A Wavenumber Integration Model of Underwater Acoustic Propagation in Arbitrary Horizontally Stratified Media Based on a Spectral Method

Houwang Tu\textsuperscript{a}, Yongxian Wang\textsuperscript{b,*}, Wei Liu\textsuperscript{b}, Shuqing Ma\textsuperscript{b}, Xiaodong Wang\textsuperscript{a}, Wenbin Xiao\textsuperscript{a}

\textsuperscript{a}College of Computer, National University of Defense Technology, Changsha, 410073, China
\textsuperscript{b}College of Meteorology and Oceanography, National University of Defense Technology, Changsha, 410073, China

Abstract
The wavenumber integration method is considered to be the most accurate algorithm of arbitrary horizontally stratified media in computational ocean acoustics. Compared with normal modes, it contains not only the discrete spectrum of the wavenumber but also the components of the continuous spectrum, eliminating errors in the model approximation for horizontally stratified media. Traditionally, analytical and semianalytical methods have been used to solve the depth-separated wave equation of the wavenumber integration method, and numerical solutions have generally focused on the finite difference method and the finite element method. In this paper, an algorithm for solving the depth equation with the Chebyshev–Tau spectral method combined with the domain decomposition strategy is proposed, and a numerical program named WISpec is developed accordingly. The algorithm can simulate both the sound field excited by a point source and the sound field excited by a line source. The key idea of the algorithm is first to discretize the depth equations of each layer by using the Chebyshev–Tau spectral method and then to solve the equations of each layer simultaneously by combining boundary and interface conditions. Several representative numerical experiments are devised to test the accuracy of ‘WISpec’. The high consistency of the results of different models running under the same configuration proves that the numerical algorithm proposed in this paper is accurate, reliable, and numerically stable.

Keywords: Chebyshev–Tau spectral method, wavenumber integration, horizontally stratified, computational ocean acoustics

\textsuperscript{*}Corresponding author: yxwang@nudt.edu.cn
Email addresses: tuhouwang@nudt.edu.cn (Houwang Tu), liuwei@nudt.edu.cn (Wei Liu)
1. Introduction

The wavenumber integration method is basically a numerical implementation of the integral transform technique for horizontally stratified media [1]. This method, which does not make any approximations to the Helmholtz equation, completely avoids approximation error and is considered the most accurate method for simulating sound propagation in horizontally stratified media.

The normal mode model is often compared with the wavenumber integration method because the mathematical basis of the two is the same; the difference is that the evaluation of the integral adopts different strategies. The normal mode model uses complex contour integration to reduce the integral representation to a sum of residues, whereas the wavenumber integration method evaluates the integrals directly by numerical quadrature [1, 2]. The wavenumber spectrum of a general waveguide is a mixture of discrete and continuous parts. The discrete spectrum in such cases leads to a representation involving a sum of modes, while the continuous spectrum involves an integral over a continuum of points in wavenumber space. In other words, the normal modes contain only a limited number of discrete wavenumbers that contribute greatly to the sound field while ignoring the continuous spectrum that may have a great error on the sound field, especially the near field. For the case where the horizontal wavenumbers are near the branch cut, the normal mode model may fail to find the root, thus reducing the accuracy of the sound field. Therefore, the wavenumber integration method is generally considered to be more accurate than the normal mode model.

The principle of wavenumber integration for horizontally stratified media was first introduced into ocean acoustics by Pekeris [3]. He used simple two- and three-layer structures to model sound propagation in horizontally stratified media. Later, Ewing, Jardetzky and Press used this method to study seismic propagation in waveguides with few layers [4]. The wavenumber integration technique performs a series of integral transformations on the Helmholtz equation to simplify the original partial differential equation into a series of ordinary differential equations of depth coordinates. These equations are then solved analytically in each layer in such a way that the amplitudes are undetermined, the undetermined amplitudes are determined by matching boundary conditions at the interfaces, and finally, the corresponding sound field is determined by evaluating the inverse integral transform. For the initially proposed ocean environment with few layers, it is easy to solve the system of linear equations analytically by expressing the boundary conditions in terms of undetermined sound field amplitudes. However, for more complicated ocean environments, the undetermined coefficients method is not applicable, and numerical methods are usually employed.

The earliest algorithm for simulating depth-dependent sound fields is the propagator matrix approach (PMA) proposed by Thomson [5] and Haskell [6]. The advantage of the PMA is that it is recursive and thus requires only a small amount of memory, but the disadvantage is that it requires a very time-consuming correction scheme to ensure numerical stability. Furthermore, the PMA is not well suited to problems where the field has to be determined at more than a single receiver depth [11]. Kennett reviewed the PMA [7] and proposed the invariant embedding approach (IEA) [8]. The advantages of the IEA are inherent numerical stability, simplicity of the recurrence algorithms and direct suitability for reflectivity modeling. In addition, the IEA has definite interpretational advantages for crustal seismology in particular. However, the IEA is not well suited to the solutions of the global problem of interest in ocean acoustics, where sources and receivers lie within the layering [9]. At present, the most widely used method for solving depth equations is the direct global matrix (DGM) approach proposed by Schmidt [10]. In this
approach, the sound field of each layer is represented as the superposition of the sound field generated by the sound source and the undetermined sound field satisfying the homogeneous depth equation, and the relationship between the sound fields of each layer is controlled by the continuity condition of the interface. Then, the depth equations in local layers are assembled into a global matrix, and after adding boundary conditions, the sound field in all layers can be obtained simultaneously by solving the global linear equations \([11]\). The most important advantage of the DGM approach is its unconditional stability, obtained at no additional computational cost, yielding very efficient numerical solution of the depth-separated wave equations in all layers simultaneously \([12]\). However, the memory requirement of the DGM approach is proportional to the number of layers. When the acoustic parameters vary greatly with depth or the frequency of the sound source is very high \([13]\), a denser configuration of layers is required to treat the acoustic parameters of each layer as constants. The size of the global matrix produced by the DGM approach then becomes unacceptable, especially for small personal computers with limited memory.

Among the methods for numerically solving differential equations, in addition to the widely used finite difference and finite element methods, spectral methods are a kind of niche but efficient new method. Spectral methods have high accuracy and fast convergence speed \([14–20]\) and have been rapidly developed in acoustics \([21, 22]\), especially computational ocean acoustics. In recent years, new algorithms of normal modes \([23–29]\), coupled modes \([30–32]\) and parabolic equation models \([33–35]\) based on spectral methods have been successively proposed. In this paper, a Chebyshev–Tau spectral method is used to numerically solve the depth-separated wave equation. In the model designed in this paper, the Chebyshev–Tau spectral method does not physically discretize the ocean environment in the vertical direction; that is, it does not use piecewise linear approximation to address ocean environmental parameters, so there is no physical discretization error. In addition, the algorithm has no factors that make the solution divergent, so it has good stability. A corresponding numerical program is developed for the algorithm. Several classic numerical experiments verify the accuracy and illustrate the capability of the algorithm and program devised in this article.

