Towards a Holographic Model of D-Wave Superconductors

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Abstract

The holographic model for S-wave high $T_c$ superconductors developed by Hartnoll, Herzog and Horowitz is generalised to describe D-wave superconductors. The 3+1 dimensional gravitational theory consists of a symmetric, traceless second-rank tensor field and a $U(1)$ gauge field in the background of the AdS black hole. Below $T_c$ the tensor field which carries the $U(1)$ charge undergoes the Higgs mechanism and breaks the $U(1)$ symmetry of the boundary theory spontaneously. The phase transition characterised by the D-wave condensate is second order with the mean field critical exponent $\beta = 1/2$. As expected, the AC conductivity is isotropic below $T_c$ and the system becomes superconducting in the DC limit but has no hard gap.
I. INTRODUCTION

One of the unsolved mysteries in modern condensed matter physics is the mechanism of the high temperature superconducting (HTSC) cuprates \[1\]. These materials are layered compounds with copper-oxygen planes and are doped Mott insulators with strong electronic correlations. The pairing symmetry is unconventional and there is a strong experimental evidence showing that it is D-wave \[2\]. It is speculated that the pairing between electrons is mediated via strong anti-ferromagnetic spin fluctuations in the system.

A prominent strong coupling theory is proposed by Anderson, called the resonant valence bond (RVB) theory, which describes liquid state with spin-singlets. Upon hole doping, the Néel order is destroyed and give rise to superconductivity \[3\]. Several gauge theories have been proposed to formulate the RVB physics, by enforcing the double occupation constraint in the strong coupling limit \[4\]. The problem is difficult due to the strong-coupling nature of the theory. Although significant progress has been made in the past few years, alternative approaches may be valuable to tackle the problem.

One alternative approach is the holographic correspondence between a gravitational theory and a quantum field theory, which first emerged under AdS/CFT correspondence \[5–7\]. This method has provided a useful and complimentary framework to describe strong interaction systems without a sign problem (see e.g. \[8–15\]). In the original top-down approach, both the gravity side and the field theory side of the theories are precisely known. Later applications assume that the correspondence exists among different pair of theories and try to make predictions from one side of the correspondence. More specifically, in this bottom-up approach, usually the gravity side of the theory is explicitly constructed with the desired symmetries, then physical observables (matrix elements) of the field theory side are predicted through the above mentioned correspondence.

Recently, a gravitational model of hairy black holes \[16, 17\] have been used to model S-wave HTSC \[18–21\]. In those class of models the Abelian symmetry of a complex scalar field is spontaneously broken (i.e. the Higgs mechanism) below some critical temperature. The Meissner effect was soon observed by including magnetic field in the background \[22, 23\]. The effect of superconducting condensate on the holographic fermi surface has been studied by calculating fermionic spectral function \[24, 26\]. Interestingly,
the properties of spectral function appeared to have similar behaviour to that found in the angle resolved photo-emission experiment. Motivated by all of these s-wave studies, holographic dual to P-wave superconductor has been proposed by coupling a $SU(2)$ Yang-Mills field to the black hole, where a vector hair develops in the superconducting phase [27–30]. Behaviour of fermionic spectral function has also been studied in those p-wave superconducting background [31]. So far, the bottom-up construction of holographic superconductor assuming the existence of gauge/gravity duality has been discussed. However, in the string theory framework people also have studied top-down approach considering various D-brane configurations in the AdS black hole background [32].

In this work, we try to construct a minimal gravitational model that models D-wave HTSC. We replace the complex scalar field in [18] by a tensor field whose condensate breaks the symmetry spontaneously below $T_c$ and the condensate becomes zero and the symmetry is restored above $T_c$. The critical exponent $\beta$ gives the mean field value $1/2$. The real part of the conductivity computed from linear response has a delta function at zero frequency which corresponds to static superconductivity below $T_c$. Above $T_c$, the delta function disappears as expected and the conductivity becomes constant in frequency. It is expected that there is no “hard gap” in the real part of the conductivity and the conductivity should be isotropic even though the condensate is not (for a model calculation, see [33]). Both features are seen in our results.

