Quantization of line bundles on lagrangian subvarieties

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To Volodya Drinfeld on the occasion of his 60th Birthday

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Abstract We apply the technique of formal geometry to give a necessary and sufficient condition for a line bundle supported on a smooth Lagrangian subvariety to deform to a sheaf of modules over a fixed deformation quantization of the structure sheaf of an algebraic symplectic variety.

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1 Introduction

1.1. Let $X$ be a smooth algebraic symplectic variety over a field $\mathbb{k}$ of characteristic zero. Let $\omega$ denote the symplectic 2-form and $\{-,-\}$ the associated Poisson bracket on $X$. 
$\mathcal{O}_X$, the structure sheaf of $X$. A formal quantization of $X$ is, by definition, a sheaf $\mathcal{O}_\hbar$ on $X$ (in the Zariski topology) of flat associative $\mathbb{K}[[\hbar]]$-algebras (which is complete and separated in the linear topology of a $\mathbb{K}[[\hbar]]$-module), equipped with an isomorphism $\mathcal{O}_\hbar/h\mathcal{O}_\hbar \cong \mathcal{O}_X$ and such that $\frac{1}{\hbar}(ab - ba) \mod \hbar = \{a \mod \hbar, b \mod \hbar\}$ for all $a, b \in \mathcal{O}_\hbar$.

We will be interested in the problem of quantization of $\mathcal{O}_X$-modules. Thus, given a coherent $\mathcal{O}_X$-module $\mathcal{L}$ and a formal quantization $\mathcal{O}_\hbar$ of $X$, we are looking for $\mathcal{O}_\hbar$-modules $\mathcal{L}_\hbar$, flat over $\mathbb{K}[[\hbar]]$, such that $\mathcal{L}_\hbar/h\mathcal{L}_\hbar \cong \mathcal{L}$. A necessary condition for the existence of such an $\mathcal{L}_\hbar$ is provided by the fundamental integrability of characteristics theorem, due to Gabber [6]. It says that if $\mathcal{L}$ admits a flat deformation to an $\mathcal{O}_\hbar/h^3\mathcal{O}_\hbar$-module then the (smooth locus of) every irreducible component of Supp $\mathcal{L}$, the support of the coherent sheaf $\mathcal{L}$, must be a coisotropic subvariety of $X$.

In this paper we restrict ourselves to a special case where the support of $\mathcal{L}$ is a smooth Lagrangian subvariety $Y \subset X$ and, moreover, $\mathcal{L}$ is a direct image to $X$ of (the sheaf of sections of) a line bundle $L$ on $Y$. Our main result provides a complete classification of all formal quantizations $\mathcal{L}_\hbar$, of $\mathcal{L}$, in terms of the Atiyah-Chern class of $L$ and a ‘noncommutative period map’ introduced by Bezrukavnikov and Kaledin [4].

In more detail, let $Y$ be a smooth Lagrangian subvariety of $X$, let $\Omega^{\geq 1}_Y$ denote the truncated de Rham complex $0 \to \Omega^1_Y \to \Omega^2_Y \to \cdots$, a subcomplex of the algebraic de Rham complex $(\Omega^*_Y, d)$. Thus, one has a canonical map $H^2(\Omega^{\geq 1}_Y) = H^2(Y, \Omega^{\geq 1}_Y) \to H^2(Y, \Omega^*_Y) = H^2_{DR}(Y)$, $\alpha \mapsto \alpha_{DR}$. For any line bundle $L$ on $Y$ there is an associated Atiyah class $c_1(L) \in H^2(\Omega^{\geq 1}_Y)$ such that its image $c_1(L)_{DR}$ is the usual first Chern class of $L$.

In Sect. 5.3 we construct a class $At(\mathcal{O}_\hbar, Y) \in H^2(Y, \Omega^{\geq 1}_Y)$ canonically associated with any quantization $\mathcal{O}_\hbar$ of $\mathcal{O}_X$. This class corresponds to a natural Atiyah algebra

$$
0 \to \mathcal{O}_Y \to \mathcal{O}_Y^{\mathcal{O}_\hbar}(\mathcal{O}_Y, \mathcal{O}_Y) \to T_Y \to 0,
$$

where $\mathcal{O}_Y$ is viewed as an $\mathcal{O}_\hbar$-module via the projection $\mathcal{O}_\hbar \to \mathcal{O}_X$.

On the other hand, Bezrukavnikov and Kaledin have constructed, see [4, Definition 4.1], a class

$$
\text{per}(\mathcal{O}_\hbar) = [\omega] + h\omega_1(\mathcal{O}_\hbar) + h^2\omega_2(\mathcal{O}_\hbar) + h^3\omega_3(\mathcal{O}_\hbar) + \cdots \in H^2_{DR}(X)[[\hbar]].
$$

where $[\omega]$ stands for the de Rham cohomology class of the closed 2-form $\omega$ and $\omega_i(\mathcal{O}_\hbar) \in H^2_{DR}(X)$. We show in Lemma 5.8 that the two constructions are compatible in the sense that one has

$$
At(\mathcal{O}_\hbar, Y)_{DR} = i_Y^*(\omega_1(\mathcal{O}_\hbar)),
$$

where $i_Y^*: H^2_{DR}(X) \to H^2_{DR}(Y)$ is the restriction map induced by the imbedding $i_Y: Y \hookrightarrow X$.

Now, let $K_Y = \Omega^\dim_Y$ denote the canonical bundle. Our main result reads
Theorem 1.4 Let \((X, \omega)\) be an algebraic symplectic manifold, let \(i_Y : Y \hookrightarrow X\) be a closed imbedding of a smooth Lagrangian subvariety, and let \(L\) be the sheaf of sections of a line bundle on \(Y\). Then, we have

(i) The sheaf \((i_Y)_* L\) admits a quantization, i.e., there exists a complete flat left \(\mathcal{O}_\hbar\)-module \(\mathcal{L}_\hbar\) such that \(\mathcal{L}_\hbar / \hbar \mathcal{L}_\hbar \cong (i_Y)_* L\) if and only if the following two conditions hold:

\[
c_1(L) - \frac{1}{2} c_1(K_Y) = \text{At}(\mathcal{O}_\hbar, Y) \quad \text{holds in } H^2(\Omega^{\geq 1}_Y);
\]

\[
i_Y^* \omega_i (\mathcal{O}_\hbar) = 0 \quad \text{holds in } H^2_{\text{DR}}(Y), \quad \forall \ i \geq 2.
\]

(ii) If the set \(Q(X, \omega, Y)\) of isomorphism classes of quantizations of line bundles on \(Y\) is non-empty, then this set affords a free and transitive action of the group of isomorphism classes of \((\mathcal{O}_Y[[\hbar]])^*\)-torsors on \(Y\) with a flat algebraic connection.

For a line bundle \(L\) the Atiyah-Chern class \(c_1(L) \in H^2(\Omega^{\geq 1}_Y)\) measures the obstruction to the existence of a flat algebraic connection on \(L\). Therefore, in a special case where \(\text{At}(\mathcal{O}_\hbar, Y) = 0\) and \(i_Y^* \omega_i (\mathcal{O}_\hbar) = 0, \ i \geq 2,\) our theorem claims that \(L\) admits a deformation quantization if and only if the line bundle \(L \otimes^2 \otimes K_Y\) has a flat algebraic connection. In such a case, one says, abusing terminology somewhat, that \(L\) is a square root of the canonical class.

The existence of quantization for square roots of the canonical bundle seems to have been first discovered (without proof) by Kashiwara [7], in the framework of complex analytic contact geometry. Later on, D’Agnolo and Schapira [5] established a similar result for Lagrangian submanifolds of a complex analytic symplectic manifold. In the \(C^\infty\)-context, some closely related constructions can be found in the work of Nest and Tsygan [9].

Remark 1.5 Our main result can be applied in a slightly more general setting where \(Y\) is a (possibly singular) normal subvariety of \(X\) such that \(Y^\text{reg}\), the smooth locus of \(Y\), is a Lagrangian submanifold, and where \(L\) is a coherent \(\mathcal{O}_Y\)-module isomorphic to the direct image of a line bundle \(L^\text{reg}\) on \(Y^\text{reg}\). In this case, our theorem tells when \(L^\text{reg}\) admits a deformation quantization \(\mathcal{L}^\text{reg}_\hbar\). This is a sheaf on \(X \setminus (Y \setminus Y^\text{reg})\) and the direct image of \(\mathcal{L}^\text{reg}_\hbar\) to \(X\) is a coherent \(\mathcal{O}_\hbar\)-module, since \(\text{dim}(Y \setminus Y^\text{reg}) \leq \text{dim} Y - 2\). It is clear that the latter module provides a deformation quantization of the original sheaf \(L\).

2 Basic constructions

2.1. Fix \(n \geq 1\) and a \(2n\)-dimensional vector space \(v\) equipped with a symplectic form \(\omega \in \wedge^2(v^*)\).

Associated with \(\omega\), there is a Heisenberg Lie algebra with an underlying vector space \(v \oplus \mathbb{R}\hbar\), where \(\hbar\) denotes a fixed base element, and the Lie bracket is defined by the formulas.
\[ [x, y] = \omega(x, y) \cdot \hbar, \quad [x, \hbar] = 0, \quad \forall x, y \in \mathfrak{v}. \]

Let \( D \) be the universal enveloping algebra of this Heisenberg Lie algebra. Thus, \( D \) is an associative \( \mathbb{k}[\hbar] \)-algebra, also known as the ‘homogeneous version of the Weyl algebra’. The algebra \( D \) comes equipped with a natural grading \( D = \mathfrak{D}_{i \geq 0} D^i \) such that the vector space \( \mathfrak{v} \) is placed in degree 1 and the element \( \hbar \) is assigned grade degree 2. Let \( \mathcal{D} = \prod_{i \geq 0} D^i \) and, for any \( j \geq 0 \), put \( \mathcal{D}^\geq j = \prod_{i \geq j} D^i \). Thus, \( \mathcal{D} \) is an associative \( \mathbb{k}[[\hbar]] \)-algebra equipped with a descending filtration \( \mathcal{D} = \mathcal{D}^\geq 0 \supset \mathcal{D}^\geq 1 \supset \cdots \), by two-sided ideals. This filtration makes \( \mathcal{D} \) a complete topological algebra with \( \mathcal{D}^\geq 1 \) being the unique maximal ideal of \( \mathcal{D} \).

Let \( \frac{1}{\hbar} \mathcal{D} \) denote a free rank one \( \mathcal{D} \)-submodule of \( \mathbb{k}((\hbar)) \otimes_{\mathbb{k}[\hbar]} \mathcal{D} \) generated by the element \( \frac{1}{\hbar} \mathcal{D}^\geq i+2 \). Let \( \mathfrak{g}' := \frac{1}{\hbar} \mathcal{D}^\geq i+j \), where \([a, b] = ab - ba\) denotes the commutator in the algebra \( \mathbb{k}((\hbar)) \otimes_{\mathbb{k}[\hbar]} \mathcal{D} \), so that \( \mathfrak{g}' \) is a graded Lie algebra. We have \( \mathfrak{g}^{-2} = \mathbb{k} \) and \( \mathfrak{g}^{-1} = \mathfrak{v} \). The vector space \( \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2} \subset \mathfrak{g} \) is a Lie subalgebra isomorphic to the Heisenberg algebra.

The homogeneous component \( \mathfrak{g}^0 \subset \mathfrak{g} \) is also a Lie subalgebra. There is a canonical direct sum decomposition \( \mathfrak{g}^0 = \mathbb{k}_\mathfrak{g} \oplus [\mathfrak{g}^0, \mathfrak{g}^0] \), where the first summand, the image of the imbedding \( \varepsilon : \mathbb{k} \hookrightarrow \mathfrak{g}^0 \), \( c \mapsto \frac{1}{\hbar} (c \hbar) \), is the center of the Lie algebra \( \mathfrak{g}^0 \). The commutator map \([-,-] : \mathfrak{g}^0 \times \mathfrak{g}^{-1} \to \mathfrak{g}^{-1} \) gives an action of the Lie algebra \( \mathfrak{g}^0 \) on \( \mathfrak{g}^{-1} = \mathfrak{v} \). The center \( \mathbb{k}_\mathfrak{g} \subset \mathfrak{g}^0 \) acts trivially; the action of the second summand yields a canonical Lie algebra isomorphism

\[ \sigma : \mathfrak{sp}(\mathfrak{v}) \sim \left[ \mathfrak{g}^0, \mathfrak{g}^0 \right] \subset \frac{1}{\hbar} \mathcal{D}^2, \tag{2.1} \]

where \( \mathfrak{sp}(\mathfrak{v}) \) is the Lie algebra of the symplectic group \( S\mathfrak{p}(\mathfrak{v}) \).

We put \( \mathfrak{g} = \frac{1}{\hbar} \mathcal{D} \slash \frac{1}{\hbar} \mathbb{k}_\mathfrak{g} \), a quotient of the Lie algebra \( \frac{1}{\hbar} \mathcal{D} \) by a central subalgebra. The filtration on \( \mathcal{D} \) induces a descending filtration \( \mathfrak{g} = \mathfrak{g}^\geq -1 \supset \mathfrak{g}^\geq 0 \supset \mathfrak{g}^\geq 1 \supset \cdots \), where \( \mathfrak{g}^\geq i = \prod_{j \geq i} \mathfrak{g}^j \). Here, \( \mathfrak{g}^\geq 0 \) is a Lie subalgebra of \( \mathfrak{g} \) and \( \mathfrak{g}^\geq 1 \) is a pronilpotent Lie ideal of \( \mathfrak{g}^\geq 0 \), moreover, the natural map \( \mathfrak{g}^0 \to \mathfrak{g} \) provides a canonical ‘Levi decomposition’:

\[ \mathfrak{g}^\geq 0 = \mathbb{k}_\mathfrak{g} \oplus (\mathfrak{sp}(\mathfrak{v}) \ltimes \mathfrak{g}^\geq 1). \tag{2.2} \]

2.2. We recall the following definition, see [1].

**Definition** A Harish-Chandra pair over \( \mathbb{k} \) is a pair \((G, \mathfrak{h})\) where \( G \) is a connected affine (pro)algebraic group, \( \mathfrak{h} \) is a Lie algebra equipped with a \( G \)-action, and with a Lie algebra embedding \( \mathfrak{g} \to \mathfrak{h} \) of the Lie algebra \( \mathfrak{g} \) of \( G \) such that the adjoint action of \( \mathfrak{g} \) on \( \mathfrak{h} \) via the imbedding equals the differential of the given \( G \)-action.

The reason for introducing the notion of a Harish-Chandra pair is that there exist infinite-dimensional Lie algebras that cannot be integrated to algebraic groups. However, they have Lie subalgebras that can be integrated to an algebraic group. Here is an important example.

