On the existence and multiplicity of solutions for a fourth order discrete BVP

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Abstract

We investigate the existence and multiplicity of solutions for fourth order discrete boundary value problems via critical point theory.

1 Introduction

Difference equations have been applied as mathematical models in diverse areas, such as finance, insurance, economy, disease control, biology, physics, mechanics, computer science - see [1]. It is important to know the conditions which guarantee the existence and multiplicity of solutions.

For fixed $a, b \in \mathbb{Z}$ we define $\mathbb{Z}[a, b] = [a, b] \cap \mathbb{Z}$. and $\Delta$ is the forward difference operator 
\[ \Delta x(k) = x(k + 1) - x(k). \]

In this note we consider a Dirichlet boundary value problem (briefly BV P) for a fourth order discrete equation

\[
\begin{align*}
\Delta^2(p(k)\Delta^2y(k-2)) + \Delta(q(k)\Delta y(k-1)) + f(k, y(k)) &= 0 \text{ for } k \in \mathbb{Z}[1, N] \\
y(-1) = y(0) = y(N + 1) = y(N + 2) &= 0
\end{align*}
\] (1)

Where $N \geq 1$, $f : \mathbb{Z}[1, N] \times \mathbb{R} \to \mathbb{R}$ is continuous on its second variable for all $k \in \mathbb{Z}[1, N]$, $p : \mathbb{Z}[1, N+2] \to \mathbb{R}$, $q : \mathbb{Z}[1, N+1] \to \mathbb{R}$. By a solution to a problem (1) we mean such a function $y : \mathbb{Z}[-1, N+2] \to \mathbb{R}$ which satisfies the difference equation and the given boundary conditions. The main purpose of this paper is to study the multiplicity of solutions to BV P(1) and obtain that is has at least $2N$ distinct solutions assuming some conditions. Our results are based on [4] and [3] by extending these to the case of fourth order discrete equations.

Let's mention, far from being exhaustive, the following recent papers on discrete BVPs investigated via critical point theory, [7], [8], [9], [10], [11], [12], [13], [14]. These papers employ in the discrete setting the variational techniques already known for continuous problems of course with necessary modifications.
The tools employed cover the Morse theory, mountain pass methodology, linking arguments.

For the sake of convinience lets recall some basic facts and definitions used in this note, \[2\]

Let \(X\) be a real Banach space and let \(T : X \to \mathbb{R}\). For fixed \(x, h \in X\), symbol \(\delta T(x; h)\) stands for the Gateaux derivative of \(T\) at point \(x\) and direction \(h\), while \(T'(x)\) stands for the Frechet or strong derivative of \(T\) at point \(x\). By \(C^1(X, \mathbb{R})\) we denote the set of continously (Frechet) differentable functionals on \(X\).

If \(X = X_1 \times \ldots \times X_n\) where \(X_i\) are one-dimensional Banach spaces we say that \(T\) has partial derivative at point \(x \in X\) on \(X_i\) if there exists \(L_x \in L(X, \mathbb{R})\) such that
\[
\lim_{h \to 0} \frac{||T(x + he_i) - T(x) - L_x h||}{||h||} = 0
\]
Where \(e_i \in X_i\) is the unit vector. The following lemma from \[15\] gives us an useful relation between continuity of partial derivatives and being \(C^1\).

**Lemma 1.1** Let \(T : U \to \mathbb{R}\) where \(U \subset X\) is an open set. Then \(T\) is continously differentable on \(U\) if and only if functions
\[
U \ni x \to J'_X(x) \in L(X_i, \mathbb{R}),
\]
are continous on \(U\) for \(i = 1, \ldots, n\).

We call \(x_0\) a critical point of \(T\) if and only if \(\delta T(x_0, h) = 0\) for all \(h \in X\). Functional \(T\) satisfies the Palais-Smale condition (P.S. condition for short) if any sequence \((x_n \in X : n \in \mathbb{N})\) for which \(\{T(x_n) : n \in \mathbb{N}\}\) is bounded and \(T'(x_n) \to \theta\) as \(n \to \infty\) possesses a convergent subsequence.

We recall theorems that will be used in research of multiplicity of solution: the mountain pass theorem and Clark’s theorem which will be essential in proving our main result.

