CAT(0) EXTENSIONS OF RIGHT-ANGLED COXETER GROUPS

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Abstract. We show that any split extension of a right-angled Coxeter group $W$ by a generating automorphism of finite order acts faithfully and geometrically on a CAT(0) metric space.

1. Introduction

An isometric group action is faithful if its kernel is trivial, and it is geometric if it is cocompact and properly discontinuous. A finitely generated group $G$ is a CAT(0) group if there exists a CAT(0) metric space $X$ equipped with a faithful geometric $G$-action. The CAT(0) property is not an invariant of the quasi-isometry class of a group (see, for example, [1, 6] and [3, p. 258]). Whether or not it is an invariant of the abstract commensurability class of a group is as yet unknown. Attention was brought to this matter in [3]. In this article we illustrate that answering this question for any family of CAT(0) groups may require a variety of techniques.

It is well-known that an arbitrary right-angled Coxeter group $W$ is a CAT(0) group because it acts faithfully and geometrically on a CAT(0) cube complex $X$. It is also well-known that the automorphism group $\text{Aut}(W)$ is generated by three types of finite-order automorphisms. As a natural source of examples we consider split extensions of right-angled Coxeter groups by finite cyclic groups, where in each case the cyclic group acts on $W$ as the group generated by one of these various generating automorphisms. Our theorem is the following:

**Theorem 1.1.** Suppose $W$ is a right-angled Coxeter group and $\phi \in \text{Aut}(W)$ is either an automorphism induced by a graph automorphism,
a partial conjugation, or a transvection. Let $m$ denote the order of $\phi$. Then the group $G = W \rtimes_{\phi} \mathbb{Z}/m\mathbb{Z}$ is a CAT(0) group.

What is most interesting is that $G$ is a CAT(0) group for different reasons in each of the three cases. When $\phi$ is an automorphism induced by a graph automorphism, the left-multiplication action $W \rtimes X$ extends to an action $G \rtimes X$; when $\phi$ is a partial conjugation, $G$ is itself a right-angled Coxeter group; when $\phi$ is a transvection, $G$ is not a right-angled Coxeter group and the action $W \rtimes X$ cannot extend to all of $G$, but we can explicitly construct a new CAT(0) space $Y$ and describe a faithful geometric action $G \rtimes Y$.

After necessary background material is described in Section 2, the three cases of the theorem are treated, in turn, in Sections 3, 4 and 5.

We also note that, in each case of the theorem, we take an extension $W \rtimes H$ where $H \leq \text{Aut}(W_{\Gamma})$ is finite. In [4], we give an example in which $H$ is infinite and $W_{\Gamma} \rtimes H$ is not a right-angled Coxeter group. We currently do not know whether such extensions with infinite $H$ are CAT(0) or not. Since this question does not address the abstract commensurability of the CAT(0) property, we will not address it further in this paper.

2. Right-angled Coxeter groups and their automorphisms

In this section we briefly recall a very small part of the rich combinatorial and geometric theory of right-angled Coxeter groups. The interested reader may consult [5] for a thorough account of the more general subject of Coxeter groups from the geometric group theory point of view.

Fix an arbitrary finite simple graph $\Gamma$ with vertex set $S$ and edge set $E$. The right-angled Coxeter group defined by $\Gamma$ is the group $W = W_{\Gamma}$ generated by $S$, with relations declaring that the generators all have order 2, and adjacent vertices commute with each other. The pair $(W, S)$ is called a right-angled Coxeter system. As described in [5, Proposition 7.3.4, p. 130], we construct a cube complex $X = X(W, S)$ inductively as follows:

- The set of vertices is indexed by $W$, say $X^0 = \{v_w \mid w \in W\}$.
- To complete the construction of the one-skeleton $X^1$ we add edges of unit length so that vertices $v_u, v_w$ are adjacent if and only if $u^{-1}w \in S$.
- For each $k \geq 2$, we construct the $k$-skeleton by gluing in Euclidean unit cubes of dimension $k$ whenever $X^{k-1}$ contains the $(k-1)$-skeleton of such a cube.

