THE LATTICE OF CYCLIC FLATS OF A MATROID

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Abstract. A flat of a matroid is cyclic if it is a union of circuits. The cyclic flats of a matroid form a lattice under inclusion. We study these lattices and explore matroids from the perspective of cyclic flats. In particular, we show that every lattice is isomorphic to the lattice of cyclic flats of a matroid. We give a necessary and sufficient condition for a lattice \( Z \) of sets and a function \( r : Z \to \mathbb{Z} \) to be the lattice of cyclic flats of a matroid and the restriction of the corresponding rank function to \( Z \). We apply this perspective to give an alternative view of the free product of matroids and we show how to compute the Tutte polynomial of the free product in terms of the Tutte polynomials of the constituent matroids. We define cyclic width and show that this concept gives rise to minor-closed, dual-closed classes of matroids, two of which contain only transversal matroids.

1. Introduction

A flat of a matroid is cyclic if it is a (possibly empty) union of circuits. Cyclic flats have played several important roles in matroid theory, starting with the theory of transversal matroids (see, for example, [4, 13]). The cyclic flats of a matroid \( M \), ordered by inclusion, form a lattice, \( Z(M) \): the join of two cyclic flats \( A \) and \( B \) in \( Z(M) \) is \( \text{cl}(A \cup B) \) and their meet is the union of all circuits contained in \( A \cap B \). (See Figure 1.) This paper studies the lattice of cyclic flats and offers insights into some topics in matroid theory from this perspective.

Figure 1 shows that properties of the lattice of flats, such as being graded, may fail in the lattice of cyclic flats. Indeed, in Section 2 we prove that every lattice is isomorphic to the lattice of cyclic flats of some matroid. Figure 1 also shows that the collection of cyclic flats alone does not determine the matroid. However, the cyclic flats together with their ranks determine the matroid [4, Proposition 2.1].

Section 3 gives a necessary and sufficient condition for a collection \( Z \) of sets and a function \( r : Z \to \mathbb{Z} \) to be the lattice of cyclic flats of a matroid and the restriction of the corresponding rank function to \( Z \). This axiom scheme for matroids is used in Section 4 to give a simple, alternative perspective on the free product [5, 6]. Section 5 introduces the concept of cyclic width and shows that the class of matroids of cyclic width \( k \) or less is closed under minors and duals. We explore these classes for small values of \( k \). The final section contains open problems.

We assume familiarity with matroid theory. We follow the notation of [15]. We call a matroid bitransversal if it is both transversal and cotransversal (or a strict gammoid). Recall that the free extension \( M + e \) of \( M \) by \( e \), for \( e \notin E(M) \), is the matroid on \( E(M) \cup e \) whose circuits are the circuits of \( M \) together with the sets \( B \cup e \) for bases \( B \) of \( M \). Free coextension is the dual operation, \((M^* + e)^*\).
Our lattice theory notation follows [8] except that we use \(0_Z\) and \(1_Z\) for the least and greatest elements of a lattice \(Z\). As is true of a lattice of flats and a lattice of cyclic flats, the elements of most lattices in this paper are sets and the order relation is containment; we use the term lattice of sets for such a lattice. The meet and join operations of a lattice of sets need not be intersection and union. All lattices considered in this paper are finite.

We close this introduction by recalling that cyclic flats link two notions of duality. Since cyclic flats are unions of circuits as well as intersections of hyperplanes, \(X\) is a cyclic flat of \(M\) if and only if \(E(M) - X\) is a cyclic flat of \(M^*\). Thus, as noted in [4, 13], \(Z(M^*)\) is isomorphic to the dual (as a lattice) of \(Z(M)\).

2. The Lattice of Cyclic Flats

By considering matroids in which every element is in a nontrivial parallel class, it follows that every geometric lattice is isomorphic to the lattice of cyclic flats of some matroid. The main result of this section is that for any lattice \(Z\), there is a matroid \(M\) (in fact, a bitransversal matroid) for which the lattice \(Z(M)\) of cyclic flats is isomorphic to \(Z\). We also show that the counterpart of this result does not hold for matroids that are representable over a fixed finite field.

We will use fundamental transversal matroids and their simplex representation ([4] [13] or [15, Section 12.2]). A fundamental (or principal) transversal matroid is a matroid \(M\) that has a basis \(B\) such that each cyclic flat of \(M\) is spanned by a subset of \(B\). To construct a geometric representation of \(M\), let \(B\) be \(\{v_1, v_2, \ldots, v_n\}\) and put \(v_1, v_2, \ldots, v_n\) at the \(n\) vertices of an \(n\)-simplex in \(\mathbb{R}^n\). For each \(x\) in \(E(M) - B\), place \(x\) freely in the (possibly empty) face of the simplex spanned by \(C(x, B) - x\), where \(C(x, B)\) is the fundamental circuit of \(x\) with respect to \(B\). Note that if \(X\) is a rank-\(k\) cyclic flat of \(M\), then \(X\) consists of the points in a \((k-1)\)-dimensional face of the simplex. Conversely, any matroid with a geometric representation on a simplex that has elements at each vertex and for which each rank-\(k\) cyclic flat consists of the points in a \((k-1)\)-dimensional face of the simplex is a fundamental transversal matroid. Fundamental transversal matroids are indeed transversal matroids; in fact, every transversal matroid is a deletion of a fundamental transversal matroid.

It is well known that these matroids are bitransversal.

We are ready to treat the main result of this section.
**Theorem 2.1.** Every lattice is isomorphic to the lattice of cyclic flats of a bi-transversal matroid.

*Proof.* Let $Z$ be a lattice. We construct a fundamental transversal matroid $M$ for which $Z(M)$ is isomorphic to $Z$. We first realize $Z$ as a lattice of sets. Let $B$ be the set of elements in $Z$ other than $1_Z$. For $z \in Z$, let $V_z$ be $\{y : y \geq z\}$. Thus, $V_z \subseteq B$. The map $z \mapsto V_z$ is an isomorphism between $Z$ and the lattice of sets $\{V_z : z \in Z\}$ since it is a bijection and $x \leq z$ in $Z$ if and only if $V_x \subseteq V_z$.

Let $M$ be the following fundamental transversal matroid on $B \cup \{s_z : z \in Z\}$. Place the elements of $B$ at the vertices of a $|B|$-simplex in $\mathbb{R}^{|B|}$ and, for each $z$ in $Z$, put $s_z$ freely in the face spanned by $V_z$. By construction, $V_z$ spans a cyclic flat of $M$, namely, $V_z \cup \{s_z : x \leq z\}$. With the isomorphism above, the theorem follows upon showing that these are the only cyclic flats of $M$.

