Classification of four and six dimensional Drinfel’d superdoubles

A. Eghbali,* A. Rezaei-Aghdam† and F. Heidarpour

Department of Physics, Faculty of science, Azarbaijan University of Tarbiat Moallem , 53714-161, Tabriz, Iran

February 23, 2011

Abstract

Using adjoint representation we firstly classify two and three dimensional Lie super-bialgebras obtain from decomposable Lie superalgebras. In this way we complete the classification obtained by Eghbali et al., [J. Math. Phys. 51, 073503 (2010); e-print arXiv:0901.4471 [math-ph]]. Then we classify all four and six dimensional Drinfel’d superdoubles.

*e-mail: a.eghbali@azaruniv.edu
†Corresponding author. e-mail: rezaei-a@azaruniv.edu
1 Introduction

Lie super-bialgebras [1], as the underlying symmetry algebras, play an important role in the integrable structure of \( AdS/CFT \) correspondence [2]. Similarly, one can consider Poisson-Lie T-dual sigma models on Poisson-Lie supergroups [3]. In this way and by considering that there is a universal quantization for Lie super-bialgebras [4], one can assign an important role to the classification of Lie super-bialgebras (especially low dimensional Lie super-bialgebras) from both physical and mathematical point of view. Until now there are distinguished and nonsystematic ways for obtaining low dimensional Lie super-bialgebras (see, for example, [5, 6, 7]). In Ref. [8], using the adjoint representation of Lie superalgebras, we have given a systematic way for obtaining and classification of low dimensional Lie super-bialgebras and applied this method to the classification of two and three dimensional Lie super-bialgebras related to indecomposable Lie superalgebras of Ref. [9]. In the continuation of that work we tried to classify all Lie superalgebras of Drinfeld’s superdoubles of these Lie super-bialgebras. Then, we saw Ref. [10] in which they have tried to do this work for Lie super-bialgebras related to two and three dimensional decomposable and indecomposable Lie superalgebras [9]. But unfortunately there are some incorrect results in their paper. Firstly, because they use same notation for nonisomorphic Lie superalgebras and incorrect automorphism Lie supergroups for some Lie superalgebras so the number of their Lie super-bialgebras are incorrect. Secondly, and importantly they use nonstandard basis in determination of Lie superalgebras of Drinfeld’s superdoubles. For these reasons some of their principal results (theorem 3) are incorrect. For obtaining correct results one must use standard basis for Lie superalgebras and superdeterminant for isomorphism matrices. Thirdly, their mixed commutation relations are not compatible with ad-invariant form \(< . , . > \) on \( D \). For these reasons, we are motivated to write this paper to complete the results of Ref. [8] and classify two and three dimensional Lie super-bialgebras related to decomposable Lie superalgebras. Furthermore by use of standard basis for Lie superalgebras of Drinfeld’s superdoubles we classify all four and six dimensional Drinfeld’s superdoubles as theorems 1-3.

The paper is organized as follows. In section two, we review and rewrite correctly the results of Ref. [8] about two and three dimensional Lie super-bialgebras; then by obtaining Lie super-bialgebras for decomposable Lie superalgebras we complete that work. In section three, we find nonisomorphic four and six dimensional Drinfeld’s superdoubles by use of standard basis, the results are summarized as three theorems. In appendix A we give some notations about supermatrices and supertensors in the standard basis. In appendix B we give solutions of super Jacobi and mixed super Jacobi identities for dual Lie superalgebras of decomposable Lie superalgebras. The differences between the list of Lie super-bialgebras of Ref. [8] with those of Ref. [10] are given in appendix C. In appendix D we give the (anti)commutation relations of four and six dimensional Drinfeld’s superdoubles. The details of differences between theorem 3 and those of Ref. [10] are given in appendix E. Finally, the isomorphism matrices of four and six dimensional Drinfeld’s superdoubles are listed in appendix F.

2 Two and three dimensional Lie super-bialgebras

Let us first review some basic definitions and notations about Lie superalgebras and Lie super-bialgebras (see, Refs. [1] and [8]).

Definition 1: A Lie superalgebra \( \mathfrak{g} \) is a graded vector space \( \mathfrak{g} = \mathfrak{g}_B \oplus \mathfrak{g}_F \) with gradings; \( grade(\mathfrak{g}_B) = 0, \) \( grade(\mathfrak{g}_F) = 1; \) such that Lie bracket satisfies the super antisymmetric and super Jacobi identities, i.e., in a graded basis \( \{ X_i \} \) of \( \mathfrak{g} \) if we put [11]

\[
[X_i, X_j] = f_{ij}^k X_k,
\]

then

\[
(-1)^{(i+j+k)} f_{ij}^m f_{kl}^i + f_{ij}^m f_{lk}^i + (-1)^{(i+j)} f_{ik}^m f_{lj}^i = 0,
\]

so that

\[
f_{ij}^k = -(-1)^{ij} f_{ji}^k.
\]
Note that, in the conventional basis, $f^B_{BB}$ and $f^F_{BF}$ are real c-numbers and $f^B_{FF}$ are pure imaginary c-numbers and other components of structure constants $f^i_{jk}$ are zero \[^{12}\], i.e., we have

$$f^k_{ij} = 0, \quad \text{if} \quad \text{grade}(i) + \text{grade}(j) \neq \text{grade}(k) \quad (\text{mod}2). \quad (4)$$

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra and $\mathfrak{g}^*$ be its dual vector space and let $(\cdot, \cdot)$ be the canonical pairing on $\mathfrak{g}^* \oplus \mathfrak{g}$.

**Definition 2:** A Lie super-bialgebra structure on a Lie superalgebra $\mathfrak{g}$ is a super skew-symmetric linear map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ (the super cocommutator) so that [1]

1) $\delta$ is a super one-cocycle, i.e.
$$\delta([X, Y]) = (ad_X \otimes I + I \otimes ad_X)\delta(Y) - (-1)^{|X||Y|}(ad_Y \otimes I + I \otimes ad_Y)\delta(X) \quad \forall X, Y \in \mathfrak{g},$$
where $|X||Y|$ indicates the grading of $X(Y)$;
2) the dual map $\delta^\flat : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a Lie superbracket on $\mathfrak{g}^*$, i.e.,
$$(\xi \otimes \eta, \delta(X)) = (\delta^\flat(\xi \otimes \eta), X) = ([\xi, \eta]_\ast, X) \quad \forall X \in \mathfrak{g}; \quad \xi, \eta \in \mathfrak{g}^*.$$  

The Lie super-bialgebra defined in this way will be indicated by $(\mathfrak{g}, \delta)$.

**Proposition 1:** Let $(\mathfrak{g}, \delta)$ be a Lie super-bialgebra. There exists a unique Lie superalgebra structures with the following commutation relations on the vector space $\mathfrak{g} \oplus \mathfrak{g}$ such that $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie superalgebras and the natural scalar product on $\mathfrak{g} \oplus \mathfrak{g}$ is invariant

$$[x, y]_\mathcal{D} = [x, y], \quad [x, \xi]_\mathcal{D} = -(\cdot)^{[x][y]} ad_x^\ast y + ad_y^\ast x, \quad [\xi, \eta]_\mathcal{D} = [\xi, \eta]_\mathfrak{g}^*, \forall x, y \in \mathfrak{g}; \quad \xi, \eta \in \mathfrak{g}^*,$$  

where

$$< ad_x y, \xi > = -(-1)^{|x||y|} y, \quad < ad_y^\ast x, \eta >, \quad < ad_x^\ast \xi, x > = -(-1)^{|\xi||\eta|} \eta, \quad < ad_y \xi, x > .$$

The Lie superalgebra $\mathcal{D} = \mathfrak{g} \oplus \mathfrak{g}$ is called Drinfel’d superdouble.

**Proposition 2:** If there exists an automorphism $A$ of $\mathfrak{g}$ such that

$$\delta' = (A \otimes A) \circ \delta \circ A^{-1},$$
then the super one-cocycles $\delta$ and $\delta'$ of the Lie superalgebra $\mathfrak{g}$ are equivalent. In this case the two Lie super-bialgebras $(\mathfrak{g}, \delta)$ and $(\mathfrak{g}, \delta')$ are equivalent (as in the bosonic case [13]).

**Definition 3:** A Manin super triple [1] is a triple of Lie superalgebras $(\mathcal{D}, \mathfrak{g}, \tilde{\mathfrak{g}})$ together with a non-degenerate ad-invariant super symmetric bilinear form $<\cdot, \cdot >$ on $\mathcal{D}$, such that

1) $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are Lie sub-superalgebras of $\mathcal{D}$,
2) $\mathcal{D} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$ as a supervector space,
3) $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are isotropic with respect to $<\cdot, \cdot>$, i.e.,

$$< X_i, X_j > = < \tilde{X}_i, \tilde{X}_j > = 0, \quad \delta^{ij} = < X_i, \tilde{X}_j > = (-1)^{|i|} < \tilde{X}_j, X_i > = (-1)^{|j|} \delta^{ji},$$

where $\{X_i\}$ and $\{\tilde{X}_i\}$ are basis of Lie superalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, respectively. Note that in the above relation $\delta^{ij}_i$ is the ordinary delta function. There is a one-to-one correspondence between Lie super-bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ and Manin super triple $(\mathcal{D}, \mathfrak{g}, \tilde{\mathfrak{g}})$ with $\mathfrak{g}^* = \tilde{\mathfrak{g}}$. [1]. If we choose the structure constants of Lie superalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ as

$$[X_i, X_j] = f^k_{ij} X_k, \quad [\tilde{X}_i, \tilde{X}_j] = \tilde{f}^{ij}_k \tilde{X}_k,$$
then ad-invariance of the bilinear form $<\cdot, \cdot>$ on $\mathcal{D} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$ implies that

$$[X_i, \tilde{X}_j] = (-1)^j \tilde{f}^{ij}_k X_k + (-1)^i f^j_{ki} \tilde{X}^k.$$  

(13)
Clearly, using the Eqs. (6), (11) and (12) we have
\[ \delta(X_i) = (-1)^{jk} \tilde{f}^{jk} X_j \otimes X_k. \] (14)

note that the appearance of \((-1)^{jk}\) in this relation is due to the definition of natural inner product between \(g \otimes g\) and \(g^* \otimes g^*\) as \((X^i \otimes \tilde{X}^j, X^k \otimes X^l) = (-1)^{jk} \delta^i_j \delta^l_k\).

As a result, if we apply this relation in the super one-cocycle condition (5), super Jacobi identities (2) for the dual Lie superalgebra and the following mixed super Jacobi identities are obtained:
\[ f^m_{\ jk} \tilde{f}^{jl} \ m = f^i_{\ mk} \tilde{f}^{im}_{\ j} + f^j_{\ jm} \tilde{f}^{im}_{\ k} + (-1)^{ik} f^i_{\ jm} \tilde{f}^{jm}_{\ k} + (-1)^{jk} f^j_{\ mk} \tilde{f}^{im}_{\ j}. \] (15)

This relation can also be obtained from super Jacobi identity of \(D\). In Ref. [8] we find and classify all two and three dimensional Lie super-bialgebras for all two and three dimensional indecomposable Lie superalgebras of Ref. [9]. The method of classification is new and indeed it is improvement and generalization of the method of Ref. [10] to the Lie superalgebras. Note that in Ref. [15] unfortunately there is not standard and logically method for obtaining of low dimensional Lie bialgebras. In this method by use of adjoint representation for super Jacobi and mixed super Jacobi identities (2) and (15) we write these relations in the matrix form and solve them for finding dual Lie superalgebras by direct calculation; then by use of automorphism Lie supergroups of Lie superalgebras we classify all non isomorphic two and three dimensional Lie super-bialgebras [5]. Here the list of two and three dimensional indecomposable Lie superalgebras [9] and their related Lie super-bialgebras [8] are given in table 1 and tables 4, 5 and 6, respectively; note that as we use DeWitt notation and standard basis here, the structure constants \(C^p_{jk}\) must be pure imaginary. Furthermore in this section we consider decomposable Lie superalgebras as in the following table 2 and obtain related Lie super-bialgebras by use of the method mentioned in Ref. [8].

