Construction of solutions for delay evolution equations in a Banach space: A delayed Dyson-Phillips series

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Abstract
Functional evolution equations have many applications in modeling many physical processes. First, using the generation theorem, we prove that a fundamental solution of the delay evolution equation is a strongly continuous semigroup of linear operators. Second, we introduce a closed-form representation of a fundamental solution using a delayed Dyson-Phillips series. Then, we establish the analytical representation of the classical solutions of linear homogeneous/nonhomogeneous evolution equations with a discrete delay in a Banach space. In the special case, we consider delay evolution equations with permutable/nonpermutable linear bounded operators and derive crucial results in terms of noncommutative analysis. Furthermore, we prove that a fundamental solution of a functional evolution equation with bounded linear operator coefficients is a uniformly continuous semigroup of linear operators. Finally, we present an example in the context of a one-dimensional heat equation with a discrete delay to demonstrate the applicability of our theoretical results and we give some comparisons with existing results.

Keywords: Delay evolution equation; perturbation theory; a discrete delay; a delayed Dyson-Phillips series; heat equation with delay

1. Introduction
Some real-world problems can be modeled more accurately by incorporating time-delays or time-lags. Differential equations involving delays are called functional differential equations or time-delay systems. Time-delay systems are central to all areas of science, especially in the physical and biological sciences, such as vibration theory, control theory, and biophysics [1, 2, 3, 4].

One of the main problems of qualitative theory for functional differential equations is the explicit representation in closed-form for solutions of linear delay differential equations. Let us briefly summarize what has been done to solve the problem of representing solutions of linear time-delay systems via delayed matrix functions. We mention the pioneering work [5, 6] where the first results for linear differential systems with constant delay were found. In [5], Khusainov and Shuklin have studied relative controllability and stabilization problems for a linear control system with a single delay using a delayed exponential matrix function. In [6], Khusainov et al. have proposed an exact analytical representation of solutions for a linear differential system with permutable matrix coefficients and a constant delay. In [7, 8], the classical results are extended to time-delay systems of fractional order with permutable [7] and nonpermutable [8] matrices with the help of delayed Mittag-Leffler matrix functions. For more general results, using the Laplace transform technique, Mahmudov has proposed in [9], a closed-form representation of solutions for linear delay differential equations using a multiple delayed perturbation of the Mittag-Leffler matrix functions. Moreover, Huseynov and Mahmudov [10] have analyzed linear matrix coefficient systems of the neutral differential equations of integer and fractional orders with two incommensurate constant delays in terms of several aspects: positivity criterion, existence & uniqueness, and stability analysis.

In recent years, there has been increasing interest in the investigation of delayed partial differential equations [11]-[15]. Partial differential equations of parabolic type with a delayed argument are commonly...
used to model and study various problems arising in the investigation of population dynamics in ecological systems [14]. In [11], Khusainov et al. have considered the existence and uniqueness of a classical solution of the initial-boundary value problem for the one-dimensional heat equation with a constant delay. For the construction of a solution, the authors in [11] have used the method of separation of variables or Fourier method. In [12], Khusainov et al. have extended the obtained results in [11], and studied a general linear nonhomogeneous one-dimensional heat equation with delay in both lower and higher order terms subject to nonhomogeneous initial and boundary conditions. In [13], Samoilenko and Serheeva have described an algorithm for the construction of global solutions of the delayed one-dimensional heat equation with time-varying coefficients. Furthermore, Khusainov et al. in [14] have investigated an exact controllability results for a one-dimensional heat equation with time-delay under nonhomogeneous Dirichlet boundary conditions. Demchenko et al. in [15] have studied the problem of optimal control process described by a one-dimensional heat equation with a single constant delay and a distributed control function. Note that the solutions in [15] are expressed in the form of series with respect to eigenfunctions of the Sturm-Liouville problem.

Perturbation theory for strongly continuous operator families is a useful tool for evolution equations (partial differential equations) in modeling many physical phenomena. Perturbation of linear operators in a Banach space has been studied to a considerable extent, most notably by Phillips [17], Travis & Webb [18], and Lutz [19]. In [17], Phillips has first studied the implication for a linear abstract Cauchy problem a Banach space has been studied to a considerable extent, most notably by Phillips [17], Travis & Webb [18], and Lutz [19]. In [17], Phillips has first studied the implication for a linear abstract Cauchy problem

\[
\frac{d}{dt} u(t) = (\mathcal{A}_0 + \mathcal{A}_1) u(t), \quad t > 0,
\]

\[
u(0) = x \in D(\mathcal{A}_0),
\]

where \(\mathcal{A}_1 \in \mathcal{L}(\mathcal{X})\) is a linear bounded operator.

In [18], Travis and Webb have established sufficient conditions for perturbed cosine operator families. In [19], Lutz has intended to perturb the infinitesimal generator by adding it a linear time-varying bounded operator \(\mathcal{A}_1(\cdot) : \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})\) and studying some perturbation properties for infinitesimal generators of strongly continuous cosine operator families. In terms of fractional sense, some perturbation results for fractional abstract initial value problems of order \(1 < \alpha < 2\) in [20, 21, 22] have been studied in several aspects and the obtained results agree with the classical ones when \(\alpha = 2\). Moreover, Chen and Lu in [23] have studied the perturbation properties of nilpotent semigroups and applied them to heat exchanger equations.

In general partial differential equations with delay are written in an abstract way as follows [22]:

\[
\begin{align*}
\frac{d}{dt} u(t) &= \mathcal{A} u(t) + \Phi u(t), \quad t > 0, \\
u(0) &= x,
\end{align*}
\]

where

- \(x \in \mathcal{X}, \mathcal{X}\) is a Banach space,
- \(\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{X} \to \mathcal{X}\) is a closed and densely-defined linear operator,
- \(\Phi : \mathcal{W}^{1,p}([-\tau, 0], \mathcal{X}) \to \mathcal{X}\) is the ”delay” operator which is a bounded linear operator,
- \(u(\cdot) : [-\tau, \infty) \to \mathcal{X}\) and \(u_t(\cdot) : [-\tau, 0] \to \mathcal{X}\) is the ”history” function defined by \(u_t(s) := u(t + s)\) for \(s \in [-\tau, 0]\),
- \(f(\cdot) : \mathcal{L}^p([-\tau, 0], \mathcal{X})\) for \(1 \leq p < \infty\).

Bátkai and Piazzera in [24] have proved the equivalency between the partial differential equation with delay [12] and the following abstract Cauchy problem associated to the operator \((\mathcal{A}, D(\mathcal{A}))\):

\[
\begin{align*}
\frac{d}{dt} U(t) &= \mathcal{A} U(t), \quad t > 0, \\
U(0) &= (t) \in D(\mathcal{A}),
\end{align*}
\]
on the product state space \( X := \mathcal{X} \times \mathbb{L}^p ([-\tau, 0], \mathcal{X}) \) where \( U(t) = (u(t)) \in X \) for \( t \in \mathbb{R}_+ \), and a linear operator \( \mathcal{A} := \begin{pmatrix} A & \Phi \\ 0 & \frac{d}{dt} \end{pmatrix} \) with domain

\[
\mathcal{D}(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{D}(\mathcal{A}) \times \mathbb{W}^{1,p} ([-\tau, 0], \mathcal{X}) : f(0) = x \right\}.
\]

Then, the authors in [24] using the perturbation theorem of Miyadera-Voigt, give sufficient conditions for \((\mathcal{A}, \mathcal{D}(\mathcal{A}))\) to be the infinitesimal generator of a \( C_0 \)-semigroup on a Banach space \( \mathcal{X} \). Thus, in [24], they have considered \( \mathcal{A} \) as the sum \( \mathcal{A}_0 + \mathcal{A}_1 \) where \( \mathcal{A}_0 := \begin{pmatrix} A & 0 \\ 0 & \frac{d}{dt} \end{pmatrix} \) with domain \( \mathcal{D}(\mathcal{A}_0) := \mathcal{D}(\mathcal{A}) \) and

\[ \mathcal{A}_1 := \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{D}(\mathcal{A}_0), \mathcal{X}); \]

under appropriate conditions that \((A, \mathcal{D}(A))\) should be the infinitesimal generator of a strongly continuous semigroup \( \mathcal{S}(t) \) for \( t \in \mathbb{R}_+ \), \( \mathcal{A}_0 \) becomes an infinitesimal generator and in accordance with the perturbation theorem of Miyadera-Voigt, \( \mathcal{A}_0 + \mathcal{A}_1 \) is an infinitesimal generator as well. Furthermore, Kunisch and Schappacher in [26] have derived similar results for more special cases of abstract functional differential equation. In [27], Kaiser have studied the existence and uniqueness of solutions for linear partial differential equations with delay in \( \mathbb{L}^p \) spaces using an approach which is used in [24] for integrated semigroups.

Furthermore, by means of approximation methods, Pinto et al. in [28] have established an approximation of a mild solution of a semi-linear first-order abstract differential equation with delay in a Banach space.

Motivated by Phillips [17] and Bátkai & Piazzera [24], we consider the following abstract Cauchy problem for an evolutionary equation with a discrete delay (with \( s = -\tau \)) in a Banach space \( \mathcal{X} \):

\[
\begin{aligned}
\frac{d}{dt} u(t) &= \mathcal{A}_0 u(t) + \mathcal{A}_1 u(t-\tau) + g(t), \quad t > 0, \quad \tau > 0, \\
v(t) &= \varphi(t), \quad -\tau \leq t \leq 0,
\end{aligned}
\]

where \( \mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \rightarrow \mathcal{X} \) is a unbounded linear operator and \( \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \) is a linear bounded operator on \( \mathcal{X} \). Moreover, \( \varphi(\cdot) : [-\tau, 0] \rightarrow \mathcal{X} \) is describing the prehistory of the system and a forcing term \( g(\cdot) : \mathbb{R}_+ \rightarrow \mathcal{X} \) is describing the external forces. Unlike the authors in [24], we consider abstract differential equation with a discrete delay on the state space \( \mathcal{X} \) not the product state space. With the help of Generation Theorem (general case of Hille-Yosida theorem), we have showed that \( \mathcal{A}_0 + \mathcal{A}_1 \) where \( \mathcal{A}_1 := \mathcal{A}_1 e^{-\lambda_0 \tau} \in \mathcal{L}(\mathcal{X}) \) is the infinitesimal generator of a semigroup family \( \{S(t; \tau), t \in \mathbb{R}_+\} \) which is a fundamental solution of delay evolution equation (3.1) with unit initial conditions. By making use of the perturbation theorem of Phillips [17], we derive closed-form representation of a fundamental solution to (3.1) via a delayed Dyson-Phillips series. In addition, the strongly continuous semigroup \( \{S(t; \tau), t \in \mathbb{R}_+\} \) coincides with the delayed perturbation of an operator-valued exponential function for an abstract delay evolution equation with bounded linear operator coefficients which forms a uniformly continuous \( C_0 \)-semigroup on a Banach space \( \mathcal{X} \). As an application of our theoretical results, we derive an explicit representation of a mild solution to one-dimensional heat equation with a discrete delay and we give some comparisons between the closed-form solutions with existing works such as [28, 11].

The paper includes significant updates in the theory of abstract delay differential equations and is outlined as follows. Section 2 is a preparatory section where we recall main definitions and results from functional analysis, operator theory and evolution equations. Section 3 is devoted to closed-form representation of a fundamental solution of (3.1) in terms of a delayed Dyson-Phillips series. Meanwhile, we showed that a fundamental solution of abstract Cauchy problem (3.1) is a strongly continuous semigroup of linear operators by making use of the well-known Generation Theorem. In addition, we provide analytical representation of classical solutions to linear homogeneous and nonhomogeneous evolution equations with a discrete delay. In Section 4, we consider functional evolution equations with linear bounded operators and their some special cases. Thus, we propose closed-form representation of solutions to time-delay dynamical systems with permutable and nonpermutable linear bounded operators. Moreover, we justify that a fundamental solution of a delay evolution equation with bounded linear operator coefficients is a uniformly continuous semigroup of linear operators. Finally, Section 5 is devoted to the presentation of an illustrative example on the one-dimensional heat equation with delay to show the efficiency and validity of the theoretical results and we give some important comparisons with existing works.
2. Preliminary concept

We embark on this section by briefly presenting some important facts from operator theory, functional analysis, and abstract differential equations which are used throughout the paper.

Let us fix some notations. Let $\mathcal{X}$ be a Banach space with norm $\| \cdot \|$. We denote by $\mathcal{L}(\mathcal{X})$ the Banach algebra of all bounded linear operators on $\mathcal{X}$ and becomes a Banach space for the norm $\| \mathcal{T} \| = \sup \{ \| \mathcal{T} x \| : \| x \| \leq 1 \}$, for any $\mathcal{T} \in \mathcal{L}(\mathcal{X})$. The identity and zero (null) operators on $\mathcal{X}$ are denoted by $\mathcal{I} \in \mathcal{L}(\mathcal{X})$ and $\Theta \in \mathcal{L}(\mathcal{X})$, respectively.

Let $\mathbb{R}_+ := [0, \infty)$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{C}$ be a set of complex numbers, $J = [0, T] = \{0\} \cup \bigcup_{i=0}^{n} (i\tau, (i+1)\tau] \subset \mathbb{R}$ where $T = (n+1)\tau$ for a fixed $n \in \mathbb{N}_0$ and $\tau > 0$, and $\mathbb{I} = [-\tau, 0] \subset \mathbb{R}$.

For the construction of the delayed operator-valued functions we will use the following indicator (characteristic) function $1_{t>n\tau} : \mathbb{R} \to \mathbb{I}$:

$$1_{t>n\tau} := \begin{cases} 1, & t > n\tau, \\ 0, & t \leq n\tau, \end{cases} \quad (2.1)$$

where $\tau > 0$, $n \in \mathbb{N}_0$.

Throughout the paper we will use the following functional spaces:

- $C(\mathbb{R}_+, \mathcal{X})$ denotes the Banach space of continuous $\mathcal{X}$-valued functions $g(\cdot) : \mathbb{R}_+ \to \mathcal{X}$ equipped with an infinity norm $\|g\|_\infty = \sup_{t \in \mathbb{R}_+} \|g(t)\|$;
- $C^n(\mathbb{R}_+, \mathcal{X})$ denotes the Banach space of $n$-times continuously differentiable $\mathcal{X}$-valued functions $g(\cdot) : \mathbb{R}_+ \to \mathcal{X}$ endowed with a norm $\|g\|_{C^n} = \sum_{i=0}^{n} \sup_{t \in \mathbb{R}_+} \|g^{(i)}(t)\|$;
- $L^1(\mathbb{R}_+, \mathcal{X})$ denotes the Banach space of equivalence classes of all measurable $\mathcal{X}$-valued functions $g(\cdot) : \mathbb{R}_+ \to \mathcal{X}$ which are Lebesgue integrable and normed by $\|g\|_{L^1} = \int_{\mathbb{R}_+} \|g(s)\| \, ds < \infty$;
- $L^2(\mathbb{R}_+, \mathcal{X})$ denotes the Banach space of equivalence classes of all measurable $\mathcal{X}$-valued functions $g(\cdot) : \mathbb{R}_+ \to \mathcal{X}$ which are Lebesgue integrable and normed by $\|g\|_{L^2} = \left( \int_{\mathbb{R}_+} \|g(s)\|^2 \, ds \right)^{1/2} < \infty$.

**Definition 2.1.** A family $\mathcal{T}(t)$ for $t \in \mathbb{R}_+$ of linear operators on a Banach space $\mathcal{X}$ is called a strongly continuous semigroup or $C_0$-semigroup if the following properties hold true:

(i) $\mathcal{T}(0) = \mathcal{I}$;
(ii) $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$ for all $t, s \in \mathbb{R}_+$;
(iii) For every $x \in \mathcal{X}$, $t \mapsto \mathcal{T}(t)x$ is a continuous function from $\mathbb{R}_+$ into $\mathcal{X}$.

