Spin Calogero–Moser systems for the cyclic quiver

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Abstract. The method of construction of classical integrable systems on quiver varieties is considered in the case of cyclic quiver. For two types of framing we obtained integrable spin generalisations of the Calogero–Moser systems associated with the complex reflection groups $S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$. This method gives Lax matrices of the systems in explicit form. In some particular cases the first Hamiltonians of the system are written explicitly. The paper is based on the work of the author with Oleg Chalykh (the University of Leeds).

1. Introduction
The first spin generalisation of the Calogero–Moser systems appeared in [7]. These were classical integrable systems defined by introducing spin interaction into the Hamiltonian of the rational Calogero–Moser system of type $A_n$. For the crystallographic root systems (or, equivalently, for semi-simple Lie algebras) integrable spin generalizations were found in [10]. In the rational case a grater class of integrable spin Calogero–Moser systems for classical series of root systems and, more generally, of complex reflection groups can be obtained from the representation theory of quivers (directed graphs): there are natural Poisson commuting Hamiltonians on quiver varieties. The latter were introduced by Nakajima for geometrical construction of the universal enveloping algebras for Kac–Moody algebras and their quantum versions [11].

As it is well-known, the (complexified) phase space of the Calogero–Moser system of type $A_n$ can be completed to the so-called Calogero–Moser space introduced by Wilson in order to classify rational solutions of the Kadomtsev–Petviashvili (KP) hierarchy by using the adelic Grassmannians [14]. The Calogero–Moser spaces can be represented as quiver varieties for a one-loop quiver. More generally, the completed phase spaces of the Calogero–Moser systems for classical series (also called Calogero–Moser spaces) are isomorphic to the quiver varieties associated to the cyclic quivers [5].

This paper is based on the work [1], where the theory of quiver varieties was applied to obtain solutions of generalised KP hierarchies. The solutions of hierarchies suppose dependence of infinite number of time variables. Each time variable appears from a Hamiltonian flow on the corresponding quiver variety. Therefore, to obtain the solutions of the hierarchy we need to find infinite number of appropriate Poisson commuting Hamiltonians on the quiver variety. It is proved that there are enough number (half of dimension of the variety) of functionally independent Hamiltonians among them. The obtained integrable systems can be interpreted as spin generalisations of the Calogero–Moser systems for classical series with completed phase spaces. In the present work these systems are described without mention of hierarchies.

The quiver varieties can be formulated in terms of representations of (deformed) preprojective algebras for quivers [3]. These algebras were introduced in [2]. To define the quiver variety
associated with a quiver $Q$ one need to consider the preprojective algebra for the framed quiver $Q_\zeta$ obtained from $Q$ by framing with respect to the framing vector $\zeta \in \mathbb{Z}^I_{\geq 0}$ where $I$ is the set of the vertices of the quiver $Q$.

The Calogero–Moser systems for classical series are obtained by considering cyclic quivers $Q$ and a shortest framing vector $\zeta$, i.e. a vector with components $\zeta_i$ such that $\sum_{i \in I} \zeta_i = 1$. Replacement of this framing vector by a vector $\zeta$ such that $\sum_{i \in I} \zeta_i \geq 2$ gives integrable spin versions of these systems. Diversity of the obtained spin systems is mainly provided by the diversity of the forms of the vector $\zeta$. We consider two cases in details: $\zeta$ with the identical components and $\zeta$ with only one non-zero component.

The plan of the paper is as follows. In Section 2 we recall the definition of the Calogero–Moser system associated with a root system and describe the Calogero–Moser space for $A_n$ case in details. In Section 3 we introduce the quiver varieties in terms of quivers and their preprojective algebras. In Sections 4 and 5 we obtain the spin systems by considering the case of cyclic quiver with two types of the framing vector mentioned above.

We mainly follow the notations of the paper [13].

2. Calogero–Moser systems and Calogero–Moser spaces

First we define the rational Calogero–Moser systems. The $A_{n-1}$ case is considered in details. We introduce the Calogero–Moser spaces and show how these spaces are related to the Calogero–Moser systems.

Let $R \subset \mathbb{R}^n \subset \mathbb{C}^n$ be a (reduced) root system in the sense of [9]. Suppose that $R \neq \emptyset$. Let $W$ be the corresponding (real) reflection group and $c_\alpha \in \mathbb{C}$ ($\alpha \in R$) be such that $c_{w \alpha} = c_\alpha$ for any $w \in W$, $\alpha \in R$. Denote by $(x,y)$ the standard scalar product of the vectors $x,y \in \mathbb{C}^n$. Let $p_\alpha, x_\alpha$ be the natural Darboux coordinates on the symplectic affine space $T^*\mathbb{C}^n$. Consider the function

$$H = \sum_{\alpha=1}^n p_\alpha^2 - \frac{1}{2} \sum_{\alpha \in R} d_\alpha^2 \frac{(\alpha,\alpha)}{(\alpha,\alpha)^2},$$

where $x \in \mathbb{C}^n$ is a vector with components $x_\alpha$. The function $H$ is a regular $W$-invariant function on the affine subvariety $T^*\mathbb{C}^n_{\reg} \subset T^*\mathbb{C}^n$ where $\mathbb{C}^n_{\reg} = \{ x \in \mathbb{C}^n \mid (\alpha, x) \neq 0 \ \forall \ \alpha \in R \}$. It can be regarded as a Hamiltonian of an integrable system on the phase space $T^*\mathbb{C}^n_{\reg}$. Namely, there are $n$ functionally independent regular $W$-invariant functions $H^\text{CM}_k (k = 1, \ldots, n)$ on $T^*\mathbb{C}^n_{\reg}$ such that $\{H^\text{CM}_k, H^\text{CM}_l\} = 0$ and $H$ is one of these functions [12], [8]. Due to the $W$-invariance we can regard that the phase space of the system is the orbit space $T^*\mathbb{C}^n_{\reg}/W$.

