Comments on $T\bar{T}$, $J\bar{T}$ and String Theory

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These notes contain some aspects of the holographic duality between the perturbative superstring on current-current deformations of $AdS_3 \times S^1 \times \mathcal{N}$ and single-trace $T\bar{T}$ and $J\bar{T}$ deformed $CFT_2$. 

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1. Introduction

Perturbative string theory on solvable worldsheet CFT backgrounds – current-current deformations of the superstring on $AdS_3 \times S^1 \times \mathcal{N}$ – allows one to conjecture a holographic duality with solvable irrelevant deformations of CFT$_2$, e.g. ‘single-trace’ $T\bar{T}$ and $J\bar{T}$ deformed CFT$_2$; we shall describe some aspects of these in the next sections.

Motivation:
- This construction provides holographic duals for a large class of vacua of string theory in asymptotically flat linear dilaton spacetimes, and sheds light on the UV behavior of $T\bar{T}$ deformed CFT$_2$; it may provide a step towards holography in flat spacetime.
- The holography above is likely to have interesting consequences for the physics of horizons/singularities/closed timelike curves (CTC’s) in string theory.

Comments:
* By a ‘single-trace’ (ST) deformation, in the context of this note, we mean e.g. a deformation in the block $\mathcal{M}$ of $\text{Symm}^N(\mathcal{M})$; its precise definition is the deformation of the boundary CFT$_2$ dual to the perturbative superstring theory on the deformed $AdS_3 \times S^1 \times \mathcal{N}$.
* The holographic conjecture ‘predicted,’ in particular, the spectrum of $J\bar{T}$ deformed CFT$_2$, which was confirmed in field theory (with reasonable assumptions).
* We also have a holographic duality conjecture predicting the spectrum of a CFT$_2$ deformed by $-tT\bar{T} + \mu J\bar{T}$ (which is partly confirmed with reasonable assumptions). Concretely, the energy $E(t, \mu; R)$ and momentum $P(R)$ of states in the deformed CFT$_2$, on a cylinder with radius $R$, are given in terms of the left and right-handed scaling dimensions of the states in the the original theory, $h, \bar{h}$, the original holomorphic $U(1)$ charge, $q$, and the central charge $c$ of the CFT$_2$, by

$$ER = n + \frac{1}{2A} \left( -B - \sqrt{B^2 - 4AC} \right); \quad PR = n, \quad (1.1)$$

where

$$A = \frac{1}{16\pi R^2} (\pi \mu^2 - 8t), \quad B = -1 + \frac{\mu}{2R} q - \frac{t}{\pi R^2}, \quad C = 2 \left( \bar{h} - \frac{c}{24} \right); \quad n = h - \bar{h}. \quad (1.2)$$

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2 In terms of non-local theories.
3 By ‘$AdS_3$’ we mean either $SL(2, R)$, and/or global $AdS_3$ and/or $\{\text{massless BTZ}\}=\{AdS_3$ in Poincaré coordinates, with a compact boundary spatial direction$\}$, and/or $H_3^+$. 
4 More predictions are e.g. those concerning the entanglement entropy in various cases.
5 One may skip the concrete prediction; it will be discussed in section 9.
The plan of this note is the following. In section 2, we recall some ‘experimental’ results and comment on the qualitative reasons that the spectrum of the perturbative superstring on $AdS_3$ has the pattern of a symmetric orbifold $CFT_2$, and in section 3, we recall how it is obtained in the Ramond sector of the boundary $CFT_2$, namely, for the superstring on the massless BTZ worldsheet background. In section 4, we present a heuristic argument for a holographic duality between the superstring on certain backgrounds, e.g. on

$$\mathcal{M}_3 \equiv J^- \bar{J}^- \text{ deformed } AdS_3 \ ,$$

and ST-$T\bar{T}$ deformed $CFT_2$, and in sections 5 and 6, we recall some properties of the $\mathcal{M}_3$ $CFT_2$ background and the spectrum of the superstring on (1.3), respectively. In sections 7 and 8, we conjecture, similarly, a holographic duality between the superstring on $WAdS_3$ and ST-$J\bar{T}$ deformed $CFT_2$, where $WAdS_3$ is the KK reduction to 3d of

$$KJ^- \text{ deformed } AdS_3 \times S^1 \ ,$$

with $K(z)$ being the $U(1)$ holomorphic current of the $S^1 CFT_2$. Finally, in section 9, we present some preliminary results concerning a holographic duality conjecture between the superstring on a family of backgrounds, combining (1.3) and (1.4), namely,

$$\lambda J^- \bar{J}^- + \epsilon KJ^- \text{ deformed } AdS_3 \times S^1 \ ,$$

and ST-$\{-tT\bar{T} + \mu J\bar{T}\}$ deformed $CFT_2$, and in section 10, we list some open problems.

There is a handful of works on $T\bar{T}$, $J\bar{T}$ and string theory; see e.g. [2-43].

The main results in sections 2–8 are based on [4,10,25,31,32] and references therein; in particular, many comments in this file are taken from there. The study in sections 7 and 8 was motivated by [12] and was done independently in [26].

On the other hand, the results in section 9 are new [1].

**Note added:** The new results in (1.1),(1.2) were obtained independently in [44].

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6 It is possible to read section 4 prior to sections 2,3 (and then to read section 3 before 2).

7 ‘ST-$T\bar{T}$’ denotes ‘Single-Trace $T\bar{T}$,’ namely, $T\bar{T}$ in perturbative String Theory (and, similarly, for other deformations below).

8 One may read section 9 before sections 7,8; the latter, as well as sections 5,6, can be regarded as comments on special cases of section 9.
2. Comments on superstring theory on $AdS_3$ and $\mathcal{M}^N/S_N$

The spectrum of the boundary $CFT_2$ on the cylinder dual to the perturbative superstring theory on $AdS_3$ has the structure of $\mathcal{M}^N/S_N$; here we list some known results regarding this pattern of the spectrum, and comment on them:

* The Ramond sector of the dual $CFT_2$ on the cylinder is given by the superstring on $M = 0$ BTZ; it will be described in the next section.

* The spectrum of the superstring on $M = 0$ BTZ was obtained e.g. in section 5.1 of [25], and is the same as that of a symmetric product CFT, $\mathcal{M}^N/S_N$, where the block $\mathcal{M}$ is the $c = 6k$ CFT associated with one string in the BTZ background and the winding $w$ labels the twisted sector; it will be described in the next section.

* The qualitative reason to expect a relation between the string spectrum and the symmetric orbifold is the following. The states in the type II superstring on massless $BTZ \times \mathcal{N}$ can be thought of as describing strings moving in the radial direction in a particular state of excitation. These strings are free (at large $N$, or small string coupling, $g_s^2 \sim 1/N$), and the fact that they can be described by a symmetric product is very similar to that utilized in matrix string theory.