### Table 1: Two and three dimensional indecomposable Lie superalgebras (Ref. [10]).

| Type | \(g\) | Bosonic basis | Fermionic basis | Non-zero (anti)commutation relations | Comments |
|------|------|--------------|----------------|--------------------------------------|---------|
| (1,1) | \(B\) | \(X_1\) | \(X_2\) | \([X_1, X_2] = X_2\) | Trivial |
| \((A_{1,1} + A)\) | \(X_1\) | \(X_2\) | \([X_2, X_2] = iX_1\) | Nontrivial |
| (2,1) | \(C^1_p\) | \(X_1, X_2\) | \(X_3\) | \([X_1, X_2] = X_2, [X_1, X_3] = pX_3\) | \(p \neq 0\), Trivial |
| \(C^2_p\) | \(X_1, X_2\) | \(X_2, X_3\) | \([X_1, X_2] = X_2, [X_1, X_3] = \frac{1}{2} X_3, \{X_3, X_3\} = iX_2\) | Nontrivial |
| (1,2) | \(C^3\) | \(X_1\) | \(X_2, X_3\) | \([X_1, X_2] = X_2, [X_1, X_3] = pX_3\) | \(0 < |p| \leq 1\), Trivial |
| \(C^4\) | \(X_1\) | \(X_2, X_3\) | \([X_1, X_2] = X_2, [X_1, X_3] = X_2 + X_3\) | Trivial |
| \(C^5_p\) | \(X_1\) | \(X_2, X_3\) | \([X_1, X_2] = pX_2 - X_3, [X_1, X_3] = X_2 + pX_3\) | \(p \geq 0\), Trivial |
| \((A_{1,1} + 2A)^1\) | \(X_1\) | \(X_2, X_3\) | \([X_2, X_2] = iX_1, [X_3, X_3] = iX_1\) | Nilpotent, Nontrivial |
| \((A_{1,1} + 2A)^2\) | \(X_1\) | \(X_2, X_3\) | \([X_2, X_2] = iX_1, [X_3, X_3] = -iX_1\) | Nilpotent, Nontrivial |

### Table 2: Three dimensional decomposable Lie superalgebras.

| Type | \(g\) | Bosonic basis | Fermionic basis | Non-zero (anti) commutation relations | Comments |
|------|------|--------------|----------------|--------------------------------------|---------|
| (2,1) | \((B + A_{1,1})\) | \(X_1, X_2\) | \(X_3\) | \([X_1, X_3] = X_3\) | Solvable, Trivial |
| \((2A_{1,1} + A) = (A_{1,1} + A) \oplus A_{1,0}\) | \(X_1, X_2\) | \(X_3\) | \([X_3, X_3] = iX_1\) | Nilpotent, Nontrivial |
| \(C^4_p = C^4_{p=0}\) | \(X_1, X_2\) | \(X_3\) | \([X_1, X_2] = X_2\) | Solvable, Trivial |
| (1,2) | \(C^5_p = C^5_{p=0} = B \oplus A_{0,1}\) | \(X_1\) | \(X_2, X_3\) | \([X_1, X_2] = X_2\) | Solvable, Trivial |
| \((A_{1,1} + 2A)^0 = (A_{1,1} + A) \oplus A_{0,1}\) | \(X_1\) | \(X_2, X_3\) | \([X_2, X_2] = iX_1\) | Nilpotent, Nontrivial |
Note that as discussed above, for this purpose we need automorphism Lie supergroups of these Lie superalgebras where we have obtained and listed in the following table 3.

Table 3: Automorphism Lie supergroups of the two and three dimensional decomposable Lie superalgebras.

| g          | Automorphism Lie supergroups | Comments |
|------------|-----------------------------|----------|
| (B + A₁,₁) | \[
\begin{pmatrix}
1 & a & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{pmatrix}, b, c \in \mathbb{R} - \{0\}, a \in \mathbb{R}
\] |          |
| (2A₁,₁ + A) | \[
\begin{pmatrix}
a^2 & 0 & 0 \\
c & b & 0 \\
0 & 0 & a
\end{pmatrix}, a, b \in \mathbb{R} - \{0\}, c \in \mathbb{R}
\] |          |
| C₁₀        | \[
\begin{pmatrix}
1 & a & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{pmatrix}, b, c \in \mathbb{R} - \{0\}, a \in \mathbb{R}
\] |          |
| C₂₀        | \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & a & 0 \\
0 & 0 & b
\end{pmatrix}, a, b \in \mathbb{R} - \{0\}
\] |          |
| (A₁,₁ + 2A)⁰ | \[
\begin{pmatrix}
a^2 & 0 & 0 \\
0 & a & b \\
0 & 0 & c
\end{pmatrix}, a, c \in \mathbb{R} - \{0\}, b \in \mathbb{R}
\] |          |

The solutions of super Jacobi and mixed super Jacobi identities and isomorphism matrices C related to these decomposable Lie superalgebras are listed in appendix B. The related Lie super-bialgebras are listed in tables 4 and 6. Note that in tables 4-6, \(I_{(m,n)}\) represent the Abelian Lie superalgebras with \(m(n)\) bosonic/fermionic generators.

Table 4: Three dimensional Lie super-bialgebras of the type (2,1).

| g          | ĝ                      | Non-zero (anti) commutation relations of ĝ | Comments |
|------------|-------------------------|--------------------------------------------|----------|
| (2A₁,₁ + A) | \(I_{(2,1)}\)          |                                            |          |
| (B + A₁,₁) | \(I_{(2,1)}\)          |                                            |          |
| (B + A₁,₁),i | \(\hat{X}^2, \hat{X}^3\) = \(\hat{X}^3\) |                                            |          |
| (2A₁,₁ + A) | \(\hat{X}^3, \hat{X}^3\) = \(i\hat{X}^1\) |                                            |          |
| (2A₁,₁ + A),i | \(\hat{X}^3, \hat{X}^3\) = \(\hat{X}^3\) |                                            |          |
| C₁₀        | \(I_{(2,1)}\)          |                                            |          |
| (2A₁,₁ + A) | \(\hat{X}^3, \hat{X}^3\) = \(i\hat{X}^1\) |                                            |          |
| (2A₁,₁ + A),i | \(\hat{X}^3, \hat{X}^3\) = \(\hat{X}^3\) |                                            |          |
| (2A₁,₁ + A),ii | \(\hat{X}^3, \hat{X}^3\) = \(i\hat{X}^2\) |                                            |          |
| C₁₋₀,i      | \(\hat{X}^1, \hat{X}^2\) = \(\hat{X}^1\), \(\hat{X}^2, \hat{X}^3\) = \(\frac{i}{2}\hat{X}^3\) | p ∈ \(\mathbb{R}\) |          |
| C₁₋₀,ii     | \(\hat{X}^1, \hat{X}^2\) = \(\hat{X}^2\), \(\hat{X}^3\) = \(\frac{i}{2}\hat{X}^3\) | p ∈ \(\mathbb{R}\) |          |
| C₁₋₀,i      | \(\hat{X}^1, \hat{X}^2\) = \(\hat{X}^1\), \(\hat{X}^2, \hat{X}^3\) = \(\frac{i}{2}\hat{X}^3\) | p ∈ \(\mathbb{R}\) |          |
| C₁₋₀,ii     | \(\hat{X}^1, \hat{X}^2\) = \(\hat{X}^2\), \(\hat{X}^3\) = \(\frac{i}{2}\hat{X}^3\) | p ∈ \(\mathbb{R}\) |          |
| C₁₋₀,i      | \(\hat{X}^1, \hat{X}^2\) = \(\hat{X}^1\), \(\hat{X}^2, \hat{X}^3\) = \(\frac{i}{2}\hat{X}^3\) | p ∈ \(\mathbb{R}\) |          |
| C₁₋₀,ii     | \(\hat{X}^1, \hat{X}^2\) = \(\hat{X}^2\), \(\hat{X}^3\) = \(\frac{i}{2}\hat{X}^3\) | p ∈ \(\mathbb{R}\) |          |
Table 5: Two dimensional Lie super-bialgebras of the type (1,1).

| g       | \( \hat{g} \) | (Anti) Commutation relations of \( \hat{g} \) |
|---------|---------------|---------------------------------|
| \((1,1 + A)\) | \( I_{(1,1)} \) | \((\hat{X}^2, \hat{X}^2) = i\hat{X}^1\) |
| B       | \( I_{(1,1)} \) | \((\hat{X}^2, \hat{X}^2) = -i\hat{X}^1\) |

Table 6: Three dimensional Lie super-bialgebras of the type (1,2), where \( \epsilon = \pm 1 \).

| g       | \( \hat{g} \) | Comments |
|---------|---------------|----------|
| \( C^2_\epsilon \) | \( I_{(2,1)} \) | \((A_{1,1} + 2A)^0, (A_{1,1} + 2A)^1, (A_{1,1} + 2A)^2 \) |
| \(-1 \leq p < 1\) | \( I_{(2,2)} \) | \((A_{1,1} + 2A)^0, (A_{1,1} + 2A)^1, (A_{1,1} + 2A)^2 \) |
| \( C^3 \) | \( I_{(1,1)} \) | \((A_{1,1} + 2A)^0, (A_{1,1} + 2A)^1, (A_{1,1} + 2A)^2 \) |
| \( C^4 \) | \( I_{(1,2)} \) | \((A_{1,1} + 2A)^0, (A_{1,1} + 2A)^1, (A_{1,1} + 2A)^2 \) |
| \( C^2_\epsilon \) | \( I_{(1,2)} \) | \((A_{1,1} + 2A)^0, (A_{1,1} + 2A)^1, (A_{1,1} + 2A)^2 \) |

For three dimensional dual Lie superalgebras \((A_{1,1} + 2A)^0\), \((A_{1,1} + 2A)^1\) and \((A_{1,1} + 2A)^2\), respectively, we have the following anticommutation relations:

\[
\{ \hat{X}^2, \hat{X}^2 \} = i\alpha \hat{X}^1, \quad \{ \hat{X}^2, \hat{X}^3 \} = i\beta \hat{X}^1, \quad \{ \hat{X}^3, \hat{X}^3 \} = i\gamma \hat{X}^1, \quad \alpha, \beta, \gamma \in \mathbb{R}.
\] (16)

Note that these Lie superalgebras are non-isomorphic and they differ in the bound of their parameters. The details of differences between these lists with lists of Ref. [10] are given in appendix C. These differences cause the differences between our theorem 3 with that of Ref. [10].
3 Four and six dimensional Drinfel’d superdoubles

Here we consider how many of four and six dimensional Lie superalgebras $D$ of Drinfel’d superdoubles $D$ where related respectively to 4 two dimensional and 70 three dimensional [17] Lie super-bialgebras of tables 4-6 are isomorphic. Indeed here we classify Lie superalgebras $D$; but for simply connected Lie supergroups $D$ this classification is equivalent to the classification of Drinfel’d superdoubles $D$. Furthermore for each Lie superalgebra $D$ with Lie super-bialgebras $(g, ̃g)$ there is other decomposition $(g, ̃g)$ as their dual which must be considered. Now we consider the Manin super triple $(D, g, ̃g)$ with the commutation relations (12) and (13) for the Lie superalgebra $D$. In general for writing these commutation relations in the standard basis for $D$ we must first omit the $i$ coefficient from these relations, then by use of $\{T_1, ..., T_{m+n}\}$ bosonic and $\{T_{m+n+1}, ..., T_{n+̃n}\}$ fermionic basis for $D = g_{m+n} \oplus ̃g_{n+̃n}$ we write the (anti)commutation relations for $D$ in the standard basis by multiply the $i$ to the fermion-fermion anticommutation relations coefficients. In this way we write the (anti)commutation relations of all four and six dimensional Drinfel’d superdoubles [these are given in appendix D]. Now we use the fact that two Lie superalgebras $D = D_B + D_\mathcal{F}$ and $D' = D_B + D_\mathcal{F}'$ are isomorphic if there are isomorphism between $D_B \rightarrow D_B$ and $D_\mathcal{F} \rightarrow D_\mathcal{F}'$. This means that in the standard basis there are block diagonal isomorphism matrix $C$ between $D$ and $D'$ such that its superdeterminant [see appendix A for definition of superdeterminant] is non zero and satisfy in the following relation

\[ (-1)^{KL+MJ} C Y^M C^st = Y^N C^M_N, \]

(17)

where $(Y^M)_{KL} = -F^M_{KL}$ are the adjoint representations and $F^M_{KL}$ are the structure constants of Lie superalgebra $D$ in the standard basis $\{T_1, ..., T_{m+n}\}$, $(L, M, K = 1, ..., n + ̃n)$ and the indices $K$ and $L$ correspond to the row and column of matrix $Y^M$, respectively, and $J$ denotes the column of matrix $C^st$ [here superscript $st$ stands for supertranspose]. In this way we classify all four and six dimensional Drinfel’d’s superdoubles. We perform this work by use of MAPLE 10 program. The results are written as the following three theorems. The isomorphism matrices are given in appendix B. Note that the Drinfel’d’s superdoubles related for Manin super triple $(D, g_{(m,n)}, ̃g_{(0,n,̃n)})$ are shown as $Dsd_{(m+n,m+n)}$.

**Theorem 1:** Every four dimensional Drinfel’d’s superdoubles of the type $(2,2)$ belongs to the one of the following 3 classes and allows decomposition into all Lie super-bialgebras listed in the class and their duals.

- $Dsd^1_{(2,2)} : (I_{(1,1)}, I_{(1,1)})$
- $Dsd^2_{(2,2)} : ((A_{1,1} + A), I_{(1,1)})$
- $Dsd^3_{(2,2)} : (B, I_{(1,1)}), (B, (A_{1,1} + A)), (B, (A_{1,1} + A)).$

**Theorem 2:** Every six dimensional Drinfel’d’s superdoubles of the type $(4,2)$ belongs to the one of the following 9 classes and allows decomposition into all Lie super-bialgebras listed in the class and their duals.

