Modular Hamiltonian of Excited States in Conformal Field Theory

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We present a novel replica trick that computes the relative entropy of two arbitrary states in conformal field theory. Our replica trick is based on the analytic continuation of partition functions that break the \( \mathbb{Z}_n \) replica symmetry. It provides a method for computing arbitrary matrix elements of the modular Hamiltonian corresponding to excited states in terms of correlation functions. We show that the quantum Fisher information in vacuum can be expressed in terms of two-point functions on the replica geometry. We perform sample calculations in two-dimensional conformal field theories.

In recent years entanglement theory has found numerous applications in the study of quantum phases of matter, relativistic field theories and gravity. Most of these applications focus on an entanglement measure in bipartite pure states known as the entanglement entropy. Unfortunately, in relativistic field theories entanglement entropy suffers from ultraviolet divergences. In gauge theories the definition of entanglement entropy is ambiguous \([1]\). In this letter, we present a method to compute in field theory another measure called relative entropy that is provably ultraviolet finite, universal and free of gauge ambiguities \([1][2]\).

Relative entropy is a measure of distinguishability between two states and has nice monotonicity and positivity properties. It appears naturally in the definition of entanglement measures for mixed states such as mutual information and the relative entropy of entanglement \([3]\). The relative entropy of two states \( \psi \) and \( \phi \) is pure.

The relative entropy of the density matrix \( \phi \) with respect to \( \psi \) is defined to be

\[
S(\phi\|\psi) = tr(\phi \log \phi) - tr(\phi \log \psi).
\]

Note that relative entropy is ill-defined when \( \psi \) is pure. The relative entropy of two states can be thought of as the expectation value of the difference of the modular Hamiltonians of the two states

\[
S(\phi\|\psi) = \langle \phi | H(\psi) - H(\phi) | \psi \rangle = tr((\phi - \psi)H(\psi)) - \Delta S.
\]

Here the positive Hermitian operator \( H(\psi) = -\log \psi \) is the modular Hamiltonian of \( \psi \), and \( \Delta S \) is the difference of the entanglement entropies of \( \phi \) and \( \psi \). If we formally define the generalized free energy function \( F_{\psi}(\phi) = tr(\phi H(\psi)) - S(\phi) \), then the relative entropy is the free energy difference between the two states

\[
S(\phi\|\psi) = F_{\psi}(\phi) - F_{\psi}(\psi).
\]

The function \( F_{\psi} \) has all the properties one expects from free energy in a thermodynamic theory where \( \psi = e^{-H(\psi)} \) plays the role of the equilibrium state \([4][6]\). Note that \( F_{\psi} \) achieves its minimum on the equilibrium state \( \psi \).

In this letter, we construct a class of field theory partition functions that are proportional to \( tr(\phi \psi^{n-1}) \). Their analytic continuation provides the relative entropy and the modular Hamiltonian of density matrices in excited states \( |\phi\rangle \) and \( |\psi\rangle \) reduced to the subsystem. While the formalism presented here applies to all quantum field theories we specialize to conformal field theories to have access to more computational tools.

According to the operator-state correspondence in conformal field theory (CFT) there is a one-to-one map between wave-functionals and operators in the Hilbert space. In radial quantization, the wave-functional of an excited state \( |\psi\rangle \) is found by performing a Euclidean path-integral with the corresponding operator \( \Psi \) inserted. Restricting to subsystem \( A \) the state is described by a density matrix \( \psi_A \); see figure \([1]\). To simplify notation we suppress the subsystem index \( A \), and use \( \psi \) to refer to the reduced state.