* In different words, the background corresponding to the boundary $CFT_2$ on the cylinder in its Ramond sector (i.e. with unbroken supersymmetry on the cylinder), is obtained by replacing $AdS_3$ by the $M = J = 0$ BTZ black hole. The strings and fivebranes that create the background are mutually BPS in this case. Thus, their potential is flat. This means that there is a continuum of states corresponding to strings moving radially away from the fivebranes. These states are described by a symmetric product, as in matrix string theory.

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9 By ‘perturbative string theory,’ we mean a theory with a worldsheet $CFT_2$ background (in particular, in the case of a sigma-model, it involves only NS-NS background fields), and with a parametrically small string coupling, $g_s \ll 1$; below, by ‘string theory,’ we mean a perturbative string theory of this type.

10 At large $N$, namely, up to corrections in $g_s^2 \sim 1/N$.

11 $N$ is the number of BPS fundamental strings with their worldvolume in the $R_t \times S^1$ of a linear dilaton background, $R_t \times S^1 \times R_\phi \times \mathcal{N}$, whose near-horizon geometry is $M = 0$ BTZ$\times\mathcal{N}$.

12 Generically, with the NS5-branes wrapped around a singular four-cycle in a Calabi-Yau fourfold and/or considering generic supersymmetric linear dilaton throats, instead of fivebranes, as in section 4.
The NS sector of the dual CFT$_2$ on the cylinder is given by the superstring on global AdS$_3$. The perturbative superstring spectrum in this sector was investigated in the superstring followups of [15], e.g., [16,17]; for states in the continuous representations, it has the same structure as that of a symmetric product CFT, $M^N/S_N$, where the block $M$ is the $c = 6k$ CFT associated with one string in the AdS$_3$ background and the winding $w$ labels the twisted sector.

* The heuristic arguments concerning the symmetric product structure in the R sector of the theory do not apply directly to the NS sector, e.g. since the discrete states live in the interior of AdS$_3$.

3. Superstring theory on $M = 0$ BTZ

In this section, we study the spectrum of perturbative string theory on the massless BTZ background. We start, in subsection 3.1, by reviewing the spectrum of the worldsheet theory, and in subsection 3.2, we review the spectrum of the spacetime theory.

3.1. The spectrum of the worldsheet theory

The worldsheet sigma-model Lagrangian on $M = 0$ BTZ is

$$\mathcal{L} = 2k \left( \partial \phi \bar{\partial} \phi + e^{2\phi} \partial \bar{\gamma} \bar{\partial} \gamma \right) , \quad (3.1)$$

where

$$\gamma = \gamma_1 + \gamma_0 \, , \quad \bar{\gamma} = \gamma_1 - \gamma_0 \, ; \quad \gamma_1 \simeq \gamma_1 + 2\pi R \, . \quad (3.2)$$

It is convenient to rewrite it in the Wakimoto form

$$\mathcal{L}_W = \beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma} + \partial \phi \bar{\partial} \phi - \sqrt{\frac{2}{k}} \hat{R} \phi - e^{-\sqrt{\frac{2}{k}} \phi} \beta \bar{\beta} \, . \quad (3.3)$$
Near the boundary, $\mathcal{L}_W$ becomes free,

$$\mathcal{L} = \beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma} + \mathcal{L}_\phi , \quad (3.4)$$

where

$$\mathcal{L}_\phi = \partial \phi \bar{\partial} \phi - \sqrt{\frac{2}{k}} \hat{R} \phi , \quad (3.5)$$

and it is thus useful for calculations.

It will be further useful for us, following [48], to bosonize the $\beta$-$\gamma$ system as

$$\begin{align*}
\gamma &= i \phi_- \quad \bar{\gamma} = i \bar{\phi}_- \quad ; \\
\beta &= i \partial \phi_+ \quad \bar{\beta} = i \bar{\partial} \bar{\phi}_+ ,
\end{align*} \quad (3.6)$$

where $\phi_\pm$ are canonically normalized light-like fields,

$$\phi_+(z) \phi_-(w) \sim \ln(z - w) . \quad (3.7)$$

In terms of $\phi_\pm$, (3.4) takes the form

$$\mathcal{L} = - \partial \phi_+ \bar{\partial} \phi_- - \partial \phi_- \bar{\partial} \phi_+ + \mathcal{L}_\phi . \quad (3.8)$$

To obtain the massless BTZ orbifolding, one introduces the twist fields,

$$t^w = e^{iw(\phi_+ + \bar{\phi}_+)} , \quad w \in \mathbb{Z} , \quad (3.9)$$

and imposes mutual locality w.r.t. to $t^w$ as well as adding the $w$ twisted sectors via OPE’s of the untwisted operators with (3.9). The vertex operators thus obtained take the form

$$V_{BTZ} = e^{\sqrt{\frac{2}{k}} j(\phi_+ + \bar{\phi}_+)} V^w_{E_L, R} , \quad (3.10)$$

where

$$V^w_{E_L, R} = e^{iw\phi_+ + iE_L \phi_-} e^{iw\bar{\phi}_+ + iE_R \bar{\phi}_-} , \quad (3.11)$$

with

$$E_L, E_R = \frac{R}{2} (E + P, E - P) ; \quad P = \frac{n}{R} , \quad n \in \mathbb{Z} . \quad (3.12)$$

Comments:

17 W.l.g. we consider states with $w > 0$; those with negative $w$ amount to their conjugates (namely, outgoing versus ingoing or annihilation versus creation).

18 We present only those of interest, for simplicity.
* $R$ sets the scale of the problem; one can think of it as the radius of compactification of the geometric coordinate $\gamma_1$, (3.2).

* In the Wakimoto representation, one has

$$J^- = \beta = i\partial\phi_+ ; \quad \bar{J}^- = \bar{\beta} = i\bar{\partial}\phi_+ ,$$

(3.13)

where $J^-$ ($\bar{J}^-$) is the null holomorphic (anti-holomorphic) current of the affine $SL(2, R)_L$ ($SL(2, R)_R$) symmetry of the underlying $SL(2, R)$ WZW model ((3.1) prior to compactifying $\gamma_1$, namely, $AdS_3$ in Poincaré coordinates).

* $V_{BTZ}$, (3.10), thus correspond to eigenstates of $J^-$ and $\bar{J}^-$ with eigenvalues $E_L$ and $E_R$, respectively.

* $E$ and $P$ are the energy and momentum of these states, respectively.

* States which carry real radial momentum $P_\phi$ have

$$j = -\frac{1}{2} + is , \quad s \in R ,$$

(3.14)

with $s$ proportional to $P_\phi$; they amount to the principal continuous representations of the underlying $SL(2, R)$.

* The integer $w$ labels different twisted sectors. These sectors are constructed in a way analogous to [45,46], but here the spectral flow/twist is in the $J^-$ direction, whereas there it was in the $J^3$ direction.

The left and right-handed scaling dimensions $\Delta_{L,R}$ of $V_{BTZ}$ are

$$\Delta_{L,R} = -wE_{L,R} - \frac{j(j+1)}{k} , \quad \Delta_R - \Delta_L = wn .$$

(3.15)

We will next use them to calculate the spectrum of the spacetime theory.

