Field equations and particle motion in covariant emergent gravity

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Abstract

We derive the full set of field equations based on Hossenfelder’s recent covariant formulation of the emergent gravity model, along with an exact solution. The solution describes a static, spherically-symmetric spacetime with a non-trivial vector field which plays the role of dark matter under the emergent gravity paradigm. Equations of motion of relativistic test masses are derived and are shown to reduce to Modified Newtonian Dynamics with additional relativistic corrections. It is also shown that the presence of the vector field gives an additional positive contribution to the bending angle in the deflection of light.

1 Introduction

Recently, Verlinde [1,2] proposed an interpretation of gravity where it is an emergent process arising out of some underlying microscopic structure. The entropy of the microscopic degrees of freedom appears as the gravitational force in the macroscopic regime. While this idea is fairly new and not free from criticisms [3], it carries many similar features to other approaches attempting to view spacetime as an emergent property arising from (quantum) non-gravitational systems, such as the holographic entanglement entropy [4–6] entanglement renormalisation [7,8], and exact holographic mapping [9,10]. Most of the examples mentioned here are either inspired by or related to the famous AdS/CFT correspondence.

One of the main drawbacks of Verlinde’s emergent gravity is that the results are calculated only in the Newtonian limit, and the model does not provide a Lagrangian
from which we may derive equations of motion for its variables. This shortcoming has been recently addressed by Hossenfelder who provided a covariant Lagrangian \cite{11} in accordance to Verlinde’s model. We shall henceforth refer to this Lagrangian as the Covariant Emergent Gravity (CEG) Lagrangian.

In this formulation, the theory of emergent gravity is modeled as a typical Einstein-Hilbert action with source terms associated with a vector field $u^\mu$, which was christened the *imposter field* in Ref. \cite{11}. This imposter field captures the effects of the microscopic degrees of freedom that manifests itself at macroscopic length scales and would tentatively play the role of dark matter and possibly even dark energy. This was further solidified in Ref. \cite{12} where the Newtonian limit of this CEG Lagrangian is shown to reproduce the acceleration of Modified Newtonian Dynamics (MOND) and is fitted against the galactic rotation curves.

However, the physics arising out of the CEG Lagrangian was also mainly considered in the non-relativistic limit in which the imposter field equations are solved against a flat spacetime, or in the probe limit in which the imposter field is solved on a fixed background metric and back-reactions to the spacetime are ignored. A cosmological spacetime was indeed considered in full relativistic treatment, and a limiting case to the de Sitter solution was obtained. Soon thereafter, Ref. \cite{13} pointed out a typo in Eq. (22) of \cite{11}, and they introduced a small modification to the Lagrangian to obtain another de Sitter limit from a cosmological metric. This CEG Lagrangian should be another relativistic completion to MOND, though so far the derivation of MOND was performed directly in the non-relativistic case of its equations of motion for the imposter field. Therefore, this derived MOND relation would not be able to account for relativistic corrections such as orbital precessions and gravitational lensing.

The aim of this paper is to address these issues with a full relativistic treatment of the equations arising from the CEG Lagrangian. In particular, we find that a full variation of the action without neglecting any terms produces a stress tensor that is different from \cite{11} and \cite{13}. The difference could be traced to a particular term in the action proportional to $\delta \Gamma^\Lambda_{\mu\nu}$. A full agreement with \cite{13} is recovered if this term is zero. With this full set of field equations, we obtain an exact solution representing a static, spherically-symmetric spacetime where the imposter field has zero spatial components. We may interpret this solution as a black hole, at least in the sense that the solution contains a curvature singularity hidden behind a horizon. Indeed, by turning off the imposter field, the spacetime reduces to the Schwarzschild solution.

With the exact solution at hand, we are able to describe the motion of relativistic test masses beyond the Newtonian limit. By Verlinde’s and Hossenfelder’s construction, the test masses should feel a force coming from the presence of the imposter field. As such, the motion of test particles are no longer described by geodesics of the spacetime, but rather geodesics of an *unphysical* spacetime where the metric is modified by the imposter
field. From its fully relativistic description, we are able to take the Newtonian limit to (re)derive the MOND acceleration along with additional relativistic corrections.

The relativistic solution also allows us to consider gravitational lensing in the spacetime. If we assume that photons are not affected by the imposter field, then the deflection of light could only be caused by the spacetime curvature. Since our solution captures the backreaction effects of the imposter field on spacetime curvature, the deflection of light due to the imposter field occurs only indirectly via this backreaction. As such this model additionally predicts a different amount of lensing compared to standard GR. We shall see below that the strength of the backreaction and the imposter field force on test masses are governed by two independent parameters. Thus one might be hopeful that this additional parameter might accommodate how a typical MOND description is not able to account for strong lensing [14], or, perhaps more generally, how lensing accounts for baryonic matter in the presence of this imposter field [15].

The rest of the paper is organised as follows. In Sec. 2 we review the covariant action under emergent gravity and derive its full set of equations of motion. In Sec. 3 we derive an exact solution starting from a general spherically-symmetric ansatz for the spacetime, and by assuming that the imposter field has zero spatial components. Once having the exact solution, we consider the motion of test masses in Sec. 4, and of photons in Sec. 5. We end the paper with some concluding remarks in Sec. 6.