Assume the subset $V$ of $B$ spans a cyclic flat $F$ of $M$. For any $s_z$ in $F - V$, we have $C(B, s_z) = V_z \cup s_z \subseteq F$. Also, each element $v$ of $V$ is in such a fundamental circuit, for if $v$ were not, then all elements of $F - V$ would be in $cl(V - v)$, so $F - v$ would be a flat, contrary to $F$ being cyclic. Thus, $F$ is the closure of $\bigcup \{V_z : s_z \in F - V\}$. Note that $V_{V_z} = V_z \cup V_z$, so $F$ is $cl(V_z')$, where $z'$ is $\bigvee \{z : s_z \in F - V\}$. Thus, as needed, the cyclic flats of $M$ are the sets $V_z \cup \{s_x : x \leq z\}$. $\square$

After (re)discovering Theorem 2.1 we learned of the work of Sims [18] extending Dilworth’s embedding theorem [7, Theorem 14.1] to rank-finite independence spaces and lattices of finite length. Her approach has much in common with Dilworth’s and gives the lattice of cyclic flats as a sublattice of the lattice of flats. Dilworth’s proof was very influential in the development of the theory of submodular functions. We note that a minor modification of the proof above gives a short, geometric proof of a strengthening of Dilworth’s theorem: for each finite lattice $Z$, there is a transversal matroid $M$ for which (i) $Z$ is isomorphic to $Z(M)$ and (ii) $Z(M)$ is a sublattice of the lattice of flats of $M$. Indeed, adapt the construction as follows: for each $z \in Z$, instead of putting one point freely in the flat spanned by $V_z$, put a set $S_z$ of $|V_z| + 1$ points freely in this flat, where $S_z \cap S_{z'} = \emptyset$ for $z \neq x$, and then delete $B$ to get a transversal matroid $M$. The cyclic flats of $M$ are the sets $F_x = \bigcup_{y \leq z} S_y$; also, $F_x \cap F_y = F_x \wedge y$, so the meet in $Z(M)$ is that of the lattice of flats; the joins in these lattices always agree, thus giving the sublattice assertion.

Since transversal matroids are representable over the reals, the counterpart of Theorem 2.1 holds for real matroids. It is natural to ask what other classes of matroids have a counterpart of this result. Our next goal (Theorem 2.3) is to show that matroids representable over GF($q$) do not have this property.

The proof of Theorem 2.3 uses the class of nested matroids, which also appears in Section 5. A *nested matroid* is a matroid that can be obtained from the empty matroid by iterating the operations of adding isthmuses and taking free extensions. Nested matroids have appeared several times in the literature (see 2, Section 4) for references. They are a subclass of bitransversal matroids.

The following observations about the operations used to construct nested matroids will be useful. For a single-element extension $M'$ of $M$, the lattices $Z(M)$ and $Z(M')$ are equal if and only if the element of $E(M') - E(M)$ is an isthmus of $M'$. The lattice $Z(M + e)$ of the free extension of $M$ is formed from $Z(M)$ by removing $E(M)$, if it is in $Z(M)$, and adjoining $E(M) \cup e$ to $Z(M)$. If $Z$ is the
lattice of cyclic flats of some matroid and $T$ is disjoint from $1_Z$, then the following construction gives all matroids $M$ for which $Z(M)$ is $Z \cup \{1_Z \cup T\}$:

1. start with any matroid $M_0$ for which $E(M_0)$ is $1_Z$ and $Z(M_0)$ is $Z$,
2. partition $T$ into two nonempty subsets $T_i$ and $T_j$,
3. take the direct sum of $M_0$ and the free matroid on $T_i$ to get $M_1$,
4. take free extensions of $M_1$ by the elements in $T_j$ to get $M_2$, and finally
5. let $M$ be the direct sum of $M_2$ and any free matroid.

Note that in steps (2)–(4), only the cardinalities $|T_i|$ and $|T_j|$, not the elements in these sets, matter. These observations give us the following result, which is essentially Lemma 2 of [10]. Recall that a chain is a linearly ordered set.

**Lemma 2.2.** A matroid $M$ is nested if and only if $Z(M)$ is a chain.

We now show that some lattices do not arise as the lattice of cyclic flats of any matroid that is representable over GF($q$).

**Theorem 2.3.** For a prime power $q$, if $Z(M)$ is a chain with $q + 2$ or more elements, then $M$ is not representable over GF($q$).

**Proof.** We show that if $Z(M)$ is a chain with $k + 2$ elements, then $M$ has $U_{k,k+2}$ as a minor; thus if $k \geq q$, then $M$ is not representable over GF($q$). Let the cyclic flats of $M$ be $X_0 \subset X_1 \subset \cdots \subset X_{k+1}$. By Lemma 2.2, $M$ is a nested matroid, so $M|X_j$, for $1 \leq j \leq k + 1$, is obtained by partitioning $X_j - X_{j-1}$ into nonempty sets $I_j$ and $F_j$, taking the direct sum of $M|X_{j-1}$ and the free matroid on $I_j$, and then taking free extensions by the elements in $F_j$. Therefore the minor

$$M|X_{k+1}/I_{k+1}\setminus(X_0 \cup F_1 \cup F_2 \cup \cdots \cup F_{k-1})$$

has just two cyclic flats, namely, $\emptyset$ and $I_1 \cup I_2 \cup \cdots \cup I_k \cup F_k \cup F_{k+1}$, and so is a uniform matroid: that the rank and nullity of this uniform minor are at least $k$ and $2$, respectively, completes the proof. □

3. **An Axiom Scheme for Cyclic Flats**

In this section we formulate the definition of a matroid using cyclic flats and their ranks. The mixed character of this axiom scheme sets it apart from most others: it is built jointly on lattice properties (but using only some of the flats) and partial information about the rank function.

We will use the following characterization of independent sets and circuits in terms of cyclic flats and their ranks.

**Lemma 3.1.** (i) A set $I \subseteq E(M)$ is independent in $M$ if and only if $|I \cap X| \leq r(X)$ for every cyclic flat $X$ of $M$.

(ii) A set $C \subseteq E(M)$ is a circuit of $M$ if and only if $C$ is minimal with the property that there is a cyclic flat $X$ with $C \subseteq X$ and $|C| = r(X) + 1$.