Let $\mathbb{C}[x_1, \ldots, x_n]^W$ be the algebra of $W$-invariant regular functions on $\mathbb{C}^n$. The Chevalley’s theorem says that it is freely generated by $n$ homogeneous polynomials $P_1(x), \ldots, P_n(x) \in \mathbb{C}[x_1, \ldots, x_n]^W$ of degrees $d_1, \ldots, d_n$ and if we suppose that $d_1 \leq d_2 \leq \ldots \leq d_n$, then these degrees do not depend on the choice of the polynomials $P_k(x)$ (see e.g. [9]). There are unique Hamiltonians $H^\text{CM}_k$ Poisson commuting with $H$ such that each $H^\text{CM}_k$ has leading term $P_k(p_1, \ldots, p_n)$ with respect to momenta and is homogeneous of degree $-d_k$ with respect to the transformation $x_\alpha \rightarrow \kappa x_\alpha$, $p_\alpha \rightarrow \kappa^{-1} p_\alpha$ with $\kappa \in \mathbb{C}^\times$ (see, e.g. [6, Theorem 2.24]). Note that $d_1 = \ldots = d_{n-r} = 1$ and $d_{n-r+1} = 2$, where $r$ is the rank of $R$. Moreover, one can choose $P_{n-r+1}(x) = \sum_{a=1}^n x_a^2$. As consequence, $H^\text{CM}_{n-r}, \ldots, H^\text{CM}_{n-1}$ are linear functions of $p_\alpha$ and $H^\text{CM}_{n-1} = H$.

Consider the case of the root system of type $A_{n-1}$. In this case $c_\alpha$ does not depend on $\alpha \in R$ and can be put to be 1. The Hamiltonians are defined on the phase space $T^*\mathbb{C}^n_{\reg}$ where $\mathbb{C}^n_{\reg} = \{ x \in \mathbb{C}^n \mid x_a \neq x_b (a \neq b) \}$. The leading term of $H^\text{CM}_k$ with respect to momenta is
\[ P_k(p_1, \ldots, p_n) = \sum_{a=1}^{n} p_a^k. \] The first two Hamiltonians have the form

\[ H_1^{CM} = \sum_{a=1}^{n} p_a, \quad H = H_2^{CM} = \sum_{a=1}^{n} p_a^2 - 2 \sum_{a<b} \frac{1}{(x_a - x_b)^2}. \]

The Calogero–Moser space \( C_n \) defined in [14] is the quotient

\[ C_n := \{ (X, Y, v, w) \mid [X, Y] + vw = 1_n \}/GL(n, \mathbb{C}), \]

where \( X, Y \) are \( n \times n \) complex matrices, \( v \in \mathbb{C}^n, w \in (\mathbb{C}^n)^* \) and the elements \( g \in GL(n, \mathbb{C}) \) act as \( g(X, Y, v, w) = (gXg^{-1}, gYg^{-1}, gv, wg^{-1}) \). This is a smooth affine variety with the symplectic structure \( \omega = \text{tr}(dY \wedge dX) + dv \wedge dv \). The functions \( H_k = \text{tr}(Y^k) \) are regular Poisson commuting functions \( \{H_k, H_l\} = 0 \).

Let \( C'_n \subset C_n \) be the set of points \( (X, Y, v, w) \) with diagonalisable matrix \( X \). Substituting \( X = \text{diag}(x_1, \ldots, x_n) \) to the relation \([X, Y] + vw = 1_n\) we obtain

\[ (x_a - x_b)Y_{ab} + v_aw_b = \delta_{ab}, \quad a, b = 1, \ldots, n. \]

For \( a = b \) this gives \( v_a \neq 0, w_a = 1/v_a \) (\( a = 1, \ldots, n \)). Hence \( x_a \neq x_b \) whenever \( a \neq b \). Thus by applying \( g = \text{diag}(v_1^{-1}, \ldots, v_n^{-1}) \) we obtain

\[
X = \begin{pmatrix}
  x_1 & 0 \\
  \vdots & \ddots \\
  0 & x_n
\end{pmatrix}, \quad v = \begin{pmatrix}
  1 \\
  \vdots \\
  1
\end{pmatrix},
\]

\[
Y = \begin{pmatrix}
  p_1 & -(x_a - x_b)^{-1} \\
  \vdots & \ddots \\
  -(x_b - x_a)^{-1} & p_n
\end{pmatrix}, \quad w = (1, \ldots, 1)
\]

for some \( p_1, \ldots, p_n \in \mathbb{C} \). The variables \( \{x_1, \ldots, x_n; p_1, \ldots, p_n\} \) are local Darboux coordinates on \( C_n \): \( \omega = \sum_{a=1}^{n} dp_a \wedge dx_a \). Restricting the functions \( H_k = \text{tr}(Y^k) \) \( k = 1, \ldots, n \) to \( C'_n \) we obtain the Hamiltonians \( H_k^{CM} \) of the Calogero–Moser system of type \( A_{n-1} \). The matrix \( Y \) defined by the formula (5) is the Lax matrix for this system.

Since the numbers \( x_1, \ldots, x_n \) are pairwise different and defined up to a permutation, we obtain \( C'_n \simeq T^* \mathbb{C}^n_{\text{reg}}/S_n \). Thus the Calogero–Moser space \( C_n \) is a completed phase space of the Calogero–Moser system of type \( A_{n-1} \).

