Abstract. We consider a mass-critical system of nonlinear Schrödinger equations
\[
\begin{align*}
    i\partial_t u + \Delta u &= \bar{u}v, \\
    i\partial_t v + \kappa \Delta v &= u^2,
\end{align*}
\] (t,x) ∈ ℝ × ℝ^4,

where (u,v) is a C^2-valued unknown function and κ > 0 is a constant. If κ = 1/2, we say the equation satisfies mass-resonance condition. We are interested in the scattering problem of this equation under the condition M(u,v) < M(φ,ψ), where M(u,v) denotes the mass and (φ,ψ) is a ground state. In the mass-resonance case, we prove scattering by the argument of Dodson [5]. Scattering is also obtained without mass-resonance condition under the restriction that (u,v) is radially symmetric.

1. Introduction.

1.1. System of NLS. We consider
\[
\begin{align*}
    i\partial_t u + \frac{1}{2m} \Delta u &= \lambda \bar{u}v, \\
    i\partial_t v + \frac{1}{2M} \Delta v &= \mu u^2,
\end{align*}
\] (t,x) ∈ ℝ × ℝ^d,

\[ (u,v)|_{t=0} = (u_0,v_0) ∈ L^2(ℝ^d)^2, \]

where (u,v) is a C^2-valued unknown function, m and M are positive constants, and λ,μ ∈ ℂ \ {0} are constants. From the viewpoint of physics, (1.1) is related to the

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Raman amplification in a plasma. See [2] for details. Furthermore (1.1) is regarded
as a non-relativistic limit of the system of nonlinear Klein–Gordon equations under
the mass-resonance condition $M = 2m$ (see [11]). Under the assumption $\lambda = c\mu$
for some $c > 0$, the solution to (1.1) conserves the mass and the energy, defined
respectively by
\[
\text{Mass} = \| u(t) \|_2^2 + c\| v(t) \|_2^2,
\]
\[
\text{Energy} = \frac{1}{2m} \| \nabla u(t) \|_2^2 + \frac{c}{4M} \| \nabla v(t) \|_2^2 + \Re \left( \lambda \int_{\mathbb{R}^d} u^2(t, x)\overline{v(t, x)} \, dx \right).
\]
To use these conservation laws, we impose $\lambda = c\mu$ for some $c > 0$. Then the system
(1.1) is reduced to the following system by some changes of variables:
\[
\begin{aligned}
&i\partial_t u + \Delta u = \pi v, \\
i\partial_t v + \kappa \Delta v = u^2,
\end{aligned}
\tag{1.2}
\]
where $\kappa$ is a positive constant. If $(u, v)$ is a solution to (1.2), then $(\lambda^\kappa u, \lambda^\kappa v)$ also solves (1.2) for any $\lambda > 0$. This property is called scaling
symmetry. Setting $s_c := d/2 - 2$, then $\|(u, v)\|_{H^{s_c}}$ is invariant under this scaling
transformation. In particular, if $d = 4$, the mass is invariant under the scaling
transformation. From this fact, if $d = 4$, we say the system (1.2) is mass-critical.
Similarly, we call the case $d = 5$ and $d = 6$ $\dot{H}^{\frac{d}{2}}$-critical and energy-critical, respectively. In this paper, we treat the following mass-critical system of NLS:
\[
\begin{aligned}
i\partial_t u + \Delta u = \pi v, \\
i\partial_t v + \kappa \Delta v = u^2,
\end{aligned}
\tag{1.3}
\]
where $\kappa$ is a positive constant. If $\kappa = 1/2$, (1.3) has Galilean invariance, i.e., if
$(u, v)$ is a solution to (1.3), then $(e^{ix\xi}e^{-it|\xi|^2}u(t, x - 2t\xi), e^{ix\xi}e^{-2it|\xi|^2}v(t, x - 2t\xi))$ is also a solution to (1.3) for any $\xi \in \mathbb{R}^d$. Note that in our case, the mass and the
energy are respectively defined by
\[
\begin{aligned}
M(u, v)(t) &:= \| u(t) \|_2^2 + \| v(t) \|_2^2, \\
E(u, v)(t) &:= \| \nabla u(t) \|_2^2 + \frac{\kappa}{2} \| \nabla v(t) \|_2^2 + \Re \int_{\mathbb{R}^d} u^2(t, x)\overline{v(t, x)} \, dx.
\end{aligned}
\]
It is well known that local well-posedness and small-data scattering hold for the
equation (1.3) in $L^2(\mathbb{R}^4)^2$. In this paper, we are interested in the behavior of the large solutions.

1.2. Known results for the single mass-critical NLS. In this subsection, we introduce known results about the following mass-critical single NLS:
\[
\begin{aligned}
i\partial_t u + \Delta u = \mu|u|^\frac{4}{d} u, \\
u(0) = u_0 \in L^2(\mathbb{R}^d),
\end{aligned}
\tag{1.4}
\]
where $u$ is a $\mathbb{C}$-valued unknown function and $\mu \in \{-1, 1\}$. If $\mu = +1$, we say
(1.4) is defocusing and if not, (1.4) is called focusing. In the defocusing case, any
$L^2$-solution $u$ of (1.4) exists globally and scatters to a free solution, which means that
\[
e^{-it\Delta} u(t) \to u_\pm \text{ in } L^2(\mathbb{R}^d) \text{ as } t \to \pm \infty \text{ for some } u_\pm \in L^2(\mathbb{R}^d).
\]
This fact was proved by Dodson in [4], [6], and [7] in the case \(d \geq 3, d = 2\), and \(d = 1\), respectively. On the other hand, in the focusing case, the existence of non-scattering solution is known. Indeed, the existence and uniqueness of the ground state \(Q\), which is the positive radial solution to

\[ Q - \Delta Q = Q^{1+\frac{2}{d}}, \]

is known (see [1], [15]). In [5], Dodson proved that a solution to (1.4) with \(\mu = -1\) exists globally and scatters to a free solution under the condition that \(\|u(0)\|_{L^2_x} < \|Q\|_{L^2_x}\). Concerning the system (1.3), we expect a similar scattering result to the focusing single NLS (1.4) with \(\mu = -1\), at least, when \(\kappa = 1/2\), where the system has the Galilean invariance. In this paper, we also treat the case of \(\kappa \neq \frac{1}{2}\).

1.3. Main results. Before stating our main results, we introduce some definitions.

**Definition 1.1.** (i) Let \(0 \in I \subset \mathbb{R}\) be an interval. We say that \((u, v) : I \times \mathbb{R}^4 \to \mathbb{C}^2\) is a (strong) solution to (1.3) if it lies in the class \((C(I; L^2(\mathbb{R}^4)) \cap L^3_{loc}(I; L^3(\mathbb{R}^4)))^2\) and obeys the Duhamel formula

\[
\begin{align*}
    u(t) &= e^{i\Delta} u_0 - i \int_0^t e^{i(t-t')\Delta} \left[ u(t') v(t') \right] dt', \\
    v(t) &= e^{i\kappa \Delta} v_0 - i \int_0^t e^{i(t-t')\kappa \Delta} \left[ u^2(t') \right] dt'
\end{align*}
\]

for all \(t \in I\).

(ii) We say that (1.3) admits global spacetime bounds for a set \(D \subset L^2(\mathbb{R}^4)^2\) of initial data if there exists a function \(C : [0, \infty) \to [0, \infty)\) such that for any \((u_0, v_0) \in D\) a solution to (1.3) exists on \(I = \mathbb{R}\) and obeys

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^4} \left( |u(t, x)|^3 + |v(t, x)|^3 \right) dx dt \leq C(M(u_0, v_0)).
\]

(iii) We say that the global solution \((u, v)\) to (1.3) scatters forward (backward) in time if there exist \((u_\pm, v_\pm) \in L^2(\mathbb{R}^4)^2\) such that

\[
(u(t), v(t)) - (e^{i\Delta} u_\pm, e^{i\kappa \Delta} v_\pm) \to 0 \quad \text{in} \quad L^2(\mathbb{R}^4)^2 \quad \text{as} \quad t \to \pm \infty. \quad (1.5)
\]

Next, we recall the result of Hayashi, Ozawa, and Tanaka [11]. Consider the solution to (1.3) which has the following form:

\[
(u, v) = \left( e^{it\phi}(x), e^{2it\psi}(x) \right) \quad (1.6)
\]

with real-valued functions \(\phi, \psi\).

If (1.6) is a solution of (1.3), then \((\phi, \psi)\) should satisfy the following system of elliptic equations:

\[
\begin{align*}
    -\phi + \Delta \phi &= \phi \psi, \\
    -2\psi + \kappa \Delta \psi &= \phi^2,
\end{align*}
\]

\(x \in \mathbb{R}^4\). \quad (1.7)

Associated functional for (1.7) is defined by

\[
I(\phi, \psi) := \|\nabla \phi\|_{L^2_x}^2 + \frac{\kappa}{2} \|\nabla \psi\|_{L^2_x}^2 + \|\phi\|_{L^2_x}^2 + \|\psi\|_{L^2_x}^2 + \int_{\mathbb{R}^4} \phi^2 \psi \, dx.
\]

Using this functional we give the definition of ground states.
Lemma 1.3 (Existence of a ground state, sharp Gagliardo-Nirenberg inequality). Let \( \kappa > 0 \).

(i) There exists at least one ground state of (1.7).

(ii) Let \((\phi, \psi)\) be a ground state of (1.7). Then, it holds that
\[
\left| \text{Re} \int_{\mathbb{R}^4} u^2(x) \overline{v(x)} \, dx \right| \leq \left( \frac{M(u, v)}{M(\phi, \psi)} \right)^{1/2} \left( \| \nabla u \|_{L^2}^2 + \frac{\kappa}{2} \| \nabla v \|_{L^2}^2 \right)
\]
for any \((u, v) \in H^1(\mathbb{R}^4)^2\). Moreover, equality is attained by the ground state.

Note that \( M(\phi, \psi) \) does not depend on the choice of a ground state \((\phi, \psi)\). From Lemma 1.3, we see that the energy is positive and controls the \( H^1 \) norm of \( H^1 \) solutions to (1.3) for (nonzero) initial data \((u_0, v_0)\) satisfying
\[
M(u_0, v_0) < M(\phi, \psi). \tag{1.8}
\]
In particular, the initial value problem (1.3) is globally well-posed in \((H^1)^2\) for initial data satisfying (1.8).

However, it is not clear whether the global well-posedness and scattering in \((L^2)^2\) for initial data satisfying (1.8) hold or not. We first give the following answer when \( \kappa = 1/2 \).

Theorem 1.4. If \( \kappa = 1/2 \), then (1.3) is globally well-posed and scattering holds in \( L^2(\mathbb{R}^4)^2 \) for initial data obeying (1.8).

The Galilean invariance plays an important role in the proof of Theorem 1.4. On the other hand, the system (1.3) does not have the Galilean invariance when \( \kappa \neq 1/2 \). Nevertheless, we get the following theorem for \( \kappa \neq 1/2 \) under the additional assumption of radial symmetry.

Theorem 1.5. Let \( \kappa \neq 1/2 \). If initial data is radially symmetric and satisfies (1.8), then (1.3) is globally well-posed and scattering holds in \( L^2(\mathbb{R}^4)^2 \).

1.4. Idea of proof. It is known that the global existence and scattering are equivalent to that \( S_{\text{max}}(u, v) \) is finite, where \( I_{\text{max}} \) is the maximal lifespan of the solution and
\[
S_I(u, v) := \int_I \int_{\mathbb{R}^4} \left( |u(t, x)|^2 + |v(t, x)|^2 \right) \, dx \, dt.
\]
To prove Theorems 1.4 and 1.5, it is enough to show \( S_I(u, v) < C(M(u, v)) \) for any solution \((u, v)\) on a time interval \( I \) satisfying (1.8). With this in mind, let
\[
L(M) := \sup \{ S_I(u, v) \, | \, (u, v) : I \times \mathbb{R}^4 \to \mathbb{C}^2 \text{ solves (1.3), } M(u, v) \leq M \}.
\]
Note that \( L : [0, \infty) \to [0, \infty) \) is nondecreasing and, from the stability result (see Section 2), continuous. Since the standing wave solution (1.6) with \((\phi, \psi)\) a ground state of (1.7) given in Lemma 1.3 is defined globally and \( S_R = \infty \), we see that \( L(M(\phi, \psi)) = \infty \). By the continuity of \( L \), there exists critical mass \( M_c \) such that
\[
L(M) < \infty \text{ for } M < M_c, \quad L(M) = \infty \text{ for } M \geq M_c.
\]
The small data theory in Section 2 ensures that \( M_c > 0 \). Note that \( L(M(\phi, \psi)) = \infty \) implies that \( M_c \leq M(\phi, \psi) \). It then suffices to show that \( M_c < M(\phi, \psi) \) would
lead to a contradiction. Furthermore we define the critical mass for the radially symmetric case. Set $L_{\text{rad}} : [0, \infty) \to [0, \infty]$ by

$$L_{\text{rad}}(M) := \sup \left\{ S_I(u, v) \mid (u, v) : I \times \mathbb{R}^4 \to \mathbb{C}^2 \text{ is a radial solution to (1.3)} \right\}.$$

such that $M(u, v) \leq M$.

Then by the stability result in Section 2, $L_{\text{rad}}$ is continuous and so there exists radially symmetric critical mass $M_{c, \text{rad}}$ such that

$$L_{\text{rad}}(M) < \infty \text{ for } M < M_{c, \text{rad}}, \quad L_{\text{rad}}(M) = \infty \text{ for } M \geq M_{c, \text{rad}}.$$

Since the ground state is radially symmetric, we obtain $M_{c, \text{rad}} \leq M(\phi, \psi)$. Note also that by the definition of $M_c$ it holds $M_c \leq M_{c, \text{rad}}$.

Our aim is to derive a contradiction by supposing $M_c < \overline{M}(\phi, \psi)$ with the mass-resonance condition and $M_{c, \text{rad}} < \overline{M}(\phi, \psi)$ without it. Toward contradiction, we construct the minimal blow-up solution which has critical mass $M_c$ (in the radial case critical mass is $M_{c, \text{rad}}$) by the profile decomposition for the system. When we construct the minimal blow-up solution in the case $\kappa \neq 1/2$, we need the radial assumption due to the lack of Galilean invariance. After that we refine the minimal blow-up solution to apply the argument of Dodson [5]. We exclude two possible scenarios which are called rapid frequency cascade and quasi-soliton. We eliminate the rapid frequency cascade scenario by additional regularity of the minimal blow-up solution, which comes from the long time Strichartz estimate. To exclude the quasi-soliton scenario, we rely on the estimate based on the virial identity (cf. [11, 17]) which is called the frequency localized interaction Morawetz estimate in [5].

The mass-critical case $d = 4$ is quite different from the $\dot{H}^{\frac{d}{2}}$-critical case $d = 5$. Hamano [9] gave the threshold for scattering or blow-up below the ground state in $\dot{H}^{\frac{d}{2}}$-critical case $d = 5$ and $H^1$ setting under the mass-resonance condition. To prove scattering, Hamano used the argument of Kenig–Merle [12] which is organized by stability, profile decomposition, construction of critical element, and rigidity of it. There are two differences between his argument and ours. One is regularity of initial data. More precisely, Hamano assumed $H^1$ regularity to solve $\dot{H}^{\frac{d}{2}}$-critical problem, while we only assume the minimal regularity $L^2$. The other is a variety of parameters in the profile decomposition from the lack of compactness in $L^2$. Indeed, the translation in the frequency side and the scaling transformation additionally breaks the compactness in $L^2$. The radial assumption is used to remove the former in the case of $\kappa \neq 1/2$.

**Organization of this paper.** In Section 2, we prepare the stability result and bilinear Strichartz estimate which are used in Section 3 and Section 4, respectively. In Section 3, we introduce the profile decomposition for the system and use it to construct the minimal blow-up solution. In Section 4, we prove the long time Strichartz estimate. In Sections 5 and 6, we treat the rapid frequency cascade scenario and the quasi-soliton scenario, respectively.

**2. Preliminaries.** First, we collect some notations. Let $\|(u, v)\|_X := \|(u, v)\|_{X \times X}$ for any function space $X$ and $(u, v) \in X \times X =: X^2$. We denote the (spatial) Fourier transform of a function $f$ by $\hat{f}$ or $\mathcal{F}f$.

Let $\theta : [0, \infty) \to [0, 1]$ be a smooth non-increasing function such that $\theta \equiv 1$ on $[0, 1]$ and $\text{supp } \theta \subset [0, 2]$. For each number $N > 0$, we define the Littlewood-Paley
projection operators by
\[
P_{\leq N} := \mathcal{F}^{-1} \theta \left( \left| \frac{x}{N} \right| \right) \mathcal{F}, \quad P_{N} := P_{\leq N} - P_{\leq N/2}, \quad P_{> N} := \text{Id} - P_{\leq N},
\]
\[
P_{< N} := P_{\leq N} - P_{N} = P_{\leq N/2}, \quad P_{\geq N} := \text{Id} - P_{< N} = P_{> N} + P_{N} = P_{> N/2}.
\]
These operators are bounded uniformly in \(N\) on \(L^p(\mathbb{R}^d)\) for any \(1 \leq p \leq \infty\), and they commute with each other, as well as with differential operators and Fourier multipliers such as \(i\partial_t + \Delta\) and \(e^{it\Delta}\). It is easy to see that \(\text{supp} \ P_{\leq N} f \subset \{ |\xi| \leq 2N \}\), \(\text{supp} \ P_{N} f \subset \{ |\xi| \leq N/2 \}\), \(\text{supp} \ P_{> N} f \subset \{ |\xi| \geq N \}\) for any function \(f\), and in particular, that \(P_{> N}(P_{\leq N/4} f \cdot P_{\leq N/4} g) = 0\) for any \(f, g\). We also define the Fourier projections with the frequency center \(\xi_0 \in \mathbb{R}^d\) as
\[
P_{|\xi-\xi_0| \leq N} := \mathcal{F}^{-1} \theta \left( \left| \frac{x}{N} - \xi_0 \right| \right) \mathcal{F}, \quad P_{|\xi-\xi_0| > N} := \text{Id} - P_{|\xi-\xi_0| \leq N}.
\]

2.1. Local well posedness. We follow the argument of Cazenave and Weissler for (1.4) to develop a standard local theory.

**Theorem 2.1** ([10, 11]). Let \((u_0, v_0) \in L^2(\mathbb{R}^4)^2\). Then there exists a unique maximal-lifespan solution \((u, v) : I \times \mathbb{R}^4 \to \mathbb{C}^2\) to (1.3). This solution also has the following properties:

- **(Local existence)** \(I\) is an open neighborhood of 0.
- **(Mass conservation)** \(M(u, v)(t)\) is conserved for all \(t \in I\).
- **(Blow-up criterion)** If sup \(I\) is finite, then there exists \(t_0 \in I\) such that
  \[
  \|u\|_{L^1_{t,x}(t_0, \sup I) \times \mathbb{R}^4} + \|v\|_{L^1_{t,x}(t_0, \sup I) \times \mathbb{R}^4} = \infty.
  \]

(In this case we say that \((u, v)\) blows up forward in time.) A similar statement holds in the negative time direction.

- **(Scattering)** If sup \(I = +\infty\) and \((u, v)\) does not blow up forward in time, then there exists \((u_+, v_+) \in L^2(\mathbb{R}^4)^2\) such that (1.5) holds. Conversely, given \((u_+, v_+) \in L^2(\mathbb{R}^4)^2\) there exists a unique solution to (1.3) in a neighborhood of \(+\infty\) so that (1.5) holds. A similar statements hold in the negative time direction.

- **(Small data global existence)** There exists \(\eta > 0\) such that for any \((u_0, v_0) \in L^2(\mathbb{R}^4)^2\) with \(M(u_0, v_0) \leq \eta\), the solution to (1.3) with initial value \((u(0), v(0)) = (u_0, v_0)\) exists globally and satisfies following bound:
  \[
  \int_{\mathbb{R}^{1+4}} |u(t, x)|^3 + |v(t, x)|^3 \, dxdt \lesssim M(u_0, v_0)^{3/2}.
  \]  

**Remark 2.2.** Note that scattering is equivalent to finiteness of \(L^3_{t,x}\)-norm on the maximal lifespan for any maximal-lifespan solution to (1.3). This is a consequence of Strichartz estimate and standard continuity argument.

2.2. Stability result. In this section, we prepare the stability result. Note that the stability result follows regardless of the condition for the coefficients \(\lambda\) and \(\mu\).

**Theorem 2.3** (Mass-critical stability result). Let \(I\) be an interval and let \((\tilde{u}, \tilde{v})\) be an approximate solution to (1.1) in the sense that
\[
\begin{aligned}
i\partial_t \tilde{u} + \frac{i}{2m} \Delta \tilde{u} &= \lambda \tilde{v} + e_1, \\
i\partial_t \tilde{v} + \frac{i}{2M} \Delta \tilde{v} &= \mu \tilde{u}^2 + e_2
\end{aligned}
\]
for some functions \( (e_1, e_2) \). Assume that
\[
\|(\tilde{u}, \tilde{v})\|_{L^\infty_t L^2_x(I \times \mathbb{R}^4)} \leq A,
\]
\[
\|(\bar{u}, \bar{v})\|_{L^1_{t,x}(I \times \mathbb{R}^4)} \leq L
\]
for some \( A, L > 0 \). Let \( t_0 \in I \) and let \((u_0, v_0)\) obey
\[
\|(u_0 - \tilde{u}(t_0), v_0 - \tilde{v}(t_0))\|_{L^2_x(\mathbb{R}^4)} \leq A'
\]
for some \( A' > 0 \). Moreover, assume the smallness conditions:
\[
\|(e_1, e_2)\|_{L^\frac{3}{2}_{t,x}(I \times \mathbb{R}^4)} \leq \varepsilon,
\]
where \( \varepsilon \) is a small constant.

Proof. The proof is very similar to the one of [19, Lemma 3.6], and so we omit the detail. \( \square \)

2.3. Bilinear Strichartz estimate. Finally, we recall the bilinear Strichartz estimate.

**Lemma 2.4.** Let \( M, N > 0 \), and let \( f, g \) be functions on \([a, b) \times \mathbb{R}^4\) with the support properties
\[
\supp \hat{f}(t, \cdot) \subset \{ |\xi| < M \}, \quad \supp \hat{g}(t, \cdot) \subset \{ |\xi| > N \}
\]
for all \( t \in [a, b) \). Let \( \theta_1, \theta_2 \in \mathbb{R} \setminus \{0\} \). Then, we have
\[
\|fg\|_{L^2([a, b) \times \mathbb{R}^4)} \lesssim \frac{M^{3/2}}{N^{1/2}} \|\hat{f}\|_{S^0_{\dot{a}}([a, b) \times \mathbb{R}^4)} \|g\|_{S^0_{\dot{a}}([a, b) \times \mathbb{R}^4)},
\]
where \( \|u\|_{S^0_{\dot{a}}([a, b) \times \mathbb{R}^4)} := \|(u(a))_{\dot{a}}\|_{L^2([a, b) \times \mathbb{R}^4)} + \|((i\partial_a + \theta \Delta)u)_{\dot{a}}\|_{L^{3/2}_{t,a}([a, b) \times \mathbb{R}^4)} \) and the implicit constant depends only on \( \theta_1, \theta_2 \).

Proof. Let \( a = 0 \) by time translation. We show only the homogeneous case:
\[
\|e^{it\theta_1 \Delta} \phi \cdot e^{it\theta_2 \Delta} \psi\|_{L^2([0, b) \times \mathbb{R}^4)} \lesssim \frac{M^{3/2}}{N^{1/2}} \|\phi\|_{L^2(\mathbb{R}^4)} \|\psi\|_{L^2(\mathbb{R}^4)},
\]
(2.2)
for \( \phi, \psi \in L^2(\mathbb{R}^4) \) satisfying \( \supp \hat{\phi} \subset \{ |\xi| < M \} \) and \( \supp \hat{\psi} \subset \{ |\xi| > N \} \). Once (2.2) is established, the general case follows by the same argument as [21, Lemma 2.5].

To show (2.2), we basically follow the argument in [3, Lemma 3.4]. Clearly, we may restrict frequencies to single dyadic regions;
\[
\supp \hat{\phi} \subset \{ |\xi| \sim M \}, \quad \supp \hat{\psi} \subset \{ |\xi| \sim N \}.
\]
Moreover, we may assume $|\theta_1| M \ll |\theta_2| N$, since otherwise the claim follows from the $L^4_t L^{8/3}_x$-Strichartz estimate and the Sobolev embedding $W^{1,8/3}_x \hookrightarrow L^8_x$. By a suitable decomposition with respect to the angle and rotation, we may further assume that

$$\text{supp } \hat{\psi} \subset \{ \xi = (\xi^1, \xi^2) \in \mathbb{R}^{1+3} \mid |\xi| \sim N, \xi^1 \geq |\xi|^2 \}.$$ 

By duality, $(2.2)$ is equivalent to

$$\left| \iint W(-\theta_1|\xi|^2 - \theta_2|\xi^2|^2, \xi^1 + \xi^2) \hat{\phi}(\xi^1) \hat{\psi}(\xi^2) \, d\xi^1 \, d\xi^2 \right| \lesssim \frac{M^{3/2}}{N^{1/2}} \left\| \hat{\phi} \right\|_{L^2(\mathbb{R}^3)} \left\| \hat{\psi} \right\|_{L^2(\mathbb{R}^3)} \left\| W \right\|_{L^2(\mathbb{R}^{1+4})}. \tag{2.3}$$

Changing variables as $(\xi^1_1, \xi^1_2, \xi^2) \mapsto (u, \xi^1_1, v) := (-\theta_1|\xi|^2 - \theta_2|\xi^2|^2, \xi^1_1, \xi^1_2 + \xi^2)$ with the assumptions on the Fourier supports of $\phi$ and $\psi$, by which we have $dv \, du \, dx = J_0 \, dx \, dv \, du$, we encounter

$$\left| \iint J^{-\frac{1}{2}} \mathbf{1}_{|\xi^1_1| \leq M} \left( \hat{\xi}^1_1 \right) W(u, v) H(u, \xi^1_1, v) J^{-\frac{1}{2}} \, du \, d\xi^1_1 \, dv \right|,$n

$$H(u, \xi^1_1, v) = \hat{\phi}(\xi^1_1) \hat{\psi}(\xi^2).$$

We apply the Cauchy-Schwarz inequality in $(u, \xi^1_1, v)$, which yields

$$N^{-\frac{1}{2}} \left\| \mathbf{1}_{|\xi^1_1| \leq M} \right\|_{L^2(\mathbb{R}^3)} \left\| W \right\|_{L^2(\mathbb{R}^{1+4})} \left( \iint |H(u, \xi^1_1, v)|^2 \, J^{-1} \, du \, d\xi^1_1 \, dv \right)^{1/2},$$

and then change back to the original variables to obtain $(2.3)$. \qed

3. Minimal mass blow-up solution.

3.1. Inverse Strichartz inequality, linear profile decomposition. In this subsection, we prepare the profile decomposition for the system of NLS.

First we give some notations.

**Definition 3.1** (Symmetry group). Fix $d \geq 1$ and $\kappa > 0$. For any phase $\theta \in \mathbb{R}/2\pi \mathbb{Z}$, position $x_0 \in \mathbb{R}^d$, frequency $\xi_0 \in \mathbb{R}^d$, and scaling parameter $\lambda \in (0, \infty)$, we define the unitary transformation $g_\kappa(\theta, \xi_0, x_0, \lambda_0): L^2(\mathbb{R}^d)^2 \rightarrow L^2(\mathbb{R}^d)^2$ and $h(\theta, \xi_0, x_0, \lambda): L^2(\mathbb{R}^d)^2 \rightarrow L^2(\mathbb{R}^d)^2$ by

$$h(\theta, \xi_0, x_0, \lambda)(\phi_1, \phi_2)(x) := \lambda^{-\frac{1}{2}} e^{i\theta_0 x} e^{i\xi_0 x} \phi(\lambda^{-1}(x - x_0)),$n

$$[g_\kappa(\theta, \xi_0, x_0, \lambda)(\phi_1, \phi_2)](x) := \left( h(\theta, \xi_0, x_0, \lambda) \phi_1, h(\frac{\theta}{\kappa}, \frac{\xi_0}{\kappa}, x_0, \lambda) \psi \right).$$

Furthermore we set the group $G_\kappa$ by

$$G_\kappa := \{ g_\kappa(\theta, \xi_0, x_0, \lambda) \mid (\theta, \xi_0, x_0, \lambda) \in \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_> \}.$$n

For a convenience, we also set $G := G_\frac{1}{2}$ and $g(\theta, \xi_0, x_0, \lambda) := g_\frac{1}{2}(\theta, \xi_0, x_0, \lambda).$

Next, we refer to the following theorem which is called Inverse Strichartz inequality.