- $Dsd^1_{(4,2)} : (I_{(2,1)}, I_{(2,1)})$
- $Dsd^2_{(4,2)} : ((2A_{1,1} + A), I{(2,1)})$
- $Dsd^3_{(4,2)} : (B, (2A_{1,1} + A) + (B, A_{1,1}, I{(2,1)}), (B, (2A_{1,1} + A), (B, (2A_{1,1} + A) + 1))$
- $Dsd^4_{(4,2)} : (C^0_{p=0}, I{(2,1)}), (C^1_{p=0}, C^1_{p=0}).$
- $Dsd^5_{(4,2)} : (C^0_{p=0}, (2A_{1,1} + A) + (C^0_{p=0}, I{(2,1)}), (C^1_{p=0}, (2A_{1,1} + A) + 1), (C^1_{p=0}, (2A_{1,1} + A) + 1)).$
- $Dsd^6_{(4,2)} : (C^0_{p=0}, (2A_{1,1} + A) + 1), (C^1_{p=0}, (2A_{1,1} + A) + 1), k \in \mathbb{R} \setminus \{0\}$
- $Dsd^7_{(4,2)} : (C^0_{p=0}, (2A_{1,1} + A) + 1, (C^0_{p=0}, I{(2,1)}), (C^1_{p=0}, C^1_{p=0}), (C^0_{p=0}, C^1_{p=0}), (C^1_{p=0}, C^0_{p=0}), (C^1_{p=0}, C^1_{p=0}), (C^0_{p=0}, C^1_{p=0}), (C^1_{p=0}, C^0_{p=0})).$
- $Dsd^8_{(4,2)} : (C^0_{p=0}, (2A_{1,1} + A) + 1, (C^0_{p=0}, I{(2,1)}), (C^1_{p=0}, C^1_{p=0}), (C^0_{p=0}, C^1_{p=0}), (C^1_{p=0}, C^0_{p=0}), (C^1_{p=0}, C^1_{p=0}), (C^0_{p=0}, C^1_{p=0}), (C^1_{p=0}, C^0_{p=0})).$
- $Dsd^9_{(4,2)} : (C^0_{p=0}, (2A_{1,1} + A) + 1, (C^0_{p=0}, I{(2,1)}), (C^1_{p=0}, C^1_{p=0}), (C^0_{p=0}, C^1_{p=0}), (C^1_{p=0}, C^0_{p=0}), (C^1_{p=0}, C^1_{p=0}), (C^0_{p=0}, C^1_{p=0}), (C^1_{p=0}, C^0_{p=0})).$
Theorem 3: Every six dimensional Drinfeld's superdoubles of the type (2, 4) belongs to the one of the following 13 classes and allows decomposition into all Lie super-bialgebras listed in the class and their duals.

\[ Dsd^1_{(2,4)} : (I_{(1,2)}, I_{(1,2)}) \]

\[ Dsd^2_{(2,4)} : (C^0_0, (A_{1,1} + 2A)^0_0, 0, 0), (C^0_0, (A_{1,1} + 2A)^2_0, 2, 2), (C^0_0, (A_{1,1} + 2A)^1_0, 0, 0), (C^0_0, (A_{1,1} + 2A)^2_0, 1, 0), \]

\[ Dsd^3_{(2,4)} : (C^0_0, (A_{1,1} + 2A)^3_0, 0, 0), (C^0_0, (A_{1,1} + 2A)^0_0, 1, 0), (C^0_0, (A_{1,1} + 2A)^2_0, 1, 0), \]

\[ Dsd^4_{(2,4)} : (C^3, (A_{1,1} + 2A)^0_0, 0, 0), (C^3, (A_{1,1} + 2A)^1_0, 0, 0), (C^3, (A_{1,1} + 2A)^2_0, 0, 0), (C^3, (A_{1,1} + 2A)^1_0, 0, 0), \]

\[ Dsd^5_{(2,4)} : (C^4, (A_{1,1} + 2A)^0_0, 0, 0), (C^4, (A_{1,1} + 2A)^1_0, 0, 0), (C^4, (A_{1,1} + 2A)^0_0, 0, 0), \]

\[ Dsd^6_{(2,4)} : (C^4, (A_{1,1} + 2A)^0_0, 0, 0), (C^4, (A_{1,1} + 2A)^1_0, 0, 0), (C^4, (A_{1,1} + 2A)^0_0, 0, 0), \]

\[ Dsd^7_{(2,4)} : (C^6, (A_{1,1} + 2A)^0_0, 0, 0), (C^6, (A_{1,1} + 2A)^1_0, 0, 0), (C^6, (A_{1,1} + 2A)^0_0, 0, 0), \]

\[ p > 0 \]

\[ Dsd^8_{(2,4)} : (C^7, (A_{1,1} + 2A)^0_0, 0, 0), (C^7, (A_{1,1} + 2A)^1_0, 0, 0), (C^7, (A_{1,1} + 2A)^0_0, 0, 0), \]

\[ Dsd^9_{(2,4)} : (A_{1,1} + 2A)^0, I_{(1,2)} \]

\[ Dsd^{10}_{(2,4)} : (A_{1,1} + 2A)^1, I_{(1,2)} \]
4 Conclusion

We classify two and three dimensional Lie super-bialgebras obtained from decomposable Lie superalgebras. We also classified all four and six dimensional Drinfel’d superdoubles as three theorems. Using this classification one can investigate Poisson-Lie T plurality of sigma models over Lie supergroups. Obtaining the modular spaces of these Drinfel’d superdoubles is the other open problem.

Acknowledgments

We would like to thank F. Darabi for carefully reading the manuscript and useful comments.

Appendix A: Some notations about supermatrices and supertensors

In this appendix, We consider the standard basis for the supervector spaces so that in writing the basis as a column matrix, we first present the bosonic base and then the fermionic one. The transformation of standard basis and its dual basis can be written as follows:

\[ e'_i = (-1)^j K^j_i e_j, \quad e'^{ij} = K^{-stij} e^j, \]

where the transformation matrix \( K \) has the following block diagonal representation [12]

\[ K = \begin{pmatrix} A & C \\ D & B \end{pmatrix}, \]

(18)

where \( A, B \) and \( C \) are real submatrices and \( D \) is pure imaginary submatrix [12]. Here we consider the matrix and tensors having a form with all upper and lower indices written in the right hand side.

1. The transformation properties of upper and lower right indices to the left one for general tensors are as follows:

\[ T_{i...k} = T_{i...k}, \quad T^{i...k} = (-1)^j T^{i...k}. \]

(19)

2. For supertransposition we have

\[ L^{stij} = (-1)^{ij} L^{ij}, \quad L^{stij} = (-1)^{ij} L^{ij}, \]

\[ M^{stij} = (-1)^{ij} M^{ij}, \quad M^{stij} = (-1)^{ij} M^{ij}. \]

(20)

3. For superdeterminant we have

\[ \text{sdet} \left( \begin{pmatrix} A & C \\ D & B \end{pmatrix} \right) = \det(A - CB^{-1}D)(\det B)^{-1}, \]

(21)

when \( \det B \neq 0 \) and

\[ \text{sdet} \left( \begin{pmatrix} A & C \\ D & B \end{pmatrix} \right) = (\det(B - DA^{-1}C))^{-1} (\det A), \]

(22)

when \( \det A \neq 0 \).

Appendix B: Solutions of super Jacobi and mixed super Jacobi identities \((sJ - msJ)\) for dual of decomposable Lie superalgebras and isomorphism matrices \( C \)

1. Solutions of \((sJ - msJ)\) for dual Lie superalgebras of \((B + A_{1,1})\) are

\[ i) \quad \tilde{f}^{23}_{3} = \alpha, \quad \alpha \in \mathbb{R}, \]

\[ ii) \quad \tilde{f}^{33}_{1} = i\beta, \quad \beta \in \mathbb{R}. \]
Isomorphism matrix $C$ between solution (i) and $(B + A_{1,1})$ is as follows:

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & 0 & 0 \\ 0 & 0 & c_{33} \end{pmatrix}, \quad c_{12}, c_{21}, c_{33} \in \mathbb{R} - \{0\}; \quad c_{11} \in \mathbb{R},$$

where $c_{12} = \frac{1}{f_{j_{1,2}}^1}$ and by imposing that $C$ must be the transformation matrix (18), we then have $c_{13} = 0$.

For solution (ii) we have isomorphism matrix between this solution and $(2A_{1,1} + A)$ as follows:

$$C = \begin{pmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \quad c_{11}, c_{22}, c_{33} \in \mathbb{R} - \{0\}; \quad c_{21} \in \mathbb{R},$$

where $c_{11} = -ic_{33}^2\tilde{f}_{j_{1,1}}^3$ and by imposing that $C$ must be the transformation matrix, we then have $c_{31}, c_{32} = 0$.

2. Solutions of $(sJ - msJ)$ for dual Lie superalgebras of $C_0^1$ are

   i) $\tilde{f}_{j_{1,1}}^1 = \alpha$, $\tilde{f}_{j_{1,2}}^2 = \beta$, $\alpha, \beta \in \mathbb{R}$,

   ii) $\tilde{f}_{j_{1,1}}^3 = \gamma$, $\gamma \in \mathbb{R}$,

   iii) $\tilde{f}_{j_{1,1}}^4 = i\delta$, $\delta \in \mathbb{R}$.

   Isomorphism matrix $C$ between solution (i) and $(C_0^1)$ is as follows:

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix}, \quad c_{33} \in \mathbb{R} - \{0\}; \quad c_{11}c_{22} \neq c_{21}c_{12},$$

where $c_{21} = \frac{f_{j_{1,2}}^2}{f_{j_{1,1}}^1}c_{22}, c_{12} = \frac{c_{21}}{f_{j_{1,1}}^1} - 1$ and by imposing that $C$ must be the transformation matrix, we then have $c_{13} = 0$. Isomorphism matrix in the solution (ii) is a special case of the above isomorphism matrix and for solution (iii) we have $sdetC = 0$.

3. Solutions of $(sJ - msJ)$ for dual Lie superalgebras of $C_p^1$ ($p \in \mathbb{R}$) are

   i) $\tilde{f}_{j_{1,1}}^1 = \alpha$, $\tilde{f}_{j_{1,2}}^2 = p\alpha$, $p \in \mathbb{R}$, $\alpha \in \mathbb{R}$,

   ii) $\tilde{f}_{j_{1,1}}^3 = \tilde{f}_{j_{1,2}}^3 = \beta$, $p = 1$, $\beta \in \mathbb{R}$,

   iii) $\tilde{f}_{j_{1,1}}^4 = i\gamma$, $p \in \mathbb{R}$, $\gamma \in \mathbb{R}$,

   iv) $\tilde{f}_{j_{1,1}}^4 = i\lambda$, $\tilde{f}_{j_{1,2}}^4 = i\eta$, $p = \frac{1}{2}$, $\lambda, \eta \in \mathbb{R}$,

   v) $\tilde{f}_{j_{1,1}}^1 = \mu$, $\tilde{f}_{j_{1,2}}^3 = -\frac{\mu}{2}$, $\tilde{f}_{j_{1,1}}^3 = i\nu$, $p = -\frac{1}{2}$, $\mu, \nu \in \mathbb{R}$.

For solutions i) and ii) we have $sdetC = 0$. For solution iii) isomorphism matrix between this solution and $(2A_{1,1} + A)$ is as follows:

$$C = \begin{pmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \quad c_{11}, c_{22}, c_{33} \in \mathbb{R} - \{0\}; \quad c_{21} \in \mathbb{R},$$

where $c_{11} = -ic_{33}^2\tilde{f}_{j_{1,1}}^3$ and by imposing that $C$ must be the transformation matrix, we then have $c_{31}, c_{32} = 0$.

For solution iv) isomorphism matrix between this solution and $(2A_{1,1} + A)$ is as follows:

$$C = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \quad c_{33} \in \mathbb{R} - \{0\}; \quad c_{11}c_{22} \neq c_{21}c_{12},$$

where $c_{11} = -ic_{33}^2\tilde{f}_{j_{1,1}}^3$, $c_{12} = -ic_{33}^2\tilde{f}_{j_{1,2}}^3$ and by imposing that $C$ must be the transformation matrix, we then have $c_{31}, c_{32} = 0$. 

10
For solution $i$) isomorphism matrix between $(C_p^1, (2A_{1,1} + A))$ is the same of isomorphism matrix in the solution $iii$) with the same of conditions.

4. Solutions of $(sJ - msJ)$ for dual Lie superalgebras of $C_f^1, C_p^2 (-1 \leq p < 1)$, $C^3, C^4$ and $C_p^5 (p \geq 0)$ are 

$$\tilde{f}_{22}^1 = i\alpha, \quad \tilde{f}_{23}^1 = i\beta, \quad \tilde{f}_{33}^1 = i\gamma, \quad \alpha, \beta, \gamma \in \mathbb{R}.$$  

For the above set of solutions we have the following isomorphism matrices which map $C_f^1, C_p^2, C^3, C^4$ and $C_p^5$ into $(A_{1,1} + 2A)^0$

$$C_1 = \begin{pmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & c_{23} \\ c_{31} & 0 & c_{33} \end{pmatrix}, \quad c_{11}, c_{22}, c_{33} \in \mathbb{R} - \{0\}; \quad c_{23} \in \mathbb{R},$$

where $c_{11} = -ic_{22}^2\tilde{f}_{22}^1$ and $\tilde{f}_{22}^1 = \tilde{f}_{33}^1 = 0$.

$$C_2 = \begin{pmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \quad c_{11} \in \mathbb{R} - \{0\}; \quad c_{22}c_{33} \neq c_{23}c_{32},$$

where $\tilde{f}_{23}^1 = \frac{ic_{11}c_{22}^2}{c_{22}c_{33}^2 - c_{23}c_{31}^2}$, $\tilde{f}_{23}^1 = -\frac{c_{21}}{c_{32}}\tilde{f}_{33}^1$ and $\tilde{f}_{22}^1 = (\frac{c_{32}}{c_{c1}})^2\tilde{f}_{33}^1$.

$$C_3 = \begin{pmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & 0 \end{pmatrix}, \quad c_{11}, c_{23}, c_{32} \in \mathbb{R} - \{0\}; \quad c_{22} \in \mathbb{R},$$

where $\tilde{f}_{33}^1 = \frac{ic_{11}}{c_{23}}, \tilde{f}_{23}^1 = \tilde{f}_{22}^1 = 0$.

