A new method to solve non-homogeneous wave equations of electromagnetic fields by fourier’s triple integral transform

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Abstract. This paper uses Fourier’s triple integral transform method to simplify the calculation of the non-homogeneous wave equations of the time-varying electromagnetic field. By adding several special definite conditions to the wave equation, it becomes a mathematical problem of definite condition. Then by using Fourier’s triple integral transform method, this three-dimension non-homogeneous partial differential wave equation is changed into an ordinary differential equation. Through the solution to this ordinary differential equation, the expression of the relationship between the time-varying scalar potential and electromagnetic wave excitation source is developed precisely. This method simplifies the solving process effectively.

1 Introduction
To find the relationship between the electric and magnetic fields and their excitation source (charge density $\rho$ and the current density $\mathbf{J}_e$) in time-varying electromagnetic fields, we introduce the time-varying potentials, usually called retarded potentials which include the time-varying scalar potential $\phi$ and vector potential $\mathbf{A}$. The Lorentz gauge ($\nabla \cdot \mathbf{A} = -\mu \epsilon \frac{\partial \phi}{\partial t}$) can help to simplify the relationship between retarded potentials and their excitation sources as 

$$\nabla^2 \mathbf{A} - \frac{1}{v^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}_e$$

(1)

$$\nabla^2 \phi - \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon}$$

(2)

Where $v = \frac{1}{\sqrt{\mu \epsilon}}$ is the electromagnetic wave velocity in the medium, and $\mu$ and $\epsilon$ are medium permeability and permittivity respectively. We commonly call Equation (1) and Equation (2) non-homogeneous wave equations of electromagnetic fields or D’Alembert equations about retarded potentials. And these are three-dimension non-homogeneous partial differential wave equations. If we solve these non-homogeneous wave equations and obtain the retarded potentials, we can use them to get electric field intensity $\mathbf{E}$ and magnetic flux density $\mathbf{B}$ by equations $\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$.

2 Ordinary solution to non-homogeneous wave equations of electromagnetic fields
Here we first introduce the ordinary solution to non-homogeneous wave equations in the common electromagnetic text books, and then we will use the Fourier’s triple integral transform method to solve these equations. We often solve Equation (2) to obtain the scalar potential $\phi$ first, and then solve the vector potential $\mathbf{A}$. The solving process of the vector potential $\mathbf{A}$ is similar to the scalar potential $\phi$. Therefore, this article focuses on the solving process of the scalar potential $\phi$. We summarize the ordinary solution of non-homogeneous wave Equation (2) as follows: it is based on that Poisson
equation solution of electrostatic field and the homogeneous wave equation are the special cases of non-homogeneous wave Equation (2) So it can be expected that the solution of Equation (2) has the similar form of Poisson equation but is volatile. Special solution of Poisson equation about $\phi$ in Electrostatic field is

$$\phi(x, y, z) = \frac{1}{4\pi\varepsilon} \int_V \frac{\rho(x', y', z')}{R} dV' ,$$  \hspace{1cm} (3)

Where $x$, $y$ and $z$ are the coordinates of field point $M(x, y, z)$, and $x'$, $y'$ and $z'$ are the coordinates of source point $M'(x', y', z')$, $R$ is the distance between source point and field point, $V'$ is the charge volume shown in Figure (1). Here the excitation source is the charges. Solution to $\phi$ should have the form of Equation (3), furthermore, in an infinite homogeneous medium, $\phi$ is volatile, so it can be expressed as[1]

$$\phi = \frac{f(t - \frac{R}{v})}{R} .$$  \hspace{1cm} (4)

Considering Equation (3) and Equation (4), the time-varying scalar potential $\phi$ caused by all charges in $V'$ is

$$\phi(x, y, z, t) = \frac{1}{4\pi\varepsilon} \int_V \frac{\rho(x', y', z', t - \frac{R}{v})}{R} dV' .$$  \hspace{1cm} (5)

Equation (5) is a fluctuation of velocity $v$ along the $R$ direction. So we can say the solution above is just based on a qualitative prediction which is based on the characteristics of the wave equation and the expression of the scalar potential $\phi$ in the electrostatic field. Strictly speaking, we should solve the non-homogeneous wave equations by strict detailed mathematical derivation to get more convincing solutions. Thus below we will add necessary boundary conditions to this three-dimension non-homogeneous partial differential wave equation, and make it a mathematical problem of definite condition, and use Fourier’s triple integral transform method to transform this equation to an ordinary differential equation and solve it.

