ON NODAL SEXTIC FIVEFOLD

IVAN CHELTSOV

Abstract. We prove the birational superrigidity and nonrationality of a hypersurface in $\mathbb{P}^6$ of degree 6 having at most isolated ordinary double points.

1. Introduction.

In many cases the only known way to prove the nonrationality of a Fano variety is to prove its birational rigidity. Many counterexamples to the Lüroth problem are obtained by proving the birational rigidity of Fano 3-folds (see [13]). Moreover, birational rigidity is the only known way to prove the nonrationality of an explicitly given Fano $n$-fold for $n > 3$.

Birational rigidity is proved in the following cases:

- for some smooth Fano 3-folds (see [13], [12], [14]);
- for many singular Fano 3-folds (see [20], [22], [11], [9], [8], [17]);
- for many smooth Fano $n$-folds (see [18], [23], [25], [2], [26], [27], [30], [10], [3], [4]), $n > 3$;
- for some singular Fano $n$-folds (see [20], [22], [28], [29], [4]), $n > 3$.

Let $X$ be a hypersurface in $\mathbb{P}^6$ of degree 6 such that the only singularities of $X$ are isolated ordinary double points. Then $-K_X \sim \mathcal{O}_{\mathbb{P}^6}(1)$, the variety $X$ is a Fano 5-fold with $\mathbb{Q}$-factorial terminal singularities and $\text{rk} \text{Pic}(X) = 1$ (see [1]). In this paper we prove the following result.

Theorem 1. The hypersurface $X$ is birationally superrigid.

In the smooth case the claim of Theorem 1 is proved in [2]. In fact, one can use Theorem 1 to construct explicit examples of nonrational singular hypersurfaces.

Example 2. The singularities of the hypersurface

$$x_0^2(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 \subset \mathbb{P}^6 \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_6]).$$

consist of a single ordinary double point, which implies that it is nonrational by Theorem 1.

Example 3. Let $X$ be a hypersurface

$$\sum_{i=0}^{2} a_i(x_0, \ldots, x_6)b_i(x_0, \ldots, x_6) = 0 \subset \mathbb{P}^6 \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_6]),$$

where $a_i$ and $b_i$ are general homogeneous polynomials of degree 3. Then $X$ has 729 isolated ordinary double points. In particular, the hypersurface $X$ is nonrational by Theorem 1.

It should be pointed out that the claim of Theorem 1 can be considered as a five-dimensional generalization of the birational rigidity of a $\mathbb{Q}$-factorial quartic 3-fold having isolated ordinary

1 All varieties are assumed to be projective, normal and defined over $\mathbb{C}$.

2 Let $V$ be a Fano variety with terminal $\mathbb{Q}$-factorial singularities and $\text{rk} \text{Pic}(V) = 1$. Then $V$ is called birationally rigid if it is not birational to the following varieties: a variety $Y$ such that there is a morphism $\tau : Y \to Z$ whose general fiber has negative Kodaira dimension and $\dim(Y) \neq \dim(Z) \neq 0$; a Fano variety of Picard rank 1 having terminal $\mathbb{Q}$-factorial singularities that is not biregular to $V$. The variety $V$ is called birationally superrigid if it is birationally rigid and $\text{Bir}(V) = \text{Aut}(V)$.

3 A priori the method of J.Kollár can be applied to construct explicit examples of nonrational Fano varieties, but a posteriori there is only one case of such explicit application (see [10], [7]).
double points (see [13, 20, 17]). The claim of Theorem 1 is relevant to [28] and [29], but one cannot use [28] and [29] to produce explicit examples of nonrational Fano hypersurfaces.

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2. The Noether–Fano–Iskovskikh inequality.

Let \( X \) be a Fano variety with terminal \( \mathbb{Q} \)-factorial singularities such that \( \text{rk} \text{Pic}(X) = 1 \), but the variety \( X \) is not birationally superrigid. Then the following result holds (see [5]):

**Theorem 4.** There is a linear system \( \mathcal{M} \) on the variety \( X \) whose base locus has codimension at least 2, and the singularities of the log pair \( (X, \gamma \mathcal{M}) \) are not canonical, where \( \gamma \) is a positive rational number such that the equivalence \( K_X + \gamma \mathcal{M} \sim \mathbb{Q} 0 \) holds.

In the rest of the section we prove Theorem 4. Let \( \rho : X \dashrightarrow Y \) be a birational map such that the rational map \( \rho \) is not biregular and one of the following holds:
- the variety \( Y \) is a Fano variety with terminal \( \mathbb{Q} \)-factorial singularities such that the equality \( \text{rk} \text{Pic}(Y) = 1 \) holds (the Fano case);
- the variety \( Y \) is smooth, and there is a morphism \( \tau : Y \rightarrow Z \) whose general fiber has negative Kodaira dimension and \( \dim(Y) \neq \dim(Z) \neq 0 \) (the fibration case).

Let us consider a commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\alpha} & X \\
\downarrow{\beta} & & \downarrow{\rho} \\
Y & \xrightarrow{} & &
\end{array}
\]

such that \( W \) is smooth, \( \alpha \) and \( \beta \) are birational morphisms. In the Fano case let \( \mathcal{D} \) be the complete linear system \(|-rK_Y|\) for \( r \gg 0 \), in the fibration case let \( \mathcal{D} \) be the complete linear system \(|\tau^*(H)|\), where \( H \) is a very ample divisor on \( Z \). Let \( \mathcal{M} \) be a proper transform on the variety \( X \) of the linear system \( \mathcal{D} \). Now choose a positive rational number \( \gamma \) such that the equivalence \( K_X + \gamma \mathcal{M} \sim \mathbb{Q} 0 \) holds. Suppose that the singularities of the log pair \( (X, \gamma \mathcal{M}) \) are not canonical. Let us show that this assumption leads to a contradiction.

