THE GRAVIGUT ALGEBRA IS NOT A SUBALGEBRA OF $E_8$, BUT $E_8$ DOES CONTAIN AN EXTENDED GRAVIGUT ALGEBRA

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Abstract. The GraviGUT algebra is defined as the semidirect sum of $\text{spin}(11, 3)$ together with its positive chirality 64-dimensional irrep. Recently, Lisi constructed a particular embedding of the GraviGUT algebra into the quaternionic real form of $E_8$. This is not a true Lie algebra embedding since the 64-dimensional irrep of $\text{spin}(11, 3)$ must be regarded merely as a subset of $E_8$, and not a subalgebra of $E_8$. We prove the stronger statement that the complexified GraviGUT algebra cannot be embedded into the complex algebra $E_8$. We then modify Lisi’s construction to create true Lie algebra embeddings of Extended GraviGUT algebras into $E_8$. We classify these embeddings up to inner automorphism. We make no claims on the physical significance of the modified construction.

1. Introduction

The Standard Model of particle physics, with gauge group $U(1) \times SU(2) \times SU(3)$, attempts to describe all particles and all forces, except gravity. Grand Unified Theories (GUT) attempt to unify the forces and particles of the Standard Model. The three main GUTs are Georgi and Glashow’s $SU(5)$ theory, Georgi’s $\text{Spin}(10)$ theory, and the Pati-Salam model based on the Lie group $SU(2) \times SU(2) \times SU(4)$ [2].

In [8], Lisi attempts to construct a unification which includes gravity. In this construction, Lisi first embeds gravity and the standard model into $\text{spin}(11, 3)$. He then embeds $\text{spin}(11, 3)$ together with the positive chirality 64-dimensional $\text{spin}(11, 3)$ irrep into the quaternionic real form of $E_8$. Lisi refers to the embedded Lie algebra as the GraviGUT algebra. As Lisi points out, while $\text{spin}(11, 3)$ is embedded as a subalgebra of the quaternionic real form of $E_8$, the whole GraviGUT algebra is actually not a subalgebra. In particular, the 64-dimensional $\text{spin}(11, 3)$ irrep is not a subalgebra. We will explicitly prove below that

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this representation is not closed under the Lie bracket when viewed as a subset of $E_8$.

In this article, we construct true Lie algebra embeddings of Extended GraviGUT algebras into $E_8$: the two Extended GraviGUT algebras will be subalgebras of $E_8$, unlike the GraviGUT algebra. Indeed, the Extended GraviGUT algebras are defined to be isomorphic to the subalgebras generated by the GraviGUT algebra, or its negative chirality counterpart, with the algebra structure they inherit from Lisi's embedding into $E_8$. We shall work over the complex numbers so that $\text{spin}(11, 3)_C \cong \mathfrak{so}(14)_C \cong D_7$, the Lie algebra of complex $14 \times 14$ matrices $N$ satisfying $N^{\text{tr}} = -N$.

The Extended GraviGUT algebras are defined as follows:

\begin{equation}
D_7 \in (V(\lambda_6) \oplus V(\lambda_1)) \text{ and } D_7 \in (V(\lambda_7) \oplus V(\lambda_1)).
\end{equation}

Note that $V(\lambda_6)$ and $V(\lambda_7)$ are the 64-dimensional irreps of $D_7$. They are associated to the fundamental weights $\lambda_6$ and $\lambda_7$, which correspond to the two “horns” on the Dynkin diagram for $D_7$. They are interchanged by an outer automorphism, and, in this sense, are closely related. It is a matter of convention which is which, so we establish our results for both. Note also that the sums above are direct as $D_7$ irreps, but not as subalgebras. The $D_7$ representations $V(\lambda_6) \oplus V(\lambda_1)$ and $V(\lambda_7) \oplus V(\lambda_1)$ are endowed with nonabelian nilpotent algebraic structures, which are described below (Section 5).

In this article, we will not only describe embeddings of the Extended GraviGUT algebras into $E_8$, but we will also classify such embeddings, up to inner automorphism.

We make no claims about the physical significance of the Extended GraviGUT algebras, but rather, from a mathematical point of view, we sought to eliminate one of the criticisms of Lisi’s GraviGUT construction, namely that the embedding into $E_8$ is not an algebra embedding. For a description of Lisi’s theory see [8] or [9]. For a critique of Lisi’s theory involving the GraviGUT algebra see [3].

The article is organized as follows. Section 2 contains relevant background on Lie algebras and their representations: in particular, it deals with the complex, simple Lie algebras $D_7$ and $E_8$. Section 3 presents additional notation and terminology. In Section 4 we describe the classification of embeddings of $D_7$ into $E_8$, which will be used in the following section.

Finally, in Section 5 we classify the embeddings of the Extended GraviGUT algebras into $E_8$. Section 5 also contains an explicit description of the algebraic structure of $V(\lambda_6) \oplus V(\lambda_1)$ and $V(\lambda_7) \oplus V(\lambda_1)$ within the Extended GraviGUT algebras.
The results of this article extend results of the authors in [4], which examined embedding abelian extensions of orthogonal Lie algebras into $E_6$, $E_7$, and $E_8$. Much of the work in the present article applies work by the authors in [4].

2. THE COMPLEX LIE ALGEBRAS $D_7$ AND $E_8$, AND THEIR REPRESENTATIONS

The special orthogonal algebra $D_7$ is the complexification of $\text{spin}(11,3)$. It is the Lie algebra of complex $14 \times 14$ matrices $N$ satisfying $N^{tr} = -N$. The dimension of $D_7$ is 91 and its rank is 7. The Lie group corresponding to $D_7$ arises naturally as the symmetry group of a projective space over $\mathbb{R}$ [1].

$E_8$ is the complex, exceptional Lie algebra of rank 8. It is 248 dimensional. Like $D_7$, $E_8$ has a close connection to the Riemannian geometry of projective spaces (for details, we refer the reader to [1]).

