On the Lorentz invariance of the Square root Klein-Gordon
Equation: the correction of the Born rule

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Abstract

In this paper, we investigate the Lorentz invariance of the square root Klein-Gordon equation.
This equation with interpreting $|\psi|^2$ as position probability density is not Lorentz invariance but
we can make this equation Lorentz invariant with the help of new proper interpretation of the
wave function. In this regard we consider the relation between the probability density and the
wave function as $\rho = \psi \psi^* + (\hat{D} \psi)^* \hat{D} \psi$ and show appropriate selection of the linear operator $\hat{D}$ the
square root Klein-Gordon equation will be Lorentz invariant. In fact the solutions of this equation
is ”equivalent” to the positive energy solutions of the Dirac equation.

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I. INTRODUCTION

The simplest Relativistic generalization of the Schrodinger equation could be derived from direct substitution of $p \rightarrow -i\hbar \nabla$ and $E \rightarrow i\hbar \frac{\partial}{\partial t}$ in the expression of relativistic energy-momentum relation $E = \sqrt{c^2p^2 + m_0^2c^4}$. Such procedure leads to

$$i\hbar \frac{\partial \psi}{\partial t} = \sum_{k=0}^{\infty} a_k \hbar^{2k} \nabla^{2k} \psi,$$

where $\nabla^{2k} \equiv \nabla^2 \nabla^2 ... \nabla^2 (k \ times)$ (1)

this equation is the so called ”Square Root Klein-Gordon Equation”, in which $a_k$ are identical with coefficients in the expansion of Energy in terms of momentum:

$$E = \sqrt{c^2(p)^2 + m_0^2c^4} = \sum_{k=0}^{\infty} a_k p^{2k}, \quad a_k := (-1)^k c(m_0c)^{1-2k}$$ (2)

In order to circumvent the problem of the divergence of above expansion for $p > m_0$, the integro-differential form of Square Root Equation can be used:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \int_{-\infty}^{+\infty} K(x-x')\psi(x')dx'$$ (3)

where the integral kernel, K can be expressed in terms of the Neumann functions,$N_1(x)$, as follows:

$$K(x) := \frac{N_1(ix/l_0) - N_1(-ix/l_0)}{4ix/l_0}$$ (4)

If the eq.(2) is convergent, the eq.(1) will be equivalent with the eq.(3). Also the equation (3) is uniquely derived from the following assumption:

The equation is linear

The Equation is first order with respect to time.

For plane waves the equation leads to the relativistic dispersion relation($\omega = \frac{1}{\hbar} \sqrt{c^2(hk)^2 + m_0^2c^4}$).

Historical background of the equation (1) is related to the early years of the relativistic quantum mechanics. In 1927 Wayl proposed using of square root operator, ($\sqrt{-c^2\hbar^2\nabla^2 + m_0^2c^4}$), to formulate the relativistic quantum mechanics[1-2], however he didn’t develop his idea to a comprehensive theory. On the other hand, other pioneers of quantum mechanics used different methods to formulate relativistic quantum mechanics that leads to another wave equations (e.g. Dirac and Klein-Gordon equations and etc). Consequently, the ”square root Klein- Gordon equation” didn’t used in the formulation of
Relativistic Quantum Mechanics. But in recent years, theoretical characteristics and integral representations of this equation were the matter of special interest [2-5].

Also recently this equation was used to describe some phenomena and study some problems in relativistic regime such as: Relativistic Harmonic Oscillator [6-7], Waves in Relativistic Quantum Plasma [8], Relativistic Bound state [9-15], Relativistic Strings [16-17] and Relativistic Bohmian Mechanics [18]. Independent of historical process, due to non-locality of the equation (1), it has been discarded in the formulation of relativistic quantum mechanics [19-23]. The non-locality of this equation is due to the time evaluation of the wave function in each time is dependent to the value of the wave function on all over of space. It can be seen in integral representation of this equation clearly (equation(3))

Usually the mathematical non-locality of this equation is considered as incompatibility with special relativity, but it should be noticed that the Lorentz invariance of equations is strict requirements of special relativity and not their ”mathematical specious locality”. Checking Lorentz invariance of this equation is too complicated and therefore can not be easily specified, because of its high derivatives [23-24]. However because of the inequality in the order of space and time derivatives, this equation in the literature usually is referred to as a frame dependent equation and Incompatible with special relativity [13,19-26]. This paper by discussing two following hints we will show this impression is Incorrect. Firstly, the concept of ”non-locality” is different from concept of ”Lorentz invariance” Secondly, transformation (and interpretation) of the wave function should be taken into account in the survey of Lorentz invariance of this equation. In explaining first hint, we note that ”non-locality” is a general concept which may be used in different meanings.But the ”Lorentz invariance” -based on special relativity- is an explicit mathematical constraint which equations of each physical theory should satisfy it. However the concept of ”locality” is corresponds to concepts of ”Lorentz invariance” in special relativity, but these are not necessarily consequence of each other. We note that some versions of quantum mechanics are non-local and Lorentz invariant, for example the relativistic versions of Bohmian mechanics have this property [27-31]. Independently from the previous example, Bell inequality implies that every hidden variable theory must be non-local[32-33] . Even Dirac equation -which is compatible with special relativity- has kind of ”non-locality in time”[3]. On the other hand, we note that ”transformation of wave function” is the most important aspect of the procedure of checking the Lorentz invariance of the square root klein -Gordon equation that mostly neglected.
Note that to determination of transformation of all parts of equation on the Lorentz transformation are needed. And we can’t opine about Lorentz invariance of wave equation only based on transformation of differential operators. It should be noted that transformation of complex quantity such as wave function should be determine on the basis of the it’s physical interpretation and its relation with observable quantity. In this regard, transformation of the wave function should be determined on the basis of the relation of the wave function with position probability density $\rho$ (and also current density $J$). It means which transformation of the physical Quantity such as $\psi$ is determine on the Lorentz transformation. For example $\rho$ and $J$ together are transform as a four vector. Therefore by knowing the mathematical relationship between wave function and probability density-the physical interpretation of the wave function- we can find wave function transformation on the Lorentz transformation. So for checking the Lorenz invariance of the wave equation, first we must determine the physical interpretation of the wave function. Because the transformation of the wave function on the Lorentz transformation is undetermined before interpretation of the wave function. Checking the Lorentz invariance of the wave equation is meaningless without Knowing the wave function transformation. Therefore in the next sections as a first step, we take $|\psi|^2$ as position probability density and Check out the Lorentz invariance of the square root Klein-Gordon equation based on this interpretation.

