Neutrino induced vorticity, Alfvén waves and the normal modes

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(Dated: March 14, 2018)

We consider plasma consisting of electrons and ions in presence of a background neutrino gas and develop the magnetohydrodynamic equations for the system. We show that electron neutrino interaction can induce vorticity in the plasma even in the absence of any electromagnetic perturbations if the background neutrino density is left-right asymmetric. This induced vorticity support a new kind of Alfvén wave whose velocity depends on both the external magnetic field and on the neutrino asymmetry. The normal mode analysis show that in the presence of neutrino background the Alfvén waves can have different velocities. We also discuss our results in the context of dense astrophysical plasma such as magnetars and show that the difference in the Alfvén velocities can be used to explain the observed pulsar kick. We discuss also the relativistic generalization of electron fluid in presence of asymmetric neutrino background.

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I. INTRODUCTION

It is important to study the characteristics of plasma in presence of neutrinos, since such systems are important in understanding various physical phenomena during the evolution of early Universe as well as the systems like core core-collapsing supernovae and magnetars (see for e.g. [1] for a brief overview). The presence of the cosmic neutrino background can influence cosmic microwave anisotropy and matter clustering [2, 3] and it can also influence dynamics of the primordial magnetic field [4–6]. There exists several studies in literature where the neutrino plasma interaction has been analysed in a variety of physical situations. Non-linear coupling of intense neutrino flux with collective plasma oscillations is studied in the Ref. [7]. Authors have shown that a neutrino flux as intense as that in supernovae core can cause parametric instabilities in the surrounding plasma. Effect of a neutrino medium in the evolution of lepton plasma had been studied invoking ponderomotive description [8, 9]. In these cases it was shown that the ponderomotive force is proportional to the gradient of neutrino density and the electrons are repelled from the regions where neutrino density is large. Interaction of very large number of neutrinos with collective plasma and oscillation and the excitation of plasma turbulence is considered in the Ref. [10]. Different kinds of plasma neutrino interactions using the ponderomotive force description and the effect on collective plasma properties can be found in the references, [11–17]. In the fore mentioned ponderomotive force description, it was assumed that the neutrino field satisfy the naive Klein-Gordon equation with appropriate interaction terms. Thus in this formalism the information about the chiral structure of the weak interaction is absent. Here we note that by Silva et. al in Refs. [18] and [19] the problem of neutrino driven streaming instability, which in turn can generate an significant energy transfer from neutrino to the plasma, was considered in the kinetic theory formalism. Formulation to study the plasma interaction with intense neutrino beam using the field theory techniques is developed in [20]. Photon polarization tensor in a medium consistent with gauge and Lorentz invariance can be found in [21]. In this work it is shown that, in presence of a medium, the photon polarization tensor can have anti-symmetric part indicating $P$ and $CP$ violations. Further studies of such effect in presence of neutrinos for different physical scenarios are explored in [22, 23].

In the context of early Universe, it has been shown by Shukla at. el. [8] that the ponderomotive force of non-uniform intense neutrino beam can be responsible for large scale
quasi-stationary magnetic field. In fact, later was the first one to suggest magnetic field generation in plasma due to plasma-neutrino interactions. Further, large-scale magnetic field generation at the time of neutrino decoupling due to the evolution of plasma in presence of asymmetric neutrino background is studied in [24] and [25]. This field can act as a seed for generation of the galactic magnetic field via the galactic dynamo mechanism (see e.g [26] to read about galactic dynamo mechanism). It is to be noted that at finite lepton/baryon density the loop corrections to the photon polarization tensor are non-vanishing. With these corrections the photon polarization tensor acquires a non-zero parity odd contribution $\Pi_2(k)$ where, $k$ is the wave vector. A finite and non-zero values of $\Pi_2(k)$ in the photon polarization tensor means that there can be single field derivative terms in the effective Lagrangian and free energy, which dominates the kinetic energy part of the free energy which is having double derivative term. For e.g. the free energy for a static gauge field can be written as $\mathcal{F}[A] = \int d^3p A_i(k)\Pi_{ij}(k)A_j(-k)$ and with parity violating interactions $\Pi_{ij}(k)$ can have a contribution $i\Pi_2(p^2)\epsilon_{ijl}k_l$. Thus a non-zero value of $\Pi_2(0)$ means a term $\Pi_2(0)A \cdot \nabla \times A$ in the expression for free energy. This in turn means that there can be a generation of a large scale ($k \to 0$) magnetic field by an instability arising due to non-zero values of parity-odd contributions $\Pi_2(0)$ to the polarization tensor [24]. In the ref. [28], thermal field theory calculations were carried out to study the corrections to the photon polarization in presence of a background neutrino which is asymmetric in left-right number densities. Authors have shown that the axial part $\Pi_2$ is proportional to the neutrino asymmetry parameter and argued that the contribution to $\Pi_2$ due to the plasma which is interacting with the neutrino gas is $\sim 10^1$ times larger than the contribution to $\Pi_2$ through the correction due to the virtual process. In ref. [29] using a kinetic theory approach it was shown that the photon polarization tensor can have the parity-odd contribution $\Pi_2(k)$ due to the asymmetric neutrino background in both the collision less and collision dominated regime. In the collision dominated regime the result for $\Pi_2(k)$ using the kinetic approach agrees with that in ref. [28]. In a recent work [35] authors have calculated the effective potential or refractive index for the cosmic neutrino background (CNB) and future experimental implications have been discussed.

Further, recent theoretical calculations showed that the asymmetry in the neutrino density can be transmuted to the fluid helicity for sufficiently large electron neutrino interaction [32]. This neutrino induced vorticity can act as axial chemical potential for
the chiral electrons. This phenomenon can induce *helical plasma instability* that generate strong magnetic field [32]. In this work the plasma particles are considered to be massless and chirally polarized. Moreover, it was assumed, in this work, that the neutrino mean free path $l_\nu$ is much smaller than the system dynamics at the length scale $L$ i.e. $L \gg l_\nu$. This allows one to write the equations for the neutrino hydrodynamics [32]. Though this assumption is justifiable for a core collapsing supernova, it is hard to be satisfied in other scenarios like the early Universe. Electroweak plasma in a rotating matter is studied in [28].