Let \( \text{Der}(\mathcal{D}) \) denote the Lie algebra of \( \mathbb{k}[[\hbar]] \)-linear continuous derivations of the ring \( \mathcal{D} \) and \( \text{Der}^\geq j(\mathcal{D}) \subset \text{Der}(\mathcal{D}) \) the space of derivations \( \delta : \mathcal{D} \to \mathcal{D} \) such that
$\delta(D^{\geq 1}) \subset D^{\geq j+1}$. It is clear that $\text{Der}^{\geq 0}(D)$ is a Lie subalgebra and $\text{Der}^{\geq 1}(D)$ is a pronilpotent Lie ideal of that subalgebra.

Let $\text{Aut}(D)$ be the group of $\mathbb{k}[[\hbar]]$-linear automorphisms of the algebra $D$. The group $\text{Aut}(D)$ has the natural structure of a proalgebraic group with Lie algebra $\text{Lie}(\text{Aut}(D)) = \text{Der}^{\geq 0}(D)$. The pair $(\text{Aut}(D), \text{Der}(D))$ is a Harish-Chandra pair.

The group $Sp(v)$ acts on the Heisenberg algebra $v \oplus \mathbb{k}\hbar$, hence also on the objects $D$, $\mathcal{G}$, $\mathcal{G}^{\geq 1}$, and $\mathcal{G}^{\geq 1}$, by automorphisms. In particular, one has a natural group homomorphism $\Theta_D : Sp(v) \to \text{Aut}(D)$ and an associated Lie algebra homomorphism $\theta_D : \mathfrak{sp}(v) \to \text{Der}(D)$. It is well-known, and easy to verify, that the map $\theta_D$ is related to the map in (2.1) by the equation

$$\theta_D(a)(x) = [\sigma(a), x], \quad \forall a \in \mathfrak{sp}(v), x \in \frac{1}{\hbar} D. \tag{2.3}$$

Next, let $\mathcal{G}^{\geq 1}$ be a pronilpotent group associated with $\mathcal{G}^{\geq 1}$, a pronilpotent Lie algebra. We have the natural central imbedding $\hbar \mathbb{k}[[\hbar]] \hookrightarrow \mathcal{G}^{\geq 1}$, of pronilpotent Lie algebras. This induces an injective homomorphism $1 + \hbar \mathbb{k}[[\hbar]] \hookrightarrow \mathcal{G}^{\geq 1}$, of the corresponding pronilpotent groups.

Following [4], we consider the group $\mathcal{G} := \mathbb{k}^\times_G \times (Sp(v) \ltimes \mathcal{G}^{\geq 1})$. Here, the semidirect product $Sp(v) \ltimes \mathcal{G}^{\geq 1}$ is taken with respect to the natural $Sp(v)$-action on $\mathcal{G}^{\geq 1}$ by automorphisms. The group $\mathcal{G}$ has the structure of a proalgebraic group.

A cartesian product of the natural maps $\mathbb{k} \hookrightarrow \mathbb{k}_G$ and $1 + \hbar \mathbb{k}[[\hbar]] \hookrightarrow \mathcal{G}^{\geq 1}$ provides, via the obvious isomorphism $\mathbb{k}[[\hbar]]^\times = \mathbb{k}^\times \times (1 + \hbar \mathbb{k}[[\hbar]])$, a central imbedding $\mathbb{k}[[\hbar]]^\times \hookrightarrow \mathcal{G}$, of proalgebraic groups. By construction, one also has a canonical imbedding $\Sigma : Sp(v) \hookrightarrow \mathcal{G}$. One may view the group $\mathbb{k}_G^\times \times \Sigma(Sp(v))$ as a Levi subgroup of $\mathcal{G}$. By (2.2), we have an isomorphism $\mathcal{G} = \mathbb{k} \oplus (\mathfrak{sp}(v) \oplus \mathcal{G}^{\geq 1}) \cong \mathcal{G}^{\geq 0}$. Furthermore, it follows from Eq. (2.3) that the natural imbedding $\mathcal{G} \cong \mathcal{G}^{\geq 0} \hookrightarrow \mathcal{G}$ makes the pair $(\mathcal{G}, \mathcal{G})$ a Harish-Chandra pair.

The space $D$ is a Lie ideal in $\frac{1}{\hbar} D$. Hence, there is a well-defined action $\frac{1}{\hbar} D \times D \to D$, $a \times x \mapsto [a, x]$. This action descends to a Lie algebra homomorphism $\varphi_D : \mathcal{G} \to \text{Der}(D)$ with kernel $\frac{1}{\hbar} \mathbb{k}[[\hbar]]/\mathbb{k} = \mathbb{k}[[\hbar]]$. It is easy to see that for all $i \geq 0$, one has $\varphi_D(\mathcal{G}^{\geq 1}) \subset \text{Der}^{\geq (i)}(D)$. For $i = 1$, exponentiating the map $\mathcal{G}^{\geq 1} \to \text{Der}^{\geq 1}(D)$, of pronilpotent Lie algebras, yields a homomorphism $\Phi^1_D : \mathcal{G}^{\geq 1} \to \text{Aut}^1(\mathcal{G})$, of the corresponding pronilpotent groups. One can further extend the latter homomorphism to a homomorphism

$$\Phi_D : \mathcal{G} = \mathbb{k}_G^\times \times (Sp(v) \ltimes \mathcal{G}^{\geq 1}) \to \text{Aut}(D), \quad c \times (g \ltimes u) \mapsto \Theta(g) \Phi^1_D(u).$$

The differential of $\Phi_D$ agrees with the homomorphism $\varphi_D : \mathcal{G}^{\geq 0} \to \text{Der}^{\geq 0}(D)$, of Lie algebras. Moreover, as has been observed in [4], the maps $\Phi_D$ and $\varphi_D$ give rise to a central extension

$$1 \to \langle \mathbb{k}[[\hbar]]^\times, \mathbb{k}[[\hbar]] \rangle \to \langle \mathcal{G}, \mathcal{G} \rangle^{(\Phi_D, \varphi_D)} \to \langle \text{Aut}(D), \text{Der}(D) \rangle \to 1 \tag{2.4}$$

of Harish-Chandra pairs, see [4, Section 3.3].
2.3. It is often convenient to choose a basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ of $\mathfrak{v}$, such that

$$\omega(x_i, x_j) = 0 = \omega(y_i, y_j), \quad \omega(x_i, y_j) = \delta_{ij}, \quad \forall i, j = 1, \ldots, n. \quad (2.5)$$

The algebra $\mathcal{D}$ is (topologically) generated by $\hbar$ and the basis elements of $\mathfrak{v}$ subject to the commutation relations

$$[x_i, x_j] = 0 = [y_i, y_j], \quad [y_j, x_i] = \delta_{ij} \hbar, \quad [\hbar, x_i] = [\hbar, y_j] = 0, \quad \forall i, j.$$ 

In particular, one has a canonical algebra imbedding $\mathbb{k}[[x_1, \ldots, x_n, \hbar]] \hookrightarrow \mathcal{D}$, resp. $\mathbb{k}[[y_1, \ldots, y_n, \hbar]] \hookrightarrow \mathcal{D}$. Further, one proves the following identity in $\mathcal{D}$:

$$y_i \cdot f - f \cdot y_i = \hbar \cdot \partial_i f, \quad \forall f \in \mathbb{k}[[x_1, \ldots, x_n]], \quad i = 1, \ldots, n, \quad (2.6)$$

where we have used the notation $\partial_i = \frac{\partial}{\partial x_i}$.

Let $\mathcal{A} = \mathcal{D}/\mathcal{hD}$. This is a complete topological commutative algebra that comes equipped with a Poisson bracket defined by the formula

$$\{a \text{ mod } \hbar, b \text{ mod } \hbar\} = (\frac{1}{\hbar} [a, b]) \text{ mod } \hbar, \quad \forall a, b \in \mathcal{D}.$$ 

There is a natural isomorphism $\mathcal{A} \cong \mathbb{k}[[x_1, \ldots, x_n, y_1, \ldots, y_n]]$, of topological algebras. The resulting Poisson bracket on $\mathbb{k}[[x_1, \ldots, x_n, y_1, \ldots, y_n]]$ has the standard form

$$\{f, g\} = \sum_{1 \leq i, j \leq n} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_j} \right). \quad (2.7)$$

The algebra $\mathcal{D}$ may be viewed as a quantization of the Poisson algebra $\mathcal{A}$.

From now on, we fix a Lagrangian subspace $\mathfrak{x} \subset \mathfrak{v}$. Let $\mathcal{M} = \mathcal{D}/\mathcal{D}\mathfrak{x}$. This is a left $\mathcal{D}$-module and we have $\mathcal{M}/\hbar \mathcal{M} \cong \mathcal{A}/\mathcal{A}\mathfrak{x}$ as an $\mathcal{A}$-module. Let $1_{\mathcal{M}} = 1 \text{ mod } \mathcal{D}\mathfrak{x}$ denote the generator of $\mathcal{M}$.

We will assume (as we may) that the symplectic basis of $\mathfrak{v}$, cf. (2.5), is chosen in such a way that $y_1, \ldots, y_n$ is a basis of $\mathfrak{x}$. It follows readily that the composite $\mathbb{k}[[x_1, \ldots, x_n, \hbar]] \hookrightarrow \mathcal{D} \twoheadrightarrow \mathcal{D}/\mathcal{D}\mathfrak{x} = \mathcal{M}$ is an isomorphism of $\mathbb{k}[[x_1, \ldots, x_n, \hbar]]$-modules. Using (2.6) one finds that the action of $y_j$ on $\mathcal{M}$ goes, via the isomorphism, to the operator $\hbar \partial_j$ on $\mathbb{k}[[x_1, \ldots, x_n, \hbar]]$.

The group $P := \{g \in Sp(\mathfrak{v}) \mid g(\mathfrak{x}) \subset \mathfrak{x}\}$ is a parabolic subgroup of $Sp(\mathfrak{v})$. Let $\mathfrak{p} = Lie P \subset sp(\mathfrak{v})$ be the corresponding parabolic subalgebra, $\mathfrak{u}$ the nilradical of $\mathfrak{p}$. Restriction to the subspace $\mathfrak{x} \subset \mathfrak{v}$ gives a map $\mathfrak{p} \rightarrow gl(\mathfrak{x})$, $a \mapsto a|_{\mathfrak{x}}$, that induces a canonical isomorphism $\mathfrak{p}/\mathfrak{u} \cong gl(\mathfrak{x})$, of Lie algebras.

By (2.1), we have the map $sp(\mathfrak{v}) \rightarrow \mathcal{D}_2^2$, $a \mapsto \hbar \sigma(a)$. The following formula is well-known.

**Lemma 2.8** For any $a \in \mathfrak{p}$, we have $(\hbar \sigma(a))(1_{\mathcal{M}}) = \frac{1}{\hbar} \text{Tr}(a|_{\mathfrak{x}}) \cdot 1_{\mathcal{M}}$.

**Proof** Using the basis $x_1, \ldots, x_n, y_1, \ldots, y_n$, the Lie algebra $\mathfrak{p}$ may be described as a subalgebra of $sp(\mathfrak{v})$ formed by the matrices
Lemma 2.9 Let $M$ be a complete topological finitely generated left $D$-module without $\hbar$-torsion. Then, any isomorphism $M/\hbar M \cong \mathcal{A}/\mathcal{A} \xi$, of $\mathcal{A}$-modules, can be lifted to an isomorphism $M \cong \mathcal{M}$ of $D$-modules.

Proof We will use the identification $\mathcal{M} \cong \mathbb{k}[[x_1, \ldots, x_n, \hbar]]$, resp. $\mathcal{A}/\mathcal{A} \xi \cong \mathbb{k}[[x_1, \ldots, x_n]]$. Let $1_M \in M$ be any element that maps to $1 \in \mathbb{k}[[x_1, \ldots, x_n]]$ under the composition $M \twoheadrightarrow M/\hbar M \cong \mathbb{k}[[x_1, \ldots, x_n]]$. Then, the map $p : \mathbb{k}[[x_1, \ldots, x_n, \hbar]] \to M, u \mapsto u(1_M)$ is a $\mathbb{k}[[x_1, \ldots, x_n, \hbar]]$-linear map that induces a bijection $\mathbb{k}[[x_1, \ldots, x_n]] \to M/\hbar M$. It follows by Nakayama’s lemma that the map $p$ is an isomorphism of $\mathbb{k}[[x_1, \ldots, x_n, \hbar]]$-modules. Therefore, for each $j = 1, \ldots, n$, there exists a unique element $f_j \in \mathbb{k}[[x_1, \ldots, x_n, \hbar]]$ such that $y_j(1_M) = f_j(1_M)$.

Furthermore, the power series $f_j$ is divisible by $\hbar$, since $y_j(1_M) \mod \hbar M = 0$. Thus, we have $f_j = \hbar \cdot g_j$ for some $g_j \in \mathbb{k}[[x_1, \ldots, x_n, \hbar]]$.

For any $1 \leq i, j \leq n$, using the commutation relation in (2.6), we find

$$y_i y_j(1_M) = y_i(f_j(1_M)) = \hbar \partial_i f_j(1_M) + f_j(y_i(1_M)) = \hbar^2 (\partial_i g_j(1_M) + g_j \cdot g_i(1_M)).$$

Since $y_i y_j = y_j y_i$, we deduce that $\partial_i g_j = \partial_j g_i$ for all $i, j$. Hence, there exists a formal power series $g \in \mathbb{k}[[x_1, \ldots, x_n, \hbar]]$ such that we have $g_j = \partial_j g$ for all $j$. Furthermore, we may (and will) choose $g$ to be contained in the ideal generated by the elements $x_1, \ldots, x_n$. Thus, the element $e^{-g}$ is a well-defined element of $\mathbb{k}[[x_1, \ldots, x_n, \hbar]]$.

We put $m := e^{-g}(1_M) \in M$. We compute

$$y_i(m) = y_i(e^{-g}(1_M)) = \hbar \partial_i (e^{-g})(1_M) + e^{-g}(y_i(1_M)) = -\hbar \partial_i g \cdot e^{-g}(1_M) + e^{-g} \cdot h g_i(1_M) = -\hbar g_i \cdot e^{-g}(1_M) + e^{-g} \cdot h g_i(1_M) = 0.$$

Hence the map $D \to M, u \mapsto u(m)$ descends to a map $F : D/\mathcal{D} \xi \to M$. The latter map is an isomorphism of rank 1 free $\mathbb{k}[[x_1, \ldots, x_n, \hbar]]$-modules, since $e^{-g} \in \mathbb{k}[[x_1, \ldots, x_n, \hbar]]$ is an invertible element. We conclude that $F$ is an isomorphism of $D$-modules that lifts the isomorphism $\mathcal{A}/\mathcal{A} \xi \cong M/\hbar M$. \hfill $\square$
3 Comparison of Harish-Chandra pairs

3.1. We introduce various Harish-Chandra pairs canonically associated with the Lagrangian subspace \( \mathfrak{r} \subset \mathfrak{v} \). We will use the notation \( K = \mathbb{k}[[\hbar]] \).

