Hausdorff measure of SLE curves

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Abstract

In this paper we prove that the Hausdorff $d$-measure of $SLE_\kappa$ is zero when $d = 1 + \frac{8}{\kappa}$.

1 Introduction

A number of measures on paths or clusters on two-dimensional lattices arising from critical statistical mechanical models are believed to exhibit some kind of conformal invariance in the scaling limit. Schramm introduced a one-parameter family of such processes, now called the (chordal) Schramm-Loewner evolution with parameter $\kappa$ ($SLE_\kappa$), and showed that these give the only possible limits for conformally invariant processes in simply connected domains satisfying a certain “domain Markov property”. He defined the process as a measure on curves from 0 to $\infty$ in $\mathbb{H}$, and then used conformal invariance to define the process in other simply connected domains.

Since then, the geometric properties of SLE has been studied extensively. In [1], Beffara proved that the Hausdorff dimension of the SLE path is $d = 1 + \min\{a, \frac{8}{\kappa}\}$. His method is based on a certain two-point estimate and can not give the $d$-measure. In [6], Lawler and Sheffield defined a $d$-dimensional measure of SLE for some values of $\kappa$. They conjectured the Hausdorff $d$-measure of SLE is zero. Later, in [8] and [5], the construction of natural parametrization was extended to all $\kappa < 8$ and basic properties of it were studied. In this paper, by using the properties of natural parametrization, we want to prove this conjecture.

Theorem 1.1. Suppose $\gamma$ is an $SLE_\kappa$ curve in $\mathbb{H}$ from 0 to $\infty$ and $d = 1 + \min\{\frac{8}{\kappa}, 1\}$. Then we have

$$\mathcal{H}^d(\gamma) = 0.$$
Here we define $\mathcal{H}^d$ and we start SLE theory in the next section. There are number of places where we can find the definition of Hausdorff measure, for example [2].

If $V \subset \mathbb{R}^n$ and $\alpha, \epsilon > 0$, let

$$\mathcal{H}^\alpha_\epsilon(V) = \inf \sum_{n=1}^{\infty} [\text{diam}(U_n)]^\alpha,$$

where the infimum is over all countable collection of sets $U_1, U_2, \ldots$ with $V \subset \bigcup U_n$ and $\text{diam}(U_n) < \epsilon$. It is easy to see this is an outer measure. The Hausdorff $\alpha$ - measure is defined by

$$\mathcal{H}^\alpha(V) = \lim_{\epsilon \to 0^+} \mathcal{H}^\alpha_\epsilon(V).$$

Since $\mathcal{H}^\alpha_\epsilon(V)$ is decreasing in $\epsilon$, the limit exists with infinity as a possible value. Note that $\mathcal{H}^\alpha$ is an outer measure. Also it is easy to check that if $\mathcal{H}^\alpha(V) < \infty$, then $\mathcal{H}^\beta(V) = 0$ for $\beta > \alpha$, and if $\mathcal{H}^\alpha(V) > 0$, then $\mathcal{H}^\beta(V) = \infty$ for $\beta < \alpha$. The Hausdorff dimension of $V$ is defined by

$$\text{dim}_h(V) = \inf \{\alpha : \mathcal{H}^\alpha(V) = 0\} = \sup \{\alpha : \mathcal{H}^\alpha(V) = \infty\}.$$

By Beffara’s result we get $\mathcal{H}^{d-\epsilon}(\gamma) = \infty$ and $\mathcal{H}^{d+\epsilon}(\gamma) = 0$.

## 2 SLE and natural parametrization

### 2.1 Schramm-Loewner evolution (SLE) and the Green’s function

In this section we review the definition of chordal Schramm-Loewner evolution and introduce our notation. See [2] for more details.

