Height estimates for constant mean curvature graphs in $\text{Nil}_3$ and $\text{P}SL_2(\mathbb{R})$

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Abstract

In this paper we obtain height estimates for compact, constant mean curvature vertical graphs in the homogeneous spaces $\text{Nil}_3$ and $\widehat{\text{P}SL_2(\mathbb{R})}$. As a straightforward consequence, we announce a structure-type result for proper graphs defined on relatively compact domains.

1 Introduction

In the last decades, height estimates have become a powerful tool when studying the global behavior of a certain class of surfaces in some ambient space, see for instance [AEG, He, HLR, KKMS, KKS, Me, Ro, RoSa]. Heinz [He] proved that if $M$ is a compact graph in the Euclidean space $\mathbb{R}^3$ with positive constant mean curvature $H$, ($H$-surface in the following), and boundary $\partial M$ lying in a plane $\Pi$, then the maximum height that $M$ can reach from the plane is $1/H$. This estimate is optimal, since it is achieved by the $H$-hemisphere intersecting orthogonally $\Pi$. Applying Alexandrov reflection technique with respect to parallel planes to $\Pi$ yields that a compact embedded $H$-surface in $\mathbb{R}^3$ with boundary in $\Pi$ has height from that plane at most $2/H$.

These height estimates for $H$-surfaces in $\mathbb{R}^3$ were the cornerstone for Meeks [Me] in his global study of $H$-surfaces in $\mathbb{R}^3$; for example, he showed that there do not exist properly embedded $H$-surfaces with one end in $\mathbb{R}^3$, and if a properly embedded $H$-surface has two ends, then the surface stays at bounded distance from a straight line. Later, Korevaar, Kusner and Solomon [KKS] proved that a properly embedded $H$-surface lying inside a solid cylinder, must be rotationally symmetric and hence a cylinder.

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or an onduloid. Moreover, they proved that each end of a properly embedded $H$-surface with finite topology in $\mathbb{R}^3$ is asymptotic to a cylinder or an onduloid. This study on properly embedded $H$-surfaces in $\mathbb{R}^3$ concludes in the so called structure Theorem.

As a straightforward adaptation from the Euclidean case, Korevaar, Kusner, Meeks and Solomon [KKMS] obtained optimal bounds for the height of compact $H$-graphs in the hyperbolic space $\mathbb{H}^3$ with boundary lying in a totally geodesic surface. The result is also extendible to compact embedded $H$-surfaces in $\mathbb{H}^3$, as a direct consequence of Alexandrov reflection technique with respect to totally geodesic surfaces in $\mathbb{H}^3$, the analogous to Euclidean planes. In the formulation of the problem in both $\mathbb{R}^3$ and $\mathbb{H}^3$ the plane where $\partial M$ lies can be chosen without specifying its orthogonal direction, as $\mathbb{R}^3$ and $\mathbb{H}^3$ are isotropic spaces; in general, a riemannian manifold is isotropic if its isometry group acts transitively in the tangent bundle.

In this context, the product spaces $\Sigma^2 \times \mathbb{R}$ defined as the riemannian product of a complete riemannian surface $\Sigma^2$ and the real line $\mathbb{R}$ are closely related to the space forms in the sense that they are highly symmetric. Following the ideas developed in [He], Hoffman, de Lira and Rosenberg [HLR] obtained height estimates for compact embedded $H$-surfaces with boundary in a slice $\Sigma^2 \times \{t_0\}$. This result in the product spaces was improved by Aledo, Espinar and Gálvez [AEG], exhibiting sharp bounds for the height of compact, embedded $H$-surfaces in $\Sigma^2 \times \mathbb{R}$ with boundary in a slice, improving the previous results by characterizing when equality held. As happened in $\mathbb{R}^3$ and $\mathbb{H}^3$, the $H$-graph in $\Sigma^2 \times \mathbb{R}$ with boundary in a slice $\Sigma^2 \times \{t_0\}$ attaining the maximum height over $\Sigma^2 \times \{t_0\}$, corresponds to the rotational $H$-hemisphere intersecting orthogonally $\Sigma^2 \times \{t_0\}$.

