A ubiquitous unifying degeneracy in two-body microlensing systems

Keming Zhang1✉, B. Scott Gaudi2 and Joshua S. Bloom1

While gravitational microlensing by planetary systems1,2 provides unique vistas on the properties of exoplanets3, observations of a given two-body microlensing event can often be interpreted with multiple distinct physical configurations. Such ambiguities are typically attributed to the close–wide4,5 and inner–outer6 types of degeneracy, which arise from transformation invariances and symmetries of microlensing caustics. However, there remain unexplained inconsistencies (see, for example, ref. 7) between the aforementioned theories and observations. Here, leveraging a fast machine learning inference framework8, we present the discovery of the offset degeneracy, which concerns a magnification-matching behaviour on the lens axis and is formulated independently of caustics. This offset degeneracy unifies the close–wide and inner–outer degeneracies, generalizes to resonant topologies and, upon reanalysis, not only appears ubiquitously in previously published planetary events with twofold degenerate solutions, but also resolves prior inconsistencies. Our analysis demonstrates that degenerate caustics do not strictly result in degenerate magnifications and that the commonly invoked close–wide degeneracy essentially never arises in actual events. Moreover, it is shown that parameters in offset-degenerate configurations are related by a simple expression. This suggests the existence of a deeper symmetry in the equations governing two-body lenses than previously recognized.

In search of new types of microlensing degeneracy, we analysed the posterior parameter distribution of a large number of simulated two-body microlensing events that exhibited multimodal solutions. With over 100 planetary microlensing events observed so far, new degeneracies have indeed been serendipitously found in routine data analysis (see, for example, ref. 7). However, while an exhaustive search on examples of multimodal event posteriors to constrain the existence of unknown degeneracies is plausible, such an endeavour has been computationally prohibitive with the current status quo microlensing data analysis approaches. Thankfully, the recent application of likelihood-free inference (LFI) (see ref. 10 for an overview) to two-body microlensing7 has accelerated calculation of microlensing posteriors to a matter of seconds, thus allowing posteriors for any trajectory shown: the magnification differs everywhere in most cases of apparent close–wide degeneracies do not exactly match with the NDE alone in mere seconds. Following ref. 8, we trained a NDE on 691,257 events simulated in the context of the Roman Space Telescope microlensing survey11 so that our results would be directly relevant. The posteriors for a large number of randomly generated events are then produced with the NDE. To identify events with multimodal solutions, we applied a clustering algorithm14, which separates each posterior into discrete modes. The exact maximum likelihood solution within each posterior mode is then calculated with an optimization algorithm (Methods).

Visual inspection of multimodal NDE posteriors revealed three apparent regimes of degeneracy: the inner–outer degeneracy, the close–wide degeneracy and degeneracies that involve the resonant caustic, which have also been previously observed (see, for example, refs. 8, 13) and studied15. The close–wide degeneracy states that the central caustic shape is invariant under the \(s \rightarrow 1/s\) transformation for \(|1-s| \ll q^{1/3}\) (ref. 13) and \(q \ll 1\) (Extended Data Fig. 1a,c), where \(q\) refers to the planet-to-star mass ratio and \(s\) refers to their projected separation normalized to the angular Einstein radius \((\theta_E = \sqrt{GM/e^2})\), which is the characteristic microlensing angular scale. Here, \(e = 4G/(c^2a_0)\), \(M\) is the total lens mass and \(a_0 = au/D_A\) is the lens–source relative parallax. Interestingly, we found that most cases of apparent close–wide degeneracies do not exactly abide by the expected \(s \rightarrow 1/s\) relation even though most are in the \(|1-s| \gg q^{1/3}\) regime, where it is expected to hold. We also noticed that for degenerate events involving one resonant caustic, the source trajectory always passed to the front end of the resonant caustic for wide–resonant degenerate events, and the back end for close–resonant degenerate events.

