STABLE MAPS AND BRANCH DIVISORS

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0. Introduction

Let $f : X \to Y$ be a surjection of nonsingular projective varieties of the same dimension. The ramification divisor $R$ of $f$ on $X$ is defined by requiring the sequence

\[ 0 \to f^*\omega_Y \to \omega_X \to \omega_X|_R \to 0 \]

(1)

to be exact. The branch divisor $br(f)$ on $Y$ is then defined by pushing forward: $br(f) = f_*(R)$. The support of $br(f)$ is the locus of points $y \in Y$ such that $f$ is not étale in any neighborhood of $f^{-1}(y)$.

If $f : C \to D$ is a degree $d$ map of nonsingular curves, then $br(f)$ is a divisor on $D$ of degree

\[ r = 2g(C) - 2 - d(2g(D) - 2) \]

by the Riemann-Hurwitz formula. Let $M_g(D,d)$ the moduli stack of degree $d$ maps from nonsingular genus $g = g(C)$ curves to $D$. The branch divisor yields a morphism of Deligne-Mumford stacks

\[ \gamma : M_g(D,d) \to \text{Sym}^r(D). \]

(2)

For moduli points $[f : C \to D] \in M_g(D,d)$, $\gamma([f]) = br(f)$. A natural extension of $\gamma$ to the compactification by stable maps

\[ M_g(D,d) \subset \overline{M}_g(D,d) \]

is the main result of the paper.

Consider first the following situation. Let $f : X \to Y$ be a projective morphism of $S$-schemes where:

(i) $X$ is a local complete intersection over $S$ of relative dimension $n$.
(ii) $Y$ is smooth over $S$ of relative dimension $n$.
(iii) All geometric fibers of $X$ over $S$ are reduced.

Under these conditions, a functorial relative Cartier divisor $br(f)$ on $Y$ over $S$ is constructed in Section 2. The divisor $br(f)$ is supported on the locus of points $y \in Y$ such that $f$ is not étale in any neighborhood of $f^{-1}(y)$. In this generality, $br(f)$ need not be an effective Cartier divisor. However, $br(f)$ is invariant under base change and coincides
with the branch divisor defined by (1) when $X \to S$ is smooth and every component of $X$ dominates one of $Y$.

The branch divisor $\text{br}(f)$ is constructed by studying the complex

$$Rf_*[f^*\omega_Y/S \to \omega_X/S],$$

well-defined up to isomorphism in $D^{-\text{coh}}(Y)$. By generalizing to complexes a classical construction of Mumford for sheaves ([Mu], §5.3), we can associate to (3) a Cartier divisor on $Y$. Section 1 contains the required generalization of Mumford’s results.

In Section 3, we apply our branch divisor construction to the universal family:

$$F : \mathcal{C} \to D \times \overline{M}_g(D,d)$$

over the moduli stack of stable maps $\overline{M}_g(D,d)$ for $d > 0$. Certainly this universal family (as a Deligne-Mumford stack) satisfies conditions (i-iii). It is shown $\text{br}(F)$ in this case is an effective relative Cartier divisor on $D \times \overline{M}_g(D,d)$ of relative degree $r$. The branch divisor $\text{br}(F)$ then yields a canonical morphism

$$\gamma : \overline{M}_g(D,d) \to \text{Sym}^r(D)$$

extending (2).

The morphism $\gamma$ has an appealing point theoretic description on the boundary of $\overline{M}_g(D,d)$. Let $[f : C \to D]$ be a moduli point where $C$ is a singular curve. Let $N \subset C$ be the cycle of nodes of $C$. Let $\nu : \tilde{C} \to C$ be the normalization of $C$. Let $A_1, \ldots, A_n$ be the components of $\tilde{C}$ which dominate $D$, and let $\{a_i : A_i \to D\}$ denote the natural maps. As $a_i$ is a surjective map between nonsingular curves, the branch divisor $\text{br}(a_i)$ is defined by (1). Let $B_1, \ldots, B_b$ be the components of $\tilde{C}$ contracted over $D$, and let $f(B_j) = p_j \in D$. We prove the formula:

$$\gamma([f]) = \text{br}(f) = \sum_i \text{br}(a_i) + \sum_j (2g(B_j) - 2)[p_j] + 2f_*(N).$$

It is easy to see that formula (5) associates an effective divisor of degree $r$ on $D$ to every moduli point $[f]$. However, the construction of $\gamma$ as a scheme-theoretic morphism requires the relative branch divisor results over arbitrary reducible, nonreduced bases $S$.

In Section 4, the morphism $\gamma$ is used to study the classical Hurwitz numbers $H_{g,d}$ via Gromov-Witten theory. $H_{g,d}$ is the number of nonsingular, genus $g$ curves expressible as $d$-sheeted covers of $\mathbb{P}^1$ with a fixed general branch divisor. The Hurwitz numbers were first computed in [Hu] by combinatorical techniques. A simple analysis of the moduli
space of stable maps to $\mathbb{P}^1$ shows:

\begin{equation}
H_{g,d} = \int_{\overline{M}_g(P^1,d)} \gamma^* (\xi^{2g-2+2d}),
\end{equation}

where $\xi$ is the hyperplane class on $\text{Sym}^{2g-2+2d}(\mathbb{P}^1) = \mathbb{P}^{2g-2+2d}$. It is then possible to directly evaluate the integral (6) using the virtual localization formula [GrP] to obtain a Hodge integral expression for the Hurwitz numbers:

\begin{equation}
H_{g,d} = \frac{(2g-2+2d)!}{d!} \int_{\overline{M}_g,d} \frac{1 - \lambda_1 + \lambda_2 - \lambda_3 + \ldots + (-1)^g \lambda_g}{\prod_{i=1}^d (1 - \psi_i)},
\end{equation}

for $(g,d) \neq (0,1), (0,2)$. The integral on the right is taken over the moduli space of pointed stable curves $\overline{M}_{g,d}$. The classes $\psi_i$ and $\lambda_j$ are the cotangent line classes and the Chern classes of the Hodge bundle respectively. The values $H_{0,1} = 1$ and $H_{0,2} = 1/2$ are degenerate cases from the point of view of the right side of (7).

A proof of formula (7) has been announced independently by Ekedahl, Lando, Shapiro, and Vainshtein using very different methods [ELSV]. In fact, the formula of [ELSV] also accounts for particular non-simply branched cases which appear again to be equal to vertex integrals in the virtual localization formula. However, the connection in the non-simply branched cases is not clear.

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1. **Perfect torsion complexes**

1.1. **Cartier divisors.** The base field $\mathbb{C}$ of complex numbers will be fixed for the entire paper. However, all the results of Sections 1.1 - 2.2 are valid over any algebraically closed base field. The characteristic 0 condition is required for generic smoothness in the construction of the branch divisor.

Let $A$ be an algebra of finite type over $\mathbb{C}$. Let $S \subset A$ be the multiplicative system of elements which are not zero divisors. Recall, the set of zero divisors of $A$ equals the union of all associated primes of $A$ ([Ma], p.50). A prime ideal $p \subset A$ is depth 0 if all non-units of $A_p$ are zero divisors. The associated primes of $A$ are exactly the depth 0 primes ([Ma], p.102). Let $K(A) = S^{-1}(A)$ be the total quotient ring of $A$. It is easy to check for $f \in A$, $K(A_f) = K(A)_f$.

