Abstract

In this article we survey recent progress on mathematical results on gas flow in pipe networks with a special focus on questions of control and stabilization. We briefly present the modeling of gas flow and coupling conditions for flow through vertices of a network. Our main focus is on gas models for spatially one-dimensional flow governed by hyperbolic balance laws. We survey results on classical solutions as well as weak solutions. We present results on well-posedness, controllability, feedback stabilization, the inclusion of uncertainty in the models and numerical methods.

Keywords. Hyperbolic Balance Laws, Stabilization, Exact Controllability, Modeling of Gas Flow, Finite-Volume Schemes, Optimal control, Uncertainty

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1 Introduction

Gas transport on networks has been an active research topic for the past decade due to an increased demand in the sustainable use of natural resources and deregulation of energy markets. A particular focus has been on improved mathematical models of the physical flow for efficient simulation and for control purposes. In this article we contribute to the discussion by reviewing mathematical results on modeling, control and numerical methods for partial differential equations. Continuum mechanical models for gas flow in a single pipe are typically of the following type

\[ \frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho v \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho v \\ \rho v^2 + p(\rho) \end{pmatrix} = - \begin{pmatrix} 0 \\ f \rho v |v| + g \sin(\alpha)\rho \end{pmatrix} \]  

(1)

where \( \rho \) is the gas density, \( v \) the gas velocity, \( f \) is a friction parameter, \( p(\rho) \) denotes the pressure as a function of \( \rho \), \( g \) is the gravitational constant and \( \alpha \) is the slope of the pipe and where a spatially one-dimensional model is chosen due to the particularities of pipe flow with low Mach number. Those also imply that temperature can be neglected. However, the model is capable of capturing transient phenomena driven by the need of simulating transient gas flow. Those flow patterns appear e.g. in the case of starting up gas power plants. If one is interested in average states of the gas network simplified models might be sufficient. The model (1) is posed on a single pipe and it is coupled by suitable transmission conditions to a flow model on networked pipes. The coupling typically leads to boundary conditions for the hyperbolic partial differential equation. Valves, compressor or generator stations are actuators of the control system. Those act pointwise (not distributed) in space and they are modelled by control through the boundary conditions and
the corresponding coupling conditions in the networked system. Regarding detailed flow models, numerical challenges and further results, we cannot provide a complete list of reference at this point but refer to \cite{19, 114, 121, 31, 7, 104}, the references therein and the forthcoming sections. In the mathematical literature the study of hyperbolic balance laws, like (1), on metric graphs or networks has been studied over the past decade and we refer to the articles \cite{25, 32, 112, 44} and references therein for further details.

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2 Modeling of Gas Flow

We are interested in the evolution equation (2) below on a network (or metric graph), consisting of a collection of one-dimensional manifolds connected at nodes. A general network is represented by a directed graph $G = (\mathcal{E}, \mathcal{V})$ composed by a finite number of edges $\mathcal{E}$ connected by vertices or junctions. We use a directed graph to model the network but the discussion is not limited to this. For the sake of simplicity, we restrict the attention to a network that is composed by $n$ edges labeled by $j$ and modeled by the real interval $I_j := (0, L_j)$ where we could include the case $L_j = \infty$. We remark that this simplification does not imply a loss of generality due to the fact that the hyperbolic systems (2) has the property that waves propagate with finite speed. The density and the flux on edge $j \in \mathcal{E}$ will be denoted $\rho_j$ and $q_j = (\rho v)_j$. Without loss of generality the vertex is situated at $x_* = 0$ for all $j \in \mathcal{E}$. Note that in this particular network $v_j > 0$ implies the flow is emerging from the vertex. For a large, general network in order to achieve this property it might be necessary to transform the orientation of the flow at a vertex of the graph. This is achieved on the adjacent edge $j$ using the transformation $x \rightarrow -x$ and $q_j(t, x) \rightarrow -q_j(t, x)$. Note that this does not change the balance law (2).

2.1 Models for Gas Flow on Edges

For readability we skip the edge index in this section. Different scales might be relevant when modeling gas flow in pipe networks and we refer e.g. to \cite{7, 31, 78} for a detailed analysis of the involved scales as well as a corresponding hierarchy of suitable models. Here, we focus on a fine scale model for gas flow as given by the isentropic Euler equations

\begin{equation}
\partial_t \left( \frac{\rho}{\rho v} \right) + \partial_x \left( \rho v^2 + p(\rho) \right) = - \begin{pmatrix} 0 \\ f \rho v |v| + g \sin(\alpha) \rho \end{pmatrix}
\end{equation}

for the gas density $\rho = \rho(t, x)$ and the gas velocity $v = v(t, x)$. The gas flux is defined by $q(t, x) = (\rho v)(t, x)$ and $t \geq 0, x \in I$. The parameters $f$ and $\alpha = \alpha(x)$ are the friction coefficient and the slope of the pipe. The number $g$ denotes the gravitational force. It has been shown that often
it is not necessary to consider temperature variations \([104, 105]\). Recent publication focuses on realistic pressure laws extending the isentropic choice
\[ p(\rho) = \kappa \rho^\gamma \] (3)
for \( \kappa > 0 \) and \( \gamma \in [1, 3] \) towards
\[ p(\rho) = z(p)\rho^\gamma \] (4)
for some polynomial \( \rho \to z(\rho) \) of order at most two \([51, 67]\). For the \( z \)-factor, often an affine linear model of the form
\[ z(p) = R\Theta (1 + \alpha p) \] (5)
is used, where \( \alpha \in (-0.9, 0) \). Equation (5) is sufficiently accurate within the network operating range, see \([67, 69]\). Often ideal gas corresponding to \( \gamma = 1 \) is considered for control and stabilization. Equation (2) is accompanied by initial data
\[ \rho(0, x) = \rho_0(x) \] (6)
and boundary data at \( x = 0 \) obtained through coupling conditions discussed in detail in the forthcoming section. In the case of \( f = \alpha = 0 \) the system (2) can be obtained formally and rigorously \([97, 23, 84]\) through kinetic relaxation. For \( f = f(t, x, \xi) \in \mathbb{R}^2 \) with \( \xi \in \mathbb{R} \) the BGK model for a fixed parameter \( \epsilon > 0 \) reads
\[ \partial_t f(t, x, \xi) + \partial_x \xi f(t, x, \xi) = \frac{1}{\epsilon} \left( M[f](t, x, \xi) - f(t, x, \xi) \right). \] (7)
The Maxwellian \( M[f] = (M_1, M_2) \in \mathbb{R}^2 \) is given by
\[ M_1(t, x, \xi) = \chi(\rho(t, x), \xi - v(t, x)), \quad M_2(t, x, \xi) = \chi(\rho(t, x), \xi - v(t, x)) \left( (1 - \theta) v + \theta \xi \right) \] (8)
where \( \chi \) is a rational polynomial and \( (\rho, v) \) are the moments of the kinetic distributions, i.e.,
\[ \rho(t, x) = \int f_1(t, x, \xi) d\xi, \quad (\rho v)(t, x) = \int f_2(t, x, \xi) d\xi. \] (9)
In \([84]\) it has been shown that for fixed \( \epsilon > 0 \) there exist a solution \( f = f_\epsilon \) to equation (7) subject to suitable initial and boundary conditions. Furthermore, the sequence of associated densities and fluxes given by equation (9) converges towards a solution to equation (2) also in the presence of coupling conditions \((19)\). For further details we refer to \([84, \text{Theorem } 2.1}\).

