WKB approximation for the Polymer quantization of the Taub Model

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We develop a suitable technical algorithm to implement a separation of the Minisuperspace configurational variables into quasi-classical and purely quantum degrees of freedom, in the framework of a Polymer Quantum Mechanics reformulation of the canonical dynamics. We then implement the obtained general scheme to the specific case of a Taub Universe, in the presence of a free massless scalar field. In particular, we identify the quasi-classical variables in the Universe volume and a suitable function of the scalar field, while the purely quantum degree of freedom corresponds to the Universe anisotropy. We demonstrate that the Taub cosmology is associated to a cyclical Universe, oscillating between a minimum and maximum volume turning points, respectively. The pure quantum Universe anisotropy exactly fulfills the Ehrenfest theorem and it always has a finite mean values.

I. INTRODUCTION

One of the most puzzling shortcomings of the Universe representation in modern Cosmology is the presence of an initial singularity, predicted by the Einstein equation, but undoubtedly it is an unphysical ultraviolet divergence to be somehow regularized [1, 21, 22].

Various non-singular cosmological models can be constructed on a classical and quantum level, see for instance [2] but the emergence of a Bounce Cosmology can be attributed to the implementation of Loop Quantum Gravity on a cosmological setting, see [3]. When a metric approach is considered, the most natural way to deal with a singularity-free cosmological model, relies on the implementation of a Polymer Quantum Mechanics approach to the Minisuperspace [3, 4]. This approach is, de facto, a discretization procedure of the considered configurational variables (in cosmology they are Universe scale factors), which turn out to live on a graph and can have only a discrete spectrum, for a picture of the literature in merit, see [4–7].

From the side of the quantum physics of space-time, an highly non-trivial question concerns the absence of a parametric (external) time variable, when the canonical method is implemented [8–11].

Among many different proposal to construct a suitable clock in quantum gravity [12], it stands the WKB approach proposed in [13], see also [14]. The proposed scenario relies on a Born-Oppenheimer approximation, in which some Minisuperspace variables behaves slowly and are quasi-classical degrees of freedom, becoming a good clock for the fully quantum and rapidly changing variables. In other words, the time dependence of the wave function of the quantum part is recovered by its dependence on the quasi-classical variables, in turn linked to the coordinate time.

The present work explores the possibility to deal with a cosmological model in which the singularity is regularized via a Polymer Quantum Mechanics approach and a time dependence of the Universe wave function is defined via a Born-Oppenheimer decomposition of the quantum dynamics. The non trivial technical question we address here is to reconcile the momentum representation of the quantum dynamics, mandatory for a Polymer quantization, as developed in [4] for the continuum limit and the WKB scheme, thought in the coordinate representation. The crucial point is that the potential term emerging in the Minisuperspace model is, in general, non quadratic in the configurational variables, like instead in general is the Kinetic part of the Hamiltonian in the momenta. To overcome this difficulty, we introduce a suitable and general algorithm and then we implement it in the particular and important case of a Taub Cosmological model [1, 15].

The classical Taub solution links a non-singular expanded universe to a singular point of the space-time curvature, as it naturally arises because it is nothing more than a Bianchi IX model with two equal cosmic scale factors (the spatial geometry is the same of a closed Robertson-Walker geometry).

The cosmological model resulting from our regularization is a very intriguing paradigm: we get an evolutionary quantum picture, whose Ehrenfest description corresponds to a (non-singular) cyclical Universe.

Our study of the Taub cosmology in the presence of a scalar field is performed using Misner-Chitré-like variables [16]. The quasi-classical variables are identified in the scalar field and in the one that is most directly linked to the Universe volume, actually in the adopted variables the isotropic metric component and the anisotropies are somehow mixed together. The quantum degree of freedom is identified in the relic anisotropy coordinate of the Taub model, a suitable redefinition of the variables is also necessary during the technical derivation.

The resulting evolutionary (Schrödinger) equation for this anisotropy variable has, in the spirit of the Ehrenfest theorem, two main physical implications: i) the Taub model is reduced to a cyclical Universe, evolving between a minimum and maximum value of the Universe variables, offering an intriguing paradigm for the physical implementation of a cosmological history: clearly the max-
In the classical domain of the Universe dynamics, while the Bounce turning point has a pure quantum character, in the sense of a Polymer regularization; ii) the Universe anisotropy is always finite in value as a result of the singularity regularization and its specific value in the Bounce turning point depends on the initial conditions of the system, but in principle, it can be restricted to small enough values to make the Bounce dynamics unaffected by their behavior, i.e., the applicability of the Born-Oppenheimer approximation is ensured in the spirit of the analysis provided in [17].

The paper is structured as follows: two section in order to introduce the Minisuperspace, the Bianchi Models and the Polymer Quantum Mechanics, one section to generalize the Vilenkin approach in both the representations, a section in which we will implement the generalized approach to the Taub model in both the Classical and Polymer Quantum Mechanics, at the end there will be a section where we discuss the obtained results, the conclusions and then the appendix.

II. MINISUPERSPACE AND BIANCHI MODELS

The idea of the Minisuperspace was born from the possibility of reducing the general problem of the quantum gravity to the simple case of a space-time highly symmetric, with a dynamics in a finite dimensions scheme, and the quantization to a natural Dirac’s prescription for the Universe wave function.

For the purpose of this paper we will limit ourselves to the Homogeneous Universes that are described by the Bianchi Models; those models represent all the possible universes that are homogeneous but anisotropic. There are 9 different models but the most studied are Bianchi I, V and IX, that contain respectively the Flat, the Open and the Close FRW model once taken the isotropic limit.

Alexander Vilenkin chose to study the dynamics of the Primordial Universe in a minisuperspace scheme, so that the metric depends only on the coordinates and the wave function is approximated by its WKB expansion where the classical variables are treated differently from the quantum ones. The Ukrainian physicist studied a method relatively elegant and linear that allows us to introduce a lattice structure in the system and see which are discrete and which are not, so fundamentally one has to introduce a lattice structure in the system and see where it leads. There are two possibilities of implementation for these requirements, depending on the choice of the polarization of the wave functions, namely the $q$-polarization and the $p$-polarization; for the purpose of this paper we choose the latter because the equations are easier to study.

The problem is that when one associates a discrete character to one of the variables, the Weyl algebra assures that the operator associated to its conjugate variable doesn’t exist. This creates a lot of problems when one tries to quantize the system, one of the most serious is that it’s necessary to decide a range of reliability which is possible to approximate that operator and never leave it.

