DUALITIES FOR SPIN REPRESENTATIONS

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Abstract. Let $S$ be the spinor representation of $U_q\mathfrak{so}_N$, for $N$ odd and $q^2$ not a root of unity. We show that the commutant of its action on $S^{\otimes n}$ is given by a representation of the nonstandard quantum group $U'_{q^2}\mathfrak{so}_n$. For $N$ even, an analogous statement also holds for $S = S_+ \oplus S_-$ the direct sum of the irreducible spinor representations of $U'_q\mathfrak{so}_N$, with the commutant given by $U'_q\mathfrak{so}_n$, a $\mathbb{Z}/2$-extension of $U'_q\mathfrak{so}_n$. Similar statements also hold for fusion tensor categories with $q$ a root of unity.

The decomposition of tensor powers of the vector representations of classical Lie groups was successfully studied in papers by Schur, Weyl and Brauer. These classical duality results were extended to Drinfeld-Jimbo quantum groups ([14], [4], [24]) which have had applications in a number of fields such as tensor categories, low-dimensional topology and von Neumann algebras. But it seems that an intrinsic description of the commutant of the action of spin groups on tensor powers of spinor representations has been found only fairly recently in [27]. The current paper deals with the missing cases in that paper. Moreover, we give simpler and more general proofs also for the cases already covered.

The inspiration for the approach in the paper [27] came from a paper by Hasegawa [10], see Section 6.1 for more details. He showed (as a special case of a far more general construction) that the obviously commuting actions of $O(N)$ and $SO(n)$ on $\mathbb{R}^N \otimes \mathbb{R}^n$ can be extended to commuting actions of the corresponding spin groups on the Clifford algebra $Cl(Nn) \cong Cl(N)^{\otimes n}$. For $N$ even, this vector space isomorphism can actually be made into an algebra homomorphism, with $Cl(N) \cong \text{End}(S)$ for the spinor representation $S$ of $O(N)$. Hence the commutant of the $Pin(N)$ action on $S^{\otimes n}$ is given by a representation of $Spin(n)$. This result could be extended to prove a duality on $S^{\otimes n}$ between a semidirect product $U$ of the quantum group $U_q\mathfrak{so}_N$ with $\mathbb{Z}/2$ and a non-standard $q$-deformation $U'_q\mathfrak{so}_n$ of the universal enveloping algebra $U\mathfrak{so}_n$. The latter has been studied before in particular by Klimyk and his coauthors, see e.g. [8], and it has also appeared in work of Noumi and Sugi tani [20] and Letzter [17].

But for $N$ odd, it was not possible to prove such a simple fact, due to the fact that the Clifford algebra is not simple in that case. Our main new result in this paper is a direct description of the commutant of the action of $U_q\mathfrak{so}_N$ on $S^{\otimes n}$ via a representation of the nonstandard orthogonal quantum group $U'_{q^2}\mathfrak{so}_n$. This can be quite easily seen for $q = 1$ using the element $C = \frac{1}{2} \sum_i e_i \otimes e_i \subset Cl(N)^{\otimes 2}$, where $\{e_i\}$ is an orthonormal basis for $\mathbb{R}^N$. We then extend this result to the quantum group case by using Hayashi’s $q$-Clifford algebra.

Here is the paper in more detail: We first review some basics about Clifford algebras and spin representations $S$ and produce a canonical element $C \in \text{End}(S^{\otimes 2})$ which commutes with
the spin group action. We review basic facts about the algebras $U'_q\mathfrak{so}_n$ and their representations in the second section. In particular, we show that we obtain a representation of $U'_{-1}\mathfrak{so}_n$ on $S^\otimes n$ which commutes with the pin group action. In the third section we review Hayashi’s spin representations of $U_q\mathfrak{so}_N$ into his $q$-Clifford algebra and we construct the $q$-analogs of the commuting elements $C$. It is shown in the following section that these can be used to define a representation of $U'_{-q^2}\mathfrak{so}_n$ on $S^\otimes n$. We then obtain first and second fundamental theorems for $\text{End}_U(S^\otimes n)$, where $U$ can be $U_q\mathfrak{so}_N$ or, for $N$ even, it can also be $U_q\mathfrak{so}_N \rtimes \mathbb{Z}/2$. We conclude with some remarks about related work and applications.

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1. Clifford Algebras and spinor representations

1.1. Basic Definitions. Let $V$ be a finite dimensional real inner product space. Then its Clifford algebra is the associative complex algebra generated by the elements of $V$ subject to the relation

$$v \cdot w + w \cdot v = 2(v, w)1.$$  

Let $\{e_1, ..., e_N\}$ be an orthonormal basis of the inner product space $V$. Then the Clifford algebra $Cl = Cl(N)$ corresponding to $V$ can also be defined via generators, denoted by $e_i$ as well, and relations

$$e_ie_j + e_je_i = 2\delta_{ij}1, \quad \text{for } 1 \leq i, j \leq N.$$  

If $N = 2k$ is even, it will be convenient to use a second presentation in terms of generators $\psi_j$ and $\psi_j^\dagger$, $1 \leq j \leq k$ defined by

$$\psi_j = \frac{1}{2}(e_{2j-1} + ie_{2j}), \quad \psi_j^\dagger = \frac{1}{2}(e_{2j-1} - ie_{2j}).$$  

Then a generator $\psi_j$ anticommutes with all other generators except $\psi_j^\dagger$ where we have

$$\psi_j^\dagger\psi_j + \psi_j\psi_j^\dagger = 1, \quad 1 \leq j \leq k.$$  

1.2. Spinor representations. Let us assume $N = 2k$ to be even. It is well-known that $Cl(N)$ has dimension $2^N$ and it is isomorphic to $M_{2^{N/2}}$, where $M_d$ denotes the $d \times d$ matrices. Let $S$ be a simple $Cl(N)$-module, of dimension $2^{N/2}$. The action of an element $g$ in the orthogonal group $O(N)$ on $V$ induces an automorphism $\alpha_g$ on $Cl(N)$, for each $g \in O(N)$. As any automorphism of the $d \times d$ matrices is inner, we obtain a projective representation $g \mapsto u_g$ of $O(N)$ on the $Cl(N)$-module $S$. This projective representation can be made into an honest representation of the universal covering groups $Pin(N)$ of $O(N)$. By restriction, the module $S$ becomes a $Pin(N - 1)$-module, which we will denote by $\hat{S}$. It decomposes into the direct sum of two simple projective $O(N - 1)$-modules $\hat{S}_+ \oplus \hat{S}_-$. These two modules are isomorphic.
as projective $SO(N-1)$-modules and simple. We need the following relations, which are easy to prove:

**Lemma 1.1.** Let $N = 2k$ be even. Define $f_r = (-i)^{r(r-1)/2}e_1e_2 \ldots e_r \in Cl(N)$, for $1 \leq r \leq N$. Then we have

(a) $f_re_i = (-1)^re_if_r$ for $i > r$ and $f_re_i = (-1)^{r-1}e_if_r$ for $i \leq r$,
(b) $f_r^2 = 1$ and $f_rf_s = (-1)^{r(s-r)}f_sf_r$ for $r < s$.
(c) $f_2 = e_1e_2 = (\psi_1\psi_1^\dagger - \psi_1^\dagger\psi_1)$ and $f_N = (-1)^{k-1}\prod_{j=1}^{k}(\psi_j\psi_j^\dagger - \psi_j^\dagger\psi_j)$.
(d) $\alpha_g(f_r) = det(g)f_r$ and $u_gf_r = det(g)f_ru_g$ for $g \in O(r)$, $1 \leq r \leq N$.

**Proof.** Parts (a) - (c) are straightforward. Part (d) is checked by an explicit calculation for $N = 2$. The same calculation also works for $g \in O(N)$ which is the identity matrix except for a $2 \times 2$ diagonal block. As such matrices generate $O(N)$, the claim follows.

1.3. **Explicit description.** We will need a more explicit description of the spin module $S$. Observe that $Cl = Cl_+ + Cl_-$, where $Cl_+$ and $Cl_-$ are the subalgebras generated by 1 and the elements $\psi_i^\dagger$ for $Cl_+$, and by 1 and the elements $\psi_i$ for $Cl_-$. We define $S = Cl/I$ for $N = 2k$ even, where $I$ is the left ideal generated by the $\psi_i$'s. $S$ has a basis $x(m)$, where $m \in \mathbb{R}^k$ with $m_i \in \{0,1\}$ for $1 \leq i \leq k$, and where

$$x(m) = (\psi_1^\dagger)^{m_1}(\psi_2^\dagger)^{m_2} \ldots (\psi_k^\dagger)^{m_k} \mod I.$$  

If we assign to the highest weight vector $m(0)$ the weight $\epsilon = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, the vector $x(m)$ would have the weight $\mu$ with $\mu_i = \frac{1}{2} - m_i$. We also define the quantity

$$m\{r\} = \sum_{j=1}^{r} m_j.$$  

Then it follows that $\psi_j^\dagger x(m) = 0$ if $m_j = 1$, and

$$\psi_j^\dagger x(m) = (-1)^{m_j-1}x(m^j), \quad \text{if } m_j = 0,$$

where $m^j$ coincides with $m$ except in the $j$-th coordinate, which is replaced by $1 - m_j$. Observe that the weight of $x(m^j)$ differs from the one for $x(m)$ only in the $j$-th coordinate, by a sign. The action of $\psi_j$ is given by the adjoint of $\psi_j^\dagger$.

