BASES IN SYSTEMS OF SIMPLICES AND CHAMBERS

TATIANA ALEKSEYEVSKAYA

Abstract. We consider a finite set $E$ of points in the $n$-dimensional affine space and two sets of objects that are generated by the set $E$: the system $\Sigma$ of $n$-dimensional simplices with vertices in $E$ and the system $\Gamma$ of chambers. The incidence matrix $A = ||a_{\sigma,\gamma}||$, $\sigma \in \Sigma$, $\gamma \in \Gamma$, induces the notion of linear independence among simplices (and among chambers). We present an algorithm of construction of bases of simplices (and bases of chambers). For the case $n = 2$ such an algorithm was described in [1]. However, the case of $n$-dimensional space required a different technique. It is also proved that the constructed bases of simplices are geometrical (according to [1]).

1. Introduction.

Let $E = (e_1, e_2, \ldots, e_N)$, $N > n$, be a finite set of points in an $n$-dimensional affine space $V$. Let $P = \text{conv}(E)$ be the convex hull of $E$. Let $\sigma = \sigma(e_{i_1}, \ldots, e_{i_{n+1}})$ be the $n$-dimensional simplex with the vertices $e_{i_1}, \ldots, e_{i_{n+1}} \in E$. Denote by $\Sigma$ the set of all such simplices $\sigma$. All the simplices $\sigma$ (as a rule overlapping) cover the polytope $P$. The simplices $\sigma$ divide the polytope $P$ into a finite number of chambers $\gamma$ (see Definition (1.1)). Denote by $\Gamma$ the set of all chambers $\gamma$ in $P$.

**Definition 1.1.** Let $\sigma \in \Sigma$ and $\tilde{\sigma}$ be the boundary of $\sigma$. Let $\tilde{\Sigma} = \bigcup_{\sigma \in \Sigma} \tilde{\sigma}$ and $\tilde{P} = P \setminus \tilde{\Sigma}$. Let $\gamma$ be a connected component of $\tilde{P}$ and $\gamma$ be closure of $\gamma$. We call $\gamma$ a chamber and $\tilde{\gamma}$ an open chamber.

Let $A$ be the incidence matrix between simplices and chambers, i.e.

$$||a_{\sigma,\gamma}|| = 1 \text{ iff } \gamma \subseteq \sigma.$$  

Consider the linear space $V_\Sigma$ generated by the rows of $A$ and the linear space $V_\Gamma$ generated by the columns of $A$ over some field of characteristic 0. Due to one-to-one correspondence between the rows of $A$ and the simplices $\sigma \in \Sigma$, we can speak about a linear combination of simplices instead of a linear combination of the corresponding rows of $A$. An important question is to construct bases of simplices and bases of chambers, i.e. bases in $V_\Sigma$ (or in $V_\Gamma$) that consists of simplices (or chambers) and not of their linear combinations.

A basis of simplices can be also defined as follows. Let $\phi_\sigma(x)$ be the characteristic function of a simplex $\sigma$, i.e.

$$\phi_\sigma(x) = 1, \ x \in \sigma \text{ and } \phi_\sigma(x) = 0, \ x \notin \sigma.$$  

*Date:* October 1, 2018.

Supported by the Gabriella and Paul Rosenbaum Foundation; part of these results was obtained at MSRI supported by NSF grant DMS 9022140.
A basis of simplices is a maximal subset of simplices such that their characteristic functions \( \phi_{\sigma}(x) \) are linearly independent.

In this paper we will describe (Section 2) the inductive algorithm of constructing bases of simplices and bases of chambers in the \( n \)-dimensional affine space; the algorithm uses the case \( n = 2 \) (see [1]) as the first step of induction. In Sections 3, 4 we prove that the set \( B \) of simplices and the set \( B' \) of chambers constructed by the algorithm are indeed bases in \( V_\Sigma \) and in \( V_\Gamma \) respectively.

2. Construction of a basis of simplices and a basis of chambers.

2.1. A special ordering of points \( e \in E \) and related polytopes. Let \( E = \{e_1, \ldots, e_N\} \) be a set of points in an \( n \)-dimensional affine space \( V^n \), \( \Sigma \) the set of \( n \)-dimensional simplices \( \sigma \) with the vertices in \( E \) and \( \Gamma \) the set of chambers \( \gamma \) (defined in Introduction.)

We will define an ordering of points \( e_i \in E \) which is essential in the construction.

Lemma 2.1. Let \( E = \{e_1, e_2, \ldots, e_N\} \) be a finite set of points in the \( n \)-dimensional affine space. There exists an ordering \( e_i_1, e_i_2, \ldots, e_i_N, \ e_{i_k} \in E, \) such that for \( k = 1, \ldots, N \)

\[
\text{conv}(e_{i_1}, \ldots, e_{i_k}) \cap \text{conv}(e_{i_{k+1}}, \ldots, e_{i_N}) = \emptyset, 
\]

and there exists a hyperplane \( H_k \) which separates the polytopes

\[
F_k = \text{conv}(e_{i_1}, \ldots, e_{i_k}) 
\]

and

\[
P_k = \text{conv}(e_{i_{k+1}}, \ldots, e_{i_N}) = \text{conv}(E \setminus \{e_{i_1}, \ldots, e_{i_k}\}).
\]

We assume also \( F_0 = \emptyset, P_N = \emptyset, P_0 = \text{conv}(E) = P, F_N = \text{conv}(E) = P. \)

This lemma is proved, for example, in [1]. The ordering \( e_1, \ldots, e_N \) satisfying Lemma 2.1 yields a shelling of the polytope \( P, \) see [1].

Let the ordering \( e_1, \ldots, e_N \) satisfy (1). Consider the sequence of polytopes \( P_0 = P, P_1, \ldots, P_N = \emptyset \) defined by formula (3). We have \( P_0 \supset P_1 \supset \ldots \supset P_N. \) Let us denote

\[
S_k = \overline{P_{k-1} \setminus P_k},
\]

where \( \overline{A} \) means the closure of the set \( A. \) Let \( \text{int}(S) \) be the interior of \( S. \)

It is easy to check that the following statements are true:

1. Each ordering \( e_1, \ldots, e_N \) determines a decomposition of the polytope \( P:\)

\[
P = \bigcup_{i=1}^{N-n} S_i,
\]

where \( \text{int}(S_i) \cap \text{int}(S_j) = \emptyset \) for \( i \neq j. \)

2. \( \text{int}(S_k) \cap E = \emptyset \) for \( k = 1, \ldots, N - n. \)
3. The polytope $S_k$ is part of the convex cone with the vertex $e_k$ and bounded by some part $L_k$ of the boundary of $P_{k-1}$, (where $L_k = S_k \cap P_{k-1}$). Note that the polytope $S_k$ is not necessarily convex\footnote{The polytopes $S_k$ are considered in more detail in [4].}.

