AN ELLIPTIC CONIC BUNDLE IN $\mathbb{P}^4$ ARISING FROM A STABLE RANK-3 VECTOR BUNDLE

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0. Introduction

In this paper we give a cohomological proof for the existence of a new family of smooth surfaces in $\mathbb{P}^4$. In fact, we show:

**Theorem 0.1.** (i) Let $X \subset \mathbb{P}^4$ be an elliptic conic bundle with degree $d = 8$ and sectional genus $\pi = 5$. Then the ideal sheaf of $X$ arises as a cokernel

$$0 \to 4\mathcal{O}(-1) \to \mathcal{G} \to \mathcal{J}_X(3) \to 0,$$

where $\mathcal{G}$ is a rank-5 vector bundle on $\mathbb{P}^4$ with Chern-classes

$$c_1 = -1, c_2 = 2, c_3 = -2 \quad \text{and} \quad c_4 = -3.$$

$\mathcal{G}$ is isomorphic to the cohomology bundle of a monad

$$(M) \quad 0 \to \Omega^3(3) \xrightarrow{\alpha} \Omega^2(2) \oplus \Omega^1(1) \xrightarrow{\beta} \mathcal{O} \to 0,$$

with

$$\alpha = \left( \begin{array}{c} e_4 \\ e_0 \wedge e_2 + e_1 \wedge e_3 \end{array} \right)$$

and

$$\beta = \left( \begin{array}{c} e_0 \wedge e_2 + e_1 \wedge e_3 \\ -e_4 \end{array} \right),$$

where $e_0, \ldots, e_4$ is a basis of the underlying vector space $V$ of $\mathbb{P}^4$. In particular, $\mathcal{G}$ is uniquely determined up to isomorphisms and coordinate transformations.

(ii) Conversely, if $\mathcal{G}$ is the cohomology bundle of the monad $(M)$ as in (i), then $\mathcal{G}(1)$ is globally generated. Therefore the dependancy locus of four general sections of $\mathcal{G}(1)$ is a smooth surface $X \subset \mathbb{P}^4$. In fact, $X$ is an elliptic conic bundle with invariants as above. □

For a geometric construction of our surfaces we refer to [Ra].

The new surfaces are missing in a series of classification papers. First of all, they are falsely ruled out in the classification of smooth degree 8 surfaces in $\mathbb{P}^4$ by Okonek [Ok2] and, independently, Ionescu [Io]. As a consequence, they are e.g. also missing in the first version of two papers which are concerned with the classification of conic bundles in $\mathbb{P}^4$ [ES], [BR].

The correct result is:
Theorem 0.2. [ES], [BR] Let $X \subset \mathbb{P}^4$ be a smooth surface ruled in conics. Then $X$ is either rational or an elliptic conic bundle with $d = 8$ and $\pi = 5$. In the first case $X$ is either a Del Pezzo surface of degree 4 or a Castelnuovo surface. □

This result is important in the context of adjunction theory [So], [SV], [VdV]. Recall, that for a smooth surface $X \subset \mathbb{P}^4$ the adjunction map is defined unless $X$ is a plane or a scroll. If the adjunction map is defined, then it has a 2-dimensional image unless $X$ is a Del Pezzo surface or a conic bundle. Therefore the classification of scrolls in $\mathbb{P}^4$ [La], [Au] and the above result imply:

Corollary 0.3. Let $X \subset \mathbb{P}^4$ be a smooth surface of degree $d \geq 9$. Then the adjunction map is defined and has a 2-dimensional image. □

The new family is one of a few known families of irregular smooth surfaces in $\mathbb{P}^4$. In fact, up to pullbacks via finite covers $\mathbb{P}^4 \to \mathbb{P}^4$, our surfaces are the first such surfaces which do not possess a Heisenberg symmetry (compare [ADHPR]). Moreover, they provide a counterexample to a conjecture of Ellingsrud and Peskine. According to this conjecture there should be no irregular $m$-ruled surface in $\mathbb{P}^4$ for $m \geq 2$.

We first came across the elliptic conic bundles when studying a stable rank-3 vector bundle $\mathcal{E}$ on $\mathbb{P}^4$ with Chern classes

$$c_1 = 4, c_2 = 8 \quad \text{and} \quad c_3 = 8.$$ 

$\mathcal{E}$ has been found by the stratification theoretical method of the third author (compare [Sa] for this method). The dependancy locus of two sections of $\mathcal{E}$ is a smooth surface of the desired type. In fact, $\mathcal{E}$ is a cokernel

$$0 \to 2\mathcal{O} \to \mathcal{G}(1) \to \mathcal{E} \to 0,$$

where $\mathcal{G}$ is the rank-5 vector bundle from Theorem 0.1.

Our paper is organized as follows: In Section 1 we review Beilinson’s theorem [Bei] in the context of smooth surfaces in $\mathbb{P}^4$. In Section 2 we follow the cohomological approach of Okonek and find the rank-5 vector bundle $\mathcal{G}$ and thus the elliptic conic bundles via Beilinson’s generalized monads. In Section 3 we take a quick look at the stratification theoretical method of the third author by studying some examples. In particular, we will construct the rank-3 bundle $\mathcal{E}$ and we will explain its relation to the rank-5 bundle $\mathcal{G}$.

