New inequalities for sector matrices applying Garg–Aujla inequalities

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Abstract
In this paper, we give new singular value inequalities and determinant inequalities including the inverse of \( A, B, \) and \( A + B \) for sector matrices. We also give the matrix inequalities for sector matrices with a positive multilinear map. Our obtained results give generalizations for the known results.

Keywords Sector matrix · Accretive matrix · Singular value inequality · Determinant inequality · Kantorovich constant · Positive linear/multilinear map

Mathematics Subject Classification 15A45 · 15A15

1 Introduction and preliminaries
Let \( M_n \) and \( M_n^+ \) denote the set of all \( n \times n \) matrices and the set of all \( n \times n \) positive semi-definite matrices with entries in \( \mathbb{C} \), respectively. \( A \geq 0 \) means \( A \in M_n^+ \). \( A > 0 \) also means \( A \in M_n^+ \) and \( A \) is invertible. For \( A \in M_n \), the famous Cartesian decomposition of \( A \) is presented as

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$A = RA + i\mathcal{I}A,$

where the matrices $RA = \frac{A + A^*}{2}$ and $\mathcal{I}A = \frac{A - A^*}{2i}$ are the real and imaginary parts of $A$, respectively. The matrix $A \in \mathbb{M}_n$ is called an accretive, if $RA$ is a positive definite. The matrix $A \in \mathbb{M}_n$ is called an accretive-dissipative, if both $RA$ and $i\mathcal{I}A$ are positive definite. For $\alpha \in \left[0, \frac{\pi}{2}\right]$, define a sector as follows:

$$S_\alpha = \{ z \in \mathbb{C} : Rez > 0, |Imz| \leq \tan \alpha(Rez) \}.$$ 

Here, we recall that the numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{ x^*Ax : x \in \mathbb{C}^n, x^*x = 1 \}.$$ 

The matrix $A \in \mathbb{M}_n$ is called a sector, if the numerical range of $A$ is contained in a sector $S_\alpha$. In other words, $W(A) \subset S_\alpha$ for some $\alpha \in \left[0, \frac{\pi}{2}\right]$. Clearly, any sector matrix is accretive with extra information about the angle $\alpha$. The sector matrix can be regarded as a kind of generalizations of the positive definite matrix, in the sense that a sector matrix becomes a positive definite matrix when $\alpha = 0$.

In this paper, we study singular value inequalities and determinant inequalities for sector matrices. We also study the inequalities for a positive linear and multilinear map.

In the paper [9], Garg and Aujla obtained the following inequalities, where the symbol $s_j(X)$ for $j = 1, \ldots, n$, represents $j$th largest singular value of $X \in \mathbb{M}_n$.

For $k = 1, \ldots, n$ and $1 \leq r \leq 2$

$$\prod_{j=1}^k s_j(|A + B|^r) \leq \prod_{j=1}^k s_j(|I_n + |A|^r|) \prod_{j=1}^k s_j(|I_n + |B|^r|),$$

(1.1)

and

$$\prod_{j=1}^k s_j(I_n + f(|A + B|)) \leq \prod_{j=1}^k s_j(I_n + f(|A|)) \prod_{j=1}^k s_j(I_n + f(|B|),$$

(1.2)

where $A, B \in \mathbb{M}_n$ and $f : [0, \infty) \to [0, \infty)$ is an operator concave function. By taking $A, B \geq 0$, $r = 1$ and $f(t) = t$ in the inequalities (1.1) and (1.2), we have for $k = 1, \ldots, n$

$$\prod_{j=1}^k s_j(A + B) \leq \prod_{j=1}^k s_j(I_n + A) \prod_{j=1}^k s_j(I_n + B),$$

(1.3)

$$\prod_{j=1}^k s_j(I_n + A + B) \leq \prod_{j=1}^k s_j(I_n + A) \prod_{j=1}^k s_j(I_n + B).$$

(1.4)

Before we state our results, we here summarize some lemmas which will be necessary to prove our results in this paper. We should note that the expression $s_j(RA)$
may be replaced by \( \lambda_j(RA) \), where \( \lambda_j(X) \) represents the \( j \)th largest eigenvalue of \( X \in M_n \), in Lemmas 1.1 and 1.2. Also, we may replace \( s_j(\cdot) \) by \( \lambda_j(\cdot) \) in (1.3) and (1.4). Throughout this paper, we use the symbol \( s_j(\cdot) \) even when we can use \( \lambda_j(\cdot) \), since we think that it is better outlook to read this paper.

**Lemma 1.1** ([3, Proposition III.5.1]) For \( A \in M_n \), we have \( s_j(RA) \leq s_j(A) \). Thus, we have \( \det(RA) \leq |\det A| \) for an accretive matrix \( A \in M_n \).

**Lemma 1.2** ([7, Theorem 3.1], [14, Lemma 2.6]) Let \( A \in M_n \) with \( W(A) \subset S_2 \). We have \( s_j(A) \leq \sec^2(\alpha) s_j(R(A)) \) and \( |\det A| \leq \sec^o(\alpha) |\det RA| \).