**Definition 2.2.** A semigroup $\mathcal{T}(t)$ for $t \in \mathbb{R}_+$ is said to be uniformly continuous with respect operator norm $\| \cdot \|$ associated with $\mathcal{X}$, if

$$\lim_{t \to 0_+} \| \mathcal{T}(t) - \mathcal{I} \| = 0.$$ 

**Definition 2.3.** A semigroup $\mathcal{T}(t)$ for $t \in \mathbb{R}_+$ is said to be strongly continuous with respect operator norm $\| \cdot \|$ associated with $\mathcal{X}$, if

$$\lim_{t \to 0_+} \| \mathcal{T}(t)x - x \| = 0, \quad \forall x \in \mathcal{X}.$$ 

**Definition 2.4.** The infinitesimal generator $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \to \mathcal{X}$ of a strongly continuous semigroup $\mathcal{T}(t)$ for $t \in \mathbb{R}_+$ given by

$$\mathcal{A}_0 x := \lim_{t \to 0_+} \frac{\mathcal{T}(t)x - x}{t}, \quad x \in \mathcal{D}(\mathcal{A}_0),$$

where

$$\mathcal{D}(\mathcal{A}_0) := \left\{ x \in \mathcal{X} : \lim_{t \to 0_+} \frac{\mathcal{T}(t)x - x}{t} \text{ exists} \right\}.$$
Lemma 2.1 \((2)\). For the infinitesimal generator \(A_0 : D(A_0) \subseteq \mathcal{X} \to \mathcal{X}\) of a strongly continuous semigroup \(\mathcal{F}(t), t \in \mathbb{R}_+\), the following properties hold true:

(i) \(A_0 : D(A_0) \subseteq \mathcal{X} \to \mathcal{X}\) is a closed and densely defined linear operator on \(\mathcal{X}\), i.e., \(\overline{D(A_0)} = \mathcal{X}\);

(ii) If \(x \in D(A_0)\), then \(\mathcal{F}(t)x \in D(A_0)\) and

\[
\frac{d}{dt}\mathcal{F}(t)x = A_0\mathcal{F}(t)x = \mathcal{F}(t)A_0x, \quad t \in \mathbb{R}_+.
\]

(iii) For every \(t \in \mathbb{R}_+\) and \(x \in \mathcal{X}\), one has

\[
\int_0^t \mathcal{F}(s)xds \in D(A_0).
\]

(iv) For every \(t \in \mathbb{R}_+\) the following identities hold true:

\[
\mathcal{F}(t)x - x = A_0 \int_0^t \mathcal{F}(s)xds, \quad x \in \mathcal{X}
\]

\[
= \int_0^t \mathcal{F}(s)A_0xds, \quad x \in D(A_0).
\]

Theorem 2.1 \((2)\). For every strongly continuous semigroup \(\mathcal{F}(t), t \in \mathbb{R}_+\), there exists constants \(\omega \in \mathbb{R}\) and \(M \geq 1\) such that

\[
\|\mathcal{F}(t)\| \leq Me^{\omega t}, \quad t \in \mathbb{R}_+.
\] (2.2)

Moreover, if \(\text{Re}(\lambda_0) > \omega\), then \(\lambda_0 \in \rho(A_0)\), and the resolvent of a semigroup is given by

\[
\mathcal{R}(\lambda_0; A_0)x = \int_0^\infty e^{-\lambda_0 t}\mathcal{F}(t)xdt, \quad x \in \mathcal{X}.
\] (2.3)

Theorem 2.2 (Generation Theorem, \(2\)). Let \(A_0 : D(A_0) \subseteq \mathcal{X} \to \mathcal{X}\) be a linear operator on a Banach space \(\mathcal{X}\) and let \(\omega \in \mathbb{R}, M \geq 1\) be constants. Then the following properties are equivalent:

(i) \(A_0\) generates a strongly continuous semigroup \(\mathcal{F}(t), t \in \mathbb{R}_+\) satisfying

\[
\|\mathcal{F}(t)\| \leq Me^{\omega t}, \quad t \in \mathbb{R}_+.
\]

(ii) \(A_0\) is closed, densely-defined, and for every \(\lambda_0 > \omega\) one has \(\lambda_0 \in \rho(A_0)\) and

\[
\left\| \left( (\lambda_0 - \omega)\mathcal{R}(\lambda_0; A_0) \right)^n \right\| \leq M, \quad \forall n \in \mathbb{N}.
\]

(iii) \(A_0\) is closed, densely-defined, and for every \(\lambda_0 \in \mathbb{C}\) with \(\text{Re}(\lambda_0) > \omega\) one has \(\lambda_0 \in \rho(A_0)\) and

\[
\|\mathcal{R}(\lambda_0; A_0)^n\| \leq \frac{M}{(\text{Re}(\lambda_0) - \omega)^n}, \quad \forall n \in \mathbb{N}.
\]

Theorem 2.3 \((17)\). A necessary and sufficient condition that a closed, densely-defined linear operator \(A_0 : D(A_0) \subseteq \mathcal{X} \to \mathcal{X}\) generates a semigroup \(\mathcal{F}(t)\) on \(\mathbb{R}_+\) is that there exists numbers \(M \geq 1\), and \(\omega \in \mathbb{R}\) such that

\[
\|\mathcal{R}(\lambda_0; A_0)\| \leq \frac{M}{\text{Re}(\lambda_0) - \omega}, \quad \text{Re}(\lambda_0) > \omega.
\]
**Definition 2.5.** A one-parameter semigroup $T(t)$ for $t \in \mathbb{R}_+$ on a Banach space $X$ is called a uniformly continuous semigroup if

$$
\mathbb{R}_+ \ni t \mapsto T(t) \in \mathcal{L}(X)
$$

is continuous with respect to the uniform operator topology on $\mathcal{L}(X)$.

**Theorem 2.4.** (2) Every uniformly continuous semigroup $T(t)$ for $t \in \mathbb{R}_+$ on a Banach space $X$ is of the form

$$
T(t) := e^{\mathcal{A}_0 t} = \sum_{k=0}^{\infty} \frac{\mathcal{A}_0^k t^k}{k!}, \quad t \in \mathbb{R}_+, 
$$

for some bounded linear operator $\mathcal{A}_0 \in \mathcal{L}(X)$.

**Theorem 2.5.** (2) For a strongly continuous semigroup $T(t)$, $t \in \mathbb{R}_+$ on a Banach space $X$ with generator $\mathcal{A}_0 : D(\mathcal{A}_0) \subseteq X \to X$, the following assertions are equivalent:

(i) The generator $\mathcal{A}_0$ is bounded, i.e., there exists $M > 0$ such that $\|y\| \leq M \|x\|$ for all $x \in D(\mathcal{A}_0)$;

(ii) The domain $D(\mathcal{A}_0)$ is all of $X$, i.e., $D(\mathcal{A}_0) = X$;

(iii) The domain $D(\mathcal{A}_0)$ is closed in $X$;

(iv) The semigroup $T(t)$ for $t \in \mathbb{R}_+$ is uniformly continuous.

In each case, the semigroup is given by

$$
T(t) = e^{\mathcal{A}_0 t} = \sum_{k=0}^{\infty} \frac{\mathcal{A}_0^k t^k}{k!}, \quad t \in \mathbb{R}_+. 
$$

**Definition 2.6.** (2) Let $X$ be a Banach space, $\mathcal{A}_0 : D(\mathcal{A}_0) \subseteq X \to X$ be a linear (unbounded) operator, and $x \in X$.

(i) The initial value problem

$$
\begin{align*}
 u'(t) &= \mathcal{A}_0 u(t), \quad t > 0, \\
 u(0) &= x,
\end{align*}
$$

is called the abstract Cauchy problem associated to $\mathcal{A}_0$ with initial value $x$;

(ii) A function $u(\cdot) : \mathbb{R}_+ \to X$ is called a classical solution of (2.6) if $u(\cdot)$ is continuously differentiable, $u(t) \in D(\mathcal{A}_0)$ for all $t \in \mathbb{R}_+$, and (2.6) holds.

**Theorem 2.6.** (2) Let $\mathcal{A}_0 : D(\mathcal{A}_0) \subseteq X \to X$ be the infinitesimal generator of a strongly continuous semigroup $T(t)$ for $t \in \mathbb{R}_+$. Then, for every $x \in D(\mathcal{A}_0)$, the function

$$
u(\cdot) : \mathbb{R}_+ \ni t \mapsto u(t) := T(t)x \in D(\mathcal{A}_0)$$

is the unique classical solution of (2.6) with initial value $x$.

Actually, there is no hope to have a classical solution of (2.6) if the initial value $x$ is not in $D(\mathcal{A}_0)$. This suggests that more general concepts of solutions might be useful.

**Definition 2.7.** (2) A continuous function $u(\cdot) : \mathbb{R}_+ \to X$ is called a mild solution of (2.6) if

$$
\int_0^t u(s)ds \in D(\mathcal{A}_0) \quad \text{for all } t \in \mathbb{R}_+
$$

and

$$
u(t) = x + \mathcal{A}_0 \int_0^t u(s)ds, \quad t \in \mathbb{R}_+. 
$$

So, Theorem 2.6 can be generalized to mild sense as follows.

**Theorem 2.7.** (2) Let $\mathcal{A}_0 : D(\mathcal{A}_0) \subseteq X \to X$ be the infinitesimal generator of a strongly continuous semigroup $T(t)$ for $t \in \mathbb{R}_+$. Then, for every $x \in X$, the function

$$
u(\cdot) : \mathbb{R}_+ \ni t \mapsto u(t) := T(t)x \in X$$

is the unique mild solution of (2.6) with initial value $x$. 

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Definition 2.8. \[2\] Let \( \mathcal{X} \) be a Banach space and assume that \( f(\cdot) : \mathbb{R}_+ \to \mathcal{X} \) is a measurable, exponentially bounded function of exponent \( \omega \in \mathbb{R} \), i.e., \( \|f(t)\| \leq Me^{\omega t} \) for all \( t \in \mathbb{R}_+ \) and some constant \( M > 0 \). Then, we define its Laplace transform \((\mathcal{L}f)(\cdot) : \{\lambda \in \mathbb{C} : Re(\lambda) > \omega\} \to \mathcal{X}\) by

\[
(\mathcal{L}f)(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt.
\] (2.8)

Theorem 2.8 \([2]\). Assume that \( f(\cdot), g(\cdot) \in \mathbb{C}^{(\mathbb{R}_+, \mathcal{L}(\mathcal{X}))} \) are exponentially bounded. If \((\mathcal{L}f)(\lambda) = (\mathcal{L}g)(\lambda)\) for sufficiently large \( Re(\lambda) \), then \( f = g \).

Definition 2.9. \([2]\) Let \( X \) be a Banach space and assume that \( f(\cdot), g(\cdot) : \mathbb{R}_+ \to \mathcal{L}(X) \). Then, we define convolution of \( f \) and \( g \) by

\[
(f * g)(t) = \int_0^t f(t - s) g(s) ds = \int_0^t f(s) g(t - s) ds, \quad t \in \mathbb{R}_+.
\] (2.9)

Theorem 2.9 \([2]\). Let \( f(\cdot), g(\cdot) : \mathbb{R}_+ \to \mathcal{L}(X) \) be strongly continuous and exponentially bounded functions of exponent \( \omega \in \mathbb{R} \). Then their convolution \( f * g \) is exponential bounded of exponent \( \omega \) and

\[
(\mathcal{L}(f * g))(\lambda) = (\mathcal{L}f)(\lambda)(\mathcal{L}g)(\lambda).
\] (2.10)

Lemma 2.2 \([2]\). Let \( J \) be some real interval and \( P(\cdot), Q(\cdot) : J \to \mathcal{L}(X) \) be two strongly continuous operator-valued functions defined on \( J \). In addition, suppose that \( P(x) : J \to X \) and \( Q(x) : J \to X \) are differentiable for all \( x \in \mathcal{D} \) for some subspace of \( X \), which is invariant under \( Q \). Then \((PQ)(t)x := P(t)Q(t)x\), is differentiable for \( x \in \mathcal{D} \) and

\[
\frac{d}{dt}(P(\cdot)Q(\cdot)x)(t_0) = \frac{d}{dt}(P(\cdot)Q(t_0)x)(t_0) + P(t_0) \left( \frac{d}{dt}Q(\cdot)x \right)(t_0).
\] (2.11)

3. Main results: a delayed Dyson-Phillips series

In this section, first, we consider the following abstract Cauchy problem for a linear homogeneous evolution equation with a discrete delay in a Banach space \( \mathcal{X} \):

\[
\begin{aligned}
    \frac{d}{dt}u(t) &= \mathcal{A}_0 u(t) + \mathcal{A}_1 u(t - \tau), \quad t > 0, \quad \tau > 0, \\
    u(t) &= \psi(t), \quad -\tau \leq t \leq 0,
\end{aligned}
\] (3.1)

where \( \mathcal{A}_0 : \mathcal{D} (\mathcal{A}_0) \subseteq \mathcal{X} \to \mathcal{X} \) is the infinitesimal generator of a \( C_0 \)-semigroup of linear operators \( \mathcal{T}(t) \) for \( t \in \mathbb{R}_+ \) and \( \mathcal{A}_1 : \mathcal{X} \to \mathcal{X} \) is a bounded linear operator on \( \mathcal{X} \). Our main aim is to determine an explicit representation of the classical solution of an abstract Cauchy problem \( (3.1) \).

A classical solution of the abstract Cauchy problem \( (3.1) \) is understood as an operator-valued function \( x(\cdot) \in \mathcal{D} (\mathcal{A}_0) \) defined for \( t \geq -\tau \), continuously differentiable for \( t > 0 \), and satisfying initial conditions \( x(t) := \psi(t) \in \mathcal{D} (\mathcal{A}_0) \) for \( -\tau \leq t \leq 0 \).

The fundamental solution of the homogeneous abstract initial value problem \( (3.1) \) which is a main part of the classical solution of delay evolution equation can be defined as follows.

If \( \mathcal{S}(\cdot; \tau) \in \mathcal{L}(\mathcal{X}) \) satisfies

\[
\begin{cases}
    \frac{d}{dt} \mathcal{S}(t; \tau) = \mathcal{A}_0 \mathcal{S}(t; \tau) + \mathcal{A}_1 \mathcal{S}(t - \tau; \tau), \quad t > 0, \quad \tau > 0, \\
    \mathcal{S}(t; \tau) = \begin{cases}
        \mathcal{I}, & t = 0, \\
        \Theta, & -\tau \leq t < 0,
    \end{cases}
\end{cases}
\] (3.2)

then \( \mathcal{S}(t; \tau) \) for \( t \in \mathbb{R}_+ \) is called the corresponding fundamental solution of abstract differential equation with a constant delay \( (3.1) \).
A fundamental solution is an operator-valued function which can be found with the help of Laplace transform technique. Let \( \lambda_0 \in \rho(\mathcal{A}_0) \). Then, applying Laplace integral transform each side of (3.2) with unit initial conditions and using integration by substitution formula for operator-valued functions [2], we obtain

\[
\lambda_0 \int_0^\infty e^{-\lambda_0 t} \mathcal{F}(t; \tau) dt - \mathcal{A} = \mathcal{A}_0 \int_0^\infty e^{-\lambda_0 t} \mathcal{F}(t; \tau) dt + \mathcal{A}_1 \int_0^\infty e^{-\lambda_0 (t+\tau)} \mathcal{F}(t; \tau) dt - \mathcal{A}_1 \int_0^\infty e^{-\lambda_0 t} \mathcal{F}(t; \tau) dt
\]

\[
= \mathcal{A}_0 \int_0^\infty e^{-\lambda_0 t} \mathcal{F}(t; \tau) dt + \mathcal{A}_1 \int_0^\infty e^{-\lambda_0 (t+\tau)} \mathcal{F}(t; \tau) dt - \mathcal{A}_1 \int_0^\infty e^{-\lambda_0 t} \mathcal{F}(t; \tau) dt.
\]

Therefore, for sufficiently large \( \text{Re} \lambda_0 \), the Laplace transform of \( \mathcal{F}(\cdot; \tau) \in \mathcal{L}(\mathcal{X}) \) is defined by

\[
\int_0^\infty e^{-\lambda_0 t} \mathcal{F}(t; \tau) dt = \left( \lambda_0 \mathcal{A} - \mathcal{A}_0 - \hat{\mathcal{A}}_1 \right)^{-1}, \quad \hat{\mathcal{A}}_1 := \mathcal{A}_1 e^{-\lambda_0 \tau} \in \mathcal{L}(\mathcal{X}) \cdot \tag{3.3}
\]

Moreover, the right-hand side of (3.3) is the perturbation of the infinitesimal generator \( \mathcal{A}_0 \) with a bounded linear operator \( \hat{\mathcal{A}}_1 \in \mathcal{L}(\mathcal{X}) \):

\[
\left( \lambda_0 \mathcal{A} - \mathcal{A}_0 - \hat{\mathcal{A}}_1 \right)^{-1} = \mathcal{R} \left( \lambda_0; \mathcal{A}_0 + \hat{\mathcal{A}}_1 \right). \tag{3.4}
\]

Therefore, the Laplace transform of a fundamental solution of \( \mathcal{A}(\cdot; \tau) \in \mathcal{L}(\mathcal{X}) \) can be determined by the resolvent of \( \mathcal{A}_0 + \hat{\mathcal{A}}_1 \) as follows:

\[
\int_0^\infty e^{-\lambda_0 t} \mathcal{F}(t; \tau) dt = \mathcal{R} \left( \lambda_0; \mathcal{A}_0 + \hat{\mathcal{A}}_1 \right). \tag{3.5}
\]

The following lemma is given in more general case for closed linear operators and plays a significant role in the proof of Theorem 3.1.

