String Theory in the Penrose Limit of AdS$_2 \times S^2$

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Abstract  
The string theory in the Penrose limit of AdS$_2 \times S^2$ is investigated. The specific Penrose limit is the background known as the Nappi-Witten spacetime, which is a plane-wave background with an axion field. The string theory on it is given as the Wess-Zumino-Novikov-Witten (WZNW) model on non-semi–simple group $H_4$. It is found that, in the past literature, an important type of irreducible representations of the corresponding algebra, $h_4$, were missed. We present this “new” representations, which have the type of continuous series representations. All the three types of representations of the previous literature can be obtained from the “new” representations by setting the momenta in the theory to special values. Then we realized the affine currents of the WZNW model in terms of four bosonic free fields and constructed the spectrum of the theory by acting the negative frequency modes of free fields on the ground level states in the $h_4$ continuous series representation. The spectrum is shown to be free of ghosts, after the Virasoro constraints are satisfied. In particular we argued that there is no need for constraining one of the longitudinal momenta to have unitarity. The tachyon vertex operator, that correspond to a particular state in the ground level of the string spectrum, is constructed. The operator products of the vertex operator with the currents and the energy-momentum tensor are shown to have the correct forms, with the correct conformal weight of the vertex operator.
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I. INTRODUCTION

The problem of how to define the string theory on a general curved manifold and to
determine the physical spectrum is a long standing problem, with only partial successes.
Definition of the string theory in a background independent way is still elusive, and for
specific backgrounds one has different methods in one’s disposal in order to determine the
physical spectrum and the associated physical quantities.

One specific kind of curved backgrounds are group manifolds. String theory on a gen-
eral group manifold is defined through the Wess-Zumino-Novikov-Witten (WZNW) model
\cite{1,2,3}. The power of this formulation comes from the fact that the string action on such
backgrounds has infinite dimensional current algebra as its symmetry structure. The well–
developed current algebra techniques allows one, in principle, to find the physical spectrum
of the theory, and then utilizing the powerful conformal field theory (CFT) methods one
can compute the correlation functions and the other physically relevant quantities from the
theory. Straightforward as it might seem, however, things are not so when one begins to
analyze the WZNW model on various kinds of group manifolds. The compact groups, e.g.
SU(2) \cite{4,3}, does not turn out to be much problematic. Expanding the analytic symme-
try currents in terms of modes, one finds that these modes obey a Kac-Moody algebra \cite{3}.
Then one constructs the spectrum of the theory as infinite dimensional representations of
the Kac-Moody symmetry algebra of the theory. Since the metric of the group manifold of a
compact group is positive definite, one does not confront with the problem of negative norm
states. Seeing the success of the method on SU(2) group manifold one expects the spectrum
of the WZNW model on a general compact group manifold can be determined with a certain
ease by using the know–how one gained from the analysis of WZNW model on SU(2) group
manifold.

When one wants to play the same game on a non-compact group manifold things are
not so easy however. Due to curved time coordinates, things get complicated more than
one initially assumes. The group manifold of the non–compact simple group SL(2,R) is an
example of this \cite{3,6,7}. Another example of a curved spacetime which is also a manifold
of a group is the four dimensional exact plane–wave background of Nappi and Witten \cite{8}.
The group, whose manifold is the Nappi–Witten (NW) spacetime, is a non-semi–simple
group however. The corresponding non-semi–simple algebra is the Heisenberg algebra with
a rotation operator added, which is denoted as $h_4$. It is also possible to view this algebra
as the centrally extended two dimensional Euclidean algebra, denoted as $E^c_2$. We denote the
 corresponding group as $H_4$.

The motivation to construct string theory on NW spacetime was just the one of accom-
plishing the formulation of string theory on a manifold with curved time and curved space
coordinates. However, after the work of Berenstein, Nastase and Maldacena (BMN) \cite{9}
the motivation has changed. In \cite{9} it is conjectured that the string theory on a plane-wave back-
ground is dual to some large charge limit of a gauge theory. This conjecture is related to the AdS/CFT conjecture, since plane-wave backgrounds can be obtained as Penrose limits of AdS spacetimes. In the post–era the motivation became to determine the string theory in NW spacetime and to check the BMN conjecture in this most easy setting. It is possible to obtain the NW spacetime as the Penrose limit of AdS spacetime as we will show in the next section. It is conjectured in that type 0A string theory on AdS spacetime is dual to a conformal quantum mechanics on the boundary of AdS. In analogy with BMN limit, we expect the bosonic string theory we quantize here would be dual to some large charge limit of conformal quantum mechanics on circle.

Since Nappi and Witten’s proposal of string theory on NW background as a WZNW model on a non-semi-simple group there have been a lot of work on the solution of this \( H_4 \) WZNW model. The main theme of all this works were the same. Find either quasi-free or free field representations of the \( \hat{h}_4 \) current algebra and then built the spectrum of string theory as an infinite dimensional representation of the current algebra. In this approach, like in SU(2) case, the symmetry currents are assumed to be analytical and expanded into modes. Then the spectrum is built, as usual, by applying the negative frequency modes of the currents on the ground level states, which constitute an irreducible representation of the corresponding Lie algebra \( h_4 \). The irreducible representations of the Lie algebra \( h_4 \) were determined in and the authors of that paper noted the extreme similarity of the irreducible representations of \( h_4 \) with the irreducible representations of \( \text{SL}(2,\mathbb{R}) \). It turned out that \( h_4 \) has two infinite dimensional discrete series representations which are conjugate to each other. Other than those, there is a third, infinite dimensional continuous series representation. Since the appearance of in all the subsequent works it is assumed that these are the only possible irreducible representations of \( h_4 \) and there are no other irreducible representations.

Specifically, from the point of view of dispersion relation, it seems that in both discrete series representations the transverse momenta in the \( x \) and \( \bar{x} \) directions are not taken into account. Whereas in the specific continuous series representation, since the momentum in \( u \) direction is taken equal to zero, there is only the contribution of transverse momenta, both of which having the same value. Therefore, these representations describe a string which moves in either of the two dimensional planes in NW spacetime, either in \((x, \bar{x})\) plane or in \((u, v)\) plane. Hence, none of these representations could adequately describe the true motion of a string in NW background. In this paper we present an irreducible representation of the Lie algebra \( h_4 \) which is missed in and in all the subsequent publications on \( H_4 \) WZNW model. This irreducible representation is the principal continuous series type, and therefore it makes the similarity of irreducible representations of \( h_4 \) to those of \( \text{SL}(2,\mathbb{R}) \) more pronounced. We also determined the conjugate representation to the new continuous series representation. The specific continuous series representation and both discrete series representations presented in are shown as special cases of the new representations we found. The irreducible representations of the Lie algebra \( h_4 \) will be presented in subsection
together with the newly found ones. Then we are going to construct the spectrum of the string following the approach of Bars [6], [7] on WZNW model on SL(2,R) manifold. This spectrum is different than the previous proposals in the literature [14]–[20], but by setting appropriate momenta to zero, all the other spectrums can be shown to be just special cases of the spectrum we determined. We observed that there is no need to impose $p_u < 1$ condition claimed in [21], [16]–[18]. We will comment on the possible reason why this condition is encountered. After that we will present the vertex operator, which corresponds to the states in the lowest level of the infinite dimensional representation of the Kac–Moody algebra $\hat{h}_4$.

We are going to make various checks in order to show that the new representation of $h_4$ as the spectrum of quantum string in NW spacetime is the correct choice. We will successfully show that the energy of the ground level states and the conformal weight of the ground level vertex operator are indeed equal to the dispersion relation, which is the same as the negative of the eigenvalue of the second Casimir operator in the new irreducible representation.

Why were these continuous series type representations missed in the previous analyses [14]–[20]? The reason could be that, when one writes the action in terms of $\sigma$—model coordinates, one observes that the string moving in NW spacetime feels a *velocity dependent* harmonic oscillator potential in the transverse coordinates [22]. Due to this *velocity dependent* harmonic oscillator potential, one immediately expects that the spectrum will be quantized, as in the particle case. Then, it is further assumed that the spectrum also needs to contain a lowest weight or highest weight state as in the particle case. However, this last assumption is not correct in the string case. This is because the string Hamiltonian is equal to $L_0 + \bar{L}_0$, but *not* to the number operator. Therefore one does not need to stop at the state $|0, \vec{p}\rangle$. States $|-n, \vec{p}\rangle$, with $n > 0$, do not have negative energy, besides the same energy as the states $|n, \vec{p}\rangle$, in the string case.

How can we check independently that we found the correct spectrum? In the case of string theory on flat spacetime, it is possible to impose the light–cone gauge and analyze the solution of the string $\sigma$—model in that gauge. It is shown by Horowitz and Steif in [22] that in order to be able to impose the string light–cone gauge on a curved background, there should be a covariantly constant null vector on that curved background. This means that the curved background should be a pp-wave spacetime in order to implement the light–cone gauge of string theory. The group manifolds of SU(2) and SL(2,R) do not contain such covariantly constant null vector. Therefore, for those WZNW models we cannot make an independent check of the spectrum through the string light–cone gauge. Whereas in the case of $H_4$ WZNW model the string $\sigma$—model action is solved in the light–cone gauge some time ago in [16]. However, in these work as well, the authors needed to express the string spectrum as a representation of the $\hat{h}_4$ Kac–Moody algebra, and they have used the incomplete set of irreducible representations of the $h_4$ Lie algebra similar to all the other works on $H_4$ WZNW model. This is one problem with their result. The other one is that they could not write a modular invariant one–loop partition function, even though it is expected that the string light–cone will ensure the complete determination of the physical spectrum of the quantum
string, free of negative norm states. The problem with the modular invariance is addressed, but not resolved in [16]. A possible reason for the appearance of such a problem is stated later in [23]. These authors noticed that in the string light–cone gauge in NW spacetime, the coordinates on the string worldsheet, \( \sigma \) and \( \tau \), are treated non-equivalently. They claim that the modular invariance might be attained if one treats them equally, possibly in covariant gauge. Therefore, even though the string light–cone gauge is possible to implement in the NW spacetime according to [22], the presence of the background axion field gives rise to non-equivalent treatment of the string worldsheet coordinates in the light–cone gauge and subsequently causes the partition function not to be modular invariant. Thus it could be concluded that even in the \( H_4 \) WZNW model, though implementable, the string light–cone gauge has intrinsic problems and thus can not be assumed as the guide to decide the correct physical spectrum of the quantum string.

