DENSITIES OF CURRENTS ON NON-KÄHLER MANIFOLDS AND COMPLEX DYNAMICS

DUC-VIET VU

ABSTRACT. The aim of this paper is to generalize the theory of densities of closed positive currents by Dinh-Sibony to non-Kähler manifolds. As an application, we generalize known upper bounds for the number of isolated periodic points of meromorphic self-maps of Kähler manifolds to the case of self-maps of non-Kähler manifolds in a class $G$ including all compact complex surfaces. We also show that the dynamical degrees and algebraic entropy of meromorphic self-maps of compact complex surfaces are finite bi-meromorphic invariant.

Classification AMS 2010: 32U40, 32H50, 37F05.

Keywords: Periodic points, topological entropy, tangent current, density of currents, non-Kähler manifold, Gauduchon metric.

1. INTRODUCTION

A fundamental problem in the pluripotential theory and complex geometry, which was posed by Demailly [8], is to define in a reasonable way the intersection of two closed positive currents. Although, the intersection of currents of bi-degree $(1,1)$ is well understood (see [7, 28, 3]), the case of currents of higher bi-degree still remains challenging.

A recent remarkable progress in this research direction is the theory of densities of currents on Kähler manifolds given Dinh-Sibony [21]. This theory generalizes the super-potential theory of currents also due to Dinh-Sibony and the classical intersection of currents of bi-degree $(1,1)$ mentioned above, see [14, 35] for details. It has deep applications in complex dynamics and foliations. We refer the reader to [22, 10, 36, 14, 11, 13, 34] for details.

The first aim of this paper is to extend the notion of tangent currents, hence, densities of currents by Dinh-Sibony to non-Kähler manifolds. We then recover their properties as in [21]. As we will see later, this generalisation not only is natural (see comments after Theorem 1.1 below) but also has immediate applications to complex dynamics on non-Kähler manifolds.

Let $X$ be a compact complex manifold of dimension $k$. Let $T$ be a closed positive $(p,p)$-current on $X$ and $V$ a smooth complex submanifold of $X$. Roughly speaking, the tangent currents to $T$ along $V$ are closed positive currents on the projective compactification of the normal bundle $E$ of $V$ in $X$ encoding the infinitesimal behaviour of $T$ along the normal directions to $V$. As a particular case, when $V$ is a point and $T$ is a complex analytic sets passing through $V$, then there is only one tangent current to $T$ along $V$ which is characterized by the tangent cone of $T$ at $V$ and the multiplicity of $V$ in $T$ (which is the Lelong number of $T$ at $V$).

Date: February 5, 2019.
Given two closed positive currents $T_1, T_2$ on $X$, the density currents associated to $T_1, T_2$ are tangent currents to $T_1 \otimes T_2$ along the diagonal $\Delta$ of $X \times X$. Intuitively, these density currents contain all of informations about the intersection of $T_1, T_2$. The following is a consequence of our main results (see Theorems 2.9 and 3.1) showing the existence of tangent currents in a more general situation than in [21].

**Theorem 1.1.** Let $X$ be a complex manifold of dimension $k$. Let $T$ be a closed positive $(p, p)$-current on $X$ and $V$ a smooth complex submanifold of $X$. Then the following two properties hold:

(i) if $T$ is the current of integration along an analytic subset of $X$, then there exists a unique tangent current to $T$ along $V$. In particular, there exists a unique density current associated to two analytic subsets on $X$,

(ii) if $X$ is compact and there exists a Hermitian metric $\omega$ on $X$ for which $dd^c \omega^j = 0$ on $V$ for $1 \leq j \leq k - p - 1$, then there exist tangent currents to $T$ along $V$ and every such current is $V$-conic.

Property (i) shows us that since the density current associated to two analytic subsets on any complex manifold is unique, it might be a good notion for the intersection of two analytic sets of arbitrary dimensions in an arbitrary complex manifold.

Recall here that a $V$-conic current means a current in $E$ which is invariant by the multiplication by non-zero complex numbers along fibers of $E$. We would like to emphasize that in Property (ii) of Theorem 1.1 the closedness property of $\omega$ is only required to hold on $V$. This is crucial for our applications. Another remark is that if $p = k - 1$, the hypothesis on $\omega$ is always satisfied because $k - p - 1 = 0$.

In the theory of closed positive currents for compact Kähler manifolds, we use very often the property that the mass of the wedge product of closed positive currents (when they are well-defined) is equivalent to the norm of the cup product of their de Rham cohomology classes. The key fact illustrates a strong link between closed positive currents on compact Kähler manifolds and their de Rham cohomology classes. In the non-Kähler situation, that property is no longer true. One can see it simply by looking at a manifold with a vanishing Betti number of even degree, for example a standard Hopf surface. So in order to study currents efficiently in the non-Kähler situation, we should find a way to bypass the use of their cohomology classes. In this spirit, we will establish Proposition 2.7 below which is a version of the semi-continuity theorem for tangent currents in [21, Th. 4.11]. That result is a key for our applications.

We now present some applications of the above results to the study of dynamics of self-maps on non-Kähler manifolds. The case of self-maps with dominant topological degree on non-Kähler manifolds was studied in [37]. We refer to the last paper for examples of dynamical systems on non-Kähler manifolds.

Let us now introduce the following class $\mathcal{G}$ of complex manifolds which is a main object of our study. Let $\mathcal{G}$ be the set of compact complex manifolds $X$ possessing a Hermitian metric $\omega$ such that $dd^c \omega^j = 0$ for $1 \leq j \leq k - 1$, where $k : = \dim X$. The Hermitian metric $\omega$ with the last properties has been studied by Fino-Tomassini in [27, 25]. This notion is related to the anestho-Kähler metric introduced by Jost-Yau in [32] and strong KT metrics surveyed in [26].

Clearly, $\mathcal{G}$ contains every compact Kähler manifold. By a result of Gauduchon [29], every $k$-dimensional compact complex manifold admits a Gauduchon metric $\omega$, i.e, $\omega$ is
a Hermitian metric with \( \dd c \omega^{k-1} = 0 \). Hence every compact complex surface belongs to \( \mathcal{G} \). We refer to [27] for more examples of manifolds in \( \mathcal{G} \). We remark however that Hopf manifolds of dimension \( > 2 \) is not in \( \mathcal{G} \), see [23, Th. 8.3].

We would like to emphasize a key difference between \( \mathcal{G} \) and the class of Kähler manifolds is that in contrast to the Kähler case, we don’t know whether the product of two manifolds in \( \mathcal{G} \) is in \( \mathcal{G} \). It is very likely that this is not true, see [23, Th. 8.3].

We would like to emphasize a key difference between \( \mathcal{G} \) and the class of Kähler manifolds is that in contrast to the Kähler case, we don’t know whether the product of two manifolds in \( \mathcal{G} \) is in \( \mathcal{G} \). It is very likely that this is not true, see [27] for some related comments. That problem is a crucial difficulty when studying the dynamics of self-maps on \( X \in \mathcal{G} \) because in order to study dynamical properties of self-maps of \( X \), we often have to work on the Cartesian products \( X^n \) of \( X \) with \( n \in \mathbb{N} \).

Let us enter the details now. Let \( X \) be a compact complex manifold of dimension \( k \).

Let \( f \) be a dominant meromorphic self-map of \( X \). We only need the dominance property of \( f \) to define the iterates \( f^n \). So this assumption is superfluous if \( f \) is holomorphic. Let \( \omega \) be a strictly positive Hermitian \((1,1)\)-form on \( X \).

For \( 0 \le q \le k \), put

\[
d_q(f) := \limsup_{n \to \infty} \left( \int_X (f^n)^* \omega^q \wedge \omega^{k-q} \right)^{1/n}, \quad h_a(f) := \max_{0 \le q \le k} \{ \log d_q(f) \}.
\]

We will write \( d_q \) for \( d_q(f) \) if no confusion arises. We can see easily that \( d_q \) are independent of the choice of \( \omega \). The number \( d_0 \) is always 1 and \( d_k \) is the topological degree of \( f \).

When \( f \) is holomorphic, these numbers are finite because the differential of \( f \) is of norm uniformly bounded on \( X \). We call \( d_q \) the \( q \)th dynamical degree of \( f \) for \( 0 \le q \le k \) and \( h_a(f) \) the algebraic entropy of \( f \).

When \( X \) is Kähler, the numbers \( d_j, h_a(f) \) are crucial finite bi-meromorphic invariants of \( f \); see [17, 18, 16]. Let \( P_n \) be the number of isolated periodic points of \( f \) of period \( n \) and \( h_t(f) \) the topological entropy of \( f \). Our next main result is the following.

**Theorem 1.2.** Let \( X \in \mathcal{G} \) and \( f \) a dominant meromorphic self-map of \( X \). Then we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log P_n \le h_a(f). \tag{1.1}
\]

When \( X \) is of dimension 2, then the algebraic entropy \( h_a(f) \) of \( f \) is finite and is a bi-meromorphic invariant of \( f \) and

\[
h_t(f) \le h_a(f) < \infty. \tag{1.2}
\]

We don’t know whether \( d_q \) are finite for general \( X \in \mathcal{G} \) of dimension \( > 2 \). When \( X \) is a Kähler manifold of arbitrary dimension, (1.1) is proved by Dinh-Nguyên-Truong [13, Th. 1.1]. The upper bound (1.2) was proved by Gromov [30] for holomorphic self-maps of compact Kähler manifolds and by Dinh-Sibony [17, 18] for meromorphic self-correspondences of compact Kähler manifolds. The proofs in these last papers use, in an essential way, a regularisation theorem for closed positive currents in [20] which is not available in the non-Kähler case.

The key ingredient in the proof of Theorem 1.2 is our study of the intersection of analytic subsets on non-Kähler manifolds based on the above-mentioned generalization of the theory of densities of currents on Kähler manifold in [21]. A direct consequence of Theorem 1.2 is the following result.

**Corollary 1.3.** Every holomorphic self-map of compact complex manifolds in \( \mathcal{G} \) is an Artin-Mazur map, i.e., the number of isolated periodic points of \( f \) of period \( n \) grows at most exponentially as \( n \) tends to \( \infty \).
The last corollary illustrates an interesting difference between complex dynamics and real dynamical systems because as showed by Kaloshin in [33], there are large families of diffeomorphisms of compact differentiable manifolds whose sets of isolated periodic points grows faster than any given sequence of integers. Note however that Artin-Mazur [2] proved that the set of Artin-Mazur maps of a compact differentiable manifold is dense in the set of $\mathcal{C}^m$ maps with $m \geq 1$.

We now turn our attention to self-maps with dominant topological degrees. Recall that $f$ is said to have a dominant topological degree if $d_k(f) > d_j(f)$ for $0 \leq j \leq k - 1$. The dynamics of a such map has been thoroughly investigated in the Kähler case, see [19] and references therein for more information. The non-Kähler case was studied in [37] where it is proved that the equilibrium measure $\mu_f$ of $f$ exists and satisfies well-known properties as in the Kähler case. Here is our next main result.

**Theorem 1.4.** Let $X \in \mathcal{G}$ and $f$ a dominant meromorphic self-map of $X$ with a dominant topological degree. Then we have

$$P_n \leq e^{nh_a(f)} + o(e^{nh_a(f)}).$$

In the Kähler case, the bound (1.3) was proved by Dinh, Nguyên and Truong [11] for correspondences with dominant topological degrees and by Dinh, Nguyên and the author of this paper [14] for holomorphic correspondences with simple actions on the cohomology ring. The proof of both results use the semi-continuity of the total tangent class of a closed positive current along a submanifold of a compact Kähler manifold, see [21, Th. 4.11]. As we explained above, the de Rham cohomology class of a closed positive current on a non-Kähler manifold can be zero. Thus, even if some variant of that semi-continuity property still holds in our setting (for this we don’t know), this doesn’t give our expected estimate. The key in the proof of (1.3) is Proposition 2.7 in Section 2 which serves as a substitute of the semi-continuity theorem of Dinh-Sibony.

All of above results still hold for meromorphic correspondences. But in order to keep the presentation as simple as possible, we don’t elaborate it here. Finally, we would like to note that many dynamical properties in the Kähler case can be generalized to holomorphic self-maps on manifolds in $\mathcal{G}$. We refer to Remark 4.3 and Corollary 4.4 in Section 4 for details.

The paper is organized as follows. In Section 2, we present a generalization of the theory of tangent currents to non-Kähler manifolds. In Section 3, we use the last theory to study the intersection of analytic sets and prove Theorem 1.2. In Section 4 Theorem 1.4 is proved.

**Acknowledgments.** The author would like to express his gratitude to Tien-Cuong Dinh and Taeyong Ahn for fruitful discussions on the paper [21]. This research is supported by a postdoctoral fellowship of Alexander von Humboldt Foundation.
$V$ and one on $V$. Hence, the hypothesis on $T$ in fact makes no restriction in our study. Let $\text{supp} T$ be the support of $T$. Let $[V]$ be the current of integration along $V$.

Denote by $\pi : E \to V$ the normal bundle of $V$ in $X$ and $\overline{E} := \mathbb{P}(E \oplus \mathbb{C})$ the projective compactification of $E$. The hypersurface at infinity $H_\infty := \overline{E} \setminus E$ of $\overline{E}$ is naturally isomorphic to $\mathbb{P}(E)$ as fiber bundles over $V$. We also have a canonical projection $\pi_\infty : E \setminus V \to H_\infty$.

Let $U$ be an open subset of $X$ with $U \cap V \neq \emptyset$. Let $\tau$ be a smooth diffeomorphism from $U$ to an open neighborhood of $V \cap U$ in $E$ which is identity on $V \cap U$ such that the restriction of its differential $d\tau$ to $E|_{V \cap U}$ is identity. Such a map is called an admissible map. When $U$ is a small enough tubular neighborhood of $V$, there always exists an admissible map $\tau$ by [21, Le. 4.2]. In general, $\tau$ is not holomorphic. When $U$ is a small enough local chart, we can choose an admissible holomorphic map by using suitable holomorphic coordinates on $U$.

For $\lambda \in \mathbb{C}^*$, let $A_\lambda : E \to E$ be the multiplication by $\lambda$ on fibers of $E$. Consider the family of closed currents $(A_\lambda)_* \tau_* T$ on $E|_{V \cap U}$ parameterized by $\lambda \in \mathbb{C}^*$.

**Definition 2.1.** [21] [35] A tangent current $T_\infty$ of $T$ along $V$ is a closed positive current on $E$ such that there are a sequence $(\lambda_n) \subset \mathbb{C}^*$ converging to $\infty$ and a collection of admissible holomorphic maps $\tau_j : U_j \to E$ for $j \in J$ satisfying the following two properties.

(i) $V \subset \bigcup_{j \in J} U_j$,

(ii) $T_\infty := \lim_{n \to \infty} (A_{\lambda_n})_* (\tau_j)_* T$ on $\pi^{-1}(U_j \cap V)$ for every $j \in J$.

When $X$ is Kähler and $\text{supp} T \cap V$ is compact, the above definition of tangent currents agrees with that given in [21] and it is proved there that tangent currents always exist and are independent of the choices of $\tau_j$. This crucial fact also holds in our setting, see Lemma 2.2 below. By this reason, the sequence $(\lambda_n)$ is called the defining sequence of $T_\infty$. Before introducing a weaker assumption (Hypothesis (H) below) guaranteeing the existence of tangent currents, we will give some notations and auxiliary results.