**Proof.** Statement (i) is immediate. As the minimal dependent sets, the circuits are the minimal sets $C$ with $|C \cap X| > r(X)$ for some cyclic flat $X$; since $X$ can be $\text{cl}(C)$, this property can be replaced by the simpler one in statement (ii). □

To motivate our main result, we note three properties of the lattice $Z(M)$ and the rank function $r$ of $M$. First, $0_{Z(M)}$, which is $\text{cl}(\emptyset)$, has rank $0$. Next, if $X$ and $Y$ are in $Z(M)$ and $X \subset Y$, then $0 < r(Y) - r(X) < |Y - X|$ since $M|Y$ has no isthmuses. Finally, since for cyclic flats $X$ and $Y$ we have $r(X \lor Y) = r(X \cup Y)$
and \( r(X \cap Y) = r(X \cap Y) + |(X \cap Y) - (X \cap Y)| \), we get the following specialization of semimodularity:

\[
r(X) + r(Y) \geq r(X \cup Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|.
\]

We next show that these properties give an axiom scheme for matroids.

**Theorem 3.2.** Let \( Z \) be a collection of subsets of a set \( S \) and let \( r \) be an integer-valued function on \( Z \). There is a matroid for which \( Z \) is the collection of cyclic flats and \( r \) is the rank function restricted to the sets in \( Z \) if and only if

1. **(Z0)** \( Z \) is a lattice under inclusion,
2. **(Z1)** \( r(0_Z) = 0 \),
3. **(Z2)** \( 0 < r(Y) - r(X) < |Y - X| \) for all sets \( X, Y \in Z \) with \( X \subset Y \), and
4. **(Z3)** for all sets \( X, Y \) in \( Z \),

\[
r(X) + r(Y) \geq r(X \cup Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|.
\]

**Proof.** We have seen that these properties are necessary; we focus on sufficiency. Assume \( Z \) and \( r \) satisfy properties (Z0)–(Z3). We define a collection \( C \) (following Lemma 3.1) that we show is the set of circuits of a matroid on \( S \). We then show that the cyclic flats of this matroid are the sets in \( Z \) and that \( r \) gives their ranks.

**Lemma 3.1.** The collection \( C \) of all minimal subsets \( C \) of \( S \) for which there is a set \( X \in Z \) with \( C \subseteq X \) and \( |C| = r(X) + 1 \) is the collection of circuits of a matroid.

**Proof of Lemma 3.1.** We focus on circuit elimination since the other two circuit axioms clearly hold. Thus, let \( C \) and \( C' \) be distinct sets in \( C \) with \( a \in C \cap C' \); let \( X \) and \( X' \) in \( Z \) contain \( C \) and \( C' \), respectively, with \( |C| = r(X) + 1 \) and \( |C'| = r(X') + 1 \).

To show that \( (C \cup C') - a \) contains a set in \( C \), it suffices to prove the inequality

\[
|C \cup C' - a| > r(X \cup X').
\]

Property (Z3) and the equalities \( |C| = r(X) + 1 \) and \( |C'| = r(X') + 1 \) give

\[
|C \cup C' - a| = |C| + |C'| - |C \cap C'| - 1
\]

\[
> r(X) + r(X') - |C \cap C'|
\]

\[
\geq r(X \cup X') + r(X \wedge X') + |(X \cap X') - (X \wedge X')| - |C \cap C'|.
\]

Thus, the desired inequality follows upon showing that the sum of the last three terms is nonnegative, that is, \(|C \cap C'| \leq r(X \wedge X') + |(X \cap X') - (X \wedge X')|\). Now \(|(C \cap C') \cap (X \wedge X')| \leq r(X \wedge X') \) since \( X \wedge X' \) is in \( Z \) and no subset of \( C \cap C' \) is in \( C \), so

\[
|C \cap C'| = |(C \cap C') \cap (X \wedge X')| + |(C \cap C') \cap (X \wedge X')|,
\]

and therefore circuit elimination holds.

Let \( M_Z \) be the matroid on \( S \) that has \( C \) as its collection of circuits and let \( r_Z \) be its rank function. The next step in essence identifies some elements of \( Z \) as the closures of circuits.

**Theorem 3.2.** For any circuit \( C \) of \( M_Z \), there is a unique set \( X \in Z \) with \( C \subseteq X \) and \( r(X) = |C| - 1 \). Furthermore, if \( Y \in Z \) and \( C \subseteq Y \), then \( X \subseteq Y \).
Proof of (3.2.2). By the definition of $C$, there is a set $X$ in $Z$ with $C \subseteq X$ and $r(X) = |C| - 1$. Uniqueness follows by showing that if $Y \in Z$ and $C \subseteq Y$, then $X \subseteq Y$. Assume instead $X \not\subseteq Y$. Thus, $X \land Y \subseteq X$. We have $C \not\subseteq X \land Y$, for otherwise, since $|C| > r(X \land Y) + 1$, some proper subset of $C$ would be in $C$, which is impossible. Property (Z3) and the equality $r(X) = |C| - 1$ give

$$r(Y) + |C| - 1 \geq r(X \lor Y) + r(X \land Y) + (|X \cap Y| - (X \land Y)).$$

No proper subset of $C$ is in $C$, so $r(X \land Y) \geq |C \cap (X \land Y)|$. Since $C \subseteq X \land Y$, we have $|X \land Y| \geq |C - (X \land Y)|$. Thus, the last two terms in inequality (11) contribute at least $|C|$, so inequality (11) implies $r(Y) - 1 \geq r(X \lor Y)$, contrary to property (Z2). Thus, $X \subseteq Y$. □

We use $\bar{C}$ to denote the unique set in (3.2.2).

The next step is used to prove (3.2.4), that $r$ and $r_Z$ agree on the sets in $Z$.

(3.2.3) For all $X, Y$ in $Z$, the union of $(X \cap Y) - (X \land Y)$ and any basis of $X \land Y$ is a basis for $X \cap Y$, so

$$r_Z(X \cap Y) = r_Z(X \land Y) + (|X \cap Y| - (X \land Y)).$$

Proof of (3.2.3). Assume $X$ and $Y$ are incomparable, otherwise the result is trivial. Let $B$ be a basis of $X \land Y$ and let $B'$ be $B \cup ((X \cap Y) - (X \land Y))$. Clearly $B'$ spans $X \cap Y$, so it suffices to show that $B'$ is independent. Assume, to the contrary, that $B'$ contains a circuit $C$. Thus, $C$ is contained in both $X$ and $Y$, so $\bar{C}$ and, consequently, $(X \land Y) \lor \bar{C}$ are contained in both $X$ and $Y$. However, $(X \land Y) \lor \bar{C}$ contains elements in $(X \cap Y) - X \land Y$ since $C$ must, so $X \land Y \not\subseteq (X \land Y) \lor \bar{C}$. This contradicts the definition of meet and so proves the claim. □

(3.2.4) If $X$ is in $Z$, then $r(X) = r_Z(X)$.