Before presenting the details of our work, we outline and summarize the main results. In the next section we will show that the NW spacetime is a Penrose limit of \( \text{AdS}_2 \times \text{S}^2 \) spacetime and then we are going to discuss the WZNW model on it in general terms. We will discuss the irreducible representations of the non-semi–simple algebra \( h_4 \) and present the new representations. The wave functional, which gives a functional representation of the group corresponding to the new representations of the algebra, will be given in coherent state basis and then the dispersion relation will be determined. In section III, we are going to start with the free-field realization of the symmetry currents. Then we will construct the spectrum by using the modes of free fields and show that no ghosts remain in the spectrum after the Virasoro constraints are satisfied. Depending on the values of transverse or longitudinal momenta the string spectrum is described as one of the representations of Kac–Moody algebra \( \hat{h}_4 \). After the determination of the string spectrum, we will construct the “tachyon” vertex operator in position and momentum spaces in section IV. The form of vertex operator in momentum space is useful to perform operator products with the currents and the energy-momentum tensor. We will present the correct quantum ordering of the vertex operator and show that it has correct conformal dimension. We will conclude with a summary and comments on possible future research.

II. WZNW MODEL IN THE NAPPI–WITTEN SPACETIME

*Plane fronted waves with parallel rays* [24], abbreviated as *pp-waves*, are the most general solutions to the Einstein equations in four dimensions, with a covariantly constant null Killing vector [25],[22]. Their metric are generally given as

\[
 ds^2 = 2dudv + dxd\bar{x} + f(u, x, \bar{x})du^2. \tag{2.1}
\]

If the function \( f \) is quadratic in \( x \) and \( \bar{x} \), such spacetimes are called exact plane waves [26],[22].
The Nappi–Witten (NW) spacetime is a special case of this, where the function \( f \) is also independent of \( u \). It is the four dimensional exact pp-wave background together with an axion field and can be obtained as a Penrose limit of different geometries. It can be obtained as the Penrose limit of the near horizon geometry of NS5 branes, \( M_6 \times R \times S^3 \), by booting along the null geodesic spinning along an equator of the three sphere, or from the near horizon geometry of a NS5 brane wrapped on \( S^2 \). Considering the NW spacetime as the Penrose limit of \( M_6 \times R \times S^3 \), in \cite{27}, the string theory on NW spacetime is also considered as the corresponding limit of \( R \times SU(2) \) WZNW model on \( R \times S^3 \). The level of WZNW model on \( R \) is taken negative so as to interpret this coordinate as the time coordinate. However, as will be seen the zero-grade algebra, \( h_4 \), of the current algebra of WZNW model on NW spacetime has some infinite dimensional representations which cannot be thought as some contraction of representations of \( U(1) \times SU(2) \) group. In this paper we are going to consider the NW spacetime as a Penrose limit of \( AdS_2 \times S^2 \) background. Since the quantization of string theory on \( AdS_2 \times S^2 \) is not known \cite{29}, we will not be able to check our results by comparing them to some limit of the string theory results on \( AdS_2 \times S^2 \). However, the quantization of string theory on NW spacetime may help to define string theory perturbatively on \( AdS_2 \times S^2 \).

A. Geometry of the Penrose Limit of \( AdS_2 \times S^2 \)

Depending on the choice of the null geodesic, the Penrose limit of \( AdS_2 \times S^2 \) background turns out to be either Minkowski spacetime or a plane–wave background \cite{12}. The null geodesic, along which we are going to boost the momentum of a test particle in order to obtain the NW spacetime, passes through the center of \( AdS_2 \) and spins around the equator of \( S^2 \). In global coordinates the metric of \( AdS_2 \times S^2 \) is

\[
ds^2 = R \left( -\cosh^2 \rho \, dt^2 + dp^2 + \cos^2 \theta \, d\phi^2 + d\theta^2 \right).
\]

(2.2)

where the coordinates \((\rho, t)\) describe the \( AdS_2 \) part, and the coordinates \((\theta, \phi)\) describe the \( S^2 \) part of the geometry. We take \( AdS_2 \) and \( S^2 \) parts to have the same radius of curvature \( R \). The Penrose limit is then the metric seen by a highly boosted particle, that is it is moving in the vicinity of a null geodesic. To find the specific Penrose limit we desire, we redefine the coordinates:

\[
u = -(t + \phi), \quad v = \frac{R}{2} (t - \phi), \quad x^+ = \sqrt{R} (\rho + i\theta), \quad x^- = \sqrt{R} (\rho - i\theta).
\]

(2.3)

and then take the \( R \to \infty \) limit. Then the metric becomes

\[
ds^2 = 2dudv + dx^+ dx^- - x^+ x^- du^2.
\]

(2.4)

which has the standard form of metric of pp-wave backgrounds. To see that this metric is the same as the metric of Nappi–Witten background as given in \cite{8} we do one more redefinitions:

\[
x^+ = e^{iu} (a_1 + ia_2) \quad \text{and} \quad x^- = e^{-iu} (a_1 - ia_2).
\]

(2.5)
These turn the metric into

\[ ds^2 = 2dudv + da_k da_k + \epsilon_{jk} da_j da_k du + bdu^2, \]  
(2.6)

which is the metric of the Nappi–Witten background [8], after one also performs the shift \( v \to v + \frac{b}{2} u \). Nappi–Witten background is not just a plane–wave background, there is also an appropriate magnetic field (axion field), to cancel the energy of the wave [8, 14]. Due to the presence of non-zero magnetic field in the background, this spacetime sometimes is called as “Hpp-wave spacetime” [19]. Like metric, the antisymmetric \( B_{\mu\nu} \) field may also be obtained via the same limiting procedure from the NS field of an appropriate string theory model in AdS\(_2 \times S^2\) spacetime [11]. We will not concern with the form of that string theory model in AdS\(_2 \times S^2\) spacetime in this paper. In the following section we will describe the WZNW model in Nappi–Witten background and read off the antisymmetric tensor field by identifying the WZNW action with the \( \sigma \)–model action written in this background.

B. \( H_4 \) WZNW Model

String theory on group manifolds are analyzed through the WZNW model. The action of the WZNW model consists of two parts. The first part is the straightforward generalization of the Polyakov action of string theory on flat spacetime into the curved background of the group manifold. This part of the action is invariant under the corresponding Lie group. The second part of the action, which is required in order to make the full action invariant under the two dimensional conformal symmetry [2], enhances the symmetry of the action from the Lie group to the corresponding Kac–Moody group. Due to the equations of motion, the Noether currents that one derives from the WZNW action separate into holomorphic and anti-holomorphic parts. Each set of currents (holomorphic and anti-holomorphic) separately obey an operator product algebra which is an affine current algebra. Expanding the currents in terms of modes, one can rewrite this current algebra as a Kac–Moody algebra. This Kac–Moody algebra is nothing but the affinization of the Lie algebra associated to the isometry group of that group manifold.

In this section we are going to review the WZNW model in the Nappi–Witten background, which is first described in [3]. The \( \sigma \)–model in Nappi–Witten background can be written as the WZNW model on the group manifold of a non-semi–simple group. The corresponding non-semi–simple algebra is the Heisenberg algebra with a rotation operator added, which rotates both the momentum and the position operators to each other,

\[ [P_1, P_2] = T, \quad [J, P_1] = P_2, \quad [J, P_2] = -P_1, \quad [T, \text{ any}] = 0. \]  
(2.7)

Under this view, the algebra is denoted as \( h_4 \), higher dimensional generalizations, \( h_{2n+2} \), of which contains \( n \) momentum and \( n \) position operators. It is also possible to view this algebra as the centrally extended two dimensional Euclidean algebra, denoted as \( E_2^c \) [14].
The corresponding group can be denoted as either $H_4$ or NW group, to emphasize that it is half of the isometry group of NW spacetime. In this paper we will denote it as $H_4$ due to the future possibility of extending this work to the case of higher Heisenberg groups $H_{2n+2}$ as in [19]. For the manifold of non-semi–simple group $H_4$ we will use the terms “$H_4$ manifold” and “NW spacetime” randomly.

The WZNW action which describes the closed strings on $H_4$ manifold is given by

$$S = \frac{k}{4\pi} \int_\Sigma d^2\sigma \, Tr \left( g^{-1} \partial^\alpha g \, g^{-1} \partial_\alpha g \right) - \frac{k}{12\pi} \int_B d^3\varsigma \, \epsilon_{\alpha\beta\gamma} \, Tr \left( g^{-1} \partial^\alpha g \, g^{-1} \partial^\beta g \, g^{-1} \partial^\gamma g \right).$$  \tag{2.8}

where $g$ is the group element of $H_4$ and $g^{-1} dg = (g^{-1} \partial_\sigma g) d\sigma + (g^{-1} \partial_\tau g) d\tau$ is the pull-back of the left invariant one-form on the $H_4$ group manifold to the closed string worldsheet. To obtain the WZNW action in terms of the coordinates of the background we parametrize the $H_4$ group element as in [8],

$$e^{a_1 P_1 + a_2 P_2 + u J_L + v T},$$  \tag{2.9}

where all the coordinates, $a_1$, $a_2$, $u$, $v$, are real. In [8] comparing the resulting form of the action with the $\sigma$–model action

$$S = \int_M d^2\sigma \left( G_{MN} \eta_{\alpha\beta} \partial^\alpha X^M \partial^\beta X^N + B_{MN} \epsilon_{\alpha\beta} \partial^\alpha X^M \partial^\beta X^N \right)$$  \tag{2.10}

the background space-time metric is found as given in equ. (2.6) and the antisymmetric tensor field is read off to be $B_{12} = u = -B_{21}$ with all the other components of $B_{MN}$ are zero. Then the magnetic field, $H_{u12} = \partial_u B_{12}$, is everywhere constant.

The generators of the isometry group of the background can be found by noting that the WZNW action (2.8) is invariant under the transformation $g \rightarrow g_L g g_R$. Then the form of symmetry generators are determined as

$$T = \partial_v,$$
$$J_L = \partial_u + (a_1 \partial_2 - a_2 \partial_1), \quad J_R = \partial_u,$$
$$P^1_L = \partial_1 + \frac{1}{2} a_2 \partial_v, \quad P^1_R = \cos u \partial_1 + \sin u \partial_2 + \frac{1}{2} (\sin u a_1 - \cos u a_2) \partial_v,$$
$$P^2_L = -\partial_2 + \frac{1}{2} a_1 \partial_v, \quad P^2_R = -\sin u \partial_1 + \cos u \partial_2 + \frac{1}{2} (\cos u a_1 + \sin u a_2) \partial_v.$$  \tag{2.11}

Therefore, the isometry group of NW spacetime is seven dimensional. It contains two commuting, left and right, groups of isometry, which are both $H_4$. Since $T$ is the central element of the non-semi–simple algebra, it is the same for left and right action. In particular, $T$ generates translations in the $v$ direction, $J_L - J_R$ generates rotations in the $(a_1, a_2)$–plane, and $J_R$ generates translations in the $u$ direction. The remaining generators generate some “twisted translations” [10].

