Abstract. Here we prove that every symmetric separable state admits a convex decomposition into symmetric pure product states. The same proof shows that every antisymmetric state is entangled. The decomposition sheds new light on numerical ranges useful to study ground state problems of infinite bosonic systems.

1. Introduction

Bosons are quantum mechanical particles described by density operators acting on the symmetric tensor product of a one-particle Hilbert space [6]. A classic result by Størmer [23] shows the $k$-particle marginal of an infinite bosonic system is a convex combination of symmetric pure product states. This result is known as the *quantum de Finetti theorem* in statistical mechanics [17, 26] and in quantum information theory [10, 16]. Besides the de Finetti theorem, the interrelation between symmetry and correlation is a more general topic in quantum information theory [4, 11, 20].

The mentioned symmetric pure product states are separable states, that is to say, they are convex combinations of product states. Here we prove the converse statement: every symmetric separable state admits a convex decomposition into symmetric pure product states. The idea of the proof is that a) the set of symmetric states is a face of the set of all quantum states and b) all one-particle marginals of a symmetric state are identical. We are aware of two references to this problem, a conjecture by Qian and Chu [20] and an incomplete proof of an erroneous, stronger result [9]. Upon a sign change, our proof shows every antisymmetric state is entangled.

The motivation behind this work is to examine linear images $\Pi_{\text{sym}}$ of the set of symmetric pure product states. As per the quantum de Finetti theorem, the ground state energy of a local energy operator of an infinite bosonic system is the distance of the origin from a supporting hyperplane to $\Pi_{\text{sym}}$. Studying the convex hull $\text{conv}(\Pi_{\text{sym}})$ is therefore pivotal to understanding ground state phenomena of bosonic systems [7, 8, 25].

Quantum information theory uses numerical ranges to represent linear images of subsets of quantum states [8, 12, 18, 22]. We show the set $\text{conv}(\Pi_{\text{sym}})$ is a linear image of the set of symmetric separable states, a face of the set of separable states. This would enable studies of $\text{conv}(\Pi_{\text{sym}})$ by taking advantage of methods used with the set of separable states [2, 3, 5, 15].

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In Section 2 we recall that the set of quantum states supported on a subspace is a face of the set of all quantum states. Section 3 contains the main proofs, a question regarding the number of terms in the decomposition of a symmetric separable state, and a comment on the mentioned incomplete proof. In Section 4 we discuss numerical ranges, which we connect to the ground state energy of an infinite bosonic system in Section 5.

2. Decomposition of states supported on subspaces

We recall aspects of the convex geometry of the set of quantum states and point out that every state that contributes to a convex decomposition of a state supported on a subspace is itself supported on the same subspace.

Let $\mathcal{H}$ be a finite-dimensional Hilbert space, $\mathcal{B}(\mathcal{H})$ the Banach algebra of linear operators on $\mathcal{H}$, and $\mathcal{S}(\mathcal{H})$ the compact, convex set of density operators on $\mathcal{H}$, that is to say, positive operators with trace one, also known as states or quantum states. If $\mathcal{H} = \mathbb{C}^n$ for some $n \in \mathbb{N}$, we identify $\mathcal{B}(\mathcal{H})$ with the set $M_n$ of $n$-by-$n$ matrices, and $\mathcal{S}(\mathcal{H})$ with the compact, convex set of positive semi-definite matrices of trace one.

A projector on $\mathcal{H}$ is an idempotent, self-adjoint operator $P \in \mathcal{B}(\mathcal{H})$; this means that $P = P^2 = P^*$. The set of projectors on $\mathcal{H}$ is partially ordered by the relation $P \preceq Q$, which signifies that $Q - P$ is a positive operator. It is well known that the mapping that sends a projector $P$ on $\mathcal{H}$ to the image $\text{im}(P) = \{P(\varphi) : \varphi \in \mathcal{H}\}$ is a lattice isomorphism from the set of projectors on $\mathcal{H}$ to the set of subspaces of $\mathcal{H}$ partially ordered by inclusion.