II. A HOLOGRAPHIC MODEL FOR D-WAVE HTSC

Our goal is to consider a minimal (3+1 dimensional) holographic model that gives rise to (2+1 dimensional) D-wave superconductivity. The construction will be similar to that of the S-wave case [18] with a spontaneous local $U(1)$ symmetry breaking in the bulk leading to a spontaneous breaking of global $U(1)$ symmetry at the boundary. Thus, strictly speaking, the boundary theory is a super-fluid. One can still study the current-current correlator which could be interpreted as the conductivity.

To have a D-wave condensate at the boundary, we introduce a charged tensor field in our dual gravity theory. Assuming the D-wave condensate originating from electron-electron pairing, the D-wave nature gives a symmetric wave function for the pair. So, wave function for this electron pair has to be a spin singlet such that its total wave function is anti-symmetric. A $3 \times 3$ symmetric traceless tensor has 5 components which can be
used to describe a D-wave state. We will promote this symmetric traceless tensor field to include time components and denote the field as $B_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$), i.e. $B_{\mu\nu} = B_{\nu\mu}$ and $B_\mu^{\mu} = 0$. However, it is important to note that the interacting higher spin fields, in general, require to satisfy additional constraints in addition to the equations of motion to remove the unphysical degrees of freedom, see the discussion in [34–36] for example.

Observing that there is no available consistent model for our purpose, we would like to propose a truncated model which has sufficient ingredients to catch some features of D-wave superconductor. It would be an important but difficult task to construct a complete theory which we would like to postpone for the future in order to attain our simple goal through the present exercise.

The desired complete action including gravity, $U(1)$ gauge field, tensor field and other auxiliary fields, may take the following form [37]:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ \left( R + \frac{6}{L^2} \right) + \mathcal{L}_m + \mathcal{L}_a \right\},$$

$$\mathcal{L}_m = -\frac{L^2}{q^2} \left[ (D_\mu B_{\nu\gamma})^* D^\mu B^{\nu\gamma} + m^2 B_{\mu\nu}^* B^{\mu\nu} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right],$$

where $R$ is the Ricci scalar, the $6/L^2$ term gives a negative cosmological constant and $L$ is the AdS radius which will be set to unity in the units that we use. $\kappa^2 = 8\pi G_N$ is the gravitational coupling. $D_\mu$ is the covariant derivative in the black hole background ($D_\mu = \partial_\mu + iA_\mu$ in flat space), and $q$ and $m^2$ are the charge and mass squared of $B_{\mu\nu}$, respectively. The terms associated with auxiliary fields are included in $\mathcal{L}_a$. This Lagrangian is the same as that appears in [34] in the flat space limit. However in the subsequent studies after [34] it appeared that construction of higher spin field Lagrangian coupled to gravity or $U(1)$ gauge in a gauge invariant way is non-trivial even in Minkowski space [38]. But later on in the context of gravitational interaction [39] it has been shown that this task may have a solution in AdS space. Also there exists a recent attempt to construct gauge invariant $U(1)$ charged massive spin 2 particles in AdS space at linear approximation [40]. In spite of all these studies fully consistent formulation of interacting higher spin gauge fields is still lacking. Keeping this in mind that the action in (1) without $\mathcal{L}_a$ contains spurious degrees of freedom that may cause the instability. We will proceed by assuming that the constrains can eliminate the instabilities but still allow the D-wave condensation.

$\mathcal{L}_m$ might look more familiar with the rescaling $B_{\mu\nu} \rightarrow qB_{\mu\nu}$ and $A_\mu \rightarrow qA_\mu$. Here we also concentrate on the “probe limit” [18] where the back-reaction to the background can
be ignored. This limit is exact when \( q \to \infty \). In the probe limit, \( \mathcal{L}_m \) can be treated as a perturbation on top of the 3+1 dimensional AdS black hole background:

\[
ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2(dx^2 + dy^2),
\]

where \( g(r) = r^2 - \frac{r_0^3}{r} \) and \( r_0 \) is the horizon size. The Hawking temperature for this black hole \( T = \frac{3m}{4\pi} \).

