Cosmological dynamics of scalar field with non-minimal kinetic term

H. Kröger\textsuperscript{a,}\ast, G. Melkonyan\textsuperscript{a,†}, S.G. Rubin\textsuperscript{b,c,‡}

\textsuperscript{a} Département de Physique, Université Laval, Québec
\textsuperscript{b} Moscow State Engineering Physics Institute, Moscow, Russia
\textsuperscript{c} Center for Cosmoparticle Physics "Cosmion", Moscow, Russia

We investigate dynamics of scalar field with non-minimal kinetic term. Nontrivial behavior of the field in the vicinity of singular points of kinetic term is observed. In particular, the singular points could serve as attractor for classical solutions.

key words: kinetic term, scalar, gravity, inflation, vacuum energy

I. INTRODUCTION

In this paper we investigate dynamics of scalar field theory with the kinetic term dependent on the field value. This hypothesis provides an additional resources in an explanation of observational data. The small value of cosmological term, consistent with recent experimental data\textsuperscript{1} can be explained using a non-canonical form of the kinetic term in the scalar field (like in the quintessential model\textsuperscript{2,3}). Substantial review of different ways of origin of the cosmological term can be found in\textsuperscript{4}. A non-trivial kinetic term could be responsible for a new coupling between adiabatic and entropy perturbations\textsuperscript{5}.

Such a model may be considered as the restricted scalar-tensor theory (STT) of gravity\textsuperscript{6}. There is much current interest in the STT, because it allows for explanations of variety of cosmological phenomena like, e.g., inflation at a low energy scale\textsuperscript{7} and dark matter caused by phantom fields\textsuperscript{8}. This theory, being a generalization of general relativity, possesses a much richer structure then the latter. The model has a sector of a scalar field and it seems natural to relate this component to the inflation. Historically, the first internally self-consistent inflationary model\textsuperscript{9} appears to be equivalent to the special limiting case of STT\textsuperscript{10}.

The exact form of STT action has not been derived yet from a more fundamental theory. Thus a large number of phenomenological potentials and kinetic terms are considered in current models. Some of those can be excluded being inconsistent with astrophysical data. Still a substantial number of them remains topical. In this connection two questions can be raised and need to be answered: (i) What shape of potential is realized in our universe? Finding an answer would require that the theoretical prediction from such model potential would fit most observational data. Suppose one would find answer, then the next question is: (ii) Why is this particular shape of potential (or kinetic term) realized in nature? What are the underlying reasons?

Some theoretical hints on the form of the potential have been given by supergravity, which predicts an infinite power series expansion in the scalar field potential\textsuperscript{11}. Its minima, if they exist, correspond to the stationary states of the field. In the low energy regime it is reasonable to retain only a few terms (lowest powers in the Taylor expansion) of the scalar field\textsuperscript{12}. However, the potential caused, e.g., by supergravity could correspond to a function with infinite set of the potential minima. There is still no physical law which limits the potential to possess only a finite number of minima. In the vicinity of each of the minima the potential has an individual form and hence the universe associated with such minimum may be individually different from others. Our own universe is associated with a particular potential minimum, not necessarily located at $\varphi = 0$. The supposition that the potential possesses infinite number of randomly distributed minima is self-consistent\textsuperscript{13}. A similar behavior may hold also for the kinetic term of STT.

The observed smallness of the value of the $\Lambda$ term is explained usually on the basis of a more fundamental theory like supergravity or the anthropic principle. Our point of view is that we have to merge these approaches. The more fundamental theory supplies us with an infinite set of minima of the potential. These minima having an individual shape are responsible for the formation of those universes used in the anthropic picture.

Actually, any way of introducing a scalar field leads to a theory similar to STT provided that quantum corrections are taken into account. Indeed, STT arises necessarily in ordinary quantum field theory with a scalar field. For example, it is a standard result that the kinetic term of the effective action acquires a multiplier which is a function of the field\textsuperscript{14}.

