WELL-POSEDNESS AND EXPONENTIAL STABILITY OF A THERMOELASTIC SYSTEM WITH INTERNAL DELAY

SMAIN MOULAY KHATIR AND FARHAT SHEL

Abstract. The presence of a delay in a thermoelastic system destroys the well-posedness and the stabilizing effect of the heat conduction [17]. To avoid this problem we add to the system, at the delayed equation, a Kelvin-Voigt damping. At first, we prove the well-posedness of the system by the semigroup theory. Next, under appropriate assumptions, we prove the exponential stability of the system by introducing a suitable Lyapunov functional.

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1. Introduction

Let us consider the following thermoelastic system with delay

\[
\begin{align*}
u_{tt}(x,t) - \alpha u_{xx}(x,t - \tau) + \gamma \theta_x(x,t) &= 0, & \text{in } (0,\ell) \times (0,\infty), \\
\theta_t(x,t) - \kappa \theta_{xx}(x,t) + \gamma u_{xt}(x,t) &= 0, & \text{in } (0,\ell) \times (0,\infty), \\
u(0,t) = u(\ell,t) = \theta_x(0,t) = \theta_x(\ell,t) &= 0, & t \geq 0
\end{align*}
\]
(1.1)

where \(\alpha, \gamma, \kappa\) and \(\ell\) are some positive constants. The functions \(u = u(x,t)\) and \(\theta = \theta(x,t)\) describe respectively the displacement and the temperature difference, with \(x \in (0,\ell)\) and \(t \geq 0\). Moreover, \(\tau > 0\) is the time delay. Racke proved in [17] that, under some initial and boundary conditions, the system (1.1) is not well posed and unstable even if \(\tau\) is relatively small. However, it is well known that, in the absence of delay, the damping through the heat conduction is strong

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obtained an exponential decay result under the assumption \( \tau \rightarrow 0 \) in \([12]\), the authors dropped the time delay in the harmonic term of the elastic equation in (1.1) where the initial data (where a “prime” denotes a one-dimensional derivative with respect to “t” and where \( \tau \).

This idea arises from \([2]\) where the authors added a Kelvin-Voigt damping term to the abstract equation, a Kelvin-Voigt damping of the form \(-\beta u_{xx}(x, t)\) for some real positive number \( \beta \), which eventually depends on \( \alpha, \gamma, \kappa \) and \( \tau \). Then our system takes the form

\[
\begin{align*}
      & u_{tt}(x, t) - \alpha u_{xx}(x, t - \tau) - \beta u_{xxt}(x, t) + \gamma \theta_x(x, t) = 0, & \text{in } \Omega \times (0, \infty), \\
      & \theta_t(x, t) - \kappa \theta_{xx}(x, t) + \gamma u_{xt}(x, t) = 0, & \text{in } \Omega \times (0, \infty), \\
      & u(0, t) = u(\ell, t) = 0, & \text{in } (0, \infty), \\
      & \theta_x(0, t) = \theta_x(\ell, t) = 0, & \text{in } (0, \infty), \\
      & u_x(x, t - \tau) = f_0(x, t - \tau), & \text{in } \Omega \times (0, \tau), \\
      & u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \theta(x, 0) = \theta_0(x), & \text{in } \Omega.
\end{align*}
\]

where the initial data \((u_0, u_1, f_0, \theta_0)\) belongs to a suitable space and with \( \Omega = (0, \ell) \). We mainly investigate well-posedness and exponential stability of such initial-boundary value problem.

In order to solve the problem, additional conditions or control terms have been used, we refer to \([7, 5, 13, 4, 1, 14]\), see also \([11]\) and references therein. In this paper we add to the delayed equation, a Kelvin-Voigt damping of the form \(-\beta u_{xx}(x, t)\) for some real positive number \( \beta \), which eventually depends on \( \alpha, \gamma, \kappa \) and \( \tau \). Then our system takes the form

\[
\begin{align*}
      & u''(t) + aBB^*u'(t) + BB^*u(t - \tau) = 0, & \text{in } (0, \infty), \\
      & u(0) = u_0, \quad u'(0) = u_1, & \text{in } (0, \tau), \\
      & B^*u(t - \tau) = f_0(t - \tau), & \text{in } (0, \tau), \\
\end{align*}
\]

where a ”prime” denotes a one-dimensional derivative with respect to ”t” and where \( B : \mathcal{D}(B) \subset H_1 \rightarrow H \) is a linear unbounded operator from a Hilbert space \( H_1 \) to a Hilbert space \( H \), such that \( B^* \), the adjoint of \( B \), satisfies some properties of coercivity and compact embedding. They obtained an exponential decay result under the assumption \( \tau \leq a \).

In \([12]\), the authors dropped the time delay in the harmonic term of the elastic equation in (1.1) and added a delay term of the form \( \int_{\tau_1}^{\tau_2} \mu(s) \theta_{xx}(x, t - s\tau)ds \) in the heat equation, where \( \tau_1 \) and \( \tau_2 \) are non-negative constants such that \( \tau_1 < \tau_2 \) and \( \mu : [\tau_1, \tau_2] \rightarrow \mathbb{R} \) is a bounded function. They proved an exponential decay result under the condition \( \int_{\tau_1}^{\tau_2} |\mu(s)|ds < \kappa \).

We define the energy of a solution of problem (1.2) as

\[
E(t) := \frac{1}{2} \int_{\Omega} (u_t^2(x, t) + \alpha u_{xx}^2(x, t) + \theta^2(x, t)) \, dx + \xi \int_{\Omega} \int_0^1 u_x^2(x, t - \tau \rho) \, d\rho \, dx.
\]
where $\xi > 0$ is a parameter fixed later on.

The paper is organized as follow. In section 2 we first formulate the problem into an appropriate Hilbert space, and then we study the well-posedness of the system using semigroup theory. In section 3, we prove, using Lyapunov’s method, a result of exponential stability of system (1.2).