4. \[ P = S_1 \cup \ldots \cup S_{k-1} \cup P_{k-1} \]

2.2. A map from the polytope $P$ to the hyperplane $H_k$. Let an ordering of points $e_1, \ldots, e_N$ satisfy (\[\text{[9]}\]). Consider the point $e_k \in P_{k-1}$ and the corresponding hyperplane $H_k$. For a segment $(e_k, e_i)$, $i = k + 1, \ldots, N$ let us denote
\[ e^k_i = (e_k, e_i) \cap H_k. \]

Thus, on the hyperplane $H_k$ we obtained the set $E_k$ of points $e^k_1, e^k_2, \ldots, e^k_N$. (Note that $N_k \leq N - k$ since the points $e \in E$ are not necessarily in general position.) In the hyperplane $H_k$ we use the points $e^k \in E_k$ to construct simplices $\sigma^k$ and chambers $\gamma^k$ in the same way as it was done for the set $E$ in $V^n$. Let $\Sigma_k$ be the set of all these simplices $\sigma^k$ and $\Gamma_k$ the set of all chambers $\gamma^k$.

Let $\sigma^k = \sigma(e^k_{i_1}, \ldots, e^k_{i_n}) \in \Sigma_k$. We denote by $\overrightarrow{e_k e^k_{i_j}}$ the ray starting at $e_k$ and passing through the point $(e^k_{i_j})$. Consider the following map:
\[ \mu : \sigma^k \mapsto \sigma, \text{ where } \sigma = \sigma(e_k, e_{i_1}, \ldots, e_{i_n}), \]
and where $e_{i_j} \in E$ is the nearest point to the point $e_k$ on the ray $\overrightarrow{e_k e^k_{i_j}}$. It is clear that $\sigma = \mu(\sigma^k) \in \Sigma$. The map $\mu$ is an injection and has the following easy to check property.

**Proposition 2.2.** Let $\sigma^k, \sigma^0_k \in \Sigma_k$ be open simplices such that $\sigma^k \cap \sigma^0_k = \emptyset$. Then $\mu(\sigma^k) \cap \mu(\sigma^0_k) = \emptyset$.

Consider the set of points $E_k$ in the hyperplane $H_k$. Let us reorder $e^k \in E_k$ according to (\[\text{[9]}\]). Let $e^k_1, e^k_2, \ldots, e^k_N$ be such an ordering. Similarly to formulas (\[\text{[9]}\]) and (\[\text{[9]}\]) we denote
\[ P^k = P^k_0 = \text{conv}(E_k), \quad P^k_j = \text{conv}(e^k_{j+1}, \ldots, e^k_N), \quad S^k_j = P^k_{j-1} \setminus P^k_j \]
where $j = 1, \ldots, N_k$. For these $(n-1)$-dimensional polytopes the formulas analogous to (\[\text{[9]}\]) and (\[\text{[9]}\]) hold:
\[ P^k = \bigcup_j S^k_j, \]
where $\text{int}(S^k_i) \cap \text{int}(S^k_j) = \emptyset$ for $i \neq j$, and
\[ P^k = S^k_1 \cup \ldots \cup S^k_{j-1} \cup S^k_j. \]

On the polytopes $S^k_j$ and $P^k_j$ let us define the following map $\tau$: $\tau(S^k_j)$ is an $n$-dimensional cone with the vertex $e_k$ and generated by the rays $\overrightarrow{e_k x}$, where $x \in S^k_j$. The map $\tau(P^k)$ is defined similarly. Clearly, the following decomposition holds:
\[ \tau(P^k) = \bigcup_i \tau(S^k_i), \]

where \( \text{int}(\tau(S^k_i)) \cap \text{int}(\tau(S^k_j)) = \emptyset \) for \( i \neq j \).

2.3. Algorithm of construction of the set \( B \) of simplices and the set \( B' \) of chambers. We will construct the set \( B \subset \Sigma \) of simplices and the set \( B' \) of chambers and will prove in Sections 3 and 4 that the set \( B \) is a basis in \( V_\Sigma \) and the set \( B' \) is a basis in \( V_\Gamma \). The algorithm of construction of \( B \) and \( B' \) is inductive on the dimension \( n \) of the affine space \( V^n \).

Let an ordering of points \( e_1, \ldots, e_N \) satisfy (4). For each point \( e_k \) we construct a set \( B_k \) of simplices \( \sigma \in \Sigma \) and a set \( B'_k \) of chambers \( \gamma \in \Gamma \). Then we define \( B = \bigcup_k B_k \) and \( B' = \bigcup_k B'_k \).

**First step** \( (n = 2) \). The points \( e_1, \ldots, e_N \) lie on the affine plane \( V^2 \). Let us denote by \( q \) an edge of a simplex \( \sigma \in \Sigma \) and by \( Q \) the set of all edges of all simplices \( \sigma \in \Sigma \). Consider the point \( e_k \in P_{k-1} \). In \( P_{k-1} \) from the point \( e_k \) there are following edges \( q_i = (e_k, e_i) \), where \( i \in (k+1, \ldots, N) \). Note that since the points \( e \in E \) are not necessarily in general position, some of these edges may coincide and several points \( e_i, i \in (k+1, \ldots, N) \) may lie on the same edge.

Let \( q_i = (e_k, e_i) \) and \( q_j = (e_k, e_j) \), where \( i, j \in (k+1, \ldots, N) \), be two neighbor edges with the vertex \( e_k \) (i.e. there is no other edge \( q = (e_k, e_m) \), \( m \in (k+1, \ldots, N) \) which lies between \( q_i \) and \( q_j \)). Let the point \( e_i \) be the nearest point of \( E \) to the point \( e_k \) on the edge \( q_i \) and, respectively, \( e_j \) the nearest point to the point \( e_k \) on the edge \( q_j \). Let \( \sigma = \sigma(e_k, e_i, e_j) \). We define \( B_k \) as the set of all such simplices \( \sigma \). We define then
\[ B = \bigcup_k B_k. \]

In Figure 1 there is an example of a set \( B \) of simplices constructed according to this algorithm.