2 Action and equations of motion

Let us briefly review the essential features of Hossenfelder’s CEG action. The basic variables are the metric $g_{\mu\nu}$ and the imposter field $u^\mu$. By dimensional analysis, Hossenfelder argues that the Lagrangian should be of the form $\chi^{3/2}$, where $\chi$ is a term that is quadratic in derivatives of $u$, namely, $\chi \sim (\nabla u)^2$. Therefore, the Lagrangian for the imposter field should take the form

$$\mathcal{L}_\theta = \frac{\alpha}{16\pi G} \chi^{3/2} - \mathcal{V}(u), \quad (1)$$

where $\mathcal{V}(u)$ is a potential described as a function of $u = \sqrt{-u_\mu u^\mu}$ and was chosen differently in previous literature. In particular, Hossenfelder chose $\mathcal{V}(u) \propto u^2$ in [11], while Dai and Stojkovic had $\mathcal{V}(u) \propto u^4$ in [13]. The parameter $\alpha$ characterises the coupling strength of the imposter field to gravity, and is expected to be of an order of inverse cosmological length scales. At this stage, the term $\chi$ is only required to be quadratic in $\nabla u$, which leads to three possible contractions

$$\chi = a \nabla_\sigma u^\sigma \nabla_\lambda u^\lambda + b \nabla_\sigma u_\lambda \nabla^\sigma u^\lambda + c \nabla_\sigma u_\lambda \nabla^\lambda u^\sigma. \quad (2)$$
In Refs. [11] and [13], a specific choice was made for the coefficients of the kinetic terms, namely $\bar{a} = \frac{4}{3}$, $\bar{b} = \bar{c} = -\frac{1}{2}$. However, in the following, we shall regard $\bar{a}$, $\bar{b}$, and $\bar{c}$ as arbitrary coefficients which may take other possible values.

Next, we wish to determine the Lagrangian to describe the interaction between normal matter and the imposter field. The main idea of Verlinde and Hossenfelder is that the effects normally attributed to dark matter is due to forces arising from the interaction between normal matter and the imposter field. In other words, the imposter field couples with the stress tensor of normal matter defined by

$$ T_{\mu\nu} = \mathcal{L}_m g_{\mu\nu} - 2\frac{\delta \mathcal{L}_m}{\delta g_{\mu\nu}}, \quad (3) $$

where $\mathcal{L}_m$ is the Lagrangian of normal matter. To construct the specific form of the interaction Lagrangian/action, we revisit the main idea of Verlinde and Hossenfelder, in which normal matter feels an effective metric of the form

$$ \tilde{g}_{\mu\nu} = g_{\mu\nu} - \beta \frac{u_{\mu}u_{\nu}}{u}. \quad (4) $$

On this basis, we can construct the interaction Lagrangian by considering the motion of a time-like test particle of mass $m$ (made with normal matter) with trajectory $x^\mu(\tau)$, where $\tau$ parameterises the trajectory. The action and corresponding stress tensor for the particle are

$$ I_m = \frac{m}{2} \int d\tau g^{\mu\nu} \dot{x}_\mu \dot{x}_\nu, \quad T_{\mu\nu}(y) = -\frac{m}{\sqrt{-g(y)}} \int d\tau \delta(x - y) \dot{x}_\mu \dot{x}_\nu, \quad (5) $$

where over-dots denote derivatives with respect to $\tau$. If the test particle (made from normal matter) is to feel an effective metric (4), our desired (matter) + (interaction) should be

$$ I_m + I_{\text{int}} = \frac{m}{2} \int d\tau \tilde{g}_{\mu\nu} \dot{x}_\mu \dot{x}_\nu = \frac{m}{2} \int d\tau \left(g_{\mu\nu} - \beta \frac{u_{\mu}u_{\nu}}{u}\right) \dot{x}_\mu \dot{x}_\nu. \quad (6) $$

This would be achieved if the interaction takes the form

$$ I_{\text{int}} = \frac{\beta}{2} \int d^4 x \sqrt{-g} \frac{u^\mu u^\nu}{u} T_{\mu\nu} = \int d^4 x \sqrt{-g} \mathcal{L}_{\text{int}}, \quad (7) $$

where

$$ \mathcal{L}_{\text{int}} = \frac{\beta}{2} \frac{u^\mu u^\nu}{u} T_{\mu\nu}. \quad (8) $$

\footnote{See, for instance, [16].}
Assembling the pieces together, the CEG model is described by the action

\[ I = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi G} R + \mathcal{L}_{\text{m}} + \mathcal{L}_\theta + \mathcal{L}_{\text{int}} \right). \] (9)

Here, we note that in our derivation of \( \mathcal{L}_{\text{int}} \), we have introduced a small modification to Hossenfelder’s action. Namely, there is an additional factor of \(-\frac{1}{2}\) when comparing Eq. (9) to Eq. (6) of [11]. (In the present notation, \( \beta = 1/L \), where \( L \) is the notation used by Hossenfelder.) However, we argue that this factor is necessary for Eq. (6), or equivalently, Eq. (5) of [11], to hold.

In the following, we find it easier to keep track of the terms by manipulating the symmetric and anti-symmetric parts of \( \chi \) separately. As such we follow [11] and consider the strain tensor defined by

\[ \epsilon_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu, \] (10)

in addition to an anti-symmetric combination

\[ F_{\mu\nu} = \nabla_\mu u_\nu - \nabla_\nu u_\mu. \] (11)

We shall also redefine our coefficients in Eq. (2) by

\[ a = \frac{\bar{a}}{2}, \quad b = \frac{\bar{b} + \bar{c}}{2}, \quad c = \frac{\bar{b} - \bar{c}}{2}. \] (12)

In terms of these quantities, Eq. (2) becomes

\[ \chi = \frac{a}{2} (\epsilon^\sigma_\sigma)^2 + \frac{b}{2} \epsilon^\sigma_\sigma \epsilon_\sigma_\lambda + \frac{c}{2} F^\sigma_\sigma F^\lambda_\lambda. \] (13)