We should note that \( |\det A| \leq \sec^2(\alpha) |\det RA| \) holds from \( s_j(A) \leq \sec^2(\alpha) s_j(R(A)) \) consequently. However, Lin proved the better bound as above. We give the proof of \( |\det A| \leq \sec^o(\alpha) |\det RA| \) along to [14, Lemma 2.6] for the convenience to the readers. It is stated in [14, Lemma 2.2] and proved in [22, Theorem 2.1] that a sector matrix \( A \) has a decomposition such as \( A = XZ + X^* \) with an invertible matrix \( X \) and the diagonal matrix \( Z = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \) with \( |\theta_j| \leq \alpha \) for all \( j = 1, \ldots, n \) and \( \alpha \in [0, \pi/2] \). We first found that \( |\det Z| = |e^{i\theta_1} \cdots e^{i\theta_n}| \leq 1 \) and

\[
\sec(\alpha) \det(Z) = \text{diag}\left(\frac{\cos \theta_1}{\cos \alpha}, \ldots, \frac{\cos \theta_n}{\cos \alpha}\right)
\]

which implies \( \sec(\alpha) \det(R(Z)) \geq 1 \), since

\[
\cos \theta_j \geq \cos \alpha \quad \text{for} \quad |\theta_j| \leq \alpha \quad \text{and} \quad \alpha \in [0, \pi/2].
\]

Thus, we have \( \sec^o(\alpha) \det(R(Z)) \cdot |\det(XX^*)| \geq |\det(XX^*)| \geq |\det Z| \cdot |\det(XX^*)| = |\det(ZXX^*)| = |\det A| \) which shows \( \sec^o(\alpha) \det(RA) \geq |\det A| \), since

\[
\det(R(Z)) \cdot |\det(XX^*)| = |\det(X(R(Z)X^*))| = |\det(R(A))| = |\det RA|.
\]

**Lemma 1.3** ([15, Lemma 2, Lemma 3]) Let \( A \in M_n \) with \( W(A) \subset S_2 \). Then, we have \( \mathfrak{R}(A^{-1}) \leq \mathfrak{R}^{-1}(A) \leq \sec^2(\alpha) \mathfrak{R}(A^{-1}) \). The first inequality holds for an accretive matrix \( A \in M_n \).

**Lemma 1.4** ([5, Theorem 1], [1, Corollary 1], [2, Lemma 2.3]) Let \( A, B \in M_n \) be positive definite and \( r > 0 \). Then, we have the following:

(i) \( \|AB\| \leq \frac{1}{4}\|A + B\|^2 \),

(ii) \( \|A^r + B^r\| \leq \|(A + B)^r\| \) for \( r \geq 1 \),

(iii) \( A \leq rB \Leftrightarrow \|A^\frac{1}{r}B^{-\frac{1}{r}}\| \leq r^2 \).

**Lemma 1.5** ([13, Lemma 2.9]) Let \( X \in M_n \) and \( r > 0 \). Then

\[
|X| \leq rI_n \Leftrightarrow \|X\| \leq r \Leftrightarrow \begin{bmatrix} rI_n & X \\ X^* & rI_n \end{bmatrix} \geq 0.
\]
Throughout this paper, we use the famous Kantorovich constant $K(h) := \frac{(h + 1)^2}{4h}$ for $h > 0$. The constant with $h := M/m$ was originally appeared in [10, p.142] as the so-called Kantorovich inequality.

2 Singular value and determinant inequalities

We first review the Tan-Xie inequality for sector matrices $A, B \in M_n$ and $v \in [0, 1]$ given in [19, Theorem 2.4]

$$\cos^2(x)\Re(A!_v B) \leq \Re(A^\sharp_v B) \leq \sec^2(x)\Re(A\nabla_v B), \tag{2.1}$$

where $A!_v B = ((1 - v)A^{-1} + vB^{-1})^{-1}$, $A^\sharp_v B = \frac{\sin v\pi}{\pi} \int_0^\infty t^{v-1}(A^{-1} + tB^{-1})^{-1}dt$, $A\nabla_v B = (1 - v)A + vB$ are the weighted operator harmonic mean, geometric mean, and arithmetic mean, respectively. The weighted geometric mean for accretive operators $A, B$ in the above was introduced in [18, Definition 2.1] which coincides with $A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ when $A, B$ are strictly positive operators. It also becomes $A^\sharp_v B := \frac{2}{\pi} \int_0^\infty \left( tA^{-1} + t^{-1}B^{-1} \right)^{-1}dt$ for $v = 1/2$, which was introduced in [6]. We use the symbols $!, \sharp$ and $\nabla$ instead of $!_{1/2}, \sharp_{1/2}$ and $\nabla_{1/2}$, respectively, for simplicity. The above double inequality (2.1) can be regarded as a generalization of the operator Young inequality

$$A!_v B \leq A^\sharp_v B \leq A\nabla_v B, \quad (A, B \geq 0, \quad 0 \leq v \leq 1).$$

From (2.1), we easily find that

$$\Re(A + B)^{-1} \leq \frac{\sec^4(x)}{4} \Re(A^{-1} + B^{-1}) \tag{2.2}$$

by putting $v = \frac{1}{2}$, $A^{-1} := A$ and $B^{-1} := B$.

However, we can improve the inequality (2.2) by the following lemma.

Lemma 2.1 Let $A, B \in M_n$ with $W(A), W(B) \subset S_2$, and $0 \leq v \leq 1$. Then

$$\Re(A\nabla_v B)^{-1} \leq \sec^2(x)\Re(A^{-1}\nabla_v B^{-1}). \tag{2.3}$$

Proof The calculations show that
\[\Re((1-v)A + vB)^{-1} \leq (\Re((1-v)A + vB))^{-1}\]
\[= ((1-v)\Re A + v\Re B)^{-1}\]
\[\leq (1-v)\Re^{-1}A + v\Re^{-1}B\]
\[\leq \sec^2(x)((1-v)\Re^{-1}A + v\Re^{-1}B)\]
\[= \sec^2(x)\Re(1-v)A^{-1} + vB^{-1}).\]

The first and the third inequality are due to Lemma 1.3. The second inequality is due to the operator convexity of \(t^{-1}\) on \((0, \infty)\).