2. Well-posedness of the problem

We introduce, as in [2], the new variable

\begin{equation}
\tag{2.1}
z(x, \rho, t) = u_x(x, t - \tau \rho), \quad \text{in } \Omega \times (0, 1) \times (0, \infty),
\end{equation}

Clearly, $z(x, \rho, t)$ satisfies

\begin{align*}
\tag{2.2}
\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) &= 0, \quad x \in \Omega, \ \rho \in (0, 1), \ t \in (0, +\infty), \\
\tag{2.3}
z(x, 0, t) &= u_x(x, t), \quad x \in \Omega, \ t \in (0, +\infty).
\end{align*}

Then, problem (1.2) takes the form

\begin{align*}
\tag{2.4}
&u_{tt}(x, t) - \alpha z_x(x, 1, t) - \beta u_{xxt}(x, t) + \gamma \theta_x(x, t) = 0, \quad \text{in } \Omega \times (0, \infty), \\
\tag{2.5}
&\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \quad \text{in } \Omega \times (0, 1) \times (0, +\infty), \\
\tag{2.6}
&\theta_t(x, t) - \kappa \theta_{xx}(x, t) + \gamma u_{x}(x, t) = 0, \quad \text{in } \Omega \times (0, \infty), \\
\tag{2.7}
u(0, t) = u(\ell, t) = 0, \quad \text{in } (0, \infty), \\
\tag{2.8}
&\theta_x(0, t) = \theta_x(\ell, t) = 0, \quad \text{in } (0, \infty), \\
\tag{2.9}
z(x, 0, t) = u_x(x, t), \quad \text{in } \Omega \times (0, \infty), \\
\tag{2.10}
u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \ \theta(x, 0) = \theta_0(x), \quad \text{in } \Omega, \\
\tag{2.11}
z(x, \rho, 0) = f_0(x, -\tau \rho), \quad \text{in } \Omega \times (0, 1).
\end{align*}

Observe that it follows from (2.6)-(2.8) that $\int_{\Omega} \theta_t(x, t)dx = 0$ that is, $\int_{\Omega} \theta(x, t)dx$ is conservative all the time. Without loss of generality, we assume that $\int_{\Omega} \theta(x, t)dx = 0$. Otherwise, we can make the substitution $\tilde{\theta}(x, t) = \theta(x, t) - \frac{1}{T} \int_{\Omega} \theta_0(x)dx$, in fact $(u, v, z, \theta)$ and $(u, v, z, \tilde{\theta})$ satisfy the same system (2.4)-(2.11).

Let

$$
\mathcal{H} = \left\{(f, g, p, h) \in H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0, 1)) \mid \int_{\Omega} h(x)dx = 0\right\}.
$$

Equipped with the following inner product: for any $U_k = (f_k, g_k, p_k, h_k) \in \mathcal{H}, \ k = 1, 2$,

$$
\langle U_1, U_2 \rangle_{\mathcal{H}} = \int_{\Omega} (\alpha f_{1x}(x)f_{2x}(x) + g_1(x)g_2(x) + h_1(x)h_2(x))\,dx + \xi \int_{\Omega} \int_{0}^{1} p_1(x, \rho)p_2(x, \rho)d\rho\,dx,
$$

$\mathcal{H}$ is a Hilbert space.
Define

\[ U := (u, u_t, z, \theta) \]

then, problem (1.2) can be formulated as a first order system of the form

\[ \begin{cases} 
U' = AU, \\
U(0) = (u_0, u_1, f_0(-\cdot, -\cdot), \theta_0)
\end{cases} \tag{2.12} \]

where the operator \( A \) is defined by

\[
\begin{pmatrix}
u \\
z \\
\theta
\end{pmatrix} = 
\begin{pmatrix}
(\alpha z(., 1) + \beta v_x) - \gamma \theta_x \\
-\frac{1}{\tau} z_p \\
-\gamma v_x + \kappa \theta_{xx}
\end{pmatrix}
\]

with domain \( D(A) = \{ U = (u, v, z, \theta) \in \mathcal{H} \cap \left[ H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega; H^1(0, 1)) \times H^2(\Omega) \right] \mid z(., 0) = u_x \text{ and } (\alpha z(., 1) + \beta v_x) \in H^1(\Omega) \} \)

in the Hilbert space \( \mathcal{H} \).

For to establish the existence of solution, we will prove that the operator \( A \) generates a \( C_0 \)-semigroup, and to do this, we will prove that \( A - mId \) generates \( C_0 \)-semigroup (of contractions), for an appropriate real number \( m \), function of \( \xi, \alpha, \beta \) and \( \tau \). Then we apply the bounded perturbation theorem (Sect. III.1 of [8]). In fact, we begin by the following result

**Lemma 2.1.** If \( \xi > \frac{2\alpha\beta^2}{\beta} \), then there exists \( m \in \mathbb{R} \) such that \( A - mId \) is dissipative maximal.

**Proof.** Take \( U = (u, v, z, h) \in D(A) \).

\[
\langle AU, U \rangle_{\mathcal{H}} = \alpha \int_{\Omega} v_x(x) u_x(x) dx + \int_{\Omega} ((\alpha z(., 1) + \beta v_x) - \gamma \theta_x) v(x) dx \\
+ \int_{\Omega} (-\gamma v_x + \kappa \theta_{xx}) (x) \theta(x) dx - \frac{\xi}{\tau} \int_{\Omega} \int_{0}^{1} z_p(x, \rho) z(x, \rho) d\rho dx.
\tag{2.13}
\]

Integrating by parts, using boundary conditions of \( u, v \) and \( \theta \) to get

\[
\int_{\Omega} ((\alpha z(., 1) + \beta v_x) - \gamma \theta_x) v(x) dx + \int_{\Omega} (-\gamma v_x + \kappa \theta_{xx}) (x) \theta(x) dx \\
= -\alpha \int_{\Omega} z(x, 1) v(x) dx - \beta \int_{\Omega} v_x^2(x) dx - \kappa \int_{\Omega} \theta_x^2(x) dx.
\]