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19 In the perturbative superstring on global $AdS_3$, [45,46], one can construct physical, discrete bound states in the principal discrete representations of $SL(2, R)$; in massless BTZ, the eigenvalues of $J^-$ can take any real value for both the continuous and discrete representations, and thus all the states are part of a continuum.

20 One can think of $w$ as the winding number of the string around the circle on the boundary of BTZ; this is particularly clear in the semiclassical approach of section 4 in [28].
3.2. The spectrum of the spacetime theory

Consider the type II superstring on massless BTZ x N, which corresponds to a Ramond ground state of the dual CFT₂. Let

\[ V_{\text{phys}} = e^{-\varphi - \bar{\varphi}} V_{\text{BTZ}} V_N \]  

be a physical vertex operator of the theory. The on-shell condition reads:

\[ \Delta_{L,R} + N_{L,R} - \frac{1}{2} = 0, \]  

where \( N_{L,R} \) are the left and right-handed scaling dimensions of \( V_N \). Plugging (3.15) into (3.17), one finds the dispersion relations

\[ E_{L,R} = \frac{1}{w} \left[ - \frac{j(j+1)}{k} + N_{L,R} - \frac{1}{2} \right]. \]  

The states that satisfy (3.18) can be thought of as describing a string that winds \( w \) times around the spatial circle in the BTZ geometry, and is moving with a certain momentum (proportional to \( s \), (3.14)) in the radial direction, in a particular state of transverse oscillation. Equation (3.18) gives the energy and momentum of such a state.

The spectrum (3.18) is the same as that of \( \mathcal{M}^N/S_N \), where the block \( \mathcal{M} \) is the \( c = 6k \) CFT₂ associated with one string in the BTZ background, and the winding \( w \) labels the twisted sectors. The string state (3.16) corresponds to an operator with dimension \( h_w \) in \( \mathcal{M}^w/Z_w \) which, when acting on the Neveu-Schwartz vacuum of this CFT₂, creates a Ramond sector state on the cylinder, with energy

\[ E_L = h_w - \frac{k w}{4}; \quad E_R = \bar{h}_w - \frac{k w}{4}. \]  

Comments:

* Plugging (3.19) into (3.18), one gets an equation that can be written as

\[ h_w = \frac{h_1}{w} + \frac{k}{4} \left( w - \frac{1}{w} \right), \]  

which describe the dimensions of the operators in the \( Z_w \) twisted sector, \( h_w \), in terms of those in the sector with \( w = 1, h_1 \). A similar equation holds for the right-movers.

* (3.20) is the expression for the dimension of operators in the \( Z_w \) twisted sector of a symmetric product CFT₂, \( \mathcal{M}^N/S_N \), where the block \( \mathcal{M} \) has central charge \( c_\mathcal{M} = 6k \).

* More comments, which are relevant here, appear already in section 2.

\[ \text{This is only a particular class of operators, which is sufficient for simplicity.} \]

\[ \text{Note that (3.17) with (3.15), and assuming that the GSO projection eliminates the tachyon, imply, in particular, that states in the } w = 0 \text{ sector are necessarily discrete states; in the NS sector of the dual CFT₂, such short strings states are equivalent to winding one states [45-47].} \]
4. $T\bar{T}$ in perturbative string theory – heuristic

Suppose that we have a consistent string theory, say, a perturbative type II superstring, on a worldsheet $CFT_2$ background that has the following properties:

1. It looks asymptotically like
   \[ R_t \times S^1_R \times R_\phi \times \mathcal{N}, \quad (4.1) \]
   where the radial direction $\phi$ has an asymptotic linear dilaton $\Phi = \frac{Q}{2} \phi = -\frac{1}{\sqrt{2k}} \phi$.

2. In the IR it is $AdS_3 \times \mathcal{N}$, with the boundary of $AdS_3$ being the cylinder $R_t \times S^1_R$.

Comments:

(a) By ‘$AdS_3$ whose boundary is the cylinder’ we mean either $AdS_3$ in global coordinates or $M = 0$ BTZ.

(b) For the superstring we need to consider both; the former amounts to the NS sector and the latter to the Ramond sector of the dual $CFT_2$.

(c) $M_3 \times \mathcal{N}$, $(1.3)$, is an example of the latter, which will be described in the next section; it can be obtained from $(4.1)$ by adding to it $N$ fundamental strings (F1) whose worldvolume lies in $R_t \times S^1_R$.

Heuristically, the physical spectrum of such a string theory is obtained as follows:

* Property (1) gives rise, in particular, to
   \[ -\frac{j(j+1)}{k} + N_{L,R} - \frac{1}{2} = \frac{\alpha'}{4} (E_t^2 - p_{L,R}^2), \quad (4.2) \]
   where $E_t$ is the total energy and
   \[ (p_L, p_R) = \left( \frac{wR}{\alpha'} + \frac{n}{R}, \frac{wR}{\alpha'} - \frac{n}{R} \right) \quad (4.3) \]
   is the Narain momentum of a string with winding $w$ and momentum $n$ on the asymptotic circle, $S^1_{x \approx x + 2\pi R}$, moving with momentum governed by the quantum number $j$.

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23 It implies, in particular, that the theory on the $AdS_3$ cap decouples from the linear dilaton throat $R_t \times S^1_R \times R_\phi$ when $R/l_s \rightarrow \infty$, where $R/l_s$ is the radius of the circle in string-length units.

24 In the IR, namely, taking the near-horizon limit of the strings, one gets the geometry of $M = 0$ BTZ $\times \mathcal{N}$, and the F1’s creating this background turn to the $N$ long strings in the superstring theory on $M = 0$ BTZ, each of which is associated with a block $\mathcal{M}$ of the $\mathcal{M}^N/\mathcal{S}_N$, as was mentioned in sections 2 and 3.

25 We shall use standard perturbative string theory conventions, e.g., the string tension is $T = 1/2\pi \alpha'$ and the string length scale $l_s$ is related to $\alpha'$ via $\alpha' = l_s^2$; when $\alpha'$ does not appear, it was set to $\alpha' = 2$. 

8
in the radial direction, in a particular state of transverse left and right-handed levels, \(N_{L,R}\); eq. (4.2) is obtained from the mass-shell relation of physical vertex operators whose asymptotic behavior in (4.1) is

\[
V_{\text{phys}} \rightarrow e^{\varphi} e^{-\bar{\varphi}} V_{N_{L,R}} e^{-iE_t t} e^{ip_{L,R} x} e^{Q_j \phi}.
\] (4.4)

It will also be useful to write the total energy \(E_t\) as

\[
E_t = E + \frac{wR}{\alpha'}
\] (4.5)

with \(E\) thus being the energy of the state relative to the energy of a BPS string wound \(w\) times around the circle with radius \(R\), due to its tension \(2\pi wRT = wR/\alpha'\).