A subset $F$ of a convex set $C$ is a face of $C$ if $F$ is convex and whenever $x \in F$ and $x = (1 - \lambda)y + \lambda z$ for some $\lambda \in (0, 1)$ and $y, z \in C$, then $y$ and $z$ are also in $F$. A point $x \in C$ is called an extreme point of $C$ if $\{x\}$ is a face of $C$.

Let $\mathcal{K} \subset \mathcal{H}$ be a subspace. We say an operator $A \in \mathcal{B}(\mathcal{H})$ is supported on $\mathcal{K}$ if the images $\text{im}(A) = \{A(\varphi) : \varphi \in \mathcal{H}\}$ of $A$ and $\text{im}(A^*) = \{A^*(\varphi) : \varphi \in \mathcal{H}\}$ of the adjoint $A^*$ are included in $\mathcal{K}$. Let

$$
(1) \quad \mathcal{S}(\mathcal{H}, \mathcal{K})
$$

denote the set of states on $\mathcal{H}$ that are supported on $\mathcal{K}$. It is well known that the mapping that sends a subspace $\mathcal{K}$ of $\mathcal{H}$ to the set $\mathcal{S}(\mathcal{H}, \mathcal{K})$ is a lattice isomorphism from the set of subspaces of $\mathcal{H}$ to the set of faces of $\mathcal{S}(\mathcal{H})$ partially ordered by inclusion [1]. All faces of the set of quantum states $\mathcal{S}(\mathcal{H})$ are compact, convex sets.

In particular, the set of extreme points of the set of quantum states $\mathcal{S}(\mathcal{H})$, also called pure states on $\mathcal{H}$, is the set of projectors of rank one,

$$
(2) \quad \text{ext}(\mathcal{S}(\mathcal{H})) = \{|\varphi\rangle\langle\varphi| : \varphi \in \mathcal{H}, \langle\varphi|\varphi\rangle = 1\}.
$$

In quantum information theory, the projector onto the span of a unit vector $\varphi \in \mathbb{C}^n$ is denoted by $|\varphi\rangle\langle\varphi|$; both $\varphi$ and $|\varphi\rangle\langle\varphi|$ are referred to as pure states.

Recall that the relative interior of a convex subset $C$ of a Euclidean vector space is the interior of $C$ with respect to the topology of the affine hull of $C$.

**Lemma 1.** Let $\mathcal{K} \subset \mathcal{H}$ be a subspace and let $\rho \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ be a state supported on $\mathcal{K}$. Let $\lambda_1, \ldots, \lambda_d > 0$ be strictly positive real numbers that sum up to one and let $\rho = \sum_{i=1}^d \lambda_i \rho_i$ be a convex combination of states $\rho_1, \ldots, \rho_d \in \mathcal{S}(\mathcal{H})$. 

Then each state $\rho_i$ is supported on $K$. If $\rho_i = |\varphi_i\rangle\langle\varphi_i|$ is the pure state corresponding to a unit vector $\varphi_i \in \mathcal{H}$, then $\varphi_i$ lies in $K$ for $i = 1, \ldots, d$.

Proof. By Theorem 6.9 of [21], the state $\rho$ belongs to the relative interior of the convex hull $C = \text{conv}(|\rho_1, \ldots, \rho_d\rangle\langle\rho_1, \ldots, \rho_d|$ of the points $\rho_1, \ldots, \rho_d$. Since $\rho$ lies in the face $\mathcal{S}(\mathcal{H}, K)$, Theorem 18.1 of [21] shows that the whole convex set $C$ is included in $\mathcal{S}(\mathcal{H}, K)$. This proves the first statement; the second statement follows as the image of $|\varphi_i\rangle\langle\varphi_i|$ is spanned by $\varphi_i$. □

The spectral projections of a state are supported on the same subspace as the state.