We can start by defining abstract kets $|\mu\rangle$ labelled by a real number. These shall belong to the Hilbert space $H_{poly}$. From these states, we define a generic state that correspond to a choice of a finite collection of numbers $\mu_i \in \mathbb{R}$ with $i = 1, 2, \ldots, N$. Associated to this choice, there are $N$ vectors $|\mu_i\rangle$, so we can take a linear combination of them $|\psi\rangle = \sum_{i=1}^{N} a_i |\mu_i\rangle$. The fundamental kets are orthonormal and the Polymer Hilbert Space $H_{poly}$ is non-separable.

There are two basic operators on this Hilbert Space, namely the label operator $\hat{\epsilon}$ and the displacement oper-
Let’s further advance and introduce the case of a grade-$n$ polynomial as the exponent

$$D_p^\mu [e^{ap}] = D_p^\mu \left[ \sum_{k=0}^\infty \frac{(ap)^k}{k!} \right] = a^\mu \sum_{k=0}^\infty \frac{(ap)^{k-\mu}}{(k+1-\mu)} = a^\mu E_{\mu}^{ap}$$

Let’s define another function for the purpose of this paper

$$ Ln_\mu \left[ E_{\mu}^{ap} \right] \equiv ap .$$

When everything is taken into account it must be said that as soon as we put a generic function in the place of the exponential of a polynomial, all the maths starts to degrade because the initial definition has a lot of problems that are solved only in the case of polynomial functions.

In the following this generalized derivative will be often used because we will only consider functions that are related to polynomial.

A. Ordinary Case

The Hamilton-Jacobi equation is described by the first order expansion of (1). In order to obtain it, it’s necessary to expand the exponential in its power series and take only the right order terms. We get:

$$\psi(p) = A(p) \left[ 1 + \frac{i}{\hbar} S(p) - \frac{1}{2\hbar^2} S^2 \right] \phi(p,q)$$

and so equation (1) becomes at the lowest order:

$$g^{\alpha\beta} \left( \frac{\partial S}{\partial p_\gamma} \right) p_\alpha p_\beta A(p) \left( - \frac{1}{2\hbar^2} S^2 \right) \phi(p,q) +$$

with the due simplifications and introducing the notation

$$\left( \frac{\partial}{\partial p_\gamma} \right) \equiv (\partial_\gamma)$$

we obtain

$$g^{\alpha\beta} \left( \frac{\partial S}{\partial p_\gamma} \right) p_\alpha p_\beta \frac{S^2}{\hbar^2} + 2U \left( \frac{\partial S}{\partial p_\gamma} \right) = 0 .$$

that reproduce exactly the Hamilton-Jacobi equation of the classical case once we identify

$$\left( \frac{\partial}{\partial p_\gamma} \right)$$

with $h^\gamma$.

At the next order we get two separate equations given that, as in the case analyzed by Vilenkin, we can exploit the adiabatic approximation. Let’s start analyzing the first equation for the amplitude $A$ and then the one for the quantum wavefunction. Studying the general case, we don’t have the explicit forms of the metric and the potential term, and so we can’t let them act directly on the wavefunction; what we can do is, instead, multiply by the identity both of the terms defining

$$\mathbb{I} = (i\hbar\partial_\gamma)^{-1} (i\hbar\partial_\gamma) .$$

valid in the regime $p \ll 1/\mu$.

One can ask why this representation is so important, the answer is that there are systems which don’t admit a standard description and if we can define a continuum limit of the Polymer then it’s possible to quantize the system with this representation and find its dynamics, at least approximately. Studying systems with exact solutions like the harmonic oscillator and the free particle, it has been established that the results given by both representation are exactly the same once the continuum limit of the Polymer Quantum Mechanics is taken.

There are, indeed, great expectations for this new representation, especially in Quantum Cosmology where all the new theories (such as the Loop Quantum Cosmology) suggest the existence of a finite inferior limit of the volume of the Universe and so the presence of a lattice structure in the space-time.

IV. GENERALIZED VILENKIN APPROACH

In this paragraph I will extend the study of Vilenkin [13] to the case of a totally general homogeneous universe. We will start from the Wheeler-De Witt equation in the momenta base that is written as:

$$\left[ g^{\alpha\beta} \left( \frac{\partial S}{\partial p_\gamma} \right) p_\alpha p_\beta - U \left( \frac{\partial S}{\partial p_\gamma} \right) - H_\mu \right] \psi(p) = 0$$

and the Action $S$ is in the ADM form and the wave function will be:

$$\psi(p) = A(p) e^{\frac{i}{\hbar} S(p)} \phi(p,q) .$$

The first step to achieve the generalized approach is to introduce a generalization of the Derivative operator that will greatly help in the following. Let’s start from the simplest cases:

$$D_p^\mu [p^\nu] = \frac{\Gamma(\nu + 1)}{\Gamma(\nu - \mu + 1)} p^{\nu-\mu}$$

$$D_p^\mu [e^{ap}] = D_p^\mu \left[ \sum_{k=0}^\infty \frac{(ap)^k}{k!} \right] = \sum_{k=0}^\infty \frac{(ap)^{k-\mu}}{\Gamma(k+1-\mu)} = E_{ap}^\mu ;$$

where $E_{ap}^\mu$ is the generalized exponential function defined by:

$$E_{ap}^\mu \equiv p^\mu e^{ap} \gamma^\mu (\mu, ap) ,$$

$$\gamma^\mu (\mu, ap) \equiv e^{-ap} \sum_{j=0}^\infty \frac{(ap)^j}{\Gamma(\mu + j + 1)} .$$

Let’s further advance and introduce the case of a grade-$n$ polynomial as the exponent

$$D_p^\mu [e^{ap}] = D_p^\mu \left[ \sum_{k=0}^\infty \frac{(ap)^k}{k!} \right] = a^\mu \sum_{k=0}^\infty \frac{(ap)^{k-\mu}}{(k+1-\mu)} = a^\mu E_{ap}^\mu .$$

Let’s define another function for the purpose of this paper

$$ Ln_\mu \left[ E_{ap}^\mu \right] \equiv ap .$$

When everything is taken into account it must be said that as soon as we put a generic function in the place of the exponential of a polynomial, all the maths starts to degrade because the initial definition has a lot of problems that are solved only in the case of polynomial functions.

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with the due simplifications and introducing the notation

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we obtain

$$g^{\alpha\beta} \left( \frac{\partial S}{\partial p_\gamma} \right) p_\alpha p_\beta \frac{S^2}{\hbar^2} + 2U \left( \frac{\partial S}{\partial p_\gamma} \right) = 0 .$$

that reproduce exactly the Hamilton-Jacobi equation of the classical case once we identify

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with $h^\gamma$.

