On special identities for dialgebras
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For every variety of algebras over a field, there is a natural definition of a corresponding variety of dialgebras (Loday-type algebras). In particular, Lie dialgebras are equivalent to Leibniz algebras. We use an approach based on the notion of an operad to study the problem of finding special identities for dialgebras. It is proved that all polylinear special identities for dialgebras can be obtained from special identities for corresponding algebras by means of a simple procedure. A particular case of this result confirms the conjecture by Bremner, Felipe, and Sánchez-Ortega [M.R. Bremner, R. Felipe, and J. Sánchez-Ortega, Jordan triple disystems, Comput. Math. Appl. 63 (2012), pp. 1039–1055].

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1. Introduction

The notion of a Leibniz algebra, appeared first in [2] and later independently in [14], gave rise to a series of research devoted to the theory of dialgebras. By definition, a (left) Leibniz algebra is a linear space with a bilinear operation $[,]$ which satisfies the Jacobi identity in the form $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$, i.e. the operator of left multiplication $[x,\cdot]$ is a derivation. This is one of the most studied noncommutative analogues of Lie algebras.

Various classes of dialgebras appeared in the literature since they are related to Leibniz algebras in the same way as the corresponding classes of ordinary algebras are related to Lie algebras. Associative dialgebras were introduced in [16] as analogues of associative enveloping algebras for Leibniz algebras. Alternative dialgebras appeared in [13] in the study of universal central extensions for Leibniz algebras, Jordan dialgebras (first under the name of quasi-Jordan algebras) were proposed in [19] (see also [3,11]). All dialgebras of these classes are linear spaces equipped with two bilinear operations $\cdot$ and $\cdot$ such that

\[(x \cdot y) \cdot z = (x \cdot y) \cdot z, \quad x \cdot (y \cdot z) = x \cdot (y \cdot z).\]  

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These identities are common for associative, alternative, Jordan dialgebras mentioned above, and they also hold for Leibniz algebras provided that \( a \dashv b = \{a, b\} \). Other defining identities of these varieties initially appeared from \textit{a posteriori} considerations motivated by relations with Leibniz algebras. For example, a dialgebra is associative if, in addition to (1), the following identities hold:

\[
\begin{align*}
    x \dashv (y \dashv z) &= (x \dashv y) \dashv z, \\
    x \dashv (y \dashv z) &= (x \dashv y) \dashv z,
\end{align*}
\]

Then the same space with respect to the new operation \([a, b] = a \dashv b - b \dashv a\) is a Leibniz algebra. A systematical study of this relation between Leibniz algebras and associative dialgebras may be found in [15].

An idea of a more conceptual approach to the definition of what should be called a dialgebra associated with a given variety \( \Psi \) of ordinary algebras was proposed in [7] in the case of associative algebras: it was shown that the operad governing the variety of associative dialgebras in the sense of [16] coincides with the Hadamard product \( \Psi \otimes \text{Perm} \), where \( \Psi \) is the operad governing the variety of associative algebras and \( \text{Perm} \) is the operad governing associative algebras satisfying the left commutativity relation \((xy)z = (yx)z = 0\).

For an arbitrary variety \( \Psi \) of ordinary algebras with one binary operation (governed by an operad \( \mathcal{P} \)), the algorithm proposed in [11,17] allows to deduce the defining identities for the class of \( (\mathcal{D}) \) -algebras governed by the operad \( \mathcal{P} \otimes \text{Perm} \) starting with the defining identities of \( \Psi \). In [4], this algorithm was generalized to the case of arbitrary varieties of algebras of any type (i.e. linear spaces with a family of polylinear operations of arbitrary arity). In this note, we will show that this generalized algorithm also leads to the class of \( \mathcal{P} \otimes \text{Perm} \)-algebras. This is why, we denote by \( \mathcal{P} \otimes \text{Perm} \) by \( \mathcal{D} \mathcal{P} \).

This observation leads us to a unified view on the elementary properties and relations on various classes of dialgebras. In particular, a morphism of operads \( \omega: \mathcal{P} \to \mathcal{R} \) always gives rise to a functor from the category of \( \mathcal{R} \)-algebras to the category of \( \mathcal{P} \)-algebras. So are the well-known functors:

| \( \mathcal{R} \) | \( \mathcal{P} \) | \( \omega \) |
|-------|-------|-----|
| Associative | Lie | \( x_1 x_2 \mapsto x_1 x_2 - x_2 x_1 \) |
| Associative | Jordan | \( x_1 x_2 \mapsto x_1 x_2 + x_2 x_1 \) |
| Alternative | Jordan | \( x_1 x_2 \mapsto x_1 x_2 + x_2 x_1 \) |
| Alternative | Mal’cev | \( x_1 x_2 \mapsto x_1 x_2 - x_2 x_1 \) |
| Associative | Jordan triple system | \( \langle x_1, x_2, x_3 \rangle \mapsto x_1 x_2 x_3 + x_3 x_2 x_1 \) |
| Jordan | Jordan triple system | \( \langle x_1, x_2, x_3 \rangle \mapsto (x_1 x_2) x_3 - (x_1 x_3) x_2 + x_1 (x_2 x_3) \) |

For each triple \((\mathcal{P}, \mathcal{R}, \omega)\) the following \textit{speciality problem} makes sense: Whether the variety generated by all those \( \mathcal{P} \)-algebras obtained from \( \mathcal{R} \)-algebras coincides with the class of all \( \mathcal{P} \)-algebras? If no, what are the identities separating these classes (special identities)?

Note that the varieties of \( \mathcal{D} \mathcal{R} \)- and \( \mathcal{D} \mathcal{P} \)-algebras are related by a functor raising from the morphism \( \omega \otimes \text{id}: \mathcal{D} \mathcal{P} = \mathcal{P} \otimes \text{Perm} \to \mathcal{R} \otimes \text{Perm} = \mathcal{D} \mathcal{R} \). The purpose of this note is to show that the speciality problem for dialgebras raising from the triple \((\mathcal{D} \mathcal{P}, \mathcal{D} \mathcal{R}, \omega \otimes \text{id})\) can always be solved modulo the same problem for ordinary algebras.
2. The BSO algorithm

Let us start with the construction from [4], assuming the base field \( k \) is of zero characteristic.

Let \( A \) be an associative algebra over \( k \) equipped with new \( n \)-ary operation

\[
\omega(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)},
\]

where \( \alpha_\sigma \in k \).

Choose an index \( i \in \{1, \ldots, n\} \). One may rewrite (2) as

\[
\omega(x_1, \ldots, x_n) = \sum_{j=1}^{n} \sum_{S_j^n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(j-1)} x_i x_{\sigma(j+1)} \cdots x_{\sigma(n)},
\]

where \( S_j^n \) is a set of permutations such that \( \sigma(j) = i \).

Denote by \( \text{Id}(\omega) \) the set of all polylinear identities satisfied by all \( n \)-ary algebras obtained in this way from associative ones.

Starting from the identities \( \text{Id}(\omega) \), one may canonically construct a set of identities \( \text{Id}(\omega)^{(2)} \) of type \( \{\omega_1, \ldots, \omega_n\} \), where each \( \omega_i \) is an \( n \)-ary operation. The algorithm of such a construction was described in [4] (as a KP algorithm), and also in Section 6.

On the other hand, consider the following operations on an associative dialgebra \( D \):

\[
\omega_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} \sum_{S_j^n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(j-1)} x_i x_{\sigma(j+1)} \cdots x_{\sigma(n)},
\]

\( i = 1, \ldots, n \) (the bracketing is not essential here). The family of the \( n \)-ary operations \( \omega_1, \ldots, \omega_n \) obtained are denoted by \( \text{BSO}(\omega) \). Let \( \text{Id}(\text{BSO}(\omega)) \) stand for the set of all polylinear identities satisfied by all algebras with \( n \)-ary operations \( \text{BSO}(\omega) \) obtained in this way from associative dialgebras.

