Research Article

Instantaneous and Noninstantaneous Impulsive Integrodifferential Equations in Banach Spaces

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Received 12 February 2020; Revised 5 April 2020; Accepted 7 April 2020; Published 11 May 2020

Abstract and Applied Analysis

Hindawi

Volume 2020, Article ID 2690125, 8 pages

https://doi.org/10.1155/2020/2690125

1. Introduction

The existence of mild solutions is developed in [1, 2] for some semilinear functional differential equations. There has been a significant development in functional evolution equations in recent years (see the monographs [3–5], the papers [6–12], and the references therein).

The study of an abstract nonlocal Cauchy problem was initiated by Byszewski [13] in 1991. Evolution equations with nonlocal initial conditions were motivated by physical problems. As a matter of fact, it is demonstrated that the evolution equations with nonlocal initial conditions have better effects in applications than the classical Cauchy problems. For example, it was pointed in [14, 15] that the nonlocal problems are used to represent mathematical models for evolution of various phenomena, such as nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution, and nonlocal combustion. Due to nonlocal problems having a wide range of applications in real-world applications, evolution equations with nonlocal initial conditions were studied by many authors. Xue [16] studied the existence of mild solutions for semilinear differential equations with nonlocal initial conditions in separable Banach spaces. Xue discussed the semilinear nonlocal differential equations when the semigroup \( T(t) \) generated by the coefficient operator is compact and the nonlocal term \( g \) is not compact. Fan and Li [4] discussed the existence for impulsive semilinear differential equations with nonlocal conditions by using Sadovskii’s fixed point theorem and Schauder’s fixed point theorem.

Recently, several researchers obtained other results by application of the technique of measure of noncompactness (see [17–19] and the references therein).

Impulsive differential equations have become more important in recent years in some mathematical models of real phenomena, especially in biological or medical domains, and in control theory (see, for example, the monographs [20–22] and the papers [23–25]). In this paper, we first discuss the existence of mild solutions for the following nonlocal problem of impulsive integrodifferential equations

\[
\begin{align*}
\quad u'(t) &= Au(t) + \int_0^t Y(t-s)u(s)ds + f(t, u(t)); \quad t \in I_k, k = 0, \ldots, m,
\quad u(t_k^+) = u(t_k^-) + L_k(u(t_k^-)) ; \quad k = 1, \ldots, m,
\quad u(0) + g(u) &= u_0 \in E,
\end{align*}
\]

where \( I_0 = [0, t_1], I_k := (t_k, t_{k+1}], k = 1, \ldots, m, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T, f : I_k \times E \to E, L_k : E \to E, k = 1, \ldots, m, g : PC \to E \) are given functions, the set \( PC \) is given later, \( E \) is a real (or complex) Banach space with
norm \( \| \cdot \| \), \( u'(t) = du/dt \), \( A : D(A) \subset E \rightarrow E \) generates a \( C_0 \)-semigroup on the Banach space \( E \), and \( Y(t) \) is a closed linear operator on \( E \) with \( D(A) \subset D(Y) \).

In [26–29] the authors initially offered to study some classes of impulsive differential equations with noninstantaneous impulses. Motivated by the above papers, we next discuss the existence of mild solutions for the following nonlocal problem of noninstantaneous impulsive integrodifferential equations:

\[
\begin{align*}
\begin{cases}
&u'(t) = Au(t) + \int_0^1 Y(t-s)u(s)ds + f(t, u(t)); \quad t \in I_k, \ k = 0, \cdots, m, \\
&u(t) = g_k(t, u(t_k^+)); \quad t \in I_k, \ k = 1, \cdots, m, \\
&u(s_k) + g(u) = u_k \in E; \quad k = 0, \cdots, m,
\end{cases}
\end{align*}
\]

(2)

where \( I_0 = [0, t_1], J_k = (t_k, s_k], I_k = (s_k, t_{k+1}]; k = 1, \cdots, m, f : I_k \times E \rightarrow E, g_k : I_k \times E \rightarrow E \) are given functions such that \( g_k(t, u(t_k^+)))_{t \in [0, T]} = u_k \in E; k = 1, \cdots, m, g : \mathcal{P}E \rightarrow E \) is a given function, the set \( \mathcal{P}E \) is given later, and \( 0 = s_0 < t_1 \leq s_1 < t_2 < \cdots < s_{m-1} < t_m = T \).