For the particular case when the base $\Sigma^2$ is a complete simply connected surface with constant curvature $\kappa$, the spaces arising are the product spaces $M^2(\kappa) \times \mathbb{R}$. Such product spaces belong to a two parameter family of homogeneous, simply connected 3-dimensional manifolds, the $E(\kappa, \tau)$ spaces. In Section 2, we will introduce these spaces and give a geometric sense to the constants $\kappa$ and $\tau$. For instance, the product spaces correspond to the case $\tau = 0$ in the $\mathbb{E}(\kappa, \tau)$ family. In the last decade, the theory of immersed surfaces in the $\mathbb{E}(\kappa, \tau)$ spaces, and more specifically constant mean curvature and minimal surfaces, have become a fruitful theory focusing the attention of many geometers. See [AbRo1, AbRo2, Da, DHM, FeMi] for an outline of the development of this theory.

Our objective in this paper is to obtain height estimates for vertical $H$-graphs in the spaces Nil$_3$ and $\widetilde{PSL}_2(\mathbb{R})$, which correspond to the particular choices in the $\mathbb{E}(\kappa, \tau)$ family of $\kappa = 0, \tau > 0$ and $\kappa < 0, \tau > 0$, respectively. We obtain the desired height estimates by means of well behavior of the stability of $H$-surfaces with respect of the limit of a sequence of $H$-surfaces with uniformly bounded second fundamental form. This well behavior of the stability of $H$-surfaces has been exploited widely in the literature; see the proof of the Main Theorem in [RST] for a global understanding of this technique in arbitrary complete 3-manifolds with bounded sectional curvature.
2 Homogeneous 3-dimensional spaces with 4-dimensional isometry group.

Let $\mathbb{M}^2(\kappa)$ be the complete, simply connected surface of constant curvature $\kappa \in \mathbb{R}$. The family of homogeneous, simply connected 3-dimensional manifolds $E$ with a 4-dimensional isometry group, can be defined as a family of riemannian submersions $\pi : E \rightarrow \mathbb{M}^2(\kappa)$. The fibre that passes through a point $p \in \mathbb{M}^2(\kappa)$ is defined as $\pi^{-1}(p)$, and translations along these fibres are ambient isometries generated by the flow of a unitary Killing vector field, $\xi$. The Killing vector field is related to the Levi-Civita connection $\nabla$ of $E$ and the cross product by the formula

$$\nabla_X \xi = \tau X \times \xi,$$

where $\tau$ is a constant named the bundle curvature. Both $\kappa$ and $\tau$ satisfy $\kappa - 4\tau^2 \neq 0$.

After a change of orientation of $E$, we can suppose that $\tau > 0$. These spaces are denoted by $E(\kappa, \tau)$, where $\kappa, \tau$ are the constants defined above. Depending on the value of $\kappa$ and $\tau$, we obtain the following geometries.

- If $\tau = 0$, then we recover the product spaces $\mathbb{M}^2(\kappa) \times \mathbb{R}$, i.e. up to scaling, the space $S^2 \times \mathbb{R}$ if $\kappa > 0$, and the space $H^2 \times \mathbb{R}$ if $\kappa < 0$.

- If $\tau > 0$ and $\kappa = 0$, the $E(\kappa, \tau)$ space arising is the Heisenberg group $\text{Nil}_3$, the Lie group of matrices

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} ; \ a, b, c \in \mathbb{R} \right\},$$

endowed with a one-parameter family of left-invariant metrics.

- When $\tau > 0$ and $\kappa < 0$, we obtain the space $\widetilde{\text{PSL}}_2(\mathbb{R})$, the universal cover of the positively oriented isometries of the hyperbolic plane $\mathbb{H}^2$, endowed with a one-parameter family of left-invariant metrics.

- When $\tau > 0$ and $\kappa > 0$, the $E(\kappa, \tau)$ spaces are the Berger spheres, a family of one-parameter metrics defined on the three dimensional sphere $S^3$. This metrics are obtained in such a way that the Hopf fibration is still a Riemannian fibration.

