Approximate Path Integral Solution for a Dirac Particle in a Deformed Hulthén Potential1

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Abstract—The problem of a Dirac particle moving in a deformed Hulthén potential is solved in the framework of the path integral formalism. With the help of the Biedenharn transformation, the construction of a closed form for the Green’s function of the second-order Dirac equation is done by using a proper approximation to the centrifugal term and the Green’s function of the linear Dirac equation is calculated. The energy spectrum for the bound states is obtained from the poles of the Green’s function. A Dirac particle in the standard Hulthén potential ($q = 1$) and a Dirac hydrogen-like ion ($q = 1$ and $a \to \infty$) are considered as particular cases.

Keywords: Hulthén potential, Rosen–Morse potential, Green’s function, path integral, bound states

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1. INTRODUCTION

The Hulthén potential has been widely used as an approximation of the interaction potential between two bodies in a number areas in physics, including nuclear and particle physics [1], atomic physics [2] and molecular physics [3] and also in quantum chemistry [4]. It is one of the important class of exponential-type potentials that behave like a Coulomb potential for small values of $r$ and decrease exponentially for large values of $r$. There are many studies in which different approaches such as the standard method based on the Schrödinger equation [5, 6], the supersymmetric quantum mechanics technique [7, 8] and the asymptotic iteration method [9] have been employed in obtaining exact or approximate solutions of the wave equation with the Hulthén potential for the s- and l-waves in the framework of non-relativistic quantum mechanics. Furthermore, a path integral treatment of this potential has been given in [10]. Nevertheless, in the presence of a strong fields, the relativistic effects on a particle under the influence of this potential could become notable. The s-wave Dirac equation with a special Hulthén potential has been claimed exactly solved by using a constraint [11, 12]. Subsequently the bound and scattering solutions of the K-wave Dirac equation with the standard Hulthén potential have been obtained in [13] by adopting a suitable approximation of the centrifugal potential term and through a technique similar to Biedenharn’s [14].

The purpose of the present work is to solve approximately the problem of a relativistic particle of mass $\mu$, charge $e$ and spin $\frac{1}{2}$ moving in a deformed Hulthén potential of the form

$$V_q(r) = -\frac{V_0}{e^{r/a} - q},$$

where $V_0$ describes the depth of the potential well, $a$ is the range of the potential and $q$ is a deformation parameter ($q \geq 1$). The introduction of the parameter $q$ can serve as an additional parameter in describing inter-atomic interactions, and especially in three-dimensional problem, it allows to establish the center of mass location of a molecule at a certain distance from the coordinate origin. In addition the potential $V_q(r)$ contains the standard Hulthén potential ($q = 1$) and the Coulomb potential ($a \to \infty$, $q = 1$) as special cases.

Here, we want to show firstly that even the problem of s-waves for a Dirac particle can be solved only approximately contrary to what was claimed by some authors [11, 12] and secondly, as to our knowledge, there is no path integral discussion for the relativistic Hulthén potential, treating the relativistic deformed Hulthén potential (1) by path integration, we wish to enlarge the list of problems of relativistic particles with spin $\frac{1}{2}$ in external electric or magnetic fields [15–18] which have been studied in the framework of path integrals.
The organisation of the paper is as follows: in section 2, we first use an approximation scheme of exponential type depending on the \( q \)-deformation parameter instead the centrifugal potential term and we diagonalize the effective Hamiltonian of the iterated form of the Dirac equation with the help of the Biedenharn transformation [14]. Then, the Green’s function for the second order Dirac equation can be expanded into partial waves in spherical coordinates. In section 3, the radial part of the Green’s function is converted into a path integral for the \( q \)-deformed Rosen–Morse potential by using the generalized Duru–Kleinert method [22]. The procedure yields a closed form for the radial Green’s function. In section 4, we construct the Green’s function of the linear Dirac equation and we obtain the energy spectrum for the \( \kappa \) states. In section 5, the Green’s function and the energy spectrum associated with the standard Hulthén potential and for the Coulomb potential are presented as special cases. The section 6 will be a conclusion.

2. DIRAC EQUATION WITH A DEFORMED HULTHÉN POTENTIAL

Choosing the atomic units \( \hbar = c = 1 \), the Green’s function \( G(r'', r') \) for the vector potential \( V_q(r) \) satisfies the Dirac equation

\[
(\mu - M)G(r'', r') = \delta(r'' - r'),
\]

(2)

where \( \mu \) is the mass of a charged particle of spin \( \frac{1}{2} \), and

\[
M = -\beta \alpha P + \beta (E - V_q(r)),
\]

(3)

in which \( \alpha \) and \( \beta \) are the usual Dirac matrices and \( E \) is the energy.

