A note on group actions on algebraic stacks

Matthieu Romagny

Stockholms Universitet, Matematiska institutionen, 106 91 Stockholm, Sweden
e-mail address: romagny@matematik.su.se

Abstract: we give the basic definitions of group actions on (algebraic) stacks, and prove the existence of fixed points and quotients as (algebraic) stacks.

1 Introduction

For people using algebraic stacks, it is inevitable, one day or another, to meet a group acting on it. Thus it is no surprise that group actions on algebraic stacks must be studied. What is more surprising is that, up to now, no systematic treatment has been done. On many occasions, such actions appear in the literature, for example the action of a torus on a stack of stable maps in [Ko], (3.2) and the action of the symmetric group $\mathfrak{S}_d$ on a stack of multisectons in [L-MB], (6.6). We can also mention the increasing need of these concepts in (orbifold) Gromov-Witten theory. But certainly the most natural instance of an action on an algebraic stack comes when we consider a scheme $X$ with an action of an algebraic group $G$. Indeed, then the normalizer of $G$ in $\text{Aut}(X)$ (= the automorphism group scheme of $X$) acts on the quotient stack $[X/G]$; it is in some sense "what remains" of the symmetries of $X$.

Another natural example of group action on an algebraic stack comes from Hurwitz stacks (this is the example that initially motivated this note). These are stacks parameterizing covers of algebraic curves. If one restricts attention to Galois covers of group $G$, then the automorphism group of $G$ acts on the resulting stack by twisting the action of $G$. If the curves have marked points then the symmetric group acts on the stack by permutation of the points. Recently these stacks have been extensively studied. One should mention works of Bertin [Be], Ekedahl [E], Wewers [We], Abramovich-Corti-Vistoli [ACV].

The aim of this note is to give basic definitions of actions, and existence theorems for fixed points and quotients. Here is an overview of both its contents and organization. Section 2 is rather informal and prepares the definitions related to actions on stacks in section 3. The "main course" is in section 4 where we prove algebraicity of fixed point stacks under a proper flat group scheme, and of quotients under a separated flat group scheme (in this section, groups are finitely presented). Having in mind the application to covers of curves in mixed characteristic, where the framework of finite constant groups is outdated by the last developments (such as works of Raynaud, Henrio, Wewers, Saïdi and others), we study particularly closely the case of (arbitrary) finite flat group schemes.

Notations. In the article, a scheme $S$ is fixed once for all. Most schemes, spaces, and stacks will be over $S$, and quite often the mention of $S$ (for instance in fibred products) will be omitted. If $X$ is an $S$-scheme and $T \to S$ a base change, we often write $X_T = X \times_S T$. We write categories in calligraphic letters (such as $\mathcal{C}$) and 2-categories in fraktur letters (such as $\mathfrak{C}$). A category fibred in groupoids over $S$ is simply called a groupoid over $S$. In such a groupoid $M$, the functor of isomorphisms between two objects $x, y \in M(T)$ is denoted $\text{Isom}_T(x, y)$ or $\text{Isom}_{X_T}(x, y)$ if mention of $M$ is needed. An algebraic stack is an algebraic stack in the sense of [L-MB], def. 4.1.

Acknowledgments. I thank B. Toen, L. Moret-Bailly and A. Vistoli for various discussions and indications. I thank Stockholm University for hospitality and the EAGER Network for financial support.

Key words: algebraic stack, group action.
Mathematics Subject Classification: 14A20, 14L30.
2 Preliminaries

We need first to recall some basics concerning diagrams in a 2-category. Loosely speaking, a diagram in a 2-category $\mathcal{C}$ is a set of objects, with a set of 1-morphisms between certain pairs of objects, and a set of 2-morphisms between certain pairs of 1-morphisms (understood, with same source and target). We will write $\mathcal{D} = \{M, f, \alpha\}$ to indicate that $M$ ranges through the set of objects, $f$ ranges through the set of 1-morphisms and $\alpha$ ranges through the set of 2-morphisms of the diagram $\mathcal{D}$. Notice that, by the set of 1-morphisms of the diagram, we mean a set which is saturated under composition, i.e. including all possible compositions we can make with the original 1-morphisms, and similarly with the 2-morphisms.

We call circuit a pair of morphisms of $\mathcal{D}$ with same source and target. A circuit commutes if its two morphisms coincide. Here is a first example:

$$
\begin{array}{c}
M \xrightarrow{f} M' \xrightarrow{f'} M'' \\
\downarrow h \downarrow \alpha \quad \quad \quad \quad \quad \downarrow h'' \\
N \xrightarrow{g} N' \xrightarrow{g'} N''
\end{array}
$$

Here $(gh, h'f)$ is a circuit of 1-morphisms (or 1-circuit), and $\alpha$ is a 2-morphism between $gh$ and $h'f$, so that the 1-circuit commutes if $\alpha = \text{id}$. A second example may be given by the same diagram, plus a 2-morphism $\alpha'' : g'gh \Rightarrow h''f'f$ attached to the exterior rectangular 1-circuit; then there is a 2-circuit

$$
\begin{array}{c}
g'\alpha
\end{array}
\xleftarrow{\alpha''}
\begin{array}{c}
g'h'f
\end{array}
\xrightarrow{\alpha'f}
\begin{array}{c}
g'f
\end{array}
\xrightarrow{\alpha''}
\begin{array}{c}
h''f'f
\end{array}
$$

If $\ast$ denotes the composition of 2-morphisms, we will write this 2-circuit $(\alpha'', \alpha' \ast \alpha)$, though to be rigorous we should write $(\alpha'', \alpha'f \ast g'\alpha)$. A diagram in $\mathcal{C}$ is said to be 2-commutative if any of its 2-circuits commutes, i.e. we have $\alpha'' = \alpha' \ast \alpha$ in the example above. Given a diagram in $\mathcal{C}$, by forgetting the 2-morphisms we get a diagram in the underlying category $\text{Cat}(\mathcal{C})$ of $\mathcal{C}$. Sometimes we refer to diagrams of $\mathcal{C}$ as 2-diagrams, and diagrams of $\text{Cat}(\mathcal{C})$ as 1-diagrams. So for example we will say that a 2-diagram in $\mathcal{C}$ is 1-commutative if the associated 1-diagram is 1-commutative.

