A FAMILY OF MULTIPLY WARPED PRODUCT SEMI-RIEMANNIAN EINSTEIN METRICS

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Abstract. In this paper, we characterize multiply warped product semi-Riemannian manifolds when the base is conformal to an n-dimensional pseudo-Euclidean space. We prove some conditions on warped product semi-Riemannian manifolds to be an Einstein manifold which is invariant under the action of an (n − 1)-dimensional translation group. After that we apply this result for the case of Ricci-flat multiply warped product space when the fibers are Ricci-flat. We also discuss the existence of infinitely many Ricci-flat multiply warped product spaces under the same action with null like vector.

1. Introduction. A semi-Riemannian manifold \((M^n, g)\) with dimension \((n \geq 3)\) is said to be an Einstein manifold if there exists a real constant \(\lambda\) such that Ricci tensor satisfies the condition \(\text{Ric}(X,Y) = \lambda g(X,Y)\), for each smooth vector fields \(X\) and \(Y\) on \(M\). B. O’Neill gave the definition of warped product space (see,[15]). Let \(f : B \rightarrow (0, \infty)\), \(f \in C^\infty(B)\). The warped product \(M = B \times_f F\) of two semi-Riemannian manifolds is a product manifold furnished with metric tensor \(g_M = g_B \oplus f^2 g_F\), defined by \(g = \pi^*(g_B) \oplus (f \circ \pi)^2 \sigma^*(g_F)\), where \(\pi\) and \(\sigma\) are the natural projections on base \(B\) and fiber \(F\) respectively. The function \(f\) is called the warping function and * denote the pull-back operator on tensors.

A multiply warped product space is the generalization of warped product space. A multiply warped product space \((M, g)\) is the product manifold of the form \(M = B \times_{h_1} F_1 \times_{h_2} F_2 \ldots \times_{h_m} F_m\) furnished with the metric

\[
g = g_B \oplus h_1^2 g_{F_1} \oplus h_2^2 g_{F_2} \oplus h_3^2 g_{F_3} \ldots \oplus h_m^2 g_{F_m},
\]

defined by

\[
g = \pi^*(g_B) \oplus (h_1 \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus (h_2 \circ \pi)^2 \sigma_2^*(g_{F_2}) \ldots \oplus (h_m \circ \pi)^2 \sigma_m^*(g_{F_m}),
\]
where for each $i \in \{1, 2, \ldots, m\}$, smooth functions $h_i : B \to (0, \infty)$ are called warping functions, $\pi$ and $\sigma_i$ are the natural projections on base $B$ and fibers $F_i$ respectively.

R. L. Bishop and B. O’Neill in [4], first studied about warped product space for construct the examples of Riemannian manifolds with negative curvature. In [1], [2]), authors expressed the exact solutions of Einstein’s field equation in term of warped products and also studied the geodesic equations for such type of spaces. In connection. In 2000, B. Unal [19] derived covariant derivative formulas for multiply warped product space and Einstein warped product space with quarter symmetric connection. In 2000, J. Choi [10] investigated the curvature of a multiply warped product with warped products and also studied the geodesic equations for such type of spaces. In 2000, J. Choi [10] investigated the curvature of a multiply warped product with $C^0$-warping functions and represented the interior Schwarzschild space–time as a multiply warped product space–time with warping functions. In 2005, F. Dobarro and B. Unal [11], studied Ricci-flat and Einstein-Lorentzian multiply warped products and provided some results on generalized Kanser space–times. In 2016, D. Dumitru [12], calculated warping functions for multiply generalized Robertson-Walker space-time to be an Einstein manifold when all fibers are Ricci flat. In 2017, F. Gholami, F. Darabi and A. Haji-Badali [13] studied multiply warped product metrics and reduced Einstein equation for generalized Friedmann-Robrtson-Walker spacetime.

In 2017, Sousa and Pina [18] studied warped product semi-Riemannian Einstein manifolds under consideration that base is conformal to an $n$-dimensional pseudo-Euclidean space and invariant under the action of an $(n-1)$-dimensional group. More recently, in [5] authors generalized the work of Sousa and Pina (see, [18]) for Quasi-Einstein manifolds.