The D’Alembertian in this background can be found as the Casimir of $H_4$ group by using the above forms of the symmetry generators (either left or right ones). We will do this in subsection II-D and then by applying it on wave functional in a specific representation we are going to derive the dispersion relation.
C. Representations of the Non-semi–simple Algebra $h_4$

In WZNW model, the spectrum of string theory on a specific group manifold is built on the unitary irreducible representations of the corresponding Lie algebra of the group. The irreducible representations of the Lie algebra constitute the ground state level of the string. On this level all the states have the same ground state energy. The excited states in higher levels are then constructed by applying either the negative frequency modes of the currents or the negative frequency modes of the free fields, depending on the approach. Therefore the first step in constructing the physical spectrum of string theory on a group manifold is to analyze the unitary irreducible representations of the associated Lie algebra. In this subsection we are going to analyze the unitary irreducible representations of the non-semisimple algebra $h_4$.

The unitary irreducible representations of $h_4$ algebra are first discussed in \cite{14}. The authors of these papers noticed that the unitary representations of $h_4$ algebra has very similar structure to the representations of the $\mathfrak{sl}(2,R)$ algebra. However, they have missed one important type of unitary representations, inclusion of which makes the similarity between unitary representations of $h_4$ and $\mathfrak{sl}(2,R)$ Lie algebras truly remarkable.

To find the unitary irreducible representations we first rewrite the $h_4$ algebra in a different form as

\[
\begin{align*}
[P^+, P^-] &= -2iT, \quad [J, P^+] = -iP^+, \quad [J, P^-] = iP^-, \quad [T, \text{any}] = 0, \\
\end{align*}
\]

where $P^+ = P^1 + iP^2$ and $P^- = P^1 - iP^2$. There are two Casimir operators of this algebra: the central element $T$ and the quadratic Casimir, which, in this basis for the algebra, has the form given by

\[
C = \frac{1}{2} (P^+P^- + P^-P^+) + 2JT. \tag{2.13}
\]

The hermiticity properties of the generators are as follows

\[
(P^+)^\dagger = -P^-, \quad J^\dagger = -J, \quad T^\dagger = -T. \tag{2.14}
\]

Since the generator $T$ is an central element and, therefore, commutes with every other member of the algebra, we can set it to a constant value

\[
T |n, \vec{p}\rangle = -ip_u |n, \vec{p}\rangle, \tag{2.15}
\]

where $\vec{p} = (p_u, p_v, p^+, p^-)$ is a vector of parameters. By notating a state of the representations like this we anticipate the possible parameters that will appear in specific representations. These parameters will turn out to be the same as the independent momenta components in NW$_4$ spacetime. Therefore, the states $|n, \vec{p}\rangle$ in this basis can either be treated as states in a “number” basis, as it will be seen in a moment, or states in a momentum basis.
To find the number basis consider an eigenstate of the Cartan generator,

\[ J |0, \vec{p}\rangle = -ip_v |0, \vec{p}\rangle. \tag{2.16} \]

From the algebra it can easily be deduced that \( P^+ \) behaves as an annihilation operator and \( P^- \) behaves as a creation operator in a number basis, in which one of the elements is the state \(|0, \vec{p}\rangle\). The \( P^+ \) and \( P^- \) generators act on \(|0, \vec{p}\rangle\) as

\[ P^\pm |0, \vec{p}\rangle = \sqrt{C + 2p_u p_v \pm p_u} |\mp 1, \vec{p}\rangle, \tag{2.17} \]

where \( C \) is the specific value of the quadratic Casimir operator in the particular representation. According to different values of \( C \) we obtained the following classification of the representations of the Lie algebra \( h_4 \).

1. **Lowest–weight representations** \((V^{p_u p_v}, p_u > 0)\): In such representations of the algebra, there is a so called lowest–weight state on which \( P^+ \) has zero eigenvalue: \( P^+ |0, \vec{p}\rangle = 0 \). The eigenvalues of the generators on a general state in these representations are

\[ T |n, \vec{p}\rangle = -ip_u |n, \vec{p}\rangle, \tag{2.18} \]
\[ J |n, \vec{p}\rangle = i(-p_v + n) |n, \vec{p}\rangle, \]
\[ P^+ |n, \vec{p}\rangle = i\sqrt{2p_u n} |n - 1, \vec{p}\rangle, \]
\[ P^- |n, \vec{p}\rangle = i\sqrt{2p_u (n + 1)} |n + 1, \vec{p}\rangle. \]

The spectrum of \(-iJ\) is \(\{-p_v + n\}, n \in \mathbb{N}\) and the quadratic Casimir operator has the eigenvalue

\[ C = -2p_u p_v - p_u. \tag{2.19} \]

These representations were called “Type III” in [14].

2. **Highest–weight representations** \((\tilde{V}^{p_u p_v}, p_u < 0)\): In these representations of the algebra, there is a so called highest–weight state on which \( P^- \) has zero eigenvalue: \( P^- |0, \vec{p}\rangle = 0 \). The eigenvalues of the generators on a general state in these representations are

\[ T |n, \vec{p}\rangle = -ip_u |n, \vec{p}\rangle, \tag{2.20} \]
\[ J |n, \vec{p}\rangle = i(-p_v + n) |n, \vec{p}\rangle, \]
\[ P^+ |n, \vec{p}\rangle = i\sqrt{2p_u (n - 1)} |n - 1, \vec{p}\rangle, \]
\[ P^- |n, \vec{p}\rangle = i\sqrt{2p_u n} |n + 1, \vec{p}\rangle. \]

The spectrum of \(-iJ\) is \(\{-p_v - n\}, n \in \mathbb{N}\) and the quadratic Casimir operator has the eigenvalue

\[ C = -2p_u p_v + p_u. \tag{2.21} \]

These representations were called “Type II” in [14].
Continuous series representations \( (C_{p^+, p^-}^{p_u, p_v}) \): Other than highest-weight and lowest-weight representations it is possible to construct a series of representations parametrized by two numbers \( p^+ \) and \( p^- \). Here these parameters are complex and they are complex conjugates of each other. In the past \( p_u \) have always been set to zero when analyzing this kind of representations \([14]–[20]\), however we found that there are also the following continuous series representations of \( h_4 \) Lie algebra:

\[
T |n, \vec{p}\rangle = -ip_u |n, \vec{p}\rangle,
\]
\[
J |n, \vec{p}\rangle = i(-p_v + n) |n, \vec{p}\rangle,
\]
\[
P^+ |n, \vec{p}\rangle = i\sqrt{p^+p^- + 2p_u n} |n-1, \vec{p}\rangle,
\]
\[
P^- |n, \vec{p}\rangle = i\sqrt{p^+p^- + 2p_u (n+1)} |n+1, \vec{p}\rangle.
\]

These representations are the true analog of the principal continuous series of \( sl(2, R) \).

The spectrum of \(-iJ\) is \( \{ -p_v + n \}, \ n \in \mathbb{Z} \) and the quadratic Casimir operator has the eigenvalue

\[
C = -2p_u p_v - p^+ p^- - p_u. \tag{2.23}
\]

There are special cases of these representations. One can set \( p^+ = p^- \neq \alpha \), therefore decreasing number of parameters to one. We denote such special cases as \( C_{\alpha}^{p_u, p_v} \). One can also set \( p_u = 0 \) together with \( p^+ = p^- = \alpha \), in which case one obtains the representations called “type I” in \([14]\) and denoted as \( V_0^{0, p_v} \) in \([18]\). One also observes that, after setting \( p^+p^- = 0 \), the lowest-weight representations, \( V^{p_u, p_v} \), \( p_u > 0 \), are special cases of continuous series representations.

Conjugate continuous series representations \( (\tilde{C}_{p^+, p^-}^{p_u, p_v}) \): These are again two parameter continuous series representations. The eigenvalues of the generators on a general state in these representations are

\[
T |n, \vec{p}\rangle = -ip_u |n, \vec{p}\rangle,
\]
\[
J |n, \vec{p}\rangle = i(-p_v + n) |n, \vec{p}\rangle,
\]
\[
P^+ |n, \vec{p}\rangle = i\sqrt{p^+p^- + 2p_u (n-1)} |n-1, \vec{p}\rangle,
\]
\[
P^- |n, \vec{p}\rangle = i\sqrt{p^+p^- + 2p_u n} |n+1, \vec{p}\rangle.
\]

Here again one can set \( p^+ \) and \( p^- \) to special values and obtain special cases of these representations. For example setting \( p^+p^- = 0 \) one obtains the highest-weight representations, \( \tilde{V}_{p_u, p_v} \), \( p_u < 0 \).

In these representations the spectrum of \(-iJ\) is again \( \{ -p_v + n \}, \ n \in \mathbb{Z} \) and the quadratic Casimir operator has the eigenvalue

\[
C = -2p_u p_v - p^+ p^- + p_u. \tag{2.25}
\]

Trivial representation: Setting \( p_u = 0 = p_v \) one obtains the trivial representation \( V^{0,0} \).
D. Wave Functional and the Dispersion Relation

In this section we are going to define Klauder–Perelomov generalized coherent states [30] for the continuous series representation. Then we are going to find the wave functional as the matrix element of the group element in the coherent state basis. The same analysis is performed for the highest and lowest weight representations in [18] by using Barut–Girardello type coherent states [31]. We refer the reader to [18] for the form of wave functions in those representations.

Klauder–Perelomov generalized coherent states are constructed by utilizing the dynamical symmetry group of the particular system. For the continuous series representation of the $H_4$ group they are defined by

$$|\lambda\rangle = e^{\lambda p^+ - \bar{\lambda} p^-} |0, \vec{p}\rangle. \quad (2.26)$$

The action of generators of $H_4$ group on $|\lambda\rangle$ can be easily determined as

$$T |\lambda\rangle = -ip_u |\lambda\rangle, \quad \quad (2.27)$$

$$J |\lambda\rangle = -i (p_v + \lambda \partial_\lambda - \bar{\lambda} \partial_{\bar{\lambda}}) |\lambda\rangle, \quad \quad (2.27)$$

$$P^+ |\lambda\rangle = (\partial_\lambda + \bar{\lambda} p_u) |\lambda\rangle, \quad \quad (2.28)$$

$$P^- |\lambda\rangle = (-\partial_{\bar{\lambda}} + \lambda p_u) |\lambda\rangle,$$

and the inner product in coherent state basis is found to be

$$(\kappa|\lambda\rangle = e^{-p_u(\lambda\bar{\lambda} + \kappa\bar{\kappa} + 2\Lambda\kappa)} \begin{pmatrix} 1 + \frac{p^+ p^-}{2p_u}, 1; 2p_u(\lambda + \kappa)(\bar{\lambda} + \bar{\kappa}) \end{pmatrix}_1 F_1 (\alpha, \gamma; w), \quad (2.28)$$

where $1 F_1 (\alpha, \gamma; z)$ is the confluent hypergeometric function.

