Magnetic field contribution to the last electron–photon scattering

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Abstract

When the cosmic microwave photons scatter electrons just prior to the decoupling of matter and radiation, magnetic fields do contribute to the Stokes matrix as well as to the scalar, vector and tensor components of the transport equations for the brightness perturbations. The magnetized electron–photon scattering is hereby discussed in general terms by including, for the first time, the contribution of magnetic fields with arbitrary direction and in the presence of the scalar, vector and tensor modes of the geometry. The propagation of relic vectors and relic gravitons is discussed for a varying magnetic field orientation and for different photon directions. The source terms of the transport equations in the presence of the relativistic fluctuations of the geometry are also explicitly averaged over the magnetic field orientations and the problem of a consistent account of the small-scale and large-scale magnetic field is briefly outlined.

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1. Formulation of the problem

The last electron–photon scattering is customarily discussed without the additional complication of a magnetic field. In numerical codes as well as in analytical estimates, the collisional contributions are evaluated as if electrons and ions were free right before last scattering [1]. The relativistic fluctuations of the geometry are included in the classic transport problem [1] either by using specific gauges [2] or with fully gauge-invariant methods. The resulting equations including both the source terms coming from electron–photon scattering and the relativistic fluctuations of the geometry form the set of transport equations which can be solved within various approaches either by truncating the system at a specific (maximal) multipole [2] or by using the integration along the line of sight [3]. One of the consequences of the consistent solution of the system of transport equations are estimates of the temperature...
and polarization inhomogeneities of the cosmic microwave background (CMB). The recent WMAP 7 data [5–10] are able to constrain the vanilla $\Lambda$CDM scenario (where $\Lambda$ stands for the dark-energy component and CDM for the cold dark matter component). In the near future the $\Lambda$CDM scenario will be tested not only in its minimal version but also in its non-minimal extensions ranging from the addition of a stochastic background of relic tensor modes of the geometry to large-scale magnetic fields [11, 12].

In recent years there has been mounting evidence of the role played by magnetic fields at large scales [11, 12]. Why should magnetic fields be assumed in various processes ranging from star formation to cluster dynamics and completely neglected prior to last scattering? Why are magnetic fields overlooked in CMB physics while they are observed in galaxies clusters, superclusters and high-redshift quasars? There are no reasons for doing so unless one would implicitly assume that large-scale magnetism suddenly arose between hydrogen recombination and, say, the gravitational collapse of the protogalaxy. While it might well be that the latter situation is the one preferred by nature, it would be nice to have some direct empirical evidence less biased by speculations. To comply with the latter program, a specific approach has been tailored through the last few years [12] (see also [13]). The idea is, in a nutshell, to introduce consistently large-scale magnetic fields in all the steps leading to the estimate of CMB anisotropies and polarization. So far the program undertaken in [12] led to various results:

- the large-scale magnetic fields have been included both at the level of the initial conditions as well as the level of the evolution equations for the standard adiabatic mode and for the other entropic initial conditions [13];
- the temperature and polarization anisotropies induced by the magnetized (adiabatic and entropic) initial conditions have been computed [14];
- the parameters of the magnetized background have been estimated (for the first time) in [15] by using the TT and TE correlations measured by the WMAP collaboration.

There exist other approaches to the interplay between large-scale magnetic fields and CMB anisotropies (see [17–22] for an incomplete list of references; see [12] for a more thorough account of earlier results). The common characteristic of those approaches has been to neglect the scalar modes of the geometry and to focus the attention to the tensor and vector modes. The recent results [13–16] show, in contrast with previous guesses, that large-scale magnetic fields alter the initial conditions and the dynamics of the scalar modes of the geometry. They consequently distort, in a computable manner, the temperature and polarization anisotropies. A limitation common to nearly all studies on pre-decoupling magnetism has been so far the total absence of the effect of the magnetic field in the process of electron–photon scattering. In [13–15], for instance, the magnetic fields are included in the initial conditions and in all the relevant governing equations. The electron–photon scattering, however, is assumed to take place as if the magnetic fields were absent. The potential smallness of the effects does not justify its neglect since diverse small effects are often claimed to be detectable because of the purported control we now have on CMB foregrounds [23].

The consistent inclusion of magnetic fields in electron–photon scattering modifies qualitatively the standard lore since the geodesics of electrons and ions are modified by the presence of the Lorentz force term in curved backgrounds. Absent the contribution of the magnetic field, the motion of electrons and ions depends only upon the incident electric field, but when the magnetic field is included the classic treatment (see, for instance [1]) must be adapted to the new situation.

1 Following the standard shorthand terminology the TT correlations denote the temperature autocorrelations while the TE correlations denote the cross-correlation between the temperature and the E-mode polarization.
The neglect of the role of the magnetic field in the electron–photon scattering has been recently relaxed in the guiding center approximation [24–27], and for a specified magnetic field orientation. The argument for keeping the direction fixed was essentially practical and in this paper a general treatment will be developed along a twofold perspective:

- the orientation of the magnetic field will be kept arbitrary so that the matrix elements either in the Jones or in the Mueller calculus will depend not only upon the directions of the incident and of the outgoing radiation but also on the magnetic field orientation;
- after including the magnetic field in the scattering process the source terms for the transport equations of the scalar, vector and tensor modes of the geometry will be deduced explicitly.

The latter analysis is still lacking both in present and earlier literature. The magnetized electron–photon scattering is often required in diverse astrophysical situations like in the physics of magnetized sun spots [30], or in the theory of synchrotron emission [31, 32] whose results cannot be directly used since at last scattering electrons and ions are notoriously non-relativistic. Conversely some studies involving directly the Thomson scattering in a magnetized environment [33] do not incorporate the fluctuations of the geometry and are also obtained using a preferential magnetic field orientation.

The layout of this paper is therefore the following. In section 2 the tenets of the Mueller and Jones calculus will be reviewed and the matrix elements for the magnetized electron–photon scattering presented. Section 3 introduces the scalar, vector and tensor components of the brightness perturbations and the calculation of the collisionless part of the transport equations. The full scalar, vector and tensor transport equations will be discussed, respectively, in sections 4, 5 and 6. Section 7 contains the concluding remarks. Explicit expressions involving all the relevant matrix elements both in the Jones and in the Mueller approaches have been collected in appendices A and B.

2. Mueller and Jones calculus

In the Mueller calculus the Stokes parameters are organized in a four-dimensional (Mueller) column vector whose components are exactly the four Stokes parameters, i.e. $I, Q, U$ and $V$. In the Jones calculus the electric fields of the wave are organized in a two-dimensional column vector and the Stokes parameters are effectively derived quantities (see [29] for an introduction to the Mueller and Jones approaches). Hereunder a hybrid approach shall be employed. The polarization tensor $P_{ij} = P_{ji} = E_i E_j^*$ can be organized in a Stokes matrix whose explicit form is

$$
P = \frac{1}{2} \begin{pmatrix} I + Q & U - iV \\ U + iV & I - Q \end{pmatrix} = \frac{1}{2} (I + U \sigma_1 + V \sigma_2 + Q \sigma_3),$$

(2.1)

where $I$ denotes the identity matrix while $\sigma_1, \sigma_2$ and $\sigma_3$ are the three Pauli matrices. Sometimes the Stokes matrix $\mathcal{P}$ is separated in a traceless part (i.e. the polarization matrix) supplemented by the identity matrix multiplying the intensity of the radiation field; this separation shall not be employed here. The orientation of the coordinate system is illustrated in figure 1. The radial, azimuthal and polar directions are

$$\hat{r} = (\cos \vartheta \sin \varphi, \sin \vartheta \sin \varphi, \cos \varphi),$$

$$\hat{\vartheta} = (\cos \varphi \cos \vartheta, \sin \varphi \cos \vartheta, -\sin \vartheta),$$

$$\hat{\varphi} = (-\sin \varphi, \cos \varphi, 0),$$

(2.2)

implying that $\hat{r} \times \hat{\vartheta} = \hat{\varphi}$. Photons propagate radially and $\hat{n} = (\vartheta, \varphi)$ denotes the direction of the scattered photon while $\hat{n}' = (\vartheta', \varphi')$ is the direction of the incoming photon; similarly $\mu = \cos \vartheta$ and $\nu = \cos \vartheta'$. 

3
When the photons impinge the electrons in a magnetized environment the magnetic field can be treated in the guiding center approximation. Denoting with $\vec{B}$ the comoving magnetic field intensity the guiding center approximation [34, 35] stipulates

$$B_i(\vec{x}, \tau) \simeq B_i(\vec{x}_0, \tau) + (x^j - x^j_0) \partial_j B_i + \cdots,$$

(2.3)

where the ellipses stand for the higher orders in the gradients leading, both, to curvature and drift corrections which will be neglected in this investigation. The scales one must therefore compare are $|\vec{x}_0| = L_0$, $|\vec{x} - \vec{x}_0| = L$, $\lambda_{\gamma}^{(\text{rec})}$ (the wavelength of the incident radiation at the recombination epoch) and $H_{\text{rec}}^{-1}$ (i.e. the Hubble rate at recombination). It is easy to appreciate that $\lambda_{\gamma}^{(\text{rec})} = O(\mu \text{m})$ implying that

$$H_{\text{rec}}^{-1} \simeq L \gg L_0 \gg \lambda_{\gamma}^{(\text{rec})}. $$

(2.4)

Equation (2.4) implies that, for the purposes of the contribution of the magnetic field to the Stokes matrix, the spatial gradients can be neglected while they cannot be neglected when estimating the effects of the large-scale inhomogeneities of the magnetic field. In spite of the fact that the contribution of the spatial gradients can be neglected in the first approximation, still the direction of the magnetic field should be appropriately taken into account. Consequently it is necessary to introduce a local basis which will define for us the magnetic field direction:

$$\hat{e}_1 = (\cos \alpha \cos \beta, \sin \alpha \cos \beta, -\sin \beta),$$

$$\hat{e}_2 = (-\sin \alpha, \cos \alpha, 0),$$

$$\hat{e}_3 = (\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta).$$

(2.5)

The basis of equation (2.5) is local since it accounts for the direction of the magnetic field over the typical scales involved in the electron–photon scattering. Once the direction of the local magnetic field has been fixed, the motion of the electrons and of the ions will follow the appropriate geodesics holding for charged particles in a gravitational field.

For the calculation of the scattering matrix the magnetic field can be aligned along $\hat{e}_3$ but since $\hat{e}_3$ has an arbitrary orientation with respect to the fixed coordinate system, the magnetic

Figure 1. Schematic view of the relation between the coordinate system defining the scattered radiation field and the local frame of reference defining the direction of the magnetic field.
field itself will have an arbitrary orientation. Consider, as an example, a situation which will be treated later on in greater detail, i.e. the case where a relic vector mode of the geometry\(^2\) propagates along the direction \(\hat{k}\). Since \(\hat{k}\) also coincides, by definition, with the direction of the Fourier wavevector the whole problem will be characterized by

- \((\hat{n} \cdot \hat{k})\), i.e. the projection of the photon momentum along the direction of propagation of the relic vector;
- \((\hat{e}_3 \cdot \hat{k})\), i.e. the projection of the magnetic field direction along the direction of propagation of the relic vector.