In this work it is shown that electric current can be induced in the direction of rotation axis due to the parity violating nature of the interaction. This phenomenon is called *galvano-rotational effect* (GRE). In a recent work [34], spin paramagnetic deformation of neutron star has been studied and authors have calculated the ellipticity of a strongly magnetized neutron star using the spin magneto hydrodynamic equations developed in [36].

In the present work we are interested in developing magnetohydrodynamic description of the plasma in presence of the left-right asymmetric neutrino background. The expression of the interaction Lagrangian of a charged lepton field and the asymmetric neutrinos suggests that the neutrino can couple with spin of the electron [27, 28]. It is interesting to note here that there exist a lot of literatures in the usual electron-ion plasmas where the dynamics of spin degree can play a significant role. For example it was suggested that a spin polarized plasma in a fusion reactor can yield higher nuclear reaction cross section [41] and the spin depolarization process in the plasma can remain small [42]. Effect of spin dynamics using single particle description, valid for a dilute gas, is studied in the context of laser plasma interaction in the Ref. [43]. The collective effects within the framework of spin-magnetohydrodynamics has been studied in Refs. [36, 37] (for general discussion see [44]). These works can have applications in studying environments with a strong external magnetic field like pulsar and magnetars. In the present work we generalize the spin-magnetohydrodynamics considered in [36] to incorporate the effect of asymmetric neutrino background.

The report is organised in the following way. In section II we consider the low energy Lagrangian for our system and the equations of motion and spin evolution equations are derived invoking the non-relativistic approximations. MHD equations are considered in section III. Velocity perturbations and electromagnetic perturbations in a magnetized plasma interacting with neutrino background is considered in section IV. In section V we apply our theory to neutron star to calculate the kick and section VI is about summary and conclusions.
II. THE LAGRANGIAN AND NON-RELATIVISTIC APPROXIMATION

Lagrangian density for lepton field interacting with background neutrino is given by,

\[ \mathcal{L} = \bar{\psi} [i\gamma^\mu \partial_\mu \psi - \gamma_\mu (f^\mu_L P_L + f^\mu_R P_R) - m] \psi \]  

(1)

where, \( m \) is mass of the lepton, \( \gamma^\mu = (\gamma^0, \gamma) \) are the Dirac matrices and \( P_{L,R} = \frac{1 + \gamma^5}{2} \) are the chiral projection operator with \( \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \).

\( f^\mu_{L,R} = (f^0_{L,R} \, f_{L,R}) \) are the neutrino currents and they are regarded as an external macroscopic quantities.

An explicit form of \( f^\mu_{L,R} \) can be calculated from effective Lagrangian \( \mathcal{L}_{eff} \) \( [27, 28] \)

\[ \mathcal{L}_{eff} = -\sqrt{2} G_F \sum_\alpha \bar{\nu}_\alpha \gamma^\mu \frac{(1 - \gamma^5)}{2} \nu_\alpha [\bar{\psi} \gamma_\mu (a^0_L P_L + a^0_R P_R) \psi] \]

(2)

where, label \( \alpha \) denotes neutrino species \( \alpha = e, \mu, \tau \) and \( G_F = 1.17 \times 10^{-11} \text{MeV}^{-2} \) is the Fermi constant. The coefficients \( a^0_L \) & \( a^0_R \) are given by

\[ a^0_L = \delta_{\alpha,e} + \sin^2 \theta_W - 1/2, \quad a^0_R = \sin^2 \theta_W \]

(3)

with \( \theta_W \) being the Weinberg angle. Next, we assume that \( \nu \bar{\nu} \) form an isotropic background gas. This in turn means that in averaging over the neutrino ensemble, only non-zero quantity will be \( < \bar{\nu}_\alpha \gamma^0 (1 - \gamma^5) \nu > = 2(n_{\nu_\alpha} - n_{\bar{\nu}_\alpha}) \). Number densities of neutrinos and anti-neutrinos can be calculated using corresponding Fermi-Dirac distribution function

\[ n_{\nu_\alpha, \bar{\nu}_\alpha} = \int \frac{d^3p}{(2\pi)^3 e^{\beta \nu_\alpha (|p| + \nu_\alpha)} + 1} \]

(4)

where \( \beta \) is the inverse temperature. Using Eqns. (1-4) one obtains

\[ f^0_L = 2\sqrt{2} G_F [\Delta n_{\nu_e} + (\sin^2 \theta_W - 1/2) \sum_\alpha \Delta n_{\nu_\alpha}], \]

(5)

\[ f^0_R = 2\sqrt{2} G_F \sin^2 \theta_W \sum_\alpha \Delta n_{\nu_\alpha}. \]

(6)

Thus the equation of motion obtained from Eqn. (1) can be written as

We provide a brief summery of relativistic generalization of the theory in appendix A.
\[
\frac{\partial \psi}{\partial t} = [\alpha \cdot \hat{p} \psi + \beta m - (f^0_L P_L + f^0_R P_R)] \psi. 
\]

(7)

Writing \( \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \) in the Eqn. (7) and following the standard procedure \cite{38}, Hamiltonian for the large component of the spinor can be obtained as,

\[
H = \frac{1}{2m} (\sigma \cdot p)(\sigma \cdot p) + \frac{\Delta f^0}{2m} (\sigma \cdot p) + \frac{f^0}{2} + O(f^2_{L,R})
\]

where, \( f^0 = f^0_L + f^0_R \) and \( \Delta f^0 = f^0_L - f^0_R \). In the above equation, we have neglected terms proportional to \( G_F^2 \). In the presence of external electromagnetic field, momentum \( p \) has to be replaced by \( p - e A \). Thus the Hamiltonian for charged fermion in interacting with an external electromagnetic field and background neutrino is given by,

\[
H = \frac{(p - e A)^2}{2m} - \mu \cdot B + e A^0 + \frac{\Delta f^0}{2m} \sigma \cdot (p - e A) + \frac{f^0}{2}
\]

(8)

where, \( \mu = \frac{e g}{4m} \sigma \) is the electron magnetic moment and \( g \) is the Land` e g-factor. The first three terms on the right hand side are well known and very well studied in the literature. The fourth and fifth terms are due to the neutrino background. The last term might contribute to the energy of the system, but it will not enter into the equations of motion as the neutrino background considered to be constant. If the neutrino background vary with space and time, this term would modify force equation as \( F \propto \nabla f = \nabla \psi^\nu \psi_\nu \). This force is called ponderomotive force. Such a scenario was studied in Ref. \cite{9}, however in their formalism the fourth term was not considered.