In performing the variation of the action, a crucial ingredient involves the variation of \( \chi \), which is given by

\[ \delta \chi = A_{\mu\nu} \delta g^{\mu\nu} + 2B^{\mu\nu} (\nabla_\mu \delta u_\nu - u_\lambda \delta \Gamma^\lambda_{\mu\nu}), \] (14)

where

\[ A_{\mu\nu} = a\epsilon^\lambda_\lambda \epsilon_{\mu\nu} + b\epsilon_{\mu\lambda} \epsilon^\lambda_\nu + cF_{\mu\lambda} F^\lambda_\nu, \] (15a)

\[ B^{\mu\nu} = a\epsilon^\lambda_\lambda g^{\mu\nu} + b\epsilon^{\mu\nu} + cF^{\mu\nu}. \] (15b)
Hence, the variation of the action is

\[
\delta I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ 3\alpha \chi^{1/2} B_{\mu\nu} \nabla_\mu \delta u_\nu + 16\pi G \frac{dV}{du} \frac{dV}{u} \delta u_\nu \\
+ 8\pi G \beta \left( \frac{u^\sigma u^\lambda T_{\sigma\lambda} u^\nu}{u^3} + \frac{2u^\lambda T^{\nu}_{\lambda}}{u} \right) \delta u_\nu \\
+ \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 8\pi GT_{\mu\nu} \right) \delta g^{\mu\nu} + g_{\mu\nu} \nabla_\sigma \nabla^\sigma \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu} \\
+ 8\pi G \beta \left( \frac{u^\lambda u^\sigma T_{\lambda\sigma} u^\mu u_\nu}{2u^3} - \frac{u^\lambda u^\sigma T_{\lambda\sigma}}{2u} g_{\mu\nu} + \frac{2u_\mu u^\lambda T^{\mu}_{\lambda}}{u} \right) \delta g^{\mu\nu} \\
+ \left[ \alpha \chi^{1/2} \left( \frac{3}{2} A_{\mu\nu} - \frac{1}{2} \chi g_{\mu\nu} \right) + 8\pi G V g_{\mu\nu} + 8\pi G \frac{dV}{du} u_\mu u_\nu \right] \delta g^{\mu\nu} \\
+ \frac{3\alpha}{2} u_\mu \chi^{1/2} B_{\sigma\nu} \nabla_\sigma \delta g^{\mu\nu} + \frac{3\alpha}{2} u_\nu \chi^{1/2} B_{\mu\sigma} \nabla_\sigma \delta g^{\mu\nu} - \frac{3\alpha}{2} u_\lambda \chi^{1/2} B_{\mu\nu} \nabla_\lambda \delta g^{\mu\nu} \right\}. 
\]

(16)

Note that we did not vary \( T_{\mu\nu} \) itself in \( L_{\text{int}} \).

Before proceeding, let us take a moment to draw a comparison between Eq. (16) and the results of [11,13], particularly the terms involving the variation \( \delta g^{\mu\nu} \). Clearly the third line in Eq. (16) is the Einstein tensor and the stress tensor due to normal matter (plus its corresponding boundary term). The fourth line is precisely \( (T_{\text{int}})_{\mu\nu} \) as given by [13]. The fifth line has the term

\[-\alpha \chi^{1/2} \left( \frac{3}{2} A_{\mu\nu} - \frac{1}{2} \chi g_{\mu\nu} \right) = -\alpha \chi^{1/2} \left( \frac{3a}{2} \epsilon^\lambda \chi \epsilon_{\mu\nu} + \frac{3b}{2} \epsilon_{\mu\lambda} \epsilon^\lambda_{\nu} + \frac{3c}{2} F_{\mu\lambda} F_{\nu}^\lambda - \frac{1}{2} \chi g_{\mu\nu} \right).\]

(17)

Now, Hossenfelder's choice of parameters were \( \bar{a} = \frac{4}{3} \), \( \bar{b} = \bar{c} = -\frac{1}{2} \). Via Eq. (12), this corresponds to \( a = \frac{2}{3} \), \( b = -\frac{1}{2} \), and \( c = 0 \). With this choice, the above equation becomes

\[\frac{\alpha}{2} \chi^{1/2} \left( -2\epsilon^\lambda \chi \epsilon_{\mu\nu} + \frac{3}{2} \epsilon_{\mu\lambda} \epsilon^\lambda_{\nu} + \chi g_{\mu\nu} \right).\]

(18)

This term, when added to \( 8\pi G \left( V g_{\mu\nu} + \frac{dV}{du} u_\mu u_\nu \right) \), is precisely \( (T_s)_{\mu\nu} \) as given by [13]. We have already reproduced all the stress tensor components of [13], but the last line of Eq. (16) is still unaccounted for!

To see where this line came from, we recall that the last term of Eq. (14) involves the variation

\[\delta \Gamma^\lambda_{\mu\nu} = -\frac{1}{2} \left( g_{\mu\sigma} \nabla_\nu \delta g^\sigma^\lambda + g_{\nu\sigma} \nabla_\mu \delta g^\sigma^\lambda - g_{\mu\sigma} g_{\nu\rho} \nabla^\lambda \delta g^{\sigma\rho} \right).\]

(19)

This term contributes to the last line of Eq. (16). If \( \delta \Gamma^\lambda_{\mu\nu} = 0 \), or if the last line of Eq. (16)
is not present, the variation $\frac{\delta I}{\delta g_{\mu \nu}} = 0$ reproduces exactly the stress tensor given in [13]. However, we are unable to find any justification to neglect these terms.