The points of the set \( C_n \setminus C'_n \) correspond to the non-diagonalisable \( X \). In this case we can transform \( X \) to its Jordan normal form and each eigenvalue will correspond to only one Jordan block. These cases are interpreted as collisions of the particles on the complex plane.

The notion of the Calogero–Moser systems for root systems \( R \) (or, equivalently, for real reflection groups \( W \)) can be generalised to the case of complex reflection groups \( W \) in terms of Dunkl operators [4]. The corresponding completed phase spaces can be constructed from Cherednik algebras associated to this groups \( W \). The classical series of the complex reflection groups are reduced to the case of the groups \( W = S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n \) (see [5]). For \( m = 1 \) and \( m = 2 \) these are real reflection groups of the types \( A_{n-1} \) and \( B_n \) respectively. For \( m \geq 3 \) they are purely complex reflection groups. The Calogero-Moser systems for these groups have description in terms of quiver varieties for the cyclic quiver (see Section 5).
3. Quiver varieties

First we recall the notions of quiver, of its path algebra and of the preprojective algebra. Then we consider the spaces of representations of these algebras and introduce quiver varieties as in [3].

Quiver $Q$ is a directed graph. That is $Q = (I, Q, t, h)$ where $I$ is the set of vertices, $Q$ is the set of edges and $t, h: Q \to I$ are the maps such that $t(a)$ and $h(a)$ are the tail and the head of an edge $a \in Q$. We will suppose the sets $I$ and $Q$ to be finite, and we will write $a: i \to j$ or $i \overset{a}{\to} j$, when $t(a) = i$ and $h(a) = j$.

Path algebra $CQ$ is the $C$-algebra generated by $1_i$ ($i \in I$) and $a \in Q$ with relations $1_i1_j = \delta_{i,j}1_i$, $a1_i = \delta_{i,t(a)}1_i$, $1_ja = \delta_{j,h(a)}a$. Any non-zero element $a_1 \cdots a_2a_1 \in CQ$, where $a_2 \in Q \forall k$, is called a path of the length $\ell \geq 0$. All the paths form a basis of $CQ$. Zero length paths $1_i$ are called trivial paths.

To give a representation of the algebra $CQ$ we need to assign a vector space $V_i$ to each $i \in I$ and a linear map $V_a: V_{t(a)} \to V_{h(a)}$ to each $a \in Q$. If all $V_i$ are finite-dimensional and $\alpha_i = \dim_C V_i$, then the representation $V = (V_i, V_a)$ is called representation of the dimension $\alpha = (\alpha_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$. For a fixed vector $\alpha = (\alpha_i)_{i \in I}$ the representations $V = (V_i, V_a)$ such that $V_i = C^{\alpha_i}$ form a vector space $\text{Rep}(CQ, \alpha) = \prod_{i \in Q} \text{Hom}(C^{\alpha_{t(i)}}, C^{\alpha_{h(i)}})$, where $\text{Hom}(C^n, C^m)$ is a space of $m \times n$ complex matrices.

For a quiver $Q = (I, Q, t, h)$ consider the double quiver $\overline{Q} = (I, \overline{Q}, t, h)$, obtained by adjoining a reverse edge $a^* : j \to i$ for every edge $a: i \to j$ in $Q$ (the extended maps $t, h: \overline{Q} \to I$ are denoted by the same letters). Let $\lambda \in C^I$ be a vector with components $\lambda_i$ ($i \in I$). Define preprojective algebra $\Pi^\lambda(Q)$ as the quotient of the path algebra $C\overline{Q}$ by the relations

$$\sum_{a \in Q} a a^* - \sum_{a \in Q} a^* a = \lambda_i 1_i \quad (i \in I).$$

The set of representations of this algebra of dimension $\alpha \in \mathbb{Z}_{\geq 0}^I$ is an affine subvariety $\text{Rep}(\Pi^\lambda(Q), \alpha) \subset \text{Rep}(C\overline{Q}, \alpha)$. Its points are $C\overline{Q}$-modules $V = (V_i, V_a, V_{a^*})$ such that $V_i = C^{\alpha_i}$ and

$$\sum_{a \in Q} V_a V_a^* - \sum_{a \in Q} V_{a^*} V_a = \lambda_i 1_i \quad (i \in I),$$

where $1_n = \text{id}_{C^n}$ is the identity matrix of size $n$. Note that $\text{Rep}(\Pi^\lambda(Q), \alpha) = \emptyset$ unless $\lambda \cdot \alpha = 0$, where $\lambda \cdot \alpha := \sum_{i \in I} \lambda_i \alpha_i$. One can prove it by taking trace in (7) and summing by $i \in I$.

Let the group $GL(\alpha) := \prod_{i \in I} GL(\alpha_i, C)$ act on $\text{Rep}(C\overline{Q}, \alpha)$ by the formula $V_a \to g_{h(a)} V_a g_{t(a)}^{-1}$, where $a \in \overline{Q}$ and $g_i \in GL(\alpha_i, C)$ are components of the acting element $g \in GL(\alpha)$. This action preserves the subvariety $\text{Rep}(\Pi^\lambda(Q), \alpha)$. Two elements of $\text{Rep}(\Pi^\lambda(Q), \alpha)$ are isomorphic as $\Pi^\lambda(Q)$-modules if and only if they belong to the same $GL(\alpha)$-orbit. In general, the space of orbits $\text{Rep}(\Pi^\lambda(Q), \alpha)/GL(\alpha)$ do not have a structure of a variety.