Our main aim in the article is to discuss group actions on algebraic stacks; of course this will just be an action on the underlying stack, and even on the underlying groupoid. Thus we simplify the approach by looking first at the case of the 2-category $\text{Grpd}/S$ of groupoids over $S$. In this particular 2-category, 2-commutativity means "(1-)commutativity up to (given) isomorphisms".

Let $M$ be such a groupoid, and $G$ be a functor in groups over $S$. We denote by $m$ the multiplication of $G$, and by $e$, or sometimes simply 1, its unit section. As is natural, one defines an action of $G$ on $M$ to be a morphism of groupoids $\mu : G \times M \to M$ satisfying the usual commutative diagrams concerning compatibility with respect to the unit section of $G$ and to associativity of the multiplication. Using the more natural notion of 2-commutativity in $\text{Grpd}/S$, we are led to 2-commutative diagrams

$$
\begin{array}{c}
G \times G \times M \xrightarrow{m \times \text{id}_M} G \times M \\
\downarrow \alpha \quad \quad \quad \quad \quad \downarrow \mu \\
G \times M \xrightarrow{\alpha \times \text{id}_M} M
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{c}
G \times M \xrightarrow{\mu} M \\
\downarrow e \times \text{id}_M \quad \quad \quad \quad \quad \quad \downarrow \text{id}_M \\
M \xrightarrow{\alpha \times \text{id}_M} M
\end{array}
$$

where $\alpha : \mu \circ (\text{id} \times \mu) \Rightarrow \mu \circ (m \times \text{id})$ and $\alpha : \mu \circ (e \times \text{id}) \Rightarrow \text{id}$ are 2-isomorphisms. Let’s make the usual notational convention that, when we have an action of $G$ on $M$, and sections $x \in M$ and $g \in G$ over a scheme $T$, we will write $g.x$ or $gx$, rather than $\mu(g, x)$; for an arrow $\varphi : x \to y$ we will write $g.\varphi$ or $g\varphi$ rather than $\mu(g, \varphi)$. So what the above 2-commutative diagrams mean is that we are given isomorphisms in $M$, natural in $(g, h, x)$,

$$
\alpha_{g,h}^x : g.(h.x) \sim (gh).x \quad \quad \quad \quad \text{and} \quad \quad \quad \quad \alpha^x : 1.x \sim x
$$

2
Similarly, a morphism of $G$-groupoids $f : M \rightarrow N$ should be given by a 2-commutative diagram

\[
\begin{array}{ccc}
G \times M & \xrightarrow{\mu} & M \\
\downarrow{\sigma} & & \downarrow{f} \\
G \times N & \xrightarrow{\nu} & N
\end{array}
\]

(2)

where $\sigma$ can be written more concretely by

\[
\sigma^x_g : g.f(x) \sim f(g.x)
\]

Furthermore, if we want a triple $(\mu, \alpha, a)$ as above to define an action, the isomorphisms $\alpha^x_{g,h}$ should be compatible with associativity in $G$. In the same vein, a pair $(f, \sigma)$ will give rise to a morphism if $\sigma$ is compatible to $\alpha, a$ and the corresponding 2-isomorphisms $\beta, b$ for $N$. We arrive at the following provisional definition of an action.

**Definition 2.1** Let $M$ be a groupoid over $S$ and $G$ a functor in groups over $S$.

(i) An action of $G$ on $M$ is a triple $(\mu, \alpha, a)$ where $\mu : G \times M \rightarrow M$ is a morphism of groupoids satisfying the above 2-commutative diagrams (1), and such that for all $x$ and $g, h, k$ we have

\[
\alpha^x_{g,h,k} \circ g.\alpha^x_h,k = \alpha^x_{g,h,k} \circ \alpha^x_{g,h}
\]

and

\[
1.a^x = a^x_{1,1}
\]

We also say that $M$ is a $G$-groupoid. If $\alpha$ and $a$ are the identity 2-isomorphisms, we say that the action (or the $G$-groupoid) is strict. Usually we simply note $M$ for $(M, \mu, \alpha, a)$.

(ii) A morphism of $G$-groupoids between $(M, \mu, \alpha, a)$ and $(N, \nu, \beta, b)$ is a pair $(f, \sigma)$ where $f : M \rightarrow N$ is a morphism of groupoids over $S$ satisfying the above 2-commutative diagram (2), and such that for all $x$ and $g, h$ we have

\[
f(\alpha^x_{g,h}) \circ \sigma^x_{g,h} = \sigma^x_{g,h} \circ \beta^f(x)
\]

and

\[
f(1.a) = 1.b^f(x)
\]

(iii) An isomorphism of $G$-groupoids is a morphism of $G$-groupoids which is also an equivalence of categories fibred over $S$.

The 2-category of $G$-groupoids over $S$ is denoted $G\text{-Grpd}/S$. Before getting a headache trying to check if these are really the good compatibilities, note that the key point in making these definitions is the following remark. Once we make the definition of a $G$-groupoid, we can recognize the data $(M, \mu, \alpha, a)$ as giving exactly what is called a lax presheaf in groupoids $\mathcal{F}$ over $C = B_0G$, where $B_0G$ is the groupoid associated to $G$, i.e. the groupoid whose fibre over a scheme $T/S$ has only one object, and morphisms the elements of $G(T)$. This lax presheaf (see for instance [Ho], Appendix B) is described as follows:

(i) To an object of $B_0G$, i.e. a scheme $T$ over $S$, is associated the groupoid $\mathcal{F}(T) = M(T)$.

(ii) To a morphism of $B_0G$, i.e. an element $g \in G(T)$, is associated the functor $\mu(g^{-1}, .) : M(T) \rightarrow M(T)$ denoted $\mu_g$.

(iii) For each $g, h \in G(T) (= pair of composable arrows)$, there is a natural transformation $\mu_g \circ \mu_h \sim \mu_{hg}$ given by $\alpha^{-1, h^{-1}}_{g,h}$.

So now, the definition of a morphism of $G$-groupoids is just a translation of the definition of a morphism of lax presheaves as we find it in [Ho]. This link with lax presheaves also explains why in fact, given a $G$-groupoid $M$, we will always be able to find an equivalent $G$-groupoid $\mathcal{M}^{str}$ such that the 2-isomorphisms $\alpha$ and $a$ are the identities.

**Proposition 2.2** There is a "strictification" functor $G\text{-Grpd}/S \rightarrow G\text{-Grpd}/S$ sending any $G$-groupoid to an isomorphic $G$-groupoid with strict action.