Imposing that $C_1, C_2$ and $C_3$ must be the transformation matrices, we have $c_{21}, c_{31} = 0$.

Appendix C: The details of differences between the lists of Lie super-bialgebras of Ref. [8] with Ref. [10]

| $g$ | $g$ of Ref. [8] | $g$ of Ref. [10] |
|-----|----------------|------------------|
| $C_p^2$ | $\mathbb{U}(1,2)$ | $A_{12}$ |
| $-1 \leq p < 1$ | $(A_{1,1} + 2A)^0_{k,0,0}$ | $N_{12}^{\nu,\delta,0} |_{\nu=0}$ |
| | $(A_{1,1} + 2A)^0_{0,0}^k$ | $N_{12}^{\nu,\delta,k} |_{\nu=0}$ |
| | $(A_{1,1} + 2A)^0_{1,1,1}$ | $N_{12}^{\nu,\delta,k^1} |_{\nu=0}$ |
| | $(A_{1,1} + 2A)^0_{-1,-1,-1}$ | Not exist |
| | $(A_{1,1} + 2A)^1_{k,\epsilon} |_{0<k<1}$ | Not exist |
| | $(A_{1,1} + 2A)^1_{-1,-1} |_{0<k<1}$ | Not exist |
| | $(A_{1,1} + 2A)^2_{k,\epsilon} |_{0<k<1}$ | Not exist |
| | $(A_{1,1} + 2A)^2_{k,\epsilon} |_{0<k<1}$ | Not exist |
| | $(A_{1,1} + 2A)^2_{1,0,1}$ | $N_{12}^{\nu,0,0} |_{\nu=1}$ |
| | $(A_{1,1} + 2A)^2_{0,1,0}$ | $N_{12}^{\nu,0,0} |_{\nu=1}$ |
| | $(A_{1,1} + 2A)^2_{0,1,0}$ | $N_{12}^{\nu,0,0} |_{\nu=1}$ |
| $g$ | $g$ of Ref. [8] | $g$ of Ref. [10] |
|-----|----------------|----------------|
| $C_2^2$ | $I_{(1,2)}$ | $A_{12}$ |
| | $(A_{1,1} + 2A)^0_{0,0,\epsilon}$ | $A_{12}^{0,0,\epsilon}$ |
| | $(A_{1,1} + 2A)^1_{0,0,\epsilon}$ | $A_{12}^{1,0,\epsilon}$ |
| | $(A_{1,1} + 2A)^2_{1,0,-1}$ | $A_{12}^{1,0,-1}$ |
| | $(A_{1,1} + 2A)^2_{1,0,1}$ | Not exist |
| $C_3^4$ | $I_{(1,2)}$ | $A_{12}$ |
| | $(A_{1,1} + 2A)^0_{1,0,0}$ | $A_{12}^{1,0,0}$ |
| | $(A_{1,1} + 2A)^0_{0,0,1}$ | $A_{12}^{0,0,1}$ |
| | $(A_{1,1} + 2A)^1_{1,0,1}$ | $A_{12}^{1,0,1}$ |
| | $(A_{1,1} + 2A)^1_{1,0,-1}$ | Not exist |
| | $(A_{1,1} + 2A)^2_{0,\epsilon,0}$ | $A_{12}^{0,\epsilon,0}$ |
| $C_5^5$ | $I_{(1,2)}$ | $A_{12}$ |
| | $(A_{1,1} + 2A)^0_{0,0,\epsilon}$ | $A_{12}^{0,0,\epsilon}$ |
| | $(A_{1,1} + 2A)^1_{0,0,1}{|_{0<k\leq 1}}$ | $A_{12}^{k,0,1}$ |
| | $(A_{1,1} + 2A)^1_{1,0,1}{|_{0<k\leq 1}}$ | Not exist |
| | $(A_{1,1} + 2A)^1_{0,0,-1}{|_{0<k\leq 1}}$ | $A_{12}^{k,0,-1}$ |
| | $(A_{1,1} + 2A)^2_{0,\epsilon,0}{|_{0<k\leq 1}}$ | Not exist |
| | $(A_{1,1} + 2A)^2_{0,\epsilon,1}{|_{0<k\leq 1}}$ | Not exist |

Note that in Ref. [10] $C_{p=0}^5$ and $C_{p=-1}^2$ appear because of incorrect automorphism supergroups.
Appendix D: Four and six dimensional Drinfel’d superdoubles (anti)commutation relations

1. (Anti)Commutation relations for Lie superalgebras $\mathcal{D}$ of the type $(2, 2)$

   $\mathcal{D} = B \oplus I_{(1,1)}$ :
   
   $[T_1, T_3] = T_3, \quad [T_1, T_4] = T_4, \quad \{T_3, T_4\} = -iT_2.$

   $\mathcal{D} = (A_{1,1} + A) \oplus I_{(1,1)}$ :
   
   $[T_3, T_2] = -T_4, \quad \{T_3, T_3\} = iT_1.$

   $\mathcal{D} = B \oplus (A_{1,1} + A)$ :
   
   $[T_1, T_3] = T_3, \quad [T_1, T_4] = -T_3 - T_4, \quad \{T_3, T_4\} = -iT_2, \quad \{T_4, T_4\} = iT_2.$

   $\mathcal{D} = B \oplus (A_{1,1} + A).i$ :
   
   $[T_1, T_3] = T_3, \quad [T_1, T_4] = T_3 - T_4, \quad \{T_3, T_4\} = -iT_2, \quad \{T_4, T_4\} = -iT_2.$

2. (Anti)Commutation relations for Lie superalgebras $\mathcal{D}$ of the type $(4, 2)$

   $\mathcal{D} = (2A_{1,1} + A) \oplus I_{(2,1)}$ :
   
   $[T_5, T_3] = -T_6, \quad \{T_5, T_3\} = iT_1.$

   $\mathcal{D} = (B + A_{1,1}) \oplus I_{(2,1)}$ :
   
   $[T_1, T_5] = T_5, \quad [T_1, T_6] = -T_6, \quad \{T_5, T_6\} = -iT_3.$

   $\mathcal{D} = C_p^1 \oplus I_{(2,1)} : \quad p \in \mathbb{R}$
   
   $[T_1, T_2] = T_2, \quad [T_1, T_4] = -T_4, \quad [T_1, T_5] = pT_5, \quad [T_1, T_6] = -pT_6,$
   
   $[T_2, T_4] = T_3, \quad [T_5, T_4] = -T_6, \quad \{T_5, T_5\} = iT_2, \quad \{T_5, T_6\} = -iT_3.$

   $\mathcal{D} = C_{\frac{1}{2}}^1 \oplus I_{(2,1)}$ :
   
   $[T_1, T_2] = T_2, \quad [T_1, T_4] = -T_4, \quad [T_1, T_5] = \frac{1}{2}T_5, \quad [T_1, T_6] = -\frac{1}{2}T_6,$
   
   $[T_2, T_4] = T_3, \quad [T_5, T_4] = -T_6, \quad \{T_5, T_5\} = iT_2, \quad \{T_5, T_6\} = -\frac{1}{2}T_3.$

   $\mathcal{D} = (B + A_{1,1}) \oplus (B + A_{1,1}).i$ :
   
   $[T_1, T_5] = T_5, \quad [T_1, T_6] = -T_6, \quad [T_4, T_6] = T_6,$
   
   $[T_5, T_4] = T_5, \quad \{T_5, T_6\} = iT_2 - iT_3.$

   $\mathcal{D} = (B + A_{1,1}) \oplus (2A_{1,1} + A)$ :
   
   $[T_1, T_5] = T_5, \quad [T_1, T_6] = -T_5 - T_6, \quad \{T_5, T_5\} = -iT_3, \quad \{T_6, T_6\} = iT_3.$

   $\mathcal{D} = (B + A_{1,1}) \oplus (2A_{1,1} + A).i$ :
   
   $[T_1, T_5] = T_5, \quad [T_1, T_6] = T_5 - T_6, \quad \{T_5, T_6\} = -iT_3, \quad \{T_6, T_6\} = -iT_3.$
\( D = C_0^1 \oplus C_{0,k}^1 : \; k \in \mathbb{R} - \{0\} \)
\[
\begin{align*}
[T_1, T_2] &= T_2, \quad & [T_1, T_4] &= -T_4, \quad & [T_2, T_3] &= kT_2, \\
[T_3, T_4] &= kT_4, \quad & [T_2, T_3] &= -kT_1 + T_3.
\end{align*}
\]

\( D = C_p^1 \oplus (2A_{1,1} + A) : \; p \in \mathbb{R} \)
\[
\begin{align*}
[T_1, T_2] &= T_2, \quad & [T_1, T_4] &= -T_4, \quad & [T_1, T_5] &= pT_5, \quad & [T_2, T_4] &= T_3, \\
[T_1, T_6] &= -T_5 - pT_6, \quad & \{T_5, T_6\} &= -ipT_3, \quad & \{T_6, T_6\} &= iT_3.
\end{align*}
\]

\( D = C_p^1 \oplus (2A_{1,1} + A).i : \; p \in \mathbb{R} \)
\[
\begin{align*}
[T_1, T_2] &= T_2, \quad & [T_1, T_4] &= -T_4, \quad & [T_1, T_5] &= pT_5, \quad & [T_2, T_4] &= T_3, \\
[T_1, T_6] &= T_5 - pT_6, \quad & \{T_5, T_6\} &= -ipT_3, \quad & \{T_6, T_6\} &= -iT_3.
\end{align*}
\]

\( D = C_p^1 \oplus C_{-1}^{1,i} : \; p \in \mathbb{R} \)
\[
\begin{align*}
[T_1, T_2] &= T_2, \quad & [T_1, T_3] &= T_2, \quad & [T_1, T_4] &= -T_1 - T_4, \quad & [T_1, T_5] &= pT_5, \\
[T_1, T_6] &= -pT_6, \quad & [T_2, T_4] &= T_3, \quad & [T_3, T_4] &= T_3, \quad & [T_4, T_6] &= pT_6, \\
[T_5, T_4] &= pT_5, \quad & \{T_5, T_6\} &= ipT_2 - ipT_3.
\end{align*}
\]

\( D = C_p^1 \oplus C_{-1}^{1,i} : \; p \in \mathbb{R} \)
\[
\begin{align*}
[T_1, T_2] &= T_2, \quad & [T_1, T_3] &= T_2, \quad & [T_1, T_4] &= -T_1 - T_4, \quad & [T_1, T_5] &= pT_5, \\
[T_1, T_6] &= -\frac{1}{2}T_6, \quad & [T_2, T_4] &= T_3, \quad & [T_3, T_4] &= T_3, \quad & [T_4, T_6] &= \frac{1}{2}T_6, \\
[T_5, T_4] &= \frac{1}{2}T_5 - T_6, \quad & \{T_5, T_5\} &= iT_2, \quad & \{T_5, T_6\} &= \frac{i}{2}T_2 - \frac{i}{2}T_3.
\end{align*}
\]

\( D = C_p^1 \oplus C_{-1}^{1,i} : \; p \in \mathbb{R} \)
\[
\begin{align*}
[T_1, T_2] &= T_2, \quad & [T_1, T_3] &= -T_2, \quad & [T_1, T_4] &= T_1 - T_4, \quad & [T_1, T_5] &= \frac{1}{2}T_5, \\
[T_1, T_6] &= -\frac{1}{2}T_6, \quad & [T_2, T_4] &= T_3, \quad & [T_3, T_4] &= -T_3, \quad & [T_4, T_6] &= -\frac{1}{2}T_6, \\
[T_5, T_4] &= -\frac{1}{2}T_5 - T_6, \quad & \{T_5, T_5\} &= iT_2, \quad & \{T_5, T_6\} &= -\frac{i}{2}T_2 - \frac{i}{2}T_3.
\end{align*}
\]