Let $X$ be a scheme (always taken here to be quasi-projective over $\mathbb{C}$). We distinguish the points of $X$ (integral subschemes) from the
geometric points of $X$ ($\text{Spec}(\mathbb{C})$ subschemes). Let $\mathcal{K}$ be the sheaf of rings on $X$ defined by associating $K(A_i)$ to the basis of all affine open sets $\text{Spec}(A_i)$ of the Zariski topology of $X$. The equality

$$\Gamma(\text{Spec}(A_i), \mathcal{K}) = K(A_i)$$

follows from the property $K(A_f) = K(A)f$. Let $\mathcal{K}^*$ denote the sheaf of invertible elements of $\mathcal{K}$. A Cartier divisor is an element of $\Gamma(X, \mathcal{K}^*/O^*)$. This discussion follows Hartshorne ([Ha1], §II.6).

A Cartier divisor is defined by the data $\{(f_i, W_i)\}$ where the open sets $W_i = \text{Spec}(R_i)$ cover $X$ and $f_i \in K(R_i)^*$, $f_i/f_j \in \Gamma(W_i \cap W_j, O^*)$.

A Cartier divisor $D$ is effective if there exist defining data as above satisfying $f_i \in R_i \subset K(R_i)$. An effective Cartier divisor naturally defines a locally free ideal sheaf of $O_X$.

**Lemma 1.** Let $U \subset X$ be an open set containing all depth 0 points of $X$. Let $f \in \Gamma(U, O_U^*)$. Then, $f$ defines a canonical element of $\Gamma(X, \mathcal{K}^*)$.

**Proof.** Let $Z = U^c \subset X$. Let $\{W_i = \text{Spec}(R_i)\}$ be an open affine cover of $X$. Let $U_i = U \cap W_i$, $Z_i = Z \cap W_i$, and $f_i = f|_{U_i}$. Let $I \subset R_i$ be the radical ideal determined by closed set $Z_i$. Since $Z_i$ contains no depth 0 points, $I$ must contain a element $x$ of $R_i$ which is not a zero divisor. Since $\text{Spec}((R_i)_x) \subset U_i$, we see $f_i$ is naturally an element of $(R_i)^*_x$. As $K(R_i)$ is obtained from $(R_i)_x$ by further localization, $f_i$ yields a canonical element of $K(R_i)^*_x$. These local sections over $W_i$ patch to yield a canonical element of $\Gamma(X, \mathcal{K}^*)$. \hfill \square

1.2. **The divisor construction (local).** We recall here a construction of Mumford ([Mu], §5.3). For our general branch divisor construction, we must extend these results from sheaves to complexes.

Let $D_{\text{coh}}(X)$ be the derived category of bounded (from above) complexes of quasi-coherent $O_X$-modules with coherent cohomology on a scheme $X$. We will identify a sheaf with a complex in degree zero; we will identify a morphism with a complex in degrees $[-1, 0]$. By convention, free and locally free sheaves will have finite rank. An object $E^\bullet$ of $D_{\text{coh}}(X)$ is perfect if it is locally isomorphic to a finite complex of locally free sheaves. $E^\bullet$ is torsion if for all $i \in \mathbb{Z}$ the support of $H^i(E^\bullet)$ does not contain any point of depth zero of $X$. 

4
Let \( E^\bullet = [E^a \to E^{a+1} \to \ldots \to E^b] \) be a finite complex of free sheaves on \( X \), and let \( \text{rank}(E^i) = r_i \). Let

\[
\Lambda(E^\bullet) = \bigotimes_{i=a}^b (\Lambda^{r_i} E^i)^{(-1)^i}.
\]

Following [Mu], a choice of an explicit isomorphism \( E^i = O_X^{r_i} \) for each \( i \) yields an isomorphism

\[
\psi : \Lambda(E^\bullet) \to O_X.
\]

Because of the choice of the trivializations of \( E^i \), \( \psi \) is determined only up to multiplication by a section of \( O_X^* \). However, if \( E^\bullet \) is exact, there is a canonical isomorphism

\[
\kappa : \Lambda(E^\bullet) \to O_X.
\]

These isomorphisms \( \psi \) and \( \kappa \) will together determine a Cartier divisor in the torsion case.

Let \( E^\bullet = [E^a \to E^{a+1} \to \ldots \to E^b] \) be a finite torsion complex of free sheaves on \( X \). We define the associated Cartier divisor \( \text{div}(E^\bullet) \) following [Mu]. Let \( U \) be the complement of the union of the supports of \( H^i(E^\bullet) \). On \( U \) there is a canonical isomorphism

\[
\kappa_U : \Lambda(E^\bullet)|_U \to O_U.
\]

Let \( \psi : \Lambda(E^\bullet) \to O_X \) be an isomorphism defined by trivializations as above. Then, \( \psi_U \circ (\kappa|_U)^{-1} \) is a section of \( O_U^* \). As \( U \) contains all points of depth zero of \( X \), we obtain a unique section \( f \) of \( K_X^* \) by Lemma 1. Different trivializations of \( E^i \) over \( X \) change the isomorphism \( \psi \) by multiplication by an element of \( O_X^* \). Hence, \( f \) yields a well-defined section of \( K_X^*/O_X^* \). Let \( \text{div}(E^\bullet) \) denote this canonically associated Cartier divisor on \( X \). Note if \( E^\bullet \) is an exact finite complex of free sheaves, then \( \text{div}(E^\bullet) \) is zero.

1.3. The divisor construction (global). Let \( \phi : E^\bullet \to F^\bullet \) be a chain map of complexes. The mapping cone of \( \phi \) is the complex \( M(\phi)^\bullet \) with sheaves \( M(\phi)^i = E^{i+1} \oplus F^i \) and differentials \( M(\phi)^i \to M(\phi)^{i-1} \) determined by \( (e, f) \mapsto (de, df + (-1)^i \phi(e)) \) (where \( d \) denotes differentials on \( E^\bullet \) and \( F^\bullet \)). Note there are natural morphisms of complexes

\[
E^\bullet \to F^\bullet \to M(\phi)^\bullet \to E^\bullet[1]
\]

where \( F^i \to M(\phi)^i \) is given by \( f \mapsto (0, f) \) and \( M(\phi)^i \to E^{i+1} \) is defined by \( (e, f) \mapsto e \). Any sequence of morphisms \( E \to F \to G \to E[1] \) in \( D_{\text{coh}}^{-}(X) \) which is isomorphic to \( (8) \) in \( D_{\text{coh}}^{-}(X) \) is called a distinguished triangle.
The morphisms (8) induce a long exact sequence of cohomology
\[ \cdots \rightarrow H^i(E) \rightarrow H^i(F) \rightarrow H^i(M(\phi)) \rightarrow H^{i+1}(E) \rightarrow \cdots \]
In particular, if \( \phi \) is a quasi-isomorphism, then \( M(\phi) \) is exact.