Remark 1. Every state $\rho \in \mathcal{S}(\mathcal{H}, K)$ is the convex combination of at most $\dim_{\mathbb{C}}(K)$ pure states supported on $K$. It suffices to write $\rho$ as a convex combination of its spectral projections. Lemma 1 shows the spectral projections are supported on $K$.

3. Decomposition of Symmetric Separable States

We show that every symmetric separable state is a convex combination of symmetric pure product states. The same proof shows all antisymmetric states are entangled. We comment on the paper [9] at the end of the section.

We label the units of a many-particle system by a finite set $\nu$. Let $\mathcal{H}$ be a finite-dimensional Hilbert space, the one-particle Hilbert space. The Hilbert space of the subsystem $\mu$ is the tensor product $\mathcal{H}^{\otimes \mu} = \bigotimes_{i \in \mu} \mathcal{H}$, the identity operator of which we denote by $\mathbb{1}_{\mu}$, for all subsets $\mu \subset \nu$. We replace $\nu$ with its cardinality $|\nu|$ if convenient, for example to write symmetric product states $\sigma^{[\nu]} = \sigma^{\nu} = \bigotimes_{i \in \nu} \sigma$ where $\sigma \in \mathcal{S}(\mathcal{H})$ is a state on the one-particle Hilbert space.

Writing the complement as $\overline{\nu} = \nu \setminus \mu$, we define the partial trace over the subsystem $\overline{\nu}$ as the linear map $\text{Tr}_{\overline{\nu}} : \mathcal{B}(\mathcal{H}^{\otimes \nu}) \to \mathcal{B}(\mathcal{H}^{\otimes \mu})$, where $\text{Tr}_{\overline{\nu}}(A)$ is characterized by the conditions

$$\text{Tr} (\text{Tr}_{\overline{\nu}}(A)B) = \text{Tr} (A(B \otimes \mathbb{1}_{\overline{\nu}})), \quad B \in \mathcal{B}(\mathcal{H}^{\otimes \mu}),$$

for all operators $A \in \mathcal{B}(\mathcal{H}^{\otimes \nu})$. If $\rho$ is a state on $\mathcal{H}^{\otimes \nu}$, then $\text{Tr}_{\overline{\nu}}(\rho)$ is a state on $\mathcal{H}^{\otimes \mu}$, called the marginal of $\rho$ in the subsystem $\mu$.

A state $\rho \in \mathcal{S}(\mathcal{H}^{\otimes \nu})$ is a product state (with respect to $\mathcal{H}$ and $\nu$) if it can be written in the form $\rho = \bigotimes_{i \in \nu} \rho_i$, where $\rho_i \in \mathcal{S}(\mathcal{H})$ for all $i \in \nu$. Using pure states (2) we define the set of pure product states by

$$\mathcal{S}_{\text{pure,prod}}(\mathcal{H}, \nu) = \left\{ \bigotimes_{i \in \nu} \rho_i \mid \rho_i \in \text{ext}(\mathcal{S}(\mathcal{H})) \forall i \in \nu \right\}.$$

A state $\rho \in \mathcal{S}(\mathcal{H}^{\otimes \nu})$ is separable if $\rho$ is a convex combination of product states. Otherwise, $\rho$ is entangled. We denote the set of separable states by

$$\mathcal{S}_{\text{sep}}(\mathcal{H}, \nu).$$

The possibility to decompose each factor of a product state into pure states and the distributive law show $\mathcal{S}_{\text{sep}}(\mathcal{H}, \nu)$ is the convex hull of $\mathcal{S}_{\text{pure,prod}}(\mathcal{H}, \nu)$. In fact, $\mathcal{S}_{\text{pure,prod}}(\mathcal{H}, \nu)$ is the set of extreme points of $\mathcal{S}_{\text{sep}}(\mathcal{H}, \nu)$ as every pure product state is an extreme point of $\mathcal{S}(\mathcal{H}^{\otimes \nu})$ and hence of $\mathcal{S}_{\text{sep}}(\mathcal{H}, \nu)$. 