Let $g$ denote $D_7$ or $E_8$. Let $k = 7$ or 8 when $g = D_7$ or $E_8$, respectively. We may define $g$ by a set of generators $\{H_i, X_i, Y_i\}_{1 \leq i \leq k}$ together with the Chevalley-Serre relations [7]:

\[
\begin{align*}
[H_i, H_j] &= 0, \\
[H_i, X_j] &= M^g_{ij} X_j, \\
[H_i, Y_j] &= -M^g_{ij} Y_j, \\
[X_i, Y_j] &= \delta_{ij} H_i, \\
(\text{ad } X_i)^{1-M^g_{ij}}(X_j) &= 0, \\
(\text{ad } Y_i)^{1-M^g_{ij}}(Y_j) &= 0, \quad \text{when } i \neq j.
\end{align*}
\]

Here $1 \leq i, j \leq k$, and $M^g$ is the Cartan matrix of $g$. The $X_i$, for $1 \leq i \leq k$, correspond to the simple roots. We write $H$ for the Cartan subalgebra spanned by $\{H_i\}$.

For future reference, the Dynkin diagrams of $D_7$ and $E_8$, indicating the numbering of simple roots, are given in Figure [1]

**Figure 1.** Dynkin diagrams of $D_7$ and $E_8.

$$
D_7 \quad \circ - \circ - \circ - \circ - \circ - \circ - \circ \\
1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7
$$

$$
E_8 \quad \circ - \circ - \circ - \circ - \circ - \circ - \circ - \circ \\
1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8
$$

We now briefly describe the finite-dimensional, irreducible representations (irreps) of $D_7$ and $E_8$, with $g$ and $k$ defined as above. For
where $M$ is the Cartan matrix of $g$. The $\lambda_i$ are the fundamental weights, and their indexing corresponds with that of the Dynkin diagram of type $D_7$ or $E_8$ in Figure 1.

For each $\lambda = \lambda_1 \lambda_1 + \ldots + \lambda_k \lambda_k \in H^*$ with nonnegative integers $m_1, \ldots, m_k$, there exists an irrep of $g$ with highest weight $\lambda$, denoted $V_g(\lambda)$. The irreps $V_g(\lambda_i)$ for $1 \leq i \leq k$ are the fundamental representations.

The irrep $V_g(\lambda)$ is realized as the quotient of $\mathcal{U}(g)$ by the left ideal, $J_\lambda$, generated by $X_i, H_i - \lambda(H_i), Y_i^{1+\lambda(H_i)}$, $1 \leq i \leq k$ (here the action of $\mathcal{U}(g)$ on itself and on $V_g(\lambda)$ is given by left multiplication). We will denote the element $1 + J_\lambda$ of $V_g(\lambda)$ by $\tilde{\lambda}$. Then one can show that $V_g(\lambda)$ is generated by \{ $Y_i \cdots Y_i \tilde{\lambda} : l \in \mathbb{N}, i_1, \ldots, i_l \in \{1, \ldots, k\}$ \}. The weight of $\tilde{\lambda}$ is $\lambda - \Sigma_{j=1}^l \alpha_{ij}$.

Each irrep of $g$ is equivalent to $V_g(\lambda)$, where $\lambda = \lambda_1 \lambda_1 + \ldots + \lambda_k \lambda_k$ for some nonnegative integers $m_1, \ldots, m_k$. A useful formula for the dimension of the $D_7$ module $V(\lambda_k)$ is as follows [5]:

$$
\dim V(\lambda_k) = \begin{cases} 
\binom{14}{k}, & \text{if } 1 \leq k < 6; \\
2^6, & \text{if } k = 6, 7.
\end{cases}
$$

3. ADDITIONAL DEFINITIONS AND NOTATION

The following definitions and notation will be used in this article:

- Let $X_{a_i}$ correspond to a simple root of $E_8$ and $1 \leq a_i \leq 8$. We then define

$$
X_{a_1, a_2, a_3, \ldots, a_m} \equiv [[...[[X_{a_1}, X_{a_2}], X_{a_3}], \ldots], X_{a_m}].
$$

$Y_{a_1, a_2, a_3, \ldots, a_m}$ is defined analogously.

- Let $\varphi : D_7 \hookrightarrow E_8$ be an embedding. Further, let $W$ be an element of $E_8$. Then, $[W]_{\varphi(D_7)}$ is the $D_7$ representation generated by $W$ with respect to the adjoint action of $\varphi(D_7)$. When the embedding $\varphi$ is clear, as will be the case below, we simply write $[W]_{D_7}$.

- Let $W_1, \ldots, W_m \in E_8$. Then

$$
\langle W_1, \ldots, W_m \rangle_{E_8}
$$

is the Lie subalgebra of $E_8$ generated by $W_1, \ldots, W_m$.

- Let $V$ be a representation of $D_7$. Then, a lift of the embedding $\varphi : D_7 \hookrightarrow E_8$ to $D_7 \in V$ is an embedding $\tilde{\varphi} : D_7 \in V \hookrightarrow E_8$ such that $\tilde{\varphi}$ restricted to $D_7$ is equal to $\varphi$. That is, $\tilde{\varphi}|_{D_7} = \varphi$. 
Let $\varphi$ and $\varphi'$ be Lie algebra embeddings of $g'$ into $g$. Then $\varphi$ and $\varphi'$ are equivalent if there is an inner automorphism $\rho : g \to g$ such that $\varphi = \rho \circ \varphi'$, and we write $\varphi \sim \varphi'$.  

Two embeddings $\varphi$ and $\varphi'$ of $g'$ into $g$ are linearly equivalent if for each representation $\pi : g \to \mathfrak{gl}(V)$ the induced $g'$ representations $\pi \circ \varphi, \pi \circ \varphi'$ are equivalent, and we write $\varphi \sim_L \varphi'$.

Clearly equivalence of embeddings implies linear equivalence, but the converse is not in general true.

We define equivalence and linear equivalence of subalgebras much as we did for embeddings:

- Two subalgebras $g'$ and $g''$ of $g$ are equivalent if there is an inner automorphism $\rho$ or $g$ such that $\rho(g') = g''$.
- Two subalgebras $g'$ and $g''$ of $g$ are linearly equivalent if for every representation $\pi : g \to \mathfrak{gl}(V)$ the subalgebras $\pi(g'), \pi(g'')$ of $\mathfrak{gl}(V)$ are conjugate under $GL(V)$.

