On a Kählerian space-time manifold

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Abstract In this paper, the theory of space-time in 4-dimensional Kähler manifold has been studied. We have also discussed the Einstein equation with cosmological constant in perfect fluid Kählerian space-time manifold. By taking conformally flat perfect fluid Kähler space-time manifold, we have obtained interesting results related to sectional curvatures. In last two sections we have studied weakly symmetric and weakly Ricci symmetric perfect fluid Kähler space-time manifolds.

Keywords: Kählerian space-time manifold. Einstein equation. conformally flat manifold. weakly symmetric manifold. weakly Ricci symmetric manifold

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1 Introduction

The study of space-time is associated with 4-dimensional semi-Riemannian manifolds equipped with Lorentz metric $g$ having signature $(-,+,+,+)$. B. O’Neill [12] discussed the application of semi-Riemannian geometry in the theory of relativity. The curvature structure of the space-time is studied by V. R. Kaigorodov [9] in 1983. After then, these ideas of general relativity of space-time are extended by A. K. Roychoudhury, S. Banerji and A. Banerji [14], M. C. Chaki and S. Roy [2], A. A. Shaikh, Dae Won Yoon and S. K. Hui [15], U. C. De and G. C. Ghosh [3] and many other Differential Geometers and Physicist.

In 2004 U. C. De and G. C. Ghosh [5] considered the weakly Ricci symmetric space-time manifold and obtained some results related to it. The weakly symmetric and weakly Ricci symmetric manifolds are introduced by L. Tamassy and T. Q. Binh [1] [10]. Also, M. Prvanovic [13] and U. C. De and S. Bandyopdhayay [3] explained it with examples. L. Tamassy, U. C. De and T. Q. Binh [17] also found some interesting results related to weakly symmetric and weakly Ricci symmetric Kähler manifolds.

An $n$-dimensional Riemannian manifold is said to be a weakly symmetric if the curvature tensor $R$ of type $(0,4)$ of the manifold satisfies

$$\langle \nabla_X R \rangle(Y, Z, U, V) = A(X)R(Y, Z, U, V)$$

$$+ B(Y)R(X, Z, U, V) + C(Z)R(Y, X, U, V)$$

$$+ D(U)R(Y, Z, X, V) + E(V)R(Y, Z, U, X),$$

(1.1)
and if the Ricci tensor $S$ of the manifold satisfies

\[(1.2) \quad (\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + C(Z)S(Y, X),\]

then manifold is called weakly Ricci symmetric manifold. Where $A, B, C, D, E$ are simultaneously non-vanishing 1-forms and $X, Y, Z, U, V$ are vector fields.

In 1995, Prvanovic [13] proved that if the manifold be a weakly symmetric manifold satisfying (1.1) then $B = C = D = E$. In this paper we have taken $B = C = D = E = \omega$, therefore, the equations (1.1) and (1.2) can be written as

\[(1.3) \quad (\nabla_X R)(Y, Z, U, V) = A(X)R(Y, Z, U, V)\]
\[\quad + \omega(Y)R(X, Z, U, V) + \omega(Z)R(Y, X, U, V)\]
\[\quad + \omega(U)R(Y, Z, X, V) + \omega(V)R(Y, Z, U, X),\]

and

\[(1.4) \quad (\nabla_X S)(Y, Z) = A(X)S(Y, Z) + \omega(Y)S(X, Z) + \omega(Z)S(Y, X),\]

where $g(X, \rho) = \omega(X)$ and $g(X, \alpha) = A(X)$.

Recently, U. C. De and A. De [4], A. De, C. Özgür and U. C. De [7] extended the theory of space-time by studying the application of almost pseudo-conformally symmetric Ricci-recurrent manifolds and conformally flat almost pseudo Ricci-symmetric manifolds respectively. In 2012 and 2014, S. Mallick and U. C. De [6, 11] gave examples of weakly symmetric and conformally flat weakly Ricci symmetric space-time manifolds and proved that a conformally flat weakly Ricci symmetric space-time with non-zero scalar curvature is the Robertson-Walker space-time and the vorticity and the shear vanish. Also, they have shown that a weakly symmetric perfect fluid space-time with cyclic parallel Ricci tensor can not admit heat flux. In 2014, U. C. De and Ljubica Velimirovic [8] explained the space-time with semi-symmetric energy-momentum tensor and proved that a space-time manifold with semi-symmetric energy momentum tensor is Ricci semi-symmetric.

In the consequence of this type of development we have studied the general relativity of space-time in 4-dimensional Kähler manifold and this type of manifold is called the Kählerian space-time manifold.