II. THE INTERPRETATION OF THE ROOT SQUARE KLEIN-GORDON EQUATION BASED ON THE BORN RULE

At this stage we consider the Born rule as a physical interpretation of the wave function so by using the square root the Klein-Gordon equation and its conjugated complex the continuity equation will be

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0 \quad (5)$$

Where the current density, $J_B$, is given by:

$$J_B = \frac{i}{\hbar} \sum_{k=1}^{\infty} a_k \hbar^{2k} \sum_{l=0}^{2k-1} (-1)^l \partial^l \psi^* \partial^{2k-l-1} \psi \quad (6)$$

The K=1 in above relation is non-relativistic current density and show the consistency of the relation (6) in non-relativistic limits. By using the Fourier transform of the wave function,
the probability current density in momentum space can be obtained. We start from the Fourier transform of the wave function:

$$
\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \psi(x) e^{-\frac{i\hbar px}{\hbar}} dx
$$

Therefore the Born probability density ($\rho_B$) can be rewritten as:

$$
\rho_B = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi^*(p')\phi(p') e^{i(p'-p'')x} dp'' dp'
$$

Then, taking the derivative of Equation (8) with respect to time, and using the time evolution equation of the ,

$$
i\hbar \frac{\partial \phi(p,t)}{\partial t} = E(p)\phi(p,t) ; \quad E(p) = \sqrt{c^2(p)^2 + m_0^2 c^4}
$$

We have:

$$
\frac{\partial \rho_B}{\partial t} = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \frac{E(p'') - E(p')}{-i\hbar} \right] \phi^*(p')\phi(p') e^{i(p'-p'')x} dp'' dp'
$$

With comparing the equation (10) with continuum equation (5), we have:

$$
J_B = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U(p',p'') \phi^*(p')\phi(p') e^{i(p'-p'')x} dp'' dp'
$$

Where $U(p',p'') = \frac{E(p'')-E(p')}{-i\hbar}$. Note that the equation (11) is a general form for the Born current density in momentum representation. As the simplest example, we evaluate the position and current densities for a plane wave by using the equations (8) and (11):

$$
\psi(x) = Aexp\left(\frac{i\hbar px}{\hbar}\right) \rightarrow \phi(p') = A\delta(p' - p)
$$

$$
J_B = |\psi|^2 = |A|^2
$$

$$
J_B = |A|^2 \frac{dE(p')}{dp'}|_{p' = p} = |A|^2 \frac{pc^2}{\sqrt{p^2c^2 + m_0^2c^4}}
$$

According to the relativistic relation between velocity and momentum, these results are consistent with the $J = \rho u$. The relation (14) in position space can be derived from insertion of a flat wave (12) in equation (6). Now that we specified the relation between the wave function with a probability density of a position associated with the current density of the probability, so then we want to investigate the Lorentz invariant of the square root Klein-Gordon equation, but due to there are a high order derivative in eq.(1) and (6) so the investigations will have a mathematical complexity. Therefore to clarify the processes, at first we review the Gelileo invariant of the Schrodinger equation.
III. AN OVERVIEW OF A GALILEO INVARIANT OF SCHRODINGER EQUATION

In this section, we will review the Galileo invariant of Schrodinger equation. We consider the Schrodinger equation of a free particle in one dimension along with the conventional interpretation of the possibility

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial t^2}$$

(15)

$$\rho_{NR} = |\psi|^2 = R^2$$

(16)

$$J_{NR} = \frac{1}{i\hbar} (\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t})$$

(17)

Where $\rho_{NR}$ and $J_{NR}$ are the density of probability and the non-relativistic current density of probability respectively. We want to find the transformation of the wave function under Galileo transformation based on the above interpretation:

$$\rho'_{NR} = \rho_{NR} \rightarrow |\psi|^2 = |\psi'|^2$$

(18)

therefore, $\psi$ and $\psi'$ can only differ from each other by a phase function $\Lambda(x,t)$:

$$\psi' = \psi e^{\Lambda(x,t)}$$

(19)

By using the polar form of wave function, $\psi = Re^{\frac{i}{\hbar}S}$, the equation (18) leads to:

$$R' = R$$

(20)

$$S' = S + \hbar \Lambda(x,t)$$

(21)

One can use the equation of transformation of probability current density under Galilean transformation to find the unknown function $\Lambda(x,t)$:

$$J'_{NR} = J_{NR} - \rho v \rightarrow \frac{R^2}{m} \frac{\partial S'}{\partial x} = \frac{R^2}{m} \frac{\partial S}{\partial x} - R^2 v$$

(22)

Where $v$ is the relative velocity of two frames. With substituting the equations (20) and (21) in the equation (22) we get:

$$\Lambda(x,t) = -\frac{mv}{\hbar} x + C(x)$$

(23)
where $C(t)$ is an indefinite function of time. Now that the transformation of wave function is (almost) specified, so we want to examine whether the Schrodinger equation to be invariant under the Galilean transformation. In fact, it can be straightforwardly shown that the Schrodinger equation will be invariant under the Galilean transformations, provided that $C(t)$ is equal to $(mv^2t)/2$. Of course, we could assume that the Schrodinger equation is invariant under Galilean transformations and then we get the function $\Lambda(x,t)$ instead of using equation (22) to determine the function $\Lambda(x,t)$. Both of these processes lead to the same results and show that the nonrelativistic Schrodinger equation is invariant under Galilean transformations [34]. Finally, we note that the process of mathematical proof of the Galilean invariant Schrodinger equation in existence of appropriate transformation for the wave function is summarized which the three equations (15),(16) and (17) hold invariant under Galilean transformations.

