On the odd girth and the circular chromatic number of generalized Petersen graphs

Amir Daneshgar · Meysam Madani

Abstract A class $G$ of simple graphs is said to be girth-closed (odd-girth-closed) if for any positive integer $g$ there exists a graph $G \in G$ such that the girth (odd-girth) of $G$ is $\geq g$. A girth-closed (odd-girth-closed) class $G$ of graphs is said to be pentagonal (odd-pentagonal) if there exists a positive integer $g^*$ depending on $G$ such that any graph $G \in G$ whose girth (odd-girth) is greater than $g^*$ admits a homomorphism to the five cycle (i.e. is $C_5$-colourable). Although, the question “Is the class of simple 3-regular graphs pentagonal?” proposed by Nešetřil (Taiwan J Math 3:381–423, 1999) is still a central open problem, Gebleh (Theorems and computations in circular colourings of graphs, 2007) has shown that there exists an odd-girth-closed subclass of simple 3-regular graphs which is not odd-pentagonal. In this article, motivated by the conjecture that the class of generalized Petersen graphs is odd-pentagonal, we show that finding the odd girth of generalized Petersen graphs can be transformed to an integer programming problem, and using the combinatorial and number theoretic properties of this problem, we explicitly compute the odd girth of such graphs, showing that the class is odd-girth-closed. Also, we obtain upper and lower bounds for the circular chromatic number of these graphs, and as a consequence, we show that the subclass containing generalized Petersen graphs $\text{Pet}(n, k)$ for which either $k$ is even, $n$ is odd and $n \equiv k-1 \pm 2$ or both $n$ and $k$ are odd and $n \geq 5k$ is odd-pentagonal. This in particular shows the existence of nontrivial odd-pentagonal subclasses of 3-regular simple graphs.

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Amir Daneshgar
daneshgar@sharif.ir

1 Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11155-9415, Tehran, Iran
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1 Introduction

In this article, for two integers \( m \leq n \), the notation \( m \mid n \) indicates that \( m \) divides \( n \), the greatest common divisor is denoted by \( \gcd(m, n) \) and also, we define

\[
[m, n] \overset{\text{def}}{=} \{m, m + 1, \ldots, n\}.
\]

The size of any finite set \( A \) is denoted by \(|A|\).

Hereafter, we only consider finite simple graphs as \( G = (V(G), E(G)) \), where \( V(G) \) is the vertex set and \( E(G) \) is the edge set. An edge with end vertices \( u \) and \( v \) is denoted by \( uv \). Moreover, for any graph \( G \), the (resp. odd) girth (resp. \( g_{\text{odd}}(G) \)) is the length of the shortest (resp. odd) cycle of \( G \). The complete graph on \( n \) vertices and the cycle of length \( n \) are denoted by \( K_n \) and \( C_n \), respectively (up to isomorphism).

Throughout the article, a homomorphism \( f : G \rightarrow H \) from a graph \( G \) to a graph \( H \) is a map \( f : V(G) \rightarrow V(H) \) such that \( uv \in E(G) \) implies \( f(u)f(v) \in E(H) \).

If there exists a homomorphism from \( G \) to \( H \) then sometimes one may say that the graph \( G \) is \( H \)-colourable and one may simply write \( G \rightarrow H \) (for more on graph homomorphisms and their central role in graph theory see Hell and Nešetřil (2004)).

Let \( n \) and \( d \) be positive integers such that \( n \geq 2d \). Then the circular complete graph \( K_{\frac{n}{d}} \) is the graph with the vertex set \( \{0, 1, \ldots, n - 1\} \) in which \( i \) is connected to \( j \) if and only if \( d \leq |i - j| \leq n - d \). A graph \( G \) is said to be \((n, d)\)-colourable if \( G \) admits a homomorphism to \( K_{\frac{n}{d}} \). The circular chromatic number \( \chi_c(G) \) of a graph \( G \) is the minimum of those ratios \( \frac{n}{d} \) for which \( \gcd(n, d) = 1 \) and \( G \) admits a homomorphism to \( K_{\frac{n}{d}} \) (see Vince 1988; Zhu 2001, 2006 for more on circular chromatic number and its properties). Also, note that \( C_{2k+1} \simeq K_{\frac{2k+1}{k}} \), and consequently

\[
\chi_c(C_{2k+1}) = 2 + \frac{1}{k}.
\]

For a graph \( G \), the graph \( G^{\frac{1}{r}} \) is defined as the graph obtained from \( G \) by replacing each edge by a path of length \( d \) (i.e. subdividing each edge by \( d - 1 \) vertices). The \( r \)-th power of a graph \( G \), denoted by \( G^r \), is a graph on the vertex set \( V(G) \), in which two vertices are connected by an edge if there exists a walk of length at most \( r \) between them in \( G \). Also, the fractional power of a graph is defined as

\[
G^\frac{r}{\pi} \overset{\text{def}}{=} \left(G^{\frac{1}{r}}\right)^r.
\]

Note that for simple graphs, \( G \rightarrow H \) implies that \( G^r \rightarrow H^r \) for any positive integer \( r > 0 \). Hence, Problem 1 is closely related to the study of the chromatic number of the third power of sparse 3-regular simple graphs (see Daneshgar and Hajiabolhassan 2008; Daneshgar et al. 2016; Hajiabolhassan and Taherkhani 2010, 2014).
The Petersen graph is an icon in graph theory. Generalized Petersen graphs are introduced in Coxeter (1950) and given the name along with a standard notation by Watkins in Watkins (1969). The generalized Petersen graph, Pet\((n, k)\), is defined as follows.

**Definition 1** In Watkins’ notation, the generalized Petersen graph Pet\((n, k)\) for positive integers \(n\) and \(k\), where \(2 < 2k \leq n\), is a graph on \(2n\) vertices, defined as follows,

\[
V(\text{Pet}(n, k)) \overset{\text{def}}{=} \{u_0, u_1, \ldots, u_{n-1}\} \cup \{v_0, v_1, \ldots, v_{n-1}\},
\]

\[
E(\text{Pet}(n, k)) \overset{\text{def}}{=} \left( \bigcup_{i=0}^{n-1} \{u_i, u_{i+1}\} \right) \cup \left( \bigcup_{i=0}^{n-1} \{u_i, v_i\} \right) \cup \left( \bigcup_{i=0}^{n-1} \{v_i, v_{i+k}\} \right),
\]

where + stands for addition modulo \(n\) (see Fig. 1).

Some basic properties of generalized Petersen graphs are as follows (e.g. see Fox et al. 2012).

- Except Pet\((2k, k)\), all generalized Petersen graphs are 3-regular.
- Pet\((n, k)\) is bipartite if and only if \(n\) is even and \(k\) is odd, otherwise Pet\((n, k)\) is 3-chromatic.
- Pet\((n, k)\) is isomorphic to Pet\((n, m)\), if and only if either \(m \not\equiv \pm k \mod n\) or \(mk \equiv \pm 1 \mod n\) (see Boben et al. 2005; Horvat et al. 2012; Steimle and Staton 2009 and references therein).
- Pet\((n, k)\) is vertex transitive if and only if \((n, k) = (10, 2)\) or \(k^2 n \equiv \pm 1 \mod n\) (see Frucht et al. 1971).
- Pet\((n, k)\) is edge-transitive (see Frucht et al. 1971) if and only if

\[
(n, k) \in \{(4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)\}.
\]
• Pet\((n, k)\) is a Cayley graph (see Nedela and Škoviera 1995) if and only if \(k^2 \equiv 1\).  
• Every generalized Petersen graph is a unit distance graph Žitnik et al. (2012) (a graph formed from a collection of points in the Euclidean plane by connecting two points by an edge whenever the distance between the two points is exactly one).  
• Some specially named generalized Petersen graphs are the Petersen graph Pet\((5, 2)\), the Dörer graph Pet\((6, 2)\), the Möbius–Kantor graph Pet\((8, 3)\), the dodecahedron Pet\((10, 2)\), the Desargues graph Pet\((10, 3)\) and the Nauru graph Pet\((12, 5)\).

We know when generalized Petersen graphs are Hamiltonian and we also know about their automorphism groups (the interested reader is referred to the exiting literature for more on these graphs).

1.1 Motivations and main results

Let us call a class \(G\) of simple graphs girth-closed (resp. odd-girth-closed) if for any positive integer \(g\) there exists a graph \(G \in G\) such that the girth (resp. odd-girth) of \(G\) is \(\geq g\). A girth-closed (resp. odd-girth-closed) class \(G\) of graphs is said to be pentagonal (resp. odd-pentagonal) if there exists a positive integer \(g^*\) depending on \(G\) such that any graph \(G \in G\) whose girth (resp. odd-girth) is greater than \(g^*\) admits a homomorphism to the five cycle (i.e. is \(C_5\)-colourable). The following question of Nešetřil has been the main motivation for a number of contributions in graph theory.

Problem 1 Nešetřil (1999) Is the class of simple 3-regular graphs pentagonal?

Note that every simple 3-regular graph except \(K_4\) admits a homomorphism to \(K_3 \cong C_3\) by Brooks’ theorem. On the other hand, it is quite interesting to note that the answer is negative if we ask the same question with the five cycle \(C_5\) replaced by \(C_7\), \(C_9\) or \(C_{11}\) (see Catlin 1988; Ghebleh 2007; Hatami 2005; Kostochka et al. 2001; Wanless and Wormald 2001 for this and the background on other negative results).

