ITO’S CONJECTURE AND THE COSET CONSTRUCTION FOR
\( W^k(\mathfrak{sl}(3|2)) \)

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Abstract. Many \( W \)-(super)algebras which are defined by the generalized Drinfeld-Sokolov reduction are also known or expected to have coset realizations. For example, it was conjectured by Ito that the principal \( W \)-superalgebra \( W^k(\mathfrak{sl}(n+1|n)) \) is isomorphic to the coset of \( V^l(\mathfrak{gl}_n) \) inside \( V^l(\mathfrak{gl}_{n+1}) \otimes \mathcal{E}(n) \) for generic values of \( l \). Here \( \mathcal{E}(n) \) denotes the rank \( n \) bc-system, which carries an action of \( V^1(\mathfrak{gl}_n) \), and \( k \) and \( l \) are related by \( (k+1)(l+n+1) = 1 \). This conjecture is known in the case \( n = 1 \), which is somewhat degenerate, and we shall prove it in the first nontrivial case \( n = 2 \). As a consequence, we show that the simple quotient \( W_k(\mathfrak{sl}(3|2)) \) is lisse and rational for all positive integers \( l > 1 \). These are new examples of rational \( W \)-superalgebras.

1. Introduction

Let \( \mathfrak{g} \) be a basic classical Lie superalgebra over \( \mathbb{C} \), \( f \) a nilpotent element with even parity, \( k \) a complex number and \( \Gamma : \mathfrak{g} = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}} \mathfrak{g}_j \) a good grading for \( f \). Then the (affine) \( W \)-algebras \( W^k(\mathfrak{g}, f; \Gamma) \) are defined as \( \frac{1}{2} \mathbb{Z}_{\geq 0} \)-graded vertex superalgebras by generalized Drinfeld-Sokolov reductions associated with \( \mathfrak{g}, f, \Gamma \). By [KW3], we have the Miura map

\[ \mu : W^k(\mathfrak{g}, f; \Gamma) \rightarrow V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_1) \],

where \( V^{\tau_k}(\mathfrak{g}_0) \) is the affine vertex superalgebra of \( \mathfrak{g}_0 \) with its invariant bilinear form \( \tau_k \) (see (2.2) in [Ge] for the definition of \( \tau_k \)), and \( \Phi(\mathfrak{g}_1) \) is the neutral vertex superalgebra associated with \( \mathfrak{g}_1 \) (see [KRW] for the definition of \( \Phi(\mathfrak{g}_1) \)). If \( k \) is generic, the image of \( \mu \) may be described as the intersection of kernels of screening operators, and thus \( \mu \) is injective, see [Ge]. By [KW3], there exists a filtration on \( W^k(\mathfrak{g}, f; \Gamma) \) such that the graded vector space is isomorphic to \( V^{\tau_k}(\mathfrak{g}_f) \), where \( \mathfrak{g}_f \) is the centralizer of \( f \) in \( \mathfrak{g} \). In particular, if \( \{ u_i \}_{i=1}^{\dim \mathfrak{g}_f} \) is a basis of \( \mathfrak{g}_f \) with \( u_i \in \mathfrak{g}_{j_i} \), the \( W \)-algebra \( W^k(\mathfrak{g}, f; \Gamma) \) has a set \( \{ J_{u_i} \}_{i=1}^{\dim \mathfrak{g}_f} \) of strong generators, and then \( J_{u_i} \) has the conformal degree \( 1 - j_i \) and the same parity as \( u_i \).

It is often useful to give an alternative realization of \( W^k(\mathfrak{g}, f; \Gamma) \) using the coset construction. Recall that given a vertex algebra \( \mathcal{V} \) and a vertex subalgebra \( \mathcal{A} \subseteq \mathcal{V} \), the coset of \( \mathcal{A} \) in \( \mathcal{V} \) is defined by

\[ \text{Com}(\mathcal{A}, \mathcal{V}) := \{ v \in \mathcal{V} | [a(z), v(w)] = 0, \forall a \in \mathcal{A} \}. \]

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This was introduced by Frenkel and Zhu in [FZ], and it generalizes earlier constructions in representation theory [KP] and physics [GKO]. Suppose that $L$ and $L'$ are the Virasoro elements of $\mathcal{V}$ and $\mathcal{A}$, respectively. Then under mild hypotheses, $L - L'$ is the Virasoro element of $\text{Com}(\mathcal{A}, \mathcal{V})$, and $v \in \mathcal{V}$ lies in $\text{Com}(\mathcal{A}, \mathcal{V})$ if and only if $L'_{-1}v = 0$; see Theorems 5.1 and 5.2 of [FZ]. These results also hold if $\mathcal{V}$ and $\mathcal{A}$ are vertex superalgebras, and if the conformal degree grading is by $\frac{1}{2}N$ rather than $N$. It is expected that $\text{Com}(\mathcal{A}, \mathcal{V})$ will inherit properties of $\mathcal{A}$ and $\mathcal{V}$ such as lisseness and rationality, although general results of this kind are not yet known. We say the vertex subalgebras $\mathcal{A}, \mathcal{B} \subseteq \mathcal{V}$ form a dual pair if $\mathcal{B} = \text{Com}(\mathcal{A}, \mathcal{V})$ and $\mathcal{A} = \text{Com}(\mathcal{B}, \mathcal{V})$; in this case, $\mathcal{A} \otimes \mathcal{B}$ is conformally embedded in $\mathcal{V}$.

Given a simple Lie algebra $\mathfrak{g}$, let $\mathcal{W}^k(\mathfrak{g})$ denote the universal $\mathcal{W}$-algebra of $\mathfrak{g}$ associated to this principal nilpotent element and principal gradation. We denote by $\mathcal{W}_k(\mathfrak{g})$ its unique simple graded quotient. It was proven by Arakawa [Ar2, Ar3] that $\mathcal{W}_k(\mathfrak{g})$ is lisse and rational when $k$ is a non-degenerate admissible level. These $\mathcal{W}$-algebras are called the minimal series principal $\mathcal{W}$-algebras since in the case $\mathfrak{g} = \mathfrak{sl}_2$ they are exactly the minimal series Virasoro vertex algebras. They are not necessarily unitary, but if $\mathfrak{g}$ is simply laced, there exists a sub-series called the discrete series which were conjectured for many years to be unitary.

