Mixed Quantum States with Variable Planck Constant

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Abstract

Recent cosmological measurements tend to confirm that the fine structure constant $\alpha$ is not immutable and has undergone a tiny variation since the Big Bang. Choosing adequate units, this could also reflect a variation of Planck’s constant $h$. The aim of this Letter is to explore some consequences of such a possible change of $h$ for the pure and mixed states of quantum mechanics. Surprisingly enough it is found that not only is the purity of a state extremely sensitive to such changes, but that quantum states can evolve into classical states, and vice versa. A complete classification of such transitions is however not possible for the moment being because of yet unsolved mathematical difficulties related to the study of positivity properties of trace class operators.

Key words: density operator; mixed states; variable Planck constant

1 Introduction

The variability of physical “constants” is a possibility that cannot be outruled and which has being an active area of research for some time in cosmology and astrophysics. In fact, since Paul Dirac [1,2] suggested in 1937 the “Large Numbers Hypothesis” that some constants of Nature could vary in space and time, the topic has remained a subject of fascination which has motivated numerous theoretical and experimental researches [3,4].

The difficulty is not only of an experimental nature but also involves delicate issues related to the choice of units. It is anyway problematic to discuss the proposed rate of change (or lack thereof) of a single dimensional physical constant in isolation [5]. The reason for this is that the choice of a system of units may arbitrarily select any physical constant as its basis, making the question of which constant is undergoing change an artefact of the choice of
units: these issues have to be studied in depth. In the present Letter we begin
by shortly discussing some recent advances on the topic of varying natural
constants; we thereafter focus on the quantum mechanical consequences of
possible changes in Planck’s constant using the Wigner formalism.

2 On the Variability of Constants of Nature

2.1 The fine structure constant

Some scientists have suggested that the fine structure constant $\alpha = e^2/\hbar c$
might not be constant, but could vary over time and space. The quest for
testing this hypothesis is ongoing. The history actually started in a quite
romantic way, with the story of the Oklo natural nuclear reactor found in
a uranium mine in Central Africa in 1972 (see [6,7] for accounts of these
findings). The measurements that were made in Oklo give limits on variation of
the fine-structure constant over the period since the reactor was running that
is ca.1.8 billion years, which is much less than the estimated age of Universe. In
1999, a team of astronomers headed by J. Webb reported that measurements
of light absorbed by very distant quasars suggest that the value of the fine-
structure constant was once slightly different from what it is today. These
experiments, made using Keck and VLT telescopes in Hawaii, put an upper
bound on the relative change per year, at roughly $10^{-17}$ per year. In spite of
many criticisms, Webb and his collaborators seem to be progressing towards
a confirmation of this variation [8]. Anyway, as Feng and Yan [9] empha-
size, space-time variations of $\alpha$ in cosmology is a new phenomenon beyond the
standard model of physics which, if proved true, must mean that at least one
of the three fundamental constants $e, \hbar, c$ that constitute it must vary. This is a
delicate issue, related to choices of units, as will be discussed in a moment. Also
see Kraiselburd et al. [10] who analyze the consistency of different astronomical
data of the variation in the fine-structure constant obtained with Keck and
VLT telescopes.

2.2 Planck’s constant

Planck’s constant is a central number for modern physics, and it also appears
in the fine structure constant. To test whether Planck’s constant is really
constant, Mohageg and Kentosh [11,12] set out to measure possible spatial
discrepancies using the freely available data obtained from the same GPS sys-
tems that car drivers use to find their way home. The story goes as follows
[13-14]: GPS relies on the most accurate timing devices we currently possess:
atomic clocks. These clocks count the passage of time according to frequency of the radiation that atoms emit when their electrons jump between different energy levels. Kentosh and Mohageg looked through a year’s worth of GPS data of seven highly stable GPS satellites and found that the corrections depended in an unexpected way on a satellite’s distance above the Earth. This small discrepancy could be due to atmospheric effects or random errors, but it could also arise from a position-dependent Planck’s constant. If \( h \) changes from place to place, so do the frequencies, and thus the “ticking rate”, of atomic clocks. And any dependence of \( h \) on location would then translate as a tiny timing discrepancy between different clocks. So, what did they discover? After careful analysis of the data they obtained, Kentosh and Mohageg concluded that \( h \) is identical at different locations to an accuracy of seven parts in a thousand. Their results, which have been largely commented (and criticized) in the media, are however controversial; see for instance Berengut and Flambaum’s rebuttal [15], and Kentosh and Mohageg’s reply [16].

