Reidemeister Moves and Groups

Vassily Olegovich Manturov

Abstract

Recently, the author discovered an interesting class of knot-like objects called free knots. These purely combinatorial objects are equivalence classes of Gauss diagrams modulo Reidemeister moves (the same notion in the language of words was introduced by Turaev [Tu1], who thought all free knots to be trivial). As it turned out, these new objects are highly non-trivial, see [Ma1], and even admit non-trivial cobordism classes [Ma2]. An important issue is the existence of invariants where a diagram evaluates to itself which makes such objects “similar” to free groups: an element has its minimal representative which “lives inside” any representative equivalent to it.

In the present paper, we consider generalizations of free knots by means of (finitely presented) groups. These new objects have lots of non-trivial properties coming from both knot theory and group theory: functoriality, coverings, etc. This connection allows one not only to apply group theory to various problems in knot theory but also to apply Reidemeister moves to the study of (finitely presented) groups.

Groups appear naturally in this setting when graphs are embedded in 2-surfaces.

Keywords: Group, Graph, Reidemeister Move, Knot, Free Knot.

AMS MSC 05C83, 57M25, 57M27

1 Introduction. Basic Definitions and Notation

Knots and virtual knots are encoded by 4-valent graphs $\Gamma$ with a framing (opposite edge structure at vertices) and some decorations at vertices (the structure of overpasses and underpasses) modulo Reidemeister moves. When skipping decorations of crossings, we get to certain equivalence classes of virtual knots, called free knots. If such a graph $\Gamma$ is treated as the image of immersion of a curve $\gamma$ in an oriented 2-surface $\Sigma$ in such a way that $\Gamma\setminus\gamma$ admits a checkerboard colouring, then there is a natural presentation of the quotient group $\pi_1(\Sigma)/\langle[\gamma]\rangle$ by the normal closure of the element corresponding to $\gamma$ where vertices are generators and regions are relators; for more details, see [IMN, FM].

*The author is partially supported by Laboratory of Quantum Topology of Chelyabinsk State University (Russian Federation government grant 14.Z50.31.0020), by RF President NSh 1410.2012.1, and by grants of the Russian Foundation for Basic Research, 13-01-00830, 14-01-91161, 14-01-31288.
Thus, considering $\Gamma$ as a homotopy class of $\gamma$, vertices of $\Gamma$ get some natural interpretations in terms of homotopy classes. When performing Reidemeister moves to $\Gamma$, labels of vertices undergo natural transformations.

When considering more abstract graphs unrelated to any surfaces, one gets the notion of a free knot (in the case of many components $\gamma_i$, one can talk about a free link). A free knot is a 1-component free link.

Free knots possess highly non-trivial invariants demonstrating the following principle:

if a free link diagram $K$ is complicated enough, then every diagram $K'$ equivalent to $K$ contains $K$ inside.

This result is achieved by using the parity bracket $[\cdot]$, an invariant of free links valued in diagrams of free links such that $[K] = K$ for diagrams which are large enough, see ahead. This allowed the author to get new approaches to various problems in virtual [Ma5, Ma6] and classical knot theory [ChM, KM], and various questions of topology.

Thus, free links are interesting objects by themselves; on the other hand, vertices of their diagrams (framed 4-graphs) can be labeled by elements of certain groups.

This leads to the notion of group free knots or $G$-free knots where $G$ is a (finitely presented) group. When forgetting labelings of vertices of framed 4-graphs, we get usual (classical,virtual, flat or free) knot theory, when embedding a framed 4-graph into an oriented 2-surface, we get a natural labeling.

When applying Reidemeister moves to diagrams within a 2-surface, we naturally get moves for labeled diagrams.

Thus, considering various questions concerning equivalence classes of group free knots, we can study arbitrary groups. Indeed, every group $G$ leads to its own $G$-labeled free knot theory, and depending on which graphs turn out to be equivalent in this group, one may judge about the structure of the group.

Let us now pass to formal definitions of the objects, we are going to deal with.

**Remark 1.** Throughout the rest of the text, all groups are assumed to be finitely presented.

We shall deal with framed 4-graphs which naturally appear as shadows of knot diagrams.

**Remark 2.** We say “4-graph” instead of “4-valent graph”, because we also admit objects to have circular components, i.e., circles which are disjoint from the graph.