4. Embedding $D_7$ into $E_8$

In [4], the authors presented the following well-known “natural” embedding of $D_7$ into $E_8$:

$$\varphi : D_7 \hookrightarrow E_8$$

\begin{align*}
H_{8-i} &\mapsto H_{i+1} \\
X_{8-i} &\mapsto X_{i+1} \\
Y_{8-i} &\mapsto Y_{i+1},
\end{align*}

where $1 \leq i \leq 7$. This embedding may be visualized as a “natural” subgraph of the Dynkin diagram of $E_8$ which is isomorphic to the Dynkin diagram of $D_7$ (see Figure 1).

In [11], Minchenko showed that there is a unique subalgebra isomorphic to $D_7$ in $E_8$, up to inner automorphism. Hence, the only way to get new embeddings of $D_7$ into $E_8$ other than the $\varphi$ described in Eq. (4) is to compose $\varphi$ with an outer automorphism of $D_7$. However, it was shown by the authors in [4] that outer automorphisms of $D_7$ do not produce new embeddings of $D_7$ into $E_8$. We thus have the following theorem [4].

**Theorem 4.1.** The map $\varphi : D_7 \hookrightarrow E_8$ defined in Eq. (4) is the unique embedding of $D_7$ into $E_8$, up to inner automorphism.
5. Embedding the Extended GraviGUT algebras into $E_8$

In [4], the authors computed the following decomposition of $E_8$ with respect to the adjoint action of $\varphi(D_7)$:

$$
E_8 \cong_{D_7} V(\lambda_2) \oplus V(\lambda_1) \oplus V(\lambda_6) \oplus V(\lambda_7) \oplus V(\lambda_1) \oplus V(0) \\
\cong_{D_7} [X_{74}]_{D_7} \oplus [X_{120}]_{D_7} \oplus [Y_1]_{D_7} \oplus [X_{112}]_{D_7} \oplus [Y_{97}]_{D_7} \oplus [H]_{D_7},
$$

where

$$
\begin{align*}
X_{74} &= X_{4,5,6,7,8,2,3,4,5,6,7}, \\
X_{112} &= -X_{3,4,2,1,5,4,3,6,5,4,7,2,6,5,8,7,6,4,5,3,4,2}, \\
X_{120} &= X_{8,7,6,5,4,3,2,1,4,5,6,7,3,4,5,6,2,4,5,3,4,2,1,3,4,5,6,7,8}, \\
Y_{97} &= -Y_{5,4,2,3,6,4,1,3,5,4,7,2,6,5,4,3,1}, \\
H &= 4H_1 + 5H_2 + 7H_3 + 10H_4 + 8H_5 + 6H_6 + 4H_7 + 2H_8.
\end{align*}
$$

Lemma 5.1. The following are 78-dimensional, nonabelian nilpotent subalgebras of $E_8$:

$$
[Y_1]_{D_7} \oplus [Y_{97}]_{D_7}, \quad [X_{112}]_{D_7} \oplus [X_{120}]_{D_7}.
$$

Note that the sums are direct as $D_7$ irreps, but not as subalgebras of $E_8$. Further, the subalgebras $[Y_{97}]_{D_7}$ and $[X_{120}]_{D_7}$ of $E_8$ are abelian.

Proof. The authors showed in [4] that $[Y_{97}]_{D_7}$ and $[X_{120}]_{D_7}$ are abelian subalgebras of $E_8$.

The positive roots of $E_8$, as explicitly described in Appendix A, give us a triangular decomposition of $E_8$: $E_{8,+} \oplus E_{8,0} \oplus E_{8,-}$. In Appendix A we also explicitly describe bases for the representations $[X_{120}]_{D_7}$, $[X_{112}]_{D_7}$, $[Y_{97}]_{D_7}$, and $[Y_1]_{D_7}$. Noting that $\varphi(D_7) = [X_{74}]_{D_7}$, we also give bases for $\varphi(D_{7,+})$ and $\varphi(D_{7,-})$. Each of these bases consists of all positive root vectors, or all negative root vectors.

The bases given in Appendix A imply

$$
\begin{align*}
E_{8,+} &= \varphi(D_{7,+}) \oplus [X_{112}]_{D_7} \oplus [X_{120}]_{D_7}, \\
E_{8,-} &= \varphi(D_{7,-}) \oplus [Y_1]_{D_7} \oplus [Y_{97}]_{D_7}.
\end{align*}
$$

And of course

$$
\begin{align*}
[[X_{112}]_{D_7} \oplus [X_{120}]_{D_7}, [X_{112}]_{D_7} \oplus [X_{120}]_{D_7}] \subseteq E_{8,+}, \\
[[Y_1]_{D_7} \oplus [Y_{97}]_{D_7}, [Y_1]_{D_7} \oplus [Y_{97}]_{D_7}] \subseteq E_{8,-}.
\end{align*}
$$

In Appendix A we see that the positive root vector $X_\alpha$ is in the basis of $[X_{112}]_{D_7}$ or $[X_{120}]_{D_7}$ if $\alpha^1 \neq 0$, where $\alpha^1$ is the first entry of $\alpha$. If $\alpha^1 = 0$, then $X_\alpha$ is in the basis of $\varphi(D_{7,+})$.

Thus, if $X_\alpha$ and $X_{\alpha'}$ are positive root vectors in the basis of $[X_{112}]_{D_7}$ or $[X_{120}]_{D_7}$ such that $[X_\alpha, X_{\alpha'}] \neq 0$, then this product is a nonzero
scalar multiple of $X_{\alpha_0 + \alpha'}$, where $(\alpha + \alpha')^1 \neq 0$, so that $X_{\alpha_0 + \alpha'}$ is an
element of $[X_{112}]_{D_7} \oplus [X_{120}]_{D_7}$. Therefore,

\begin{equation}
([X_{112}]_{D_7} \oplus [X_{120}]_{D_7}, [X_{112}]_{D_7} \oplus [X_{120}]_{D_7}) \subseteq [X_{112}]_{D_7} \oplus [X_{120}]_{D_7}.
\end{equation}

In a similar manner we show

\begin{equation}
([Y_1]_{D_7} \oplus [Y_{97}]_{D_7}, [Y_1]_{D_7} \oplus [Y_{97}]_{D_7}) \subseteq [Y_1]_{D_7} \oplus [Y_{97}]_{D_7}.
\end{equation}

Thus $[Y_1]_{D_7} \oplus [Y_{97}]_{D_7}$ and $[X_{112}]_{D_7} \oplus [X_{120}]_{D_7}$ are subalgebras of $E_8$.