A 4-dimensional space-time manifold is said to be a Kählerian space-time manifold if the following conditions hold:

\[(1.5) \quad F^2(X) = -X,\]
\[(1.6) \quad g(\overline{X}, \overline{Y}) = g(X, Y),\]
\[(1.7) \quad (\nabla_X F)(Y) = 0,\]

where, $F$ is a tensor field of type (1,1) such that $F(X) = \overline{X}$, $g$ is a Riemannian metric and $\nabla$ is a Levi-Civita connection.

We know that in a Kähler manifold the Ricci tensor $S$ satisfies

\[(1.8) \quad S(\overline{X}, \overline{Y}) = S(X, Y).\]
2 Perfect fluid Kähler space-time manifold

We know that the Einstein equation with cosmological constant for the perfect fluid space-time is given by

\( S(X, Y) - \frac{R}{2} g(X, Y) + \lambda g(X, Y) = k[(\sigma + p)\omega(X)\omega(Y) + pg(X, Y)] \),

where \( k \) is the gravitational constant, \( \sigma \) is energy density, \( p \) is isotropic pressure of the fluid and \( \omega \) is 1-form defined by \( \omega(X) = g(X, \rho) \) for time-like vector field \( \rho \). The time-like vector field \( \rho \) is called velocity of the fluid and satisfies \( g(\rho, \rho) = -1 \). Also, the energy density \( \sigma \) and the pressure \( p \) can be described in the sense that if \( \sigma \) vanishes then content matter of the fluid is not pure and if the pressure \( p \) vanishes then the fluid is dust.

Now replacing \( X \) and \( Y \) by \( \overline{X} \) and \( \overline{Y} \) respectively in (2.1) and using (1.6) and (1.8), we get

\( S(X, Y) - \frac{R}{2} g(X, Y) + \lambda g(X, Y) = k[(\sigma + p)\omega(\overline{X})\omega(\overline{Y}) + pg(X, Y)] \).

Subtracting (2.1) from (2.2), we have

\( k(\sigma + p)[\omega(\overline{X})\omega(\overline{Y}) - \omega(X)\omega(Y)] = 0. \)

Putting \( Y = \rho \) in (2.3), we obtained

\( k(\sigma + p)\omega(X) = 0, \)

since \( k \neq 0 \) and \( \omega(X) \neq 0 \), we have

\( \sigma + p = 0, \)

which shows the fluid behaves like a cosmological constant. Also from (2.5), we have \( \sigma = -p \) which represents a rapid expansion of space-time in cosmology and known as Inflation.

Now using (2.5), equation (2.1) gives

\( S(X, Y) = (\frac{R}{2} - \lambda + k.p)g(X, Y). \)

Putting \( X = Y = e_i, 1 \leq i \leq 4 \) in (2.6) and taking summation over \( i \), we can easily obtained

\( \lambda - k.p = \frac{R}{4}. \)

From (2.6) and (2.7), we get

\( S(X, Y) = \frac{R}{4}g(X, Y). \)

Hence from above discussion, we conclude the following:

**Theorem 2.1.** If \( M \) be a perfect fluid Kähler space-time manifold satisfying the Einstein equation with cosmological constant then

(i) the Einstein equation is independent of the isotropic pressure \( p \), energy density \( \sigma \), cosmological constant \( \lambda \) and the gravitational constant \( k \).

(ii) the space-time manifold is an Einstein manifold.
Since an Einstein manifold is the manifold of constant scalar curvature \( r \), therefore, equation (2.7) implies the pressure \( p \) is constant and hence from (2.5) we have the energy density \( \sigma \) is constant.

It is well known [12] that the Energy equation for the perfect fluid is given by

\[(2.9)\quad \rho \sigma = -(\sigma + p) \text{div} p. \]

Using (2.5) in (2.9), we get

\[(2.10)\quad \rho \sigma = 0. \]

Above equation implies \( \sigma = 0 \) as \( \rho \neq 0 \). Because if \( \rho = 0 \) then we have contradiction that \( g(\rho, \rho) = -1 \). But then the equation (2.5) gives \( p = 0 \) and hence the energy momentum tensor \( T(X, Y) = (\sigma + p)\omega(X)\omega(Y) + pg(X, Y) \) vanishes.

Thus we conclude:

**Theorem 2.2.** If \( M \) be a perfect fluid Kähler space-time manifold satisfying the Einstein equation with cosmological constant then

(i) the energy density \( \sigma \) and the isotropic pressure \( p \) vanish i.e. the content matter of the fluid is not pure and the perfect fluid is dust.

(ii) the energy momentum tensor \( T(X, Y) \) vanishes i.e. the space-time is vaccum.

Since the velocity vector field \( \rho \) is constant, we have

\[(2.11)\quad \text{div} \rho = 0 \quad \text{and} \quad \nabla \rho \rho = 0. \]

Hence from equations (2.11), we can state:

**Theorem 2.3.** If \( M \) be a perfect fluid Kähler space-time manifold satisfying Einstein equation with cosmological constant then the expansion scalar and the acceleration vector vanish.