For a simple Lie algebra $\mathfrak{g}$ and $l \in \mathbb{C}$, let $V^l(\mathfrak{g})$ be the universal affine vertex algebra associated with $\mathfrak{g}$ of level $l$, and let $L_l(\mathfrak{g})$ be its unique simple quotient which is graded by conformal degree. We denote by $u(z)$ the generating fields of $V^l(\mathfrak{g})$ for $u \in \mathfrak{g}$. By abuse of notation, we also denote by $u(z)$ the generating fields of $L_l(\mathfrak{g})$. There exists a diagonal homomorphism

$$V^{l+1}(\mathfrak{g}) \to V^l(\mathfrak{g}) \otimes L_1(\mathfrak{g}), \quad u(z) \mapsto u(z) \otimes 1 + 1 \otimes u(z), \quad u \in \mathfrak{g},$$

and if we identify $V^{l+1}(\mathfrak{g})$ with its image inside $V^l(\mathfrak{g}) \otimes L_1(\mathfrak{g})$, $\text{Com}(V^{l+1}(\mathfrak{g}), V^l(\mathfrak{g}) \otimes L_1(\mathfrak{g}))$ is well defined. The following result was proven in [ACL].

**Theorem 1.1.** Let $\mathfrak{g}$ be simply laced and let $k, l$ be complex numbers related by

$$k + h^\vee = \frac{l + h^\vee}{1 + h^\vee + 1},$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

1. For generic values of $k$, we have a vertex algebra isomorphism

$$\mathcal{W}^k(\mathfrak{g}) \cong \text{Com}(V^{l+1}(\mathfrak{g}), V^l(\mathfrak{g}) \otimes L_1(\mathfrak{g})), \quad \text{and } V^{l+1}(\mathfrak{g}) \text{ and } \mathcal{W}^k(\mathfrak{g}) \text{ form a dual pair inside } V^l(\mathfrak{g}) \otimes L_1(\mathfrak{g}).$$

2. Suppose that $l$ is an admissible level for $\hat{\mathfrak{g}}$. Then $k$ defined above is a non-degenerate admissible level for $\hat{\mathfrak{g}}$ so that $\mathcal{W}_k(\mathfrak{g})$ is a minimal series $\mathcal{W}$-algebra. We have a vertex algebra isomorphism

$$\mathcal{W}_k(\mathfrak{g}) \cong \text{Com}(L_{l+1}(\mathfrak{g}), L_l(\mathfrak{g}) \otimes L_1(\mathfrak{g})), \quad \text{and } L_{l+1}(\mathfrak{g}) \text{ and } \mathcal{W}_k(\mathfrak{g}) \text{ form a dual pair in } L_l(\mathfrak{g}) \otimes L_1(\mathfrak{g}).$$

This was conjectured in [BBSS] in the case of discrete series, which correspond to the case $k \in \mathbb{N}$, and by Kac and Wakimoto [KW1, KW2] for arbitrary minimal series $\mathcal{W}$-algebras. The conjectural character formula of $\text{FEK}$ for minimal series representations of $\mathcal{W}$-algebras that was proved in [Ar1], together with the character formula of [KW2] of branching rules, proves the matching of characters. This provided strong evidence for the conjecture before [ACL] appeared. Theorem 1.1 immediately implies the unitarity of the discrete series $\mathcal{W}$-algebras. It also has
the following striking application. Suppose that \( \mathfrak{g} \) is simply laced and \( k \) is an admissible level for \( \mathfrak{g} \). In [CHY], a certain subcategory of the Bernstein-Gelfand-Gelfand category \( \mathcal{O} \) for the corresponding affine Lie algebra \( \mathfrak{g} \) was considered. Using Theorem 1.1, it was shown that this category has a natural ribbon vertex tensor category structure, which is modular under some mild arithmetic conditions on \( k \). This gives the first examples of such modular tensor categories for vertex algebras that are not lisse.

Since Theorem 1.1 has many applications, it is an important problem to find coset realizations of other families of \( \mathcal{W} \)-(super)algebras. One such family involves the \( \mathcal{W} \)-superalgebra \( \mathcal{W}^k(\mathfrak{sl}(n+1|n)) \) associated to the principal nilpotent element in the even part of \( \mathfrak{sl}(n+1|n) \). This was conjectured in the physics literature by Ito's conjecture holds in the first nontrivial case \( N = 2 \) superconformal vertex algebra [KRW, CL]. Our main result in this paper is that Ito's conjecture holds in the first nontrivial case \( n = 2 \). As a corollary, we show that the simple \( \mathcal{W} \)-superalgebras \( \mathcal{W}_k(\mathfrak{sl}(3|2)) \) are lisse and rational, where \((k + 1)(l + 3) = 1\) and \( l > 1 \) is a positive integer.

2. Free field realization of \( \mathcal{W}^k(\mathfrak{sl}(n+1|n)) \)

We introduce free field realizations of the \( \mathcal{W} \)-superalgebra \( \mathcal{W}^k(\mathfrak{sl}(n+1|n)) \) associated to the principal nilpotent element in the even part of \( \mathfrak{sl}(n+1|n) \). In this paper, we follow the same notations for vertex (super)algebras as in Section 2 in [CL]. Let \( \{e_{i,j}\}_{i,j=1}^{2n+1} \) be the standard basis of \( \mathfrak{gl}(n+1|n) \). Denote by \( \bar{h} = \bigoplus_{i=1}^{2n+1} \mathbb{C}e_{i,i} \) a Cartan subalgebra of \( \mathfrak{gl}(n+1|n) \) and by \( \bar{h}^* = \bigoplus_{i=1}^{2n+1} \mathbb{C}e_i \) the dual of \( \bar{h} \), where \( \{e_i\}_{i=1}^{2n+1} \) is the dual basis of \( \bar{h}^* \), i.e. \( e_i(e_{j,j}) = \delta_{i,j} \). Let \( h = \{h \in \bar{h} \mid \text{str}(h) = 0\} \) be a Cartan subalgebra of \( \mathfrak{sl}(n+1|n) \subset \mathfrak{gl}(n+1|n) \), where \( \text{str} \) denotes the super trace. We identify \( h^* \) with \( \bar{h} \) by the super trace and denote by \( h^* \ni \lambda \mapsto t_\lambda \in h \), i.e. \( \lambda(h) = \text{str}(t_\lambda h) \) for all \( h \in h \). Set a non-degenerate symmetric bilinear form \( \langle \lambda | \lambda' \rangle = \text{str}(t_\lambda t_{\lambda'}) \) on \( h^* \) for \( \lambda, \lambda' \in h^* \). Then \( \Delta = \{e_i - e_j \mid 1 \leq i \neq j \leq 2n+1\} \) is the root system of \( \mathfrak{sl}(n+1|n) \) associated with \( h \). Define a set \( \Pi = \{\alpha_i \mid i = 1, \ldots, 2n\} \) of simple roots by \( \alpha_{2i-1} = e_i - e_{i+n+1} \) and \( \alpha_{2i} = e_{i+n+1} - e_i+1 \) for \( i = 1, \ldots, n \). Since \( (\alpha_i|\alpha_j) = (-1)^{i+j+1}\delta_{j,i+1} \) for all \( i \leq j \), all simple roots are odd and isotropic.