The purpose of this article is to extend the work of Sousa and Pina (see [18]) for multiply warped space $M = (\mathbb{R}^{n+3}, \tilde{g}) \times h_1 F_1^{m_1} \times h_2 F_2^{m_2} \times \cdots \times h_l F_l^{m_l}$ with metric $\tilde{g} = \tilde{g} + h_1^2 g_{F_1} \cdots + h_l^2 g_{F_l}$, where $\tilde{g} = \frac{1}{\nu} g$, $g$ is pseudo-Euclidean metric on $\mathbb{R}^n$ with coordinates $x = (x_1, \ldots, x_n)$, $g_{ij} = \delta_{ij} \varepsilon_i$ and for each $s \in \{1, 2, \ldots, l\}$, $h_s, \varphi : \mathbb{R}^n \to \mathbb{R}$ are smooth functions, where $h_s$ are positive smooth functions, $F_s^{m_s}$ is a semi-Riemannian manifolds with constant Ricci curvatures $\lambda_{F_s}$, $m_s \geq 1$. We will classify multiply warped product Einstein manifold when the base is locally conformally flat.

We organize the paper as follows: In Theorem 2.1, we compute the necessary and sufficient conditions for the multiply warped product metric $\tilde{g} = \tilde{g} + h_1^2 g_{F_1} \cdots + h_l^2 g_{F_l}$ to be an Einstein. In Theorem 2.2, we apply the concept of Theorem 2.1 by considering $\xi = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i \in \mathbb{R}$ as a basic invariant for an $(n-1)$-dimensional translation group, and $\varphi$, $h_s$ are invariant under the action of an $(n-1)$-dimensional translation group for all $s \in \{1, 2, \ldots, l\}$. We show that these conditions are different depending on the direction of $\alpha = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}$, being null or not. In Theorem 2.3, we prove necessary and sufficient conditions for the multiply warped product $(M, \tilde{g})$ to be a Ricci-flat manifold when fibers $F_s$ are Ricci-flat manifolds and the direction of $\alpha$ is timelike or spacelike. In the last Theorem, we see that if the direction
of \( \alpha \) is null, then there are infinitely many solutions. More explicitly, for any given positive differentiable functions \( h_s \), the function \( \varphi(\xi) \) satisfy a linear ordinary differential equation of second order. We clarify this fact by giving an example. We give explicit solutions for Einstein field equation, in the vacuum case, that are not differential equation of second order. We clarify this fact by giving an example. We give explicit solutions for Einstein field equation, in the vacuum case, that are not locally conformally flat.

2. Main results. In the following theorems, \( \varphi_{x,x,x_j} \) and \( h_{s,x,x_j} \) denote the second order derivatives of \( \varphi \) and \( h_s \) with respect to \( x, x_j \), for all \( s \in \{1, 2, \ldots \} \).

**Theorem 2.1.** Let \((\mathbb{R}^n, g)\) be a semi-Euclidean space, \( n \geq 3 \), with coordinates \( x = (x_1, \ldots, x_n) \) and \( g_{ij} = \delta_{ij} \varepsilon_1 \). Consider a multiply warped product \( M = (\mathbb{R}^n, \bar{g}) \times h_1 F_1^{m_1} \times h_2 F_2^{m_2} \times \cdots \times h_l F_l^{m_l} \) with metric \( \bar{g} = \frac{1}{\varepsilon_2} g \), and for each \( s \in \{1, 2, \ldots \} \), \( F_s^{m_s} \) are semi-Riemannian Einstein manifolds with constant Ricci curvatures \( \lambda_{F_s^{m_s}} \), \( m_s \geq 1 \), and \( h_s, \varphi : \mathbb{R}^n \to \mathbb{R} \) are smooth functions, where \( h_s \) are positive smooth functions. Then the multiply warped product space \((M, \bar{g})\) is an Einstein manifold with constant Ricci curvature \( \lambda \) if and only if for each \( s \in \{1, 2, \ldots \} \), \( h_s \) and \( \varphi \) satisfies:

\[
(n - 2) \varphi_{x,x,x_j} - \sum_{k=1}^{l} \frac{m_k}{h_k} [\varphi_{x,x,x_j} + \varphi_{x,x} h_{k,x} + \varphi_{x,x} h_{k,x}] = 0, \quad i \neq j,
\]

\[
\varphi \left\{ (n - 2) \varphi_{x,x,x} - \sum_{k=1}^{l} \frac{m_k}{h_k} [\varphi_{h_{k,x},x} + 2 \varphi_{x,x} h_{k,x}] \right\} + \varepsilon_i \left\{ \varphi \sum_{r=1}^{n} \varepsilon_r \varphi_{x,x,x} \right\} - (n - 1) \sum_{r=1}^{n} \varepsilon_r \varphi_{x,x} + \varphi \sum_{k=1}^{l} \sum_{r=1}^{n} \varepsilon_r \frac{m_k}{h_k} \varphi_{x,x} h_{k,x} = \lambda \varepsilon_i,
\]

\[
- \varphi^2 h_s \sum_{r=1}^{n} \varepsilon_r h_{s,x,x} + (n - 2) \varphi h_s \sum_{r=1}^{n} \varepsilon_r \varphi_{x,x} h_{s,x} - (m_s - 1) \varphi^2 \sum_{r=1}^{n} \varepsilon_r h_{s,x},
\]

\[
- \varphi^2 h_s \sum_{k=1, k \neq s}^{l} \sum_{r=1}^{n} \varepsilon_r \frac{m_k}{h_k} h_{k,x} h_{s,x} = \lambda h_s^2 - \lambda_{F_s^{m_s}}.
\]