For an affine variety $M$ denote by $\mathbb{C}[M]$ the algebra of all regular functions on $M$. Let a group $G$ act on $M$. Denote by $\mathbb{C}[M]^G$ the subalgebra of $G$-invariant regular function. Define categorical quotient as the affine variety $M//G := \text{Spec}\mathbb{C}[M]^G$. In particular, the variety $N_\lambda(\alpha) = \text{Rep}(\Pi^\lambda(Q), \alpha)/GL(\alpha)$ can be identified with the set of $GL(\alpha)$-orbits of all the semi-simple modules $V \in \text{Rep}(\Pi^\lambda(Q), \alpha)$. We call the variety $N_\lambda(\alpha)$ moduli space for the quiver $Q$.

Let $p(\alpha) = 1 + \sum_{a \in Q} \alpha_{t(a)} \alpha_{h(a)} - \sum_{i \in I} \alpha_i^2$. The following result is known by [3, Corollary 1.4 and Lemma 6.5].
Proposition 1. If a generic module \( V \in \text{Rep}(\Pi^λ(Q), α) \) is simple, then \( N_λ(α) \) is an irreducible affine variety of dimension \( 2p(α) \). It is smooth at the points corresponding to the simple modules. In particular, if all \( V \in \text{Rep}(\Pi^λ(Q), α) \) are simple, then the variety \( N_λ(α) \) coincides with the usual orbit space \( \text{Rep}(\Pi^λ(Q), α)/\text{GL}(α) \), it is smooth and connected in this case.

Identifying \( \text{Rep}(\mathbb{C}Q, α) = T^* \text{Rep}(\mathbb{C}Q, α) \) we obtain a \( \text{GL}(α) \)-invariant symplectic structure on this space:

\[
ω = \sum_{a \in Q} \text{tr}(dV_a^* \wedge dV_a).
\]

(8)

The varieties \( N_λ(α) \) can be obtained by Hamiltonian reduction from the symplectic space \( \text{Rep}(\mathbb{C}Q, α) \) with the action of the group \( \text{GL}(α) \) (see [3]). As consequence, when \( N_λ(α) \) is smooth it has a structure of the symplectic variety. This structure is given by the same formula (8) understood in a proper way (see [13] for details).

Define the notion of framing of a quiver \( Q = (I, E, t, h) \) by a framing vector \( ζ \in \mathbb{Z}^I_{≥0} \). Let us add to the quiver \( Q \) one more vertex \( ∞ \) and \( ζ \) edges \( b_i: ∞ \rightarrow i \) \((r = 1, \ldots, ζ_i)\) for each \( i \in I \). In this way we obtain a new quiver \( Q_ζ \) with the set of vertices \( I_∞ = \{∞\} \cup I \). It is called framed quiver.

Consider the case \( V_∞ = \mathbb{C}^1 \). Given vectors \( α \in \mathbb{Z}^I_{≥0} \) and \( λ \in \mathbb{C}^I \) can be uniquely extended to the vectors \( α \in \mathbb{Z}^{I∞}_{≥0} \) and \( λ \in \mathbb{C}^{I∞} \) such that \( α_∞ = 1 \) and \( λ \cdot α = 0 \):

\[
α = (1, α), \quad λ = (-λ \cdot α, λ).
\]

(9)

Quiver varieties \( M_λ(α, ζ) \) associated with the quiver \( Q \) are the moduli space \( N_λ(α) \) for the quiver \( Q_ζ \):

\[
M_λ(α, ζ) = \text{Rep}(\Pi^λ(Q_ζ), α)/\text{GL}(α).
\]

(10)

Since \( α_∞ = 1 \), we have \( \text{GL}(α) = \mathbb{C}^× \times \text{GL}(α) \). Note that, though the subgroup \( \text{GL}(α_∞, \mathbb{C}) = \mathbb{C}^× \) acts non-trivially, the quotient over \( \text{GL}(α) \) in (10) is reduced to the quotient over the subgroup \( \text{GL}(α) \).

Example 1. Consider the one-loop quiver \( Q \) with one vertex 0 and one edge \( a_0: 0 \rightarrow 0 \), and its framing for \( ζ = 1 \):

\[
Q : \begin{array}{c}
\underbrace{0}
\end{array}, \quad Q_ζ : \begin{array}{c}
\underbrace{0}
\end{array}
\]

By putting \( λ = 1, α = n \) (and \( ζ = 1 \)) in \( M_λ(α, ζ) \) we obtain the Calogero–Moser space \( C_n \) defined in Section 2. Indeed, after the identification \( X = V_{α_0}, Y = V_{α_n}, v = V_{b_{0,1}}, w = V_{b_{0,1}} \) the equations (7) are rewritten as \([X, Y] + vw = 1_n \) and \( vw = n \) (the latter equation follows from the former one by taking trace). This identifies \( C_n \) with the quiver variety \( M_1(α, ζ) = M_1(n, 1) \) for the one-loop quiver \( Q \).

4. Integrable systems for the cyclic quiver with the framing \( ζ = (d, d, \ldots, d) \)

Let us first consider the case of cyclic quiver (with general framing). Then we describe the quiver varieties for the particular framing vector \( ζ = (d, d, \ldots, d) \) in more details.

The cyclic quiver with \( m ≥ 1 \) vertices is the quiver \( Q = (I, Q, t, h) \) where \( I = \mathbb{Z}/m\mathbb{Z} = \{0, 1, \ldots, m - 1\}, Q = \{a_i \mid i = 0, \ldots, m - 1\}, t(a_i) = i, h(a_i) = i + 1 \) (since \( I \) is identified with
the cyclic group $\mathbb{Z}/m\mathbb{Z}$, the expressions like $i + 1$ are understood modulo $m$):

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
Q : \\
0 \\
\end{array}
\end{array}
\end{array}$$

Note that for $m = 1$ the cyclic quiver is nothing but the one-loop quiver considered above.