\( D = C_p^1 \oplus C_{-1}^{1,i} : \; p \in \mathbb{R} \)
\[
\begin{align*}
[T_1, T_2] &= T_2, \quad & [T_1, T_3] &= T_2, \quad & [T_1, T_4] &= -T_1 - T_4, \quad & [T_1, T_5] &= \frac{1}{2}T_5, \\
[T_1, T_6] &= -\frac{1}{2}T_6, \quad & [T_2, T_4] &= T_3, \quad & [T_3, T_4] &= T_3, \quad & [T_4, T_6] &= -\frac{1}{2}T_6, \\
[T_5, T_4] &= -\frac{1}{2}T_5 - T_6, \quad & \{T_5, T_5\} &= iT_2, \quad & \{T_5, T_6\} &= -\frac{i}{2}T_2 - \frac{i}{2}T_3, \quad & \{T_6, T_6\} &= iT_3.
\end{align*}
\]
\[ D = C^1_2 \oplus C^1_{\frac{1}{2}, k} : k \in \mathbb{R} - \{0\} \]

\[
[T_1, T_2] = T_2, \quad [T_1, T_4] = -T_4, \quad [T_1, T_5] = \frac{1}{2} T_5, \quad [T_1, T_6] = -\frac{1}{2} T_6, \\
[T_2, T_6] = -kT_5, \quad [T_2, T_4] = -kT_1 + T_3, \quad [T_3, T_4] = kT_4, \quad [T_3, T_6] = \frac{k}{2} T_6, \\
[T_2, T_3] = kT_2, \quad [T_5, T_3] = \frac{k}{2} T_5, \quad [T_5, T_4] = -T_6, \quad \{T_3, T_3\} = iT_2, \\
\{T_5, T_6\} = \frac{i}{2} T_1 - \frac{i}{2} T_3, \quad \{T_6, T_6\} = i k T_4.
\]

3. (Anti)Commutation relations for Lie superalgebras \( D \) of the type \((2, 4)\). Note that \( \tilde{G}_{0, \alpha, \beta, \gamma} \) is one of the dual Lie superalgebras \((A_{1,1} + 2A_0)_{0, \alpha, \beta, \gamma} \), \((A_{1,1} + 2A_1)_{1, \alpha, \beta, \gamma} \) or \((A_{1,1} + 2A_2)_{2, \alpha, \beta, \gamma} \).

\[ D = C^2_p \oplus \tilde{G}_{0, \alpha, \beta, \gamma} : \quad p \in [-1, 1] \]

\[
[T_1, T_3] = T_3, \quad [T_1, T_4] = pT_4, \quad \{T_4, T_6\} = -i p T_2, \quad [T_1, T_5] = -\alpha T_3 - \beta T_4 - T_5, \\
\{T_3, T_5\} = -i T_2, \quad \{T_5, T_3\} = i \alpha T_2, \quad \{T_5, T_6\} = i \beta T_2, \quad [T_1, T_6] = -\gamma T_4 - \beta T_3 - pT_6, \\
\{T_6, T_6\} = i \gamma T_2.
\]

\[ D = C^3 \oplus \tilde{G}_{1, \alpha, \beta, \gamma} : \]

\[
[T_1, T_4] = T_3 + T_4, \quad \{T_4, T_5\} = -i T_2, \quad \{T_5, T_3\} = i \alpha T_2, \quad [T_1, T_5] = -\alpha T_3 - \beta T_4 - T_5, \\
\{T_5, T_6\} = i \beta T_2, \quad \{T_6, T_6\} = i \gamma T_2.
\]

\[ D = C^4 \oplus \tilde{G}_{1, \alpha, \beta, \gamma} : \]

\[
[T_1, T_3] = T_3 + T_4, \quad [T_1, T_3] = T_3, \quad \{T_3, T_5\} = -i T_2, \quad [T_1, T_5] = -\alpha T_3 - \beta T_4 - T_5 - T_6, \\
\{T_4, T_5\} = -i T_2, \quad \{T_4, T_6\} = -i T_2, \quad \{T_5, T_5\} = i \alpha T_2, \quad [T_1, T_6] = -\gamma T_4 - \beta T_3 - T_6, \\
\{T_5, T_6\} = i \beta T_2, \quad \{T_6, T_6\} = i \gamma T_2.
\]

\[ D = C^5_p \oplus \tilde{G}_{0, \alpha, \beta, \gamma} : \quad p \geq 0 \]

\[
[T_1, T_3] = pT_3 - T_4, \quad \{T_3, T_5\} = -i p T_2, \quad \{T_3, T_6\} = i T_2, \quad [T_1, T_5] = -\alpha T_3 - \beta T_4 - pT_5 - T_6, \\
[T_1, T_4] = T_3 + pT_4, \quad \{T_4, T_5\} = -i T_2, \quad \{T_4, T_6\} = -i p T_2, \quad [T_1, T_6] = -\gamma T_4 - \beta T_3 - pT_6 + T_5, \\
\{T_5, T_5\} = i \alpha T_2, \quad \{T_5, T_6\} = i \beta T_2, \quad \{T_6, T_6\} = i \gamma T_2.
\]

\[ D = (A_{1,1} + 2A)^0 \oplus I_{(1,2)} : \]

\[
[T_3, T_2] = -T_5, \quad \{T_3, T_3\} = iT_1.
\]

\[ D = (A_{1,1} + 2A)^1 \oplus I_{(1,2)} : \]

\[
[T_3, T_2] = -T_5, \quad [T_4, T_2] = -T_6, \quad \{T_3, T_3\} = iT_1, \quad \{T_4, T_4\} = iT_1.
\]

\[ D = (A_{1,1} + 2A)^2 \oplus I_{(1,2)} : \]

\[
[T_3, T_2] = -T_5, \quad [T_4, T_2] = T_6, \quad \{T_3, T_3\} = iT_1, \quad \{T_4, T_4\} = -iT_1.
\]
Appendix E: Comparison of theorem 3 with those of Ref. [10]

1. Comparison of $\mathcal{Dsd}_v^{1,p}$ with $DD_{(1,2)}II_p$:
   In $DD_{(1,2)}II_p$ the Drinfel’d superdoubles $\left( C_p^2 \right)_{(A_1,1)+2A} \not\exists \mathcal{Dsd}_v^{1,p}$ with $DD_{(1,2)}II_p$.
   \[ (C^2_p, (A_1,1)+2A)^{0,0,0,0} \mid_{1<1} \not\exists \mathcal{Dsd}_v^{1,p} \]

2. Comparison of $\mathcal{Dsd}_v^{4,p}$ with $DD_{(1,2)}II_1$:
   In $DD_{(1,2)}II_1$ the Drinfel’d superdoubles $\left( C_4^2 \right)_0 (A_1,1)+2A$ do not exist. On the other hand, in $DD_{(1,2)}II_1$ the Drinfel’d superdoubles $\left( C_2^2 \right)_{(A_1,1)+2A}^{0,0,0,0} \not\exists \mathcal{Dsd}_v^{4,p} \mid_{1<1} \not\exists \mathcal{Dsd}_v^{4,p}$.

3. There are no differences between $\mathcal{Dsd}_v^{3,p}$ with $DD_{(1,2)}II_0$.

4. There are no differences between $\mathcal{Dsd}_v^{4,p}$ with $DD_{(1,2)}II_1$.

5. Comparison of $\mathcal{Dsd}_v^{5,p}$ with $DD_{(1,2)}II_0$:
   In $DD_{(1,2)}II_1$ the Drinfel’d superdoubles $\left( C_5^2 \right)_0 (A_1,1)+2A$ do not exist. On the other hand, in $DD_{(1,2)}II_0$ the Drinfel’d superdoubles $\left( C_3^2 \right)_{(A_1,1)+2A}^{0,0,0,0} \not\exists \mathcal{Dsd}_v^{5,p} \mid_{1<1} \not\exists \mathcal{Dsd}_v^{5,p}$.

6. Comparison of $\mathcal{Dsd}_v^{6,p}$ with $DD_{(1,2)}II_1$:
   In $DD_{(1,2)}II_1$ the Drinfel’d superdoubles $\left( C_6^2 \right)_0 (A_1,1)+2A$ do not exist. On the other hand, in $DD_{(1,2)}II_1$ the Drinfel’d superdoubles $\left( C_4^2 \right)_{(A_1,1)+2A}^{0,0,0,0} \not\exists \mathcal{Dsd}_v^{6,p} \mid_{1<1} \not\exists \mathcal{Dsd}_v^{6,p}$.

7. Comparison of $\mathcal{Dsd}_v^{7,p}$ with $DD_{(1,2)}II_0$:
   In $DD_{(1,2)}II_0$ the Drinfel’d superdoubles $\left( C_7^2 \right)_0 (A_1,1)+2A$ do not exist. On the other hand, in $DD_{(1,2)}II_0$ the Drinfel’d superdoubles $\left( C_5^2 \right)_{(A_1,1)+2A}^{0,0,0,0} \not\exists \mathcal{Dsd}_v^{7,p} \mid_{1<1} \not\exists \mathcal{Dsd}_v^{7,p}$.

8. Comparison of $\mathcal{Dsd}_v^{8,p}$ with $DD_{(1,2)}II_1$:
   In $DD_{(1,2)}II_1$ the Drinfel’d superdoubles $\left( C_8^2 \right)_0 (A_1,1)+2A$ do not exist. On the other hand, in $DD_{(1,2)}II_1$ the Drinfel’d superdoubles $\left( C_6^2 \right)_{(A_1,1)+2A}^{0,0,0,0} \not\exists \mathcal{Dsd}_v^{8,p} \mid_{1<1} \not\exists \mathcal{Dsd}_v^{8,p}$.

9. Comparison of $\mathcal{Dsd}_v^{9,p}$ with $DD_{(1,2)}II_0$:
   In $DD_{(1,2)}II_0$ the Drinfel’d superdoubles $\left( C_9^2 \right)_0 (A_1,1)+2A$ do not exist. On the other hand, in $DD_{(1,2)}II_0$ the Drinfel’d superdoubles $\left( C_7^2 \right)_{(A_1,1)+2A}^{0,0,0,0} \not\exists \mathcal{Dsd}_v^{9,p} \mid_{1<1} \not\exists \mathcal{Dsd}_v^{9,p}$.

10. Comparison of $\mathcal{Dsd}_v^{10,p}$ with $DD_{(1,2)}II_1$:
   In $DD_{(1,2)}II_1$ the Drinfel’d superdoubles $\left( C_{10}^2 \right)_0 (A_1,1)+2A$ do not exist. On the other hand, in $DD_{(1,2)}II_1$ the Drinfel’d superdoubles $\left( C_8^2 \right)_{(A_1,1)+2A}^{0,0,0,0} \not\exists \mathcal{Dsd}_v^{10,p} \mid_{1<1} \not\exists \mathcal{Dsd}_v^{10,p}$.
Appendix F: Isomorphisms of the Drinfeld’s superdoubles of the types (2, 2), (4, 2) and (2, 4)

Isomorphisms of the Drinfeld’s superdoubles of the type (2, 2)

\[ D_{sd}^{3}(2, 2) : \]
\[
(B, I_{(1, 1)}) \rightarrow (B, (A_{1, 1} + A))
\]
\[
C = \begin{pmatrix}
1 & c & 0 & 0 \\
0 & ab & 0 & 0 \\
0 & 0 & a & b \\
0 & 0 & -\frac{2}{b} & b
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c \in \mathbb{R},
\]

\[ (B, I_{(1, 1)}) \rightarrow (B, (A_{1, 1} + A).i) \]
\[
C = \begin{pmatrix}
1 & c & 0 & 0 \\
0 & ab & 0 & 0 \\
0 & 0 & a & 0 \\
0 & 0 & \frac{2}{a} & b
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c \in \mathbb{R}.
\]

Isomorphisms of the Drinfeld’s superdoubles of the type (4, 2)

\[ D_{sd}^{3}(4, 2) : \]
\[
((B + A_{1, 1}), I_{(2, 1)}) \rightarrow ((B + A_{1, 1}), (B + A_{1, 1}).i)
\]
\[
C = \begin{pmatrix}
1 & e & f & g & 0 & 0 \\
0 & d & a - bc & n & 0 & 0 \\
0 & d & a & n & 0 & 0 \\
-1 & m & r & s & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & c
\end{pmatrix}, \quad mn - ds \neq 0; \quad b, c \in \mathbb{R} - \{0\}; \quad a, d, e, ..., s \in \mathbb{R},
\]

\[ ((B + A_{1, 1}), I_{(2, 1)}) \rightarrow ((B + A_{1, 1}), (2A_{1, 1} + A)) \]
\[
C = \begin{pmatrix}
1 & c & d & e & 0 & 0 \\
0 & m & r & v & 0 & 0 \\
0 & 0 & ab & 0 & 0 & 0 \\
0 & u & s & n & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & -\frac{2}{a} & b
\end{pmatrix}, \quad mn - uv \neq 0; \quad a, b \in \mathbb{R} - \{0\}; \quad c, d, e, ..., v \in \mathbb{R},
\]

\[ ((B + A_{1, 1}), I_{(2, 1)}) \rightarrow ((B + A_{1, 1}), (2A_{1, 1} + A).i) \]
\[
C = \begin{pmatrix}
1 & c & d & e & 0 & 0 \\
0 & m & r & v & 0 & 0 \\
0 & 0 & ab & 0 & 0 & 0 \\
0 & u & s & n & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & \frac{2}{a} & b
\end{pmatrix}, \quad mn - uv \neq 0; \quad a, b \in \mathbb{R} - \{0\}; \quad c, d, e, ..., v \in \mathbb{R},
\]
$$Dsd_{(4,2)}^{p=0}:$$

$$(C_{p=0}^1, I_{(2,1)}) \rightarrow (C_{p=0}^1, C_{-p=0}^1)$$

$$C = \begin{pmatrix}
1 & -a & a(b + c) - d & c & 0 & 0 \\
0 & e & -bc & 0 & 0 & 0 \\
0 & e & eb & 0 & 0 & 0 \\
-1 & a & d & b & 0 & 0 \\
0 & 0 & 0 & 0 & m & s \\
0 & 0 & 0 & 0 & r & n
\end{pmatrix}, \quad mn - rs \neq 0; \ b + c \neq 0; \ e \in \mathbb{R} - \{0\}; \ a, ..., s \in \mathbb{R},$$