**Proof.** For each \( s \in \{1, 2, \ldots \} \), assume that \( m_s > 1 \). Let \( X_1, X_2, \ldots, X_n \in \mathcal{L}(\mathbb{R}^n) \) and \( Y^s_p \in \mathcal{L}(F_s^{m_s}) \), for each \( s \in \{1, 2, \ldots \} \) and \( p \in \{1, 2, \ldots, m_s \} \) \((\mathcal{L}(\mathbb{R}^n) \) and \( \mathcal{L}(F_s^{m_s}) \) are respectively the lift of vector field on \( \mathbb{R}^n \) and \( F_s^{m_s} \) to \( M \) respectively, for each \( s \in \{1, 2, \ldots \} \)). Then from proposition 2.5 of [11], it follows that

\[
\begin{aligned}
\text{Ric}_g(X_i, X_j) &= \text{Ric}_g(X_i, X_j) - \sum_{k=1}^{l} \frac{m_k}{h_k} H^g_{hk}(X_i, X_j), \quad \forall i, j \in \{1, 2, \ldots, n\}, \\
\text{Ric}_g(X_i, Y^s_p) &= 0, \quad \forall i \in \{1, 2, \ldots, n\}, s \in \{1, 2, \ldots \} \text{ and } p \in \{1, 2, \ldots, m_s \}, \\
\text{Ric}_g(Y^s_p, Y^s_q) &= 0, \quad \forall s \neq t \text{ where } s, t \in \{1, 2, \ldots \}, p \in \{1, 2, \ldots, m_s \} \\
\text{and } q \in \{1, 2, \ldots, m_t \}, \\
\text{Ric}_g(Y^s_p, Y^s_q) &= \text{Ric}_{g_{F_s^{m_s}}}(Y^s_p, Y^s_q) - \left\{ \frac{\Delta h_{s}}{h_{s}} + (m_s - 1) \frac{\bar{g}(\nabla h_{s}, \nabla h_{s})}{h_{s}^2} \right\} \bar{g}(Y^s_p, Y^s_q), \quad \forall s \in \{1, 2, \ldots \} \\
\text{and } p, q \in \{1, 2, \ldots, m_s \}.
\end{aligned}
\]
Note that if \( \bar{g} = \frac{1}{\varphi^2} g \) then from (ex. [3]) we get

\[
Ric_{\bar{g}} = \frac{1}{\varphi^2} [(n - 2)\varphi H^\varphi_{\bar{g}} + \{\varphi \Delta_g \varphi - (n - 1)|\nabla_g \varphi|^2\} g].
\]

As \( g(X_i, X_j) = \delta_{ij}\varepsilon_i \), thus the above equation implies that

\[
Ric_{\bar{g}}(X_i, X_j) = \frac{1}{\varphi^2} \{(n - 2)\varphi H^\varphi_{\bar{g}}(X_i, X_j)\} \quad \forall \, i \neq j \text{ where } i, j \in \{1, 2, \ldots n\}.
\]

\[
Ric_{\bar{g}}(X_i, X_i) = \frac{1}{\varphi^2} \{(n - 2)\varphi H^\varphi_{\bar{g}}(X_i, X_i) + \{\varphi \Delta_g \varphi - (n - 1)|\nabla_g \varphi|^2\} \varepsilon_i\} \quad \forall \, i \in \{1, 2, \ldots n\}.
\]

Since \( H^\varphi_{\bar{g}}(X_i, X_j) = \varphi, x_i x_j, \Delta_g \varphi = \sum_{r=1}^{n} \varepsilon_r \varphi, x_r x_r \) and \( |\nabla_g \varphi|^2 = \sum_{r=1}^{n} \varepsilon_r \varphi^2_{x_r} \), we have

\[
\begin{align*}
Ric_{\bar{g}}(X_i, X_j) &= \frac{(n-2)\varphi, x_i x_j}{\varphi}, \quad \forall \, i \neq j \text{ where } i, j \in \{1, 2, \ldots n\}, \\
Ric_{\bar{g}}(X_i, X_i) &= \frac{(n-2)\varphi, x_i x_i + \varepsilon_i \sum_{r=1}^{n} \varepsilon_r \varphi, x_r x_r}{\varphi} - (n-1)\varepsilon_i \sum_{r=1}^{n} \varepsilon_r \frac{\varphi^2_{x_r}}{\varphi^2}.
\end{align*}
\]