Integrating by parts in \( \rho \), we get

\[
\int_{\Omega} \int_{0}^{1} z_p(x, \rho) z(x, \rho) d\rho dx = \frac{1}{2} \int_{\Omega} (z^2(x, 1) - z^2(x, 0)) dx.
\]

Then (2.13) become

\[
\langle AU, U \rangle_{\mathcal{H}} = \alpha \int_{\Omega} v_x(x) u_x(x) dx - \alpha \int_{\Omega} z(x, 1) v_x(x) dx \\
- \beta \int_{\Omega} v_x^2(x) dx - \kappa \int_{\Omega} \theta_x^2(x) dx - \frac{\xi}{2\tau} \int_{\Omega} (z^2(x, 1) - z^2(x, 0)) dx,
\]
from which follows, using the Young’s inequality and that \( z(x, 0) = u_x(x) \),
\[
\langle AU, U \rangle_H \leq (\alpha \varepsilon - \beta) \int \Omega \nu^2(x) dx + \left( \frac{\alpha}{2\varepsilon} - \frac{\xi}{2\tau} \right) \int \Omega z^2(x, 1) dx + \left( \frac{\alpha}{2\varepsilon} + \frac{\xi}{2\tau} \right) \int \Omega u^2(x) dx
\]
\[
- \kappa \int \Omega \theta^2_x(x) dx.
\]
Choosing \( \alpha \varepsilon = \frac{\beta}{2} \), or equivalently, \( \varepsilon = \frac{\beta}{2\alpha} \), we get
\[
\langle AU, U \rangle_H \leq -\frac{\beta}{2} \int \Omega \nu^2(x) dx + \left( \frac{\alpha^2}{\beta} - \frac{\xi}{2\tau} \right) \int \Omega z^2(x, 1) dx + \left( \frac{\alpha^2}{\beta} + \frac{\xi}{2\tau} \right) \int \Omega u^2(x) dx - \kappa \int \Omega \theta^2_x(x) dx.
\]
Then we choose \( \xi > 0 \) such that
\[
\frac{\alpha^2}{\beta} - \frac{\xi}{2\tau} < 0,
\]
that is, \( \xi > \frac{2\alpha^2}{\beta} \). Furthermore we take
\[
m = \frac{\alpha^2}{\beta} + \frac{\xi}{2\tau} > 2\alpha^2 \beta,
\]
to get
\[
\langle (A - mI) U, U \rangle_H \leq -\frac{\beta}{2} \int \Omega \nu^2(x) dx + \left( \frac{\alpha^2}{\beta} - \frac{\xi}{2\tau} \right) \int \Omega z^2(x, 1) dx - \kappa \int \Omega \theta^2_x(x) dx \leq 0
\]
which means that the operator \( A - mI \) is dissipative.

Now, we will prove the maximality of \( A - mI \). It suffices to show that \( \lambda I - A \) is surjective for a fixed \( \lambda > m \). Given \( (f, g, p, h) \in H \), we look for \( U = (u, v, z, \theta) \in D(A) \), solution of
\[
\begin{pmatrix}
u \\
v \\
z \\
\theta
\end{pmatrix}
= \begin{pmatrix} f \\ g \\ p \\ h \end{pmatrix},
\]
that is verifying
\[
\begin{cases}
\lambda u - v = f, \\
\lambda v - (\alpha z(., 1) + \beta v_x)_x + \gamma \theta_x = g, \\
\lambda z - \frac{1}{\tau} z_\rho = p, \\
\lambda \theta + \gamma v_x - \kappa \theta_{xx} = h.
\end{cases}
\]
(2.14)
Suppose that we have found \( u \) with the appropriate regularity. Then,
\[
v = \lambda u - f.
\]
(2.15)
To determine \( z \), recall that \( z(., 0) = u_x \), then, by (2.14), we obtain
\[
z(., \rho) = e^{-\lambda \tau \rho} u_x + \tau e^{-\lambda \tau \rho} \int_0^\rho p(s) e^{\lambda \tau s} ds,
\]
(2.16)
and, in particular
\[
z(x, 1) = e^{-\lambda \tau} u_x + z_0,
\]
(2.17)
with \( z_0 \in L^2(\Omega) \) defined by
\[
z_0 = \tau e^{-\lambda \tau} \int_0^1 p(s) e^{\lambda \tau s} ds.
\]
Now, Multiplying (2.14)₂ and (2.14)₄ respectively by \( w \in H₀¹(Ω) \) and \( φ \in H²(Ω) \) such that \( φ_x(0) = φ_x(ℓ) = 0 \), we obtain after some integrations by parts taking into account boundary conditions on \( v, θ \) and \( w \),

\[
λ \int_Ω vwdx + \int_Ω (αz(.,1) + βv_x) w_x dx + γ \int_Ω θ_x wdx = \int_Ω gwdx
\]

and

\[
λ \int_Ω θφdx - γ \int_Ω vφ_x dx + κ \int_Ω θ_xφ_x dx = \int_Ω hφdx.
\]

Substituting (2.15) and (2.17) into (2.18) and (2.19), we get

\[
λ^2 \int_Ω uwdx + (αe^{-λτ} + λβ) \int_Ω u_xw_x dx + γ \int_Ω θ_xwdx = \int_Ω (g + λf)wdx + \int_Ω (f_x - αz_0)w_x dx
\]

and

\[
λ \int_Ω θφdx - λγ \int_Ω uφ_x dx + κ \int_Ω θ_xφ_x dx = \int_Ω (h - γf)φdx.
\]