In principle, one can compute the logarithm of density matrix directly from the path-integral by taking the logarithm of a path-ordered operator using the so-called Magnus expansion, however in practice this is too hard. Here, we propose an alternative method to compute matrix elements of the modular Hamiltonian of excited states from the analytic continuation of correla-

\[\footnote{This is a consequence of positivity of relative entropy: \( F_{\psi}(\phi) \geq F_{\psi}(\psi) \)}\]
finding relative entropies and the diagonal elements of the modular Hamiltonian of excited states

\[ S(\phi|\psi) = \partial_n \log \left[ \frac{\text{tr}(\phi^n)\text{tr}(\psi)^{n-1}}{\text{tr}(\phi^n\psi^{n-1})}\right]_{n \to 1} \]

\[ \langle \phi|H(\psi) - H(\sigma_0)|\phi \rangle = \partial_n \log \left[ \frac{\langle \phi|\psi_0^n|\phi\rangle\text{tr}(\psi)^{n-1}}{\langle \phi|\psi^{-1}|\phi\rangle\text{tr}(\phi^n)^{n-1}}\right]_{n \to 1}, \]

where \( \sigma_0 \) is the reduced density matrix in vacuum. We subtract the vacuum modular Hamiltonian so that we have ultra-violet finite quantities at any \( n \). The off-diagonal elements of the modular Hamiltonian are obtained from its diagonal element in superposition states; see supplementary material.

Each of the terms inside the logarithm above can be expressed as a Euclidean path-integral with operator insertions on a replicated or the original geometry \[13\]. For instance, consider the terms \( \text{tr}(\rho\psi^{-1}) \). Sewing \( n \) copies of the density matrix cyclically along the boundary of their subsystems we obtain \( n \)-sheeted replica manifold \( \mathcal{R}_n \) and \( 2n \) operator insertions (figure 2)

\[ \text{tr}(\phi^n\psi^{-1}) = Z(\mathcal{R}_n)\langle \Phi(z'_n)|\Phi(z_n)\mathcal{O}_{\phi}^{n-1}\rangle_{\mathcal{R}_n} \]

\[ \mathcal{O}_{\phi}^{(n)} = \prod_{i=1}^{n} \Phi(z'_i)\Phi(z_i) \]

where \( z_i \) and \( z'_i \) are points \( z \) and \( z' \) on the \( i \)th sheet of \( \mathcal{R}_n \). It is important to note that plugging \([6] \) in \([5] \) all partition function terms cancel and we are left only with correlation functions at any \( n \) which are free of ultraviolet divergences.

Written explicitly in terms of correlation functions we find the main results of this section:

\[ S(\phi|\psi) = \]

\[ \partial_n \log \left[ \frac{\langle O_{\phi}^{(n)}\rangle_{\mathcal{R}_n} \langle \Phi(z'_1)|\Phi(z_1)\rangle^{n-1}_{\mathcal{R}_n}}{\langle \Phi(z'_1)|\Phi(z_n)\mathcal{O}_{\phi}^{(n-1)}\rangle_{\mathcal{R}_n} \langle \Phi(z'_1)|\Phi(z_1)\rangle^{n-1}_{\mathcal{R}_1}}\right]_{n \to 1} \]

\[ \langle \phi|H(\psi) - H(\sigma_0)|\phi \rangle = \]

\[ \partial_n \log \left[ \frac{\langle \Phi(z'_n)|\Phi(z_n)\mathcal{O}_{\phi}^{(n-1)}\rangle_{\mathcal{R}_n} \langle \Phi(z'_1)|\Phi(z_1)\rangle^{n-1}_{\mathcal{R}_1}}{\langle \Phi(z'_1)|\Phi(z_n)\mathcal{O}_{\phi}^{(n-1)}\rangle_{\mathcal{R}_n} \langle \Phi(z'_1)|\Phi(z_1)\rangle^{n-1}_{\mathcal{R}_1}}\right]_{n \to 1} \]

where

\[ X_c = E_{\Phi\Phi}(\mathcal{O}_{\phi}^{(n-1)}) + |c|^2 E_{\chi\chi}(\mathcal{O}_{\phi}^{(n-1)}) \]

\[ + c E_{\Phi\chi}(\mathcal{O}_{\phi}^{(n-1)}) + h.c. \]

\[ E_{\Phi\chi}(\mathcal{O}) = \frac{\langle \Phi(z'_n)|\chi(z_n)\mathcal{O}\rangle_{\mathcal{R}_n}}{\sqrt{\langle \Phi(z'_1)|\Phi(z_1)\rangle_{\mathcal{R}_1} \langle \chi(z'_1)|\chi(z_1)\rangle_{\mathcal{R}_1}}} \]

