COMPRESSIBLE EULER EQUATIONS INTERACTING WITH INCOMPRESSIBLE FLOW

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ABSTRACT. We investigate the global existence and large-time behavior of classical solutions to the compressible Euler equations coupled to the incompressible Navier-Stokes equations. The coupled hydrodynamic equations are rigorously derived in [1] as the hydrodynamic limit of the Vlasov/incompressible Navier-Stokes system with strong noise and local alignment. We prove the existence and uniqueness of global classical solutions of the coupled system under suitable assumptions. As a direct consequence of our result, we can conclude that the estimates of hydrodynamic limit studied in [1] hold for all time. For the large-time behavior of the classical solutions, we show that two fluid velocities will be aligned with each other exponentially fast as time evolves.

1. Introduction. In this paper, we are concerned with the global existence and large-time behavior of classical solutions to the compressible isothermal Euler equations coupled to the incompressible Navier-Stokes equations in the periodic domain $T^3$. More precisely, let $\rho(x, t)$ and $u(x, t)$ be the density and the velocity of compressible fluid, respectively, and $v(x, t)$ be the velocity of incompressible fluid at a domain $(x, t) \in T^3 \times \mathbb{R}_+$. In this situation, our coupled hydrodynamic equations can be described as follows:

$$
\begin{align*}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \quad x \in T^3, \quad t > 0, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x \rho &= -\rho(u - v), \quad x \in T^3, \quad t > 0, \\
\partial_t v + v \cdot \nabla_x v + \nabla_x p - \mu \Delta_x v &= \rho(u - v), \quad x \in T^3, \quad t > 0, \\
\nabla_x \cdot v &= 0, \quad x \in T^3, \quad t > 0,
\end{align*}
$$

(1.1)

with initial data

$$(\rho, u, v)|_{t=0} =: (\rho_0, u_0, v_0), \quad \nabla_x \cdot v_0 = 0. \quad (1.2)$$

Since the total mass is conserved in time, without loss of generality, we may assume that $\rho$ is a probability density function, i.e., $\|\rho(\cdot, t)\|_{L^1(T^3)} = 1$. We also assume that the viscosity coefficient $\mu = 1$ for simplicity. In fact, a more general condition on the viscosity coefficient $\mu > 0$ does not yield any difficulties for our analysis.

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Recently, in [1], the author and his collaborators studied coupled kinetic-fluid equations with local alignment forces and rigorously derived the coupled hydrodynamic equations (1.1) when the noise and local alignment are strong enough. More specifically, let $f = f(x, \xi, t)$ be the one-particle distribution function at a spatial periodic domain $(x, \xi) \in T^3 \times \mathbb{R}^3$ at time $t$ and $v = v(x, t)$ be the velocity of fluid, then the following kinetic-fluid equations are considered in [1]:

\[
\begin{align*}
\partial_t f^\varepsilon + \xi \cdot \nabla_x f^\varepsilon + \nabla_\xi \cdot ((v^\varepsilon - \xi) f^\varepsilon) &= \frac{1}{\varepsilon} \nabla_\xi \cdot (\nabla_\xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon), \\
\partial_t v^\varepsilon + v^\varepsilon \cdot \nabla_x v^\varepsilon + \nabla_x p^\varepsilon - \mu \Delta_x v^\varepsilon &= -\int_{\mathbb{R}^3} (v^\varepsilon - \xi) f^\varepsilon d\xi, \\
\nabla_x \cdot v^\varepsilon &= 0,
\end{align*}
\] (1.3)

where

\[
\rho^\varepsilon := \int_{\mathbb{R}^3} f^\varepsilon(x, \xi) d\xi \quad \text{and} \quad \rho^\varepsilon u^\varepsilon := \int_{\mathbb{R}^3} \xi f^\varepsilon(x, \xi) d\xi. \quad (1.4)
\]

In [1], the authors showed that as long as there exists a unique classical solution to the system (1.1) the weak solutions $(f^\varepsilon, v^\varepsilon)$ to the system (1.3) satisfying a natural entropy inequality converge to $(\rho_{\rho, u}, v)$ where $\rho_{\rho, u}$ denotes the Maxwellian distribution with the density $\rho$ and the velocity $u$ as $\varepsilon$ goes to zero, i.e.,

\[
f^\varepsilon(x, \xi, t) \to \frac{\rho(x, t)}{(2\pi)^{3/2}} e^{-\frac{|x - u(x, t)|^2}{2}} \quad \text{and} \quad v^\varepsilon(x, t) \to v(x, t) \quad \text{as} \quad \varepsilon \to 0,
\]

where $\rho$ and $u$ are the $\varepsilon \to 0$ limits of (1.4). Moreover, $(\rho, u, v)$ solve the system (1.1). Without the interactions with the fluid, the rigorous hydrodynamic limit of the kinetic equations via relative entropy arguments and the global existence of classical solutions to the limiting system are investigated in [2, 11]. For the coupling with other fluids without the local alignment force, global existence of weak solutions to Vlasov-Fokker-Planck/Navier-Stokes equations is treated in [14], and the global existence of classical solutions near equilibrium to Vlasov-Fokker-Planck/Euler equations is discussed in [3, 7]. We also refer to [4, 9, 10, 15] for the hydrodynamic limit of kinetic-fluid equations.

In this paper, we are interested in the drag forcing effect which comes from the coupling term $\rho(u - v)$ in the system (1.1) on the regularity and large-time behavior of solutions. Without the interactions with the incompressible fluid, the system (1.1)$_1$-(1.1)$_2$ becomes compressible Euler equations, and it is well-known that this system has the formation of singularities such as $\delta$-shock in finite time no matter how smooth initial data are. Thus it is natural to consider the measure solutions for the global well-posedness. We refer the readers to [5] and references therein for the general survey of the Euler equations. The issue of development of the singularity presents new challenges to the global existence theory of classical solutions. For the global existence of the unique classical solution to the coupled system (1.1), we reinterpret the drag forcing term as the relative damping. To be more precise, we reduce the system (1.1) to a symmetric system (see (1.5) below), and show that the drag forcing term can prevent the formation of singularities in the compressible Euler equations if the initial data are small enough in an appropriate norm. Since the drag forcing term does not give the real damping effect, we carefully analyze the coupling term with the help of the viscous term in the incompressible Navier-Stokes equations. As a direct consequence of this result, we can conclude that the hydrodynamic limit studied in [1] holds for all time. For the large-time behavior, we employ a similar strategy which is recently proposed in [6] for the
Vlasov/compressible Navier-Stokes equations. We construct a Lyapunov function measuring local variances of fluids around their local averages and the distance between local averaged velocities. We show the emergence of alignment between two fluid velocities as time evolves using our proposed Lyapunov function.

Before stating our main results on the global existence and large-time behavior of classical solutions, we introduce several simplified notations. For a function $f(x)$, we denote by $\|f\|_{L^p}$ the usual $L^p(\mathbb{T}^3)$-norm. $f \lesssim g$ represents that there exists a positive constant $C > 0$ such that $f \leq C g$. We also denote by $C$ a generic positive constant depending only on the norms of the data, but independent of $t$, and drop $x$-dependence of differential operators $\nabla_x$, that is, $\nabla f := \nabla_x f$ and $\Delta f := \Delta_x f$. For any nonnegative integer $k$, $H^k$ denotes the $k$-th order $L^2$ Sobolev space. $C^k([0,T];E)$ is the set of $k$-times continuously differentiable functions from an interval $[0,T] \subset \mathbb{R}$ into a Banach space $E$, and $L^p(0,T;E)$ is the set of the $L^2$ functions from an interval $(0,T)$ to a Banach space $E$. $\nabla^k$ denotes any partial derivative $\partial^\alpha$ with multi-index $\alpha, |\alpha| = k$.

For the global existence of a unique classical solution, by setting $n := \ln \rho$, we first reformulate the system (1.1) into the symmetric system as follows:

\[
\begin{align*}
\partial_t n + \nabla n \cdot u + \nabla \cdot u &= 0, \quad x \in \mathbb{T}^3, \quad t > 0, \\
\partial_t u + u \cdot \nabla u + \nabla n &= -(u - v), \quad x \in \mathbb{T}^3, \quad t > 0, \\
\partial_t v + v \cdot \nabla v + \nabla p - \Delta v &= e^n (u - v), \quad x \in \mathbb{T}^3, \quad t > 0, \\
\nabla \cdot v &= 0, \quad x \in \mathbb{T}^3, \quad t > 0.
\end{align*}
\]

(1.5)

with initial data

\[
(n, u, v)|_{t=0} = : (n_0 = \ln \rho_0, u_0, v_0), \quad \nabla \cdot v_0 = 0, \quad x \in \mathbb{T}^3.
\]

(1.6)

We notice that the Cauchy problem for the symmetric system (1.5) has a unique classical solution if and only if the system (1.1) has a unique classical solution (see Proposition 1 for details). From this observation, we prove the global existence of classical solutions to the system (1.5) under suitable assumptions on the initial data (1.6).

**Theorem 1.1.** Let $s > \frac{5}{2}$. Given the initial condition $(U_0, v_0) \in H^{s+1}(\mathbb{T}^3) \times H^{s+1}(\mathbb{T}^3)$ with $\|e^{0t}\|_{L^1} = 1$ and $\|U_0\|_{H^{s+1}} + \|v_0\|_{H^{s+1}} \leq \epsilon_0 \ll 1$, there exists a unique global-in-time classical solution $(U, v)$ to the system (1.5)-(1.6) satisfying $\|e^{nt}\|_{L^1} = 1$ for all $t \geq 0$ and

\[
\begin{align*}
U &\in C((0,\infty); H^{s+1}(\mathbb{T}^3)) \cap C^1([0,\infty); H^{s}(\mathbb{T}^3)), \\
v &\in C((0,\infty); H^{s+1}(\mathbb{T}^3)) \cap C^1([0,\infty); H^{s-1}(\mathbb{T}^3)),
\end{align*}
\]

where $U := U(x, t) := (n(x, t), u(x, t))^t$ and $U_0 := U_0(x) := (n_0(x), u_0(x))^t$.

In order to state our second result on the large-time behavior of classical solutions, we define a total fluctuated energy function $\mathcal{L}(t)$ by

\[
\mathcal{L}(t) := \int_{\mathbb{T}^3} \rho |u - m_c|^2 dx + \int_{\mathbb{T}^3} (\rho - 1)^2 dx + \int_{\mathbb{T}^3} |v - v_c|^2 dx + |m_c - v_c|^2,
\]

\[
\mathcal{L}_0 := \mathcal{L}(0),
\]

where $m_c$ and $v_c$ are averaged quantities given by

\[
m_c := \int_{\mathbb{T}^3} \rho u^2 dx \quad \text{and} \quad v_c := \int_{\mathbb{T}^3} v^2 dx.
\]
Theorem 1.2. Let \((\rho, u, v)\) be any global classical solutions to the system (1.1)-(1.2). Assume that the followings hold.

(i) \(\int_{T^3} \rho_0(x) \, dx = 1\), i.e., \(\int_{T^3} \rho(x, t) \, dx = 1\) for all \(t \geq 0\).

(ii) \(\rho \in [0, \bar{\rho}]\) for some \(\bar{\rho} > 0\), and \(u \in L^\infty(T^3 \times \mathbb{R}_+)\).

