Mathematical modeling of one type of three-link robot manipulator

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Abstract. Determining the parameters and conditions under which, when the robot manipulators are operating, the set of optimality criteria can be achieved is an urgent task. In this regard, the problem arises of mathematical modeling of the executive system of a manipulation robot. The article discusses manipulators having rotational, translational and rotational kinematic pairs of the fifth class. Three possible kinematic schemes for such manipulators are given. To describe the movement of the manipulation robot, generalized coordinates are introduced that uniquely determine its configuration. The angles of relative rotations and linear relative displacements of the links were taken as generalized coordinates.

A Cartesian rectangular coordinate system was associated with each link of the manipulator and with the fixed base. The problem of determining the relative position of the links and their position in the inertial space was solved by converting one coordinate system to another. As a result, the basic kinematic relations were found that determine the position, speed, and acceleration of the centers of mass of the links in a fixed coordinate system for robot manipulators having one translational and two rotational degrees of mobility. Based on these relations, the problem of constructing equations of dynamics in the form of Lagrange equations of the second kind was solved. The advantage of this form of recording is the closed form of expressions that determine the dynamics of the system, which makes it possible to correctly take into account the internal forces acting in the system. For each of the manipulators considered, systems of three second-order nonlinear differential equations were obtained that describe their dynamics. Due to the need to assess the load moment occurring on the control drive shaft, external generalized forces were reduced to the corresponding generalized coordinate.

Mathematical modeling of various production processes and mechanisms is currently an urgent task in connection with the need to determine the conditions and parameters under which the established optimality criteria will be achieved. Robot manipulators are widely used, including logging and woodworking industries. In this regard, the goal of the work was to find analytical models of the dynamics of a three-link robot manipulator with rotational, translational and rotational kinematic pairs of the fifth class. All possible kinematic schemes of such manipulators are presented in Figure 1(a-c).

Consider a three-link manipulator, the structural kinematic scheme of which is shown in Figure 1. Handle the task of finding the basic kinematic relations that determine the position, speed, and acceleration of the manipulator links and its dynamics equations [1]. All kinematic pairs of the
specified manipulator are fifth-class kinematic pairs. To uniquely determine its configuration, it is necessary to enter generalized coordinates. If we assume that all the links \( i = 1, 2, 3 \) are solids, then as the generalized coordinates \( q = (q_1, q_2, q_3)^T \) the angles of relative rotations and linear relative displacement of the links are taken. Suppose that the manipulator is located on a fixed base, with which is connected to the Cartesian coordinate system \( O_0x_0y_0z_0 \). Each link of the manipulator is connected to the Cartesian coordinate system \( O_i x_i y_i z_i \).

Figure 1. Kinematic schemes of manipulators with rotational, translational and rotational degrees of mobility.

The choice of a Cartesian coordinate system associated with link \( i \) is as follows:

1) The \( z_i \) axis is directed along the axis of the kinematic pair \( (i, i + 1) \).
2) The \( x_i \) axis is the common perpendicular to the \( z_i \) and \( z_{i-1} \) axes.
3) The \( y_i \) axis is selected so that the resulting coordinate system is right.

With the grip of the manipulator the coordinate system \( O_3 x_3 y_3 z_3 \) is associated. The \( z_3 \) axis characterizes the orientation of the capture axis and is directed along the axis of the last link. The remaining axes are selected according to the above rules. As is known [2], the transformation of the coordinate system \( O_{i-1} x_{i-1} y_{i-1} z_{i-1} \) into the coordinate system \( O_i x_i y_i z_i \) can be carried out by the following sequentially performed operations:

1) Rotation by the angle \( \theta_i \) around the \( z_{i-1} \) axis until the \( x_{i-1} \) and \( x_i \) axes are parallel.
2) Movement by \( s_i \) along the \( z_{i-1} \) axis until the \( x_{i-1} \) and \( x_i \) axis are aligned.
3) Movement by \( a_i \) along the \( x_{i-1} \) axis until the origin \( O_i \) and \( O_{i-1} \) coincide.
4) Rotation by the angle \( \alpha_i \) in relation to \( x_i \) until all coordinate axes are aligned.