Further, they are nilpotent since they are contained in $E_{8\pm}$ or $E_{8\pm}$, respectively. In the proof of Theorem 5.4 below (see Eqs. (21) and (22)), we give an explicit example illustrating that these subalgebras are not abelian (although this is also obvious by dimension considerations: there are no 78-dimensional abelian subalgebras of $E_8$). □

**Lemma 5.2.** The following are not subalgebras of $E_8$:

\begin{equation}
[Y_{97}]_{D_7} \oplus [X_{112}]_{D_7}, \quad [X_{120}]_{D_7} \oplus [Y_1]_{D_7}.
\end{equation}

*Proof.* Referring to the bases of $[X_{120}]_{D_7}$ and $[Y_1]_{D_7}$ described in Appendix A, we have $Y_{112} \in [Y_1]_{D_7}$, and of course $X_{120} \in [X_{120}]_{D_7}$. However, $[Y_{112}, X_{120}]$ is a nonzero multiple of $X_{17}$, which is not in $[X_{120}]_{D_7} \oplus [Y_1]_{D_7}$. Hence $[X_{120}]_{D_7} \oplus [Y_1]_{D_7}$ is not a subalgebra. Similarly $[Y_{97}]_{D_7} \oplus [X_{112}]_{D_7}$ is not a subalgebra. □

Note in particular, that from Lemma 5.1 and 5.2 we now can explicitly define the nonabelian nilpotent algebraic structure of $V(\lambda_7) \oplus V(\lambda_1)$ and $V(\lambda_7) \oplus V(\lambda_1)$ in the Extended GraviGUT algebras in terms of subalgebras of $E_8$:

\begin{equation}
V(\lambda_6) \oplus V(\lambda_1) \cong [Y_{97}]_{D_7} \oplus [Y_1]_{D_7},
\end{equation}

\begin{equation}
V(\lambda_7) \oplus V(\lambda_1) \cong [X_{120}]_{D_7} \oplus [Y_1]_{D_7},
\end{equation}

where, in each case, the sum is direct as $D_7$ irreps, but not as subalgebras. Hence, our definition of the Extended GraviGUT algebras has proceeded in reverse order. Beginning with an embedding of $D_7$ into $E_8$, we identified subsets of $E_8$ that had the required $D_7$ representation structure. We then defined the algebraic structure of $V(\lambda_6) \oplus V(\lambda_1)$ and $V(\lambda_7) \oplus V(\lambda_1)$ in the Extended GraviGUT algebras to have the algebraic structure inherited from the algebraic structure of $E_8$.

We now proceed to the classification of embeddings of the GraviGUT algebras into $E_8$. A lift of $\varphi : D_7 \hookrightarrow E_8$ to $D_7 \hookrightarrow (V(\lambda_6) \oplus V(\lambda_1))$ is completely determined by its definition on the highest weight vectors.
of \( V(\lambda_6) \) and \( V(\lambda_1) \). Call these vectors \( u \) and \( v \), respectively. Hence, for any \( \alpha, \beta \in \mathbb{C}^* \), the following is a lift of \( \varphi \) to \( D_7 \in (V(\lambda_6) \oplus V(\lambda_1)) \):

\[
\widetilde{\varphi}^{\alpha,\beta}_6 : D_7 \in (V(\lambda_6) \oplus V(\lambda_1)) \mapsto E_8
\]

\[
\begin{array}{ccl}
u & \mapsto & \alpha Y_1 \\
v & \mapsto & \beta Y_{97}.
\end{array}
\]

Similarly, if \( u \) and \( v \) are highest weight vectors of \( V(\lambda_7) \) and \( V(\lambda_1) \), respectively, then the following is a lift of \( \varphi : D_7 \mapsto E_8 \) to \( D_7 \in (V(\lambda_7) \oplus V(\lambda_1)) \):

\[
\widetilde{\varphi}^{\alpha,\beta}_7 : D_7 \in (V(\lambda_7) \oplus V(\lambda_1)) \mapsto E_8
\]

\[
\begin{array}{ccl}
u & \mapsto & \alpha X_{12} \\
v & \mapsto & \beta X_{120},
\end{array}
\]

where \( \alpha, \beta \in \mathbb{C}^* \).

**Theorem 5.3.** All embeddings of the Extended GraviGUT algebras into \( E_8 \) are given by \( \widetilde{\varphi}^{\alpha,\beta}_6 \), and \( \widetilde{\varphi}^{\alpha,\beta}_7 \), for all \( \alpha, \beta \in \mathbb{C}^* \). These embeddings are classified according to the rules,

\[
\widetilde{\varphi}^{\alpha,\beta}_6 \sim \widetilde{\varphi}^{\alpha',\beta'}_6 \iff \alpha^2 \beta = \alpha'^2 \beta' \quad \text{and} \quad \widetilde{\varphi}^{\alpha,\beta}_7 \sim \widetilde{\varphi}^{\alpha',\beta'}_7 \iff \alpha^2 \beta = \alpha'^2 \beta'.
\]

**Proof.** First note that by Theorem 4.1, all embeddings of the Extended GraviGUT algebras must come from lifts of \( \varphi \), and hence, considering Lemmas 3.1 and 3.2. Eqs. (14) and (15) define all embeddings of the Extended GraviGUT algebras into \( E_8 \).

Consider two embeddings \( \widetilde{\varphi}^{\alpha,\beta}_6 \) and \( \widetilde{\varphi}^{\alpha',\beta'}_6 \). Define inner automorphisms of \( E_8 \) as follows:

\[
\begin{aligned}
\rho : & \quad X_1 \mapsto \alpha X_1, \quad Y_1 \mapsto \frac{1}{\alpha} Y_1, \\
& \quad X_i \mapsto X_i, \quad Y_i \mapsto Y_i, \\
\rho' : & \quad X_1 \mapsto \alpha' X_1, \quad Y_1 \mapsto \frac{1}{\alpha'} Y_1, \\
& \quad X_i \mapsto X_i, \quad Y_i \mapsto Y_i,
\end{aligned}
\]

for \( 2 \leq i \leq 8 \). Then \( \rho \circ \widetilde{\varphi}^{\alpha,\beta}_6 = \widetilde{\varphi}^{1/\alpha,\beta}_6 \), and \( \rho' \circ \widetilde{\varphi}^{\alpha',\beta'}_6 = \widetilde{\varphi}^{1/\alpha',\beta'}_6 \). Hence \( \widetilde{\varphi}^{\alpha,\beta}_6 \sim \widetilde{\varphi}^{1/\alpha,\beta}_6 \), and \( \widetilde{\varphi}^{\alpha',\beta'}_6 \sim \widetilde{\varphi}^{1/\alpha',\beta'}_6 \).