For $\lambda = (\lambda_0, \ldots, \lambda_{m-1}) \in \mathbb{C}^f$ denote $|\lambda| := \sum_{i \in I} \lambda_i$. We suppose that $\lambda$ is regular in the sense that

$$n|\lambda| \neq \lambda_i + \cdots + \lambda_{j-1} \quad \text{for all } n \in \mathbb{Z} \text{ and } 1 \leq i \leq j \leq m \quad (11)$$

(in particular, $|\lambda| \neq 0$). Then every module $V \in \text{Rep}(\Pi^\lambda(Q_\zeta), \alpha)$ is simple (see [1], [13]), so due to Proposition 1 the quiver varieties $M_\lambda(\alpha, \zeta)$ associated with the cyclic quiver $Q$ are smooth and connected.

Consider first the framing $\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_{m-1})$ where $\zeta_i = d$ for all $i \in I$. That is $\zeta = d\delta$ where $\delta \in \mathbb{Z}^f_{\neq 0}$ has components $\delta_i = 1$. The framed quiver has the form

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
Q_{\delta \delta} : \\
0 \\
\end{array}
\end{array}
\end{array}$$

For a module $V \in \text{Rep}(\Pi^\lambda(Q_\zeta), \alpha)$ consider the matrices

$$X_i = V_{a_i} : V_i \rightarrow V_{i+1}, \quad Y_i = V_{a_i^*} : V_{i+1} \rightarrow V_i \quad (12)$$

(where $V_i = \mathbb{C}^n$) and also the matrices $v_i \in \text{Hom}(\mathbb{C}^d, \mathbb{C}^{n_i})$, $w_i \in \text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^d)$ with entries $(v_i)_{l,r} = (V_{b_{ir}})_{l}$ and $(w_i)_{r,l} = (V_{b_{r}^{*l}})_{l}$. These are matrix-valued functions of $V$ and each point $V \in \text{Rep}(\Pi^\lambda(Q_\zeta), \alpha)$ can be given by their values. In these terms the equations (7) take the form

$$X_{i-1}Y_{i-1} - Y_iX_i + v_iw_i = \lambda_i1_{a_i} \quad (i = 0, 1, \ldots, m - 1), \quad (13)$$

(since $\lambda \cdot \alpha = 0$, the equation (7) for $i = \infty$ follows from (13)). Thus a point of $M_\lambda(\alpha, d\delta)$ correspond to a collection of matrices $(X_i, Y_i, v_i, w_i)$, satisfying (13), up to the action of $\text{GL}(\alpha)$.  

6
Consider the following regular functions on $\text{Rep} \left( \Pi^\lambda(Q_{\zeta}), \alpha \right)$:

$$J_\ell(A) = \sum_{i \in I} \text{tr}(A w_{i-\ell} Y_{i-\ell} \cdots Y_{i-1} v_i),$$  \hspace{1cm} (14)

where $\ell \in \mathbb{Z}_{\geq 0}$ and $A$ is a constant $d \times d$ matrix. Due to their $\text{GL}(\alpha)$-invariance they can be considered as regular functions on $M_\lambda(\alpha, \delta)$. Their Poisson brackets are

$$\{ J_\ell(A), J_{\ell'}(B) \} = -J_{\ell + \ell'}([A, B]),$$  \hspace{1cm} (15)

so they span a Lie subalgebra $L$ in the Poisson algebra $\mathbb{C}[M_\lambda(\alpha, \delta)]$. The functions (14) with diagonal matrices $A$ span a maximal commutative subalgebra in $L$. Substituting $A = E_{rr}$ where $E_{rr}$ are diagonal matrix units, we obtain the basis of this subalgebra:

$$H_{\ell,r} = J_l(E_{rr}) = \sum_{i \in I} w_{i-\ell,r} Y_{i-\ell} \cdots Y_{i-1} v_i \hspace{1cm} (\ell \geq 0, \hspace{0.2cm} r = 1, \ldots, d),$$  \hspace{1cm} (16)

where $v_{i,r} \in \mathbb{C}^{\alpha_i}$ and $w_{i,r} \in (\mathbb{C}^{\alpha_i})^*$ are vectors and covectors with components $(v_{i,r})_l = (v_{i,r})_{lr}$, $(w_{i,r})_I = (w_{i,r})_{rl}$. Since $\{ H_{k,r}, H_{\ell,a} \} = 0$, one can consider $H_{\ell,r}$ as Hamiltonians. In this way we obtain a Hamiltonian system on the quiver variety $M_\lambda(\alpha, \zeta)$. It is convenient to regard the sums $H_{mk} = \sum_{r=1}^d H_{mk,r} = J(1_r)$ ($k = 1, \ldots, n$) as "main" Hamiltonians. By substituting (13) to $H_{mk} = \sum_{i \in I} \text{tr}(Y_{i-\ell,m} \cdots Y_{i-1} Y_{i-1}) = \sum_{i \in I} \text{tr}(Y_{i-\ell,m} \cdots Y_{i-1} Y_{i-1})$ we derive

$$H_{mk} = |\lambda| \text{tr} \left( (Y_0 Y_1 \cdots Y_{m-1})^k \right).$$  \hspace{1cm} (17)

We also have $\sum_{r=1}^d H_{0,r} = \lambda \cdot \alpha$, so the Hamiltonians $H_{0,r}$ are algebraically dependent.