**Lemma 2.** Let \( E^\bullet \) and \( F^\bullet \) be finite torsion complexes of free sheaves, and let \( \phi : E^\bullet \to F^\bullet \) be a chain map. Then, the mapping cone \( G^\bullet \) of \( \phi \) is also a finite torsion complex of free sheaves, and
\[
\text{div}(F^\bullet) = \text{div}(E^\bullet) + \text{div}(G^\bullet).
\]

**Proof.** \( G^\bullet \) is certainly a finite complex of free sheaves. Let \( Z \subset X \) be the union of the supports of the cohomology sheaves of \( E^\bullet \) and \( F^\bullet \). As the latter supports do not contain points of depth zero, neither does \( Z \). Let \( U = Z^c \). Both \( E^\bullet \) and \( F^\bullet \) are exact on \( U \), so \( \phi|_U \) is a quasi-isomorphism and \( G^\bullet|_U \) is also exact. Hence, \( G^\bullet \) is torsion.

There is a canonical isomorphism of \( \Lambda(F^\bullet) \) with \( \Lambda(E^\bullet) \otimes \Lambda(G^\bullet) \), which proves the lemma.

**Corollary 1.** Let \( E^\bullet_1 \) and \( E^\bullet_2 \) be finite torsion complexes of free sheaves. If they are isomorphic in \( D^-_{\text{coh}}(X) \), then the induced Cartier divisors \( \text{div}(E^\bullet_1) \) and \( \text{div}(E^\bullet_2) \) are equal.

**Proof.** If \( E^\bullet_1 \) and \( E^\bullet_2 \) are isomorphic in \( D^-_{\text{coh}}(X) \), then there exists an object \( L^\bullet \in D^-_{\text{coh}}(X) \) and chain maps \( L^\bullet \to E^\bullet_1 \) which are quasi-isomorphisms. We may prove the Corollary locally on \( X \). Locally, we can find a free complex \( F^\bullet \) with a chain map \( F^\bullet \to L^\bullet \) which is a quasi-isomorphism ([Ha1], Lemma 12.3). As \( E^\bullet_1 \) are finite and free, \( F^\bullet \) may be cut-off from below to yield a finite and free complex with quasi-isomorphisms: \( F^\bullet_{\text{cut}} \to E^\bullet_1 \).

It is therefore enough to prove the Corollary in case there exists a quasi-isomorphism \( \phi : E^\bullet_1 \to E^\bullet_2 \), but then it follows from Lemma 4. 

Let \( E^\bullet \) be a perfect torsion complex on \( X \). As \( E^\bullet \) is locally a finite torsion complex of free sheaves, Cartier divisors may be associated locally to \( E^\bullet \) via local trivializations and the construction of Section 1.2. By Corollary 1, these locally associated divisors agree and define a canonical Cartier divisor \( \text{div}(E^\bullet) \) on \( X \).

**Proposition 1.** Let \( E^\bullet \) be a perfect torsion complex on \( X \). Then \( \text{div}(E^\bullet) \) satisfies the following properties:

(i) \( \text{div}(E^\bullet) \) depends only on the isomorphism class of \( E^\bullet \) in \( D^-_{\text{coh}}(X) \).
(ii) If \( F \) is a coherent torsion sheaf on \( X \) admitting locally a finite free resolution, then \( \text{div}(F) \) is the divisor constructed in [Mu].

Moreover, \( \text{div}(F) \) is an effective Cartier divisor.
(iii) If $D$ is an effective Cartier divisor in $X$, $\text{div}(O_D) = D$.
(iv) The divisor is additive for distinguished triangles.
(v) If $f : X' \to X$ is a base change, such that $f^*E^*$ is torsion, then $f^*(\text{div}(E^*))$ is a Cartier divisor. Moreover, in this case
$$\text{div}(f^*(E^*)) = f^*(\text{div}(E^*))$$
(vi) $\text{div}(E^*[−1]) = −\text{div}(E^*)$.
(vii) If $L$ is a line bundle on $X$, $\text{div}(E^*) = \text{div}(E^* \otimes L)$.

Proof. For the most part, these properties are simple consequences of the construction. Property (i) follows immediately from local considerations and Corollary [1]. The equivalence with Mumford’s construction (ii) is true by definition. The effectivity of $\text{div}(F)$ is a subtle issue proven in [Mu]. An easy computation using the isomorphism between $[O_D]$ and $[O_X(-D) \to O_X]$ in $D^{-\text{coh}}(X)$ proves (iii). Lemma [3] and local analysis together imply (iv). Property (v) may be checked locally on $X$ and $Y$ where the divisor construction is seen to be compatible with the definition of the pull-back of Cartier divisors. Properties (vi) and (vii) are trivial consequences of the definitions. Property (vi) shows $\text{div}(E^*)$ is not an effective Cartier divisor for all perfect torsion complexes.

The following example will be required. Let $X$ be a projective scheme, and let $Y$ be a nonsingular curve. Let $f : X \to Y$ be a constant morphism with image $y \in Y$.

**Lemma 3.** For any coherent sheaf $F$ on $X$, $Rf_*(F)$ is a perfect torsion complex in $D^{-\text{coh}}(Y)$, and $\text{div}(Rf_*(F)) = \chi(F)[y]$.

**Proof.** $Rf_*(F)$ defines a complex in $D^{-\text{coh}}(Y)$ with coherent cohomology, nonzero in finitely many degrees. By the nonsingularity of $Y$, $Rf_*(F)$ is perfect. That $Rf_*(F)$ is torsion is clear. As $Rf_*(F)$ is exact outside of $y$, $\text{div}(Rf_*(F))$ is a multiple of the point $[y]$. The Lemma then follows from a local calculation.

1.4. **Torsion criterion.** Let $q : Y \to S$ be a smooth morphism with irreducible fibers. Let $\text{Ass}(Y)$ and $\text{Ass}(S)$ be the sets of depth 0 points of the schemes $Y$ and $S$ respectively. A point $p$ of $S$ corresponds to an integral subscheme $V_p \subset S$. Since $q$ is smooth with irreducible fibers, $q^{-1}(V_p)$ is an integral subscheme of $Y$ determining a point $q$ of $Y$. Let $\iota(p) = q$.

**Lemma 4.** $\iota(\text{Ass}(S)) = \text{Ass}(Y)$. 


Proof. The Lemma may be checked locally on \( Y \) and \( S \), so we may take \( Y = \text{Spec}(B) \) and \( S = \text{Spec}(A) \). Since \( q \) is smooth, \( q \) is flat. If \( M \) is a Noetherian \( R \)-module, let \( \text{Ass}_R(M) \) denote the set of primes of \( R \) associated to \( M \). An algebraic result from Bourbaki is now required (also [Ma], Theorem 12):

\[
\text{Ass}_B(B) = \bigcup_{p \in \text{Ass}_A(A)} \text{Ass}_B(B/pB).
\]

(9)

As discussed above, \( pB \subset B \) is a prime ideal. Hence \( \text{Ass}_B(B/pB) = \{pB\} \). Moreover, \( \iota(p) = pB \) by definition.

Let \( E^\bullet \) be a perfect object of \( D^-_{\text{coh}}(Y) \). We will require the following criterion for torsion.

**Lemma 5.** Let \( q : Y \to S \) be a smooth morphism with irreducible fibers. If for every geometric point \( s \in S \), the complex \( i^*_s(E^\bullet) \) is torsion on \( Y_s \) (where \( i_s : Y_s \to Y \) is the inclusion), then \( E^\bullet \) is torsion on \( Y \).

