Asymptotically flat Fredholm bundles and assembly

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Almost flat finitely generated projective Hilbert C*-module bundles were successfully used by Hanke and Schick to prove special cases of the Strong Novikov Conjecture. Dadarlat later showed that it is possible to calculate the index of a K-homology class \( \eta \in K_*(M) \) twisted with an almost flat bundle in terms of the image of \( \eta \) under Lafforgue’s assembly map and the almost representation associated to the bundle. Mishchenko used flat infinite-dimensional bundles equipped with a Fredholm operator in order to prove special cases of the Novikov higher signature conjecture.

We show how to generalize Dadarlat’s theorem to the case of an infinite-dimensional bundle equipped with a continuous family of Fredholm operators on the fibers. Along the way, we show that special cases of the Strong Novikov Conjecture can be proven if there exist sufficiently many almost flat bundles with Fredholm operator.

To this end, we introduce the concept of an asymptotically flat Fredholm bundle and its associated asymptotic Fredholm representation, and prove an index theorem which relates the index of the asymptotic Fredholm bundle with the so-called asymptotic index of the associated asymptotic Fredholm representation.

1 Introduction

Since Novikov made his famous higher signature conjecture [Nov70], special cases of the signature were successfully proven using very different techniques. Already Lusztig [Lus72] realized that one can use index theory and families of flat vector bundles to prove Novikov’s conjecture for free abelian groups. Later, Mishchenko [Mis74] used families of infinite-dimensional flat bundles, equipped with Fredholm operators on the fibers, to prove that Novikov’s conjecture holds for nonpositively curved manifolds. Kasparov [Kas95] showed that the higher signature conjecture follows from what was later called the Strong Novikov Conjecture, namely the conjecture that the analytic assembly map
\( \mu_{BG}: K_0(BG) \rightarrow K_0(C^*G) \) is rationally injective, where \( C^*G \) is any C*-algebra completion of the complex group algebra \( CG \). This analytic assembly map is defined in terms of a certain flat Hilbert C*-module bundle, the Mishchenko bundle. Connes, Gromov, and Moscovici [CGM90] realized that it is sufficient to consider not only flat bundles, but bundles which have a very small curvature in order to prove Novikov’s higher signature conjecture.

Hanke and Schick [HS06; HS07; HS08; Han12] proved that also the Strong Novikov Conjecture holds if there is a sufficient supply of almost flat finitely generated projective Hilbert C*-module bundles.

Now let us consider the following situation: Let \( p: \widetilde{M} \rightarrow M \) be a covering space, and suppose that there exists a sequence of compactly supported Hilbert \( B_n \)-module bundles \( E_n \rightarrow M \) with compatible connections \( ^\nabla \) such that the curvatures of \( E_n \) tend to zero as \( n \) goes to infinity, and such that every \( E_n \) detects the lift \( p^* \text{ch}(\eta) \) of the Chern character of a K-homology class \( \eta \in K_*(M) \). In the case where all of the C*-algebras \( B_n \) are equal to \( \mathbb{C} \), Hanke and Schick [HS07] developed a method to construct finitely generated projective Hilbert C*-module bundles \( E_n \rightarrow M \) with compatible connection which detect \( \text{ch}(\eta) \) and whose curvatures tend to zero. Note that this is easy if \( M \rightarrow M \) is a finite cover: then one can simply take as fiber over \( x \in M \) the direct sum of the fibers of the pre-images \( \bar{x} \in p^{-1}(x) \). The construction is, however, not trivial at all if \( M \rightarrow M \) is an infinite cover.

Hanke’s and Schick’s construction does not work if the C*-algebras \( B_n \) are arbitrary C*-algebras. Indeed, their construction makes essential use of the algebra of trace-class operators on Hilbert spaces, i.e., on Hilbert \( C \)-modules, and for \( B_n \neq \mathbb{C} \) there is no appropriate replacement for this algebra. However, in this case one can still consider the bundles \( E_n \) which have as fibers the direct sums of the fibers of the pre-images. The fibers of these bundles will then be infinite-dimensional over the C*-algebra \( B_n \). Since \( E \) is assumed to be compactly supported, there is a trivialization at infinity which induces a map from \( E_n \) to a trivial bundle which is unitary modulo compact operators.

In other words, we obtain a graded Hilbert \( B_n \)-module bundle \( E_n \) together with an odd self-adjoint operator \( F_n: E_n \rightarrow E_n \) whose square equals the identity modulo a fiberwise compact operator. Furthermore, \( F_n \) commutes with parallel transport in \( E_n \), modulo compact operators because we assumed that parallel transport in \( E_n \) is trivial at infinity. Such a pair \( (E_n, F_n) \) will be called a Fredholm bundle, and the family \( (E_n, F_n)_{n \in \mathbb{N}} \) will be called an asymptotically flat Fredholm bundle because the curvatures of the bundles \( E_n \) tend to zero as \( n \) goes to infinity. Now one can consider the generalized Fredholm index \( \text{ind } F_n \in K_0(C(X) \otimes B_n) \), and observe that \( \langle \eta, \text{ind } F_n \rangle \neq 0 \) for all \( n \). We will prove in Theorem [Dadarlat] that in this situation again \( \mu_{\text{BS}_{1}(M;*)} \psi \eta \neq 0 \in K_*(C^*\pi_1(M;*)) \), which generalizes the results of Hanke and Schick.

Dadarlat proved that one can calculate the index of a K-homology class \( \eta \in K_0(M) \) twisted with an \( \epsilon \)-flat Hilbert C*-module bundle \( E \rightarrow M \) in terms of Lafforgue’s [Laf02b].

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1In particular, we assume that there is a trivialization at infinity such that parallel transport is trivial with respect to this trivialization.
ℓ¹-assembly map \( \mu^M_M \) and the almost representation associated to \( E \) via parallel transport. The point here is that \( \epsilon \) is a number which only depends on \( \eta \) and on a concrete representation of \( \mu^M_M(\eta) \) as formal difference of equivalence classes of projections, but which is independent of the bundle \( E \). We will generalize Dadarlat’s theorem to the case of almost flat Fredholm bundles in Theorem 10.3.

The key step in the proof of Theorem 9.2 and Theorem 10.3 is that one can associate to an asymptotically flat Fredholm bundle \( (E_n, F_n)_{n \in \mathbb{N}} \) an asymptotic Fredholm representation \( (W_n, \rho_n, \hat{F}_n)_{n \in \mathbb{N}} \) of the fundamental group \( \pi_1(M; \ast) \). Such an asymptotic Fredholm representation, with respect to a finite presentation \( \pi_1(M; \ast) = \langle L \mid R \rangle \), consists for every \( n \) of a graded Hilbert \( B_n \)-module \( W_n \), an odd operator \( \hat{F}_n \), and a group homomorphism \( \rho_n : \text{Fr}(L) \to L_{B_n}(W_n) \) defined on the free group generated by the set of generators \( L \), such that \( \rho_n \) and \( \hat{F}_n \) satisfy certain compatibility conditions, and such that \( \| \rho_n(r) - \text{id} \| \) tends to zero as \( n \to \infty \) for all relations \( r \in R \). If \( (E_n, F_n)_{n \in \mathbb{N}} \) is an asymptotically flat Fredholm bundle, then we take \( W_n \) to be the fiber of \( E_n \). The homomorphism \( \rho_n \) is defined using parallel transport in \( E_n \) as in [CGM90], and \( \hat{F}_n \) is the restriction of \( F_n \) to a fiber.

Let us assume now that all C*-algebras \( B_n \) are equal to a single C*-algebra \( B \)—this reduction step turns out to always be possible. We will then associate to an asymptotic Fredholm representation its asymptotic index

\[
\text{asind} \left( (W_n, \rho_n, \hat{F}_n)_{n \in \mathbb{N}} \right) \in D(\Sigma C^*\pi_1(M; \ast), B).
\]

Here \( D(-, -) \) denotes Thom森’s D-theory group [Tho03], a variant of Connes’s and Higson’s E-theory [CH90b]. D-theory is defined in terms of so-called discrete asymptotic homomorphisms which will make it a natural object to work with when we consider asymptotic representations.

In Theorem 8.1 we will show how one can calculate the index \( \text{ind} F_n \) for an asymptotically flat Fredholm bundle \( (E_n, F_n)_{n \in \mathbb{N}} \) with underlying C*-algebra \( B \) in terms of the asymptotic index of the associated asymptotic Fredholm representation, at least if \( n \) is sufficiently large. This calculation will then be used to prove our applications in Theorem 9.2 and Theorem 10.3.

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2 Almost flat Fredholm bundles

The key concept in this paper is that of an almost flat Fredholm bundle. In order to motivate the definition, consider a closed Riemannian manifold $M$. Let $E \rightarrow M$ be a smooth Hermitian vector bundle (i.e. a complex vector bundle equipped with a smoothly varying Hermitian inner product on the fibers), and let $\nabla$ be a connection on $E$ which is compatible with the metric in the sense that parallel transport preserves the inner product. Finally let $R^\nabla$ be the curvature tensor associated to $\nabla$. Thus,

$$R^\nabla(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]}s$$

for smooth sections $X, Y \in C^\infty(TM)$ and $s \in C^\infty(E)$. We define $\|R^\nabla\|$ to be the supremum of $\|R^\nabla(X,Y)s\|$ over all points $p \in M$, all pairs of orthonormal tangent vectors $X, Y \in T_pM$, and all $s \in E_p$ with $\|s\| = 1$. The value $\|R^\nabla\|$ has the following geometric significance [cf. Hun19, Proposition 2.7]:

**Proposition 2.1.** Let $E$ and $\nabla$ be as above. Let $f : D^2 \rightarrow M$ be a piecewise smooth map, and denote by $T : E_{f(1)} \rightarrow E_{f(1)}$ the parallel transport map along the curve $\tau \mapsto f(e^{2\pi i\tau})$. Then

$$\|T - \text{id}\| \leq \text{vol}(f) \cdot \|R^\nabla\|,$$

where

$$\text{vol}(f) = \int_{D^2} \|\partial_s f(s,t) \wedge \partial_t f(s,t)\| d(s,t)$$

is a number which depends only on $f$ and not on $E$ or $\nabla$.

Now suppose that $M$ is equipped with a smooth triangulation. This means that $M$ is homeomorphic to the geometric realization of a simplicial complex, and that the simplices are smoothly embedded in $M$. Recall that every point $p \in M$ can then be written uniquely as a convex combination

$$p = \sum_{v \in V_M} \lambda_v(p) \cdot v,$$

where $V_M$ denotes the set of vertices of $M$. Recall further that the open star around a vertex $v \in V_M$ is the set of all points $p \in M$ with $\lambda_v(p) > 0$.

We return to the Hermitian bundle $E \rightarrow M$ with compatible connection $\nabla$. For every vertex $v \in V_M$ we consider a smooth triangulation $\Phi_v : S_v \times \mathbb{C}^\ell \rightarrow E|_{S_v}$ which is constructed in such a way that $\Phi_v(x, \xi) \in E_x$ is obtained by parallel transporting $\Phi_v(v, \xi) \in E_v$ along the linear path joining $v$ and $x$. For $v, v' \in V_M$ we consider the transition function $\Psi_{v',v} : S_v \cap S_{v'} \rightarrow U(\ell)$. This function is defined by the equation

$$\Phi_v(x, \xi) = \Phi_{v'}(x, \Psi_{v',v}(x)\xi)$$

for all $x \in S_v \cap S_{v'}$ and $\xi \in W$. If $x' \in S_v \cap S_{v'}$ is another point, then $\Psi_{v',v}(x)^{-1}\Psi_{v',v}(x')$ is essentially given by parallel transport along the concatenation $\gamma$ of the linear paths.
joining \( v \) to \( x' \), \( x' \) to \( v' \), \( v' \) to \( x \), and \( x \) to \( v \). Obviously, \( \gamma \) bounds a disc whose area is bounded above by \( C d(x, x') \) for some constant \( C > 0 \) depending only on the Riemannian manifold \( M \) and the triangulation. Thus, Proposition 2.1 implies that the transition functions \( \Psi_{v', v} \) are \( C \| \mathcal{R} \| - \text{Lipschitz} \). In particular, the diameter of the images of \( \Psi_{v', v} \) is bounded by \( C' \| \mathcal{R} \| \) where \( C' \) does not depend on \( E \) or \( \nabla \). This motivates the following definition [cf. Hun19, Definition 2.3]:

**Definition 2.2.** An \( \epsilon \)-flat Hermitian vector bundle over a simplicial complex \( X \) is a Hermitian vector bundle \( E \to X \), together with trivializations \( \Phi_v : S_v \times \mathbb{C}^\ell \to E|_{S_v} \) over the open stars of \( X \), such that each of the images of the transition functions \( \Psi_{v', v} \) has its diameter bounded by \( \epsilon \).

Note that in contrast to [Hun19, Definition 2.3], we only demand a bound on the diameter of the image of \( \Psi_{v', v} \), rather than a bound the Lipschitz constants of \( \Psi_{v', v} \). Furthermore, we consider trivializations over the open stars \( S_v \) here, rather than trivializations over the simplices of \( X \).

**Remark 2.3.** These changes to the definition actually do not make a big difference. Indeed, as we will recall in section 5, an almost flat bundle (regardless of the precise definition) induces an almost representation of the fundamental group of the base space. On the other hand, given an almost representation of the fundamental group, by [Hun19, Theorem 8.7] one can construct an almost flat bundle in the sense of [Hun19, Definition 2.3] which has an almost representation close to the given almost representation. By [Hun19, Theorem 8.8] the newly constructed bundle is isomorphic to the bundle that we started with.

**Definition 2.2** can also be extended to the case of Hilbert C*-module bundles [Hun19]. Recall from [Sch05] that for a C*-algebra \( B \), a Hilbert \( B \)-module bundle over a space \( X \) is a fiber bundle \( E \to X \) with typical fiber a Hilbert \( B \)-module \( W \), and with structure group given by the isometric isomorphisms of \( W \). Now as before, we define an \( \epsilon \)-flat Hilbert \( B \)-module bundle over a simplicial complex \( X \) to be a Hilbert \( B \)-module bundle, together with trivializations over the open stars, such that the diameters of the images of the transition functions is uniformly bounded by \( \epsilon \). In [Sch05], a definition for connections on Hilbert C*-module bundles is given, and the above example from Riemannian geometry carries over to Hilbert \( B \)-module bundles.

We will also need to consider the case of graded Hilbert C*-module bundles, where the fibers are \( \mathbb{Z}_2 \)-graded Hilbert C*-modules, and where the local trivializations are assumed to preserve the grading.