**Lemma 3.1.** Let \( \mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \rightarrow \mathcal{X} \) be closed linear operator on \( \mathcal{X} \) and assume \( \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \) is such that \( \| \mathcal{A}_1 \mathcal{R}(\lambda_0 ; \mathcal{A}_0) \| < e^{\text{Re} \lambda_0 \tau} \) for some \( \lambda_0 \in \rho(\mathcal{A}_0) \). Then, \( \mathcal{A} + \hat{\mathcal{A}}_1 \) where \( \hat{\mathcal{A}}_1 := \mathcal{A}_1 e^{-\lambda_0 \tau} \in \mathcal{L}(\mathcal{X}) \) is a closed linear operator with domain and \( \mathcal{R} \left( \lambda_0; \mathcal{A}_0 + \hat{\mathcal{A}}_1 \right) \) exists, and the following identity holds true:

\[
\mathcal{R} \left( \lambda_0; \mathcal{A}_0 + \hat{\mathcal{A}}_1 \right) = \sum_{n=0}^\infty \mathcal{R}(\lambda_0; \mathcal{A}_0) \left[ \mathcal{A}_1 \mathcal{R}(\lambda_0 ; \mathcal{A}_0) \right]^n. \tag{3.6}
\]

**Proof.** Since the Neumann series \( \sum_{n=0}^\infty \left[ \mathcal{A}_1 \mathcal{R}(\lambda_0 ; \mathcal{A}_0) \right]^n \) converges under the hypotheses \( \| \mathcal{A}_1 \mathcal{R}(\lambda_0 ; \mathcal{A}_0) e^{-\lambda_0 \tau} \| < 1 \), we note that

\[
\mathcal{R} \equiv \sum_{n=0}^\infty \mathcal{R}(\lambda_0; \mathcal{A}_0) \left[ \mathcal{A}_1 \mathcal{R}(\lambda_0 ; \mathcal{A}_0) \right]^n
\]

\[
= \mathcal{R}(\lambda_0; \mathcal{A}_0) \sum_{n=0}^\infty \left[ \mathcal{A}_1 \mathcal{R}(\lambda_0 ; \mathcal{A}_0) e^{-\lambda_0 \tau} \right]^n
\]

\[
= \mathcal{R}(\lambda_0; \mathcal{A}_0) \left( \mathcal{A} - \mathcal{A}_1 \mathcal{R}(\lambda_0 ; \mathcal{A}_0) e^{-\lambda_0 \tau} \right)^{-1},
\]

\[
\frac{\mathcal{R}(\lambda_0; \mathcal{A}_0)}{1 - \mathcal{A}_1 \mathcal{R}(\lambda_0 ; \mathcal{A}_0)} = \sum_{n=0}^\infty \mathcal{A}_1^n \mathcal{R}(\lambda_0; \mathcal{A}_0),
\]

\[
\mathcal{R}(\lambda_0; \mathcal{A}_0) = \frac{\mathcal{R}(\lambda_0; \mathcal{A}_0)}{1 - \mathcal{A}_1 \mathcal{R}(\lambda_0 ; \mathcal{A}_0)} = \sum_{n=0}^\infty \mathcal{A}_1^n \mathcal{R}(\lambda_0; \mathcal{A}_0).
\]
and by using the relation \[(3.4)\], we derive that
\[
\left(\lambda_0 \mathcal{A} - \mathcal{A}_0 - \mathcal{A}_1 e^{-\lambda_0 \tau}\right) = \left(\mathcal{A} - \mathcal{A}_1 e^{-\lambda_0 \tau}\right) \mathcal{A}(\lambda_0; \mathcal{A}_0)^{-1} \mathcal{A}(\lambda_0; \mathcal{A}_0) \left(\mathcal{A} - \mathcal{A}_1 e^{-\lambda_0 \tau}\right)^{-1} = \mathcal{A}. \quad (3.7)
\]

Furthermore, the range of \(\mathcal{A}\) is precisely \(\mathcal{D}(\mathcal{A}_0)\) since the range of \(\left(\mathcal{A} - \mathcal{A}_1 e^{-\lambda_0 \tau}\right)^{-1}\) is \(\mathcal{X}\). Thus, for a given \(x \in \mathcal{D}(\mathcal{A}_0)\) there exists a \(y\) such that \(x = \mathcal{A} y\). Therefore, by \((3.7)\), we attain that
\[
\mathcal{A}(\lambda_0 \mathcal{A} - \mathcal{A}_0 - \mathcal{A}_1 e^{-\lambda_0 \tau}) x = \mathcal{A}(\lambda_0 \mathcal{A} - \mathcal{A}_0 - \mathcal{A}_1 e^{-\lambda_0 \tau}) \mathcal{A} y = \mathcal{A} y = x,
\]
so that \(\mathcal{A}\) is both a left and a right inverse. The proof is complete. \(\square\)

With the help of following theorem, we will show that a fundamental solution \(\mathcal{F}(t; \tau), t \in \mathbb{R}_+\) of \((3.1)\) is a strongly continuous semigroup generated by \(\mathcal{A} : = \mathcal{A}_0 + \mathcal{A}_1\).

**Theorem 3.1.** If \(\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \rightarrow \mathcal{X}\) is the infinitesimal generator of a strongly continuous semigroup of linear operators \(\mathcal{F}(t)\) on \(\mathbb{R}_+\), and if \(\mathcal{A}_1 \in \mathcal{L}(\mathcal{X})\), then \(\mathcal{A}_0 + \mathcal{A}_1\) where \(\mathcal{A}_1 : = \mathcal{A} e^{-\lambda_0 \tau}\) on \(\mathcal{D}(\mathcal{A}_0)\) is like-wise the infinitesimal generator of a semigroup of operators \(\mathcal{F}(t; \tau)\) on \(\mathbb{R}_+\).

**Proof.** By Theorem 2.2 ( Theorem 3.8, \[2\], p. 77 ), in the case of \(n = 1\), for \(\lambda_0 \in \mathbb{C}\) with \(Re(\lambda_0) > \omega\), \(\mathcal{R}(\lambda_0; \mathcal{A}_0)\) satisfies the following inequality:
\[
\|\mathcal{R}(\lambda_0; \mathcal{A}_0)\| \leq \frac{M}{Re(\lambda_0) - \omega}.
\]

Hence, for \(Re(\lambda_0) > \omega_1 := \omega + M \|\mathcal{A}_1\| e^{-\omega \tau}\), we have
\[
\|\mathcal{A}_1 \mathcal{R}(\lambda_0; \mathcal{A}_0)\| \leq \frac{M \|\mathcal{A}_1\| e^{-Re(\lambda_0) \tau}}{Re(\lambda_0) - \omega} < 1,
\]
and by Lemma 3.3 \(\mathcal{R}(\lambda_0; \mathcal{A}_0 + \mathcal{A}_1)\) exists and is equal to \(\sum_{n=0}^{\infty} \mathcal{R}(\lambda_0; \mathcal{A}_0) \left[\mathcal{A}_1 \mathcal{R}(\lambda_0; \mathcal{A}_0)\right]^n\). It remains to show that \(\mathcal{A} : = \mathcal{A}_0 + \mathcal{A}_1\) is also a generator of a semigroup of operators \(\mathcal{F}(t; \tau)\) on \(\mathbb{R}_+\). To do this, we now formulate
\[
\|\mathcal{R}(\lambda_0; \mathcal{A})\| = \|\mathcal{R}(\lambda_0; \mathcal{A}_0 + \mathcal{A}_1)\| \leq \|\mathcal{R}(\lambda_0; \mathcal{A}_0)\| \left[\frac{1}{1 - \|\mathcal{A}_1\| \|\mathcal{R}(\lambda_0; \mathcal{A}_0) e^{-\lambda_0 \tau}\|}\right] \leq \frac{M}{Re(\lambda_0) - \omega} \left[\frac{Re(\lambda_0) - \omega - \|\mathcal{A}_1\| Me^{-\omega \tau}}{Re(\lambda_0) - \omega_1}\right] = \frac{M}{Re(\lambda_0) - \omega_1},
\]
where we have used the fact that
\[
\|\mathcal{R}(\lambda_0; \mathcal{A}_0) e^{-\lambda_0 \tau} x\| = \left\|\int_0^\infty e^{-\lambda_0 (t+\tau)} T(t) x dt\right\| = \left\|\int_\tau^{\infty} e^{-\lambda_0 (t-\tau)} T(t-\tau) x dt\right\| = \left\|\int_0^{\infty} e^{-\lambda_0 t} T(t-\tau) 1_{t>\tau} x dt\right\|.
\]
Therefore, by Theorem 2.3 [17], the fundamental solution to (3.1) is a strongly continuous semigroup generated by \( \mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 \) where \( \mathcal{A}_1 = \mathcal{A} e^{-\lambda_0 \tau} \). The proof is complete. \( \square \)

So, by the virtue of Lemma 3.1 and Theorem 3.1, the Laplace transform of a semigroup \( \mathcal{S}^t(\cdot; \tau) \in \mathcal{L}(\mathcal{X}) \) can be estimated by the following formula:

\[
\int_0^\infty e^{-\lambda_0 t} \mathcal{S}(t; \tau) dt = \sum_{n=0}^\infty \mathcal{R}(\lambda_0; \mathcal{A}_0) \left[ \mathcal{A}_1 \mathcal{R}(\lambda_0; \mathcal{A}_0) \right]^n e^{-n\lambda_0 \tau}.
\]  

(3.8)

Remark 3.1. Whenever \( \tau = 0 \), our results coincide with the results for a perturbed abstract differential equations which is investigated by Phillips in [17] with the help of perturbation theory. Note that in this case \( \tau = 0, \mathcal{A} := \mathcal{A}_0 + \mathcal{A}_1 \) is the infinitesimal generator of a semigroup \( \mathcal{S}(t) \) for \( t \in \mathbb{R}_+ \) and resolvent of \( \mathcal{S}(\cdot) \in \mathcal{L}(\mathcal{X}) \) is defined by

\[
\int_0^\infty e^{-\lambda_0 t} \mathcal{S}(t) dt = \mathcal{R}(\lambda_0; \mathcal{A}) = \sum_{n=0}^\infty \mathcal{R}(\lambda_0; \mathcal{A}_0) \left[ \mathcal{A}_1 \mathcal{R}(\lambda_0; \mathcal{A}_0) \right]^n.
\]

The closed-form representation of a fundamental solution \( \mathcal{S}(t; \tau) \in \mathcal{L}(\mathcal{X}), t \in \mathbb{R}_+ \) to delay evolution equation (3.1) can be expressed with the help of a delayed Dyson-Phillips series.

Theorem 3.2. Let \( \mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \to \mathcal{X} \) be the infinitesimal generator of a strongly continuous semigroup of linear operators \( \mathcal{S}(t) \) on \( \mathbb{R}_+ \) and \( \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \). Then, the abstract Cauchy problem for functional evolution equation (3.1) admits an uniquely determined fundamental (classical) solution \( \mathcal{S}(\cdot; \tau) \in \mathcal{C}^1(\mathbb{R}_+, \mathcal{D}(\mathcal{A}_0)) \) which is satisfying unit initial conditions \( \mathcal{S}(t; \tau) = \Theta, -\tau < t < 0 \) and \( \mathcal{S}(0; \tau) = \mathcal{S} \) given by the formula

\[
\mathcal{S}(t; \tau) = \sum_{n=0}^\infty \mathcal{S}_n(t - n\tau) \mathbb{1}_{t > n\tau}, \quad t \in \mathbb{R}_+,
\]  

(3.9)

where

\[
\mathcal{S}_0(t) = \mathcal{S}(t),
\]

\[
\mathcal{S}_n(t - n\tau) = \int_{n\tau}^t \mathcal{S}(t - s) \mathcal{A}_1 \mathcal{S}_{n-1}(s - n\tau) ds, \quad n \in \mathbb{N}.
\]  

(3.10)

Proof. It is obvious that \( \mathcal{S}_0(t) \) is strongly continuous on \( \mathbb{R}_+ \) with \( \|\mathcal{S}_0(t)\| \leq M e^{\omega t} \). Suppose \( \mathcal{S}_n(t - n\tau) \) is like-wise strongly continuous on \( [n\tau, t] \) and that

\[
\|\mathcal{S}_n(t - n\tau)\| \leq M (M \|\mathcal{A}_1\|)^n \frac{(t - n\tau)^n}{n!} e^{\omega(t - n\tau)}.
\]  

(3.11)

Then \( \mathcal{S}(t - s) \mathcal{A}_1 \mathcal{S}_n(s - (n + 1)\tau) \) will be strongly continuous on \( [(n + 1)\tau, t] \) so that the integral defining \( \mathcal{S}_{n+1}(t - (n + 1)\tau) \) exists in the strong topology. Furthermore, by (3.10), we have

\[
\|\mathcal{S}_{n+1}(t - (n + 1)\tau)\| \leq M \|\mathcal{A}_1\| \int_{(n + 1)\tau}^t e^{\omega(t - s)} \|\mathcal{S}_n(s - (n + 1)\tau)\| ds
\]

\[
\leq M (M \|\mathcal{A}_1\|)^{n+1} e^{\omega(t - (n + 1)\tau)} \int_{(n + 1)\tau}^t \frac{(s - (n + 1)\tau)^n}{n!} ds
\]

\[
= M (M \|\mathcal{A}_1\|)^{n+1} e^{\omega(t - (n + 1)\tau)} \left[ \frac{(s - (n + 1)\tau)^{n+1}}{(n+1)!} - \frac{(t - (n + 1)\tau)^{n+1}}{(n+1)!} \right]_{(n + 1)\tau}^t
\]

\[
= M (M \|\mathcal{A}_1\|)^{n+1} e^{\omega(t - (n + 1)\tau)} \frac{(t - (n + 1)\tau)^{n+1}}{(n+1)!}.
\]  

(3.12)
From these bounds it follows that the series \( \mathcal{F}(t; \tau) \) representing in (3.9) are convergent on any compact subsets of \( \mathbb{R}_+ \) with respect to the strong operator topology. Finally, for \( t_1 < t_2 \), we have

\[
\| \mathcal{F}_{n+1}(t_2 - (n + 1)\tau) - \mathcal{F}_{n+1}(t_1 - (n + 1)\tau) \| \leq \int_{(n+1)\tau}^{t_1} \left\| \left( \mathcal{F}(t_2 - s) - \mathcal{F}(t_1 - s) \right) \omega t \mathcal{F}_n(s - (n + 1)\tau) \right\| ds
\]

\[ + \int_{t_1}^{t_2} \| \mathcal{F}(t_2 - s) \| \| \omega t \mathcal{F}_n(s - (n + 1)\tau) \| ds. \tag{3.12} \]

As \( t_1 \to t_2 \), the integrand in the first term on the right of (3.12) converges to zero boundedly, the integrand of the second term is bounded and the second integral converges to zero boundedly, too. It follows that \( \mathcal{F}_{n+1}(t - (n + 1)\tau) \mathcal{F}_{1 > (n+1)\tau} \) is strongly continuous on \( \mathbb{R}_+ \). By mathematical induction principle, \( \mathcal{F}_{n}(t - n\tau) \mathcal{F}_{1 > n\tau} \) is well-defined, strongly continuous, and satisfies (3.11) for all \( n \in N_0 \). Hence, the delayed Dyson-Phillips series \( \mathcal{F}(t; \tau) \) is a strongly continuous function on \( \mathbb{R}_+ \) with values \( \mathcal{L}(\mathcal{F}) \) and majorized by the series expansion of \( Me^{-\omega t} \) where \( \omega_1 := \omega + M \| \omega t \| e^{-\omega t} \) as below:

\[
\| \mathcal{F}(t; \tau) \| = \left\| \sum_{n=0}^{\infty} \mathcal{F}_n(t - n\tau) \mathcal{F}_{1 > n\tau} \right\| \leq \sum_{n=0}^{\infty} \| \mathcal{F}_n(t - n\tau) \|
\]

\[
\leq M \sum_{n=0}^{\infty} \left( M \| \omega t \| \right)^n \frac{(t - n\tau)^n}{n!} e^{\omega(t - n\tau)}
\]

\[
\leq M \sum_{n=0}^{\infty} \left( M \| \omega t \| e^{-\omega t} \right)^n \frac{t^n}{n!} e^{\omega t}
\]

\[
= Me^{(\omega + M \| \omega t \| e^{-\omega t})t}
\]

\[
= Me^{\omega t}.
\]

Then, for \( \text{Re}(\lambda_0) > \omega_1 \), we can attain that

\[
\int_{0}^{\infty} e^{-\lambda_0 t} \mathcal{F}(t; \tau) dt = \int_{0}^{\infty} e^{-\lambda_0 t} \sum_{n=0}^{\infty} \mathcal{F}_n(t - n\tau) \mathcal{F}_{1 > n\tau} dt
\]

\[
= \sum_{n=0}^{\infty} \int_{0}^{\infty} e^{-\lambda_0 t} \mathcal{F}_n(t - n\tau) \mathcal{F}_{1 > n\tau} dt
\]

\[
= \sum_{n=0}^{\infty} \int_{n\tau}^{\infty} e^{-\lambda_0 s} \mathcal{F}_n(t - n\tau) dt,
\]

where the interchanging of the summation and integration is justified by the convergence of the series in the strong operator topology.

Now, it is a consequence of the strong convergence of the integral and of the Fubini’s theorem for an iterated integration \( 2 \) that

\[
\int_{n\tau}^{\infty} e^{-\lambda_0 t} \left[ \mathcal{F}_n(t - n\tau) \right] dt
\]

\[
= \int_{n\tau}^{\infty} e^{-\lambda_0 t} \int_{n\tau}^{t} \left[ \mathcal{F}(t - s) \omega t \mathcal{F}_{1 > n\tau} (s - n\tau) \right] ds dt
\]

\[
= \int_{n\tau}^{\infty} e^{-\lambda_0 t} \int_{n\tau}^{\infty} \left[ \mathcal{F}(t - s) \omega t \mathcal{F}_{1 > n\tau} (s - n\tau) \right] dt ds
\]

\[
= \int_{n\tau}^{\infty} e^{-\lambda_0 s} \int_{s}^{\infty} e^{-\lambda_0 (t-s)} \left[ \mathcal{F}(t - s) \omega t \mathcal{F}_{1 > n\tau} (s - n\tau) \right] dt ds
\]

\[11\]
On the other hand, by Theorem 3.1, we have

\[ \Re \left( \lambda; \omega_0 \right) \omega_1 \mathcal{F}_{n-1}(s-n\tau) \]

hence by the uniqueness theorem of Laplace transform \[2\], these two functions are equal for any fundamental solution of (3.1). To do this, with the help of Leibniz integral rule \[29\], we differentiate last expression

\[ \tau \quad t < 0. \]

Since \( T \lambda dt \), we have \( U(t) = \omega_1 \), \( \mathcal{F}(t; \tau) = \mathcal{F} \) and \( \mathcal{F}(t; \tau) = \Theta, -\tau \leq t < 0 \), i.e., the unit initial conditions are satisfied. Applying \(5.10\), \( \mathcal{T}(t; \tau) \) and interchanging of the summation and integration, it follows that

\[ \mathcal{W}(t) = \mathcal{F}(t; \tau)x = \sum_{n=0}^{\infty} \mathcal{F}_n(t-n\tau)1_{t>n\tau}x = \mathcal{F}_0(t)x + \sum_{n=1}^{\infty} \mathcal{F}_n(t-n\tau)1_{t>n\tau}x \]

\[ = \mathcal{F}(t)x + \sum_{n=1}^{\infty} \int_0^t \mathcal{F}(s)\omega_1 \mathcal{F}_{n-1}(s-n\tau)1_{s>n\tau}x ds \]

\[ = \mathcal{F}(t)x + \sum_{n=1}^{\infty} \int_0^t \mathcal{F}(s)\omega_1 \mathcal{F}_{n-1}(s-n\tau)1_{s>n\tau}x ds \]

\[ = \mathcal{F}(t)x + \sum_{n=1}^{\infty} \int_0^t \mathcal{F}(s)\omega_1 \mathcal{F}_{n-1}(s-n\tau)1_{s>n\tau}x ds \]

\[ = \mathcal{F}(t)x + \sum_{n=1}^{\infty} \int_0^t \mathcal{F}(s)\omega_1 \mathcal{F}_{n-1}(s-n\tau)1_{s>n\tau}x ds \]

\[ = \mathcal{F}(t)x + \sum_{n=1}^{\infty} \int_0^t \mathcal{F}(s)\omega_1 \mathcal{F}_{n-1}(s-n\tau)1_{s>n\tau}x ds \]

\[ = \mathcal{F}(t)x + \sum_{n=1}^{\infty} \int_0^t \mathcal{F}(s)\omega_1 \mathcal{F}_{n-1}(s-n\tau)1_{s>n\tau}x ds \]

\[ = \mathcal{F}(t)x + \sum_{n=1}^{\infty} \int_0^t \mathcal{F}(s)\omega_1 \mathcal{F}_{n-1}(s-n\tau)1_{s>n\tau}x ds \]

\[ = \mathcal{F}(t)x + \sum_{n=1}^{\infty} \int_0^t \mathcal{F}(s)\omega_1 \mathcal{F}_{n-1}(s-n\tau)1_{s>n\tau}x ds \]