2. Mathematical Modeling

For a horizontally stratified ocean environment, the interfaces at different depths are all parallel planes, the layer properties are functions of depth only, and the field is independent of azimuthal angle, as shown in Fig. 1. For this range-independent problem, the Helmholtz equation takes the following form \([11]\):

\[
\left[ \rho(z) \nabla \cdot \left( \frac{1}{\rho(z)} \nabla \right) + k^2(z) \right] \psi(r, z) = F(r, z) \tag{1}
\]

where \(\psi(r, z)\) denotes the displacement potential, \(F(r, z)\) is the body force, \(k\) is the wavenumber, \(k = 2\pi f/c(1 + i\eta)\), \(\eta = (40\pi \log_{10} e)^{-1}\), \(f\) is the frequency of the sound source, and \(c\) and \(\alpha\) are the sound speed and attenuation of the medium, respectively. The derivations of the sound fields for point and line sources discussed below are based on the Helmholtz equation.

2.1. Integral transformation for point source problems

For a point sound source, the waveguide it excites is usually solved in cylindrical coordinates. The sound field is related only to the depth and horizontal range away from the sound source, so in a cylindrical coordinate system, we let the \(z\)-axis pass through the sound source and go
vertically downward, and the r-axis is parallel to the sea surface, as illustrated in Fig. 1. The Helmholtz equation (Eq. (1)) in the cylindrical coordinate system is taken in the following form:

\[
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \rho(z) \frac{\partial}{\partial z} \left( \frac{1}{\rho(z)} \frac{\partial}{\partial z} \right) + k^2(z) \right] \psi(r,z) = -\frac{\delta(r)\delta(z - z_s)}{2\pi r} \tag{2}
\]

where \(z_s\) is the depth of the sound source.

We consider using the following Hankel transform for the above equation:

\[
\psi(r,z) = \int_0^\infty \Psi(kr,z) J_0(kr) k \, dk \tag{3a}
\]

\[
\Psi(kr,z) = \int_0^\infty \psi(r,z) J_0(kr) \, r \, dr \tag{3b}
\]

Specifically, the following operation is applied to Eq. (2):

\[
\int_0^\infty (\cdot) J_0(kr) \, r \, dr
\]

Therefore, we can easily obtain the following depth-separated wave equation:

\[
\left[ \rho(z) \frac{d}{dz} \left( \frac{1}{\rho(z)} \frac{d}{dz} \right) + \left( k^2 - k_f^2 \right) \right] \Psi(k_f, z) = -\frac{\delta(z - z_s)}{2\pi} \tag{4}
\]

This equation is an ordinary differential equation in depth and can be solved numerically or analytically. Conventionally, the solution strategy for Green’s function \(\Psi(k_f, z)\) is to first physically
discretize the ocean environment in the depth direction [5–7, 12]. The ocean environment is divided into sufficiently thin layers, and the acoustic parameters of each layer are regarded as depth-independent constants, which evidently introduce errors. In the next section, we introduce a Chebyshev–Tau spectral method to numerically solve the depth-separated wave equation, which is a high-precision numerical method that does not involve physical discretization.

After the depth-dependent Green’s function \( \Psi(k_x, z) \) is found at a discrete number of wavenumbers for the selected receiver depths, Eq. (3a) is evaluated, yielding the total displacement potential \( \psi(r, z) \) at the selected depths and ranges.

2.2. Integral transformation for line source problems

An infinitely long line sound source is often used to verify the accuracy of models in computational ocean acoustics. We also consider the solution of this common model. The line source problem is usually introduced in a Cartesian coordinate system, still letting the \( z \)-axis pass through the sound source and vertically downward; the \( x \)-axis is parallel to the sea surface, and the sound source penetrates the \( xo\z \)–plane perpendicularly. The main structure is still as shown in Fig. 1 except that the \( r \)-axis is replaced by the \( x \)-axis. Therefore, the Helmholtz equation of the line source in the Cartesian coordinate system can be written in the following form [11]:

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{1}{\rho(z)} \frac{\partial}{\partial z} \left( \frac{1}{\rho(z)} \frac{\partial}{\partial z} \right) + k^2(z) \right] \psi(x, z) = -\delta(x)\delta(z - z_s) \tag{5}
\]

We apply the following Fourier transform to Eq. (5):

\[
\psi(x, z) = \int_{-\infty}^{\infty} \Psi(k_x, z) e^{i k_x x} \, dk_x \tag{6a}
\]

\[
\Psi(k_x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(x, z) e^{-i k_x x} \, dx \tag{6b}
\]

Specifically, the following operator is applied to the above formula:

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} (\cdot) e^{-i k_x x} \, dx
\]

The following depth-separated wave equation is thus obtained:

\[
\left[ \Delta + \frac{1}{\rho(z)} \frac{\partial^2}{\partial z^2} \right] \Psi(k_x, z) = -\frac{\delta(z - z_s)}{2\pi} \tag{7}
\]

Solving the depth-separated wave equation provides the depth-dependent Green’s function \( \Psi(k_x, z) \). After obtaining \( \Psi(k_x, z) \), the total sound field can be synthesized by Eq. (6a), as discussed for the point source.

Comparison of Eqs. (4) and (7) indicates that the depth-separated wave equations for the point source and line source have exactly the same form, except that \( r \) is replaced by \( x \) and \( k_r \) is replaced by \( k_x \). Therefore, Green’s function of the depth-separated equation can be used not only as the integral kernel function for the point source but also for the line source. We take only Eq. (4) as an example for the solution of the depth-separated wave equation below.
2.3. Interface conditions and boundary conditions

In the ocean environment shown in Fig. 1, the interfaces \( \{h_l \}_{l=1}^{\ell-1} \) with discontinuous environmental parameters in seawater need to satisfy the interface conditions. The sound pressure must be continuous, yielding:

\[
\rho(h_l^+\Psi(k_r, h_l^+) = \rho(h_l^-\Psi(k_r, h_l^-)), \quad l = 1, 2, \ldots, \ell - 1
\]  

(8)

The normal particle velocity must also be continuous:

\[
\frac{d\Psi}{dz} \Bigg|_{z=h_l^+} = \frac{d\Psi}{dz} \Bigg|_{z=h_l^-}, \quad l = 1, 2, \ldots, \ell - 1
\]

(9)

where the superscripts \(-\) and \(+\) indicate the interfaces from above and below, respectively.

