Fair Algorithms for Multi-Agent Multi-Armed Bandits

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Abstract

We propose a multi-agent variant of the classical multi-armed bandit problem, in which there are $N$ agents and $K$ arms, and pulling an arm generates a (possibly different) stochastic reward to each agent. Unlike the classical multi-armed bandit problem, the goal is not to learn the “best arm”, as each agent may perceive a different arm as best for her. Instead, we seek to learn a fair distribution over arms. Drawing on a long line of research in economics and computer science, we use the Nash social welfare as our notion of fairness. We design multi-agent variants of three classic multi-armed bandit algorithms, and show that they achieve sublinear regret, now measured in terms of the Nash social welfare.

1 Introduction

In the classic (stochastic) multi-armed bandit (MAB) problem, a principal has access to $K$ arms, and pulling an arm $j$ generates a stochastic reward from an unknown distribution with an unknown mean $\mu_j^*$. If these mean rewards were known a priori, the principal could just pull a best arm, $\arg\max_j \mu_j^*$, every time. However, the principal has no prior information about the quality of the arms. Hence, she uses a learning algorithm which pulls an arm $j'$ in round $t$, observes the stochastic reward generated, and uses that information to learn the best arm over time. The performance of such an algorithm is measured in terms of its cumulative regret up to a horizon $T$, which is the total difference between the best mean reward and the mean reward of the arms pulled by round $T$, i.e., $\sum_{t=1}^{T}(\max_j \mu_j^* - \mu_{j'}^*)$.

This problem can model a situation where the principal is deliberating a policy decision, and the arms are the different alternatives she can implement. However, in many real-life scenarios, making a policy decision affects not one, but several agents. For example, imagine a company making a decision that affects all its employees, or a conference deciding the structure of its review process, which affects various research communities. This can be modeled by a multi-agent variant of the multi-armed bandit (MA-MAB) problem, in which there are $N$ agents, and pulling an arm $j$ generates a stochastic reward to each agent $i$ from an unknown distribution with an unknown mean $\mu_{i,j}^*$.

Before pondering about learning the “best arm”, we must ask what the best arm even means in this context. Indeed, the “best arm” for one agent may not be the best for another. A first attempt may be to associate some “aggregate quality” to each arm: for example, the quality of arm $j$ may be defined as the total mean reward it gives to all agents, i.e., $\sum_i \mu_{i,j}^*$. This would nicely reduce our problem to the classic multi-armed bandit problem, for which we have an armory of available solutions [1]. However, this approach

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We consider three classic algorithms for the multi-armed bandit problem: Explore-First, Epsilon-Greedy, and UCB [1]. All three algorithms attempt to balance exploration (pulling arms only to learn their rewards) and exploitation (using the information learned so far to pull “good” arms). Explore-First performs exploration followed by exploitation to achieve regret bounds. Here, \( \tilde{O} \) hides log factors, and we note that the focus is on the exponent of \( T \) rather than on the exponent of \( K \), as the horizon \( T \) is often large while the number of arms \( K \) is constant.

We propose natural multi-agent variants of the three algorithms, which take the Nash social welfare objective into account and select a distribution over arms in each round rather than a single arm. For Explore-First, we derive \( \tilde{O}(N^{2/3}K^{1/3}T^{2/3}) \) regret bound, which recovers the aforementioned single-agent bound with an additional factor of \( N^{2/3} \). We also show that changing a parameter of the algorithm yields a different regret bound, which reduces \( N^{2/3} \) to \( N^{1/3} \) at the expense of increasing \( K^{1/3} \) to \( K^{2/3} \), thus offering a tradeoff to the principal. We derive the same bounds for Epsilon-Greedy, although through a much more intricate analysis. Finally, for UCB we derive \( \tilde{O}(NKT^{1/2}) \) and \( \tilde{O}(N^{1/2}K^{3/2}T^{1/2}) \) regret bounds; while we recover the \( \sqrt{T} \) factor as in the single-agent case, the dependence on \( K \) worsens in our bounds. That said, we cannot hope to improve the dependence on \( T \) further because the \( \sqrt{T} \) factor is known to be optimal even in the single-agent case [1].
Deriving these regret bounds for the multi-agent case requires overcoming two key difficulties that do not appear in the single-agent case. First, our goal is to optimize a complex function, the Nash social welfare, rather than simply selecting the best arm. This requires a Lipschitz-continuity analysis of the Nash social welfare function and the use of new tools such as the McDiarmid’s inequality which are not needed in the standard analysis. Second, the optimization is over an infinite space (the set of distributions over arms) rather than over a finite space (the set of arms). Thus, certain tricks such as a simple union bound no longer work; we use the concept of $\delta$-covering, used heavily in the Lipschitz bandit framework [18], in order to address this.

1.2 Related Work

Since the multi-armed bandit problem was introduced by Thompson [19], many variants of it have been proposed, such as sleeping bandit [20], contextual bandit [21], dueling bandit [22], Lipschitz bandit [18], etc. However, all these variants involve a single agent who is affected by the decisions. We note that other multi-agent variants of the multi-armed bandit problem have been explored recently [23, 24]; however, their focus is on getting the agents to cooperate to solve the classic multi-armed bandit problem to maximize the reward to a single principal.

Another key aspect of our framework is the focus on fairness. Recently, several papers have focused on fairness in the multi-armed bandit problem. For instance, Joseph et al. [25] design a UCB variant which guarantees what they refer to as meritocratic fairness to the arms, i.e., that a worse arm is never preferred to a better arm regardless of the algorithm’s confidence intervals for them. Liu et al. [26] require that similar arms be treated similarly, i.e., two arms with similar mean rewards be selected with similar probabilities. Gillen et al. [27] focus on satisfying fairness with respect to an unknown fairness metric. And Patil et al. [28] assume that there are external constraints requiring that each arm be pulled in at least a certain fraction of the rounds, and design algorithms that achieve low regret subject to this constraint. However, all these papers seek to achieve fairness with respect to the arms. In contrast, in our work, the arms are inanimate (e.g. policy decisions), and we seek fairness with respect to the agents, who are separate from the arms.

From a computational social choice theoretic perspective, maximizing the Nash social welfare when the rewards are known upfront is well-studied [7, 29, 30]. Our work can be viewed as performing Nash social welfare maximization when noisy information can be queried regarding the rewards.

2 Preliminaries

For $n \in \mathbb{N}$, define $[n] = \{1, \ldots, n\}$. Let $N, K \in \mathbb{N}$. In the multi-agent multi-armed bandit (MA-MAB) problem, there is a set of agents $[N]$ and a set of arms $[K]$. For each agent $i \in [N]$ and arm $j \in [K]$, there is a reward distribution $D_{i,j}$ with mean $\mu_{i,j}$ and support $[0, 1]$; when arm $j$ is pulled, each agent $i$ observes an independent reward sampled from $D_{i,j}$. Let us refer to $\mu^* = (\mu^*_{i,j})_{i \in [N], j \in [K]} \in [0, 1]^{N \times K}$ as the (true) reward matrix.

Policies: As mentioned in the introduction, pulling an arm deterministically may be favorable to one agent, but disastrous to another. Hence, we are interested in probability distributions over arms, which we refer to as policies. The $K$-simplex, denoted $\Delta^K$, is the set of all policies. For a policy $p \in \Delta^K$, $p_j$ denotes the probability with which arm $j$ is pulled. Note that due to linearity of expectation, the expected reward to agent $i$ under policy $p$ is $\sum_{j=1}^K p_j \cdot \mu^*_{i,j}$.