Thus, aside from the uniqueness proof, it remains only to show that \( \mathcal{W}(t) = S(t; \tau)x \) for \( x \in \mathcal{D}(\omega_0) \) is a fundamental solution of (3.1). To do this, with the help of Leibniz integral rule \[29\], we differentiate last expression as follows:

\[ \frac{d}{dt} \mathcal{W}(t) = \frac{d}{dt} \mathcal{F}(t; \tau)x = \frac{d}{dt} \left[ \mathcal{F}(t)x + \int_0^t \mathcal{F}(s)\omega_1 \mathcal{F}(s-\tau)x ds \right] \]

\[ = \omega_0 \mathcal{F}(t)x + \omega_0 \int_0^t \mathcal{F}(s)\omega_1 \mathcal{F}(s-\tau)x ds + \omega_1 \mathcal{F}(t-\tau)x \]

\[ = \omega_0 \mathcal{F}(t;x) + \omega_1 \mathcal{F}(t-\tau;x) \]

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\[ = \mathcal{A}_0 \mathcal{U}(t) + \mathcal{A}_1 \mathcal{U}(t - \tau), \quad t > 0. \]

In the uniqueness proof, it will be sufficient to show that if for \( x \in \mathcal{D}(\mathcal{A}_0) \), \( \mathcal{V}(t; \tau)x \) solves the abstract functional differential equation (3.1) with zero initial conditions \( \mathcal{V}(t; \tau)x = 0 \) for \( -\tau \leq t \leq 0 \), then \( \mathcal{V}(t; \tau)x \equiv 0 \) for all \( t \in \mathbb{R}_+ \), since \( \mathcal{A}_0 \) is densely defined in \( \mathcal{X} \). In other words, it is sufficient to consider a continuously differentiable function \( \mathcal{V}(t) \) on \( \mathbb{R}_+ \) to \( \mathcal{D}(\mathcal{A}_0) \) such that \( \mathcal{V}(t) = 0 \) for \( -\tau \leq t \leq 0 \) and \( \frac{d\mathcal{V}}{dt} = \mathcal{A}_0 \mathcal{V}(t) + \mathcal{A}_1 \mathcal{V}(t - \tau), \quad t > 0. \). Operating on both sides of this equation by \( \mathcal{F}(t-s) \) and integrating on \([0, t]\) gives

\[
\int_0^t \mathcal{F}(t-s) \frac{d}{ds} \mathcal{V}(s)ds = \int_0^t \mathcal{F}(t-s)\mathcal{A}_0 \mathcal{V}(s)ds + \int_0^t \mathcal{F}(t-s)\mathcal{A}_1 \mathcal{V}(s-\tau)ds, \quad t > 0. \tag{3.13}
\]

It can be easily shown for \( s \in [0, t] \) that

\[
\frac{d}{ds} \left[ \mathcal{F}(t-s) \mathcal{V}(s) \right] = -\mathcal{F}(t-s)\mathcal{A}_0 \mathcal{V}(s) + \mathcal{F}(t-s) \frac{d}{ds} \mathcal{V}(s). \tag{3.14}
\]

Therefore, by virtue of (3.13), (3.14) and fundamental theorem of calculus - II \([1]\) (FTC-2), we attain that

\[
\mathcal{V}(t) = \int_0^t \mathcal{F}(t-s)\mathcal{A}_0 \mathcal{V}(s)ds + \int_0^t \mathcal{F}(s)\mathcal{A}_1 \mathcal{V}(s-\tau)ds, \quad t > 0. \tag{3.15}
\]

Let \( \mathcal{V}(t) := \sup \{u(t + h) : h \in [-\tau, 0]\}. \) For a fixed \( \tau > 0 \) we are setting \( m_t = \sup \{||\mathcal{V}(s)|| : s \in [0, t]\} \) and we see that for \( m_t > 0 \)

\[
m_t \leq Mm_t \|\mathcal{A}_1\| \frac{t}{\omega} e^{\omega t} < m_t,
\]

if \( t > 0 \) is chosen sufficiently small. This implies that \( m_t = 0 \). Thus, \( \mathcal{V}(t) = 0 \) on \([0, t_0]\) with \( t_0 > 0 \). Iteration of this argument leads to \( \mathcal{V}(t) \equiv 0 \) on \( \mathbb{R}_+ \).

It is worth noting that if a solution \( \mathcal{V}(t; \tau), t \in \mathbb{R}_+ \) to (3.1) exists, then the method of successive approximations which we employed will lead to this solution. For operating on both sides of (3.1) by \( \mathcal{F}(t-s) \) and integrating on \([0, t]\) gives for \( x \in \mathcal{D}(\mathcal{A}_0) \):

\[
\int_0^t \mathcal{F}(t-s) \frac{d}{ds} \mathcal{V}(s; \tau)xds = \int_0^t \mathcal{F}(t-s)\mathcal{A}_0 \mathcal{V}(s; \tau)xds + \int_0^t \mathcal{F}(t-s)\mathcal{A}_1 \mathcal{V}(s-\tau; \tau)xds, \quad t > 0. \tag{3.16}
\]

It can be easily shown for \( s \in [0, t] \) that

\[
\frac{d}{ds} \left[ \mathcal{F}(t-s) \mathcal{V}(s; \tau)x \right] = -\mathcal{F}(t-s)\mathcal{A}_0 \mathcal{V}(s; \tau)x + \mathcal{F}(t-s) \frac{d}{ds} \mathcal{V}(s; \tau)x. \tag{3.17}
\]

Therefore, from (3.16) and (3.17) and by FTC-2, we obtain that

\[
\mathcal{V}(t; \tau)x = \mathcal{F}(t)x + \int_0^t \mathcal{F}(t-s)\mathcal{A}_1 \mathcal{V}(s-\tau; \tau)xds, \quad t > 0. \tag{3.18}
\]

On the other hand, the method of successive approximations yields \( \mathcal{V}(t; \tau), t \in \mathbb{R}_+ \) is also satisfying

\[
\mathcal{V}(t; \tau)x = \mathcal{F}(t)x + \int_0^t \mathcal{F}(t-s)\mathcal{A}_1 \mathcal{V}(s-\tau; \tau)xds, \quad t > 0. \tag{3.19}
\]

The difference \( \mathcal{V}(t) = \mathcal{V}(t; \tau)x - \mathcal{V}(t; \tau)x, t \in \mathbb{R}_+ \) satisfies (3.15) and vanishes at the points \( t \in \mathbb{R}_+ \). Thus, the uniqueness argument shows that this difference is identically zero for any \( t \in \mathbb{R}_+ \). The proof is complete. \( \square \)
Remark 3.2. Whenever \( \tau = 0 \), our results coincide with the results for a perturbed evolution equations (1.1) which is studied by Phillips in [17] with the help of perturbation theory. Note that in this case (where \( \tau = 0 \)), for \( x \in \mathcal{D}(\mathcal{A}_0) \), the classical solution of the abstract Cauchy problem for a perturbed evolution equation can be represented via a Dyson-Phillips series as follows:

\[
    u(t) = \mathcal{I}(t)x = \sum_{n=0}^{\infty} \mathcal{J}_n(t)x, \quad t \in \mathbb{R}_+,
\]

where

\[
    \mathcal{J}_0(t) = \mathcal{I}(t), \quad \mathcal{J}_n(t) = \int_0^t \mathcal{I}(t-s)\mathcal{A}_n \mathcal{J}_{n-1}(s)ds, \quad n \in \mathbb{N}.
\]

If we consider abstract differential equation with a discrete delay (3.1) on \( J = [0, T] \) where \( T = (n+1)\tau \) for a fixed \( n \in \mathbb{N}_0 \), then we can introduce a piece-wise construction for a fundamental solution \( \mathcal{I}(:\tau) \in \mathcal{L}(\mathcal{X}) \).

Corollary 3.1. Let \( \mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \to \mathcal{X} \) be the infinitesimal generator of a strongly continuous semigroup of linear operators \( \mathcal{I}(t) \) on \( J \) and \( \mathcal{A}_0 \in \mathcal{L}(\mathcal{X}) \). Then, the abstract Cauchy problem for functional evolution equation (3.1) admits an uniquely determined fundamental (classical) solution \( \mathcal{I}(t; \tau) : J, \mathcal{D}(\mathcal{A}_0) \to \mathcal{X} \) which is satisfying unit initial conditions \( \mathcal{I}(t; 0) = \Theta, -\tau \leq t < 0 \) and \( \mathcal{I}(0; \tau) = \mathcal{I} \) given by the formula

\[
    \mathcal{I}(t; \tau) = \sum_{k=0}^{n} \mathcal{J}_k(t-k\tau), \quad k\tau < t \leq (k+1)\tau, \quad n \in \mathbb{N}_0, \tag{3.20}
\]

where

\[
    \mathcal{J}_0(t) = \mathcal{I}(t), \quad \mathcal{J}_k(t-k\tau) = \int_{k\tau}^{t} \mathcal{I}(t-s)\mathcal{A}_k \mathcal{J}_{k-1}(s-k\tau)ds, \quad k = 0, 1, \ldots, n, \quad n \in \mathbb{N}_0.
\]

In the particular case, by using the powers of \( \mathcal{A}(\lambda_0; \mathcal{A}_0) \), we can derive the following elegant representation formula for a fundamental solution \( \mathcal{I}(t; \tau), \quad t \in \mathbb{R}_+ \) of (3.1).

Theorem 3.3. Let \( \mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \to \mathcal{X} \) be the infinitesimal generator of a strongly continuous semigroup \( \mathcal{I}(t) \) for \( t \in \mathbb{R}_+ \). If a strongly continuous semigroup \( \mathcal{I}(t), \quad t \in \mathbb{R}_+ \) commutes with \( \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \), then, the strongly continuous semigroup \( \mathcal{I}(t; \tau), \quad t \in \mathbb{R}_+ \) generated by \( \mathcal{A} := \mathcal{A}_0 + \mathcal{A}_1 \) where \( \mathcal{A}_1 = \mathcal{A}_1 e^{-\lambda_0 \tau} \) can be represented by

\[
    \mathcal{I}(t; \tau) = \sum_{n=0}^{\infty} \mathcal{A}_1^n \left( \frac{t-n\tau}{n!} \right) \mathcal{I}(t-n\tau)\mathbb{I}_{t>n\tau}, \quad t \in \mathbb{R}_+. \tag{3.21}
\]

Proof. Since a \( C_0 \)-semigroup \( \mathcal{I}(t) \in \mathcal{L}(\mathcal{X}) \) commutes with \( \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \) for \( t \in \mathbb{R}_+ \), then by Theorem 3.1 the following identity holds true:

\[
    \mathcal{A}(\lambda_0; \mathcal{A}_0 + \mathcal{A}_1) = \int_0^\infty e^{-\lambda_0 t} \mathcal{I}(t; \tau)dt
    = \sum_{n=0}^{\infty} \mathcal{A}(\lambda_0; \mathcal{A}_0) \left[ \mathcal{A}_1 \mathcal{A}(\lambda_0; \mathcal{A}_0) \right]^n e^{-n\lambda_0 \tau}
    = \sum_{n=0}^{\infty} \mathcal{A}_1^n \left[ \mathcal{A}(\lambda_0; \mathcal{A}_0) \right]^{n+1} e^{-n\lambda_0 \tau}. \tag{3.22}
\]
Corollary 3.2. Let the fundamental solution of (3.1) be the infinitesimal generator of a strongly continuous semigroup in a Banach space $X$. The proof is complete.

Theorem 3.4. For $Re(\lambda_0) > \omega$ and $n \in \mathbb{N}$ it is well-known that [2, Corollary 1.11, p. 56]:

$$\mathcal{A}(\lambda_0; \mathcal{A}_0)^{n+1} x = \int_0^\infty e^{-\lambda_0 t} \frac{t^n}{n!} \mathcal{I}(t)x dt, \quad x \in \mathcal{X}.$$  (3.23)

By (3.23) and using integration by substitution for operator-valued functions [2], we get

$$\left[\mathcal{A}(\lambda_0; \mathcal{A}_0)^{n+1} e^{-\lambda_0 \tau} x = \int_0^\infty e^{-\lambda_0 (t-n\tau)} \frac{(t-n\tau)^n}{n!} \mathcal{I}(t-n\tau) \mathbb{1}_{t>n\tau} x dt, \quad x \in \mathcal{X}. \right.$$  (3.24)

Therefore, from (3.22) and (3.24), we derive a desired result:

$$\mathcal{I}(t; \tau)x = \sum_{n=0}^{\infty} \mathcal{A}^n \tau (t-k\tau)^n \mathcal{I}(t-k\tau) \mathbb{1}_{t>k\tau} x, \quad x \in \mathcal{X}. \quad (3.25)$$

The proof is complete.

If we consider this particular case on $J = [0, T]$ where $T = (n+1)\tau$ for a fixed $n \in \mathbb{N}_0$, then we can introduce a piece-wise construction for a fundamental solution $\mathcal{I}(.; \tau) \in \mathcal{L}(\mathcal{X})$ as below.

Corollary 3.3. Let $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \rightarrow \mathcal{X}$ be the infinitesimal generator of a strongly continuous semigroup $\mathcal{I}(t)$ for $t \in \mathbb{R}$. If a strongly continuous semigroup $\mathcal{I}(t)$, $t \in \mathbb{R}$ commutes with $\mathcal{A}_1 \in \mathcal{L}(\mathcal{X})$, then, the fundamental solution of (3.1) can be represented by

$$\mathcal{I}(t; \tau)x = \sum_{k=0}^{n} \mathcal{A}^k \tau (t-k\tau)^k \mathcal{I}(t-k\tau), \quad k\tau < t \leq (k+1)\tau, \quad n \in \mathbb{N}_0. \quad (3.26)$$

For abstract differential equations (1.1), this result will be a product of the uniformly continuous semigroup with the strongly continuous semigroup in a Banach space $\mathcal{X}$.

Corollary 3.4. Let $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \rightarrow \mathcal{X}$ be the infinitesimal generator of a strongly continuous semigroup $\mathcal{I}(t)$ for $t \in \mathbb{R}_+$. If a strongly continuous semigroup $\mathcal{I}(t)$, $t \in \mathbb{R}_+$ commutes with $\mathcal{A}_1 \in \mathcal{L}(\mathcal{X})$, then, for $x \in \mathcal{D}(\mathcal{A}_0)$, the classical solution $u(\cdot) \in C^1(\mathbb{R}_+, \mathcal{D}(\mathcal{A}_0))$ to (1.1) can be expressed by

$$u(t) = \mathcal{I}(t)x = e^{\mathcal{A}_1 t} \mathcal{I}(t)x = \sum_{n=0}^{\infty} \mathcal{A}^n \tau (t-n\tau)^n \mathcal{I}(t)x, \quad t \in \mathbb{R}_+. \quad (3.27)$$

Next, we derive two explicit representation formulae for the classical solution of the abstract initial value problem to linear homogeneous functional evolution equation (3.1) via the method of variation of constants formula.

Theorem 3.4. Let $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \rightarrow \mathcal{X}$ be infinitesimal generator of a strongly continuous semigroup of linear operators $\mathcal{I}(t)$ on $\mathbb{R}_+$, $\mathcal{A}_1 \in \mathcal{L}(\mathcal{X})$ and the initial function $\varphi(\cdot) \in C^1(\mathbb{R}_+, \mathcal{D}(\mathcal{A}_0))$. Then, the classical solution $u(\cdot) \in C^1(\mathbb{R}_+, \mathcal{D}(\mathcal{A}_0))$ of linear homogeneous functional evolution equation (3.1) with initial conditions $u(t) = \varphi(t)$, $t \in \mathbb{R}$ can be represented in the integral form

$$u(t) = \mathcal{I}(t+\tau; \tau)\varphi(-\tau) + \int_{-\tau}^{0} \mathcal{I}(t-s; \tau) [\varphi'(s) - \mathcal{A}_0 \varphi(s)] ds, \quad t \geq \tau, \quad (3.28)$$

$$u(t) = \mathcal{I}(t; \tau)\varphi(0) + \int_{-\tau}^{0} \mathcal{I}(t-\tau-s; \tau) \mathcal{A}_1 \varphi(s) ds, \quad t \geq 0. \quad (3.29)$$
Proof. By using the variation of constants formula, any solution \( u(\cdot) \in \mathcal{D}(\mathcal{A}_0) \) of linear homogeneous delay evolution equation (3.1) should be satisfied in the form

\[
u(t) = \mathcal{I}(t + \tau; \tau)c + \int_{-\tau}^{0} \mathcal{I}(t - s; \tau)g(s)\,ds, \quad t \geq -\tau,
\]

where \( c \) is an unknown operator, \( g(\cdot) : \mathbb{R} \to \mathcal{I} \) is an unknown continuously differentiable operator-valued function and that it satisfies the initial conditions \( u(t) = \varphi(t) \) for \( t \in \mathbb{R} \):

\[
\varphi(t) = \mathcal{I}(t + \tau; \tau)c + \int_{-\tau}^{0} \mathcal{I}(t - s; \tau)g(s)\,ds, \quad t \in \mathbb{R}.
\]  
(3.30)

If \( t = -\tau \), then in accordance with the following results

\[
\mathcal{I}(-\tau - s; \tau) = \begin{cases} 
0, & -\tau < s \leq 0, \\
\mathcal{I}, & s = -\tau,
\end{cases}
\]

and \( \mathcal{I}(0; \tau) = \mathcal{I}(0) = \mathcal{I} \),

we acquire \( c = \varphi(-\tau) \in \mathcal{D}(\mathcal{A}_0) \).