Following [21], a bi-Lipschitz map $\tilde{\tau}$ from $U$ to an open neighborhood of $U \cap V$ in $E$ is said to be almost-admissible if $\tilde{\tau}|_{U \cap V} = \text{id}$, $\tilde{\tau}$ is smooth outside $V$ and on every local chart $(W, x = (x', x''))$ near $V \cap U$ with $V \cap W = \{x'' = 0\}$ then

$$
\tilde{\tau}(x) = (x' + O^*|x''|, x'' + O^*|x''|^2),
$$

and

$$
d\tilde{\tau}(x) = (dx' + O^*|x''|, dx'' + O^{**}|x''|^2),
$$

where for every positive integer $m$, $O^*|x''|^m$ means a function which is continuous outside $V$ and is equal to $O(|x''|^m)$ as $x'' \to 0$; $O^{**}|x''|^m$ means the sum of $1$-forms with $O^*|x''|^m$ coefficients and a combination of $dx''$, $d\overline{x''}$ with $O^*|x''|^{m-1}$.

Note that $\tilde{\tau}, T$ is well-defined as a closed current on $\tilde{\tau}(U) \setminus (U \cap V)$ which is of locally finite mass near $U \cap V$ because $\tilde{\tau}$ is bi-Lipschitz and smooth outside $V$. We extend $\tilde{\tau}, T$ to be a current of order $0$ on $\tilde{\tau}(U)$ by putting $\tilde{\tau}, T := 0$ on $U \cap V$. Although $\tilde{\tau}, T$ is actually closed (see [24, 4.1.14]), we will not need that fact in the sequel. When $\tilde{\tau}$ is smooth, it
is clear that $\tilde{\tau}_{*}T$ is the usual pushforward of $T$ by $\tilde{\tau}$ because $T$ has no mass on $V$. We will need to use both admissible and almost-admissible maps.

**Lemma 2.2. ([35])** Let $T_{n}$ be a tangent current of $T$ along $V$ with the defining sequence $(\lambda_{n})_{n \in \mathbb{N}}$. Then for any almost-admissible map $\tilde{\tau} : U \rightarrow E$, we have

$$T_{n} = \lim_{n \rightarrow \infty} (A_{\lambda_{n}})_{*} \tilde{\tau}_{*} T$$
onumber

on $\pi^{-1}(U \cap V)$.

**Proof.** We follow closely the arguments from [35]. Let $\tau_{j}, U_j$ with $j \in J$ be as above. Fix a $j \in J$. Without loss of generality, we can suppose that $U = U_j$ is a local chart. Put $\tau := \tau_{j}$. Let $(x', x'')$ be local coordinates on $U$ for which $V \cap U = \{x'' = 0\}$. Identify $E$ with $(V \cap U) \times \mathbb{C}^{k-l}$, recall here that $l = \text{dim } V$. Since both $\tau$ and $\tilde{\tau}$ are almost-admissible, we have

$$\tau_{*} \tau_{*} = \tau_{*} \tilde{\tau}_{*} \tau_{*} \tilde{\tau}_{*}$$

(2.1)

as $x'' \rightarrow 0$. Let $\Phi$ be a smooth form with compact support in $\pi^{-1}(U \cap V) \subset E$ and denote

$$\Phi_{\lambda} := (A_{\lambda})_{*} \Phi.$$ 

Notice that $|x''| \lesssim |\lambda|^{-1}$ on the support of $\tau_{*} \Phi_{\lambda} - \tilde{\tau}_{*} \Phi_{\lambda}$. Using this fact and (2.1), on $U \setminus V$, we have

$$\tau_{*} \Phi_{\lambda} - \tilde{\tau}_{*} \Phi_{\lambda} = \frac{1}{|\lambda|} \tau_{*} A_{\lambda}^{*} \Psi_{\lambda}$$

where $\Psi_{\lambda}$ are forms on $\pi^{-1}(U \cap V) \setminus V$ supported in a fixed compact subset of $E$ and the coefficients of $\Psi_{\lambda}$ are uniformly bounded on $\pi^{-1}(U \cap V)$ in $\lambda$. Let $\Omega$ be a positive form with compact support on $\pi^{-1}(U \cap V)$ such that $\Psi_{\lambda} \leq \Omega$ on $\pi^{-1}(U \cap V) \setminus V$ for every $\lambda$. Since $(A_{\lambda})_{*} \tau_{*} T$ is positive and $T$ has no mass on $V$, we have

$$\left| \langle T, \tau_{*} \Phi_{\lambda} - \tilde{\tau}_{*} \Phi_{\lambda} \rangle \right| = \left| \langle T, 1_{U \setminus V} (\tau_{*} \Phi_{\lambda} - \tilde{\tau}_{*} \Phi_{\lambda}) \rangle \right| \leq |\lambda|^{-1} \left| \langle T, \tau_{*} A_{\lambda}^{*} \Psi_{\lambda} \rangle \right| \leq |\lambda|^{-1} \left| \langle (A_{\lambda})_{*} \tau_{*} T, \Psi_{\lambda} \rangle \right| \leq |\lambda|^{-1} \left| \langle (A_{\lambda})_{*} \tau_{*} T, \Omega \rangle \right|.$$ 

By the hypothesis, $\lim_{\lambda_{n} \rightarrow \infty} (A_{\lambda_{n}})_{*} \tau_{*} T = T_{\infty}$. Thus the mass of $(A_{\lambda_{n}})_{*} \tau_{*} T$ on compact sets is bounded uniformly in $\lambda_{n}$. This gives

$$\left| \langle T, \tau_{*} \Phi_{\lambda_{n}} - \tilde{\tau}_{*} \Phi_{\lambda_{n}} \rangle \right| \leq C|\lambda_{n}|^{-1}.$$ 

for some constant $C$ independent of $n$. Hence $\lim_{n \rightarrow \infty} (A_{\lambda_{n}})_{*} \tilde{\tau}_{*} T = T_{\infty}$. The proof is finished.

For two closed positive currents $T_{1}, T_{2}$ on $X$. Consider the tensor current $T_{1} \otimes T_{2}$ on $X \times X$. A density current associated to $T_{1}, T_{2}$ is a tangent current of $T_{1} \otimes T_{2}$ along the diagonal $\Delta$ of $X \times X$. Consider a particular case where $T_{1} := T$ and $T_{2} := [V]$. We will show that a density current associated to $T_{1}[V]$ corresponds naturally to a tangent current of $T$ along $V$.

Observe that we have natural identifications $T(X^{2}) \approx TX \times TX$ between vector bundles, where $TX$ is the tangent bundle of $X$ and $\Delta \approx X$. Since $V \subset X \approx \Delta$, there is a canonical inclusion $\iota$ from $TV$ to $(TX \times \{0\})|_{\Delta}$ which is a subbundle of $T(X^{2})|_{\Delta}$. Let $F$ be the image of $\iota(TV)$ in the normal bundle $E_{\Delta} = T(X^{2})/T\Delta$. Put $\Delta_{V} := \{(x, x) \in X^{2} :$
Let $x \in V$. Let $E_{\Delta, V}$ be the restriction of $E_{\Delta}$ to $\Delta_V$. Observe that $F$ is a subbundle of $E_{\Delta, V}$ of rank $l$ and the natural map

$$\Psi : E_{\Delta, V}/F \to E = TX/TV$$

is an isomorphism. Let $p_V : E_{\Delta, V} \to E_{\Delta, V}/F$ be the natural projection.

**Lemma 2.3.** ([21] Le. 5.4) If $T_\infty$ is a tangent current of $T$ along $V$, then the current $p_V^*\Psi^*T_\infty$ is a tangent current of $T \otimes [V]$ along $\Delta$. Conversely, every tangent current of $T \otimes [V]$ along $\Delta$ can be written as $p_V^*\Psi^*T_\infty$ for some tangent current $T_\infty$ of $T$ along $V$.

**Proof.** Firstly notice that every tangent current of $T \otimes [V]$ along $\Delta$ is supported on $p^\Delta_\Delta^{-1}((\text{supp}T \times V) \cap \Delta)$, where $p^\Delta_\Delta$ is the natural projection from $E_{\Delta}$ to $\Delta$. Hence, a such current is supported on $E_{\Delta, V}$.

Let $T_\infty$ be a tangent current of $T$ along $V$ and $(\lambda_n)$ its defining sequence. Consider a local chart $(U, x)$ of $X$ with $U = U' \times U''$ and $x = (x', x'')$ so that $V \cap U$ is given by the equation $x'' = 0$ and $0 \in U$. We then obtain an induced local chart $U \times U$ with coordinates $(x, y)$ on $X \times X$ with $x = (x', x'')$ and $y = (y', y'')$. The diagonal $\Delta$ is given by the equation $x = y$ on $U \times U$. Put $z = (z', z'') := x - y$, $z' = x' - y'$, $z'' = x'' - y''$. Thus, for an open subset $U_1 = U_1' \times U_1''$ of $U$ small enough containing $0$, $(U_1^2, (x, z))$ is also a local chart on $X^2$ with $\Delta' = \{z = 0\}$.

Using the local coordinates $(x, z)$, we identify the tangent bundle of $X^2$ on $U_1^2$ with $U_1^2 \times \mathbb{C}^2k$ and $E_\Delta$ with $U_1 \times \mathbb{C}^k$ which is embedded in $U_1^2 \times \mathbb{C}^2k$ as $U_1 \times \{0\} \times \mathbb{C}^k$. Similarly, we also identify $TX$ on $U$ with $U \times \mathbb{C}^k$ and $E$ with $U' \times \mathbb{C}^{k-1}$. With these identifications, we see that

$$E_{\Delta, V} \approx U' \times \mathbb{C}^k, \quad F \approx U' \times \mathbb{C}^l \times \{0\}.$$

It follows that

$$p_V : U' \times \mathbb{C}^k \to U' \times \{0\} \times \mathbb{C}^{k-1}, \quad \Psi : U' \times \{0\} \times \mathbb{C}^{k-1} \to U' \times \mathbb{C}^{k-1}.$$

We also have that the identity maps $id_U : U \to U$ and $id_{U_1^2} : U_1^2 \to U_1^2$ are (local) holomorphic admissible maps for $V, \Delta$ on $X, X^2$ respectively. By definition of $T_\infty$, we get $T_\infty = \lim_{n \to \infty} (A_{\lambda_n})_* T$ on $U' \times \mathbb{C}^{k-1}$. Thus

$$p_V^* \Psi^* T_\infty = \lim_{n \to \infty} p_V^* \Psi^* (A_{\lambda_n})_* T = \lim_{n \to \infty} (A_{\lambda_n})_* p_V^* \Psi^* T$$

because $(A_{\lambda_n})_* = A_{\lambda_n}^{-1}$ and $A_{\lambda_n}$ commutes with the vector bundle maps $p_V, \Psi$. We now prove that $(A_{\lambda_n})_* (T \otimes [V])$ is convergent on $U_1^2 \times \mathbb{C}^{k-1}$. Let $\Phi = \Phi_0(x') \wedge \Phi_1(x'') \wedge \Phi_2(\lambda_n z') \wedge \Phi_3(\lambda_n z'')$ be a smooth form with compact support in $U_1 \times \mathbb{C}^k$. The set of forms $\Phi$ is dense in $\mathcal{C}^\infty$-topology in the space of smooth forms with compact support. We consider first the case where $\Phi_1$ is a function in $x''$. Without loss of generality, we can assume $\Phi_1(0) = 1$. We have

$$\langle (A_{\lambda_n})_* (T \otimes [V]), \Phi \rangle = \langle T \otimes [V], \Phi_0(x') \wedge \Phi_1(x'') \wedge \Phi_2(\lambda_n z') \wedge \Phi_3(\lambda_n z'') \rangle$$

$$= \langle T(x), \Phi_0(x') \wedge \Phi_1(x'') \wedge \Phi_3(\lambda_n x'') \wedge \int_{y' \in V} \Phi_2(\lambda_n (x' - y')) \rangle$$

$$= \langle T(x), \Phi_0(x') \wedge \Phi_3(\lambda_n x'') \wedge \int_{(y',0) \in V} \Phi_2(\lambda_n (x' - y')) \rangle + O(|\lambda_n|^{-1})$$
because \( x'' \to 0 \) as \( \lambda_n \to \infty \) and \((A_{\lambda_n})_* T\) is of uniformly bounded mass on compact subsets of \( U' \times \mathbb{C}^{k-l} \). Observe that

\[
\int_{(y',0) \in V} \Phi_2(\lambda_n(x' - y')) = \int_{z' \in \lambda_n^{-1}(x' - U'_1)} \Phi_2(\lambda_n z') = \int_{C_l} \Phi_2(v') = \int_{C_l} \Phi_2(v')
\]

for every \( x' \) in a fixed compact set if \( n \) big enough because \( \text{supp} \Phi_2 \subset \mathbb{C}^l \) which is contained in \( \lambda_n^{-1}(x' - U'_1) \) if \( |\lambda_n| \) is big. This together with (2.4) implies

\[
\langle (A_{\lambda_n})_*(T \otimes [V]), \Phi \rangle = \langle T(x), \Phi_0(x') \wedge \Phi_3(\lambda_n x'') \rangle + o_n \to \infty (1)
\]

\[
= \langle (A_{\lambda_n})_* T, \Phi_0(x') \wedge \Phi_3(x'') \rangle + o_n \to \infty (1).
\]

Notice that

\[
\Phi|_{E_{\Delta', V}} = \Phi_0(x') \wedge \Phi_2(z') \wedge \Phi_3(z'')
\]

because \( \Phi_1(0) = 1 \). Using this and (2.3) gives

\[
\langle p_{1'}^* \Psi^* T_{\infty}, \Phi|_{E_{\Delta', V}} \rangle = \langle T_{\infty}, \Psi_* (p_{1'}^*)_*(\Phi|_{E_{\Delta', V}}) \rangle
\]

\[
= \langle T_{\infty}, \Psi_* (p_{1'}^*)_*(\Phi_0(x') \wedge \Phi_2(z') \wedge \Phi_3(z'')) \rangle
\]

\[
= \langle T_{\infty}, \Phi_0(x') \wedge \Phi_3(x'') \int_{C_l} \Phi_2 \rangle.
\]

Comparing (2.6) and (2.5) gives \( \lim_{n \to \infty} (A_{\lambda_n})_*(T \otimes [V]) = p_{1'}^* \Psi^* T_{\infty} \). Consider now \( \Phi_1 \) is a form of degree \( \geq 1 \). Then \( \Phi|_{E_{\Delta', V}} = 0 \). It follows that \( \langle p_{1'}^* \Psi^* T_{\infty}, \Phi \rangle = 0 \). On the other hand, we also see from (2.4)-(2.5) that \( \langle (A_{\lambda_n})_*(T \otimes [V]), \Phi \rangle \to 0 \) as \( n \to \infty \). Consequently, \( \lim_{n \to \infty} (A_{\lambda_n})_*(T \otimes [V]) = p_{1'}^* \Psi^* T_{\infty} \) holds in both cases.

We now assume that \( T_{\infty} := \lim_{n \to \infty} (A_{\lambda_n})_*(T \otimes [V]) \) exists. Then, by choosing \( \Phi_1(x'') \equiv 1 \) in the above defining formula of \( \Phi \) and using (2.4)-(2.6), we obtain that \((A_{\lambda_n})_* T\) is of uniformly mass on compact subsets of \( U' \times \mathbb{C}^{k-l} \). Hence, there is a subsequence \( (\lambda_n') \) of \( (\lambda_n) \) for which \((A_{\lambda_n'})_* T \to T_{\infty} \) for some \( T_{\infty} \).