A couple of relaxations of Problem 1 have already been considered in the literature. The following relaxation has been attributed to (see Ghebleh 2007) and to the best of our knowledge is still open.

Problem 2 Is it true that every cubic graph with sufficiently large girth has circular chromatic number strictly <3?

Also, Ghebleh introduced the following relaxation of Problem 1 and answered it negatively by constructing the class of spiderweb graphs whose circular chromatic numbers are equal to 3 (see Ghebleh 2007 for more on spiderweb graphs).

Problem 3 Ghebleh (2007) Is the class of simple 3-regular graphs odd-pentagonal?

Although, to answer Problem 1 negatively it is sufficient to introduce a subclass of simple 3-regular graphs which is not pentagonal, it is still interesting to study pentagonal subclasses of 3-regular graphs, even if the problem has a negative answer, since this study will definitely reveal structural properties that can influence the graph homomorphism problem.
Clearly, to analyze the above mentioned problems one has to have a good understanding about the cycle structure of the graphs, where it seems that the answers are strongly related to the number theoretic properties of the cycle lengths of the graphs being studied.

This article may be classified into the positive part of the above scenario which is related to the study of pentagonal and odd-pentagonal subclasses of 3-regular graphs on the one hand, and the study of subclasses of 3-regular graphs with circular chromatic number \(<3\) on the other\(^1\). Also, note that other variants of the positive scenario have already been studied either using stronger average degree conditions or topological conditions on the sparse graphs. For instance, we have

**Theorem A** Borodin et al. (2008) The class of all simple graphs as \(G\) for which every subgraph has average degree \(<12/5\), is pentagonal (actually with \(g^* = 3\)).

Also, using results of Borodin et al. (2004) we deduce that,

**Theorem B** Borodin et al. (2004) The class of planar graphs, projective planar graphs, graphs that can be embedded on the torus or Klein bottle are pentagonal.

Along the same lines we also have,

**Theorem C** Galluccioa et al. (2001) For every fixed simple graph \(H\), the class of \(H\)-minor free graphs is pentagonal.

Our major motivation to study the cycle structure of generalized Petersen graphs was, of course, related to our belief that the class of generalized Petersen graphs is odd-pentagonal. Needless to say, some results and techniques used in the sequel are related to the specific properties of generalized Petersen graphs and may be of independent interest.

Our main results in this article can be divided into two categories. The first category is concerned about the odd girth of generalized Petersen graphs, discussed in Sect. 2, where the following theorem presents an explicit description of this parameter as our main result.

**Theorem 1** Let \(\mathcal{O}\) be the set of odd numbers, and define,

\[
\text{Ind}(n, k) = \begin{cases} 
\{ t \mid t \text{ is odd, } 0 \leq t \leq \min \left\{ \frac{2k}{\gcd(n,k)}, \left\lfloor \frac{(k-1)^2}{n} \right\rfloor + 1 \right\} \} & \text{if } k \text{ and } n \text{ are odd,} \\
\{ t \mid 0 < t \leq \min \left\{ \frac{2k}{\gcd(n,k)}, \left\lfloor \frac{(k-1)^2}{n} \right\rfloor \right\} \} & \text{if } \frac{n}{\gcd(n,k)} \text{ is odd and } k \text{ is even,} \\
\{ t \mid 0 < t \leq \min \left\{ \frac{k}{\gcd(n,k)}, \left\lfloor \frac{(k-1)^2}{n} \right\rfloor \right\} \} & \text{if } \frac{n}{\gcd(n,k)} \text{ and } k \text{ are even.}
\end{cases}
\]

and

\[
\mathcal{G} \overset{\text{def}}{=} \bigcup_{t \in \text{Ind}(n,k)} \left\{ tn + (1 - k) \left\lfloor \frac{tn}{k} \right\rfloor + 2, (1 + k) \left\lceil \frac{tn}{k} \right\rceil - tn + 2 \right\}.
\]

\(^1\) See Ghebleh (2007) and references therein for other related results and the background. Also see Pan and Zhu (2002) for a similar approach.
Then we have

\[ g_{\text{odd}}(\text{Pet}(n, k)) = \min \left( \bigcup \left( \left\{ \frac{n}{\gcd(n, k)}, k + 3 \right\} \cup \mathcal{G} \right) \right). \]

Using Theorem 1 one may prove that the class of generalized Petersen graphs is odd-girth closed. This is a consequence of the following Proposition that provides a bit more information about some asymptotic properties of the odd girth of these graphs.

**Proposition 1** The odd girth of a non-bipartite generalized Petersen graph, Pet(n, k), is either equal to \( k + 3 \) or satisfies the following inequality

\[
\max \left( \frac{n}{k}, (\min\{ \gcd(n, k - 1), \gcd(n, k + 1) \} + 2) \right) \leq g_{\text{odd}}(\text{Pet}(n, k)) \leq \frac{n}{k} \text{par}(k) + k + 1,
\]

where \( \text{par}(k) \) is the parity function which is equal to one when \( k \) is odd and is zero otherwise.

Note that by Proposition 1 the odd girth of Pet(n, k) tends to infinity either if both \( \frac{n}{k} \) and \( k + 3 \) tend to infinity, or if \( \min\{ \gcd(n, k - 1), \gcd(n, k + 1) \} \) tends to infinity. However, the converse may not be true.

For instance, for fixed and odd \( k \), the odd girth of Pet(jk, k) tends to infinity, when \( j \) is odd and tends to infinity. On the other hand, for a fixed even integer \( j \), the odd girth of Pet(jk, k) by Theorem 1 is equal to \( k + 3 \) and tends to infinity, when \( k \) is even and tends to infinity. Note that in this case \( \frac{n}{k} \) is fixed. Both of these observations show that, \( g_{\text{odd}}(\text{Pet}(n, k)) \) may tend to infinity while one of the parameters \( \frac{n}{k} \) or \( k \) is fixed.

Moreover, note that for \( n = (k - 1)(k + 1) + 1 \), if \( k \) tends to infinity, the odd girth \( g_{\text{odd}}(\text{Pet}(n, k)) \) tends to infinity since both \( k \) and \( \frac{n}{k} \) tend to infinity, while in this case \( \min\{ \gcd(n, k - 1), \gcd(n, k + 1) \} = 1 \) is fixed.

It is easy to verify that in Watkins’ notation \( u_0u_1v_1v_{k+1}u_{k+1}u_kv_kv_0u_0 \) is a closed walk in Pet(n, k), implying that the girth of generalized Petersen graphs are always \( \leq 8 \). On the other hand, by Proposition 1 generalized Petersen graphs are odd-girth closed.

**Corollary 1** The class of generalized Petersen graphs is odd-girth closed but not girth-closed.

The second category of our results is concerned with circular chromatic number of generalized Petersen graphs, discussed in Sect. 3. The key observation in this regard is the fact that the information we already extract about the odd cycles of Pet(n, k) will help to prove bounds on the clique-number of some powers of these graphs, or helps to prove existence and nonexistence results on homomorphisms to powers of cycles, which in turn will give rise to some bounds on the circular chromatic number of generalized Petersen graphs.

In this regard, in Sect. 3, first, we obtain some lower bounds for the circular chromatic number of generalized Petersen graphs (see Propositions 3, 4 and Corollaries 4
and 5), and after that we concentrate on upper bounds (see Proposition 6 and Corollary 6) which will lead to our second main result as follows.

**Theorem 2** Let \( C \) be the subclass of the class of generalized Petersen graphs for which one of the following conditions holds.

(a) \( \text{Pet}(n, k) \), where \( k \) is even, \( n \) is odd and \( n^{k-1} \equiv \pm 2 \).

(b) \( \text{Pet}(n, k) \), where both \( n \) and \( k \) are odd and \( n \geq 5k \).

Then \( C \) is odd-pentagonal.

It is instructive to note that by results of Boben et al. (2005), Horvat et al. (2012) and Steimle and Staton (2009), \( \text{Pet}(n, k) \) is isomorphic to \( \text{Pet}(n, m) \), if and only if either \( m \not\equiv \pm k \) or \( mk \equiv \pm 1 \). Hence, although this fact cannot be used to prove a similar result for a larger class of graphs \( C \), but one may use this to construct a larger set of pairs \((n, k)\) for which the corresponding class of generalized Petersen graphs is odd-pentagonal. Clearly, this study motivates the interesting problem of characterizing conditions under which we have

\[
\text{Pet}(n, k) \rightarrow \text{Pet}(n', k').
\]

In subsequent sections, first, in Sect. 2 we prove Theorem 1 that explicitly describes the odd-girth of generalized Petersen graphs. This result will give rise to the proofs of Proposition 1 and Corollary 1 indicating that the class of generalized Petersen graphs is odd-girth closed (while it is not girth-closed). Then we will use the data to study the circular chromatic number of these graphs in Sect. 3 and prove Theorem 2 as a partial evidence to our motivating question.