IV. INVESTIGATION OF LORENTZ INVARIANT OF KLEIN-GORDON EQUATION ON THE BORN INTERPRETATION

Considering the explanations in the previous section, it is clear that due to examination of the Lorentz invariance of the square root Klein-Gordon equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \sum_{k=0}^{\infty} a_k \hbar^{2k} \frac{\partial^{2k} \psi}{\partial x^{2k}}$$  \hspace{1cm} (24)

Together with the Born Probabilistic interpretation

$$\rho_B = |\psi|^2$$  \hspace{1cm} (25)

$$J_B = \frac{i}{\hbar} \sum_{k=1}^{\infty} a_k \hbar^{2k} \sum_{l=0}^{2k-1} (-1)^l \frac{\partial^l \psi^* \partial^{2k-l-1} \psi}{\partial x^l \partial x^{2k-l-1}}$$  \hspace{1cm} (26)

Can be performed in two ways. In the first method the transformation of the wave function can be found according to the transformation of the $\rho_B$ and $J_B$ under Lorentz transformations and the equations (25) and (26), then having the transformation of the wave function we directly investigate the Lorentz invariant of the square Klein-Gordon equation. In the second method, the square root Klein-Gordon equation is invariant under Lorentz transformations and then we will find transformation of the wave function, and after that check whether such a transformation lead to relativistic wave function transformation for and or not ? namely
\( \rho_B \) and \( J_B \) together will form a ”four-vector” or not? however due to the nonrelativistic calculation were performed, transformation of the wave function can not be identified alone by using the equations (24), (25) and (26). to prove the Lorentz invariant of the square root Klein-Gordon equation we should find the transformation which simultaneously the equations (24), (25) and (26) becomes the equations with the prime parameters.

\[
i \hbar \frac{\partial \psi'}{\partial t'} = \sum_{k=0}^{\infty} a_k \hbar^{2k} \frac{\partial^{2k} \psi'}{\partial x'^{2k}} \quad (27)
\]

\[
\rho'_B = |\psi'|^2 \quad (28)
\]

\[
J'_B = \frac{i}{\hbar} \sum_{k=1}^{\infty} a_k \hbar^{2k} \sum_{l=0}^{2k-1} (-1)^l \frac{\partial \psi'^*}{\partial x'^l} \frac{\partial^{2k-l-1} \psi'}{\partial x'^{2k-l-1}} \quad (29)
\]

In principle, this transformation means Lorentz invariant of the square root Klein-Gordon equation with the Born rule. It seems very difficult to find the convert wave function with this method but due to the complexity of the mathematical form of the above equations. So we start with the simplest cases and gradually make the calculations more generally.

A. A plane wave

As the simplest example, suppose \( \psi \) as a plane wave,

\[
\psi(x, t) = A e^{i(Hx - E(p)t)} \quad , \quad E(p) = \sqrt{p^2 + c^2 + m_0^2c^4} \quad (30)
\]

We want to find a convenient transformation for wave function in its given form that leads to simultaneous Lorentz invariance of equations (24), (25) and (26). Such a wave in the moving frame, \( S' \), also takes the plane-wave form:

\[
\psi'(x', t') = A' e^{i(H'x' - E(p')t')} \quad , \quad E(p') = \sqrt{p'^2 + c^2 + m_0^2c^4} \quad (31)
\]

However it is possible that the amplitude of two waves are not equal. Also it is natural that the momentums \( p \) and \( p' \) are related to each other based on the relativistic transformations of momentum,

\[
p' = \gamma (p - v \sqrt{p^2c^2 + m_0^2c^4}) \quad , \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \quad (32)
\]

\( \psi \) and \( \psi' \) are the plane wave of particles with momentums \( p \) and \( p' \) in the two different frameworks. \( ? \) (implied). In order to evaluate the Lorentz invariant, there is enough to
consider a transformation which is compatible with the Born probabilistic interpretation. This transformation corresponding to finding the relationship between $A$ and $A'$. So at first we calculate $\rho_B$ and $J_B$ in the both frameworks:

$$\rho_B = |A|^2 \quad (33)$$

$$\rho_B' = |A'|^2 \quad (34)$$

$$J_B = |A|^2 \frac{pc^2}{\sqrt{p^2c^2 + m_0^2c^4}} \quad (35)$$

$$J_B' = |A'|^2 \frac{p'c^2}{\sqrt{p'^2c^2 + m_0^2c^4}} \quad (36)$$

We know that $\rho_B$ and $J_B$ should transform as a "four-vector", namely:

$$\rho' = \gamma(\rho_B - \frac{v^2}{c^2}J_B) \quad (37)$$

$$J'_B = \gamma(J_B - v\rho_B) \quad (38)$$

Where $\gamma$ is commonly defined as:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (39)$$

And $v$ is the relative velocity of the two frames. Therefore, with direct substitution of equations (33) to (36) in the equations (37) and (38) we have:

$$|A'|^2 = \gamma|A|^2 - \frac{v}{c^2}(|A|^2 - \frac{pc^2}{\sqrt{p^2c^2 + m_0^2c^4}}) \quad (40)$$

$$|A'|^2 \frac{p'c^2}{\sqrt{p'^2c^2 + m_0^2c^4}} = -\gamma |A|^2 + \gamma(|A|^2) \frac{pc^2}{\sqrt{p^2c^2 + m_0^2c^4}} \quad (41)$$

By substituting the momentums in equations (40) and (41) in terms of velocities, we get at these simpler:

$$|A'|^2 = \gamma|A|^2(1 - \frac{vu}{c^2}) \quad (42)$$
\[ |A'|^2 u^2 = \gamma |A|^2(u - v) \]  \tag{43}

Where \( u = pc^2(p^2c^2 + m_0^2c^4)^{\frac{1}{2}}, u' = p'c^2(p'^2c^2 + m_0^2c^4)^{\frac{1}{2}} \). From each of the above equations can be calculated \(| A^2 | \) in terms of \(| A^2 | \), which means it is finding the transformation of the wave function. It is necessary that the equations (42) and (43) are compatible to each other to the square root Klein-Gordon equation with \( \rho_B = |\psi|^2 \) is Lorentz invariant. For in this case the transformation for the wave function is found which the \( \rho_B \) and \( J_B \) transform as a four vector. Clearly the equations (42) and (43) are equivalent to each other if

\[ u' = \frac{u - v}{1 - \frac{uv}{c^2}} \]  \tag{44}

Equation (44) is the same as the relativistic transformation of the velocities whose truth is trivial.

\[ |A'|^2 = -\gamma |A|^2(1 - \frac{vu}{c^2}) \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \]  \tag{45}

Therefore we found a suitable transformation for a wave function in this particular case (single plane wave) but it should be noted that the requirement of Lorentz invariant of the square root of Klein-Gordon equation there must be an appropriate transformation not only for a wave function, but for an arbitrary wave function which the equation holds invariant and also, \( \rho_B \) and \( J_B \) convert as a four-vector. In the next section by considering the wave function of the superposition of two waves, we will investigate the Lorentz invariant of the square root Klein-Gordon Equation in more complex than the single plane wave. With regard to the Born interpretation of the wave function, we will see that the existence of interference effects violates the Lorentz invariant of the square root Klein-Gordon equation.