**Definition 1.** A 4-valent graph $\Gamma$ is framed if at every vertex $X$, the four half-edges incident to it, are split into two sets; (half)edges from the same set are called opposite; (half)edges which are not opposite are called adjacent; such a choice is called framing of $\Gamma$.

**Definition 2.** By a rotating cycle of a framed 4-graph $\Gamma$ we mean either a circular component of $\Gamma$ or a sequence of vertices (possibly, non-distinct) $v_i$ and
distinct edges $e_i, i = 0, \ldots, k - 1$, such that $e_i$ connects $v_i$ and $v_{i+1}$ and at $v_i$, the edges $e_i$ and $e_{i-1}$ are non opposite. Here indices are counted modulo $k$.

**Definition 3.** When talking about the number of components, we separately count **circular components** and separately count **unicursal components**; by the latter, we mean an equivalence class of edges called by elementary equivalence relation where every two edges opposite at some vertex are treated as opposite.

Thus, the disjoint diagram of the disjoint sum of the simplest 2-vertex Hopf link and the unknot has three components: a circular one corresponding to the unknot and the two unicursal components of the Hopf link.

**Definition 4.** A **source-sink structure** of a framed 4-graph $\Gamma$ is an orientation of all edges of $\Gamma$ such that at every vertex $X$ of it some two opposite (half)edges are emanating and the other two are incoming, see Fig. 1.

Circular edges are just assumed to be oriented.

**Definition 5.** Let $G$ be a framed 4-graph, and let $X$ be a vertex of $G$, let $a, b$ be one pair of opposite (half)edges, and let $c, d$ be the other pair of opposite (half)edges at $X$; by the **vertex removal** we mean the operation of deleting the vertex $X$ from $G$ and connecting $a$ to $b$ and connecting $c$ to $d$.

**Remark 3.** Note that if a connected framed 4-graph admits a source-sink structure then there exist exactly two such structures which differ by the overall orientation change.

**Notation 1.** We adopt the following convention. Whenever we draw an immersion of a framed 4-graph in $\mathbb{R}^2$, we depict its vertices by solid circles, those points which are encircled are artifacts of projection caused by immersion. At every vertex, the framing is assumed to be **induced from the plane**: (half)edges which are locally opposite on the plane are opposite.

Note that if at a vertex $X$ of a framed 4-graph a half-edge $a$ is opposite to a half-edge $c$ and a half-edge $b$ is opposite to a half-edge $d$, then then when drawing on the plane, the counterclockwise order of edges can be $a, b, c, d$ or $a, d, c, b$. 

This leads to the two “moves” for planar diagrams of 4-graphs which do not change the framed 4-graph: the detour move and the virtualization move. The detour move removes one piece of an edge with all encircled crossings inside it and redraws it elsewhere with new encircled crossings with itself and other edges, see Fig. 2; the virtualization move changes the local counterclockwise order at a vertex without changing its framing; one can represent this move by flanking the classical crossings by two encircled intersection points of $a, b$ and $c, d$.

**Remark 4.** In the sequel, when drawing some transformation on the plane, we shall depict only the changing part of the move assuming the remaining part to be fixed.

**Example 1.** Consider the framed 4-graph with one vertex $A$ and two loops, $p$ and $q$ such that $p$ is opposite to itself at $A$ and $q$ is opposite to itself at $A$. Then this graph admits no source-sink structure because whatever orientation of the edge $p$ we take, it will be emanating from one side and incoming from the other side.

**Definition 6.** We say that a framed 4-graph is good if it admits a source-sink structure.

We say that a good framed 4-graphs oriented if a source-sink structure of it is selected.

Likewise, we define good (resp., oriented) $G$-framed 4 graphs $(\Gamma, f)$ if $\Gamma$ is good (resp., oriented).

**Remark 5.** We shall refer to oriented $G$-framed 4-graphs simply as $G$-graphs. Moreover, we shall often omit $f$ from the notation if the definition of $f$ is clear from the context.

**Definition 7.** Let $G$ be a group. An oriented $G$-free link is an equivalence class of $G$-graphs by the following three Reidemeister moves shown in Fig. 3.

Here we assume that for the first Reidemeister move, the vertex taking part in this move is marked by the unit element of the group, for the second Reidemeister moves, the two elements are inverse to each other, and for the third move we have some three elements $a, b, c$ in the left hand side such that $a \cdot b \cdot c = 1$, and the opposite elements $a^{-1}, b^{-1}, c^{-1}$ in the right hand side, see Fig. 3.