\textsuperscript{\ast} Email: hkroger@phy.ulaval.ca
\textsuperscript{†} Email: gmelkony@phy.ulaval.ca
\textsuperscript{‡} Email: sergeirubin@mtu-net.ru
Here we consider an action with a non-trivial kinetic term of the following form
\[ S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} + \frac{1}{2} K(\varphi) \partial_\mu \varphi \partial^\mu \varphi - V(\varphi) \right]. \] (1)

In this paper we concentrate on the influence of singular points of the kinetic term on the scalar field dynamics. The singular points are not exceptional cases. E.g., the well known Brans - Dicke model does contain a singularity at zero value of the field \( \varphi \). If a multiplier of the Ricci scalar equals to zero at some point in the Jordan frame, the kinetic term will be singular in the Einstein frame (see e.g. [10]). Quintessential models with negative power law are another well known example for the potential of such a sort.

By a suitable change of variable the function \( K \) can be transformed to become \( K(\varphi) = \pm 1 \). Such behavior is physical during some inflationary period [20] but not at some recent epoch where the field fluctuates around the singular point. The similar problem is discussed in refs. [21], [22] and [23] in the framework of more general case of STT.

The general form of the kinetic term considered in this paper is
\[ K(\varphi) = \frac{M^n}{(\varphi - \varphi_s)^n}. \] (2)

This form is correct at least in the vicinity of the singularity if \( n > 0 \), or zero if \( n < 0 \). Here \( M \) is some parameter (its value will be discussed below).

The equation of motion has the form
\[ K(\varphi) [\ddot{\varphi} + 3H \dot{\varphi}] + \frac{1}{2} K(\varphi) \dot{\varphi}^2 + V(\varphi)' = 0 \] (3)
in the Friedmann-Robertson-Walker universe where \( H \) denotes the Hubble parameter. Keeping in mind expression (2) we have
\[ \ddot{\varphi} + 3H \dot{\varphi} - \frac{n}{2(\varphi - \varphi_s)} \dot{\varphi}^2 + V(\varphi_s)'(\varphi - \varphi_s)^n/M^n = 0. \] (4)

We observe that \( \varphi = \varphi_s \) is a stationary solution for any smooth potential \( V \) and \( n > 0 \). The cosmological energy density of the vacuum is connected usually with one of the potential minima. In the case considered here it is not like this - it could be located at the singular point of the kinetic term \( K(\varphi) \).

In the following we limit ourselves by considering the first two terms in Taylor series of the potential and shift the field variable so that the potential acquires the simple form
\[ V(\varphi) = V_0 + m^2 \varphi^2/2. \] (5)

To facilitate a more detailed analysis, a new auxiliary variable \( \chi \) is useful. We suggest the substitution of variables \( \varphi \rightarrow \chi \) in the following manner
\[ d\chi = \pm \sqrt{K(\varphi)}d\varphi, \quad K(\varphi) > 0. \] (6)

An action in terms of the auxiliary field \( \chi \) has the form
\[ S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} + sgn(\varphi - \varphi_s) \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - U(\chi) \right], \] (7)
where the potential \( U(\chi) \equiv V(\varphi(\chi)) \) 'partly discontinues' function. Its form depends on the form of initial potential \( V(\varphi) \), the form of kinetic term and a position of the singularities. Now let us consider particular cases of \( K(\varphi) \).

## II. THE CASE \( n = 1 \)

The kinetic term \( K(\varphi) = M/(\varphi - \varphi_s) \) and Eq.(6) reads
\[ \varphi = \varphi_s + sgn(\varphi - \varphi_s) \chi^2/4M \] (8)
with the potential
\[ U(\chi) \equiv V(\varphi(\chi)) = V_0 + \frac{m^2}{2} (\varphi_s + sgn(\varphi - \varphi_s) \chi^2/4M)^2; \varphi_s > 0; |\chi| < \infty. \] (9)
FIG. 1: The position of the singular point of the kinetic term.

FIG. 2: The form of the potential for the case $n = 1$. Chosen branches are: $\varphi > \varphi_s \leftrightarrow \chi > 0; \varphi < \varphi_s \leftrightarrow \chi < 0$. In the latter case the auxiliary field $\chi$ behaves like phantom field moving classically to the local maximum.