**Lemma 3.2** (Inverse Strichartz inequality). Fix $d \geq 1$. Let $\{u_n\}_n \subset L^2(\mathbb{R}^d)$ be a bounded sequence. We also assume that

$$A := \lim_{n \to \infty} \| u_n \|_{L^4_t(\mathbb{R}^d)}^2 > 0,$n

$$\varepsilon := \lim_{n \to \infty} \left\| e^{itA} u_n \right\|_{L^2_t(\mathbb{R}^{1+d})}^2 > 0.$$
Then, passing to a subsequence if necessary, there exist \( \{\xi_n, y_n, \lambda_n, s_n\} \subset \mathbb{R}^d \times \mathbb{R}^d \times (0, \infty) \times \mathbb{R} \), and \( \phi \in L^2(\mathbb{R}^d) \) such that

\[
h(0, \xi_n, y_n, \lambda_n)^{-1} e^{i s_n \Delta} u_n \rightharpoonup \phi \quad \text{in} \quad L^2(\mathbb{R}^d),
\]

\[
\|\phi\|_{L^2}^2 \gtrsim A^2 \left( \frac{\varepsilon}{A} \right)^{2(d+1)(d+2)}.
\]

**Proof.** See for instance [13, Proposition 4.25]. \( \square \)

Using above Lemma, we give the inverse Strichartz inequality for the system.

**Proposition 3.3** (Inverse Strichartz inequality for the system). Fix \( d \geq 1, \kappa > 0 \), and \( \{(u_n, v_n)\} \subset L^2_x(\mathbb{R}^d)^2 \). Suppose also that

\[
A := \lim_{n \to \infty} \| (u_n, v_n) \|_{L^2_x(\mathbb{R}^d)},
\]

\[
\varepsilon := \lim_{n \to \infty} \| (e^{it\Delta} u_n, e^{it\Delta} v_n) \|_{L^2_{t,x}((\mathbb{R}^d)^d)} > 0.
\]

Then, after passing to a subsequence if necessary, there exist \( (\phi, \psi) \in L^2_x(\mathbb{R}^d)^2 \), \( \{\lambda_n\} \subset (0, \infty) \), \( \{\xi_n\} \subset \mathbb{R}^d \), and \( \{s_n, y_n\} \subset \mathbb{R}^{1+d} \) such that

\[
\left[ \begin{array}{c}
ge_n(0, \xi_n, y_n, \lambda_n)^{-1} (e^{i s_n \Delta} u_n, e^{i s_n \Delta} v_n) \\
i \rightarrow (\phi, \psi) \end{array} \right] \quad \text{weakly in} \quad L^2_x(\mathbb{R}^d)^2,
\]

\[
\lim_{n \to \infty} \left[ \| u_n \|_{L^2_x}^2 - \| u_n - \phi_n \|_{L^2_x}^2 \right] = \| \phi \|_{L^2_x}^2,
\]

\[
\lim_{n \to \infty} \left[ \| v_n \|_{L^2_x}^2 - \| v_n - \psi_n \|_{L^2_x}^2 \right] = \| \psi \|_{L^2_x}^2,
\]

\[
\| \phi \|_{L^2_x}^2 + \| \psi \|_{L^2_x}^2 \gtrsim A^2 \left( \frac{\varepsilon}{A} \right)^{2(d+1)(d+2)},
\]

where \( \phi_n \) and \( \psi_n \) are defined by

\[
\phi_n := e^{-i s_n \Delta} h(0, \xi_n, y_n, \lambda_n) \phi, \quad \psi_n := e^{-i s_n \Delta} h(0, \frac{\xi_n}{\kappa}, y_n, \lambda_n) \psi.
\]

**Proof.** Passing to subsequences if necessary, we may assume that

\[
A_1 := \lim_{n \to \infty} \| u_n \|_{L^2_x}, \quad A_2 := \lim_{n \to \infty} \| v_n \|_{L^2_x},
\]

\[
\varepsilon_1 := \lim_{n \to \infty} \| e^{it\Delta} u_n \|_{L^2_{t,x}((\mathbb{R}^d)^d)}^{2(d+2)}, \quad \varepsilon_2 := \lim_{n \to \infty} \| e^{it\Delta} v_n \|_{L^2_{t,x}((\mathbb{R}^d)^d)}^{2(d+2)}.
\]

For simplicity, we suppose that \( \varepsilon_1 \leq \varepsilon_2 \). Then we can apply Lemma 3.2 to \( \{v_n\} \).

Therefore, passing to a subsequence if necessary, there exist \( \{\xi_n\}, \{y_n\}, \{\lambda_n\}, \{s_n\} \), and \( \psi \in L^2(\mathbb{R}^d) \) satisfying stated properties. Since \( \{h(0, \kappa \xi_n, y_n, \lambda_n)^{-1} e^{i s_n \Delta} u_n\}_n \) is bounded in \( L^2(\mathbb{R}^d) \), passing to a subsequence if necessary, there exists \( \phi \in L^2(\mathbb{R}^d) \) such that

\[
h(0, \kappa \xi_n, y_n, \lambda_n) e^{i s_n \Delta} u_n \rightharpoonup \phi \quad \text{weakly in} \quad L^2(\mathbb{R}^d).
\]

Then (3.1) and (3.2) follow immediately. We can prove (3.3) as follows:

\[
\| \phi \|_{L^2}^2 + \| \psi \|_{L^2}^2 \gtrsim A^2 \left( \frac{\varepsilon_2}{A_2} \right)^{2(d+1)(d+2)} \gtrsim A^2 \left( \frac{\varepsilon}{A} \right)^{2(d+1)(d+2)}.
\]

\( \square \)

Now we are ready to give the profile decomposition.

**Theorem 3.4** (Profile decomposition). Fix \( d \geq 1 \) and \( \kappa > 0 \). Let \( \{(u_n, v_n)\} \) be a bounded sequence in \( L^2(\mathbb{R}^d)^2 \). Then, passing to a subsequence if necessary, there
exist \( J^* \in \mathbb{Z}_{\geq 0} \cup \{ \infty \}, \{(\phi^j, \psi^j)\}_{j=1}^{J^*}, \{g^j_n = g_n(\theta^j_n, \xi^j_n, x^j_n, \lambda^j_n)\} \subset G_\kappa, \{W^J_n = (w^J_n, \zeta^J_n)\}_{j=1}^{J^*} \subset L^2(\mathbb{R}^d)^2, \) and \( \{t^J_n\}_{j=1}^{J^*} \subset \mathbb{R} \) such that

\[
(u_n, v_n) = \sum_{j=1}^{J} g^j_n U_n(t^J_n)(\phi^j, \psi^j) + W^J_n, \quad (3.4)
\]

\[
\lim_{j \to J^*} \lim_{n \to \infty} \|U_n(t)W^J_n\|_{L^{2(d+2)}(\mathbb{R}^d)}^{2(d+2)} = 0, \quad (3.5)
\]

\[
U_n(-t^J_n)(g^J_n)^{-1}W^J_n \to 0 \quad \text{weakly in } L^2(\mathbb{R}^d)^2 \quad \text{for each } 1 \leq j \leq J, \quad (3.6)
\]

\[
\lim_{n \to \infty} \|u_n\|_{L^2}^2 - \sum_{j=1}^{J} \|\phi^j\|_{L^2}^2 - \|w^J_n\|_{L^2}^2 = 0, \quad (3.7)
\]

\[
\lim_{n \to \infty} \|v_n\|_{L^2}^2 - \sum_{j=1}^{J} \|\psi^j\|_{L^2}^2 - \|\zeta^J_n\|_{L^2}^2 = 0, \quad (3.8)
\]

where we set \( U_n(t) = (e^{it\Delta}, e^{it\Delta}). \) Furthermore for each \( j \neq k \) it follows that

\[
\frac{\lambda^j_n}{\lambda^k_n} + \frac{\lambda^j_n}{\lambda^k_n} + \frac{\lambda^j_n}{\lambda^k_n} |\xi^j_n - \xi^k_n|^2 + \frac{|t^J_n(\lambda^j_n)^2 - t^J_n(\lambda^k_n)^2|}{\lambda^j_n, \lambda^k_n} + \frac{|\lambda^j_n - \lambda^k_n|^2}{\lambda^j_n, \lambda^k_n} \to \infty. \quad (3.9)
\]

**Proof.** This theorem follows by using Proposition 3.3 and the standard induction argument. For the detail see [14]. \( \square \)

### 3.2. Construction of a minimal blow-up solution

In this subsection we fix \( \kappa = 1/2 \). Note that (1.3) has Galilean invariance in this case.

First we show the existence of a minimal blow-up solution which does not necessarily have additional properties (Theorem 3.8 below).

**Definition 3.5.** Let \( g = g(\theta, \xi_0, x_0, \lambda) \in G \) and \( (u, v) : I \times \mathbb{R}^4 \to \mathbb{C}^2 \). Then we define \( T_g(u, v) : \lambda^2 I \times \mathbb{R}^4 \to \mathbb{C}^2 \) as follows:

\[
T_g(u, v)(t, x) := \left( \begin{array}{c}
\lambda^{-2} e^{it\theta} e^{it\xi_0} e^{-it\xi_0^2} u(\lambda^{-2} t, \lambda^{-1}(x - x_0 - 2t\xi_0)) \\
\lambda^{-2} e^{it\theta} e^{it\xi_0} e^{-it\xi_0^2} v(\lambda^{-2} t, \lambda^{-1}(x - x_0 - 2t\xi_0))
\end{array} \right).
\]

For a solution \( (u, v) \) to (1.3), \( T_g(u, v) \) is a solution to (1.3) with initial data \( g(u_0, v_0) \) since we now assume \( \kappa = 1/2 \).

**Definition 3.6 (Almost periodic modulo symmetries).** Let \( (u, v) : I \times \mathbb{R}^4 \to \mathbb{C}^2 \) be a solution to (1.3). \( (u, v) \) is said to be almost periodic modulo symmetries if there is a function \( g : I \to G \) such that the set \( \{g(t)u(t) \mid t \in I\} \) is precompact in \( L^2(\mathbb{R}^4)^2 \), or equivalently, if there exist functions \( N : I \to \mathbb{R}_{>0}, x : I \to \mathbb{R}^4, \xi : I \to \mathbb{R}^4, \) and \( C : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) such that

\[
\sup_{t \in I} \left[ \int_{|x - x(t)| \geq C(\eta) N(t)} |u(t, x)|^2 \, dx + \int_{|\xi - \xi(t)| \geq C(\eta) N(t)} |\dot{u}(t, \xi)|^2 \, d\xi \right] \leq \eta, \quad (3.10)
\]

\[
\sup_{t \in I} \left[ \int_{|x - x(t)| \geq C(\eta) N(t)} |v(t, x)|^2 \, dx + \int_{|\xi - 2\xi(t)| \geq C(\eta) N(t)} |\dot{v}(t, \xi)|^2 \, d\xi \right] \leq \eta
\]

\[1\text{In that book, it was only treated the energy critical case. But we can similarly treat the mass critical system.}\]
for any \( \eta > 0 \). The functions \( N(\cdot), x(\cdot), \xi(\cdot) \), and \( C(\cdot) \) are called the frequency scale function, the spatial center function, the frequency center function, and the compactness modulus function, respectively.

Note that the choice of \( N(\cdot), x(\cdot), \xi(\cdot) \) is not unique. For instance, if another function \( \tilde{N} : I \to \mathbb{R}_+ \) satisfies \( C^{-1} \leq N(t)/\tilde{N}(t) \leq C \) on \( I \) for some \( C > 1 \), then we can replace \( N(t) \) with \( \tilde{N}(t) \). In fact, it turns out that we can choose these functions to be continuous with respect to \( t \), although we will not do so here.

**Remark 3.7.** Note that a solution \((u, v)\) is almost periodic modulo symmetries if and only if \( \{G(u(t), v(t)) \mid t \in I\} \) is precompact in \( G \setminus L^2(\mathbb{R}^4)^2 \). The proof of this fact is very similar to the one of [8, Lemma A.2]. For the convenience, we give the proof in Appendix A.

**Theorem 3.8.** Assume \( \kappa = 1/2 \). There exists a maximal lifespan solution \((u, v)\) to (1.3) with \( M(u, v) = M_c \), which is almost periodic modulo symmetries and which blows up both forward and backward in time.

Before proving this theorem, we prepare the following proposition.

**Proposition 3.9.** Assume \( \kappa = 1/2 \). Let \((u_n, v_n): I_n \times \mathbb{R}^4 \to \mathbb{C}^2\) be a sequence of solutions to (1.3) and \( t_n \in I_n \) be a sequence of times such that \( M(u_n, v_n) \leq M_c \), \( M(u_n, v_n) \to M_c \), and

\[
\lim_{n \to \infty} S_{\geq t_n}(u_n, v_n) = \lim_{n \to \infty} S_{\leq t_n}(u_n, v_n) = +\infty,
\]

where

\[
S_{\leq t_n}(u, v) := \int_{[\inf t_n, t_n]} \int_{\mathbb{R}^4} |u|^3 + |v|^3 \, dxdt, \quad S_{\geq t_n}(u, v) := \int_{[t_n, \sup t_n]} \int_{\mathbb{R}^4} |u|^3 + |v|^3 \, dxdt.
\]

Then \( G(u_n(t_n), v_n(t_n)) \) has a subsequence which converges in \( G \setminus L^2_2(\mathbb{R}^4)^2 \) topology.

**Proof.** By time translation symmetry, we may assume \( t_n = 0 \) for any \( n \in \mathbb{R} \). Applying Theorem 3.4 to the bounded sequence \{\((u_n(0), v_n(0))\)\} and passing to a subsequence if necessary, we obtain the profile decomposition

\[
\begin{pmatrix}
  u_n(0) \\
  v_n(0)
\end{pmatrix} = \sum_{j=1}^J g_n^j \begin{pmatrix}
  e^{it_n^j \Delta} \phi^j \\
  e^{it_n^j \Delta} \psi^j
\end{pmatrix} + W_n^j
\]

with stated properties, where we may assume \( t_n^j \equiv 0 \) or \( t_n^j \to \pm \infty \) as \( n \to \infty \). We define nonlinear profile \((a^j, b^j): I^j \times \mathbb{R}^4 \to \mathbb{C}^2\) associated to \((\phi^j, \psi^j)\) as follows:

- If \( t_n^j \equiv 0 \), we define \((a^j, b^j)\) to be the maximal-lifespan solution with \((a^j(0), b^j(0)) = (\phi^j, \psi^j)\).
- If \( t_n^j \to \infty \), we define \((a^j, b^j)\) to be the maximal-lifespan solution which scatters forward in time to \((e^{it \Delta} \phi^j, e^{it \Delta} \psi^j)\).
- If \( t_n^j \to -\infty \), we define \((a^j, b^j)\) to be the maximal-lifespan solution which scatters backward in time to \((e^{it \Delta} \phi^j, e^{it \Delta} \psi^j)\).

Finally, for each \( j, n \geq 1 \) we define \((a^j_n, b^j_n): I^j_n \times \mathbb{R}^4 \to \mathbb{C}^2\) by

\[
\begin{pmatrix}
  a^j_n \\
  b^j_n
\end{pmatrix} := T_{g_n^j} \begin{pmatrix}
  a^j(\cdot + t_n^j) \\
  b^j(\cdot + t_n^j)
\end{pmatrix},
\]

\footnote{For \((\phi, \psi) \in L^2 \times L^2\), we set \(G(\phi, \psi) := \{g(\phi, \psi) \mid g \in G\}\).}
In order to lead a contradiction, we define the approximate solution \( \varepsilon > 0 \) for some \( t \). Note that by the asymptotic orthogonality conditions (3.9), (3.15), and (3.17), moreover by small data scattering, we get are defined globally and obey the estimates.

If not, then we have \( \lim sup_{j \to \infty} M(a^j_n, b^j_n) = M(\phi^j, \psi^j) < M_c \). By passing to subsequences for each \( J \), we obtain that \( M(a^j_n, b^j_n) = M(\phi^j, \psi^j) \leq M_c - \varepsilon \).

Moreover by small data scattering, we get

\[
S_\mathbb{R}(a^j_n, b^j_n) \leq L(M(\phi^j, \psi^j)) \leq C_0 M(\phi^j, \psi^j)^{\frac{3}{2}} \leq C_1 M(\phi^j, \psi^j).
\]

In order to lead a contradiction, we define the approximate solution

\[
\begin{pmatrix} u^j_n \\ v^j_n \end{pmatrix} := \sum_{j=1}^J \begin{pmatrix} a^j_n \\ b^j_n \end{pmatrix} + \begin{pmatrix} e^{it_\lambda t} \phi^j_n \\ e^{it_\lambda t} \psi^j_n \end{pmatrix}.
\]

Note that by the asymptotic orthogonality conditions (3.9), (3.15), and (3.17),

\[
\lim sup_{J \to J^*} \lim sup_{n \to \infty} S_\mathbb{R}(a^J_n, b^J_n) \leq 2^3 \lim sup_{J \to J^*} \lim sup_{n \to \infty} S_\mathbb{R}(J, \sum_{j=1}^J a^j_n, \sum_{j=1}^J b^j_n) \leq C_2 \lim_{J \to J^*} \sum_{j=1}^J M(\phi^j, \psi^j) \leq C_2 M_c.
\]
where $c_{1,n}^J := \sum_{j=1}^J b_n^J a_n^{J,j} - v_n^J \bar{a}_n^J$, $c_{2,n}^J := \sum_{j=1}^J (a_n^J)^2 - (u_n^J)^2$. To apply Theorem 2.3, we prepare the following lemmas:

**Lemma 3.10.** For any $J \geq 1$, the following holds:

$$\lim_{n \to \infty} M(u_n^J(0), v_n^J(0)) = 0.$$  

**Proof.** This follows immediately by (3.13), (3.14), and the definition of $(u_n^J, v_n^J)$. □

**Lemma 3.11.** The following holds:

$$\lim_{J \to J^*} \lim_{n \to \infty} \| (c_{1,n}^J, c_{2,n}^J) \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})} = 0.$$  

**Proof.** By the definition, we have

$$\| (c_{1,n}^J, c_{2,n}^J) \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})} = \| \sum_{j=1}^J b_n^J \bar{a}_n^J - v_n^J \bar{u}_n^J \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})} + \| \sum_{j=1}^J (a_n^J)^2 - (u_n^J)^2 \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})}.$$  

Using the triangle inequality and the Hölder inequality, we get

$$I = \| \sum_{j=1}^J b_n^J \bar{a}_n^J - \sum_{j=1}^J b_n^J + e^{i\Delta t \zeta_n^J} (J) \sum_{j=1}^J a_n^J + e^{it\Delta w_n^J} \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})}$$

$$\leq \sum_{j \neq k} \| a_k^J \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})} + \sum_{j=1}^J \| b_n^J \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})} + \sum_{j=1}^J \| a_n^J \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})} + \| e^{i\Delta t \zeta_n^J} \|^2_{L^4_{t,x}(\mathbb{R}^{1+4})} + \| e^{it\Delta w_n^J} \|^2_{L^4_{t,x}(\mathbb{R}^{1+4})}.$$  

$$II = \| \sum_{j=1}^J (a_n^J)^2 - \sum_{j=1}^J a_n^J + e^{it\Delta w_n^J} \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})}$$

$$\leq \sum_{j \neq k} \| a_k^J \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})} + 2 \sum_{j=1}^J \| a_n^J \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})} + \| e^{it\Delta w_n^J} \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})}.$$  

From asymptotic orthogonality (3.9), first terms of $I, II$ converge to zero. Moreover the other terms converge to zero by (3.5) and (3.18).

**Lemma 3.12.** There exists a constant $M_1 > 0$ such that

$$\lim_{J \to J^*} \limsup_{n \to \infty} \| (u_n^J, v_n^J) \|^2_{L^\infty_t L^2_x(\mathbb{R}^{1+4})} \leq M_1.$$  

**Proof.** By Lemma 3.10, for each $J \geq 1$

$$\limsup_{n \to \infty} \| (u_n^J(0), v_n^J(0)) \|^2_{L^2(\mathbb{R}^{1+4})} \leq \sup_{n \geq 1} \| (u_n(0), v_n(0)) \|^2_{L^2(\mathbb{R}^{1+4})}. \quad (3.19)$$

By Strichartz’s estimate we have

$$\| (u_n^J, v_n^J) \|^2_{L^\infty_t L^2_x(\mathbb{R}^{1+4})} \leq \| (u_n^J(0), v_n^J(0)) \|^2_{L^2(\mathbb{R}^{1+4})}$$

$$+ C(\| (e_{1,n}^J, e_{2,n}^J) \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})} + \| (u_n^J, v_n^J) \|^2_{L^2_{t,x}(\mathbb{R}^{1+4})}).$$

Combining this with (3.18), (3.19) and, Lemma 3.11, we obtain the result. □
Combining these Lemmas 3.10-3.12 and Theorem 2.3, we obtain boundedness of \( \{S_R(u_n,v_n)\} \) which contradicts (3.11). Therefore we establish (3.16). Then we see \( J^* = 1 \). Consequently, the profile decomposition simplifies to

\[
\begin{pmatrix}
  u_n(0) \\
  v_n(0)
\end{pmatrix} = g_n \begin{pmatrix}
  e^{it_n \Delta \phi} \\
  e^{it_n \Delta \psi}
\end{pmatrix} + \begin{pmatrix}
  w_n \\
  \zeta_n
\end{pmatrix}
\]  

(3.20)

for some \( t_n \in \mathbb{R} \) such that either \( t_n \equiv 0 \) or \( t_n \to \pm \infty \), some \( g_n \in G \), some \((\phi, \psi) \in L^2_2(\mathbb{R}^4)^2 \) of mass \( M(\phi, \psi) = M_c \), and some \((w_n, \zeta_n)\) with \( M(w_n, \zeta_n) \to 0 \). If \( t_n \equiv 0 \), then

\[_0^n M(u_n(0), v_n(0)) - g_n(\phi, \psi) = M(w_n, \zeta_n) \to 0.

This implies that \( G(u_n(0), v_n(0)) \) converges in \( G \setminus L^2_2(\mathbb{R}^4)^2 \). So we consider the case \( t_n \to \pm \infty \). The case \( t_n \to -\infty \) can be treated similarly, and so we omit it. By the Strichartz inequality we have

\[_n S_R(e^{it \Delta \phi}, e^{it \Delta \psi}) \lesssim \| (\phi, \psi) \|_{L^2_2(\mathbb{R}^4)} < \infty

and so

\[S_{\geq 0}(e^{it \Delta \phi}, e^{it \Delta \psi}) = S_{\geq t_n}(e^{it \Delta \phi}, e^{it \Delta \psi}) \to 0 \text{ as } n \to \infty.

Let \( \theta_n, \xi_n, x_n, \lambda_n \) be parameters of \( g_n \) and set \( h_{1,n} := h(\theta_n, \xi_n, x_n, \lambda_n), h_{2,n} := h(2\theta_n, 2\xi_n, x_n, \lambda_n) \). Then we establish

\[S_{\geq 0}(e^{it \Delta h_{1,n} e^{it \Delta \phi}, e^{it \Delta h_{2,n} e^{it \Delta \psi}}}) = S_{\geq 0}(T_{h_{1,n}} e^{it \Delta \phi}, T_{h_{2,n}} e^{it \Delta \psi}) = S_{\geq 0}(e^{it \Delta \phi}, e^{it \Delta \psi}) \to 0,

as \( n \to 0 \). Since \( S_R(e^{it \Delta v_n}, e^{it \Delta \zeta_n}) \to 0 \), we see from (3.20) that

\[\lim_{n \to \infty} S_{\geq 0}(e^{it \Delta u_n(0), e^{it \Delta v_n(0)}) = 0.

Applying Theorem 2.3 (using \((0, 0)\) as the approximate solution), we conclude that

\[\lim_{n \to \infty} S_{\geq 0}(u_n, v_n) = 0.

This contradicts (3.11).

Proof of Theorem 3.8. Since \( L(M_c) = \infty \), we can find a sequence \( (u_n, v_n) : I_n \times \mathbb{R}^4 \to \mathbb{C}^2 \) of maximal-lifespan solutions with

\[M(u_n, v_n) \leq M_c, \quad \lim_{n \to \infty} S_{t_n}(u_n, v_n) = +\infty.

Then there exist \( t_n \in I_n \) such that

\[\lim_{n \to \infty} S_{\geq t_n}(u_n, v_n) = \lim_{n \to \infty} S_{\leq t_n}(u_n, v_n) = \infty.

By time translation invariance we may take \( t_n = 0 \). Invoking Proposition 3.9 and passing to a subsequence if necessary, we find \( g_n \in G \) such that \( g_n(u_n(0), v_n(0)) \to (u_0, v_0) \) in \( L^2_2(\mathbb{R}^4)^2 \) for some \( (u_0, v_0) \in L^2_2(\mathbb{R}^4)^2 \).

By applying \( T_{g_n} \) to \( (u_n, v_n) \) we may take \( g_n = I \) for all \( n \in \mathbb{N} \). Then we have \((u_n(0), v_n(0)) \to (u_0, v_0) \) and so \( M(u_0, v_0) \leq M_c \). Let \((u, v) : I \times \mathbb{R}^4 \to \mathbb{C}^2 \) be the maximal-lifespan solution with initial data \((u(0), v(0)) = (u_0, v_0) \). We claim that \((u, v)\) blows up both forward and backward in time. Indeed, if \((u, v)\) does not blow up forward in time, then \([0, \infty) \subset I \) and \( S_{\geq 0}(u, v) < \infty \). By Theorem 2.3 this implies

4See (3.5) in Theorem 3.4.

5From \( S_{t_n}(u_n, v_n) \to \infty \) and \( M(u_n, v_n) \leq M_c \), passing to a subsequence if necessary, we have \( \lim_{n \to \infty} M(u_n, v_n) = M_c \).
that for sufficiently large \( n \), we have \( [0, \infty) \subset I_n \) and \( \limsup_{n \to \infty} S_{\geq 0}(u_n, v_n) < \infty \). This is a contradiction.

It remains to show that the solution \((u, v)\) is almost periodic modulo symmetries. Consider an arbitrary sequence
\[
\{(u(s_n), v(s_n))\}_n \subset (u, v)[I].
\]
Since \((u, v)\) blows up both forward and backward in time, we have
\[
S_{\geq s_n}(u, v) = S_{\leq s_n}(u, v) = \infty.
\]
Applying Proposition 3.9 once again, we see that \((u_n, v_n)\) has a convergent subsequence in \( G \setminus L^2(\mathbb{R}^d)^2 \). Thus the orbit \( \{G(u(t), v(t)) \mid t \in I\} \) is precompact in \( G \setminus L^2(\mathbb{R}^d)^2 \).

### 3.3. Further refinements.

In this subsection, we again let \( \kappa = 1/2 \). In the following, we consider to refine given solution in Theorem 3.8 as we can apply the argument of [5]. For this aim we give the some definitions and lemmas. Following argument is based on [13] and [18].

**Definition 3.13** (Convergence of solutions). Let \( (u_n, v_n) : I_n \times \mathbb{R}^4 \to \mathbb{C}^2 \) be a sequence of solutions to (1.3), let \((u, v) : I \times \mathbb{R}^4 \to \mathbb{C}^2\) be another solution, and \( K \) be a compact time interval. We say that \((u_n, v_n)\) converges uniformly to \((u, v)\) on \( K \) if \( K \subset I \), \( K \subset I_n \) for sufficiently large \( n \), and \((u_n, v_n) \to (u, v)\) in \( C(K, L^2(\mathbb{R}^d)^2) \cap L^3_{t,x}(K \times \mathbb{R}^d)^2 \) as \( n \to \infty \). We say that \((u_n, v_n)\) converge locally uniformly to \((u, v)\) if \((u_n, v_n)\) converges uniformly to \((u, v)\) on every compact interval \( K \subset I \).