Let \( \mathcal{B} \) be a proper transform on \( W \) of the linear system \( \mathcal{M} \). Then

\[
\sum_{i=1}^{k} a_i F_i \sim_{\mathbb{Q}} \alpha^*(K_X + \gamma \mathcal{M}) + \sum_{i=1}^{k} a_i F_i \sim_{\mathbb{Q}} K_W + \gamma \mathcal{B} \sim_{\mathbb{Q}} \beta^*(K_Y + \gamma \mathcal{D}) + \sum_{i=1}^{l} b_i G_i,
\]

where \( F_i \) is a \( \beta \)-exceptional divisor, \( G_i \) is an \( \alpha \)-exceptional divisor, \( a_i \) is a nonnegative rational number, and \( b_i \) is a positive rational number. Let \( n \) be a sufficiently big and sufficiently divisible natural number. Then

\[
h^0\left( \mathcal{O}_W \left( \sum_{j=1}^{k} n a_j F_j \right) \right) = h^0\left( \mathcal{O}_W \left( \beta^*(nK_Y + n\gamma \mathcal{D}) + \sum_{i=1}^{l} nb_i G_i \right) \right),
\]

but \( h^0(\mathcal{O}_W(\beta^*(nK_Y + n\gamma \mathcal{D}) + \sum_{i=1}^{l} nb_i G_i)) = 0 \) in the fibration case. Hence, the fibration case is impossible. In the Fano case the equality \( h^0(\mathcal{O}_W(\beta^*(nK_Y + n\gamma \mathcal{D}) + \sum_{i=1}^{l} nb_i G_i)) = 1 \) implies that \( \gamma = 1/r \). Thus, we have

\[
\sum_{i=1}^{k} a_i F_i \sim_{\mathbb{Q}} \sum_{i=1}^{l} b_i G_i,
\]

and it follows from Lemma 2.19 in [15] that \( \sum_{i=1}^{k} a_i F_i = \sum_{i=1}^{l} b_i G_i \), which implies that the singularities of the log pair \( (X, \gamma \mathcal{M}) \) are terminal.
There is a rational number $\mu > \gamma$ such that both log pairs $(X, \mu\mathcal{M})$ and $(X, \mu\mathcal{B})$ have terminal singularities. Hence, we have

$$\alpha^*(K_X + \mu\mathcal{M}) + \sum_{i=1}^{k} a'_iF_i \sim_{\mathbb{Q}} K_W + \mu\mathcal{B} \sim_{\mathbb{Q}} \beta^*(K_Y + \mu\mathcal{D}) + \sum_{i=1}^{l} b'_iG_i,$$

where $a'_i$ and $b'_i$ are positive rational numbers. Let $n$ be a sufficiently big and divisible natural number, and $\psi : W \rightarrow U$ be a map given by the linear system $|nK_W + n\mu\mathcal{B}|$. Then $\psi \circ \beta^{-1}$ is an isomorphism, because the divisor $n(K_Y + \mu\mathcal{D})$ is very ample, but the divisor $\sum_{i=1}^{l} nb'_iG_i$ is effective and $\beta$-exceptional. Similarly, we get $\psi \circ \alpha^{-1}$ is an isomorphism. Hence, the birational map $\rho$ is an isomorphism, which is a contradiction. Thus, we proved Theorem 3.

3. The lemma of Corti.

Let $X$ be a variety, $O$ be an isolated ordinary double point on $X$, $B_X$ be an effective $\mathbb{Q}$-Cartier divisor on the variety $X$, $\pi : W \rightarrow X$ be a blow up of $O$, $E$ be a $\pi$-exceptional divisor, $B_W$ be a proper transform of the divisor $B_X$ on the variety $W$. Then the equivalence

$$\pi^*(B_X) \sim_{\mathbb{Q}} B_W + \text{mult}_O(B_X)E$$

holds, where $\text{mult}_O(B_X)$ is a non-negative rational number. Suppose that $\dim(X) \geq 3$ and the singularities of the log pair $(X, B_X)$ are not canonical in the point $O$. Then elementary calculations imply $\text{mult}_O(B_X) > 1/2$. The following result is implied by Theorem 3.10 in [6].

Lemma 5. The inequality $\text{mult}_O(B_X) > 1$ holds.

In the rest of the section we prove Lemma 5. Suppose that $\text{mult}_O(B_X) \leq 1$. Let us show that this assumption leads to a contradiction. Replacing the divisor $B_X$ by $(1 - \epsilon)B_X$ for some positive sufficiently small rational $\epsilon$, we may assume that $\text{mult}_O(B_X) < 1$. Moreover, taking sufficiently general hyperplane sections of $X$, we may assume that $\dim(X) = 3$ due to Theorem 17.6 in [15].

Lemma 6. Let $S$ be a surface $\mathbb{P}^1 \times \mathbb{P}^1$, and $B_S$ be an effective divisor on $S$ of bi-degree $(a, b)$, where $a$ and $b$ are rational numbers in $(0, 1)$. Then the log pair $(S, B_S)$ has log-terminal singularities.

Proof. Suppose that the singularities of $(S, B_S)$ are not log-terminal. Then the locus of log canonical singularities $\text{LCS}(S, B_S)$ is not empty and consists of points of the surface $S$. Hence, the locus $\text{LCS}(S, F + B_S)$ is not connected, where $F$ is a sufficiently general fiber of the projection of the surface $S$ to $\mathbb{P}^1$. The later contradicts Theorem 17.4 in [15].