In the mentioned example we can choose, without loss of generality, \(\hat{k} = \hat{z}\) and the two physical polarizations of the relic vector will then be defined in the \(\hat{x}-\hat{y}\) plane. In this situation \(\cos \theta = \hat{k} \cdot \hat{n}\) and \(\cos \alpha = \hat{k} \cdot \hat{e}_3\). The direction \(\hat{e}_3\) does not coincide, in general, with \(\hat{z}\). For instance if \(\alpha = \beta = -\pi/2\), \(\hat{e}_3\) coincides with \(\hat{e}_y\), while for \(\alpha = 0\) and \(\beta = \pi/2\), \(\hat{e}_3\) coincides with \(\hat{e}_y\). This simple example shows explicitly that since the direction of \(\hat{e}_3\) is arbitrary, the orientation of the magnetic field is also generic. Such an arbitrariness entails the dependence of the scattering matrix upon two supplementary angles. In total the Stokes matrix will then depend overall upon six angles: \((\theta, \varphi)\) (for the directions of the scattered photons), \((\theta', \varphi')\) (for the directions of the incident photons) and \((\alpha, \beta)\) for the magnetic field direction. The schematic relation between the direction of the scattered radiation and the local frame defined by equation (2.5) is summarized in figure 1. The (thick) dashed line denotes the direction of \(\hat{n}\), i.e. the direction of propagation of the radiation field. The direction of \(\hat{e}_3\) is arbitrary and it is determined by the angles \(\alpha\) and \(\beta\) illustrated in figure 1.

In the dipole approximation the scattered electric field can be computed as the composition of the scattered electric fields due to the electrons and to the ions:

\[
\vec{E}^{\text{out}} = -\epsilon \frac{\vec{r} \times [\vec{r} \times \vec{a}(e)]}{r^3}, \quad \vec{E}^{\text{out}} = \epsilon \frac{\vec{r} \times [\vec{r} \times \vec{a}(i)]}{r^3},
\]

where \(\vec{a}(e)\) and \(\vec{a}(i)\) are, respectively, the accelerations for the electrons and for the ions. In the local frame defined by equation (2.5) the vector \(\vec{A} = (\vec{a}(e) - \vec{a}(i))\) can be decomposed as \(\vec{A} = (A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3)\). Denoting with \(E_1 = (\vec{E} \cdot \hat{e}_1)\), \(E_2 = (\vec{E} \cdot \hat{e}_2)\) and \(E_3 = (\vec{E} \cdot \hat{e}_3)\) the components of the electric fields of the incident radiation in the local frame we have that from the geodesics of electrons and ions

\[
A_1 = \frac{\omega_{pe}^2}{4\pi n_0}\zeta(\omega)[\Lambda_1 E_1 - i f_e \Lambda_2 E_2],
\]
\[
A_2 = \frac{\omega_{pe}^2}{4\pi n_0}\zeta(\omega)[\Lambda_1 E_2 + i f_e \Lambda_2 E_1],
\]
\[
A_3 = -\frac{\omega_{pe}^2}{4\pi n_0}\Lambda_3 E_3,
\]

where because of the global neutrality of the plasma, \(n_0 = n_0 d^3\) is the common comoving concentration of electrons and ions; \(\omega_{pe,e}\) and \(\omega_{pe,i}\) denote respectively the Larmor and plasma

\(^2\) The same discussion, with due differences, can be repeated in the case of the scalar or tensor modes of the geometry. Here the case of the vector modes is just selected for the sake of illustration.
frequencies for electrons (and ions)
\[ \omega_{\text{Be}, i} = \frac{e \mathcal{B} \cdot \mathbf{\hat{e}}_i}{m_{\text{Be}, i} a}, \quad \omega_{\text{pe}, i} = \frac{4 \pi e^2 n_0}{m_{\text{e}, i} a}, \]
where \( m_{\text{e}, i} \) denote either the electron or the ion mass depending upon the relative subscript and \( a(\tau) \) is the scale factor of a conformally flat geometry of Friedmann–Robertson–Walker type whose line element and metric tensor are defined as
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(\tau)[d\tau^2 - d\mathbf{x}^2], \quad g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}. \]

In equations (2.7), (2.8) and (2.9) the functions \( \Lambda_i \) (with \( i = 1, 2, 3 \)) as well as \( \zeta(\omega) \) and all depend upon the angular frequency of the photon (i.e. \( \omega = 2\pi v \)) and are defined as
\[
\Lambda_1(\omega) = 1 + \left( \frac{\omega^2}{\omega_{\text{Be}}^2} \right) \left( \frac{\omega^2 - \omega_{\text{Be}}^2}{\omega_{\text{Be}}^2} \right), \\
\Lambda_2(\omega) = 1 - \left( \frac{\omega^2}{\omega_{\text{Be}}^2} \right) \left( \frac{\omega^2 - \omega_{\text{Be}}^2}{\omega_{\text{Be}}^2} \right), \\
\Lambda_3(\omega) = 1 + \left( \frac{\omega^2}{\omega_{\text{Be}}^2} \right), \\
\zeta(\omega) = \frac{\omega^2}{\omega_{\text{Be}}^2} - \frac{1}{f_c^2(\omega) - 1}, \\
f_c(\omega) = \left( \frac{\omega_{\text{Be}}}{\omega} \right).
\]

The scale factor \( a(\tau) \) appears explicitly in equations (2.10) since the mass of the (nonrelativistic) species breaks the conformal invariance of the system of equations. Indeed, equations (2.7), (2.8) and (2.9) follow from the geodesics of charged species in the conformally flat metric of equation (2.11) where, for a generic massive particle, the mass shell condition implies that \( g_{\mu\nu} P^\mu P^\nu = m^2 \) (\( P^\mu = m u^\mu \) is the canonical momentum and \( u^\mu \) the four-velocity). Recalling that the comoving three-momentum \( \mathbf{\tilde{q}} \) is defined as \( \mathbf{\tilde{q}} = a \mathbf{\tilde{p}} \) where \( \delta_{ij} p^i p^j = -g_{ij} P^i P^j \), the comoving three-velocity is given by \( \mathbf{\tilde{v}} = \mathbf{\tilde{q}}/\sqrt{q^2 + m^2 a^2} \). Since the electrons are non-relativistic at last scattering \( \mathbf{\tilde{q}} = m a \mathbf{\tilde{v}} \) and this is, ultimately, the rationale for the appearance of the scale factors in the explicit expressions of the Larmor and plasma frequencies for the electrons and for the ions\(^3\). The numerical value of \( f_c(\omega) \) for typical cosmological parameters is given by
\[ f_c(\omega) = \left( \frac{\omega_{\text{Be}}}{\omega} \right) = 2.79 \times 10^{-12} \left( \frac{B}{\text{nG}} \right) \left( \frac{\text{GHz}}{v} \right)(z_s + 1) \ll 1, \]
where \( z_s \) is the redshift to last scattering, i.e. \( z_s = 1090.79^{+0.94}_{-0.92} \) according to the WMAP-7 yr data \([10]\). In equation (2.13) \( B = |\mathbf{\tilde{e}}_3 \cdot \mathbf{\tilde{B}}| \); grossly speaking the typical values of \( v \) and \( B \) appearing in equation (2.13) do correspond, respectively, to the (very minimal) value of the frequency channel of CMB experiments and to the maximal value of the comoving magnetic field allowed by the distortions of the temperature autocorrelations and of the cross-correlations between temperature and polarization. The evolution of the Stokes matrix \( \mathcal{P} \) can be formally written as\(^4\)
\[ \frac{d\mathcal{P}}{dr} + \epsilon' \mathcal{P} = \frac{3\epsilon'}{16\pi} \int M(\Omega, \Omega', \alpha, \beta) \mathcal{P}(\Omega, \Omega') M(\Omega, \Omega', \alpha, \beta), \]
\(^3\) For more explicit discussions of these points see \([15, 24]\). Note that equation (A.12) of \([24]\) contains few trivial typos which have been corrected in the archive version of the same paper.
\(^4\) The dagger in equation (2.14) defines, as usual, the complex conjugate of the transposed matrix.
where \(d\Omega' = d\cos \theta' \; d\psi'\) and, defining the rate of electron–photon scattering \(\Gamma_{ye}\),

\[
\epsilon' = a \Gamma_{ye} = a \bar{\hbar}_0 n_e \sigma_{ey}, \quad \sigma_{ey} = \frac{8}{3} \pi r_e^2, \quad r_e = \frac{e^2}{m_e} \tag{2.15}
\]
is the differential optical depth. On the right-hand side of equation (2.14), \(M(\Omega, \Omega', \alpha, \beta)\) is a two by two matrix and the four entries of the matrix \(M(\Omega, \Omega', \alpha, \beta)\) are separately reported in appendix A. From the matrix elements of \(M(\Omega, \Omega', \alpha, \beta)\) it is immediately possible to derive the evolution equations for the Stokes parameters in the Mueller form, namely

\[
\frac{dI}{d\tau} + \epsilon' I = \frac{3\epsilon'}{32\pi} \int d\Omega' \; T'(\Omega, \Omega', \alpha, \beta) T(\Omega'), \tag{2.16}
\]

where \(I\) is a column matrix whose entries are, respectively, \(I, Q, U\) and \(V\). The entries of the \(4 \times 4\) Mueller matrix will be denoted as \(T_{ij}(\Omega, \Omega', \alpha, \beta)\) where \(i\) and \(j\) run over the various Stokes parameters \(I, Q, U\) and \(V\) and are reported in appendix A (see, in particular, equations (A.11)–(A.26)). Finally, in terms of the matrix the evolution equations of the different Stokes parameters can be formally written as

\[
\frac{dI}{d\tau} + \epsilon' I = \frac{3\epsilon'}{32\pi} \int d\Omega' \; T_{ij}(\Omega, \Omega', \alpha, \beta), \tag{2.17}
\]

\[
\frac{dQ}{d\tau} + \epsilon' Q = \frac{3\epsilon'}{32\pi} \int d\Omega' \; T_{ij}(\Omega, \Omega', \alpha, \beta), \tag{2.18}
\]

\[
\frac{dU}{d\tau} + \epsilon' U = \frac{3\epsilon'}{32\pi} \int d\Omega' \; T_{ij}(\Omega, \Omega', \alpha, \beta), \tag{2.19}
\]

\[
\frac{dV}{d\tau} + \epsilon' V = \frac{3\epsilon'}{32\pi} \int d\Omega' \; T_{ij}(\Omega, \Omega', \alpha, \beta), \tag{2.20}
\]

where in all the integrands on the right-hand side of equations (2.17), (2.18), (2.19) and (2.20) the matrix elements \(T_{ij}(\Omega, \Omega', \alpha, \beta)\) are functions of both the angles of the incident radiation \(\Omega' = (\theta', \psi')\), the angles of the scattered radiation \(\Omega = (\theta, \psi)\) and the orientation of the magnetic field defined by the angles \(\alpha\) and \(\beta\). The explicit relations between \(T_{ij}(\Omega, \Omega', \alpha, \beta)\) and the matrix elements \(M_{ij}(\Omega, \Omega', \alpha, \beta)\) are reported in appendix A. It is worth pointing out that the explicit expression of equation (2.20) is frequently omitted in standard treatments of CMB anisotropies. This is because the circular polarizations are simply set to zero from the very beginning. In the present case a nonvanishing amount of circular polarization can be generated from the properties of the magnetized electron–photon scattering and it is therefore also important to keep track of the \(V\)-mode polarization as recently emphasized in a related context [24, 25, 28] (also see the introductory section for a more extended discussion).