## 2 Odd girth of generalized Petersen graph

In this section, we prove Theorem 1 and Proposition 1. Our main strategy to obtain this is based on the key observation that the problem of finding the odd girth of \( \text{Pet}(n, k) \) can be reduced to finding the solutions of the following integer program,

\[
\begin{align*}
\min & \quad u + v_+ + v_- \\
\text{s.t.} & \quad u + k(v_+ - v_-) = tn, \\
& \quad u + v_+ + v_- = 2r + 1, \\
& \quad r, u, v_+, v_- \geq 0,
\end{align*}
\]

To start, let \( C \) be a cycle of length \( \ell \) in \( \text{Pet}(n, k) \) with a fixed orientation as

\[
C = w_0 w_1 w_2 \ldots w_i w_0.
\]

We define,

- \( u_+(C) \equiv |\{w_j w_{j+1} \mid \exists i \in [0, n - 1], (w_j, w_{j+1}) = (u_i, u_{i+1})\}|, \)
- \( u_-(C) \equiv |\{w_j w_{j+1} \mid \exists i \in [0, n - 1], (w_j, w_{j+1}) = (u_i, u_{i-1})\}|, \)

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and consider the potential function

\[ \varphi(C) \overset{\text{def}}{=} \{ w_j, w_{j+1} | \exists i \in [0, n-1], (w_j, w_{j+1}) = (v_i, v_{i+k}) \}, \]

\[ \varphi(C) \overset{\text{def}}{=} \{ w_j, w_{j+1} | \exists i \in [0, n-1], (w_j, w_{j+1}) = (v_i, v_{i-k}) \}, \]

\[ b(C) \overset{\text{def}}{=} \{ w_j, w_{j+1} | \exists i \in [0, n-1], (w_j, w_{j+1}) = (u_i, v_i) \} \]

in which addition, +, and subtraction, −, are in \( \mathbb{Z}_n \). Let us list the following basic facts for further reference.

**Lemma 1** For any cycle \( C \) of length \( \ell \) along with a fixed orientation in \( \text{Pet}(n, k) \), we have,

(a) Parameters \( u_+(C), u_-(C), v_+(C), v_-(C), \) and \( b(C) \) are all non-negative.
(b) \( u_+(C) + u_-(C) + v_+(C) + v_-(C) + b(C) = \ell. \)
(c) \( u_+(C) + u_-(C) \leq n \) and \( v_+(C) + v_-(C) \leq n. \)
(d) \( u_+(C) - u_-(C) + k(v_+(C) - v_-(C)) \equiv 0. \)
(e) \( b(C) \) is an even number. Also, if \( u_+(C) + u_-(C) \neq 0 \) and \( v_+(C) + v_-(C) \neq 0, \)
then \( b(C) \geq 2. \)

**Proof** Statements (a), (b) and (c) are clear by definitions. For (d), according to the fixed orientation of the cycle \( C \), let

\[ C = w_0 w_1 w_2 \ldots w_{\ell} w_0 \]

and consider the potential function \( \varphi \) on the vertex set of \( \text{Pet}(n, k) \) to \([0, n-1]\), that is equal to \( i \) exactly on \( u_i \) and \( v_i \) (in Watkins’ notation). Without loss of generality, let \( \varphi(w_0) = 0. \) Note that

\[ 0 = \sum_C (\varphi(w_{j+1}) - \varphi(w_j) \overset{\text{def}}{=} (u_+(C) + k v_+(C)) - (u_-(C) + k v_-(C)). \]

Statement (e) is also clear by considering the fact that the number of transitions between the parts consisting of \( u_i \)'s and \( v_i \)'s is always an even number. Also, if the cycle \( C \) has some vertices in both parts, then \( b(C) \neq 0. \)

Now, we consider some properties of the solutions of the integer program \((*)\). We will see that the optimization problem \((*)\) may have some trivial solutions. In this regard, a solution \((u, v_+, v_-)\) of \((*)\) is said to be a trivial solution if either \( u = 0 \) or \( v_+ + v_- = 0. \) Evidently, other solutions are referred to as nontrivial solutions. Moreover, note that \( r \) and \( t \) are uniquely determined by a solution \((u, v_+, v_-)\). Hence, sometimes, when there is no ambiguity, we may talk about a solution \((u, v_+, v_-)\) or a solution \((u, v_+, v_-, r, t)\), when we want to explicitly refer to parameters \( r \) and \( t \). We may also say that \((u, v_+, v_-, r, t)\) is feasible if the parameters satisfy conditions of \((*)\). Note that \((*)\) has a solution if and only if its feasible set is non-empty.

**Lemma 2** If \((u, v_+, v_-)\) is a solution of \((*)\), then

(a) Either \( v_- = 0 \) or \( v_+ = 0. \)
(b) If \( n \) is even, then \( k \) is even.
(c) \(0 \leq u + v_+ + v_- \leq n\).

(d) If \(k\) is odd, then \(u < k\).

(e) If \(k\) is even, then either \((u, v_+, v_-, t) = (k, 0, 1, 0)\) or \(u < k\).

(f) \((n, k)|u\).

Proof

(a) Let \((u, v_+, v_-)\) be a solution with \(v_- \neq 0\) and \(v_+ \neq 0\), and define the new parameters \(v'_+\) and \(v'_-\) as,

\[
\begin{align*}
v'_+ &= v_+ - v_- , v'_- = 0 \text{ if } v_+ - v_- \geq 0 , \\
v'_- &= v_- - v_+ , v'_+ = 0 \text{ if } v_+ - v_- < 0 ,
\end{align*}
\]

and note that \((u, v'_+, v'_-)\) is a solution satisfying \(u + v'_+ + v'_- < u + v_+ + v_-\), which is a contradiction.

(b) Suppose that \(n\) is even and \(k\) is odd. Since \(u + k(v_+ - v_-) = tn\) is even, either both \(u\) and \((v_+ - v_-)\) are even or both are odd. Thus,

\[
u + v_+ + v_- = (u + v_+ - v_-) + 2v_-
\]

is even, which is a contradiction.

(c) If \(n\) is odd then \((u', v'_+, v'_-) = (n, 0, 0)\) is feasible. Also, if \(n\) is even, then by part (b), \(k\) is even, and one may verify that \((u', v'_+, v'_-) = (k, 0, 1)\) is feasible. In both cases, since we have an objective \(\leq n\) and \((u, v_+, v_-)\) is a minimizer, we have \(u + v_+ + v_- \leq n\).

(d) By contradiction assume that \(u \geq k\). Using (a) we may assume \(v_- = 0\) or \(v_+ = 0\) and define

\[
\begin{align*}
u' &= u - k , v'_- = v_- = 0 , v'_+ = v_+ - 1 , \text{ if } v_+ > 0 , \\
u' &= u - k , v'_- = v_- + 1 , v'_+ = 0 , \text{ if } v_+ = 0 .
\end{align*}
\]

Now note that \((u', v'_+, v'_-)\) is feasible for (\(*\)), satisfying

\[
u' + v'_+ + v'_- \leq u + v_+ + v_- - (k - 1) < u + v_+ + v_- ,
\]

which is a contradiction.

(e) If \(k\) is even, then \((u', v'_+, v'_-) = (k, 0, 1)\) is feasible. Now, if \((u, v_+, v_-)\) is a solution giving rise to an odd objective \(< k + 3\) and \(u > k\), then we should have \(k < u \leq u + (v_+ + v_-) \leq k + 1\). Consequently, \(u = k + 1\), and we may conclude that \(v_+ + v_- = 0\) and \(k + 1 = u = u + k(v_+ - v_-) = tn > 2k \geq k + 2\), which is a contradiction.

(f) By the definition of (\(*\)), we have \(u = tn - k(v_+ - v_-)\). Since \(\gcd(n, k)\) divides the right hand side, it divides \(u\) too.

The following simple observation as a corollary of Lemma 2(a) is sometimes quite useful. Note that by Lemma 2(a) one may define \(v \equiv v_+ - v_-\) and talk about a \((u, v, r, t)\) solution of the system (\(*\)). Clearly, for such a solution if \(v \geq 0\) then one has \((u, v_+ = v, v_- = 0, r, t)\) as a solution and if \(v \leq 0\) then one has \((u, v_+ = 0, v_- =\)
as a solution. Hereafter, we may freely talk about \((u, v, r, t)\) as a solution, adapting this convention. Also, note that by Lemma 1(a) for any solution \((u, v_+, v_-)\) we have,

\[ |v| = |v_+ - v_-| = v_+ + v_- . \]

On the other hand, the set of solutions of \((*)\) is equal to the set of solutions of the following minimization problem,

\[
\begin{align*}
\min & \quad u + |v| \\
\text{subject to} & \quad u + |v| = 2r + 1 \\
& \quad u + k v = t n \\
& \quad u \geq 0.
\end{align*}
\]

Now, by substitution for the variable \(u\) we find the following minimization problem whose set of solutions is equal to the set of solutions of \((*)\).

\[
\begin{align*}
\min & \quad t n - k v + |v| \\
\text{subject to} & \quad t n - k v + |v| = 2r + 1 \\
& \quad t n - k v \geq 0.
\end{align*}
\]

The following result is our first step toward a clarification of relationships between the set of solutions of \((*)\) and the odd girth of \(\text{Pet}(n, k)\).

**Theorem 3** The following statements are equivalent,

(a) The feasible set of \((*)\) is non-empty.
(b) \(\text{Pet}(n, k)\) has an odd cycle.