Proof of (3.2.4). By the definition of $C$, any subset of $X$ with more than $r(X)$ elements contains a circuit, so $r_Z(X) \leq r(X)$. Thus, it suffices to show that $X$ contains an independent set of size $r(X)$. This statement clearly holds for $0_Z$. Let $X$ be a minimal set in $Z$ for which this property has not yet been established. We consider two cases, according to whether $X$ covers one or more elements of $Z$.

First assume $X$ covers only $Y$ in $Z$. Now $Y$ contains an independent set $J$ with $|J| = r(Y)$ by the choice of $X$. Let $I$ be the union of $J$ and any set of $(X - r(Y)$ elements of $X - Y$. If $I$ were dependent, it would contain a circuit $C$ with at most $r(X)$ elements. Thus, $r(C) < r(X)$, so, by (3.2.2), $\bar{C} \subseteq X$. Since $J$ is independent, $\bar{C} \not\subseteq Y$. These conclusions contradict the assumption that $X$ covers only one set in $Z$. Thus, $I$ is an independent subset of $X$ with $r(X)$ elements, as desired.

Now assume $X$ covers $X_1$ and $X_2$ (and perhaps more sets) in $Z$. Let $B'$ be a basis of $X_1 \land X_2$ and let $B''$ be $B' \cup ((X_1 \cap X_2) - (X_1 \land X_2))$; by (3.2.3), $B''$ is a basis of $X_1 \land X_2$. Choose bases $B_1$ of $X_1$ and $B_2$ of $X_2$, both containing $B''$, and let $B$ consist of $B_1$ and any $r(X) - r(X_1)$ elements of $B_2 - B''$. (By condition (Z3), $|B_2 - B''| \geq |r(X) - r(X_1)|$.) Thus, $|B| = r(X)$. We claim that $B$ is independent.

Assume, to the contrary, that $B$ contains a circuit $C$. Since $C \not\subseteq X_1$, we have $X_1 \subseteq X_1 \lor \bar{C} \subseteq X$; since $X$ covers $X_1$, we have $X_1 \lor \bar{C} = X$. Also, $r(X_1 \land \bar{C}) = r_Z(X_1 \land \bar{C})$ since $X_1 \land \bar{C} \subseteq X$. Applying property (Z3) to $X_1$ and $\bar{C}$ and using
The next two steps show that the sets in \( Z \) are cyclic flats of \( M_Z \).

(3.2.5) Each set \( X \) in \( Z \) is a flat of \( M_Z \).

Proof of (3.2.5). To show this, let \( x \) be in a circuit \( C \) of \( M_Z \) with \( C \subseteq X \cup x \); we need to show that \( x \) is in \( X \). Property (Z3) applied to \( X \) and \( \bar{C} \), together with (3.2.3) and (3.2.4), give the inequalities

\[
\begin{align*}
  r(X) + |C| - 1 &\geq r(X) + r(X_1 \land \bar{C}) + |(X_1 \cap \bar{C}) - (X_1 \land \bar{C})| \\
  &= r(X) + r_\mathcal{Z}(X_1 \land \bar{C}) + |(X_1 \cap \bar{C}) - (X_1 \land \bar{C})| \\
  &= r(X) + r_\mathcal{Z}(X_1 \cap \bar{C}) \\
  &\geq r(X) + |X_1 \cap C| \\
  &\geq r(X) + |C| - (r(X) - r(X_1)) \\
  &= r(X_1) + |C|.
\end{align*}
\]

(The last inequality follows by the construction of \( B \).) This contradiction implies that \( B \) contains no circuit and hence is independent, as needed.

(3.2.6) Each set \( X \) in \( Z \) is a union of circuits and so is a cyclic flat.

Proof of (3.2.6). The claim holds for \( 0_Z \) since, by property (Z1), its elements are loops. Let \( X \) be a minimal set in \( Z \) for which the claim has not yet been verified and let \( Y \) cover the set \( Y \) in \( Z \). Since \( Y \) is a union of circuits, we need only show that each element \( x \) in \( X \) is in a circuit that is contained in \( X \). Let \( B \) be a basis of \( Y \) and let \( I \) be a set of \( r(X) - r(Y) \) elements of \( X - (Y \cup x) \), which exists by property (Z2). The subset \( B \cup I \cup x \) of \( X \) has \( r(X) + 1 \) elements and so must contain a circuit \( C \); we claim that \( x \) is in \( C \). Assume, to the contrary, that \( x \) is not in \( C \). Therefore \( r(X) - r(Y) \geq |C - Y| \). Since \( \bar{C} \subseteq X \) and \( C \) contains some elements of \( I \cup x \), we have \( Y \subseteq Y \vee \bar{C} \subseteq X \); that \( X \) covers \( Y \) forces \( Y \vee \bar{C} = X \). Applying property (Z3) to \( Y \) and \( \bar{C} \), we obtain

\[
\begin{align*}
  r(Y) + |C| - 1 &\geq r(X) + r(Y \land \bar{C}) + |(Y \cap \bar{C}) - (Y \land \bar{C})| \\
  &= r(X) + r_\mathcal{Z}(Y \cap \bar{C}) \\
  &\geq r(X) + |Y \cap C|.
\end{align*}
\]

Combining this with the inequality \( r(X) - r(Y) \geq |C - Y| \) noted above gives the contradiction \( |C| - 1 \geq |C| \). Thus, \( x \) is in \( C \), so \( X \) is a union of circuits.

Lastly, we show that the cyclic flats of \( M_Z \) are in \( Z \).