If \( W, W' \) are Hilbert \( B \)-modules, we denote by \( \mathcal{L}_B(W, W') \) the set of adjointable operators \( W \to W' \), and by \( \mathcal{K}_B(W, W') \subset \mathcal{L}_B(W, W') \) the set of \( B \)-compact operators. As usual, we put \( \mathcal{L}_B(W) = \mathcal{L}_B(W, W) \) and \( \mathcal{K}_B(W) = \mathcal{K}_B(W, W) \). A generalized Fredholm operator is an operator \( F \in \mathcal{L}_B(W, W') \) such that \( F^* F - \text{id} \) and \( FF^* - \text{id} \) are compact.
If \( F \in \mathcal{L}_\mathcal{B}(W,W') \) is a generalized Fredholm operator, then we may consider the \( \mathbb{Z}_2 \)-graded Hilbert \( \mathcal{B} \)-module \( \hat{W} = W \oplus W' \). Then the odd self-adjoint operator
\[
\hat{F} = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix} \in \mathcal{L}_\mathcal{B}(\hat{W})
\]
satisfies \( \hat{F}^2 - \text{id} \in \mathcal{K}_\mathcal{B}(\hat{W}) \). Conversely, if \( \hat{F} \in \mathcal{L}_\mathcal{B}(\hat{W}) \) is an odd self-adjoint operator such that \( \hat{F}^2 - \text{id} \) is compact, then \( \hat{F} \) is of the form described above, for some generalized Fredholm operator \( F \in \mathcal{L}_\mathcal{B}(\hat{W}) \). This shows why it can be useful to consider graded Hilbert modules. Motivated by this observation, we can now formulate the definition of an \( \epsilon \)-flat Fredholm bundle.

**Definition 2.4.** An \( \epsilon \)-flat Fredholm bundle over a simplicial complex \( X \) consists of the following data:

- An \( \epsilon \)-flat graded Hilbert \( \mathcal{B} \)-module bundle \( E \to X \), where \( \mathcal{B} \) is a unital C*-algebra and where the typical fiber of \( E \) is a countably generated Hilbert \( \mathcal{B} \)-module \( W \),
- A map \( F_E : E \to E \) with the following property: For each vertex \( v \in V_X \) there exists a continuous map \( F_v : S_v \to \mathcal{L}_\mathcal{B}(W) \) such that
  \[
  F_E(\Phi_v(x,\xi)) = \Phi_v(x,F_v(x)\xi),
  \]
  and such that \( F_v \) takes values in the set of odd self-adjoint operators on \( W \).

Furthermore, we assume that \( F_v(x)^2 - \text{id} \) is compact for all \( x \in S_v \), and that \( F_v(x) - F_{v'}(x') \in \mathcal{K}_\mathcal{B}(W) \) for all \( v,v' \in V_X \), \( x \in S_v \), and \( x' \in S_{v'} \).

The motivating example for almost flat Hilbert module bundles can be extended to give important examples of almost flat Fredholm bundles. Indeed, let \( E \to M \) be a smooth Hilbert \( \mathcal{B} \)-module bundle over a smoothly triangulated manifold, equipped with a compatible connection \( \nabla \), and let \( F : E \to E \) be a smooth bundle map such that \( F^2 - \text{id} \) is fiberwise compact, and such that \( F \) commutes with parallel transport on \( E \) up to compact operators. We choose the trivializations \( \Phi_v : S_v \times W \to E|_{S_v} \) in such a way that for any two vertices \( v,v' \in V_M \), the map \( \Phi_v(v,-) \) is obtained from \( \Phi_{v'}(v',-) \) by parallel transport along a fixed smooth curve in \( M \). With these data, \( (E,F) \) becomes a \( C\epsilon \)-flat Fredholm bundle over \( M \).

### 3 The generalized Fredholm index

In this section, we will review the generalized Fredholm index. The key idea underlying the index is the following: If \( F \in \mathcal{L}_C(\mathcal{H},\mathcal{H}') \) is a Fredholm operator between two Hilbert spaces (i.e. Hilbert \( C \)-modules) \( \mathcal{H} \) and \( \mathcal{H}' \), then the kernel and the cokernel of \( F \) are finite-dimensional and the index of \( F \) is defined to be the number
\[
\text{ind } F = \dim \ker F - \dim \text{coker } F \in \mathbb{Z}.
\]
If the underlying C*-algebra is arbitrary, then various complications arise. Firstly, a generalized Fredholm operator \( F \in \mathcal{L}_B(W,W') \)
need not have finitely generated projective kernel and cokernel, although there always exists a compact perturbation $F'$ of $F$ such that $\ker F'$ and $\operatorname{coker} F'$ are finitely generated projective Hilbert $B$-modules. Secondly, while finitely generated projective Hilbert $C$-modules are classified by their dimension, the same is not true if $C$ is replaced by an arbitrary unital $C^*$-algebra $B$. However, the K-theory group $K_0(B)$—which can be defined to be the Grothendieck group of the semigroup of equivalence classes of finitely generated Hilbert $B$-modules—is a natural target for the index

$$\operatorname{ind} F = [\ker F'] - [\operatorname{coker} F'] \in K_0(B)$$

where $F' - F \in K_B(W, W')$ is compact and $\ker F'$ and $\operatorname{coker} F'$ are finitely generated projective Hilbert $B$-modules. It is well-known that $\operatorname{ind} F \in K_0(B)$ does not depend on the choice of compact perturbation $F'$ [Weg93, Chapter 17].

The generalized Fredholm index fits very well into the framework of Kasparov’s KK-theory, in a sense that we will describe next. Basic references for KK-theory include [Bla98; JT91]. KK-theory is a bivariant functor on the category of $C^*$-algebras, but we will only need to consider the case where the first of the $C^*$-algebras is equal to $C$. Thus, we will use the notation $KK(B)$ for the group which is usually denoted by $KK(C, B)$. The group $KK(B)$ is constructed from what we will call a Kasparov $B$-module.

For the rest of this section, we fix a unital $C^*$-algebra $B$.

**Definition 3.1.** A *Kasparov $B$-module* is a triple $(W, p, F)$ where

- $W$ is a graded countably generated Hilbert $B$-module,
- $p \in L_B(W)$ is an even projection, and
- $F \in L_B(W)$ is an odd operator,

such that $[p, F], p(F^2 - \operatorname{id}), p(F - F^*) \in K_B(W)$.

Denote by $I = [0, 1]$ the unit interval. A *homotopy* of Kasparov $B$-modules is a triple $(W, (p_\tau)_{\tau \in I}, (F_\tau)_{\tau \in I})$ where $\tau \mapsto p_\tau$ is a continuous path of even projections and $\tau \mapsto F_\tau$ is a continuous path of odd operators such that every $(W, p_\tau, F_\tau)$ is a Kasparov $B$-module. The Kasparov $B$-modules $(W, p_0, F_0)$ and $(W, p_1, F_1)$ are called *homotopic*.

A Kasparov $B$-module $(W, p, F)$ is called *degenerate* if $[p, F], p(F^2 - \operatorname{id}), p(F - F^*) = 0$. Two Kasparov $B$-modules $(W, p, F)$ and $(W', p', F')$ are called *equivalent* if there exists a unitary equivalence $U : W \to W'$ of graded Hilbert $B$-modules such that $p' = UpU^*$ and $F' = UFU^*$.

By Kasparov’s Stabilization Theorem [Weg93, Theorem 15.4.6], $W \oplus \mathcal{H}_B$ is unitarily isomorphic to $\mathcal{H}_B$, where $\mathcal{H}_B$ is the standard Hilbert $B$-module.

Thus, every equivalence class of Kasparov $B$-modules has a representative $(W, p, F)$ where $W \subset \mathcal{H}_B$ is a Hilbert $B$-submodule. We denote by $\mathcal{E}(B)$ the set of equivalence classes of Kasparov $B$-modules. Direct sum gives $\mathcal{E}(B)$ an abelian monoid structure.
with zero element given by \(0 = [(0, 0, 0)]\). We consider the equivalence relation \(\sim\) on \(\mathcal{E}(B)\) which is generated by homotopy and the addition of degenerate modules, and put \(KK(B) = \mathcal{E}(B)/\sim\). It is clear that the monoid structure on \(\mathcal{E}(B)\) induces a monoid structure on \(KK(B)\). In fact, \(KK(B)\) is an abelian group \([\text{Bla98}, \text{Chapter 17}]\).

There are various different ways in which the definition of the groups \(W, \ldots\) can be simplified, and the following is the general setup for these simplifications. Let \(S \subset \mathcal{E}(B)\) be a subsemigroup. By abuse of notation, we will write \((E, p, F) \in S\) whenever the class of \((E, p, F)\) in \(\mathcal{E}(B)\) is contained in \(S\). We define \(\sim_S\) to be the equivalence relation on \(S\) which is generated by homotopies \((W, (p_t), (F_t))\) such that all \((W, p_t, F_t) \in S\), and by addition of degenerate modules \((W, p, F) \in S\). Now we call \(S\) \emph{ample} if the natural map \(S/\sim_S \to \mathcal{E}(B)/\sim = KK(B)\), which is induced by the inclusion \(S \to \mathcal{E}(B)\), is bijective.

Consider the following subsemigroups:

- \(\mathcal{C}(B) \subset \mathcal{E}(B)\) is the set of equivalence classes of Kasparov \(B\)-modules of the form \((W, p, F)\) where \(F = F^*\) and \(\|F\| \leq 1\),
- \(\mathcal{H}(B) \subset \mathcal{E}(B)\) is the set of equivalence classes of Kasparov \(B\)-modules of the form \((\mathcal{H}_B, p, F)\)\(^4\),
- \(\mathcal{U}(B) \subset \mathcal{E}(B)\) is the set of equivalence classes of Kasparov \(B\)-modules of the form \((W, \text{id}, F)\),
- \(\mathcal{Q}(B) \subset \mathcal{E}(B)\) is the set of equivalence classes of Kasparov \(B\)-modules of the form \((W, p, F)\) with \(F = F^* = F^{-1}\).

The following statement is well-known, and possesses generalizations to the case of \(KK(A, B)\) for arbitrary \(C^*\)-algebras \(A\) \([\text{cf. Bla98, Chapter 17}]\).

**Proposition 3.2.** The subsemigroups \(\mathcal{C}(B), \mathcal{H}(B), \mathcal{U}(B), \) and all of their intersections, are ample. Similarly, \(\mathcal{Q}(B)\) and \(\mathcal{Q}(B) \cap \mathcal{H}(B)\) are ample. \(\square\)

In order to describe the relation to the generalized Fredholm index, we consider the description of \(KK(B)\) corresponding to the intersection \(\mathcal{U}(B) \cap \mathcal{C}(B) \cap \mathcal{H}(B)\). Thus, an element of \(KK(B)\) is represented by a triple \((\mathcal{H}_B, \text{id}, F)\) for some odd operator \(F \in \mathcal{L}_B(\mathcal{H}_B)\) such that \(F = F^*, \|F\| \leq 1\), and such that \(F^2 - \text{id}\) is compact. Thus,

\[
F = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix}
\]

for a generalized Fredholm operator \(F_0: \mathcal{H}_B \to \mathcal{H}_B\), where \(\mathcal{H}_B\) is the ungraded standard Hilbert \(B\)-module. The operator \(F\) is degenerate if and only if \(F_0\) is a unitary isomorphism. The following theorem is well-known.

**Theorem 3.3** ([\text{Weg93}, Theorem 17.3.11]). The map \(\text{ind}: KK(B) \to K_0(B)\) which associates to \([\mathcal{H}_B, \text{id}, F] \in KK(B)\) the index \(\text{ind} F_0 \in K_0(B)\) is a group isomorphism. \(\square\)

\(^4\)Note that \(\mathcal{H}_B \oplus \mathcal{H}_B \cong \mathcal{H}_B\) by Kasparov’s Stabilization Theorem, so that indeed \(\mathcal{H}(B)\) is a subsemigroup of \(\mathcal{E}(B)\).
Let $p \in M_n(B)$ be a projection. Then one easily calculates that $\text{ind}[pB^n \oplus 0, \text{id}, 0] = [p] \in K_0(B)$. This description of pre-images implies the following recognition principle for the generalized Fredholm index.

**Corollary 3.4.** If $\text{ind}': KK(B) \to K_0(B)$ is a group homomorphism which satisfies $\text{ind}'[pB^n \oplus 0, \text{id}, 0] = [p]$ for all projections $p \in M_n(B)$ and $n \in \mathbb{N}$, then $\text{ind} = \text{ind}'$.  

We will next use Corollary 3.4 to give a new description of the generalized Fredholm index. In order to do this, we use the description of $KK(B)$ via the ample submonoid $\mathcal{Q}(B) \cap \mathcal{H}(B)$. Thus, we represent an element of $KK(B)$ by a triple $(\mathcal{H}_B, p, F)$ where $F = F^* = F^{-1}$. Consequently, $F \in \mathcal{L}_B(\mathcal{H}_B)$ is an odd self-adjoint unitary, $p \in \mathcal{L}_B(\mathcal{H}_B)$ is an even projection, and $[F, p] \in K_B(\mathcal{H}_B)$. Such a triple is degenerate if and only if $[F, p] = 0$.

To any odd self-adjoint unitary $F \in \mathcal{L}_B(\mathcal{H}_B)$ we associate the $C^*$-algebra

$$Q_F = \{ x \in \mathcal{L}_B^{ev}(\mathcal{H}_B) : [F, x] \in K_B(\mathcal{H}_B) \}.$$  

Of course, $(\mathcal{H}_B, p, F)$ is a Kasparov $B$-module if and only if $p$ is a projection in $Q_F$. We can write

$$F = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix} \in \mathcal{L}_B(\mathcal{H}_B) = \mathcal{L}_B(H_B \oplus H_B)$$

for a unitary operator $F_0 \in \mathcal{L}_B(H_B)$. An element in $\mathcal{L}_B^{ev}(\mathcal{H}_B)$ is of the form $x = x_0 \oplus x_1$ for operators $x_0, x_1 \in \mathcal{L}_B(H_B)$, and $x_0 \oplus x_1 \in Q_F$ if and only if $F_0 x_0 F_0^* - x_1 \in K_B(H_B)$. This shows that there is a split short exact sequence

$$0 \longrightarrow K_B(H_B) \xrightarrow{i_F} Q_F \xrightarrow{\pi_F} \mathcal{L}_B(H_B) \longrightarrow 0$$

of $C^*$-algebras, where the maps are given by $i_F(x) = x \oplus 0$, $\pi_F(x \oplus y) = y$, and $s_F(y) = F_0 y F_0^* \oplus y$. Since the functor $K_0$ is split exact, there is an associated split exact sequence

$$0 \longrightarrow K_0(K_B(H_B)) \xrightarrow{(i_F)_*} K_0(Q_F) \xrightarrow{(\pi_F)_*} K_0(\mathcal{L}_B(H_B)) \longrightarrow 0$$

of $K$-theory groups. Then $(\pi_F)_* (\text{id} - (s_F)_*(\pi_F)_*) = 0$, so there exists a unique group homomorphism $\rho_F : K_0(Q_F) \to K_0(K_B(H_B))$ with $(i_F)_* \rho_F = \text{id} - (s_F)_*(\pi_F)_*$. Now for each projection $p \in Q_F$ we put

$$\text{ind}_p(F) = \rho_F[p] \in K_0(K_B(H_B)) \cong K_0(B).$$

We define $\text{ind}'(\mathcal{H}_B, p, F) = \text{ind}_p(F)$ for $(\mathcal{H}_B, p, F) \in \mathcal{Q}(B) \cap \mathcal{H}(B)$. The main result of this section will be that $\text{ind}': KK(B) \to K_0(B)$ is a well-defined group homomorphism and in fact that $\text{ind} = \text{ind}'$. We begin with a very useful easy observation.
Lemma 3.5. Let

\[
\begin{array}{c}
0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \xrightarrow{f} 0 \\
0 \rightarrow A' \xrightarrow{i'} B' \xrightarrow{\pi'} C' \xrightarrow{f} 0
\end{array}
\]

be a commutative diagram of split short exact sequences of abelian groups. Let \( \rho: B \rightarrow A \) be the unique homomorphism with \( i \rho = \text{id} - s \pi \), and let \( \rho': B' \rightarrow A' \) be the unique homomorphism with \( i' \rho' = \text{id} - s' \pi' \). Then \( \rho' f = f|_A \rho \).