$$Dsd_{(4,2)}^{p}: \quad p \in \mathbb{R} - \{0\},$$

$$(C_{p}^1, I_{(2,1)}) \rightarrow (C_{p}^1, I_{(2,1)})$$

$$C = \begin{pmatrix}
1 & d & e & f & 0 & 0 \\
0 & -\frac{bc}{d} & \frac{bc}{d} & 0 & 0 & 0 \\
0 & 0 & -bc & 0 & 0 & 0 \\
0 & 0 & -ad & a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b \\
0 & 0 & 0 & 0 & c & 0
\end{pmatrix}, \quad a, b, c \in \mathbb{R} - \{0\}; \ d, e, f \in \mathbb{R},$$

$$(C_{p}^1, I_{(2,1)}) \rightarrow (C_{p}^1, C_{-p}^1)$$

$$C = \begin{pmatrix}
1 & -e & bce - n & bc - a & 0 & 0 \\
0 & d & ad - bc & 0 & 0 & 0 \\
0 & d & ad & 0 & 0 & 0 \\
-1 & e & n & a & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & c
\end{pmatrix}, \quad b, c, d \in \mathbb{R} - \{0\}; \ a, e, n \in \mathbb{R},$$

$$(C_{p}^1, I_{(2,1)}) \rightarrow (C_{p}^1, (2A_{1,1} + A))$$

$$C = \begin{pmatrix}
1 & d & e & f & 0 & 0 \\
0 & \frac{ab}{c} & -\frac{ab}{c} & 0 & 0 & 0 \\
0 & 0 & ab & 0 & 0 & 0 \\
0 & 0 & -cd & c & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & -\frac{a}{2p} & b
\end{pmatrix}, \quad a, b, c \in \mathbb{R} - \{0\}; \ d, e, f \in \mathbb{R},$$

$$(C_{p}^1, I_{(2,1)}) \rightarrow (C_{p}^1, (2A_{1,1} + A).i)$$

$$C = \begin{pmatrix}
1 & d & e & f & 0 & 0 \\
0 & \frac{ab}{c} & -\frac{ab}{c} & 0 & 0 & 0 \\
0 & 0 & ab & 0 & 0 & 0 \\
0 & 0 & -cd & c & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & -\frac{a}{2p} & b
\end{pmatrix}, \quad a, b, c \in \mathbb{R} - \{0\}; \ d, e, f \in \mathbb{R},$$
\[ D_{sd}^7(4,2) : \]
\[ (C^1_{p=0^+}(2A_{1,1} + A)) \rightarrow (C^1_{p=0^-}(2A_{1,1} + A), i) \]
\[
C = \begin{pmatrix}
1 & c & e & d & 0 & 0 \\
0 & -\frac{a^2}{b^2} & \frac{a^2}{b^2} & 0 & 0 & 0 \\
0 & 0 & -a^2 & 0 & 0 & 0 \\
0 & 0 & -bc & b & 0 & 0 \\
0 & 0 & 0 & 0 & -a & 0 \\
0 & 0 & 0 & 0 & n & a \\
\end{pmatrix}, \quad a, b \in \mathbb{R} \setminus \{0\}; \quad c, d, e, n \in \mathbb{R},
\]

\[ \vdots \]

\[ D_{sd}^7(4,2) : \]
\[ (C^1_{p=0^+}(2A_{1,1} + A)) \rightarrow (C^1_{p=0^-}(2A_{1,1} + A), ii) \]
\[
C = \begin{pmatrix}
1 & a & e & c & 0 & 0 \\
0 & 0 & ad & d & 0 & 0 \\
0 & 0 & -db^2 & 0 & 0 & 0 \\
0 & b^2 & cb^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -bd & 0 \\
0 & 0 & 0 & 0 & b & -bc \\
\end{pmatrix}, \quad b, d \in \mathbb{R} \setminus \{0\}; \quad a, c, e \in \mathbb{R},
\]

\[ \vdots \]
Isomorphisms of the Drinfeld’s superdoubles of the type (2, 4)

\[
\mathcal{D}^{sd}_{(2,4)}:
\]

\[
\left( C^2_{0}, (A_{1,1} + 2A)^0_{0,0,\epsilon_1} \right) \rightarrow \left( C^2_{0}, (A_{1,1} + 2A)^0_{\epsilon_2}, \epsilon_2 \right), \quad \epsilon_1, \epsilon_2 = \pm 1
\]

\[
C = \begin{pmatrix}
1 & d & 0 & 0 & 0 & 0 \\
0 & \frac{\epsilon_1 a^2}{\epsilon_2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\epsilon_1 a^2}{\epsilon_2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\epsilon_1 a}{\epsilon_2} & 0 & 0 \\
0 & 0 & \frac{-\epsilon_1 a^2}{\epsilon_2} & -\epsilon_1 a & b & 0 \\
0 & 0 & \frac{-\epsilon_1 a^2}{\epsilon_2} & c & 0 & a
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\]

\[
\left( C^2_{0}, (A_{1,1} + 2A)^0_{0,0,\epsilon_1} \right) \rightarrow \left( C^2_{0}, (A_{1,1} + 2A)^1_{\epsilon_2, \epsilon_2} \right), \quad \epsilon_1, \epsilon_2 = \pm 1; \quad -1 < k < 1
\]

\[
C = \begin{pmatrix}
1 & d & 0 & 0 & 0 & 0 \\
0 & \frac{\epsilon_1 a^2}{\epsilon_2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\epsilon_1 a^2}{\epsilon_2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\epsilon_1 a}{\epsilon_2} & 0 & 0 \\
0 & 0 & \frac{-\epsilon_1 a^2}{\epsilon_2} & -\frac{\epsilon_1 ka}{\epsilon_2} & b & 0 \\
0 & 0 & \frac{-\epsilon_1 a^2}{\epsilon_2} & c & 0 & a
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\]

\[
\left( C^2_{0}, (A_{1,1} + 2A)^0_{0,0,\epsilon_1} \right) \rightarrow \left( C^2_{0}, (A_{1,1} + 2A)^2_{0,1,\epsilon_2} \right), \quad \epsilon_1, \epsilon_2 = \pm 1
\]

\[
C = \begin{pmatrix}
1 & d & 0 & 0 & 0 & 0 \\
0 & \frac{\epsilon_1 a^2}{\epsilon_2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-\epsilon_1 a^2}{\epsilon_2} & -\epsilon_1 a & 0 & 0 \\
0 & 0 & \frac{-\epsilon_1 a^2}{\epsilon_2} & \frac{\epsilon_1 ka}{\epsilon_2} & 0 & 0 \\
0 & 0 & \frac{-\epsilon_1 a^2}{\epsilon_2} & c & 0 & a
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\]

\[
\left( C^2_{0}, (A_{1,1} + 2A)^0_{0,0,\epsilon_1} \right) \rightarrow \left( C^2_{0}, (A_{1,1} + 2A)^2_{1,2,-\epsilon_2} \right), \quad \epsilon_1, \epsilon_2 = \pm 1; \quad k \in \mathbb{R}
\]

\[
C = \begin{pmatrix}
1 & d & 0 & 0 & 0 & 0 \\
0 & \frac{-\epsilon_1 a^2}{\epsilon_2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-\epsilon_1 a^2}{\epsilon_2} & -\epsilon_1 a & 0 & 0 \\
0 & 0 & \frac{-\epsilon_1 a^2}{\epsilon_2} & \frac{\epsilon_1 ka}{\epsilon_2} & 0 & 0 \\
0 & 0 & \frac{-\epsilon_1 a^2}{\epsilon_2} & \frac{\epsilon_1 ka}{\epsilon_2} & b & 0 \\
0 & 0 & \frac{-\epsilon_1 a^2}{\epsilon_2} & c & 0 & a
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\]

\[
\mathcal{D}^{sd}_{(2,4)}^{p=0}:
\]

\[
\left( C^2_{0}, (A_{1,1} + 2A)^p_{0,0,\epsilon_1} \right) \rightarrow \left( C^2_{0}, (A_{1,1} + 2A)^2_{0,1,\epsilon_2} \right), \quad \epsilon = \pm 1
\]

\[
C = \begin{pmatrix}
1 & e & 0 & 0 & 0 & 0 \\
0 & -ab & 0 & 0 & 0 & 0 \\
0 & 0 & -a & 0 & 0 & 0 \\
0 & 0 & 0 & -c & 0 & -d \\
0 & 0 & c & b & 0 & d \\
0 & 0 & a & h & 0 & g
\end{pmatrix}, \quad \text{hd} - gc \neq 0; \quad a, b \in \mathbb{R} - \{0\}; \quad c, d, e, g, h \in \mathbb{R},
\]
\[
\begin{align*}
&\left(C^2_0, (A_1, 1 + 2A)^0_{e_1, 0, 0}\right) \longrightarrow \left(C^2_0, (A_1, 1 + 2A)^2_{e_2, 1, 0}\right), \quad \epsilon_1, \epsilon_2 = \pm 1 \\
&C = \\
&\begin{pmatrix}
1 & e & 0 & 0 & 0 & 0 \\
0 & -ab & 0 & 0 & 0 & 0 \\
0 & 0 & -a & 0 & 0 & 0 \\
0 & 0 & 0 & -c & 0 & -d \\
0 & 0 & \frac{1}{2}a & \frac{a}{c} & c & b & d \\
0 & 0 & 0 & h & 0 & g \\
\end{pmatrix}, \quad \text{hd} - gc \neq 0; \quad a, b \in \mathbb{R} - \{0\}; \quad c, d, e, g, h \in \mathbb{R},
\end{align*}
\]

\[
\begin{align*}
&\left(C^2_0, (A_1, 1 + 2A)^0_{e_1, 0, 0}\right) \longrightarrow \left(C^2_0, I_{1(2)}\right), \quad \epsilon = \pm 1 \\
&C = \\
&\begin{pmatrix}
1 & e & 0 & 0 & 0 & 0 \\
0 & ab & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & d \\
0 & 0 & \frac{1}{2}a & \frac{a}{c} & c & b & d \\
0 & 0 & 0 & h & 0 & g \\
\end{pmatrix}, \quad \text{hd} - gc \neq 0; \quad a, b \in \mathbb{R} - \{0\}; \quad c, d, e, g, h \in \mathbb{R},
\end{align*}
\]

\[
\mathcal{D} s d^1_{(2,4)}^{p=1}: \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \四方…
\[
\left(C_{p,1}^2, (A_{1,1} + 2A)_{0,0,1}^0 \right) \rightarrow \left(C_{p-1,1}^2, (A_{1,1} + 2A)_{0,0,-1}^2 \right), \quad \epsilon_1, \epsilon_2 = \pm 1
\]

\[
C = \begin{pmatrix}
-1 & c & 0 & 0 & 0 & 0 \\
0 & -ab & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{c}{2} & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & d & \frac{ab}{2} & \epsilon_2 \frac{ab}{2} \\
0 & 0 & 0 & 0 & \frac{c}{2} & \frac{a}{2}
\end{pmatrix}, \quad a, b, d \in \mathbb{R} - \{0\}; \quad c, e \in \mathbb{R},
\]

\[
\left(C_{p,1}^2, (A_{1,1} + 2A)_{0,0,1}^0 \right) \rightarrow \left(C_{p-1,1}^2, (A_{1,1} + 2A)_{0,0,-1}^2 \right), \quad \epsilon_1, \epsilon_2 = \pm 1
\]

\[
C = \begin{pmatrix}
-1 & c & 0 & 0 & 0 & 0 \\
0 & \epsilon_2(2b - \epsilon_1 a) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{c}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{c}{2} & 0 & 0 \\
0 & 0 & 0 & d & \frac{ab}{2} & \epsilon_2 \frac{ab}{2} \\
0 & 0 & 0 & 0 & \frac{c}{2} & \frac{a}{2}
\end{pmatrix}, \quad 2b - \epsilon_1 a \neq 0; \quad a, d \in \mathbb{R} - \{0\}; \quad b, c, e \in \mathbb{R},
\]

\[
\left(C_{p,1}^2, (A_{1,1} + 2A)_{0,0,1}^0 \right) \rightarrow \left(C_{p,1}^2, I_{1,1} \right), \quad \epsilon = \pm 1
\]

\[
C = \begin{pmatrix}
1 & c & 0 & 0 & 0 & 0 \\
0 & ab & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{a}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{a}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & d & 0 \\
0 & 0 & 0 & 0 & \frac{c}{2} & \frac{a}{2}
\end{pmatrix}, \quad a, b, d \in \mathbb{R} - \{0\}; \quad c, e \in \mathbb{R},
\]

\[
\left(C_{p,1}^2, (A_{1,1} + 2A)_{0,0,1}^0 \right) \rightarrow \left(C_{p-1,1}^2, I_{1,1} \right), \quad \epsilon = \pm 1
\]

\[
C = \begin{pmatrix}
1 & c & 0 & 0 & 0 & 0 \\
0 & -ab & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{a}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{a}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & d & 0 \\
0 & 0 & 0 & 0 & \frac{c}{2} & \frac{a}{2}
\end{pmatrix}, \quad a, b, d \in \mathbb{R} - \{0\}; \quad c, e \in \mathbb{R},
\]