As in S-wave case \([18]\), electric field can exist in the bulk by the appropriate choice of boundary conditions. The charged tensor field, which can be considered as charged particles, experiences a force under the electric field, with positive (negative) charges repelled (attracted) away from (toward) the black hole. One the other hand, the black hole tries to pull all the charged particles in it. At lower \( T \), the black hole is smaller and the gravitational pull is weaker. Thus, the positively charged particles have a bigger chance to stay outside the horizon and form the condensate. At very large \( T \), the gravitational force from the large black hole is strong enough to pull all the charged particles into the horizon such that there is no condensate. Thus, we have a phase transition.

We are interested in describing the D-wave SC in the continuum such that there is a condensate on the \( x-y \) plane on the boundary with translational invariance. Rotational symmetry is broken down to \( Z(2) \) with the condensate changing its sign under a \( \pi/2 \) rotation on the \( x-y \) plane. To incorporate these features, we use an ansatz for the \( B_{\mu\nu} \) and the gauge field \( A_\mu \), i.e.,

\[
B_{\mu\nu} = \text{diagonal } (0, 0, f(r), -f(r)), \quad A = \phi(r)dt.
\]

After plugging in this ansatz, we have the equation of motion for \( B \)

\[
r^2 f''(r) + r \left[ \frac{g'(r)}{g(r)} - 2 \right] f'(r) + \left\{ \frac{r^2 \phi^2(r)}{g(r)^2} - \frac{m^2 r + 2g'(r)r}{g(r)} \right\} f(r) = 0,
\]

and the corresponding Maxwell’s equation is

\[
r^2 \phi''(r) + 2r \phi'(r) - \frac{4f^2(r)\phi(r)}{r^2 g(r)} = 0,
\]

where the \( ' \) is the derivative with respect to \( r \).

We would like to choose the solution such that \( \phi(r) \) has the asymptotic form

\[
\phi(r) \to \mu + \frac{\rho}{r}
\]
near the boundary \((r \to \infty)\), where \(\mu\) is interpreted as the chemical potential and \(\rho\) as the charge density in the boundary theory. Here, we will first assume this and then show that indeed this can be satisfied later. Now, the asymptotic of Eq.(4) has the asymptotic form

\[
r^2 f''(r) - (m^2 + 4) f(r) \simeq 0
\]

near the boundary, which yields

\[
f(r) \to f_0 r^{\Delta_+} + f_1 r^{\Delta_-},
\]

\[
\Delta_{\pm} = \frac{1 \pm \sqrt{17 + 4m^2}}{2}.
\]

If we interpret \(f_0\) as the source and \(f_1\) as the vacuum expectation value (VEV) of the operator that couples to \(B\) at boundary theory, we need \(m^2 \geq -4\) (and \(\Delta_- \leq 0\)) such that the \(f_1\) term is constant or vanishing at the boundary. After setting the source \(f_0 = 0\) and using \(\Delta_- \leq 0\), Eq.(5) indeed gives the asymptotic solution of Eq.(6). Note that the \(f_0 r^{\Delta_+}\) term does not impose a constraint on \(m^2\) by requiring that the third term on the LHS of Eq.(5) to be smaller than the other two terms since we have imposed \(f_0 = 0\). One way to see this is to do the integration of the differential equations from the boundary, then \(f(r) \to f_1 r^{\Delta_-}, \phi(r) \to \mu + \rho r\) satisfy the asymptotic behaviors of Eqs.(4) and (5). The order parameter of the boundary theory can be read off from the asymptotic behavior of \(B\),

\[
\langle O_{ij} \rangle = \begin{pmatrix} f_1 & 0 \\ 0 & -f_1 \end{pmatrix}
\]

where \((i,j)\) are the indexes in the boundary coordinates \((x,y)\).