**Definition 3.14.** Let \((u, v)\) be a solution to (1.3) which is almost periodic modulo symmetries with parameters \( N(t), x(t), \xi(t) \). We say that \((u, v)\) is normalized if the lifespan \( I \) contains zero and
\[
N(0) = 1, \ x(0) = \xi(0) = 0.
\]
More generally, we can define the normalization of a solution \((u, v)\) at time \( t_0 \in I \) by
\[
(u^{[t_0]}, v^{[t_0]}) := T_{g(0, -\xi(t_0)/N(t_0), -x(t_0)/N(t_0), N(t_0))}(u(\cdot + t_0), v(\cdot + t_0)).
\]
Note that \((u^{[t_0]}, v^{[t_0]})\) is a normalized solution which is almost periodic modulo symmetries and has lifespan
\[
I^{[t_0]} := \{ s \in \mathbb{R} \mid t_0 + sN(t_0)^{-2} \in I \}.
\]
The parameters of \((u^{[t_0]}, v^{[t_0]})\) are given by
\[
N^{[t_0]}(s) := \frac{N(t_0 + sN(t_0)^{-2})}{N(t_0)},
\]
\[
\xi^{[t_0]}(s) := \frac{\xi(t_0 + sN(t_0)^{-2}) - \xi(t_0)}{N(t_0)},
\]
\[
x^{[t_0]}(s) := N(t_0)[x(t_0 + sN(t_0)^{-2}) - x(t_0)] - 2 \frac{\xi(t_0)}{N(t_0)} s.
\]

**Lemma 3.15** ([13], Theorem 5.15). Let \((u_n, v_n)\) be a sequence of solutions to (1.3) with lifespans \( I_n \), which is almost periodic modulo symmetries with parameters \( N_n, x_n, \xi_n \) and compactness modulus function \( C \). Suppose that \((u_n, v_n)\) converges locally uniformly to a non-zero solution \((u, v)\) with lifespan \( I \). Then \((u, v)\) is almost periodic modulo symmetries with some parameters \( N(t), x(t), \xi(t) \) and the same compactness modulus function \( C \). In particular we may take \( N(t) = \limsup_{n \to \infty} N_n(t) \).
Proof. We first show that

\[
0 < \liminf_{n \to \infty} N_n(t) \leq \limsup_{n \to \infty} N_n(t) < \infty, \tag{3.21}
\]

\[
\limsup_{n \to \infty} |x_n(t)| + \limsup_{n \to \infty} |\xi_n(t)| < \infty, \tag{3.22}
\]

for all \( t \in I \). If (3.21) is failed for some \( t \in I \), passing to a subsequence if necessary, we have

\[
\lim_{n \to \infty} N_n(t) = 0 \quad \text{or} \quad \lim_{n \to \infty} N_n(t) = +\infty.
\]

Then we have \( u_n(t), v_n(t) \to 0 \) weakly in \( L^2_x(\mathbb{R}^4) \). Indeed, taking \( \phi \in C_{0}^{\infty}(\mathbb{R}^4) \) and \( \varepsilon > 0 \) arbitrarily, we have

\[
\left| \int_{\mathbb{R}^4} u_n(t) \phi \ dx \right|^2 \lesssim \|\phi\|_{L^2_x}^2 \varepsilon + \left( \int_{|x-x(t)|<C(\varepsilon)/N_n(t)} |u_n(t)| \phi \ dx \right)^2
\]

\[
\lesssim \|\phi\|_{L^2_x}^2 \varepsilon + \|\phi\|_{L^\infty} \|u_n(t)\|_{L^2(C(\varepsilon)/N_n(t))^4}^2 \to \|\phi\|_{L^2_x}^2 \varepsilon
\]

if \( N_n(t) \to +\infty \) as \( n \to \infty \). In the case \( N_n(t) \to 0 \), we can get the result by similar argument and using Plancherel’s theorem. This contradicts the nonzero assumption of \((u, v)\). We can easily obtain (3.22) by similar argument, and so we omit the detail.

Passing to a subsequences if necessary,\(^6\) we may assume

\[
N(t) := \lim_{n \to \infty} N_n(t)
\]

\[
\xi(t) := \lim_{n \to \infty} \xi_n(t)
\]

\[
x(t) := \lim_{n \to \infty} x_n(t).
\]

Then we can easily prove almost periodicity of \((u, v)\) with parameters \( N(t), \xi(t), x(t) \).

\[\square\]

Lemma 3.16. Let \((u_n, v_n)\) be a sequence of normalized maximal-lifespan solutions with lifespans \( 0 \in I_n = (-T_n^-, T_n^+) \), which are almost periodic modulo symmetries on \([0, T_n^+)\) with parameters \( N_n(t), x_n(t), \xi_n(t) \) and a uniform compactness modulus function \( C \). Assume that we also have

\[
0 < \inf_n M(u_n, v_n) \leq \sup_n M(u_n, v_n) < \infty.
\]

Then, after passing to a subsequence if necessary, there exists a non-zero maximal lifespan solution \((u, v)\) with lifespan \( 0 \in I = (-T^-, T^+) \) that is almost periodic modulo symmetries on \([0, T^+)\), such that \((u_n, v_n)\) converge locally uniformly to \((u, v)\) on \( I \).

Proof. By almost periodicity and the assumption \( \sup_n M(u_n, v_n) < \infty \), \( \{(u_n, v_n)(0)\} \) is precompact in \( L^2(\mathbb{R}^4)^2 \), and so passing to a subsequence if necessary, we may assume that

\[
(u_n(0), v_n(0)) \to (u_0, v_0) \quad \text{in} \quad L^2(\mathbb{R}^4)^2 \quad \text{for some} \quad (u_0, v_0).
\]

Let \((u, v) : I \to \mathbb{C}^2\) be the maximal lifespan solution with initial data \((u, v)(0) = (u_0, v_0)\). Then we can apply the stability result, and so we establish \((u_n, v_n) \to (u, v)\) locally uniformly on \( I \). Almost periodicity of \((u, v)\) on \([0, T^+)\) follows from Lemma 3.15. \[\square\]

\(^6\)Note that subsequences depend on \( t \).
Lemma 3.17. Let \((u, v)\) be a non-zero maximal-lifespan solution with lifespan \(I = (-T^-, T^+)\) that is almost periodic modulo symmetries on \([0, T^+)\) with parameters \(N(t), x(t), \xi(t)\). Then there exists a small number \(\delta = \delta(u, v) > 0\) such that for every \(t_0 \in [0, T^+)\) we have
\[
[0, t_0 + \delta N(t_0)^{-2}] \subset [0, T^+)
\]
and
\[
N(t) \sim_{u, v} N(t_0) \quad \text{whenever} \quad \max\{0, t_0 - \delta N(t_0)^{-2}\} \leq t \leq t_0 + \delta N(t_0)^{-2}.
\]

Proof. First we prove \((3.23)\) by contradiction. If not there exist sequences \(t_n \in [0, T^+)\) and \(\delta_n > 0\) such that
\[
t_n + \delta_n N(t_n)^{-2} \notin [0, T^+), \quad \delta_n \to 0.
\]
Applying Theorem 3.16 there exists the maximal-lifespan solution \((u, v) : I = (-T^-, T^+) \to \mathbb{C}^2\) such that
\[
(u^{[t_n]}, v^{[t_n]}) \to (u, v) \quad \text{locally uniformly on} \quad I,
\]
where \((u^{[t_n]}, v^{[t_n]}) : I^{[t_n]} \to \mathbb{C}^2\) is normalized solution at time \(t_n\). Then there exists \(n_0 \in \mathbb{N}\) such that \(\delta_n \in [0, T^+)\) for all \(n \geq n_0\), and so there exists \(m(n) \in \mathbb{N}\) such that \(t_m + \delta_n N(t_m)^{-2} \in [0, T^+)\) for \(n \geq n_0, \ m \geq m(n)\). This is a contradiction.

Next, we prove also \((3.24)\) by contradiction. If not we can find sequences \(t_n, t'_n \in [0, T^+)\) such that
\[
s_n := (t'_{n} - t_{n})N(t_{n})^2 \to 0
\]
and
\[
\frac{N(t'_n)}{N(t_n)} \to 0 \quad \text{or} \quad +\infty \quad \text{as} \quad n \to \infty.
\]
Then we easily get \((u^{[t_n]}(s_n), v^{[t_n]}(s_n)) \rightharpoonup (0, 0)\) weakly in \(L^2(\mathbb{R}^4)^2\). On the other hand, by Lemma 3.16, there exists a maximal-lifespan non-zero solution \((u, v) : I \to \mathbb{C}^2\) such that \((u^{[t_n]}, v^{[t_n]}) \to (u, v)\) locally uniformly on \(I\). Then we have
\[
(u^{[t_n]}, v^{[t_n]})(s_n) \to (u(0), v(0)) \quad \text{in} \quad L^2(\mathbb{R}^4)^2 \quad \text{as} \quad n \to \infty, \quad \text{and} \quad (u, v) = (0, 0)\]. This is a contradiction.

Corollary 3.18. Let \((u, v)\) be a maximal lifespan solution with lifespan \(0 \in I = (-T^-, T^+)\) that is almost periodic modulo symmetries on \([0, T^+)\) with frequency scale function \(N : [0, T^+) \to \mathbb{R}_{>0}\). If \(T^+ < \infty\), then it follows that \(\lim_{t \to T^+} N(t) = \infty\). If \(T^+ = \infty\), then we have \(\frac{N(t)}{\min\{N(0), t^{-1/2}\}} \to 0\) \(\forall t \in [0, \infty)\).

Proof. If \(T^+ < \infty\), the result follows easily by Lemma 3.17. Consider the case \(T^+ = \infty\). Take any \(t \in [0, \infty)\). If \(\max\{0, t - \delta N(t)^{-2}\} = 0\), then by Lemma 3.17, we obtain \(N(t) \geq N(0)\). On the other hand, if \(0 < t < \delta N(t)^{-2}\), then we get easily that \(N(t) \geq \frac{\delta^{1/2}}{t^{1/2}}\).

Theorem 3.19 (Existence of minimal mass blow-up solution). There exists a maximal lifespan solution \((u, v)\) with lifespan \(I = (-T^-, T^+)\) that is almost periodic modulo symmetries on \([0, \infty)\) and blows up forward in time satisfying \(M(u, v) = M_c, N(0) = \sup\{N(t) : t \in [0, \infty)\} = 0\), and \(x(0) = \xi(0) = 0\).

Proof. Let \((\tilde{u}, \tilde{v}) : J \to \mathbb{C}^2\) be an almost periodic solution constructed in Theorem 3.8 with frequency scale function \(\tilde{N} : J \to \mathbb{R}_{>0}\). Take a sequence \(\{J_n\}\) of compact intervals satisfying \(J_n \not\supset J\). Since \(\sup\{\tilde{N}(t) : t \in J_n\} < \infty\), there exist \(t_n \in J_n\) such that
\[
\sup_{t \in J_n} \tilde{N}(t) \leq 2\tilde{N}(t_n).
\]
Note that $M(\tilde{u}^{[n]}, \tilde{v}^{[n]}) = M(u, v) > 0$. Since $(\tilde{u}^{[n]}(0), \tilde{v}^{[n]}(0))$ is precompact in $L^2(\mathbb{R}^4)^2$, passing to a subsequence if necessary, it follows that

$$(\tilde{u}^{[n]}(0), \tilde{v}^{[n]}(0)) \to (u_0, v_0) \quad \text{in } L^2(\mathbb{R}^4)^2$$

for some $(u_0, v_0)$.

Let $(u, v) : I \times \mathbb{R}^4 \to \mathbb{C}^2$ be the maximal-lifespan solution with $(u(0), v(0)) = (u_0, v_0)$ and $(u^n, v^n) : I_n \times \mathbb{R}^4 \to \mathbb{C}^2$ be the maximal-lifespan solution with $(u^n(0), v^n(0)) = (\tilde{u}^{[n]}(0), \tilde{v}^{[n]}(0))$. If $0 \in K$ is compact subinterval of $I = (-T^-, T^+)$, from the stability result we see that

$$(u^n, v^n) \to (u, v) \quad \text{uniformly on } K.$$

In particular $\limsup_{n \to \infty} \| (u^n, v^n) \|_{L^4_t, (J_n \times \mathbb{R}^4)} < \infty$. On the other hand, we have

$$\| (u^n, v^n) \|_{L^4_t, (J_n \times \mathbb{R}^4)} = \| (u, v) \|_{L^4_t, (J_n \times \mathbb{R}^4)} \to \infty.$$

Therefore we obtain $J_{[n]}^{\bigcirc} \not\subset K$ for sufficiently large $n$. Since $0 \in K$ is an arbitrary compact subinterval of $I$, after passing to a subsequence, we may assume one of the following holds:

- For every $t \in (0, T^+)$, $t \in J_{[n]}^{\bigcirc}$ for all sufficiently large $n$.
- For every $t \in (0, T^-)$, $t \in J_{[n]}^{\bigcirc}$ for all sufficiently large $n$.

By time reversal symmetry, it suffices to consider the former possibility. Then it follows that

$$(\tilde{u}^{[n]}, \tilde{v}^{[n]}) \to (u, v) \quad \text{locally uniformly on } [0, T^+).$$

Applying Lemma 3.15, we see that $(u, v)$ is almost periodic modulo symmetry with frequency scale function $N_0(t) = \limsup_{n \to \infty} \tilde{N}^{[n]}(t) \leq 2$. By Corollary 3.18, then we get $T^+ = \infty$. Setting

$$N(t) := \begin{cases} 
N_0(t)/\sup_{s \in [0, \infty)} N_0(s) & t > 0 \\
1 & t = 0,
\end{cases}$$

we have

$$\min\left\{ \frac{1}{\sup_{s \in [0, \infty)} N_0(s)}, \frac{1}{N_0(0)} \right\} \leq \frac{N(t)}{N_0(t)} \leq \max\left\{ \frac{1}{\sup_{s \in [0, \infty)} N_0(s)}, \frac{1}{N_0(0)} \right\}$$

and so we can replace $N_0$ with $N$. It remains to show that $(u, v)$ blows up forward in time. If not $(u, v)$ scatters to $(e^{it\Delta}u^+, e^{i\frac{d}{2}\Delta}v^+)$ as $t \to \infty$ for some $(u^+, v^+) \in L^2(\mathbb{R}^4)^2$. Furthermore we can show that $(e^{-it\Delta}u(t), e^{-i\frac{d}{2}\Delta}v(t)) \to (0, 0)$ weakly in $L^2(\mathbb{R}^4)^2$. Indeed, for any $(\phi, \psi) \in C_0^\infty(\mathbb{R}^4)^2$, we can calculate as follows:

$$|e^{-it\Delta}u(t), \phi|^2 \leq \eta \| \phi \|^2_{L^2} + \| u(t) \|^2_{L^2} \| e^{it\Delta} \phi \|^2_{L^2(B(x(t), C(t)/N(t)))},$$

$$|e^{-i\frac{d}{2}\Delta}v(t), \psi|^2 \leq \eta \| \psi \|^2_{L^2} + \| v(t) \|^2_{L^2} \| e^{i\frac{d}{2}\Delta} \psi \|^2_{L^2(B(x(t), C(t)/N(t)))}.$$

By the dispersive estimate and Corollary 3.18, we obtain the desired result. Therefore we get $(u^+, v^+) = (0, 0)$. This is a contradiction.

3.4. Minimal blow-up solution in the radial case. In this subsection we introduce the minimal blow-up solution in the radial case.

**Theorem 3.20** (Minimal blow-up solution in the radial case). There exists a maximal-lifespan solution $(u, v)$ with lifespan $I = (-T^-, \infty)$ that is radial, almost periodic modulo symmetries on $[0, \infty)$, and blows up forward in time satisfying $M(u, v) = M_{e, \text{rad}}, N(0) = \sup_{t \in [0, \infty)} = 1$, and $x \equiv \xi \equiv 0$, where $M_{e, \text{rad}}$ is radially symmetric critical mass defined in Section 1.
The proof of this theorem is parallel to the proof of Theorem 3.19. However, we can not use Galilean invariance and so we remove the parameter $\xi_n^j$ in the profile decomposition by using radial assumption. Indeed, for a sequence of radial $L^2$ function, we may refine profile decomposition as follows:

**Proposition 3.21** (Profile decomposition for radially symmetric sequence). Consider the case $d \geq 2$. Let $\{(u_n, v_n)\} \subset L^2_{\text{rad}}(\mathbb{R}^d)^2$ be bounded. Then in a profile decomposition given in Theorem 3.4, we can replace all $\xi_n^j$ and $x_n^j$ by zero. Furthermore we may take $W_n^j$ and $(\phi^j, \psi^j)$ are radially symmetric.

Proof. The proof of this fact is very similar to Theorem 7.3 in [20] but we give it in Appendix B. 

In Theorem 3.21 we may also assume $\theta_n^j \equiv 0$ after modifying the remainder term. Then we can prove Theorem 3.20 by quite similar argument in the proof of Theorem 3.19 because we do not need to use Galilean invariance.

3.5. Properties of almost periodic solutions. We collect various properties of almost periodic solutions (cf. [13], Lemma 5.13–Proposition 5.23). Proofs will be given in Appendix C.

**Lemma 3.22.** Let $(u, v) : I \times \mathbb{R}^d \to \mathbb{C}^2$ be a non-zero solution to (1.3) that is almost periodic modulo symmetries. Let $J$ be a subinterval of $I$ such that $S_J(u, v) < \infty$. Then, there exists $C = C(u, v, S_J(u, v)) > 0$ such that we have

$$\sup_{t \in J} N(t) \leq C \inf_{t \in J} N(t).$$

**Lemma 3.23.** Let $(u, v) : I \times \mathbb{R}^d \to \mathbb{C}^2$ be a non-zero solution to (1.3) that is almost periodic modulo symmetries. Let $J$ be a subinterval of $I$ such that $S_J(u, v) < \infty$. Then, there exists $C = C(u, v, S_J(u, v)) > 0$ such that we have

$$\sup_{t_1, t_2 \in J} |\xi(t_1) - \xi(t_2)| \leq C \sup_{t \in J} N(t).$$

**Lemma 3.24.** Let $(u, v) : I \times \mathbb{R}^d \to \mathbb{C}^2$ be a solution to (1.3) that is almost periodic modulo symmetries. Let $J$ be a subinterval of $I$ such that $S_J(u, v) < \infty$. Then, for any $\eta > 0$ there exists $R = R(u, v, \eta) > 0$ such that we have

$$\int_J \int_{|x - x(t)| \geq \frac{\eta}{\sqrt{3}}} \left(|u|^3 + |v|^3\right) dx dt$$

$$+ \left\| P_{|\xi - \xi(t)| \geq R(t)u} \right\|^3_{L^3(J \times \mathbb{R}^d)} + \left\| P_{|\xi - 2\xi(t)| \geq R(t)v} \right\|^3_{L^3(J \times \mathbb{R}^d)}$$

$$\leq \eta(1 + S_J(u, v)).$$

**Lemma 3.25.** Let $(u, v) : I \times \mathbb{R}^d \to \mathbb{C}^2$ be a non-zero solution to (1.3) that is almost periodic modulo symmetries. Let $J$ be a subinterval of $I$. Then, there exists $C = C(u, v) > 0$ such that we have

$$C^{-1} \int_J N(t)^2 dt \leq S_J(u, v) \leq 1 + C \int_J N(t)^2 dt. \quad (3.25)$$

The following is an immediate consequence of Lemma 3.25 and its proof.

**Corollary 3.26.** Let $(u, v) : I \times \mathbb{R}^d \to \mathbb{C}^2$ be a non-zero solution to (1.3) that is almost periodic modulo symmetries. Let $J$ be a subinterval of $I$ such that $0 <
$S_J(u,v) < \infty$. Then, there exists $C = C(u,v, S_J(u,v)) > 0$ such that we have
\[
C^{-1} \leq \int J N(t)^2 \, dt \leq C.
\]
In particular, from Lemma 3.22 we have
\[
\sup_{t \in J} N(t) \sim_{u,v,S_J(u,v)} \int_J N(t)^3 \, dt.
\]
Now, since the minimal mass blow-up solution $(u,v)$ given in Theorem 3.19 or 3.20 satisfies $(u(t),v(t)) \neq 0$ for all $t \geq 0$ and $S_{(0,\infty)}(u,v) = \infty$, there is a unique sequence $\{t_k\}_{k=0}^{\infty} \subset [0,\infty)$ such that
\[
0 = t_0 < t_1 < t_2 < \cdots, \quad S_{(t_k,t_{k+1})}(u,v) = 1 \quad \text{for any} \quad k \geq 0.
\]
We also have $t_k \to \infty$ as $k \to \infty$, because for any compact interval $J \subset [0,\infty)$ it follows from Lemma 3.25 and $N(t) \leq 1$ for $t \geq 0$ that $S_J(u,v) < \infty$. We call these subintervals $J_k := (t_k, t_{k+1})$ the characteristic intervals.

Then, we see that $N(t)$ and $\xi(t)$ given in Theorem 3.19 or 3.20 can be taken so that
\[
\begin{cases}
N(t), \xi(t) \text{ are constant on each characteristic interval } J_k, \quad \xi(0) = 0, \\
N(0) = \sup_{t \geq 0} N(t) = 1, \quad N(t) \in \{1, C_0^{-1}, C_0^{-2}, \ldots\} \quad (t \geq 0), \\
N(t_{k+1}) \in \{C_0^{-1} N(t_k), N(t_k), C_0 N(t_k)\} \quad (k \geq 0)
\end{cases}
\tag{3.26}
\]
for some $C_0 = C_0(u,v) > 1$. In fact, Lemmas 3.22 and 3.23 show that (3.10) still holds if we modify $N(t)$ and $\xi(t)$ on $[0,\infty)$ to
\[
\bar{N}(t) := \sum_{k=0}^{\infty} C_0^{\left\lfloor \log N(t_k) \right\rfloor} \mathbf{1}_{J_k}(t), \quad \bar{\xi}(t) := \sum_{k=0}^{\infty} \xi(t_k) \mathbf{1}_{J_k}(t),
\]
for any $C_0 > 1$. (Note that $C_0^{-1} N(t_k) < C_0^{\left\lfloor \log N(t_k) \right\rfloor} \leq N(t_k)$.) These functions $\bar{N}(t), \bar{\xi}(t)$ also satisfy the same properties as $N(t), \xi(t)$, namely, $\bar{N}(0) = \sup_{t \geq 0} \bar{N}(t) = 1$ and $\bar{\xi}(0) = 0$. Moreover, if $C_0 = C_0(u,v)$ is sufficiently large, it holds that $\bar{N}(t_{k+1}) \in \{C_0^{-1} \bar{N}(t_k), \bar{N}(t_k), C_0 \bar{N}(t_k)\}$ for any $k \geq 0$.

Therefore, in what follows we additionally assume (3.26). These additional properties will be useful later.

Furthermore, in the case $\xi(t) \neq 0$, we see from Lemmas 3.23 and 3.26 that
\[
|\xi(t_k) - \xi(t_{k+1})| \lesssim_{u,v} N(t_k) \sim_{u,v} \int_{t_k} N(t)^3 \, dt.
\]
Hence, we can take a constant $C_* = C_*(u,v) \gg 1$ such that
\[
|\xi(t_k) - \xi(t_{k+1})| \leq 2^{-10} C_* N(t_k), \quad |\xi(t_k) - \xi(t_{k+1})| \leq 2^{-10} C_* \int_{t_k} N(t)^3 \, dt \tag{3.27}
\]
for any $k \geq 0$. We set $C_* : = 1$ when $\xi(t) \equiv 0$.

4. Long-time Strichartz estimate. This section is devoted to the following estimate.

**Theorem 4.1** (Long-time Strichartz estimate). Let $(u,v)$ be the minimal mass blow-up solution of (1.3) on a time interval $[0,\infty)$ given in Theorem 3.19 or 3.20. (Hence, we consider the particular situation that either $\kappa = 1/2$ or $\xi(t) \equiv 0$ holds.)
Then, there exists a bounded non-increasing function \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \rho(N) \to 0 \) as \( N \to \infty \) such that the following holds:

Let \( J \subset [0, \infty) \) be an arbitrary interval which is a union of finite number of characteristic intervals \( \{ J_k \} \), and let \( K := \int_J N(t)^3 \, dt (\leq \infty) \). Then, for any \( N \leq C_\ast K \) (with \( C_\ast \) given in (3.27)) we have

\[
\left\| P_{[\xi(t) > N]} u \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} + \left\| P_{[\xi(t) > N]} v \right\|_{L^4(\mathbb{R}^d)} \leq \left( \frac{K}{N} \right)^{1/2} \rho(N). \tag{4.1}
\]

**Remark 4.2.** The minimality of the mass of \((u, v)\) is not used in the proof of Theorem 4.1, and in fact we can obtain a similar estimate for general almost periodic solutions satisfying \( N(t) \leq 1 \). However, when \( \xi(t) \not\equiv 0 \) we still need to assume \( \kappa = 1/2 \) so that the system has the Galilean invariance.

### 4.1. Non-radial, mass-resonance case

Let us consider the case of general \( \xi(t) \) under the assumption \( \kappa = 1/2 \). We first derive recursive bounds. Let \( J \subset [0, \infty) \) be an interval, and define the functions \( A_J, S_J \) on \( 2^\mathbb{Z} \) as

\[
A_J(N) := \left\| P_{[\xi(t) > N]} u \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} + \left\| P_{[\xi(t) > N]} v \right\|_{L^4(\mathbb{R}^d)},
\]

\[
S_J(N) := \left\| P_{[\xi(t) > N]} u \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} + \left\| P_{[\xi(t) > N]} v \right\|_{L^4(\mathbb{R}^d)}.
\]

Note that \( A_J(N) \leq A_{\mathbb{R}_+}(N) \leq 1 \) for any \( N \), and \( A_{\mathbb{R}_+}(N) \) tends to 0 as \( N \to \infty \) by (3.10) and \( N(t) \leq 1 \), while \( S_J(N) \) can be infinite if \( J \) contains infinitely many characteristic intervals.

**Lemma 4.3.** Let \( \kappa = 1/2 \), and let \((u, v)\) be as in Theorem 4.1. Then, there exists \( C_1 = C_1(u, v) > 0 \) such that the following inequalities hold: Let \( J \subset [0, \infty) \) be an arbitrary interval which is a union of (possibly infinite) \( \{ J_k \} \) such that \( K := \int_J N(t)^3 \, dt \leq \infty \).

(i) For any \( R \geq C_\ast \) and \( 0 < N \leq RK \) such that \( S_J(2^{-6}N) < \infty \), we have

\[
A_J(N) + S_J(N) \leq C_1 \delta(R) S_J(2^{-6}N) + C_1 R^{\frac{2}{3}} \left( \frac{K}{N} \right)^{\frac{1}{3}} \left( 1_{(0, R)}(N) + A_J(2^{-8}N)^{\frac{1}{2}} \right). \tag{4.2}
\]

(ii) For any \( R \geq C_\ast, N \geq \max \{ RK, R \} \) such that \( S_J(2^{-8}N) < \infty \), we have

\[
A_J(N) + S_J(N) \leq \inf_k \left\| P_{[\xi(t_k) > N/4]} u(t_k) \right\|_{L^2(\mathbb{R}^d)} + \inf_k \left\| P_{[\xi(t_k) > N/4]} v(t_k) \right\|_{L^2(\mathbb{R}^d)} + C_1 \delta(R) S_J(2^{-6}N) + C_1 R^{\frac{2}{3}} \left( \frac{K}{N} \right)^{\frac{1}{3}} \left( A_J(2^{-8}N) + S_J(2^{-8}N) \right). \tag{4.3}
\]

Here,

\[
\delta(R) := \left\| P_{[\xi(t) > 2^{-10}RN(t)]} u \right\|_{L^\infty L^2(\mathbb{R}, \times \mathbb{R}^d)} + \left\| P_{[\xi(t) > 2^{-10}RN(t)]} v \right\|_{L^\infty L^2(\mathbb{R}, \times \mathbb{R}^d)}
\]

\[
+ \| u \|_{L^\infty L^2([t \in \mathbb{R}_+, \|x - x(t)\| > R/N(t)])} + \| v \|_{L^\infty L^2([t \in \mathbb{R}_+, \|x - x(t)\| > R/N(t)])}.
\]

In particular, \( \delta(R) \to 0 \) as \( R \to \infty \) by (3.10).