2. Preliminaries

By \( B(E) \), we denote the space of the bounded linear operator from \( E \) into itself. Let \( C(I) = C(I, E) \) be the Banach space of continuous functions from \( I = [0, T] \) into \( E \). Let \( L^\infty(I) \) be the Banach space of measurable functions \( v : I \rightarrow \mathbb{R} \) that are essentially bounded and equipped with the norm

\[
\|v\|_{L^\infty} = \inf \{ c > 0 : |v(t)| \leq c, \ a.e. t \in I \}.
\]

Consider the Banach space

\[
\mathcal{PC} = \{ u : I \rightarrow E : u \in C(I_k); \ k = 0, \cdots, m, \text{ and there exist } u(t_k^+) \text{ and } u(t_k^+) \text{; } k = 1, \cdots, m, \text{ with } u(t_k^+) = u(t_k) \}
\]

with the norm

\[
\|u\|_{\mathcal{PC}} = \sup_{t \in I} \|u(t)\|.
\]

A semigroup of bounded linear operators \( T(t) \) is uniformly continuous if

\[
\lim_{t \to 0} \|T(t) - I\|_E = 0.
\]

Here, \( I \) denotes the identity operator in \( E \).

We note that if a semigroup \( T(t) \) is of class \( (C_0) \), then it satisfies the growth condition

\[
\|T(t)\|_{B(E)} \leq Me^{\beta t}, \quad 0 \leq t < \infty
\]

with some constants \( M > 0 \) and \( \beta \geq 0 \).

If, in particular, \( M = 1 \) and \( \beta = 0 \), i.e., \( \|T(t)\|_{B(E)} \leq 1 \), for \( t \geq 0 \), then the semigroup \( T(t) \) is called a contraction semigroup. For more details on strongly continuous operators, we refer the reader to the books [30, 31].

Let \( \mathcal{M}_X \) denote the class of all bounded subsets of a metric space \( X \).

Definition 1. Let \( X \) be a complete metric space. A map \( \mu : \mathcal{M}_X \rightarrow [0, \infty) \) is called a measure of noncompactness on \( X \) if it satisfies the following properties for all \( B, B_1, B_2 \in \mathcal{M}_X \):

(a) \( \mu(B) = 0 \) if and only if \( B \) is precompact (regularity)

(b) \( \mu(B) = \mu(B) \) (invariance under closure)

(c) \( \mu(B_1 \cup B_2) = \max \{ \mu(B_1), \mu(B_2) \} \) (semiadditivity)

Definition 2 [32]. Let \( X \) be a Banach space and let \( \Theta X \) be the family of bounded subsets of \( X \). The Kuratowski measure of noncompactness is the map \( \mu : \Theta X \rightarrow [0, \infty) \) defined by

\[
\mu(M) = \inf \{ \varepsilon > 0 : M \subset \bigcup_{j=1}^{n} M_j, \text{ diam}(M_j) \leq \varepsilon \},
\]

where \( M \in \Theta X \).

For our purpose, we will need the following fixed point theorem:

Theorem 3. (Monch’s fixed point theorem [33]). Let \( D \) be a bounded, closed, and convex subset of a Banach space such that \( 0 \in D \) and let \( \mu : D \rightarrow D \) be a continuous mapping of \( D \) into itself. If the implication

\[
V = \text{conv}(N(V)) \quad \text{or} \quad V = N(V) \cup \{0\} \Rightarrow V \text{ is compact},
\]

holds for every subset \( V \) of \( D \), then \( N \) has a fixed point.

3. Mild Solutions with Instantaneous Impulses

In this section, we are concerned with the existence results of the problem (1).