In engineering application and connected with discrete decisions often stationary solutions to equation (2) and (4) are considered. The existence of steady states in the general case has been established in \([58, 67]\). Those stationary solutions are important for stabilization of gas flow. Furthermore, they also appear in the design of optimal controls due to the turnpike phenomenon. This will be discussed in detail in the section below.

In the domain of operation of the gas pipeline networks (low Mach numbers and large pressure) the flow model can also modelled by a degenerate parabolic model as studied e.g. by \([8, 91]\). Formally, the model is obtained from equation (2) by neglecting the momentum term \( \rho v^2 \) and the time derivative \( \partial_t (\rho v) \). After small computations one obtains a degenerate parabolic equation in the pressure \( p \) in the isentropic case.

For further models we refer to the literature, e.g. \([7, 31]\).

### 2.2 Models for the Flow Through Vertices

The modeling of gas flow in pipes as spatially one-dimensional flow has severe implications on the modeling of the dynamics at the vertex. In particular, the modeling solely relies on the traces of the gas density \( \rho_j(t, 0+) \) and gas flux \( q_j(t, 0+) \) for all adjacent edge \( j \) and possibly a control input \( u(t) \).

In general the modeling of the vertex is hence given by a nonlinear function \( \Psi : \mathbb{R}^+ \times (\mathbb{R}^2)^n \to \mathbb{R}^n \nabla \)
\[ \Psi \left( t, \rho_1(t, 0+), q_1(t, 0+), \ldots, \rho_n(t, 0+), q_n(t, 0+) \right) = u(t) \ a.e. \ t \geq 0. \] (10)
Thus the one-dimensional framework allows a rather simple modeling structure given by the form of $\Psi$, but very fast numerical integration. While a model in one space dimension well describes the dynamics within a pipe, it hardly covers geometry effects at a junction, which is clearly an intimately 3D phenomenon. As a consequence, the literature offers several different choices for a coupling or nodal condition $\Psi$ and equation (10), depending on the specific needs of each particular situation. In engineering literature, the nodal conditions $\Psi$ are typically accompanied by parameters whose values are empirically justified. A numerical study of one- and two-dimensional situations can be found e.g. in [12, 74, 82].

A detailed well-posedness analysis of equation (10) and (2) is deferred to the next section. However, it is important to notice that the condition (10) implicitly describes possible boundary conditions for the hyperbolic differential equation (2). This has implications on the modeling of suitable functions $\Psi$ as seen below.

A further important aspect in the modeling of flows through vertices is the control action, here denoted by a given function $u(t)$. Those functions may model the closure of valves or the supplied power for compressor stations [75]. Most examples in the literature consider explicit control actions as shown in equation (10) even so an implicit dependence on $u$ is possible. Additionally, the coupling condition might dependent explicitly on time $t \geq 0$ when fatigue of material is of importance.

In the following we turn to typical examples for $\Psi$. First, consider the case of subsonic data, i.e.,

$$\lambda_1(\rho_j(t, 0+), q_j(t, 0+)) < 0 < \lambda_2(\rho_j(t, 0+), q_j(t, 0+)).$$

(11)

Here, $\lambda_k(\rho, q)$ are the $k = 1, 2$ eigenvalues of the Jacobian of the flux function $f$. In the subsonic case we have therefore locally in time and phase space $n$ boundary conditions to be determined by equation (10). A similar consideration in the supersonic case is possible and has been investigated in [61]. In general, the boundary conditions are not explicit but rather obtained by physical modeling considerations.

The first element of $\Psi$ typically models the conservation of mass through the vertex and henceforth reads

$$\Psi_1(t, \rho_1, q_1, \ldots, \rho_n, q_n) = \sum_{j=1}^{n} q_j.$$  

(12)

Except for a coupling in the gas-to-power setting [79] there is no control active at the first component of $\Psi$. The other components of $\Psi$ may impose different physically desirable properties.

A condition typically used in the engineering community [104, 11] is to assume equal pressure $p$ at the vertex, i.e.,

$$\Psi_j(t, \rho_1, q_1, \ldots, \rho_n, q_n) = p(\rho_j) - p(\rho_1) \quad j = 2, \ldots, n.$$  

(13)

An alternative condition proposed in [11, 98] is to assume the continuity of the dynamic pressure or equality of momentum flux, i.e.,

$$\Psi_j(t, \rho_1, q_1, \ldots, \rho_n, q_n) = \left(\frac{q_j^2}{\rho_j} + p(\rho_j)\right) - \left(\frac{q_1^2}{\rho_1} + p(\rho_1)\right) \quad j = 2, \ldots, n.$$  

(14)

In [98] also geometric information has been included in the the previous two conditions and an analytical comparison of qualitative properties has been conducted in [99]. The conditions (13) and (14) might lead to a production of energy at the vertex as observed in [109, 102]. A condition that preserves the energy is to assume the equality of stagnation enthalpy or equality of Bernoulli invariant

$$\Psi_j(t, \rho_1, q_1, \ldots, \rho_n, q_n) = \frac{1}{2} \left(\frac{q_j}{\rho_j}\right)^2 + p'(\rho_j) - \frac{1}{2} \left(\frac{q_1}{\rho_1}\right)^2 + p'(\rho_1) \quad j = 2, \ldots, n.$$  

(15)

This condition implies the conservation of energy at the vertex. As noted in [85], the equation (2) itself dissipates energy and this might be also a desirable property of the coupling condition. An
Implicit condition ensuring this property has been introduced in [85, Definition 1]. The proposed condition is implicit in the sense that only the resulting boundary values \((\rho_k, q_k)(t, 0^+)\) are given but not necessarily an explicit function \(\Psi\). Hence, the starting point are constant initial data \((\hat{\rho}_k, \hat{q}_k)\) for \(k = 1, \ldots, n\) adjacent to the vertex. Then, there exist boundary values \((\rho_k, q_k)(t, 0^+)\) with the following properties.

