A posteriori error estimates for linear problems in Cosserat elasticity

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Abstract. Paper is addressed to reliability and efficiency of functional a posteriori error estimates for Cosserat elasticity in the process of adaptive mesh refinements. A new variant of such error estimate is proposed. Numerical investigations continue previous work of authors. It is shown that functional approach can be not only efficiently implemented for refinements (with MATLAB), but it provides upper bounds for the energy norm of errors without significant overestimations.

1. Introduction
The theory of a posteriori error control for PDE’s is one of important directions of modern computational mathematics with applications to mechanical engineering. Error estimators or indicators provide some quantitative measure of discretization errors (and, for some cases, round-off errors), which appear in numerical simulations, and indicate parts of a computational domain with large errors for further mesh refinements. For classical elasticity theory, there is a significant amount of literature including decades of research, achievements, numerical testing and comparison of different approaches to a posteriori error control (see, e.g. [1–4] and recent paper [5] for reviews).

The functional approach to a posteriori error control for problems of continuum mechanics has been developed starting from [6] by S. Repin and colleagues. For further investigations for classical linear and nonlinear problems, one can see [4, 7–9] and some papers cited therein. Cosserat continuum is one of well-known generalizations of the classical theory (see, for example, [10]). Using rotational degrees of freedom independently of translational, it takes into account a material microstructure. Such models are under intensive developments from 60-s of XX century, but still there is an explicit lack of publications, which are focused on a posteriori error control for computed approximations – see [5,11–14], and this work requires further efforts.

2. Problem statement
The energy functional for Cosserat elasticity in 2D depends on the triple of variables: two in-plane displacements $u_x, u_y$ (as for the classical elasticity) and the rotation $\omega$ about out-of-plane axis. It has the form
\[ f(u_x, u_y, \omega) = \int_{\Omega} \left( \mu (u_{xx}^2 + u_{yy}^2) + \frac{\mu_c}{2} (u_{xy} + u_{yx})^2 + \frac{\lambda}{2} (u_{xx} + u_{yy})^2 \right) \, d\Omega \]
\[ + \int_{\Omega} \left( \frac{\mu_c}{2} (u_{yx} - u_{xy} - 2\omega)^2 + 2B (\omega_x^2 + \omega_y^2) \right) \, d\Omega - \int_{\Omega} \left( f_x u_x + f_y u_y \right) \, d\Omega \]
\[ - \int_{\Gamma_S} (t_x u_x + t_y u_y) \, d\Gamma, \]
where \( \Omega \) is bounded connected domain with a Lipschitz continuous boundary \( \Gamma \), \( \mu \) and \( \lambda \) are standard elastic constants, \( \mu_c \) and \( B \) are constants, related to a microstructure of a material. Vectors \((f_x, f_y)\) and \((t_x, t_y)\) represent volumetric and surface loading, respectively. Boundary \( \Gamma \) is assumed to be divided into two parts \( \Gamma_D \) and \( \Gamma_S \), with known displacements and boundary tractions, respectively. All elements of solution are considered as elements of respective Sobolev space \( H_0^1 \) with zero trace on \( \Gamma_D \) (for all details, see [13]).

Using standard arguments of the Calculus of Variations one can consider deviations from the exact solution \( u_x + \alpha v_x, u_y + \alpha v_y, \omega + \alpha \theta \) with an arbitrary real number \( \alpha \) and obtain integral identities for trial functions \( v_x, v_y \) and \( \theta \)
\[ \int_{\Omega} (2\mu v_{xx} v_x + \mu (u_{xx} + u_{yy}) v_x + \lambda (u_{xx} + u_{yy}) v_{xx}) \, d\Omega - \int_{\Omega} \mu_c (u_{yx} - u_{xy} - 2\omega) v_{xy} \, d\Omega \]
\[ = \int_{\Omega} f_x v_x \, d\Omega + \int_{\Gamma_S} t_x v_x \, d\Gamma; \]
\[ \int_{\Omega} (2\mu v_{yy} v_y + \mu (u_{xx} + u_{yy}) v_y + \lambda (u_{xx} + u_{yy}) v_{yy}) \, d\Omega + \int_{\Omega} \mu_c (u_{yx} - u_{xy} - 2\omega) v_{yx} \, d\Omega \]
\[ = \int_{\Omega} f_y v_y \, d\Omega + \int_{\Gamma_S} t_y v_y \, d\Gamma; \]
\[ \int_{\Omega} (4B \nabla \omega \cdot \nabla \theta - 2\mu_c (u_{yx} - u_{xy} - 2\omega) \theta) \, d\Omega = 0. \]