**Proof.** We again may take \( Y = \text{Spec}(B) \) and \( S = \text{Spec}(A) \). Let \( q = \iota(p) \) be a depth 0 point of \( Y \). By Lemma 4, all depth 0 points of \( Y \) may be so expressed. Let \( y \in V_q \) be a geometric point of \( Y \) with \( s = q(y) \) satisfying: \( i^*_s(E^\bullet) \) has cohomology supported away from \( y \) in \( Y_s \). Such a \( y \) can be found since \( V_q \) contains fibers of \( q \). As \( E^\bullet \) is perfect on \( Y \), we can take a finite locally free representative

\[
E^\bullet = [E^a \to E^{a+1} \to \cdots \to E^b]
\]

locally at \( y \in Y \). Since the fiber sequence

\[
0 \to E^a_y \to E^{a+1}_y \to \cdots \to E^b_y \to 0
\]

is exact by the torsion condition on \( i^*_s(E^\bullet) \), \( E^\bullet \) is exact in a Zariski neighborhood of \( y \) in \( Y \). In particular, the point \( y \) does not lie in the support of the cohomology of \( E^\bullet \) on \( Y \). Since \( y \) is in the closure of the point \( q \), we see \( q \) does not lie in the cohomology support.

We first note \( i^*_s(E^\bullet) \) is the pull-back in the derived category. For a complex of free objects (or, more generally a complex of \( S \)-flat objects), this pull-back is determined by the simple pull-back of sheaves. Second, we note the irreducibility hypothesis on the fibers of \( q : Y \to S \) can be easily removed in Lemma 5 by generalizing Lemma 4 slightly. We leave the details to the reader.

2. **Branch divisors**

2.1. **Notation.** Let \( X, Y, \) and \( S \) be schemes. Let

\[
p : X \to S, \quad q : Y \to S
\]
be morphisms satisfying:

(i) $X$ is a local complete intersection over $S$ of relative dimension $n$.
(ii) $Y$ is smooth over $S$ of relative dimension $n$.
(iii) All geometric fibers of $X$ over $S$ are reduced.

Let $f : X \to Y$ be a projective morphism over $S$. This data will be fixed for the entire section. We will construct a relative Cartier divisor $br(f)$ on $Y$ generalizing the standard branch divisor.

2.2. Direct images. We review here the natural map

$$Rf_* : D^-_{coh}(X) \to D^-_{coh}(Y)$$

obtained from direct images. Let $U$ be an $f$-relative Cech cover of $X$ (over every affine open in $Y$, $U$ restricts to a usual Cech covering). For any quasi-coherent sheaf $E$ on $X$, let $C^\bullet(U,E)$ be the associated Cech complex of quasi-coherent sheaves on $Y$. Let $E^\bullet$ be an object of $D^-_{coh}(X)$. Then, $Rf_*(E^\bullet)$ is defined to be the simple complex on $Y$ obtained from the double complex $C^p(U,E^q)$. The complex $Rf_*(E^\bullet)$ is certainly bounded from above. Moreover, the cohomology of $Rf_*(E^\bullet)$ may be computed by a spectral sequence with $E_2$ term $R^p f_*(H^q(E^\bullet))$. Since, $R^p f_*(H^q(E^\bullet))$ is a grid of coherent sheaves on $Y$ with only finitely many objects on each line of slope $-1$, the cohomology of $Rf_*(E^\bullet)$ is coherent. Hence, $Rf_*(E^\bullet)$ defines an element of $D^-_{coh}(Y)$. To show this construction is well-defined in the derived category, see [Ha2].

**Lemma 6.** $Rf_* : D^-_{coh}(X) \to D^-_{coh}(Y)$ carries perfect complexes to perfect complexes.

**Proof.** The statement is local, so we assume $Y$ is affine. Since $f$ is projective and $E^\bullet$ is perfect, we can assume $E^\bullet$ is a finite complex of locally free sheaves globally on $X$. By Lemma 5.8 of [Mu], each of the Cech sheaves $C^p(U,E^q)$ has finite Tor-dimension and hence admits a finite flat resolution by quasi-coherent sheaves on $Y$. Therefore $Rf_*(E^\bullet)$ is isomorphic in the derived category to a finite complex of quasi-coherent flat sheaves and hence is Tor-finite. As $Rf_*(E^\bullet)$ is bounded from above and has coherent cohomology, we can construct an isomorphic complex of locally free sheaves, indexed in $(-\infty, a]$ for some $a$. Then the Tor-finiteness implies the cut-off the complex below at a sufficient negative value will be locally free: the added sheaf will be flat and finitely generated, hence locally free. 

We now study the required base change properties. Let $\psi : \tilde{Z} \to Z$ be a projective morphism of schemes. We assume $Z$ has enough locally
frees (certainly $Z$ quasi-projective over $C$ suffices). The functor $\psi^*$ induces a natural derived functor

$$L\psi^* : D^{-}_{coh}(Z) \to D^{-}_{coh}(\widetilde{Z})$$

which sends perfect complexes to perfect complexes.

Let $\phi : \tilde{S} \to S$ be a base change of schemes and consider the Cartesian diagram:

$$
\begin{array}{ccc}
\tilde{X} & \longrightarrow & \tilde{Y} \\
\downarrow \phi_X & & \downarrow \phi_Y \\
X & \longrightarrow & Y
\end{array}
$$

\[f\]

In this case, $L\phi_X^*$ and $L\phi_Y^*$ may be defined on complexes of $S$-flat sheaves by $\phi_X^*$ and $\phi_Y^*$ respectively.

**Lemma 7.** For each complex $E^\bullet \in D^{-}_{coh}(X)$, there is a natural isomorphism

(10) \[L\phi_Y^*(Rf_*(E^\bullet)) \to R\tilde{f}_*(L\phi_X^*(E^\bullet)).\]

of complexes in $D^{-}_{coh}(\tilde{Y})$.

**Proof.** As $f$ is projective, $E^\bullet$ may be taken to be a complex of locally free sheaves (bounded from above). Let $\mathcal{U}$ be an $f$-relative Cech covering of $X$. Then the pull-back covering $\tilde{\mathcal{U}}$ is a $\tilde{f}$-relative Cech covering of $\tilde{X}$ (as may be checked locally on $Y$). As $E^\bullet$ is a locally free complex, $L\phi_X^*(E^\bullet)$ is just $\phi_X^*(E^\bullet)$ in $D^{-}_{coh}(\tilde{X})$. Hence $R\tilde{f}_*(L\phi_X^*(E^\bullet))$ is represented by the simple complex on $\tilde{Y}$ associated to

(11) \[C^p(\tilde{\mathcal{U}}, \phi_X^*E^q).\]

On the other hand, $Rf_*(E^\bullet)$ is the simple complex on $Y$ associated to the double complex

(12) \[C^p(\mathcal{U}, E^q).\]

The double complex (11) is easily seen to be the $\phi_Y$ pull-back of the complex (12). As a consequence, there is a natural map

(13) \[L\phi_Y^*(C^p(\mathcal{U}, E^q)) \to C^p(\tilde{\mathcal{U}}, \phi_X^*E^q).\]

As $X$ is flat over $S$, the complex (12) is also $S$-flat. Hence, the map (13) is a quasi-isomorphism. \qed
2.3. **The branch divisor construction.** Let $\omega_{X/S}$ and $\omega_{Y/S}$ denote the relative dualizing sheaves of the structure maps $p$ and $q$ respectively. After constructing a natural perfect torsion complex
\[ E^\bullet = [f^*\omega_{Y/S} \to \omega_{X/S}], \]
the branch divisor is defined by $br(f) = div(Rf_*(E^\bullet))$ on $Y$.