To solve Eq. (4), it is necessary to impose boundary conditions at the sea surface \( z = 0 \) and the seabed \( z = H \). Considering the large difference in impedance between seawater and air, the sea surface is usually taken as the perfectly reflected boundary, that is, the pressure-release boundary:

\[
\psi(r, 0) = 0 \iff \Psi(k_r, 0) = 0
\]

(10)

For the lower boundary condition, the pressure-release seabed is also considered:

\[
\psi(r, H) = 0 \iff \Psi(k_r, H) = 0
\]

(11)

When the lower boundary is perfectly rigid, the boundary condition is taken as:

\[
\frac{d\psi}{dz} \Bigg|_{z=H} = 0 \iff \frac{d\Psi}{dz} \Bigg|_{z=H} = 0
\]

(12)

When modeling the ocean environment, the acoustic half-space boundary shown in Fig. 1 is typically found in practice. Next, we deduce the boundary condition that the acoustic half-space should satisfy. Since the energy in the acoustic half-space has only downward waves and no upward waves, the general solution of the displacement potential satisfies the following form:

\[
\Psi_{\infty}(k_r, z) = \beta e^{ik_z(z-H)}, \quad k_z = \sqrt{k_r^2-k_{\infty}^2}, \quad z \in [H, +\infty]
\]

(13)

where \( \beta \) is the magnitude of the horizontal wavenumber and \( k_{\infty} = \frac{2\pi f}{c_{\infty}}(1+i\eta_{\alpha_{\infty}}) \). The interface conditions still need to be satisfied at \( z = H \). Thus,

\[
\rho_{\infty}\Psi_{\infty}(k_r, H) = \rho_l(H)\Psi_l(k_r, H)
\]

\[
\frac{d\Psi_{\infty}}{dz} \Bigg|_{z=H} = \frac{d\Psi_l}{dz} \Bigg|_{z=H}
\]

(14a)

(14b)

The following is easily obtained from Eq. (13):

\[
\frac{d\Psi_{\infty}}{dz} = ik_z\Psi_{\infty}
\]

(15)

Substituting the above formula into Eq. (14b) and noting Eq. (14a), the boundary condition that needs to be satisfied on the boundary of the acoustic half-space can be obtained:

\[
\rho_{\infty} \frac{d\Psi_l}{dz} \Bigg|_{z=H} + \rho_l(H) \sqrt{k_r^2-k_{\infty}^2} \Psi_l \Bigg|_{z=H} = 0
\]

(16)
Here, $\rho_0 \to 0$ and $\rho_\infty \to \infty$ correspond to perfectly free and perfectly rigid seabeds, respectively. Note that the inhomogeneous term on the right-hand side of Eq. (4) contains $\delta(z - z_s)$ and that the singularity necessitates special treatment of the equation at the depth of the sound source. We add a virtual interface at the depth of the sound source. The acoustic parameters at the virtual interface are continuous, so the sound pressure must also be continuous (Eq. (8)). Due to the singularity of the sound source, the normal particle velocity cannot be constrained using the continuity condition at the discontinuous interface (Eq. (9)). A natural idea is to integrate both sides of Eq. (4) in a very small neighborhood of $z_s$ so that $\delta(z - z_s)$ can be eliminated.

\[
\int_{z_s - \epsilon}^{z_s + \epsilon} \left[ \rho(z) \frac{d}{dz} \left( \frac{1}{\rho(z)} \frac{d\Psi}{dz} \right) + \left( k^2 - k_r^2 \right) \Psi \right] dz = -\frac{1}{2\pi} \tag{17}
\]

Since $\epsilon \to 0$, the above equation translates to:

\[
\frac{d\Psi}{dz} \bigg|_{z_s - \epsilon} = -\frac{1}{2\pi} \iff \frac{d\Psi}{dz} \bigg|_{z_s^+} - \frac{d\Psi}{dz} \bigg|_{z_s^-} = -\frac{1}{2\pi} \tag{18}
\]

where $z_s^+$ and $z_s^-$ represent the layers below and above the depth of the sound source, respectively. This is the interface condition that the displacement potential at the depth of the sound source needs to satisfy.

3. Wavenumber Integration

3.1. Calculation of the sound pressure

After Green’s function of Eq. (4) or (7) is obtained, the corresponding point source displacement potential field $\psi(r, z)$ can be obtained through the inverse Hankel transform of Eq. (3a), or the line source displacement potential field $\psi(x, z)$ can be obtained through the inverse Fourier transform of Eq. (6a). However, in the actual numerical evaluation, when using Eq. (3a) or (6a) to compute the displacement potential field of a point source or a line source, only the finite interval $[k_{\text{min}}, k_{\text{max}}]$ and discrete $M$ points can be used for numerical integration; $k_{\text{min}}$ and $k_{\text{max}}$ are the lower and upper limits of numerical integration, respectively. Undersampling the spikes of Green’s function with a limited number of discrete points would introduce large errors. In addition, waveguide problems have poles on or close to the real wavenumber axis. Fortunately, the aliasing problem can be eliminated by simply moving the integral contour out into the complex plane [1]. According to Cauchy’s theorem, the integral between two points on the complex plane does not vary with the change in the integral contour provided that the integrand is analytical between the contours. Thus, the contour offset $\epsilon$ can be introduced, as shown in Fig. 2. When the points are chosen where the kernels are small and the contour offset $\epsilon \ll k_{\text{max}} - k_{\text{min}}$, then the contributions from the vertical sections become insignificant compared to the integral along the horizontal section.

When the sound source is a point source, substituting $\tilde{k} = k_r - i\epsilon$ into Eq. (3a) can yield:

\[
\psi(r, z) = \int_0^{\infty} \Psi(k_r - i\epsilon, z) J_0 [(k_r - i\epsilon) r] (k_r - i\epsilon) dk_r \tag{19}
\]

When the sound source is a line source, substituting $\tilde{k} = k_x - i\epsilon$ into Eq. (6a) can yield:

\[
\psi(x, z) = \int_{-\infty}^{\infty} \Psi(k_x - i\epsilon, z) \exp [i (k_x - i\epsilon) x] dk_x \\
= 2 \int_0^{\infty} \Psi(k_x - i\epsilon, z) \cos [(k_x - i\epsilon) x] dk_x \tag{20}
\]
The value of \( \epsilon \) is not extremely critical. However, if \( \epsilon \) is too large, the contributions of the two vertical parts of the contour are nonnegligible. However, an excessively small value requires a very large number of sampling points \( M \). For most practical purposes, an attenuation of the wrap-around by 60 dB is more than sufficient [1]. The corresponding value of \( \epsilon \) is:

\[
\epsilon = \frac{3 \Delta k_r}{2 \pi \log_{10} e}, \quad \Delta k_r = \frac{k_{\text{max}} - k_{\text{min}}}{M - 1}
\]  

The integrals in Eqs. (19) and (20) above become the rectangular integrals of the following equations in the actual numerical calculations:

\[
\psi(r, z) = \sum_{k_r = k_{\text{min}}}^{k_{\text{max}}} \Psi(k_r, z) J_0[(k_r - i \epsilon) r] (k_r - i \epsilon)
\]  

\[
\psi(x, z) = 2 \Delta k_x \sum_{k_x = k_{\text{min}}}^{k_{\text{max}}} \Psi(k_x, z) \cos[(k_x - i \epsilon) x]
\]  

\( \Delta k_x \) has the same form as \( \Delta k_r \) in Eq. (21). The above numerical integration can be easily written in the form of matrix multiplication, which greatly improves the actual computational efficiency and is computationally attractive. In addition, we should choose the parameters of numerical integration with great care, such as \( \Delta r, k_{\text{min}}, k_{\text{max}} \) and the farthest range \( r_{\text{max}} \) of the sound field of interest. Techniques related to evaluating integrals (e.g., quadrature schemes, fast field techniques) are universal; they are not the main innovations of this article and therefore are not described in detail.