Fix a root vector \( e_{i,j} - e_{j,i} = e_{i,j} \) for \( i \neq j \). Let

\[
    f = f_{\text{prim}} := \sum_{i=1}^{2n-1} e_{-\alpha_i - \alpha_{i+1}} = \sum_{1 \leq i \leq 2n, i \neq n+1} e_{i+1,i}
\]

be a principal nilpotent element in the even part of \( \mathfrak{sl}(n+1|n) \), and

\[
    x := \sum_{i=1}^{n} \frac{i}{2} (t_{\alpha_{2i}} + t_{\alpha_{2n-2i+1}}) = \sum_{i=1}^{n+1} \left( \frac{n}{2} - i + 1 \right) e_{i,i} + \sum_{i=1}^{n} \left( \frac{n}{2} - i + \frac{1}{2} \right) e_{i+n+1,i+n+1}
\]

a semisimple element of \( \mathfrak{sl}(n+1|n) \) in \( h \). Then \( \text{ad}(x) \) defines a good grading \( \Gamma = \Gamma_x \) on \( \mathfrak{sl}(n+1|n) \) for \( f_{\text{prim}} \) such that all positive roots has non-negative degree. The corresponding weighted Dynkin diagram is the following:
\[
\begin{array}{cccc}
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{2n-1} \alpha_{2n}
\end{array}
\]

See [Ho] for good gradings of \(\mathfrak{sl}(n+1|n)\). We have \(\mathfrak{g}_0 = \mathfrak{h}\) and \(\tau_k(u|v) = (k+1)\text{str}(uv)\) for all \(u, v \in \mathfrak{h}\). Thus, \(\mathcal{H} = V^{\tau_k}(\mathfrak{g}_0)\) is the Heisenberg vertex algebra generated by fields \(\alpha_i(z)\) of conformal degree 1 for \(i = 1, \ldots, 2n\) which satisfy that

\[
\alpha_i(z)\alpha_j(w) \sim \frac{(-1)^{i+j}(k+1)\delta_{j+i+1}}{(z-w)^2}
\]

for all \(i \leq j\). Since \(\mathfrak{g}_\frac{1}{2} = \bigoplus_{i=1}^{2n} C\alpha_i\), and \(\text{str}(f(e_{\alpha_i}, e_{\alpha_j})) = (-1)^{i+j}\delta_{j+i+1}\) for all \(i \leq j\), \(\Phi(\mathfrak{g}_\frac{1}{2})\) is the vertex superalgebra generated by odd fields \(\Phi_i(z) = \Phi_{\alpha_i}(z)\) of conformal degree \(\frac{1}{2}\) which satisfy that

\[
\Phi_i(z)\Phi_j(w) \sim \frac{(-1)^{i+j}\delta_{j+i+1}}{z-w}
\]

for all \(i \leq j\). By [Ge], for generic \(k\), we have an isomorphism

\[(2.1) \quad \mathcal{W}^k(\mathfrak{sl}(n+1|n)) := \mathcal{W}^k(\mathfrak{sl}(n+1|n), f_{\text{prin}}, \Gamma_x) \simeq \bigcap_{i=1}^{2n} \ker \mathcal{Q}_i,
\]

where

\[
\mathcal{Q}_i = \int : e^{-\frac{i}{k+1} f \alpha_i(z)} \Phi_i(z) : dz
\]

for all \(i\) are screening operators acting on \(\mathcal{H} \otimes \Phi(\mathfrak{g}_\frac{1}{2})\). If \(k \neq -1\), \(\mathcal{W}^k(\mathfrak{sl}(n+1|n))\) is conformal with central charge \(-3n(kn+k+n)\).

Replacing \(k\) by a indeterminate \(k\) in \(T = \mathbb{C}[k, (k+1)^{-\frac{1}{2}}]\), we define the \(\mathcal{W}\)-algebra \(\mathcal{W}^k(\mathfrak{sl}(n+1|n))\) over \(T\), which also satisfies \((2.1)\). Let

\[
\begin{align*}
G_+^{(n)} &= \frac{1}{\sqrt{k+1}} \sum_{i=1}^{n} \left( \sum_{j=i}^{n} : \alpha_{2i-1} \Phi_{2j} : + i(k+1)\partial \Phi_{2i} \right), \\
G_-^{(n)} &= \frac{1}{\sqrt{k+1}} \sum_{i=1}^{n} \left( \sum_{j=1}^{i} : \alpha_{2i} \Phi_{2j-1} : + (n-i+1)(k+1)\partial \Phi_{2i-1} \right), \\
H^{(n)} &= (G_+^{(n)})_{(1)} (G_-^{(n)}), \quad L^{(n)} = (G_+^{(n)})_{(0)} (G_-^{(n)}) - \frac{1}{2} H^{(n)}.
\end{align*}
\]

Then \(H^{(n)}(z), G_+^{(n)}(z), L^{(n)}(z)\) generate a copy of \(N = 2\) superconformal vertex algebra, and belong to \(\bigcap_{i=1}^{2n} \ker \mathcal{Q}_i\). Since the specialization of \(\mathcal{W}^{k}(\mathfrak{sl}(n+1|n))\) at \(k = k \in \mathbb{C}\setminus\{1\}\) coincides with \(\mathcal{W}^{k}(\mathfrak{sl}(n+1|n))\), \(\mathcal{W}^{k}(\mathfrak{sl}(n+1|n))\) has a copy of \(N = 2\) algebra. As shown in Section 7 in [KRW], in the case \(n = 1\), \(\mathcal{W}^{k}(\mathfrak{sl}(2|1))\) is just isomorphic to the \(N = 2\) algebra.
3. The Kazama-Suzuki coset