- Mass is conserved at the junction:
  \[
  \sum_{k=1}^{d} q_k(t, 0^+) = 0, \text{ a.e. } t \geq 0. \tag{16}
  \]

- There exists \(\rho_* \geq 0\) such that for each fixed \(k = 1, \ldots, n\) the boundary values \((\rho_k, q_k)(t, 0^+)\) are equal to the restriction to \(x > 0\) of the (unique) weak entropy solution of equation (2) in the sense of Lax with initial condition
  \[
  (\rho, q)(0^+, x) = \begin{cases} 
  (\hat{\rho}_k, \hat{q}_k), & x > 0, \\
  (\rho_*, 0), & x < 0.
  \end{cases} \tag{17}
  \]

The existence of \(\rho_* > 0\) is proven in [85, Lemma 1], the boundary values dependent continuously on \(\rho_*\) [85, Proposition 2] and no assumption on subsonic initial data is required. Further, the previous construction can be shown to be decrease entropy at the vertex for a large class of symmetric entropies including the physical energy. The condition (17) can also be derived by using the kinetic formulation of the isentropic Euler equations (7). Coupling conditions for \(f_k = f_k(t, x, \xi)\) where \(f_k\) is the kinetic particle density on edge \(k = 1, \ldots, n\) can be formulated using a coupling condition of similar type as (10), i.e.,

\[
\Psi(t, f_1(t, 0^+, \cdot), \ldots, f_n(t, 0^+, \cdot))(\xi) = u(t) \text{ a.e. } t \geq 0, \xi \in \mathbb{R}. \tag{18}
\]

It has been shown in [84, Theorem 1] that among all coupling conditions \(\Psi\) that conserve the total mass, the condition that dissipates the most energy is given by

\[
\Psi_k(t, f_1(t, 0^+, \cdot), \ldots, f_n(t, 0^+, \cdot)) = M(\rho_*(t), 0, \xi), k = 1, \ldots, n. \tag{19}
\]

Hence, in the formal limit \(\epsilon \to 0\), we expect that at the vertex a state with zero velocity and some (unknown) density \(\rho_*\) prevails. This is precisely the condition imposed by equation (17).

In the uncontrolled case \(u \equiv 0\) a qualitative and quantitative comparison of the previous conditions has been shown in [85, Section 7]. Conditions (13), (14) lead to boundary values resulting in an energy decay of the order of \(10^{-2}\), while condition (15) conserves the energy and condition (17) leads to a decay of the order of \(10^{-1}\). In the case of condition (13) no waves emerge from the vertex while for all other conditions except for (17) an (nonphysical) shock wave on \(k = 2, 3\) appears. On the other hand, rarefaction waves on edges \(k = 2, 3\) are observed for condition (17).

Next, we turn to the modeling of controls. The main control is due to compressor stations in gas networks. In terms of the formulation of equation (10) they are modelled by considering \(n = 2\), equation (12) and

\[
\Psi_2(t, \rho_1, q_1, \rho_2, q_2) = q_2 \left( \frac{p(\rho_2)}{p(\rho_1)} - 1 \right)^{(\gamma-1)/\gamma} \tag{20}
\]

for some \(\gamma \in (1, 3)\). Furthermore, \(u(t) = (0, \Pi(t))\) where \(\Pi(t)\) is the supplied compressor energy at time \(t\), see e.g. [101, 81, 35, 59, 110]. This framework naturally leads to various control problems, where the open-loop control has to be chosen to satisfy suitable optimality criteria that will be discussed in the forthcoming sections. In contrast to the compressor control, control of valves can not add energy to the system and only one-way flow is possible through valves. In [41] a
mathematical model for this situation has been proposed and it amounts to prescribe a desired flow \( q_u \). The control \( u \) acts in such way that the valve keeps the flow at a constant value \( q_u \) if possible; otherwise it is closed. Other types of valves where bi-directional flow is possible have also been considered e.g. in [40].

Finally and for sake of completeness, we recall that the above conditions have been partly extended to the case of the full \( 3 \times 3 \) system of Euler equations in [38, 39].

3 Well-Posedness of Mathematical Models For Fixed Control Action

In this section we consider the case of a fixed given control action \( u = u(t) \) and recall well-posedness results for classical and weak solutions.

3.1 Classical Solutions

Tatsien Li and his collaborators have been very active in the study of semi-global classical solutions to the mixed initial-boundary value problem of one-dimensional quasilinear hyperbolic systems and the investigation of exact controllability in this framework, see for example [95, 96]. As an example for a quasilinear initial boundary value problem, let us consider the following existence result for semi-global classical solutions has the following structure:

**Theorem 3.1** Let a finite (arbitrarily large) time \( T > 0 \) be given. Then there exists a number \( \varepsilon(T) > 0 \) such that for all initial data and boundary data for which the maximal \( C^1 \)-norm is less than \( \varepsilon(T) \) and that satisfy the corresponding \( C^1 \)-compatibility conditions there exists a unique classical solution on the time-interval \( [0, T] \), i.e. a continuously differentiable function that satisfies the initial conditions, the boundary conditions and the partial differential equation. Moreover, there exists a constant \( C_0(T) > 0 \) such that the \( C^1 \)-norm of the solution is bounded a priori by the product of \( C_0(T) \) and the maximal \( C^1 \)-norm of the initial data and the boundary data.
The semi-global existence results are proved by rewriting the problem with integral equations along the characteristic curves (see [87]) whose slopes are given by the eigenvalues of the system matrix. In the case of (21) the eigenvalues are $v \pm c(\rho)$. Using the notation $c^2 = \frac{\partial p}{\partial \rho}$, the eigenvalues for (1) can also be written as $\lambda_{\pm} = v \pm c$.

Starting from this formulation, similar as in the Picard iteration a map is defined such that every fixed point of this map solves the initial boundary value problem. Then the convergence of the fixed point iteration is shown and the assertion follows. The a priori bound is shown using Gronwall’s Lemma.

Note that the fixed point iteration also allows to consider solutions that are only required to have Lipschitz regularity, see [68]. The characteristic curves are often referred to only as characteristics. The method of characteristics is a classical method for the solution of one-dimensional quasilinear hyperbolic systems. In order to obtain a well-posed problem, the boundary conditions have to be chosen according to the signs of the slopes of the characteristics. In the operation of gas networks, subsonic flow occurs, that is the velocity of the gas is smaller than the sound speed in the gas. This implies that one of the eigenvalues is positive in each point and the other one is negative in each point. Hence one family of characteristic curves travels from the left-hand side to the right-hand side and the other family of characteristics travels from the right-hand side to the left-hand side. Therefore and due to the structure of the Riemann invariants, for subsonic flow on a single pipe, at each end the value of one physical variable can be prescribed by the boundary conditions.

For the study of control problems, it is often useful to work in an Hilbert space. This is the reason why solution of $H^2$-regularity are of interest that are more regular than classical solutions. Solutions of this type have been studied in [14] where an existence result is given in Appendix B. It has the same structure as Theorem 3.1 but with the $C^1$-norm replaced by the $H^2$-norm. In particular, the compatibility conditions remain unchanged.