### 3 Conformally flat perfect fluid Kähler space-time manifold

The Weyl conformal curvature tensor \( C \) on an \( n \)-dimensional manifold \( M \) is defined by

\[(3.1)\quad C(X, Y, Z, T) = R(X, Y, Z, T) - \frac{1}{(n-2)}[S(Y, Z)g(X, T) - S(X, Z)g(Y, T) + S(X, T)g(Y, Z) - S(Y, T)g(X, Z)] + \frac{r}{(n-1)(n-2)}[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)]. \]

If the manifold be conformally flat, then above equation implies

\[(3.2)\quad R(X, Y, Z, T) = \frac{1}{2}[S(Y, Z)g(X, T) - S(X, Z)g(Y, T) + S(X, T)g(Y, Z) - S(Y, T)g(X, Z)] - \frac{r}{6}[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)]. \]
Putting the value of Ricci tensor $S$ from (2.8) in (3.2), we have

\[ R(X, Y, Z, T) = \frac{r}{12}[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)]. \]

If the 3-dimensional distribution of the manifold orthogonal to $\rho$ is denoted by $\rho^\perp$ then by putting $Z = Y$ and $T = X$ in (3.3), we can write

\[ R(X, Y, Y, X) = \frac{r}{12}[g(X, X)g(Y, Y) - g(X, Y)g(X, Y)], \]

where $X, Y \in \rho^\perp$.

Putting $Y = \rho$ in (3.4), we have

\[ R(X, \rho, \rho, X) = -\frac{r}{12}g(X, X), \]

where $X \in \rho^\perp$.

If we denote the sectional curvatures determined by $X, Y$ and $X, \rho$ by $K(X, Y)$ and $K(X, \rho)$ respectively then from (3.4) and (3.5), we have

\[ K(X, Y) = \frac{R(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)g(X, Y)} = \frac{r}{12}; \]

and

\[ K(X, \rho) = \frac{R(X, \rho, \rho, X)}{g(X, X)g(\rho, \rho) - g(X, \rho)g(X, \rho)} = \frac{r}{12}. \]

Thus we have:

**Theorem 3.1.** If $M$ be a conformally flat perfect fluid Kähler space-time manifold satisfying the Einstein equation with cosmological constant then the sectional curvature determined by $X, Y$ and $X, \rho$ are same and equal to $\frac{r}{12}$ i.e. $K(X, Y) = K(X, \rho) = \frac{r}{12}$.

Karcher [10] has defined that a Lorentzian manifold is said to be infinitesimally spatially isotropic relative to the velocity vector field $\rho$ if the Riemannian curvature tensor $R$ satisfies

\[ R(X, Y, Z, T) = a[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)], \]

and

\[ R(X, \rho, \rho, Y) = b.g(X, Y), \]

where $a$ and $b$ are real valued functions and $X, Y, Z, T \in \rho^\perp$. Putting $Y = Z = \rho$ in (3.3), we get

\[ R(X, \rho, \rho, T) = -\frac{r}{12}g(X, T). \]

Hence from (3.3) and (3.10), we can state:

**Theorem 3.2.** If $M$ be a conformally flat perfect fluid Kähler space-time manifold satisfying the Einstein equation with cosmological constant then the manifold is infinitesimally spatially isotropic relative to the velocity vector field $\rho$. 

Thus we have:
4 Weakly symmetric perfect fluid Kähler space-time manifold

If $M$ be a weakly symmetric Kähler manifold then we have

\[(4.1) \quad R(\overline{Y}, \overline{Z}, U, V) = R(Y, Z, U, V).\]

Taking covariant derivative of (4.1), we get easily

\[(4.2) \quad (\nabla_X R)(\overline{Y}, \overline{Z}, U, V) = (\nabla_X R)(Y, Z, U, V).\]

Using (1.3) in (4.2), we have

\[(4.3) \quad \omega(Y) R(X, Z, U, V) + \omega(Z) R(Y, X, U, V) = \omega(Y) R(X, Z, U, V) + \omega(Z) R(Y, X, U, V).\]

Putting $Z = U = e_i, 1 \leq i \leq 4$ in (4.3) and taking summation over $i$, we obtained

\[(4.4) \quad \omega(Y) S(X, V) - R(Y, X, V, \rho) = \omega(Y) S(X, V) + R(Y, X, V, \overline{\rho}).\]

By using (2.8), equation (4.4) implies

\[(4.5) \quad \frac{r}{4} \omega(Y) g(X, V) - R(Y, X, V, \rho) = \frac{r}{4} \omega(Y) g(X, V) + R(Y, X, V, \overline{\rho}).\]