Let \( \mathcal{J} \subset \mathcal{D} \) be the preimage of the ideal \( \mathcal{A}\mathfrak{r} \subset \mathcal{A} \) under the natural algebra projection \( \mathcal{D} \twoheadrightarrow \mathcal{D}/\hbar \mathcal{D} = \mathcal{A} \). Thus \( \mathcal{J} \) is a two-sided ideal of \( \mathcal{D} \). Let \( \text{Aut}(\mathcal{D})_{\mathcal{J}} \), resp. \( \text{Der}(\mathcal{D})_{\mathcal{J}} \), be the subset of \( \text{Aut}(\mathcal{D}) \), resp. \( \text{Der}(\mathcal{D}) \), formed by the maps \( f : \mathcal{D} \to \mathcal{D} \) such that \( f(\mathcal{J}) \subset \mathcal{J} \). The pair \( \langle \text{Aut}(\mathcal{D})_{\mathcal{J}}, \text{Der}(\mathcal{D})_{\mathcal{J}} \rangle \) is a Harish-Chandra subpair of \( \langle \text{Aut}(\mathcal{D}), \text{Der}(\mathcal{D}) \rangle \).

Let \( \langle \mathcal{G}_{\mathcal{J}}, \mathfrak{g}_{\mathcal{J}} \rangle \) be the preimage of the pair \( \langle \text{Aut}(\mathcal{D})_{\mathcal{J}}, \text{Der}(\mathcal{D})_{\mathcal{J}} \rangle \) under the morphism \( \langle \Phi_{\mathcal{D}}, \varphi_{\mathcal{D}} \rangle \) in (2.4). Thus, by construction, one has a central extension

\[
1 \to \langle K^X, K \rangle \longrightarrow \langle \mathcal{G}_{\mathcal{J}}, \mathfrak{g}_{\mathcal{J}} \rangle \xrightarrow{\langle \Phi_{\mathcal{D}}, \varphi_{\mathcal{D}} \rangle} \langle \text{Aut}(\mathcal{D})_{\mathcal{J}}, \text{Der}(\mathcal{D})_{\mathcal{J}} \rangle \to 1 \quad (3.1)
\]

Below, we will identify the space \( \frac{1}{\hbar} \mathcal{J} \subset \frac{1}{\hbar} \mathcal{D} \) with its image in \( \mathfrak{g} \) under the composite map \( \frac{1}{\hbar} \mathcal{J} \hookrightarrow \frac{1}{\hbar} \mathcal{D} \to \frac{1}{\hbar} \mathcal{D}/\frac{1}{\hbar} D^0 = \mathfrak{g} \), which is injective since \( \mathcal{J} \cap D^0 = 0 \).

**Lemma 3.2** We have \( \mathfrak{g}_{\mathcal{J}} = \frac{1}{\hbar} \mathcal{J} \).

**Proof** Let \( a \in \mathfrak{g} \) and put \( f := (ha) \mod h \mathcal{D} \). We view \( f \) as an element of \( \mathcal{A} \) without constant term. Then, since \( \mathcal{J}/\hbar \mathcal{D} = \mathcal{A}\mathfrak{r} \), an inclusion \( \{a, \mathcal{J}\} \subset \mathcal{J} \) is equivalent to \( \{f, \mathcal{A}\mathfrak{r}\} \subset \mathcal{A}\mathfrak{r} \). Using formula (2.7), one shows that the latter inclusion holds if and only if one has \( f = c + f' \) for some constant \( c \in \mathbb{k} \) and some \( f' \in \mathcal{A}\mathfrak{r} \). We must have \( c = 0 \). Hence, \( a \in \frac{1}{\hbar} \mathcal{J} \). \( \square \)

Let \( \text{Der}(\mathcal{D}, \mathcal{M}) \) be the Lie algebra of derivations, resp. \( \text{Aut}(\mathcal{D}, \mathcal{M}) \) the group of automorphisms, of the pair \( \langle \mathcal{D}, \mathcal{M} \rangle \). By definition, an element of \( \text{Der}(\mathcal{D}, \mathcal{M}) \) is a pair \( (^D\delta, ^M\delta) \) where \( ^D\delta \in \text{Der}(\mathcal{D}) \) and \( ^M\delta : \mathcal{M} \to \mathcal{M} \) is a continuous \( \mathbb{k}[[\hbar]] \)-linear map such that

\[
^M\delta(um) = ^D\delta(um) + u(^M\delta(m)), \quad \forall u \in \mathcal{D}, \, m \in \mathcal{M}. \quad (3.3)
\]

Similarly, an element of \( \text{Aut}(\mathcal{D}, \mathcal{M}) \) is a pair \( (^D\varphi, ^M\varphi) \) where \( ^D\varphi \in \text{Aut}(\mathcal{D}) \) and \( ^M\varphi : \mathcal{M} \to \mathcal{M} \) is a continuous \( \mathbb{k}[[\hbar]] \)-linear bijection such that \( ^M\varphi(um) = ^D\varphi(u) \cdot ^M\varphi(m) \).

Thus, \( \langle \text{Aut}(\mathcal{D}, \mathcal{M}), \text{Der}(\mathcal{D}, \mathcal{M}) \rangle \) is a Harish-Chandra pair.

The assignment \( c \mapsto (\text{Id}_\mathcal{D}, c \cdot \text{Id}_\mathcal{M}) \) gives a natural central imbedding \( \varepsilon_{\text{Aut}} : K^X \hookrightarrow \langle \text{Aut}(\mathcal{D}, \mathcal{M}) \rangle \). Similarly, one has a central Lie algebra imbedding \( \varepsilon_{\text{Der}} : K \hookrightarrow \langle \text{Der}(\mathcal{D}, \mathcal{M}) \rangle \) given by \( a \mapsto (0, a \cdot \text{Id}_\mathcal{M}) \). This gives an injective morphism \( \varepsilon = \langle \varepsilon_{\text{Aut}}, \varepsilon_{\text{Der}} \rangle : \langle K^X, K \rangle \hookrightarrow \langle \text{Aut}(\mathcal{D}, \mathcal{M}), \text{Der}(\mathcal{D}, \mathcal{M}) \rangle \) of Harish-Chandra pairs.

Further, it is clear that forgetting the action on \( \mathcal{M} \) yields a morphism of Harish-Chandra pairs:

\[
\mathbf{F} = (\mathbf{F}_{\text{Aut}}, \mathbf{F}_{\text{Der}}) : \langle \text{Aut}(\mathcal{D}, \mathcal{M}), \text{Der}(\mathcal{D}, \mathcal{M}) \rangle \to \langle \text{Aut}(\mathcal{D}), \text{Der}(\mathcal{D}) \rangle.
\]

**Lemma 3.4** One has an inclusion

\[
\mathbf{F}(\langle \text{Aut}(\mathcal{D}, \mathcal{M}), \text{Der}(\mathcal{D}, \mathcal{M}) \rangle) \subset \langle \text{Aut}(\mathcal{D})_{\mathcal{J}}, \text{Der}(\mathcal{D})_{\mathcal{J}} \rangle. \quad (3.5)
\]
Proof. Observe that the ideal $J$ is equal to $\text{Ann}(\mathcal{M}/\hbar \mathcal{M})$, the annihilator of the $\mathcal{D}$-module $\mathcal{M}/\hbar \mathcal{M}$, since $\mathcal{M}/\hbar \mathcal{M} = \mathcal{A}/\mathcal{A}\hbar$. Further, it follows from Eq. (3.3) that, for any pair $(\delta, \delta^\mathcal{M}) \in \text{Der}(\mathcal{D}, \mathcal{M})$, the map $\mathcal{M}\delta$ takes the annihilator of $\mathcal{M}/\hbar \mathcal{M}$ to itself, that is, takes $J$ to $J$. We deduce that the image of the map $F_{\text{Der}}: \text{Der}(\mathcal{D}, \mathcal{M}) \to \text{Der}(\mathcal{D})$ is contained in $\text{Der}(\mathcal{D})J$. A similar argument yields the inclusion involving the map $F_{\text{Aut}}$, proving (3.5).

Next we note that since $J = \text{Ann}(\mathcal{M}/\hbar \mathcal{M})$, one has $J \mathcal{M} \subset \hbar \mathcal{M}$. Hence, there is a well-defined action $\frac{1}{\hbar}J \times \mathcal{M} \to \mathcal{M}$. For $a \in \frac{1}{\hbar}J$, let $\varphi_{\mathcal{M}}(a) : \mathcal{M} \to \mathcal{M}$ denote the map $m \mapsto am$. It is immediate to check that Eq. (3.3) holds for the pair of maps $(\varphi_{\mathcal{D}}(a), \varphi_{\mathcal{M}}(a))$, i.e., this pair gives an element of $\text{Der}(\mathcal{D}, \mathcal{M})$. We deduce that the assignment $a \mapsto (\varphi_{\mathcal{D}}(a), \varphi_{\mathcal{M}}(a))$ yields a map $\varphi_{\mathcal{D}, \mathcal{M}} : \frac{1}{\hbar}J \to \text{Der}(\mathcal{D}, \mathcal{M})$. Note that $\varphi_{\mathcal{D}, \mathcal{M}}|_K = \epsilon_{\text{Der}}$.

Lemma 3.6 The map $\varphi_{\mathcal{D}, \mathcal{M}}$ is a Lie algebra isomorphism.

Proof. Let $\varphi_{\mathcal{J}}$ be the restriction of the map $\varphi_{\mathcal{D}}$ to the subalgebra $\mathfrak{G}_{\mathcal{J}} \subset \mathfrak{G}$. Using Lemma 3.2 we obtain the following diagram

\[
\begin{array}{ccc}
K^\mathcal{C} & \longrightarrow & \frac{1}{\hbar}J \\
\varphi_{\mathcal{D}, \mathcal{M}} \downarrow & & \varphi_{\mathcal{J}} \\
\text{Der}(\mathcal{D}, \mathcal{M}) & \longrightarrow & \text{Der}(\mathcal{D})J
\end{array}
\]

(3.7)

It is immediate from definitions that the square in the diagram commutes. The map $\varphi_{\mathcal{J}}$ being surjective, it follows that the map $F_{\text{Der}}$ is surjective. The kernel of this map is formed by the pairs $(0, \mathcal{M}\delta)$ where the map $\mathcal{M}\delta : \mathcal{M} \to \mathcal{M}$ commutes with the $\mathcal{D}$-action, i.e., is a $\mathcal{D}$-module endomorphism. All $\mathcal{D}$-module endomorphisms of $\mathcal{M}$ are given by multiplication by an element of $K$. We deduce that $\text{Ker} F_{\text{Der}} = \varphi_{\mathcal{D}, \mathcal{M}}(K)$. Also, since $\text{Ker} \varphi_{\mathcal{J}} = K$ we get $\text{Ker} \varphi_{\mathcal{D}, \mathcal{M}} \subset K$. Furthermore, it is clear that multiplication by a nonzero element of $K$ gives a nonzero endomorphism of $\mathcal{M}$. We conclude that the map $\varphi_{\mathcal{D}, \mathcal{M}}$ is injective.

Now, let $\delta = (\mathcal{D}\delta, \mathcal{M}\delta)$ be an element of $\text{Der}(\mathcal{D}, \mathcal{M})$. The map $\varphi_{\mathcal{J}}$ being surjective, there exists $a \in \frac{1}{\hbar}J$ such that we have $\mathcal{D}\delta = \varphi_{\mathcal{J}}(a)$. We know that $\varphi_{\mathcal{D}, \mathcal{M}}(a) \in \text{Der}(\mathcal{D}, \mathcal{M})$ and it is clear that $F_{\text{Der}}(\delta - \varphi_{\mathcal{D}, \mathcal{M}}(a)) = 0$. It follows that $\delta - \varphi_{\mathcal{D}, \mathcal{M}}(a) = \varphi_{\mathcal{D}, \mathcal{M}}(c)$ for some $c \in K$ and, hence, we have $\delta = \varphi_{\mathcal{D}, \mathcal{M}}(a + c)$. We deduce that the injective map $\varphi_{\mathcal{D}, \mathcal{M}}$ is also surjective, proving the lemma.

As a consequence of the above proof we obtain

Corollary 3.8 The following sequence is exact

$$1 \to (K^\times, K) \xrightarrow{\epsilon} (\text{Aut}(\mathcal{D}, \mathcal{M}), \text{Der}(\mathcal{D}, \mathcal{M})) \xrightarrow{F} (\text{Aut}(\mathcal{D})J, \text{Der}(\mathcal{D})J) \to 1.$$

3.2. Below, we identify the subgroup $\{\pm 1\} \subset \mathbb{K}^\times$ with its images in $\epsilon_{\text{Aut}}(\mathbb{K}^\times)$ and $\mathbb{K}^\times_{\mathcal{G}}$, respectively.

The main result of this section is the following
Proposition 3.9 There is an isomorphism
\[ \Phi_{\mathcal{D}, \mathcal{M}} : \mathcal{G}_J / \{ \pm 1 \} \cong \text{Aut}(\mathcal{D}, \mathcal{M}) / \{ \pm 1 \}, \]
of proalgebraic groups, that fits into a commutative diagram
\[
\begin{array}{ccc}
\left\{ \frac{K^x}{\{ \pm 1 \}}, K \right\} & \xrightarrow{\Phi_{\mathcal{D}, \mathcal{M}}} & \left\{ \frac{\mathcal{G}_J}{\{ \pm 1 \}}, \mathcal{G}_J \right\} \\
\text{Id} & \cong & \left\{ \frac{\Phi_{\mathcal{D}, \mathcal{M}} \Phi_{\mathcal{D}, \mathcal{M}}}{\mathcal{G}_J}, \text{Der}(\mathcal{D}, \mathcal{J}) \right\} \\
\left\{ \frac{K^x}{\{ \pm 1 \}}, K \right\} & \implies & \left\{ \frac{\text{Der}(\mathcal{D}, \mathcal{M}), \text{Der}(\mathcal{D}, \mathcal{M})}{\{ \pm 1 \}} \right\} \\
\end{array}
\]
of central extensions of Harish-Chandra pairs.