* Property (2) gives rise, in particular, to

\[
-j\left(\frac{j+1}{k}\right) + N_{L,R} - \frac{1}{2} = w \left( h_w - \frac{kw}{4}, \bar{h}_w - \frac{kw}{4} \right),
\] (4.6)

for all \(w \geq 1\), where \(h_w - \frac{kw}{4} (\bar{h}_w - \frac{kw}{4})\) is the left-handed (right-handed) energy (in units of \(R\)) of the states corresponding to (4.4) in the IR theory, namely, in the \(CFT_2\) dual to the superstring theory on the \(AdS_3\) cap (with a cylindrical boundary); eq. (4.6) is obtained by taking the limit \(R/l_s \rightarrow \infty\) in (4.2), which gives rise to

\[
E_{L,R}(R/l_s \rightarrow \infty) \rightarrow \frac{1}{w} \left[ -j\left(\frac{j+1}{k}\right) + N_{L,R} - \frac{1}{2} \right],
\] (4.7)

with

\[
E_{L,R} = \frac{1}{2}(ER \pm n),
\] (4.8)

and since, on the other hand, property (2) implies that \(27\)

\[
E_{L,R}(R/l_s \rightarrow \infty) \rightarrow \left( h_w - \frac{kw}{4}, \bar{h}_w - \frac{kw}{4} \right).
\] (4.9)

The consequence of (4.7) and (4.9) is the equality (4.6).

* Together, (4.2) and (4.6) give

\[
\frac{\alpha'}{4} \left( E_t^2 - p_{L,R}^2 \right) = w \left( h_w - \frac{kw}{4}, \bar{h}_w - \frac{kw}{4} \right),
\]

which (with (4.3) and (4.5)) we rewrite as

\[
\left( E + \frac{R_w}{\alpha'} \right)^2 - \left( \frac{R_w}{\alpha'} \right)^2 = \frac{2}{\alpha'} \left( h_1 + \bar{h}_1 - \frac{k}{2} \right) + \left( \frac{n_w}{R_w} \right)^2,
\] (4.10)

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26 We present only a certain class of observables, for simplicity.

27 For \(M = 0\) BTZ, it is presented in (3.18), (3.19), and for global \(AdS_3\), it follows from (4.5).
\[ h_1 - \bar{h}_1 = n_w, \] (4.11)

with \( R_w = wR, n_w = wn \); it means that the spectrum of strings with winding \( w \) and momentum \( n \) is the same as that of a string singly wound around a circle with radius \( R_w \) and momentum \( n_w \). This agrees with the spectrum in the \( Z_w \) twisted sector of

\[ (\mathcal{M}_{-TT})^N/S_N; \quad t = \pi \alpha', \] (4.12)

with \( Z_w \) acting via cyclic permutation on the \( w \) copies \( \mathcal{M} \).

Comments:

(i) Note that in the IR limit, \( R/l_s \rightarrow \infty \), (4.10) reduces to the well-known results in string theory on \( AdS_3 \) \[45,46\], such as

\[ h_w = h_1 w + k \left( w - \frac{1}{w} \right), \] (4.13)

describing long strings winding \( w \) times around the boundary circle, \[28\] which is the well-known expression for the dimension of operators in the \( Z_w \) twisted sector of a symmetric product \( CFT_2, \mathcal{M}^N/S_N \), where the block \( \mathcal{M} \) has central charge \[29\]

\[ c_{\mathcal{M}} = 6k. \] (4.14)

(ii) More qualitative reasons that support the symmetric product structure are discussed in the previous sections.

(iii) For the superstring on \( \mathcal{M}_3 \times \mathcal{N}, \) (1.3), we proved the relations (4.10), (4.11) in a couple of ways \[10,25\]; some of this will be described in the next two sections. Here, we point out that we know \[49\] that adding \( J^-(z)\bar{J}^-(\bar{z}) \) to the Lagrangian of the \( AdS_3 \) worldsheet theory \[34\] is the same as adding \( D(x, \bar{x}) \) to the Lagrangian of the boundary theory,

\[ \int d^2xD(x, \bar{x}) \simeq \int d^2zJ^-(z)\bar{J}^-(\bar{z}), \] (4.15)

where the operator \( D(x, \bar{x}) \) has the following properties:

(a) \( D(x, \bar{x}) \) transforms under \( T(x) \) and \( \bar{T}(\bar{x}) \) as a quasi-primary operator of dimension \( (2, 2) \); its OPE’s with \( T(x) \) and \( \bar{T}(\bar{x}) \) is the same as that of \( TT \),

(b) but \( D(x, \bar{x}) \) is a ‘single-trace’ operator – it is a massive mode of the dilaton-graviton sector of string theory on \( AdS_3 \).

* These points ‘dismystify’ the result (4.10)-(4.12).

\[28\] As it had to, (4.9).

\[29\] In harmony with the discussion in sections 2 and 3.

\[30\] Namely, to the \( SL(2, R) \) WZW model and/or its Euclidean \( H^+_3 \) version.
5. $M_3$: $J^- \bar{J}^-$ deformed $AdS_3$

Deforming (3.1) by adding to it

$$\delta \mathcal{L} \simeq \lambda J^- \bar{J}^- ,$$

(5.1)

where $J^- \simeq e^{2\phi} \partial \bar{\gamma}$ and $\bar{J}^- \simeq e^{2\phi} \bar{\partial} \gamma$ are the null holomorphic and anti-holomorphic currents of the $SL(2,R)$ WZW theory, one finds [50] a sigma-model background with a metric, dilaton and $B$-field:

$$ds^2 = k \left( d\phi^2 + f d\gamma d\bar{\gamma} \right) ,$$

(5.2)

$$e^{2\Phi} = g^2 e^{-2\phi} f , \quad B_{\gamma \bar{\gamma}} = kf/2 ,$$

(5.3)

$$f^{-1} = f_1 = \lambda + e^{-2\phi} ,$$

(5.4)

which we refer to as $M_3$.

From the discussion in section 3, in the representation (3.8), the $J^- \bar{J}^-$ deformation of the sigma model on $AdS_3$ takes the form (at large $\phi$)

$$\mathcal{L} = -\partial \phi_+ \bar{\partial} \phi_- - \partial \phi_- \bar{\partial} \phi_+ + \lambda \partial \phi_+ \bar{\partial} \phi_+ + \mathcal{L}_\phi ;$$

(5.5)

the deformation [31] thus acts on the two dimensional space labeled by $(\phi_+, \phi_-)$ by changing the metric $G_{\mu\nu}$ from $\eta_{+-} = -1$, to

$$G = \begin{pmatrix} \lambda & -1 \\ -1 & 0 \end{pmatrix} .$$

(5.6)

The geometry of $M_3$ depends on the sign of the deformation parameter $\lambda$ in (5.4):

* For $\lambda > 0$, one finds a smooth asymptotically linear dilaton flat three-dimensional space-time, compactified on a circle with radius $R$, and capped in the infrared region by a locally $AdS_3$ space; we will refer to this background as $M_3^{(+)}$.