The Dirac Eq. (2) can be iterated by writing the Green’s function \( G(r'', r') \) as

\[
G(r'', r') = (\mu + M)g(r'', r'),
\]

(4)

where \( g(r'', r') \) is the Green’s function defined as the solution of the second-order Dirac equation

\[
(\mu^2 - M^2)g(r'', r') = \delta(r'' - r').
\]

(5)

Using the Schwinger integral representation [19], the solution of the equation (5) can be written as follows:

\[
g(r'', r') = \frac{1}{2\pi} \int_0^\infty d\Lambda \langle r'' \rangle \exp(-i\Lambda a^2(\epsilon - q/\sqrt{a^2(e^\epsilon q - a^2)})),
\]

(6)

where the integrand \( \langle r'' \rangle \exp(-i\Lambda a^2(\epsilon - q/\sqrt{a^2(e^\epsilon q - a^2)}) \) is similar to the propagator of a quantum system evolving in \( \Lambda \) time from \( r' \) to \( r'' \) with the effective Hamiltonian

\[
H = \frac{1}{2}(\mu^2 - M^2).
\]

(7)

Since the potential (1) has spherical symmetry, these are the polar coordinates which are best suited to find the explicit expression of the Green’s function (4). Before approaching the construction of the Green’s function (6), we note that we can make some simplifications. Using the radial momentum operator

\[
P_r = \frac{1}{r}(rP_r - i),
\]

and the velocity operator \( \alpha_r = \frac{\alpha r}{r} \), the operator \( M \) takes the form

\[
M = -\beta \alpha_r P_r + i\alpha_r \frac{K}{r} + \beta(E - V_q(r)),
\]

(8)

and the effective Hamiltonian (7) then becomes

\[
H = \frac{1}{2}\left(P_r^2 + \frac{K^2}{r^2} - \beta K - i\alpha_r \frac{dV_q(r)}{dr} - (E - V_q(r))^2 + \mu^2\right).
\]

(9)

Note that \( \beta \) has eigenvalues \( \hat{\beta} = \pm 1 \) and

\[
K^2 = J^2 + \frac{1}{4},
\]

where \( J = L + \frac{1}{2} \) is the total angular momentum operator. The eigenvalues of the operator \( J^2 \) are \( j(j + 1) \), where \( j = l \pm \frac{1}{2} \) except for the s-states \( (l = 0) \), in which \( j \) can take only the value \( \frac{1}{2} \). Thus, the operator \( K^2 \) has eigenvalues \( \kappa^2 = \left(j + \frac{1}{2}\right)^2 \), so that

\[
\kappa = \pm \left(j + \frac{1}{2}\right) = \pm 1, \pm 2, \pm 3, \ldots.
\]

On the other hand, the Hulthén-type potential in (9) depends on the arbitrary real deformation parameter \( q \) and the Green’s function (6) cannot be evaluated exactly because of the presence of the centrifugal potential term. However, if \( r/a \ll 1 \), it is easy to show that

\[
\frac{qe^{r/2a}}{a^2(e^{q/2a} - q)} + \frac{1}{12a^2}
\]

can be used as a good approximation to \( \frac{1}{r^2} \) in the centrifugal term only when the parameter \( q \geq 1 \). This can be seen by performing a Taylor’s expansion of \( \frac{qe^{r/2a}}{a^2(e^{q/2a} - q)} \), or immediately, by plotting \( \frac{1}{r^2} \) and \( \frac{qe^{r/2a}}{a^2(e^{q/2a} - q)} \) as functions of \( r/a \) for typical values of \( q \) in figure. The coordinate \( r \) is scaled in units of \( a \). This figure shows that these expressions have similar behavior in the interval \( 0, +\infty \) when \( q \geq 1 \). For \( q < 1 \), the curves cut the \( y \)-axis. Then, by means of this approximation, the effective Hamiltonian (9) can be written in the form:
\[ H = \frac{1}{2} \left( \mathbf{P}^2 + \Gamma (\Gamma + 1) \frac{q e^{r/a}}{a^2 (e^{r/a} - q)^2} + V_0 \left( \frac{V_0 - 2E}{q} \right) + \frac{1}{12a^2} \mathbf{K} (\mathbf{K} - \mathbf{\beta}) + \mu^2 - E^2 \right), \]  

where

\[ \Gamma = -\left( \beta \mathbf{K} + \frac{i a V_0}{q} \right) \alpha_r \]  

is the Martin–Glauber operator \([20, 21]\). In analogy with the work of Biedenharn on the Coulomb problem, we have found a similarity transformation \( S \) defined by

\[ S = \exp \left[ \frac{i}{2} \beta \alpha_r \tanh^{-1} \left( \frac{a V_0}{q \mathbf{K}} \right) \right] \]  

which diagonalizes \( \Gamma \). This gives

\[ \Gamma_s = S \Gamma S^{-1} = -\beta \mathbf{K} \left[ 1 - \left( \frac{a V_0}{q \mathbf{K}} \right)^2 \right]^{1/2}, \]  

whose eigenvalues are, in view of those of \( \beta \) and \( \mathbf{K}^2 \):

\[ \gamma = \pm \left[ \kappa^2 - \left( \frac{a V_0}{q} \right)^2 \right]^{1/2}. \]  

