FRANKS’ LEMMA FOR $C^2$-MAÑÉ PERTURBATIONS OF RIEMANNIAN METRICS AND APPLICATIONS TO PERSISTENCE

AYADI LAZRAG, LUDOVIC RIFFORD AND RAFAEL O. RUGGIERO
(Communicated by Federico Rodriguez Hertz)

ABSTRACT. We prove a uniform Franks’ lemma at second order for geodesic flows on a compact Riemannian manifold and apply the result in persistence theory. Our approach, which relies on techniques from geometric control theory, allows us to show that Mañé (i.e., conformal) perturbations of the metric are sufficient to achieve the result.

1. INTRODUCTION

One of the most important tools of $C^1$ generic and stability theories of dynamical systems is the celebrated Franks’ Lemma [15]:

Let $M$ be a smooth (i.e., of class $C^\infty$) compact manifold of dimension $n \geq 2$ and let $f : M \to M$ be a $C^1$ diffeomorphism. Consider a finite set of points $S = \{p_1, p_2, \ldots, p_m\}$, let $\Pi = \bigoplus_{i=1}^m T_{p_i}M$, and let $\Pi' = \bigoplus_{i=1}^m T_{f(p_i)}M$. Then there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$ there exists $\delta = \delta(\epsilon) > 0$ such that the following holds: let $L = (L_1, L_2, \ldots, L_m) : \Pi \to \Pi'$ be an isomorphism such that

$$\| L_i - D_{p_i}f \| < \delta \quad \forall i = 1, \ldots, m,$$

then there exists a $C^1$ diffeomorphism $g : M \to M$ satisfying

1. $g(p_i) = f(p_i)$ for every $i = 1, \ldots, m$,
2. $D_{p_i}g = L_i$ for each $i = 1, \ldots, m$,
3. the diffeomorphism $g$ is in the $\epsilon$ neighborhood of $f$ in the $C^1$ topology.

In a few words, the lemma asserts that given a collection $S$ of $m$ points $p_i$ in the manifold $M$, any isomorphism from $\Pi$ to $\Pi'$ can be the collection of the differentials of a diffeomorphism $g$, $C^1$ close to $f$, at each point of $S$ provided that the isomorphism is sufficiently close to the direct sum of the maps $D_{p_i}f$, $i = 1, \ldots, m$. The sequence of points is particularly interesting for applications in dynamics when the collection $S$ is a subset of a periodic orbit. The idea of the proof of the lemma is quite elementary: we conjugate the isomorphisms $L_i$ by the exponential map of $M$ in suitably small neighborhoods of the points $p_i$'s and then glue (smoothly) the diffeomorphism $f$ outside the union of such neighborhoods with this collection of conjugate-to-linear maps. So the proof
strongly resembles an elementary calculus exercise: given a $C^1$ function $h: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, and a number $c$ close to $h'(x_0)$, we can glue $h$ outside a small neighborhood $U$ of $x_0$ with the linear function in $U$ $\sigma(x) = h(x_0) + c(x - x_0)$; so $\sigma(x_0) = h(x_0)$ and $\sigma'(x_0) = c$; to get a new function that is $C^1$ close to $h$.

Franks’ Lemma admits a natural extension to flows, and its important applications in the study of stable dynamics gave rise to versions for more specific families of systems, like symplectic diffeomorphisms and Hamiltonian flows [35, 41]. It is clear that for specific families of systems the proof of the lemma should be more difficult than just gluing conjugates of linear maps by the exponential map since this surgery procedure in general does not preserve specific properties of systems, like preserving symplectic forms in the case of symplectic maps. Franks’ Lemma was extensively used by R. Mañé in his proof of the $C^1$ structural stability conjecture [23], and we could claim with no doubts that it is one of the pillars of the proof together with C. Pugh’s $C^1$ closing lemma [30, 31] (see Newhouse [27] for the proof of the $C^1$ structural stability conjecture for symplectic diffeomorphisms).

A particularly challenging problem is to obtain a version of Franks’ Lemma for geodesic flows. First of all, a typical perturbation of the geodesic flow of a Riemannian metric in the family of smooth flows is not the geodesic flow of another Riemannian metric. To ensure that perturbations of a geodesic flow are geodesic flows as well the most natural way to proceed is to perturb the Riemannian metric in the manifold itself. But then, since a local perturbation of a Riemannian metric changes all geodesics through a neighborhood, the geodesic flow of the perturbed metric changes in tubular neighborhoods of vertical fibers in the unit tangent bundle. Since local perturbations of the metric are not quite local for the geodesic flow, the usual strategy applied in generic dynamics of perturbing a flow in a flowbox without changing the dynamics outside the box does not work. This poses many interesting, technical problems in the theory of local perturbations of dynamical systems of geometric origin, the famous works of Klingenberg-Takens [18] and Anosov [3] (the bumpy metric theorem) about generic properties of closed geodesics are perhaps the two best known examples. Moreover, geodesics in general have many self-intersections so the effect of a local perturbation of the metric on the global dynamics of perturbed orbits is unpredictable unless we know a priori that the geodesic flow enjoys some sort of stability (negative sectional curvatures, Anosov flows for instance).

The family of metric perturbations which preserves a compact piece of a given geodesic is the most used to study generic theory of periodic geodesics. This family of perturbations is relatively easy to characterize analytically when we restrict ourselves to the category of conformal perturbations or, more generally, to the set of perturbations of Lagrangians by small potentials. Recall that a Riemannian metric $h$ in a manifold $M$ is conformally equivalent to a Riemannian metric $g$ in $M$ if there exists a positive, $C^\infty$ function $b: M \to \mathbb{R}$ such that $h_x(v, w) = b(x)g_x(v, w)$ for every $x \in M$ and $v, w \in T_xM$. Given a $C^\infty$, Tonelli Lagrangian $L: TM \to \mathbb{R}$ defined in a compact manifold $M$, and a $C^\infty$
function \(u : M \rightarrow \mathbb{R}\), the function \(L_u(p, v) = L(p, v) + u(p)\) gives another Tonelli Lagrangian. The function \(u\) is usually called a potential because of the analogy between this kind of Lagrangian and mechanical Lagrangians.

By Maupertuis’ principle (see for example [12]), the Lagrangian associated to a metric \(h\) in \(M\) that is conformally equivalent to \(g\) is of the form

\[
L(p, v) = \frac{1}{2} g_p(v, v) + u(p)
\]

for some function \(u\). Since the Lagrangian of a metric \(g\) is given by the formula \(L_g(p, v) = \frac{1}{2} g_p(v, v)\), we get \(L_h(p, v) = L_g(p, v) + u(p)\). Now, given a compact part \(\gamma : [0, T] \rightarrow M\) of a geodesic of \((M, g)\), the collection of potentials \(u : M \rightarrow \mathbb{R}\) such that \(\gamma[0, T]\) is still a geodesic of \(L(p, v) = L_g(p, v) + u(p)\) contains the functions whose gradients vanish along the subset of \(T_\gamma(t)M\) which are perpendicular to \(\gamma'(t)\) for every \(t \in [0, T]\) (see for instance [37, Lemma 2.1]). Lagrangian perturbations of Tonelli Lagrangians of the type \(L_h(p, v) = L_g(p, v) + u(p)\) were used extensively by R. Mañé to study generic properties of Tonelli Lagrangians and applications to Aubry-Mather theory (see for instance [24, 25]). Mañé’s idea proved to be very fruitful and insightful in Lagrangian generic theory, and opened a new branch of generic theory that is usually called Mañé’s genericity. Recently, Rifford-Ruggiero [34] gave a proof of Klingenberg-Takens and Anosov \(C^1\) genericity results for closed geodesics using control theory techniques applied to the class of Mañé type perturbations of Lagrangians. Control theory ideas simplify a great deal the technical problems involved in metric perturbations and at the same time show that Mañé type perturbations attain full Hamiltonian genericity. This result, combined with a previous theorem by Oliveira [28], led to the Kupka-Smale Theorem for geodesic flows in the family of conformal perturbations of metrics.

These promising applications of control theory to the generic theory of geodesic flows motivate us to study Franks’ Lemma for conformal perturbations of Riemannian metrics or equivalently, for Mañé type perturbations of Riemannian Lagrangians. Before stating our main theorem, let us recall first some notions and basic results about geodesic flows. The geodesic flow of a Riemannian manifold \((M, g)\) will be denoted by \(\phi_t\); the flow acts on the unit tangent bundle \(T_1M\); a point \(\theta \in T_1M\) has canonical coordinates \(\theta = (p, v)\), where \(p \in M\) and \(v \in T_pM\); and \(\gamma_\theta\) denotes the unit speed geodesic with initial conditions \(\gamma_\theta(0) = p, \gamma_\theta'(0) = v\). Let \(N_\theta \subset T_\theta T_1M\) be the plane of vectors that are perpendicular to the geodesic flow with respect to the Sasaki metric (see for example [38]). The collection of these planes is preserved by the action of the differential of the geodesic flow: \(D_\theta \phi_t(N_\theta) = N_{\phi_t(\theta)}\) for every \(\theta\) and \(t \in \mathbb{R}\).

Let us consider a geodesic arc of length \(T\)

\[
\gamma_\theta : [0, T] \rightarrow M,
\]

and let \(\Sigma_0\) and \(\Sigma_T\) be local transverse sections for the geodesic flow which are tangent to \(N_\theta\) and \(N_{\phi_T(\theta)}\) respectively. Let \(\mathbb{P}_g(\Sigma_0, \Sigma_T, \gamma_\theta)\) be a Poincaré map going from \(\Sigma_0\) to \(\Sigma_T\). In horizontal-vertical coordinates of \(N_\theta\), the differential
$D_\theta \phi_T$ that is the linearized Poincaré map

$$P_g(\gamma_\theta)(T) := D_\theta \phi_T^* (\Sigma_0, \Sigma_T, \gamma_\theta)$$

is a symplectic endomorphism of $\mathbb{R}^{2n-2} \times \mathbb{R}^{2n-2}$. This endomorphism can be expressed in terms of the Jacobi fields of $\gamma_\theta$ which are perpendicular to $\gamma_\theta'(t)$ for every $t$:

$$P_g(\gamma_\theta)(T)(j(0), \dot{j}(0)) = (J(T), \dot{J}(T)),$$

where $\dot{J}$ denotes the covariant derivative along the geodesic. We can identify the set of all symplectic endomorphisms of $\mathbb{R}^{2n-2} \times \mathbb{R}^{2n-2}$ with the symplectic group

$$\text{Sp}(n-1) := \{ X \in \mathbb{R}^{(2n-2) \times (2n-2)} ; X^* J X = J \},$$

where $X^*$ denotes the transpose of $X$ and

$$J = \begin{pmatrix} 0 & I_{n-1} \\ -I_{n-1} & 0 \end{pmatrix}.$$ 

Given a geodesic $\gamma_\theta : [0, T] \to M$, an interval $[t_1, t_2] \subset [0, T]$, and $\rho > 0$, we denote the open geodesic cylinder along $\gamma_\theta([t_1, t_2])$ of radius $\rho$ by $\mathcal{C}_{g}(\gamma_\theta([t_1, t_2]); \rho)$. It is the open set defined by

$$\mathcal{C}_{g}(\gamma_\theta([t_1, t_2]); \rho) := \{ p \in M | d_g(p, \gamma_\theta([t_1, t_2])) < \rho \},$$

where $d_g$ denotes the geodesic distance with respect to $g$. We note that by compactness of $M$, there is $\bar{\rho} > 0$ such that any geodesic of length $2\bar{\rho}$ has no self-intersection, which makes any cylinder of the form $\mathcal{C}_{g}(\gamma_\theta([t_1, t_2]); \rho)$ a genuine open cylinder provided $t_2 - t_1$ and $\rho$ are small enough. Our first result is the following.

**Theorem 1.1** (Franks’ Lemma). Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $\geq 2$. For every $T \geq \bar{t}$ there exist $\delta_T, K_T > 0$ such that the following property holds:

For every unit geodesic $\gamma_\theta : [0, T] \to M$ and every $\bar{t} \in [0, T]$ with $[\bar{t}, \bar{t} + \bar{t}] \subset [0, T]$, there is $\bar{\rho} > 0$ such that for every $\delta \in (0, \delta_T)$, for each symplectic map $A$ in the open ball (in $\text{Sp}(n-1)$) centered at $P_g(\gamma_\theta(T))$ of radius $\delta$ and for every $\rho \in (0, \bar{\rho})$, there exists a $C^\infty$ metric $h$ in $M$ that is conformal to $g$, $h_p(v, w) = (1 + \sigma(p))g_p(v, w)$, such that

1. the geodesic $\gamma_\theta : [0, T] \to M$ is still a geodesic of $(M, h)$,
2. $\text{supp}(\sigma) \subset \mathcal{C}_{g}(\gamma_\theta([\bar{t}, \bar{t} + \bar{t}]); \rho)$,
3. $P_h(\gamma_\theta(T)) = A$, and
4. the $C^2$ norm of the function $\sigma$ is less than $K_T\sqrt{\delta}$.

Moreover, if there is a finite set $\Sigma = \bigcup_{\ell \geq 1} c_\ell(I_\ell)$ of smooth curves $c_\ell : I_\ell \to M$ (each $I_\ell$ is a compact interval) which are all transverse to $\gamma_\theta([0, T])$, then, taking $\bar{\rho} > 0$ smaller if necessary, the above property holds with the following strengthened version of 2:

1. $\text{supp}(\sigma) \subset \mathcal{C}_{g}(\gamma_\theta([\bar{t}, \bar{t} + \bar{t}]); \rho)$ $\sim$ $\Sigma$.
Theorem 1.1 improves a previous result by Contreras [7, Theorem 7.1] which gives a controllability result at first order under an additional assumption on the curvatures along the initial geodesic. Other proofs of Contreras’ Theorem can also be found in [40] and [21]. The Lazrag proof follows already the ideas from geometric control introduced in [34] to study controllability properties at first order. Our new Theorem 1.1 shows that controllability holds at second order without any assumption on curvatures along the geodesic. Its proof amounts to study how small conformal perturbations of the metric \( g \) along \( \Gamma := \gamma_\theta([0, T]) \) affect the differential of \( \mathbb{P}_g(\Sigma_0, \Sigma_T, \gamma_\theta) \). This can be seen as a problem of local controllability along a reference trajectory in the symplectic group. As in [34], the idea is to see the Hessian of the conformal factor along the initial geodesic as a control and to obtain Theorem 1.1 as a uniform controllability result at second order for a control system in the symplectic group \( \text{Sp}(n - 1) \). We apply Franks’ Lemma to extend some results concerning the characterization of hyperbolic geodesic flows in terms of the persistence of some \( C^1 \) generic properties of the dynamics. These results are based on well known steps towards the proof of the \( C^1 \) structural stability conjecture for diffeomorphisms.

Let us first introduce some notations. Given a smooth compact Riemannian manifold \( (M, g) \), we say that a property \( P \) of the geodesic flow of \( (M, g) \) is \( \epsilon \)-\( C^k \) persistent from Mañé’s viewpoint if for every \( C^\infty \) function \( f : M \to \mathbb{R} \) whose \( C^k \) norm is less than \( \epsilon \) we have that the geodesic flow of the metric \( (M, (1 + f)g) \) has property \( P \) as well. By Maupertuis’ principle, this is equivalent to the existence of an open \( C^k \)-ball of radius \( \epsilon' > 0 \) of functions \( q : M \to \mathbb{R} \) such that for every \( C^\infty \) function in this open ball the Euler-Lagrange flow of the Lagrangian \( L(p, v) = \frac{1}{2} g_p(v, v) - q(p) \) in the level of energy equal to 1 has property \( P \). This definition is inspired by the definition of \( C^{k-1} \) persistence for diffeomorphisms: a property \( P \) of a diffeomorphism \( f : M \to M \) is called \( \epsilon \)-\( C^{k-1} \) persistent if the property holds for every diffeomorphism in the \( \epsilon \)-\( C^{k-1} \) neighborhood of \( f \). It is clear that if a property \( P \) is \( \epsilon \)-\( C^1 \) persistent for a geodesic flow then the property \( P \) is \( \epsilon' \)-\( C^2 \) persistent from Mañé’s viewpoint for some \( \epsilon' \).