Note that the existence result for semi-global classical solutions Theorem 3.1 can be extended to the case of networked systems that are defined on finite graphs if the node conditions uniquely define the necessary boundary input data for each adjacent pipe, see [68].

### 3.2 Weak Solutions

Even for smooth initial data $(\rho_0, q_0)$ (6) it is known that there may exists a time $t > 0$ such that solution $(\rho(t, \cdot), g(t, \cdot))$ to (2) may develop discontinuities [113]. Hence, the notion of weak solutions has been introduced to treat solutions of lower regularity compared with solutions in the previous section. We refer to [113, 24] for details on solutions to the Cauchy problem (2),(6) of systems of conservation and balance laws. Over several publications those results have extended to networked systems (2), (6) and (10) and refer to [25] for an overview and to for $2 \times 2$ hyperbolic balance laws with fixed control action to [36]. Next, we briefly recall the basic definition of weak solution and well-posedness following [98, 37, 36].

In order to present the notion of weak solutions for a single vertex located at $x = 0$ we define $u_j = (\rho_j, g_j)$ as density and flux on the adjacent edge $j = 1, \ldots, n$. Further, we define $f(u) = (q, \frac{c^2}{\rho} + p(\rho))$ with $p(\rho)$ given by equation (3). Further, we consider a source term $g(t, x, u) = (0, -f\rho v|v| - g\sin(\alpha)\rho)$. Then, (2), (6), (10) reads

$$
\begin{align*}
\partial_t u_1 + \partial_x f(u_1) &= g(t, x, u_1) \\
\vdots \\
\partial_t u_n + \partial_x f(u_n) &= g(t, x, u_n)
\end{align*}
$$

(26)

coupled through the nodal condition

$$
\Psi(u_1(t, 0+), \cdots, u_n(t, 0+)) = u(t).
$$

(27)

Here, for every $j \in \{1, \ldots, n\}$, $u_j : [0, T) \times I \rightarrow \Omega_j$, $T \in (0, +\infty]$, $I = (0, \infty)$, and $\Omega_j$ is a subset
of $\mathbb{R}^2$. We supplement (26) and (27) with the initial condition
\[
\left\{
\begin{array}{l}
u_1(0, x) = u_{1,0}(x), \quad x > 0, \\
\vdots \\
u_n(0, x) = u_{n,0}(x), \quad x > 0,
\end{array}
\right.
\] (28)
where, for every $j \in \{1, \ldots, n\}$, $u_{j,0} : I_j \to \Omega_j$ are given functions. For brevity we introduce the notation
\[
\tilde{u} = \begin{bmatrix}
u_1 \\
\vdots \\
u_n
\end{bmatrix}, \quad \tilde{f}(u) = \begin{bmatrix} f(u_1) \\
\vdots \\
f(u_n)
\end{bmatrix}, \quad \tilde{g}(t, x, u) = \begin{bmatrix} g(t, x, u_1) \\
\vdots \\
g(t, x, u_n)
\end{bmatrix},
\] (29)
and we rewrite (26)-(27)-(28) in the form
\[
\partial_t \tilde{u} + D\tilde{f}(\tilde{u}) = \tilde{g}(t, x, \tilde{u}), \quad \Psi(\tilde{u}(t, 0^+)) = u(t), \quad \tilde{u}(0, x) = \tilde{u}_0.
\] (30)

**Definition 3.2** Fix $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n) \in \prod_{j=1}^n \Omega_j$ and $T \in [0, +\infty]$. Assume $u \in BV(\mathbb{R}^+; \mathbb{R}^n)$. A weak solution to the Cauchy problem (30) on $[0, T]$ is a function $\tilde{u} \in C^0([0, T]; \tilde{u} + L^1(\mathbb{R}^+; \Omega^n))$ such that the following conditions hold.

1. For all $\phi \in C^\infty_c([-\infty, T[ \times \mathbb{R}^+; \mathbb{R})$ and for $j \in \{1, \ldots, n\}$
   \[
   \int_0^T \int_{\mathbb{R}^+} (\nu_j \partial_x \phi - g(t, x, \nu_j)\phi + f(\nu_j)\partial_x \phi) \, dx \, dt + \int_{\mathbb{R}^+} \nu_{j,0}(x)\phi(0, x) \, dx = 0.
   \] (31)

2. For a.e. $t \in (0, T)$, $\Psi(\tilde{u}(t, 0^+)) = u(t)$.

The weak solution $\tilde{u}$ is an entropy solution if for any convex entropy-entropy flux pair $(\eta_j, Q_j)$, for all $\phi \in C^\infty_c([-\infty, T[ \times \mathbb{R}^+; \mathbb{R})$ and for all $j = 1, \ldots, n$
\[
\int_0^T \int_{\mathbb{R}^+} (\eta_j(\nu_j)\partial_x \phi - g(t, x, \nu_j)D\eta_j(\nu_j)\phi + Q(\nu_j)\partial_x \phi) \, dx \, dt \geq 0.
\] (32)

Well-posedness of equation (30) is obtained for initial data having small total variation and under suitable assumptions on $\tilde{f}$ and $\tilde{g}$ as given by [36, Assumption F and G]. Those assumptions are precisely as in the case of the Cauchy problem. In the following we only recall the additional assumptions on $\Psi$ specific to the case of a network. We consider solutions in the space
\[
X = \left(\tilde{u}_* + L^1(\mathbb{R}^+; \Omega), u_* + L^1(\mathbb{R}^+; \Omega^n)\right), \quad \Omega = \prod_{j=1}^n \Omega_j.
\] (33)
where the constant states $(\tilde{u}_*, u_*)$ are such that they fulfill the coupling condition
\[
\Psi(\tilde{u}_*) = u_*.
\] (34)
A solution is then obtained for data $(\tilde{u}_0, u) = (\tilde{u}_*, u_* + (L^1(\mathbb{R}^+; \Omega), L^1(\mathbb{R}^+; \Omega^n))$ having small norm
\[
TV(\tilde{u}_0) + TV(u) + \|\Psi(\tilde{u}_0(0^+)) - u(0^+)\|_{L^1(\mathbb{R}^+; \mathbb{R}^n)} \leq \delta,
\] (35)
for some possibly small $\delta > 0$. In the following we denote by $r_2(u)$ the right eigenvector of $Df(u)$ corresponding to the second characteristic family. Then, the assumption on $\Psi$ is as follows: Assume $\Psi \in C^1(\Omega; \mathbb{R}^n)$ such that
\[
\det \left[ D_1\Psi(\tilde{u}_*) r_2(\tilde{u}_1, u) D_2\Psi(\tilde{u}_*) r_2(\tilde{u}_2, u) \ldots D_n\Psi(\tilde{u}_*) r_2(\tilde{u}_n, u) \right] \neq 0.
\] (36)
Under the assumptions (35) and (36) the result [36, Theorem 2.3] yields existence of a weak solution in the sense of Definition 3.2.