If $N = 2k + 1$ is odd, we can make $S$ as before into a $Cl(2k + 1)$-module as follows. The actions of $e_i$ with $i \leq 2k$ resp $\psi_j$ and $\psi_j^\dagger$ with $j \leq k$ is as before. The action of $e_{2k+1}$ is given by

$$e_{2k+1}x(m) = \pm f_{2k}x(m) = \pm(-1)^{m(k)}x(m),$$

where we obtain a $Cl(2k + 1)$ action for each choice of the sign.
1.4. **Tensor products.** Let $S$ be the $\text{Cl}(N)$-module as described in the previous subsection, for both $N$ odd and $N$ even. In the following, we will identify elements of $\text{Cl}(N)$ with its image in $\text{End}(S)$. We define elements $C \in \text{End}(S \otimes 2)$ by

$$C = \frac{1}{2} \sum_{i=1}^{N} e_i \otimes e_i.$$ 

Using the definitions in 1.1, we obtain

$$\frac{1}{2}(e_{2j-1} \otimes e_{2j-1} + e_{2j} \otimes e_{2j}) = \psi_j \otimes \psi_j^\dagger + \psi_j^\dagger \otimes \psi_j.$$ 

Using this and the definitions in the last section, we can also write for $N = 2k + 1$

$$(1.2) \quad C = \frac{1}{2} f_{2k} \otimes f_{2k} + \sum_{j=1}^{k} \psi_j \otimes \psi_j^\dagger + \psi_j^\dagger \otimes \psi_j,$$

while for $N = 2k$ even $C$ is as above without the first summand. We will also need the well-known isomorphisms of $\text{Pin}(N)$ modules given by

$$(1.3) \quad N = 2k : \quad S \otimes 2 \cong \bigoplus_{j=0}^{2k} \Lambda^j V, \quad N = 2k + 1 : \quad S \otimes 2 \cong \bigoplus_{j=0}^{k} \Lambda^j V,$$

where $V = \mathbb{C}^N$ is the vector representation of $O(N)$.

**Lemma 1.2.** (a) If $N = 2k$, the element $C$ has exactly the integer eigenvalues $j$ satisfying $-k \leq j \leq k$.

(b) If $N = 2k + 1$, the element $C$ has the eigenvalues $\varepsilon(-1)^{j(k + \frac{1}{2} - j)}$, $1 \leq j \leq k$ with the sign $\varepsilon$ depending on the choice of the $\text{Cl}(N)$-module $S$.

**Proof.** Observe that

$$(\psi_j \otimes \psi_j^\dagger + \psi_j^\dagger \otimes \psi_j) x(m) \otimes x(n) = (-1)^{m(j-1)-n(j-1)} x(\bar{m}^j) \otimes x(\bar{n}^j),$$

if $m_j + n_j = 1$, and it is equal to 0 otherwise. Let now $\rho = (k - i)_i \in \mathbb{R}^k$, and let $\bar{m}$ be defined by $\bar{m}_j = 1 - m_j$, $1 \leq j \leq k$. Let

$$(1.4) \quad v = \sum_{m} (-1)^{(m,\rho)} x(m) \otimes x(\bar{m}).$$

Then the $x(m) \otimes x(\bar{m})$ coordinate of $Cv$ is given by

$$(-1)^{(m,\rho)} \sum_{j} (-1)^{2m(j-1)-(j-1)+\bar{m}^j-m,\rho} = (-1)^{(m,\rho)} (-1)^{k-1} k.$$ 

This shows that $(-1)^{k-1} k$ is an eigenvalue for $C$. One obtains an eigenvector with eigenvalue $(-1)^k k$ by multiplying the $x(m) \otimes x(\bar{m})$-coordinate of the vector $v$ by the scalar $(-1)^{(m,\rho)}$. One can similarly define eigenvectors $v$ with eigenvalues $\pm j$ by the sum of vectors $x(m) \otimes x(n)$.
where \( m_i + n_i = 1 \) for \( 1 \leq i \leq j \) and where \( m(i) = n(i) = 0 \) for \( i > j \). These are all possible eigenvalues in view of 1.3 and Lemma 1.3.

It follows from Lemma 1.1,(c) that \( f_N x(m) \otimes f_N x(n) = (-1)^{m(k)+n(n)} x(m) \otimes x(n) \). It follows from this and statement (a) that the eigenvalues of \( C \) are given by \((-1)^{j(k-\frac{1}{2}-j)}, 1 \leq j \leq k\).

**Lemma 1.3.** The element \( C \) commutes with the action of \( \text{Pin}(N) \) on \( S^{\otimes 2} \). Moreover, if \( C_1 = C \otimes 1 \in \text{End}(S^{\otimes 3}) \) and \( C_2 = 1 \otimes C \in \text{End}(S^{\otimes 3}) \) then we have

\[
C_1^2 C_2 + 2 C_1 C_2 C_1 + C_2 C_1^2 = C_2, \quad C_2^2 C_1 + 2 C_2 C_1 C_2 + C_1 C_2^2 = C_1.
\]

**Proof.** As \( g.(\sum_{i=1}^N e_i \otimes e_i) = \sum_{i=1}^N e_i \otimes e_i \in V^{\otimes 2} \) for any \( g \in O(N) \), the corresponding element \( \sum_{i=1}^N e_i \otimes e_i \in \mathcal{U}(\mathfrak{n})^{\otimes 2} \) commutes with the action of \( \text{Pin}(N) \) on \( S^{\otimes 2} \). The first statement follows from this and Lemma 1.1,(d).

Let \([A,B]^+_+ = AB + BA \) for any two elements \( A,B \) of a ring. Then it follows from the relations that

\[
\sum_{i,j=1}^N [e_i, e_j]_+ = \sum_{i,j=1}^N 2 \delta_{i,j} 1.
\]

Using this, we obtain

\[
[C_1, C_2]_+ = \frac{1}{4} \sum_{i,j=1}^N e_i \otimes [e_i, e_j]_+ \otimes e_j = \frac{1}{2} \sum_{i=1}^N e_i \otimes 1 \otimes e_i.
\]

We obtain in the same fashion

\[
[C_1, [C_1, C_2]_+]_+ = \frac{1}{4} \sum_{i,j=1}^N [e_i, e_j]_+ \otimes e_i \otimes e_j = \frac{1}{4} \sum_{i,j=1}^r 2 \delta_{ij} 1 \otimes e_i \otimes e_j = C_2.
\]

This proves the first identity in the statement. The proof of the second identity goes the same way.

2. **Representations of \( U_q^\prime \mathfrak{so}_n \)**

We assume throughout this paper all the algebras to be defined over the field of complex numbers, with \( q \) not being a root of unity. See Section 5.5 for more general rings.

2.1. **The algebras \( U_q^\prime \mathfrak{so}_n \) and \( U_q^\prime \mathfrak{o}_n \).** We shall need a \( q \)-deformation \( U_q^\prime \mathfrak{so}_n \) of the universal enveloping algebra of the Lie algebra \( \mathfrak{so}_n \). It is defined via generators \( B_i, 1 \leq i < n \) and relations \( B_i B_j = B_j B_i \) for \( |i-j| > 1 \) and

\[
B_i^2 B_{i \pm 1} - (q + q^{-1}) B_i B_{i \pm 1} B_i + B_{i \pm 1} B_i^2 = B_{i \pm 1},
\]

with the choice of sign in the indices the same for all terms. This algebra was defined, independently from each other, by Gavrilenk and Klimyk [8], by Letzter [17] and by Noumi and Sugitani [20]. Its finite-dimensional representations were classified by Klimyk and collaborators (see [12] and references there).
Theorem 2.1. Let \( q \) not be a root of unity. Then there are two series of finite-dimensional irreducible representations of \( U'_q\mathfrak{so}_n \), where \( k = \lfloor n/2 \rfloor \):

(a) The classical representations are \( q \)-deformations of the representations of \( \mathfrak{so}_n \). They are labeled by the dominant integral weights of \( \mathfrak{so}_n \). They are given by all vectors \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), where all coefficients are either integers or they all are congruent to \( \frac{1}{2} \) mod \( \mathbb{Z} \), and such that \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0 \) (for \( n \) odd), or \( \lambda_1 \geq \lambda_2 \geq \ldots \geq |\lambda_k| \) (for \( n \) even). They have the same dimensions as the corresponding \( \mathfrak{so}_n \) representations.

(b) The nonclassical representations are labelled by all dominant integral weights of \( \mathfrak{so}_n \) whose coefficients are not integers. Their dimensions are \( 2^{-k} \) times the dimension of the corresponding classical representations for \( n \) odd, and \( 2^{1-k} \) times the dimension of the corresponding classical representations for \( n \) even. For each such weight, we have \( 2^{n-1} \) non-equivalent representations of \( U'_q\mathfrak{so}_n \). They can be obtained from each other by multiplying the matrices for various generators by \(-1\). For \( n \) even, it suffices to consider only representations with highest weights \( \lambda \) for which \( \lambda_k > 0 \), together with the just mentioned operation of sign changes.

Remark 2.2. The construction of the representations of \( U'_q\mathfrak{so}_n \) by Klimyk et al essentially is a \( q \)-version of the construction of representations of \( \mathfrak{so}_n \) via Gelfand-Zetlin bases, see [19]. This approach takes advantage of the fact that a simple \( \mathfrak{so}_n \)-module, viewed as an \( \mathfrak{so}_{n-1} \)-module decomposes into a direct sum of mutually nonisomorphic simple \( \mathfrak{so}_{n-1} \) modules, see e.g. [19], Section 4.1 and 4.2 for details. This also determines the decomposition of a simple classical \( U'_q\mathfrak{so}_n \) module into a direct sum of mutually nonisomorphic \( U'_q\mathfrak{so}_{n-1} \) modules.