We introduce vector-tensor notations
\[ \mathbf{U} := (u_x, u_y), \mathbf{V} := (v_x, v_y), P(\mathbf{V}, \theta) := v_{yx} - v_{xy} - 2\theta, \mathbf{F} := (f_x, f_y), \mathbf{T} := (t_x, t_y); \]
\[ \varepsilon(\mathbf{V}) := \begin{bmatrix} v_{xx} & \frac{1}{2} (v_{xy} + v_{yx}) \\ \frac{1}{2} (v_{xy} + v_{yx}) & v_{yy} \end{bmatrix}, \sigma^{sym}(\mathbf{V}) = \text{Le}(\mathbf{V}) := 2\mu \varepsilon(\mathbf{V}) + \lambda \text{tr} \varepsilon(\mathbf{V}) \mathbf{I}; \]
where := means equality by definition; \( \varepsilon \) and \( \sigma^{sym} \) are standard strain and stress tensors, respectively; \( \text{tr}, \mathbf{I}, \) and \( \text{L} \) – are the trace operator, the unit tensor, and the tensor of elastic moduli, respectively. In vector form, we arrive at the system (1), which is the starting point for derivation of functional-type a posteriori error estimates
\[ \int_{\Omega} \left( \text{Le}(\mathbf{U}) \varepsilon(\mathbf{V}) + 4B \nabla \omega \cdot \nabla \theta + \mu_c P(\mathbf{U}, \omega) P(\mathbf{V}, \theta) \right) \, d\Omega = \int_{\Omega} \mathbf{F} \cdot \mathbf{V} \, d\Omega + \int_{\Gamma_S} \mathbf{T} \cdot \mathbf{V} \, d\Gamma. \] (1)

Also, such a representation provides some prerequisites for similar analysis in 3D.

3. Error estimates
Let us consider an arbitrary pair of elements \((\mathbf{U}, \mathbf{W})\) from the same functional spaces as the exact solution, but without specification of the approach used to calculate this approximate solution. For estimate construction we also introduce one additional, generally, non-symmetric tensor \( \tau \) as a free element that can be provided with mixed-FEM approximations suitable for space \( H(\text{Div}) \) – the Hilbert space of square summable vector-functions with square summable divergence. Then, we can use the following integration by parts formula, where \( \tau^{sym} \) and \( \tau^{skew} \) are symmetric and skew-symmetric parts of the tensor \( \tau \):

\[ \int_{\Omega} \left( \text{Le}(\mathbf{U}) \varepsilon(\mathbf{V}) + 4B \nabla \omega \cdot \nabla \theta + \mu_c P(\mathbf{U}, \omega) P(\mathbf{V}, \theta) \right) \, d\Omega = \int_{\Omega} \mathbf{F} \cdot \mathbf{V} \, d\Omega + \int_{\Gamma_S} \mathbf{T} \cdot \mathbf{V} \, d\Gamma. \] (1)
Then, we modify system (1) with deviations from exact values \( \mathbf{U} - \tilde{\mathbf{U}} \) and \( \omega - \tilde{\omega} \) and easily arrive at

\[
\int_{\Omega} \left( \frac{1}{2} \mathbf{L}(\mathbf{U} - \tilde{\mathbf{U}}) : \mathbf{ε}(\mathbf{V}) + 4\mathbf{B} \nabla \mathbf{V} \cdot \mathbf{ω} \right) \cdot \nabla \mathbf{θ} + \mu_c P(\mathbf{U} - \tilde{\mathbf{U}}, \omega - \tilde{\omega}) P(\mathbf{V}, \mathbf{θ}) \right) \, d\Omega
\]