**Proof.** Since the time-dependent projection operator \( P_{[\xi(t) > N]} \) does not commute with \( i\partial_t + \Delta \), we need to freeze the frequency center \( \xi(t) \) before applying the Strichartz estimate.

(i) Note that the both sides of (4.2) are finite by the assumption \( S_J(2^{-6}N) < \infty \). We make a special decomposition of \( J = \cup_k J_k \). Let \( \{ B_j \} \) be the collection of the
characteristic intervals $J_k$ for which $N(t_k) > N/R$. Since
\[ \sum_k N(t_k) \sim \sum_k \int_{J_k} N(t)^3 \, dt = K \]
by Lemma 3.26, the number of $B_j$ is at most $O(RK/N)$. Also note that such a characteristic interval does not exist if $N \geq R$.

On each $B_j$, we estimate the left hand side of (4.2) crudely by $O(1)$ using the Duhamel formula and the Strichartz estimates. Then, the contribution from $\bigcup B_j$ by Lemma 3.26, the number of each of which is a union of characteristic intervals, such that for each $G_l$ it holds that
\[ \frac{N}{R} < \sum_{k; J_k \subseteq G_l} N(t_k) \leq \frac{2N}{R}, \]
or that
\[ \sum_{k; J_k \subseteq G_l} N(t_k) \leq \frac{N}{R} \sup G_l = \inf B_j \text{ for some } B_j \quad \text{or} \quad \sup G_l = \sup J. \]
This is possible because $N(t_k) \leq N/R$ for all $k$ such that $J_k \subseteq J \setminus \bigcup B_j$. Since $N \leq RK$, we have
\[ \#G_l \leq C \frac{RK}{N} + \#B_j + 1 \lesssim \frac{RK}{N}. \]
(3.27) and the assumption $R \geq C_\ast$ imply that if $t, t^* \in G_l$,
\[ |\xi(t) - \xi(t^*)| \leq 2^{-10} C_\ast \sum_{k; J_k \subseteq G_l} N(t_k) \leq 2^{-9} N. \]
(Recall that $\xi(t) \equiv \xi(t_k)$ on $J_k$.) We thus have
\[ \{\xi - \xi(t) \geq N\} \subset \{\xi - \xi(t^*) \geq N/2\} \]
for any $t, t^* \in G_l$, which implies that $P_{\xi - \xi(t)}>N = P_{\xi - \xi(t)}>N P_{\xi - \xi(t^*)}>N/4$. In the same manner, we have $P_{\xi - \xi(t)}>N/4 = P_{\xi - \xi(t^*)}>N/4 P_{\xi - \xi(t^*)}>2^{-4} N$.

Let us focus on the estimate for $u$; the argument for $v$ is analogous. For each $G_l$, using the Duhamel formula and the Strichartz estimates, we have
\[ \|P_{\xi - \xi(t)}>N u\|_{L^\infty L^2 L^4(G_l \times \mathbb{R}^4)} \lesssim \|P_{\xi - \xi(t^*)}>N/4 u\|_{L^\infty L^2 L^4(G_l \times \mathbb{R}^4)} \]
\[ \lesssim \|P_{\xi - \xi(t^*)}>N/4 u(t_\ast)\|_{L^2(\mathbb{R}^4)} + \|P_{\xi - \xi(t^*)}>N/4(\pi u)\|_{L^2 L^{4/3}(G_l \times \mathbb{R}^4)} \]
\[ \lesssim \|P_{\xi - \xi(t^*)}>N/4 u(t_\ast)\|_{L^2(\mathbb{R}^4)} + \|P_{\xi - \xi(t^*)}>2^{-4} N(\pi u)\|_{L^2 L^{4/3}(G_l \times \mathbb{R}^4)}, \]
where $t_\ast \in G_l$ is an arbitrary point in $G_l$ and the implicit constant does not depend on $t_\ast$. By square-summing the above estimate over $G_l$'s and applying the bound on $\#G_l$, we obtain
\[ \|P_{\xi - \xi(t)}>N u\|_{L^\infty L^2 L^4(\bigcup G_l \times \mathbb{R}^4)} \]
\[ \lesssim R^{\frac{1}{2}} \left( \frac{K}{N} \right)^{\frac{1}{2}} A_J(2^{-2} N) + \left( \sum_{k; J_k \subseteq \bigcup B_j} \|P_{\xi - \xi(t_k)}>2^{-4} N(\pi u)\|_{L^2 L^{4/3}(J_k \times \mathbb{R}^4)}^2 \right)^{\frac{1}{2}}. \]
Note that $A_J(2^{-2} N) \lesssim A_J(2^{-8} N)^{1/3}$, since $A_J$ is bounded and non-increasing in $N$. 

All we have to do is the estimate of $\|P_{|\xi|>rN}(\overline{u})\|_{L^2L^{4/3}(J_k \times \mathbb{R}^4)}$ for each $J_k \subset J \cup B_j$, on which $N(t) = N(t_k) \leq N/R$. The following properties of the Galilean transforms are easily verified.

**Lemma 4.4.** For a smooth function $\phi$ on $\mathbb{R}^4$, let $P_{\phi}$ be the Fourier multiplier defined by $P_{\phi}f := \mathcal{F}_{\xi}^{-1}\phi \mathcal{F}_{\xi}f$. Also define the shift operator $\tau(\xi_0)$ and the Galilean transforms $G^{u}_{\xi_0}, G^{v}_{\xi_0}$ for $\xi_0 \in \mathbb{R}^4$ by

$$[\tau(\xi_0)f](\xi) := f(\xi + \xi_0),$$

$$[G^{u}_{\xi_0}u](t,x) := e^{ix \cdot \xi_0}e^{-it|\xi_0|^2}u(t,x - 2\xi_0 t),$$

$$[G^{v}_{\xi_0}v](t,x) := e^{ix \cdot \xi_0}e^{-2it|\xi_0|^2}v(t,x - 2\xi_0 t).$$

Then, the following holds for $x_0, \xi_0 \in \mathbb{R}^4$.

1. $P_{\phi}e^{ix \cdot \xi_0}f = e^{ix \cdot \xi_0}P_{\tau(\xi_0)}f$, $P_{\phi}\tau(x_0)f = \tau(x_0)P_{\phi}f$.

2. $\|P_{\phi}G^{u}_{\xi_0}u\| = \|P_{\tau(\xi_0)}u\|$, $\|P_{\phi}G^{v}_{\xi_0}v\| = \|P_{\tau(\xi_0)}v\|

   for the $L^pL^q(I \times \mathbb{R}^4)$-norm with any $1 \leq p, q \leq \infty$ and $I \subset \mathbb{R}$.

3. $G^{u}_{\xi_0}(\overline{u}) = \mathcal{F}_{\xi_0}^*uG^{u}_{\xi_0}v$, $G^{v}_{\xi_0}(u^2) = (G^{u}_{\xi_0}u)^2$.

4. $(i\partial_t + \Delta)G^{u}_{\xi_0}u = \mathcal{F}_{\xi_0}^*u\mathcal{F}_{\xi_0}G^{u}_{\xi_0}v$, $(i\partial_t + \frac{1}{2}\Delta)G^{v}_{\xi_0}v = (G^{u}_{\xi_0}u)^2$ if $(u, v)$ solves (1.3) with $\kappa = 1/2$.

Hence, we evaluate

$$\|P_{|\xi|>rN}(\overline{u})\|_{L^2L^{4/3}(J_k \times \mathbb{R}^4)} = \|P_{2^{-sN}}(\overline{G^{u}_{-\xi(t_k)}uG^{v}_{-\xi(t_k)}v})\|_{L^2L^{4/3}(J_k \times \mathbb{R}^4)} \leq \|P_{2^{-sN}}G^{u}_{-\xi(t_k)}uP_{2^{-sN}}G^{v}_{-\xi(t_k)}v\|_{L^2L^{4/3}(J_k \times \mathbb{R}^4)}$$

$$+ \|P_{2^{-sN}}G^{u}_{-\xi(t_k)}uG^{v}_{-\xi(t_k)}v\|_{L^2L^{4/3}(J_k \times \mathbb{R}^4)}$$

$$+ \|G^{u}_{-\xi(t_k)}uP_{2^{-sN}}G^{v}_{-\xi(t_k)}v\|_{L^2L^{4/3}(J_k \times \mathbb{R}^4)}. \quad (4.4)$$

In the last inequality we have used the identities

$$u\overline{v} = P_{2^{-sN}}uP_{2^{-sN}}v - P_{2^{-sN}}uP_{2^{-sN}}v + P_{2^{-sN}}u\overline{v} + \overline{u}P_{2^{-sN}}v,$$

and the Bernstein inequality.

Since $N \geq RN(t_k)$, we have $P_{2^{-sN}} = P_{2^{-sN}}P_{2^{-N}RN(t_k)}$ and obtain a bound on (4.4) as

$$\|P_{2^{-sN}}G^{u}_{-\xi(t_k)}u\|_{L^\infty L^2(J_k \times \mathbb{R}^4)} = \|P_{2^{-sN}}G^{v}_{-\xi(t_k)}v\|_{L^\infty L^2(J_k \times \mathbb{R}^4)}.$$

Then, summing up over $J_k$’s, we obtain the total contribution from (4.4) as

$$C\delta(R)S J(2^{-sN}).$$
We make further decomposition for (4.5) as
\[
\begin{align*}
(4.5) & \leq \left\| P_{>2^{-6}N} G_{\xi(t)}^u u G_{\xi(t)} v \right\|_{L^2 L^4/(\{t \in J_k, |x+2\xi(t)|-x(t)| > R/N(t_k)\})} \\
+ & \left\| P_{>2^{-6}N} G_{\xi(t)}^u u P_{>2^{-9}N} G_{\xi(t)} v \right\|_{L^2 L^4/(\{t \in J_k, |x+2\xi(t)|-x(t)| \leq R/N(t_k)\})} \\
+ & \left\| P_{>2^{-6}N} G_{\xi(t)}^u u P_{\leq 2^{-9}N} G_{\xi(t)} v \right\|_{L^2 L^4/(\{t \in J_k, |x+2\xi(t)|-x(t)| \leq R/N(t_k)\})}.
\end{align*}
\] (4.7) (4.8)

For (4.7) and (4.8), we estimate as
\[
\left\| P_{|\xi(t)| > 2^{-6}N} u \right\|_{L^2 L^4(J_k \times \mathbb{R}^d)} \left( \left\| v \right\|_{L^\infty L^2(\{t \in J_k, |x-x(t)| > R/N(t_k)\})} \right.
+ \left. \left\| P_{|\xi(t)| > 2^{-10}R N(t_k)} v \right\|_{L^\infty L^2(J_k \times \mathbb{R}^d)} \right).
\] (4.9)

We sum up this bound over \( J_k \)'s and obtain \( C\delta(R) J(2^{-6}N) \). Finally, we apply the bilinear Strichartz estimate (Lemma 2.4) to (4.9).\footnote{The Fourier supports of \( P_{|\xi(t)| > 2^{-6}N} u(t) \) and \( P_{|\xi(t)| \leq 2^{-9}N} v(t) \) are not necessarily separated when \( \xi(t_k) \neq 0 \). To apply the bilinear Strichartz estimate, we exploit the Galilean transformations to adjust the frequency centers to the origin. This is the only part where we essentially use the Galilean invariance of the system in the proof of Theorem 4.1.} We have
\[
(4.9) \lesssim \frac{R}{N(t_k)} \left\| P_{>2^{-6}N} G_{\xi(t)}^u u P_{\leq 2^{-9}N} G_{\xi(t)} v \right\|_{L^2(J_k \times \mathbb{R}^d)}
\lesssim \frac{R}{N(t_k)} \left( \frac{RN(t_k)}{N^{1/2}} \right)^{3/2} \left\| P_{>2^{-6}N} G_{\xi(t)}^u u \right\|_{S^0(J_k \times \mathbb{R}^d)} \left\| P_{\leq 2^{-9}N} G_{\xi(t)} v \right\|_{S^0(J_k \times \mathbb{R}^d)},
\]
where
\[
\begin{align*}
\left\| u \right\|_{S^0(J_k \times \mathbb{R}^d)} & := \left\| u(t_k) \right\|_{L^2(\mathbb{R}^d)} + \left\| (i\partial_t + \Delta) u \right\|_{L^{3/2}(J_k \times \mathbb{R}^d)}, \\
\left\| v \right\|_{S^0(J_k \times \mathbb{R}^d)} & := \left\| v(t_k) \right\|_{L^2(\mathbb{R}^d)} + \left\| (i\partial_t + \frac{1}{2} \Delta) v \right\|_{L^{3/2}(J_k \times \mathbb{R}^d)}.
\end{align*}
\]

Since \( (i\partial_t + \frac{1}{2} \Delta) P_{\leq 2^{-9}N} G_{\xi(t)} v = P_{\leq 2^{-9}N} (G_{\xi(t)}^u u)^2 \), the last norm is bounded by
\[
\begin{align*}
\left\| P_{\leq 2^{-9}N} G_{\xi(t)}^u v \right\|_{L^2(\mathbb{R}^d)} & + \left\| P_{\leq 2^{-9}N} (G_{\xi(t)}^u u)^2 \right\|_{L^{3/2}(J_k \times \mathbb{R}^d)} \\
& \lesssim \left\| v(t_k) \right\|_{L^2(\mathbb{R}^d)} + \left\| u \right\|_{L^3(J_k \times \mathbb{R}^d)}^2 \lesssim 1.
\end{align*}
\]

Similarly, we have
\[
\begin{align*}
\left\| P_{\leq 2^{-9}N} G_{\xi(t)} u \right\|_{S^0(J_k \times \mathbb{R}^d)}
& = \left\| P_{|\xi(t)| > 2^{-6}N} u(t_k) \right\|_{L^2(\mathbb{R}^d)} + \left\| P_{>2^{-6}N} G_{\xi(t)}^u u G_{\xi(t)} v \right\|_{L^{3/2}(J_k \times \mathbb{R}^d)}.
\end{align*}
\]

For the second term on the right-hand side, we make a similar decomposition as (4.4)–(4.6). Applying the Hölder inequality, interpolation, and that \( J(t_k) \subseteq 1 \)
for any $M > 0$, we have

$$
\| P_{> -N^2} G_{\xi(t)}^{\nu} u \|_{L^{3/2}(J_k \times \mathbb{R}^4)}
\lesssim \| P_{> -N^2} G_{\xi(t)}^{\nu} u \|_{L^{3/2}(J_k \times \mathbb{R}^4)}
+ \| P_{> -N^2} G_{\xi(t)}^{\nu} u \|_{L^{3/2}(J_k \times \mathbb{R}^4)}
\lesssim \left( \| P_{\xi}(\xi(t)) > 2^{N} u \|_{L^3(J_k \times \mathbb{R}^4)} + \| P_{\xi}(\xi(t)) > 2^{N} v \|_{L^3(J_k \times \mathbb{R}^4)} \right)
\times \left( \| u \|_{L^3(J_k \times \mathbb{R}^4)} + \| v \|_{L^3(J_k \times \mathbb{R}^4)} \right)
\lesssim A_{J_k}(2^{-N} N)^{3/2} S_{J_k}(2^{-N} N)^{3/2} \lesssim A_{J_k}(2^{-N} N)^{3/2}.
$$

Thus, the total contribution from (4.9) is bounded by

$$
\frac{R^{5/2}}{N^{1/2}} \left( \sum_k N(t_k) \right)^{3/2} \left[ A_{J}(2^{-N} N) + A_{J}(2^{-N} N)^{3/2} \right] \lesssim R^{2} \left( \frac{K}{N} \right)^{3/2} A_{J}(2^{-N} N)^{3/2}.
$$

This completes the estimate for (4.5). (4.6) can be treated in a similar manner, and we have (4.2).

(ii) Since $N \geq R$ and $N(t) \leq 1$, we have no $B_j$. Moreover, from (3.27) and $N \geq C, K$ we have

$$
|\xi(t) - \xi(t^*)| \leq 2^{-10} C_s \sum_k J_k(t) \int_{J_k} N(t)^3 \, dt = 2^{-10} C_s K \leq 2^{-10} N
$$

for any $t, t^* \in J$. Similarly to the estimate on $G_t$ in (i), we show

$$
\| P_{\xi}(\xi(t)) > 2^{N} u \|_{L^\infty L^2 \cap L^4(J_k \times \mathbb{R}^4)}
\lesssim \inf_k \| P_{\xi}(\xi(t_k)) > 2^{N} u(t_k) \|_{L^2(J_k \times \mathbb{R}^4)}
+ \left( \sum_{k, J_k \subset J} \| P_{\xi}(\xi(t_k)) > 2^{N} (\Pi v) \|_{L^2 L^{4/3}(J_k \times \mathbb{R}^4)} \right)^{1/2}.
$$

The estimate on the nonlinear term is the same as before, except that at the last step we use

$$
A_{J_k}(2^{-N} N)^{3/2} S_{J_k}(2^{-N} N)^{3/2} \lesssim A_{J_k}(2^{-N} N) + S_{J_k}(2^{-N} N).
$$

This implies (4.3).

We are now ready to prove Theorem 4.1 in the case of $\kappa = 1/2$.

**Proof of Theorem 4.1 ($\kappa = 1/2$).** The proof will be done via an induction on $N$. Note that $\mathcal{S}_J(N) < \infty$ for any $N$, since $J$ consists of finitely many $J_k$’s.

We begin with the base case. We always have a crude bound $\mathcal{S}_J(N) \lesssim (\# J_k)^{1/2}$, which can be seen by applying the Duhamel formula and the Strichartz estimates on each $J_k$. This bound is acceptable as long as $N$ is so small that $\# J_k \leq K/N$. Hence, there exists $C_2 = C_2(u, v) > 0$ such that

$$
\mathcal{S}_J(N) \leq C_2 \left( \frac{K}{N} \right)^{3/2}, \quad N \leq \frac{K}{\# J_k}, \quad (4.10)
$$
With the base case in mind, we impose the following condition on the function \( \rho \):
\[
\rho(N) \geq C_2, \quad N \leq \frac{K}{\# J_k}. \tag{4.11}
\]
On the other hand, if we had (4.1), then the formula (4.2) would imply that
\[
S_J(N) \leq \left[ 8C_1 \delta(R) \rho(2^{-6}N) + C_1 R^2 \bar{\rho}(N) \right] \left( \frac{K}{N} \right)^{\frac{1}{2}},
\]
where
\[
\bar{\rho}(N) := 1_{(0,R)}(N) + A_{\# J_k}(2^{-8}N)^{\frac{3}{4}}.
\]
(Note that \( \bar{\rho}(N) \) is non-increasing, \( \sup_{N>0} \bar{\rho}(N) \leq 1 \) and \( \lim_{N \to \infty} \bar{\rho}(N) = 0 \). We thus need to define \( \rho(N) \) so that
\[
\rho(N) \geq 8C_1 \delta(R) \rho(2^{-6}N) + C_1 R^2 \bar{\rho}(N), \quad N > 0. \tag{4.12}
\]
We first fix \( R \geq C_s \) sufficiently large so that \( 8C_1 \delta(R) \leq \frac{1}{2} \), and then define the function \( \rho_{N_*} \) for \( N_* \in 2^\mathbb{Z} \) by
\[
\rho_{N_*}(N) := \begin{cases} 
2C_1 R^2 \sup_{M>0} \bar{\rho}(M), C_2 \}, & N \leq N_* , \\
C_1 R^2 \bar{\rho}(N) + \frac{1}{2} \rho_{N_*}(2^{-6}N), & N > N_* \text{ (recursively).}
\end{cases}
\]
It is easy to verify that \( \rho_{N_*} \) is bounded uniformly in \( N_* \), and that (4.11), (4.12) hold if \( N_* \geq K/\# J_k \). Moreover, \( \rho_{N_*} \) is non-increasing and thus has a limit \( \rho_{N_*}(\infty) \geq 0 \) as \( N \to \infty \). Letting \( N \to \infty \) in the above recursive formula, we have \( \rho_{N_*}(\infty) = \frac{1}{2} \rho_{N_*}(\infty) \), concluding \( \rho_{N_*}(\infty) = 0 \).

Now, since \( K = \int_J N(t)^3 \, dt \leq \int_J N(t)^2 \, dt \lesssim S_J(u,v) = \# J_k \) by Lemma 3.25, the quantity \( K/\# J_k \) has an upper bound \( N_0 \) which is independent of \( J \). Therefore, we finally define \( \rho := \rho_{N_0} \). The claimed estimate (4.1) can be shown by an induction on \( N \), with the base case (4.10) and the recursive formula (4.2), noticing (4.11) and (4.12).

This completes the proof. \( \square \)

4.2. Radial case. Here, we consider the case \( \xi(t) \equiv 0 \). Since we do not need the Galilean invariance, the same argument as above can be applied for any \( \kappa > 0 \). Moreover, it turns out that we do not have to consider the intervals \( \{G_i\} \). Note that the estimate of (4.8)–(4.9) is slightly different; we decompose \( v \) as \( P_{> \varepsilon N} v + P_{\leq \varepsilon N} v \) with \( \varepsilon > 0 \) sufficiently small depending on \( \kappa \), so that we can apply Lemma 2.4. As a result, we show the following:

**Lemma 4.5.** Let \((u,v)\) be as in Theorem 4.1, and assume that \( \xi(t) \equiv 0 \). Then, there exists \( C_1 = C_1(u,v) > 0 \) such that the following inequalities hold: Let \( J \subset [0,\infty) \) be an arbitrary interval which is a union of (possibly infinite) \( \{J_k\} \) such that \( K := \int_J N(t)^3 \, dt < \infty \).

(i) For any \( R > 0 \) and \( 0 < N \leq K \) such that \( S_J(2^{-2}N) < \infty \), we have
\[
A_J(N) + S_J(N) \leq C_1 \delta(R) S_J(2^{-2}N)
+ C_1 R^2 \left( \frac{K}{N} \right)^{\frac{1}{2}} \left( 1_{(0,R)}(N) + A_J(2^{-4}N)^{\frac{3}{4}} \right). \tag{4.13}
\]
(ii) For any $R > 0$, $N \geq \max\{K, R\}$ such that $S_J(2^{-4}N) < \infty$, we have
$$A_J(N) + S_J(N) \leq \inf_k \|P_{> N}(u, v)(t_k)\|_{L^2(\mathbb{R}^4)} + C_1 \delta(R) S_J(2^{-2}N)$$
\[+ C_1 R^{\frac{3}{2}} \left( \frac{K}{N} \right)^{\frac{1}{2}} \left( A_J(2^{-4}N) + S_J(2^{-4}N) \right). \tag{4.14} \]

Here,
$$\delta(R) := \|P_{> \varepsilon R N(t)}(u, v)\|_{L^\infty L^2(\mathbb{R}^+ \times \mathbb{R}^4)} + \|(u, v)\|_{L^\infty L^2\{|t| < R/N(t)\}}$$
with $0 < \varepsilon = \varepsilon(K) \ll 1$.

Theorem 4.1 can be shown by the same argument as before using Lemma 4.5 (i).

5. Additional regularity: Rapid frequency cascade scenario. For the minimal mass blow-up solution $(u, v)$ given in Theorem 3.19 or 3.20, the following two scenarios are possible:

- $\int_0^\infty N^3(t) \, dt < \infty$ (Rapid frequency cascade scenario),
- $\int_0^\infty N^3(t) \, dt = \infty$ (Quasi-soliton scenario).

In this section we shall derive additional regularity for the former case from Theorem 4.1, Lemma 4.3 (ii) or Lemma 4.5 (ii) and use it to preclude this scenario.

**Theorem 5.1.** In the rapid frequency cascade scenario, the solution $(u, v)$ is in $L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^4)^2)$ for any $s > 0$ and, with $K := \int_0^\infty N(t)^3 \, dt < \infty$, satisfies
$$\|u\|_{L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^4)^2)} + \|v\|_{L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^4)^2)} \lesssim_s (K)^s.$$  

**Proof.** Let us focus on the case $\kappa = 1/2$; for the case $\xi(t) \equiv 0$ the claim is shown similarly by means of Lemma 4.5 (ii) instead of Lemma 4.3 (ii). We continue to use the notation in the preceding section.

In order to prove Theorem 5.1 it suffices to show that, for any $s > 0$, there exist $C_s > 0$, $C'_s \geq C_s$ depending on $u, v, s$ such that
$$A_J(N) \leq C_s \left( \frac{K}{N} \right)^s, \quad N \geq C'_s K. \tag{5.1}$$

In fact, if we have (5.1), then $\|P_{> 4N}u\|_{L^\infty L^2(\mathbb{R}^+ \times \mathbb{R}^4)} + \|P_{> 4N}v\|_{L^\infty L^2(\mathbb{R}^+ \times \mathbb{R}^4)} \lesssim C_s(K/N)^s$ for $N \geq C'_s K$, since (3.27) implies that for any $t \geq 0$
$$|\xi(t)| = |\xi(t) - \xi(0)| \leq 2^{-10} C_s \sum_{k: J_k \subset [0, t]} \int_{J_k} N(k)^3 \, dt \leq 2^{-10} C_s K \leq 2^{-10} N. \tag{5.2}$$

Therefore, we have
$$\|u\|_{L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^4)^2)} \leq \|P_{< 4C'_s K} u\|_{L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^4)^2)} + \sum_{N \geq C'_s K} \|P_{N} u\|_{L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^4)^2)}$$
$$\lesssim_s (K)^s \|u\|_{L^\infty L^2(\mathbb{R}^+ \times \mathbb{R}^4)} + \sum_{N \geq C'_s K} (N)^s \left( \frac{K}{N} \right)^{2s} \lesssim_s (K)^s,$$
and similarly for $v$, proving Theorem 5.1.

To show (5.1), we first observe that $A_{\mathbb{R}^+}(C_s K) + S_{\mathbb{R}^+}(C_s K) \lesssim 1$. In fact, the monotone convergence theorem reduces to showing it for any compact interval $J \subset \mathbb{R}^+$, which follows from Theorem 4.1. (We do not exploit the decaying factor $\rho(N)$ here.) In particular, we have $A_{\mathbb{R}^+}(N) + S_{\mathbb{R}^+}(N) \lesssim 1$ for any $N \geq C_s K$. 


Next, we set \( R \geq C_4 \) and \( C_3 \geq \max\{R, RK^{-1}\} \) sufficiently large (according to \( s \)) so that
\[
2^{8s}C_1\delta(R) \leq \frac{1}{2}, \quad 2^{8s}C_1R^2\left(\frac{K}{2^{8s}C_3K}\right)^{\frac{1}{2}} \leq \frac{1}{2}
\]
Since \( \inf N(t_k) = 0 \) by \( \int_0^\infty N(t)^3\,dt < \infty \), it holds
\[
\inf_k \|P\|_{L^2(R^4)} > N(u(t_k)) = \inf_k \|P\|_{L^2(R^4)} > \frac{\omega}{\infty} N(t_k) u(t_k)\|L^2(R^4) = 0
\]
for any \( N \) by the almost orthogonality (3.10), and similarly for \( v \). From Lemma 4.3 (ii), we have, for \( N \geq 2^sC_3K \),
\[
A_{R,4}(N) + S_{R,4}(N) \leq \frac{1}{2}\left\{ 2^{-6s}S_{R,4}(2^{-6}N) + 2^{-8s}\left(A_{R,4}(2^{-8}s) + S_{R,4}(2^{-8}s)\right) \right\}
\]
Now, we choose \( C_4 = C_4(u,v,s) > 0 \) large enough so that
\[
A_{R,4}(2^{j}C_4K) + S_{R,4}(2^{j}C_4K) \leq 2^{-q}C_4, \quad j = 0, 1, \ldots, 7.
\]
Then, it is easy to show by induction that (5.1) holds with \( C_s = C_4' \) and \( C_s' = C_3 \).