Recall that

\[
H^h_{\bar{g}}(X_i, X_j) = h_{s, x_i x_j} - \sum_{r=1}^{n} \bar{\Gamma}^r_{ij} h_{s, x_r}, \quad \forall \, s \in \{1, 2, \ldots l\},
\]

where \( \bar{\Gamma}^r_{ij} \) are the Christoffel symbols of the metric \( \bar{g} \). For \( i, j, r \) different, we have

\[
\bar{\Gamma}^r_{ij} = 0, \quad \bar{\Gamma}^i_{ij} = \frac{\varphi, x_i}{\varphi}, \quad \bar{\Gamma}^r_{ii} = \varepsilon_i \varepsilon_r \frac{\varphi, x_r}{\varphi}, \quad \bar{\Gamma}^i_{ii} = -\frac{\varphi, x_i}{\varphi}.
\]

Hence \( \forall \, s \in \{1, 2, \ldots l\} \),

\[
\begin{align*}
H^h_{\bar{g}}(X_i, X_j) &= h_{s, x_i x_j} + \frac{\varphi, x_i}{\varphi} h_{s, x_i} + \frac{\varphi, x_j}{\varphi} h_{s, x_j}, \quad \forall \, i \neq j \text{ where } i, j \in \{1, 2, \ldots n\}, \\
H^h_{\bar{g}}(X_i, X_i) &= h_{s, x_i x_i} + 2 \frac{\varphi, x_i}{\varphi} h_{s, x_i} - \varepsilon_i \sum_{r=1}^{n} \varepsilon_r \frac{\varphi, x_r}{\varphi} h_{s, x_r}.
\end{align*}
\]

After using (5) and (6) in the first equation of system (4), we obtain

\[
Ric_{\bar{g}}(X_i, X_j) = (n - 2)\frac{\varphi, x_i x_j}{\varphi} - \sum_{k=1}^{l} \frac{m_k}{h_k} [h_{k, x_i x_j} + \frac{\varphi, x_i}{\varphi} h_{k, x_i} + \frac{\varphi, x_j}{\varphi} h_{k, x_j}], \quad \forall \, i \neq j \tag{7}
\]

and

\[
Ric_{\bar{g}}(X_i, X_i) = \frac{(n-2)\varphi, x_i x_i + \varepsilon_i \sum_{r=1}^{n} \varepsilon_r \varphi, x_r x_r}{\varphi} - (n-1)\varepsilon_i \sum_{r=1}^{n} \varepsilon_r \frac{\varphi^2_{x_r}}{\varphi^2} - \sum_{k=1}^{l} \frac{m_k}{h_k} \{h_{k, x_i x_i} + 2 \frac{\varphi, x_i}{\varphi} h_{k, x_i} - \varepsilon_i \sum_{r=1}^{n} \varepsilon_r \frac{\varphi, x_r}{\varphi} h_{k, x_r}\}. \tag{8}
\]
Next,
\[
\begin{aligned}
\{ \text{Ric}_{\tilde{g}^{F_{m s}}}(Y^s_p, Y^s_q) &= \lambda_{F_{m s}} g_{F_{m s}}(Y^s_p, Y^s_q), \\
\tilde{g}(Y^s_p, Y^s_q) &= h^2 g_{F_{m s}}(Y^s_p, Y^s_q), \\
\Delta_{\tilde{g}} h_s &= \varphi^2 \sum_{r=1}^n \varepsilon_r h_{s, x_r x_r} - (n-2) \varphi \sum_{r=1}^n \varepsilon_r \varphi_{x_r} h_{s, x_r}, \\
\tilde{g}(\nabla h_s, \nabla h_s) &= \varphi^2 \sum_{r=1}^n \varepsilon_r h_{s, x_r}^2. 
\end{aligned}
\tag{9}
\]

Substituting (9) in the fourth equation of system (4), we have
\[
\text{Ric}_{\tilde{g}}(Y^s_p, Y^s_q) = \gamma_{pq} g_{F_{m s}}(Y^s_p, Y^s_q),
\tag{10}
\]
where
\[
\gamma_{pq} = \lambda_{F_{m s}} - \varphi^2 h_s \sum_{r=1}^n \varepsilon_r h_{s, x_r x_r} + (n-2) \varphi h_s \sum_{r=1}^n \varepsilon_r \varphi_{x_r} h_{s, x_r} - (m_s - 1) \varphi^2 h_s \sum_{k=1}^l \sum_{r=1}^n \varepsilon_r \frac{m_k}{h_k} h_{k, x_r} h_{s, x_r},
\]
\forall \ p, q \in \{1, 2, \ldots, m_s\}. Applying the equations (7), (8), (10) and the second, third equations of (4), implies that \((M, \tilde{g})\) is an Einstein manifold if and only if the equations (1), (2), and (3) are satisfied. Hence prove the theorem.