\[
= \int_{\Omega} \mathbf{F} \cdot \mathbf{V} \, d\Omega + \int_{\Gamma_S} \mathbf{T} \cdot \mathbf{V} \, d\Gamma
\]

\[
- \int_{\Omega} \left( \mathbf{L} \left( \mathbf{U} \right) : \mathbf{ε}(\mathbf{V}) + 4\mathbf{B} \nabla \mathbf{V} \cdot \mathbf{ω} \right) \cdot \nabla \mathbf{θ} + \mu_c P(\mathbf{U}, \omega - \tilde{\omega}) P(\mathbf{V}, \mathbf{θ}) \right) \, d\Omega.
\]

Setting \( \mathbf{V} = \mathbf{U} - \tilde{\mathbf{U}} \) and \( \mathbf{θ} = \omega - \tilde{\omega} \) yields the following error representation:

\[
\| (\mathbf{U} - \tilde{\mathbf{U}}, \omega - \tilde{\omega}) \|^2 = \int_{\Omega} \left( \frac{1}{2} \mathbf{L}(\mathbf{U} - \tilde{\mathbf{U}}) : \mathbf{ε}(\mathbf{U} - \tilde{\mathbf{U}}) + 2\mathbf{B} \nabla (\omega - \tilde{\omega}) \right)^2 + \frac{\mu_c}{2} P^2(\mathbf{U} - \tilde{\mathbf{U}}, \omega - \tilde{\omega}) \right) \, d\Omega
\]

\[
= \int_{\Omega} \frac{1}{2} (\mathbf{F} - \mathbf{Div} : \mathbf{U} - \tilde{\mathbf{U}}) \, d\Omega + \int_{\Omega} \frac{1}{2} \mathbf{t}^{\text{skew}} : \nabla (\mathbf{U} - \tilde{\mathbf{U}}) \, d\Omega
\]

\[
+ \int_{\Gamma_S} \frac{1}{2} (\mathbf{T} - \mathbf{τ}) \cdot (\mathbf{U} - \tilde{\mathbf{U}}) \, d\Gamma + \int_{\Omega} \frac{1}{2} \left( \mathbf{t}^{\text{sym}} - \mathbf{L} \mathbf{ε}(\mathbf{U}) \right) : \mathbf{ε}(\mathbf{U} - \tilde{\mathbf{U}}) \, d\Omega
\]

\[
- \int_{\Omega} 2\mathbf{B} \nabla \tilde{\omega} \cdot \nabla (\omega - \tilde{\omega}) \, d\Omega - \int_{\Omega} \frac{\mu_c}{2} P(\tilde{\mathbf{U}}, \tilde{\omega}) P(\mathbf{U} - \tilde{\mathbf{U}}, \omega - \tilde{\omega}) \, d\Omega
\]

Using Young-Fenchel, Cauchy-Schwarz and Hölder’s inequalities, one can obtain a new error estimate. It has the following structure:

\[
\| (\mathbf{U} - \tilde{\mathbf{U}}, \omega - \tilde{\omega}) \| \leq M = D + R,
\]

where \( M \) is the functional error majorant, \( D \) is its main term, namely

\[
D^2 = \int_{\Omega} \left( \frac{1}{2} \nabla^{-1} \left( \mathbf{t}^{\text{sym}} - \mathbf{L} \mathbf{ε}(\tilde{\mathbf{U}}) \right) : \left( \mathbf{t}^{\text{sym}} - \mathbf{L} \mathbf{ε}(\tilde{\mathbf{U}}) \right) + \frac{\mu_c}{2} P^2(\tilde{\mathbf{U}}, \tilde{\omega}) + 2\mathbf{B} \nabla \tilde{\omega} \right) \, d\Omega,
\]

and \( R \) is a residual term as in [13] with one additional summand with \( \mathbf{t}^{\text{skew}} \). The first summand of \( D \) represents errors in constitutive relations. Term \( R \) includes mesh-independent constants and represents some proper balance of the equilibrium equations. This estimate differs from results of [12,13] – the majorant is coincident with the "non-symmetric" majorant for the classical elasticity from [8] for the limit case of zero microstructure constants. Additionally to the global error estimate, part \( D \) of the functional \( M \) can be used as an indicator of the local error distribution according to contributions on each finite element. Therefore, it can provide a basis for construction of adaptive algorithms. Generally, adaptive algorithms for FEM are based on four steps: solve, estimate, mark and refine (see, for example, [15,16]). In our case, the estimation can be done with \( M \) and marking with \( D \).