**Lemma 8.** There is a natural morphism $f^*\omega_{Y/S} \to \omega_{X/S}$.

**Proof.** The canonical morphism $f^*\Omega_{Y/S} \to \Omega_{X/S}$ induces a morphism
\[ f^*\omega_{Y/S} = \Lambda^n f^*\Omega_{Y/S} \to \Lambda^n \Omega_{X/S}. \quad (14) \]
Locally on $X$, we have an $S$-embedding $X \to M$, where $M$ is smooth of relative dimension $n + r$ over $S$ and $X$ is a local complete intersection. Let $I = I_{X/M}$. There is an exact sequence
\[ 0 \to I/I^2 \to \Omega_{M/S} \otimes \mathcal{O}_X \to \Omega_{X/S} \to 0, \]
where $I/I^2$ and $\Omega_{M/S} \otimes \mathcal{O}_X$ are locally free sheaves on $X$ of ranks $r$ and $n + r$. This sequence yields a morphism
\[ \Lambda^n \Omega_{X/S} \otimes \Lambda^r(I/I^2) \to \Lambda^{n+r} \Omega_{M/S} \otimes \mathcal{O}_X. \quad (15) \]
On the other hand, there is a canonical isomorphism
\[ \omega_{X/S} \cong \text{Hom}(\Lambda^r(I/I^2), \Lambda^{n+r} \Omega_{M/S} \otimes \mathcal{O}_X). \quad (16) \]
The morphisms (15) and (16) above induce a morphism
\[ \Lambda^n \Omega_{X/S} \to \omega_{X/S}. \quad (17) \]
It is easily checked the locally defined morphism (17) is canonical and hence yields a global morphism on $X$. The Lemma is established by composing (14) with (17). \qed

**Lemma 9.** Let $E^\bullet = [f^*\omega_{Y/S} \to \omega_{X/S}]$. Then $Rf_*(E^\bullet)$ is a perfect torsion complex in $D_{\text{coh}}(Y)$.

**Proof.** Since $E^\bullet$ is perfect, $Rf_*(E^\bullet)$ is perfect by Lemma 6. To prove $Rf_*(E^\bullet)$ is torsion on $Y$, we may use Lemmas 5 and 7 to reduce to the case in which $S$ is a geometric point. Then, by property (iii), $X$ is reduced. Let $\nu : \tilde{X} \to X$ be a resolution of singularities. Let $Z_1$ be the image in $Y$ of the singular locus of $X$, and let $Z_2 \subset Y$ be the locus where $f \circ \nu$ is not étale. Let $Z = Z_1 \cup Z_2$. As $Rf_*(E^\bullet)$ is exact on $Y \setminus Z$, the cohomology of $Rf_*(E^\bullet)$ is supported on $Z$. Since $Z$ is a closed subset $Y$ of dimension at most $n - 1$, $Z$ does not contain any point of depth zero (a generic point of a component of $Y$). \qed
Definition. Let \( br(f) = \text{div}(Rf_*(E^•)) \). We call \( br(f) \) the generalized branch divisor of \( f \).

Base change. Let \( \phi : \tilde{S} \to S \) be any morphism. Properties (i-iii) hold for \( \tilde{X} \to \tilde{Y} \to \tilde{S} \), and
\[
\phi_Y^*(br(f)) = br(\tilde{f}).
\]

Proof. By Lemma 7, \( L\phi_\ast Rf_*(E^•) \to R\tilde{f}_\ast L\phi_X^*(E_j^•) \). The result then follows from Proposition \( \Box \).

By the base change property, the generalized branch divisor \( br(f) \) is a relative Cartier divisor on \( Y \). By relative we mean here the restriction to any geometric fiber of \( Y \) over \( S \) is a well-defined Cartier divisor.

If \( p : X \to S \) is smooth and every component of \( X \) dominates a component of \( Y \), then \( br(f) \) is the standard branch divisor of \( f \).

3. Stable maps

3.1. Moduli points. Let \( \overline{M}_g(D, d) \) be the moduli space of genus \( g \), degree \( d > 0 \) stable maps to a nonsingular curve \( D \). Let
\[
F : C \to D \times \overline{M}_g(D, d)
\]
be the universal family of maps over \( \overline{M}_g(D, d) \). These objects and morphisms naturally lie in the category of Deligne-Mumford stacks. We could instead utilize equivariant constructions in the category of schemes to study these universal objects (See [FuP], [GrP]). In any case, conditions (i-iii) of Section 2.1 are valid for \( (18) \). Hence, there exists a relative Cartier divisor \( br(F) \) on \( D \times \overline{M}_g(D, d) \) over \( D \).

Let \( [f : C \to D] \in \overline{M}_g(D, d) \) be a moduli point. We first calculate \( br(f) \) on \( D \). Let \( \nu : \tilde{C} \to C \) be the normalization map, and let \( \tilde{f} = f \circ \nu \). Let \( N \) be the singular locus of \( C \) (\( N \) is the union of the nodal points).

There are canonical exact sequences
\[
0 \to f^*\omega_D \to \nu_*\tilde{f}^*\omega_D \to \mathcal{O}_N \otimes f^*\omega_D \to 0,
\]
\[
0 \to \nu_*\omega_{\tilde{C}} \to \omega_{\tilde{C}} \to \mathcal{O}_N \to 0.
\]

We will use these sequences to express the branch divisor \( br(f) \) as a sum over component contributions.

Lemma 10. \( br(f) = br(\tilde{f}) + 2f_*(N) \).

Proof. Since \( \nu \) is a finite map,
\[
R\tilde{f}_*(|[\tilde{f}^*\omega_D \to \nu_*\omega_{\tilde{C}}]) \to Rf_*(|\nu_*\tilde{f}^*\omega_D \to \nu_*\omega_{\tilde{C}})).
\]
Using (19) and (20), there are a natural distinguished triangles in $D_{coh}(C)$:

$$\left[ f^*\omega_D \to \nu_*\omega_{\tilde{C}} \right] \to \left[ \nu_*f^*\omega_D \to \nu_*\omega_{\tilde{C}} \right] \to \left[ \mathcal{O}_N \otimes f^*\omega_D \to 0 \right],$$

$$\left[ f^*\omega_D \to \nu_*\omega_{\tilde{C}} \right] \to \left[ f^*\omega_D \to \omega_{\tilde{C}} \right] \to \left[ 0 \to \mathcal{O}_N \right].$$

Push-forward by $Rf_*$ preserves distinguished triangles. By (21) and the first triangle,

$$br(\tilde{f}) = divRf_*([f^*\omega_D \to \nu_*\omega_{\tilde{C}}]) - f_*(N)$$

(using also properties (iv) and (vi) of Proposition 4). The second triangle yields

$$br(f) = divRf_*([f^*\omega_D \to \nu_*\omega_{\tilde{C}}]) + f_*(N).$$

The Lemma now follows.