Some remarks are in order. The boundary traces of \( \tilde{u}(t, \cdot) \) are well-defined, since in fact we have \( u_j(t, \cdot) \in BV(R^+; \Omega_j) \). However, the known result is only a perturbation result around the state \( (\tilde{u}_0, u) \). Theorem 2.3 [36] also shows continuous dependence of \( \tilde{u} \) with respect to the initial data \( (\tilde{u}_0, u) \). This leads to an existence result for suitable optimal control problems shown in the next section. It is important to mention that the stated nodal conditions in the previous section all fulfill condition (36) if \( \tilde{u}_0(x) \) is subsonic for all \( x \in R^+ \).

4 Control and Controllability

4.1 Optimal Control

In the context of weak solutions (3.2) existence of optimal controls has been established in [36]. The result essentially follows by the continuous dependence of solutions \( u \in X \) on \( (\tilde{u}_0, u) \) where the space \( X \) is given by equation (33).

**Theorem 4.1** Let \( n \in N, n \geq 2 \). Assume that \( \tilde{f} \) satisfies (F) at \( \tilde{u} \), and \( \tilde{G} \) satisfies (G). Fix a map \( \Psi \in C^1(\Omega; R^n) \) satisfying (36) and let \( u_\ast = \Psi(\tilde{u}_\ast) \). For a fixed \( \tilde{u}_\ast \in H \) assume that

\[
J_0 : \left\{ u | [0, T] : u \in \left( u_\ast + L^1(R^+; R^n) \right) \text{ and } (u_\ast, u) \in D \right\} \to R, \text{ and } J_1 : D \to R
\]

are non negative and lower semicontinuous with respect to the \( L^1 \) norm. Then, the cost functional

\[
J(u) = J_0(u) + \int_0^T J_1(\tilde{u}(\tau, \cdot)) d\tau
\]

(37)

admits a minimum on \( \left\{ u | [0, T] : u \in \left( u_\ast + L^1(R^+; R^n) \right) \text{ and } (\tilde{u}_0, u) \in D \right\} \).

Here, we denote by \( \tilde{u}(t, \cdot) \) the weak solution in the sense of Definition (3.2) with data \( (\tilde{u}_0, u) \). The definition of the sets \( D_0, U_0 \) and \( D_0 \) are given in [36]. In particular, the sets \( U_0 \) and \( D_0 \) involve the assumption (35) on sufficiently small TV norm. The assumption F and G on flux and source term are as for the existence of solution to a Cauchy problem and omitted here.

In the context of gas networks a typical cost functional \( J \) measures the distance to a given desired pressure \( \tilde{p} \) on a certain part \( I_i = [x_1, x_2] \) of of pipe \( i \). If also oscillations in the control \( u(\cdot) \) should be penalised the resulting functional reads

\[
J(u) = TV(u) + \int_0^T \int_{I_i} |p(\rho_i(t, x)) - \tilde{p}| dxdt.
\]

(38)

This functional fulfills the assumptions of Theorem 4.1. Note that as a possible substitute for the tracking term for the pressure in the objective function, also box constraints for the pressure are of interest.

Since the mapping \( \tilde{u}_0 \to \tilde{u} \) is non-differentiable in any \( L^p, p \geq 1 \), optimality conditions are not straightforward. An alternative notion of differentiability has been introduced in [26, 27, 116]. However, the extension to boundary control is still subject to active research [106].

In [63], the instantaneous control of mixed-integer PDE-constrained gas transport problems has been studied. Zero-one decisions occur in a natural way in the operation of gas networks in decisions as whether to switch on or off a compressor or whether to open or close a valve. As a first step towards the solution of transient optimal control problems with decisions of this type, an instantaneous control approach is suggested, where in the time-discrete problem, in each time step the control is chosen in such a way that the integrand in the objective function for the next time step is minimized. Approximation of the nonlinearities by piecewise linear functions leads to large mixed integer linear optimization problems where a solution close to global optimality is possible.
4.2 Controllability

The results about the exact boundary controllability of general one-dimensional quasilinear hyperbolic systems that have been obtained in the framework of classical solutions (see e.g. [96]) can be applied to (1). It is typical for hyperbolic systems, that due to the finite speed of information flow exact controllability is only possible after a sufficiently large set-up time, see [86]. In a nutshell, exact controllability is only possible after the flow of information from the boundary input has reached each point of the space interval on both families of characteristics. Exact boundary controllability on tree-like networks has also been studied in [52].

In the context of gas flow through pipelines, apart from the classical exact controllability, also the controllability of nodal profiles is of importance, since in the operation of gas pipeline networks, customer satisfaction is achieved by generating the desired pressure and flow rate profiles at the nodes where the customers are located. Stated as a controllability problem, the problem of the system operators is to steer the system in such a way that at the boundary nodes, after a finite time the desired nodal profiles are reached exactly during a certain time interval.

Note that in this notion of controllability of nodal profiles, in contrast to the classical notion of exact controllability, not the full state is prescribed at a fixed time, but instead the boundary trace of the state is prescribed on a certain time interval. Of course one of the variables can simply be prescribed by the boundary conditions, so the task is to drive simultaneously also the other variable to the values of the desired nodal profile. This notion of exact controllability of nodal profiles has been discussed in [62] in the framework of classical solutions. The constructions in the proofs are based upon the exchange of the roles of time and space, that allows that the desired nodal profiles can play the role of virtual initial conditions. Recently, there has been a lot of research activity in the analysis of the exact controllability of nodal profiles in the framework of classical solutions of general networked quasilinear hyperbolic systems, see [93, 53, 94]. For quasilinear wave equations it has been studied in [119] and for the Saint Venant system in [120].

4.3 Feedback Stabilization

If a desired stationary state is known, the problem arises, whether the system state in the gas network can be stabilized towards the desired state exponentially fast by suitably chosen boundary feedback laws. The corresponding analysis can be based upon suitably chosen Lyapunov functions. In particular Laypunov functions in terms of Riemann invariants with exponential weights have been used successfully, see e.g. [18] in order to show the exponential decay of the $L^2$-norm for a linearized system with linear feedback laws in terms of the Riemann invariants. The corresponding analysis for the quasilinear system on a star-shaped network is given in [56].

Extensions to the case of time delay in the feedback control have been presented in [55].

The boundary feedback stabilization of the Saint-Venant system by a proportional feedback control is studied in [72] using a Lyapunov function in physical coordinates. A similar analysis for general hyperbolic density velocity systems (21) that also include (1) is presented in [15]. Here linear boundary conditions of the form

$$v(t, 0) = k_0 \rho(t, 0), \ v(t, L) = k_L \rho(t, L)$$

(39)

with real feedback parameters $k_0$, $k_L$ are considered. In [15], intervals are defined in terms of the desired state for which the feedback law (39) leads to exponential decay of the $L^2$-norm of the distance between the current and the desired state. Note that in order to extend the semi-global solutions to global solutions, it is useful to work with solutions with $H^2$ regularity and to show that the $H^2$-norm of the solutions decays exponentially fast. This is the reason why in [15, 72] also $H^2$ Lyapunov functions are presented and the exponential decay with respect to the $H^2$-norm is shown.