It is useful to note that the action and the equations of motion are invariant under the scaling

\[
(t, r, x, y) \to \left( \frac{t}{c}, cr, x/c, y/c \right),
\]

\[
(r_0, T, g(r)) \to \left( cr_0, cT, c^2 g(r) \right),
\]

\[
(f(r), \phi(r)) \to \left( c^2 f(r), c\phi(r) \right).
\]

Thus, we can always scale \(\mu \to 1\). This also helps to keep track of the scaling dimension for observables, e.g., the scaling dimensions for \(\mu, \rho\), and \(f_1\) are 1, 2, and \(2 - \Delta_-\), respectively.
FIG. 1: (color online) The dimensionless D-wave condensate $f_1/\mu^{2-\Delta-}$ shown as a function of $T/T_c$ for $m^2 = -1/4$. The condensate goes to zero at $T = T_c \propto \mu$. The critical exponent $f_1 \to c(T_c - T)^\beta$ for $T_c - T \to 0^+$ is of the mean field value $\beta = 1/2$.

In Fig. 1, we show the numerical result between the dimensionless quantities $f_1/\mu^{2-\Delta-}$ and $T/T_c$ for $m^2 = -1/4$. It is a second order phase transition. Numerically, the critical exponent $\beta$ defined as $f_1 \to c(T_c - T)^\beta$ for $T_c - T \to 0^+$ is very close to the mean field value $\beta = 1/2$. Below we show that only the values $\beta = 1/2, 3/2, 5/2, \ldots$ satisfy the equations of motion. Thus, without fine tuning, one would get the mean field value $\beta = 1/2$.

Now we present the derivation. The metric $g(r)$ is a smooth function of $\epsilon = T_c - T$, while $\phi$ and $f$ can be expanded as

$$g(r, T) = a_g(r) + b_g(r) \epsilon + O(\epsilon^2),$$
$$\phi(r, T) = \epsilon^k (a_\phi(r) + b_\phi(r) \epsilon + O(\epsilon^2)), $$
$$f(r, T) = \epsilon^n (a_f(r) + b_f(r) \epsilon + O(\epsilon^2)).$$

Since, Eq.(4) is a linear equation in $f$ and the pre-factors of $f''(r)$ and $f'(r)$ are polynomials of $\epsilon$, the pre-factor of $f(r)$ has to be a polynomial of $\epsilon$ as well in order the satisfy the equation. This implies $k$ is an integer. At $T_c$ and at the boundary, $\phi$ gives the value of chemical potential which is finite. This yields $k = 0$.

Analogously, the Maxwell equation, Eq.(5), is linear in $\phi$. The pre-factors of $\phi''(r)$ and $\phi'(r)$ are polynomials of $\epsilon$ and thus the pre-factor of $\phi(r)$ is required to be a polynomial of $\epsilon$. This yields $2n$ to be an integer. We also know that $n > 0$ for a second order phase

\[ \text{(11)} \]
transition, and hence only $\beta = 1/2, 3/2, \ldots$ are allowed.

III. CONDUCTIVITY

In this section, we compute the conductivity of this D-wave HTSC by linear response. The conductivity tensor $\sigma_{ij}$ can be defined through the linear response relation

$$J_i = \sigma_{ij}E_j, \hspace{1cm} (12)$$

where $i, j = 1, 2$,. $J$ and $E$ are the electric current and electric field, respectively. Following the approach of [18], we perturb the gauge field by $\delta A = e^{-i\omega t}A_x(r)dx$. To get a consistent set of equations, we also need to perturb $\delta B_{rx} = \delta B_{xr} = ib_{rx}(r)e^{-i\omega t}$ and $\delta B_{tx} = \delta B_{xt} = b_{tx}(r)e^{-i\omega t}$ respectively. The resulting equations of motion are

$$gA''_x + g'A_x + \left(\frac{\omega^2}{g} - \frac{4f^2}{r^4}\right)A_x = 0, \hspace{1cm} (13)$$

$$gb_{rx}'' + 2g'b_{rx}' + \left[\frac{g''}{2} - \frac{g'}{r} - 5\frac{g}{r^2} + r_0^3 + \frac{(\omega - \phi)^2}{g}\right]b_{rx} = -\frac{2f}{r^3}A_x - \frac{(\omega - \phi)g'b_{tx}}{g}, \hspace{1cm} (14)$$

$$gb_{tx}'' + \left[\frac{g''}{2} - \frac{g'}{r} - \frac{g}{r^2} + r_0^3 + \frac{(\omega - \phi)^2}{g}\right]b_{tx} = (\omega - \phi)g'b_{rx}. \hspace{1cm} (15)$$