**Problem 2.1** [4] Let \( \text{char} \ k = 0 \). Prove that for every choice of \( \omega \) we have \( \text{Id}(\text{BSO}(\omega)) = \text{Id}(\omega)^{(2)} \).

Next, suppose \( \text{char} \ k = p > 0 \). Then the relation in Problem 2.1 is not valid in general, but it is reasonable to state

**Problem 2.2** [4] For \( \text{char} \ k = p > 0 \) and \( d < p \), prove that for every choice of \( \omega \) we have \( \text{Id}_d(\text{BSO}(\omega)) = \text{Id}_d(\omega)^{(2)} \), where \( \text{Id}_d(\cdot) \) stands for the subset of identities of degree \( d \) in \( \text{Id}(\cdot) \).

In this article, we solve these problems.

3. Preliminaries in operads

In this section, we state the necessary notions of the operad theory following mainly [8] with a particular accent on the operads governing varieties of algebras.

A *language* \( \Omega \) is a set of functional symbols \( \{f_i \mid i \in I\} \) equipped with an *arity function* \( \nu: f_i \mapsto n_i = \nu(f_i) \in \mathbb{N} \). An \( \Omega \)-*algebra* is a linear space \( A \) over a base field \( k \).
endowed with linear maps \( f_i^j: A^\otimes n_i \rightarrow A, \ i \in I \ [12] \). Below, we will use the term algebra of type \( \Omega \) for an \( \Omega \)-algebra to avoid confusion. In this article, we assume \( n_i \geq 2 \).

Denote by \( \mathcal{F}_\Omega(X) \) the free algebra of type \( \Omega \) generated by the countable set \( X = \{ x_1, x_2, \ldots \} \). The linear basis of this algebra consists of all terms of type \( \Omega \) in variables from \( X \). Let us call such terms monomials, their linear combinations (elements of the free algebra) are called polynomials.

For every \( n \in \mathbb{N} \) consider the space \( \mathcal{F}_\Omega(n) \) of all polylinear polynomials of degree \( n \) in \( x_1, \ldots, x_n \). The composition

\[
\gamma_{m_1, \ldots, m_n}: \mathcal{F}_\Omega(n) \otimes \mathcal{F}_\Omega(n_1) \otimes \cdots \otimes \mathcal{F}_\Omega(n_m) \rightarrow \mathcal{F}_\Omega(m_1 + \cdots + m_n)
\]

of such maps is naturally defined by the rule

\[
\gamma_{m_1, \ldots, m_n}(f; g_1, \ldots, g_n) = f(g_1(x_1, \ldots, x_{m_1}), g_2(x_{m_1+1}, \ldots, x_{m_1+m_2}), \ldots),
\]

where \( f(x_1, \ldots, x_n) \in \mathcal{F}_\Omega(n), g_i(x_1, \ldots, x_{m_i}) \in \mathcal{F}_\Omega(m_i), i = 1, \ldots, n \); the result belongs to \( \mathcal{F}_\Omega(m_1 + \cdots + m_n) \). The simplest term \( x_1 \in \mathcal{F}_\Omega(1) \) behaves as an identity with respect to this composition. Symmetric groups \( S_n \) act on \( \mathcal{F}_\Omega(n) \) by permutations of variables.

The collection of spaces \( \{ \mathcal{F}_\Omega(n) \}_{n \in \mathbb{N}} \) together with the above-mentioned composition rule, identity element and \( S_n \)-action is a particular case of an operad which is natural to call the free operad \( \mathcal{F}_\Omega \) generated by \( \Omega \).

Given two operads \( \mathcal{P} \) and \( \mathcal{R} \), a morphism \( \alpha: \mathcal{P} \rightarrow \mathcal{R} \) is just a family of \( S_n \)-linear maps \( \{ \alpha(n) \}_{n \in \mathbb{N}} \),

\[
\alpha(n): \mathcal{P}(n) \rightarrow \mathcal{R}(n)
\]

preserving the composition and the identity element. The kernel of \( \alpha \) is the collection of subspaces (even \( S_n \)-submodules) \( \ker \alpha(n) \subseteq \mathcal{P}(n), \ n \in \mathbb{N} \), which is closed with respect to compositions in the obvious sense. Such a family of subspaces is called an operad ideal in \( \mathcal{P} \).

To define a morphism \( \pi \) from \( \mathcal{F}_\Omega \) to an operad \( \mathcal{P} \) it is enough to determine \( \pi(n_i)(f_i), \ f_i = f_i(x_1, \ldots, x_{n_i}) \in \mathcal{F}_\Omega(n_i) \), where \( f_i \) range through the language \( \Omega \), \( n_i = \nu(f_i) \). Moreover, every family \( g_i \in \mathcal{P}(n_i), \ i \in I \), defines a unique morphism of operads \( \pi: \mathcal{F}_\Omega \rightarrow \mathcal{P} \) such that \( \pi(n_i)(f_i) = g_i \).

Every linear space \( A \) gives rise to an operad \( \mathcal{E}(A) \), the operad of endomorphisms of \( A \). Namely, \( \mathcal{E}(A)(n) = \text{Hom}(A^\otimes n, A), \ n \in \mathbb{N} \), compositions and \( S_n \)-actions are defined in the ordinary way.

A structure of an algebra of type \( \Omega \) on a linear space \( A \) may be identified with a morphism of operads \( \alpha: \mathcal{F}_\Omega \rightarrow \mathcal{E}(A) \) such that \( \alpha(n_i)(f_i) = f_i^j, \ i \in I \). Conversely, every morphism \( \alpha: \mathcal{F}_\Omega \rightarrow \mathcal{E}(A) \) defines a structure of an algebra on \( A \).

Suppose \( \Psi \) is a variety of algebras of type \( \Omega \) defined by polynlinear identities (this is a generic case if \( \text{char} \mathbb{k} = 0 \)). Then the following consideration makes sense.

Let \( \mathcal{T}(\Psi) \) be the ideal of identities (T-ideal) in \( \mathcal{F}_\Omega(X) \) corresponding to the variety \( \Psi \). Denote \( \mathcal{P}(n) = \mathcal{F}_\Omega(n)/(\mathcal{T}(\Psi) \cap \mathcal{F}_\Omega(n)) \), \( n \in \mathbb{N} \). The composition rule and \( S_n \)-actions are well-defined on the family \( \{ \mathcal{P}(n) \}_{n \in \mathbb{N}} \), so this collection is also an operad. Such an operad is said to be the governing operad for the variety \( \Psi \). There exists a natural quotient morphism \( \pi: \mathcal{F}_\Omega \rightarrow \mathcal{P} \). If \( S \) is a defining family of polynlinear identities of the variety \( \Psi \), then the kernel of \( \pi \) is exactly the operad ideal generated by \( S \) in \( \mathcal{F}_\Omega \).
Every algebra $A$ from the variety $\mathfrak{P}$ is determined by a composition $\pi \circ \tilde{a}$, where $\tilde{a}$ is a morphism from $\mathcal{P}$ to $\mathcal{E}(A)$. Thus, $A$ is defined by a morphism of operads $\mathcal{P} \to \mathcal{E}(A)$. Conversely, every morphism of this kind defines an algebra structure on $A$, and the obtained algebra belongs to $\mathfrak{P}$.

In general, given an operad $\mathcal{P}$, a $\mathcal{P}$-algebra is a pair $(A, \alpha)$ of a linear space $A$ and a morphism of operads $\alpha: \mathcal{P} \to \mathcal{E}(A)$.

4. Conformal algebras

The notion of a conformal algebra was introduced in [10] as a tool for studying vertex operator algebras. In a more general context, a conformal algebra is a pseudo-algebra over the polynomial algebra $k[T]$ in one variable [1]. Here we consider the last approach for the arbitrary set of operations $\Omega$.