When the displacement potential field is obtained by the above numerical integration, the sound pressure field can be obtained by the following formula [36]:

\[
p(r, z) = \rho(z) \omega^2 \psi(r, z)
\]  

\[
p(x, z) = \rho(z) \omega^2 \psi(x, z)
\]

where \( \omega = 2 \pi f \).
3.2. Calculation of the transmission loss (TL)

The TL of a point source is defined as:

\[
\text{TL}(r, z) = -20 \log_{10} \left| \frac{p(r, z)}{p_0(R = 1)} \right| \tag{24}
\]

where

\[
p_0(R) = \rho_s \omega^2 \frac{e^{ik_s R}}{4\pi R} \tag{25}
\]

The TL of a line source is defined as:

\[
\text{TL}(x, z) = -20 \log_{10} \left| \frac{p(x, z)}{p_0(X = 1)} \right| \tag{26}
\]

where

\[
p_0(X) = \rho_s \omega^2 \frac{1}{4} \mathcal{H}_0^{(1)}(k_s) \tag{27}
\]

Here, \( p_0(1) \) is the acoustic pressure 1 m from the source, \( \rho_s \) and \( k_s \) are the density and wavenumber of the medium at the location of the source, respectively, and \( \mathcal{H}_0^{(1)}(\cdot) \) denotes the first type of Hankel function.

4. Numerical Discretization

4.1. Chebyshev–Tau spectral method

Here, we employ the Chebyshev–Tau spectral method to solve for the depth-separated wave equation Eq. (4). The Chebyshev spectral method is a spectral method that uses the Chebyshev polynomial as a basis function \([18]\), so it is necessary to introduce the Chebyshev polynomial here.

\[
\begin{align*}
T_0(t) &= 1, \\
T_1(t) &= t, \\
T_{i+1}(t) &= 2tT_i(t) - T_{i-1}(t), \quad i \geq 1
\end{align*}
\tag{28}
\]

Chebyshev polynomials are a class of orthogonal polynomials whose orthogonality is defined as follows \([20]\):

\[
\int_{-1}^{1} T_i(t)T_j(t) \frac{dt}{\sqrt{1 - t^2}} = \begin{cases} 
0, & i \neq j \\
\pi, & i = j = 0 \\
\frac{\pi}{2}, & i = j \geq 1
\end{cases} \tag{29}
\]

Since the Chebyshev polynomial \( T_i(t) \), that is, the basis function, is defined in \( t \in [-1, 1] \), the equation to be solved, Eq. (4), must first be scaled to \( t \in [-1, 1] \) as:

\[
\mathcal{L}\Psi(t) = 0, \quad \mathcal{L} = \frac{4}{\Delta h^2} \rho(t) \left( \frac{1}{\rho(t)} \frac{d}{dt} \frac{d}{dt} + k_s^2(t) - k^2 \right) \tag{30}
\]

where \( \Delta h \) denotes the length of the domain and \( \mathcal{L} \) represents the differential operator. Note that the above formula is applicable only to the depth of the nonsound source; the addition of Eq. (18) is required for the depth of the sound source due to the singularity.
Next, the function to be determined, \( \Psi(t) \), is transformed into the spectral space spanned by the basis functions \( \{ T_i(t) \}_{i=0}^{\infty} \). Furthermore, the expression of spectral coefficients \( \{ \hat{\Psi}_i \}_{i=0}^{\infty} \) can also be obtained from the orthogonality of the Chebyshev polynomial [19].

\[
\Psi(t) = \sum_{i=0}^{\infty} \hat{\Psi}_i T_i(t) \iff \Psi_i = \frac{2}{\pi d_i} \int_{-1}^{1} \frac{T_i(t) \Psi(t)}{\sqrt{1-t^2}} dt,
\]

\[
d_i = \begin{cases} 2, & i = 0 \\ 1, & i > 0 \end{cases}
\]

(31)

The integral on the right side of the above equation is usually calculated using the Gauss–Chebyshev–Lobatto numerical quadrature [18].

Since it is impossible to expand to infinite terms in the actual calculation, only a limited first \((N+1)\) terms can be retained, as follows [17]:

\[
\Psi(t) \approx \Psi_N(t) = \sum_{i=0}^{N} \hat{\Psi}_i T_i(t) \quad (32)
\]

\( \Psi_N(t) \) is a function approximation, which becomes increasingly accurate as \( N \) increases. The truncation of the infinite term expansion described above inevitably introduces errors, which means that Eq. (30) no longer strictly holds. Substituting \( \Psi_N(t) \) into Eq. (30) yields a residual [15], which we call \( R_N \).

\[
R_N = L \Psi_N(t) \quad (33)
\]

Some principle must be adopted to minimize \( R_N \) so that the above spectral expansion can achieve higher accuracy. In the Tau-type spectral method, the basis function is used as the weight function, and then the inner product of the weight function and the residual is forced to be equal to 0 [37].

\[
\int_{-1}^{1} \frac{L \Psi_N(t) T_i(t)}{\sqrt{1-t^2}} dt = 0, \quad i = 0, 1, \cdots, N
\]

(34)

This constraint on the residual is the essence of the weighted residual method [20]. In mathematical monographs, the above equation is generally called the weak form [18] of Eq. (30). Taking into account the orthogonality of the Chebyshev polynomial and Eq. (31), the above equation becomes:

\[
\hat{L} \hat{\Psi}_i = 0, \quad i = 0, 1, \cdots, N
\]

(35)

where \( \hat{L} \) represents the \( L \) operator on the spectral space.

The next most important thing is \( L \), that is, the transformation of the \( L \) operator to the spectral space. The \( L \) operator has a derivative term. According to the characteristics of the Chebyshev polynomial, the following is straightforward to prove:

\[
\hat{\Psi}'_i = \frac{2}{c_i} \sum_{\substack{j=0 \text{ even} \atop j+i \text{ odd}}}^{N} \hat{\Psi}_j, \quad c_0 = 2, c_{i>1} = 1 \implies \hat{\Psi}' = D_N \hat{\Psi}
\]

(36)

Then, the derivative term is transformed into a differential matrix \( D_N \), which is related only to the truncation order \( N \) and is completely unrelated to \( \Psi \). The term is obtained by the relationship between the Chebyshev polynomial and its derivatives [19].