**Proof**: Let $M$ be a $G$-groupoid, and define a $G$-groupoid $\mathcal{M}^{str}$ in the following way:
(i) the sections of $M^{\text{str}}$ over a scheme $T$ are pairs $(g, x)$ with $g \in G(T)$, $x \in M(T)$,
(ii) the arrows in $M^{\text{str}}$ between $(g, x)$ and $(h, y)$ are arrows $\varphi : x \to (g^{-1}h).y$ in $M(T)$,
(iii) composition of two arrows $\varphi : (g, x) \to (h, y)$ and $\psi : (h, y) \to (k, z)$ is given by
\[ x \xrightarrow{\varphi} (g^{-1}h).y \xrightarrow{(g^{-1}h).\psi} (g^{-1}h).(h^{-1}k).z \xrightarrow{\alpha_{g,h}^{-1}k^{-1}} (g^{-1}k).z \]
There is a strict action of $G$ on $M^{\text{str}}$: an element $\gamma \in G(T)$ sends an object $(g, x)$ to $(\gamma g, x)$, and sends an arrow $\varphi : x \to (g^{-1}h).y$ to the same arrow as a morphism between $(\gamma g, x)$ and $(\gamma h, y)$. Furthermore it is clear that $M^{\text{str}}$ is functorial in $M$.

It only remains to check that $M$ and $M^{\text{str}}$ are isomorphic. We define a morphism of groupoids $u : M^{\text{str}} \to M$ by mapping an object $(g, x)$ to $g.x$, and an arrow $(g, x) \to (h, y) \to (k, z)$ represented by $\varphi : x \to (g^{-1}h).y$ to the composition $\alpha_{g,h}^{-1}k^{-1} \circ (g, \varphi)$. Clearly, $u$ is a $G$-morphism. Furthermore it is essentially surjective because any object $x$ in $M$ is isomorphic via $x^e$ to $1.x$. Finally it is straightforward to see that it is fully faithful, so $u$ is an isomorphism.

Now assume that the scheme $S$, viewed as the category of $S$-schemes, is endowed with a Grothendieck topology. In practice, for us this will be the fppf or étale topology. Then it is clear that an action of $G$ on a groupoid $M$ extends uniquely to an action on the associated stack $\tilde{M}$. We could make, in the context of stacks over $S$, statements similar to all ones in this section, taking associated stacks at the right moments. (For example the groupoid $BG$ would be replaced by the stack of $G$-torsors $BG$.)

In the next section we develop the basics of the theory of actions on stacks, starting from the idea that we can restrict to considering strict actions. This is made legitimate by proposition 2.2. Note that the theory could certainly be developed with general weak actions, at the cost of substantial technical complications. This seems unnecessary: the practice will show that all constructions we wish to make will yield strict actions, and if that were not the case, strictifying at the right place would bring back to this context.

## 3 Group actions on stacks

Below is the minimal number of definitions that we will need for $G$-stacks. There are two ways to present these concepts, according to whether we define morphisms before commutative diagrams, or after them. We take the first option and explain the second in a remark.

**Definition 3.1** Let $M$ be a stack over $S$ and $G$ a group in groups over $S$. Let $m$ denote the multiplication of $G$, and $e$ its unit section.

(i) An action of $G$ on $M$ is a morphism of stacks $\mu : G \times M \to M$ with 1-commutative diagrams

\[
\begin{array}{ccc}
G \times G \times M & \xrightarrow{m \times \text{id}} & G \times M \\
\text{id} \times \mu & \downarrow & \downarrow \mu \\
G \times M & \xrightarrow{\mu} & M
\end{array}
\]
\[
\begin{array}{ccc}
G \times M & \xrightarrow{\mu} & M \\
\epsilon \times \text{id} & \downarrow & \downarrow \text{id} \\
M & \xrightarrow{\text{id}} & M
\end{array}
\]

We say that $(M, \mu)$ is a $G$-stack.

(ii) A 1-morphism of $G$-stacks, or $1$-$G$-morphism, between $(M, \mu)$ and $(N, \nu)$ is a pair $(f, \sigma)$ where $f : M \to N$ is a morphism of stacks with a 2-commutative diagram

\[
\begin{array}{ccc}
G \times M & \xrightarrow{\mu} & M \\
\text{id} \times f & \downarrow \sigma & \downarrow f \\
G \times N & \xrightarrow{\nu} & N
\end{array}
\]

such that for all sections $x \in M(T)$ and $g, h \in G(T)$ over a scheme $T$, the isomorphisms $\sigma^x_g : g.f(x) \simeq f(g.x)$ satisfy the cocycle relation $\sigma^h_{g} \circ g.\sigma^x_h = \sigma^x_{gh}$. Composition of 1-morphisms of $G$-stacks is defined in the obvious way, namely, $(f_2, \sigma_2) \circ (f_1, \sigma_1) = (f_3, \sigma_3)$ where $f_3 = f_2 \circ f_1$ and $\sigma^x_{3,g} = f_2(\sigma^x_{1,g}) \circ \sigma^f(x)_{2,g}$. 


(iii) A 2-morphism of $G$-stacks, or $2G$-morphism, between 1-morphisms $(f_1, \sigma_1)$ and $(f_2, \sigma_2)$ is a 2-morphism of stacks $\tau : f_1 \Rightarrow f_2$ compatible with the $\sigma_i$ i.e. such that for all sections $x \in M(T)$ and $g \in G(T)$ over a scheme $T$, we have $\sigma_2^g \circ g \cdot \tau = \tau \cdot x \circ \sigma_1^g$. In this way we have defined a 2-category of $G$-stacks over $S$, which will be denoted by $G\text{-}\text{St}/S$ or simply $G\text{-}\text{St}$ if the base $S$ is clear. In particular, given two $G$-stacks $M, N$ there is the stack $\text{Hom}_{G\text{-}\text{St}}(M, N)$ of 1-morphisms and 2-morphisms between them.

(iv) Two actions $\mu, \mu'$ on the stack $M$ are said to be equivalent if there exists $\sigma$ such that the pair $(\text{id}, \sigma)$ is a 1-$G$-morphism between $(M, \mu)$ and $(M, \mu')$.

(v) An isomorphism of $G$-stacks is a 1-$G$-morphism which is also an equivalence of groupoids over $S$.

