GENERALIZATIONS
OF RAD-SUPPLEMENTED MODULES

Engin Kaynar, Ergül Türkmen, and Yıldız Aydın

Abstract. Let $R$ be an associative ring with identity. We introduce the notion of semi-$\tau$-supplemented modules, which is adapted from srs-modules, for a preradical $\tau$ on $R$-$\text{Mod}$. We provide basic properties of these modules. In particular, we study the objects of $R$-$\text{Mod}$ for $\tau = \text{Rad}$. We show that the class of semi-$\tau$-supplemented modules is closed under finite sums and factor modules. We prove that, for an idempotent preradical $\tau$ on $R$-$\text{Mod}$, a module $M$ is semi-$\tau$-supplemented if and only if it is $\tau$-supplemented. For $\tau = \text{Rad}$, over a local ring every left module is semi-Rad-supplemented. We also prove that a commutative semilocal ring whose semi-Rad-supplemented modules are a direct sum of $w$-local left modules is an artinian principal ideal ring.

1. Introduction

Throughout this study, $R$ will be an associative ring with identity and all modules are unitary left $R$-modules, unless otherwise specified. Let $M$ be such a module over the ring $R$. By $R$-$\text{Mod}$ we denote the category of left $R$-modules. The notation $N \subseteq M$ means that $N$ is a submodule of $M$. A functor $\tau: R$-$\text{Mod} \to R$-$\text{Mod}$ is said to be a preradical if $\tau(M) \subseteq M$ for every $M \in R$-$\text{Mod}$ and for every homomorphism $f: M \to N$ in $R$-$\text{Mod}$, we have $f(\tau(M)) \subseteq \tau(N)$. A preradical $\tau$ is called radical if $\tau(M/\tau(M)) = 0$ for every left $R$-module $M$. A module $M$ is called $\tau$-torsion (respectively, $\tau$-torsion free) if $\tau(M) = M$ (respectively, $\tau(M) = 0$).

A nonzero submodule $N$ of a module $M$ is called essential, written by $N \triangleright M$, if $N \cap K \neq 0$ for every nonzero submodule $K$ of $M$. Dually, a proper submodule $S$ of $M$ is called small, denoted by $S \ll M$, if $S + K = M$ implies that $K = M$, where $K$ is a submodule of $M$. By $\text{Rad}(M)$ we will denote the Jacobson radical for a module $M$. If $M = \text{Rad}(M)$, then it is called radical. A nonzero module $M$ is said to be hollow if every proper submodule is small in $M$, and it is said to be local if it is hollow and finitely generated. $M$ is local if and only if it is finitely generated and $\text{Rad}(M)$ is maximal (see [6 2.12 §2.15]).

2010 Mathematics Subject Classification: Primary 16D10; Secondary 16N80.

Key words and phrases: preradical, Jacobson radical, Rad-supplement, $\tau$-supplement.

Communicated by Zoran Petrović.
For two submodules $N$ and $K$ of a module $M$, $K$ is said to be supplement of $N$ in $M$ (or $N$ is said to have a supplement $K$) if $M = N + K$ and $N \cap K \ll K$. $M$ is called supplemented if every submodule of $M$ has a supplement in $M$. Since every direct summand of a module has a supplement, supplemented modules are a proper generalization of semisimple modules. Hollow modules are supplemented.

Al-Takhman, Lomp and Wisbauer [1] generalize supplemented modules to $\tau$-supplemented modules for a preradical $\tau$ for $R$-Mod. A module $M$ is called $\tau$-supplemented if every submodule $N$ of $M$ has a $\tau$-supplement $K$ in $M$, that is, $M = N + K$ and $N \cap K \subseteq \tau(K)$ where $\tau$ is a preradical for $R$-Mod. Instead of a preradical $\tau$ for $R$-Mod, we can use the radical $\text{Rad}$ on $R$-Mod. A module $M$ is called Rad-supplemented if every submodule $N$ of $M$ has a Rad-supplement $K$ in $M$. Since the Jacobson radical of any module is the sum of all small submodules, every supplement submodule is Rad-supplement, and so supplemented modules are Rad-supplemented. Also, a noetherian Rad-supplemented module is supplemented.