Note that for the first Reidemeister move the choice of the source-sink orientation does not matter; it only matters whether these orientations agree for the
edges touching the boundary of the picture; for the third Reidemeister move, it is crucial to require that the cyclic order of the three vertices along the source-sink orientation is $a, b, c$ and not $a, c, b$.

**Definition 8.** Let $G$ be a finitely presented group. By a $G$-framed-4-graph we mean a pair $(\Gamma, f)$ where $\Gamma$ is a framed 4-graph and $f$ is the map from the set of vertices of $\Gamma$ to $G$.

Let $\Gamma$ be a $G$-graph. Let $\gamma$ be a rotating cycle on $\Gamma$ with $k$ vertices (a vertex is counted twice if it is passed twice). Then $\gamma$ inherits a natural orientation from the source-sink structure of $\Gamma$. If we choose a starting point on $\gamma$, we can write down the elements of $G$ corresponding to the vertices of the group $G : [\gamma] = \gamma_1, \cdots, \gamma_k \in G$. If no starting point is given, we can obtain a conjugacy class of $\gamma$.

**Definition 9.** The minimal crossing number of a $G$-free link $L$ is the minimal number of vertices of oriented $G$-framed 4-graphs representing $L$.

This definition means that every group $G$ leads to a well defined free knot theory corresponding to this group. Even for the trivial group, this theory is extremely interesting: it gives rise to the theory of even free knots (i.e. free knots whose diagrams admit source-sink structure), which has non-trivial invariants reproducing itself \[Ma1\], \[Ma3\].
Thus, one can use this theory in both directions: it is possible to study groups by means of free knots and their generalizations and to study various knot theories by using group-valued labeling of their vertices.

**Remark 6.** A similar theory can be constructed for all free knots, not necessary having source-sink structure, however, in this way one should overcome some difficulties with orientations corresponding to relations for Reidemeister moves. We shall touch on this question in another paper.

Note also that, in the case of the $\mathbb{Z}_2$-group, the theory can be constructed even in the case of arbitrary graphs (see ahead).

The paper is organized as follows. In the next section, we present the three main tools for constructing invariants of our object, the parity bracket, the parity delta, and the covering.

In the third section, we shall consider various examples where the group labeling arises in topological situation, and which relations on groups does this group labeling are imposed in concrete situation. We shall mostly concentrate on the case of group coming from a 2-surface.

We conclude the paper by a list of unsolved problems relating Reidemeister moves with group theory.

## 2 Basic Invariants

We are now ready to construct our first invariants of groups. Let $G$ be a finite group.

**Definition 10.** Let $\mathcal{R}(G)$ be the $\mathbb{Z}_2$-module freely generated by all $G$-knots.

**Definition 11.** Let $G$ be a group. By $\text{Set}(G, k)$ we mean the formal integral linear combination of all $G$-links $K$ having minimal crossing number $k$.

From definition, it follows that $\text{Set}(G, k)$ is an invariant of the group $G$.

**Remark 7.** For the case of infinite $G$, the number of all $G$-links $K$ having minimal crossing number $k$ can be infinite.

**Example 2.** Let $G$ be the trivial group. Then the $G$-link theory coincides with the theory of (having source-sink structure and oriented) free links.

One of the main phenomena in virtual knot theory is the local crossing information. Having two (or many) different types of crossings which behave nicely under Reidemeister moves, we can enhance many invariants by localizing some information at crossings; moreover, this allows one to reduce the study of objects to the study of their diagrams.

In the case of the $\mathbb{Z}_2$-group, the main invariants are called the parity bracket, the parity projection, and parity covering. We are going to describe the parity bracket and to generalize the projection for the case of arbitrary groups.
The group bracket, unlike the parity bracket, does not reduce the study of knots to the study of their diagrams, but rather is a functorial mapping from the category of $G$-knots to $G$-knots.

Actually, iterative use of projections, and coverings and other tools lead to enhancement of many invariants constructed combinatorially.

In a similar way, one can enhance various combinatorial invariants in the case of arbitrary group $G$. We shall touch on these problems in subsequent papers.

