar to those described in Appendix B is used). The result, expressed through the pump field intensity $I \sim |E|^2$, is the following:

$$V_{\text{min}} = 0.5 + c^3 f_1(c)s + f_2(c) \left( \frac{I - I_{\text{min}}}{I_{\text{th}}} \right)^2, \quad (20)$$

where

$$I_{\text{min}} = I_{\text{th}} [1 + f_3(c) s], \quad (21)$$

$$c = \sqrt{\cos(\arg \Lambda)}, \quad f_1(c) = 0.0164, \quad f_2(c) = 0.113 + 0.00221c - 0.330c^2 + 0.371c^3 - 0.132c^4, \quad f_3(c) = -2.219 + 0.217c + 2.83c^2.$$  

Note that for fixed damping rates $\gamma_i$ $c$ is a decreasing function of detuning, and $c = 1$ at exact resonance. Formula (20) is valid for $c^3 \gg s$ (i.e. not too close to the interjacent regime $c = 0$) and $(I - I_{\text{min}})/I_{\text{th}} \ll s^2$.

Let us discuss the results (20), (21) in more detail. Positiveness of $f_1(c)$ means that the minimal value of $V_{\text{min}}$ doesn’t drop below 0.5. This result differs from the studied case of the perfect two-mode squeezing generated in an undamped NOPA, where $V_{\text{min}}$ vanishes. Evidently, in our system the larger limiting value for $V_{\text{min}}$ is due to dissipation and cavity feedback effects. Then we see that the point of maximal two-mode squeezing $V_{\text{min}}$ is located close to the generation threshold $I_{\text{th}}$. Function $f_2(c)$ is negative for values of $c$ close to zero, however it increases and becomes positive as $c$ approaches 1. Thus the point of maximal two-mode squeezing can be located both below and above the generation threshold of monostable dynamics.

Formulae (20) and (21) are an example of a result that couldn’t be obtained with the help of linear treatment of quantum fluctuations. Although the linearized theory gives correct expressions for $V_{\text{min}}$ in the below- and above-threshold ranges, it not only fails to link those solutions through the narrow range of the generation threshold of monostable regime, but also turns entirely invalid in the interjacent and bistable regimes.

In the light of experimental investigations of CV quadrature entanglement in NOPA and NOPO, the obtained value for the highest achievable squeezing of 3 dB ($V_{\text{min}} = 0.5$) may seem inadequate. Indeed, the measurements of the frequency spectra of the variances of the output quadrature-phase amplitudes reveal up to 3.7 dB squeezing for NOPA, and up to 4.9 dB squeezing for NOPO operating above threshold. However, we remind that our results pertain to the full squeezing of the system and not the spectral component squeezing. Moreover, as noted in [15], states with even perfect squeezing of a certain spectral component may be non-squeezed in the full sense. Note, that complete study of that question is beyond the scope of this paper.

Our results allow us to explain qualitatively the growth of $V_{\text{min}}$ in the critical region of bistability. We remind that the minimization of the variance $V$ is performed through a specific selection of phases of the quadrature amplitudes $X_k$ and $Y_k$ - see formula (18). In the bistable regime the state can be regarded as a mixture $\rho = \rho_w \rho_w + p_s \rho_s$ of two states corresponding to the weakly and strongly excited semiclassical stable solutions. What if both $\rho_w$ and $\rho_s$ are highly squeezed two-mode states, but the minimizing phases are rather different for them? In this case the minimal variance for their mixture will be achieved for an intermediate value of the phases, which
is good for none of them. The analysis of the phase of \( \langle a_1 a_2 \rangle \) suggests that this is indeed the reason. Fig. 4 shows that the minimizing phase changes very abruptly in the bistable critical region. In this context we note that in the critical bistable regime the state is extremely different from Gaussian and therefore the condition \( V_{\text{min}} > 1 \) does not witness against entanglement since the criterion provides only sufficient condition for CV entanglement in this case.