For the properties and characterizations of (Rad-) supplemented modules and in general $\tau$-supplemented modules we refer to [1, 4, 10].

In [11], Zöschinger studied on modules whose Jacobson radical have a supplement and termed these modules radical supplemented. He determined the structure of these modules over local Dedekind domains. Büyükaşk and Türkmen called a module $M$ strongly radical supplemented (for shortly srs) if every submodule $N$ of $M$ with $\text{Rad}(M) \subseteq N$ have a supplement $K$ in $M$ (see [5]). They gave the various properties of srs-modules in the same paper. In particular, it was shown in [5, Proposition 2.3] that every finite sum of srs-modules is srs. By [5, Proposition 3.3], over a local Dedekind domain a module is radical supplemented if and only if it is srs.

In this paper, we introduce the notion of semi-$\tau$-supplemented modules, which is adapted from srs-modules, for a preradical $\tau$ on $R$-Mod. We provide basic properties of these modules. In particular, we study on the objects of $R$-Mod for $\tau = \text{Rad}$. We show that the class of semi-$\tau$-supplemented modules is closed under finite sums and factor modules. We prove that, for an idempotent preradical $\tau$ on $R$-Mod, a module $M$ is semi-$\tau$-supplemented if and only if it is $\tau$-supplemented. Let $\tau = \text{Rad}$. Any direct sum of $w$-local modules is semi-Rad-supplemented. It follows that over a local ring every left module is semi-Rad-supplemented. We give some counterexamples to separate classes of semi-Rad-supplemented modules, Rad-supplemented modules and srs-modules (see Example 3.3). We have that the following proper implications on modules hold:

$$\text{supplemented} \supseteq \text{srs} \supseteq \text{Rad-supplemented} \supseteq \text{semi-Rad-supplemented}$$

We also prove that a commutative semilocal ring whose semi-Rad-supplemented modules are a direct sum of $w$-local left modules is an artinian principal ideal ring.
2. Semi-$\tau$-Supplemented Modules

Let $\tau$ be a preradical on $R$-$\text{Mod}$. We call a module $M$ semi-$\tau$-supplemented if every submodule $N$ of $M$ with $\tau(M) \subseteq N$ has a $\tau$-supplement in $M$. By definitions, every $\tau$-supplemented module is semi-$\tau$-supplemented. In this section, we obtain the various properties of semi-$\tau$-supplemented modules. We prove that, for an idempotent preradical $\tau$ on $R$-$\text{Mod}$, a module $M$ is semi-$\tau$-supplemented if and only if it is $\tau$-supplemented.

Recall from [2] that a module $M$ is $\tau$-local if it is $\tau$-torsion or $\tau(M)$ is maximal.

**Lemma 2.1.** Every $\tau$-local module is semi-$\tau$-supplemented.

**Proof.** Let $M$ be a $\tau$-local module. If $M$ is $\tau$-torsion, it is clear. Suppose that $\tau(M)$ is the maximal submodule of $M$. Then $M = \tau(M) + M$ and $\tau(M) \cap M \subseteq \tau(M)$. Thus it is semi-$\tau$-supplemented. □

**Corollary 2.1.** Let $M$ be a module and $N$ be a maximal submodule of $M$. Then every $\tau$-supplement of $N$ is semi-$\tau$-supplemented.

**Proof.** It follows from [2] Lemma 2.2] and Lemma 2.1. □

Now we show that the finite sum of semi-$\tau$-supplemented modules is semi-$\tau$-supplemented. For this fact, we use the standard lemma (see, [1] 2.3(1)).

**Lemma 2.2.** Let $M$ be an $R$-module and $M_1, U$ be submodules of $M$ such that $M_1$ is semi-$\tau$-supplemented, $\tau(M) \subseteq U$ and $M_1 + U$ has a $\tau$-supplement $V$ in $M$. Then, $M_1 \cap (U + V)$ has a $\tau$-supplement $L$ in $M_1$ and $V + L$ is a $\tau$-supplement of $U$ in $M$.

**Theorem 2.1.** Let $M_1$ and $M_2$ be semi-$\tau$-supplemented modules. If $M = M_1 + M_2$, then $M$ is semi-$\tau$-supplemented.