At the next order we get two separate equations given that, as in the case analyzed by Vilenkin, we can exploit the adiabatic approximation. Let’s start analyzing the first equation for the amplitude $A$ and then the one for the quantum wavefunction. Studying the general case, we don’t have the explicit forms of the metric and the potential term, and so we can’t let them act directly on the wavefunction; what we can do is, instead, multiply by the identity both of the terms defining

$$\mathbb{I} = (i\hbar\partial_\gamma)^{-1} (i\hbar\partial_\gamma) .$$
The desired equation can be obtained at the next order of the expansion in $\hbar$. Multiplying by the identity defined above and having the exotic derivative acting only on the amplitude while the normal one acts on the exponential term we obtain:

$$g^{\alpha \beta} p_{\alpha} p_{\beta} \left[ (\partial^{-1} A) \left( \partial e^{\frac{k}{\hbar} S} \right) \right] \phi - U \left[ (\partial A) \left( \partial^{-1} e^{\frac{k}{\hbar} S} \right) \right] \phi = 0 \tag{14}$$

Those are not the only terms at the right order so we multiply again the equation by the identity and we get:

$$ig^{\alpha \beta} p_{\alpha} p_{\beta} \left[ \{2 \partial^{-1} (\partial^{-1} A) (\partial S) + [\{\partial^{-2} A\} (\partial^2 S)] \} e^{\frac{k}{\hbar} S} - 2 (\partial A) \left( \partial^{-1} e^{\frac{k}{\hbar} S} \right) \right] = 0 \tag{15}$$

And this is the equation for the amplitude $A(p)$.

Now we analyze the equation for the pure quantum wavefunction. As in the above case, we multiply the initial equation by the identity, but this time the important part, in order to obtain the Schrödinger equation, is when the exotic derivative acts on the exponential and the normal one acts on the quantum term; before we approach the real calculation it’s opportune expanding the action in its power series: $S(p) = \sum_{k=0}^{\infty} c_k (t) p^k (t)$.

Let’s start applying the exotic derivative on the exponential using the definition:

$$(ih \partial_{\gamma})^{-1} e^{\frac{k}{\hbar} S} \sum_{k=0}^{\infty} c_k p^k = (ih \partial_{\gamma})^{-1} \prod_{k=0}^{\infty} \sum_{j=0}^{\infty} \left[ \left( \frac{i c_k p^k}{\hbar} \right)^j \right] = \frac{1}{i \hbar} \prod_{k=1}^{\infty} \left[ \sum_{k=1}^{\infty} \frac{\Gamma(j+2)}{\Gamma(j+1)} \right] = \frac{1}{i \hbar c} \prod_{k=0}^{\infty} E_{-1}^{\frac{k}{\hbar} c_k p^k} \tag{16}$$

where $\Gamma$ is the Euler Gamma Function, for the sake of notation I have defined $c = \sum_{k=1}^{\infty} \frac{k}{\hbar} c_k$ and I have introduced the generalized exponential function defined above in (7). The equation

$$\hat{H}_0 \left( \partial^{-1} e^{\frac{k}{\hbar} S} \right) (\partial \phi) = \hat{H}_q e^{\frac{k}{\hbar} S} \phi \tag{17}$$

becomes, exploiting (16),

$$\hat{H}_0 \frac{1}{c} \prod_{k=0}^{\infty} E_{-1}^{\frac{k}{\hbar} c_k p^k} (\partial \phi) = \hat{H}_q e^{\frac{k}{\hbar} S} \sum_{k=0}^{\infty} c_k p^k \phi \tag{18}$$

where $\hat{H}_0$ is the classical part of the WD. We highligh in particular the property of one of the terms in equation (18):

$$\frac{1}{c} \prod_{k=0}^{\infty} E_{-1}^{\frac{k}{\hbar} c_k p^k} = \frac{F(p)}{F'(p)} = \frac{1}{\partial_t \ln (p)} \tag{19}$$

and so we obtain:

$$\frac{1}{\partial_t \ln (p)} \{ F(p) \} (\partial_t \phi) = \hat{H}_q \phi \tag{20}$$

We can rewrite the p-derivative of the logarithm as its time derivative times $\frac{1}{\partial_p}$, thanks to the properties of the differentials, and it ensures that it’s possible to obtain the time derivative even of the quantum terms. Let’s see how it can be done

$$\frac{1}{\partial_p} \{ F(p) \} \partial_t \phi = \hat{H}_0 (\partial_t \phi) = \hat{H}_q \phi \tag{21}$$

The time derivative of the logarithm, $\partial_t \ln (p)$, can be written as $\frac{1}{\hbar} \partial_t$ and so we obtain the equation:

$$-i \hbar \frac{1}{\hbar} \frac{\partial \phi}{\partial t} = \hat{H}_0^{-1} \hat{H}_q \phi \tag{22}$$

We can take all the temporal dependence of the above equation and define a new time derivative in $\tau$ in order to get

$$-i \hbar \frac{1}{\hbar} \frac{\partial \phi}{\partial \tau} = \hat{H}_0^{-1} \hat{H}_q \phi \tag{23}$$

if we define $\tau$ such that

$$\frac{\partial \phi}{\partial \tau} = \frac{1}{\hbar} \frac{\partial \phi}{\partial t} \tag{24}$$

and after we take all the other terms to the second member we obtain:

$$-i \hbar \frac{\partial \phi}{\partial \tau} = \hat{H}_1 \phi \tag{25}$$

that is the desired Schrödinger’s equation for the quantum wavefunction. In equation (25) I have defined

$$\hat{H}_1 = \hat{H}_0^{-1} \hat{H}_q \tag{26}$$

B. Polymer Case

As seen in the section about the Polymer quantum mechanics, imposing a Polymer quantization means assuming a discrete structure for some of the variables of the phase space. The consequence of this fact is that it’s not possible to associate to the conjugated variables quantum differentials operators as in the ordinary case. The Polymer paradigm, to solve this problem, consists in the substitution $p \rightarrow \frac{1}{\mu} \sin (\mu p)$. As a consequence, the Polymer version of the WD equation is:

$$\left[ \frac{\hbar^2}{\mu^2} g^{0 \beta} \sin \left( \frac{\mu p_0}{\hbar} \right) \sin \left( \frac{\mu p_0}{\hbar} \right) - U - \hat{H}_q \right] \psi = 0 \tag{27}$$
Expanding it at the lowest order and using the power series of the exponential we find the Hamilton-Jacobi equation for the Polymer case:

\[-\frac{\hbar^2}{\mu^2} g^{\alpha \beta} \sin \left( \frac{\mu p_\alpha}{\hbar} \right) \sin \left( \frac{\mu p_\beta}{\hbar} \right) \frac{A S^2 \phi}{2 \hbar^2} - U A \phi = 0\]

\[\Rightarrow \frac{1}{\mu^2} g^{\alpha \beta} \sin \left( \frac{\mu p_\alpha}{\hbar} \right) \sin \left( \frac{\mu p_\beta}{\hbar} \right) S^2 + 2U = 0 .\]