The decomposition of a simple non-classical \( U'_q\mathfrak{so}_n \) module \( V_\lambda \) into a direct sum \( \bigoplus V_\mu \) of simple \( U'_q\mathfrak{so}_{n-1} \) modules can be described similarly: If \( n = 2k \) is even, \( \mu \) runs through all weights \( \mu \) satisfying

\[
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \ldots \geq \mu_{k-1} \geq \lambda_k > 0,
\]

where for \( n = 2k + 1 \) odd we have

\[
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \ldots \geq \mu_{k-1} \geq \lambda_k > \mu_k > 0.
\]

All quantities in these inequalities are half-integers, i.e. congruent to 1/2 mod \( \mathbb{Z} \). For classical representations, the restriction rules are almost the same. We only need to replace \( \lambda_k \) by \( |\lambda_k| \) for \( n = 2k \), and \( \mu_k \) by \( |\mu_k| \) for \( n = 2k + 1 \) in the inequalities above, see [19].

We will also need an analog of the full orthogonal group in this setting. We define the algebra \( U'_q\mathfrak{o}_n \) by adding an additional generator \( F \) to the generators of \( U'_q\mathfrak{so}_n \) with the relations

\[
F^2 = 1, \quad FB_1 = -B_1F \quad \text{and} \quad FB_i = B_iF \quad \text{for } i > 1.
\]

Remark 2.3. 1. It is well-known that the Lie algebra \( \mathfrak{so}_n \) can be defined via generators \( L_i = E_{i,i+1} - E_{i+1,i}, 1 \leq i < n \), with \( E_{i,j} \) matrix units. It is then easy to check that the maps \( B_j \mapsto \sqrt{-1} E_j, 1 \leq j < n \) and \( F \mapsto \text{diag}(-1, 1, 1, \ldots, 1) \) define representations of \( U'_q\mathfrak{so}_n = U\mathfrak{so}_n \) and \( U'_q\mathfrak{o}_n \) respectively.
2. It is also clear from the representation in the first remark that any irreducible representation of the group $O(N)$ defines a representation of $U_q\mathfrak{g}_n$ by viewing the image of $F$ as a group element, and identifying $U_q\mathfrak{g}_n$ with the universal enveloping algebra of $\mathfrak{g}_n$. We will see in this paper that these representations also exist and remain irreducible for generic $q$.

2.2. Homomorphism onto Temperley-Lieb algebras. The Temperley-Lieb algebra $TL_n$ is given by generators $e_i$, $1 \leq i < n$ and relations $e_i^2 = e_i$, $e_i e_j = e_j e_i$ for $|i-j| > 1$ and $e_i e_{i+1} e_i = \frac{1}{q+q^{-1}} e_i$. It is well-known that $\text{End}_{U_q\mathfrak{g}_n}(V^{\otimes n})$ is isomorphic to $TL_n$ in our parametrization for $V = \mathbb{C}^2$. The proof of the following proposition is a straightforward, if moderately tedious calculation. It is a special case of our main results Theorem 5.2 and 5.3.

**Proposition 2.4.** The map $B_i \mapsto \frac{1}{q+q^{-1}} - (q + q^{-1}) e_i$, $1 \leq i < n$ defines an algebra homomorphism from $U_q\mathfrak{g}_n$ onto $TL_n$, which induces non-classical representations of $U'_q\mathfrak{g}_n$.

2.3. Representations of $U'_q\mathfrak{g}_3$. Similarly as for the case of the Lie algebra $\mathfrak{sl}_2$, it is easy to write down explicit irreducible representations for $U'_q\mathfrak{g}_3$. The $(N+1)$-dimensional irreducible classical representation $V_N$ of $U'_q\mathfrak{g}_3$ can be described as follows. We fix a basis $\{v_j, 0 \leq j \leq N\}$ of weight vectors. Then the actions of $B_1$ and $B_2$ are given by

$$B_1 v_j = [N/2 - j] v_j, \quad B_2 v_j = v_{j+1} + \alpha_{j-1,j} v_{j-1},$$

where

$$\alpha_{j-1,j} = \frac{[N+1-j][j]}{(q^{N/2-j} + q^{-N/2})(q^{N/2-j+1} + q^{-N/2-1})}.$$  

Similarly, for $N$ odd, we can describe an $(N+1)/2$-dimensional simple non-classical representation of $U'_q\mathfrak{g}_3$ with respect to a basis $\{v_j, 0 \leq j \leq (N-1)/2\}$ by

$$B_1 v_j = [N/2 - j]^+ v_j, \quad B_2 v_j = v_{j+1} + \alpha^{+}_{j-1,j} v_{j-1}, \quad j < (N-1)/2,$$

where

$$\alpha^{+}_{j-1,j} = \frac{[N+1-j][j]}{(q^{N/2-j} - q^{-N/2})(q^{N/2-j+1} - q^{-N/2-1})}.$$  

If $j = (N-1)/2$, the action on $v_j$ by $B_1$ is as in 2.4, while we have

$$B_2 v_{(N-1)/2} = \pm \frac{[N+1/2]}{i(q^{1/2} - q^{-1/2})} v_{(N-1)/2} + \alpha^{+}_{(N-3)/2,(N-1)/2} v_{(N-3)/2},$$

where the representation with the minus sign in the formula above is equivalent to the representation with the plus sign, after replacing $B_2$ with $-B_2$. We will consider representations of the algebra $U'_q\mathfrak{g}_n$. If we choose $(-q^2)^{1/2} = iq$, the eigenvalues of $B_1$ in the $(N+1)/2 = k+1$-dimensional nonclassical representation will be $(-1)^{k-j} (q^{N-2j} - q^{2j-N})/(q^2 - q^{-2})$, $0 \leq j \leq k$.

**Lemma 2.5.** (a) We can make $V_N$ into a $U'_q\mathfrak{g}_3$-module by leaving the action of $B_2$ the same, and replacing $B_1$ by $\tilde{B}_1$ whose action on $V_N$ is given by $\tilde{B}_1 v_j = (-1)^j B_1 v_j$.

(b) If $N$ is even, the representation of $U'_q\mathfrak{g}_3$ in (a) is isomorphic to its classical representation with highest weight $[N/2]$. If $N$ is odd, it decomposes into the direct sum of two
non-classical irreducible representations of $U_q\mathfrak{so}_3$ with highest weight $[N/2]^+ = i(q^{N/2} + q^{-N/2})/(q - q^{-1})$.

Proof. Obviously the actions of $B_1^2$ and $\tilde{B}_1^2$ on $V_N$ are the same, while one checks easily that $\tilde{B}_1 B_2 \tilde{B}_1 v_j = -B_1 B_2 B_1 v_j$ for all basis vectors of $V_N$. This implies (a), using Theorem 2.1.

If $N$ is even, it is easy to check that $v_N$ is a highest weight vector with weight $[N/2]_q$ for the $U_q\mathfrak{so}_3$ action on $V_N$ given via $\tilde{B}_1$ and $B_2$ which remains irreducible. For $N$ odd, $\tilde{B}_1$ acts via the same scalar on $v_j$ as on $v_N-j$. Hence we obtain two highest weight vectors (in the sense of [28]) with highest weight $[N/2]_q$, where the $q$-number here is defined for $-q$. It follows that $V_N$ decomposes into the direct sum of two irreducible non-equivalent $U_q\mathfrak{so}_3$-modules (see e.g. [28], Theorem 3.8 for details).

2.4. Homomorphisms. Let $S$ be the spinor module as described in Section 1.3 and let $C$ and $\tilde{C}$ be as in Section 1.4. We define elements

$$C_i = 1 \otimes 1 \otimes ... \otimes C \otimes ... \otimes 1 \in \text{End}(S^{\otimes n}), \quad 1 \leq i < n,$$

$$\tilde{C}_i = 1 \otimes 1 \otimes ... \otimes \tilde{C} \otimes ... \otimes 1 \in \text{End}(S^{\otimes n}), \quad 1 \leq i < n, \quad i \text{ odd},$$

where the element $C$ on the right hand side acts on the $i$-th and $(i+1)$-st factor of $S^{\otimes n}$. Moreover, we define $\tilde{C}_{2i} = C_{2i}$ for $1 \leq 2i < n$.

Proposition 2.6. For both $N$ odd and even, the map $B_i \mapsto C_i$ defines a homomorphism of $U_q\mathfrak{so}_n$ into $\text{End}_{\text{Pin}(N)}(S^{\otimes n})$. For $N$ even, the map $B_i \mapsto \tilde{C}_i$ for $i$ odd and $B_i \mapsto C_i$ for $i$ even defines a homomorphism of $U_q\mathfrak{so}_n$ into $\text{End}_{\text{Pin}(N)}(S^{\otimes n})$.

Proof. The first statement follows from Lemma 1.3 and the discussion before this proposition. The second statement was already shown in [27]. It can also be deduced from the first statement using Lemma 2.5.