In order to find the equation of motion for a charged particle in an electromagnetic field and the neutrino background, one can use Eqn. (8) and the Heisenberg equation \( \dot{O} = i[\hat{H}, \hat{O}] \) and write:

\[
v = \frac{p - e A}{m} + \frac{\Delta f^0}{2m} \sigma
\]

(9)

where we wrote \( \dot{x} = v \).

\[
\dot{p} = \frac{e}{m} (p - e A)_k \nabla A_k + \frac{e g}{m} \nabla (s \cdot B) - e \nabla A^0
\]

(10)

where we have defined \( s = \sigma/2 \) and

\[
\dot{s} = \mu_B (s \times B) - \Delta f^0 (s \times v)
\]

(11)
From the equations \[9\] we get
\[
\ddot{x} = \frac{e}{m} \left[ E + v \times B \right] + \frac{e\Delta f^0}{2m^2} (s \times B) + \frac{eg}{2m^2} \nabla (s \cdot B)
\] (12)

III. THE HYDRODYNAMIC EQUATIONS

In this section we follow the methods developed in Ref. \[36\] to derive the hydrodynamic equations from the quantum Lagrangian for spin half particles. We consider a system of electrons and ions in presence of homogeneous neutrino background. The neutrino background assumed to have a left-right asymmetry. Furthermore we treat electrons as quantum particles and ions as classical particles so that we can neglect the spin dynamics and other quantum effects for ions.

For simplicity, first let us consider the Eqn. \[8\] without the neutrino interaction term. We can decompose the wave function as
\[
\psi_\alpha = \sqrt{n_\alpha} e^{iS_\alpha} \chi_\alpha
\]
where \(n_\alpha\) is the density, \(S_\alpha\) is the phase and \(\chi_\alpha\) is a two component spinor in which the spin-1/2 information is contained. Inserting this decomposition and considering the real and imaginary parts of the resulting equation we get the continuity and momentum conservation equation for the “species \(\alpha\)” as,
\[
\frac{\partial n_\alpha}{\partial t} + \nabla \cdot \left( n_\alpha \mathbf{v}_\alpha \right) = 0
\] (13)

and
\[
m_\alpha \left( \frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \right) \mathbf{v}_\alpha = q_\alpha \left( \mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B} \right) + 2\mu (\nabla \otimes \mathbf{B}) \cdot \mathbf{s}_\alpha - \nabla Q_\alpha - \frac{1}{m_\alpha n_\alpha} \nabla \cdot \left( n_\alpha \Sigma_\alpha \right)
\] (14)

Velocity is defined via \(m_\alpha \mathbf{v}_\alpha = \mathbf{j}_\alpha / \psi^\dagger \psi\), from which we obtain
\[
m_\alpha \mathbf{v}_\alpha = (\nabla S - i\chi_\alpha^\dagger \nabla \chi_\alpha) - q_\alpha \mathbf{A}
\] (15)

and,
\[
\mathbf{s}_\alpha = \frac{1}{2} \chi_\alpha^\dagger \mathbf{\sigma} \chi_\alpha
\] (16)

The quantity \(Q_\alpha\) is known as the quantum potential (Bohm potential) defined as
\[ Q_\alpha = -\frac{1}{2m_\alpha \sqrt{n_\alpha}} \nabla^2 \sqrt{n_\alpha} \]  

and \( \Sigma_\alpha \) is the symmetric spin gradient tensor.

\[ \Sigma_\alpha = \nabla s_{(\alpha) a} \otimes \nabla s_{(\alpha)}^a \]  

where \( a = 1, 2, 3 \). By contracting the Pauli equation with \( \psi^\dagger \sigma \), one can obtain the spin evolution equation as

\[ \frac{\partial}{\partial t} + v_\alpha \cdot \nabla) s_\alpha = 2\mu(s_\alpha \times B) + \frac{s_\alpha \times [\partial_a(n_\alpha \partial^a s_\alpha)]}{m_\alpha n_\alpha} \]  

In presence of neutrino background, the continuity equation remain unchanged. But both momentum conservation and spin evolution equations are modified in the following way.

\[ m_\alpha \left( \frac{\partial}{\partial t} + v_\alpha \cdot \nabla \right) v_\alpha = q_\alpha(E + v_\alpha \times B) + 2\mu(\nabla \otimes B) \cdot s_\alpha - \frac{1}{m_\alpha n_\alpha} \nabla \cdot (n_\alpha \Sigma_\alpha) + \frac{\Delta f^0}{2m} s_\alpha \times B \]  

and

\[ \frac{\partial}{\partial t} + v_\alpha \cdot \nabla) s_\alpha = 2\mu(s_\alpha \times B) - \frac{\Delta f^0}{2} s_\alpha \times v_\alpha + \frac{s_\alpha \times [\partial_a(n_\alpha \partial^a s_\alpha)]}{m_\alpha n_\alpha} \]  

Now, in order to define hydrodynamic quantities, we need to specify how to calculate the expectation values. Suppose that we have \( N \) wave function with same kind of particles with magnetic moment \( \mu \) charge \( q \) and mass \( m \) so that the wave function for the entire system can be factorised as \( \psi = \psi_1(\psi_2...\psi_N) \). Then we can define the total particle density for charge \( q \) as, \( n_q = \sum_\alpha n_\alpha \) and the expectation value of any quantity \( f \) as \( <f> = \sum_\alpha \frac{\psi_{\alpha}^* f \psi_{\alpha}}{n_q} \).