Nash social welfare: The Nash social welfare is defined as the product of (expected) rewards to the agents. Given $\mu = (\mu_{i,j})_{i \in [N], j \in [K]}$, and policy $p \in \Delta^K$, define $\text{NSW}(p, \mu) = \prod_{i=1}^N \left( \sum_{j=1}^K p_j \cdot \mu_{i,j} \right)$. Thus, the (true) Nash social welfare under policy $p$ is $\text{NSW}(p, \mu^*)$. Hence, if we knew $\mu^*$, we would pick an optimal policy $p^* \in \arg \max_{p \in \Delta^K} \text{NSW}(p, \mu^*)$. However, because we do not know $\mu^*$ in advance, our algorithms will often

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1We need the support of the distribution to be non-negative and bounded, but the upper bound of 1 is without loss of generality. All our bounds scale linearly with the upper bound on the support.
produce an estimate \( \hat{\mu} \), and use it to choose a policy; the quantity \( \text{NSW}(p, \hat{\mu}) \) will play a key role in our algorithms and their analysis.

**Algorithms:** An algorithm for the MA-MAB problem chooses a policy \( p^t \) in each round \( t \in \mathbb{N} \). Then, an arm \( a^t \) is sampled according to policy \( p^t \), and for each agent \( i \in [N] \), a reward \( X^t_{i,a^t} \) is sampled independently from distribution \( D_{i,a^t} \). At the end of round \( t \), the algorithm learns the sampled arm \( a^t \) and the reward vector \( (X^t_{i,a^t})_{i \in [N]} \), which it can use to choose policies in the later rounds.

**Reward estimates:** All our algorithms maintain an estimate of the mean reward matrix \( \mu^* \) at every round. For round \( t \) and arm \( j \in [K] \), let \( n^t_j = \sum_{s=1}^{t-1} \mathbb{1}[a^s = j] \) denote the number of times arm \( j \) is pulled at the beginning of round \( t \), and let \( \hat{\mu}^t_{i,j} = \frac{1}{n^t_j} \sum_{s \in [t-1]: a^s = j} X^s_{i,j} \) denote the average reward experienced by agent \( i \) from the \( n^t_j \) pulls of arm \( j \) thus far. Our algorithms treat these as an estimate of \( \mu^*_{i,j} \) available at the beginning of round \( t \). Let \( \hat{\mu}^t = (\hat{\mu}^t_{i,j})_{i \in [N], j \in [K]} \).

**Regret:** Recall that \( p^* \) is an optimal policy that has the highest Nash social welfare. The instantaneous regret in round \( t \) due to an algorithm choosing \( p^t \) is \( r^t = \text{NSW}(p^*, \mu^*) - \text{NSW}(p^t, \mu^*) \). The (cumulative) regret in round \( T \) due to an algorithm choosing \( p^1, \ldots, p^T \) is \( R^T = \sum_{t=1}^{T} r^t \). We note that \( R^T \) and \( r^t \) are defined for a specific algorithm, which will be clear from the context. We are interested in bounding the expected regret \( \mathbb{E}[R^T] \) of an algorithm at round \( T \), where the expectation is over the randomness involved in sampling the arms \( a^t \) and the agent rewards \( (X^t_{i,a^t})_{i \in [N]} \) for \( t \in [T] \).

**\( \delta \)-Covering:** Given a metric space \((X, d) \) and \( \delta > 0 \), a set \( S \subseteq X \) is called a \( \delta \)-cover if for each \( x \in X \), there exists \( s \in S \) with \( d(x, s) \leq \delta \). That is, from each point in the metric space, there is a point in the \( \delta \)-cover that is no more than \( \delta \) distance away. We will heavily use the fact that there exists a \( \delta \)-cover of \((\Delta^K, \|\cdot\|_1) \) (i.e. the \( K \)-simplex under the \( L_1 \) distance) with size at most \((1 + 2/\delta)^K \) [31, p. 126], which follows from a simple discretization of the simplex.

### 3 Explore-First

**Algorithm 1:** Explore-First

**Input:** Number of agents \( N \), number of arms \( K \), horizon \( T \)

**Parameters:** Exploration period \( L \)

// Pull each arm \( L \) times

for \( t = 1, \ldots, K \cdot L \) do

\( j \leftarrow [t/L] \) // Exploration

\( p^t \leftarrow \text{policy that puts probability 1 on arm } j \) // Pull arm \( j \) deterministically

end

Compute the estimated reward matrix \( \hat{\mu} \triangleq \hat{\mu}^{K \cdot L + 1} \) of the rewards observed so far

Compute \( \hat{\mu} \in \arg \max_{p \in \Delta^K} \text{NSW}(p, \hat{\mu}) \)

for \( t = K \cdot L + 1, \ldots, T \) do

\( p^t \leftarrow \hat{\mu} \) // Exploitation

end

Perhaps the simplest algorithm (with a sublinear regret bound) in the classic single-agent MAB framework is Explore-First. It is composed of two distinct stages. The first stage is exploration, during which the

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2The algorithms we study do not introduce any further randomness in choosing the policies.
algorithm pulls each arm $L$ times. At the end of this stage, the algorithm computes the arm $\hat{a}$ with the best estimated mean reward, and in the subsequent exploitation stage, pulls arm $\hat{a}$ in every round. The algorithm is horizon-dependent, i.e., it takes the horizon $T$ as input and sets $L$ as a function of $T$. Setting $L = \Theta \left( K^{-\frac{4}{3}} T^{\frac{2}{3}} \log^\frac{4}{3}(T) \right)$ yields regret bound $E[R_T] = O \left( K^{\frac{4}{3}} T^{\frac{2}{3}} \log^\frac{4}{3}(T) \right)$ [1].

In our multi-agent variant, presented as Algorithm 1, the exploration stage pulls each arm $L$ times as before. However, at the end of this stage, the algorithm computes, not an arm $\hat{a}$, but a policy $\hat{p}$ with the best estimated Nash social welfare. During exploitation, it then uses policy $\hat{p}$ in every round. With an almost identical analysis as in the single-agent setting, we recover the aforementioned regret bound with an additional $N^{2/3}$ factor for $N$ agents.

Using a novel and more intricate argument, we show that a different tradeoff between the exponents of $N$ and $K$ can be obtained, where $N^{2/3}$ is reduced to $N^{1/3}$ at the expense of increasing $K^{1/3}$ to $K^{2/3}$ (and adding a logarithmic term). The same approach, but with more tricks, is presented in later sections to analyze more sophisticated algorithms. Thus, for the reader’s convenience, we defer the proof of the next result to the appendix.

Theorem 1. Explore-First is horizon-dependent, and has the following expected regret at round $T$.

- When $L = \Theta \left( N^{\frac{2}{3}} K^{-\frac{4}{3}} T^{\frac{2}{3}} \log^\frac{4}{3}(N K T) \right)$, $E[R_T] = O \left( N^{\frac{2}{3}} K^{\frac{4}{3}} T^{\frac{2}{3}} \log^\frac{4}{3}(N K T) \right)$.

- When $L = \Theta \left( N^{\frac{1}{3}} K^{-\frac{1}{3}} T^{\frac{2}{3}} \log^\frac{2}{3}(N K T) \right)$, $E[R_T] = O \left( N^{\frac{1}{3}} K^{\frac{2}{3}} T^{\frac{2}{3}} \log^\frac{2}{3}(N K T) \right)$.