The inequality $\text{mult}_O(B_X) < 1$ and the equivalence

$$K_W + B_W \sim_{\mathbb{Q}} \pi^*(K_X + B_X) + (1 - \text{mult}_O(B_X))E,$$

imply that there is a proper irreducible subvariety $Z \subset E$ such that the log pair $(W, B_W)$ is not canonical in the generic point of $Z$. Hence the singularities of the log pair $(E, B_W|_E)$ are not log terminal by Theorem 17.6 in [15], which is impossible by Lemma 6.

4. Main inequalities.

Let $X$ be a variety, $O$ be an isolated ordinary double point on $X$, $\mathcal{M}$ be a linear system on the variety $X$ having no base components, and $r = \dim(X) \geq 4$. Let $\pi : V \rightarrow X$ be a blow up of $X$ at the point $O$, $E$ be a $\pi$-exceptional divisor, and let $B$ be a proper transform of the linear system $\mathcal{M}$ on the variety $V$. Then the divisor $E$ can be identified with a smooth quadric hypersurface in $\mathbb{P}^r$, and the equivalence

$$B \sim \pi^*(\mathcal{M}) - \text{mult}_O(\mathcal{M})E$$

holds for some natural number $\text{mult}_O(\mathcal{M})$. It should be pointed out that $\text{mult}_O(\mathcal{M})$ is different from the scheme-theoretic multiplicity of a general surface of $\mathcal{M}$ in the point $O$. 

3
Let $S_1$ and $S_2$ be general divisors in the linear system $\mathcal{M}$, and $H_i$ be a general hyperplane section of $X$ passing through $O$, where $i = 1, \ldots, r-2$. We can define $\text{mult}_O(S_i)$ and $\text{mult}_O(H_i)$ in the same way as we defined the number $\text{mult}_O(\mathcal{M})$. Let $\hat{S}_i$ and $\hat{H}_i$ be proper transforms on the variety $V$ of the divisors $S_i$ and $H_i$ respectively. Then we can put

$$\text{mult}_O(S_1 \cdot S_2) = 2\text{mult}_O^2(S_i) + \sum_{P \in E} \text{mult}_P(\hat{S}_1 \cdot \hat{S}_2) \cdot \text{mult}_P(\hat{H}_1) \cdots \text{mult}_P(\hat{H}_{r-2}).$$

**Remark 7.** The inequality $\text{mult}_O(S_1 \cdot S_2) \geq 2\text{mult}_O^2(S_i) + \text{mult}_Z(\hat{S}_1 \cdot \hat{S}_2)$ holds for any irreducible subvariety $Z \subset E$ of codimension one.

**Example 8.** Let $X$ be a hypersurface in $\mathbb{P}^6$ of degree 6 such that the singularities of the hypersurface $X$ consist of a finite number of isolated ordinary double points, and let $O$ be a singular point of the variety $X$. Then the groups $\text{Cl}(X)$ and $\text{Pic}(X)$ are generated by a hyperplane section $H$ of the hypersurface $X$ (see [1]), which implies that $S_i \sim nH$ for some natural number $n$. Moreover, the inequality $\text{mult}_O(S_1 \cdot S_2) \leq 6n^2$ holds.

Suppose that the singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$ are not canonical in the point $O$, but they are canonical in a punctured neighborhood of the point $O$.

**Lemma 9.** Suppose that $\dim(X) \geq 6$. Then $\text{mult}_O(S_1 \cdot S_2) > 6n^2$.

**Proof.** We prove the inequality $\text{mult}_O(S_1 \cdot S_2) > 6n^2$ only when $\dim(X) = 6$, because the proof in the case $\dim(X) \geq 7$ is similar. So suppose that $\dim(X) = 6$. Then

$$K_V + \frac{1}{n}B \sim_\mathbb{Q} \pi^*(K_X + \frac{1}{n}\mathcal{M}) + \left(4 - \frac{\text{mult}_O(\mathcal{M})}{n}\right)E.$$

Put $\tilde{X} = \cap_{i=1}^3 H_i$ and $\tilde{\mathcal{M}} = \mathcal{M}|_X$. Then $O$ is an isolated ordinary double point on $\tilde{X}$, and the singularities of the log pair $(\tilde{X}, \frac{1}{n}\tilde{\mathcal{M}})$ are not log canonical in the point $O$ by Theorem 17.6 of the paper [15]. Let $\tilde{\pi} : \tilde{V} \to \tilde{X}$ be a blow up of the point $O$, and $\tilde{E}$ be an exceptional divisor of the birational morphism $\tilde{\pi}$. Then the diagram

$$\begin{array}{ccc}
\tilde{V} & \xrightarrow{\tilde{\pi}} & V \\
\downarrow & & \downarrow \pi \\
\tilde{X} & \xleftarrow{\tilde{\pi}} & X
\end{array}$$

is commutative, where the 3-fold $\tilde{V}$ is identified with a proper transform of the subvariety $\tilde{X}$ on the variety $V$. In particular, we have $\tilde{E} = E \cap \tilde{V}$. The generality of $H_i$ implies

$$\text{mult}_O(\tilde{\mathcal{M}}) = \text{mult}_O(\mathcal{M}),$$

and we may assume that $\text{mult}_O(\mathcal{M}) < 2n$, because otherwise $\text{mult}_O(S_1 \cdot S_2) > 6n^2$.