However, note that \( \widetilde{\varphi}^{1/\alpha,\beta}_6 \sim \widetilde{\varphi}^{1/\alpha',\beta'}_6 \) if and only if \( \frac{\beta}{\alpha} = \frac{\beta'}{\alpha'} \) since any inner automorphism relating these two embeddings must fix \( Y_{97} \). Hence we have established

\[
\widetilde{\varphi}^{\alpha,\beta}_6 \sim \widetilde{\varphi}^{\alpha',\beta'}_6 \iff \alpha^2 \beta = \alpha'^2 \beta'.
\]

Considering again the two inner automorphisms of Eq. (17), we have \( \widetilde{\varphi}^{\alpha,\beta}_7 = \rho \circ \widetilde{\varphi}^{\alpha',\beta'}_7 \), and \( \widetilde{\varphi}^{\alpha',\beta'}_7 = \rho' \circ \widetilde{\varphi}^{\alpha,\beta}_7 \). Hence \( \widetilde{\varphi}^{\alpha,\beta}_7 \sim \widetilde{\varphi}^{1/\alpha,\beta}_7 \), and \( \widetilde{\varphi}^{\alpha',\beta'}_7 \sim \widetilde{\varphi}^{1/\alpha',\beta'}_7 \).
If \( \vartheta \) is an inner automorphism of \( E_8 \) such that \( \vartheta \circ \tilde{\varphi}_{7}^{1/2} = \tilde{\varphi}_{7}^{1/2} \), then \( \vartheta \) fixes \( X_i \) and \( Y_i \) for \( 2 \leq i \leq 8 \), and also \( X_{112} \). We have

\[
\begin{align*}
\{ \cdots [X_{112}, Y_2], Y_4, Y_3, Y_5, Y_6, Y_7, Y_8], Y_4, Y_5] \\
\} = X_1,
\end{align*}
\]

so that \( \vartheta \) fixes \( X_1 \). Hence \( \vartheta(X_{120}) = X_{120} \), so that \( \frac{\beta}{\alpha} = \frac{\beta'}{\alpha'} \). The opposite implication is obvious. Hence we have established

\[
(20) \quad \tilde{\varphi}_{7}^{\alpha, \beta} \sim \tilde{\varphi}_{7}^{\alpha', \beta'} \Leftrightarrow \alpha^2 \beta = \alpha'^2 \beta'.
\]

□

We end by showing that the GraviGUT algebra of \( [8] \) cannot, strictly speaking, be embedded into a real form of \( E_8 \). That is, \( \text{spin}(11, 3) \oplus V \), where \( V \) is a 64-dimensional irrep of \( \text{spin}(11, 3) \) endowed with an algebra structure, cannot be embedded in any real form of \( E_8 \).

**Theorem 5.4.** The GraviGUT algebra cannot be embedded into the quaternionic real form of \( E_8 \), or any other real form of \( E_8 \).

**Proof.** The complexification of an embedding of the GraviGUT algebra into a real form of \( E_8 \) is an embedding of an extension of the complex Lie algebra \( D_7 \) by a non-abelian 64-dimensional subalgebra into the complex Lie algebra \( E_8 \). Recall also that this 64-dimensional subalgebra is an irreducible module with respect to the action of \( D_7 \).

Hence, it suffices to show that an extension of \( D_7 \) by a non-abelian 64-dimensional subalgebra cannot be embedded into \( E_8 \). Since there is a unique embedding of \( D_7 \) into \( E_8 \), up to inner automorphism (Theorem 4.1), it suffices to show that the natural embedding \( \varphi : D_7 \hookrightarrow E_8 \) cannot be lifted to the above-mentioned extension of \( D_7 \). With reference to \( [5] \), there are two possible extension of \( D_7 \) to consider: \( D_7 \in V(\lambda_6) \) or \( D_7 \in V(\lambda_7) \).

We first consider the 64-dimensional \( D_7 \) irrep \( V(\lambda_6) \cong [Y_1]_{D_7} \) in the decomposition of \( E_8 \). From Appendix \( [A] \) we see \( Y_{51} \) and \( Y_{59} \in [Y_1]_{D_7} \). Further,

\[
(21) \quad [Y_{59}, Y_{51}] = Y_{97}.
\]

However, \( Y_{97} \) is not an element of \( [Y_1]_{D_7} \). Hence, the representation \( [Y_1]_{D_7} \) is not a subalgebra of \( E_8 \). Thus, we cannot lift \( \varphi \) to \( D_7 \in V(\lambda_6) \).

Similarly, the representation \( V_{D_7}(\lambda_7) \cong [X_{112}]_{D_7} \) is not a subalgebra of \( E_8 \). From Appendix \( [A] \) we see \( X_{84} \) and \( X_{94} \in [X_{112}]_{D_7} \). Further,

\[
(22) \quad [X_{84}, X_{94}] = X_{120}.
\]
However, \( X_{120} \) is not an element of \([X_{112}]_{D_7}\). Hence, the representation \([X_{112}]_{D_7}\) is not a subalgebra of \(E_8\). Thus, we cannot lift \(\varphi\) to \(D_7 \in V(\lambda_7)\).

□

**Remark 5.5.** Theorem 5.4 established that an embedding of \(D_7\) into \(E_8\) cannot be lifted to an extension of \(D_7\) by a 64-dimensional, non-abelian algebra. It is interesting to note that such an embedding would also be impossible had the algebra been abelian: The maximal dimension of an abelian subalgebra of \(E_8\) is 36 \([10]\).

