A Modified Kirchhoff plate theory for Free Vibration analysis of functionally graded material plates using meshfree method

Vuong Nguyen Van Do1*

1Faculty of Civil Engineering, Ton Duc Thang University, 19 Nguyen Huu Tho St. Dist. 7, HCMC 700000, Viet Nam.

Corresponding author (*): donguyenvanvuong@tdt.edu.vn

Abstract. In this paper, a modified Kirchhoff theory is presented for free vibration analyses of functionally graded material (FGM) plate based on modified radial point interpolation method (RPIM). The shear deformation effects are taken account into modified theory to ignore the locking phenomenon of thin plates. Due to the proposed refined plate theory, the number of independent unknowns reduces one variable and exists with four degrees of freedom per node. The simulated free vibration results employed by the modified RPIM are compared with the other analytical solutions to verify the effectiveness and the accuracy of the developed mesh-free method. Detail parametric studies of the proposed method are then conducted including the effectiveness of thickness ratio, boundary condition and material inhomogeneity on the sample problems of square plates. Results illustrated that the modified mesh-free RPIM can effectively predict the numerical calculation as compared to the exact solutions. The obtained numerical results are indicated that the proposed method are stable and well accurate prediction to evaluate with other published analyses.

Keywords: functionally graded material (FGM) plate, Mesh-free, modified Radial point interpolation, locking free, a normalization of cubic function.

1. Introduction

In the recent years, the functionally graded materials (FGM) were the first proposed by Bever and Duwez [1] and have been applied in a variety of engineering industries because of their distinctive material properties, which vary continuously through the thickness. These are the special types with their characteristics can be tailored for many application and various working environments. In the typically plates, the smoothness and continuously changing in microstructure from the top to the bottom layer can made the materials displayed in distinct phases regarding to ceramic and metal. The combination by two phases is demonstrated that the ceramic with low thermal conductivity can well resist the harms of thermal stress and surface corrosion due to the high temperature effects [2]. Meanwhile the metal-rich surface is well capable of the highly impacted loading on the structures. In addition, matrix materials can reduce and not be prone to the initial cracks growing into the material sections and debonding of fiber composites at extremely thermal forces. Due to the advantages of mechanical behaviours of functional graded materials (FGM), the significant number of researches had been conducted to examine the mechanical responses of FG shells and plates [3]. Therefore, the analyses of displacement, stress, dynamic problems of natural frequencies and buckling responses are really necessary for FG material behaviours.

Besides the advantages of material properties and behaviours of FGM plates, many plate theories have been researched. The typical method such as the classical plate theory (CPT) [4] with Kirchhoff-
Love assumptions for thin plate gave acceptable results. Nevertheless, the effects of shear deformations were not incorporated into the model, then it leads to totally unsuitable numerical calculations in thick plates. The large number of significant theories had been written in FEM [5] including the shear deformation to overcome the shortcomings of classical plate theory. For instances, the finite element formulations were derived by Reddy [6] for FG plates based on third-order shear deformation theory [TSDT]. The FSDT and kp-Ritz method were investigated by Zhao et al. [7] for bending and vibration analyses. The FSDT and four node finite element using the exact neutral surface position were derived by Singha et al. [8] to calculate the nonlinear bending responses of FG plate with different boundary conditions. However, these methods are over-stiffness when the plates become too thin. It is also known that the HSDTs with FEM are complicated and increased the higher variables in numerical analysis due to the use of linear shape function.

With these disadvantages of FE analysis, the mesh-free method developed by Belytschko [9] could be available to obtain the better accuracy results. Until today, the later various successful methods were established to capture the defects of conventional finite element computation during the meshing generation. In the present study, the cubic form taken account into the radial point shape function is the target to eliminate the dependence of coordinate meshes as well as disappearance of any coefficients in the chosen form. The high accuracy in analyzing for FG plates of the dynamic vibrations is employed by a modified Kirchhoff theory. The verifications of numerical examples are shown a good validation between the proposed method and the other published from the various reported literatures. Results demonstrate that the mesh-free RPIM can effectively predict the dynamic vibration of the FGM isotropic plates.