Here, we have assumed that \( \langle \psi|\phi \rangle = 0 \).

\[ \text{tr}(\phi^n\psi^{-1}) \]

(1)

In two dimensional CFTs, a finite temperature state on a line also has reduced states with local modular Hamiltonian \([14] \).
QUANTUM FISHER INFORMATION

Our replica trick connects the modular Hamiltonian of excited states to analytic continuation of 2n-point correlation functions. Apart from integrable models and large central charge theories obtaining analytic expressions for 2n-point functions is an intractable problem. However, as we show in this section a great simplification occurs once we focus on near-vacuum states.

Let us first consider a one-parameter family of states 
\[ \frac{\langle \phi | + \epsilon | X \rangle}{\sqrt{1 + \epsilon^2 x^2}} \] 
perturbed around \(|\phi\rangle\) in perpendicular direction \(|X\rangle\). The reduced density matrix on subsystem A expanded in \(\epsilon\) has the form
\[ \phi + \epsilon \rho_X^{(1)} + \epsilon^2 \rho_X^{(2)} + O(\epsilon^3) \]
Relative entropy is a smooth non-degenerate function of two states. Hence, the relative entropy of two nearby states expanded in \(\epsilon\) vanishes to the first order. The coefficient of the second order term, \(F_\phi(X,Y)\), is called the quantum Fisher information at point \(\phi\) in the space of density matrices:
\[ S(\phi + \epsilon \rho_X \parallel \phi) = \epsilon^2 F_\phi(X, X) + O(\epsilon^3). \]
This function defines a metric on the space of perturbations to state \(\phi\)
\[ 2F_\phi(X, Y) = F_\phi(X + Y, X + Y) - F_\phi(X, X) - F_\phi(Y, Y). \]
Quantum Fisher information is a local measure of distinguishability, and is intimately connected with uncertainty relations [14]. Consider the relative entropy of two nearby states. Our replica trick in [8] implies
\[ F_\phi(X, X) = \partial_n \left[ \sum_{m=0}^{n-2} \frac{\langle \langle \{ X, \Phi \}^{2n} \{ X, \Phi \}^{2(n-m-2)} \rangle_{\alpha, \beta} \langle \Phi \rangle \langle \Phi \rangle}{\langle \Phi \rangle \langle X \rangle \langle X \rangle} \right]_{n \to 1} \]
where \(X\) and \(\Phi\) denotes the operators that create the perturbations corresponding to \(|X\rangle\) and \(|\phi\rangle\), respectively. The location of operator insertions are the same as [6].

For near vacuum states, we replace \(\Phi\) in [8] with identity. The quantum Fisher information takes the form of an analytic continuation of two-point functions on the replica geometry; see figure 3
\[ F_\alpha(X, X) = \partial_n \left[ \sum_{z = z'} \sum_{z = z''} K_\alpha(z^+, z^-) \right]_{n \to 1} \]
\[ K_\alpha(z^+, z^-) = \sum_{m=1}^{n} \frac{\langle X(z^+)X(z^{m+1}) \rangle_{\alpha, \beta}}{\langle X(z_0)X(z'_0) \rangle_{\alpha, \beta}}. \]
This implies that in arbitrary dimensions the vacuum Fisher information of any primary excitation reduced to a ball or radius \(R\) is universal in the sense that it depends only on energy and subsystem size. In the remaining of this letter, we provide examples of relative entropies, modular Hamiltonians and quantum Fisher information in two-dimensional CFTs computed using the method above.