**Theorem 1.2** \[5\] Let \(T \in C^1(X, \mathbb{R})\) satisfy the P.S. condition. Assume that \(T(\theta) = 0\), \(\Omega \subset X\) is an open set containing \(\theta\), \(x_1 \notin \Omega\). If
\[
\max\{T(\theta), T(x_1)\} < \inf_{x \in \partial \Omega} T(x)
\]
then
\[
c = \inf_{h \in \Gamma} \max_{t \in [0, 1]} J(h(t))
\]
is the critical value of \(J\), where
\[
\Gamma = \{ h : [0, 1] \to E : h is continuous, h(0) = \theta, h(1) = x_1 \}.
\]

**Theorem 1.3** \[6\] Let \(X\) be a real Banach space and let \(T \in C^1(X, \mathbb{R})\) be even, bounded from below and satisfying the P.S condition. Suppose \(T(\theta) = 0\) and that there exist a set \(K \subset X\) such that \(K\) is homeomorphic to \(S^{n-1}\) \((n-1)\)-dimensional sphere) by an odd map, and \(\sup K < 0\). Then \(T\) possesses at least \(n\) distinct pairs of critical points.
2 Main results

Solutions to \(u\) are obtained in space
\[
E = \{ y : \mathbb{Z}[-1, N + 2] \to \mathbb{R} | y(-1) = y(0) = y(N + 1) = y(N + 2) = 0 \}
\]
considered with a norm
\[
||y|| = \sqrt{\sum_{k=1}^{N} y(k)^2}
\]

Lemma 2.1 For all \(y \in E\) let
\[
J(y) = \sum_{k=1}^{N+2} \frac{p(k)}{2} (\Delta^2 y(k-2))^2 - \sum_{k=1}^{N+1} \frac{q(k)}{2} (\Delta y(k-1))^2 + \sum_{k=1}^{N} F(k, y(k))
\]
Where \(F(k, s) = \int_0^s f(k, t) dt\). Then \(J \in C^1(E, \mathbb{R})\). Point \(y_0\) is a solution to \(u\) if and only if it is a critical point of \(J\).

Proof. We denote by \(\varphi : \mathbb{R} \to \mathbb{R}\) the function \(\varphi(\varepsilon) = J(y + \varepsilon h)\) for \(y, h \in E\) and \(\varepsilon \in \mathbb{R}\). Then
\[
\varphi(\varepsilon) = \sum_{k=1}^{N+2} \frac{p(k)}{2} (\Delta^2 (y + \varepsilon h)(k-2))^2 - \sum_{k=1}^{N+1} \frac{q(k)}{2} (\Delta (y + \varepsilon h)(k-1))^2 + \sum_{k=1}^{N} F(k, (y + \varepsilon h)(k))
\]
\(\varphi(\varepsilon)\) is differentiable therefore \(J\) is differentiable in the sense of Gateaux. Moreover, we have
\[
\delta J(y; h) = \sum_{k=1}^{N+2} \frac{p(k)}{2} (\Delta^2 y(k-2)) \Delta^2 h(k-2) - \sum_{k=1}^{N+1} \frac{q(k)}{2} (\Delta y(k-1)) \Delta h(k-1) + \sum_{k=1}^{N} \Delta f(k, y(k)) h(k) = \sum_{k=1}^{N} (\Delta^2 (p(k) \Delta^2 y(k-2)) + (q(k)y(k-1)) + f(k, y(k)) h(k))
\]
Since \(h\) is arbitrary fixed it follows that \(y_0 \in E\) is a critical point of \(J\) if and only if \(y_0\) satisfies BVP \(u\). To prove that assume first that \(y_0\) is a critical point of \(J\), i.e. \(\delta J(y_0; h) = 0\) for all \(h \in E \setminus \{0\}\) and put
\[
h(k) = \Delta^2 (p(k) \Delta^2 y_0(k-2)) + (q(k)y_0(k-1)) + f(k, y_0(k)) \text{ for } k = 1, \ldots, N.
\]
Then \(\delta J(y_0; h) = \sum_{k=1}^{N} (\Delta^2 (p(k) \Delta^2 y_0(k-2)) + (q(k)y_0(k-1)) + f(k, y_0(k)) h(k)) = 0\) and therefore
\[
\Delta^2 (p(k) \Delta^2 y_0(k-2)) + (q(k)y_0(k-1)) + f(k, y_0(k)) = 0 \text{ for } k = 1, \ldots, N.
\]
On the other side if \( y_0 \) is solution to the above system then \( \delta J(y_0; h) = 0 \) for every \( h \in E \setminus \{ 0 \} \).