Suppose that $\gamma : (0, \infty) \to \mathbb{H} = \{x+iy : y > 0\}$ is a curve with $\gamma(0+) \in \mathbb{R}$ and $\gamma(t) \to \infty$ as $t \to \infty$. Let $H_t$ be the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$. Using the Riemann mapping theorem, one can see that there is a unique conformal transformation

$$g_t : H_t \to \mathbb{H}$$

satisfying $g_t(z) - z \to 0$ as $z \to \infty$. It has an expansion at infinity

$$g_t(z) = z + \frac{a(t)}{z} + O(|z|^{-2}).$$

$a(t)$ equals $\text{hcap}(\gamma(0, t])$ where $\text{hcap}(A)$ denotes the half plane capacity from infinity for a bounded set $A$ such that $\mathbb{H}/A$ is simply connected. It can be defined by

$$\text{hcap}(A) = \lim_{y \to \infty} y \mathbb{E}^{iy}[\text{Im}(B_\tau)],$$

where $B$ is a complex Brownian motion and $\tau = \inf \{t : B_t \in \mathbb{R} \cup A\}$. There are other ways to define it but we mainly use this one.
We assume that $\gamma$ is a noncrossing curve so $V_t = g_t(\gamma(t))$ is well defined. Then $g_t$ satisfies the (chordal) Lowner equation
\[
\dot{g}_t(z) = \frac{a}{g_t(z) - V_t}, \quad g_0(z) = z, \quad (1)
\]
where $V_t = g_t(\gamma(t))$ is a continuous function.

Conversely, one can start with a continuous real-valued function $V_t$ and define $g_t$ by (1). For $z \in \mathbb{H} \setminus \{0\}$, the function $t \mapsto g_t(z)$ is well defined up to time $T_z := \sup\{t : \text{Im}[g_t(z)] > 0\}$.

The (chordal) Schramm-Loewner evolution (SLE$_\kappa$) (from 0 to $\infty$ in $\mathbb{H}$) is the solution to (1) where $V_t = -B_t$ is a standard Brownian motion and $a = 2/\kappa$. By [10], there exists a random non-crossing curve $\gamma$, which is also called SLE, such that $g_t$ comes from the curve $\gamma$ as above. Moreover, if $H_t$ denotes the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$, then
\[
H_t = \{z \in \mathbb{H} : T_z > t\}.
\]
We write
\[
f_t(z) = g_t^{-1}(z + V_t).
\]

We will use the scaling property of SLE that we recall in a proposition.

**Proposition 2.1.** Suppose $U_t$ is a standard Brownian motion and let $g_t$ be the solution to the Lowner equation (1) with $V_t = U_t$ producing the SLE$_\kappa$ curve $\gamma(t)$. Let $r > 0$ and define
\[
\hat{\gamma}(t) = r^{-1} \gamma(r^2 t), \quad \hat{g}_t(z) = r^{-1} g_{t \cdot r^2 t}(rz), \quad \hat{U}_t = r^{-1} U_{t \cdot r^2 t}.
\]
Then $\hat{\gamma}(t)$ has the distribution of SLE$_\kappa$. Indeed, $\hat{g}_t(z)$ is the solution to (1) with $V_t = \hat{U}_t$.

SLE$_\kappa$ in other simply connected domains is defined by conformal invariance. To be more precise, suppose that $D$ is a simply connected domain and $w_1, w_2$ are distinct points in $\partial D$. Let $F : \mathbb{H} \to D$ be a conformal transformation of $\mathbb{H}$ onto $D$ with $F(0) = w_1, F(\infty) = w_2$. Then the distribution of
\[
\hat{\gamma}(t) = F \circ \gamma(t),
\]
is that of SLE$_\kappa$ in $D$ from $w_1$ to $w_2$. Although the map $F$ is not unique, the scaling invariance of SLE$_\kappa$ in $\mathbb{H}$ shows that the distribution is independent of the choice. This measure is often considered as a measure on paths modulo reparameterization, but we can also consider it as a measure on parameterized curves.