Consider the case $\alpha = n \delta$, i.e. $\alpha_i = n \hspace{0.2cm} \forall i \in I$. Then, from Proposition 1 we obtain $\dim M_\lambda(n \delta, \delta) = 2nmd$. In this case $X_i$ and $Y_i$ are square matrices of size $n$. For a generic point $(X_i, Y_i, v_i, w_i)$ of $M_\lambda(n \delta, \delta)$ the matrix $X_{m-1} \cdots X_1 X_0$ is invertible and diagonalisable. Since the collection of matrices is defined up to the action of $\text{GL}(\alpha)$ we can diagonalise it: $X_{m-1} \cdots X_1 X_0 = \text{diag}(x_1^m, \ldots, x_n^m)$ where $x_i$ are non-zero complex numbers. Moreover, by action of $\text{GL}(\alpha)$ one can make the matrices $X_i$ to be

$$X_i = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \hspace{1cm} (i = 0, \ldots, m - 1).$$  \hspace{1cm} (18)

Let $\varphi_a \in \text{Hom}(\mathbb{C}^m, \mathbb{C}^d)$ and $\psi_a \in \text{Hom}(\mathbb{C}^d, \mathbb{C}^m)$ ($a = 1, \ldots, n$) are matrices with components

$$(\varphi_a)_{ri} = (v_{i,r})_a, \hspace{1cm} (\psi_a)_{ir} = (w_{i,r})_a.$$  \hspace{1cm} (19)

Then substituting (18) and (19) to (13) we derive

$$(Y_i)_{aa} = \frac{1}{m} p_a + \frac{1}{x_a} \sum_{l=1}^{m-1} \frac{m-l}{m} \left( \lambda_l - (\psi_a \varphi_a)_{ll} \right) - \frac{1}{x_a} \sum_{l=1}^i \left( \lambda_l - (\psi_a \varphi_a)_{ll} \right),$$  \hspace{1cm} (20)

$$(Y_i)_{ab} = -\sum_{j=0}^{m-1} \frac{x_d x_a}{x_a - x_b} \frac{x_a}{x_a - x_b} (\psi_b \varphi_a)_{l-j,i-j} \hspace{1cm} (a \neq b),$$  \hspace{1cm} (21)
where \( i = 0,\ldots,m-1, \ a, b = 1,\ldots,n \) and \( p_i \) are arbitrary complex numbers (see details in [1]). For instance, one deduces that the eigenvalues \( x_a^m \) are pairwise different. Thus we expressed the matrices \( X_i, Y_i, v_i, w_i \) through the variables
\[
x_a, \quad (\varphi_a)_{ri}, \quad p_a, \quad (\psi_a)_{ir}.
\] (22)
The numbers \( x_a \) and \( p_a \) are interpreted as coordinate and momentum of the \( a \)-th particle, while the matrices \( \varphi_a \) and \( \psi_a \) characterise its internal degrees of freedom and are interpreted as spin variables.

The total number of variables (22) is \( 2n + 2nmd \), which is more than the dimension of \( M_\lambda(n\delta,d\delta) \). In fact, we can exclude one component of each \( \varphi_a \) and \( \psi_a \). Indeed, the formulae (18) and (13) also imply \( \text{tr}(\varphi_a \psi_a) = |\lambda| \) \( \forall a = 1,\ldots,n \). In a generic point, a chosen component of each \( \varphi_a \), for example \( (\varphi_a)_{10} \), does not vanish. The transformation \( g = (g_\ell) \in \text{GL}(\alpha) \) with \( g_\ell = \text{diag} \left((\varphi_\ell)_0^{10},\ldots,(\varphi_\ell)_0^{10}\right) \) does not change matrices (18). By applying \( g \) one can make these components to be unity and express the corresponding component of \( \psi_a \):
\[
(\varphi_a)_{10} = 1, \quad (\psi_a)_{01} = |\lambda| - \sum_{r=2}^{d} (\varphi_a)_{r0} (\psi_a)_{0r} - \sum_{i=1}^{m-1} \sum_{r=1}^{d} (\varphi_a)_{ri} (\psi_a)_{ir}.
\] (23)

In this way we obtain local coordinates (22) where \( (i, r) \neq (0, 1) \). They are Darboux coordinates:
\[
\omega = \sum_{i \in I} \text{tr}(dY_i \wedge dX_i) + \sum_{i \in I} \text{tr}(dw_i \wedge dv_i) = \sum_{a=1}^{n} dp_a \wedge dx_a + \sum_{a=1}^{n} \text{tr}(d(\psi_a) \wedge d(\varphi_a)),
\]
where due to (23) the last term has the form
\[
\text{tr} \left( d(\psi_a) \wedge d(\varphi_a) \right) = \sum_{r=2}^{d} (\psi_a)_{0r} \wedge d(\varphi_a)_{r0} + \sum_{i=1}^{m-1} \sum_{r=1}^{d} (\psi_a)_{ir} \wedge d(\varphi_a)_{ri}.
\]

By using these coordinates (or rather their dual version) one can prove the following fact [1].

**Theorem 1.** The Hamiltonians \( H_{\ell,r} \in \mathbb{C}[M_\lambda(n\delta,d\delta)] \) \( \ell = 1,\ldots, nm, \ r = 1,\ldots,d \) are functionally independent, so they define an integrable system on the quiver variety \( M_\lambda(n\delta,d\delta) \).

**Remark 1.** The Hamiltonians \( H_{\ell,r} \) \( \ell = 0,\ldots, nm - 1, \ r = 1,\ldots,d \) are algebraically dependent, since \( \sum_{r=1}^{d} H_{0,r} = n|\lambda| \). One can obtain \( nmd \) independent Hamiltonians by replacing one of \( H_{0,r} \) in this family, for instance, by \( H_{mn} = \sum_{r=1}^{d} H_{mn,r} \).