On the interval \(-\tau \leq t \leq 0\), we split (3.30) into two integrals:

\[
\varphi(t) = \mathcal{I}(t + \tau; \tau)c + \int_{-\tau}^{t} \mathcal{I}(t - s; \tau)g(s)\,ds + \int_{t}^{0} \mathcal{I}(t - s; \tau)g(s)\,ds, \quad t \in \mathbb{R}.
\]

Furthermore, by making use of the following relations

\[
\mathcal{I}(t - s; \tau) = \begin{cases} 
0, & t < s \leq 0, \\
\mathcal{I}, & s = t,
\end{cases}
\]

\[
\mathcal{I}(t - s; \tau) = \mathcal{I}(t - s), \quad -\tau \leq s \leq t \quad \text{and} \quad \mathcal{I}(t + \tau; \tau) = \begin{cases} 
\mathcal{I}(t + \tau), & -\tau < t \leq 0, \\
\mathcal{I}, & t = -\tau,
\end{cases}
\]

one can easily derive that

\[
\varphi(t) = \mathcal{I}(t + \tau)c + \int_{-\tau}^{t} \mathcal{I}(t - s; \tau)g(s)\,ds, \quad t \in \mathbb{R}.
\]  
(3.31)

After differentiating the equation (3.31) by FTC-1, we derive

\[
\varphi'(t) = \mathcal{A}_0 \mathcal{I}(t + \tau)c + \mathcal{A}_0 \int_{-\tau}^{t} \mathcal{I}(t - s; \tau)g(s)\,ds + g(t) = \mathcal{A}_0 \varphi(t) + g(t), \quad t \in \mathbb{R}.
\]

Hence, it follows that \( g(t) = \varphi'(t) - \mathcal{A}_0 \varphi(t), t \in \mathbb{R} \).

Next, we prove the equivalency of the Equations (3.28) and (3.29). To do this, we use the well-known integration by parts formula for operator-valued functions [2]:

\[
\int_{-\tau}^{0} \mathcal{I}(t - s; \tau)\varphi'(s)\,ds = \int_{-\tau}^{0} \mathcal{I}(t - s; \tau)d\varphi(s)
\]

\[
= \mathcal{I}(t - s; \tau)d\varphi(s) \bigg|_{s=0}^{s=-\tau} - \int_{-\tau}^{0} \frac{\partial}{\partial s} \mathcal{I}(t - s; \tau)\varphi(s)\,ds
\]
\[ \mathcal{F}(t; \tau) \varphi(0) - \mathcal{F}(t + \tau; \tau) \varphi(-\tau) + \int_{-\tau}^{0} \mathcal{F}(t - s; \tau) \mathcal{A}_{0} \varphi(s) \, ds + \int_{-\tau}^{0} \mathcal{F}(t - \tau - s; \tau) \mathcal{A}_{1} \varphi(s) \, ds. \]

Therefore, the desired results hold:

\[ u(t) = \mathcal{F}(t + \tau; \tau) \varphi(-\tau) + \int_{-\tau}^{0} \mathcal{F}(t - s; \tau) [\varphi'(s) - \mathcal{A}_{0} \varphi(s)] \, ds, \quad t \geq -\tau, \]

\[ = \mathcal{F}(t; \tau) \varphi(0) + \int_{-\tau}^{0} \mathcal{F}(t - \tau - s; \tau) \mathcal{A}_{1} \varphi(s) \, ds, \quad t \geq 0. \]

The proof is complete. \( \square \)

**Remark 3.3.** These results are natural extension of the results in [32] which is derived for linear system of first-order delayed differential equations with matrix coefficients.

Secondly, we consider the following abstract Cauchy problem for a linear nonhomogeneous functional evolution equation in a Banach space \( \mathcal{X} \):

\[
\begin{cases}
    \frac{d}{dt} u(t) = \mathcal{A}_0 u(t) + \mathcal{A}_1 u(t - \tau) + g(t), & t > 0, \quad \tau > 0, \\
    u(t) = \varphi(t), & -\tau \leq t \leq 0,
\end{cases}
\]

where \( \mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \to \mathcal{X} \) is the infinitesimal generator of a strongly continuous semigroup of linear operators \( \mathcal{F}(t) \) for \( t \in \mathbb{R}_+ \), \( \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \) is a bounded linear operator on \( \mathcal{X} \) and a forced term \( g(\cdot) : \mathbb{R}_+ \to \mathcal{X} \) is a continuous operator-valued function on \( \mathbb{R}_+ \).

We will find a closed-form representation of the analytical solution to linear nonhomogeneous evolution equation with a constant delay by applying superposition principle for linear abstract differential equations [1].

Consider the following two abstract Cauchy problems for functional evolution equations in a Banach space \( \mathcal{X} \):

\[
\begin{cases}
    \frac{d}{dt} u(t) = \mathcal{A}_0 u(t) + \mathcal{A}_1 u(t - \tau), & t > 0, \quad \tau > 0, \\
    u(t) = \varphi(t), & -\tau \leq t \leq 0,
\end{cases}
\]

\[
\begin{cases}
    \frac{d}{dt} u(t) = \mathcal{A}_0 u(t) + \mathcal{A}_1 u(t - \tau) + g(t), & t > 0, \quad \tau > 0, \\
    u(t) \equiv 0, & -\tau \leq t \leq 0.
\end{cases}
\]

The following fact is clear from linearity of the problem, but it is a very useful principle in solving linear nonhomogeneous abstract differential equations with time-delay [1].

It is known that if \( u_1(t) \) is a solution of homogeneous abstract Cauchy problem (3.33) with nonhomogeneous initial conditions \( u_1(t) = \varphi(t) \), \( t \in \mathbb{I} \) and \( u_2(t) \) is a solution of nonhomogeneous abstract Cauchy problem (3.32) with homogeneous initial conditions \( u_2(t) = 0 \), \( t \in \mathbb{I} \), then \( u(t) := u_1(t) + u_2(t) \) is a solution of nonhomogeneous abstract Cauchy problem (3.32) with nonhomogeneous initial conditions \( u(t) = \varphi(t) \), \( t \in \mathbb{I} \). Note that the solution \( u_1(t) \) of (3.33) have studied in Theorem 3.3. In other words, to achieve our aim we need to find a closed-form representation of \( u_2(t) \) which is a particular solution of abstract initial value problem (3.32).

Before finding a particular solution to (3.32), we prove the following auxiliary lemma plays a crucial role in the proof of Theorem 3.3.

**Lemma 3.2.** Let \( \mathcal{F}(t) \) be a strongly continuous semigroup of linear operators on \( \mathbb{R}_+ \). For a strongly continuous operator-valued function \( f(t) \) on \( \mathbb{R}_+ \) to \( \mathcal{X} \), \( g(t) = \int_{0}^{t} \mathcal{F}(t-s) f(s) \, ds = \int_{0}^{t} \mathcal{F}(s) f(t-s) \, ds \) exists and
is itself strongly continuous on \( \mathbb{R}_+ \) to \( \mathcal{X} \). If \( f(t) \) is a strongly continuously differentiable operator-valued function on \( \mathbb{R}_+ \), then \( g(t) \) is also a strongly continuously differentiable on \( \mathbb{R}_+ \) and satisfies the following relations:

\[
\frac{dg(t)}{dt} = f(t) + \mathcal{A}_0 \int_0^t \mathcal{I}(t-s)f(s)ds, \quad t \in \mathbb{R}_+, \tag{3.35}
\]

\[
= \mathcal{I}(t)f(0) + \int_0^t \mathcal{I}(t-s)f'(s)ds, \quad t \in \mathbb{R}_+. \tag{3.36}
\]

**Proof.** Since \( \|\mathcal{I}(t)\| \leq M e^{\omega t} \), it is obvious that \( \mathcal{I}(t-s)f(s) \) is strongly continuous in \( s \in [0, t] \) whenever the same is true of \( f(s) \). In this case, \( g(t) = \int_0^t \mathcal{I}(t-s)f(s)ds \) will exist in the strong topology and by the symmetry property of convolution operator [2] it will be equal to \( \int_0^t \mathcal{I}(s)f(t-s)ds \). According to the well-known Leibniz integral rule [29], we obtain that

\[
\frac{dg(t)}{dt} = \frac{d}{dt} \int_0^t \mathcal{I}(t-s)f(s)ds = f(t) + \int_0^t \frac{d}{dt} \mathcal{I}(t-s)f(s)ds
\]

\[
=f(t) + \mathcal{A}_0 \int_0^t \mathcal{I}(t-s)f(s)ds, \quad t \in \mathbb{R}_+. \tag{3.37}
\]

Next, we prove the equivalence of the Equations (3.35) and (3.36). To do this, we apply the integration by parts formula for operator-valued functions [2]:

\[
\int_0^t \mathcal{I}(t-s)f'(s)ds = \int_0^t \mathcal{I}(t-s)df(s) = \mathcal{I}(t-s)f(s)|_{s=t}^{s=0} + \mathcal{A}_0 \int_0^t \mathcal{I}(t-s)f(s)ds
\]

\[
= f(t) - \mathcal{I}(t)f(0) + \mathcal{A}_0 \int_0^t \mathcal{I}(t-s)f(s)ds, \quad t \in \mathbb{R}_+. \tag{3.38}
\]

As a consequence, from (3.37) and (3.38), we attain desired results:

\[
\frac{dg(t)}{dt} = f(t) + \mathcal{A}_0 \int_0^t \mathcal{I}(t-s)f(s)ds, \quad t \in \mathbb{R}_+, \tag{3.39}
\]

\[
= \mathcal{I}(t)f(0) + \int_0^t \mathcal{I}(t-s)f'(s)ds, \quad t \in \mathbb{R}_+.
\]

The proof is complete.

**Theorem 3.5.** Let \( \mathcal{I}(t) \) be a strongly continuous semigroup of linear operators with infinitesimal generator \( \mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \to \mathcal{X} \) on \( \mathbb{R}_+ \). Let \( \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \) and \( g(t) \) be strongly continuously differentiable operator-valued function on \( \mathbb{R}_+ \) to \( \mathcal{X} \). Then, there exists a unique continuously differentiable operator-valued function \( w(\cdot; \tau) : \mathbb{R}_+ \to \mathcal{D}(\mathcal{A}_0) \) which is a particular solution of (3.32) with zero initial conditions \( w(t; \tau) = 0, t \in 1 \). This solution has a closed-form

\[
w(t; \tau) = \sum_{n=0}^{\infty} w_n(t - n\tau)1_{t > n\tau}, \quad t \in \mathbb{R}_+, \tag{3.39}
\]
where
\[ w_0(t) = \int_0^t \mathcal{T}(t-s)g(s)ds, \]
\[ w_n(t-n\tau) = \int_{n\tau}^t \mathcal{T}(t-s)\mathcal{A}_1 w_{n-1}(s-n\tau)ds, \quad n \in \mathbb{N}. \]

**Proof.** Since \( g(\cdot) \in C^1(\mathbb{R}_+, \mathcal{F}) \), this implies that \( w_0(t) \) is strongly continuously differentiable and hence by induction that \( w_n(t-n\tau) \) is like-wise for any \( n \in \mathbb{N} \). In fact, by Lemma 3.2 we have
\[ w'_0(t) = \mathcal{T}(t)g(0) + \int_0^t \mathcal{T}(t-s)g'(s)ds, \]
\[ w'_n(t-n\tau) = \int_{n\tau}^t \mathcal{T}(t-s)\mathcal{A}_1 w'_{n-1}(s-n\tau)ds, \quad n \in \mathbb{N}. \]

By using (2.2) it is easy to acquire the following estimations for \( n \in \mathbb{N}_0 \):
\[ \|w_n(t-n\tau)\| \leq M^{n+1} \|\mathcal{A}_1\|^{n} Ne^{\omega(t-n\tau)} \frac{(t-n\tau)^{n+1}}{(n+1)!}, \]
\[ \|w'_n(t-n\tau)\| \leq M^{n+1} \|\mathcal{A}_1\|^{n} Ne^{\omega(t-n\tau)} \left[ \frac{(t-n\tau)^n}{n!} + \frac{(t-n\tau)^{n+1}}{(n+1)!} \right], \]
where \( N := \sup_{t \in \mathbb{R}_+} \{\|g(t)\|, \|g'(t)\|\} \).

If we set \( w(t; \tau) = \sum_{n=0}^\infty w_n(t-n\tau)1_{t > n\tau} \) for \( t \in \mathbb{R}_+ \), then as in Theorem 3.2, \( w(t; \tau) \) is strongly continuously differentiable on \( \mathbb{R}_+ \) to \( \mathcal{D}(\mathcal{A}_0) \) and \( w'(t; \tau) = \sum_{n=0}^\infty w'_n(t-n\tau)1_{t > n\tau}, t \in \mathbb{R}_+ \). Furthermore, \( w(t; \tau) \equiv 0, \quad t \in \mathbb{I} \).

From the definition of \( w_n(t-n\tau), n \in \mathbb{N}_0 \) and convergence of the series \( \sum_{n=0}^\infty w_n(t-n\tau)1_{t > n\tau} \) in the strong operator topology on \( \mathbb{R}_+ \), it follows that
\[ w(t; \tau) = \sum_{n=0}^\infty w_n(t-n\tau)1_{t > n\tau} = w_0(t) + \sum_{n=1}^\infty w_n(t-n\tau)1_{t > n\tau} \]
\[ = w_0(t) + \sum_{n=1}^\infty \int_{n\tau}^t \mathcal{T}(t-s)\mathcal{A}_1 w_{n-1}(s-n\tau)1_{s > n\tau} ds \]
\[ = w_0(t) + \sum_{n=1}^\infty \int_0^t \mathcal{T}(t-s)\mathcal{A}_1 w_{n-1}(s-n\tau)1_{s > n\tau} ds \]
\[ = w_0(t) + \int_0^t \mathcal{T}(t-s)\mathcal{A}_1 \sum_{n=1}^\infty w_{n-1}(s-n\tau)1_{s > n\tau} ds \]
\[ = w_0(t) + \int_0^t \mathcal{T}(t-s)\mathcal{A}_1 \sum_{n=0}^\infty w_{n}(s-n\tau-\tau)1_{s > (n+1)\tau} ds \]
\[ = w_0(t) + \int_0^t \mathcal{T}(t-s)\mathcal{A}_1 w(s-\tau; \tau) ds, \quad t > 0. \quad (3.40) \]
Since \( w(t; \tau) \) is strongly continuously differentiable on \( \mathbb{R}_+ \), we can differentiate \((3.40)\) term-wise by considering \( \mathcal{T}(0) = \mathcal{S} \), as follows:

\[
\frac{d}{dt} w(t; \tau) = \frac{d}{dt} \left( w_0(t) + \int_0^t \mathcal{T}(t-s) \mathcal{A}_1 w(s-\tau; \tau) ds \right) \\
= \frac{d}{dt} \left( \mathcal{T}(t) * g(t) \right) + \frac{d}{dt} \left( \mathcal{T}(t) * (\mathcal{A}_1 w(t-\tau; \tau)) \right) \\
= \mathcal{A}_0 \mathcal{T}(t) * g(t) + g(t) + \mathcal{A}_0 \mathcal{T}(t) * (\mathcal{A}_1 w(t-\tau; \tau)) + \mathcal{A}_1 w(t-\tau; \tau) \\
= \mathcal{A}_0 w_0(t) + g(t) + \mathcal{A}_0 \int_0^t \mathcal{T}(t-s) \mathcal{A}_1 w(s-\tau; \tau) ds + \mathcal{A}_1 w(t-\tau; \tau) \\
= \mathcal{A}_0 w(t; \tau) + \mathcal{A}_1 w(t-\tau; \tau) + g(t), \quad t > 0.
\]

This shows immediately that \( w(t; \tau) \) is a particular solution of linear nonhomogeneous abstract Cauchy problem \((3.32)\). The uniqueness of a particular solution follows precisely as in the uniqueness proof of Theorem \((3.2)\). The proof is complete.

**Remark 3.4.** Whenever \( \tau = 0 \), our results coincide with the results for a particular solution of perturbed linear nonhomogeneous evolution equations \((1.1)\) which is studied by Phillips in \([17]\). Note that in this case (where \( \tau = 0 \)), the particular solution of the abstract linear nonhomogeneous Cauchy problem \((1.1)\) can be expressed via a Dyson-Phillips series as below:

\[
w(t) = \sum_{n=0}^{\infty} w_n(t), \quad t \in \mathbb{R}_+, 
\]

where

\[
w_0(t) = \int_0^t \mathcal{T}(t-s) g(s) ds, \\
w_n(t) = \int_0^t \mathcal{T}(t-s) \mathcal{A}_1 w_{n-1}(s) ds, \quad n \in \mathbb{N}.
\]

If we consider linear nonhomogeneous abstract differential equation with delay \((1.4)\) on \( J = [0, T] \) where \( T = (n+1)\tau \) for a fixed \( n \in \mathbb{N}_0 \), then we can introduce a piece-wise construction for a particular solution \( w(\cdot; \tau) \in \mathcal{C}^1(\mathbb{R}_+, \mathcal{D}(\mathcal{A}_0)) \) as follows.