The rest of the section is devoted to the proof of the proposition.
Let \( \mathcal{G}^{\leq 1}_J = \mathcal{G}^{\leq 1} \cap \mathcal{G}_J \). It is clear that \( \mathcal{G}^{\leq 0}_J \) is a Lie subalgebra of \( \mathcal{G}_J \), and \( \mathcal{G}^{\geq 1}_J \) is a pronilpotent Lie ideal of \( \mathcal{G}^{\leq 0}_J \). Thus there is a pronilpotent normal subgroup \( \mathcal{G}^{\geq 1}_J \subset \mathcal{G}_J \) that corresponds to the ideal \( \mathcal{G}^{\geq 1}_J \).

Claim 3.10 We have
\[ \mathcal{G}_J = \mathbb{k}^\times_{\mathcal{G}} \times (\Sigma(P) \ltimes \mathcal{G}^{\geq 1}_J), \]
where \( \Sigma \) was introduced in the paragraph following (2.3) and the group \( \mathcal{G}_J \) in the beginning of this section. In particular, one has \( \text{Lie} \mathcal{G}_J = \mathcal{G}^{\leq 0}_J \).

Proof of Claim. Observe first that, for any \( g \in \text{Aut}(\mathcal{D}) \), we have \( g(D^{\geq 1}) \subset D^{\geq 1} \) and \( g(D^{\leq 2}) \subset D^{\leq 2} \), since \( D^{\geq 2} = hD^{\geq 1} + (D^{\leq 1})^2 \). It follows that \( g \) induces an automorphism \( \tau(g) \) of the vector space \( D^{\geq 1}/D^{\leq 2} = v \). The map \( g \mapsto \tau(g) \) yields a homomorphism \( \tau : \text{Aut}(\mathcal{D}) \to Sp(v) \). Clearly, one has \( \tau(\text{Aut}(\mathcal{D})_J) \subset P \). We also have the homomorphism \( \Phi_{\mathcal{D}} : \mathcal{G} \to \text{Aut}(\mathcal{D}) \) such that \( \Phi_{\mathcal{D}}(k^\times_{\mathcal{G}}) = 1 \) and \( \Phi_{\mathcal{D}}(\mathcal{G}_J) \subset \text{Aut}(\mathcal{D})_J \), by definition. Thus, there is a well-defined composition \( \mathcal{G}_J/k^\times_{\mathcal{G}} \xrightarrow{\Phi_{\mathcal{D}}} \text{Aut}(\mathcal{D})_J \xrightarrow{\tau} P \), to be denoted by \( \tau_J \). Note that \( \mathcal{G}^{\geq 1}_J \subset \ker(\tau_J) \).

It is clear that we have \( \text{Lie} \mathcal{G}_J \subset \mathcal{G}^{\leq 0}_J \). Furthermore, the Lie algebra map \( \tau_{J,\text{Lie}} : \text{Lie}(\mathcal{G}_J/k^\times_{\mathcal{G}}) \to p \) induced by the group homomorphism \( \tau_J \) is equal to the composition of natural maps
\[ \text{Lie}(\mathcal{G}_J/k^\times_{\mathcal{G}}) \hookrightarrow \mathcal{G}^{\geq 0}_J/k_{\mathcal{G}} \hookrightarrow \mathcal{G}^{\leq 0}_J/(k_{\mathcal{G}} \oplus \mathcal{G}^{\geq 1}_J) = (\mathcal{G}^{0} \cap \mathcal{G}_J)/k_{\mathcal{G}} \cong p, \]
where we have used that \( \mathcal{G}^{0} \cap \mathcal{G}_J = k_{\mathcal{G}} \oplus \sigma(p) \), so \( (\mathcal{G}^{0} \cap \mathcal{G}_J)/k_{\mathcal{G}} \cong p \).

Using the inclusion \( \mathcal{G}^{\geq 1}_J \subset \ker(\tau_J) \), we deduce
\[ \text{Lie} \mathcal{G}^{\geq 1}_J \subset \text{Lie}(\ker(\tau_J)) \subset \ker(\tau_{J,\text{Lie}}) = \mathcal{G}^{\geq 1}_J = \text{Lie} \mathcal{G}^{\geq 1}_J. \]
This implies an equality $\text{Ker}(\tau_J) = G_J^{\geq 1}$, since both groups are prounipotent. The proof is completed by observing that the map $\Sigma|_P$ provides a section $P \to G_J \to G_J/\kappa \mathcal{G}$ of the map $\tau_J$. \hfill \Box

We use the isomorphism $\varphi_{D,M}$ and put $\text{Der}^{\geq 1}(D, M) := \varphi_{D,M}(\mathcal{G}^{\geq 1}_J)$. Thus, $\text{Der}^{\geq 0}(D, M)$ is a Lie subalgebra of $\text{Der}(D, M)$, resp. $\text{Der}^{\geq 1}(D, M)$ is a pronilpotent ideal of $\text{Der}^{\geq 0}(D, M)$. Let $\text{Aut}^{\geq 1}(D, M)$ be a prounipotent subgroup of the group $\text{Aut}(D, M)$ corresponding to the ideal $\text{Der}^{\geq 1}(D, M)$.

Next, we observe that, for any $g \in P$, the left ideal $\mathcal{D}_{\tau}$ is stable under the map $\Theta_{D}(g) : \mathcal{D} \to \mathcal{D}$. Hence, this map descends to a map $\Theta_{M}(g) : M \to M$. The assignment $g \mapsto (\Theta_{D}(g), \Theta_{M}(g))$ provides a canonical homomorphism $\Theta_{D,M} : P \to \text{Aut}(D, M)$.

**Claim 3.11** We have $\text{Aut}(D, M) = \varepsilon_{\text{Aut}}(\kappa^x) \times (\Theta_{D,M}(P) \times \text{Aut}^{\geq 1}(D, M))$, in particular, one has $\text{Lie Aut}(D, M) = \text{Der}^{\geq 0}(D, M)$.

Using the Claim, we see that the first projection of the direct product in the RHS of the isomorphism above provides a canonical homomorphism

$$\chi : \text{Aut}(D, M) \to \kappa^x. \quad (3.12)$$

**Sketch of Proof of Claim.** First one shows, similar to the Lie algebra case, that $\text{Ker}(F_{\text{Aut}}) = \varepsilon_{\text{Aut}}(K^x)$. Further, we know that $\tau(\text{Aut}(D,J)) \subset P$ and $\text{Ker}(\tau)$ is a pronilpotent group. It follows that $\tau \circ F_{\text{Aut}}(\text{Aut}(D, M)) \subset P$ and that $\text{Ker}(\tau \circ F_{\text{Aut}})/\varepsilon_{\text{Aut}}(\kappa^x)$ is a pronilpotent group. The proof is now completed by showing that the Lie algebra of the latter group equals the Lie algebra of the group $\text{Aut}^{\geq 1}(D, M)$.

We leave details to the reader.

Write $a \mapsto \theta_{D,M}(a) = (\theta_D(a), \theta_M(a))$ for the Lie algebra homomorphism $\theta_{D,M} : p \to \text{Der}(D, M)$ induced by the group homomorphism $\Theta_{D,M}$. There is a diagram

$$\begin{array}{ccc}
\mathfrak{p} & \xrightarrow{\sigma} & \frac{1}{h} J \\
\downarrow \theta_{D,M} & & \downarrow \varphi_{D,M}
\end{array} \xrightarrow{\psi_{D,M}} \text{Der}(D, M) \quad (3.13)
$$

It is immediate from definitions that $\theta_{D} = \varphi_{D} \circ \sigma$. However, the corresponding diagram involving the module $M$ does not commute; indeed, one has an equation

$$(\varphi_M \circ \sigma)(a) = \theta_M(a) + \frac{1}{2} \text{Tr}(a|_M) \cdot \text{Id}_M, \quad \forall a \in \mathfrak{p}. \quad (3.14)$$

To prove (3.14), note that for $u \in \mathcal{D}$, we have $\theta_{D}(a)(1_M) = \theta_{M}(a)(u(1_M))$, by (3.3). Hence, using (2.3) and Lemma 2.8, we compute
\[(\varphi_M \circ \sigma)(a)(u(1_M)) = (\sigma(a)u)(1_M) = [\sigma(a), u](1_M) + u(\sigma(a)(1_M)) = \theta_D(a)(u(1_M)) + u(\frac{1}{2} \text{Tr}(a) \cdot 1_M) = \theta_M(a)(u(1_M)) + \frac{1}{2} \text{Tr}(a) \cdot u(1_M).\]

This proves Eq. (3.14) since any element of \(M\) has the form \(u(1_M)\) for some \(u \in D\). \(\square\)

**Proof of Proposition 3.9.** The isomorphism \(\varphi_{D, M} : G^\geq \sim \text{Der}^\geq(D, M)\), of pronilpotent Lie algebras, can be exponentiated to an isomorphism \(\Phi^1_{D, M} : G^\geq \sim \text{Aut}^\geq(D, M)\), of the corresponding pronilpotent groups. We define a map \(\kappa^\geq_G/[\pm 1] \times (\Sigma(P) \ltimes G^\geq_J) \rightarrow \varepsilon_{\text{Aut}}(\kappa^\geq)/[\pm 1] \times (\Theta_{D, M}(P) \ltimes \text{Aut}^\geq(D, M))\) by the formula

\[c \times (\Sigma(p) \ltimes g) \mapsto \varepsilon_{\text{Aut}}(c \cdot \sqrt{\det(p)}) \times (\Theta_{D, M}(p) \ltimes \Phi^1_{D, M}(g)).\]

It is clear that this map is an isomorphism of proalgebraic groups. Thanks to Claims 3.10 and 3.11, the above map gives an isomorphism

\[\Phi_{D, M} : G_J/[\pm 1] \sim \text{Aut}(D, M)/[\pm 1].\]

Moreover, Eq. (3.14) insures that the map \(\text{Lie} G_J \rightarrow \text{Lie} \text{Aut}(D, M)\), the Lie algebra homomorphism induced by \(\Phi_{D, M}\), is equal to the map \(\varphi_{D, M}|_{G^\geq_J} : G^\geq_J \rightarrow \text{Der}^\geq(D, M)\). The latter map is a Lie algebra isomorphism. We conclude that the pair of maps \((\Phi_{D, M}, \varphi_{D, M})\) yields an isomorphism of Harish-Chandra pairs as required in the statement of Proposition 3.9. \(\square\)

**4 Harish-Chandra torsors**

4.1. We will use the notation \(\otimes := \otimes_k\) and write \(T_Y\) for the tangent sheaf of a smooth variety \(Y\).

The following definition, that has been used in [4], is due to Beilinson and Drinfeld [2].

**Definition** Let \(Y\) be a smooth algebraic variety and \((G, g)\) a Harish-Chandra pair. A transitive Harish-Chandra torsor (or transitive torsor for short) over \(Y\) is a \(G\)-torsor \(\mathcal{P}\) over \(Y\) equipped with a \(G\)-equivariant Lie algebra homomorphism \(g \rightarrow H^0(\mathcal{P}, T_P)\) that extends the map \(\text{Lie} G \rightarrow H^0(\mathcal{P}, T_P)\), the differential of the \(G\)-action on \(\mathcal{P}\), and induces an isomorphism \(g \otimes \mathcal{O}_P \simeq T_P\), of locally free sheaves on \(\mathcal{P}\).

Let \(a\) be a vector space viewed as an additive algebraic group. The Lie algebra of this group is identified with \(a\), so the pair \((a, a)\) is a Harish-Chandra pair. Further, let

\[1 \rightarrow (a, a) \rightarrow (\tilde{G}, \tilde{g}) \rightarrow (G, g) \rightarrow 1.\]

be a central extension of Harish-Chandra pairs, to be denoted \(c\).
In [4, Proposition 2.7], the authors associate with any transitive \((G, \mathfrak{g})\)-torsor \(\mathcal{P}\) on \(Y\) a class \(\text{Loc}(\mathcal{P}, c) \in \mathfrak{a} \otimes H^2_{\text{DR}}(Y)\), sometimes also denoted \(\text{Loc}(\mathcal{P}, \tilde{G}, \tilde{\mathfrak{g}})\), such that the existence of a lift of \(\mathcal{P}\) to a transitive torsor \(\tilde{\mathcal{P}}\) over \(\langle \tilde{G}, \tilde{\mathfrak{g}} \rangle\) is equivalent to the vanishing of the class \(\text{Loc}(\mathcal{P}, c)\).

We now recall the construction of \(\text{Loc}(\mathcal{P}, c)\) since some functorial properties of the construction will be used later in the paper.

**Construction:** We start with the following exact sequence of \(\mathfrak{g}\)-modules:

\[
0 \to \mathfrak{a} \to U_+(\tilde{\mathfrak{g}})/(\mathfrak{a} \cdot U_+(\tilde{\mathfrak{g}})) \to U(\mathfrak{g}) \to \mathbb{k} \to 0, \tag{4.2}
\]

where \(U(-)\) denotes the universal enveloping algebra of a Lie algebra and \(U_+(-)\) its augmentation ideal. Note that the adjoint action of the Harish-Chandra pair \(\langle \tilde{G}, \tilde{\mathfrak{g}} \rangle\) on itself factors through \(\langle G, \mathfrak{g} \rangle\). Therefore, (4.2) is an extension of \(\mathfrak{g}\)-modules. This extension provides an explicit representative for the class in \(H^2(\mathfrak{g}, \mathfrak{a}) = \text{Ext}^2_{\mathfrak{g}}(\mathbb{k}, \mathfrak{a})\) that corresponds to the Lie algebra central extension \(\mathfrak{a} \hookrightarrow \tilde{\mathfrak{g}} \to \mathfrak{g}\).

The \(G\)-action on each term in (4.2) gives an associated vector bundle on \(Y\) corresponding to the \(G\)-torsor \(\mathcal{P}\). Moreover, the \(g\)-action provides each of these vector bundles with a flat connection. Further, the exact sequence (4.2) induces an exact sequence of the associated vector bundles, which is compatible with the connections. The latter exact sequence gives an extension class in \(\text{Ext}^2_{\text{locysts}}(\mathcal{O}_Y, \mathfrak{a} \otimes \mathcal{O}_Y)\), where \(\text{Ext}^2_{\text{locysts}}\) denotes the Ext-group in the category of vector bundles with flat connections, i.e., the category of local systems (not necessarily of finite rank, in general). One defines \(\text{Loc}(\mathcal{P}, c) \in \mathfrak{a} \otimes H^2_{\text{DR}}(Y)\) to be the element that corresponds to the extension class via the canonical isomorphism \(\text{Ext}^2_{\text{locysts}}(\mathcal{O}_Y, \mathfrak{a} \otimes \mathcal{O}_Y) \cong \mathfrak{a} \otimes H^2_{\text{DR}}(Y)\).

The following result is immediate from the above construction of the class \(\text{Loc}(\mathcal{P}, c)\).

**Corollary 4.3** Fix a Harish-Chandra pair \(\langle G, \mathfrak{g} \rangle\) and a central extension \(c\) as in (4.1).