Taking \(v = \frac{1}{2}\) in (2.3), we have
\[\Re(A + B)^{-1} \leq \frac{\sec^2(x)}{4} \Re(A^{-1} + B^{-1}),\] (2.4)

which improves the inequality (2.2). We use the inequality (2.4) to prove the following Theorems 2.2 and 2.6. From the process of the proof in [19, Theorem 2.4], we have for \(A, B \in \mathbb{M}_n\) with \(W(A), W(B) \subseteq S_x\)
\[\Re(A^+_x, B) \leq \sec^2(x)(\Re(A)_x^+, \Re(B)).\] (2.5)

For the convenience to the readers, we give the proof of (2.5). Indeed, we have
\[\Re(A^+_x, B) = \frac{\sin v\pi}{\pi} \int_0^{\infty} t^{v-1} \Re^{-1}(A^{-1} + tB^{-1}) dt\]
\[\leq \frac{\sin v\pi}{\pi} \int_0^{\infty} t^{v-1} \sec^2(x)(\Re^{-1}(A) + t\Re^{-1}(B))^{-1} dt\]
\[= \sec^2(x)\Re(A)_x^+, \Re(B).\]

The above inequality can be proven by the use of Lemma 1.3. Actually, we have the following from the second inequality in Lemma 1.3:
\[\Re(A^{-1}) + t\Re(B^{-1}) \geq \cos^2(x)(\Re^{-1}(A) + t\Re^{-1}(B)),\]

which implies that
\[(\Re(A^{-1}) + t\Re(B^{-1}))^{-1} \leq \sec^2(x)(\Re^{-1}(A) + t\Re^{-1}(B))^{-1}.\]

Thus, we reach to
\[\Re^{-1}(A^{-1} + tB^{-1}) \leq \sec^2(x)(\Re^{-1}(A) + t\Re^{-1}(B))^{-1},\]
since for any \(t \geq 0\)
\[\Re^{-1}(A^{-1} + tB^{-1}) = (\Re(A^{-1}) + t\Re(B^{-1}))^{-1}.\]

On the other hand, by [12, Corollary 3.1], we have
\[ \Re(A)^v \leq \Re(A) \nabla_v \Re(B) - 2r_{\min}(\Re(A \nabla B) - \Re(A)^v \Re(B)) \]

for \( r_{\min} := \min\{1 - v, v\} \) with \( v \in [0, 1] \). Thus

\[
\Re(A)^v \leq \sec^2(\beta)(\Re(A) \nabla_v \Re(B)) - 2r_{\min} \sec^2(\beta)(\Re(A \nabla B) - \Re(A)^v \Re(B)) \tag{2.6}
\]

which shows that (2.6) is a refinement of the second inequality of (2.1). From now on, we study some singular value inequalities. By a consequence of (2.3) with Lemma 1.1 and 1.2, we also see the inequalities

\[
\prod_{j=1}^{k} s_j(A,!v,B) \leq \sec^{2k}(\beta) \prod_{j=1}^{k} s_j(\Re(A,!v,B)) \leq \sec^{4k}(\beta) \prod_{j=1}^{k} s_j(\Re(A \nabla_v B)) \\
\leq \sec^{4k}(\beta) \prod_{j=1}^{k} s_j(A \nabla_v B).
\]

We aim to obtain the singular value inequalities including the inverse of \( A, B, \) and \( A + B \).

**Theorem 2.2** Let \( A, B \in M_n \) with \( W(A), W(B) \subset S_\beta \). Then, we have for \( k = 1, \ldots, n \)

\[
\prod_{j=1}^{k} s_j(A + B)^{-1} \leq \frac{\sec^{4k}(\beta)}{4k} \prod_{j=1}^{k} s_j(I_n + A^{-1}) \prod_{j=1}^{k} s_j(I_n + B^{-1}), \tag{2.7}
\]

\[
\prod_{j=1}^{k} s_j(I_n + (A + B)^{-1}) \\
\leq \sec^{2k}(\beta) \prod_{j=1}^{k} s_j(I_n + \frac{\sec^2(\beta)}{4} A^{-1}) \prod_{j=1}^{k} s_j(I_n + \frac{\sec^2(\beta)}{4} B^{-1}). \tag{2.8}
\]

**Proof** Since sum of two sector matrices and inverse of every sector matrix are also sector, \( (A + B)^{-1} \) is a sector matrix. On the other hand, every sector matrix is an accretive. Thus, we calculate the following chain of inequalities:
\[
\prod_{j=1}^{k} s_j(A + B)^{-1} \leq \sec^{2k}(x) \prod_{j=1}^{k} s_j(\Re(A + B)^{-1}) \quad \text{(by Lemma 1.2)}
\]

\[
\leq \frac{\sec^{4k}(x)}{4^k} \prod_{j=1}^{k} s_j(\Re(A^{-1} + B^{-1})) \quad \text{(by (2.4))}
\]

\[
= \frac{\sec^{4k}(x)}{4^k} \prod_{j=1}^{k} s_j(\Re(A^{-1}) + \Re(B^{-1}))
\]

\[
\leq \frac{\sec^{4k}(x)}{4^k} \prod_{j=1}^{k} s_j(I_n + \Re(A^{-1})) \prod_{j=1}^{k} s_j(I_n + \Re(B^{-1})) \quad \text{(by (1.3))}
\]

\[
= \frac{\sec^{4k}(x)}{4^k} \prod_{j=1}^{k} s_j(I_n + A^{-1}) \prod_{j=1}^{k} s_j(I_n + B^{-1})
\]

\[
\leq \frac{\sec^{4k}(x)}{4^k} \prod_{j=1}^{k} s_j(I_n + A^{-1}) \prod_{j=1}^{k} s_j(I_n + B^{-1}) \quad \text{(by Lemma 1.1).}
\]