There are other recent work dealing with the possibility of a varying Planck’s constant. In [17] Seshavatharam and Lakshminarayana have discussed the possibility of viewing Planck’s constant as a cosmological variable; they argue that using the cosmological rate of change in Planck’s constant, the future cosmic acceleration can be verified from the ground based laboratory experiments. Mangano et al. [18] consider the possibility that the Planck constant is characterized by stochastic fluctuations. Their study is motivated by Dirac’s suggestion that fundamental constant are dynamical variables and by conjectures on quantum structure of spacetime at small distances; they assume that there is a time-dependence of \( h \) consisting in Gaussian random fluctuations around its constant mean value, with a typical correlation time scale.

We notice that the quantity \( h \) is also fundamental in the “positive” sense [19]: it is the quantum of the angular momentum \( J \) and a natural unit of the action \( S \). When \( J \) or \( S \) are close to \( h \), the whole realm of quantum mechanical phenomena appears (this is very important because it makes clear the relationship between \( h \) and minimal action and the notion of quantum blob we have developed elsewhere [20,21,22]).

### 2.3 Choice of units: a delicate problem

M. Duff has remarked [19,40,5] that all the fundamental physical dimensions could be expressed using only one unity: mass. Duff first noticed the obvious, namely that length can be expressed as times using \( c \), the velocity of light, as a conversion factor. One can therefore take \( c = 1 \), and measure lengths in seconds. The second step is to use the relation \( E = h \nu \) which relates energy to a frequency, that is to the inverse of a time. We can thus measure a time using
the inverse of energy. But energy is equivalent to mass as shown by Einstein, so that time can be measured by the inverse of mass. Thus, setting $c = h = 1$ we have reduced all the fundamental dimensions to one: mass. A further step consists in choosing a reference mass such that the gravitational constant is equal to one: $G = 1$. Summarizing, we have got a theoretical system of units in which $c = h = G = 1$. There are other ways to define irreducible unit systems. Already Stoney [23], noting that electric charge is quantized, derived units of length, time, and mass in 1881 by normalizing $G, c,$ and $e$ to unity; see the Wikipedia article [24] for a complete review of standard choices of units.

3 The Dependency of Density Matrices on Planck’s Constant

3.1 The $\eta$-Wigner distribution

A mixed quantum state is the datum of a countable set of pairs $S = \{ (|\psi_j\rangle, \alpha_j) : j \in \mathbb{N} \}$ where the $\psi_j$ are normalized elements of some Hilbert space $\mathcal{H}$ and the $\alpha_j$ are positive real numbers playing the role of probabilities: $\sum_j \alpha_j = 1$. The datum of $S$ is equivalent to that of the density matrix

$$\hat{\rho} = \sum_j \alpha_j |\psi_j\rangle \langle \psi_j|; \quad (1)$$

the $\hat{\rho}_j = |\psi_j\rangle \langle \psi_j|$ are the orthogonal projectors on the ray generated by $\psi_j$; they are identified with the pure states $|\psi_j\rangle$. Let us now be more specific, and choose once for all $\mathcal{H} = L^2(\mathbb{R}^n)$ (the square integrable functions on the configuration space $\mathbb{R}^n$). To the density matrix $\hat{\rho}$ let us associate the $\eta$-Wigner distribution