Remark 2.1. We note the following about this construction:

- The dimension of $X$ equals the number of vertices in the largest clique in $\Gamma$. 
The barycentric subdivision of $X$ is the well-known Davis complex $\Sigma = \Sigma(W, S)$. By a result of Gromov, $\Sigma$, and hence also $X$, is a CAT(0) metric space (see [5, Theorem 12.3.3, p. 235] for a generalization due to Moussong).

By construction, the geometry of $X$ is determined entirely by its 1-skeleton $X^1$. It follows that a permutation $\sigma$ of the vertex set $X^0$ determines an isometry of $X$ if it respects the adjacency relation. In particular, for all $w \in W$ the map $v_u \mapsto v_{uw}$ extends to an isometry $\Phi_w \in \text{Isom}(X)$. The map $w \mapsto \Phi_w$ is a faithful geometric action $W \curvearrowright X$ known as the left-multiplication action.

From the graph $\Gamma$ we may infer the existence of certain finite-order automorphisms of $W$. For each vertex $a \in S$, we write $Lk_a$ for the set of vertices adjacent to $a$, and $\text{St}(a)$ for $Lk(a) \cup \{a\}$.

- Each graph automorphism $f \in \text{Aut}(\Gamma)$ restricts to a permutation of $S$ which determines an automorphism $\phi_f \in \text{Aut}(W)$.
- For each union of non-empty connected components $D$ of $\Gamma \setminus \text{St}(a)$, the map
  \[ s \mapsto \begin{cases} \text{asa} & s \in D, \\ s & s \in S \setminus D, \end{cases} \]
  determines an automorphism of $W$ called the partial conjugation with acting letter $a$ and domain $D$.
- If $a, d \in S$ are such that $\text{St}(d) \subseteq \text{St}(a)$, then the rule
  \[ s \mapsto s \text{ for all } s \in S \setminus \{d\}, \quad d \mapsto da, \]
  determines an automorphism of $W$ called the transvection with acting letter $a$ and domain $d$.

Together, the automorphisms induced by graph automorphisms, the partial conjugations and the transvections comprise a generating set for $\text{Aut}(W)$ [7]. We note that partial conjugations and transvections are involutions, and graph automorphisms have finite order.

In what follows, $\phi \in \text{Aut}(W)$ shall always denote a non-trivial automorphism of finite order $m$, and $G$ shall denote the semi-direct product $G = W \rtimes_{\phi} \mathbb{Z}/m\mathbb{Z}$. So $G$ is presented by:

\[ P_1 = \langle S \cup \{z\} \mid s^2 = 1 \text{ for all } s \in S, [s, t] = 1 \text{ for all } \{s, t\} \in E, z^m = 1, zsz^{-1} = \phi(s) \text{ for all } s \in S \rangle. \]

3. WHEN $\phi$ IS INDUCED BY A GRAPH AUTOMORPHISM

Suppose $\phi$ is induced by a graph automorphism $f \in \text{Aut}(\Gamma)$. Then the map $v_w \mapsto v_{\phi(w)}$ preserves the adjacency relation in $X^1$, and hence determines an isometry $\Phi \in \text{Isom}(X)$. By simple computation the reader
may confirm that the relations in the presentation $P_1$ are satisfied when each $s \in S$ is replaced by $\Phi_s$, and $z$ is replaced by $\Phi$. Hence the rule

$$s \mapsto \Phi_s \text{ for all } s \in S, z \mapsto \Phi,$$

determines an action $G \acts X$. We leave the reader to confirm that the action is faithful and geometric, and hence Theorem 1.1 holds in the first of the three cases.

In fact, a stronger result holds for similar reasons.

**Lemma 3.1.** If $H \subseteq \text{Aut}(\Gamma)$ is the group of graph automorphisms and $H$ is the corresponding subgroup of $\text{Aut}(W)$, then the natural action $W \acts X$ extends to a faithful geometric action $W \acts H \acts X$.

4. **When $\phi$ is a partial conjugation**

Now suppose that $\phi$ is the partial conjugation with acting letter $a$ and domain $D$. Recall that $v_w$ denotes the vertex of $X$ indexed by the group element $w \in W$. For any $d \in D$, $v_1$ and $v_d$ are adjacent in $X^1$, but $v_{\phi(1)}$ and $v_{\phi(d)}$ are not. Since the map $v_w \mapsto v_{\phi(w)}$ does not respect adjacency in $X^1$, the left-multiplication action $W \acts X$ does not naturally extend to an action $G \acts X$. However, $G$ is itself a right-angled Coxeter group, and hence also a $\text{CAT}(0)$ group.