### B. Superposition of two plane wave

We now consider the wave function as a superposition of two plane waves.

\[ \psi(x, t) = A_1 e^{i\frac{\pi}{\hbar}(p_1x - E(p_1)t)} + A_2 e^{i\frac{\pi}{\hbar}(p_2x - E(p_2)t)} \]  \tag{46}

\[ \psi(x', t') = A'_1 e^{i\frac{\pi}{\hbar}(p'_1x' - E(p'_1)t')} + A'_2 e^{i\frac{\pi}{\hbar}(p'_2x' - E(p'_2)t')} \]  \tag{47}

The time dependence of the wave functions in equations (46) and (47), establish the square root Klein-Gordon equation in both frameworks. Then to prove the Lorentz invariant of the
equations (24), (25) and (26), we just have to find a transformation for the wave function which \( \rho_B \) and \( J_B \) transform as a four-vector under Lorentz transformations. In the process whenever necessary a polar form factors have been used:

\[
A_1 = |A_1| e^{i\delta_1}, \quad A_2 = |A_1| e^{i\delta_2}, \quad A_1' = |A_1'| e^{i\delta_1'}, \quad A_2' = |A_1'| e^{i\delta_2'}
\]

Substituting the wave functions (46) and (47) in the equations (8) and (11) leads to:

\[
\rho_B = |A_1|^2 + |A_2|^2 + 2|A_1||A_2| \cos(\Delta_{12}(x, t) + \delta_{12}) \tag{48}
\]

\[
\rho_B' = |A_1'|^2 + |A_2'|^2 + 2|A_1'||A_2'| \cos(\Delta_{12}'(x', t') + \delta_{12}') \tag{49}
\]

\[
J_B = u_1|A_1|^2 + u_2|A_2|^2 + 2|A_1||A_2|u_{12} \cos(\Delta_{12}(x, t) + \delta_{12}) \tag{50}
\]

\[
J_B' = u_1'|A_1'|^2 + u_2'|A_2'|^2 + 2|A_1'||A_2'|u_{12}' \cos(\Delta_{12}'(x', t') + \delta_{12}') \tag{51}
\]

In which we used the shorthand notations \( U_{12} = U(p_2, p_1), U'_{12} = U(p_2', p_1'), \delta_{12} = \delta_2 - \delta_1 \) and \( \delta_{12}' = \delta_2' - \delta_1' \). Also the quantities \( \Delta_{12}(x, t) \) and \( \Delta_{12}'(x', t') \) are defined as follows:

\[
\Delta_{12}(x, t) = (p_2 - p_1)x - (E(p_2 - E(p_1))t \tag{52}
\]

\[
\Delta_{12}'(x', t') = (p_2' - p_1')x - (E(p_2' - E(p_1'))t \tag{53}
\]

It is clear that the quantity \( \Delta_{12} \) is a scalar by its definition:

\[
\Delta_{12}'(x', t') = \Delta_{12}(x, t) \tag{54}
\]

Then by direct substitution of equations (48)-(51) in the relativistic probability density transformation, \( \rho_B' = \gamma(\rho_B - v/c^2 J_B) \), we arrive at following equations:

\[
\delta_{12}' - \delta_{12} = \delta_2 - \delta_1 \tag{55}
\]

\[
|A_1'||A_2'| = \gamma(1 - \frac{vU_{12}}{c^2})|A_1||A_2| \tag{56}
\]

\[
\sum_{i=1}^{2} |A_i|^2 = \sum_{i=1}^{2} \frac{\gamma}{c^2}(c^2 - vu_i)|A_i|^2 \tag{57}
\]
Also, direct substitution of equations (48)-(51) into the relativistic current probability density transformation, \( J_B' = \gamma(J_B - v\rho_B) \), leads to:

\[
\delta_2' - \delta_1' = \delta_2 - \delta_1 
\]

(58)

\[
|A_1'|^2|A_2'|^2 = \frac{\gamma(U_{12} - v)}{U_{12}}|A_1|^2|A_2|^2 \]

(59)

\[
\sum_{i=1}^{2} u_i'|A_i'|^2 = \sum_{i=1}^{2} \gamma(u_i - v)|A_i|^2 
\]

(60)

In the above relation \( u_1, u_2, u'_1 \) and \( u'_2 \) are the velocities corresponding to \( p_1, p_2, p'_1 \) and \( p'_2 \). Here the equations (56) and (57) are equivalent with (59) and (60). Note that from each of these sets \( |A'_1|^2 \) and \( |A'_2|^2 \) can be calculated in terms of \( |A_1|^2 \) and \( |A_2|^2 \). Due to the Lorentz invariance formalism, the results of these calculation should be identical based on both equations. But instead of doing this calculation, we will review the compatibility of the system of equations with another way. At first we calculate \( |A'_1|^2|A'_2|^2 \) from each of equations (56) and (59) and for compatibility we have

\[
U_{12}' = \frac{U_{12} - v}{1 - \frac{vu}{c^2}} 
\]

(61)

By putting the defined functions in equation (61), we can see that this equality is maintained with a long but straightforward calculation. The equation (61) means that the "velocity" obeys the relativistic velocity transformations. It is easy to show that the system of equations (56) and (57) are not consistent with equations (59) and (60) and does not have solutions because the equations (57) and (60) are linear equations for two variables \( |A'_1|^2 \) and \( |A'_2|^2 \) that their solutions are as follows:

\[
|A'_1|^2 = \gamma(1 - \frac{vu_1}{c^2})|A_1|^2 
\]

(62)

\[
|A'_2|^2 = \gamma(1 - \frac{vu_2}{c^2})|A_2|^2 
\]

(63)

But the above solutions are inconsistent with equation (56), because the direct substitution of equations (62) and (63) into equation (56) leads to the following incorrect equality:

\[
\gamma(1 - \frac{vu_1}{c^2})(1 - \frac{vu_2}{c^2}) = (1 - \frac{vU_{12}}{c^2}) 
\]