$$P_\eta(x, p) = \left(\frac{1}{2\pi\eta}\right)^n \int \langle x + \frac{1}{2} y | \hat{\rho} | x - \frac{1}{2} y \rangle e^{-\frac{1}{\eta} p y} \, d^n y \quad (2)$$

where $\eta$ is a non-zero real parameter. For the choice $\eta = \hbar$ the function $P_\hbar = P$ is just the usual Wigner distribution [26,25] commonly used in quantum mechanics. Using the definition (11) of the density matrix, we can rewrite formula (2) in the more explicit form

$$P_\eta(x, p) = \sum_j \alpha_j W_\eta \psi_j(x, p) \quad (3)$$

where

$$W_\eta \psi_j(x, p) = \left(\frac{1}{2\pi\eta}\right)^n \int e^{-\frac{1}{\eta} p y} \psi_j(x + \frac{1}{2} y) \psi_j^*(x - \frac{1}{2} y) \, d^n y \quad (4)$$

is the $\eta$-Wigner transform of $\psi_j$; when $\eta = \hbar$ we recover the usual Wigner transform $W_\hbar \psi_j = W \psi_j$. 

4
Exactly as the operator $\hat{\rho}$ is the Weyl transform of $P$, it is also the $\eta$-Weyl transform of $P_\eta$ in the sense that $\hat{\rho}$ has the harmonic decomposition

$$\hat{\rho} = \iint F_\eta \rho(x, p) e^{-\frac{i}{\hbar}(x\hat{x} + p\hat{p})} d^n x d^n p$$  \hspace{1cm} (5)

where $F_\eta \rho$ is the $\eta$-Fourier transform of $\rho$:

$$F_\eta \rho(x, p) = \left(\frac{1}{2\pi \eta}\right)^n \iint e^{-\frac{i}{\eta}(xx' + pp')} \rho(x', p') d^n x' d^n p'.$$  \hspace{1cm} (6)

Formula (5) shows that $(2\pi \eta)^n \rho$ is the Weyl symbol of the operator $\hat{\rho}$ \cite{33, 42, 34}, that is, we have

$$\hat{\rho} \psi(x) = \iint e^{\frac{i}{\eta}(x-y)} \rho \left(\frac{1}{2}(x + y), p\right) \psi(y) d^n y d^n p$$  \hspace{1cm} (7)

for every wavefunction $\psi$.

We are allowing the parameter $\eta$ to take negative values. The change of a positive $\eta$ to the negative value $-\eta$ has a simple physical interpretation: it corresponds to a reversal of the arrow of time. We have the following explicit formula relating $W_\eta \psi$ and $W_{-\eta} \psi$:

$$W_\eta \psi = (-1)^n W_{-\eta} (\psi^*);$$  \hspace{1cm} (8)

this is readily proved by making the substitution $y \mapsto -y$ in the integral in formula (4). An easy (but important) consequence of this equality is that the marginal properties

$$\int W_\eta \psi(x, p) d^n x = |F_\eta \psi(p)|^2, \quad \int W_\eta \psi(x, p) d^n p = |\psi(x)|^2$$  \hspace{1cm} (9)

hold for all the $\psi \in L^2(\mathbb{R}^n)$ such that $\psi \in L^1(\mathbb{R}^n)$ and $F_\eta \psi \in L^1(\mathbb{R}^n)$; here

$$F_\eta \psi(p) = \left(\frac{1}{2\pi |\eta|}\right)^{n/2} \int e^{-\frac{i}{\eta} p \hat{x}} \psi(x) d^n x.$$