Then the wave functional as the matrix element of the group element of $H_4$ is found to have the form

$$\Psi_{\kappa, \lambda}^{p,v,p^+,p^-} (u, v, a^+, a^-) = (\kappa| g |\lambda\rangle = (\kappa| e^{a^+ p^- + a^- p^+} e^{uJ + vT} |\lambda\rangle = e^{-i(p_u + v p_u)} e^{-p_u(-a^+ a^- - 2a^+ \Lambda + \bar{\Lambda} + \kappa \bar{\kappa} + 2(a^+ a^-)\kappa)} _1 F_1 (\alpha, 1; w), \quad (2.29)$$

where $\Lambda = e^{-iu} \lambda$, $\bar{\Lambda} = e^{iu} \bar{\lambda}$, $\alpha = 1 + \frac{p^+ p^-}{2p_u}$ and $w = 2p_u(\lambda + \kappa + a^-)(\bar{\lambda} + \bar{\kappa} - a^+)$.

This wave functional should obey the wave equation $(\Box - m^2) \Psi = 0$, where the D’Alembertian operator is given as the Casimir operator of the left or the right isometry groups. The generators of the isometry group are given in Equ. (2.11). Using the left or right generators one finds

$$\Box = 2\partial_u \partial_u + \partial^2_1 + \partial^2_2 + (a_1 \partial_2 - a_2 \partial_1) \partial_v + \frac{1}{4} (a_1^2 + a_2^2) \partial^2_v. \quad (2.30)$$

This formula is given in $(a_1, a_2)$ coordinate system. Since the wave functional in coherent state basis is given in $(a^+, a^-)$ coordinate system, we translate the D’Alembertian operator
into \((a^+, a^-)\) coordinate system by using \(a^+ = \frac{1}{2} (a_1 + ia_2)\) and \(a^- = \frac{1}{2} (a_1 - ia_2)\). We find

\[
\Box = 2 \partial_v \partial_a + \partial \tilde{\partial} + i \left( a^+ \partial_+ - a^- \partial_- \right) \partial_v + a^+ a^- \partial_v^2.
\]  

(2.31)

An arbitrary wave functional \(\Psi_{n_L, n_R}\) can be obtained by acting \(n_L\) and \(n_R\) times the raising (lowering) operators \(P^+_L = P^1_L + i P^2_L\) and \(P^-_R = P^1_R + i P^2_R\), respectively, on the wave functional \(\Psi (2.29)\). Then the action of \((\Box - m^2)\) on an arbitrary wave functional \(\Psi_{n_L, n_R}\) gives the dispersion relation as

\[
2 p_a p_v + p^+ p^- + p_a (n_L + n_R + 1) + m^2 = 0.
\]

(2.32)

III. COVARIANT QUANTIZATION AND THE PHYSICAL SPECTRUM

A. Free Field Realization of \(\hat{h}_4\) Current Algebra

In this section we are going to realize the holomorphic part of the current algebra of \(H_4\) WZNW model in terms of four bosons, \(\theta^+, \theta^-, \phi, \varphi\). Likewise the anti-holomorphic currents are going to be realized by another four bosons, \(\tilde{\theta}^+, \tilde{\theta}^-, \tilde{\phi}, \tilde{\varphi}\). In the following we will often omit the analysis of anti-holomorphic part. In those instances it should be understood that the anti-holomorphic part works the same way as holomorphic part with essentially trivial modifications. We start by writing the group element as

\[
g = g_L(z) g_R(\bar{z}), \quad g_L(z) = e^{\theta^+ p^- - e^{\phi} J^+ T e^{\theta^- p^+}}
\]

\[
g_R(\bar{z}) = e^{\tilde{\theta}^+ p^- - e^{\tilde{\phi} J^+ T e^{\tilde{\theta}^- p^+}}}
\]

(3.1)

in terms of holomorphic and anti-holomorphic fields. Here in terms of worldsheet coordinates \(z = e^{i(\tau + \sigma)}\) and \(\bar{z} = e^{i(\tau - \sigma)}\). Notice that \(z\) and \(\bar{z}\) are independent complex variables. The symmetry currents of the WZNW model are given in terms of the group element by the relations \(J_L (z) = g \partial_z g^{-1}\) for the holomorphic currents and \(J_R (\bar{z}) = g^{-1} \partial_{\bar{z}} g\) for the anti-holomorphic currents. In terms of special form of the group element \(g(z, \bar{z}) = g_L(z) g_R(\bar{z})\) the currents become

\[
J_L (z) = g_L \partial_z g_L^{-1}, \quad J_R (\bar{z}) = g_R^{-1} \partial_{\bar{z}} g_R.
\]

(3.2)

where \(\partial\) and \(\tilde{\partial}\) stand for \(\partial_z\) and \(\partial_{\bar{z}}\), respectively. Then using the above realization of left and right group elements we find the currents in terms of holomorphic and anti-holomorphic fields as

\[
T_L (z) = - \partial \phi, \quad T_R (z) = \tilde{\partial} \tilde{\phi}
\]

\[
J_L (z) = - \partial \varphi - i : \theta^+ \theta^- :, \quad J_R (\bar{z}) = \tilde{\partial} \tilde{\varphi} + i : \tilde{\theta}^- \tilde{\theta}^+ :
\]

\[
P^+_L (z) = - 2 \partial \theta^+ + 2i \varphi \theta^+, \quad P^+_R (z) = \tilde{\theta}^+.
\]

\[
P^-_L (z) = - \theta^-, \quad P^-_R (z) = 2 \tilde{\theta}^- - 2i \tilde{\varphi} \tilde{\theta}^-.
\]

(3.3)

where \(\theta^- (z) = 2 \partial \theta^- e^{-i \phi}\) and \(\tilde{\theta}^+ (\bar{z}) = 2 \tilde{\partial} \tilde{\theta}^+ e^{-i \tilde{\phi}}\) are bosonic free fields. At this point we would like to remind the reader that we did not call any of the bosonic fields in the realization
of the group element as free fields. The free fields in our construction are $\theta^+, \vartheta^-, \phi$ and $\varphi$ from the point of view of holomorphic currents and $\bar{\theta}^-, \bar{\vartheta}^+, \bar{\phi}$ and $\bar{\varphi}$ from the point of view of anti-holomorphic currents. With these points of view, the fields $\theta^-$ and $\bar{\theta}^+$ are composite fields and are given in terms of free fields as

$$\theta^- (z) = x^-_0 + \frac{1}{2} \int^z dz' \vartheta^- (z') e^{i\phi(z')} , \quad (3.4)$$

$$\bar{\vartheta}^+ (\bar{z}) = \bar{x}^+_0 + \frac{1}{2} \int^{\bar{z}} d\bar{z}' \vartheta^+ (\bar{z}') e^{i\bar{\phi}(\bar{z}')}. \quad (3.5)$$

These expressions will be used when we discuss about the operator products of vertex operator with the currents and with the energy-momentum tensor.

We should warn the reader that in the present realization the pairs of fields $\theta^+, \vartheta^-$ and $\bar{\theta}^-, \bar{\vartheta}^+$ are not ghost fields, they have bosonic character. In fact the mode expansions of all the holomorphic fields are as follows

$$\phi (z) = \phi_0 + i\alpha^- \ln z - i \sum_{n \neq 0} \frac{1}{n} \alpha^- z^{-n} \quad (3.6)$$

$$\varphi (z) = \varphi_0 - i\alpha^+ \ln z + i \sum_{n \neq 0} \frac{1}{n} \alpha^+ z^{-n} \quad (3.7)$$

$$\vartheta^- (z) = i \sum_{n=-\infty}^{\infty} \beta^- z^{-n-1} \quad (3.8)$$

$$\theta^+ (z) = x^+_0 - i\beta_0 \ln z + i \sum_{n \neq 0} \frac{1}{n} \beta^+ z^{-n} \quad (3.9)$$

The anti-holomorphic free fields have similar mode expansions. The free fields $\phi (z)$ and $\varphi (z)$ are Hermitian, whereas $\vartheta^- (z)$ and $\partial \theta^+ (z)$ are Hermitian conjugates of each other:

$$[\phi (z)]^\dagger = \phi \left( \frac{1}{z} \right), \quad [\varphi (z)]^\dagger = \varphi \left( \frac{1}{z} \right), \quad [z \vartheta^- (z)]^\dagger = \frac{1}{z} \partial \theta^+ \left( \frac{1}{z} \right). \quad (3.10)$$

Then the Hermicity of the modes follows as

$$(\alpha^-)_n^\dagger = \alpha^-_n, \quad (\alpha^+_n)^\dagger = \alpha^+_n, \quad (\beta^-_n)^\dagger = \beta^+_n, \quad (\beta^+_n)^\dagger = \beta^-_n. \quad (3.11)$$

We emphasize that under Hermitian conjugation the $\alpha^-$ and $\alpha^+$ modes are conjugate to their kinds, however the $\beta^-$ and $\beta^+$ modes switch to each other.