The first part of the proof then implies that \( T_{\infty}' = p_{1'}^* \Psi^* T_{\infty} \). Hence, \( T_{\infty} \) is the unique limit current of the sequence \((A_{\lambda_n})_* T\). In other words, \( \lim_{n \to \infty} (A_{\lambda_n})_* T = T_{\infty} \) and \( T_{\infty}' = p_{1'}^* \Psi^* T_{\infty} \). This finishes the proof.

Let \( \sigma : \hat{X} \to X \) be the blowup along \( V \) of \( X \) and \( \hat{V} := \sigma^{-1}(V) \) the exceptional hypersurface. Recall that \( \hat{V} \) is naturally biholomorphic to \( \mathbb{P}(E) \). Let \( \sigma_E : \overline{E} \to \hat{E} \) the blowup along \( V \) of \( \overline{E} \). The restriction of \( \sigma_E \) to \( \hat{E} := \sigma_E^{-1}(E) \) is the blowup along \( V \) of \( E \). The projection \( \pi \) induces naturally a vector bundle projection \( \pi_E \) from \( \hat{E} \) to \( E \). The last map can be extended to a projection \( \pi_E \) from \( \overline{E} \) to \( E \). The vector bundle \( \sigma_E : \hat{E} \to \sigma_E^{-1}(V) \) is naturally identified with the normal bundle of \( \hat{V} \) in \( \hat{X} \). Hence we can identify \( \sigma_E^{-1}(V) \) with \( \hat{V} \) and use \( \hat{E} \) as the normal bundle of \( \hat{V} \) in \( \hat{X} \).

Given any smooth admissible map \( \tau : U \to E \), by [21, Le. 4.3], we can lift \( \tau \) to a bi-Lipschitz almost-admissible map \( \hat{\tau} \) with

\[
\sigma_E \circ \hat{\tau} = \tau \circ \sigma_E.
\]

Observe that the hypersurface at infinity \( \hat{H}_\infty \) of \( \overline{E} \) is biholomorphic to that of \( E \) via \( \sigma_E \).

We use \( \hat{\pi}_\infty \) to denote the natural projection from \( \overline{E} \\hat{\setminus} \hat{V} \) to \( \hat{H}_\infty \). Since the rank of \( \overline{E} \) over
\( \hat{\nabla} \) is 1, we can extend \( \hat{n}_\infty \) to a projection from \( \hat{E} \) to \( \hat{H}_\infty \). Let \( \hat{T} \) be the pull-back of \( T \) on \( \hat{X} \setminus \hat{V} \) by \( \sigma|_{\hat{X} \setminus \hat{V}} \). We assume from now on the following.

(H): \( \hat{T} \) has locally finite mass near \( \hat{V} \) and there are countably many holomorphic admissible maps \( \hat{\tau}_j: \hat{U}_j \to \hat{E} \) with \( j \in J \) such that \( \hat{V} \subset \cup_{j \in J} \hat{U}_j \), \((A_\lambda)s(\hat{\tau}_j)\hat{T} \) is of uniformly bounded mass on compact subsets of \( \pi_{\hat{E}}^{-1}(\hat{U}_j \cap \hat{V}) \) as \( |\lambda| \to \infty \) for every \( j \in J \).

We will prove in Theorem 2.9 at the end of this section that the last assumption is satisfied if \( \text{supp} T \cap V \) is compact and there exists a Hermitian form \( \omega \) on \( X \) with \( d\bar{d}\omega^j = 0 \) on \( V \) for \( 1 \leq j \leq k-p-1 \). This generalizes the criteria given in [21, Th. 4.6, Le. 3.12] for Kähler manifold \( X \) where the above form \( \omega \) is a Kähler form on \( X \). Another interesting case where (H) is satisfied is when \( T \) is a current of integration along an analytic subset of \( X \), see Theorem 3.1 in the next section.

Note that since \( \hat{T} \) has locally finite mass near \( \hat{V} \), it can be extended trivially through \( \hat{V} \) to be a closed positive current on \( \hat{X} \). We still use \( A_\lambda \) to denote the multiplication by \( \lambda \in \mathbb{C}^* \) in fibers of \( \hat{E} \) or \( \hat{E} \).

By a diagonal argument, Hypothesis (H) ensures the existence of a tangent current \( \hat{T}_\infty \) to \( \hat{T} \) along \( \hat{V} \) associated with a sequence \((A_\lambda_n) \subset \mathbb{C} \to \infty \). The following result is essentially contained in [21].

**Proposition 2.4.** For any smooth admissible map \( \tau: U \to E \), the mass \((A_\lambda)_s\tau_*T \) on compact subsets of \( E|_{V \cup U} \) is uniformly bounded in \( \lambda \in \mathbb{C}^* \) with \( |\lambda| \geq 1 \). Every tangent current \( T_\infty \) to \( T \) along \( V \) satisfies

\[
(2.8) \quad T_\infty = (\sigma_E)_s\hat{T}_\infty
\]

for some tangent current \( \hat{T}_\infty \) of \( \hat{T} \) along \( \hat{V} \). There exists a closed positive current \( \hat{S}_\infty \) on \( \hat{H}_\infty \) such that

\[
(2.9) \quad \hat{T}_\infty = \hat{\pi}_*\hat{S}_\infty, \quad T_\infty = \pi_*S_\infty
\]

for \( S_\infty := (\sigma_E)_s\hat{S}_\infty \).

Since \( \pi_\infty \) is only a submersion from \( E \setminus V \) to \( H_\infty \), in the second equality of (2.9), the current \( \pi_*S_\infty \) is à priori a closed positive current on \( E \setminus V \) which can be extended to be a current on \( E \) trivially through \( V \) because it has locally bounded mass there. A direct consequence of (2.9) is that \( T_\infty \) can be extended to be a closed positive current on \( E \setminus V \) having no mass on \( V \). We still have that the de Rham cohomology of every tangent current \( T_\infty \) is the same as in [21]. But we don’t need to use that fact in this paper.

**Proof.** Let \( \tau \) be as in the statement and \( \hat{\tau}: \hat{U} \to \hat{E} \) the lift of \( \tau \) to \( \hat{U} = \sigma_{\hat{E}}^{-1}(U) \) as above. By (2.7) and the fact that \( T, \hat{T} \) have no mass on \( V, \hat{V} \) respectively, we have

\[
(2.10) \quad \langle (A_\lambda)_s\hat{\tau}_*\hat{T}, \sigma_{\hat{E}}^*\Phi \rangle = \langle (A_\lambda)_s\tau_*T, \Phi \rangle
\]

for every smooth form \( \Phi \) with compact support in \( E|_{V \cup U} \). Recall that \( \hat{\tau} \) is almost-admissible. By Lemma 2.2 and (H), the mass of \((A_\lambda)_s\hat{\tau}_*\hat{T} \) is uniformly bounded on compact subsets of \( \pi_{\hat{E}}^{-1}(\hat{U} \cap \hat{V}) \). Using this and (2.10) implies the first desired assertion.
Let $T_\infty$ be a tangent current with the defining sequence $(\lambda_n)$. By using a subsequence of $(\lambda_n)$ if necessary, we can assume also that $(A_{\lambda_n})_\ast \tau_\ast T$ converges to a tangent current $\hat{T}_\infty$ of $\hat{T}$ along $\hat{V}$. Substituting $\lambda = \lambda_n$ in (2.10) and letting $n \to \infty$ give

$$\langle \hat{T}_\infty, \sigma^c \hat{\Phi} \rangle = \langle T_\infty, \hat{\Phi} \rangle.$$  

Hence, the equality (2.8) follows.

We claim that $\hat{T}_\infty$ have no mass on $\hat{V}$. To see it, let $(\hat{U}_1' \times \hat{U}_1'', (\hat{x}', \hat{x}_k))$ be a relatively compact local chart of $\hat{X}$ with $\hat{x}' = (\hat{x}_1, \ldots, \hat{x}_{k-1})$ so that $\hat{V}$ is given by $\hat{x}_k = 0$. Since the restriction of $\hat{T}_\infty$ to $\hat{V}$ is a closed positive current, it is the pushforward of a closed positive current on $\hat{V}$. It follows that the mass of $\hat{T}$ on $\hat{V} \cap (\hat{U}_1' \times \hat{U}_1'')$ is

$$\lesssim \langle \hat{T}_\infty, 1_{\hat{U}_1' \times \hat{U}_1''}(dd^\ast\|\hat{x}'\|^k) \rangle = \lim_{n \to \infty} \langle \hat{T}, A_{\lambda_n}^\ast (1_{\hat{U}_1' \times \hat{U}_1''}(dd^\ast\|\hat{x}'\|^k)) \rangle = \lim_{n \to \infty} \langle \hat{T}, (1_{\hat{U}_1' \times (\lambda_n \hat{U}_1'')})(dd^\ast\|\hat{x}'\|^k) \rangle = 0$$

because $\hat{T}$ has no mass on $\hat{V}$.

We now prove (2.9). To this end, we first check that $\hat{T}_\infty$ is $\hat{V}$-conic, i.e., $(A_t)^\ast \hat{T}_\infty = \hat{T}_\infty$ for every $t \in \mathbb{C}^\ast$. Let the local coordinates be as above. Let $\hat{\Phi} = \Phi_1(\hat{x}') \wedge \Phi_2(\hat{x}_k)$ be a smooth form with compact support on $\hat{U}_1' \times \hat{C}$. Observe that $(A_t)^\ast \Phi_2$ and $\Phi_2$ belongs to the same cohomology class with compact support in $\hat{C}$ because their integrals over $\hat{C}$ are equal.

Since $H^2_\ast(\mathbb{C}, \mathbb{C})$ is of dimension 1, there exists a smooth 1-form $\Theta(x_k)$ with compact support on $\hat{C}$ for which $(A_t)^\ast \Phi_2 - \Phi_2 - d\Theta$ is a 1-form in $x_k$, i.e,

$$(A_t)^\ast \Phi_2 - \Phi_2 - d\Theta = a(x_k)dx_k + b(x_k)d\tau_k.$$  

Using this, the closedness of $T$ and Cauchy-Schwarz inequality, we obtain that

$$\left| \langle (A_\lambda)^\ast T, \Phi_1 \wedge ((A_t)^\ast \Phi_2 - \Phi_2) \rangle \right| = \left| \langle (A_\lambda)^\ast T, \Phi_1 \wedge ((A_t)^\ast \Phi_2 - \Phi_2 - d\Theta) \rangle + d\Phi_1 \wedge \Theta \rangle \right| \lesssim \langle (A_\lambda)^\ast T, (dd^\ast\hat{x})^{k-l} \rangle \langle (A_\lambda)^\ast T, 1_{\supp \Phi}(dd^\ast\hat{x})^{k-l} \rangle.$$  

The first term in the right-hand side of the last inequality is uniformly bounded in $\lambda$, whereas the second one converges to 0 as $\lambda \to \infty$ because $(dd^\ast\hat{x})$ is invariant by $A_\lambda$ and $\supp (A_\lambda)^\ast \Phi$ converges to $V$. Letting $\lambda \to \infty$ gives $(A_\lambda)^\ast T_\infty = T_\infty$.

Observe that if $\theta$ is a smooth closed positive form on $\hat{U}_1 := \hat{U}_1' \times \hat{U}_1''$ then $\hat{T}_\infty \wedge \pi_{\hat{E}}^\ast(\theta|_{\hat{V}})$ is a tangent current of $\hat{T} \wedge \theta$ along $\hat{V}$ on $\hat{U}_1$ with the same defining sequence $(\lambda_n)$. Choose a such $\theta$ of bidegree $(k-p, k-p)$. We obtain that $\mu := \hat{T}_\infty \wedge \pi_{\hat{E}}^\ast(\theta|_{\hat{V}})$ is a nonnegative measure on $\pi_{\hat{E}}^{-1}(\hat{U}_1 \cap \hat{V})$ which has no mass on $\hat{V}$ because $\hat{T}_\infty$ has no mass on $\hat{V}$.

On the other hand, since $\hat{T}_\infty \wedge \pi_{\hat{E}}^\ast(\theta|_{\hat{V}})$ is $\hat{V}$-conic, for any smooth positive cut-off function $\chi(\hat{x}', \hat{x}_k)$ supported on $\hat{U}_1$, we have

$$\langle \mu, \chi \rangle = \langle \mu, A_1^\ast \chi \rangle$$  

for every $t \in \mathbb{C}^\ast$. Letting $t \to \infty$ in the last equality and observing that the limit supremum of the uniformly bounded functions $A_1^\ast \chi$ as $|t| \to \infty$ is a function supported on $\pi_{\hat{E}}(\supp \chi) \subset \hat{V} \cap \hat{U}_1$, we obtain

$$|\langle \mu, \chi \rangle| \lesssim \|1_{\hat{V} \cap \hat{U}_1} \mu\| = 0.$$
Thus, \( \mu = 0 \). Or in other words, \( \hat{T}_\infty \wedge \pi^*_E(\theta|_\nu) = 0 \) for every smooth closed positive form \( \theta \) of bidegree \( (k - p, k - p) \). It follows that in the local coordinates \((\hat{x}', t)\) of \( \hat{E} \), the current \( \hat{T}_\infty \) must have the following form:

\[
\hat{T}_\infty = \sum_{I, I'} \alpha_{I, I'}(\hat{x}', t)d\hat{x}'_I \wedge d\hat{x}'_{I'},
\]

for some Radon measures \( \alpha_{I, I'} \) on \( \hat{E} \), where the sum is taken over \( I, I' \) with \( I, I' \subset \{1, \ldots, k - 1\} \) and \( I, I' \) are of cardinality \( p \). Since \( \hat{T}_\infty \) is closed, \( \alpha_{I, I'} \) is independent of \( t \). As a result, we obtain a current \( \hat{S}_\infty \) on \( \hat{U}_1 \) for which \( \hat{T}_\infty = \pi^*_E \hat{S}_\infty \) on \( \pi^{-1}_E(\hat{U}_1) \). The last formula tells us that \( \hat{S}_\infty \) is independent of local coordinates and local charts. Hence, \( \hat{S}_\infty \) is a well-defined closed positive current on \( \hat{V} \) for which \( \hat{T}_\infty = \pi^*_E \hat{S}_\infty \).

Now notice that \( (\pi_E)|_{\hat{H}_\infty} \) is a biholomorphism between \( \hat{H}_\infty \) and \( \hat{V} \). So we can identify these two submanifolds via that biholomorphism. We then view \( S_\infty \) as a current on \( \hat{H}_\infty \). Observe now the fiber of \( \pi_E \) at \( \hat{x} \in V \) only differs to the fiber of \( \hat{S}_\infty \) at \( (\pi_E)|_{\hat{H}_\infty}^{-1}(\hat{x}) \) at two points. This implies that \( \hat{T}_\infty = \hat{S}_\infty \hat{S}_\infty \). This proves the first equality of (2.9). The second equality follows directly from the first one and the following formulae:

\[
T_\infty = [(\sigma_E|_{\hat{E}\setminus \nu})^{-1}]^*(\hat{T}_\infty), \quad \hat{\pi}_\infty \circ (\sigma_E|_{\hat{E}\setminus \nu})^{-1} = (\sigma_E|_{\hat{E}\setminus \nu})^{-1} \circ \pi_\infty.
\]

This ends the proof. \( \square \)

Recall that \( [\hat{V}] \) is a current of bidegree \((1, 1)\). Thus, \( [\hat{V}] \) can be represented as the sum \( d\hat{u} + \beta \) for some smooth form \( \beta \) and some quasi-plurisubharmonic function \( \hat{u} \) on \( \hat{X} \). A such \( \hat{u} \) is called a potential of \( [\hat{V}] \).