2. The Kirchhoff theory for FGM plates

2.1. The Kirchhoff theory

Considering the Kirchhoff plate theory with the domain $\Omega$ presented in the $R^2$ given for the mid-plane of the plate, the displacement variables of $u_0, v_0,$ and bending displacement $w^b$ belong to three $x; y; z$ directions. The conventional Kirchhoff model is not considered the effect of shear deformation, therefore it is only suitable for thin plate analysis. To improve this model for analysis both moderately thick and thin plates, a modified transverse displacement can be separated into bending and shear parts. A modified formulation of Kirchhoff theory is described by

$$
\begin{align*}
\hat{u}(x, y, z) &= u_0(x, y) - z \frac{\partial w^b}{\partial x}, \\
\hat{v}(x, y, z) &= v_0(x, y) - z \frac{\partial w^b}{\partial y}, \\
\hat{w}(x, y, z) &= w^b(x, y) + w'(x, y)
\end{align*}
$$

The displacement field for proposed theory in Eq. (1) included four unknown variables of $u_0, v_0, w^b, w'$. In comparison with the Reissner-Mindlin FSDT theory, the number of independent unknowns of the present theory are reduced one variable.

The relationship of strains and displacement equations is described by

$$
\varepsilon_s = [\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}]^T = \varepsilon_0 + z\varepsilon_c
$$

$$
\gamma = [\gamma_{yz}, \gamma_{zy}]^T = \varepsilon_c
$$

(2)
where,

\[ \mathbf{e}_0 = \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} + \frac{\partial u_0}{\partial y} \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} -\frac{\partial^2 w^b}{\partial x^2} \\ -\frac{\partial^2 w^b}{\partial y^2} - 2\frac{\partial^2 w^b}{\partial x \partial y} \end{bmatrix} \text{ and } \mathbf{e}_s = \begin{bmatrix} w_x^s \\ w_y^s \end{bmatrix} \] 

The dynamic equation of modified Kirchhoff theory based on Ref. [11] can be obtained by the weak form as

\[ \int_{\Omega} \delta \left( \mathbf{e}^T \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{bmatrix} \mathbf{e}_0 \right) d\Omega + \int_{\Omega} \delta \left( \mathbf{u}_0 \right)^T \mathbf{P} d\Omega + \int_{\Omega} \delta \left( \mathbf{u}_s \right)^T \left( I_1, I_2, I_s \right) \left[ \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_s \end{bmatrix} \right] d\Omega = 0 \] 

where

\[ (I_1, I_2, I_s) = \int_{-h/2}^{h/2} \rho(z)(1, z, z^2) dz \quad \mathbf{u}_0 = [u v w]^T \quad \mathbf{u}_s = \begin{bmatrix} \frac{\partial w_x}{\partial x} \\ \frac{\partial w_y}{\partial y} \end{bmatrix}^T \] 

and

\[ A_i B_i D_q = \int_{-h/2}^{h/2} (I_1, z, z^2) Q_i dz \quad D_q = k_s \int_{-h/2}^{h/2} G_q dz \] 

The material matrices are given as

\[ Q = \frac{E_s}{1 - \nu_s^2} \begin{bmatrix} 1 & \nu_s & 0 \\ \nu_s & 1 & 0 \\ 0 & 0 & (1 - \nu_s) / 2 \end{bmatrix}, \quad G = \frac{E_s}{2(1 + \nu_s)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \] 

in which \( E_s, \nu_s \) are effective Young’s modulus and Poisson’s ratio, respectively. Those parameters can be discussed in the next subsection. And \( k_s \) denotes the transverse shear correction coefficient because transverse shear strains in Eq. (2) are constant through the thickness of plate. For FGM, it can be defined by Ref. [16].

\[ k_s = \frac{5}{6 - (\nu_s V_s + \nu_m V_m)} \] 

### 2.2. Problem formulation for isotropic FGM plates

The isotropic homogeneous plate is divided into two different material phases in which ceramic and metal are arranged at the top and bottom layers, respectively. In this paper, the homogenized formulation of the rule of mixture is utilized to homogenize the material parameters. The volume fraction of ceramic phase is described as [6].
\[ V_c(z) = \left( 0.5 + \frac{z}{h} \right)^n, \quad z \in \left[ -\frac{h}{2}, \frac{h}{2} \right] \text{ and } V_m = 1 - V_c \]  

where \( m \) and \( c \) are defined to the metal and ceramic. The scale parameter \( n \) is representative to the gradation of material properties varying through the thickness. The following estimation of the effective material characteristics is homogenized by

\[ P_e = P_c V_c(z) + P_m V_m(z) \]  

where \( P, P_m \) define the material constituents of the ceramic and the metal, respectively. The Young’s modulus (\( E \)), Poisson’s ratio (\( \nu \)), and density (\( \rho \)) are material properties which are presented by Eq. (10).