As seen above at the next order we find two separate equations because of the adiabatic approximation. In order to find those equations we will use the same method of the last section with the identity defined by

\[\ll = (i \hbar \partial_{\gamma_{Pol}})^{-1} (i \hbar \partial_{\gamma_{Pol}}) .\]

Using the same notation of the last section we get:

\[
\frac{\hbar^2}{\mu^2} g^{\alpha \beta} \sin \left( \frac{\mu p_\alpha}{\hbar} \right) \sin \left( \frac{\mu p_\beta}{\hbar} \right) \phi - U \| \psi = 0 \Rightarrow \\
\frac{\hbar^2}{\mu^2} g^{\alpha \beta} \sin \left( \frac{\mu p_\alpha}{\hbar} \right) \sin \left( \frac{\mu p_\beta}{\hbar} \right) \left( \partial_{Pol}^{-1} A \right) \left( \partial_{Pol} E_+ \right) \phi - U I \left( \left[ \left( \partial_{Pol} A \right) \left( \partial_{Pol} E_+ \right) \right] \phi = 0 \right.
\]

\[
\Rightarrow \frac{\hbar^2}{\mu^2} g^{\alpha \beta} \sin \left( \frac{\mu p_\alpha}{\hbar} \right) \sin \left( \frac{\mu p_\beta}{\hbar} \right) \left( \partial_{Pol}^{-1} \right) \left( \partial_{Pol} \right) \left( \left[ \left( \partial_{Pol} A \right) \left( \partial_{Pol} E_+ \right) \right] \phi = 0 \right. - U \left( \partial_{Pol}^{-1} \right) \left( \partial_{Pol} \right) \left( \left[ \partial_{Pol} A \right) \left( \partial_{Pol} E_+ \right) \right] \phi = 0 .
\]

If we write explicitly the known terms we obtain:

\[
i \hbar \frac{\hbar^2}{\mu^2} g^{\alpha \beta} \sin \left( \frac{\mu p_\alpha}{\hbar} \right) \sin \left( \frac{\mu p_\beta}{\hbar} \right) \left[ 2 \left( \partial_{Pol}^{-1} A \right) \left( \partial_{Pol} S \right) + \left( \partial_{Pol}^{-2} A \right) \left( \partial_{Pol}^2 S \right) \right] E_+ - U \hbar \left[ 2 \left( \partial_{Pol} A \right) \left( \partial_{Pol}^{-1} E_+ \right) \right] = 0
\]

This is the equation for the Polymer amplitude \( A \).

Although the calculation made till now demonstrates that the equations that we obtain in both the representations are the same taking into account the correction introduced by the passage from one to the other, let’s see what happen to the quantum wavefunction. The method is exactly the same of the last section since if \( f \left( \sin \left( p \right) \right) = f \left( p \right) \). The equation

\[\hat{H}_{0_{Pol}} \left( \partial_{Pol}^{-1} e^{\frac{i}{\hbar} S} \right) \left( \partial_{Pol} \phi \right) = \hat{H}_{q_{Pol}} e^{\frac{i}{\hbar} S} \phi \]

becomes, exploiting (16),

\[\hat{H}_{0_{Pol}} \frac{1}{\epsilon} \prod_{k=0}^{\infty} E_{\epsilon k}^{-1} \left( \partial_{\gamma} \phi \right) = \hat{H}_{q_{Pol}} e^{\frac{i}{\hbar} \sum_{k=0}^{\infty} c_k p_k} \phi , \]

where with \( \hat{H}_{0_{Pol}} \) has been indicated the classical part of the WD in the Polymer representation. In this particular case the generalized exponential function contains all the Polymer correction and it is substantially different from the ordinary one. We highlight, even in this case, the property of one of the terms of the equations in (31):

\[\frac{1}{\epsilon} \prod_{k=0}^{\infty} E_{\epsilon k}^{-1} = \frac{F \left( p \right)}{F' \left( p \right)} = \frac{1}{\partial_{\gamma} L_{n-1} \left( F \left( p \right) \right)} \]

and so we get:

\[\hat{H}_{0_{Pol}} \frac{1}{\epsilon} \prod_{k=0}^{\infty} E_{\epsilon k}^{-1} \left( \partial_{\gamma} \phi \right) = \hat{H}_{q_{Pol}} e^{\frac{i}{\hbar} S} \phi . \]

Taking into account the properties of the differentials, we can rewrite the p-derivative of the logarithm as the time derivative of the logarithm times \( \frac{\partial t}{\partial p} \).

\[\hat{H}_{0_{Pol}} \frac{1}{\epsilon} \prod_{k=0}^{\infty} E_{\epsilon k}^{-1} \left( \partial_{\gamma} \phi \right) \partial_{\gamma} t = \hat{H}_{q_{Pol}} e^{\frac{i}{\hbar} S} \phi . \]

The time derivative of the logarithm, \( \partial_{\gamma} L_{n-1} \left( F \left( p \right) \right) \) can be written as \( \frac{i}{\hbar} \mathcal{D}_{Pol} \) and so we get the equation:

\[\frac{1}{\epsilon} \prod_{k=0}^{\infty} E_{\epsilon k}^{-1} \left( \partial_{\gamma} \phi \right) = \hat{H}_{0_{Pol}} e^{\frac{i}{\hbar} S} \phi . \]

We can take all the temporal dependence of the above equation and define a new time derivative in \( \tau \) in order to get

\[-i \hbar \frac{1}{\mathcal{D}_{Pol}} \left( \partial_{\tau} \phi \right) = \hat{H}_{0_{Pol}} e^{\frac{i}{\hbar} S} \phi , \]

if we define \( \tau_{Pol} \) such that

\[\frac{\partial}{\partial \tau_{Pol}} = \frac{1}{\mathcal{D}_{Pol}} \left( \frac{\partial}{\partial t} \right) \]

(37)
and after we take all the other terms to the second member we obtain:

\[-i\hbar \frac{\partial \phi}{\partial \tau_{P\text{pol}}} = \hat{H}_1_{\text{pol}} \phi, \quad (38)\]

where

\[\hat{H}_1_{\text{pol}} = \hat{H}_q_{\text{pol}} \hat{H}_p_{\text{pol}} \]

The equation above is the desired Schrödinger equation and it’s equivalent to the ordinary case. Clearly both in the time variable and in the terms of the Hamiltonian there is the Polymer correction, but formally they are the same.