(64)
Incorrectness of the equality (64) could straightforwardly be investigated by substituting the definition of the quantity $U_{12}$ into it. Therefore, in case of superposition of two plane waves, there is no correct transformation for the wave function which makes the square root Klein-Gordon equation along with the Born rule $\rho = |\psi|^2$ into a Lorentz invariant formalism. In free space, the single waves can be described as a stream of classical particles with constant velocity and it can be found the appropriate transformation for the wave function but due to interference effects, the superposition of two waves can not be described as a classical fluid with independent densities and velocities. Therefore by considering the superposition of two plane waves, we are going to the level of quantum mechanics and investigating the Lorentz invariant of the square root Klein-Gordon equation in this case. As we have shown by considering the Born rule does not exist an appropriate transformation for a wave function. A similar result is obtained for a superposition of an arbitrary number of plane waves and so the final result will be as follows: For an arbitrary wave packet - which is made of the superposition of plane waves- no transformation will be found which keeps the Lorentz invariant of the square root Klein-Gordon equation and simultaneously is consistent with giving $|\psi|^2$ as position probability density! The above result shows that for the generalization of relativistic quantum mechanics, the square root Klein-Gordon equation or the Born rule $\rho = |\psi|^2$ or both of them must be removed. In the conventional formulation of relativistic quantum mechanics to construct a Lorentz invariant theory, the square root Klein-Gordon equation was excluded. An interesting question that arises is that: can be found a Lorentz invariant theory with preserving the square root Klein -Gordon equation and the corrected Born rule? In this paper, we show that the answer is yes and some results of this theory will be examined.

V. NEW INTERPRETATION OF LORENTZ INVARIANT OF THE SQUARE ROOT KLEIN-GORDON EQUATION

A. Relativistic correction of the Born rule

In this section, by modification of Born rule, we want to introduce a correct Lorentz invariant interpretation of the square root Klein-Gordon equation. Hereafter, most of our calculations are in momentum representation, because it is mathematically less complicated.
To suggest a more general form for probabilistic interpretation of wave function, we import an indefinite function $F(p', p'')$ in the momentum representation of Born rule (that is, equation (8))

$$\rho = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(p', p'') \phi^*(p'')\phi(p')e^{i\frac{\hbar}{\gamma}(p''-p')x}dp''dp'$$  \hspace{1cm} (65)

The above relation is one of the simplest generalizations of the Born rule in the momentum representation. By substituting the eq(65) in the continuity equation and using the root Klein-Gordon equation, the following expression for the probability current density is achieved:

$$J = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(p', p'')U(p'', p')\phi^*(p'')\phi(p')e^{i\frac{\hbar}{\gamma}(p''-p')x}dp''dp'$$  \hspace{1cm} (66)

Although the relation (65) is not the most general possibility to generalize the Born rule but it can use as a general because we don’t consider a specific form for the function. Only the equation (65) leads to the true probability density so the function $F$ is required to meet the following constraints:

$$F^*(p', p'') = F(p'', p')$$  \hspace{1cm} (67)

Now we are going to find an explicit expression for the function $F(p', p'')$, with supposing the Lorentz invariant of the square root Klein-Gordon equation. For this purpose we will obtain the probability density and the current density of superposition of two plane waves (46) and (47), by using the equations (65) and (66). Then, by substituting these results in the equation of the probability density transformation, $\rho' = \gamma(\rho - \frac{\gamma}{c^2}J)$, we arrive at following equations:

$$F'_{12} | A'_1 || A'_2 | = \gamma(1 - \frac{vU_{12}}{c^2}) | A_1 || A_2 | F_{12}$$  \hspace{1cm} (68)

$$\sum_{i=1}^{2} F'_{ii} | A'_i | = \sum_{i=1}^{2} \frac{\gamma}{c^2}(c^2 - vu_i) | A_i |^2 F_{ii}$$  \hspace{1cm} (69)

Where the shorthand notations $F_{ij} = F(p_i, p_j)$ and $F'_{ij} = F(p'_i, p'_j)$ were used for simplicity. Also, in this case the equation of transformation of current density, $J' = \gamma(J - v\rho)$, leads to:

$$F'_{12} | A'_1 || A'_2 | = \frac{\gamma(U_{12} - v)}{U'_{12}} | A_1 || A_2 | F_{12}$$  \hspace{1cm} (70)
\[ \sum_{i=1}^{2} u_i F_{ii} | A_i | = \sum_{i=1}^{2} \gamma (u_i - v) | A_i |^2 F_{ii} \] (71)

Naturally, the difference between the equations (56), (57), (59) and (60) with equations (68) to (71) is only in the presence of \( \gamma \). From the equation (61) we can conclude that the equations (68) and (70) are together equivalent. It is only necessary to investigate the compatibility of the equations (68), (69) and (71). \( A'_i \) and \( A'_i' \) can be calculated in terms of \( A_i \) and \( A_i' \) from equations (69) and (71):

\[ | A'_i |^2 = \frac{F_{11}}{F'_{11}} (1 - \frac{vu_1}{c^2}) | A_1 |^2 \] (72)

\[ | A'_2 |^2 = \frac{F_{22}}{F'_{22}} (1 - \frac{vu_2}{c^2}) | A_2 |^2 \] (73)

As a result, we will only be needing the consistency of the equations (72) and (73) with the equation (68). For this purpose, first we will square the equation (68) and substitute the equations (72) and (73) in it; which leads to:

\[ \left( \frac{F(p'_1, p'_2)}{F(p_1, p_2)} \right)^2 \frac{F(p_1, p_1)}{F(p'_1, p'_1)} \frac{F(p_2, p_2)}{F(p'_2, p'_2)} \frac{(1 - \frac{vU_{12}}{c^2})}{(1 - \frac{vu_1}{c^2})(1 - \frac{vu_2}{c^2})} \] (74)

Since \( u_1 \), \( u_2 \) and \( U_{12} \) are substituted by \( p_1 \) and \( p_2 \), the right hand side of the above equation in terms of \( p_1 \) and \( p_2 \) clearly defined. As a result, equation (74) is a mathematical constraint and for compatibility the function \( F \) be satisfies in equations (68), (69) and (71). Applies to the function \( F \) in equation (74) ensures that \( \text{and} \) \( J \) which are defined by equations (65) and (66 form a four-vector. In other words, the definitions (65) and (66) lead to Lorentz invariant interpretation of the square root Klein-Gordon equation. The calculations of the previous section showed that \( F(p_1, p_2) = 1 \) (which = \( |\psi|^2 \) leads) does not apply in the above equation. This equation has no separable solution as \( F(p_1, p_2) = F(p_1) F(p_2) \) because the right hand side of eq. (74) is equal to 1 for such a choice for the function \( F \) and again a false statement (64) will get. In fact, the general solution of the equation (74) has the form (Appendix 1):

\[ F(p_1, p_2) = F_D g(p_1) g(p_2) G(\xi) \] (75)