Also, if \((u, v_+, v_-, r, t)\) is a solution of \((*)\), then the odd girth of \(\text{Pet}(n, k)\) is equal to \(2r + 1\) if \((u, v_+, v_-, r, t)\) is a trivial solution (i.e. either \(u = 0\) or \(v_+ + v_- = 0\)). The odd girth of \(\text{Pet}(n, k)\) is equal to \(2r + 3\), if all solutions of \((*)\) are nontrivial.

**Proof**  \(\Rightarrow (b)\) : Let \((u, v_+, v_-, r, t)\) be a feasible point of the system \((*)\). First, we show that there exists an odd cycle \(C\) in \(\text{Pet}(n, k)\) of length \(\ell\), where

\[ \ell \leq \begin{cases} 2r + 1, & \text{if } u = 0 \text{ or } v_+ + v_- = 0, \\ 2r + 3 & \text{otherwise}. \end{cases} \]

Consider the following three cases.

(i) If \(v_+ + v_- = 0\), then we have \(n \equiv 0\), \(u = 2r + 1 \neq 0\) and \(0 \leq u \leq n\).

Consequently, \(2r + 1 = u = n\), and we have an odd cycle (the outer cycle \(u_0u_1 \ldots u_{n-1}u_0\)) of length \(2r + 1 = n\) (see Fig. 2a).

(ii) If \(u = 0\), then without loss of generality suppose that \(v_- \leq v_-\), now

\[ v \overset{\text{def}}{=} v_+ - v_- = v_+ + v_- - 2v_- = 2r + 1 - 2v_- = 2r' + 1 > 0 \]

and

\[ k(v_+ - v_-) \overset{n}{=} 0. \]
Constructing an odd cycle given $v_+, v_-$ and $u$ with $u + k(v_+ - v_-) \equiv 0$ in three cases: (a) $v_+ + v_- = 0$, (b) $u = 0$, (c) $v_+ + v_- \neq 0$ and $u \neq 0$

Consequently, one can consider the odd closed walk $v_0 v_+ v_2 v_3 \ldots v_{(v-1)k} v_0$. The length of this closed walk is equal to $v = 2r + 1$, and contains at least one odd cycle of length $\leq 2r + 1$ (see Fig. 2b).

(iii) If $v_+ + v_- \neq 0$ and $u \neq 0$, consider the closed walk

$$\begin{align*}
&u_0 u_1 \ldots u_v v_{v+1} v_{v+2k} \ldots v_{u+(v_+-v_-)k} v_0 u_0, \text{ if } v_+ \geq v_-, \\
&u_0 u_1 \ldots u_v v_{v-1} v_{v-2k} \ldots v_{u-(v_-+v_+1)k} v_0 u_0, \text{ if } v_+ < v_-.
\end{align*}$$

Note that in either case we have a closed walk of length $u + k|v_+ - v_-| + 2 \leq 2r + 3$ that contains an odd cycle of length $\leq 2r + 3$ (see Fig. 2c).

• $(b) \Rightarrow (a)$: Let $C_\mu$ be a minimum odd cycle in $\text{Pet}(n, k)$ and note that for one of the two orientations of this cycle we have $u_+(C_\mu) \geq u_-(C_\mu)$. Fix this orientation for the rest of this proof. Also, let

$$u \overset{\text{def}}{=} u_+(C_\mu) - u_-(C_\mu), \quad v_+ \overset{\text{def}}{=} v_+(C_\mu), \quad \text{and } v_- \overset{\text{def}}{=} v_-(C_\mu).$$

Note that since $l(C_\mu) = u + v_+ + v_- + 2u_-(C_\mu) + b(C_\mu)$ is odd, and $b(C_\mu)$ is even, then $u + v_+ + v_- \geq 2r + 1$ and consequently, $(u, v_+, v_-)$ is feasible for $(*)$.

Now, fix a minimum odd cycle $C_\mu$ with parameters $(u, v_+, v_-, r)$ along with its corresponding orientation for which $u_+(C_\mu) \geq u_-(C_\mu)$. To determine the length of $C_\mu$ we consider the following two cases.

1. There is a trivial solution $(u^*, v_+^*, v_-^*, r^*)$.

By $(a) \Rightarrow (b)$, there exists an odd cycle of length $\leq 2r^* + 1$ and consequently, $l(C_\mu) \leq 2r^* + 1$.

Also, by $(b) \Rightarrow (a)$, parameters $(u, v_+, v_-, r)$ are feasible for $(*)$ and

$$l(C_\mu) \geq u + v_+ + v_- = 2r + 1 \geq 2r^* + 1.$$

Hence, $l(C_\mu) \geq 2r^* + 1$ which shows that $l(C_\mu) = 2r^* + 1$ in this case.
(2) There is not any trivial solution and \((u^*, v^*_+, v^*_-, r^*)\) is a non-trivial solution. Similarly, by \((a) \Rightarrow (b)\), we have \(l(C_{\mu}) \leq 2r^* + 3\). Also, using \((b) \Rightarrow (a)\), parameters \((u, v^*_+, v^*_-, r)\) are feasible for (*) and consequently,

\[
l(C_{\mu}) \geq u + v^*_+ + v^*_- + b(C_{\mu}) = 2r + 1 + b(C_{\mu}) \geq 2r^* + 1 + b(C_{\mu}).
\]

Note that if \(l(C_{\mu}) < 2r^* + 3\), then since \(l(C_{\mu})\) is odd, by the above inequality we have

\[
2r^* + 1 \geq l(C_{\mu}) \geq 2r^* + 1 + b(C_{\mu}),
\]

which implies that \(b(C_{\mu}) = 0\), and consequently by Lemma 1(e) either \(u = 0\) or \(v^*_+ + v^*_- = 0\). But since \((u, v^*_+, v^*_-, r)\) is feasible and \(2r + 1 \leq 2r^* + 1\) one concludes that \((u, v^*_+, v^*_-)\) is a trivial solution which is a contradiction. Therefore, \(l(C_{\mu}) \geq 2r^* + 3\) and consequently, \(l(C_{\mu}) = 2r^* + 3\). (As a byproduct we have also proved that \(b(C_{\mu}) \geq 2\) in this case.)

A couple of remarks on the proof of the previous theorem are instructive. First, note that by the argument appearing in part (i) of the \((a) \Rightarrow (b)\) section of the proof we may conclude that for any trivial solution \((u, v^*_+, v^*_-)\) if \(v^*_+ + v^*_- = 0\) then \(u\) is odd and equal to \(n\). This shows that if \(n\) is even then we have \(v^*_+ + v^*_- \neq 0\).

On the other hand, if \(u = 0\) then one may note that \((u = 0, v_+ = 0, v^- = \frac{n}{\gcd(n,k)})\) is a solution. This not only shows that \(\frac{n}{\gcd(n,k)}\) being an even number implies that \(u \neq 0\), but also the above argument shows that if \(u + v^*_+ + v^*_- \notin \left\{n, \frac{n}{\gcd(n,k)}\right\}\) then \((u, v^*_+, v^*_-)\) is a nontrivial solution.

Also, note that by the proof of the last part of the theorem, if \((u, v^*_+, v^*_-, r, t)\) is a solution of (*) and the odd girth of Pet\((n, k)\) is equal to \(2r + 3\), then any cycle of minimum length will intersect both outer and inner cycles of Pet\((n, k)\).

To see more, note that if \(n\) is even and \(k\) is odd, then by Lemma 2(b) the feasible set of the system (*) is empty. Hence, by Theorem 3 Pet\((n, k)\) is bipartite since it does not have any odd cycle.

Moreover, if \(n\) is odd, then \(u = n\) and \(v^- = v^*_+ = 0\) constitute a feasible set of parameters for the system (*), implying the existence of an odd cycle by Theorem 3. Also, if both \(n\) and \(k\) are even, then similarly, \(u = k\) and \(v^*_+ = 0, v^- = 1\), constitute a feasible set of parameters for the system (*), again implying the existence of an odd cycle by Theorem 3.

These facts re-establish the well-known fact that the generalized Petersen graph, Pet\((n, k)\), is bipartite if and only if \(n\) is even and \(k\) is odd.

Now, we set for a proof of Theorem 1, but first we will be needing the following simple lemma.

**Lemma 3** Consider the equation \(u + kv = tn\), in which \(k \in \mathbb{N}, n \in \mathbb{N}\) and \(t \in \mathbb{Z}\) are constants and \(u \in \mathbb{N}\) and \(v \in \mathbb{Z}\) are unknowns. Then

(i) If \(t = 0\) and \(0 \leq u \leq k\) then the only solution of this equation other than the trivial solution \((u, v) = (0, 0)\) is \((u, v) = (k, -1)\).
(ii) If \( t \neq 0 \) and \( 0 \leq u < k \) then the solution is uniquely determined as follows,

\[
\begin{align*}
\{ u &= tn - k \left\lfloor \frac{tn}{k} \right\rfloor, \quad v = \left\lfloor \frac{tn}{k} \right\rfloor, \quad t > 0, \\
\{ u &= k \left\lceil -\frac{tn}{k} \right\rceil + tn, \quad v = -\left\lceil -\frac{tn}{k} \right\rceil, \quad t < 0.
\end{align*}
\]

Proof Part (i) is clear. For part (ii) one may easily verify that the provided expressions satisfy the equation and its conditions. On the other hand, let’s assume that we have two solutions, namely,

\[ u + k v = tn, \quad \text{and} \quad u' + k v' = tn. \]

Without loss of generality, we may assume \( u \geq u' \), and consequently, \((u - u') + k(v - v') = 0\). But by part (i) and the fact that \( 0 \leq u - u' < k \) we have \( u - u' = v - v' = 0 \), proving the uniqueness.