Definition 4 [10]. A resolvent operator for the Cauchy problem

\[
\begin{align*}
\begin{cases}
&u'(t) = Au(t) + \int_0^t Y(t-s)u(s)ds; \quad t \in [0, \infty), \\
&u(0) = u_0 \in E,
\end{cases}
\end{align*}
\]

(9)
is a bounded linear operator-valued function \( R(t) \in B(E) \); \( t \geq 0 \), verifying the following conditions:

(i) \( R(0) = I \) (the identity map of \( E \)) and \( \| R(t) \| \leq N e^{\alpha t} \) for some constants \( N > 0 \), and \( \nu \in \mathbb{R} \)

(ii) For each \( u \in E \), \( R(t)u \) is strongly continuous for \( t \geq 0 \)

(iii) \( R(t) \) is bounded for \( t \geq 0 \). For \( u \in D(A) \), \( R(t)u \in C(\mathbb{R}^+, D(A)) \) \( \cap C^1(\mathbb{R}^+, E) \) and

\[
R'(t)u = AR(t)u + \int_0^t Y(t-s)R(s)uds = R(t)Au + \int_0^t R(t-s)Y(s)uds; t \in [0, \infty)
\]

Let us introduce the following hypotheses:

\( (R_1) \) The operator \( A \) is the infinitesimal generator of a uniformly continuous semigroup \( (S(t))_{t \geq 0} \)

\( (R_2) \) For all \( t \geq 0 \), \( Y(t) \) is a closed linear operator from \( D(A) \) to \( E \) and \( Y(t) \in B(E) \). For any \( u \in E \), the map \( t \mapsto Y(t)u \) is bounded differentiable and the derivative \( t \mapsto Y'(t)u \) is bounded uniformly continuous on \( \mathbb{R}^+ \).

**Theorem 5** [10, 34]. Assume that \( (R_1) \) and \( (R_2) \) hold. Then, there exists a unique uniformly continuous resolvent operator for the Cauchy problem (9).

**Definition 6** [34]. By a mild solution of the problem (1), we mean a function \( u \in PC \) that satisfies

\[
u(t) = R(t)[u_0 - g(u)] + \int_0^t R(t-s)f(s, u(s))ds + \sum_{0 \leq t_i < t} R(t-t_i)L_i(u(t_i)); t \in I.
\]

The following hypotheses will be used in the sequel.

\( H_1 \) The function \( t \mapsto f(t, u) \) is measurable on \( I \) for each \( u \in E \), and the function \( u \mapsto f(t, u) \) is continuous on \( E \) for a.e. \( t \in I \).

\( H_2 \) There exists a function \( p \in L^\infty(I) \), such that

\[
\| f(t, u) \| \leq p(t)(1 + \| u \|); \quad \text{for a.e.t} \in I, \text{and each } u \in E
\]

\( H_3 \) There exist positive constants \( q^*, l_k^*; k = 0, \cdots, m \), such that

\[
\| g(u) \| \leq q^*(1 + \| u \|)_{PC}; \quad \text{for each } u \in PC
\]

and

\[
l_k^*(1 + \| u \|); \quad k = 0, \cdots, m, \text{ for a.e.t} \in I, \text{and each } u \in E,
\]

\( H_4 \) For each bounded set \( B \subset E \), we have

\[
\mu(f(t, B)) \leq p(t)\mu(B), \mu(L_k(B)) \leq l_k^*\mu(B); \quad k = 0, \cdots, m,
\]

and for each bounded set \( B_1 \subset PC \), we have

\[
\mu(g(B_1)) \leq q^* \sup_{t \in I} \mu(B_1(t)),
\]

where \( B_1(t) = \{ u(t); u \in B_1 \}; t \in I \).