Non-zero frequency modes of the free fields have the following commutation relations with each other:

$$[\alpha^-_n, \alpha^+_m] = n \delta_{n+m}, \quad [\beta^-_n, \beta^+_m] = n \delta_{n+m}. \quad (3.12)$$

We also choose the following commutation relations between the zero modes of the free fields:

$$[\phi_0, \alpha^-_0] = -i, \quad [\varphi_0, \alpha^+_0] = i, \quad [x^+_0, \beta^-_0] = i. \quad (3.13)$$
These zero modes are normal ordered as
\[
: \beta_0^+ x_0^- : = x_0^+ \beta_0^- \quad \Rightarrow \quad \langle \beta_0^- x_0^+ \rangle = -i, \quad : x_0^+ \beta_0^- : = x_0^+ \beta_0^- \quad \Rightarrow \quad \langle x_0^+ \beta_0^- \rangle = 0,
\]
\[
: \alpha_0^- \varphi_0 : = \varphi_0 \alpha_0^- \quad \Rightarrow \quad \langle \alpha_0^- \varphi_0 \rangle = -i, \quad : \varphi_0 \alpha_0^- : = \varphi_0 \alpha_0^- \quad \Rightarrow \quad \langle \varphi_0 \alpha_0^- \rangle = 0,
\]
\[
: \alpha_0^+ \phi_0 : = \phi_0 \alpha_0^+ \quad \Rightarrow \quad \langle \alpha_0^+ \phi_0 \rangle = i, \quad : \phi_0 \alpha_0^+ : = \phi_0 \alpha_0^+ \quad \Rightarrow \quad \langle \phi_0 \alpha_0^+ \rangle = 0. \quad (3.14)
\]

Then the contractions of the bosonic free fields are calculated to be equal to
\[
\langle \phi(z) \varphi(w) \rangle = \ln(z - w), \quad (3.15)
\]
\[
\langle \varphi(z) \phi(w) \rangle = \ln(z - w), \quad (3.16)
\]
\[
\langle \vartheta^-(z) \theta^+(w) \rangle = \frac{1}{z - w}, \quad (3.17)
\]
\[
\langle \theta^+(z) \vartheta^-(w) \rangle = -\frac{1}{z - w}. \quad (3.18)
\]

Now we can calculate the operator products among currents and check that we have a correct realization of the currents in terms of free fields. The operator products come out to be the correct ones:
\[
T_L(z) J_L(w) \sim \frac{1}{(z - w)^2}, \quad (3.19)
\]
\[
J_L(z) P_L^+(w) \sim -i \frac{1}{z - w} P_L^+(w)
\]
\[
J_L(z) P_L^-(w) \sim i \frac{1}{z - w} P_L^-(w)
\]
\[
P_L^+(z) P_L^-(w) \sim \frac{2}{(z - w)^2} - \frac{2i}{z - w} T_L(w)
\]

These are of course the singular parts of the operator products.

Next we calculate the energy–momentum tensor. In terms of the affine currents the energy–momentum tensor is defined in Sugawara form as
\[
\mathcal{T}_L(z) = \frac{1}{2} L_{ij} : \mathcal{J}_L^i \mathcal{J}_L^j :,
\]
where sum over \( i \) and \( j \) indices is implicit and : : means that one should subtract any singular part from the OPE of current–current products. We determine the form of \( L_{ij} \) by requiring the affine symmetry currents to be primary with respect to the energy–momentum tensor. This means that the currents should have conformal weight one and, therefore, their OPE with the energy–momentum tensor should have the form
\[
\mathcal{T}_L(z) \mathcal{J}_L^j(w) = \frac{1}{(z - w)^2} \mathcal{J}_L^j(w) + \frac{1}{z - w} \partial_w \mathcal{J}_L^j(w) \quad (3.21)
\]
If we did not required this condition, then energy–momentum tensor would have to obey a Virasoro master equation \[32, 14, 33\]. We find that the currents are primary if the matrix \( L_{ij} \) is the same as the non-degenerate bilinear form \( \Omega_{ij} \) for the group \( H_4 \). Then the
holomorphic energy–momentum tensor is calculated as

\[ T_L(z) = \frac{1}{2} \left\{ T_L(z) J_L(z) + J_L(z) T_L(z) + \frac{1}{2} \left[ P^+_L(z) P^-_L(z) + P^-_L(z) P^+_L(z) \right] \right\} : \partial \phi(z) \partial \phi(z) : + : \partial \theta^+(z) \partial \theta^-(z) : + \frac{i}{2} \partial^2 \phi(z) \]  

(3.22)

In the CFT analysis of the spectrum of string theory, the zero mode in the Laurent expansion of the holomorphic energy–momentum tensor plays the role of one of the parts of the Hamiltonian. The other part comes symmetrically from the anti-holomorphic energy–momentum tensor. To find the so-called Virasoro generators we expand \( T_L(z) \) as

\[ T_L(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}. \]  

(3.23)

Then using the mode expansions of the free fields the forms of \( L_n \) and \( L_0 \) are determined as

\[ L_n = \sum_{m=-\infty}^{\infty} \left( : \alpha^{-}_{n-m} \alpha^{+}_m : + : \beta^{-}_{n-m} \beta^{+}_m : \right) + \frac{\alpha_n}{2}, \]  

(3.24)

\[ L_0 = \frac{\alpha_0}{2} + \alpha_0^+ \alpha_0^- \beta_0^+ \beta_0^- + \sum_{m>0} \left( \alpha^{-}_{m} \alpha^{+}_m + \alpha^{-}_{-m} \alpha^{+}_m + \beta^{-}_{m} \beta^{+}_m + \beta^{-}_{-m} \beta^{+}_m \right). \]  

(3.25)

\( L_n \)'s obey Virasoro algebra with central charge equal to 4. Interestingly, this charge, at the same time, is equal to the dimension of the \( H_4 \) manifold and is independent of the Kac–Moody level \( k \), unlike in the cases of WZNW models on semi-simple group manifolds. To make the bosonic string theory on \( H_4 \) manifold critical, one also needs an internal CFT with \( c = 22 \). We are not going to tell anything more about the internal CFT in this paper. We will determine the spectrum without referring to any special properties of such internal CFT.

B. Virasoro Constraints and the Physical Spectrum

In the previous section we have successfully mapped the affine currents to a system of four free fields. Then, in this section, we determine the spectrum of bosonic strings on NW spacetime as a Fock space created by the negative frequency modes of four bosonic free fields as

\[ \prod_{i,j,k,l=1}^{\infty} \left( \alpha^{-}_i \right)^{e^{1,i}} \left( \alpha^{+}_j \right)^{e^{2,j}} \left( \beta^{-}_k \right)^{e^{3,k}} \left( \beta^{+}_l \right)^{e^{4,l}} \left| n; p_u, p_v, p^+, p^- \right> \]  

(3.26)

where \( \left| n; p_u, p_v, p^+, p^- \right> \equiv \left| n, \vec{p} \right> \) are the states in the base level and the powers \( e_{\mu,i} \) (\( \mu = 1, \ldots, 4 \)) are positive integers or zero. This Fock space is equivalent to the continuous series representation of the Kac–Moody algebra \( \hat{h}_4 \). Since we have four free bosons, this spectrum is no different than the spectrum of string theory in 4 dimensional flat space-time. States created by \( \beta^{-}_{-m}, \beta^{+}_{-m} \) and \( \alpha^{-}_m = \frac{1}{\sqrt{2}} \left( \alpha^{+}_{-m} + \alpha^{+}_{m} \right) \) have positive norms, but there are ghosts
in this spectrum due to time-like oscillator modes $\alpha^0_m = \frac{1}{\sqrt{2}} (\alpha^+_m - \alpha^-_m)$. However, the situation is no worse than the case in string theory in flat space-time. Here, again, we expect that the Virasoro constraints,

$$ (L_n - a\delta_n) \left| \text{phys} \right\rangle = 0 \quad (n \geq 0) , \quad (3.27) $$

where $a = 1$ in the critical case, will be enough to clean the spectrum from unwanted negative-norm states.

We first look at the meaning of Virasoro constraints on the ground level states in the critical case. Since the positive frequency modes of free fields annihilate these states, we find that the eigenvalues of $L_m$ ($m \geq 0$) operators are

$$ L_m |n, \vec{p}\rangle = 0 \quad (m > 0) , \quad (3.28) \\
L_0 |n, \vec{p}\rangle = \left( \alpha^-_0 \alpha^+_0 + \beta^-_0 \beta^+_0 + \frac{\alpha^-_0}{2} \right) |n, \vec{p}\rangle = \left( p_u p_v + \frac{p^- p^+}{2} + \frac{p_u}{2} \right) |n, \vec{p}\rangle = |n, \vec{p}\rangle . \quad (3.29) $$

Due to the constraint $L_n = \bar{L}_n$, the anti-holomorphic Virasoro generators have the same eigenvalues on the ground level states. Then, since the covariant Hamiltonian of bosonic string theory is given by the sum $H = L_0 + \bar{L}_0$ of holomorphic and anti-holomorphic zero frequency Virasoro generators, the eigenvalue of the Hamiltonian on ground level states is found to be the same as the dispersion relation derived in subsection II D. Therefore all the states in the ground level have the same energy. This is also true for the higher levels: All the states in the same level of continuous series representation of $\hat{h}_4$ Kac–Moody algebra have the same energy. This will cause the same regularization problem in the one-loop partition function of closed strings as in the SL(2,R) WZNW model.

From the constraint (3.29) we read the eigenvalues of the zero modes on ground level states as

$$ \alpha^-_0 |n, \vec{p}\rangle = p_u |n, \vec{p}\rangle , \quad \alpha^+_0 |n, \vec{p}\rangle = p_v |n, \vec{p}\rangle , \quad (3.30) \\
\beta^-_0 |n, \vec{p}\rangle = \frac{p^-}{\sqrt{2}} |n, \vec{p}\rangle , \quad \beta^+_0 |n, \vec{p}\rangle = \frac{p^+}{\sqrt{2}} |n, \vec{p}\rangle . \quad (3.31) $$

At a higher level $l$ the $(L_0 - a) \left| \text{phys} \right\rangle = 0$ constraint gives the mass shell condition as

$$ p_u p_v + \frac{p^+ p^-}{2} + \frac{p_u}{2} + l = a \quad (3.32) $$

Next, continuing à la Bars [6] we determine the physical states at level 1. We find the following combinations of states of Fock space obey the physical state conditions (3.27):

$$ |\Phi^\alpha\rangle = \left[ \left( p_v + \frac{1}{2} \right) \alpha^-_{-1} - p_u \alpha^+_{-1} \right] |n, \vec{p}\rangle , $$$$ |\Phi^\beta\rangle = \frac{1}{\sqrt{2}} \left( p^+ \beta^-_{-1} - p^- \beta^+_{-1} \right) |n, \vec{p}\rangle , \quad (3.33) $$
\[ |\Phi_m\rangle = \left[ p^+ p^- \left( p_u \alpha_+ + \left( p_v + \frac{1}{2} \right) \alpha_- \right) - \frac{2p_u p_v + p_u}{\sqrt{2}} \left( p^+ \beta_+ - 1 \right) \right] |n, \vec{p}\rangle \]

These states are orthogonal to each other and we found the norms of these states as

\[
\langle \Phi_\alpha | \Phi_\alpha \rangle = (2p_u p_v - p_u) = p^+ p^- + 2(1-a) > 0, \\
\langle \Phi_\beta | \Phi_\beta \rangle = p^+ p^- > 0, \\
\langle \Phi_m | \Phi_m \rangle = 2p^+ p^- (p^+ p^- + 2(1-a)) (1-a),
\]

where we used the mass-shell condition at level 1,

\[ p_u p_v + \frac{p^+ p^-}{2} + \frac{p_u}{2} + 1 = a \]

(3.35)

to obtain the the last relation in (3.34). As it is seen, the norms of the first two states in (3.34) are positive. Whereas, the norm of the last state in (3.34) is positive only for \( a < 1 \). It becomes a null state in the critical case, \( a = 1 \), and a ghost for \( a > 1 \). This is the same situation as in the 4 dimensional flat bosonic string theory if we take it as a part of string theory in 26 dimensions. We are encountering the same kind of situation here, because we mapped our theory to the theory of four bosonic free fields as in the case of 4 dimensional flat string theory.