**Proof.** Let $\tau(M) \subseteq U \subseteq M$. Since $M = M_1 + M_2$, $M_1 + (M_2 + U)$ has the trivial $\tau$-supplement 0 in $M$. So by Lemma 2.2, $M_2 + U$ has a $\tau$-supplement in $M$. Again applying Lemma 2.2 we obtain a $\tau$-supplement for $U$ in $M$. Hence $M$ is semi-$\tau$-supplemented. □

**Corollary 2.2.** A finite direct sum of semi-$\tau$-supplemented modules is semi-$\tau$-supplemented.

A module $M$ is said to be a duo module if every submodule $N$ of $M$ is fully invariant [8]. Now we prove that arbitrary direct sums of semi-$\tau$-supplemented modules are semi-$\tau$-supplemented, under a certain condition: namely, when $M$ is a duo module. The proof of the next result is the same as [7] Theorem 1).

**Theorem 2.2.** Let $M_i$ $(i \in I)$ be any collection of semi-$\tau$-supplemented modules in $R$-$\text{Mod}$ and $M = \bigoplus_{i \in I} M_i$. If $M$ is a duo module, then it is a semi-$\tau$-supplemented module.

**Proposition 2.1.** If $M$ is a semi-$\tau$-supplemented module, then every factor module of $M$ is semi-$\tau$-supplemented.
Proof. For any submodule $N$ of $M$, let $U/N \subseteq M/N$ with $\tau(M/N) \subseteq U/N$. Since $(\tau(M) + N)/N \subseteq \tau(M/N)$, we can write $\tau(M) \subseteq U$. By the hypothesis, $U$ has a $\tau$-supplement $V$ in $M$, that is, $M = U + V$ and $U \cap V \subseteq \tau(V)$ for some submodule $V$ of $M$. So $M/N = U/N + (V + N)/N$. Therefore,

$$U/N \cap (V + N)/N = [U \cap (V + N)]/N = (U \cap V + N)/N \subseteq (\tau(V) + N)/N \subseteq \tau(V + N)/N$$

and so $(V + N)/N$ is a $\tau$-supplement of $U/N$ in $M/N$. Hence $M/N$ is semi-$\tau$-supplemented. □

Recall that a module $M$ is weakly supplemented if every submodule $N$ of $M$ has a weak supplement $K$ in $M$, that is, $M = N + K$ and $N \cap K \ll M$ [6, 17.8].

Lemma 2.3. Let $M$ be a semi-$\tau$-supplemented module. Suppose that $\tau(M)$ is a small submodule of $M$. Then $M$ is $\tau$-supplemented. In particular, $M$ is weakly supplemented.

Proof. Let $U$ be any submodule of $M$. Then, $\tau(M) \subseteq \tau(M) + U$. It follows from the hypothesis that $\tau(M) + U$ has a $\tau$-supplement $V$ in $M$. So $M = (\tau(M) + U) + V$ and $(\tau(M) + U) \cap V \subseteq \tau(V)$. Since $\tau(M) \ll M$, we get $M = U + V$. Therefore $U \cap V \subseteq (\tau(M) + U) \cap V \subseteq \tau(V)$, we obtain that $U \cap V \subseteq \tau(V)$. Hence $V$ is a $\tau$-supplement of $U$ in $M$.

Since $U \cap V \subseteq \tau(V) \subseteq \tau(M)$, it follows from [10, 19.3(4)] that $U \cap V$ is small in $M$. Hence $V$ is a weak supplement of $U$ in $M$. This means that $M$ is a weakly supplemented module. □

Corollary 2.3. Let $M$ be a $\tau$-torsion free module. Then the following statements are equivalent:

1. $M$ is (semi) $\tau$-supplemented.
2. $M$ is semisimple.

Proof. Clearly, we have the implications (2) $\Rightarrow$ (1): since $M$ is $\tau$-torsion free, $\tau(M) = 0$. It follows from Lemma 2.3 that $M$ is (semi)-$\tau$-supplemented.

(1) $\Rightarrow$ (2) is obvious. □

Corollary 2.4. Let $M$ be a semi-$\tau$-supplemented module. Suppose that $\tau$ is radical. Then $M/\tau(M)$ is semisimple and $\text{Rad}(M) \subseteq \tau(M)$.