(iii) An initial total energy \(E_0 := \int_{T^3} \rho_0 |u_0|^2 \, dx + \int_{T^3} (\rho_0 - 1)^2 \, dx + \int_{T^3} |v_0|^2 \, dx\) is sufficiently small.

Then we have

\[ L(t) \lesssim L_0 e^{-Ct} \quad t \geq 0, \]

where \(C\) is a positive constant independent of \(t\).

Remark 1. 1. The smallness assumption on the initial data \(U_0\) and \(v_0\) is necessary. More precisely, we need the smallness of solution \(v\) in order to control the convection term in the incompressible Navier-Stokes equations (1.5) \(3\)-(1.5) \(4\), and the smallness of solutions \(n\) and \(u\) are required to avoid the formation of singularity in compressible Euler equations (1.5) \(1\)-(1.5) \(2\).

2. For the estimate of large-time behavior, we do not require that \(L^\infty(T^3)\)-norms of \(\rho\) and \(u\) should be small, we need only the small total initial energy. We also notice that

\[ E_0 = \int_{T^3} \rho_0 |u_0 - m_c(0)|^2 \, dx + \int_{T^3} (\rho_0 - 1)^2 \, dx + \int_{T^3} |v_0 - v_c(0)|^2 \, dx \]

\[ + |v_c(0)|^2 + |m_c(0)|^2 \]

\[ = L_0 + 2v_c(0) \cdot m_c(0), \]

\[ \geq L_0 - 2E_0, \]

where we used \(|v_c(0) \cdot m_c(0)| \leq E_0\). Hence we have \(L_0 \leq 3E_0\), and this implies that the initial total fluctuated energy function \(L_0\) is also small.

3. The unique global classical solution obtained in Theorem 1.1 satisfies the assumptions in Theorem 1.2.

The rest of this paper is organized as follows. In Section 2, we provide a priori energy estimates for the system (1.1) and several useful lemmas. We also discuss the local existence of the unique classical solution and the relation between Cauchy problems (1.1)-(1.2) and (1.5)-(1.6). Section 3 is devoted to give the details of the proof for Theorem 1.1. We present the major energy estimates to extend the local existence to the global one. Finally in Section 4, we establish the large-time behavior of global classical solutions showing the alignment between two fluid velocities exponentially fast.

2. Preliminaries.

2.1. A priori energy estimates and useful inequalities. We first present basic energy estimates which show conservations of mass and total momentum, and a dissipation of total energy of the system (1.1).
Lemma 2.1. Let \((\rho, u, v)\) be any global classical solutions to the system (1.1)-(1.2). Then we have

\[\begin{align*}
(i) & \quad \frac{d}{dt} \int_{T^3} \rho \, dx = 0, \quad \frac{d}{dt} \left( \int_{T^3} \rho u \, dx + \int_{T^3} v \, dx \right) = 0. \\
(ii) & \quad \frac{d}{dt} \left( \frac{1}{2} \int_{T^3} \rho |u|^2 \, dx + \int_{T^3} \rho \ln \rho \, dx + \frac{1}{2} \int_{T^3} |v|^2 \, dx \right) \\
& \quad + \int_{T^3} \nabla v^2 \, dx + \int_{T^3} \rho |u - v|^2 \, dx = 0.
\end{align*}\]

Proof. First we can easily obtain the estimates (i). For the estimate of (ii), we find

\[\frac{d}{dt} \int_{T^3} \rho |u|^2 \, dx = - \int_{T^3} u \cdot \nabla \rho \, dx - \int_{T^3} \rho (u - v) \cdot u \, dx, \quad (2.8)\]

and

\[\frac{d}{dt} \int_{T^3} |v|^2 \, dx + \int_{T^3} \nabla v^2 \, dx = \int_{T^3} \rho (u - v) \cdot v \, dx. \quad (2.9)\]

We also notice that

\[\int_{T^3} u \cdot \nabla \rho \, dx = \frac{d}{dt} \int_{T^3} \rho \ln \rho \, dx, \quad (2.10)\]

due to the continuity equation (1.1). Thus by combining (2.8)-(2.10), we have the estimate (ii).

We next provide elementary estimates for the pressure of the compressible Euler equations.

Lemma 2.2. Let \(\bar{\rho} > 0\) and \(\rho \in C^1(T^3 \times [0,T])\). Then we have

\[\frac{d}{dt} \int_{T^3} \rho \ln \rho \, dx = \frac{d}{dt} \int_{T^3} \rho \left( \int_1^\rho \frac{h-1}{h^2} \, dh \right) \, dx.\]

Proof. A straightforward computation yields the desired result.

Lemma 2.3. Let \(\bar{\rho} > 0\) and \(\rho \in C^1(T^3 \times [0,T])\). Then there exist positive constants \(c_1, c_2 > 0\) we have

\[c_1 (\rho - 1)^2 \leq f(\rho) \leq c_2 (\rho - 1)^2, \quad \rho \in [0, \bar{\rho}],\]

where \(f(\rho) := \rho \int_1^\rho \frac{h-1}{h^2} \, dh\).

Proof. We set

\[g(\rho) := \frac{\rho \int_1^\rho \frac{h-1}{h^2} \, dh}{(\rho - 1)^2}.\]

Then we can easily find that

\[\lim_{\rho \to 0} g(\rho) = 1 > 0 \quad \text{and} \quad \lim_{\rho \to 1} g(\rho) = \frac{1}{2} > 0.\]

Thus we deduce that \(g(\rho)\) is a continuous function on \([0, \bar{\rho}]\) with \(g(\rho) > 0\), and this completes the proof.
Remark 2. Let \((\rho, u, v)\) be any global classical solutions to the system (1.1)-(1.2). Suppose \(\rho \in [0, \bar{\rho}]\). Then it follows from Lemmas 2.1-2.3 that
\[
\int_{\mathbb{T}^3} \rho |u|^2 dx + \int_{\mathbb{T}^3} (\rho - 1)^2 dx + \int_{\mathbb{T}^3} |v|^2 dx \\
+ \int_0^t \int_{\mathbb{T}^3} |\nabla v|^2 dx ds + \int_0^t \int_{\mathbb{T}^3} \rho|u - v|^2 dx ds \\
\lesssim \int_{\mathbb{T}^3} \rho_0 |u_0|^2 dx + \int_{\mathbb{T}^3} (\rho_0 - 1)^2 dx + \int_{\mathbb{T}^3} |v_0|^2 dx.
\]
We set
\[E(t) := \int_{\mathbb{T}^3} \rho |u|^2 dx + \int_{\mathbb{T}^3} (\rho - 1)^2 dx + \int_{\mathbb{T}^3} |v|^2 dx,\]
then we have
\[E(t) \lesssim E_0.\]

In the following two lemmas, we recall Sobolev inequalities and provide an equivalence relation between \(\|\ln f\|_{L^2}\) and \(\|f - 1\|_{L^2}\) under suitable assumptions.

Lemma 2.4. (i) For any pair of functions \(f, g \in (H^k \cap L^\infty)(\mathbb{T}^3)\), we obtain
\[\|\nabla^k (fg)\|_{L^2} \lesssim \|f\|_{L^\infty} \|\nabla^k g\|_{L^2} + \|\nabla^k f\|_{L^2} \|g\|_{L^\infty}.\]
Furthermore if \(\nabla f \in L^\infty(\mathbb{T}^3)\) we have
\[\|\nabla^k (fg) - f \nabla^k g\|_{L^2} \lesssim \|\nabla f\|_{L^\infty} \|\nabla^k-1 g\|_{L^2} + \|\nabla f\|_{L^2} \|g\|_{L^\infty}.\]
(ii) Let \(k \in \mathbb{N}, p \in [1, \infty], h \in C^k(\mathbb{T}^3)\). Then there exists a positive constant \(c = c(k, p, h)\) such that
\[\|\nabla^k h(w)\|_{L^p} \leq c \|w\|_{L^\infty}^{k-1} \|\nabla^k w\|_{L^p},\]
for all \(w \in (W^{k,p} \cap L^\infty)(\mathbb{T}^3)\).

Lemma 2.5. For \(0 < a \leq f(x) \leq b\) with \(a \leq 1 \leq b\), there exist positive constants \(c_3, c_4 > 0\) such that
\[c_3 \int_{\mathbb{T}^3} (f - 1)^2 dx \leq \int_{\mathbb{T}^3} (\ln f)^2 dx \leq c_4 \int_{\mathbb{T}^3} (f - 1)^2 dx,
\]
where \(c_3\) and \(c_4\) depend only on \(a\) and \(b\), respectively.

Proof. Note that \(\ln x \leq x - 1\) for all \(x \in (-\infty, \infty)\). For \(1 \leq f(x) \leq b\), we easily find that \((\ln f)^2 \leq (f - 1)^2\). On the other hand, for \(a \leq f(x) \leq 1\) we set \(h_a(x) := \frac{\ln a}{1 - a}(1 - x)\). Then a straightforward computation yields \(h_a(f) \leq \ln f\), and this deduces \((\ln f)^2 \leq h_a(f)^2\). Thus we combine these two estimates to obtain
\[\int_{\mathbb{T}^3} (\ln f)^2 dx = \left(\int_{\{x \in \mathbb{T}^3 : h_a \geq f(x) \geq 1\}} + \int_{\{x \in \mathbb{T}^3 : 1 \geq f(x) \geq a\}}\right) (\ln f)^2 dx \leq \max\left\{1, \left(\frac{\ln a}{1 - a}\right)^2\right\} \int_{\mathbb{T}^3} (f - 1)^2 dx.
\]
Similarly, we set \(h_b(x) = \frac{\ln b}{b - 1}(x - 1)\). Then we find
\[\ln f^2 \geq \left\{\begin{array}{ll}
(f - 1)^2 & \text{for } 0 \leq f \leq 1, \\
h_b(f)^2 & \text{for } 1 \leq f \leq b.
\end{array}\right.\]
and this implies
\[ \int_{T_3} (\ln f)^2 dx \geq \min \left\{ 1, \left( \frac{\ln b}{b-1} \right)^2 \right\} \int_{T_3} (f-1)^2 dx. \]

This completes the proof. \(\square\)

**Remark 3.** It follows from Remark 2 and Lemma 2.5 that
\[ \int_{T_3} e^n|u|^2 dx + \int_{T_3} n^2 dx + \int_{T_3} |v|^2 dx + \int_0^t \int_{T_3} |\nabla v|^2 dx ds + \int_0^t \int_{T_3} e^n|u-v|^2 dx ds \]
\[ \lesssim \int_{T_3} e^{n_0}|u_0|^2 dx + \int_{T_3} n_0^2 dx + \int_{T_3} |v_0|^2 dx, \]
for \(n \in L^\infty(T^3 \times \mathbb{R}_+).\)

### 2.2. Local existence.

**Theorem 2.6.** Let \(s > \frac{5}{2},\) and suppose \((U_0, v_0) \in H^{s+1}(T^3) \times H^{s+1}(T^3).\) Then there exists a positive constant \(T_0 > 0\) such that the system (1.5)-(1.6) admits a unique solution \(U \in C([0, T_0]; H^{s+1}(T^3)) \cap C^1([0, T_0]; H^s(T^3)),\) \(v \in C([0, T_0]; H^{s+1}(T^3)) \cap C^1([0, T_0]; H^{s-1}(T^3)).\)

**Proof.** Since local existence theories for a type of conservation laws have been well developed, we omit the proof here. We refer to [12, 13] for the readers who are interested in it. \(\square\)

The proof of the following proposition is straightforward, and the positivity of the density in the proposition below is obtained from the corresponding positivity of the initial density by using the method of characteristics. For more details, we refer to [16].