Let $\mathcal{E}$ be the $bc$-system, i.e., the vertex superalgebra generated by odd fields $b(z)$ and $c(z)$ which satisfy

$$b(z)c(w) \sim \frac{1}{z-w}, \quad b(z)b(w) \sim 0 \sim c(z)c(w).$$

Set $\mathcal{E}(n) = \mathcal{E}^n$, and denote by $b_i(z), c_i(z)$ the generating fields $b(z), c(z)$ of the $i$-th component in $\mathcal{E}(n)$. Then $b_i(z), c_i(z)$ are both primary of conformal degree $\frac{1}{2}$ with respect to the Virasoro element

$$L(z) = \frac{1}{2} \sum_{i=1}^{n} \{ : b_i(z) \partial c_i(z) : - : (\partial b_i(z)) c_i(z) : \},$$

which has central charge $n$. We have a $V^1(\mathfrak{gl}_n)$-module structure on $\mathcal{E}(n)$ defined by

$$e_{i,j}(z) \mapsto : b_i(z) c_j(z);$$

which in fact descends to an action of the simple quotient $L_1(\mathfrak{gl}_n)$. Here $\{ e_{i,j} \}_{i,j=1}^{n}$ denotes the standard basis of $\mathfrak{gl}_n$. Set $h_i = e_{i,i} - e_{i+1,i+1}$. Since $V^1(\mathfrak{gl}_n) \subset V^1(\mathfrak{sl}_{n+1})$, we have an embedding

$$V^l + 1(\mathfrak{gl}_n) \hookrightarrow V^l(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n)$$

defined by the diagonal action, where the field $I(z)$ on $V^l + 1(\mathfrak{gl}_n)$ corresponding to a central element $I = \sum_{i=1}^{n} e_{i,i} \in \mathfrak{gl}_n$ is mapped to $\varpi_n(z) + \sum_{i=1}^{n} : b_i(z) c_i(z) :$ and $\varpi_n = \frac{1}{n+1} \sum_{i=1}^{n} i h_i \in \mathfrak{sl}_{n+1}$, which is a semisimple element corresponding to the $n$-th fundamental weight of $\mathfrak{sl}_{n+1}$. Define a coset vertex superalgebra

$$C^l(n) := \text{Com}(V^l + 1(\mathfrak{gl}_n), V^l(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n)),$$

which has central charge $c = \frac{3n-1}{n+1}$. This coset is known as the Kazama-Suzuki coset \cite{KS}, and it was conjectured by Ito \cite{I} to be isomorphic to $W^k(\mathfrak{sl}(n+1)\mathfrak{n})$ for generic $k$, where $(k+1)(l+n+1) = 1$. Here we discuss some features of this coset that hold for all $n \geq 3$. As shown in Example 7.11 of \cite{CL}, $C^l(n)$ has a minimal strong generating set consisting of even fields in conformal degrees

$$1, 2, 2, 3, 3, \ldots, n, n, n+1,$$

and odd fields in conformal degrees

$$\frac{3}{2}, \frac{5}{2}, \frac{5}{2}, \ldots, \frac{2n+1}{2}, \frac{2n+1}{2}. $$

Also, the Heisenberg field, the Virasoro field, and two odd fields in conformal degree $\frac{3}{2}$ generate a copy of the $N = 2$ superconformal vertex algebra. In the case $n = 1$, $C^l(1)$ is just the $N = 2$ algebra; see Lemma 8.6 of \cite{CL}.

**Lemma 3.1.** For $n \geq 2$, and generic values of $l$, we have a conformal embedding

$$\mathcal{H}_0 \otimes W^r(\mathfrak{sl}_n) \otimes C^l(n) \hookrightarrow C^l(n), \quad r = -n + \frac{l+n}{l+n+1}. $$

Here $\mathcal{H}_0$ is the rank one Heisenberg algebra, $W^r(\mathfrak{sl}_n)$ is the principal $W$-algebra of $\mathfrak{sl}_n$ at level $r$, and

$$G^l(n) = \text{Com}(V^l(\mathfrak{gl}_n), V^l(\mathfrak{sl}_{n+1})), $$

which was called a generalized parafermion algebra in \cite{I}. \hfill \square
**Proof.** Corresponding to the natural embedding $\mathfrak{gl}_n \hookrightarrow \mathfrak{sl}_{n+1}$ is an embedding $V^l(\mathfrak{gl}_n) \hookrightarrow V^l(\mathfrak{sl}_{n+1})$, so $C^l(n)$ contains a copy of the coset $\text{Com}(V^{l+1}(\mathfrak{gl}_n), V^l(\mathfrak{gl}_n) \otimes \mathcal{E}(n))$, which is isomorphic to

$$\mathcal{H}_0 \otimes \text{Com}(V^{l+1}(\mathfrak{sl}_n), V^l(\mathfrak{sl}_n) \otimes \mathcal{E}(n)) \cong \mathcal{H}_0 \otimes \text{Com}(V^{l+1}(\mathfrak{sl}_n), V^l(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n)).$$

By Theorem 1.1 this coset is isomorphic to $\mathcal{H}_0 \otimes \mathcal{W}^r(\mathfrak{sl}_n)$.

It is also clear from the definition of $C^l(n)$ that it contains a copy of $\mathcal{G}^l(n)$; what remains to show is that this copy of $\mathcal{G}^l(n)$ commutes with $\mathcal{H}_0 \otimes \mathcal{W}^r(\mathfrak{sl}_n)$ and that $\mathcal{G}^l(n)$ is a conformal embedding. But this is clear from Theorems 5.1 and 5.2 of [FZ], since the Virasoro elements for the cosets $\text{Com}(\mathcal{H}_0, C^l(n))$, $\text{Com}(\mathcal{W}^r(\mathfrak{sl}_n), C^l(n))$, and $\text{Com}(\mathcal{G}^l(n), C^l(n))$ pairwise commute, and their sum is the total Virasoro element for $C^l(n)$.

**Remark 3.2.** By Theorem 8.1 of [L], $\mathcal{G}^l(n)$ is of type $\mathcal{W}(2, 3, \ldots, n^2 + 3n + 1)$ for generic values of $l$. In the case $n = 1$, $\mathcal{G}^l(1)$ coincides with the parafermion algebra of $\mathfrak{sl}_2$, which is of type $\mathcal{W}(2, 3, 4, 5)$ [DLY].