The third author, Nobuo Sasakura, died in June 1997. His death has been a great loss to us.

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Notations 0.4. \( \mathbb{P}^4 = \mathbb{P}^4(x) = \mathbb{P}(V) \) will be the projective space of lines in a 5-dimensional \( \mathbb{C} \)-vector space \( V \) and
\[
R = \mathbb{C}[x_0, \ldots, x_4] = \bigoplus_{m \geq 0} S^m V^* \]
its homogeneous coordinate ring. We write \( \Omega^i = \Lambda^i \mathbb{P}^4 \) for the \( i \)th bundle of differentials on \( \mathbb{P}^4 \). If \( X \) is a smooth surface in \( \mathbb{P}^4 \), then we denote by
- \( H \) its hyperplane class
- \( d = H^2 \) its degree
- \( K = K_X \) its canonical divisor
- \( \pi = \frac{1}{2} H \cdot (H + K) + 1 \) its sectional genus
- \( q = p_g - p_a \) its irregularity
- \( \chi = \chi(O_X) = 1 - q + p_g \) the Euler characteristic of its structure sheaf.

1. Beilinson’s theorem

In this section we review Beilinson’s theorem [Bei] in the context of smooth surfaces in \( \mathbb{P}^4 \). The general theorem tells us that the derived category of coherent sheaves on \( \mathbb{P}^n \) is generated by the twisted bundles of differentials \( \Omega^i(i) \), \( 0 \leq i \leq n \). More concretely, every coherent sheaf on \( \mathbb{P}^n \) is the cohomology of a certain generalized monad involving bundles of differentials. In the next section we apply Beilinson’s theorem in order to study our surfaces in \( \mathbb{P}^4 \) from a cohomological point of view.

Theorem 1.1. [Bei] For any coherent sheaf \( S \) on \( \mathbb{P}^4 \) there is a complex \( \mathcal{K}^\cdot \) with
\[
\mathcal{K}^i \cong \bigoplus_j H^{i+j}(\mathbb{P}^4, S(-j)) \otimes \Omega^j(j),
\]
such that
\[
H^i(\mathcal{K}^\cdot) = \begin{cases} S & i = 0 \\ 0 & i \neq 0 \end{cases}. \quad \square
\]
The differentials of \( \mathcal{K}^\cdot \) are given by matrices with entries in the exterior algebra \( \Lambda V \) over the underlying vector space \( V \) of \( \mathbb{P}^4 \).

Remark 1.2. The Koszul complex on \( \mathbb{P}^4 = \mathbb{P}(V) \) is the exact sequence
\[
0 \leftarrow O(-1) \otimes V^* \otimes O(-1) \otimes \Lambda^2 V^* \otimes O(-2) \otimes \Lambda^3 V^* \otimes O(-5)) \leftarrow 0,
\]
defined by contraction with the tautological subbundle
\[
O(-1) \rightarrow V \otimes O.
\]
The syzygy bundles of this complex are the bundles of differentials:
\[
(ker \kappa_i) \cong \Omega^i, \quad 0 \leq i \leq 4.
\]
Via the short exact sequences associated to the Koszul complex we may compute the intermediate cohomology modules of the $\Omega^i$:

$$H^j_*(\mathbb{P}^4, \Omega^i) \cong \begin{cases} \mathbb{C} & 1 \leq j = i \\ 0 & 1 \leq j \neq i \leq 3 \end{cases}.$$  

In the same way we obtain for $0 \leq i, j \leq 4$:

$$\text{Hom}(\Omega^i(i), \Omega^j(j)) \cong \Lambda^{i-j}V,$$

the isomorphism being defined by contraction. The composition of homomorphisms coincides with multiplication in $\Lambda V$. □

We may therefore express conditions on a coherent sheaf on $\mathbb{P}(V)$ as conditions on certain matrices with entries in $\Lambda V$. For example we need:

**Remark 1.3.** Let

$$t\Omega^i(i) \xrightarrow{A} s\Omega^{i-1}(i-1)$$

be a vector bundle homomorphism, i.e., let $A$ be a $s \times t$-matrix with entries in $V$. Then a necessary condition for $A$ to be surjective is: If $(a_1, \ldots, a_t)$ is a non-trivial linear combination of the rows of $A$, then

$$\dim \text{span}(a_1, \ldots, a_t) \geq i + 1 :$$

$A$ is surjective iff its dual map is pointwise injective iff

$$s\Lambda^{i-1}V \wedge x \xrightarrow{A^t} t\Lambda^iV \wedge x$$

is injective for any $\langle x \rangle \in \mathbb{P}(V)$. □

In order to apply Beilinson’s theorem to the ideal sheaf (suitably twisted) of a smooth surface $X \subset \mathbb{P}^4$ we need information on the dimensions $h^i\mathcal{J}_X(m)$. First we recall Riemann-Roch:

**Proposition 1.4.** Let $X \subset \mathbb{P}^4$ be a smooth surface. Then

$$\chi(\mathcal{J}_X(m)) = \chi(\mathcal{O}_{\mathbb{P}^4}(m)) - \binom{m+1}{2}d + m(\pi - 1) - 1 + q - p_g .$$