The transformation of the coordinate system \( O_{i-1} x_{i-1} y_{i-1} z_{i-1} \) into \( O_i x_i y_i z_i \) is determined by the matrix:

\[
A_i = A_{\theta_i} \cdot A_{s_i} \cdot A_{a_i} = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i \cos \alpha_i & \sin \alpha_i \sin \theta_i & a_i \cos \theta_i \\
\sin \theta_i & \cos \theta_i \cos \alpha_i & -\cos \alpha_i \sin \theta_i & a_i \sin \theta_i \\
0 & \sin \alpha_i & \cos \alpha_i & s_i \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\alpha(i) \\ r_i \\ 0 \\ 0 \\ 0 \\ 1
\end{bmatrix}
\]

For the manipulator under consideration, these matrixes have the form:

\[
A_1 = \begin{bmatrix}
\cos q_1 & -\sin q_1 & 0 & 0 \\
\sin q_1 & \cos q_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
\alpha(1) \\ r_1 \\ 0 \\ 0 \\ 0 \\ 1
\end{bmatrix}
\]
Inject the coordinate system \( O_i^0 x_i^0 y_i^0 z_i^0 \), the beginning of which coincides with the center of mass of the link \( i \)

\[
A_{ic} = \begin{bmatrix}
\alpha(i) & r_{ic} \\
\cdots & \cdots \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

The following vectors determine the position of the centers of mass of the links in the coordinate system associated with the link.

\[
r_{1c} = \begin{bmatrix} 0, 0, \frac{l_1}{2} \end{bmatrix}^T; r_{2c} = \begin{bmatrix} 0, \frac{l_2}{2}, q_2 \end{bmatrix}^T
\]

\[
r_{3c} = \begin{bmatrix} -\frac{l_3}{2} \sin q_3, \frac{l_3}{2} \cos q_3, 0 \end{bmatrix}
\]

The position of the center of mass of link \( i \), and the orientation of the axes of the system \( O_i^0 x_i^0 y_i^0 z_i^0 \) in a fixed coordinate system is determined by the matrixes \( A_{0ic} \)

\[
A_{0ic} = A_{1c}
\]

\[
A_{02c} = A_1 \cdot A_{2c} = \begin{bmatrix}
\alpha(0,2) & r_{02c} \\
\cdots & \cdots \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
A_{03c} = A_1 \cdot A_{2} \cdot A_{3c} = \begin{bmatrix}
\alpha(0,3) & r_{03c} \\
\cdots & \cdots \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Based on this, it is possible to determine the linear velocities of the centers of mass of the links in a fixed coordinate system associated with the base:

\[
\begin{align*}
\mathbf{v}_{01c} &= \mathbf{\dot{r}}_{01c} = \begin{bmatrix} 0, 0, 0 \end{bmatrix}^T \\
\mathbf{v}_{02c} &= \mathbf{\dot{r}}_{02c} = \begin{bmatrix} -\frac{l_2}{2}\cos q_1\dot{q}_1, -\frac{l_2}{2}\sin q_1\dot{q}_1, \dot{q}_2 \end{bmatrix}^T \\
\mathbf{v}_{03c} &= \mathbf{\dot{r}}_{03c} = \begin{bmatrix} -\frac{l_2}{2}\cos q_1\sin q_2\dot{q}_3 + \frac{l_2}{2}\sin q_1\cos q_3\dot{q}_3 - \frac{l_2}{2}\cos q_1\dot{q}_3, -\frac{l_2}{2}\cos q_1\sin q_3\dot{q}_1 - \frac{l_2}{2}\cos q_1\cos q_3\dot{q}_3 - \frac{l_2}{2}\sin q_1\dot{q}_3, \dot{q}_2 \end{bmatrix}^T
\end{align*}
\] (11)

To determine the projections of the angular velocity vector of the link \(i\) on the axis of the associated coordinate system, the formulas can be used:

\[
\begin{align*}
\omega_{ix} &= \alpha_{13}(0, i)\dot{\alpha}_{12}(0, i) + \alpha_{23}(0, i)\dot{\alpha}_{22}(0, i) + \alpha_{33}(0, i)\dot{\alpha}_{32}(0, i) \\
\omega_{iy} &= \alpha_{13}(0, i)\dot{\alpha}_{11}(0, i) + \alpha_{23}(0, i)\dot{\alpha}_{21}(0, i) + \alpha_{33}(0, i)\dot{\alpha}_{31}(0, i) \\
\omega_{iz} &= \alpha_{12}(0, i)\dot{\alpha}_{11}(0, i) + \alpha_{22}(0, i)\dot{\alpha}_{22}(0, i) + \alpha_{32}(0, i)\dot{\alpha}_{32}(0, i)
\end{align*}
\] (12) to (15)