The \( L \) operator also contains a product term, and the spectral transformation of the product of the two functions satisfies the following relationship:

\[
(\hat{v}\hat{\Psi})_i \approx \frac{1}{2} \sum_{m+n=i}^{N} \Psi_m \hat{v}_n + \frac{1}{2} \sum_{m+n=i}^{N} \hat{\Psi}_m v_n \iff (\hat{v}\hat{\Psi}) \approx C_v \hat{\Psi}
\]

(37)
where \( v = v(t) \) is any continuous function on \( t \in [-1, 1] \) used, for example. Similarly, the relationship between the spectral coefficients of the product of two functions and the spectral coefficients of the individual functions is represented by a matrix \( C_v \), which is related only to \( v \) and not to \( \Psi \) [14][15].

4.2. Discretization

According to the above analysis, Eq. (30) is discretized into the following matrix–vector form in Chebyshev spectral space:

\[
\left( \frac{4}{\Delta h^2} C_r D_N C_{1/p} D_N + C_{k^2} - \hat{k}^2 E \right) \hat{\Psi} = 0
\]

(38)

where \( E \) is the identity matrix. The above equation is equivalent to Eq. (35), where \( \hat{\Psi} \) is a column vector composed of \( \{ \hat{\Psi}_l \}_{l=0}^{N_l} \). Eq. (38) is a set of linear equations, but the boundary conditions are not imposed at this time.

For the waveguide in Fig. 1, the depth-separated wave equation must be established in all discontinuous layers. A single set of basis functions cannot span all layers since the Chebyshev polynomial corresponding to the interfaces \( \{ h_l \}_{l=1}^{\ell} \) is not continuously differentiable. Thus, we apply the domain decomposition strategy [38] to Eq. (4) and split the domain into \( \ell \) subintervals:

\[
\Psi_l(z) = \Psi_l(t) \approx \sum_{i=0}^{N_l} \hat{\Psi}_l, i T_i(t), t = \frac{2z}{h_l-1-h_l} + \frac{h_l + h_{l-1}}{h_l - h_{l-1}}, \quad z \in [h_{l-1}, h_l]
\]

(39)

\( N_l \) is the spectral truncation order in the \( l \)-th layer, and \( \{ \hat{\Psi}_l \}_{l=0}^{N_l} \) is the spectral coefficient in the \( l \)-th layer. Similar to Eq. (38), the depth-separated wave equation in the \( l \)-th layer can be discretized into the following matrix–vector form:

\[
A_l \hat{\Psi}_l = 0, \quad A_l = \frac{4}{(h_l - h_{l-1})^2} C_r D_N C_{1/p} D_N + C_{k^2} - \hat{k}^2 E
\]

(40)

where \( A_l \) is a square matrix of order \( (N_l + 1) \) and \( \hat{\Psi}_l \) is a column vector composed of \( \{ \hat{\Psi}_l \}_{l=0}^{N_l} \). Since the interface conditions are related to the adjacent layers, a total of \( \ell \) Eqs. (40) of \( l = 1, \ldots, \ell \) should be solved simultaneously, which is expressed as follows:

\[
\begin{bmatrix}
A_1 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & A_\ell
\end{bmatrix}
\begin{bmatrix}
\hat{\Psi}_1 \\
\hat{\Psi}_2 \\
\vdots \\
\hat{\Psi}_\ell
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

(41)

Note that when \( z_s \) is not on the interface, we set up a virtual interface for it as described above. Eq. (40) that needs to be satisfied for the two layers above and below the virtual interface can also be organized into Eq. (41), but the total number of layers becomes \( (\ell + 1) \) at this time.

\[
\begin{bmatrix}
A_1 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & A_{\ell+1}
\end{bmatrix}
\begin{bmatrix}
\hat{\Psi}_1 \\
\hat{\Psi}_2 \\
\vdots \\
\hat{\Psi}_{\ell+1}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

(42)
where $A_s$ and $\hat{\Psi}_s$ represent Eq. $\{40\}$ on the layer at the depth of the sound source.

The interface conditions in Eqs. $\{8\}$ and $\{9\}$ and boundary conditions in Eqs. $\{10\}$–$\{12\}$, $\{16\}$ must also be expanded to the spectral space and explicitly added to Eq. $\{42\}$. In addition, on the virtual interface at the depth of the sound source, the intermittent condition that the normal particle velocity needs to satisfy, i.e., Eq. $\{18\}$, is also added to Eq. $\{42\}$. After considering the virtual interface, the seawater media comprise a total of $(\ell + 1)$ layers, so there are $\ell$ interfaces leading to the $2\ell$ interface conditions. With the addition of the boundary conditions at sea surface ($z = 0$) and seabed ($z = H$), there are $2(\ell + 1)$ conditions to apply. Next, we describe the imposition of boundary conditions and interface conditions in detail. For the convenience of description, we define the following intermediate row vectors:

$$s_i = [s_{0i}, s_{1i}, s_{2i}, \ldots, s_{Ni}], \quad q_i = [q_{0i}, q_{1i}, q_{2i}, \ldots, q_{Ni}]$$

where $s_i = T_j(-1) = (-1)^i$, $q_i = T_j(+1) = 1$. Thus, the interface conditions and boundary conditions of Eqs. $\{8\}$–$\{12\}$, $\{16\}$ and $\{18\}$ can be transformed and expressed as:

$$\rho(h_{s_i})s_i \hat{\Psi}_{i+1} - \rho(h_{s_i})q_i \hat{\Psi}_i = 0 \quad (43a)$$

$$s_i D_{N_i+1} \hat{\Psi}_{i+1} / (h_i - h_{i+1}) - q_i D_{N_i+1} \hat{\Psi}_i / (h_{i-1} - h_i) = 0 \quad (43b)$$

$$q_i \hat{\Psi}_i = 0 \quad (43c)$$

$$s_{i+1} \hat{\Psi}_{i+1} = 0 \quad (43d)$$

$$s_{i+1} D_{N_{i+1}} \hat{\Psi}_{i+1} = 0 \quad (43e)$$

$$2 \rho(h_{s_i})s_{i+1} D_{N_{i+1}} / (h_i - h_{i+1}) + \rho(H) \sqrt{k_s^2 - k_s^2 s_{i+1}} \hat{\Psi}_{i+1} = 0 \quad (43f)$$

$$s_{i+1} D_{N_{i+1}} \hat{\Psi}_{i+1} / (h_i - z_{i+1}) - q_i D_{N_{i+1}} \hat{\Psi}_i / (z_i - h_{i+1}) = -\frac{1}{4\pi} \quad (43g)$$

where $h_i$ and $h_{i+1}$ represent the depth of the interfaces above and below the sound source, respectively.