Set

\[
p^* = \| p \|_{L^\infty}, \text{ and } M = \sup_{t \in I} \| R(t) \|_{B(E)}.
\]

**Theorem 7.** Assume that the hypotheses \( (R_1), (R_2), (H_1) \) \((H_4)\) hold. If

\[
\ell := M\left(q^* + Tp^* + \sum_{k=0}^m l_k^*\right) < 1,
\]

then problem (1) has at least one mild solution defined on \( I \).

**Proof.** Transform problem (1) into a fixed point problem. Consider the operator \( N : PC \longrightarrow PC \) defined by

\[
(Nu)(t) = R(t)[u_0 - g(u)] + \int_0^t R(t-s)f(s, u(s))ds + \sum_{0 \leq t_i < t} R(t-t_i)L_i(u(t_i)); \quad t \in I.
\]

Let \( \rho > 0 \), such that

\[
\rho \geq \frac{M(\| u_0 \| + q^* + Tp^* + \sum_{k=0}^m l_k^*)}{1 - M(q^* + Tp^* + \sum_{k=0}^m l_k^*)},
\]

and consider the ball \( B_\rho = B(0, \rho) = \{ u \in PC : \| u \|_{PC} \leq \rho \} \).

For any \( u \in B_\rho \) and each \( t \in I \), we have

\[
\| (Nu)(t) \| \leq \| R(t) \|_{B(E)}\| u_0 \| + \| g(u) \| + \int_0^t \| R(t-s) \|_{B(E)}\| f(s, u(s)) \| ds
\]

\[
+ \sum_{0 \leq t_i < t} \| R(t-t_i) \|_{B(E)}\| L_i(u(t_i)) \|
\]

\[
\leq M\left[\| u_0 \| + (1 + \rho)\left(q^* + Tp^* + \sum_{k=0}^m l_k^*\right)\right] \leq \rho.
\]

Thus,

\[
\| N(u) \|_{PC} \leq \rho.
\]

This proves that \( N \) transforms the ball \( B_\rho \) into itself.

We shall show that the operator \( N : B_\rho \longrightarrow B_\rho \) satisfies all the assumptions of Theorem 3. The proof will be given in three steps.
Step 1. \( N : B_p \rightarrow B_p \) is continuous.

Let \( \{ u_n \}_{n \in \mathbb{N}} \) be a sequence such that \( u_n \rightarrow u \) as \( n \rightarrow \infty \) in \( B_p \). Then, for each \( t \in I \), we have

\[
\| (Nu_n)(t) - (Nu)(t) \| \leq \| R(t) - R(\tau) \|_{B(E)} (\| u_n \| + \| g(u_n) \|) \\
+ \int_0^\tau \| R(t-s) - R(\tau - s) \|_{B(E)} \| f(s, u(s)) \| ds \\
+ \int_\tau^T \| R(t-s) \|_{B(E)} \| f(s, u(s)) \| ds \\
+ \sum_{0 < t_i < t} \| R(t-t_i) - R(\tau - t_i) \|_{B(E)} \| L_i(u(t_i)) \|.
\]

(22)

Since \( u_n \rightarrow u \) as \( n \rightarrow \infty \) and \( f, g, L_i \) are continuous, the Lebesgue-dominated convergence theorem implies that

\[
\| N(u_n) - N(u) \|_{pc} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

(23)

Step 2. \( N(B_p) \) is bounded and equicontinuous.

Since \( N(B_p) \subset B_p \) and \( B_p \) is bounded, then \( N(B_p) \) is bounded.

Next, let \( t, \tau \in I, \tau < t \) and let \( u \in B_p \). Thus, we have

\[
\| (Nu)(t) - (Nu)(\tau) \| \leq \| R(t) - R(\tau) \|_{B(E)} (\| u_0 \| + \| g(u) \|) \\
+ \int_0^\tau \| R(t-s) - R(\tau - s) \|_{B(E)} \| f(s, u(s)) \| ds \\
+ \int_\tau^T \| R(t-s) \|_{B(E)} \| f(s, u(s)) \| ds \\
+ \sum_{0 < t_i < t} \| R(t-t_i) - R(\tau - t_i) \|_{B(E)} \| L_i(u(t_i)) \|.
\]