**Remark 3.2** We can restate definitions (ii) and (iii) using only diagrams in the 2-category of stacks. Indeed, let $D = \{M, f, \alpha\}$ be a diagram of stacks where $M, f, \alpha$ range through objects, 1-morphisms, and 2-morphisms of $D$ respectively. Consider

(a) the diagram $G \times D := \{G \times M, \text{id}_G \times f, \text{id}_{id_G} \times \alpha\},$

(b) the diagram $G \times G \times D := \{G \times G \times M, \text{id}_{G \times G} \times f, \text{id}_{id_{G \times G}} \times \alpha\}.$

Assume that the objects are in fact $G$-stacks $(M, \mu)$ and for any objects $(M, \mu), (N, \nu)$ and any 1-morphism $f : M \to N$ we are given a 2-isomorphism $\sigma : \nu \circ (\text{id}_G \times f) \Rightarrow f \circ \mu$. Then we can form a new diagram $G \times G \times D \to G \times D \to D$. Precisely, at the stage $G \times G \times D \to D$ the 1-morphisms are the $\mu$'s, the 2-morphisms are the $\sigma$'s. At the stage $G \times G \times D \to G \times D$, the 1-morphisms are the $(\text{id}_G \times \mu)$'s, the 2-morphisms are the $(\text{id}_G \times \sigma)$'s. Then we can define a 2-commutative diagram of $G$-stacks to be given by data $(D, \{\mu\}, \{\sigma\})$ such that $G \times G \times D \to G \times D \to D$ is a 2-commutative diagram of stacks. In particular, we can now define the notions of 1-morphisms of $G$-stacks and 2-morphisms between 1-morphisms of $G$-stacks, by the 2-commutativity of the following two elementary diagrams:

\[
\begin{array}{ccc}
(M, \mu) & \xrightarrow{(f, \sigma)} & (N, \nu) \\
\downarrow & & \downarrow \\
(f_2, \sigma_2)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
(M, \mu) & \xrightarrow{(f_1, \sigma_1)} & (N, \nu) \\
\downarrow & & \downarrow \\
(f_2, \sigma_2)
\end{array}
\]

which means checking 2-commutativity of the following "prisms" \\

\[
\begin{array}{ccc}
G \times G \times M & \xrightarrow{\mu} & M \\
\downarrow & & \downarrow f \\
G \times G \times N & \xrightarrow{\nu} & N
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
G \times G \times M & \xrightarrow{\mu} & M \\
\downarrow & & \downarrow f \\
G \times G \times N & \xrightarrow{\nu} & N
\end{array}
\]

Observe that if we already know that all pairs $(f, \sigma)$ in a diagram are 1-morphisms of $G$-stacks, then we only have to check 2-commutativity of the "lower stage" of the prisms. \hfill \Box

The 2-category $G\text{-}\text{St}/S$ has arbitrary projective and inductive limits. In particular $G\text{-}\text{St}/S$ has fibred products, defined in the obvious way, so we have the notion of a 2-cartesian square. Any stack $M$ over $S$ gives a trivial $G$-stack $(M, pr_2)$ and this gives a 2-functor $\nu : \text{St} \to G\text{-}\text{St}$. The invariants and coinvariants are the 2-adjoints of this functor:

**Definition 3.3** Let $G$ be a sheaf in groups over $S$ and $M$ a $G$-stack over $S$.

(i) A stack of fixed points $M^G$ is a stack that represents the 2-functor $\text{St}^G \to \text{Cat}$ defined by

\[
F(N) = \text{Hom}_{G\text{-}\text{St}}(\nu(N), M)
\]

(the latter is the stack of (iii), and $\text{Cat}$ is the 2-category of categories).
(ii) A quotient stack $\mathcal{M}/G$ is a stack that represents the 2-functor $\mathcal{O}t \to \mathcal{C}at$ defined by

$$F(N) = \mathcal{H}om_{G,\mathcal{O}t}(\mathcal{M}, \iota(N))$$

**Remark 3.4** There are in fact several candidates for the notion of a trivial action, needed to define fixed points and quotients. The trivial $G$-stacks as defined in [ET](iii) form a full 2-subcategory of $G-\mathcal{O}t$, denoted $\mathcal{T}-G-\mathcal{O}t$. Its essential image in $G-\mathcal{O}t$ defines the essentially trivial $G$-stacks and we will denote it by $\mathcal{T}-G-\mathcal{O}t$. The final picture of the factorization of $\iota$ is:

$$\mathcal{O}t \xrightarrow{\text{ess. surjective}} \mathcal{T}-G-\mathcal{O}t \xrightarrow{\sim} \mathcal{T}-G-\mathcal{O}t \subseteq G-\mathcal{O}t$$

In definition 3.3 we could have chosen $N$ among either of these categories of trivial $G$-stacks. Here the crucial point is to note the uncommon feature of $G-\mathcal{O}t$ that, unlike most usual categories where we can consider group actions (e.g. sets, modules, algebras, varieties, schemes,...), the quotient of an object with trivial $G$-action is not the object itself. The example of $\mathcal{M} = S$ and $G$ acting trivially is in the mind of everyone: then the stack-quotient is $S/G = BG$, and more generally, for any $\mathcal{M}$ with trivial $G$-action we should have $\mathcal{M}/G = \mathcal{M} \times BG$. For this reason it would be meaningless to choose trivial objects among a full subcategory "?" of $G-\mathcal{O}t$, because then we would have

$$\text{Hom}_{G-\mathcal{O}t}(S, N) = \text{Hom}(BG, N) = \text{Hom}_{G-\mathcal{O}t}(BG, N) = \text{Hom}(BG \times BG, N) = \ldots$$

and so on. This is why $\mathcal{O}t$ is the only possibility.

**Proposition 3.5** Let $G$ be a sheaf in groups over $S$ and $\mathcal{M}$ a $G$-stack over $S$. Then there exists a stack of fixed points $\mathcal{M}^G$, and its formation commutes with base change on $S$. The essentially trivial stacks of remark 3.4 are the stacks $(\mathcal{M}, \mu)$ isomorphic to $\iota(\mathcal{M}^G)$.