We also note that if we have a 64-dimensional representation \(V\) that is not irreducible, then embedding \(D_7 \in V\) into \(E_8\) is still not possible. The summands in the direct sum decomposition \((5)\) of \(E_8\) as a \(\varphi(D_7)\) module have dimensions 1, 14, 14, 64, 64, 91. Hence, the only 64-dimensional \(D_7\) submodules of \(E_8\) are \([Y_1]_{D_7}\) and \([X_{112}]_{D_7}\) considered in the proof of Theorem 5.4.

### 6. Conclusions

In \([8]\), Lisi constructed an embedding of the GraviGUT algebra into a real form of \(E_8\). Over the complex numbers, this is an embedding of \(D_7\) together with a 64-dimensional irrep of \(D_7\) into the complex Lie algebra \(E_8\). Under this construction, the embedded 64-dimensional, irreducible representation of \(D_7\) must be regarded merely as a subset of \(E_8\), since it is not a subalgebra of \(E_8\).

In this article, we constructed embeddings into \(E_8\) of the Extended GraviGUT algebras \(D_7 \in (V(\lambda_6) \oplus V(\lambda_1))\) and \(D_7 \in (V(\lambda_7) \oplus V(\lambda_1))\). These embeddings are true Lie algebra embeddings. In particular, the embedded \(D_7\) representations \(V(\lambda_7) \oplus V(\lambda_1)\) and \(V(\lambda_6) \oplus V(\lambda_1)\) are subalgebras of \(E_8\), each of which is nonabelian and nilpotent (see Lemma 5.1). Next, we classified embeddings of the Extended GraviGUT algebras into \(E_8\), up to inner automorphism (Theorem 5.3).

The complexifications of the GraviGUT algebra of \([8]\) and of its negative chirality counterpart, \(D_7 \in V(\lambda)\), with \(\lambda = \lambda_6\) or \(\lambda_7\), are quotients of the corresponding Extended GraviGUT algebras \(D_7 \in (V(\lambda) \oplus V(\lambda_1))\). Each can also be regarded as a quotient of any of the copies of the appropriate Extended GraviGUT algebra embedded in \(E_8\). Roughly speaking, the mathematics appears to say that you cannot embed the GraviGUT algebra into \(E_8\) without an extra piece.

It is interesting to note that we may perform a construction with \(E_7\) that is analogous to that constructed in the present article. In particular, we believe we may create the four nonabelian nilpotent extensions
of $D_6$ given by

$$
\begin{align*}
D_6 &\in (V(\lambda_5) \oplus V(0)), & D_6 &\in (V(\lambda_5) \oplus V(0)), \\
D_6 &\in (V(\lambda_5) \oplus V(\lambda_5)), & D_6 &\in (V(\lambda_6) \oplus V(\lambda_6)), \\
\end{align*}
$$

where the sums are direct as $D_6$ irreps but not as subalgebras. The subalgebra structure is inherited from $E_7$.

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**Appendix A. The Representations**

In this appendix we describe the representations $[X_{74}]_{D_7}$, $[X_{120}]_{D_7}$, $[X_{112}]_{D_7}$, $[Y_{97}]_{D_7}$, and $[Y_1]_{D_7}$ from Eq. (23).
Let $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_8$ be a set of simple roots for $E_8$. To any positive root $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \cdots + a_8\alpha_8$ we may associate a vector $[a_1, a_2, a_3, ..., a_8] \in \mathbb{Z}_{\geq 0}^8$. With this convention, the positive roots of $E_8$, as computed with GAP \cite{gap}, are as follows:

\begin{align*}
\alpha_1 &= [1, 0, 0, 0, 0, 0, 0], & \alpha_2 &= [0, 1, 0, 0, 0, 0, 0], \\
\alpha_3 &= [0, 1, 0, 0, 0, 0, 0], & \alpha_4 &= [0, 0, 1, 0, 0, 0, 0], \\
\alpha_5 &= [0, 0, 0, 1, 0, 0, 0], & \alpha_6 &= [0, 0, 0, 0, 1, 0, 0], \\
\alpha_7 &= [0, 0, 0, 0, 0, 1, 0], & \alpha_8 &= [0, 0, 0, 0, 0, 0, 1], \\
\alpha_9 &= [1, 0, 1, 0, 0, 0, 0], & \alpha_{10} &= [0, 1, 0, 1, 0, 0, 0], \\
\alpha_{11} &= [0, 0, 1, 1, 0, 0, 0], & \alpha_{12} &= [0, 0, 0, 1, 1, 0, 0], \\
\alpha_{13} &= [0, 0, 0, 0, 1, 1, 0], & \alpha_{14} &= [0, 0, 0, 0, 0, 1, 1], \\
\alpha_{15} &= [0, 0, 0, 0, 0, 1, 1], & \alpha_{16} &= [0, 0, 0, 0, 0, 1, 1], \\
\alpha_{17} &= [0, 0, 0, 1, 1, 0, 0], & \alpha_{18} &= [0, 0, 0, 0, 1, 1, 0], \\
\alpha_{19} &= [0, 0, 0, 0, 0, 1, 1], & \alpha_{20} &= [0, 0, 0, 0, 0, 0, 0], \\
\alpha_{21} &= [0, 0, 0, 0, 0, 1, 1], & \alpha_{22} &= [0, 0, 0, 0, 0, 0, 0], \\
\alpha_{23} &= [0, 0, 0, 0, 0, 0, 0], & \alpha_{24} &= [0, 0, 0, 0, 0, 0, 0], \\
\alpha_{25} &= [0, 0, 1, 1, 1, 0, 0], & \alpha_{26} &= [0, 0, 0, 1, 1, 1, 0], \\
\alpha_{27} &= [0, 0, 0, 1, 1, 1, 0], & \alpha_{28} &= [0, 0, 0, 0, 1, 1, 1], \\
\alpha_{29} &= [0, 0, 0, 0, 0, 0, 0], & \alpha_{30} &= [0, 0, 0, 0, 0, 0, 0], \\
\alpha_{31} &= [0, 0, 1, 1, 1, 1, 0], & \alpha_{32} &= [0, 0, 1, 2, 1, 1, 0], \\
\alpha_{33} &= [0, 0, 1, 1, 1, 1, 0], & \alpha_{34} &= [0, 0, 0, 0, 0, 0, 0], \\
\alpha_{35} &= [0, 0, 1, 1, 1, 1, 1], & \alpha_{36} &= [0, 0, 0, 0, 0, 0, 0], \\
\alpha_{37} &= [0, 0, 0, 0, 0, 0, 0], & \alpha_{38} &= [0, 0, 0, 0, 0, 0, 0], \\
\alpha_{39} &= [0, 1, 0, 1, 1, 1, 1], & \alpha_{40} &= [0, 1, 0, 1, 1, 1, 1], \\
\alpha_{41} &= [0, 1, 1, 1, 1, 1, 0], & \alpha_{42} &= [0, 1, 1, 2, 1, 1, 0], \\
\alpha_{43} &= [0, 1, 1, 1, 1, 1, 1], & \alpha_{44} &= [0, 1, 1, 2, 1, 1, 1], \\
\alpha_{45} &= [0, 1, 1, 2, 1, 1, 0], & \alpha_{46} &= [0, 1, 1, 2, 1, 0, 0], \\
\alpha_{47} &= [0, 1, 1, 2, 1, 1, 1], & \alpha_{48} &= [0, 1, 1, 2, 0, 0, 0], \\
\alpha_{49} &= [0, 1, 1, 2, 1, 1, 0], & \alpha_{50} &= [0, 1, 1, 1, 1, 1, 1], \\
\alpha_{51} &= [0, 1, 1, 2, 1, 0, 0], & \alpha_{52} &= [0, 1, 1, 2, 1, 0, 0], \\
\alpha_{53} &= [0, 1, 1, 2, 1, 1, 1], & \alpha_{54} &= [0, 1, 1, 2, 1, 1, 1], \\
\alpha_{55} &= [0, 1, 1, 2, 1, 1, 1], & \alpha_{56} &= [0, 1, 1, 2, 1, 1, 1], \\
\alpha_{57} &= [0, 1, 1, 2, 2, 1, 1], & \alpha_{58} &= [0, 1, 1, 2, 2, 1, 1], \\
\alpha_{59} &= [0, 1, 1, 2, 2, 1, 0], & \alpha_{60} &= [0, 1, 1, 2, 2, 1, 0], \\
\alpha_{61} &= [0, 1, 1, 2, 2, 1, 1], & \alpha_{62} &= [0, 1, 1, 2, 2, 1, 1], \\
\alpha_{63} &= [0, 1, 1, 2, 2, 1, 0], & \alpha_{64} &= [0, 1, 1, 2, 2, 1, 0], \\
\alpha_{65} &= [0, 1, 1, 2, 2, 1, 1], & \alpha_{66} &= [0, 1, 1, 2, 2, 1, 1], \\
\alpha_{67} &= [0, 1, 1, 2, 2, 1, 1], & \alpha_{68} &= [0, 1, 1, 2, 2, 1, 1], \\
\alpha_{69} &= [0, 1, 1, 2, 2, 1, 0], & \alpha_{70} &= [0, 1, 1, 2, 2, 0, 0], \\
\alpha_{71} &= [0, 1, 1, 2, 2, 0, 0], & \alpha_{72} &= [0, 1, 1, 2, 2, 0, 0], \\
\end{align*}
\[ \alpha_{73} = [1, 1, 2, 2, 2, 1, 1], \quad \alpha_{74} = [0, 1, 1, 2, 2, 2, 2, 1], \]
\[ \alpha_{75} = [1, 2, 2, 3, 2, 1, 1, 0], \quad \alpha_{76} = [1, 1, 2, 3, 2, 2, 1, 0], \]
\[ \alpha_{77} = [1, 1, 2, 3, 2, 1, 1, 1], \quad \alpha_{78} = [1, 2, 2, 2, 2, 2, 1, 1], \]
\[ \alpha_{79} = [1, 1, 1, 2, 2, 2, 2, 1], \quad \alpha_{80} = [1, 2, 3, 2, 2, 2, 1, 0], \]
\[ \alpha_{81} = [1, 2, 3, 2, 1, 1, 1], \quad \alpha_{82} = [1, 2, 3, 3, 2, 1, 0], \]
\[ \alpha_{83} = [1, 1, 2, 3, 2, 2, 1, 1], \quad \alpha_{84} = [1, 2, 2, 3, 2, 2, 1], \]
\[ \alpha_{85} = [1, 2, 3, 3, 2, 1, 1], \quad \alpha_{86} = [1, 2, 3, 3, 2, 2, 1, 0], \]
\[ \alpha_{87} = [1, 1, 2, 3, 3, 2, 1, 1], \quad \alpha_{88} = [1, 2, 3, 2, 2, 2, 1], \]
\[ \alpha_{89} = [1, 2, 2, 4, 3, 2, 1, 0], \quad \alpha_{90} = [1, 2, 3, 3, 2, 2, 1, 1], \]
\[ \alpha_{91} = [1, 2, 2, 3, 2, 2, 2, 1], \quad \alpha_{92} = [1, 2, 3, 3, 2, 2, 1], \]
\[ \alpha_{93} = [1, 2, 3, 4, 3, 2, 1, 0], \quad \alpha_{94} = [1, 2, 3, 3, 2, 1, 1], \]
\[ \alpha_{95} = [1, 2, 2, 3, 3, 2, 2, 1], \quad \alpha_{96} = [1, 2, 3, 3, 3, 2, 1], \]
\[ \alpha_{97} = [2, 2, 3, 4, 3, 2, 1, 0], \quad \alpha_{98} = [1, 2, 3, 4, 3, 2, 2, 1], \]
\[ \alpha_{99} = [1, 2, 2, 4, 3, 2, 2, 1], \quad \alpha_{100} = [1, 2, 3, 4, 3, 2, 1], \]
\[ \alpha_{101} = [2, 2, 3, 4, 3, 2, 1, 1], \quad \alpha_{102} = [1, 2, 3, 4, 3, 2, 2, 1], \]
\[ \alpha_{103} = [1, 2, 2, 4, 3, 3, 2, 1], \quad \alpha_{104} = [2, 2, 3, 4, 3, 2, 2, 1], \]
\[ \alpha_{105} = [1, 2, 3, 4, 3, 3, 2, 1], \quad \alpha_{106} = [1, 2, 4, 3, 4, 3, 2, 1], \]
\[ \alpha_{107} = [2, 2, 3, 4, 3, 3, 2, 1], \quad \alpha_{108} = [2, 3, 4, 3, 4, 3, 2, 1], \]
\[ \alpha_{109} = [2, 2, 3, 4, 3, 2, 1, 1], \quad \alpha_{110} = [1, 2, 3, 5, 4, 3, 2, 1], \]
\[ \alpha_{111} = [2, 2, 3, 5, 4, 3, 2, 1], \quad \alpha_{112} = [1, 3, 3, 5, 4, 3, 2, 1], \]
\[ \alpha_{113} = [2, 3, 3, 5, 4, 3, 2, 1], \quad \alpha_{114} = [2, 2, 4, 5, 4, 3, 2, 1], \]
\[ \alpha_{115} = [2, 3, 4, 5, 4, 3, 2, 1], \quad \alpha_{116} = [2, 3, 4, 6, 4, 3, 2, 1], \]
\[ \alpha_{117} = [2, 3, 4, 6, 5, 4, 3, 2, 1], \quad \alpha_{118} = [2, 3, 4, 6, 5, 4, 2, 1], \]
\[ \alpha_{119} = [2, 3, 4, 6, 5, 4, 3, 1], \quad \alpha_{120} = [2, 3, 4, 6, 5, 4, 3, 2]. \]