Another consequence is that the Moyal identity

$$\int W_\eta \psi(z) W_\eta \phi(z) d^n z = \left(\frac{1}{2\pi |\eta|}\right)^n |\langle \psi | \phi \rangle|^2$$  \hspace{1cm} (10)

holds for all square integrable functions $\psi$ and $\phi$ and all $\eta \neq 0$ (we are writing for short $z = (x, p)$). In fact formula (10) is well known \cite{26, 25} when $\eta > 0$ (it suffices to replace $\hbar$ with $\eta > 0$ in the proof); when $\eta < 0$ formula (8) shows that

$$\int W_\eta \psi(z) W_\eta \phi(z) d^n z = \int W_{-\eta} \psi^*(z) W_{-\eta} \phi^*(z) d^n z$$

and since $-\eta > 0$ this leads us back to the former case:

$$\int W_{-\eta} \psi^*(z) W_{-\eta} \phi^*(z) d^n z = \left(\frac{1}{2\pi |\eta|}\right)^n |\langle \psi | \phi \rangle|^2$$
since 〈ψ∗|φ∗〉 = 〈ψ|φ〉∗. Notice that the Moyal identity implies that the functions ψ and φ are orthogonal if and only if their η-Wigner transforms are.

3.2 Transitions of quantum states

Consider the following situation: we have an unknown quantum system, on which we perform a quorum of measurements (for instance, by a homodyne quantum tomography [27]) in order to determine its (quasi)probability distribution \( \rho(z) = \rho(x,p) \). The latter allows us then to infer the density matrix \( \hat{\rho} \) using the Weyl correspondence [17]. It should however be clear that the result will depend on the value of Planck’s constant. Suppose that \( \rho(z) \) is the Wigner distribution in the usual sense (i.e. with \( \eta = \hbar \)) of a density matrix. Then there exists normalized square integrable functions \( \psi_1, \psi_2, \ldots \) and positive constants \( \alpha_1, \alpha_2, \ldots \) summing up to one and such that

\[
\rho(z) = \sum_j \alpha_j W\psi_j
\]

and \( \rho(z) \) is thus the Wigner distribution of a density matrix \( \hat{\rho} \). Can \( \rho(z) \) be the \( \eta \)-Wigner distribution of another density operator \( \hat{\rho}_\eta \) that is, can we have

\[
\rho(z) = \sum_j \beta_j W_\eta \phi_j
\]

where the \( \phi_j \) are normalized and \( \beta_j \geq 0, \sum_j \beta_j = 1 \)? We are going to see that this indeed possible, but only if some severe conditions are imposed to the probabilities \( \beta_j \). We begin by making a remark that will considerably simplify our argument. If \( \hat{\rho} \) is a density matrix, it is a trace class operator and is hence compact. But it then follows from the spectral decomposition theorem that \( \rho(z) \) can be written in the form \( \rho(z) = \sum_j \alpha'_j W\psi'_j \) with \( \alpha'_j \geq 0, \sum_j \alpha'_j = 1 \) and the vectors \( \psi'_j \) forming an orthonormal system (the \( \alpha'_j \) are the eigenvalues of \( \hat{\rho} \) and the \( \psi'_j \) are the corresponding normalized eigenfunctions); there is thus no restriction to assume that the vectors \( \psi_j \) in (11) are orthonormal. The same argument applies to \( \hat{\rho}_\eta \) so we can also assume that the vectors \( \phi_j \) are orthonormal. So let us assume that

\[
\sum_j \alpha_j W\psi_j = \sum_j \beta_j W_\eta \phi_j;
\]