### 3.1. Simple quotients.

Let $C_l(n)$ denote the unique simple graded quotient of $C^l(n)$. Suppose first that $l > 1$ is a positive integer. Then the maps

$$V^l(\mathfrak{gl}_n) \hookrightarrow V^l(\mathfrak{sl}_{n+1}), \quad V^{l+1}(\mathfrak{gl}_n) \hookrightarrow V^l(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n)$$

induce maps of simple vertex algebras

$$(3.2) \quad L_l(\mathfrak{sl}_n) \otimes \mathcal{H}_0 \hookrightarrow L_l(\mathfrak{sl}_{n+1}), \quad L_1(\mathfrak{sl}_n) \otimes \mathcal{H}_0 \hookrightarrow L_l(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n).$$

In fact, these maps extend to embeddings

$$L_l(\mathfrak{sl}_n) \otimes V_L \hookrightarrow L_l(\mathfrak{sl}_{n+1}), \quad L_{l+1}(\mathfrak{sl}_n) \otimes V_L \hookrightarrow L_l(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n). \quad (3.3)$$

Here $V_L$ is the lattice vertex algebra associated to the rank one lattice

$$L = \sqrt{n(n+1)(n+l+1)} \mathbb{Z},$$

which is an extension of $\mathcal{H}_0$. It follows from Theorem 8.1 of [CL] that the simple quotient $\mathcal{G}_l(n)$ of $\mathcal{G}^l(n)$ coincides with the coset

$$\text{Com}(L_l(\mathfrak{sl}_n) \otimes V_L, L_l(\mathfrak{sl}_{n+1})).$$

Similarly,

$$C_l(n) = \text{Com}(L_{l+1}(\mathfrak{sl}_n) \otimes V_L, L_l(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n)).$$

In particular, $C_l(n)$ is a vertex superalgebra extension of

$$V_L \otimes \mathcal{W}_r(\mathfrak{sl}_n) \otimes \mathcal{G}_l(n).$$

It was shown in [ACL] that

$$\mathcal{G}_l(n) \cong \mathcal{W}_s(\mathfrak{sl}_n), \quad r = -n + \frac{l + n}{1 + l + n}, \quad s = -l + \frac{l + n}{1 + l + n},$$

which is lisse and rational by [Ar2] [Ar3]. It follows that $C_l(n)$ is a rational vertex superalgebra; see Corollary 14.1 of [ACL].

**Remark 3.3.** The case $l = 1$ is degenerate because $L_1(\mathfrak{sl}_n) \otimes \mathcal{H}_0$ is conformally embedded in $L_1(\mathfrak{sl}_{n+1})$ [AKMPP], so that $\mathcal{G}_1(n) \cong \mathbb{C}$. In this case, $C_1(n)$ is an extension of $V_L \otimes \mathcal{W}_r(\mathfrak{sl}_n)$, where $L = \sqrt{n(n+1)(n+2)} \mathbb{Z}$ and $r = -n + \frac{1}{2} \frac{n}{n+2}$, so it is also rational and lisse.
For \( l > 1 \), the key ingredient in proving the rationality of \( C_l(n) \) is that \( G_l(n) \) is coincident with a type \( A \) principal \( W \)-algebra. In fact, the levels \( l \) where \( G_l(n) \) is isomorphic to \( W_s(\mathfrak{sl}_m) \) for some \( m \geq 3 \) and some level \( s \in \mathbb{C} \), have been classified; see Theorem 10.4 of [L]. In addition to the above family when \( l = m \), we have the following two additional families:

1. For \( n > 1 \), \( m \geq 3 \), \( m \neq n + 1 \), and \( l = -(n + 1) + \frac{m-n+1}{m-1} \), we have

\[
G_l(n) \cong W_s(\mathfrak{sl}_m), \quad s = -m + \frac{-1 - n + m}{-1 + m}, \quad -m + \frac{-1 + m}{-1 - n + m},
\]

which has central charge \( c = \frac{(1+n-m+n)(m+n-1)}{1+n-m} \). Note that the level \( l \) of \( V^l(\mathfrak{sl}_{n+1}) \) is admissible if \( m \geq 2(n+1) \), and \( m-1 \) and \( m-(n+1) \) are relatively prime.

2. For \( n > 1 \), \( m \geq 3 \), \( m \neq n \), and \( l = -(n + 1) + \frac{1+n}{1+m} \), we have

\[
G_l(n) \cong W_s(\mathfrak{sl}_m), \quad s = -m + \frac{-n + m}{1 + m}, \quad -m + \frac{1 + m}{-n + m},
\]

which has central charge \( c = \frac{n(m-1)(1+2m+nm)}{1+m} \). Note that if \( m + 1 \) and \( n + 1 \) are relatively prime the level \( l \) is boundary admissible for \( \mathfrak{sl}_{n+1} \).

**Question 3.4.** For the above levels \( l \), do the maps

\[
V^l(\mathfrak{gl}_n) \hookrightarrow V^l(\mathfrak{sl}_{n+1}), \quad V^{l+1}(\mathfrak{gl}_n) \hookrightarrow V^l(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n)
\]

induce maps of simple vertex algebras

\[
(3.4) \quad L_l(\mathfrak{sl}_n) \otimes \mathcal{H}_0 \hookrightarrow L_l(\mathfrak{sl}_{n+1}), \quad L_{l+1}(\mathfrak{sl}_n) \otimes \mathcal{H}_0 \hookrightarrow L_l(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n)?
\]

**Remark 3.5.** If the maps (3.4) exist, Theorem 8.1 of [CL] would imply that

\[
G_l(n) = \text{Com}(L_l(\mathfrak{sl}_n) \otimes \mathcal{H}_0, L_l(\mathfrak{sl}_{n+1})), \quad C_l(n) = \text{Com}(L_{l+1}(\mathfrak{sl}_n) \otimes \mathcal{H}_0, L_l(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n)),
\]

and we would have a conformal embedding

\[
\mathcal{H}_0 \otimes W_r(\mathfrak{sl}_n) \otimes W_s(\mathfrak{sl}_m) \hookrightarrow C_l(n), \quad r = -n + \frac{l + n}{1 + l + n}.
\]

**4. Ito’s Conjecture for \( n = 2 \)**

In this section, we shall prove our main result, which is that Ito’s conjecture holds for \( n = 2 \). For this, we need the explicit free field realization of \( W^k(\mathfrak{sl}(3|2)) \). We introduce two even fields \( H, S \) and two odd fields \( G_{\pm} \) on \( \mathcal{H} \otimes \Phi(\mathfrak{g}_+^\mathbb{Z}) \) defined as
follows:

\[ H = 2\alpha_1 - \alpha_2 + \alpha_3 - 2\alpha_4 - :\Phi_1\Phi_2 : - :\Phi_1\Phi_4 : - :\Phi_3\Phi_4 :, \]
\[ G_+ = :\alpha_1\Phi_2 : + :\alpha_4\Phi_4 : + :\alpha_3\Phi_4 : + (k + 1)\partial\Phi_1 + 2(k + 1)\partial\Phi_4, \]
\[ G_- = :\alpha_2\Phi_1 : + :\alpha_4\Phi_4 : + 2(k + 1)\partial\Phi_1 + (k + 1)\partial\Phi_3, \]
\[ S = \frac{3}{2}(\alpha_2^2 : + : \alpha_1\alpha_3 : - : \alpha_1\alpha_4 : - \frac{1}{2} : \alpha_2^2 : - 2 : \alpha_2\alpha_3 : + : \alpha_2\alpha_4 : - \frac{1}{2} : \alpha_3^2 : + : \alpha_1\alpha_3 : - : \alpha_1\alpha_4 : - \frac{1}{2} : \alpha_3^2 : ) \]
\[ + : \alpha_2^2 : - : \alpha_1\Phi_2 : + \frac{1}{2}(2k + 3) : \Phi_1\partial\Phi_2 : + \frac{1}{2}(2k + 3) : \Phi_1\partial\Phi_4 : + \frac{1}{2}(4k + 3) : (\partial\Phi_3)\Phi_4 : \]
\[ - \frac{1}{2}(2k + 3) : (\partial\Phi_1)\Phi_2 : - \frac{1}{2}(2k + 3) : (\partial\Phi_1)\Phi_4 : + \frac{1}{2}(4k + 3) : (\partial\Phi_3)\Phi_4 : \]
\[ + (k + 1)\partial(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4) ). \]

**Lemma 4.1.** \( H, S, G_\pm \) belong to \( \bigcap_{i=1}^4 \text{Ker} \Phi_i. \)

**Proof.** Direct calculations. \( \square \)

Set

\[ L = \frac{1}{k + 1}G_{+(0)}G_- - \frac{1}{2}\partial H, \quad W_2 = \frac{1}{2}L + (k + 1)S, \quad Q_\pm = G_{+(0)}S + \frac{1}{4}\partial G_\pm, \]
\[ W_3 = G_{+(0)}Q_- - \frac{1}{4}(k + 1)(2\partial S + \partial L + 6 : HL : - 2 : H^3 : ), \]

where \( k \neq -1, \) and we denote by \( A(z) = \sum_{n \in \mathbb{Z}} A(n)z^{-n-1} \) the field corresponding to \( A \) for any \( A \in \mathcal{H} \otimes \Phi(g_2^\pm) \). Then \( L \) is a Virasoro element of \( \mathcal{H} \otimes \Phi(g_2^\pm) \) with central charge \(-6(3k+2)\). Even primary elements \( H, W_2, W_3 \) have conformal degree 1, 2, 3, and odd primary elements \( G_+, G_-, Q_+, Q_- \) have conformal degree \( \frac{5}{2}, \frac{3}{2}, \frac{5}{2}, \frac{3}{2} \) respectively.

**Proposition 4.2.** For generic \( k \), the elements \( H, L, W_2, W_3, G_\pm, Q_\pm \) strongly generate the \( \mathcal{W} \)-algebra \( \mathcal{W}^k(\mathfrak{sl}(3|2)) \) in \( \mathcal{H} \otimes \Phi(g_2^\pm) \).

**Proof.** For generic \( k \), let \( \mathcal{V} = \bigcap_{i=1}^4 \text{Ker} \Phi_i \) be a vertex subalgebra of \( \mathcal{H} \otimes \Phi(g_2^\pm) \). By Lemma 4.1, \( \mathcal{V} \) is isomorphic to \( \mathcal{W}^k(\mathfrak{sl}(3|2)) \). By Lemma 4.2, all elements \( H, L, W_2, W_3, G_\pm, Q_\pm \), belong to \( \mathcal{V} \). Since \( \dim \mathfrak{sl}(3|2)^{\text{form}} \) = 8, and \( \dim \mathfrak{sl}(3|2)^{\text{form}} \cap \mathfrak{sl}(3|2)_j \) is equal to 1 for \( j = 0, -2 \) and is equal to 2 for \( j = -1, -\frac{3}{2} \), \( \mathcal{V} \) is of type \( \mathcal{W}(1, \frac{3}{2}, 2, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}, \frac{3}{2}, \frac{3}{2}). \) Let \( \mathcal{V}_j \) be the subspace of \( \mathcal{V} \) with conformal degree \( j \) and \( \mathcal{U}_j = \mathcal{V}_{<j} \cap \text{Span}[A_{1(-n_1)} \cdots A_{s-1(-n_s)}A_s \in \mathcal{V} \mid A_i \in \mathcal{V}_{<j}, n_i \geq 1, s \in \mathbb{Z}_{\geq 1}], \) where \( \mathcal{V}_{<j} = \bigoplus_{p<j} \mathcal{V}_p. \) Then \( \dim \mathcal{V}_{<0} = 0 = \dim \mathcal{V}_{\frac{3}{2}} = \dim \mathcal{V}_{\frac{5}{2}}, \) \( \dim \mathcal{V}_{\frac{1}{2}} = \mathcal{C}H, \mathcal{V}_{\frac{5}{2}} = \mathcal{C}G_+ \oplus \mathcal{C}G_-, \mathcal{V}_{\frac{3}{2}} = \mathcal{C}L \oplus \mathcal{C}W_2, \mathcal{V}_{\frac{1}{2}} = \mathcal{C}Q_+ \oplus \mathcal{C}Q_-, \mathcal{V}_{\frac{3}{2}} = \mathcal{C}W_3 \oplus \mathcal{U}_3 \) and \( \mathcal{V}_3 = \mathcal{C}W_3 \oplus \mathcal{U}_3. \) Therefore \( H, L, W_2, W_3, G_\pm, Q_\pm \) strongly generate \( \mathcal{V} \simeq \mathcal{W}^k(\mathfrak{sl}(3|2)) \). \( \square \)

**Corollary 4.3.** If \( k \) is generic, the vertex subalgebra of \( \mathcal{H} \otimes \Phi(g_2^\pm) \) (weakly) generated by \( H, S, G_\pm \) is isomorphic to \( \mathcal{W}^k(\mathfrak{sl}(3|2)) \).