**Proposition 1.** For any fixed \(T > 0,\) if \((\rho, u, v) \in C^1(T^3 \times [0, T])\) solves the system (1.1)-(1.2) with \(\rho > 0,\) then \((n, u, v) \in C^1(T^3 \times [0, T])\) solves the system (1.5)-(1.6) with \(e^n > 0,\) Conversely, if \((n, u, v) \in C^1(T^3 \times [0, T])\) solves the system (1.5)-(1.6) with \(e^n > 0,\) then \((\rho, u, v) \in C^1(T^3 \times [0, T])\) solves the system (1.1)-(1.2) with \(\rho > 0.\)

### 3. Global existence of classical solutions.

#### 3.1. A priori estimates.

In this part, we present the a priori estimates for the global existence of the classical solutions to the system (1.5)-(1.6). For this, we first introduce a norm \(W^m(f, g)\) for \(f, g \in H^m(T^3)\) which is recursively defined by
\[ W^m(f, g) := \|\nabla^m f\|_{L^2}^2 + \|\nabla^m g\|_{L^2}^2 + 2W^{m-1}(f, g) \quad \text{with} \quad W^0(f, g) := \|f\|_{L^2}^2 + \|g\|_{L^2}^2. \]

(3.11)

for \(m \geq 1.\) Then it is clear to get \(W^m(f, g) \approx \|f\|_{H^m}^2 + \|g\|_{H^m}^2\) in the sense that there exists a positive constant \(C > 0\) such that
\[ \frac{1}{C} (\|f\|_{H^m}^2 + \|g\|_{H^m}^2) \leq W^m(f, g) \leq C (\|f\|_{H^m}^2 + \|g\|_{H^m}^2). \]

**Lemma 3.1.** Let \(T > 0\) be given and \(s > \frac{5}{2}.\) Suppose \(U \in C([0, T]; H^{s+1}(T^3)) \cap C^1([0, T]; H^s(T^3))\) and \(v \in C([0, T]; H^{s+1}(T^3)) \cap L^2(0, T; H^{s+2}(T^3)),\) and \((U, v)\) is a solution to the system (1.5)-(1.6). Furthermore we assume
\[ \|U\|_{L^\infty(0, T; H^{s+1})} + \|v\|_{L^\infty(0, T; H^{s+1})} \leq \epsilon_1, \]

(3.12)
for some $0 < \epsilon_1 \ll 1$. Then we have

\[
\frac{d}{dt} W^{s+1}(U, v) \leq - (1 - 2\delta_0 - C\epsilon_1) \sum_{k=0}^{s} 2^k \| \nabla^{s-k} u \|^2_{L^2} - (1/2 - \delta_0) \| \nabla^{s+1} u \|^2_{L^2} \\
- \sum_{k=0}^{s+1} 2^{k+1} \| \nabla^{s+1-k} v \|^2_{L^2} - 2 \| \nabla^{s+2} v \|^2_{L^2} + C\epsilon_1 \| \nabla U \|^2_{H^s} + C \| v \|^2_{L^2},
\]

where $\delta_0$ is a positive constant.

**Proof.** From the assumption (3.12), we can find a positive constant $\delta_0 \ll \frac{1}{2}$ satisfying $|e^n - 1| < \delta_0$ for all $x \in \mathbb{T}^3$.

- $L^2$-estimates: We first notice

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} (n^2 + |u|^2) \, dx = - \frac{1}{2} \int_{\mathbb{T}^3} u \cdot (\nabla n^2 + \nabla |u|^2) \, dx - \int_{\mathbb{T}^3} u \cdot (u - v) \, dx \\
\leq \| u \|_{L^2} \| U \|_{L^\infty} \| \nabla U \|_{L^2} - \frac{3}{4} \| u \|^2_{L^2} + \| v \|^2_{L^2} \\
\leq - \frac{1}{2} \left( \frac{3}{2} - \epsilon_1 \right) \| u \|^2_{L^2} + \epsilon_1 \| \nabla U \|^2_{L^2} + \| v \|^2_{L^2},
\]

and this deduces

\[
\frac{d}{dt} \| U \|^2_{L^2} + \left( \frac{3}{2} - \epsilon_1 \right) \| u \|^2_{L^2} \leq \epsilon_1 \| \nabla U \|^2_{L^2} + 2 \| v \|^2_{L^2}. \tag{3.13}
\]

For the estimate of $v$, we can easily find that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |v|^2 \, dx + \int_{\mathbb{T}^3} |\nabla v|^2 \, dx \leq \frac{1}{2} \int_{\mathbb{T}^3} e^n |u|^2 \, dx - \frac{1}{2} \int_{\mathbb{T}^3} e^n |v|^2 \, dx \\
\leq \frac{1 + \delta_0}{2} \| u \|^2_{L^2} - \frac{1 - \delta_0}{2} \| v \|^2_{L^2},
\]

where we used $1 - \delta_0 < e^n < 1 + \delta_0$. Thus we obtain

\[
\frac{d}{dt} \| v \|^2_{L^2} + 2 \| \nabla v \|^2_{L^2} + (1 - \delta_0) \| v \|^2_{L^2} \leq (1 + \delta_0) \| u \|^2_{L^2}. \tag{3.14}
\]

We combine (3.13) and (3.14) to deduce

\[
\frac{d}{dt} \left( \| U \|^2_{L^2} + \| v \|^2_{L^2} \right) + \left( \frac{1}{2} - \delta_0 - \epsilon_1 \right) \| u \|^2_{L^2} + (1 - \delta_0) \| v \|^2_{L^2} + 2 \| \nabla v \|^2_{L^2} \leq \epsilon_1 \| \nabla U \|^2_{L^2} + 2 \| v \|^2_{L^2}. \tag{3.15}
\]

- $\dot{H}^k$-estimates for $1 \leq k \leq s + 1$: It follows from (1.5) that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\nabla^k n|^2 + |\nabla^k u|^2 \, dx \\
= - \int_{\mathbb{T}^3} \nabla^k n \cdot (\nabla^k (\nabla n \cdot u) + \nabla^k (\nabla \cdot u)) \, dx - \int_{\mathbb{T}^3} \nabla^k u \cdot (\nabla^k (u \cdot \nabla u) + \nabla^{k+1} n) \, dx \\
- \int_{\mathbb{T}^3} \nabla^k u \cdot (\nabla^k u - \nabla^k v) \, dx
\]
where \([\cdot, \cdot]\) denotes the commutator operator, i.e., \([A, B] = AB - BA\). Then we now estimate \(I_i, i = 1, \ldots, 4\) as follows.

\[
I_1 \leq \frac{1}{2} \|\nabla u\|_{L^\infty} (\|\nabla^n u\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2) \lesssim \|\nabla U\|_{L^\infty} \|\nabla^k U\|_{L^2}^2 \leq C\epsilon_1 \|\nabla^k U\|_{L^2}^2,
\]

\[
I_2 \lesssim \|\nabla^k n\|_{L^2} (\|\nabla^k u\|_{L^2} \|\nabla n\|_{L^\infty} + \|\nabla^k n\|_{L^2} \|\nabla u\|_{L^\infty}) \lesssim \|\nabla U\|_{L^\infty} \|\nabla^k U\|_{L^2}^2 \leq C\epsilon_1 \|\nabla^k U\|_{L^2}^2,
\]

\[
I_3 \lesssim \|\nabla^k u\|_{L^2}^2 \|\nabla u\|_{L^\infty} \lesssim \|\nabla U\|_{L^\infty} \|\nabla^k U\|_{L^2}^2 \leq C\epsilon_1 \|\nabla^k U\|_{L^2}^2,
\]

\[
I_4 \leq -\frac{3}{4} \|\nabla^k u\|_{L^2}^2 + \|\nabla^k v\|_{L^2}^2.
\]

This yields

\[
\frac{d}{dt} \|\nabla^k U\|_{L^2}^2 + \frac{3}{2} \|\nabla^k u\|_{L^2}^2 \leq C\epsilon_1 \|\nabla^k U\|_{L^2}^2 + 2 \|\nabla^k v\|_{L^2}^2. \tag{3.16}
\]

Similarly, we find

\[
\frac{1}{2} \frac{d}{dt} \int_{T^3} |\nabla^k v|^2 dx + \int_{T^3} |\nabla^{k+1} v|^2 dx
\]

\[
= -\int_{T^3} \nabla^k v \cdot (v \cdot \nabla^{k+1} v) dx - \int_{T^3} \nabla^k v \cdot [\nabla^k, v \cdot \nabla] v dx
\]

\[
- \int_{T^3} e^n \nabla^k v \cdot \nabla^k (u - v) dx - \int_{T^3} \nabla^k v \cdot (\nabla^k (e^n (u - v))) - e^n \nabla^k (u - v)) dx
\]

\[
=: \sum_{i=1}^4 J_i,
\]

where \(J_i, i = 1, \ldots, 4\) are estimated by

\[
J_1 \leq \frac{1}{2} \|\nabla v\|_{L^\infty} \|\nabla^k v\|_{L^2}^2 \leq C\epsilon_1 \|\nabla^k v\|_{L^2}^2,
\]

\[
J_2 \lesssim \|\nabla n\|_{L^\infty} \|\nabla^k v\|_{L^2}^2 \leq C\epsilon_1 \|\nabla^k v\|_{L^2}^2,
\]

\[
J_3 \leq \frac{1 + \delta_0}{2} \|\nabla^k u\|_{L^2}^2 - \frac{1 - \delta_0}{2} \|\nabla^k v\|_{L^2}^2,
\]

\[
J_4 \lesssim \|\nabla^k n\|_{L^2} (\|\nabla^k n\|_{L^2} \|u - v\|_{L^\infty} + \|\nabla n\|_{L^\infty} \|\nabla v\|_{L^2}) \leq C\epsilon_1 \|\nabla^k v\|_{L^2}^2 + C\epsilon_1 \|\nabla^k n\|_{L^2}^2 + C\epsilon_1 \|\nabla^{k-1} (u - v)\|_{L^2}.
\]

Here we used

\[
\|\nabla^k e^n (u - v)\|_{L^2} \lesssim \|\nabla^k e^n\|_{L^2} \|u - v\|_{L^\infty} + \|\nabla e^n\|_{L^\infty} \|\nabla^{k-1} (u - v)\|_{L^2}
\]

\[
\lesssim \|\nabla^k n\|_{L^2} \|u - v\|_{L^\infty} + \|\nabla n\|_{L^\infty} \|\nabla^{k-1} (u - v)\|_{L^2}.
\]
Then by combining the estimates above we get

$$
\frac{d}{dt} \|\nabla v\|_{L^2}^2 + 2\|\nabla^{k+1} v\|_{L^2}^2 + (1 - \delta_0 - C\epsilon_1)\|\nabla v\|_{L^2}^2 \\
\leq (1 + \delta_0)\|\nabla v\|_{L^2}^2 + C\epsilon_1\|\nabla U\|_{L^2}^2 + C\epsilon_1\|\nabla^{k-1}(u - v)\|_{L^2}^2.
$$