squaring both sides of this equality we get

\[
\sum_{j,k} \alpha_j \alpha_k W\psi_j W\psi_k = \sum_{j,k} \beta_j \beta_k W_\eta \phi_j W_\eta \phi_k.
\]
Now, by the Moyal formula (10) we have
\[
\int W\psi_j(z)W\psi_k(z)\,dn\,z = \left(\frac{1}{2\pi\hbar}\right)^n \delta_{jk}
\]
\[
\int W_\eta\phi_j(z)W_\eta\phi_k(z)\,dn\,z = \left(\frac{1}{2\pi|\eta|}\right)^n \delta_{jk}
\]
hence, integrating both sides of the equality (14), we are led to the condition
\[
\left(\frac{1}{2\pi\hbar}\right)^n \sum_j \alpha_j^2 = \left(\frac{1}{2\pi|\eta|}\right)^n \sum_j \beta_j^2.
\]
This equality can be interpreted in terms of the purity \(\text{Tr}(\hat{\rho}^2)\) and \(\text{Tr}(\hat{\rho}_\eta^2)\) of the states \(\hat{\rho} = \sum_j \alpha_j\) and \(\hat{\rho}_\eta\). In fact, since the vectors \(\psi_j\) are orthonormal it follows from formula (11) that
\[
\hat{\rho}^2 = \left(\sum_j \alpha_j |\psi_j\rangle\langle\psi_j|\right)^2
= \sum_{j,k} \alpha_j \alpha_k |\psi_j\rangle\langle\psi_k| |\psi_k\rangle\langle\psi_j|
= \sum_j \alpha_j^2 |\psi_j\rangle\langle\psi_j|
\]
hence \(\text{Tr}(\hat{\rho}^2) = \sum_j \alpha_j^2\) and, similarly, \(\text{Tr}(\hat{\rho}_\eta^2) = \sum_j \beta_j^2\). The condition (15) can therefore be rewritten in the simple form
\[
|\eta|^n \text{Tr}(\hat{\rho}^2) = h^n \text{Tr}(\hat{\rho}_\eta^2).
\]
This interesting formula shows that the purity of a mixed quantum state crucially depends on the value of Planck’s constant. For instance if \(\text{Tr}(\hat{\rho}) = \text{Tr}(\hat{\rho}_\eta^2) = 1\) then we must have \(|\eta| = h\): no pure state remains pure if we change Planck’s constant (except for a change of sign in \(h\) corresponding to time reversal). More generally, if \(\hat{\rho}\) is a pure state \(|\psi\rangle\langle\psi|\) then formula (16) becomes \(\text{Tr}(\hat{\rho}_\eta^2) = (|\eta|/h)^n\) hence \(\hat{\rho}_\eta\) can be a mixed quantum state only if \(|\eta| < h\) and any decrease of \(|\eta|\) leads to a loss of purity.

4 Gaussian Mixed States

Let us study in some detail the possible transitions of Gaussian states; in addition to their intrinsic interest and importance, Gaussian states are the only one whose dependence on Planck’s constant is fully understood for the moment being (see however Dias and Prata’s analysis of non-Gaussian pure states).
4.1 A very simple example

Consider the centered normal probability distribution on $\mathbb{R}^2$ defined by

$$
\rho_{X,P}(x,p) = \frac{1}{2\pi\sigma_X\sigma_P} \exp \left[ -\frac{1}{2} \left( \frac{x^2}{\sigma_X^2} + \frac{p^2}{\sigma_P^2} \right) \right]
$$

(17)

where $\sigma_X, \sigma_P > 0$; the associated variances are $\sigma_X^2$ and $\sigma_P^2$ and the covariance is $\sigma_{XP} = 0$. Suppose now that the value of Planck’s constant is *hic et nunc* $\hbar$. It is known that $\rho_{X,P}$ is the Wigner distribution of a density operator $\hat{\rho}_{X,P}$ if and only if it satisfies the Heisenberg inequality $\sigma_X\sigma_P \geq \frac{1}{2}\hbar$ where $\hbar = \hbar/2\pi$. If we have $\sigma_X\sigma_P = \frac{1}{2}\hbar$ (which we assume from now on) then $\rho_{X,P}$ is the Wigner distribution of the coherent state

$$
\psi_X(x) = (2\pi\sigma_X^2)^{-1/4}e^{-x^2/2\sigma_X^2}
$$

(18)