How do these $2(\ell + 1)$ conditions apply to Eq. $\{42\}$? A natural idea is to replace the last two rows of the $A_j$ to $A_{\ell+1}$ block matrix with the boundary/interface conditions that the corresponding layers need to satisfy. Doing so reduces the original spectral accuracy of each layer from order $N_i$ to order $(N_i - 2)$, but this problem can be compensated by increasing the value of $N_i$. The spectral coefficients $[\hat{\Psi}_i]_{[1]}^{\ell+1}$ of each layer of Green’s function can be obtained by solving Eq. $\{42\}$ after adding boundary constraints. The numerical solution of Green’s function can be obtained by performing the inverse Chebyshev transform (Eq. $\{32\}$) of $[\hat{\Psi}_i]_{[1]}^{\ell+1}$ sequentially and then stacking into a single column vector.

5. Numerical Simulation

Here, we present a program named WISpec (Wavenumber Integration based on the Spectral method) developed based on the above algorithm and verify the accuracy of the algorithm through several numerical experiments.

5.1. Analytical example: ideal fluid waveguide

The ideal fluid waveguide is a very simple example with an analytical solution. It consists of a layer of homogeneous seawater and upper/lower boundaries; the sea surface is usually perfectly...
free, and the bottom can be perfectly free or rigid. The ideal fluid waveguide of the perfectly free seabed has an analytical solution of the following form [1]:

\[ p(r, z) = \frac{2}{H} i \sum_{m=1}^{\infty} \sin (k_m z_s) \sin (k_m z) H_0^{(1)}(k_m r) \]  

(44a)

\[ k_m z_s = \frac{m \pi}{H}, \quad k_m r = \sqrt{k^2 - k_m^2}, \quad m = 1, 2, 3, \ldots \]  

(44b)

The analytical solution of the sound field of the perfectly rigid seabed is the same as that of Eq. (44), except that the vertical wavenumber becomes:

\[ k_m z_s = \left( m - \frac{1}{2} \right) \frac{\pi}{H}, \quad m = 1, 2, 3, \ldots \]  

(45)

In this example, the sound source frequency is \( f = 20 \) Hz, we take the sea depth \( H = 100 \) m, \( z_s = 36 \) m, the density \( \rho = 1 \) g/cm³, the speed of sound \( c = 1500 \) m/s, and the maximum horizontal range is \( r_{\text{max}} = 3000 \) m. The number of discrete points in the wavenumber domain is taken as \( M = 2048 \), the integral interval is \([0, 2k_0]\) (\( k_0 \) is the wavenumber in water), and the spectral truncation order is \( N = 20 \).

Fig. 3(a) and 3(b) show the wavenumber spectrum of the ideal fluid waveguide with the perfectly free seabed calculated by WISpec. Two peaks (propagating modes) appear at the positions of \( k = \frac{0.077678}{m^{-1}} \) and \( 0.055412 \) m⁻¹. The presence of peaks in the wavenumber spectrum indicates that modes are excited by the source. The analytical solution of the discrete spectrum of the ideal fluid waveguide with a perfectly free seabed is listed in Table 1. The wavenumbers at the peaks are very close to the analytical solution. At this time, the number of discrete points in the wavenumber domain is only 2048. The more discrete points there are in the wavenumber domain, the closer the wavenumbers at the peaks are to the analytical solution. Figs. 3(c) to 3(e) show the pseudocolor and line graphs of the sound field calculated by the analytical solution and WISpec, respectively. The sound field calculated by WISpec is very consistent with the analytical solution, with negligible error, as shown in Fig. 3(f).

**Table 1: Analytical solution of the discrete modes for ideal fluid waveguides (\( f = 20 \) Hz, \( H = 100 \) m).**

| Order of mode | Free seabed | Rigid seabed |
|---------------|-------------|--------------|
| 1             | 0.077 662   | 0.082 290    |
| 2             | 0.055 412   | 0.069 266    |
| 3             | /           | 0.029 153    |

Similarly, Fig. 4(a) and 4(b) show the wavenumber spectrum of the ideal fluid waveguide with the perfectly rigid seabed calculated by WISpec; three peaks appear at the positions of \( k = \frac{0.082343}{m^{-1}}, \frac{0.069247}{m^{-1}} \) and \( 0.029139 \) m⁻¹. This matches the analytical solution in Table 1 very well, and the sound fields shown in Figs. 4(c) to 4(f) lead to the same conclusion as in Fig. 3, namely, that WISpec can calculate the sound field very accurately.

Fig. 5 displays the variation trend of the error of the sound fields calculated by WISpec and SCOOTER with the truncation order/number of piecewise linear elements. Here, to show the accuracy of the spectral method in solving the depth-separated wave equation, WISpec and SCOOTER use the same wavenumber integration parameters. The integration interval of the two programs is \([0, 2k_0]\), and \( M = 2048 \) sampling points are used in both programs. Therefore, the
Figure 3: Wavenumber spectrum of the ideal fluid waveguide with a perfectly free bottom calculated by WISpec (a), wavenumber spectrum at a depth of $z = 46$ m (b); sound fields calculated by the analytical solution (c) and WISpec (d); TLs (e) and error of WISpec (f) along the $r$-direction at a depth of $z = 46$ m.
Figure 4: Wavenumber spectrum of the ideal fluid waveguide with a perfectly rigid bottom calculated by WISpec (a), wavenumber spectrum at a depth of $z = 46$ m (b); sound fields calculated by the analytical solution (c) and WISpec (d); TLs (e) and error of WISpec (f) along the $r$-direction at a depth of $z = 46$ m.
Figure 5: Variation trend of the error of the TL field of ideal fluid waveguides calculated by WISpec and SCOOTER with $N$. In WISpec, $N$ is the spectral truncation order; in SCOOTER, $N$ is the number of piecewise linear elements.

The error of the numerical sound fields mainly comes from the error of Green’s function. The numerical sound fields are given in the form of a discrete grid with 3000 discrete points horizontally from 1 to 3000 m and 401 discrete points vertically from 0 to 100 m. The error of the numerical sound field is calculated by:

$$\text{TL}_{\text{error}} = \frac{\sum_{z=1}^{n_z} \sum_{r=1}^{n_r} |\text{TL}_{i,j} - \text{TL}_{i,j}|}{n_z \times n_r}$$

(46)

where $n_z$ and $n_r$ are the number of discrete points in the vertical and horizontal directions, respectively, and $\text{TL}_{i,j}$ represents the analytical solution for the TL at $(z_i, r_j)$. Fig. 5(a) clearly shows that as $N$ increases, the error of the sound field rapidly converges to a very low level and remains stable, which also proves that the spectral method indeed maintains the advantage of exponential convergence in solving depth-separated equations. In Fig. 5(b), the error of SCOOTER decreases linearly at the beginning and then stabilizes on the order of $10^{-1}$ dB, while WISpec can converge to the order of $10^{-2}$ dB, which further illustrates the high-precision properties of the spectral method.