(3.2.7) Each cyclic flat \( X \) of \( M_Z \) is in \( Z \).
Proof of (3.2.7). The assertion clearly holds for the least cyclic flat of $M_Z$. Let $X$ be a minimal cyclic flat of $M_Z$ for which the claim has not yet been verified and assume that $X$ covers $Y$ in $Z(M_Z)$. Thus, $Y$ is in $Z$. Let $B$ be a basis of $Y$ and let $T$ be any set of $r_Z(X) - r_Z(Y) + 1$ elements of $X - Y$. Since $X$ is cyclic and covers $Y$, it follows that $B \cup T$ contains a circuit $C$ and that $C \cap (X - Y)$ is $T$. Note that $X = c_{M_Z}(Y \cup C)$ since $X$ covers $Y$ and contains $C$. Since $Y$ and $C$ are in $Z$, their join $Y \vee C$ in $Z$ is a cyclic flat of $M_Z$ by (3.2.6), and this flat must contain the smallest flat that contains $Y$ and $C$, which is $X$. We will show that $Y \vee C$ is $X$, which implies that $X$ is in $Z$, as claimed. It suffices to prove the inequality $r(C \vee Y) \leq r_Z(X)$. Note that in the semimodular inequality in $M_Z$,

$$r_Z(C \cup Y) + r_Z(C \cap Y) \leq r_Z(C) + r_Z(Y),$$

the right side is $|C|-1+r(Y)$, that is, $\left(|C\cap Y| + r_Z(X) - r(Y)\right) + r(Y)$, which is $|C\cap Y| + r_Z(X)$. The left side is clearly at least as large, so we have

$$(2) \quad r_Z(C \cup Y) + r_Z(C \cap Y) = r_Z(C) + r_Z(Y).$$

By (3.2.3) and property (Z3), we have

$$r(C \cup Y) + r_Z(C \cap Y) = r(C \cup Y) + r(C \cap Y) + |(C \cap Y) - (C \cup Y)| \leq r(C) + r(Y),$$

which, with Eqn. (2) and statement (3.2.4), gives $r(C \cap Y) \leq r_Z(C \cup Y)$. Now $r_Z(C) = r_Z(C \vee Y)$, so $r_Z(X) = r_Z(C \cup Y) = r_Z(C \cup Y)$, so the desired inequality

$$r(C \cap Y) \leq r_Z(X)$$

follows. \hfill \Box

Thus, as needed, from $Z$ and $r : Z \rightarrow \mathbb{Z}$, we have constructed a matroid $M_Z$ for which $Z(M_Z)$ is $Z$ and $r$ is the restriction of the rank function of $M_Z$ to $Z$. \hfill \Box

In Theorem 3.2, we rediscovered a result of Sims [17, Chapter 3, Theorem 2.2]. Indeed, she proves the theorem for rank-finite independence spaces in the case that $S$ is in $Z$. (Our proof also applies in that context; both approaches also apply when there are only finitely many elements in $S$ that are not in the greatest set in $Z$.) Her proof consists of defining a function $r'$ on the subsets of $S$ by

$$r'(A) = \min\{r(F) + |A - F| : F \in Z\}$$

and then proving (i) $r'$ satisfies the rank axioms of a rank-finite independence space (specifically, $r'(\emptyset) = 0$, the unit increase property, semimodularity, and that each subset $A$ of $S$ has a finite subset $B$ with $r'(A) = r'(B)$), (ii) $r(F) = r'(F)$ for $F \in Z$, and (iii) the sets in $Z$ are precisely the cyclic flats of the resulting independence space. One intermediate result that proves quite useful is that $r(G) < r(F) + |G - F|$ for all pairs of distinct sets $F$ and $G$ in $Z$.

We close this section by noting that viewing matroid operations from the perspective of cyclic flats can suggest extensions of familiar operations. For instance, relaxing a circuit-hyperplane $H$ consists of removing $H$ from the lattice of cyclic flats. It follows from Theorem 3.2 that one can remove any cyclic flat that, like a circuit-hyperplane, is comparable only to the least and greatest cyclic flats.

4. THE FREE PRODUCT FROM THE PERSPECTIVE OF CYCLIC FLATS

Crapo and Schmitt [5] introduced a noncommutative matroid operation called the free product. Many interesting properties and applications are treated in [5, 6]. In this section we show that cyclic flats give a transparent path to the free product.
The lattice of cyclic flats

We connect this approach with the definition in [5], which uses independent sets, and we use a corollary of this link to compute Tutte polynomials of free products.

Our starting point for the free product is the description of the cyclic flats given in [6, Proposition 6.1]. The new observation is that from this description of the cyclic flats, it is obvious that properties (Z0)–(Z3) of Theorem 3.2 hold. Thus, cyclic flats give a simple way to define the free product.

**Theorem 4.1.** Let $M$ and $N$ be matroids on disjoint ground sets. Set

$$Z' = \{X : X \in Z(M), X \notin E(M)\} \cup \{E(M) \cup Y : Y \in Z(N), Y \neq \emptyset\}.$$  

Let $Z$ be $Z' \cup \{E(M)\}$ if $M$ has no isthmuses and $N$ has no loops; otherwise let $Z$ be $Z'$. Set $r(X) = r_M(X)$ and $r(E(M) \cup Y) = r(M) + r_N(Y)$. The pair $(Z, r)$ satisfies conditions (Z0)–(Z3) of Theorem 3.2 and so defines a matroid on $E(M) \cup E(N)$.

We take this as the definition of the free product $M \boxtimes N$. Note that if $M$ has an isthmus and $N$ has a loop, then $Z$ is isomorphic to the linear sum of $Z(M)$ and $Z(N)$; otherwise the difference between $Z$ and the linear sum is that the greatest element of $Z(M)$ and the least element of $Z(N)$ are, in effect, identified in $Z$. Only property (Z3) requires even a minor observation: the only nontrivial instances of this property are either instances of this property in $M$ or shifts by $2r(M)$ of instances of this property in $N$.

Two cases of the free product, treated in [5, 6], deserve special mention. The lattice of cyclic flats of $M \boxtimes U_{0,1}$ agrees with that of $M$ except that $E(M) \cup E(U_{0,1})$ is a cyclic flat in $M \boxtimes U_{0,1}$ while $E(M)$ is not; ranks are unchanged; this is the free extension of $M$. The lattice of cyclic flats of $U_{1,1} \boxtimes M$ is formed from that of $M$ by augmenting every nonempty set by $E(U_{1,1})$; ranks increase by 1; this is the free coextension of $M$. Theorem 4.1 and this view of free coextension give the following informal geometric description of the free product: $M \boxtimes N$ is formed by taking $r(M)$ free coextensions of $N$, say by $e_1, e_2, \ldots, e_{r(M)}$, gluing $M$ freely into the flat spanned by $e_1, e_2, \ldots, e_{r(M)}$, and then deleting $e_1, e_2, \ldots, e_{r(M)}$.