Proof. By Proposition 2.4 we get that $M/\tau(M)$ is semi-$\tau$-supplemented. Since $\tau$ is radical, $M/\tau(M)$ is a $\tau$-torsion free module. Hence $M/\tau(M)$ is semisimple by Corollary 2.3. It follows that $\text{Rad}(M/\tau(M)) = 0$. Thus $\text{Rad}(M) \subseteq \tau(M)$. □

Proposition 2.2. Let $M$ be a semi-$\tau$-supplemented module. If $\tau(M)$ is $\tau$-supplemented, then $M$ is $\tau$-supplemented.

Proof. Let $U \subseteq M$. By the hypothesis, $\tau(M) + U$ has a $\tau$-supplement in $M$. Since $\tau(M)$ is $\tau$-supplemented, by Lemma 2.2 $U$ has a $\tau$-supplement in $M$. □
A preradical $\tau$ is said to be idempotent if $\tau(\tau(M)) = \tau(M)$ for every left $R$-module $M$. For an example of an idempotent preradical on $R$-$\mathcal{M}$, we consider an idempotent ideal $I$ of a ring $R$ and put $\tau^I(M) = IM$ each $M \in R$-$\mathcal{M}$. Then, $\tau^I$ is an idempotent preradical for $R$-$\mathcal{M}$.

**Corollary 2.5.** Let $\tau$ be an idempotent preradical on $R$-$\mathcal{M}$. Then an $R$-$\mathcal{M}$ module $M$ is semi-$\tau$-supplemented if and only if it is $\tau$-supplemented.

**Proof.** We only need to show that $M$ is semi-$\tau$-supplemented, then it is $\tau$-supplemented.

Let $N \subseteq M$. Let us look at the submodule $N + \tau(M)$. By the assumption, there exists a submodule $K$ such that

$$(N + \tau(M)) + K = M \quad \text{and} \quad (N + \tau(M)) \cap K \subseteq \tau(K).$$

Let $K_1 = \tau(M) + K$. We have $N + K_1 = M$ and we only need to prove that $N \cap K_1 \subseteq \tau(K_1)$, or, more explicitly, that $N \cap (\tau(M) + K) \subseteq \tau(\tau(M) + K)$. So, let $x \in N \cap (\tau(M) + K)$. This means that $x \in N$ and there exist elements $m' \in \tau(M)$ and $k \in K$ such that $x = m' + k$. From this, we get that $k = x - m'$. Since $x \in N$ and $m' \in \tau(M)$, we get $k \in (N + \tau(M)) \cap K$. Since $(N + \tau(M)) \cap K \subseteq \tau(K)$, it follows that $k \in \tau(K)$. So, $x \in \tau(M) + \tau(K) = \tau(\tau(M)) + \tau(K) \subseteq \tau(\tau(M) + K)$, which concludes our proof. The last inclusion follows from the fact that $\tau(A) + \tau(B) \subseteq \tau(A + B)$, since $\tau$ is a preradical.

By $P_\tau(M)$ we denote the sum of all $\tau$-torsion submodules of an $R$-module $M$. It is clear that $P_\tau(M)$ is the largest $\tau$-torsion submodule of $M$. Note that $P_\tau(M) \subset \tau(M)$ and $P_\tau$ is an idempotent preradical for $R$-$\mathcal{M}$, whenever $\tau$ is a radical on $R$-$\mathcal{M}$.

**Theorem 2.3.** Let $M$ be a module. Suppose that $\tau$ is a radical on $R$-$\mathcal{M}$. Then it is semi-$\tau$-supplemented if and only if $M/P_\tau(M)$ is semi-$\tau$-supplemented.