3. Quantum groups

3.1. $q$-Clifford algebra. We follow the paper [11] by Hayashi, with some minor modifications. For a somewhat more conceptual approach to $q$-Clifford algebras, see [7] and Section 6.3. The $q$-Clifford algebra $Cl_q(2k)$ coincides with the ordinary Clifford algebra in the sense that it is again generated by elements $\psi_i$ and $\psi_i^\dagger$, $1 \leq i \leq k$ satisfying the relations for the usual Clifford algebra. In particular, $\psi_i^\dagger \psi_i$ and $\psi_i \psi_i^\dagger$ are idempotents which annihilate each other and add up to 1. The dependency on $q$ will be reflected by additional elements $\omega_i$ defined by

$$\omega_i = \psi_i \psi_i^\dagger + q^{-1} \psi_i^\dagger \psi_i. \tag{3.1}$$

Then it is clear that $\omega_i^\pm 1 \omega_j^\pm 1 = \omega_j^\pm 1 \omega_i^\pm 1$ and that

$$\omega_i \psi_i = \psi_i = q \psi_i \omega_i, \quad \psi_i^\dagger \omega_i = \psi_i^\dagger = q \omega_i \psi_i^\dagger. \tag{3.2}$$
3.2. Homomorphisms into \( q \)-Clifford algebras. We are now defining maps from the quantum groups \( U_q\mathfrak{so}_{2k} \) and \( U_q\mathfrak{so}_{2k+1} \), i.e. of Lie type \( D_k \) and \( B_k \) into \( Cl_q(N) \). Here we use the definition of the quantum groups as in [18] or in [13], Section 4.3. In particular, the inner product on the weight lattice is normalized such that \( (\alpha, \alpha) = 2 \) for every short root. The maps appeared before in [11]. However, our normalizations are not always the same as in that paper, so we give the explicit maps below as follows: For \( 1 \leq i \leq k-1 \) we define
\[
K_i \mapsto \omega_i^2 \omega_{i+1}^{-2}, \quad E_i \mapsto \psi_i \psi_{i+1}^\dagger, \quad F_i \mapsto \psi_{i+1} \psi_i^\dagger.
\]
For type \( B_k \), we also define
\[
K_k \mapsto q \omega_k^2, \quad E_k \mapsto \psi_k f_{2k}, \quad F_k \mapsto f_{2k} \psi_k^\dagger,
\]
while for type \( D_k \) we define
\[
K_k \mapsto q^2 \omega_{k-1}^2 \omega_k^2, \quad E_k \mapsto \psi_{k-1} \psi_k, \quad F_k \mapsto \psi_{k-1}^\dagger \psi_k^\dagger.
\]

**Proposition 3.1.** (see [11]) (a) The assignments in 3.3 and 3.5 define a representation of \( U_q\mathfrak{so}_N, N = 2k \) even into \( Cl_q(N) \cong \text{End}(S) \).
(b) The assignments in 3.3 and 3.4 define a representation of \( U_q\mathfrak{so}_{2k+1} \) into \( Cl_q(2k) \cong \text{End}(S) \).

**Proof.** Statement (a) was proved in [11]. The assignments in (b) differ from the ones in [11] only by multiplying the images of \( E_k \) and \( F_k \) by \( f_{2k} \) from the right and from the left. It is not hard to check that this still satisfies the quantum group relations, as \( f_{2k}^2 = 1 \) and \( f_{2k} \) commutes with the images of the lower indexed generators.

**Remark 3.2.** 1. Observe that we have defined a representation of \( U_q\mathfrak{so}_N \) for \( N = 2k \). This will make it easier to deal with the odd- and even-dimensional cases at the same time.
2. One can check that for \( N = 2k + 1 \) odd the vector \( x(0) \) is a highest weight vector with weight \( \varepsilon = \frac{1}{2} (1,1, \ldots, 1) \), and that the vector \( x(m) \) has weight \( \mu \) with \( \mu_i = \frac{1}{2} - m_i, 1 \leq i \leq k \). If \( N = 2k \) even, we also have the highest weight \( \varepsilon_- = x(m) \) with \( m_i = \delta_{i,k} \). The weight of \( x(m) \) can be determined as in the odd-dimensional case.

3.3. Commuting objects, Lie type \( D \). We now define the \( q \)-deformations of the operators \( C \) of the previous section for quantum groups. As before, we define them as elements of \( Cl_q(N)^{\otimes 2} \) acting on \( \text{End}(S^{\otimes 2}) \). For Lie type \( D_k \), we define
\[
C = \sum_{i=1}^k \Omega_{i-1}^{-1} \psi_i \otimes \Omega_{i-1} \psi_i^\dagger + \Omega_{i-1}^{-1} \psi_i^\dagger \otimes \Omega_{i-1} \psi_i,
\]
where \( \Omega_r = \prod_{j=1}^r \omega_j^2, 1 \leq r \leq k \). We leave it to the reader to check that
\[
(\Omega_{j-1}^{-1} \otimes \Omega_{j-1})(\psi_j \otimes \psi_j^\dagger + \psi_j^\dagger \otimes \psi_j)x(m) \otimes x(n) = (-q^2)^{m(j-1)-n(j-1)} x(m^j) \otimes x(n^j),
\]
if \( m_j + n_j = 1 \), and it is equal to 0 otherwise.

**Lemma 3.3.** The operator \( C \) defined in Eq 3.6 commutes with the action of \( U_q\mathfrak{so}_{2k} \) on \( S^{\otimes 2} \).
Proof. Let us first do this for type $D_2$. We shall use the coproduct defined by
\[ \Delta(E_i) = K_i^{1/2} \otimes E_i + E_i \otimes K_i^{-1/2}; \]
it is well-known that this is equivalent to the coproduct defined in [18] and [13], using the automorphism defined by $E_i \mapsto E_i K_i^{1/2}$, $F_i \mapsto K_i^{-1/2} F_i$, $K_i \mapsto K_i$, $1 \leq i \leq k$. Using Def. 3.3, we obtain
\[ \Delta(E_1) = \omega_1 \omega_2^{-1} \otimes \psi_1 \psi_2^\dagger + \psi_1 \psi_2 \otimes \omega_1^{-1} \omega_2. \]
Let $C_1 = \psi_1 \otimes \psi_1^\dagger + \omega_1^{-2} \psi_2 \otimes \omega_1^2 \psi_2$. We then obtain
\[ [\Delta(E_1), C_1] = \omega_1 \omega_2^{-1} \psi_1 \otimes \psi_1 \psi_2^\dagger - \psi_1 \psi_1 \omega_2^{-1} \otimes \psi_1 \psi_2^\dagger \]
\[ + \psi_1 \psi_2 \omega_1^{-2} \psi_2 \otimes \omega_1^{-1} \omega_2 \omega_2^2 \psi_2^\dagger - \omega_1^{-2} \psi_2 \psi_1 \psi_2^\dagger \otimes \omega_1^2 \psi_2^\dagger \omega_1^{-1} \omega_2 \]
\[ = - \omega_2^{-1} \psi_1 \otimes (\psi_1 \psi_2^\dagger + q \psi_1 \psi_1^\dagger \psi_1) \psi_2^\dagger + \psi_1 q \psi_2 \psi_2^\dagger + \psi_2 \psi_2^\dagger \otimes \omega_1 \psi_2^\dagger \]
\[ = - \omega_2^{-1} \psi_1 \otimes \omega_1 \psi_1^\dagger + \omega_2^{-1} \psi_1 \otimes \omega_1 \psi_1^\dagger = 0, \tag{3.8} \]
where we used the relations 3.1 and 3.2 after the definition of the $q$-Clifford algebra. One similarly also shows that the commutant of $\Delta(E_1)$ with $\psi_1^\dagger \otimes \psi_1 + \omega_1^2 \psi_2 \otimes \omega_2^2 \psi_2^\dagger$ is equal to 0. This shows that $[\Delta(E_1), C] = 0$. The statement for $F_1$ can be shown by a similar calculation, or it can be deduced by the following argument: The transpose map $^T$ induced by
\[ \psi_i \mapsto \psi_i^T = \psi_i^\dagger, \quad \psi_i^\dagger \mapsto (\psi_i^\dagger)^T = \psi_i \]
induces an algebra antiautomorphism of $Cl_q$ which induces on the image of $U_q \mathfrak{so}_N$ the algebra anti-automorphism defined by
\[ E_i \mapsto E_i^T = F_i, \quad F_i \mapsto F_i^T = E_i, \quad K_i \mapsto K_i^T = K_i, \]
which is compatible with the Hopf algebra structure. As $C^T = C$, it follows
\[ [\Delta(F_1), C] = [\Delta(E_1^T), C^T] = -[\Delta(E_1), C]^T = 0. \]
The commutation relation of $C$ with $\Delta(E_2)$ is shown by a similar calculation, from which follows the claim for $F_2$ by the previous argument.

For the general case, one observes that $\Delta(E_i)$ trivially commutes with all summands of $C$ except the ones indexed by $i$ and $i+1$. The proof that $\Delta(E_i)$ commutes with these remaining summands is essentially the same as the one for the case $D_2$.

3.4. Commuting objects. Lie type $B$. For Lie type $B_k$, we define the operator $C \in \text{End}(S\otimes S)$ by (compare with Section 1.4 for $q = 1$)
\[ C = \frac{1}{[2]} (\Omega_k^{-1} f_{2k} \otimes \Omega_k f_{2k}) + \sum_{i=1}^{k} \Omega_{i-1}^{-1} \psi_i \otimes \Omega_{i-1} \psi_i^\dagger + \Omega_{i-1}^{-1} \psi_i^\dagger \otimes \Omega_{i-1} \psi_i. \tag{3.9} \]

Lemma 3.4. The operator $C$ defined in Eq 3.9 commutes with the action of $U_q \mathfrak{so}_{2k+1}$ on $S\otimes S$. 
We also have
definition of 

We will use relations 3.1 and 3.2, which also imply $\psi$ commutes with $C$. It is now easy to check that $\Delta(E_i)$ also commutes with the first summand in the definition of $C$.