We can give a unified model for the $E(\kappa, \tau)$ spaces; when $\kappa \leq 0$ the model is global and when $\kappa > 0$ we get the universal cover of $E(\kappa, \tau)$ minus one fibre. We endow $\mathbb{R}^3$ (if $\kappa \geq 0$) and $(\mathbb{D}(2/\sqrt{-\kappa}) \times \mathbb{R})$ (if $\kappa < 0$) with the metric

$$ds^2 = \lambda^2(dx^2 + dy^2) + (dz + \tau \lambda(ydx - xdy))^2,$$

where $\lambda$ is defined as

$$\lambda = \frac{4}{4 + \kappa(x^2 + y^2)}.$$
The riemannian submersion is given by the projection onto the first two coordinates, where we identify the basis $\mathbb{M}^2(\kappa)$ with $\{z = 0\} \subset \mathbb{E}(\kappa, \tau)$. The vector field $\partial_z$ is the unitary Killing vector field whose flow generates the \textit{vertical translations}. The integral curves of this flow are the fibres of the submersion, and they are complete geodesics. The fields given by

$$E_1 = \frac{1}{\lambda} \partial_x - \tau y \partial_z, \quad E_2 = \frac{1}{\lambda} \partial_y + \tau x \partial_z, \quad E_3 = \partial_z,$$

are an orthonormal basis at each point. In this framework, the angle function of an immersed, orientable surface $M$ is defined as $\nu = \langle \eta, \partial_z \rangle$, where $\eta$ is a unit normal vector field defined on $M$.

Henceforth, we will denote simply by $(\mathbb{E}, \langle \cdot, \cdot \rangle)$ to any of the $\mathbb{E}(\kappa, \tau)$ spaces with the model given above. A \textit{horizontal plane} is a subset of $\mathbb{E}$ of the form $\{z = z_0; \ z_0 \in \mathbb{R}\}$, where $z_0$ is called the height of the plane. Every horizontal plane is a minimal surface, and when $\tau = 0$ they are totally geodesic copies of $\mathbb{M}^2(\kappa)$ that differ one from the other by a vertical translation. A \textit{vertical graph} in $\mathbb{E}$ is a surface $M$ with the property that intersects each fibre of the submersion at most one. As a matter of fact, each vertical graph in $\mathbb{E}$ can be parametrized as

$$M = \{(x, y, f(x, y)); \ (x, y) \in \Omega\},$$

for a certain smooth function $f$ defined in a domain $\Omega$ contained in some horizontal plane $\{z = z_0\}, \ z_0 \in \mathbb{R}$. Note that after a vertical translation, the domain of a vertical graph can be contained in any horizontal plane. A graph is compact if $\Omega$ is compact and $f$ extends to $\partial \Omega$ continuously. The \textit{boundary} of a compact graph is defined as $\partial M = f(\partial \Omega)$. A compact graph $M$ has boundary in a horizontal plane if its boundary $\partial M$ has constant height. This is equivalent to the fact that $f$ restricted to $\partial \Omega$ is a constant function.

### 2.1 Stability of $H$-surfaces in the $\mathbb{E}(\kappa, \tau)$ spaces

It is a well known fact that an $H$-surface $M$ immersed in an arbitrary riemannian 3-manifold, is a critical point for the area functional associated to compactly supported variations of the surface that preserve the enclosed volume constant. Equivalently, $M$ is an $H$-surface if and only if it is a critical point for the functional $\text{Area-2HVol}$ [BCE]. The second variation of this functional is given by the quadratic form

$$Q(f, f) = - \int_M \left( \Delta_M f + f(|\sigma|^2 + \text{Ric}(\eta)) \right) f dA, \ \forall f \in C_0^\infty(M) \quad (2.1)$$

where $\Delta_M$ is the Laplace-Beltrami operator of the surface $M$, $|\sigma|^2$ is the squared length of the second fundamental form of $M$, $\eta$ is the unit normal of $M$ and $\text{Ric}(\eta)$ is the \textit{Ricci curvature} along the direction $\eta$. Equation \((2.1)\) can be rewritten by defining the elliptic operator

$$\mathcal{L} = \Delta_M + |\sigma|^2 + \text{Ric}(\eta) \quad (2.2)$$


and thus (2.1) is equivalent to
\[ Q(f,f) = -\int_M f \mathcal{L} f \, dA, \quad \forall f \in C^\infty_0(M). \] (2.3)