Under the transformation (12), Eq. (4) is thus put in the form:

\[ \tilde{G}(\mathbf{r}, \mathbf{r}') = (\mu + \mathbf{M}_s) \tilde{g}(\mathbf{r}, \mathbf{r}'), \]  

where

\[ \mathbf{M}_s = \mathbf{S} \mathbf{M} \mathbf{S}^{-1}, \]  

and

\[ \tilde{g}(\mathbf{r}, \mathbf{r}') = \frac{i}{2} \int_{0}^{\infty} d\Lambda \langle \mathbf{r}' | \exp(-i \mathbf{H}_s \Lambda) | \mathbf{r} \rangle, \]  

with

\[ \mathbf{H}_s = \mathbf{S} \mathbf{H} \mathbf{S}^{-1}. \]  

The integrand in Eq. (17) can be expressed into partial wave expansion as follows:

\[ \langle \mathbf{r}' | \exp(-i \mathbf{H}_s \Lambda) | \mathbf{r} \rangle = \frac{1}{r' r}, \]  

\[ \times \sum_{\lambda} \langle \theta', \varphi', \lambda | r' | \exp(-i \mathbf{H}_s \Lambda) | r \rangle \langle \lambda | \theta, \varphi \rangle, \]  

where we have denoted by \( | \lambda \rangle = | j, m, \kappa, \tilde{\beta} \rangle \) the simultaneous eigenvectors of the operators \( J^2, J_z, K, \beta \), and by \( r' | \exp(-i \mathbf{H}_s \Lambda) | r \rangle \) the radial propagator, in which

\[ H_s = \frac{1}{2} \left( \mathbf{P}^2 + \lambda (\lambda + 1) \frac{q e^{r/a}}{a^2 (e^{r/a} - q)^2} + V_0 \left( \frac{V_0 - 2E}{q} \right) + \frac{1}{12a^2} \mathbf{K} (\mathbf{K} - \mathbf{\beta}) + \mu^2 - E^2 \right), \]  

with

\[ \lambda = |\gamma| + \frac{1}{2} (\text{sgn} \gamma - 1). \]  

and by \( \langle \theta, \varphi, j, m, \kappa, \tilde{\beta} \rangle \) the dependence in terms of angles \( \theta, \varphi \), spin and \( \tilde{\beta} \) variables which can be written explicitly as:

\[ \left\{ \begin{array}{c} \langle \theta, \varphi, j, m, \kappa, -1 \rangle = \frac{1}{\Omega^m_{\kappa} (\theta, \varphi)}, \\
\langle \theta, \varphi, j, m, \kappa, 1 \rangle = \frac{1}{\Omega^m_{-\kappa} (\theta, \varphi)}, \\
\end{array} \right. \]  

where \( \Omega^m_{\kappa} (\theta, \varphi) \) denotes the hyperspherical harmonic

\[ \Omega^m_{\kappa} (\theta, \varphi) = \sum_{\lambda} (-1)^{\lambda} sgn \lambda \left( \begin{array}{c} \kappa + m + \frac{1}{2} \left( \begin{array}{c} \frac{1}{2} \end{array} \right) \right) \right) \chi_{\lambda}^m (\theta, \varphi) \]  

A plot of the approximation to the centrifugal term for various values of \( q \).
in which $\chi_{\frac{1}{2}}$ and $\chi_{\frac{1}{2}}$ are the spin wavefunctions, and $\text{sgn} \kappa = \pm \text{sgn} \gamma$ for $\beta = \mp 1$.