Where \( F_D \) and \( \xi \) are defined as follows:

\[ F_D = 1 + \left( \frac{p_1 c}{mc^2 + E(p_1)} \right) \left( \frac{p_2 c}{mc^2 + E(p_2)} \right) \] (76)
\[ \xi = \ln \left( \frac{p_2^c + E(p_2)}{p_1^c + E(p_1)} \right) \]  

(77)

\( \xi \) and \( g \) are arbitrary functions. Of course, the function must satisfy the following equation:

\[ G(0) = 1 \]  

(78)

Moreover, as we noted before, for the probability density defined based on the function \( F(p_1, p_2) \) to be real, truth (establishment) of the condition (67) is necessary. In the general case, this condition requires the absolute realness of function \( g \) or its absolute imaginariness and in addition to this, the function must satisfy the following condition (appendix 1):

\[ G(-\xi) = G^*(\xi) \]  

(79)

Clearly, the general solution (75) is a very broad set because of arbitrariness of the functions \( g \) and \( G \). In this paper we derive the simplest possible solution for \( F(p_1, p_2) \) supposing \( g(p_1) = g(p_2) = G(\xi) \):

\[ F(p_1, p_2) = F_D = 1 + \left( \frac{p_1^c}{mc^2 + E(p_1)} \right) \left( \frac{p_2^c}{mc^2 + E(p_2)} \right) \]  

(80)

However with directly substitution, we can see that the above relation applies in the equation (74). The particular solution of the eq.(80) will be represented with \( F_D(p_1, p_2) \). We note that \( F_D(p_1, p_2) \) in nonrelativistic limit \( \frac{p}{mc} \ll 1 \) approximate to the unity \( F(p_1, p_2) \to 1 \), so that the compatibility with the Born rule be guaranteed in nonrelativistic limit.

The probability density and current density corresponding to \( F_D(p_1, p_2) \) represent with \( \rho_D \) and \( J_D \) respectively. Now we want to calculate the and explicitly in terms of the wave function (in position representation). For this purpose, we use the Taylor expansion of \( \frac{pc}{mc^2 + E(p)} \) around \( p = 0 \):

\[ \frac{pc}{mc^2 + E(p)} = \sum_{i=0}^{\infty} D_i p^i \]  

(81)

We rewrite the equation (80) as follows:

\[ F_D(p_1, p_2) = 1 + \sum_{i=0}^{\infty} D_i p_1^i \sum_{j=0}^{\infty} D_j p_2^j \]  

(82)

Where \( D_i \) are the expansion coefficients. We can find \( \rho_D \) and \( J_D \) in the position representation to be by direct substitution of the equation (82) in the (65) and (66):

\[ \rho_D = \psi^* \psi + (\hat{D} \psi)^* (\hat{D} \psi) \]  

(83)
\[ J_D = \psi^*(\hat{D}\psi) + \psi(\hat{D}\psi)^* \]  

(84)

Where the linear operator \( \hat{D} \) defined as:

\[ \hat{D} = \sum_j (-i\hbar)^j D_j \frac{\partial^j}{\partial x^j} \]  

(85)

Clearly, the nonrelativistic limit of the equations (83) and (84) are just the Schrodinger’s probability and current densities, respectively, in fact the equations (83) and (84) can be considered as relativistic generalization of the Born probabilistic interpretation.

B. Investigation of deviation from the Born rule in relativistic level

We are going to examine the deviation from the Born rule by using the equations (83) and (84) as the relativistic correction. At first with an example, we obtain the quantitative estimation of the deviation of ” the relativistic probability density, \( \rho_D \)” from the Born rule \( (\rho_B = \psi^*\psi) \). For this purpose, we consider a particle in a one-dimensional potential well with \( x = 0 \) to \( x = L \):

\[ \sum_{k=0}^{\infty} a_k h^{2k} \frac{\partial^{2k}}{\partial x^{2k}} \psi_n = E_n \psi_n \]  

(86)

With the boundary conditions:

\[ \psi_n(0) = \psi_n(L) = 0 \]  

(87)

Leads to the following energy spectrum and steady states:

\[ E_n = \sqrt{c^2 \left( \frac{n\pi \hbar}{L} \right)^2 + m_0^2 c^4} \]  

(88)

\[ \psi_n = A_n \sin(\frac{n\pi \hbar}{L}) \]  

(89)

If we accept the equations (83) and (84) are the physical interpretation of the wave function so the probabilistic content of energy function is different from the nonrelativistic case. To view this explicitly differences by substituting the wave function (84) in equation (83) the probability density of position is calculated.

\[ \rho_D = |A_n|^2 \left[ \sin^2 \left( \frac{n\pi \hbar}{L} \right) + \left( \sum_{j=0} \left( \frac{n\pi \hbar}{L} \right)^j \right)^2 \cos^2 \left( \frac{n\pi \hbar}{L} \right) \right] \]  

(90)
Considering the equation (81) one can simplify the above equation to read:

\[
\rho_D = |A_n|^2 \sin^2\left(\frac{n\pi \hbar}{L}\right) + \left(\frac{n\pi L_c/L}{1 + \sqrt{1 + (n\pi L_c/L)^2}}\right)^2 \cos^2\left(\frac{n\pi \hbar}{L}\right)
\]

(91)

Where \( L_c \) represents the Compton wavelength of particle, \( L_c = \frac{\hbar}{mc} \). The quantity \( A_n \) can be obtained from Normalization condition, \( \int_0^L \rho_D dx = 1 \), to have the following form:

\[
|A_n|^2 = \sqrt{\frac{2}{L}} \left[ 1 + \left(\frac{n\pi L_c/L}{1 + \sqrt{1 + (n\pi L_c/L)^2}}\right)^2 \right]^{-\frac{1}{2}}
\]

(92)

In Figure 1, the probability density of the position for the first excited state for different size boxes (compared to the Compton length) is plotted. As seen from Figure, it is clear; If the box size is smaller or equal to the Compton wavelength of the particle, the probability density is considerably deviate from the Born rule (\( \rho = |\psi|^2 \)). So to see this effects for particle such as an electron it is necessary the electrons locked up in a box with dimensions of less than \( 10^{-13}m \). The above results show that even in the weak relativistic limit, the deviation of the probability density of the position from the Born probability density is not negligible but it has not been noticed in many articles that the square root Klein-Gordon equation is used to describe some phenomena. For example, in [8], the quantum dynamics of plasmas in the relativistic regime have been formulated based on the square root Klein-Gordon equation and the properties of wave propagation in such plasmas are calculated. In this paper, \( \rho = e \psi^* \psi \) is used as the charge density in the Poisson equation \( \nabla^2 \phi = -\frac{\varepsilon_0}{\varepsilon_0} |\psi|^2 \). But as we have shown in this article, such an interpretation of the wave function leads to a theory of non-Lorentz invariant. Thus for a consistent calculations, it is necessary to calculate the right-hand side of the Poisson equation to use the correction to the Born rule as equation (78). This correction changes the properties of the wave dispersion in quantum plasmas (calculated in [8]). This example shows that the deviation of the probability density of the position from the Born rule can cause observable effects in the relativistic limit.