Let \( X_{\alpha_i} = X_i \), and \( Y_{\alpha_i} = Y_i \) be a choice of positive (resp. negative) root vector corresponding to the root \( \alpha_i \).

A basis of \( V(\lambda_1) = [X_{120}]_{D_T} \) is given by the 14 positive root vectors:

\[(24) \quad X_{97}, \quad X_{101}, \quad X_{104}, \quad X_{107}, \quad X_{109}, \quad X_{111}, \quad X_{113}, \quad X_{114}, \quad X_{115}, \quad X_{116}, \quad X_{117}, \quad X_{118}, \quad X_{119}, \quad X_{120}. \]

A basis of \( V(\lambda_1) = [Y_{97}]_{D_T} \) is given by the 14 negative root vectors:

\[(25) \quad Y_{97}, \quad Y_{101}, \quad Y_{104}, \quad Y_{107}, \quad Y_{109}, \quad Y_{111}, \quad Y_{113}, \quad Y_{114}, \quad Y_{115}, \quad Y_{116}, \quad Y_{117}, \quad Y_{118}, \quad Y_{119}, \quad Y_{120}. \]
A basis of $V(\lambda_7) = [X_{112}]_{D_7}$ is given by the 64 positive root vectors:

| X_{1}, X_{9}, X_{16}, X_{23}, X_{24}, X_{30}, X_{31}, X_{37}, X_{38}, |
| X_{39}, X_{44}, X_{45}, X_{46}, X_{47}, X_{51}, X_{52}, X_{53}, X_{54}, |
| X_{57}, X_{58}, X_{59}, X_{60}, X_{63}, X_{64}, X_{65}, X_{66}, X_{67}, |
| X_{69}, X_{70}, X_{71}, X_{72}, X_{73}, X_{75}, X_{76}, X_{77}, X_{78}, |
| X_{79}, X_{80}, X_{81}, X_{82}, X_{83}, X_{84}, X_{85}, X_{86}, X_{87}, |
| X_{88}, X_{89}, X_{90}, X_{91}, X_{92}, X_{93}, X_{94}, X_{95}, X_{96}, |
| X_{98}, X_{99}, X_{100}, X_{102}, X_{103}, X_{105}, X_{106}, X_{108}, X_{110}, |

A basis of $V(\lambda_6) = [Y_1]_{D_7}$ is given by the 64 negative root vectors:

| Y_{1}, Y_{9}, Y_{16}, Y_{23}, Y_{24}, Y_{30}, Y_{31}, Y_{37}, Y_{38}, |
| Y_{39}, Y_{44}, Y_{45}, Y_{46}, Y_{47}, Y_{51}, Y_{52}, Y_{53}, Y_{54}, |
| Y_{57}, Y_{58}, Y_{59}, Y_{60}, Y_{63}, Y_{64}, Y_{65}, Y_{66}, Y_{67}, |
| Y_{69}, Y_{70}, Y_{71}, Y_{72}, Y_{73}, Y_{75}, Y_{76}, Y_{77}, Y_{78}, |
| Y_{79}, Y_{80}, Y_{81}, Y_{82}, Y_{83}, Y_{84}, Y_{85}, Y_{86}, Y_{87}, |
| Y_{88}, Y_{89}, Y_{90}, Y_{91}, Y_{92}, Y_{93}, Y_{94}, Y_{95}, Y_{96}, |
| Y_{98}, Y_{99}, Y_{100}, Y_{102}, Y_{103}, Y_{105}, Y_{106}, Y_{108}, Y_{110}, |

Note that $\varphi(D_7) = [X_{74}]_{D_7}$. We describe bases of $\varphi(D_{7,+})$ and $\varphi(D_{7,-})$, respectively:

| X_{2}, X_{3}, X_{4}, X_{5}, X_{6}, X_{7}, X_{8}, X_{10}, X_{11}, |
| X_{12}, X_{13}, X_{14}, X_{15}, X_{17}, X_{18}, X_{19}, X_{20}, X_{21}, |
| X_{22}, X_{25}, X_{26}, X_{27}, X_{28}, X_{29}, X_{32}, X_{33}, X_{34}, |
| X_{35}, X_{36}, X_{40}, X_{41}, X_{42}, X_{43}, X_{48}, X_{49}, X_{50}, |
| X_{55}, X_{56}, X_{61}, X_{62}, X_{68}, X_{74}, |

| Y_{2}, Y_{3}, Y_{4}, Y_{5}, Y_{6}, Y_{7}, Y_{8}, Y_{10}, Y_{11}, |
| Y_{12}, Y_{13}, Y_{14}, Y_{15}, Y_{17}, Y_{18}, Y_{19}, Y_{20}, Y_{21}, |
| Y_{22}, Y_{25}, Y_{26}, Y_{27}, Y_{28}, Y_{29}, Y_{32}, Y_{33}, Y_{34}, |
| Y_{35}, Y_{36}, Y_{40}, Y_{41}, Y_{42}, Y_{43}, Y_{48}, Y_{49}, Y_{50}, |
| Y_{55}, Y_{56}, Y_{61}, Y_{62}, Y_{68}, Y_{74}, |

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