**Theorem 1.2.** Let \( (M, g) \) be a smooth compact Riemannian manifold of dimension \( \geq 2 \) such that the periodic orbits of the geodesic flow are \( C^2 \) persistently hyperbolic from Mañé’s viewpoint. Then the closure of the set of periodic orbits of the geodesic flow is a hyperbolic set.

An interesting application of Theorem 1.2 is the following extension of Theorem A in [36]: \( C^1 \) persistently expansive geodesic flows in the set of Hamiltonian flows of \( T_1 M \) are Anosov flows. We recall that a non-singular smooth flow \( \phi_t : Q \to Q \) acting on a complete Riemannian manifold \( Q \) is \( \epsilon \)-expansive if given \( x \in Q \) we have that for each \( y \in Q \) such that there exists a continuous surjective function \( \rho : \mathbb{R} \to \mathbb{R} \) with \( \rho(0) = 0 \) satisfying
\[
 d(\phi_t(x), \phi_{\rho(t)}(y)) \leq \epsilon \quad \forall t \in \mathbb{R},
\]
for every \( t \in \mathbb{R} \) then there exists \( t(y) \), \( |t(y)| < \epsilon \), such that \( \phi_{t(y)}(x) = y \). A smooth non-singular flow is called expansive if it is expansive for some \( \epsilon > 0 \). Anosov
flows are expansive, and it is not difficult to get examples which show that the converse of this statement is not true. Theorem 1.2 yields the following.

**Theorem 1.3.** Let \((M,g)\) be a smooth compact Riemannian manifold, suppose that either \(M\) is a surface or \(\dim M \geq 3\) and \((M,g)\) has no conjugate points, and assume that the geodesic flow is \(C^2\) persistently expansive from Mañé’s viewpoint. Then the geodesic flow is Anosov.

The proof of the above result requires the set of periodic orbits to be dense. Such a result follows from expansiveness on surfaces [36] and from the absence of conjugate points in any dimension. If we drop the assumption of the absence of conjugate points we do not know whether periodic orbits of expansive geodesic flows are dense (and so if the geodesic flow in Theorem 1.3 is Anosov). This is a difficult, challenging problem.

The paper is organized as follows. In the next section, we introduce some preliminaries which describe the relationship between local controllability and some properties of the End-Point mapping and we introduce the notions of local controllability at first and second order. We recall a result of controllability at first order (Proposition 2.1) already used in [34] and state a result (Propositions 2.2) at second order whose proof is given in Section 2.5. In Section 3, we provide the proof of Theorem 1.1, and the proof of Theorems 1.2 and 1.3 are given in Section 4.

### 2. Preliminaries in control theory

Our aim here is to provide sufficient conditions for first and second order local controllability results. These kind of results could be developed for nonlinear control systems on smooth manifolds. For the sake of simplicity, we restrict our attention here to the case of affine control systems on the set of (symplectic) matrices. We refer the interested reader to [1, 9, 17, 20, 33] for a further study in control theory.

#### 2.1. The End-Point mapping.**

Let us consider a **bilinear control system** on \(M_{2m}(\mathbb{R})\) (with \(m,k \geq 1\)), of the form

\[
\dot{X}(t) = A(t)X(t) + \sum_{i=1}^{k} u_i(t)B_iX(t), \quad \text{for a.e. } t,
\]

where the **state** \(X(t)\) belongs to \(M_{2m}(\mathbb{R})\), the **control** \(u(t)\) belongs to \(\mathbb{R}^k\), \(t \in [0,T] \rightarrow A(t)\) (with \(T > 0\)) is a smooth map valued in \(M_{2m}(\mathbb{R})\), and \(B_1, \ldots, B_k\) are \(k\) matrices in \(M_{2m}(\mathbb{R})\). Given \(\bar{X} \in M_{2m}(\mathbb{R})\) and \(\bar{u} \in L^2([0,T];\mathbb{R}^k)\), the Cauchy problem

\[
\dot{X}(t) = A(t)X(t) + \sum_{i=1}^{k} \bar{u}_i(t)B_iX(t) \quad \text{for a.e. } t \in [0,T], \ X(0) = \bar{X},
\]

possesses a unique solution \(X_{\bar{X},\bar{u}}(\cdot)\). The **End-Point mapping** associated with \(\bar{X}\)
in time $T > 0$ is defined as

$$E_{\tilde{X}, T} : L^2([0, T]; \mathbb{R}^k) \rightarrow M_{2m}(\mathbb{R}),$$

$$u \mapsto X_{\tilde{X}, u}(T).$$

It is a smooth mapping whose differential can be expressed in terms of the linearized control system (see [33]). Given $\tilde{X} \in M_{2m}(\mathbb{R}),$ $\tilde{u} \in L^2([0, T]; \mathbb{R}^k),$ and setting $\tilde{X}(\cdot) := X_{\tilde{X}, \tilde{u}}(\cdot),$ the differential of $E_{\tilde{X}}$ at $\tilde{u}$ is given by the linear operator

$$D_{\tilde{u}}E_{\tilde{X}, T} : L^2([0, T]; \mathbb{R}^k) \rightarrow M_{2m}(\mathbb{R}),$$

$$\nu \mapsto Y(T),$$

where $Y(\cdot)$ is the unique solution to the linearized Cauchy problem

$$\begin{cases}
\dot{Y}(t) = A(t)Y(t) + \sum_{i=1}^{k} \nu_i(t)B_i(t)\tilde{X}(t) & \text{for a.e. } t \in [0, T], \\
Y(0) = 0.
\end{cases}$$

Note that if we denote by $S(\cdot)$ the solution to the Cauchy problem

$$\begin{cases}
\dot{S}(t) = A(t)S(t) \\
S(0) = I_{2m}
\end{cases} \quad \forall t \in [0, T],$$

then there holds

$$D_{\tilde{u}}E_{\tilde{X}, T}(\nu) = \sum_{i=1}^{k} S(T) \int_{0}^{T} \nu_i(t)S(t)^{-1}B_i\tilde{X}(t) \, dt,$$

for every $\nu \in L^2([0, T]; \mathbb{R}^k)$.

Let $Sp(m)$ be the symplectic group in $M_{2m}(\mathbb{R})$ $(m \geq 1),$ that is, the smooth submanifold of matrices $X \in M_{2m}(\mathbb{R})$ satisfying $X^*JX = J$. Denote by $\mathcal{S}(2m)$ the set of symmetric matrices in $M_{2m}(\mathbb{R})$. The tangent space to $Sp(m)$ at the identity matrix is given by

$$T_{I_{2m}}Sp(m) = \left\{ Y \in M_{2m}(\mathbb{R}) \mid JY \in \mathcal{S}(2m) \right\}.$$

Therefore, if there holds

$$\begin{cases}
JA(t), JB_1, \ldots, JB_k \in \mathcal{S}(2m) & \forall t \in [0, T],
\end{cases}$$

then $Sp(m)$ is invariant with respect to (1), that is, for every $\tilde{X} \in Sp(m)$ and $\tilde{u} \in L^2([0, T]; \mathbb{R}^k),$ $X_{\tilde{X}, \tilde{u}}(t) \in Sp(m)$ for every $t \in [0, T]$. In particular, this means that for every $\tilde{X} \in Sp(m),$ the End-Point mapping $E_{\tilde{X}, T}$ is valued in $Sp(m).$ Given $\tilde{X} \in Sp(m)$ and $\tilde{u} \in L^2([0, T]; \mathbb{R}^k),$ we are interested in local controllability properties of (1) around $\tilde{u}$. The control system (1) is called controllable around $\tilde{u}$ in $Sp(m)$ (in time $T$) if for every final state $X \in Sp(m)$ close to $X_{\tilde{X}, \tilde{u}}(T)$ there is a control $u \in L^2([0, T]; \mathbb{R}^k)$ which steers $\tilde{X}$ to $X,$ that is, such that $E_{\tilde{X}, T}(u) = X.$ Such a property is satisfied as soon as $E_{\tilde{X}, T}$ is locally open at $\tilde{u}.$ Our aim in the next sections is to give an estimate from above on the size of $\|u\|_{L^2}$ in terms of $\|X - X_{\tilde{X}, \tilde{u}}(T)\|$. 
2.2. First order controllability results. Given $T > 0$, $\bar{X} \in \text{Sp}(m)$, a smooth mapping $t \in [0, T] \mapsto A(t) \in M_{2m}(\mathbb{R})$, $k$ matrices $B_1, \ldots, B_k \in M_{2m}(\mathbb{R})$ satisfying (5), and $\bar{u} \in L^2([0, T]; \mathbb{R}^k)$, we say that the control system (1) is controllable at first order around $\bar{u}$ in $\text{Sp}(m)$ if the End-Point mapping $E^{\bar{X}, T} : L^2([0, T]; \mathbb{R}^k) \to \text{Sp}(m)$ is a submersion at $\bar{u}$, that is, if the linear operator $D_{\bar{u}}E^{\bar{X}, T} : L^2([0, T]; \mathbb{R}^k) \to T_{\bar{X}(T)}\text{Sp}(m)$ is surjective with $\bar{X}(T) := X_{\bar{X}, \bar{u}}(T)$. By the Inverse Function Theorem, it is equivalent to existence of $\mu, K > 0$ such that for every $X \in B[\bar{X}(T), \mu] \cap \text{Sp}(m)$, there is $u \in L^2([0, T]; \mathbb{R}^k)$ satisfying

$$E^{\bar{X}, T}(u) = X \quad \text{and} \quad \|u - \bar{u}\|_{L^2} \leq K \|X - \bar{X}(T)\|_1.$$  

The following sufficient condition for first order controllability is given in [34, Proposition 2.1] (see also [21, 20]).

**Proposition 2.1.** Let $T > 0$, $t \in [0, T] \mapsto A(t)$ be a smooth mapping, and let $B_1, \ldots, B_k \in M_{2m}(\mathbb{R})$ be matrices in $M_{2m}(\mathbb{R})$ satisfying (5). Define the $k$ sequences of smooth mappings

$$\left\{B^0_i\right\}_1^k, \ldots, \left\{B^j_i\right\}_1^k : [0, T] \to T_{L^2m}\text{Sp}(m)$$

by

$$ B^0_i(t) := B_i, \quad B^j_i(t) := B^{j−1}_i(t) + B^{j−1}_i(t)A(t) − A(t)B^{j−1}_i(t) \quad \forall j \in \mathbb{N}^∗, $$

for every $t \in [0, T]$ and every $i \in [1, \ldots, k]$. Assume that there exists some $\tilde{t} \in [0, T]$ such that

$$\text{span}\left\{B^j_i(\tilde{t}) \mid i \in [1, \ldots, k], j \in \mathbb{N}\right\} = T_{L^2m}\text{Sp}(m).$$

Then for every $\bar{X} \in \text{Sp}(m)$, the control system (1) is controllable at first order around $\bar{u} \equiv 0$.

In fact, the control system which is relevant in the present paper is not always controllable at first order. For example, it is the case if we consider the control system (see (21)) coming from the canonical flat metric on the torus of dimension $\geq 3$. We leave the reader to check that in this case the assumption (8) does not hold which by analyticity implies that controllability of (21) at first order fails (see [9]). So, we need sufficient condition for controllability at second order.

2.3. Second-order controllability results. Using the same notations as above, we say that the control system (1) is controllable at second order around $\bar{u}$ in $\text{Sp}(m)$ if there are $\mu, K > 0$ such that for every $X \in B[\bar{X}(T), \mu] \cap \text{Sp}(m)$, there is $u \in L^2([0, T]; \mathbb{R}^k)$ satisfying

$$E^{\bar{X}, T}(u) = X \quad \text{and} \quad \|u - \bar{u}\|_{L^2} \leq K \|X - \bar{X}(T)\|^{1/2}. $$

Controllability at second order is weaker than controllability at first order (compare (6)-(9)). It requires a study of the End-Point mapping at second order.
Recall that given two matrices $B, B' \in M_{2m}(\mathbb{R})$, the bracket $[B, B']$ is the matrix in $M_{2m}(\mathbb{R})$ defined as

$$[B, B'] := BB' - B'B.$$

The following result is the key point in the proof of Theorem 1.1. It provides sufficient conditions for controllability at second order of (1) around $\bar{u} \equiv 0$ in a uniform manner. Its proof will be given in Sections 2.5. For the sake of simplicity we restrict here our attention to control systems of the form (1) satisfying (10)-(11). More general results can be found in [20].

**Proposition 2.2.** Let $\Theta$ be a compact set of parameters and $T > 0$ be fixed, assume that for each $\theta \in \Theta$ there are a smooth mapping $t \in [0, T] \to A^\theta(t)$ and $k$ matrices $B^\theta_1, \ldots, B^\theta_k$ in $M_{2m}(\mathbb{R})$ satisfying (5) (with $A(t) = A^\theta(t)$) and such that

$$B^\theta_i B^\theta_j = 0 \quad \forall i, j = 1, \ldots, k. \quad (10)$$

Define for every $\theta \in \Theta$ the $k$ sequences of smooth mappings $\{B^\theta_1(0), \ldots, B^\theta_k(0)\} : [0, T] \to T_{I_{2m}} \text{Sp}(m)$ as in (7) and assume that the properties

$$[B^\theta_i(0), B^\theta_j] \in \text{span} \{B^\theta_r(0) \mid r = 1, \ldots, k, s \geq 0\} \quad \forall i = 1, \ldots, k, \forall j = 1, 2, \quad (11)$$

and

$$\text{span} \{B^\theta_i(0), [B^\theta_i(0), B^\theta_j(0)] \mid i, l = 1, \ldots, k \text{ and } j = 0, 1, 2\} = T_{I_{2m}} \text{Sp}(m). \quad (12)$$

are satisfied for every $\theta \in \Theta$. Assume, moreover, that the sets

$$\{B^\theta_i \mid i = 1, \ldots, k, \theta \in \Theta\} \subset M_{2m}(\mathbb{R})$$

and

$$\{t \in [0, T] \to A^\theta(t) \mid \theta \in \Theta\} \subset C^2([0, T]; M_{2m}(\mathbb{R}))$$

are compact. Then, for every compact set $\mathcal{X} \subset \text{Sp}(m)$, there are $\mu, K > 0$ such that for every $\theta \in \Theta$, every $\bar{X} \in \mathcal{X}$ and every $X \in B\{\bar{X}, \mu\} \cap \text{Sp}(m)$ ($\bar{X}$ denotes the solution at time $T$ of the control system (1) with parameter $\theta$ starting from $\bar{X}$ with control $\bar{u} \equiv 0$), there is $u \in L^2([0, T]; \mathbb{R}^k)$ satisfying

$$E^\bar{X},T_\theta(u) = X \quad \text{and} \quad \|u\|_{L^2} \leq K \left|X - \bar{X}(T)\right|^{1/2} \quad (13)$$

($E^\bar{X},T_\theta$ denotes the End-Point mapping associated with (1) and parameter $\theta$). Moreover, if some set $\mathcal{S} \subset L^2([0, T]; \mathbb{R}^k)$ is dense in $L^2([0, T]; \mathbb{R}^k)$, then for every $\theta \in \Theta$, every $\bar{X} \in \mathcal{X}$ and every $X \in B\{\bar{X}, \mu\} \cap \text{Sp}(m)$, there are a finite number of controls $u^1, \ldots, u^S$ in $\mathcal{S}$ such that any $u$ satisfying property (13) has the form

$$u = \sum_{s=1}^{S} \lambda^s u^s \quad \text{with} \quad \lambda = (\lambda^1, \ldots, \lambda^S) \in \mathbb{R}^S \quad \text{and} \quad |\lambda| \leq K \left|X - \bar{X}(T)\right|^{1/2}. \quad (14)$$

Our proof is based on a series of results on openness properties of $C^2$ mappings near critical points in Banach spaces which was developed by Agrachev and his co-authors [1].
2.4. Some sufficient condition for local openness. Here we are interested in the study of mappings \( F: \mathcal{U} \rightarrow \mathbb{R}^N \) of class \( C^2 \) in an open set \( \mathcal{U} \) in some Banach space \( X \). We call critical point of \( F \) any \( u \in \mathcal{U} \) such that \( D_u F: \mathcal{U} \rightarrow \mathbb{R}^N \) is not surjective. We call the quantity \( \text{corank}(u) := N - \dim(\text{Im}(D_u F)) \) the corank of \( u \). If \( Q: \mathcal{U} \rightarrow \mathbb{R} \) is a quadratic form, its negative index is defined by
\[
\text{ind}_-(Q) := \max \{ \dim(L) \mid Q_{L,\sim|0|} < 0 \}.
\]
The following quantitative result whose proof can be found in [1, 20, 33] provides a sufficient condition at second order for local openness. (We denote by \( B_X(\cdot, \cdot) \) the balls in \( X \) with respect to the norm \( \| \cdot \|_X \).