For this example we have: \( B_1 = (153, 134, 142); \) \( B_2 = (253, 236, 264); \) \( B_3 = (356, 364); \) \( B_4 = (456). \)
With the point $e_k$ we also associate the following set of chambers $B'_k$. In each simplex $\sigma$ let us choose one chamber adjacent to the point $e_k$. We define the set $B'_k$ as the set of all such chambers and define

$$B' = \bigcup_k B'_k.$$ 

In Figure 2 the chambers from the set $B'$ are shaded.

Suppose that we have described the construction in $V^{n-1}$. Let us describe it in $V^n$.

Let the points $e_i \in E$ be in $V^n$ and let $e_1, \ldots, e_N$ be an ordering satisfying (1). Consider the point $e_k$ and the hyperplane $H_k$ from Lemma 2.1. In the hyperplane $H_k$ we have the set of points $E_k$ (see (7)), the set of simplices $\Sigma_k$ and the set of chambers $\Gamma_k$. Let us reorder the points $e_i$ so that the ordering $e_1^k, e_2^k, \ldots, e_N^k$ satisfies (1).

By the induction hypothesis, we can construct a set of simplices in $H_k$, i.e. the set $\tilde{B}_k \subset \Sigma_k$ and the set of chambers in $H_k$, i.e. the set $\tilde{B}'_k \subset \Gamma_k$. Then we define the set $B_k$ of simplices in $V^n$ as $B_k = \{\mu(\sigma^k), \forall \sigma^k \in \Sigma_k\}$, where the map $\mu$ is defined by (8). Finally, we define $B = \bigcup_k B_k$.

We have already constructed the set $\tilde{B}'_k$ of chambers in the hyperplane $H_k$. Consider $\gamma^k \in \tilde{B}'_k$. According to the algorithm in $H_k$ the chamber $\gamma^k$ was chosen at a certain step $j$, $j \in (1, \ldots, N_k)$ and the point $e^k_j$ is a vertex of $\gamma^k$. Besides, there is one simplex $\sigma^k \in \tilde{B}_k$ such that $e^k_j$ is a vertex of $\sigma^k$ and $\gamma^k \subset \sigma^k$. Let us choose a chamber $\gamma \in \Gamma$ such that

1) $\gamma \subset \mu(\sigma^k);$  
2) $\gamma$ is adjacent to the point $e_k$ and to the edge $(e_k, e_j)$.

---

2 In case of an affine plane in each simplex $\sigma$ there is only one chamber adjacent to the point $e_k$ since every edge of a chamber $\gamma \in \Gamma$ necessarily lies on some edge $q \in Q$. 

[Figure 2. (There is a mistake in the figure. One shaded chamber should be in a different place.)]
Thus, with each chamber $\gamma^k \in \tilde{B}'_k$ we associate a chamber $\gamma \in \Gamma$. Denote by $B'_k$ the set of all such chambers $\gamma$ corresponding to $\gamma^k \in \tilde{B}'_k$ and define

$$B' = \bigcup_k B'_k.$$ 

In Figure 3 there is a fragment of a configuration of points in the 3-dimensional space. In the plane $H_k$, separating the points $e_1, \ldots, e_k$ and $e_{k+1}, \ldots, e_N$, there are five points which are reordered according to condition (1). For this ordering in the plane $H_k$ we construct the basis

$$\tilde{B}_k = \{(e_i, e_i, e_{i_5}), (e_i, e_{i_5}, e_{i_3}), (e_i, e_{i_5}, e_{i_2}), (e_i, e_{i_5}, e_{i_3}), (e_i, e_{i_5}, e_{i_2})\}$$

of simplices and the basis $\tilde{B}'_k$ of chambers (are shaded). The set $B_k$ consists of the following simplices:

$$B_k = \{(e_k e_{i_1} e_{i_4} e_{i_5}), (e_k e_{i_1} e_{i_4} e_{i_5}), (e_k e_{i_1} e_{i_4} e_{i_5}), (e_k e_{i_1} e_{i_4} e_{i_5}), (e_k e_{i_1} e_{i_4} e_{i_5})\}$$

where $\tilde{e}_j = \mu(e_j)$.

Note that a chamber from $B'_k$ cannot be seen in Figure 3 since we need to take into account the points $e_1, \ldots, e_{k-1}$ which are below the plane $H_k$.

3. Linear independence of simplices $\sigma \in B$ and linear independence of chambers $\gamma \in B'$.

Let $B \subset \Sigma$ and $B' \subset \Gamma$ be the sets constructed in Section 2.
Theorem 3.1. The simplices $\sigma \in B$ are linearly independent in $V_2$ and the chambers $\gamma \in B'$ are linearly independent in $V_1$.

First let us prove the following proposition.

Proposition 3.2. Let $\tilde{A}$ be a submatrix of the incidence matrix $A$ corresponding to the rows $\sigma \in B$ and the columns $\gamma \in B'$. The columns and rows of the matrix $\tilde{A}$ can be ordered in such a way that: 1) $\tilde{A}$ is a block matrix $\parallel A_{ik} \parallel$, where $A_{ik} = 0$ for $i > k$, and 2) each diagonal element $a_{\sigma,\gamma}$ of the matrix $\tilde{A}$ equals 1, i.e. $a_{\sigma,\gamma} = 1$.

Proof. Let us recall that by the algorithm the set $B$ of simplices was constructed as $B = \bigcup B_k$ and the set $B'$ of chambers as $B' = \bigcup B'_k$, where $B_k, B'_k$ correspond to the point $e_k$ in the ordering $e_1, \ldots, e_N$. Thus, the matrix $\tilde{A}$ is a block-matrix $\parallel A_{ik} \parallel$, where

$$
A_{ik} = \{a_{\sigma,\gamma}, \sigma \in B_i, \gamma \in B'_k\}.
$$

Consider a diagonal block $A_{kk}$ ($k = 1, \ldots, N - n$). By the construction, to each simplex $\sigma \in B_k$ there corresponds a chamber $\gamma \in B'_k$ such that $\gamma \subseteq \sigma$, therefore, $a_{\sigma,\gamma} = 1$. Thus, if we choose the corresponding orderings of columns in $B'_k$ and rows in $B_k$, we obtain "1" on the main diagonal in the block $A_{kk}$ and, therefore, any diagonal element $a_{\sigma,\gamma}$ of the matrix $\tilde{A}$ is such that $a_{\sigma,\gamma} = 1$.