The operator \( \mathcal{L} \) is the Jacobi operator, or stability operator of \( M \). An orientable immersion \( M \) in an \( \mathbb{E}(\kappa, \tau) \) space is stable if and only if
\[ -\int_M f \mathcal{L} f \, dA \geq 0, \quad \forall f \in C^\infty_0(M). \]

The non-vanishing functions \( f \in C^\infty(M) \) lying in the kernel of \( \mathcal{L} \) are called Jacobi functions. If \( M \) is an orientable immersed surface in an \( \mathbb{E}(\kappa, \tau) \) space and \( \nu \) denotes the angle function of \( M \), then \( \nu \) is a Jacobi function for the stability operator \( \mathcal{L} \) [Da], i.e. the elliptic equation \( \mathcal{L}\nu = 0 \) holds. This equation reads as
\[ \Delta_M \nu + \nu \left( (1 - \nu^2)(\kappa - 4\tau^2) + |\sigma|^2 + 2\tau^2 \right) = 0. \] (2.4)

A classical theorem due to Fischer-Colbrie [Fi] asserts that the existence of a positive Jacobi function defined on a surface \( M \) is equivalent to the stability of the surface. As a matter of fact, each vertical \( H \)-graph in an \( \mathbb{E}(\kappa, \tau) \) space is stable, since either the function \( \nu \) or \( -\nu \) is always positive.

## 3 Height estimates

In this section, \( H \) will denote a positive constant and \( (\mathbb{E}, \langle \cdot, \cdot \rangle) \) will be either the space \( \text{Nil}_3 \) or \( \widetilde{PSL}_2(\mathbb{R}) \) with the corresponding metric. In particular, as in both spaces we have \( \kappa \leq 0 \), \( \mathbb{E} \) is given by the global model defined in Section 2. The theorem that we prove is the following:

**Theorem 1** Let \( H \) be a positive constant and suppose that
\[ 4H^2 + \kappa > 0. \]
Then, there exists a constant \( C = C(H, \kappa, \tau) > 0 \), such that for every compact vertical \( H \)-graph \( M \) in \( \mathbb{E} \) with boundary in a horizontal plane, the height that \( M \) reaches over that plane is at most \( C \).

In the space \( \widetilde{PSL}_2(\mathbb{R}) \), the hypothesis \( 4H^2 + \kappa > 0 \) has a relevant geometric sense, since this condition for \( H \) and \( \kappa \) ensures the existence of a rotationally symmetric \( H \)-sphere.

Before proving Theorem 1 we will prove a technical Lemma that guarantees a uniformly bound of the second fundamental form for \( H \)-graphs in \( \mathbb{E} \).
Lemma 2 Let \( \{M_n\} \) be a sequence of compact vertical \( H \)-graphs in \( \mathbb{E} \), with \( \partial M \subset \{z = 0\} \) and with heights to \( \{z = 0\} \) tending to infinity, and fix \( d > 0 \). Then, there exists a positive constant \( \Lambda \) such that the uniform estimate holds:

\[
|\sigma_n(p)| < \Lambda, \quad \forall p \in M^*_n, \quad \forall n \in \mathbb{N},
\]

where \( M^*_n \) is the subset of \( M_n \) defined as

\[
M^*_n = \{ p \in M_n; \; d(p, \partial M_n) > 2d \}.
\]