\[
D_{sd_{(2,4)}}^p: \quad p \in (-1, 1) - \{0\}
\]

\[
\left(C_{p,1}^2, (A_{1,1} + 2A)_{0,0,1}^0 \right) \rightarrow \left(C_{p,1}^2, (A_{1,1} + 2A)_{0,0,1}^0 \right), \quad \epsilon_1, \epsilon_2 = \pm 1
\]

\[
C = \begin{pmatrix}
1 & d & 0 & 0 & 0 & 0 \\
0 & -ab & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{a}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{c}{2} + \epsilon_1 \frac{ab}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & b
\end{pmatrix}, \quad a, b, c \in \mathbb{R} - \{0\}; \quad d \in \mathbb{R},
\]

\[
\left(C_{p,1}^2, (A_{1,1} + 2A)_{0,0,1}^0 \right) \rightarrow \left(C_{p,1}^2, (A_{1,1} + 2A)_{0,0,-1}^2 \right), \quad \epsilon_1, \epsilon_2 = \pm 1
\]

\[
C = \begin{pmatrix}
1 & d & 0 & 0 & 0 & 0 \\
0 & -ab & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{a}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{c}{2} + \epsilon_1 \frac{ab}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & b
\end{pmatrix}, \quad a, b, c \in \mathbb{R} - \{0\}; \quad d \in \mathbb{R},
\]
\[
\begin{align*}
(C_p, (A_{1,1} + 2A)^{0}_{1,0,0}) &\rightarrow (C_p, (A_{1,1} + 2A)^{0}_{1,2,\\epsilon_2}), \quad \epsilon_1, \epsilon_2 = \pm 1 \\
C &= \begin{pmatrix}
1 & d & 0 & 0 & 0 \\
0 & -\frac{(p+1)ab}{c} & 0 & 0 & 0 \\
0 & 0 & -\frac{(p+1)ab}{c} & 0 & 0 \\
0 & 0 & 0 & -\frac{\epsilon_2}{2} & b \\
0 & 0 & 0 & 0 & \frac{2\epsilon_2}{c} \\
\end{pmatrix}, \quad a, b, c \in \mathbb{R} - \{0\}; \quad d \in \mathbb{R},
\end{align*}
\]

\[
\begin{align*}
(C_p, (A_{1,1} + 2A)^{0}_{1,1,0}) &\rightarrow (C_p, (A_{1,1} + 2A)^{1}_{1,2,\\epsilon_2}), \quad -1 < k < 1; \quad \epsilon_1, \epsilon_2 = \pm 1 \\
C &= \begin{pmatrix}
1 & d & 0 & 0 & 0 \\
1 + p & ab & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & \frac{\epsilon_2}{2} & b \\
0 & 0 & 0 & 0 & \frac{2\epsilon_2}{c} \\
\end{pmatrix}, \quad a, b, c \in \mathbb{R} - \{0\}; \quad d \in \mathbb{R},
\end{align*}
\]

\[
\begin{align*}
(C_p, (A_{1,1} + 2A)^{0}_{1,0,0}) &\rightarrow (C_p, (A_{1,1} + 2A)^{2}_{0,1,0}), \quad \epsilon = \pm 1 \\
C &= \begin{pmatrix}
1 & d & 0 & 0 & 0 \\
0 & -ab(1 + p) & 0 & 0 & 0 \\
0 & 0 & -a(1 + p) & 0 & 0 \\
0 & 0 & 0 & \frac{ab}{a} & b \\
0 & 0 & 0 & 0 & c \\
\end{pmatrix}, \quad a, b, c \in \mathbb{R} - \{0\}; \quad d \in \mathbb{R},
\end{align*}
\]

\[
\begin{align*}
(C_p, (A_{1,1} + 2A)^{0}_{1,0,0}) &\rightarrow (C_p, (A_{1,1} + 2A)^{2}_{2,1,0}), \quad \epsilon_1, \epsilon_2 = \pm 1 \\
C &= \begin{pmatrix}
1 & d & 0 & 0 & 0 \\
0 & -ab(1 + p) & 0 & 0 & 0 \\
0 & 0 & -a(1 + p) & 0 & 0 \\
0 & 0 & 0 & \frac{ab}{a} & b \\
0 & 0 & 0 & 0 & c \\
\end{pmatrix}, \quad a, b, c \in \mathbb{R} - \{0\}; \quad d \in \mathbb{R},
\end{align*}
\]

\[
\begin{align*}
(C_p, (A_{1,1} + 2A)^{0}_{1,0,0}) &\rightarrow (C_p, (A_{1,1} + 2A)^{2}_{0,1,2}), \quad \epsilon_1, \epsilon_2 = \pm 1 \\
C &= \begin{pmatrix}
1 & d & 0 & 0 & 0 \\
0 & -ab(1 + p) & 0 & 0 & 0 \\
0 & 0 & -a(1 + p) & 0 & 0 \\
0 & 0 & 0 & \frac{ab}{a} & b \\
0 & 0 & 0 & 0 & c \\
\end{pmatrix}, \quad a, b, c \in \mathbb{R} - \{0\}; \quad d \in \mathbb{R},
\end{align*}
\]

\[
\begin{align*}
(C_p, (A_{1,1} + 2A)^{0}_{1,0,0}) &\rightarrow (C_p, (A_{1,1} + 2A)^{2}_{2,1,\epsilon_2}), \quad s \in \mathbb{R}; \quad \epsilon_1, \epsilon_2 = \pm 1 \\
C &= \begin{pmatrix}
1 & d & 0 & 0 & 0 \\
0 & -\frac{\epsilon_2}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{\epsilon_2}{2} & 0 & 0 \\
0 & 0 & \epsilon_2(\epsilon_1a - 2b) & \frac{\epsilon_2}{c} & a \\
0 & 0 & b & \frac{\epsilon_2}{2} & c \\
\end{pmatrix}, \quad \epsilon_1a - 2b \neq 0; \quad a, c \in \mathbb{R} - \{0\}; \quad b, d \in \mathbb{R},
\end{align*}
\]
\[
\begin{align*}
(C^2, (A_{1,1} + 2A)^0_{0,0,0}) &\rightarrow (C^2, I_{1,1}), \quad \epsilon = \pm 1 \\
C &= \begin{pmatrix}
1 & d & 0 & 0 & 0 & 0 \\
0 & ab & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 \\
\end{pmatrix}, \quad a, b, c \in \mathbb{R} - \{0\}; \quad d \in \mathbb{R},
\end{align*}
\]

\[
Dsd^4_{(2,4)}:
\]

\[
\begin{align*}
(C^3, (A_{1,1} + 2A)^0_{0,0,0}) &\rightarrow (C^3, (A_{1,1} + 2A)^2_{0,0,0}), \quad \epsilon = \pm 1 \\
C &= \begin{pmatrix}
a & b & 0 & 0 & 0 & 0 \\
2cd & 0 & 0 & 0 & 0 \\
0 & ac & 0 & 0 & 0 \\
0 & e & c & 0 & f \\
0 & g & d & 2d & h \\
0 & ad & ec & 0 & 2ad & ef \\
\end{pmatrix}, \quad a, c, d \in \mathbb{R} - \{0\}; \quad b, e, f, g, h \in \mathbb{R},
\end{align*}
\]

\[
\begin{align*}
(C^3, (A_{1,1} + 2A)^0_{1,0,0}) &\rightarrow (C^3, I_{1,1,2}) \\
C &= \begin{pmatrix}
a & b & 0 & 0 & 0 & 0 \\
0 & cd & 0 & 0 & 0 & 0 \\
0 & ac & 0 & 0 & 0 & 0 \\
0 & e & c & 0 & f \\
0 & g & d & d & h \\
0 & ad & ec & 0 & 2ad & ef \\
\end{pmatrix}, \quad a, c, d \in \mathbb{R} - \{0\}; \quad b, e, f, g, h \in \mathbb{R},
\end{align*}
\]

\[
\begin{align*}
(C^3, (A_{1,1} + 2A)^0_{1,0,0}) &\rightarrow ((A_{1,1} + 2A)^2, I_{1,1,2}) \\
C &= \begin{pmatrix}
0 & c(c - 2b) & 0 & 0 & 0 & 0 \\
a & d & 0 & 0 & 0 & 0 \\
0 & e & b & c & g \\
0 & f & c - b & c & h \\
0 & a(b - c) & 0 & 0 & -ac \\
0 & ab & 0 & 0 & ac \\
\end{pmatrix}, \quad c - 2b \neq 0; \quad a, c \in \mathbb{R} - \{0\}; \quad b, d, e, f, g, h \in \mathbb{R},
\end{align*}
\]

\[
Dsd^4_{(2,4)}:
\]

\[
\begin{align*}
(C^3, (A_{1,1} + 2A)^0_{0,0,1}) &\rightarrow (C^3, (A_{1,1} + 2A)^1_{0,0,1}), \quad \epsilon = \pm 1 \\
C &= \begin{pmatrix}
a & b & 0 & 0 & 0 & 0 \\
0 & e^{2a^2} & 0 & 0 & 0 & 0 \\
0 & ac & 0 & 0 & 0 & 0 \\
0 & d & 0 & 0 & 0 & 0 \\
0 & e & \frac{e^2 - a^2c^2}{2a^2c} & \frac{e^2 + a^2c^2}{2a^2c} & \frac{\epsilon c}{a^2} & -\epsilon \frac{d}{a} \\
0 & -e^{2a^2d^2} - e & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad a, c \in \mathbb{R} - \{0\}; \quad b, d, e \in \mathbb{R},
\end{align*}
\]

\[
\begin{align*}
(C^3, (A_{1,1} + 2A)^0_{0,0,1}) &\rightarrow (C^3, (A_{1,1} + 2A)^2_{0,0,1}), \quad \epsilon = \pm 1 \\
C &= \begin{pmatrix}
a & b & 0 & 0 & 0 & 0 \\
0 & -e^{2a^2} & 0 & 0 & 0 & 0 \\
0 & -e^{2a^2} & 0 & 0 & 0 & 0 \\
0 & e^{2a^2} & 0 & 0 & 0 & 0 \\
0 & e & \frac{e^2 - a^2c^2}{2a^2c} & \frac{e^2 + a^2c^2}{2a^2c} & c & d \\
0 & 0 & \frac{a(e^2 - a^2c^2d^2)}{2ac} & 0 & ac & 0 \\
\end{pmatrix}, \quad a, c \in \mathbb{R} - \{0\}; \quad b, d, e \in \mathbb{R},
\end{align*}
\]
$D_{sd}^{0}_{(2,4)}$:

\[
(C^4, (A_{1,1} + 2A)^{\epsilon_1, \epsilon_2} \rightarrow (C^4, (A_{1,1} + 2A)^{\epsilon_2} \rightarrow (C^4, (A_{1,1} + 2A)^{\epsilon_1, \epsilon_2}) \quad \epsilon_1, \epsilon_2 = \pm 1
\]

\[
C = \begin{pmatrix}
1 & b & 0 & 0 & 0 \\
0 & ac & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 \\
0 & 0 & -\frac{ad}{c} & a & 0 \\
0 & 0 & \frac{2c^2 - ac_2 + \epsilon_2 c}{\epsilon_2} & \frac{c}{\epsilon_2} & \epsilon_2 \\
0 & 0 & \frac{c}{\epsilon_2} & \epsilon_2 & 0 & c
\end{pmatrix}, \quad a, c \in \mathbb{R} - \{0\}; \quad b, d \in \mathbb{R},
\]

\[
(C^4, (A_{1,1} + 2A)^{\epsilon_1, \epsilon_2} \rightarrow (C^4, (A_{1,1} + 2A)^{\epsilon_2} \rightarrow (C^4, (A_{1,1} + 2A)^{\epsilon_1, \epsilon_2}) \quad \epsilon_1, \epsilon_2 = \pm 1
\]

\[
C = \begin{pmatrix}
1 & c & 0 & 0 & 0 \\
0 & ab & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 \\
0 & 0 & -\frac{ad}{c} & a & 0 \\
0 & 0 & \frac{a_2b + d + 2b(ma - c_1)}{a_2} & \frac{a_2}{a_2} & \frac{a}{a_2} \\
0 & 0 & \frac{a_2}{a_2} & \frac{a}{a_2} & 0 & b
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\]

\[
(C^4, (A_{1,1} + 2A)^{\epsilon_1, \epsilon_2} \rightarrow (C^4, (A_{1,1} + 2A)^{\epsilon_2} \rightarrow (C^4, (A_{1,1} + 2A)^{\epsilon_1, \epsilon_2}) \quad \epsilon_1, \epsilon_2 = \pm 1
\]

\[
C = \begin{pmatrix}
1 & c & 0 & 0 & 0 \\
0 & ab & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 \\
0 & 0 & -\frac{ad}{c} & a & 0 \\
0 & 0 & \frac{a_2b + d + 2b(ma - c_1)}{a_2} & \frac{a_2}{a_2} & \frac{a}{a_2} \\
0 & 0 & \frac{a_2}{a_2} & \frac{a}{a_2} & 0 & b
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\]