Consider a block $A_{ik}$, where $i > k$. We need to prove that $a_{\sigma,\gamma} = 0$, where $\sigma \in B_i$ and $\gamma \in B'_k$. Let $\sigma \in B_i$ and $\gamma \in B'_i \cup \ldots \cup B'_{i-1}$. Then $\sigma \in P_{i-1} = \text{conv}(e_i, \ldots, e_N)$. From the algorithm for the construction of the set $B'$ it is easy to see that $\gamma \in S_1 \cup \ldots \cup S_{i-1}$. Due to formulas (7) and (8) we conclude that $\gamma \notin \sigma$ and, therefore, $a_{\sigma,\gamma} = 0$. Thus, the matrix $\tilde{A}$ is upper triangular as a block matrix. \hfill $\Box$

Proof of Theorem 3.1. We will show that the submatrix $\tilde{A}$ defined above is an upper triangular matrix. Due to Proposition 3.2 it is sufficient to prove that a block $A_{kk}$ (where $k = 1, \ldots, N - n$) of the matrix $\tilde{A}$ is upper triangular.

Let us consider the case $n = 2$. Let $e_1, \ldots, e_N$ be an ordering satisfying (1). By the construction, the simplices $\sigma \in B_k$ lie between the neighbor edges starting at the point $e_k$, therefore, the open simplices $\sigma \in B_k$ are disjoint. Besides, there is exactly one chamber $\gamma \in B'_k$ (i.e. with the vertex $e_k$) such that $\gamma \in \sigma$, therefore, the block $A_{kk}$ is the identity matrix.

Since for a general $n$ the notations are cumbersome we consider in detail the case $n = 3$ which already contains all the technique. Let $e_1, \ldots, e_N$ be an ordering satisfying (1). Consider a point $e_k$ and the block $A_{kk}$ of the matrix $\tilde{A}$. We recall (see Section 2.3) that a simplex $\sigma \in B_k$ is defined as $\mu(\sigma^k)$, where the 2-dimensional simplex $\sigma^k \in \bar{B}_k$ lies in the plane $H_k$ separating the points $e_1, \ldots, e_k$ and $e_{k+1}, \ldots, e_N$.

The set $\bar{B}_k$ of simplices is constructed according to the algorithm in $H_k$. For this we sort the points $e_i^k$ (see (1)) according to condition (1). Let $e_{k1}, \ldots, e_{kn}^k$ be such an ordering. Then $\bar{B}_k = \bigcup_i \bar{B}_{k,i}$, where $\bar{B}_{k,i}$ is the set of 2-dimensional simplices which were chosen in the algorithm for the point $e_i^k$. We have a similar equality for...
chambers: \( \tilde{B}'_k = \bigcup_i \tilde{B}'_{k,i} \), where \( \tilde{B}'_{k,i} \) is the set of 2-dimensional chambers chosen in the algorithm for the point \( e^k_i \).

Due to the inductive construction of simplices \( \sigma \in B_k \) and chambers \( \gamma \in B'_k \) we obtain also the following formulas:

\[
B_k = \bigcup_i B_{k,i}
\]
and

\[
B'_k = \bigcup_i B'_{k,i}.
\]

This means that the block \( A_{kk} \) consists, in turn, of blocks \( B_{11,k1} \):

\[
B_{11,k1} = \{ a_{\sigma,\gamma} : \sigma^k \in \tilde{B}_{k,i} \},
\]
where \( \sigma = \mu(\sigma^k) \) and the chamber \( \gamma \) corresponds \( \| \) to the chamber \( \gamma^k \subset H_k \).

1\(^{\circ} \). Consider a diagonal block \( B_{k1,k1} \) of the block \( A_{kk} \), i.e. all \( a_{\sigma,\gamma} \), where \( \sigma \in B_{k,k1} \) and \( \gamma \in B'_{k,k1} \).

Let \( \sigma, \sigma_0 \in B_{k,k1} \). Then the corresponding simplices \( \sigma^k \) and \( \sigma^0_k \) were constructed on the plane \( H_k \) from the same point \( e^k_1 \). By the algorithm the open simplices \( \sigma^k, \sigma^0_k \) are disjoint. Then (see Proposition 2.2) \( \mu(\sigma^k) \cap \mu(\sigma^0_k) = \emptyset \), where \( \mu \) is defined by formula \( (\text{11}) \). But \( \sigma = \mu(\sigma^k) \), and \( \sigma_0 = \mu(\sigma^0_k) \). Obviously, for any chamber \( \gamma \) such that \( \gamma \subset \sigma \), we have \( \gamma \not\subset \sigma_0 \). This means that the diagonal block \( B_{k1,k1} \) is the identity matrix.

2\(^{\circ} \). Let us show that \( B_{11,k1} = 0 \) for \( i_1 > k_1 \). Indeed, let \( \sigma \in B_{k,i_1} \) and \( \gamma \in B'_{k,k1} \). Then from the construction we have \( \sigma^k \in \tilde{B}_{k,i_1} \) and \( \sigma^k \in P^k_{i_1-1} = \text{conv}(e^k_1, \ldots, e^k_{N_k}) \) (see \( (\text{11}) \)).

Concerning \( \gamma \) we know that there is a simplex \( \sigma_0 \in B_{k,k1} \) such that:

1) \( \gamma \in \sigma_0 \);
2) \( \gamma \) is adjacent to the point \( e^k_i \);
3) \( \gamma \) is adjacent to the edge \( (e^k_i, e^k_{k_1}) \).

Since \( \sigma_0 \in B_{k,k1} \) we have \( \sigma^0_k \in \tilde{B}_{k,k1} \) and \( \sigma^0_k \subset P^k_{k_1-1} \).

For \( i_1 > k_1 \) we can rewrite the formula \( (\text{11}) \) as follows:

\[
P^k = S^k_1 \cup \ldots \cup S^k_{k_1} \cup S^k_{k_1+1} \cup \ldots \cup S^k_{i_1-1} \cup P^k_{i_1-1},
\]
where all the open polytopes \( S^k_j \) and \( P^k_{i_1-1} \) are disjoint. Note that if \( i_1 = k_1 + 1 \) then \( S^k_{k_1} = S^k_{i_1-1} \).

Due to decomposition \( (\text{12}) \) we obtain

\[
\tau(P^k) = \tau(S^k_1) \cup \ldots \cup \tau(S^k_{k_1}) \cup \tau(S^k_{k_1+1}) \cup \ldots \cup \tau(S^k_{i_1-1}) \cup \tau(P^k_{i_1-1}),
\]
where all the corresponding open cones are disjoint. We have \( \sigma = \mu(\sigma^k) \subset \tau(P^k_{i_1-1}) \).