Suppose that there exists a minimal mass blow-up solution \( (u,v) \) in the rapid frequency cascade scenario. Since \( \int_0^\infty N(t)^3\,dt < \infty \), there is a sequence of time \( \{t_n\} \) such that \( N(t_n) \to 0 \) as \( n \to \infty \).

We first consider the radial case: \( \xi(t) \equiv 0 \) and \( \kappa > 0 \) is arbitrary. From the almost periodicity of \( (u,v) \), we see that
\[
\sup_n \|P_{> R}(u,v)(t_n)\|_{L^2} = o(1) \quad (R \to \infty).
\]
By Theorem 5.1 with \( s = 2 \) (in fact \( s = 1 + \varepsilon \) is sufficient) we obtain
\[
\limsup_{n \to \infty} \|(u,v)(t_n)\|_{H^1} \\
\leq \limsup_{n \to \infty} \left( \|P_{> R}(u,v)(t_n)\|_{H^1} + \|P_{\leq R}(u,v)(t_n)\|_{H^1} \right) \\
\leq \limsup_{n \to \infty} \left( \|(u,v)(t_n)\|^{1/2}_{H^2} \|P_{> R}(u,v)(t_n)\|^{1/2}_{L^2} + RN(t_n) \right) \\
= \langle K \rangle o(1) \quad (R \to \infty),
\]
which implies that \( \|(u,v)(t_n)\|_{H^1} \to 0 \) as \( n \to \infty \). Using the Gagliardo-Nirenberg inequality we have \( E((u,v)(t_n)) \to 0 \) as \( n \to \infty \), and then \( E((u,v)(0)) = 0 \) by the energy conservation. This combined with \( M(u,v) = M_{\xi,\text{rad}} < M(\phi,\psi) \) and the sharp Gagliardo-Nirenberg inequality (Lemma 1.3 (ii)) shows that \( \|(u,v)(0)\|_{H^1} = 0 \). By the Sobolev embedding we conclude that \( \|(u,v)(0)\|_{L^4} = 0 \), which clearly contradicts the fact that \( (u,v) \) is a non-zero solution.

For the case of general \( \xi(t) \) and \( \kappa = 1/2 \), we apply the Galilean transformation and argue with \( (u_n,v_n) := T_{\theta(0,0,-\xi(t_n),1)}(u,v) \) instead of \( (u,v) \). Then, we have
\[
\sup_n \|P_{> R}(u_n,v_n)(t_n)\|_{L^2} = o(1) \quad (R \to \infty)
\]
and
\[
\|(u_n,v_n)(t_n)\|_{H^2} = ||\cdot|^2(\hat{u}(t_n, \cdot + \xi(t_n)), \hat{v}(t_n, \cdot + 2\xi(t_n)))\|_{L^2} \lesssim \langle K \rangle^2
\]
by Theorem 5.1 together with (5.2), and hence the above argument implies that \( \|(u_n,v_n)(t_n)\|_{H^1} \to 0 \) as \( n \to \infty \), which yields \( E(u_n(0),v_n(0)) \to 0 \). Since
\[
M(u_n,v_n) = M(u,v) < M(\phi,\psi),
\]
the sharp Gagliardo-Nirenberg inequality shows that \( \| (u_n, v_n)(0) \|_{H^1} \to 0 \), and then
\( \| (u_n, v_n)(0) \|_{L^q} \to 0 \) as \( n \to \infty \). But now \( \| (u_n, v_n)(0) \|_{L^q} = \| (u, v)(0) \|_{L^q} \), and we have the same conclusion \( \| (u, v)(0) \|_{L^q} = 0 \).

Therefore, the rapid frequency cascade scenario is impossible.

6. Virial argument: Quasi-soliton scenario. Now we consider the scenario \( \int_0^\infty N(t)^3 \, dt = \infty \).

6.1. Radial case. Let \((u, v)\) be the minimal mass blow-up solution \((u, v)\) given in Theorem 3.20. We recall here that \( x(t) = \xi(t) \equiv 0 \), while \( \kappa > 0 \) is arbitrary.

We use the cut-off function \( \theta : [0, \infty) \to [0, 1] \) introduced in Section 2. Recall that \( \theta \) is smooth, non-increasing, and satisfies \( \theta \equiv 1 \) on \([0, 1] \), \( \theta \equiv 0 \) on \([2, \infty) \). Let

\[
\Theta(r) := \frac{1}{r} \int_0^r \theta(s) \, ds \quad (r > 0), \quad \Theta(0) := \theta(0) = 1.
\]

We notice that
\[
0 \leq \theta(r) \leq \Theta(r) \leq \min\{1, 2/r\},
\]
\[
0 \leq -\Theta'(r) = \frac{(\Theta - \theta)(r)}{r} \begin{cases} \leq r^{-2} & (r > 1), \\ = 0 & (0 \leq r \leq 1). \end{cases}
\]

We will derive a contradiction by taking a close look at the quantity

\[
\mathcal{M}(t) := \int_{\mathbb{R}^4} \Theta \left( \frac{\tilde{N}(t)|x|}{L} \right) \tilde{N}(t) x \cdot \text{Im} \left[ \bar{U} \nabla U + \frac{1}{2} \bar{V} \nabla V \right](t, x) \, dx
\]

on some time interval \([0, T] \) which is a union of characteristic intervals. Here, \( L > 0 \) is a positive large constant to be chosen later and \((U, V) := P_{\leq K}(u, v)\) with \( K := \int_0^T \tilde{N}(t)^3 \, dt < \infty \). (We need to localize to low frequencies so that \( \mathcal{M}(t) \) will be finite for \( L^2 \) solutions.) Also, a \( C^1 \) function \( \tilde{N} : [0, \infty) \to \mathbb{R}_+ \) is a variant of the frequency scale function \( \tilde{N}(\cdot) \) of \((u, v)\) which will also be defined in the proof, and we assume for now that

\[
\begin{cases}
\tilde{N}(t) \leq N(t) (\leq 1) & \text{for any } t \geq 0; \\
\sup_{t \in J_k} \tilde{N}(t) \leq C_0 \inf_{t \in J_k} \tilde{N}(t) & \text{for any } k \geq 0; \\
on \text{on each } J_k, \tilde{N}(t) \text{ is monotone and } |\tilde{N}'(t)| \leq \frac{2C_0\tilde{N}(t)}{|J_k|},
\end{cases}
\]

(6.2)

where \( C_0 = C_0(u, v) > 1 \) is the constant given in (3.26).

Remark 6.1. Recall that Dodson’s argument for (1.4) in [5] essentially used the virial identity; if \( u \) is a nontrivial solution of (1.4) and \( \| u \|_{L^2} < \| Q \|_{L^2} \), then

\[
\frac{d^2}{dt^2} \| xu(t) \|_{L^2}^2 = \frac{d}{dt} \left( 4 \text{Im} \int_{\mathbb{R}^d} x \cdot u(t) \nabla u(t) \right) = 16E_{\text{NLS}}(u(t)) > 0,
\]

where

\[
E_{\text{NLS}}(u) = \frac{1}{2} \| \nabla u \|_{L^2}^2 - \frac{1}{2s} \| u \|_{L^{2s}}^2, \quad 2s = 2 + \frac{4}{d}
\]

is the conserved energy for (1.4). In our case, the following analogue is valid; for a solution \((u, v)\) of (1.3),

\[
\frac{d}{dt} \left( 4 \text{Im} \int_{\mathbb{R}^d} x \cdot \left( \bar{u}(t) \nabla u(t) + \frac{1}{2} \bar{v}(t) \nabla v(t) \right) \right) = 8E(u, v)(t). \quad (6.3)
\]
Roughly speaking, $\mathfrak{M}(t)$ is a modification of the virial quantity $\text{Im} \int x \cdot (\overline{u} \nabla v + \nabla \overline{v} u)$, for which the time derivative is always positive and away from zero by (6.3) and the minimality of the mass $M(u, v) = M_{\text{rad}} < M(\phi, \psi)$. If we assume $N(t) \equiv 1$, then $\mathfrak{M}(T) - \mathfrak{M}(0) \geq T = K$, whereas the almost periodicity suggests that the solution is in some sense localized to low frequencies uniformly in $t \geq 0$; we can in fact show that $\| \nabla (U, V) \|_{L^\infty((0, T), L^2)} = o(K)$ as $K \to \infty$. Since the weight function $\Theta(|x|/L)$ is bounded, we obtain $\mathfrak{M}(t) = o(K)$ uniformly in $t$, which contradicts the fact $\mathfrak{M}(T) - \mathfrak{M}(0) \geq K$ for $K$ sufficiently large. We will explain later the idea for the case that $N(t)$ varies, where we need to introduce a time-dependent weight function with a carefully chosen $\tilde{N}(t)$.

$(U, V)$ solves the following perturbed system:

$$
i \partial_t U + \Delta U = \overline{U} V + F, \quad i \partial_t V + \kappa \Delta V = U^2 + G,$$

where we have written $F := P_{\leq K}(\overline{u} v) - \overline{U} V$, $G := P_{\leq K}(u^2) - U^2$.

Using equations, we calculate time derivative of the momentum density as

$$
\partial_t \text{Im} \left( \overline{U} \partial_j U + \frac{1}{2} \nabla \partial_j V \right) = -2 \partial_k \text{Re} \left( \partial_j U \overline{\partial_k U} + \frac{\kappa}{2} \partial_j V \overline{\partial_k V} \right) - \frac{1}{2} \partial_j \text{Re} (U^2 \overline{V}) + \frac{1}{2} \partial_j \Delta \left( |U|^2 + \frac{\kappa}{2} |V|^2 \right) + \{ (U, V), (F, G) \}_p, j,
$$

where (and hereafter) we always take summation with respect to repeated indices and

$$
\{ (U, V), (F, G) \}_p, j := \text{Re} (\overline{F} \partial_j U - \overline{U} \partial_j F) + \frac{1}{2} \text{Re} (\overline{G} \partial_j V - \nabla \partial_j G) + 2 \text{Re} (\overline{F} \partial_j U + \frac{1}{2} \overline{G} \partial_j V) - \partial_j \text{Re} (\overline{F} U + \frac{1}{2} \overline{G} V).
$$

Then, we have

$$
\partial_t \mathfrak{M}(t) = \int \partial_t \left[ \Theta \left( \frac{|A|}{L} \right) A_j \right] \text{Im} \left[ \overline{U} \partial_j U + \frac{1}{2} \nabla \partial_j V \right] (t, x) \, dx
$$

$$
+ 2 \int \partial_k \left[ \Theta \left( \frac{|A|}{L} \right) A_j \right] \text{Re} \left[ \partial_j U \overline{\partial_k U} + \frac{\kappa}{2} \partial_j V \overline{\partial_k V} \right] (t, x) \, dx
$$

$$
+ \frac{1}{2} \int \partial_j \left[ \Theta \left( \frac{|A|}{L} \right) A_j \right] \text{Re} \left[ U^2 \overline{V} \right] (t, x) \, dx
$$

$$
- \frac{1}{2} \int \partial_j \Delta \left[ \Theta \left( \frac{|A|}{L} \right) A_j \right] \left[ |U|^2 + \frac{\kappa}{2} |V|^2 \right] (t, x) \, dx
$$

$$
+ \int \Theta \left( \frac{|A|}{L} \right) A_j \{ (U, V), (F, G) \}_p, j (t, x) \, dx,
$$

where we have written $A := \tilde{N}(t)x$. We now see

$$
\partial_t \left[ \Theta \left( \frac{|A|}{L} \right) A_j \right] = \theta \left( \frac{|A|}{L} \right) \tilde{N}'(t) x_j,
$$

$$
\partial_k \left[ \Theta \left( \frac{|A|}{L} \right) A_j \right] = \left[ \theta \left( \frac{|A|}{L} \right) \delta_{jk} + (\Theta - \theta) \left( \frac{|A|}{L} \right) B_{jk} \right] \tilde{N}(t), \quad B_{jk} := \delta_{jk} - \frac{x_j x_k}{|x|^2},
$$

$$
\partial_j \left[ \Theta \left( \frac{|A|}{L} \right) A_j \right] = \left[ 4 \theta \left( \frac{|A|}{L} \right) + 3 (\Theta - \theta) \left( \frac{|A|}{L} \right) \right] \tilde{N}(t).
$$
Therefore, we obtain
\[ \partial_t S(t) = \tilde{N}(t) \int \theta \left( \frac{|A|}{L} \right) x_j \text{Im} \left[ \overline{U} \partial_j U + \frac{1}{2} \nabla \partial_j V \right](t, x) dx \]  
(6.4)
\[ + 2 \tilde{N}(t) \int \theta \left( \frac{|A|}{L} \right) \left[ |\nabla U|^2 + \frac{\kappa}{2} |\nabla V|^2 \right](t, x) dx \]  
(6.5)
\[ + 2 \tilde{N}(t) \int (\Theta - \theta) \left( \frac{|A|}{L} \right) B_{jk} \text{Re} \left[ \partial_j U \partial_k U + \frac{\kappa}{2} \partial_j V \partial_k V \right](t, x) dx \]  
(6.6)
\[ + 2 \tilde{N}(t) \int \Theta \left( \frac{|A|}{L} \right) x_j \{(U, V), (F, G)\}_{p,j}(t, x) dx. \]  
(6.10)

We shall prove the following:

**Theorem 6.2.** There exists a large constant \( L > 0 \) and a \( C^1 \) function \( \tilde{N} : [0, \infty) \rightarrow \mathbb{R}_+ \) satisfying (6.2) such that \( S(t) \) defined as above satisfies
\[ \int_0^T \frac{dS(t)}{dt} dt \geq K \]
for any sufficiently large \( T \).

Note that we can choose \( T \) for which \( K = K(T) = \int_0^T N(t)^3 dt \) is arbitrarily large, since we are dealing with the case \( \int_0^\infty N(t)^3 dt = \infty \).

**Proof.** Let us begin with the estimate on
\[ \int \left[ |\nabla U|^2 + \frac{\kappa}{2} |\nabla V|^2 \right](t, x) dx. \]

We can find the energy of the system in it, so by the minimality assumption \( M(u, v) = M_c < M(\phi, \psi) \) and the sharp Gagliardo-Nirenberg inequality (Lemma 1.3 (ii)) we expect these terms to be positive and the main part of \( \int_0^T \frac{dS(t)}{dt} dt \). To show that, we introduce another cutoff \( \chi : [0, \infty) \rightarrow [0, 1] \), a smooth non-increasing function satisfying \( \chi(r) = 1 \) on \( [0, \frac{1}{2}] \) and \( \chi(r) = 0 \) on \( [1, \infty) \), so that \( \theta \equiv 1 \) on the support of \( \chi \). For a small \( \varepsilon > 0 \), we have\(^8\)
\[ \int \left[ (1 - \varepsilon) \chi^2 \left( \frac{|A|}{L} \right) \left( |\nabla U|^2 + \frac{\kappa}{2} |\nabla V|^2 \right) \right] dx \]
\[ \geq \int \left[ (1 - \varepsilon) \chi^2 \left( \frac{|A|}{L} \right) \left( |\nabla U|^2 + \frac{\kappa}{2} |\nabla V|^2 \right) \right] dx \]
\(8\) We keep a small amount of the \( \dot{H}^1 \) norm for later use.
Consequently, we have
\[
\int_0^T \left[ |\nabla \tilde{U}|^2 + \frac{\kappa}{2} |\nabla \tilde{V}|^2 \right] dt \geq \frac{1}{C} \int_0^T \tilde{N}(t) \left( \frac{|A|}{L} \right)^3 \left[ |\nabla U|^2 + \frac{\kappa}{2} |\nabla V|^2 \right] dx dt
\]
which is, via the Gagliardo-Nirenberg inequality for a single function, bounded from below by
\[
c\varepsilon \int [|	ilde{U}|^3 + |	ilde{V}|^3] dx \geq c\varepsilon \int_{|x| \leq \frac{1}{2N(t)}} [U]^3 + |V|^3 dx.
\]

Consequently, we have
\[
\int_0^T \left[ (6.5) + (6.7) \right] dt \geq \frac{1}{C} \int_0^T \tilde{N}(t) \left( \frac{|A|}{L} \right)^3 \left[ |\nabla U|^2 + \frac{\kappa}{2} |\nabla V|^2 \right] dx dt
\]
for some \( C = C(\eta_0, \kappa) \gg 1 \).

We next observe that \((6.6) \geq 0\). This follows from the fact that the matrix \((B_{jk}(x))_{1 \leq j, k \leq 4}\) is non-negative for any \( x \).
All the remaining terms are considered as error terms. Among them, \((6.8) + (6.9)\) is easy to handle. In fact, we see
\[
\left| \int_0^T (6.8) \, dt \right| \leq 3 \int_0^T \tilde{N}(t) \int_{|x| \geq \frac{1}{N(t)}} \left[ |U|^3 + |V|^3 \right] \, dx \, dt,
\]
\[
\left| \int_0^T (6.9) \, dt \right| \leq \frac{C(\kappa)}{L^2} \int_0^T \tilde{N}(t)^3 \, dt.
\]

Let us turn to the control of \((6.4)\) including the time derivative of \(\tilde{N}(t)\). By the Cauchy-Schwarz inequality, for \(\epsilon > 0\) we have
\[
\left| \int_0^T (6.4) \, dt \right| \leq 2L \int_0^T \frac{\tilde{N}(t)}{N(t)} \int \theta \left( \frac{|A|}{L} \right) \left[ |U| |\nabla U| + \frac{1}{2} |V| |\nabla V| \right] \, dx \, dt
\]
\[
\leq \epsilon \int_0^T \tilde{N}(t) \int \theta \left( \frac{|A|}{L} \right) \left[ |\nabla U|^2 + \frac{\kappa}{2} |\nabla V|^2 \right] \, dx \, dt + C(\kappa) \frac{L^2}{\epsilon} \int_0^T \frac{\tilde{N}(t)^2}{N(t)^3} \, dt.
\]

Taking \(\epsilon = \epsilon(\eta_0, \kappa) > 0\) sufficiently small, together with the estimates obtained so far and \(\tilde{N}(t) \leq N(t)\), we come to
\[
\int_0^T \left[ (6.4) + \cdots + (6.9) \right] \, dt
\]
\[
\geq \frac{1}{C} \int_0^T \tilde{N}(t) \int_{|x| \leq \frac{1}{N(t)}} \left[ |U|^3 + |V|^3 \right] \, dx \, dt - 5 \int_0^T \tilde{N}(t) \int_{|x| \geq \frac{1}{N(t)}} \left[ |U|^3 + |V|^3 \right] \, dx \, dt
\]
\[
- \frac{C}{T^2} \int_0^T \tilde{N}(t)^3 \, dt - CL^2 \int_0^T \frac{\tilde{N}(t)^2}{N(t)^3} \, dt
\]
\[
\geq \frac{1}{C} \int_0^T \tilde{N}(t) \int_{|x| \leq \frac{1}{N(t)}} \left[ |U|^3 + |V|^3 \right] \, dx \, dt - 5 \int_0^T \tilde{N}(t) \int_{|x| \geq \frac{1}{N(t)}} \left[ |U|^3 + |V|^3 \right] \, dx \, dt
\]
\[
- \frac{C}{T^2} \int_0^T \tilde{N}(t)N(t)^2 \, dt - CL^2 \int_0^T \frac{\tilde{N}(t)^2}{N(t)^3} \, dt
\]
for some \(C = C(\eta_0, \kappa) \gg 1\).

Now, we deduce from the almost periodicity and Lemma 3.24 that
\[
\int_{J_k} \int_{|x| \leq \frac{1}{N(t)}} \left[ |U|^3 + |V|^3 \right] \, dx \, dt \sim \int_{J_k} \int_{\mathbb{R}^4} \left[ |u|^3 + |v|^3 \right] \, dx \, dt = 1
\]

for sufficiently large \(K, L\), uniformly in \(k \geq 0\), and
\[
\sup_k \int_{J_k} \int_{|x| > \frac{1}{N(t)}} \left[ |U|^3 + |V|^3 \right] \, dx \, dt = o(1) \quad (K, L \to \infty).
\]

Since we are assuming \(\tilde{N}(t) \sim \tilde{N}(t_k)\) for \(t \in J_k\) uniformly in \(k \geq 0\), it follows from Corollary 3.26 that
\[
\int_0^T \tilde{N}(t) \int_{|x| \leq \frac{1}{N(t)}} \left[ |U|^3 + |V|^3 \right] \, dx \, dt \sim \sum_{k; J_k \subset [0,T]} \tilde{N}(t_k) \sim \int_0^T \tilde{N}(t)N(t)^2 \, dt
\]
for sufficiently large \(K, L\) and
\[
\int_0^T \tilde{N}(t) \int_{|x| > \frac{1}{N(t)}} \left[ |U|^3 + |V|^3 \right] \, dx \, dt = \int_0^T \tilde{N}(t)N(t)^2 \, dt \cdot o(1) \quad (K, L \to \infty).
\]
Lemma 6.3. Let \( \tilde{N} : [0, T) \to \mathbb{R}_+ \) be a \( C^1 \) function satisfying (6.2). Then, there exist constants \( C > 1 \) and \( L \gg 1 \) depending only on \((u, v)\) and \( \kappa \) such that we have
\[
\int_0^T \left[ (6.4) + \cdots + (6.9) \right] dt \geq \frac{1}{C} \int_0^T \tilde{N}(t)N(t)^2 dt - C \int_0^T \frac{\tilde{N}'(t)^2}{\tilde{N}(t)^3} dt \tag{6.12}
\]
whenever \( K > 0 \) is sufficiently large.

From now on we fix such an \( L \gg 1 \). Our next task is to find an appropriate function \( \tilde{N}(t) \) satisfying (6.2) and
\[
\int_0^T \frac{\tilde{N}'(t)^2}{\tilde{N}(t)^3} dt \ll \int_0^T \tilde{N}(t)N(t)^2 dt. \tag{6.13}
\]
This is the key step in the proof of Theorem 6.7, and we follow the argument of Dodson [5]. To see the idea, we observe that (6.2) and Corollary 3.26 imply
\[
L \int \left[ \tilde{N}'(t)^2 \right] dt \lesssim \sum_{k; J_k \subset [0, T)} \frac{1}{N(t_k)^3} \frac{\tilde{N}(t_k)}{|J_k|} \int_{J_k} |\tilde{N}'(t)| dt
\]
\[
\sim \sum_{k; J_k \subset [0, T)} \frac{N(t_k)^2}{\tilde{N}(t_k)^2} |\tilde{N}(t_k) - \tilde{N}(t_{k+1})|.
\]
Let us consider the case \( \tilde{N}(t) \sim N(t) \). (In fact, we can easily construct \( \tilde{N}(t) \) satisfying (6.2) and \( C_0^{-1} \leq \tilde{N}(t)/N(t) \leq 1 \).) We have
\[
\sum_{k; J_k \subset [0, T)} \frac{N(t_k)^2}{\tilde{N}(t_k)^2} |\tilde{N}(t_k) - \tilde{N}(t_{k+1})| \lesssim \sum_{k; J_k \subset [0, T)} N(t_k)
\]
\[
\sim \int_0^T N(t)^3 dt \sim \int_0^T \tilde{N}(t)N(t)^2 dt,
\]
which is not sufficient to conclude (6.13). However, if we consider the extremal case where \( N(t) \) is monotone on the whole interval \([0, T)\), then we can construct \( \tilde{N}(t) \) which is also monotone on \([0, T)\) and
\[
\sum_{k; J_k \subset [0, T)} \frac{N(t_k)^2}{\tilde{N}(t_k)^2} |\tilde{N}(t_k) - \tilde{N}(t_{k+1})| \lesssim \tilde{N}(0) - \tilde{N}(T) \leq 1 \ll K \sim \int_0^T \tilde{N}(t)N(t)^2 dt,
\]
since we can take \( K \) arbitrarily large. This observation suggests that (6.13) is easier to achieve if \( N(t) \) is less oscillatory. The idea for constructing \( \tilde{N}(t) \) is that we deform \( N(t) \) to be less undulating by leveling ‘peaks’ in the graph of \( N(t) \).

Lemma 6.4. There exists a sequence of \( C^1 \) functions \( \tilde{N}_m : [0, \infty) \to \mathbb{R} \) \((m = 0, 1, 2, \ldots)\) such that each \( \tilde{N}_m \) satisfies (6.2) and
\[
\sup_{t \geq 0} \frac{N(t)}{N_m(t)} \leq C_0^{m+2},
\]
\[
\sum_{k; J_k \subset [0, T)} \frac{N(t_k)}{N_m(t_k)^2} |\tilde{N}_m(t_k) - \tilde{N}_m(t_{k+1})| \lesssim \left( \frac{C_0}{K} + \frac{1}{m+1} \right) \int_0^T \tilde{N}_m(t)N(t)^2 dt
\]
for any \( T > 0 \).
Proof. Recall that $N(t)$ is a step function associated with the partition $\{J_k\}_{k=0}^{\infty}$ of $[0, \infty)$ and satisfies (3.26). Define the $C_0^{2-\delta}$-valued step functions $N_m(t)$ ($m = 0, 1, 2, \ldots$) associated with $\{J_k\}$ inductively in $m$ as follows: Let $N_0(t) := N(t)$. Then, for given $N_m(t)$, define $N_{m+1}(t)$ by

$$
N_{m+1}(t_k) := \begin{cases} 
C_0^{-1}N_m(t_k) & \text{if } J_k \subset \text{a peak in } N_m, \\
N_m(t_k) & \text{otherwise},
\end{cases}
$$

for $k = 0, 1, 2, \ldots$.

Then, the total variation of $N_{m+1}(t)$ is estimated as

$$
\sum_{k=0}^{\infty} N_{m+1}(t_k) 1_{J_k}(t).
$$

where for a positive integer $l$ we call a union of consecutive $l$ characteristic intervals $[t_{k_0}, t_{k_0+l})$ a peak of length $l$ in $N_m$ if $N_m(t) = N_m(t_{k_0})$ on $[t_{k_0}, t_{k_0+l})$ and $N_m(t_{k_0-1}) = N_m(t_{k_0}) = C_0^{-1}N_m(t_{k_0})$. It is easily verified that $C_0^{1-l}N_m(t) \leq N_m(t) \leq C_0^{-1}N_m(t)$, and $N_m(t_{k_0})/N_m(t_{k_0}) \in \{C_0^{-1}, 1, C_0\}$ for any $m, k \geq 0$. Moreover, we claim that: (i) Every peaks in $N_m$ has length $\geq 2m + 1$; (ii) If $N_m(t_k) \neq N(t_k)$, then $N_m(t_k) = N_m(t_{k+1})$. (i) follows from the definition. To verify (ii), assume $N_m(t_k) \neq N(t_k)$ for some $m, k \geq 0$. Then, there exists $0 < m' \leq m$ such that $N_m'(t_k) = C_0^{-1}N_m(t_k)$, i.e., $J_k$ is included in a peak of $N_m', \text{ which implies } N_m'(t_k) = N_m'(t_{k+1}).$ We see that this coincidence of the value of $N_m'$ at consecutive points $t_k, t_{k+1}$ persists throughout the construction procedure of $\{N_m\}$.