3. Brightness perturbations

The brightness perturbations, i.e. the fluctuations of the Stokes parameters in comparison to their equilibrium values can be decomposed as

\[
\Delta_X(\vec{x}, \tau) = \Delta_X^{(s)}(\vec{x}, \tau) + \Delta_X^{(v)}(\vec{x}, \tau) + \Delta_X^{(t)}(\vec{x}, \tau), \tag{3.1}
\]

where \(X = I, Q, U, V\) denotes, generically, one of the four Stokes parameters and where the superscripts refer, respectively, to the scalar, vector and tensor modes of the geometry. Before discussing, in detail, the contribution of the scalar, vector and tensor modes to the collisionless part of the transport equations it is appropriate to derive the general form of the full (perturbed) transport equations from the exact expression reported in equations (2.17)–(2.20). To avoid
lengthy expressions the attention will be concentrated only on the intensity, i.e. equation (2.17) which can be written as

\[ I(\vec{x}, \tau, q, \eta) = f_0(q)[1 + f^{(1)}(\vec{x}, \tau, q, \eta)] \]

\[ = f_0(q) \left[ 1 - \frac{\partial \ln f_0}{\partial q} \Delta_I(\vec{x}, \tau, \eta) \right], \quad (3.2) \]

where \( f_0(q) \) is the (unperturbed) Bose–Einstein distribution, \( q \) is the modulus of the comoving three-momentum and \( \eta \) denotes, as usual, the direction of the photon. In what follows the dependence of the brightness perturbations on \( \eta \) will not be indicated in explicit terms so that, for instance, we shall write \( \Delta_I(\vec{x}, \tau) \) instead of \( \Delta_I(\vec{x}, \tau, \eta) \). Note that \( \Delta_I(\vec{x}, \tau) \) does not depend on \( q \) and this is the advantage of using \( \Delta_I(\vec{x}, \tau) \) instead of \( f^{(1)}(\vec{x}, \tau, q, \eta) \). In terms of \( f^{(1)}(\vec{x}, \tau, q, \eta) \) and \( \Delta_I(\vec{x}, \tau) \) the left-hand side of equation (2.17) can be written, respectively, as

\[ \frac{dI}{d\tau} + \epsilon \frac{dI}{d\tau} = f_0(q) \left[ \frac{\partial f^{(1)}}{\partial \tau} + \hat{n} \frac{\partial f^{(1)}}{\partial q} + \left( \frac{dq}{d\tau} \right)_{\eta} \frac{\partial \ln f_0}{\partial q} + \epsilon \frac{dI}{d\tau} \right] \]

\[ = \frac{\partial \ln f_0}{\partial \ln q} \left[ \frac{\partial y_I^{(y)}}{\partial \tau} + \hat{n} \frac{\partial y_I^{(y)}}{\partial q} + \left( \frac{dq}{d\tau} \right)_{\eta} + \frac{1}{q} \frac{dq}{d\tau} \right] + \epsilon \frac{dI}{d\tau}, \quad (3.3) \]

where the superscripts denote, respectively, the scalar vector and tensor modes, i.e. \( y = s, v, t \).

Within the same notations employed to derive equation (3.3), the right-hand side of equation (2.17) can be written as

\[ \frac{3\epsilon'}{32\pi} \int d\Omega' \mathcal{F}_I(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} \]

\[ + \mathcal{T}_{II}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} + \mathcal{T}_{IQ}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} + \mathcal{T}_{IV}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} \]

\[ = \frac{\partial \Delta_I^{(y)}}{\partial \tau} + \hat{n} \frac{\partial \Delta_I^{(y)}}{\partial q} + \epsilon \frac{dI}{d\tau} + \left( \frac{dq}{d\tau} \right)_{\eta} + \frac{1}{q} \frac{dq}{d\tau} \]

\[ = \frac{3\epsilon'}{32\pi} \int d\Omega' \left[ \mathcal{T}_{II}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} + \mathcal{T}_{IQ}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} + \mathcal{T}_{IV}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} \right] \]

\[ + \mathcal{T}_{II}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} + \mathcal{T}_{IV}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} \], \quad (3.4) \]

By putting together the second equality of equations (3.3) and (3.4) the explicit form of equation (2.17) becomes

\[ \frac{dI}{d\tau} + \epsilon \frac{dI}{d\tau} = \frac{3\epsilon'}{32\pi} \int d\Omega' \left[ \mathcal{T}_{II}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} + \mathcal{T}_{IQ}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} + \mathcal{T}_{IV}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} \right] \]

\[ + \mathcal{T}_{II}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} + \mathcal{T}_{IV}(\Omega, \Omega', \alpha, \beta) \Delta_I^{(y)} \], \quad (3.5) \]

which is the full expression of one of the transport equations written in terms of the brightness perturbations. The length of equation (3.5) explains, in what follows, that more compact notations shall be preferred (like the ones introduced, hereunder, in equations (3.10)–(3.12)). Equations (3.4) and (3.5) demonstrate in concrete terms the relevance of equations (2.17)–(2.20) and of the results of appendix B for the derivation of the explicit expressions which shall be used in this and in the following sections. Finally, the last term on the left-hand side of equation (3.5) accounts for the contribution of the scalar, vector and tensor modes of the geometry to the total time derivative of the comoving three-momentum. The next task of this section will be to compute the contributions of the scalar, vector and tensor modes to the total time derivative of the comoving three-momentum.

The scalar, vector and tensor components of the brightness perturbations are affected, respectively, by the scalar, vector and tensor inhomogeneities of the geometry and of the
various sources. Assuming the conformally flat background introduced in equation (2.11), the fluctuations of the metric can be written, in general terms, as

\[ \delta g_{\mu\nu}(\vec{x}, \tau) = \delta g_{\mu\nu}(\vec{x}, \tau) + \delta v g_{\mu\nu}(\vec{x}, \tau) + \delta g_{\mu\nu}(\vec{x}, \tau), \]

where \( \delta s, \delta v, \delta t \) denote the inhomogeneity preserving, separately, the scalar, vector and tensor nature of the fluctuations. The scalar modes of the geometry are parametrized in terms of four independent functions \( \psi(\vec{x}, \tau), \phi(\vec{x}, \tau), E(\vec{x}, \tau) \) and \( F(\vec{x}, \tau) \):

\[
\begin{align*}
\delta g_{00}(\vec{x}, \tau) &= 2a^2(\tau)\phi(\vec{x}, \tau), \\
\delta g_{0i}(\vec{x}, \tau) &= -a^2(\tau)\partial_i F(\vec{x}, \tau), \\
\delta g_{ij}(\vec{x}, \tau) &= 2a^2(\tau)[\psi(\vec{x}, \tau)\delta_{ij} - \partial_i \partial_j E(\vec{x}, \tau)].
\end{align*}
\]

By setting \( E \) and \( F \) to zero the gauge freedom is completely fixed and this choice pins down the longitudinal (or conformally Newtonian) gauge. The vector modes are described by two partial derivations with respect to the spatial coordinates will be instead denoted by \( \partial_i \), and there will be subjected to the conditions \( \partial_i Q^i = 0 \) and \( \partial_i W^i = 0 \). It will be convenient, for the present purposes, to choose the gauge \( Q_i = 0 \). The tensor modes of the geometry are parametrized in terms of a rank-2 tensor in three spatial dimensions, i.e.

\[
\begin{align*}
\delta g_{ij}(\vec{x}, \tau) &= -a^2h_{ij}, \\
\partial_i h^i_j(\vec{x}, \tau) &= h^i_j(\vec{x}, \tau) = 0,
\end{align*}
\]

which is automatically invariant under infinitesimal coordinate transformations. The following shorthand notation\(^5\) will be adopted

\[
\begin{align*}
L_i^{(s)}(\vec{n}, \vec{x}, \tau) &= \partial_t \Delta_i^{(s)} + \hat{n}^i \partial_i \Delta_i^{(s)} + \epsilon' \Delta_i^{(s)} + \frac{1}{q} \left( \frac{dq}{d\tau} \right)^i, \\
L_i^{(v)}(\vec{n}, \vec{x}, \tau) &= \partial_t \Delta_i^{(v)} + \hat{n}^i \partial_i \Delta_i^{(v)} + \epsilon' \Delta_i^{(v)} + \frac{1}{q} \left( \frac{dq}{d\tau} \right)^i, \\
L_i^{(t)}(\vec{n}, \vec{x}, \tau) &= \partial_t \Delta_i^{(t)} + \hat{n}^i \partial_i \Delta_i^{(t)} + \epsilon' \Delta_i^{(t)} + \frac{1}{q} \left( \frac{dq}{d\tau} \right)^i,
\end{align*}
\]

where \( q = \hat{n}_i q^i \) and where the scalar, vector and tensor contributions to the derivatives of the modulus of the comoving three-momentum are given, respectively, by

\[
\begin{align*}
\left( \frac{dq}{d\tau} \right)^i &= -q \partial_i \psi + q \hat{n}^i \partial_i \phi, \\
\left( \frac{dq}{d\tau} \right)^v &= \frac{q}{2} \hat{n}^i \hat{n}^j (\partial_i \partial_j W_j + \partial_i \partial_j W_i), \\
\left( \frac{dq}{d\tau} \right)^t &= -q \frac{\hat{n}^i}{\hat{n}^0} \partial_t h_{ij}.
\end{align*}
\]

The identities of equations (3.13), (3.14) and (3.15) can be derived from the inhomogeneities of equations (3.7)–(3.9) by recalling the definition of comoving three momentum (see the discussion after equation (2.11)) and by using the relations

\[
\frac{dx^i}{d\tau} = \frac{P^i}{P^0} = \frac{q^i}{q} = \hat{n}^i,
\]