(i) Let \(f : \langle H, \mathfrak{h} \rangle \to \langle G, \mathfrak{g} \rangle\) be a morphism of Harish-Chandra pairs and let \(f^*(c)\) denote the extension of \(\langle H, \mathfrak{h} \rangle\) by \((\mathfrak{a}, \mathfrak{a})\) obtained by pull-back of (4.1) via \(f\). Then, in \(\mathfrak{a} \otimes H^2_{\text{DR}}(Y)\), we have \(\text{Loc}(\mathcal{P}, f^*(c)) = \text{Loc}(\mathcal{P}, c)\).

(ii) Let \(f : \mathfrak{a} \to \mathfrak{a}'\) be a linear map, and let \(f_*(c)\) be the extension of \(\langle G, \mathfrak{g} \rangle\) by \((\mathfrak{a}', \mathfrak{a}')\) obtained by push-out of (4.1) via \(f\). Then, we have \(\text{Loc}(\mathcal{P}, f_*(c)) = (f \otimes \text{Id})(\text{Loc}(\mathcal{P}, c))\), where \(f \otimes \text{Id} : \mathfrak{a} \otimes H^2_{\text{DR}}(Y) \to \mathfrak{a}' \otimes H^2_{\text{DR}}(Y)\) is the map induced by \(f\).

4.2. Fix a central extension \(c\) of Harish-Chandra pairs

\[
c : 1 \to \langle \mathbb{k}^X, \mathbb{k} \rangle \to \langle \tilde{G}, \tilde{\mathfrak{g}} \rangle \to \langle G, \mathfrak{g} \rangle \to 1. \tag{4.4}
\]

and a splitting \(\iota : G \to \tilde{G}\) of the projection \(\tilde{G} \to G\) (the Lie algebra projection \(\tilde{\mathfrak{g}} \to \mathfrak{g}\) is, however, not assumed to be split, in general). The splitting induces a group isomorphism \(\tilde{G} \cong \mathbb{k}^X \times G\) (that may depend on the choice of \(\iota\)).

In the above setting, to any transitive Harish-Chandra \(\langle G, \mathfrak{g} \rangle\)-torsor \(f : Z \to Y\) one associates a class \(\alpha(Z, c, \iota) \in H^2(\Omega_{Y}^{\leq 1})\) as follows.
Tensoring the Lie algebra central extension $\mathfrak{k} \hookrightarrow \widetilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ by $\mathcal{O}_Z$ and using the isomorphism $\mathfrak{g} \otimes \mathcal{O}_Z \cong T_Z$ yields an exact sequence
\begin{equation}
0 \rightarrow \mathcal{O}_Z \rightarrow \widetilde{\mathfrak{g}} \otimes \mathcal{O}_Z \xrightarrow{\eta} T_Z \rightarrow 0,
\end{equation}
of $\mathcal{O}_Z$-modules. As explained in [1, §1.2.2], there is a natural Lie algebroid structure on $\widetilde{\mathfrak{g}} \otimes \mathcal{O}_Z$ such that the corresponding anchor map is the map $\eta$ in (4.5). Further, the group $G$ acts on $\widetilde{\mathfrak{g}}$ and on $Z$, giving the sheaf $\widetilde{\mathfrak{g}} \otimes \mathcal{O}_Z$ a $G$-equivariant structure. Moreover, the composition of Lie algebra maps
\[ \text{Lie } G \xrightarrow{\text{dt}} \text{Lie } \widetilde{G} \hookrightarrow \widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}} \otimes 1 \hookrightarrow \widetilde{\mathfrak{g}} \otimes \mathcal{O}_Z \]
gives a partial splitting of $\eta$. Using the terminology of [1, §2.1.3], the above data give the sheaf $\widetilde{\mathfrak{g}} \otimes \mathcal{O}_Z$ the structure of a $G$-equivariant Picard algebroid on $Z$. Applying equivariant descent for Picard algebroids, as explained in [1, §1.8.9], one obtains from $\widetilde{\mathfrak{g}} \otimes \mathcal{O}_Z$ a Picard algebroid $\widetilde{\mathfrak{g}}_Y$ on $Y$. Explicitly, $\widetilde{\mathfrak{g}}_Y$ is the quotient of $[f_*(\widetilde{\mathfrak{g}} \otimes \mathcal{O}_Z)]^G$ by the image of $[f_*(\text{Lie } G \otimes \mathcal{O}_Z)]^G$.

We define $\alpha(Z, c, \iota) \in H^2(\Omega^*_{\mathcal{Y}Z} \otimes \iota)$ to be the Atiyah class of $\widetilde{\mathfrak{g}}_Y$.

**Remark 4.6** The class $\alpha(Z, c, \iota)$ only depends on the action of $G$ on $\widetilde{G}$ and the differential of $\iota$.

Let $\mathcal{P}$ be $(\widetilde{G}, \mathfrak{g})$-torsor on $Y$. Given an extension (4.4), one obtains a $G$-torsor $\mathbb{k}^X \setminus \mathcal{P} \rightarrow Y$, a push-out of $\mathcal{P}$ via the projection $\widetilde{G} \rightarrow G$. Further, given a section $\iota : G \rightarrow \widetilde{G}$, as above, one obtains a group decomposition $\widetilde{G} \simeq \mathbb{k}^X \times G \rightarrow \mathbb{k}^X$. Hence, there is $\mathbb{k}^X$-torsor $G \setminus \mathcal{P} \rightarrow Y$, a push-out of $\mathcal{P}$ via the first projection $\widetilde{G} \simeq \mathbb{k}^X \times G \rightarrow \mathbb{k}^X$.

**Lemma 4.7** The assignment $\mathcal{P} \mapsto (\mathbb{k}^X \setminus \mathcal{P}, G \setminus \mathcal{P})$ yields a bijective map from the set (of isomorphism classes) of transitive $(\widetilde{G}, \mathfrak{g})$-torsors on $Y$ onto the set (of isomorphism classes) of pairs $(Z, L)$, where $Z$ is a transitive $(G, \mathfrak{g})$-torsor and $L$ is a $\mathbb{k}^X$-torsor on $Y$ satisfying an equation $\alpha(Z, c, \iota) = c_1(L)$.

**Proof** Given any $\widetilde{G}$-torsor $\mathcal{P}$, put $L := G \setminus \mathcal{P}$, $Z := \mathbb{k}^X \setminus \mathcal{P}$, and let the group $\mathbb{k}^X \times G$ act on $L \times_Y Z$ by $t \times g : l \times z \mapsto tl \times g(z)$ (i.e., $\mathbb{k}^X$ acts on the $L$ factor and $G$ on the $Z$ factor). Then the map $p \mapsto (Gp) \times (\mathbb{k}^X p)$ gives a $G$-equivariant isomorphism $\mathcal{P} \to L \times_Y Z$. Conversely, given a $G$-torsor $Z$ and a $\mathbb{k}^X$-torsor $L$, put $\mathcal{P} := Z \times_Y L$ and let $\widetilde{G}$ act on $\mathcal{P}$ as above. It is clear that this makes $\mathcal{P} \to Y$ a $\widetilde{G}$-torsor.

Thus, proving the lemma amounts to showing that the equation $c_1(L) = \alpha(Z, c, \iota)$ insures that there exists a $G$-equivariant Lie algebroid structure on $\mathfrak{g} \otimes \mathcal{P}$ which is compatible with the second projection $\mathcal{P} = L \times_Y Z \rightarrow Z$ (in the sense that the projection of $\mathfrak{g}$ onto its quotient mod $\mathbb{k}$ agrees with the differential $T_{\mathcal{P}} \rightarrow \text{pr}^*T_Z$). It suffices to construct the corresponding anchor map $\eta : \mathfrak{g} \otimes \mathcal{P} \rightarrow T_{\mathcal{P}}$. To this end, let $\mathbb{A}(\mathcal{P}/Z) := (\text{pr}_*T_Z)^{\mathbb{k}^X}$ be the Atiyah algebra of the $\mathbb{k}^X$-torsor $\text{pr} : \mathcal{P} \rightarrow Z$. Since this torsor $\mathcal{P} \rightarrow Z$ is a pull-back of the torsor $L \rightarrow Y$, for the Atiyah classes we have
\[ \mathbb{A}(\mathcal{P}/Z) = \text{pr}^*c_1(L) = \text{pr}^*\alpha(Z, c, \iota). \]
The class on the right equals, by construction of $\alpha(Z, c, t)$, the class of the extension (4.5) (we use the fact that Atiyah classes on $Y$ are in bijective correspondence with $G$-equivariant Atiyah classes on $Z$). We deduce that there is an isomorphism $\tilde{\mathfrak{g}} \otimes \mathcal{O}_Z \to \text{At}(P/Z)$, of $G$-equivariant Lie algebroids on $Z$. We define $\eta$ to be this isomorphism. \hfill $\Box$

4.3. We return to the setup of § 3.2.

We put $\tilde{\mathcal{G}}_J = \mathcal{G}_J / k^x_\mathcal{G}$, resp. $\tilde{\mathfrak{g}}_J = \mathfrak{g}_J / k_J$. From Claim 3.10 we deduce a natural isomorphism $\tilde{\mathcal{G}}_J \cong \Sigma(P) \ltimes \tilde{\mathcal{G}}_{J^1}^\sim$ and hence a direct product decomposition

$$
\tilde{\mathcal{G}}_J \cong k^x \times (\Sigma(P) \ltimes \tilde{\mathcal{G}}_{J^1}^\sim) \cong k^x \times \tilde{\mathcal{G}}_{J}.
$$

The above decomposition yields a splitting $\eta_J : \tilde{\mathcal{G}}_J \to \mathcal{G}_J$ of the canonical projection $\mathcal{G}_J \to \tilde{\mathcal{G}}_J$. Observe next the assumptions formulated at the beginning of Sect. 4.2 hold in the case where $G = \tilde{\mathcal{G}}_J$, $\tilde{\mathfrak{g}} = \mathfrak{g}_J$, and $\eta_J$ is the natural imbedding.

We also consider a Harish-Chandra pair

$$
\langle \overline{\text{Aut}}(\mathcal{D}, \mathcal{M}), \overline{\text{Der}}(\mathcal{D}, \mathcal{M}) \rangle := \langle \text{Aut}(\mathcal{D}, \mathcal{M}) / \varepsilon_{\text{Aut}}(k^x), \text{Der}(\mathcal{D}, \mathcal{M}) / \varepsilon_{\text{Der}}(k) \rangle.
$$

The isomorphism of Claim 3.11, yields a direct product decomposition

$$
\text{Aut}(\mathcal{D}, \mathcal{M}) \cong k^x \times \overline{\text{Aut}}(\mathcal{D}, \mathcal{M}).
$$

and we denote by $\eta_{\text{Der}} : \overline{\text{Aut}}(\mathcal{D}, \mathcal{M}) \to \text{Aut}(\mathcal{D}, \mathcal{M})$ the homomorphism that results from the imbedding of the second factor. Again, the data $G = \overline{\text{Aut}}(\mathcal{D}, \mathcal{M})$, $\tilde{\mathfrak{g}} = \text{Der}(\mathcal{D}, \mathcal{M})$, and $\eta_{\text{Der}}$, satisfy the assumptions formulated at the beginning of Sect. 4.2.

The isomorphism $\langle \Phi_{\mathcal{D}, \mathcal{M}}, \varphi_{\mathcal{D}, \mathcal{M}} \rangle$, of Proposition 3.9, induces an isomorphism $\langle \tilde{\Phi}_{\mathcal{D}, \mathcal{M}}, \tilde{\varphi}_{\mathcal{D}, \mathcal{M}} \rangle$, of Harish-Chandra pairs, in the following diagram

$$
\begin{array}{ccc}
\langle K^x / k^x, K / k \rangle \overset{\text{Id}}{\longrightarrow} \langle \tilde{\mathcal{G}}_J, \tilde{\mathfrak{g}}_J \rangle & \cong & \langle \text{Aut}(\mathcal{D})_J, \text{Der}(\mathcal{D})_J \rangle \\
\langle K^x / k^x, K / k \rangle \overset{\text{Id}}{\longrightarrow} \langle \overline{\text{Aut}}(\mathcal{D}, \mathcal{M}), \overline{\text{Der}}(\mathcal{D}, \mathcal{M}) \rangle & \cong & \langle \text{Aut}(\mathcal{D})_J, \text{Der}(\mathcal{D})_J \rangle
\end{array}
$$

of central extensions of Harish-Chandra pairs.

Remark 4.11 We note that because of noncommutativity of diagram (3.13), the map $\varphi_{\mathcal{D}, \mathcal{M}}$ does not respect the Lie algebra decompositions resulting from decompositions (4.8) and (4.9), respectively. Therefore, the pair of maps $(\tilde{\Phi}_{\mathcal{D}, \mathcal{M}}, \varphi_{\mathcal{D}, \mathcal{M}}) : \langle \tilde{\mathcal{G}}_J, \mathfrak{g}_J \rangle \to \langle \overline{\text{Aut}}(\mathcal{D}, \mathcal{M}), \text{Der}(\mathcal{D}, \mathcal{M}) \rangle$ does not form a morphism of Harish-Chandra pairs.

Given a $\langle \overline{\text{Aut}}(\mathcal{D}, \mathcal{M}), \overline{\text{Der}}(\mathcal{D}, \mathcal{M}) \rangle$-torsor $Z$, let $\tilde{\Phi}_* Z$ denote the $\langle \tilde{\mathcal{G}}_J, \tilde{\mathfrak{g}}_J \rangle$-torsor obtained by transporting the torsor structure via the isomorphism $(\tilde{\Phi}_{\mathcal{D}, \mathcal{M}}, \tilde{\varphi}_{\mathcal{D}, \mathcal{M}})$, of
Harish-Chandra pairs. We also denote by $c_{\text{Der}}$, resp. $c_{\mathcal{J}}$, the natural central extension of $\langle \text{Aut}(D, M), \text{Der}(D, M) \rangle$, resp. $\langle \tilde{\mathcal{G}}_{\mathcal{J}}, \tilde{\mathcal{G}}_{\mathcal{J}} \rangle$, by $\langle k^\times, k \rangle$.

**Proposition 4.12** For any $\langle \text{Aut}(D, M), \text{Der}(D, M) \rangle$-torsor $Z$, on $Y$, we have in $H^2(\Omega^+_{=1}Y)$

$$
\alpha(Z, c_{\text{Der}}, \iota_{\text{Der}}) - \alpha(\Phi_*Z, c_{\mathcal{J}}, \iota_{\mathcal{J}}) = \frac{1}{2} c_1(L_Z),
$$

where $L_Z$ is the $k^\times$-torsor on $Y$ induced from $Z$ via the composition of homomorphisms:

$$
\mathcal{A}(D, M) \to \text{Aut}(D, \mathcal{J}) \to \Sigma(P) \to k^\times
$$

where the last arrow is the character $\Sigma(p) \mapsto \det(p|_x)$.