The symmetric group $S_\nu$ of the finite set $\nu$ acts by linear automorphisms on the many-particle Hilbert space $\mathcal{H}^{\otimes \nu}$ as per the law

$$\sigma \left( \bigotimes_{i \in \nu} \varphi_i \right) = \bigotimes_{i \in \nu} \varphi_{\sigma^{-1}(i)}, \quad \varphi_i \in \mathcal{H} \ \forall i \in \nu,$$

for all permutations $\sigma \in S_\nu$. The $\nu$-fold symmetric tensor product of the Hilbert space $\mathcal{H}$ is defined by

$$\mathcal{H}^{\otimes \nu}_{\text{sym}} = \left\{ \varphi \in \mathcal{H}^{\otimes \nu} \mid \sigma(\varphi) = \varphi \ \forall \sigma \in S_\nu \right\}. \quad (5)$$

It is easy to write down a basis for the symmetric tensor product.

**Remark 2.** Let $|0\rangle, |1\rangle, \ldots, |n-1\rangle$ be an orthonormal basis of $\mathcal{H}$. Let $T = (T_0, \ldots, T_{n-1})$ be an $n$-tuple of non-negative integers that sum up to the cardinality $|\nu|$ of $\nu$ and define

$$\varphi_T = c_T \sum_{\sigma \in S_\nu} \sigma \left( |0\rangle |0\rangle |1\rangle \ldots |n-1\rangle |n-1\rangle \right)_{T_0 T_1 \ldots T_{n-1}},$$

where $c_T = (|\nu|! \prod_{i=0}^{n-1} T_i!)^{-1/2}$ is a normalization constant. The orthonormal set $\{\varphi_T\}$ is a basis of the symmetric tensor product $\mathcal{H}^{\otimes \nu}_{\text{sym}}$, because $\{\varphi_T\}$ spans the image of the projector $\mathcal{H}^{\otimes \nu} \to \mathcal{H}^{\otimes \nu}_{\text{sym}}$ defined by $\frac{1}{|\nu|!} \sum_{\sigma \in S_\nu} \sigma$, see [13, Section 4.10]. For example, the set of Dicke states [19] with $k$ excitations,

$$\varphi_{(|\nu|-k,k)} = (|\nu|! (|\nu|-k)! k!)^{-1/2} \sum_{\sigma \in S_\nu} \sigma \left( |0\rangle |0\rangle |1\rangle \ldots |n-1\rangle \right)_{|\nu|-k k}, \quad k = 0, \ldots, |\nu|,$$

is an orthonormal basis of the $\nu$-fold symmetric tensor product of $\mathbb{C}^2$.

**Definition 1 (Symmetric States).** A state $\rho \in \mathcal{S}(\mathcal{H}^{\otimes \nu})$ is symmetric (with respect to $\mathcal{H}$ and $\nu$) if $\rho$ is supported on the symmetric tensor product $\mathcal{H}^{\otimes \nu}_{\text{sym}}$ introduced in equation (5). Using the notation from equation (1) and (4), we denote the set of symmetric separable states by

$$\mathcal{S}_{\text{sym,sep}}(\mathcal{H}, \nu) = \mathcal{S}(\mathcal{H}^{\otimes \nu}_{\text{sym}}) \cap \mathcal{S}_{\text{sep}}(\mathcal{H}, \nu). \quad (6)$$

Every convex combination of symmetric pure product states $\sigma^{\otimes \nu}$ is a symmetric separable state. Here we prove the converse statement that every symmetric separable state is a convex combination of symmetric pure product states, thereby confirming a conjecture by Qian and Chu [20].