Using these arguments we define the total fluid velocity \( V_q = <v_\alpha> \) and \( S_q = <s_\alpha> \). In order to simplify further calculations, we redefine these quantities such that \( w_\alpha = v_\alpha - V_q \) and \( S_\alpha = s_\alpha - S_q \), satisfying \( <w_\alpha> = 0 \) and \( <S_\alpha> = 0 \). Now taking the ensemble average of the equations (13), (20) and (21) we get the following expressions.

\[ \frac{\partial n_q}{\partial t} + \nabla \cdot (n_q V_q) = 0 \]  

\[ m_q n_q \left( \frac{\partial}{\partial t} + V_q \cdot \nabla \right) V_q = q n_q \left( E + V_q \times B \right) - \nabla \cdot \Pi - \nabla P + C_{qi} + F_Q + F_{ve} \]
\[ n_q \left( \frac{\partial}{\partial t} + V_q \cdot \nabla \right) S_q = 2 \mu_B n_q S_q \times B - \frac{\Delta f^0}{2} S_q \times V_q + \Omega_s - \nabla \cdot K_q + K_{\nu e} \]  

(24)

Where, \( \Pi \) is the traceless anisotropic part of the pressure tensor and \( P \) is the homogeneous part. \( C_{qi} \) represents the collision between particle with charge \( q \) and ion denoted using the letter \( i \) and the quantum force density \( F_Q \) and force due to the interaction with the neutrino background \( F_{\nu e} \) has the definitions,

\[ F_Q = 2 \mu_B n_q \left( \nabla \otimes B \right) \cdot S_q - n_q \left( \nabla Q_\alpha \right) - \frac{1}{m} \nabla \cdot \left( n_q \Sigma \right) - \frac{1}{m} \nabla \cdot \left( n_q \bar{\Sigma} \right) \]  

(25)

and

\[ F_{\nu e} = n_e \epsilon_{ijk} \delta f^0 \left( \nabla \otimes w_{ak} \right) \]  

(26)

The quantities \( \Omega_s, \Sigma \) and \( \bar{\Sigma} \) depends on the spin of the particles and their precise definitions can be found in the Ref. [36]. \( K_q = \left( S_\alpha \otimes w_\alpha \right) \) is the spin thermal coupling and \( K_{\nu e} = \epsilon_{ijk} \frac{\delta f^0}{2} \left( S_{\alpha j} w_{ak} \right) \) is the thermal spin coupling induced by neutrino interaction.

In the following sections, we will replace the subscript \( q \) with \( e \) and \( i \) for electrons and ions respectively. Since we are considering ions as classical particle, we can neglect the contributions from spin and other quantum effects for ions. Thus, the fluid equations for ions read,

\[ \frac{\partial n_i}{\partial t} + \nabla \cdot \left( n_i V_i \right) = 0 \]  

(27)

\[ m_i n_i \left( \frac{\partial}{\partial t} + V_i \cdot \nabla \right) V_i = q_i n_i \left( E + V_i \times B \right) - \nabla \cdot \Pi_i - \nabla P_i + C_{iq} \]  

(28)

Note that there is no spin evolution equation for ions. Therefore whatever spin contributions governing the dynamics of the system are only due to the spin of the electrons. Now we can construct the single fluid equations from the above equations for electrons and ions. In order to do that we define the total mass density, \( \rho = (m_e n_e + m_i n_i) \), the centre of mass velocity of the fluid \( \rho V = (m_e n_e V_e + m_i n_i V_i) \) and the current density \( j = (-en_e V_e + Zn_i V_i) \) and assuming quasi-neutrality \( n_e = Z n_i \), one can immediately obtain the continuity equation,

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0 \]  

(29)

and momentum conservation equation
\[
\rho \left( \frac{\partial}{\partial t} + V \cdot \nabla \right) V = j \times B - \nabla \cdot \Pi - \nabla P + F_Q + F_{ve} \tag{30}
\]

Note that, with the assumption of quasi-neutrality we can write \( n_e = \rho / (m_e + m_i) \) and \( V_e = V - m_i j / Ze \rho \). Therefore we can express the quantum terms in terms of the total density and, centre of mass velocity of the fluid and current. Thus the spin evolution equation becomes,

\[
\rho \left( \frac{\partial}{\partial t} + V \cdot \nabla \right) S = \frac{m_i}{Ze \rho} j \cdot \nabla S + 2 \mu \rho S \times B - \left( m_e + m_i / Z \right) \nabla \cdot K_e + \left( m_e + m_i / Z \right) \Omega_s \tag{31}
\]

in general, for a magnetized medium with magnetization density \( \mathbf{M} \) we can write the free current density \( j = \nabla \times B - \mathbf{J}_M \), where \( \mathbf{J}_M = \nabla \times \mathbf{M} \) is the magnetization current density. Note that, here we have discarded the displacement current term \( \partial E / \partial t \).

In order to simplify the further calculation, we consider only the transverse waves in that case the Bohm potential i.e. \( < Q_\alpha > \) term in Eqn. (25) can be dropped \[44\]. Further all the other terms in Eqn. (25) are second order in the spin variable and of order \( \hbar^2 \). We neglect these terms. However, \( F_{ve} \) term in Eqn. (23) and the spin dynamic Eqn. (24) are order \( \hbar \). These terms are retained in the calculation. In such a situation can write the total force density exerted on the fluid element as,

\[
F^i = -\partial^i \left( \frac{B_0^2}{2} - \mathbf{B} \cdot \mathbf{M} \right) + \mathbf{B} \cdot \nabla H^i - \partial^i P - \partial^i \Pi^{ij} \tag{32}
\]

For an isotropic plasma, the trace-free part of the pressure tensor \( \Pi^{ij} \) is zero. It is worth noting that the spacial part of the stress tensor take the form \( T^{ij} = -H^i B^j + \left( B^2 / 2 - \mathbf{B} \cdot \mathbf{M} \right) \delta^{ij} \) \[39\], apart from the pressure terms. Thus the total force density on a magnetized fluid element can be written as, \( F^i = -\partial^i T^{ij} \). Therefore the momentum conservation equation takes the form,

\[
\rho \left( \frac{\partial}{\partial t} + V \cdot \nabla \right) V = -\nabla\left( \frac{B_0^2}{2} - \mathbf{B} \cdot \mathbf{M} \right) + \left( \mathbf{B} \cdot \nabla \right) H - \nabla P - \nabla \cdot \Pi \tag{33}
\]

Following the procedure in Ref. \[40\] we can write,
\[ j \sim \frac{\sigma m_i}{\rho e} \nabla P + \sigma (E + V \times B) + \frac{\sigma m_i}{\rho e} j \times B + \frac{\sigma m_i}{\rho e} F_Q - n_0 \frac{\sigma m_i}{\rho} \left( \frac{\Delta f^0}{2 m_e} \right) S \times B \]  

(34)

Taking \( \rho \sim n_0 m_i \) the expression for total current can be written as,

\[ j \sim \sigma \left( E + V \times B \right) - \sigma \left( \frac{\Delta f^0}{2 m_e} \right) S \times B + j_M \]  

(35)

For the above expression for the hydrodynamic current, the time evolution for the magnetic field \( B \) is given by

\[ \frac{\partial B}{\partial t} = -\eta \nabla \times \left( \nabla \times B \right) + \nabla \times \left( V \times B \right) - \left( \frac{\Delta f^0}{2 m_e} \right) \nabla \times \left( S \times B \right) + \eta \nabla \times j_M \]  

(36)

Where, \( \eta = 1/\sigma \) is the resistivity.