4 Epsilon-Greedy

**Algorithm 2: $\epsilon^t$-Greedy**

- **Input:** Number of agents $N$, number of arms $K$
- **Parameters:** Exploration probabilities $\epsilon^t$ for $t \in \mathbb{N}$
  
  $\text{curr} \leftarrow 1$ // Next arm to pull during exploration

  for $t = 1, 2, \ldots,$ do
  
  Toss a coin with success probability $\epsilon^t$
  
  if success then // Exploration
  
  // Round-robin among arms during exploration
  
  $p^t \leftarrow$ policy that puts probability 1 on arm $\text{curr}$ // Pull it deterministically
  
  $\text{curr} \leftarrow \text{curr} + 1$ // When $\text{curr}$ becomes $K + 1$, reset to 1
  
  else // Exploitation
  
  Compute the estimated reward matrix $\hat{\mu}^t$ from the rewards observed so far
  
  $p^t \leftarrow \arg \max_{p \in \Delta^K} \text{NSW}(p, \hat{\mu}^t)$

  end

end

A slightly more sophisticated algorithm than Explore-First is Epsilon-Greedy, which is presented as Algorithm 2. It spreads out exploration instead of performing it all at the beginning. Specifically, at each round $t$, it performs exploration with probability $\epsilon^t$, and exploitation otherwise. Exploration cycles through the arms in a round-robin fashion, while exploitation uses the policy $p^t$ with the highest Nash social welfare under the current estimated reward matrix (rather than choosing a single estimated best arm as in the classical algorithm). The key advantage of Epsilon-Greedy over Explore-First is that it is horizon-independent.
However, in the \( \hat{\mu} \) computed in Explore-First at the end of exploration, each \( \hat{\mu}_{i,j} \) is the average of \( L \) iid samples, where \( L \) is fixed. In contrast, in the \( \hat{\mu} \) computed in Epsilon-Greedy in round \( t \), each \( \hat{\mu}_{i,j}^{t} \) is the average of \( n_{j}^{t} \) iid samples. The fact that \( n_{j}^{t} \) is itself a random variable and the \( \hat{\mu}_{i,j}^{t} \)-s are correlated through the \( n_{j}^{t} \)-s prevents a direct application of certain statistical inequalities, thus complicating the analysis of Epsilon-Greedy. To address this, we first present a sequence of useful lemmas that apply to any algorithm, and then use them to prove the regret bounds of Epsilon-Greedy and later UCB.

### 4.1 Useful Lemmas

We begin by focusing on the Nash social welfare function \( \text{NSW}(p, \mu) \). We are interested in how much the function can change when its argument change. To that end, the following lemma translates the difference in a product to a sum of point-wise differences that are easier to deal with.

**Lemma 1.** Let \( a_{i}, b_{i} \in [0, 1] \) for \( i \in [N] \). Then, \( \left| \prod_{i=1}^{N} a_{i} - \prod_{i=1}^{N} b_{i} \right| \leq \sum_{i=1}^{N} |a_{i} - b_{i}|. \)

**Proof.** We prove this using induction on \( N \). For \( N = 1 \), the lemma trivially holds. Suppose it holds for \( N = n + 1 \), we have

\[
\left| \prod_{i=1}^{n+1} a_{i} - \prod_{i=1}^{n+1} b_{i} \right| = \left| \prod_{i=1}^{n} a_{i} - b_{n+1} \prod_{i=1}^{n} a_{i} + b_{n+1} \prod_{i=1}^{n} a_{i} - \prod_{i=1}^{n+1} b_{i} \right|
\leq \left( \prod_{i=1}^{n} a_{i} \right) \left| a_{n+1} - b_{n+1} \right| + b_{n+1} \left| \prod_{i=1}^{n} a_{i} - \prod_{i=1}^{n} b_{i} \right|
\leq |a_{n+1} - b_{n+1}| + \sum_{i=1}^{n} |a_{i} - b_{i}| = \sum_{i=1}^{n+1} |a_{i} - b_{i}|
\]

where the second transition is due to the triangle inequality, and the third transition holds due to the induction hypothesis and because \( a_{i}, b_{i} \in [0, 1] \) for each \( i \).

Using Lemma 1, we can easily analyze Lipschitz-continuity of \( \text{NSW}(p, \mu) \) when either \( p \) or \( \mu \) changes and the other is fixed. First, we consider change in \( p \) with \( \mu \) fixed.

**Lemma 2.** Given a reward matrix \( \mu \in [0, 1]^{N \times K} \) and policies \( p^{1}, p^{2} \in \Delta^{K} \), we have

\[
\left| \text{NSW}(p^{1}, \mu) - \text{NSW}(p^{2}, \mu) \right| \leq N \cdot \|p^{1} - p^{2}\|_{1} = N \cdot \sum_{j \in [K]} |p^{1}_{j} - p^{2}_{j}|.
\]

**Proof.** Using Lemma 1, we have

\[
\left| \text{NSW}(p^{1}, \mu) - \text{NSW}(p^{2}, \mu) \right| \leq \sum_{i \in [N]} \sum_{j \in [K]} \left( p^{1}_{j} - p^{2}_{j} \right) \cdot \mu_{i,j} \leq N \cdot \sum_{j \in [K]} |p^{1}_{j} - p^{2}_{j}|,
\]

where the final transition is due to the triangle inequality and because \( \mu_{i,j} \in [0, 1] \) for each \( i, j \).

Next, we consider change in \( \mu \) with \( p \) fixed.

**Lemma 3.** Given a policy \( p \in \Delta^{K} \), and reward matrices \( \mu^{1}, \mu^{2} \in [0, 1]^{N \times K} \), we have

\[
\left| \text{NSW}(p, \mu^{1}) - \text{NSW}(p, \mu^{2}) \right| \leq \sum_{i \in [N]} \sum_{j \in [K]} p_{j} \cdot |\mu_{i,j}^{1} - \mu_{i,j}^{2}|.
\]
Proof. Again, using Lemma 1, we have

\[ |\text{NSW}(p, \mu^*) - \text{NSW}(p, \tilde{\mu}^t)| \leq \sum_{i \in [N]} \sum_{j \in [K]} p_i \cdot (\mu_{i,j}^1 - \mu_{i,j}^2) \leq \sum_{i \in [N], j \in [K]} p_j \cdot |\mu_{i,j}^1 - \mu_{i,j}^2|, \]

where the last transition is due to the triangle inequality.

Recall that \( \mu^* \) and \( \tilde{\mu}^t \) denote the true reward matrix and the estimated reward matrix at the beginning of round \( t \), respectively. Our goal is to find an upper bound on the quantity \( |\text{NSW}(p, \mu^*) - \text{NSW}(p, \tilde{\mu}^t)| \) that, with high probability, holds at every \( p \in \Delta^K \) simultaneously. To that end, we first need to show that \( \tilde{\mu}^t \) will be close to \( \mu^* \) with high probability.

Recall that random variable \( n_j^t \) denotes the number of times arm \( j \) is pulled by an algorithm before round \( t \), and \( \tilde{\mu}_{i,j}^t \) is an average over \( n_j^t \) independent samples. Hence, we cannot directly apply Hoeffding’s inequality, but we can nonetheless use standard tricks from the literature.

**Lemma 4.** Define \( r_j^t = \sqrt[2]{\frac{\log(KNt)}{n_j^t}} \), and event

\[ \mathcal{E}^t \triangleq \forall i \in [N], j \in [K] : |\tilde{\mu}_{i,j}^t - \mu_{i,j}^*| \leq r_j^t. \]

Then, for any algorithm and any \( t \), we have \( \Pr[\mathcal{E}^t] \geq 1 - \frac{2}{t} \).