5.2. Analytical example: pseudolinear waveguide

A pseudolinear waveguide is a waveguide whose sound speed profile follows [11]:

$$c(z) = \frac{1}{\sqrt{n_z + b}}$$

(47)

A pseudolinear waveguide has an analytical solution involving Airy functions ($Ai(\cdot)$ and $Bi(\cdot)$) and their first derivatives ($Ai'(\cdot)$ and $Bi'(\cdot)$), and the horizontal wavenumbers $k_r$ are the roots of the following transcendental equation [39]:

$$Ai(\beta)Bi'(\gamma) - Ai'(\beta)Bi(\gamma) = 0$$

(48a)

$$\beta = (\omega^2 a)^{-2/3} \left[ k_r^2 - \omega^2 b \right]$$

(48b)

$$\gamma = (\omega^2 a)^{-2/3} \left[ k_r^2 - \omega^2 (aH + b) \right]$$

(48c)
In this example, the seabed is perfectly rigid; we take the sea depth $H = 100$ m, $a = 5.94 \times 10^{-10}$ $s^2/m^3$, and $b = 4.16 \times 10^{-7}$ $s^2/m^3$, the sound source frequency is $f = 50$ Hz, and $M = 4096$. Table 2 lists the discrete modes calculated by the three methods, and the spectral truncation order used by WISpec is $N = 20$. The results of WISpec are very consistent with the analytical solution. Even if there is a certain error, a large part is due to the limited number of discrete points.

Table 2: Discrete modes of the pseudolinear waveguide calculated by the three methods ($f = 50$ Hz, $H = 100$ m).

| Mode order | Analytic solution | WISpec   | SCOOTER |
|------------|-------------------|----------|---------|
| 1          | 0.2130            | 0.2130   | 0.2131  |
| 2          | 0.2044            | 0.2045   | 0.2044  |
| 3          | 0.1943            | 0.1943   | 0.1944  |
| 4          | 0.1785            | 0.1785   | 0.1785  |
| 5          | 0.1549            | 0.1549   | 0.1547  |
| 6          | 0.1188            | 0.1188   | 0.1189  |
| 7          | 0.0477            | 0.0478   | 0.0478  |

5.3. Pekeris waveguide

The Pekeris waveguide is a classic waveguide in ocean acoustics; the ocean environment of the Pekeris waveguide consists of a layer of homogeneous water and an acoustic half-space below it. In this example, the same configuration as the ideal fluid waveguide is used, except that the sound source frequency is $f = 50$ Hz, the density in the acoustic half-space is $\rho_\infty = 1.5$ g/cm$^3$, the speed of sound is $c_\infty = 2000$ m/s, and the attenuation is $\alpha_\infty = 0.5$ dB/\(\lambda\). Since there is no analytical solution for this example, we present the results of SCOOTER [40] and NM-CT [25, 27] in Fig. 6 for reference. The former is a wavenumber integration model based on the finite element method, and the latter is a normal mode model based on the spectral method. The sound fields calculated by the three programs are basically the same, but the two programs based on the wavenumber integration model have a higher degree of agreement. Whether on the sound field or the TL line diagram, the sound field calculated by the NM-CT is still somewhat different. This also proves that for the Pekeris waveguide, the normal mode model generates a certain error in the near field due to ignoring the continuous spectrum.

In addition to point sources, WISpec can also calculate the sound field of line sources. Replacing the sound source with a line source in this example results in a sound field, as displayed in Fig. 7. The sound fields calculated by SCOOTER, WISpec and KRAKENC (a normal mode model based on the finite difference method [41]) are still very similar, but there are slight differences in the near field. Note that when the sound source is a line source, the difference in the sound field calculated by WISpec and KRAKENC in the far field is smaller than that of the point source. To facilitate the comparison with SCOOTER and KRAKENC, the sound field of the line source is normalized as $p_0 = i\rho_0\omega^2H_0(1)/(1/4)$ instead of Eq. (27).

5.4. Munk waveguide

The Munk waveguide is a typical example of deep-sea acoustic propagation problems. Here, the ocean environment consists of a layer of seawater with a sound speed profile of the Munk profile and a homogeneous half-space below, as schematically shown in Fig. 8(a). In this experiment, the frequency of the sound source is $f = 50$ Hz and $z_s = 100$ m, the sea depth is $H = 5000$ m.
Figure 6: Sound fields of the Pekeris waveguide calculated by SCOOTER (a), WISpec (b) and NM-CT (c); TLs along the $r$-direction at a depth of $z = 46$ m (d).
Figure 7: Sound fields of the Pekeris waveguide of the line source calculated by SCOOTER (a), WISpec (b) and KRAKENC (c); TLs along the x-direction at a depth of $z = 46$ m (d).
The number of discrete points in the wavenumber domain is taken as \( M = 55000 \), the integral interval is \([0, 2k_0]\), and the spectral truncation order is \( N = 500 \).

Fig. 9 illustrates the sound fields of the Munk waveguide calculated by SCOOTER, WISpec and NM-CT. The results of the three programs are very similar, and there is almost no difference in the sound fields. This result demonstrates the accuracy of WISpec and demonstrates that normal modes are an excellent approximation model for long-range propagation.

### 5.5. Bucker waveguide

The Bucker waveguide is a benchmark for ocean acoustic propagation models. As shown in Fig. 8(b), the sound speed contrast is very small, yielding a small number of normal modes with real propagation wavenumbers. On the other hand, this environment is characterized by a strong density contrast at the bottom, and the density contrast yields a significant number of virtual modes close to the real wavenumber axis. Therefore, normal mode models ignoring the continuous spectrum are not able to provide accurate predictions of the TL. However, wavenumber integration has no restrictions on the density contrast or on the spectral composition and is therefore capable of providing an exact solution for this waveguide. In this experiment, the sound source frequency is taken as \( f = 100 \text{ Hz} \), \( z_s = 30 \text{ m} \), and the sea depth is \( H = 240 \text{ m} \).
Figure 9: Sound fields of the Munk waveguide calculated by SCOOTER (a), WISpec (b) and KRAENC (c); TLs along the $r$-direction at a depth of $z = 1000$ m (d).
Figure 10: Sound fields of the Bucker waveguide calculated by SCOOTER (a), WISpec (b) and KRAENC (c); TLs along the $r$-direction at a depth of $z = 100$ m (d).
The number of discrete points in the wavenumber domain is taken as $M = 4096$, the integral interval is $[0, 2k_0]$, and the spectral truncation order is $N = 40$. Fig. 10 lists the sound fields of the Bucker waveguide calculated using SCOOTER, WISpec and KRAKENC. Compared with the KRAKENC results, the results of the two wavenumber integration programs, SCOOTER and WISpec, are more similar. As shown in Fig. 10(d) even over a range of 2000 m from the sound source, the sound field predicted by the normal mode program still has a certain error with the wavenumber integration models. This clearly proves that in the Bucker waveguide, the influence of the continuous spectrum is notably important even over very long ranges.