**Lemma 4.1.** If $\phi$ is a partial conjugation with acting letter $a$ and domain $D$, then $G$ is itself a right-angled Coxeter group.

We will omit the details of the proof, which may be found in [4]. In that paper, we engage more broadly with the problem of identifying a right-angled Coxeter presentation in a given group (or proving that no such presentation exists). We find various families of extensions of right-angled Coxeter groups which are again right-angled Coxeter, and these include Lemma 4.1 as a special case.

Here we will give a description of how to construct the defining graph $\Lambda$ for $G$ based on the original graph $\Gamma$. The procedure is as follows:

1. Add a new vertex labeled $x$, which we connect to everything in $\Gamma \setminus D$.
2. Replace the label of vertex $a$ with the label $ax$, and add edges connecting $ax$ to each vertex in $D$.

An example is shown in Figure 1.

5. **When $\phi$ is a transvection**

Finally, we suppose that $\phi$ is the transvection with acting letter $a$ and domain $d$. Recall that this means that $\text{St}(d) \subseteq \text{St}(a)$, and $\phi$ is determined
Figure 1. $\Lambda$ is the defining graph of $W_\Gamma \rtimes \langle x \rangle$, where $x$ has acting letter $a_5$ and domain $\{a_6\}$.

by the rule:

$d \mapsto da$, and $s \mapsto s$ for all $s \in S \setminus \{d\}$.

We note that $v_1$ and $v_d$ are adjacent in $X^1$, but $v_{\phi(1)}$ and $v_{\phi(d)}$ are not. Since the map $v_w \mapsto v_{\phi(w)}$ does not respect adjacency in $X^1$, the left-multiplication action $W \circ X$ does not naturally extend to an action $G \circ X$. In fact, a stronger statement is true. It follows from [5, Section 13.2] that $\operatorname{Fix}(d)$ is a codimension 1 subspace of $\Sigma$, and $\operatorname{Fix}(da)$ is codimension 2. Hence there is no isometry of $X$ which can conjugate the isometry representing $d$ to give the isometry representing $da$, so the left-multiplication action $W \circ X$ cannot be extended in any way to an action $G \circ X$.

We also note that $G$ does not embed in a right-angled Coxeter group since $G$ contains an element of order 4. Since $xdx = ad$, we have that $(xd)^2 = a$ and $xd$ has order 4. In a right-angled Coxeter group, any non-trivial element of finite order is an involution.

It seems that to show that $G$ is a CAT(0) group, we must identify a new CAT(0) space $Y$, and describe a faithful geometric action $G \circ Y$. The key to our success in doing exactly this is the existence of a certain finite-index subgroup of $W$ which is itself a right-angled Coxeter group.

Although the existence of such a subgroup is well-known (see [2, Example 1.4], for example, where the analogous subgroup is used in the context of right-angled Artin groups), we provide the details here for completeness.

Let $h_a : W \to \mathbb{Z} / 2\mathbb{Z}$ denote the homomorphism determined by the rule: $a \mapsto 1$, and $s \mapsto 0$ for all $s \in S \setminus \{a\}$. Let $U$ denote the kernel of $h_a$, and let

$$S' = (S \setminus \{a\}) \cup \{asa \mid s \in S \setminus \operatorname{St}(a)\}.$$ 

**Lemma 5.1.** The pair $(U, S')$ is a right-angled Coxeter system, and hence $U$ is a right-angled Coxeter group. Further, conjugation by $a$ in $W$ restricts to an automorphism $\theta \in \operatorname{Aut}(U)$ induced by a permutation of $S'$; this automorphism is trivial if and only if $a$ is central in $W$. 


Proof. If \( a \) is central in \( W \), then \( S' = S \setminus \{a\} \), and the result is evident. In this case conjugation by \( a \) restricts to the trivial automorphism of \( U \), and hence is the automorphism of \( U \) induced by the trivial permutation of \( S' \).