**Proof:** We define \( M^*_n := \{ p \in M_n; \; d(p, \partial M_n) > 2d \} \). As the heights of \( M_n \) over \( \{z = 0\} \) tend to infinity, and the ambient distance is always lower or equal than the intrinsic distance, it is clear that \( M^*_n \) is a non-empty, possibly non-connected, graph over \( \{z = 0\} \) for \( n \) large enough. Arguing by contradiction, suppose that such estimate does not hold. Then, there exists a sequence of points \( p_n \in M^*_n \) such that \( |\sigma_n(p_n)| \to \infty \). After passing to a subsequence, we may suppose that each \( p_n \in M^*_n \) satisfies \( |\sigma_n(p_n)| > n, \; \forall n \in \mathbb{N} \). We will follow the ideas exhibited in the proof of Theorem 1.1. in [MePR]. Consider the compact intrinsic ball \( D_n = B_{M_n}(p_n, d) \) in \( M_n \), which by construction lies at a positive distance from \( \partial M_n \). Let \( q_n \) be the maximum on \( D_n \) of the function

\[
g_n(x) = |\sigma_n(x)|d_{M_n}(x, \partial D_n).
\]

Clearly, \( q_n \) is an interior point as \( g_n \) vanishes on \( \partial D_n \). Define \( \lambda_n = |\sigma_n(q_n)| \) and \( r_n = d_{M_n}(q_n, \partial D_n) \). Then,

\[
\lambda_n r_n = g_n(q_n) \geq g_n(p_n) = |\sigma_n(p_n)|d_{M_n}(p_n, \partial D_n) > nd.
\]

(3.1)

In particular, \( \lambda_n r_n \to \infty \) as \( n \to \infty \).

Denote by \( \tilde{M}_n \) to the image of \( M_n \) under the isometry of \( \mathbb{E} \) that takes the point \( q_n \) to the origin of \( \mathbb{E} \) denoted by \( \sigma \). Now, we consider geodesic coordinates around \( \sigma \), and fix \( t > 0 \). As \( \lambda_n \to \infty \), the sequence \( \{\lambda_n B_\mathbb{E}(\sigma, t/\lambda_n)\} \) converges to the open ball \( \mathbb{B}(t) \) of \( \mathbb{R}^3 \) with its usual flat metric. For this fixed \( t \), let \( m(t) \in \mathbb{N} \) be large enough such that \( t/\lambda_m < r_m/2 \). For all \( n > m(t) \), we define \( \mathcal{B}_n(t) = \lambda_n B_{M_n}(\sigma, t/\lambda_n) \) which is a vertical graph and with angle function equal to \( \nu_n \). By construction, the distances from \( \sigma \) to \( \partial \mathcal{B}_n(t) \) converge to \( t \). Also, the mean curvature of \( \mathcal{B}_n \) is equal to \( H_n = H/\lambda_n \). Note that for every \( z_n \in B_{M_n}(q_n, r_n/2) \), a straightforward computation yields

\[
d_{M_n}(q_n, \partial D_n) \leq 2d_{M_n}(z_n, \partial D_n).
\]

(3.2)

According to \( (3.2) \), and because \( t/\lambda_n < r_n/2 \), we have the following estimate for the second fundamental form of each \( \mathcal{B}_n(t) \)

\[
|\sigma_{\mathcal{B}_n(t)}(z_n)| = \frac{|\sigma_n(z_n)|}{\lambda_n} = \frac{g_n(z_n)}{\lambda_n d_{M_n}(z_n, \partial D_n)} \leq \frac{g_n(q_n)}{\lambda_n d_{M_n}(z_n, \partial D_n)} = \frac{d_{M_n}(q_n, \partial D_n)}{d_{M_n}(z_n, \partial D_n)} \leq 2.
\]

(3.3)

This implies that the length of the second fundamental form of each \( \mathcal{B}_n(t) \) is uniformly bounded. Also, each \( \mathcal{B}_n(t) \) is the graph of a solution to an elliptic PDE for \( H_n \), and the
sequence $H_n = H/\lambda_n$ converges to zero. Moreover, the graphs $\mathcal{B}_n(t)$ have uniform $C^2$ estimates (since the lengths of their second fundamental forms are uniformly bounded, by (3.3)), and the length of the second fundamental form of each $\mathcal{B}_n(t)$ at the origin is equal to 1. Finally, each $\mathcal{B}_n(t)$ is contained in $\lambda_n D(0, t/\lambda_n)$.