**Proof.** Fix \( t \). For \( i \in [N], j \in [K], \) and \( \ell \in [t] \), let \( \tau_{i,j}^\ell \) denote the average reward to agent \( i \) from the first \( \ell \) pulls of arm \( j \), and define \( \tau_j^t = \sqrt{\frac{\log(KNt)}{\ell}} \). Then, by Hoeffding’s inequality, we have

\[ \forall i \in [N], j \in [K], \ell \in [t] : \Pr \left[ |\tau_{i,j}^\ell - \mu_{i,j}| > \tau_j^t \right] \leq \frac{2}{(NKt)^2}. \]

By the union bound, we get

\[ \Pr \left[ \forall i \in [N], j \in [K], \ell \in [t] : |\tau_{i,j}^\ell - \mu_{i,j}| \leq \tau_j^t \right] \geq 1 - \frac{2}{NKt}. \]

Because \( n_j^t \in [t] \) for each \( j \in [K] \), the above event implies our desired event \( \mathcal{E}^t \). Hence, we have that \( \Pr[\mathcal{E}^t] \geq 1 - 2/(NKt) \geq 1 - 2/t \).

Conditioned on \( \mathcal{E}^t \), we wish to bound \( |\text{NSW}(p, \mu^*) - \text{NSW}(p, \tilde{\mu}^t)| \) simultaneously at all \( p \in \Delta^K \). We provide two such (incomparable) bounds, which will form the crux of our regret bound analysis. The first bound is a direct application of the Lipschitz-continuity analysis from Lemma 3.

**Lemma 5.** Conditioned on \( \mathcal{E}^t \), we have that

\[ \forall p \in \Delta^K : |\text{NSW}(p, \tilde{\mu}^t) - \text{NSW}(p, \mu^*)| \leq N \cdot \sum_{j \in [K]} p_j \cdot r_j^t. \]

**Proof.** Conditioned on \( \mathcal{E}^t \), we have \( |\tilde{\mu}_{i,j}^t - \mu_{i,j}^*| \leq r_j^t \) for each \( j \in [K] \). In that case, it is easy to see that the upper bound from Lemma 3 becomes \( N \cdot \sum_{j \in [K]} p_j \cdot r_j^t \).

The factor of \( N \) in Lemma 5 stems from analyzing how much \( \tilde{\mu}^t \) may deviate from \( \mu^* \) conditioned on \( \mathcal{E}^t \), in the worst case. However, even after conditioning on \( \mathcal{E}^t \), \( \tilde{\mu}^t \) remains a random variable. Hence, one may expect that its deviation, and thus the difference \( |\text{NSW}(p, \tilde{\mu}^t) - \text{NSW}(p, \mu^*)| \), may be smaller in expectation. Thus, to derive a different bound than in Lemma 5, we wish to apply McDiarmid’s inequality. However, there are two issues in doing so directly.
• McDiarmid’s inequality bounds the deviation of $\text{NSW}(p, \hat{\mu}^t)$ from its expected value. If $\hat{\mu}^t$ consisted of independent random variables, like in Explore-First, this would be equal to $\text{NSW}(p, \mu^*)$. However, in general, these variables may be correlated through $n^t_i$. We use a conditioning trick to address this issue.

• We cannot hope to apply McDiarmid’s inequality at each $p \in \Delta^K$ separately and use the union bound because $\Delta^K$ is infinite. So we apply it at each $p$ in a $\delta$-cover of $\Delta^K$, apply the union bound, and then translate the guarantee to nearby $p \in \Delta^K$ using the Lipschitz-continuity analysis from Lemma 2.

The next result is one of the key technical contributions of our work with a rather long proof.

**Lemma 6.** Define the event

$$\mathcal{H}^t \triangleq \forall p \in \Delta^K : |\text{NSW}(p, \hat{\mu}^t) - \text{NSW}(p, \mu^*)| \leq \sqrt{4NK \log(NKt)} \cdot \sum_{j \in [K]} p_j \cdot r_j^t + \frac{4}{t}.$$

Then, for any algorithm and any $t$, we have $\Pr[\mathcal{H}^t | \mathcal{E}^t] \geq 1 - 2/t$.

**Proof.** Fix $p \in \Delta^K$. Fix $\delta > 0$, and let $\mathcal{P}$ be a $\delta$-cover of the policy simplex $\Delta^K$ with $|\mathcal{P}| \leq (1 + 2/\delta)^K$ [31, p. 126].

Conditioned on $\mathcal{E}^t$ (i.e. $|\hat{\mu}^t_{i,j} - \mu^*_{i,j}| \leq r_j^t = \frac{\log(NKt)}{n_j^t}$, $\forall i \in [N], j \in [K]$), we wish to derive a high probability bound on $|\text{NSW}(p, \hat{\mu}^t) - \text{NSW}(p, \mu^*)|$. We can bound the deviation of $\text{NSW}(p, \hat{\mu}^t)$ from its expected value. However, unlike in the case of Explore-First, we cannot directly claim that the expected value is $\text{NSW}(p, \mu^*)$ because $\hat{\mu}^t$ consists of random variables that may be correlated through the random variable $n^t = (n^t_1, \ldots, n^t_K)$ taking values in $[t]^K$. Thus, we further condition on the value taken by $n^t$.

Specifically, fix $\ell = (\ell_1, \ldots, \ell_K) \in [t]^K$. For each $j \in [K]$, define $\gamma_j^t = \sqrt{\log(NKt) / \ell_j}$. We use $\mathcal{L}$ to denote the event that $n^t$ takes the value $\ell$.

**Evaluating conditional expectation:** Conditioned on $\mathcal{L}$, each $\hat{\mu}^t_{i,j}$ is independent and satisfies $\mathbb{E}[\hat{\mu}^t_{i,j}] = \mu^*_{i,j}$. Since expectation decomposes over sums and products of independent random variables, we have $\mathbb{E}[\text{NSW}(p, \hat{\mu}^t) | \mathcal{L}] = \text{NSW}(p, \mu^*)$.

We next argue that further conditioning on the high probability event $\mathcal{E}^t$ does not change the expectation by much. Formally,

$$\begin{align*}
|\text{NSW}(p, \mu^*) - \mathbb{E}[\text{NSW}(p, \hat{\mu}^t) | \mathcal{E}^t \land \mathcal{L}]| &= \left| \mathbb{E}[\text{NSW}(p, \hat{\mu}^t) | \mathcal{E}^t \land \mathcal{L}] - \mathbb{E}[\text{NSW}(p, \hat{\mu}^t) | \mathcal{E}^t \land \mathcal{L}] \right| \\
&= \mathbb{P}[-\mathcal{E}^t] \cdot \left| \mathbb{E}[\text{NSW}(p, \hat{\mu}^t) | (-\mathcal{E}^t) \land \mathcal{L}] - \mathbb{E}[\text{NSW}(p, \hat{\mu}^t) | \mathcal{E}^t \land \mathcal{L}] \right| \\
&\leq \mathbb{P}[-\mathcal{E}^t] \cdot \frac{2}{t},
\end{align*}$$

where the penultimate transition holds because NSW is bounded in $[0, 1]$, and the final transition is due to Lemma 4.