Let $\mathcal{B}$ be a proper transform of $\mathcal{M}$ on the variety $V$, and $\tilde{\mathcal{B}}$ be a proper transform of the linear system $\tilde{\mathcal{M}}$ on the 3-fold $\tilde{V}$. Then $\tilde{\mathcal{B}} = \mathcal{B}|_V$ and we have

$$K_V + \frac{1}{n}B + \left(\frac{\text{mult}_O(\mathcal{M})}{n} - 1\right)E + \hat{H}_1 + \hat{H}_2 + \hat{H}_3 \sim_\mathbb{Q} \tilde{\pi}^*(K_X + \frac{1}{n}\mathcal{M} + H_1 + H_2 + H_3)$$

and

$$K_{\tilde{V}} + \frac{1}{n}\tilde{\mathcal{B}} + \left(\frac{\text{mult}_O(\tilde{\mathcal{M}})}{n} - 1\right)\tilde{E} \sim_\mathbb{Q} \tilde{\pi}^*(K_{\tilde{X}} + \frac{1}{n}\tilde{\mathcal{M}}),$$

but $\text{mult}_O(\mathcal{M}) < 2n$ implies the existence of irreducible subvarieties $\Omega \subset E$ and $\tilde{\Omega} \subset \tilde{E}$ such that the singularities of the log pair $(V, \frac{1}{n}\mathcal{B} + (\text{mult}_O(\mathcal{M})/n - 1)\tilde{E})$ are not log canonical in the generic point of $\Omega$, the singularities of the log pair $(\tilde{V}, \frac{1}{n}\tilde{\mathcal{B}} + (\text{mult}_O(\tilde{\mathcal{M}})/n - 1)\tilde{E})$ are not log canonical in the generic point of $\tilde{\Omega}$, and $\tilde{\Omega} \subset \Omega \cap \tilde{V}$. We have $\tilde{\Omega} = \Omega \cap \tilde{V}$ when $\dim(\tilde{\Omega}) > 0$, and we may assume that $\Omega$ and $\tilde{\Omega}$ have the greatest possible dimensions among all subvarieties.
Proof. We have mult by the Lefschetz theorem. Hence, the inequality \( \dim(\tilde{\Omega}) \leq \text{codimension} 3 \) that is contained in the smooth quadric hypersurface \( O \), which implies mult by Lemma 10.

Suppose that \( \dim(\tilde{\Omega}) = 0 \). Then \( \tilde{\Omega} = \Omega \cap \tilde{V} \).

Let \( \Delta \) be an effective divisor on the variety \( X \) passing through the point \( O \) and \( \hat{\Delta} \) be its proper transform on the variety \( V \). Suppose that the divisor \( \Delta \) does not contain irreducible components of the cycle \( S_1 \cdot S_2 \), and the divisor \( \hat{\Delta} \) does not contain irreducible components of the cycle \( \hat{S}_1 \cdot \hat{S}_2 \). Then we can put

\[
\text{mult}_O(S_1 \cdot S_2 \cdot \Delta) = 2\text{mult}_O^2(S_i)\text{mult}_O(\Delta) + \sum_{P \in E} \text{mult}_P(\hat{S}_1 \cdot \hat{S}_2 \cdot \hat{\Delta})\text{mult}_P(\hat{H}_1) \cdots \text{mult}_P(\hat{H}_{r-3}),
\]

which implies \( \text{mult}_O(\hat{S}_1 \cdot \hat{S}_2 \cdot \hat{\Delta}) = \text{mult}_O(S_1|_{\Delta} \cdot S_2|_{\Delta}) \) in the case when the point \( O \) is an isolated ordinary double point on the divisor \( \Delta \).

Lemma 10. Suppose that \( \dim(X) = 4 \). Then there is a line \( \Lambda \subset E \subset \mathbb{P}^4 \) such that the strict inequality \( \text{mult}_O(S_1 \cdot S_2 \cdot \Delta) > 6n^2 \) holds if \( \Lambda \subset \hat{\Delta} \), and \( O \) is an ordinary double point on \( \Delta \).

Proof. We have \( \text{mult}_O(M) > n \) by Lemma 5, but

\[
K_V + \frac{1}{n} \mathcal{B} \sim_{\mathbb{Q}} \pi^*(K_X + \frac{1}{n} \mathcal{M}) + \left( 2 - \frac{\text{mult}_O(M)}{n} \right) E.
\]

Suppose that \( O \) is an ordinary double point on \( \Delta \). Put \( \bar{S}_i = S_i|_{\Delta} \) and \( \bar{M} = M|_{\Delta} \). Then the log pair \( (\Delta, \frac{1}{n} \bar{M}) \) is not log canonical in the point \( O \) by Theorem 17.6 in [15].

Let \( \tilde{\pi} : \Delta \to \Delta \) be a blow up of \( O \), and \( \tilde{E} \) is a \( \tilde{\pi} \)-exceptional divisor. Then the diagram

\[
\begin{array}{ccc}
\tilde{\Delta} & \xrightarrow{\tilde{\pi}} & \tilde{V} \\
\downarrow \pi & & \downarrow \pi \\
\Delta & \xrightarrow{\pi} & X
\end{array}
\]

is commutative, where we can identify \( \tilde{\Delta} \) with \( \hat{\Delta} \), and \( \tilde{E} = E \cap \tilde{\Delta} \) can be considered as a nonsingular quadric hypersurface in \( \mathbb{P}^3 \). The inequality \( \text{mult}_O(\bar{M}) \geq 2n \) gives

\[
\text{mult}_O(S_1 \cdot S_2 \cdot \Delta) = \text{mult}_O(\tilde{S}_1 \cdot \tilde{S}_2) \geq 8n^2,
\]

hence, we may assume that \( \text{mult}_O(\bar{M}) < 2n \).

Let \( \bar{M} \) be a proper transform of the linear system \( \bar{M} \) on \( \tilde{\Delta} \). Then \( \text{mult}_O(\bar{M}) < 2n \) implies the existence of an irreducible subvariety \( \Xi \subset \tilde{E} \) such that the singularities of the log pair

\[
\left( \tilde{\Delta}, \frac{1}{n} \bar{M} + (\text{mult}_O(\bar{M})/n - 1) \bar{E} \right).
\]
are not log canonical in the generic point of $\Xi$.