(3.17)

Thus we obtain from (3.16) and (3.17) that

$$
\frac{d}{dt} (\|\nabla U\|_{L^2}^2 + \|\nabla v\|_{L^2}^2) \\
+ (1/2 - \delta_0)\|\nabla v\|_{L^2}^2 + 2\|\nabla^{k+1} v\|_{L^2}^2 + (1 - \delta_0 - C\epsilon_1)\|\nabla v\|_{L^2}^2 \\
\leq C\epsilon_1\|\nabla U\|_{L^2}^2 + C\epsilon_1\|\nabla^{k-1}(u - v)\|_{L^2}^2 + 2\|\nabla v\|_{L^2}^2.
$$

(3.18)

• $H^{s+1}$-estimates: We now consider $W^{s+1}(U, v)$ defined in (3.11). Then it follows from (3.15) and (3.18) that

$$
\frac{d}{dt} W^{s+1}(U, v) \\
= \sum_{k=0}^s 2^k \frac{d}{dt} [(\|\nabla^{s+1-k} U\|_{L^2}^2 + \|\nabla^{s+1-k} v\|_{L^2}^2) + 2^{k+1} \frac{d}{dt} (\|U\|_{L^2}^2 + \|v\|_{L^2}^2) ] \\
\leq 2^{s+1}\epsilon_1\|\nabla U\|_{L^2}^2 + C\epsilon_1\sum_{k=0}^s 2^k \|\nabla^{s+1-k} U\|_{L^2}^2 + 2^{s+2}\|v\|_{L^2}^2 \\
- \sum_{k=0}^s 2^{k+1}\|\nabla^{s+2-k} v\|_{L^2}^2 + \sum_{k=0}^s 2^{k+1}\|\nabla^{s+1-k} v\|_{L^2}^2 - 2^{s+2}\|v\|_{L^2}^2 \\
- \sum_{k=0}^s 2^k (1 - \delta_0 - C\epsilon_1)\|\nabla^{s+1-k} v\|_{L^2}^2 + C\epsilon_1\sum_{k=0}^s 2^k \|\nabla^{s-k} v\|_{L^2}^2 - 2^{s+1}(1 - \delta_0)\|v\|_{L^2}^2 \\
- \sum_{k=0}^s 2^{k-1}(1 - 2\delta_0)\|\nabla^{s+1-k} u\|_{L^2}^2 + C\epsilon_1\sum_{k=0}^s 2^k \|\nabla^{s-k} u\|_{L^2}^2 - 2^s(1 - 2\delta_0 - 2\epsilon_1)\|u\|_{L^2}^2 \\
= \sum_{i=1}^{12} K_i
$$

(3.19)

where $K_i, i = 1, \cdots, 12$ are estimated as follows.

$$
\sum_{i=1}^3 K_i \leq C\epsilon_1 \sum_{k=0}^s \|\nabla^{s+1-k} U\|_{L^2}^2 + C\|v\|_{L^2}^2,
$$

$$
\sum_{i=4}^6 K_i = -2\|\nabla^{s+2} v\|_{L^2}^2 - \sum_{k=0}^s 2^{k+1}\|\nabla^{s+1-k} v\|_{L^2}^2 \\
= -2\|\nabla^{s+2} v\|_{L^2}^2 - \sum_{k=0}^{s+1} 2^{k+1}\|\nabla^{s+1-k} v\|_{L^2}^2 + 2^{s+2}\|v\|_{L^2}^2,
$$

$$
\sum_{i=7}^9 K_i = -(1 - \delta_0 - C\epsilon_1)\|\nabla^{s+1} v\|_{L^2}^2 - (1 - \delta_0 - C\epsilon_1)\sum_{k=0}^s 2^{k+1}\|\nabla^{s-k} v\|_{L^2}^2 \\
= -(1 - \delta_0 - C\epsilon_1)\sum_{k=0}^{s+1} 2^k \|\nabla^{s+1-k} v\|_{L^2}^2,
\[ \sum_{i=10}^{12} K_i = -\left(\frac{1}{2} - \delta_0\right)\|\nabla^{s+1} u\|_{L_2}^2 - (1 - 2\delta_0 - C\epsilon_1) \sum_{k=0}^{s} 2^k \|\nabla^{s-k} u\|_{L_2}^2. \]  

(3.20)

Combining (3.19) and (3.20), we have

\[ \frac{d}{dt} W^{s+1}(U,v) \leq -(1 - 2\delta_0 - C\epsilon_1) \sum_{k=0}^{s} 2^k \|\nabla^{s-k} u\|_{L_2}^2 - (1 - 2\delta_0 - C\epsilon_1) \sum_{k=0}^{s} 2^k \|\nabla^{s-k} v\|_{L_2}^2 \]

\[ - \sum_{k=0}^{s+1} 2^{k+1} \|\nabla^{s+1-k} v\|_{L_2}^2 - 2 \|\nabla^{s+2} v\|_{L_2}^2 + C\epsilon_1 \sum_{k=0}^{s} \|\nabla^{s+1-k} U\|_{L_2}^2 + C\|v\|_{L_2}^2, \]

and this completes the proof. \qed

We next provide the estimates of \( U_t \) and \( v_t \). Note that the regularities of \( U_t \) and \( v_t \) are different. More precisely, we will estimate \( \|U_t\|_{H^s} \) and \( \|v_t\|_{H^{s-1}} \) in the lemma below.

Lemma 3.2. Let \( T > 0 \) be given and \( s > \frac{5}{2} \). Suppose \( U \in C([0,T];H^{s+1}(T^3)) \cap C^1([0,T];H^s(T^3)) \) and \( v \in C([0,T];H^{s+1}(T^3)) \cap C^1([0,T];H^{s-1}(T^3)) \), and \((U,v)\) is a solution to the system (1.5)-(1.6). Furthermore we assume

\[ \|U\|_{L^\infty(0,T;H^{s+1})} + \|U_t\|_{L^\infty(0,T;H^s)} + \|v\|_{L^\infty(0,T;H^{s+1})} + \|v_t\|_{L^\infty(0,T;H^{s-1})} \leq \epsilon_1, \]

for some \( 0 < \epsilon_1 \ll 1 \). Then we have

\[ \frac{d}{dt} \left( W^{s-1}(U_t,v_t) + \frac{1}{2} \|\nabla U_t\|_{L_2}^2 \right) \leq - (1 - 2\delta_0) \sum_{k=0}^{s-1} 2^{k-1} \|\nabla^{s-k} u_t\|_{L_2}^2 - (1 - \delta_0 - C\epsilon_1) \sum_{k=0}^{s-1} 2^k \|\nabla^{s-k} v_t\|_{L_2}^2 \]

\[ - \sum_{k=0}^{s} \left( 2^{k+1} - C\epsilon_1 \right) \|\nabla^{s-1-k} v_t\|_{L_2}^2 - \|\nabla v_t\|_{L_2}^2 - \frac{3}{4} \|\nabla^2 u_t\|_{L_2}^2 \]

\[ + C\epsilon_1 \|U_t\|_{H^s}^2 + C\epsilon_1 \|\nabla U\|_{H^s}^2 + C\epsilon_1 \|\nabla v\|_{H^{s-1}}^2 + C\|v_t\|_{L_2}^2, \]

where \( \delta_0 \) is the positive constant determined in Lemma 3.1.

Proof. By differentiating (1.5) with respect to \( t \), we find

\[ n_{tt} = -\nabla n_t \cdot u - \nabla n \cdot u_t - \nabla \cdot u_t, \]

\[ u_{tt} = -u_t \cdot \nabla u - u \cdot \nabla u_t - \nabla n_t - u_t + v_t, \]

\[ v_{tt} = -v_t \cdot \nabla v - v \cdot \nabla v_t - \nabla p_t + \Delta v_t + e^n n_t (u - v) + e^n (u_t - v_t). \]

(3.21)

For the zeroth-order estimates, we multiply (3.21)_1 and (3.21)_2 by \( n_t \) and \( u_t \), respectively, and integrating over \( T^3 \) to obtain