Since \( f \) is continous on its second variable it follows that \( \delta J(y, h) \) is continous. It holds that \( J'_{X_i}(\cdot) = \delta J(y, e_i) \), where \( X_i = \{ y \in E : y(k) = 0 \text{ for } k \neq i \} \) and \( e_i \in X_i \) are unit vectors for \( i = 1, \ldots, N \). Thus all partial derivate are continous. Then from lemma \( \boxed{1} \) it follows that \( J \in C^1(E, \mathbb{R}) \). \( \blacksquare \)

**Lemma 2.2** For all \( y \in E \) it holds that

\[
\sum_{k=1}^{N+1} (\Delta y(k-1))^2 \leq 4||y||^2 \quad \text{and} \quad \sum_{k=1}^{N+2} (\Delta^2 y(k-2))^2 \leq 16||y||^2
\]

**Proof.** Since \(-2ab \leq a^2 + b^2\) for all \( a, b \in \mathbb{R} \) we have

\[
\sum_{k=1}^{N+1} (\Delta y(k-1))^2 = \sum_{k=1}^{N+1} (y(k) - y(k-1))^2 = \\
\sum_{k=1}^{N+1} (y(k)^2 - 2y(k)y(k-1) + y(k-1)^2) \leq \sum_{k=1}^{N+1} (2y(k)^2 + 2y(k-1)^2) = \sum_{k=1}^{N} 2y(k)^2 + \sum_{k=1}^{N} 2y(k-1)^2 \leq 4||y||^2
\]

and

\[
\sum_{k=1}^{N+2} (\Delta^2 y(k-2))^2 = \sum_{k=1}^{N+2} (\Delta y(k-1) - \Delta y(k-2))^2 \\
\leq 4 \sum_{k=1}^{N+1} (\Delta y(k-1))^2 \leq 16||y||^2.
\]

\( \blacksquare \)

**Lemma 2.3** Let \( A_{m \times m} \) be a symmetric and positive-defined real matrix and let \( B_{m \times n} \) be a real matrix. Then \( B^T AB \) is positive defined if and only if \( \text{Rank}(B) = n \).

**Proof.** If \( B^T AB \) is positive defined for all \( x \in \mathbb{R}^n \setminus \{ \theta \} \) we have

\[
(Bx)^T A(Bx) = x^T B^T ABx > 0,
\]

hence \( Bx \neq \theta \), and therefore \( \text{Rank}(B) = n \). Assume that \( \text{Rank}(B) = n \). Then for all \( x \in \mathbb{R}^n \setminus \{ \theta \} \) it holds that \( Bx \neq \theta \) and \( (Bx)^T A(Bx) > 0 \) since \( A \) is positive defined, hence \( B^T AB \) is positive defined. \( \blacksquare \)

Define \( v(k) = \Delta y(k) = y(k+1) - y(k) \) for \( k \in \mathbb{Z}[0, N] \). Then \( v = V \tilde{y} \), where \( v = [v(0), v(1), \ldots, v(N)]^T \), \( \tilde{y} = [y(1), \ldots, y(N)]^T \), and

\[
V = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
-1 & 1 & 1 & \cdots & 1 \\
1 & -1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & -1 & 1
\end{bmatrix}_{(N+1) \times N}
\]
Note that
\[\sum_{k=1}^{N+1} (\Delta y(k-1))^2 = v^T v = (V\tilde{y})^T (V\tilde{y}) = \tilde{y}^T V^T V \tilde{y}\]

By lemma (2.3) with \(A = I_{(N+1)\times(N+1)}\) it follows that \(V^T V\) is positive-defined. Therefore all eigenvalues of \(V^T V\) are real and positive. Denote henceforth by \(\lambda_1\) the smallest eigenvalue of \(V^T V\). Then it follows that for all \(y \in E\).
\[\sum_{k=1}^{N+1} (\Delta y(k))^2 = \tilde{y}^T V^T V \tilde{y} \geq \lambda_1 \tilde{y}^T \tilde{y} = \lambda_1 ||y||^2\]