Using (22), the propagator (19) can be written as:

$$\langle r' | \exp(-iH_0 \Lambda) | r \rangle = \frac{1}{r'} \sum_{j,k} \langle r' | \exp(-iH_2 \Lambda) | r \rangle \times \Omega^{j,k}_{\kappa}(\theta^*, \varphi^*, \theta', \varphi') \beta^2,$$

where

$$\Omega^{j,k}_{\kappa}(\theta^*, \varphi^*, \theta', \varphi') = \sum_{\lambda} \Omega^{\lambda}_{\kappa}(\theta^*, \varphi^*) \Omega^{\lambda'}_{\kappa}(\theta', \varphi').$$

(24)

Inserting (24) into (17), we obtain

$$g(r', r) = \frac{1}{r'^2} \sum_{j,k} g_{j,k}(r'', r') \Omega^{j,k}_{\kappa}(\theta^*, \varphi^*, \theta', \varphi') \beta^2$$

(25)

with the radial Green’s function given by

$$g_{j,k}(r'', r') = \frac{i}{2} \int_0^\infty d\Lambda \langle r'' | \exp \left[ -\frac{i}{2} \left( P_r^2 + \frac{q \lambda (\lambda + 1) \epsilon^r_{/a}}{a^2 (\epsilon^r_{/a} - q)^2} + \frac{V_0}{q} + 2E \right) + \epsilon^2 \right] \Lambda | r' \rangle$$

(26)

where

$$\epsilon = \sqrt{\mu^2 - E^2 + \frac{\kappa(\kappa - 1)}{2a^2}}.$$  

(27)

The expression (27) can now be solved by the path integral technique.

### 3. PATH INTEGRAL SOLUTION

It is easy to express (27) in the form of a path integral,

$$g_{j,k}(r'', r') = \frac{i}{2} \int_0^\infty dS' P_{j,k}(r'', r'; S'),$$

(28)

where

$$P_{j,k}(r'', r'; S') = f_k(r'') f_j(r') \langle r'' | \exp [-\frac{i}{2} f_L(r)] \times \left[ \left( P_r^2 + \frac{q \lambda (\lambda + 1) \epsilon^r_{/a}}{a^2 (\epsilon^r_{/a} - q)^2} + \frac{V_0}{q} + 2E \right) + \epsilon^2 \right] \times f_L(r') | r' \rangle = f_k(r'') f_L(r') \times \lim_{N \to \infty} \prod_{n=1}^N \left[ \int dr_n \prod_{n=1}^{N+1} \int d(P_r) \exp \left[ i \sum_{n=1}^{N+1} \mathcal{A}_n \right] \right],$$

(29)

with the short-time action in configuration space given by

$$\mathcal{A}_n = -\frac{\langle \Delta r_n \rangle^2}{2 \epsilon \sqrt{f(r_n) f(r_{n-1})}} - \frac{\epsilon^2}{2} \left[ \frac{q \lambda (\lambda + 1) \epsilon^r_{/a}}{a^2 (\epsilon^r_{/a} - q)^2} + \frac{V_0}{q} + 2E \right] + \epsilon^2 \sqrt{f(r_n) f(r_{n-1})}.$$  

(30)

To evaluate the path integral (34), we perform the following space transformation:

$$r \in \left[ r_0; +\infty \right] \rightarrow \tilde{z} \in \left[ -\infty; +\infty \right]$$

(31)

defined by

$$r = a \ln \left[ \exp(2 \tilde{z}/a) + q \right],$$

(32)

accompanied by the appropriate regulating function

$$f(r(\tilde{z})) = \frac{e^{2 \tilde{z}/a}}{\cosh^2(\tilde{z}/a)} = g^2(\tilde{z}).$$

(33)
Under these transformations, the kernel (34) takes the form:

\[
P_{j,k}(r'', r'; S') = \left[ f(r'') f(r') \right]^{\frac{1}{2}} \lim_{N \to \infty} \prod_{n=1}^{N} \frac{1}{2 i \pi \varepsilon_n} \left[ \int d \xi_n \right] \exp \left\{ \sum_{n=1}^{N} \left[ \frac{(\Delta \xi_n)^2}{8 \varepsilon_n} \right] + \frac{1}{2} \right\} (39)
\]

where

\[
(\Delta \xi_n)^4 = \frac{\xi_n}{g} \left( \frac{g''}{g} \right)^2 \left( \Delta \xi_n \right)^4 - \frac{\varepsilon_n}{a^2} \left[ (\tilde{\varepsilon})^2 + \lambda (\lambda + 1) + \frac{2 \varepsilon_n}{a^2} \cosh^2(\frac{\Delta \xi_n}{a}) \right]
\]

(40)

and

\[
\tilde{\varepsilon} = a \sqrt{\frac{1}{2} - E'^2 + \frac{\kappa (\kappa - \beta)}{12a^2}},
\]

(41)

In Eqs. (38) and (39), we have used the deformed hyperbolic functions introduced for the first time by Arai [23] and denoted by

\[
cosh_q x = \frac{1}{2} (e^x + q e^{-x}),
\]

\[
sinh_q x = \frac{1}{2} (e^x - q e^{-x}),
\]

\[
tanh_q x = \frac{\sinh_q x}{\cosh_q x},
\]

(42)

where \( q \) is a real parameter.