A similar analysis with a stabilizing Neumann feedback of the form

$$v_x(t, 0) = v_x^*(0) + k_0 v_t(t, 0), \ v(t, L) = v^*(L)$$

(39)
for the quasilinear wave equation

\[ v_{tt} = (c^2 - v^2) v_{xx} - 2v \left( v_{tx} + (v_x)^2 \right) - 2v_t v_x - 2f |v| \left( v_t + \frac{3}{2} v v_x \right) \]  \tag{40}

for the velocity that is derived from (1) is given in [64]. Here \( v^* \) is a desired stationary state for (40) and \( k_0 > 0 \) is a feedback parameter. The Lyapunov function that is used to show the exponential decay has the form \( \mathcal{L}(t) = \mathcal{L}_1(t) + \mathcal{L}_2(t) \) with

\[
\mathcal{L}_1(t) = \int_0^L k \left[ (c^2 - v^2) (v - v^*)^2 + v_t^2 \right] - 2 \exp \left( -\frac{x}{L} \right) \left[ v (v - v^*)^2 + v_t (v - v^*)_x \right] \, dx, \tag{41}
\]

\[
\mathcal{L}_2(t) = \int_0^L k \left[ (c^2 - v^2) (v - v^*)^2 + v_x^2 \right] - 2 \exp \left( -\frac{x}{L} \right) \left[ v (v - v^*)^2_{xx} + v_t (v - v^*)_{xx} \right] \, dx. \tag{42}
\]

A typical result then yields under suitable assumptions on \( k_0 \) the following estimate

\[ \mathcal{L}(t) \leq \mathcal{L}(0) \exp(-\nu t), \quad t \geq 0. \]  \tag{43}

Here, the constant \( \nu > 0 \) is typically not known explicitly and may depend among others on \( L \). We refer to the section on numerical results for estimates of \( \nu \) in the discrete case.

The boundary feedback stabilization by proportional-integral (PI) control is analyzed in [16] using a Lyapunov function in physical coordinates. The considered boundary conditions have the differential form

\[ \rho(t, 0) v(t, 0) = Q_0(t), \quad \rho(t, L) v(t, L) = \kappa_L ((1 + k_L)\rho(t, L) - Z(t)), \quad Z'(t) = \alpha_L (\rho^*(L) - \rho(t, L)). \]  \tag{44}

where \( \rho^* \) denotes the desired stationary state.

Other techniques applied to stabilize linear hyperbolic balance rely on the backstepping technique and we refer e.g. to [4, 88, 89] for further references and details. When using the backstepping technique an important aspect is the design of observers in order to determine the feedback law. State estimation and observer design have been studied in the context of system of hyperbolic equations in [3, 21, 2, 5]. Similar techniques as for the stabilization of gas dynamics have been used to stabilize St. Venant flow on general networks [108, 107, 70]. The partial differential equations (2) have a similar structure, except that the pressure is given by

\[ p(\rho) = \frac{g}{2} \rho^2 \]  \tag{45}

and that there are several models available for the friction term [17, 42]. The system (2) can also be written in quasi-linear form using the variables \( (\rho, v) \) instead of \( (\rho, q) \). For classical solutions both systems are equivalent. Stabilization in terms of the variables \( (\rho, v) \) has been discussed recently e.g. in [71].

The boundary feedback stabilization for the degenerate parabolic model from [8] is studied in [57]. Also in this contribution the analysis is based upon a suitably chosen Lyapunov function. The suggested feedback law has the form

\[ p(t, 0) = p_0, \quad q(t, L) = \kappa_L p(t, L) \]  \tag{46}

where \( p_0 > 0 \) is a desired pressure value and \( \kappa_L \) is a feedback parameter.

Note that in the analysis of the closed loop systems presented in this section, smallness assumptions for the initial state are used. In the analysis, these assumptions imply that no shocks are generated in the system. In order to take this into account, in the practical application of the feedback laws, it is important to keep in mind that it is often not clear whether these smallness assumptions are satisfied for the given initial states.
5 Uncertainty Quantification

In the operation of gas networks, the treatment of uncertainties plays a decisive role, since customer demands are uncertain. In practice, in a procedure with several steps the customers first buy the option to book their gas consumption later within a certain defined range. Then in a second step, precise quantities are nominated on a day-ahead market. A detailed description of the European entry-exit gas market is given in [50].

In order to take into account the uncertainty, random boundary data are included in the model. In [49], a method for the computation of the probability of feasible load constellations in a stationary gas network with uncertain demand is given. A network with a single entry and several exits with uncertain loads is studied. The feasible flows have to satisfy given pressure bounds in the pipes.

The numerical method is based upon a spherical radial decomposition that is used both for the computation of the probabilities and the corresponding derivatives with respect to the control. Gradient formulae for nonlinear probabilistic constraints with non-convex quadratic forms are presented in [118].

In order to include the information on uncertainty in the optimization problems, probabilistic constraints of the form

$$P\left(g(x, \omega) \geq 0\right) \geq p$$  \hspace{1cm} (47)

are useful. Here the parameter $p \in (0, 1)$ is a probability threshold that can be chosen by the decision maker a priori according to his preferences, $x$ denotes the decision variable and $\omega$ is a random variable. The deterministic form of (47) is a classical inequality constraint $g(x) \leq 0$.

For optimization problems, the structure of the corresponding set of feasible controls is relevant. In general, for the problems with probabilistic constraints this is not a convex set. However, in [66] it has been shown that under weak assumptions the feasible set is star-shaped, which is an important result that implies that in the spherical radial decomposition on each ray at most one interval has to be considered in the computation of the probability. In [111] this computational approach is compared with a more general collocation method that is based upon kernel density estimators.

In [1], the approach with probabilistic constraints is generalized to a setting that also allows to take the dynamic case into account. In this paper, for a decision variable $x$, a set $\mathcal{U}$ (this could for example be a time interval) a desired probability threshold $p \in (0, 1)$ and a random variable $\omega$ probabilistic constraints of the form

$$P\left(g(x, \omega, y) \geq 0 \forall y \in \mathcal{U}\right) \geq p$$  \hspace{1cm} (48)

are considered and referred to as probust constraints. This is useful for example to define a model where the probability that the state satisfies the pressure bounds throughout the time-interval is required to be at least $p$. Note that (48) is a stronger requirement than the constraint

$$P\left(g(x, \omega, y) \geq 0 \forall y \in \mathcal{U}\right) \geq p \forall y$$  \hspace{1cm} (49)

which does not guarantee that for a feasible decision $x$ the constraint is satisfied with probability $p$ uniformly for all $y \in \mathcal{U}$.