The situation is illustrated by Figure 1. If the physical field $\varphi < \varphi_s$, than the auxiliary field $\chi$ behaves like the phantom field; if the field $\varphi$ fluctuates around $\varphi_s$, the substitution is meaningless.

The inflationary scenario on the basis of potential is well known and we discuss only its final stage. If $\varphi_s = 0$ and $V_0 = 0$ exactly, we obtain a potential $U \sim \chi^4$. If $\varphi_s > 0$, the auxiliary field $\chi$ oscillates, according to classical equation, with damping around its final position at $\chi = 0$ - see Figure 2.

For the physical field $\varphi$, the motion looks like oscillations around the point $\varphi_s$. Finally, the field is captured in the vicinity of the singular point, supplying an energy density of the vacuum equal to

$$\Lambda = V_0 + \frac{1}{2} m^2 \varphi_s^2,$$

(10)

rather than $\Lambda = V_0$ as it could be expected from expression.

The problem of smallness of the vacuum energy density, $\Lambda = 10^{-123} M_P^4$, is still topical. In other words, what physical reason adjusts the location of the singular point $\varphi_s$ such that it lies in the extremely tiny interval $\sqrt{-2V_0/m^2} \div \sqrt{-2V_0/m^2 + 2\Lambda/m^2}$? In the view of the above discussion, we could expect that some minima $\varphi_m$ of the potential could incorporate critical point(s) $\varphi_s$ at specific distance from this minimum. Now the problem is reformulated as: “what part of an infinite amount of the minima contains singular points located so closely to the minima?” It seems plausible that this part is very small, but not zero, due to an infinite number of minima. Only this part is important - it represents those vacua where galaxies could be formed.
FIG. 3: The form of the potential of the auxiliary field

III. THE CASE \( n = 2 \)

In this case, the inflationary period is more interesting than in the previous case, so we will discuss it more thoroughly. The kinetic term has the form \( K(\varphi) \equiv K_s(\varphi) = M^2/(\varphi - \varphi_s)^2 \). The new auxiliary field is connected with the physical one by

\[
\chi = M \ln \left| \frac{\varphi - \varphi_s}{\varphi_s} \right|
\]

so that the potential has the form

\[
U(\chi) = \frac{1}{2} m^2 \varphi_s^2 (1 + \text{sgn}(\varphi_s) \cdot \text{sgn}(\varphi - \varphi_s) \cdot e^{\chi/M})^2 + V_0.
\]

If \( \varphi_s > 0 \) and \( \varphi > \varphi_s \), the potential mimics the quintessential model with nonzero vacuum energy density \( \Lambda = \frac{1}{2} m^2 \varphi_s^2 + V_0 \).

The case \( \varphi_s < 0 \) and \( \varphi > \varphi_s \) is much more interesting. This potential is highly asymmetric so that a behavior of inflaton is rather different at \( \chi < 0 \) and at \( \chi > 0 \). Let us suppose that inflation starts with \( \chi_{in} > 0 \), i.e. point 1 in Figure 3. The picture is similar to the improved quintessence potential [26], without problems of the radiation-dominated stage during Big Bang nucleosynthesis. A modern epoch is characterized by a large negative value of the field \( \chi \). It slowly varies along the flat part of the potential with exponentially slow variation of the vacuum energy density around the value \( \Lambda \).

It is worth to discuss another possibility, when the inflation starts at \( \chi = \chi_{in} < 0 \), see Figure 3, point 2. The form of the potential \( \chi \to -\infty \) is rather plain being suitable for slow rolling, so it is reasonable to expect some new results. Indeed, our estimations indicate that the inflation could take place at the parameter values \( M \sim M_P \) (Planck scale), \( m \sim \varphi_s \sim 10^{-3} M_P \) (GUT scale) and \( |\chi| \sim M_P \ln 5 \). Evidently, this inflationary model does not require fine tuning of its parameters. The inflation ends at the minimum of the potential, \( \chi = 0 \) and the final energy density of the vacuum becomes \( \Lambda = V_0 \).