If $\gamma(t)$ is an SLE$_\kappa$ curve with transformations $g_t$ and driving function $U_t$, we write $\gamma_t = \gamma(0, t], \gamma = \gamma_\infty$, and let $H_t$ be the unbounded component of $\mathbb{H} \setminus \gamma_t$. If $z \in \mathbb{H}$ and $t < T_z$, we let
\[
Z_t(z) = g_t(z) - U_t, \quad S_t(z) = \sin[\arg Z_t(z)], \quad \Upsilon_t(z) = \frac{\text{Im}[g_t(z)]}{|g_t(z)|}.
\]
More generally, if $D$ is a simply connected domain and $z \in D$, we let $\Upsilon_D(z)$ denote $(1/2)$ times the conformal radius of $D$ with respect to $z$, that is, if $F : \mathbb{D} \to D$ is a conformal
transformation with \( F(0) = z \), then \( |F'(0)| = 2 \Upsilon_D(z) \). Using the Schwarz lemma and the Koebe (1/4)-theorem, we see that
\[
\frac{\Upsilon_D(z)}{2} \leq \text{dist}(z, \partial D) \leq 2 \Upsilon_D(z).
\]

It is easy to check that if \( t < T_z \), then \( \Upsilon_t(z) \) as given in (2) is the same as \( \Upsilon_{H_t}(z) \). Also, if \( z \not\in \gamma \), then \( \Upsilon(z) := \Upsilon_{T_{\gamma^-}}(z) = \Upsilon_D(z) \) where \( D \) denotes the connected component of \( \mathbb{H} \setminus \gamma \) containing \( z \). Similarly, if \( w_1, w_2 \) are distinct boundary points on a simply connected domain \( D \) and \( z \in D \), we define
\[
S_D(z; w_1, w_2) = \sin[\arg f(z)],
\]
where \( f : D \to \mathbb{H} \) is a conformal transformation with \( f(w_1) = 0, f(w_2) = \infty \). If \( t < T_z \), then \( S_t(z) = S_{H_t}(z; \gamma(t), \infty) \). If \( f : D \to f(D) \) is a conformal transformation,
\[
S_D(z; w_1, w_2) = S_{f(D)}(f(z); f(w_1), f(w_2)).
\]

If \( \partial_1, \partial_2 \) denote the two components of \( \partial D \setminus \{w_1, w_2\} \), then
\[
S_D(z; w_1, w_2) \propto \min \{\text{hm}_D(z, \partial_1), \text{hm}_D(z, \partial_2)\}.
\]

Here, and throughout this paper, \( \text{hm} \) will denote harmonic measure; that is, \( \text{hm}_D(z, K) \) is the probability that a Brownian motion starting at \( z \) exits \( D \) at \( K \).

Let
\[
G(z) = |z|^{d-2} \sin^{\frac{d-2}{2}}(\arg z) = \text{Im}(z)^{d-2} \sin^{4a-1}(\arg z),
\]
denote the (chordal) Green’s function for \( \text{SLE}_\kappa \) (in \( \mathbb{H} \) from 0 to \( \infty \)). This function first appeared in \([10]\) and the combination \((d, G)\) can be characterized up to a multiplicative constant by the scaling rule \( G(rz) = r^{d-2} G(z) \) and the fact that
\[
M_t(z) := |g_t'(z)|^{2-d} G(Z_t(z))
\]
is a local martingale. In general, if \( D \) is a simply connected domain with distinct \( w_1, w_2 \in \partial D \), we define
\[
G_D(z; w_1, w_2) = \Upsilon_D(z)^{d-2} S_D(z; w_1, w_2)^{4a-1}.
\]

The Green’s function satisfies the conformal covariance rule
\[
G_D(z; w_1, w_2) = |f'(z)|^{2-d} G_{f(D)}(f(z); f(w_1), f(w_2)).
\]

Note that if \( t < T_z \), then
\[
M_t(z) = G_{H_t}(z; \gamma(t), \infty).
\]
The local martingale \( M_t(z) \) is not a martingale because it “blows up” at time \( t = T_z \). If we stop it before that time, it is actually a martingale. To be precise, suppose that
\[
\tau = \tau_{t, z} = \inf\{t : \Upsilon_t(z) \leq \epsilon\}
\]
Then for every \( \epsilon > 0 \), \( M_{t \wedge \tau}(z) \) is a martingale. The following is proved in \([3]\) (the proof there is in the upper half plane, but it immediately extends by conformal invariance).
Proposition 2.2. Suppose $\kappa < 8$, $z \in D, w_1, w_2 \in \partial D$ and $\gamma$ is a chordal SLE$_\kappa$ path from $w_1$ to $w_2$ in $D$. Let $D_\infty$ denote the component of $D \setminus \gamma$ containing $z$. Then, as $\epsilon \downarrow 0$,

$$
P\{\Upsilon_{D_\infty}(z) \leq \epsilon\} \sim c_* \epsilon^{2-d} G_D(z), \quad c_* = 2 \left[\int_0^\pi \sin^4 x \, dx\right]^{-1}.
$$

2.2 Two-sided radial SLE

In this subsection we give a quick review of another variation of SLE which is called *two-sided radial SLE$_\kappa$ through $z$*. We can think about it as SLE$_\kappa$ conditioned to go through $z$ and stop at $T_z$. Because this is an event of probability zero, we get it through Girsanov transformation.