**Applying McDiarmid’s inequality:** We first decompose $\hat{\mu}^t$ into $N$ random variables: for each $i \in [N]$, let $\hat{\mu}_i^t = (\hat{\mu}_{i,j}^t)_{j \in [K]}$. To apply McDiarmid’s inequality, we need to analyze the maximum amount $c_i$ by which changing $\hat{\mu}_i^t$ can change $\text{NSW}(p, \hat{\mu}^t)$. Fix $i \in [N]$, and fix all the variables except $\hat{\mu}_i^t$. Conditioned on $\mathcal{E}^t \land \mathcal{L}$, each $\hat{\mu}_{i,j}^t$ can change by at most $2\gamma_j^t$. Hence, using Lemma 3, we have that $c_i \leq 2 \sum_{j \in [K]} p_j \cdot \gamma_j^t$. Now, applying McDiarmid’s inequality, we have

$$\Pr \left[ |\text{NSW}(p, \hat{\mu}^t) - \mathbb{E}[\text{NSW}(p, \hat{\mu}^t) | \mathcal{E}^t \land \mathcal{L}] | \geq \epsilon \mid \mathcal{E}^t \land \mathcal{L} \right] \leq 2e^{-2 \epsilon^2 / \sum_{i \in [N]} c_i^2} = 2e^{-2 \epsilon^2 / \left( \sum_{j \in [K]} p_j \gamma_j^t \right)^2}.$$
Using Equation (1), and setting $\epsilon = \sqrt{2N \log(|\mathcal{P}|t)} \cdot \sum_{j \in [K]} p_j \cdot \delta_j^t$, we have that for each $\ell \in [t]^K$,

$$\Pr \left[ |\text{NSW}(p, \hat{\mu}^t) - \text{NSW}(p, \mu^*)| \geq \sqrt{2N \log(|\mathcal{P}|t)} \cdot \sum_{j \in [K]} p_j \cdot \delta_j^t + \frac{2}{t} \right] = \Theta \left( \frac{1}{|\mathcal{P}|t} \right).$$

Removing the conditioning on $n^t$: We can now remove the conditioning on $n^t$ as follows.

$$\Pr \left[ |\text{NSW}(p, \hat{\mu}^t) - \text{NSW}(p, \mu^*)| \geq \sqrt{2N \log(|\mathcal{P}|t)} \cdot \sum_{j \in [K]} p_j \cdot r_j^t + \frac{2}{t} \right] \leq \sum_{\ell \in [t]^K} \Pr[n^t = \ell] \cdot \frac{2}{|\mathcal{P}|t} = \frac{2}{|\mathcal{P}|t}.$$

Extending to all policies in $\mathcal{P}$: Using the union bound, we have that

$$\Pr \left[ \forall p \in \mathcal{P} : |\text{NSW}(p, \hat{\mu}^t) - \text{NSW}(p, \mu^*)| \leq \sqrt{2N \log(|\mathcal{P}|t)} \cdot \sum_{j \in [K]} p_j \cdot r_j^t + \frac{2}{t} \right] \geq 1 - \frac{2}{t}.$$

Extending to all policies in $\Delta^K$: For $p \in \Delta^K$, let $\overline{p} = \arg \min_{p' \in \mathcal{P}} \|p - p'\|_1$. Then, since $\mathcal{P}$ is a $\delta$-cover, we have $\|p - \overline{p}\|_1 \leq \delta$. Thus, due to Lemma 2, we have

$$|\text{NSW}(p, \hat{\mu}^t) - \text{NSW}(p, \mu^*)| \leq \sum_{\mu \in \{\mu^t, \mu^*\}} |\text{NSW}(p, \mu) - \text{NSW}(\overline{p}, \mu)| + |\text{NSW}(\overline{p}, \hat{\mu}^t) - \text{NSW}(\overline{p}, \mu^*)| \leq 2N\delta + |\text{NSW}(\overline{p}, \hat{\mu}^t) - \text{NSW}(\overline{p}, \mu^*)|.$$

Setting $\delta = \frac{1}{Nt}$, we have

$$\Pr \left[ \forall p \in \Delta^K : |\text{NSW}(p, \hat{\mu}^t) - \text{NSW}(p, \mu^*)| \leq \sqrt{2N \log(|\mathcal{P}|t)} \cdot \sum_{j \in [K]} p_j \cdot r_j^t + \frac{4}{t} \right] \geq 1 - \frac{2}{t}.$$

Substituting $|\mathcal{P}| \leq (1 + 2/\delta)^K \leq (3/\delta)^K$ with $\delta = \frac{1}{Nt}$ yields the desired bound.

Finally, we need the following lemma for technical purposes.

Lemma 7. For constant $p \in \mathbb{R}$, $\sum_{t=1}^{T} tp$ is $\Theta(\log T)$ if $p = -1$, and $\Theta(T^{p+1})$ otherwise.

4.2 Analysis of Epsilon-Greedy

We can now use these lemmas to derive the regret bounds for Epsilon-Greedy.

Theorem 2. Epsilon-Greedy is horizon-independent, and has the following expected regret at any round $T$.

- If $\epsilon^t = \Theta \left( N^{\frac{1}{p}} K^{\frac{1}{p(p+1)}} \log^{\frac{1}{p}} (NKt) \right)$ for all $t$, $\mathbb{E}[R_T] = O \left( N^{\frac{1}{p}} K^{\frac{1}{p(p+1)}} \log^{\frac{1}{p}} (NKt) \right)$.
\begin{itemize}
    \item If $\epsilon^t = \Theta \left( N^2 K^2 t^{-2} \log^2 (NKt) \right)$ for all $t$, $\mathbb{E}[R^T] = \Theta \left( N^2 K^2 T^2 \log^2 (NKT) \right)$.
\end{itemize}

**Proof.** Fix $t \in [T]$. Let $b^t$ denote the number of times Epsilon-Greedy performs exploration up to round $t$. Note that $\mathbb{E}[b^t] = \sum_{s=1}^{t} \epsilon^s \geq t \epsilon^t$, where the last step follows from the fact that $\epsilon^t$ is monotonically decreasing in both cases of the theorem. Let $\theta > 0$ be a constant such that $\epsilon^t \geq \theta \cdot t^{-1/3}$ in both cases of the theorem.

Define the event $B^t \triangleq b^t \geq \gamma \cdot t \epsilon^t$, where $\gamma = 1 - 1/\theta$. Then, by Hoeffding’s inequality, we have

$$
\Pr[\neg B^t] \leq e^{-2(1-\gamma)^2 t \epsilon^t} = e^{-2t^{1/3}} \leq e^{-\log t} = \frac{1}{t}.
$$

(2)

Because the algorithm performs round-robin during exploration, conditioned on $B^t$, we have that $n_j^t \geq \frac{\mu_j^t}{n_j^t}$ for each arm $j$, which implies $r_j^t \leq \sqrt{\frac{K \log (NKt)}{\gamma \cdot t \epsilon^t}}$ for each $j$. Thus, conditioned on $B^t$, we have

$$
\forall p \in \Delta^K : \sum_{j \in [K]} p_j \cdot r_j^t \leq \max_j r_j^t \leq \sqrt{\frac{K \log (NKt)}{\gamma \cdot t \epsilon^t}}.
$$

(3)

We are now ready to use the bounds from Lemmas 5 and 6. We focus on the event

$$
C^t_{\alpha} \triangleq \forall p \in \Delta^K : |\text{NSW}(p, \mu^*) - \text{NSW}(p, \hat{\mu}^t)| \leq \alpha^t \cdot \sum_{j \in [K]} p_j \cdot r_j^t + \frac{4}{t}.
$$

Conditioned on $\mathcal{E}^t \land \mathcal{H}^t$, note that $C^t_{\alpha}$ holds for $\alpha^t = N$ due to Lemma 5, and for $\alpha^t = \sqrt{4NK \log (NKt)}$ due to Lemma 6.