\[ \frac{1}{2} \frac{d}{dt} \left( \int_{T^3} |n_t|^2 + |u_t|^2 \, dx \right) \]
\[
\begin{align*}
= & \frac{1}{2} \int_{\Omega} (\nabla \cdot u)(n_t^2 + |u_t|^2) \, dx - \int_{\Omega} n_t(\nabla n \cdot u_t) \, dx \\
& - \int_{\Omega} u_t \cdot (u_t \cdot \nabla u) \, dx - \int_{\Omega} u_t \cdot (u \cdot \nabla u_t) \, dx - \int_{\Omega} u_t \cdot (u_t - v_t) \, dx \\
=: & \sum_{i=1}^{5} I_i,
\end{align*}
\]
where \( I_i, i = 1, \ldots, 5 \) are estimated by
\[
\begin{align*}
I_1 & \leq \frac{1}{2} \|\nabla u\|_{L^\infty} \|U_t\|_{L^2}^2 \leq C\epsilon_1 \|U_t\|_{L^2}^2, \\
I_2 & \leq \|\nabla n\|_{L^\infty} \|n_t\|_{L^2} \|u_t\|_{L^2} \leq C\epsilon_1 \|U_t\|_{L^2}^2, \\
I_3 & \leq \|\nabla u\|_{L^\infty} \|u_t\|_{L^2} \leq C\epsilon_1 \|U_t\|_{L^2}^2, \\
I_4 & \leq \|\nabla u\|_{L^\infty} \|u_t\|_{L^2}^2 \leq C\epsilon_1 \|U_t\|_{L^2}^2, \\
I_5 & \leq \frac{3}{4} \|u_t\|_{L^2}^2 + \|v_t\|_{L^2}^2.
\end{align*}
\]
This deduces
\[
\frac{d}{dt} \|U_t\|_{L^2}^2 + \frac{3}{2} \|u_t\|_{L^2}^2 \leq C\epsilon_1 \|U_t\|_{L^2}^2 + 2\|v_t\|_{L^2}^2. \tag{3.22}
\]
Similar fashion to the above, we also find
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} & \int_{\Omega} |v_t|^2 \, dx + \int_{\Omega} |\nabla v_t|^2 \, dx \\
= & \int_{\Omega} (v_t \cdot (v_t \cdot \nabla v) - v_t \cdot (v \cdot \nabla v_t)) \, dx \\
& + \int_{\Omega} e^n v_t \cdot n_t(u - v) \, dx + \int_{\Omega} e^n (u_t - v_t) \cdot v_t \, dx \\
\leq & C \|\nabla v\|_{L^\infty} \|v_t\|_{L^2}^2 + (1 + \delta_0) \|u - v\|_{L^\infty} \|n_t\|_{L^2} \|v_t\|_{L^2} \\
& + \frac{1 + \delta_0}{2} \|u_t\|_{L^2}^2 - \frac{1 - \delta_0}{2} \|v_t\|_{L^2}^2 \\
\leq & C\epsilon_1 \|U_t\|_{L^2}^2 + \frac{1 + \delta_0}{2} \|u_t\|_{L^2}^2 - \frac{1 - \delta_0 - C\epsilon_1}{2} \|v_t\|_{L^2}^2.
\end{align*}
\tag{3.23}
\]
We now combine (3.22) and (3.23) to get
\[
\begin{align*}
\frac{d}{dt} \left( \|U_t\|_{L^2}^2 + \|v_t\|_{L^2}^2 \right) & + \left( \frac{1}{2} - \delta_0 \right) \|u_t\|_{L^2}^2 + (1 - \delta_0 - C\epsilon_1) \|v_t\|_{L^2}^2 + 2\|\nabla v_t\|_{L^2}^2 \\
& \leq C\epsilon_1 \|U_t\|_{L^2}^2 + 2\|v_t\|_{L^2}^2. \tag{3.24}
\end{align*}
\]
For the high-order estimates, we take \( \nabla^k \)-derivatives of the (3.21) for \( 1 \leq k \leq s \) to find
\[
\begin{align*}
\nabla^k u_t & = -u \cdot \nabla^{k+1} n_t - [\nabla^k, u \cdot \nabla] u_t - \nabla^k(\nabla n \cdot u_t) - \nabla^k(\nabla \cdot u_t), \\
\nabla^k u_t & = -\nabla^k (u \cdot \nabla u_t) - u \cdot \nabla^{k+1} u_t - [\nabla^k, u \cdot \nabla] u_t - \nabla^{k+1} u_t - \nabla^k n_t + \nabla^k v_t, \\
\nabla^k v_t & = -\nabla^k (v_t \cdot \nabla v) - v \cdot \nabla^{k+1} v_t - [\nabla^k, v \cdot \nabla] v_t - \nabla^{k+1} v_t + \nabla^k \Delta v_t \\
& + \nabla^k (e^n n_t \cdot (u - v)) + e^n \nabla^k (u_t - v_t) + [\nabla^k, e^n](u_t - v_t).
\end{align*}
\tag{3.25}
Then one can obtain that
\[
\frac{1}{2} \frac{d}{dt} \left( \int_{T^3} |\nabla^2 n_t|^2 dx + \int_{T^3} |\nabla^k u_t|^2 dx \right)
\]
\[
= - \int_{T^3} \nabla^k n_t \cdot (u \cdot \nabla^{k+1} n_t) \, dx - \int_{T^3} \nabla^k n_t \cdot [\nabla^k, u \cdot \nabla] u_t \, dx
\]
\[
- \int_{T^3} \nabla^k n_t \cdot (\nabla^k (v \cdot u_t)) \, dx - \int_{T^3} \nabla^k u_t \cdot (\nabla^k (u_t \cdot \nabla u)) \, dx
\]
\[
- \int_{T^3} \nabla^k u_t \cdot (u \cdot \nabla^{k+1} u_t) \, dx - \int_{T^3} \nabla^k u_t \cdot [\nabla^k, u \cdot \nabla] u_t \, dx
\]
\[
- \int_{T^3} \nabla^k u_t \cdot (\nabla^k u_t - \nabla^k v_t) \, dx
\]
\[
= \sum_{i=1}^{7} J_i,
\]
where \( J_i, i = 1, \ldots, 7 \) are estimated as follows.

\[ J_1 \leq \| \nabla u \|_{L^\infty} \| \nabla^2 n_t \|_{L^2}^2 \leq C \epsilon_1 \| \nabla^k U_t \|_{L^2}^2, \]
\[ J_2 \leq \| \nabla^k n_t \|_{L^2} \left( \| \nabla^k u_t \|_{L^2} \| \nabla u_t \|_{L^\infty} + \| \nabla^k u_t \|_{L^2} \| \nabla u_t \|_{L^\infty} \right)
\]
\[
\leq C \epsilon_1 \| \nabla^k U_t \|_{L^2}^2 + C \epsilon_1 \| \nabla^k U \|_{L^2}^2, \]
\[ J_3 \leq \| \nabla^k n_t \|_{L^2} \left( \| \nabla^{k+1} n_t \|_{L^2} \| u_t \|_{L^\infty} + \| \nabla n \|_{L^\infty} \| \nabla^k u_t \|_{L^2} \right)
\]
\[
\leq C \epsilon_1 \| \nabla^k U_t \|_{L^2}^2 + C \epsilon_1 \| \nabla^{k+1} U \|_{L^2}^2, \]
\[ J_4 \leq \| \nabla^k u_t \|_{L^2} \left( \| \nabla^k u_t \|_{L^2} \| u_t \|_{L^\infty} + \| \nabla u_t \|_{L^\infty} \| \nabla^{k+1} u_t \|_{L^2} \right)
\]
\[
\leq C \epsilon_1 \| \nabla^k U_t \|_{L^2}^2 + C \epsilon_1 \| \nabla^{k+1} U \|_{L^2}^2, \]
\[ J_5 \leq \| \nabla u \|_{L^\infty} \| \nabla^k u_t \|_{L^2}^2 \leq C \epsilon_1 \| \nabla^k U_t \|_{L^2}^2, \]
\[ J_6 \leq \| \nabla^k u_t \|_{L^2} \left( \| \nabla^k u_t \|_{L^2} \| u_t \|_{L^\infty} + \| \nabla u_t \|_{L^\infty} \| \nabla^k u_t \|_{L^2} \right)
\]
\[
\leq C \epsilon_1 \| \nabla^k U_t \|_{L^2}^2 + C \epsilon_1 \| \nabla^k U \|_{L^2}^2, \]
\[ J_7 \leq - \frac{3}{4} \| \nabla^k u_t \|_{L^2}^2 + \| \nabla^k v_t \|_{L^2}^2. \]

This yields
\[
\frac{d}{dt} \| \nabla^k U_t \|_{L^2}^2 + \frac{3}{2} \| \nabla^k u_t \|_{L^2}^2 \leq C \epsilon_1 \| \nabla^k U_t \|_{L^2}^2 + C \epsilon_1 \| \nabla^k U \|_{L^2}^2 + C \epsilon_1 \| \nabla^{k+1} U \|_{L^2}^2 + 2 \| \nabla^k v_t \|_{L^2}^2, \tag{3.26}
\]

For the estimate of \( v_t \), it follows from (3.25) that
\[
\frac{1}{2} \frac{d}{dt} \int_{T^3} |\nabla^k v_t|^2 dx + \int_{T^3} |\nabla^{k+1} v_t|^2 dx
\]
\[
= - \int_{T^3} \nabla^k v_t \cdot (\nabla^k (v_t \cdot \nabla v)) \, dx - \int_{T^3} \nabla^k v_t \cdot (v \cdot \nabla^{k+1} v_t) \, dx
\]
\[
- \int_{T^3} \nabla^k v_t \cdot (\nabla^k v_t \cdot \nabla^k v_t) \, dx + \int_{T^3} \nabla^k v_t \cdot (\nabla^k u_t \cdot (u - v)) \, dx
\]
\[
+ \int_{T^3} \nabla^k v_t \cdot (\nabla^k (u_t - v_t)) \, dx + \int_{T^3} \nabla^k v_t \cdot (\nabla^k (u_t - v_t)) \, dx
\]
We estimate $K_i$, $i = 1, \ldots , 6$ as follows.

$$K_1 \lesssim \|v_t\|_{L^2} \left( \|v_t\|_{L^2} \|v\|_{L^\infty} + \|v_t\|_{L^\infty} \|v\|_{L^2} \right)$$

$$\leq C_1 \|v_t\|_{L^2}^2 + C_1 \|v\|_{L^2}^2,$$

$$K_2 \leq \|v\|_{L^\infty} \|v_t\|_{L^2} \leq C_1 \|v_t\|_{L^2}^2,$$

$$K_3 \lesssim \|v_t\|_{L^2} \left( \|v_t\|_{L^2} \|v_t\|_{L^\infty} + \|v_t\|_{L^\infty} \|v\|_{L^2} \right)$$

$$\leq C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2,$$

$$K_4 \lesssim \|v_t\|_{L^2} \left( \|v_t\|_{L^2} \|v_t\|_{L^\infty} + \|v_t\|_{L^\infty} \|v\|_{L^2} \right)$$

$$\leq C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2,$$

$$K_5 \leq \frac{1 + \delta_0}{2} \|v_t\|_{L^2}^2 - \frac{1 - \delta_0}{2} \|v_t\|_{L^2}^2,$$

$$K_6 \lesssim \|v_t\|_{L^2} \left( \|v_t\|_{L^2} \|v_t\|_{L^\infty} + \|v_t\|_{L^\infty} \|v_t\|_{L^2} \right)$$

$$\leq C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2,$$

where we used the following estimates for $K_4$ with the help of Lemma 2.4

$$\|v_t\|_{L^2} \lesssim \|v_t\|_{L^2} \|v_t\|_{L^\infty} \|v_t\|_{L^\infty} \leq C_1.$$

Thus we obtain

$$\frac{d}{dt} \left( \|v_t\|_{L^2}^2 + \|v_t\|_{L^2}^2 + \|v_t\|_{L^2}^2 + \|v_t\|_{L^2}^2 \right)$$

$$\leq C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2,$$

Combining (3.26) and (3.27), we get

$$\frac{d}{dt} \left( \|v_t\|_{L^2}^2 + \|v_t\|_{L^2}^2 \right)$$

$$\leq C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2 + C_1 \|v_t\|_{L^2}^2,$$

Then we again consider $W^{s-1}(U_t, v_t)$. Using the similar arguments in the proof of Lemma 3.1, we find from (3.24) and (3.28) that

$$\frac{d}{dt} \left( \|v_t\|_{L^2}^2 + \|v_t\|_{L^2}^2 \right)$$

$$= \sum_{k=0}^{s-2} \left( \|v_t\|_{L^2}^2 + \|v_t\|_{L^2}^2 \right) + 2^{s-1} \frac{d}{dt} \left( \|U_t\|_{L^2}^2 + \|v_t\|_{L^2}^2 \right)$$
Hence we have

\[
\frac{d}{dt} \left( \mathcal{W}^s(U_t, v_t) + \frac{1}{2} \| \nabla U_t \|_{L^2}^2 \right)
\]

\[
\leq -(1 - 2\delta_0) \sum_{k=0}^{s-1} 2^{k-1} \| \nabla^{s-1-k} u_t \|_{L^2}^2 - (1 - \delta_0 - C\epsilon_1) \sum_{k=0}^{s-1} 2^k \| \nabla^{s-1-k} v_t \|_{L^2}^2
\]

\[
- \sum_{k=0}^{s-1} (2^{k+1} - C\epsilon_1) \| \nabla^{s-1-k} v_t \|_{L^2}^2 - \| \nabla^s v_t \|_{L^2}^2 - \frac{3}{4} \| \nabla^s u_t \|_{L^2}^2
\]

\[
+ C\epsilon_1 \| U_t \|_{H^{s-1}}^2 + C\epsilon_1 \| \nabla v \|_{H^{s-1}}^2 + C \| v_t \|_{L^2}^2.
\]

This concludes the desired result.

In the following lemma, we provide the relation between \( \| (n_t, \nabla n) \|_{H^s} \) and \( \| (u_t, \nabla u) \|_{H^s} \).