Putting $X = V = e_i, 1 \leq i \leq 4$ in (4.5) and taking summation over $i$, we get

\[(4.6) \quad S(Y, \rho) = \frac{r}{2} \omega(Y).\]

or

\[(4.7) \quad S(Y, \rho) = \frac{r}{2} g(Y, \rho).\]

Replacing $\rho$ by $\overline{\rho}$ in (4.7), we can write

\[(4.8) \quad S(Y, \overline{\rho}) = \frac{r}{2} g(Y, \overline{\rho}).\]

Hence from (4.7) and (4.8), we can state:

**Theorem 4.1.** If $M$ be a weakly symmetric perfect fluid Kähler space-time manifold satisfying the Einstein equation with cosmological constant then $\rho$ and $\overline{\rho}$ are the eigen vector of the Ricci tensor $S$ with respect to eigen value $\frac{r}{2}$.

Now putting $X = V = e_i, 1 \leq i \leq 4$ in (1.3) and summing over $i$, we have

\[(4.9) \quad (\text{div} R)(Y, Z) U = R(Y, Z, U, \alpha) + \omega(Y) S(Z, U) + \omega(Z) S(Y, U) + R(Y, Z, U, \rho).\]

From (2.8), it can be easily obtained

\[(4.10) \quad (\nabla_Y S)(Z, U) = \frac{r}{4} (\nabla_Y g)(Z, U) = 0.\]
Using (4.10) in Bianchi second identity, we can write

\[(4.11) \quad (\text{div} R)(Y, Z)U = (\nabla_Y S)(Z, U) - (\nabla_Z S)(Y, U) = 0.\]

By using (4.11), the equation (4.9) gives

\[(4.12) \quad R(Y, Z, U, \alpha) + \omega(Y)S(Z, U) - \omega(Z)S(Y, U) + R(Y, Z, U, \rho) = 0.\]

Putting \(Z = U = e_i, 1 \leq i \leq 4\) in (4.12) and taking summation over \(i\), we have

\[(4.13) \quad S(Y, \alpha) + r\omega(Y) = 0.\]

Using (2.8) in (4.13), we can write

\[(4.14) \quad \frac{r}{4}g(Y, \alpha) + rg(Y, \rho) = 0.\]

Replacing \(Y\) by \(\rho\) in (4.14), we get

\[(4.15) \quad r[g(\alpha, \rho) - 4] = 0,\]

which implies either \(r = 0\) or \(g(\alpha, \rho) = 4\).

Hence, from above discussion we have

**Theorem 4.2.** If \(M\) be a weakly symmetric perfect fluid Kähler space-time manifold satisfying the Einstein equation with cosmological constant then either the manifold is of zero scalar curvature or the associated vector fields \(\alpha\) and \(\rho\) are related by \(g(\alpha, \rho) = 4\).

5 Weakly Ricci symmetric perfect fluid Kähler space-time manifold

From equation (4.10) it is clear that if the manifold is perfect fluid Kähler space-time manifold then \((\nabla_X S)(Y, Z) = 0\), therefore from (1.4), we have

\[(5.1) \quad A(X)S(Y, Z) + \omega(Y)S(Z, X) + \omega(Z)S(X, Y) = 0,\]

for weakly Ricci symmetric perfect fluid Kähler space-time manifold. Using (2.8) in (5.1), we can write

\[(5.2) \quad \frac{r}{4}[A(X)g(Y, Z) + \omega(Y)g(Z, X) + \omega(Z)g(X, Y)] = 0.\]

Equation (5.2) implies either scalar curvature \(r = 0\) or

\[(5.3) \quad A(X)g(Y, Z) + \omega(Y)g(Z, X) + \omega(Z)g(X, Y) = 0.\]

Now, if (5.3) holds then by replacing \(Y\) and \(Z\) by \(\bar{Y}\) and \(\bar{Z}\) in (5.3) and using (1.6), we get

\[(5.4) \quad A(X)g(Y, Z) + \omega(\bar{Y})g(Z, X) + \omega(\bar{Z})g(X, \bar{Y}) = 0.\]

Subtracting (5.3) from (5.4), we have

\[(5.5) \quad \omega(\bar{Y})g(X, \bar{Z}) - \omega(Y)g(X, Z) + \omega(\bar{Z})g(X, \bar{Y}) - \omega(Z)g(X, Y) = 0.\]
Putting $X = Z = e_i, 1 \leq i \leq 4$ in (5.5) and taking summation over $i$, we get

\begin{equation}
(5.6) \quad \omega(Y) = 0,
\end{equation}

which is not possible because $g(\rho, \rho) = -1$.

Hence, we can state that:

**Theorem 5.1.** There does not exist a weakly Ricci symmetric perfect fluid Kähler space-time manifold satisfying the Einstein equation with cosmological constant having non-zero scalar curvature tensor.

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