In [43], as a step towards the treatment of the full transient gas pipeline network flow using probabilistic constraints, the optimal Neumann boundary control of a vibrating string with uncertain initial data and probabilistic terminal constraints is analyzed and a numerical method is provided.

In [65], the quasilinear wave equation (40) is considered with the feedback law at $x = 0$ and uncertain boundary data $\omega(t)$ at $x = L$, that is with boundary conditions of the form

$$v_x(t, 0) = v^*_x(0) + k_v v(t, 0), \quad v(t, L) = v^*(L) + v^\omega(t).$$

It is shown that if the noise $v^\omega$ decays exponentially fast, the $H^1$-norm of the difference between $v$ and the desired stationary state $v^*$ decays exponentially fast.
In contrast to the probabilistic approach to robust control that we have presented above, the more conservative classical approach in robust optimization is to consider a certain range (called the uncertainty-set) for the uncertain parameters and optimize subject to the constraint that all elements of the uncertainty set are feasible (i.e., in particular the worst case scenario). For gas networks, this approach has been studied in [6]. In order to make the approach less costly, in [6] a two-stage approach is proposed. For the two-stage model, the problem variables are classified as here-and-now variables that have to be decided at once before the uncertainty is realized and wait-and-see or adjustable variables whose values can be chosen later after the uncertainty is realized.

Another aspect of uncertainty is that also physical parameters that appear in the pde are uncertain. As a contribution to this topic, in [13] the problem to identify uncertain friction parameters that vary along the pipes is studied. The consequences of uncertain friction for the transport of natural gas through passive networks of pipelines have also been studied in [73].

6 Numerical Methods For Simulation and Control

The type of equation (2) is a hyperbolic balance law in one-spatial dimension. Hence, there is a vast literature on possible numerical schemes to discretize system (2) and we refer e.g. to [92] for an overview. The network structure imposes few particularities that will be reviewed in this section with particular focus on finite volume schemes. For simplicity we also use a regular spatial grid. In order to present numerical discretization in compact notation we introduce $u = (\rho, q)$ and $g = (0, -f \rho v|v| - g \sin(\alpha) \rho)$.

6.1 Discretization of Coupling Conditions for Finite Volume Schemes

Equation (2) are approximated numerically using a finite volume method with numerical cell size $x_{i+1} - x_i = \Delta x > 0$ and time step $t^{m+1} - t^m = \Delta t$, chosen such the CFL condition [90] $\lambda_{\text{max}} \Delta t \leq \Delta x$ is satisfied. Here, $\lambda_{\text{max}}$ is the maximum absolute value of the eigenvalues of Jacobian of $f$. The following discretization is done for each component $u = (u_1, u_2)$ separately. Within a finite volume method the cell average $\bar{U}_{j,i}^m$ of $u_j$ for $j = 1, 2$ of cell $i$ at time $t^m$ is given by

$$\bar{U}_{j,i}^m := \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_j(x, t^m) dx.$$

The evolution of the cell average over time $\Delta t$ is

$$\bar{U}_{j,i}^{m+1} = \bar{U}_{j,i}^m - \frac{\Delta t}{\Delta x} ((F_j)_{i+\frac{1}{2}}^m - (F_j)_{i-\frac{1}{2}}^m) + \bar{G}_{j,i}^m,$$

where in Godunov’s method [48] $(F_j)_{i+\frac{1}{2}}^m = F_j(\bar{U}_{j,i}^m, \bar{U}_{j,i+1}^m)$ denotes the numerical flux of component $u_j(t, x)$, $j = 1, 2$ through the boundary of the cells $i$ and $i + 1$. Further, $\bar{G}_{j,i}^m$ is an approximation to $\frac{1}{\Delta x} \int_{t^m}^{t^{m+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g_j(t, x, u_j(x, t)) dx dt$ obtained by a suitable quadrature rule. Within Godunov’s method the exact solution to a Riemann problem posed at the cell boundary $i + \frac{1}{2}$ is used to define the numerical flux $(F_j)_{i+\frac{1}{2}}^m$. Many other numerical fluxes have been proposed and we refer to the literature [92, 115] for further details. Here, we focus on Godunov’s method as a basic first-order method. Therein, an approximation to $u_j(t, x)$ is then obtained by the piecewise constant reconstruction

$$u_j(t, x) = \sum_i \sum_m \bar{U}_{j,i}^m \chi[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^m, t^{m+1}](t, x).$$
obtained by the following method. Assume at time $t^m$ the cell averages in the first cell $i = 0$ corresponding to $x = 0$ of the connected edges are $U_{j,0}^m$ for $j = 1, \ldots, n$. Denote by $σ \rightarrow L_κ(u_σ, σ)$ the $κ$–th Lax curve through the state $u_σ$ for $κ = 1, 2$. Then, we solve for $(σ^*_1, \ldots, σ^*_n)$ using Newton’s method the nonlinear system

$$\Psi \left(t^m, L_2(U_{1,0}^m, σ_1), \ldots, L_n(U_{n,0}^m, σ_n)\right) = u(t^m),$$

where $u(t^m)$ denotes the discretized control at time $t^m$. Under suitable assumptions on $Ψ$ a unique solution $(σ^*_1, \ldots, σ^*_n)$ to equation (52) exists. Then, a boundary value $U_{j,0}^{m+1}$ at time $t^{m+1}$ is given by equation (50) at $i = 0$

$$U_{j,-1}^m := L_2(U_{j,0}^m, σ^*_j).$$

The given construction yields a first–order approximation to the coupling condition (10).

For finite-volume schemes of higher-order additional values at the boundary are required. In all recent publications [9, 22, 30, 103] the additional information required for reconstruction is obtained using a Lax-Wendroff type approach. This amounts to differentiate condition (52) with respect to time and solve the additional equations for information on the slope of a reconstructed solution $t \rightarrow u(t, 0+)$. It can be shown [9] that this construction allows to preserve the desired order.

Recently in [100] the interplay of the discretization order of the numerical flux $F$ and the source term $G$ has been investigated. In the case of spatially one-dimensional flow a numerical discretization has been proposed that allows to obtain steady-states up to machine precision even for large spatial grids $Δx$. This technique has been known as well–balanced schemes for the Cauchy problem but could be extended to the case of network equations.