Similarly, we have

\[
\prod_{j=1}^{k} s_j(I_n + (A + B)^{-1})
\]

\[
\leq \sec^{2k}(x) \prod_{j=1}^{k} s_j(\Re(I_n + (A + B)^{-1})) \quad \text{(by Lemma 1.2)}
\]

\[
\leq \sec^{2k}(x) \prod_{j=1}^{k} s_j \left( I_n + \frac{\sec^{2}(x)}{4} \Re(A^{-1} + B^{-1}) \right) \quad \text{(by (2.4))}
\]

\[
= \sec^{2k}(x) \prod_{j=1}^{k} s_j \left( I_n + \frac{\sec^{2}(x)}{4} \Re(A^{-1}) + \frac{\sec^{2}(x)}{4} \Re(B^{-1}) \right)
\]

\[
\leq \sec^{2k}(x) \prod_{j=1}^{k} s_j \left( I_n + \frac{\sec^{2}(x)}{4} \Re(A^{-1}) \right) \prod_{j=1}^{k} s_j \left( I_n + \frac{\sec^{2}(x)}{4} \Re(B^{-1}) \right)
\]

(by (1.4))

\[
= \sec^{2k}(x) \prod_{j=1}^{k} s_j \left( \Re(I_n + \frac{\sec^{2}(x)}{4} A^{-1}) \right) \prod_{j=1}^{k} s_j \left( \Re(I_n + \frac{\sec^{2}(x)}{4} B^{-1}) \right)
\]

\[
\leq \sec^{2k}(x) \prod_{j=1}^{k} s_j \left( I_n + \frac{\sec^{2}(x)}{4} A^{-1} \right) \prod_{j=1}^{k} s_j \left( I_n + \frac{\sec^{2}(x)}{4} B^{-1} \right)
\]

(by Lemma 1.1).
Remark 2.3 We may claim that Theorem 2.2 is a non-trivial result, since the inequality (1.4) is true whenever \( f \) is an operator concave function. However, inequalities (2.7) and (2.8) with \( \alpha = 0 \) are true, although the function \( f(t) = t^{-1} \) for \( t > 0 \) is not an operator concave. Therefore, we found the upper bound \( \prod_{j=1}^{k} s_j(I_n + (A + B)^{-1}) \) without using (1.2).

We also note that we can obtain the inequality (2.7) for the special case \( A, B > 0 \) from (1.3) in the following. Since \( A!B \leq A \nabla B \)

\[
\prod_{j=1}^{k} 2s_j(A!B) \leq \prod_{j=1}^{k} 2s_j(A \nabla B) \leq \prod_{j=1}^{n} s_j(I_n + A) \prod_{j=1}^{n} s_j(I_n + B).
\]

If we put \( A := A^{-1} \) and \( B := B^{-1} \), then we get (2.7) for \( \alpha = 0 \).

Note that the following proposition has already been proven in [17, Eq.(15)].

Proposition 2.4 ([17]) Let \( A, B \in \mathbb{M}_n \) with \( W(A), W(B) \subset S_{2\alpha} \). Then, we have for \( k = 1, \ldots, n \)

\[
\prod_{j=1}^{k} s_j(A + B) \leq \prod_{j=1}^{k} s_j(I_n + \sec^2(\alpha)A) \prod_{j=1}^{k} s_j(I_n + \sec^2(\alpha)B). \quad (2.9)
\]

Next, we study some determinant inequalities in the rest of this section. On the determinant inequality, the following is well known [23, Theorem 7.7]

\[
\det(A + B) \geq \det A + \det B, \quad (A, B \geq 0). \quad (2.10)
\]

With this, we have the following inequality for sector matrices \( A \) and \( B \):

\[
|\det(A + B)| \geq \det(\Re(A + B)) \quad \text{(by Lemma 1.1)}
\]

\[
= \det(\Re(A) + \Re(B)) \geq \det(\Re(A)) + \det(\Re(B)) \quad \text{(by (2.10))}
\]

\[
\geq \cos^n(\alpha)(|\det(A)| + |\det(B)|) \quad \text{(by Lemma 1.2).}
\]

If \( A, B \geq 0 \), that is, \( \alpha = 0 \), then (2.11) becomes (2.10). Also, (2.11) is a reverse of [21, Eq.(13)]. Of course, (2.11) is trivial for \( A, B \geq 0 \), since \( \cos(\alpha) \leq 1 \) for \( \alpha \in \left[0, \frac{\pi}{2}\right] \).

For further inequalities on determinant, we give the following remark.

