1. INTRODUCTION. In this note we present solutions to two problems which appeared in the American Mathematical Monthly. Although it may appear that one problem is more general than the other, the two problems seem to cover different situations but both give sufficient conditions for a real valued function not to be periodic over the real line. However the two problems are related most intimately because they can be proved using essentially the same technique. In the end, we introduce a new problem which actually implies both. These problems appeared in ‘04–’05 and their solutions in ‘06 and ‘07 (see [4], [5], [7] and [10]). The proofs included here are based on a particular case of the well-known Stolz-Cesàro Lemma and on the fact that every continuous periodic function on $\mathbb{R}$ must be uniformly continuous. The use of the latter idea is not new as it was used in the published solutions of these problems. On the other hand, the use of Stolz-Cesàro Lemma, is just another good example where an old tool of analysis appears unexpectedly (see [2], [6], [8], [9], and [13]). L’Hospital’s rule, which is very well known to calculus students is its “differentiable” counterpart.

The version of Stolz-Cesàro Lemma we are going to employ here is stated next.

**Lemma 1.** Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $\{b_n\}$ is increasing and convergent to infinity. If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ then $\limsup_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \infty$.

We are going to include its classical idea of proof for completeness. Let us assume to the contrary that $\gamma := \limsup_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} < \infty$. Then for an arbitrary $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} \leq \gamma + \epsilon,$$

or

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\begin{align*}
a_{n+1} - a_n & \leq (b_{n+1} - b_n)(\gamma + \epsilon), \\
\text{for all } n \geq n_0. \text{ Adding up inequalities as in (1) for } n = k...l, \ l > k \geq n_0, \text{ we obtain} \\
a_{l+1} - a_k & \leq (b_{l+1} - b_k)(\gamma + \epsilon).
\end{align*}

Eventually \( b_{l+1} \) is going to be a positive number so we can divide the last inequality by \( b_{l+1} \) and then let \( l \to \infty \). Using the hypothesis we obtain \( \infty \leq \gamma + \epsilon \) which is a contradiction. □

Next, we are including the two original problems.

**Problem 11111.** Let \( f \) and \( g \) be nonconstant, continuous periodic functions mapping \( \mathbb{R} \) into \( \mathbb{R} \). Is it possible that the function \( h \) on \( \mathbb{R} \) given by \( h(x) = f(xg(x)) \) is periodic?

The second problem seems to be more general but it is not clear to us at this point if this is indeed the case. We are going to discuss the relationship between the two problems briefly but it is not our purpose to get into the details of a thorough analysis.

**Problem 11174.** Let \( f \) and \( g \) be nonconstant, continuous functions mapping \( \mathbb{R} \) into \( \mathbb{R} \) satisfying the following conditions:

1. \( f \) is periodic.
2. There is a sequence \( \{x_n\}_{n \geq 1} \) such that \( \lim_{n \to \infty} x_n = \infty \) and \( \lim_{n \to \infty} \left| \frac{g(x_n)}{x_n} - \frac{g(y_n)}{y_n} \right| = \infty \).
3. \( f \circ g \) is not constant on \( \mathbb{R} \).

Determine whether \( h = f \circ g \) can be periodic.

Both problems have a negative answer. The function \( h_1(x) = \sin(x \cos x) \) gives obvious choices for \( f \) and \( g \) that satisfy the conditions in the first problem but it does not seem to be an example (at least in an obvious way) good for the second problem. On the other hand the function \( h_2(x) = \sin(x^2) \) gives rise to an \( f \) and \( g \) that satisfy the conditions of the second problem but it is hard to imagine that \( h_2(x) = \hat{f}(x\hat{g}(x)) \) for some \( \hat{f} \) and \( \hat{g} \) nonconstant, continuous periodic functions. It is an interesting question whether or not, for example, \( h_1 \) can be covered by Problem 11111.