**Theorem 2.3.** Let \( F: \mathcal{U} \rightarrow \mathbb{R}^N \) be a mapping of class \( C^2 \) on an open set \( \mathcal{U} \subset X \) and \( \bar{u} \in \mathcal{U} \) be a critical point of \( F \) of corank \( r \). Assume that
\[
\text{ind}_-(\lambda^*(D_{\bar{u}}^2 F)|_{\ker(D_{\bar{u}} F)}) \geq r \quad \forall \lambda \in \left( \text{Im}(D_{\bar{u}} F) \right) \perp \sim \{0\}.
\]
Then there exist \( \bar{c}, c \in (0,1) \) such that for every \( \epsilon \in (0, \bar{c}) \) the following property holds: for every \( u \in \mathcal{U}, z \in \mathbb{R}^N \) with
\[
\|u - \bar{u}\|_X < \epsilon, \quad |z - F(u)| < c \epsilon^2,
\]
there are \( w_1, w_2 \in X \) such that \( u + w_1 + w_2 \in \mathcal{U} \),
\[
z = F(u + w_1 + w_2), \quad w_1 \in \ker(D_u F), \quad \|w_1\|_X < \epsilon, \quad \|w_2\|_X < \epsilon^2.
\]
In the above statement, \( (D_{\bar{u}}^2 F)|_{\ker(D_{\bar{u}} F)} \) refers to the quadratic mapping from \( \ker(D_{\bar{u}} F) \) to \( \mathbb{R}^N \) defined by
\[
(D_{\bar{u}}^2 F)|_{\ker(D_{\bar{u}} F)} (v) := D_{\bar{u}}^2 F \cdot (v, v) \quad \forall v \in \ker(D_{\bar{u}} F).
\]
Again, the proof of Theorem 2.3 which follows from previous results by Agrachev-Sachkov [1] and Agrachev-Lee [2] can be found in [20, 33].

2.5. Proof of Proposition 2.2. Let us first assume that there are no parameters, that is, that \( \Theta \) is a singleton. First, we claim that it is sufficient to treat the case \( \bar{X} = I_{2m} \). As a matter of fact, if \( X_u: [0, T] \rightarrow \text{Sp}(m) \subset M_{2m}(\mathbb{R}) \) is solution to the Cauchy problem
\[
\dot{X}_u(t) = A(t)X_u(t) + \sum_{i=1}^{k} u_i(t)B_iX_u(t) \quad \text{for a.e. } t \in [0, T], \quad X_u(0) = I_{2m},
\]
then for every \( \bar{X} \in \text{Sp}(m) \), the trajectory \( \{X_u \bar{X} : [0, T] \rightarrow M_{2m}(\mathbb{R})\} \) starts at \( \bar{X} \) and satisfies
\[
\frac{d}{dt} (X_u(t) \bar{X}) = A(t) (X_u(t) \bar{X}) + \sum_{i=1}^{k} u_i(t)B_i (X_u(t) \bar{X}) \quad \text{for a.e. } t \in [0, T].
\]
So any trajectory of (1), that is, any control, steering \( I_{2m} \) to some \( X \in \text{Sp}(m) \) gives rise to a trajectory, with the same control, steering \( \bar{X} \in \text{Sp}(m) \) to \( \bar{X}X \in \text{Sp}(m) \). Since right-translations in \( \text{Sp}(m) \) are diffeomorphisms, we infer that local controllability at second order around \( \bar{u} \equiv 0 \) from \( \bar{X} = I_{2m} \) implies controllability at second order around \( \bar{u} \equiv 0 \) for any \( \bar{X} \in \text{Sp}(m) \). Then the existence of
We infer that $\overline{X} = I_{2m}$ (in the sequel we omit the lower index and simply write $I$). We recall that $\overline{X} : [0, T] \to \text{Sp}(m) \subset M_{2m}(\mathbb{R})$ denotes the solution of (17) associated with $u = \bar{u} \equiv 0$ while $X_u : [0, T] \to \text{Sp}(m) \subset M_{2m}(\mathbb{R})$ stands for a solution of (17) associated with some control $u \in L^2([0, T]; \mathbb{R}^k)$. Furthermore, we may also assume that the End-Point mapping $E^{I,T} : L^2([0, T]; \mathbb{R}^k) \to \text{Sp}(m)$ is not a submersion at $\bar{u}$ because it would imply controllability at first order around $\bar{u}$ and so at second order, as desired.

We equip the vector space $M_{2m}(\mathbb{R})$ with the scalar product defined by

$$P \cdot Q = \text{tr}(P^*Q) \quad \forall P, Q \in M_{2m}(\mathbb{R}).$$

Let us fix $P \in T_{\overline{X}(T)}\text{Sp}(m)$ such that $P$ belongs to $\text{Im}(D_0 E^{I,T}) \subseteq [0]$ with respect to our scalar product (note that $\text{Im}(D_0 E^{I,T}) \subseteq [0]$ is nonempty since $D_0 E^{I,T} : L^2([0, T]; \mathbb{R}^k) \to T_{\overline{X}(T)}\text{Sp}(m)$ is assumed to be not surjective).

**Lemma 2.4.** For every $t \in [0, T]$, we have

$$\text{tr}\left[P^* S(T) S(t)^{-1} B_i^j(t) S(t)\right] = 0 \quad \forall j \geq 0, \forall i = 1, \ldots, k.$$

*Proof of Lemma 2.4.* Since $P$ belongs to $\text{Im}(D_0 E^{I,T}) \subseteq [0]$, we have (remember (4))

$$\text{tr}\left[P^* S(T) \int_0^T S(t)^{-1} \sum_{i=1}^k u_i(t)B_i \overline{X}(t) d t\right] = 0 \quad \forall u \in L^2([0, T]; \mathbb{R}^k).$$

We infer that

$$\text{tr}\left[P^* S(T) S(t)^{-1} B_i S(t)\right] = 0 \quad \forall i \in \{1, \ldots, k\}, \forall t \in [0, T].$$

We conclude by noticing that

$$\frac{d^j}{d t^j}\left(S(t)^{-1} B_i S(t)\right) = S(t)^{-1} B_i^j(t) S(t)$$

for every $t \in [0, T]$.

We check easily that for every $u \in L^2([0, T]; \mathbb{R}^k)$ the second derivative of $E^{I,T}$ at 0 is given by (see [33])

$$D_0^2 E^{I,T}(u) = 2S(T) \int_0^T S(t)^{-1} \sum_{i=1}^k u_i(t)B_i \varphi(t) d t,$$

where

$$\varphi(t) := \sum_{i=1}^k S(t) \int_0^T S(s)^{-1} u_i(s)B_i \overline{X}(s) d s.$$

Then we infer that for every $u \in L^2([0, T]; \mathbb{R}^k)$,

(18) \hspace{1cm} $P \cdot D_0^2 E^{I,T}(u)$

$$= 2 \sum_{i,j=1}^k \int_0^T \int_0^t u_i(t)u_j(s) \text{tr}\left[P^* S(T) S(t)^{-1} B_i S(t) S(s)^{-1} B_j S(s)\right] d s d t.$$
It is useful to work with an approximation of the quadratic form $P \cdot D_0^2 E^{1:T}$. For every $\delta > 0$, we see the space $L^2([0, \delta]; \mathbb{R}^k)$ as a subspace of $L^2([0, T]; \mathbb{R}^k)$ by the canonical immersion

$$u \in L^2([0, \delta]; \mathbb{R}^k) \mapsto \tilde{u} \in L^2([0, T]; \mathbb{R}^k),$$

with $\tilde{u}(t) := u(t)$ if $t \in [0, \delta]$ and $\tilde{u}(t) := 0$ otherwise. For the sake of simplicity, we keep the same notation for $\tilde{u}$ with $\tilde{\delta}$

Then by (18) we infer that for any $Q_\delta$ with $\tilde{\delta}$

where $Q_\delta : L^2([0, \delta]; \mathbb{R}^k) \to \mathbb{R}$ is defined by

$$Q_\delta(u) := 2 \sum_{i,j=1}^k \int_0^\delta \int_0^t u_i(t)u_j(s)) tr\left[ P^* S(T) \mathcal{E}_{i,j}(t,s) \right] ds \, dt \quad \forall u \in L^2([0, \delta]; \mathbb{R}^k),$$

with

$$\mathcal{E}_{i,j}(t,s) = \left( sB_iB_j^1(0) + tB_i^1(0)B_j + \frac{s^2}{2}B_iB_j^2(0) + \frac{t^2}{2}B_i^2(0)B_j + tsB_i^1(0)B_j^1(0) \right)$$

for any $t, s \in [0, T]$.

**Proof of Lemma 2.5.** Setting for every $i, j = 1, \ldots, k$,

$$\mathcal{E}_i(t) := B_i + tB_i^1(0) + \frac{t^2}{2}B_i^2(0) \quad \forall t \in [0, T]$$

and using (10), we check that for any $t, s \in [0, T]$, $\mathcal{E}_i(t)\mathcal{E}_j(s) = \mathcal{E}_{i,j}(t,s) + \Delta_{i,j}(t,s)$, with

$$\Delta_{i,j}(t,s) := \frac{t^2s^2}{2}B_i^2(0)B_j^1(0) + \frac{ts^2}{2}B_i^1(0)B_j^2(0) + \frac{t^2s^2}{4}B_i^2(0)B_j^2(0).$$

Moreover, remembering that $\frac{d^i}{dt^i} (S(t)^{-1}B_iS(t)) = S(t)^{-1}B_i(t)S(t)$ for every $t \in [0, T]$, we have

$$S(t)^{-1}B_iS(t) = \mathcal{E}_i(t) + O(t^3).$$

Then by (18) we infer that for any $\delta \in (0, \infty)$ and any $u \in L^2([0, \delta]; \mathbb{R}^k)$,

$$P \cdot D_0^2 E^{1:T}(u) - Q_\delta(u) = 2 \sum_{i,j=1}^k \int_0^\delta \int_0^t u_i(t)u_j(s) tr\left[ P^* S(T) \left( O(t^3)B_j(s) + B_i(t)O(t^3) + O(t^3)O(t^3) + \Delta_{i,j}(t,s) \right) \right] ds \, dt.$$

We conclude easily by Cauchy-Schwarz inequality.$\square$

Returning to the proof of Proposition 2.2, we now want to show that the assumption (15) of Theorem 2.3 is satisfied. We are in fact going to show that a stronger property holds, namely that the index of the quadratic form in (15) goes to infinity as $\delta$ tends to zero.
**Lemma 2.6.** For every integer \( N > 0 \), there are \( \delta > 0 \) and a space \( L_δ \subset L^2([0, \delta]; \mathbb{R}^k) \) of dimension larger than \( N \) such that the restriction of \( Q_δ \) to \( L_δ \) satisfies

\[
Q_δ(u) \leq -2C\|u\|^2_2\delta^4 \quad \forall u \in L_δ.
\]

**Proof of Lemma 2.6.** Using the notation

\[
h_1 \circ h_2 = h_1(t) \circ h_2(s) := \int_0^\delta \int_0^t h_1(t)h_2(s) \, ds \, dt,
\]

for any pair of continuous functions \( h_1, h_2 : [0, \delta] \rightarrow \mathbb{R} \), we check that for every \( u \in L^2([0, \delta]; \mathbb{R}^k) \),

\[
(19) \quad \frac{1}{2} Q_δ(u) = \sum_{i,j=1}^k (u_i \circ (su_j)) \text{ tr } [P^* S(T)B_iB_j^1(0)]
\]

\[
+ \sum_{i,j=1}^k \left( (tu_i) \circ u_j \right) \text{ tr } [P^* S(T)B_i^2(0)B_j] + \sum_{i,j=1}^k \left( u_i \circ \left( \frac{1}{2} u_i \right) \right) \text{ tr } [P^* S(T)B_iB_j^2(0)]
\]

\[
+ \sum_{i,j=1}^k \left( \left( \frac{1}{2} u_i \right) \circ u_j \right) \text{ tr } [P^* S(T)B_i^2(0)B_j] + \sum_{i,j=1}^k \left( (tu_i) \circ (su_j) \right) \text{ tr } [P^* S(T)B_i^1(0)B_j^1(0)]
\]

\[
= S_1 + S_2 + S_3 + S_4 + S_5.
\]

Fix \( i, j \in \{1, \ldots, k\} \) with \( i \neq j \) and take \( v = (v_1, \ldots, v_k) \in L^2([0, \delta]; \mathbb{R}^k) \) such that

\[
v_i(t) = 0 \quad \forall t \in [0, \delta], \forall i \in \{1, \ldots, k\} \sim [i, j].
\]

The sum of the first two terms in the right-hand side of (19) is given by (we set \( \hat{P} := P^* S(T) \))

\[
S_1 + S_2 = \left( v_i \circ (sv_j) \right) \text{ tr } [\hat{P}B_iB_j^1(0)] + \left( v_j \circ (sv_i) \right) \text{ tr } [\hat{P}B_jB_i^1(0)]
\]

\[
+ \left( v_i \circ (sv_j) \right) \text{ tr } [\hat{P}B_iB_j^1(0)] + \left( v_j \circ (sv_i) \right) \text{ tr } [\hat{P}B_jB_i^1(0)]
\]

\[
+ \left( (tv_i) \circ v_j \right) \text{ tr } [\hat{P}B_i^1(0)B_j] + \left( (tv_j) \circ v_i \right) \text{ tr } [\hat{P}B_j^1(0)B_i]
\]

\[
+ \left( (tv_i) \circ v_j \right) \text{ tr } [\hat{P}B_i^1(0)B_j] + \left( (tv_j) \circ v_i \right) \text{ tr } [\hat{P}B_j^1(0)B_i].
\]

Note that by integration by parts, we have

\[
\left( v_i \circ (sv_j) \right) \text{ tr } [\hat{P}B_iB_j^1(0)] + \left( (tv_i) \circ v_j \right) \text{ tr } [\hat{P}B_iB_j^1(0)B_i]
\]

\[
= \left( \int_0^\delta v_i(s) \, ds \right) \left( \int_0^\delta sv_j(s) \, ds \right) \text{ tr } [\hat{P}B_iB_j^1(0)] + \left( tv_i \circ v_j \right) \text{ tr } [\hat{P}B_iB_j^1(0), B_i].
\]

Using (11) with \( i = i \) and Lemma 2.4 we obtain \( \text{ tr } [\hat{P}[B_i^1(0), B_i]] = 0 \), and consequently

\[
\left( v_i \circ (sv_j) \right) \text{ tr } [\hat{P}B_iB_j^1(0)] + \left( tv_i \circ v_j \right) \text{ tr } [\hat{P}B_iB_j^1(0)B_i]
\]

\[
= \left( \int_0^\delta v_i(s) \, ds \right) \left( \int_0^\delta sv_j(s) \, ds \right) \text{ tr } [\hat{P}B_iB_j^1(0)].
\]
Similarly, we have

\[
\begin{align*}
\left( v_j \odot (sv_j) \right) \text{tr}[\tilde{P} B_j B_j^1(0)] &+ \left( (tv_j) \odot v_j \right) \text{tr}[\tilde{P} B_j^1(0)B_j^2] \\
&= \left( \int_0^\delta v_j(s) \, ds \right) \left( \int_0^\delta sv_j(s) \, ds \right) \text{tr}[\tilde{P} B_j B_j^1(0)].
\end{align*}
\]