This shows that if we have a nontrivial solution, and \( u \neq k \) then either \( u = tn - k \left\lfloor \frac{tn}{k} \right\rfloor \), \( v = \left\lfloor \frac{tn}{k} \right\rfloor \) or \( u = k \left\lceil -\frac{tn}{k} \right\rceil - tn \), \( v = -\left\lceil -\frac{tn}{k} \right\rceil \) for some positive integer \( t \).

Proof (of Theorem 1)

Let \( \text{Pet}(n, k) \) be a non-bipartite graph. We consider the following two cases.

- **There exist a trivial solution** \((u, v, r, t)\) for (*):
  
  We show that in this case the odd girth is equal to \( \frac{n}{\gcd(n, k)} \). Note that if \((u, v) = (0, \frac{n}{\gcd(n, k)})\), then by the remarks mentioned after Theorem 3 the odd girth is equal to \( \frac{n}{\gcd(n, k)} \).
  
  On the other hand, if \( v = 0 \), then \( u = n \) is odd and equal to the odd girth by Theorem 3. Also, note that in this case \((u = 0, v_+ = \frac{n}{\gcd(n, k)}, v_- = 0)\) is feasible for (*) and consequently, \( n \leq \frac{n}{\gcd(n, k)} \). This implies \( \gcd(n, k) = 1 \) showing that the odd girth is equal to \( n \).

- **There is not any trivial solution and** \((u, v, r, t)\) **is a solution of (*)**:
  
  First, we prove the following claims. Recall that \( \text{par}(k) \) is the parity function which is equal to one when \( k \) is odd and is zero otherwise.

  (i) There exists a nontrivial solution \((u, v, r, t)\) for which \( |t| \leq \left\lfloor \frac{(k-1)^2}{n} \right\rfloor + \text{par}(k) \).

  We consider the following cases.

  - \( k \) and \( n \) are odd.

  Note that \((u' = n - k \left\lfloor \frac{n}{k} \right\rfloor, v_+ = \left\lfloor \frac{n}{k} \right\rfloor, v_- = 0)\) is feasible for (*), and consequently, for any nontrivial solution \((u, v, r, t)\) we have

    \[
    u + |v| \leq n - \left\lfloor \frac{n}{k} \right\rfloor (k - 1) < n - \left( \frac{n}{k} - 1 \right) (k - 1) = \frac{n}{k} + k - 1.
    \]

    Hence,

    \[
    \|r|n = |u + kv| \leq u + |v|k < u + \left( \frac{n}{k} + k - 1 - u \right) k
    \]

    \[
    = u(1 - k) + n + k^2 - k \leq n + (k - 1)^2,
    \]
implying that

\[ |t| \leq \left\lfloor \frac{(k-1)^2}{n} \right\rfloor + 1. \]

\(- k \text{ is even.}

Note that \((u' = k, v'_+ = 0, v'_- = 1, t' = 0)\) is feasible for \((*)\). If this is a solution then the inequality is trivially satisfied. Otherwise, for any nontrivial solution \((u, v, r, t)\) we have \(u + |v| < k + 1\), implying that \(u + |v| \leq k - 1\) since \(k\) is even. On the other hand, \(u \geq 1\) implies that \(|v| \leq k - 2\), and consequently, applying Lemma 3 we have,

\[ |t| n = |u + k| v| \leq u + |v| k \leq (u + |v|) + (k - 1)|v| \leq k - 1 + (k - 1)(k - 2) = (k - 1)^2. \]

This shows that

\[ |t| \leq \left\lfloor \frac{(k-1)^2}{n} \right\rfloor . \]

(ii) There exists a nontrivial solution \((u, v, r, t)\) for which \(|t| \leq \frac{2k}{\gcd(n,k)}\). Moreover, if \(\frac{n}{\gcd(n,k)}\) is even, then there exists a nontrivial solution \((u, v, r, t)\) for which \(|t| \leq \frac{k}{\gcd(n,k)}\).

First, assume that for positive integers \(\alpha\) and \(\beta < t\) we have \(t = \beta + \frac{\alpha k}{\gcd(n,k)}\). Then one may verify that,

\[
\begin{align*}
tn + (1 - k) \left\lceil \frac{n}{k} \right\rceil &= \left(\frac{\alpha k}{\gcd(n,k)} + \beta\right) n + (1 - k) \left[ \left( \frac{\alpha k}{\gcd(n,k)} + \beta \right) n \right] \\
&= \left( \beta n + (1 - k) \left\lceil \frac{\beta n}{k} \right\rceil \right) + \left( \left( \frac{\alpha k}{\gcd(n,k)} \right) n + (1 - k) \left( \frac{\alpha n}{\gcd(n,k)} \right) \right) \\
&= \left( \beta n + (1 - k) \left\lceil \frac{\beta n}{k} \right\rceil \right) + \left( \frac{\alpha n}{\gcd(n,k)} \right) \\
(1 + k) \left\lceil \frac{tn}{k} \right\rceil - tn &= (1 + k) \left[ \left( \frac{\alpha k}{\gcd(n,k)} + \beta \right) n \right] - \left( \frac{\alpha k}{\gcd(n,k)} + \beta \right) n \\
&= \left( 1 + k \right) \left\lceil \frac{\beta n}{k} \right\rceil - \beta n \\
&+ \left( (1 + k) \left( \frac{\alpha n}{\gcd(n,k)} \right) - \left( \frac{\alpha k}{\gcd(n,k)} \right) n \right) \\
&= \left( 1 + k \right) \left\lceil \frac{\beta n}{k} \right\rceil - \beta n + \left( \frac{\alpha n}{\gcd(n,k)} \right) \\
\end{align*}
\]

To prove the claim we proceed by contradiction. First, note that by parts \((d)\) and \((e)\) of Lemma 2, either \((u, v, t) = (k, -1, 0)\) or \(0 \leq u < k\). Clearly, the
claims are true for $t = 0$, hence we may assume that $t \neq 0$ and $0 \leq u < k$. On the other hand by Lemma 3 we know that $u + v$ is equal to

$$
\begin{cases}
u + v = tn + (1 - k) \left\lfloor \frac{tn}{k} \right\rfloor, & t > 0, \\
u + v = (1 + k) \left\lceil \frac{-tn}{k} \right\rceil + tn, & t < 0
\end{cases}
$$

as the minimum value of the solutions of ($\ast$). But by setting $\alpha = 1$ when $\frac{n}{\gcd(n, k)}$ is even and $\alpha = 2$ when $\frac{n}{\gcd(n, k)}$ is odd, and the above computation we see that one finds the minimum values

$$
\left( \beta n + (1 - k) \left\lfloor \frac{\beta n}{k} \right\rfloor \right)
$$

or

$$
\left( (1 + k) \left\lceil \frac{\beta n}{k} \right\rceil - \beta n \right)
$$

for the solutions of ($\ast$) which are strictly smaller, and this is a contradiction.

Now, the rest of the proof is clear by the above case studies since the set $\text{Ind}(n, k)$ essentially categorizes different cases and the minimization provides that odd girth by Theorem 3.

Note that Theorem 1 not only provides an explicit expression for the odd girth of $\text{Pet}(n, k)$, but also the provided expression is also effectively computable in many cases. The following corollary and example will show some special cases.

**Corollary 2** For any odd number $n$, there exists two generalized Petersen graphs $\text{Pet}(2n + 1, 2d + 1)$ and $\text{Pet}(2n + 1, 2d)$ with odd girth $n$.

**Example 1** Let us consider the following special cases as a couple of concrete examples.

- The odd girth of $\text{Pet}(n, 2)$ is equal to 5, except for the case $n = 6$, for which odd girth is equal to 3.

  The case $n = 5$ is well-known. If $n > 5$ then $n > (k - 1)^2$, and consequently, for $\frac{n}{(n, 2)} \geq 5$, we have $|t| \leq \left\lfloor \frac{(k - 1)^2}{n} \right\rfloor = 0$. Hence, in this case the odd girth is equal to $k + 3 = 5$. The only case for which we have $\frac{n}{(n, 2)} < 5$ is $n = 6$, in which case the odd girth is equal to $\frac{6}{(6, 2)} = 3$.

- For $\text{Pet}(n, 3)$, we have

  $$
g_{\text{odd}}(\text{Pet}(n, 3)) = \begin{cases}
\frac{n}{3}, & n \equiv 1, \\
\frac{n+8}{3}, & n \equiv 2, \\
\frac{n+10}{3}, & n \equiv 3.
\end{cases}
$$

Note that for $n \geq 5$ we have $|t| \leq \left\lfloor \frac{(k-1)^2}{n} \right\rfloor + 1 = 1$. Therefore, we just need to consider the special cases $t = 1, t = -1$ and the trivial solutions.
The following result is essentially a direct consequence of Theorem 1, but we provide some direct proofs for clarity and simplicity.