C. Conserved Current

We analyze now the probability current defined from the equation (1) in order to obtain the continuity equa-

\[\partial_\delta J^\delta = \frac{i}{2} \hbar \alpha \beta \gamma \left( \partial_{\delta}^{-1} (\delta^\gamma \psi^*) (\delta_\gamma \psi) - (\delta_\gamma \psi^*) (\delta_{\delta}^{-1} \psi) - \psi^* \psi + \psi^* \psi - \psi^* \psi + \psi^* \psi + \psi^* \psi \right) + \]

\[+ \frac{i}{2} \hbar \partial_\delta \left( \partial_\gamma \left[ (\delta_{\delta}^{-1} \psi^*) (\delta_\gamma \psi) - (\delta_\gamma \psi^*) (\delta_{\delta}^{-1} \psi) \right] - \psi^* \psi + \psi^* \psi - \psi^* \psi + \psi^* \psi \right) + \]

\[+ \frac{i}{2} \hbar \partial_\delta \left( \partial_\gamma \left[ (\delta_{\delta}^{-1} \psi^*) (\delta_\gamma \psi) - (\delta_\gamma \psi^*) (\delta_{\delta}^{-1} \psi) \right] - \psi^* \psi + \psi^* \psi - \psi^* \psi + \psi^* \psi \right) ; \]

with the due simplifications and defining

\[\Lambda \equiv \left[ (\delta_{\delta}^{-1} \psi^*) (\delta_\gamma \psi) - (\delta_\gamma \psi^*) (\delta_{\delta}^{-1} \psi) \right] \]

we obtain the following equation:

\[\partial_\delta J^\delta = \frac{i}{2} \hbar \alpha \beta \gamma \left[ 4\Lambda + (\delta_{\delta}^{-1} \partial_{\delta}^{-1} \psi^*) (\delta_\gamma \partial_\gamma \psi) - (\delta_\gamma \partial_\gamma \psi^*) (\delta_{\delta}^{-1} \partial_{\delta}^{-1} \psi) + (\delta_\gamma \partial_{\delta}^{-1} \psi^*) (\delta_{\delta}^{-1} \partial_{\delta}^{-1} \psi) - (\delta_{\delta}^{-1} \partial_{\delta}^{-1} \psi) (\delta_\gamma \partial_{\delta}^{-1} \psi) \right] + \]

\[+ \frac{i}{2} \hbar \partial_\delta \left( \partial_\gamma \left[ \Lambda \right] \right) \partial_{\delta}^{-1} \Lambda . \]

The last two terms within the square brackets of the above equation are null for the properties of the gen-

eralized derivative while the last line of the right hand side reproduce exactly the equation of motion and so it’s null.

From the analysis of the term in \(\Lambda\) it is evident that the only terms at the right order in \(\hbar\) are:

\[\Lambda = i \left( \partial_{\delta}^{-1} |A|^2 \right) (\partial_\delta S) |\phi|^2 + \]

\[+ |A|^2 \left( \partial_{\delta}^{-1} E_+ \right) E_+ \psi^* (\partial_\gamma \phi) - |A|^2 \left( \partial_{\delta}^{-1} E_- \right) (\partial_\gamma \phi^*) \phi , \]

\[\quad (43)\]

with the notation \(E_\pm = e^{-i \psi} \). A property very impor-

tant of the generalized derivative is, as in the ordinary one, the Leibniz law, that applied in this case gives the relation

\[\left( \partial_{\delta}^{-1} E_- \right) E_+ + E_- \left( \partial_{\delta}^{-1} E_+ \right) = D_p^{-1} (E_+ E_+) = D_p^{-1} (1) = p \]

\[\quad (44)\]

and so it is possible to express one term of the left hand side as a function of the other, in order to maintain the initial ordering we choose the relation \(E_- \left( \partial_{\delta}^{-1} E_+ \right) = \)
\[ p - (\partial_{\gamma}^{-1} E_{-}) E_{+} \] and we get
\[ \Lambda = i \left( \partial_{\gamma}^{-1} |A|^2 \right) (\partial_{\gamma} S) |\phi|^2 + |A|^2 \left( \partial_{\gamma}^{-1} E_{-} \right) E_{+} \left( \partial_{\gamma} |\phi|^2 \right) \] \tag{45}

the term that contains \( p \) is of a different order and so it can be neglected.

As for the second term on the right hand side of the first line of the equation (42) the only term of the right order is \( i \left( \partial_{\delta}^{-1} \partial_{\gamma}^{-1} |A|^2 \right) (\partial_{\delta} \partial_{\gamma} S) |\phi|^2 \). At the end we can say that the dominant terms of the equation (42) reduce to:
\[ \partial_{\delta} J^\delta = i \left( \left( \partial_{\delta}^{-1} |A|^2 \right) (\partial_{\delta} S) |\phi|^2 + \left( \partial_{\delta}^{-1} \partial_{\gamma}^{-1} |A|^2 \right) (\partial_{\delta} \partial_{\gamma} S) |\phi|^2 \right) + |A|^2 \left( \partial_{\delta}^{-1} E_{-} \right) E_{+} \left( \partial_{\gamma} |\phi|^2 \right). \] \tag{46}

Those are the equations (15) and (17) for the Universe wavefunction and for its complex conjugate derived before. Considering their definitions the term on the right hand side it’s identically null and so even in the case of this study there is a conserved probability current. This demonstration is valid for both Standard and Polymer Quantum mechanics once taken the correct assumptions.

\section*{V. APPLICATION TO THE TAUB MODEL}

In this section I will applicate the results of the previous sections to the Taub Model (one of the particular cases of Bianchi IX model), the result will be a quantum wavefunction for the Universe that will allow us to infer the behavior of the Early Universe.

Althoug usually the best choice for this kind of study are the Misner Variables \((\alpha, \beta_{\bot}, \beta_{\bot})\) for their immediate physical interpretation: \(\alpha\) is related to the volume of the Universe, while the \(\beta\) are related to the two physical degree of freedom of the Gravitational Field. For the following discussion I chose another set of variables more complicate and with a not immediate physical sense, the Misner-Chitrè variables. They enable us to study the dynamics of the sistem in the so-called \textit{Poincaré Half Plane} that eliminate the dynamics of the potential’s wall. In particular the two set of variables have the following relations [20]:
\[ \alpha - \alpha_0 = -e^\gamma \frac{1 + u + u^2 + v^2}{\sqrt{3}v} \]  
\[ \beta_{\bot} = e^\gamma \frac{1 + 2u + 2u^2 + 2v^2}{2\sqrt{3}v} \]  
\[ \phi = e^\gamma \frac{1 - 2u}{2v}. \] \tag{47}

In order to make the Vilenkin Approach works it’s necessary to insert a term of matter, for the purpose of this study I chose the Scalar Field.