Let $k_*$ be a positive integer, and let $\{(t_{k_{j}}, t_{k_{j}+l_{j}})\}_{j=1}^{j_{\cdot}(m, k_{\cdot})}$ be the set of all peaks of $N_m$ included in the interval $[0, t_{k_{\cdot}})$. ($j_{\cdot}(m, k_{\cdot})$ is the number of peaks in $N_m$ before $t_{k_{\cdot}}$, and for the $j$-th peak in $N_m$ we denote by $k_{j}$ and $l_{j}$ the index of the beginning characteristic interval and the length, respectively. Note that this set is possibly empty.) Then, the total variation of $N_m$ on $[0, t_{k_{\cdot}}]$ is estimated as

$$
\sum_{k=0}^{k_{\cdot}-1} |N_m(t_k) - N_m(t_{k+1})| \leq N_m(0) + N_m(t_{k_{\cdot}}) + 2 \sum_{j=1}^{j_{\cdot}(m, k_{\cdot})} N(t_{k_{j}}).
$$

Hence, the properties (i), (ii) imply that

$$
\sum_{k=0}^{k_{\cdot}} \frac{N(t_k)}{N_m(t_k)} |N_m(t_k) - N_m(t_{k+1})| \leq 2 + 2 \sum_{j=1}^{j_{\cdot}(m, k_{\cdot})} N(t_{k_{j}}) \leq 2 + 2 \sum_{k=0}^{k_{\cdot}} N_m(t_k) \leq 1 + \frac{1}{m+1} \int_0^{t_{k_{\cdot}}} N_m(t)N(t)^2 \, dt.
$$

Finally, we can construct a $C^1$ function $\tilde{N}_m(t)$ on $[0, \infty)$ satisfying (6.2) such that $C_0^{-2}N_m(t) \leq \tilde{N}_m(t) \leq N_m(t)$ and $\tilde{N}_m(t_k) = C_0^{-1}N_m(t_k)$ ($k \geq 0$). (For instance, it suffices to connect the points $\{(t_k, N_m(t_k))\}_{k \geq 0}$ on the graph smoothly and multiply it by $C_0^{-1}$.) This $\tilde{N}_m$ also satisfies

$$
\sum_{k; J_k \subset [0, T)} \frac{N(t_k)^2}{\tilde{N}_m(t_k)^2} |\tilde{N}_m(t_k) - \tilde{N}_m(t_{k+1})| \leq C_0^{3} \sum_{k; J_k \subset [0, T)} \frac{N(t_k)^2}{\tilde{N}_m(t_k)^2} |N_m(t_k) - N_m(t_{k+1})|.
$$
which is also deduced from Theorem 4.1 as
\[ N(t) \leq C(\varepsilon) \leq C(\varepsilon(t)) \]
for any \( T > 0 \), as desired.

From Lemma 6.4 we deduce the following results.

**Corollary 6.5.** For any \( \varepsilon > 0 \) there exists \( C(\varepsilon) > 0 \) and a \( C^1 \) function \( \tilde{N}_\varepsilon : [0, \infty) \rightarrow \mathbb{R}_+ \) satisfying (6.2) such that
\[
\sup_{t \geq 0} \frac{N(t)}{\tilde{N}_\varepsilon(t)} \leq C(\varepsilon),\quad \int_0^T \frac{\tilde{N}_\varepsilon(t)^2}{N(t)^3} dt \leq \left( \varepsilon + \frac{C(\varepsilon)}{K} \right) \int_0^T \tilde{N}_\varepsilon(t)N(t)^2 dt
\]
for any \( T > 0 \).

Now we choose \( \varepsilon = \frac{1}{4K^2} \) with the constants \( C \) given in (6.12), and fix the function \( \tilde{N} \) to be \( \tilde{N}_\varepsilon \) given in Corollary 6.5. Then, whenever \( K \) is sufficiently large and \( C(\varepsilon) \leq \varepsilon K \), we have
\[
\int_0^T \left[ (6.4) + \cdots + (6.9) \right] dt \geq \frac{K}{2C \cdot C(\varepsilon)}. \quad (6.14)
\]

All we have to do is the estimate for the error term (6.10) arising from frequency localization by \( P_{\leq K} \). To conclude the proof of Theorem 6.2, we claim that
\[
\int_0^T (6.10) dt = o(K) \quad (K \rightarrow \infty). \quad (6.15)
\]

The long-time Strichartz estimate (Theorem 4.1) now plays an essential role, since each of \( F, G \) contains at least one high-frequency function \( P_{> K/4} \) for instance,
\[
F := P_{\leq K}(\overline{uv}) - P_{\leq K}uP_{\leq K}v = \overline{uv} - \overline{P_{\leq K}uP_{\leq K}v} - P_{> K}(\overline{uv}) = P_{> K}u\overline{v} + P_{\leq K}\overline{uP_{> K}v} - P_{> K}(P_{> K}(\overline{uP_{> K}v}) - P_{> K}(P_{\leq K}\overline{uP_{> K}v}).
\]

First, we see that
\[
\|P_{> K/4}(u, v)\|_{L^2L^4([0, T] \times \mathbb{R}^4)} = o(1), \quad K \rightarrow \infty, \quad (6.16)
\]
which is a direct consequence of Theorem 4.1. We prepare one more estimate:
\[
\||\nabla|^s P_{\leq K}(u, v)\|_{L^2L^4([0, T] \times \mathbb{R}^4)} \lesssim K^s \quad \text{for } s > 1/2, \quad (6.17)
\]
which is also deduced from Theorem 4.1 as
\[
\||\nabla|^s P_{\leq K}(u, v)\|_{L^2L^4} \lesssim \sum_{N < K} N^s \|P_{> N}(u, v)\|_{L^2L^4} \lesssim \sum_{N < K} N^s \left( \frac{K}{N} \right)^{1/2} \lesssim K^s.
\]

To verify (6.15), we begin with observing that
\[
\int_0^T (6.10) dt = 2 \int_0^T \left[ \Re \left( \theta \right) \right] A_j \left( \frac{|A_j|}{L} \right) \frac{1}{2} \overline{G} \partial_j V dx dt
\]
\[
+ \int_0^T \tilde{N}(t) \int \left( \theta + 3\theta \right) \left( \frac{|A_j|}{L} \right) \Re \left[ \overline{F} U + \frac{1}{2} \overline{G} V \right] dx dt.
\]

\[\text{This is the only point where we exploit the decaying factor } \rho(N) \text{ in the long-time Strichartz estimate.}\]
so that \( K = 6.2 \). For this subsection, we use the Fourier projection operator \( P \) and introduce the interaction-type modification of \( \kappa \) invariance of the system and thus restrict ourselves to the mass-resonance case \( \theta \) as before. First, we define the smooth radial function \( \Theta(t) = 1 \) for \( 0 \leq t \leq N(t) \) and \( \Theta(0) = 1 \) for \( t > N(t) \) by

\[
\Theta(t) = \frac{N(t)}{N(t) + 1} + \frac{1}{N(t) + 1}.
\]

Noticing \( \Theta(r) \leq 2/r \) and using (6.16)–(6.17), we estimate the first integral by

\[
4L\| (F, G) \|_{L^2 L^{4/3}} \| \nabla (U, V) \|_{L^2 L^4} \\
\lesssim \| P_{> K/4} (u, v) \|_{L^2 L^4} \| (u, v) \|_{L^\infty L^2} \| \nabla P_{\leq K} (u, v) \|_{L^2 L^4} = o(K).
\]

The second integral is estimated by the Sobolev embedding and (6.16)–(6.17) as

\[
4\left( \int_0^T \bar{N}(t)^{1/3} \| (F, G) \|_{L^2 L^{4/3}} \| (U, V) \|_{L^6 L^4} \right) \\
\lesssim K^{1/3} \| P_{> K/4} (u, v) \|_{L^2 L^4} \| (u, v) \|_{L^\infty L^2} \| P_{\leq K} (u, v) \|_{L^2 L^4}^{2/3} \| P_{\leq K} (u, v) \|_{L^2 L^4}^{1/3} \\
\lesssim o(K^{1/3}) \cdot \| \nabla \|^{2/3} P_{\leq K} (u, v) \|_{L^2 L^2}^{2/3} \| \nabla \|^{1/3} P_{\leq K} (u, v) \|_{L^2 L^2} = o(K).
\]

This is the end of the proof of Theorem 6.2.

In view of Theorem 6.2, the quasi-soliton scenario is precluded once we show

\[
\sup_{t \in [0, T]} | \mathcal{M}(t) | = o(K), \quad K \to \infty.
\]

Now, by the almost periodicity of \( (u, v) \) and the fact \( N(t) \leq 1 \), we have

\[
\| \nabla P_{\leq K} (u, v) \|_{L^\infty L^2} \lesssim K^{1/2} \| P_{\leq K^{1/2}} (u, v) \|_{L^\infty L^2} + K \| P_{> K^{1/2}} (u, v) \|_{L^\infty L^2} = o(K)
\]

as \( K \to \infty \). This and the boundedness of the weight function imply (6.18).

So far, the proof of Theorem 1.5 has been completed.

### 6.2. Non-radial, mass-resonance case

In non-radial case, we need the Galilean invariance of the system and thus restrict ourselves to the mass-resonance case \( \kappa = 1/2 \).

To treat the situation where the spatial center function \( x(t) \) varies, we follow [5] and introduce the interaction-type modification of \( \mathcal{M}(t) \) of the form

\[
\mathcal{M}(t) := \int \int_{\mathbb{R}^4 \times \mathbb{R}^4} \Theta_L (\bar{N}(t) |x - y|) \bar{N}(t)(x - y) \cdot \text{Im} \left[ U \nabla U + \frac{1}{2} \nabla V \right] (t, x) \\
\times \left[ |U|^2 + |V|^2 \right](t, y) \, dx \, dy
\]

on some time interval \([0, T]\) which is a union of characteristic intervals. \( \bar{N} : [0, \infty) \to \mathbb{R}_+ \) is the same \( C^1 \) function as we used in the radial case, which satisfies (6.2). In this subsection, we use the Fourier projection operator \( P_{\leq C, K} \) instead of \( P_{\leq K} \), and so \((U, V) := P_{\leq C, K} (u, v)\), where the constant \( C_* = C_*(u, v) > 0 \) is given in (3.27) so that

\[
|\xi(t)| \leq 2^{-10} C_* \sum_{k : J_k \subset [0, t]} \int_{T_k} N(t)^3 \, dt \leq 2^{-10} C_* K, \quad t \in [0, T).
\]

Definition of the weight function \( \Theta_L \) is quite different from the radial case. Let \( L \) be a large positive number to be chosen later, and let \( \theta : [0, \infty) \to [0, 1] \) be the same as before. First, we define the smooth radial function \( \vartheta_L : \mathbb{R}^4 \to [0, 1] \) by

\[
\vartheta_L (x) := \theta(\max\{0, |x| - L + 2\}), \quad x \in \mathbb{R}^4.
\]

Note that \( \vartheta_L \) is non-increasing in \( |x| \) and satisfies \( \vartheta_L (x) = 1 \) for \( 0 \leq |x| \leq L - 1 \) and \( \vartheta_L (x) = 0 \) for \( |x| \geq L \). Then, define the functions \( \vartheta_L, \Theta_L : [0, \infty) \to [0, \infty) \) by

\[
\vartheta_L (r) := \int_{\mathbb{R}^4} \vartheta_L (r e_1 - z) \vartheta_L (z) \frac{dz}{|z|^4}, \quad \Theta_L (r) := \frac{1}{r} \int_0^r \vartheta_L (s) \, ds \quad (\Theta_L (0) := \vartheta_L (0)),
\]

SCATTERING FOR MASS CRITICAL NLS SYSTEM 6335
where \( e_1 := (1, 0, 0, 0) \in \mathbb{R}^4 \). It is worth noticing that, since \( \vartheta_L \) is radially symmetric, \( \theta_L \) satisfies
\[
\theta_L(|x-y|) = \int_{\mathbb{R}^4} \vartheta_L(x-z) \vartheta_L(y-z) \frac{dz}{L^4}, \quad x, y \in \mathbb{R}^4.
\] (6.19)

This helps us separate two spatial variables in the analysis of the interaction-type quantity \( \mathcal{M}(t) \).

It follows from the definition that \( \vartheta_L \) and \( \Theta_L \) are non-negative, bounded uniformly in \( L \), smooth (on \((0, \infty)\), for \( \Theta_L \)) functions such that \( \vartheta_L \equiv 0 \) outside \([0, 2L]\) and \( \Theta_L(r) \lesssim \min\{1, L/r\} \). Moreover, we can show that:

**Lemma 6.6.** The following holds.

(i) \( \theta_L \) is non-increasing. In particular, \( \Theta_L \geq \vartheta_L \geq 0 \).

(ii) \( |\theta_L'(r)| \lesssim \min\{1/L, r/L\} \), \( |\theta_L''(r)| \lesssim 1/L \).

(iii) \( \Theta_L \) is non-increasing and
\[
0 \leq -\Theta_L'(r) = \frac{\Theta_L(r) - \vartheta_L(r)}{r} \lesssim \min\left\{\frac{L}{r^2}, \frac{1}{L}, \frac{r}{L}\right\}.
\]

**Proof.** (i) Although this is intuitively obvious, the proof is rather long. We see that
\[
\theta_L'(r) = \int (\partial_{x_1} \partial_L)(r e_1 - z) \vartheta_L(z) \frac{dz}{L^4} = \int \int (\partial_{x_1} \partial_L)(\zeta, -z') \vartheta_L(r - \zeta, z') \, d\zeta \, \frac{dz'}{L^4},
\]
\[
= \int \left( \int_0^\infty (\partial_{x_1} \partial_L)(\zeta, -z') \vartheta_L(r - \zeta, z') \, d\zeta \right. \\
\left. + \int_0^\infty (\partial_{x_1} \partial_L)(-\eta, -z') \vartheta_L(r + \eta, z') \, d\eta \right) \frac{dz'}{L^4}.
\]

Observe that if \( \varrho \) is an even function on \( \mathbb{R} \) which is non-increasing in \([0, \infty)\), then it holds that \( \varrho(t_0+t) \leq \varrho(t_0-t) \) for any \( t_0, t \geq 0 \).\(^{10}\) We apply this to the second integral of the above with \( \varrho = \vartheta_L(\cdot, z') \), \( t_0 = r \) and \( t = \eta \). Note that \((\partial_{x_1} \vartheta_L)(-\eta, -z') \geq 0 \) for any \( \eta > 0 \) and \( z' \in \mathbb{R}^4 \), because \( \vartheta_L \) is radially symmetric and non-increasing in the radial direction. Then, we obtain
\[
\theta_L'(r) \leq \int \left( \int_0^\infty (\partial_{x_1} \vartheta_L)(\zeta, -z') \vartheta_L(r - \zeta, z') \, d\zeta \\
+ \int_0^\infty (\partial_{x_1} \vartheta_L)(-\eta, -z') \vartheta_L(r + \eta, z') \, d\eta \right) \frac{dz'}{L^4}.
\]

Since \((\partial_{x_1} \vartheta_L)(\cdot, -z')\) is an odd function, the right hand side is equal to zero, and thus \( \theta_L \) is non-increasing.

(ii) We observe that
\[
\theta_L'(r) = \int (\partial_{x_1} \vartheta)((r, 0, 0, 0) - z) \vartheta(z) \frac{dz}{L^4}, \quad \theta_L''(r) = \int (\partial_{x_2}^2 \vartheta)((r, 0, 0, 0) - z) \vartheta(z) \frac{dz}{L^4}
\]
are of \( O(L^{-1}) \) uniformly in \( r \), since \( |\text{supp } \partial_{x_1} \vartheta| \lesssim L^3 \). Integrating the second quantity with respect to \( r \) we obtain \( |\theta_L''(r)| \lesssim r/L \), since \( \theta_L'(0) = \int (\partial_{x_1} \vartheta)((r, 0, 0, 0) - z) \vartheta(z) \frac{dz}{L^4} = 0 \).

\(^{10}\) If \( 0 \leq t \leq t_0 \), then \( 0 \leq t_0 - t \leq t_0 + t \), which implies \( \varrho(t_0 + t) \leq \varrho(t_0 - t) \). If \( t > t_0 \), then \( 0 < t - t_0 \leq t_0 + t \), so we have \( \varrho(t_0 + t) \leq \varrho(t - t_0) = \varrho(t_0 - t) \).
We now compute \( \partial_t \mathcal{M}(t) \). Derivatives of the weight functions are given by

\[
\partial_t \left[ \Theta_L(N(t)|x|) \tilde{N}(t)x_j \right] = \theta(N(t)|x|) \tilde{N}'(t)x_j,
\]
\[
\partial_k \left[ \Theta_L(N(t)|x|) \tilde{N}(t)x_j \right] = \left[ \theta(L(N(t)|x|) \delta_{jk} + (\Theta_L - \theta_L)(\tilde{N}(t)|x|) \left( \delta_{jk} - \frac{x_j x_k}{|x|^2} \right) \right] \tilde{N}(t),
\]
\[
\partial_j \left[ \Theta_L(N(t)|x|) \tilde{N}(t)x_j \right] = \left[ 4\theta_L(\tilde{N}(t)|x|) + 3(\Theta_L - \theta_L)(\tilde{N}(t)|x|) \right] \tilde{N}(t),
\]
and the time derivative of mass density is

\[
\partial_t (|U|^2 + |V|^2) = -2 \partial_j \left[ \overline{U} \partial_j U + \frac{1}{2} \nabla \partial_j V \right] - 2 \{(U, V), (F, G)\}_m,
\]
\[
\{(U, V), (F, G)\}_m := \text{Im}(\overline{U} \nabla U + \frac{1}{2} \nabla \overline{V} \nabla V).
\]

Then, introducing the notation

\[
m[U, V] := |U|^2 + |V|^2, \quad p[U, V] := \text{Im} \left[ \overline{U} \nabla U + \frac{1}{2} \nabla \overline{V} \nabla V \right],
\]
\[
e_2[U, V] := |\nabla U|^2 + \frac{1}{4} |\nabla V|^2, \quad e_3[U, V] := \text{Re} [U^2 \overline{V}],
\]
we write down \( \partial_t \mathcal{M}(t) \) as

\[
\partial_t \mathcal{M}(t)
\]
\[
= \tilde{N}'(t) \int \theta_L(|A|)(x - y) \cdot p[U, V](t, x) m[U, V](t, y) \, dx \, dy + 2 \tilde{N}(t) \int \theta_L(|A|) e_2[U, V](t, x) m[U, V](t, y) \, dx \, dy
\]
\[
+ 2 \tilde{N}(t) \int (\Theta_L - \theta_L)(|A|) B_{jk} \text{Re} \left[ \partial_j U \overline{\partial_k U} + \frac{1}{4} \partial_j V \overline{\partial_k V} \right] (t, x) m[U, V](t, y) \, dx \, dy
\]
\[
+ 2 \tilde{N}(t) \int \theta_L(|A|) e_3[U, V](t, x) m[U, V](t, y) \, dx \, dy
\]
\[
+ \frac{3}{2} \tilde{N}(t) \int (\Theta_L - \theta_L)(|A|) e_3[U, V](t, x) m[U, V](t, y) \, dx \, dy
\]
\[
- \frac{1}{2} \tilde{N}(t) \int \Delta_x \left[ (\theta_L + 3\Theta_L)(|A|) \right] \left[ |U|^2 + \frac{1}{4} |V|^2 \right] (t, x) m[U, V](t, y) \, dx \, dy
\]
+ \tilde{N}(t) \int \Theta_L(|A|)(x - y) \cdot \{ (U, V), (F, G) \} \varphi(t, x) \mathbf{m}[U, V](t, y) \, dx \, dy \quad (6.26)
- 2\tilde{N}(t) \int \theta_L(|A|) p[U, V](t, x) \cdot p[U, V](t, y) \, dx \, dy \quad (6.27)
- 2\tilde{N}(t) \int (\Theta_L - \theta_L)(|A|) B_{jk} p_j[U, V](t, x) p_k[U, V](t, y) \, dx \, dy \quad (6.28)
- 2\tilde{N}(t) \int \Theta_L(|A|)(x - y) \cdot p[U, V](t, x) \{ (U, V), (F, G) \} \varphi(t, y) \, dx \, dy, \quad (6.29)

where in this subsection we define $A := \tilde{N}(t)(x - y)$, $B_{jk} := \delta_{jk} - \frac{(x - y)_k(x - y)_j}{|x - y|^2}$.

Our goal is to establish the following:

**Theorem 6.7.** There exists a large constant $L > 0$ and a $C^1$ function $\tilde{N} : [0, \infty) \to \mathbb{R}_+$ satisfying (6.2) such that $\mathcal{M}(t)$ defined as above satisfies
\[
\int_0^T \frac{d\mathcal{M}}{dt}(t) \, dt \geq K
\]
for any sufficiently large $T$.

**Proof.** The main part of $\theta_t \mathcal{M}(t)$ will be (6.21)+(6.23)+(6.27), which corresponds to (6.5)+(6.7) in the radial case. The new term (6.27) can be concealed by an appropriate gauge transformation. To see this, we write $\theta_L(|A|)$ in the integral form (6.19), then
\[
(6.21) + (6.23) + (6.27) = 2\tilde{N}(t) \int_{\mathbb{R}^4} \mathcal{E}[U, V](t, z) \frac{dz}{L^4},
\]
where
\[
\mathcal{E}[U, V](t, z) := \int_{\mathbb{R}^4} \vartheta_L(\tilde{N}(t)x - z) (\varepsilon_2 + \varepsilon_3) [U, V](t, x) \, dx \int \vartheta_L(\tilde{N}(t)x - z) \mathbf{m}[U, V](t, x) \, dx
- \left| \int \vartheta_L(\tilde{N}(t)x - z) p[U, V](t, x) \, dx \right|^2.
\]

We observe that the gauge transformation $(U, V) \mapsto (e^{-ix\xi_0 U}, e^{-2ix\xi_0 V})$, $\xi_0 \in \mathbb{R}^4$ keeps the densities $\mathbf{m}, \varepsilon_3$ invariant and changes $p, \varepsilon_2$ as
\[
p[e^{-ix\xi_0 U}, e^{-2ix\xi_0 V}] = p[U, V] - \xi_0 \mathbf{m}[U, V],
\]
\[
\varepsilon_2[e^{-ix\xi_0 U}, e^{-2ix\xi_0 V}] = \varepsilon_2[U, V] - 2\xi_0 \cdot p[U, V] + |\xi_0|^2 \mathbf{m}[U, V]. \quad (6.30)
\]

From this, we see that the quantity $\mathcal{E}[U, V](t, z)$ (for fixed $t, z$) is invariant under such gauge transformations.\footnote{This is not true if $\kappa \neq 1/2$. Hence, it seems difficult to deal with the term (6.27) without assuming $\kappa = 1/2$.} We choose $\xi_0 = \xi_0(t, z)$ for $(t, z) \in [0, T) \times \mathbb{R}^4$ so that the last term in $\mathcal{E}[U, V](t, z)$ will vanish;
\[
\xi_0(t, z) := \frac{\int \vartheta_L(\tilde{N}(t)x - z) p[U, V](t, x) \, dx}{\int \vartheta_L(\tilde{N}(t)x - z) \mathbf{m}[U, V](t, x) \, dx}. \quad (6.31)
\]
and $\xi_0(t, z) := 0$ if the denominator vanishes (in this case the numerator is also equal to zero). We thus consider the estimate on

$$2\tilde{N}(t) \int \left[ \int \vartheta_L(\tilde{N}(t)x - z) (\epsilon_2 + \epsilon_3)[U_*, V_*](t, x, z) \, dx \right] \frac{dz}{L^4}$$

which is a smooth radial function, non-increasing in $0 \leq |x|$. Following the previous argument, we have

$$\tilde{N}(t) \int \left[ \int \vartheta_L(\tilde{N}(t)x - z) \epsilon_2[U_*, V_*] \, dx \right] \frac{dz}{L^4}$$

$$= 2\varepsilon \tilde{N}(t) \int \left[ \int \vartheta_L(\tilde{N}(t)x - z) \epsilon_2[U_*, V_*] \, dx \right] \frac{dz}{L^4}$$

$$= 2\varepsilon \tilde{N}(t) \int \left[ \int \vartheta_L(\tilde{N}(t)x - z) \left( \epsilon_2 + \epsilon_3 \right)[U_*, V_*] \, dx \right] \frac{dz}{L^4},$$

where $(U_*, V_*)(t, x, z) := (e^{-ix \cdot \xi_0(t, z)} U(t, x), e^{-2ix \cdot \xi_0(t, z)} V(t, x))$ and $\varepsilon > 0$ is a small constant.

As we did in the radial case, we introduce another cutoff $\chi_L : \mathbb{R}^4 \rightarrow [0, 1]$ by

$$\chi_L(x) := \theta(\max(0, |x| - L + 3)),$$

which is a smooth radial function, non-increasing in $|x|$, satisfies $\chi_L(x) = 1$ for $0 \leq |x| \leq L - 2$, $\chi_L(x) = 0$ for $|x| \geq L - 1$, so that $\vartheta_L \equiv 1$ on the support of $\chi_L$. Following the previous argument, we have

$$\int \vartheta_L(\tilde{N}(t)x - z) (1 - \varepsilon) \epsilon_2 + \epsilon_3[U_*, V_*](t, x, z) \, dx$$

$$\geq \int \left[ \int \frac{1 - \varepsilon}{1 + \varepsilon} \epsilon_2 + \epsilon_3[\tilde{U}_*, \tilde{V}_*](t, x, z) \, dx \right]$$

$$- 1 - \varepsilon \tilde{N}(t)^2 \int |(\nabla \chi_L)(\tilde{N}(t)x - z)|^2 \left[ |U|^2 + \frac{1}{4} |V|^2 \right](t, x) \, dx$$

$$- \int [\vartheta_L(1 - \chi_L^2)](\tilde{N}(t)x - z) [[U]^3 + [V]^3](t, x) \, dx,$$

where $(\tilde{U}_*, \tilde{V}_*) := \chi_L(\tilde{N}(t)x - z)(U_*, V_*)$. By the sharp Gagliardo-Nirenberg inequality for the function $(\tilde{U}_*, \tilde{V}_*)(t, x, z)$ and

$$\frac{M(\tilde{U}_*(t, x, z), \tilde{V}_*(t, x, z))}{M(\phi, \psi)} \leq \frac{M_e}{M(\phi, \psi)} = \eta_0 \in (0, 1),$$

we can select $\varepsilon = \varepsilon(\eta_0) > 0$ so that

$$\int \left[ \int \frac{1 - \varepsilon}{1 + \varepsilon} \epsilon_2 + \epsilon_3[\tilde{U}_*, \tilde{V}_*](t, x, z) \, dx \right] \geq \varepsilon \int \chi_L^2(\tilde{N}(t)x - z) [[U]^3 + [V]^3](t, x) \, dx.$$

Therefore, we have the following lower bound:

$$(6.21) + (6.23) + (6.27)$$

$$\geq c(\eta_0) \tilde{N}(t) \int \left[ \int \vartheta_L(\tilde{N}(t)x - z) \epsilon_2[U_*, V_*](t, x, z) \, dx \right]$$

$$\times \int \vartheta_L(\tilde{N}(t)x - z) m[U, V](t, x) \, dx \frac{dz}{L^4}.$$
\[+ c(\eta_0)\bar{N}(t) \iint \left[ \int \chi_L^2(\bar{N}(t)x - z) \vartheta_L(\bar{N}(t)y - z) \frac{dz}{L^4} \right] \times \left[ |U|^3 + |V|^3 \right](t, x) m[U, V](t, y) \, dx \, dy - C(\eta_0)\bar{N}(t)^3 \iint \left[ \int |\nabla \chi_L|^2(\bar{N}(t)x - z) \vartheta_L(\bar{N}(t)y - z) \frac{dz}{L^4} \right] \times m[U, V](t, x) m[U, V](t, y) \, dx \, dy \]

\[- 2\bar{N}(t) \iint \left[ \int (\vartheta_L(1 - \chi_L^3))(\bar{N}(t)x - z) \vartheta_L(\bar{N}(t)y - z) \frac{dz}{L^4} \right] \times \left[ |U|^3 + |V|^3 \right](t, x) m[U, V](t, y) \, dx \, dy.\]

From the support property of \(\vartheta_L\) and \(\chi_L\), we see that if \(|x - y| \leq \frac{L - 2}{2\bar{N}(t)}\), then

\[\int \chi_L^2(\bar{N}(t)x - z) \vartheta_L(\bar{N}(t)y - z) \frac{dz}{L^4} \geq \int_{|z - \bar{N}(t)x| \leq (L - 2)/2} \frac{dz}{L^4} \gtrsim 1.\]

On the other hand, since the supports of \(|\nabla \chi_L|^2\) and \(\vartheta_L(1 - \chi_L^3)\) are of measure \(O(L^3)\), we have

\[
\int |\nabla \chi_L|^2(\bar{N}(t)x - z) \vartheta_L(\bar{N}(t)y - z) \frac{dz}{L^4} + \int (\vartheta_L(1 - \chi_L^3))(\bar{N}(t)x - z) \vartheta_L(\bar{N}(t)y - z) \frac{dz}{L^4} \lesssim \frac{1}{L}
\]

for any \(t, x, y\). Therefore, by \(\bar{N}(t) \leq N(t)\), we have

\[(6.21) + (6.23) + (6.27) \geq c(\eta_0)\bar{N}(t) \iint \left[ \int \vartheta_L(\bar{N}(t)x - z) \epsilon_2[U_*, V_*](t, x, z) \, dx \right] \times \left[ \int \vartheta_L(\bar{N}(t)x - z) \, m[U, V](t, x) \, dx \right] \frac{dz}{L^4} + c(\eta_0)\bar{N}(t) \int_{|x - \bar{N}(t)x| \leq \frac{L - 2}{2\bar{N}(t)}} \left[ |U|^3 + |V|^3 \right](t, x) \, dx \int_{|x - \bar{N}(t)x| \leq \frac{L - 2}{2\bar{N}(t)}} m[U, V](t, x) \, dx \]

\[- C(\eta_0)\frac{\bar{N}(t)^2}{L} \int_{\mathbb{R}^4} \left[ |U|^3 + |V|^3 \right](t, x) \, dx.
\]

For the second term on the right-hand side, we use \(|\xi(t)| \leq 2^{-10}C_* K\) and \(N(t) \leq 1\) to see that for \(q = 2, 3\),

\[\|(U, V)(t)\|_{L^q(|x - \xi(t)| \leq \frac{L - 2}{2\bar{N}(t)})} \geq \|(u, v)(t)\|_{L^q(\mathbb{R}^4)} - \|(u, v)(t)\|_{L^q(|x - \xi(t)| \leq \frac{L - 2}{2\bar{N}(t)})} - C\left\| \left( P_{|\xi - \xi(t)| > \frac{L - 2}{2\bar{N}(t)}} N(t) u(t) + P_{|\xi - 2\xi(t)| > \frac{L - 2}{2\bar{N}(t)}} N(t) v(t) \right) \right\|_{L^q(\mathbb{R}^4)},\]

which, combined with the almost periodicity and Lemma 3.24, implies

\[\inf_{t \in [0, T]} \int_{|x - \xi(t)| \leq \frac{L - 2}{2\bar{N}(t)}} m[U, V](t, x) \, dx \geq 1,
\]

\[\inf_{k \geq 0} \int_{k T} \int_{|x - \xi(t)| \leq \frac{L - 2}{2\bar{N}(t)}} \left[ |U|^3 + |V|^3 \right](t, x) \, dx \, dt \geq 1\]

for \(K\) and \(L\) sufficiently large. Hence, similarly to the radial case, we obtain the following:
Lemma 6.8. Let $\tilde{N} : [0, T) \to \mathbb{R}_+$ be a $C^1$ function satisfying (6.2). Then, there exist a constant $C > 1$ depending only on $(u, v)$ such that we have
\[
\int_0^T \left( (6.21) + (6.23) + (6.27) \right) dt
\geq \frac{1}{C} \int_0^T \tilde{N}(t) \int \left[ \int \partial_L (\tilde{N}(t)x - z) \epsilon_x[U_*, V_*(t,x, z)] dx \right.
\times \left. \int \partial_L (\tilde{N}(t)x - z) m[U, V](t,x, z) dx \right] \frac{dz}{L^4} dt
\]
for sufficiently large $K, L > 0$, where $(U_*, V_*) := (e^{-ix\xi_0 U}, e^{-2ix\xi_0 V})$ with $\xi_0(t, z)$ given in (6.31).