Proof. We have \( i' \rho' f = f - s' \pi' f = f - s' \overline{f} \pi = f - f s \pi = f \rho = i' f|_A \rho \). The claim follows because \( i' \) is injective. \( \square \)

By applying Lemma 3.5 several times, one can prove straightforwardly that the map \( \text{ind}' \) is well-defined. For example, let us show that \( \text{ind}_{U^*pU}(U^*FU) = \text{ind}_p(F) \) whenever \( U \in \mathcal{L}_B(H_B) \) is an even unitary.

Since \( U \) is even, we can write \( U = U_0 \oplus U_1 \) for unitaries \( U_0, U_1 \in \mathcal{L}_B(H_B) \). Let \( \text{Ad}_{U_k}: \mathcal{L}_B(H_B) \rightarrow \mathcal{L}_B(H_B) \) be the maps given by \( \text{Ad}_{U_k}(x) = U_k^*xU_k \). Then we have a commutative diagram

\[
\begin{array}{c}
0 \rightarrow \mathcal{K}_B(H_B) \xrightarrow{i_F} Q_F \xrightarrow{\pi_F} \mathcal{L}_B(H_B) \xrightarrow{0} \\
0 \rightarrow \mathcal{K}_B(H_B) \xrightarrow{i_{U^*FU}} Q_{U^*FU} \xrightarrow{\pi_{U^*FU}} \mathcal{L}_B(H_B) \xrightarrow{0}
\end{array}
\]

of split short exact sequences of C*-algebras. Let \( \rho: \mathcal{K}_0(Q_F) \rightarrow \mathcal{K}_0(\mathcal{K}_B(H_B)) \) and \( \rho': \mathcal{K}_0(Q_{U^*FU}) \rightarrow \mathcal{K}_0(\mathcal{K}_B(H_B)) \) be the respective splitting homomorphisms. Then Lemma 3.5 shows that

\[
(\text{Ad}_{U_0})_* \text{ind}_p(F) = (\text{Ad}_{U_0})_* \rho[p] = \rho'(\text{Ad}_{U_0} \oplus \text{Ad}_{U_1})_*[p] = \rho'[U^*pU] = \text{ind}_{U^*pU}(U^*FU).
\]

The claim follows from the fact that \( (\text{Ad}_{U_0})_* \) equals the identity on \( \mathcal{K}_0(\mathcal{K}_B(H_B)) \).

Theorem 3.6. The maps \( \text{ind}: KK(B) \rightarrow K_0(B) \) and \( \text{ind}': KK(B) \rightarrow K_0(B) \) coincide.
Proof. By Corollary 3.3 it suffices to show that ind\([pB^n \oplus 0, id, 0]\) = \([p]\) if \(p \in M_n(B)\) is any projection. Therefore, we have to represent \([pB^n \oplus 0, id, 0]\) by a Kasparov \(B\)-module in \(Q(B) \cap \mathcal{H}(B)\). The construction goes as follows: Firstly, we add on the degenerate module \(((1-p)B^n \oplus 0, 0, 0)\) to obtain \((B^n \oplus 0, p, 0)\). Next, add on \((0 \oplus B^n, 0 \oplus 0, 0)\) and perturb compactly to obtain
\[
\left( B^n \oplus B^n, p \oplus 0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).
\]
Finally, stabilize by adding \((\mathcal{H}_B, 0, \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix})\), and obtain
\[
\left( (B^n \oplus H_B) \oplus (B^n \oplus H_B), (p \oplus 0) \oplus 0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).
\]
This is equivalent to \((\mathcal{H}_B, p' \oplus 0, F)\) for some \(p' \in K_B(H_B)\) such that \([p'] = [p] \in K_0(B)\). Now in the short exact sequence
\[
0 \longrightarrow K_0(K_B(H_B)) \xrightarrow{(i_F)_*} K_0(Q_F) \xrightarrow{(\pi_F)_*} K_0(L_B(H_B)) \longrightarrow 0
\]
we have \((i_F)_*[p'] = [p' \oplus 0] = (id - s_{P \pi_F})_*[p' \oplus 0] = (i_F)_* \text{ind}_{p' \oplus 0}(F)\), and therefore
\[
\text{ind}'[pB^n \oplus 0, id, 0] = \text{ind}_{p' \oplus 0}(F) = [p'] = [p] \text{ as claimed.}
\]

4 The index bundle

In this section, we will review the definition of the index of a bundle of Fredholm operators. We will then use Theorem 3.3 in order to give a different description of this index.

The general setup is as follows: Let \(X\) be a compact Hausdorff space, and let \(p: E \to X\), \(p': E' \to X\) be Hilbert \(B\)-module bundles. A Hilbert \(B\)-module bundle morphism from \(E\) to \(E'\) is a map \(F: E \to E'\) between the total spaces such that \(p' \circ F = p\), and such that for all local trivializations \(\Phi: U \times X \to E|_U\) and \(\Phi': U' \times X \to E'|_{U'}\) there exists a continuous map \(f: U \cap U' \to L_B(W, W')\) with
\[
f\Phi(x, \xi) = \Phi'(x, f(x)\xi)
\]
for all \(x \in U \cap U'\) and \(\xi \in W\). We denote by \(L_B(E, E')\) the set of Hilbert \(B\)-module bundle morphisms \(E \to E'\), and abbreviate \(L_B(E) = L_B(E, E)\). By definition, \(F \in L_B(E, E')\) restricts to adjointable maps \(F_x \in L_B(E_x, E'_x)\) on the fibers, and taking fiberwise adjoints defines a morphism \(F^* \in L_B(E', E)\). Obviously, \(id^* = id\) and \((GF)^* = F^* \circ G^*\). Of course, \(F: E \to E'\) is an isometric isomorphism if and only if \(F \in L_B(E, E')\), \(FF^* = id\), and \(F^*F = id\).
We will also need to consider the Hilbert $C(X;B)$-module $\Gamma(E)$ of continuous sections of $E \to X$. Postcomposition with $F \in \mathcal{L}_B(E,E')$ defines an adjointable operator $F_* \in \mathcal{L}_{C(X;B)}(\Gamma(E),\Gamma(E'))$, and its adjoint is given by postcomposition with $F^*$. Furthermore, $G_*F_* = (GF)_*$ and $\text{id}_* = \text{id}$. We will need the following observation, which is a simple consequence of compactness of $X$:

**Lemma 4.1.** Let $p: E \to X$ and $p': E' \to X$ be Hilbert $B$-module bundles over a compact Hausdorff space $X$, and let $G: E \to E'$ be a map such that every $x \in X$ possesses a neighborhood $U_x \subset X$, local trivializations $\Phi: U_x \times W_x \to E|U_x$, $\Phi': U'_x \times W'_x \to E'|U'_x$, and a continuous map $G_x: U_x \to \mathcal{K}_B(W_x,W'_x)$ with $G(\Phi_x(y,\xi)) = \Phi'_x(y,G_x(y)\xi)$ for all $y \in U_x$ and all $\xi \in W_x$. Then $G_* \in \mathcal{K}_{C(X;B)}(\Gamma(E),\Gamma(E'))$.

**Definition 4.2.** A bundle of (generalized) Fredholm operators over $X$ consists of a graded Hilbert $B$-module bundle $E \to X$, where $B$ is a unital $\mathbb{C}^*$-algebra and the fibers of $E$ are countably generated, and of an odd self-adjoint bundle morphism $F \in \mathcal{L}_B(E)$ with the property that $F^2 - \text{id}$ restricts to compact operators on all fibers.

It is clear that an $e$-flat Fredholm bundle, as defined before, is a bundle of Fredholm operators. Now if $(E,F)$ is a bundle of Fredholm operators, then Lemma 4.1 implies that $F^2_* - \text{id} \in \mathcal{K}_{C(X;B)}(\Gamma(E))$.

**Proposition 4.3.** Let $(E,F)$ be a bundle of Fredholm operators over a compact Hausdorff base space $X$. Then the triple $\hat{E} = (\Gamma(E),\text{id},F_*)$ is a Kasparov $C(X;B)$-module in the sense of definition 3.1, and therefore defines a class $[\hat{E}] \in KK(C(X;B))$. We define

$$\text{ind} F = \text{ind}([\hat{E}] \in K_0(C(X;B)).$$

**Proof.** Since $F_*$ is odd and self-adjoint, and since $F^2_* - \text{id}$ is compact, it only remains to prove that $\Gamma(E)$ is countably generated, which again is a straightforward consequence of the compactness of $X$.

The index $\text{ind} F$ can be described as follows: Since $F \in \mathcal{L}_B(E)$ is odd and self-adjoint, we can write

$$F = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix} \in \mathcal{L}_B\left(E^{(0)} \oplus E^{(1)}\right)$$

with respect to the grading $E = E^{(0)} \oplus E^{(1)}$. Thus $F_0 \in \mathcal{L}_B(E^{(0)},E^{(1)})$. One can now perturb $F_0$ compactly to a partial isometry $F_1 \in \mathcal{L}_B(E^{(0)},E^{(1)})$, so that $p = \text{id} - F_1^*F_1$ and $q = \text{id} - F_1F_1^*$ are fiberwise projections, whose images then form bundles $E_p = pE^{(0)}$ and $E_q = qE^{(1)}$ of finitely generated projective Hilbert $B$-modules over $X$. Now the definition of the generalized Fredholm index implies that

$$\text{ind} F = |E_p| - |E_q|,$$

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thus recovering the definition of the index bundle introduced by Jänich \cite{Jän65}.

By Theorem 3.6 we also have \( \text{ind} F = \text{ind}'[E] \in K_0(C(X; B)) \). In the remainder of this section, we will use this observation in order to give yet another description of the index of an \( \epsilon \)-flat Fredholm bundle over a simplicial complex. In order to do this, it is useful to modify the Kasparov \( C(X; B) \)-module \( E \) in such a way that the relevant index-theoretic information is completely contained in the projection entry. This is done by a well-known procedure \cite[cf. Bla98, Section 17]{Bla98}, which we will describe next.

We fix a unital C*-algebra \( B \), a countably generated graded Hilbert \( B \)-module \( W \), and an odd operator \( F \in \mathcal{L}_B(W) \). Let \( \phi: \mathbb{R} \to \mathbb{R} \) be the function given by

\[
\phi(t) = \begin{cases} 
-1, & t \leq -1, \\
t, & -1 \leq t \leq 1, \\
1, & t \geq 1.
\end{cases}
\]

Since \( \frac{1}{2}(F + F^*) \) is self-adjoint, we may define another operator \( C(F) = \phi(\frac{1}{2}(F + F^*)) \) by continuous functional calculus in the C*-algebra \( \mathcal{L}_B(W) \). It is easy to see that \( C(F) \) is an odd self-adjoint operator with \( \|C(F)\| \leq 1 \). We put

\[
Q(F) = \left( \begin{array}{cc} C(F) & \sqrt{1 - C(F)^2} \\
1 - C(F) & -C(F) \end{array} \right) \in \mathcal{L}_B(W \oplus W^{\text{op}}),
\]

which is an odd self-adjoint unitary\footnote{The graded Hilbert \( B \)-module \( W^{\text{op}} \) is given by the Hilbert \( B \)-module \( W \), but with opposite grading.} and

\[
U(F) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\
Q(F) \end{array} \right) \in \mathcal{L}_B ((W \oplus W^{\text{op}}) \oplus (W \oplus W^{\text{op}})^{\text{op}}),
\]

which is an even self-adjoint unitary. The crucial properties of this construction are summarized in the following proposition whose proof is straightforward.

**Proposition 4.4.** Let \( W \) and \( F \) be as above, and put

\[
T = \left( \begin{array}{cc} 0 & 1 \\
1 & 0 \end{array} \right) \in \mathcal{L}_B ((W \oplus W^{\text{op}}) \oplus (W \oplus W^{\text{op}})^{\text{op}}).
\]

Assume furthermore that \( H \in \mathcal{L}_B(W) \) is such that \( [H,F], [H,F^*], H(F^2 - \text{id}) \), and \( H(F^* - F) \) are all contained in \( \mathcal{K}_B(W) \). Then also the operators

\[
((H \oplus 0) \oplus 0) \cdot (U(F)^*TU(F) - (F \oplus (-F)) \oplus ((-F) \oplus F))
\]

and

\[
(U(F)^*TU(F) - (F \oplus (-F)) \oplus ((-F) \oplus F)) \cdot ((H \oplus 0) \oplus 0)
\]

are compact. In particular, the commutator \( [(H \oplus 0) \oplus 0, U(F)^*TU(F)] \) is compact. \( \square \)
Now let $B$ be a unital C*-algebra, and let $(E,F_E)$ be an $\epsilon$-flat Fredholm bundle over a finite simplicial complex $X$, where the fiber $W$ of $E$ is a countably generated Hilbert $B$-module. Let $V_X = \{v_1, \ldots, v_n\} \subset X$ be the (finite) set of vertices of $X$. We use the abbreviation $S_k = S_{v_k}$ for the open star around $v_k$. Consider the local trivializations $\Phi_k = \Phi_{v_k} : S_k \times W \to E|_{S_k}$ and the transition functions $\Psi_{jk} = \Psi_{v_j,v_k} : S_j \cap S_k \to \mathcal{L}_B(W)$.

Thus, 
$$\Phi_k(x, \xi) = \Phi_j(x, \Psi_{jk}(x)\xi)$$

for all $x \in S_j \cap S_k$ and $\xi \in W$. Similarly, we put $F_k = F_{v_k} : S_k \to \mathcal{L}_B(W)$, so that 
$$F_E(\Phi_k(x, \xi)) = \Phi_k(x, F_k(x)\xi)$$

for all $x \in S_k$ and $\xi \in W$. Further, we consider the operator $F = F_1(v_1) \in \mathcal{L}_B(W)$.

We fix an even isometric isomorphism $U : W \oplus \mathcal{H}_B \to \mathcal{H}_B$ (which exists by Kasparov’s Stabilization Theorem). Now we define

$$\Psi_{jk}(x) = U(\Psi_{jk}(x) \oplus 0)U^* \in \mathcal{L}_B(\mathcal{H}_B)$$

for $x \in S_j \cap S_k$, and

$$F' = U(F \oplus \text{id})U^* \in \mathcal{L}_B(\mathcal{H}_B).$$

Using the assumption that $F - F_k(x)$ is compact for all $k$ and all $x \in S_k$, one obtains

**Lemma 4.5.** For all $j$, all $k$, and all $x \in S_j \cap S_k$, the operator $[\Psi_{jk}(x), F']$ is compact. \( \square \)

We write $\mathcal{H}_B' = (\mathcal{H}_B \oplus \mathcal{H}_B'^{\text{op}}) \oplus (\mathcal{H}_B \oplus \mathcal{H}_B'^{\text{op}})^{\text{op}}$. Then the construction described above yields an even self-adjoint unitary $U(F') \in \mathcal{L}_B(\mathcal{H}_B')$, and we define

$$\Psi_{jk}''(x) = U(F')((\Psi_{jk}'(x) \oplus 0) \oplus 0)U(F')^* \in \mathcal{L}_B(\mathcal{H}_B')$$

for $x \in S_j \cap S_k$. Let $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{L}_B(\mathcal{H}_B')$ be as in Proposition 4.4. Since $[\Psi_{jk}'(x), F']$ is compact by Lemma 4.5, also $[U(F')^*\Psi_{jk}'(x)U(F'), (F' \oplus (-F')) \oplus ((-F') \oplus F')]$ is compact, and Proposition 4.4 implies that 

$$[U(F')^*\Psi_{jk}''(x)U(F'), U(F')^*TU(F')]$$

is compact as well. Thus, $[\Psi_{jk}''(x), T] \in \mathcal{K}_B(\mathcal{H}_B')$ for all $x \in S_j \cap S_k$.