The dynamics of this model near the singularity reduces to the one of a particle that hit continuously the walls of a pseudo-triangular box [18] [19]; the cosmological singularity is reached when the trajectory ends in one of the corner of the box. This model consists in taking one preferencial direction in the \(\beta\)-plane, and so only one of the walls of the Bianchi IX Universe that the particle hits only one time and then goes directly in the opposite corner. This means that the Misner \(\beta\) is identically null and so the Misner-Chitrè \(\beta\) is always constant and equal to \(-1/2\), implying that the conjugate momentum \(p_{\beta}\) is always zero.

In the chosen variables the Super-Hamiltonian constraint \(\mathcal{H} = 0\) leads to a WD equation without all the terms in \(p_{\beta}\). In this case the metric assumes the simple form
\[ ds^2 = \frac{\epsilon}{v^2} \left[ du^2 + dv^2 \right]. \] \tag{48}

In order to make the math easier we change again variables, introducing
\[ v = \rho \sin (2\delta) \]  
\[ u = \rho \cos (2\delta), \] \tag{49}

with \(0 < \rho < \infty\) and \(0 < \delta < \pi\). If we insert them in the metric it’s simple to verify that (48) becomes
\[ ds^2 = \epsilon \left[ \frac{d\rho^2}{\rho^2 \sin^2 (2\delta)} + \frac{8d\delta^2}{\sin^2 (2\delta)} \right]. \] \tag{50}

If now we define \(dx = d\rho/\rho \epsilon \ d\theta = d\delta/\sin (2\delta)\) and integrate then we find two variables with the same limits of the Misner-Chitrè ones
\[ x = \log |\rho|, \quad -\infty < x < \infty \]  
\[ \theta = \frac{1}{2} \log |\tan (\delta)|, \quad -\infty < \theta < \infty ; \] \tag{51}

with a few calculations it’s possible to rewrite the term \(\sin^2 (2\delta)\) present in (50) as a function of the new variable \(\theta\) only as
\[ \sin^2 (2\delta) = 4 \sin^2 (\delta) \cos^2 (\delta) = 4 \sin^2 \left( \arctan \left( e^{2\theta} \right) \right) \cos^2 \left( \arctan \left( e^{2\theta} \right) \right) \]  
\[ = \frac{4 e^{4\theta}}{e^{4\theta} + 1} \left( \frac{1}{1 + \cosh (4\theta)} \right) = \frac{2}{1 + \cosh (4\theta)} \] \tag{52}

where I used the formula \(\sin^2 \left( \arctan \left( x \right) \right) = \frac{x^2}{x^2 + 1}\) and the definition of the hyperbolic cosine. With these substitutions the metric becomes
\[ ds^2 = \epsilon \left[ \frac{2d\rho^2}{1 + \cosh (4\theta)} + 8d\theta^2 \right], \] \tag{53}

and so the Hamiltonian of the system becomes
\[ H = \left[ -p_x^2 - \frac{p_y^2}{8} + \frac{1 + \cosh (4\theta)}{2} p_x^2 \right]. \] \tag{54}
A. Ordinary Case

Let’s analyze this Hamiltonian (54) in order to get the equations for the dynamics of the system, we derive them via the Ehrenfest Theorem as

\[
\{\langle \dot{r} \rangle = \frac{1}{i\hbar} \{[r, H]\} = \frac{1}{i\hbar} \{[r, -p_\tau^2/8]\} = -2p_\tau,
\langle \dot{\theta} \rangle = \frac{1}{i\hbar} \{[\theta, H]\} = \frac{1}{i\hbar} \{[\theta, -p_\theta]\} = -\frac{p_\theta}{4},
\langle \dot{z} \rangle = \frac{1}{i\hbar} \{[z, H]\} = \frac{1}{i\hbar} \{[z, 1/2(1 + \cosh(4\theta))p_x^2]\} = (1 + \cosh(4\theta))\frac{p_x}{2},
\]

with the same method we derive also the equations for the conjugated momenta finding

\[
\langle p_\tau \rangle = 0,
\langle p_\theta \rangle = 0,
\langle p_x \rangle = 0.
\]  

(55)

Those are the equations that describe the dynamics of the Universe.

Now I will adapt the Vilenkin approach to the Taub Universe. First of all I will use a wavefunction in the form

\[
\psi(p_\tau, p_\theta, p_x) = A(p_\tau, p_\theta) e^{i\frac{\chi}{\hbar}}\phi(p_\theta, p_x)
\]

where, again, all the calculations will be inserted in the appendix (respectively as:

\[
\psi(p_\tau, p_\theta, p_x) = A(p_\tau, p_\theta) e^{i\frac{\chi}{\hbar}}\phi(p_\theta, p_x)
\]

and we put it in (59) we can solve it and we find

\[
E = \frac{p_x^2}{2} + \frac{2}{\mu^2} \frac{1 - \cos \mu p_x}{1 - \cos \mu p_\theta} \frac{\chi}{\hbar} \frac{1}{\mu}
\]

(60)

B. Polymer Case

Let’s go back to the Hamiltonian (54) and use the Polymer Quantum Dynamics instead of the classical one. If we want the Hamilton equations we must remember that in this case the canonical commutator is \([\hat{x}_i, \hat{p}_i] = i\hbar \cos (\mu p_i)\). The Wheeler-DeWitt equation in this case is in the form:

\[
\begin{align*}
\left\{ -\frac{2}{\mu^2} \left[ 1 - \cos (\mu p_\tau) \right] - \frac{1}{4\mu^2} \left[ 1 - \cos (\mu p_\theta) \right] \right\} \Psi + \left\{ \frac{1 + \cosh(4\theta)}{\mu^2} \right\} \left[ 1 - \cos (\mu p_x) \right] \Psi &= 0.
\end{align*}
\]

(61)

With the same calculations of the previous section we find the equations for the dynamics of the particle Universe

\[
\begin{align*}
\langle \dot{x} \rangle &= \frac{C_x}{\mu} \sin (2\mu p_x), \quad \langle \dot{p}_x \rangle = 0,
\langle \dot{\theta} \rangle &= \frac{C_\theta}{\mu} \sin (2\mu p_\theta), \quad \langle \dot{p}_\theta \rangle = 0,
\langle \dot{\tau} \rangle &= \frac{C_\tau}{\mu} \sin (2\mu p_\tau), \quad \langle \dot{p}_\tau \rangle = 0.
\end{align*}
\]

(62)

Those are the equations that describe the dynamics of the Early Universe.