Note that the term in \((\Delta \xi_n)^4\) appearing in the action contained in Eq. (39) contributes significantly to the path integral. It can be estimated by using the Mcloughlin and Schulman procedure [24] and replaced by

\[
\left\langle (\Delta \xi_n)^4 \right\rangle = \int \frac{d(\Delta \xi_n)}{(\Delta \xi_n)^4} \left\{ \frac{1}{2 i \pi \varepsilon_n} \right\}^3 \times \exp \left\{ \frac{i}{2 \varepsilon_n} (\Delta \xi_n)^2 \right\} = -3 \varepsilon_n^2.
\]

(43)

After changing \( \xi_n/a \) into \( y_n \) and \( \varepsilon_n \) into \( a^2 \varepsilon_n \), one obtains for the radial Green’s function (29) the following expression:

\[
G_{RM}(y'', y'; \tilde{E}) = \frac{\Gamma(M_1 - L_{\varepsilon}) \Gamma(L_{\varepsilon} + M_1 + 1)}{\Gamma(1 + M_1 + M_2 + 1) \Gamma(M_1 - M_2 + 1)} \times \left\{ \frac{1 - \tanh_q y' - \tanh_q y''}{2} \right\}^{M_1 + M_2} \left\{ 1 + \tanh_q y' + \tanh_q y'' \right\}^{2} \times \frac{1 - \tanh_q y''}{2} \times \frac{1 + \tanh_q y'}{2} \times \frac{1 - \tanh_q y'}{2} \times \frac{1 + \tanh_q y''}{2} \times \frac{1 - \tanh_q y''}{2}
\]

(50)

where

\[
g_{j,k}(r'', r') = \frac{1}{2} \left[ f(r'') f(r') \right] G_{RM}(y'', y'; \tilde{E}),
\]

(44)

with

\[
G_{RM}(y'', y'; \tilde{E}) = i \int_0^\infty d \sigma \exp(i \tilde{E} \sigma) P^{RM}_{j,k}(y'', y'; \sigma),
\]

(45)

is the propagator for the general Rosen–Morse potential [25] defined in terms of \( q \)-deformed hyperbolic functions as:

\[
V^{RM}_{j,k}(y) = A \tanh_q y - \frac{B}{\cosh_q^2 y}; \quad y \in \mathbb{R},
\]

(48)

where the constants \( A \) and \( B \) are given by

\[
\begin{align*}
A &= \tilde{\varepsilon}^2 - \lambda (\lambda + 1) - \frac{1}{4}, \\
B &= \frac{1}{2} \left( q \tilde{\varepsilon}^2 - \omega^2 - \frac{q^2}{4} \right).
\end{align*}
\]

(49)

Since its path integral solution is well-established [26–28], we immediately can write down the closed expression for the Green’s function as:

\[
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\]

where we have used the notation

\[
\begin{aligned}
L_E &= -\frac{1}{2} + \left( \frac{1}{16} + 2E_{PR} \right) \frac{1}{2} \\
E_{PR} &= -\frac{1}{2} \left( \frac{2\tilde{e}^2 - \frac{\omega^2}{q} - \frac{1}{16} \right) \\
M_{1,2} &= \tilde{e} \pm \left( \lambda + \frac{1}{2} \right),
\end{aligned}
\]

\( \gamma = ; =, \omega \) is the hypergeometric function and the symbols \( y_\geq \) and \( y_\leq \) denote \( \max(y^\nu, y^\gamma) \) and \( \min(y^\nu, y^\gamma) \), respectively.

\[
4. \text{ GREEN'S FUNCTION AND ENERGY SPECTRUM}
\]

Having completed the path integration, we substitute (50) into (44) and then into (26). Then, after transforming the variable \( y = \frac{\xi}{a} \) to the radial variable by (37) and taking into account the notation (51), we get the following expression for the complete Green’s function of the second order Dirac equation:

\[
\tilde{G}(r^*, r') = \frac{a}{r^*} \sum_{j,k} \frac{\Gamma(1 + \lambda + \tilde{\epsilon} - \frac{\omega^2}{q}) \Gamma(1 + \lambda + \tilde{\epsilon} + \frac{\omega^2}{q})}{\Gamma(2\tilde{\epsilon} + 1) \Gamma(2\lambda + 2)} \\
\times \left[ q^2 e^{-r^*|a|} \left( 1 - qe^{-r^*|a|} \right)^{\lambda+1} \times 2F_1 \left( 1 + \lambda + \frac{\omega^2}{q}, 1 + \lambda + \tilde{\epsilon} + \frac{\omega^2}{q}, 2\lambda + 2; 1 - qe^{-r^*|a|} \right) \Omega_{\kappa \kappa} (\theta^\mu, \varphi^\mu, \theta', \varphi') \beta^2. \right]
\]