To finish the proof, recall that we have (with $N = 2k$)

$$
\Delta(E_k) = q^{1/2}\omega_k \otimes \psi_k f_N + \psi_k f_N \otimes q^{-1/2}\omega_k^{-1}.
$$

We will use relations 3.1 and 3.2, which also imply $\psi_{k-1}\Omega_{k-1} = q^{-2}\Omega_{k-1}\psi_{k-1}$. We obtain

$$
[\Delta(E_k), \Omega_k^{-1} f_N \otimes \Omega_{k-1} f_N] =
q^{1/2} \omega_k \Omega_k^{-1} f_N \otimes [\psi_k f_N \Omega_k f_N - \Omega_k f_N \psi_k f_N]
+ [\psi_k f_N \Omega_k^{-1} f_N - \Omega_k^{-1} f_N \psi_k f_N] \otimes q^{-1/2} \omega_k^{-1} \Omega_k f_N
= \omega_k \Omega_k^{-1} f_N \otimes (q^{-3/2} + q^{1/2}) \Omega_k \psi_k
+ (q^{3/2} + q^{-1/2}) \Omega_k^{-1} \psi_k \otimes \Omega_k f_N \omega_k^{-1}
= (q + q^{-1}) (q^{-1/2} \omega_k \Omega_k^{-1} f_N \otimes \Omega_k \psi_k + q^{1/2} \Omega_k^{-1} \psi_k \otimes \Omega_k f_N \omega_k^{-1} - \Omega_k^{-1} f_N \otimes \Omega_k \psi_k + q^{-1/2} \omega_k^{-1} \Omega_k f_N \psi_k)
$$

(3.10)

We also have

$$
[\Delta(E_k), (\Omega_k^{-1} \otimes \Omega_{k-1}) (\psi_k \otimes \psi_k^\dagger + \psi_k^\dagger \otimes \psi_k)] =
q^{1/2} \omega_k \Omega_k^{-1} \psi_k \otimes \psi_k f_N \Omega_{k-1} \psi_k^\dagger - \Omega_k^{-1} \psi_k \Omega_{k-1} \psi_k^\dagger q^{1/2} \omega_k \otimes \Omega_k^{-1} \psi_k \psi_k f_N
+ \psi_k f_N \Omega_k^{-1} \psi_k^\dagger \otimes q^{-1/2} \omega_k^{-1} \Omega_{k-1} \psi_k - \Omega_k^{-1} \psi_k^\dagger \psi_k f_N \otimes q^{-1/2} \Omega_{k-1} \psi_k \omega_k^{-1}
= - \Omega_k^{-1} \psi_k \otimes \Omega_{k-1} f_N (q^{1/2} \psi_k \psi_k^\dagger + q^{-1/2} \psi_k^\dagger \psi_k)
- (q^{-1/2} \psi_k \psi_k^\dagger + q^{1/2} \psi_k^\dagger \psi_k) \Omega_k^{-1} f_N \otimes \Omega_{k-1} \psi_k
= - q^{1/2} \Omega_k^{-1} \psi_k \otimes \Omega_{k-1} f_N \omega_k - q^{-1/2} \omega_k^{-1} \Omega_k^{-1} f_N \otimes \Omega_{k-1} \psi_k.
$$

(3.11)

Obviously, $\Delta(E_k)$ commutes with the first $k - 1$ summands under the summation sign in the definition of $C$. It follows from the identity $\omega_k \Omega_k^{-1} = \omega_k^{-1} \Omega_k^{-1}$ and the last two calculations that $\Delta(E_k)$ also commutes with the remaining summands of $C$. The commutation with $\Delta(F_k)$ can be deduced from this using the transposition map $T$ as in the proof of Lemma 3.3.

4. Relations

The main result of this section will be to give an algebraic description of the centralizer of the action of $U_q\mathfrak{so}_N$ on $S^{\otimes n}$. It is possible, and fairly straightforward, to extend the proof in [27] to the additional cases treated here. However, that proof was somewhat indirect. So we decided to give another proof here which, basically, is a direct calculation. While not quite as straightforward as the proof for $q = 1$ in Lemma 1.3, it would still seem to be an improvement over the one in [27].
4.1. Basic relations. Let \( a, b \) be any elements in an associative algebra, and let \( v \) be any invertible element in its ground ring. Then we define

\[
\text{lhs}_q(a; b) = a^2b + (v + v^{-1})aba + ba^2.
\]

It will be convenient to introduce the notation

\[
c_{i,+} = \Omega_{i-1}^{-1} \psi_i, \quad c_{i,-} = \Omega_{i-1}^{-1} \psi_i^\dagger, \quad d_{i,+} = \Omega_{i-1} \psi_i, \quad d_{i,-} = \Omega_{i-1} \psi_i^\dagger.
\]

Using this, we can write the commuting operators \( C \) from Lemma 3.3 and Lemma 3.4 as

\[
C = \sum_{i=1}^k C(i) \quad \text{and} \quad \tilde{C}(k+1) = \frac{1}{2} \Omega_k^{-1} f_N \otimes \Omega_k f_N.
\]

It is straightforward to check the following relations, where \( \varepsilon, \kappa \in \{\pm\} \) and \( q^{2\varepsilon} = q^{\pm 2} \):

\[
d_{i,\varepsilon}c_{j,\kappa} = \begin{cases} 
-q^{2\varepsilon} c_{j,\kappa} d_{i,\varepsilon} & i < j, \\
-q^{2\varepsilon} c_{j,\kappa} d_{i,\varepsilon} & i > j.
\end{cases}
\]

We obtain from the relations so far the following equation which will be useful later:

\[
d_{i,\varepsilon}d_{i, -\varepsilon}c_{j,\kappa} + (q^2 + q^{-2})d_{i,\varepsilon}c_{j,\kappa}d_{i, -\varepsilon} + c_{j,\kappa}d_{i,\varepsilon}d_{i, -\varepsilon} =
\]

\[
= \begin{cases} 
0 & i > j, \\
(1 - q^{4\varepsilon})d_{i,\varepsilon}d_{i, -\varepsilon}c_{j,\kappa} & i < j.
\end{cases}
\]

4.2. Technical lemma.

Lemma 4.1. Using notation defined in 4.1 we have

(a) \( \text{lhs}_{q^2}(C \otimes 1; 1 \otimes C) = \sum_{i,k} \text{lhs}_{q^2}(C(i) \otimes 1; 1 \otimes C(k)), \)

(b) \[
\text{lhs}_{q^2}(C(i) \otimes 1; 1 \otimes C(k)) = \begin{cases} 
0 & i > k, \\
(\Omega_{i-1}^{-2} \otimes \Omega_i^2 \otimes 1)(1 \otimes C(k)) & i = k, \\
([\Omega_{i-1}^{-2} \otimes \Omega_i^{-2} \otimes \Omega_i^2] \otimes 1)(1 \otimes C(k)) & i < k.
\end{cases}
\]

Proof. It will be convenient to write \( C = \sum_u c_u \otimes d_u \) for (a). (This is not quite consistent with our previous notation, but should not lead to confusion.) It then follows that

\[
\text{lhs}_{q^2}(C \otimes 1; 1 \otimes C) = \sum_{u,v,w} c_u c_v \otimes [d_u d_v c_w + (q^2 + q^{-2})d_u c_w d_v + c_w d_u d_v] \otimes d_w.
\]

Let \( c_u = c_{i,\varepsilon} \) and let \( c_v = c_{j,\varepsilon} \) with \( i \neq j \). Observe that the claim is proved if for given index \( w \) the summand for our given indices \( u \) and \( v \) cancels with the one with \( c_u = c_{j,\varepsilon} \) and \( c_v = c_{i,\varepsilon} \).
Let \( d_u = d_{i,-\varepsilon} \) and \( d_v = d_{j,-\varepsilon} \), and \( c_u \) and \( c_v \) as at the beginning of this paragraph. It follows from 4.3 that
\[
c_u c_v = -q^{\pm 2\varepsilon} c_v c_u, \quad d_u d_v = -q^{\pm 2\varepsilon} d_u d_v,
\]
with matching signs in the exponents. Let us choose the labeling such that \( c_u c_v = -q^2 c_v c_u \), and hence also \( d_u d_v = -q^2 d_u d_v \). Using the commutation relations above these two summands add up to
\[
c_u c_v \otimes (q^2 + q^{-2})[q^{-2} d_u d_v c_w + d_u c_w d_v - q^{-2} d_u d_v c_w + q^{-2} c_w d_u d_v] \otimes d_w.
\]
Then our claim will follow if we can show that the middle factor \( M \) in this tensor product is equal to 0.

First observe that \( d_u d_v = -q^2 d_u d_v \) implies that \( d_u = d_{i,+} \) and \( d_v = d_{j,\pm} \) or \( d_v = d_{i,-} \) and \( d_u = d_{j,\pm} \) with \( i < j \). Let us consider the case \( d_u = d_{i,+} = \omega_{i-1} \psi_i \). Then one checks that \( c_w d_u = -q^{-2} d_u c_w \) is only possible for \( c_w = c_{a,-} \) for \( a < i \). As \( i < j \), we then also have \( c_w d_v = -q^{-2} d_v c_w \), which forces \( M = 0 \). One similarly shows in the second case with \( d_u = d_{i,-} \) that \( c_w d_u = -q^{-2} d_u c_w \) would imply \( c_w d_v = -q^{-2} d_v c_w \). This completes the proof for claim (a).

For part (b), we only need to consider the cases with \( c_u = c_{i,\varepsilon} \) and \( c_v = c_{i,-\varepsilon} \), as \( c_{i,\varepsilon}^2 = 0 \). Using 4.3, one checks that
\[
c_{i,\varepsilon} c_{i,-\varepsilon} \otimes [d_{i,-\varepsilon} d_{i,\varepsilon} c_{j,\kappa} + (q^2 + q^{-2}) d_{i,-\varepsilon} c_{j,\kappa} d_{i,\varepsilon} + c_{j,\kappa} d_{i,-\varepsilon} d_{i,\varepsilon}] \otimes d_{j,-\kappa} =
\]
\[
\begin{cases}
0 & i > j, \\
(1 - q^{4\varepsilon})(\Omega_{i-1}^{-2} \psi_i^\varepsilon \psi_i^{-\varepsilon} \otimes \Omega_{i-1}^2 \psi_i^{-\varepsilon} \psi_i^\varepsilon \otimes 1)(1 \otimes c_{j,\kappa} \otimes d_{j,-\kappa}) & i < j.
\end{cases}
\]
Adding up these quantities for all possible choices of \( \varepsilon \) and \( \kappa \), we obtain 0 for \( i > j \). Using
\[
(1 - q^4) \psi_i^\varepsilon \psi_i^{-\varepsilon} \psi_i^\varepsilon + (1 - q^{-4}) \psi_i^{-\varepsilon} \psi_i^\varepsilon \psi_i^{-\varepsilon} = 1 \otimes 1 - \omega_i^4 \otimes \omega_i^{-4},
\]
we similarly obtain the claim for \( i < j \). The claim for \( i = j \) follows from a direct calculation.