As we have already mentioned, every linear space $A$ gives rise to the operad $\mathcal{E}(A)$. A similar construction exists for left unital modules over a cocommutative bialgebra $H$. Suppose $M$ is such a module, then denote

$$\mathcal{E}^*(M)(n) = \text{Hom}_{H\text{-mod}}(M^{\otimes n}, H^{\otimes n} \otimes_H M).$$

Hereafter, the symbol $\otimes$ without a subscript stands for the tensor product of spaces over the base field. The space $H^{\otimes n}$ is considered as the outer product of regular right $H$-modules, i.e.

$$(h_1 \otimes \cdots \otimes h_n) \cdot h = \sum_{(h)} h_1 h_{(1)} \otimes \cdots \otimes h_n h_{(n)},$$

where $\sum_{(h)} h_{(1)} \otimes \cdots \otimes h_{(n)}$ is the value of $n$-iterated coproduct on $h$. Compositions $\gamma_{m_1, \ldots, m_n}$ of such maps and the action of $S_n$ on $\mathcal{E}^*(M)(n)$ were defined in [1] (see also [11]) (one needs cocommutativity of $H$ to ensure the action of $S_n$ is well-defined).

A conformal algebra over $H$ is a pair $(C, \alpha)$, where $C$ is an $H$-module as above and $\alpha: \mathcal{F}_\Omega \to \mathcal{E}^*(C)$ is a morphism of operads. If $\alpha$ splits into $\mathcal{F}_\Omega \to \mathcal{P} \overset{\beta}{\to} \mathcal{E}^*(C)$ then $C$ is said to be a $\mathcal{P}$-conformal algebra.

A simple but important example of a conformal algebra may be constructed as follows. Let $(A, \alpha)$ be a $\mathcal{P}$-algebra. Consider the free $H$-module $C = H \otimes A$ and define $\beta = \text{Cur} \alpha: \mathcal{P} \to \mathcal{E}^*(C)$ by the rule

$$\beta(n)(f): (h_1 \otimes a_1) \otimes \cdots \otimes (h_n \otimes a_n) \mapsto (h_1 \otimes \cdots \otimes h_n) \otimes_H \alpha(f)(a_1 \otimes \cdots \otimes a_n),$$

$f \in \mathcal{P}(n)$, $h_k \in H, a_k \in A$. This is a morphism of operads, and the $\mathcal{P}$-conformal algebra $(C, \beta)$ obtained is denoted by $(\text{Cur} A, \text{Cur} \alpha)$, the current conformal algebra over $A$.

The correspondence $A \mapsto \text{Cur} A$ is a functor from the category of $\mathcal{P}$-algebras to the category of $\mathcal{P}$-conformal algebras: every morphism $\varphi$ between $\mathcal{P}$-algebras can be continued by $H$-linearity to the morphism $\text{Cur} \varphi$ of the corresponding current algebras.

5. The operad $\text{Perm}$

The operad $\text{Perm}$ introduced in [7] is given by a family of spaces $\text{Perm}(n) = k^n$ with natural composition rule

$$\gamma_{m_1, \ldots, m_n} : e_k^{(m)} \otimes e_{j_1}^{(m_1)} \otimes \cdots \otimes e_{j_n}^{(m_n)} = e_{m_1 + \cdots + m_k + j_k}^{(m_1 + \cdots + m_n)},$$
where $\xi_k^{(n)}$, $k = 1, \ldots, n$, is the standard basis of $k^n$, $n \in \mathbb{N}$. Symmetric groups $S_n$ act on $\text{Perm}(n)$ by permutations of coordinates.

Let $\mathcal{P}$ be an operad. Denote by $\text{di} \mathcal{P}$ the Hadamard product $\mathcal{P} \otimes \text{Perm}$: $\text{di} \mathcal{P}(n) = \mathcal{P}(n) \otimes \text{Perm}(n)$, compositions and $S_n$-action are defined in the component-wise way.

Let us fix a cocommutative bialgebra $H$, and let $\varepsilon$ stand for its counit. A left unital $H$-module $C$ is in particular a linear space over the base field $k$. For every $n \in \mathbb{N}$ consider $k$-linear maps $\mu_n^k$, $k = 1, \ldots, n$, from $H^{\otimes n} \otimes H C$ to $C$ defined by

$$\mu_n^k : (h_1 \otimes \cdots \otimes h_n) \otimes H c \mapsto \varepsilon(h_1 \cdots h_n)h_k c.$$  

**Lemma 5.1** If $(C, \alpha)$ is a $\mathcal{P}$-conformal algebra then the family of maps $\{\alpha^{(0)}(n)\}_{n \in \mathbb{N}}$ $\alpha^{(0)}(n): \text{di} \mathcal{P}(n) \to \mathcal{E}(C)(n)$, defined by

$$\alpha^{(0)}(n)(f \otimes \xi_k^{(n)}) = \alpha(n)(f) \circ \mu_n^k,$$

$f \in \mathcal{P}(n)$, $k = 1, \ldots, n, a_j \in C$, defines a morphism $\alpha^{(0)}$ of operads.

**Proof** First, note that $\alpha^{(0)}(n)$ is $S_n$-linear. Indeed,

$$\alpha^{(0)}(n) : (f \otimes \xi_k^{(n)})^\sigma = f^\sigma \otimes \xi_{\alpha(k)}^{(n)} \mapsto \alpha(n)(f)^\sigma \circ \mu_n^k(\alpha) = (\alpha(n)(f) \circ \mu_n^k)^\sigma$$

since the action of $\sigma$ on $\mathcal{E}^*(C)$ permutes the arguments of $\alpha(n)(f)$ together with tensor factors in $H^{\otimes n} \otimes HC$ (see [11]).

Next, this is obvious that $\alpha^{(0)}(1)$ preserves the identity.

Finally, consider a composition $\gamma_{m_1, \ldots, m_n}(f; g_1, \ldots, g_n)$ in $\mathcal{P}$. By abuse of notations, assume

$$\alpha(m_l)(g_l) : a_l^{(1)} \otimes \cdots \otimes a_{m_l}^{(1)} \mapsto F_l^{(1)} \otimes H b_l^{(1)}, \quad F_l^{(1)} \in H^{\otimes m_l},$$

$l = 1, \ldots, n$, and

$$\alpha(n)(f) : b^{(1)} \otimes \cdots \otimes b^{(n)} \mapsto G \otimes H c, \quad G \in H^{\otimes n},$$

$a_j^{(l)}, b_l^{(l)}, c \in C$. Then by definition

$$\alpha(m_1 + \cdots + m_n)(\gamma_{m_1, \ldots, m_n}(f; g_1, \ldots, g_n)) :$$

$$a_1^{(1)} \otimes \cdots \otimes a_{m_1}^{(1)} \otimes \cdots \otimes a_1^{(n)} \otimes \cdots \otimes a_{m_n}^{(n)}$$

$$\mapsto (F_1^{(1)} \otimes \cdots \otimes F_n^{(n)} \otimes H 1)(\Delta^{[m_1]} \otimes \cdots \otimes \Delta^{[m_n]})G \otimes H c,$$

where $\Delta^{[m]}(h) = \sum_{(h)} h_1 \otimes \cdots \otimes h_n, h \in H$.