**Proof:** From the definition we must have $\mathcal{H}om_{G-\mathcal{O}t}(N, \mathcal{M}^G) = \mathcal{H}om_{G-\mathcal{O}t}(\iota(N), \mathcal{M})$. From the particular case $N = S$ we deduce $\mathcal{M}^G = \mathcal{H}om_{G-\mathcal{O}t}(S, \mathcal{M}^G) = \mathcal{H}om_{G-\mathcal{O}t}(\iota(S), \mathcal{M})$. This is the stack of $G$-invariant sections of $\mathcal{M}$, whose objects over a base $T$ are pairs $(x, \{\alpha_g\}_{g \in G(T)})$ where $x \in \mathcal{M}(T)$ and $\alpha_g : x \to g.x$ are isomorphisms such that $g.\alpha_h \circ \alpha_g = \alpha_{gh}$ for all sections $g, h \in G(T)$. The second assertion follows because, by definition, an essentially trivial $G$-stack is a $G$-stack $(\mathcal{M}, \mu)$ such that there exists an isomorphism $(\mathcal{M}, \mu) \simeq \iota(\mathcal{N})$ for some $\mathcal{N}$. Taking fixed points and then $\iota$, we obtain the result.

**Proposition 3.6** Let $G$ be a sheaf in groups over $S$ and $\mathcal{M}$ a $G$-stack over $S$. Then there exists a quotient stack $\mathcal{M}/G$, and its formation commutes with base change on $S$.

**Proof:** We define a prestack $\mathcal{P}$ as follows: sections of $\mathcal{P}(T)$ are sections of $\mathcal{M}(T)$, and morphisms in $\mathcal{P}(T)$ between $x$ and $y$ are pairs $(g, \phi)$ with $g \in G(T)$ and $\phi : g.x \to y$ a morphism in $\mathcal{M}(T)$. Let $\mathcal{M}/G$ be the stack associated to $\mathcal{P}$. It is straightforward to check the universal 2-property.

## 4 Group actions on algebraic stacks

In this section we will prove algebraicity of fixed points and quotients for certain algebraic groups $G$ acting on algebraic stacks. We consider the category $G-\text{AlgGal}/S$ of algebraic $G$-stacks over $S$: this is defined to be the full subcategory of $G-\mathcal{O}t/S$ of $G$-stacks whose underlying stack is algebraic. In particular all definitions of $\text{ET}$ apply, so we do not have to rewrite them. The definitions of $\text{ET}$ carry on in an obvious way, namely the algebraic stack of fixed points represents a 2-functor $\text{AlgGal}^G \to \mathcal{C}at$, and the quotient algebraic stack represents a 2-functor $\text{AlgGal} \to \mathcal{C}at$.

Before we go further we recall a few examples:
Examples 4.1 (i) Let $G$ be a flat, separated group scheme of finite presentation over $S$. Then the sheaf $\text{Aut}(G)$ acts on the stack of $G$-torsors $BG$ by twisting the action: given $\theta \in \text{Aut}(G)$ and $E \to T$ a $G$-torsor over $T/S$, the twisted action is defined by $g \ast e = \theta(g).e$.

(ii) Let $M_g,n$ be the stack of stable curves of genus $g$ with $n$ marked points. Then the symmetric group $S_n$ acts on it by permuting the marked points.

(iii) Let $M_g(n)$ be the stack of smooth curves of genus $g$ together with a level $n$ structure, i.e., an isomorphism $\varphi : C[n] \isom (\mathbb{Z}/n\mathbb{Z})^g$. Then $G = \text{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$ acts on $M_g(n)$ by twisting the level structure.

(iv) Let $X_1(N)$ be the stack of elliptic curves together with a "point of $N$-torsion" (see [KaMa]). Then $G = (\mathbb{Z}/n\mathbb{Z})^N$ acts on $X_1(N)$ by acting on the point of $N$-torsion.

4.1 Fixed points

Theorem 4.2 Let $G$ be a proper, flat group scheme of finite presentation over $S$. Let $M$ be an algebraic $G$-stack, with diagonal locally of finite presentation over $S$. Then the fixed point stack $M^G$ (prop. 4.3) is algebraic (so it is a fixed point stack in $\mathcal{AlgSt})$. The morphism $\epsilon : M^G \to M$ is representable, separated and locally of finite presentation. The formation of $M^G$ commutes with base change on $S$.

Proof: It is enough to show that the morphism $M^G \to M$ is representable with the desired properties. So let $f : T \to M$ be a 1-morphism, corresponding to an object $x \in M(T)$. The fibre product $M^G \times_M T$ is the sheaf whose sections over $T'/T$ are collections of isomorphisms $\{\alpha_g : x \simeq g.x\}_{g \in G(T')}$ such that for all sections $g,h \in G(T')$ we have $g.\alpha_h \circ \alpha_g = \alpha_{gh}$. Denote by $x_1$ and $x_2$ the objects of $M(G \times T)$ corresponding to the 1-morphisms $pr_2 \circ (id_G \times f)$ and $\mu \circ (id_G \times f)$. Reformulating what we said above, there is a closed immersion, locally of finite presentation, $M^G \times_M T \to \text{Hom}_T(G_T, \text{Isom}_{G_T}(x_1,x_2))$. With our assumptions, a section of this Hom sheaf gives, via its graph, a closed subspace of $G_T \times \text{Isom}_{G_T}(x_1,x_2)$ which is proper and flat over the base, being isomorphic to $G_T$ via the first projection. So the sheaf is representable by the corresponding open constructible subspace of the Hilbert space, which is algebraic, separated and locally of finite presentation by (Artin cor. 6.2). The result follows.

Remarks 4.3 (i) If $M$ is representable, then $M^G$ is representable also, and so the fixed points of $M$ as a space or as a stack are the same (in general the Yoneda functor from spaces into stacks commutes with projective limits when they exist, but not with inductive limits; see also [11]).

(ii) For an essentially trivial $G$-stack $(N, \nu)$ (see [11] and [5]), arbitrary $G$-morphisms $(f, \sigma) : N \to M$ still factor through $M^G$, because $(N, \nu) \simeq (N^G, pr_2)$. This factorization is of course not unique.

If we relax the assumption of properness of $G$, it does not seem plausible that we can say much on representability of $\epsilon : M^G \to M$, at least if the diagonal of $M$ is not flat. If we put conditions on $G$ and on the diagonal of $M$ such as flatness or smoothness, then it may be that using arguments such as these developed in SGA4, tome 2, we obtain representability of $M^G \to M$ in some cases.

In this case, it is not possible to deduce that the space $\text{Hom}_T(G_T, \text{Isom}_{G_T}(x_1,x_2))$ in the proof above is finite, or even proper, because $G$ may be ramified. However we will deduce the corresponding property for $\epsilon$ by giving a slightly different construction of $M^G$. We start with a lemma:

Lemma 4.4 Let $Q$ be a finite flat scheme of finite presentation over $S$. Let $M$ be an algebraic stack locally of finite presentation over $S$. Then the stack $\text{Hom}_S(Q,M)$ of morphisms of stacks from $Q$ to $M$ is algebraic and locally of finite presentation over $S$.