**Example 2.** [1] Let \( m = 2 \) and \( d = 1 \). Then \( \lambda = (\lambda_0, \lambda_1) \) and the integrable system on \( M_\lambda(n\delta, \delta) \) is defined by the Hamiltonians \( H_{\ell,1} \) \( \ell = 1,\ldots, 2n \). Let us denote \( Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and
\[
F_\pm = \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}.
\]
In this case we have \( \varphi_a \in \mathbb{C}^2 \) and \( \psi_a \in \mathbb{C}^2 \) with components \( (\varphi_a)_i = (\varphi_a)_{ii} \) and \( (\psi_a)_i = (\psi_a)_{i1} \) \( (i = 0, 1) \). The first two Hamiltonians have the form
\[
H_{1,1} = w_0 Y_0 v_1 + w_1 Y_1 v_0 = \frac{1}{2} \sum_{a=1}^{n} \left( (\varphi_a F_+ \psi_a) p_a + \frac{1}{x_a} (\varphi_a F_- \psi_a) (\lambda_1 + (\varphi_a)_{1}(\psi_a)_{1}) \right) + \frac{1}{2} \sum_{a \neq b} \left( \frac{(\varphi_a F_+ \psi_b) (\varphi_b \psi_a)}{x_a - x_b} + \frac{(\varphi_a F_- \psi_b) (\varphi_b \psi_a)}{x_a + x_b} \right).
\]
\[ H_2 = H_{2,1} = |\lambda| \text{tr}(Y_0 Y_1) = \frac{|\lambda|}{4} \sum_{a=1}^{n} \left( \frac{p_a^2}{x_a^4} - \frac{1}{x_a^2} (\lambda_1 - (\varphi_a)_1(\psi_a)_1)^2 \right) + \]
\[ + \frac{|\lambda|}{4} \sum_{a \neq b} \left( \frac{(\varphi_a \psi_b)(\varphi_b \psi_a)}{(x_a - x_b)^2} + \frac{(\varphi_a Z \psi_b)(\varphi_b Z \psi_a)}{(x_a + x_b)^2} \right) + \]
\[ + \frac{|\lambda|}{2} \sum_{a \neq b} \frac{(\varphi_a)_1(\psi_b)_1(\varphi_b)_0(\psi_a)_0 - (\varphi_a)_0(\psi_b)_0(\varphi_b)_1(\psi_a)_1)}{x_a^2 - x_b^2}. \]

In a generic point one can take \((\varphi_a)_0 = 1, (\psi_a)_0 = |\lambda| - (\varphi_a)_1(\psi_a)_1\), so there are exactly two independent "spin" variables for each particle: \((\varphi_a)_1\) and \((\psi_a)_1\).

5. Framing \(\zeta = (d,0,\ldots,0)\)

Here we consider another framing for the cyclic quiver \(Q\) defined above and related integrable systems on the corresponding quiver varieties. Namely, let all the additional edges go from \(\infty\) to only one vertex \(i_0 \in I\). Without loss of generality we suppose that \(i_0 = 0\). Then the corresponding framed quiver \(Q_\zeta\) is

\[ Q_{d\varepsilon_0} : \]

where \(\varepsilon_0 \in \mathbb{Z}^I_{>0}\) is the vector with components \((\varepsilon_0)_i = \delta_{i0}\).

As in previous case we define the matrix-valued functions \(X_i \in \text{Hom}(\mathbb{C}^\alpha, \mathbb{C}^{\alpha+1}_i), Y_i \in \text{Hom}(\mathbb{C}^\alpha_{i+1}, \mathbb{C}^\alpha_i), v_0 \in \text{Hom}(\mathbb{C}^d, \mathbb{C}^0), w_0 \in \text{Hom}(\mathbb{C}^0, \mathbb{C}^d)\) by the formulae (12), \((v_0)_{i,r} = (V_{0r})_i\) and \((w_0)_{r,t} = (V_{rt})_0\). In these notations we define the following functions on \(M_\lambda(\alpha, d\varepsilon_0)\):

\[ J_{mk}(A) = \text{tr}(Aw_0(Y_0 Y_1 \cdots Y_{m-1})^k v_0) \quad (k \in \mathbb{Z}_{\geq 0}, \quad A \in \text{End}(\mathbb{C}^d)), \quad (25) \]

where \(\text{End}(\mathbb{C}^d)\) is the algebra of the constant complex \(d \times d\) matrices.

**Proposition 2.** The variety \(M_\lambda(\alpha, d\varepsilon_0)\) can be identified with the subvariety in \(M_\lambda(\alpha, d\delta)\) defined by the equations

\[ v_i = 0, \quad w_i = 0 \quad (i = 1, \ldots, m - 1). \quad (26) \]

For \(\ell\) not divisible by \(m\) the restriction of the function \(J_\ell(A)\) defined by (14) vanishes on this subvariety. For \(\ell = mk\) its restriction coincides with (25). The Hamiltonian flows defined by the functions \(J_{mk}\) preserves subvariety \(M_\lambda(\alpha, d\varepsilon_0)\) if \(\ell = mk\).

**Proof.** Equations (26) define the inclusion \(\text{Rep}(P^\lambda(Q_{d\varepsilon_0}), \alpha) \subset \text{Rep}(P^\lambda(Q_{d\delta}), \alpha)\). Since the action of \(\text{GL}(\alpha)\) preserves the equations (26), this inclusion induces the inclusion of quotients. The rest is proven by direct calculations. \(\square\)
In particular, restriction of the Hamiltonians (16) to are local coordinates on

Let \( \phi \) written in the compact form:

where \( i \in \{1, \ldots, m\} \). The analogue of Theorem 1 for this case is also proved in [1].
Theorem 2. The Hamiltonians $H_{mk,r} \in \mathbb{C}[M_\lambda(n\delta,d\varepsilon_0)]$ ($k = 1, \ldots, n$, $r = 1, \ldots, d$) are functionally independent and, hence, they define an integrable system on $M_\lambda(n\delta,d\varepsilon_0)$.