The physical state combinations at higher levels can be constructed, therefore, just as in the case of 4 dimensional flat string theory. Thus there is no need to repeat that construction here. It is possible to prove the non-existence of negative norm states in the spectrum just as in the flat string theory \[34\]. We would like to emphasize that the spectrum we are proposing is free of ghosts without the requirement of an extra constraint. Such a constraint, which is \( p_u < 1 \), is mentioned first in an unpublished paper by Forgacs et al. \[21\], and later again in \[16\], \[17\], \[18\]. Since it is very similar to the analogous constraint in the case of SL(2,R) WZNW model, it seemed as a plausible constraint. However, in the next short subsection we comment on the origin of this constraint. We are going to claim that the \( p_u < 1 \) condition is an artifact of the formalism previously used.

1. Comments on \( p_u < 1 \) Condition

To understand the \( p_u < 1 \) condition we map the theory into 2 bosonic free fields and 2 bosonic ghosts as in \[18\]. Ghosts, together, construct a \( \beta - \gamma \) system. Then the currents in terms of these fields are given as

\[
T_L (z) = -\partial \phi, \\
J_L (z) = -\partial \phi - i : \gamma \beta :, \\
P^+_L (z) = -2 \partial \gamma + 2i \partial \phi \gamma, \\
P^-_L (z) = -\beta,
\]

(3.36)
where the bosonic ghost fields, $\beta$ and $\gamma$, have the mode expansions:

\[
\beta(z) = \sum_n \beta_n z^{-n-1}, \quad \gamma(z) = \sum_n \gamma_n z^{-n-1}
\] (3.37)

with commutation relation among the modes being $[\beta_n, \gamma_m] = \delta_{n+m}$. The commutation relations of the modes of the bosonic free fields are as in subsection III A. The modes of currents in terms of modes of the bosonic and the ghost fields can be easily determined as

\[
T_n = -i\alpha_n^-, \\
J_n = i\alpha_n^+ - i \sum_m \beta_{n-m} \gamma_m, \\
P_n^+ = 2n\gamma_n - 2 \sum_m \alpha_{n-m} \gamma_m, \\
P_n^- = -\beta_n
\] (3.38)

The Hermiticity properties of the modes of currents can be deduced from the Hermiticity properties of the zero frequency modes of currents, which are the generators of the Lie algebra $h_4$. Then the Hermiticity properties of the modes are

\[
(T_n)^\dagger = -T_{-n}, \quad (J_n)^\dagger = -J_{-n}, \quad (P_n^+)^\dagger = P_{-n}, \quad (P_n^-)^\dagger = P_{-n}^+
\] (3.39)

Now, consider the excited state $|\Phi\rangle = P_{-1}^- |\text{base}\rangle$. Its norm is

\[
\langle \Phi | \Phi \rangle = (2 - 2p_u).
\] (3.40)

In the calculation we used the fact that $\alpha_0^- |\text{base}\rangle = iT_0 |\text{base}\rangle = -p_u |\text{base}\rangle$. Therefore, for the norm of the state $|\Phi\rangle$ to be positive, $p_u < 1$ condition is required. Thus this condition is an artifact of the formalism in which the currents are realized in terms of bosonic and ghost fields, and the states are constructed by using the modes of currents, but not the modes of the set of free fields, to which the currents are mapped.

C. Monodromy

Since we realize the affine currents of $H_4$ WZNW model in terms of bosonic free fields, the currents, when expanded in terms of modes of these bosonic free fields, contain logarithmic cuts and consequently not analytic. We recall that the equations of motion derived from WZNW action require only that the currents separate into the holomorphic and the anti-holomorphic parts. There are no constraints that require the currents to be analytic form the start. However, their matrix elements in the physical sector of the spectrum should be analytic, in fact
periodic. This is due to the boundary condition in the closed string sector, which require that the fields on the string worldsheet to be periodic in the $\sigma$ variable. The corresponding requirement on the radial quantization is that the fields should be invariant under the monodromy transformations as $z \rightarrow z e^{i2\pi n}$. That is instead of taking the currents as periodic from the beginning, the periodicity is implemented in the Hilbert space [6]. Since the energy-momentum tensor [3.22] turned out to be analytic and therefore it was possible to mode expand it in terms of Virasoro generators, the monodromy operation commutes with the Virasoro generators and can be implemented in Hilbert space without any problems.

The monodromy condition on the currents means that

$$\langle \text{phys}' | J_L (z e^{i2\pi n}) | \text{phys} \rangle = \langle \text{phys}' | J_L (z) | \text{phys} \rangle$$

(3.42)

From the form of the currents in terms of free fields (3.3) one finds that the currents undergo a linear transformation under the monodromy:

$$T_L (z e^{i2\pi n}) = T_L (z),$$
$$J_L (z e^{i2\pi n}) = J_L (z) + i2\pi n \beta_0^+ P_L^- (z),$$
$$P_L^+ (z e^{i2\pi n}) = P_L^+ (z) - i4\pi n \beta_0^+ T_L (z),$$
$$P_L^- (z e^{i2\pi n}) = P_L^- (z).$$

(3.43)

Since this transformation is linear, it can be written as an adjoint action and since $P_L^- (z)$ does not change under monodromy, its zero mode should be the generator of the transformation:

$$J_L (z e^{i2\pi n}) = e^{-i2\pi n \beta_0^+ \beta_0^-} J_L (z) e^{i2\pi n \beta_0^+ \beta_0^-}.$$

(3.44)

where $\beta_0^+$ just acts as a number and $-i\beta_0^-$ is the zero mode of $P_L^- (z)$. Then, physical states which satisfy the monodromy condition (3.42) are the states that are invariant under the monodromy transformation

$$e^{-i2\pi n \beta_0^+ \beta_0^-} | \text{phys} \rangle = | \text{phys} \rangle.$$

(3.45)

Implementing this condition on the states in the Fock space

$$e^{-i2\pi n \beta_0^+ \beta_0^-} \prod_{i,j,k,l=1}^{\infty} (\alpha_{-i})^{\epsilon_{1,i}} (\alpha_{+j})^{\epsilon_{2,j}} (\beta_{-k})^{\epsilon_{3,k}} (\beta_{+l})^{\epsilon_{4,l}} | n; p_u, p_v, p^+, p^- \rangle$$

one finds that the monodromy condition does not affect the form of the physical states at the higher levels, but the product of momenta in transverse coordinates is quantized in terms of non-negative integers:

$$e^{-i2\pi n \beta_0^+ \beta_0^-} | n, \vec{p} \rangle = | n, \vec{p} \rangle \quad \Longrightarrow \quad \beta_0^+ \beta_0^- = 1/2 p^+ p^- = r \geq 0.$$

(3.47)

Here the reason we take $r$ to be a non-negative integer is that according to mass-shell condition the $p^+ p^-$ product is non-negative. Then, the mass-shell condition at excitation level $l$ becomes

$$p_u p_v + r + \frac{p_u}{2} + l = a.$$

(3.48)
IV. VERTEX OPERATOR

In the previous section, we have shown that the physical spectrum of the quantum string on NW spacetime is the affine continuous series representation of the Kac–Moody algebra \( \hat{h}_4 \). According to the state-operator correspondence hypothesis, to each state in the physical spectrum, there corresponds a vertex operator. In this section, we are going to write the most basic vertex operator that corresponds to the \( |0, \vec{p}\rangle \) state in ground level of the string spectrum. This “tachyon” vertex operator can be written as the group element in the continuous series representation of \( \hat{h}_4 \) as

\[
V(g) = e^{a^+ \hat{P}^- + a^- \hat{P}^+} e^{u \hat{J} + v \hat{T}},
\]

(4.1)

or from the group property, in any representation we can write

\[
V(g) = V(g_L)V(g_R)\]

(4.2)

where \( V(g_L) \) and \( V(g_R) \) are constructed from free fields of previous section. Here \( \hat{T}, \hat{J}, \hat{P}^+ \) and \( \hat{P}^- \) are some operator representations of the generators of \( h_4 \). We found that the following position-momentum basis representation of generators correspond to the continuous series representation of the algebra \( h_4 \):

\[
\begin{align*}
\hat{T} &\equiv -ip_u \\
\hat{J} &\equiv -ip_v + \hat{P}^- \hat{x}^+ \\
\hat{P}^+ &\equiv 2p_u \hat{x}^+ + 2i \frac{r}{p^+} \\
\hat{P}^- &\equiv ip^- - 2p_u \hat{x}^- + 2i \frac{r}{p^-}
\end{align*}
\]

(4.3)

In these representations of the generators, \((\hat{x}^+, \hat{p}^-)\) and \((\hat{x}^-, \hat{p}^+)\) are canonically conjugate pairs. \( r \) is a quantum number of the states in the continuous series representation of \( h_4 \) as determined by the monodromy considerations. It should be noted that in the representation on the left there is only the \((\hat{x}^+, \hat{p}^-)\) conjugate pair, and in the representation on the right there is only the \((\hat{x}^-, \hat{p}^+)\) conjugate pair. In order to prevent possible confusion we remark that these \( \hat{x}^+, \hat{p}^- \), etc. operators should not be thought as the zero modes of the free fields \((3.6-3.9)\). They are just a convenient device for a representation of generators in the continuous series representation of \( h_4 \). Using the canonical commutation relations \([\hat{x}^+, \hat{p}^-] = i = [\hat{x}^-, \hat{p}^+]\) it can easily be shown that \( \hat{T}, \hat{J}, \hat{P}^+ \) and \( \hat{P}^- \) obey correct commutation relations of the algebra \( h_4 \). To be sure that these operator representations really correspond to the continuous series representation of \( h_4 \) we also calculate the quadratic Casimir operator. For either form of the generators it is

\[
\hat{C} = \frac{1}{2} \left( \hat{P}^+ \hat{P}^- + \hat{P}^- \hat{P}^+ \right) + \hat{J} \hat{T} + \hat{T} \hat{J}
\]

\[
= -2p_up_v - 2r - p_u.
\]

(4.4)

Recall that in subsection III C from monodromy considerations we have found that \( r = \frac{1}{2} p^+ p^- \). Therefore this representation of generators correspond to the continuous series representation of \( h_4 \).
We now consider states on which either \( \hat{P}^+ \) or \( \hat{P}^- \) operators are diagonal. These states are the momentum basis states \( \langle p^-, p_u, p_v | \equiv \langle p^- | \), on which \( \hat{P}^- \equiv i\hat{p}^- \) is diagonal, and \( |p^+, p_u, p_v \rangle \equiv |p^+ \rangle \), on which \( \hat{P}^+ \equiv i\hat{p}^+ \) is diagonal. In the momentum basis we have \( \langle p^- | \hat{x}^+ = i\frac{\partial}{\partial p^-} \langle p^- | \) and \( \hat{x}^- | p^+ \rangle = -i\frac{\partial}{\partial p^+} | p^+ \rangle \) consistent with the commutation rules. The Fourier transform of these states correspond to diagonalizing the operators \( \hat{x}^+ \) and \( \hat{x}^- \) in the position basis states \( \langle x^+, p_u, p_v | \equiv \langle x^+ | \) and \( |x^-, p_u, p_v \rangle \equiv |x^- \rangle \), respectively. In the position basis \( \hat{P}^- \) and \( \hat{P}^+ \) operate as \( \hat{P}^- \langle x^+ | \hat{P}^- = -i\frac{\partial}{\partial x^+} \langle x^+ | \) and \( \hat{P}^+ | x^- \rangle = i\frac{\partial}{\partial x^-} | x^- \rangle \), respectively. 