Another important issue in the virtual knot theory is that many invariants (like linking numbers or writhe numbers) become picture-valued: instead of some count of crossings (with signs), we are allowed to count pictures arising at these points.

### 2.1 The Parity Bracket

In the present section, we will concentrate on the case of the group $\mathbb{Z}_2$.

We will define the parity bracket which is well-defined for $\mathbb{Z}_2$-graphs and which realizes the principle “if a knot (link) diagram is complicated enough then it reproduces itself”.

In this case, complicated enough will mean irreducible and odd, see ahead, however, for other groups $G$ there are many other situation where it works.

A $\mathbb{Z}_2$-labeling $f$ of a $G$-graph $\Gamma$ is usually referred to as parity.

This bracket was first defined in [Ma1] for the so-called Gaussian parity which is trivial for all good framed 4-graphs. Nevertheless, all the results stated below (as well as their proofs) remain true whatever parity we take.

**Definition 12.** By a smoothing of a framed 4-graph at a vertex we mean the result of deleting this vertex and repasting the half-edges into two edges in one of the two possible ways $\xrightarrow{\leftrightarrow}, \xrightarrow{\leftrightarrow}$.

**Remark 8.** Note that the smoothing may result in new circular components of a free graph. Note that a smoothing of a good framed 4-graph is good.

Denote by $\mathcal{G}$ the space of $\mathbb{Z}_2$-linear combinations of the following objects. We consider the linear space of all framed 4-graphs (not necessarily good ones) subject to the second Reidemeister moves.

By a bigon of a framed 4-graph $\Gamma$ we mean two edges $e_1, e_2$ which connect some two vertices $v_1$ and $v_2$ and are non-opposite at both these vertices; bigons appear in the second Reidemeister moves. A framed 4-graph is irreducible if it has no bigons inside.

**Remark 9.** It can be easily seen that the equivalence classes modulo the second Reidemeister moves are characterized by its minimal representatives, i.e., framed 4-graphs without “bigons”. Thus, we shall use the term “graph” when talking about an element from $\mathcal{G}$ assuming the minimal representative of its element.
Let $K$ be a framed good $\mathbb{Z}_2$ 4-graph. The bracket invariant $[\cdot]$ (see [Ma1]) is given by

$$[K] = \sum_{s \text{ even, 1-component}} K_s \in \mathfrak{S},$$

where the sum is taken by all smoothings $s$ at even vertices which lead to one-component diagrams $K_s$. This sum is considered as an element from $\mathfrak{S}$.

The following theorem is proved in [Ma1]

**Theorem 1.** $[\cdot]$ is an invariant of free knots; in other words, if two framed 4-graphs $\Gamma_1$ and $\Gamma_1'$ are equivalent then $[\Gamma_1] = [\Gamma_2] \in \mathfrak{S}$.  

Let us call a $\mathbb{Z}_2$-graph *odd* if all vertices of it are odd.

It follows from the definition of the bracket that if $\Gamma$ is odd and irreducible then $[\Gamma] = [\Gamma]$. In particular, this means that for every irreducible and odd $\mathbb{Z}_2$-graph $\Gamma$ and every graph $\Gamma'$ equivalent to it, $\Gamma$ can be obtained from $\Gamma'$ by means of smoothings at some vertices.

### 2.2 Turaev’s Delta and its Generalizations

Actually, many integer-valued invariants of topological objects can be calculated as certain (algebraic) sums taken over certain reference points (e.g., some intersection points in some configuration spaces etc). It turns out that in many situations which happen in low-dimensional topology, these points can be themselves endowed with certain (topological or combinatorial) information which makes them responsible for non-triviality of the object itself and various properties of it. Moreover, in some cases these points can be associated with objects similar to the initial one.

This switch from numbers to pictures changes the situation crucially. The (algebraic) sums instead of being just integer-valued invariants of the initial object, transform into functorial mappings from objects to similar objects.

For curves in 2-surfaces, there are two operations, the multiplication and the comultiplication. The first one is due to W.Goldman [Goldman], and the second one is due to V.G.Turaev [1]. The main idea is that we can get knots from 2-component links and 2-component links from knots by taking smoothings at some crossings. Then we take sums over such crossings and get an invariant map. We are mostly interested in the second operation and call it and its generalization “Turaev’s delta”.