\[ u^k(x) = \sum_{i=1}^{m} R_i(x) a_i + \sum_{j=1}^{m} p_j(x) b_j \]  

And the constraint condition
\[
\sum_{i=0}^{n} p_j(x_i) a_i = 0, \quad j = 1, 2, ..., m
\]  \tag{12}

where \( R_i(x) \) is the radial basis function (RBF) which describes the distance of interpolation point \( x \) and a node \( x_i \), \( n \) is the number of nodes in the sub-domain around of \( x \), \( p_j(x) \) is the monomial in the space coordinates \( x^T = [x, y] \), \( m \) is the number of polynomial basis functions, coefficients of \( a_i \) and \( b_j \) are interpolation constants. These are determined by rewriting the Eqs. (11) and (12) in the matrix form.

\[
G a_0 = \begin{bmatrix} R & P \end{bmatrix} \begin{bmatrix} a \\ \end{bmatrix} = \begin{bmatrix} u_i \\ 0 \end{bmatrix}
\]  \tag{13}

where \( a_i^T = [a^T, b^T] = (a_1, a_2, ..., a_n, b_1, b_2) \), \( u_i^T = (u_x, u_y, ..., u_z) \)

To avoid calculating \( G^{-1} \) due to the symmetric matrices of \( R \) and \( G \), the algorithm of Eq. (13) can be solved as following

\[
a = R^{-1} u_x - R^{-1} P b \]  \tag{14}

Substituting above equation to the Eq. (12), gives

\[
b = S_a u_x
\]  \tag{15}

where \( S_a = [P^T R^{-1} P]^{-1} P^T R^{-1} \)

Substituting Eq. (15) into Eq. (14), the expression can be written as \( a = S_a u_x \), where \( S_a = R^{-1} [I_{n,n} - P S_a] \). Finally, the displacement field can be approximated by general basic form of radial point interpolation

\[
u^i(x) = [R^T(x) S_a + P^T(x) S_a] u_x
\]  \tag{16}

In this study, the aim of the present work is to utilize a proposed interpolation function as polynomial form to build the shape function without any supporting fixing coefficients. The proposed normalized cubic function is formulated by

\[
R(r) = \begin{cases} 
\frac{1}{2} - 4 \left( \frac{\theta}{r_s} \right)^3 r_{ij}^2 + 4 \left( \frac{\theta}{r_s} \right)^2 r_{ij}^3 \text{ for } 0 \leq \frac{r_{ij}}{r_s} \leq \frac{1}{2} \\
\frac{1}{2} - 4 \left( \frac{\theta}{r_s} \right)^2 r_{ij}^3 - 4 \left( \frac{\theta}{r_s} \right) r_{ij}^2 \text{ for } \frac{1}{2} \leq \frac{r_{ij}}{r_s} \leq 1 \\
0 \text{ otherwise}
\end{cases}
\]  \tag{17}
where a denominator $r_i$ is taken to be a maximum distance between a pair of nodes in the support domain $\Omega_i$. In mesh-free method, a set of nodes are determined as a support domain. The circle support domain can be utilized with radius defined as in Eq. (18)

$$d_m = \alpha d_c$$  

(18)

where $d_c$ is a length parameter which relates to the distance of nodes, which is equal to nodal distance in regular mesh. $\alpha$ is the scale factor.

### 3.2. RPIM interpolation for discrete equations

Supposing that the number of total nodes converges in $\Omega \in \Omega$ is $n$ and the amount of nodes $n$ in the support domain corresponding to the interest node is at centered $x$ with the radius $d_m$. The RPIM approximations for geometric shape and the displacement field $u$ in the middle plane of the plate can be implemented following as

$$x^d = \sum_{i}^{n} \phi_i(x) x_i, \quad u^d(x) = \sum_{i}^{n} \phi_i(x) q_i$$  

(19)

where $x^d = (x \quad y)$ is the coordinate vector of the real geometry of plate. In Eq. (19), $\phi_i(x)$ is the normalized RPIM basis function, $q_i = [u_{i0} \quad v_{i0} \quad w_{i0}^b \quad w_{i0}^s]^T$ is nodal vector of the four unknown variables for an arbitrary node $i$ and $x_i$ is the nodal value.