Keeping all the terms in Eq. (16) and performing integration by parts, the variation of the action is

$$\delta I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ -\delta u_c \nabla_{\mu} \left( \chi^{1/2} B^{\mu \nu} \right) + 16\pi G \frac{dV}{du} \frac{\delta u}{\delta u} \right. $$

$$+ 8\pi G \frac{\delta \nu \delta u^3 \frac{\lambda^\nu}{\mu \nu}}{u^3} \delta u_{\mu} + \left( R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} - 8\pi G T_{\mu \nu} \right) \frac{\delta g_{\mu \nu}}{u^3}$$

$$+ 8\pi G \frac{\delta \nu \delta u^3 \frac{\lambda^\nu}{\mu \nu}}{u^3} \delta u_{\mu} + \left( R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} - 8\pi G T_{\mu \nu} \right) \frac{\delta g_{\mu \nu}}{u^3}$$

$$+ \left[ \alpha \chi^{1/2} \left( \frac{3}{2} A_{\mu \nu} - \frac{1}{2} \chi^\nu \right) \delta g_{\mu \nu} \right] + \left( \frac{3}{2} A_{\mu \nu} - \frac{1}{2} \chi^\nu \right) \delta g_{\mu \nu}$$

$$- \frac{3\alpha}{2} \left[ \nabla_{\sigma} \left( \frac{u_{\mu} \chi^\nu (B_{\mu}^\nu) + \nabla_{\nu} \left( u_{\mu} \chi^\nu (B_{\mu}^\nu) - \nabla_{\lambda} \left( u_{\mu} \chi^\nu (B_{\mu}^\nu) \right) \right) \right) \right] \left. \right\}.$$  \(\text{(20)}\)

where the last two lines are total derivatives which only provide contributions to the boundary of the spacetime. The third-last line is proportional to $\delta g_{\mu \nu}$, and would contribute to the stress tensor, ultimately modifying the stress tensor used by [13] and [11].

The variation $\frac{\delta I}{\delta u^\mu} = 0$ gives the equation of motion for $u^\mu$, which we shall refer to as the imposter equation,

$$\frac{3\alpha}{16\pi G} \nabla_{\mu} \left[ \chi^{1/2} \left( a e^\sigma g_{\mu \nu} + b e^\nu g_{\mu \nu} + c F_{\mu \nu} \right) \right] = \frac{dV}{du} \frac{\delta u}{\delta u} + \frac{\beta}{2} \left( \frac{2u^{\lambda T_{\lambda}}}{u^3} + \frac{u^{\sigma} u^{\lambda T_{\sigma} T_{\lambda}}}{u^3} \right).$$  \(\text{(21)}\)

Finally, the variation $\frac{\delta I}{\delta g_{\mu \nu}} = 0$ gives us the Einstein equation\(^2\)

$$R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = 8\pi G \left[ T_{\mu \nu} + \frac{1}{2} \beta u^\lambda T_{\lambda \mu \nu} \left( \frac{u_{\mu} u_{\nu}}{u^2} \right) + \frac{dV}{du} \frac{u_{\mu} u_{\nu}}{u^3} - \nabla g_{\mu \nu} \right]$$

$$- \alpha \varepsilon^{\mu \nu \lambda} \left( \frac{3}{2} F_{\mu \lambda} F_{\nu}^\lambda - \frac{1}{4} F^{\sigma \lambda} F_{\sigma \lambda} g_{\mu \nu} \right)$$

$$+ \alpha \chi^{1/2} \left( \frac{a}{2} (e^\sigma)^2 g_{\mu \nu} + \frac{b}{4} (e^\sigma)^2 \varepsilon_{\sigma \lambda} g_{\mu \nu} - \frac{3b}{4} \varepsilon^\sigma g_{\mu \nu} - \frac{3b}{4} \varepsilon_{\sigma \lambda} g_{\mu \nu} + \frac{3b}{4} F_{\sigma \mu} \varepsilon_{\nu}^\lambda \right)$$

$$- \frac{3\alpha}{2} u^\lambda \nabla_{\lambda} \left[ \chi^{1/2} \left( a e^\sigma g_{\mu \nu} + b e^\nu g_{\mu \nu} \right) \right] - 3\alpha c \left[ \nabla_{\sigma} \left( \chi^{1/2} F_{\sigma (\mu}^\nu \right) u_{\nu}) \right].$$  \(\text{(22)}\)

where we have used Eq. (21) to simplify some terms involving the potential and stress tensor.

\(^2\)Our notation for symmetrisation is $A_{(\mu \nu)} = \frac{1}{2} (A_{\mu \nu} + A_{\nu \mu})$. 

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In Ref. [11], Hossenfelder argued that the parameter choice $\bar{a} = \frac{4}{3}, \bar{b} = \bar{c} = -\frac{1}{2}$ was obtained by enforcing the conservation of the stress tensor in a de Sitter background and assuming a constant imposter field with zero spatial components. Since our present stress tensor is modified due to the non-trivial variation $\delta \Gamma^\lambda_{\mu
u}$, we should revisit this statement.

Applying Hossenfelder’s argument to the right-hand side of (22) for $T_{\mu\nu} = 0$, and $V = 0$, we find that

$$3a + b = 0 \quad \rightarrow \quad 3\bar{a} = -(\bar{b} + \bar{c}).$$

(23)

However, we note that this holds only by assuming an empty de Sitter background, and may not hold in general. In practice, it might be more convenient to solve the equations of motion for arbitrary $\bar{a}$, $\bar{b}$, and $\bar{c}$, first, then use stress-energy conservation to find a constraint among the parameters.

### 3 Exact solution

We now attempt find an exact solution in the absence of normal matter ($T_{\mu\nu} = 0$), and $V = 0$. We consider a spherically-symmetric ansatz of the form

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2d\Omega^2_{(2)},$$

(24a)

$$u^\mu = \phi(r)\delta^\mu_t,$$

(24b)

where $f$, $h$, and $\phi$ are scalar functions that depend only on $r$. Under this ansatz, we find that a simple exact solution is possible for the case $b = 0$, $c = -1$ and $h(r) = 1/f(r)$, given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2_{(2)}, \quad u^\mu = \phi(r)\delta^\mu_t,$$

(25a)

$$f(r) = 1 - \frac{2M}{r} - \frac{\alpha q^3}{r} \ln \left( \frac{r}{r_g} \right),$$

(25b)

$$\phi(r) = \frac{q}{f(r)} \ln \left( \frac{r}{r_0} \right).$$

(25c)

This solution is parametrised by $M$, $q$, and $r_0$. The parameter $r_g$ can be absorbed by rescaling $M$, though we will keep it so that the argument of the logarithmic function appears explicitly dimensionless.