\(^5\) The partial derivatives with respect to \( \tau \) will be denoted by \( \partial_i \); the partial derivatives with respect to the spatial coordinates will be instead denoted by \( \partial_i \) with \( i = 1, 2, 3 \).
where $P^i$ and $P^0$ are the space-like and time-like components of the canonical momentum obeying, for the photons, $g_{ab} P^a P^b = 0$. The notation introduced in equations (3.10), (3.11), (3.12) for the fluctuations of the intensity can also be generalized to the linear and circular polarizations:

$$\mathcal{L}^{(1)}_X(\tilde{n}, \tilde{i}, \tau) = \partial_i \Delta_\tilde{X}^{(1)} + \tilde{n}^i \partial_i \Delta_\tilde{X}^{(1)} + \epsilon^i \Delta_\tilde{X}^{(1)}, \tag{3.17}$$

where the subscript can coincide, alternatively, with $Q, U$ and $V$ (i.e. $X = Q, U, V$) and the superscript denotes the transformation properties of the given fluctuation (i.e. $y = s, v, t$). The fluctuations of the geometry would seem to affect only the brightness perturbation for the intensity but such a conclusion would be incorrect: in the presence of a magnetic field the evolution equations of the four brightness perturbations are all coupled by the collision term which does not only contain the intensity of the radiation field but a weighted sum of the four brightness perturbations integrated over the directions of the incident radiation. Consequently the polarization of the metric fluctuations will also impact on all the four brightness perturbations. The conventions on the Fourier transform and polarizations of the scalar, vector and tensor modes will be, in short,

$$\phi(\tilde{x}, \tau) = \frac{1}{(2\pi)^{3/2}} \int d^3 k \phi(\tilde{k}, \tau) e^{i \tilde{k} \cdot \tilde{x}}, \tag{3.18}$$

$$\psi(\tilde{x}, \tau) = \frac{1}{(2\pi)^{3/2}} \int d^3 k \psi(\tilde{k}, \tau) e^{i \tilde{k} \cdot \tilde{x}}, \tag{3.19}$$

$$W_i(\tilde{x}, \tau) = \frac{1}{(2\pi)^{3/2}} \int d^3 k W_i(\tilde{k}, \tau) e^{i \tilde{k} \cdot \tilde{x}}, \quad \partial_i W_i(\tilde{x}, \tau) = 0, \tag{3.20}$$

$$h_{ij}(\tilde{x}, \tau) = \frac{1}{(2\pi)^{3/2}} \int d^3 k h_{ij}(\tilde{k}, \tau) e^{i \tilde{k} \cdot \tilde{x}}, \quad \partial_j h_{ij}(\tilde{x}, \tau) = h_j(\tilde{x}, \tau) = 0. \tag{3.21}$$

The vector and the tensor polarizations can be decomposed, respectively, as

$$W_i(\tilde{k}, \tau) = \sum_\lambda \epsilon^i_{\lambda}(\tilde{k}) W_{\lambda}(\tilde{k}, \tau) = \hat{a}_i W_{\lambda}(\tilde{k}, \tau) + \tilde{b}_i W_{\phi}(\tilde{k}, \tau), \tag{3.22}$$

$$h_{ij}(\tilde{k}, \tau) = \sum_\lambda \epsilon_{ij}^{\lambda}(\tilde{k}) h_{\lambda}(\tilde{k}, \tau) = \epsilon_{ij}^{\lambda}(\tilde{k}) h_{\lambda}(\tilde{k}, \tau) + \epsilon_{ij}^{\lambda}(\tilde{k}) h_{\phi}(\tilde{k}, \tau), \tag{3.23}$$

where $\tilde{k}$ denotes the direction of propagation and the two orthogonal directions $\hat{a}$ and $\tilde{b}$ are such that $\hat{a} \times \tilde{b} = \tilde{k}$. Supposing that the direction of propagation of the relic tensor is oriented along $\tilde{k}$, the two tensor polarizations are defined in terms of $\hat{a}_i$ and $\tilde{b}_i$ as

$$\epsilon_{ij}^{\lambda}(\tilde{k}) = \hat{a}_i \hat{a}_j - \tilde{b}_i \tilde{b}_j, \quad \epsilon_{ij}^{\lambda}(\tilde{k}) = \hat{a}_i \tilde{b}_j + \tilde{b}_i \hat{a}_j. \tag{3.24}$$

The projections of the vector and of the tensor polarizations on the direction of photon propagation $\tilde{n}$ are

$$\tilde{n}^i W_i(\tilde{k}, \tau) = [\tilde{n}^i \hat{a}_i W_{\lambda}(\tilde{k}, \tau) + \tilde{n}^i \tilde{b}_i W_{\phi}(\tilde{k}, \tau)], \tag{3.25}$$

$$\tilde{n}^i \nabla_i h_{ij}(\tilde{k}, \tau) = \{[(\hat{a} \cdot \hat{a})^2 - (\hat{a} \cdot \tilde{b})^2] h_{\lambda}(\tilde{k}, \tau) + 2(\hat{a} \cdot \hat{a})(\hat{a} \cdot \tilde{b}) h_{\phi}(\tilde{k}, \tau)\}. \tag{3.26}$$

Choosing the direction of propagation of the relic vector and of the relic tensor along the $\hat{z}$ axis, the unit vectors $\hat{a}$ and $\tilde{b}$ will coincide with the remaining two Cartesian directions and the related Fourier amplitudes will satisfy
Following the notation of equations (3.27) and (3.28) the scalar transport equations can be formally expressed as:

\[
\mathcal{L}^{(s)}_{v}(\mu, \varphi, \bar{k}, \tau) = e^r \mathcal{L}^{(s)}_{v}(\mu, \varphi, \bar{k}, \tau) + \frac{3e' \mu}{32\pi} \int_{-1}^{1} dv \int_{0}^{2\pi} d\varphi' \mathcal{F}^{(s)}(\mu, \nu, \varphi', \alpha, \beta),
\]

\[
\mathcal{L}^{(s)}_{Q}(\mu, \varphi, \bar{k}, \tau) = 3e' \frac{\mu}{32\pi} \int_{-1}^{1} dv \int_{0}^{2\pi} d\varphi' \mathcal{F}_{Q}^{(s)}(\mu, \nu, \varphi', \alpha, \beta),
\]

\[
\mathcal{L}^{(s)}_{U}(\mu, \varphi, \bar{k}, \tau) = 3e' \frac{\mu}{32\pi} \int_{-1}^{1} dv \int_{0}^{2\pi} d\varphi' \mathcal{F}_{U}^{(s)}(\mu, \nu, \varphi', \alpha, \beta),
\]

Similarly equation (3.30) becomes, in Fourier space,

\[
\mathcal{L}^{(s)}_{X}(\mu, \varphi, \bar{k}, \tau) = \partial_{\tau} \Delta^{(s)}_{X} + (i\mu + \epsilon') \Delta_{X}^{(s)}.
\]

The explicit form of the transport equations for the scalar, vector and tensor modes of the geometry will be scrutinized in the three forthcoming sections.

### 4. Scalar modes

Following the notation of equations (3.31) and (3.34) the scalar transport equations can be formally expressed as

\[
\mathcal{L}^{(s)}_{v}(\mu, \varphi, \bar{k}, \tau) = e^r \mathcal{L}^{(s)}_{v}(\mu, \varphi, \bar{k}, \tau) + \frac{3e' \mu}{32\pi} \int_{-1}^{1} dv \int_{0}^{2\pi} d\varphi' \mathcal{F}_{v}^{(s)}(\mu, \nu, \varphi', \alpha, \beta),
\]

\[
\mathcal{L}^{(s)}_{Q}(\mu, \varphi, \bar{k}, \tau) = 3e' \frac{\mu}{32\pi} \int_{-1}^{1} dv \int_{0}^{2\pi} d\varphi' \mathcal{F}_{Q}^{(s)}(\mu, \nu, \varphi', \alpha, \beta),
\]

\[
\mathcal{L}^{(s)}_{U}(\mu, \varphi, \bar{k}, \tau) = 3e' \frac{\mu}{32\pi} \int_{-1}^{1} dv \int_{0}^{2\pi} d\varphi' \mathcal{F}_{U}^{(s)}(\mu, \nu, \varphi', \alpha, \beta),
\]
\[ \mathcal{L}_V^{(s)}(\mu, \varphi, \bar{k}, \tau) = \frac{3\epsilon'}{32\pi} \int_0^1 dv \int_0^{2\pi} d\psi' \mathcal{F}_V^{(s)}(\mu, v, \varphi, \alpha, \beta), \]  

(4.4)

where \( \psi' \) denotes the scalar component of the baryon velocity field. In equations (4.1)–(4.4) the source terms involve the integration over the incoming photon directions. Both the integration over \( \psi' \) and \( v \) can be performed explicitly and the final expressions are rather lengthy, as easily imaginable. To make the explicit equations more manageable without losing any relevant information it is useful, in the following part of this section, to write the results already in the physical limit, i.e. owing to the numerical values of the plasma and Larmor frequencies and recalling equation (2.12)

\[ \Lambda_1(\omega) = \Lambda_2(\omega) = \Lambda_3(\omega) = 1 + \mathcal{O}(m_e/m_\nu), \quad \zeta(\omega) \simeq -1 + f^2_s(\omega) + \mathcal{O}(f^4_s). \]  

(4.5)

The scalar source terms depend upon the explicit form of the matrix elements appearing in equations (4.7), (4.8), (4.9) and (4.10). The integration over \( \psi' \) can be performed explicitly. Using the notation

\[ \mathcal{T}_{ab}(\mu, v, \varphi, \alpha, \beta) = \int_0^{2\pi} d\psi' \mathcal{T}_{ab}(\mu, v, \varphi, \psi', \alpha, \beta), \]  

(4.6)

the final results are reported, for completeness and future perusal, in appendix B. According to equations (4.1)–(4.4), the expressions reported in equations (B.1)–(B.16) must be integrated over \( v \). For the \( v \) integration it is useful to expand the various brightness perturbations in a series of Legendre polynomials \( P_\ell(v) \)

\[ \Delta_X(v, k, \tau) = \sum_\ell (-i)^\ell (2\ell + 1) P_\ell(v) \Delta_X(\ell, k, \tau). \]  

(4.7)

The integration over \( v \) will then have the net result of expressing the source terms in terms of a limited number of multipoles of the intensity and of the polarization. In explicit terms the source terms can be expressed, for each brightness perturbation, as an expansion in \( f_s(\omega) \):

\[ \partial_\tau \Delta_1^{(i)} + (ik\mu + \epsilon') \Delta_1^{(i)} = \partial_\tau \psi' - ik\mu \varphi' + \epsilon' A_I + \epsilon' f_s(\omega) B_I + \epsilon' f^2_s(\omega) C_I, \]  

(4.8)

\[ \partial_\tau \Delta_2^{(i)} + (ik\mu + \epsilon') \Delta_2^{(i)} = \epsilon' A_Q + \epsilon' f_s(\omega) B_Q + \epsilon' f^2_s(\omega) C_Q, \]  

(4.9)

\[ \partial_\tau \Delta_3^{(i)} + (ik\mu + \epsilon') \Delta_3^{(i)} = \epsilon' A_U + \epsilon' f_s(\omega) B_U + \epsilon' f^2_s(\omega) C_U, \]  

(4.10)

\[ \partial_\tau \Delta_4^{(i)} + (ik\mu + \epsilon') \Delta_4^{(i)} = \epsilon' A_V + \epsilon' f_s(\omega) B_V + \epsilon' f^2_s(\omega) C_V, \]  

(4.11)

where, for \( X = I, Q, U, V, A_X \) denotes the leading-order result, \( B_X \) denotes the next-to-leading order (NLO) correction while \( C_X \) denotes the next-to-next-to-leading (NNLO) term. Defining with \( S_p \) the usual combination of the quadrupole of the intensity and of the monopole and quadrupole of the linear polarization (see, e.g. [2–4])

\[ S_p = \Delta_{12} + \Delta_{Q0} + \Delta_{Q2}, \]  