**Proposition 2.5.** Assume that \( \hat{T} \wedge [\hat{V}] \) is well-defined in the classical sense, that means that potentials of \( [\hat{V}] \) are locally integrable with respect to the trace measure of \( \hat{T} \). Then the tangent current to \( T \) along \( V \) is unique and is given by

\[
T_\infty = \pi^*_E(\hat{T} \wedge [\hat{V}]),
\]

where recall that we identified \( \hat{V} \) with \( \mathcal{P}(E) \) and identified \( \hat{T} \wedge [\hat{V}] \) with a current on \( \hat{V} \).

**Proof.** By [35], there is a unique tangent current \( R \) of \( \hat{T} \wedge [\hat{V}] \) along the diagonal \( \Delta_{\hat{X}} \) of \( \hat{X} \times \hat{X} \) and \( R \) is equal to \( \pi^*_{\Delta_{\hat{X}}}(\hat{T} \wedge [\hat{V}]) \) by identifying \( \hat{X} \) with \( \Delta_{\hat{X}} \), where \( \pi_{\Delta_{\hat{X}}} \) is the projection from the normal bundle \( E_{\Delta_{\hat{X}}} \) onto \( \Delta_{\hat{X}} \). Hence, by Lemma 2.3, there exists uniquely a tangent current of \( \hat{T} \) along \( V \) which is given by \( \pi^*_E(\hat{T} \wedge [\hat{V}]) \). This combined with (2.9) yields that

\[
S_\infty = \hat{T} \wedge [\hat{V}].
\]

The desired equality then follows. The proof is finished. \( \square \)

The following simple result plays a key role for our purposes later.

**Lemma 2.6.** Let \( Y \) be a complex manifold and \( Z \) a complex hypersurface of \( Y \). Let \( R_n \) be a sequence of closed positive currents of bidimension \((q, q)\) on \( Y \) converging to a current \( R_\infty \). Assume that

(i) there exists a compact \( K \) of \( Y \) for which \( \text{supp} R_n \cap Z \subset K \) for every \( n \),
\((ii)\) \(R_n \wedge [Z] \) and \((1_{Y \setminus Z} R_\infty) \wedge [Z] \) are classically well-defined for every \(n\), where \(1_{Y \setminus Z} \) is the characteristic function of \(Y \setminus Z\).

Then for every smooth \(2(q - 1)\)-form \(\Omega\) on \(Y\) with \(\dd c \Omega = 0\) on \(Z\), we have

\begin{equation}
\lim_{n \to \infty} \langle R_n \wedge [Z], \Omega \rangle = \langle (1_{Y \setminus Z} R_\infty) \wedge [Z], \Omega \rangle.
\end{equation}

**Proof.** By \((i)\) there is a cut-off function \(\chi\) compactly supported on \(Y\) such that \(0 \leq \chi \leq 1\) and \(\chi \equiv 1\) on an open neighborhood \(W\) of \(Z\) with \(W \cap \text{supp} R_n \subseteq K'\) for every \(n\) and some fixed compact \(K'\) independent of \(n\). Write \([Z] = \dd c u + \xi\) for some smooth closed form \(\xi\) and some quasi-p.s.h. function \(u\) which is smooth outside \(Z\) and has a log singularity near \(Z\). Since the support of \(R_n \wedge [Z]\) is contained in \(W\), we get

\[
\langle R_n \wedge [Z], \Omega \rangle = \langle R_n \wedge [Z], \chi \Omega \rangle = \langle R_n \wedge (\dd c u + \xi), \chi \Omega \rangle = \langle R_n \wedge \xi, \chi \Omega \rangle + \int_Y R_n \wedge (u \chi \dd c \Omega) + \int_Y u R_n \wedge (\dd c \chi \wedge \Omega - d^c \chi \wedge d\Omega + d\chi \wedge d^c \Omega).
\]

Denote by \(I_1, I_2, I_3\) respectively the first, second and third terms in the right-hand side of the last formula. Clearly,

\[
I_1 + I_3 \to \langle R_\infty \wedge \xi, \chi \Omega \rangle + \int_Y u (1_{Y \setminus Z} R_\infty) \wedge (\dd c \chi \wedge \Omega - d^c \chi \wedge d\Omega + d\chi \wedge d^c \Omega)
\]

because \(u\) is smooth outside \(Z\) and \(d\chi, d^c \chi\) vanish near \(Z\). On the other hand, since \(\dd c \Omega = 0\) on \(Z\), the form \(u \chi \dd c \Omega\) is continuous on \(Y\) and equal to 0 on \(Z\). This implies that

\[
I_2 \to \int_Y R_\infty \wedge (u \chi \dd c \Omega) = \int_Y (1_{Y \setminus Z} R_\infty) \wedge (u \chi \dd c \Omega).
\]

The desired limit \((2.13)\) then follows. The proof is finished. \(\square\)

As mentioned in Introduction, in a general non-Kähler compact manifold \(X\), the de Rham cohomology class of a positive closed current can be vanished. So the use cohomology classes of closed positive currents is not efficient as usual. Consequently, we don’t know whether a semi-continuity property similar to \([21, \text{Th. 4.11}]\) still holds. Proposition \([2.7]\) below will serve as a substitute for that semi-continuity theorem. Although its hypothesis is more restrictive, this result is still enough to have meaningful applications in complex dynamics as we will see later.

Let \(\omega\) be a positive definite Hermitian form on \(X\). If \(\text{codim} V \geq 2\), let \(\tilde{\omega}_h\) be a Chern form of \(O(-\tilde{V})\) whose restriction to each fiber of \(\tilde{V} \approx \mathbb{P}(E)\) is strictly positive, otherwise we simply put \(\tilde{\omega} := 0\). By scaling \(\omega\) if necessary, we can assume that \(\tilde{\omega} := \sigma^* \omega + \tilde{\omega}_h > 0\). Since \(\sigma_* \tilde{\omega}_h = \sigma_* \tilde{\omega} - \omega \geq -\omega\), there exists a quasi-p.s.h. function \(\varphi\) on \(\tilde{X}\) such that

\begin{equation}
\sigma_* \tilde{\omega}_h = \dd c \varphi + \eta
\end{equation}

for some smooth closed form \(\eta\). The function \(\varphi\) is a crucial object for us as it is in \([21]\). By multiplying \(\tilde{\omega}_h\) by a strictly positive constant, we have

\[
\sigma^* \sigma_* \tilde{\omega}_h = \tilde{\omega}_h + [\tilde{V}]
\]
if $\text{codim} V \geq 2$. Indeed one only needs to check it locally. Hence, we can reduce this question to the Kähler case where the desired identity is already known. Thus we have
\begin{equation}
|\varphi(x) - \log \text{dist}(x, V)| \lesssim 1
\end{equation}
on compact subsets of $X$ provided that $\text{codim} V \geq 2$.

**Proposition 2.7.** Let $X$ be a complex manifold. Let $T_n$ be a sequence of closed positive currents of bidimension $(q, q)$ converging to a current $T$. Assume that
(i) $T_n$ has no mass on $V$ and $\text{supp} T_n \cap V \subset K$ for some compact $K$ independent of $n$,
(ii) $\hat{T}_n, \hat{T}$ are of locally bounded mass near $\hat{V}$ (hence they can be extended trivially through $\hat{V}$) and the products $\hat{T}_n \wedge \hat{V}, \hat{T}_\infty \wedge \hat{V}$ are well-defined in the classical sense for every $n$,
(iii) $\text{ddc} \omega^j = 0$ on $V$ for $1 \leq j \leq q - 1$.

Then we have
\begin{equation}
\lim_{n \to \infty} \langle \sigma_*(\hat{T}_n \wedge \hat{V} \wedge \hat{\omega}^j), \omega^{q-j-1} \rangle = \langle \sigma_*(\hat{T} \wedge \hat{V} \wedge \hat{\omega}^j), \omega^{q-j-1} \rangle
\end{equation}
as $n \to \infty$ for $0 \leq j \leq q - 1$.

Note that if $q \geq 3$, Condition (iii) is equivalent to that $\text{ddc} \omega = \partial \omega \wedge \bar{\partial} \omega = 0$ on $V$. If $T_n$ are currents of integration along analytic sets, then the assumption of Proposition 2.7 on $T_n$ is automatically satisfied. This is the case in our application to the problem of estimating the number of isolated periodic points of meromorphic self-maps later.

**Proof.** By extracting a subsequence, we can assume that $\hat{T}_n \to \hat{T}'$. We have $1_{\hat{X} \setminus \hat{V}} \hat{T}' = \hat{T}$. By (iii), observe that $\text{ddc} \hat{\omega}^j = 0$ on $\hat{V}$ for $1 \leq j \leq q - 1$. This allows us to apply Lemma 2.6 to $Y := \hat{X}, Z := \hat{V}, R_n := \hat{T}_n$ and $\Omega := \hat{\omega}^j \wedge \sigma^* \omega^{q-j-1}$. The desired limit (2.16) follow immediately. The proof is finished. \hfill $\square$

Although we will not need the following remark later, we present it here because it might be useful elsewhere.

**Remark 2.8.** Consider the case where $\omega$ is Kähler. Then in the proof of Proposition 2.7 we can choose $\Omega$ to be a smooth closed form. As a result, we get
\begin{equation}
\lim_{n \to \infty} \{\sigma_*(\hat{T}_n \wedge \hat{V} \wedge \hat{\omega}^j)\} = \{\sigma_*(\hat{T} \wedge \hat{V} \wedge \hat{\omega}^j)\}
\end{equation}
as cohomology classes with compact support in $V$. In other words, under our assumption, with the notation as in [21] Th. 4.11], for every $r$ the continuity property $\kappa^V_r(T_n) \to \kappa^V_r(T)$ as $n \to \infty$ holds.

The following result generalizes [21] Th. 4.6].

**Theorem 2.9.** Let $X$ be a complex manifold. Let $T$ be a closed positive current of bidimension $(q, q)$ on $X$ for $q \geq 0$ and $V$ a smooth submanifold of $X$. Assume that
(i) $\text{supp} T \cap V$ is compact and $T$ has no mass on $V$,
(ii) there exists a Hermitian form $\omega$ on $X$ for which $\text{ddc} \omega^j = 0$ on $V$ for $1 \leq j \leq q - 1$.

Then Assumption (H) holds for $X, V, T$. Moreover, given any compact $K$ in $X$, such that $\text{supp} T \cap V \subset K$, there exists a constant $c$ independent of $T$ for which
\begin{equation}
\|\hat{T}\| \leq c\|T\|, \quad \|T_\infty\| \leq c\|T\|,
\end{equation}
for every tangent current $T_\infty$ of $T$ along $V$. 
Here for every current \( S \) of order 0 on \( X \), \( \|S\| \) denotes the mass of \( S \).

**Proof.** If \( q = 0 \), then \( T \) is a measure having no mass on \( V \) and \( T_\infty = 0 \). The desired assertions obviously hold. Consider now \( q \geq 1 \). We first show that the mass of \( \hat{T} \) is locally bounded near \( \hat{\nu} \). Let \( W, W_T \) be open neighborhoods of \( V, \text{supp} T \) respectively such that \( W_T \cap W \) is relatively compact in \( X \). Fix \( \omega \) as in Assumption (\( ii \)) and \( \hat{\sigma}, \hat{\omega} \) as above.

Put \( \hat{W} := \sigma^{-1}(W) \) and \( \hat{W}_T := \sigma^{-1}(W_T) \). Clearly \( \hat{W}_T \cap \hat{W} \in \hat{X} \). If \( V \) is a hypersurface, the first inequality of (2.17) is clear because \( \hat{T} = T \). Consider codim \( V \geq 2 \). Since \( \hat{T} \) has no mass on \( \hat{V} \), using the fact that \( \sigma_* \hat{\omega}_h \) is smooth outside \( V \) and \( \sigma_* \sigma^* \omega = \omega \), we have

\[
\int_{\hat{W}_T \cap \hat{W}} \hat{T} \wedge \hat{\omega}^q = \int_{(W_T \cap W) \setminus V} T \wedge \sigma_*(\hat{\omega}^q) = \int_{(W_T \cap W) \setminus V} T \wedge (\dd^c \varphi + \eta + \omega)^q \leq \int_{(W_T \cap W) \setminus V} T \wedge (\dd^c \varphi + c\omega)^q,
\]

for some positive constant \( c \) independent of \( T \) with \( \dd^c \varphi + c\omega \geq 0 \).

For a positive constant \( M \), put \( \varphi_M := \max\{\varphi, -M\} \). Note that \( \dd^c \varphi_M + c\omega \geq 0 \). Since the positive current \( T \wedge (\dd^c \varphi_M + c\omega)^q \) converges to \( T \wedge (\dd^c \varphi + c\omega)^q \) on \( X \setminus V \) as \( M \to \infty \), using (2.18), we get

\[
||\hat{T}||_{W_T \cap \hat{W}} \leq \liminf_{M \to \infty} \int_{(W_T \cap W) \setminus V} T \wedge (\dd^c \varphi_M + c\omega)^q \leq \liminf_{M \to \infty} A_M,
\]

where

\[
A_M := \int_X T \wedge (\dd^c \varphi_M + c\omega)^q.
\]

Using Assumption (\( ii \)), we observe that \( A_M \) can be written as a linear combination of

\[
A_{M,j} := \int_X T \wedge (\dd^c \varphi_M)^j \wedge \omega_j \quad (0 \leq j \leq q)
\]

with coefficients of absolute values bounded a constant independent of \( T \), where \( \omega_j \) is a smooth \((q - j, q - j)\)-form depending only on \( \omega \) such that \( \dd^c \omega_j = 0 \) on \( V \). So we only need to bound \( A_{M,j} \). We will prove that

\[
|A_{M,j}| \lesssim \|T\|
\]

for \( 0 \leq j \leq q \) by induction on \( j \). When \( j = 0 \), (2.20) is also clear. Assume that (2.20) holds for every \( 1, \ldots, j - 1 \). This in particular implies

\[
\|T \wedge (\dd^c \varphi_M)^{j - 1}\| \lesssim \|T\|.
\]

Since \( T \wedge (\dd^c \varphi_M)^{j - 1} \) can be written as a linear combination of \( T \wedge (\dd^c \varphi_M + c\omega)^s \) for \( 0 \leq s \leq l - 1 \), we deduce from (2.21) that

\[
\|T \wedge (\dd^c \varphi_M)^l\| \lesssim \|T\|.
\]
By (2.15), for $M$ big enough we have $\text{supp} T \cap \text{supp} \varphi_M \subset W \cap W$ which is compact in $X$. Thus using Stokes' theorem, one obtains

$$A_{M,j} = \int_X \varphi_M \dd^{*}(T \wedge \omega^j) \wedge (\dd^{c} \varphi_M)^{j-1}$$

$$= \int_X T \wedge (\dd^{c} \varphi_M)^{j-1} \wedge \varphi_M \dd^{c} \omega_j \leq \|\varphi_M \dd^{c} \omega_j\|_{\mathcal{C}^0} \|T \wedge (\dd^{c} \varphi_M)^{j-1}\|$$

$$\lesssim \|\varphi_M \dd^{c} \omega_j\|_{\mathcal{C}^0} \|T\|$$

because of (2.22). By (2.15) and the fact that $\dd^{c} \omega_j = 0$ on $V$, the $\mathcal{C}^0$-norm of the form $\varphi_M \dd^{c} \omega_j$ is bounded independently of $M$. Thus we get (2.20) for $j$. It follows that $A_M \lesssim \|T\|$ for $M$ big enough. This combined with (2.19) gives $\|\hat{T}\| \lesssim \|T\|$.