IV. THE CASE \( n = -1 \)

A nontrivial situation takes place if the kinetic function has zero value at some point, i.e. \( K(\varphi) = (\varphi - \varphi_s)/M \). Eq.6 leads to the classical equation

\[
(\varphi_s - \varphi) \cdot (\dot{\varphi} + 3H\varphi) - \frac{1}{2} \dot{\varphi}^2 + M \cdot V(\varphi)' = 0
\]

This equation is quite uncommon. Indeed, if the zero point \( \varphi_s \) of the kinetic term does not coincide exactly with the position of the potential minimum at \( \varphi = 0 \), the point \( \varphi = \varphi_s \) is not a stationary solution of this equation. On the other hand, the point \( \varphi_s \) is some kind of attractor. Namely, if the field value is larger than \( \varphi_s \), then \( K(\varphi) > 0 \) and we have standard rolling of the field down to the singular point \( \varphi_s \). If the field value is smaller than \( \varphi_s \), then \( K(\varphi) < 0 \). The field behaves like the phantom field [24], climbing up to the potential and thus tending toward the
point $\varphi_s$. Classically, the situation looks very strange - the singular point attracts the solution, but forbids it to stay there forever. Evidently, the field fluctuates stochastically around the singular point. Additional discussion on this problem can be found in the paper [27].

Let the initial field value $\varphi = \varphi_{in} > \varphi_s$. An appropriate variable substitution looks like

$$\varphi = \varphi_s + \text{sgn}(\varphi - \varphi_s) \cdot \gamma |\chi|^{2/3}, \quad \gamma \equiv (3\sqrt{m}/2)^{2/3} \quad \varphi > \varphi_s.$$  

The potential of the auxiliary field $\chi$ becomes

$$U(\chi) = \frac{1}{2} m^2 (\varphi_s + \text{sgn}(\varphi - \varphi_s) \cdot \gamma |\chi|^{2/3})^2 + V_0.$$  

(13)

$U(\chi)$ is finite at $\chi = 0$ but its derivative is singular,

$$U'_{\chi \to +0} = -\frac{2}{3} m^2 \varphi_s^2 \gamma |\chi|^{-1/3}.$$  

The potential (13) behaves like $\chi^{4/3}$ at large field values. It leads to standard inflation with moderate fine tuning of the parameters. If $\varphi_s > 0$, the field $\varphi$ will oscillate around critical point and energy density.

**V. DISCUSSION**

It is shown that the region where the kinetic term changes its sign gives new possibilities for the scalar field dynamics. It takes place even for the simplest form of potential. Depending on the position of the singular point of the kinetic term specific forms of the potential of the auxiliary field can be obtained. One of the main result is that the scalar field could be located in the singular points of the kinetic term rather than in minima of the potential. Another interesting result is that if the kinetic term becomes zero at the singular point, the behavior of scalar field acquires stochastic character.

The problem of the cosmological $\Lambda$ - term acquires another sense. One has to explain the extremal proximity of the singular point of the kinetic term and zero point of the potential. Such situation seems absolutely accidental. Moreover, the probability of such event to occur is very small. The problem can be solved in the framework of the random potential and a kinetic term of the scalar field discussed in the Introduction. In this case we have infinite number of singular points of the kinetic term. Fluctuations of the scalar field which were created at high energies moves classically to stationary points. Those of them who reach stationary points with appropriate energy density could form a universe similar to our Universe. This energy density ($\sim 10^{-123}M_P^4$) is the result of small value of a concrete potential minimum or small value of the difference $\varphi_s - \varphi_0$, $\varphi_0$ is a zero of the potential. The fraction of such universes is extremely small, but nevertheless is finite because of infinite number of stationary states.

The abundance of the variants discussed in the paper is based on the simplest forms of the potential and the kinetic terms. Nevertheless, we reproduced the quintessence feature, proposed a new possibility for inflation without fine tuning and discussed a fluctuating state of the scalar field as a final state which be could realized in the Universe.

One of us (S.G.R) is grateful to K. A. Bronnikov and V. M. Zhuravlev for discussion. The work was partially performed in the framework of Russian State contract 40.022.1.1.1106, RFBR grant 02 – 02 – 17490 and of grant of Russian Universities UR.02.01.026.

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