Suppose \( a \) is not central in \( W \). An alternative presentation for \( W \) may be constructed from the standard Coxeter presentation for \( W \) by the following Tietze transformations:

- For each vertex \( s \in S' \setminus \text{St}(a) \), introduce a new generator \( \hat{s} \), the defining relation \( asa = \hat{s} \), and redundant relations \( a\hat{s}a = s \) and \( \hat{s}^2 = 1 \).
- For each pair of adjacent vertices \( s, t \in S' \setminus \text{St}(a) \), introduce the redundant relation \( \hat{s}\hat{t} = \hat{t}\hat{s} \).
- For each pair of adjacent vertices \( s \in S' \setminus \text{St}(a) \) and \( t \in \text{Lk}(a) \), introduce the redundant relation \( \hat{s}t = t\hat{s} \).
- For each vertex \( x \in \text{Lk}(a) \), we rewrite the relation \( xa = ax \) as \( axa = x \).

The resulting presentation of \( W \) is:

\[
P_2 = \langle S' \cup \{a\} \mid x^2 = 1 \text{ for all } x \in S',
\]

\[
[s, t] = 1 \text{ for all } \{s, t\} \in E \text{ such that } s, t \neq a,
\]

\[
[\hat{s}, \hat{t}] = 1 \text{ for all } \{s, t\} \in E \text{ such that } s, t \in S' \setminus \text{St}(a),
\]

\[
[\hat{s}, t] = 1 \text{ for all } \{s, t\} \in E \text{ such that } s \in S' \setminus \text{St}(a) \text{ and } t \in \text{Lk}(a),
\]

\[
a^2 = 1, asa = s \text{ for all } s \in \text{Lk}(a),
\]

\[
asa = \hat{s} \text{ and } a\hat{s}a = s \text{ for all } s \in S' \setminus \text{St}(a)\rangle.
\]

Evidently, this is the presentation of a semi-direct product in which the non-normal factor is \( \langle a \rangle \), the normal factor is a right-angled Coxeter group with generating set

\[
S' = (S \setminus \{a\}) \cup \{\hat{s} \mid x \in S' \setminus \text{St}(a)\},
\]

and \( a \) acts on the normal factor as the automorphism \( \theta \) induced by permuting the generators according to the rule

\[
x \mapsto \hat{x} \text{ and } \hat{x} \mapsto x \text{ for all } x \in S' \setminus \text{St}(a), y \mapsto y \text{ for all } y \in \text{Lk}(a).
\]

The action of \( a \) on \( U \) is non-trivial because \( S \neq \text{St}(a) \). □

We now have the following refined decomposition of \( G \):

\[
G = \langle U \rtimes_\theta \langle a \rangle \rangle \rtimes_\phi \langle z \rangle.
\]

A presentation \( P_3 \) for \( G \) is obtained from the presentation \( P_2 \) for \( W \) by appending the generator \( z \) and relations

\[
z^2 = 1, zsz = s \text{ for all } s \in S' \setminus \{d\}, zdz = da, za\hat{z} = a.
\]
It follows that for each \( g \in G \), there exist unique choices \( u_g \in U \), and \( \epsilon_g, \delta_g \in \{0, 1\} \), such that \( g = u_g a^{\epsilon_g} z^{\delta_g} \). We shall write \( Y \) for the CAT(0) cube complex on which \( U \) acts geometrically and faithfully as defined in Section 2 and we write \( p : G \to U \) for the projection map \( g \to u_g \).

The projection map is not a homomorphism because for \( s \in S \setminus \St(a) \) we have \( p(a)p(s)p(a) = s \neq s' = p(s') \). Even so, it allows us to parlay the left-multiplication action of \( G \) on itself into an action of \( G \cap Y \).

**Lemma 5.2.** For all \( g \in S' \cup \{a, z\} \), the rule

\[
v_u \mapsto v_{p(\phi u)} \quad \text{for all } u \in U,
\]

respects adjacency in \( Y \), and hence determines an isometry \( \Phi_g \in \text{Isom}(Y) \).

**Proof.** Let \( u \in U \), \( s \in S' \) and \( g \in S' \cup \{a, z\} \). To prove the result we must establish that \( v_{p(\phi u)} \) and \( v_{p(\phi us)} \) are adjacent. For this it suffices to show that \( (p(\phi u))^{-1}p(\phi us) \in S' \).