In conclusion, the sum of the first two terms in the right-hand side of (19) can be written as

\[
S_1 + S_2 = \left( v_j \odot (sv_j) \right) \text{tr}[\tilde{P} B_j B_j^1(0)] + \left( v_j \odot (sv_j) \right) \text{tr}[\tilde{P} B_j^1(0)] \\
+ \left( \int_0^\delta v_i(s) \, ds \right) \left( \int_0^\delta sv_i(s) \, ds \right) \text{tr}[\tilde{P} B_i^1(0)] + \left( (tv_j) \odot v_j \right) \text{tr}[\tilde{P} B_j^1(0)B_j^2] \\
+ \left( (tv_j) \odot v_j \right) \text{tr}[\tilde{P} B_j^1(0)B_j^2] + \left( \int_0^\delta v_j(s) \, ds \right) \left( \int_0^\delta sv_j(s) \, ds \right) \text{tr}[\tilde{P} B_j B_j^1(0)].
\]

By the same arguments as above, the sum of the third and fourth terms in the right-hand side of (19) can be written as

\[
S_3 + S_4 = \left( v_j \odot \left( \frac{s^2v_j}{2} \right) \right) \text{tr}[\tilde{P} B_j B_j^2(0)] + \left( v_j \odot \left( \frac{s^2v_j}{2} \right) \right) \text{tr}[\tilde{P} B_j^2(0)] \\
+ \left( \int_0^\delta v_i(s) \, ds \right) \left( \int_0^\delta \frac{s^2v_i(s)}{2} \, ds \right) \text{tr}[\tilde{P} B_i B_i^2(0)] \\
+ \left( \left( \frac{t^2v_j}{2} \right) \odot v_j \right) \text{tr}[\tilde{P} B_j^2(0)B_j] + \left( \left( \frac{t^2v_j}{2} \right) \odot v_j \right) \text{tr}[\tilde{P} B_j^2(0)B_j] \\
+ \left( \int_0^\delta v_j(s) \, ds \right) \left( \int_0^\delta \frac{s^2v_j(s)}{2} \, ds \right) \text{tr}[\tilde{P} B_j B_j^2(0)],
\]

and the fifth (and last) part of \( \frac{1}{2} Q_\delta(v) \) is given by

\[
S_5 = \left( (tv_j) \odot (sv_j) \right) \text{tr}[\tilde{P} B_j B_j^1(0)] \\
+ \left( \int_0^\delta sv_i(s) \, ds \right) \left( \int_0^\delta sv_j(s) \, ds \right) \text{tr}[\tilde{P} B_j B_j^1(0)] \\
+ \frac{1}{2} \left( \int_0^\delta sv_i(s) \, ds \right)^2 \text{tr}[\tilde{P} B_j B_j^1(0)]^2 + \frac{1}{2} \left( \int_0^\delta sv_j(s) \, ds \right)^2 \text{tr}[\tilde{P} B_j B_j^1(0)]^2.
\]

We now need the following technical result whose proof is given in Appendix.

**Lemma 2.7.** Denote by \( \mathcal{L}_{i,j} \) the set of \( v = (v_1, \ldots, v_k) \in L^2([0, 1]; \mathbb{R}^k) \) such that

\[
v_i(t) = 0 \quad \forall t \in [0, 1], \forall i \in \{1, \ldots, k\} \sim \{i, j\},
\]

\[
\int_0^1 v_i(s) \, ds = \int_0^1 sv_i(s) \, ds = \int_0^1 v_j(s) \, ds = \int_0^1 sv_j(s) \, ds = 0,
\]

\[
v_i \odot (sv_j) = v_j \odot (sv_i) = v_i \odot (s^2v_j) = v_j \odot (s^2v_i) = 0, \quad \text{and} \quad (tv_j) \odot (sv_j) > 0.
\]
Then, for every integer \( N > 0 \), there are a vector space \( L_{i,j}^N \subset L^2_{i,j} \cup \{0\} \) of dimension \( N \) and a constant \( K(N) > 0 \) such that

\[
(tv_i) \cap (sv_j) \geq \frac{1}{K(N)} \|v\|_{L^2}^2 \quad \forall \, v \in L_{i,j}^N.
\]

Let us now show how to conclude the proof of Lemma 2.6. Recall that \( P \in T_{\overline{X}(T)} \text{Sp}(m) \) was fixed in \( \{ \text{Im}(D_0 E^{I,T}) \} \perp 0 \) and that by Lemma 2.4, we know that (taking \( t = 0 \))

\[
\text{tr}\left( P^* S(T) B_j^1(0) \right) = 0 \quad \forall \, j \geq 0, \forall \, i \in 1, ..., k.
\]

Consequently, using (12) we infer that there are \( \bar{i}, \bar{j} \in \{1, ..., k\} \) with \( \bar{i} \neq \bar{j} \) such that

\[
\text{tr}\left( P^* S(T) \left[ B_{\bar{i}}^1(0), B_{\bar{j}}^1(0) \right] \right) < 0.
\]

Let \( N > 0 \) an integer be fixed, \( L_{i,j}^N \subset L^2_{i,j} \cup \{0\} \) of dimension \( N \) and the constant \( K(N) > 0 \) given by Lemma 2.7, for every \( \delta \in (0, t) \) denote by \( L_0^N \) the vector space of \( u \in L^2([0, \delta]; \mathbb{R}^k) \subset L^2([0, T]; \mathbb{R}^k) \) such that there is \( v \in L_{i,j} \) satisfying \( u(t) = v(t/\delta) \) for every \( t \in [0, \delta] \). For every \( v \in L_{i,j} \), the control \( u_\delta : [0, T] \rightarrow \mathbb{R}^k \) defined by

\[
u_\delta(t) := v(t/\delta) \quad t \in [0, \delta]
\]

belongs to \( L_0^N \) and by an easy change of variables,

\[
\|u_\delta\|^2 = \int_0^T |u_\delta(t)|^2 \, dt = \int_0^\delta |u_\delta(t)|^2 \, dt = \delta \int_0^1 |v(t)|^2 \, dt = \delta \|v\|^2.
\]

Moreover, it satisfies

\[
Q_\delta(u_\delta) = 2 \left( (tv_i) \cap (sv_j) \right) \delta^3 \text{tr}\left( P^* S(T) \left[ B_{\bar{i}}^1(0), B_{\bar{j}}^1(0) \right] \right).
\]

Then we infer that

\[
\frac{Q_\delta(u_\delta)}{\|u_\delta\|_{L^2}^2 \delta^4} = \frac{2 \left( (tv_i) \cap (sv_j) \right)}{\delta \|v\|_{L^2}^2} \text{tr}\left( P^* S(T) \left[ B_{\bar{i}}^1(0), B_{\bar{j}}^1(0) \right] \right)
\leq \frac{2}{\delta \delta K(N)} \text{tr}\left( P^* S(T) \left[ B_{\bar{i}}^1(0), B_{\bar{j}}^1(0) \right] \right).
\]

We get the result for \( \delta > 0 \) small enough. \( \square \)

We can now conclude the proof of Proposition 2.2. First we note that given \( N \in \mathbb{N} \) strictly larger than \( m(2m + 1) \), if \( L \subset L^2([0, T]; \mathbb{R}^k) \) is a vector space of dimension \( N \), then the linear operator

\[
(D_0 E^{I,T})_{|L} : L \rightarrow T_{\overline{X}(T)} \text{Sp}(m) \subset M_{2m}(\mathbb{R})
\]

has a kernel of dimension at least \( N - m(2m + 1) \), which means \( \ker(D_0 E^{I,T}) \cap L \) has dimension at least \( N - m(2m + 1) \). Then, thanks to Lemma 2.6, for every integer \( N > 0 \), there are \( \delta > 0 \) and a subspace \( L_\delta \subset L^2([0, \delta]; \mathbb{R}^k) \subset L^2([0, T]; \mathbb{R}^k) \) such
that the dimension of $\tilde{L}_\delta := L_\delta \cap \ker(D_0 E^{1,T})$ is larger than $N$ and the restriction of $Q_\delta$ to $\tilde{L}_\delta$ satisfies
\[ Q_\delta(u) \leq -2C\|u\|_{L^2}^2\delta^4 \quad \forall u \in \tilde{L}_\delta. \]
By Lemma 2.5, we have
\[ P \cdot D_0^2 E^{1,T}(u) \leq Q_\delta(u) + C\delta^4 \|u\|_{L^2}^2 \quad \forall u \in \tilde{L}_\delta. \]
Then we infer that
\[ P \cdot D_0^2 E^{1,T}(u) \leq -C\delta^4 \|u\|_{L^2}^2 < 0 \quad \forall u \in \tilde{L}_\delta. \]

Note that since $E^{1,T}$ is valued in $\text{Sp}(m)$, which is a submanifold of $M_{2m}(\mathbb{R})$, assumption (15) is not satisfied and Theorem 2.3 does not apply.

Let $\Pi : M_{2m}(\mathbb{R}) \to T_{\tilde{X}(T)} \text{Sp}(m)$ be the orthogonal projection onto $T_{\tilde{X}(T)} \text{Sp}(m)$. Its restriction to $\text{Sp}(m)$, $\tilde{\Pi} := \Pi_{|\text{Sp}(m)}$, is a smooth mapping whose differential at $\tilde{X}(T)$ is equal to the identity of $T_{\tilde{X}(T)} \text{Sp}(m)$, so it is an isomorphism. Thanks to the Inverse Function Theorem (for submanifolds), $\tilde{\Pi}$ is a local $C^{\infty}$-diffeomorphism at $\tilde{X}(T)$. Hence there exists $\mu > 0$ such that the restriction of $\tilde{\Pi}$ to $B(\tilde{X}(T), \mu) \cap \text{Sp}(m)$
\[ \tilde{\Pi}_{|B(\tilde{X}(T), \mu) \cap \text{Sp}(m)} : B(\tilde{X}(T), \mu) \cap \text{Sp}(m) \to \tilde{\Pi}(B(\tilde{X}(T), \mu) \cap \text{Sp}(m)) \]
is a smooth diffeomorphism. The map $E^{1,T}$ is continuous so
\[ \mathcal{U} \quad \text{is an open set of } L^2([0, T]; \mathbb{R}^k) \quad \text{containing } \tilde{u} = 0. \]
Define the function $F : \mathcal{U} \to T_{\tilde{X}(T)} \text{Sp}(m)$ by $F := \tilde{\Pi} \circ E^{1,T} = \Pi \circ E^{1,T}$. The mapping $F$ is $C^2$, and we have
\[ F(\tilde{u}) = \tilde{X}(T), \quad D_{\tilde{a}} F = D_{\tilde{a}} E^{1,T}, \quad \text{and} \quad D_{\tilde{a}}^2 F = \Pi \circ D_{\tilde{a}}^2 E^{1,T}. \]
Let us check that $F$ satisfies assumption (15). For every $P \in T_{\tilde{X}(T)} \text{Sp}(m)$ such that $P$ belongs to $(\text{Im}(D_0 F))^\perp \sim \{0\}$ and every $v \in L^2([0, T]; \mathbb{R}^k)$, we have
\[ P \cdot D_{\tilde{a}}^2 E^{1,T}(v) = P \cdot \Pi \circ D_{\tilde{a}}^2 E^{1,T}(u) + P \cdot \left( D_{\tilde{a}}^2 E^{1,T}(u) - \Pi \circ D_{\tilde{a}}^2 E^{1,T}(u) \right). \]
But
\[ D_{\tilde{a}}^2 E^{1,T}(u) - \Pi \circ D_{\tilde{a}}^2 E^{1,T}(u) \in \left( T_{\tilde{X}(T)} \text{Sp}(m) \right)^\perp, \]
hence
\[ P \cdot D_{\tilde{a}}^2 E^{1,T}(u) = P \cdot D_0^2 F(u). \]
Therefore, by (20), assumption (15) is satisfied. Consequently, thanks to Theorem 2.3 there exist $\tilde{c}, c \in (0, 1)$ such that for every $\tilde{c} \in (0, \tilde{c})$ the following property holds: for every $u \in \mathcal{U}, Z \in T_{\tilde{X}(T)} \text{Sp}(m)$ with
\[ \|u - \tilde{u}\|_{L^2} \leq \|u\|_{L^2} < \epsilon, \quad |Z - F(u)| < c\epsilon^2, \]
there are $w_1, w_2 \in L^2([0, T]; \mathbb{R}^k)$ such that $u + w_1 + w_2 \in \mathcal{U}$,
\[ Z = F(u + w_1 + w_2), \quad w_1 \in \ker(D_0 F), \quad \|w_1\|_{L^2} < \epsilon, \quad \|w_2\|_{L^2} < \epsilon^2. \]
Apply the above property with \( u = \tilde{u} \equiv 0 \) and \( X \in \text{Sp}(m) \) such that

\[
|X - \tilde{X}(T)| = \frac{c\varepsilon^2}{2} \quad \text{with} \quad \varepsilon < \tilde{\epsilon}.
\]

Set \( Z := \Pi(X) \), then we have (\( \Pi \) is an orthogonal projection so it is 1-lipschitz)

\[
|Z - F(\tilde{u})| = \|\Pi(X) - \Pi(\tilde{X}(T))\| \leq |X - \tilde{X}(T)| = \frac{c\varepsilon^2}{2} < c\varepsilon^2.
\]

Therefore by the above property, there are \( w_1, w_2 \in L^2([0, T]; \mathbb{R}^k) \) such that \( \tilde{u} := w_1 + w_2 \in \mathcal{U} \) satisfies

\[
Z = F(\tilde{u}) \quad \text{and} \quad \|\tilde{u}\|_{L^2} \leq \|w_1\|_{L^2} + \|w_2\|_{L^2} \leq \varepsilon + \varepsilon^2.
\]

Since \( \tilde{\Pi}|_{B(\tilde{X}(T), \mu) \cap \text{Sp}(m)} \) is a local diffeomorphism, taking \( \varepsilon > 0 \) small enough, we infer that

\[
X = E^{i,T}(\tilde{u}) \quad \text{and} \quad \|\tilde{u}\|_{L^2} \leq 2\varepsilon = 2\sqrt{\frac{2}{c}} |X - \tilde{X}(T)|^{1/2}.
\]

In conclusion, the control system (1) is controllable at second order around \( \tilde{u} \equiv 0 \) and we have (13).

Assume now that we are given a set \( \mathcal{S} \subset L^2([0, T]; \mathbb{R}^k) \) which is dense in \( L^2([0, T]; \mathbb{R}^k) \). If the End-Point mapping \( E^{i,T} : L^2([0, T]; \mathbb{R}^k) \rightarrow \text{Sp}(m) \) is a submersion at \( \tilde{u} = 0 \), then by density of \( \mathcal{S} \) and continuity of \( D_0 E^{i,T} \) there are \( u^1, \ldots, u^S \in \mathcal{S} \) such that the restriction of \( D_0 E^{i,T} \) to \( \text{span}\{u^1, \ldots, u^S\} \) is an isomorphism. We infer that the mapping

\[
\begin{align*}
\mathbb{R}^S & \quad \longrightarrow \quad \text{Sp}(m) \\
\lambda = (\lambda^1, \ldots, \lambda^S) & \quad \longrightarrow \quad E^{i,T}(\sum_{s=1}^S \lambda^s u^s)
\end{align*}
\]

is a local diffeomorphism at the origin and the controllability at first order around \( \tilde{u} \equiv 0 \) follows easily. If \( E^{i,T} : L^2([0, T]; \mathbb{R}^k) \rightarrow \text{Sp}(m) \) is not a submersion at \( \tilde{u} = 0 \), then a restriction of the function \( F : \mathcal{U} \rightarrow T_{\tilde{X}(T)}\text{Sp}(m) \) defined above to some subset of \( \mathcal{U} \) of finite dimension contained in \( \text{span}(\mathcal{S}) \) satisfies the assumptions of Theorem 2.3 and the controllability at second order with controls in \( \text{span}(\mathcal{S}) \) (property (14)) follows as above. In the case when \( \Theta \) is a singleton (no parameters), the uniformity of \( K, \mu \) with respect to \( \tilde{X} \in \mathcal{X} \) follows by compactness of \( \mathcal{X} \).