This uniformly bound on the length of the second fundamental forms of $\mathcal{B}_n(t)$ allows us to prove that a subsequence of the surfaces $\mathcal{B}_n(t)$ converges uniformly on compact sets to a minimal surface $M_\infty(t) \subset \mathbb{B}(t)$ in $\mathbb{R}^3$. First, Proposition 2.3 in [RST] ensures us that there exist positive constants $\delta, \mu$, only depending on the bound given by (3.3), with the property that a neighbourhood of the origin in $\mathcal{B}_n(t)$ can be seen as a graph of a function $u_n$ defined in a disk $D_n(\delta)$ of its tangent plane, and such that $||u_n||_{C^2(D_n(\delta))} \leq \mu$. Denote by $\{(\eta_n)_\sigma\}$ to the image of the unit normal of $\mathcal{B}_n(t)$ at the origin. As all the images $(\eta_n)_\sigma$ lie in the unit sphere of the tangent space at $\sigma$, after passing to a subsequence we may assume that $\{(\eta_n)_\sigma\} \to \eta_\sigma$, with $\eta_\sigma$ being a constant unitary vector in $T_0 \mathbb{E}$. After making $\delta$ smaller (resp. $\mu$ larger) if necessary, we ensure that for $n$ large enough an open neighbourhood of the origin in $\mathcal{B}_n(t)$ is the graph of a function $z = u_n(x, y)$ over $D(0, \delta)$ and $||u_n||_{C^2(D(0, \delta))} \leq \mu$. As the mean curvatures of $\mathcal{B}_n(t)$ are uniformly bounded, they actually converge to zero, by Schauder estimates we conclude that the functions $u_n$ are uniformly bounded in the $C^{2, \alpha}$ topology in any disk $D(0, \delta_0) \subset D(0, \delta)$. Now a similar diagonal argument as the one used in the last part of Theorem 2.17 in [BGM], applying Arzela-Ascoli theorem, ensures us that a subsequence of the graphs $\mathcal{B}_n(t)$ converges uniformly on compact sets in the $C^2$ topology to a minimal surface $M_\infty(t)$ contained in $\mathbb{B}(t)$. Notice that $\bigcup_{t \geq 1} \mathbb{B}(t) = \mathbb{R}^3$, with its usual flat metric. Now it is clear that $M_\infty = \bigcup_{t \geq 1} M_\infty(t)$ is a complete minimal surface in $\mathbb{R}^3$, passing through the origin and with $|\sigma_{M_\infty}(0)| = 1$. In particular, as the angle functions of the graphs $\tilde{\mathcal{B}}_n$ are negative functions, the angle function of $M_\infty$ satisfies $\nu_\infty \leq 0$. This implies that if we denote by $\eta_\infty$ the Gauss map of the minimal surface $M_\infty$, then $\eta_\infty(M_\infty) \subset \mathbb{S}_2^2$, where $\mathbb{S}_2^2$ is the lower closed hemisphere of $\mathbb{S}^2$. By a classical result of Osserman, according to which the Gauss map image of a complete non-planar minimal surface in $\mathbb{R}^3$ is dense in $\mathbb{S}^2$., we deduce that $M_\infty$ is a plane. This contradicts the fact that the norm of the second fundamental form of all the $\mathcal{B}_n(t)$ at the origin is equal to 1, which completes the proof of Lemma 2.

Now, we stand in position to prove Theorem 1.