Now we use the Vilenkin approach in this case, from equation (61) we can derive the Hamilton-Jacobi equation, the equation for the amplitude of the wavefunction and the Shrödinger equation for the quantum variables respectively

\[
\begin{align*}
1 \mu^2 \sin^2 (\mu p_\tau) S^2 - \frac{\hbar^2}{4\mu^2} \sin^2 (\mu p_\theta) &= 0,
\end{align*}
\]

(63)

\[
\frac{2}{\mu^2} \left[ 1 - \cos (\mu p_\tau) \right] \left\{ 8 \left( \frac{\partial^{-1}A}{\partial E} \right) \left( \partial^2 E \right) \right\} + \frac{\hbar}{2\mu^2} \left[ 1 - \cos (\mu p_\theta) \right] \left\{ \left( \partial A \right) \left( \partial^{-1} E \right) \right\} = 0,
\]

(64)

\[
\begin{align*}
- \frac{\hbar \mu^2}{\partial z} = \frac{[1 - \cos (\mu p_\tau)]}{\partial z} \frac{\chi}{\mu^2}.
\end{align*}
\]

(65)

The variable \( z \) is a time-variable defined by \( \frac{\partial}{\partial z} = -\frac{2[1 - \cos (\mu p_\tau)]}{\mu^2 (\partial p_\tau)} \frac{\partial}{\partial p_\tau} \). If we consider a quantum part of the Universe wavefunction in the form \( \chi = e^{iEz}\phi(p_\theta, p_x) \).
Universe wavefunction in the form \( \chi = e^{i\mathcal{H}z} \phi(p_\theta, p_z) \) and we put it in (65) we can solve it and we find

\[
k = k(\mu) = \frac{1}{\mu^2} [1 - \cos (\mu p_z)] \leq k_{\text{max}} = \frac{2}{\mu^2}
\]

\[
\phi_{k,\mu}(p_z) = C_1 \delta(p_z - p_{k,\mu}) + C_2 \delta(p_z + p_{k,\mu}) \quad (66)
\]

\[
\phi_{k,\mu}(x) = \frac{1}{\sqrt{2\pi}} e^{-ip_{k,\mu}x} (C_1 + C_2 e^{2ip_{k,\mu}x}).
\]

We can notice that those are the same results of the previous section once taken into account the Polymer modifications, moreover we can also notice that the eigenvalue here has an upper limit and this will be very important in the dynamics of the Universe.

\[
\langle \tau \rangle = -2p_z z
\]

\[
\langle \theta \rangle = -\frac{p_\theta z}{4}
\]

\[
\langle x \rangle = [1 + \cosh (4\theta(z))] p_z z
\]

for the ordinary case and

\[
\langle \tau \rangle = \frac{C_r}{\mu} \sin (2\mu p_\tau) \ z
\]

\[
\langle \theta \rangle = \frac{C_\theta}{\mu} \sin (2\mu p_\theta) \ z
\]

\[
\langle x \rangle = \frac{C_x [1 + \cosh (4\theta(z))] \sin (2\mu p_z) \ z}{\mu}
\]

for the Polymer case. In the figures (1) and (2) are shown the dynamics, in both cases we obtained formerly the same equations and the dynamics is the same. The Universe starts at a point with finite volume, evolves towards the potential wall and then goes toward the singularity of the model (\(z=0\)) and then goes straight into the singularity without the possibility to evade it. The anisotropies, instead, explode near the singularity and are practically null near the wall.

If now we integrate the time-variables defined in the Shrodinger equations above and substitute into the Hamilton equations we obtain a two new systems of equations, in particular we get with simple maths (shown in the appendix)

\[
z = \mp \frac{\text{Const}}{p_\theta}
\]

for the ordinary case and

\[
z = \mp \frac{\text{Const}}{p_\theta}
\]

for the Polymer case. If we insert this information in the Hamilton equations (67) and (68) we get the plots shown in figure (3) and (4) that allow us to state that the Taub model can be reduced to a singularity-free model with a cyclical behavior in both volume and anisotropies. In those graphs it’s possible to highlight the main differences between the two representations, in the ordinary case the singularity is unavoidable, while in the Polymer approach there is a periodic behavior of the Universe variables, and so the singularity is regularized.

VI. DISCUSSION

We now analyze the equations that we found in the previous section, in particular the Hamilton equations (55) (56) and (62) obtained in the two different cases. If we integrate those systems we obtain the following equations for the volume of the Universe \(\tau\), the scalar field \(\theta\) and the anisotropies \(x\)

\[
\langle \tau \rangle = -2p_z z
\]

\[
\langle \theta \rangle = -\frac{p_\theta z}{4}
\]

\[
\langle x \rangle = [1 + \cosh (4\theta(z))] p_z z
\]

for the ordinary case and

\[
\langle \tau \rangle = \frac{C_r}{\mu} \sin (2\mu p_\tau) \ z
\]

\[
\langle \theta \rangle = \frac{C_\theta}{\mu} \sin (2\mu p_\theta) \ z
\]

\[
\langle x \rangle = \frac{C_x [1 + \cosh (4\theta(z))] \sin (2\mu p_z) \ z}{\mu}
\]

for the Polymer case. In the figures (1) and (2) are shown the dynamics, in both cases we obtained formerly the same equations and the dynamics is the same. The Universe starts at a point with finite volume, evolves towards the potential wall and then goes toward the singularity of the model (\(z=0\)) and then goes straight into the singularity without the possibility to evade it. The anisotropies, instead, explode near the singularity and are practically null near the wall.

If now we integrate the time-variables defined in the Shrodinger equations above and substitute into the Hamilton equations we obtain a two new systems of equations, in particular we get with simple maths (shown in the appendix)

\[
z = \mp \frac{\text{Const}}{p_\theta}
\]
VII. CONCLUSION

We developed a technical algorithm to implement the WKB approach to the quantum Minisuperspace dynamics [13] within the Polymer representation of quantum mechanics [4]. One of the difficulties of the analysis above consisted in the necessity to deal with the momenta representation of the quantum dynamics, the only viable for the Polymer quantization procedure, as approached in the continuum limit. The point is that the potential term of the Minisuperspace Hamiltonian is, in general, not quadratic in the Minisuperspace variable, like the kinetic part is in the momenta.