In the same way we put \(w(k) = \Delta^2 y(k)\) for \(k \in Z[-1, N]\). Then \(w = W\tilde{y}\), where \(w = [w(-1), w(0), ..., w(N)]^T\) and
\[
W = \begin{bmatrix}
1 & -2 & 1 \\
-2 & 1 & -2 & \ddots & 1 \\
& \ddots & \ddots & \ddots & \ddots & 1 \\
& & \ddots & -2 & 1 \\
& & & 1 & -2 \\
& & & & 1
\end{bmatrix}_{(N+2)\times N}
\]

Likewise in previous case by lemma (2.3) \(W^T W\) is positive defined. Denote henceforth by \(\lambda_2\) the smallest eigenvalue of \(W^T W\) Then for all \(y \in E\) it holds that
\[\sum_{k=1}^{N+2} (\Delta^2 y(k-2))^2 = \tilde{y}^T W^T W \tilde{y} \geq \lambda_2 ||y||^2\]

So we have proven the following lemma:

**Lemma 2.4** For all \(y \in E\) it follows that
\[\sum_{k=1}^{N+1} (\Delta y(k-1))^2 \geq \lambda_1 ||y||^2 \quad \text{and} \quad \sum_{k=1}^{N+2} (\Delta^2 y(k-2))^2 \geq \lambda_2 ||y||^2\]

Define \(p_{\min} = \min_{k \in Z[1,N+2]} \{p(k)\}\), \(p_{\max} = \max_{k \in Z[1,N+2]} \{p(k)\}\) and \(q_{\min}\), \(q_{\max}\) in the same manner, and let for \(p : Z[1,N+2] \rightarrow \mathbb{R}\) and \(q : Z[1,N+1] \rightarrow \mathbb{R}\):
\[
\eta'(p) = \begin{cases}
\lambda_2 & \text{if } p_{\min} \geq 0 \\
16 & \text{if } p_{\min} < 0
\end{cases}
\]
\[
\eta(q) = \begin{cases}
\lambda_1 & \text{if } q_{\max} < 0 \\
4 & \text{if } q_{\max} \geq 0
\end{cases}
\]
Theorem 2.5 Assume that 
1) there exist $m > 0$ such that $sf(k,s) \geq 0$ for $|s| \geq m$ and $k \in \mathbb{Z}[1,N]$ 
2) $\eta'(p) p_{\min} - \eta(q) q_{\max} > 0$

Then $J$ coercive and BVP (1) has at least one solution. Moreover if there exist $k_0 \in \mathbb{Z}[1,N]$ such that $f(k_0,0) \neq 0$ the solution is non-zero.

Proof. For all $k \in \mathbb{Z}[1,N]$ it follows that 
\[
\int_{0}^{s} f(k,t)dt \geq \int_{0}^{s} -\text{sgn}(s)|f(k,t)|dt \geq \int_{-m}^{m} -\text{sgn}(s)|f(k,t)|dt > -\infty
\]
since $f$ is continuous on second variable and $[-m,m]$ bounded.

Put $C_1 = \sum_{k=1}^{N} -\int_{m}^{-m} |f(k,t)|dt$. We have 
\[
J(y) = \sum_{k=1}^{N+2} \frac{p_k}{2}(\Delta^2 y(k-2))^2 - \sum_{k=1}^{N+1} \frac{q_k}{2}(\Delta y(k-1))^2 + \sum_{k=1}^{N} F(k,y(k)) \geq \sum_{k=1}^{N+2} \frac{p_{\min}}{2}(\Delta^2 y(k-2))^2 - \sum_{k=1}^{N+1} \frac{q_{\max}}{2}(\Delta y(k-1))^2 + C_1 \geq \frac{1}{2}(\eta'(p)p_{\min} - \eta(q)q_{\max})||y||^2 + C_1.
\]

And from 2) it follows that $J$ is coercive, and since it is coercive and $C^1$ there exist critical point $y_0 \in E$ such that $J(y_0) = \min_{y \in E} J(y)$. Therefore BVP (1) has a solution. If $f(k_0,0) \neq 0$, for $y = \theta$ we have 
\[
\Delta^2 (p(k_0)\Delta^2 y(k_0 - 2)) + \Delta (q(k_0)\Delta y(k_0 - 1)) + f(k_0,y(k_0)) = f(k_0,0) \neq 0
\]
hence $\theta$ is not a solution to BVP (1).