**Theorem 1.** The set of symmetric separable states is the convex hull of the set of symmetric pure product states,

$$\mathcal{S}_{\text{sym,sep}}(\mathcal{H}, \nu) = \text{conv} \left( \{ \sigma^{\otimes \nu} \mid \sigma \in \text{ext}(\mathcal{S}(\mathcal{H})) \} \right). \quad (7)$$

The symmetric pure product states are the extreme points of $\mathcal{S}_{\text{sym,sep}}(\mathcal{H}, \nu)$,

$$\text{ext} \left( \mathcal{S}_{\text{sym,sep}}(\mathcal{H}, \nu) \right) = \{ \sigma^{\otimes \nu} \mid \sigma \in \text{ext}(\mathcal{S}(\mathcal{H})) \}. \quad (8)$$

**Proof.** In order to prove (7) it suffices to show that every symmetric separable state $\rho$ is a convex combination of symmetric pure product states. Being a separable state, $\rho$ admits a convex decomposition $\rho = \sum_{i=1}^{d} \lambda_i |\varphi_i\rangle \langle \varphi_i|$ with positive coefficients $\lambda_1, \ldots, \lambda_d > 0$ into pure product states

$$\varphi_i = \bigotimes_{j \in \nu} \psi_{i,j} \in \mathcal{H}^{\otimes \nu},$$
where \( \psi_{i,j} \in H \) is a pure state for all \( i = 1,\ldots,d \) and \( j \in \nu \). Since \( \rho \) is supported on the symmetric tensor product \( H^\otimes \nu \), Lemma 1 shows \( \varphi_i \) lies in the symmetric tensor product \( H^\otimes \nu \) for all \( i = 1,\ldots,d \). This implies the one-particle marginals of \( \varphi_i \) are identical to a pure state \( \psi_i \in H \),

\[
|\psi_i \rangle \langle \psi_i| = |\psi_{i,j} \rangle \langle \psi_{i,j}| = \text{Tr}_{\{j\}}(|\varphi_i \rangle \langle \varphi_i|), \quad j \in \nu,
\]

for all \( i = 1,\ldots,d \), which shows \( \rho = \sum_{i=1}^d \lambda_i \varphi_i \) is a symmetric pure product state. Thus, \( \rho \) is an extreme point of \( S_{\text{sym,sep}}(H,\nu) \) and proves the claim.

The inclusion "\( \subset \)" in equation (7) implies that every extreme point of \( S_{\text{sym,sep}}(H,\nu) \) must be a symmetric pure product state. Conversely, every symmetric pure product state is an extreme point of \( S(H^\otimes \nu) \) and hence of \( S_{\text{sym,sep}}(H,\nu) \), which yields equation (8).

**Question 1.** How many symmetric pure product states are required in the decomposition of a symmetric separable state \( \rho \)? The rank of \( \rho \) may not be an upper bound on \( d \) (as in Remark 1) because the spectral projections of \( \rho \) may be entangled. Examples of separable two-qubit states of rank three that require four pure product states in the decomposition are described in [24]. How is the situation in the symmetric setting?

The \( \nu \)-fold **antisymmetric tensor product** of the Hilbert space \( H \) is defined by

\[
\left\{ \varphi \in H^\otimes \nu \mid \sigma(\varphi) = \text{sgn}(\sigma)|\varphi \quad \forall \sigma \in \mathcal{S}_\nu \right\},
\]

where \( \text{sgn}(\sigma) \) is the sign of a permutation \( \sigma \in \mathcal{S}_\nu \), which is +1 if \( \sigma \) is a composition of an even number of transpositions, and −1 else. A state on \( H^\otimes \nu \) is antisymmetric if it is supported on the antisymmetric tensor product.

**Theorem 2.** If \( \nu \) has at least two elements, then every antisymmetric state is entangled.

**Proof.** The same reasoning as in Theorem 1 shows every antisymmetric separable state is a convex combination of symmetric pure product states. Hence, the existence of an antisymmetric separable state would lead to a contradiction. \( \Box \)

The paper [9] contains an incomplete proof of an erroneous, stronger version of Theorem 1.

**Remark 3.** Chen et al. are claiming in Theorem 14 of [9] that every symmetric separable state \( \rho \in S_{\text{sym,sep}}(\mathbb{C}^n,\nu) \) admits a convex decomposition into symmetric pure product states \(|\varphi^\otimes \nu \rangle \langle \varphi^\otimes \nu|\), where \( \varphi \in \mathbb{R}^n \) is a pure state with real coefficients.