Let us show that \( \gamma \subset \tau(S^k_{k_1}) \). Since \( \gamma \subset \sigma_0 \), then \( \gamma \subset \mu(\sigma^0_k) \), i.e. \( \gamma \subset \tau(P^k_{k_1-1}) \).

\(^{3}\)Note that in the algorithm there is no direct correspondence between the chambers \( \gamma \) and \( \gamma^k \); given a chamber \( \gamma^k \subset H_k \) we find the corresponding simplex \( \sigma^k \subset H_k \), then in the simplex \( \sigma = \mu(\sigma^k) \) we choose a certain chamber (see Section 2.3).
Note that $P_{k_1-1}^k = \mathcal{S}_{k_1}^k \cup \mathcal{S}_{k_1+1}^k \cup \ldots$ (see formulas (7) and (11)). Then $\gamma \subset \tau(\mathcal{S}_k^k) \cup \tau(\mathcal{S}_{k_1+1}^k) \cup \ldots$. Any chamber $\gamma \in \Gamma$ may lie in only one of these cones. Since $\gamma$ is adjacent to the edge $(e_k, e_{k_1})$ we obtain $\gamma \subset \tau(\mathcal{S}_k^k)$. Then $\gamma \not\subset \tau(P_{i_1}^{k_1})$, i.e. $\gamma \not\subset \sigma$. We have proved that $a_{\gamma,\gamma} = 0$ for $\sigma \in B_{k_1}$ and $\gamma \in B_{0}^{k_1}$ if $i_1 > k_1$.

This completes the proof (for $n = 3$) that the matrix $\bar{A}$ is an upper triangular matrix.

In case of $n$-dimensional space an element $a_{\sigma,\gamma}$ of the matrix $\bar{A}$ belongs to a sequence of enclosed blocks which can be denoted as $A_i, i = 1, 2, \ldots$ (One can check that the "depth" of the enclosed blocks, i.e. the length of the sequence, is $n - 1$.)

1°. Let us show that a diagonal block of maximal depth is the identity matrix. Let $\mathcal{Z}_{k_{n-2}, k_{n-2}}$ be such a block. Then $\mathcal{Z}_{k_{n-2}, k_{n-2}}$ lies inside all diagonal blocks $A_{k_0, k_0}, B_{k_1, k_0}, C_{k_2, k_1}, \ldots$ of the matrix $\bar{A}$. Let $\sigma, \sigma_0$ be two simplices from the block $\mathcal{Z}_{k_{n-2}, k_{n-2}}$. This means that $\sigma$ and $\sigma_0$ have the same sequence of corresponding points in the inductive construction. Let us denote these points by $e_{k_0}^{(1)}, e_{k_1}^{(2)}, \ldots, e_{k_{n-2}}^{(n-2)}$, where $e_{k_0} \in E$, $e_{k_1}^{(1)} \in H_{k_0}$, $e_{k_2}^{(2)}$ lies in the $(n-2)$-dimensional plane which corresponded to the point $e_{k_1}^{(1)}$ in the algorithm, and so on; finally, the point $e_{k_{n-2}}^{(n-2)}$ lies in the 2-dimensional plane which corresponded to the point $e_{k_{n-3}}^{(n-3)}$. On this latter plane there are two simplices $\bar{\sigma}, \bar{\sigma}_0$ which were both chosen for the point $e_{k_{n-2}}^{(n-2)}$.

According to the algorithm for the plane the open simplices $\bar{\sigma}, \bar{\sigma}_0$ are disjoint. Therefore (see Proposition 2.2), for the 3-dimensional open simplices we have $\mu(\sigma) \cap \mu(\sigma_0) = \emptyset$, where $\mu$ is a map defined by formula (8) for the point $e_{k_{n-3}}^{(n-3)}$ instead of $e_k$. Then in order to obtain 4-dimensional simplices, the map $\mu$ for the point $e_{k_{n-4}}^{(n-4)}$ was applied to the simplices $\mu(\sigma), \mu(\sigma_0)$ and so on. Finally, the map $\mu$ for the point $e_{k_0}$ gives the simplices $\sigma, \sigma_0 \in \Sigma$. Clearly, we have $\sigma \cap \sigma_0 = \emptyset$. Therefore, for any chamber $\gamma \subset \sigma$ we have $\gamma \not\subset \sigma_0$, i.e. the block $\mathcal{Z}_{k_{n-2}, k_{n-2}}$ is the identity matrix.

2°. In order to prove that the matrix $\bar{A}$ is upper triangular it suffices to check that for $i > j$ any block $\mathcal{X}_{i,j} \in (B_{i_1,k_1}, C_{i_2,k_2}, \ldots)$ satisfies the condition $\mathcal{X}_{i,j} = 0$. In Proposition 3.2 we have proved that for $i > k$ a block $A_{ik} = 0$. By applying the arguments from 2° of $n = 3$ case, we can prove that in a diagonal block $A_{kk}$ we have $B_{i_1,k_1} = 0$ for $i_1 > k_1$. Similarly, $C_{i_2,k_2} = 0$ for $i_2 > k_2$, etc.

It follows from 1° and 2° that the submatrix $\bar{A}$ of the incidence matrix $A$ is an upper triangular matrix.

4. THE SET $B$ IS A GEOMETRICAL BASIS OF SIMPLICES.

In this section we will prove that the set $B$ constructed by the algorithm is a geometrical basis. First we repeat some of the definitions and theorems from [8] and [11].

Let $E = (e_1, \ldots, e_N)$ be a set of points in the $n$-dimensional affine space $V^n$, $\Sigma$ the set of simplices and $\Gamma$ the set of chambers (defined in the Introduction).
Consider a subset \( S \subseteq E \) consisting of \( n + 2 \) points and such that \( S \) contains at least \( n + 1 \) points in general position. Denote
\[
(14) \quad f = \{ \sigma : \sigma(e_{i_1}, \ldots, e_{i_{n+1}}) \in \Sigma \text{ and } e_{i_k} \in S \}
\]
Thus, with each \( S \) we associate a subset \( f \subset \Sigma \) (clearly, \( f \neq \emptyset \)). Let \( F \) be the set of all such \( f \) corresponding to all possible \( S \subseteq E \).