Now it remains to show that $(A_{\lambda})_{*}(\hat{\tau})_* \hat{T}$ is of uniformly mass on compact subsets of $\pi^{-1}_E(\hat{U}_j)$ for some suitable holomorphic admissible maps $\hat{\tau}_j : \hat{U}_j \to \hat{E}$ with $\hat{V} \subset \cup \hat{U}_j$. Let $(\hat{U}_j)_j$ be a family of local charts biholomorphic to $\mathbb{D}^k$ covering $\hat{V} \cap W_T$ and $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_k)$ local coordinates on $\hat{U}_j$ such that $\hat{V} \cap \hat{U}_j$ is given by $\hat{x}_1 = 0$. Identify $\hat{E}$ with $\hat{V} \times \mathbb{C}$. Let $\hat{\tau}_j : \hat{U}_j \to \hat{V} \times \mathbb{C}$ be the identity map.

Let $\rho := d\hat{x}_1$ or $d\hat{x}_1$. We need to prove that the mass of $\hat{T} \wedge \rho$ on $Z_\lambda := \{ |\hat{x}_1| \leq |\lambda| \}$ is $O(|\lambda|^{-1})$ and the mass of $\hat{T} \wedge \dd^{c}|\hat{x}_1|^2$ on $Z_\lambda$ is $O(|\lambda|^{-2})$. By the Cauchy-Schwarz inequality,

$$\|\hat{T} \wedge \rho\|_{Z_\lambda} \lesssim \|\hat{T}\|_{W_T \cap W} \|\hat{T} \wedge \dd^{c}|\hat{x}_1|^2\|_{Z_\lambda}^{1/2}.$$

Thus, it is enough to estimate $\|\hat{T} \wedge \dd^{c}|\hat{x}_1|^2\|_{Z_\lambda}$. This is already done in [21]. We reproduce arguments here for the readers’ convenience.

Put $\hat{W} := \sigma^{-1}(W)$. Let $u$ be a quasi-p.s.h. function on $\hat{X}$ such that $u$ vanishes outside $\hat{W}$ and $u$ is a potential of $[\hat{V}]$ on a small enough open neighborhood $\hat{W}_1 \Subset \hat{W}$ of $\hat{V}$ i.e, $[\hat{V}] = \dd^{c}u + \beta$, for some smooth form $\beta$ on $\hat{W}$. Note that $\hat{W} \cap \text{supp} T$ is relatively compact in $\hat{X}$. We have

$$|u(\hat{x}) - \log |\hat{x}_1|| \leq A$$

on $\hat{U}_j$ for every $j$ and some constant $A$ independent of $j$.

Put $s := \log |\lambda|$. We only need to consider $|\lambda|$ big, say, $|\lambda| \geq \epsilon^{3A}$. Thus $s \leq - \log |\lambda| + 2A$ on $Z_\lambda$. Let $\chi_\lambda$ be a convex increasing function bounded from below on $\mathbb{R}$ such that $\chi_\lambda(t) = t$ for $t \geq - \log |\lambda| + 3A$ and $0 \leq \chi' \leq 1$ and

$$\chi''_\lambda(t) = e^{2t + 2 \log |\lambda| - 5A}$$

for $t \leq - \log |\lambda| + 2A$. Put $\phi_\lambda := \chi_\lambda \circ u$ which is bounded and supported on $\hat{W}$. Hence we get

(2.23)

$$\text{supp} \phi_\lambda \cap \text{supp} \hat{T} \Subset \hat{X}.$$ We also have

(2.24)

$$|\phi_\lambda(\hat{x})| \lesssim |\log |\hat{x}_1||.$$

Direct computations (see [21], Le. 2.11) give

$$\dd^{c}||\hat{x}_1||^2 \leq c|\lambda|^{-2}(\dd^{c}\phi_\lambda + \omega)$$
on $Z_\lambda$ for some constant $c$ independent of $\lambda$. This implies that
\[
\|\hat{T} \wedge dd^c|\hat{x}|_1^2\|_{Z_\lambda} = \int_{Z_\lambda} T \wedge dd^c|\hat{x}|_1^2 \wedge \hat{\omega}^{q-1} \lesssim |\lambda|^{-2} \int_{\hat{W}_{r_1} \cap \hat{W}} T \wedge (dd^c\phi_\lambda + \hat{\omega}) \wedge \hat{\omega}^{q-1}.
\]
Observe that $dd^c(\hat{\omega}^{q-1}) = 0$ on $\hat{V}$. This together with (2.24) and (2.23) allows us to argue as before to obtain that
\[
\|\hat{T} \wedge dd^c|\hat{x}|_1^2\|_{Z_\lambda} \lesssim |\lambda|^{-2} \|\hat{T}\|_{\hat{W}_{r_1} \cap \hat{W}} \lesssim |\lambda|^{-2}\|T\|.
\]
The proof is finished. \qed

3. Intersection of analytic sets

In this section, we study the intersection of analytic sets by using the theory of tangent currents developed in the last section. Our first main result in this section is Theorem 3.1 below saying that the tangent current of an analytic subset along a smooth submanifold on every complex manifold always exists and is unique. Let $X$ be a complex manifold. We emphasize that there is neither compactness assumption nor Kähler condition on $X$. Let $V, E, \sigma, \hat{X}, \hat{V}, \pi_\infty$ be as in the last section.

**Theorem 3.1.** Let $V_1 \not\subset V$ be an analytic subset of $X$ and $\hat{V}_1$ the strict transform of $V_1$ in $\hat{X}$. Then, the tangent current of $[V_1]$ along $V$ is unique and is given by the pull-back of $[\hat{V}_1] \wedge [\hat{V}]$ by $\pi_\infty$. As a consequence, for analytic subsets $V_1, V_2$ of $X$, the density current associated to $[V_1], [V_2]$ is unique.

**Proof.** Clearly, $\hat{V}_1$ is not a subset of $\hat{V}$. Thus, $\hat{V}_1$ intersects $\hat{V}$ properly because $\hat{V}$ is a hypersurface. We deduce that the wedge product $[\hat{V}_1] \wedge [\hat{V}]$ is well-defined in the classical sense, see [28, 7]. The desired assertion then follows immediately from Proposition 2.5. The proof is finished. \qed

Put $l_1 := \dim V_1$. Denote by $W$ the set of irreducible components of $\hat{V}_1 \cap \hat{V}$. These components are of dimension $(l_1 - 1)$. Write
\[
[V_1] \wedge [\hat{V}] = \sum_{\hat{W} \in W} \alpha_{\hat{W}} \hat{W},
\]
for some nonnegative numbers $\alpha_{\hat{W}}$. Recall that given hypersurfaces $D_1, \ldots, D_m$ and an analytic subset $D$, if $D_1, \ldots, D_m, D$ intersects properly, then the intersection $D_1 \wedge \cdots \wedge D_m \wedge D$ in the sense of the pluripotential theory is the same as that defined in the classical sense of the theory of the intersection of analytic sets. The reason is that the both definitions enjoy the same continuity property, see [6, p. 212]. Thus $\alpha_{\hat{W}}$ is equal to the usual multiplicity along $\hat{W}$ of the proper intersection $\hat{V}_1 \cap \hat{V}$. In particular, $\alpha_{\hat{W}}$ is a strictly positive integer for every $\hat{W} \in W$.

**Definition 3.2.** For every isolated point $x \in V_1 \cap V$, its multiplicity $\nu_x$ is defined to be
\[
\nu_x = \sum_{\hat{W}: \sigma(\hat{W}) = \{x\}} \alpha_{\hat{W}}.
\]
Note that in the above definition there is no assumption on the dimensions of $V_1, V$. 
Lemma 3.3. Assume that $V_1, V$ are of complementary dimensions. The following two properties hold:

(i) for any isolated point $x$ in $V_1 \cap V$, the multiplicity $\nu_x$ defined above is equal to the usual multiplicity of $x$ in the intersection $V_1 \cap V$. Moreover, the only irreducible component $\hat{W}$ of $\hat{V}_1 \cap \hat{V}$ such that $x \in \sigma(\hat{W})$ is $\sigma^{-1}(x)$.

(ii) for every compact $K$ of $X$ and for every positive $(l_1 - 1, l_1 - 1)$-form $\Phi$ on $\hat{X}$ whose restriction to each fiber of $\hat{V} \approx \mathbb{P}(E)$ is of mass 1 on that fiber, we have

\begin{equation}
\sum_{x \in V_1 \cap V \cap K} \nu_x \delta_x \leq \sigma_*(\hat{[V_1]} \wedge \hat{[V]} \wedge \Phi).
\end{equation}

Proof. Let $T_\infty$ be the tangent current of $T := [V_1]$ along $V$. Let $x$ be an isolated point in the intersection $V_1 \cap V$ and $\nu_x'$ its multiplicity defined in the classical sense. It is already observed in [11, Le. 2.2] that in a small enough local chart around $x$ we have

$T_\infty = \nu_x'[\pi^{-1}(x)]$,

where $T_\infty$ is the tangent current of $T$ along $V$. This can be seen directly from the classical definition of the multiplicity of $x$. Since $T_\infty = \pi_\infty^*([\hat{V}_1] \wedge [\hat{V}])$, we deduce that $\nu_x' = \nu_x$. Assertion (i) follows immediately.

The inequality (3.1) is deduced directly from the fact that

$\sum_x \nu_x[\sigma^{-1}(x)] \leq \hat{V}_1 \wedge \hat{V}$,

where the sum is taken over isolated points in $V_1 \cap V$. The proof is finished. \qed

The following is a direct consequence of Theorem 2.9 and Lemma 3.3.

Proposition 3.4. Let $X$ be a complex manifold and $V$ a smooth submanifold of $X$ and $K$ a compact in $X$. Let $q$ be a positive integer. Assume that there exists a Hermitian form $\omega$ on $X$ for which $\dd^c \omega^j = 0$ on $V$ for $1 \leq j \leq q - 1$. Then there exists a constant $c$ such that for any $q$-dimensional analytic subset $V_1$ of $X$ with $V_1 \cap V \subset K$, we have

$|V_1 \cap V| \leq c \text{vol}(V_1 \cap K)$,

where $|V_1 \cap V|$ denotes the number of isolated points counted with multiplicity in the intersection $V_1 \cap V$.

We now study the contribution of other irreducible components of higher dimension of $V_1 \cap V$ in the tangent current of $V_1$ along $V$.

Lemma 3.5. Let $W_1$ be an irreducible component of $V_1 \cap V$. Assume that $W_1 \cap \text{Reg} V_1 \neq \emptyset$. Then there exists a unique $\hat{W}_1 \in \mathcal{W}$ such that $W_1 = \sigma(\hat{W}_1)$.

Proof. Let $l'_1 := \dim W_1$. Let $W'_1 := \text{Reg} W_1 \cap \text{Reg} V_1$ which is an analytic subset of $\text{Reg} V_1$ of dimension $l'_1$. We will show that

Claim: There is an open subset $W''_1$ of $W'_1$ in the usual topology such that

$A := \sigma^{-1}(W''_1) \cap \hat{V}_1 \cap \hat{V}$

is a connected fiber bundle over $W''_1$ whose fibers are immersed analytic subsets of $\hat{V}$ of dimension at most $(l_1 - 1 - l'_1)$. 

Suppose first that Claim is proved. We will show how to finish the proof. Let \( \widetilde{W} \in \mathcal{W} \) so that \( \sigma(\widetilde{W}) \) has a non-empty intersection with \( W''_1 \). Let \( x_0 \) be in \( \sigma(\widetilde{W}) \cap W''_1 \).

Clearly, \( \sigma(\widetilde{W}) \) is an irreducible analytic subset in \( V_1 \cap V \). Thus we have either \( W_1 = \sigma(\widetilde{W}) \) or \( \sigma(\widetilde{W}) \) is a proper analytic subset of \( W_1 \). If the latter case occurs, the dimension of the fibers of the projection \( \widetilde{W} \to \sigma(\widetilde{W}) \) is \( (l_1 - 1) - l'_1 \). This is a contradiction because according to Claim, the dimension of \( \sigma^{-1}(x_0) \cap \tilde{V}_1 \cap V \) is \( (l_1 - 1 - l'_1) \) which is strictly smaller than the dimension of the fiber of \( x_0 \) of the projection \( \widetilde{W} \to \sigma(\widetilde{W}) \). Thus \( W_1 = \sigma(\widetilde{W}) \) and \( A \) is an open subset of \( \widetilde{W} \). If we have another \( \widetilde{W}' \) for which \( \sigma(\widetilde{W}') \) has a non-empty intersection with \( W''_1 \), then \( A \) is also an open subset of \( \widetilde{W}' \). We deduce that \( \widetilde{W} = \widetilde{W} \) because of the irreducibility of \( \widetilde{W}' \). The desired assertion follows.

We now prove Claim. In order to do so, we need to understand the set of limit points of \( \sigma^{-1}(x^n) \) for any sequence \( (x^n) \subset V_1 \) converging to \( V \) because that set is exactly \( \tilde{V}_1 \cap \tilde{V} \).

To this end, we will construct below a family of holomorphic discs which is crucial to describe the desired limit sets.

Let \( x_0 \in W'_1 \). Consider a local chart \( \Psi_1 : U_1 \approx \mathbb{D}^{l_1} \times \mathbb{D}^{l_1 - l'_1} \) of \( \text{Reg} V_1 \) around \( x_0 \) with local coordinates \( y = (y_1, y_2) \) for which \( \Psi_1(W'_1 \cap U_1) = \{y_2 = 0\} \). We take \( U_1 \) small enough such that \( U_1 \cap V = U \cap W_1 \). For \( t \in \mathbb{D} \) and \( \tau = (\tau_1, \tau_2) \in \mathbb{D}^{l_1} \times (\mathbb{D}^{l_1 - l'_1} \setminus \{0\}) \), put

\[
\varphi_\tau(t) := \Psi_1^{-1}((\tau_1, 0) + t(0, \tau_2)).
\]

We obtain a family of holomorphic discs \( \varphi_\tau : \mathbb{D} \to \text{Reg} V_1 \) parameterized by \( \tau \) and each disc \( \varphi_\tau \) intersects \( W'_1 \) only at its center. Moreover, for \( \tau_1 \) fixed, \( x_0(\tau_1) := \varphi_{\tau_1, \tau_2}(0) \) is independent of \( \tau_2 \) and is diffeomorphic in \( \tau_1 \).

Now we will examine how \( \varphi_\tau \) is lifted to \( \hat{X} \). Observe that

\[
\varphi_\tau(\mathbb{D}) \cap V = \{\varphi_\tau(0)\}.
\]

Consider a local chart \( U = U' \times U'' \) around \( x_0 \) of \( X \) with coordinates \( x = (x', x'') \) for which \( V \cap U = \{x'' = 0\} \). We write \( \varphi_\tau = (\varphi'_\tau, \varphi''_\tau) \) in the local coordinates \( x = (x', x'') \).

One shouldn’t confuse the coordinates \( x \) with \( y \) above because \( x \) are coordinates on \( X \) whereas \( y \) are coordinates on \( V_1 \).