The relationship between in-plane/shear strain and nodal displacements is obtained by substituting Eq. (19) into Eq. (3) by

$$\varepsilon_0 = \sum_{i=1}^{n} B_i^0 q_i; \quad \varepsilon_1 = \sum_{i=1}^{n} B_i^1 q_i; \quad \varepsilon_s = \sum_{i=1}^{n} B_i^s q_i$$  

(20)

in which

$$B_i^0 = \begin{bmatrix} \phi_{i,x} & 0 & 0 & 0 \\ 0 & \phi_{i,y} & 0 & 0 \\ \phi_{i,y} & \phi_{i,x} & 0 & 0 \end{bmatrix}, \quad B_i^1 = \begin{bmatrix} 0 & 0 & -\phi_{i,xx} & 0 \\ 0 & 0 & -\phi_{i,yy} & 0 \\ 0 & 0 & -2\phi_{i,xy} & 0 \end{bmatrix} \quad \text{and} \quad B_i^s = \begin{bmatrix} 0 & 0 & 0 & \phi_{i,x} \\ 0 & 0 & 0 & \phi_{i,y} \end{bmatrix}$$  

(21)

Also, the displacement fields $u_0$ and $u_1$ by substituting Eq. (19) into Eq. (5), can be expressed as follows

$$u_0 = \sum_{i=1}^{n} N_i^0 q_i; \quad u_1 = \sum_{i=1}^{n} N_i^1 q_i$$  

(22)
where

\[
\begin{bmatrix}
\phi_{x} & 0 & 0 \\
0 & \phi_{x} & 0 \\
0 & 0 & \phi_{y}
\end{bmatrix} \begin{bmatrix}
N_{y}^0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 & -\phi_{x} \\
0 & 0 \\
0 & -\phi_{y}
\end{bmatrix} \begin{bmatrix}
N_{x}^0 \\
0 \\
0
\end{bmatrix}
\] (23)

The derivatives of the transverse displacements including components of \( w_{x}, w_{y} \) are also given by

\[
\begin{bmatrix}
w_{x} & + w'_{x} \\
w_{y} & + w'_{y}
\end{bmatrix} = \sum_{i=1}^{n} B_{i}^{x} q_{i}
\] (24)

where

\[
B_{i}^{x} = \begin{bmatrix}
0 & 0 & \phi_{x,i} \\
0 & 0 & \phi_{y,i}
\end{bmatrix}
\] (25)

Finally, substituting Eqs. (20), (22) and (24) into Eqs.(4) for free vibration can be derived by following equations

\[
(K - \omega^{2}M)q = 0
\] (26)

where \( \omega \in \mathbb{R} \) are the natural frequency and the critical buckling parameter, respectively.

where

\[
K = \int_{\Omega} \begin{bmatrix}
[B^{n}]^{T} & [A] & [B^{n}] \\
[B^{x}] & [B^{y}] & [B^{x}] + [B^{y}]^{T} [D^{y}] [B^{x}]
\end{bmatrix} d\Omega
\]

\[
M = \int_{\Omega} \begin{bmatrix}
[N_{y}]^{T} & [I_{1}] & [I_{2}] & [N_{y}] \\
[I_{1}] & [I_{2}] & [N_{y}]
\end{bmatrix} d\Omega
\] (27)

4. Numerical results and discussion

This section reports several non-dimensional fundamental frequencies \( \bar{\omega} = \omega h \sqrt{\rho / E} \) of simply supported (SSSS) plates to confirm the accuracy and the effectiveness of the proposed modified Kirchhoff theory using meshfree method. The material properties in which the first set is of Aluminum-Ziconia (Al/ZrO2), while the second one is of aluminum and alumina (Al/Al2O3) can be found in Table 1. The effective characteristics of functional graded materials are assumed to homogenize by the rule of mixture. For the numerical calculations, FGM plate was discretized by 23×23 regular nodes with 23×23 quadrilateral background mesh for the integration and the scaling factor \( (\alpha) \) of 2.4 was used to determine the size of the influence domain. The free vibration analyses for the isotropic Al/AL2O3 FG thin plate of length to thickness ratio, \( a/h= 100 \) tabulated in Table 2 have proved the comparison between the proposed theory and the conventional Kirchhoff plate theory.
As a result, the numerical solutions are acceptably given and effectively predicted by the present method, however, the higher discrepancy of conventional Kirchhoff plate theory is observed as validating to the analytical results [16]. Furthermore, considering the numerical results for the moderately thick FG Al/ZrO$_2$ plate of a/h=5 shown in Table 3, the computed frequencies from suggested theory are well matched to the exact 3D solutions [14] and slightly higher than others reported by SSDT[17] and Quasi-3D [15]. It is noting that the accuracy of the numerical results for the thin to moderately thick plates is dependent on the modified Kirchhoff theory irrespective of different FG material constituents, meanwhile, almost results from CPT theory are always over-estimated due to the shear part not be taken account into the calculation.