We now show that the independent sets of $M \boxtimes N$, which were used to define this operation in [5, Proposition 1], can be identified easily from the cyclic flat perspective. We let $\nu(X)$ denote the nullity, $|X| - r(X)$, of $X$.

**Theorem 4.2.** Let $I_M$ be the collection of independent sets of $M$. The collection of independent sets of $M \boxtimes N$ is given by

$$\{X \cup Y : X \in I_M, Y \subseteq E(N), \text{ and } \nu_N(Y) \leq r(M) - |X|\}.$$  

**Proof.** By Lemma 3.1 for $X \subseteq E(M)$ and $Y \subseteq E(N)$ the set $X \cup Y$ is independent in $M \boxtimes N$ if and only if $|X \cup Y| \cap Z \leq r(Z)$ for every cyclic flat $Z$ of $M \boxtimes N$. This condition, with $Z$ ranging over $Z(M)$ and $(if it is in $Z(M \boxtimes N)) E(M) \cup e_N(\emptyset)$, is equivalent to $X$ being independent in $M$. The other cyclic flats of $M \boxtimes N$ have the form $E(M) \cup Z'$ where $Z'$ is in $Z(N)$. Since $r(E(M) \cup Z') = r(M) + r_N(Z')$, the inequality of interest is equivalent to $|X| + |Y \cap Z'| \leq r(M) + r_N(Z')$, or $|Y \cap Z'| - r_N(Z') \leq r(M) - |X|$. The proof is completed by noting that the maximum of $|Y \cap Z'| - r_N(Z')$ over all cyclic flats of $N$ is $\nu_N(Y)$. Indeed, we have $|Y \cap Z'| - r(N(Z')) \leq \nu_N(Y \cap Z') \leq \nu_N(Y)$, and if $Z'$ is the largest cyclic flat of $N$ contained in $e_N(Y)$, then $|Y \cap Z'| - r_N(Z')$ is $\nu_N(Y)$.

By [6, Proposition 7.2], the free product $M \boxtimes N$ is the freeset matroid (in the weak order) on $E(M) \cup E(N)$ whose restriction to $E(M)$ is $M$ and whose contraction to
E(N) is N; also, the direct sum M ⊕ N is the least matroid in the weak order with these specified minors. Both operations are simple from the perspective of cyclic flats: the lattice of cyclic flats of M ⊕ N is isomorphic to the direct product of the lattices of cyclic flats of M and N and, as noted above, the lattice of cyclic flats of M □ N is related to the linear sum of those for M and N.

Recall that the Tutte polynomial of a matroid M is given by
\[ t(M; x, y) = \sum_{A \subseteq E(M)} (x-1)^{r(M)-r(A)}(y-1)^{r(A)}. \]

We now show that, given the Tutte polynomials of two matroids, it is easy to compute the Tutte polynomial of their free product. However, unlike the simple formula \( t(M \oplus N; x, y) = t(M; x, y)t(N; x, y) \) for direct sums, for the free product we get a formula for each coefficient in the Tutte polynomial. We use the next lemma [5, Proposition 3.5], which follows from Theorem 4.2.

**Lemma 4.3.** For \( X \subseteq E(M) \) and \( Y \subseteq E(N) \), we have
\[ r_{M \boxtimes N}(X \cup Y) = r_M(X) + r_N(Y) + \min\{r(M) - r_M(X), \nu_N(Y)\}. \]

**Theorem 4.4.** The Tutte polynomial of a free product can be computed in polynomial time in the size of its ground set from the Tutte polynomials of its factors.

**Proof.** First note that \( r(M \boxtimes N) = r(M) + r(N) \), so for \( X \subseteq E(M) \) and \( Y \subseteq E(N) \), Lemma 4.3 gives
\[ r(M \boxtimes N) - r_{M \boxtimes N}(X \cup Y) = r(M) - r_M(X) + r(N) - r_N(Y) - \min\{r(M) - r_M(X), \nu_N(Y)\} \]
and
\[ \nu_{M \boxtimes N}(X \cup Y) = \nu_M(X) + \nu_N(Y) - \min\{r(M) - r_M(X), \nu_N(Y)\}. \]

We prove the theorem using the Whitney rank generating function \( R(M; x, y) \), which is \( t(M; x+1, y+1) \). Thus, the coefficient of \( x^i y^j \) in \( R(M; x, y) \) is the number of subsets \( A \) of \( E(M) \) with \( r(M) - r_M(A) = i \) and \( \nu_M(A) = j \). Let \( R(M; x, y) \) and \( R(N; x, y) \) be
\[ R(M; x, y) = \sum_{0 \leq i \leq r(M)} a_{ij} x^i y^j \quad \text{and} \quad R(N; x, y) = \sum_{0 \leq k \leq r(N)} b_{kl} x^k y^l. \]

The number of coefficients in these polynomials is at most \((\nu(M) + 1)(r(M) + 1)\) and \((\nu(N) + 1)(r(N) + 1)\), respectively. By the equations in the first sentence, the coefficient of \( x^p y^q \) in \( R(M \boxtimes N; x, y) \) is
\[ \sum_{i,j,k,l} a_{ij} b_{kl}. \]

Thus, each of the \((|E(M)| + |E(N)| + 1)^2 \) or fewer coefficients of \( R(M \boxtimes N; x, y) \) can be computed with polynomially many steps from \( R(M; x, y) \) and \( R(N; x, y) \), so \( R(M \boxtimes N; x, y) \) can be computed from \( R(M; x, y) \) and \( R(N; x, y) \) in polynomial time in \(|E(M)| + |E(N)|\). \(\square\)
5. Cyclic Width

Order-theoretic properties of the lattice of cyclic flats can be used to define some classes of matroids; this section begins to explore some such classes.

The width of a lattice is the maximal cardinality of an antichain (i.e., a set of incomparable elements) in the lattice, so we define the cyclic width of a matroid $M$ to be the width of $\mathcal{Z}(M)$. For instance, the matroids in Figure 1 have cyclic width 4. We use $CW(k)$ to denote the class of all matroids whose cyclic width is at most $k$. By Lemma 2.2, the class of nested matroids is $CW(1)$. We first show that $CW(k)$ is closed under several basic matroid operations.