![Figure 1: Distribution of volume charge](image)

### 3 Solving wave equations of electromagnetic field by Fourier’s triple integral transform

In order to facilitate the calculation, so let $v = 1/\sqrt{\mu\varepsilon} = a$ and $a^2 \rho(x', y', z', t) = \varepsilon f(x', y', z', t)$. Substituting these two equations into Equation (2), we get

$$\frac{\partial^2 \phi}{\partial t^2} - a^2 \nabla^2 \phi = f(x', y', z', t) \quad \begin{cases} \scriptstyle -\infty < x, y, z < +\infty, t > 0 \end{cases} .$$  \hspace{1cm} (6)

Assuming when $t \leq 0$, every field quantities is zero, so we give Equation (6) boundary conditions and get a problem of definite condition as[2]
\[ \frac{\partial^2 \varphi}{\partial t^2} - a^2 \nabla^2 \varphi = f(x', y', z', t) \]  \hspace{1cm} (7)

\[ \varphi_{t=0} = 0, \frac{\partial \varphi}{\partial t}_{t=0} = 0 \]

Equation (7) is an equation of mathematical physics which is three-dimensional second order partial differential equations with constant coefficients. We will choose Fourier’s triple integral transform method to transform this partial differential equation into ordinary differential equation, so that the calculation is greatly simplified. For convenience, here we list two important mathematical theorems used below.

3.1 The definition of Fourier integral transform

When the function \( F(t) \) is in the infinite interval \((-\infty, +\infty)\), if \( \int_{-\infty}^{\infty} |F(t)|dt \) converges, it can be expressed as a Fourier integral [2].

Considering function \( F(x, y, z) \), we use Fourier’s triple integral transform method and obtain

\[ F(x, y, z) = \iiint_{-\infty}^{\infty} \tilde{F}(w_1, w_2, w_3) e^{j(w_1 x + w_2 y + w_3 z)} dw_1 dw_2 dw_3 \]  \hspace{1cm} (8)

\[ \tilde{F}(w_1, w_2, w_3) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} F(x, y, z) e^{-j(w_1 x + w_2 y + w_3 z)} dx dy dz \]  \hspace{1cm} (9)

Then we define two separate position vectors \( r = xi + yj + zk \) and \( w = w_1 i + w_2 j + w_3 k \). So Equation (8) and Equation (9) become

\[ F(r) = \iiint_{-\infty}^{\infty} \tilde{F}(w)e^{j(w \cdot r)} dw_1 dw_2 dw_3 \]  \hspace{1cm} (10)

\[ F(w) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} F(r)e^{-j(w \cdot r)} dx dy dz \]

3.2 Screening characteristic of \( \delta \) Function

Let \( \varphi(x) \) is an arbitrary continuous function at \( x = x_0 \), we have \( \int_{-\infty}^{\infty} \varphi(x) \delta(x - x_0) dx = \varphi(x_0) \) [3].

3.3 Equation and its Solution

Let \( w = |w|, r = |r| \), the integral is [2]

\[ l = \frac{1}{4\pi^2} \iiint_{-\infty}^{\infty} \frac{1}{jw} (e^{j(w \cdot r)} - e^{-j(w \cdot r)}) e^{j(w \cdot r)} dw_1 dw_2 dw_3 \]  \hspace{1cm} (10)

\[ = \frac{1}{r} [\delta(r - at) - \delta(r + at)] \]

Where the integrating interval is the whole space and \( a \) is a real number. Assuming the solution of Equation (7) is

\[ \varphi(r, t) = \iiint_{-\infty}^{\infty} \tilde{\varphi}(w, t)e^{j(w \cdot r)} dw_1 dw_2 dw_3 \]  \hspace{1cm} (11)

We notice

\[ \frac{\partial^2 \varphi}{\partial x^2} = \iiint_{-\infty}^{\infty} w_1^2 \tilde{\varphi}(w, t)e^{j(w \cdot r)} dw_1 dw_2 dw_3 \]
\[
\frac{\partial^2 \varphi}{\partial y^2} = -\iiint_{\infty} w_2^2 \bar{\varphi}(w,t)e^{j(w \cdot r)} \, dw_1dw_2dw_3 \\
\frac{\partial^2 \varphi}{\partial z^2} = -\iiint_{\infty} w_3^2 \bar{\varphi}(w,t)e^{j(w \cdot r)} \, dw_1dw_2dw_3
\]

Where \( w^2 = w_1^2 + w_2^2 + w_3^2 \). Substituting Equation (11) into Equation (7), we get an ordinary differential equation with variable \( t \) as

\[
\frac{\partial^2 \bar{\varphi}}{\partial t^2} = \int_{-\infty}^{0} f(r',t)e^{-j(w \cdot r)} \, dr' - \int_{0}^{+\infty} a'(r',t)e^{-j(w \cdot r)} \, dr' \bigg|_{t=0} = 0 \\
\bar{\varphi}(w,0) = 0, \quad \frac{\partial \bar{\varphi}}{\partial t} \bigg|_{t=0} = 0 
\]

Where \( \bar{f}(w,t) \) is the Fourier’s triple integral transform of non-homogeneous terms, namely

\[
\bar{f}(w,t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} f(r',t)e^{j(w \cdot r')} \, dr' \, dy' \, dz'
\]

Where \( r' = x'i + y'j + z'k \). And its inverse transform is \( f(r',t) = \iint_{-\infty}^{\infty} \bar{f}(w,t)e^{-j(w \cdot r')} \, dw \, dr' \, dy' \, dz' \).