Table 1. Material properties

| E (GPa) | Al     | ZrO$_2$ | Al$_2$O$_3$ |
|---------|--------|---------|-------------|
| $\nu$   | 0.3    | 0.3     | 0.3         |
| $\rho$ (kg/m$^3$) | 2707   | 5700    | 3800        |

Table 2. The natural frequency $\bar{\omega}=\omega h \sqrt{\rho_n / E_n}$ of SSSS Al/Al$_2$O$_3$ square plate with a/h=100.

| Model          | a/h=1 | a/h=2 |
|----------------|-------|-------|
| Analytical [14]|       |       |
| Present        |       |       |
| CPT            |       |       |

Table 3. The natural frequency $\bar{\omega}=\omega h \sqrt{\rho_n / E_n}$ of SSSS Al/ZrO$_2$ square plate with a/h=5.

| Model          | $\varepsilon$ | $n$ |
|----------------|---------------|-----|
| Exact [14]     | -             | 0   |
| Quasi-3D [15]  | $\neq 0$      | 0.2192 0.2211 0.2225 |
| SSDT [17]      | 0             | 0.2184 0.2202 0.2215 |
| SSDT [17]      | $\neq 0$      | 0.2193 0.2212 0.2225 |
| Present        | 0             | 0.2462 0.2223 0.2234 0.2223 |
| CPT            | 0             | 0.2462 0.2542 0.2483 0.2500 0.2529 |

Table 4. Reveals the first ten normalized frequencies in the parametric study of length to thickness ratios such as a/h=5, 10, 20. The normalized frequencies $\bar{\omega}$ of the first mode are high accuracy and closed to exact results within thick to moderately thin plates, while higher modes are reasonable agreement could be observed. Figure 2 illustrates the six dynamic vibration mode shapes of simply supported FG plate. Note that the deformations are magnified by 1000 times for clarity.

Table 4. The natural frequency $\bar{\omega}$ of SSSS Al/ZrO$_2$ square plate with various ratios a/h.
Finally, parametric studies were performed to scrutinize the free vibration characteristics of the FGM plate. Table 5 and Table 6 respectively displays the effects, in that order, of the plate length-to-thickness ratio and boundary conditions of SSSC, SCSC. The letter ‘S’ is defined for simply supported and ‘C’ is denoted for the clamped edge. The material of FG Al/Al₂O₃ are also given in Table 1.

The first normalized natural frequencies \( \omega = \sqrt{\frac{\rho}{h E_x}} \) are exhibited in Table 5 regarding to several gradient indices \( n (0; 0.5; 1; 4; 10) \) evidence accurate estimation of natural frequencies comparing to the Refs of FSDT-IGA, s-FSDT-IGA [12] and analytical results [11, 13], respectively. The non-dimensional frequencies are decreased when the index values \( n \) are increased. The similar trends could be seen regarding to different length to thickness \( a/h \) ratios. Furthermore, considering the effects of the different boundary conditions on the normalized frequencies, the three different length to thickness ratios \( a/h = 5, 10, 20 \) and inhomogeneous indices \( n = 0, 0.5, 1, 2, 5, 10 \) of Al/Al₂O₃ FG plates are adopted. A good agreement can be found for the computed results of proposed method verified by the exact analytical solutions [13]. Remarkable influence of clamped increasing on the edges could make the higher dynamic vibration magnitudes as closed locking in Table 6 due to gradual increasing of the stiffness of the FG plates.