Let us fix some $k \in \{1, \ldots, n\}$, $j_k \in \{1, \ldots, m_l\}$, $l = 1, \ldots, n$. Suppose $F_l^{(1)} = h_1^{(1)} \otimes \cdots \otimes h_{l-1}^{(1)} \otimes h_l^{(k)}$, $G = h_1 \otimes \cdots \otimes h_n$. Then by the properties of the counit $\varepsilon$ the image of the right-hand side of (4) under $\mu_n^{m_1 + \cdots + m_n + k}$ is equal to

$$\varepsilon(F_1^{(1)}) \cdots \varepsilon(F_n^{(n)}) \varepsilon(F_k^{(k)}) \varepsilon(G_k)h_k^{(k)} h_k c,$$

where $G_k = h_1 \cdots h_n$ and $F_k^{(k)}$ is defined similarly.
On the other hand, let us compute the composition
\[ \gamma_{m_1, \ldots, m_n}(\alpha^{(0)}(n)(f \otimes e_k^{(n)}); \alpha^{(0)}(m_1)(g_1 \otimes e_j^{(m_1)}), \ldots, \alpha^{(0)}(m_n)(g_n \otimes e_j^{(m_n)})) \]
in \( E(C) \). By (3), we have
\[ \alpha^{(0)}(m_1)(g_1 \otimes e_j^{(m_1)}) : a_1^{(l)} \otimes \cdots \otimes e_{m_1}^{(l)} \mapsto \varepsilon(F_{h_1}) h_{j_1}^{(l)} h_{j_1}^{(1)} \].

The \( H^{\otimes n} \)-linearity of \( \alpha(n)(f) \) implies
\[ \alpha(n)(f) : \varepsilon(F_{h_1}) h_{j_1}^{(1)} h_{j_1}^{(1)} \otimes \cdots \otimes \varepsilon(F_{h_n}) h_{j_n}^{(n)} h_{j_n}^{(n)} \]
\[ \mapsto \varepsilon(F_{h_1}) \cdots \varepsilon(F_{h_n}) (h_{j_1}^{(1)} \otimes \cdots \otimes h_{j_n}^{(n)}) G \otimes_H c. \quad (6) \]

It is now obvious that the image of the right-hand side of (6) under \( \mu_n^k \) coincides with (5). \( \blacksquare \)

Hence, every \( \mathcal{P} \)-conformal algebra \((C, \alpha)\) gives rise to a \( \text{di} \mathcal{P} \)-algebra \( C^{(0)} = (C, \alpha^{(0)}) \). The correspondence \( C \mapsto C^{(0)} \) is obviously a functor from the category of \( \mathcal{P} \)-conformal algebras to the category of \( \text{di} \mathcal{P} \)-algebras.

6. Dialgebras

Suppose \( \mathcal{P} \) is a quotient operad of \( \mathcal{F}_\Omega \), \( \pi \) is the corresponding morphism. It is easy to see that \( \text{di} \mathcal{P} \) is a quotient of \( \text{di} \mathcal{F}_\Omega = \mathcal{F}_\Omega \otimes \text{Perm} \) with respect to the morphism \( \pi \otimes \text{id} \). In general, it is hard to determine the generators and defining relations of the Hadamard product of two operads, but due to the nice properties of \( \text{Perm} \) this is easy to do for \( \text{di} \mathcal{P} \).

Consider the free operad \( \mathcal{F}_{\Omega^{(2)}} \), where the language \( \Omega^{(2)} \) is constructed in the following way. If \( \Omega = \{f_i \mid i \in I\}, n_i = \nu(f_i) \), then \( \Omega^{(2)} = \{f_i^{(k)} \mid i \in I, k = 1, \ldots, n_i\}, \nu(f_i^{(k)}) = n_i \).

Define the morphism \( \zeta_{\Omega} : \mathcal{F}_{\Omega^{(2)}} \to \text{di} \mathcal{F}_\Omega \) in the following way: \( \zeta_{\Omega}(n) \) maps \( f_i^{(k)}(x_1, \ldots, x_n) \in \mathcal{F}_{\Omega^{(2)}}(n) \) to \( f_i(x_1, \ldots, x_n) \otimes e_k^{(n)} \). The composition of \( \zeta_{\Omega} \) with \( \pi \otimes \text{id} \) provides a morphism \( \pi^{(2)} : \mathcal{F}_{\Omega^{(2)}} \to \text{di} \mathcal{P} \).

**Lemma 6.1** For every \( n \in \mathbb{N} \) the linear maps \( \zeta_{\Omega}(n) \) and \( \pi^{(2)}(n) \) are surjective.

**Proof** To prove the surjectivity of \( \zeta_{\Omega} \) (and hence of \( \pi^{(2)} \)) it is enough to show that \( \text{di} \mathcal{F}_\Omega \) is generated by \( f_i \otimes e_k^{(n)} \), \( i \in I, k = 1, \ldots, n_i \). In the binary case it was actually done in [11,18]. The general case can be processed analogously. It is enough to construct a section \( \rho(n) : (\mathcal{F}_\Omega \otimes \text{Perm})(n) \to \mathcal{F}_{\Omega^{(2)}}(n) \) such that \( \rho(n) \circ \zeta_{\Omega} (n) = \text{id} \). It was done in [4]. Let us recall here the construction in terms of planar trees. Every monomial \( f \in \mathcal{F}_\Omega(n) \) can be identified with a planar tree with \( n \) leaves (variables) labelled by numbers \( 1, \ldots, n \) and vertices labelled by symbols from \( \Omega \), the degree (number of outgoing branches) of a vertex labelled by \( f_i \in \Omega \) is equal to \( n_i \). Then \( f \otimes e_k^{(n)} \) may be considered as a tree with \( k \)th emphasized vertex. To get \( \rho(n)(f \otimes e_k^{(n)}) \) we should add superscripts to the labels of vertices in the tree corresponding to \( f \) in the following way. If a \( k \)th (counting from the left-hand side) outgoing branch of a vertex labelled by \( f_i \in \Omega \) contains the emphasized leaf then the label is replaced
with \( f_i^k \). If neither of the outgoing branches in this vertex contain the emphasized leaf then the label is replaced with \( f_i^1 \).

Suppose \( S \) is a set of polylinear polynomials such that the kernel of \( \pi \) is generated by \( S \) (e.g. if \( P \) is a governing operad for a variety \( \Psi \) then \( S \) consists of its defining identities). Consider the operad ideal \( J(S) \) in \( \mathcal{F}_{\Omega(2)} \) generated by

\[
f^k(x_1, \ldots, x_{j-1}, g^l(x_j, \ldots, x_{j+m-1}), x_{j+m}, \ldots, x_{n+m-1}) - f^k(x_1, \ldots, x_{j-1}, g^l(x_j, \ldots, x_{j+m-1}), x_{j+m}, \ldots, x_{n+m-1})
\]

\[
f, g \in \Omega, \ n = \nu(f), \ m = \nu(g), \ k, j = 1, \ldots, n, \ k \neq j, \ l, p = 1, \ldots, m,
\]

and

\[
s^k(x_1, \ldots, x_n), \ s \in S \cap \mathcal{F}_\Omega(n), \ n \in \mathbb{N}, \ k = 1, \ldots, n,
\]

where \( s^k = \rho(n)(s \otimes \varepsilon^{(n)}_k) \). Denote by \( \mathcal{P}^{(2)} \) the quotient operad of \( \mathcal{F}_{\Omega(2)} \) with respect to \( J(S) \), and let \( \tilde{\pi}^{(2)} \) be the corresponding morphism from \( \mathcal{F}_{\Omega(2)} \) to \( \mathcal{P}^{(2)} \).

It is easy to see that \( J(S) \) is contained in the kernel of \( \pi^{(2)} \). Thus we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{F}_{\Omega(2)} & \xrightarrow{\zeta_\Omega} & \mathcal{F}_\Omega \otimes \text{Perm} \\
\tilde{\pi}^{(2)} \downarrow & & \downarrow \pi \otimes \text{id} \\
\mathcal{P}^{(2)} & \longrightarrow & \mathcal{P} \otimes \text{Perm} \\
& \xrightarrow{\text{pr}} & \mathcal{P}
\end{array}
\]

where pr stands for the natural projection.