□

Moreover one knows:

**Proposition 1.5.** [DES] Let $X \subset \mathbb{P}^4$ be a smooth, non-general type surface which is not contained in any cubic hypersurface. Then we have the following table for the $h^i\mathcal{J}_X(m)$:

| $i$ | 0 | 0 | 0 | N + 1 | $p_g$ | 0 | 0 | 0 | 0 | 0 | 0 |
|-----|---|---|---|-------|-------|---|---|---|---|---|---|
| 0   | 0 | 0 | 0 | 0     | 0     | 0 | 0 | 0 | 0 | 0 | 0 |
| $N + 1$ | $p_g$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | q | $h^2\mathcal{J}_X(1)$ | $h^2\mathcal{J}_X(2)$ | $h^2\mathcal{J}_X(3)$ | $h^2\mathcal{J}_X(4)$ | $h^2\mathcal{J}_X(1)$ | $h^2\mathcal{J}_X(2)$ | $h^2\mathcal{J}_X(3)$ | $h^2\mathcal{J}_X(4)$ | $h^2\mathcal{J}_X(1)$ | $h^2\mathcal{J}_X(2)$ | $h^2\mathcal{J}_X(3)$ | $h^2\mathcal{J}_X(4)$ |
| 0 | 0 | 0 | $h^4\mathcal{J}_X(2)$ | $h^4\mathcal{J}_X(3)$ | $h^4\mathcal{J}_X(4)$ | $h^4\mathcal{J}_X(2)$ | $h^4\mathcal{J}_X(3)$ | $h^4\mathcal{J}_X(4)$ | $h^4\mathcal{J}_X(2)$ | $h^4\mathcal{J}_X(3)$ | $h^4\mathcal{J}_X(4)$ |
| 0 | 0 | 0 | 0 | 0 | $h^0\mathcal{J}_X(4)$ | 0 | 0 | 0 | 0 | 0 | 0 |

where

$$N = \pi - q + p_g - 1 .$$

□

In the sequel we represent a zero in a cohomology table by an empty box.
2. A monad construction

In this section we show, that every elliptic conic bundle in \( \mathbb{P}^4 \) with \( d = 8 \) and \( \pi = 5 \) is given as the dependancy locus of four sections in a unique rank-5 vector bundle \( \mathcal{G}(1) \). We give a monad construction for \( \mathcal{G} \), which conversely provides a proof for the existence of our surfaces since \( \mathcal{G}(1) \) is globally generated.

We first mention a classification result concerning the numerical invariants of those surfaces we are interested in here. This result is a consequence of generalized Serre correspondence [Ok1, Theorem 2.2] and adjunction theory [So], [SV], [VdV].

**Proposition 2.1.** [Ok2] Let \( X \subset \mathbb{P}^4 \) be a smooth surface with \( d = 8 \) and \( \pi = 5 \). Then \( p_g = 0 \) and \( q \leq 1 \). Moreover:

(i) If \( q = 0 \), then \( X \) is the blow-up of \( \mathbb{P}^2 \) in eleven points,

\[
X = \mathbb{P}^2(p_0, \ldots, p_{10}),
\]

embedded by the linear system (with obvious notations)

\[
H = 7L - E_0 - \sum_{i=1}^{10} 2E_i.
\]

(ii) If \( q = 1 \), then the adjunction map \( \Phi_{|K+H|} \) exhibits \( X \) as a conic bundle over an elliptic normal curve \( C \) in \( \mathbb{P}^3 \). There are precisely eight singular fibres of \( \Phi_{|K+H|} \). These singular fibres are pairs of (-1)-lines \( E_i, \tilde{E}_i \) with \( E_i \cdot \tilde{E}_i = 1 \).

**Proof of (ii).** By classification (see [La] and [Au]) \( X \) is not a scroll. It follows from [So], [VdV] that \( \mathcal{O}_X(K + H) \) is spanned, i.e., that the adjunction map is a well-defined morphism \( X \to \mathbb{P}^N \) with \( N = \pi - q + p_g - 1 = 3 \). We have \( H \cdot K = 0 \) since \( \pi = 5 \). So we obtain \( K^2 = -8 \) and thus \( (K + H)^2 = 0 \) from the double point formula (cf. [Ha, Appendix A, 4.1.3])

\[
d^2 - 10d - 5H \cdot K - 2K^2 + 12\chi = 0.
\]

By [So] \( \Phi_{|K+H|} \) exhibits \( X \) as a conic bundle over a smooth elliptic curve in \( \mathbb{P}^3 \) of degree \( \pi - 1 = 4 \). The singular fibres of \( \Phi_{|K+H|} \) are pairs of (-1)-lines \( E_i, \tilde{E}_i \) with \( E_i \cdot \tilde{E}_i = 1 \). By blowing down one irreducible component for each singular fiber we obtain a ruled surface \( Y \) of irregularity \( q = 1 \) together with a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Phi_{|K+H|}} & C \subset \mathbb{P}^3 \\
\downarrow & & \downarrow \\
Y & & \end{array}
\]