The question that arises at this stage is: The deviation from the Born in the relativistic level is just only for a probability distribution of position or the probability distribution of the other quantities deviated from the Born rule? To answer this question, we note that there are several methods for extracting the Born rule for other quantities from the default setting of this principle is in position. As a specific example, \( |\phi|^2 \) as the probability density of momentum have been obtained by analyzing the process of measuring the time of flight with the assumption \( \rho = |\psi|^2 \) [27]. In general, the establishment of Born rule on
FIG. 1: This figure shows the general behavior of the relativistic probability density of the particle in a box. The blue-dashed line represents the nonrelativistic probability density $\rho_B = |\psi|^2$ and red line represents relativistic probability density at the first excited state ($n = 2$) for a particle in a box. If the size of the box is large relative to the Compton wavelength of a particle then $\rho_D$ to be consistent with $\rho_B$ and the size of box is smaller than the Compton wavelength of the particle, $\rho_D$ is deviated from $\rho_B$. In fact $L \rightarrow 0$ so $\rho_D \rightarrow 1$ and the probability density of the particle is uniform.

Other observable quantities can be derived according to the causal theory of measurement with the assumption $\rho = |\psi|^2$ [27, 35]. In all such demonstration, the measurement of other quantities related to measurement of position and thus the establishment of the Born rule for the probability density of position has a key role in the derivation of this principle for the other observable quantities. Consequently, deviation of probability density of position from the Born rule can deviate probability density of other quantities from the Born rule.

Most important results of the formalism of quantum mechanics is rooted in the establishment of the Born rule. For example, the statistical interpretation of uncertainty principle relation of position-momentum is rooted in the acceptance of $|\psi|^2$ as a probability density of position, and $|\phi|^2$ as a probability density of momentum. So if the Born rule in relativistic level is not established, the uncertainty principle in the relativistic case will be doubtful and then the Heisenberg uncertainty relations in the relativistic should be corrected. However,
the need to reform the Heisenberg uncertainty relations even in the early years of the relativistic formulation of quantum mechanics has been expressed. For example, in 1930 Landau referred to some restrictions of the application of the Heisenberg uncertainty relations in relativistic limit [36]. However, this is not considered in the formulation of relativistic quantum mechanics (or less is considered). Also Aldaya. et al have used the commutative relations to describe the relativistic harmonic oscillators differently than the Heisenberg commutative relation which involve modification of the uncertainty relations in the relativistic level [37-39]. Although we have not provided the explicit expressions for the relativistic correction of uncertainty relations but note that the corrected form of the Born rule , which is presented in this paper - can be used as a tool to find the relativistic correction of uncertainty relation. It should be noted that this relativistic corrections play an important role in the analysis of some phenomena, for example, the usual Heisenberg uncertainty relations is used to analyzing the neutrino oscillations on the wave packet approach (used to calculate the coherence length of the wave packet) [40-43]. While considering the speed of neutrinos, regardless of the relativistic corrections to the uncertainty principle is not allowed on this phenomenon and therefore such computation should also be reviewed.

C. A relationship with the Dirac equation

It’s usual to think of the square root Klein-Gordon equation as a description for particles having zero spin. Although, similar to Dirac’s method some physicists use a four-vector and rewrite the square root Klein-Gordon equation as a way to generalize this equation for particles with spin [4 and 44]. As an example the following generalization of the square root Klein-Gordon equation is used as a description for particles having spin: \[ i\hbar \frac{\partial \psi}{\partial t} = \left[ \beta \sqrt{c^2 (e^2 - \frac{e^2}{c^2} A^2) - e\hbar c \Sigma \cdot B + m^2 c^4} \right] \psi \] (93)

In which \( \beta \) and \( \Sigma \) are Dirac matrices:

\[ \beta = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \] (94)

Where \( I \) are Pauli matrices and the identity matrix respectively. In this section we shall show that the square root Klein-Gordon equation can be used to provide a consistent description of the particle with the spin1/2 even without the Pauli matrices. \( F_D \) is more
than a simple answer to the equation (74). In fact $D$ and $J_D$ together (sort of) are equivalent to the four Dirac current and consideration of the $D$ and $J_D$ are considered as a probabilistic interpretation of the wave function and then the square root Klein-Gordon equation converts to isomorphic formalism with the positive energy solutions of the Dirac equation (in one dimension). The index D refers to the connection with the Dirac equation. To illustrate this point, we introduce as a four-component wave function as follows:

$$
\psi = \begin{pmatrix}
\psi \\
0 \\
\hat{D}\psi \\
0
\end{pmatrix}
$$

(95)

By using the Fourier transformation, $\Psi$ in terms of wave function in momentum space takes the form:

$$
\Psi(x, t) = \int_{-\infty}^{+\infty} \phi(p) U_p dp
$$

(96)

Where $U_p$ is defined as

$$
U_p = \begin{pmatrix}
1 \\
0 \\
\frac{pc}{mc^2 + E(p)} \\
0
\end{pmatrix} e^{i(px - E(p)t)}
$$

(97)

It should be reminded that the plane wave $U_p$ represents the positive energy solution of the Dirac equation for a free particle in one dimension and as a result represents the wave packet which satisfies the Dirac equation. Also, current and probability densities $\rho_D$ and $J_D$ alongside together are the same as Dirac’s four-vector current, $J_D^\mu = \psi^\dagger \gamma^\mu \psi$:

$$
\rho_D = \Psi^* \Psi = \psi^\dagger \psi + (\hat{D}\psi)^* (\hat{D}\psi)
$$

(98)

$$
J_D = \Psi^\dagger \gamma^1 \Psi = \psi^*(\hat{D}\psi) + \psi(\hat{D}\psi)^*
$$

(99)

Then the square root Klein-Gordon equation with a probabilistic interpretation of the wave function on the basis of equations (83) and (84) becomes the similar formalism of the Dirac equation with positive energy solutions. Such a profound correspondence between the square
root Klein-Gordon equation with Dirac equation is considerable and this result is different from the conventional approach which the square root Klein-Gordon equation describes zero spin particles. So that with the corrected interpretation of the wave function, we can use the square root Klein-Gordon equation to describe the particles of spin without the Pauli matrices.