EXAMPLES IN TWO DIMENSIONS

Relative entropy of excited states: Consider a free massless boson CFT in two dimensions on a circle of radius \(R\) and a subsystem at \(A = (-l/2, l/2)\). We are interested in the excited states obtained by the action of chiral vertex operators on vacuum at past infinity: \(|\alpha\rangle = V_\alpha |\Omega\rangle = e^{i\alpha \phi} |\Omega\rangle\), where \(\phi\) is the boson field. The dimension of this operator is \((h, \bar{h} = (\alpha^2/2, 0))\). Here \(x = l/R\) is the dimensionless parameter. In supplementary material it is shown that in two dimensions one can equally use correlators on cylinder, full complex plane or a strip in our formulae in [25] for relative entropy and modular Hamiltonian. The conformal factors found from the change of coordinates vanish in the limit of \(n \to 1\). In a free theory with a non-degenerate ground state all correlation functions are determined by Wick’s theorem (8) [15]:
\[ \langle V_\alpha \mathcal{O}_\beta^{(n-1)} V_\alpha \rangle_S = (2 \sin(\pi x/n))^2 (1 - \alpha^2 - g_\alpha^{(n-2)2 - 2\alpha \beta}) \]
where \(S\) refers to correlators on a strip of width \(2\pi\), and \(g_n = \frac{\sin(\pi x)}{n \sin(\pi x/n)}\). For holomorphic excitations \(|V_\alpha\rangle \sim V_{-\alpha}\). Therefore,
\[ S(\alpha \parallel \beta) = \partial_n \log \left( \frac{\langle \mathcal{O}_\alpha^{(n-1)} \rangle_S (V_{-\beta} V_\beta)^{n-1}}{\langle V_{-\alpha} V_\alpha \rangle^{n-1}} \right) \]
\[ = (\alpha - \beta)^2 (1 - \pi x \cot(\pi x)). \]
The analytic continuation used above is justified in supplementary material. When \(\beta = 0\) this matches the result previously found using a \(Z_n\)-symmetric replica trick in [12]: \(S(\alpha \parallel 0) = \alpha^2 (1 - \pi x \cot(\pi x))\). Interestingly, the answer in [5] is symmetric in its arguments, \(S(\alpha \parallel \beta) = S(\beta \parallel \alpha)\). These excited states further have the property
that $S(\alpha) = S(\beta) = S(\sigma_0)$, where $\sigma_0$ is the vacuum density matrix. Hence, we find $\text{tr}(\rho_\alpha H_\beta) = \text{tr}(\rho_\beta H_\alpha)$ for all $\alpha$ and $\beta$.

**Modular Hamiltonian of excited states:** In the free $c = 1$ CFT Wick contractions imply that a correlator is zero unless $\sum_i \alpha_i = 0$. For all $\alpha \neq \gamma$ we have $\langle V_\alpha V_\gamma \mathcal{O}_\beta^{(n-1)} \rangle_S = 0$. As a result, $X_c$ in (25) is independent of $c$, and we find that the modular operator $H_\beta$ has no off-diagonal terms in the $|\alpha\rangle$ basis.

The diagonal elements are

$$\langle \alpha | H(\beta) - H(\sigma_0) | \alpha \rangle = \beta(\beta - 2\alpha)(1 - \pi x \cot(\pi x)).$$

Note that in the limit $\alpha = \beta$ this reproduces $-\beta^2(1 - \pi x \cot(\pi x)) = -S(\beta|\sigma_0)$ as it should. In the limit $\beta = 0$ the difference of modular Hamiltonians is the zero operator and hence the answer should vanish as it does.