Our aim is to show that the kernels of \( \tilde{\pi}^{(2)} \) and \( \pi^{(2)} \) are equal.

Suppose \( (A, \alpha) \) is a \( \mathcal{P}^{(2)} \)-algebra. By abuse of notations, let us identify \( f^k \in \mathcal{F}_{\Omega(2)}(n) \) and their images under \( \tilde{\pi}^{(2)}(n) \) in \( \mathcal{P}^{(2)}(n) \).

Let \( A_0 \) be the \( \mathbb{K} \)-linear span of all

\[
\alpha(n)(f^p_i - f^l_i)(a_1, \ldots, a_n), \ i \in I, \ a_j \in A, \ p, l = 1, \ldots, n_j.
\]

It follows from (7) that \( A_0 \) is an ideal in the algebra \( A \). Indeed, for every \( f \in \Omega \)

\[
\alpha(n)(f^k)(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n) = 0, \ b \in A_0,
\]

if \( j \neq k \) (\( f \in \Omega, \ n = \nu(f), \ k = 1, \ldots, n \)). For \( j = k \), one may add \( \alpha(n)(f^q)(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n) \) with \( q \neq k \) (which is zero) and make sure the result is again in \( A_0 \).

Denote \( \tilde{A} = A/A_0 \). The morphism \( \alpha \) induces a morphism \( \tilde{\alpha} : \mathcal{P}^{(2)} \to \mathcal{E}(\tilde{A}) \), such that \( (\tilde{A}, \tilde{\alpha}) \) is the quotient \( \mathcal{P}^{(2)} \)-algebra. In this algebra, the values of algebraic operations \( \alpha(n)(f^k), \ f \in \Omega, \ n = \nu(f), \ k \in \{1, \ldots, n\} \) do not depend on \( k \), so this is actually an algebra of language \( \Omega \):

\[
f^k(\tilde{a}_1, \ldots, \tilde{a}_n) = \overline{\alpha(n)(f^k)(a_1, \ldots, a_n)}, \ f \in \Omega, \ n = \nu(f),
\]

where \( a_j \in A, \ \tilde{a} = a + A_0 \in \tilde{A} \). Moreover, it is obvious that \( \tilde{A} \) is actually a \( \mathcal{P} \)-algebra.
Consider the formal direct sum of spaces $\hat{A} = \hat{A} \oplus A$ and define algebraic operations of language $\Omega$ on $\hat{A}$ as follows:

$$g^{\hat{A}}(z_1, \ldots, z_n) = \begin{cases} g^{\hat{A}}(z_1, \ldots, z_n), & z_i \in \hat{A} \text{ for all } i = 1, \ldots, n; \\ a(n)(g^k(a_1, \ldots, a_n)), & z_i = \hat{a}_i \in \hat{A} \text{ for all } i \neq k, \ z_k = a_k \in A; \\ 0, & \text{more than one } z_i \in A, \end{cases}$$

g \in \Omega, \ \nu(g) = n. \text{ Denote by } \hat{\alpha} \text{ the corresponding morphism from } F_{\Omega} \text{ to } E(\hat{A}).$

The definition of the canonical section $\rho$ from Lemma 6.1 and induction on $n$ imply that for every $s \in F_{\Omega}(n), \ s^{\hat{A}} := \hat{\alpha}(n)(s),$ we have

$$s^{\hat{A}}(z_1, \ldots, z_n) = \rho(n)(s \otimes e_k^{(n)}(a_1, \ldots, a_n)) \in A \subseteq \hat{A}$$

if $z_i = \hat{a}_i \in \hat{A}$ for all $i \neq k$ and $z_k = a_k \in A.$ Therefore, every $s \in S$ is an identity on $\hat{A},$ so $(\hat{A}, \hat{\alpha})$ is actually a $P$-algebra.

**Theorem 6.2** The kernels of $\pi^{(2)}$ and $\hat{\pi}^{(2)}$ coincide, so the operads $P^{(2)}$ and $\hat{\pi}$ are equivalent.

**Proof** We have already seen that the kernel of $\hat{\pi}^{(2)}$ is contained in the kernel of $\pi^{(2)}.$ Conversely, assume there exists an identity that holds for all $\hat{\pi}$-algebras but does not hold for some $P^{(2)}$-algebra $(A, \alpha).$

Consider the $P$-algebra $(\hat{A}, \hat{\alpha})$ constructed above and fix a bialgebra $H$ with a nonzero $T \in H$ such that $e(T) = 0.$ For example, one may consider the group algebra $H = \mathbb{C}Z_2.$

The current conformal algebra $\text{Cur } \hat{A} = H \otimes \hat{A}$ is a $P$-conformal algebra. ByLemma 5.1, $(\text{Cur } \hat{A})^{(0)}$ is a $\hat{\pi}$-algebra. Note that

$$A \rightarrow (\text{Cur } \hat{A})^{(0)}, \ a \mapsto 1 \otimes \hat{a} + T \otimes a, \ a \in A$$

is an injective homomorphism of $\Omega^{(2)}$-algebras. Hence, $A$ is in fact $\hat{\pi}$-algebra and thus satisfies all identities that hold for the class of such algebras. ■

**Corollary 6.3** Every di $P$-algebra $A$ is embedded into $(\text{Cur } \hat{A})^{(0)}$ over an appropriate bialgebra $H, \hat{A}$ is a $P$-algebra.

**Lemma 6.4** Consider $t = t(x_1, \ldots, x_d) \in \pi(n).$ Then $t = t_1 \otimes e_1^{(n)} + \cdots + t_n \otimes e_n^{(n)}, \ t_k \in \pi(n).$ Let $(A, \alpha)$ be a $P$-algebra. Then the dialgebra $(\text{Cur } A)^{(0)}$ satisfies the identity $t = 0$ if and only if $A$ satisfies all identities $t_k = 0, k = 1, \ldots, d.$

**Proof** It follows from the construction that if $\alpha(n)(t_k) = 0$ for all $k = 1, \ldots, n$ then $(\text{Cur } \alpha)(n)(t_k) = 0,$ and hence $(\text{Cur } \alpha)^{(0)}(n)(t) = 0.$

Conversely, consider $g = (\text{Cur } \alpha)^{(0)}(n)(t) \in E(H \otimes A)(n)$ and compute

$$b_k = g(1 \otimes a_1, \ldots, T \otimes a_k, \ldots, 1 \otimes a_n), \quad k = 1, \ldots, n,$$

for all $a_1, \ldots, a_n \in A.$ On one hand, $b_k = 0$ since $(\text{Cur } A)^{(0)}$ satisfies the identity $t = 0.$ On the other hand, $b_k = T \otimes \alpha(n)(t_k)(a_1, \ldots, a_n),$ so $A$ satisfies $t_k = 0$ for all $k.$ ■

### 7. Morphisms of operads and functors

If $P$ and $R$ are two operads then every morphism $\alpha: P \rightarrow R$ gives rise to a functor from the category of $R$-algebras to the category of $P$-algebras. Namely, if $(A, \beta)$ is an
$\mathcal{R}$-algebra then the same space $A$ with respect to the composition $\alpha \circ \beta : \mathcal{P} \to \mathcal{E}(A)$ is a $\mathcal{P}$-algebra. The correspondence $(A, \beta) \mapsto (A, \alpha \circ \beta)$ is obviously functorial. The construction that appears in Problem 2.1 is a particular case of such a functor.

Indeed, assume $\Omega$ and $\Xi$ are two languages, $\mathcal{F}_\Omega$ and $\mathcal{F}_\Xi$ are two corresponding free operads. Suppose $\pi: \mathcal{F}_\Omega \to \mathcal{P}$ and $\rho: \mathcal{F}_\Xi \to \mathcal{R}$ are two quotient morphisms to operads $\mathcal{P}$ and $\mathcal{R}$ governing some varieties of algebras.