In principle, we can also add other $\delta B_{\mu\nu}$ components in the perturbation. However, those components do not couple to $\delta A_\mu$ to the quadratic order in the action. So, if we set the initial condition of our system to be in the ground state before $\delta A_\mu$ perturbation being turned on then those extra $B$ field perturbations will not be produced. However, in the full stability analysis, those $\delta A_\mu$ independent perturbations are important. We will defer this stability analysis to future study.

Eq. (13) is very similar to the S-wave case and is decoupled from $\delta B_{\mu\nu}$. Near the boundary, we have

$$r^2A''_x + 2r A'_x \simeq 0, \hspace{1cm} (16)$$

which yields the asymptotic form

$$A_x \rightarrow A_{x,0} + \frac{A_{x,1}}{r}, \hspace{1cm} (17)$$
where $A_{x,0}$ is the $x$-component gauge field at the boundary whose time derivative gives $E_x$, and $A_{x,1}$ is the expectation value of the current operator $J_x$. The ratio of $J_x$ and $E_x$ is the frequency dependent conductivity

$$\sigma(\omega) \equiv \sigma_{xx}(\omega) = -\frac{iA_{x,1}}{\omega A_{x,0}}. \quad (18)$$

The fact that Eq. (13) depends only on $A_x$ implies

$$\sigma_{yx}(\omega) = 0. \quad (19)$$

This is dictated by the reflection symmetry with respect to the $y = 0$ plane.

We are now focusing on the case $m^2 > -2$, where the asymptotic forms of Eqs. (14) and (15) are particularly simple:

$$r^2 b_{rx}'' + 4rb_{rx}' - (m^2 + 6) b_{rx} \simeq 0, \quad (20)$$

$$r^2 b_{tx}'' - (m^2 + 4) b_{tx} \simeq 0. \quad (21)$$

These two equations can be solved with

$$b_{rx} \to b_{rx,0}r^{\Delta_+} + b_{rx,1}r^{\Delta_-},$$

$$b_{tx} \to b_{tx,0}r^{\Delta_+} + b_{tx,1}r^{\Delta_-}, \quad (22)$$

where $\Delta_\pm = \frac{3 \pm \sqrt{33+4m^2}}{2}$ and $\Delta_\pm$ is defined in Eq. (15). Here, we also identify $b_{rx,0}$ and $b_{tx,0}$ as the source terms and $b_{rx,1}$ and $b_{tx,1}$ are the normalizable fluctuations.

Near the horizon, $g(r) = 3r_0dr + O(dr^2)$ with $dr = r - r_0$ and $\phi(r) = O(dr)$. The equations of motion become

$$9dr^2A_x'' + 9drA_x' + \frac{\omega^2}{r_0^2}A_x = 0, \quad (23)$$

$$9dr^2b_{rx}'' + 18drb_{rx}' + \frac{\omega^2}{r_0^2}b_{rx} = -\frac{\omega}{r_0^2}drb_{tx} - \frac{6f(r_0)}{r_0^4}drA_x, \quad (24)$$

$$9dr^2b_{tx}'' + \frac{\omega^2}{r_0^2}b_{tx} = 9\omega drb_{rx}. \quad (25)$$

The solutions near the horizon are

$$A_x \to \bar{A}_{x,1}dr^{-i\frac{\omega}{r_0}} + \bar{A}_{x,2}dr^{i\frac{\omega}{r_0}},$$

$$b_{tx} \to 3ir_0\left(\bar{b}_{rx,1}dr^{-i\frac{\omega}{r_0}+1} - \bar{b}_{rx,2}dr^{-i\frac{\omega}{r_0}} - \bar{b}_{rx,3}dr^{i\frac{\omega}{r_0}+1} + \bar{b}_{rx,4}dr^{i\frac{\omega}{r_0}}\right),$$