We fix an even isometric isomorphism $V : \mathcal{H}_B' \to \mathcal{H}_B$ (which again exists since $\mathcal{H}_B'$ is countably generated), and consider $T' = VT'V^* \in \mathcal{L}_B(\mathcal{H}_B)$. Then $T'$ is an odd self-adjoint unitary, and we may consider the C*-algebra

$Q = Q_{T'} = \{ x \in \mathcal{L}_B^{ev}(\mathcal{H}_B) : [x, T'] \in \mathcal{K}_B(\mathcal{H}_B) \}$

as in the definition of $\text{ind}'$. Since we have seen that $[\Psi_{jk}''(x), T]$ is compact, it follows that

$$\tilde{P}^E(x) = \left( \sqrt{\lambda_j(x)\lambda_k(x)}(V\Psi_{jk}''(x)V^*) \right)_{j,k}$$
is contained in \( M_n(Q) \) for all \( x \in X \), where the \( \lambda_j \) are the barycentric coordinates on \( X \). In addition, \( \tilde{P}_E(x) \) is self-adjoint because \( \Psi''_{jk}(x)^* = \Psi''_{kj}(x) \) for all \( j, k \). Finally, a calculation shows that \( \tilde{P}_E(x)^2 = \tilde{P}_E(x) \), so that \( \tilde{P}_E(x) \) is a projection. Thus, \( \tilde{P}_E \) defines a projection in \( C(X; M_n(Q)) \cong M_n(C(X; Q)) \), and hence a class \([\tilde{P}_E] \in K_0(C(X; Q)) \cong K_0(C(X) \otimes Q)\). We have already seen that there is a split short exact sequence

\[
0 \longrightarrow \mathcal{K}_B(H_B) \xrightarrow{i_T} Q \xrightarrow{\pi_T} \mathcal{L}_B(H_B) \longrightarrow 0,
\]

so the sequence

\[
0 \longrightarrow C(X) \otimes \mathcal{K}_B(H_B) \xrightarrow{id \otimes i_T} C(X) \otimes Q \xrightarrow{id \otimes \pi_T} C(X) \otimes \mathcal{L}_B(H_B) \longrightarrow 0,
\]

is split exact as well because \( C(X) \) is nuclear. Thus, we get an associated split exact sequence in K-theory, and therefore a group homomorphism \( \rho_X : K_0(C(X) \otimes Q) \to K_0(C(X) \otimes \mathcal{K}_B(H_B)) \) such that \((id \otimes i_T)_* \circ \rho_X = \id - (id \otimes s_T \pi_T)_*\). Of course, \( C(X) \otimes \mathcal{K}_B(H_B) \) is naturally isomorphic to \( C(X; B) \otimes \mathcal{K} \), so in particular there is a natural isomorphism \( K_0(C(X) \otimes \mathcal{K}_B(H_B)) \cong K_0(C(X; B) \otimes \mathcal{K}) \cong K_0(C(X; B)) \). Now we obtain the following description of the index of the \( \epsilon \)-flat Fredholm bundle \((E, F_E)\).

**Theorem 4.6.** Under the above isomorphisms, \( \text{ind} F_E = \rho_X[\tilde{P}_E] \in K_0(C(X; B)). \)

**Proof.** From the definition and from Theorem 3.6 we have that \( \text{ind} F_E = \text{ind}'[\hat{E}] \). Thus, we will first replace \( \hat{E} \) by a Kasparov \( C(X; B) \)-module in \( \mathcal{Q}(C(X; B)) \cap \mathcal{H}(C(X; B)) \) in order to be able to calculate \( \text{ind}'[\hat{E}] \). In fact, consider

\[
T'_{X} = \id_X \times (T' \oplus \cdots \oplus T') \in \mathcal{L}_B(X \times (\mathcal{H}_B \oplus \cdots \oplus \mathcal{H}_B))
\]

where \( T' \in \mathcal{L}_B(\mathcal{H}_B) \) is as in the definition of the C*-algebra \( Q \).

One can now show that the triple

\[
\hat{E}' = (\Gamma(X \times (\mathcal{H}_B \oplus \cdots \oplus \mathcal{H}_B)), (\hat{P}_E)_*, (T'_{X})_*)
\]

is a Kasparov \( C(X; B) \)-module in \( \mathcal{Q}(C(X; B)) \cap \mathcal{H}(C(X; B)) \), and that \([\hat{E}] = [\hat{E}'] \in KK(C(X; B))\). Thus, \( \text{ind} F_E = \text{ind}'[\hat{E}'] \in K_0(C(X; B)) \).

It is then straightforward to relate this result to \( \rho_X[\tilde{P}_E] \). For details we refer to [Hun20, Theorem 4.5.7].

### 5 Asymptotic Fredholm representations

Up to now, we have not used that \( \epsilon \) in the definition of an \( \epsilon \)-flat Fredholm bundle is actually small. In this section, we will consider \( \epsilon \)-flat Fredholm bundles where \( \epsilon \) tends to zero in the sense of the following definition.
**Definition 5.1.** An asymptotically flat Fredholm bundle is a sequence of \( \epsilon_n \)-flat Fredholm bundles \((E_n, F_n)\) with the same underlying unital C*-algebra \(B\), such that \(\lim_{n \to \infty} \epsilon_n = 0\).

If \(E \to M\) is a flat smooth Hilbert \(B\)-module bundle with connection over a smooth manifold \(M\), then parallel transport along a curve \(\gamma\) is invariant under homotopy of \(\gamma\) relative to its endpoints. This construction yields a representation of the fundamental group of \(M\) on the fibers of \(E\). It turns out that a similar construction is feasible for asymptotically flat Fredholm bundles as well. This yields the concept of an almost representation associated to an almost flat bundle. This connection between almost flat bundles and almost representations has first been observed by Connes, Gromov, and Moscovici [CGM90], and was further analyzed by Manuilov and Mishchenko [MM01], Mishchenko and Teleman [MT05], Hanke [Han12], Carrión and Dadarlat [CD18], and the author of this paper [Hun19].

In order to describe the construction of the almost representation associated to an almost flat bundle, let \(E \to X\) be an \(\epsilon\)-flat Hilbert \(B\)-module bundle over a simplicial complex \(X\), with typical fiber \(W\). Consider vertices \(v_0\) and \(v_1\) which bound an edge in \(X\). In particular, \(\frac{1}{2}(v_0 + v_1) \in S_{v_0} \cap S_{v_1}\), and we may define the transport operator along the (oriented) edge \([v_0, v_1]\) to be the even unitary operator \(T_{(v_0, v_1)} = \Psi_{v_1, v_0}(\frac{1}{2}(v_0 + v_1)) \in \mathcal{L}_B(W)\).

We can also define transport along simplicial paths in \(X\). A simplicial path in \(X\) is a finite sequence \(\Gamma = (v_0, v_1, \ldots, v_k)\) of vertices in \(X\), such that \([v_i, v_{i+1}]\) is an edge for all \(i\). One should imagine \(\Gamma\) as the continuous path which is the concatenation of the linear paths joining \(v_i\) and \(v_{i+1}\). We define the transport operator along such a simplicial path \(\Gamma\) to be

\[
T_\Gamma = T_{(v_0, v_1)} \circ \cdots \circ T_{(v_{n-1}, v_n)} \in \mathcal{L}_B(W).
\]

Of course, each \(T_{(v_0, v_1)}\) and hence also each \(T_\Gamma\) is an isometric automorphism of \(W\). Furthermore, \(T_\Gamma\) is graded if \(E\) is a graded almost flat Hilbert \(B\)-module bundle.

If \(\Gamma = (v_0, \ldots, v_k)\) and \(\Gamma' = (v_k, \ldots, v_{k+\ell})\) are two simplicial paths such that the endpoint of \(\Gamma\) equals the starting point of \(\Gamma'\), then the concatenation of \(\Gamma\) and \(\Gamma'\) is the simplicial path \(\Gamma' \# \Gamma = (v_0, \ldots, v_{k+\ell})\). It is clear from the definition that \(T_{\Gamma' \# \Gamma} = T_{\Gamma'} T_\Gamma\). The opposite of a simplicial path \(\Gamma = (v_0, \ldots, v_n)\) is the simplicial path \(\bar{\Gamma} = (v_n, \ldots, v_0)\), and clearly \(T_{\bar{\Gamma}} = T_\Gamma^*\).

The most important feature of the transport operators is that they satisfy an analogue of Proposition 2.1.

**Theorem 5.2 (Hun19, Theorem 3.4).** Let \(\Gamma = (v_0, \ldots, v_n)\) be a contractible simplicial loop in \(X\), in the sense that the concatenation of linear paths joining \(v_i\) and \(v_{i+1}\) form a loop in \(X\) which is contractible. Then there are constants \(C(\Gamma) > 0\) and \(\delta(\Gamma) > 0\), depending only on \(X\) and \(\Gamma\), such that the following holds:

Let \(E \to X\) be an \(\epsilon\)-flat Hilbert \(B\)-module bundle where \(\epsilon \leq \delta\). Then the transport operator \(T_\Gamma \in \mathcal{L}_B(W)\) satisfies \(\|T_\Gamma - \text{id}\| \leq C \cdot \epsilon\). \(\square\)
Now suppose that $X$ is connected and that $\pi_1(X; v_0) = \langle L \mid R \rangle$ is finitely presented, where $v_0 \in X$ is a base vertex. This means that $L \subset \pi_1(X; v_0)$ is a finite subset which generates $\pi_1(X; v_0)$, and that $R \subset \text{Fr}(L)$ is a finite subset of the free group generated by $L$ such that the normal subgroup $\langle R \rangle \subset \text{Fr}(L)$ generated by $R$ equals the kernel of the projection $\text{Fr}(L) \to \pi_1(X; v_0)$.

It follows from the simplicial approximation theorem that every $g \in L$ can be represented by a simplicial loop $\Gamma_g$ in $X$, based at $v_0$. We fix a choice of these simplicial loops. Now suppose that $E \to X$ is an $\epsilon$-flat Hilbert $B$-module bundle over $X$, with typical fiber $W$. We consider the group homomorphism $\rho_E: \text{Fr}(L) \to U(W)$ which is uniquely determined by $\rho_E(g) = \Gamma_{\rho}(g)$ for all $g \in L \subset \text{Fr}(L)$. Here $U(W)$ is the group of isometric automorphisms of $W$.

**Proposition 5.3.** There exist constants $C, \delta > 0$ which depend on $X$, the basepoint $v_0$, the finite presentation $\pi_1(X; v_0) = \langle L \mid R \rangle$, and the representing simplicial loops $\Gamma_g$, but not on the $\epsilon$-flat bundle $E \to X$, such that $\|\rho_E(r) - \text{id}\| < C\epsilon$ for all $r \in R$ if $\epsilon \leq \delta$.

**Proof.** Write $r \in R$ as a product $r = g_1 \cdots g_n$ of elements $g_k \in L \cup L^{-1}$. Put $\Gamma_k = \Gamma_{g_k}$ if $g_k \in L$, and $\Gamma_k = \Gamma_{g_k^{-1}}$ otherwise. Then $\rho_E(r) = \Gamma_1 \ast \cdots \ast \Gamma_n$. Since $r$ is contained in the kernel of the map $\text{Fr}(L) \to \pi_1(X; v_0)$, the simplicial loop $\Gamma(r) = \Gamma_1 \ast \cdots \ast \Gamma_n$ is contractible. Therefore, Theorem 5.2 implies that $\|T_{\Gamma(r)} - \text{id}\| \leq C_r \cdot \epsilon$ if $\epsilon \leq \delta_r$, for some constants $C_r, \delta_r > 0$. The claim follows with $\delta = \min_{r \in R} \delta_r$ and $C = \max_{r \in R} C_r$.  

This property of $\rho_E$ can be formalized as follows:

**Definition 5.4 (CGM90).** An $\epsilon$-almost representation of a finitely presented group $G = \langle L \mid R \rangle$ on a Hilbert $B$-module $W$ is a group homomorphism $\rho: \text{Fr}(L) \to U(W)$ such that $\|\rho(r) - \text{id}\| \leq \epsilon$ for all $r \in R$.

Therefore, Proposition 5.3 can be reformulated by saying that $\rho_E: \text{Fr}(L) \to U(W)$ is a $C\epsilon$-representation if $E \to X$ is an $\epsilon$-flat bundle with $\epsilon \leq \delta$. We will mainly be interested in almost flat bundles and almost representations in the limit $\epsilon \to 0$, which can be formalized as follows.

**Definition 5.5.** An asymptotic representation $\text{MM01}$ of a finitely presented group $G = \langle L \mid R \rangle$ over the C*-algebra $B$ is a sequence $(W_n, \rho_n)_{n \in \mathbb{N}}$ of $\epsilon_n$-representations $\rho_n: \text{Fr}(L) \to U(W_n)$ with $\epsilon_n \to 0$, where the $W_n$ are all Hilbert $B$-modules.

Similarly, an asymptotically flat Hilbert $B$-module bundle over a simplicial complex $X$ is a sequence $(E_n)_{n \in \mathbb{N}}$ of $\epsilon_n$-flat Hilbert $B$-module bundles $E_n \to X$, such that $\epsilon_n \to 0$.

Now Proposition 5.3 implies that the almost representations associated to an asymptotically flat Hilbert $B$-module bundle $(E_n)_{n \in \mathbb{N}}$ form an asymptotic representation $(W_n, \rho_{E_n})_{n \in \mathbb{N}}$ of the fundamental group of $X$. Of course, this asymptotic representation...
depends on the choice of generating set $L$ and on the representing curves $\Gamma_g$. However, these choices lead to equivalent asymptotic representations in a sense that we will describe next.

**Lemma 5.6.** Let $G$ be a group with two finite presentations $G = \langle L_1 \mid R_1 \rangle$ and $G = \langle L_2 \mid R_2 \rangle$. For $k = 1, 2$ we denote by $\pi_k : \Fr(L_k) \to G$ the canonical projections. For $k = 1, 2$ and $n \in \mathbb{N}$ let $\rho_{k,n} : \Fr(L_k) \to U(W_n)$ be almost representations such that $(W_n, \rho_{k,n})_{n \in \mathbb{N}}$ are asymptotic representations for $k = 1, 2$. Then the following are equivalent:

1. There exist set-theoretic sections $s_k : G \to \Fr(L_k)$ of the projections $\pi_k$ such that
   \[
   \lim_{n \to \infty} \|\rho_{1,n}(s_1(g)) - \rho_{2,n}(s_2(g))\| = 0
   \]
   for all $g \in G$.