\[
(C^4, (A_{1,1} + 2A)^{\epsilon_1, \epsilon_2} \rightarrow (C^4, (A_{1,1} + 2A)^{\epsilon_2} \rightarrow (C^4, (A_{1,1} + 2A)^{\epsilon_1, \epsilon_2}) \quad \epsilon_1, \epsilon_2 = \pm 1
\]

\[
C = \begin{pmatrix}
1 & c & 0 & 0 & 0 \\
0 & ab & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 \\
0 & 0 & -\frac{ad}{c} & a & 0 \\
0 & 0 & \frac{a_2b + d + c^2}{2b} & \frac{a}{2b} & \frac{a}{2b} \\
0 & 0 & \frac{a}{2b} & \frac{a}{2b} & 0 & b
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\]

\[
(C^4, (A_{1,1} + 2A)^{\epsilon_1, \epsilon_2} \rightarrow (C^4, I_{(1,2)}) \quad \epsilon = \pm 1
\]

\[
C = \begin{pmatrix}
1 & c & 0 & 0 & 0 \\
0 & ab & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 \\
0 & 0 & -\frac{ad}{c} & a & 0 \\
0 & 0 & \frac{c_2b - \epsilon_2 c}{2b} & \frac{b}{2b} & \frac{b}{2b} \\
0 & 0 & \frac{b}{2b} & \frac{b}{2b} & 0 & b
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\]

\[
(C^4, (A_{1,1} + 2A)^{\epsilon_1, \epsilon_2} \rightarrow (C^4, I_{p = -1, (A_{1,1} + 2A)^{\epsilon_2, k, \epsilon_2}}) \quad k \in (-1, 1) - \{0\}; \quad \epsilon_1, \epsilon_2 = \pm 1
\]

\[
C = \begin{pmatrix}
1 & c & 0 & 0 & 0 \\
0 & k & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 \\
0 & 0 & \frac{c_2 b - \epsilon_2 c}{2b} & \frac{b}{2b} & \frac{b}{2b} \\
0 & 0 & \frac{b}{2b} & \frac{b}{2b} & 0 & b
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\[
\begin{align*}
(C^4, (A_{1, 1} + 2A)^0_{\epsilon_1, 0, 0}) & \rightarrow (C^2_{p^{\pm 1}}, (A_{1, 1} + 2A)^0_{\epsilon_2, \epsilon_2}), \quad \epsilon_1, \epsilon_2 = \pm 1 \\
C &= \begin{pmatrix}
1 & c & 0 & 0 & 0 & 0 \\
0 & \epsilon_2 ab & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b \\
0 & 0 & \frac{\sqrt{2}}{d} (\epsilon_1 b - a) & 0 & \epsilon_2 b & \frac{ka}{\epsilon_2} \\
0 & 0 & \frac{\epsilon_2}{d} & -\epsilon_2 a & 0 & \epsilon_2 b
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\end{align*}
\]

\[
(C^4, (A_{1, 1} + 2A)^0_{\epsilon_1, 0, 0}) \rightarrow (C^2_{p^{\pm 1}}, (A_{1, 1} + 2A)^2_{\epsilon_2, 1, 0}), \quad \epsilon = \pm 1 \\
C &= \begin{pmatrix}
1 & c & 0 & 0 & 0 & 0 \\
0 & -ab & 0 & 0 & 0 & 0 \\
0 & 0 & -b & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & -\frac{ad}{b} & a & 0 & \epsilon_2 \frac{ka}{b} \\
0 & 0 & \epsilon_1 \frac{a}{d} & 0 & b & d
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\]

\[
(C^4, (A_{1, 1} + 2A)^0_{\epsilon_1, 0, 0}) \rightarrow (C^2_{p^{\pm 1}}, (A_{1, 1} + 2A)^2_{\epsilon_2, 1, 0}), \quad \epsilon_1, \epsilon_2 = \pm 1 \\
C &= \begin{pmatrix}
1 & c & 0 & 0 & 0 & 0 \\
0 & -ab & 0 & 0 & 0 & 0 \\
0 & 0 & -b & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & \epsilon_1 \frac{a}{d} & 0 & a & -\frac{ad}{b} \\
0 & 0 & \frac{\epsilon_2}{d} & b & 0 & \epsilon_2 \frac{ka}{b}
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\]

\[
(C^4, (A_{1, 1} + 2A)^0_{\epsilon_1, 0, 0}) \rightarrow (C^2_{p^{\pm 1}}, (A_{1, 1} + 2A)^2_{\epsilon_2, s, -\epsilon_2}), \quad \epsilon_1, \epsilon_2 = \pm 1 \\
C &= \begin{pmatrix}
-1 & c & 0 & 0 & 0 & 0 \\
0 & -2sabe & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2be\epsilon \epsilon_2 \\
0 & 0 & \epsilon_2 \frac{-\epsilon_2 a d}{2b} & sa & 0 & b \\
0 & 0 & \epsilon_2 (s \epsilon_1 b - \frac{ka}{b}) & 0 & 2sbe \epsilon_2 & d
\end{pmatrix}, \quad a, b \in \mathbb{R} - \{0\}; \quad c, d \in \mathbb{R},
\]

\[
Dsd_9^{\gamma, p=0}; \\
(C^0, (A_{1, 1} + 2A)^0_{\epsilon_1, 0, 0}) \rightarrow (C^2, (A_{1, 1} + 2A)^0_{\epsilon_2, 0, 0}), \quad \epsilon_1, \epsilon_2 = \pm 1 \\
C &= \begin{pmatrix}
1 & c & 0 & 0 & 0 & 0 \\
0 & \frac{a^2}{\epsilon_1} (k + \epsilon_2) & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & -a & 0 & 0 & 0 \\
0 & 0 & -\epsilon_2 \frac{a}{\epsilon_1} & b & 0 & \frac{s b}{\epsilon_1 (k + \epsilon_2)} \\
0 & 0 & -b & \epsilon_2 \frac{a}{\epsilon_1} & -\frac{s a}{\epsilon_1 (k + \epsilon_2)} & 0
\end{pmatrix}, \quad a \in \mathbb{R} - \{0\}; \quad b, c \in \mathbb{R},
\]

26
\[ (C_0^5, (A_{1,1} + 2A)_0^5, I_{0,0,1}) \rightarrow (C_0^5, (A_{1,1} + 2A)_0^5) \], \quad \epsilon_1, \epsilon_2 = \pm 1

\[ C = \begin{pmatrix}
1 & c & 0 & 0 & 0 & 0 \\
0 & a^2(s + \epsilon_2) & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & -a & 0 & 0 & 0 \\
0 & 0 & -\epsilon_2^b & b & 0 & \frac{2}{c_1}(s + \epsilon_2) \\
0 & 0 & -b & \epsilon_2^b & -\frac{2}{c_1}(s + \epsilon_2) & 0
\end{pmatrix} \], \quad a \in \mathbb{R} - \{0\}; \quad b, c \in \mathbb{R},

\[ D_{sd}^p \quad p > 0 \]

\[ (C_0^5, (A_{1,1} + 2A)_0^5, I_{0,0,1}) \rightarrow (C_0^5, (A_{1,1} + 2A)_0^5) \], \quad \epsilon_1, \epsilon_2 = \pm 1

\[ C = \begin{pmatrix}
1 & c & 0 & 0 & 0 & 0 \\
0 & a^2(k-\epsilon_2) - 4ab(1+p^2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & -b & a^2 + 2b(1+2p^2) & 0 & \frac{a(k-\epsilon_2) - 4b(1+p^2)}{c_1} \\
0 & 0 & -\frac{a^2 + 2b}{2p} & \frac{a^2 + 2b(1+2p^2)}{2p} & \frac{a(k-\epsilon_2) - 4b(1+p^2)}{c_1} & 0
\end{pmatrix} \], \quad a \in \mathbb{R} - \{0\}; \quad b, c \in \mathbb{R},

\[ (C_0^5, (A_{1,1} + 2A)_0^5, I_{0,0,1}) \rightarrow (C_0^5, (A_{1,1} + 2A)_0^5) \], \quad \epsilon_1, \epsilon_2 = \pm 1

\[ C = \begin{pmatrix}
1 & c & 0 & 0 & 0 & 0 \\
0 & a^2(s-\epsilon_2) - 4ab(1+p^2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & -b & a^2 + 2b(1+2p^2) & 0 & \frac{a(s-\epsilon_2) - 4b(1+p^2)}{c_1} \\
0 & 0 & -\frac{a^2 + 2b}{2p} & \frac{a^2 + 2b(1+2p^2)}{2p} & \frac{a(s-\epsilon_2) - 4b(1+p^2)}{c_1} & 0
\end{pmatrix} \], \quad a \in \mathbb{R} - \{0\}; \quad b, c \in \mathbb{R},

\[ (C_0^5, (A_{1,1} + 2A)_0^5, I_{0,0,1}) \rightarrow (C_0^5, I_{1,2}) \], \quad \epsilon = \pm 1

\[ C = \begin{pmatrix}
-1 & c & 0 & 0 & 0 & 0 \\
0 & a^2b(1+p^2) & 0 & 0 & 0 & 0 \\
0 & 0 & -a & 2ap & \frac{4ap^2}{c} & \frac{4ap^2}{c} \\
0 & 0 & a & \frac{4ap^2}{c} & \frac{4ap^2}{c} & \frac{4ap^2}{c} \\
0 & 0 & -b & pb & 0 & 0 \\
0 & 0 & pb & b & 0 & 0
\end{pmatrix} \], \quad a, b \in \mathbb{R} - \{0\}; \quad c \in \mathbb{R},

\[ D_{sd}^p \quad p > 0 \]

\[ (C_0^5, (A_{1,1} + 2A)_0^5, I_{0,0,1}) \rightarrow (C_0^5, I_{1,2}) \], \quad \epsilon = \pm 1

\[ C = \begin{pmatrix}
1 & e & 0 & 0 & 0 & 0 \\
0 & ab - cd & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{cd}{2} - a & 0 & 0 & 0 \\
0 & 0 & a - \frac{cd}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{cd}{2} - e & 0 & 0 & 0 \\
0 & 0 & e - c & \frac{cd}{2} & 0 & 0
\end{pmatrix} \], \quad ab - cd \neq 0; \quad a, b, c, d, e \in \mathbb{R}.
References

[1] N. Andruskiewitsch, *Lie Superalgebras and Poisson-Lie supergroups*, Abh. Math. Sem. Univ. Hamburg 63 (1993) 147-163.

[2] N. Beisert and E. Spill, *The classical r-matrices of AdS/CFT and its Lie bialgebras structure*, [arXiv:0708.1762 [hep-th]].

[3] A. Eghbali, A. Rezaei-Aghdam, *Poisson-Lie T-dual sigma models on supermanifolds*, JHEP 09 (2009) 094, [arXiv:0901.1592v3 [hep-th]].

[4] N. Geer, *Etingof-Kazdan quantization of Lie super bialgebras*, [arXiv:math.QA/0409563].

[5] C. Juszczak and J. T. Sobczyk, *Classification of low dementional Lie super-bialgebras*, J. Math. Phys. 39, (1998) 4982-4992.

[6] C. Juszczak, *Classification of osp(2|2) Lie super-bialgebras*, [arXiv:math.QA/9906101].

[7] G. Karaali, *A New Lie Bialgebra Structure on sl(2,1)*, Contemp. Math. 413 (2006), pp.101-122.

[8] A. Eghbali, A. Rezaei-Aghdam and F. Heidarpour, *Classification of two and three dimensional Lie super-bialgebras*, J. Math. Phys. 51, 073503 (2010); [arXiv:0901.4471v [math-ph]].

[9] N. Backhouse, *A classification of four-dimensional Lie superalgebras*, J. Math. Phys. 19 (1978) 2400-2402.

[10] L. Hlavaty, J. Vysoky, *Manin supertriples and Drinfel’d superdoubles in low dimensional*, [arXiv:0908.0997v2 [math-ph]].

[11] Note that the bracket of one boson with one boson or one fermion is usual commutator but for one fermion with one fermion is anticommutator. Furthermore we identify grading of indices by the same indices in the power of (-1), for example grading(i) ≡ i; this is the notation that DeWitt applied in his book (Ref. [12]).

[12] B. DeWitt, *Supermanifolds*, (Cambridge University Press 1992).

[13] A. Rezaei-Aghdam, M. Hemmati, A. R. Rastkar, *classification of real three-dimensional Lie bialgebras and their Poisson-Lie groups*, J. Phys. A:Math.Gen. 38 (2005) 3981-3994, [arXiv:math-ph/0412092].

[14] M.A. Jafarizadeh and A. Rezaei-Aghdam, *Poisson-Lie T-duality and Bianchi type algebras*, Phys. Lett. B 458 (1999) 477-490, [arXiv:hep-th/9903152].

[15] L. Snobl, L. Hlavaty, *Classification of 6-dimensional real Drinfeld doubles*, Int.J.Mod.Phys. A17 (2002) 4043-4068, [arXiv:math.QA/0202210v2].

[16] The Lie superalgebra A is one dimensional Abelian Lie superalgebra with one fermionic generator where Lie superalgebra A_{1,1} is its bosonization. Furthermore, C^1_{\frac{1}{2}} is different from C^1_{p} for p = \frac{1}{2} and we show the latter by C^1_{\frac{1}{2}}.

[17] Note that 17 of them are two bosons-one fermion and 53 of them are one boson-two fermions Lie super-bialgebras.