Weight the path by the local martingale $M_t(z)$ that we defined in the last section. By Girsanov theorem, we get that the driving function of the Lowner equation in (1) satisfies

$$
dV_t = \frac{(1-4a)X_t(z)}{|Z_t(z)|^2} \, dt + dW_t,
$$

where $W_t$ is a standard Brownian motion in the weighted measure. We should consider the fact that the above equation is valid until $T_z$, the first time that we hit $z$. We can justify this definition by the following proposition which is in [7].

Proposition 2.3. There exists $u > 0$, $c < \infty$ such that the following is true. Suppose $\gamma$ is the chordal SLE$_\kappa$ path from 0 to $\infty$ and $z \in \mathbb{H}$. For $\epsilon \leq \Im(z)$, consider $\tau_\epsilon$ in (7). Suppose $\epsilon' < 3\epsilon/4$. Let $\mu_1, \mu_2$ be the two probability measures on $\{\gamma(t) : 0 \leq t < \tau_\epsilon\}$ corresponding to chordal SLE$_\kappa$ conditiones on the event $\{\tau_{\epsilon'} < \infty\}$ and the two sided radial SLE$_\kappa$ through $z$. Then $\mu_1, \mu_2$ are mutually absolutely continuous with respect to each other and the Radon-Nikodym derivative satisfies

$$
\left|\frac{d\mu_2}{d\mu_1} - 1\right| < c(\epsilon'/\epsilon)^u.
$$

Whenever we want to consider measure with respect to two-sided radial SLE we use $\mathbb{P}_z$.

2.3 Natural parametrization

As we mentioned above the natural parametrization defined in [6] is a candidate for the right parametrization of SLE curves which arise in the scaling limits of various discrete models. It has the scaling exponent which is $d$. Also under some mild conditions, it is the only parametrization that we can get ([6 Proposition 2.2]). The definition goes as the following.

Suppose that $D$ is a bounded domain and $z, w$ are distinct boundary points. Let $\gamma$ denote an SLE$_\kappa$ curve from $z$ to $w$ in $D$ and $t$ is the capacity parametrization inherited from $\mathbb{H}$. Put

$$
\Psi_t(D) = \int_{D_t} G_{D_t}(\zeta; \gamma(t), w) \, dA(\zeta).
$$
For simplicity, assume that $\Psi_0(D) < \infty$. Because $G_{D_t}(z)$ is a positive local martingale, we get that $\Psi_t(D)$ is a supermartingale. By the Doob-Meyer decomposition, under some conditions which we have in this case, we get a unique increasing process $\Theta_t(D)$ such that $\Psi_t(D) + \Theta_t(D)$ is martingale. $\Theta_t(D)$ is called Natural parametrization for SLE$_\kappa$ in $D$.

The original definition given in [6] is only valid in $\mathbb{H}$. They use some cutoff argument instead of assuming $\Psi_0(D) < \infty$ (they define it on a bounded subdomain in $\mathbb{H}$ instead of whole $\mathbb{H}$). In [5] the definition is extended to every domain and basic properties of it are studied. We recall the results that we need here.

**Proposition 2.4.** If $F : \mathbb{H} \to D$ is a conformal map, $z, w \in \partial D$, such that $F(0) = z$ and $F(\infty) = w$, then we get

$$\hat{\Theta}_t(D) = \int_0^t |F'(\gamma(s))|^d d\Theta_s,$$

where $\hat{\Theta}$ and $\Theta$ are the natural parametrization in $D$ and $\mathbb{H}$ respectively.