Note that $a$ has so far been irrelevant in our calculations, since the kinetic term where $a$ is the coefficient identically vanishes. However, the particular choice $a = b = 0$ corresponds to an anti-symmetric combination for $\chi$, and thus the field is similar to that of a gauge boson. As such we might consider $\chi$ to be something akin to a Maxwell-type entity. In fact, one may have already noticed that the metric (25a) is indeed similar to a
black hole solution in non-linear Maxwell theory where the power of the Maxwell invariant is $3/2$ [17, 18], which is precisely the power of $\chi$ in the Lagrangian.

In any case, the imposter field $u$ should not be interpreted as an electromagnetic potential, as it couples to matter in a very different way — clearly the field $u$ exerts forces on uncharged particles, as intended in the construction of emergent gravity. Unlike the vector potential in non-linear Maxwell theory, the imposter field $u$ will contribute to the gravitational potential that is argued to explain the galactic rotation curves in the dark matter problem [12]. Furthermore, the inclusion of matter fields introduces $I_{\text{int}}$ to the action, and will lead to very different results from non-linear Maxwell theory. We shall explore the effect of $u$ on particles in further detail in the following sections.

Before closing this section, let us briefly mention the physical properties of the spacetime. Firstly, the spacetime is asymptotically flat, though the metric functions include a term $\sim \ln(r)/r$ which dies off more slowly as compared to the pure Schwarzschild case. The horizon is located at $r = r_+$ for which $f(r_+) = 0$. It may be more convenient to parametrise the solution with $r_+$ in place of $M$, where $M$ can be recovered by

$$M = \frac{1}{2} \left( r_+ - \alpha q^3 \ln \frac{r_+}{r_g} \right).$$  \hspace{1cm} (26)

In terms of $r_+$, the surface gravity and horizon area are given by

$$\kappa = \frac{r_+ - \alpha q^3}{2r_+^2}, \quad A = r_+^2 \Omega_{(2)},$$  \hspace{1cm} (27)

where $\Omega_{(2)} = 4\pi$ is the area of a unit two-sphere. The Kretschmann invariant and Ricci scalar are respectively

$$R_{\sigma\mu\nu\rho}R^{\sigma\mu\nu\rho} = \frac{1}{r^6} \left\{ 48M^2 + 8M\alpha q^3 \left( 6 \ln \frac{r}{r_g} - 5 \right) ight.$$  
$$+ \alpha^2 q^6 \left[ 12 \left( \ln \frac{r}{r_g} \right)^2 - 20 \ln \frac{r}{r_g} + 13 \right] \right\},$$  \hspace{1cm} (28)

$$R = \frac{\alpha q^3}{r^3},$$  \hspace{1cm} (29)

indicating the presence of a curvature singularity at $r = 0$. It can be easily checked that the solution (25) saturates the Null Energy Condition $R_{\mu\nu}k^\mu k^\nu \geq 0$ where $k^\mu$ is any null vector satisfying $k^\mu k_\mu = 0$. 

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4 Motion of test masses

4.1 Equations of motion and the Newtonian limit

As discussed in Sec. 2, Hossenfelder’s model [11] was designed to be a covariant description of [2], where matter feels an effective metric of the form \( \tilde{g}_{\mu\nu} = g_{\mu\nu} - \beta \frac{u_{\mu}u_{\nu}}{u} \). Therefore, the motion of a test particle is no longer a geodesic curve of \( g_{\mu\nu} \), but rather that of the unphysical spacetime \( \tilde{g}_{\mu\nu} \).

In light of this, for the motion of particles described by a curve \( x^\mu(\tau) \) parametrised by \( \tau \). As we have mentioned in Sec. 2, the action for a particle of mass \( m \) is described by Eq. (6). Let us write the Lagrangian here for convenience:

\[
\mathcal{L} = \frac{m}{2} \tilde{g}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{m}{2} \left( g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \beta \frac{u_{\mu}u_{\nu}}{u} \dot{x}^\mu \dot{x}^\nu \right),
\]

where over-dots denote derivatives with respect to \( \tau \). Applying the Euler-Lagrange equation, or equivalently, extremising \( \int d\tau \mathcal{L} \), leads to

\[
\left( \delta^\kappa_{\mu} - \beta \frac{u_{\mu}u_{\kappa}}{u} \right) \ddot{x}^\mu + \Gamma^{\kappa}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \beta C^{\kappa}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu,
\]

where

\[
C^{\kappa}_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} \left[ \partial_\mu \left( \frac{u_{\lambda}u_{\nu}}{u} \right) + \partial_\nu \left( \frac{u_{\lambda}u_{\mu}}{u} \right) - \partial_\lambda \left( \frac{u_{\mu}u_{\nu}}{u} \right) \right]
\]

At this stage, it is important to reiterate that \( x^\mu(\tau) \) that solves Eq. (31) is a geodesic of an unphysical metric \( \tilde{g}_{\mu\nu} = g_{\mu\nu} - \beta \frac{u_{\mu}u_{\nu}}{u} \), and therefore \( \dot{x}^\mu \) is no longer a parallel-transported vector in the spacetime \( g_{\mu\nu} \). Nevertheless, as a geodesic of \( \tilde{g}_{\mu\nu} \), it is a vector that is parallel-transported along that unphysical metric, and therefore the inner product \( g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \) is constant along the geodesic. For time-like geodesics, we can then always rescale \( \tau \) to make the magnitude of this constant be unity, giving us a first integral

\[
\left( g_{\mu\nu} - \beta \frac{u_{\mu}u_{\nu}}{u} \right) \dot{x}^\mu \dot{x}^\nu = -1.
\]