Besides being a wrong statement, Theorem 14 in [9] offers only an incomplete proof of Theorem 1. The authors begin by observing that every symmetric separable state has a convex decomposition \( \sum_{i=1}^d \lambda_i |\varphi_i \rangle \langle \varphi_i| \) into pure product states \( \varphi_i \in H^\otimes \nu \). Afterward, they assert that \( \varphi_i \) is a symmetric state without providing evidence. We solve this problem in Theorem 1 using convex geometry (Lemma 1).
Theorem 14 of [9] is clearly untenable. To each pure state $\varphi \in \mathbb{C}^n$ would correspond a real pure state $\psi \in \mathbb{R}^n$ such that $\varphi \otimes \varphi = e^{i\theta} \psi \otimes \psi$, where $\theta \in [0, 2\pi)$ is a phase. All coefficients of $\varphi$ would lie on the real line through the origin in $\mathbb{C}$ spanned by $e^{i\theta/2}$; a counterexample is $\varphi = \frac{1}{\sqrt{2}}$.

Chen et al. justify this fallacy with the claim that every symmetric state would be invariant under transposition (see [9, Lemma 4]). This would again imply that the coefficients of every vector $\varphi \in \mathbb{C}^n$ lie on a real line through the origin of $\mathbb{C}$, as hermitian matrices invariant under transposition are real. Chen et al., ignoring the Bell states $|00\rangle + \sqrt{2}|11\rangle$ and $|01\rangle + \sqrt{2}|10\rangle$, also state that every symmetric state is invariant under partial transposition and that all symmetric many-qubit states are separable.

4. Joint Numerical Ranges

Numerical ranges, originally a topic of matrix theory, have become indispensable tools in quantum information theory [8, 12, 18, 22] to study linear images of certain subsets of quantum states. We show two notions of symmetric numerical ranges to be equal.

Let $A_1, \ldots, A_m \in B(\mathcal{H}^\otimes \nu)$ be a sequence of hermitian operators. Using the sets of pure states (2), pure product states (3), and separable states (4), we define the numerical ranges [8, 12]

$\Lambda = \{ \text{Tr}(\rho A_i)_{i=1}^m \mid \rho \in \text{ext}(\mathcal{G}(\mathcal{H}^\otimes \nu)) \}$, \hspace{1cm} (joint numerical range)

$\Pi = \{ \text{Tr}(\rho A_i)_{i=1}^m \mid \rho \in \mathcal{G}_{\text{pure,prod}}(\mathcal{H}, \nu) \}$, \hspace{1cm} (−"− product numerical range)

$\Theta = \{ \text{Tr}(\rho A_i)_{i=1}^m \mid \rho \in \mathcal{G}_{\text{sep}}(\mathcal{H}, \nu) \}$, \hspace{1cm} (−"− separable −"− −"−)

Symmetric numerical ranges have been studied, too [8]. Using symmetric pure product states, we define the set

(9) $\Pi_{\text{sym}} = \{ \text{Tr}(\sigma^\otimes \nu A_i)_{i=1}^m \mid \sigma \in \text{ext}(\mathcal{G}(\mathcal{H})) \}$,

which we call joint symmetric product numerical range. Employing the set of symmetric separable states (6), we define the set

$\Theta_{\text{sym}} = \{ \text{Tr}(\rho A_i)_{i=1}^m \mid \rho \in \mathcal{G}_{\text{sym,sep}}(\mathcal{H}, \nu) \}$,

which we call joint symmetric separable numerical range.

Recalling equation (3), we observe the set of separable states is the convex hull of the set of pure product states; therefore $\Theta = \text{conv}(\Pi)$. The analogue is true for symmetric numerical ranges.

**Corollary 1.** The joint symmetric separable numerical range $\Theta_{\text{sym}}$ is the convex hull of the joint symmetric product numerical range $\Pi_{\text{sym}}$.