To explore potential connections among these apparently discrete regimes of degeneracies, and to better understand the reason why the expected \(s \rightarrow 1/s\) relation of the close–wide degeneracy is almost never satisfied, we examined maps of magnification differences between pairs of lenses with the same mass ratio \((g = 2 \times 10^{-4})\), keeping lens B fixed at \(s_A = 1/1.1\) and changing the projected separation \(s_B\) of lens A. The sequence of magnification difference maps in Fig. 1a–h immediately reveals the continuous evolution of a vertically extended ring structure where the magnification difference vanishes (see also Extended Data Figs. 2 and 3). This null ring originates near the primary star and grows increasingly large with increasing deviation from the close–wide degenerate configuration of \(s_A = 1/\sqrt{s_B}\) at which point the null contracts to a singular point (see Extended Fig. 4 for a zoom-in). We may thus expect null-passing trajectories (cyan arrows in Fig. 1a–h) to have degenerate magnifications, which is confirmed by light curves shown in Fig. 11–p.

It is also immediately clear from Fig. 11 why the close–wide pair of configurations \((s_A = 1/s_B)\) does not result in degenerate magnifications for any trajectory shown: the magnification differs everywhere on the lens axis except for the singular null point. Thus for any given trajectory, close to or far from the central caustic, one can always move the null to the location of the source by shifting the planet.
The numerically determined $x_{null}$ (Fig. 2) shows that deviations from this analytic prescription are consistently less than 5% except for extreme separation ($|\log_{10}(s)| \gtrsim 0.5$) cases where sources do not pass close to either caustic and therefore do not yield substantial planetary perturbation of practical interest. This expression can be interpreted as the midpoint between the locations $x_{c} = s_{A,B} - 1/s_{A,B}$ of the planetary caustics, which arises from the perturbative picture of planetary microlensing. However, the fact that such an expression holds well into the resonant regime, for which there are no planetary caustics at all, and persists through caustic topology changes, suggests the existence of much deeper symmetries in the gravitational lens equation for mass ratios of $q \ll 1$ than had previously been appreciated, and should be explored in future work.

We now consider the relationship between the offset degeneracy and the two previously known mathematical degeneracies. First, the offset degeneracy is a magnification degeneracy while the two previous degeneracies are caustic degeneracies. Our analysis demonstrates that degenerate caustics do not strictly result in degenerate magnifications. Furthermore, by setting $x_{null} = 0$ in equation (1), we immediately recover the $s_{A} = 1/s_{B}$ relation of the close–wide degeneracy. This suggests that the close–wide degeneracy is more suitably viewed as a transition point of the offset degeneracy where the central caustics happen to be degenerate. On the other hand, while the inner–outer degeneracy implies an expression similar to equation (1), it arises from the symmetry of the Chang–Refsdal approximation to the planetary caustics. However, cases attributed to the inner–outer degeneracy are often not in the pure Chang–Refsdal regime, in which case the planetary caustics are asymmetrical. Also, even in the Chang–Refsdal regime, in observed events the source trajectory is fixed and passes equidistant from two different planetary caustics, rather than two sides of the same caustic. Therefore, the offset degeneracy not only resolves inconsistencies and unifies the two previously known degeneracies into a generalized regime, but also relaxes the $|1 - s| \gg q^{1/3}$ condition required by both cases.
planetary caustics: $(q) \leq 1$ and $s_q + q/(1+q)$ are found outside the box $s_q > 1$—are found outside the box $s_q = 1/s_B$ (gold star). 0 corresponds to $s_q = 1$ and 1 corresponds to the asymptotic inner–outer degenerate case where $s_q = s_B$ (brown hexagon). The coordinate origin is set to $s_q/(1 + q)$ from the primary for $s < 1$ and $s^2 q/(1 + q)$ for $s > 1$, which describe the location of the central caustic and account for the non-differentiability at $s_q = 1$.

Because of this unifying feature, we expected the offset degeneracy to be ubiquitous in past events with twofold degenerate solutions and speculate that a large number of cases may have been mistakenly attributed to the close–wide degeneracy. Therefore, we systematically searched for previously published events with twofold degenerate solutions satisfying $q_A \approx q_B \ll 1$ (see Methods). We found 23 such events, and then first compared the intercept of the source trajectory on the star–planet axis to the location of the null predicted with equation (1). We also invert equation (1) to predict one degenerate $s_A$ from the other $s_B$:

$$s_A = \frac{1}{2} \left( 2x_0 - (s_B - 1/s_B) + \sqrt{(2x_0 - (s_B - 1/s_B))^2 + 4} \right), \quad (2)$$