Suppose that $\Xi$ is a curve. Let $\hat{S}_i$ be a proper transform of $S_i$ on $\hat{\Delta}$. Then
$$\text{mult}_O(\hat{S}_1 \cdot \hat{S}_2) \geq 2 \text{mult}_O(\mathcal{M})^2 + \text{mult}_\Xi(\hat{S}_1 \cdot \hat{S}_2),$$
but Theorem 3.1 of [B] applied to the log pair $(\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\text{mult}_O(\mathcal{M})/n - 1)\hat{E})$ in the generic point of $\Xi$ implies that the inequality
$$\text{mult}_\Xi(\hat{S}_1 \cdot \hat{S}_2) > 4(2n^2 - n\text{mult}_O(\hat{\mathcal{M}}))$$
holds. Hence, the inequalities
$$\text{mult}_O(\hat{S}_1 \cdot \hat{S}_2) > 2 \text{mult}_O^2(\mathcal{M}) + 4(2n^2 - n\text{mult}_O(\mathcal{M})) \geq 6n^2$$
hold. Thus, we may assume that $\Xi$ is a point.

Suppose that the divisor $\Delta$ is a sufficiently general hyperplane section of $X$ passing through the point $O$. Then applying Theorem 17.4 of [I] to the log pair $(\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\text{mult}_O(\mathcal{M})/n - 1)\hat{E})$ and the morphism $\hat{\pi}$ we see that one of the following holds:
- the singularities of the log pair $(V, \frac{1}{n}B + (\text{mult}_O(\mathcal{M})/n - 1)E)$ are not log canonical in the generic point of some surface that is contained in the divisor $E$;
- there is a line $\Lambda \subset E \subset \mathbb{P}^4$ such that the singularities of $(V, \frac{1}{n}B + (\text{mult}_O(\mathcal{M})/n - 1)E)$ are not log canonical in the generic point of line $\Lambda$ and $\Xi = \Lambda \cap \hat{\Delta}$.

In the case when the singularities of the log pair $(V, \frac{1}{n}B + (\text{mult}_O(\mathcal{M})/n - 1)E)$ are not log canonical in the generic point of some surface contained in $E$, the previous arguments implies the inequality $\text{mult}_O(\hat{S}_1 \cdot \hat{S}_2) > 6n^2$. Thus, we may assume that there is a line $\Lambda \subset E \subset \mathbb{P}^4$ such that $\Xi = \Lambda \cap \hat{\Delta}$ and the singularities of the log pair $(V, \frac{1}{n}B + (\text{mult}_O(\mathcal{M})/n - 1)E)$ are not log canonical in the generic point of the curve $\Lambda$. It should be pointed out that the line $\Lambda$ does not depend on the choice of the divisor $\Delta$. Therefore, we may assume that the divisor $\Delta$ is chosen under the additional assumption $\Lambda \subset \Delta$, where we identified $\Delta$ with $\hat{\Delta}$.

The singularities of the log pair $(\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\text{mult}_O(\mathcal{M})/n - 1)\hat{E})$ are not log canonical in the generic point of $\Lambda$ by Theorem 17.6 in [I], because the boundary $\frac{1}{n}B + (\text{mult}_O(\mathcal{M})/n - 1)E$ is effective due to the inequality $\text{mult}_O(\mathcal{M}) > n$. Hence, we can apply Theorem 3.1 of [B] to the log pair $(\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\text{mult}_O(\mathcal{M})/n - 1)\hat{E})$ in the generic point of $\Lambda$ to obtain the inequalities
$$\text{mult}_O(\hat{S}_1 \cdot \hat{S}_2) > 2 \text{mult}_O^2(\hat{\mathcal{M}}) + 4(2n^2 - n\text{mult}_O(\hat{\mathcal{M}})) \geq 6n^2,$$
which conclude the proof. □

Finally, let us prove the following result.

**Lemma 11.** Suppose that $\dim(X) = 5$ Then $\text{mult}_O(S_1 \cdot S_2) > 6n^2$.

**Proof.** Put $\tilde{X} = H_1 \cap H_2$ and $\tilde{\mathcal{M}} = \mathcal{M}|_{\tilde{X}}$. Then $O$ is an isolated ordinary double point on $\tilde{X}$, and the singularities of the log pair $(\tilde{X}, \frac{1}{n}\tilde{\mathcal{M}})$ are not log canonical in the point $O$ by Theorem 17.6 of the paper [I]. Let $\tilde{\pi} : \tilde{V} \to \tilde{X}$ be a blow up of the point $O$, and $\tilde{E}$ be an exceptional divisor of the morphism $\tilde{\pi}$. Then we can identify $\tilde{V}$ with a proper transform of $\tilde{X}$ on $\tilde{V}$. We have
$$\text{mult}_O(S_1 \cdot S_2) \geq 2 \text{mult}_O^2(\tilde{\mathcal{M}}) > 6n^2$$
in the case when $\text{mult}_O(\mathcal{M}) \geq 2n$. Hence, we may assume that $\text{mult}_O(\mathcal{M}) < 2n$.