Remark 2.5

(i) For \( A, B \in \mathbb{M}_n \) with \( W(A), W(B) \subset S_{\alpha} \), we have
\[ |\det(A)||\det(B)| \leq \sec^n(z)(\det(\mathcal{R}(A))! \det(\mathcal{R}(B))) \quad \text{(by Lemma 1.2)} \]
\[ \leq \sec^n(z)(\det(\mathcal{R}(A))\nabla \det(\mathcal{R}(B))) \]
\[ \leq \sec^n(z)(|\det(A)|\nabla |\det(B)|) \quad \text{(by Lemma 1.1)}. \]

(2.12)

(ii) For \( A, B \in \mathbb{M}_n \) with \( W(A), W(B) \subset S_\pi \), such that \( 0 < ml_n \leq \mathcal{R}(A), \mathcal{R}(B) \leq MI_n \), we have
\[ |\det(A)||\det(B)| \geq \det(\mathcal{R}(A))! \det(\mathcal{R}(B)) \quad \text{(by Lemma 1.1)} \]
\[ \geq K^{-2}(h)(\det(\mathcal{R}(A))\nabla \det(\mathcal{R}(B))) \]
\[ \geq K^{-2}(h)\cos^n(z)(|\det(A)|\nabla |\det(B)|) \quad \text{(by Lemma 1.2)}. \]

In the second inequality, we used the scalar inequality \( a\nabla b \leq K^2(h)a!b \) for \( 0 < m \leq a, b \leq M \) with \( h := M/m \).

We here aim to obtain the determinant inequalities including the inverse of \( A, B \), and \( A + B \) as shown in Theorem 2.2.

**Theorem 2.6** Let \( A, B \in \mathbb{M}_n \) with \( W(A), W(B) \subset S_\pi \). Then
\[ |\det(A + B)^{-1}| \leq \frac{\sec^3(z)}{4^n} |\det(I_n + A^{-1})| \cdot |\det(I_n + B^{-1})| \quad \text{(2.14)} \]
and
\[ |\det(I_n + (A + B)^{-1})| \leq \sec^n(z)\left|\det\left(I_n + \frac{\sec^2(z)}{4}A^{-1}\right)\right| \cdot \left|\det\left(I_n + \frac{\sec^2(z)}{4}B^{-1}\right)\right|. \quad \text{(2.15)} \]

**Proof** The following direct calculations imply the results, since \( (A + B)^{-1} \) and \( A^{-1} + B^{-1} \) are sector:
\[ |\det(A + B)^{-1}| \leq \sec^n(x) \det(\Re(A + B)) \quad \text{(by Lemma 1.2)} \]
\[ \leq \frac{\sec^n(x)}{4^n} \det(\Re(A^{-1} + B^{-1})) \quad \text{(by (2.4))} \]
\[ = \frac{\sec^n(x)}{4^n} \det(\Re(A^{-1}) + \Re(B^{-1})) \]
\[ \leq \frac{\sec^n(x)}{4^n} \det(I_n + \Re(A^{-1})) \det(I_n + \Re(B^{-1})) \quad \text{(by } k = n \text{ in (1.3))} \]
\[ = \frac{\sec^n(x)}{4^n} \det(\Re(I_n + A^{-1})) \det(\Re(I_n + B^{-1})) \]
\[ \leq \frac{\sec^n(x)}{4^n} |\det(I_n + A^{-1})||\det(I_n + B^{-1})| \quad \text{(by Lemma 1.1).} \]

Similarly
\[ |\det(I_n + (A + B)^{-1})| \]
\[ \leq \sec^n(x) \det(\Re(I_n + (A + B))^{-1}) \quad \text{(by Lemma 1.2)} \]
\[ = \sec^n(x) \det(I_n + \Re(A + B)^{-1}) \]
\[ \leq \sec^n(x) \det\left(I_n + \frac{\sec^2(x)}{4} \Re(A^{-1} + B^{-1})\right) \quad \text{(by (2.4))} \]
\[ = \sec^n(x) \det\left(I_n + \frac{\sec^2(x)}{4} \Re(A^{-1}) + \frac{\sec^2(x)}{4} \Re(B^{-1})\right) \]
\[ \leq \sec^n(x) \det\left(I_n + \frac{\sec^2(x)}{4} \Re(A^{-1})\right) \det\left(I_n + \frac{\sec^2(x)}{4} \Re(B^{-1})\right) \quad \text{(by } k = n \text{ in (1.4))} \]
\[ = \sec^n(x) \det\left(\Re\left(I_n + \frac{\sec^2(x)}{4} A^{-1}\right)\right) \det\left(\Re\left(I_n + \frac{\sec^2(x)}{4} B^{-1}\right)\right) \]
\[ \leq \sec^n(x) \left|\det\left(I_n + \frac{\sec^2(x)}{4} A^{-1}\right)\right| \cdot \left|\det\left(I_n + \frac{\sec^2(x)}{4} B^{-1}\right)\right| \quad \text{(by Lemma 1.1).} \]

\[ \square \]

**Remark 2.7** Under the special assumption, such that \( A, B > 0 \), the inequalities (2.14) and (2.15) are trivially derived from the inequalities (1.3) and (1.4) with \( k = n \), respectively. Indeed, from (1.3) and \( A!B \leq A \nabla B \), we have
\[ 2^n \det(A!B) \leq 2^n \det(A \nabla B) \leq \det(I_n + A) \cdot \det(I_n + B). \]
By putting \( A := A^{-1}, B := B^{-1} \) in the above inequality, we have
\[ 2^n \det(A^{-1}!B^{-1}) \leq \det(I_n + A^{-1}) \cdot \det(I_n + B^{-1}), \]
which is equivalent to the inequality (2.14) for \( x = 0 \), taking an absolute value in
both sides.

Similarly, we have
\[
\det(I_n + 2A \! B) \leq \det(I_n + 2A \nabla B) \leq \det(I_n + A) \cdot \det(I_n + B)
\]
from (1.4), and \( A \! B \leq A \nabla B \). By putting \( A := \frac{1}{4} A^{-1}, \ B := \frac{1}{4} B^{-1} \) above, we have
\[
\det\left(I_n + (A + B)^{-1}\right) \leq \det \left( I_n + \frac{1}{4} A^{-1} \right) \cdot \det \left( I_n + \frac{1}{4} B^{-1} \right),
\]
which is equivalent to the inequality (2.15) for \( \alpha = 0 \), taking an absolute value in both sides.