Let $A_1, \ldots, A_a$ be the components of $\tilde{C}$ which dominate $D$, and let $B_1, \ldots, B_b$ be the components of $\tilde{C}$ contracted over $D$. Let

$$\{a_i : A_i \to D\}, \quad \{b_j : B_j \to D\}$$

denote the natural restrictions of $f$. As $a_i$ is a surjective map between nonsingular curves, the branch divisor $br(a_i)$ is defined by (1). Let $b_j(B_j) = p_j \in D$.

**Lemma 11.** Let $b : B \to p \in D$ be a contracted component. Then, $br(b) = (2g(B) - 2)[p]$.

*Proof.* The complex $[b^*\omega_D \to \omega_B]$ is isomorphic to $[\mathcal{O}_B \to \omega_B]$ with the zero map. By Lemma 3, $divRf_*(\mathcal{O}_B) = \chi(\mathcal{O}_B)[p] = (1 - g(B))[p]$, and $divRf_*(\omega_B) = \chi(\omega_B)[p] = (g(B) - 1)[p]$. As there is a distinguished triangle in $D_{coh}(B)$:

$$\mathcal{O}_B \to [\omega_B] \to [\mathcal{O}_B \to \omega_B],$$

we find $br(b) = (2g(B) - 2)[p]$. \qed

**Lemma 10** and **11** prove:

$$br(f) = \sum_i br(a_i) + \sum_j (2g(B_j) - 2)[p_j] + 2f_*(N). \quad (22)$$

The only negative contributions in (22) occur for contracted genus 0 components of $\tilde{C}$. However, by stability such components must contain at least 3 nodes of $C$. The Cartier divisor $br(f)$ is therefore effective for every moduli point $[f : C \to D]$. \[13\]
3.2. Universal effectivity. The effectivity of \( br(F) \) over each closed point of \( \overline{M}_g(D, d) \) does not guarantee \( br(F) \) is an effective Cartier divisor on \( D \times \overline{M}_g(D, d) \). The latter effectivity will now be established.

Let \( \pi : C \to \overline{M}_g(D, d) \) be the structure map of the universal curve. There is a canonical exact sequence on \( C \):

\[
0 \to K \to F^* \omega_D \to \omega_\pi \to Q \to 0.
\]

The following vanishing statement will be proven in Section 3.3.

**Lemma 12.** \( R^0 F_*(K) = 0 \) and \( R^1 F_*(Q) = 0 \).

Let \( E^* = [F^* \omega_D \to \omega_\pi] \) in \( D^-_{coh}(C) \). By definition,

\[
br(F) = \text{div}(RF_*(E^*)).
\]

The cohomology of \( RF_*(E^*) \) may be computed via a spectral sequence with \( E_2 \) term:

\[
\begin{array}{cc}
R^1 F_*(K) & R^1 F_*(Q) \\
R^0 F_*(K) & R^0 F_*(Q)
\end{array}
\]

where the grading is -1 for the bottom left corner, 0 for the diagonal, and +1 for the top right corner. By Lemma 13, we see \( RF_*(E^*) \) has cohomology only at grade 0. Hence, locally on \( D \times \overline{M}_g(D, d) \), the complex \( RF_*(E^*) \) is isomorphic in the derived category to a finite resolution of the coherent torsion sheaf \( H^0(RF_*(E^*)) \). By Mumford’s effectivity result (Proposition 1.ii), \( \text{div}(RF_*(E^*)) \) is an effective Cartier divisor on \( D \times \overline{M}_g(D, d) \).

As \( br(F) \) is effective and \( \pi \)-relatively effective, \( br(F) \) determines a \( \pi \)-flat subscheme of \( D \times \overline{M}_g(D, d) \). The relative degree of \( br(f) \) is \( r = 2g - 2 - d(2g(D) - 2) \). We have proven:

**Theorem 1.** The branch divisor \( br(F) \) induces a morphism:

\[
\gamma : \overline{M}_g(D, d) \to \text{Hilb}^r(D) = \text{Sym}^r(D).
\]

3.3. Proof of Lemma 12. We follow here the notation of Section 3.1. The first step in the proof is:

**Lemma 13.** The vanishings \( R^0 F_*(K) = 0 \), \( R^1 F_*(Q) = 0 \) are equivalent to the vanishings \( R^0 \pi_*(K) = 0 \), \( R^1 \pi_*(Q) = 0 \) respectively.

**Proof.** Let \( p : D \times \overline{M}_g(D, d) \to \overline{M}_g(D, d) \) be the projection. Consider first \( K \). Since \( \pi = p \circ F \), there is a spectral sequences with \( E_2 \) term:

\[
\begin{array}{cc}
R^1 p_*(R^0 F_*(K)) & R^1 p_*(R^1 F_*(K)) \\
R^0 p_*(R^0 F_*(K)) & R^0 p_*(R^1 F_*(K))
\end{array}
\]
which calculates the sheaves $R^i\pi(K)$. As both $R^0F_*(K)$ and $R^1F_*(K)$ have support finite over $\overline{M}_g(D,d)$, the first row of the above spectral sequence vanishes. Hence,

$$R^0\pi_*(K) = R^0p_*(R^0F_*(K)).$$

Moreover as the support of $R^0F_*(K)$ is $p$-finite, $R^0F_*(K)$ vanishes if and only if $R^0p_*(R^0F_*(K))$ does. The proof for $Q$ is identical as the supports of the sheaves $R^iF_*(Q)$ are also $p$-finite.

**Lemma 14.** $R^0\pi_*(K) = 0$.

*Proof.* Let $[f : C \to D]$ be a moduli point. Let $[f] \in U \subset \overline{M}_g(D,d)$ where $U$ is an open set. Let $z \in \Gamma(\pi^{-1}(U),K)$. The element $z$ is naturally a section of $F^*\omega_D$ over $\pi^{-1}(U)$ which lies in the kernel of

$$F^*\omega_D \to \omega_{\pi}.$$

Let $Spec(A) \subset \overline{M}_g(D,d)$ be any Artinian local subscheme supported at $[f]$. We will show the restriction of $z$ to the closed scheme $C_A = \pi^{-1}(Spec(A))$ vanishes for all such Artinian local subschemes. This vanishing suffices to prove $z = 0$ over a Zariski open neighborhood of $[f]$ by the Theorem on formal functions (see [Ha1]).

For notational simplicity, let $L = F^*\omega_D$ on $C$. Let $B \subset C$ be the union of subcurves contracted by $f$. Since $L|_B$ is trivial, we find the vanishing condition: a section of $\Gamma(C,L_{[f]})$ which has support on $B$ must vanish identically.