6. Remarks and Conclusion

6.1. Remarks

The above simulation experiments confirm that WISpec is a robust and accurate program and that the spectral method is effective in solving the depth-separated wave equation. From the above analysis, we can directly summarize the following features of the algorithm and program developed in this article:

1. Depending on the range-depth requirement, the evaluation of the Green’s function may have to be performed a substantial number of times. The solution of the depth-separated wave equation is parallelizable because it is independent for different wavenumbers.
2. When applying the spectral method to solve the depth-separated wave equation, as shown in Eq. (38), there is no need to use piecewise linear elements to approximate the environmental parameters, i.e., eliminating the need for subdividing the environment into homogeneous layers, thus avoiding error caused by physical discretization in the vertical direction.
3. The depth-separated wave equation discretized by the Chebyshev–Tau spectral method (see Eq. (42)) yields a block diagonal matrix, and in many cases, the Chebyshev matrix is quasi-diagonally dominant; this sparsity makes it easy to solve efficiently.
4. WISpec can still maintain the advantage that the error decreases exponentially with increasing $N$ in solving the depth-separated wave equation, and this method can often obtain higher accuracy than the low-order finite difference scheme.
5. The algorithm and program designed in this paper can calculate the sound field excited by both point and line sources.

6.2. Conclusion

In this paper, we developed a novel wavenumber integration model that can solve for a two-dimensional sound field in an arbitrary horizontally stratified ocean environment based on a Chebyshev–Tau spectral method. First, the Helmholtz equation is transformed into the wavenumber domain by the Hankel/Fourier transformation. Since horizontally stratified media are considered, the wavenumber kernel function satisfies the depth-separated wave equation. The algorithm first samples the wavenumbers in a preset interval $[k_{\text{min}}, k_{\text{max}}]$ and solves the depth-dependent Green’s function $\Psi(k_r, z)$ in parallel for the discrete wavenumbers obtained by sampling. After the wavenumber kernel function is obtained, the inverse Hankel/Fourier transform is applied to synthesize the sound field in the physical space.

This algorithm is the first to use a spectral method to solve the depth-separated wave equation. Spectral methods use the idea of function approximation to control accuracy and the idea of the weighted residual to discretize the equation. When the ocean environment parameters are
sufficiently smooth, the solution of Green’s function converges exponentially. The results of numerical simulations verify the accuracy and reliability of the model and code. The robust and high-precision Chebyshev–Tau spectral method avoids the possible instability of traditional algorithms for the depth-separated wave equation.

In terms of its application scope, this model requires that the ocean environment be independent in the $r/x$ direction, which limits the practicality of WISpec to a certain extent. Therefore, developing a high-precision wavenumber integration model based on the spectral method to solve the range-dependent waveguide has a bright future. In addition, elastic sediment is a more accurate model of the real ocean environment. In the future, WISpec can be further improved to enable the prediction of more complicated ocean acoustic fields.

Acknowledgments

This work was supported by the National Key Research and Development Program of China [grant number 2016YFC1401800] and the National Natural Science Foundation of China [grant number 61972406].

References

[1] F. B. Jensen, W. A. Kuperman, M. B. Porter, H. Schmidt, Computational Ocean Acoustics, Springer-Verlag, New York, 2011. doi:10.1007/978-1-4419-8678-8

[2] P. C. Etter, Underwater Acoustic Modeling and Simulation, CRC Press, Boca Raton, USA, 2018. doi:10.1201/9781315166346

[3] C. L. Pekeris, Theory of propagation of explosive sound in shallow water, Geological Society of America Memoirs 27 (1) (1948) 1–117. doi:10.1130/mem27-2-p1

[4] W. M. Ewing, W. S. Jardetzky, P. Press, Elastic Wave in Layered Media, McGraw-Hill, New York, 1957. doi:10.1785/0150030290

[5] W. T. Thomson, Transmission of elastic waves through a stratified solid medium, Journal of Applied Physics 21 (2) (1950) 89–93. doi:10.1063/1.1699629

[6] N. A. Haskell, The dispersion of surface waves in multilayered media, Bulletin of the Seismological Society of America 43 (1) (1953) 17–34. doi:10.1785/0150030290

[7] B. L. N. Kennett, Seismic Wave Propagation in Stratified Media, ANU Press, Canberra, 2009. doi:10.22457/bodem05.2009

[8] B. L. N. Kennett, Reflections, rays and reverberations, Bulletin of the Seismological Society of America 64 (6) (1974) 1685–1699. doi:10.1785/1500688

[9] B. L. N. Kennett, N. J. Kerry, Seismic waves in a stratified half-space, Geophysical Journal of the Royal Astronomical Society 57 (3) (1979) 557–583. doi:10.1111/j.1365-246X.1979.tb06779.x

[10] H. Schmidt, User’s guide of SAFARI, seismo-acoustic fast field algorithm for range-independent environments (1988). URL https://openlibrary.cmre.nato.int/handle/20.500.12489/281

[11] H. Schmidt, F. B. Jensen, A full wave solution for propagation in multilayered viscoelastic media with application to gaussian beam reflection at fluid–solid interfaces, The Journal of the Acoustical Society of America 77 (3) (1985) 813–825. doi:10.1121/1.392050

[12] H. Schmidt, G. J. Tango, Efficient global matrix approach to the computation of synthetic seismograms, Geophysical Journal International 84 (2) (1986) 331–359. doi:10.1111/j.1365-246X.1986.tb04369.x

[13] F. B. Jensen, On the use of stair steps to approximate bathymetry changes in ocean acoustic models, The Journal of the Acoustical Society of America 104 (3) (1998) 1310–1315. doi:10.1121/1.422430

[14] S. A. Orszag, Comparison of pseudospectral and spectral approximation, Studies in Applied Mathematics 51 (3) (1972) 253–299. doi:10.1002/sapm1972513253

[15] D. Gottlieb, S. A. Orszag, Numerical Analysis of Spectral Methods, Theory and Applications, Society for Industrial and Applied Mathematics, Philadelphia, USA, 1977. doi:10.1137/1.9781561147042

[16] C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang, Spectral Methods in Fluid Dynamics, Springer-Verlag, Berlin, Germany, 1988. doi:10.1007/978-3-642-84108-8

[17] B. Guo, Spectral Methods and Their Applications, World Scientific, 1998. doi:10.1142/3662
[40] M. B. Porter, *SCOOTER*: A finite element FFP code (2022).
URL https://oalib-acoustics.org/models-and-software/acoustics-toolbox/

[41] M. B. Porter, The kraken normal mode program (2001).
URL https://oalib-acoustics.org/models-and-software/normal-modes/