In the general case, the result follows by compactness of \( \Theta \) and of the sets

\[
\left\{ B^0_i \mid i = 1, \ldots, k, \theta \in \Theta \right\} \subset M_{2m}(\mathbb{R})
\]

and

\[
\left\{ t \in [0, T] \mapsto A^0(t) \mid \theta \in \Theta \right\} \subset C^2([0, T]; M_{2m}(\mathbb{R})).
\]

We refer the reader to [20, 33] for further details.
3. Proof of Theorem 1.1

Let $T \geq \bar{t}$, $\gamma_\theta : [0, T] \to M$ be a unit speed geodesic of length $T$, and $\bar{t} \in [0, T]$ with $[\bar{t}, \bar{t} + \bar{t}] \subset [0, T]$ be fixed. Set

$$\tilde{\theta} = (\bar{p}, \bar{v}) := (\gamma_\theta(\bar{t}), \gamma_\theta'(\bar{t})), \quad \tilde{\theta} = (p, \bar{v}) := (\gamma_\theta(\bar{t}), \gamma_\theta'(\bar{t} + \bar{t})),$$

and consider local transverse sections $\Sigma_0, \tilde{\Sigma}, \Sigma_T \subset T_1M$ respectively tangent to $N_\theta, N_{\bar{\theta}}, N_{\bar{\theta}}, N_T$. Then we have

$$P_{\Sigma}(\gamma_\theta)(T) = D_\theta P_{\Sigma}(\Sigma_0, \Sigma_T, \gamma_\theta)$$

$$= D_\Sigma P_{\Sigma}(\tilde{\Sigma}, \Sigma_T, \gamma_\theta) \circ D_\theta P_{\Sigma}(\tilde{\Sigma}, \Sigma_0, \gamma_\theta) \circ D_\theta P_{\Sigma}(\Sigma_0, \tilde{\Sigma}, \gamma_\theta).$$

Since the sets of endomorphisms of $\text{Sp}(m)$ of the form $D_\theta P_{\Sigma}(\tilde{\Sigma}, \Sigma_T, \gamma_\theta)$ and $D_\theta P_{\Sigma}(\Sigma_0, \tilde{\Sigma}, \gamma_\theta)$, that is, the differential of Poincaré maps associated with geodesics of lengths $T - \bar{t} - \bar{t}$ and $\bar{t}$, are compact and the left and right translations in $\text{Sp}(m)$ are diffeomorphisms and since moreover self-intersections are transverse, it is sufficient to prove Theorem 1.1 with $T = \bar{t}$. More exactly, it is sufficient to show that there are $\delta, \bar{K} > 0$ such that for every $\delta \in (0, \bar{\delta})$, the following property holds: let $\gamma = \gamma_\theta : [0, \bar{t}] \to M$ be a unit geodesic in $M$ and $\Sigma = \cup_{\ell=1}^{\bar{L}} c_\ell(I_\ell)$ be a finite set of smooth curves $c_\ell : I_\ell \to M$ (each $I_\ell$ is a compact interval) which are all transverse to $\gamma([0, \bar{t}])$, let $U$ be the open ball centered at $P_{\Sigma}(\gamma)(\bar{t})$ of radius $\delta$ in $\text{Sp}(n-1)$; then for each symplectic map $A \subset U$ there exists a $C^\infty$ metric $h$ in $M$ that is conformal to $g$, $h_p(v, w) = (1 + \sigma(p))^2 g_p(v, w)$, such that

1. The geodesic $\gamma : [0, \bar{t}] \to M$ is still a geodesic of $(M, h)$,
2. $\text{supp}(\sigma) \subset E_\Sigma \{\gamma([0, \bar{t}]); \rho\} \sim \Sigma$,
3. $h_p(\gamma(\bar{t})) = A$,
4. the $C^2$ norm of the function $\sigma$ is less than $K \sqrt{h}$. 

Let $\gamma = \gamma_\theta : [0, \bar{t}] \to M$ a unit geodesic in $M$ and and $\Sigma = \cup_{\ell=1}^{\bar{L}} c_\ell(I_\ell)$ a finite set of smooth curves be fixed. Since $\gamma$ has no self intersection, we can consider a Fermi coordinate system $\Phi(t, x_1, x_2, \ldots, x_{n-1})$, $t \in (0, \bar{t})$, $(x_1, x_2, \ldots, x_{n-1}) \in (-\delta, \delta)^{n-1}$ along $\gamma([0, \bar{t}])$, where $t$ is the arc length of $\gamma$, and the coordinate vector fields $e_1(t), \ldots, e_{n-1}(t)$ of the system are orthonormal and parallel along $\gamma$. Denote by $t_1 < \ldots < t_{\bar{v}}$ the set of times in $(0, \bar{t})$ such that

$$\gamma([0, \bar{t}]) \cap \Sigma = \{\gamma(t_1), \ldots, \gamma(t_{\bar{v}})\}$$

and set $t_0 := 0, t_{\bar{v}+1} := \bar{t}$. Let us consider the family of smooth functions $P_{ij} : \mathbb{R}^{n-1} \to \mathbb{R}$ with $i, j = 1, \ldots, n - 1$ defined by

$$P_{ij}(y_1, y_2, \ldots, y_{n-1}) := y_i y_j Q(|y|) \quad \forall i \neq j \in \{1, \ldots, n - 1\}$$

and

$$P_{ii}(y_1, y_2, \ldots, y_{n-1}) := \frac{y_i^2}{2} Q(|y|) \quad \forall i \in \{1, \ldots, n - 1\},$$

where $Q$ is a smooth function such that $Q(y) = \frac{1}{2} y^2$ for $y \in \mathbb{R}$.
for every \( y = (y_1, y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-1} \) where \( Q : \mathbb{R} \to [0, +\infty) \) is a smooth cutoff function satisfying

\[
\begin{cases}
Q(\lambda) = 1 & \text{if } |\lambda| \leq 1/3 \\
Q(\lambda) = 0 & \text{if } |\lambda| \geq 2/3.
\end{cases}
\]

Define in addition for every \( a \in (0, 1/100) \) the function \( \omega_a : \mathbb{R} \to [0, +\infty) \) by

\[
\omega_a(s) := 1 - Q\left(\frac{(t - t_j)}{a}\right) \quad \forall t \in [t_j, t_{j+1}], \forall j = 0, \ldots, v
\]

and \( \omega_a = 0 \) outside \([0, \bar{t}]\). By construction, \( \omega_a \) is smooth, vanishes in the neighborhood of each \( t_j \) \( (j = 0, \ldots, v + 1) \), and is equal to one on a large subinterval of each interval \([t_j, t_{j+1}] \) \( (j = 0, \ldots, v) \).

Given a radius \( \rho > 0 \) with \( \mathcal{C}_g \left( \gamma((0, \bar{t})]; \rho \right) \subset \Phi \left( I_r \times (-\delta, \delta)^{n-1} \right) \), \( a \in (0, 1/100) \), and a family of smooth functions \( u = (u_{ij})_{i \leq j = 1, \ldots, n-1} : \mathbb{R} \to \mathbb{R} \) we define a family of smooth perturbations

\[
\sigma_{ij}^{a, \rho, u} \left( \Phi \left( I_r \times (-\delta, \delta)^{n-1} \right) \right)
\]

with support in \( \Phi((0, \bar{t})] \times (-\delta, \delta)^{n-1} \) by

\[
\sigma_{ij}^{a, \rho, u} \left( \Phi \left( I_r \times (-\delta, \delta)^{n-1} \right) \right) := \rho^2 \omega_a(t) u_{ij}(t) P_{ij} \left( \frac{x_1}{\rho}, \frac{x_2}{\rho}, \ldots, \frac{x_{n-1}}{\rho} \right),
\]

for every \( p = \Phi \left( (0, \bar{t})] \times (-\delta, \delta)^{n-1} \right) \) and we define \( \sigma^{a, \rho, u} : M \to \mathbb{R} \) by

\[
\sigma^{a, \rho, u} := \sum_{i,j=1}^{n-1} \sigma_{ij}^{a, \rho, u}.
\]

The following result follows easily (see also [37, Lemmas 2.1 and 2.2]). It relies on the construction and on well known formulae for the conformal connection and sectional curvatures. The notation \( \partial_l \) with \( l = 0, 1, \ldots, n-1 \) stands for the partial derivative in coordinates \( x_0 = t, x_1, \ldots, x_{n-1} \) and \( H\sigma^{a, \rho, u} \) denotes the Hessian of \( \sigma^{a, \rho, u} \) with respect to \( g \).

**Lemma 3.1.** The following properties hold:

1. \( \text{supp}(\sigma^{a, \rho, u}) \subset \mathcal{C}_g \left( \gamma((0, \bar{t})]; \rho \right) \),
2. \( \sigma^{a, \rho, u}(\gamma(t)) = 0 \) for every \( t \in [0, \bar{t}] \),
3. \( \sigma^{a, \rho, u}(\Phi(t, \hat{x})) = 0 \) for every \( t \in [t_j - a/3, t_j + a/3] \), every \( j = 0, \ldots, v + 1 \) and \( \hat{x} \in (-\delta, \delta)^{n-1} \),
4. \( \partial_l \sigma^{a, \rho, u}(\gamma(t)) = 0 \) for every \( t \in [0, \bar{t}] \) and \( l = 0, 1, \ldots, n-1 \),
5. \( (H\sigma^{a, \rho, u})_{i,0}(\gamma(t)) = 0 \) for every \( t \in [0, \bar{t}] \) and \( i = 1, \ldots, n-1 \),
6. \( (H\sigma^{a, \rho, u})_{i,j}(\gamma(t)) = \omega_a(t) u_{ij}(t) \) for every \( t \in [0, \bar{t}] \) and \( i, j = 1, \ldots, n-1 \),
7. \( \|\sigma^{a, \rho, u}\|_{C^2} \leq C \left( \frac{\rho}{a} + \frac{\rho^2}{a^2} \right) \|u\|_{C^2} \) for some universal constant \( C > 0 \),
8. \( \|u - \omega_a u\|_{L^2} < Cva\|u\|_{L^2} \),
9. The conformal metric

\[
h := \left(1 + \sigma^{a, \rho, u} \right)^2 g.
\]

has the following properties:

- **The** \((M, g)\) **-geodesic** \( \gamma : [0, \bar{t}] \to M \) **is a geodesic of** \((M, h)\) .
The sectional curvatures of \((M, h)\) at \(\gamma(t), t \in [0, \bar{t}]\), with respect to the orthonormal basis \(\dot{\gamma}(t), e_1, e_2, \ldots, e_{n-1}\) are
\[
R^h_{ij}(t) := R^h(e_i(t), \dot{\gamma}(t), e_j(t), \dot{\gamma}(t)) = R_{ij}(t) - \omega_{\sigma}(t)u_{ij}(t),
\]
where \(R_{ij}(t) := R(e_i(t), \dot{\gamma}(t), e_j(t), \dot{\gamma}(t))\), and \(R, R^h\) are respectively the curvature tensors of \((M, g)\) and \((M, h)\).

Set \(m := n - 1\) and \(k := m(m + 1)/2\). Let us study the effect of a function \(u = (u_{ij})_{i,j=1,\ldots,n-1} : [0, \bar{t}] \rightarrow \mathbb{R}^{n(n-1)/2}\) on the symplectic mapping \(P_h(\gamma)(\bar{t})\). By the Jacobi equation, we have
\[
P_h(\gamma)(\bar{t})(J(0), J(0)) = (J(\bar{t}), J(\bar{t})),
\]
where \(J : [0, \bar{t}] \rightarrow \mathbb{R}^m\) is solution to the Jacobi equation
\[
\dot{J}(t) + R^h(t)J(t) = 0 \quad \forall t \in [0, \bar{t}],
\]
where \(R^h(t)\) is the \(m \times m\) symmetric matrix whose coefficients are the \(R_{ij}(t)\). In other terms, \(P_h(\gamma)(\bar{t})\) is equal to the \(2m \times 2m\) symplectic matrix \(X(\bar{t})\) given by the solution \(X : [0, \bar{t}] \rightarrow \text{Sp}(m)\) at time \(\bar{t}\) of the following Cauchy problem (compare [34, Sect. 3.2] and [21]):
\[
X(t) = A(t)X(t) + \sum_{i,j=1}^m u_{ij}(t)\mathcal{S}(i,j)X(t) \quad \forall t \in [0, \bar{t}], \quad X(0) = I_{2m},
\]
where the \(2m \times 2m\) matrices \(A(t), \mathcal{S}(i,j)\) are defined by
\[
A(t) := \begin{pmatrix} 0 & I_m \\ -R(t) & 0 \end{pmatrix} \quad \forall t \in [0, \bar{t}] \quad \text{and} \quad \mathcal{S}(i,j) := \begin{pmatrix} 0 & 0 \\ (E(i,j) & 0) \end{pmatrix},
\]
where the \(E(i,j), 1 \leq i \leq j \leq m\) are the symmetric \(m \times m\) matrices defined by
\[
(E(i,j))_{k,l} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \quad \forall i, j, k, l = 1, \ldots, m.
\]
Since our control system has the form (1), all the results gathered in Section 2 apply. So, Theorem 1.1 will follow from Proposition 2.2. First by compactness of \(M\) and regularity of the geodesic flow, the compactness assumptions in Proposition 2.2 are satisfied. It remains to check that assumptions (10), (11), and (12) hold.

First we check immediately that
\[
\mathcal{S}(i,j)\mathcal{S}(k,l) = 0 \quad \forall i, j, k, l \in \{1, \ldots, m\} \text{ with } i \leq j, k \leq l,
\]
so assumption (10) is satisfied. Since the \(\mathcal{S}(i,j)\) do not depend on time, we check easily that the matrices \(B^0_{ij}, B^1_{ij}, B^2_{ij}\) associated to our system are given by (remember that we use the notation \([B, B'] := BB' - B'B)\)
\[
B^0_{ij}(t) = B_{ij} := \mathcal{S}(i,j), \quad B^1_{ij}(t) = [\mathcal{S}(i,j), A(t)], \quad B^2_{ij}(t) = \left[[\mathcal{S}(i,j), A(t)], A(t)\right],
\]
for every \(t \in [0, \bar{t}]\) and any \(i, j = 1, \ldots, m\) with \(i \leq j\). An easy computation yields for any \(i, j = 1, \ldots, m\) with \(i \leq j\) and any \(t \in [0, \bar{t}]\),
\[
B^1_{ij}(t) = [\mathcal{S}(i,j), A(t)] = \begin{pmatrix} -E(ij) & 0 \\ 0 & E(ij) \end{pmatrix}
\]
and
\[ B^2_{ij}(t) = \begin{bmatrix} 0 & -E(ij)R(t) - R(t)E(ij) \\ -E(ij)R(t) - R(t)E(ij) & 0 \end{bmatrix}. \]

Then we get for any \( i, j = 1, \ldots, m \) with \( i \leq j \),
\[ [B^1_{ij}(0), B_{ij}] = 2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \subseteq \text{span}\{B^0_{rs}(0) \mid r \leq s\} \]
and
\[ [B^2_{ij}(0), B_{ij}] = 2 \begin{bmatrix} -(E(ij))^2 & 0 \\ 0 & (E(ij))^2 \end{bmatrix} \subseteq \text{span}\{B^1_{rs}(0) \mid r \leq s\}. \]

So assumption (11) is satisfied. It remains to show that (12) holds. We first notice that for any \( i, j, k, l = 1, \ldots, m \) with \( i \leq j, k \leq l \), we have
\[ [B^1_{ij}(0), B^1_{kl}(0)] = \begin{bmatrix} [\mathcal{E}(i, j), A(0)] & [\mathcal{E}(k, l), A(0)] \end{bmatrix} \begin{bmatrix} E(ij) & 0 \\ 0 & E(kl) \end{bmatrix}, \]
with
\[ E(ij), E(kl) = \delta_{il}F(jk) + \delta_{jk}F(il) + \delta_{ik}F(jl) + \delta_{jl}F(ik), \]
where \( F(pq) \) is the \( m \times m \) skew-symmetric matrix defined by
\[ (F(pq))_{rs} := \delta_{rp}\delta_{sq} - \delta_{rq}\delta_{sp}. \]

It is sufficient to show that the space \( S \subset M_{2m}(\mathbb{R}) \) given by
\[ S := \text{span}\{B^0_{ij}(0), B^1_{ij}(0), B^2_{ij}(0), [B^1_{kl}(0), B^1_{rs}(0)] \mid i, j, k, l, r \} \subset T_{\text{lin}} \text{Sp}(m) \]
has dimension \( p := 2m(2m + 1)/2 \). First since the set matrices \( \mathcal{E}(i, j) \) with \( i, j = 1, \ldots, m \) and \( i \leq j \) forms a basis of the vector space of \( m \times m \) symmetric matrices \( \mathcal{S}(m) \) we check easily by the above formulas that the vector space
\[ S_1 := \text{span}\{B^0_{ij}, B^2_{ij} \mid i, j \} = \text{span}\{\mathcal{E}(i, j), \mathcal{E}(i, j), A(0), A(0) \mid i, j \} \]
has dimension \( 2(m(m + 1)/2) = m(m + 1) \). We check easily that the vector spaces
\[ S_2 := \text{span}\{B^1_{ij} \mid i, j \} \]
and
\[ S_3 := \text{span}\{[B^1_{ij}, B^1_{kl}] \mid i, j, k, l \} \]
are orthogonal to \( S_1 \) with respect to the scalar product \( P \cdot Q = \text{tr}(P^*Q) \). So, we need to show that \( S_2 + S_3 \) has dimension \( p - m(m + 1) = m^2 \). By the above formulas, we have
\[ S_2 = \text{span}\begin{bmatrix} -E(ij) & 0 \\ 0 & E(ij) \end{bmatrix} \]
and
\[ S_3 = \text{span}\begin{bmatrix} E(ij) & 0 \\ 0 & E(ij) \end{bmatrix}. \]
and in addition $S_2$ and $S_3$ are orthogonal. The first space $S_2$ has the same dimension as $\mathcal{S}(m)$, that is, $m(m+1)/2$. Moreover, by (22) for every $i \neq j, k = i,$ and $l \notin \{i, j\}$, we have

$$[E(ij), E(kl)] = F(jl).$$

The space spanned by the matrices of the form

$$\begin{pmatrix} F(jl) & 0 \\ 0 & F(jl) \end{pmatrix},$$

with $1 \leq j < l \leq m$, has dimension $m(m-1)/2$. This shows that $S_3$ has dimension at least $m(m-1)/2$ and so $S_2 \neq S_3$ has dimension $m^2$. In conclusion, the control system (21) satisfies all the assumptions of Proposition 2.2.