2. Equation (1) holds for all pairs of set-theoretic sections $s_k : G \to \Fr(L_k)$ of $\pi_k$.

**Proof.** The claim follows straightforwardly from the observation that
\[
\lim_{n \to \infty} \|\rho_{k,n}(r) - \id_{W_n}\| = 0
\]
for all $r \in \ker \pi_k$, which in turn follows from the fact that $r$ can be written as a product of conjugates of elements in $R \cup R^{-1}$. \hfill \qed

If two asymptotic representations $(\rho_{1,n})_{n \in \mathbb{N}}$ and $(\rho_{2,n})_{n \in \mathbb{N}}$ satisfy the two equivalent conditions of Lemma 5.6 then they are called *asymptotically equivalent*. It is clear from the definition that asymptotic equivalence is an equivalence relation.

**Proposition 5.7.** Let $X$ be a connected simplicial complex, let $v_0 \in X$ be a base vertex, and suppose that $G = \pi_1(X; v_0)$ has two finite presentations $G = \langle L \mid R \rangle$ and $G' = \langle L' \mid R' \rangle$. For each $g \in L$ let $\Gamma_g$ be a simplicial loop representing $g$, and for each $g' \in L'$ let $\Gamma'_{g'}$ be a simplicial loop representing $g'$.

Consider an asymptotically flat Hilbert $B$-module bundle $(E_n)_{n \in \mathbb{N}}$, and let $\rho_{E,n} : \Fr(L) \to U(W_n)$ and $\rho'_{E,n} : \Fr(L') \to U(W_n)$ be the associated almost representations. Then the asymptotic representations $(\rho_{E,n})_{n \in \mathbb{N}}$ and $(\rho'_{E,n})_{n \in \mathbb{N}}$ are asymptotically equivalent.

**Proof.** Choose sections $s : G \to \Fr(L)$ and $s' : G \to \Fr(L')$, and consider $g \in G$. Then there exist simplicial loops $\Gamma$ and $\Gamma'$ in $X$, both representing $g$, such that $\rho_{E,n}(s(g)) = T^n_\Gamma$ and $\rho'_{E,n}(s'(g)) = T'^n_{\Gamma'}$ for all $n \in \mathbb{N}$, where $T^n_\Gamma$ and $T'^n_{\Gamma'}$ denote the transport operators along $\Gamma$ and $\Gamma'$ in $E_n$, respectively. In particular, $\Gamma' \ast \Gamma$ is a contractible simplicial loop in $X$. Thus,
\[
\lim_{n \to \infty} \|\rho_{E,n}(s(g)) - \rho'_{E,n}(s'(g))\| = \lim_{n \to \infty} \|T^n_{\Gamma' \ast \Gamma} - \id\| = 0
\]
by Theorem 5.2. \hfill \qed
Now let us introduce the datum of a Fredholm operator into the picture of asymptotic representations.

**Definition 5.8.** Let $B$ be a unital $C^*$-algebra. An $\epsilon$-**Fredholm representation** of a finitely presented group $G = \langle L \mid R \rangle$ is an $\epsilon$-representation $\rho: Fr(L) \to U(W)$ by even operators on a graded Hilbert $B$-module $W$, together with an odd operator $F \in \mathcal{L}_B(W)$ such that $F^2 - \text{id}$, $F^* - F$, $[\rho(g), F]$, and $[\rho(g), F^*]$ are compact operators for all $g \in L$.

An asymptotic Fredholm representation of $G = \langle L \mid R \rangle$ is a sequence of $\epsilon_n$-Fredholm representations $(W_n, \rho_n, F_n)_{n \in \mathbb{N}}$, all of which have the same underlying unital $C^*$-algebra $B$, such that $\epsilon_n \to 0$.

Similarly, an asymptotically flat Fredholm bundle is a sequence $(E_n, F_n)_{n \in \mathbb{N}}$ of $\epsilon_n$-flat Fredholm bundles such that $\epsilon_n \to 0$.

Not surprisingly, we can associate to an almost flat Fredholm bundle over a finite connected simplicial complex $X$ an almost Fredholm representation of the fundamental group of $X$. Indeed, if $(E, F_E)$ is an $\epsilon$-flat Fredholm bundle, then we put $\hat{F}_E = F_{v_0}(v_0) \in \mathcal{L}_B(W)$, which is of course just the operator $F$ in the definition of the projection $\hat{P}_E$ which appeared in Theorem 4.6.

**Lemma 5.9.** If $\Gamma$ is an arbitrary simplicial path in $X$, then $[T_\Gamma, \hat{F}_E] \in K_B(W)$.

**Proof.** It suffices to prove this in the case where $\Gamma = (v, v')$. Now the statement follows immediately from the fact that $\hat{F} - F_v(\frac{1}{2}(v + v'))$ and $\hat{F} - F_{v'}(\frac{1}{2}(v + v'))$ are compact by definition of an almost flat Fredholm bundle.

Thus, $(W, \rho_E, \hat{F}_E)$ is a $C\epsilon$-Fredholm representation of $\pi_1(X; v_0)$ if $\epsilon$ is sufficiently small. In particular, $(W_n, \rho_n, \hat{F}_n)_{n \in \mathbb{N}}$ is an asymptotic Fredholm representation of $\pi_1(X; v_0)$ if $(E_n, F_n)_{n \in \mathbb{N}}$ is an asymptotically flat Fredholm bundle, where $W_n$ denotes the fiber of $E_n$ and where $\rho_n$ is the almost representation associated to $E_n$.

### 6 D-theory and E-theory

Later we will associate an asymptotic index to an asymptotically flat Fredholm representation. This asymptotic index takes its value in the Thomsen D-theory group $D(\Sigma C^*G, B)$ [Tho03]. We will recall the definition and a few basic properties of D-theory groups and the related E-theory groups of Connes and Higson in this section.

Let $B$ be a $C^*$-algebra, which we first assume to be separable. The discrete asymptotic algebra over $B$ is the $C^*$-algebra

$$\mathcal{A}_B = \frac{C_b(\mathbb{N}, B)}{C_0(\mathbb{N}, B)}$$
where \( C_b(\mathbb{N}, B) \) is the C*-algebra of bounded sequences in \( B \), and \( C_0(\mathbb{N}, B) \) is the ideal of those sequences which tend to zero at infinity. Obviously, \( A_\delta \) is an endofunctor on the category of C*-algebras. A **discrete asymptotic homomorphism** from \( A \) to \( B \) is a \( * \)-homomorphism \( A \to A_\delta B \). Two such discrete asymptotic homomorphisms \( f, g : A \to A_\delta B \) are called **asymptotically homotopic** if there is an asymptotic homomorphism \( H : A \to A_\delta IB \) with \( f = A_\delta \text{ev}_0 \circ H \) and \( g = A_\delta \text{ev}_1 \circ H \). Here \( IB = C([0,1], B) \) is the C*-algebra of continuous \( B \)-valued functions on the unit interval, and \( \text{ev}_\tau : IB \to B \) is the map given by evaluation at \( \tau \in [0,1] \).

We denote by \([A,B]_\delta\) the set of asymptotic homotopy classes of discrete asymptotic homomorphisms, and put

\[
D(A,B) = [\Sigma A \otimes K, \Sigma^2 B \otimes K]_\delta,
\]

where \( \Sigma A = C_0(\mathbb{R}; A) \cong C_0(\mathbb{R}) \otimes A \) is the suspension of \( A \) and \( K \) is the C*-algebra of compact operators on a separable Hilbert space.

Similarly, we recall the definition Connes’s and Higson’s E-theory groups \([CH90]\) as described in \([GHT00]\): The **asymptotic algebra** over \( B \) is the C*-algebra

\[
\mathcal{A}B = \frac{C_b(P,B)}{C_0(P,B)},
\]

where \( C_b(P,B) \) is the C*-algebra of bounded continuous \( B \)-valued functions on \( P = [0,\infty) \subset \mathbb{R} \), and \( C_0(P,B) \) is the ideal of those continuous functions which vanish at infinity. As before, an **asymptotic homomorphism** from \( A \) to \( B \) is a \( * \)-homomorphism \( A \to \mathcal{A}B \), and also asymptotic homotopy is defined as before. We denote by \([A,B]\) the set of asymptotic homotopy classes of asymptotic homomorphisms from \( A \) to \( B \), and put

\[
E(A,B) = [\Sigma A \otimes K, \Sigma B \otimes K].
\]

Note that there is a double suspension in front of \( B \) in the definition of \( D(A,B) \) but only a single suspension in the definition of \( E(A,B) \). The advantage of this definition is that one now gets associative products \( E(A,B) \times E(B,C) \to E(A,C), E(A,B) \times D(B,C) \to D(A,C), D(A,B) \times E(B,C) \to D(A,C), \) and \( D(A,B) \times D(B,C) \to D(A,C) \) \([Tho03]\), at least if all of the appearing C*-algebras are separable. All the composition products are written as \((f,g) \mapsto g \circ f\): For example, if \( f \in D(A,B) \) and \( g \in E(B,C) \) then we write \( g \circ f \in D(A,C) \) for their product. In addition, there are natural maps \( D(A,B) \to E(A,B) \) which are compatible with the products. Both the E-theory groups and the D-theory groups carry natural abelian group structures such that the various products are bilinear.

It should be noted that the construction of the products above require that the occurring C*-algebras are separable. However, the definition of the E-theory groups and the D-theory groups and all the products can be extended to non-separable C*-algebras by considering \([A,B] = \lim A'[A'B] \) where the limit is taken over the partially ordered set of all separable C*-subalgebras \( A' \subset A \), and by putting \( E(A,B) = [\Sigma A \otimes K, \Sigma B \otimes K] \)

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in this case. Of course, the two definitions agree for separable C*-algebras, and most of the properties (in particular, all properties that we will need) of the E-theory groups continue to hold in this setting. Analogously, we define $[A,B]_\delta = \lim_{A'} [A',B]_\delta$ and define $D(A,B)$ in terms of $[-,-]_\delta$ for non-separable C*-algebras. For details about this construction we refer to the author’s doctoral dissertation [Hum20, Section 3.7].

For a C*-algebra $B$ there is a natural map $\kappa_B : B \to A_B$ given by $\kappa_B(b) = [t \mapsto b]$. In particular, if $f : A \to B$ is an arbitrary *-homomorphism then there is an associated element $\kappa(f) \in E(A,B)$ given by

$$\kappa(f) = [\kappa_{\Sigma B \otimes K} \circ (\Sigma f \otimes \text{id}_K)].$$

If in addition $g : \Sigma B \otimes K \to A(\Sigma C \otimes K)$ is an asymptotic homomorphism, then also $g \circ (\Sigma f \otimes \text{id}_K) : \Sigma A \otimes K \to A(\Sigma C \otimes K)$ is an asymptotic homomorphism which represents an element in $E(A,C)$, and one can easily show, using the definition of the E-theory product, that

$$g \cdot \kappa(f) = [g \circ (\Sigma f \otimes \text{id}_K)]$$

in $E(A,C)$. Similarly, (2) holds in $D(A,C)$ if $g : \Sigma B \otimes K \to A(\Sigma C \otimes K)$ is a discrete asymptotic homomorphism.

Essentially by definition, D-theory and E-theory are stable in the following sense: Let $P \in K$ be any rank-one projection, and consider the map $f : B \to B \otimes K$ given by $f(b) = b \otimes P$. Then $f$ induces isomorphisms $E(A,B) \to E(A,B \otimes K)$, $E(B \otimes K,C) \to E(B,C)$ and $D(A,B) \to D(A,B \otimes K)$, $D(B \otimes K,C) \to D(B,C)$.

Both E-theory and D-theory allow the following tensor product construction: Let $A,B,C$ be arbitrary C*-algebras. Then there is a natural group homomorphism $\otimes \otimes \text{id}_C : E(A,B) \to E(A \otimes C,B \otimes C)$ where $\otimes$ denotes the maximal tensor product. This group homomorphism is defined as follows: Represent an element of the domain $E(A,B)$ by an asymptotic homomorphism $f : \Sigma A \otimes K \to A(\Sigma B \otimes K)$. Then $[f] \otimes \text{id}_C \in E(A \otimes C,B \otimes C)$ is the class which is represented by the asymptotic homomorphism

$$f \otimes \text{id}_C : \Sigma A \otimes C \otimes K \cong \Sigma A \otimes K \otimes C \to A(\Sigma B \otimes K) \otimes C \to A(\Sigma B \otimes C \otimes K).$$

In an analogous fashion one can also define $- \otimes \text{id}_C : D(A,B) \to D(A \otimes C,B \otimes C)$. Of particular importance are the suspension maps $\Sigma : E(A,B) \to E(\Sigma A,\Sigma B)$ and $\Sigma : D(A,B) \to D(\Sigma A,\Sigma B)$, which are isomorphisms by [GHT00, Proposition 6.17] and [Tho03, Theorem 4.2], respectively.

There is another important fact about E-theory which we will need in the following section.

**Proposition 6.1** ([CH90a, Proposition 5.1]). Let

$$0 \longrightarrow J \xrightarrow{i} A \xrightarrow{\pi} B \longrightarrow 0$$

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be a short exact sequence of C*-algebras, and let \( \sigma: B \to A \) be a splitting, i.e. a \(*\)-homomorphism such that \( \pi \circ \sigma = \text{id}_B \). Then there exists an element \( \sigma \in E(A, J) \) such that \( \sigma \cdot \kappa(\iota) = \kappa(\text{id}_J) \) and \( \kappa(\iota) \cdot \sigma + \kappa(s \circ \pi) = \kappa(\text{id}_A) \).

Finally, we will need a few concrete calculations of E-theory and D-theory groups, namely of the groups \( E(C, B) \) and \( D(\Sigma, B) \) where \( \Sigma = \Sigma C_0(\mathbb{R}) \) is the suspension algebra.

For E-theory, the relevant calculation goes back to [Ros82, Theorem 4.1] which far predates the development of E-theory. Let us assume for simplicity that \( B \) is unital. Let \( p \in M_n(B) \) be a projection. Then we can consider the \(*\)-homomorphism \( f_p: C \to B \otimes K \) which is determined by the property that \( f_p(1) = p \). Now we define a map

\[ \Phi_B: K_0(B) \to E(C, B) \]

by \( \Phi_B[p] = \kappa(f_p) \in E(C, B \otimes K) \cong E(C, B) \). For the proof of the following statement we refer to [GHT00, Proposition 2.19].

**Theorem 6.2.** The map \( \Phi_B \) is a group isomorphism for all unital C*-algebras \( B \).

For D-theory, there is a similar isomorphism (although the proof is more complicated), which was already hinted at in Thomsen’s paper [Tho03]. Namely, if \( B \) is unital then we define a map

\[ \Psi_B: \prod_{n \in \mathbb{N}} K_0(B) \to D(\Sigma, B) \]

by the prescription \( \Psi_B(([p_n])_{n \in \mathbb{N}}) = \left[ \Sigma^2 f_{(p_n)} \otimes \text{id}_K \right] \in [\Sigma^2 C \otimes K, \Sigma^2 B \otimes K \otimes K]_\delta = D(\Sigma, B \otimes K) \cong D(\Sigma, B) \), where \( f_{(p_n)}: C \to A_\delta(B \otimes K) \) is the unique discrete asymptotic homomorphism such that \( f_{(p_n)}(1) = [n \mapsto p_n] \). Thus, \( \Sigma^2 f_{(p_n)} \otimes \text{id}_K(\phi \otimes \psi \otimes T) = [n \mapsto \phi \otimes \psi \otimes p_n \otimes T] \) for all \( \phi, \psi \in \Sigma \) and \( T \in K \).