By (3.3), we get

\[
t = 0 \iff \varphi_\tau(t) \in V \iff \varphi''_\tau(t) = 0
\]

which implies

\[
\varphi'_{\tau_1, \tau_2}(t) = x_0(\tau_1) + O_\tau(t).
\]

Let \( k_0 \) be the smallest positive integer for which there exist \( \tau_1^0 \in \mathbb{D}^{l_1} \) and \( \tau_2^0 \in \mathbb{D}^{l_1 - l'_1} \setminus \{0\} \) satisfying \( \partial_t^{k_0} \varphi''_{\tau_1^0, \tau_2^0}(0) \neq 0 \). By the choice of \( k_0 \), we have \( \partial_t^j \varphi''_{\tau_1, \tau_2}(0) = 0 \) for every \( 0 \leq j \leq k_0 - 1 \) and every \( \tau_1, \tau_2 \). The continuity of \( \varphi \) in \( \tau \) implies that

\[
\partial_t^{k_0} \varphi''_{\tau}(0) \neq 0
\]

for \( \tau \) in a small enough neighborhood \( Z^0 := Z_{\tau_1^0} \times Z_{\tau_2^0} \) of \( (\tau_1^0, \tau_2^0) \). Using (3.4) and (3.6), one gets

\[
\varphi''_{\tau}(t) = t^{k_0} \left[ \partial_t^{k_0} \varphi''_{\tau}(0) + O_\tau(t) \right].
\]
Let $V_{1,\varphi}$ be the image of the union of the images of $\varphi_{\tau}$ over $\tau \in Z^0$. By the defining formula of $\varphi_{\tau}$, the set $V_{1,\varphi}$ is an open subset of $\text{Reg} V_1$ and

$$W''_1 := V_{1,\varphi} \cap W'_1 = \{x_0(\tau_1) : \tau_1 \in \mathbb{Z}_{x_0}^0\}$$

is an open subset of $W'_1$.

Let $L_1$ be the set of limit points of $\sigma^{-1}(x^n)$ for any sequence $(x^n) \subset V_1 \setminus V$ converging to $W''_1$. Observe that $L_1$ is a closed subset of $\sigma^{-1}(W''_1)$. Since $V_{1,\varphi}$ is open in $V_1$ and its intersection with $V$ is $W'_1$, the set $L_1$ is also the set of limit points of $\sigma^{-1}(x^n)$ for any sequence $(x^n) \subset V_{1,\varphi} \setminus V$ converging to $W''_1$. Denote by $L_{2, \tau}$ the set of limit points of $\sigma^{-1}(\varphi_{\tau}(t_n))$ for any sequence $(t_n) \subset \mathbb{D}^*$ converging to 0. Put

$$L_2 := \bigcup_{\tau \in Z^0} L_{2,\tau}.$$  

Obviously, we have $L_2 \subset L_1$. We will prove that $L_1 = L_2$. It remains to prove the converse inclusion $L_1 \subset L_2$.

Over $U$, the blowup $\hat{X} \cap \sigma^{-1}(U)$ is simply the set

$$U' \times \{(x'', [x'']) : x'' \in U'' \setminus \{0\}\},$$

where $[x'']$ denotes the point in $\mathbb{P}^{k-l-1}$ induced by $x''$. We infer that $L_{2, \tau}$ is the limit set of $\left(\varphi_{\tau}(t), [\varphi_{\tau}'(t)]\right)$ as $t \to 0$. The formula (3.7) tells us that

$$(3.8) \quad \lim_{t \to 0} (\varphi_{\tau}(t), [\varphi_{\tau}'(t)]) = (\varphi_{\tau}(0), [\partial_t \varphi_{\tau}(0)])$$

which is continuous with respect to $\tau \in Z^0$.

Let $(x^n) \in V_{1,\varphi} \setminus V$ converging to $x^\infty \in W''_1$. Thus there are $(t^n) \subset \mathbb{D}^*$ converging to 0 and $(\tau^n) \in Z^0$ converging to $\tau^\infty$ such that $x^n = \varphi_{\tau^n}(t_n)$ for every $n$. Using (3.8) and the $C^1$ continuity of $\varphi$ in $\tau$ yields that

$$x^\infty = \lim_{n \to \infty} \sigma^{-1}(\varphi_{\tau^n}(t_n)).$$

Hence $L_1 = L_2$.

Now we will determine the fibers of the projection $L_2 \to W''_1$. Let $F_x$ be the fiber of the last projection over $x \in W''_1$. By the construction of $W''_1$, there is $\tau_1$ for which $x = x_0(\tau_1)$. By the injectivity of $x_0(\tau_1)$ and the definition of $L_2$, we get

$$F_x = \{[\partial_t^{k_0} \varphi_{\tau_1, \tau_2}'(0)] : \tau_2 \in \mathbb{Z}_{x_0}^2\}.$$  

By (3.2), we see that

$$\varphi_{\tau_1, \tau_2}'(t) = \Phi(\theta_1^{k_0}(\tau_1, 0) + t(0, \tau_2))$$

for some smooth map $\Phi$. Taking derivatives in $t$ in the last equality shows that $\partial_t^{k_0} \varphi_{\tau_1, \tau_2}'(0)$ is a homogeneous polynomial in $\tau_2$ if $\tau_1$ is fixed. Consequently,

$$[\partial_t^{k_0} \varphi_{\tau_1, \lambda \tau_2}'(0)] = [\partial_t^{k_0} \varphi_{\tau_1, \tau_2}'(0)]$$

for every $\lambda \in \mathbb{C}$ closed to 1. It follows that $F_x$ is an immersed analytic subsets of dimension at most $(l_1 - 1 - l_1')$ in $\hat{X}$. Then Claim follows. The proof is finished.  

We don’t know whether the above lemma still holds if we don’t have the condition that $\text{Reg} V_1 \neq \emptyset$. 

\[\square\]
Proposition 3.6. Let \( W_1 \) be the set of irreducible components \( W_1 \) of \( V_1 \cap V \) such that \( W_1 \cap \operatorname{Reg} V_1 \neq \emptyset \). Let \( K \) be a compact subset of \( X \). Then there is a constant \( c \) independent of \( V_1 \) for which
\[
\sum_{W_1 \in W} \operatorname{vol}(W_1 \cap K) \leq c \| \hat{V}_1 \wedge \hat{V} \|_{\sigma^{-1}(K)}
\]

Proof. Let \( \omega \) be a Hermitian metric on \( X \) and \( \hat{\omega}, \hat{\omega}_h \) as in the last section. Let \( \hat{W}_1 \) be as in Lemma 3.5. Put \( l'_1 := \dim W_1 \). For \( x \in W_1 \), let \( F_x \) be the fiber over \( x \) of \( \hat{W}_1 \to W_1 \). We have \( \dim F_x \geq (l_1 - 1 - l'_1) \) and the equality occurs for \( x \) in some open Zariski subset \( W_1' \) of \( W_1 \).

Since \( \hat{\omega}_h^{l_1-1} \succeq \omega^{l'_1} \wedge \hat{\omega}_h^{l_1-1-l'_1} \), we have
\[
\operatorname{vol}(\hat{W}_1 \cap \sigma^{-1}(K)) \geq \int_{\hat{W}_1 \cap \sigma^{-1}(K)} \omega^{l'_1} \wedge \hat{\omega}_h^{l_1-1-l'_1} = \int_{x \in W_1 \cap K} \omega^{l'_1} \int_{F_x} \hat{\omega}_h^{l_1-1-l'_1}
\]
The second integral in the right-hand side of the last equality is equal to the cup product of the cohomology classes of \( F_x \) and \( (\hat{\omega}_h|_{\sigma^{-1}(x)})^{l_1-1-l'_1} \) in \( \sigma^{-1}(x) \approx \mathbb{P}^{l_1-1} \) which is thus \( \geq c_0 \) for some strictly positive constant \( c_0 \) independent of \( V_1 \). It follows that
\[
\operatorname{vol}(\hat{W}_1 \cap \sigma^{-1}(K)) \geq \int_{W_1 \cap K} \omega^{l'_1} = \operatorname{vol}(W_1 \cap K).
\]

Consequently,
\[
\sum_{W_1 \in W} \operatorname{vol}(W_1 \cap K) \leq \sum_{W \in W} \operatorname{vol}(\hat{W} \cap \sigma^{-1}(K)) \leq \| \hat{V}_1 \wedge V \|_{\sigma^{-1}(K)}.
\]

This finishes the proof. \( \square \)

Proposition 3.7. Let \( X \) be a compact complex surface, \( Y \) and \( Z \) two compact complex manifolds. Assume that \( Y, Z \) admit Hermitian pluriclosed metrics. Let \( \Delta \) be the diagonal of \( X^2 \) and \( \Delta_2 := Y \times \Delta \times Z \). Let \( V_1, V_2 \) be complex analytic subsets of dimension 2 of \( Y \times X \), \( X \times Z \) respectively. If \( W_1, \ldots, W_m \) are irreducible components of \( (V_1 \times V_2) \cap \Delta_2 \) such that \( W_1 \cap (\operatorname{Reg} V_1 \times \operatorname{Reg} V_2) \cap \Delta_2 \) is nonempty, then we have
\[
\sum_{j=1}^m \operatorname{vol}(W_j) \leq c_X \operatorname{vol}(V_1) \operatorname{vol}(V_2),
\]
for some constant \( c_X \) depending only on \( X \).

Recall that a Hermitian metric \( \omega \) is pluriclosed if \( \ddc \omega = 0 \).

Proof. Since \( X, Y, Z \) admits Hermitian pluriclosed metrics, there exist positive definite Hermitian forms \( \omega, \omega_Y, \omega_Z \) on \( X, Y, Z \) respectively such that \( \ddc \omega = \ddc \omega_Y = \ddc \omega_Z = 0 \). Put \( X_2 := Y \times X \times X \times Z \). Let \( \sigma_2 : \tilde{X}_2 \to X_2 \) be the blowup of \( X_2 \) along \( \Delta_2 \) and \( \sigma : \tilde{X} \times X \to X \times X \) the blowup of \( X \times X \) along \( \Delta \). We see that \( \tilde{X}_2 = Y \times \tilde{X} \times \tilde{X} \times Z \) and \( \sigma_2 = (\operatorname{id}_Y, \sigma, \operatorname{id}_Z) \) because of the choice of \( \Delta_2 \).

Let \( \Delta \) be the exceptional hypersurface of \( \sigma \) and \( \hat{\Delta} \) the exceptional hypersurface of \( \sigma_2 \). By the above observation, \( \hat{\Delta}_2 = Y \times \hat{\Delta} \times Z \). Let \( \hat{\omega}_h \) be the Chern form of a Hermitian metric on \( O(-\hat{\Delta}) \) whose restriction to \( \hat{\Delta} \) is Fubiny-Study form on \( \hat{\Delta} \approx \mathbb{P}(N\Delta) \). Denote by \( p_j \) the projection from \( X_2 \) to the \( j \)th component for \( 1 \leq j \leq 4 \). Put
\[
\omega_2 := p_1^* \omega_Y + p_2^* \omega + p_3^* \omega + p_4^* \omega_Z.
\]
By rescaling $\omega$, we can assume that \( \hat{\omega} := \hat{\omega}_h + \rho^* \omega + \rho^* \omega > 0 \). Hence \( \hat{\omega}_2 := \hat{\rho}^* \hat{\omega}_h + \sigma^* \omega_2 > 0 \) as well, where \( \hat{\rho} \) is the natural projection from \( \hat{X}_2 \) to \( \hat{X} \times \hat{X} \).

Theorem 3.1 tells us that the tangent current to \( T := [V_1] \otimes [V_2] \) along \( \Delta_2 \) is unique and given by \( \pi^*_\infty([\hat{T}] \wedge [\hat{\Delta}_2]) \), where \( \hat{T} \) is the strict transform of \( T \) in \( \hat{X}_2 \), and \( \pi^*_\infty \) is the projection from \( \mathbb{P}(N\Delta_2 \oplus \mathbb{C}) \) to \( \hat{\Delta}_2 \approx \mathbb{P}(N\Delta_2) \). On the other hand, Proposition 3.6 implies that

\[
\sum_{j=1}^m \vol(W_j) \lesssim \int_{\hat{X}_2} \hat{T} \wedge [\hat{\Delta}_2] \wedge \hat{\omega}_2^3 =: A.
\]

Thus in order to get (3.9), we only need to bound the last integral. Denote by \((x_1, \ldots, x_4)\) a general point in \( X^4 \). Write \([\hat{\Delta}] = \dd^c \varphi + \eta \)' for some smooth form \( \varphi \)' and some quasi-s.h. function \( u \) on \( \hat{X} \times \hat{X} \). It follows that \([\hat{\Delta}_2] = \dd^c \hat{\rho}^* u + \hat{\rho}^* \eta \).

Let \( \varphi, \eta \) be as in the proof of Theorem 2.9, i.e., \( \sigma_i \hat{\omega}_h = \dd^c \varphi + \eta \). Note that

\[
\dd^c(\varphi \circ \sigma) = \hat{\omega}_h - \sigma^* \eta + c[\hat{\Delta}]
\]

for some strictly positive constant \( c \). Multiplying \( \hat{\omega}_h \) by \( c^{-1} \) allows us to assume that \( c = 1 \). Hence, \( \varphi \circ \sigma - u \) is a smooth function on \( \hat{X} \times \hat{X} \). This together with the Chern-Levine-Nirenberg gives

\[
\|\hat{T} \wedge \dd^c(u \circ \hat{\rho} - \hat{\varphi})\| \lesssim \|\hat{T}\|,
\]

where \( \hat{\varphi} := \varphi \circ \sigma \circ \hat{\rho} \) and we recall the wedge product in the last inequality is defined classically, i.e., \( u \) (hence \( \varphi \circ \sigma \)) is integral with respect to \( \hat{T} \). Let \( c_1 \) be a positive constant such that

\[
\eta' \leq c_1 \hat{\omega}, \quad \dd^c(\varphi \circ \sigma) + c_1 \hat{\omega} \geq 0.
\]

Using this, (3.11) and the fact that \( \hat{\varphi} \) is integrable with respect to \( \hat{T} \), we see that

\[
A \lesssim \|\hat{T}\| + \int_{\hat{X}_2} \hat{T} \wedge (\dd^c \hat{\varphi} + c_1 \hat{\omega}_2) \wedge \hat{\omega}_2^3 = \|\hat{T}\| + \lim \inf_{M \to \infty} \int_{\hat{X}_2} \hat{T} \wedge (\dd^c \hat{\varphi}_M + c_1 \hat{\omega}_2) \wedge \hat{\omega}_2^3,
\]

where \( \hat{\varphi}_M := \max\{\hat{\varphi}, -M\} \). Denote by \( A_M \) the last integral. Since \( \hat{T} \wedge \dd^c \hat{\varphi}_M \) has no mass near \( \hat{\Delta}_2 \), we get

\[
A_M = \int_{X_2 \setminus \Delta_2} T \wedge (\dd^c \varphi_M + c_1(\sigma_2) \hat{\omega}_2) \wedge ((\sigma_2)_* \hat{\omega}_2)^3.
\]

Notice that \( \varphi, \varphi_M \) are functions of \((x_2, x_3)\) and \((\sigma_2)_* \hat{\omega}_2 = \dd^c \varphi + \eta + \sum_{j=1}^4 p_j^* \omega \) and \( \eta \) is a closed smooth form. Let \( c_2 \) be a positive constant such that \( (c_2 - c_1) \omega_2 \geq c_1 \eta \). Thus for \( \varphi_{M_1} := \max\{\varphi_M + c_1 \varphi, -M_1\} \), we have \( \dd^c \varphi_{M_1} + c_2 \omega_2 \geq 0 \). Using this and (3.13) gives

\[
A_M \leq \lim \inf_{M_1 \to \infty} \int_{X_2} T \wedge \Phi_{M_1},
\]

where

\[
\Phi_{M_1} := (\dd^c \varphi_{M_1} + c_2 \omega_2) \wedge (\dd^c \varphi_{M_1} + c_2 \omega_2)^3.
\]
Put \( \omega_{21} := p_1^* \omega_Y, \omega_{22} := p_2^* \omega, \omega_{23} := p_3^* \omega \) and \( \omega_{24} := p_4^* \omega_Z \). Since \( \omega_2 = \sum_{j=1}^4 \omega_{2j} \), we can write \( \Phi_{M_1} \) as a linear combinations of forms

\[
\Phi_{M_1; s', s, 1} := (dd^c \varphi_{M_1})^{s'} \wedge (dd^c \varphi_{M_1})^s \wedge \omega_{2j} \wedge \omega_{2j}^l
\]

with \( l = (l_1, \ldots, l_4) \) and

\[
s' + s + l_1 + l_2 + l_3 + l_4 = 4, \quad 0 \leq s' \leq 1.
\]

So to bound \( A_M \), we only need to bound \( \langle T, \Phi_{M_1; s', s, 1} \rangle \). If \( s' + s = 4 \) or \( 3 \), then

\[
\langle T, \Phi_{M_1; s', s, 1} \rangle = 0
\]

because of Stokes’ theorem and \( dd^c \omega = 0 \). Recall that \( T = [V_1] \otimes [V_2] \). If \( s' + s = 0 \), then \( \langle T, \Phi_{M_1; s', s, 1} \rangle \) is bounded by the mass of \( T \). On the other hand, if \( l_1 + l_2 = 1 \), we can apply Stokes’ theorem to \( \langle [V_1], \Phi_{M_1; s', s, 1} \rangle \) to show that this product is equal to 0 because \( dd^c (\omega_{21}^l \wedge \omega_{22}^l) = 0 \). Hence \( \langle T, \Phi_{M_1; s', s, 1} \rangle = 0 \). If \( l_3 + l_4 = 1 \), we obtain the same conclusion. So it remains to treat the case where \( (l_1 + l_2 - 1)(l_3 + l_4 - 1) \neq 0 \) and \( s' + s = 1 \) or 2.