We proposed a procedure to construct the semi-classical WKB limit in the momentum representation, which is, in principle, applicable to any Minisuperspace system. Such an algorithm has the aim to implement the concept of a cut-off on the quantum dynamics of the Universe, by separating the dynamics into a quasi-classical evolution of a set of configurational variables, e.g. the Universe volume, and those ones rapidly evolving in a fully quantum picture of the dynamics. According to the original idea proposed in [13], we arrive to define a Schroedinger-like equation for the quantum subsystem, allowing a consistent interpretation of the wavefunction.

Then, we applied the general procedure constructed above, to the particular case of a taub cosmology, as described in the framework of Misner-Chitré-like variables. We consider as quasi-classical variables the most closely resembling the Universe volume and a suitable function of the free massless scalar field included in the dynamics. As purely quantum variable, we adopt that one most closely resembling the Universe anisotropy.

As a result, we get a consistent cosmological picture, describing a cyclical Universe in which a quantum anisotropy exactly verifies the Ehrenfest theorem and is regularized, i.e. its amplitude is always finite. The obtained cosmological paradigm is of significant interest in view of constructing a realistic global (quantum and classical) dynamics of the Universe, being characterized by a regular minimum volume turning point (the Big-Bounce), where the possibility for an interpretation of the anisotropy wavefunction can be coherently pursued. Furthermore, such a resulting model has a maximum volume turning point, living in the pure classical region of the dynamics for all configurational coordinates and allowing for the emergence of cyclical closed Universe dynamics, slightly generalizing the positive curved Robertson-Walker geometry, but removing the singular point in which the Big-Bang takes place for the Standard Cosmological Model [1, 21, 22].

Appendix A: The equation for the Amplitude

From the WD equation the only terms which contribute to the equation for the amplitude of the Universe wavefunction come from, in a simplified notation,

\[ -p_\tau^2 - \frac{p_\theta^2}{8} \right] A\chi = 0 \tag{A1} \]
If we multiply by the identity defined in the paper we obtain

$$\left[-p^2_r - \frac{p^2_0}{8}\right] \partial^{-1} \left[ (\partial A) E \chi + A(\partial E) \chi + AE(\partial \chi) \right] = 0$$

(A2)

with some maths we get

$$\left[-p^2_r - \frac{p^2_0}{8}\right] [3AE \chi + (\partial A)(\partial^{-1} E) \chi + (\partial A)(\partial E) \chi + A(\partial E)(\partial^{-1} E) \chi + (\partial^{-1} A) E(\partial \chi) + A(\partial^{-1} E)(\partial \chi)] = 0.$$  

(A3)

Those are not the only terms, so we have to multiply again by the same identity, to make easier the calculations we define $\hat{P} = \left[-p^2_r - \frac{p^2_0}{8}\right]$, and so we get

$$\hat{P} \partial^{-1} [5(\partial A) E \chi + 5A(\partial E) \chi + 5AE(\partial \chi) + (\partial^2 A)(\partial^{-1} E) \chi + 2(\partial A)(\partial^{-1} E)(\partial \chi) + (\partial^2 A)(\partial^{-1} E) \chi] + + \hat{P} \partial^{-1} [(\partial^{-1} A) (\partial^2 E) \chi + 2(\partial^{-1} A) (\partial E) \chi + A(\partial E) (\partial^{-1} E) \chi + (\partial^{-1} A) E (\partial^2 \chi) + A(\partial^{-1} E) (\partial^2 \chi)] = 0$$

(A4)

and so we obtain

$$\hat{P} \left[15AE \chi + 8(\partial A)(\partial^{-1} E) \chi + 8(\partial A)(\partial E) \chi + 8(\partial^{-1} A)(\partial E) \chi + 8A(\partial E)(\partial^{-1} E) \chi + 8(\partial^{-1} A) E(\partial \chi) + 8A(\partial^{-1} E)(\partial \chi) \right] + + \hat{P} \left[(\partial^2 A)(\partial^{-2} E) \chi + 2(\partial^2 A)(\partial^{-1} E) \chi + 2(\partial A)(\partial^{-2} E)(\partial \chi) + (\partial^2 A)(\partial^{-2} E)(\partial \chi) + 2(\partial A)(\partial^{-1} E)(\partial^2 \chi) + (\partial^{-2} A)(\partial^2 E) \chi \right] + + \hat{P} \left[2(\partial^{-1} A)(\partial^2 E)(\partial^{-1} E) \chi + 2(\partial^{-2} A)(\partial E)(\partial \chi) + A(\partial^2 E)(\partial^{-1} E) \chi + (\partial^{-2} A)(\partial^2 E)(\partial \chi) + 2(\partial^{-1} A)(\partial^{-1} E)(\partial^2 \chi) + A(\partial^{-2} E)(\partial^2 \chi) \right] = 0$$

(A5)

From this equation, if we take only the dominant terms of the right order we find, with the due simplifications

$$-\hbar p^2_0 \left[(\partial A)(\partial^{-1} E)\right] - p^2_0 \left[(\partial^{-2} A)(\partial^2 S)E + 8(\partial^{-1} A)(\partial S)E \right] = 0$$

(A6)

that is the equation for the Amplitude of the Universe wave function that we were searching for.

---

**Appendix B: Momenta of the distribution**

The Ehrenfest equations state that the equations for the momenta are identically null in both cases, so we studied every order of their momenta distribution and find out that being zero the first order, in this case every order is null because if we calculate them we find

$$\langle \dot{p}_i \rangle = 0, \quad \sigma^2_{p_i} = \langle p^2_i \rangle - \langle p_i \rangle^2 = 0$$  

(B1)

The second order is null because the second term in the right-hand side is zero and for the first term the commutator between the variable and the Hamiltonian is null. Every other momenta of every order contains combinations of those terms and so they are identically null.

**Appendix C: Time Variable**

From the definition of the time variable in the Shrodinger we can easily get the relationship between it and the scalar field (used in this work as time variable).

We begin from the ordinary case

$$\frac{\partial}{\partial z} = -\frac{p^2_0}{\hbar p_0} \frac{\partial}{\partial \mu p_0}, \quad -\frac{dp}{2p^2} dz = z = \frac{\hbar}{\mu p_0}$$

(C1)

where we have exploited the Hamilton-Jacobi equation for $S$. We now derive the same relationship in the Polymer case.

$$\frac{\partial}{\partial z} = \frac{1}{\mu^2 \partial \mu p_0} \sin^2(\mu p_0) \frac{\partial}{\partial \mu p_0}, \quad \frac{\sin^2(\mu p_0)}{\mu^2} dz = (\partial_{\mu} S) dp_0 \Rightarrow \frac{\sin^2(\mu p_0)}{\mu^2} z = \frac{\hbar}{2 \sin(\mu p_0)}$$

(C2)

where again we inserted the Hamilton-Jacobi equation for $S$. 

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