where $x_0 = u_0/\sin(\alpha)$ is the intercept of the source trajectory on the binary axis, $u_0$ is the impact parameter and $\alpha$ is the angle of the source trajectory with respect to the binary axis. As shown in Fig. 3, the source trajectory always passes through the null location on the star–planet axis as predicted by equation (1). Additionally, equation (2) accurately predicts one degenerate solution from the other. The fact that equation (1) applies for a wide range of $\alpha$ confirms that the offset degeneracy accommodates oblique trajectories, although proximity to planetary caustics might break the degeneracy (for example, KMT-2016-BLG-13977). Thus we conclude that equations (1) and (2) will be useful in the analysis of future events with offset-degenerate solutions.

Given its apparent ubiquity, it is reasonable to ask why the offset degeneracy has only been discovered over two decades after the first in-depth explorations of degeneracies in two-body microlens-

![Fig. 2](image_url) Deviation ($\Delta x_{\text{null}}$) of numerically derived, exact null position from the analytic form (equation (1)) for changing $s_A$ against three values of fixed $s_B < 1$, normalized to the separation between the two (implied) planetary caustics: $(s_A - 1/s_A) - (s_B - 1/s_B)$; $\Delta x_{\text{null}}$ is calculated for $q = 2 \times 10^{-4}$ but was found to be independent of $q$ for $q \ll 1$ (Extended Data Fig. 5). The x axis shows $\log_{10}(s_A)$ scaled to $\log_{10}(s_B)$ such that $-1$ corresponds to the close–wide degenerate case of $s_A = 1/s_B$ (gold star). 0 corresponds to $s_A = 1$ and 1 corresponds to the asymptotic inner–outer degenerate case where $s_A = s_B$ (brown hexagon). The coordinate origin is set to $s_A/(1 + q)$ from the primary for $s < 1$ and $s^2 q/(1 + q)$ for $s > 1$, which describe the location of the central caustic and account for the non-differentiability at $s_A = 1$.

![Fig. 3](image_url) Offset degeneracy reanalysis of 23 systematically selected events in the literature with twofold degenerate solutions. a. The source trajectory always passes close to the null intercept on the star–planet axis ($x_{\text{null}}$), as predicted by equation (1). The x axis shows the source trajectory intercept on the star–planet axis, calculated from $x_0$ and $\alpha$. The y axis shows the prediction for $x_{\text{null}}$ using equation (1) and reported values of $s_A$ and $s_B$. Event labels as shown in the legend are the event abbreviations: for example, KMT162397 means KMT-2016-BLG-2397. The inset shows a zoom-in of the central boxed region. b. The x and y axes show the smaller and larger values of the degenerate solutions, referred to as $s_{\text{true}}$. Circles are reported values of $s_{\text{true}}$ whereas triangles are $s_{\text{true}}$ values predicted with equation (2) of the offset degeneracy and $s_{\text{true}}$, $\alpha$ and $u_0$. The colour coding follows the legend in a. Circles and triangles largely coincide for all cases, demonstrating the predictive power of the offset degeneracy. Sizes of circles and triangles are scaled to the expected null location, $x_0 = u_0/\sin(\alpha)$, to show the correlation between larger size and greater distance from the dash–dotted diagonal line, which represents the exact close–wide degeneracy where $s_{\text{true}} = 1/s_{\text{true}}$. Cases typically understood as inner–outer—$s_{\text{true}} > 1$ or $s_{\text{true}} < 1$—are found outside the box bounded by the dashed lines. Cases close to the dashed lines but far from their conjunction correspond to resonant-close/wide degeneracies. Cases within the dashed box and not on the diagonal line do not belong to either close–wide or inner–outer degeneracies. The inset shows a zoom-in of the region boxed by solid lines. Error bars are marginalized 1$\sigma$ posterior intervals. Uncertainties for the predicted $x_{\text{null}}$ are propagated from the uncertainties of whichever of $s_{\text{true}}$ and $s_{\text{true}}$ gives rise to a smaller uncertainty on $x_{\text{null}}$. 