The conditions in Problem 11174 can be weakened to obtain:

**Theorem 1.** Let \( f \) and \( g \) be nonconstant, continuous functions mapping \( \mathbb{R} \) into \( \mathbb{R} \) and satisfying the following conditions:

(i) \( f \) is periodic.
(ii) there exist sequences \( \{x_n\}_{n \geq 1} \) and \( \{y_n\}_{n \geq 1} \) such that

\[ \inf_{n} |x_n - y_n| > 0 \quad \text{and} \quad \lim_{n \to \infty} \left| \frac{g(x_n) - g(y_n)}{x_n - y_n} \right| = \infty. \]
Under these assumptions the function \( h = f \circ g \) cannot be periodic.

2. SOME FACTS FROM REAL ANALYSIS. Let us begin with this next fact about continuous functions on compact sets (Theorem 4.19 in [12]).

Theorem 2. Every continuous function on a compact set in a metric space is uniformly continuous.

We recall that a function on some domain \( \mathcal{D}(f) \subset \mathbb{R} \) is uniformly continuous if for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for every \( x, y \in \mathcal{D}(f) \) for which \( |x - y| < \delta \) we have \( |f(x) - f(y)| < \varepsilon \).

An easy consequence of this theorem is:

Corollary 1. Every continuous periodic function on \( \mathbb{R} \) is uniformly continuous.

This can be seen by applying Theorem 2 to the restriction of a periodic continuous function on \( \mathbb{R} \) to the compact set \( [0, 2T] \) where \( T > 0 \) is a period of \( f \) and then taking the \( \delta' \) given by the theorem corresponding to an arbitrary \( \varepsilon \). This \( \delta' \) is good for the interval \( [0, 2T] \) as the theorem insures but then \( \delta := \min\{\delta', T\} \) is actually good for \( \mathbb{R} \) as one can easily verify.

The idea of our proofs is to show that the function \( h \) is not uniformly continuous. As a result of Corollary 1 we see that \( h \) cannot be periodic.

Let us see how Problem 11111 follows from Theorem 1. Since \( g \) is assumed to be continuous and periodic but not constant we can find \( a \) and \( b \) such that \( g(a) - g(b) \neq 0 \). Assume \( T > 0 \) is a period of \( g \). Then we consider \( x_n = a + nT \) and \( y_n = b + nT \). Then \( |x_n - y_n| = |a - b| > 0 \) and

\[
\lim_{n \to \infty} \frac{|x_n g(x_n) - y_n g(y_n)|}{|x_n - y_n|} = |a - b|^{-1} \lim_{n \to \infty} |ag(a) - bg(b) + nT(g(a) - g(b))| = \infty,
\]

which says that \( x \xrightarrow{g} xg(x) \) and \( f \) satisfy the conditions (i) and (ii) in Theorem 1 and so \( f \circ \hat{g} = h \) is not periodic. So we have a solution for Problem 11111.

To show that Problem 11174 follows from Theorem 1 we need the weaker version of the Stolz-Cesàro Lemma as stated in the introduction as Lemma 1.

Now, let us assume the \( f, g \) and \( \{x_n\} \) satisfy the conditions 1-3 in Problem 11174. We can find a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \), so that \( x_{n_{k+1}} - x_{n_k} \geq 1 \) for all \( k \), and for which either

\[
\lim_{k \to \infty} \frac{g(x_{n_k})}{x_{n_k}} = \infty \text{ or } \lim_{k \to \infty} \frac{g(x_{n_k})}{x_{n_k}} = -\infty.
\]

Without loss of generality we may assume the first situation because the other case is going to follow from this one by changing \( g \) with \(-g\) and
f with $x \xrightarrow{\text{odd}} f(-x)$ ($x \in \mathbb{R}$). By Lemma 1 we see that
\[ \limsup_{k \to \infty} \frac{g(x_{n_k+1}) - g(x_k)}{x_{n_k+1} - x_k} = \infty \]
which proves the existence of the two sequences in (ii) as in Theorem 1. Hence, Theorem 1 can be applied to $f$ and $g$ and get that $h = f \circ g$ is not periodic. This settles Problem 11174.