(24)

Hence, we get

\[
\| (Nu)(t) - (Nu)(\tau) \| \leq (\| u_0 \| + q^* (1 + \rho)) \| R(t) - R(\tau) \|_{B(E)} \\
+ p^* (1 + \rho) \int_0^\tau \| R(t-s) - R(\tau - s) \|_{B(E)} ds \\
+ M p^* (1 + \rho) (t-\tau) + \sum_{0 < t_i < t} I_i^* (1 + \rho) \| R(t-t_i) \|_{B(E)}.
\]

(25)

As the resolvent operator \( R(\cdot) \) is uniformly continuous, the right-hand side of the above inequality tends to zero as \( \tau \rightarrow t \).

Step 3. The implication (8) holds.

Now let \( V \) be a subset of \( B_p \) such that \( V \subset N(V) \cup \{ 0 \} \). \( V \) is bounded and equicontinuous, and therefore, the function \( t \mapsto v(t) = \mu(V(t)) \) is continuous on \( I \). By \( (H_3) \) and the properties of the measure \( \mu \), for each \( t \in I \), we have

\[
v(t) \leq \mu((NV(t) \cup \{ 0 \}) \leq \mu((NV(t)) \\
\leq \| R(t) \|_{B(E)} q^* \sup \{ V(t) \} + \int_0^T \| R(t-s) \|_{B(E)} \| p(s) \| \| V(s) \| ds \\
+ \sum_{k=0}^{m} \| R(t-t_k) \|_{B(E)} l_k(t) \| H(t) \| \\
\leq M q^* \| \nu \|_{\infty} + M p^* \int_0^T \| v(s) \| ds + M \sum_{k=0}^{m} l_k^* \| v(t) \| \\
\leq M \left( q^* + T p^* + \sum_{k=0}^{m} l_k^* \right) \| \nu \|_{\infty}.
\]

(26)

Hence,

\[
\| \nu \|_{\infty} \leq 0 \| \nu \|_{\infty}.
\]

(27)

From (17), we get \( \| \nu \|_{\infty} = 0 \), that is, \( v(t) = \mu(V(t)) = 0 \), for each \( t \in I \), and then \( V(t) \) is relatively compact in \( E \). In view of the Ascoli-Arzelà theorem, \( V \) is relatively compact in \( B_p \). Applying now Theorem 3, we conclude that \( N \) has a fixed point which is a mild solution of our problem (1).

4. Mild Solutions with Noninstantaneous Impulses

In this section, we are concerned with the existence results of problem (2). Denote by

\[
PC = \{ y : I \rightarrow E : y \in C([0, t_1] \cup \{ t_1, s_2 \} \cup \{ s_2, t_2 \}, E) \}, k = 1, \ldots, m,
\]

and there exist \( y(t^*_k), y(t^*_k'), y(s^*_k), \) and \( y(s^*_k) \), \( k = 1, \ldots, m \) with \( y(t^*_k) = y(t^*_k) \) and \( y(s^*_k) = y(s^*_k) \), the Banach space equipped with the standard supremum norm.

Definition 8 [34]. By a mild solution of problem (2), we mean a function \( u \in PC \) that satisfies

\[
\begin{cases}
  u(t) = R(t)[u_0 - g(u)] + \int_0^t R(t-s)f(s, u(s))ds; & t \in I, k = 0, \ldots, m, \\
  u(t) = g_k(t, u(t^*_k)); & t \in J_k, k = 1, \ldots, m.
\end{cases}
\]

(29)

The following hypotheses will be used in the sequel:

\( H_{01} \) The functions \( t \mapsto f(t, u) \) and \( t \mapsto g_k(t, u) \) are measurable on \( I_k \) and \( J_k \), respectively, for each \( u \in E \), and the functions \( u \mapsto f(t, u) \) and \( u \mapsto g_k(t, u) \) are continuous on \( E \) for a.e. \( t \in I_k \) and \( J_k \), respectively.