Proof: Let’s note $\mathcal{H} := \text{Hom}_S(Q,M)$ and $n = [Q : S]$. Notice that, given an $S$-scheme $T$, we have $\mathcal{H}(T) = M(Q \times T)$. From this and the fact that $Q$ is affine, after algebraicity is proved it will follow that $\mathcal{H}$ is locally of finite presentation over $S$ because given a filtering inductive system of $S$-algebras $A_i$, we have isomorphisms

$$\lim \mathcal{H}(A_i) \simeq \lim M(Q \otimes A_i) \simeq M(\lim Q \otimes A_i) \simeq M(Q \otimes \lim A_i) \simeq \mathcal{H}(\lim A_i)$$
Now we show that the diagonal of $\mathcal{H}$ is representable, separated and quasi-compact. It is enough to study the sheaf $\text{Isom}_{\mathcal{H}}(x, y)$ for two fixed objects $x, y \in \mathcal{H}(T)$. These correspond to objects $\eta \in \mathcal{M}(Q \times T)$ and $\xi \in \mathcal{M}(Q \times T)$, and

$$\text{Isom}_{\mathcal{H}}(x, y) = \text{Hom}_{\mathcal{M}}(Q_T, \text{Isom}_{\mathcal{M}}(\eta, \xi))$$

Here the sheaf $I := \text{Isom}_{\mathcal{M}}(\eta, \xi)$ is representable and of finite presentation over $Q_T$ (it is locally of finite presentation because $\mathcal{M}$ is, by [EGA], I, 6.2.6. which extends to stacks). It keeps these properties as a $T$-sheaf. Let us introduce the functor $H_n$ which is the component of the full Hilbert functor of $Q_T \times I$ parametrizing 0-dimensional subspaces of length $n$. It is representable by a separated algebraic space locally of finite presentation (Artin cor. 6.2), and in fact the length $n$ component is quasi-compact because $Q_T \times I$ is. Now, the graph of a morphism $Q_T \to I$ defines a point in $H_n$ (by separation of $I$), such that the restriction of the first projection $Q_T \times I \to Q_T$ is an isomorphism. The sheaf $\text{Isom}_{\mathcal{H}}(x, y)$ is thus isomorphic to the corresponding constructible open subspace of $H_n$. By constructibility this open immersion is quasi-compact ([EGA], 0II, 9.15), and, of course, separated.

Now let $U \to M$ be an atlas; we can choose $U$ separated. Then I claim that $V := \text{Hom}_S(Q, U)$ will be an atlas for $\mathcal{H}$. First, by Artin’s result again $V$ is representable and locally of finite presentation. As $\mathcal{H}$ is also locally of finite presentation this shows that the map $V \to \mathcal{H}$ has the same property. Thus we only have to prove that it is formally smooth and surjective.

To prove surjectivity take an algebraically closed field $k$ and a morphism $\text{Spec}(k) \to \mathcal{H}$ i.e. a morphism $f : Q_k \to \mathcal{M}_k$. Then $Q_k$ is an artinian scheme, hence a sum of local artinian $k$-schemes, so we reduce to the local case. By surjectivity of $U \to M$, the image of the underlying point of $Q_k$ lifts to $U_k$, and by smoothness the whole morphism lifts.

It remains to prove formal smoothness. Let $A \to A_0$ be a surjection of artinian rings with nilpotent kernel. Assume we have a 2-commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{\delta} & \mathcal{H} \\
\downarrow \delta & & \downarrow \delta \\
\text{Spec}(A_0) & & \mathcal{M}_{A_0}
\end{array}$$

meaning that we have

$$\begin{array}{ccc}
U_{A_0} & \xrightarrow{\delta} & Q_{A_0} \\
\downarrow \delta & & \downarrow \delta \\
A & \to & A_0
\end{array}$$

As $Q_{A_0}$ is artinian, by smoothness of $U_A \to M_A$, the map $Q_{A_0} \to U_{A_0} \to U_A$ immediately lifts to $Q_A \to U_A$, and we are done. 

\begin{rem}
(i) If $Q = S[\varepsilon]/\varepsilon^2$ we recover the tangent stack $T(M/S)$, and the lemma gives a proof of its algebraicity which is simpler than in [LM], chap. 17. If $Q = S[x]/x^n$ we get the stack of $n$-truncated arcs in $M$. If $Q$ is sum of $n = [Q : S]$ copies of $S$ then $\text{Hom}_S(Q, M) = M^n$ so the result is trivial.

(ii) The result of [OS] of representability of Quot functors for Deligne-Mumford stacks does not allow to derive algebraicity of $\text{Hom}_S(Q, M)$ because one can not express this stack as an open substack of the "Hilbert space" (graph morphisms of algebraic stacks are no longer closed immersions, not even for $M$ separated).
\end{rem}

\begin{prop}
Let $G$ be a finite, flat group scheme of finite presentation over $S$. Let $\mathcal{M}$ be an algebraic $G$-stack, locally of finite presentation over $S$. Then the morphism $\mathcal{M}^G \to \mathcal{M}$ is furthermore quasicompact, and enjoys any property enjoyed by the diagonal of $\mathcal{M}$, by closed immersions, and stable by composition. In particular it is proper if $\mathcal{M}$ is separated.
\end{prop}

\begin{proof}
Throughout, we will omit the description of the morphisms of the different stacks introduced, since they are obvious and quite lengthy to write completely. By the lemma applied to $Q = G$ the stack $\mathcal{H} = \text{Hom}(G, M)$ is algebraic. We now define two morphisms $a, b : M \to \mathcal{H}$. Let $x \in \mathcal{M}(T)$, corresponding to a morphism $f : T \to M$, and look at the compositions

$$G \times T \xrightarrow{\text{id} \times f} G \times \mathcal{M} \xrightarrow{\mu_{pr_2}} \mathcal{M}$$

\end{proof}
Then we define $a(x) = (\mu \circ (\text{id}_G \times f))^*(x)$ and $b(x) = (\text{pr}_2 \circ (\text{id}_G \times f))^*(x) = x_{G_T}$. In more naive terms, $a(x) = (g \mapsto g.x)$ and $b(x) = (g \mapsto x)$. Now look at the fibre product defined by the diagram

\[
\begin{array}{c}
N \\
\downarrow \\
M \quad a \\
\downarrow \\
\mathcal{H}
\end{array}
\]