(4.12)

the leading-order contribution for the brightness perturbations is then given by

\[ A_I = \Delta_{10} + \mu v_b - \frac{P_2(\mu)}{2} S_p, \]  

(4.13)

\[ A_Q = \frac{3}{4} (1 - \mu^2) S_p, \]  

(4.14)

\[ A_U = 0, \quad A_V = -\frac{3}{2} \mu \Delta V_1, \]  

(4.15)
where the notation $\vec{v}(s) = \vec{k}v_b$ has been employed for the scalar component of the Doppler term. The NLO contribution to the right-hand side of equations (4.8), (4.9), (4.10) and (4.11) is

$$B_I = -\frac{3}{2} i [1 + \mu^2 + \mu \sqrt{1 - \mu^2} \cos (\phi - \alpha) \sin \beta] \Delta_{V1}$$  \hspace{1cm} (4.16)

$$B_Q = -\frac{3}{2} i \Delta_{V1} [(\mu^2 - 1) \cos \beta + \mu \sqrt{1 - \mu^2} \cos (\phi - \alpha) \sin \beta],$$  \hspace{1cm} (4.17)

$$B_U = \frac{3}{2} i \Delta_{V1} \sqrt{1 - \mu^2} \sin \beta \sin (\phi - \alpha),$$  \hspace{1cm} (4.18)

$$B_V = \left[ \mu \cos \beta - \frac{\sqrt{1 - \mu^2}}{2} \cos (\phi - \alpha) \sin \beta \right] \Delta_{I0}$$

$$- \left[ 2 \mu \cos (\phi - \alpha) \sin^2 \beta + \frac{\sqrt{1 - \mu^2}}{2} \sin \beta \right] \Delta_{I2}$$

$$- \left[ \frac{\mu}{2} \cos \beta - \frac{\sqrt{1 - \mu^2}}{4} \cos (\phi - \alpha) \sin \beta \right] (\Delta_{Q2} + \Delta_{Q0}).$$  \hspace{1cm} (4.19)

Finally, the NNLO contribution to the right-hand side of equations (4.8), (4.9), (4.10) and (4.11) is

$$C_I = [(\mu^2 + 1) + \mu \sqrt{1 - \mu^2} \cos (\phi - \alpha) \sin 2\beta] \left( \frac{\Delta_{I0}}{2} - \Delta_{I2} \right)$$

$$+ \frac{1}{2} [\mu^2 (\Delta_{I0} + \Delta_{I2}) + (\Delta_{I2} - 2 \Delta_{I0})] \sin^2 \beta \cos^2 (\phi - \alpha) - \frac{\cos 2\beta}{2} (1 + 3 \mu^2)$$

$$- \left\{ (1 - \mu^2) \sin^2 (\phi - \alpha) + 1 + \frac{1}{4} \cos 2(\phi - \alpha - \beta) + \cos 2(\phi - \alpha + \beta) \right\} \right] \Delta_{Q0}.$$

$$C_Q = \left\{ \frac{\mu^2}{8} - \frac{1}{2} + \frac{\sin^2 \beta}{8} [4 - 2(2\mu^2 + 1) \cos^2 (\phi - \alpha)] + \frac{\mu \sqrt{1 - \mu^2}}{2} \cos (\phi - \alpha) \sin 2\beta \right\} \Delta_{I0}$$

$$- \left\{ \frac{\mu^2 - 1}{4} + \frac{\sin^2 \beta}{2} [1 + \mu^2] \cos^2 (\phi - \alpha) \right\} \Delta_{I2}$$

$$+ \left\{ \frac{\mu^2}{8} - \frac{\mu^2 + 1}{8} \cos 2(\phi - \alpha) + \frac{3}{8} (1 - \mu^2) \cos 2\beta \right\} \Delta_{Q0}.$$

Finally, the NNLO contribution to the right-hand side of equations (4.8), (4.9), (4.10) and (4.11) is

$$C_V = \left\{ \frac{\mu^2}{8} - \frac{1}{2} + \frac{\sin^2 \beta}{8} [4 - 2(2\mu^2 + 1) \cos^2 (\phi - \alpha)] + \frac{\mu \sqrt{1 - \mu^2}}{2} \cos (\phi - \alpha) \sin 2\beta \right\} \Delta_{I0}$$

$$- \left\{ \frac{\mu^2 - 1}{4} + \frac{\sin^2 \beta}{2} [1 + \mu^2] \cos^2 (\phi - \alpha) \right\} \Delta_{I2}$$

$$+ \left\{ \frac{\mu^2}{8} - \frac{\mu^2 + 1}{8} \cos 2(\phi - \alpha) + \frac{3}{8} (1 - \mu^2) \cos 2\beta \right\} \Delta_{Q0}.$$

Finally, the NNLO contribution to the right-hand side of equations (4.8), (4.9), (4.10) and (4.11) is

$$C_U = \left\{ \frac{\mu^2}{8} - \frac{1}{2} + \frac{\sin^2 \beta}{8} [4 - 2(2\mu^2 + 1) \cos^2 (\phi - \alpha)] + \frac{\mu \sqrt{1 - \mu^2}}{2} \cos (\phi - \alpha) \sin 2\beta \right\} \Delta_{I0}$$

$$- \left\{ \frac{\mu^2 - 1}{4} + \frac{\sin^2 \beta}{2} [1 + \mu^2] \cos^2 (\phi - \alpha) \right\} \Delta_{I2}$$

$$+ \left\{ \frac{\mu^2}{8} - \frac{\mu^2 + 1}{8} \cos 2(\phi - \alpha) + \frac{3}{8} (1 - \mu^2) \cos 2\beta \right\} \Delta_{Q0}.$$
\[ C_L = \sin (\phi - \alpha) \left[ \frac{1 - \mu^2}{2} \sin 2\beta - \mu \cos (\phi - \alpha) \sin ^2 \beta \right] \Delta_{L0} \]
\[ \quad + \sin (\phi - \alpha) \left[ \mu \cos (\phi - \alpha) \sin ^2 \beta - \frac{1 - \mu^2}{4} \sin 2\beta \right] \Delta_{L2} \]
\[ \quad + \frac{\sin \beta \sin (\phi - \alpha)}{2} \left[ \sqrt{1 - \mu^2} \cos \beta + 2\mu \cos (\phi - \alpha) \sin \beta \right] (\Delta_{Q2} + \Delta_{Q0}), \quad (4.22) \]
\[ C_V = -\frac{3}{2} i \left[ \mu \cos ^2 \beta + \sqrt{1 - \mu^2} \cos (\phi - \alpha) \sin \beta \cos \beta \right] \Delta_{V1}. \quad (4.23) \]

Several cross-checks on the obtained results have been made; they will be swiftly mentioned and can be directly reproduced by using the results reported in appendices A and B:

- it has been verified explicitly at the level of the exact expressions (i.e. without implementing the limit of equation (4.5)) the equations must be independent of the \( \alpha \) and \( \beta \) once \((\hat{\epsilon} \cdot \hat{B}) \rightarrow 0\): this is exactly what happens;
- it has been verified that in the limit \( \alpha = \beta = 0\) the exact expressions must reproduce the partial results already obtained in [25, 27, 28]; with equations (4.8)–(4.11) the few typos present in the published version of [25] are corrected;
- by averaging of the source terms over \( \alpha \) and \( \beta \) terms proportional to \( f_s(\omega) \) should automatically disappear without performing any specific limit: this is what will be explicitly shown in the remaining part of this section.

The remaining part of the section is devoted to the averaging of the source terms over the magnetic field directions as suggested in the last point of the above list of items. By integrating over \( \alpha \) and \( \beta \) the source functions appearing on the right-hand side of equations (4.1), (4.2), (4.3) and (4.4), the evolution equations for the brightness perturbations read

\[ L_I^{(s)}(\mu, \varphi, \bar{k}, \tau) = \epsilon \hat{n} v^{(s)}_I + \frac{3\epsilon^2}{128\pi^2} \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{2\pi} \sin \beta \, d\beta \int_{0}^{2\pi} \, d\alpha F^{(s)}_I(\mu, v, \varphi, \varphi', \alpha, \beta), \quad (4.24) \]
\[ L_Q^{(s)}(\mu, \varphi, \bar{k}, \tau) = \frac{3\epsilon^2}{128\pi^2} \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{2\pi} \sin \beta \, d\beta \int_{0}^{2\pi} \, d\alpha F^{(s)}_Q(\mu, v, \varphi, \varphi', \alpha, \beta), \quad (4.25) \]
\[ L_U^{(s)}(\mu, \varphi, \bar{k}, \tau) = \frac{3\epsilon^2}{128\pi^2} \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{2\pi} \sin \beta \, d\beta \int_{0}^{2\pi} \, d\alpha F^{(s)}_U(\mu, v, \varphi, \varphi', \alpha, \beta), \quad (4.26) \]
\[ L_V^{(s)}(\mu, \varphi, \bar{k}, \tau) = \frac{3\epsilon^2}{128\pi^2} \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{2\pi} \sin \beta \, d\beta \int_{0}^{2\pi} \, d\alpha F^{(s)}_V(\mu, v, \varphi, \varphi', \alpha, \beta), \quad (4.27) \]

where the factor \( 128\pi^2 \) accounts for the \( 4\pi \) factor arising from the average over the solid angle spanned by \( \alpha \) and \( \beta \). By performing the averages explicitly, the evolution equations of the four brightness perturbations read

\[ \partial_\tau \Delta_I^{(s)} + (i\kappa \mu + \epsilon') \Delta_I^{(s)} = \partial_\tau \psi - i\kappa \mu \phi + \epsilon' \left[ \Delta_{I0} + \mu v_b - \frac{P_2(\mu)}{2} S_p \right] + f_s^2 \left( \frac{2}{3} \Delta_{I0} + \frac{P_2(\mu)}{6} S_p \right) \]
\[ \partial_\tau \Delta_Q^{(s)} + (i\kappa \mu + \epsilon') \Delta_Q^{(s)} = \epsilon' \left( f_s^2 - 3 \right) \left( \mu^2 - 1 \right) S_p, \quad (4.28) \]
\[ \partial_t \Delta_U^{(s)} + (ik\mu + \epsilon') \Delta_U^{(s)} = 0, \]  
\[ \partial_t \Delta_V^{(s)} + (ik\mu + \epsilon') \Delta_V^{(s)} = -\frac{i\epsilon'}{2}(3 + f_\epsilon^2)\Delta V_1. \]  

The results of this section make quantitatively clear that if the magnetic field has a predominant direction over typical scales comparable with the wavelengths of the scattered photons, then the circular polarization is larger than in the case where, over the same physical scales the magnetic field is randomly oriented. It has been argued in [24, 25] (see also [28]) that over small angular scales the maximal amount of circular polarization arises when there is a strong alignment of the magnetic field along the direction of propagation of the photon which coincides, for large multipoles, with the third Cartesian direction.

The present results improve and confirm, at once, the assumptions made in the analytic and numerical estimates of magnetized CMB anisotropies of [13–15]. Indeed, the scalar fluctuations of the geometry obey a set of evolution equations where large-scale magnetic fields contribute in many respects. These equations will not be repeated here and can be found in [13–15]. The present results improve on the transport equations used there and pave the way for a more consistent account the effects of pre-decoupling magnetic fields both at large as well as at small angular scales.