However, we have to state that the above derivations are true for the case \( A, B \geq 0 \) and would like to emphasize that Theorem 2.6 is valid for sector matrices \( A, B \) which are more general condition than \( A, B > 0 \).

It is quite natural to consider the lower bound. We give a result for this question.

**Proposition 2.8** Let \( A, B \in \mathbb{M}_n \) with \( W(A), W(B) \subset S_\alpha \). If we have \( 0 < m \mathbb{R} \mathbb{R}(A^{-1}) \leq \mathbb{R}(B^{-1}) \leq M \mathbb{R} \mathbb{R}(A^{-1}) \), then
\[
|\det(A \! B)| \geq \frac{\cos^{3n}(x) \kappa^{-n}}{2^n} (|\det A| + |\det B|), \tag{2.16}
\]
where \( \kappa := \max\{K^2(m), K^2(M)\} \) and \( K(x) := \frac{(x + 1)^2}{4x} \) for \( x > 0 \).

**Proof** Since \( K(x) \geq 1 \) for \( x > 0 \), we have the scalar inequality \( \frac{1 + x}{2} \leq K^2(x) \frac{2x}{x + 1} \) for \( x > 0 \). By the standard functional calculus, we have
\[
\mathbb{R}(A^{-1}) \nabla \mathbb{R}(B^{-1}) \leq \kappa \ \mathbb{R}(A^{-1}) \! \mathbb{R}(B^{-1}), \tag{2.17}
\]
under the assumption \( 0 < m \mathbb{R} \leq \mathbb{R}(A^{-1})^{-1/2} \mathbb{R}(B^{-1}) \mathbb{R}(A^{-1})^{-1/2} \leq M \mathbb{R} \mathbb{R} \). The inequality (2.17) implies
\[
\mathbb{R}^{-1}(A + B) \geq \frac{\kappa^{-1}}{4} \left( \mathbb{R}^{-1}(A) + \mathbb{R}^{-1}(B) \right), \tag{2.18}
\]
putting \( A^{-1} =: A \) and \( B^{-1} =: B \). Thus, we have the following calculations:
\[ |\text{det}(A + B)^{-1}| \geq \text{det}(\Re(A + B)^{-1}) \quad \text{(by Lemma 1.1)} \]
\[ \geq \cos^{2n}(z) \text{det}(\Re^{-1}(A + B)) \quad \text{(by Lemma 1.3)} \]
\[ \geq \frac{\cos^{2n}(z)\kappa^{-n}}{4^n} \text{det}(\Re^{-1}(A) + \Re^{-1}(B)) \quad \text{(by (2.18))} \]
\[ \geq \frac{\cos^{2n}(z)\kappa^{-n}}{4^n} \text{det}(\Re^{-1}(A) + \Re^{-1}(B^{-1})) \quad \text{(by Lemma 1.3)} \]
\[ \geq \frac{\cos^{2n}(z)\kappa^{-n}}{4^n} \left( |\text{det}(A^{-1})| + |\text{det}(B^{-1})| \right) \quad \text{(by (2.10))} \]
\[ \geq \frac{\cos^{3n}(z)\kappa^{-n}}{4^n} \quad \text{(by Lemma 1.2)}, \]

which implies (2.16) by putting \( A^{-1} := A \) and \( B^{-1} := B \). \( \square \)

Closing this section, we give a few comments on our results, Theorem 2.2. For the special case, \( \alpha = 0 \) in Theorem 2.2, then we have \( A, B > 0 \). Then, two inequalities (2.7) and (2.8) give upper bounds for any \( k = 1, 2, \ldots, n \) respectively

\[ \prod_{j=1}^{k} \lambda_j(A + B)^{-1} \leq \prod_{j=1}^{k} \hat{\lambda}_j \left( I_n + A^{-1} \right) \prod_{j=1}^{k} \hat{\lambda}_j \left( I_n + B^{-1} \right) \]

and

\[ \prod_{j=1}^{k} \lambda_j(I_n + (A + B)^{-1}) \leq \prod_{j=1}^{k} \lambda_j \left( I_n + \frac{1}{4}A^{-1} \right) \prod_{j=1}^{k} \hat{\lambda}_j \left( I_n + \frac{1}{4}B^{-1} \right) \]

since \( A^{-1}, B^{-1} > 0 \) and \( (A + B)^{-1} > 0 \).

Therefore, it is of interest to consider the following singular value inequalities hold or not for any non-singular \( A, B, A + B \in \mathbb{M}_n \) and any \( k = 1, \ldots, n \):

\[ \prod_{j=1}^{k} s_j(A + B)^{-1} \leq \prod_{j=1}^{k} s_j(I_n + A^{-1}) \prod_{j=1}^{k} s_j(I_n + B^{-1}) \]

and

\[ \prod_{j=1}^{k} s_j \left( I_n + (A + B)^{-1} \right) \leq \prod_{j=1}^{k} s_j(I_n + A^{-1}) \prod_{j=1}^{k} s_j(I_n + B^{-1}). \]

However, the above inequalities do not hold in general. We give counter-examples. First, take \( k = 1 \) and
\[
A := \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 3 \\ 1 & 3 & 20 \end{pmatrix}, \quad B := \begin{pmatrix} 100 & 2 & -3 \\ 2 & 1 & 4 \\ -3 & 4 & 1 \end{pmatrix}.
\]

By the numerical computations, we have
\[
s_1(A + B)^{-1} \simeq 3.07774, \quad s_1\left(I_3 + (A + B)^{-1}\right) \simeq 2.07774, \quad s_1(I_3 + A^{-1})s_1(I_3 + B^{-1}) \simeq 1.82851.
\]

Thus, the following norm inequality does not hold in general:
\[
\min \left\{ \| (A + B)^{-1} \|, \| I_n + (A + B)^{-1} \| \right\} \leq \| I_n + A^{-1} \| \cdot \| I_n + B^{-1} \|
\]
for any non-singular hermitian \( A, B, A + B \in \mathbb{M}_n \).