Given a unit geodesic $\gamma : [0, \bar{\tau}] \to M$ and a finite set of smooth curves $\Sigma = \bigcup_{\ell=1}^{L} \xi_{\ell}(\bar{I}_\ell)$ all of which are transverse to $\gamma([0, \bar{\tau}])$, the set of smooth controls

$$\nu = \{nu_{ij}\}_{i \leq j = 1, \ldots, n-1} : [0, \bar{\tau}] \to \mathbb{R}^{n(n-1)/2}$$

of the form $\nu = \omega_{ij}(t)u$ with $u = (u_{ij})_{i \leq j = 1, \ldots, n-1} : [0, \bar{\tau}] \to \mathbb{R}^{n(n-1)/2}$ a smooth function and $a \in (0, 1/100)$ is dense in $L^2([0, \bar{\tau}]; \mathbb{R}^{n(n-1)/2})$ (by Lemma 3.1.8). Thus, by Proposition 2.2 there is a finite number of smooth functions $u^1, \ldots, u^s : [0, \bar{\tau}] \to \mathbb{R}^{n(n-1)/2}$ such that any symplectic matrix $A$ close to $P_{0}(\gamma)(\bar{\tau})$ has the form $X(\bar{\tau})$ where $X$ is solution to the Cauchy problem (21) associated with a smooth control of the form $\sum_{s=1}^{S} \lambda^s \omega_{ij} u^i$ with $|\lambda| \leq K|A - P_{0}(\gamma)(\bar{\tau})|^{1/2}$ and $a_1, \ldots, a_S \in (0, 1/100)$. Since the factors $\sigma^{a}, \rho, u^s$ vanish above each intersection of $\Sigma$ with $\gamma([0, \bar{\tau}])$ (by Lemma 3.1.3, we infer easily that $\sigma := \sum_{s=1}^{S} \lambda^s \sigma^{a}, \rho, u^s$ solves the required properties 1, 2', 3, 4 for $\rho > 0$ small enough). This concludes the proof of Theorem 1.1.

**Remark 3.2.** By Proposition 2.1, the control system (21) can be shown to be controllable at first order along $\nu \equiv 0$ if (8) holds. This amounts to verifying that some assumption on the curvature along $\gamma$ is satisfied, see [20, 21].

4. **Proofs of Theorems 1.2 and 1.3**

Let us start with the proof of Theorem 1.2, namely, if the periodic orbits of the geodesic flow of a smooth compact manifold $(M, g)$ of dimension $\geq 2$ are $C^2$ persistently hyperbolic from Mañé’s viewpoint then the closure of the set of periodic orbits is a hyperbolic set. Recall that an invariant set $\Lambda$ of a smooth flow $\psi : Q \to Q$ acting without singularities on a complete manifold $Q$ is called hyperbolic if there exist constants $C > 0, \lambda \in (0, 1)$, and a direct sum decomposition $T_{0}Q = E^s(p) \oplus E^u(p) \oplus X(p)$ for every $p \in \Lambda$, where $X(p)$ is the subspace tangent to the orbits of $\psi$, such that

1. $\|D\psi_{t}(W)\| \leq C\lambda^{t}\|W\|$ for every $W \in E^s(p)$ and $t \geq 0$,
2. $\|D\psi_{t}(W)\| \leq C\lambda^{-t}\|W\|$ for every $W \in E^u(p)$ and $t \leq 0$.

In particular, when the set $\Lambda$ is the whole $Q$ the flow is called Anosov. The proof follows the same steps of the proof of Theorem B in [36] where the same conclusion is obtained supposing that the geodesic flow is $C^1$ persistently expansive in the family of Hamiltonian flows.
4.1. **Dominated splittings and hyperbolicity.** Let $F^2(M, g)$ be the set of Riemannian metrics in $M$ conformal to $g$ endowed with the $C^2$ topology such that all closed orbits of their geodesic flows are hyperbolic.

The first step of the proof of Theorem 1.2 is closely related with the notion of dominated splitting introduced by Mañé.

**Definition 4.1.** Let $\phi_t : Q \rightarrow Q$ be a smooth non-singular flow acting on a complete Riemannian manifold $Q$ and let $\Omega \subset Q$ be an invariant set. We say that $\Omega$ has a dominated splitting in $\Omega$ if there exist constants $\delta \in (0, 1)$, $m > 0$ and invariant subspaces $S(\theta), U(\theta)$ in $T_\theta \Omega$ such that for every $\theta \in \Omega$,

1. if $X(\theta)$ is the unit vector tangent to the flow then $S(\theta) \oplus U(\theta) \oplus X(\theta) = T_\theta Q$;
2. $\| D_\theta \phi_m |_{S(\theta)} \| \cdot \| D\phi_m |_{U(\phi_m(\theta))} \| \leq \delta$.

The invariant splitting of an Anosov flow is always dominated, but the converse may not be true in general. However, for geodesic flows the following statement holds.

**Theorem 4.2.** Any continuous, Lagrangian, invariant, dominated splitting in a compact invariant set for the geodesic flow of a smooth compact Riemannian manifold is a hyperbolic splitting. Therefore, the existence of a continuous Lagrangian invariant dominated splitting in the whole unit tangent bundle is equivalent to the Anosov property in the family of geodesic flows.

This statement is proved in [36] not only for geodesic flows but for symplectic diffeomorphisms. Actually, the statement extends easily to a Hamiltonian flow in a nonsingular energy level (see also Contreras [6]).

The following step of the proof of Theorem 1.2 relies on the connection between persistent hyperbolicity of periodic orbits and the existence of invariant dominated splittings. One of the most remarkable facts about Mañé’s work about the stability conjecture (see Proposition II.1 in [22]) is to show that persistent hyperbolicity of families of linear maps is connected to dominated splittings, the proof is pure generic linear algebra (see Lemma II.3 in [22]). Then Mañé observes that Franks’ Lemma allows us to reduce the study of persistently hyperbolic families of periodic orbits of diffeomorphisms to persistently hyperbolic families of linear maps. Let us explain briefly Mañé’s result and see how its combination with Franks’ Lemma for geodesic flows implies Theorem 1.2.

Let $GL(n)$ be the group of linear isomorphisms of $\mathbb{R}^n$. Let $\psi : \mathbb{Z} \rightarrow GL(n)$ be a sequence of such isomorphisms, we denote by $E^s_j(\psi)$ the set of vectors $v \in \mathbb{R}^n$ such that

$$\sup_{n \geq 0} \left\{ \left\| \Pi_{i=0}^n \psi_{j+i} \right\| v \right\} < \infty,$$

and by $E^u_j(\psi)$ the set of vectors $v \in \mathbb{R}^n$ such that

$$\sup_{n \geq 0} \left\{ \left\| \Pi_{i=0}^n \psi_{j-1-i} \right\|^{-1} v \right\} < \infty.$$

Let us say that the sequence $\psi$ is hyperbolic if $E^s_j(\psi) \oplus E^u_j(\psi) = \mathbb{R}^n$ for every $j \in \mathbb{Z}$. Actually, this definition is equivalent to requiring the above direct sum
We say that the family \( \psi \) is hyperbolic if every sequence in the family is hyperbolic. Let us call by periodically equivalent two families \( \psi, \eta \) for which given any \( \alpha \), the minimum periods of \( \psi^\alpha \) and \( \eta^\alpha \) coincide. Following Mañé, we say that the family \( \{\psi^\alpha, \alpha \in \Lambda\} \) is uniformly hyperbolic if there exists \( \epsilon > 0 \) such that every periodically equivalent family \( \eta^\alpha \) such that \( d(\psi, \eta) < \epsilon \) is also hyperbolic. The main result concerning uniformly hyperbolic families of linear maps is the following symplectic version of Lemma II.3 in [22].

**Theorem 4.3.** Let \( \{\psi^\alpha, \alpha \in \Lambda\} \) be a uniformly hyperbolic family of periodic linear sequences of symplectic isomorphisms of \( \mathbb{R}^n \). Then, there exist constants \( K > 0 \), \( m \in \mathbb{N} \), and \( \lambda \in (0, 1) \) such that

1. if \( \alpha \in \Lambda \) and \( \psi^\alpha \) has minimum period \( n \geq m \), then

\[
\prod_{j=0}^{k-1} \left\| (\Pi_{i=0}^{m-1} \psi_{m j+i}^\alpha)_{E^u_{m j+i}(\psi^\alpha)} \right\| \leq K \lambda^k \quad \text{and} \quad \prod_{j=0}^{k-1} \left\| (\Pi_{i=0}^{m-1} \psi_{m j+i}^\alpha)^{-1} \right\|_{E^u_{m j+i}(\psi^\alpha)} \leq K \lambda^k, \]

where \( k \) is the integer part of \( \frac{n}{m} \); 

2. for all \( \alpha \in \Lambda \), \( j \in \mathbb{Z} \),

\[
\left\| (\Pi_{i=0}^{m-1} \psi_{j+i}^\alpha)_{E^u_{j+i}(\psi^\alpha)} \right\| \cdot \left\| (\Pi_{i=0}^{m-1} \psi_{j+i}^\alpha)^{-1} \right\|_{E^u_{j+i}(\psi^\alpha)} \leq \lambda; \]

3. for every \( \alpha \in \Lambda \),

\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln \left( \left\| (\Pi_{i=0}^{m-1} \psi_{m j+i}^\alpha)_{E^u_{m j+i}(\psi^\alpha)} \right\| \right) < 0 \quad \text{and} \quad \limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln \left( \left\| (\Pi_{i=0}^{m-1} \psi_{m j+i}^\alpha)^{-1} \right\|_{E^u_{m j+i}(\psi^\alpha)} \right) < 0.
\]

The proof of Theorem 4.3 is based on Mañé's Lemma II.3 in [22], which is proved for linear isomorphisms without the symplectic assumption. The proof of this lemma is long and involved, and it relies basically on generic properties of linear cocycles. So the proof of Theorem 4.3 consists of following step by step the proof of Lemma II.3 in [22] and checking that it holds for symplectic cocycles. To see a detailed proof we refer to an extended version of this paper in arxiv (Section 4.3) (see also [20]).
Now we are ready to combine Franks’ Lemma from Mañé’s viewpoint and Theorem 4.3 to get a geodesic flow version of Theorem 4.3. Recall that by construction of $\bar{\tau} > 0$, every closed geodesic (with respect to $g$) has period greater than $\bar{\tau}$.

Let $\text{Per}(g)$ be the set of periodic points of the geodesic flow of $(M, g)$. Given a periodic point $\theta \in \text{Per}(g)$ with period $T(\theta)$, consider a family of local sections $\Sigma_\theta^i$, $i = 0, 1, \ldots, k_\theta = \lfloor T(\theta) / \bar{\tau} \rfloor$ ($\lfloor T(\theta) / \bar{\tau} \rfloor$ is the integer part of $T(\theta) / \bar{\tau}$), with the following properties:

1. $\Sigma_\theta^i$ contains the point $\phi_{i \bar{\tau}}(\theta)$ for every $i = 0, 1, \ldots, k_\theta - 1$,
2. $\Sigma_\theta^i$ is perpendicular to the geodesic flow at $\phi_{i \bar{\tau}}(\theta)$ for every $i$.

Let us consider the sequence of symplectic isomorphisms

$$\psi_{\theta, g} = \{ A_{\theta, i, g}, i \in \mathbb{Z} \}$$

1. For $i = nk_\theta + s$, where $n \in \mathbb{Z}$, $0 \leq s < k_\theta - 1$, let

$$A_{\theta, i, g} = D_{\phi_{i \bar{\tau}}(\theta)} \phi_{\bar{\tau}} : T_{\phi_{i \bar{\tau}}(\theta)} \Sigma_\theta^i \to T_{\phi_{(i+1) \bar{\tau}}(\theta)} \Sigma_\theta^{i+1},$$

2. For $i = nk_\theta - 1$, where $n \in \mathbb{Z}$, let

$$A_{\theta, i, g} = D_{\phi_{nk_\theta - \bar{\tau}}(\theta)} \phi_{\bar{\tau} + r_0} : T_{\phi_{nk_\theta - \bar{\tau}}(\theta)} \Sigma_\theta^{nk_\theta - 1} \to T_{\theta} \Sigma_\theta^0$$

where $T(\theta) = k_\theta \bar{\tau} + r_0$.

Notice that the sequence $\psi_{\theta, g}$ is periodic and let

$$\psi_g = \{ \psi_{\theta, g}, \theta \in \text{Per}(g) \}.$$ 

The family $\psi_g$ is a collection of periodic sequences, and by Franks’ Lemma from Mañé’s viewpoint (Theorem 1.1) we have

**Lemma 4.4.** Let $(M, g)$ be a compact Riemannian manifold. If $g$ is in the interior of $F^2(M, g)$ then the family $\psi_g$ is uniformly hyperbolic.

**Proof.** Let $\delta, \bar{\tau} > 0$ be given by Franks’ Lemma (Theorem 1.1 with $T = \bar{\tau}$). If $g$ is in the interior of $F^2(M, g)$ then there exists an open $C^2$ neighborhood $U$ of $g$ in the set of metrics that are conformally equivalent to $g$ such that every closed orbit of the geodesic flow of $(M, h)$, where $h \in U$, is hyperbolic. In particular, given a periodic point $\theta \in T_1 M$ for the geodesic flow of $(M, g)$, the set of metrics $h_\theta \in U$ for which the orbit of $\theta$ is still a periodic orbit for the geodesic flow of $(M, h_\theta)$ have the property that this orbit is hyperbolic as well for the $h_\theta$-geodesic flow. By Theorem 1.1, for any $\delta \in (0, \bar{\delta})$, the $(K \sqrt{\delta})$-$C^2$ open neighborhood of the metric $g$ in the set of its conformally equivalent metrics covers a $\delta$-open neighborhood of symplectic linear transformations of the derivatives of the Poincaré maps between the sections $\Sigma^i, \Sigma^i_{s+1}$ defined above. Then consider $\delta > 0$ such that the $(K \sqrt{\delta})$-$C^2$ open neighborhood of the metric $g$ is contained in $U$, and we get that the family $A_{\theta, i, g}$ is uniformly hyperbolic. Since this holds for every periodic point $\theta$ for the geodesic flow of $(M, g)$ the family $\psi_g$ is uniformly hyperbolic.