\( H_{02} \) There exist functions \( p, l_k \in L^{\infty}(I) \); \( k = 0, \ldots, m \), such that

\[
\| f(t, u) \| \leq p(t)(1 + \| u \|); \quad \text{for a.e.} \ t \in I_k, \ \text{and each} \ u \in E,
\]

(30)
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\[\|g_k(t,u)\| \leq L_k(t)(1+\|u\|) \quad ; \quad k=1,\ldots,m, \text{ for a.e.t } I_k, \text{ and each } u \in E. \quad (31)\]

There exists a positive constant \( q^* \), such that
\[\|g(u)\| \leq q^* (1+\|u\|_{\mathcal{P}}) \quad ; \quad \text{ for a.e.t } I, \text{ and each } u \in \mathcal{P}. \quad (32)\]

For each bounded set \( B \subset E \) and for each \( t \in I \), we have
\[\mu(f(t,B)) \leq p(t)\mu(B), \quad \mu(g_k(t,B)) \leq L_k(t)\mu(B); \quad k=0,\ldots,m \quad (33)\]

and for each bounded set \( B_0 \subset \mathcal{P} \), we have
\[\mu(g(B_0)) \leq q^* \sup_{t \in I} \mu(B_0(t)), \quad (34)\]

where \( B_0(t) = \{ u(t) : u \in B_0 \} ; t \in I. \)

Set
\[p^* = \|p\|_{L^\infty}, \quad l^* = \max_{k=0,\ldots,m} \|l_k\|_{L^\infty}, \quad M = \sup_{t \in I} \|R(t)\|_{\mathcal{B}(E)}. \quad (35)\]

**Theorem 9.** Assume that the hypotheses \((R_1),(R_2),(H_{01})\) \(-\quad (H_{01})\) hold. If
\[\nu = \max\{l^*, M(q^*+Tp^*)\} < 1, \quad (36)\]

then problem (2) has at least one mild solution defined on \( I. \)

**Proof.** Transform problem (2) into a fixed point problem. Consider the operator \( N : \mathcal{P} \rightarrow \mathcal{P} \) defined by
\[
\begin{cases}
(Nu)(t) = R(t)(u_k - g(u)) \quad + \int_0^t R(t-s)f(s,u(s))ds; \quad t \in I_k, k=0,\ldots,m; \\
(Nu)(t) = g_k(t,u(t_k)) \quad ; \quad t \in I_k, k=1,\ldots,m.
\end{cases}
\]

Let \( L > 0 \), such that
\[L \geq \frac{M\|u_k\|+q^*+Tp^*}{1-M(q^*+Tp^*)}. \quad (38)\]

For any \( u \in \mathcal{P} \) and each \( t \in I_k \), we have
\[
\|Nu(t)\| \leq \|R(t)\|_{\mathcal{B}(E)}\|u_k\|+\|g(u)\| \\
+ \int_0^t \|R(t-s)\|_{\mathcal{B}(E)}\|f(s,u(s))\|ds \\
\leq M\|u_k\|+(1+L)(q^*+Tp^*) \leq L.
\]

Thus,
\[\|N(u)\|_{\mathcal{P}} \leq L. \quad (40)\]

Next, for each \( t \in I_k \); \( k=1,\ldots,m \), it is clear that
\[\|(Nu)(t)\|_E \leq l^*. \quad (41)\]

Hence,
\[\|N(u)\|_{\mathcal{P}} \leq \max\{L,l^*\} = \rho. \quad (42)\]

This proves that \( N \) transforms the ball \( B_\rho := \{ u \in \mathcal{P} : \|u\|_{\mathcal{P}} \leq \rho \} \) into itself.

We shall show that the operator \( N : B_\rho \rightarrow B_\rho \) satisfies all the assumptions of Theorem 3. The proof will be given in three steps.