### 6.2 Discretization of Stabilization Problems

The theoretical decay rates established in the previous section will be complemented by corresponding numerical results following [10]. Therein, a discrete stabilization result for a first–order spatial discretization of a quasi–linear system for $u ∈ \mathbb{R}^d$

$$∂_t u + Df(u)∂_x u = 0, \quad u(x, 0) = u_0(x), \quad u(t, 0) = K u(t, 1)$$

for $Df(u)_{i,j} = δ_{i,j}λ_j(u) > 0$ and $K_{i,j} = δ_{i,j}κ_i > 0$ has been established. A finite–volume discretization of the quasi-linear equation (54) is given by (50) where the flux $F$ is chosen as Upwind flux due to the fixed direction $λ_j(u) > 0$. This leads to the following discretization for $j = 1, \ldots, d, i = 0, \ldots, N$ and $m = 0, \ldots, K$

$$U_{j,i}^{m+1} = U_{j,i}^m - \frac{Δt}{Δx} λ_j(U_{i-1,i}^m, \ldots, U_{i,d,i}^m) \left(U_{j,i}^m - U_{j,i-1}^m\right),$$

and for $j = 1, \ldots, d, i = 0, \ldots, N$ and $m ≥ 0$ the initial boundary conditions are given by

$$U_{j,0}^m = \frac{1}{Δx} \int_{I_j} u_{j,0}(x) dx \quad \text{and} \quad U_{j,-1}^m = κ_j U_{j,N}^m.$$

We further require that initial and boundary conditions are compatible, i.e.,

$$U_{j,-1}^m = κ_j U_{j,N}^m,$$

and that in the vicinity of $u ∈ B_δ(0) ⊂ \mathbb{R}^d$ the CFL condition (58) holds. For given $δ > 0$, $Δt$ is chosen such that

$$\frac{Δt}{Δx} \max_{j=1,\ldots,d} \max_{u ∈ B_δ(0)} |λ_j(u)| \leq 1.$$
For discrete initial data \( \overline{U}_{j,i}^0 \in B_\delta(0) \), having small and bounded discrete gradients discrete exponential stability of solutions to (55) has been established in [10, Theorem 2]. Under suitable assumptions we obtain for discrete Lyapunov function

\[
L^m = \Delta x \sum_{i=0}^N \sum_{j=1}^d \left( \overline{U}_{j,i}^m \right)^2 \exp(-\mu_j x_i), \ m \geq 0
\]

exponential decay in time

\[
L^m \leq \exp(-\nu t^m) L^0, \ m \geq 0.
\]

We do not review all required assumptions but recall that the exponential decay only holds provided that

\[
0 < \kappa_j^2 \leq \frac{D_j^\text{min}}{D_j^\text{max}} \quad \text{and} \quad \mu_j \leq \log \left( \frac{\sqrt{\kappa_j D_j^\text{min}}}{D_j^\text{max}} \right),
\]

where for \( j = 1, \ldots, d \) we define

\[
0 < D_j^\text{min} := \min_{u \in B_\delta(0)} \frac{\Delta t}{\Delta x} \lambda_j(u) \leq \max_{u \in B_\delta(0)} \frac{\Delta t}{\Delta x} \lambda_j(u) \leq D_j^\text{max} \leq 1.
\]

The corresponding decay rate \( \nu \) is given by

\[
\nu = \min_{j=1, \ldots, d} \left( D_j^\text{min} \exp(-\mu_j \Delta x) \mu_j \frac{\Delta x}{2\Delta t} \right).
\]

The discrete scheme allows for explicit decay rate \( \nu \) which is also independent of the grid since \( \frac{\Delta t}{\Delta x} \) is fixed. Further, several extensions e.g. to \( \lambda_j(u) < 0 \) and other boundary conditions are possible, see [10]. In particular, in the linear case, \( \lambda_j(u) = \lambda_j \), the constant \( D_j^\text{min} = D_j^\text{max} \). In this case

\[
\nu = \min_{j=1, \ldots, d} \left( \frac{1}{2} \lambda_j \mu_j \right) \quad \text{for} \quad \Delta x \to 0.
\]

The results on discrete \( L^2 \)-stability have been extended to \( H^s \)-norm for any \( s \geq 2 \) in the case of linear flux \( Df(u) = Au \) and linear source terms \( g(u) = Gu \) in the recent paper [45].

### 6.3 Numerical Methods for Optimal Control Problems

As mentioned in the theoretical results, in the case of optimal boundary control problems optimality conditions are not straightforward to obtain. The main reason is the lack of differentiability in \( L^1 \) of the control to state mapping. A remedy in the case of systems has been introduced in [28, 29] and for scalar equations in [116, 117] using a novel differential. A numerical implementation of those conditions remains challenging due to the required resolution of the fine structure of the solution to the system (2), (6) and (10). However, several approximations of the optimality system have been studied recently in the literature and we refer to [60, 47, 46, 34, 76, 77, 80, 13, 33] for further details. For classical solutions, the evaluation of derivatives in the optimal nodal control of networked hyperbolic systems has been studied in [54].

### 7 Open Problems

In this section we briefly mention open problems that might be relevant for the future development of modeling, control and numerics for gas networks.

From a modeling perspective it would be interesting to extend the current models in at least two directions. It has been observed that in the operation of gas networks the quality of the injected gas varies. This leads to the transport of a mixture of different gases and the development of suitable models for gas mixtures is certainly interesting and a current research topic. On the other hand, future simulations and control of interconnected energy systems like e.g. coupled gas and electricity networks will be of importance. From a modeling point of view this requires suitable model couplings, from a numerical point of view this requires to treat multi-scale phenomena due to the different involved time scales. Preliminary results in this direction have been obtained but
the full control of coupled infrastructure is still at large. This might also possibly require the development of novel control paradigms as well as reduced models.

From a control point of view a recent topic of interest for the operation of large-scale gas networks is the turnpike phenomenon. Turnpike results for the complex system dynamics can justify a combination of steady state models with dynamic models. A survey on large time horizon control and turnpike properties for wave equations is presented in [122]. Due to the turnpike phenomenon, often the control time can be divided into dynamic phases at the beginning and the end of the time interval and a static phase between them. In the dynamic phase one goal is to develop novel efficient feedback controls. A second aspect is the development of suitable Lyapunov functions and feedback controls for weak solutions, like BV functions, in order to treat effects like closing valves and shock waves. Moreover, it would be of interest to have feedback laws that can stabilize the system for a large set of initial states, that is to weaken the smallness assumptions for the initial state. A further aspect in the control of complex systems is to include reinforcement learning of dynamics that does not require any mathematical model. An application of those techniques towards gas networks is still an open problem.

For numerical computations novel methods to treat large-scale networks might need to be developed in order to obtain practical relevance. Here, tools like model-order reduction or suitable adaptive schemes might be of importance. See [20] for a model with a differential-algebraic equation. For the efficient computation of gradients on a network an adjoint calculus is desirable. While formally this system [81] can be derived major obstacles appear due to the non-differentiability of the control to state mapping. Here, efficient schemes would be desirable.

Finally, results obtained for energy networks might also lead to new insights for networks appearing in different transport processes like e.g. blood flow, vehicular traffic flow or production engineering.

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