This proposition basically shows the $d$-dimensional scaling that we expect from natural parametrization. The next proposition establishes additivity of the natural length.

**Proposition 2.5.** Consider $\gamma$ as an SLE curve from $0$ to $\infty$ in $\mathbb{H}$ and $r > 0$. Put

$$\gamma'(t) = \gamma(t + r)$$

Then if $\tau$ is a stopping time and $0 < s < t$, we have

$$\Theta_{t+\tau} - \Theta_{s+\tau} = \Theta^\tau_t - \Theta^\tau_s$$

where $\Theta^\tau$ is natural parametrization in the domain $H_\tau$.

### 3 Proof of Theorem 1.1

To prove Theorem 1.1, we follow the proof in [9] for Brownian motion. The formulation of [9] is more complicated here by the lack of Markov property for SLE.

Take $A = [0, 1] \times [0, 1]$ and take $z^0 = -\frac{1}{2} + 2i$. We prove that $\mathcal{H}^d(\gamma \cap (A + z^0)) = 0$ with probability 1. The same proof works for $[-m, m] \times [i/m, mi]$ for any $m$, which gives the theorem. From here, by $B^0$ when $B \subset A$, we mean $B + z^0$. Define

$$\mu(B^0) = \int_0^\infty 1_{B^0}(\tilde{\gamma}(s)) ds \quad \text{for } B \subset A,$$

where $\tilde{\gamma}(s)$ is the reparametrization of $\gamma(s)$ by natural parametrization. Fix a big integer $l$ throughout the section which we determine later (something around $10^6$ should be enough). Let $D_k$ be the set of all squares of the form

$$[n_1 l^{-k}, (n_1 + 1) l^{-k}] \times [n_2 l^{-k}, (n_2 + 1) l^{-k}],$$

where $n_1, n_2$ are integers.
where \( n_i \in \{0, 1, \ldots, l^k - 1\} \). As above, when we write \( D_k^{0} \) we mean \( D_k + z^0 \). Take \( \varepsilon > 0 \).

Fix \( m < M \). We will find them as functions of \( \varepsilon \) in the proof. We say \( D^0 \in D_k^{0} \) is big (with respect to \( \varepsilon \)) if

\[
\mu(D^0) > \frac{l^{-dk}}{\varepsilon}.
\]

Consider the following cover \( C(m, M, \varepsilon) = C \) of \( \gamma \cap A^0 \).

- Maximal big squares \( D^0 \in D_k^{0} \), \( m \leq k < M \), i.e., all big squares that are not contained in another big one for \( m \leq k < M \).
- Squares in \( D_M^{0} \) that are not contained in any big square \( D_k \) for \( m \leq k < M \) but intersect \( \gamma \).

This is a cover with sets of diameter at most \( \sqrt{2}l^{-m} \). Using this cover we can see that

\[
\mathcal{H}^d(\gamma \cap A^0) \leq \lim_{\varepsilon \downarrow 0} \sum_{k=m}^{M-1} \sum_{D^0 \in D_k^{0} \cap C} l^{-dk} + \sum_{D^0 \in D_M^{0} \cap C} l^{-dM} = \lim_{\varepsilon \downarrow 0}[Y_1 + Y_2],
\]

where

\[
Y_1 = \sum_{k=m}^{M-1} \sum_{D^0 \in D_k^{0} \cap C} l^{-dk},
\]

and \( Y_2 \) is the rest. For each \( \varepsilon > 0 \) we will find \( m \) and \( M \) such that

\[
\lim_{\varepsilon \downarrow 0} E[Y_1 + Y_2] = 0.
\]