5. Vector modes

In the case of the vector modes of the geometry the integration over \( \phi' \) of the source functions cannot be easily performed as in the case of the scalar modes of the geometry (see section 4). Each of the two vector polarizations induce a different angular dependence in the corresponding brightness perturbations. In this paper three categories of circular and linear polarizations can be defined:

- the linear and circular polarizations of the scattered (and incident) photons (already described in sections 2 and 3) which are described by the four Stokes parameters or by the appropriate Stokes matrix;
- the linear and circular polarizations of the relic vector waves which we will discuss in this section and which have already been introduced, respectively, in equations (3.25) and (3.27);
- the linear and circular polarizations of the relic tensor waves introduced, respectively, in equations (3.26) and (3.28) and discussed in section 4.

The linear and circular polarizations of the relic tensor and vector waves are just equivalent basis for the description of the tensor and vector modes of the geometry. To avoid potential confusions the vector and the tensor waves will always be treated in the basis of the linear polarizations. In full analogy with the treatment of section 4 the evolution equations for the vector components of the brightness perturbations can be formally written as

\[ \mathcal{L}_I^{(v)}(\mu, \varphi, \bar{k}, \tau) = \epsilon' n^{I(v)} + \frac{3\epsilon'}{32\pi} \int_{-1}^{1} dv \int_{0}^{2\pi} d\phi' \mathcal{F}^{(v)}(\mu, v, \varphi, \phi', \alpha, \beta), \]  
\[ \mathcal{L}_Q^{(v)}(\mu, \varphi, \bar{k}, \tau) = \frac{3\epsilon'}{32\pi} \int_{-1}^{1} dv \int_{0}^{2\pi} d\phi' \mathcal{F}^{(v)}(\mu, v, \varphi, \phi', \alpha, \beta), \]  
\[ \mathcal{L}_U^{(v)}(\mu, \varphi, \bar{k}, \tau) = \frac{3\epsilon'}{32\pi} \int_{-1}^{1} dv \int_{0}^{2\pi} d\phi' \mathcal{F}^{(v)}(\mu, v, \varphi, \phi', \alpha, \beta), \]
\[ \mathcal{L}_V^{(v)}(\mu, \varphi, \tilde{k}, \tau) = \frac{3\varepsilon'}{32\pi} \int_{-1}^{1} dv \int_{0}^{2\pi} d\varphi' \mathcal{F}_V^{(v)}(\mu, v, \varphi, \alpha, \beta), \]  

(5.4)

where \( v_{i}^{(v)} \) denotes the vector component of the baryon velocity field. The polarizations of the baryon velocity will follow the same kind of decomposition illustrated for the vector of the geometry in equation (3.27). The relative directions of the magnetic field intensity and of the photon propagation determine the polarization of the outgoing radiation. Following the strategy described in section 2 the direction of propagation of the relic vector wave can be fixed and the direction of the magnetic field varied at wish.

Consider first the case where the magnetic field is oriented along the same direction of the vector wave and suppose, without loss of generality, that the vector propagates along \( k = \hat{z} \). Since, in this case, \( \alpha = \beta = 0 \) and equations (A.1), (A.2), (A.3) and (A.4) will lead, respectively, to the following matrix elements:

\[ M_{11}(\mu, v, \varphi, \varphi') = \zeta \mu v \Lambda_1 \cos(\varphi' - \varphi) - \sqrt{1 - v^2} \Lambda_3 - i \Lambda_2 f_\xi \mu v \sin(\varphi' - \varphi), \]
\[ M_{12}(\mu, v, \varphi, \varphi') = -\zeta \mu \Lambda_1 \sin \Delta \varphi - i \Lambda_2 f_\xi \mu \cos(\varphi' - \varphi), \]
\[ M_{21}(\mu, v, \varphi, \varphi') = \zeta v \Lambda_1 \sin \Delta \varphi + i f_\xi \Lambda_2 v \cos(\varphi' - \varphi), \]
\[ M_{22}(\mu, v, \varphi, \varphi') = \zeta \Lambda_1 \cos(\varphi' - \varphi) - i f_\xi \Lambda_2 \zeta \sin(\varphi' - \varphi). \]

(5.5)

Using equation (5.5) the source terms of equations (5.1), (5.2), (5.3) and (5.4) can be computed. The explicit form of equations (5.1)–(5.4) is rather lengthy: instead of writing all the equations, equation (5.1) will just be written for illustration with the purpose of demonstrating how the different vector polarizations induce a specific azimuthal dependence in the vector brightness perturbations. Equation (5.1) written in the basis of the linear polarizations reads

\[ \partial_\tau \Delta_I^{(v)} + (ik \mu + \varepsilon') \Delta_I^{(v)} + i \mu \sqrt{1 - \mu^2} \{ \cos \varphi \partial_\tau W_a + \sin \varphi \partial_\tau W_b \} = \varepsilon' \sqrt{1 - \mu^2} \{ \cos \varphi v_a + \sin \varphi v_b \} + \frac{3\varepsilon'}{32\pi} \int_{-1}^{1} dv \int_{0}^{2\pi} d\varphi' \mathcal{F}_I^{(v)}(\mu, v, \varphi, \varphi', \alpha, \beta), \]

(5.6)

where, according to equation (A.7), the integrand of the source term acts on the vector components of the various brightness perturbations and it is given by

\[ \mathcal{F}_I^{(v)}(\mu, v, \varphi, \varphi', \alpha, \beta) = T_{I1}(\mu, v, \varphi, \varphi', \alpha, \beta)\Delta_I^{(v)}(\varphi, \varphi') + T_{I2}(\mu, v, \varphi, \varphi', \alpha, \beta)\Delta_\varphi^{(v)}(\varphi, \varphi') + T_{I3}(\mu, v, \varphi, \varphi', \alpha, \beta)\Delta_\varphi^{(v)}(\varphi, \varphi') + T_{I4}(\mu, v, \varphi, \varphi', \alpha, \beta)\Delta_{\varphi\varphi}^{(v)}(\varphi, \varphi'). \]

(5.7)

After inspection of all the four expressions appearing in equations (5.1), (5.2), (5.3) and (5.4) it can be checked that the consistent ansatz for the four brightness perturbations is given by

\[ \Delta_I^{(v)}(\varphi, \mu, k, \tau) = \sqrt{1 - \mu^2} \{ \cos \varphi M_\mu(k, \tau) + \sin \varphi M_\nu(k, \tau) \}, \]

(5.8)

\[ \Delta_\varphi^{(v)}(\varphi, \mu, k, \tau) = \mu \sqrt{1 - \mu^2} \{ \cos \varphi N_\mu(k, \tau) + \sin \varphi N_\nu(k, \tau) \}, \]

(5.9)

\[ \Delta_\varphi^{(v)}(\varphi, \mu, k, \tau) = \sqrt{1 - \mu^2} \{ - \sin \varphi N_\mu(k, \tau) + \cos \varphi N_\nu(k, \tau) \}, \]

(5.10)

\[ \Delta_{\varphi\varphi}^{(v)}(\varphi, \mu, k, \tau) = \sqrt{1 - \mu^2} \{ \cos \varphi V_\mu(k, \tau) + \sin \varphi V_\nu(k, \tau) \}. \]

(5.11)

The equations obeyed by \( M_\mu, N_\mu \) and \( V_\mu \) are the same as the ones obeyed by \( M_\nu, N_\nu \) and \( V_\nu \) and they can be written, for a generic linear polarization, as

\[ \partial_\tau M + (ik \mu + \varepsilon') M + i \mu \partial_\tau W = \varepsilon' \nu + \varepsilon' \mu \zeta \Lambda_1 \Lambda_3 \Sigma_1^{(v)} + \varepsilon' \mu f_\xi \Lambda_2 \Lambda_3 \Sigma_2^{(v)}, \]

(5.12)
By expanding equation (4.7), \( \partial_t N + (ik\mu + e')N = \epsilon' \xi \Lambda_1 \Lambda_3 \Sigma_1^{(v)} + \epsilon' f_2 \Lambda_2 \Lambda_3 \Sigma_2^{(v)}, \)

\( \partial_t V + (ik\mu + e')V = \epsilon' f_2 \xi \Lambda_2 \Lambda_3 \Sigma_1^{(v)} + \epsilon' \xi \Lambda_1 \Lambda_3 \Sigma_2^{(v)}, \)

where the two newly defined source functions \( \Sigma_1^{(v)} \) and \( \Sigma_2^{(v)} \) are given by

\[
\Sigma_1^{(v)}(k, \tau) = \frac{3}{8} \int_{-1}^{1} [v(v^2 - 1) M(k, v, \tau) + (v^4 - 1) N(k, v, \tau)] \, dv,
\]

\[
\Sigma_2^{(v)}(k, \tau) = \frac{3}{8} \int_{-1}^{1} (v^2 - 1) / V(k, v, \tau) \, dv.
\]

The source functions appearing in equations (5.15) and (5.16) can be made more explicit by expanding \( M(v, k, \tau), N(v, k, \tau) \) and \( V(v, k, \tau) \) with the same conventions employed in equation (4.7):

\[
M(v, k, \tau) = \sum_\ell (-i)^\ell (2\ell + 1) P_\ell(v) M_\ell(k, \tau),
\]

\[
N(v, k, \tau) = \sum_\ell (-i)^\ell (2\ell + 1) P_\ell(v) N_\ell(k, \tau),
\]

\[
V(v, k, \tau) = \sum_\ell (-i)^\ell (2\ell + 1) P_\ell(v) V_\ell(k, \tau),
\]

where, following the same conventions of equation (4.7), \( M_\ell, N_\ell \) and \( V_\ell \) denote the \( \ell \)th multipole of the corresponding quantity. The result of the integration over \( v \) is therefore

\[
\Sigma_1^{(i)}(k, \tau) = \frac{6}{35} N_2 - \frac{3}{7} N_2 - \frac{N_0}{6} + \frac{3}{10} i (M_1 + M_3),
\]

\[
\Sigma_2^{(i)}(k, \tau) = -\frac{V_2}{2} - \frac{V_4}{4}.
\]

The same considerations developed in the basis of the linear vector polarizations can be repeated in the case of the left and right polarized waves. Bearing in mind equation (3.30), equations (5.8)–(5.11) can be written as

\[
\Delta_J^{(v)}(\psi, \mu, k, \tau) = 2i \sqrt\frac{\pi}{3} \left[ Y_1^{-1}(\mu, \psi) M_L(k, \tau) - Y_1^{-1}(\mu, \psi) M_R(k, \tau) \right],
\]

\[
\Delta_Q^{(v)}(\psi, \mu, k, \tau) = 2 \mu \sqrt\frac{\pi}{3} \left[ Y_1^{-1}(\mu, \psi, k, \tau) N_L(k, \tau) - Y_1^{-1}(\mu, \psi) N_R(k, \tau) \right],
\]

\[
\Delta_U^{(v)}(\psi, \mu, k, \tau) = -2i \sqrt\frac{\pi}{3} \left[ Y_1^{-1}(\mu, \psi) N_L(k, \tau) + Y_1^{-1}(\mu, \psi, k, \tau) N_R(k, \tau) \right],
\]

\[
\Delta_V^{(v)}(\psi, \mu, k, \tau) = 2 \sqrt\frac{\pi}{3} \left[ Y_1^{-1}(\mu, \psi) V_L(k, \tau) - Y_1^{-1}(\mu, \psi) V_R(k, \tau) \right].
\]

For sufficiently small angular scales (i.e. for sufficiently large multipoles) the microwave sky degenerates into a plane. In this situation microwave photons propagate, for all practical purposes, along the \( \hat{z} \) axis and instead of the spherical decomposition based on spherical harmonics one can safely use a plane-wave decomposition. Since the wavelength of the
In this case we can already expect, in comparison with the situation then, in the language of equation (2 hereunder. Should however be scrutinized more carefully and this is the purpose of the discussion reported hereunder.