In the next theorem, we will convert the equations (1), (2), and (3) in the form of \(\varphi(\xi)\) and \(h_s(\xi)\), where \(s \in \{1, 2, \ldots, l\}\) and \(\xi = \sum_{i=1}^n \alpha_i x_i, \alpha_i \in \mathbb{R}\), whenever \(\sum_{i=1}^n \varepsilon_i \alpha_i^2 \neq 0\), without loss of generality we may assume that \(\sum_{i=1}^n \varepsilon_i \alpha_i^2 = \pm 1\).

**Theorem 2.2.** Let \((\mathbb{R}^n, g)\) be a semi-Euclidean space, \(n \geq 3\), with coordinates \(x = (x_1, \ldots, x_n)\) and \(g_{ij} = \delta_{ij} \varepsilon_i\). Consider a multiply warped product \(M = (\mathbb{R}^n, \tilde{g}) \times_{h_1} F_{m_1}^{F_{m_2}} \times_{h_2} F_{m_2}^{F_{m_3}} \times_{h_3} F_{m_3}^{F_{m_4}} \ldots \times_{h_l} F_{m_l}^{F_{m_l}}\), with metric \(\tilde{g} = \tilde{g} \oplus h_1^2 g_{F_{m_1}} \oplus h_2^2 g_{F_{m_2}} \oplus \ldots \oplus h_l^2 g_{F_{m_l}}, \) where \(\tilde{g} = \varphi^2 g\) and for each \(s \in \{1, 2, \ldots, l\}\), \(F_{m_s}\) are semi-Riemannian Einstein manifolds with constant Ricci curvatures \(\lambda_{F_{m_s}}\), and smooth functions \(\varphi(\xi)\) and \(h_s(\xi)\), where

\(\xi = \sum_{i=1}^n \alpha_i x_i, \alpha_i \in \mathbb{R}\) and \(\sum_{i=1}^n \varepsilon_i \alpha_i^2 = \varepsilon_{i_0} \) or \(\sum_{i=1}^n \varepsilon_i \alpha_i^2 = 0\). Then the multiply warped product space \((M, \tilde{g})\) is an Einstein manifold with constant Ricci curvature \(\lambda\) if and only if for each \(s \in \{1, 2, \ldots, l\}\), \(h_s\) and \(\varphi\) satisfies:

\(\xi\)
\[
\begin{aligned}
(n-2) \varphi'' - \sum_{k=1}^l \frac{m_k}{h_k} [\varphi h_k' + 2 \varphi' h_k''] = 0, \\
\sum_{r=1}^n \varepsilon_r \alpha_r^2 \left\{ \varphi \varphi' - (n-1) \varphi \varphi'' + \sum_{k=1}^l \frac{m_k}{h_k} \varphi \varphi' h_k' \right\} = \lambda, \\
\sum_{r=1}^n \varepsilon_r \alpha_r^2 \left\{ - \varphi^2 h_s h_s'' + (n-2) \varphi \varphi' h_s h_s' - (m_s - 1) \varphi^2 h_s'^2 \\
- \sum_{k=1}^l \frac{m_k}{h_k} \varphi^2 h_s h_s' h_k' \right\} = \lambda h_s^2 - \lambda_{F_{m_s}},
\end{aligned}
\tag{11}
\]
wherever \( \sum_{i=1}^{n} \varepsilon_i \alpha_i^2 \neq 0 \), and

\[(ii) \quad (n-2)\varphi'' - \sum_{k=1}^{l} \frac{m_k}{h_k}[\varphi h_k'' + 2\varphi' h_k'] = 0, \quad \text{and} \quad \lambda = \lambda_{F_{x\alpha}} = 0, \quad (12)\]

whenever \( \sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = 0 \).

\[\text{Proof.} \quad \text{We are assuming that for all } s \in \{1, 2, \ldots, l\}, \varphi(s) \text{ and } h_s(s), \text{ are functions of } \xi, \text{ where } \xi = \sum_{i=1}^{n} \alpha_i x_i, \alpha_i \in \mathbb{R} \text{ and } \sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = \varepsilon_{i_0} \text{ or } \sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = 0. \]