Summing (2.20), and (2.21) multiplied by \( \frac{1}{λ} \), we get

\[
b((u, θ), (w, φ)) = F(w, φ)
\]

with

\[
b((u, θ), (w, φ)) = \int_Ω [λ²uw + (αe^{-λτ} + λβ) u_xw_x] dx + \int_Ω \left( θφ + \frac{κ}{λ} θ_xφ_x \right) dx + γ \int_Ω (θ_xw - wφ_x) dx
\]

and

\[
F(w, φ) = \int_Ω (g + λf)wdx + \int_Ω (f_x - αz_0)w_x dx + \frac{1}{λ} \int_Ω (h - γf)φdx.
\]

The space

\[\mathcal{F} := \{(w, φ) ∈ H₀¹(Ω) × H²(Ω) \mid θ_x(0) = θ_x(ℓ) = 0\}\]

equipped with the inner product

\[\langle (w₁, φ₁), (w₂, φ₂) \rangle_\mathcal{F} = \int_Ω (w₁w₂ + w₁xw₂x + φ₁φ₂ + φ₁xφ₂x) dx,\]

is a Hilbert space; the bilinear form \( b \) on \( \mathcal{F} \times \mathcal{F} \) and the linear form \( F \) on \( \mathcal{F} \) are continuous. Moreover, for every \((w, φ) ∈ \mathcal{F}\),

\[|b((w, φ), (w, φ))| ≥ c||φ||_H²\]

with \( c := \min(λ², (αe^{-λτ} + λβ), 1, \frac{λ}{κ}) > 0 \).

By the Lax-Milgram lemma, equation (2.22) has a unique solution \((u, θ) ∈ \mathcal{F}\). Immediately, from (2.16), we have that \( v \in H₀¹(Ω) \). Now, if we consider \((w, φ) ∈ \{0\} × D(Ω)\) in (2.22) we deduce that equation (2.14)₄ holds true. The function \( z \), defined by (2.17), belongs to \( L¹(Ω, H¹(0,1)) \) and satisfies (2.14)₃ and \( z(.,0) = u_x \).

The functions \( z(.,1) \) and \( u_x \) belong to \( L²(Ω) \), then we take \((w, φ) ∈ D(Ω) × \{0\}\) in (2.22) to deduce that \( αz(.,1) + βv_x \) belongs to \( H₀¹(Ω) \) and that equation (2.14)₁ holds true.
Let \( \tilde{\theta} = \theta - \frac{1}{t} \int_0^t \theta_0 \, dx \), then we have that \( U = (u, v, z, \tilde{\theta}) \) belongs to \( \mathcal{D}(A) \), and \( \mathcal{A}U = (f, g, p, h) \).
Thus, \( \lambda I - A \) is surjective for every \( \lambda > 0 \).

In conclusion the operator \( A - mI \) generates a \( C_0 \)-semigroup of contraction. By the bounded perturbation theorem (Sect. III.1 of [8]), we have

**Lemma 2.2.** The operator \( A \) generates a \( C_0 \)-semigroup on \( \mathcal{H} \).

Finally, the well-posedness result follows from semigroup theory.

**Theorem 2.3.** For any initial datum \( U_0 \in \mathcal{H} \) there exists a unique solution \( U \in C([0, +\infty), \mathcal{H}) \) of problem (2.12). Moreover, if \( U_0 \in \mathcal{D}(A) \), then \( U \in C([0, +\infty), \mathcal{D}(A)) \cap C^1([0, +\infty), \mathcal{H}) \).

### 3. Exponential stability

Based on Lyapunov method, we prove that the system (1.2) is exponentially stable for some \( \beta > 0 \). More precisely:

**Theorem 3.1.** There exists \( \beta_0 > 0 \) such that for every \( \beta \geq \beta_0 \), the system (1.2) is exponentially stable.

**Proof.** We take as Lyapunov function
\[
V(t) := N_1 V_1(t) + \alpha N_2 V_2(t) + N_3 V_3(t) + N_4 V_4(t) + N_5 V_5(t) + N_6 V_6(t)
\]
where
\[
\begin{align*}
V_1(t) & := \frac{1}{2} \| u_t \|^2 = \frac{1}{2} \int_\Omega u_t^2 \, dx, \\
V_2(t) & := \frac{1}{2} \| u_x \|^2 = \frac{1}{2} \int_\Omega u_x^2 \, dx, \\
V_3(t) & := \frac{1}{2} \| \theta \|^2 = \frac{1}{2} \int_\Omega \theta^2 \, dx, \\
V_4(t) & := \int_0^t e^{-2\lambda \rho} \| z(\cdot, \rho, t) \|^2 \, d\rho = \int_0^t e^{-2\lambda \rho} \int_\Omega z^2(x, \rho, t) \, dx \, d\rho, \\
V_5(t) & := -\int_0^t e^{-\lambda \rho} f(\rho) \langle z(\cdot, \rho, t), u_x \rangle \, d\rho = -\int_0^t e^{-\lambda \rho} f(\rho) \int_\Omega z(x, \rho, t) u_x(x, t) \, dx \, d\rho \\
V_6(t) & := \langle u, u_t \rangle = \int_\Omega uu_t \, dx.
\end{align*}
\]

\( f \) is a real function defined on \([0, 1]\) and that will be determined later. The constants \( N_1, N_2, N_3, N_4, N_5 \) and \( N_6 \) are positive numbers to be fixed later too.

Denote by \( \tilde{V}(t) \) the energy defined by
\[
\tilde{V}(t) := N_1 V_1(t) + \alpha N_2 V_2(t) + N_5 V_5(t) + N_4 V_4(t).
\]

It is clear that \( \tilde{V}(t) \) is equivalent to \( E(t) \). Then for a suitable choice of \( f \) we will prove that we can find \( \{N_1, ..., N_6\} \) and \( \beta > 0 \) such that the following two assumptions are satisfied:

(A1) \( V(t) \) is equivalent to \( \tilde{V}(t) \),

...
(A2) \( V'(t) \leq -n_0 \tilde{V}(t) \), for some positive number \( n_0 \).