**Proof.** Theorem 1 proves the set of symmetric separable states $\mathcal{G}_{\text{sym,sep}}(\mathcal{H}, \nu)$ is the convex hull of the set $\{ \sigma^\otimes \nu \mid \sigma \in \text{ext}(\mathcal{G}(\mathcal{H})) \}$ of symmetric pure product states. The claim follows as $\Theta_{\text{sym}}$ is the image of the former set and $\Pi_{\text{sym}}$ is the image of the latter set under the same linear map. \(\square\)

Corollary 1 clarifies a relationship between two numerical ranges.

**Remark 4.** Chen et al. [8] have defined the symmetric version of the joint separable numerical range $\Theta$ to be the convex hull $\text{conv}(\Pi_{\text{sym}})$ of the joint
symmetric product numerical range \( \Pi_{\text{sym}} \). This brings to mind the equation \( \Theta = \text{conv}(\Pi) \). The analogous equation \( \Theta_{\text{sym}} = \text{conv}(\Pi_{\text{sym}}) \) shows the set \( \text{conv}(\Pi_{\text{sym}}) \) is a linear image of the set of symmetric separable states \( \mathcal{S}_{\text{sym,sep}}(\mathcal{H}, \nu) \), thereby corroborating \( \text{conv}(\Pi_{\text{sym}}) \) is a natural choice for a symmetric version of \( \Theta \).

Corollary 1 offers a new approach to study the set \( \text{conv}(\Pi_{\text{sym}}) \).

**Remark 5.** The identity \( \text{conv}(\Pi_{\text{sym}}) = \Theta_{\text{sym}} \) allows us to study the set \( \text{conv}(\Pi_{\text{sym}}) \) via the set of symmetric separable states \( \mathcal{S}_{\text{sym,sep}}(\mathcal{H}, \nu) \), the preimage of the numerical range \( \Theta_{\text{sym}} \). The convex set \( \mathcal{S}_{\text{sym,sep}}(\mathcal{H}, \nu) \) is a face of the set of separable states \( \mathcal{S}_{\text{sep}}(\mathcal{H}, \nu) \), for which algebraic tools are available [2, 3, 5, 15]. Studying linear images of \( \mathcal{S}_{\text{sym,sep}}(\mathcal{H}, \nu) \) in the context of statistical mechanics is interesting already for two qubits [7, 8], where every separable state has a positive partial transpose [14].

5. THE QUANTUM DE FINETTI THEOREM

This section highlights the role the joint symmetric separable numerical range \( \Theta_{\text{sym}} \) plays in the statistical mechanics of bosonic systems.

Symmetric states on the many-particle Hilbert space \( \mathcal{H}^{\otimes \nu} \) are called *bosonic states* in physics [6, 8, 17]. Due to the symmetry, the marginals of a symmetric state \( \rho \in \mathcal{S}(\mathcal{H}^{\otimes \nu}, \mathcal{H}^{\otimes \nu}_{\text{sym}}) \) are themselves symmetric states. The marginals of \( \rho \) depend on the subsystems only through their sizes,

\[
\text{Tr}_{\mu}(\rho) = \text{Tr}_{\eta}(\rho) \quad \text{for all } \mu, \eta \subset \nu, |\mu| = |\eta|.
\]

Thus, one studies the set of *\( k \)-particle \( N \)-representable density operators*

\[
\mathcal{P}^{(k)}_N = \left\{ \text{Tr}_{\{1, \ldots, k\}}(\rho) \mid \rho \in \mathcal{S}(\mathcal{H}^{\otimes N}, \mathcal{H}^{\otimes N}_{\text{sym}}) \right\} \subset \mathcal{S}(\mathcal{H}^{\otimes k}, \mathcal{H}^{\otimes k}_{\text{sym}})
\]

with reference to the total number \( N = |\nu| \) of units and the size \( k \) of the subsystem. According to a classic result by Størmer [23], also known as the *quantum de Finetti theorem* [8, 17, 26], we have

\[
\bigcap_{N \geq k} \mathcal{P}^{(k)}_N = \text{conv} \left\{ \{\sigma^{\otimes k} \mid \sigma \in \text{ext}(\mathcal{S}(\mathcal{H}))\} \right\}, \quad k \in \mathbb{N}.
\]