784

NATURE ASTRONOMY | VOL 6 | JULY 2022 | 782–787 | www.nature.com/natureastronomy
ing events were made in ref. 4. One reason may be the early strategic focus on high-magnification ($\mu \leq 1$) events in ref. 19, where deviations from $s \approx 1/5$ were small, and the cause was not explored in detail. Recently, deviations from $s \approx 1/5$ in semiresonant topology events have led to explicit discussions on the applicability of the close–wide degeneracy in the resonant regime and potential connections to the inner–outer degeneracy$^{11,14}$. Nevertheless, as we have shown, the resonant condition itself does not cause the deviation from $s \approx 1/5$, but only allows it to be noticeable (Methods). To our advantage, the novel technique of ref. 8 based on machine learning presented us with a large number of degenerate events in the non-resonant $[1 \to s] \gg q_{1,2}^{1/3}$ regime that deviated from the $s \approx 1/5$ expectation, but also did not conform to the inner–outer degeneracy. These ‘intermediate’ offset-degenerate events ultimately allowed us to recognize the continuous and unifying nature of the offset degeneracy, showcasing another instance of new theoretical insight guided by machine learning (cf. ref. 7). As the next-generation surveys further expand the sensitivity limit from space$^{28}$, the offset degeneracy will increasingly manifest.

**Methods**

The Z21 fast inference technique. Zhang et al.$^8$ (Z21 hereafter) presented an LFI approach to binary microlensing analysis that allowed an approximate posterior for a given event to be computed in seconds on a consumer-grade GPU, compared with the hours–to-days timescale on CPU clusters that are typically required for status quo approaches. We summarize the Z21 approach at the high level here, and defer to the paper for details.

The Z21 method is likelihood free in that it does not iteratively perform simulations to compute the likelihood, which is typical for sampling-based inference methods. Instead, Z21 directly learns the posterior probability as a conditional distribution $p(\theta|\mathbf{xi})$ with an NDE, where $\theta$ are the NDE parameters, $\theta$ is the binary microlensing (2L1S) parameters and $\mathbf{xi}$ the input light curve. The NDE is essentially a mapping that takes a light curve as input and produces a specific number of discrete posterior samples. Such a mapping is trained on a large number of simulations $(x, \theta)$ with parameters drawn from a wide prior, and the $\theta$ are optimized to maximize the expectation of this conditional probability under the training set data distribution. The mapping learned can thus be applied to any given event unseen during training as long as it is within the prespecified prior.

This specific approach to LFI is called amortized neural posterior estimation, where ‘amortized’ refers to the process of paying all the simulation cost upfront so that inferences of future events do not require additional simulations. After training, the NDE alone generates posterior samples for any future event at a rate of $\approx 10^5$ s$^{-1}$ on a consumer-grade GPU, or $\approx 10^6$ s$^{-1}$ on an eight-core CPU, effectively carrying out an inference in real time. Z21 demonstrated that, although not exact, the neural posterior places accurate constraints on all parameters nearly 100% of the time, except for the parameter that quantifies the effect of a finite-sized source. This is because substantial finite-source effects only occur when the source approaches sufficiently close to the caustics, which is satisfied by only a small subset of events.

With a focus on the next-generation, space-based microlensing survey planned for the Roman Space Telescope$^{15}$, here we developed a training set in a similar fashion as the Z21 training set, but with a caustic-centred coordinate system rather than a centre-of-mass (COM) coordinate system. This is because the COM coordinate system is high inefficiency for producing planetary-caustic passing events with randomly drawn source trajectories with respect to the COM. In addition, for wide binary ($q > 1/3$) events, the time to closest approach $(t_{\text{CA}})$ to the COM could have an arbitrarily large offset from the time of peak magnification, which can lead to the missing of solution modes (Section 4.3 of Z21). The caustic-centred coordinate system, on the other hand, efficiently spans the entire 2L1S parameter space that allows for substantial deviation from a single central light curve.

We generated a total of 228,892 events centred on the planetary caustic and 960,000 events centred on the central caustic, and further removed those that are consistent with a single-lens model by fitting each light curve to such a model and adopting a $\Delta \gamma = 140$ cutoff (Z21). This resulted in a training set of 691,257 simulations, including 137,644 planetary-caustic events and 553,863 central-caustic events.