4. Numerical tests with adaptive mesh refinements

In this section we provide two examples: the first one is taken from [5] and is supplemented with new results with mesh adaptation and the global error estimation; the second example is a test problem with Dirichlet boundary conditions and \( x-y \) symmetry. Adaptive algorithms are implemented in MATLAB using FEM approximation from [17] for computations of the error majorant and indicator.

4.1. Example 1 (square domain with a hole)

For the first example, we consider the square domain with side 16.2 mm with a circular hole with radius 0.216 mm in the center. The left edge is fully clamped and the tensile loading of 1 MPa is applied to the right edge.

The solution of Cosserat problem (the total displacement in figure 1a) is compared to the classical linear elastic solution. The difference between solutions is moderate (figure 1b). For Cosserat linear elasticity adaptive meshes are constructed using majorant-based error indicator. Corresponding results
for uniformly refined nested meshes and for adaptive ones are collected in table 1. The resulting adaptive meshes are presented on figure 2.

Table 1 has the following structure: the first block of results corresponds to the uniform mesh refinement with no adaptation. The initial mesh (first column) is provided by a standard MATLAB tool and remains the same for all refinement algorithms. Nodes, elements and relative errors are collected in corresponding table rows. Relative errors are computed with the so-called reference solution – an approximate solution obtained on a fine mesh. It is very time-consuming to calculate the reference solution; therefore, it is provided only for numerical experiments on validation and comparison of different approaches.

In the second block of table 1, results for adaptation with the reference error indicator are presented. The reference indicator is the local contributions from each element to the energy norm of the difference between solutions on coarse and fine meshes.

In the third block of the table, results for majorant-based adaptation process are collected. The functional-type error majorant is used for upper error estimation. The ratio between the error majorant $M$ and the error norm is used as a standard quality measure for error control. This parameter is usually called the efficiency index – it is denoted by $I_{\text{eff}}$.

Some of these results were discussed earlier in [5]. With some algorithm modifications and an appropriate choice of constants in the majorant, the guaranteed upper bounds for the energy norm are calculated and their efficiency indexes are collected in table 1. Figure 5a shows a decrease in the error norm, the majorant $M$ and its first summand $D$ with uniform mesh refinement.

**Figure 1.** Example 1. Solution for the Cosserat elasticity – displacement vector sum (a) and deviation from the classical one (b).

For adaptive mesh refinements the bulk criterion is chosen: all elements are sorted by the indicator value and then elements with maximum values for which the sum of values is 30% of the total sum are refined. Using this criterion we refine a significant number of elements on each step to avoid hundreds of mesh adaptation steps. The influence of the choice of a refinement criterion is discussed in [18] for classical linear elasticity problems.

Due to using bulk refinement criterion, for some adaptation steps the majorant-based indicator seems to work better than the reference one. To obtain the “perfect” reference mesh one should refine one element at each step, which is not effective from a practical point of view.

The results show that for considered parameters, geometry and loading the majorant-based error indicator leads to final adaptive meshes, which are similar to reference ones. For uniformly refined meshes even with more than 70000 nodes the relative error is more than 2%, as for adaptive meshes it is less than 2% on meshes with about 6500 nodes. This result illustrates the advantage of using adaptive refinement strategies based on reliable error estimation.
Table 1. Example 1. Sequences of refined meshes and error estimates.

| Uniform refinement | 1 | 2 | 3 | 4 | 5 |
|---------------------|---|---|---|---|---|
| Mesh No.            |   |   |   |   |   |
| Nodes               | 295| 1147| 4522| 17956| 71560|
| Elements            | 557| 2228| 8912| 35648| 142592|
| Relative error, %   | 11.96| 9.16| 6.59| 4.42| 2.69|