By comparing \( K_X^2 = -8 \) with \( K_Y^2 = 8(1 - q) = 0 \) we find that \( \Phi \) has precisely eight singular fibers. \( \square \)

The existence of surfaces of type (i) has been verified by Alexander [Al]. Okonek [Ok2] mistakenly claimed that surfaces \( X \) of type (ii) do not exist. Essentially, he claimed that the generalized monad of Beilinson for the twisted ideal sheaf \( \mathcal{I}_X(2) \) cannot exist. He gave no argument, however, and in fact such a monad can be easily constructed. It is even more convenient to apply Beilinson’s theorem to \( \mathcal{I}_X(3) \).
Proposition 2.2. Let $X \subset \mathbb{P}^4$ be a smooth surface with $d = 8$, $\pi = 5$ and $q = 1$. Then:

(i) $\mathcal{J}_X$ has the following cohomology table:

```
   i
   ┌─┬─┬─┬─┐
   │ │ │ │ │
   ├──┼─┼─┼─┤
   │ 4 │1 │1 │ │
   ├──┼─┼─┼─┘
   │ 1 │1 │1 │6 │
   └─┴─┴─┴─┘
```

In particular, $X$ is cut out by 6 quartics.

(ii) There exists an exact sequence

$$0 \to 4\mathcal{O}(-1) \to \mathcal{G} \to \mathcal{J}_X(3) \to 0,$$

where $\mathcal{G}$ is a rank-5 vector bundle on $\mathbb{P}^4$ with Chern-classes

$$c_1 = -1, c_2 = 2, c_3 = -2 \quad \text{and} \quad c_4 = -3.$$

$\mathcal{G}$ is isomorphic to the cohomology bundle of a monad

$$(M) \quad 0 \to \Omega^3(3) \xrightarrow{\alpha} \Omega^2(2) \oplus \Omega^1(1) \xrightarrow{\beta} \mathcal{O} \to 0,$$

with

$$\alpha = \left( \begin{array}{c} e_4 \\ e_0 \wedge e_2 + e_1 \wedge e_3 \end{array} \right)$$

and

$$\beta = \left( e_0 \wedge e_2 + e_1 \wedge e_3 - e_4 \right),$$

where $e_0, \ldots, e_4$ is a basis of the underlying vector space $V$ of $\mathbb{P}^4$. In particular, $\mathcal{G}$ is uniquely determined up to isomorphisms and coordinate transformations.

(iii) $\mathcal{G}$ has a minimal free resolution of type

$$0 \leftarrow 10\mathcal{O}(-1) \leftarrow 5\mathcal{O}(-2) \oplus \mathcal{O}(-3) \leftarrow 4\mathcal{O}(-4) \oplus \mathcal{O}(-5) \leftarrow 0.$$ 

So $\mathcal{J}_X$ has syzygies of type

$$0 \leftarrow \mathcal{J}_X \leftarrow 6\mathcal{O}(-4) \oplus 5\mathcal{O}(-6) \leftarrow 4\mathcal{O}(-5) \oplus \mathcal{O}(-7) \oplus \mathcal{O}(-8) \leftarrow 0.$$
Proof. (i) $X$ is a ruled surface by Proposition 2.1, (ii). In particular, $X$ is not of general type. Moreover, $X$ is not contained in a cubic hypersurface by general classification results (see [Ro], [Au] and [Ko]). Therefore $\mathcal{J}_X$ has a cohomology table of type

\[
\begin{array}{cccccc}
& & & & & a \\
4 & 1 & 1 & * & * & * \\
* & * & * \\
* & \\
\end{array}
\]

by Proposition 1.5. Beilinson’s theorem applied to $\mathcal{J}_X (2)$ implies the existence of a surjective map $\Omega^1(1) \to a\mathcal{O}$. By Remark 1.3 this is only possible if $a = h^2 J_X (2) = 0$. Then $h^2 J_X (m) = 0$ for every $m \geq 2$. In particular, $h^1 J_X (2) = h^1 J_X (3) = 1$ by Riemann-Roch. Applying the same argument again gives $h^1 J_X (m) = 0$ for every $m \geq 4$ and $h^0 J_X (4) = 6$.

(ii) The adjoint linear system

\[
H^0(\mathcal{O}_X (K + H)) \cong \text{Ext}^1(J_X (3), \mathcal{O}(-1)) =: W
\]

has dimension 4 and generates $\mathcal{O}_X (K + H) \cong \omega_X (1)$ (compare the proof of Proposition 2.1). Therefore the identity in

\[
W^* \otimes W \cong \text{Ext}^1(J_X (3), W^* \otimes \mathcal{O}(-1))
\]

defines an extension as in the assertion with a locally free sheaf $\mathcal{G}$. By construction and (i), $\mathcal{G}(-3)$ has the following cohomology table:

\[
\begin{array}{cccccc}
& & & & & 1 \\
1 & 1 & & & & \\
& & 1 & 1 & & \\
& & & & & \\
\end{array}
\]