**Definition 4.1.** We say that a point \( p \) of affine space is visible from a point \( e \), \( e \neq p \), with respect to a simplex \( \sigma \) if the open segment \((e, p)\) is disjoint with \( \sigma \), i.e. \((e, p) \cap \sigma = \emptyset \).

We say also that a subset \( S \) of points is visible from the point \( e \) if every point of this subset is visible from \( e \).

**Theorem 4.2.** Let \( \sigma \in \Sigma \) be a simplex and \( e \in E \) a point that is not a vertex of \( \sigma \). There is the following linear relation in \( \text{V}_\Sigma \) among simplices:
\[
(15) \quad \sigma = \sum_{q_i \in Q^+} \sigma(q_i, e) - \sum_{q_i \in Q^-} \sigma(q_i, e),
\]
where
- \( Q^+ \) is the set of all facets (i.e. \((n-1)\)-dimensional faces) \( q_i \) of the simplex \( \sigma \) that are not visible from \( e \) (with respect to \( \sigma \));
- \( Q^- \) is the set of all facets \( q_i \) of the simplex \( \sigma \) that are visible from \( e \) (with respect to \( \sigma \));
- \( \sigma(q_i, e) \) is the \( n \)-dimensional simplex spanned by the facet \( q_i \) of the simplex \( \sigma \) and the point \( e \).

**Definition 4.3.** Let \( B \subset \Sigma \). We say that an element \( \sigma \notin B \) is expressed in one step “in terms of the set \( B \) using \( F \)” if there exists \( f \in F \) such that \( \sigma \in f \), and \( f \setminus \sigma \subseteq B \).

We say that an element \( \sigma \notin B \) can be expressed in \( k \) steps in terms of the set \( B \) using \( F \) if there exists a sequence \( \sigma_1, \ldots, \sigma_k \), \( \sigma_i \in \Sigma \), such that \( \sigma_k = \sigma \) and \( \sigma_1 \) is expressed in one step in terms of the set \( B \) (using \( F \)), \( \sigma_2 \) is expressed in one step in terms of the set \( B \cup \sigma_1, \ldots, \sigma_k \) is expressed in one step in terms of the set \( B \cup \sigma_1 \ldots \cup \sigma_{k-1} \).

**Definition 4.4.** A subset \( B \subset \Sigma \) is a geometrical basis in \( \text{V}_\Sigma \) with respect to \( F \) if it satisfies the following conditions:
1) \( B \) is a basis in \( \text{V}_\Sigma \);
2) for any \( \sigma \in \Sigma \), \( \sigma \notin B \) there exists \( k \) such that \( \sigma \) can be expressed in \( k \) steps in terms of \( B \) using \( F \).

Let \( B \subset \Sigma \) be the set of simplices constructed by the algorithm of Section 1.

**Theorem 4.5.** The set \( B \) is a geometrical basis in \( \text{V}_\Sigma \) with respect to the system \( F \) defined by formula (14).

\(^4\)This theorem is stated in [3] in another form. Here we use the important geometric notion of visibility.
Proof. Due to Theorem 3.1 the simplexes \( \sigma \in B \) are linearly independent, therefore it is sufficient to prove that any simplex \( \sigma \in \Sigma \) can be expressed in a finite number of steps in terms of \( B \) with respect to the system \( F \). This will be proved by induction on the dimension \( n \) of the space \( V^n \).

**Induction on \( n \). First step (\( n = 2 \)).** The theorem is proved in [3].

**Passing from \( n - 1 \) to \( n \).** Suppose that the statement is true for \( V^{n-1} \). Let us prove it for \( V^n \).

Let the points \( e_i \in E \) be in \( V^n \) and an ordering \( e_1, \ldots, e_N \) satisfy (1). Let us denote by \( \Sigma^i \) the set of simplexes \( \sigma \in \Sigma \) such that \( \sigma \) has the point \( e_i \) as the vertex with the minimal number. We have

\[
\Sigma = \Sigma^1 \cup \ldots \cup \Sigma^{N-n} = \bigcup_{i=n}^{N-1} \Sigma^{N-i},
\]

where \( \Sigma^i \cap \Sigma^j = \emptyset, \ i \neq j \). We also have \( \Sigma^N = \Sigma^{n-1} = \ldots = \Sigma^{N-n+1} = \emptyset \). The set \( \Sigma^{N-n} \) either contains only one simplex \( \sigma(e_N, e_{N-1}, \ldots, e_{N-n}) \) or \( \Sigma^{N-n} = \emptyset \).

We need to prove that any simplex \( \sigma \in \Sigma \) can be expressed in terms of \( B \) using \( F \). Due to the partition (16) we can prove this statement by induction on \( i \) considering cases when \( \sigma \in \Sigma^{N-i} \), where \( i = n, \ldots, N - 1 \).

**Induction on \( i \).** First step. Let us check the first nontrivial step of induction. Let \( \Sigma^{N-i_0} \) (where \( i_0 \geq n \)) be the first nonempty set, i.e.

\[
\Sigma^{N-i_0} = \{ \sigma(e_N, \ldots, e_{N-i_0+1}, e_{N-i_0}) \},
\]

where the points \( e_N, \ldots, e_{N-i_0+1} \) lie in an \((n-1)\)-dimensional hyperplane, while the point \( e_{N-i_0} \) does not lie in this hyperplane. It is clear from the algorithm that the simplex \( \sigma(e_N, \ldots, e_{N-i_0+1}, e_{N-i_0}) \) belongs to the set \( B_{N-i_0} \subset B \) and, therefore, can be expressed in terms of \( B \) in 0 steps.

**Passing from \( i - 1 \) to \( i \).** Suppose that any simplex \( \sigma \in \Sigma^{N-(i-1)} \) can be expressed in terms of \( B \) using \( F \). Let us show that any simplex \( \sigma_0 \in \Sigma^{N-i} \) can also be expressed in terms of \( B \) using \( F \).

Consider the point \( e_{N-i} \) and the hyperplane \( H = H_{N-i} \) of the algorithm. According to the algorithm we mark in \( H \) the set \( E_{N-i} \) of points \( e_1^{N-i}, e_2^{N-i}, \ldots, e_{N-i}^{N-i} \), where \( e_j^{N-i} = (e_{N-i}, e_j) \cap H \). To any vertex of \( \sigma_0 \) other than \( e_{N-i} \), there corresponds a point from \( E_{N-i} \), therefore, to the simplex \( \sigma_0 \) there corresponds one \((n-1)\)-dimensional simplex \( \sigma_0^{N-i} \) in \( H \).