We can take the non-relativistic limit by having \( \dot{t} \gg \dot{x}^i \). Further assuming the spacetime is static and the \( tt \)-component of the metric is of the form \( g_{tt} \simeq -1 + 2\psi_N \), Eq. (31) reduces to

\[
m \frac{d^2\tilde{r}}{dt^2} = -\nabla (\psi_N + \psi_u).
\]
functions in (25), the potentials are

\[
\psi_N = \frac{M}{r} + \frac{1}{2} \alpha q^3 \ln \left( \frac{r}{r_g} \right), \tag{35}
\]

\[
\psi_u = -\frac{\beta q}{2} \ln \left( \frac{r}{r_0} \right). \tag{36}
\]

We see that \(\psi_N\) is the gravitational potential obtained by taking the weak field limit of the spacetime geometry, and \(\psi_u\) is the potential due to the imposter field directly exerting a force on the moving particles. Interestingly, since this limit is obtained from the exact solution where the mass-energy of the imposter field contributes to the spacetime curvature, \(\psi_N\) has an additional term \(\frac{1}{2} \alpha q^3 \ln \left( \frac{r}{r_g} \right)\), which is a relativistic correction coming from this back-reaction. Incidentally, a \(\ln(r)/r\) term was also obtained by [19] is reproduced within \(\psi_N\) from a different context.

Differentiating the total potential \(\psi = \psi_N + \psi_u\), we find that the acceleration of the particle is

\[
a = -\frac{M}{r^2} - \frac{\beta q}{2r} + \frac{\alpha q^3}{2r^2} - \frac{\alpha q^3}{2r^2} \ln \left( \frac{r}{r_g} \right). \tag{37}
\]

The first two terms reproduces the MOND relation obtained in [12] with the MOND parameter expressed in terms of the present notation as

\[
\sqrt{a_0} = \frac{\beta q}{2\sqrt{M}}, \tag{38}
\]

and the last two terms are due to the backreaction of the imposter field to the metric, which will be negligible if \(\alpha q^3\) is sufficiently small. Let us then regard these terms as the relativistic correction to MOND under CEG.

Having this non-relativistic limit allows us to make a few statements on some of the parameters of our solution. Firstly, for Eq. (37) to appropriately contribute to the galactic rotation curves, we require \(\beta q > 0\). To further make contact with the Lagrangian in [11], we have \(\beta = 1/L\). This sets \(q\) to be positive. The quantity \(\alpha q^3\) may perhaps be constrained by the rotation curve data of the galaxies via Eq. (37).

### 4.2 Relativistic test mass

We now consider the test mass in a fully relativistic treatment. For the present exact solution, the particle Lagrangian is

\[
\mathcal{L} = \frac{m}{2} \left[ -(f + \beta f^{3/2} \phi) \dot{t}^2 + \frac{r^2}{f} + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right]. \tag{39}
\]
We can use the spherical symmetry of the spacetime to fix the coordinate system such that the motion is confined to the plane \( \theta = \frac{\pi}{2} = \text{constant} \), and we need not consider \( \theta \) henceforth.

With the Lagrangian (39), we can now proceed to obtain the equations of motion. Since \( t \) and \( \varphi \) are cyclic variables, we have the first integrals

\[
\dot{t} = \frac{E}{f (1 + \beta \sqrt{f} \phi)}, \quad \dot{\varphi} = \frac{L}{r^2},
\]

where \( E \) and \( L \) may be interpreted as the energy and angular momentum of the particle, respectively. Applying the Euler-Lagrange equation to the coordinate \( r \) gives

\[
\ddot{r} = \frac{f'}{2f} \dot{r}^2 - \frac{(f + \beta f^{3/2} \phi)'}{2f (1 + \beta \sqrt{f} \phi)^2} E^2 + \frac{L^2 f}{r^3}.
\]

As argued in the previous subsection, inner products of vectors are preserved if they are parallel-transported in \( \tilde{g}_{\mu\nu} \) instead of \( g_{\mu\nu} \). Therefore Eq. (33) gives a constraint

\[
\dot{r}^2 = \frac{E^2}{1 + \beta \sqrt{f} \phi} - \left( \frac{L^2}{r^2} + 1 \right) f.
\]

Because of the logarithmic functions appearing in \( f \) and \( \phi \), it will generally be difficult to integrate the equations of motion exactly. Nevertheless, we could still extract some qualitative features by inspecting Eq. (42), in addition to solving the equations numerically.

For instance, given a particle of a specific energy and angular momentum in a spacetime with parameters \( M, q, \alpha, \beta, \) and \( r_0 \), we can look for the presence of stable bound orbits by finding a finite range where \( \dot{r}^2 > 0 \) in Eq. (42). If this range includes the horizon \( r_+ \), then this particle may eventually fall into the black hole. On the other hand, if the range extends to infinity, the particle is unbound and may escape. However, if we find a finite range of \( r \) with positive \( \dot{r}^2 \), the particle is in a stable bound orbit. An example of a bound orbit is shown in Fig. 1.