Theorem 2.6 Assume that 1), 2) of theorem (2.5) hold, $f$ is non-decreasing on $s$ for $k \in [1,N]$ and $p_{\min}\eta'(p) > q_{\max}\eta(q)$ Then BVP (1) has exactly one solution.

Proof. By assumptions 1) and 2) of theorem (2.5) there exists at least one solution to BVP (1). Since $f$ is non-decreasing on $F$ is convex on $s$ for $k \in [1,N]$, and therefore $y \rightarrow \sum_{k=1}^{N} F(k,y(k))$ is convex. Let for $y \in E$, 
\[
I(y) = \sum_{k=1}^{N+2} \frac{p(k)}{2}(\Delta^2 y(k-2))^2 - \sum_{k=1}^{N+1} \frac{q(k)}{2}(\Delta y(k-1))^2.
\]

For arbitrary fixed $y,h \in E$, $h \neq 0$ we have
\[ I(y + h) - I(y) = \sum_{k=1}^{N+2} \frac{p(k)}{2}[2\Delta^2 h(k-1)\Delta^2 y(k-2) + \Delta^2 h(k-2)^2] - \sum_{k=1}^{N+2} \frac{q(k)}{2}[2\Delta^2 y(k-1) + \Delta(\Delta h(k-1)^2)] - \sum_{k=1}^{N} [\Delta^2 p(k)\Delta^2 y(k-2)) + \Delta(q(k)y(k-1)) = \]

\[ \sum_{k=1}^{N+2} p(k)\Delta^2 y(k-2)\Delta^2 h(k-2) + \sum_{k=1}^{N+2} \frac{q(k)}{2}\Delta^2 h(k-2)^2 - \sum_{k=1}^{N+1} q(k)\Delta y(k-1)\Delta h(k-1) \]

and

\[ I(y + h) - I(y) - \delta I(y; h) = \]

\[ \sum_{k=1}^{N+2} p(k)\Delta^2 y(k-2)\Delta^2 h(k-2) + \sum_{k=1}^{N+2} \frac{q(k)}{2}\Delta^2 h(k-2)^2 - \sum_{k=1}^{N+1} q(k)\Delta y(k-1)\Delta h(k-1) \]

\[ \sum_{k=1}^{N+2} \frac{1}{2}p(k)\Delta^2 h(k-2)^2 - \sum_{k=1}^{N+1} \frac{1}{2}q(k)\Delta h(k-1)^2 \geq \]

\[ \frac{1}{2} \sum_{k=1}^{N+2} p_{\min}\Delta^2 h(k-2)^2 - \sum_{k=1}^{N+1} q_{\max}\Delta h(k-1)^2 \geq \frac{1}{2}(p_{\min}'(p) - q_{\max}'(q)) ||h||^2 > 0 \]

hence \( I \) is strongly convex. Therefore \( J \) is sum of convex and strongly convex functionals, hence it is strongly convex. That implies that the solution to (I) is unique. \( \blacksquare \)

Put

\[ \alpha_1 = \eta(q)q_{\max} - \eta'(p)p_{\min} \]

**Lemma 2.7** Assume that

1) \( \min_{k \in [1, N]} \lim_{s \to \pm \infty} \frac{f(k,s)}{s} > \alpha_1 \)

2) there exists \( S > 0 \) such that \( f(k, -s) \leq -f(k, s) \).

Then \( J \) is coercive and BVP (I) has at least one solution.

**Proof.** For every \( s < -S \) and fixed \( k \) we have:

\[
\begin{align*}
    f(k, -s) & \leq -f(k, s) \\
    \frac{f(k, -s)}{-s} & \geq \frac{-f(k, s)}{-s} \\
    \frac{f(k, -s)}{-s} & \geq \frac{f(k, s)}{s} \\
    \lim_{s \to -\infty} \frac{f(k, s)}{s} & \geq \lim_{s \to -\infty} \frac{f(k, s)}{s} > \alpha_1.
\end{align*}
\]
Therefore \( \min_{k \in \mathbb{Z}[1,N]} \lim_{|s| \to +\infty} \frac{f(k,s)}{s} > \alpha_1 \). Take a number \( \beta \) such that

\[
\min_{k \in \mathbb{Z}[1,N]} \lim_{|s| \to +\infty} \frac{f(k,s)}{s} \geq \beta > \alpha_1.
\]

for \( \varepsilon = \frac{1}{2}(\beta - \alpha_1) > 0 \), there exist a constant \( r > 0 \)

\[
\frac{f(k,s)}{s} \geq \beta - \varepsilon \quad \text{for} \quad |s| \geq M, k \in \mathbb{Z}[1,N],
\]

\[
F(k,s) \geq \frac{\beta - \varepsilon}{2} s^2 \quad \text{for} \quad |s| \geq M, k \in \mathbb{Z}[1,N]\]