For planetary-caustic events, $u_0$ is randomly sampled from 0 to 50 times the caustic size. For central-caustic events, $u_0$ is randomly sampled from 0 to 2. Compared with Z21, we expanded the source flux fraction, defined as $f_s = \frac{\text{saturate}(\mathbf{xi})}{\log\text{flux}(\mathbf{xi})}$, to probe more deeply into the severely blended regime. Other aspects of event simulation are the same as in Z21 and the reader is referred to Section 3 of Z21 for details.

Identifying degeneracies in Z21 posteriors. Z21 provided three example events with degenerate posteriors where light curve realizations from each degenerate mode are almost indistinguishable from one another, a confirmation of the effectiveness in modelling light curves with degenerate solutions. While the posterior modes in Z21 were identified manually, in this work we automate the degeneracy-finding process.

To work with posterior distributions that vary in scale, position and shape, we first fitted and applied a parametric, monotonic ‘power’ transformation$^{16}$ to the LFI-generated posterior samples for each simulated light curve. This transformation normalizes each marginal parameter distribution to an approximately Gaussian. To automatically identify degenerate events, we used the HDSCAN algorithm$^7$ to perform clustering on the transformed posterior samples. The HDSCAN algorithm is a density-based, hierarchical clustering method, which required, for our task, minimal hyperparameter tuning. The output of HDSCAN is a suggested cluster label for each posterior sample, including the labelling for outlier/noise samples. Events with more than one cluster are identified as degenerate events.

Although the NDE posteriors are accurate enough for a qualitative study of degeneracies, we nevertheless refined each solution mode to the maximum likelihood value. The approximate posterior allows us to make use of bounded optimization algorithms to quickly locate the exact solution. We use a parallel implementation$^8$ of the L-BFGS-B optimization algorithm$^8$ to quickly solve the best-fit solutions. The entire process from light curve to degenerate exact solutions takes a few minutes for each event, with the last refinement step costing the most time.

**Comparison with events in the literature.** We demonstrate the ubiquity of the offset degeneracy by performing a thorough investigation of 2L1S events in the literature with reported degenerate posteriors. We first filter through events on the NASA microlensing exoplanet archive, which contains 112 planets and 306 entries with reported 2L1S parameters (retrieved 23 August 2021). Each entry reports one solution for a given event.

Events from adaptive-optics follow-up papers of published events, as well as duplicate entries with identical 2L1S solutions, are first removed. Triple-lens events with detections of two planets—OGLE-2006-BLG-109 and OGLE-2018-BLG-1011—are also removed. Planets with reported higher-order effects (parallax, xallarap) are also removed, as such effects often exhibit additional degeneracies and may complicate the application of the offset degeneracy. We further remove two-fold degenerate events with $\Delta \gamma > 10$ where one solution is significantly favoured. This leaves us with 20 planets with exactly two solutions and 12 with more than two solutions.

Among the 20 planets with exactly two solutions$^{29,30}$, six are excluded: KMT-2016-BLG-1107$^+$ because it is a different type of degeneracy (two distinct source trajectories crossing the $s < 1$ planetary caustic, one of which is parallel to and does not intersect with the binary axis), OGLE-2017-BLG-0373$^-$ because it is an accidental degeneracy without complete temporal coverage of the caustic entrance/exit and KMT-2019-BLG-0371$^+$ because of the large mass ratio ($q = 0.1$) and the fact that the offset degeneracy only strictly manifests when $q < 1$. We also exclude OGLE-2016-BLG-1220 and OGLE-2017-BLG-0337 because both cases $s_{\text{peak}}$ $\approx 4$ makes it difficult to include in Fig. 3 scale-wise, and because both cases are deep in the $[1 \to s] \gg q_{1,2}^{1/3}$ limit, and are thus already well characterized by the inner–outer degeneracy. Similarly, MOA-2007-BLG-400$^-$ is also deep in the $[1 \to s] \gg q_{1,2}^{1/3}$ limit and represents one of the few instances where the source passes almost exactly the location of the primary star, thus allowing a degenerate pair of central caustics to manifest. However, the large uncertainty of $s_{\text{CA}}$ $\approx 2.0$ translates into an uncertainty in $s_{\text{peak}}$ that is orders of magnitude larger than the size of the central caustic, and it makes it uninformative to include here.