Beilinson’s theorem implies that $\mathcal{G}$ is the cohomology bundle of a monad with bundles of differentials as in the assertion. Expressing the monad conditions in terms of the exterior algebra $\Lambda V$ shows, that, up to isomorphisms, also the arrows of the monad are of the claimed type (compare Remark 1.2 and Remark 1.3).
(iii) The kernel $\mathcal{K} = \ker \beta$ is a rank-9 bundle which fits into a commutative diagram with exact rows and columns as follows:

\[
\begin{array}{ccccccc}
0 & 0 & \Omega^2(2) & \Omega^1(1) & \beta & \mathcal{O} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & U \otimes \mathcal{O} & (\Lambda^2V^* \oplus \Lambda^1V^*) \otimes \mathcal{O} & \beta & \mathcal{O} & 0 \\
\phi & & & & & & \\
(\Lambda^1V^* \oplus \Lambda^0V^*) \otimes \mathcal{O}(1) = (\Lambda^1V^* \oplus \Lambda^0V^*) \otimes \mathcal{O}(1) & & & & & & \\
\downarrow & & \downarrow & & & & \\
\mathcal{O}(2) & = & \mathcal{O}(2) & & & & \\
\downarrow & & \downarrow & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

With respect to suitably chosen bases of $U$ and $\Lambda^1V^* \oplus \Lambda^0V^*$ the map $\phi$ is given by the matrix

\[
\phi = \begin{pmatrix}
x_4 & 0 & 0 & 0 & x_1 & x_3 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\
0 & x_4 & 0 & 0 & -x_0 & 0 & x_2 & 0 & 0 & x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & x_4 & 0 & 0 & 0 & -x_1 & x_3 & -x_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_4 & 0 & -x_0 & 0 & -x_2 & 0 & -x_1 & 0 & 0 & 0 & 0 \\
-x_0 & -x_1 & -x_2 & -x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & x_4 & x_0 & x_1 & x_2 & x_3
\end{pmatrix}
\]

Resolving $\phi$ shows that $\mathcal{K}$ has syzygies of type

\[0 \leftarrow \mathcal{K} \leftarrow 15 \mathcal{O}(-1) \quad 5 \mathcal{O}(-2) \quad \mathcal{O}(-3)\]

Comparing with the Koszul resolution of $\Omega^3(3)$ gives the result. \qed

**Remark 2.3.** Conversely, by starting with $(M)$ and $\mathcal{J}$, we obtain a proof for the existence of smooth surfaces in $\mathbb{P}^4$ with $d = 8$, $\pi = 5$ and $q = 1$ : Since $\mathcal{J}(1)$ is globally generated, Kleiman’s Bertini-type theorem [Kl] implies that the dependancy locus of 4 general sections of $\mathcal{J}(1)$ is indeed a smooth surface (with invariants as above). \qed

**Remark 2.4.** (i) Let $X \xrightarrow{\Phi} C$ be an elliptic conic bundle as above. Ionescu [Io] studied the union $W$ of the planes of the conics which are the fibres of $\Phi$. He showed, that $W$ is a quartic hypersurface in $\mathbb{P}^4$ which is a cone with vertex a point. The base $S$ of the cone is an
elliptic ruled quartic surface in $\mathbb{P}^3$ with two double lines $l_1$ and $l_2$. The rulings of $S$ form a divisor of class $(2,2)$ in the $\mathbb{P}^1 \times \mathbb{P}^1 \cong l_1 \times l_2$ of lines joining $l_1$ and $l_2$. Ionescu considered the cone over a pair of rulings through a point on $l_1$ but overlooked the presence of $l_2$. His residual quadric at the end of [Io, 6.4] is the double plane over $l_2$.

(ii) The vertex of the cone $W$ is the distinguished point $\langle e_4 \rangle$ in our construction. The projection from $\langle e_4 \rangle$ maps $X$ 2:1 to $S$. The conics on $X$ are mapped 2:1 onto the rulings of $S$.

**Remark 2.5.** In the minimal free resolution of $S$ (and thus also of $\mathcal{J}_X$) there are two maps which are given by four linear forms. In fact, for both maps, these forms generate the ideal of the distinguished point $\langle e_4 \rangle$. □

### 3. Vector bundles via stratifications

In this section we take a quick look at the approach of the third author to the construction of vector bundles on $\mathbb{P}^n$. In particular, we construct a stable rank-3 bundle $\mathcal{E}$ on $\mathbb{P}^4$ such that the dependancy locus of two general sections of $\mathcal{E}$ is an elliptic conic bundle. This rank-3 bundle was the starting point of our investigations. It is related to the rank-5 bundle $\mathcal{S}$ from Section 2 via an exact sequence

$$0 \to 2\mathcal{O} \to \mathcal{S}(1) \to \mathcal{E} \to 0.$$ 

Let $\mathcal{E}$ be a rank-$r$ vector bundle on $\mathbb{P}^n$ with first Chern class $c_1$. Minimal systems of generators $s_1, \ldots, s_l$ of $H^0_*\mathcal{E}$ and $\sigma_1, \ldots, \sigma_k$ of $H^0_\mathcal{E}^\vee$ give rise to a commutative diagram

$$\begin{array}{ccc}
\mathcal{L} & \xrightarrow{S} & \mathcal{K} \\
\downarrow & & \downarrow \\
\mathcal{E} & \to & 0 \\
0 & \to & 0,
\end{array}$$