**IV. NEUTRINO INDUCED VORTICITY, ALFVÉN WAVE AND NORMAL MODES**

In this section, we consider a very simple scenario. A background magnetic field \( B_0 = B_0 \hat{z} \) is applied to the plasma. As a result, there is a non-zero constant magnetization in the system even in the absence of any perturbations, which also imply that \( S \times B = 0 \) for the plasma at equilibrium. In this case the spin of the electrons align anti-parallel to the magnetic field to reduce the energy and therefore we can assume the equilibrium magnetization density \( M_0 \) to take the form \( M_0 = -\mu_B n_e S_0 = \mu_B n_e \xi \left( \frac{\mu_B B_0}{T_e} \right) \hat{z} \), where \( \xi(x) = \tanh(x) \) is the Brillouin function. For the following discussions we make the approximation \( \xi \left( \frac{\mu_B B_0}{T_e} \right) \sim \left( \frac{\mu_B B_0}{T_e} \right) \) so that \( S_0 \sim -\frac{1}{2} \frac{\mu_B B_0}{T_e} \hat{z} \). Furthermore we assume that there are no electromagnetic perturbations in the system and the fluid velocity enters into the governing equations as perturbation. That is, \( E = 0, B = B_0 \hat{z}, V = \delta V \) and \( S = S_0 + \delta S \). With these assumptions, up to linear order in perturbations, we use the hydrodynamic equations in the following form.

\[ \frac{\partial \delta S}{\partial t} = 2 \mu_B \delta S \times B_0 - \left( \frac{\Delta f^0}{2 m_e} \right) S_0 \times \delta V \]  

(37)

\[ \rho_0 \frac{\partial \delta V}{\partial t} = -\frac{\mu_B n_e}{T} \Delta f^0 \nabla (B_0 \cdot \delta V) + \frac{\mu_B n_e}{T} \Delta f^0 (B_0 \cdot \nabla) \delta V \]  

(38)
From the equation (37) we get \( \frac{\partial S}{\partial t} \cdot B_0 = 0 \). In order to satisfy this conditions, we choose \( \delta S \cdot B_0 = 0 \). We also take the space-time dependence of the perturbations to be of the following form,

\[
\delta V(t, x) = \delta V_{\omega, k} e^{-i(\omega t - k \cdot x)}, \quad \delta S(t, x) = \delta S_{\omega, k} e^{-i(\omega t - k \cdot x)}
\]  

(39)

With these assumptions we get,

\[
\delta S_{\omega, k} = -\left( \frac{\Delta f_0}{2\Omega_e} \right) |S_0| \delta V_{\omega, k}
\]

(40)

Where, \( \Omega_e = \frac{eB_0}{m_e} \). To obtain the above expression we have assumed that \( \frac{\omega^2}{\Omega_e^2} \ll 1 \). Thus, from the equations (38), and (40), we get

\[
-i\omega \delta V_{\omega, k} = \left( \frac{\mu B n_e}{T} \Delta f_0 \right) \Omega_{\omega, k} \times B_0
\]

(41)

Where, \( \Omega_{\omega, k} = i k \times \delta V_{\omega, k} \) is the vorticity in the Fourier space. Note that, in the above expressions, we have kept the terms only up to linear in \( \Delta f_0 \). From the above expression we can see that vorticity term will not contribute to the fluid dynamics if \( \Delta f_0 = 0 \). Therefore we conclude that \( \Omega \) is induced via the electron neutrino interaction. From the Eqn. (41) we can obtain the dispersion relations. For the case \( k \parallel B_0 \),

\[
\omega = -\left( \frac{\mu B n_e B_0}{\rho_0} \right) \left( \frac{\Delta f_0}{T} \right) k
\]

(42)

Group velocity of this new mode is given by,

\[
v_g = \frac{d\omega}{dk} = \left| \left( \frac{\mu B n_e B_0}{\rho_0} \right) \left( \frac{\Delta f_0}{T} \right) \right|
\]

\[
\sim 2\sqrt{2} \left( \frac{\mu B n_e B_0}{\rho_0} \right) \left( \frac{G_F}{T} \right) \left| n_{\nu e} - n_{\bar{\nu} e} \right|
\]

(43)

(44)

The Eqn. (42) corresponds to a new type of transverse mode propagating in the direction parallel to the background magnetic field, induced by asymmetry in the neutrino background. The wave velocity not just depend on the strength of magnetic field but also on the neutrino asymmetry. This new mode is similar to the one found in very high energy plasma with the chiral-anomaly \[32\]. In contrast to Ref. \[32\], in our work the electrons are not considered to be chirally polarized. However, the parity violating interaction in our work arises due to neutrino-electron interaction. Further, the effect of dissipation can easily be introduced by incorporating contribution of the finite shear viscosity \(-ik^2 \eta_{vis}\) and the resistivity \(-i\sigma_1 B_0^2\)
into the dispersion relation (42), where \( \eta_{vis} \) is the kinematic viscosity and \( \sigma_1 = \sigma/\rho_0 \) with \( \sigma \) being the resistivity.