Let $\tilde{\mathcal{B}}$ be a proper transform of the linear system $\mathcal{M}$ on $\tilde{V}$. Then $\mathcal{B} = \mathcal{B}|_{\mathcal{V}}$ and we have
$$K_{\mathcal{V}} + \frac{1}{n}B + \left(\frac{\text{mult}_O(\mathcal{M})}{n} - 1\right)E + \tilde{H}_1 + \tilde{H}_2 \sim_{\tilde{Q}} \tilde{\pi}^*(K_{\tilde{X}} + \frac{1}{n}\tilde{\mathcal{M}} + H_1 + H_2),$$
but $K_{\mathcal{V}} + \frac{1}{n}B + (\text{mult}_O(\mathcal{M})/n - 1)\tilde{E} \sim_{\tilde{Q}} \tilde{\pi}^*(K_{\tilde{X}} + \frac{1}{n}\tilde{\mathcal{M}})$. Therefore, there are proper irreducible subvarieties $\Omega \subset E$ and $\tilde{\Omega} \subset \tilde{E}$ such that $\tilde{\Omega} \subset \Omega \cap \tilde{V}$ and the following holds:
- the log pair $(\mathcal{V}, \frac{1}{n}\mathcal{B} + (\text{mult}_O(\mathcal{M})/n - 1)\mathcal{E})$ is not log canonical in $\Omega$;
- the log pair $(\tilde{\mathcal{V}}, \frac{1}{n}\tilde{\mathcal{B}} + (\text{mult}_O(\mathcal{M})/n - 1)\tilde{E})$ is not log canonical in $\tilde{\Omega}$.
We may assume that $\Omega$ and $\tilde{\Omega}$ have the greatest possible dimensions among all subvarieties having such properties. Therefore, we have $\tilde{\Omega} = \Omega \cap \tilde{V}$ in the case when $\dim(\tilde{\Omega}) \geq 1$.

Suppose that $\dim(\Omega) \geq 1$ holds. Then $\dim(\tilde{\Omega}) = 3$ and we can apply Theorem 3.10 of [6] to the log pair $(\tilde{V}, \frac{1}{n} \tilde{B} + (\text{mult}_O(\mathcal{M})/n - 1)E)$ in the generic point of $\Omega$. Therefore, we have

$$\text{mult}_O(\hat{S}_1 \cdot \hat{S}_2) > 4(2n^2 - n\text{mult}(\mathcal{M})),$$

which implies that the inequalities

$$\text{mult}_O(S_1 \cdot S_2) \geq 2\text{mult}_O^2(\mathcal{M}) + \text{mult}_O(\hat{S}_1 \cdot \hat{S}_2) > 6n^2$$

hold. Therefore, we may assume that $\dim(\tilde{\Omega}) = 0$.

Applying Theorem 17.4 of [15] to $(\tilde{V}, \frac{1}{n} \tilde{B} + (\text{mult}_O(\mathcal{M})/n - 1)E)$ and $\tilde{\pi}$ we see that the locus

$$LCS\left(\tilde{V}, \frac{1}{n} \tilde{B} + (\text{mult}_O(\mathcal{M})/n - 1)\tilde{E}\right)$$

consists of a single point $\tilde{\Omega}$ in the neighborhood of the divisor $\tilde{E}$. Hence, the subvariety $\tilde{\Omega}$ is a plane in $\mathbb{P}^5$. In fact, the subvariety $\Omega$ can not be a plane\(^4\). Let us prove the latter by using the arguments of the original proof of Lemma 5 (see Theorem 3.10 in [4]).

Let $\tilde{X}$ be a general hyperplane section of $X$ passing through $O$ that is locally given as

$$xy +zt = 0 \subset \mathbb{C}^5 \cong \text{Spec}(\mathbb{C}[x,y,z,t,u])$$

in the neighborhood of $O$, which is given by $x = y = z = t = u = 0$. Then $\tilde{X}$ has non-isolated singularities, but we can apply the previous arguments to $\tilde{X}$. Namely, let $\tilde{V}$ be a proper transform of the variety $\tilde{X}$ on $V$, and $\tilde{\pi} : \tilde{V} \to \tilde{X}$ be the induced birational morphism. Then

$$K_{\tilde{V}} + \frac{1}{n} \tilde{B} + (\text{mult}_O(\mathcal{M})/n - 2)\tilde{E} \sim_{\text{Q}} \tilde{\pi}^*(K_{\tilde{X}} + \frac{1}{n} \mathcal{M}|_{\tilde{X}}),$$

where $\tilde{B} = B|_{\tilde{V}}$, and $\tilde{E}$ is the exceptional divisor of $\tilde{\pi}$, which is a cone over $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $\tilde{S}_x$ and $\tilde{S}_y$ be irreducible reduced Weil divisors on the variety $\tilde{X}$ that are given by the equations $x = t = 0$ and $y = t = 0$ respectively. Then $\tilde{S}_x$ and $\tilde{S}_y$ are not $\text{Q}$-Cartier divisors, but the divisor $\tilde{S}_x + \tilde{S}_y$ is Cartier and given by the equation $t = 0$. Moreover, the equivalence

$$K_{\tilde{V}} + \frac{1}{n} \tilde{B} + (\text{mult}_O(\mathcal{M})/n - 1)\tilde{E} + \tilde{H}_x + \tilde{H}_y \sim_{\text{Q}} \tilde{\pi}^*(K_{\tilde{X}} + \frac{1}{n} \mathcal{M}|_{\tilde{X}} + \tilde{S}_x + \tilde{S}_y),$$

holds, where $\tilde{H}_x$ and $\tilde{H}_y$ are proper transforms of $\tilde{S}_x$ and $\tilde{S}_y$ on the variety $\tilde{V}$. Then

$$\text{LCS}\left(\tilde{V}, \frac{1}{n} \tilde{B} + (\text{mult}_O(\mathcal{M})/n - 1)\tilde{E}\right) = \tilde{\Omega},$$

where $\tilde{\Omega} = \Omega|_{\tilde{V}}$, because we can apply the previous arguments to $(\tilde{X}, \frac{1}{n} \mathcal{M}|_{\tilde{X}} + \tilde{S}_x + \tilde{S}_y)$ due to the generality in the choice of $\tilde{X}$. Note, that $\tilde{\Omega}$ is a line on the quadric cone $\tilde{E} \subset \mathbb{P}^4$.