Therefore, there exists a constant \( C \) such that

\[
F(k,s) \geq \frac{\beta - \varepsilon}{2} s^2 + C \quad \text{for} \quad s \in \mathbb{R}, k \in \mathbb{Z}[1,N]
\]

Using this inequality we obtain the following:

\[
J(y) = \sum_{k=1}^{N+2} \left( \frac{p(k)(\Delta^2 y(k-2))}{2} - \sum_{k=1}^{N+1} \frac{q(k)}{2}(\Delta y(k-1))^2 + \sum_{k=1}^{N} F(k,y(k)) \right) \geq \sum_{k=1}^{N+2} \frac{p_{\min}}{2}(\Delta^2 y(k-2))^2 - \sum_{k=1}^{N+1} \frac{q_{\max}}{2}(\Delta y(k-1))^2 + \sum_{k=1}^{N} \frac{\beta - \varepsilon}{2} y(k)^2 + NC \geq \frac{1}{2}(\eta'(p)p_{\min} - \eta(q)q_{\max} + \beta - \varepsilon)||y||^2 + NC
\]

Then it follows from definition of \( \alpha_1 \) that \( \eta'(p)p_{\min} - \eta(q)q_{\max} + \alpha_1 + \varepsilon = \varepsilon > 0 \) and therefore \( J \) is coercive.

Proof. Note that \( J(\theta) = 0 \). By 1) and 2) of lemma (2.7) and \( J \) is coercive and since it is also continuous it satisfies P.S. condition. Therefore there exists a critical point \( y_1 \).

Take a number \( \beta \) such that

\[
\max_{k \in \mathbb{Z}[1,N]} \lim_{|s| \to 0} \frac{f(k,s)}{s} < \beta < \alpha_2.
\]

Theorem 2.8 Assume that \( \dim E > 1, p_{\max} > 0, 1) \) and \( 2) \) of lemma (2.7) hold and that

\[
\max_{k \in \mathbb{Z}[1,N]} \lim_{s \to 0} \frac{f(k,s)}{s} < \alpha_2.
\]

Then BVP (1) has at least two solutions.

Proof. Note that \( J(\theta) = 0 \). By 1) and 2) of lemma (2.7) and \( J \) is coercive and since it is also continuous it satisfies P.S. condition. Therefore there exists a critical point \( y_1 \).