Next, we consider the effect of electro-magnetic perturbations. That is we take the perturbations in the following form.

\[
V = \delta V, \quad B = B_0 \hat{z} + \delta B, \quad E = \delta E.
\]

(45)

For this case, linearized hydrodynamic equations, Eqn.(29) and (33) takes the form,

\[
\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \delta V = 0
\]

\[
\rho_0 \frac{\partial \delta V}{\partial t} = -\nabla \left( B_0 \cdot \delta B - M_0 \cdot \delta M \cdot B_0 \right) + B_0 \cdot \nabla \delta H - \nabla P
\]

(47)

And the spin evolution equation becomes,

\[
\left( \frac{\partial S}{\partial t} \right) = 2 \mu_B B \cdot \delta V
\]

(48)

Where, \( S = S_0 + \delta S \). For a perfectly conducting medium (\( \eta \to 0 \)), the equation (36) becomes,

\[
\frac{\partial \delta B}{\partial t} = \nabla \times \left( \delta V \times B_0 \right) - \left( \frac{\Delta f^0}{2m_e} \right) \nabla \times \left( \delta S \times B_0 + S_0 \times \delta B \right)
\]

(49)

Following the same procedure in the last section with same assumptions, we get the expression for \( \delta S \) as,

\[
\delta S_{\omega,k} = \frac{\mu_B}{T} \delta B_{\omega,k} - \frac{\Delta f^0}{T} \delta V_{\omega,k}
\]

(50)

Where, \( \omega_p^2 \) is the plasma frequency. Using the equations (46), (47) and (50) and using the approximation \( M_0 = -\mu_B n_e S_0 = \mu_B n_e \eta \left( \frac{\mu_B B_0}{m_e} \right) \hat{z} \) we get,

\[
- i \omega \rho_0 \delta V_{\omega,k} = i \left[ \frac{\omega_p^2}{m_e T} \right] B_0 \times \left( k \times \delta B_{k,\omega} \right) - \left[ \frac{\mu_B n_e}{T} \Delta f^0 \right] B_0 \times \Omega_{k,\omega}
\]

(51)

Where, \( \Omega_{k,\omega} = i k \times \delta V \) is the vorticity in the Fourier space. We can see that the last term in the Eqn.(51) is proportional to the neutrino asymmetry of the background. Expression for velocity in the Fourier space as,

\[
\delta V_{k,\omega} = \left( \frac{B_0 \cdot k \left( \frac{\omega_p^2}{m_e T} \right) - 1}{\omega \rho_0 T} \right) \left( \frac{1}{\omega} \Delta f^0 \right) \left( B_0 \cdot \delta V_{\omega,k} \right) - \left( B_0 \cdot k \right) \delta V_{\omega,k}
\]

(52)

Note that, we have neglected the contributions from the pressure terms in the above expression. Taking \( k \) in the direction of background magnetic field and assuming \( B_0 \cdot \delta V = 0 \), we get the following dispersion relation,

\[
\omega = -\frac{\tilde{v}_A}{\sqrt{\rho_0 \alpha}} \frac{\mu_B n_e}{2T} \Delta f^0 k + \tilde{v}_A k
\]

(53)
where, \( \alpha = (1 - \frac{\omega^2}{m_e^2}) \) and \( \tilde{v}_A = v_A \alpha^{1/2} \) is the spin-modified Alfvén velocity \[36\]. Here we note that the quantity \( \alpha \) describes the spin corrections and in the absence of spin dynamics \( \alpha = 1 \). It is clear from the Eqn. \[53\] that the group velocity \( v_g \) can have two values given by,

\[
v_g^\pm = \tilde{v}_A \left| 1 \pm \frac{1}{\sqrt{\rho_0 \alpha}} \frac{\mu_B n_e}{2T} \Delta f^0 \right|
\]

which is absent in the absence of any neutrino asymmetry \( (\Delta f^0 = 0) \). Thus we can have two different group velocities for the Alfvén waves propagating parallel or anti-parallel to \( B_0 \).

For finite value of conductivity, we have to take into account of the first and last terms of the Eqn.(36) and the dispersion relation can be obtained from,

\[
\omega^2 + \omega \left[ i \alpha \eta k^2 + \frac{\mu_B n_e}{T \sqrt{\rho_0 \alpha}} \Delta f^0 \tilde{v}_A k \right] - \tilde{v}_A^2 k^2 = 0
\]

Solving for \( \omega \) we get,

\[
\omega = -\frac{1}{2} \left[ i \alpha \eta k^2 + \frac{\mu_B n_e}{T \sqrt{\rho_0 \alpha}} \Delta f^0 k \right] \pm \frac{1}{2} \sqrt{\left[ i \alpha \eta k^2 + \frac{\mu_B n_e}{T \sqrt{\rho_0 \alpha}} \Delta f^0 k \right]^2 + 4 \tilde{v}_A^2 k^2}
\]

We can see that in the absence of any neutrino asymmetry and \( \eta \), the Eqn.\[56\] reduces to \( \omega^2 = \tilde{v}_A^2 k^2 \), which is the same in magneto hydrodynamics with spin corrections as obtained in \[36\].

V. NEUTRINO ASYMMETRY AND THE PULSAR KICK

We use our formalism for a qualitative calculation of observed pulsar kick \[45,47\]. There are several attempts to explain the reason for the kick, for eg. see the references \[48,51\]. Recently there have been attempts to explain the pulsar kick using anomalous hydrodynamic theories (for eg. see Ref.\[52\]), but the exact reason for the pulsar kick is not yet resolved.

We note that, the energy flux associated with the wave is equal to the energy density in the wave times the group velocity \[53\], which is the Poynting vector \( P = E \times B \) in our case \[53,54\]. The Poynting vector can be expressed in the form,

\[
P = (\omega A^2) k
\]

where \( A \) is the magnitude of the vector potential \( A_{\omega,k} \). Using the Eqn. \[53\] we write,
\begin{align}
|P| &= k^2 A^2 \tilde{v}_A \left( 1 \pm \frac{1}{\sqrt{\rho_0 \alpha}} \frac{\mu B n_e}{2I} \Delta f^0 \right) \\
&= \left( k^2 A^2 \right) v_g \tag{58}
\end{align}

From Eqn. \((59)\), we infer that the energy density associated with the wave is \(k^2 A^2\).\n
Further we note from the Eqn. \((58)\) that the energy transported in the direction of the background field \(B_0\) and opposite to \(B_0\) are different due to the parity violation within the system. An excess amount of energy is transported in the direction of magnetic field. This excess amount of energy transported per unit area per unit time is given by,

\[
\Delta P = \left( k^2 A^2 \tilde{v}_A \right) \left( \frac{\Delta f^0}{T} \frac{\mu B n_e}{\sqrt{\rho_0 \alpha}} \right) \tag{60}
\]

Which is essentially the momentum carried by the excess photons leaving the pulsar per unit area per unit time. Therefore the change in velocity experienced by the pulsar can be expressed as,

\[
\Delta V_{\text{NS}} = \frac{\Delta P}{M_{\text{NS}}} \times \Delta t \times (\text{area}) \tag{61}
\]

Where \(M_{\text{NS}} \sim 10^{30} \text{ kg}\) is the mass of neutron star and \(\Delta t\) is the time span we assume for the kick to last, which is approximately 10 seconds. The radius of the neutron star \(R_{\text{NS}}\) is approximately 10 km. Taking \(k \sim A \sim T\), \(\Delta n_{\nu e} \sim 1.6 \times 10^8 \text{ (MeV)}^3\), \(T \sim 10^{12} \text{ K}\) and \(B_0 \sim (10^{15} - 10^{16}) \text{ Gauss}\), we get \(\Delta V_{\text{NS}} \sim (10^2 - 10^3) \text{ km/s}\), which is within the order of magnitude of observed pulsar kicks.