Let \( \Sigma_{N-i} \) be the set of all \((n-1)\)-dimensional simplexes with the vertices in \( E_{N-i} \). We have \( \sigma_0^{N-i} \in \Sigma_{N-i} \). Applying the algorithm in the hyperplane \( H \), we construct the set \( \tilde{B}_{N-i} \) of simplexes, \( \tilde{B}_{N-i} \subset \Sigma_{N-i} \). Due to the assumption of the induction on \( n \), the simplex \( \sigma_0^{N-i} \in \Sigma_{N-i} \) can be expressed in terms of \( \tilde{B}_{N-i} \) using \( F' \) (where \( F' \) is the set of all subsets \( f' \) defined in \( H \) by formula (14)).

First, let us show that the simplex \( \mu(\sigma_0^{N-i}) \), where \( \mu \) is the map defined by (8) for \( k = N - i \), can be expressed in terms of \( B \) using \( F \).

As we have proved, the simplex \( \sigma_0^{N-i} \) can be expressed in terms of \( \tilde{B}_{N-i} \) using \( F' \) in a finite number of steps, for example, in \( k \) steps. This means that there
exists a sequence of simplices \( \sigma_1^{N-i}, \ldots, \sigma_k^{N-i} \), where \( \sigma_k^{N-i} = \sigma_0^{N-i} \) and a sequence of \( f'_1, \ldots, f'_k \), \( f'_i \in F' \), such that

\[
(17) \quad f'_1 \setminus \sigma_1^{N-i} \subseteq \tilde{B}_{N-i},
\]

\[
(18) \quad f'_2 \setminus \sigma_2^{N-i} \subseteq (\tilde{B}_{N-i} \cup \sigma_1^{N-i}),
\]

\[
\ldots
\]

\[
(19) \quad f'_k \setminus \sigma_k^{N-i} \subseteq (\tilde{B}_{N-i} \cup \sigma_1^{N-i} \cup \ldots \cup \sigma_k^{N-i}).
\]

Consider formula (17). It means that the simplex \( \sigma_1^{N-i} \) can be expressed in terms of \( \tilde{B}_{N-i} \) using \( F' \). Let us show that this implies that the simplex \( \mu(\sigma_1^{N-i}) \) can be expressed in terms of \( B_{N-i} \) using \( F \). Indeed, according to the definition, the set \( f'_1 \) contains all the simplices with vertices in some \( n+1 \) points \( e_{i_1}^{N-i}, \ldots, e_{i_{n+1}}^{N-i} \in E_{N-i} \).

One of these simplices is \( \sigma_1^{N-i} \). For simplicity of notations let us assume that \( \sigma_1^{N-i} = \sigma(\overline{e_{i_1}^{N-i}}, \overline{e_{i_2}^{N-i}}, \ldots, \overline{e_{i_{n+1}}^{N-i}}, \overline{e_{N-i}}) \), where \( \overline{e} \) means that the point \( e \) is not a vertex of the simplex. Thus, \( f'_1 \) consists of the following simplices:

\[
\sigma_1^{N-i} = \sigma(\overline{e_{i_1}^{N-i}}, \overline{e_{i_2}^{N-i}}, \ldots, \overline{e_{i_{n+1}}^{N-i}}),
\]

\[
\sigma(\overline{e_{i_1}^{N-i}}, \overline{e_{i_2}^{N-i}}, \ldots, \overline{e_{i_{n+1}}^{N-i}}),
\]

\[
\ldots
\]

\[
\sigma(\overline{e_{i_1}^{N-i}}, \overline{e_{i_2}^{N-i}}, \ldots, \overline{e_{i_{n+1}}^{N-i}}),
\]

where each simplex except \( \sigma_1^{N-i} \) belongs to \( \tilde{B}_{N-i} \) due to formula (17).

To \( f'_1 \in F' \) let us associate an element \( f_1 \in F \). For this consider \( n+2 \) points \( e_{N-i}, e_{i_1}^\mu, \ldots, e_{i_{n+1}}^\mu \in E \), where \( e_{i_j}^\mu \) is the point of \( E \) which is the closest to the point \( e_{N-i} \) among all points of \( E \) lying on the ray \( \overline{(e_{N-i}, e_{i_j}^{N-i})} \). Then \( f_1 \) is the set of all simplices with the vertices in these \( n+2 \) points. Thus, \( f_1 \) consists of the following simplices:

\[
\sigma(\overline{e_{i_1}^\mu}, \overline{e_{i_2}^\mu}, \ldots, \overline{e_{i_{n+1}}^\mu}),
\]

\[
\sigma(e_{N-i}, \overline{e_{i_1}^\mu}, \overline{e_{i_2}^\mu}, \ldots, \overline{e_{i_{n+1}}^\mu}),
\]

\[
\sigma(e_{N-i}, \overline{e_{i_1}^\mu}, \overline{e_{i_2}^\mu}, \ldots, e_{i_{n+1}}^\mu),
\]

\[
\ldots
\]

\[
\sigma(e_{N-i}, \overline{e_{i_1}^\mu}, \overline{e_{i_2}^\mu}, \ldots, \overline{e_{i_{n+1}}^\mu}).
\]

The point \( e_{N-i} \) is not the vertex of the simplex \( \sigma(e_{i_1}^\mu, \ldots, e_{i_{n+1}}^\mu) \). Therefore, \( \sigma(e_{i_1}^\mu, \ldots, e_{i_{n+1}}^\mu) \in \Sigma^{N-(i-1)} \). By induction hypothesis for \( i \) this simplex can be expressed in terms of \( B \) using \( F \).