* For $\lambda < 0$, the background, which we will denote by $M_3^{(-)}$, looks as follows. In the infrared region in the radial coordinate, it approaches $AdS_3$. As one moves towards the UV, the geometry is deformed, and at some value of the radial coordinate, that depends on $\lambda$, one encounters a singularity. The region between the IR $AdS_3$ and the singularity looks like the region between the horizon and the singularity of a black hole.

31 The $\lambda$ in the geometry (5.4) and the one in the representation (5.5) differ by a factor of $R^2/2\alpha'$, which we ignore here.
Proceeding past the singularity, the geometry approaches a linear dilaton spacetime. From the point of view of an observer living in that spacetime, the singularity in question is naked. Also, the role of space and time on the boundary are flipped when passing the singularity. Thus, the region past the singularity has CTC’s.

* While the backgrounds $M_3^{(+)}$ and $M_3^{(-)}$ look rather different, their constructions in string theory are very similar. As described in [10], the worldsheet theory corresponding to both can be obtained via null gauging of the worldsheet CFT on $R_t \times S^1_R \times AdS_3$. For $\lambda > 0$ ($\lambda < 0$), the gauging involves an axial (vector) symmetry. Therefore, it is natural to expect both of them to give rise to good string backgrounds.

6. Superstring theory on $M_3$ and $(M_{-tT})^N/S_N$

According to the discussion of sections 2, 3 and 4, fundamental string excitations of the massless BTZ background are described by the symmetric product CFT $M^N/S_N$, where $N$ is the number of strings, and $M$ the CFT describing one string in this background. One of the interesting aspects of this picture is that, if it is correct, then the deformation by $J^-\bar{J}^-$ corresponds to deforming the symmetric product to $(M_t)^N/S_N$, where $M_t$ is the CFT $M$ deformed by the $tT\bar{T}$ deformation of [3,4]. Thus, from comment (iii) in section 4, one expects the string theory analysis of the spectrum to yield in this case the same results as that of the spectrum of $TT\bar{T}$ deformed CFT in the above papers.

The fact that this is the case was shown in [10], using the construction of the $M_3$ theory as a null gauging of $R_t \times S^1_R \times AdS_3$. In this section, following [25], we will use instead the techniques reviewed in section 3 to derive the same results; the latter will be useful also for more general theories, e.g. those in sections 7, 8 and 9.

6.1. The spectrum of the worldsheet theory

The spectrum of a theory with a general constant metric, such as (5.6), is a familiar problem in string theory, in the context of toroidal compactifications, where it gives rise to the Narain moduli space. The slight novelty here is that the deformation involves time, but we can still use techniques developed in the Narain context, and we will do that below.

After the deformation, the scaling dimensions of the vertex operators (3.10), (3.11), which become operators in the sigma model on $M_3$, are given by

$$\Delta_{L,R} = \frac{1}{2} P^2_{L,R} - \frac{j(j + 1)}{k},$$

(6.1)
where

\[ P_{L,R} = (n^t + m^t (B \mp G)) e^*, \quad P^2 \equiv PP^t, \quad e^*(e^*)^t = \frac{1}{2} G^{-1}, \quad (6.2) \]

with

\[ n^t = (n_+, n_-) = \frac{1}{\sqrt{2}} (2w, ER), \]
\[ m^t = (m_+, m_-) = \frac{1}{\sqrt{2}} (PR, 0). \quad (6.3) \]

Substituting (6.2) and (6.3) in (6.1), we get that operators of the form (3.10), (3.11) in \( \mathcal{M}_3 \) have left and right scaling dimensions

\[ \Delta_{L,R} = -wE_{L,R} - \frac{\lambda R^2}{8} \left( E^2 - P^2 \right) - \frac{j(j+1)}{k}, \quad \Delta_R - \Delta_L = wn, \quad (6.4) \]

with \( E_{L,R} \) given in terms of the energy and momentum \( E, P \) and the radius \( R \) in (3.12).

This equation generalizes (3.15) to \( \lambda \neq 0 \).

6.2. The spectrum of the spacetime theory

We can use the results of subsection 6.1 to calculate the spectrum of the spacetime theory, as we did in the undeformed case, \( \lambda = 0 \), in section 3. Using the mass-shell condition (3.17), we find the dispersion relation

\[ h_w - \frac{k w}{4} = E_L + \frac{\lambda R^2}{8w} \left( E^2 - P^2 \right), \]
\[ \bar{h}_w - \frac{k w}{4} = E_R + \frac{\lambda R^2}{8w} \left( E^2 - P^2 \right), \quad (6.5) \]

where \( h_w, \bar{h}_w \) are properties of the undeformed theory (e.g., they can be obtained by setting \( \lambda = 0 \) in (3.5), and using the dispersion relations of the undeformed theory, (3.18)).

It is interesting to compare the spectrum (6.5) to the field theory analysis of \( TT \) deformed \( CFT_2 \) [5,6]. It is easy to see that the two agree, if we take the boundary \( CFT_2 \) to be the symmetric product \( \mathcal{M}^N/S_N \), interpret the deformation (5.5) to be the \( TT \)

32 See e.g. the review [51], around (2.4.12) (with \( L \leftrightarrow R \)). The antisymmetric background \( B \) is zero in the present example; we keep it for the cases considered in the next sections.

33 The light-cone momentum is \( (n_+, n_-) = \frac{1}{\sqrt{2}} (n_0 + n_1, n_0 - n_1) \), and similarly for the light-cone winding \( m \).
deformation in $\mathcal{M}$, and take the coupling $\lambda$ in the string theory problem to be related to the $tT\bar{T}$ coupling, $t$, via

$$t = \frac{\pi}{2} \lambda R^2.$$  \hspace{1cm} \text{(6.6)}

For $w = 1$, the spectrum (6.5) is just that of $T\bar{T}$ deformed $CFT_2$ for the deformed block $\mathcal{M}$, $\mathcal{M}_t$. For $w > 1$, it is that of the $Z_w$ twisted sector of the symmetric product $(\mathcal{M}_t)^N/S_N$ \cite{10}.

Comments:

* The energy of states in the $tT\bar{T}$ deformed $CFT_2$ of the block $\mathcal{M}_t$ takes the form (for $P = 0$, namely, $\hbar - h = n = 0$, for simplicity):

$$E(t, R) = \frac{\pi R}{t} \left[-1 + \sqrt{1 + \frac{4t}{\pi R^2} \left(h - \frac{c_\mathcal{M}}{24}\right)}\right]. \hspace{1cm} \text{(6.7)}$$

Some properties of the spectrum are the following:

* In the IR, namely, for $h - \frac{c_\mathcal{M}}{24} \ll \frac{R^2}{t}$, the energy (6.7) is parametrically close to that of the CFT $\mathcal{M}$, $ER \simeq 2 \left(h - \frac{c_\mathcal{M}}{24}\right)$, as it should.