Let $\omega: \mathcal{F}_\Omega \to \mathcal{F}_\Xi$ be a morphism of operads. The family of maps $\omega(n): \mathcal{F}_\Omega(n) \to \mathcal{F}_\Xi(n)$ determines (and can be completely determined by) an interpretation of operations from $\Omega$ via operations from $\Xi$. We say that $\omega$ induces a morphism $\tilde{\omega}: \mathcal{P} \to \mathcal{R}$ if and only if $\text{Ker} \, \pi(n) \subseteq \text{Ker} \, (\omega(n) \circ \rho(n))$ for all $n \in \mathbb{N}$.

Example 7.1 Let $\Omega$ and $\Xi$ contain one binary operation denoted by $\lambda$ in $\Omega$ and $\mu$ in $\Xi$, then the morphism $\omega$ determined by the rule $\lambda = \mu - \mu^{(12)}$ (12) $\in S_2$, induces a morphism $\tilde{\omega}$ from the operad Lie (governing the variety of Lie algebras) to the operad As (associative algebras). If $(A, \alpha)$ is an As-algebra then the pair $(A, \tilde{\omega} \circ \alpha)$ is exactly the adjoint Lie-algebra of $A$ (usually denoted by $A^{\text{adj}}$).

The Hadamard product $\tilde{\omega} \otimes \text{id} : \text{diLie} \to \text{diAs}$ defines the corresponding functor from the category of associative dialgebras to the category of Leibniz algebras (Lie dialgebras) [15].

A similar relation holds for Mal’cev dialgebras [6] and alternative dialgebras [13].

Example 7.2 Let $\text{JTS}$ be the operad governing the variety of Jordan triple systems (see, e.g. [9]). Then there exists an $\tilde{\omega}: \text{JTS} \to \text{As}$ defined as follows: if $\tau = (\cdot, \cdot, \cdot) \in \text{JTS}(3)$ is the triple operation on JTS-algebras and $\mu \in \text{As}(2)$ is the product on associative algebras then $\tilde{\omega}(\tau) = \gamma_{1,2}(\mu; \text{id}, \mu) + \gamma_{1,2}(\mu; \mu, \text{id})^{(13)}$. This is the well-known construction of a Jordan triple system on an associative algebra: $(a, b, c) = abc + cba$.

In [4], the notion of a Jordan triple disystem (JTD) was introduced in such a way that $\text{JTD} = \text{JTS}^{(2)}$, in our notations. Theorem 6.2 immediately implies $\text{JTD} = \text{diJTS} = \text{JTS} \otimes \text{Perm}$. Hence, $\tilde{\omega} \otimes \text{id} : \text{JTD} \to \text{diAs}$ defines a structure of a Jordan triple disystem on an associative dialgebra (cf. [4, Theorem 5.10]).

Example 7.3 Let Jord stand for the operad governing the variety of Jordan algebras, $\mu \in \text{Jord}(2)$ is the commutative operation. Then there exists a morphism $\tilde{\omega}: \text{JTS} \to \text{Jord}$ defined by $\tilde{\omega}(\tau) = \gamma_{2,1}(\mu; \mu, \text{id}) - \gamma_{2,1}(\mu; \mu, \text{id})^{(23)} + \gamma_{1,2}(\mu; \text{id}, \mu)$, i.e. $(a, b, c) = (ab)c - (ac)b + a(bc)$.

The notion of a Jordan dialgebra was studied in [3,11,19] (see also [5]). As in the previous example, Theorem 6.2 gives a new proof of the relationship between Jordan dialgebras and Jordan triple disystems [4, Theorem 7.3].

Let us fix two quotient morphisms $\pi: \mathcal{F}_\Omega \to \mathcal{P}$, $\rho: \mathcal{F}_\Xi \to \mathcal{R}$, and a morphism $\omega: \mathcal{F}_\Omega \to \mathcal{F}_\Xi$ inducing a morphism from $\mathcal{P}$ to $\mathcal{R}$ which is also denoted by $\omega$ for simplicity.

Definition 7.4 A $\mathcal{P}$-algebra $(A, \alpha)$ is called $\omega$-special if there exists an $\mathcal{R}$-algebra $(A, \beta)$ such that $\alpha = \omega \circ \beta$. The same notion makes sense for conformal algebras over an arbitrary comutative bialgebra $H$.

Lemma 7.5 If $(A, \alpha)$ is an $\omega$-special $\mathcal{P}$-algebra then $(\text{Cur} \, A, \text{Cur} \, \alpha)$ is an $\omega$-special $\mathcal{P}$-conformal algebra.
Then the claim follows from the observation
\[ \omega \circ \text{Cur} \beta = \text{Cur} (\omega \circ \beta). \] (9)

Indeed, for every \( f \in P(n) \) the pseudo-linear maps \( (\omega \circ \text{Cur} \beta)(n)(f) \in E^*(H \otimes A) \) are completely defined by their values at \( (1 \otimes a_1, \ldots, 1 \otimes a_n) \), \( a_i \in A \). By the definition of \( \text{Cur} \), we have \( (\text{Cur} \beta)(n)(g)(1 \otimes a_1, \ldots, 1 \otimes a_n) = (1 \otimes \ldots \otimes 1) \otimes H (\beta(n)(g))(a_1, \ldots, a_n) \) for every \( g \in R(n) \), in particular, for \( g = \omega(n)(f) \). This is now easy to see that left- and right-hand sides of (9) coincide at every \( f \in P(n) \).

**Lemma 7.6** If \((C, \alpha)\) is an \( \omega \)-special \( P \)-conformal algebra then \((C, \alpha^{(0)})\) is an \((\omega \otimes \text{id})\)-special \( \text{di} P \)-algebra.

**Proof** It is enough to show that \((\omega \circ \beta)^{(0)} = (\omega \otimes \text{id}) \circ \beta^{(0)}\) for every \( \beta: R \to E^*(C) \). Relation (3) implies
\[ (\omega \circ \beta)^{(0)}(n): f \otimes e_k^{(n)} \mapsto (\omega \circ \beta)(n)(f) \circ \mu_n^k = \beta(n)(\omega(n)(f)) \circ \mu_n^k \]
for every \( f \in P(n), k = 1, \ldots, n \). On the other hand,
\[ ((\omega \otimes \text{id}) \circ \beta^{(0)}(n): f \otimes e_k^{(n)} \mapsto \beta^{(0)}(n)(\omega(n)(f) \otimes e_k^{(n)}) = \beta(n)(\omega(n)(f)) \circ \mu_n^k. \]

The class of all \( \omega \)-special \( P \)-algebras is closed under Cartesian products. Therefore, the class of all homomorphic images of all subalgebras of \( \omega \)-special \( P \)-algebras is a variety \( \Xi \). Consider the set of polylinear identities that hold on this variety and define the corresponding operad \( S^\omega P \). This is a quotient operad of \( P \), and there exists a morphism \( S^\omega P \to S^\omega P \). Every \( P \)-algebra from \( \Xi \) is an \( S^\omega P \)-algebra, but the converse may not be true if \( \text{char} \ k \neq 0 \).

**Lemma 7.7** Consider the class of all quotient operads \( P' \) of \( P \) satisfying the following property: for every morphism \( \alpha: R \to E(A) \) there exists a morphism \( \alpha': P' \to E(A) \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{F}_\Omega & \xrightarrow{\pi} & P \\
\downarrow & & \downarrow \omega \\
\mathcal{F}_\Xi & \xrightarrow{\rho} & R \\
\end{array}
\begin{array}{ccc}
& \xrightarrow{\pi'} & P' \\
& \downarrow {\alpha'} & \\
& \alpha & \xrightarrow{\alpha} E(A)
\end{array}
\]
is commutative. Then \( S^\omega P \) is a quotient of all such \( P' \).