Proof (of Proposition 1) For the upper bounds note that

- If \( k \) is even and \( g_{odd}(\text{Pet}(n, k)) \neq k + 3 \) then by feasibility of \((u, v_+, v_-) = (k, 0, 1)\) we have \( g_{odd}(\text{Pet}(n, k)) \leq k + 1 \).
- If \( k \) is odd then \( n \) is also odd since otherwise the graph will be bipartite. Also, considering a feasible point uniquely determined by Lemma 3 for \( t = 1 \), by and Theorem 3 we have

\[
g_{odd}(\text{Pet}(n, k)) \leq n + (1 - k) \left\lfloor \frac{n}{k} \right\rfloor + 2 \leq n + (1 - k) \left( \frac{n}{k} - 1 \right) + 2 = \frac{n}{k} + k + 1.
\]

For the lower bounds, first, and by contradiction, let \((u, v_+, v_-, r, t)\) be a solution of \((*)\) such that \( u + v_- + v_+ < \frac{n}{k} \), then

\[
\begin{align*}
    u + k(v_+ - v_-) &\leq u + k(v_+ + v_-) \leq k(u + v_+ + v_-) < k\frac{n}{k} = n, \\
    u + k(v_+ - v_-) &\geq u - k(v_+ + v_-) \geq -k(u + v_+ - v_-) > -k\frac{n}{k} = -n.
\end{align*}
\]

So \(-n < u + k(v_+ - v_-) = tn < n\), thus we have \( u + k(v_+ - v_-) = t = 0 \).

Now, if \( k \) is even this contradicts Lemma 2(e). On the other hand, if \( k \) and \( n \) are odd, since \( u + k(v_+ - v_-) = 0 \), either both \( u \) and \( v_+ - v_- \) are odd or both of them are even. Hence, \( u + v_+ + v_- = (u + v_+ - v_-) + 2v_- \) is even, which is contradiction, since we already know that \( u + v_+ + v_- = 2r + 1 \).

For the second term in the lower bound, consider a solution of \((**)*\) as \( tn - kv + |v| \geq 1 \). Then one may verify that,

\[
    tn - kv + |v| = \begin{cases} 
    tn - kv + v = tn - (k - 1)v & v > 0 \\
    tn - kv - v = tn - (k + 1)v & v < 0 \\
    tn & v = 0 \\
    \gcd(n, k - 1) & v > 0 \\
    \gcd(n, k + 1) & v < 0 \\
    k + 1 \geq \gcd(n, k - 1) + 2 & v = 0.
    \end{cases}
\]

Therefore, if \( v = 0 \) then the solution is greater than or equal to \( \gcd(n, k - 1) + 2 \) (note that by Definition 1 we know that \( 2 < 2k \leq n \)). Otherwise, the odd girth is greater than or equal to \( \min\{\gcd(n, k - 1), \gcd(n, k + 1)\} \).

3 On circular chromatic number of Pet\((n, k)\)

In this section we prove some lower bounds and upper bounds for the circular chromatic number of generalized Petersen graphs and eventually we will prove Theorem 2. To start, let us recall a couple of known results.
Theorem D Hajiabolhassan (2009) Let $q$ be a non-negative integer and also let $G$ and $H$ be two graphs, then

$$G^{\frac{1}{2q+1}} \rightarrow H \iff G \rightarrow H^{2q+1}. $$

Theorem E Hajiabolhassan and Taherkhani (2014) Let $G$ be a non-bipartite graph with circular chromatic number $\chi_c(G)$. Then for any positive integer $s$, we have

$$\chi_c\left(G^{\frac{1}{2s+1}}\right) = \frac{(2s+1)\chi_c(G)}{s\chi_c(G) + 1}. $$

Proposition 2 Let $C \overset{\text{def}}{=} w_0w_1w_2 \ldots w_{s-1}w_0$ be an odd cycle of the graph $G$, where $u, v \in V(G) \setminus \{w_0, w_1, w_2, \ldots, w_{s-1}\}$. Also, let $u$ be adjacent to $w_i$ and $v$ be adjacent to $w_j$ in $G$ for some $0 \leq i < j \leq s - 1$. Then in the power graph $G^{s-2}$, 

(a) $w_i$ and $w_j$ are adjacent, 
(b) If $j - i \not\equiv \pm 1$, then $u$ is adjacent to $w_j$, 
(c) If $j - i \not\equiv \pm 2$, then $u$ is adjacent to $v$.

Proof Since $s - 2$ is odd, in each case we show that there exists an odd path of length $\leq s - 2$ between the corresponding vertices in $G$.

(a) If $j - i$ is odd, then we have an odd walk $w_iw_{i+1} \ldots w_j$ with length $\leq s - 2$. If $j - i$ is even consider the odd walk $w_jw_{j+1} \ldots w_{s-1}w_0w_1 \ldots w_i$.
(b) If $j - i$ is odd, then we have an odd walk $w_jw_{j+1} \ldots w_{s-1}w_0w_1 \ldots w_iu$, of length $s - j + i + 1 \leq s - 2$. If $j - i$ is even consider the odd walk $uw_iw_{i+1} \ldots w_j$.
(c) If $j - i$ is odd, then we have an odd walk $uw_iw_{i+1} \ldots w_jv$ of length $j - i + 2 \leq s - 2$. If $j - i$ is even consider the odd walk $vw_jw_{j+1} \ldots w_{s-1}w_0w_1 \ldots w_iu$ of length $s - j + i + 2 \leq s - 2$.

3.1 Lower bounds

In this section we prove some lower bounds for the circular chromatic number of generalized Petersen graphs.

Proposition 3 Suppose that the system $(\ast)$ has no trivial solution. Then, if $(u, v_+, v_-, r)$ is a (non-trivial) solution,

$$K_{4r+2} \rightarrow \text{Pet}(n, k)^{2r+1}. $$
Proof We prove the result for \( u = tn - k \left\lfloor \frac{tn}{k} \right\rfloor \), \( v = \left\lfloor \frac{tn}{k} \right\rfloor \) and \( 2r + 1 = v + u \) (i.e. \( t > 0 \)). (for \( u = k \left\lceil -\frac{tn}{k} \right\rceil + tn, v = -\left\lfloor -\frac{tn}{k} \right\rfloor \) (i.e. when \( t < 0 \) one may use a similar argument.)

Consider the set

\[
P = \{ u_0, \ldots, u_u \} \cup \{ v_0, \ldots, v_v \} \cup \{ u_{u+k}, \ldots, u_{u+(v-1)k} \} \cup \{ v_{u+k}, \ldots, v_{u+(v-1)k} \},
\]

as a subset of \( V(\text{Pet}(n, k)) \) and note that \( |P| = 4r + 2 \). Now, it is enough to show that \( P \) constitutes a clique in \( \text{Pet}(n, k)^{2r+1} \). Clearly, for this, it is enough to show that there exists an odd path of length \( \leq 2r + 1 \) between any pair of vertices in \( P \).

For each \( 0 \leq i \leq n - 1 \) consider the \((2r + 3)\)-cycle, \( C_i \), defined as follows

\[
C_i \overset{\text{def}}{=} u_i u_{i+1} \ldots u_{i+u} v_{i+u} v_{i+u+k} v_{i+u+2k} \ldots v_{i+u+(v-1)k} v_i u_i,
\]

where summations are in modulo \( n \). Applying Proposition 2(a) to \( C_0 \) we find out that

\[
\{ u_0, \ldots, u_u \} \cup \{ v_{u+k}, \ldots, v_{u+(v-1)k} \} \cup \{ v_0, v_v \}
\]

constitutes a clique in \( \text{Pet}(n, k)^{2r+1} \).

We claim that, if \( i, j \in \{ 0, 1, 2, \ldots, u \} \), and \( h, \ell \in \{ 0, 1, 2, 3, \ldots, v - 1 \} \), then there exists an odd path of length less than or equal to \( 2r + 1 \) between the following pairs of vertices in \( \text{Pet}(n, k) \).

- \( v_i \) to \( u_j \). Applying Proposition 2(a) to \( C_i \) the claim is true for \( j \geq i \). For \( j < i \) use the same argument on \( C_{i-u} \).

- \( v_i \) to \( v_j \).

For \( j > i \) note that \( v_j \) is connected to \( u_j \) in \( C_i \) and the distance between \( v_j \) and \( u_j \) in \( C_i \) is greater than one. Hence, Proposition 2(b) proves this case. If \( j < i \) then use the same reasoning on \( C_{i-u} \).

- \( u_{u+hk} \) to \( v_{u+tk} \).

If \( \ell \geq h \) invoke Proposition 2(a) for \( C_{hk} \). If \( \ell < h \) consider \( C_{(h-v)k} \), note that \( u_{u+hk} \) is connected to \( v_{u+tk} \) in this cycle, and invoke Proposition 2(b).

- \( u_{u+hk} \) to \( u_{u+tk} \).

Note that if \( |\ell - h| \neq 2 \), we may invoke Proposition 2(c) in \( C_0 \). Also, if \( |\ell - h| = 2 \), we may applying Proposition 2(b) on \( C_{hk} \) if \( \ell > h \). If \( |\ell - h| = 2 \) and \( \ell < h \), consider \( C_{(h-v)k} \) and invoke Proposition 2(b).

- \( u_{u+hk} \) to \( v_j \).

If \( h \neq 0 \), consider \( C_0 \) and Proposition 2(c), (note that in this case the distance between \( v_{u+hk} \) and \( u_j \) is not equal to 2). Also, the case \( h = 0 \) is the case for \( u_i \) and \( v_j \) mentioned before.