* On the other hand, in the UV, when $t > 0$, the large energy behavior of (6.7) is $E \simeq \sqrt{\frac{4\pi}{t} \left(h - \frac{c_\mathcal{M}}{24}\right)}$, hence, the entropy, \cite{34}

$$S((E(h))) \simeq 4\pi \sqrt{\frac{c_\mathcal{M}}{6} \left(h - \frac{c_\mathcal{M}}{24}\right)} \simeq \sqrt{\frac{2\pi c_\mathcal{M}t}{3} E} = \beta_H E,$$  \hspace{1cm} \text{(6.8)}

is a Hagedorn one, with $\beta_H$ being the circumference of the circle, $2\pi R$, at the point where the ground state energy, $E(h = 0)$, becomes complex; see \cite{8} for a detailed discussion.

* The high energy behavior of the entropy of the deformed symmetric product CFT, $(\mathcal{M}_t)^N/S_N$, was shown \cite{33} to agree with the Bekenstein-Hawking entropy of black holes in the deformed geometry induced by the single-trace deformation, $\mathcal{M}_3^{(+)}$.

* From the holographic duality above, the Hagedorn behavior is clear, since the UV completion provided by string theory on $\mathcal{M}_3^{(+)}$ is a 2d Little String Theory.

\footnote{34 We use the fact that the degeneracy of states does not change when we turn on $t$, and thus $S(E(h)) = S(h)$, where the latter is the Cardy entropy of the CFT $\mathcal{M}$.}

\footnote{35 Using the fact that the maximal entropy configuration in $(\mathcal{M}_t)^N/S_N$ is one in which the total energy $E$ is divided equally in each of the $N$ blocks, namely, $S_{\text{total}}(E) = NS_M(E/N)$; see \cite{8} for more details.}
When \( t < 0 \), there is a maximal value, \( E_{max} = \frac{\pi R}{|t|} \), above which the energies develop an imaginary piece.

Many aspects of the discussion of the partition sum in \([31]\) have a natural interpretation in the above string theory construction:

\((+)\) For example, we found that for \( t > 0 \), the spectrum of the \( tT\bar{T} \) deformed theory does not receive non-perturbative corrections. This is natural in the string theory construction since \( \mathcal{M}_3^{(+)} \) is a smooth space. The explicit calculation above shows that the states in string theory on \( \mathcal{M}_3^{(+)} \) described by the symmetric product do indeed have a smooth limit as \( t \to 0^+ \).

\((-)\) On the other hand, for \( t < 0 \), we found that the partition sum of the theory has a non-perturbative ambiguity, which corresponds to states with energies that diverge as \( t \to 0^- \). It would be interesting to understand these and other features of the field theory discussion from the string theory perspective. It is tempting to speculate that states whose energies have a good perturbative limit correspond in the bulk to wavefunctions that in some sense live in the region between the horizon and the singularity, while those whose energies diverge in the limit \( t \to 0 \) live in the region beyond the singularity. Analyzing this could shed light on whether the singularity of the space-time \( \mathcal{M}_3^{(-)} \) is resolved in string theory, and how. We hope to return to this subject in future work.

7. \( WAdS_3: \ K\bar{T}^- \) deformed \( AdS_3 \times S^1 \)

The worldsheet sigma-model Lagrangian on \( M = 0 \ BTZ\times S^1 \) is

\[
\mathcal{L} = 2k \left( \partial\phi\partial\bar{\phi} + e^{2\phi} \partial\bar{\gamma}\partial\gamma + \partial y\partial\bar{y} \right),
\]

where \( \gamma_1 \), \((3.2)\). Deforming \((7.1)\) by adding to it

\[
\delta\mathcal{L} \simeq \epsilon K\bar{\gamma}^- , \quad K \simeq i\partial y ,
\]

\(36\) The above comments suggest, however, that the other branch of the solutions to the quadratic equation for \( E \), namely, \((6.7)\) with a minus in front of the square root, is unambiguous in string theory.

\(37\) I am very telegraphic here; more details can be found in \([25]\) and references therein.
and KK reducing the resulting sigma-model background to 3d, one finds the geometry

\[ ds^2 = k \left( d\phi^2 + e^{2\phi} d\gamma d\bar{\gamma} - \epsilon^2 e^{4\phi} d\gamma^2 \right), \quad (7.3) \]

with a gauge field, \( A_\gamma = 2\sqrt{k} \epsilon e^{2\phi} \), and a B-field, \( B_{\gamma\bar{\gamma}} = k \epsilon e^{2\phi}/2 \).

In the representation (3.8), the \( K\bar{J}^- \) deformation of the sigma model on \( AdS_3 \) takes the form (at large \( \phi \))

\[ \mathcal{L} = -\partial\phi_+ \bar{\partial}\phi_- - \partial\phi_- \bar{\partial}\phi_+ + \partial y \bar{\partial} y + 2\epsilon \partial y \bar{\partial} \phi_+ + \mathcal{L}_\phi ; \quad (7.4) \]

this background involves a non-trivial metric and B-field background in the three-dimensional spacetime labeled by \((\phi_+, \phi_-, y)\),

\[ G = \begin{pmatrix} 0 & -1 & \epsilon \\ -1 & 0 & 0 \\ \epsilon & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -\epsilon \\ 0 & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix}. \quad (7.5) \]

Comments:

* The background (7.3) is the simplest example of warped \( AdS_3 \) (\( W AdS_3 \)); it appears e.g. as the simplest example in the context of AdS/cold atoms correspondence [52,53], and may have applications as a toy laboratory for the Kerr/CFT correspondence [54].

* The geometry (7.3) has no curvature singularity and, in particular, the scalar curvature is an \( \epsilon \)-independent constant, \( R \simeq -1/k \); there are, however, CTC’s at large values of the radial coordinate, when \( e^{2\phi} > 1/\epsilon^2 \).

* We know [49] that adding \( K(z)\bar{J}^- (\bar{z}) \) to the Lagrangian of the worldsheet theory is the same as adding \( A(x, \bar{x}) \) to the Lagrangian of the boundary theory,

\[ \int d^2 x A(x, \bar{x}) \simeq \int d^2 z K(z)\bar{J}^- (\bar{z}) , \quad (7.6) \]

where the operator \( A(x, \bar{x}) \) has the following properties:

(a) \( A(x, \bar{x}) \) has dimension \((1, 2)\).

(b) It transforms under \( J(x) \) and \( \bar{T}(\bar{x}) \) of the boundary theory dual to string theory on \( AdS_3 \times S^1 \) like \( J\bar{T} \).

(c) but \( A(x, \bar{x}) \) is a ‘single-trace’ operator – it is a marginal operator in the worldsheet theory.

In the following section, we shall compute the spectrum of the superstring on the \( W AdS_3 \) background (7.3), and thus ‘predict’ the spectrum of \( J\bar{T} \) deformed \( CFT_2 \).