However, if we forget about 2-surfaces and deal with curves in general position, we can take care not about their homotopy classes, but rather about free knots or links which appear at these crossings (see [IMN] for free knots). But if we deal not with bare free knots, we can enhance the definition and take into account some group information.

Let us be more specific.

Let $\gamma$ be a one-component framed 4-graph.

At each crossing $c$, one can perform the following oriented smoothing $\begin{array}{c|c}
\text{X} \\
\hline
\text{Y} \end{array} \rightarrow \begin{array}{c|c}
\text{Y} \\
\hline
\text{X} \end{array}$. Let us denote the result of such smoothing at $c$ by $\gamma_c$.  

8
Then $\gamma_c$ is a framed 4-graph with two unicursal components. Consider the linear space $\mathcal{M}$ generated by free two-component links. Now let $\mathcal{L}$ be the quotient of links in $\mathcal{M}$ by those links having one trivial component. Let $\Delta(\gamma) = \sum_c \gamma_c$.

**Theorem 2.** $\Delta$ is a well-defined map from the set of free knots to $\mathcal{L}$.

The proof follows from a consideration of all Reidemeister moves. Looking carefully at Reidemeister moves, we see that when taking sum in $\mathcal{M}$, the invariance Reidemeister move requires the two-component free link with one trivial component to be zero.

**Remark 10.** For the case of 2-surfaces, we have homotopy classes of curves which are represented by curves in general position; the latter have intersection points which correspond to vertices of the framed 4-graph.

If we had not curves in 2-surfaces but $G$-knots it would be sufficient to define our $\Delta_G$ to be

$$\Delta_G = \sum_{c, l(c) \neq 1} \gamma_c,$$

to be the sum over all crossings $c$ with non-trivial label $l(c)$ of the resulting two-component free links.

Note that the invariance under the third Reidemeister move does not allow one to consider the summands as $G$-links.

However, one can split $\Delta_G$ with respect to elements of $\Delta$. For every element $g \neq 1 \in G$ we define $\Delta_g = \Delta_g = \Delta_{g^{-1}}$ to be

$$\sum_{c, l(c) \in \{g, g^{-1}\}} \gamma_c,$$

Then all $\Delta_g$ are invariants of the initial link and $\Delta_G$ naturally splits into the sum over all classes of non-trivial inverse elements.

As we shall see later, the structure of $G$-knots naturally appears for curves in 2-surfaces.

**Theorem 3.** For each $g$, the mapping $\Delta_g$ is invariant.

Consequently, $\Delta_G$ is an invariant mapping.

### 2.3 The group bracket

As we have seen before, $\mathbb{Z}_2$-knots admit a natural bracket $[\cdot]$ which takes all $\mathbb{Z}_2$-knots into $\mathbb{Z}_2$-linear combinations of equivalence classes of free knots modulo second Reidemeister moves. This actually means that we get from knots to graphs.
This actually happens because there exist exactly one non-trivial element in $\mathbb{Z}_2$.

For arbitrary group $G$, there is a similar bracket operation which takes $G$-knots to equivalence classes of diagrams modulo moves; however, the target space will consist of equivalence classes of $G$-graphs modulo not only second Reidemeister moves but also third Reidemeister moves. Thus, the group bracket unlike the pairing bracket, can not be treated as a graph-valued invariant, but rather, as a functorial map for $G$-knots, similar to Turaev’s delta.

Let us be more specific.

Let $\mathcal{S}_G$ be the set of $\mathbb{Z}_2$-equivalence classes of $G$-graphs with no vertex marked by the unit element of the group, modulo the second and the third Reidemeister moves.

Let $K$ be a $G$-graph. Consider the set of all vertices of $K$ marked by the unit element of the group $G$. We define smoothings $s$ at unit vertices of $K$ as before. Now, we set

$$[K]_G = \sum_{\text{even, 1-component}} K_s \in \mathcal{S}_G.$$  

**Theorem 4.** The map $\mathcal{S}_G$ is an invariant of $G$-knots.

**Proof.** We have to check the invariance under the three Reidemeister moves.

Let $K$ and $K'$ be two $G$-graphs which differ by a Reidemeister move at some vertex. We shall show that $[K] = [K']$.

Assume first $K'$ is obtained from $K$ by the first Reidemeister move which adds one vertex $v$ with trivial label. Then $[K']$ contains summands where $v$ yields a local loop. They do not count in the bracket since they have at least two components.