There are natural ways to desingularize $\tilde{X}$ and $\tilde{V}$. Indeed, consider a commutative diagram

where we have the following notations:

\(^4\)The referee pointed out to the author that the subvariety $\Omega$ can not be a plane. We follow the arguments of the referee to conclude the proof of Lemma 5.
• \( \tilde{\phi} \) is a blow up of the ideal sheaf of the curve \( x = y = z = t = 0 \);
• \( \tilde{\alpha}_x \) and \( \tilde{\alpha}_y \) are blow ups of the ideal sheaves of \( \tilde{S}_x \) and \( \tilde{S}_y \) respectively;
• \( \tilde{\beta}_x \) and \( \tilde{\beta}_y \) are blow ups of the exceptional surfaces of \( \tilde{\alpha}_x \) and \( \tilde{\alpha}_y \) respectively;
• \( \tilde{\xi}, \tilde{\beta}_x, \tilde{\beta}_y \) are blow ups of the fibers of \( \phi, \tilde{\alpha}_x, \tilde{\alpha}_y \) over the point \( O \) respectively;
• \( \tilde{\psi} \) is a blow up of the ideal sheaf of the proper transform of \( x = y = z = t = 0 \);
• \( \tilde{\gamma}_x \) and \( \tilde{\gamma}_y \) are blow ups of the ideal sheaves of \( \tilde{H}_x \) and \( \tilde{H}_y \) respectively;
• \( \tilde{\delta}_x \) and \( \tilde{\delta}_y \) are blow ups of the exceptional surfaces of \( \tilde{\gamma}_x \) and \( \tilde{\gamma}_y \) respectively.

The varieties \( \tilde{W}, \tilde{W}_x, \tilde{W}_y, \tilde{U}, \tilde{U}_x, \tilde{U}_y \) are smooth by construction. Moreover, the birational morphisms \( \tilde{\alpha}_x, \tilde{\alpha}_y, \tilde{\gamma}_x, \tilde{\gamma}_y \) are small\(^5\), and \( \pi \circ \tilde{\psi} = \tilde{\phi} \circ \tilde{\xi} \). Let \( \tilde{F} \) be the \( \tilde{\xi} \)-exceptional divisor. Then

\[
\tilde{F} \cong \mathbb{P}\left( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1) \right),
\]

where \( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1) \) is a hyperplane section of the quadric \( \mathbb{P}^1 \times \mathbb{P}^1 \) with respect to the natural embedding into \( \mathbb{P}^3 \). The induced morphism \( \tilde{\xi}|_{\tilde{F}} \) is the natural projection to \( \mathbb{P}^1 \times \mathbb{P}^1 \), the induced morphisms \( \tilde{\eta}_x \circ \tilde{\delta}_x|_{\tilde{F}} \) and \( \tilde{\eta}_y \circ \tilde{\delta}_y|_{\tilde{F}} \) are projections to \( \mathbb{P}^1 \), the morphisms \( \tilde{\delta}_x|_{\tilde{F}} \) and \( \tilde{\delta}_y|_{\tilde{F}} \) are contractions of the exceptional section of \( \tilde{F} \) to curves, and \( \tilde{\psi}|_{\tilde{F}} \) is the contraction of the exceptional section of the surface \( \tilde{F} \) to the vertex of the cone \( \tilde{E} \), where \( \tilde{E} = \tilde{\psi}(\tilde{F}) \).

The subvariety \( \tilde{\Omega} \) is a line on the quadric cone \( \tilde{E} \subset \mathbb{P}^4 \) that does not pass through the vertex of the quadric cone \( \tilde{E} \), but \( (\tilde{H}_x + \tilde{H}_y) \cdot \tilde{\Omega} = 1 \). We may assume that \( \tilde{H}_x \cdot \tilde{\Omega} = 0 \) and \( \tilde{H}_y \cdot \tilde{\Omega} = 1 \).

Let \( \tilde{D}_x \) and \( \tilde{D}_y \) be the proper transforms of \( \tilde{H}_x \) and \( \tilde{H}_y \) on \( \tilde{U}_y \) respectively, and \( \tilde{\Gamma} \) be the proper transform of \( \tilde{\Omega} \) on the variety \( \tilde{U}_y \). Then \( \tilde{D}_x \cdot \tilde{\Gamma} = 0 \) and \( \tilde{D}_y \cdot \tilde{\Gamma} = 1 \). Moreover, we have

\[
K_{\tilde{U}_y} + \frac{1}{n} \tilde{D} + (\text{mult}_O(\mathcal{M})/n - 1) \tilde{G} + \tilde{D}_x + \tilde{D}_y \sim_{\tilde{\mathbb{Q}}} (\tilde{\pi} \circ \tilde{\gamma}_y)^* \left( K_X + \frac{1}{n} M|_X + \tilde{S}_x + \tilde{S}_y \right),
\]

where \( \tilde{D} \) and \( \tilde{G} \) are proper transforms of the linear system \( \tilde{B} \) and exceptional divisor \( \tilde{E} \) on the variety \( \tilde{U}_y \). The morphism \( \tilde{\eta}_y \) contracts the divisor \( \tilde{G} \), but the morphism \( \tilde{\eta}_y|_{\tilde{G}} \) is a \( \mathbb{P}^2 \)-bundle.