3. PROOF OF THEOREM 1. Let us start with $f$ and $g$ satisfying (i) and (ii) of Theorem 1. Because $g$ is continuous and by property (ii) we see that the interval $I_n := g([x_n, y_n])$ (or $I_n := g([y_n, x_n])$, for $n$ large enough, must be an interval that has length greater than the period $T$ of $f$. Hence the range of $f$ is the same as the range of $h = f \circ g$.

Since $f$ is assumed nonconstant then $h$ is nonconstant. Therefore we can choose $\alpha$ and $\beta$ such that $f(g(\alpha)) \neq f(g(\beta))$ and then we let $\epsilon_0 = |f(g(\alpha)) - f(g(\beta))| > 0$. As we said in the introduction the key idea is to prove that $h$ is not uniformly continuous. More precisely, we want to show that the definition of uniform continuity is not satisfied for this $\epsilon_0$.

We fix $n \in \mathbb{N}$ large enough to insure that $|I_n| > 2T$ and denote by $\sharp(g(\alpha))$ the number of integer values of $k$ for which $g(\alpha) + kT$ is in $I_n$. Then, it is easy to see that
\[ \sharp(g(\alpha)) > \frac{|g(x_n) - g(y_n)|}{T} - 1 > 1. \]

Similarly we denote by $\sharp(g(\beta))$, the number of integers $k$ for which $g(\beta) + kT$ is in $I_n$. Again, we have $\sharp(g(\beta)) > \frac{|g(x_n) - g(y_n)|}{T} - 1 > 1$.

It is clear that the values $g(\alpha) + kT$ ($k \in \mathbb{Z}$) interlace with those of $g(\beta) + kT$ ($k \in \mathbb{Z}$). Using again the fact that $g$ is continuous, by repeated application of the Intermediate Value Theorem we can find two finite sequences $u_k$ and $v_k$ in the interval $[x_n, y_n]$ (or $[y_n, x_n]$) both increasing and interlacing such that $g(u_k) = g(\alpha) + kT$ and $g(v_k) = g(\beta) + s_kT$ with $l_k, s_k \in \mathbb{Z}$. The number of the intervals of the form $[u_k, v_k]$ (or $[v_k, u_k]$), etc.) is at least
\[ M := \min[2(\sharp(g(\alpha)) - 1), 2(\sharp(g(\beta)) - 1)] \geq 2. \]

These intervals form a partition of a subinterval of $J_n := [x_n, y_n]$ (or $J_n := [y_n, x_n]$) of length $|x_n - y_n|$. It follows that at least one of these intervals has to have length less than or equal to $\frac{|x_n - y_n|}{M}$.

We denote such an interval by $[\zeta_n, \eta_n]$ and notice that
\[ |\zeta_n - \eta_n| \leq \frac{|x_n - y_n|}{M} < \frac{|x_n - y_n|}{2|g(x_n) - g(y_n)|/T} - 1 = \frac{1}{\frac{2}{T} \frac{|g(x_n) - g(y_n)|}{|x_n - y_n|} - \frac{4}{|x_n - y_n|}} \to 0 \text{ as } n \to \infty, \]
and \( |f(g(\zeta_n)) - f(g(\eta_n))| = \epsilon_0 \). For an arbitrary but fixed \( \delta > 0 \), we choose \( n \) even bigger so that \( |\zeta_n - \eta_n| < \delta \). This can be done because of (2). For such an \( n \) we still have \( |h(\zeta_n) - h(\eta_n)| \geq \epsilon_0 \) which proves that \( h \) is not uniformly continuous.

In the end we would like to leave the reader with a natural question: can Theorem 1 be generalized to almost periodic functions? There are various concepts of almost periodicity but we are going to include here as an example only Bohr’s definition:

A continuous real valued function \( F \) defined on \( \mathbb{R} \) is said to be almost periodic if for each \( \epsilon > 0 \) there exists an \( L > 0 \) such that every interval of length \( L \) contains an \( \epsilon \)-period, i.e. a number \( T \) such that \( |F(x + T) - F(x)| < \epsilon \) for all \( x \in \mathbb{R} \).

What we find encouraging when it comes to new developments, related to the above questions, is the fact that every almost periodic function is also uniformly continuous (3).

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