where $\mathcal{L}$ and $\mathcal{K}$ are direct sums of line bundles of ranks $l$ and $k$ resp. $S$ is a $k \times l$-matrix with polynomial entries. The entries in the $j$-th column of $S$ define the zero-scheme $X_{s_j} = \{s_j = 0\}$. The entries in the $i$-th row of $S$ define the zero-scheme $X_{\sigma_i} = \{\sigma_i = 0\}$. Suppose that $\mathcal{K}$ has a direct summand of type $r\mathcal{O}(m)$, and let $\sigma_{i_1}, \ldots, \sigma_{i_r}$ be the corresponding elements of $H^0_\mathcal{E}^\vee$. By projecting onto this direct summand we may represent $\mathcal{E}$ as a subsheaf of $r\mathcal{O}(m)$:

$$\begin{array}{ccc}
\mathcal{L} & \xrightarrow{T} & r\mathcal{O}(m) \\
\downarrow & & \downarrow \\
\mathcal{E} & \to & 0 \\
0 & \to & 0.
\end{array}$$

In fact, $\theta$ is a monomorphism of vector bundles outside the divisor defined by the form $f = \sigma_{i_1} \wedge \cdots \wedge \sigma_{i_r} \in H^0(\mathbb{P}^n, \mathcal{O}((r \cdot m - c_1))) \cong H^0(\mathbb{P}^n, \Lambda^r \mathcal{E}^\vee(m))$. The $j$-th column of
$T$ represents the section $t_j := H^0(\theta)(s_j)$ of $r\mathcal{O}(m)$, and we have the relations of forms

$t_{j_1} \wedge \cdots \wedge t_{j_r} = f \cdot (s_{j_1} \wedge \cdots \wedge s_{j_r})$. In order to detect properties of $\mathcal{E}$ it is often enough to study the matrix $T$. For example, one may verify that $\mathcal{E}$ is indeed a vector bundle by looking at the ideal $I$ generated by the maximal minors of $T$ and by checking that the ideal quotient $(I : f)$ defines the empty subset of $\mathbb{P}^n$.

Conversely, one can construct vector bundles $\mathcal{E}$ on $\mathbb{P}^n$ by starting with convenient matrices $T$. In the examples below $f$ is of the form $(f_{\text{red}})^{(r-1)}$, where $f_{\text{red}} = x_0 \cdots x_{c_1}$ is a product of coordinates, and $T$ is of the form $T = (T' \ T'') : \mathcal{E} \to r\mathcal{O}(c_1)$, where $T'$ is the appropriate identity matrix multiplied by $f_{\text{red}}$. In each example the minimal free resolution of $\mathcal{E}$ is obtained by resolving $T$ (use e.g. Macaulay [BS]).

**Example 3.1.** The Nullcorrelation bundle $\mathcal{E}$ on $\mathbb{P}^3$ (compare e.g. [Ba]).

In this case $r = 2$, $c_1 = 2$, and we may choose $f = f_{\text{red}} = x_0 x_1$ and

$$T = \begin{pmatrix} x_0 x_1 & 0 & x_0 x_3 & x_1 x_2 & x_2 x_3 \\ x_0 x_1 & x_0 x_3 & x_1 x_2 & x_2 x_3 & x_0 x_2 + x_1 x_3 \end{pmatrix}.$$  

Note that the sections of $\mathcal{E}^\vee(2) \cong \mathcal{E}$ corresponding to the rows of $T$ vanish along the union of two skew lines and a double line resp. The minimal free resolution of $\mathcal{E}$ is of type

$$0 \leftarrow \mathcal{E} \leftarrow 5\mathcal{O} \leftarrow 4\mathcal{O}(-1) \leftarrow \mathcal{O}(-2) \leftarrow 0. \quad \Box$$

**Example 3.2.** The Horrocks-Mumford bundle $\mathcal{E}$ on $\mathbb{P}^4$ [HM].

In this case $r = 2$, $c_1 = 5$, and we may choose $f = f_{\text{red}} = x_0 x_1 x_2 x_3 x_4$ and

$$T = (T_0 \ T_1 \ T_2 \ T_3),$$

with $T_0, \ldots, T_3$ as follows:

$$T_0 = \begin{pmatrix} \gamma_0 & 0 & \gamma_3 & \gamma_4 \\ 0 & \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix},$$

where $\gamma_0, \ldots, \gamma_4$ are the five Horrocks-Mumford quintics

$$\gamma_0 = y_0 y_1 y_2 y_3 y_4 \quad \gamma_1 = \sum_{i \in \mathbb{Z}_5} y_i y_{i+2} y_{i+3}^2 \quad \gamma_2 = \sum_{i \in \mathbb{Z}_5} y_i^3 y_{i+2} y_{i+3} \quad \gamma_3 = \sum_{i \in \mathbb{Z}_5} y_i^3 y_{i+1} y_{i+4} \quad \gamma_4 = \sum_{i \in \mathbb{Z}_5} y_i y_{i+1} y_{i+4}^2.$$