- \( u_{u+hk} \) to \( u_i \).

If \( h = 0 \) this is the case of \( u_i \) and \( u_j \) mentioned before. If \( h \neq 0 \), the distance between \( v_{u+hk} \) and \( u_i \) in \( C_0 \) is not equal to 1. Therefore, we can apply Proposition 2(b).
v_{u+hk} to v_i.
If h = 0 this is the case of v_i and v_j. Also, for i = 0 we can apply Proposition 2(a) on C_0. If \( i \neq 0 \) and \( h \neq 0 \), the distance between \( v_{u+hk} \) and \( u_i \) in \( C_0 \) is not equal to 1. Hence, we can apply Proposition 2(b).

**Corollary 3** If the system (\(*)\) does not have any trivial solution and the odd girth of \( \text{Pet}(n, k) \) is equal to \( 2r + 3 \), then

\[
\text{Pet}(n, k) \not\rightarrow C_{2r+3}.
\]

**Proof** By contradiction assume that

\[
\text{Pet}(n, k) \rightarrow C_{2r+3}.
\]

Then by Proposition 3 and the functorial property of the power construction we have

\[
K_{2r+2} \rightarrow \text{Pet}(n, k)^{2r+1} \rightarrow C_{2r+3}^{2r+1} = K_{2r+3},
\]

which is a contradiction.

Now, we may deduce our first lower bound.

**Corollary 4** Let \( n \) and \( k \) be odd and suppose that the system (\(*\)) has no trivial solution. If \((u, v_+, v_-, r)\) is a (non-trivial) solution of (\(*\)), then

\[
2 + \frac{4r}{4r^2 + 2r + 1} \leq \chi_c(\text{Pet}(n, k)).
\]

**Proof** By Theorem D we have

\[
K_{2r+2} \rightarrow \text{Pet}(n, k)^{2r+1} \Rightarrow K_{\frac{1}{2r+1}}^{2r+1} \rightarrow \text{Pet}(n, k).
\]

Therefore, by Theorem E we have,

\[
\chi_c(\text{Pet}(n, k)) \geq \chi_c(K_{\frac{1}{2r+1}}) = \frac{(2r+1)\chi_c(K_{2r+2})}{r\chi_c(K_{2r+2}) + 1} = \frac{(2r+1)(4r+2)}{r(4r+2) + 1} = 2 + \frac{4r}{4r^2 + 2r + 1}.
\]

**Example 2** Consider the special case in which \( n \leq 4k^2 + 1 \), \( n \equiv 1 \) and \( n - 1 \) is even. Then (\*) has no trivial solution and \( g_{\text{odd}}(\text{Pet}(n, k)) \leq \frac{n - 1}{2k} + 3 \). Therefore, by setting \( 2r + 1 = \frac{n - 1}{2k} + 1 \), we have

\[
\chi_c(\text{Pet}(n, k)) \geq 2 + \frac{4r}{4r^2 + 2r + 1}.
\]
In particular, if \( n = 4sk + 1 \), where \( u \leq k \), by setting \( r = s \), we have,
\[
\chi_c(\text{Pet}(4sk + 1, 2k)) \geq 2 + \frac{4s}{4s^2 + 2u + 1}.
\]
For example, for \( u = 1 \), we have
\[
\chi_c(\text{Pet}(4k + 1, 2k)) \geq 2 + \frac{4}{7},
\]
implying \( \text{Pet}(4k + 1, 2k) \nrightarrow C_5 \).

**Example 3** (1) Let \( k = 3 \). For any odd \( n \), where \( 3 \nmid n \), by Theorem 1, we have
\[
2r + 1 = \begin{cases} 
  n - 2 \frac{n-1}{3} = \frac{n+2}{3} & n \equiv 1, \\
  n - 2 \frac{n-2}{3} = \frac{n+4}{3} & n \equiv 2.
\end{cases}
\]
Therefore,
\[
\chi_c(\text{Pet}(n, 3)) \geq \begin{cases} 
  2 + \frac{6n-6}{n^2+n+7} & n \equiv 1, \\
  2 + \frac{6n+6}{n^2+5n+13} & n \equiv 2.
\end{cases}
\]
For instance, \( \chi_c(\text{Pet}(11, 3)) \geq 2 + \frac{72}{189} \). Note that since
\[
\chi_c(C_7) = 2 + \frac{1}{3} < 2 + \frac{72}{189} = \chi_c(\text{Pet}(11, 3))
\]
one may conclude that \( \text{Pet}(11, 3) \nrightarrow C_7 \). Similarly,
\[
\chi_c(\text{Pet}(7, 3)) \geq 2 + \frac{36}{63} > 2 + \frac{1}{2} = \chi_c(C_5),
\]
and we have \( \text{Pet}(7, 3) \nrightarrow C_5 \).

(2) If \( n = 21 \) and \( k = 5 \), by Theorem 1 we have \( 2r + 1 = 5 \), and consequently,
\[
\chi_c(\text{Pet}(21, 5)) \geq 2 + \frac{8}{21} > 2 + \frac{1}{3},
\]
implying \( \text{Pet}(21, 5) \nrightarrow C_7 \).

**Proposition 4** Let \( \overline{K}_{\frac{2n}{4k+4}} \) be the complement of the circular complete graph \( K_{\frac{2n}{4k+4}} \).
Then, for \( n > 2k \), we have
\[
\overline{K}_{\frac{2n}{4k+4}} \nrightarrow \text{Pet}(n, 2k)^{2k+1}.
\]
Proof Reorganize the vertices of Pet\((n, 2k)\)^{2k+1} as

\[ x_0 = u_0, x_1 = v_0, x_2 = u_1, x_3 = v_1, \ldots x_{2n-2} = u_{n-1}, x_{2n-1} = v_{n-1}. \]

In what follows we prove that each \(x_i\) is connected to \(x_{i+\ell}\) for \(1 \leq \ell \leq 4k + 1\) where summation is modulo \(n\). This clearly implies that there is a subgraph in Pet\((n, 2k)\)^{2k+1} isomorphic to \(K_{4k+4}\).

Note that, by symmetry, it is enough to prove the claim for \(u_0\) and \(v_0\). Hence, in what follows, we show that for any \(j \in \{1, 2, \ldots, 2k\}\) and in Watkins’ notation, there exists an odd path of length \(\leq 2k + 1\) between the following pairs of vertices in Pet\((n, 2k)\).

1. **\(u_0\) to \(u_j\)**
   - Apply Proposition 2(a) in \(C_0\).

2. **\(v_0\) to \(u_j\)**
   - Apply Proposition 2(a) in (see Fig. 3)
   
   \[ C_0 = u_0 u_1 \ldots u_{2k} v_{2k} v_0 u_0. \]

3. **\(v_0\) to \(v_j\)**
   - If \(j = 2k\) we apply Proposition 2(a) in \(C_0\). Also, if \(j \neq 2k\), the distance between \(v_0\) and \(u_j\) is not equal to 1 in \(C_0\), and consequently, one may invoke Proposition 2(b).

4. **\(u_0\) to \(v_j\)**
   - If \(j = 2k\) we apply Proposition 2(a) in \(C_0\). Also, if \(j \neq 2k\), and \(j \neq 1\) the distance between \(u_0\) and \(u_j\) is not equal to 1 in \(C_0\), and consequently, one may invoke Proposition 2(b). On the other hand, if \(j = 1\) we apply Proposition 2(a) in (see Fig. 3)
   
   \[ C_{n-2k-1} = u_{n-2k-1} u_{n-2k} \ldots u_0 u_1 v_1 v_{n-2k-1} v_{n-2k-1}. \]

5. **\(v_0\) to \(u_{2k+1}\)**
   - Consider the odd path \(v_0 v_{2k} u_{2k} u_{2k+1}\) (see Fig. 3).

We will need the following result to obtain our second lower bound.
Theorem F  Yang (2005) Let $K_{p/q}$ be the complement of the circular complete graph $K_{p/q}$. Then, for $p/q \geq 2$, we have

$$\chi_c(K_{p/q}) = \frac{p}{\left\lfloor \frac{p}{q} \right\rfloor}.$$  

Corollary 5 If $g_{odd}(Pet(n, 2k)) = 2k + 3$, we have

$$\frac{2n(2k + 1)}{2kn + \left\lfloor \frac{2n}{4k+2} \right\rfloor} \leq \chi_c(Pet(n, 2k)).$$

Proof  By Theorem D we have,

$$\left( K_{\frac{2n}{4k+2}} \right)^{1/4k+1} \rightarrow Pet(n, 2k) \iff K_{\frac{2n}{4k+2}} \rightarrow Pet(n, 2k)^{2k+1}.$$  

Proposition 4 implies that $K_{\frac{2n}{4k+2}} \rightarrow Pet(n, 2k)^{2k+1}$, and consequently,

$$\left( K_{\frac{2n}{4k+2}} \right)^{1/4k+1} \rightarrow Pet(n, 2k).$$  

But since $n > 2k$ by Theorem F, we have

$$\chi_c(Pet(n, 2k)) \geq \chi_c\left( \left( K_{\frac{2n}{4k+2}} \right)^{1/4k+1} \right) = \frac{2n(2k + 1)}{2kn + \left\lfloor \frac{2n}{4k+2} \right\rfloor}.$$  

In Ghebleh (2007), the following lower bound has been obtained for the circular chromatic number of generalized Petersen graphs.