Next, we consider (6.22) + (6.28) with the matrix $B = (B_{jk})_{1 \leq j, k \leq 4}$, which corresponds to (6.6) in the radial case. It turns out that the new term (6.28) can be absorbed into the positive term (6.22).

Lemma 6.9. We have (6.22) + (6.28) $\geq 0$.

Proof. By symmetry it suffices to show that
\[
\sum_{j,k=1}^4 B_{jk} \left\{ \text{Re} \left[ \partial_j U \partial_k U + \frac{1}{4} \partial_j V \partial_k V \right] (x) m[U, V](y) \right. \\
+ m[U, V](x) \text{Re} \left[ \partial_j U \partial_k U + \frac{1}{4} \partial_j V \partial_k V \right] (y) - 2 p_j[U, V](x) p_k[U, V](y) \right\} \geq 0.
\]
Since $B_{jk}$ is real valued and $B_{jk} = \sum_l B_{lj} B_{lk}$, the left hand side is equal to
\[
\left[ |B\nabla U(x)|^2 + \frac{1}{4} |B\nabla V(x)|^2 \right] m[U, V](y) + m[U, V](x) \left[ |B\nabla U(y)|^2 + \frac{1}{4} |B\nabla V(y)|^2 \right]
- 2 \text{Im} \left[ \overline{U(x)} B\nabla U(x) + \frac{1}{2} \overline{V(x)} B\nabla V(x) \right] \cdot \text{Im} \left[ \overline{U(y)} B\nabla U(y) + \frac{1}{2} \overline{V(y)} B\nabla V(y) \right],
\]
which is bounded from below by
\[
\left[ |B\nabla U(x)|^2 + \frac{1}{4} |B\nabla V(x)|^2 \right] \left[ |U(y)|^2 + |V(y)|^2 \right]
+ \left[ |U(x)|^2 + |V(x)|^2 \right] \left[ |B\nabla U(y)|^2 + \frac{1}{4} |B\nabla V(y)|^2 \right]
- 2 \left( |U(x)||B\nabla U(y)| \cdot |U(y)||B\nabla U(x)| + \frac{1}{2} |V(x)||B\nabla V(y)| \cdot \frac{1}{2} |V(y)||B\nabla V(x)|
+ \frac{1}{2} |U(x)||B\nabla V(y)| \cdot |V(y)||B\nabla U(x)| + |V(x)||B\nabla U(y)| \cdot \frac{1}{2} |U(y)||B\nabla V(x)| \right)
= \left( |U(x)||B\nabla U(y)| - |U(y)||B\nabla U(x)| \right)^2
+ \left( \frac{1}{2} |V(x)||B\nabla V(y)| - \frac{1}{2} |V(y)||B\nabla V(x)| \right)^2
+ \left( \frac{1}{2} |U(x)||B\nabla V(y)| - |V(y)||B\nabla U(x)| \right)^2
+ \left( |V(x)||B\nabla U(y)| - \frac{1}{2} |U(y)||B\nabla V(x)| \right)^2 \geq 0,
\]
as claimed. \qed
The others will be error terms. We begin with:

**Lemma 6.10.** We have

\[
\int_0^T \left( (6.24) + (6.25) \right) dt = \int_0^T \tilde{N}(t)N(t)^2 dt \cdot o(1) \quad (K, L \to \infty).
\]

**Proof.** These two terms correspond to (6.8) + (6.9) in the radial case, but we need more careful treatment.

For (6.24) we use Lemma 6.6 (iii) and the tightness in \( L^\infty \) and \( L^2_{t,x} \). On each characteristic interval \( J_k \), we split the integral as follows:

\[
\int_{J_k} |(6.24)| \, dt
\]

\[
\lesssim \tilde{N}(t) \int_{J_k} \left[ \int \int_{|x-x(t)|>L^{1/2}/\tilde{N}(t)} + \int \int_{|y-x(t)|>L^{1/2}/\tilde{N}(t)} + \int \int_{|x-x(t)|\leq L^{1/2}/\tilde{N}(t)} \right]
\]

\[
(\Theta_L - \theta_L) \left( \tilde{N}(t)|x-y| \right) \left[ |U|^2[V]|(t,x)\right] m[U, V](t,y) \, dx \, dy \, dt.
\]

In the first and the second term we bound \( \Theta_L - \theta_L \) by 1, while in the last one we see from Lemma 6.6 (iii) that \( (\Theta_L - \theta_L) (\tilde{N}(t)|x-y|) \lesssim L^{1/2}/L = L^{-1/2} \). Since \( \tilde{N}(t) \leq N(t) \), we apply the almost periodicity and Lemma 3.24 to conclude that the above is \( \tilde{N}(t_k) \cdot o(1) (K, L \to \infty) \) uniformly in \( k \geq 0 \). The claim for (6.24) then follows from Corollary 3.26.

We next consider (6.25), in which the weight function has additional two derivatives. Since

\[
\Delta[\theta_L + 3\Theta_L](|x|) = |\theta''_L + 3\Theta''_L||x| + 3|\theta'_L + 3\Theta'_L||x|, \quad x \in \mathbb{R}^4
\]

and \( \Theta''_L(r) = r^{-1}(\theta''_L(r) - 2\Theta'_L(r)) \), we see from Lemma 6.6 (ii) and (iii) that \( \Delta_x[(\theta_L + 3\Theta_L)|(A)|] = O(\tilde{N}(t)^2/L) \), and that

\[
\int_0^T |(6.25)| \, dt \lesssim \frac{1}{L} \int_0^T \tilde{N}(t)^3 \, dt \leq \frac{1}{L} \int_0^T \tilde{N}(t)N(t)^2 \, dt,
\]

as claimed. \( \square \)

As a result, we still have the same lower bound given in Lemma 6.8 if we add the terms (6.22), (6.28), (6.24), and (6.25).

We now fix \( L \gg 1 \) and proceed to the control of (6.20), which is parallel to that of (6.4). From the relation (6.30), for fixed \( t, z \) we have

\[
\int \int \frac{\partial_L(\tilde{N}(t)x-z)}{\partial_L(\tilde{N}(t)y-z)} (x-y) \cdot p[U, V](t,x) m[U, V](t,y) \, dx \, dy.
\]

\[
= \int \int \frac{\partial_L(\tilde{N}(t)x-z)}{\partial_L(\tilde{N}(t)y-z)} (x-y) \cdot p[U_*, V_0](t,x, z) m[U, V](t,y) \, dx \, dy
\]

\[
+ \xi_0(t,z) \int \int \frac{\partial_L(\tilde{N}(t)x-z)}{\partial_L(\tilde{N}(t)y-z)} (x-y) m[U, V](t,x) m[U, V](t,y) \, dx \, dy,
\]

where the last integral is equal to zero by symmetry. Since the triangle inequality implies

\[
\frac{\partial_L(\tilde{N}(t)x-z)}{\partial_L(\tilde{N}(t)y-z)} |x-y| \leq \frac{2L}{N(t)} \frac{\partial_L(\tilde{N}(t)x-z)}{\partial_L(\tilde{N}(t)y-z)}
\]

\[
\lesssim \frac{1}{L} \int_0^T \tilde{N}(t)^3 \, dt \leq \frac{1}{L} \int_0^T \tilde{N}(t)N(t)^2 \, dt,
\]

as claimed. \( \square \)
for any $t, x, y, z$, we estimate (6.20) similarly to the radial case with the Cauchy-Schwarz as

$$\|(6.20)\| \leq \varepsilon \tilde{N}(t) \int \left[ \int \partial_L (\tilde{N}(t)x - z) \varepsilon_2[U_x, V_x](t, x, z) \, dx \times \int \partial_L (\tilde{N}(t)y - z) \, m[U, V](t, y) \, dy \right] \frac{dz}{L^4} + \frac{CL^2 \tilde{N}'(t)^2}{\varepsilon} \tilde{N}(t)^3$$

for any $\varepsilon > 0$. We can choose $\varepsilon$ small and $\tilde{N}(t)$ via Corollary 6.5 to make the contribution from (6.20) smaller than the lower bound given in Lemma 6.8. Consequently, we have

$$\int_0^T \left( (6.20) + \cdots + (6.25) + (6.27) + (6.28) \right) dt \gtrsim K, \quad (6.32)$$

whenever $K$ is sufficiently large.

There are only (6.26) + (6.29) remaining. It is then sufficient to prove

$$\int_0^T \left( (6.26) + (6.29) \right) dt = o(K) \quad (K \to \infty), \quad (6.33)$$

which corresponds to (6.15) in the radial case.

Here, instead of (6.16)–(6.17) we can obtain the following estimates from Theorem 4.1:

$$\left\| (e^{-ix \xi(t)} P_{\xi + \xi(t)} u, e^{-2ix \xi(t)} P_{\xi + \xi(t)} v) \right\|_{L^2 L^4([0,T] \times \mathbb{R}^4)} = o(1), \quad K \to \infty, \quad (6.34)$$

$$\left\| \nabla^s (e^{-ix \xi(t)} P_{\xi + \xi(t)} u, e^{-2ix \xi(t)} P_{\xi + \xi(t)} v) \right\|_{L^2 L^4([0,T] \times \mathbb{R}^4)} \lesssim K^s, \quad s > 1/2. \quad (6.35)$$

For (6.34) we notice that $|\xi(t)| \leq 2^{-10} C_s K$ for $t \in [0, T)$, which implies

$$\left\| P_{\xi - \xi(t)} (u, v) \right\|_{L^2 L^4} \lesssim \left\| P_{[\xi - \xi(t)] > \Delta} (u, v) \right\|_{L^2 L^4}. \quad (6.36)$$

(6.35) is shown as follows:

$$\left\| \nabla^s e^{-ix \xi(t)} P_{\xi + \xi(t)} u \right\|_{L^2 L^4} = \left\| \nabla^s P_{[\xi + \xi(t)] \leq C_s K e^{-ix \xi(t)} u} \right\|_{L^2 L^4} \lesssim \sum_{N < C_s K} N^s \left\| P_N e^{-ix \xi(t)} u \right\|_{L^2 L^4} = \sum_{N < C_s K} N^s \left\| e^{-ix \xi(t)} P_{[\xi - \xi(t)] > N} u \right\|_{L^2 L^4} \sim \left\| e^{-ix \xi(t)} P_{[\xi - \xi(t)] > N} u \right\|_{L^2 L^4}$$

and similarly for $v$.

To prove (6.33), we observe that (6.26) + (6.29) is invariant under the gauge transform

$$(U, V, F, G) \mapsto (U^*, V^*, F^*, G^*) := (e^{-ix \xi(t)} U, e^{-2ix \xi(t)} V, e^{-ix \xi(t)} F, e^{-2ix \xi(t)} G).$$
Hence, by an integration by parts in $x$ we see that
\[
\int_0^T \left( (6.26) + (6.29) \right) dt \\
= 2 \int_0^T \iint \Theta_L(|A|) A \cdot \Re \left[ F^* \nabla U^* + \frac{1}{2} G^* \nabla V^* \right] (t,x) m[U,V](t,y) \, dx \, dy \, dt \\
+ \int_0^T \bar{N}(t) \iint [\theta_L + 3\Theta_L](|A|) \Re \left[ F^* U^* + \frac{1}{2} \nabla G^* V^* \right] (t,x) m[U,V](t,y) \, dx \, dy \, dt \\
- 2 \int_0^T \iint \Theta_L(|A|) A \cdot \Im \left[ F^* U^* + G^* V^* \right] (t,x) \, dx \, dy \, dt.
\] (6.38)

Recall that the weight functions satisfy $|\Theta_L(|A|)A| \lesssim L$, $|\theta_L + 3\Theta_L| \lesssim 1$.

Since $(F^*, G^*)$ has at least one of $e^{-ix \cdot \xi(t)} P_{> C_k, K} u$ and $e^{-2ix \cdot \xi(t)} P_{> C_k, K} v$; for instance,
\[
F^* = e^{-ix \cdot \xi(t)} P_{> C_k, K} u e^{-2ix \cdot \xi(t)} P_{> C_k, K, v} + e^{-ix \cdot \xi(t)} P_{< C_k, K} u e^{-2ix \cdot \xi(t)} P_{> C_k, K} v
\]
\[
- P_{\xi(t) > C_k, K} \left[ e^{-ix \cdot \xi(t)} P_{> C_k, K} u e^{-2ix \cdot \xi(t)} P_{> C_k, K} v \right]
\]
we can estimate (6.36) and (6.37) using (6.34)–(6.35) similarly to the radial case.

We focus on the last integral (6.38), which does not have a counterpart in the radial case. Notice that $\Im \left[ F^* U^* + G^* V^* \right]$ has two types of products of three functions; two highs and a low,
\[
\left[ e^{-ix \cdot \xi(t)} P_{> C_k, K} u \text{ or } e^{-2ix \cdot \xi(t)} P_{> C_k, K} v \right]^2 \\
\times \left[ e^{-ix \cdot \xi(t)} P_{< C_k, K} u \text{ or } e^{-2ix \cdot \xi(t)} P_{< C_k, K} v \right],
\]
or two lows and a high,
\[
\left[ e^{-ix \cdot \xi(t)} P_{> C_k, K} u \text{ or } e^{-2ix \cdot \xi(t)} P_{> C_k, K} v \right] \\
\times \left[ e^{-ix \cdot \xi(t)} P_{< C_k, K} u \text{ or } e^{-2ix \cdot \xi(t)} P_{< C_k, K} v \right]^2.
\]

The former case is easier to treat. In fact, we use the Hölder inequality in $(t,y)$ as $L^2 L^4 \cdot L^2 L^4 \cdot L^\infty L^2$ and apply (6.34) to obtain a bound of $o(K)$:
\[
\|(U^*, V^*)\|_{L^\infty L^2} \|\nabla (U^*, V^*)\|_{L^\infty L^2} \\
\times \left\| \left( e^{-ix \cdot \xi(t)} P_{> C_k, K} u, e^{-2ix \cdot \xi(t)} P_{> 2C_k, K} v \right) \right\|_{L^2 L^4}^2 \|\nabla (U^*, V^*)\|_{L^\infty L^2}.
\]

For the latter case, we extract one derivative from the high-frequency functions as
\[
\left( e^{-ix \cdot \xi(t)} P_{> C_k, K} u, e^{-2ix \cdot \xi(t)} P_{> C_k, K} v \right)
\]
\[
= -\nabla \cdot \nabla (-\Delta)^{-1} \left( e^{-ix \cdot \xi(t)} P_{> C_k, K} u, e^{-2ix \cdot \xi(t)} P_{> C_k, K} v \right)
\]
and integrate it by parts (in y). When the derivative moves to one of the low-frequency functions, we obtain a bound like
\[
\| (U^*, V^*) \|_{L^\infty L^2} \| \nabla (U^*, V^*) \|_{L^\infty L^2} \times \| \nabla |^{-1} (e^{-ix\xi(t)} P_{> \xi K} u, e^{-2ix\xi(t)} P_{> \xi K} v) \|_{L^2 L^4} \times \| \nabla (U^*, V^*) \|_{L^2 L^4} \| (U^*, V^*) \|_{L^\infty L^2},
\]
which is again \( o(K) \) by (6.34)–(6.35). If the derivative moves to the weight function \( \Theta_L(\| A \|) \lambda \), then we have another \( \tilde{N}(t) \) as (6.37), and the resulting bound will be
\[
\| \tilde{N} \|_{L^2} \| (U^*, V^*) \|_{L^\infty L^2} \| \nabla (U^*, V^*) \|_{L^\infty L^2} \times \| \nabla |^{-1} (e^{-ix\xi(t)} P_{> \xi K} u, e^{-2ix\xi(t)} P_{> \xi K} v) \|_{L^2 L^4} \times \| (U^*, V^*) \|_{L^\infty L^2} \| (U^*, V^*) \|_{L^\infty L^2}.\]
Similarly to (6.37), the Sobolev embedding and (6.34)–(6.35) show that this is also \( o(K) \).

This completes the proof of (6.33), and hence that of Theorem 6.7. \( \square \)

We finish the proof of Theorem 1.4 by showing
\[
\sup_{t \in [0, T)} | \mathcal{M}(t) | = o(K), \quad K \to \infty,
\]
which precludes the quasi-soliton scenario for the non-radial case. To prove that, first observe that \( \mathcal{M}(t) \) is also invariant under the gauge transformation \( (U, V) \mapsto (U^*, V^*) \), which can be seen in the same manner as (6.20) is under \( (U, V) \mapsto (U_*, V_*) \). Now, the almost periodicity of \( (u, v) \) yields
\[
\| \nabla U^* \|_{L^\infty L^2} = \| \nabla P_{|\xi+\xi(t)| \leq \xi K} e^{-ix\xi(t)} u \|_{L^\infty L^2} \lesssim \| \nabla P_{|\xi+\xi(t)| \leq 4\xi K} e^{-ix\xi(t)} u \|_{L^\infty L^2} \lesssim K^{1/2} \| P_{|\xi| \leq 1/2} e^{-ix\xi(t)} u \|_{L^\infty L^2} \lesssim K^{1/2} \| P_{|\xi+\xi(t)| > 1/2} e^{-ix\xi(t)} u \|_{L^\infty L^2} \leq K^{1/2} \| P_{|\xi+\xi(t)| > 1/2} e^{-ix\xi(t)} u \|_{L^\infty L^2} \leq K \to \infty,
\]
and similarly for \( V^* \). An application of the Cauchy-Schwarz finally leads us to the conclusion.

**Appendix A. Proof of Remark 3.7.** Here, we give the proof of Remark 3.7 on the basis of the argument in [8].

**Lemma A.1.** For all \( \phi \in L^2_2(\mathbb{R}^d) \) and \( (\theta_0, \xi_0, x_0, \lambda_0) \in \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R}^d \times \mathbb{R}^d \times (0, \infty) \), it follows that
\[
\mathcal{F} h(\theta_0, \xi_0, x_0, \lambda_0) \phi = h(\theta_0 + \xi_0 x_0, -x_0, \xi_0, \lambda_0^{-1}) \mathcal{F} \phi.
\]

**Proof.** This follows by changing variables and density argument. \( \square \)

**Definition A.2.** On the basis of above lemma, we set as follows:
\[
\hat{h}(\theta, \xi_0, x_0, \lambda_0) := h(\theta + x_0 \xi_0, -x_0, \xi_0, \lambda_0^{-1}),
\hat{\hat{h}}(\theta, \xi_0, x_0, \lambda_0) := h(\theta + x_0 \xi_0, x_0, -\xi_0, \lambda_0^{-1}).
\]

Under this notation, it follows that \( \hat{\hat{h}} \circ \hat{h} = \hat{\hat{h}} \circ \hat{h} = \hat{h} = h \).
Lemma A.3. For any \( \{g_n\}_n \subset G \), precisely one of the following statements holds:

1. \( g_n \rightarrow 0 \) in WOT.
2. \( g_n \rightarrow g \) in SOT for some \( g \in G \) after passing to a subsequence if necessary.

Proof. Let \( g_n = g(\theta_n, \xi_n, x_n, \lambda_n) \). First we consider the case any one of \( \lambda_n, \lambda_n^{-1}, |\xi_n|, |x_n| \) converges to infinity as \( n \rightarrow \infty \). Take subsequence of \( \{g_n\} \) arbitarily and use same symbol \( \{g_n\} \).

Case 1. \( \lambda_n \rightarrow \infty \). Then for any \( \phi, \psi \in C_0^\infty(\mathbb{R}^d) \), we can calculate as follows:

\[
\begin{align*}
| (h(\theta_n, \xi_n, x_n, \lambda_n)\phi, \psi)_{L^2} | & \leq (\lambda_n)^{-\frac{d}{2}} \| \phi \|_{L^\infty} \| \psi \|_1 \rightarrow 0 \text{ as } n \rightarrow \infty, \\
| (h(2\theta_n, 2\xi_n, x_n, \lambda_n)\phi, \psi)_{L^2} | & \leq (2\lambda_n)^{-\frac{d}{2}} \| \phi \|_{L^\infty} \| \psi \|_1 \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{align*}
\]

By density argument we get \( g_n \rightarrow 0 \) in WOT.

Case 2. \( \lambda_n^{-1} \rightarrow \infty \). This case is similar to Case 1. (Note that \( g_n \) is unitary operator.)

Case 3. \( \lambda_n \) and \( \lambda_n^{-1} \) \( \rightarrow \infty \) and \( |x_n| \rightarrow \infty \). In this case , passing to a subsequence if necessary we may assume \( \lambda_n \rightarrow \lambda_0 \) for some \( \lambda_0 \in (0, \infty) \). Then it follows that

\[
\begin{align*}
| (h(\theta_n, \xi_n, x_n, \lambda_n)\phi, \psi)_{L^2} | & \leq \lambda_n^{-\frac{d}{2}} \int_{\mathbb{R}^d} |\phi(\frac{x - x_n}{\lambda_n})| |\psi(x)| \, dx \rightarrow 0 \text{ as } n \rightarrow \infty, \\
| (h(2\theta_n, 2\xi_n, x_n, \lambda_n)\phi, \psi)_{L^2} | & \leq (2\lambda_n)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |\phi(\frac{x - x_n}{\lambda_n})| |\psi(x)| \, dx \rightarrow 0 \text{ as } n \rightarrow \infty
\end{align*}
\]

for any \( \phi, \psi \in C_0^\infty(\mathbb{R}^d) \). We can get the result by density argument.

Case 4. \( \lambda_n \) and \( \lambda_n^{-1} \) \( \rightarrow \infty \) and \( |\xi_n| \rightarrow \infty \). This case can be treated as Case 3. (Use Plancherel’s theorem and dominated convergence theorem.)

Finally we consider the case each parameters are bounded. Passing to subsequences if necessary, we may assume that

\[
(\lambda_n, \xi_n, x_n, \theta_n) \rightarrow (\lambda_0, \xi_0, x_0, \theta_0) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}/2\pi \mathbb{Z}).
\]

Then it follows easily that \( g_n \rightarrow g(\theta_0, \xi_0, x_0, \lambda_0) \) in SOT.

\[ \square \]

Lemma A.4. For each \( (\phi, \psi) \in L^2(\mathbb{R}^d)^2 \), \( G(\phi, \psi) \) is closed in \( L^2(\mathbb{R}^d)^2 \).

Proof. If \( (\phi, \psi) = (0, 0) \), the result follows easily. So we assume \( (\phi, \psi) \neq (0, 0) \). Take any \( \{g_n\} \subset G \) and \( (\psi_0, \phi_0) \in L^2(\mathbb{R}^d)^2 \) satisfying \( g_n(\phi, \psi) \rightarrow (\phi_0, \psi_0) \) in \( L^2(\mathbb{R}^d)^2 \). Then we get \( 0 < \| (\phi, \psi) \|_{(L^2)^2} = \| (\phi_0, \psi_0) \|_{(L^2)^2} \). Therefore by Lemma A.3, passing to a subsequence if necessary, we establish

\[ g_n \rightarrow g_0 \text{ in SOT for some } g_0 \in G. \]

Then it follows that \( g_n(\phi, \psi) \rightarrow g_0(\phi, \psi) \) and so \( (\phi_0, \psi_0) = g_0(\phi, \psi) \in G(\phi, \psi) \). \[ \square \]

Definition A.5. We denote by \( \mathcal{O} \) the quotient topology of \( G \setminus L^2(\mathbb{R}^d)^2 \). Let \( \pi : L^2(\mathbb{R}^d)^2 \rightarrow G \setminus L^2(\mathbb{R}^d)^2 \) be the canonical projection.