Let $\tilde{\mu}^t \in \arg \max_{p \in \Delta^K} \text{NSW}(p, \hat{\mu}^t)$. We wish to bound the regret $\text{NSW}(p^*, \mu^*) - \text{NSW}(\tilde{\mu}^t, \mu^*)$ that Epsilon-Greedy incurs when performing exploitation in round $t$ by choosing policy $\hat{\mu}^t$. Conditioned on $\mathcal{E}^t \land \mathcal{H}^t \land B^t$, we have

$$
\text{NSW}(p^*, \mu^*) - \text{NSW}(\tilde{\mu}^t, \mu^*)
= (\text{NSW}(p^*, \mu^*) - \text{NSW}(p^*, \hat{\mu}^t)) + (\text{NSW}(p^*, \hat{\mu}^t) - \text{NSW}(\tilde{\mu}^t, \hat{\mu}^t)) + (\text{NSW}(\tilde{\mu}^t, \hat{\mu}^t) - \text{NSW}(\tilde{\mu}^t, \mu^*)
\leq \sum_{p \in \{p^*, \tilde{\mu}^t\}} |\text{NSW}(p, \mu^*) - \text{NSW}(p, \hat{\mu}^t)| \leq 2\alpha^t \sqrt{\frac{K \log (NKt)}{\gamma \cdot t \epsilon^t}} + \frac{8}{t},
$$

(4)

where the penultimate transition holds because $\hat{\mu}^t$ is the optimal policy under $\tilde{\mu}^t$, so $\text{NSW}(p^*, \hat{\mu}^t) \leq \text{NSW}(\tilde{\mu}^t, \hat{\mu}^t)$, and the final transition follows from Equation (3) and the fact that $\mathcal{E}^t \land \mathcal{H}^t$ imply $C^t_{\alpha}$.

We are now ready to analyze the expected regret of Epsilon-Greedy at round $T$. We have

$$
\mathbb{E}[R^T] = \sum_{t=1}^{T} \mathbb{E}[R^t] \leq \sum_{t=1}^{T} \mathbb{E} \left[ \epsilon^t \cdot 1 + (1 - \epsilon^t) \cdot (\text{NSW}(p^*, \mu^*) - \text{NSW}(\tilde{\mu}^t, \mu^*)) \right]
\leq \sum_{t=1}^{T} \left( \epsilon^t + \Pr \left[ \mathcal{E}^t \land \mathcal{H}^t \land C^t_{\alpha} \right] \cdot \mathbb{E} \left[ \text{NSW}(p^*, \mu^*) - \text{NSW}(\tilde{\mu}^t, \mu^*) \mid \mathcal{E}^t \land \mathcal{H}^t \land C^t_{\alpha} \right] \right)
\leq \sum_{t=1}^{T} \left( \epsilon^t + 2\alpha^t \sqrt{\frac{K \log (NKt)}{\gamma \cdot t \epsilon^t}} + \frac{8}{t} + \frac{5}{t^2} \right),
$$

where the final transition holds due to Equation (4), Lemma 4, Lemma 6, and Equation (2).

\footnote{Technically, $n_j^t \geq \left[ \frac{\mu_j^t}{n_j^t} \right]$ for each arm $j$, but we omit the floor for the ease of presentation.}
To obtain the first regret bound, we set \( \epsilon^t = \Theta \left( N^{2/3} K^{2/3} t \log^2(NKt) \right) \) and \( \alpha^t = N \), and obtain

\[
E[R_T] = O \left( N^{2/3} K^{2/3} \log^2(NKt) \sum_{t=1}^T t^{-1/3} + \sum_{t=1}^T \frac{1}{t} \right) = O \left( N^{2/3} K^{2/3} T \log^2(NKt) \right).
\]

To obtain the second regret bound, we set \( \epsilon^t = \Theta \left( N^{1/3} K^{1/3} t^{-1} \log \left( \frac{N K T}{n} \right) \right) \) and \( \alpha^t = \sqrt{4NK \log(NKt)} \), and obtain

\[
E[R_T] = O \left( N^{1/3} K^{1/3} \log \left( \frac{N K T}{n} \right) \sum_{t=1}^T t^{-1/3} + \sum_{t=1}^T \frac{1}{t} \right) = O \left( N^{1/3} K^{1/3} T \log \left( \frac{N K T}{n} \right) \right).
\]

In both cases, we use Lemma 7 at the end.

5 UCB

Algorithm 3: UCB

| Input: Number of agents \( N \), number of arms \( K \) |
| **Parameters**: Confidence parameter \( \alpha^t \) for each \( t \in \mathbb{N} \) |
| // Pull each arm once |
| for \( t = 1, \ldots, K \) do |
| \( p^t \leftarrow \) policy that puts probability 1 on arm \( t \) // Pull arm \( t \) deterministically |
| end |
| for \( t = K + 1, \ldots \) do |
| Compute the estimated reward matrix \( \hat{\mu}^t \) |
| \( p^t \leftarrow \arg \max_{p \in \Delta^K} \text{NSW}(p, \hat{\mu}^t) + \alpha^t \sum_{j \in [K]} p_j \cdot r_j^t \), where \( r_j^t \triangleq \sqrt{\frac{\log(NKt)}{n_j}} \). |
| end |

In the classical multi-armed bandit setting, UCB first pulls each arm once. Afterwards, it merges exploration and exploitation cleverly by pulling, in each round, an arm maximizing the sum of its estimated reward and a confidence interval term similar to \( r_j^t \) in Algorithm 3. Our multi-agent variant similarly selects a policy that maximizes the estimated Nash social welfare plus a confidence term for a policy, which simply takes a linear combination of the confidence intervals of the arms. We show that this achieves the desired \( \sqrt{T} \) dependence on the horizon. As this dependence is optimal even for the single-agent setting \([1]\), one cannot hope to improve it beyond logarithmic factors.

**Theorem 3.** UCB is horizon-independent, and has the following expected regret at any round \( T \).

- If \( \alpha^t = N \) for all \( t \), \( E[R_T] = O \left( NKT^{2/3} \log(NKT) \right) \).
- If \( \alpha^t = \sqrt{4NK \log(NKt)} \) for all \( t \), \( E[R_T] = O \left( N^{1/3} K^{1/3} T^{2/3} \log \left( \frac{N K T}{n} \right) \right) \).

**Proof.** Let us again focus on the event

\[
C_t^\alpha \triangleq \forall p \in \Delta^K : |\text{NSW}(p, \mu^*) - \text{NSW}(p, \hat{\mu}^t)| \leq \alpha^t \cdot \sum_{j \in [K]} p_j \cdot r_j^t + \frac{4}{t}.
\]
As argued in the proof of Theorem 2, note that conditioned on \( \mathcal{E}^t \land \mathcal{H}^t \), \( C^t_{\alpha} \) holds for \( \alpha^t = N \) due to Lemma 5, and for \( \alpha^t = \sqrt{4NK \log(NKt)} \) due to Lemma 6. Next, conditioned on \( C^t_{\alpha} \), we have that

\[
\text{NSW}(p^t, \mu^t) \leq \text{NSW}(p^t, \hat{\mu}^t) + \alpha^t \sum_{j \in [K]} p_j \cdot r_j^t + \frac{4}{t}
\]

\[
\leq \text{NSW}(p^t, \hat{\mu}^t) + \alpha^t \sum_{j \in [K]} p_j \cdot r_j^t + \frac{4}{t}
\]

\[
\leq \text{NSW}(p^t, \mu^t) + 2\alpha^t \sum_{j \in [K]} p_j \cdot r_j^t + \frac{8}{t},
\]

where the first and the last transition are from conditioning on \( C^t_{\alpha} \), and the second transition is because \( p = p^t \) maximizes the quantity \( \text{NSW}(p, \hat{\mu}^t) + \alpha^t \sum_{j \in [K]} p_j \cdot r_j^t \) in the UCB algorithm.