4. \text{ GREEN’S FUNCTION AND ENERGY SPECTRUM}

We now proceed to evaluate the Green’s function for the linear Dirac equation (15). To do this, we first note that the transformed operator \( M_\gamma \) applied to \( \kappa \)-states can be put in the form:

\[
M_\gamma = i \beta \gamma_\alpha \left( \frac{d}{dr^*} + \frac{1 - \gamma \gamma}{r} + aV_0 E \right) \gamma_\beta
\]

\[
+ \frac{\kappa E_\gamma}{\gamma} \beta + \frac{aV_0 E}{q r^*} - aV_0 \beta \gamma - \frac{aV_0 E}{q r^*} \gamma_\beta \gamma
\]

Then, on account of the relations

\[
\text{Then, on account of the relations}
\]

\[
\left[ \alpha_\gamma = \sigma_\gamma \gamma^5 = i \sigma_\gamma \beta \gamma \gamma^5 \beta, \right.
\]

\[
\gamma' = \beta \alpha_\gamma, \quad i = 1, 2, 3,
\]

\[
\left. \sigma_\gamma \Omega_{\kappa \kappa} (\theta^\mu, \varphi^\mu, \theta', \varphi') = -\Omega_{\kappa \kappa} (\theta^\mu, \varphi^\mu), \right.
\]

we arrive at the following expression:

\[
\tilde{G}(r^*, r') = \frac{a}{r^*} \sum_{j,k} \frac{\Gamma(1 + \lambda + \tilde{\epsilon} - \frac{\omega^2}{q}) \Gamma(1 + \lambda + \tilde{\epsilon} + \frac{\omega^2}{q})}{\Gamma(2\tilde{\epsilon} + 1) \Gamma(2\lambda + 2)} \\
\times U(r^*) \left( \frac{\mu - \frac{\kappa E_\gamma}{\gamma} + \frac{aV_0 E}{q r^*}}{\gamma} - aV_0 \beta \gamma - \frac{aV_0 E}{q r^*} \right) \Omega_{\kappa \kappa} (\theta^\mu, \varphi^\mu, \theta', \varphi') \beta^2
\]

\[
- \left( \frac{d}{dr^*} + \frac{1 + \beta \gamma}{r^*} - aV_0 E \right) \sum_{j,k} \frac{\Gamma(1 + \lambda + \tilde{\epsilon} + \frac{\omega^2}{q}) \Gamma(1 + \lambda + \tilde{\epsilon} + \frac{\omega^2}{q}) }{\Gamma(2\tilde{\epsilon} + 1) \Gamma(2\lambda + 2)} \\
\times 2F_1 \left( 1 + \lambda + \tilde{\epsilon} - \frac{\omega^2}{q}, 1 + \lambda + \tilde{\epsilon} + \frac{\omega^2}{q}, 2\lambda + 2; 1 - qe^{-r^*|a|} \right) \Omega_{\kappa \kappa} (\theta^\mu, \varphi^\mu, \theta', \varphi') \alpha_\gamma \alpha_\gamma
\]

\[
\text{where}
\]

\[ U(r^*) = \left( qe^{-r^*|a|} \right)^{\lambda+1} \left( 1 - qe^{-r^*|a|} \right)^{\lambda+1} \]

\[ \times 2F_1 \left( 1 + \lambda + \tilde{\epsilon} - \frac{\omega^2}{q}, 1 + \lambda + \tilde{\epsilon} + \frac{\omega^2}{q}, 2\lambda + 2; 1 - qe^{-r^*|a|} \right) \]

\[ \times 2F_1 \left( 1 + \lambda + \tilde{\epsilon} + \frac{\omega^2}{q}, 1 + \lambda + \tilde{\epsilon} + \frac{\omega^2}{q}, 2\lambda + 2; 1 - qe^{-r^*|a|} \right). \]

\[
\text{and}
\]

\[ U(r^*) = \left( qe^{-r^*|a|} \right)^{\lambda+1} \left( 1 - qe^{-r^*|a|} \right)^{\lambda+1} \]

\[ \times 2F_1 \left( 1 + \lambda + \tilde{\epsilon} - \frac{\omega^2}{q}, 1 + \lambda + \tilde{\epsilon} + \frac{\omega^2}{q}, 2\lambda + 2; 1 - qe^{-r^*|a|} \right) \]