Theorem G  Ghebleh (2007) For any $n > 2k$, we have

$$2 + \frac{2}{2k + 1} \leq \chi_c(Pet(n, 2k)).$$  

The following proposition shows that the lower bound of Corollary 5 can be strictly greater than the lower bound presented in Theorem G.

Proposition 5 Let $n \equiv y$ and $0 < y \leq 2k + 1 - \frac{n}{2k+2}$, then

$$\frac{2n(2k + 1)}{2kn + \left\lfloor \frac{n}{2k+1} \right\rfloor} > 2 + \frac{2}{2k + 1}.$$  

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Proof Since \( y \leq 2k + 1 - \frac{n}{2k + 2} \), if we set \( u = \left\lfloor \frac{n}{2k + 2} \right\rfloor \), then

\[
n = u(2k + 2) + y \Rightarrow n = u(2k + 1) + (u + y).
\]

Since \( 2k + 2 \nmid n \) we have \( u < \frac{n}{2k + 2} \), and consequently,

\[
u + y < \left( \frac{n}{2k + 2} \right) + \left( 2k + 1 - \frac{n}{2k + 2} \right) = 2k + 1.
\]

Therefore, \( \left\lfloor \frac{n}{2k + 1} \right\rfloor = u = \left\lfloor \frac{n}{2k + 2} \right\rfloor \). Finally,

\[
\frac{2n(2k + 1)}{2kn + \left\lfloor \frac{n}{2k+1} \right\rfloor} = \frac{2n(2k + 1)}{2kn + \left\lfloor \frac{n}{2k+2} \right\rfloor} \geq \frac{2n(2k + 1)}{2kn + \frac{n}{2k+2}} = 2 + \frac{2}{2k + 1}.
\]

But, equality holds if and only if \( 2k + 2 \mid n \), which is impossible since \( n \equiv 2k+2 \) and \( y \neq 0 \).

It must be noted that there exist many pair of numbers \((n, k)\), such that \( 4k^2 - 1 < n < 4k^2 + 6k + 2 \) for which conditions of both Proposition 5 and Corollary 5 hold simultaneously (e.g. \( 5 \leq n < 12 \) for \( \text{Pet}(n, 2) \), or \( 16 \leq n < 30 \) for \( \text{Pet}(n, 4) \)).

### 3.2 Upper bounds

Let us define the graph \( \text{Pb}(n, k) \) on the cycle \( C_n \) by connecting vertices having distance \( k \) (i.e. the graph is obtained by contracting edges \( u, v_i \) in \( \text{Pet}(n, k) \) for all \( 0 \leq i < n \)). Also, let \( C_r^n \) be the \( r \)-th power of \( C_n \).

**Proposition 6** (a) If \( n \) is odd, \( k \) is even, \( n \equiv \pm 2 \) and \( s = \frac{(n-4)(k-2)}{2(k-1)} \), then there exists a homomorphism

\[
\sigma : \text{Pb}(n, k) \rightarrow K_n^s
\]

(b) For any generalized Petersen graph \( \text{Pet}(n, k) \), there exists a homomorphism

\[
\sigma : \text{Pet}(n, k) \rightarrow \text{Pb}(n, k).
\]

(c) If \( n \) and \( k \) are odd and \( n > 2k + 1 \), there exists a homomorphism

\[
\sigma : \text{Pet}(n, k) \rightarrow C_n^k.
\]
Proof (a) Assume that \( n \) is odd, \( k \) is even and \( n^{k-1} \equiv -2 \), (for \( n^{k-1} \equiv 2 \) one may use the same argument on the reverse direction for \( \text{Pet}(n, k) \)). Also, let

\[
V(\text{Pb}(n, k)) = \{x_0, x_1, \ldots, x_{n-1}\}.
\]

Note that if \( j(k-1) = n+2 \), then

\[
\frac{k}{2} j(k-1) = \frac{k}{2} n + k \equiv k, \quad \left( \frac{k}{2} j - 1 \right)(k-1) = \frac{k}{2} n + 1 \equiv 1 \quad (1),
\]

and consequently, since \( n \) and \( (k - 1) \) are relatively prime, we can define

\[
\sigma(x_{i(k-1)}) \overset{\text{def}}{=} y_i,
\]

where operations are in \( \mathbb{Z}_n \).

By the hypothesis, \( s = \frac{(n-4)(k-2)}{2(k-1)} \) and

\[
\frac{(n-4)(k-2)}{2(k-1)} = s \leq \frac{k(n+2)}{2(k-1)} \Rightarrow k(n+2) - 1 \leq n - s
\]

\[
= n - \frac{(n-4)(k-2)}{2(k-1)}.
\]

Hence, using (1) and symmetry, one may verify that the edge \( x_0x_1 = x_0x_{\left(\frac{k}{2} j-1\right)(k-1)} \)

is mapped to \( y_0y_{\frac{k}{2} j-1} = y_0y_{\frac{k(n+2)}{2(k-1)}} \in E(K_p) \), and also, \( x_0x_k = x_0x_{\frac{k}{2} j} \)

is mapped to \( y_0y_{\frac{k}{2} j} = y_0y_{\frac{k(n+2)}{2(k-1)}} \in E\left(\frac{K_p}{2}\right) \).

(b) Again, using Watkins’ notation for \( \text{Pet}(n, k) \), we can define

\[
\sigma_v(v_i) \overset{\text{def}}{=} \sigma(u_{i+1}) \overset{\text{def}}{=} x_i,
\]

which is a homomorphism.

(c) Here, let

\[
V(C_n^k) = \{x_0, x_1, \ldots, x_{n-1}\},
\]

and define \( \sigma(v_i) \overset{\text{def}}{=} \sigma_v(u_{i+1}) \overset{\text{def}}{=} x_i \). Consider the following three cases:

– Edges of type \( u_iu_{i+1} \), are mapped to \( x_{i+1}x_{i+2} \),
– Edges of type \( v_i v_{i+k} \) are mapped to \( x_i x_{i+k} \),
– Edges of type \( v_i u_j \) are mapped to \( x_i x_{i+1} \),

which prove that \( \sigma \) is a homomorphism.

Using Proposition 6 we obtain the following upper bounds for the circular chromatic number of generalized Petersen graphs.
Corollary 6 (a) Let $k > 2$ be even, $n$ be odd and $n^{k-1} \equiv \pm 2$. Then

$$\chi_c(\text{Pet}(n, k)) \leq \frac{2n(k - 1)}{(n - 4)(k - 2)}.$$ 

(b) If $n$ and $k$ are odd and $n > 2k + 1$, then

$$\chi_c(\text{Pet}(n, k)) \leq \frac{2n}{n - k}.$$ 

Proof (a) By Proposition 6(b),

$$\text{Pet}(n, k) \rightarrow \text{Pb}(n, k),$$

and by Proposition 6(a), for $s = \frac{(n-4)(k-2)}{2(k-1)}$ we have a homomorphism

$$\text{Pb}(n, k) \rightarrow K_{\frac{n}{s}}.$$ 

Thus, there exists a homomorphism

$$\text{Pet}(n, k) \rightarrow K_{\frac{n}{s}},$$

and consequently,

$$\chi_c(\text{Pet}(n, k)) \leq \chi_c(K_{\frac{n}{s}}) = \frac{2n(k - 1)}{(k - 2)(n - 4)}.$$ 

(b) First, one may verify that $\chi_c(C_n^k) = \frac{2n}{n-k}$, by considering the following homomorphism $\eta : C_n^k \rightarrow K_{\frac{n}{s}}$,

$$\eta_V(x_i) = \begin{cases} y_i & \text{if } i \text{ is even} \\ y_{i+\frac{k}{2}} & \text{if } i \text{ is odd} \end{cases},$$

in which $V\left(K_{\frac{n}{s}}\right) = \{y_0, y_1, \ldots, y_{n-1}\}$.

Now, by Proposition 6(c), there exists a homomorphism

$$\sigma : \text{Pet}(n, k) \rightarrow C_n^k.$$ 

Thus, we have

$$\chi_c(\text{Pet}(n, k)) \leq \chi_c(C_n^k) = \frac{2n}{n - k}.$$
Note that Corollary 6 extends the following result of Ghebleh (2007).

**Theorem H**  
Ghebleh (2007) For all odd $n > 9$.

\[ \chi_c(Pet(n, 3)) \leq \frac{2n}{n - 3}. \]

Also, Corollary 6 can be used to prove our main result on the existence of an odd-pentagonal subclass of generalized Petersen graphs.

**Proof** (of Theorem 2)

(a) Since $k$ is even, if $g_{\text{odd}}(Pet(n, k))$ tends to infinity, then by Proposition 1, $k$ tends to infinity, and consequently, if $n$ is odd and $n \overset{k-1}{\equiv} \pm 2$, by Corollary 6 the circular chromatic number $\chi_c(Pet(n, k))$ tends to 2 because $\frac{2n(k-1)}{(n-4)(k-2)}$ tends to 2.

(b) If both $n$ and $k$ are odd and $n \geq 5k$, by Corollary 6 we have,

\[ \chi_c(Pet(n, k)) \leq \frac{2n}{n - k} \leq \frac{5}{2}. \]

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