**Proof** In the notation of 4.2 we observe that for both central extensions $c_{\mathcal{J}}, c_{\text{Der}}$ the Lie algebra part $\tilde{\mathcal{g}}$ can be identified with $\frac{1}{p} \mathcal{J}$. Although this identification of Lie algebras does not extend to a homomorphism on the group parts, only on their quotients by the image of $\{\pm 1\}$, for the construction of the $\alpha$ classes we only need the adjoint action of $G$ on $\tilde{\mathcal{g}}$, which does match for the two extensions, and the differentials $dt : \mathcal{G} \to \tilde{\mathcal{G}}$, which are different for $c_{\text{Der}}$ and $c_{\mathcal{J}}$, respectively.

In fact, identifying $\text{Der}(D, M)$ with $\mathcal{G}_{\mathcal{J}}$ and applying (3.14) we see that $(dt_{\mathcal{J}} - dt_{\text{Der}})$ is the composition

$$
\text{Lie}(\mathcal{A}(D, M)) \to \text{Lie}(\text{Aut}(D, \mathcal{J})) \to \mathfrak{p} \to k \subset \mathcal{G}_{\mathcal{J}}
$$

where $\mathfrak{p} \to k$ is given by $\frac{1}{2} \text{Tr}(a|_{\mathcal{J}})$. We will denote this composition also by $\frac{1}{2} \text{Tr}(a|_{\mathcal{J}})$. Therefore we see that the classes $\alpha(Z, c_{\text{Der}}, \iota_{\text{Der}})$ and $\alpha(\Phi_*Z, c_{\mathcal{J}}, \iota_{\mathcal{J}})$ arise from the same Lie algebroid $\tilde{\mathcal{G}}_{\mathcal{J}} \otimes \mathcal{O}_Z$ but equipped with different $\text{Aut}(D, M)$-equivariant structures. More precisely, the group action on the sheaf is the same in both cases but the partial connection along the fibers of the projection $Z \to Y$ differs by the map $\frac{1}{2} \text{Tr} : \text{Lie}(\mathcal{A}(D, M)) \to k$ described above.

Using the group structure on the set of isomorphism classes of Lie algebroids, it suffices to check that $\frac{1}{2} c_1(L_Z)$ is the class of the trivial Lie algebroid $\mathcal{O}_Z \oplus \mathcal{T}_Z$ with the equivariant structure in which the canonical embedding

$$
\text{Lie}(\mathcal{A}(D, M)) \otimes \mathcal{O}_Z \hookrightarrow \text{Der}(D, M) \otimes \mathcal{O}_Z \cong \mathcal{T}_Z \hookrightarrow \mathcal{O}_Z \oplus \mathcal{T}_Z
$$

is adjusted by $\frac{1}{2} \text{Tr} \otimes \mathcal{O}_Z$. Furthermore since the group of isomorphism classes of Lie algebroids is a vector space over a field of characteristic zero, it suffices to show the statement without both factors $\frac{1}{2}$. Then the trace map $\text{Tr} : \text{Lie}(\mathcal{A}(D, M)) \to k$ integrates to the group homomorphism $\mathcal{A}(D, M) \to k^\times$ described in the statement.
of the proposition. Applying equivariant descent with respect to the kernel \( \mathcal{U} \) of \( \text{Aut}(D, M) \to \kappa^\times \), we reduce the statement to the assertion that \( c_1(L_Z) \) is the class of the Atiyah algebra of \( \mathcal{U} \setminus Z = L_Z \), which holds by definition of the first Chern class \( c_\zeta(L_Z) \) with values in \( H^2(Y, \Omega^\geq_1) \).

\[ \square \]

5 Torsors associated with a quantization

5.1. Let \( X \) be a smooth symplectic variety and for any point \( x \in X \) let \( \hat{\mathcal{O}}_x \) denote the completion of the local ring at \( x \). A choice of a formal symplectic coordinate system at \( x \) is equivalent to a choice of a topological \( \kappa[[h]] \)-algebra isomorphism \( \eta : \hat{\mathcal{O}}_x \cong A \) of Poisson algebras. Composing \( \eta \) with an automorphism of \( A \) yields another isomorphism \( \hat{\mathcal{O}}_x \cong A \). This shows that the pairs \( (x, \eta) \), as above, form the set of (closed) points of a transitive Harish-Chandra \( \langle \text{Aut}(A), \text{Der}(A) \rangle \)-torsor \( \mathcal{P}_X \), on \( X \) (both automorphisms are derivations are assumed to preserve the Poisson structure on \( A \)).

Next, let \( \mathcal{O}_h \) be a formal quantization of \( \mathcal{O}_X \) and let \( \mathcal{O}_{x,h} \) denote the completion of \( \mathcal{O}_h \) at a point \( x \in X \). The algebra \( \mathcal{O}_{x,h} \) is isomorphic to \( D \) as a topological \( \kappa[[h]] \)-algebra. Furthermore, the pairs \( (x, \eta_h) \), where \( x \in X \) and \( \eta_h : D \to \mathcal{O}_{x,h} \) is an isomorphism of topological \( \kappa[[h]] \)-algebras, form the set of (closed) points of a transitive Harish-Chandra \( \langle \text{Aut}(D), \text{Der}(D) \rangle \)-torsor \( \mathcal{P}_h \), on \( X \).

The algebra projection \( D \twoheadrightarrow D/hD = A \) induces a canonical projection \( \langle \text{Aut}(D), \text{Der}(D) \rangle \to \langle \text{Aut}(A), \text{Der}(A) \rangle \), of Harish-Chandra pairs. According to an observation of [4], a choice of deformation quantization \( \mathcal{O}_h \), of \( \mathcal{O}_X \), is equivalent to a choice of lift of the torsor \( \mathcal{P}_X \), over \( \langle \text{Aut}(A), \text{Der}(A) \rangle \), to a transitive Harish-Chandra torsor \( \mathcal{P}_h \), over \( \langle \text{Aut}(D), \text{Der}(D) \rangle \) (we just choose an identification of the completion \( \mathcal{O}_{x,h} \), of \( \mathcal{O}_h \) at \( x \), with \( D \)).

From now on, we fix a quantization \( \mathcal{O}_h \) and an associated torsor \( \mathcal{P}_h \) as above. Recall that the period map assigns to the quantization \( \mathcal{O}_h \) a class \( \text{per}(\mathcal{O}_h) \), of the form (1.2). This class is defined in terms of the torsor \( \mathcal{P}_h \) as follows.

First, one introduces a proalgebraic group \( \mathcal{G}^+ := \kappa \times \langle Sp(v) \rangle \), which is almost isomorphic to the group \( \mathcal{G} \) except that the multiplicative group \( \kappa_\mathcal{G}^+ \), in the center of \( \mathcal{G} \), is replaced by a copy of the additive group \( \kappa \). There is, in fact, a copy of the additive group \( \kappa[[h]] \) contained in the center of \( \mathcal{G}^+ \). The imbedding \( \kappa[[h]] \subseteq \mathcal{G}^+ \) is defined, via the natural group decomposition \( \kappa[[h]] = \kappa \times h\kappa[[h]] \), to be a cartesian product of the imbedding \( \kappa \subseteq \mathcal{G}^+ \), as the first factor, and the composition \( h\kappa[[h]] \to 1 + h\kappa[[h]] \subseteq \mathcal{G}^+ \), where the first map is the exponential map and the second map was discussed after formula (2.3). Further, using that the morphism \( \langle \Phi_D, \varphi_D \rangle \) vanishes on the subpair \( \langle \kappa_\mathcal{G}^+, \kappa_\mathcal{G} \rangle \subseteq \langle \mathcal{G}, \mathcal{G} \rangle \), one constructs an ‘additive counterpart’ of (2.4), a central extension of Harish-Chandra pairs of the form:

\[ 1 \to \langle \kappa[[h]], \kappa[[h]] \rangle \to \langle \mathcal{G}^+, \frac{1}{h}D \rangle \to \langle \text{Aut}(D), \text{Der}(D) \rangle \to 1, \quad (5.1) \]

of Harish-Chandra pairs. Then, one defines \( \text{per}(\mathcal{O}_h) := \text{Loc}(\mathcal{P}_h, \mathcal{G}^+, \frac{1}{h}D) \), the obstruction class to lifting the torsor \( \mathcal{P}_h \), over \( \langle \text{Aut}(D), \text{Der}(D) \rangle \), to a transitive torsor over the Harish-Chandra pair \( \langle \mathcal{G}^+, \frac{1}{h}D \rangle \).
Let \( i_Y : Y \hookrightarrow X \) be a closed imbedding of a smooth Lagrangian subvariety and \( I_Y \) the ideal sheaf of \( Y \). Let \( \mathcal{J}_Y \) be the preimage of the ideal \( I_Y \) under the natural projection \( \mathcal{O}_h \to \mathcal{O}_X \). For \( y \in Y \), let \( I_{Y,y} \), resp. \( \mathcal{J}_Y \), denote the completion of \( I_Y \), resp. \( \mathcal{J}_Y \), at \( y \). It is clear that one can choose a \( \mathbb{k}[[\hbar]] \)-algebra isomorphism \( \eta : \mathcal{O}_{y,h} \cong \mathcal{D} \) in such a way that \( \eta(\mathcal{J}_y) = \mathcal{J} \). Moreover, the pairs \((y, \eta)\) form the set of (closed) points of a transitive torsor \( \mathcal{P}_\mathcal{J} \) over the Harish-Chandra pair \( \langle \text{Aut}(\mathcal{D})_\mathcal{J}, \text{Der}(\mathcal{D})_\mathcal{J} \rangle \) (from transitivity, observe that the bracket with \( \frac{1}{\hbar}\mathcal{J} \) preserves \( \mathcal{J} \) and that the conormal bundle to \( Y \) is identified with its tangent bundle, due to the Lagrangian condition).

Write \( q^\times : K^\times \to K^\times /\mathbb{k}^\times \), resp. \( q^+ : K \to K /\mathbb{k} \), for the natural projections. Let

\[
1 \to \langle K^\times /\mathbb{k}^\times, K /\mathbb{k} \rangle \to \langle \tilde{\mathcal{G}}_\mathcal{J}, \tilde{\mathcal{S}}_\mathcal{J} \rangle \to \langle \text{Aut}(\mathcal{D})_\mathcal{J}, \text{Der}(\mathcal{D})_\mathcal{J} \rangle \to 1 \tag{5.2}
\]

be a push-out of the extension (2.4) via the morphism \( \langle q, q^+ \rangle : \langle K^\times, K \rangle \to \langle K^\times /\mathbb{k}^\times, K /\mathbb{k} \rangle \). Associated with the central extension (5.2), one has the obstruction class \( \text{Loc}(\mathcal{P}_\mathcal{J}, \tilde{\mathcal{G}}_\mathcal{J}, \tilde{\mathcal{S}}_\mathcal{J}) \) for lifting the torsor \( \mathcal{P}_\mathcal{J} \) to a transitive torsor over \( \langle \tilde{\mathcal{G}}_\mathcal{J}, \tilde{\mathcal{S}}_\mathcal{J} \rangle \).

**Lemma 5.3** In \( H^2(Y)[[\hbar]] \), we have

\[
\text{Loc}(\mathcal{P}_\mathcal{J}, \tilde{\mathcal{G}}_\mathcal{J}, \tilde{\mathcal{S}}_\mathcal{J}) = i_\mathcal{Y}^*(h^2 \omega_2(\mathcal{O}_h) + h^3 \omega_3(\mathcal{O}_h) + \cdots)
\]

**Proof** First of all, it is immediate to see that the \( \langle \text{Aut}(\mathcal{D}), \text{Der}(\mathcal{D}) \rangle \)-torsor \( i_\mathcal{Y}^*\mathcal{P}_h \), the restriction of the torsor \( \mathcal{P}_h \) to the subvariety \( Y \), is induced from the torsor \( \mathcal{P}_\mathcal{J} \) via the imbedding \( \langle \text{Aut}(\mathcal{D})_\mathcal{J}, \text{Der}(\mathcal{D})_\mathcal{J} \rangle \hookrightarrow \langle \text{Aut}(\mathcal{D}), \text{Der}(\mathcal{D}) \rangle \).

Let \( e \) be a pull-back of the central extension (5.1) with respect to this embedding and let \( \langle \tilde{\mathcal{G}}_\mathcal{J}^+, \frac{1}{\hbar}\mathcal{D}_\mathcal{J} \rangle \) be the preimage of the Harish-Chandra pair \( \langle \text{Aut}(\mathcal{D})_\mathcal{J}, \text{Der}(\mathcal{D})_\mathcal{J} \rangle \) under the morphism \( \phi^+ \) in (5.1). Applying Corollary 4.3(i) we deduce that the obstruction for lifting the torsor \( \mathcal{P}_\mathcal{J} \) to a \( \langle \tilde{\mathcal{G}}_\mathcal{J}^+, \frac{1}{\hbar}\mathcal{D}_\mathcal{J} \rangle \)-torsor is equal to \( i_\mathcal{Y}^*(\omega_1 + h^2 \omega_2 + h^3 \omega_3 + \cdots) \). The variety \( Y \) being Lagrangian, we have \( i_\mathcal{Y}^*(\omega_1) = 0 \).

Note that the Lie algebra \( \frac{1}{\hbar}\mathcal{D}_\mathcal{J} \) breaks up into a direct sum \( \frac{1}{\hbar}\mathcal{k} \oplus \frac{1}{\hbar}\mathcal{J} \). Let

\[
1 \to \langle K /\mathcal{k}, K /\mathcal{k} \rangle \to \langle \tilde{\mathcal{G}}_\mathcal{J}^+ /\mathcal{k}^\times, \frac{1}{\hbar}\mathcal{J} \rangle \to \langle \text{Aut}(\mathcal{D})_\mathcal{J}, \text{Der}(\mathcal{D})_\mathcal{J} \rangle \to 1,
\]

be the push-out of the resulting extension via the morphism \( \langle q^+, q^+ \rangle : \langle K, K \rangle \to \langle K /\mathcal{k}, K /\mathcal{k} \rangle \). Applying part (ii) of Corollary 4.3, we conclude that the obstruction for lifting the torsor \( \mathcal{P}_\mathcal{J} \) to a torsor over \( \langle \tilde{\mathcal{G}}_\mathcal{J}^+ /\mathcal{k}^\times, \frac{1}{\hbar}\mathcal{J} \rangle \) is equal to \( i_\mathcal{Y}^*(h^2 \omega_2 + h^3 \omega_3 + \cdots) \) (i.e., the term \( i_\mathcal{Y}^*(h^2 \omega_1) \) disappears when we take the quotient by \( \langle \mathcal{k}, \mathcal{k} \rangle \)).