Thus, we have

\[\varphi_{x_i} = \varphi' \alpha_i, \quad \varphi_{x_i x_j} = \varphi'' \alpha_i \alpha_j \]

and

\[h_{s x_i} = h'_s \alpha_i, \quad h_{s x_i x_j} = h''_s \alpha_i \alpha_j. \]

After using the above expressions in (1), we have

\[(n-2)\varphi'' - \sum_{k=1}^{l} \frac{m_k}{h_k}[\varphi h_k'' + 2\varphi' h_k'] = 0, \quad \forall i \neq j. \quad (13)\]

If there exist \( i \neq j \) such that \( \alpha_i \alpha_j \neq 0 \) then this equation implies that

\[(n-2)\varphi'' - \sum_{k=1}^{l} \frac{m_k}{h_k}[\varphi h_k'' + 2\varphi' h_k'] = 0. \quad (13)\]

Similarly equation (2), implies that

\[\varphi \alpha_i^2 \{ (n-2)\varphi'' - \sum_{k=1}^{l} \frac{m_k}{h_k}[\varphi h_k'' + 2\varphi' h_k'] \} + \varepsilon_i \left[ \sum_{r=1}^{n} \varepsilon_r \alpha_r^2 \{ \varphi \varphi' - (n-1)\varphi'^2 + \sum_{k=1}^{l} \frac{m_k}{h_k} \varphi \varphi' h_k' \} \right] = \varepsilon_i \lambda. \]

Substituting equation (13), the above equation reduces to

\[\sum_{r=1}^{n} \varepsilon_r \alpha_r^2 \{ \varphi \varphi' - (n-1)\varphi'^2 + \sum_{k=1}^{l} \frac{m_k}{h_k} \varphi \varphi' h_k' \} = \lambda. \quad (14)\]

Similarly, the equation (3) reduces to

\[\sum_{r=1}^{n} \varepsilon_r \alpha_r^2 \{ \varphi \varphi' - (n-2)\varphi h_s h_s' - (m_s - 1)\varphi'^2 h_s'^2 \]

\[ - \sum_{k=1, k \neq s}^{l} \frac{m_k}{h_k} \varphi^2 h_s h_s' h_k' \} = \lambda h_s^2 - \lambda_{F^{m_s}}. \quad (15)\]

Thus if we consider the case \( \sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = \varepsilon_{i_0} \) then equations (13), (14) and (15), provided the system of equations (11). On the other hand using \( \sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = 0 \), in
In this case, since the equations (13), (14) and (15), we get (12). Now, if $\alpha_i \alpha_j = 0$ for all $i \neq j$, then $\xi = x_{i_0}$ the equation (1) trivially satisfied $\forall$ $i \neq j$. Now for the other equations we will consider two cases:

* $i \neq i_0$

In this case, since $\alpha_i = 0$ $\forall$ $i \neq i_0$ the equation (2), implies

$$
\varepsilon_{i_0} \alpha_{i_0}^2 \left\{ \varphi \varphi' - (n - 1) \varphi'^2 + \sum_{k=1}^{l} \frac{m_k}{h_k} \varphi \varphi' h_k' \right\} = \lambda
$$

(16)

and equation (3) will convert into equation (17)

$$
\varepsilon_{i_0} \alpha_{i_0}^2 \left\{ - \varphi^2 h_s h_s'' + (n - 2) \varphi \varphi' h_s h_s' - (m_s - 1) \varphi^2 h_s'^2 - \sum_{k=1,k \neq s}^{l} \frac{m_k}{h_k} \varphi^2 h_s h_s' h_k' \right\} = \lambda h_s^2 - \lambda F_s^{m_s}.
$$

(17)

* $i = i_0$

In this case, since $\alpha_i = 0$ $\forall$ $i \neq i_0$ the equation (2) provides

$$
\alpha_{i_0}^2 \varphi \left\{ (n - 2) \varphi'' - \sum_{k=1}^{l} \frac{m_k}{h_k} \varphi h_k'' + 2 \varphi' h_k' \right\} + \varepsilon_{i_0} \alpha_{i_0}^2 \left\{ \varphi \varphi' - (n - 1) \varphi'^2 + \sum_{k=1}^{l} \frac{m_k}{h_k} \varphi \varphi' h_k' \right\} = \lambda \varepsilon_{i_0}
$$

(18)

and equation (3) again will convert into equation (17). Hence from (16) and (18) we get the first equation of (11). This conclude the proof of the theorem.