Therefore, applying Theorem 4.3 to the sequence $\psi_g$ we obtain the following.
**Theorem 4.5.** Suppose that there exists an open neighborhood $V(e)$ of $g$ in $F^2(M, g)$. Then there exist constants $K > 0$, $D \geq \bar{r}$, $\lambda \in (0, 1)$ such that

1. For every periodic point $\theta$ with minimum period $\omega \geq D$, we have
   \[ \prod_{i=0}^{k-1} \| D\phi_D|_{E^u(\phi_D^i(\theta))} \| \leq K\lambda^k \quad \text{and} \quad \prod_{i=0}^{k-1} \| D\phi_{-D}|_{E^u(\phi_{-D}^i(\theta))} \| \leq K\lambda^k, \]
   where $E^s(\tau) \oplus E^u(\tau) = N_\tau$ is the hyperbolic splitting of the geodesic flow of $(M, g)$ at a periodic point $\tau$ and $k = \lfloor \omega/D \rfloor$;

2. There exists a continuous Lagrangian, invariant, dominated splitting
   \[ T_\theta T_1 M = G^s(\theta) \oplus G^u(\theta) \oplus X(\theta) \]
   in the closure of the set of periodic orbits of $\phi_1$, which extends the hyperbolic splitting of periodic orbits: if $\theta$ is periodic then $G^s(\theta) = E^s(\theta)$, $G^u(\theta) = E^u(\theta)$.

Theorem 4.5 improves Theorem 2.1 in [36] where the same conclusions are claimed assuming that the geodesic flow of $(M, g)$ is in the $C^1$ interior of the set of Hamiltonian flows all of whose periodic orbits are hyperbolic.

Hence, the proof of Theorem 1.2 follows from the combination of Theorems 4.2 and Theorem 4.5.

**4.2. Proof of Theorem 1.3.** Let $E^2(M, g)$ be the set of Riemannian metrics in $M$ conformally equivalent to $g$, endowed with the $C^2$ topology, whose geodesic flows are expansive. The main result of the subsection is an improved version of Proposition 1.1 in [36]. Theorem 1.3 will follow easily from Theorem 4.6 and the results of the previous section.

**Theorem 4.6.** The interior of $E^2(M, g)$ is contained in $F^2(M, g)$.

We just give an outline of the proof based on [36]. The argument is by contradiction. Suppose that there exists $h$ in the interior of $E^2(M, g)$ whose geodesic flow has a nonhyperbolic periodic point $\theta$. Let $\Sigma$ be a cross section of the geodesic flow at $\theta$ tangent to $N_\theta$. The derivative of the Poincaré return map has some eigenvalues in the unit circle. By the results of Rifford-Ruggiero [34] every generic property in the symplectic group is attained by $C^2$ perturbations by potentials of $(M, h)$ preserving the orbit of $\theta$. This means that there exists $\bar{h}$ $C^2$-close to $h$ and conformally equivalent to it such that the orbit of $\theta$ is still a periodic orbit of the geodesic flow of $(M, \bar{h})$ and the derivative of the Poincaré map $\hat{P} : \Sigma \to \Sigma$ has generic unit circle eigenvalues. By the Central Manifold Theorem of Hirsch-Pugh-Shub [16] there exists a central invariant submanifold $\Sigma_0 \subset \Sigma$ such that the return map $P_0$ of the geodesic flow of $(M, \bar{h})$ is tangent to the invariant subspace associated to the eigenvalues of $D\hat{P}$ in the unit circle. Moreover, we can suppose by the $C^k$ Mañé-generic version of the Klingenberg-Takens Theorem due to Carballo-Gonçalves [5] that the Birkhoff normal form of the Poincaré map at the periodic point $\theta$ is generic. So we can apply the Birkhoff-Lewis fixed point Theorem due to Moser [26] to deduce that given $\delta > 0$ there exists infinitely many closed orbits of the geodesic flow of $(M, \bar{h})$. 


in the $\delta$-tubular neighborhood of the orbit of $\theta$. This clearly contradicts the expansiveness of the geodesic flow of $(M, \tilde{h}) \in E^2(M, g)$.

In the case where $(M, g)$ is a closed surface, we know that the expansiveness of the geodesic flow implies the density of the set of periodic orbits in the unit tangent bundle (see [36] for instance). So if $g$ is in the interior of $E^2(M, g)$ the closure of the set of periodic orbits of the geodesic flow of $(M, g)$ is a hyperbolic set by Theorem 1.2, and since this set is dense its closure is the unit tangent bundle, and therefore the geodesic flow is Anosov. If the dimension of $M$ is arbitrary, then we know that if $(M, g)$ has no conjugate points the expansiveness of the geodesic flow implies the density of periodic orbits as well, so we can extend the above result for surfaces.

**APPENDIX A. PROOF OF LEMMA 2.7**

First of all, we observe that given $L_{i,j}^N$, the existence of $K(N)$ follows by homogeneity and continuity of the mapping $v \in L_{i,j}^N \mapsto \{tv_i \cup (sv_j) \in \mathbb{R}$. Let us now demonstrate the existence of $L_{i,j}^N$ by induction over $N$. In fact, setting $f = v_i, g = v_j$, it is sufficient to show that the set $\mathcal{L}$ of $w = (f, g) \in L^2([0,1]; \mathbb{R}^2)$ with $f$ and $g$ polynomials satisfying

$$
\begin{align*}
\int_0^1 f(s)ds &= \int_0^1 sf(s)ds = \int_0^1 sg(s)ds = 0 \\
\int_0^1 g(s)ds &= \int_0^1 sf(s)ds = \int_0^1 sg(s)ds = 0
\end{align*}
$$

contains (adding the origin) vector spaces $L^N$ of any dimension. Assume that we proved the existence of a vector space $L^N \subset \mathcal{L} \cup \{0\}$ of dimension $N \geq 0$. Let $\{(f_1, g_1), \ldots, (f_N, g_N)\}$ be a basis of $L^N$ (if $N = 0$ we take $(f_1, g_1) = (0,0)$ only). We need to find a pair $(f, g)$ such that for any $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$, the pair

$$
\left( \beta f + \sum_{i=1}^N \alpha_i f_i, \beta g + \sum_{i=1}^N \alpha_i g_i \right)
$$

satisfies (23). By bilinearity of the $\circ$ product, this amounts to saying that

$$
\begin{align*}
\beta^2 f \circ (sg) + \beta \sum_{i=1}^N \alpha_i f_i \circ (sg) &= 0 \\
\beta^2 f \circ (s^2 g) + \beta \sum_{i=1}^N \alpha_i f_i \circ (s^2 g) &= 0 \\
\beta^2 g \circ (sf) + \beta \sum_{i=1}^N \alpha_i g_i \circ (sf) &= 0 \\
\beta^2 g \circ (s^2 f) + \beta \sum_{i=1}^N \alpha_i g_i \circ (s^2 f) &= 0
\end{align*}
$$

(24)

and

$$
\begin{align*}
\int_0^1 f(s)ds &= 0 \\
\int_0^1 g(s)ds &= 0 \\
\int_0^1 sf(s)ds &= 0 \\
\int_0^1 sg(s)ds &= 0
\end{align*}
$$

(25)

and

$$
\beta^2 (tf) \circ (sg) + \beta \sum_{i=1}^N \alpha_i (tf_i) \circ (sg) + \beta \sum_{i=1}^N \alpha_i (tf_i) \circ (sg) + \sum_{i=1}^N \alpha_i^2 (tf_i) \circ (sg) \neq 0.
$$

(26)
In fact, any pair \((f, g)\) satisfying (23) and the systems

\[
\begin{align*}
\begin{cases}
  f \circ (sg_1) = 0 & f_1 \circ (sg) = 0 \\
  f \circ (s^2 g_1) = 0 & f_1 \circ (s^2 g) = 0 \\
  g_1 \circ (sf) = 0 & g \circ (sf_1) = 0 \\
  g_1 \circ (s^2 f) = 0 & g \circ (s^2 f_1) = 0 \\
  (t f) \circ (sg) = 0 & (t f_1) \circ (sg) = 0
\end{cases}
\end{align*}
\]

(27)

provides a solution. First we claim that there is a polynomial \(f_0\) satisfying the left systems in (25) and (27). As a matter of fact, \(f_0\) has to belong to the intersection of \(2 + 5N\) hyperplanes in \(\mathbb{R}_d[X]\). Such an intersection is not trivial if \(d\) is large enough. The function \(f = f_0\) being fixed, we need now to find a polynomial \(g\) solution to the four last equations of system in (23) and to the right part, and to the right systems in (25) and (27). Thus \(g\) needs to belong to the intersection of \(6 + 5N\) hyperplanes and to satisfy the right part in (23). Let

\[
f_i(t) = \sum_{p=0}^{P} a_p^i t^p, \quad g_i(t) = \sum_{q=0}^{P} b_q^i t^q, \quad f_0(t) = \sum_{p=0}^{d} a_p^0 t^p, \quad g(t) = \sum_{q \in \mathbb{Z}} b_q t^q,
\]

where \(P\) is the maximum of the degrees of \(f_1, \ldots, f_N, g_1, \ldots, g_N\) and \(d\) is the degree of \(f_0\). We have

\[
\begin{align*}
\begin{cases}
  f_i^1 g(s) ds = \sum_{q=0}^{P} \frac{1}{q+1} b_q^i & f_i \circ (s^2 g) = \sum_{q=0}^{P} \frac{1}{q+2} b_q^i \\
  f_i^0 g(s) ds = \sum_{q=0}^{P} \frac{1}{q+2} b_q^0 & g \circ (sf) = \sum_{q=0}^{P} \frac{1}{q+1} b_q^0 \\
  f_i^1 g(s) ds = \sum_{q=0}^{P} \frac{1}{q+1} b_q^1 & g \circ (s^2 f) = \sum_{q=0}^{P} \frac{1}{q+2} b_q^1
\end{cases}
\end{align*}
\]

and

\[
(t f_i) \circ (sg) = \sum_{q=0}^{P} \sum_{p=0}^{P} \alpha_{p+1,q+1}^i a_p^i b_q
\]

for every \(l = 1, \ldots, N\), and moreover

\[
(t f_0) \circ (sg) = \sum_{q=0}^{P} \sum_{p=0}^{P} \alpha_{p+1,q+1}^0 a_p^0 b_q.
\]

We need to show that the kernel of the linear form (given by (30))

\[
\Phi_{f_0} : (b_q) \in \mathbb{R}_d[X] \mapsto \sum_{q=0}^{P} \sum_{p=0}^{P} \alpha_{p+1,q+1}^0 a_p^0 b_q
\]

does not contain the intersection of the kernels of the \(2 + 4(N+1) + N = 5N + 6\) linear forms given by (28)-(29). If this is the case, for every integer \(d \geq 0\), any choice of \(f_0\) in \(\mathbb{R}_d[X]\), and any integer \(d' \geq 0\), there are \(C = 5N + 6\) real numbers (not all zero)

\[
\lambda_1^{d'}, \ldots, \lambda_5^{d'}, \lambda_6^{0,d'}, \lambda_7^{0,d'}, \lambda_8^{0,d'}, \lambda_9^{0,d'}, \lambda_{10}^{d'}, \lambda_{11}^{d'}.
\]
such that for every integer $q \in \{0, \ldots, d\}$,

$$
\sum_{p=0}^{d} \alpha_{p+1,q+1} a_p^0 - \sum_{l=1}^{N} \lambda_{1}^{l,d'} \left[ \sum_{p=0}^{l} \alpha_{p,q+1} a_p^l \right] + \sum_{l=1}^{N} \lambda_{2}^{l,d'} \left[ \sum_{p=0}^{l} \alpha_{p,q+2} a_p^l \right] \\
+ \sum_{l=1}^{N} \lambda_{3}^{l,d'} \left[ \sum_{p=0}^{l} \alpha_{q,p+1} a_p^l \right] + \sum_{l=1}^{N} \lambda_{4}^{l,d'} \left[ \sum_{p=0}^{l} \alpha_{q,p+2} a_p^l \right] \\
+ \sum_{l=1}^{N} \lambda_{5}^{l,d'} \left[ \sum_{p=0}^{l} \alpha_{p+1,q+1} a_p^l \right] + \sum_{l=1}^{N} \lambda_{6}^{l,d'} \left[ \sum_{p=0}^{l} \alpha_{q,p+1} a_p^l \right] \\
+ \lambda_{7}^{0,d'} \left[ \sum_{p=0}^{d} \alpha_{p,q+2} a_p^0 \right] + \lambda_{8}^{0,d'} \left[ \sum_{p=0}^{d} \alpha_{q,p+1} a_p^0 \right] \\
+ \lambda_{9}^{0,d'} \left[ \sum_{p=0}^{d} \alpha_{q,p+2} a_p^0 \right] + \frac{\lambda_{10}^{d'}}{q+1} + \frac{\lambda_{11}^{d'}}{q+2}.
$$

Observe that the above equality can be written as

$$
0 = \sum_{p=0}^{d} \left[ \left( \sum_{l=1}^{N} \lambda_{3}^{l,d'} a_p^l \right) - a_p^0 \right] \alpha_{p+1,q+1} + \sum_{p=0}^{d} \left[ \left( \sum_{l=1}^{N} \lambda_{1}^{l,d'} a_p^l \right) + \lambda_{7}^{0,d'} a_p^0 \right] \alpha_{p,q+1} \\
+ \sum_{p=0}^{d} \left[ \left( \sum_{l=1}^{N} \lambda_{2}^{l,d'} a_p^l \right) + \lambda_{8}^{0,d'} a_p^0 \right] \alpha_{q,p+1} + \sum_{p=0}^{d} \left[ \left( \sum_{l=1}^{N} \lambda_{4}^{l,d'} a_p^l \right) + \lambda_{9}^{0,d'} a_p^0 \right] \alpha_{q,p+2} + \frac{\lambda_{10}^{d'}}{q+1} + \frac{\lambda_{11}^{d'}}{q+2}.
$$