and $\hat{\rho}_{X,P}$ is then just the pure-state density matrix $|\psi_X\rangle\langle\psi_X|$. Notice that $\hbar$ does not appear explicitly in the function (18). Suppose now that we move the distribution $\rho_{X,P}$ in space-time, to a location where $\hbar$ has a new value $\eta > 0$. Then $\sigma_X$ and $\sigma_P$ must satisfy the new Heisenberg inequality $\sigma_X\sigma_P \geq \frac{1}{2}\eta$ to qualify $\rho_{X,P}$ as a Wigner distribution, which implies that we must have $\eta \leq \hbar$ since we have fixed $\sigma_X\sigma_P$ equal to $\frac{1}{2}\hbar$. Physically this means that if we decrease the value of Planck’s constant then $\hat{\rho}_{X,P}$ is the density operator of a (mixed) quantum state, but if we increase its value so that $\eta > \hbar$ then the Gaussian $\rho_{X,P}$ can only be viewed as the probability density of a classical state – it no longer represents a quantum state. In this case we are witnessing a transition from the quantum world to the classical world.

4.2 General multi-mode Gaussians

Let us now consider Gaussians of the type

$$
\rho_\Sigma(z) = (2\pi)^{-n}\sqrt{|\det \Sigma|}e^{-\frac{1}{2}z^T\Sigma^{-1}z^2}
$$

(19)

where $\Sigma$ is a positive definite symmetric (real) $2n \times 2n$ matrix (the “covariance matrix”). We have $\rho \geq 0$ and

$$
\int \rho_\Sigma(z)d^{2n}z = 1
$$

hence the function $\rho$ can always be viewed as a classical probability distribution. It is the $\eta$-Wigner distribution of a density matrix if and only if $\Sigma$ satisfies the positivity condition

$$
\Sigma + \frac{i\eta}{2}J \geq 0
$$

(20)
where \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \) is the standard \( 2n \times 2n \) symplectic matrix (this condition means that all the eigenvalues of \( \Sigma + (i\eta/2)J \) are \( \geq 0 \)). This can be proven in several ways \([30,31,32]\), but none of the available proofs is really elementary. Condition (20) can be restated in several different ways. The most convenient is to use the symplectic eigenvalues of the covariance matrix. Observing that the product \( J\Sigma \) has the same eigenvalues as the antisymmetric matrix \( \Sigma_1/2J\Sigma_1/2 \) (because they are conjugate) its eigenvalues are pure imaginary numbers \( \pm i\lambda_1^\sigma, \pm i\lambda_2^\sigma, \ldots, \pm i\lambda_n^\sigma \) where \( \lambda_j^\sigma > 0 \) for \( j = 1, 2, \ldots, n \). The set \( \{\lambda_1^\sigma, \lambda_2^\sigma, \ldots, \lambda_n^\sigma\} \) is called the symplectic spectrum of \( \Sigma \). Now, there exists a symplectic matrix \( S \) (i.e. a matrix such that \( S^TJS = J \)) diagonalizing \( \Sigma \) as follows:

\[
\Sigma = S^TDS \quad D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}
\]

where \( \Lambda \) is the diagonal matrix with non-zero entries the positive numbers \( \lambda_1^\sigma, \lambda_2^\sigma, \ldots, \lambda_n^\sigma \) (this is called a symplectic, or Williamson, diagonalization of \( \Sigma \) \([30,33,32]\)). Since \( S^TJS = J \) we have

\[
\Sigma + \frac{i\eta}{2}J = S^TDS + \frac{i\eta}{2}J = S^T(D + \frac{i\eta}{2}J)S
\]