Second, we can show that the following determinantal inequality:
\[
\min \left\{ | \det((A + B)^{-1})|, | \det(I_n + (A + B)^{-1})| \right\} \leq | \det(I_n + A^{-1})| \cdot | \det(I_n + B^{-1})|
\]
also does not hold in general for any non-singular hermitian \( A, B, A + B \in \mathbb{M}_n \). Indeed, we take a counter-example for the above inequality as
\[
A := \begin{pmatrix} 1 & -1 & 2.5 \\ -1 & 2 & -2 \\ 2.5 & -2 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} -1 & 1 & -3 \\ 1 & -1 & 1 \\ -3 & 1 & -1 \end{pmatrix}.
\]

Then, we have
\[
| \det((A + B)^{-1})| = 4, \quad | \det(I_3 + (A + B)^{-1})| = 2, \quad | \det(I_3 + A^{-1})| \cdot | \det(I_3 + B^{-1})| \simeq 1.84091
\]
by the numerical computations.

### 3 Matrix inequalities for positive multilinear maps

In the paper [16], the authors obtained the following result for two accretive operators \( A, B \) on a Hilbert space:
\[
\Re(A) \not\subset \Re(B) \leq \Re(A \# B).
\]

The authors extended the above inequality as follows [18]:
\[
\Re(A) \not\subset \Re(B) \leq \Re(A \#_{\nu} B), \quad (3.1)
\]
where \( 0 \leq \nu \leq 1 \).

A linear map \( \Phi : \mathbb{M}_n \to \mathbb{M}_l \) is said to be a positive if \( \Phi(A) \succeq 0 \) whenever \( A \succeq 0 \) and \( \Phi \) is called a normalized if \( \Phi(I_n) = I_l \).
A \leq B \Rightarrow A^2 \leq B^2 \text{ is not true in general. However, we have the following useful fact.}

**Lemma 3.1** \((8, \text{Theorem 6})\) If \(A, B \in \mathbb{M}_n\) satisfy \(0 \leq A \leq B\) and \(0 < mI_n \leq A \leq MI_n\) with \(h := M/m\), then we have \(A^2 \leq K(h)B^2\).

We have the following squared inequalities for (2.5), (3.1) and the second inequality in Lemma 1.3 by a direct consequence of Lemma 3.1, with \(K(1/h) = K(h)\).

**Proposition 3.2** \(\text{Let } 0 \leq v \leq 1.\)

(i) Let \(A, B \in \mathbb{M}_n\) with \(W(A), W(B) \subset S_2\), such that \(0 < mI_n \leq \mathcal{R}(A), \mathcal{R}(B) \leq MI_n\) with \(h := M/m\). Then, we have

\[
\mathcal{R}^2(A_{\#v}B) \leq \sec^4(z)K(h)(\mathcal{R}(A_{\#v}B))^2.
\]

(ii) Let \(A, B \in \mathbb{M}_n\) be accretive, such that \(0 < mI_n \leq \mathcal{R}(A), \mathcal{R}(B) \leq MI_n\) with \(h := M/m\). Then, we have

\[
(\mathcal{R}(A_{\#v})^2 \mathcal{R}(B))^2 \leq K(h)\mathcal{R}^2(A_{\#v}B).
\]

(iii) Let \(A \in \mathbb{M}_n\) with \(W(A) \subset S_2\), such that \(0 < mI_n \leq \mathcal{R}(A) \leq MI_n\) with \(h := M/m\). Then, for every normalized positive linear map \(\Phi\)

\[
\Phi^2(\mathcal{R}^{-1}(A)) \leq \sec^4(z)K(h)\Phi^2(\mathcal{R}^{-1}(A)).
\]

**Corollary 3.3** \(\text{Let } A \in \mathbb{M}_n\) with \(W(A) \subset S_2\), such that \(0 < mI_n \leq \mathcal{R}(A) \leq MI_n\) with \(h := M/m\). Then, for every normalized positive linear map \(\Phi\)

\[
|\Phi(\mathcal{R}^{-1}(A))\Phi^{-1}(\mathcal{R}(A^{-1})) + \Phi^{-1}(\mathcal{R}(A^{-1}))\Phi(\mathcal{R}^{-1}(A))| \leq 2 \sec^2(z)K^{1/2}(h).
\]

**Proof** By the use of Lemma 1.4 (iii) with Proposition 3.2 (iii), we have

\[
\|\Phi(\mathcal{R}^{-1}(A)\Phi^{-1}(\mathcal{R}(A^{-1})))\| \leq \sec^2(z)K^{1/2}(h).
\]