**Quantum Fisher metric around the vacuum:** Consider an arbitrary two-dimensional conformal field theory on a circle. The vacuum Fisher information is given by equation (9). After some algebra we find

$$F_{\sigma}(X, X) = \partial_n \left[ 2s_n(0) + s_n(x) + s_n(-x) \right]_{|n \to 1}$$

$$s_n(x) = \left( \frac{\sin^2(\pi x)}{n^2} \right)^{h + \frac{h}{n}} \frac{1}{n} \sum_{m=1}^{n-1} \sin(\pi (m + x)/n)^{-2(h + \frac{h}{n})}.$$  

For simplicity we expand in small $x$ to find:

$$F_{\sigma} \simeq \partial_n \left[ 4 \left( \frac{\pi x}{n} \right)^{2(h + \frac{h}{n})} \frac{1}{n} \sum_{m=1}^{n-1} \sin(\pi m/n)^{-2(h + \frac{h}{n})} \right]_{|n \to 1}$$

$$= 2(\pi x)^{2(h + \frac{h}{n})} \sqrt{\pi} \Gamma(h + \frac{h}{n} + 1) \Gamma(h + h + \frac{3}{2}),$$  \hspace{1cm} (11)

where we have used the analytic continuation found in [10].

**Multiple intervals:** The replica trick developed here can be applied to subsystems with multiple intervals. As an example we focus on mutual information in vacuum

$$S(\sigma_{AB} \| \sigma_A \otimes \sigma_B) = I(A : B),$$  \hspace{1cm} (12)

where $A$ and $B$ are non-overlapping intervals. According to [5] the relative entropy is the analytic continuation of the vacuum partition functions on manifolds $Z_{AB}$ and $Z_{A,B}$ illustrated in figure 4

$$I(A : B) = \lim_{n \to 1} \frac{1}{n - 1} \left( \log Z_{AB}^n - \log Z_{A,B}^n \right).$$  \hspace{1cm} (13)

The first partition function $Z_{AB}$ corresponds to Renyi entropies of $\sigma_{AB}$. Therefore, from (12) all we need to check is

$$\partial_n Z_{n,AB} |_{n \to 1} = S(A) + S(B).$$  \hspace{1cm} (14)

Riemann-Hurwitz formula tells us that $Z_{AB}$ has genus $(n-1)$ and $Z_{A,B}$ is simply the Riemann sphere. Following [19] we compute the path-integral over these manifolds using twist operators in an orbifold theory with replica copies of the fields. In particular, up to normalization $Z_{n,AB}$ is the correlation function

$$\langle \sigma(1\ldots n)(u_A)\sigma(n\ldots 1)(v_A)\sigma(n\ldots 2n-1)(u_B)\sigma(2n-1\ldots n)(v_B) \rangle$$

in a $(2n - 1)$ replica theory. Here $u_A$ and $v_A$ are the endpoint of interval $A$, and going around the twist operator $\sigma(1\ldots m)$ the replica fields transform as $(X^1, X^2, \ldots X_m, X^{m+1}, \ldots X^{2n-1}) \rightarrow (X^1, X^2, \ldots X^m, X^{m+1}, \ldots X^{2n-1})$. Inserting a resolution of identity on $n^\text{th}$ sheet splits the correlator into a sum over the product of sphere one-point functions. (see figure 5). Sphere one-point function are zero unless $\Phi_\ell$ is the identity operator. In other words,

$$S_{n,AB} = S_A^n + S_B^n$$  \hspace{1cm} (15)

which is the sum of Renyi entropies of intervals $A$ and $B$; and hence (14) follows.

**DISCUSSION**

In this letter, we have developed a replica trick that takes advantage of breaking the replica symmetry to access the modular Hamiltonian of excited states. In the absence of the $Z_n$ replica symmetry Renyi’s are not monotonic in $n$, hence our method cannot be used to obtain lower or upper bounds on relative entropy. The
applicability of this method crucially relies on our ability to analytically continue correlation functions in $n$. According to the Carlson theorem [17], in order to find the unique analytic continuation of Renyi’s at integer $n$ one needs to further fix the behavior at $n \to \pm i \infty$. We postpone a careful study of this asymptotic choice and its physical implications to future work.