**Step 1.** \( N : B_\rho \rightarrow B_\rho \) is continuous.

Let \( \{u_n\}_{n \in \mathbb{N}} \) be a sequence such that \( u_n \rightharpoonup u \) as \( n \rightarrow \infty \) in \( B_\rho \). Then, for each \( t \in I_k \); \( k=1,\ldots,m \), we have
\[
\|Nu_n(t) - (Nu)(t)\| \leq \|R(t)\|_{\mathcal{B}(E)}\|g_k(t,u_n(t_k)) - g_k(t,u(t_k))\|,
\]

and for each \( t \in I_k \); \( k=0,\ldots,m \), we have
\[
\|Nu_n(t) - (Nu)(t)\| \leq \|R(t)\|_{\mathcal{B}(E)}\|g_k(t,u_n(t_k)) - g_k(t,u(t_k))\|,
\]

Since \( u_n \rightharpoonup u \) as \( n \rightarrow \infty \) and \( f, g, g_k \) are continuous, the Lebesgue-dominated convergence theorem implies that
\[\|N(u_n) - N(u)\|_{\mathcal{P}} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (45)\]

**Step 2.** \( N(B_\rho) \) is bounded and equicontinuous.

Since \( N(B_\rho) \subset B_\rho \) and \( B_\rho \) is bounded, then \( N(B_\rho) \) is bounded.

Next, let \( t, \tau \in I_k, \tau < t \) and let \( u \in B_\rho \). Thus, we have
\[
\|Nu(t) - (Nu)(\tau)\| \leq \|R(t) - R(\tau)\|_{\mathcal{B}(E)}\|u_k\|+\|g(u)\| \\
+ \int_0^\tau \|R(t-s) - R(\tau-s)\|_{\mathcal{B}(E)}\|f(s,u(s))\|ds \\
+ \int_\tau^t \|R(t-s)-R(t-s)\|_{\mathcal{B}(E)}\|f(s,u(s))\|ds.
\]

Hence, we get
\[
\|Nu(t) - (Nu)(\tau)\| \leq (\|u_k\|+q^*(1+\rho))\|R(t) - R(\tau)\|_{\mathcal{B}(E)} \\
+ p^*(1+\rho)\int_0^\tau \|R(t-s) - R(\tau-s)\|_{\mathcal{B}(E)}ds.
\]

(47)
As \( t \to \tau \), the right-hand side of the above inequality tends to zero.

**Step 3. The implication (8) holds.**

Now let \( V \) be a subset of \( B_\rho \) such that \( V \subset N(V) \cup \{0\} \). \( V \) is bounded and equicontinuous, and therefore, the function \( t \to v(t) = \mu(V(t)) \) is continuous on \( I \). By \((H_{03})\) and the properties of the measure \( \mu \), for each \( t \in I_k \), we have

\[
v(t) \leq \mu((NV)(t) \cup \{0\}) \leq \mu((NV)(t)) \leq \eta^* \|v\|_{\infty} \leq \ell \|v\|_{\infty}.
\]

Thus, for each \( t \in I \), we get

\[
v(t) \leq \ell \|v\|_{\infty},
\]

\[\text{by the Ascoli-Arzelà theorem, } V \text{ is relatively compact in } E. \]

Applying now Theorem 3, we conclude that the point which is a mild solution of problem (2).

**5. Examples**

Let

\[
H = L^2([0, \pi]) = \left\{ u : [0, \pi] \longrightarrow \mathbb{R} : \int_0^\pi |u(x)|^2 \, dx < \infty \right\},
\]

be the Hilbert space with the scalar product \( \langle u, v \rangle = \int_0^\pi u(x)v(x) \, dx \). It is known that \( H \) is a Banach space with the norm

\[
\|u\|_2 = \left( \int_0^\pi |u(x)|^2 \, dx \right)^{1/2}.
\]