In order to do this first we estimate \( E[Y_2] \). Put \( D^0 = D_M^{0} \subset D_{M-1}^{0} \subset \cdots \subset D_m^{0} \) with \( D_k^{0} \in D_k^{0} \). For any square \( D \), Let \( D^{*} \) be the square with same center as \( D \) and 100 times its sidelength. Put \( \tau_k = \inf\{t > 0 : \gamma(t) \in D_k^{0}\} \) and \( \tau_k^{*} = \inf\{t > 0 : \gamma(t) \in D_k^{0}^{*}\} \). We claim that we can take \( l \) such that, for every \( \varepsilon > 0 \), there is \( q = q_\varepsilon < 1 \) that

\[
P\left[\int_{\tau_k^{*}}^{\tau_{k+1}} 1_{D_k^{0}}(\tilde{\gamma}(s))ds < \frac{l^{-dk}}{\varepsilon} |\tau_{D^0} < \infty, F_{\tau_k}^{*}\right] < q,
\]

for every \( m \leq k < M - 1 \). We will prove this later as Lemma 3.1. Given this, if \( \tau(D^0) < \infty \), then the probability that \( D^0 \) is not contained in any big cube is less than \( q^{M-m} \), that is

\[
P[D^0 \in C | \tau(D^0) < \infty] < q^{M-m},
\]

Combining this with the fact that \( P[\tau(D^0) < \infty] \propto l^{M(d-2)} \) by Proposition 2.22 we get

\[
E[Y_2] < cl^{-Md}2Mq^{M-m}l^{M(d-2)} = cq^{M-m}.
\]

To estimate \( E[Y_1] \), we have
\[ E[Y_1] = E \left[ \sum_{k=m}^{M-1} c l^{-dk} \sum_{D^0 \in C \cap D_k^0} 1\{ \mu(D^0) > l^{-dk} \frac{1}{\epsilon} \} \right] < c \epsilon E \left[ \sum_{k=m}^{M-1} \sum_{D^0 \in C \cap D_k^0} \mu(D^0) \right] < c \epsilon E[\mu(A^0)]. \]  

We have the first inequality because of

\[ l^{-dk} 1\{ \mu(D^0) > l^{-dk} \frac{1}{\epsilon} \} < \epsilon \mu(D^0). \]

Because \( E[\mu(A^0)] < \infty \), then by (10) and (11), we can find \( m \) and \( M \) such that

\[ \lim_{\epsilon \downarrow 0} E[Y_1 + Y_2] = 0. \]

We can also take \( m \to \infty \) as \( \epsilon \to 0 \). Therefore, we get \( \mathcal{H}^d(\gamma \cap A^0) \to 0 \) in \( L^1 \) and then we get a subset which goes to zero w.p.1 which is what we want. In the rest of the section we prove (9) to complete the proof of Theorem 1.1. By scaling it is enough to prove the following lemma.

**Lemma 3.1.** Suppose \( D \subset D_1 \subset D_2 \) are three squares in \( \mathbb{H} \) with centers at \( O, O_1 \) and \( O_2 \) respectively, and sides parallel to axes. Suppose the sidelength of \( D_2 \) is 1 and the sidelength of \( D_1 \) is \( 1/l \). Also put \( D_1^* \) and \( D_2^* \) as above. We can take \( l \) such that for every \( N > 0 \) there exists \( q > 0 \) such that

\[ \mathbb{P} \left[ \Theta_{r(D_1)}(D_2) - \Theta_{r(D_2)}(O_2) > N | \tau_D < \infty, \mathcal{F}_{r(D_2)} \right] > q. \]  

Here \( D_r(V) = \int_0^r 1(\gamma(s) \in V) ds \).

**Proof.** Let \( \tau' = \inf\{ t | T_t(O_2) = 20 \} \). Note that

\[ \mathcal{Y}_{r(D_2)}(O_2) > 25 \quad 5 < \text{dist}(O_2, \gamma_{\tau'}) < 80, \]  

by (3). Define the event \( E = \{ S_{\tau'}(O_2) > 1/4 \} \). Recall that \( \mathbb{P}_z^* \) is the measure of two-sided radial through \( z \). By Lemma 2.2 in [8], there is \( p_1 > 0 \), such that

\[ \mathbb{P}_z^*[E | \mathcal{F}_{r(D_2)}] > p_1. \]

Note that until time \( \tau' \), \( \mathbb{P}_O^* \) and \( \mathbb{P}_z^* \) are mutually absolutely continuous with bounded Radon-Nykodim derivative. Therefore

\[ \mathbb{P}_O^*[E | \mathcal{F}_{r(D_2)}] > p'_1, \]

for some \( p'_1 \). So by Proposition 2.3

\[ \mathbb{P}[E | \mathcal{F}_{r(D_2)} \cap \tau(D) < \infty] > p_2, \]
for some $p_2 > 0$. Put $Z_{\tau'} : H_{\tau'} \to \mathbb{H}$ as in [2]. If $U = D_1^* \cup D_2$, then there is a universal $	heta_0 > 0$ such that if $z \in U$ then $\sin(\text{Arg}(Z_{\tau'}(z))) > \theta_0$ on the event $E$.