Let $Spec(A) \subset \overline{M}_g(D,d)$ be an Artinian local subscheme as above. Let $z_A$ be the restriction of $z$ to $C_A$. Let $m \subset A$ be the maximal ideal. We note $z_A$ must have support on $B$ as the sheaf map $L_A \to \omega_{\pi_A}$ is an isomorphism on the open set $B^c \subset C_A$. By the flatness of $\pi$, there is an exact sequence

$$0 \to mL_A \to L_A \to L_{[f]} \to 0$$

on $C_A$. By the vanishing condition we see $z_A \in \Gamma(C_A,mL_A)$. We then use the exact sequences

$$0 \to m^{k+1}L_A \to m^kL_A \to m^{k}/m^{k+1} \otimes L_{[f]} \to 0$$

to inductively prove $z_A \in \Gamma(C_A,m^kL_A)$ for all $k$. Thus $z_A = 0$ by the Artinian condition.

**Lemma 15.** $R^1\pi_*(Q) = 0$.

*Proof.* From sequence (23), we obtain:

$$R^1\pi_*(F^*\omega_D) \to R^1\pi_*(\omega_{\pi}) \to R^1\pi_*(Q) \to 0$$
on $\overline{M}_g(D, d)$. It suffices to prove $i$ is a surjection of sheaves. As before, let $[f : C \to D]$ be a moduli point. Consider the standard diagram:

$$
\begin{array}{c}
R^1\pi_*(F^*\omega_D)[f] \xrightarrow{i[f]} R^1\pi_*(\omega_{\pi})[f] \\
\downarrow s \quad \downarrow t \\
H^1(C, F^*\omega_D) \xrightarrow{j} H^1(C, \omega_C).
\end{array}
$$

Here, the top line denotes the fiber of the sheaves at the point $[f]$. As $R^1\pi_*(\omega_{\pi})$ is locally free and Serre dual to $R^0\pi_*(\mathcal{O}_C)$ on $\overline{M}_g(D, d)$, the map $t$ is an isomorphism. As $F^*\omega_D$ is $\pi$-flat, we may apply the cohomology and base change Theorem (see [Ha1]) to deduce $s$ is surjective. (As $R^2\pi_*(F^*\omega_D)[f] \to H^2(C, F^*\omega_D)$ is trivially surjective and $R^2\pi_*(F^*\omega_D)$ is locally free, the surjectivity of $s$ follows.) Surjectivity of $i$ locally at $[f]$ is equivalent to the surjectivity of $i|[f]$ by Nakayama’s Lemma. Therefore, the Lemma may be proven by showing $j$ is surjective.

It suffices finally to prove $H^1(C, Q) = 0$. Again, let $B \subset C$ be the union of subcurves contracted by $f$. Let $I_B \subset \mathcal{O}_C$ be the ideal sheaf of $B$. As the map $F^*\omega_D \to \omega_C$ is 0 on $B$, we see:

$$Image(F^*\omega_D) \subset I_B \otimes \omega_C.$$

Hence, there is a sequence

$$0 \to T \to Q \to \omega_C|_B \to 0$$

where $T$ is easily seen to be a torsion sheaf. Then,

$$h^1(C, Q) = h^1(C, \omega_C|_B) = h^0(C, Hom(\omega_C|_B, \omega_C)).$$

The last equality is by Serre duality. As $B$ is a proper subcurve, the last cohomology group certainly vanishes. 

Lemma 13-15 combine to prove Lemma 12.

### 4. Hurwitz numbers

#### 4.1. Integrals

Let $g \geq 0$ and $d > 0$ be integers. Let $b$ be a fixed general divisor of degree $2g - 2 + 2d$ on $\mathbb{P}^1$. Let $H_{g,d}$ be the number of nonsingular genus $g$ curves expressible as $d$ sheeted covers of $\mathbb{P}^1$ with branch divisor $b$. There is a long history of the study of $H_{g,d}$ in geometry and combinatorics. The first approach to these numbers via the combinatorics of the symmetric group was pursued by Hurwitz in [Hu].
Proposition 2. The Hurwitz numbers are integrals in Gromov-Witten theory:

\[ H_{g,d} = \int_{\overline{M}_g(P^1,d)} \gamma^*(\xi^{2g-2+2d}), \]

where \( \xi \) is the hyperplane class on \( \text{Sym}^{2g-2+2d}(P^1) = P^{2g-2+2d} \).

Proof. We first prove the locus \( M_g(P^1,d) \subset \overline{M}_g(P^1,d) \) is nonsingular (of the expected dimension). It suffices to prove the obstruction space \( \text{Obs}(f) \) vanishes. Let \([f : C \to P^1]\) be a moduli point with \( C \) nonsingular. There is a canonical right exact sequence:

\[ H^1(C, T_C) \xrightarrow{i} H^1(C, f^*T_{P^1}) \rightarrow \text{Obs}(f) \rightarrow 0. \]

Since \( d > 0 \), the sheaf map \( T_C \to f^*T_{P^1} \) has a torsion quotient. Hence, \( i \) is surjective and \( \text{Obs}(f) = 0 \). The virtual class of \( \overline{M}_g(P^1,d) \) must then restrict to the ordinary fundamental class of the open set \( M_g(P^1,d) \).

Let \( r = 2g - 2 + 2d \). Let \( p_1, \ldots, p_r \in P^1 \) be distinct points. By the computation of \( \gamma \) on singular curves (22), we find \( \gamma^{-1}(\sum p_i) \subset M_g(P^1,d) \). By Bertini’s Theorem applied to

\[ \gamma : M_g(P^1,d) \to P^r, \]

a general divisor \( \sum p_i \) intersects the stack \( M_g(P^1,d) \) transversely via \( \gamma \) in a finite number of points. These points are simply the finitely many Hurwitz covers ramified over \( \sum p_i \).

The first approach to the Hurwitz numbers is via divisor linear equivalences in \( \overline{M}_g(P^1,d) \). In genus 0 and 1, the divisor of maps

\[ D_p \subset \overline{M}_g(P^1,d) \]

ramified over a fixed point \( p \in P^1 \) may be expressed in terms of the boundary divisors of \( \overline{M}_g(P^1,d) \): \( D_p = \sum \alpha_i \Delta_i \). The equation

\[ H_{g,d} = D_p \cap \gamma^*(\xi^{r-1}) = \sum \alpha_i \Delta_i \cap \gamma^*(\xi^{r-1}) \]

then immediately yields recursive relations for \( H_{g,d} \):

\[ H_{0,d} = \frac{2d - 3d}{d} \sum_{i=1}^{d-1} \binom{2d - 4}{2i} i^2 (d - i)^2 H_{0,i} H_{0,d-i}, \]

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\[ H_{1,d} = \frac{d}{6} \binom{d}{2}(2d-1)H_{0,d} + \sum_{i=1}^{d-1} \binom{2d-2}{2i-2} (4d-2)i^2(d-i)H_{0,i}H_{1,d-i}. \]

The above recursions were derived by the second author and T. Graber. R. Vakil has extended these formulas in genus 0 and 1 by refining the method to include non-simple branching. We omit the proofs here since a uniform treatment may be found in [Va]. Following the shape of these equations, the recursion

\[ H_{2,d} = d^2 \left( \frac{97}{136}d - \frac{20}{17} \right) H_{1,d} + \sum_{i=1}^{d-1} \binom{2d}{2i-2} (8d - \frac{115}{17} i)i(d-i)H_{0,i}H_{2,d-i} + \sum_{i=1}^{d-1} \binom{2d}{2i} \left( \frac{11697}{34} i(d-i) - \frac{3899}{68} d^2 i(d-i) \right) H_{1,i}H_{1,d-i} \]

was conjectured by the second author and T. Graber in 1997. Using a completely different combinatorial approach, Goulden and Jackson have recently proven the genus 2 conjecture in [GoJ].