**Example 1.** Consider the following problem of impulsive integro-differential equations

\[
\begin{cases}
\frac{\partial}{\partial t} z(t, x) = \frac{\partial^2}{\partial x^2} z(t, x) + Q(t, z(t, x)) + \int_0^t \int_0^s (b(t-s) \frac{\partial^2}{\partial x^2} z(s, x)) \, ds \, ds,
\end{cases}
\]

where \( t \in [0, 2], PC := PC([0, 2], H), \)

\[
Q(t, z(t, x)) = \frac{e^z}{1 + \|z\|_2^2} (1 + z(t, x)); \quad t \in [0, 1] \cup (1, 2],
\]

\[
L_1(z(1^+), x) = \frac{z(1^+, x)}{3e^z(1 + \|z(1^+), x\|_2)},
\]

and

\[
g(z) = \int_0^\tau K(x, y) \frac{e^{-y}}{1 + \|z\|_{PC}} \, dy,
\]

where \( \int_0^\pi K(x, y) \, dy < \infty. \)

We define the strongly elliptic operator \( A : D(A) \subset H \longrightarrow H \) by

\[
Au = \mathcal{A}(x, D)u = \sum_{|\mu| \leq 2m} a_{\mu}(x)D^{\mu}u,
\]

where \( a_{\mu} \in C^{2m}([0, \pi]) \) and \( D(A) = H^{2m}([0, \pi]) \cap H^m_0([0, \pi]). \)

It is well known (see [31]) that \( A \) generates a uniformly continuous semigroup \( T(t) \); \( t \geq 0 \) in the Hilbert space \( H \).

For \( x \in [0, \pi], \) we have

\[
\begin{align*}
\phi(t)(x) &= \mathcal{A}(x, D)u \quad t \in [0, 1] \cup (1, 2],
\end{align*}
\]

\[
f(t, u(t))(x) = Q(t, z(t, x)); \quad t \in [0, 1] \cup (1, 2],
\]

\[
Y(t) = b(t)A,
\]

\[
u_0(x) = 1 + x^2; \quad x \in [0, \pi].
\]
Thus, under the above definitions of $f$, $u_0$, and $A$, system (54) can be represented by problem (1). Furthermore, more appropriate conditions on $Q$ ensure the hypotheses $(R_1)$, $(R_2)$, $(H_1)$ – $(H_5)$. Consequently, Theorem 7 implies that problem (54) has at least one mild solution on $[0, 2]$.

**Example 2.** Consider now the following problem of impulsive integrodifferential equations

\[
\begin{aligned}
\frac{\partial}{\partial t} z(t, x) &= \frac{\partial^2}{\partial x^2} z(t, x) + Q(t, z(t, x)) + \int_0^t b(t - s) \frac{\partial^2}{\partial x^2} z(s, x) ds; \quad t \in [0, 1] \cup (2, 3], x \in [0, \pi], \\
z(t, x) &= g_1(t, z(1^-, x)); \quad t \in (1, 2], x \in [0, \pi], \\
z(t, 0) &= z(t, \pi) = 0; \quad t \in [0, 1] \cup (2, 3], \\
z(0, x) + g(z) &= 1 + e^x; \quad x \in [0, \pi], \\
z(2, x) + g(z) &= 2 + e^x; \quad x \in [0, \pi],
\end{aligned}
\]

where $t \in [0, 3]$, $PC := PC([0, 3], H)$,

\[
Q(t, z(t, x)) = \frac{ct^2}{1 + \|z\|_2^2} \left( e^z + \frac{1}{e^{x(z) + 5}} \right) (1 + z(t, x)); \quad t \in [0, 1] \cup (2, 3],
\]

\[
g_1(t, z(1^-, x)) = \frac{z(1^-, x)}{(3x^2)(1 + \|z(1^-)\|_2^2)}; \quad t \in (1, 2], x \in [0, \pi],
\]

\[
g(z) = \int_0^\pi K(x, y) \frac{e^{-y}}{1 + \|z\|_{\infty}} dy,
\]

with $\int_0^\pi K(x, y) dxdy < \infty$.

Again, as the above example, simple computations show that all conditions of Theorem 9 are satisfied. It follows that problem (59) has at least one mild solution on $[0, 3]$.

**Data Availability**

There is no data used in this work.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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