If \( g \in S' \), then

\[
(p(\phi u))^{-1}p(\phi us) = (gu)^{-1}gu = s \in S'.
\]

If \( g = a \), then

\[
(p(au))^{-1}p(aus) = (p(\theta(u)a))^{-1}p(\theta(us)a)
\]

\[
= (\theta(u))^{-1}\theta(us)
\]

\[
= \theta(s) \in S'.
\]

Finally, we consider the case \( g = z \). We note that if \( d \) occurs an even number of times in any word for \( u \), then \( a \) occurs an even number of times in any word for \( \phi(u) \), and \( p(zu) = \phi(u) \). If, on the other hand, \( d \) occurs an odd number of times in any word for \( u \), then \( a \) occurs an odd number of times in any word for \( \phi(u) \), and \( p(zu) = \phi(u)a \). The parity of \( d \) in a group element \( u \in U \) is identified by the homomorphism \( h_d : U \to \mathbb{Z}/2\mathbb{Z} \) determined by the rule \( d \mapsto 1 \), and \( s \mapsto 0 \) for all \( s \in S' \setminus \{d\} \). Therefore, we consider cases based on the value of \( h_d(u) \), and whether or not \( s = d \).

If \( h_d(u) = 0 \) and \( s \neq d \), then

\[
(p(zu))^{-1}p(zus) = (\phi(u))^{-1}\phi(us) = s \in S'.
\]

If \( h_d(u) = 0 \) and \( s = d \), then

\[
(p(zu))^{-1}p(zud) = (\phi(u))^{-1}\phi(ud)a = \phi(d)a = d \in S'.
\]

If \( h_d(u) = 1 \) and \( s \neq d \), then

\[
(p(zu))^{-1}p(zus) = (\phi(u)a)^{-1}\phi(us)a = a\phi(u)^{-1}\phi(us)a = asa = \theta(s) \in S'.
\]

If \( h_d(u) = 1 \) and \( s = d \), then

\[
(p(zu))^{-1}p(zud) = (\phi(u)a)^{-1}\phi(ud) = a\phi(u)^{-1}\phi(u)da = ada = d \in S'.
\]
Adjacency is respected in all cases, so the result holds in the case that 
\( g = z, \) and thus \( \Phi_g \) is an isometry of \( Y \) as required. \( \square \)

In summary, we have that \( G \) is presented by:

\[
P_3 = \langle S' \cup \{a, z\} \mid x^2 = 1 \text{ for all } x \in S', \\
[s, t] = 1 \text{ for all } \{s, t\} \in E \text{ such that } s, t \neq a, \\
[s', t'] = 1 \text{ for all } \{s', t'\} \in E \text{ such that } s', t' \in S' \setminus \text{St}(a), \\
[s, t'] = 1 \text{ for all } \{s, t'\} \in E \text{ such that } s \in S' \setminus \text{St}(a) \text{ and } t' \in \text{Lk}(a), \\
a^2 = 1, \; asa = s \text{ for all } s \in \text{Lk}(a), \\
asa = \hat{s} \text{ and } a\hat{s}a = s \text{ for all } s \in S' \setminus \text{St}(a), \\
z^2 = 1, \; zsz = s \text{ for all } s \in S' \setminus \{d\}, \; zdz = da, \; zaz = a \rangle;
\]

and

\[
\Phi_s(v_u) = v_{su} \text{ for all } s \in S', \\
\Phi_a(v_u) = v_{\theta(u)}, \\
\Phi_z(v_u) = v_{\phi(u)} \text{ if } h_d(u) = 0, \\
\Phi_z(v_u) = v_{\phi(u)a} \text{ if } h_d(u) = 1.
\]

**Lemma 5.3.** The map

\[ g \mapsto \Phi_g \text{ for all } g \in S' \cup \{a, z\}, \]

determines a geometric action \( G \circlearrowleft Y \) which extends the left-multiplication action \( U \circlearrowleft Y \). If \( a \) is not central in \( W \), the action is faithful. If \( a \) is central in \( W \), the kernel is the subgroup generated by \( \{a, z\} \).