Let \( \tilde{Y} \) be a general fiber of \( \tilde{\eta}_y|_{\tilde{G}} \). Then \( \tilde{Y} \cap \tilde{D}_x \) is a line in \( \tilde{Y} \cong \mathbb{P}^2 \), the intersection \( \tilde{\Gamma} \cap \tilde{Y} \) is a point that is not contained in \( \tilde{Y} \cap \tilde{D}_x \), and \( \tilde{Y} \cap \tilde{D}_y = \emptyset \). Therefore, in the neighborhood of the fiber \( Y \) of the morphism \( \tilde{\eta}_y \) the locus of log canonical singularities

\[
\text{LCS}\left( \tilde{U}_y, \frac{1}{n} \tilde{D} + (\text{mult}_O(\mathcal{M})/n - 1) \tilde{G} + \tilde{D}_x + \tilde{D}_y \right)
\]

consists of \( \tilde{\Gamma} \) and \( \tilde{D}_x \), which contradicts Theorem 17.4 in [15], because \( \tilde{\Gamma} \cap \tilde{D}_x = \emptyset \). \quad \square

5. The proof of Theorem 1

Let \( X \) be a hypersurface in \( \mathbb{P}^6 \) of degree 6 having at most isolated ordinary double points, which is not birationally superrigid. Let us show that this assumption leads to a contradiction.

It follows from Theorem 1 that there is a linear system \( \mathcal{M} \) on the hypersurface \( X \) that does not have fixed components such that the singularities of the log pair \( (X, \frac{1}{m} \mathcal{M}) \) are not canonical, where \( m \) is a natural number such that the rational equivalence \( \mathcal{M} \sim -mK_X \) holds.

Let \( Z \) be a proper irreducible subvariety of \( X \) such that the log pair \( (X, \frac{1}{m} \mathcal{M}) \) is not canonical in the generic point of \( Z \), and \( Z \) has maximal dimension among the subvarieties of \( X \) with such property. Then \( \dim(Z) \leq 1 \) by Theorem 2 in [21].

Suppose that either \( \dim(Z) \neq 0 \) or \( Z \) is a smooth point of the hypersurface \( X \). Let \( P \) be any sufficiently general point of \( Z \), and \( V \) be a sufficiently general hyperplane section of \( X \) passing through the point \( P \), and \( \mathcal{B} = \mathcal{M}|_V \). Then \( V \) is a smooth hypersurface in \( \mathbb{P}^5 \) of degree 6, and the singularities of \( (V, \frac{1}{m} \mathcal{B}) \) are not canonical in \( P \) by Theorem 16.7 of [15]. Let \( S_1 \) and \( S_2 \) be sufficiently general divisors in \( \mathcal{B} \), and \( F = S_1 \cdot S_2 \). Then

\[
\dim\{ O \in F \mid \text{mult}_O(F) > m \} \leq 1
\]

A birational morphism is called small if it does not contract any divisor.
by Proposition 5 in [27]. Let \( Y \) be a sufficiently general hyperplane section of \( V \) passing through the point \( P \), and \( \mathcal{P} = B|_Y \). Then \( Y \) is a smooth hypersurface in \( \mathbb{P}^4 \) of degree 6, and
\[
\dim \{ O \in F \cap Y \mid \text{mult}_O(F|_Y) > m \} \leq 0
\]
by Proposition 4.5 in [10]. On the other hand, the singularities of the log pair \( (Y, \frac{1}{m}\mathcal{P}) \) are not log canonical in \( P \) by Theorem 17.6 of [15]. Let \( \eta : \mathbb{P}^4 \dasharrow \mathbb{P}^2 \) be a general projection. Then
\[
\eta(P) \in \text{LCS}(\mathbb{P}^2, \frac{1}{4m^2}\eta_*[F|_Y])
\]
by Theorem 1.1 in [10]. Moreover, it follows from the inequality [12] and Proposition 4.7 in [10] that the singularities of the log pair \( (\mathbb{P}^2, \frac{1}{4m^2}\eta_*[F|_Y]) \) are log terminal in a punctured neighborhood of the point \( \eta(P) \). Hence, the locus LCS(\( \mathbb{P}^2, L + \frac{1}{4m^2}\eta_*[F|_Y] \)) is not connected for a sufficiently general line \( L \subset \mathbb{P}^2 \), which is impossible by Theorem 17.4 of [15], because
\[
K_{\mathbb{P}^2} + L + \frac{1}{4m^2}\eta_*[F|_Y] \sim Q - \frac{1}{2}L.
\]
Therefore, we proved that \( Z \) is a singular point of \( X \). Let \( \pi : U \to X \) be a blow up of the point \( Z \), and \( E \) be a \( \pi \)-exceptional divisor. Then \( \text{mult}_Z(\mathcal{M}) > m \) by Lemma [5] but
\[
K_U + \frac{1}{m}\mathcal{H} \sim Q \pi^*(K_X + \frac{1}{m}\mathcal{M}) + \left( 3 - \frac{1}{m}\text{mult}_Z(\mathcal{M}) \right)E,
\]
where \( \mathcal{H} \) is a proper transform of \( \mathcal{M} \) on \( U \). Let \( M_1 \) and \( M_2 \) be sufficiently general divisors in the linear system \( \mathcal{M} \). Then the inequality
\[
\text{mult}_Z(M_1 \cdot M_2) > 6m^2
\]
holds by Lemma [11]. Hence, we have
\[
6m^2 = M_1 \cdot M_2 \cdot H_1 \cdot H_2 \cdot H_3 \geq \text{mult}_Z(M_1 \cdot M_2) > 6m^2,
\]
where \( H_i \) is a sufficiently general hyperplane section of the hypersurface \( X \) that passes through the point \( Z \). The obtained contradiction proves Theorem [1].

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Steklov Institute of Mathematics
8 Gubkin street, Moscow 117966
Russia
cheltsov@yahoo.com

School of Mathematics
The University of Edinburgh
Kings Buildings, Mayfield Road
Edinburgh EH9 3JZ, UK
icheltsov@ed.ac.uk