**Proof** Given \( f \in P(n) \), if \( \pi'(n)(f) = 0 \) then \( \omega \circ \alpha(n)(f) = 0 \) for every \( \alpha \). For the free countably generated \( R \)-algebra \((A, \alpha)\) each \( \alpha(n) \) is injective, so \( \text{Ker} \pi'(n) \subseteq \text{Ker} \omega(n) \).

By the definition, \( S^\omega P \) satisfies the condition on \( P' \) described above. Hence, the kernel of the quotient morphism \( S^\omega: P \to S^\omega P \) contains all \( f \in P(n) \) such that \( \omega(n)(f) = 0 \) in \( R(n) \). Therefore, \( \text{Ker} \pi'(n) \subseteq \text{Ker} \omega(n) \subseteq \text{Ker} S^\omega(n) \).

**Corollary 7.8** The kernel of \( S^\omega: P \to S^\omega P \) coincides with the kernel of \( \omega: P \to R \).
8. Speciality of algebras

In this section, we state the solution of Problems 2.1 and 2.2. First, let us reformulate the Problem 2.1 in a more general framework.

On the one hand, \( \omega : \mathcal{F}_\Omega \rightarrow \mathcal{F}_\Xi \) gives rise to

\[
\omega \otimes \text{id} : \text{di} \mathcal{F}_\Omega = \mathcal{F}_\Omega \otimes \text{Perm} \rightarrow \mathcal{F}_\Xi \otimes \text{Perm} = \text{di} \mathcal{F}_\Xi.
\]

Thus, we may define \((\omega \otimes \text{id})\)-special di \( \mathcal{P} \)-algebras and a variety \( \mathcal{E}^{(2)} \) generated by them. Consider the operad \( S^{\omega \otimes \text{id}} \mathcal{P} \) defined by all polylinear identities that hold on \( \mathcal{E}^{(2)} \). This is a generalization of the BSO procedure from [4]. In the case of zero characteristic, we can conclude that every \( S^{\omega \otimes \text{id}} \mathcal{P} \)-algebra is a homomorphic image of a subalgebra of an \((\omega \otimes \text{id})\)-special di \( \mathcal{P} \)-algebra.

On the other hand, the variety of \( S^\omega \mathcal{P} \)-algebras (defined by polylinear identities) gives rise to the corresponding variety of di \( S^\omega \mathcal{P} \)-algebras, where di \( S^\omega \mathcal{P} = S^\omega \mathcal{P} \otimes \text{Perm} \), as above.

Both operads di \( S^\omega \mathcal{P} \) and \( S^{\omega \otimes \text{id}} \mathcal{P} \) are quotients of \( \mathcal{P} \otimes \text{Perm} \) and, therefore, of \( \mathcal{F}_\Omega \otimes \text{Perm} \).

In these terms, Problems 2.1 and 2.2 are particular cases of the following theorem.

**Theorem 8.1**

1. If \( \text{char} \ k = 0 \), then \( \text{di} S^\omega \mathcal{P} = S^{\omega \otimes \text{id}} \mathcal{P} \).
2. If \( \text{char} \ k = p > 0 \), then the identities of degree \( d < p \) that hold on di \( S^\omega \mathcal{P} \)-algebras are the same that hold on \( S^{\omega \otimes \text{id}} \mathcal{P} \)-algebras.

**Proof**

(1) We will show that the class of di \( S^\omega \mathcal{P} \)-algebras coincides with the class of \( S^{\omega \otimes \text{id}} \mathcal{P} \)-algebras.

\( \mathcal{E} \): To compare varieties, we may compare the sets of defining identities. Suppose \( f = f(x_1, \ldots, x_n) \in \mathcal{F}_\Omega(n) \) is a polylinear identity that holds on the variety of all di \( S^\omega \mathcal{P} \)-algebras. Hence, \( \zeta_{\Omega}(n)(f) = \sum_{k=1}^{n} f_k \otimes \varsigma^{(n)}_k \in \text{Ker} (S^\omega \otimes \text{id})(n) \), so \( f_k \in \text{Ker} S^\omega (n) = \text{Ker} \omega(n) \) for all \( k = 1, \ldots, n \) (Corollary 7.8). Therefore, \((\omega \otimes \text{id})(n) (\zeta_{\Omega}(n)(f)) = 0 \), i.e. \( \zeta_{\Omega}(n)(f) \) is an identity of the variety of all \( S^{\omega \otimes \text{id}} \mathcal{P} \)-algebras. Since the kernel of \( \zeta_{\Omega}(n) \) annihilates in di \( \mathcal{P} \), we obtain that \( f \) is an identity of the variety of all \( S^{\omega \otimes \text{id}} \mathcal{P} \)-algebras. Such a relation between identities implies the claim.

Note that this part of the proof does not depend on the characteristic of the base field.

\( \mathcal{E} \subseteq \): Assume \( A \) is a di \( S^\omega \mathcal{P} \)-algebra. Then by Corollary 6.3, \( A \subseteq (\text{Cur} \, \hat{A})^{(0)} \), where \( \hat{A} \) is a \( S^\omega \mathcal{P} \)-algebra. Hence \( \hat{A} = \varphi(B_1) \), where \( \varphi \) is a homomorphism of \( \mathcal{P} \)-algebras, \( B_1 \subseteq B \) and \( B \) is an \( \omega \)-special \( \mathcal{P} \)-algebra. Then \( \text{Cur} \, B \) is an \( \omega \)-special conformal \( \mathcal{P} \)-algebra by Lemma 7.5. Therefore, \((\text{Cur} \, B)^{(0)} \) is an \((\omega \otimes \text{id})\)-special di \( \mathcal{P} \)-algebra by Lemma 7.6. Since \((\text{Cur} \, B_1)^{(0)} \subseteq (\text{Cur} \, B)^{(0)} \) and \( \text{Cur} \, A = \text{Cur} \, \varphi(\text{Cur} \, B_1) \), we have \((\text{Cur} \, \hat{A})^{(0)} = (\text{Cur} \, \varphi)^{(0)}((\text{Cur} \, B_1)^{(0)}), i.e. (\text{Cur} \, \hat{A})^{(0)} \) is a homomorphic image of a subalgebra in an \((\omega \otimes \text{id})\)-special di \( \mathcal{P} \)-algebra. Hence, \( A \) belongs to the variety of \( S^{\omega \otimes \text{id}} \mathcal{P} \)-algebras.

(2) We have to compare the sets of polylinear identities of degree \( d < p \) that hold on all algebras from \( \mathcal{E}^{(2)} \) and on all di \( S^\omega \mathcal{P} \)-algebras. Denote the first set by \( \text{Id}_d(\mathcal{E}^{(2)}) \) and the latter one by \( \text{Id}_d(\text{di} S^\omega \mathcal{P}) \).
The embedding \( \text{Id}_d(\text{di} S^P) \subseteq \text{Id}_d(\mathcal{E}^{(2)}) \) has already been proved in part (1). It remains to prove the converse.

Let \( X = \{x_1, x_2, \ldots \} \) be a countable set of variables. Denote by \( \overline{X} = \{\overline{x}_1, \overline{x}_2, \ldots \} \) a copy of \( X \). Consider the free \( S^P(\mathcal{X} \cup \overline{X}) \) generated by \( X \) and \( \overline{X} \). Since the class of \( S^P \)-algebras is a variety defined by homogeneous identities, the notion of degree is well defined for its elements. Denote by \( \text{deg}_X f \) the degree of \( f \in S^P(\mathcal{X} \cup \overline{X}) \) with respect to all elements from \( X \).

Consider the \( S^P \)-algebra

\[
F(X) = S^P(\mathcal{X} \cup \overline{X})/J,
\]

where \( J \) is the linear span of all homogeneous \( f \in S^P(\mathcal{X} \cup \overline{X}) \) such that \( \text{deg}_X f \geq 2 \).