$$b_{rx} \to \bar{b}_{rx,1}dr^{-i\frac{\omega}{r_0}} + \bar{b}_{rx,2}dr^{-i\frac{\omega}{r_0}-1} + \bar{b}_{rx,3}dr^{i\frac{\omega}{r_0}} + \bar{b}_{rx,4}dr^{i\frac{\omega}{r_0}-1}. \quad (26)$$
FIG. 2: (color online) The real (left plot) and imaginary (right plot) of conductivity shown as a function for frequency $\omega$ for different temperatures. Above $T_c$, $\text{Re}[\sigma(\omega)] = 1$ while $\text{Im}[\sigma(\omega)] = 0$. Below $T_c$, $\text{Re}[\sigma(\omega)]$ has a $\delta(\omega)$ delta function whose height decrease in $T$ and vanishes at $T_c$. The right most curve has the lowest $T$, which implies the zero temperature gap $\omega_g/T_c \simeq 13$. (The construction in [18] for S-wave gives the value 8 for this gap.)

The ingoing wave boundary condition [41, 42], which sets the wave falling into the horizon, demands $\bar{a}_{x,2} = \bar{b}_{r,x,3} = \bar{b}_{r,x,4} = 0$. We further set the divergent term $\bar{b}_{r,x,2} = 0$ to keep the action finite. Now, $b_{r,x,0(1)}$ and $b_{tx,0(1)}$ in the Eq.(22) are linear combinations of $\bar{a}_{x,1}$ and $\bar{b}_{r,x,1}$. So we have both normalizable and non-normalizable solutions for $b_{tx}$ and $b_{r,x}$ perturbations. The divergent source terms $b_{r,x,0}$ and $b_{tx,0}$ near the boundary can be cancelled by counter terms [19].

In Fig. 2, we plot the real and imaginary part of $\sigma(\omega)$ for different $T$. The behaviours are similar to that of S-wave HTSC. $\text{Re}[\sigma(\omega)]$ has a delta function behaviour at $\omega = 0$ corresponding to infinite DC conductivity when $T < T_c$. On the other hand for $T \geq T_c$, the delta function and $\text{Im}[\sigma(\omega)]$ disappear and $\text{Re}[\sigma(\omega)]$ becomes $\omega$ independent. There is no “hard gap” in our dual boundary superconducting system because $\text{Re}[\sigma(\omega)]$ does not vanish even for arbitrary small $\omega$. One can also read off this soft gap from the plot, i.e. $\omega_g/T_c \simeq 13$. It is larger than the one obtained in the construction for S-wave [18], where $\omega_g/T_c \simeq 8$. This may imply our D-wave pairing requires higher energy than the S-wave one.

Unlike the case for S-wave superconductor, the vanishing of the gap is actually expected in the D-wave case. In the BCS-type theory (see, e.g. [43]), the lowest dimensional D-wave operator for two fermion pairing is $O_{ij} = \psi^T \left( \vec{\partial}_i \vec{\partial}_j - \vec{\partial}^2 \delta_{ij}/2 \right) \psi$, where $\vec{\partial}_i = \vec{\partial}_i - \vec{\partial}_i$ is the relative momentum between the two fermion which is invariant under
Galilean transformation. The leading order Lagrangian in a weakly interaction theory is

\[ \mathcal{L} = \mathcal{L}_0 - c \left( O_{ij}^\dagger + J_{ij}^\dagger \right) \left( O_{ij} + J_{ij} \right) + c J_{ij}^2, \]  

(27)

where \( \mathcal{L}_0 \) is the free Lagrangian, \( c \) is the coupling and \( J_{ij} \) is an external source. Under a Hubbard Stratanovich transformation, the Lagrangian can be rewritten as

\[ \mathcal{L}' = \mathcal{L}_0 + \left[ B_{ij}^\dagger \left( O_{ij} + J_{ij} \right) + \left( O_{ij}^\dagger + J_{ij}^\dagger \right) B_{ij} \right] + \frac{B_{ij}^\dagger B_{ij}}{c} + c J_{ij}^2, \]  