Using Lemma 1.5 with (3.6), we have

\[
\begin{bmatrix}
K^{1/2}(h) \sec^2(z) & \Phi(\mathcal{R}^{-1}(A))\Phi^{-1}(\mathcal{R}(A^{-1})) \\
(\Phi(\mathcal{R}^{-1}(A))\Phi^{-1}(\mathcal{R}(A^{-1})))^* & K^{1/2}(h) \sec^2(z)
\end{bmatrix} \geq 0
\]

and

\[
\begin{bmatrix}
K^{1/2}(h) \sec^2(z) & \Phi^{-1}(\mathcal{R}(A^{-1}))\Phi(\mathcal{R}^{-1}(A)) \\
(\Phi^{-1}(\mathcal{R}(A^{-1}))\Phi(\mathcal{R}^{-1}(A)))^* & K^{1/2}(h) \sec^2(z)
\end{bmatrix} \geq 0.
\]

Summing up above two matrices, and then dividing by 2 and using Lemma 1.5, we get the desired result. \(\square\)
A map $\Phi : \mathbb{M}_n^k := \mathbb{M}_n \times \cdots \times \mathbb{M}_n \to \mathbb{M}_l$ is said to be a multilinear whenever it is linear in each of its variable and also is called a positive if $A_i \geq 0$ for $i = 1, \ldots, k$ implies that $\Phi(A_1, \ldots, A_k) \geq 0$. Moreover, $\Phi$ is called a normalized if $\Phi(I_n, \ldots, I_n) = I_l$.

**Lemma 3.4** ([11]) Let $A_i \in \mathbb{M}_n(1 \leq i \leq k)$, such that $0 < mI_n \leq A_i \leq MI_n$ with $h := M/m$. Then, for every positive multilinear map $\Phi$

$$\Phi(A_1 \ldots, A_k^{-1}) \leq K(h^k)\Phi(A_1, \ldots, A_k)^{-1}. \quad (3.7)$$

Let $A_i \in \mathbb{M}_n(1 \leq i \leq k)$ be accretive such that $0 < mI_n \leq \Re(A_i) \leq MI_n$. Then, by (3.7) and Lemma 1.3, we get

$$\Phi(\Re(A_1^{-1}), \ldots, \Re(A_k^{-1})) \leq K(h^k)\Phi(\Re A_1, \ldots, \Re A_k)^{-1}, \quad (3.8)$$

where $\Phi$ is a positive multilinear map. In the following, we present a square of (3.8).

**Theorem 3.5** Let $A_i \in \mathbb{M}_n(1 \leq i \leq k)$ be accretive such that $0 < mI_n \leq \Re(A_i) \leq MI_n$ with $h := M/m$. Then, for every positive multilinear map $\Phi$

$$\Phi^2(\Re(A_1^{-1}), \ldots, \Re(A_k^{-1})) \leq K^2(h^k)\Phi^{-2}(\Re A_1, \ldots, \Re A_k). \quad (3.9)$$

**Proof** If we apply [11, Lemma 2.5] with $r = -1$, then we have

$$M^k m^k \Phi(\Re^{-1}(A_1), \ldots, \Re^{-1}(A_k)) + \Phi(\Re A_1, \ldots, \Re A_k) \leq M^k + m^k. \quad (3.10)$$

On the other hand, we have $\Re^{-1}(A_i) - \Re(A_i^{-1}) \geq 0$ by Lemma 1.3; therefore, we have

$$\Phi(\Re^{-1}(A_1), \ldots, \Re^{-1}(A_k)) \geq \Phi(\Re(A_1^{-1}), \ldots, \Re(A_k^{-1})). \quad (3.11)$$

From (3.10) and (3.11), we obtain

$$M^k m^k \Phi(\Re(A_1^{-1}), \ldots, \Re(A_k^{-1})) + \Phi(\Re A_1, \ldots, \Re A_k) \leq M^k + m^k. \quad (3.12)$$

By applying Lemma 1.4 (i) and (3.12), respectively, it follows that:

$$M^k m^k \|\Phi(\Re(A_1^{-1}), \ldots, \Re(A_k^{-1}))\Phi(\Re(A_1), \ldots, \Re(A_k))\|$$

$$\leq \frac{1}{4} \|M^k m^k \Phi(\Re(A_1^{-1}), \ldots, \Re(A_k^{-1})) + \Phi(\Re(A_1), \ldots, \Re(A_k))\|^2 \quad (3.13)$$

$$\leq \frac{1}{4} (M^k + m^k)^2.$$ 

This completes the proof, by Lemma 1.4 (iii).

**Remark 3.6** Theorem 3.5 gives a general result in the following sense.

(i) If we put $k = 1$, then Theorem 3.5 recovers [20, Theorem 2.9].
For a special case, such that $A_i \geq 0$ $(1 \leq i \leq k)$, Theorem 3.5 recovers [11, Theorem 2.6].

**Remark 3.7** Let $A_i \in M_n(1 \leq i \leq k)$ be accretive such that $0 < ml_n \leq \Re(A_i) \leq Ml_n$. If $0 \leq p \leq 2$, then $0 \leq \frac{p}{2} \leq 1$. By Theorem 3.5 and the Löwner–Heinz inequality (see, e.g., [23, Theorem 7.10]), we have

$$\Phi^p(\Re(A_1^{-1}), \ldots, \Re(A_k^{-1})) \leq K^p(h^k)\Phi^{-p}(\Re A_1, \ldots, \Re A_k).$$ (3.14)

for every positive normalized multilinear map $\Phi : M_n^k \to M_l$ and Kantorovich constant $K(h)$ with $h = \frac{M}{m}$. If $p > 2$, then using a similar method in Theorem 3.5 and using Lemma 1.4 (ii), we get

$$\Phi^p(\Re(A_1^{-1}), \ldots, \Re(A_k^{-1})) \leq K^p(h^k)\Phi^{-p}(\Re A_1, \ldots, \Re A_k).$$ (3.15)

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