The correlation functions needed to compute the modular operator of an excited state are $2n$-point functions. There are not many examples of CFTs for which we have access to high-point correlators. One class of such CFTs are free theories which we briefly discussed. Another class are CFTs with large central charge, where one can reduce the calculation of $n$-point functions of heavy operators to a classical monodromy problem for differential equations that correlation functions satisfy [18].

In holographic theories, the vacuum Fisher information in spherical subsystems was recently shown to be dual to canonical energy in gravity [20]. This confirms the CFT calculation of this quantity and canonical energy in gravity [20]. This confirms the

\[ \log \left( \frac{X_c}{E_{\Phi \Phi}(1) + |c|^2 E_{\chi \chi}(1)} \right) \]

From (17) we find

\[ \langle \chi | H(\psi) - H(\sigma_0) | \xi_c \rangle = \partial_n \left[ \log \left( \frac{X_c}{X_{n-1}} \right) + i \log \left( \frac{X_{n-1}}{X_n} \right) \right] \]

\[ \langle \xi_c | A[\Phi] | \xi_c \rangle = \frac{\langle \Phi A \Phi \rangle}{\langle \Phi \Phi \rangle} + \frac{\langle \chi A \chi \rangle}{\langle \chi \chi \rangle} + \frac{c_A \langle \Phi A \chi \rangle}{\sqrt{\langle \Phi \Phi \rangle \langle \chi \chi \rangle}} + h.c. \]

We are interested in off-diagonal elements of the modular operator which are obtained from its diagonal element in superposition states

\[ \langle \chi | A \phi \rangle = \frac{1}{2} \left( A(1) - A(-1) + i A(1+i) - i A(-1+i) \right) \]

where the boundary conditions are $\phi(\omega = 1) = \phi^*$. It is immediate that a superposition state of $|\phi\rangle$ and $|\chi\rangle$ is given by

\[ c_\psi \langle \phi_n | \psi \rangle + c_\chi \langle \phi_n | \chi \rangle = \int_{\partial_d} D\phi e^{-S(\phi)}(c_\psi \Psi(0) + c_\chi \chi(0)). \]

The complex conjugate state $\langle \phi |$ is created by the action of $[\phi(z, \bar{z})]_i^{-1} = \bar{z}^{-2h} z^{-2h} \phi(1/\bar{z}, 1/z)$. A simple way to keep track of normalization for superposition states is to normalize term by term

\[ \langle \xi_c | A [\Phi] | \xi_c \rangle = \frac{\langle \Phi A \Phi \rangle}{\langle \Phi \Phi \rangle} + \frac{\langle \chi A \chi \rangle}{\langle \chi \chi \rangle} + \frac{c_A \langle \Phi A \chi \rangle}{\sqrt{\langle \Phi \Phi \rangle \langle \chi \chi \rangle}} + h.c. \]

We are interested in off-diagonal elements of the modular operator which are obtained from its diagonal element in superposition states

\[ \langle \chi | A | \phi \rangle = \frac{1}{2} \left( A(1) - A(-1) + i A(1+i) - i A(-1+i) \right) \]

where $\langle \xi_c | A | \xi_c \rangle = A(c)$ and $|\xi_c\rangle = \frac{1}{\sqrt{1+|c|^2}}(|\phi\rangle + c|\chi\rangle)$.

Note that $|\phi\rangle$ and $|\chi\rangle$ are orthogonal: $\langle \chi | \Phi \rangle = 0$.