Take a number \( \beta \) such that

\[
\max_{k \in \mathbb{Z}[1,N]} \lim_{s \to 0} \frac{f(k,s)}{s} \leq \beta < \alpha_2.
\]
For $\varepsilon = \frac{1}{2}(\alpha_2 - \beta) > 0$, there exist a constant $\delta > 0$ such that
\[
\frac{f(k, s)}{s} \leq \beta + \varepsilon, \text{ for all } |s| \leq \delta \text{ and } k \in Z[1, N]
\]
there exist $\delta > 0$ such that
\[
\frac{f(k, s)}{s} \leq \beta + \varepsilon
\]
\[
F(k, s) \leq \frac{1}{2}(\beta + \varepsilon)s^2
\]
for all $|s| \leq \delta$ and $k \in Z[1, N]$. Let
\[
\Omega = \{y \in E : ||y|| < \delta\}.
\]
Then $\partial \Omega = \{y \in E : ||y|| = \delta\}$. Note that for all $y \in \partial \Omega$ and $k \in Z[1, N]$ $|y(k)| \leq \delta$. and
\[
J(y) = \sum_{k=1}^{N+2} \left( \frac{p(k)}{2}(\Delta^2 y(k-2))^2 - \sum_{k=1}^{N+1} \frac{g(k)}{2}(\Delta y(k-1))^2 + \sum_{k=1}^{N} F(k, y(k)) \right)
\]
\[
\leq \frac{1}{2}(16p_{max}-q_{min}\xi(q)+\beta+\varepsilon)||y||^2 = \frac{1}{2}(16p_{max}-q_{min}\xi(q)+\alpha_2-\varepsilon)||y||^2 < 0.
\]
and since $\partial \Omega$ is compact
\[
\sup\{J(y) : y \in \partial \Omega\} < 0.
\]
Since $J$ is coercive, there exist $y_1$ with $||y_1|| > \delta$ such that $\min\{0, J(y_1)\} = 0$. Therefore
\[
\min\{0, J(y_1)\} > \sup\{J(y) : y \in \partial \Omega\}.
\]
and $-J$ satisfies all assumptions or theorem (1.2) what implies that there exist $y_2 \in E$ such that $J'(y_2) = 0$, and
\[
J(y_2) = \sup_{h \in \Gamma} \min_{t \in \{0, 1\}} J(h(t)).
\]
Assume that $y_1 = y_2$, That is
\[
J(y_2) = \sup_{h \in \Gamma} \min_{t \in \{0, 1\}} J(h(t)) = \inf_{y \in E} J(y) = J(y_1).
\]
Therefore for all $h \in \Gamma$
\[
\min_{t \in \{0, 1\}} J(h(t)) \leq \inf_{y \in E} J(y).
\]
and since $h([0, 1]) \subseteq E$
\[
\min_{t \in \{0, 1\}} J(h(t)) \geq \inf_{y \in E} J(y)
\]
Therefore

\[ \min_{t \in [0,1]} J(h(t)) = \inf_{y \in E} J(y) \]

Since \( \dim E > 1 \) there exist \( h_1, h_2 \in \Gamma \) such that \( h_1([0,1]) \cap h_2([0,1]) = \emptyset \).

From the continuity of \( h_1 \) and \( h_2 \) there exist \( t_1, t_2 \in (0,1) \) such that \( h_1(t_1) = \min_{t \in [0,1]} J(h_1(t)) \) and \( h_2(t_2) = \min_{t \in [0,1]} J(h_2(t)) \). Therefore \( h_1(t_1) \) and \( h_2(t_2) \) are two different critical points, i.e. they are two different solutions to BVP (I).

\[ \blacksquare \]

**Theorem 2.9** Assume that \( f \) is odd on its second variable for all \( k \in \mathbb{Z}[1, N] \) and that

1) \( p_{\min} \geq 0 \)
2) \( \min_{k \in \mathbb{Z}[1, N]} \lim_{s \to \infty} \frac{f(k, s)}{s} > \alpha_1 \)
3) \( \max_{k \in \mathbb{Z}[1, N]} \lim_{s \to 0^+} \frac{f(k, s)}{s} < \alpha_2 \)

Then BVP (I) has at least \( 2N \) distinct solutions.

**Proof.** Since \( f \) is odd, assumption 2) of lemma (2.7) is satisfied. From this and 2) by lemma (2.7) it follows that \( J \) is coercive (and therefore bounded from below on \( E \) and satisfying the P.S. condition). Since \( f \) is odd on its second variable, \( J \) is even and \( f(k, s) \) for \( s \in \mathbb{R} \) and \( k = 1, ..., N \). Therefore there exist limits \( \lim_{s \to 0^+} \frac{f(k, s)}{s} \), \( k = 1, ..., N \). Take a number \( \beta \) such that

\[ \max_{k \in \mathbb{Z}[1, N]} \lim_{s \to 0^+} \frac{f(k, s)}{s} \leq \beta < \alpha_2. \]

For \( \varepsilon = \frac{1}{2}(\alpha_2 - \beta) > 0 \), there exist a constant \( \delta > 0 \) such that

\[ \frac{f(k, s)}{s} \leq \beta + \varepsilon \]
\[ f(k, s) \leq \frac{\beta + \varepsilon}{s} \]
\[ F(k, s) \leq \frac{\beta + \varepsilon}{s^2} \]

for all \( |s| \leq \delta \) and \( k \in \mathbb{Z}[1, N] \).