**Proposition 4.2.** Let \( C_1 = C \otimes 1 \) and \( C_2 = 1 \otimes C \). Then we have
\[
C_1^2 C_2 + (q^2 + q^{-2}) C_1 C_2 C_1 + C_2 C_1^2 = C_2.
\]

**Proof.** It follows from Lemma 4.1(b) that for fixed \( j \)
\[
\sum_{i=1}^{k} l s_{q^2}(C(i) \otimes 1; 1 \otimes C(j)) = 1 \otimes C(j).
\]
The claim follows from this and Lemma 4.1(a) for type \( D \). For Lie type \( B_k \), we have to add \( \tilde{C}(k+1) \) to the expression for type \( D_k \), see 4.2. Setting \( c_{k+1,\varepsilon} = \Omega_k^{-1} f_N \) and \( d_{k+1,\varepsilon} = \Omega_k f_N \), one checks that Eq 4.3 and 4.4 also hold if one of the indices is \( k+1 \). One deduces the results of Lemma 4.1(a) and, except for \( i = j = k+1 \), also of part (b) from this. Observe that \( c_{k+1,\varepsilon} \) commutes with \( d_{k+1,\varepsilon} \). One calculates that
\[
ls_{q^2}(\tilde{C}(k+1) \otimes 1; 1 \otimes \tilde{C}(k+1)) = (\Omega_k^{-2} \otimes \Omega_k^2 \otimes 1)(1 \otimes \tilde{C}(k+1)).
\]
The claim can be deduced from this, using Lemma 4.1, as it was done for type \( D \).
5. First and Second Fundamental Theorem

5.1. Preliminaries. We consider the spinor module $S$ as in Section 1.2 for $U = U_q\mathfrak{so}_{2k+1}$, for $U = U_q\mathfrak{so}_{2k}$ and for $U = U_q\mathfrak{so}_{2k} \rtimes \mathbb{Z}/2$, where the $\mathbb{Z}/2$ action is given by the diagram automorphism permuting the generators given by the end vertices of the Dynkin diagram next to the triple vertex. It has dimension $2^k$ in all these cases, and its weights are given by all possible vectors $\omega = (\pm \frac{1}{2}, \pm \frac{1}{4}, ..., \pm \frac{1}{2^k}) \in \mathbb{R}^k$. As all weights have multiplicity 1, tensoring an irreducible highest weight module $V_\lambda$ by $S$ is given by

$$V_\lambda \otimes S \cong \oplus \mu V_\mu,$$

where the summation goes over all dominant weights $\mu$ of the form $\mu = \lambda + \omega$. If $U = U_q\mathfrak{so}_{2k} \rtimes \mathbb{Z}/2$, this has to be slightly modified, see [27]. A first fundamental theorem has been proved for $\text{End}_U(S^{\otimes n})$ for this case as well as for $U = U_q\mathfrak{so}_{2k+1}$ in [27], Theorem 3.3. We will review the method used there, a modification of which will be used also for the missing case to be proved here. In all of these cases, we have

$$S^{\otimes n} = S^{\otimes n}_{\text{old}} \oplus S^{\otimes n}_{\text{new}},$$

where $S^{\otimes n}_{\text{old}}$ is a direct sum of irreducible modules $V_\lambda$ which have already appeared in smaller tensor powers of $S$ (for which $\lambda_1 < n/2$) and where $S^{\otimes n}_{\text{new}}$ is a direct sum of irreducible modules $V_\lambda$ which have not appeared before (for which $\lambda_1 = n/2$). The following proposition is a slight generalization of [26], Prop. 4.10. We assume $C$ to be a $\mathbb{C}$-linear rigid tensor category, see e.g. [23] for precise definitions. The only property we will need is the fact that for every object $X$ in $C$ there exists an object $\tilde{X}$ and morphisms $\iota_X : 1 \to X \otimes \tilde{X}$, $\tilde{d}_X : X \otimes \tilde{X} \to 1$ such that $\text{Tr}(a) = \tilde{d}_X(a \otimes 1)\iota_X$ for any $a \in \text{End}(X)$. It is well-known that this holds for $C = \text{Rep}U_q\mathfrak{so}_N$.

Proposition 5.1. Let $V$ be a self-dual rigid object in the braided spherical tensor category $C$ and let $p = \iota_V \tilde{d}_V$. Let $\mathcal{E}_n = \text{End}_C(V^{\otimes n})$. Then $\text{End}(V^{\otimes n})_{\text{odd}} = (\mathcal{E}_{n-1} \otimes 1)p_{n-1}(\mathcal{E}_{n-1} \otimes 1)$.

Proof. The $p$ in the statement can be normalized to be a projection. It then satisfies exactly the same properties as the $p$ in the proof of Proposition 4.10 in [26] for $k = 2$, even if $V$ is not necessarily simple. The claim follows from this.

5.2. First fundamental theorem. We study $S^{\otimes n}_{\text{new}}$ by induction on the rank of $U$. It follows from the relations that the subalgebra of $U_q\mathfrak{so}_N$, generated by $E_i$, $F_i$ and $K_i^{\pm 1}$, $2 \leq i \leq k$ is isomorphic to $U_q\mathfrak{so}_{N-2}$, for $N = 2k$ or $N = 2k + 1$. It will be convenient to denote $U_q\mathfrak{so}_N$ or $U_q\mathfrak{so}_N \rtimes \mathbb{Z}/2$ by $U(N)$, the spinor module $S$ of $U(N)$ by $S(N)$, and the centralizer algebra $\text{End}_U(S^{\otimes n})$ by $\mathcal{E}_n^{(N)}$. We have the following well-known facts which are easy to check:

(a) We have the isomorphism of $U(N-2)$ modules $S(N) \cong S(N-2)_{1} \oplus S(N-2)_{2}$, where $S(N-2)_{1}$ is spanned by the weight vectors with weights $(\frac{1}{2}, \omega')$, with $\omega'$ a weight of $S(N-2)$.

(b) Let $v \in S(N-2)_{1}^{\otimes n} \subset S(N)^{\otimes n}$. Then $v$ is a highest weight vector for $U(N-2)$ of weight $\lambda'$ if and only if it is a highest weight vector for $U(N)$ of weight $(\frac{1}{2}, \lambda')$. 


Theorem 5.2. (First Fundamental Theorem) The algebra $\text{End}_U(S^\otimes n)$ is generated by the elements $a_i$, $i = 1, 2, \ldots, n - 1$, with $a \in \text{End}_U(S^\otimes 2)$.

Proof. We only need to consider the case $U = U_q\mathfrak{so}_{2k}$. The statement was proved in [27], Theorem 3.3 for the other cases.

For getting the induction on the rank $k$ going, we define $\text{Spin}(2)$ to be the $\mathbb{Z}/2$ cover of $SO(2)$. Its irreducible representations are labeled by half integers. In this case $S$ is the direct sum of two 1-dimensional representations with weights $\pm \frac{1}{2}$, which is obviously self-dual. Let $f \in \text{End}(S)$ act via $\pm 1$ on the vectors with weights $\pm \frac{1}{2}$. It follows from the tensor product rules that $f \otimes 1_{n-1}$ again acts via $\pm 1$ on the 1-dimensional representations with weights $\pm \frac{1}{2}$. The tensor product rules also show that this is $S_{\text{new}}^\otimes n$. The claim now follows for $SO(2)$ from this and Proposition 5.1 by induction on $n$.

We similarly prove the claim for $N = 2k > 2$ by induction on $n$. For $n = 1$, $S$ is the direct sum of two irreducible modules $S_{\pm}$, on which the endomorphism $f$ acts via $\pm 1$. The claim follows for $\text{End}_U(S^\otimes n)_{\text{odd}}$ from Proposition 5.1 by induction assumption on $n - 1$.

It follows from observation (b) before this theorem that we have a surjective map from $\text{End}_{U(N)}(S^\otimes n)$ onto $\text{End}_{U(N-2)}(S^\otimes n)$, given by restriction from $S^\otimes n$ to $S_1^\otimes n$. Indeed, the simple module of $\text{End}_{U(N)}(S^\otimes n)$ consisting of highest weight vectors of weight $(\frac{n}{2}, \lambda')$ coincides with the simple module of $\text{End}_{U(N-2)}(S_1^\otimes n)$ consisting of highest weight vectors of weight $\lambda'$. Its kernel is $\text{End}_{U(N)}(S^\otimes n)_{\text{odd}}$. By induction assumption, $\text{End}_{U(N-2)}(S_1^\otimes n)_{\text{odd}} \cong \text{End}_{U(N)}(S_{\text{new}}^\otimes n)$ is generated by $\text{End}_{U(N)}(S_1^\otimes n)$ in $U_{\text{new}}$. But the latter is just the restriction of $\text{End}_{U(N)}(S^\otimes 2)$ to $S_1^\otimes 2$. Hence $\text{End}_{U(N)}(S^\otimes 2)$ also generates $\text{End}_{U(N)}(S_{\text{new}}^\otimes n)$. The proof for $N$ odd goes the same way, with the case $N = 3$ already proved in Proposition 2.4.

5.3. Second fundamental theorem. Recall that the algebra $U'_q\mathfrak{so}_n$ was defined by adding an additional generator $F$ to $U_q\mathfrak{so}_n$ with the relations $F^2 = 1$, $FB_1 = -B_1F$ and $FB_i = B_iF$ for $i > 1$. Also recall that the finite dimensional representations of the group $O(n)$ are labeled by all Young diagrams whose first two columns contain at most $n$ boxes, see [29] or also e.g. [27].