Suppose that the direction of propagation of the vector wave is not parallel to the magnetic field direction but orthogonal. If the relic vector propagates along the magnetic field direction, then, in the language of equation (2.5), \( \hat{k} \parallel \hat{e}_3 \) implying \( \alpha = \beta = 0 \). If the relic vector propagates orthogonally to the magnetic field direction then we can set \( \alpha = \beta = -\pi/2 \) implying that \( \hat{k} \perp \hat{e}_3 \). The direction of \( \hat{k} \) will still be chosen to be the \( \hat{z} \) axis so that \((\hat{k} \cdot \hat{n}) = \mu = \cos \delta \). In the case \( \alpha = \beta = -\pi/2 \) equations (A.1), (A.2), (A.3) and (A.4) read

\[
M_{11}(\mu, \nu, \nu') = \zeta \Lambda_1 \sqrt{1 - \mu^2} \sqrt{1 - \nu^2} + \zeta \Lambda_1 \mu \nu \cos \nu' \cos \nu - \Lambda_3 \mu \nu \sin \nu \sin \nu' + i f_e \Lambda_2 \mu \nu \sin \nu' - \mu \sqrt{1 - \nu^2} \cos \nu \sin \nu',
\]

\[
M_{12}(\mu, \nu, \nu') = -\zeta \mu \Lambda_1 \cos \nu' - \Lambda_3 \mu \cos \nu' \sin \nu - i f_e \Lambda_2 \sqrt{1 - \mu^2} \cos \nu',
\]

\[
M_{21}(\mu, \nu, \nu') = -\Lambda_3 \nu \cos \nu' - \zeta \Lambda_1 \nu \cos \nu' \sin \nu + i f_e \Lambda_2 \sqrt{1 - \nu^2},
\]

\[
M_{22}(\mu, \nu, \nu') = \zeta \Lambda_1 \sin \nu' \sin \nu - \Lambda_3 \cos \nu' \cos \nu.
\]

In this case we can already expect, in comparison with the situation \( \hat{k} \parallel \hat{e}_3 \), that the transport equations differ depending upon the specific vector polarization. The solution of the system can indeed be written as

\[
\Delta_I^{(v)}(\phi, \mu, k, \tau) = \sqrt{1 - \mu^2} \cos \phi M_{a}(k, \tau) + \sin \phi M_b(k, \tau),
\]

\[
\Delta_Q^{(v)}(\phi, \mu, k, \tau) = \mu \sqrt{1 - \mu^2} \cos \phi N_a(k, \tau) + \sin \phi N_b(k, \tau),
\]

\[
\Delta_U^{(v)}(\phi, \mu, k, \tau) = \sqrt{1 - \mu^2} \sin \phi N_a(k, \tau) + \cos \phi N_b(k, \tau),
\]

\[
\Delta_V^{(v)}(\phi, \mu, k, \tau) = \sqrt{1 - \mu^2} \sin \phi V_a(k, \tau) + \mu V_b(k, \tau).
\]

The azimuthal factorization of equations (5.30)–(5.33) is not arbitrary and it is dictated by the specific form of the system of equations (5.1)–(5.4) in the case when \( \hat{k} \perp \hat{e}_3 \). Starting with the polarization \( W_b \) the corresponding evolution equations for \( M_a, N_a \) and \( V_a \) are given by

\[
\partial_\tau M_a + (ik \mu + \epsilon') M_a + i \mu \partial_\tau W_a - \epsilon' v_a = -\epsilon' \mu \zeta^2 (\Lambda_1^2 - f_e^2 \Lambda_2^2) \Sigma_{1a}^{(v)},
\]

\[
\partial_\tau N_a + (ik \mu + \epsilon') N_a = -\epsilon' \mu \zeta^2 (\Lambda_1^2 - f_e^2 \Lambda_2^2) \Sigma_{1a}^{(v)},
\]

\[
\partial_\tau V_a + (ik \mu + \epsilon') V_a = 0,
\]

where

\[
\Sigma_{1a}^{(v)} = \frac{3}{8} \int_{-1}^{1} dv [v^2 - 1] M_a + (v^4 - 1) N_a.
\]

Consider then the case of the polarization \( W_b \). The evolution equations are, in this second case,

\[
\partial_\tau M_b + (ik \mu + \epsilon') M_b + i \mu \partial_\tau W_b - \epsilon' v_b = \epsilon' \mu \zeta \Lambda_1 \Lambda_3 \Sigma_{1b}^{(v)} - \frac{3}{4} f_e \zeta \Lambda_2 \Lambda_3 \mu \int_{-1}^{1} v^2 V_b,
\]

\[
\partial_\tau N_b + (ik \mu + \epsilon') N_b = \epsilon' \zeta \Lambda_1 \Lambda_3 \Sigma_{1b}^{(v)} = \frac{3}{4} f_e \zeta \Lambda_2 \Lambda_3 \int_{-1}^{1} v^2 V_b dv,
\]
\[ \partial_t \mathcal{V}_b + (i k \mu + e') \mathcal{V}_b = e' f_{\delta \mu \xi} \Lambda^2 \Lambda^3 \Sigma_{ib}^{(v)} - \frac{3}{4} f_{\xi} \Lambda_1 \Lambda_3 \mu \int_{-1}^{1} v^2 \mathcal{V}_b \, dv, \] (5.40)

where

\[ \Sigma_{ib}^{(v)} = \frac{3}{8} \int_{-1}^{1} dv [v(v^2 - 1) M_b + (v^4 - 1) N_b]. \] (5.41)

The two sets of equations reported in equations (5.34)–(5.36) and in equations (5.37)–(5.40) show various interesting features which can be summarized as follows:

- if \( \hat{\epsilon}_3 \parallel \hat{k} \) (i.e. \( \alpha = \beta = 0 \)) the evolution equations of the two vector polarizations are independent insofar as they can be given different initial conditions prior to decoupling but the evolution equations of the corresponding brightness perturbations are the same;
- if \( \hat{\epsilon}_3 \perp \hat{k} \) (e.g. \( \alpha = \beta = -\pi/2 \)) the two linear polarizations are equally independent but obey different evolution equations as it is clear by comparing equations (5.34)–(5.36) with equations (5.38)–(5.40);
- the equation for the \( a \)-vector polarization (i.e. equations (5.34)–(5.36)) lead to linear and circular photon polarizations which are a factor \( \mathcal{O}(f_{\epsilon}) \) smaller than the corresponding equations for the \( b \)-vector polarization (see equations (5.37)–(5.40)).

When \( \hat{k} \perp \hat{\epsilon}_3 \) and \( \hat{k} = \hat{z} \) we also have that \( \hat{a} = \hat{x} \) and \( \hat{b} = \hat{y} \). But if \( \alpha = \beta = -\pi/2 \), then \( \hat{\epsilon}_3 = \hat{y} \). Therefore, the amount of magnetically induced linear and circular photon polarization is larger when the magnetic field and the vector polarization are oriented along the same direction. In figure 2 where the geometric set-up of the vector problem is summarized. The wiggly line represents pictorially a vector wave propagating in the direction \( \hat{k} \) which has been taken to be aligned with the \( \hat{z} \) axis. Always in figure 2 the shaded plane denotes the polarization plane of the vector wave spanned by the two unit vectors \( \hat{a} \) and \( \hat{b} \). Finally \( \hat{n} \) denotes the direction of propagation of the photons. If the direction of the magnetic field is parallel to the direction in which the vector modes propagate (thick arrow in figure 2), the photons do not inherit a computable amount of circular polarization and, furthermore, the two linear vector polarizations will lead to the same transport equations for the brightness perturbations.

Conversely, if the magnetic field is parallel to one of the two vector polarizations (thick dashed arrows in figure 2) the transport equations for the two linear vector polarizations will be different. If the linear vector polarization is aligned with the magnetic field intensity (for instance \( \hat{B} \parallel \hat{b} \)) the transport equations for \( M_b, N_b \) and \( \mathcal{V}_b \) will lead to a \( V \)-mode polarization larger than the one generated by the other vector polarization and described in terms of \( M_b, N_b, \) and \( \mathcal{V}_b \).

The source terms for the evolution equations of the vector modes can be averaged over the orientations of the magnetic field, i.e.

\[ \mathcal{L}_{I}^{(v)}(\mu, \varphi, \tilde{k}, \tau) = \epsilon^{c} \hat{n} \mathcal{V}_{i}^{(v)}, \] (5.42)

\[ \mathcal{L}_{Q}^{(v)}(\mu, \varphi, \tilde{k}, \tau) = \frac{3 \epsilon^{c}}{128 \pi^{2}} \int_{-1}^{1} dv \int_{0}^{2\pi} d\varphi' \int_{0}^{\pi} \sin \beta \, d\beta \int_{0}^{2\pi} d\alpha \mathcal{F}_{Q}^{(v)}(\mu, \nu, \varphi', \alpha, \beta), \] (5.43)

\[ \mathcal{L}_{U}^{(v)}(\mu, \varphi, \tilde{k}, \tau) = \frac{3 \epsilon^{c}}{128 \pi^{2}} \int_{-1}^{1} dv \int_{0}^{2\pi} d\varphi' \int_{0}^{\pi} \sin \beta \, d\beta \int_{0}^{2\pi} d\alpha \mathcal{F}_{U}^{(v)}(\mu, \nu, \varphi', \alpha, \beta), \] (5.44)

\[ \mathcal{L}_{V}^{(v)}(\mu, \varphi, \tilde{k}, \tau) = \frac{3 \epsilon^{c}}{128 \pi^{2}} \int_{-1}^{1} dv \int_{0}^{2\pi} d\varphi' \int_{0}^{\pi} \sin \beta \, d\beta \int_{0}^{2\pi} d\alpha \mathcal{F}_{V}^{(v)}(\mu, \nu, \varphi', \alpha, \beta). \] (5.45)
The direct computation of the averaged source terms leads to the same expression for both vector polarizations. Denoting with $\mathcal{M}$ either $\mathcal{M}_a$ or $\mathcal{M}_b$ (and similarly for $\mathcal{N}$ and $\mathcal{V}$), equations (5.42), (5.43), (5.44) and (5.45) lead to the following triplet of equations:

$$
\partial_\tau \mathcal{M} + (ik\mu + \epsilon')\mathcal{M} + i\mu\partial_\tau W - \epsilon' v = \frac{\mu}{15} \epsilon' \left[ (5f_\epsilon^2 \Lambda_2^2 - 7\Lambda_1^2) + 6\zeta \Lambda_1 \Lambda_3 - 2\Lambda_2^2 \right] \Sigma_1^{(v)},
$$

(5.46)

$$
\partial_\tau \mathcal{N} + (ik\mu + \epsilon')\mathcal{N} = \frac{\mu}{15} \epsilon' \left[ (5f_\epsilon^2 \Lambda_2^2 - 7\Lambda_1^2) + 6\zeta \Lambda_1 \Lambda_3 - 2\Lambda_2^2 \right] \Sigma_1^{(v)},
$$

(5.47)

$$
\partial_\tau \mathcal{V} + (ik\mu + \epsilon')\mathcal{V} = \frac{\epsilon'}{3} \zeta \left[ 2\Lambda_1 \Lambda_3 - \zeta \left( \Lambda_2^3 + f_\epsilon^2 \Lambda_2^2 \right) \right] \Sigma_2^{(v)}.
$$

(5.48)

As in the case of equations (4.28)–(4.31) if the magnetic field has a predominant direction over typical scales comparable with the wavelengths of the scattered photons, the circular polarization of the photons induced by the vector modes is larger than in the case where, over the same physical scales, the magnetic field does not have a specific orientation. This conclusion can be reached by comparing equations (5.46), (5.47) and (5.48) to equations (5.37)–(5.40) obtained in the case when the magnetic field is oriented along one of the two polarizations of the relic vector.