**Corollary 3.4.** Let \( \mathcal{T}(t) \) be a strongly continuous semigroup of linear operators with infinitesimal generator \( \mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \to \mathcal{X} \) on \( J \). Let \( \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \) and \( g(t) \) be strongly continuously differentiable operator-valued function on \( J \) to \( \mathcal{X} \). Then, there exists a unique continuously differentiable operator-valued function \( w(\cdot; \tau) : J \to \mathcal{D}(\mathcal{A}_0) \) which is a particular solution of \((3.32)\) with zero initial conditions \( w(t; \tau) = 0, \quad t \in [0, 1] \). This solution has a closed-form

\[
w(t; \tau) = \sum_{k=0}^{n} w_k(t - k\tau), \quad k\tau < t \leq (k+1)\tau, \quad n \in \mathbb{N}_0, \quad (3.41)
\]

where

\[
w_0(t) = \int_0^t \mathcal{T}(t-s) g(s) ds, \\
w_k(t - k\tau) = \int_{k\tau}^t \mathcal{T}(t-s) \mathcal{A}_1 w_{k-1}(s - n\tau) ds, \quad k = 0, 1, \ldots, n, \quad n \in \mathbb{N}.
\]
The particular solution $w(t; \tau)$, $t \in \mathbb{R}_+$, of (3.32) can also be put in a more suggestive form. By Theorem 3.6 for a fixed $s \in \mathbb{R}_+$, a fundamental solution of the abstract Cauchy problem for linear homogeneous functional evolution equation (3.1) can be expressed by $\mathcal{F}(t-s; \tau) = \sum_{n=0}^{\infty} \mathcal{F}_n(t-s-n\tau)1_{t>s+n\tau}$, for $t \in \mathbb{R}_+$ where $\mathcal{F}_0(t-s) = \mathcal{F}(t-s)$ and $\mathcal{F}_n(t-s-n\tau) = \int_{s+n\tau}^{t} \mathcal{F}(t-\sigma)\mathcal{A}_1 \mathcal{F}_{n-1}(\sigma-s-n\tau) d\sigma$, $n \in \mathbb{N}$. Moreover, by making use of the well-known Fubini’s theorem for an iterated integration [2], we attain:

$$\int_{0}^{t} \mathcal{F}_n(t-s-n\tau)1_{t>s+n\tau} g(s) ds = \int_{0}^{t} \int_{s+n\tau}^{t} \mathcal{F}(t-\sigma)\mathcal{A}_1 \mathcal{F}_{n-1}(\sigma-s-n\tau)1_{\sigma>s+n\tau} g(s) d\sigma ds$$

$$= \int_{0}^{t} \mathcal{F}(t-\sigma)\mathcal{A}_1 \sum_{n=0}^{\infty} \mathcal{F}_n(t-s-n\tau)1_{\sigma>s+n\tau} g(s) d\sigma, \quad n \in \mathbb{N}.$$ 

Hence, by mathematical induction principle, we get

$$w_n(t-n\tau)1_{t>n\tau} = \int_{0}^{t} \mathcal{F}_n(t-s-n\tau)1_{t>s+n\tau} g(s) ds, \quad n \in \mathbb{N}.$$ 

Finally, in accordance with the convergence of the series in the strong operator topology we derive that

$$w(t; \tau) = \sum_{n=0}^{\infty} w_n(t-n\tau)1_{t>n\tau} = \sum_{n=0}^{\infty} \int_{0}^{t} \mathcal{F}_n(t-s-n\tau)1_{t>s+n\tau} g(s) ds$$

$$= \int_{0}^{t} \sum_{n=0}^{\infty} \mathcal{F}_n(t-s-n\tau)1_{t>s+n\tau} g(s) ds$$

$$= \int_{0}^{t} \mathcal{F}(t-s)g(s) ds.$$ (3.42)

The following theorem present the construction of formula of the classical solution to linear nonhomogeneous abstract Cauchy problem (3.32). The proof is straightforward, so, we pass over it here.

**Theorem 3.6.** Let $\mathcal{F}(t)$ be a strongly continuous semigroup of linear operators with infinitesimal generator $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ on $\mathbb{R}_+$. Let $\mathcal{A}_1 \in \mathcal{L}(\mathcal{H})$ and $g(t)$ be strongly continuously differentiable operator-valued function on $\mathbb{R}_+$ to $\mathcal{H}$. Then, there exists a unique continuously differentiable solution $u(\cdot) : \mathbb{R}_+ \rightarrow \mathcal{D}(\mathcal{A}_0)$ to (3.32) which have closed-forms

$$u(t) = \mathcal{F}(t+\tau; \tau)\varphi(-\tau) + \int_{-\tau}^{0} \mathcal{F}(t-s; \tau) [\varphi(s) - \mathcal{A}_0 \varphi(s)] ds + w(t; \tau), \quad t \geq -\tau,$$ (3.43)

$$= \mathcal{F}(t; \tau)\varphi(0) + \int_{-\tau}^{0} \mathcal{F}(t-\tau-s; \tau)\mathcal{A}_1\varphi(s) ds + w(t; \tau), \quad t \geq 0,$$ (3.44)

where $\mathcal{F}(\cdot; \tau)$ is given by as in (3.9) and $w(\cdot; \tau)$ are given by as in (3.39) or (3.42), respectively.

If we consider linear nonhomogeneous abstract initial value problem (1.4) on $J = [0, T]$ where $T = (n+1)\tau$ for a fixed $n \in \mathbb{N}_0$, then we can introduce a piece-wise construction for a unique classical solution $u(\cdot) \in C^1(J, \mathcal{D}(\mathcal{A}_0))$ as below.
Corollary 3.5. Let \( \mathcal{I}(t) \) be a strongly continuous semigroup of linear operators with infinitesimal generator \( \mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \to \mathcal{X} \) on \( \mathcal{J} \). Let \( \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \) and \( g(t) \) be strongly continuously differentiable operator-valued function on \( \mathcal{J} \) to \( \mathcal{X} \). Then, there exists a unique continuously differentiable classical solution \( u(\cdot) : \mathcal{J} \to \mathcal{D}(\mathcal{A}_0) \) to (3.32) which has a closed-form

\[
 u(t) = \mathcal{I}(t + \tau; \tau)\varphi(-\tau) + \int_{-\tau}^{0} \mathcal{I}(t - s; \tau) [\varphi'(s) - \mathcal{A}_0\varphi(s)] ds + w(t; \tau),
\]

where \( \mathcal{A}(\cdot; \tau) \) is given by \( (3.22) \) and \( w(\cdot; \tau) \) are given by as in \( (3.11) \) or \( (3.12) \), respectively.

Remark 3.5. If we consider the solution to (1.4) in the mild sense, then it will be \( \mathcal{I}(\cdot; \tau) \in C(\mathcal{R}_+, \mathcal{X}) \) with values in \( \mathcal{X} \).

4. Delay evolution equations with bounded linear operators

In this section, we consider different important cases of delay evolution equations (3.1) with bounded linear operators \( \mathcal{A}_0, \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \) in a Banach space \( \mathcal{X} \). In this case, the domain \( \mathcal{D}(\mathcal{A}_0) \) coincides with the state space \( \mathcal{X} \), i.e., \( \mathcal{D}(\mathcal{A}_0) = \mathcal{X} \), and a one-parameter semigroup \( \mathcal{I}(t), t \in \mathcal{R}_+ \) which is continuous with respect to uniform operator topology defined by

\[
 \mathcal{I}(t) = e^{\mathcal{A}_0 t} = \sum_{k=0}^{\infty} \frac{\mathcal{A}_0^k t^k}{k!}.
\]

For \( n \in \mathbb{N} \) and \( \tau > 0 \), we define the following sequence of operator-valued functions via a recursive way:

\[
 \mathcal{A}_0(t) := e^{\mathcal{A}_0 t}, \quad t \in \mathcal{R}_+,
\]

\[
 \mathcal{A}_1(t - \tau) := \begin{cases} \int_{\tau}^{t} e^{\mathcal{A}_0 (t-s)} \mathcal{A}_1 \mathcal{I}_0(s - \tau) ds, & t > \tau, \\ \Theta, & t \leq \tau. \end{cases}
\]

\[
 \mathcal{A}_n(t - n\tau) := \begin{cases} \int_{n\tau}^{t} e^{\mathcal{A}_0 (t-s)} \mathcal{A}_1 \mathcal{I}_{n-1}(s - n\tau) ds, & t > n\tau, \\ \Theta, & t \leq n\tau. \end{cases}
\]

Therefore, in this case, the delayed Dyson-Phillips series becomes the delayed perturbation of an operator-valued exponential function on \( \mathcal{R}_+ \).

Definition 4.1. The delayed perturbation of an operator-valued exponential function \( \mathcal{I}(\cdot; \tau) : \mathcal{R} \to \mathcal{L}(\mathcal{X}) \) generated by bounded linear operators \( \mathcal{A}_0, \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \) is defined by

\[
 \mathcal{I}(t; \tau) := \begin{cases} \Theta, & -\infty < t < 0, \\ \mathcal{I}_0, & t = 0, \\ e^{\mathcal{A}_0 t} + \mathcal{A}_1(t - \tau) + \cdots + \mathcal{A}_n(t - n\tau), & n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0. \end{cases}
\]

The following theorem deals with the fundamental solution of delay evolution equation with non-permutable bounded linear operators.

Theorem 4.1. Let \( \mathcal{A}_0, \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \) with non-zero commutator \([\mathcal{A}_0, \mathcal{A}_1] := \mathcal{A}_0 \mathcal{A}_1 - \mathcal{A}_1 \mathcal{A}_0 \neq 0\). Then a delayed perturbation of an operator-valued exponential function \( \mathcal{I}(\cdot; \tau) \in \mathcal{L}(\mathcal{X}) \) can be represented by
In a recursive way, for $n$ we get:

$$\mathcal{J}(t; \tau) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_k^0 a_k^m (t \tau^k \frac{(t - l \tau)^k}{k!}, \quad l \tau < t \leq (l + 1) \tau, \quad n \in \mathbb{N}_0, \quad (4.2)$$

where $a_k^0 a_k^m (l \tau) \in \mathcal{L}(\mathcal{X})$, $k, l \in \mathbb{N}_0$ is given by

$$a_k^0 a_k^m (0) := a_k^m, \quad a_k^0 a_k^m (l \tau) := \sum_{m=0}^{k} a_k^{m-m} a_k^m a_k^m ((l - 1) \tau), \quad k, l \in \mathbb{N}. \quad (4.3)$$

**Proof.** By making use of the definition of a delayed operator-valued exponential function $\mathcal{J}(\cdot; \tau) \in \mathcal{L}(\mathcal{X})$ (4.1), we derive the basis case for $n = 0$:

$$\mathcal{J}_0(t) = e^{a_0^0 t} = \sum_{k=0}^{\infty} a_k^0 \frac{t^k}{k!} = \sum_{k=0}^{\infty} a_k^0 (0) \frac{t^k}{k!}, \quad t > 0, \quad a_k^0 a_k^m (0) := a_k^m, \quad k \in \mathbb{N}_0.$$

For $n = 1$, by making use of the well-known Cauchy product formula and interchanging the order of summation and integration which is permissible in accordance with the uniform convergence of the series (2.4), we get:

$$\mathcal{J}_1(t - \tau) = \int_{\tau}^{t} e^{a_0^0 (t-s)} a_1 e^{a_0^0 (s-\tau)} ds$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_k^0 a_k^m a_k^m \int_{\tau}^{t} (t-s)^k (s-\tau)^m \frac{m!}{k!} ds$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_k^0 a_k^m \frac{(t-\tau)^{k+m+1}}{(k+m+1)!}$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{k} a_k^{m-m} a_k^m \frac{(t-\tau)^{k+1}}{(k+1)!}$$

$$= \sum_{k=1}^{\infty} \sum_{m=0}^{k} a_k^{m-m} a_k^m \frac{(t-\tau)^{k}}{k!}$$

$$= \sum_{k=1}^{\infty} a_k^0 a_k^m (0) \frac{(t-\tau)^{k}}{k!}, \quad t > \tau, \quad a_k^0 a_k^m (0) := \sum_{m=1}^{k} a_k^{m-m} a_k^m a_k^m (0), \quad k = 1, 2, \ldots$$

In a recursive way, for $n = 2$, we derive that

$$\mathcal{J}_2(t - 2\tau) = \int_{2\tau}^{t} e^{a_0^0 (t-s)} a_1^2 e^{a_0^0 (s-2\tau)} ds$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_k^0 a_k a_k^m \frac{(t-2\tau)^{k+m+1}}{(k+m+1)!}$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{k} a_k^{m-m} a_k^m a_k^m \frac{(t-2\tau)^{k+m+1}}{(k+m+1)!}$$
\[ \sum_{k=0}^{\infty} \sum_{m=0}^{k} q_k^0 \mathcal{A}_k \mathcal{Q}^{\mathcal{A}_m, \mathcal{A}_l}(\tau) \frac{(t-2\tau)^{k+1}}{(k+1)!} \]

\[ = \sum_{k=0}^{\infty} \sum_{m=1}^{k} q_k^0 \mathcal{A}_k \mathcal{Q}^{\mathcal{A}_m, \mathcal{A}_l}(\tau) \frac{(t-2\tau)^{k+1}}{(k+1)!} \]

\[ = \sum_{k=1}^{\infty} \sum_{m=0}^{k-1} q_k^0 \mathcal{A}_k \mathcal{Q}^{\mathcal{A}_m, \mathcal{A}_l}(\tau) \frac{(t-2\tau)^{k+1}}{(k+1)!} \]

\[ = \sum_{k=2}^{\infty} \sum_{m=0}^{k-1} q_k^0 \mathcal{A}_k \mathcal{Q}^{\mathcal{A}_m, \mathcal{A}_l}(\tau) \frac{(t-2\tau)^{k+1}}{(k+1)!} \]

Eventually, for the \( n \)-th case, it yields that for \( t > n\tau \), we have

\[ \mathcal{S}_n(t - n\tau) = \int_{n\tau}^{t} e^{\mathcal{A}_0(t-s)} \mathcal{A}_1 \mathcal{S}_{n-1}(s - n\tau) ds \]

\[ = \sum_{k=n}^{\infty} q_k^{\mathcal{A}_0, \mathcal{A}_1}(n\tau) \frac{(t - n\tau)^k}{k!} \]

\[ = \sum_{k=n}^{\infty} q_k^{\mathcal{A}_0, \mathcal{A}_1}(n\tau) \frac{(t - n\tau)^k}{k!} \]

Therefore, since the delayed perturbation of an operator-valued exponential function \( \mathcal{S}(\cdot; \tau) \in \mathcal{L}(\mathcal{H}) \) is a summation of the above steps, we attain a desired result:

\[ \mathcal{S}(t; \tau) = \sum_{k=0}^{\infty} q_k^{\mathcal{A}_0, \mathcal{A}_1}(0) \frac{t^k}{k!} + \ldots + \sum_{k=n}^{\infty} q_k^{\mathcal{A}_0, \mathcal{A}_1}(n\tau) \frac{(t - n\tau)^k}{k!} \]

\[ = \sum_{k=1}^{\infty} \sum_{l=0}^{n} q_k^{\mathcal{A}_0, \mathcal{A}_1}(l\tau) \frac{(t - l\tau)^k}{k!} \]

The proof is complete. \( \square \)

**Remark 4.1.** In the particular case, these results are attained for a system of nonhomogeneous linear difference equations with linear parts given by noncommutative matrices in [33]. Further, it is interesting to note that the operator (matrix) construction \( q_k^{\mathcal{A}_0, \mathcal{A}_1}(l\tau), k, l \in \mathbb{N}_0 \) is satisfying the following crucial properties:

- \( q_k^{\mathcal{A}_0, \mathcal{A}_1}(l\tau) = q_0^{\mathcal{A}_0, \mathcal{A}_1}(l\tau) + \mathcal{A}_1 q_k^{\mathcal{A}_0, \mathcal{A}_1}((l-1)\tau), \quad k, l \in \mathbb{N}_0 \),

- If \( \mathcal{A}_0 \mathcal{A}_1 = \mathcal{A}_1 \mathcal{A}_0 \), then \( q_k^{\mathcal{A}_0, \mathcal{A}_1}(l\tau) = \left( \frac{1}{t_l} \right) q_0^{\mathcal{A}_0, \mathcal{A}_1} l \), \( k, l \in \mathbb{N}_0 \).

To get more information regarding the properties of \( q_k^{\mathcal{A}_0, \mathcal{A}_1} \in \mathcal{L}(\mathcal{H}), k, l \in \mathbb{N}_0 \), we refer to [33].

By Theorem 4.1 we can introduce another suggestive explicit formula for the delayed perturbation of an operator-valued exponential function (4.2) as follows.
Definition 4.2. The delayed perturbation of an operator-valued exponential function \( \mathcal{J}(\cdot; \tau) \in \mathcal{L}(\mathcal{X}) \) generated by bounded linear operators \( \mathcal{A}_0, \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \) is defined by for \( n \in \mathbb{N}_0 \)

\[
\mathcal{J}(t; \tau) := \begin{cases} 
\varTheta, & -\tau \leq t < 0, \\
\mathcal{J}, & t = 0, \\
\sum_{k=0}^{\infty} \mathcal{A}_0^k \mathcal{A}_1 (t-k\tau) \frac{(t-k\tau)^k}{k!} + \cdots + \sum_{k=n}^{\infty} \mathcal{A}_0^k \mathcal{A}_1 (n\tau) \frac{(t-n\tau)^k}{k!}, & n\tau < t \leq (n+1)\tau.
\end{cases}
\]

The construction of \( \mathcal{D}_{k+1}(n\tau) \) for \( k, n \in \mathbb{N}_0 \) for nonpermutable bounded linear operators plays a role of kernel in this definition. With the help of formulas (4.3), simple calculations show that

\[
\begin{array}{cccccc}
\mathcal{D}_{k+1}(l\tau) & l = 0 & l = 1 & l = 2 & l = 3 & \ldots & l = n \\
\hline
k = 0 & \mathcal{J} & \varTheta & \varTheta & \varTheta & \cdots & \varTheta \\
1 = 1 & \mathcal{A}_0 & \mathcal{A}_1 & \varTheta & \varTheta & \cdots & \varTheta \\
k = 2 & \mathcal{A}_0^2 & \mathcal{A}_0 \mathcal{A}_1 + \mathcal{A}_1 \mathcal{A}_0 & \varTheta & \varTheta & \cdots & \varTheta \\
k = 3 & \mathcal{A}_0^3 & \mathcal{A}_0 (\mathcal{A}_0 \mathcal{A}_1 + \mathcal{A}_1 \mathcal{A}_0) + \mathcal{A}_1 \mathcal{A}_0^2 & \varTheta & \varTheta & \cdots & \varTheta \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \mathcal{A}_0^n \\
k = n & \mathcal{A}_0^n & \cdots & \cdots & \cdots & \varTheta & \cdots & \mathcal{A}_0^n \\
\end{array}
\]

Remark 4.2. It is easily seen that if \( l \geq k + 1 \) for \( k \geq 0 \), then \( \mathcal{D}_{k+1}(l\tau) = \varTheta \).