Note that according to the algorithm,

\[
\sigma(e_{N-i}, e_{i_1}^\mu, \ldots, \overline{e_{i_j}^\mu}, \ldots, \overline{e_{i_{n+1}}^\mu}) = \mu(\sigma(\overline{e_{i_1}^{N-i}}, \ldots, \overline{e_{i_j}^{N-i}}, \ldots, \overline{e_{i_{n+1}}^{N-i}}))
\]
for any $j = 1, \ldots, n + 1$. Therefore, we have
\[
\sigma(e_{N-i}, e_{i_1}^\mu, \ldots, e_{i_{n+1}}^\mu) = \mu(\sigma_1^{N-i}).
\]

Since each simplex $\sigma(e_{N-i}^{N-i}, e_{i_1}^\mu, \ldots, e_{i_{n+1}}^\mu)$, $j = 2, \ldots, n + 1$, belongs to $\tilde{B}_{N-i}$, each simplex $\sigma(e_{N-i}, e_{i_1}^\mu, \ldots, e_{i_{n+1}}^\mu)$ belongs to $B_{N-i}$ according to the construction of $B_{N-i}$. Thus, $f_1 \setminus \mu(\sigma_{1}^{N-i}) \subseteq (B_{N-i} \cup \sigma(e_{i_1}^\mu, \ldots, e_{i_{n+1}}^\mu)$, where, as we have already mentioned, the simplex $\sigma(e_{i_1}^\mu, \ldots, e_{i_{n+1}}^\mu)$ can be expressed in terms of $F$ using $F$. We have proved that $\mu(\sigma_{1}^{N-i})$ can be expressed in terms of $B$ using $F$.

Considering consecutively formulas (18) – (19) we can prove similarly that the simplex $\mu(\sigma_{0}^{N-i})$ can be expressed in terms of $B$ using $F$.

Consider the simplex $\sigma_0$. The following cases are possible:

1) $\sigma_0 = \mu(\sigma_{0}^{N-i})$; in this case the proof is already finished.

2) $\sigma_0 \neq \mu(\sigma_{0}^{N-i})$. Let $\sigma_0^{N-i} = \sigma(e_{j_1}^{N-i}, \ldots, e_{j_n}^{N-i})$. This case can only occur if there is a vertex of $\sigma_0$ which lies on some ray $(e_{N-i}, e_{j_k}^{N-i})$, where $k \in (1, \ldots, n)$, and which is not the closest point from $E$ to the point $e_{N-i}$.

Let us show that since the simplex $\mu(\sigma_{0}^{N-i})$ can be expressed in terms of $B$ using $F$ then the simplex $\sigma_0$ can be also expressed in terms of $B$ using $F$.

For simplicity of notations let us denote here vertices of the simplices $\sigma_0$ and $\mu(\sigma_{0}^{N-i})$ as follows:
\[
\sigma_0 = \sigma(e_{N-i}, e_0^1, \ldots, e_n^0), \quad \mu(\sigma_0^{N-i}) = \sigma(e_{N-i}, e_1^1, \ldots, e_n^\mu).
\]

We can also assume that in this notation the points $e_k^0$ and $e_k^\mu$ lie on the same ray $(e_{N-i}, e_{j_k}^{N-i})$.

First, let us consider the case when the simplices $\sigma_0$ and $\mu(\sigma_0^{N-i})$ differ only by one point, i.e. $e_k^0 \neq e_k^\mu$ for some $k$ and we have

\[
\sigma_0 = \sigma(e_{N-i}, e_1^\mu, \ldots, e_{k-1}^\mu, e_k^0, e_{k+1}^\mu, \ldots, e_n^\mu), \quad \mu(\sigma_0^{N-i}) = \sigma(e_{N-i}, e_1^1, \ldots, e_{k-1}^\mu, e_k^\mu, e_{k+1}^\mu, \ldots, e_n^\mu).
\]

Then for the $n + 2$ points $e_{N-i}, e_1^\mu, \ldots, e_{k-1}^\mu, e_k^0, e_{k+1}^\mu, \ldots, e_n^\mu, e_k^\mu \in E$ there exists $f \in F$ which consists of the following simplices:
\[
\begin{align*}
\sigma(e_{N-i}, e_1^\mu, \ldots, e_{k-1}^\mu, e_k^0, e_{k+1}^\mu, \ldots, e_n^\mu) &= \mu(\sigma_0^{N-i}), \\
\sigma(e_{N-i}, e_1^\mu, \ldots, e_{k-1}^\mu, e_k^\mu, e_{k+1}^\mu, \ldots, e_n^\mu) &= \sigma_0, \\
\sigma(e_{N-i}, e_1^0, \ldots, e_{k-1}^\mu, e_k^\mu, e_{k+1}^\mu, \ldots, e_n^\mu) &= \sigma_0, \\
\sigma(e_{N-i}, e_1^\mu, \ldots, e_{k-1}^\mu, e_k^\mu, e_{k+1}^\mu, \ldots, e_n^\mu) &= \mu(\sigma_0^{N-i}).
\end{align*}
\]
Since the points $e_{N-i}, e^0_k, e^\mu_k$ lie on the same ray $(e_{N-i}, e_j)$, all the simplices except $\sigma_0$ and $\mu(\sigma_0^{N-i})$ are not $n$-dimensional and do not belong to $\Sigma$. This implies that the simplex $\sigma_0$ can be expressed in terms of $B$ using $f \in F$.

If the simplices $\sigma_0$ and $\mu(\sigma_0^{N-i})$ differ by two points, then we consider first a simplex $\sigma'$ which differs from the simplex $\mu(\sigma_0^{N-i})$ only by one point and express it in terms of $B$ using $F$. Then by applying the above arguments to the simplices $\sigma'$ and $\sigma_0$ we can express the simplex $\sigma_0$ in terms of $B$ using $F$. By repeating this process we prove that the simplex $\sigma_0$ can be expressed in terms of $B$ using $F$. Thus, the set $B$ constructed by the algorithm is a geometrical basis in $V_\Sigma$.

A pair of a basis $e_1, \ldots, e_n$ in the space $V$ and a basis $f_1, \ldots, f_n$ in the dual space $V'$ is called a triangular pair if $(e_i, f_k) = 0$ for $i > k$ and $(e_i, f_i) = 1$.

Theorems 4.3 and 3.1 imply

**Theorem 4.6.** The set $B' \subset \Gamma$ constructed by the algorithm is a basis in $V_\Gamma$. The basis of simplices $B$ and the basis of chambers $B'$ form a triangular pair.

**References**

[1] T.V. Alekseyevskaya, *Combinatorial bases in systems of simplices and chambers*, Discrete Mathematics 157 (1996) 15–37.

[2] T.V. Alekseyevskaya, I.M. Gelfand, *Incidence Matrices, Geometrical Bases, Combinatorial Prebases and Matroids*, will appear in Discrete Mathematics.

[3] T.V. Alekseyevskaya, I.M. Gelfand, A.V. Zelevinsky, *An arrangement of real hyperplanes and the partition function connected with it*, Soviet Math. Doklady 36 (1988) 589–593.

[4] G.Danaraj, V.Klee, *Shellings of spheres and polytopes*, Duke Math. J., 41 (1974), 443-451.