\[ \times 2F_1 \left( 1 + \lambda + \tilde{\epsilon} + \frac{\omega^2}{q}, 1 + \lambda + \tilde{\epsilon} + \frac{\omega^2}{q}, 2\lambda + 2; 1 - qe^{-r^*|a|} \right). \]
The poles of the Green’s function (56), coming from the first \( \Gamma \)-function in the numerator, are
\[
1 + \lambda + \tilde{\varepsilon} - \sqrt{\tilde{\varepsilon}^2 - \omega^2} = -n_r, \quad n_r = 0, 1, 2, \ldots. \tag{59}
\]
They determine the bound-state energies. Converting (59) into energy by using (40) and (41) yields
\[
\left( E_{n_r, \kappa} - \frac{V_0}{2q} \right)^2 = \frac{(n_r + \lambda + 1)^2}{(n_r + \lambda + 1)^2 + \left( \frac{aV_0}{q} \right)^2} \times \left[ \mu^2 + \frac{1}{12a^2} \kappa (\kappa - \tilde{\beta}) \right] - \frac{1}{4a^2} (n_r + \lambda + 1)^2
\]
for \( q \geq 1 \) and where \( \lambda \) has been given by (21).

5. PARTICULAR CASES

5.1. First Case: Standard Hulthén Potential

By setting \( q = 1 \) in the expression (1), we obtain the standard Hulthén potential
\[
V(r) = -\frac{V_0}{e^{r/a} - 1}. \tag{61}
\]
The parameters (14) and (41) can thus be written
\[
\gamma = \pm \frac{\kappa}{\sqrt{\kappa^2 - (aV_0)^2}}, \quad \omega = a\sqrt{V_0(V_0 - 2E)}, \tag{62, 63}
\]
The Green’s function satisfying the linear Dirac equation can be deduced from (56):
\[
\tilde{G}(r''', r') = \frac{a}{r'''} \sum_{j, k} \frac{1}{r'''} \frac{[1 + \lambda + \tilde{\varepsilon} - \sqrt{\tilde{\varepsilon}^2 - \omega^2}] \Gamma[1 + \lambda + \tilde{\varepsilon} + \sqrt{\tilde{\varepsilon}^2 - \omega^2}]}{\Gamma(2\varepsilon + 1)\Gamma(2\lambda + 2)} U(r') \left( \mu - \frac{\kappa E + \tilde{V}_0 \tilde{\beta}}{\gamma} - \frac{aV_0 \tilde{\beta}}{\gamma} \right) \left( \frac{1}{r'''} \Omega_{\kappa, \kappa}(\theta'', \phi'', \theta', \phi') \right) \left( \frac{d}{dr'''} + \frac{1 + \tilde{\beta}r'''}{r'''} - \frac{aV_0 \tilde{\beta}}{\gamma} \right) \left( r'''' \right) \Omega_{\kappa - \kappa}(\theta'', \phi'', \theta', \phi') \alpha_{\alpha_3} \alpha_{\alpha_3}, \tag{64}
\]
with
\[
U(r') = \left( e^{-r/a} \right)^{\lambda} \left( 1 - e^{-r/a} \right)^{\lambda + 1} \times 2F1 \left( 1 + \lambda + \tilde{\varepsilon} - \sqrt{\tilde{\varepsilon}^2 - \omega^2}, 1 + \lambda + \tilde{\varepsilon} + \sqrt{\tilde{\varepsilon}^2 - \omega^2}, 2\lambda + 2; 1 - e^{-r/a} \right), \tag{65}
\]
and
\[
U(r'') = \left( e^{-r''/a} \right)^{\lambda} \left( 1 - e^{-r''/a} \right)^{\lambda + 1} \times 2F1 \left( 1 + \lambda + \tilde{\varepsilon} - \sqrt{\tilde{\varepsilon}^2 - \omega^2}, 1 + \lambda + \tilde{\varepsilon} + \sqrt{\tilde{\varepsilon}^2 - \omega^2}, 2\tilde{\varepsilon} + 1; e^{-r''/a} \right). \tag{66}
\]
The energy spectrum is then found from (60) to be
\[
\left( E_{n_r, \kappa} - \frac{V_0}{2q} \right)^2 = \frac{(n_r + \lambda + 1)^2}{(n_r + \lambda + 1)^2 + \left( \frac{aV_0}{q} \right)^2} \times \left[ \mu^2 + \frac{1}{12a^2} \kappa (\kappa - \tilde{\beta}) \right] - \frac{1}{4a^2} (n_r + \lambda + 1)^2, \tag{67}
\]
and
\[
\tilde{\varepsilon} = a\mu^2 - E^2, \quad 1 + \lambda + \tilde{\varepsilon} - \sqrt{(\tilde{\varepsilon})^2 - \omega^2} = 1 + \lambda_0 - \frac{Ze^2}{\sqrt{\mu^2 - E^2}}, \tag{69}
\]
where we have put \( V_0 = Ze^2/a \) with \( Ze \) the charge of the nucleus.