**Theorem 6.3 ([Hun20, Theorem 3.8.11]).** The map \( \Psi_B \) is a surjective group homomorphism with \( \ker \Psi_B = \bigoplus_{n \in \mathbb{N}} K_0(B) \), for all unital C*-algebras \( B \).

### 7 The asymptotic index of an asymptotic representation

In this section, we fix a unital C*-algebra \( B \) and an asymptotic Fredholm representation \((W_n, \rho_n, F_n)_{n \in \mathbb{N}}\). We want to construct the asymptotic index

\[ \text{asind} \left( (W_n, \rho_n, F_n)_{n \in \mathbb{N}} \right) \in D((\Sigma C^*G), B), \]

where \( C^*G \) is the maximal group C*-algebra of \( G \). The construction parallels the calculation of \( \text{ind} F_E \) for an almost flat Fredholm bundle \((E, F_E)\) in Theorem [43]. Firstly, we want to get rid of the Hilbert \( B \)-modules \( W_n \). In order to do this, we choose even isometric isomorphisms \( U_n: W_n \oplus \mathcal{H}_B \to \mathcal{H}_B \), which exist by Kasparov’s Stabilization Theorem. For all \( w \in \text{Fr}(L) \) we define

\[ \rho'_n(w) = U_n(\rho_n(w) \oplus 0)U^*_n \]
and
\[ F'_n = U_n(F_n \oplus \text{id})U_n^*. \]

Further, we define
\[ \rho_n^w(w) = U(F'_n)((\rho_n(w) + 0) \oplus 0)U(F'_n)^* \in \mathcal{L}(\mathcal{H}_B) \]
and
\[ \tilde{\rho}_n(w) = V\rho_n^w(w)V^* \in \mathcal{L}(\mathcal{H}_B) \]
where \( V : \mathcal{H}_B' \rightarrow \mathcal{H}_B \) is another even unitary isomorphism. Proposition 4.4 easily implies that \( [\rho_n^w(g), T] \) is compact for all \( g \in L \), where \( T = (\frac{0}{1}) \in \mathcal{L}(\mathcal{H}_B') \). As before, we put \( T' = VT \). Then
\[ \tilde{\rho}_n(w) \in Q = Q_{T'} = \{ x \in \mathcal{L}_B^{\text{even}}(\mathcal{H}_B) : [x, T'] \in \mathcal{K}_B(\mathcal{H}_B) \} \]
for all \( w \in \text{Fr}(L) \).

**Lemma 7.1.** There exists a unique *-homomorphism \( \rho : C^*G \rightarrow A_3Q \) such that
\[ \rho(\pi(w)) = [n \mapsto \tilde{\rho}_n(w)] \]
for all \( w \in \text{Fr}(L) \), where \( \pi : \text{Fr}(L) \rightarrow G \) is the canonical projection and where we identify \( \pi(w) \in G \) with its image in \( C^*G \).

**Proof.** Uniqueness is clear since \( C^*G \) is generated by the elements of \( G \subset C^*G \). For existence, we consider \( P_n = \tilde{\rho}_n(1) \in Q \). It follows from the definition of \( \tilde{\rho}_n \) and the fact that each \( \rho_n \) is a group homomorphism that \( \tilde{\rho}_n(ww') = \tilde{\rho}_n(w)\tilde{\rho}_n(w') \) for all \( w, w' \in \text{Fr}(L) \), and that \( \tilde{\rho}_n(w)^* = \tilde{\rho}_n(w^{-1}) \) since the analogous statement is true for \( \rho_n \). In particular, \( \tilde{\rho}_n(w)^*\tilde{\rho}_n(w) = \tilde{\rho}_n(1) = P_n = \tilde{\rho}_n(w)\tilde{\rho}_n(w)^* \), so that each \( \tilde{\rho}_n(w) \) is unitary in the unital \( C^* \)-algebra \( P_nQP_n \). Thus, each \( \tilde{\rho}_n : \text{Fr}(L) \rightarrow P_nQP_n \) is a unitary representation, so that \( \tilde{\rho} : \text{Fr}(L) \rightarrow P(A_3Q)P \), \( \tilde{\rho}(w) = [n \mapsto \tilde{\rho}_n(w)] \) is a unitary representation as well if we define \( P = [n \mapsto P_n] \in A_3Q \).

Now if \( r \in \ker \pi \) is arbitrary, we get that \( \tilde{\rho}(r) = [n \mapsto \tilde{\rho}_n(r)] = [n \mapsto P_n] = \tilde{\rho}(1) \) because \( \lim_{n \rightarrow \infty} \| \tilde{\rho}_n(r) - P_n \| = 0 \) by definition of an asymptotic representation. Therefore, \( \tilde{\rho} : \text{Fr}(L) \rightarrow P(A_3Q)P \) descends to a unitary representation \( G \rightarrow P(A_3Q)P \), which extends to a *-homomorphism \( \rho : C^*G \rightarrow P(A_3Q)P \subset A_3Q \) by the universal property of the maximal group \( C^* \)-algebra. It is clear that \( \rho(\pi(w)) = [n \mapsto \tilde{\rho}_n(w)] \) as required. \( \square \)

Now we put
\[ \hat{\rho} = \Sigma_2^d \rho \otimes \text{id}_\mathcal{K} : \Sigma_2^2C^*G \otimes \mathcal{K} \rightarrow \Sigma_2^2A_3Q \otimes \mathcal{K} \rightarrow A_3(\Sigma_2Q \otimes \mathcal{K}). \]
Therefore, \( \hat{\rho} \) represents an element \( [\hat{\rho}] \in [\Sigma_2^2C^*G \otimes \mathcal{K}, \Sigma_2^2Q \otimes \mathcal{K}]_\mathcal{K} = D(\Sigma_2^2C^*G, Q) \). We consider the split short exact sequence
\[ 0 \rightarrow \mathcal{K}_B(H_B) \xrightarrow{i_{T'}} Q \xrightarrow{\pi_{T'}} \mathcal{L}_B(H_B) \rightarrow 0. \]
By Proposition 6.1 there exists a class \( \sigma \in E(Q,K_B(H_B)) \) such that \( \kappa(i_{T'}) \bullet \sigma + \kappa(s_{T'}\pi_{T'}) = \kappa(id_Q) \) and \( \sigma \bullet \kappa(i_{T'}) = \kappa(id_{K_B(H_B)}) \).

**Definition 7.2.** The asymptotic index of the asymptotic Fredholm representation is defined to be
\[
\text{asind}((W_n,\rho_n,F_n)_{n \in \mathbb{N}}) = \sigma \bullet [\hat{\rho}] \in D(\Sigma C^*G,K_B(H_B)) \cong D(\Sigma C^*G,B).
\]

It is straightforward to show that \( \text{asind}((W_n,\rho_n,F_n)_{n \in \mathbb{N}}) \) is independent of the choices of isomorphisms \( U_n \) and \( V \), and that asymptotically equivalent asymptotic homomorphisms have the same asymptotic index.

## 8 The index of an asymptotically flat Fredholm bundle

We have gathered all the necessary preliminaries to formulate and prove our main theorem which relates the index of an asymptotically flat Fredholm bundle to the asymptotic index of the associated asymptotic Fredholm representation. Let \( X \) be a finite connected simplicial complex, let \( B \) be a unital C*-algebra, and let \((E_n,F_n)_{n \in \mathbb{N}}\) be an asymptotically flat Fredholm bundle over \( X \) with underlying C*-algebra \( B \). Choose a finite presentation \( G = \pi_1(X;v_0) = \langle L \mid R \rangle \) and simplicial loops \( \Gamma_g \) which represent the generators \( g \in L \). Let \((W_n,\rho_n,\hat{F}_n)_{n \in \mathbb{N}}\) be the asymptotic Fredholm representation which is associated to the asymptotically flat Fredholm bundle \((E_n,F_n)_{n \in \mathbb{N}}\) as described in section 5.

Consider the Mishchenko bundle
\[
M_X = (C^*G \times \tilde{X})/G,
\]
where \( \tilde{X} \) is the universal cover of \( X \) and where the quotient is taken by the diagonal action of \( G \) on the product \( C^*G \times \tilde{X} \). Then \( M_X \) is a Hilbert \( C^*G \)-module bundle over \( X \) and hence defines a class \([M_X] \in K^0(X;C^*G) \cong K_0(C(X) \otimes C^*G)\). Let \( \Phi = \Phi_{C(X) \otimes C^*G} : K_0(C(X) \otimes C^*G) \to E(C(X) \otimes C^*G) \) be the isomorphism of Theorem 6.2.

**Theorem 8.1.** Under the identification \( D(\Sigma,C(X) \otimes B) \cong \prod_{n \in \mathbb{N}} K_0(C(X) \otimes B) / \bigoplus_{n \in \mathbb{N}} K_0(C(X) \otimes B) \) of Theorem 6.2, the classes
\[
\left( \text{id}_{C(X)} \otimes \text{asind} \left( (W_n,\rho_n,\hat{F}_n)_{n \in \mathbb{N}} \right) \right) \bullet \Sigma \Phi[M_X] \in D(\Sigma,C(X) \otimes B)
\]
and
\[
[(\text{ind} F_n)_{n \in \mathbb{N}}] \in \prod_{n \in \mathbb{N}} K_0(C(X) \otimes B) / \bigoplus_{n \in \mathbb{N}} K_0(C(X) \otimes B)
\]
coincide.
Proof. We begin with a few simplifications. Denote by \( \Phi_{n,v}: S_v \times W_n \to E_n|_{S_v} \) the local trivializations of \( E_n \). Choose a maximal tree \( T \subset X \). For every vertex \( v \in V_X \) of \( X \) let \( \Gamma_v \) be the unique simple simplicial path connecting the base vertex \( v_0 \) to \( v \) inside \( T \), and put \( \Phi'_{n,v}(x,\xi) = \Phi_{n,v}(x,T_{v(x)},\xi) \). Then a straightforward calculation shows that replacing \( \Phi \) by \( \Phi' \) does not change the asymptotic Fredholm representation \( (W_n, \rho_n, \hat{F}_n)_{n \in \mathbb{N}} \), but all transport operators along paths in \( T \) are equal to the identity. Therefore, we may assume without loss of generality that parallel transport in \( T \) is trivial.

As a second simplification, note that we may choose a concrete presentation of the fundamental group and of representing loops in the definition of \( 5.7 \), and asymptotically equivalent asymptotic Fredholm representations have the same asymptotic index.

We will use the following presentation of \( G = \pi_1(X; v_0) \): For any two vertices \( v, w \in X \) such that \([v, w]\) is an edge in \( X \) we associate the simplicial loop \( \Gamma_{v,w} = \Gamma_v * (v, w) * \Gamma_w \) at \( v_0 \) and put \( \hat{\Gamma}_{v,w} \in \pi_1(X; v_0) \). Then \( L = \{ \hat{\Gamma}_{v,w} : v, w \} \) is a finite generating set for \( \pi_1(X; v_0) \), and it follows from the Seifert–van Kampen Theorem that this generating set admits a finite set \( R \) of relations.

Note that

\[
T_{\hat{\Gamma}_{(v,w)}} = T_{\hat{\Gamma}_v} T_{(v,w)} T_{\hat{\Gamma}_w} = T_{(v,w)} = \Psi_{w,v} \left( \frac{1}{2} (v + w) \right) \tag{3}
\]

if \( T \) denotes the transport operator in any of the bundles \( E_n \) because we assumed that transport in \( T \) is trivial.

Now let us return to the proof of the theorem. Consider the map \( \Psi = \Psi_{C(X) \otimes B}: \prod_{n \in \mathbb{N}} K_0(C(X) \otimes B) \to D(\Sigma, C(X) \otimes B) \). Then by Theorem 4.6 the statement of the theorem can be reformulated as

\[
\Psi \left( \left( \rho_X [\hat{P}_n] \right)_{n \in \mathbb{N}} \right) = \left( \text{id}_{C(X)} \otimes \text{asind} \left( (W_n, \rho_n, \hat{F}_n)_{n \in \mathbb{N}} \right) \right) \bullet \Sigma \Phi [M_X]. \tag{4}
\]

Recall that asind\((W_n, \rho_n, \hat{F}_n)_{n \in \mathbb{N}}\) = \( \sigma \bullet [\hat{\rho}] \) where \( \hat{\rho} = \Sigma^2 \rho \otimes \text{id}_K \) and \( \rho: C^* G \to A_\delta Q \) is such that \( \rho(\pi(w)) = [n \mapsto \hat{\rho}_n(w)] \) for all \( w \in \text{Fr}(L) \), and where \( \sigma \in E(Q, \mathcal{K}_B(H_B)) \) is the class associated to the split short exact sequence

\[
0 \longrightarrow \mathcal{K}_B(H_B) \longrightarrow Q \longrightarrow \mathcal{L}_B(H_B) \longrightarrow 0.
\]

It is straightforward to show that \( \text{id}_{C(X)} \otimes (\sigma \bullet [\hat{\rho}]) = (\text{id}_{C(X)} \otimes \sigma) \bullet (\text{id}_{C(X)} \otimes [\hat{\rho}]) \), so that the right hand side in (4) equals \( (\text{id}_{C(X)} \otimes \sigma) \bullet (\text{id}_{C(X)} \otimes [\hat{\rho}]) \bullet \Sigma \Phi [M_X] \). Consider the
of split short exact sequences of abelian groups. Since $\Psi$ is natural, the diagram commutes. Let $\rho'_X : D(\Sigma, C(X) \otimes Q)$ be the group homomorphism associated to the bottom sequence, so that $\rho'_X$ is uniquely determined by the equation

$$\kappa(id_{C(X)} \otimes i_T) \bullet \rho'_X(\eta) + \kappa(id_{C(X)} \otimes s_T \pi_{T'}) \bullet \eta = \eta$$

for all $\eta \in D(\Sigma, C(X) \otimes Q)$. However, this equation is fulfilled by the map which is given by postcomposition with $id_{C(X)} \otimes \sigma \in E(C(X) \otimes Q, C(X) \otimes K_B(H_B))$. Therefore, we must have $\rho'_X(\eta) = (id_{C(X)} \otimes \sigma) \bullet \eta$ for all $\eta \in D(\Sigma, C(X) \otimes Q)$. Now Lemma 3.5 implies that

$$\Psi \left( \left( \rho_X(\bar{P}_{E_n}) \right)_{n \in \mathbb{N}} \right) = \Psi \left( \prod_{n \in \mathbb{N}} \rho_X \left( \left( \bar{P}_{E_n} \right)_{n \in \mathbb{N}} \right) \right) = \rho'_X \circ \Psi \left( \left( \bar{P}_{E_n} \right)_{n \in \mathbb{N}} \right) = (id_{C(X)} \otimes \sigma) \bullet \Psi \left( \left( \bar{P}_{E_n} \right)_{n \in \mathbb{N}} \right).$$

Therefore, in order to prove (4) it suffices to prove that

$$\Psi \left( \left( \bar{P}_{E_n} \right)_{n \in \mathbb{N}} \right) = (id_{C(X)} \otimes \hat{\rho}) \bullet \Sigma \Phi[M_X] \in D(\Sigma, C(X) \otimes Q).$$

(5)

The left hand side in (5) is defined to be the class in $[\Sigma^2 C \otimes K, \Sigma^2 C(X) \otimes Q \otimes K \otimes K]_d \cong D(\Sigma, C(X) \otimes Q)$ of the discrete asymptotic homomorphism $\Sigma^2 g \otimes id_K$ where $g : C \to A_d(C(X) \otimes Q \otimes K)$ is determined by $g(1) = [n \mapsto \bar{P}_{E_n}]$.