The derivatives are calculated, the components of the angular acceleration vector are found:

\[
\begin{align*}
\epsilon_1 &= \begin{bmatrix} 0, 0, \dot{q}_1 \end{bmatrix}^T \\
\epsilon_2 &= \begin{bmatrix} 0, \ddot{q}_1, 0 \end{bmatrix}^T \\
\epsilon_3 &= \begin{bmatrix} \dot{q}_1\sin q_3 + \dot{q}_1\dot{q}_3\cos q_3 \\
-\ddot{q}_3 \\
\dot{q}_1\cos q_3 - \dot{q}_1\dot{q}_3\sin q_3 \end{bmatrix}
\end{align*}
\] (16) to (18)

In a similar way, the basic kinematic relations for the remaining types of manipulators were obtained. Namely, for a robot manipulator, the structural kinematic diagram of which is shown in Figure 1(b), these relations have the form:

\[
\begin{align*}
\mathbf{r}_{1c} &= \begin{bmatrix} 0, 0, \frac{l_1}{2} \end{bmatrix}^T \\
\mathbf{r}_{2c} &= \begin{bmatrix} 0, \frac{l_2}{2}, q_2 \end{bmatrix}^T \\
\mathbf{r}_{3c} &= \begin{bmatrix} \frac{l_2}{2}\cos q_3, \frac{l_2}{2}\sin q_3, 0 \end{bmatrix}^T \\
\mathbf{r}_{02c} &= \begin{bmatrix} -\frac{l_2}{2}\sin q_1, \frac{l_2}{2}\cos q_1, q_2 \end{bmatrix}^T \\
\mathbf{r}_{03c} &= \begin{bmatrix} -\frac{l_3}{2}\cos(q_1 + q_3) - l_2\sin q_1, -\frac{l_3}{2}\sin(q_1 + q_3) + l_2\cos q_1, q_2 \end{bmatrix}^T \\
\mathbf{v}_{01c} &= \mathbf{\dot{r}}_{01c} = \begin{bmatrix} 0, 0, 0 \end{bmatrix}^T
\end{align*}
\] (19) to (25)
For a robot manipulator, the structural kinematic scheme of which is shown in Figure 1(c), these relations have the form:

\[ r_{1c} = [0, 0, \frac{l_3}{2}]^T \]  
\[ r_{2c} = [0, 0, q_2]^T \]  
\[ r_{3c} = [\frac{l_3}{2} \sin q_3, -\frac{l_3}{2} \cos q_3, 0]^T \]  
\[ r_{02c} = [-q_2 \sin q_1, q_2 \cos q_1, \frac{l_3}{2} \sin q_3]^T \]  
\[ r_{03c} = [-\frac{l_3}{2} \sin q_1 \cos q_3 - q_2 \sin q_1, -\frac{l_3}{2} \cos q_1 \cos q_3 + q_2 \cos q_1, \frac{l_3}{2} \sin q_3]^T \]  
\[ v_{01c} = \dot{r}_{01c} = [0, 0, 0]^T \]  
\[ v_{02c} = \dot{r}_{02c} = [-q_2 \sin q_1 - q_2 \cos q_1 \dot{q}_1, q_2 \cos q_1 - q_2 \sin q_1 \dot{q}_1, 0]^T \]  
\[ v_{03c} = \dot{r}_{03c} = \left[-\frac{l_3}{2} \cos q_1 \cos q_3 \dot{q}_1 + \frac{l_3}{2} \sin q_1 \sin q_3 \dot{q}_3 - \dot{q}_2 \sin q_1 - q_2 \cos q_1 \dot{q}_1 \right]^T \]  
\[ \omega_{1x} = 0 \quad \omega_{1y} = -q_1 \quad \omega_{1z} = 0 \]  
\[ \omega_{2x} = \dot{q}_1 \quad \omega_{2y} = 0 \quad \omega_{2z} = 0 \]  
\[ \omega_{3x} = \cos q_3 \dot{q}_1 \quad \omega_{3y} = \dot{q}_3 \quad \omega_{3z} = \sin q_3 \dot{q}_1 \]  

The obtained kinematic relations make it possible to go on to find the equations of dynamics that characterize the relationship of the position, speed and acceleration of the links with control and disturbing forces and moments. Considering the manipulator as a holonomic mechanical system having a finite number of degrees of freedom, Lagrange equations of the second kind can be used to describe its dynamics [3-4].