**Theorem 2.3.** Let $(\mathbb{R}^n, g)$ be a semi-Euclidean space, $n \geq 3$, with coordinates $x = (x_1, ..., x_n)$ and $g_{ij} = \delta_{ij} \varepsilon_i$. Consider a multiply warped product $M = (\mathbb{R}^n, \bar{g}) \times_{h_1} F_1^{m_1} \times_{h_2} F_2^{m_2} \times_{h_l} F_l^{m_l}$ with metric $\bar{g} = \bar{g} \oplus h_1^2 g_{F_1} \oplus h_2^2 g_{F_2} \oplus \cdots \oplus h_l^2 g_{F_l}$, where $\bar{g} = \frac{1}{\varphi_0^2} g$. Consider for each $s \in \{1, 2, ..., l\}$, non-constant smooth functions $\varphi(\xi)$ and $h_s(\xi)$, where $\xi = \sum_{i=1}^{n} \alpha_i x_i, \alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = \varepsilon_{i_0}$ and $F_s$ are Ricci-flat manifolds with $m_s \geq 2$. Then the multiply warped product space $(M, \bar{g})$ is a Ricci-flat manifold if and only if for each $s \in \{1, 2, ..., l\}$, $h_s$ and $\varphi$ satisfies:

$$
\begin{align*}
(n - 2) \frac{\varphi''}{\varphi} + (n - 2) \frac{\varphi'}{\varphi}^2 &- \sum_{k=1}^{l} m_k \left( \frac{h_k'}{h_k} \right)' = 0, \\
-2 &\frac{\varphi'}{\varphi} \sum_{k=1}^{l} m_k \frac{h_k'}{h_k} = \lambda \\
(\varphi')' &- (n - 2) \frac{\varphi'}{\varphi} + \frac{\varphi'}{\varphi} \sum_{k=1}^{l} m_k \frac{h_k'}{h_k} = 0, \\
\left( \frac{h_k'}{h_k} \right)' &- (n - 2) \frac{\varphi'}{\varphi} h_s' + \frac{\varphi'}{h_s} \sum_{k=1}^{l} m_k \frac{h_k'}{h_k} = 0.
\end{align*}
$$

(19)

**Proof.** Given that $\varphi(\xi)$ and $h_s(\xi)$ are smooth functions for each $s \in \{1, 2, ..., l\}$ and where $\xi = \sum_{i=1}^{n} \alpha_i x_i, \alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = \pm 1$. Theorem (2.3) implies that $(M, \bar{g})$
is Ricci-flat if and only for each \( s \in \{1, 2, \ldots, l\} \), smooth functions \( \varphi(\xi) \) and \( h_s(\xi) \), satisfy

\[
\begin{cases}
(n - 2)\varphi'' - \sum_{k=1}^{l} \frac{m_k}{h_k} [\varphi h_k'' + 2\varphi'h_k'] = 0, \\
\varphi\varphi' - (n - 1)\varphi^2 + \sum_{k=1}^{l} \frac{m_k}{h_k} \varphi\varphi'h_k' = 0, \\
- \varphi^2 h_s h_s'' + (n - 2)\varphi\varphi'h_s h_s' - (m_s - 1)\varphi^2 h_s'^2 \\
- \sum_{k=1, k \neq s}^{l} \frac{m_k}{h_k} \varphi^2 h_s h_s'h_k' = 0.
\end{cases}
\] (20)

Note that,

\[
\frac{\varphi''}{\varphi} = (\frac{\varphi'}{\varphi})' + (\frac{\varphi'}{\varphi})^2
\] (21)

and

\[
\frac{h_k''}{h_k} = (\frac{h_k'}{h_k})' + (\frac{h_k'}{h_k})^2.
\] (22)

Dividing on both sides by \( \varphi \sum_{k=1}^{l} h_k \) in the first equation of (20), we have

\[
(n - 2)\frac{\varphi''}{\varphi} - \sum_{k=1}^{l} \frac{m_k}{h_k} \frac{h_k''}{h_k} - 2\frac{\varphi'}{\varphi} \sum_{k=1}^{l} \frac{m_k}{h_k} \frac{h_k'}{h_k} = 0.
\]

After using equations (21) and (22) in this equation, we get the first equation of (20). Now dividing on both sides by \( \varphi^2 \sum_{k=1}^{l} h_k \) in the second equation of (20) we have

\[
\frac{\varphi''}{\varphi} - (n - 1)(\frac{\varphi'}{\varphi})^2 + \frac{\varphi'}{\varphi} \sum_{k=1}^{l} \frac{m_k}{h_k} \frac{h_k'}{h_k} = 0.
\]

Using (21) in the above equation, we get the second equation of (20). Next, dividing on both sides by \( \varphi^2 h_s \sum_{k=1}^{l} h_k \) in the third equation of (20), we have

\[
- \frac{h_s''}{h_s} + (n - 2)\frac{\varphi'}{\varphi} \frac{h_s'}{h_s} - (m_s - 1)(\frac{h_s'}{h_s})^2 - \frac{h_s'}{h_s} \sum_{k=1, k \neq s}^{l} \frac{m_k}{h_k} \frac{h_k'}{h_k} = 0.
\]

Using (22) in the above equation, we get the third equation of (20). Hence prove the theorem.