Consider the four rectangles $V_1, V_2, V_3$ and $V_4$ in $D_2$ each of sidelength $1/3$ and each of them has exactly one corner of $D_2$. At least one of them has distance at least $1/4$ to $D_1^*$, say $V_1$. Under the event $E$, let $f = bZ_{\tau'}$ where $b > 0$ is chosen so that $|f'(O_2)| = 1$. We have

$$\frac{1}{c} < |f'(z)| < c,$$

for any $z \in U$ for some $c > 0$ by Distortion theorem (see [2, Section 3]). We take $l$ large enough such that $2cl^{-1} < \min\{\frac{1}{40}, \frac{1}{10000}\}$. By Propositions 2.4 and 2.5, we need to show

$$P[\Theta_{\tau^*}(f(V_1)) > N|\tau^{}(f(D)) < \infty] > p > 0,$$

where $\tau^*$ is the first time that we hit $f(D_1^*)$.

By (14), we have $n$ fixed such that $f(U) \subset [-n, n] \times [i/n, ni]$. Also we have fixed $r_1, r_2, \delta > 0$ such that we have two disks $B_{r_1}(z) \subset f(V_1)$ and $f(D_1^*) \subset B_{r_2}(w)$ and we have $\text{dist}(z, w) = r_1 + r_2 + \delta$. By taking $L = f(D)$ we need to show the following lemma to finish the proof.

**Lemma 3.2.** For every $C, n, \delta > 0$, there exists $p > 0$ such that the following holds. If $r_1 > \delta, r_2 > 0, \text{dist}(z, w) > r_1 + r_2 + \delta$,

$$B_{r_1}(z), B_{r_2}(w) \subset [-n, n] \times [\frac{i}{n}, ni],$$

and $L \subset B_{r_2}(w)$, then

$$P[\Theta_{\tau^L}(B_{r_1}(z)) > C|\tau^{} < \infty] > p.$$  

**Proof.** For the moment let us assume $z$ is fixed and later by usual compactness argument we will show how we can take $p$ independent of $z$. To show (15), we define a geometric event which has positive probability such that the natural length is big on it with positive probability. Note that we can make $r_1$ smaller so by scaling we can assume that $r_1 = 1$. Consider the following path $\iota_1$ from 0 to a point in distance $1/2$ of $z$:

$$\iota_1 = [0, \frac{i}{2n}] \cup [\frac{i}{2n}, n + \frac{i}{2n}] \cup [n + \frac{i}{2n}, n + \text{Im}(z)i] \cup [n + \text{Im}(z)i, z + 1] \cup \lambda,$$

where $\lambda$ is a ”$k$– arm spiral around $z” as described below. We will determine $k$ based on $C$ later but for now it is a fix number satisfying $\frac{1}{k^2} < \frac{\delta}{10}$. 


We define $\lambda : [0, 1] \to B_1(z)$ by $\lambda(t) = e^{2\pi i t} (1 - \frac{4}{z}) + z$. Let $\iota_2$ be the reflection of $\iota_1$ by the $y$-axis. We use $\iota_2$ if $\text{Re}(w) > \text{Re}(z)$ and otherwise we use $\iota_1$. Assume $\text{Re}(w) > \text{Re}(z)$. Basic geometry shows that

$$\text{dist}(\iota_2, B_{\delta}(w)) \geq \delta.$$  \hspace{1cm} (16)