**Table 5.** The first normalized natural frequency \( \omega = \sqrt{\frac{\rho}{h E_x}} \) of SSSS Al/Al₂O₃

| \( a/h \) | Model | \( n \) | \( 0 \) | \( 0.5 \) | \( 1 \) | \( 4 \) | \( 10 \) |
| --- | --- | --- | --- | --- | --- | --- | --- |
| 2 | Analytical solution [11] | 0.9400 | 0.8232 | 0.7476 | 0.5997 | 0.5460 |
|  | FSDT-IGA[12] | 0.9265 | 0.8060 | 0.7330 | 0.6111 | 0.5640 |
|  | s-FSDT-IGA [12] | 0.9265 | 0.8027 | 0.7267 | 0.6055 | 0.5620 |
|  | Present | 0.9267 | 0.8062 | 0.7332 | 0.6113 | 0.5641 |
|  | CPT | 0.9715 | 0.9467 | 0.9295 | 0.8839 | 0.8595 |
| 10 | Analytical solution [11] | 0.0578 | 0.0492 | 0.0443 | 0.0381 | 0.0364 |
|  | FSDT[13] | 0.0577 | 0.049 | 0.0442 | 0.0382 | 0.0366 |
|  | FSDT-IGA[12] | 0.0577 | 0.049 | 0.0442 | 0.0382 | 0.0366 |
|  | s-FSDT-IGA [12] | 0.0577 | 0.049 | 0.0441 | 0.0382 | 0.0365 |
Table 6. The natural frequency $\bar{\omega}$ of various boundary condition SSSS Al/Al$_2$O$_3$ square plate with various ratios $a/h$

| Boundary condition | $a/h$ | Model         | Power law index ($n$) |
|--------------------|-------|---------------|-----------------------|
|                    |       |               | 0         | 0.5  | 1     | 2     | 5     | 10    |
| SSSC               | 5     | Exact [13]    | 5.9625   | 5.1188 | 4.6356 | 4.1996 | 3.8916 | 3.7146 |
|                    |       | Present       | 6.0619   | 5.1957 | 4.7019 | 4.2605 | 3.9569 | 3.7818 |
|                    |       | CPT           | 6.9319   | 6.4689 | 6.3196 | 6.3042 | 6.4355 | 6.4197 |
|                    | 10    | Exact [13]    | 6.7751   | 5.7649 | 5.2039 | 4.7261 | 4.4462 | 4.2839 |
|                    |       | Present       | 6.7853   | 5.7683 | 5.2051 | 4.7276 | 4.4525 | 4.2928 |
|                    |       | CPT           | 7.0419   | 6.6263 | 6.4745 | 6.4660 | 6.6126 | 6.5978 |
|                    | 20    | Exact [13]    | 7.0526   | 5.9810 | 5.3926 | 4.9019 | 4.6382 | 4.5443 |
|                    |       | Present       | 7.0187   | 5.9501 | 5.3642 | 4.8764 | 4.6149 | 4.4630 |
|                    |       | CPT           | 7.0876   | 6.6675 | 6.5150 | 6.5084 | 6.6591 | 6.6447 |
| SCSC               | 5     | Exact [13]    | 6.7663   | 5.8409 | 5.3039 | 4.8032 | 4.4127 | 4.1865 |
|                    |       | Present       | 6.9847   | 6.0132 | 5.4538 | 4.9414 | 4.5585 | 4.3346 |
|                    |       | CPT           | 8.3208   | 7.8387 | 7.6580 | 7.6392 | 7.7974 | 7.7779 |
|                    | 20    | Exact [13]    | 8.5674   | 7.2715 | 6.5585 | 5.9612 | 5.6332 | 5.4423 |
|                    |       | Present       | 8.4740   | 7.1879 | 6.4822 | 5.8931 | 5.5725 | 5.3853 |
|                    |       | CPT           | 8.5964   | 8.0872 | 7.9026 | 7.8948 | 8.0773 | 8.0594 |
5. Conclusion

In this paper, a simple improved Kirchhoff plate theory combines with mesh-free in which a cubic function using RPIM method for FGM isotropic plates has been proposed. Based on the results of this framework, the following conclusions can be drawn:

- The normalized cubic shape function in the radial point interpolation is chosen in the present method could give the stable solutions in the numerical analysis and better than the conventional Kirchhoff theory.

- The improved Kirchhoff theory of which the transverse deflection $w$ divides into bending ($w^b$) and shear ($w^s$) components effectively predicts the FG plate from the moderately thick to thin plates.

- The numerical results in the parametric study approach to the exact solutions with closed form of the high order shear deformation theories such as HSDT, SSDT. It is worth concluding that the simple present theory is highly reliable to assess the results for many types of FGM plates.

Figure 2. First six mode shapes of Al/ZrO$_2$ FG square plate (n=1, a/h=20)
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