The entries of $T_1, \ldots, T_3$ are sextics:

$$T_1 = \left( x_{i+1} x_{i+2} x_{i+3} \ t(x_{i+2} x_{i+3} - x_{i+1} x_{i+4}) \right)_{i \in \mathbb{Z}_5},$$

$$T_2 = \left( x_{i+2} x_{i+3} x_{i+4} \ t(x_{i+2} x_{i+3} - x_{i+1} x_{i+4} + x_{i+2} x_{i+3}) \right)_{i \in \mathbb{Z}_5},$$

$$T_3 = \left( x_{i+1} x_{i+3} x_{i+4}^2 \ t(x_{i+2} x_{i+3}^2 + x_{i+2} x_{i+3}^2) \right)_{i \in \mathbb{Z}_5}.$$  

The zero-schemes of the sections of $\mathcal{E}$ are called *Horrocks-Mumford surfaces*. They are the minimal abelian surfaces in $\mathbb{P}^4$ and the degenerations of these smooth surfaces [HM] (compare [BHM]). Those sections of $\mathcal{E}^\vee(5) \cong \mathcal{E}$ corresponding to the rows of $T$ vanish along the unions of five double planes, namely $\{x_i^2 = x_{i+2}^2 = x_i x_{i+2} = x_{i+2} x_{i+3} + x_i x_{i+4} = 0\}$,
\( i \in \mathbb{Z}_5 \), and \( \{ x_i^2 = x_{i+1}^2 = x_i x_{i+1} = x_{i+1} x_{i+4}^2 + x_i x_{i+2}^2 = 0 \}, i \in \mathbb{Z}_5 \), resp. (compare e.g. [ADHPR, Section 9] for these degenerations). The minimal free resolution of \( \mathcal{E} \) is of type

\[
0 \leftarrow \mathcal{E} \leftarrow \bigoplus 150(-1) \leftarrow 350(-2) \leftarrow 200(-3) \leftarrow 20(-5) \leftarrow 0
\]

(compare e.g. [De]).

**Example 3.3.** A stable rank-2 reflexive sheaf \( \mathcal{E} \) on \( \mathbb{P}^4 \) such that \( \mathcal{E}(-3) \) has Chern classes \( c_1 = -1, c_2 = 9, c_3 = 25 \) and \( c_4 = 50 \) [ADHPR, Section 7].

In this case \( r = 2, c_1 = 5 \), and we may choose \( f = f_{\text{red}} = x_0 x_1 x_2 x_3 x_4 \) and

\[
T = (T_0 \quad T_1),
\]

with \( T_0 \) and \( T_1 \) as follows:

\[
T_0 = \begin{pmatrix} \gamma_0 & 0 & \gamma_2 & \gamma_3 \\ 0 & \gamma_0 & \gamma_2 & \gamma_3 \end{pmatrix},
\]

where \( \gamma_0, \ldots, \gamma_4 \) are the Horrocks-Mumford quintics from 3.2. The entries of \( T_1 \) are septic:

\[
T_1 = (x_{i+1} x_{i+2} x_{i+3} x_{i+4} \ t(x_{i+2} x_{i+4}^2 + x_{i+1} x_{i+3} x_{i+2} x_{i+3} x_{i+4} + x_{i+1} x_{i+2}^2))_{i \in \mathbb{Z}_5}.
\]

This time \( \mathcal{E} \) is locally free only outside the union of the 25 Horrocks-Mumford lines (compare [HM] for these lines). The zero-schemes of the sections of \( \mathcal{E} \) are non-minimal abelian surfaces in \( \mathbb{P}^4 \) and the degenerations of these smooth surfaces. They are \((5,5)\)-linked to Horrocks-Mumford surfaces. The zero-schemes of those sections of \( \mathcal{E} \cap (5) \cong \mathcal{E} \) corresponding to the rows of \( T \) consist of five components, namely \( \{ x_i = x_{i+1}^2 + x_{i+2} x_{i+4} = 0 \}, i \in \mathbb{Z}_5 \), and \( \{ x_i = x_{i+1} x_{i+2} + x_{i+3} x_{i+4} = 0 \}, i \in \mathbb{Z}_5 \), resp. The minimal free resolution of \( \mathcal{E} \) is of type

\[
0 \leftarrow \mathcal{E} \leftarrow \bigoplus 50(-2) \leftarrow 150(-3) \leftarrow 100(-4) \leftarrow 20(-5) \leftarrow 0.
\]

Compare [ADHPR, Section 7 and 9].

**Example 3.4.** A stable rank-3 vector bundle \( \mathcal{E} \) on \( \mathbb{P}^4 \) with Chern classes \( c_1 = 4, c_2 = 8 \) and \( c_3 = 8 \).