On the other hand, the class $\Sigma \Phi[M_X] \in [\Sigma^2 C \otimes K, \Sigma^2 C(X) \otimes C^*G \otimes K \otimes K]_d \cong E(\Sigma, \Sigma C(X) \otimes C^*G)$ is given by the class $\kappa(\Sigma^2 f \otimes id_K)$ where $f : C \to C(X) \otimes C^*G$ is determined by $f(1) = P^{M_X}$ where $P^{M_X}$ is a projection in a matrix algebra over $C(X) \otimes C^*G$ whose image is isomorphic to $M_X$. In particular, equation (2) on page 21 implies that the right hand side in (5) is given by the class in $E(\Sigma, C(X) \otimes B \otimes K)$ of the discrete asymptotic homomorphism $\kappa(id_{C(X)} \otimes \hat{\rho} \otimes id_K) : \Sigma \Phi[M_X] \in \Sigma \Phi[M_X]$.

Therefore, in order to prove (5) it suffices to show that $g$ and $(id_{C(X)} \otimes \hat{\rho} \otimes id_K) \circ f : C \to A_d(C(X) \otimes Q \otimes K)$ are equal, or in other words that

$$id_{C(X)} \otimes \hat{\rho} \otimes id_K(P^{M_X}) = [n \mapsto \bar{P}_{E_n}] \in A_d(C(X) \otimes Q \otimes K).$$

(6)
Let $V_X = \{v_1, \ldots, v_n\}$ be the ordering of vertices of $X$ which goes into the definition of $\tilde{P}^{E_n}$. It is easy to see that one can take

$$P^{M_X}(x, -) = \left(\sqrt{\lambda_j(x)\lambda_k(x)g(v_1,v_2)}\right)_{j,k} \in M_n(\mathcal{L}(C^*G)),$$

as the definition of $P^{M_X}$, so that

$$\text{id}_{C(X)} \otimes \rho \otimes \text{id}_K(P^{M_X}) = \left[ n \mapsto \left( x \mapsto \left(\sqrt{\lambda_j(x)\lambda_k(x)\tilde{\rho}_n(g(v_1,v_2))}\right)_{j,k}\right) \right].$$

On the other hand, the definition of $\tilde{P}^{E_n}$ is

$$\tilde{P}^{E_n}(x, -) = \left(\sqrt{\lambda_j(x)\lambda_k(x)V\Psi''_{n,jk}(x)V^*}\right)_{j,k}.$$

Therefore, we have to prove that $\lim_{n \to \infty} \|\tilde{\rho}_n(g(v_1,v_2)) - V\Psi''_{n,jk}(x)V^*\| = 0$ uniformly in $x \in S_j \cap S_k$ for all $j, k$. However, a straightforward calculation using the definition of $\tilde{\rho}_n$ and $\Psi''_{n,jk}$ shows that

$$\|\tilde{\rho}_n(g(v_1,v_2)) - V\Psi''_{n,jk}(x)V^*\| = \|\rho_n(g(v_1,v_2)) - \Psi_{n,jk}(x)\|.$$

By the definition of $\rho_n$ and by (3) we have $\rho_n(g(v_1,v_2)) = \Psi_{n,jk}(\frac{1}{2}(v_j + v_k))$. Since $E_n$ is $\epsilon_n$-flat, we have

$$\|\tilde{\rho}_n(g(v_1,v_2)) - V\Psi''_{n,jk}(x)V^*\| \leq \epsilon_n \to 0.$$

This completes the proof of (6). \qed

9 The Strong Novikov Conjecture

In the last two sections of this paper, we will give two applications of Theorem 8.1. In this section, we will show how to use the theorem in order to prove special cases of the

Strong Novikov Conjecture. We fix a finite connected simplicial complex $X$.

One can define the $K$-homology groups of a compact space $X$ by $K_j(X) = E(C(X), \Sigma_j \mathbb{C})$ for $j \in \mathbb{N}$. Bott periodicity implies that $K_j(X) \cong K_{j+2}(X)$ for all $j$, so that one actually only needs to consider $K_0(X) = E(C(X), \mathbb{C})$ and $K_1(X) = E(C(X), \Sigma)$.

Recall from [Hig00, Section 4] that the (analytic) assembly map $\mu_X: K_*(X) \to K_*(C^*\pi_1(X; v_0))$ for the complex $X$ is defined by the equation

$$\Phi(\mu_X(\eta)) = (\text{id}_{C^*\pi_1(X; v_0)} \otimes \eta) \cdot \Phi[M_X]$$

where the maps $\Phi$ are the natural isomorphisms from Theorem 6.2 and where $[M_X] \in K_0(C^*\pi_1(X; v_0) \otimes C(X))$ is the class of the Mishchenko bundle. \footnote{The original definition of the analytic assembly map, in terms of KK-theory, is due to Kasparov [Kas95, Definition 9.2]. However, the two definitions agree by [Hig00, Section 4].}

The Strong Novikov
Conjecture states that $\mu_X \otimes Q: K_*(X) \otimes \mathbb{Q} \to K_*(C^*_\pi_1(X; v_0)) \otimes \mathbb{Q}$ is injective whenever $X \cong B \pi_1(X; v_0)$ is a classifying space for its fundamental group. The main theorem of this section will show how one can prove the Strong Novikov Conjecture in the presence of sufficiently many almost flat Fredholm bundles.

Recall that the Kronecker pairing relating K-homology and K-theory is defined as follows: If $\eta \in K_j(X)$ and $\xi \in K^0(B) = K_0(C(X) \otimes B)$ are arbitrary then we define $\langle \eta, \xi \rangle \in K_j(B)$ by

$$\langle \eta, \xi \rangle = (\eta \otimes \text{id}_B) \bullet \Phi(\xi) \in E(\Sigma, \Sigma^j B) \cong K_j(B).$$

**Proposition 9.1.** Let $X$ be a finite connected simplicial complex, let $B$ be a unital $C^*$-algebra, and let $(E_n, F_n)_{n \in \mathbb{N}}$ be an asymptotically flat Fredholm bundle over $X$, with underlying $C^*$-algebra $B$. Let $(W_n, \rho_n, \hat{F}_n)_{n \in \mathbb{N}}$ be the associated asymptotic Fredholm representation.

If $\eta \in K_0(X) = E(C(X), \mathbb{C})$ then

$$\Psi((\langle \eta, \text{ind} F_n \rangle)_{n \in \mathbb{N}}) = \text{asind} \left( (W_n, \rho_n, \hat{F}_n)_{n \in \mathbb{N}} \right) \bullet \Sigma \Phi(\mu_X(\eta)) \in D(\Sigma, B)$$

where $\Psi$ is as in Theorem 6.3. Similarly, if $\eta \in K_1(X) = E(C(X), \Sigma)$ then

$$\Psi((\langle \eta, \text{ind} F_n \rangle)_{n \in \mathbb{N}}) = \Sigma \left( \text{asind} \left( (W_n, \rho_n, \hat{F}_n)_{n \in \mathbb{N}} \right) \right) \bullet \Sigma \Phi(\mu_X(\eta)) \in D(\Sigma, \Sigma B).$$

**Proof.** In the case $\eta \in K_0(X)$ we consider the commuting diagram

$$
\begin{array}{ccc}
E(C(X), \mathbb{C}) & \xrightarrow{E(C^*G \otimes C(X), C^*G)} & E(\Sigma C^*G \otimes C(X), \Sigma C^*G) \\
\downarrow & & \downarrow \\
E(B \otimes C(X), \mathbb{C}) & \xrightarrow{\text{asind} \otimes \text{id}_{C(X)}} & D(\Sigma C^*G \otimes C(X), B) \\
\end{array}
$$

where the unlabeled arrows are tensor products with the corresponding identities, and the labeled arrows are composition products with the respective elements. Of course, here $\text{asind} = \text{asind}((W_n, \rho_n, \hat{F}_n)_{n \in \mathbb{N}})$ is the asymptotic index of the asymptotic Fredholm representation associated to the asymptotically flat Fredholm bundle $(E_n, F_n)_{n \in \mathbb{N}}$.

The assembly map is the composition along the top row, under the identifications $\Phi: K_0(X) \to E(C(X), \mathbb{C})$ and $\Phi: K_0(C^*G) \to E(C, C^*G)$. By associativity of the composition product, the composition along the bottom row is given by precomposition with the element

$$(\text{asind} \left( (W_n, \rho_n, \hat{F}_n)_{n \in \mathbb{N}} \right) \otimes \text{id}_{C(X)}) \bullet \Sigma \Phi[M_X] \in D(\Sigma, B \otimes C(X)).$$
which equals $\Psi[(\text{ind } F_n)]_{n \in \mathbb{N}}$ by Theorem 8.1. Now one can show [Hun20, Corollary 3.9.4] that $\eta \in E(C(X), \Sigma)$ is mapped to

$$(\text{id}_B \otimes \eta) \cdot \Psi[(\text{ind } F_n)]_{n \in \mathbb{N}} = \Psi[(\langle \eta, \text{ind } F_n \rangle)]_{n \in \mathbb{N}} \in D(\Sigma, B)$$

under the composition along the left and bottom arrows. Thus, commutativity of the diagram completes the proof in the case $\eta \in K_0(X)$.

In the case where $\eta \in K_1(X) = E(C(C(X)), \Sigma)$, one can carry out the same argument with the diagram

$$E(C(X), \Sigma) \xrightarrow{E(C \otimes C(X), \Sigma G \otimes C(X), \Sigma G \otimes \Phi[M_X]} E(C(X), \Sigma G \otimes \Phi[M_X])$$

$$E(\Sigma G \otimes C(X), \Sigma G \otimes \Phi[M_X]) \xrightarrow{\Sigma \Phi[M_X]} E(\Sigma G \otimes C(X), \Sigma G \otimes \Phi[M_X])$$

instead of the diagram for $K_0(X)$.

Our application to the Strong Novikov Conjecture is the following.

**Theorem 9.2.** Consider a finite connected simplicial complex $X$ and a $K$-homology class $\eta \in K_*(X)$. Assume that for each $\epsilon > 0$ there exists an $\epsilon$-flat Fredholm bundle $(E, F_E)$ over $X$, with arbitrary underlying unital C*-algebra $B$, such that $\langle \eta, \text{ind } F_E \rangle \neq 0$. Then the image of $\eta$ under the analytic assembly map $\mu_X: K_* (X) \to K_* (C^* \pi_1 (X; v_0))$ is nonzero.

**Remark 9.3.** The special case of Theorem 9.2 where the bundles $E$ are all finitely generated projective has been proved and used by Hanke and Schick [HS06; HS07; HS08; Han12]. Note that in this case $F_E$ does not carry any information.

Theorem 9.2 follows quite directly from Proposition 9.1 if the C*-algebras $B$ appearing in the statement of theorem are all equal to a single unital C*-algebra $B$. Therefore, the missing step towards the proof of Theorem 9.2 is provided by the following lemma.

**Lemma 9.4.** Let $f: A \to B$ be a (not necessarily unital) *-homomorphism between unital C*-algebras. Let $(E, F_E)$ be an $\epsilon$-flat Fredholm bundle over $X$, with underlying unital C*-algebra $A$. Then there exists an $\epsilon$-flat Fredholm bundle $(f_* E, f_* F_E)$ over $X$, with underlying unital C*-algebra $B$, such that

$$\text{ind}(f_* F_E) = (\text{id}_{C(X)} \otimes f)_* \text{ind } F_E \in K_0(C(X) \otimes B).$$

Before we prove the lemma, we show how it can be used to derive Theorem 9.2.
Proof of Theorem 9.2. By the assumptions, there exists a sequence $(E_n, F_n)_{n \in \mathbb{N}}$ of $\epsilon_n$-flat Fredholm bundles over $X$, with underlying $C^*$-algebras $B_n$, such that $\langle \eta, \text{ind } F_n \rangle \neq 0 \in K_*(B_n)$ for all $n \in \mathbb{N}$. Let $\iota_n: B_n \to \prod_{n \in \mathbb{N}} B_n = B$ be the inclusions. Then $\text{ind}((\iota_n)_* F_n) = (\text{id}_{C(X)} \otimes \iota_n)_* \text{ind } F_n$ by Lemma 9.4, so that

$$\langle \eta, \text{ind}((\iota_n)_* F_n) \rangle = \langle \eta, (\text{id}_{C(X)} \otimes \iota_n)_* \text{ind } F_n \rangle = (\iota_n)_* \langle \eta, \text{ind } F_n \rangle \in K_*(B).$$

These elements of $K_*(B)$ are nonzero because each $\iota_n: B_n \to B$ is split injective (a splitting is given by the projection $B \to B_n$), so that $(\iota_n)_*: K_*(B_n) \to K_*(B)$ is injective. Therefore, we may replace the $C^*$-algebras $B_n$ by the single $C^*$-algebra $B$. Then $(B, F_n)_{n \in \mathbb{N}}$ is an asymptotically flat Fredholm bundle, and $\Psi((\langle \eta, \text{ind } F_n \rangle)_{n \in \mathbb{N}}) \neq 0$, so that indeed $\mu_X(\eta) \neq 0$ by Proposition 9.1.

Proof of Lemma 9.4. Let $W$ be the typical fiber of $E$. We may consider the Hilbert $B$-module $f_s W = W \otimes_f B$ [JT91, Section 1.2]. As a set, we put $f_s E = \bigsqcup_{x \in X} f_s E_x$, and consider the local trivializations

$$f_s \Phi_v: S_v \times f_s W \to f_s E|_{S_v}$$

which are determined by $f_s \Phi_v(x, \xi \otimes b) = \Phi_v(x, \xi) \otimes b \in f_s (E_x)$ for all $x \in S_v, \xi \in W$, and $b \in B$. Similarly, we define $f_s F_E: f_s E \to f_s E$ by

$$f_s F_E|_{E_s \otimes_f B} = F_E|_{E_s \otimes f \text{id}}: E_x \otimes_f B \to E_x \otimes_f B.$$

It is straightforward to check that $(f_s E, f_s F_E)$ is an $\epsilon$-flat Hilbert $B$-module bundle. Furthermore, one can show that the Kasparov $C(X; B)$-module

$$(\Gamma(f_s E), \text{id}, (f_s F_E)_*)$$

is unitarily equivalent to $(\Gamma(E) \otimes \text{id} \otimes f C(X; B), \text{id}, (F_E)_* \otimes \text{id})$, so that indeed

$$\text{ind } f_s F_E = \text{ind}[(\Gamma(E) \otimes \text{id} \otimes f C(X; B), \text{id}, (F_E)_* \otimes \text{id}] = (\text{id} \otimes f)_* \text{ind } \Gamma(E), \text{id}, (F_E)_* = (\text{id} \otimes f)_* \text{ind } F_E$$

as claimed.