In other words, the set of all possible \( k \)-particle marginals of an infinite bosonic system is the convex hull of the set of symmetric pure product states. By Theorem 1,

\[
(10) \quad \bigcap_{N \geq k} \mathcal{P}^{(k)}_N = \mathcal{S}_{\text{sym,sep}}(\mathcal{H}, k), \quad k \in \mathbb{N},
\]

where \( \mathcal{S}_{\text{sym,sep}}(\mathcal{H}, k) = \mathcal{S}_{\text{sym,sep}}(\mathcal{H}, \nu) \) is the set of symmetric separable states on a system \( \nu \) of size \( |\nu| = k \).

Quantum phase transitions are associated with abrupt changes of the ground state energy of an energy operator

\[
H(x) = x_1 H_1 + x_2 H_2 + \cdots + x_m H_m
\]

on \( \mathcal{H}^{\otimes N} \) under smooth changes of the parameter \( x = (x_1, \ldots, x_k) \in \mathbb{R}^m \) for large system sizes \( N \to \infty \). In many cases the energy operators \( H_1, \ldots, H_m \) are *\( k \)-local Hamiltonians* for some fixed \( k \in \mathbb{N} \). This means

\[
H_i = \sum_{\mu \subset \nu, |\mu| = k} H_{i,\mu} \otimes 1_\mathcal{P}, \quad i = 1, \ldots, m,
\]
where $H_{i,\mu} \in \mathcal{B}(\mathcal{H}^{\otimes \mu})$ interacts with at most $|\mu| = k$ units of the total system $\nu = \{1, \ldots, N\}$. The ground state energy of a bosonic system with energy operator $H(x)$ is

$$
\min_{\rho \in \mathfrak{S}(\mathcal{H}^{\otimes N}, \mathcal{H}^{\otimes \text{sym}})} \text{Tr}(H(x)\rho) = \min_{\sigma \in \mathfrak{T}_N^{(k)}} \text{Tr}(A(x)\sigma),
$$

where $A(x) = \sum_{i=1}^m x_i A_i$ and

$$
A_i = \sum_{\mu \subseteq \nu, |\mu| = k} H_{i,\mu} \in \mathcal{B}(\mathcal{H}^{\otimes k}), \quad i = 1, \ldots, m.
$$

For this sum to be well-defined, one may assume each operator $H_{i,\mu}$ is supported on the symmetric tensor product $\mathcal{H}^{\otimes \text{sym}}$ or replace $H_{i,\mu}$ with its compression onto $\mathcal{H}^{\otimes \text{sym}}$. Provided the convergence of the energy operators $A_i$ is guaranteed in the thermodynamic limit $N \to \infty$, Størmer’s result in the version of equation (10) shows the ground state energy of $H(x)$ is

$$
\min_{\sigma \in \Theta_{\text{sym},\text{sep}}(\mathcal{H},k)} \text{Tr}(A(x)\sigma) = \min_{y \in \Theta_{\text{sym}}^{\otimes k}} x_1 y_1 + x_2 y_2 + \cdots + x_m y_m,
$$

where $\Theta_{\text{sym}}$ is the joint symmetric separable numerical range (9) with respect to the operators $A_1, \ldots, A_m \in \mathcal{B}(\mathcal{H}^{\otimes k})$.

In other words, the ground state energy of $H(x)$ is the distance of the origin from the supporting hyperplane to $\Theta_{\text{sym}}$ with inner normal vector $x$. This rationale strongly motivates to study the geometry of the numerical range $\Theta_{\text{sym}}$. The key-feature of a ruled surface on the boundary of $\Theta_{\text{sym}}$ is an expression of a phase transition [7,8,25] and deserves a thorough investigation.

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