**Proof:** Arguing by contradiction, suppose that the height estimate in the statement of the theorem does not hold. Then, there exist a sequence of compact vertical $H$-graphs $M_n$, whose boundaries are contained in horizontal planes $\{z = z_n\}$, and such that if we denote by $h_n$ to the height of each $M_n$ to $\{z = z_n\}$, then $\{h_n\} \to \infty$. After a vertical translation we can suppose that all the boundaries are contained in the plane $\Pi = \{z = 0\}$. By the mean curvature comparison principle, each graph is contained in one of the half-spaces $\{z \geq 0\}$ or $\{z \leq 0\}$. After a rotation of angle $\pi$ around some horizontal geodesic contained in $\Pi$, we may assume that all the graphs $M_n$ lie above the $\Pi$, i.e. they lie in the half-space $\{z \geq 0\}$. Let $\eta_n$ be the unit normal to each $M_n$ such
that the mean curvature with respect to $\eta_n$ is $H$. In particular, each $M_n$ is downwards oriented as a consequence again of the mean curvature comparison principle, and thus every angle function $\nu_n = \langle \eta_n, \partial_z \rangle$ is a negative function on $M_n$. Again, fix some positive number $d$ and let us now denote $M_n^* := \{ p \in M_n : d(p, \partial M_n) > 2d \}$. As the heights of $M_n$ over $\Pi$ tend to infinity, it is clear that $M_n^*$ is a non-empty, possibly non-connected, graph over $\Pi$ for $n$ large enough. In this situation, Lemma 2 ensures us that there exists a positive constant $\Lambda$ in such a way that the second fundamental $\sigma_{M_n^*}$ form of each surface $M_n^*$ satisfy $|\sigma_{M_n^*}| < \Lambda$.

Consider for each $n$ the connected component $M_n^0$ of $M_n^*$ of maximum height over $\Pi$. Let $x_n \in M_n^0$ be the point where this maximum height is attained, and consider $\Phi_n$ the isometry that sends $x_n$ to the origin. Now, define $M_n^1 = \Phi_n(M_n^0)$. The length of the second fundamental form of each graph $M_n^1$ is uniformly bounded by $\Lambda > 0$, as they are obtained by translations of subsets of $M_n^*$. Moreover, the distances in $M_n^1$ of the origin to $\partial M_n^1$ diverge to $\infty$. By a similar compactness argument to the one we used in the proof of Lemma 2, we deduce that, up to a subsequence, there are compact sets $K_n \subset M_n^1$ that converge uniformly on compact sets in the $C^2$ topology to a complete, possibly non-connected, $H$-surface $M_\infty$ that passes through the origin. From now on, we will consider the connected component of $M_\infty$ that passes through the origin, and we will still denote this component by $M_\infty$. Let $\nu_\infty := \langle \eta_\infty, \partial_z \rangle$ denote the angle function of $M_\infty$, where here $\eta_\infty$ is the unit normal of $M_\infty$. Since $M_\infty$ is a limit of the downwards-oriented graphs $M_n^1$, we see that $\nu_\infty$ is non-positive. We claim that $\nu_\infty$ cannot be bounded away from zero; indeed, assume that $\nu_\infty^2 \geq c > 0$ for some $c > 0$. Consider the projection $p : M_\infty \to \mathbb{M}^2(\kappa)$, let $\langle \cdot, \cdot \rangle_{\text{proj}}$ be the induced metric on $M_\infty$ via $p$, and let $\langle \cdot, \cdot \rangle$ be the induced ambient metric on $M_\infty$.

As $\langle \cdot, \cdot \rangle$ is complete and it is well-known that $\nu_\infty^2 \langle \cdot, \cdot \rangle \leq \langle \cdot, \cdot \rangle_{\text{proj}}$, we conclude by $\nu_\infty^2 \geq c > 0$ that $\langle \cdot, \cdot \rangle_{\text{proj}}$ is also complete. In particular, $p$ is a local isometry from $(M_\infty, \langle \cdot, \cdot \rangle_{\text{proj}})$ onto $\mathbb{M}^2(\kappa)$. In these conditions, $p$ is necessarily a (surjective) covering map over the simply connected surface $\mathbb{M}^2(\kappa)$, and thus $M_\infty$ is an entire vertical graph. Let $S$ be the sphere with constant mean curvature $H$; the condition $4H^2 + \kappa > 0$ ensures us the existence of such a sphere for the case $\kappa < 0$. Then, we can translate $S$ by vertical translations until it touches $M_\infty$ in a first contact point in such a way that the unit normals of $S$ and $M_\infty$ agree. However, this situation would yield to a contradiction with the maximum principle. Therefore, there must exist a sequence of $p_n \in M_\infty$ with $\nu_\infty(p_n) \to 0$.