**Proof.** Let $M$ be a semi-$\tau$-supplemented module. It follows from Proposition 2.3 that $M/P_\tau(M)$ is semi-$\tau$-supplemented as a factor module of $M$. Conversely, suppose that $U$ is any submodule of $M$ with $\tau(M) \subset U$. Then $P_\tau(M) \subset U$. By properties of a radical, we have $\tau(M/P_\tau(M)) = \tau(M)/P_\tau(M) \subset U/P_\tau(M)$. Since $M/P_\tau(M)$ is a semi-$\tau$-supplemented module, $U/P_\tau(M)$ has a $\tau$-supplement, say $V/P_\tau(M)$, in $M/P_\tau(M)$. So

$$M/P_\tau(M) = U/P_\tau(M) + V/P_\tau(M),$$

$$U/P_\tau(M) \cap V/P_\tau(M) \subseteq \tau(V/P_\tau(M)).$$

Therefore, $M = U + V$. Note that

$$(U \cap V)/P_\tau(M) = (U/P_\tau(M)) \cap (V/P_\tau(M)) \subseteq \tau(V/P_\tau(M)) = \tau(V)/P_\tau(M)$$

and this implies $U \cap V \subseteq \tau(V)$. Consequently, $M$ is semi-$\tau$-supplemented.

Let $M$ be an $R$-module. $M$ is said to be $\tau$-reduced if $P_\tau(M) = 0$. If $\tau$ is radical, by [4] Theorem 3.1 (vii), then $P_\tau(M/P_\tau(M)) = 0$ and so $M/P_\tau(M)$ is $\tau$-reduced. Using Theorem 2.3 we obtain the following fact.
Corollary 2.6. Let $R$ be a ring and $\tau$ be a radical on $R$-$\text{Mod}$. The following statements are equivalent:

(1) Every left $R$-module is semi-$\tau$-supplemented.
(2) Every left $\tau$-reduced $R$-module is semi-$\tau$-supplemented.

3. Semi-Rad-Supplemented Modules

In this section, we shall consider $\tau = \text{Rad}$. Recall that a module $M$ is semi-Rad-supplemented if every submodule $N$ of $M$ with $\text{Rad}(M) \subseteq N$ has a Rad-supplement $K$ in $M$, that is, $M = N + K$ and $N \cap K \subseteq \text{Rad}(K)$. It is clear that Rad-supplemented modules and srs-modules are semi-Rad-supplemented. For modules with zero Jacobson radical the notions of semi-Rad-supplemented, Rad-supplemented and being srs-module coincide by Corollary 2.3. In general, semi-Rad-supplemented modules need not be Rad-supplemented and srs. Later we shall give an example of such modules (see Example 3.3).

Recall that a module $M$ is semilocal if $M/\text{Rad}(M)$ is semisimple. A ring $R$ is called semilocal if $R_R$ (or $R_R$) is a semilocal module. It is known that a commutative ring $R$ is semilocal if $R$ has only finitely many maximal ideals. Since the preradical $\text{Rad}$ is a radical on $R$-$\text{Mod}$, we obtain the following fact by Corollary 2.4.

Corollary 3.1. Semi-Rad-supplemented modules are semilocal.

Proof. It follows from Corollary 2.4.

In the following example, we show that semilocal modules need not be semi-Rad-supplemented, in general. Firstly, we need this simple lemma.

Lemma 3.1. Finitely generated semi-Rad-supplemented modules are Rad-supplemented.

Proof. Let $M$ be a finitely generated module. Then $\text{Rad}(M)$ is a small submodule of $M$. If $M$ is semi-Rad-supplemented, then it is Rad-supplemented by Lemma 2.3.

Example 3.1. Consider the localization ring $\mathbb{Z}_{(2,3)}$ containing all rational numbers of the form $\frac{a}{b}$ with $2 \nmid b$ and $3 \nmid b$ for prime integers 2, 3 in $\mathbb{Z}$. Let $M$ be the left $\mathbb{Z}_{(3,3)}$-module $\mathbb{Z}_{(2,3)}$. Then $M$ is a semilocal noetherian module, but not Rad-supplemented. By Lemma 3.1 it is not semi-Rad-supplemented.

As a proper generalization of local modules, one calls a module $M$ $w$-local if $\text{Rad}(M)$ is a maximal submodule of $M$. If $M$ is semi-Rad-supplemented, then it is Rad-supplemented by Lemma 2.3.

Example 3.1. Consider the localization ring $\mathbb{Z}_{(2,3)}$ containing all rational numbers of the form $\frac{a}{b}$ with $2 \nmid b$ and $3 \nmid b$ for prime integers 2, 3 in $\mathbb{Z}$. Let $M$ be the left $\mathbb{Z}_{(3,3)}$-module $\mathbb{Z}_{(2,3)}$. Then $M$ is a semilocal noetherian module, but not Rad-supplemented. By Lemma 3.1 it is not semi-Rad-supplemented.