\( \square \)

**Remark 1.** If we take \( M = (\mathbb{R}^n, \bar{g}) \times_f \mathbb{R} \) and \( M = (\mathbb{R}^n, \bar{g}) \times_f F^m \) then the Theorem 2.3 will convert into Theorem 1.3 and Theorem 1.4 respectively of [18]. Both Theorems provide the explicit solutions of the functions.

The next theorem shows that there are infinitely many multiply warped product \( M = (\mathbb{R}^n, \bar{g}) \times_{h_1} F_{1}^{m_1} \times_{h_2} F_{2}^{m_2} \times_{h_3} F_{3}^{m_3} \) Ricci-flat, which are invariant under the action of an \((n - 1)\)-dimensional group acting on \( \mathbb{R}^n \), when \( \alpha = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i} \) is a null like vector.
Theorem 2.4. Let for each \( s \in \{1, 2, \ldots, l\} \), \( h_s \) are positive differentiable functions, where \( \xi = \sum_{i=1}^{n} \alpha_i x_i \), and \( \sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = 0 \). Then there exists a function \( \varphi(\xi) \) satisfy (12) and \( M = (\mathbb{R}^n, \bar{g}) \times h_1 F_1^{m_1} \times h_2 F_2^{m_2} \times \cdots \times h_l F_l^{m_l} \) is a Ricci-flat manifold.

First, we present one example illustrating Theorem 2.4, after that we will prove the Theorem 2.4. Let for each \( s \in \{1, 2, \ldots, l\} \), \( h_s(\xi) = B_s e^{A_s \xi} \) where \( B_s > 0 \) and \( A_s \neq 0 \), \( \xi = \sum_{i=1}^{n} \alpha_i x_i \), and \( \sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = 0 \). In this case, the equation (12) will convert into the form

\[
(n - 2) \varphi - 2 \varphi' \sum_{k=1}^{l} m_k A_k - \varphi \sum_{k=1}^{l} m_k A_k^2 = 0.
\]

Since this is second order linear ordinary differential equation in \( \varphi \) therefore after solving, we have

\[
\varphi(\xi) = c_1 e^{\left( \frac{A + \sqrt{A^2 + B(n - 2)}}{(n - 2)} \right) \xi} + c_2 e^{\left( \frac{-A + \sqrt{A^2 + B(n - 2)}}{(n - 2)} \right) \xi},
\]

where \( c_1, c_2 \in \mathbb{R} \) and \( A = \sum_{k=1}^{l} m_k A_k, B = \sum_{k=1}^{l} m_k A_k^2 \).

Thus from Theorem 2.4, we conclude that \( M = (\mathbb{R}^n, \bar{g}) \times h_1 F_1^{m_1} \times h_2 F_2^{m_2} \times \cdots \times h_l F_l^{m_l} \) is a Ricci-flat manifold. Considering \( c_1 \) and \( c_2 \) are positive constants then we have that \( \varphi \) is globally defined on \( \mathbb{R}^n \).

**Proof.** Assume that for each \( s \in \{1, 2, \ldots, l\} \), \( h_s \) are positive differentiable functions invariant under the action of an \((n-1)\)-dimensional group, whose basic invariant \( \xi = \sum_{i=1}^{n} \alpha_i x_i \), where \( \alpha_i \in \mathbb{R} \) and \( \sum_{i=1}^{n} \varepsilon_i \alpha_i^2 = 0 \). Then it follows from Theorem 2.2 that \( M = (\mathbb{R}^n, \bar{g}) \times h_1 F_1^{m_1} \times h_2 F_2^{m_2} \times \cdots \times h_l F_l^{m_l} \) is a Ricci-flat manifold, with \( h_s \) being warping functions if and only if for each \( s \in \{1, 2, \ldots, l\} \), \( \lambda = \lambda_{h_s} = 0 \) and \( \varphi(\xi) \) satisfy the second order linear ordinary differential equation (12) determined by \( h_s \).

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