The rest of the proof will be divided into three parts:

**First part:** it concerns the second assumption (A2). We start with the following lemma

**Lemma 3.2.** Let \( V(t) \) be defined as before. By choosing a function \( f \) satisfying

\[
- e^{-2\lambda \rho} = \left( e^{-\lambda \rho} f(\rho) \right)', \quad \lambda > 0
\]

and by taking \( N_3 = N_1, \ N_6 \beta = N_2 \alpha \) and \( N_6 \alpha = \frac{f(1)e^{-\lambda}}{\tau} N_5 \) we have that for every positive real numbers \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) and \( \varepsilon_4 \),

\[
V'(t) \leq \left( -N_4 \frac{e^{-2\lambda}}{\tau} + N_1 \frac{\alpha \varepsilon_1}{2} \right) \| z(.,1,.) \|^2 \\
+ \left( -2 \frac{k_1}{\tau} N_4 + N_5 \left( \varepsilon_2 + \frac{\tau}{\varepsilon_3} \right) \right) V_4(t) \\
+ \left( N_4 \left( \frac{\alpha}{2 \varepsilon_1} - \beta \right) + N_5 \left( \frac{\varepsilon_3}{2} \Phi + \frac{\Psi \gamma c_p}{2 \alpha \tau} \right) \right) \| u_x \|^2 \\
+ \left( -N_1 \kappa + N_5 \frac{\Psi \gamma}{2 \alpha \tau \varepsilon_4} \right) \| \theta_x \|^2
\]

where \( c_p > 0 \) is the Poincaré constant associated to \( \Omega \), (it can be taken equal to \( \frac{\ell^2}{2} \)) and

\[
\Psi := f(1)e^{-\lambda}, \quad \Lambda := f(0) \quad \text{and} \quad \Phi := \int_0^1 f^2(\rho) d\rho.
\]

Furthermore,

\[
\Gamma := \int_0^1 e^{-2\lambda \rho} d\rho = \frac{1 - e^{-2\lambda}}{2\lambda}.
\]

Notice that, in view of (3.1), we have

\[
\Lambda = \Psi + \Gamma.
\]

**Proof.** Computing the derivatives of \( V_1, V_2, V_3, V_4 \) and \( V_5 \) using integration by parts, boundary conditions and Young's inequality, we have

\[
V_1'(t) = \langle u_{tt}, u_t \rangle = \langle (\alpha z(.,1,.) + \beta u_{tx}), u_t \rangle - \gamma \langle \theta_x, u_t \rangle \\
= -\langle \alpha z(.,1,.) + \beta u_{tx}, u_{tx} \rangle - \gamma \langle \theta_x, u_t \rangle \\
= -\langle \alpha z(.,1,.) + \beta \| u_{tx} \|^2 - \gamma (\theta_x, u_t) \rangle \\
\leq \left( \frac{\alpha}{2 \varepsilon_1} - \beta \right) \| u_{tx} \|^2 + \frac{\alpha \varepsilon_1}{2} \| z(.,1,.) \|^2 - \gamma \langle \theta_x, u_t \rangle,
\]

\[
V_2'(t) = \langle u_{xt}, u_x \rangle,
\]

\[
V_3'(t) = \langle \theta_t, \theta \rangle = -\gamma \langle (u_{xt}, \theta) + \kappa \theta_{xx}, \theta \rangle \\
= \gamma \langle u_t, \theta_x \rangle - \kappa \| \theta_x \|^2.
\]
The derivative of $V_4$ is
\[
V_4'(t) = 2 \int_0^1 e^{-2\lambda \rho} \langle z(\cdot, \rho, \cdot), z_t(\cdot, \rho, \cdot) \rangle \, d\rho \\
= -\frac{2}{\tau} \int_0^1 e^{-2\lambda \rho} \langle z(\cdot, \rho, \cdot), z_\rho(\cdot, \rho, \cdot) \rangle \, d\rho \\
= -\frac{e^{-2\lambda}}{\tau} \|z(\cdot, 1, \cdot)\|^2 + \frac{1}{\tau} \|u_x\|^2 - \frac{2\lambda}{\tau} \int_0^1 e^{-2\lambda \rho} \|z(\cdot, \rho, \cdot)\|^2 \, d\rho \\
\leq -\frac{e^{-2\lambda}}{\tau} \|z(\cdot, 1, \cdot)\|^2 + \frac{1}{\tau} \|u_x\|^2 - \frac{2\lambda}{\tau} V_4(t).
\]

The derivative of $V_5$ is calculated as follows
\[
V_5'(t) = \frac{1}{\tau} \int_0^1 e^{-\lambda \rho} f(\rho) \langle z_\rho(\cdot, \rho, \cdot), u_x \rangle \, d\rho - \int_0^1 e^{-\lambda \rho} f(\rho) \langle z(\cdot, \rho, \cdot), u_{xt} \rangle \, d\rho \\
= \frac{1}{\tau} e^{-\lambda} f(1) \langle z(\cdot, 1, \cdot), u_x \rangle - \frac{1}{\tau} f(0) \|u_x\|^2 \\
+ \frac{1}{\tau} \int_0^1 (e^{-\lambda \rho} f(\rho))' \langle z(\cdot, \rho, \cdot), u_x \rangle \, d\rho - \int_0^1 e^{-\lambda \rho} f(\rho) \langle z(\cdot, \rho, \cdot), u_{xt} \rangle \, d\rho.
\]
Replacing $e^{-\lambda} f(1)$ by $\Psi$, $f(0)$ by $\Lambda$ and $(e^{-\lambda} f(\rho))'$ by $-e^{-2\lambda}$, we obtain (using Young’s inequality),
\[
V_5'(t) \leq \frac{1}{\tau} \left( \frac{\Gamma}{2e_2} - \Lambda \right) \|u_x\|^2 + \frac{1}{2\tau} \left( e_2 + \frac{\tau}{e_3} \right) V_4(t) + \frac{e_3}{2} \Phi \|u_{tx}\|^2 - \frac{\Psi}{\tau} \langle z(\cdot, 1, \cdot), u_x \rangle.
\]
Finally, the derivative of $V_6$ is
\[
V_6'(t) = \|u_t\|^2 - \alpha \langle z(\cdot, 1, \cdot), u_x \rangle - \beta \langle u_x, u_{xt} \rangle + \gamma \langle u, \theta_x \rangle.
\]
To conclude, it suffices to sum up $N_1 V_1'(t)$, $\alpha N_2 V_2'(t)$, $N_3 V_3'(t)$, $N_4 V_4'(t)$, $N_5 V_5'(t)$, and $N_6 V_6'(t)$.