\(^{38}\) The \( \epsilon \) in the geometry (7.3) and the one in the representation (7.4) differ by a factor \( R/\sqrt{2}l_s \), which we ignore here.
8. Superstring theory on \( W AdS_3 \) and \( (\mathcal{M}_{\mu jT})^N / SN \)

The analogs of (3.10), (3.11) for this case are vertex operators \( V \) in the deformed massless \( BTZ \times S^1 \) background, which are obtained by multiplying (3.10) by an extra factor of \( e^{i(q_L y + q_R \bar{y})} \), where \( (q_L, q_R) \) is a Narain momentum on the \( S^1_y \). Using the standard techniques, described in section 6.1, one finds that the left and right-handed scaling dimensions of \( V \) are

\[
\Delta_{L,R} = -w E_{L,R} + \epsilon q_L E_R + \frac{1}{2} \epsilon^2 E_R^2 - \frac{1}{2} q_{L,R}^2 j(j + 1) k , \quad \Delta_R - \Delta_L = \frac{1}{2} (q_R^2 - q_L^2) + wn ,
\]

(8.1)

and repeating similar steps to those in section 3 and subsection 6.2, one finds that the spectrum of the spacetime theory is given by the dispersion relation

\[
\begin{align*}
\hbar w - \frac{k w}{4} &= E_L - \frac{\epsilon}{w} q_L E_R - \frac{\epsilon^2}{2w} E_R^2 , \\
\bar{\hbar} w - \frac{k w}{4} &= E_R - \frac{\epsilon}{w} q_L E_R - \frac{\epsilon^2}{2w} E_R^2 , \\
\hbar w - \bar{\hbar} w &= n .
\end{align*}
\]

(8.2)

Comments:

* To understand the physical content of (8.2), it is convenient to rewrite it as follows. Consider first the case \( w = 1, \) corresponding to a string with winding one. From the discussion in earlier sections, in this sector we expect to see the spectrum of a \( JT \) deformed \( CFT_2 \) with central charge \( c = 6k \) (before the deformation). In this sector, one can write (8.2) as follows:

\[
\begin{align*}
h_1 - \frac{c}{24} - \frac{1}{2} q_L^2 &= E_L(\epsilon) - \frac{1}{2} (q_L(\epsilon))^2 , \quad E_L - E_R = n ,
\end{align*}
\]

(8.3)

where \( c = 6k \), and

\[
q_L(\epsilon) = q_L + \epsilon E_R(\epsilon) .
\]

(8.4)

We see that the spectrum is completely determined by the following requirements:

1. The quantity \( E_L(\epsilon) - \frac{1}{2} (q_L(\epsilon))^2 \) is independent of \( \epsilon \).
2. The charge \( q_L \) flows according to (8.4).
3. For all \( \epsilon, E_L \) and \( E_R \) differ by the second eq. in (8.3).
4. At \( \epsilon = 0 \), one has \( q_L(0) = q_L, \ E_L(0) = h_1 - \frac{c}{24}, \ E_R(0) = \bar{\hbar}_1 - \frac{c}{24} . \)

\[39\] See section 5.3 in [25] for details.
* In a $\mu J(x)\tilde{T}(\tilde{x})$ deformed CFT$_2$, one obtains \cite{25,32} the same spectrum, with
\[
\mu = 2\epsilon R.
\] (8.5)

* For $w > 1$, the spectrum (8.2) is compatible with that of a $Z_w$ twisted sector of the symmetric product $(\mathcal{M}_\mu)^N/S_N$, where the block $\mathcal{M}_\mu$ is deformed as described above.

* The energy of states in the $w = 1$ sector takes the form (for $n = q_L = 0$, for simplicity)
\[
E(\mu, R) = \frac{8R}{\mu^2} \left[ 1 - \sqrt{1 - \frac{\mu^2}{2R^2} \left( h - \frac{c_M}{24} \right)} \right],
\] (8.6)

where $h$ is the dimension of the operator which creates the state in the undeformed CFT$_2$ block $\mathcal{M}$.

* There is a maximal value of this energy, $E_{max} = \frac{8R}{\mu^2}$, above which the energies develop an imaginary piece. \footnote{Also for generic momentum and charge, $n, q_L$, beyond a certain maximal undeformed (right-handed) energy, $\tilde{h} - c_M/24$, that depends on the charge, the deformed energy and charge become complex (this can be seen explicitly e.g. from equations presented in section 9.2).}

* The spectrum (8.6) is identical to that of the $tT\tilde{T}$ deformed case (6.7) with a negative deformation parameter, $t = -\frac{\pi\mu^2}{8}$. \footnote{The spectra are different though for non-zero momentum $P = n/R$ and/or for non-zero $U(1)$ charge $q_L$, if only since the $J\tilde{T}$ deformed theory is not Lorentz invariant.}

* It is thus interesting to compare the properties of the boundary and bulk theories in the $T\tilde{T}$ and $J\tilde{T}$ cases. As discussed in \cite{32}, on the field theory side, many properties of $J\tilde{T}$ deformed CFTs are indeed analogous to those of a $tT\tilde{T}$ deformed CFT with negative $t$. In particular, not only does the energy spectrum becomes complex in the UV, but also the partition sum has non-perturbative ambiguities in both cases. On the bulk side of the ST-$tT\tilde{T}$ case, as discussed in section 5, the background has a curvature singularity at a finite value of the radial coordinate, and closed timelike curves beyond it. In the ST-$J\tilde{T}$ case, as discussed in section 7, there is no curvature singularity, but there are closed timelike curves at large values of the radial coordinate. Thus, it is natural to conjecture that the complex energies and non-perturbative ambiguities mentioned above are related to the closed timelike curves and not to the curvature singularity. \footnote{New evidence in support to this claim is presented in the next section.}

40,41,42
* It would be interesting to understand this relation better, and in particular understand whether the theory is well defined on the cylinder after all, despite the issues with complex energies, non-perturbative ambiguities and closed timelike curves. One possible way to go about this is to further explore the string theory formulation of the theory, as a current-current deformation of string theory on $AdS_3 \times S^1$; we leave this too for future work.

9. Combining the two: spectrum versus geometry and $(\mathcal{M}_{-t\bar{T}+\mu J\bar{T}})^N/S_N$

In this section, we deform the massless BTZ$\times S^1$ Lagrangian, (7.1), by adding to it

$$\delta \mathcal{L} = \tilde{\lambda} J^- J^- + \tilde{\epsilon} K J^- ,$$

namely, (5.1) plus (7.2), inspect the 3d geometry obtained (after KK reduction) and the spectrum of the superstring on this background. The results, presented below, are the outcomes of straightforward, simple calculations, following what was done in sections 3,5,6,7,8 and, in particular, they predict the spectrum of $-t\bar{T} + \mu J\bar{T}$ deformed CFT$_2$.