**Lemma 3.3.** Let \( T > 0 \) be given and \( s > \frac{5}{2} \). Suppose \( U \in C([0, T]; H^{s+1}(\mathbb{T}^3)) \cap C^1([0, T]; H^s(\mathbb{T}^3)) \) and \( v \in C([0, T]; H^{s+1}(\mathbb{T}^3)) \cap C^1([0, T]; H^s(\mathbb{T}^3)) \), and \((U, v)\) is a solution to the system (1.5)-(1.6). Then we obtain

\[
\| n_t \|_{H^s} + \| \nabla n \|_{H^s} \leq C \left( \| u_t \|_{H^s} + \| \nabla u \|_{H^s} + \| u \|_{H^s} \right| \nabla U \|_{H^s} + \| v \|_{H^s} \right).
\]

Furthermore we have

\[
\| U_t \|_{H^s} + \| \nabla U \|_{H^s} \leq C \left( \| u_t \|_{H^s} + \| \nabla u \|_{H^s} + \| u \|_{H^s} \right| \nabla U \|_{H^s} + \| u \|_{H^s} + \| v \|_{H^s} \right).
\]

**Proof.** It follows from (1.5) that

\[
n_t = -\nabla n \cdot u - \nabla \cdot u,
\]

\[
\nabla n = -u_t - u \cdot \nabla u - (u - v).
\]

Then we easily find that for \( 0 \leq k \leq s \)

\[
\| \nabla^k n_t \|_{L^2} \leq \| \nabla^k (\nabla n \cdot u) \|_{L^2} + \| \nabla^k \nabla \cdot u \|_{L^2}
\]

\[
\leq C \left( \| \nabla n \|_{H^s} \| u \|_{L^\infty} + \| \nabla n \|_{L^\infty} \| u \|_{H^s} \right) + \| \nabla^k \nabla \cdot u \|_{L^2},
\]

and this yields that

\[
\| n_t \|_{H^s} \leq C \left( \| u \|_{H^s} \| \nabla n \|_{H^s} + \| \nabla u \|_{H^s} \right).
\]  \((3.29)\)

Similarly, we also obtain

\[
\| \nabla n \|_{H^s} \leq C \left( \| u_t \|_{H^s} + \| \nabla u \|_{H^s} \| u \|_{H^s} + \| u \|_{H^s} + \| v \|_{H^s} \right).
\]  \((3.30)\)

By combining (3.29) and (3.30), we have

\[
\| U_t \|_{H^s} + \| \nabla U \|_{H^s} \leq C \left( \| u_t \|_{H^s} + \| \nabla u \|_{H^s} + \| u \|_{H^s} \right| \nabla U \|_{H^s} + \| u \|_{H^s} + \| v \|_{H^s} \right).
\]  \(\square\)

We next show the estimates of upper bounds of \( \| (n, v) \|_{L^2} \) and \( \| v_t \|_{L^2} \).
Lemma 3.4. The followings hold.

(i) \[ \|v\|_{L^2}^2 + \|n\|_{L^2}^2 \leq C (\|u_0\|_{L^2}^2 + \|n_0\|_{L^2}^2 + \|v_0\|_{L^2}^2). \]

(ii) \[ \frac{1}{2} \|v_t\|_{L^2}^2 + \frac{d}{dt} \|\nabla v\|_{L^2}^2 \leq \|v\|_{H^{s-1}}^2 \|\nabla v\|_{L^2}^2 + C \|\nabla e^n (u - v)\|_{L^2}. \]

Proof. (i) We first notice from Remark 3 that

\[ \int_{T^3} |v|^2 dx \leq \int_{T^3} e^{n_0} |u_0|^2 dx + \int_{T^3} (e^{n_0} - 1)^2 dx + \int_{T^3} |v_0|^2 dx \]

\[ \leq C \int_{T^3} |u_0|^2 dx + C \int_{T^3} |n_0|^2 dx + \int_{T^3} |v_0|^2 dx. \] (3.31)

We also find from Lemma 2.5 and Remark 3 that

\[ \int_{T^3} |n|^2 dx \leq C \int_{T^3} (e^n - 1)^2 dx \]

\[ \leq C \left( \int_{T^3} e^{n_0} |u_0|^2 dx + \int_{T^3} (e^{n_0} - 1)^2 dx + \int_{T^3} |v_0|^2 dx \right) \] (3.32)

Then we now combine (3.31) and (3.32) to conclude the proof of (i).

(ii) It follows from (1.5) that

\[ |v_t|^2 = -(v \cdot \nabla v) \cdot v_t - \nabla p \cdot v_t + \Delta v \cdot v_t + e^n (u - v) \cdot v_t. \]

This yields

\[ \frac{1}{2} \int_{T^3} |v_t|^2 dx + \frac{d}{dt} \int_{T^3} |\nabla v|^2 dx \leq \int_{T^3} |v|^2 |\nabla v|^2 dx + \int_{T^3} e^{2n} |u - v|^2 dx \]

\[ \leq \|v\|_{H^{s-1}}^2 \int_{T^3} |\nabla v|^2 dx + C \int_{T^3} e^n |u - v|^2 dx. \]

This completes the proof. \qed

3.2. Global existence.

Lemma 3.5. Let \( T > 0 \) be given and \( s > \frac{5}{2} \). Suppose \( U \in C([0,T]; H^{s+1}(T^3)) \cap C^1([0,T]; H^s(T^3)) \) and \( v \in C([0,T]; H^{s+1}(T^3)) \cap C^1([0,T]; H^s(T^3)), \) and \((U,v)\) is a solution to the system (1.5)-(1.6). Furthermore we assume

\[ \|U\|_{L^\infty(0,T;H^{s+1})} + \|U_t\|_{L^\infty(0,T;H^s)} + \|v\|_{L^\infty(0,T;H^{s+1})} + \|v_t\|_{L^\infty(0,T;H^{s-1})} \leq \epsilon_1, \]

for some \( 0 < \epsilon_1 \ll 1. \) Then we have

\[ \sup_{0 \leq t \leq T} \left( \|U(t)\|_{H^{s+1}}^2 + \|v(t)\|_{H^s}^2 + \|U_t(t)\|_{H^{s-1}}^2 + \|v_t(t)\|_{H^{s-1}}^2 \right) \leq C \|(U_0, v_0)\|_{H^{s+1}}, \]

where \( C \) is a positive constant independent of \( t. \)

Proof. We combine the two differential inequalities obtained in Lemmas 3.1 and 3.2 to find

\[ \frac{d}{dt} \left( \mathcal{W}^{s+1}(U,v) + \mathcal{W}^{s-1}(U_t, v_t) + \frac{1}{2} \|\nabla U_t\|_{L^2}^2 \right) + \mathcal{H}(U, U_t, v, v_t) \]

\[ \leq C \epsilon_1 \left( \|U_t\|_{H^s}^2 + \|\nabla U\|_{H^s}^2 + C \|v_t\|_{L^2}^2 + C \|v\|_{L^2}^2, \right. \]
where $\mathcal{H}(U, U_t, v, v_t)$ is given by

\[
\mathcal{H}(U, U_t, v, v_t) := (1 - 2\delta_0 - C_{c1}) \sum_{k=0}^{s} 2^k \|\nabla^{s-k} u\|_{L^2}^2 + (1 - 2\delta_0) \sum_{k=0}^{s-1} 2^{k-1} \|\nabla^{s-1-k} u_t\|_{L^2}^2 \\
+ \left(\frac{1}{2} - \delta_0\right) \|\nabla^{s+1} u\|_{L^2}^2 + \frac{3}{4} \|\nabla^s u_t\|_{L^2}^2 + 2 \|\nabla^{s+2} v\|_{L^2}^2 + \|\nabla^s v_t\|_{L^2}^2 \\
+ \sum_{k=0}^{s+1} (2^{k+1} - C_{c1}) \|\nabla^{s+1-k} u\|_{L^2}^2 + \sum_{k=0}^{s-1} (2^{k+1} - C_{c1}) \|\nabla^{s-1-k} v_t\|_{L^2}^2.
\]

Note that

\[
\mathcal{W}^{s+1}(U, v) + \mathcal{W}^{s-1}(U_t, v_t) + \frac{1}{2} \|\nabla^s U_t\|_{L^2}^2 \approx \|U\|_{H^{s+1}}^2 + \|U_t\|_{H^s}^2 + \|v\|_{H^{s+1}}^2 + \|v_t\|_{H^s}^2,
\]

and

\[
\mathcal{H}(U, U_t, v, v_t) \approx \|u\|_{H^{s+1}}^2 + \|u_t\|_{H^s}^2 + \|v\|_{H^{s+1}}^2 + \|v_t\|_{H^s}^2.
\]

On the other hand, it follows from Lemma 3.3 that

\[
\|U_t\|_{H^s}^2 + \|\nabla U\|_{H^s}^2 \leq C \left(\|u_t\|_{H^s}^2 + \|u\|_{H^{s+1}}^2 + \|v\|_{H^s}^2\right),
\]

and this yields

\[
\frac{d}{dt} \left(\mathcal{W}^{s+1}(U, v) + \mathcal{W}^{s-1}(U_t, v_t) + \frac{1}{2} \|\nabla^s U_t\|_{L^2}^2 \right) + C \left(\|u_t\|_{H^{s+1}}^2 + \|u\|_{H^{s+1}}^2 + \|v\|_{H^{s+1}}^2 + \|v_t\|_{H^s}^2\right) \leq C \|v_t\|_{H^s}^2 + C \|v\|_{H^s}^2,
\]

Furthermore we use the following inequality obtained in Lemma 3.4

\[
\|U_t\|_{H^s}^2 + \|U\|_{H^{s+1}}^2 \lesssim \|u_t\|_{H^s}^2 + \|u\|_{H^{s+1}}^2 + \|v\|_{H^s}^2 + \|v_t\|_{H^s}^2 + \|U_0\|_{L^2}^2 + \|v_0\|_{L^2}^2,
\]

to have

\[
\frac{d}{dt} \left(\mathcal{W}^{s+1}(U, v) + \mathcal{W}^{s-1}(U_t, v_t) + \frac{1}{2} \|\nabla^s U_t\|_{L^2}^2 + C \|\nabla^s U_t\|_{L^2}^2 \right) + C \left(\|U\|_{H^{s+1}}^2 + \|U_t\|_{H^s}^2 + \|v\|_{H^{s+1}}^2 + \|v_t\|_{H^s}^2\right) \lesssim \|U_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\sqrt{c^n}(u - v)\|_{L^2}^2.
\]