As a proper generalization of local modules, one calls a module $M$ $w$-local if $\text{Rad}(M)$ is a maximal submodule of $M$ as in [3].

Proposition 3.1. Every $w$-local module is semi-Rad-supplemented.

Proof. Let $M$ be any $w$-local module and $\text{Rad}(M) \subseteq U \subseteq M$. Since $M$ is $w$-local, we have $U = \text{Rad}(M)$. Then, $M = U + M$ and $U \cap M \subseteq \text{Rad}(M)$ and so $M$ is a Rad-supplement of $U$ in $M$. Hence $M$ is semi-Rad-supplemented.

Theorem 3.1. Let $M$ be any direct sum of $w$-local modules. Then it is semi-Rad-supplemented.
Proof. Let $M = \bigoplus_{i \in I} M_i$ and each $M_i$ be $w$-local. Let $\text{Rad}(M) \subseteq U \subseteq M$. For $i \in I$, we have that $(M_i + \text{Rad}(M))/\text{Rad}(M) \cong M_i/\text{Rad}(M_i)$ is simple because $M_i$ is $w$-local. Note that $M/\text{Rad}(M) = \bigoplus_{i \in I} (M_i + \text{Rad}(M))/\text{Rad}(M)$. So $M/\text{Rad}(M)$ is semisimple by [6], 2.8(5). It follows that 

$$M/\text{Rad}(M) = U/\text{Rad}(M) \oplus \left( \bigoplus_{i \in J} (M_i + \text{Rad}(M))/\text{Rad}(M) \right)$$

for some $J \subseteq I$ by [10], 20.1. Let $V = \bigoplus_{i \in J} M_i$. Therefore, $M = U + V$ and $U \cap V \subseteq \text{Rad}(M)$. Since $V$ is a direct summand of $M$, $\text{Rad}(V) = V \cap \text{Rad}(M)$ and so $U \cap V \subseteq \text{Rad}(V)$. Hence $U$ has a Rad-supplement in $M$ as required. □

In the next Theorem, we characterize commutative semilocal rings in terms of semi-Rad-supplemented modules. A ring $R$ is called a left max ring if every nonzero left $R$-module has a maximal submodule, and it is called left perfect if $R$ is semilocal and a left max ring. Note that over a left max ring every nonzero left module has a small Jacobson radical. Left $V$-rings (i.e., every left simple module is injective) are left max rings.

Lemma 3.2. Let $R$ be a left max ring and $M$ be a module over this ring. Then the following statements are equivalent.

1. $M$ is semi-Rad-supplemented.
2. $M$ is Rad-supplemented.
3. $M$ is supplemented.
4. $M$ is an srs-module.

Proof. (1) ⇒ (2). By Lemma 2.3

(2) ⇒ (3). It follows from [6], 20.7(3)].

(3) ⇒ (4) and (4) ⇒ (1) are clear. □

Theorem 3.2. Let $R$ be a ring whose semi-Rad-supplemented modules are the direct sum of $w$-local $R$-modules. Then $R$ is a left max ring and every semi-Rad-supplemented $R$-module is supplemented. If $R$ is a commutative semilocal ring, then $R$ is an artinian principal ideal ring.

Proof. Since radical modules are semi-Rad-supplemented, it is enough to prove that $R$ has no radical modules. Let $N = \text{Rad}(N)$ be an $R$-module. By the assumption, we can write $N = \bigoplus_{i \in I} N_i$, where each $N_i$ is a $w$-local $R$-module. It follows that $N_i = \text{Rad}(N_i)$. Therefore, $N_i = 0$ for every $i \in I$. Thus $N = 0$. This means that $R$ is a left max ring. Applying Lemma 3.2, every semi-Rad-supplemented $R$-module is supplemented.

Let $R$ be a commutative semilocal ring. Then, $R$ is perfect. Let $M$ be any $R$-module. By [10], 43.9], $M$ is supplemented. Since $w$-local modules over left max rings are local, $M$ is the direct sum of cyclic submodules. Hence $R$ is an artinian principal ideal ring by [9], Theorem 6.7]. □

Let $n > 1$ be a positive integer. Then the ring $\mathbb{Z}_n$ is a commutative semilocal ring which satisfies the above theorem.