\textbf{VI. DISCUSSION AND CONCLUSION}

In conclusion we have developed spin magnetohydrodynamic equations in the presence of asymmetric background neutrinos and analysed the normal modes of the plasma in presence of a constant magnetic field. We have shown that a new kind of wave (Alfvén) is generated Eqn\(42\) can exist whose velocity depends on the neutrino asymmetry. Such a wave can be generated in a dense astrophysical plasma such as magnetar. For example for \(B_0^{15} \text{ Gauss},\ T \sim 10 \text{MeV}\) and \(\Delta n_{\nu e} \sim 1.6 \times 10^8 \text{(MeV)}^3\), one can estimate the velocity of the wave (in units of speed of light) around \(10^{-5}\). Similarly for the Alfvén waves (Eqn. \((52)\) ) can have
two different speeds. We have shown that the background neutrino asymmetry can change the wave-velocity in directions parallel and anti-parallel to the external magnetic field (as shown in Eqn. (54)). We have used our formalism to calculate the kick received by pulsar during its birth. An order of magnitude calculation matches with the observations $\Delta V_{NS} \simeq (10^2 - 10^3) \text{ km/s}$. In the appendix we have derived the relativistic hydrodynamical equation for the electrons using Dirac equation. For the case when the electrons are relativistic the estimate given here for the Alfvén velocity can be suppressed by a factor $1/\sqrt{2}$.

Appendix A

In the many astrophysical situations it is necessary to consider the system temperature to be greater than its rest-mass and therefore we discuss relativistic generalization of the electron fluid. For such a generalization in the context of quantum plasma one needs to start with the Dirac equation. Works by Pauli [56], Harish Chandra [57] and T. Takabayasi [58] have shown that Dirac equation can be cast into hydrodynamical form. Here we use the methodology similar to that given in [58] (see also [59]) to describe the fluid equations for relativistic electrons in presence of the asymmetric neutrino background. In the standard MHD-approximation electron contributes in defining the current whereas the ion provides the inertia and therefore significantly contributes to the fluid velocity [40, 36]. In this appendix we first derive electron fluid equations from the Dirac equation and then carry-out the MHD approximations with the non-relativistic ion fluid and obtain an expression for the relativistic corrections to the MHD current and finally discuss the changes this brings about the our (non-relativistic) results on the Alfvén waves. The subsequent derivation is rather lengthy and involved we would like to refer the readers to Ref. [58] for further details.

Following [58] we start with writing the bilinear covariants with hydrodynamic variables and establishing the relations among them using properties of the gamma matrices. And also establishing their evolution equations from moments of the corresponding Dirac equation.
We choose following bilinear covariants.

\[ \Omega = \bar{\psi} \psi \]  
\[ \bar{\Omega} = i \bar{\psi} \gamma^5 \psi \]  
\[ S^\mu = \bar{\psi} \gamma^\mu \psi \]  
\[ \bar{S}^\mu = \bar{\psi} \gamma^5 \gamma^\mu \psi \]  
\[ M^{\mu\nu} = \bar{\psi} \sigma^{\mu\nu} \psi \]  
\[ \bar{M}^{\mu\nu} = i \bar{\psi} \gamma^5 \sigma^{\mu\nu} \psi \]  

Where, \( \sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu] \). One can obtain the equations of motion for \( \psi \) and \( \bar{\psi} \) from Eqn.(1) and using these equations of motion one can write following two generic equations involving the dynamics of the above bilinear forms:

\[ i \left( \bar{\psi} \gamma^A \gamma^\mu \partial_\mu \psi + \partial_\mu \bar{\psi} \gamma^\mu \gamma^A \psi \right) - eA_\mu \bar{\psi} \{ \gamma^A, \gamma^\mu \} \psi - \frac{\Delta f_\mu}{2} \bar{\psi} \{ \gamma^A, \gamma^\mu \gamma^5 \} \psi = 0 \]  
\[ i \left( \bar{\psi} \gamma^A \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \gamma^A \psi \right) - eA_\mu \bar{\psi} \{ \gamma^A, \gamma^\mu \} \psi - \frac{\Delta f_\mu}{2} \bar{\psi} \{ \gamma^A, \gamma^\mu \gamma^5 \} \psi - 2m \bar{\psi} \gamma^A \psi = 0 \]  

Using the definition for covariant differential operator \( \delta^*_\mu (\bar{\psi} \gamma^A \psi) = i (\bar{\psi} \gamma^A \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^A \psi) - 2eA_\mu \bar{\psi} \gamma^A \psi \) we define,

\[ j_\mu = (1/2m)\delta^*_\mu \Omega \]  
\[ \bar{j}_\mu = (1/2m)\delta^*_\mu \bar{\Omega} \]  
\[ T_\mu^\nu = (1/2m)\delta^*_\mu S^{\nu} \]  
\[ \bar{T}_\mu^\nu = (1/2m)\delta^*_\mu \bar{S}^{\nu} \]  
\[ N^{\mu\nu}_\alpha = (1/2m)\delta^*_\alpha M^{\mu\nu} \]  
\[ \bar{N}^{\mu\nu}_\alpha = (1/2m)\delta^*_\alpha \bar{M}^{\mu\nu} \]  