Remark 9.5. In [Han12, Theorem 3.9], Hanke formulated and proved Theorem 9.2 in the case of finite dimensional Hilbert module bundles. However, it was observed by Mario Listing that his proof needed the assumption that the transition functions of the Hilbert module bundles are Lipschitz with small Lipschitz constant. However, the proof of Theorem 9.2 does not need this assumption, so this clarifies that Theorem 3.9 of [Han12] is indeed correct under the assumptions stated there.

Note that a simpler way of clarifying this point is to use Remark 2.3, which shows that one can get transition functions with small Lipschitz constants when provided only with an almost flat bundle in the sense of this paper or in the sense of [Han12].
10 Dadarlat’s index theorem

In this section, we will show how to use the methods of the foregoing section in order to generalize a theorem of Dadarlat [Dad12, Theorem 3.2] We consider the involutive Banach algebra $\ell^1(G)$ which is the completion of the complex group algebra $\mathbb{C}G$ with respect to the 1-norm $\|\sum_{g \in G} \lambda_g \cdot g\| = \sum_{g \in G} |\lambda_g|$. 

This algebra has a property which makes it a very natural object to work with when studying almost flat bundles. Namely, if $f: G \to V$ is any bounded map into a Banach space $V$, there is a unique extension to a bounded linear operator $\hat{f}: \ell^1(G) \to V$. In addition, we will consider $\hat{f}^{(k)}: M_k(\ell^1(G)) \to M_k(V)$ which is defined by applying $\hat{f}$ on every matrix entry.

In particular, suppose that $\rho: Fr(L) \to L_B(W)$ is an $\epsilon$-representation of $G$ with respect to some finite presentation $G = \langle L \mid R \rangle$. Choose a set-theoretic section $s: G \to Fr(L)$ of the projection map $Fr(L) \to G$. Since $\rho: Fr(L) \to L_B(W)$ is an almost representation, in particular its image is contained in the set of unitary operators on $W$. Thus, $\|\rho(w)\| \leq 1$ for all $w \in Fr(L)$, so that $\rho \circ s: G \to L_B(W)$ is bounded. Thus, the construction above yields an extension $\tilde{\rho}: \ell^1(G) \to L_B(W)$ of $\rho \circ s$, and maps $\hat{\rho}^{(k)}: M_k(\ell^1(G)) \to M_k(L_B(W))$. We observe the following fact:

**Proposition 10.1.** Let $G = \langle L \mid R \rangle$ be a finitely presented group and let $s: G \to Fr(L)$ be a projection, and fix $\delta > 0$. Then there exists a number $\epsilon = \epsilon(L, R, s, p, \delta) > 0$ such that for every $\epsilon$-representation $\rho: Fr(L) \to L_B(W)$ of $\rho \circ s$, and maps $\hat{\rho}^{(k)}: M_k(\ell^1(G)) \to M_k(L_B(W))$ satisfies $\|\hat{\rho}^{(k)}(p)^2 - \hat{\rho}^{(k)}(p)\| < \delta$ and $\|\hat{\rho}^{(k)}(p)^* - \hat{\rho}^{(k)}(p)\| < \delta$. 

In particular, if $\delta > 0$ is small enough then Proposition 10.1 implies that the self-adjoint matrix $\tilde{\rho} = \frac{1}{2}(\hat{\rho}^{(k)}(p) + \hat{\rho}^{(k)}(p)^*)$ satisfies $\|\tilde{\rho}^2 - \tilde{\rho}\| < \frac{1}{4}$. Consider the function $\psi: \mathbb{R} - \{\frac{1}{2}\} \to \mathbb{R}$ which is constantly equal to 0 on $\mathbb{R}_{<1/2}$ and constantly equal to 1 on $\mathbb{R}_{>1/2}$. Then the element $\psi(\tilde{\rho}) \in M_k(L_B(W))$, defined by continuous functional calculus, is a well-defined projection. We put

$$\rho_{\#}(p) = \psi(\tilde{\rho})$$

in this case.

**Lemma 10.2.** Let $(W, \rho, F)$ be an $\epsilon$-Fredholm representation where $\epsilon > 0$ is so small that $\rho_{\#}(p)$ as above is defined. Then $[\rho_{\#}(p), F \oplus \cdots \oplus F] \in M_k(K_B(W))$.

**Proof.** Since $F^* - F$ is compact by assumption, the set of all matrices which commute with $F \oplus \cdots \oplus F$ up to $M_k(K_B(W))$ is a C*-subalgebra of $M_k(L_B(W))$, and in particular this set is closed under continuous functional calculus. It is thus enough to prove that $[\hat{\rho}^{(k)}(p), F \oplus \cdots \oplus F] \in M_k(K_B(W))$. Equivalently, one has to prove that all entries of $\hat{\rho}^{(k)}(p)$ commute with $F$ up to $K_B(W)$. Again, by the definition of $\hat{\rho}^{(k)}$ and by an approximation argument it suffices to prove that $[\hat{\rho}(g), F] \in K_B(W)$ for all $g \in G$. 

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However, $\hat{\rho}(q) = \rho(s(g))$ commutes with $F$ up to compact operators by definition of an almost Fredholm representation.

In particular, consider the Kasparov $B$-module $W^k = W \oplus \cdots \oplus W$. Then Lemma 10.2 shows that $(W^k, \rho#(p), F \oplus \cdots \oplus F)$ defines a Kasparov $B$-module and hence a class in $KK(B)$. We will prove a generalization of [Dad12, Theorem 3.2] which states that this construction relates to the pairing of a K-homology class with an almost flat Fredholm bundle.

Lafforgue [Laf02a; Laf02b] introduced the so-called $\ell^1$-assembly map

$$\mu^\ell_1: K_0(X) \to K_0(\ell^1(G))$$

which has the property that the inclusion $i_{\ell^1}: \ell^1(G) \to C^*G$ satisfies

$$\mu_X = (i_{\ell^1})_* \circ \mu^\ell_1: K_0(X) \to K_0(C^*G).$$

For a proof of this equation, one may, for instance, replace $C^*_r(G,B)$ by $C^*G$ in [Laf02b, Proposition 1.7.6]. Now we can state and prove the main theorem of this section.

**Theorem 10.3.** Let $\eta \in K_0(X)$ be a K-homology class of a finite connected simplicial complex $X$ with finitely presented fundamental group $G = \pi_1(X; v_0) = \langle L | R \rangle$. Choose simplicial loops $\Gamma_g$ representing the generators $g \in L$. Let $p, q \in M_k(\ell^1(G))$ be projections such that $\mu^\ell_1(\eta) = [p] - [q] \in K_0(\ell^1(G))$. Then there exists a number $\epsilon > 0$ such that the following holds:

Let $(E, F_E)$ be an $\epsilon$-flat Fredholm bundle over $X$, with arbitrary underlying unital C*-algebra $B$, and let $(W, \rho, F)$ be the associated almost Fredholm representation. Then $\rho#(p)$ and $\rho#(q)$ are defined, and

$$\langle \eta, \text{ind } F_E \rangle = \text{ind} \left( [W^k, \rho#(p), F \oplus \cdots \oplus F] - [W^k, \rho#(q), F \oplus \cdots \oplus F] \right)$$

in $K_0(B)$, where $\text{ind}: KK(B) \to K_0(B)$ is the index isomorphism.

**Proof.** If $(W, \rho, F)$ is an $\epsilon$-representation of $G$ over a C*-algebra $B$ and $p \in M_k(\ell^1(G))$ is a projection, then we abbreviate

$$(W, \rho, F)#(p) = \text{ind}[W^k, \rho#(p), F \oplus \cdots \oplus F] \in K_0(B),$$

provided that $\epsilon$ is small enough such that the right hand side is defined. The proof of the theorem proceeds by contradiction. Thus, we assume that there is an asymptotically flat Fredholm bundle $(E_n, F_n)_{n \in \mathbb{N}}$ over $X$ with associated asymptotic Fredholm representation $(W_n, \rho_n, F_n)_{n \in \mathbb{N}}$ such that

$$\langle \eta, \text{ind } F_n \rangle \neq (W_n, \rho_n, F_n)#(p) - (W_n, \rho_n, F_n)#(q) \in K_0(B)$$

(7)
We define a $*$-homomorphism $f$.

Then by definition we have $\tilde{\rho}$.

Let us write $H$.

and $L$.

Let us first calculate the right hand side of (4). In order to do this, we need to analyze $W$.

the classes $(\mathcal{F}_n, \hat{n})$, $\hat{\mathcal{F}}_n$, and $T'$ as in the definition of the asymptotic index of an asymptotic Fredholm representation.

We define a $*$-homomorphism $f : \mathcal{L}_B(W) \to \mathcal{L}_B(H_B)$ by

$$f(F) = VU(\hat{F}_n)(U_n(F \oplus 0)U_n^* \oplus 0)U(\hat{F}_n)^*V^*.$$  

Then by definition we have $\tilde{\rho}_n(w) = f(\rho_n(w))$ for all $w \in \text{Fr}(L)$, and therefore

$$\hat{\rho}_n(p) = \text{id}_{M_k} \oplus f(\rho_n(p)).$$

Let us write $\rho_n(p) = (p_{j,l})_{j,l} \in M_k(\mathcal{L}_B(W))$. We abbreviate $p'_{j,l} = U_n(p_{j,l} \oplus 0)U_n^* \in \mathcal{L}_B(W)$, $\hat{F}_n'' = (\hat{F}_n \oplus (-\hat{F}_n)) \oplus (\hat{F}_n^* \oplus \hat{F}_n^*) \in \mathcal{L}_B(H_B)$, and $\tilde{F}_n = VU(\hat{F}_n)\hat{F}_n''U(\hat{F}_n)^*V^*$.

Then

$$[(W_n)^k, \rho_n(p), \hat{F}_n \oplus \cdots \oplus \hat{F}_n] = [(H_B)^k, \hat{\rho}_n(p), \tilde{F}_n \oplus \cdots \oplus \tilde{F}_n]$$

in $KK(B)$. Furthermore, it can be shown straightforwardly that $T' \oplus \cdots \oplus T'$ is a compact perturbation of $([H_B]^k, \tilde{\rho}_n(p), \tilde{F}_n \oplus \cdots \oplus \tilde{F}_n)$. In summary,

$$\begin{align*}
(W_n, \rho_n, \hat{F}_n) & = \text{ind}[(W_n)^k, \rho_n(p), \hat{F}_n \oplus \cdots \oplus \hat{F}_n] \\
& = \text{ind}([H_B]^k, \tilde{\rho}_n(p), T' \oplus \cdots \oplus T').
\end{align*}$$

A priori, each $E_n$ is a Hilbert $B_n$-module bundle where $B_n$ depends on $n$. However, as in the proof of Theorem 9.2 we may replace each $B_n$ by $B = \prod_{n \in \mathbb{N}} B_n$.  

for all $n \in \mathbb{N}$, where each $E_n$ is a Hilbert $B$-module bundle for a unital C*-algebra $B$.
where we used that $\text{ind} = \text{ind}'$ by Theorem 3.6. This index can be calculated in way which is similar to the calculation of $\text{ind}''[E']$ in the proof of Theorem 4.6. This calculation yields

$$(W_n, \rho_n, \tilde{F}_n)(p) = \rho[\tilde{\rho}_n(p)].$$

We refer to [Hun20, Theorem 5.2.4] for details.

Let $\sigma \in E(Q, \mathcal{K}_B(H_B))$ be the $\mathcal{E}$-theory element associated to the sequence

$$0 \longrightarrow \mathcal{K}_B(H_B) \longrightarrow Q \longrightarrow \mathcal{L}_B(H_B) \longrightarrow 0.$$  

by Proposition 6.1. As in the proof of Theorem 8.1 it follows from Lemma 8.3 and from the naturality of $\Psi$ that

$$\Psi \left(\left((W_n, \rho_n, \tilde{F}_n)(p)\right)_{n \in \mathbb{N}}\right) = \Psi \left(\left(\rho[\tilde{\rho}_n(p)]\right)_{n \in \mathbb{N}}\right) = \sigma \cdot \Psi \left(\left(\tilde{\rho}_n(p)\right)_{n \in \mathbb{N}}\right)$$

in $D(\Sigma, \mathcal{K}_B(H_B))$.

On the other hand, note that in the left hand side of (8) we have

$$\text{asind}((W_n, \rho_n, \tilde{F}_n)_{n \in \mathbb{N}}) = \sigma \cdot [\Sigma^2 \rho \otimes \text{id}_K],$$

where $\rho$ is as in Lemma 4.1. In particular, (8) follows if we can prove that

$$\Psi \left(\left(\tilde{\rho}_n(p)\right)_{n \in \mathbb{N}}\right) = [\Sigma^2 \rho \otimes \text{id}_K] \cdot \Sigma \Phi \left(\left(i_{i_1}\right)_*[p]\right) \in D(\Sigma, Q).$$

(9)

By the definition of $\Psi$ we have $\Psi\left(\left(\tilde{\rho}_n(p)\right)_{n \in \mathbb{N}}\right) = [\Sigma^2 f(i_{i_1})_*[p]] \otimes \text{id}_K$ where $f(i_{i_1})_*[p] : \mathbb{C} \rightarrow \mathcal{A}_g(Q \otimes K)$ is the unique *-homomorphism such that $f(i_{i_1})_*[p] (1) = [n \mapsto \tilde{\rho}_n(p)]$. Of course, we have $[n \mapsto \tilde{\rho}_n(p)] = [n \mapsto \tilde{\rho}_n^{(k)}(p)] \in \mathcal{A}_g(Q \otimes K)$ since $\lim_{n \rightarrow \infty} \|\tilde{\rho}_n^{(k)}(p) - \tilde{\rho}_n(p)\| = 0$. On the other hand, $\Phi((i_{i_1})_*[p]) = \kappa(\Sigma f(i_{i_1})_*[p] \otimes \text{id}_K)$ where $f(i_{i_1})_*[p] : \mathbb{C} \rightarrow C^*G \otimes \mathcal{K}$ is such that $f(i_{i_1})_*[p](1) = i_{i_1} \otimes \text{id}_K(p)$. Thus, (7) implies that the right hand side of (9) is given by $[\Sigma^2 h_p \otimes \text{id}_K]$ where $h_p = (\rho \otimes \text{id}_K) \circ f(i_{i_1})_*[p] : \mathbb{C} \rightarrow \mathcal{A}_g(Q \otimes K)$ is determined by $h_p(1) = \rho \otimes \text{id}_K(i_{i_1} \otimes \text{id}_K(p))$. In particular,

$$h_p(1) = [n \mapsto \tilde{\rho}_n^{(k)}(p)] = f(i_{i_1})_*[p](1)$$

by the definition of $\rho$, so that actually $h_p = f(i_{i_1})_*[p]$. This proves (9) and hence completes the proof of the theorem. \hfill \Box

**Remark 10.4.** Dadarlat’s theorem [Dad12, Theorem 3.2] is the specialization of Theorem 10.3 to the case where the bundles are not almost flat Fredholm bundles but finite-dimensional almost flat bundles (in which case the Fredholm operator is neither necessary nor carries any important information). The proof of Theorem 10.3 in this special case can be carried out replacing the use of Proposition 9.1 with the arguments of the proof of [Han12, Theorem 3.9]. Thus, we have also given a fairly simple proof of Dadarlat’s theorem which is quite different from Dadarlat’s original proof.
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