**Proof.** By definition, all elements \( L, W_2, W_3, Q_\pm \) belong to the vertex subalgebra generated by \( H, S, G_\pm \). Therefore the assertion follows from Proposition 4.2. \( \square \)
Next, we consider the coset $C^l(2)$. Suppose that $l \neq -3$. Let $\hat{H}, \hat{S}$ be even elements and $\hat{G}_\pm$ odd elements in $V^l(\mathfrak{sl}(3)) \otimes \mathcal{E}(2)$ defined as follows:

$$\hat{H} = \frac{1}{l + 3} (h_1(z) - 2h_2(z) + l : b_1 c_1 + l : b_2 c_2 :),$$

$$\hat{G}_+ = \frac{1}{l + 3} (: e_{3,1} b_1 + e_{3,2} b_2 :), \quad \hat{G}_- = \frac{1}{l + 3} (: e_{1,3} c_1 + e_{2,3} c_2 :),$$

$$\hat{S} = -\frac{3}{2(l + 3)^2} \left( \frac{1}{2} : h_1^2 + : h_1 h_2 : - : h_2^2 + : e_{1,2} e_{2,1} : - : e_{2,3} e_{3,2} : - : e_{1,3} e_{3,1} : \right)$$

$$- (2l + 3) : h_1 b_1 c_1 + (l + 2) : h_2 b_2 c_2 + (l + 1) : h_2 b_1 c_1 + (l + 1) : h_2 b_2 c_2 :$$

$$- (3l + 5) : e_{1,2} b_2 c_1 - (3l + 5) : e_{2,1} b_1 c_2 + l (2l + 3) : b_1 b_2 c_1 c_2 - \frac{1}{2} l (l + 2) : b_1 \partial c_1 :$$

$$- \frac{1}{2} l (l + 2) : b_2 \partial c_2 + \frac{1}{2} l (l + 2) : (\partial b_2) c_2 - 2 \partial h_1 + \partial h_2 \right).$$

**Lemma 4.4.** $\hat{H}, \hat{S}, \hat{G}_\pm$ belong to $C^l(2)$. 

**Proof.** Direct calculations. □

For $l \neq -3$, set $k = \frac{-4}{l + 3} - 1$. We define elements $\hat{L}, \hat{W}_2, \hat{W}_3, \hat{Q}_\pm$ in $V^l(\mathfrak{sl}(3)) \otimes \mathcal{E}(2)$ in the same way as $L, W_2, W_3, Q_\pm$, with $A$ replaced by $\hat{A}$ for $A = H, S, G_\pm$. By Lemma 4.4, all elements $\hat{L}, \hat{W}_2, \hat{W}_3, \hat{Q}_\pm$ belong to $C^l(2)$.

**Lemma 4.5.** For generic $k$, there exists a vertex superalgebra homomorphism $\gamma: W^k(\mathfrak{sl}(3))[2]) \rightarrow C^l(2)$, where $(k + 1)(l + 3) = 1$.

**Proof.** Suppose that $k$ is generic. By direct calculations, we have

$$H(z)H(w) \sim \frac{-2(3k + 2)}{(z - w)^2}, \quad H(z)S(w) \sim \frac{3(2k + 1)H(w)}{(z - w)^2}, \quad H(z)G_\pm(w) \sim \frac{\pm G_\pm(w)}{z - w},$$

$$G_+(z)G_-(w) \sim -2(k + 1)(3k + 2)(z - w)^3 + \frac{(k + 1)H(w)}{(z - w)^2} + \frac{(k + 1)(L(w) + \frac{1}{4} \partial J(w))}{z - w},$$

$$G_\pm(z)S(w) \sim \frac{-\frac{3}{2} G_\pm(w)}{(z - w)^2} + \frac{Q_\pm(w) - \frac{1}{4} \partial G_\pm(w)}{z - w}, \quad G_+(z)G_+(w) \sim 0 \sim G_-(z)G_-(w),$$

$$S(z)S(w) \sim \frac{4(3k + 2)(12k^2 + 23k + 6)}{(z - w)^4}$$

$$+ \frac{3(5k + 2)S(w) - \frac{3}{2} (k + 1)(4k + 1)L(w) - \frac{9}{4} (3k + 1) : H(w)^2 :)}{(z - w)^2}$$

$$+ \frac{\frac{3}{2} \partial (3(5k + 2)S(w) - \frac{9}{2} (k + 1)(4k + 1)L(w) - \frac{9}{4} (3k + 1) : H(w)^2 :)}{z - w}.$$ 

If $(k + 1)(l + 3) = 1$, $\hat{H}(z), \hat{S}(z), \hat{G}_\pm(z)$ also satisfy the same formulae as above by replacing $A$ by $\hat{A}$ for $A = H, S, G_\pm, L, Q_\pm$. By Corollary 1.3, $H, S, G_\pm$ are weak generators of $W^k(\mathfrak{sl}(3)[2])$. Thus, the map

$$\gamma: W^k(\mathfrak{sl}(3)[2]) \rightarrow C^l(2), \quad A \mapsto \hat{A}, \quad A = H, S, G_\pm,$$

gives a well-defined homomorphism of vertex superalgebras. □

**Theorem 4.6.** The map $\gamma$ is an isomorphism. Therefore, for generic $k$, we have

$$W^k(\mathfrak{sl}(3)[2]) \simeq C^l(2),$$
where \((k + 1)(l + 3) = 1\).

**Proof.** Since \(\mathcal{W}^{k}(\mathfrak{sl}(3|2))\) is generically simple, the map \(\gamma\) is injective for generic \(k\). It is immediate that the fields \(\hat{A}\) for \(A = H, S, G_{\pm}, L_{r}, Q_{\pm}\) strongly generate the same vertex algebra as the fields in \(C^{l}(2)\) whose limits are given by the fields in the case \(n = 2\). It follows from Corollary 6.12 of [CL] that these fields generate all of \(C^{l}(2)\) for generic values of \(l\), so \(\gamma\) is surjective as well. \(\Box\)

In the case \(n = 2\) and \(r = -2 + \frac{l+2}{2+l}\), we have \(\mathcal{W}^{r}(\mathfrak{sl}_2) \cong \text{Vir}^{c}\) where \(c = \frac{(5+l)}{(2+l)(3+l)}\). Here \(\text{Vir}^{c}\) denotes the universal Virasoro vertex algebra with central charge \(c\). Similarly, we denote by \(\text{Vir}^{c}\) the simple quotient of \(\text{Vir}^{c}\).

**Corollary 4.7.** For \(l > 1\) a positive integer and \((k + 1)(l + 3) = 1\), \(\mathcal{W}_{k}(\mathfrak{sl}(3|2))\) is lisse and rational. In particular it is an extension of the rational vertex algebra

\[
\mathcal{W}_{k}(\mathfrak{sl}(3|2)) \rightarrow C^{l}(2),
\]

which are defined over the (suitably localized) rings \(\mathbb{C}[k]\) and \(\mathbb{C}[l]\), respectively. Here \(k\) and \(l\) are related by \((k + 1)(l + 3) = 1\).