We also inspected events with more than two degenerate solutions, and found that the solutions of KMT-2019-BLG-1339$^+$ and MOA-2015-BLG-337$^-$ consist of two pairs of degeneracies, each with their distinct shape and magnification ratios. For both events, we include the pairs of solutions with planetary mass ratios ($q < 1$).

Beyond the total of 16 degenerate events retrieved from the NASA microlensing exoplanet archive and discussed above, we further looked for relevant events in the literature that are not included in the NASA exoplanet archive. Additions include the pairs of solutions with planetary mass ratios for OGLE-2011-BLG-0526$^-$ and OGLE-2011-BLG-0950$^-$ as well as five events with degenerate solutions recently reported in ref. 1. We also include OGLE-2019-BLG-0966$^\star$. This results in a final sample of 23 degenerate events.

**Range of applicability of the offset degeneracy.** When considering larger $q$, we found that the qualitative structure of the null persistence through $\gamma \to 1$ (Extended Data Figs. 3 and 5), suggesting that some form of the offset degeneracy may manifest even for $q \geq 0.1$. In this regime, there should also be a transition point similar to the close–wide degeneracy that results in $\gamma_{\text{peak}} \to 0$, but $q_{\text{peak}}$ may not hold, nor $s_{\text{peak}}$ $\approx 1/3$. For example, in the quadrupole and pure shear approximation, the analogy to the close–wide degeneracy requires $Q = \rho_q$, where $Q = (f_{\text{peak}})$ and $\rho_q = (1 + q_1 + q_2)^{1/3}$ is the quadrupole moment of the close central caustic, and $\theta = (1/s_\text{CA})f_{\text{peak}}/(1 + q_1) = \text{shear}$ of the wide central caustic$^7$. Furthermore, it is not clear if the values of $q_{\text{peak}}$ at the $s_{\text{peak}} \approx 0$ close–wide-equivalent transition point remain constant when one of $s_{\text{peak}}$ and $q_{\text{peak}}$ undergoes offset. A notable example in the literature is KMT-2019-BLG-0371$^+$, where the source trajectory passes through...
the null created by the two degenerate solutions but $q_G = 0.123$ and $q_M = 0.079$ are substantially different. The exact behaviour of the offset degeneracy for $q < 1$ should be studied in future work.

We also note that offset-degenerate, caustic-crossing events usually require nearly vertical trajectories because of the additional constraint on the caustic-crossing length. However, oblique trajectories are allowed if the change in caustic width near $s_{\text{wide}}$ is small for both solutions (for example, OGLE-2019-BLG-0960).

Relevant previous work. Inconsistencies of the close–wide and inner–outer degeneracies with degeneracies in observed events have recently been pointed out in the literature. In the analysis of the seminonsen topology event OGLE-2019-BLG-0960, the authors of ref. 1 noticed that, while the close–wide degeneracy is expected to break down as $s \to 1$, there are large numbers of resonant and seminonsen topology events involving the close–wide degeneracy where one solution has $s_{\text{close}} \approx 1$ and the other $s_{\text{wide}} < 1$, but they do not satisfy $s_{\text{close}} = 1/s_{\text{wide}}$. They further noted the conceptual similarity to the inner–outer degeneracy for these events, but again noted that this type of degeneracy too is expected to break down in the resonant regime. On the basis of these observations, they speculated that the two degeneracies merge as $s \to 1$.

While ref. 1 pointed out inconsistencies for resonant events ($|1-s| \lesssim q^{1/2}$), here we found that inconsistencies with $s_{\text{wide}} = 1/s_{\text{close}}$ persist even within the $|1-s| \gg q^{1/2}$ regime, in which the two degeneracies are derived and the caustics are well separated. We claim that this inconsistency is fundamentally because caustic degeneracies are only approximately correct in describing magnification degeneracies, irrespective of caustic topology. While small deviations from $s_{\text{close}} = 1/s_{\text{wide}}$ in early-high-magnification events tend to go unnoticed, resonant events do allow the asymmetry from $\log(s_{\text{close}}) = 0$ to be immediately noticeable. For OGLE-2019-BLG-0960, log$_{10}(s_{\text{close}}) = -0.011\log(s_{\text{wide}}) = -0.01$ differs from log$_{10}(s_{\text{close}}) = 0.01\log(s_{\text{wide}}) \approx 0.01$ by an order of magnitude.