It should be noted that the correspondence of the square root Klein-Gordon equation with the Dirac equation is not complete due to the separation of the solution of positive and negative energy. It may be avoided from some of the paradoxes of the Dirac equation that the results of the mixing the positive and negative energy solutions such as the Klein paradox. As a result, it is possible that you can present a more appropriate description for the spin- particles than other equations by using the square root Klein-Gordon equation.

VI. DISCUSSION

In the early development of relativistic quantum mechanics, the square root Klein Gordon equation is considered as a non-Lorentz invariant equation so this equation is not used in the formulation of relativistic quantum mechanics. But we have shown that with the proper physical interpretation of the wave function, can be found the Lorentz invariant interpretation of the square root Klein Gordon equation. Also our calculations were performed in the absence of external field (in one dimension), while the most doubts about the Lorentz invariant of equation are in the external field. And also the Dirac in his book [45] notes that the square root Klein Gordon equation in the presence of interaction is not Lorentz invariant and therefore should be excluded:

...although it takes into account the relation between energy and momentum required by relativity, [it] is yet unsatisfactory from the point of view of relativistic theory it is very unsymmetrical between $p_0$ and the other $p's$, so much so that one cannot generalize it in a relativistic way to the case when there is a field present. We must therefore look for a new wave equation...

In this regard, in 1963, J. Sucher tried to prove what Dirac said. He showed that the square root Klein Gordon equation is not Lorentz invariant in the presence of interactions by entering the interaction with minimal coupling ($\partial_\mu \rightarrow \partial_\mu - ieA_\mu$) and supposed that the wave function is scalar function [46]. But it should be noted that the default scalar wave
function is not required and the properties of the wave function may be more complex. In fact converting a wave function must be determined according to the physical interpretation and we have shown that with the proper interpretation of the wave function, the square root Klein-Gordon equation is Lorentz invariant in the absence of external field. So then with such an interpretation, the wave function is not scalar function. However in the presence of external fields, the default scalar wave function may be corrected before its physical interpretation. And also, Sucher proof should not be considered as a final proof of the non-Lorentz invariant of the square root Klein-Gordon equation because it is possible to construct a Lorentz invariant equation with proper physical interpretation of the wave function in the presence of interaction.

On the other hand, in 1993, J. Smith showed the square root Klein- Gordon equation by including the interaction with minimal coupling is inconsistent with the assumption that this equation is invariant under gauge transformations [37]. Smith concluded that entering interactions with minimal coupling constant will be incorrect and thus the Sucher's proof be incomplete. And may we can construct a theory which is invariant under Lorentz transformations and also gauge transformation by intering interaction with minimal coupling constant differently [47]. With the correct interpretation of the wave function, it is possible that gauge invariant is well established even taking into account the interaction of a minimal coupling. Note that the gauge invariant means the equation is invariant under the transformation of the potential as this form:

$$A'_\mu = A_\mu + \partial_\mu \Theta$$  \hspace{1cm} (100)

where $\Theta$ is an arbitrary function which represents the gauge transformation. To illustrate the gauge invariant of wave equations, transform of the wave function (Under the gauge transformations) can be considered as follows:

$$\psi' = \psi e^{i\Theta(x,t)}$$  \hspace{1cm} (101)

These may be the above transformation is not appropriate to gauge invariant of the square root of Klein Gordon equation because the physical content (testability of the empirical predictions) must be is invariant under the gauge transformations and the wave function transforms under gauge transformations must be compatible with it. For example, the Born probability density, $\rho_B = |\psi|^2$, (used for the physical interpretation the non-relativistic
Schrödinger equation) is invariant under the transforming the wave function in (101). In the non-relativistic quantum mechanics, the transforming the wave function does not change the physical content in (101). But as we have shown in the relativistic limit the Born rule is not correct and the relationship between the wave function and the probability density with $|\psi|^2$ is different. As a result, it is possible that for the physical content of the theory is invariant under the gauge transformation, it is required to the different transformation of the wave function (other than $\psi' = \psi e^{i\Theta(x,t)}$). In other words, Due to the relativistic generalization of the Born rule (equations (83) and (84)), freedom is more than a phase factor in the transforming the wave function without changing its content.

J. Smith assumed the transformation off the wave function under gauge transformation as (101) without specific interpretation of the wave function while considering the transformation such as (101) before its interpretation is not necessary. It is similar to the default scalar function for the Lorentz transformation of the wave function in the Sucher’s article [46]. In this article the transformation without the physical interpretation of the wave function and its related to observable quantities is considered. The default setting of invariant symmetry such as Lorentz and gauge invariance can be used to find the correct interpretation of the wave function in the presence of the external fields. The development of methods from this paper can be built the square root Klein-Gordon which is invariant under the gauge transformation. Finally the consistent interpretation with Lorentz and gauge invariance for the square root Klein-Gordon equation is an open question in this article.

VII. CONCLUSION

The physical interpretation of wave function is a fundamental element of the Lorentz invariant of the quantum wave function. So study of the Lorentz invariant of wave equation is meaningless independent of its physical interpretation. In fact, the interpretation of wave function makes a constraint on the wave function (under the Lorentz transformations). The conversion of the wave function involved directly in the process of the mathematical proof of the Lorentz invariant equation. This issue can be viewed from another direction: assuming the Lorentz invariant of wave function can be interpreted as mathematical constraints on the wave function and can be produced its relation to observable quantities. In this paper, by using the application of such constraints on the square root Klein Gordon equation the
relativistic generalization of the Born rule achieved. Based on the generalized probabilistic interpretation, the Lorentz invariant of the square root Klein Gordon equation in the absence of interaction have been proven. It should be noted that such an interpretation is based on a non scalar wave function, while all proof of non- Lorentz invariant of the square root Klein-Gordon equation consists default of scalar wave function (even in the presence of interactions) [46]. But since such an assumption before interpretation of the wave function is not necessary, therefore all such proofs are incomplete. It seems the main reason that cause the square root Klein-Gordon removed in the formulation of relativistic quantum mechanics can be bridged with the proper interpretation of the wave function. As a result, again re-formulation of quantum mechanics and relativistic quantum field theory based on the square root Klein Gordon equation is important. By using the square root Klein-Gordon, some paradoxical results of the Dirac equation such as the Klein tunneling are rigorous avoided

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