The quantities \( M^{\mu\nu} \) and \( \bar{M}^{\mu\nu} \) can be expressed as \( \rho^2 M^{\mu\nu} = -\bar{\Omega} (S^\mu \bar{S}^\nu - S^\nu \bar{S}^\mu) + \Omega \epsilon^{\mu\nu\kappa\lambda} S^\kappa \bar{S}^\lambda \) and \( \bar{M}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\kappa\lambda} M_{\kappa\lambda} \), where \( \rho = \sqrt{\Omega^2 + \bar{\Omega}^2} \) has the interpretation of density. From Eqns.(A7) and (A8) we obtain the evolution equation of the above defined quantities.

\[ \partial_\mu S^{\nu} = 0 \]  
\[ \partial_\mu \bar{S}^{\nu} = -2m \bar{\Omega} \]  

\[ (1/2m)\partial_\nu M^{\mu\nu} + j^\mu - S^{\mu} + \frac{\Delta f_\nu}{2m} \bar{M}^{\mu\nu} = 0 \]  
\[ (1/2m)\partial_\nu \bar{M}^{\mu\nu} + \bar{j}^\mu - \frac{\Delta f_\nu}{2m} M^{\mu\nu} = 0 \]
Next, one defines four-velocity $v_\mu = S_\mu / \rho$ and four-spin $w_\mu = \bar{S}_\mu / \rho$ in such a way that it satisfies the following constraints: $v^\mu v_\mu = 1$, $w^\mu w_\mu = -1$ and $v^\mu W_\mu = 0$. From the last constraint, it is clear that $w_0 = v \cdot w / u^0$ and thus in the rest-frame zeroth component of the four spin $w_0 = 0$. By taking the divergence of $\bar{T}^{\mu\nu}$ one obtains the following equation

$$\partial_\nu \bar{T}^{\mu\nu} = \frac{e}{m} w_\nu F^{\mu\nu} - \bar{j}_st^\mu + \delta f_\nu \left[ -\rho \sin \theta \left( v^\mu w_\nu - v^\nu w_\mu \right) + \rho \cos \theta \epsilon^{\mu\nu\kappa\lambda} v_\kappa w_\lambda \right]$$

(A19)

where we have used $M^{\mu\nu} = \left[ -\rho \sin \theta \left( v^\mu w_\nu - v^\nu w_\mu \right) + \rho \cos \theta \epsilon^{\mu\nu\kappa\lambda} v_\kappa w_\lambda \right]$ following [58] and $\bar{j}_st^\mu$ has the same standard expression as given in Refs. [58] or [59]. Besides we have used new definitions: $\cos \theta = \Omega / \rho$ and $\sin \theta = \bar{\Omega} / \rho$. Here we note that $f^\mu$ term for the neutrino current does not appear in the above equation. Equation (A19) is at the single body particle-antiparticle state level and one is required to take the fluid average for a collection of $N$ such states. This $N$ particle spinor must be written as a $4^N \times 4^N$ Slatter determinant of $N$ one-particle states this procedure had been developed in Ref. [59] and we follow it up for our calculation. We find following equation, in thermal equilibrium, for the spatial part of the spin dynamic:

$$\gamma \left( \partial_t + v \cdot \nabla \right) S = \frac{e/m}{< \cos \theta >} \left( W^0 E/2 + S \times B \right) - \gamma \Delta f^0 S \times v + \gamma \Delta f^0 < \sin \theta > < \cos \theta > S$$

(A20)

where, $\gamma = \frac{1}{\sqrt{1 - v^2}}$ and $W^0 = S \cdot v$. Here we note that as we have assumed before we have dropped the spin-thermal coupling and the non-linear spin terms. The last two terms on the right hand side gives an additional contribution to the spin dynamics of the electron-fluid dynamics given in Ref. [59]. This additional term solely depends on the neutrino background as it should be the case. Following electron relativistic hydrodynamical model in Ref. [59], we we regard $\theta$ as a constant parameter which is zero for the non-relativistic quantum case and for an extreme relativistic quantum case $\theta = \pi / 4$. For the non-relativistic spin dynamics (Eqn.(24)) can be reproduced when we take $\theta = 0$. When the electrons are at relativistic temperature one can replace $mn$ by $(\epsilon + p)$ i.e. by enthalpy density [60].

Now one can define the total mass density $\rho = (\epsilon + p) + m_i n_i$ where, $(\epsilon + p)$ represents the enthalpy density of the electrons. Since in the magnetohydrodynamic equations the inertia of the fluid is dominated by ions, the momentum of the fluid is dominated by the ion momenta [40] (see also [36]). This remains true for the relativistic electron case also provided $\rho \sim m_i n_i$ and Eqn.(33) remains valid for us. Next we derive the analogue of Eqn.(42) when the electrons are relativistic. For this consider that there is an external
magnetic field $B_0$ in $z$–direction and there is no streaming of fluid $v_0 = 0$. Also there is no electromagnetic perturbations i.e. $\delta E, \delta B = 0$. The background spin vector is anti-parallel to the external magnetic field and given by $S_0 = -\mu_B B_0/2T$ as considered before. Next one can eliminate the electron velocity in the spin equation $v_e = v - m_i j/(Ze\rho)$. Since there is no electromagnetic perturbation for this case $j = 0$ and one can obtain using Eqs. (33) one obtains the following dispersion relation:

$$\omega = -\left(\frac{\mu_B n_e}{\sqrt{\rho_0}}\right) \left(\frac{\Delta f^0}{2T} <\cos \theta>\right) k v_A$$

(A21)

where, $v_A = \frac{B_0}{\sqrt{\rho_0}}$. Here we note here that when then the last term in Eqn. (A20) does not contribute to significantly to the dispersion relation.

Similarly for the electromagnetic perturbation for the standard Alfvén waves one obtains the following dispersion relation:

$$\omega = -\frac{\bar{v}_A}{\sqrt{\rho_0\alpha}} \frac{\mu_B n_e}{2T} \Delta f^0 <\cos \theta> k \pm \bar{v}_A$$

(A22)

Here we note that both the new Alfvén waves(Eqn. (A21)) and the regular Alfvén waves (Eqn. (A22)), in the ideal MHD limit, gets correction due to the relativistic effect which is characterized by the $<\cos \theta>$ factors. Now for an ultra relativistic limit if one takes $\theta = \pi/4$ following Ref.[59], one gets $1/\sqrt{2}$ factor suppression in the speed of the new Alfvén wave compared to the non-relativistic case (with $\theta = 0$) case.

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