hence the condition \( \Sigma + \frac{i\eta}{2}J \geq 0 \) is equivalent to \( D + \frac{i\eta}{2}J \geq 0 \). Now, the characteristic polynomial of the matrix \( D + \frac{i\eta}{2}J \) is the product \( P_1(\lambda) \cdots P_n(\lambda) \) where the \( P_j \) are the second degree polynomials \( P_j(\lambda) = (\lambda_j^\sigma - \lambda)^2 - \frac{\eta^2}{4} \) hence the eigenvalues \( \lambda \) of \( D + \frac{i\eta}{2}J \) are the numbers \( \lambda = \lambda_j^\sigma \pm \frac{1}{2}\eta \); the condition \( D + \frac{i\eta}{2}J \geq 0 \) implies that all these eigenvalues \( \lambda_j \) must be \( \geq 0 \) and hence \( \lambda_j^\sigma \geq \sup\{\pm \frac{1}{2}\eta\} = \frac{1}{2}|\eta| \) for all \( j \). We have thus proven the equivalence

\[
\Sigma + \frac{i\eta}{2}J \geq 0 \iff |\eta| \leq 2\lambda_{\min}^\sigma
\]

where \( \lambda_{\min}^\sigma \) is the smallest symplectic eigenvalue of the covariance matrix \( \Sigma \). The purity of the corresponding \( \eta \)-density matrix is \([33\text{, p. 302}]\)

\[
\text{Tr}(\hat{\rho}_{\Sigma,\eta}^2) = \left(\frac{\eta}{2}\right)^n \det(\Sigma^{-1/2})
\]

hence \( \hat{\rho}_{\Sigma,\eta} \) is a pure state if and only if \( \det(\Sigma) = (\eta/2)^n \). Since \( \det(\Sigma) = \det(J\Sigma) = (\lambda_j^\sigma)^2 \cdots (\lambda_n^\sigma)^2 \) this requires that \( \lambda_j^\sigma = 1 \) for all \( j = 1, 2, \ldots, n \) in view (22). In this case the matrix \( D \) in (21) is the identity and \( \Sigma = S^T S \); the corresponding state is then a squeezed coherent state \([33,42,32,34]\): namely the image of the fiducial coherent state \( \phi_0(x) = (\pi\hbar)^{-n}e^{-|x|^2/2\hbar} \) by any of the two metaplectic operators \( \pm \hat{S} \) defined by the symplectic matrix \( S \). To summarize, we have the following situation (we assume here for simplicity that \( \eta > 0 \)): suppose that (20) holds for \( \eta = \hbar \). Then the system is a mixed...
quantum state for all $\eta \leq \hbar$; when $\hbar \leq \eta \leq 2\lambda_{\min}^\sigma$ it is still a mixed state unless $\eta = \lambda_1^\sigma = \cdots = \lambda_n^\sigma$ in which case it becomes a coherent state; when $\eta > 2\lambda_{\min}^\sigma$ we are in the presence of a classical Gaussian state.

5 Discussion

We have given necessary conditions for a quantum state to remain a quantum state if Planck’s constant undergoes a variation. To find sufficient conditions is a very difficult mathematical problem related to the study of positivity of trace class operators. Theoretical conditions allowing to test the positivity of a given trace class operators were actually developed by Kastler [35], Loupias and Miracle-Sole [36,37] in the late 1960s using the theory of $C^*$-algebras, and further studied by Narcowich and O’Connell [50,51,52,53] and Werner [39]. These conditions (the “KLM conditions”) are however of limited practical use and no major advances have been made since then (see however the paper of Dias and Prata [29] which deals with pure states). Very little is actually known about the consequences of a varying Planck constant outside the Gaussian case we discussed above; it can be shown that while the condition

$$\Sigma + \frac{i\hbar}{2} J \geq 0 \quad (24)$$

is necessary for a trace class operator $\hat{\rho}$ to be positive (and hence to be a quantum state), it is not sufficient. In fact, this condition alone (which is equivalent to the Robertson–Schrödinger inequalities) does not ensure positivity [46,32], except in the Gaussian case studied above (there are counterexamples where (24) is satisfied while the corresponding operator is non-positive [50,51,47,16]). We are discussing in a new work [38] alternative methods to study these issues using the theory of Weyl–Heisenberg frames. These methods might provide a better insight in these difficult questions.

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