Assume \( t = t(x_1, \ldots, x_d) \in \text{Id}_d(S^{(2)}) \). As an element of \( \text{di} S^P(d) \), \( t \) can be identified with \( t_1 \otimes e_1^{(d)} + \cdots + t_d \otimes e_d^{(d)} \), \( t_k \in S^P(d) \).

There exists a \( T \)-ideal \( \Phi \) in \( S^P(\mathcal{X} \cup \overline{X}) \) such that

\[
S^P(\mathcal{X} \cup \overline{X})/\Phi \simeq \Phi.
\]

Assume \( f \in \Phi \) is a homogeneous polynomial, and \( \text{deg} f < p \). Then the complete linearization of \( f \), denoted by \( L(f) \), also belongs to \( \Phi \). Thus, \( L(f) \) is a polynormal identity that holds for all algebras in \( \mathcal{E} \). Hence, \( L(f) = 0 \) in \( S^P(\mathcal{X} \cup \overline{X}) \). Since \( \text{deg} f < p \), we have \( f = 0 \) in \( S^P(\mathcal{X} \cup \overline{X}) \). Therefore, \( \text{deg} f \geq p \) for all nonzero homogeneous polynomials \( f \in \Phi \).

For every nonzero homogeneous \( f \in \Phi \) we have \( \text{deg} f \geq p \) since all identities of lower degree follow from polynormal identities that hold on \( S^P \)-algebras by definition. Hence,

\[
F_1(X) = S^P(\mathcal{X} \cup \overline{X})/J \simeq F(X)/((\Phi + J)/J) \in \mathcal{E},
\]

and so \( \text{Cur} F_1(X) \) satisfies the identity \( t = 0 \). By Lemma 6.4, \( F_1(X) \) satisfies all identities \( t_k = 0, k = 1, \ldots, d \). But if \( t_k \neq 0 \) in \( S^P(d) \) then \( t_k(x_1, \ldots, x_d) \notin \Phi + J \) in \( S^P(\mathcal{X} \cup \overline{X}) \) by the degree-related reasoning. Therefore, \( t = 0 \) in \( \text{di} S^P(d) \).

Given a triple \( (P, R, \omega) \) as above, a non-zero \( f \in P(n) \) is said to be a special identity if \( f \in \text{Ker} \omega(n) \). As a corollary of Theorem 8.1, we may conclude that in all possible settings (\( \text{di} P, \text{di} R, \omega \otimes \text{id} \)) we should not expect the existence of polynormal special identities different from \( f \otimes e^{(n)} \), where \( f \) is a special identity for \( (P, R, \omega) \). This explains, in particular, the results of [5,6,20] concerning special identities of Jordan and Mal’cev algebras.

To be more precise, we state the following corollary.

**Corollary 8.2** Let \( \text{char} \mathbb{k} = 0 \). Suppose \( f \in F_{\Omega(\mathcal{X})}(X) \), \( X = \{x_1, x_2, \ldots \} \), is a polynomial of type \( \Omega^{(2)} \). Then the following conditions are equivalent.

1. \( f = 0 \) is an identity on all \( (\omega \otimes \text{id}) \)-special \( P \)-algebras, but not an identity on all \( P \)-algebras;
2. \( \xi_\Omega(n)(L(f)) = \sum_{k=1}^n f_k \otimes e_k^{(n)} \), \( f_k \in F_{\Omega(\mathcal{X})} \), where all \( f_k \) are identities on the class of all \( \omega \)-special \( P \)-algebras and at least one of them is not an identity on the class of all \( P \)-algebras.

**Proof** Over a field of characteristic zero, \( f \) is a special identity if and only if so is \( L(f) \). If \( L(f) \) does not hold for all \( \text{di} P \)-algebras then \( (\pi(n) \otimes \text{id})\xi_\Omega(n)(L(f)) \neq 0 \), i.e.
there exists a $k$ such that $\pi(n)(f_k) \neq 0$, and so $f_k$ does not hold for all $\mathcal{P}$-algebras. On the other hand, $(\omega(n) \otimes \text{id})(\zeta_\omega(n)) = 0$, i.e. $\omega(n)(f_k) = 0$ for all $k$, and so all $f_k$ are identities for the class of all $\omega$-special $\mathcal{P}$-algebras. The proof of the converse statement is similar.

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References

[1] B. Bakalov, A. D’Andrea, and V.G. Kac, *Theory of finite pseudoalgebras*, Adv. Math. 162 (2001), pp. 1–140.
[2] A. Bloch, *On a generalization of the concept of Lie algebra*, Dokl. Akad. Nauk SSSR 165 (1965), pp. 471–473 (in Russian).
[3] M.R. Bremner, *On the definition of quasi-Jordan algebra*, Comm. Algebra 38 (2010), pp. 4695–4704.
[4] M.R. Bremner, R. Felipe, and J. Sánchez-Ortega, *Jordan triple disystems*, Comput. Math. Appl. 63 (2012), pp. 1039–1055.
[5] M.R. Bremner and L.A. Peresi, *Special identities for quasi-Jordan algebras*, Comm. Algebra 39 (2011), pp. 2313–2337.
[6] M.R. Bremner, L.A. Peresi, and J. Sánchez-Ortega, *Malcev dialgebras*, Linear Multilinear Algebra, DOI: 10.1080/03081087.2011.651721.
[7] F. Chapoton, *Un endofoncteur de la catégorie des opérades*, in *Dialgebras and Related Operads*, Lectures Notes in Mathematics, Vol. 1763, J.-L. Loday, F. Chapoton, F. Goichot, and A. Frabetti, eds., Springer-Verlag, Berlin, 2001, pp. 105–110.
[8] V. Ginzburg and M. Kapranov, *Koszul duality for operads*, Duke Math. J. 76 (1994), pp. 203–272.
[9] N. Jacobson, *Lie and Jordan triple systems*, Amer. J. Math. 71 (1949), pp. 149–170.
[10] V.G. Kac, *Vertex Algebras for Beginners*, Vol. 10, University Lecture Series, AMS, Providence, RI, 1996.
[11] P. Kolesnikov, *Varieties of dialgebras and conformal algebras*, Sib. Math. J. 49 (2008), pp. 257–272.
[12] A.G. Kurosh, *Free sums of multiple operator algebras*, Sib. Mat. Zh. 1 (1960), pp. 62–70 (in Russian).
[13] D. Liu, *Steinberg–Leibniz algebras and superalgebras*, J. Algebra 283 (2005), pp. 199–221.
[14] J.-L. Loday, *Une version non commutative des algèbres de Lie: Les algèbres de Leibniz*, Enseign. Math. (2) 39 (1993), pp. 269–293.
[15] J.-L. Loday, *Dialgebras*, in *Dialgebras and Related Operads*, Lectures Notes in Mathematics, Vol. 1763, J.-L. Loday, F. Chapoton, F. Goichot, and A. Frabetti, eds., Springer-Verlag, Berlin, 2001, pp. 1–61.
[16] J.-L. Loday and T. Pirashvili, *Universal enveloping algebras of Leibniz algebras and (co)homology*, Math. Ann. 296 (1993), pp. 139–158.
[17] A.P. Pozhidaev, *0-Dialgebras with bar-unity, ternary Leibniz algebras and Rota–Baxter algebras*, Contemp. Math. 499 (2009), pp. 245–256.
[18] B. Vallette, *Manin products, Koszul duality, Loday algebras and Deligne conjecture*, J. Reine Angew. Math. 620 (2008), pp. 105–164.

[19] R. Velásquez and R. Felipe, *Quasi-Jordan algebras*, Comm. Algebra 36 (2008), pp. 1580–1602.

[20] V. Voronin, *Special and exceptional Jordan dialgebras*, J. Algebra Appl. 11, DOI: 10.1142/S0219498811005531.