Theorem 5.3. (Second Fundamental Theorem) (a) If $N$ is odd and $U = U_q\mathfrak{so}_{2N}$, we have a surjective map $U'_q\mathfrak{so}_n \to \text{End}_U(S^\otimes n)$ defined by $B_i \mapsto C_i$, where $C_i$ is defined as in Section 2.4, using the map $C$ defined in 3.9. Its image is the direct sum of all non-classical representations of $U'_q\mathfrak{so}_n$ with highest weights $\mu$ such that $\mu_1 \leq N/2$ and in which all $B_i$s have eigenvalues contained in $\{(-1)^j(q^{2j-2} - q^{2j-N})/(q^2 - q^{-2}), \ 0 \leq j < N/2\}$.

(b) If $N$ is even and $U = U_q\mathfrak{so}_N \times \mathbb{Z}/2$, we have a surjective map $U'_q\mathfrak{so}_n \to \text{End}_U(S^\otimes n)$ defined by $B_i \mapsto C_i$, using the map $C$ defined in 3.6. Its image is the direct sum of all classical representations of $U'_q\mathfrak{so}_n$ with highest weights $\mu$ such that $\mu_1 \leq N/2$, with $\mu_1 \in \mathbb{Z}$.

(c) If $N$ is even and $U = U_q\mathfrak{so}_N$, we have a surjective map $U'_q\mathfrak{so}_n \to \text{End}_U(S^\otimes n)$ defined by $B_i \mapsto C_i$ and by $F \mapsto f \otimes 1_{n-1}$. Its image is the direct sum of irreducible representations of $U'_q\mathfrak{so}_n$ which specialize to representations of $O(N)$ labeled by Young diagrams $\mu$ whose first two columns contain $\leq n$ boxes and such that $\mu_1 \leq N/2$ for $q = 1$. 
5.4. Combinatorics and representations of \( U'_q\mathfrak{so}_n \) and \( U'_q\mathfrak{so}_n \). Our duality result implies that we can associate to each irreducible representation \( V_\lambda \) of \( \mathfrak{so}_n \) which appears in \( S^\otimes n \) an irreducible representation \( W_{\lambda^c} \) of \( U'_q\mathfrak{so}_n \) such that the multiplicity of \( V_\lambda \) in \( S^\otimes n \) is given by the dimension of \( W_{\lambda^c} \). The fact that such multiplicities are given by dimension formulas (modified for \( N \) odd) has been known for a long time, see e.g. [3]. The Young diagram (or weight) \( \lambda^c \) can be obtained from the Young diagram \( \lambda \) as its complement in a rectangle with side lengths \( N/2 \) and \( n/2 \), reflected at the line \( y = x \), to get it into usual Young diagram position. As our weights also involve half integers, and there are some additional subtleties for \( N \) even, we will spell this out in more detail in this section, even though it is not new (see e.g. [3], [10] or [27]).

Let us briefly describe the irreducible representations of \( Pin(N) \). If \( N \) is even, the irreducible representations of \( O(N) \) are labeled by Young diagrams, whose first two columns contain at most \( N \) boxes. The irreducible representations of \( Pin(N) \) which do not factor over \( O(N) \) are given by \( N/2 \)-tuples \((\lambda_i)\) with \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N/2} > 0 \) with all \( \lambda_i \equiv 1/2 \mod \mathbb{Z} \). We also view them as Young diagrams with an additional half box of width \( 1/2 \) and height \( 1 \) at the end. We can now give a precise description of the diagram \( \lambda^c \) which has already appeared before, e.g. in [3].

**Definition 5.4.** Let \( R \) be a rectangle of height \( N/2 \) and width \( n/2 \).

(a) Let \( N = 2k \) and let \( \lambda \) be a Young diagram labeling an irreducible representation of \( Pin(N) \) appearing in \( S^\otimes n \). Then its \( n \)-complement is the dominant integral weight \( \lambda^c \) of \( \mathfrak{so}_n \) such that \( \lambda^c_j \) is equal to the number of boxes in the \( j \)-th column from the right in \( R \setminus \lambda \). If \( \lambda \) does not fit into \( R \), i.e. if the first column of \( \lambda \) contains more than \( N/2 \) boxes, we define \( \lambda^c_{n/2} = n/2 - \lambda'_1 < 0 \), where \( \lambda'_1 \) is the number of boxes in the first column of \( \lambda \).

(b) If \( N = 2k \) is even and \( \lambda \) a dominant integral weight labeling an irreducible \( \mathfrak{so}_N \) module appearing in \( S^\otimes n \), we associate to it the Young diagram \( \lambda^c \) whose \( j \)-th column contains \( n/2 - \lambda_{N/2+j-1} \) boxes.

(c) For \( N \) odd, and \( \lambda \) an integral dominant weight labeling a non-classical representation of \( U'_q\mathfrak{so}_n \) by \( \lambda^c \) to be equal to the number of boxes in the \( j \)-th column from the right of \( R \setminus \lambda \).
**Proposition 5.5.** (see e.g. [3], [10]) Let \( S, U, U_q^{\mathfrak{so}}_n \) etc be as at the beginning of this section. Then the module \( S^\otimes n \) has a multiplicity one decomposition
\[
S^\otimes n \cong \bigoplus_{\lambda} V_{\lambda} \otimes W_{\lambda^e},
\]
where \( \lambda \) ranges over the equivalence classes of irreducible \( U \) modules \( V_{\lambda} \) which appear in \( S^\otimes n \), and \( \lambda^e \) is the label of the simple \( U_q^{\mathfrak{so}}_n \) resp. \( U_q^{\mathfrak{so}}_n \)-module \( W_{\lambda^e} \) associated to it in Def 5.4.

**Proof.** We give a proof for the case \( N \) odd by induction on \( n \). We define \( \lambda^r = \lambda^{(n)} \) by \( \lambda^r_i = n/2 - \lambda_i \). This describes the number of boxes in the \( i \)-th row of the (unreflected) complement of \( \lambda \) in \( R \). Let \( W_{\lambda^d} \) be the \( U_q^{\mathfrak{so}}_n \) module associated to \( \lambda \) (e.g. we could take the module of highest weight vectors in \( S^\otimes n \) with weight \( \lambda \)). It follows from the tensor product rules and induction assumption that
\[
W_{\lambda^d} \cong \bigoplus W_{\mu^{(n-1)}}
\]
as \( U_q^{\mathfrak{so}}_{n-1} \)-modules, where the summation goes over all highest weights \( \mu \) with \( V_{\mu} \subset S^\otimes n-1 \) such that \( V_{\lambda} \subset V_{\mu} \otimes S \). This implies that \( \lambda = \mu + \omega \) for some weight \( \omega \) of \( S \). As \( \omega = \epsilon - \sum_j \epsilon_{ij} \) for some subset \( \{ij\} \subset \{1, 2, ..., k\} \), it follows that
\[
\mu^{(n-1)} = (n-1)\epsilon - \mu = n\epsilon - \lambda - \sum_j \epsilon_{ij} = \lambda^{r(n)} - \sum_j \epsilon_{ij},
\]
i.e. the rows of \( \mu^{(n-1)} \) and \( \lambda^{r(n)} \) differ by at most 1. Recall that \( \lambda^{c(n)} \) is obtained from \( \lambda^{r(n)} \) by reflecting the latter at the line \( y = x \), i.e. its \( j \)-th column contains \( \lambda^{r(n)}_{(N+1)/2-j} \) boxes. It is probably easiest seen geometrically that the condition \( 0 \leq \lambda^r_i - \mu^r_i \leq 1 \), \( 1 \leq i \leq N \), is equivalent to the branching conditions 2.1 and 2.2 for \( \mu^c \) and \( \lambda^c \). Hence the simple \( U_q^{\mathfrak{so}}_n \) module \( W_{\lambda^d} \) coincides with the simple \( U_q^{\mathfrak{so}}_n \) module \( W_{\lambda^c} \) as a \( U_q^{\mathfrak{so}}_{n-1} \)-module. It is easy to see from 2.1 and 2.2 that non-isomorphic simple \( U_q^{\mathfrak{so}}_n \)-modules are still non-isomorphic if viewed as \( U_q^{\mathfrak{so}}_{n-1} \) modules. Hence \( W_{\lambda^d} \cong W_{\lambda^c} \).

The other cases can be shown similarly. Another method would be to calculate the character of \( O(N) \times SO(n) \subset O(Nn) \) on the spin module of \( O(Nn) \). This was essentially done in [10], with some minor errors. Hopefully the statement of the pairings is correct in this paper.

**Remark 5.6.** It was already observed in [3] that for \( N \) odd the multiplicity of \( V_{\lambda} \) in \( S^\otimes 2k+1 \) coincides with the dimension of the irreducible \( Sp(2k) \) module \( W_{\lambda^{r,-\varepsilon}} \), where \( \varepsilon = (\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) \in \mathbb{R}^k \). No such classical interpretation seems to be available for multiplicities in even tensor powers of \( S \).

**Corollary 5.7.** Let \( q \) not be a root of unity. Then every finite-dimensional irreducible representation of \( U_q^{\mathfrak{so}}_n \) appears in some tensor power \( S^\otimes n \) for some \( U = U_q^{\mathfrak{so}}_N \) or \( U = U_q^{\mathfrak{so}}_N \times \mathbb{Z}/2 \). Moreover, every irreducible finite dimensional representation of \( O(n) \) extends to an irreducible representation of \( U_q^{\mathfrak{so}}_n \).
Proof This was shown for classical representations with integer highest weights in [27] and also in this paper. It can be shown the same way for classical representations with half integer highest weights for \( N \) odd, using the bigger module \( \tilde{S} \) and [27] Lemma 4.2 and Proposition 4.3. Existence of non-classical representations and of representations of \( U_q^\prime \mathfrak{so}_n \) as stated follows from Proposition 5.5.