Lemma A.6. We define the metric \( d \) on \( G \setminus L^2(\mathbb{R}^d)^2 \) as follows:

\[ d(\pi(\phi_1, \psi_1), \pi(\phi_2, \psi_2)) := \inf_{g \in G} \| g(\phi_1, \psi_1) - (\phi_2, \psi_2) \|_{L^2(\mathbb{R}^d)^2}, \]

where \( (\phi_1, \psi_1), (\phi_2, \psi_2) \in L^2(\mathbb{R}^d)^2 \). Then \( \mathcal{O} = \mathcal{O}_d \), where \( \mathcal{O}_d \) denotes the topology defined by the metric \( d \).
Proof. Note that $d$ is really a metric on $G \setminus L^2(\mathbb{R}^d)^2$ by Lemma A.4. Take any $(\phi, \psi) \in G \setminus L^2(\mathbb{R}^d)^2$ and $\varepsilon > 0$. Then $B((\phi, \psi), \varepsilon) \in \mathcal{O}_{L^2(\mathbb{R}^d)^2}$ and $\pi[B((\phi, \psi), \varepsilon)] = B((\hat{\phi}(\phi, \psi), \varepsilon)$. Therefore $\pi: (L^2(\mathbb{R}^d)^2, \mathcal{O}_{L^2(\mathbb{R}^d)^2}) \to (G \setminus L^2(\mathbb{R}^d)^2, \mathcal{O}_d)$ is continuous. By the definition of the quotient topology, we get $\mathcal{O}_d \subset \mathcal{O}$. Let $U \in \mathcal{O}$ and take any $(\phi, \psi) \in L^2(\mathbb{R}^d)^2$ with $(\phi, \psi) \in U$. Then $(\phi, \psi) \in \pi^{-1}[U] \in L^2(\mathbb{R}^d)$, and so there exists $\varepsilon > 0$ such that $B((\phi, \psi), \varepsilon) \subset \pi^{-1}[U]$. Therefore we get $\pi[B((\phi, \psi), \varepsilon)] = B((\phi, \psi), \varepsilon) \subset U$. This implies $U \in \mathcal{O}_d$. \qed

Remark A.7. Let $(\phi_n, \psi_n) \subset L^2(\mathbb{R}^d)^2$ and $(\phi_0, \psi_0) \subset L^2(\mathbb{R}^d)^2$. Then the following statements are equivalent:
1. $\pi(\phi_n, \psi_n) \to \pi(\phi_0, \psi_0)$ in $G \setminus L^2(\mathbb{R}^d)^2$.
2. There exists $\{g_n\} \subset G$ such that $g_n(\phi_n, \psi_n) \to (\phi_0, \psi_0)$ in $L^2(\mathbb{R}^d)^2$.

Proof. $1 \Rightarrow 2$: Suppose $\pi(\phi_n, \psi_n) \to \pi(\phi_0, \psi_0)$. For each $n \in \mathbb{N}$, there exists $g_n \in G$ such that $\|g_n(\phi_n, \psi_n) - (\phi_0, \psi_0)\|_{L^2(\mathbb{R}^d)^2} < \inf_{g \in G} \|g(\phi_n, \psi_n) - (\phi_0, \psi_0)\|_{L^2(\mathbb{R}^d)^2} + \frac{1}{n} \to 0$ as $n \to \infty$.

$2 \Rightarrow 1$: This is clear and so we omit the proof. \qed

Lemma A.8. Let $K \subset G \setminus L^2(\mathbb{R}^d)^2$ be precompact. Assume also that
$$\exists \eta > 0 \text{ s.t. } \forall (\phi, \psi) \in \pi^{-1}[K], \infty = \|\pi(\phi, \psi)\|_{L^2(\mathbb{R}^d)^2}. \quad (A.1)$$
Then there exists a precompact set $\hat{K} \subset L^2(\mathbb{R}^d)^2$ such that $\pi[\hat{K}] = K$.

Proof. First we prove by contradiction that
$$\exists \varepsilon > 0 \text{ s.t. } \forall p \in K, \exists f(p) \in \pi^{-1}[\{p\}] \text{ s.t.}$$
$$\min\{\|f(p)\|_{L^2(B(0,1))^2}, \|\hat{f}(p)\|_{L^2(B(0,1))^2}\} \geq \varepsilon.$$ 
If not, for each $n \in \mathbb{N}$ there exists $(\phi_n, \psi_n) \in \pi^{-1}[K]$ such that
$$\sup_{g \in G} \min\{\|g(\phi_n, \psi_n)\|_{L^2(B(0,1))^2}, \|g[n(\phi_n, \psi_n)]\|_{L^2(B(0,1))^2}\} \leq \frac{1}{n}.$$ 
By precompactness of $K$ and $\phi_n \in \pi^{-1}[K]$, passing to a subsequence if necessary, there exist $g_n \in G$ and $(\phi, \psi) \in L^2(\mathbb{R}^d)^2$ such that
$$g_n(\phi_n, \psi_n) \to (\phi, \psi) \text{ in } L^2(\mathbb{R}^d)^2. \quad (A.2)$$
Take $g \in G$ arbitrarily. Then it follows that
$$\min\{\|g(\phi, \psi)\|_{L^2(B(0,1))^2}, \|\hat{F}[g(\phi, \psi)]\|_{L^2(B(0,1))^2}\} \leq \|\phi, \psi\| - g_n(\phi_n, \psi_n)\|_{L^2(\mathbb{R}^d)^2}$$
$$+ \min\{\|g_n(\phi_n, \psi_n)\|_{L^2(B(0,1))^2}, \|\hat{F}[g_n(\phi_n, \psi_n)]\|_{L^2(B(0,1))^2}\}$$
$$\to 0 \quad (n \to \infty).$$
Therefore we obtain
$$g(\phi, \psi) = (0, 0) \text{ or } \hat{F}[g(\phi, \psi)] = 0 \text{ for any } g \in G,$$
and so $(\phi, \psi) = (0, 0)$. This contradicts (A.1).

Next we set $\hat{K} := \{f(p)\mid p \in K\}$ and prove this satisfies the result. It is clear that $\pi[\hat{K}] = K$. To prove precompactness of $\hat{K}$, take $\{(\phi_n, \psi_n)\} \subset \hat{K}$ arbitrarily. Since $K$ is precompact, passing to a subsequence if necessary, there exists
Then we establish that by the same argument.

**Proof.** See for instance Lemma 3.63 in [16].

**Case 1.** Suppose \( \lambda_n^{-1} \) is unbounded. Passing to a subsequence if necessary, we may assume \( \lambda_n \to 0 \). Then it follows that

\[
\| (\phi_n, \psi_n) \|_{L^2(B(0,1))^2}^2 = \| g_n(\phi_n, \psi_n) \|_{L^2(B(x_n, \lambda_n))^2}^2 \\
\leq \| g_n(\phi_n, \psi_n) - (\phi, \psi) \|_{L^2(\mathbb{R}^2)}^2 + \| (\phi, \psi) \|_{L^2(B(x_n, \lambda_n))^2}^2 \to 0.
\]

This contradicts the definition of \( \tilde{K} \).

**Case 2.** Suppose \( \lambda_n \) is unbounded. Then passing to a subsequence if necessary, we may assume that \( \lambda_n \to \infty \). Then it follows that

\[
\| \phi_n \|_{L^2(B(0,1))} = \| \tilde{h}(\theta_n, \xi_n, x_n, \lambda_n) \phi_n \|_{L^2(B(\xi_n, \lambda_n^{-1}))} \\
\leq \| \tilde{h}(\theta_n, \xi_n, x_n, \lambda_n) \phi_n - \phi \|_{L^2(\mathbb{R}^d)} + \| \phi \|_{L^2(B(\xi_n, \lambda_n^{-1}))} \to 0,
\]

\[
\| \psi_n \|_{L^2(B(0,1))} = \| \tilde{h}(2\theta_n, 2\xi_n, x_n, \lambda_n) \psi_n \|_{L^2(2B(\xi_n, \lambda_n^{-1}))} \\
\leq \| \tilde{h}(2\theta_n, 2\xi_n, x_n, \lambda_n) \psi_n - \psi \|_{L^2(\mathbb{R}^d)} + \| \psi \|_{L^2(B(2\xi_n, \lambda_n^{-1}))} \to 0.
\]

This contradicts the definition of \( \tilde{K} \). Therefore \( \lambda_n \) and \( \lambda_n^{-1} \) are bounded.

**Case 3.** Suppose \( x_n \) is unbounded or \( \xi_n \) is unbounded. Passing to a subsequence if necessary, we may assume \( |x_n| \to \infty \) or \( |\xi_n| \to \infty \). Then we get the contradiction by the same argument.

Therefore all parameters are bounded, and so we may assume

\[
g_n^{-1} \to g_0 \quad \text{in SOT for some } g_0 \in G.
\]

Then we establish that

\[
(\phi_n, \psi_n) = g_n^{-1} g_n(\phi_n, \psi_n) \to g_0(\phi, \psi) \quad \text{in } L^2(\mathbb{R}^d)^2.
\]

**Corollary A.9.** Let \((u, v) : I \times \mathbb{R}^d : \to \mathbb{C}^2 \) be a nonzero solution to NLS. Assume \( Gu[I] := \{ g(t) \mid t \in I \} \) is precompact in \( G \setminus L^2(\mathbb{R}^d)^2 \). Then there exists \( g \in \text{Map}(I, G) \) such that \( \{ g(t)u(t) \mid t \in I \} \) is precompact in \( L^2(\mathbb{R}^d)^2 \).

**Appendix B. Proof of the profile decomposition in the radial case.**

**Lemma B.1.** Let \( \{ u_n \} \subset L^2(\mathbb{R}^d), u \in L^2(\mathbb{R}^d). \) Then, the following three conditions are equivalent.

1. \( u_n \rightharpoonup u \) weakly in \( L^2(\mathbb{R}^d) \).
2. \( e^{it\Delta} u_n \rightharpoonup e^{it\Delta} u \) weakly in \( L^{\frac{2(4+2)}{\alpha+2}}(\mathbb{R}^{1+d}) \) for some \( \alpha > 0 \).
3. \( e^{it\Delta} u_n \rightharpoonup e^{it\Delta} u \) weakly in \( L^{\frac{2(4+2)}{\alpha+2}}(\mathbb{R}^{1+d}) \) for any \( \alpha > 0 \).

**Proof.** See for instance Lemma 3.63 in [16].

**Lemma B.2.** Fix \( \kappa > 0 \). Then for any \( \{ g_n = g_\kappa(\theta_n, \xi_n, x_n, \lambda_n) \} \subset G_\kappa \) and \( \{ t_n \} \subset \mathbb{R} \), the following conditions are equivalent.

1. \( g_n U_\kappa(t_n) \rightharpoonup 0 \) in WOT.
2. Passing to a subsequence if necessary, \( g_n U_\kappa(t_n) \rightharpoonup g U_\kappa(t_0) \) in SOT for some \( g \in G_\kappa \) and \( t_0 \in \mathbb{R} \).

**Proof.** This follows from the same argument in the proof of Lemma A.3.
Lemma B.3. Let \( \{(u_n, v_n)\} \) be a bounded sequence of \( L^2(\mathbb{R}^d)^2 \), and let
\[
\begin{pmatrix}
u_n \\
u_n
\end{pmatrix} = \sum_{j=1}^{J} g_{n,j}^0 U_n(t_n^j) \begin{pmatrix} \phi^j \\ \psi^j \end{pmatrix} + W_n^j
\]
be a profile decomposition given in Theorem 3.4, where \( U_n(t) = (e^{it\Delta}, e^{it\Delta}) \). Assume also that \( [g_n U_n(t_n)]^{-1}(u_n, v_n) \rightharpoonup (\phi, \psi) =: \Psi \) weakly in \( L^2 \times L^2 \) for some \( \{g_n\} \subset G_\kappa \). {\{t_n\} \subset \mathbb{R} \text{ and } (\phi, \psi) \in L^2 \times L^2 \setminus \{(0,0)\}}. Then, after passing to a subsequence if necessary, there exists a unique \( J^* \) such that
\[
[g_n U_n(t_n)]^{-1} g_{n,j}^0 U_n(t_n^j) \rightharpoonup g U_n(t_0) \quad \text{in } SOT \text{ for some } g \in G_\kappa \text{ and } t_0 \in \mathbb{R}
\]
and
\[
(\phi, \psi) = g U_n(t_0)(\phi^{j_0}, \psi^{j_0}).
\]

Proof. If not, by Lemma B.2 we have
\[
[g_n U_n(t_n)]^{-1} g_{n,j}^0 U_n(t_n^j) \rightharpoonup 0 \quad \text{in WOT for all } j \leq J^*.
\]

Then it follows from profile decomposition that
\[
[g_n U_n(t_n)]^{-1} W_n^\ell \rightharpoonup \Psi \quad \text{for all } \ell \leq J^*.
\]

Applying Lemma B.1 and using weak lower semicontinuity of \( L^{\frac{2(d+2)}{d}}(\mathbb{R}^{1+d}) \) norm, we obtain that
\[
0 < \|U_n(t)\Psi\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^{1+d})^2} \leq \liminf_{n \to \infty} \left\|U_n(t)[g_n U_n(t_n)]^{-1} W_n^\ell\right\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^{1+d})^2}
\]
\[
= \liminf_{n \to \infty} \left\|U_n(t)W_n^\ell\right\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^{1+d})^2}.
\]

This contradicts the property of \( W_n^\ell \). Finally we show the uniqueness. If \( j_1 \neq j_0 \), then we have by asymptotic orthogonality that
\[
[g_n U_n(t_n)]^{-1} g_{n,j_1}^0 U_n(t_n^j) = [g_n U_n(t_n)]^{-1} (g_{n,j_0}^0 U_n(t_n^j)U_n(t_n^j)) U_n(t_n^j) \rightharpoonup 0 \quad \text{in WOT}.
\]

Proof of Theorem 3.21. First we show that all \( \{\lambda_n^j \xi_n^j\} \) and \( \{(\lambda_n^j)^{-1} x_n^j - 2t_n^j \lambda_n^j \xi_n^j\} \) are bounded, by contradiction. If not, there exists \( j_0 \leq J^* \), such that
\[
|\lambda_n^{j_0} \xi_n^{j_0}| + |(\lambda_n^{j_0})^{-1} x_n^{j_0} - 2t_n^{j_0} \lambda_n^{j_0} \xi_n^{j_0}| \to \infty.
\]

Since \( d \geq 2 \), we can take a sequence \( \{A_{k\ell}\} \subset SO(d) \) such that
\[
|\langle A_{k\ell} - A_m \rangle [\lambda_n^{j_0} \xi_n^{j_0}]| + |\langle A_{k\ell} - A_m \rangle [(\lambda_n^{j_0})^{-1} x_n^{j_0} - 2t_n^{j_0} \lambda_n^{j_0} \xi_n^{j_0}]| \to \infty \quad \text{for } \ell \neq m.
\]

Then we have \( \{G_n^\ell := g_n(\theta_n^{j_0}, A_x \xi_n^{j_0}, A_x x_n^{j_0}, \lambda_n^{j_0})U_n(t_n^{j_0})\} \) is asymptotically orthogonal. On the other hand, from radial property of \( \{(u_n, v_n)\} \) and a profile decomposition we have
\[
(G_n^\ell)^{-1}(u_n, v_n) = U_n(-t_n^{j_0}) g_n(\theta_n^{j_0}, A_x \xi_n^{j_0}, A_x x_n^{j_0}, \lambda_n^{j_0})^{-1} A_{k\ell}(u_n, v_n)
\]
\[
= U_n(-t_n^{j_0}) A_{k\ell}(g_n^{j_0})^{-1}(u_n, v_n) = A_{k\ell} U_n(-t_n^{j_0})(g_n^{j_0})^{-1}(u_n, v_n)
\]
\[
\to A_{k\ell}(\phi^{j_0}, \psi^{j_0}) \quad \text{weakly in } L^2 \times L^2.
\]

Therefore by Lemma B.3, there exist \( j_\ell \), \( \ell \in G_\kappa \) and \( t_\ell^j \) such that
\[
(G_n^\ell)^{-1} g^{j_\ell} U_n(t_n^{j_\ell}) \rightharpoonup g^{t_\ell} U_n(t_\ell) \quad \text{in } SOT \quad \text{and} \quad A_{k\ell}(\phi^{j_0}, \psi^{j_0}) = g^{t_\ell} U_n(t_\ell)(\phi^{t_\ell}, \psi^{t_\ell}).
\]

\[\text{Set } A(\phi, \psi) = (\phi(A), \psi(A)) \text{ for } (\phi, \psi) \in L^2(\mathbb{R}^d)^2.\]
Noting that
\[
[g^{l}_{n}U_{\kappa}(t^{j}_{n})]^{-1}g^{m}_{n}U_{\kappa}(t^{m}_{n}) = [[G^{l}_{n}]^{-1}g^{l}_{n}U_{\kappa}(t^{j}_{n})]^{-1}[[G^{m}_{n}]^{-1}G^{m}_{n}]^{-1}g^{m}_{n}U_{\kappa}(t^{m}_{n})
\]
we obtain \( j_{\ell} \neq j_{m} \) for \( \ell \neq m \). Therefore we obtain that
\[
\lim_{n \to \infty} \| (u_{n}, v_{n}) \|^{2}_{L^{2}_{x}L^{2}} \geq \sum_{\ell = 1}^{\infty} \| (\phi^{\ell}, \psi^{\ell}) \|^{2}_{L^{2}_{x}L^{2}} = \sum_{\ell = 1}^{\infty} \| (\phi^{j_{\ell}}, \psi^{j_{\ell}}) \|^{2}_{L^{2}_{x}L^{2}} = \infty.
\]
This contradicts boundedness of \( \{(u_{n}, v_{n})\} \). Therefore passing to a subsequence if necessary, we may assume that
\[
g_{n}(t^{j}_{n}, \lambda^{j}_{n}x_{n}, \lambda^{j}_{n}x_{n}, 1) \rightarrow h_{j} \quad \text{in SOT} \quad \text{for some } h_{j} \in \mathcal{G}_{\kappa}.
\]
Noting that
\[
g_{n}(0, 0, \lambda^{j}_{n}x_{n}, 1)U_{\kappa}(t^{j}_{n}) = g_{n}(0, 0, \lambda^{j}_{n}x_{n}, \lambda^{j}_{n}x_{n} - 2t_{n}^{j}\lambda^{j}_{n}x_{n}, 1)
\]
we may assume that scaling parameters and translation parameters are zero (after modifying remainder terms).

Next we prove that \( W_{n}^{J} \) and \( (\phi^{J}, \psi^{J}) \) are radially symmetric. Let \( A \in SO(d) \). Then from the profile decomposition, we obtain that
\[
\sum_{j=1}^{J} g^{l}_{n}U_{\kappa}(t^{j}_{n})(\phi^{j}, \psi^{j}) + W_{n}^{J} = \sum_{j=1}^{J} g^{l}_{n}U_{\kappa}(t^{j}_{n})A(\phi^{j}, \psi^{j}) + AW_{n}^{J}, \quad (B.1)
\]
and so for \( 1 \leq \ell \leq J \), we have
\[
\sum_{j=1}^{J} [g^{l}_{n}U_{\kappa}(t^{j}_{n})]^{-1}g^{l}_{n}U_{\kappa}(t^{j}_{n})(\phi^{j}, \psi^{j}) + [g^{l}_{n}U_{\kappa}(t^{j}_{n})]^{-1}W_{n}^{J}
\]
\[
= \sum_{j=1}^{J} [g^{l}_{n}U_{\kappa}(t^{j}_{n})]^{-1}g^{l}_{n}U_{\kappa}(t^{j}_{n})A(\phi^{j}, \psi^{j}) + A[g^{l}_{n}U_{\kappa}(t^{j}_{n})]^{-1}W_{n}^{J}.
\]
Taking a weak limit in above equation, we establish that
\[
(\phi^{l}, \psi^{l}) = A(\phi^{l}, \psi^{l}).
\]
From the equation (B.1), we also have \( W_{n}^{J} = AW_{n}^{J} \).

### Appendix C. Proof of lemmas in Section 3.5.

**Proof of Lemma 3.22.** We may assume that \( (0 <) S_{J}(u,v) \leq \epsilon_{1} \) with \( \epsilon_{1} \) sufficiently small, by cutting \( J \) into finite number of small intervals. Take arbitrary \( t_{1}, t_{2} \in J \). By the almost periodicity, we have
\[
\int_{|x-x(t_{2})| \leq \frac{R}{N_{J}(2)}} \left( |u(t_{2}, x)|^{2} + |v(t_{2}, x)|^{2} \right) \, dx \geq \frac{9}{10}M(u,v) \quad (C.1)
\]
for sufficiently large \( R > 0 \) which is independent of \( t_{1}, t_{2} \). We divide the left hand side of the above inequality with respect to frequency. Using the H"older inequality
and the Hausdorff-Young inequality,

\[ \int_{|x-x(t_2)| \leq \frac{R}{N(t_2)}} |P_{|\xi-\xi(t_1)| \leq RN(t_1)} u(t_2, x)|^2 \, dx \]

\[ \lesssim \left( \frac{R}{N(t_2)} \right)^4 \| P_{|\xi-\xi(t_1)| \leq RN(t_1)} u(t_2) \|_{L^\infty(R^4)}^2 \]

\[ \leq \left( \frac{R}{N(t_2)} \right)^4 \| \tilde{u}(t_2) \|_{L^2(|\xi-\xi(t_1)| \leq 2RN(t_1))}^2 \leq \left( \frac{R}{N(t_2)} \right)^4 (RN(t_1)) \| \tilde{u}(t_2) \|_{L^2(R^4)}^2. \]

Together with a similar estimate for \( v \) (replacing \( \xi(t_1) \) with \( 2\xi(t_1) \)), we have

\[ \int_{|x-x(t_2)| \leq \frac{R}{N(t_2)}} \left( |P_{|\xi-\xi(t_1)| \leq RN(t_1)} u(t_2, x)|^2 + |P_{|\xi-2\xi(t_1)| \leq RN(t_1)} v(t_2, x)|^2 \right) \, dx \]

\[ \lesssim R^8 \left( \frac{N(t_1)}{N(t_2)} \right)^4 M(u, v). \] (C.2)

On the other hand, we use the equations (write \( P_u := P_{|\xi-\xi(t_1)| > RN(t_1)} \) and \( P_v := P_{|\xi-2\xi(t_1)| > RN(t_1)} \))

\[ P_u u(t_2) = e^{i(t_2-t_1)\Delta} P_u u(t_1) - iP_u \int_{t_1}^{t_2} e^{i(t_2-t')\Delta} (\bar{u}(t')v(t')) \, dt', \]

\[ P_v v(t_2) = e^{i(t_2-t_1)\Delta} P_v v(t_1) - iP_v \int_{t_1}^{t_2} e^{i(t_2-t')\Delta} (u(t')^2) \, dt', \]

the Strichartz estimates, and the almost periodicity again to obtain

\[ \int_{R^4} \left( |P_u u(t_2, x)|^2 + |P_v v(t_2, x)|^2 \right) \, dx \]

\[ \leq 2 \int_{R^4} \left( |P_u u(t_1, x)|^2 + |P_v v(t_1, x)|^2 \right) \, dx + C\varepsilon_1^{4/3} \] (C.3)

\[ \leq \frac{1}{10} M(u, v) + C\varepsilon_1^{4/3} \]

for \( R \) sufficiently large. From (C.1), (C.2), (C.3), and the fact that \( M(u, v) > 0 \), we obtain the estimate

\[ \left( \frac{N(t_1)}{N(t_2)} \right)^4 \gtrsim R^{-8} \]

whenever \( R \gg 1 \) and \( \varepsilon_1 \ll 1 \), which implies the claim. \( \square \)

**Proof of Lemma 3.23.** We may assume again that \( (0 <) S_J(u, v) \leq \varepsilon_1 \) with \( \varepsilon_1 \) sufficiently small. Take \( t_1, t_2 \in J \) arbitrarily. Using the almost periodicity we have

\[ \| P_{|\xi-\xi(t_1)| \leq RN(J)} u(t_1) \|_{L^2}^2 + \| P_{|\xi-2\xi(t_1)| \leq RN(J)} v(t_1) \|_{L^2}^2 \geq \frac{9}{10} M(u, v), \]

\[ \| P_{|\xi-\xi(t_2)| \leq RN(J)} u(t_2) \|_{L^2}^2 + \| P_{|\xi-2\xi(t_2)| \leq RN(J)} v(t_2) \|_{L^2}^2 \geq \frac{9}{10} M(u, v) \]

for sufficiently large \( R > 0 \) independent of \( t_1, t_2 \), where \( N(J) := \sup_{t \in J} N(t) \). But from the Duhamel formula and the Strichartz estimates we see that

\[ \| P_{|\xi-\xi(t_2)| \leq RN(J)} u(t_2) \|_{L^2}^2 + \| P_{|\xi-2\xi(t_2)| \leq RN(J)} v(t_2) \|_{L^2}^2 \]

\[ \leq \| P_{|\xi-\xi(t_2)| \leq RN(J)} u(t_1) \|_{L^2}^2 + \| P_{|\xi-2\xi(t_2)| \leq RN(J)} v(t_1) \|_{L^2}^2 + C\varepsilon_1^{2/3}, \]
hence for sufficiently small $\varepsilon_1 > 0$,
\[
\|P_{[\xi-\varepsilon(t_2)] \leq RN(J)} u(t_1)\|_{L^2_x}^2 + \|P_{[\xi-2\varepsilon(t_2)] \leq RN(J)} v(t_1)\|_{L^2_x}^2 \gtrsim \frac{4}{5} M(u, v).
\]

Now assume that $|\xi(t_1) - \xi(t_2)| > 10RN(J)$, which would imply that $\{|\xi - \xi(t_1)| \leq 2RN(J)\} \cap \{|\xi - \xi(t_2)| \leq 2RN(J)\} = \emptyset$ and similarly for $2\xi(\cdot)$, then we had a contradiction as follows,
\[
M(u, v) \gtrsim \|P_{|\xi - \xi(t_1)| \leq RN(J)} u(t_1)\|_{L^2_x}^2 + \|P_{|\xi - 2\xi(t_1)| \leq RN(J)} v(t_1)\|_{L^2_x}^2
\]
\[
+ \|P_{|\xi - \xi(t_2)| \leq RN(J)} u(t_1)\|_{L^2_x}^2 + \|P_{|\xi - 2\xi(t_2)| \leq RN(J)} v(t_1)\|_{L^2_x}^2
\]
\[
\gtrsim \frac{17}{10} M(u, v).
\]
Therefore, we have $|\xi(t_1) - \xi(t_2)| \leq 10RN(J)$.

\begin{proof}[Proof of Lemma 3.24] By the Duhamel formula and the Strichartz estimates, we have
\[
\|u\|_{L_t^2 L_x^\infty(J \times \mathbb{R}^4)} + \|v\|_{L_t^2 L_x^\infty(J \times \mathbb{R}^4)} \lesssim u, v \ 1 + S_J(u, v)^{2/3}.
\]
The desired estimate follows from an interpolation between the above and (3.10) if $S_J(u, v) \leq 1$. When $S_J(u, v) > 1$, we first divide $J$ into $O(S_J(u, v))$ subintervals $\{J_k\}$ so that $S_{J_k}(u, v) \sim 1$ on each $J_k$, and sum up the obtained estimates on $J_k$.

\begin{proof}[Proof of Lemma 3.25] The almost periodicity with the non-zero assumption gives an $R = R(u, v) > 0$ satisfying
\[
\inf_{t \in J} \int_{|x-x(t)| \leq \frac{R}{N(t)}} (|u(t, x)|^2 + |v(t, x)|^2) \, dx \gtrsim u, v \ 1.
\]
Applying the Hölder inequality to the left hand side, we have
\[
1 \lesssim u, v \left( \frac{R}{N(t)} \right)^{\frac{4}{3}} \left( \int_{\mathbb{R}^4} (|u(t, x)|^3 + |v(t, x)|^3) \, dx \right)^{2/3}
\]
for any $t \in J$. The first inequality in (3.25) then follows after an integration in $t$.

For the second inequality in (3.25), we may focus on the case $S_J(u, v) > 1$. Applying Lemma 3.24 with $\eta = \frac{S_J(u, v) / 100}{1 + S_J(u, v)} \sim 1$, we see that there exists $R = R(u, v) > 0$ satisfying
\[
S_J(u, v) \lesssim \|P_{[\xi - \xi(t)] \leq RN(t)} u\|_{L^3(J \times \mathbb{R}^4)}^3 + \|P_{[\xi - 2\xi(t)] \leq RN(t)} v\|_{L^3(J \times \mathbb{R}^4)}^3,
\]
which is, via the Hausdorff-Young inequality followed by the Hölder, bounded by
\[
\int_J \left( (RN(t))^{2/3} (\|u(t)\|_{L^2_x} + \|v(t)\|_{L^2_x}) \right)^3 \, dt \lesssim u, v \int_J N(t)^2 \, dt,
\]
as desired.
\end{proof}

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