Adaptive mesh refinement with reference indicator

| Nodes               | 295| 357| 679| 1094| 6455| 11609|
| Elements            | 557| 670| 1274| 2058| 12507| 22662|
| Relative error, %   | 11.96| 9.70| 5.76| 4.23| 1.79| 1.36|

Adaptive mesh refinement with majorant-based indicator

| Nodes               | 295| 328| 659| 975| 6327| 10075|
| Elements            | 557| 613| 1224| 1822| 12240| 19602|
| Relative error, %   | 11.96| 9.49| 5.42| 4.46| 1.85| 1.49|
| $I_{eff}$           | 1.6| 1.4| 1.3| 1.2| 1.2| 1.2|
| $I_{eff}(D)$        | 1.6| 1.4| 1.3| 1.2| 1.2| 1.2|

4.2. Example 2 (square domain with fixed boundaries)

For example 2, we consider unit square with all boundary edges clamped, the volume force $f_x = f_y = 590$ Pa is applied. As for example 1, the solution for the Cosserat elasticity is presented on figure 3a. The difference between solutions of classical and Cosserat elasticity problems (figure 3b) is less than in example 1.
Figure 3. Example 2. Solution for the Cosserat elasticity – displacement vector sum (a) and deviation from the classical one (b).

The final adaptive meshes are presented on figure 4. Results for uniformly refined and adaptive meshes are collected in table 2. For this example the overestimation for $M$ is larger, due to the proximity of Cosserat and classical solutions. Similar effect was observed in [14]. Figure 5b illustrates the behavior of the error norm, the majorant $M$ and its first summand $D$ with uniform mesh refinements. The first summand of the majorant $D$ still can be used to indicate the error which can be seen from figure 4.

Table 2. Example 2. Sequences of refined meshes and error estimates.

| Uniform refinement | Mesh No. | 1  | 2  | 3  | 4  | 5  | 6  |
|--------------------|----------|----|----|----|----|----|----|
| Nodes              |          | 41 | 145| 545| 2113| 8321| 33025|
| Elements           |          | 64 | 256| 1024| 4096| 16384| 65536|
| Relative error, %  |          | 21.51| 12.94| 7.42| 4.21| 2.44| 1.45|

| Adaptive mesh refinement with reference indicator | Mesh No. | 1  | 2  | 3  | 4  | 5  | 6  |
|--------------------------------------------------|----------|----|----|----|----|----|----|
| Nodes                                            |          | 41 | 234| 773| 2541| 7782| 13300|
| Elements                                         |          | 64 | 421| 1447| 4833| 14910| 25635|
| Relative error, %                                |          | 21.51| 10.89| 6.41| 3.76| 2.23| 1.69|

| Adaptive mesh refinement with majorant-based indicator | Mesh No. | 1  | 2  | 3  | 4  | 5  | 6  |
|--------------------------------------------------------|----------|----|----|----|----|----|----|
| Nodes                                                 |          | 41 | 247| 861| 2030| 7866| 11976|
| Elements                                              |          | 64 | 442| 1608| 3882| 15212| 23116|
| Relative error, %                                     |          | 21.51| 10.86| 6.23| 4.23| 2.19| 1.73|
| $I_{eff}$                                             |          | 4.0| 3.7| 3.5| 3.3| 3.4| 3.5|
| $I_{eff}(D)$                                          |          | 2.1| 2.0| 1.9| 1.8| 1.9| 1.9|
Figure 4. Example 2. Adaptive mesh with reference indicator: 13300 nodes, rel. error 1.69% (a); with majorant-based indicator: 11976 nodes, rel. error 1.73% (b).

Figure 5. Error norm, majorant $M$ and its first summand $D$ behavior with uniform mesh refinement: example 1 (a), example 2 (b).

5. Conclusions
The functional approach is efficiently implemented for mesh adaptations in 2D problems of the Cosserat elasticity. New modification of the functional-type a posteriori error majorant for this problem is proposed. This estimate is consistent with the estimate for the classical linear elasticity from [8]. This approach is always reliable and reasonably efficient – it does not show high (or remarkably growing) overestimation of the true error in the process of adaptive mesh refinement. Efficiency indexes are stable and the method provides additional useful information on the behavior of the error.

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