If \( \mathcal{A}_1 = \varTheta \), then \( \mathcal{D}_{k+1}(0) = \varTheta \), \( \mathcal{D}_{k+1}(l\tau) = \varTheta \), \( l = 1, 2, \ldots, n \), and \( \mathcal{J}(\cdot; \tau) \) becomes an operator-valued exponential function, i.e., \( \mathcal{J}(t; \tau) = e^{\mathcal{A}_0 t} \), \( t \in \mathbb{R}_+ \). If \( \mathcal{A}_0 = \varTheta \), then \( \mathcal{D}_{k+1}(l\tau) = \mathcal{A}_1 \), \( \mathcal{D}_{k+1}(l\tau) = \varTheta \), \( k \neq l, \ l = 0, 1, \ldots, n \) and \( \mathcal{J}(\cdot; \tau) \) becomes the purely delayed operator-valued exponential function as below:

\[
\mathcal{J}(t; \tau) = \mathcal{A}_1 (t - \tau) + \cdots + \mathcal{A}_1^n (t - n\tau) \frac{(t - n\tau)^n}{n!} = \sum_{l=0}^{n} \mathcal{A}_1 (t - l\tau) \frac{(t - l\tau)^l}{l!}, \ l\tau < t \leq (l+1)\tau.
\]

It is obvious that operator coefficients of the first series of \( \mathcal{J}(\cdot; \tau) \) (4.4) are the elements of the first column \((l = 0)\) of the above table, operator coefficients of the second series of (4.4) are elements of the second column \((l = 1)\) of the table and so on.

Rearranging the terms we can write the delayed perturbation of an operator-valued exponential function \( \mathcal{J}(\cdot; \tau) \) as follows:

\[
\mathcal{J}(t; \tau) = \mathcal{D}_{k+1}(l\tau) + \mathcal{D}_{k+1}^2 (t)(t - \tau) + \cdots + \mathcal{D}_{k+1}^n (n\tau) \frac{(t - n\tau)^n}{n!}
\]

where \((t)_+ = \max\{0, t\} \).

Now, we introduce a shift operator also known as translation operator is an operator \( T_\tau \) for \( \tau \in \mathbb{R} \) that takes a function \( t \mapsto f(t) \) on \( \mathbb{R} \) to its translation \( t \mapsto f(t + \tau) \):

\[ T_\tau f(t) := f(t + \tau). \]
For the notation of our results, we will use the following Lagrange translation formula for shift operators \( \mathcal{T} \):
\[
\mathcal{T}_\tau f(t) = f(t + \tau) = e^{\tau \frac{d}{dt}} f(t).
\]

Using the shift operator we rewrite (1.5) as below:
\[
\mathcal{S}(t; \tau) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} k! \mathcal{Q}_{k+1}^{l\alpha_l}(l\tau) \frac{(t - l\tau)^k}{k!} = \sum_{k=0}^{\infty} k! \mathcal{Q}_{k+1}^{l\alpha_l}(l\tau) e^{-l\tau \frac{d}{dt}} \frac{(t)^k}{k!}.
\]

In the following lemma, we introduce a new relation for the construction of nonpermutable linear bounded operators with the help of translation operator.

**Lemma 4.1.** A bounded linear operator \( \mathcal{Q}_{k+1}^{l\alpha_l}(l\tau) \in \mathcal{L}(\mathcal{S}) \) satisfies the following identity:
\[
\sum_{l=0}^{k} \mathcal{Q}_{k+1}^{l\alpha_l}(l\tau) e^{-l\tau \frac{d}{dt}} = \mathcal{Q}_0 + \mathcal{Q}_1 e^{-\tau \frac{d}{dt}}, \quad k \in \mathbb{N}_0.
\]

**Proof.** To prove the identity (4.7), we use the mathematical induction principle with respect to \( k \in \mathbb{N}_0 \). For \( k = 0 \), we have \( \mathcal{Q}_1^{0\alpha_l}(0) = \mathcal{S} \). For \( k = 1 \), using the property of \( \mathcal{Q}_{k+1}^{l\alpha_l}(l\tau) \), we proceed as follows:
\[
\begin{align*}
\sum_{l=0}^{1} \mathcal{Q}_{2}^{l\alpha_l}(l\tau) e^{-l\tau \frac{d}{dt}} &= \mathcal{Q}_0 \mathcal{Q}_1^{0\alpha_l}(0) + \mathcal{Q}_1 \mathcal{Q}_1^{0\alpha_l}(\tau) e^{-\tau \frac{d}{dt}} + \mathcal{Q}_1 \mathcal{Q}_1^{0\alpha_l}(0) e^{-\tau \frac{d}{dt}} \\
&= \mathcal{Q}_0 + \mathcal{Q}_1 e^{-\tau \frac{d}{dt}}.
\end{align*}
\]

Then, assuming that the identity is true for \( k = n \) and we prove it for \( k = n + 1 \):
\[
\left( \mathcal{Q}_0 + \mathcal{Q}_1 e^{-\tau \frac{d}{dt}} \right) = \left( \mathcal{Q}_0 + \mathcal{Q}_1 e^{-\tau \frac{d}{dt}} \right) \sum_{l=0}^{k} \mathcal{Q}_{k+1}^{l\alpha_l}(l\tau) e^{-l\tau \frac{d}{dt}}
\]
\[
= \sum_{l=0}^{k} \mathcal{Q}_0 \mathcal{Q}_{k+1}^{l\alpha_l}(l\tau) e^{-l\tau \frac{d}{dt}} + \sum_{l=0}^{k} \mathcal{Q}_1 \mathcal{Q}_{k+1}^{l\alpha_l}(l\tau) e^{-(l+1)\tau \frac{d}{dt}}
\]
\[
= \sum_{l=0}^{k} \mathcal{Q}_0 \mathcal{Q}_{k+1}^{l\alpha_l}(l\tau) e^{-l\tau \frac{d}{dt}} + \sum_{l=1}^{k+1} \mathcal{Q}_1 \mathcal{Q}_{k+1}^{l\alpha_l}((l-1)\tau) e^{-l\tau \frac{d}{dt}}
\]
\[
= \mathcal{Q}_0^{k+1} + \sum_{l=1}^{k+1} \left( \mathcal{Q}_0 \mathcal{Q}_{k+1}^{l\alpha_l}(l\tau) + \mathcal{Q}_1 \mathcal{Q}_{k+1}^{l\alpha_l}((l-1)\tau) \right) e^{-l\tau \frac{d}{dt}} + \mathcal{Q}_1^{k+1} e^{-(k+1)\tau \frac{d}{dt}}
\]
\[
= \sum_{l=0}^{k+1} \left( \mathcal{Q}_0 \mathcal{Q}_{k+1}^{l\alpha_l}(l\tau) + \mathcal{Q}_1 \mathcal{Q}_{k+1}^{l\alpha_l}((l-1)\tau) \right) e^{-l\tau \frac{d}{dt}}
\]
\[
= \sum_{l=0}^{k} \mathcal{Q}_{k+2}^{l\alpha_l}(l\tau) e^{-l\tau \frac{d}{dt}}.
\]

Therefore, the identity (4.7) holds for any \( k \in \mathbb{N}_0 \). The proof is complete. \( \square \)

**Remark 4.3.** This identity is proved for multi-delayed Mittag-Leffler type matrix functions which are fundamental solutions of fractional-order time-delay systems in [9].
Therefore, by using the formula [4.6] and identity [4.7], we get a new representation formula for the delayed perturbation of an operator-valued exponential function on positive half-line as below.

**Definition 4.3.** The delayed perturbation of an operator-valued exponential function $\mathcal{S}(\cdot;\tau) : \mathbb{R}^+ \to \mathcal{L}(\mathcal{X})$ generated by bounded linear operators $\mathcal{A}_0, \mathcal{A}_1 \in \mathcal{L}(\mathcal{X})$ is defined by

$$\mathcal{S}(t;\tau) = \sum_{k=0}^{\infty} \left( \mathcal{A}_0 + \mathcal{A}_1 e^{-\tau \frac{d}{dt}} \right)^k \left( \frac{t^k}{k!} \right), \quad t \in \mathbb{R}^+. \quad (4.8)$$

**Remark 4.4.** In the special case, for delay-free dynamical systems ($\tau = 0$), the perturbed operator-valued exponential functions defined by $\mathcal{S}(t) = \sum_{k=0}^{\infty} \left( \mathcal{A}_0 + \mathcal{A}_1 \right)^k \left( \frac{t^k}{k!} \right) = e^{(\mathcal{A}_0 + \mathcal{A}_1)t}$ for $t \in \mathbb{R}^+$.

Meanwhile, we need to show that the fundamental solution $\mathcal{S}(\cdot;\tau)$ forms a uniformly continuous semigroup on a Banach space $\mathcal{X}$. To do this, we will use the formula [4.8] for a fundamental solution to (8.1).

**Theorem 4.2.** Assume that the linear operators $\mathcal{A}_0, \mathcal{A}_1 \in \mathcal{L}(\mathcal{X})$. Then, the family $\{\mathcal{S}(t;\tau), t \in \mathbb{R}^+\}$ defined by

$$\mathcal{S}(t;\tau) = \sum_{k=0}^{\infty} \left( \mathcal{A}_0 + \mathcal{A}_1 e^{-\tau \frac{d}{dt}} \right)^k \left( \frac{t^k}{k!} \right), \quad t \in \mathbb{R}^+, \quad (4.9)$$

where

$$\left( \mathcal{A}_0 + \mathcal{A}_1 e^{-\tau \frac{d}{dt}} \right)^k = \sum_{i=0}^{k} \mathcal{L}_{k+1}^{\mathcal{A}_0,\mathcal{A}_1}(\tau) e^{-i \tau \frac{d}{dt}}, \quad k \in \mathbb{N}_0,$$

forms a uniformly continuous semigroup on $\mathcal{X}$.

**Proof.** By using the formula [4.6] and [4.7], we derive that

$$\|\mathcal{S}(t;\tau)\| \leq \sum_{k=0}^{\infty} \left( \|\mathcal{A}_0\| + \|\mathcal{A}_1\| e^{-\tau \frac{d}{dt}} \right) \left( \frac{t^k}{k!} \right) \leq \sum_{k=0}^{\infty} \sum_{l=0}^{k} \left( \frac{k}{l} \right) \|\mathcal{A}_0\|^{k-l} \|\mathcal{A}_1\|^{l} \left( \frac{t^k}{k!} \right) \leq e^{\|\mathcal{A}_0\| + \|\mathcal{A}_1\| t}, \quad t > 0, \quad (4.10)$$

where we have used that

$$\mathcal{L}_{k+1}^{\mathcal{A}_0,\mathcal{A}_1}(\tau) = \left( \frac{k}{l} \right) \|\mathcal{A}_0\|^{k-l} \|\mathcal{A}_1\|^{l},$$

and hence, $\mathcal{S}(t;\tau)$ is well-defined on $\mathbb{R}^+$ with $\mathcal{S}(0;\tau) = I$.

Further, we need to show that $\mathcal{S}(\cdot;\tau)$ satisfies the following semigroup property:

$$\mathcal{S}(t;\tau)\mathcal{S}(s;\tau) = \mathcal{S}(t+s;\tau), \quad t, s \in \mathbb{R}^+. \quad (4.11)$$

To do this, we use Cauchy product formula for double series as follows:

$$\mathcal{S}(t;\tau)\mathcal{S}(s;\tau) = \sum_{k=0}^{\infty} \left( \mathcal{A}_0 + \mathcal{A}_1 e^{-\tau \frac{d}{dt}} \right)^k \left( \frac{t^k}{k!} \right) \sum_{k=0}^{\infty} \left( \mathcal{A}_0 + \mathcal{A}_1 e^{-\tau \frac{d}{dt}} \right)^k \left( \frac{s^k}{k!} \right)$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \left( \frac{k}{l} \right) \|\mathcal{A}_0\|^{k-l} \|\mathcal{A}_1\|^{l} \left( \frac{t^k}{k!} \right) \left( \frac{s^k}{k!} \right) = \sum_{k=0}^{\infty} \left( \mathcal{A}_0 + \mathcal{A}_1 e^{-\tau \frac{d}{dt}} \right)^k \left( \frac{(t+s)^k}{k!} \right) = \mathcal{S}(t+s;\tau).$$
and hence, it tends to zero as $t \to 0$. Therefore, since $\lim_{t \to 0^+} \|\mathcal{S}(t; \tau) - \mathcal{S}\| = 0$, the semigroup family $\{\mathcal{S}(t; \tau), t \in \mathbb{R}_+\}$ is uniformly continuous with respect to operator norm $\|\cdot\|$ associated with $\mathcal{S}$. The proof is complete. \qed

Furthermore, the fundamental solution - a uniformly continuous semigroup can also be put in more suggestive form.

**Theorem 4.3.** Let $\mathcal{A}_0, \mathcal{A}_1 \in \mathcal{L}(\mathcal{X})$ with non-zero commutator $[\mathcal{A}_0, \mathcal{A}_1] := \mathcal{A}_0 \mathcal{A}_1 - \mathcal{A}_1 \mathcal{A}_0 \neq 0$. Then a delayed perturbation of an operator-valued exponential function $\mathcal{S}(.; \tau) \in \mathcal{L}(\mathcal{X})$ can be represented by

$$
\mathcal{S}(t; \tau) = \sum_{k=0}^{\infty} \sum_{l=0}^{n} \mathcal{D}_{k,l}^{\mathcal{A}_0, \mathcal{A}_1} \frac{(t-l \tau)^{k+l}}{(k+l)!}, \quad l \tau < t \leq (l+1)\tau, \quad n \in \mathbb{N}_0, \quad (4.11)
$$

where $\mathcal{D}_{k,l}^{\mathcal{A}_0, \mathcal{A}_1} \in \mathcal{L}(\mathcal{X})$, $k, l \in \mathbb{N}_0$ is given by

$$
\mathcal{D}_{k,0}^{\mathcal{A}_0, \mathcal{A}_1} := \mathcal{A}_0^k, \quad k \in \mathbb{N}_0, \quad \mathcal{D}_{k,l}^{\mathcal{A}_0, \mathcal{A}_1} := \sum_{m=0}^{k} \mathcal{A}_0^{k-m} \mathcal{A}_1 \mathcal{D}_{m,l-1}^{\mathcal{A}_0, \mathcal{A}_1}, \quad k, l \in \mathbb{N}, \quad \mathcal{D}_{0,l}^{\mathcal{A}_0, \mathcal{A}_1} := \mathcal{A}_1^l, \quad l \in \mathbb{N}_0. \quad (4.12)
$$

**Proof.** By making use of the definition of a delayed operator-valued exponential function $\mathcal{S}(.; \tau) \in \mathcal{L}(\mathcal{X})$ (4.11), we derive the basis case for $n = 0$:

$$
\mathcal{S}_0(t) = e^{\mathcal{A}_0 t} = \sum_{k=0}^{\infty} \mathcal{A}_0^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} \mathcal{D}_{k,0}^{\mathcal{A}_0, \mathcal{A}_1} \frac{t^k}{k!}, \quad t > 0, \quad \mathcal{D}_{k,0}^{\mathcal{A}_0, \mathcal{A}_1} := \mathcal{A}_0^k, \quad k \in \mathbb{N}_0.
$$

For $n = 1$, by interchanging the order of summation and integration which is permissible in accordance with the uniform convergence of the series (4.3), we have

$$
\mathcal{S}_1(t - \tau) = \int_{\tau}^{t} e^{\mathcal{A}_0 (t-s)} \mathcal{A}_1 e^{\mathcal{A}_0 (s-\tau)} ds = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{A}_0^k \mathcal{A}_1 \mathcal{A}_0^m \int_{\tau}^{t} \frac{(t-s)^k (s-\tau)^m}{k! m!} ds = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{D}_{k,m}^{\mathcal{A}_0, \mathcal{A}_1} \mathcal{A}_0^m \frac{(t-\tau)^{k+m+1}}{(k+m+1)!},
$$

$$
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathcal{A}_0^k \mathcal{A}_1 \mathcal{A}_0^{n-k} \frac{(t-\tau)^n}{n!} = \mathcal{S}_1(t; \tau) = \mathcal{S}_1(t, \tau), \quad t, s \in \mathbb{R}_+.
$$
Meanwhile, this technique which is used for the construction of $Q$ operators $A$ is a summation of the above cases, we get a desired result:

**Remark**

In a similar way, for $n = 2$, we acquire that

$$
\mathcal{I}_2(t - 2\tau) = \int_{-2\tau}^{t} e^{\mathcal{A}_0(t-s)} \mathcal{A}_1(s - 2\tau) ds
$$

Finally, for the general case, it yields that

$$
\mathcal{I}_n(t - n\tau) = \int_{-n\tau}^{t} e^{\mathcal{A}_0(t-s)} \mathcal{A}_1(s - n\tau) ds
$$

Therefore, since the delayed perturbation of an operator-valued exponential function $\mathcal{I}(\cdot; \tau) \in \mathcal{L}(\mathcal{X})$ is a summation of the above cases, we get a desired result:

$$
\mathcal{I}(t; \tau) = \sum_{k=0}^{\infty} \mathcal{Q}_{k,0} e^{\mathcal{A}_0(t-n\tau)} + \sum_{k=0}^{\infty} \mathcal{Q}_{k,n} e^{\mathcal{A}_0(t - (l+1)\tau)}
$$

The proof is complete. $\square$

**Remark 4.5.** These results are new even for first-order time-delay systems with non-commutative matrices. Meanwhile, this technique which is used for the construction of $Q$ operators $A$ in $\mathcal{L}(\mathcal{X})$, $k, l \in \mathbb{N}_0$ is slightly different from the technique that is utilized in [3]. Moreover, it is interesting to note that for bounded linear operators $\mathcal{A}_0, \mathcal{A}_1 \in \mathcal{L}(\mathcal{X})$, $\mathcal{Q}_{k,l} \in \mathcal{L}(\mathcal{X})$, $k, l \in \mathbb{N}_0$ is satisfying the following crucial properties:

$$
\forall \mathcal{A}_0, \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}), \quad \mathcal{Q}_{k,l} = \mathcal{A}_0 \mathcal{Q}_{k-1,l} + \mathcal{A}_1 \mathcal{Q}_{k,l-1}, \quad k, l \in \mathbb{N}_0.
$$

$$
\exists \mathcal{A}_0, \mathcal{A}_1 \in \mathcal{L}(\mathcal{X}) \text{ s.t. } \mathcal{A}_0 \mathcal{A}_1 = \mathcal{A}_1 \mathcal{A}_0, \quad \mathcal{Q}_{k,l} = \binom{k+l}{l} \mathcal{A}_0^k \mathcal{A}_1^l, \quad k, l \in \mathbb{N}_0.
$$
To get more information about the properties of $\mathcal{L}_{k,l}^{x_0,x_1} \in \mathcal{L} (\mathcal{X})$, $k,l \in \mathbb{N}_0$, we refer to our other joint papers [30, 31].