We note that in Eq. (42), one might be concerned that the coefficient of \( E^2 \) might blow up if

\[
\beta q f^{-1/2} \ln(r/r_0) = -1
\]

for some \( r = r_* \) outside the horizon. We have established in the previous subsection that \( \beta q > 0 \). Therefore, Eq. (43) requires \( r/r_0 < 1 \) for there to be a real solution. An example for such a scenario is shown in Fig. 2, where a particle with angular momentum \( L^2 = 20 \) moves in a spacetime with parameters \( M = 1, r_0 = 10, q = 1, \alpha = 0.001, \) and \( \beta = 0.1 \). For this spacetime, the horizon is located at \( r_+ = 2.000693494 \), and \( \dot{r}^2 \) diverges.
Figure 1: A test mass trajectory of energy $E^2 = 0.94$ and $L^2 = 16$ in a spacetime of parameters $M = 1$, $r_0 = 1$, $q = 1$, $\alpha = 0.001$, and $\beta = -0.001$. The left plot shows $\dot{r}^2$ vs $r$, where positive $\dot{r}$ is seen to lie in the range $7.93 < r < 20.7$. And this range can be clearly seen in the trajectory plotted on the right in Cartesian coordinates.

Figure 2: Plots of $\dot{r}^2$ vs $r$ for trajectories with angular momentum $L^2 = 20$ in a spacetime of parameters $M = 1$, $r_0 = 10$, $q = 1$, $\alpha = 0.001$, $\beta = 0.1$ and various values of $E$. Starting from $E < E_{\text{crit}}$, there is a stable bound orbit separated from $r_*$ by a potential barrier. If the energy is increased beyond $E \geq E_{\text{crit}}$, the potential barrier vanishes, and the particle now can access $r = r_*$. For the parameters in this figure, the critical energy is approximately $E_{\text{crit}}^2 = 0.98534352$ and $r_* = 2.052188553$. 
at $r_s = 2.052188553$, which is outside the horizon.

Let us then consider particles of various energies which may encounter the position $r = r_s$. By numerical exploration, we find that for certain values of energy below some critical value $E_{\text{crit}}$, there exist a stable bound orbit with a range that does not include $r_s$. (See the dashed curve of Fig. 2.) However, if $E$ is increased, the particle can access the location $r = r_s$, where the particle acquires infinite velocity but does not go beyond $r < r_s$ as $\dot{r}^2 < 0$ for that range.

We can regard the $r = r_s$ singularity as a failure in the parametrisation of the motion in the coordinates of the unphysical metric $\tilde{g}_{\mu\nu}$. We can trace this back to the fact that at $r = r_s$, the metric function $\tilde{g}_{tt}$ becomes zero. Hence the particle has encountered a ‘pole’ in the coordinate system of the unphysical spacetime. This is analogous to the ordinary equations of motion of dynamical systems with spherical symmetry which are also singular at the north and south poles at $\theta = 0$ and $\theta = \pi$, respectively.

5 Motion of photons

At this stage, Refs. [2,11] do not indicate how massless fields couple to $u^\mu$. Lacking further information, it seems reasonable to assume that photons still travel along geodesics of the physical spacetime, and we have

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \quad (44)$$

The corresponding Lagrangian is simply $2\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ per unit energy. For the metric (25a)

$$\mathcal{L} = \frac{1}{2} \left( -ft^2 + \frac{r^2}{f} + r^2\dot{\varphi}^2 \right), \quad (45)$$

where again, we use the spherical symmetry of the spacetime to fix the coordinates such that the geodesics is confined to the plane $\theta = \frac{\pi}{2} = \text{constant}$. As in the previous subsection, we can derive the equations of motion in a similar manner, which gives us

$$\dot{t} = \frac{E}{f}, \quad \dot{\varphi} = \frac{L}{r^2}, \quad (46a)$$

$$\dot{r}^2 = E^2 - \frac{L^2}{r^2}f, \quad (46b)$$

$$\ddot{r} = \frac{f'}{2f^2}r^2 - \frac{f'E^2}{2f} + \frac{fL^2}{r^2}. \quad (46c)$$

\[ \text{Since it is the } tt\text{-component of the metric that vanishes, one might be tempted to say that } r = r_s \text{ is the ‘horizon’ of the unphysical metric. We shall not explore this analogy further, as this will not affect null trajectories if we assume photons do not couple with the imposter field, as we do in Sec. 5.} \]
Figure 3: Bending angle vs $\alpha q^3$, for $M = 1$ and impact parameters $J = 10$ (solid curve) and $J = 20$ (dashed curve).

We can find the (coordinate) distance of closest approach, $r_{\text{min}}$ by finding the largest root of $r$ satisfying $\dot{r} = 0$. This gives

$$\frac{E^2}{L^2} = \frac{f(r_{\text{min}})}{r_{\text{min}}^2}. \quad (47)$$

Further defining $u = 1/r$ and $u_{\text{min}} = 1/r_{\text{min}}$, we can calculate the deflection angle by dividing $\dot{r}$ with $\dot{\varphi}$ from Eq. (46) and integrating

$$\Delta \varphi = 2 \int_0^{u_{\text{min}}} \frac{du}{\sqrt{u_{\text{min}}^2 f(1/u_{\text{min}}) - u^2 f(1/u)}}. \quad (48)$$

In order to compare how lensing in CEG fares against the Schwarzschild ($\alpha q^3 = 0$) case, we calculate lensing in both cases for photons with the same impact parameter

$$J = \frac{r_{\text{min}}}{\sqrt{f(r_{\text{min}})}}. \quad (49)$$

The numerical results can be seen in Fig. 3, where as $\alpha q^3$ is increased, lensing is greater than the Schwarzschild case at $\alpha q^3 = 0$. As expected, the smaller impact parameter results in a larger bending angle, as the photon passes within a closer proximity to the gravitating mass.

6 Conclusion

With a fully relativistic variation of the action, we derive a full set of field equations from the CEG Lagrangian. We find that our resulting equations of motion involves a stress tensor that is different from [11] and [13], where our present stress tensor has an additional
contribution due to the variation $\delta \Gamma^\lambda_{\mu\nu}$.