The theoretical follow-up work of ref. 1 studied the behaviour of the close–wide degenerate in the resonant regime. They first clarified that, rather than $|\log(s_{\text{close}})| \gg 0$, the exact condition of the close–wide degeneracy is $|1-s| \gg q^{1/2}$, which is dependent on the mass ratio. Furthermore, even for $|1-s| \lesssim q^{1/2}$, the central caustic could still be locally invariant under $s \to 1/s$, for parts of the caustic satisfying $|1-s| \gg q^{1/2}$, where $g$ is a parametric variable that describes the position along the caustic. We note that this fact has also been observed in the earlier work of ref. 1. They concluded by suggesting that slight changes to $s_{\text{close}}$ and $q_M$ may create a local pair of degenerate models, which in some sense anticipated our discovery.

Data availability
Source data for figures 2 and 3 have been made available online. Figure 3 data are also partially available in the NASA microlensing exoplanet archive, https://exoplanetarchive.ipac.caltech.edu.

Code availability
This work utilized the public microlensing code, MulensModel*, available at https://github.com/rpoleski/MulensModel.

Received: 27 November 2021; Accepted: 30 March 2022; Published online: 23 May 2022

References
1. Mao, S. & Paczyński, B. Gravitational microlensing by double stars and planetary systems. Astrophys. J. Lett. 374, L37 (1991).
2. Gould, A. & Loeb, A. Discovering planetary systems through gravitational microlenses. Astrophys. J. 396, 104–114 (1992).
3. Gaudi, B. S. Microlensing surveys for exoplanets. Annu. Rev. Astron. Astrophys. 50, 411–453 (2012).
4. Grisetti, K. & Safirzhadeh, D. The use of high-magnification microlensing events in discovering extrasolar planets. Astrophys. J. 500, 37 (1998).
5. Dominik, M. The binary gravitational lens and its extreme cases. Astron. Astrophys. 349, 108–125 (1999).
6. Han, C. et al. MOA-2016-BLG-319Lb: microlensing planet subject to rare minor-image perturbation degeneracy in determining planet parameters. Astron. J. 156, 226 (2018).
7. Yee, J. C. et al. OGLE-2019-BLG-0960 Lb: the smallest microlensing planet. Astron. J. 162, 180 (2021).
8. Zhang, K. et al. Real-time likelihood-free inference of Roman binary microlensing events with amortized neural posterior estimation. Astron. J. 161, 262 (2021).
46. Bozza, V. Perturbative analysis in planetary gravitational lensing. *Astron. Astrophys.* 348, 311–326 (1999).
47. Poleski, R. & Yee, J. C. Modeling microlensing events with MulensModel. *Astron. Comput.* 26, 35–49 (2019).

**Acknowledgements**

K.Z. thanks the LSSTC Data Science Fellowship Program, which is funded by LSSTC, NSF Cybertraining grant 1829740, the Brinson Foundation and the Moore Foundation; his participation in the programme has benefited this work. K.Z. and J.S.B are supported by a Gordon and Betty Moore Foundation Data-Driven Discovery grant. Work by B.S.G. is supported by NASA grant NNG16PJ32C and the Thomas Jefferson Chair for Discovery and Space Exploration. We thank E. Agol and J. Lu for helpful comments on a draft of this Letter.

**Author contributions**

K.Z. and J.S.B. conceived of the degeneracy-finding search. K.Z. implemented the search and identified the offset degeneracy. J.S.B. designed and wrote the code for the cluster-finding approach. B.S.G. aided in the study and interpretation of the LFI-derived posteriors of microlensing events and helped to develop the interpretation of the offset degeneracy and place it in the context of results from the literature. K.Z., B.S.G. and J.S.B. co-wrote the manuscript.

**Competing interests**

The authors declare no competing interests.

**Additional information**

Extended data is available for this paper at https://doi.org/10.1038/s41550-022-01671-6.

Supplementary information The online version contains supplementary material available at https://doi.org/10.1038/s41550-022-01671-6.

Correspondence and requests for materials should be addressed to Keming Zhang.