(28)

where \( B \) and \( B^* \) are auxiliary fields. After integrating over the auxiliary fields, \( \mathcal{L} \) is recovered from \( \mathcal{L}' \). It is clear that \( \langle O_{ij} \rangle \) in \( \mathcal{L} \) is \( \langle B_{ij} \rangle \) in \( \mathcal{L}' \). The gap equation of \( \mathcal{L}' \) gives the dispersion relation

\[ E_k = \sqrt{\left( \frac{k^2}{2m} - \mu \right)^2 + |B_{ij} k_i k_j|^2}. \]  

(29)

The gap \( |B_{ij} k_i k_j| \propto |k_x^2 - k_y^2| \) vanishes at four nodes \( k_x^2 = k_y^2 \). So, naturally gapless excitations can contribute to conductivity. This makes the conductivity for a D-wave superconductor gapless. In the S-wave case, however, the gap is isotropic and does not vanish in any direction leading to a hard gap in conductivity.

If we change the gauge field perturbation to \( \delta A = e^{-i\omega t} (A_x(r)dx + A_y(r)dy) \), then there will be response from \( \delta B_{rx}, \delta B_{tx}, \delta B_{ry} \) and \( \delta B_{ty} \). \( A(r) = A_x(r)\hat{x} + A_y(r)\hat{y} \) satisfies the same differential equation as Eq.(13):

\[ g \mathbf{A}'' + g' \mathbf{A}' + \left( \frac{\omega^2}{g} - \frac{4f^2}{r^4} \right) \mathbf{A} = 0. \]  

(30)

This shows that the conductivity is isotropic:

\[ \sigma_{ij} (\omega) = \sigma (\omega) \delta_{ij}. \]  

(31)

This might seem surprising at the first sight because the condensate is not isotropic. However, this is a consequence of the symmetries that \( \sigma_{ij} \) has in the D-wave case. In the linear response theory, \( \sigma_{ij} \) is a current-current correlator which can be schematically denoted as \( \sigma_{ij} \sim \langle \Omega | [J_i, J_j] | \Omega \rangle \), where the matrix element denotes an ensemble average. Under a \( \pi/2 \) rotation along the \( z \)-axis \( (R) \), \( R^{-1} J_i R = \epsilon_{ij} J_j \), where \( \epsilon_{ij} \) is an anti-symmetric tensor, and assuming the ensemble average is governed by properties of the ground state which has the condensate structure of Eq.(9), so \( R \langle \Omega \rangle = -\langle \Omega \rangle \). Then, \( \sigma_{ij} \sim \langle \Omega | [J_i, J_j] | \Omega \rangle = \langle \Omega | R^{-1} [J_i, J_j] R | \Omega \rangle = \langle \Omega | [\epsilon_{ik} J_k, \epsilon_{jl} J_l] | \Omega \rangle \). This implies \( \sigma_{xx} = \sigma_{yy} \) and \( \sigma_{xy} = -\sigma_{yx} \). A similar analysis with parity operator with respect to the \( x \)-axis gives \( \sigma_{xy} = \sigma_{yx} = 0 \). Thus, we have \( \sigma_{ij} \propto \delta_{ij} \). An explicit microscopic model calculation also yields an isotropic conductivity for a D-wave superconductor.
IV. CONCLUSION

We have constructed a minimal holographic model for high $T_c$ D-wave superconductors. We follow closely the work of Hartnoll, Herzog and Horowitz on the S-wave case. The 3+1 dimensional gravitational theory consists of a symmetric, traceless second-rank tensor field and a $U(1)$ gauge field in the background of the AdS black hole. Below $T_c$, the tensor field is Higgsed to break the $U(1)$ symmetry at the boundary theory. The phase transition characterised by the D-wave condensate is second order with the mean field critical exponent $\beta = 1/2$. As expected, the AC conductivity is isotropic; below $T_c$, the system becomes superconducting in the DC limit but has no hard gap.

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