Solve Equation (12) and obtain its special solution

\[
\bar{\varphi}(w,t) = \frac{1}{j2wa} \int_{-\infty}^{t} \bar{f}(w,t)[e^{j(w \cdot r - t)} - e^{-j(w \cdot r - t)}] \, dt
\]

Substituting Equation (13) into Equation (14), and then substituting Equation (14) into Equation (11), we have

\[

\varphi(r,t) = \iiint_{\infty} \left[ \frac{1}{jw} \int_{-\infty}^{t} \bar{f}(w,t)[e^{j(w \cdot r - t)} - e^{-j(w \cdot r - t)}] \, dt \right] e^{j\varphi(r,t)} \, dr \, dy \, dz 
\]

First we calculate the integral of \( w \), considering Equation (10) and \( \delta(ax) = \frac{\delta(x)}{a} \), we notice

\[

\frac{1}{4\pi^2} \iiint_{\infty} \frac{1}{jw} \left[ e^{j(w \cdot r - t)} - e^{-j(w \cdot r - t)} \right] e^{j\varphi(r,t)} \, dw \, dr \, dy \, dz 
\]

Where \( R = |r - r'| \), thus we obtain the solution

\[

\varphi(r,t) = \frac{1}{4\pi a^2} \iiint_{\infty} \int_{-\infty}^{t} f(r',t) \frac{1}{R} \left[ \delta(t - \tau - \frac{R}{a}) - \delta(t - \tau + \frac{R}{a}) \right] \, dt \, dy' \, dz' 
\]

We also notice \( \delta(t + \frac{R}{a}) = 0 \), and take use of the screening characteristic of \( \delta \) function and obtain

\[

\varphi(r,t) = \frac{1}{4\pi a^2} \iiint_{\infty} \int_{-\infty}^{t} f(r',t) \frac{1}{R} \delta(t - \tau - \frac{R}{a}) \, dt \, dy' \, dz' 
\]

(18)
Finally we substitute \( v = 1/\sqrt{\mu \varepsilon} = a \) and \( a^2 \rho(x', y', z', t) = \varepsilon f(x', y', z', t) \) into the Equation (18), and get

\[
\phi(r, t) = \frac{1}{4\pi \varepsilon} \iiint_{V'} \frac{\rho(r', t - \frac{R}{v})}{R} dx' dy' dz'.
\] (19)

Where \( r \) corresponds to the point \( M(x, y, z) \), and \( r' \) corresponds to \( M'(x', y', z') \). \( V_{at}^M \) also can be written as \( V_{at}^M \), \( V_{at}^M \) is a sphere which center is \( M \) and the radius is \( at. R \) changes from 0 to \( at. \) Comparing Equation (19) with Equation (5), we find that both of the two solutions are the same in forms, but the only difference is that they have different integrating intervals. Shown as in Figure (2), \( V_{at}^M \) is a sphere which center is the field point \( M \) and the integrating radius changes with time \( t \) [4,5]. When the sphere has not intersected with the charge source region \( V' \), the whole interval integral is zero, which indicates that the electromagnetic wave caused by the source charge has not reached the field point \( M \). When the intersection of this sphere and the source charge region \( V' \) is \( V'' \), the whole interval integral is not zero, which indicates that the time-varying scalar potential \( \phi \) is determined by the superposition of charge effects in \( V'' \). Despite Equation (19) and Equation (5) have different integrating intervals, but both of them are essentially consistent, except for the integrating interval center of Equation (5) is the source point \( M' \).

![Figure 2: Distribution of sphere \( V_{at}^M \) and \( V' \)](image)

4 Summary
We use Fourier’s triple integral transform to solve non-homogeneous wave equations of electromagnetic fields. It transforms the three-dimensional non-homogeneous partial differential wave equation into an ordinary differential equation, so that the calculation is greatly simplified. Compared with the ordinary method based on a qualitative prediction in the common text books, we have derived the exact relationship between the excitation source and time-varying scalar potential \( \phi \) mathematically. It provides appropriate supplement and a convincing mathematical basis to the common textbooks which lack of mathematical deduction of the non-homogeneous wave equations [6].

References
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