**Theorem 5.1.** The class $CW(k)$ is closed under duals, minors, and free products.

*Proof.* Closure under duals and free products follows from the views of these operations in terms of cyclic flats given in Sections 1 and 2. For the result on minors, it suffices to show that if $M \in CW(k)$ and $x \in E(M)$, then $M \setminus x$ is in $CW(k)$. If this were false, then there would be flats $X_1, X_2, \ldots, X_{k+1}$ in $\mathcal{Z}(M \setminus x)$ with $X_i \not\subseteq X_j$ for $i \neq j$. Note that exactly one of $X_i$ and $X_i \cup x$ would be in $\mathcal{Z}(M)$; let $X_1', X_2', \ldots, X_{k+1}'$ be these flats in $\mathcal{Z}(M)$. That $M$ is in $CW(k)$ gives the inclusion $X_i' \subseteq X_j'$ for some distinct $i$ and $j$. This inclusion gives the contradiction $X_i \subseteq X_j$, so $M \setminus x$ is, as claimed, in $CW(k)$.  

Recall that the truncation $T(M)$ of $M$ is $(M + e)/e$; this is the matroid on $E(M)$ whose bases are the independent sets of $M$ of size $r(M) - 1$. The Higgs lift is the dual operation.

**Corollary 5.2.** The class $CW(k)$ is closed under free extension, free coextension, truncation, and the Higgs lift.

Note that $CW(k)$ is not closed under direct sums.

We make some remarks about the excluded minors for $CW(k)$. The excluded minors for $CW(1)$ were shown in [14] to be $P_n$, for $n \geq 2$, where $P_n$ is the (iterated) truncation to rank $n$ of $U_{n-1,n} \oplus U_{n-1,n}$. Likewise, $CW(k)$ has infinitely many excluded minors, including, for $n \geq 2$, the truncation to rank $n$ of the direct sum of $k+1$ copies of $U_{n-1,n}$; the dual is also an excluded minor. For $k > 1$, there are other excluded minors for $CW(k)$, such as the truncation to rank 3 of the $(k+1)$-whirl. Not all excluded minors for $CW(k)$ have cyclic width $k + 1$. Indeed, as $k$ grows, the difference between $k$ and the cyclic width of an excluded minor for $CW(k)$ can be arbitrarily large, as the following example shows. Let $M_n$ be the truncation to rank three of the rank-$(n + 1)$ binary projective geometry $PG(n, 2)$. We claim that $M_n$ is an excluded minor for $CW(k)$, where $k = (2^{n+1} - 4)(2^n - 1)/3$, and the cyclic width of $M_n$ exceeds $k$ by $2^n - 1$. The cyclic width of $M_n$ is the number of lines of $PG(n, 2)$, which is $(2^{n+1} - 1)(2^n - 1)/3$. Every element of $PG(n, 2)$ is on $2^n - 1$ lines, all of which have three points, so every single-element deletion of $M_n$ has cyclic width $(2^{n+1} - 1)(2^n - 1)/3 = (2^n - 1)$ (which is $k$) and every single-element contraction of $M_n$ has cyclic width $2^n - 1$.

We now turn to $CW(1)$. The class of nested matroids has many interesting properties, which partly explains why these matroids have been introduced a number of times in different contexts (see [2, Section 4]). Recall that a class of matroids is well-quasi-ordered if it contains no infinite antichain in the minor order, that is, there is no infinite set of matroids in the class none of which is isomorphic to a minor of another. Theorem 5.3 shows that $CW(1)$ is an example (apparently the
first known) of a well-quasi-ordered class of matroids that, as mentioned above, has infinitely many excluded minors. This result, along with well-known examples, shows that there is no connection between the following two properties that a minor-closed class $\mathcal{M}$ of matroids may have: (a) $\mathcal{M}$ is well-quasi-ordered; (b) $\mathcal{M}$ has a finite set of excluded minors.

To prove that nested matroids are well-quasi-ordered, we use Higman’s theorem [12], which is stated in Lemma 5.3 (see also, e.g., [19, Theorem 5.2]). Recall that if $X$ is quasi-ordered, then the set of finite sequences in $X$ is quasi-ordered as follows: for $x = x_1, x_2, \ldots, x_m$ and $y = y_1, y_2, \ldots, y_n$, set $x \leq y$ if there are integers $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ with $x_j \leq y_{i_j}$ for $1 \leq j \leq m$.

**Lemma 5.3.** If $X$ is well-quasi-ordered, then so is the set of finite sequences of $X$.

**Theorem 5.4.** Nested matroids are well-quasi-ordered.

**Proof.** We treat isomorphic matroids as equal. A nested matroid $M$ is formed from the empty matroid by applying two operations: (i) adding an isthmus and (ii) adding an element freely. Let $v_M$ be the corresponding sequence of $i$’s and $f$’s. Consider the order on $\{i, f\}$ in which $i$ and $f$ are incomparable; this is a well-quasi-order. The theorem follows from Lemma 5.3 by showing that for nested matroids $M$ and $N$, if $v_N \leq v_M$, then $N$ is a minor of $M$. Indeed, $v_N \leq v_M$ means that $v_N$ is a subsequence of $v_M$, so to obtain $N$, from $M$ remove the elements that do not contribute to this subsequence, deleting those that were added freely and contracting those that were added as isthmuses. □

A. M. H. Gerards [10] has noted that Theorem 5.4 is a concrete instance of a general result: any quasi-order $X$ that is not a well-quasi-order and that has no infinite descending chains contains a well-quasi-order $Y$ for which infinitely many elements in $X$ are minimal in $X - Y$. This follows by letting $Y$ be the set of all elements of $X$ that are smaller than at least one element of a fixed minimal bad sequence (as defined in [9]).

Note that CW(2) is not well-quasi-ordered; the excluded minors for CW(1) (the matroids $P_n$ mentioned above) are an infinite antichain in CW(2).

It appears that few matroids in CW(2) have been studied previously. Acketa [1] proved that any matroid whose lattice of cyclic flats is a product of two 2-element chains is transversal. Our next result shows that this conclusion holds for all matroids in CW(2). This result is a consequence of the following characterization of transversal matroids due to Ingleton [13], which refines a result of Mason.

**Lemma 5.5.** A matroid is transversal if and only if for every nonempty family $(X_1, \ldots, X_n)$ of cyclic flats of $M$,

$$r(X_1 \cap X_2 \cap \cdots \cap X_n) \leq \sum_{J \subseteq \{1,2,\ldots,n\}} (-1)^{|J|+1} r\left( \bigcup_{j \in J} X_j \right).$$

In Lemma 5.5, it suffices to consider antichains of cyclic flats since if $X_i \subseteq X_j$, then omitting $X_j$ does not change either side of inequality (3). Indeed, the terms on the right side that include $X_j$ cancel via the involution that adjoins or omits $X_i$.