Then we use the Gronwall’s inequality and the equivalent relation (3.33) to get

\[
\|U\|_{H^{s+1}}^2 + \|U_t\|_{H^s}^2 + \|v\|_{H^{s+1}}^2 + \|v_t\|_{H^s}^2 \lesssim C \left(\|U(\tau)\|_{H^{s+1}}^2 + \|U_t(\tau)\|_{H^s}^2 + \|v(\tau)\|_{H^{s+1}}^2 + \|v_t(\tau)\|_{H^s}^2\right) + C \int_{0}^{t} \|\sqrt{c^n}(u - v)\|_{L^2}^2 \ d\tau + C \left(\|U_0\|_{L^2}^2 + \|v_0\|_{L^2}^2\right) \lesssim \|U(\tau)\|_{H^{s+1}}^2 + \|U_t(\tau)\|_{H^s}^2 + \|v(\tau)\|_{H^{s+1}}^2 + \|v_t(\tau)\|_{H^s}^2 + \|U_0\|_{L^2}^2 + \|v_0\|_{L^2}^2,
\]

for $t \geq \tau$, and where we used

\[
\int_{0}^{t} \|\sqrt{c^n}(u - v)\|_{L^2}^2 \ d\tau \lesssim \|u_0\|_{L^2}^2 + \|e^{n_0} - 1\|_{L^2}^2 + \|v_0\|_{L^2}^2 \lesssim \|u_0\|_{L^2}^2 + \|n_0\|_{L^2}^2 + \|v_0\|_{L^2}^2.
\]
Next we notice that
\begin{align*}
\|u_t\|^2_{H^s} + \|u\|^2_{H^s} & \lesssim \|\nabla u \cdot u\|^2_{H^s} + \|\nabla u\|^2_{H^s} + \|u\cdot\nabla u\|^2_{H^s} + \|\nabla u\|^2_{H^s} + \|v\|^2_{H^s} \\
& \lesssim \|\nabla u\|^2_{H^s} + \|\nabla u\|^2_{H^s} + \|\nabla u\|^2_{H^s} + \|\nabla u\|^2_{H^s} + \|v\|^2_{H^s} \\
& \lesssim \|U\|^4_{H^{s+1}} + \|U\|^2_{H^{s+1}} + \|v\|^2_{H^s}.
\end{align*}

(3.34)

It also follows from (1.5) that for $0 \leq k \leq s-1$
\begin{align*}
\int_{T^3} \nabla^k v_t \cdot \nabla^k v_t \, dx = & - \int_{T^3} (\nabla^k (v \cdot \nabla v) \cdot \nabla^k v_t + \nabla^k p \cdot \nabla^k v_t) \, dx \\
& + \int_{T^3} (\nabla^k \Delta v + \nabla^k (e^n (u - v))) \cdot \nabla^k v_t \, dx \\
& \leq C (\|\nabla^k (v \cdot \nabla v)\|^2_{L^2} + \|\nabla^k \Delta v\|^2_{L^2} + \|\nabla^k (e^n (u - v))\|^2_{L^2} ) \\
& + \frac{1}{2} \|\nabla^k v_t\|^2_{L^2},
\end{align*}
and this deduces
\begin{align*}
\|\nabla^k v_t\|^2_{L^2} \lesssim \|\nabla^k (v \cdot \nabla v)\|^2_{L^2} + \|\nabla^k \Delta v\|^2_{L^2} + \|\nabla^k (e^n (u - v))\|^2_{L^2}.
\end{align*}

We now sum the inequality above over $0 \leq k \leq s-1$ to find
\begin{align*}
\|v_t\|^2_{H^{s-1}} & \lesssim \|v \cdot \nabla v\|^2_{H^{s-1}} + \|\Delta v\|^2_{H^{s-1}} + \|e^n (u - v)\|^2_{H^{s-1}} \\
& \lesssim \|v\|^4_{H^{s+1}} + \|v\|^2_{H^{s+1}} + \|u\|^2_{H^{s+1}} + \|v\|^2_{H^{s+1}} (\|v\|^2_{H^{s+1}} + \|u\|^2_{H^{s+1}}).
\end{align*}

(3.35)

Finally we combine (3.34) and (3.35) to obtain
\begin{align*}
\limsup_{\tau \to 0^+} (\|U(\tau)\|^2_{H^s} + \|v(\tau)\|^2_{H^{s-1}}) & \lesssim \|U_0\|^2_{H^{s+1}} + \|U_0\|^2_{H^{s+1}} + \|v_0\|^4_{H^{s+1}} \\
& + \|v_0\|^2_{H^{s+1}} + \|u_0\|^2_{H^{s+1}} + \|v_0\|^2_{H^{s+1}}.
\end{align*}

Hence we have
\begin{align*}
\sup_{0 \leq t \leq T} (\|U(t)\|^2_{H^{s+1}} + \|v(t)\|^2_{H^{s+1}} + \|U_1(t)\|^2_{H^{s}} + \|v_1(t)\|^2_{H^{s-1}}) & \leq C (\|U_0\|^2_{H^{s+1}} + \|U_0\|^2_{H^{s+1}} + \|v_0\|^2_{H^{s+1}} + \|v_0\|^2_{H^{s+1}} + \|U_0\|^2_{H^{s+1}} + \|v_0\|^2_{H^{s+1}}) \\
& \leq C (\|U_0\|^2_{H^{s+1}} + \|v_0\|^2_{H^{s+1}}),
\end{align*}
due to the smallness assumption on the initial data. Here $C$ is a positive constant independent of $t$.

Proof of Theorem 1.1. The proof is easily obtained from Theorem 2.6 and Lemma 3.5.

4. Large-time behavior. In this section, we study the large-time behavior of global classical solutions to the system (1.1)-(1.2). For this, we set a function $E(t)$ and its corresponding dissipation $D(t)$:
\begin{align*}
E(t) & := \frac{1}{2} \int_{T^3} \rho |u - m_c|^2 \, dx + \int_{T^3} \rho \frac{h - 1}{h^2} \, dhdx + \frac{1}{2} \int_{T^3} |v - v_c|^2 \, dx + \frac{1}{4} |m_c - v_c|^2, \\
D(t) & := \int_{T^3} |\nabla v|^2 \, dx + \int_{T^3} \rho |u - v|^2 \, dx.
\end{align*}
Lemma 4.1. Let \((\rho, u, v)\) be any global classical solutions to the system \((1.1)-(1.2)\). Then we have
\[
\frac{d}{dt} \mathcal{E}(t) + \mathcal{D}(t) = 0.
\]

Proof. The proof is obtained by combining the following three equalities and the equality in Lemma 2.2:

(i) \[
\frac{d}{dt} \left( \frac{1}{2} \int_{\Omega^3} \rho |u - m_c|^2 \, dx + \int_{\Omega^3} \rho \ln \rho \, dx \right) = - \int_{\Omega^3} \rho (u - v) \cdot (u - m_c) \, dx,
\]
(ii) \[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^3} |v - u_c|^2 \, dx + \int_{\Omega^3} |\nabla v|^2 \, dx = \int_{\Omega^3} \rho (u - v) \cdot (v - u) \, dx,
\]
(iii) \[
\frac{d}{dt} |m_c - v_c|^2 = -2 \int_{\Omega^3} \rho (u - v) \cdot (m_c - v_c) \, dx.
\]

Straightforward computations yield the equalities (i) and (ii). For the estimate of (iii), we use the conservation of total momentum in Lemma 2.1 to get
\[
\frac{1}{2} \frac{d}{dt} |m_c - v_c|^2 = (m_c - v_c) \cdot (m'_c - v'_c)
\]
\[
= -2 (m_c - v_c) \cdot v'_c
\]
\[
= -2 \int_{\Omega^3} \rho (u - v) \cdot (m_c - v_c) \, dx.
\]

\square

We next recall a type of Bogovskii’s result in the periodic domain which can be obtained from the estimate of elliptic regularity for Poisson’s equations. For more details, we refer to [6] (see also [8] for the general bounded domains).

Lemma 4.2. Given any \(f \in L^2_{\#}(\Omega^3) := \\left\{ f \in L^2(\Omega^3) \mid \int_{\Omega^3} f \, dx = 0 \right\}\), the following stationary transport equation with auxiliary equations
\[
\nabla \cdot \nu = f, \quad \nabla \times \nu = 0, \quad \text{and} \quad \int_{\Omega^3} \nu \, dx = 0,
\]
(4.36)

admit a solution operator \(\mathcal{B} : f \mapsto \nu\) satisfying the following properties:

1. \(\nu = \mathcal{B}[f]\) is a solution to the problem (4.36) and a linear operator from \(L^2_{\#}(\Omega^3)\) into \(H^1(\Omega^3)\), i.e.,
\[
\|\mathcal{B}[f]\|_{H^1} \leq C \|f\|_{L^2}.
\]

2. If a function \(f \in H^1(\Omega^3)\) can be written in the form \(f = \nabla \cdot g\) with \(g \in H^1(\Omega^3)^3\), then
\[
\|\mathcal{B}[f]\|_{L^2} \leq C \|g\|_{L^2}.
\]

We now set perturbed functions \(\mathcal{E}^\sigma(t)\) and \(\mathcal{D}^\sigma(t)\) as follows.
\[
\mathcal{E}^\sigma(t) := \mathcal{E}(t) + \sigma \int_{\Omega^3} \rho (u - m_c) \cdot \mathcal{B}[\rho - 1] \, dx,
\]
and
\[
\mathcal{D}^\sigma(t) := \mathcal{D}(t) + \sigma \left( \int_{\Omega^3} (\rho u \otimes u) : \nabla \mathcal{B}[\rho - 1] + (\rho - 1)^2 - \rho (u - v) \cdot \mathcal{B}[\rho - 1] \, dx \right),
\]
\[
- \sigma \left( \int_{\Omega^3} \rho (u - m_c) \cdot \mathcal{B}[\nabla \cdot (\rho u)] + (\rho m_c) \cdot \mathcal{B}[\rho - 1] \, dx \right).
\]
Then it is clear to find
\[ \frac{d}{dt} \mathcal{E}^\sigma(t) + \mathcal{D}^\sigma(t) = 0. \] (4.37)
Recall our Lyapunov function \( \mathcal{L}(t) \):
\[ \mathcal{L}(t) = \int_{\mathbb{T}^3} \rho |u - m_c|^2 dx + \int_{\mathbb{T}^3} (\rho - 1)^2 dx + \int_{\mathbb{T}^3} |v - v_c|^2 dx + |m_c - v_c|^2. \]

In the following lemma, we show that the equivalence relation between \( \mathcal{L} \) and \( \mathcal{E}^\sigma \) for \( \sigma > 0 \) small enough.

**Lemma 4.3.** Let \((\rho, u, v)\) be any global classical solutions to the system (1.1)-(1.2). Suppose \( \rho \in [0, \bar{\rho}] \). Then there exists a positive constant \( C > 0 \) such that
\[ \frac{1}{C} \mathcal{L}(t) \leq \mathcal{E}^\sigma(t) \leq C \mathcal{L}(t) \quad t \geq 0, \]
for \( \sigma > 0 \) small enough.

**Proof.** Note that
\[ \sigma \left| \int_{\mathbb{T}^3} \rho (u - m_c) : B(\rho - 1) dx \right| \leq \frac{\sigma}{2} \int_{\mathbb{T}^3} \rho |u - m_c|^2 dx + \frac{C \sigma \bar{\rho}}{2} \int_{\mathbb{T}^3} (\rho - 1)^2 dx. \]
Then this and together with the estimate in Lemma 2.3 yield
\[ \mathcal{E}^\sigma(t) \geq \frac{1 - \sigma}{2} \int_{\mathbb{T}^3} \rho |u - m_c|^2 dx + \left( c_1 - \frac{C \sigma \bar{\rho}}{2} \right) \int_{\mathbb{T}^3} (\rho - 1)^2 dx \]
\[ + \frac{1}{2} \int_{\mathbb{T}^3} |v - v_c|^2 dx + \frac{1}{4} |m_c - v_c|^2. \]
Thus by choosing \( \sigma > 0 \) small enough, we conclude our result.