All other summands in $[K']$ are in one-to-one correspondence with all summands in $[K]$.

Assume now $K'$ is obtained from $K$ by a second Reidemeister move where the two additional crossings in $K'$ are both non-trivial.

Then all summands in $[K']$ are in one-to-one correspondence with those of $[K]$, and they differ exactly by the same second Reidemeister move.

The same situation happens when $K'$ differs from $K$ by a third Reidemeister move with three vertices of non-trivial labels: we have a one-to-one correspondence of equal summands.

The cases when $K'$ differs from $K$ by a second Reidemeister move with two additional trivial crossings or by a third Reidemeister move with two additional trivial crossings are actually the same as in the case of usual bracket; for more detail, see [Ma1].

Finally, assume $K'$ differs from $K$ by a third Reidemeister move with labels $a, b, c$ such that $b = a^{-1}$ and $c = 1$ (see Fig. 3). In the right hand side, we will have labels $a^{-1}, b^{-1} = a$, and $1^{-1} = 1$.

Now we see that one smoothing at the vertex of $K$ labeled 1 coincides identically with the similar smoothing of $K$; the other smoothing of $K$ differs by the similar smoothing of $K'$ by two second Reidemeister moves.
2.4 Coverings and Projections

First, let \( \alpha : G \to H \) be a homomorphism of two groups. Then, for every \( G \)-graph \( \Gamma \), we get an \( H \)-graph \( \alpha_*(\Gamma) \) by changing each vertex label \( g \in G \) to \( \alpha(g) \in H \).

From the definition of \( G \)-knots we get to the following

**Theorem 5.** The operation \( \alpha_* \) leads to a well defined mapping from \( G \)-knots to \( H \)-knots, i.e., if \( \Gamma_1, \Gamma_2 \) are two equivalent \( G \)-graphs then their images \( \alpha_*(\Gamma_1) \) and \( \alpha_*(\Gamma_2) \) are two equivalent \( H \)-graphs.

Below, we briefly sketch the idea how to construct a covering for \( G \)-links for some subgroup \( G' \) of a group \( G \); we define the corresponding projection in the general case; as for the covering, we define it only in the abelian case; in the general case it is defined quite analogously (see also subsection 3.2 curves in 2-surfaces).

Now, let \( G' \) be a subgroup of a group \( G \). Then for every \( G \)-graph \( \Gamma \), we can define the \( G' \)-graph \( \beta_{G'}(\Gamma) \) obtained from \( \alpha_* \) by removing all vertices of \( \Gamma \) whose labels do not belong to \( G' \); the remaining vertices will keep their labels considered now as elements of \( G' \).

**Theorem 6.** The operation \( \beta_{G'}(\Gamma) \) is a well defined mapping from \( G \)-knots to \( G' \)-knots.

Thus, we have constructed a map \( \beta_{G'} \) which removes some nodes. In particular, this projection maps knots to knots.

In the simplest case when \( G = \mathbb{Z}_2, G' = \{1\} \), the parity projection map can be justified to a parity covering map. Namely, the projection map takes knots to 2-component links, and generally, \( l \)-component links to \( 2l \)-component links with two sets of \( l \) components each. Over each crossing of the initial knot \( X \), there are two crossings \( X_1 \) and \( X_2 \), of the obtained link. The odd crossings (those marked by non-trivial elements of \( \mathbb{Z}_2 \) which disappear under the projection map), will belong to two different components (sets of components) \( \text{Max}_4 \) (“mixed crossings”). Thus, over each crossing marked by the trivial element, there will be two pure crossings belonging to one component (in the case of links, to one set of components), crossings marked by the non-trivial element will be covered by two crossings, each being mixed (i.e., belonging to two different sets of components).

Now, let \( G \) be an abelian group. Let \( G' \) be a subgroup\(^1\) of \( G \) and let \( H \) be the quotient group so that we have a short exact sequence

\[
G' \xrightarrow{\beta} G \xrightarrow{\alpha} H.
\]

\(^1\)In the case of general \( G \), one should consider normal subgroups only
Let \((L, f)\) be a \(G\)-graph with \(k\) components; we denote the \(G\)-link represented by \(L, k\) by the same letter \(L\). We shall construct the covering corresponding to the inclusion \(\beta : G \rightarrow G'\), as follows.