On the other hand, one has natural isomorphisms \( \tilde{\mathcal{G}}_\mathcal{J}^+ /\mathcal{k} \cong \Sigma(P) \times \tilde{\mathcal{G}}_\mathcal{J}^+ /\mathcal{k}, \tilde{\mathcal{G}}_\mathcal{J}^+ /\mathcal{k}^\times \cong \tilde{\mathcal{G}}_\mathcal{J} /\mathcal{k}^\times, \) cf. Claim 3.10. Furthermore, it is straightforward to see by comparing the constructions that the central extension in the displayed formula above is isomorphic to the one in (5.2); the isomorphism being induced by the exponential map exp : \( K /\mathcal{k} \to K^\times /\mathcal{k}^\times \).

The isomorphism of extensions implies an equality of the corresponding obstruction classes, and the result follows from the conclusion of the preceding paragraph. \( \square \)

5.3. In this subsection, we are going to associate with the quantization \( \mathcal{O}_h \) and the Lagrangian subvariety \( Y \subset X \) a Picard algebroid of the form (1.1).
To this end, let $\mathcal{J}'_Y$ be the preimage of the ideal $I^2_Y$ under the projection $\mathcal{O}_h \to \mathcal{O}_X$, and write $\mathcal{J}'_Y := (\mathcal{J}_Y)^2$. It is clear that one has inclusions $\mathcal{J}'_Y \subset \mathcal{J}_Y \subset \mathcal{J}_Y$.

**Lemma 5.4** There are canonical isomorphisms

$$\mathcal{J}_Y / \mathcal{J}'_Y \cong \mathcal{I}_Y, \quad \mathcal{J}'_Y / \mathcal{J}'_Y \cong \mathcal{O}_Y, \quad \text{and} \quad \mathcal{J}_Y / \mathcal{J}'_Y \cong \mathcal{F}or^0_h(\mathcal{O}_Y, \mathcal{O}_Y).$$

**Proof** By definition, the projection $\text{pr} : \mathcal{O}_h \to \mathcal{O}_X$ induces an isomorphism $\mathcal{J}_Y / \mathcal{J}'_Y \to I_Y / I^2_Y$. Further, the symplectic form on $X$ provides an isomorphism between $I_Y / I^2_Y$, the conormal sheaf to $Y$, and the tangent sheaf $\mathcal{T}_Y$. The first isomorphism of the lemma follows.

To prove the second isomorphism, note that we have $\text{pr}(\mathcal{J}'_Y) = I^2_Y = \text{pr}(\mathcal{J}_Y')$. It follows that the natural imbedding $h\mathcal{O}_h \leftrightarrow \mathcal{J}'_Y$ induces an isomorphism $h\mathcal{O}_h/(\mathcal{J}'_Y \cap h\mathcal{O}_h) \to \mathcal{J}'_Y / \mathcal{J}_Y$. Clearly, one has an inclusion $h\mathcal{J}_Y \subset \mathcal{J}'_Y \cap h\mathcal{O}_h$. Furthermore, for any $y \in Y$, an explicit computation in local coordinates shows that the inclusion $h\mathcal{J}_{Y,y} \to \mathcal{J}'_{Y,y} \cap h\mathcal{O}_{h,y}$ is, in fact, an equality. It follows that $\mathcal{J}'_Y \cap h\mathcal{O}_h = h\mathcal{J}_Y$.

Thus, we deduce a chain of isomorphisms

$$\mathcal{J}'_Y / \mathcal{J}_Y \cong h\mathcal{O}_h/(\mathcal{J}'_Y \cap h\mathcal{O}_h) \cong h\mathcal{O}_h/h\mathcal{J}_Y$$

$$\cong h(\mathcal{O}_h / \mathcal{J}_Y) \cong h(\mathcal{O}_X / I_Y) \cong h\mathcal{O}_Y.$$  

The second isomorphism of the lemma follows.

To prove the third isomorphism, we use the long exact sequence of $\mathcal{F}or$-sheaves associated with the short exact sequence $\mathcal{J}_Y \hookrightarrow \mathcal{O}_h \twoheadrightarrow \mathcal{O}_Y$. A final part of that long exact sequence reads

$$\cdots \to \mathcal{F}or^0_1(\mathcal{O}_Y, \mathcal{O}_h) \to \mathcal{F}or^0_1(\mathcal{O}_Y, \mathcal{O}_Y) \xrightarrow{a} \mathcal{O}_Y \otimes_{\mathcal{O}_h} \mathcal{J}_Y$$

$$\to \mathcal{O}_Y \otimes_{\mathcal{O}_h} \mathcal{O}_h \xrightarrow{b} \mathcal{O}_Y \otimes_{\mathcal{O}_h} \mathcal{O}_Y \to 0.$$  

We have $\mathcal{F}or^0_1(\mathcal{O}_Y, \mathcal{O}_h) = 0$ and the map $b$ above is, essentially, the identity map $\mathcal{O}_Y \to \mathcal{O}_Y$. It follows from the exact sequence that the map $a$ is an isomorphism. It remains to note that one has isomorphisms $\mathcal{O}_Y \otimes_{\mathcal{O}_h} \mathcal{J}_Y = (\mathcal{O}_h / \mathcal{J}_Y) \otimes_{\mathcal{O}_h} \mathcal{J}_Y = \mathcal{J}_Y / \mathcal{J}'_Y$.  

It is easy to see using $\{I_Y, I_Y\} \subset I_Y$ that the bracket $\mathcal{O}_h \times \mathcal{O}_h \to \mathcal{O}_h$, $a \times b \mapsto \frac{1}{\hbar}(ab - ba)$ descends to a well-defined Lie bracket on $\mathcal{J}_Y / \mathcal{J}'_Y$, resp. $\mathcal{J}'_Y / \mathcal{J}'_Y$ and $\mathcal{J}_Y / \mathcal{J}'_Y$. Furthermore, the bracket on $\mathcal{J}_Y / \mathcal{J}'_Y$ goes, via the first isomorphism of Lemma 5.4, to the commutator of vector fields.

Now, there is an obvious short exact sequence

$$0 \to \mathcal{J}'_Y / \mathcal{J}'_Y \to \mathcal{J}_Y / \mathcal{J}'_Y \to \mathcal{J}_Y / \mathcal{J}'_Y \to 0.$$  

(5.5)

All the maps in this sequence respect the brackets and the image of the element $\hbar \in \mathcal{J}_Y / \mathcal{J}'_Y$ is contained in the center of the Lie algebra $\mathcal{J}_Y / \mathcal{J}'_Y$. Thus, we see from Lemma 5.4 that our short exact sequence takes the form of the extension in (1.1).
Remark 5.6 It is not difficult to see that the extension (1.1) may be interpreted in a natural way as a short exact sequence of the form

\[ 0 \to \mathcal{O}_Y \otimes_{\mathcal{O}_N} \mathcal{F}_{1h}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{F}_{1h}(\mathcal{O}_Y, \mathcal{O}_Y) \to \mathcal{F}_{1h}(\mathcal{O}_Y, \mathcal{O}_Y) \to 0, \]

where the tensor product in the first term involves the $\mathcal{O}_N$-module structure on $\mathcal{F}_{1h}(\mathcal{O}_X, -)$ induced by the $\mathcal{O}_N$-action on $\mathcal{O}_X = \mathcal{O}_N / h \mathcal{O}_N$ on the left. The above exact sequence is a noncommutative version of the Jacobi-Zariski sequence, cf. [8, §3.5.5], associated with the algebra homomorphisms $\mathcal{O}_N \to \mathcal{O}_X \to \mathcal{O}_Y$.

We let $\text{At}(\mathcal{O}_N, Y) \in H^2(\Omega^\perp_Y)$ be the Atiyah class of the extension (1.1), equivalently, of the extension (5.5).

Remark 5.7 There is an alternative construction of the class $\text{At}(\mathcal{O}_N, Y)$ in terms of Čech cocycles as follows.

6. Locally in the Zariski topology we can write $\mathcal{O}_N / h^3 \mathcal{O}_N$ as $\mathcal{O}_X + h \mathcal{O}_X + h^2 \mathcal{O}_X$ and $\mathcal{F}_Y = I_Y + h \mathcal{O}_X + h^2 \mathcal{O}_X$, $\mathcal{F}_Y' = I^2_Y + h \mathcal{O}_X + h^2 \mathcal{O}_X$. On an open subset $X_i$ the truncated product looks like

\[ a \ast b = ab + h \alpha^i_1(a, b) + h^2 \alpha^i_2(a, b) \]

while on double intersections the two direct sum splittings are related by the $\mathbb{K}[[h]]$ linear map

\[ a \mapsto a + h \beta^{ij}_1(a) + h^2 \beta^{ij}_2(a). \]

It follows from the standard associativity equations on the $\ast$ product that antisymmetrizing $\alpha^i_2$, then choosing $a, b$ only from the ideal of functions vanishing on $Y$, and finally restricting to $Y$ we obtain, due to $N^* \simeq T_Y$, a closed 2-form $\eta'_i$ on $Y_i = X_i \cap Y$. On the double intersections, since the transition functions $\beta^{ij}_1$ agree with products and since by assumption $\alpha^i_1(a, b) = \frac{1}{2} P(da, db)$, we can conclude that each $\beta^{ij}_1$ is a derivation, i.e. induced by a vector field on $X_i$. Projecting its restriction to $Y$ on the normal bundle $N$ and then using $N \simeq \Omega_Y$ we obtain 1-forms $\xi^{ij}$ on $X_i \cap Y_j$, such that $\eta^{ij}_{|Y_i \cap Y_j} - \eta^{ij}_{|Y_i \cap Y_j} = d\xi^{ij}$. Then the collection $\eta^{ij}, \xi^{ij}$ defines a class in $H^2$ of the truncated de Rham complex (i.e., the 1-forms $\xi^{ij}$ satisfy the cocycle condition on triple intersections, rather than up to a differential of function, since the cocycle condition holds for the original vector fields $\beta^{ij}$).

We now consider the setting of § 4.2 in a special case where $(G, \mathfrak{g}) = (\text{Aut}(\mathcal{D}), \text{Der}(\mathcal{D}))$ and $(\hat{G}, \mathfrak{g}) = (\hat{G} \mathcal{J}/(1 + hK), \Theta \mathcal{J}/hK)$. We have a natural extension $\tilde{c}$ as in (4.4), and also a section $\tilde{\eta} : G \to \hat{G}$ that comes from the direct product decomposition (4.8). Thus, the construction of § 4.2 associates with this data a class $\alpha(P, \tilde{c}, \tilde{\eta}) \in H^2(\Omega^\perp_Y)$.

Lemma 5.8 (i) In $H^2(\Omega^\perp_Y)$, one has an equality $\alpha(P, \tilde{c}, \tilde{\eta}) = \text{At}(\mathcal{O}_N, Y)$.

(ii) The canonical morphism $H^2(\Omega^\perp_Y) \to H^2_{DR}(Y)$ sends $\text{At}(\mathcal{O}_N, Y)$ to $i^*_Y(\omega_1(\mathcal{O}_N))$. 

Proof: By definition, the class $\alpha(\mathcal{P}_{\mathcal{J}}, \bar{\tau}, \bar{t})$ is the class of the equivariant descent of the Lie algebroid $\tilde{\mathfrak{g}} \otimes \mathcal{O}_{\mathcal{P}_{\mathcal{J}}}$ and we need to identify this with the Atiyah algebra $\mathcal{J}_Y / \mathcal{J}_Y^2$ (as Lie algebroids on $Y$). Instead, we can pull back the latter algebra to $\mathcal{P}_{\mathcal{J}}$ and identify the pullback with the quotient of $\tilde{\mathfrak{g}} \otimes \mathcal{O}_{\mathcal{P}_{\mathcal{J}}}$ by $d\tilde{t}(\text{Lie}(G)) \otimes \mathcal{O}_{\mathcal{P}_{\mathcal{J}}}$. But by definition of $\mathcal{P}_{\mathcal{J}}$ at every (closed) point of this torsor the completion of $\mathcal{J}_Y$ is identified with $\mathcal{J}$ and

$$
\mathcal{J} / \mathcal{J}_Y^2 \simeq (\frac{1}{h} \mathcal{J} \mod hK) / d\tilde{t}(\text{Lie}(\text{Aut}(\mathcal{D})_{\mathcal{J}})),
$$

as required. This finishes the proof of (i).

For (ii) recall that the class in $H^2_{\text{DR}}(Y)$ is represented by a sequence of flat bundles on $Y$ induced by $\mathcal{P}_{\mathcal{J}}$ from the sequence (4.2) with $a = k$. On the other hand, for any Atiyah algebra $\mathcal{O}_Y \to \mathcal{L} \to \mathcal{T}_Y$ the image of its class in $H^2_{\text{DR}}(Y)$ is represented by the extension

$$
0 \rightarrow \mathcal{O}_Y \rightarrow U_+(\mathcal{L})/\mathcal{O}_Y \cdot U_+(\mathcal{L}) \rightarrow \mathcal{D}_Y \rightarrow \mathcal{O}_Y \rightarrow 0
$$

where $\mathcal{D}_Y$ is the sheaf of algebraic differential operators on $Y$ and the last arrow sends an operator to its value on the constant function 1. If the Atiyah algebra in question is the equivariant descent of $\tilde{\mathfrak{g}} \otimes \mathcal{O}_{\mathcal{P}_{\mathcal{J}}}$, then the above long extension is obtained from a sequence very similar to (4.2), except that the two middle terms are replaced by their quotients by the ideal generated by the image of $\text{Lie}(G)$. But it is easy to check that these two ideals are isomorphic, hence the class in $\text{Ext}^2$ is the same, as required. □

6 Proof of the main result

6.1. We keep the notation of the previous section. In particular, we have the obstruction class $\text{per}(\mathcal{O}_h) = [\omega] + h \omega_1(\mathcal{O}_h) + h^2 \omega_2(\mathcal{O}_h) + \cdots$, associated with the extension (5.1).

Lemma 6.1. A choice of line bundle $L$ on $Y$ and its deformation quantization $L_h$ is equivalent to the choice of a lift of the torsor $\mathcal{P}_{\mathcal{J}}$ to a transitive Harish-Chandra torsor $\mathcal{P}_{\mathcal{D},\mathcal{M}}$ over $\langle \text{Aut}(\mathcal{D}), \text{Der}(\mathcal{D}, \mathcal{M}) \rangle$.