An object of $N$ is a pair $(x, \psi^x : a(x) \simeq b(x))$ where $\psi^x$ consists in isomorphisms $\psi^x_g : g.x \simeq x$. We define a closed substack $\mathcal{Z} \subset \mathcal{H}$ by considering the morphisms $\psi : G \rightarrow \mathcal{M}$ such that for all sections $g, h \in G(T)$ we have $g(\psi_h) \circ \psi_g = \psi_{gh}$. The stack $\mathcal{M}^G$ is isomorphic to the preimage of $\mathcal{Z}$ in $\mathcal{N}$. The morphism $\epsilon : \mathcal{M}^G \rightarrow \mathcal{M}$ is the first projection. Finally, it is not hard to check that $\mathcal{M}^G$ is locally of finite presentation, using that it is the case for $a$ and for $\mathcal{H}$ and its diagonal.

It remains to prove the properties of the morphism $\mathcal{M}^G \rightarrow \mathcal{M}$. First we look at the morphism $b : \mathcal{M} \rightarrow \mathcal{H}$. Let $U \rightarrow \mathcal{M}$ be a morphism, corresponding to an object $\xi \in \mathcal{M}(G \times U)$. The fibre product $\mathcal{M} \times_{\mathcal{H}} U$ is the stack of triples $(T, \eta, \alpha)$ composed of a map of schemes $T \rightarrow U$, an object $\eta \in \mathcal{M}(T)$ and an isomorphism $\alpha$ between $\eta_{G_T}$ and $\xi_{G_T}$. By fppf descent, this is none other than the functor of descent data for $\xi$ with respect to the fppf covering $G_U \rightarrow U$. It is represented by a closed sub-algebraic space of $\text{Isom}_{G_U \times_{\mathcal{H}} U}(\text{pr}_1^* \xi, \text{pr}_2^* \xi)$. This space inherits the properties such as quasi-compactness and separatedness of the diagonal of $\mathcal{M}$. It follows that $b$ has these properties, and similarly for $\mathcal{N}$ and $\mathcal{M}^G$.

In the case of a finite constant group $G$, everything is much simpler because we have $\mathcal{K}om_S(G, \mathcal{M}) = \mathcal{M}^n$ as noticed earlier, and we do not need to make assumptions on $\mathcal{M}$.

**Example 4.7** The following example shows that the morphism $\mathcal{M}^G \rightarrow \mathcal{M}$ needs not be a monomorphism of algebraic stacks, although $\psi(\mathcal{M}^G) \rightarrow \mathcal{M}$ is necessarily a monomorphism of $G$-algebraic stacks, because of the 2-universal property. Let $\mathcal{M}_{g,2}$ be the stack of smooth 2-pointed curves of genus $g$ (see \[\text{I.3.2(iii)}\]). It has an action of the symmetric group $\mathcal{S}_2$. Let $(C, a, b)$ be a curve over a base $S$, and suppose that $C$ has two distinct automorphisms $\sigma_1$ and $\sigma_2$ which exchange the marked points. Then these give two morphisms $S \rightarrow (\mathcal{M}_{g,2})^{\mathcal{O}_S}$, and the compositions $S \rightarrow \mathcal{M}_{g,2}$ are equal as morphisms of algebraic stacks. However, they are not equal as morphisms of $\mathcal{S}_2$-algebraic stacks because the maps $\sigma_1, \sigma_2$ enter in the definition of such a morphism.

**Example 4.8** The following example shows that "fixed points" and "coarse moduli space" do not commute. Let $Q = \{ \pm 1, \pm i, \pm j, \pm k \}$ be the quaternion group, of order 8. Its unique involution generates its center $Z$, and $G = Q/Z \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is not isomorphic to a subgroup of $Q$. There is a faithful action of $G$ on $Q$ by conjugation, whence an action of $G$ on $BQ$ (see \[\text{I.3.2(i)}\]). For the trivial $Q$-torsor $x : Q \rightarrow S$, for all $g$ the left multiplication by $g^{-1}$ is an isomorphism $g.x \simeq x$, and however there is no $G$-linearization so $x$ is not a fixed point. Actually $(BQ)^G$ is empty, whereas the moduli space of $BQ$ is $S$ and we have $S^G = S$ for the induced action.

**Example 4.9** The following example shows that $\mathcal{M}^G$ may not be algebraic when $G$ is not proper. If $H$ is a commutative group scheme and $G$ a group scheme acting trivially on $BH$, then an objet of $(BH)^G$ is an $H$-torsor $x$ together with a morphism $G \rightarrow \text{Aut}(x) = H$, so $(BH)^G = BH \times \text{Hom}(G, H)$. This stack is not algebraic in general, though for special groups $G, H$ it may be the case (for instance if both $G, H$ are of multiplicative type — see \[\text{SGA3}, \text{tome 2} \text{ again} \].

### 4.2 Quotients

Let $G$ be a flat, separated group scheme of finite presentation over $S$. By a $G$-torsor over an $S$-scheme $T$, we will mean an algebraic space with $G$-action $p : E \rightarrow T$ that locally on $T$ is isomorphic to the trivial $G$-space $G \times T$. In general such a torsor will not be a scheme, unless if for example $G$ is quasi-affine.

Let $M$ be a $G$-algebraic stack over $S$. In case $M = X$ is an algebraic space, the quotient of $X$ is known under the more familiar denomination of the stack of $G$-torsors with an equivariant morphism to $X$. 
It is traditionally denoted \([X/G]\), to avoid confusion with a hypothetical quotient algebraic space, but when \(M\) is a general stack no such confusion is possible so it is natural to suppress the brackets.