**Proof.** To prove that the map determines an isometric group action, we must prove that the relations in the presentation \( P_3 \) for \( G \) hold when each \( g \in S' \cup \{a, z\} \) is replaced by \( \Phi_g \). It is clear that those relations not involving either \( a \) or \( z \) remain true when each \( g \in S' \) is replaced by \( \Phi_g \). We leave the reader to verify that the following relations hold (using the
rules listed immediately before the statement of the lemma):

\[
\begin{align*}
\Phi_0^\circ & = 1, \\
\Phi_a \Phi_s \Phi_a & = \Phi_s \text{ for all } s \in \text{Lk}(a), \\
\Phi_a \Phi_s \Phi_a & = \Phi_z \text{ for all } s \in S \setminus \text{St}(a), \\
\Phi_a \Phi_z \Phi_a & = \Phi_s \text{ for all } s \in S \setminus \text{St}(a), \\
\Phi_z^2 & = 1, \\
\Phi_z \Phi_s \Phi_z & = \Phi_s \text{ for all } s \in S \setminus \{d\}, \\
\Phi_z \Phi_d \Phi_z & = \Phi_d \Phi_a, \\
\Phi_z \Phi_a \Phi_z & = \Phi_a.
\end{align*}
\]

We note that, because \(v_1 \rightarrow v_{p(q)}\), the stabilizer of \(v_1\) is a subgroup of the finite abelian group \(\langle a, z \rangle\). If \(a\) is not central in \(W\), there exists \(s \in S \setminus \text{St}(a)\). Computation shows that \(\Phi_a, \Phi_az\) do not fix \(v_s\), and \(\Phi_z\) does not fix \(v_{ds}\). Our claims about the kernel of the action follow immediately. \(\square\)

If \(a\) is central in \(W\), then there is no obvious way in which \(a\) should act non-trivially on \(Y\). We can, however, extend \(Y\) to a new space \(Y^+\) by appending two unit length edges in a “v” shape at each vertex, thereby providing pieces on which \(a\) and \(\phi\) can act non-trivially. More formally, to construct \(Y^+\) from \(Y\) we write \(v_0^0 u\) for \(v_0 u\), and we append new vertices \(v_i^1 u\) for all \(u \in U\) and \(i \in \{-1, 1\}\), and new unit length edges

\[
\{ \{v_0^0 u, v_u^{-1} \}, \{v_0^1 u, v_u^1 \} \mid \text{for all } u \in U \}.
\]

It is evident that appending such “v” shapes at each vertex does not cause the CAT(0) property to fail, hence \(Y^+\) is a CAT(0) cube complex.

**Proposition 5.4.** If \(a\) is central in \(W\), then \(G\) acts faithfully and geometrically on \(Y^+\).

**Proof.** Suppose that \(a\) is central in \(W\), i.e., that \(\text{St}(a) = \Gamma\). Then \((U, S')\) is a right-angled Coxeter system, and \(W = U \times \langle a \rangle\).

We now define a homomorphism \(\Phi: G \rightarrow \text{Isom}(Y^+)\). For each \(s \in S'\), we declare \(\Phi(s)\) to be the isometry determined by the rule:

\[
v_u^i \mapsto v_u^{i+} \text{ for all } u \in U \text{ and } i \in \{-1, 0, 1\}.
\]

We declare \(\Phi(a)\) to be the isometry determined by the rule:

\[
v_u^i \mapsto v_u^{-i} \text{ for all } u \in U \text{ and } i \in \{-1, 0, 1\}.
\]
We declare $\Phi(z)$ to be the isometry determined by the rule:

$$v_u^i \rightarrow \begin{cases} v_u^i & \text{if } h_d(u) = 0, \\ v_u^{-i} & \text{if } h_d(u) = 1, \end{cases}$$

for all $u \in U$ and $i \in \{-1, 0, 1\}$. The maps can be described informally as follows: each $s \in S'$ acts on $Y^+$ in the way which most naturally extends the left-multiplication action $U \wr Y$; $a$ flips the “v” attached to every vertex; while $z$ flips only half the “v” shapes, because it flips the “v” attached to a vertex $v_u$ if and only if $d$ has an odd parity in $u$.

It is evident that the maps described above preserve adjacency in the one-skeleton of $Y^+$, and hence determine isometries of $Y^+$. Simple computations confirm that these definitions respect the relations in the presentation $P_3$ of $G$ (some of the relations listed are vacuous). Therefore these definitions do indeed determine an isometric action $G \wr Y^+$. That the action is geometric follows easily from the fact that the left-multiplication action $U \wr Y$ is geometric. \hfill $\square$

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