**Theorem 5.6.** Matroids in CW(2) are bitransversal.

**Proof.** By duality it is enough to show that matroids in CW(2) satisfy the condition in Lemma 5.5. In CW(2), antichains of cyclic flats contain at most two
flats. Equality holds in inequality (3) for \( n = 1 \). For \( n = 2 \), inequality (3) is the semimodular inequality, \( r(X_1 \cap X_2) \leq r(X_1) + r(X_2) - r(X_1 \cup X_2) \).

It follows from the definition of nested matroids that there are, up to isomorphism, \( 2^n \) nested matroid on \( n \) elements. In contrast, the class \( CW(2) \) is superexponential, as shown by the following result due to O. Giménez [11].

**Theorem 5.7.** In \( CW(2) \), there are at least \( n! \) matroids on \( 4n + 5 \) elements that all have isomorphic lattices of cyclic flats.

**Proof.** Let \( S \) be \( \{a, a', a'', b, b', x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n, w_1, \ldots, w_n\} \). For a permutation \( \sigma \) of \( \{1, 2, \ldots, n\} \), let \( Z_\sigma \) be \( \{\emptyset, A_0, A_1, \ldots, A_n, B_0, B_1, \ldots, B_n, S\} \) where

\[
A_0 = \{a, a', a'', x_1, \ldots, x_n\}, \\
B_0 = \{b, b', y_1, \ldots, y_n\}, \\
A_i = A_{i-1} \cup \{z_i, w_i\}, \text{ for } 1 \leq i \leq n, \\
B_i = B_{i-1} \cup \{z_i, w_{\sigma(i)}\}, \text{ for } 1 \leq i \leq n.
\]

Note that the only inclusions among these sets are \( \emptyset \subset A_0 \subset \cdots \subset A_n \subset S \) and \( \emptyset \subset B_0 \subset \cdots \subset B_n \subset S \). Also, \( Z_\sigma \) is a lattice. Define \( r : Z_\sigma \to \mathbb{Z} \) by \( r(\emptyset) = 0 \), \( r(A_i) = r(B_i) = n + 1 + i \) for \( 0 \leq i \leq n \), and \( r(S) = 2n + 2 \). We show that the pair \( (Z_\sigma, r) \) satisfies conditions (Z1)–(Z3) of Theorem 3.2 and so defines a matroid \( M_\sigma \). Conditions (Z1) and (Z2) hold by construction. For condition (Z3), we need

\[
r(A_i) + r(B_j) \geq r(A_i \lor B_j) + r(A_i \land B_j) + |(A_i \land B_j) - (A_i \land B_j)|
\]

for all \( i, j \) between 0 and \( n \). The left side is \( n + i + 1 + n + j + 1 \); since \( A_i \lor B_j = S \) and \( A_i \land B_j = \emptyset \), the right side is at most \( 2n + 2 + 2 \min\{i, j\} \), so the required inequality holds. Finally, note that \( \sigma \) can be recovered from \( Z(M) \) for any matroid \( M \) isomorphic to \( M_\sigma \), so different permutations give nonisomorphic matroids. \( \square \)

The construction in this proof can be adapted (using \( k - 1 \) permutations) to show that there are at least \( (n!)^{k-1} \) nonisomorphic matroids in \( CW(k) \) on a set of \( (k + 2)n + \binom{k+2}{2} - 1 \) elements, all with isomorphic lattices of cyclic flats.

We now consider the matroids in \( CW(3) \) that are binary, ternary, or have rank 3. (See [13] or [15] for the definitions of gammoids and base-orderable matroids.)

**Theorem 5.8.** Binary matroids in \( CW(3) \) are direct sums of series-parallel networks. Rank-3 matroids in \( CW(3) \) are base-orderable. Ternary matroids in \( CW(3) \) are gammoids.

**Proof.** Since \( M(K_4) \) has cyclic width 4 and \( CW(3) \) is minor-closed, no matroid in \( CW(3) \) has an \( M(K_4) \)-minor. The first and second assertions follow since (i) binary matroids with no \( M(K_4) \)-minor are direct sums of series-parallel networks [15] Theorem 13.4.9 and (ii) a rank-3 matroid is base-orderable if and only if it has no \( M(K_4) \)-restriction [13] Theorem 14.]. The third statement follows similarly since the ternary excluded minors of ternary gammoids have cyclic width at least 4 (the excluded minors for ternary gammoids were found in [14]). \( \square \)

### 6. Open Problems

We close with several problems suggested by the topics of this paper.

Theorem 2.1 states that every lattice is isomorphic to the lattice of cyclic flats of a matroid. The problem is more subtle if we drop the phrase “isomorphic to”.

Problem 6.1. Characterize the lattices of sets that are lattices of cyclic flats of matroids.

In other words: for which lattices $\mathcal{Z}$ of sets can one find a function $r: \mathcal{Z} \rightarrow \mathcal{Z}$ so that conditions (Z1)–(Z3) of Theorem 3.2 hold?

Problem 6.2. What is the largest number of cyclic flats that a matroid on $n$ elements can have?

The direct sum of $\lfloor n/2 \rfloor$ copies of $U_{1,2}$, along with $U_{1,1}$ if $n$ is odd, has $n$ elements and $2^{\lfloor n/2 \rfloor}$ cyclic flats. For $n \geq 7$, the number of cyclic flats is greater in binary spikes (with or without a tip, according to the parity of $n$), and, for sufficiently large $n$, even greater in certain ternary matroids that are similar to spikes. We are not currently aware of any plausible candidates for matroids that may maximize the numbers of cyclic flats.

Another line of research is to investigate which properties of CW(1) have counterparts for other classes CW($k$). For instance, matroids in CW(1) and CW(2) are transversal, and those in CW(1) have presentations by particularly simple set systems (see [3, 16]). Is there an interesting description of some presentations of matroids in CW(2)? Under certain linear orders, the broken circuit complex of a matroid in CW(1) is the independence complex of another matroid in CW(1) (see [3]). Do matroids in CW(2) have a similar property? Are all matroids in CW(3) gammoids? A weaker question is this: are all matroids in CW(3) base-orderable? (This is not the case for CW(4) since $M(K_4)$ is both the smallest matroid that is not a gammoid and the smallest matroid that is not base-orderable.)

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