In summary, we have investigated CV entangling resources for all operational regimes of nondegenerate OPO in the most basic and explicit way through the P-complex probability distribution. We have studied the entanglement as two-mode squeezing and have shown that entanglement is present in the system for quantum noise of arbitrary level and, what is remarkable, for a wide range of intensity of the coherent driving field including the far-above-threshold limit. The numerical and analytical analysis for the quantum critical range of NOPO has provided a clear distinction between the degree of two-mode squeezing and have shown that entanglement is present in the system for quantum noise of arbitrary level and, what is remarkable, for a wide range of intensity of the coherent driving field including the far-above-threshold limit. The numerical and analytical analysis for the quantum critical range of NOPO has provided a clear distinction between the degree of two-mode squeezing and have shown that entanglement is present in the system for quantum noise of arbitrary level and, what is remarkable, for a wide range of intensity of the coherent driving field including the far-above-threshold limit. The numerical and analytical analysis for the quantum critical range of NOPO has provided a clear distinction between the degree of two-mode squeezing and have shown that entanglement is present in the system for quantum noise of arbitrary level and, what is remarkable, for a wide range of intensity of the coherent driving field including the far-above-threshold limit.

APPENDIX A: ASYMPTOTIC FORM OF \( V_{\text{min}} \)

Supposing that for \( p \gg |\Lambda|^2 \) \( \delta n \) is monotonous and \( \delta n \ll n_{cl} \) (these assumptions are justified during a later analysis) it can be easily shown that \( n \approx \frac{1}{2} \sqrt{p} \) and

\[
\lim_{p \to \infty} \frac{p}{n} \frac{\partial n}{\partial p} = \frac{1}{2}.
\]  

(A1)

The following exact expression is also not difficult to obtain using formula (17):

\[
|\langle a_1 a_2 \rangle| = n \sqrt{1 + Q} \left( 2n + \frac{p}{n} \frac{\partial n}{\partial p} \right),
\]  

(A2)

where

\[
Q = 2\sqrt{p - (\text{Im} \Lambda)^2} + 2\delta n + \frac{p}{n} \frac{\partial n}{\partial p}.
\]  

(A3)

\( Q \approx 2\sqrt{p} \), when \( p \to \infty \). Therefore (A2) can be expanded into series, whereupon we find

\[
|\langle a_1 a_2 \rangle| = n + \frac{nQ}{2p} \left( 2\delta n + \frac{p}{n} \frac{\partial n}{\partial p} \right) + O(p^{-\frac{3}{2}}),
\]  

(A4)

This immediately yields the asymptotic behavior of \( V_{\text{min}} \).

APPENDIX B: ASYMPTOTIC VALUE OF \( \delta n \)

We decompose the value of \( 2n_{cl} \) into the integer and fraction parts: \( 2n_{cl} = \mu + \xi \), with \( \mu = E \{2n_{cl}\} \) and \( \xi = F \{2n_{cl}\} \). After that we can write:

\[
\delta n = n - n_{cl} = \frac{F}{2N} - \frac{\xi}{2},
\]  

(B1)

where

\[
F = \sum_{j=0}^{\infty} (j - \mu) N_j,
\]  

(B2)

and \( N \) and \( N_j \) are defined with formula (7).

It can be shown that \( N_\mu \) is the largest term in \( N \) and that the values of \( N \) and \( F \) can be calculated with sufficient precision by summing only about \( O(\sqrt{\mu}) \) terms around \( N_\mu \). Within that range of indices the term \( N_{\mu+j} \) can be represented as below:

\[
N_{\mu+j} \approx \exp \left[ -\frac{j(j+1-2\xi)}{\mu} \right].
\]  

(B3)

Summation of such terms can be safely replaced with integration, with bounds extended to infinity. We finally obtain the following expression for \( N \):

\[
N \approx \sqrt{\pi} \exp \left[ \frac{(1-2\xi)^2}{4\mu} \right] N_\mu.
\]  

(B4)

Acknowledgments

Acknowledgements: We gratefully acknowledge useful discussions with H. H. Adamyan, H. K. Avetissyan, J. Bergou and C. W. Gardiner. This work was supported by the NFSAT PH 098-02/CRDF grant no. 12052 and the ISTC grant no. A-353.
Calculation of $F$ requires more accurate treatment, since it contains positive and negative parts. We ensure against errors due to subtraction of two large values rewriting $F$ as follows:

$$F = \sum_{j=1}^{O(\sqrt{\mu})} j (N_{\mu+j} - N_{\mu-j}).$$  \hspace{1cm} (B5)

The difference $N_{\mu+j} - N_{\mu-j}$ can be simplified to an appropriate approximate form, allowing to replace the sum with integration. Proceeding like with $N$ we arrive at $\delta n \to -0.125$.

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