For general \(M\) we can still define a stack whose objects are \(G\)-torsors \(p : E \to T\) with an equivariant morphism \((f, \sigma) : E \to M\). More precisely we define a stack \((M/G)^*\) whose sections over \(T\) are triples \(t = (p, f, \sigma)\) as above, and the isomorphisms between \(t\) and \(t'\) in \((M/G)^*\) are pairs \((u, \alpha)\) with a \(G\)-morphism \(u : E \to E'\) and a 2-commutative diagram of \(G\)-stacks (see \ref{fig:comm-diag}).

\[
\begin{array}{ccc}
E & \xrightarrow{u} & E' \\
\downarrow{(f, \sigma)} & & \downarrow{(f', \sigma')}
\end{array}
\]

**Theorem 4.10** Let \(G\) be a flat, separated group scheme of finite presentation over \(S\). Let \(M\) be a \(G\)-algebraic stack over \(S\). Then the quotient stack \(M/G\) (prop. \ref{prop:quotient-stack}) is isomorphic to the stack of \(G\)-torsors \((M/G)^*\), and it is algebraic (so it is a quotient stack in \(\mathfrak{AlgSt}\)). The canonical morphism \(\pi : M \to M/G\) is the universal torsor over \(M/G\). The formation of \(M/G\) commutes with base change on \(S\).

**Proof:** There are two things to show. First, we explain why \(M/G \simeq (M/G)^*\). Let \(M/G\) be the quotient as described in \ref{prop:quotient-stack} which is the stack associated to a prestack \(\mathcal{P}\). We define a morphism \(u : \mathcal{P} \to (M/G)^*\) by sending an object \(x \in M(T)\) viewed as a map \(x : T \to M\), to the trivial torsor together with the map \(G \times T \to M\) given by \(\mu \circ (\text{id} \times x)\), which is clearly equivariant. The image of a morphism \((g, \varphi) : x \to y\) in \(\mathcal{P}\) is the multiplication by \(g\) (as a map of torsors). This morphism \(u\) extends to a morphism of stacks \(u' : M/G \to (M/G)^*\). It is clearly fully faithful, and also locally essentially surjective by the definition of a torsor. So it is an isomorphism of stacks. From now on we identify \(M/G\) and \((M/G)^*\).

Second, we prove algebraicity. We keep the above notations of \(t = (p, f, \sigma)\) for sections of \(M/G\) and \(\varphi = (u, \alpha)\) for morphisms between \(t\) and \(t'\). Note that there is a morphism \(\omega : M \to BG\) obtained by forgetting the maps to \(M\). To study the diagonal of \(M/G\), we take \(t, t' \in (M/G)(T)\), then \(\omega\) induces a morphism \(\text{Isom}_\mathcal{T}(t, t') \to \text{Isom}_{BG}(E, E')\) given by \((u, \alpha) \mapsto u\). The latter space is algebraic, and the fibre of this projection above an isomorphism \(u : E \to E'\) is the closed (algebraic) subspace of \(\text{Isom}_{\text{alg}}(E, u^*E')\) of \(2\)-\(G\)-isomorphisms. This shows that \(\text{Isom}_\mathcal{T}(t, t')\) is representable, separated, quasi-compact. From the fact the morphism \(S \to BG\) is fppf, and the obvious 2-cartesian diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\pi} & S \\
\downarrow & & \downarrow \\
M/G & \xrightarrow{\text{pr}} & BG
\end{array}
\]

we deduce that \(M \to M/G\) is fppf, and by composition with an atlas of \(M\) we get an fppf presentation of \(M/G\), whence the result.

**Remark 4.11** It is clear that if \(M\) is representable, then the quotient \(M/G\) depends on if we compute it in the category of spaces or of stacks. Any algebraic space \(X\) with non-free action of a finite group \(G\) has a quotient space \(X/G\), distinct from the quotient stack.

**Example 4.12** Let \(M\) be a \(G\)-algebraic stack over \(S\), so we have morphisms \(G \times M \xrightarrow{\mu, \text{pr}_2} M\). Given a sheaf \(F\) on the smooth-étale site of \(M\), a \(G\)-linearization of \(F\) is an isomorphism \(\alpha : \mu^*F \simeq \text{pr}_2^*F\) which is compatible with associativity : \((m \times \text{id}_M)^*\alpha = (\text{id}_G \times \mu)^*\alpha\). We define a (smooth-étale) \(G\)-sheaf on \(M\) to be a pair \((F, \alpha)\) as above. We can look at the stack of invertible \(G\)-sheaves (with obvious isomorphisms of \(G\)-sheaves between them), denoted \(\mathcal{Pic}^G(M)\), and it is easy to see that we have canonical isomorphisms of stacks \(\mathcal{Pic}(M/G) \simeq \mathcal{Pic}^G(M) \simeq \mathcal{Pic}(M)^G\). In particular if \(\mathcal{Pic}(M)\) is algebraic and \(G\) is proper, flat, of finite presentation, we obtain algebraicity of the first two stacks, by theorem \ref{thm:alg-stack}. 

10
References

[ACV] D. Abramovich, A. Corti, A. Vistoli, *Twisted bundles and admissible covers*, electronic preprint [arXiv:math.AG/0106211]

[Ar] M. Artin, *Algebrization of formal moduli I*, Global Analysis, papers in honor of K. Kodaira, Spencer and Iyanaga Eds, Univ. Tokyo Press, 21-71 (1969).

[Be] J. Bertin, *Compactification des schémas de Hurwitz*, C. R. Acad. Sci. Paris, Sér. I 322, No.11, 1063-1066 (1996).

[EGA] A. Grothendieck, *Eléments de géométrie algébrique I, II, III, IV*, Publ. Math. IHÉS 4, 8, 11, 17, 20, 24, 28, 32 (1961-1967).

[E] T. Ekedahl, *Boundary behaviour of Hurwitz schemes*, in Resolution of singularities, Hauser et al. (ed.), tribute to Oscar Zariski, Obergurgl, Austria, 1997. Birkhäuser, PM 181, 285-298 (2000).

[Ho] S. Hollander, *A Homotopy Theory for Stacks*, electronic preprint [arXiv:math.AT/0110247]

[JKK] T. Jarvis, R. Kaufmann, T. Kimura, electronic preprint [arXiv:math.AG/0302316]

[Ko] M. Kontsevitch, *Enumeration of rational curves via torus actions*, Dijkgraaf et al. (ed.), The moduli space of curves, Texel Island, Netherlands, April 1994. Birkhäuser. PM 129, 335-368 (1995).

[L-MB] G. Laumon, L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 39. Springer (2000).

[OS] M. Olsson, J. Starr, *Quot Functors for Deligne-Mumford Stacks*, electronic preprint [arXiv:math.AG/0204307]

[SGA3] M. Demazure, A. Grothendieck, *Schémas en groupes* (SGA3), LNM 152. Springer-Verlag (1970).

[We] S. Wewers, *Construction of Hurwitz spaces*, thesis, Universität GH Essen, preprint 21 (1998).