We first consider \( s' + s = 1 \). We have \( \sum_j l_j = 3 \). Let \( l_1', \ldots, l_4' \) be the numbers \( l_1, \ldots, l_4 \) written in an order such that \( l_1' \geq \cdots \geq l_4' \). Hence \( l_4' = 0 \) because otherwise \( \sum_j l_j' \geq 4 \), a contradiction. We then see easily that either \( l_1' = l_2' = l_3' = 1 \) or \( l_2' = 1, l_3' = 2 \). The first case can’t happen because otherwise we will get \( (l_1 + l_2 - 1)(l_3 + l_4 - 1) = 0 \). Hence, we obtain \( l_3' = 0, l_2' = 1, l_4' = 2 \) and \( l_1' = 0 \). It follows that \( (l_2, l_3) = (0, 1) \) or \( (1, 0) \) and \( (l_1, l_4) = (0, 2) \) or \( (2, 0) \) because \( \varphi_{M_1}, \varphi_{M_1}' \) depends only on \( x_2, x_3 \) and \( \dim X = 2 \). Without loss of generality, we can suppose \( l_4 = 2 \). This combined with the fact that \( \dim V_1 = \dim V_2 = 2 \) gives

\[
\langle T, \Phi_{M_1; s', s, 1} \rangle = \int_{V_2} \omega_1^2 (x_4) \int_{V_1} (dd^c x_2 \varphi_{M_1})^{s'} \wedge (dd^c x_2 \varphi_{M_1})^s \wedge \omega_1 (x_1)^{l_2} \wedge \omega_2^{l_3} (x_2) = 0
\]

by Stokes’ theorem and \( l_2 + l_3 = 1 \).

We now consider \( s' + s = 2 \). We have \( \sum_j l_j = 2 \). Let \( l_1', \ldots, l_4' \) be as above. Arguing as above gives \( l_3' = l_4' = 0 \) and \( (l_1', l_2') = (2, 0) \) or \( (l_1', l_2') = (1, 1) \). If the latter case happens, we get either \( l_1 = l_2 = 0, l_3 = l_4 = 1 \) or \( l_1 = l_2 = 1, l_3 = l_4 = 0 \). For these both cases, the stokes’ theorem gives the \( \langle T, \Phi_{M_1; s', s, 1} \rangle = 0 \). The case where \( (l_1', l_2') = (2, 0) \) is treated similarly.

Hence, we have proved that \( \langle T \wedge \Phi_{M_1} \rangle \leq \|T\| \) independent of \( M_1 \). Combining this with \((3.14), (3.12), \) and \((3.10)\) gives the desired inequality. The constant \( c_X \) depends only on \( X \) because all of constants in the estimates we used above do so. The proof is finished.

**Proof of Theorem 1.2** By the hypothesis, there is a Hermitian metric \( \omega \) on \( X \) with \( dd^c \omega_j = 0 \) for \( 1 \leq j \leq k - 1 \). Let \( \omega_2 := p_1^* \omega + p_2^* \omega \) where \( p_1, p_2 \) are the projections from \( X^2 \) to the first and second components respectively. Let \( \Delta \) be the diagonal of \( X \). Observe that \( dd^c \omega_2 = 0 \) on \( \Delta \) for \( 1 \leq j \leq k \). Let \( \Gamma_n \) be the graph of \( f^n \) on \( X^2 \). Observe that \( [\Gamma_n] \) is a closed positive current of bidimension \( (k, k) \) on \( X^2 \). Applying Proposition 3.4 to \( q = k \); \( X^2 \) in place of \( X, V_1 := \Gamma_n, V := \Delta \) and \( K := X^2 \), we obtain that

\[
F_n = |V_1 \cap V| \leq \text{vol}(\Gamma_n).
\]

This combined with the fact that \( \lim_{n \to \infty} [\text{vol}(\Gamma_n)]^{1/n} e^{-h_0(f)} = 1 \) gives \((1.1)\).

Now assume that \( X \) is a compact complex surface. We will prove that \( h_0(f) \) is finite. To this end, we need to estimate \( \text{vol}(\Gamma_n) \). Let \( n_1, n_2 \) be positive integers. Put \( V_j := \Gamma_{n_j} \)
for \( j = 1, 2 \). Consider the intersection \((V_1 \times V_2) \cap (X \times \Delta \times X)\) in \(X^4\). Let \(p_{1,4}\) be the projection from \(X^4\) to \(X^2\) by sending \((x_1, \ldots, x_4)\) to \((x_1, x_4)\). By the definition of \(\Gamma_{n_1+n_2}\), there exists a \(k\)-dimensional irreducible component \(W\) of \((V_1 \times V_2) \cap (X \times \Delta \times X)\) such that \(\Gamma_{n_1+n_2} = p_{1,4}(W)\). Because \(\dim \Gamma_{n_1+n_2} = \dim W\), we have \(\text{vol}(\Gamma_{n_1+n_2}) \leq \text{vol}(W)\) which is

\[
\leq C_X \text{vol}(\Gamma_{n_1}) \text{vol}(\Gamma_{n_2})
\]

by Proposition 3.7. It follows that \(\limsup_{n \to \infty} [\text{vol}(\Gamma_n)]^{1/n}\) exists and is a finite number.

On the other hand, we can check directly that

\[
\max \left\{ \int_X (f^n)^* \omega^q \wedge \omega^{k-q} : 0 \leq q \leq k \right\} \leq \text{vol}(\Gamma_n) \leq \max \left\{ \int_X (f^n)^* \omega^q \wedge \omega^{k-q} : 0 \leq q \leq k \right\}.
\]

Thus, \(h_a(f) = \limsup_{n \to \infty} [\text{vol}(\Gamma_n)]^{1/n} < \infty\).

Now consider a bi-meromorphic map \(g : X \to X'\) and \(f' := g \circ f \circ g^{-1} : X' \to X'\). We need to show that \(h_a(f') = h_a(f)\). Observe that \(f^n = g \circ f^n \circ g^{-1}\). Applying similar arguments as above gives

\[
\text{vol}(\Gamma_n) \lesssim \text{vol}(\Gamma'_n) \lesssim \text{vol}(\Gamma_n),
\]

where \(\Gamma'_n\) is the graph of \(f^n\). Consequently, \(h_a(f') = h_a(f)\). In other words, \(h_a(f)\) is a bi-meromorphic invariant of \(f\).

It remains to prove (1.2). Let \(\Gamma_{[n]}\) be the graph of \((f, f^2, \ldots, f^n)\) in \(X^{n+1}\). For \(1 \leq s \leq k\) and \(M = (n_1, \ldots, n_s)\) in \(\mathbb{N}^s\) with \(1 \leq n_1 < \cdots < n_s \leq n\), denote by \(\Gamma_M\) the image of the map \((f^{n_1}, \ldots, f^{n_s})\) in \(X^s\). It was proved in [30, 13] that

\[
h_s(f) \leq \text{lov}(f) = \limsup_{n \to \infty} [\text{vol}(\Gamma_{[n]})]^{1/n}.
\]

Using an appropriate metric on \(X^n\) induced from that on \(X\), we can see that

\[
\text{vol}(\Gamma_n) \lesssim \sum_M \text{vol}(\Gamma_M),
\]

where the sum is taken over \(M = (n_1, \ldots, n_k)\) with \(0 \leq n_1 < \cdots < n_k \leq n\). Since the number of such \(M\) is \(\leq n^k\), in order to get the desired bound for \(\text{lov}(f)\), we only need to bound \(\text{vol}(\Gamma_M)\). Fix a such \(M = (n_1, \ldots, n_k)\). Recall \(k = \dim X = 2\). Thus,

\[
\text{vol}(\Gamma_M) \lesssim \sum_{0 \leq q \leq 2} \int_X (f^{n_1})^* \omega^q \wedge (f^{n_2})^* \omega^{k-q} \lesssim \sum_{0 \leq q \leq 2} \int_X (f^{n_1})^* (\omega^q \wedge (f^{n_2-(n_1)})^* \omega^{k-q}).
\]

The last term in the above inequality is equal to

\[
d_2(f)^{n_1} \int_X \omega^q \wedge (f^{n_2-n_1})^* \omega^{k-q} \leq [h_a(f) + \epsilon]^{n_1} [h_a(f) + \epsilon]^{n_2-n_1} \leq [h_a(f) + \epsilon]^n
\]

for any constant \(\epsilon > 0\) and \(n \geq n_\epsilon\). Therefore, we get

\[
\text{lov}(f) \leq h_a(f).
\]

A direct computation shows that \(\text{lov}(f) \geq h_a(f)\). It follows that \(\text{lov}(f) = h_a(f)\). This finishes the proof.
4. Maps with Dominant Topological Degrees

In this section, we study meromorphic self-maps of a $k$-dimensional compact complex manifold having a dominant topological degree, i.e., $d_k > d_j$ for $0 \leq j \leq k - 1$. For the proof of Theorem 1.4, we will need the following lemma generalizing a similar inequality due to Dinh in [9] in the Kähler case.

**Lemma 4.1.** Let $X$ be a compact complex manifold. Let $d_0, \ldots, d_k$ be the dynamical degrees of a meromorphic self-correspondence $f$ of $X$. Then given every smooth $(p, q)$-form $\Phi$, we have

\[
\limsup_{n \to \infty} \| (f^n)^* \Phi \|^{1/n} \leq \sqrt{d_p d_q}, \quad \limsup_{n \to \infty} \| (f^n)_* \Phi \|^{1/n} \leq \sqrt{d_{k-p} d_{k-q}}.
\]

**Proof.** Let $\omega$ be a Hermitian metric on $X$. The second inequality of (4.1) is a direct consequence of the first one by using $f^{-1}$ instead of $f$. Let $\Phi$ be a smooth $(p, q)$-form. Without loss of generality, we can suppose that $q \geq p$.

By using a partition of unity, we can write $\Phi$ as a sum of forms of type $\Phi' := \Phi_{(p, p)} \wedge \Phi_{(0, q-p)}$ for some positive smooth form $\Phi_{(p, p)}$ of bidegree $(p, p)$ and some $(0, q-p)$-form $\Phi_{(0, q-p)}$. Let $\Psi$ be a smooth $(k-p, k-q)$-form. Similarly, we can write $\Psi$ as a sum of forms of type $\Psi' := \Psi_{(k-q, k-q)} \wedge \Psi_{(q-p, 0)}$ for some positive form $\Psi_{(k-q, k-q)}$. It follows that in order to estimate $\langle (f^n)^* \Phi, \Psi \rangle$, it is sufficient to estimate $\langle (f^n)^* \Phi', \Psi' \rangle$.

Let $\pi_1, \pi_2$ be the natural projections from $X^2$ to the first and second components respectively. Recall $(f^n)^* \Phi = (\pi_1)_* ([\Gamma_n] \wedge \pi_2^* \Phi)$, where $\Gamma_n$ is the graph of $f^n$. Thus,

\[
\langle (f^n)^* \Phi', \Psi' \rangle = \int_{\Gamma_n} \pi_2^* \Phi \wedge \pi_1^* \Psi'.
\]

By the Cauchy-Schwarz inequality, we have

\[
\left| \langle (f^n)^* \Phi', \Psi' \rangle \right| \leq \left( \int_{\Gamma_n} \pi_2^* (\Phi_{(0, q-p)} \wedge \Phi_{(0, q-p)}) \wedge \pi_2^* (\Phi_{(p, p)} \wedge \pi_1^* \Psi_{(k-q, k-q)}) \right)^{1/2}
\]

\[
\left( \int_{\Gamma_n} \pi_1^* (\Psi_{(q-p, 0)} \wedge \Psi_{(q-p, 0)}) \wedge \pi_2^* (\Phi_{(p, p)} \wedge \pi_1^* \Psi_{(k-q, k-q)}) \right)^{1/2}
\]

which is \lesssim

\[
\left( \int_{\Gamma_n} \pi_2^* \omega^{q-p} \wedge \pi_2^* \omega^p \wedge \pi_1^* \omega^{k-q} \right)^{1/2} \left( \int_{\Gamma_n} \pi_1^* \omega^{q-p} \wedge \pi_2^* (\omega^p) \wedge \pi_1^* \omega^{k-q} \right)^{1/2}.
\]

Hence, the desired inequality follows. The proof is finished. \hfill \Box

**Theorem 4.2.** Let $X$ be a compact complex manifold of dimension $k$ and $f$ a meromorphic self-map of $X$. Let $\nu$ be a complex measure with $L^{k+1}$ density on $X$ and $\nu(X) = 1$. Assume that $d_k > d_{k-1}$. Then the sequence $d_k^{-n} (f^n)^* \nu$ converges to an invariant $PC$ probability measure $\mu_f$ of entropy $\geq \log d_k$ independent of $\nu$ such that $d_k^{-1} f_* \mu_f = \mu_f$ and

\[
\lim_{n \to \infty} \langle d_k^{-n} (f^n)^* \nu - \mu_f, \varphi \rangle = 0
\]

for every quasi-p.s.h. function $\varphi$ on $X$. 