We choose \( f = f_{\text{red}} = x_0 x_1 x_2 x_3 \) and

\[
T = (T' \quad t_4 \quad t_5 \quad t_6 \quad t_7 \quad t_8),
\]

with

\[
T' = \begin{pmatrix} x_0 x_1 x_2 x_3 & 0 & 0 \\ 0 & x_0 x_1 x_2 x_3 & 0 \\ 0 & 0 & x_0 x_1 x_2 x_3 \end{pmatrix},
\]

and

\[
t_4 = x_0 x_3 \ t(x_3^2 + x_1 x_3 - x_0 x_2 \ x_2 x_3 - x_0 x_1 - x_0 x_3),
\]
A check on the maximal minors of $T$ shows that $\mathcal{E}$ is locally free (use e.g. Macaulay [BS]). Let $X_1$, $X_2$ and $X_3$ be the zero-schemes of the sections of $\mathcal{E}^\vee(4)$ corresponding to the rows of $T$. Then $X_1$ is the union of four plane quadrics, whereas $X_2$ and $X_3$ are the unions of four double planes (use e.g. the primary decomposition package of Singular [GPS] to obtain the explicit equations). The minimal free resolution of $\mathcal{E}$ is of type

$$0 \leftarrow \mathcal{E} \leftarrow 8\mathcal{O} \xleftarrow{\oplus} 4\mathcal{O}(-1) \xleftarrow{\oplus} \mathcal{O}(-2) \xleftarrow{\oplus} 5\mathcal{O}(-2) \xleftarrow{\oplus} 4\mathcal{O}(-3) \xleftarrow{\oplus} \mathcal{O}(-4) \leftarrow 0.$$ 

The minimal free resolution of $\mathcal{E}^\vee$ is of type

$$\mathcal{O}(-2) \xleftarrow{\oplus} 0 \leftarrow \mathcal{E}^\vee \leftarrow 4\mathcal{O}(-3) \xleftarrow{\oplus} 2\mathcal{O}(-4) \xleftarrow{\oplus} 8\mathcal{O}(-4) \xleftarrow{\oplus} 12\mathcal{O}(-5) \xleftarrow{\oplus} 4\mathcal{O}(-6) \xleftarrow{\oplus} 5\mathcal{O}(-6) \xleftarrow{\oplus} 8\mathcal{O}(-7) \xleftarrow{\oplus} 3\mathcal{O}(-8) \leftarrow 0.$$

In particular, $\mathcal{E}$ is stable since the normalized bundle $\mathcal{E}(-2)$ and its dual twisted by -1 have no non-zero sections.

Let $(M)$ be the monad of the rank-5 vector bundle $\mathfrak{S}$ as in Proposition 2.2. By taking sections in the display of $(M)$ we identify $H^0\mathfrak{S}(1)$ in terms of the exterior algebra $\Lambda V^*$ as the kernel

$$0 \rightarrow H^0\mathfrak{S}(1) \rightarrow \Lambda^1 V^* \oplus \Lambda^0 V^* \xrightarrow{\beta} V^* \rightarrow 0.$$ 

Two general elements of $H^0\mathfrak{S}(1)$ are nowhere dependant. In fact,

$$\sigma_1 = \begin{pmatrix} e_0 \wedge e_1 + e_2 \wedge e_3 \\ 0 \end{pmatrix}$$

and

$$\sigma_2 = \begin{pmatrix} e_1 \wedge e_2 - e_0 \wedge e_3 \\ e_1 \wedge e_2 \wedge e_4 - e_0 \wedge e_3 \wedge e_4 \end{pmatrix}$$

give rise to an exact sequence

$$0 \rightarrow 2\mathcal{O} \xrightarrow{\sigma} \mathfrak{S}(1) \rightarrow \mathcal{E} \rightarrow 0.$$
This can be easily seen by comparing the syzygies of the cokernel of $\sigma$ with those of $E$ (use e.g. Macaulay [BS]). It follows that $E(-1)$ is isomorphic to the cohomology bundle of the monad

$$0 \to 2\mathcal{O}(-1) \oplus \Omega^3(3) \xrightarrow{(\sigma, \alpha)} \Omega^2(2) \oplus \Omega^1(1) \xrightarrow{\beta} \mathcal{O} \to 0. \quad \square$$

In each of these examples we have a stratification

$$L^{m_0} \subset \cdots \subset L^1 \subset \mathbb{P}^n, \quad m_0 = \min\{n, c_1\},$$

where we use the following notations: For any subset $I$ of $\mathbb{Z}_{m_0}$ we set $L_I = \bigcap_{i \in I} L_i$, where $L_i = \{x_i = 0\}$, and for any $m \in \mathbb{Z}_{m_0}$ we write

$$L^m = \bigcup\{L_I \mid I \in \mathcal{P}_m(\mathbb{Z}_{m_0})\},$$

where $\mathcal{P}_m(\mathbb{Z}_{m_0})$ is the collection of subsets $I \subset \mathbb{Z}_{m_0}$ whith cardinality $|I| = m$. One may use this filtration to find local frames for $E$ in a more systematic way, and to construct a filtration of sheaves

$$E^0 \cong r\mathcal{O} \subset E^1 \cdots \subset E^{m_0} \cong E,$$

such that

$$E^m/E^{m-1} \cong \bigoplus_{I \in \mathcal{P}_m(\mathbb{Z}_{m_0})} \mathcal{O}_{L_I}(d_{L_I}),$$

with convenient twists $d_{L_I}$. Via this filtration one can e.g. compute the cohomology of $E$ (use Čech cohomology). We refer to [Sa] and [SEKS] for more details.

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