5.5. Results for more general rings. It is well-known that the Drinfeld-Jimbo quantum groups can be defined over the ring \( \mathbb{Z}[q,q^{-1}] \), see [18]. It is not hard to check that the same is true for the subalgebra \( U_q^\prime \mathfrak{so}_n \subset U_q^\prime \mathfrak{sl}_n \). Also observe that the representations of \( U_q^\prime \mathfrak{so}_n \) in Theorem 5.3 can be defined over the ring \( \mathbb{Z}[q,q^{-1}] \) in type \( D \) (for \( N \) even) and over the ring \( \mathbb{Z}[q,q^{-1},[2]^{-1}] \) for \( N \) odd. Indeed, one checks easily that the matrix coefficients of the maps \( C \) in 3.6 and 3.9 with respect to the basis \( x(m) \) are in the rings as claimed. We can deduce the following result from this and Theorem 5.3:

**Proposition 5.8.** The representations of the algebras in Theorem 5.3 on \( S^\otimes n \) are well-defined also over the ring \( \mathbb{Z}[q,q^{-1}] \) for \( N \) even. For \( N \) odd, they are well-defined over the ring \( \mathbb{Z}[q,q^{-1},[2]^{-1}] \). The statements of Theorem 5.3 also hold in these cases, except possibly the surjectivity statements.

5.6. Results for fusion categories. For \( q \neq \pm 1 \) a root of unity the representation category of a quantum group \( U_q\mathfrak{g} \) is not semisimple. But one can define an important semisimple quotient tensor category \( \mathcal{F} \) of the subcategory of tilting modules of \( \text{Rep}(U_q\mathfrak{so}_N) \), see [1]. It only has finitely many simple objects up to isomorphism. We only give some very basic facts about it here, see e.g. [2] for more details.

We assume \( q \) to be a primitive \( 2\ell \)-th (for \( N \) even) respectively a primitive \( 4\ell \)-th root of unity (for \( N \) odd), with \( \ell \geq N \) in both cases. The simple objects of \( \mathcal{F} \) are labeled by the \( \ell \)-admissible integral dominant weights \( \lambda \) of \( \mathfrak{so}_N \), which are defined by

\[
\lambda_1 + \lambda_2 + N - 2 \leq \ell.
\]

Theses categories also appear as level \( \ell + 2 - N \) representations of loop groups connected with \( \text{SO}(N) \). The image of the spinor representation \( S \) is again simple and nonzero in the quotient \( \mathcal{F} \) and will be denoted by the same letter. Tensoring a simple object \( V_\lambda \) by \( S \) is given by the same rule 5.1, where we remove all representations which are not labeled by an \( \ell \)-admissible weight. With a little care, all the arguments we used above will also go through for fusion categories related to \( U_q\mathfrak{so}_N \).

**Theorem 5.9.** The statements of Theorem 5.2 and 5.3 also hold in the context of fusion categories \( \mathcal{F} \) as defined in this section. In particular, \( \text{End}_\mathcal{F}(S^\otimes n) \) is given by a representation of \( U_q^\prime \mathfrak{so}_n \) for \( N \) even, and by a representation of \( U_{-q}^\prime \mathfrak{so}_n \) for \( N \) odd.

Proof. It is known that \( S \) is a tilting module, and that \( S^\otimes n \) can be written as a direct sum of indecomposable tilting modules \( T_\lambda \) with highest weight \( \lambda \). The quotient of \( S^\otimes n \) which lies in \( \mathcal{F} \) is isomorphic to the summand consisting of those \( T_\lambda \) for which \( \lambda \) is \( \ell \)-admissible. In this case \( T_\lambda \cong V_\lambda \) is simple. These basic facts allow us to proceed by induction on \( n \) and on \( N \).
as in the proofs for Theorem 5.2 and 5.3. As there, one shows that End(S^\otimes 2) is generated by C (for N odd) and by C and F (for N even), using the fact that the eigenvalues of C for the different subrepresentations of S^\otimes 2 are distinct as in the generic case (which would not necessarily be true if q had a smaller degree). Our conditions also ensure that the q-dimensions of all simple objects in F are nonzero. This is all needed to show that also the statement about S_{\text{old}}^\otimes n in the proof of Theorem 5.2 holds in this context. We can now use the same argument as in the proof of Theorem 5.2 to show that End_F(S_{\text{new}}^\otimes n) will be isomorphic to a quotient of End_F(U(N-2))\otimes (S(N-2)_{\text{old}}^\otimes n), where F(U(N-2)) refers to the fusion category for U_q so_{N-2} at the same root of unity.

6. Related results and applications

6.1. Connection with results in [27] and Hasegawa duality. This paper is closely related to the paper [27] which in turn was inspired by the paper [10] by Hasegawa. The key observation there was (as far as this paper is concerned) the fact that the commuting actions of O(N) and SO(n) on \( \mathbb{C}^N \otimes \mathbb{C}^n \) extend to commuting actions of the corresponding spin groups on the Clifford algebra Cl(Nn) \( \cong Cl(N)^{\otimes n} \) (isomorphism of vector spaces). This suggested commuting actions of these groups and the corresponding quantum groups on the n-th tensor power of the spin representation of O(N). It was indeed shown in [27] for N even that there exist actions of \( U = U_q so_N \rtimes \mathbb{Z}/2 \) and and \( U'_q so_n \) on \( S^{\otimes n} \) which are each others commutants. As the Clifford algebra is not simple for N odd, no such simple statement could be shown in that case. The best one could do was to prove a duality statement between the action of \( U_q so_N \) and the sub-algebra of \( U'_q so_n \) generated by the elements \( B_i^2, 1 \leq i < n \) (our \( q^2 \) here corresponds to q in the parametrization in [27] for N odd). This subalgebra does not seem to allow a convenient algebraic description on its own.

In the current paper we do not apply Hasegawa’s results at all. Using the operator C, we directly show that the centralizers can be described in terms of the coideal algebra \( U'_q so_n \). This does not change much our previous description in terms of \( U'_q so_n \) for N even, see Lemma 2.5. But for N odd, it gives the desired duality result between actions of \( U_q so_N \) and \( U'_q so_n \), which now acts via its non-classical representations on \( S^{\otimes n} \). Because of the latter, the result seems to be new even in the classical case \( q = 1 \). In particular, there does not seem to be any indication of the algebra \( U'_q so_n \) in the context of Hasegawa duality or Howe duality (see next section).

6.2. Connections to q-Howe duality. Similarly as for Clifford algebras in Hasegawa duality, one obtains commuting actions of groups, say Gl(N) and Gl(n) on \( (\bigwedge \mathbb{C}^N \otimes \mathbb{C}^n) \cong (\bigwedge \mathbb{C}^N)^{\otimes n} \). For Lie type A, it has been shown by Cautis, Kamnitzer and Morrison [6] that there similarly exist commuting actions of the quantum groups \( U_q sl_N \) and \( U_q sl_n \) on \( (\bigwedge \mathbb{C}^N)^{\otimes n} \). The situation is more complicated for other Lie types, such as for the commuting actions of so_N and so_n as well as of sp_N and sp_n on the exterior algebra \( (\bigwedge \mathbb{C}^N \otimes \mathbb{C}^n) \cong (\bigwedge \mathbb{C}^N)^{\otimes n} \). It was shown by Sartori and Tubbenhauer [21] that one obtains commuting actions of \( U_q so_N \) and \( U'_q so_n \) and of \( U_q sp_N \) and \( U'_q sp_n \), where \( U'_q so_n \) is as in this paper, and \( U'_q sp_n \subset U_q sl_n \).
similarly is a coideal subalgebra. In the orthogonal case, at least for $N$ even, this result would also follow from the results in [27], as $S^2 \cong \bigwedge C^N$ as $Pin(N)$-modules. The methods in [21] are completely different from the ones in [27] and in this paper.

6.3. Connections to the $q$-Clifford algebra in [7]. A $q$-Clifford algebra has also been defined by Ding and Frenkel in [7]. Setting $\hat{\psi}_{k+1} - i = \Omega_{k+1} \psi_k$ and $\hat{\psi}_{k+1}^\dagger = \Omega_{k+1} \psi_k^\dagger$, one can show (see [7], Proposition 5.3.1) that the elements $\hat{\psi}_i$ and $\hat{\psi}_i^\dagger$ satisfy the relations of their $q$-Clifford algebra. Similarly, if one defines $\hat{\psi}_{i,-}$ and $\hat{\psi}_{i,-}^\dagger$ as before with $\Omega_{i-1}$ replaced by $\Omega_{i-1}^{-1}$, we obtain the relations of their Clifford algebra with $q$ replaced by $q^{-1}$. In particular the element $C$ would then have the somewhat more appealing form

$$C = \sum_{i=1}^k \hat{\psi}_i \otimes \hat{\psi}_i^\dagger + \hat{\psi}_i^\dagger \otimes \hat{\psi}_i$$

in the even-dimensional case. The elements $\hat{\psi}_i$, $\hat{\psi}_i^\dagger$ etc already appeared in Section 4 as $c_{i,\pm}$ and $d_{i,\pm}$. In particular, one can see the parameter $q$ appear explicitly in the relations there, see e.g. 4.3.

6.4. Existence of representations of $U_q\mathfrak{so}_n$ and $U_q\mathfrak{o}_n$. The irreducible representations of $U_q\mathfrak{so}_n$ for $q$ not a root of unity were constructed by Klimyk and his coauthors, and in special cases also by different groups, see [12] and the references there. Another approach was given by the author in [27]. In all of these cases, the constructions of the representations were somewhat involved. Our duality result provides yet another construction of all irreducible representations of $U_q\mathfrak{so}_n$ and of many irreducible representations of $U_q\mathfrak{o}_n$ if $q$ is not a root of unity, see corollary 5.7. We also obtain some new nontrivial representations at roots of unity in the fusion category setting. In our set-up, it suffices to find the linear map $C$. After that the representations are obtained in a comparatively painless way using our duality result.

6.5. New vertex models. Results in this paper were used by D. Gepner in the recent preprint [9] to construct new vertex models in connection with spin representations and to study their algebraic properties.

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