Proof of Theorem 1.4. Let \( \pi_1, \pi_2 \) be the natural projections from \( X^2 \) to the first and second components respectively. Recall that smooth forms on \( X^2 \) can be approximated by forms \( \pi_1^* \Phi_1 \land \pi_2^* \Phi_2 \) in \( C^\infty \)-topology, where \( \Phi_1, \Phi_2 \) are smooth forms on \( X \). By Lemma 2.4, we see that for smooth \( (k-p, k-q) \)-form \( \Phi_1 \) and \( (p, q) \)-form \( \Phi_2 \) on \( X \),

\[
(\text{4.3}) \quad \langle d^{-n}_k[\Gamma_n], \pi_1^* \Phi_1 \land \pi_2^* \Phi_2 \rangle = d^{-n}_k \int_X (f^n)^* \Phi_2 \land \Phi_1 = O((d_p d_q)^{n/2} d^{-n}_k) \to 0
\]

if \((p, q) \neq (k, k)\) because \( d_k > d_p \) or \( d_q \) in this case. Let \( \mu_f \) be the equilibrium measure of \( f \). On the other hand, if \( (p, q) = (k, k) \), we have

\[
\langle d^{-n}_k[\Gamma_n], \pi_1^* \Phi_1 \land \pi_2^* \Phi_2 \rangle = \langle d^{-n}_k(f^n)^* \Phi_2, \Phi_1 \rangle \to \langle \mu_f, \Phi_1 \rangle \int_X \Phi_2 = \langle \pi_1^* \mu_f, \pi_1^* \Phi_1 \land \pi_2^* \Phi_2 \rangle
\]

by Theorem 4.2. Using this and (4.3) gives

\[
d^{-n}_k[\Gamma_n] \to \pi_1^* \mu_f.
\]

Let \( \omega' \) be a Hermitian metric on \( X \) for which \( dd^c \omega'^j = 0 \) for \( 1 \leq j \leq k-1 \). This metric induces naturally a metric \( \omega := \pi_1^* \omega' + \pi_2^* \omega' \) on \( X \times X \) with \( dd^c \omega^j = 0 \) on \( \Delta \) for \( 1 \leq j \leq k-1 \). Let \( \sigma : \hat{X} \times \hat{X} \to X \times X \) be the blowup of \( X^2 \) along the diagonal \( \Delta \). Let \( \hat{\omega}_h \) be a Chern form of a Hermitian metric of \( O(-\hat{\Delta}) \) whose restriction to each fiber of the projection \( \hat{\Delta} \to \Delta \) is strictly positive and belongs to the cohomology class of a hyperplane of that fiber. By rescaling \( \omega' \), we can assume that \( \hat{\omega} := \sigma^* \omega + \hat{\omega}_h \) > 0. By our choice of \( \hat{\omega}_h \), the restriction of \( \hat{\omega}^{-1} \) to each fiber of the projection \( \hat{\Delta} \to \Delta \) is a volume form of mass 1.

Applying Proposition 2.7 to \( T_n := d^{-n}_k[\Gamma_n], T_\infty := \pi_1^* \mu_f, V := \Delta \) and \( X^2 \), we obtain

\[
\lim_{n \to \infty} \left\| \sigma_*(\hat{T}_n \land [\hat{V}] \land \hat{\omega}^{-k-1}) \right\| = \left\| \sigma_*(\hat{T}_\infty \land [\hat{V}] \land \hat{\omega}^{-k-1}) \right\|
\]

for \( 0 \leq j \leq q-1 \), where the notation is as in Proposition 2.7. This combined with (3.1) of Lemma 3.3 implies

\[
\limsup_{n \to \infty} d^{-n}_k P_n \leq \left\| \sigma_*(\hat{T}_\infty \land [\hat{V}] \land \hat{\omega}^{-k-1}) \right\| = (\hat{T}_\infty \land [\hat{V}], \hat{\omega}^{-k-1}).
\]

Denote by \( \Pi_1 : \hat{X} \times \hat{X} \to X \) the composition of \( \sigma \) and \( \pi_1 \). Observe that \( \Pi_1 \) is a submersion. Consider local coordinates \( x = (x_1, \ldots, x_k) \) on \( X \). These coordinates induce a natural coordinate system \( (x, y) \) on \( X^2 \). The diagonal \( \Delta \) is given by \( x = -y = 0 \). Put \( y' := x - y \). We obtain new local coordinates \((x, y')\). A typical local chart on \( X \times X \) can be described as \((x, y'_1, y'_2, \ldots, y'_k)\) and

\[
\sigma(x, y'_1, y'_2, \ldots, y'_k) = (x, y'_1 y'_2, \ldots, y'_1 y'_k), \quad \hat{V} = \{ y'_1 = 0 \}
\]

We deduce that the fiber of \( \Pi_1 \) in the considered local chart is parameterized by \( y'_1, y'_2, \ldots, y'_k \). Since \( \hat{T}_\infty = \sigma^* \pi_1^* \mu_f = \Pi_1^* \mu_f \), we get

\[
\langle \hat{T}_\infty \land [\hat{V}], \hat{\omega}^{-k-1} \rangle = \int_{x \in X} d\mu_f \int_{(y'_1, y'_2, \ldots, y'_k) \in \Pi_1^{-1}(x)} [\hat{V}] \land \hat{\omega}^{-k-1} = \int_{x \in X} d\mu_f \int_{\sigma^{-1}(\{x, x\})} \hat{\omega}^{-k-1} = \int_{x \in X} d\mu_f = 1.
\]
Combining this with \((4.5)\) gives \(\limsup_{n \to \infty} d_k^{-n} P_n \leq 1\). The desired inequality follows. This finishes the proof.

\[\square\]

**Remark 4.3.** Consider now \(X\) is of dimension 2 and \(f\) a meromorphic self-map of \(X\) with dominant topological degree. By Theorems 4.2 and 1.2, \(\mu_f\) is an invariant measure of maximal entropy. We can show that \(\mu_f\) is a unique measure of maximal entropy by using Gauduchon’s metric instead of a Kähler form, arguments in the proof of Theorem 1.2 and repeating arguments in Kähler case [19, Th. 1.118] or [5, Th. 2].

Now we give some more dynamical properties of \(f\).

**Corollary 4.4.** Let \(X \in \mathcal{G}\) and \(f\) a surjective holomorphic self-map of \(X\) with dominant topological degree. Then the following properties hold:

(i) every Lyapunov exponent of the equilibrium measure \(\mu_f\) of \(f\) is at least \(\frac{1}{2} \log(d_k/d_{k-1})\).

(ii) the isolated periodic points of \(f\) is equidistributed with respect to \(\mu_f\):

\[(4.6) \quad \mu_n := \frac{1}{P_n} \sum_x \nu_x \delta_x \to \mu_f,
\]

where the sum is taken over all isolated periodic points of \(f\) of period \(n\), \(\nu_x\) is the multiplicity of \(x\) and \(\delta_x\) is the Dirac mass at \(x\),

(iii) there exists a totally invariant (possibly empty) proper analytic subset \(E\) of \(f\), i.e, \(f^{-1}(E) = E\) such that \(d_k^{-n}(f^n)^* \delta_a \to \mu_f\) if and only if \(a \notin E\).

In the Kähler case, Corollary 4.4 is already known, see [19, 12, 15, 31, 4, 5]. In order to prove Corollary 4.4 we will need the following result which is more or less a trivial extension of [14, Le. 4.7].

**Lemma 4.5.** Let \(X \in \mathcal{G}\) and \(f\) a surjective holomorphic self-map of \(X\). Then the fibers of \(f\) are finite, in other words, \(f\) is a ramified covering. Moreover for every constant \(\epsilon > 0\) and every irreducible analytic subset \(Y\) of dimension \(q\) of \(X\) with \(f(Y) \subset Y\), then \(f(Y) = Y\) and the topological degree of \(f^n|_Y\) is \(\leq \text{vol}(Y)^{-1}(d_q(f) + \epsilon)^n\) for \(n \geq n_\epsilon\) big enough.

**Proof.** We will argue exactly as in the proof of [14, Le. 4.7] with the Bott-Chern cohomology in place of the de Rham cohomology. Let \(\omega\) be a Hermitian metric on \(X\) with \(dd^c_0 \omega^j = 0\) for \(1 \leq j \leq k - 1\), where \(k = \dim X\). The form \(\omega^j\) for \(1 \leq j \leq k - 1\) induces a class in the Aeppli cohomology of \(X\), see [11] for the definition of the Aeppli cohomology.

Consider the Bott-Chern cohomology \(H_{BC}^*(X)\) of \(X\) which is defined by \(H_{BC}^{*,*}(X) := \text{Ker} d/ \text{Im} dd^c\). This cohomology \(H_{BC}^{*,*}(X)\) for currents or forms is the same and its dimension is finite because \(X\) is compact, see [11] for a proof. Observe that \(f^*, f_*\) induce naturally linear endomorphisms on \(H_{BC}^{*,*}(X)\). Since \(f_* f^* \alpha = d_k \alpha\) for every smooth form \(\alpha\), the map \(f_* f^*\) acting on the Bott-Chern cohomology is just the multiplication by \(d_k\). Hence, \(f_*\) is invertible.

Suppose that there is \(x \in X\) for which \(Y := f^{-1}(x)\) is of a strictly positive dimension \(q\). Denote by \(\{Y\}_{BC}\) the class of \([Y]\) in \(H_{BC}^{q,k-q}(X)\). Using \(\int_Y \omega^q > 0\), we see that \(\{Y\}_{BC}\) is nonzero because of the duality between the Bott-Chern cohomology and the Aeppli cohomology. On the other hand, \(f_* [Y] = 0\) because \(f(Y) = \{x\}\) of dimension 0. It follows that \(f_* \{Y\}_{BC} = 0\). This is a contradiction because \(f_*\) is invertible. We conclude that the fibers of \(f\) are finite.

Let \(Y\) be an irreducible analytic subset of \(X\) with \(f(Y) \subset Y\). Since the fiber of \(f\) is finite, \(f(Y)\) is of the same dimension \(q\). Thus \(f(Y) = Y\). We have \(f^n(Y) = Y\) for every \(n\).
Let $\delta$ be the topological degree of $f^n|_Y$. Observe $(f^n)_* [Y] = \delta [Y]$ because $Y$ is irreducible. Let $\beta_1, \ldots, \beta_m$ be closed forms on $X$ such that their Bott-Chern cohomology classes form a basis of $H_{BC}^{n-q,k-q}(X)$. Let $\Phi$ be a closed form in the Bott-Chern cohomology class of $[Y]$. We can write $\Phi = \sum_{j=1}^m a_j \beta_j + dd^c \Phi'$ for some $a_j \in \mathbb{C}$ and some smooth form $\Phi'$. Consequently,

$$
\|(f^n)_* [Y]\| = \langle (f^n)_* [Y], \omega^g \rangle = \langle (f^n)_* \Phi, \omega^g \rangle \lesssim \sum_{j=1}^m |\langle (f^n)_* \beta_j, \omega^g \rangle| \lesssim (d_q + \epsilon)^n,
$$

for $n \geq n_\epsilon$ big enough. We deduce that $\delta \| [Y] \| \leq (d_q + \epsilon)^n$. This finishes the proof. 

**Proof of Corollary 4.4.** We first prove $\mu_n \rightarrow \mu_f$. We just follow the usual idea to construct good inverse branches of $f^n$. Let $Y$ be the set of critical values of $f$. The fiber $f^{-1}(x)$ has exactly $d_k$ points for $x \in X \setminus Y$. The set $Y$ is the image by $f$ of the critical set of $f$ which is a hypersurface. By Lemma 4.3 $Y$ is also a hypersurface.

Let $\epsilon$ be a small positive constant for which $d_k > d_j + \epsilon$ for $0 \leq j \leq k - 1$. By (4.7) and the fact that $d_k > d_{k-1}$, we see that

$$
R := \sum_{n \geq 0} d_k^{-n} f^n [Y]
$$

is a well-defined closed positive $(1,1)$-current on $X$. This current $R$ is called the ramification current of $f$. Put $E_1 := \{ x \in X : \nu(R, x) \geq 1 \}$ which is an analytic subset of $X$ by Siu’s semi-continuity theorem. Let $E$ be the set of $x \in X$ for which $f^{-n}(x) \in E_1$ for every $n \in \mathbb{N}$.

Arguing exactly as in the proof of [19] Pro. 1.51, Th. 1.45, we see that Property (iii) holds and $E$ is totally invariant and maximal in the sense that for every proper analytic subset $E$ of $X$ with $f^{-s}(E) \subset E$ for some $s \geq 1$ then $E \subset E$. We also have that there are at most a finitely many analytic sets in $X$ which are totally invariant. We only need to note that the proofs presented there only used the Kähler form to construct $R$ and estimate the topological degree of $f^n|_Y$; the other arguments hold without the presence of a Kähler form. By the same reason, we obtain the lower bound for the Lyapunov exponents of $\mu_f$ and the equidistribution of isolated periodic points of $f$ as in [19] Th. 1.57, Th. 1.120]. Remark that although we don’t know whether $\mu_f$ is ergodic, this issue doesn’t affect arguments in [19] Th. 1.120]. The proof is finished.  

**References**

[1] D. Angella, *Cohomological aspects in complex non-Kähler geometry*, vol. 2095 of Lecture Notes in Mathematics, Springer, Cham, 2014.

[2] M. Artin and B. Mazur, *On periodic points*, Ann. of Math. (2), 81 (1965), pp. 82–99.

[3] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math., 149 (1982).

[4] J.-Y. Briand and J. Duval, *Exposants de Lyapounov et distribution des points périodiques d’un endomorphisme de $\mathbb{C}P^k$*, Acta Math., 182 (1999).

[5] , *Deux caractérisations de la mesure d’équilibre d’un endomorphisme de $\mathbb{P}^k(\mathbb{C})$*, Publ. Math. Inst. Hautes Études Sci., (2001), pp. 145–159.

[6] E. M. Chirka, *Complex analytic sets*, vol. 46 of Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers Group, Dordrecht, 1989. Translated from the Russian by R. A. M. Hoksbergen.

[7] J.-P. Demailly, *Complex analytic and differential geometry*. http://www.fourier.ujf-grenoble.fr/~demailly
[8] ———, Courants positifs et théorie de l’intersection, Gaz. Math., (1992), pp. 131–159.

[9] T.-C. Dinh, Suites d’applications méromorphes multivaluées et courants laminaires, J. Geom. Anal., 15 (2005), pp. 207–227.

[10] T.-C. Dinh, V.-A. Nguyên, and N. Sibony, Unique ergodicity for foliations on compact Kähler surfaces. [https://arxiv.org/abs/1811.07450]

[11] T.-C. Dinh, V.-A. Nguyên, and T. T. Truong, Equidistribution for meromorphic maps with dominant topological degree, Indiana Univ. Math. J., 64 (2015), pp. 1805–1828.

[12] T.-C. Dinh, V.-A. Nguyên, and T. T. Truong, Equidistribution for meromorphic maps with dominant topological degree, Indiana Univ. Math. J., 64 (2015).

[13] T.-C. Dinh and N. Sibony, Dynamique des applications d’allure polynomiale, J. Math. Pures Appl. (9), 82 (2003).

[14] T.-C. Dinh and N. Sibony, Density of positive closed currents, a theory of non-generic intersections, J. Algebraic Geom., 27 (2018), pp. 497–551.

[15] T.-C. Dinh and N. Sibony, Oka’s inequality for currents and applications, Math. Ann., 301 (1995), pp. 399–419.
[37] D.-V. Vu, *Equilibrium measures of meromorphic self-maps on non-kähler manifolds*. arxiv:1901.04775, 2018.

**UNIVERSITY OF COLOGNE, MATHEMATICAL INSTITUTE, GERMANY**

*E-mail address*: vuduc@math.uni-koeln.de