Let $\Theta_n$ be an isometry in $E$ that takes each point $p_n$ to the origin, and define $M_n^\infty = \Theta_n(M_\infty)$, which is a sequence of complete, stable surfaces with constant mean curvature $H$ passing through the origin and whose angle functions satisfy $\nu_n^\infty \leq 0$. Again, standard elliptic theory ensures that, up to a subsequence, the surfaces $M_n^\infty$ converges to a stable $H$-surface $M_\infty^*$, passing through the origin. As this convergence is $C^2$, the angle function $\nu_\infty^*$ of $M_\infty^*$ satisfies $\nu_\infty^* \leq 0$ and $\nu_\infty^*(0) = 0$. Also, the stability operators $\mathcal{L}_n$ converge to the stability operator $\mathcal{L}_\infty$ of the limit surface $M_\infty^*$.

The maximum principle for elliptic operators applied to $\mathcal{L}_\infty$, ensures us that any non-zero solution to (2.4) changes sign around any of its zeros. As $\mathcal{L}_\infty$ also admits the zero
function as a solution, and $\nu^*_\infty$ vanishes in a point, the condition $\nu^*_\infty \leq 0$ implies that $\nu^*_\infty$ is identically zero. Therefore the limit surface $M^*_\infty$ is contained in a flat cylinder $\gamma \times \mathbb{R}$, for a planar curve $\gamma$ in $\mathbb{R}^2$ or $\mathbb{H}^2$ (depending on whether $\kappa = 0$ or $\kappa < 0$, respectively). An analytic prolongation argument ensures that the maximal surface containing $M^*_\infty$ has to be the complete flat cylinder $\gamma \times \mathbb{R}$. This cylinder is an $H$-cylinder as well, and thus the geodesic curvature of $\gamma$ satisfies $\kappa_\gamma = 2H$. This implies that $\gamma$ is a closed curve in $\mathbb{R}^2$ or $\mathbb{H}^2$ (depending if $\kappa = 0$ or $\kappa < 0$, respectively). In the cylinder $\gamma \times \mathbb{R}$, the operator $L^*_\infty$ has the expression

$$L^*_\infty = \Delta_M + \kappa^2_\gamma + \kappa.$$ 

As all the surfaces $M^n_\infty$ are stable, the limit cylinder $M^*_\infty$ is also a stable surface. But a complete, vertical $H$-cylinder in an $E(\kappa, \tau)$ is stable if and only if $\kappa^2_\gamma + \kappa \leq 0$. Thus, the limit cylinder is stable if and only if $4H^2 + \kappa$ is a negative constant, which is a contradiction with the hypothesis $4H^2 + \kappa > 0$. This contradiction completes the proof of Theorem 1.

A straightforward consequence of Theorem (1) is the following structure-type result:

**Corollary 3** Let $H$ be a positive constant and suppose that

$$4H^2 + \kappa > 0.$$ 

Then, there do not exist proper vertical $H$-graphs defined on relatively compact domains $\Omega \subset \{z = z_0\}$ in the spaces $\text{Nil}_3$ and $\tilde{\text{PSL}}_2(\mathbb{R})$.

**Proof:** Let $M$ be a proper $H$-graph over a relatively compact domain $\Omega \subset \{z = z_0\}$. After a vertical translation and a rotation of angle $\pi$ around a horizontal geodesic, we can suppose that $M$ lies in the halfspace $\{z \leq 0\}$, and intersects tangentially the plane $\{z = 0\}$. Let $C$ be the constant appearing in Theorem 1. Then, as the height of $M$ with respect to the plane $\{z = 0\}$ is unbounded, there exists some $d_0 > 0$ such that if we intersect $M$ with the halfspace $\{z \geq -d_0\}$, we obtain a compact $H$-graph with boundary lying in the plane $\{z = -d_0\}$ and with height over the plane $\{z = -d_0\}$ greater than $C$, contradicting Theorem 1.

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