\[\square\]

In view of Lemma 3.2 for the assumption (A2) to be satisfied, it suffices that
\[
(3.2) \quad -N_4 \frac{e^{-2\lambda}}{\tau} + N_1 \frac{\alpha e_1}{2} = 0,
\]
\[
(3.3) \quad n_1 := -2\lambda N_4 + \frac{N_5}{2\tau} \left( e_2 + \frac{\tau}{e_3} \right) < 0,
\]
\[
(3.4) \quad n_2 := \frac{N_4}{\tau} + \frac{N_5}{\tau} \left( \frac{\Gamma}{2e_2} - \Lambda \right) + N_5 \frac{\Psi \varepsilon_4 c_\rho}{2\alpha \tau} < 0,
\]
\[
(3.5) \quad n_3 := \frac{N_1}{\tau} \left( \frac{\alpha}{2e_1} - \beta \right) + N_5 \left( \frac{e_3}{2} \Phi + \frac{\Psi \varepsilon_4}{\alpha \tau} \right) < 0,
\]
\[
(3.6) \quad n_4 := -N_1 k + \frac{N_5}{2\alpha \tau e_4} \Psi \gamma < 0.
\]
The first condition (3.2) is equivalent to
\[
N_4 = a N_1
\]
with $a := \frac{1}{2} \alpha e_1 \tau e^{2\lambda}$. 
The second condition (3.3) means that there exists $0 < k < 1$ such that

$$N_5 = bN_4 = abN_1$$

with \( b := \frac{4\lambda k}{\varepsilon_2 \tau} \).

Note that, we have then

$$N_6 = \Psi \frac{\alpha \varepsilon_1}{\tau} N_5 = ab \Psi \frac{\alpha \varepsilon_1}{\tau} N_1 \quad \text{and} \quad N_2 = \frac{\beta}{\alpha} N_6 = \frac{ab \Psi \beta}{\alpha^2 \tau} N_1.$$ 

Replacing \( N_5 \) by \( abN_1 \) and \( a \) by \( \frac{1}{2} \alpha \varepsilon_1 \tau e^{2\lambda} \) in (3.5), then multiplying the inequality by \( \frac{\alpha}{\varepsilon_1} \), we obtain

(3.7) \[
\frac{1}{2} b \Psi \alpha e^{2\lambda} c_p + \frac{1}{4} \tau b \Psi \varepsilon_3 \alpha^2 e^{2\lambda} < \frac{\alpha}{\varepsilon_1} \left( \beta - \frac{\alpha}{2\varepsilon_1} \right) .
\]

We take \( \varepsilon_1 = \frac{\alpha}{\beta} \), then (3.7) turns into

(3.8) \[
b \Psi \alpha e^{2\lambda} c_p + \frac{1}{2} \tau b \Psi \varepsilon_3 \alpha^2 e^{2\lambda} < \beta^2 .
\]

Return back to (3.6), replacing \( N_5 \) by \( abN_1 \) to obtain

(3.9) \[
\frac{ab \Psi \gamma}{2\alpha \varepsilon_4} < \kappa .
\]

Also inequality (3.4) becomes

(3.10) \[
\frac{b \Psi \varepsilon_4 c_p}{2\alpha} < \left( \left( \Lambda - \frac{\Gamma}{2\varepsilon_2} \right) b - 1 \right) .
\]

Already, it is necessary that \( \Lambda - \frac{\Gamma}{2\varepsilon_2} > 0 \), that is \( \Lambda > \frac{\Gamma}{2\varepsilon_2} \), and \( \left( \Lambda - \frac{\Gamma}{2\varepsilon_2} \right) b - 1 > 0 \), that is,

(3.11) \[
b = \frac{A}{\Lambda - \frac{\Gamma}{2\varepsilon_2}} , \quad A > 1 ,
\]

hence, (3.10) turns into

(3.12) \[
\frac{b \Psi \varepsilon_4 c_p}{2\alpha} < (A - 1) .
\]

Combining (3.9) and (3.12) to obtain

(3.13) \[
\frac{ab \Psi \gamma}{2\alpha \varepsilon_4} < \varepsilon_4 < \frac{2\alpha}{\gamma b \Psi c_p} (A - 1) .
\]

Replacing \( a \) by \( \frac{1}{2} \alpha^2 \tau e^{2\lambda} \) in (3.13) to get

(3.14) \[
\frac{\alpha \gamma b \Psi}{4\beta \kappa} e^{2\lambda} < \varepsilon_4 < \frac{2\alpha}{\gamma b \Psi c_p} (A - 1) .
\]

Now, going back with more detail on assumption (3.11). To do this, replacing \( b \) by \( \frac{4\lambda k}{\varepsilon_2 + \varepsilon_3} \), we obtain