9.1. The 3d geometry holographic to single-trace $-t\bar{T} + \mu J\bar{T}$

The metric, dilaton, gauge field and $B$-field of the sigma-model background (7.1) deformed by (9.1), after reduction to three dimensions, are

$$ds^2 = k (d\phi^2 + f d\gamma d\bar{\gamma} - \epsilon^2 f^2 d\gamma^2) ,$$

$$e^{2\Phi} = g^2 e^{-2\phi} f , \quad A_\gamma = 2\sqrt{k} \epsilon f , \quad B_{\gamma\bar{\gamma}} = kf/2 ;$$

$$f^{-1} = f_1 = \lambda + e^{-2\phi} .$$

Comment:

* The $\lambda, \epsilon$ (in the geometry) are proportional to the $t, \mu$ (in the field theory), respectively: 

$$t = \pi \alpha' \lambda , \quad \mu = 2\sqrt{2}\ell_s \epsilon ; \quad \alpha' = \ell_s^2 .$$

43 The precise factors between $\tilde{\lambda}, \tilde{\epsilon}$ in (9.1) and $\lambda, \epsilon$, respectively, are not important here.

44 These precise relations can be read from [10,25].
In the representation \((3.8)\), the metric and \(B\)-field in the three dimensional space \((\phi_+, \phi_-, y)\) are given by the combination of \((5.6)\) and \((7.5)\), namely,

\[
G = \begin{pmatrix}
\hat{\lambda} -1 & \hat{\epsilon} \\
-1 & 0 & 0 \\
\hat{\epsilon} & 0 & 1 \\
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & -\hat{\epsilon} \\
0 & 0 & 0 \\
\hat{\epsilon} & 0 & 0 \\
\end{pmatrix}.
\]

The \(\lambda, \epsilon\) (in \((9.2)-(9.4)\)) are related to the hatted ones (in \((9.6)\)) by \(45\)

\[\{\lambda, \epsilon^2\} = \frac{R^2}{2\alpha'}\{\hat{\lambda}, \hat{\epsilon}^2\}.\]  

\(9.2. \text{ The spectrum}\)

The spectrum of the superstring on the background in the previous subsection is obtained straightforwardly following the discussion in the previous sections. One finds the dispersion relation

\[h_w - \frac{cw}{24} = E_L - \frac{\hat{\epsilon}}{w}q_LE_R + \frac{1}{2w}(\hat{\lambda}E_L - \hat{\epsilon}^2E_R)E_R,\]

\[h_w - \bar{h}_w = E_L - E_R = n,\]

hence, for \(w = 1\) (for simplicity)

\[ER = E_L + E_R = n + \frac{1}{2A}\left(-B - \sqrt{B^2 - 4AC}\right),\]  

with \(46\)

\[A = \frac{1}{4}(\hat{\epsilon}^2 - \hat{\lambda}), \quad B = -1 + \hat{\epsilon}q_L - \frac{1}{2}\hat{\lambda}n, \quad C = 2\bar{h}_1 - \frac{c}{12};\]

\[n = h_1 - \bar{h}_1, \quad c = 6k,\]

for instance, when \(n = q_L = 0\) (for simplicity), the energy of states in the \(w = 1\) sector takes the form

\[E(\lambda, \epsilon; R) = \frac{R}{\alpha'(\epsilon^2 - \lambda)} \left[1 - \sqrt{1 - \frac{4\alpha'}{R^2} (\epsilon^2 - \lambda) \left(h_1 - \frac{c}{24}\right)}\right].\]  

\(45\) The precise relation can be read from \([25]\).

\(46\) The branch of the two solutions to the quadratic equation for \(E\), that follows from \((9.8),(9.9)\), which is presented in \((9.10)\), is the one connected to the \(CFT_2\) spectrum; on the other hand, the other branch either decouple or is unphysical when the deformation is turned off: \(E(\lambda \to 0^\pm, \epsilon \to 0) \to \pm\infty\) when \(A \to 0^\pm\), where \(A = \frac{\alpha'}{2\pi}\epsilon^2 - \lambda).
Comments:
* The dispersion relation (9.8), (9.9) is the minimal combination of (6.5) and (8.2), which reduces to them when either $\epsilon = 0$ or $\lambda = 0$.
* The structure of the spectrum is, again, as in a symmetric product $\mathcal{M}^N/S_N$, with the spectrum in the block $\mathcal{M}$ given by (9.10)-(9.12).
* We are thus led to conjecture that (9.11)-(9.12) is the spectrum of a $-t\tilde{T} + \mu J\tilde{T}$ deformed $\text{CFT}_2$, $\mathcal{M}$; \cite{5} for instance, when $n = qL = 0$ (for simplicity), the energy is conjectured to take the form

$$E(t, \mu; R) = \frac{8\pi R}{\pi \mu^2 - 8t} \left[ 1 - \sqrt{1 - \frac{1}{2\pi R^2} (\pi \mu^2 - 8t) \left( h - \frac{c_M}{24} \right)} \right].$$  \hspace{1cm} (9.14)

* Note that at large $\phi$, the part of the metric (9.2) in the $\gamma, \bar{\gamma}$ directions behaves like $(\lambda d\bar{\gamma} - \epsilon^2 d\gamma) d\gamma$, \cite{50} hence, $\gamma_1$ is spacelike at large $\phi$ if $\lambda - \epsilon^2 > 0$, but if $\lambda - \epsilon^2 < 0$, then $\gamma_1$ is timelike at large $\phi$. \cite{49}
* One can thus see that the background (9.2) has closed timelike curves precisely when the spectrum (9.10)-(9.11) has the property that beyond a certain maximal undeformed (right-handed) energy, $\bar{h}_1 - c/24$, that depends on the momentum and charge, $n, qL$, the deformed energy become complex.
* In particular, when e.g. $n = qL = 0$, there is a maximal value of the deformed energy, $E_{\text{max}} = \frac{R}{\alpha (\epsilon^2 - \lambda)}$, above which the energies in (9.13) develop an imaginary piece.
* All in all, we are thus led to conjecture that complex energies in the UV are associated with closed timelike curves in the holographic string-theory geometry.

10. Open problems

(i) Is the dual theory indeed $\mathcal{M}^N/S_N$, or what precisely? \cite{47}
(ii) Does the dual theory tell us if/how string theory resolves singularities/CTC’s?

\cite{47} For $n = 0$, this was already verified (with reasonable assumptions) \cite{1].
\cite{48} And recalling that $\bar{\gamma}, \gamma$ are conjugate to $E_{L,R}$, respectively, this is in harmony with the factor $(\lambda E_L - \epsilon^2 E_R) E_R$ in the spectrum (9.8).
\cite{49} Note that the determinant of the metric (9.2) is $\epsilon$-independent and, in particular, there is thus one time for any $\epsilon$ (and $\phi$), but which of the directions is timelike does depend on $\epsilon$ (and $\phi$).
\cite{50} We find non-trivial match and confirmed predictions if we assume $\mathcal{M}^N/S_N$; but what is the precise duality?
(iii) Does string theory tell us what is the \textit{definition} of $t\overline{T}$ deformed $CFT_2$ (for both signs of $t$) and the other irrelevant deformations above?

(iv) Applications to AdS/cold atoms and Kerr/CFT correspondence?

(v) Modular invariance in general cases?

(vi) Is the conjecture in section 9 correct?

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