This covering will be a \(H\)-link having \(k \cdot |H|\) components. This link will be denoted by \(L^\beta\).

With each vertex \(X\) of \((L, f)\) with label \(g \in G\), where \(g = g'h, g' \in G', h \in H\), we shall associate \(H\) vertices \(X^h, h = \alpha(g) \in H\) indexed by elements of \(H\); all these vertices will have label \(g'\).

Now, with every edge \(XY\) connecting some two vertices \(X, Y\) of \(L\) we associate \(|H|\) edges, namely, \(X^g\) is connected to \(Y^{gh}\) where \(h = \alpha(g)\).

**Theorem 7.** This map \(L \rightarrow L^\beta\) is a well defined map from \(G\)-free links to \(G'\)-free links.

**Proof.** This theorem follows from the consideration of Reidemeister moves: when looking at the product of labels along the loop, bigon, or triangle, we see that these products are all equal to the unit of the group \(G\), which means that both “the \(G'\)-factor” and “the \(H\)-factor” of them are units of \(G'\) and \(H\), respectively. This means that the corresponding edges of \(L^\beta\) will close up to a loop, bigon or triangle, respectively. Hence, the Reidemeister move applied to \(L\) will result in the corresponding Reidemeister move applied to \(L^\beta\). \(\square\)

### 3 Further Applications and Examples

#### 3.1 Minimal Crossing Number

Consider the example shown in Fig. 4.

For every group \(G\), the framed graph \(F_1\) is equivalent to the unknot if the label on the unique vertex is equal to the unit of the group \(G\) as a \(G\)-knot.

**Theorem 8.** For every non-trivial group \(G\), there exists a non-trivial labeling \(f\) of the graph \(F_1\) such that the \(G\)-knot \((F_1, f)\) is not equivalent to the unknot.

Indeed, \(\Delta(F_1)\) has one non-trivial summand.

Now, let us consider the example shown in Fig. 5.
As well as $F_1$, the framed 4-graph $F_2$ is trivial for the 1-element group. Moreover, if we take $\mathbb{Z}_2$, then for every labeling of $F_2$, $F_2$ is not minimal for $\mathbb{Z}_2$.

**Theorem 9.** For every group $G$ containing at least three elements, there exists a labeling $f$ of vertices of $F_2$ which makes this labeled graph minimal in its $G$-knot class.

### 3.2 Curves in 2-surfaces

In the present section, we give the main example where the above theory comes from.

Let $\Sigma$ be a closed oriented 2-surface of genus $g$. We say that an embedding $f$ of a framed 4-graph $\Gamma$ in $\Sigma$ is cellular if $\Sigma \setminus \Gamma$ is the disjoint union of 2-cells.

We say that a cellular embedding $f : \Gamma \to \Sigma$ is *checkerboard* if one can colour all cells of $\Sigma$ with 2 colours in a way such that no two cells of the same colour are adjacent along an edge.

The following fact can be proved by the reader as a simple exercise.

**Exercise 1.** If a framed 4-graph $\Gamma$ admits a checkerboard cellular embedding into some $\Sigma$ then $\Gamma$ admits a source-sink structure.

Moreover, if $\Gamma$ admits a source-sink structure then every cellular embedding of $\Gamma$ in every $\Sigma$ (of any genus) is checkerboard.

Now, let us fix a framed graph $\Gamma$ and its checkerboard cellular embedding $f$ into $\Sigma = S_g$. Consider the group $G(\Gamma, f)$ given by the following presentation $P$.

The generators of $P$ are in one-to-one correspondence with vertices of $\Gamma$.

The relations of $P$ are in one-to-one correspondence with cells of $\Sigma \setminus f(\Gamma)$. More precisely, with each cell $C$, one can associate its boundary which is a *rotating cycle* on $\Gamma$. Since $\Gamma$ admits a source-sink structure, this cycle becomes naturally oriented. This means, that all vertices of this cycle are cyclically ordered. This generates a cyclic word which we take to be the relator corresponding to the cell.

Now, assume $\Gamma$ has one unicursal component. Then $f(\Gamma)$ can be thought of as the image of an immersed curve; we shall call this image $\gamma$.

In [?], I.M. Nikonov proved the following

**Theorem 10.** If $\Gamma$ has one unicursal component, then the group $G(\Gamma, f)$ is isomorphic to the quotient group of the fundamental group $\pi_1(\Sigma)$ by the free homotopy class $[\gamma]$. 