Let us write \( p^t = (p^t_1, \ldots, p^t_T) \) for the random variable denoting the policies used by the algorithm, and \( \mathbf{p}^T = (p^1, \ldots, p^T) \) to denote a specific value in \( (\Delta^K)^T \) taken by the random variable. Next, we show that regardless of the value of \( p^t \), we have bounded regret. Note that even after conditioning on the policies, there is still randomness left in sampling actions from the policies and sampling the rewards of those actions.

Specifically, for \( c = N \) and \( c = \sqrt{4NK \log(NKT)} \), we show that \( \mathbb{E}[R^T \mid p^t = \mathbf{p}^T] = \mathcal{O}(ck\sqrt{T \log(NKT)}) \) holds for every \( \mathbf{p}^T \), which also implies \( \mathbb{E}[R^T] = \mathcal{O}(ck\sqrt{T \log(NKT)}) \). Substituting the two values of \( c \) then yields the two desired regret bounds, finishing the proof.

Fix \( \mathbf{p}^T \) and let us condition on \( p^t = \mathbf{p}^T \). For \( t \in [T] \) and \( j \in [K] \), define \( q^t_j = \sum_{s=1}^t p^t_s \). Then, \( \mathbb{E}[n^t_j \mid p^t = \mathbf{p}^T] = q^t_j \). For each \( j \in [K] \), let \( T_j \) be the smallest \( t \) for which \( q^t_j \geq 2\sqrt{T \log(NKT)} \) (if no such \( t \) exists, let \( T_j = T \)); note that \( T_j \) is fixed and not a random variable. Also, we have \( q^T_j = \Theta\left(\sqrt{T \log(NKT)}\right) \) for each \( j \in [K] \).

Let us define a clean event \( \mathcal{B} \triangleq \forall j \in [K], n^T_j \geq \sqrt{T \log(NKT)} \). We first show that this is a high probability event. Indeed, using Hoeffding’s inequality, we have that for each \( j \in [K] \),

\[
\Pr[n^T_j < \sqrt{T \log(NKT)} \mid p^T = \mathbf{p}^T] \leq \Pr[n^T_j < s^T_j - \sqrt{T \log(NKT)} \mid p^T = \mathbf{p}^T] \leq \frac{1}{N^2K^2T^2}.
\]

Taking union bound over \( j \in [K] \), we have that \( \Pr[\mathcal{B}] \geq 1 - \frac{1}{N^2K^2T^2} \). Next, we bound expected regret at round \( T \) using event \( \mathcal{B} \). Indeed,\n
\[
\mathbb{E} \left[ R^T \mid p^t = \mathbf{p}^T \right]
\]

\[
= \sum_{t=1}^T \mathbb{E} \left[ \text{NSW}(p^t, \mu^t) - \text{NSW}(p^t, \mu^t) \mid p^t = \mathbf{p}^T \right]
\]

\[
\leq K + \sum_{t=K+1}^T \left( \Pr[\mathcal{E}^t \land \mathcal{H}^t \land \mathcal{B} \land \text{NSW}(p^t, \mu^t) - \text{NSW}(p^t, \mu^t) \mid p^t = \mathbf{p}^T \land \mathcal{E}^t \land \mathcal{H}^t \land \mathcal{B}] + \Pr[\neg \mathcal{E}^T \lor \neg \mathcal{H}^t \lor \neg \mathcal{B}] \cdot 1 \right)
\]

\[
\leq K + \sum_{t=K+1}^T \mathbb{E} \left[ 2\alpha^t \sum_{j \in [K]} \mu_j \cdot r_j^t + \frac{8}{t} \mid \mathcal{E}^t \land \mathcal{H}^t \land \mathcal{B} \right] + \sum_{t=K+1}^T \Pr[\neg \mathcal{E}^T \lor \neg \mathcal{H}^t \lor \neg \mathcal{B}]
\]

\[
\leq K + 2c\sqrt{\log(NKT)} \sum_{t=K+1}^T \sum_{j \in [K]} \frac{\mu_j}{\sqrt{c_j}} + \sum_{t=K+1}^T \frac{16}{t}.
\]

The penultimate transition holds for \( \alpha^t = N \) and \( \alpha^t = \sqrt{4NK \log(NKT)} \leq \sqrt{4NK \log(NKT)} \), as argued in the definition of \( C^t_{\alpha} \). In the final transition, we observe that both values of \( \alpha^t \) are at most the corresponding
values of $c$, and that conditioned on $B$, $n^j_t$ is lower bounded by $c_j$, where $c_j = 1$ if $t < T_j$, and $c_j = \sqrt{T \log(NKT)}$ if $t \geq T_j$. Hence,
\[
\mathbb{E} \left[ R^T \mid p^T = \hat{p}^T \right] \\
\leq K + 2c \sqrt{\log(NKT)} \sum_{j \in [K]} \left( \frac{T_j - 1}{T} \hat{p}^j_{T_j} + \frac{T_j}{T} \frac{\hat{p}^j_{T_j}}{\sqrt{T \log(NKT)}} \right) + O(\log T) \\
= K + 2c \sqrt{\log(NKT)} \sum_{j \in [K]} \left( \frac{T_j - 1}{T} \hat{p}^j_{T_j} + \frac{T}{\sqrt{T \log(NKT)}} \right) + O(\log T) \\
\leq K + 2c \sqrt{\log(NKT)} \sum_{j \in [K]} \left( q^{T_j}_{j} + \frac{T}{\sqrt{T \log(NKT)}} \right) + O(\log T) \\
= O\left( cK \sqrt{T \log(NKT)} \right),
\]
as desired.

6 Discussion

Our work leaves several open questions and directions for future work.

Computation. We did not formally analyze the computational complexity of our algorithms. It is easy to check that both Explore-First and Epsilon-Greedy can be implemented efficiently; the only non-trivial step in these algorithms is computing the optimal policy given an estimated reward matrix, i.e., $\arg \max_{p \in \Delta^K} NSW(p, \hat{\mu})$. Since the Nash social welfare is known to be log-concave [29], this can be solved efficiently. However, for UCB, the non-trivial step is $\arg \max_{p \in \Delta^K} NSW(p, \hat{\mu}) + \alpha \sum_{j \in [K]} p_j r^j_t$. Due to the added linear term, the objective is no longer log-concave. Hence, it is not clear if this problem can be solved efficiently. This remains a challenging open problem. However, this can also be viewed as optimizing a polynomial over a simplex, which, while NP-hard in general, is known to admit a PTAS when the degree is a constant [32, 33]. Hence, in our case, when the number of agents $N$ is a constant, the problem can be solved approximately, but it remains to be seen how this approximation translates to the regret bounds.

Logarithmic regret bound for UCB. In the classical stochastic multi-armed bandit setting, UCB has two known regret bounds with optimal dependence on $T$. There is an instance-independent bound that grows roughly as $\sqrt{T}$ (where the constants depend only on $K$, and not on the mean rewards) and an instance-dependent bound that grows roughly as $\log T$ (where the constants may depend on the mean rewards in addition to $K$). While we recover the former bound in our multi-agent version, we were not able to derive an instance-dependent logarithmic regret bound. This remains another challenging open problem.