A ring $R$ is called local if $R$ has a unique left maximal ideal.
Theorem 3.3. Let $R$ be a local ring. Then every left $R$-module is semi-Rad-supplemented.

Proof. Let $M$ be any left $R$-module. Then there exists an epimorphism $\Psi : R^I \to M$, where $I$ is an index set. Since $R$ is local, by Theorem 3.1, the free left $R$-module $R^I$ is semi-Rad-supplemented. Hence $M$ is semi-Rad-supplemented as a factor module of $R^I$ by Proposition 2.1.

The following example shows that the converse of Theorem 3.3 is not true.

Example 3.2. Let $R$ be the factor ring $\mathbb{Z}/6\mathbb{Z}$ of the ring $\mathbb{Z}$. Therefore, $R$ is an artinian principal ideal ring. Let $M$ be any left $R$-module. Then $M$ is a direct sum of local $R$-modules. It follows from Theorem 3.3 that it is semi-Rad-supplemented. However, $R$ is not a local ring because $R$ has two maximal ideals.

Now we give examples of a module, which is semi-Rad-supplemented but not Rad-supplemented.

Example 3.3. (1) For a prime integer $p \in \mathbb{Z}$, given $\mathbb{Z}_{(p)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0, p \nmid n\}$. Then $\mathbb{Z}_{(p)}$ is a local Dedekind domain. Let $F$ be the direct sum $\mathbb{Z}_{(p)}^N$ of countably many copies of $\mathbb{Z}_{(p)}$. By Theorem 3.3, we get that $F$ is semi-Rad-supplemented. On the other hand, $F$ is not Rad-supplemented according to [4, Theorem 7.1(i)].

(2) Let $p$ be a prime in $\mathbb{Z}$ and consider the left $\mathbb{Z}$-module $M = \bigoplus_{i \geq 1} \mathbb{Z}_{p^i}$ which is the sum of local $\mathbb{Z}$-modules $\mathbb{Z}_{p^i}$. Since local modules are $w$-local, we obtain that $M$ is a semi-Rad-supplemented module by Theorem 3.1. Suppose that $M$ is Rad-supplemented. Note that $M$ is reduced. By [4, Theorem 4.6 and Proposition 3.5], we get that Rad$(M)$ is small in $M$. This is a contradiction. Consequently, $M$ is not Rad-supplemented.

The Example 3.3 (2) also shows that the class of semi-Rad-supplemented modules contains properly the class of srs-modules by [5, Example 2.2].

In [3], $M$ is called cofinitely Rad-supplemented if every submodule $N$ of $M$ with $M/N$ which is finitely generated has a Rad-supplement in $M$. It was shown in [3, Theorem 3.7] that a module $M$ is cofinitely Rad-supplemented if and only if every maximal submodule has a Rad-supplement in $M$. Using the characterization we get the result:

Corollary 3.2. Semi-Rad-supplemented modules are cofinitely Rad-supplemented.

Proof. Let $M$ be a semi-Rad-supplemented module. Therefore, every maximal submodule of $M$ has a Rad-supplement in $M$. It follows from [3, Theorem 3.7] that it is cofinitely Rad-supplemented.

The following example shows that a cofinitely Rad-supplemented module need not be semi-Rad-supplemented. Let $R$ be a Dedekind domain and $M$ be an $R$-module. We denote by $T(M)$ the set of all elements $m$ of $M$ for which there exists a nonzero element $r$ of $R$ such that $rm = 0$, i.e., Ann$(m) \neq 0$. Then $T(M)$, which is a submodule of $M$, is called the torsion submodule of $M$. 