The existence of the genus 2 relation does not yet have a straightforward geometric explanation. In this sense, it is analogous to the surprisingly simple Virasoro prediction for the elliptic Gromov-Witten invariants of \( \mathbb{P}^2 \) (see [EHX], [P], [DZ]). It is not likely such simple recursive formulas occur for \( g \geq 3 \).

4.2. Localization. Let the torus \( \mathbb{C}^* \) act on \( V = \mathbb{C} \oplus \mathbb{C} \) diagonally with weights \( [0, 1] \) on a basis set \( \{ v_1, v_2 \} \). This action induces natural \( \gamma \)-equivariant actions on \( \mathcal{M}_g(\mathbb{P}(V), d) \) and

\[ \mathbb{P}^r = \text{Sym}(\mathbb{P}(V)) = \mathbb{P}((\text{Sym}^rV^*)^*). \]

Moreover, the \( \mathbb{C}^* \) action lifts equivariantly to the line bundle

\[ L = \mathcal{O}_{\mathbb{P}^r}(1). \]

The choice of equivariant lift to \( L \) will be exploited below. The integral

\[ H_{g,d} = \int_{[\mathcal{M}_g(\mathbb{P}(V), d)]^{vir}} \gamma^*(c_1(L)^r), \]

may then be evaluated via the virtual localization formula [GrP].
The connected components of the $C^*$-fixed locus of $\overline{M}_g(P(V), d)$ are indexed by a set of labelled connected graphs $\Gamma$ first studied by Kontsevich [Ko]. The vertices of these graphs lie over the fixed points $p_1, p_2 \in P^1$ and are labelled with genera (which sum over the graph to $g - h^1(\Gamma)$). The edges of the graphs lie over $P^1$ and are labelled with degrees (which sum over the graph to $d$). The virtual localization formula of [GrP] yields the equation:

$$H_{g,d} = \int_{\overline{M}_g(P(V), d)} \gamma^*(c_i(L)^r) = \sum_{\Gamma} \frac{1}{\text{Aut}(\Gamma)} \int_{\overline{M}_\Gamma} \gamma^*(c_1(L)^r) e(N^{vir}_\Gamma)$$

where $\overline{M}_\Gamma$ is a product moduli spaces of stable pointed curves and $\overline{M}_\Gamma/\text{Aut}(\Gamma)$ is the fixed locus associated to $\Gamma$ (see [GrP]). Moreover, the equivariant Euler class of the virtual normal bundle, $e(N^{vir}_\Gamma)$, is explicitly calculated in terms of the tautological $\psi$ and $\lambda$ classes on $\overline{M}_\Gamma$. Recall the Hodge integrals are the top intersection products of the $\psi$ and $\lambda$ classes on the moduli spaces of curves (see [GeP], [FP]).

For each choice of equivariant lifting of $C^*$ to $L$, formula (25) yields an explicit Hodge integral expression for $H_{g,d}$.

There are exactly $r + 1$ distinct $C^*$-fixed points of $P^r = P(\text{Sym}^r V^*)$. For $0 \leq a \leq r$, Let $p_a$ denote the fixed point $v_1^a v_2^{r-a}$. The canonical $C^*$-linearization on $L = O(1)$ has weight $w_a = r - a$ at $p_a$. Let $L_i$ denote the unique $C^*$-linearization of $L$ satisfying $w_i = 0$. We note the weight at $p_0$ of $L_i$ is $i$. We may rewrite (25) as:

$$H_{g,d} = \sum_{\Gamma} \frac{1}{\text{Aut}(\Gamma)} \int_{\overline{M}_\Gamma} \prod_{i=1}^r \gamma^*(c_1(L_i)) e(N^{vir}_\Gamma)$$

This choice of linearization for the integrand will lead to the simplest localization formula.

The morphism $\gamma$ associates a unique fixed point $p_\Gamma$ to each graph $\Gamma$:

$$\gamma(\overline{M}_\Gamma/\text{Aut}(\Gamma)) = p_\Gamma.$$}

Let $p_\Gamma = p_i \neq p_0$. Then, $\gamma(L_i)$ is a trivial bundle with trivial linearization when restricted to the fixed locus $\overline{M}_\Gamma/\text{Aut}(\Gamma)$. The $\Gamma$-contribution to the sum (26) therefore vanishes. We must only consider those graphs $\Gamma$ satisfying $p_\Gamma = p_0$ in the sum (26).

The point $p_0 = [v_2^r]$ corresponds to the divisor $r[v_1]$ on $P(V)$. It is a very strong condition for a stable map $[f : C \to P(V)]$ to have $br(f)$ supported at the single point $[v_1]$ - all nodes, collapsed components, and ramifications must lie over $[v_1]$. Hence, if $p_\Gamma = p_0$, the graph $\Gamma$ may not have any vertices of positive genus or valence greater than 1 lying over $[v_2]$. Moreover, the degrees of the edges of $\Gamma$ must all be 1. Exactly one graph $\Gamma_0$ satisfies these conditions: $\Gamma_0$ has a single genus
$g$ vertex lying over $[v_1]$ which is incident to exactly $d$ degree 1 edges (the vertices over $[v_2]$ are degenerate of genus 0).

The sum (26) contains only one term:

$$H_{g,d} = \frac{1}{\text{Aut}(\Gamma_0)} \int_{\mathcal{M}_{g,d}} \frac{\prod_{i=1}^r \gamma^*(c_1(L_i))}{e(N_{\text{vir}}^{\Gamma_0})}.$$  

By definition (see [GrP]), $\mathcal{M}_{g,d} = \overline{\mathcal{M}}_{g,d}$. Since the automorphism group of $\Gamma_0$ is the full permutation group of the edges, $\text{Aut}(\Gamma_0) = d!$. The virtual normal bundle contribution is calculated in [GrP]:

$$\frac{1}{e(N_{\text{vir}}^{\Gamma_0})} = \frac{1 - \lambda_1 + \lambda_2 - \lambda_3 + \ldots + (-1)^g \lambda_g}{\prod_{i=1}^d (1 - \psi_i)}.$$  

Finally, the integrand $\prod_{i=1}^r \gamma^*(c_1(L_i))$ restricts to a term of pure weight $r!$. We have proven:

**Theorem 2.**

$$H_{g,d} = \frac{(2g - 2 + 2d)!}{d!} \int_{\mathcal{M}_{g,d}} \frac{1 - \lambda_1 + \lambda_2 - \lambda_3 + \ldots + (-1)^g \lambda_g}{\prod_{i=1}^d (1 - \psi_i)},$$

for $(g, d) \neq (0, 1), (0, 2)$.

In genus 0, there is a well-known formula for the $\psi$ integrals:

$$\int_{\mathcal{M}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \binom{n-3}{a_1, \ldots, a_n}$$

(see [W]). The genus 0 formula

$$H_{0,d} = \frac{(2d - 2)!}{d!} d^{d-3}$$

then follows immediately from Theorem 2. Equation (28) was first found by Hurwitz.

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