Recall that for all \(k \in \mathbb{C}\setminus\{-1\}\), the specialization of \(\mathcal{W}^{k}(\mathfrak{sl}(3|2))\) at \(k = k\), coincides with \(\mathcal{W}^{k}(\mathfrak{sl}(3|2))\). Similarly, by Corollary 6.7 of [CL], the specialization of \(C^{l}(2)\) at \(l = l\) coincides with \(C^{l}(2)\) for all real numbers \(l > 2\). (In [CL], the formal variable \(l\) was chosen instead so that \(l = \sqrt{l}\) gave the specialization of \(C^{l}(2)\), but the same proof applies.) It follows that for all positive integers \(l > 1\), the isomorphism \(\mathcal{W}_{k}(\mathfrak{sl}(3|2)) \simeq C^{l}(2)\) of Theorem 4.4 holds.

Next, by Theorem 8.1 of [CL], for all positive integers \(l > 1\), the natural map

\[
C^{l}(2) \rightarrow C_{l}(2) \cong \text{Com}(L_{l+1}(\mathfrak{sl}_2) \otimes V_{\sqrt{6(3+l)l}} L_{l}(\mathfrak{sl}_3) \otimes E(2)),
\]

is surjective. Therefore \(\mathcal{W}_{k}(\mathfrak{sl}(3|2)) \cong \text{Com}(L_{l+1}(\mathfrak{sl}_2) \otimes V_{\sqrt{6(3+l)l}} L_{l}(\mathfrak{sl}_3) \otimes E(2))\) for all positive integers \(l > 1\), and the result follows from Corollary 14.1 of [ACL] in the case \(n = 2\). \(\Box\)

**Remark 4.8.** Set \((k + 1)(l + 3) = 1\) and \(c = \frac{(5+l)}{(2+l)(3+l)}\). If the maps \((5.4)\) exist for \(n = 2\), we would have a conformal embedding of simple vertex algebras

\[
\mathcal{H}_{0} \otimes \text{Vir}^{c} \otimes \mathcal{W}_{s}(\mathfrak{sl}_m) \hookrightarrow \mathcal{W}_{k}(\mathfrak{sl}(3|2)),
\]

in the following cases.

1. \(m \geq 3\), \(l = -3 + \frac{m-3}{m-1}\), and \(s = -m + \frac{3m}{1+m}\).
2. \(m \geq 3\), \(l = -3 + \frac{3}{1+m}\), and \(s = -m + \frac{2m}{1+m}\).

**References**

[AKMPP] D. Adamovic, V. G. Kac, P. Moseneder Frajria, P. Papi, and O. Perse. Finite vs. infinite decompositions in conformal embeddings. Comm. Math. Phys. 348 (2016), no. 2, 445-473.

[Ar1] T. Arakawa. Representation theory of \(W\)-algebras. Invent. Math. 169 (2), 219–320, 2007.

[Ar2] T. Arakawa. Associated varieties of modules over Kac-Moody algebras and \(C_{2}\)-cofiniteness of \(W\)-algebras. Int. Math. Res. Not. (2015) Vol. 2015 11605-11666.
[Ar3] T. Arakawa. Rationality of $W$-algebras: principal nilpotent cases. *Ann. Math.* 182(2):565–694, 2015.

[ACL] T. Arakawa, T. Creutzig, A. Linshaw. $W$-algebras as coset vertex algebras. arXiv:1801.03822.

[BBSS] F. Bais, P. Bouwknegt, M. Surridge, and K. Schoutens. Coset construction for extended Virasoro algebras. *Nuclear Phys. B* 304(2):371-391, 1988.

[C] T. Creutzig. Fusion categories for affine vertex algebras at admissible levels. arXiv:1807.00415.

[CL] T. Creutzig and A. Linshaw. Cosets of affine vertex algebras inside larger structures. *J. Algebra* 517 (2019) 396-438.

[CHY] T. Creutzig, Y-Z Huang, and J. Yang. Braided tensor categories of admissible modules for affine Lie algebras. *Comm. Math. Phys.* 362 (2018), no. 3, 827–854.

[DLY] C. Dong, C. Lam, and H. Yamada. $W$-algebras related to parafermion algebras. *J. Algebra* 322 (2009), no. 7, 2366-2403.

[FF] B. L. Feigin, E. Frenkel. Quantization of Drinfel’d-Sokolov reduction. *Phys. Lett., B* 246(1–2):75–81, 1990.

[FKW] E. Frenkel, V. Kac, and M. Wakimoto. Characters and fusion rules for $W$-algebras via quantized Drinfeld-Sokolov reduction. *Comm. Math. Phys.* 147 (1992), 295–328.

[FZ] I.B. Frenkel and Y.C. Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke Math. J.* Vol. 66, No. 1, (1992), 123-168.

[Ge] N. Genra. Screening operators for $W$-algebras. *Selecta Math. (N.S.)*, 23(3):2157–2202, 2017.

[GKO] P. Goddard, A. Kent and D. Olive, Virasoro algebras and coset space models, *Phys. Lett B* 152 (1985) 88-93.

[Ho] C. Hoyt. Good gradings of basic Lie superalgebras. *Israel J. Math.*, 192(1):251–280, 2012.

[I] K. Ito. $N = 2$ superconformal $CP(n)$ model. *Nucl. Phys. B* 370 (1992) 123.

[KP] V. Kac and D. Peterson. Spin and wedge representations of infinite-dimensional Lie algebras and groups. *Proc. Natl. Acad. Sci. USA* 78 (1981), 3308-3312.

[KRW] V. G. Kac, S.-S. Roan, M. Wakimoto. Quantum reduction for affine superalgebras. *Comm. Math. Phys.*, 241(2-3):307–342, 2003.

[KW1] V. Kac and M. Wakimoto. Classification of modular invariant representations of affine algebras. In *Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988)*, volume 7 of Adv. Ser. Math. Phys. pages 138–177. World Sci. Publ., Teaneck, NJ, 1989.

[KW2] V. Kac and M. Wakimoto. Branching functions for winding subalgebras and tensor products. *Acta Appl. Math.* 21(1-2):3–39, 1990.

[KW3] V. G. Kac, M. Wakimoto. Quantum reduction and representation theory of superconformal algebras. *Adv. Math.*, 185(2):400–458, 2004.

[KW4] V. G. Kac, M. Wakimoto. Corrigendum to: “Quantum reduction and representation theory of superconformal algebras” [Adv. Math. 185(2004)400–458]. *Adv. Math.*, 193(2):453–455, 2005.

[KS] Y. Kazama and H. Suzuki, New $N = 2$ Superconformal Field Theories and Superstring Compactification. *Nucl. Phys. B* 321 (1989) 232.

[L] A. Linshaw Universal two-parameter $\mathcal{W}_{\infty}$-algebra and vertex algebras of type $\mathcal{W}(2, 3, \ldots, N)$. arXiv:1710.02270.