**Lemma 4.4.** The averaged momentum \( m_c \) for the compressible fluid satisfies the followings.
\[ |m_c(t)|^2 \leq C E_0 \quad \text{and} \quad |m'_c(t)|^2 \leq \int_{\mathbb{T}^3} \rho |u - v|^2 dx. \]

**Proof.** It follows from Remark 2 that
\[ m_c(t) = \int_{\mathbb{T}^3} \rho u dx \leq \left( \int_{\mathbb{T}^3} \rho |u|^2 dx \right)^{1/2} \leq C E_0^{1/2}. \]
We also find
\[ \frac{d}{dt} m_c(t) = \frac{d}{dt} \int_{\mathbb{T}^3} \rho u dx = - \int_{\mathbb{T}^3} \rho (u - v) dx, \]
and this yields
\[ \left| \frac{d}{dt} m_c(t) \right| \leq \left( \int_{\mathbb{T}^3} \rho |u - v|^2 dx \right)^{1/2}. \]

**Lemma 4.5.** Let \((\rho, u, v)\) be any global classical solutions to the system (1.1)-(1.2) satisfying (1.7). Then there exists a positive constant \( C > 0 \) such that
\[ \mathcal{D}^\sigma(t) \geq C \mathcal{L}(t) \quad t \geq 0. \] (4.38)
for \( \sigma > 0 \) small enough.
Proof. We will do the estimate of each terms in $D^\sigma(t)$. We set
\[
\sum_{i=1}^{4} I_i := \sigma \left( \int_{T^3} (\rho u \otimes u) : \nabla B[\rho - 1] - \rho (u - v) \cdot B[\rho - 1] \, dx \right),
- \sigma \left( \int_{T^3} \rho (u - m_c) : B[\nabla \cdot (\rho u)] + (\rho m_c)_t \cdot B[\rho - 1] \, dx \right).
\]
By adding and subtracting, we find that
\[
\sigma \left| \int_{T^3} (\rho u \otimes u) : \nabla B[\rho - 1] \, dx \right| = \sigma \int_{T^4} \rho ((u - m_c) \otimes u) : \nabla B[\rho - 1] \, dx
+ \sigma \int_{T^3} \rho (m_c \otimes (u - m_c)) : \nabla B[\rho - 1] \, dx
+ \sigma \int_{T^3} (\rho - 1)(m_c \otimes m_c) : \nabla B[\rho - 1] \, dx
=: I_1^1 + I_2^1 + I_3^1.
\]
Here $I_1^1, i = 1, 2, 3$ are estimated as follows.
\[
I_1^1 \leq \frac{\sigma^{1/2} |B|}{2} \int_{T^3} \rho |u - m_c|^2 \, dx + C \sigma^{3/2} \int_{T^3} (\rho - 1)^2 \, dx,
I_2^1 \leq \frac{\sigma^{1/2} \rho E_0}{2} \int_{T^3} \rho |u - m_c|^2 \, dx + C \sigma^{3/2} \int_{T^3} (\rho - 1)^2 \, dx,
I_3^1 \leq C \sigma E_0 \int_{T^3} (\rho - 1)^2 \, dx.
\]
Thus we have
\[
I_1 \leq \left( \frac{\sigma^{1/2} |B|}{2} (\|u\|_{L^\infty} + E_0) \right) \int_{T^3} \rho |u - m_c|^2 \, dx + C \sigma \left( E_0 + \sigma^{1/2} \right) \int_{T^3} (\rho - 1)^2 \, dx.
\]
For the estimate of $I_2$, we easily obtain
\[
\sigma \left| \int_{T^3} \rho (u - v) B[\rho - 1] \, dx \right| \leq \frac{1}{4} \int_{T^3} \rho |u - v|^2 \, dx + C \sigma^2 \rho \int_{T^3} (\rho - 1)^2 \, dx.
\]
For the estimate of $I_3$, we deduce that
\[
\sigma \int_{T^3} \rho (u - m_c) \cdot B[\nabla \cdot (\rho u)] \, dx
= \sigma \int_{T^3} \rho (u - m_c) \cdot B[\nabla \cdot (\rho (u - m_c))] \, dx + \sigma \int_{T^3} \rho (u - m_c) \cdot B[\nabla \cdot (\rho m_c)] \, dx
= \sigma \int_{T^3} \rho (u - m_c) \cdot B[\nabla \cdot (\rho (u - m_c))] \, dx + \sigma \int_{T^3} \rho (u - m_c) \cdot B[\nabla \cdot ((\rho - 1)m_c)] \, dx
\leq C \sigma \rho \int_{T^3} \rho |u - m_c|^2 \, dx + \sigma \int_{T^3} \rho |u - m_c||B[\nabla \cdot ((\rho - 1)m_c)] \, dx
\leq C \left( \sigma + \sigma^{1/2} \right) \rho \int_{T^3} \rho |u - m_c|^2 \, dx + C \sigma^{3/2} E_0 \int_{T^3} (\rho - 1)^2 \, dx.
\]
We finally estimate $I_4$. Note that
\[
(\rho m_c)_t = m_c \rho_t + \rho m'_c = -m_c \nabla_x \cdot (\rho u) + \rho m'_c.
\]
Then we obtain
\[
\sigma \int_{T^3} (p m_c)_t \cdot \nabla \rho \, \rho - 1 \, dx
\]
\[
= \sigma \int_{T^3} \rho (u - m_c) \otimes m_c : \nabla \rho \, \rho - 1 \, dx
\]
\[
+ \sigma \int_{T^3} (\rho - 1)(m_c \otimes m_c) : \nabla \rho \, \rho - 1 \, dx + \sigma m'_c \cdot \int_{T^3} \rho \, \rho \, \rho - 1 \, dx
\]
\[
=: I_4^1 + I_4^2 + I_4^3.
\]
Here the terms \(I_4^i, i = 1, 2, 3\) can be estimated as
\[
I_4^1 \leq \sigma^{1/2} \int_{T^3} \rho |u - m_c|^2 \, dx + C \sigma^{3/2} E_0 \tilde{\rho} \int_{T^3} (\rho - 1)^2 \, dx,
\]
\[
I_4^2 \leq C \sigma E_0 \int_{T^3} (\rho - 1)^2 \, dx,
\]
\[
I_4^3 \leq \frac{1}{4} \int_{T^3} \rho |m'_c|^2 \, dx + C \rho \sigma^2 \int_{T^3} (\rho - 1)^2 \, dx
\]
\[
\leq \frac{1}{4} \int_{T^3} \rho |u - v|^2 \, dx + C \rho \sigma^2 \int_{T^3} (\rho - 1)^2 \, dx.
\]
This deduces
\[
I_4 \leq \frac{1}{4} \int_{T^3} \rho |u - v|^2 \, dx + \sigma^{1/2} \int_{T^3} \rho |u - m_c|^2 \, dx + C \sigma \left( \sigma^{1/2} E_0 \tilde{\rho} + E_0 + \rho \sigma \right) \int_{T^3} (\rho - 1)^2 \, dx.
\]
We now combine the estimates above to find
\[
D^\sigma(t) \geq \int_{T^3} |\nabla v|^2 \, dx + \frac{1}{2} \int_{T^3} \rho |u - v|^2 \, dx + C_1 \int_{T^3} (\rho - 1)^2 \, dx
\]
\[
- C_2 \sigma^{1/2} \int_{T^3} \rho |u - m_c|^2 \, dx,
\]
where \(C_1 \) and \(C_2 \) are given by
\[
C_1 := \sigma \left( 1 - C E_0 - C \sigma^{1/2} \left( 1 + E_0 (1 + \tilde{\rho}) \right) - C \sigma \tilde{\rho} \right),
\]
and
\[
C_2 := C \tilde{\rho} \left( 1 + \frac{\|u\|_{L^\infty} + E_0}{2} \right) + C \tilde{\rho} \left( \sigma^{1/2} + 1 \right) + 1.
\]
Note that the constant \(C_1 \) is positive for sufficiently small \(\sigma \) and \(E_0 \). We now estimate the following to extract good dissipation terms.
\[
\int_{T^3} \rho |u - v|^2 \, dx = \int_{T^3} \rho |u - m_c + m_c - v_c + v_c - v|^2 \, dx
\]
\[
= \int_{T^3} \rho |u - m_c|^2 \, dx + |m_c - v_c|^2 + \int_{T^3} \rho |v - v_c|^2 \, dx
\]
\[
+ 2 \int_{T^3} \rho (u - m_c) \cdot (v_c - v) \, dx.
\]
Since
\[
-4 \int_{T^3} \rho (u - m_c) \cdot (v_c - v) \, dx \leq 4 \int_{T^3} \rho |v_c - v|^2 \, dx + \int_{T^3} \rho |u - m_c|^2 \, dx,
\]
we obtain
\[
\frac{1}{2} \int_{T^3} \rho |u - m_c|^2 \, dx + |m_c - v_c|^2 \leq \int_{T^3} \rho |v - v_c|^2 \, dx + \int_{T^3} \rho |u - v|^2 \, dx. \tag{4.39}
\]
We multiply (4.39) by $\frac{1}{2} > \gamma > 0$ which will be determined later to deduce that

$$D^\sigma(t) \geq \int_{T^3} |\nabla v|^2 \, dx + \left(1 - \frac{1}{2} - \gamma \right) \int_{T^3} \rho |u - v|^2 \, dx + (\gamma - C_2 \sigma^{1/2}) \int_{T^3} \rho |u - m_c|^2 \, dx$$

$$+ \gamma |m_c - v_c|^2 + C_1 \int_{T^3} (\rho - 1)^2 \, dx - \gamma \bar{\rho} \int_{T^3} |v - v_c|^2 \, dx$$

$$\geq (\gamma - C_2 \sigma^{1/2}) \int_{T^3} \rho |u - m_c|^2 \, dx + C_1 \int_{T^3} (\rho - 1)^2 \, dx$$

$$+ C(1 - C\gamma) \int_{T^3} |v - v_c|^2 \, dx + \gamma |m_c - v_c|^2,$$

where we used Poincaré’s inequality. We now choose $\frac{1}{2} > \gamma > 0$ such that $1 > C\gamma$, and then again select $\sigma$ and $E_0$ small enough to get the positive coefficients $C_1 > 0$ and $\gamma - C_2 \sigma^{1/2} > 0$. This yields that the all coefficients above are positive. Hence we have the inequality (4.38).

**Proof of Theorem 1.2.** We first notice from Lemmas 4.3 and 4.5 that there exists a positive constant $C > 0$ such that $CE^\sigma(t) \leq D^\sigma(t)$ for $\sigma, E_0 > 0$ small enough. Then it follows from (4.37) that

$$\frac{d}{dt} E^\sigma(t) + CE^\sigma(t) \leq 0, \quad t \geq 0,$$

for sufficiently small $\sigma > 0$ and $E_0 > 0$. We now apply Gronwall’s inequality to find

$$E^\sigma(t) \leq E_0^\sigma e^{-Ct}, \quad t \geq 0.$$

Finally, we again use the equivalent relation between $E^\sigma(t)$ and $L(t)$ in Lemma 4.3 to conclude that

$$L(t) \leq C L_0 e^{-Ct}, \quad t \geq 0,$$

where $C$ is a positive constant independent of $t$. □

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