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \; ; \; Q_j = Q_{jd} + Q_{jb} + Q_{jF} \] - generalized forces;

\[ Q_{jd} = M_{jd} + M_{jc} \] - generalized forces of controlled drive;
\[ Q_{jB} = Q_{jB}' + Q_{jB}'' \] - external generalized forces.

\[ M_{jD} \text{ and } M_{jC} \] - the moments developed by the engine controlling the \( j \)th generalized coordinate and the moments of viscous and dry friction. As is known, [2], external generalized forces can be calculated by the formulas:

\[ Q_{jB}' = \sum_{i=1}^{3}(a_{31}(0,j-1)y_{j-1,i}c_{32}(0,j-1)x_{j-1,i})p_{i} \] (42)

\[ Q_{jB}'' = (a_{31}(0,j-1)y_{j-1,3}c_{32}(0,j-1)x_{j-1,3})p_{g} \] (43)

where \( x_{j-1,ic}, y_{j-1,ic} \) - components of the vector \( \tau_{j-1,ic} \), which determines the position of the center of mass of the link \( i \) in the coordinate system \( O_{i-1}x_{i-1}y_{i-1}z_{i-1} \), \( x_{j-1,3}, y_{j-1,3} \) - components of the vector \( \tau_{j-1,3} \) which sets the position of the gripper in the coordinate system \( O_{i-1}x_{i-1}y_{i-1}z_{i-1} \). Considering the actuator of the manipulator as a system of three solids, the formula is obtained:

\[ T = \sum_{i=1}^{3}T_{i} \] (44)

When calculating the kinetic energy of the link, the formula is used:

\[ T_{i} = \frac{1}{2}m_{i}v_{0ic}^{2} + \frac{1}{2}\omega_{i}^{2}I_{0i}^{0}\omega_{i} \] (45)

where \( v_{0ic} \) - the velocity vector of the center of inertia of the link \( i \) relative to the fixed coordinate system, \( \omega_{i} \) - the angular velocity vector of the link \( i \) in the associated coordinate system, \( I_{0i}^{0} \) - the inertia tensor of link \( i \) at the point \( O_{i}^{0} \). The axial moments of inertia of the links of a given configuration are known constants. When finding the equations of dynamics, only the inertial properties of the rotors of the electric motors were taken into account by adding terms of the form

\[ l_{pj}n_{j}^{2}\ddot{q}_{j}, \] where \( l_{pj} \) - the moment of inertia of the motor rotor \( j \), \( n_{j} \) - the gear ratio of the gearbox.

Based on the above assumptions, the dynamics equations were obtained in the following form:

\[ (l_{p1}n_{1}^{2} + a_{1})\ddot{q}_{1} + a_{5}\ddot{q}_{3} = -n_{1}(M_{1D} + M_{1C}) \] (46)

\[ (l_{p2}n_{2}^{2} + a_{2})\ddot{q}_{2} + a_{4}\ddot{q}_{3} = -n_{2}(M_{2D} + M_{2C}) + \frac{1}{2}\sin q_{3}(p_{2} + 2p_{g}) \] (47)

\[ (l_{p3}n_{3}^{2} + a_{3})\ddot{q}_{3} + \frac{1}{2}a_{4}\ddot{q}_{2} - \frac{1}{2}a_{5}\ddot{q}_{1} = -n_{3}(M_{3D} + M_{3C}) + \frac{1}{2}\sin q_{3}(p_{2} + 2p_{g}) \] (48)

The coefficients of the equations obtained can be calculated by the formulas:

\[ a_{1} = l_{z1} + m_{2}(\frac{1}{2})^{2} + l_{y2} + l_{x3}(\sin q_{3})^{2} + l_{z3}(\cos q_{3})^{2} - m_{3}l_{2}l_{3}\sin q_{3} + m_{3} \]

\[ +m_{3}(\frac{1}{2})^{2}(\sin q_{3})^{2} \] (49)

\[ a_{2} = m_{2} + m_{3} \] (50)

\[ a_{3} = m_{3}(\frac{1}{2})^{2} + l_{y3} \] (51)

\[ a_{4} = -l_{z3}m_{3} \sin q_{3} \] (52)

\[ a_{5} = -m_{3}l_{2}l_{3}\cos q_{3} + \left(l_{x3} - l_{2z} + m_{3}(\frac{1}{2})^{2}\right)\sin 2q_{3} \] (53)

\[ a_{6} = -m_{3}l_{3}\cos q_{3} \] (54)
Similarly, the equations of dynamics were found for a manipulator, the structural kinematic scheme of which is shown in Figure 1(b).