Proof: For $y \in Y$, let $\eta : \mathcal{O}_{Y,y} \cong \mathcal{D}$ be an isomorphism such that $\eta(\mathcal{J}_y) = \mathcal{J}$. Given a line bundle $L$ and its quantization $L_h$, let $L_y := \mathcal{O}_{Y,y} \otimes_{\mathcal{O}_Y} L$, resp. $L_{y,h} := \mathcal{O}_{Y,y} \otimes_{\mathcal{O}_h} L_h$. A choice of local section of $L$ at $y$ provides an isomorphism $L_y \cong \mathcal{O}_{Y,y}$. We obtain a chain of isomorphisms

$$
L_{y,h}/h \cong L_{y,h} \cong L_y \cong \mathcal{O}_{Y,y} \cong \mathcal{O}_{Y,y}/(h \mathcal{O}_{Y,y} + \mathcal{J}_y) \cong \mathcal{D}/\mathcal{J} \cong \mathcal{M}/h \mathcal{M},
$$

where the fourth isomorphism is induced by the isomorphism $\eta$. Thus, applying Lemma 2.9, we deduce that the $\mathcal{D}$-module $\eta^* L_{y,h}$, obtained from $L_{y,h}$ by transporting the module structure via $\eta$, is isomorphic to $\mathcal{M}$. Various choices of an isomorphism $\eta^* L_{y,h} \cong \mathcal{M}$ for all $y \in Y$ give the required lift of the torsor $\mathcal{P}_{\mathcal{J}}$ to a transitive Harish-Chandra torsor $\mathcal{P}_{\mathcal{D},\mathcal{M}}$ over $\langle \text{Aut}(\mathcal{D}), \text{Der}(\mathcal{D}, \mathcal{M}) \rangle$. 

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In the opposite direction, let $\mathcal{P}_{D,\mathcal{M}}$ a lift of $\mathcal{P}_J$. Then $\Aut(D,\mathcal{M})$ acts on $\mathcal{M}$ and, therefore, one has a vector bundle $\mathcal{M}_P$ associated with the $\Aut(D,\mathcal{M})$-module $\mathcal{M}$ and $\mathcal{P}_{D,\mathcal{M}}$, viewed as an $\Aut(D,\mathcal{M})$-torsor. Moreover, the Lie algebra action of $\Lie(\Aut(D,\mathcal{M}))$ extends to the action of the full algebra $\Der(D,\mathcal{M}) \simeq \mathfrak{g}_J$ which implies that the bundle $\mathcal{M}_P$ admits a flat algebraic connection. Now $L_\hbar$ may be recovered as the sheaf of flat sections with respect to this connection.

Finally, we note that the (non-quantized) line bundle $L$ may also be recovered from $\mathcal{P}_{D,\mathcal{M}}$. Specifically, one has an isomorphism $L \cong \mathbb{k}^\times \otimes_x \mathcal{P}_{D,\mathcal{M}}$, of $\mathbb{k}^\times$-torsors on $Y$, where $\mathbb{k}^\times \otimes_x \mathcal{P}_{D,\mathcal{M}}$ denotes the push-out of the torsor $\mathcal{P}_{D,\mathcal{M}}$ via the canonical homomorphism $x : \Aut(D,\mathcal{M}) \to \mathbb{k}^\times$, see (3.12).

By Lemma 6.1 a choice of a quantized line bundle $L_\hbar$ on $Y$ is equivalent to a choice of the following data:

- A lift of the $(\Aut(D)_J, \Der(D)_J)$-torsor $\mathcal{P}_J$ to a transitive $\mathcal{P}_{D,\mathcal{M}}$ over $(\overline{\Aut(D,\mathcal{M})}, \overline{\Der(D,\mathcal{M})})$.

- A lift of $\mathcal{P}_{D,\mathcal{M}}$ to a transitive $(\Aut(D,\mathcal{M}), \Der(D,\mathcal{M}))$-torsor $\mathcal{P}_{D,\mathcal{M}}$ such that one has an isomorphism $L \cong \mathbb{k}^\times \otimes_x \mathcal{P}_{D,\mathcal{M}}$.

**Lemma 6.4** The existence of a lift $\mathcal{P}_{D,\mathcal{M}}$, as in (6.2), is equivalent to an equation $i_Y^*(\hbar^2 \omega_2(\mathcal{O}_h) + \hbar^3 \omega_3(\mathcal{O}_h) + \cdots) = 0$ in $H^2(Y)[[\hbar]]$.

**Proof** Diagram (4.10) provides an isomorphism of central extensions of Harish-Chandra pairs. Therefore, the torsor $\mathcal{P}_J$ can be lifted to a transitive $(\overline{\Aut(D,\mathcal{M})}, \overline{\Der(D,\mathcal{M})})$-torsor if and only if it can be lifted to a transitive $(\widehat{G}_J, \mathfrak{g}_J)$-torsor. The latter holds if and only if the class $\text{Loc}(\mathcal{P}_J, \widehat{G}_J, \mathfrak{g}_J)$ vanishes, see § 4.1. The result now follows from Lemma 5.3.

6.1 Proof of Theorem 1.4

From now on, we assume that the equation $i_Y^*(\hbar^2 \omega_2(\mathcal{O}_h) + \hbar^3 \omega_3(\mathcal{O}_h) + \cdots) = 0$ holds and hence there is a torsor $\mathcal{P}_{D,\mathcal{M}}$, over $(\overline{\Aut(D,\mathcal{M})}, \overline{\Der(D,\mathcal{M})})$, as in (6.2).

**Lemma 6.5** The $(\overline{\Aut(D,\mathcal{M})}, \overline{\Der(D,\mathcal{M})})$-torsor $\mathcal{P}_{D,\mathcal{M}}$ can be lifted to a $(\Aut(D,\mathcal{M}), \Der(D,\mathcal{M}))$-torsor $\mathcal{P}_{D,\mathcal{M}}$, as in (6.3), if and only if one has: $c_1(L) = \frac{1}{2}c_1(K_Y) + \Aut(\mathcal{O}_h, Y)$.

**Proof** By (4.9) any lift $\mathcal{P}_{D,\mathcal{M}}$ must be isomorphic as an $\Aut(D,\mathcal{M})$-torsor to $L \times_Y \mathcal{P}_{D,\mathcal{M}}$ for some line bundle $L$ on $Y$. By Lemma 4.7 this $\Aut(D,\mathcal{M})$-torsor structure extends to the structure of a transitive Harish-Chandra torsor over $(\Aut(D,\mathcal{M}), \Der(D,\mathcal{M}))$ if and only if $c_1(L) = \alpha(\mathcal{P}_{D,\mathcal{M}}, c_{\Der}, t_{\Der})$ in $H^2(Y, \Omega_{\widehat{Y}}^{\geq 1})$.

From Proposition 4.12 the latter class is also equal to $\alpha(\Phi_* \mathcal{P}_{D,\mathcal{M}}, c_J, t_J) + \frac{1}{2}c_1(K_Y)$. By equivariant descent with respect to $(1 + \hbar K)$ and Lemma 5.8 we identify the first of the two terms as $\Aut(\mathcal{O}_h, Y)$. 


We conclude that the equation \( c_1(L) = \frac{1}{2} c_1(K_Y) + At(O_{\mathfrak{h}}, Y) \) holds if and only if the \((\text{Aut}(\mathcal{D}, \mathcal{M}), \text{Der}(\mathcal{D}, \mathcal{M}))\)-torsor \( \mathcal{P}_{D, \mathcal{M}} \) can be lifted to a torsor \( \mathcal{P}_{D/\mathcal{M}} \) as in (6.3).

We now discuss the set of isomorphism classes of the lifts \( \mathcal{P}_{D, \mathcal{M}} \) for a fixed \( \mathcal{P}_{\mathcal{J}} \) to, assuming it is non-empty. Assume that \( L \) is a flat \( (O_Y[[h]])^\times \) torsor (but we do not fix a choice of a flat connection), then \( \mathcal{P}_{D, \mathcal{M}}(L) := (\mathcal{P}_{D, \mathcal{M}} \times_Y L)/K^\times \) is again a torsor over \( \text{Aut}(\mathcal{D}, \mathcal{M}) \), since \( K^\times \) is central in \( \text{Aut}(\mathcal{D}, \mathcal{M}) \). The fact that this extends to a transitive Harish-Chandra torsor structure on \( \mathcal{P}_{D, \mathcal{M}}(L) \) can be established as follows. We have a direct product decomposition \( (O_Y[[h]])^\times \cong O_Y^\times \times (1 + hO_Y) \). Therefore, choosing \( L \) is equivalent to choosing a pair \((L_0, L_1)\) consisting of a flat \( O_Y^\times \)-torsor \( L_0 \) and a flat \((1 + hO_Y)\)-torsor \( L_1 \).

It is clear that the lift \( \mathcal{P}_{D, \mathcal{M}} \) can be adjusted by passing to \( \mathcal{P}_{D, \mathcal{M}}(L_1) \) (which is defined similarly), since \( L_1 \) has a trivial class in \( hH^{2}_{\text{DR}}(Y)[[h]] \). By Proposition 2.7 in [4], any lift of \( \mathcal{P}_{\mathcal{J}} \) to a \((\text{Aut}(\mathcal{D}, \mathcal{M}), \text{Der}(\mathcal{D}, \mathcal{M}))\)-torsor has the form \( \mathcal{P}_{D, \mathcal{M}}(L_1) \) for a unique \( L_1 \). Similarly, with fixed choice of \( \mathcal{P}_{D, \mathcal{M}} \), any two lifts to a \((\text{Aut}(\mathcal{D}, \mathcal{M}), \text{Der}(\mathcal{D}, \mathcal{M}))\)-torsor \( \mathcal{P}_{D, \mathcal{M}} \) differ by a twist by a unique \( L_0 \), as follows from Lemma 4.7. Hence, every lift of \( \mathcal{P}_{\mathcal{J}} \) is isomorphic to \( \mathcal{P}_{D, \mathcal{M}}(L) \) for a unique flat \((O_Y[[h]])^\times \)-torsor \( L \), as required. This finishes the proof of Theorem 1.4.

6.2 Final remarks

Given a line bundle \( L \), one \( Y \), one can consider a problem of quantization of \( L \) up to a finite order in \( h \). That is, for any \( s = 1, 2, \ldots \), one can study deformations of \( L \) to an \( O_{\mathfrak{h}}/h^{s+1}\mathfrak{h} \)-module \( L_s \), which is flat over \( \mathbb{k}[h]/h^{s+1} \). The corresponding ‘finite-order’ counterpart of Theorem 1.4 is more complicated, in a sense, than Theorem 1.4 itself.

To explain this, for each \( N \geq 1 \), consider the following condition

\[
(*_N) : \quad c_1(L) - \frac{1}{2} c_1(K_Y) = At(O_{\mathfrak{h}}, Y) \quad \& \quad i_Y^* \omega_i(O_{\mathfrak{h}}) = 0 \quad \forall i = 1, \ldots, N.
\]

Then, it turns out that one has implications

\[
(*_{s+1}) \quad \Rightarrow \quad \exists L_s \quad \Rightarrow \quad (*_{s-1}),
\]

however, none of the two implications is an equivalence, in general. The origin of this phenomenon comes from the fact that the classes in the sequence \( \omega_i(O_{\mathfrak{h}}) \), \( i = 1, 2, \ldots \), are, essentially, the obstructions to extending a torsor over the Harish-Chandra pair \( (\mathfrak{g}/\mathfrak{g}^{\leq i}, \mathfrak{g}/\mathfrak{g}^{> i}) \) to a torsor over \( (\mathfrak{g}/\mathfrak{g}^{\leq i+1}, \mathfrak{g}/\mathfrak{g}^{> i+1}) \). Thus, the sequence \( \omega_i(O_{\mathfrak{h}}) \), \( i = 1, 2, \ldots \), is closely related to the descending filtration \( D^{\leq i} \) on the algebra \( D \). On the other hand, associated with the choice of a Lagrangian subspace \( \mathfrak{r} \) there is another descending filtration, \( F^\leq_{\mathfrak{r}} D \), on \( D \). It is defined as the multiplicative filtration on the enveloping algebra of the Heisenberg algebra \( \mathfrak{v} \oplus \mathbb{k}h \) induced by the 3-step filtration.
\[ F^0 = v \oplus \mathbb{k} h \supset F^1 = v \supset F^2 = \mathfrak{r}, \]

on the vector space \( v \oplus \mathbb{k} h \). The obstructions for the existence of finite-order deformations of the line bundle \( L \) are more naturally related to the filtration \( F^i \mathfrak{g}_D \) rather than to \( D \sigma^i \).

We illustrate the above in the case \( s = 1 \). Assume for simplicity that \( Y \) is (smooth) projective and that the sheaf \( \mathcal{O}_h/\mathfrak{h}^2 \mathcal{O}_h \) splits globally as \( \mathcal{O}_X + \mathfrak{h} \mathcal{O}_X \). Then, by Hodge theory, the cohomology group \( H^2(\Omega_Y^2) \) is a subspace of \( H^2_{\text{DR}}(Y) \) and we have \( \text{At}(\mathcal{O}_h, Y) = i_Y^* \omega_1(\mathcal{O}_h) \).

By [3], a first-order deformation exists Zariski locally on \( X \) if and only if \( \omega|_Y = 0 \), which corresponds to \( i = 0 \) and a vanishing in the \( H^0(Y, \Omega^2) \) component of \( H^2_{\text{DR}}(Y) \). The local first-order deformations can be adjusted so that they are isomorphic on the double intersections of our Zariski covering, if and only if \( 2c_1(L) = c_1(K_Y) \) in \( H^1(Y, \Omega^1_Y) \), which is a part of our equation on \( i_Y^* \omega_1(\mathcal{O}_h) \) (since it is represented by a closed 2-form on \( Y \) under our assumptions and the projection onto \( H^1(Y, \Omega^1_Y) \) is zero).

Further, the isomorphisms on double intersections can be chosen to satisfy the cocycle condition on triple intersections precisely when a certain class in \( H^2(Y, \Omega^2_Y) \) vanishes. In [3] we were unable to evaluate this class explicitly, but very recently a formula for it was found by K. Behrend and B. Fantechi and we expect that the corresponding vanishing condition is the \( H^2(Y, \Omega^2_Y) \) component of \( i_Y^* \omega_2(\mathcal{O}_h) = 0 \).

Finally, by [3], the \( H^0(Y, \Omega^2_Y) \) component of the equation for \( i_Y^* \omega_1(\mathcal{O}_h) \) is precisely the condition that \( L \) admits a local second-order deformation. Again, local second-order deformations can be globalized when the projections of \( i_Y^* \omega_2(\mathcal{O}_h) \) to \( H^1(Y, \Omega^1_Y) \) and \( i_Y^* \omega_3(\mathcal{O}_h) \) to \( H^2(Y, \Omega^2_Y) \) vanish, respectively, and so on.

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