Now the matrix elements of the tachyon vertex operator can be evaluated in the position or momentum basis just as one computes \( D-\)functions in a representation of a group. To do that we find it convenient to compute the vertex operator in the position basis as follows

\[
V_{x^+, x^-}^{p_u, p_v, r}(g) = \langle x^+ | e^{a^+ \hat{P}^- + a^- \hat{P}^+ e^{u J + v T}} | x^- \rangle,
\]

since \( \hat{P}^- \) has simple action on \( \langle x^+ | \), and \( \hat{P}^+ \) has simple action on \( |x^- \rangle \). We use the same form also for the left/right vertex operators. In terms of those the full vertex operator in the momentum basis is given by

\[
V_{p^-, p^+}^{p_u, p_v, r}(z, \bar{z}) = \langle p^- | V (g_L) V (g_R) | p^+ \rangle = \int \int d\bar{p}^+ d\bar{p}^- \langle p^- | V (g_L) | \bar{p}^+ \rangle \langle \bar{p}^+ | \bar{p}^- \rangle \langle \bar{p}^- | V (g_R) | p^+ \rangle = \int \int d\bar{p}^+ d\bar{p}^- \langle \bar{p}^+ | \bar{p}^- \rangle V_{p^-, \bar{p}^+}^{p_u, p_v, r}(z) V_{\bar{p}^-, p^+}^{p_u, p_v, r}(\bar{z}).
\]

Here again \( \hat{P}^- \) has simple action on \( \langle p^- | \), and \( \hat{P}^+ \) has simple action on \( |p^+ \rangle \).

In the following the alternative representations of the tachyon vertex operator in position and momentum bases will have different usages. The form of the vertex operator in the position basis will be useful for comparison to the wave functional \([2, 29]\) on the \( H_4 \) manifold, and for possible interpretations of it in the BMN correspondence \([9]\), as it is done in \([7]\) for the case of AdS\(_3\)/CFT\(_2\) correspondence. Whereas the momentum basis form of the vertex operator will be useful in determination of operator products of the vertex operator with the basic fields in the theory. In fact we will show that with the correct quantum ordering prescription the vertex operator has correct operator products with the currents and it is a primary operator with conformal weight one. We will also show that the conformal dimension comes in the correct form, being proportional to the eigenvalue of the zero frequency Virasoro generator \( L_0 \). Due to its simpler form, the momentum basis vertex operator is also expected to be useful in the computations of the correlation functions.
A. Vertex Operator in the Position Basis

We define the vertex operator in the position basis by

\[ V_{p^+, p^-}^{x^+, x^-} (g) = \langle x^+ | e^{a^+ \hat{J}^+ + a^- \hat{J}^-} e^{u \hat{J}} | x^- \rangle = \langle x^+ | e^{a^+ \hat{P}^+} e^{u \hat{J}} | x^- \rangle. \]  

(4.7)

Note that we are not using the factorized form of the vertex operator in this basis. The vertex operator in the position basis is written in terms of the coordinates of the group manifold. This is because this representation of the vertex operator will turn out to be useful not in quantum computations, but possible semi-classical interpretations like in [7].

In order to evaluate this matrix element of the vertex operator we note that \( \hat{P}^- \equiv i \hat{p}^- \) and \( \hat{P}^+ \equiv i \hat{p}^+ \) behave as translation operators, and \( \hat{J} \) behaves as dilation operator, together with the extra factor \( e^{-iu p^+} e^{iu} \), in the position space. Therefore we obtain

\[ V_{p^+, p^-}^{x^+, x^-} (g) = \langle x^+ + a^+ | e^{u \hat{J}^+ (v - ia^- a^-)} | x^- - e^{iu} a^- \rangle \]

(4.8)

\[ = e^{-i(u p^+ (v - ia^- a^-) p^-)} e^{-iu} \langle x^+ + a^+ | x^- - e^{iu} a^- \rangle \]

(4.9)

\[ = e^{-i(u p^+ (v - ia^- a^-) p^-)} e^{-iu} \langle x^+ + a^+ | e^{-iu} (x^- - e^{iu} a^-) \rangle. \]

(4.10)

We must calculate the inner product \( \langle x^+ | x^- \rangle \). Its properties under the dilations show that it is a function of only \( x^+ x^- \), therefore we write \( \langle x^+ | x^- \rangle = f_{p^+, r} (x^+ x^-) \). Then the full vertex operator is

\[ V_{p^+, p^-}^{x^+, x^-} (g) = e^{-i(u p^+ + u p^-)} e^{-a^+ a^- p^+} e^{-iu} f_{p^+, r} (e^{-iu} (x^+ + a^+) (x^- - e^{iu} a^-)). \]

(4.11)

In order to determine \( f_{p^+, r} (x^+ x^-) \) we first insert \( \hat{P}^- \) and then \( \hat{P}^+ \) in between left and right states and we find their action on either states:

\[ \left( -2 p^+ x^- + 2 \frac{r}{\partial_x} \right) f_{p^+, r} = \langle x^+ | \hat{P}^- | x^- \rangle = \partial_+ f_{p^+, r} \]  

(4.12)

\[ -\partial_- f_{p^+, r} = \langle x^+ | \hat{P}^+ | x^- \rangle = \left( 2 p^+ x^- - 2 \frac{r}{\partial_x} \right) f_{p^+, r} \]  

(4.13)

Multiplying the equality (4.12) with \( \partial_- \) and the equality (4.13) with \( \partial_+ \) we obtain differential equations

\[ \partial_- \partial_+ f_{p^+, r} = -2 (p^+ + p^- x^- \partial_- - r) f_{p^+, r} \]

\[ \partial_- \partial_+ f_{p^+, r} = -2 (p^+ + p^- x^+ \partial_+ - r) f_{p^+, r} \]

Adding them and setting \( x^+ x^- = y \), we finally obtain the differential equation

\[ y \partial_y^2 f_{p^+, r} (y) + (1 + 2 p^+ y) \partial_y f_{p^+, r} (y) + (2 p^+ - 2 r) f_{p^+, r} (y) = 0. \]  

(4.14)

This is the Kummer’s differential equation. Its exact solution that is well behaved at the origin is the confluent hypergeometric function

\[ f_{p^+, r} (y) = e^{-2 p^+ y} F_1 \left( 1 + \frac{r}{p^+}, 1; 2 p^+ y \right). \]  

(4.15)
Then the vertex operator in the position basis becomes

\[ V_{x^+, x^-}^{p_v, p_u, r}(g) = e^{-i(xp_v + vp_u)} e^{-p_u a^+ a^-} e^{-iu} e^{-w} {}_1F_1(\alpha, 1; w), \] (4.16)

where \( \alpha = 1 + \frac{p_v^2}{2p_u} \) and \( w = 2p_u e^{-iu} (x^+ + a^+) (x^- - e^{iu} a^-) \). This form of the vertex operator resembles the wave functional (2.29). Therefore for semi-classical considerations this form will have great utility. However for quantum computations the confluent hypergeometric function would be difficult to manipulate. For those kind of computations the momentum space form of the vertex operator will be more useful, whose form is considerably simpler as it will be seen in the next subsection.

B. Vertex Operator in the Momentum Basis

In this section we are going to describe only the holomorphic part of the factorized vertex operator (4.16), \( V_{p^-, \tilde{p}^+}^{p_v, p_u, r}(z) \), in the momentum basis and show that it has correct operator products with the holomorphic currents and the holomorphic energy-momentum tensor. The anti-holomorphic part, \( \tilde{V}_{\tilde{p}^-, p^+}^{p_v, p_u, r}(\tilde{z}) \), is insensitive to these operator products and therefore we do not include it in the discussion. The operator products in the anti-holomorphic sector has the same form as the corresponding ones in the holomorphic sector. The operator products are performed after determining the correct quantum ordering of the fields and their zero modes in the expression for the vertex operator. We are going to show that the holomorphic part of the vertex operator has the correct conformal dimension, \( h_{p_v, p_u, r} = p_u p_v + r + \frac{p_v^2}{2} \).