It follows that for all \( y \in K \) and \( k \in \mathbb{Z}[1, N] \), \( F(k, y(k)) \leq \frac{\beta + \varepsilon}{2}s^2 \). Let \( 0 < \delta' < \delta \), \( K = \{ y \in E : ||y|| = \delta' \} \). Then \( K \) is homeomorphic to a \( S^{N-1} \) by an odd map. For \( k \in \mathbb{Z}[1, N] \) and \( y \in K \) we have \( |y(k)| \leq ||y|| = \delta' < \delta \). Therefore for \( y \in K \).

\[ J(y) = \sum_{k=1}^{N+2} \frac{p(k)}{2}(\Delta^2 y(k-2))^2 - \sum_{k=1}^{N+1} q(k)(\Delta y(k-1))^2 + \sum_{k=1}^{N} F(k, y(k)) \leq \frac{1}{2}((16\rho_{max}-\xi(q)q_{min}+\beta+\varepsilon)||y||^2 \leq \frac{1}{2}((16\rho_{max}-\xi(q)q_{min}+\alpha_2-\varepsilon)\delta'^2 < 0, \]

Since \( K \) is compact sup \( K = \max K < 0 \). Therefore all assumptions of theorem (1) are satisfied and it follows that \( J \) has at least \( N \) distinct pairs of critical points, that is BVP (I) has at least \( N \) distinct pairs of solutions.

Put

\[ \alpha_3 = \min\{\lambda_1q_{min}-\lambda_2p_{max}, 4q_{min}-\lambda_2p_{max}\} \]

\[ \blacksquare \]
Theorem 2.10 Assume that \( f \) is odd on its second variable for all \( k \in Z[1, N] \) and that
1) \( \max_{k \in Z[1, N]} \lim_{s \to \infty} \frac{f(k, s)}{s} > \alpha_2 \)
2) \( \min_{k \in Z[1, N]} \lim_{s \to 0} \frac{f(k, s)}{s} < \alpha_3 \)
Then BVP (1) has at least \( 2N \) distinct solutions.

**Proof.** Since \( f \) is odd and 2) holds, by lemma (2.7) \( J \) is even, bounded from below and satisfies P.S. condition. Take a number \( \beta \) such that \( \lim_{s \to 0} f(k, s) \leq \beta \).

For \( \varepsilon = \frac{1}{2}(\alpha_3 - \beta) > 0 \) there exist a constant \( \delta > 0 \) such that
\[
\frac{f(k, s)}{s} \leq \beta + \varepsilon \text{ for all } |s| \leq \delta \text{, and } k \in Z[1, N].
\]

As in proof of theorem for fixed \( \delta' \in (0, \delta) \) and each \( y \in K = \{ y \in E : \|y\| = \delta' \} \) we have
\[
J(y) = \sum_{k=1}^{N+2} \left( \frac{p(k)}{2}(\Delta^2 y(k-2))^2 - \sum_{k=1}^{N+1} \frac{q(k)}{2}(\Delta y(k-1))^2 + \sum_{k=1}^{N} F(k, y(k)) \right) \\
\leq \frac{1}{2}(\lambda_2 p_{\max} - \min_{k=1}^{N} \xi(q) y_{\min}^2 + \beta + \varepsilon) \delta'^2 < \frac{1}{2}(\lambda_2 p_{\max} - \min_{k=1}^{N} \xi(q) y_{\min}^2 + \alpha_3) \delta'^2
\]

Therefore all assumptions of Clark’s theorem are satisfied and \( J \) has at least \( N \) distinct pairs of critical points, that is BVP (1) has at least \( N \) distinct pairs of solutions.

**Example 2.11** Let us consider the following BVP:

\[
\begin{align*}
\Delta^4 y(k-2) - \Delta^2 y(k-1) + \frac{1}{3} y(k)^3 + y(k) &= 0 \text{ for } k \in Z[1, N] \\
y(-1) &= y(0) = y(N+1) = y(N+2) = 0
\end{align*}
\]

Here \( f(k, s) = \frac{1}{3}s^3 + s \) for \( k \in Z[1, N] \), \( p = q = 1 \). Then \( f(k, s) \) is odd and \( \lim_{s \to \infty} \frac{1}{3}s^3 + s = +\infty \) and \( \lim_{s \to 0} \frac{1}{3}s^3 + s = 0 < 12 = \min\{12, 16 - \lambda_1\} \). Assumptions of theorem (2.9) are satisfied therefore this BVP has at least \( 2N \) solutions.

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