In the case where $b = 0$, we considered an exact solution corresponding to a static, spherically-symmetric spacetime and an imposter field that varies logarithmically in $r$. While this system corresponds to a different stress tensor considered by [11] and [13], the weak-field limit nevertheless reproduces the MOND relations in [11, 12], along with additional relativistic corrections which was not captured in its purely non-relativistic derivation. We have also calculated the motion of fully relativistic test masses and gravitational lensing.

Verlinde’s main formulation for emergent gravity comes from arguments of entropy. In the present paper, we have obtained a black hole solution with its associated surface gravity and horizon area given in Eq. (27). It would be interesting to consider a thermodynamic analysis of this solution, perhaps along the lines of the Gibbons-Hawking path-integral method [20]. Indeed, since the metric (25a) is similar to a black hole in non-linear Maxwell theory, some of the thermodynamic analysis of Gonzales et al. [18] could be carried over. Furthermore, a non-trivial $\mathcal{V}(u)$ should probably be taken into account as well.

With regards to gravitational lensing, we have so far assumed that photons travel along null geodesics of the metric in the usual manner. In this view, the imposter field do not directly exert forces on photons, but only influence their motion indirectly through its backreaction on the metric. Our solution is asymptotically flat with no cosmological horizons. Thus we take the coordinate angle $\Delta \phi$ to be equivalent to the angle measured by an observer at infinity, therefore we need not measure bending angles at finite distances, thus avoiding the need to apply the Rindler-Ishak method [21].

While we have demonstrated that the presence of an imposter field provides a positive contribution to the bending angle, it should be noted that this is in the somewhat idealised case of a static, spherically-symmetric vacuum solution. Lensing observations are due to galaxy or galaxy clusters with non-trivial mass distribution. Furthermore, in order to draw conclusions of the theory in relation to observation, the redshift has to be taken into account. Therefore an obvious task in in extension to the present work is to recast this solution as a perturbed Robertson-Walker-type metric which would be able to account for cosmological expansion.

References

[1] E. P. Verlinde, ‘On the Origin of Gravity and the Laws of Newton’, JHEP 04 (2011) 029, [arXiv:1001.0785].

[2] E. P. Verlinde, ‘Emergent Gravity and the Dark Universe’, SciPost Phys. 2 (2017) 016, [arXiv:1611.02269].
[3] D.-C. Dai and D. Stojkovic, ‘Inconsistencies in Verlindes emergent gravity’, JHEP 11 (2017) 007, [arXiv:1710.00946].

[4] S. Ryu and T. Takayanagi, ‘Holographic derivation of entanglement entropy from AdS/CFT’, Phys. Rev. Lett. 96 (2006) 181602, [hep-th/0603001].

[5] V. E. Hubeny, M. Rangamani, and T. Takayanagi, ‘A Covariant holographic entanglement entropy proposal’, JHEP 07 (2007) 062, [arXiv:0705.0016].

[6] M. Rangamani and T. Takayanagi, ‘Holographic Entanglement Entropy’, Lect. Notes Phys. 931 (2017) 1, [arXiv:1609.01287].

[7] G. Vidal, ‘Entanglement Renormalization’, Phys. Rev. Lett. 99 (2007), no. 22 220405, [cond-mat/0512165].

[8] B. Swingle, ‘Entanglement Renormalization and Holography’, Phys. Rev. D 86 (2012) 065007, [arXiv:0905.1317].

[9] X.-L. Qi, ‘Exact holographic mapping and emergent space-time geometry’, arXiv:1309.6282.

[10] C. H. Lee and X.-L. Qi, ‘Exact holographic mapping in free fermion systems’, Phys. Rev. B 93 (2016) 035112, [arXiv:1503.08592].

[11] S. Hossenfelder, ‘Covariant version of Verlindes emergent gravity’, Phys. Rev. D 95 (2017) 124018, [arXiv:1703.01415].

[12] S. Hossenfelder and T. Mistele, ‘The Redshift-Dependence of Radial Acceleration: Modified Gravity versus Particle Dark Matter’, arXiv:1803.08683.

[13] D.-C. Dai and D. Stojkovic, ‘Comment on “Covariant version of Verlindes emergent gravity”’, Phys. Rev. D96 (2017) 108501, [arXiv:1706.07854].

[14] R. H. Sanders, ‘Resolving the virial discrepancy in clusters of galaxies with modified newtonian dynamics’, Astrophys. J. 512 (1999) L23, [astro-ph/9807023].

[15] R. Massey, T. Kitching, and J. Richard, ‘The dark matter of gravitational lensing’, Rept. Prog. Phys. 73 (2010) 086901, [arXiv:1001.1739].

[16] M. Hobson, G. Efstathiou, and A. Lasenby, ‘General Relativity: An Introduction for Physicists’. Cambridge University Press, 2006.

[17] H. Maeda, M. Hassaine, and C. Martinez, ‘Lovelock black holes with a nonlinear Maxwell field’, Phys. Rev. D 79 (2009) 044012, [arXiv:0812.2038].
[18] H. A. Gonzalez, M. Hassaine, and C. Martinez, ‘Thermodynamics of charged black holes with a nonlinear electrodynamics source’, Phys. Rev. D 80 (2009) 104008, [arXiv:0909.1365].

[19] M. Cadoni, R. Casadio, A. Giusti, W. Mück, and M. Tuveri, ‘Effective Fluid Description of the Dark Universe’, Phys. Lett. B 776 (2018) 242, [arXiv:1707.09945].

[20] G. W. Gibbons and S. W. Hawking, ‘Action Integrals and Partition Functions in Quantum Gravity’, Phys. Rev. D 15 (1977) 2752.

[21] W. Rindler and M. Ishak, ‘Contribution of the cosmological constant to the relativistic bending of light revisited’, Phys. Rev. D 76 (2007) 043006, [arXiv:0709.2948].