**Peer review information** Nature Astronomy thanks Przemek Mróz and the other, anonymous, reviewer(s) for their contribution to the peer review of this work.

**Reprints and permissions information** is available at www.nature.com/reprints.

**Publisher’s note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

© The Author(s), under exclusive licence to Springer Nature Limited 2022
Extended Data Fig. 1 | Caustics shown in green atop of maps of magnification differences from a 1-body lens, for wide (top), resonant (middle), and close (bottom) caustic topologies. Red dots indicate locations of the planet, with separations $s = 1/0.8, 1, 0.8$ from the host star, located at the origin. The mass-ratio is fixed at $q = 2 \times 10^{-3}$. Blue dashed lines represent the Einstein ring $\theta_E$, the angular size to which the projected separation ($s$) is normalised. Caustic topologies are delineated by values of $s$ for a given $q$. In the wide regime ($s \gtrsim 1 + (3/2)q^{1/3}$), there is one central caustic located near the host star and one asteroid-shaped ‘planetary’ caustic towards the location of the planet. In the close regime ($s \lesssim 1 - (3/4)q^{1/3}$), there are two small, triangular shaped ‘planetary’ caustics in addition to the central caustic that appears similar to the wide central caustic, due to the close-wide degeneracy. For values of $s$ in between these regimes, there is one six-cusped ‘resonant’ caustic. For all cases, there are lobes of excess magnification compared to a point lens near caustic cusps, and lobes of de-magnification towards the back-end of the central/resonant caustic.
Extended Data Fig. 2 | The manifestation of the offset degeneracy in source-plane magnification difference maps (top) and light curves (bottom) for fixed $s_A = 1.18 > 1$. This completes the resonant-close (b) and wide-topology inner-outer (d) cases.
Extended Data Fig. 3 | The manifestation of the offset degeneracy in source-plane magnification difference maps (top) and light curves (bottom) for fixed $s_A = 1$. (i)–(p) shows logarithmic deviations from PSPL on arbitrary scales, where green dashed curves are the changing lens A and sold blue curves are for fixed lens B. (a)–(d) and (e)–(h) show the same sequence of $s_A$ but for $q = 10^{-3}$ and $q = 10^{-2}$ to illustrate how the offset degeneracy generalises to larger mass-ratios. (a,e) reveals that the ring structure of the null is composed of two distinct null segments, where one appears to originate from the centre of the central/resonant caustic and the other from the left two cusps of the same caustic. Closer inspection shows that the null rings for (a) and (e) have different topologies: for (a) it is the left part of the null that intersects on the star–planet axis but for (e) it is the right part. This disjoint topology of the null is also seen in Fig. 1 and Extended Data Figure 4&5. The topology transition point, presumably a function of $s$ and $q$, may have mathematical implications for the offset degeneracy. Furthermore, we observe that the null segment near the star–planet axis becomes increasingly curved for $|\log(s)| \gg 0$ and $q \to 1$, which may explain how Equation 1 and the offset degeneracy in general, may break down in those limits.
Extended Data Fig. 4 | Magnification difference maps zoomed-in on the central caustic. Same $s_A = 1/1.1$ as Fig. 1. Cyan arrows indicate the location of the null. For (b) and (c), the null always crosses the two caustics at their intersection.
Extended Data Fig. 5 | Magnification difference maps which demonstrates the offset degeneracy independence on $q$ for $q \ll 1$. Lens B shares the same fixed $s_B = 1.1$ as in Fig. 1. Each row shows cases of $s_A = 0.95, 1.0, 1.16$ for $q = 10^{-2}, 10^{-4}, 10^{-6}$. The null location predicted from Equation 1 is shown in cyan crosses. For $q = 10^{-4}$ and $q = 10^{-6}$, the null shape largely remains constant where the null intersection on the star–planet axis is well predicted by the analytic prescription (Equation 1). The three cases of $q = 10^{-2}$ demonstrate how the behaviour of the null changes as $q \to 1$. In the case of $s_A = 1.16$, the null is split into two disconnected segments inside and outside of the caustic, where the analytic prediction is close to their mean location. For $s_A = 0.95$, the discrepancy from the analytic prediction may be attributed to the curvature of the null near the star–planet axis.