\[
\left(I_{p1}n_1^2 + a_1\right)\ddot{q}_1 + \frac{1}{2}a_4\dot{q}_4 + \frac{1}{2}a_5\dot{q}_5 + \frac{1}{2}a_5(q_3)^2 = -n_1(M_{1D} + M_{1C})
\]

\[
\left(I_{p2}n_2^2+a_2\right)\ddot{q}_2 = n_2(M_{2D} + M_{2C})
\]

\[
\left(I_{p3}n_3^2 + a_3\right)\ddot{q}_3 + \frac{1}{2}a_4\dot{q}_4 + \frac{1}{2}a_5\dot{q}_5 = n_3(M_{3D} + M_{3C})
\]

The coefficients of the equations obtained can be calculated by the formulas:

\[
a_1 = I_{x1} + m_2\left(\frac{l_2}{4}\right)^2 + I_{x2} + I_{x3}(\sin q_3)^2 + m_3\left(\frac{l_3}{2}\right)^2 - m_3l_2l_3\cos q_3 + m_3l_2^2
\]

\[
a_2 = m_2 + m_3
\]

\[
a_3 = m_3\left(\frac{l_3}{2}\right)^2 + I_{y3}
\]

\[
a_4 = m_3\left(\frac{l_3}{2}\right)^2 - m_3l_2l_3\cos q_3 + 2I_{y3}
\]

\[
a_5 = -m_3l_2l_3\sin q_3
\]

The dynamics of the manipulator, the structural kinematic scheme of which is shown in Figure 1(c), is described by the equations:

\[
\left(I_{p1}n_1^2 + a_1\right)\ddot{q}_1 + a_6\dot{q}_4 + a_7\dot{q}_5 = -n_1(M_{1D} + M_{1C})
\]

\[
\left(I_{p2}n_2^2 + a_2\right)\ddot{q}_2 + \frac{1}{2}a_4\dot{q}_4 + \frac{1}{2}a_5\dot{q}_5 - \frac{1}{2}a_6\dot{q}_6^2 = n_2(M_{2D} + M_{2C})
\]

\[
\left(I_{p3}n_3^2 + a_3\right)\ddot{q}_3 + \frac{1}{2}a_4\dot{q}_4 - \frac{1}{2}a_7\dot{q}_7^2 = n_3(M_{3D} + M_{3C}) - \frac{l_3}{2}\cos^2 q_3(p_3 + 2p_g)
\]

The coefficients of the equations of dynamics are determined according to the formulas:

\[
a_1 = I_{y1} + m_2(q_2)^2 + I_{x2} + I_{x3}(\sin q_3)^2 + (m_3\left(\frac{l_3}{2}\right)^2 + I_{x3})(\cos q_3)^2 - l_3\sin q_3 + m_3q_3^2
\]

\[
a_2 = m_2 + m_3
\]

\[
a_3 = m_3\left(\frac{l_3}{2}\right)^2 + I_{y3}
\]

\[
a_4 = -m_3l_2l_3\sin q_3
\]

\[
a_6 = (2m_2 + 2m_3)q_2
\]

\[
a_7 = -l_3\sin q_3 + \left(l_{x3} - l_{x3} - m_3\left(\frac{l_3}{2}\right)^2\right)\sin 2q_3
\]

\[
a_8 = -m_3l_3\cos q_3
\]

In the above ratios, the following notation is used. The values \(q_i\) are the generalized coordinates that determine the configuration of the gripper and the robot manipulator; \(m_i, l_i, p_i, l_{xi}, l_{yi}, l_z\) denote the mass, weight and axial moments of inertia of the link \(i\). \(p_l\) - the weight of the load in the gripper.

Thus, as a result of the calculations, the basic kinematic relations and equations of dynamics of three-link robot manipulators with rotational, translational and rotational degrees of mobility in the form of Lagrange equations of the second kind are obtained.
References
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