Consider the event $V$ that SLE curve $\gamma$ stays in a strip of width $\frac{1}{k}$ of $\iota_2$ until it reaches distance $1/2$ of $z$. If we change the Lowner driving function for $\iota_2$ a little, then we stay in a strip around it. So we get

$$P[V] > 0.$$  

It is easy to see by (16) that, $M_t(w); t \leq \tau$ is a first time expected time that SLE spends $B$. Consider the map $Z_{\tau_1}$ such that $\text{dist}(0, \gamma(\tau_1)) = 1/k$. Harmonic measure consideration (4), shows us that $h(z_0) = \frac{i}{k}$. By the Koebe-1/4 theorem we get $|h'(z)| \approx 1$. Now let $B = \{ z \in \mathbb{H} | \frac{1}{2\sqrt{k}} < \text{Im}(z) < \frac{2}{k}; \frac{\pi}{4} < \text{Arg}(z) < \frac{3\pi}{4} \}$. The expected time that SLE spends $B$, in the natural parametrization, is $O(k^{-d})$. So there is $p > 0$ and $c > 0$ such that

$$P\{ \Theta_\infty(B) \geq ck^{-d} \} > p.$$  

$p$ and $c$ are independent of $k$ because of the Proposition 2.4.

Since we have $|h'(z)| \approx 1$ for $z \in B$ by growth theorem, by Proposition 2.4 we get (perhaps with a different $c$)

$$P\{ \Theta_\infty(h(B)) \geq ck^{-d} \mid F_{\tau_1} \} > p > 0,$$

where $p$ and $c$ are independent of $j$ and $k$. We claim that there is a fixed $N$ large enough, independent of $k$ and $j$ with the following property: Suppose $l$ is a ray with distance $\frac{N}{k}$ of $l_0$ such that $l \cap \gamma[\tau_0, \tau_1] = \emptyset$. Consider $\overline{l'}$ as part of $l$ between arm $j$ and $j+1$ of $l$. Then,

$$P\{ \gamma \text{ hits } \overline{l'} \text{ and comes back and hit } B \mid F_{\tau_1} \} < p/2.$$  

To justify this, consider $l' = h(\overline{l'})$. By conformal invariance of Brownian motion

$$P[\text{Brownian motion starts at } i/k \text{ hits } l' \text{ before leaving } \mathbb{H}] = P[\text{Brownian motion starts at } z_0 \text{ hits } l' \text{ before leaving } \gamma_{\tau_1}].$$
The second probability is bounded by a constant times of the probability that in a strip with width $O(1/k)$, Brownian motion started at $z_0$, goes distance $N/k$ without leaving the strip. By computing this probability we can see $l'$ should be within at least distance $cN/k$ of $i/k$ for some $c$. Because SLE goes to $\infty$ we can get $N$ large enough such that

$$P\{\text{SLE hits } l' \text{ and comes back and hit } B\} < p/2,$$

then by conformal invariance of SLE we get what we want.

We should change the measure on $\gamma$ from SLE in the upper half plane to two sided radial SLE towards $w_2$. Notice that these two have the Radon-Nikodym derivative equals to $M_t(w_2)$ by the definition in Section 2.2. It is easy to see that in the return $M_t(w_2)$ is bounded by a universal constant by the conditions that we put on $\lambda$ until we exit $B_1(z)$. Then we change measure from two sided to SLE conditioned to $\tau(L) < \infty$ by Proposition 2.3.

Now consider the set of rays $l_i$, $1 \leq i \leq \frac{k}{10N}$ such that the distance between any two of them is at least $2N/k$ in any arm. Consider $l_i^j$ as the part of $l_i$ between the arm of $j$ and $j+1$ of $\gamma$. Then we can take a set $B_i^j$ for any of $l_i^j$ like above. The probability that we spend $O(k^{-d})$ in $B_i^j$ before hitting $l_{i+1}^j$ is $p > 0$ independent of the others. By this, we get with probability $1/2$, we spend at least $ck^{2-d}$ in the spiral for some $c > 0$ and if we put $k = \left(\frac{1}{\epsilon c}\right)^{\frac{1}{2-d}}$ we get what we want.

For the compactness argument, notice that if we consider another point $z'$ in the $B_{\frac{1}{4}}(z)$, the proof above works. We just cover the domain by them and compactness shows what we want.

At the end of this proof we want to raise the following question.

**Question:** What is the exact Gauge function for SLE curves?

We expect to have a log term correction but in order to answer it probably we need to have higher moments of natural parametrization which is related to the main conjecture of [5] about Minkowski content of SLE.

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