(3.15) \[
\frac{A}{\Lambda - \frac{\Gamma}{2\varepsilon_2}} = \frac{4\lambda k}{\varepsilon_2 + \varepsilon_3} ,
\]

or equivalently,

(3.16) \[
\frac{A}{4\lambda k} \varepsilon_2 - \left( \Lambda - \frac{A}{4\lambda k} \varepsilon_3 \right) \varepsilon_2 + \frac{\Gamma}{2} = 0
\]
the discriminant of such equation in $\epsilon_2$ is

$$\Delta := \left( \Lambda - \frac{A}{4\lambda k} \frac{\tau}{\epsilon_3} \right)^2 - \frac{A\Gamma}{2\lambda k}$$

which must be at least zero. In the sequel, we choose it zero. On the other hand $\epsilon_2$ is positive, then

$$\Lambda - \frac{A}{4\lambda k} \frac{\tau}{\epsilon_3} = \sqrt{\frac{A\Gamma}{2\lambda k}}$$

or equivalently

$$\frac{A}{4\lambda k} \frac{\tau}{\epsilon_3} = \Lambda - \sqrt{\frac{A\Gamma}{2\lambda k}}.$$ 

It is obvious that the left hand side of the last equation is positive, that is

$$\frac{A}{k} < 2\Lambda \frac{\Lambda^2}{\Gamma}.$$ 

Moreover, since $A > 1$ and $0 < k < 1$ we have

$$1 < \frac{A}{k} < 2\Lambda \frac{\Lambda^2}{\Gamma}.$$ 

Finally, note that

$$\epsilon_2 = \sqrt{2\Lambda \frac{k}{A}}.$$ 

**Second part:** it concerns the equivalence between $V(t)$ and $\tilde{V}(t)$. Let $\epsilon_5 > 0$ and $\epsilon_6 > 0$, we have

$$|N_5 V_5| \leq \frac{N_5}{2\epsilon_5} V_4 + \frac{N_5 \Phi \epsilon_5}{2} ||u_x||^2$$

and

$$|N_6 V_6| \leq \frac{N_6 \epsilon_6}{2} c_p ||u_x||^2 + \frac{N_6}{2\epsilon_6} ||u_t||^2.$$ 

For $V(t)$ to be equivalent to $\tilde{V}(t)$ it is sufficient that

$$\frac{N_6}{2\epsilon_6} < \frac{N_1}{2}, \quad \frac{N_5}{2\epsilon_5} < N_4,$$

$$\frac{N_6 \epsilon_6}{2} c_p + \frac{N_5 \Phi \epsilon_5}{2} < \frac{N_2 \alpha}{2}.$$ 

Using $N_6 = \frac{ab\psi}{\alpha \tau} N_3$ and $N_5 = ab N_1$ in (3.23) we get

$$\epsilon_6 > \frac{ab\psi}{\alpha \tau}, \quad \epsilon_5 > \frac{b}{2}.$$ 

We choose

$$\epsilon_6 = 2\frac{ab\psi}{\alpha \tau}, \quad \epsilon_5 = b.$$ 

Using again $N_6 = \frac{\psi}{\alpha \tau} N_3$ and $N_2 = \frac{\psi}{\alpha \tau} N_5$, inequality (3.24) becomes

$$\frac{b\psi^2}{\beta \tau} c_p + \Phi b < \frac{\beta \psi}{\alpha \tau}.$$ 

**Third part:** It is enough to examine the equations (3.24), (3.8), (3.14) and (3.25).

We take $h := \frac{\psi}{\tau}$, then $\Lambda = \psi + \Gamma = (1 + h)\Gamma = (1 + h)\frac{1 - \epsilon^{-2\lambda}}{2\lambda}$. 

**First step.** We begin by assumption (3.21) which can be translated into

\[
1 < \frac{A}{k} < (1 - e^{-2\lambda})(1 + h)^2\]

We choose \(h := e^{-2\lambda}\). Then

\[
(1 - e^{-2\lambda})(1 + h)^2 = (1 - h)(1 + h)^2 = 1 + h - h^2 - h^3 > 1
\]

for \(\lambda\) large enough. We choose \(A = 1 + h - h^2 - 2h^3 - 4h^4\) and \(k = 1 - h^4\). We have, for \(\lambda\) large enough, \(A > 1, 0 < k < 1\) and (3.26) is satisfied since

\[
1 < \frac{A}{k} = 1 + h - h^2 - 2h^3 + o(h^3) < (1 - h)(1 + h)^2
\]

**Second step.** Estimate of \(\varepsilon_2\), \(b\) and \(\varepsilon_3\) according to \(h\) and \(\lambda\) for \(\lambda\) large enough. We have

\[
\frac{k}{A} = \frac{1 + o(h^2)}{1 + h - h^2 + o(h^2)}
\]

(3.28)

\[
= 1 - h + 2h^2 + o(h^2)
\]

and

\[
\Gamma = \frac{1}{2\lambda}(1 - h)
\]

then

\[
2\lambda \Gamma \frac{k}{A} = (1 - h)(1 - h + 2h^2 + o(h^2))
\]

\[
= 1 - 2h + 3h^2 + o(h^2).
\]

Hence we obtain, using (3.22),

\[
\varepsilon_2 = 1 - h + \frac{3}{2}h^2 - \frac{1}{2}h^2 + o(h^2)
\]

\[
= 1 - h + h^2 + o(h^2).
\]

We evaluate \(b\). First,

\[
\frac{1}{2\varepsilon_2} = \frac{1}{2(1 - h + h^2 + o(h^2))}
\]

\[
= \frac{1}{2} \left( 1 + h - h^2 + (h - h^2)^2 + o(h^2) \right)
\]

\[
= \frac{1}{2} \left( 1 + h + o(h^2) \right),
\]

then

\[
1 + h - \frac{1}{2\varepsilon_2} = \frac{1}{2} (1 + h + o(h^2)),
\]

hence

\[
\frac{1}{1 + h - \frac{1}{2\varepsilon_2}} = 2(1 - h + h^2 + o(h^2)).
\]
Finally, from (3.11) and using that \( \Lambda = (1 + h) \Gamma \) we have

\[
b = \frac{A}{\Gamma (1 + h - \frac{1}{2\varepsilon^2})} = 4\lambda \frac{(1 + h - h^2 + o(h^2))(1 - h + h^2 + o(h^2))}{1 - h} \]