1. Classical Expression

The holomorphic part of the vertex operator in momentum basis is defined by

\[ V_{p^+, p^-}^{p_v, p_u, r}(z) = \langle p^- | e^{\theta^+ \tilde{p}^-} e^{\phi^+ \tilde{p}^-} e^{\theta^- \tilde{p}^+} | p^+ \rangle. \] (4.17)

In order to evaluate this we note that \( \hat{p}^- \equiv i\tilde{p}^- \) and \( \hat{p}^+ \equiv i\tilde{p}^+ \) are diagonal on states \( \langle p^- \rangle \) and \( | p^+ \rangle \), respectively, and \( \hat{J} \) behaves as dilation operator, together with the extra factor \( e^{-i\phi p_v} \), in the momentum space. Therefore we obtain

\[ V_{p^+, p^-}^{p_v, p_u, r}(z) = e^{i\theta^+ p^-} \langle p^- \mid e^{\phi^+ \tilde{p}^-} e^{\theta^- \tilde{p}^+} \mid p^+ \rangle e^{i\theta^- p^+} \]
\[ = e^{i\theta^+ p^-} e^{-i(\phi p_v + \phi p_u)} \langle p^- \mid e^{i\theta^+ p^-} \mid p^+ \rangle e^{i\theta^- p^+} \]
\[ = e^{i\theta^+ p^-} e^{-i(\phi p_v + \phi p_u)} \langle e^{i\phi p^-} \mid p^+ \rangle e^{i\theta^- p^+}. \] (4.18)

Now we define the function \( \tilde{f}_{p_u, r} (p^+ p^-) = \langle p^- \mid p^+ \rangle \). This function must be a function of the single variable \( p^+ p^- \) due to its properties under dilations. Then the holomorphic part of the vertex operator is

\[ V_{p^+, p^-}^{p_v, p_u, r}(z) = e^{i\theta^+ p^-} e^{-i(\phi p_v + \phi p_u)} \tilde{f}_{p_u, r} (e^{i\phi p^-} p^+) e^{i\theta^- p^+} \] (4.19)
In order to determine $\tilde{f}_{p_u,r}(p^+p^-)$ we insert $\hat{P}^-$ and then $\hat{P}^+$ in between left and right states and then following the same procedure as in the previous subsection we find the differential equation

$$y \partial_y \tilde{f}_{p_u,r}(y) + \left( \frac{r}{p_u} - \frac{y}{2p_u} \right) \tilde{f}_{p_u,r}(y) = 0. \tag{4.20}$$

where $y = p^+p^-$. The solution of this differential equation is found to be $\tilde{f}_{p_u,r}(y) = y^{-r/p_u} \exp(y/2p_u)$. Then the holomorphic part of the vertex operator in the momentum basis takes the form

$$V_{p^+,p^-}^{p_u,r}(z) = e^{i\theta^+p^-} e^{-i(\phi p_u + \varphi p_u)} (e^{i\phi p^+p^-})^{-r/p_u} \exp\left( e^{i\phi \frac{p^+p^-}{2p_u}} \right) e^{i\theta^+p^+}. \tag{4.21}$$

2. Quantum Ordering and Operator Products

To define the quantum expression of the vertex operator, we start by preserving the order of the operators that comes from group theoretical construction. Then we take the factors related to each non-commuting generator of the algebra as already normal ordered:

$$V_{p^+,p^-}^{p_u,r}(z) = e^{i\theta^+p^-} \langle p^- \mid e^{\phi \hat{J} + \varphi \hat{T}} \mid p^+ \rangle : e^{i\theta^+p^+} : \tag{4.22}$$

$$= e^{i\theta^+p^-} : e^{-i(\phi p_u + \varphi p_u)} (e^{i\phi p^+p^-})^{-r/p_u} \exp\left( e^{i\phi \frac{p^+p^-}{2p_u}} \right) :$$

$$\times : \exp\left( i p^+ \left( x_0^- + \frac{1}{2} \int^z dz' \theta^-(z') \phi^+(z') \right) \right) : \tag{4.23}$$

where $\theta^-(z)$ is written in terms of the canonical variables (see equ.(4.21)). Due to the definition of the canonical variables, $e^{i\theta^+p^-}$ does not need normal ordering. This gives the ordering of the operators. However, in order to be able to use the Wick theorem during the computations of the operator products we need to write the quantum expression of the vertex operator in fully normal ordered form. Using the contractions of free fields as given in equ. (3.15-3.18) this fully normal ordered form is found as follows

$$V_{p^+,p^-}^{p_u,r}(z) = : e^{i\theta^+p^-} e^{-i(\phi p_u + \varphi p_u)} (e^{i\phi p^+p^-})^{-r/p_u} \exp\left( e^{i\phi \frac{p^+p^-}{2p_u}} \right) e^{i\theta^+p^+}$$

$$\times \exp\left( i \frac{p^+}{2} \int^z d\varphi \left( \theta^-(z') - i \frac{p^-}{z - z'} \right) e^{i\phi(z') (z - z')^p_u} \right) : \tag{4.24}$$

Note that now the whole expression is between the normal ordering columns, which is the reason of the complicated additional factors. Since the initial definition (4.23) of the vertex operator is not fully normal ordered, we should show that the vacuum expectation value of it is finite. To compute this we use the last form (4.24) of the vertex operator and find

$$\langle 0 \mid V_{p^+,p^-}^{p_u,r}(z) \mid 0 \rangle = (p^+p^-)^{r/p_u} e^{p^+p^-/2p_u} \exp\left( \frac{p^+p^-}{2} \int^z d\varphi (z - z')^{p_u-1} \right) \tag{4.25}$$
which is finite.

The laborious technical details of the calculation of the operator products of the currents and the energy-momentum tensor with the vertex operator is similar to the computation presented in the appendix of [7]. Therefore, there is no need to repeat it here. We just would like to point out a very important ingredient of the computation. This is the problem of how the zero modes in the quantum expression for the vertex operator need to be ordered. We found that the following ordering of the zero modes is the correct one,

\[
\langle e^{\phi \hat{J} + \varphi \hat{T}} \rangle_{\text{zero modes}} = e^{-i\phi p_v} \left( e^{i\phi_{p^+} p^-} \right)^{-r/p_u} e^{\left( e^{i\phi_{p^+} p^-} \right) e^{-i\varphi p_u}}
\] (4.26)

With the ordering prescription given above, we found the correct operator products with the holomorphic currents as

\[
\mathcal{J}_L^i (z) \times V_{p^+, p^-, r}^{p_v, p_u, r} (w) = \frac{1}{z - w} \langle p^- | \hat{J}^i V_{p^+, p^-, r}^{p_v, p_u, r} (w) | p^+ \rangle,
\] (4.27)

where \( \hat{J}^i \) is \( \hat{T}, \hat{J}, \hat{P}^+ \) or \( \hat{P}^- \). The action of \( \hat{J}^i \) on \( \langle p^- \rangle \) is a differential operator that follows from the left side of (4.3). Thus, \( \hat{P}^- = ip^- \), \( \hat{J} = \text{dilations} \), etc. We also found that the operator product of the vertex operator with the energy-momentum tensor

\[
\mathcal{T}_L (z) \times V_{p^+, p^-, r}^{p_v, p_u, r} (w) = \frac{1}{(z - w)^2} \langle p^- | \left( \frac{1}{2} L_{ij} \hat{J}^i \hat{J}^j \right) V_{p^+, p^-, r}^{p_v, p_u, r} (w) | p^+ \rangle + \frac{1}{z - w} \partial_w V_{p^+, p^-, r}^{p_v, p_u, r} (w)
\] (4.28)

gives the correct conformal dimension

\[
h_{p_v, p_u, r} = p_u p_v + r + \frac{p_u}{2}.
\] (4.29)

This proves that the vertex operator we are proposing is the correct tachyon vertex operator that correspond to the state \( |0, \vec{p} \rangle \) of the continuous series representation of \( h_4 \).

V. CONCLUSIONS

In this paper we have investigated the physical spectrum of the string theory on NW spacetime. We started by showing that the NW spacetime is a Penrose limit of AdS\(_2 \times S^2\) spacetime and its metric has the form of an exact plane-wave background. NW spacetime also contains an anti-symmetric field whose field strength is everywhere constant. The NW spacetime can also be thought as the group manifold of the non-semi–simple group \( H_4 \). This group is equivalent to either left or right part of the isometry group. The corresponding algebra is the Heisenberg algebra with a rotation operator added. From the isometry generators it is straightforward to write the D’Alembertian on this background. We acted the D’Alembertian on a wave functional written in a specific irreducible representation of \( h_4 \) and derived the dispersion relation.
The string theory on a group manifold is described by the WZNW model. We realized the currents in the $H_4$ WZNW model in terms of four bosonic free fields and constructed the string spectrum by using the negative frequency modes of these free fields. The spectrum that is constructed by the free field modes turned out to be equivalent to affine continuous series representation of Kac–Moody algebra $\hat{h}_4$. We showed this by checking that the eigenvalue of $L_0$ operator (half of the Hamiltonian) on states is equivalent to the negative of the eigenvalue of the quadratic Casimir operator in the continuous series representation of $h_4$. Due to the indefinite signature of the metric there appeared the negative norm states in the spectrum. However, it is argued that the situation is no worse than the case in flat spacetime and the Virasoro constraints are enough to eliminate all the negative norm states from the spectrum. We claimed that the spectrum we found contains another spectrum as a sub-spectrum which is obtained when one sets the transverse momenta to zero and thus gets the affine discrete series representation as the spectrum of quantum strings. In that case we had to explain why $p_u < 1$ condition is not seen in our construction. We argued that the mentioned condition is just an artifact of the formalism used in the previous approaches [17], [18]. Since we realized the currents in terms of free bosons, there appeared logarithmic cuts in the expressions of the currents. We required that in the physical sector, the currents should be periodic due to the periodicity of $\sigma$ coordinate on closed string worldsheet. We found that this requirement results in the quantization of the product of transverse momenta.

In the last section we revived the state-operator correspondence and determined the tachyon vertex operator that corresponds to the $|0, \vec{p}\rangle$ state in the string spectrum. We expressed the classical form of the vertex operator in two different bases. The position basis expression is given for the full vertex operator. This expression is useful for comparison to the wave functional derived in the coherent state basis in the continuous series representation of $h_4$. We expect that this form of the vertex operator will also be useful to make claims about the operators in the dual gauge theory as it is made in the case of AdS$_3$/CFT$_2$ correspondence in [35], [7]. The momentum basis expression of the vertex operator is utilized for completely different purpose. We observed that the momentum space form of the vertex operator consists of simple exponentials and therefore for it a quantum ordering prescription can be given more easily than the position space expression, which contains confluent hypergeometric function. We determined the ordering prescription and showed that the quantum ordered vertex operator has the correct operator product expansions with the currents and the energy–momentum tensor. From the latter OPE we obtained the conformal dimension of the vertex operator in the expected form.

This way we have determined the physical spectrum of the string theory in the NW spacetime. A possible extension of this work could be applying the same ideas to higher dimensional $H_{2n+2}$ group manifolds. In the past literature the irreducible representations of higher $h_{2n+2}$ algebras were given in analogy with the 4 dimensional case and therefore the important continuous series representations of those higher algebras are also missed [19]. We plan to investigate the construction of string theory on $H_{2n+2}$ group manifolds again,
with the knowledge of these previously unknown representations. In view of the important changes observed in the definition of the physical string spectrum in $H_4$ case, we expect the same, if not more, new results to emerge in higher dimensional cases.

The other channel of research, as mentioned in the introduction, is to investigate the BMN dual gauge theory of the string theory we constructed. Recalling the conjecture in [13] that the string theory on $\text{AdS}_2 \times S^2$ is dual to a conformal quantum mechanics on the boundary of $\text{AdS}_2$, we expect the bosonic string theory we quantize here would be dual to some large $N$ limit of conformal quantum mechanics on circle. We plan to investigate this duality in the near future.

Acknowledgments

C. D. is supported in part by the Turkish Academy of Sciences in the framework of the Young Scientist Program (CD/TÜBA–GEBİP/2002–1–7).

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