The expectation value of the modular Hamiltonian in superposition state $\xi_c$ is:

\[ \langle \xi_c | H(\psi) - H(\sigma_0) | \xi_c \rangle = \log \left( \Psi(t_0)^* \Psi(t_0) \right) \]
Then, conditions imposed on \(\omega\) the strip coordinates \(\chi(z)\) and \(\chi(\bar{z})\) are at \(\pm h \phi - \bar{h} \phi\). On the t-plane operator insertions are at \(t_i = \pi(2i + x)/n\) and \(t'_i = \pi(2i - x)/n\), \(t_o = \pi x\), \(t'_o = -\pi x\) so \((22)\) needs to be regulated. However, the regulators cancel in the expression for \(\mathcal{J}_S\):

\[
\mathcal{J}_S(t_i) = \left( \frac{\partial z}{\partial t} \right)^{(n-1)\Delta} \left( \frac{\partial \bar{z}}{\partial t} \right)^{(n-1)\Delta} \times \prod_{i=1}^{n-1} \left( \frac{\partial z}{\partial t_i} \right)^{-\Delta} \left( \frac{\partial \bar{z}}{\partial t_i} \right)^{-\Delta},
\]

where \(\Delta = h \phi - \bar{h} \phi\). On the t-plane operator insertions are at \(t_i = \pi(2i + x)/n\) and \(t'_i = \pi(2i - x)/n\), \(t_o = \pi x\), \(t'_o = -\pi x\) so \((22)\) needs to be regulated. However, the regulators cancel in the expression for \(\mathcal{J}_S\):

\[
S(\phi||\psi) = \partial_n \log \left[ \frac{\langle \mathcal{O}_n^{(n)}(\phi) \rangle^{(n-1)}_{\mathcal{O}_n^{(n)}}}{\langle \mathcal{O}_n^{(n)}(\phi) \rangle^{(n-1)}_{\mathcal{O}_n^{(n)}}} \right]_{n \to 1}
\]

\[
\langle \phi|H(\psi) - H(\sigma_0)|\phi\rangle = \partial_n \log \left[ \frac{\langle \mathcal{O}_n^{(n)}(\phi) \rangle^{(n-1)}_{\mathcal{O}_n^{(n)}}}{\langle \mathcal{O}_n^{(n)}(\phi) \rangle^{(n-1)}_{\mathcal{O}_n^{(n)}}} \right]_{n \to 1}
\]

where the correlators can be either on the strip or the full complex plane, and we have suppressed the location of operator insertions.

**Analytic continuation**

We are interested in finding the analytic continuation of following finite sums

\[
\prod_{m=1}^{n-1} f_{n,m}(x)^{n-m}, \quad \prod_{m=1}^{n-2} f_{n,m}(x)^{n-m-1}, \quad \prod_{m=1}^{n-1} f_{n,m}(x),
\]

where \(f_{n,m}\) is given by

\[
f(n, m) = \frac{\sin^2((\pi m/n))}{\sin((\pi m + x)/n) \sin((\pi m - x)/n)}. \quad (26)
\]
The answer is given by the identities below:

\[
\prod_{m=1}^{n-1} f_{n,m}(x)^{n-m} = \left( \frac{n \sin(\pi x / n)}{\sin(\pi x)} \right)^n
\]

\[
\prod_{m=1}^{n-2} f_{n,m}(x)^{n-m-1} = \left( \frac{n \sin(\pi x / n)}{\sin(\pi x)} \right)^{n-2}
\]

\[
\prod_{m=1}^{n-1} f_{n,m}(x) = \left( \frac{n \sin(\pi x / n)}{\sin(\pi x)} \right)^2
\]

\[
\prod_{m=1}^{n-1} \sin^{(\alpha+\gamma)\beta}(\pi m / n) = \left( \frac{n \sin(\pi x / n)}{\sin(\pi x)} \right)^{(\alpha+\gamma)\beta}
\]

(27)

To prove these identities, first bring the expression on the right to the left-hand side. Then, note that both the numerator and denominator always have poles of the same degree, and therefore has no poles at any value of \(x\). Taking the limit \(x \to 0\) we find that our four expressions are equal to one for all \(x, n, \alpha, \beta\) and \(\gamma\).