(3.29)

\[= 4\lambda (1 + h + o(h^2)).\]

Now we evaluate \( \varepsilon_3 \):

First, recall that

\[
\frac{A}{2\lambda k} = \frac{1}{2\lambda} (1 + h - h^2 - 2h^3 - 3h^4 + o(h^4)),
\]

then

\[
\frac{A\Gamma}{2\lambda k} = \frac{1}{4\lambda^2} (1 - 2h^2 - h^3 - h^4 + o(h^4))
\]

and

\[
\sqrt{\frac{A\Gamma}{2\lambda k}} = \frac{1}{2\lambda} (1 - h^2 - \frac{1}{2} h^3 - h^4 + o(h^4))
\]

Using (3.19), (3.30), (3.31) and that \( \Lambda = \frac{1}{2\lambda} (1 - h^2) \), we have

\[
\varepsilon_3 = \frac{\tau}{h^3} (1 - h + o(h)).
\]

Third step. Interpretation of Inequality (3.8). First, we need to express \( \Phi \) according to \( h \) and \( \lambda \).

Since

\[
f(\rho) = e^{\lambda \rho} \left( h\Gamma + \int_\rho^1 e^{-2\lambda s} ds \right)
\]

\[
= \frac{1}{2\lambda} e^{\lambda \rho} \left( e^{-2\lambda} (1 - e^{-2\lambda}) + (e^{-2\lambda}\rho - e^{-2\lambda}) \right)
\]

\[
= \frac{1}{2\lambda} e^{\lambda \rho} \left( e^{-2\lambda\rho} - e^{-4\lambda} \right)
\]

then,

\[
f^2(\rho) = \frac{1}{4\lambda^2} e^{2\lambda \rho} \left( e^{-4\lambda\rho} - 2e^{-4\lambda\rho}e^{-2\lambda\rho} + e^{-8\lambda} \right)
\]

\[
= \frac{1}{4\lambda^2} \left( e^{-2\lambda\rho} - 2e^{-4\lambda} + e^{-8\lambda}e^{2\lambda\rho} \right).
\]

Hence,

\[
\Phi = \int_0^1 f^2(\rho) d\rho
\]

\[
= \frac{1}{4\lambda^2} \left( \frac{1 - e^{-2\lambda}}{2\lambda} - 2e^{-4\lambda} + \frac{1}{2\lambda} (e^{-6\lambda} - e^{-8\lambda}) \right)
\]

\[
= \frac{1}{8\lambda^3} (1 - h - 4\lambda h^2 + o(h^2)).
\]

Now, inequality (3.8) can be rewritten as:

\[2\alpha c_p (1 - h^2 + o(h^2)) + \frac{\tau^2 \alpha^2}{4\lambda^2 h^2} (1 - h + o(h)) < \beta^2.\]
Then, we take
\[(3.33) \quad 2 \left( \frac{c_p}{\alpha \tau^2} + \frac{e^{8\lambda}}{8\lambda^2} \right) < \left( \frac{\beta}{\alpha \tau} \right)^2 \]
with \(\lambda\) large enough.

**Fourth step.** Condition \((3.14)\) and existence of \(\varepsilon_4\). Inequality \((3.14)\) can be rewritten as:
\[
\frac{\alpha \gamma}{2\beta \kappa} (1 - h^2 + o(h^2)) < \varepsilon_4 < \frac{\alpha}{\gamma c_p} (1 - h + o(h)).
\]
Then, we take
\[(3.34) \quad \frac{\alpha \gamma}{2\beta \kappa} (1 - h^2 + o(h^2)) < \varepsilon_4 \]
and \(\varepsilon_4\) can be equal to \(\frac{\alpha}{\gamma c_p} (1 - h + o(h))\), with \(\lambda\) large enough.

**Fifth step.** Interpretation of assumption \((3.25)\). It can be rewritten as:
\[
2h \frac{c_p}{\tau \beta} (1 - h^2 + o(h^2)) + \frac{1}{\lambda h} (1 + h + o(h)) < \frac{\beta}{\tau \alpha}.
\]
It suffices to take
\[(3.35) \quad \left( 2e^{-2\lambda} \frac{c_p}{\tau \beta} + \frac{1}{\lambda} (e^{2\lambda} + 1 + o(1)) \right) < \frac{\beta}{\alpha \tau},\]
with \(\lambda\) large enough.

Note that for \(\lambda\) large enough, \(\beta = \alpha \tau e^{4\lambda}\) satisfies the three conditions \((3.33), (3.34)\) and \((3.35)\). Moreover, there exists \(\beta_0 > 0\) such that every \(\beta > \beta_0\) satisfies the three conditions \((3.33), (3.34)\) and \((3.35)\).

For every \(\beta > \beta_0\) we have
\[(3.36) \quad \dot{V}(t) \leq -n_0 V(t),\]
where \(n_0 = \min\{n_1, n_2, \frac{1}{c_p} n_3, \frac{1}{c_p} n_4\}\). Recall that \(V(t), \dot{V}(t)\) and \(E(t)\) are equivalent then, there exists \(a_0 > 0, C > 0\) such that
\[
E(t) < Ce^{-a_0 t}.
\]

\[\square\]

**Comments**

We can replace the Neumann conditions for \(\theta\)
\[
\theta_x(0, t) = \theta_x(\ell, t) = 0
\]
by the Dirichlet conditions
\[
\theta(0, t) = \theta(\ell, t) = 0,
\]
we then obtain the same results.
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UR Analysis and Control of PDEs, UR 13ES64, Department of Mathematics, Faculty of Sciences of Monastir, University of Monastir, Tunisia, Laboratory of Analysis and Control of Partial Differential Equations, Djillaly Liabes University, Sidi Bel Abbes, Algeria.

Email address: s.moulay_khatir@yahoo.fr

Email address: farhat.shel@ipeit.rnu.tn