Fairness. While maximizing the Nash social welfare is often seen as a fairness guarantee of its own, as discussed in the introduction, the policy with the highest Nash social welfare is also known to satisfy other fairness guarantees. However, it is not clear if the additive regret bounds we derive in terms of the Nash social welfare also translate to bounds on the amount by which these other fairness guarantees are violated. Considering other fairness guarantees and bounding their total violation is also an interesting direction for the future.

Multi-agent extensions. More broadly, our work opens up the possibility of designing multi-agent extensions of other multi-armed bandit problems. For example, one can consider a multi-agent dueling bandit problem, in which an algorithm asks an agent (or all agents) to compare two arms rather than report their reward for a single arm. Meaningfully defining the regret for such frameworks and designing algorithms that bound it is an exciting future direction.
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Appendix

A Proof of Theorem 1

Proof. Note that the instantaneous regret $r_t(p^i)$ in any round $t$ can be at most 1 because $\text{NSW}(p, \mu^*) \leq 1$ for every policy $p$. Thus,

$$E[R^T] = \sum_{t=1}^{T} E[r_t] \leq KL \cdot 1 + (T - KL) \cdot E[\text{NSW}(p^*, \mu^*) - \text{NSW}(\hat{p}, \mu^*)].$$

(5)

Thus, our goal is to bound $E[\text{NSW}(p^*, \mu^*) - \text{NSW}(\hat{p}, \mu^*)]$. We bound this in two ways.

In the first approach, we bound how much $\hat{\mu}$ can deviate from $\mu^*$. Specifically, we let $\epsilon = \sqrt{\frac{\log(NKT)}{L}}$ and define the event $E \triangleq \forall i \in [N], \forall j \in [K] : |\hat{\mu}_{i,j} - \mu^*| \leq \epsilon$. Since $L$ is fixed, we have $E[\hat{\mu}_{i,j}] = \mu^*$. Hence, we can directly apply Hoeffding’s inequality followed by the union bound to derive $\Pr[E] \geq 1 - 2/T^2$. Then, using Lemma 3, we have that $\text{NSW}(p, \mu^*)$ for each policy $p$. Now, we have

$$\text{NSW}(p^*, \mu^*) \leq \text{NSW}(p^*, \hat{\mu}) + N\epsilon \leq \text{NSW}(\hat{p}, \hat{\mu}) + N\epsilon \leq \text{NSW}(\hat{p}, \mu^*) + 2N\epsilon,$$

where the second transition is because $\hat{\mu} \in \arg\max_{\mu^*} \text{NSW}(p, \hat{\mu})$. Substituting this into Equation (5) and setting $L = \Theta \left( N^\frac{3}{2}K^\frac{3}{2}T^\frac{3}{2} \log^3(NKT) \right)$ yields the first regret bound.

In the second approach, we notice that for a given $p$, $E[\text{NSW}(p, \hat{\mu})] = \text{NSW}(p, \mu^*)$ because all $\hat{\mu}_{i,j}$ are independent, and expectation decomposes over sums and products of independent variables. Thus, we can use McDiarmid’s inequality to bound $\text{NSW}(p, \hat{\mu}) - \text{NSW}(p, \mu^*)$ at a given $p$. Fix a $\delta$-cover $\mathcal{P}$ of $(\Delta^K, ||\cdot||_1)$.

Fix $p \in \mathcal{P}$. We first notice that $\hat{\mu}_{i,j} = (1/L) \cdot |X_{i,j}| \cdot X_{i,j}$, where $X_{i,j}$ is the reward to agent $i$ from the $s$-th pull of arm $j$ during the exploration phase.

We thus decompose $\hat{\mu}$ into $N \cdot L$ random variables: for each $i \in [N]$ and $s \in [L]$, we let $X_{i,s} = (X_{i,j})_{j \in [K]}$. To apply McDiarmid’s inequality, we need to analyze the maximum amount $c_{i,s}$ by which changing $X_{i,s}$ can change $\text{NSW}(p, \hat{\mu})$. Using Lemma 3, it is easy to see that $c_{i,s} \leq 1/L$ for each $i \in [N]$ and $s \in [L]$. Now, applying McDiarmid’s inequality, we have

$$\Pr[|\text{NSW}(p, \hat{\mu}) - \text{NSW}(p, \mu^*)| \leq \epsilon] \leq 2e^{-\frac{\epsilon^2}{2L}}.$$

Setting $\epsilon = \sqrt{\frac{N \log(|\mathcal{P}|T)}{2L}}$, we have that for each $p \in \mathcal{P},$

$$\Pr[|\text{NSW}(p, \hat{\mu}) - \text{NSW}(p, \mu^*)| \leq \sqrt{\frac{N \log(|\mathcal{P}|T)}{2L}}] \leq \frac{2}{|\mathcal{P}|T}.$$

Using the union bound, we have that

$$\Pr \left( \forall p \in \mathcal{P} : |\text{NSW}(p, \hat{\mu}) - \text{NSW}(p, \mu^*)| \leq \sqrt{\frac{N \log(|\mathcal{P}|T)}{2L}} \right) \geq 1 - \frac{2}{T}.$$

For $p \in \Delta^K$, let $\mathbf{p} \in \arg\min_{p \in \mathcal{P}} \|p - p’\|_1$. Then, since $\mathcal{P}$ is a $\delta$-cover, we have $\|p - \mathbf{p}\|_1 \leq \delta$. Thus, due to Lemma 2, we have

$$|\text{NSW}(p, \hat{\mu}) - \text{NSW}(p, \mu^*)| \leq \sum_{\mu \in \{\hat{\mu}, \mu^*\}} |\text{NSW}(p, \mu) - \text{NSW}(\mathbf{p}, \mu)| + |\text{NSW}(\mathbf{p}, \hat{\mu}) - \text{NSW}(\mathbf{p}, \mu^*)|$$

$$\leq 2N\delta + |\text{NSW}(\mathbf{p}, \hat{\mu}) - \text{NSW}(\mathbf{p}, \mu^*)|.$$
Setting $\delta = \frac{1}{N^2 T}$, we have

$$\Pr \left[ \forall p \in \Delta^K : |\text{NSW}(p, \hat{\mu}) - \text{NSW}(p, \mu^*)| \leq \frac{2}{T} + \sqrt{\frac{N \log(|P|T)}{2L}} \right] \geq 1 - \frac{2}{T}.$$  

Next, we use the fact that

$$\text{NSW}(p^*, \mu^*) - \text{NSW}(\hat{p}, \mu^*) \leq \sum_{p \in \{p^*, \hat{p}\}} |\text{NSW}(p, \hat{\mu}) - \text{NSW}(p, \mu^*)|.$$  

Hence,

$$\Pr \left[ |\text{NSW}(p^*, \mu^*) - \text{NSW}(\hat{p}, \mu^*)| \leq \frac{4}{T} + \sqrt{\frac{2N \log(|P|T)}{L}} \right] \geq 1 - \frac{2}{T}.$$  

Next, we substitute $|P| \leq (1 + 2/\delta)^K \leq (3/\delta)^K$, $\delta = \frac{1}{N^2 T}$, and $L = \Theta \left( N^{\frac{1}{2}} K^{-\frac{1}{4}} T^{\frac{1}{3}} \log^2 (NKT) \right)$, and then substitute the derived bound in Equation (5) to get the second regret bound. \qed