Consider first the case $d = 1$. In this case the quiver variety $M_\lambda(n\delta,d\varepsilon_0)$ isomorphic to the Calogero–Moser space for the group $W = S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^d$. If $m = 1$ then we obtain the Calogero–Moser systems on the completed phase space $M_\lambda(n,1) \simeq M_1(n,1) = C_n$ ($\lambda \neq 0$) (see Example 1). If $m = 2$ then $\lambda = (\lambda_0, \lambda_1)$ and we obtain an integrable system with the Hamiltonian

$$H = \frac{4}{|\lambda|} H_2 = \sum_{a=1}^{n} \left( p_a^2 - \frac{\lambda_1^2}{x_a^2} \right) - 2|\lambda|^2 \sum_{a < b} \left( \frac{1}{(x_a - x_b)^2} + \frac{1}{(x_a + x_b)^2} \right).$$

This is the Hamiltonian (1) for the root system of type $B_n$ for some parameters $c_a$. The higher Hamiltonians are $H_{2k} = |\lambda| \text{tr}(Y_0 Y_1^k)$ ($k = 1, \ldots, n$), where

$$\begin{align*}
(Y_0)_{aa} &= \frac{1}{2} \left( p_a + \frac{\lambda_1}{x_a} \right), \\
(Y_1)_{aa} &= \frac{1}{2} \left( p_a - \frac{\lambda_1}{x_a} \right), \\
(Y_0)_{ab} &= -|\lambda| \frac{x_b}{x_a - x_b^2}, \\
(Y_1)_{ab} &= -|\lambda| \frac{x_b}{x_a - x_b^2} (a \neq b). 
\end{align*} \tag{37}$$

The role of Lax matrix is played by the $2n \times 2n$ block matrix $Y = \begin{pmatrix} 0 & Y_0 \\ Y_1 & 0 \end{pmatrix}$. In particular, $H_{2k} = |\lambda|^2 \text{tr}(Y^{2k})$. The Hamiltonians $H_{2k}$ are proportional to the Hamiltonians $H_k^{CM}$ defined in Section 2, so we obtain the Calogero–Moser system of type $B_n$ with the completed phase space $M_\lambda(n\delta,d\varepsilon_0)$ isomorphic to the corresponding Calogero–Moser space. For $m \geq 3$ the Hamiltonians (17) define the Calogero–Moser system for the complex reflection group $W = S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^d$.

Now consider the case $d \geq 2$. In this case we have also spin variables $\varphi_a$, $\psi_a$ with the conditions $\psi_a \varphi_a = |\lambda|$ and equivalence relations $(\varphi_a, \psi_a) \sim (\kappa_a \varphi_a, \kappa_a^{-1} \psi_a)$ where $\kappa_a \in \mathbb{C}^\times$. If $m = 1$ and $\lambda = 1$ we obtain the Gibbons–Herman system [7] with the “main” Hamiltonians

$$\begin{align*}
H_1 &= \sum_{a=1}^{n} p_a, \\
H_2 &= \sum_{a=1}^{n} p_a^2 - 2 \sum_{a < b} \frac{(\psi_a \varphi_b)(\psi_b \varphi_a)}{(x_a - x_b)^2}, \\
H_k &= \sum_{a=1}^{n} p_a^k + \ldots
\end{align*} \tag{38}$$

Numerator in the expression for $H_2$ can be interpreted as interaction of spins: $(\psi_a \varphi_b)(\psi_b \varphi_a) = \text{tr}(s_a s_b)$ where $s_a = \varphi_a \psi_a$ is the spin operators. By this reason this system is also called spin Calogero–Moser system (of type $A_{n-1}$). The Hamiltonians of this system have the form $H_{k,r} = w_{0,r} Y^k v_{0,r}$ with the Lax matrix

$$Y = \begin{pmatrix}
1 & \vdots & \frac{\psi_a \varphi_a}{x_a - x_b} \\
\vdots & \ddots & \vdots \\
\frac{\psi_b \varphi_b}{x_b - x_a} & \cdots & 1
\end{pmatrix}. \tag{39}$$

If $m = 2$ then the first Hamiltonian is

$$H = \frac{4}{|\lambda|} H_2 = \sum_{a=1}^{n} \left( p_a^2 - \frac{\lambda_1^2}{x_a^2} \right) - 2 \sum_{a < b} \left( \frac{1}{(x_a - x_b)^2} + \frac{1}{(x_a + x_b)^2} \right)(\psi_a \varphi_b)(\psi_b \varphi_a).$$

The Lax matrix of this system is $Y = \begin{pmatrix} 0 & Y_0 \\ Y_1 & 0 \end{pmatrix}$ where $Y_0$ and $Y_1$ are $n \times n$ matrices with the diagonal entries (37) and the following non-diagonal entries:

$$\begin{align*}
(Y_0)_{ab} &= -\frac{x_b}{x_a^2 - x_b^2} \psi_b \varphi_a, \\
(Y_1)_{ab} &= -\frac{x_a}{x_a^2 - x_b^2} \psi_b \varphi_a (a \neq b). \tag{40}
\end{align*}$$
Then the Hamiltonians are expressed as $H_{2k,r} = w_r Y^{2k} v_r$ where $w_r = (w_{0,r}, 0)$ and $v_r = \left( v_{0,r} \right)$.

This is $B_n$ type version of the Gibbons–Hermsen system. For $m \geq 3$ we obtain Gibbons–Hermsen system for the complex reflection group $S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$.

Note that the spin generalisations of the Calogero–Moser systems obtained in this section is different from those more complicated spin generalisations obtained in Section 4.

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