Example 3.4. Consider the left \( \mathbb{Z} \)-module \( M = \prod_{p \in \Omega} \mathbb{Z}/p\mathbb{Z} \), where \( \Omega \) is an infinite collection of distinct prime elements of \( \mathbb{Z} \). Then the torsion submodule \( T(M) \) of \( M \) is the submodule \( \bigoplus_{p \in \Omega} \mathbb{Z}/p\mathbb{Z} \) of \( M \). Therefore, \( \text{Rad}(M) = 0 \) and there exists a submodule \( N \) of \( M \) such that \( N/T(M) \cong \mathbb{Q} \). Since \( \mathbb{Q} \) is injective and \( \mathbb{Z} \) is a Dedekind domain, we get \( \text{Rad}(\mathbb{Q}) = \mathbb{Q} \). So any maximal submodule of \( N \) does not contain \( T(M) \). Thus every maximal submodule of \( N \) is a direct summand. This means that every maximal submodule of \( N \) has a Rad-supplement in \( N \). By [3, Theorem 3.7], \( N \) is cofinitely Rad-supplemented.

If \( N \) is Rad-supplemented, then it is semisimple according to Corollary 2.3. Hence \( \mathbb{Q} \) is semisimple as a factor module of \( N \), a contradiction.

We prove an analogue of [5, Proposition 2.14] in the following.

Proposition 3.2. Let \( M \) be an \( R \)-module. Suppose that \( M/\text{Rad}(M) \) is finitely generated. If \( M \) is cofinitely Rad-supplemented, then it is semi-Rad-supplemented.

Proof. Let \( \text{Rad}(M) \subseteq U \subseteq M \). Note that
\[
(M/\text{Rad}(M))/(U/\text{Rad}(M)) \cong M/U
\]
is finitely generated. Since \( M \) is cofinitely Rad-supplemented, \( U \) has a Rad-supplement in \( M \). Hence \( M \) is a semi-Rad-supplemented module. \( \square \)

Corollary 3.3. The following statements are equivalent for a finitely generated module \( M \).

1. \( M \) is Rad-supplemented.
2. \( M \) is semi-Rad-supplemented.
3. \( M \) is cofinitely Rad-supplemented.

Proof. (1) \( \Rightarrow \) (3) is clear.
(3) \( \Rightarrow \) (2) By Proposition 3.2
(2) \( \Rightarrow \) (1) It follows from Lemma 2.3 \( \square \)

Acknowledgement. The authors sincerely thank Engin Büyükaşık for his interest and helpful suggestions. Supported by Scientific Research Project in Amasya University FMB-BAP 15-107.

References
1. K. Al-Takman, C. Lomp, R. Wisbauer, \( \tau \)-complemented and \( \tau \)-supplemented modules, Algebra Discr. Math. 3 (2006), 1–16.
2. E. Büyükaşık, Modules whose maximal submodules have \( \tau \)-supplements, Algebra Discr. Math. 10(2) (2010), 1–9.
3. E. Büyükaşık, C. Lomp, On a recent generalization of semiperfect rings, Bull. Aust. Math. Soc. 78(2) (2008), 317–325.
4. E. Büyükaşık, E. Mermut, S. Özdemir, Rad-supplemented modules, Rend. Semin. Mat. Univ. Padova 124 (2010), 157–177.
5. E. Büyükaşık, E. Türkmen, Strongly radical supplemented modules, Ukr. Math. J. 106 (2011), 25–30.
6. J. Clark, C. Lomp, N. Vanaja, R. Wisbauer, Lifting modules. Supplements and projectivity in module theory, Frontiers in Mathematics, Birkhäuser, Basel, 2006, 406.
7. E. Türkmen, A. Pancar, *Some properties of Rad-supplemented modules*, Internat. J. Phys. Sci. 6(35) (2011), 7904–7909.
8. A. Ç. Özcan, A. Harmaci, P. F. Smith, *Duo modules*, Glasgow Math. J. 48 (2006), 533–545.
9. D.W. Sharpe, P. Vamos, *Injective Modules*, Cambridge University Press, Cambridge, 1972.
10. R. Wisbauer, *Foundations of Modules and Rings*, Gordon and Breach, 1991.
11. H. Zöschinger, *Module, die in jeder Erweiterung ein Komplement haben*, Math. Scand. 35 (1974), 267–287.

Vocational School of Technical Sciences
Amasya University
Amasya
Turkey
engin.kaynar@amasya.edu.tr

Department of Mathematics
Amasya University
Amasya
Turkey
ergulturkmen@hotmail.com

Department of Mathematics
Ondokuz Mayis University
Samsun
Turkey
yildizaydin60@hotmail.com