The following theorem is regarding a uniformly continuous semigroup - the fundamental solution of delay evolution equation with permutable bounded linear operators.

**Theorem 4.4.** Let $\mathcal{A}_0, \mathcal{A}_1 \in \mathcal{L} (\mathcal{X})$ with zero commutator $[\mathcal{A}_0, \mathcal{A}_1] := \mathcal{A}_0 \mathcal{A}_1 - \mathcal{A}_1 \mathcal{A}_0 = 0$. Then, the delayed perturbation of an operator-valued exponential function $\mathcal{J} (; \tau) \in \mathcal{L} (\mathcal{X})$ can be expressed by

$$\mathcal{J} (t; \tau) = e^{\mathcal{A}_0 t} e^{\mathcal{A}_2 (t - \tau)}, \quad \mathcal{A}_2 := \mathcal{A}_1 e^{-\mathcal{A}_0 \tau} \in \mathcal{L} (\mathcal{X}),$$

where $e^{\mathcal{A}_2 (t - \tau)} \in \mathcal{L} (\mathcal{X})$ is a delayed operator-valued exponential function defined by

$$e^{\mathcal{A}_2 (t - \tau)} := \sum_{l=0}^{n} \mathcal{A}_2^{l} \frac{(t - l \tau)^l}{l!}, \quad l \tau < t \leq (l + 1) \tau, \quad n \in \mathbb{N}_0.$$  \hspace{1cm} (4.15)

**Proof.** Since bounded linear operators $\mathcal{A}_0, \mathcal{A}_1 \in \mathcal{L} (\mathcal{X})$ are permutable, i.e., $\mathcal{A}_0 \mathcal{A}_1 = \mathcal{A}_1 \mathcal{A}_0$, we have $e^{\mathcal{A}_0 (t-s)} \mathcal{A}_1 = \mathcal{A}_1 e^{\mathcal{A}_0 (t-s)}$. By making use of this property, we derive the following cases, recursively. Firstly, for $n = 0$, we attain that

$$\mathcal{J}_0(t) = e^{\mathcal{A}_0 t}, \quad t > 0.$$

For the case of $n = 1$, we have

$$\mathcal{J}_1(t - \tau) = \int_{\tau}^{t} e^{\mathcal{A}_0 (t-s)} \mathcal{A}_1 e^{\mathcal{A}_0 (s-\tau)} ds$$

$$= e^{\mathcal{A}_0 (t-\tau)} \mathcal{A}_1 (t - \tau)$$

$$= e^{\mathcal{A}_0 t} \mathcal{A}_1 e^{-\mathcal{A}_0 \tau}(t - \tau), \quad t > \tau.$$

Similarly, for $n = 2$, it follows that

$$\mathcal{J}_2(t - 2\tau) = \int_{2\tau}^{t} e^{\mathcal{A}_0 (t-s)} \mathcal{A}_1 \mathcal{J}_2(s - 2\tau) ds$$

$$= \int_{2\tau}^{t} e^{\mathcal{A}_0 (t-s)} \mathcal{A}_1 e^{\mathcal{A}_0 (s-2\tau)} \mathcal{A}_1 (s - 2\tau) ds$$

$$= e^{\mathcal{A}_0 (t-2\tau)} \mathcal{A}_1^2 \int_{2\tau}^{t} (s - 2\tau) ds$$

$$= e^{\mathcal{A}_0 t} \mathcal{A}_1^2 e^{-2\mathcal{A}_0 \tau} \frac{(t - 2\tau)^2}{2!}, \quad t > 2\tau.$$

Eventually, for the general case, we acquire

$$\mathcal{J}_n(t - n\tau) = e^{\mathcal{A}_0 t} \mathcal{A}_1^n e^{-n\mathcal{A}_0 \tau} \frac{(t - n\tau)^n}{n!}, \quad t > n\tau, \quad n \in \mathbb{N}_0.$$

Therefore, since the delayed perturbation of an operator-valued exponential function $\mathcal{J} (; \tau) \in \mathcal{L} (\mathcal{X})$ is a summation of the above cases, we get the following result:

$$\mathcal{J} (t; \tau) = \sum_{l=0}^{n} \mathcal{J}_l(t - l \tau) = e^{\mathcal{A}_0 t} \sum_{l=0}^{n} \mathcal{A}_2^l \frac{(t - l \tau)^l}{l!}$$

$$= e^{\mathcal{A}_0 t} e^{\mathcal{A}_2 (t - \tau)}, \quad \mathcal{A}_2 := \mathcal{A}_1 e^{-\mathcal{A}_0 \tau} \in \mathcal{L} (\mathcal{X}), \quad l \tau < t \leq (l + 1) \tau, \quad n \in \mathbb{N}_0.$$
In a alternative way, we can prove this case by using Theorem 3.3 on $J = [0, T]$ where $T = (n + 1)\tau$ for a fixed $n \in \mathbb{N}_0$. Since $\mathcal{J}(t)x = e^{\varphi_0(t)}x$ for any $x \in \mathcal{X}$, we derive the following desired result:

\[
\mathcal{J}(t; \tau) = \sum_{l=0}^{n} \varphi_l(t - l\tau) \frac{(t - l\tau)!}{l!} e^{\varphi_0(t - l\tau)}
\]

\[
= \frac{e^{\varphi_0}}{t - l\tau} \sum_{l=0}^{n} \left[ \varphi_l e^{-\varphi_0 (t - l\tau)} \right]^{l} \frac{l!}{l} e^{\varphi_0 (t - l\tau)}
\]

\[
= e^{\varphi_0} e^{\varphi_2(t-\tau)}
\]

The proof is complete. 

**Remark 4.6.** Furthermore, this theorem can be proved using the property of $Q_{k,l}^{\varphi_0,\varphi_1} \in \mathcal{L}(\mathcal{X})$, $k, l \in \mathbb{N}_0$ and $X$, directly.

**Remark 4.7.** It should be note that this particular case is a natural extension of the results are attained in [9] in terms of commutative matrix coefficients by Khusainov and Shuklin. Corresponding results are derived for fractional-order time-delay systems in [11] by Huseynov and Mahmudov.

**Corollary 4.1.** Let $\mathcal{J}(t; \tau), t \in \mathbb{R}_+$ be defined as in Equation (4.1). Then the following assertions hold true:

(i) If $\mathcal{A}_0 = \Theta$, then a delayed operator-valued exponential function $\mathcal{J}(\cdot; \tau) \in \mathcal{L}(\mathcal{X})$ can be represented by

\[
\mathcal{J}(t; \tau) = e^{\varphi_0(t-\tau)} = \sum_{l=0}^{n} \varphi_0(t-l\tau) \frac{(t-l\tau)!}{l!}, \quad l\tau < t \leq (l+1)\tau, \quad n \in \mathbb{N}_0.
\]

(ii) If $\mathcal{A}_1 = \Theta$, then a delayed operator-valued exponential function $\mathcal{J}(\cdot; \tau) \in \mathcal{L}(\mathcal{X})$ can be expressed by

\[
\mathcal{J}(t; \tau) = e^{\varphi_0(t)} = \sum_{k=0}^{\infty} \varphi_0(t) \frac{t^k}{k!}, \quad t \in \mathbb{R}_+.
\]

**Proof.** The proof of this Corollary follows from directly Theorem 4.4. So, we pass over it here.

**Remark 4.8.** It is interesting to note that the first part have studied by Khusainov and Shuklin in [5] for linear differential equations with matrix coefficients and pure delay, and the second part is the classical case which is considered in [2].

**5. Application: a delayed heat equation**

Let us take $\mathcal{X} := \mathbb{L}^2([0, \pi], \mathbb{R})$. We consider the following abstract initial-boundary value problem with homogeneous Dirichlet boundary conditions for a one-dimensional heat equation with a constant delay:

\[
\begin{aligned}
&\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + u(x, t - \tau) + \psi(x, t), \quad x \in [0, \pi], \quad t > 0, \quad \tau > 0, \\
u(x, t) &= \varphi(x, t), \quad x \in [0, \pi], \quad t \in [-\tau, 0], \\
u(t, 0) &= u(t, \pi) = 0, \quad t \geq -\tau.
\end{aligned}
\]

We define the following linear operators $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{X} \rightarrow \mathcal{X}$ and $\mathcal{A}_1 := \mathcal{J} : \mathcal{X} \rightarrow \mathcal{X}$ as follows:

\[
\begin{aligned}
\mathcal{A}_0 u &= \frac{\partial^2 u}{\partial x^2}, \quad u \in \mathcal{D}(\mathcal{A}_0), \\
\mathcal{A}_1 u &= u, \quad u \in \mathcal{X},
\end{aligned}
\]

with the domains are given by

\[
\mathcal{D}(\mathcal{A}_0) := \left\{ u \in \mathcal{X} : u, \frac{\partial u}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 u}{\partial x^2} \in \mathcal{X}, u(0) = u(\pi) = 0 \right\} \quad \text{and} \quad \mathcal{D}(\mathcal{A}_1) = \mathcal{X}.
\]
It is known that \( \mathcal{A}_0 \) has discrete spectrum with eigenvalues of the form \( \lambda_n := -n^2 \), \( n \in \mathbb{N} \) and the corresponding normalized eigenvectors are given by \( u_n(x) := \sqrt{\frac{2}{\pi}} \sin(nx) \). Moreover, \( \{u_n : n \in \mathbb{N}\} \) is an orthonormal basis for \( \mathcal{X} \) and thus, \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) can be written as below:

\[
\mathcal{A}_0u = \sum_{n=1}^{\infty} -n^2(u, u_n)u_n, \quad u \in \mathcal{D}(\mathcal{A}_0),
\]

\[
\mathcal{A}_1u = \sum_{n=1}^{\infty} (u, u_n)u_n, \quad u \in \mathcal{X}.
\]

Therefore, a one-dimensional heat equation with a constant delay \( \tau \) can be formulated in abstract sense as follows:

\[
\begin{aligned}
\begin{cases}
\frac{d}{dt}u(t) = \mathcal{A}_0u(t) + \mathcal{A}_1u(t-\tau) + \psi(t), & t > 0, \quad \tau > 0, \\
u(t) = \phi(t), & -\tau \leq t \leq 0,
\end{cases}
\end{aligned}
\]  \( (5.2) \)

where \( u(t) = u(\cdot, t) \), \( \phi(t) = \phi(\cdot, t) \) and \( \psi(t) = \psi(\cdot, t) \).

Hence, by making use the formulas \( (3.43) \) and \( (4.15) \), for any \( u \in \mathcal{X} \), a fundamental solution \( \mathcal{S}(\cdot; \tau) \in \mathcal{L}(\mathcal{X}) \) of \( (5.2) \) can be represented as follows:

\[
\begin{aligned}
\mathcal{S}(t; \tau)u &= \sum_{n=1}^{\infty} e^{-n^2(t+\tau)}e_{\tau}^{b_n}(u, u_n)u_n \\
&= \sum_{n=1}^{\infty} e^{-n^2(t+\tau)}e_{\tau}^{b_n(t-\tau)}(u, u_n)u_n,
\end{aligned}
\]

where \( b_n = e^{n^2\tau}, n \in \mathbb{N} \).

So, the mild solution of abstract Cauchy problem \( (5.2) \) can be expressed with the help of a delayed perturbation of exponential function as below:

\[
\begin{aligned}
\begin{cases}
u(t) = \sum_{n=1}^{\infty} \left\{ e^{-n^2(t+\tau)}e_{\tau}^{b_n(t-\tau)} \phi(-\tau), u_n > 0 \right\} + \int_{-\tau}^{0} e^{-n^2(t-s)}e_{\tau}^{b_n(t-\tau-s)} \phi'(s) + n^2\phi(s), u_n > ds \\
+ \int_{0}^{t} e^{-n^2(t-s)}e_{\tau}^{b_n(t-\tau-s)} \psi(s), u_n > ds \right\} u_n, & t \geq -\tau, \\
\end{cases}
\end{aligned}
\]  \( (5.3) \)

Remark 5.1. Currently, in \( [28] \), Pinto et al. have studied an approximation of a mild solution \( u(\cdot) \in \mathcal{X} \) of linear homogeneous first-order abstract differential problem \( (5.2) \) with a constant delay \( (\text{where } \psi(x, t) = 0, \tau = 1, \mathcal{A}_1u = Lu, \text{for } L \in \mathbb{R}_+) \), which depends on an initial history condition \( \phi(\cdot) \in \mathcal{X} \) and unbounded closed linear operator \( \mathcal{A}_0 \) generating a \( C_0 \)-semigroup on a Banach space \( \mathcal{X} \). For \( L = 1 \), the mild solution \( u(\cdot) \) of the abstract problem \( (5.2) \) associated with \( \phi(\cdot) \), on the compact interval \([0, 1]\), for any \( t \in [0, 1] \) is given by

\[
u(t) = \sum_{n=1}^{\infty} e^{-n^2t} < \phi(0), u_n > u_n + \sum_{n=1}^{\infty} e^{-n^2t} + n^2t - \frac{1}{n^4} < \phi(0), u_n > u_n. \]  \( (5.4) \)

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In contrast to [28], we can easily deduce an exact explicit representation of mild solution $u(\cdot)$ to (3.22) by taking $\tau = 1$ in (5.3) as follows:

$$u(t) = \sum_{n=1}^{\infty} e^{-n^2} e_1^b_1(t-1) < \varphi(0), u_0 > + \int_{-1}^{0} e^{-n^2(t-1-s)} e_1^b_1(t-2-s) < \varphi(s), u_0 > ds, \quad 0 \leq t \leq 1,$$

(5.5)

where

$$e_1^b_1(t-1) = \sum_{l=0}^{n} b_l \left( \frac{t-l}{l!} \right), \quad b_n = e^{n^2}, \quad n \in \mathbb{N}.$$

Remark 5.2. Furthermore, by using Fourier method the explicit representation of a mild solution to delayed heat equation (5.1) can be represented with Fourier coefficients which is studied in [11, 12, 14]:

$$u(t,x) = \sum_{n=1}^{\infty} \left( e^{-n^2(t+\tau)} e_1^b_1(t-\tau) \Phi_n(0) + \int_{-\tau}^{0} e^{-n^2(t-s)} e_1^b_1(t-\tau-s) \left[ \Phi_n'(s) + n^2 \Phi_n(s) \right] ds \right) + \int_{0}^{t} e^{-n^2(t-s)} e_1^b_1(t-\tau-s) \Phi_n(s)ds \sin(nx), \quad x \in [0,\pi], \quad t \geq -\tau,$$

$$= \sum_{n=1}^{\infty} \left( e^{-n^2} e_1^b_1(t-\tau) \Phi_n(0) + \int_{-\tau}^{0} e^{-n^2(t-s)} e_1^b_1(t-2\tau-s) \Phi_n(s)ds \right) + \int_{0}^{t} e^{-n^2(t-s)} e_1^b_1(t-\tau-s) \Phi_n(s)ds \Psi_n(s)ds \sin(nx), \quad x \in [0,\pi], \quad t \geq 0,$$

where $\Phi_n(\cdot) : [-\tau,0] \rightarrow \mathbb{R}$ and $\Psi_n(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$ for $n \in \mathbb{N}$ are Fourier coefficients of $\varphi(x,t)$ and $\psi(x,t)$, respectively, that is

$$\Phi_n(t) = \frac{2}{\pi} \int_{0}^{\pi} \varphi(\xi,t) \sin(n\xi)d\xi, \quad t \in [-\tau,0],$$

$$\Psi_n(t) = \frac{2}{\pi} \int_{0}^{\pi} \psi(\xi,t) \sin(n\xi)d\xi, \quad t \in \mathbb{R}_+.$$

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