![Figure 5: The graph $F_2$](image-url)
This leads us to the natural source where the labels for vertices of \( \Gamma \) may come from. With every embedding \( f \) of \( \Gamma \) into \( \Sigma \), one gets a natural labeling of vertices by elements of \( G(\Gamma, f) \).

Thus, curves in \( \mathcal{S}_g \) satisfying the checkerboard colorability condition naturally obtain the group labeling, which allows one to define the group bracket and the group delta for these knots.

4 Unsolved Problems

The bracket and the Turaev delta for curves in 2-surfaces can be treated as follows.

We study curves as conjugacy classes \([\gamma]\) of elements \( \gamma \) of the fundamental group for an oriented 2-surface \( S_g \). Every summand of \( \Delta \) transform one such conjugacy class \([\gamma]\) into two conjugacy classes \([\gamma_1], [\gamma_2]\) such that \( \gamma_1 \cdot \gamma_2 = \gamma \). Here \( \gamma_1 \cup \gamma_2 \) is the result of smoothing of \( \gamma \) at some crossing \( c \).

For which groups \( G \) can one define a \( \Delta \)-like comultiplication map \( G \to \mathbb{Z}_2 G \to G \) where the sum is taken along “reference points”?

How to define these “reference points” in a way similar to crossings of a diagram? Possibly, if the comultiplication operation is expected to be defined in the language of words for some presentation of \( G \), the operation should be defined in terms of some letters selected in a specific way?

A similar question can be asked about the group bracket and the parity bracket: we expect a well-defined map from \( G \) to the direct sum of various tensor products \( G \otimes G \otimes \cdots \otimes G \).

Can we define such a bracket for elements \( g \in G \) realizing the principle \([g] = g\) and solving the word problem/ the conjugacy problem for some groups?

References

[ChM] M.W.Chrisman, V.O.Manturov, Fibered Knots and Virtual Knots, Journal of Knot Theory and Its Ramifications, Vol. 22, No. 12 (2013) 1341003 (23 pages).

[Goldman] W. Goldman, “Invariant functions on Lie groups and Hamiltonian flows of surface group representations”, Inventiones Mathematicae 85, 263–302 (1986).

[IMN] D.P.Ilyutko, V.O.Manturov, I.M.Nikonov, Parity in Knot Theory and Graph-Links, CMFD, 41 (2011), 3163

[FM] Invariants of homotopy classes of curves and graphs on 2-surfaces, Fundam. Prikl. Mat., 2013, Volume 18, Issue 4, Pages 89105

[KM] V.A.Krasnov, V.O.Manturov, Graph-Valued Invariants of Virtual and Classical Links and Minimality Problem Journal of Knot Theory and Its Ramifications Vol. 22, No. 12 (2014) 1341006 (14 pages)
[Ma1] V.O. Manturov, *Parity in Knot Theory*, Mat. Sbornik, 201:5 (210), pp. 65-110.

[Ma2] V.O. Manturov, *Parity and Cobordisms of Free Knots*, Mat. Sbornik, 203:2 (2012), pp. 45-76.

[Ma3] V. O. Manturov, “An almost classification of free knots”, *Doklady Mathematics*, 88:2 (2013), 556–558 (Original Russian Text in *Doklady Akademii Nauk*, 452:4 (2013), 371–374).

[Ma4] V.O. Manturov, New Parities and Coverings over Free Knots, (in Russian) Izvestiya RAN, Submitted

[Ma5] V.O. Manturov, Parity and Projection from virtual knots to classical knots, Journal of Knot Theory and Its Ramifications Vol. 22, No. 9 (2013) 1350044 (20 pages).

[Ma6] V.O. Manturov, Virtual Crossing Numbers for Virtual knots, Journal of Knot Theory and Its Ramifications Vol. 21, No. 13 (2012) 1240009 (13 pages)

[Tu1] V. G. Turaev, “Topology of words”, *Proc. Lond. Math. Soc.* 95:3, 360–412 (2007).

[1] V. G. Turaev, “Algebras of loops on surfaces, algebras of knots, and quantization”, *Braid Group, Knot Theory and Statistical Mechanis* (C. N. Yang and M. L. Ge, eds), *Math. Phys.* 9, World Sci. Publ., Singapore, 59–95 (1989).