In this case, it can be seen from Eqs. (40) and (41) that
\[
\begin{aligned}
1 + \lambda + \tilde{\varepsilon} - \sqrt{(\tilde{\varepsilon})^2 - \omega^2} & \approx 1 + \lambda_0 - \frac{Ze^2}{\sqrt{\mu^2 - E^2}} \\
1 + \lambda + \tilde{\varepsilon} + \sqrt{(\tilde{\varepsilon})^2 - \omega^2} & \approx 1 + \lambda_0 + \frac{Ze^2}{\sqrt{\mu^2 - E^2}} + 2a\mu^2 - E^2 \rightarrow \infty.
\end{aligned}
\]
On the other hand, by using the Gauss transformation formula (see Ref. [29], p. 1043, Eq. (9.131.2))

\[
\begin{align*}
    &{}_{2}F_{1}(a, b, c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \times {}_{2}F_{1}(a, b, a + b - c + 1; 1 - z) \\
    &\quad \times \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} (1 - z)^{c - a - b}
\end{align*}
\]

(70)

it is easy to show that, in the limit \( a \to \infty \), the closed form (52) of the Green’s function of the second order Dirac equation, for \( q = 1 \), becomes

\[
\lim_{a \to \infty} \tilde{g}(r'', r') = \tilde{g}_{0}(r'', r')
\]

\[
= \frac{1}{2r''} \sum_{j,k} \Gamma_{0} \left( 1 + l_{0} - \frac{Z^2}{\tilde{r}^{2}} \right) M_{Ze_{E}}(\lambda_{0} + \frac{1}{2}) \Omega_{k,k}^{j}(\theta'', \phi'', \theta', \phi') \beta^{j} - \frac{i}{\gamma} \sum_{j,k} \Gamma_{0} \left( 1 + l_{0} - \frac{Z^2}{\tilde{r}^{2}} \right) M_{Ze_{E}}(\lambda_{0} + \frac{1}{2}) \Omega_{k,k}^{j}(\theta'', \phi'', \theta', \phi') \beta^{j}
\]

(73)

With the Coulomb potential, the operator (53) reduces to

\[
\tilde{g}_{0}(r'', r') = (\mu + M_{s}(r')) \tilde{g}_{0}(r'', r') = \frac{1}{2r''} \sum_{j,k} \Gamma_{0} \left( 1 + l_{0} - \frac{Z^2}{\tilde{r}^{2}} \right) M_{Ze_{E}}(\lambda_{0} + \frac{1}{2}) \Omega_{k,k}^{j}(\theta'', \phi'', \theta', \phi') \beta^{j} - \frac{i}{\gamma} \sum_{j,k} \Gamma_{0} \left( 1 + l_{0} - \frac{Z^2}{\tilde{r}^{2}} \right) M_{Ze_{E}}(\lambda_{0} + \frac{1}{2}) \Omega_{k,k}^{j}(\theta'', \phi'', \theta', \phi') \beta^{j}
\]

(76)

with \( \lambda_{0} = \lambda_{0}(\gamma_{0}) \) and \( \tilde{\lambda}_{0} = \lambda_{0}(-\gamma_{0}) \).

The discrete energy spectrum is then

\[
E_{n,s} = \mu \left[ 1 + Z^2 e^4 (\eta_{e} + \lambda_{0} + 1)^{-2} \right]^{\frac{1}{2}}
\]

(77)

These results coincide with those obtained by path integration [16].

6. CONCLUSIONS

In this contribution, we have shown that the problem of a relativistic spin-\( \frac{1}{2} \) system in the presence of a deformed Hulthén potential can be solved for arbitrary \( k \)-states by using a proper approximation to \( r^{-1} \). It is worth mentioning that the problem of a Dirac particle moving in the standard Hulthén potential cannot be solved exactly even for the s-wave because of the centrifugal potential term contained in the two radial wave equations (see for example Eqs. (47) in Ref. [13]) contrary to what was stated by Guo et al. [11] and Alhaidari [12]. These authors had performed several incorrect manipulations by applying a constraint to get rid of the centrifugal term depending on the eigenvalues of
the Martin–Glauber operator [20, 21]. Consequently, their energy spectra are unsatisfactory. Furthermore, one can observe that the $s$-wave relativistic energy spectrum for the Coulomb potential cannot be found from that obtained through Alhaidari’s approach. For the standard Hulthén potential and the Coulomb potential ($q = 1$ and $a \rightarrow \infty$), our results concerning the energy spectra are identical with that obtained by solving Dirac’s equation [13] (see also Refs. [16, 31] for the Coulomb potential).

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