For every $q$, let

$$
V(q) = \{ V^1(q), \ldots, V^7(q) \} \in \mathbb{R}^{7(d+1)}
$$

with

$$
V^i(q) = \{ V^i_0(q), \ldots, V^i_d(q) \} \in \mathbb{R}^{d+1} \quad \forall i = 1, \ldots, 7,
$$

defined by

$$
V^1_{p}(q) = \alpha_{p+1,q+1}, \quad V^2_{p}(q) = \alpha_{p,q+1}, \quad V^3_{p}(q) = \alpha_{p,q+2}, \quad V^4_{p}(q) = \alpha_{q,p+1}, \quad V^5_{p}(q) = \frac{1}{(d+1)(q+1)}, \quad V^6_{p}(q) = \frac{1}{(d+1)(q+2)},
$$

for every $p = 0, \ldots, d$. The above equality means that for every $d' \geq 0$, there is a linear form $\psi_{d'}$ on $\mathbb{R}^{7(d+1)}$ of the form

$$
\psi_{d'}(V) = \sum_{p=0}^{d} \left[ \Gamma_{1,p}^{d'} - a_p^0 \right] V^1_{p} + \sum_{p=0}^{d} \left[ \Gamma_{2,p}^{d'} + \lambda_{6}^{0,d'} a_p^0 \right] V^2_{p} \\
+ \sum_{p=0}^{d} \left[ \Gamma_{3,p}^{d'} + \lambda_{7}^{0,d'} a_p^0 \right] V^3_{p} + \sum_{p=0}^{d} \left[ \Gamma_{4,p}^{d'} + \lambda_{8}^{0,d'} a_p^0 \right] V^4_{p} \\
+ \sum_{p=0}^{d} \left[ \Gamma_{5,p}^{d'} + \lambda_{9}^{0,d'} a_p^0 \right] V^5_{p} + \sum_{p=0}^{d} \lambda_{10}^{0,d'} V^6_{p} + \sum_{p=0}^{d} \lambda_{11}^{0,d'} V^7_{p}.
$$
for every $V = (V^1, \ldots, V^7) \in (\mathbb{R}^{(d+1)})^7$ such that
\[
\Psi_{d'}(V(q)) = 0 \quad \forall q \in \{0, \ldots, d'\}.
\]
For every integer $d' \geq 0$, let $\dim(d')$ be the dimension of the vector space generated by $V(0), \ldots, V(d')$. The function $d' \mapsto \dim(d')$ is nondecreasing and valued in the positive integers. Moreover, it is bounded by $7(d + 1)$. Thus it is stationary and in consequence there is $\bar{d} \geq 0$ such that for every $q > \bar{d}$,
\[
V(q) \in \text{span}\{V(0), \ldots, V(\bar{d}')\}.
\]
Therefore there is a linear form $\Psi : \mathbb{R}^{7(d+1)} \to \mathbb{R}$ of the form
\[
\Psi(V) = \sum_{p=0}^{d} \left[ \sum_{d=0}^{d} \left[ \frac{\Gamma_p + \lambda_0^0 a_p^0}{d+1} \right] \frac{\Gamma_p - a_p^0}{d+1} \right] V_p^2 + \sum_{p=0}^{d} \left[ \frac{\Gamma_p + \lambda_0^0 a_p^0}{d+1} \right] V_p^2
\]
\[
+ \sum_{p=0}^{d} \left[ \frac{\Gamma_p + \lambda_0^0 a_p^0}{d+1} \right] V_p^2 + \sum_{p=0}^{d} \left[ \frac{\Gamma_p + \lambda_0^0 a_p^0}{d+1} \right] V_p^2 + \sum_{p=0}^{d} \sum_{\lambda_{10}} V_p^2 + \sum_{p=0}^{d} \lambda_{11} V_p^7,
\]
for every $V = (V^1, \ldots, V^7) \in (\mathbb{R}^{(d+1)})^7$ such that
\[
\Psi(V(q)) = 0 \quad \forall q \in \mathbb{N}.
\]
We observe that for any integers $p, q \geq 0$,
\[
\alpha_{p,q} = \frac{1}{(q+1)(p+q+2)} = \frac{1}{p+1} \left[ \frac{1}{q+1} - \frac{1}{q+q+2} \right],
\]
then we have for all $q \in \mathbb{N}$,
\[
0 = \Psi(V(q))
\]
\[
= \sum_{p=0}^{d} \lambda_{10} \frac{1}{(q+1)} \left[ \sum_{d=0}^{d} \left[ \frac{\Gamma_p + \lambda_0^0 a_p^0}{d+1} \right] \frac{\Gamma_p - a_p^0}{d+1} \right] \frac{1}{q+1},
\]
\[
+ \sum_{p=0}^{d} \left[ \frac{\Gamma_p + \lambda_0^0 a_p^0}{d+1} \right] \frac{\Gamma_p - a_p^0}{d+1} \right] \frac{1}{q+3},
\]
\[
- \sum_{r=4}^{d+3} \Delta_r \cdot \frac{1}{q+r} \left[ \frac{\Gamma_r^7 \lambda_0^0 a_r^0}{d+1} \right] \frac{\Gamma_r^7 \lambda_0^0 a_r^0}{d+3} + \frac{\Gamma_r^1 a_r^0}{d+2} \right] \frac{1}{q+r+4},
\]
where for any $r \in \{4, \ldots, d+3\}$,
\[
\Delta_r := \frac{\Gamma_r^7 \lambda_0^0 a_r^0}{d+1} \frac{\Gamma_r^7 \lambda_0^0 a_r^0}{d+3} + \frac{\Gamma_r^1 a_r^0}{d+2} \right] \frac{1}{q+r+4},
\]
The function $\Psi$ is a rational function with infinitely many zeros, so it vanishes everywhere and in consequence all its coefficients vanish. Remember in addition that by construction,
\[
\Gamma_p^l = 0 \quad \forall p \in \{P + 1, \ldots, d\}, \forall l \in \{1, \ldots, N\}.
\]
Then we have $\Delta_r = 0$ for any $r \in \{P+5, \ldots, d+3\}$, that is,
\[
\left(\frac{\lambda_0}{r-1} - \frac{\lambda_0}{r-2}\right) a_{r-3}^0 + \left(\frac{\lambda_0}{r-1} - \frac{\lambda_0}{r-3} - \frac{1}{r-2}\right) a_{r-4}^0 = 0,
\]
and in addition the coefficient in front of $\frac{1}{q+d+4}$ vanishes, that is,
\[
\left(\frac{\lambda_7}{d+1} - \frac{\lambda_9}{d+3}\right) a_d^0 = \frac{a_d^0}{d+2}.
\]
In conclusion, if there is no vector space of dimension $N+1$ in $\mathcal{L} \cup \{0\}$, then for every polynomial $f_0 \in \mathbb{R}_d[X]$ of degree $d$ (that is, $a_d^0 \neq 0$) the linear form $\Phi_{f_0}$ contains the intersection of the kernels of the $5N+6$ linear forms given by (28)-(29). By the above discussion, this implies that there are four reals numbers $A, B, C, D$ not all zero (because $a_d^0 \neq 0$) such that
\[
\left(\frac{A}{r-1} + \frac{B}{r-2}\right) a_{r-3}^0 + \left(\frac{C}{r-1} - \frac{1}{r-2} + \frac{D}{r-3}\right) a_{r-4}^0 = 0,
\]
for any $r \in \{P+5, \ldots, d+3\}$ and in addition
\[
\left(\frac{D}{d+1} + \frac{C}{d+3}\right) a_d^0 = -\frac{a_d^0}{d+2} \implies D = -(d+1)\left(\frac{1}{d+2} + \frac{C}{d+3}\right).
\]
Note that for every $p \in \{P+2, \ldots, d\}$,
\[
\frac{C}{p+2} - \frac{1}{p+1} + \frac{D}{p} = \frac{2C(d+2)(p-d-1)(p+1) - (d+3)((2d+3)p+d+1)(p+2)}{p(p+1)(p+2)(d+2)(d+3)}.
\]
This means that the set of coefficients $(a_p^0)_{p \in \{P+1, d\}}$ belongs to the algebraic set $\mathcal{S}$ of $(d-P)$-tuples $(a_p)_{p \in \{P+1, d\}} \in \mathbb{R}^{d-N}$ for which there is $(A, B, C) \in \mathbb{R}^3$ such that
\[
\left(\frac{2C(d+2)(p-d-1)(p+1) - (d+3)((2d+3)p+d+1)(p+2)}{p(p+1)(p+2)(d+2)(d+3)}\right) a_{p-1} + \left(\frac{A}{p+2} + \frac{B}{p+1}\right) a_p = 0,
\]
for every $p \in \{P+2, \ldots, d\}$. For every triple $(A, B, C) \in \mathbb{R}^3$, denote by $\mathcal{S}(A, B, C)$ the algebraic set of $(d-P)$-tuples $(a_p)_{p \in \{P+1, d\}} \in \mathbb{R}^{d-N}$ satisfying (31). Notice that for every $(A, B, C) \in \mathbb{R}^3$, the function
\[
p \in \{P+2, \ldots, d\} \mapsto \frac{A}{p+2} + \frac{B}{p+1} = \frac{(A+B)p+(A+2B)}{(p+1)(p+2)}
\]
vansishes for at most one $p$ in $\{P+2, \ldots, d\}$. This means that given $(A, B, C) \in \mathbb{R}^3$ either we have
\[
a_p = C_p^d a_{p-1} \quad \forall p \in \{P+2, \ldots, d\},
\]
with
\[
C_p^d := \left(\frac{2C(d+2)(p-d-1)(p+1) - (d+3)((2d+3)p+d+1)(p+2)}{p(p+1)(p+2)(d+2)(d+3)}\right) / \left(\frac{(A+B)p+(A+2B)}{(p+1)(p+2)}\right),
\]
for every $p \in \{P+2, \ldots, d\}$, or there is $\bar{p} = \bar{p}(A, B, C) \in \{P+2, \ldots, d\}$ such that
\[
a_p = C_p^d a_{p-1} \quad \forall p \in \{P+2, \ldots, d\} \setminus \{\bar{p}\}
\]
and
\[
\left( \frac{2C(d + 2)(\bar{p} - d - 1)(\bar{p} + 1) - (d + 3)(2d + 3)\bar{p} + d + 1)(\bar{p} + 2)}{\bar{p}(\bar{p} + 1)(\bar{p} + 2)(d + 2)(d + 3)} \right) a_{\bar{p} - 1} = 0.
\]
Since the sets we are dealing with are algebraic (see [4, 10]), we infer that given \((A, B, C) \in \mathbb{R}^3\), the algebraic set \(\mathcal{A}(A, B, C) \subset \mathbb{R}^{d-N}\) has at most dimension three, which means that \(\mathcal{A} \subset \mathbb{R}^{d-N}\) has at most dimension six.

In conclusion, the coefficients \((a_{\bar{p}}^i)_{\bar{p} \in \{0, d\}}\) of \(f_0\) have to belong to the intersection of \(2 + 5N\) hyperplanes in \(\mathbb{R}_d[X]\), and if in addition if there is no vector space of dimension \(N + 1\) in \(\mathcal{L} \cup \{0\}\), then the \((d - P)\)-tuples \((a_{\bar{p}}^i)_{\bar{p} \in \{P + 1, d\}}\) must belong to \(\mathcal{A} \subset \mathbb{R}^{d-N}\) of dimension \(\leq 6\). But, for \(d\) large enough, the intersection of \(2 + 5N\) hyperplanes in \(\mathbb{R}_d[X]\) with the complement of an algebraic set of dimension at most \(6 + P + 1\) is non-empty. This concludes the proof of Lemma 2.7.

REFERENCES

[1] A. A. Agrachev and Yu. L. Sachkov, *Control Theory from the Geometric Viewpoint*, Encyclopaedia of Mathematical Sciences, 87, Springer-Verlag, Berlin, 2004.

[2] A. Agrachev and P. Lee, *Optimal transportation under nonholonomic constraints*, Trans. Amer. Math. Soc., **361** (2009), 6019–6047.

[3] D. V. Anosov, *Generic properties of closed geodesics*, Izv. Akad. Nauk. SSSR Ser. Mat., **46** (1982), 675–709, 896.

[4] J. Bocknak, M. Coste and M.-F. Roy, *Real Algebraic Geometry*, Ergebnisse des Mathematik und ihrer Grenzgebiete (3), 36, Springer-Verlag, Berlin, 1998.

[5] M. Carballo and J. A. G. Miranda, *Jets of closed orbits of Mañé’s generic Hamiltonian flows*, Bull. Braz. Math. Soc. (N. S.), **44** (2013), 219–232.

[6] G. Contreras, *Partially hyperbolic geodesic flows are Anosov*, C. R. Math. Acad. Sci. Paris, **334** (2002), 585–590.

[7] G. Contreras, *Geodesic flows with positive topological entropy, twist maps and hyperbolicity*, Ann. of Math. (2), **172** (2010), 761–808.

[8] G. Contreras and R. Iturriaga, *Convex Hamiltonians without conjugate points*, Ergodic Theory Dynam. Systems, **19** (1999), 901–952.

[9] J.-M. Coron, *Control and nonlinearity*, Mathematical Surveys and Monographs, 136, American Mathematical Society, Providence, RI, 2007.

[10] M. Coste, Ensembles semi-algébriques, in *Real Algebraic Geometry and Quadratic Forms* (Rennes, 1981), Lecture Notes in Math., 939, Springer, Berlin-New York, 1982, 109–138.

[11] M. do Carmo, *Riemannian Geometry*, Birkhäuser Boston Inc., Boston, MA, 1992.

[12] B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, *Modern Geometry–Methods and Applications, Part I. The Geometry of Surfaces, Transformation Groups, and Fields*, Second edition, Graduate Texts in Mathematics, 93, Springer-Verlag, New York, 1992.

[13] A. Figalli and L. Rifford, *Closing Aubry sets I*, Comm. Pure Appl. Math., **68** (2015), 210–285.

[14] A. Figalli and L. Rifford, *Closing Aubry sets II*, Comm. Pure Appl. Math., **68** (2015), 345–412.

[15] J. Franks, * Necessary conditions for stability of diffeomorphisms*, Trans. Amer. Math. Soc., **158** (1971), 301–308.

[16] M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics, Vol. 583, Springer-Verlag, Berlin-New York, 1977.

[17] V. Jurjevic, *Geometric Control Theory*, Cambridge Studies in Advanced Mathematics, 52, Cambridge University Press, Cambridge, 1997.

[18] W. Klingenberg and F. Takens, *Generic properties of geodesic flows*, Math. Ann., **197** (1972), 323–334.
[19] R. Kulkarni, Curvature structures and conformal transformations, *J. Diff. Geom.*, 4 (1970), 425–451.

[20] A. Lazrag, *Control Theory and Dynamical Systems*, Thesis, 2014.

[21] A. Lazrag, A geometric control proof of linear Franks’ lemma for geodesic flows, preprint, 2014.

[22] R. Mañé, An ergodic closing lemma, *Ann. of Math.* (2), 116 (1982), 503–540.

[23] R. Mañé, A proof of the $C^1$ stability conjecture, *Inst. Hautes Études Sci. Publ. Math.*, 66 (1988), 161–210.

[24] R. Mañé, On the minimizing measures of Lagrangian dynamical systems, *Nonlinearity*, 5 (1992), 623–638.

[25] R. Mañé, *Global Variational Methods in Conservative Dynamics*, IMPA, Rio de Janeiro, 1993.

[26] J. Moser, Proof of a generalized form of a fixed point theorem due to G. D. Birkhoff, in *Geometry and Topology* (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), Lecture Notes in Mathematics, Vol. 597, Springer, Berlin, 1977, 464–494.

[27] S. Newhouse, Quasi-elliptic periodic points in conservative dynamical systems, *Amer. J. Math.*, 99 (1977), 1061–1087.

[28] E. Oliveira, Generic properties of Lagrangians on surfaces: The Kupka-Smale theorem, *Discrete Contin. Dyn. Syst.*, 21 (2008), 351–369.

[29] M. Paternain, Expansive geodesic flows on surfaces, *Ergodic Theory Dynam. Systems*, 13 (1993), 153–165.

[30] C. C. Pugh, The closing lemma, *Amer. J. Math.*, 89 (1967), 956–1009.

[31] C. C. Pugh, An improved closing lemma and a general density theorem, *Amer. J. Math.*, 89 (1967), 1010–1021.

[32] C. Pugh and C. Robinson, The $C^1$ closing lemma, including Hamiltonians, *Ergodic Theory Dynam. Systems*, 3 (1983), 261–313.

[33] L. Rifford, *Sub-Riemannian Geometry and Optimal Transport*, Springer Briefs in Mathematics, Springer, Cham, 2014.

[34] L. Rifford and R. Ruggiero, Generic properties of closed orbits of Hamiltonian flows from Mañé’s viewpoint, *Int. Math. Res. Not. IMRN*, 22 (2012), 5246–5265.

[35] C. Robinson, Generic properties of conservative systems I and II, *Amer. J. Math.*, 92 (1970), 562–603 and 897–906.

[36] R. Ruggiero, Persistently expansive geodesic flows, *Comm. Math. Phys.*, 140 (1991), 203–215.

[37] R. Ruggiero, On the creation of conjugate points, *Math. Z.*, 208 (1991), 41–55.

[38] T. Sakai, *Riemannian Geometry*, Translated from the 1992 Japanese original by the author, Translations of Mathematical Monographs, 149, American Mathematical Society, Providence, RI, 1996.

[39] C. Villani, *Optimal Transport. Old and new*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 338, Springer-Verlag, Berlin, 2009.

[40] D. Visscher, A new proof of Franks’ lemma for geodesic flows, *Discrete Contin. Dyn. Syst.*, 34 (2014), 4875–4895.

[41] T. Vivier, *Robustly Transitive 3-Dimensional Regular Energy Surfaces are Anosov*, Ph.D. Thesis, preprint, Dijon, 2005.