It must be finally noticed that equations (5.46), (5.47) and (5.48) are fully analog to equations (4.28), (4.29), (4.30) and (4.31) with the only difference that while equations (5.46)–(5.48) are exact, equations (4.28)–(4.31) are the result of an expansion where the ion mass is neglected in comparison with the electron mass; the tenets of the latter expansion have been carefully spelled out in equation (4.5). The expansion of equation (4.5) can also be used in equations (5.46)–(5.48) with the result that

$$
\partial_\tau \mathcal{M} + (ik\mu + \epsilon')\mathcal{M} + i\mu\partial_\tau W - \epsilon' v = -\mu \epsilon' \left( 1 - \frac{11}{15} f_\epsilon^2 \right) \Sigma_1^{(v)},
$$

(5.49)

$$
\partial_\tau \mathcal{N} + (ik\mu + \epsilon')\mathcal{N} = -\mu \epsilon' \left( 1 - \frac{11}{15} f_\epsilon^2 \right) \Sigma_1^{(v)}.
$$

(5.50)
\[ \partial_t \mathcal{V} + (ik\mu + \epsilon') \mathcal{V} = -\epsilon'(1 - f_\epsilon^2) \Sigma_2^{(v)}. \] (5.51)

In the case of the scalar modes, in contrast, we are somehow forced on the approximate result since the exact expressions are rather lengthy. A final point should be mentioned. Why the equations for the scalar modes involve four different equations whereas the equations for the vectors involve instead only three equations? The reason for this is that the scalar equations also include the explicit equation for \( \Delta_{U}^{(\alpha)} \). Both \( \Delta_{U}^{(\alpha)} \) and \( \Delta_{\mu}^{(\alpha)} \) are not invariant under two-dimensional rotations around the direction of propagation of the radiation (i.e. \( \hat{n} \)). This means that we can always choose \( \Delta_{U}^{(\alpha)} = 0 \).

6. Tensor modes

Recalling the notations introduced in equations (3.31)–(3.34), the evolution equations for the tensor components of the brightness perturbations shall be written, in general terms, as

\[ \mathcal{L}_I^{(\alpha)}(\mu, \nu, \tilde{k}, \tau) = \frac{3\epsilon'}{32\pi} \int_{-1}^{1} d\nu' \int_{0}^{2\pi} d\phi' \mathcal{F}_I^{(\alpha)}(\mu, \nu, \phi', \alpha, \beta), \] (6.1)

\[ \mathcal{L}_Q^{(\alpha)}(\mu, \nu, \tilde{k}, \tau) = \frac{3\epsilon'}{32\pi} \int_{-1}^{1} d\nu' \int_{0}^{2\pi} d\phi' \mathcal{F}_Q^{(\alpha)}(\mu, \nu, \phi', \alpha, \beta), \] (6.2)

\[ \mathcal{L}_U^{(\alpha)}(\mu, \nu, \tilde{k}, \tau) = \frac{3\epsilon'}{32\pi} \int_{-1}^{1} d\nu' \int_{0}^{2\pi} d\phi' \mathcal{F}_U^{(\alpha)}(\mu, \nu, \phi', \alpha, \beta), \] (6.3)

\[ \mathcal{L}_V^{(\alpha)}(\mu, \nu, \tilde{k}, \tau) = \frac{3\epsilon'}{32\pi} \int_{-1}^{1} d\nu' \int_{0}^{2\pi} d\phi' \mathcal{F}_V^{(\alpha)}(\mu, \nu, \phi', \alpha, \beta). \] (6.4)

Consider first the case where the propagation of the relic graviton is parallel to the direction of the magnetic field intensity. The direction of propagation of the tensor wave can be chosen, without loss of generality, as \( \tilde{k} = \hat{z} \). Therefore, equation (2.5) implies that \( \alpha = \beta = 0 \), i.e. \( \tilde{k} \parallel \hat{z} \). To illustrate the azimuthal dependence of the problem it is instructive to write down equation (6.1) in explicit terms:

\[ \partial_t \Delta_{I}^{(\alpha)} + (ik\mu + \epsilon') \Delta_{I}^{(\alpha)} - \frac{1}{2}(1 - \mu^2)[\cos 2\phi \partial_\tau h_{\tilde{z}} + \sin 2\phi \partial_\tau h_{\tilde{z}}] \]

\[ = \frac{3\epsilon'}{32\pi} \int_{-1}^{1} d\nu \int_{0}^{2\pi} d\phi' \mathcal{F}_I^{(\alpha)}(\mu, \nu, \phi', \alpha, \beta), \] (6.5)

where, according to equation (A.7), the integrand of the source term acts on the vector components of the various brightness perturbations and it is given by

\[ \mathcal{F}_I^{(\alpha)}(\mu, \nu, \phi', \alpha, \beta) = T_{I\mu}(\mu, \nu, \phi', \alpha, \beta)\Delta_{I}^{(\alpha)}(\nu, \phi') + T_{I\nu}(\mu, \nu, \phi', \alpha, \beta)\Delta_{I}^{(\alpha)}(\nu, \phi') \]

\[ + T_{I\nu}(\mu, \nu, \phi', \alpha, \beta)\Delta_{I}^{(\alpha)}(\nu, \phi'), \] (6.6)

The remaining three equations (i.e. equations (6.2), (6.3) and (6.4)) have a similar structure but the contribution of the tensor modes of the geometry is absent. The azimuthal dependence can be decoupled from the radial dependence and the brightness perturbations will be

\[ \Delta_{I}^{(\alpha)}(\mu, \nu, \tilde{k}, \tau) = (1 - \mu^2)[\cos 2\phi Z_{\tilde{z}}(\mu, \nu, \tilde{k}, \tau) + \sin 2\phi Z_{\tilde{z}}(\mu, \nu, \tilde{k}, \tau)], \] (6.7)

\[ \Delta_{Q}^{(\alpha)}(\mu, \nu, \tilde{k}, \tau) = (1 + \mu^2)[\cos 2\phi T_{\tilde{z}}(\mu, \nu, \tilde{k}, \tau) + \sin 2\phi T_{\tilde{z}}(\mu, \nu, \tilde{k}, \tau)], \] (6.8)

\[ \Delta_{U}^{(\alpha)}(\mu, \nu, \tilde{k}, \tau) = 2\mu[-\sin 2\phi T_{\tilde{z}}(\mu, \nu, \tilde{k}, \tau) + \cos 2\phi T_{\tilde{z}}(\mu, \nu, \tilde{k}, \tau)], \] (6.9)
\[ \Delta_V^{(1)}(\varphi, \mu, k, \tau) = 2\mu \{ \cos 2\varphi V_0(\mu, k, \tau) + \sin 2\varphi S_0(\mu, k, \tau) \}. \] (6.10)

In the case \( \hat{k} \parallel \hat{e}_3 \) the symmetry of the system implies necessarily an ansatz in the form of equations (6.7), (6.8), (6.9) and (6.10). Starting with the explicit form of equation (6.5), it is immediately clear that the form of \( \Delta_V^{(1)}(\varphi, \mu, k, \tau) \) is constrained by the \( \varphi \) dependence appearing on the left-hand side of equation (6.5). The integrand on the right-hand side of equation (6.5), i.e., equation (6.6), also contains \( \Delta_V^{(1)}(\varphi', v, k, \tau) \) whose explicit form is univocally determined by observing that the integral over \( \varphi' \) must match with the \( \varphi \) dependence appearing on the left-hand side of equation (6.5). But the obtained ansatz for \( \Delta_V^{(1)}(\varphi, \mu, k, \tau) \) and \( \Delta_V^{(1)}(\varphi, \mu, k, \tau) \) can be inserted back into equation (6.2): this step will constructively determine the explicit form of \( \Delta_V^{(1)}(\varphi, \mu, k, \tau) \). Equation (6.3) will finally determine the explicit form of \( \Delta_V^{(1)}(\varphi, \mu, k, \tau) \) whose \( \varphi \) dependence will have to be consistent with equation (6.4). The result of this procedure, expressed by equations (6.7), (6.8), (6.9) and (6.10), determines the radial evolution and, in particular, the following set of equations [27]:

\[
\begin{align*}
\partial_r Z + (ik \mu + \epsilon') Z &= -\frac{1}{2} \partial_r h = \epsilon' \xi^2(\omega) \left[ \Lambda_1^1(\omega) - f_2^2(\omega) \Lambda_2^1(\omega) \right] \Sigma^{(1)}, \\
\partial_r T + (ik \mu + \epsilon') T + \epsilon' T &= -\epsilon' \xi^2(\omega) \left[ \Lambda_1^1(\omega) - f_2^2(\omega) \Lambda_2^1(\omega) \right] \Sigma^{(1)}, \\
\partial_r S + (ik \mu + \epsilon') S &= 0,
\end{align*}
\] (6.11-6.12)

where \( Z, T \) and \( S \) denote either the \( \oplus \) or the \( \otimes \) polarization. By expanding \( Z, T \) in series of Legendre polynomials

\[
\begin{align*}
Z(v, k, \tau) &= \sum_\ell (-i)^\ell (2\ell + 1) P_\ell(v) Z_\ell(k, \tau), \\
T(v, k, \tau) &= \sum_\ell (-i)^\ell (2\ell + 1) P_\ell(v) T_\ell(k, \tau),
\end{align*}
\] (6.14-6.15)

the source term \( \Sigma^{(1)} \) can also be expressed as

\[
\Sigma^{(1)} = \frac{3}{32} \int_0^1 dv [(1 - v^2)^2 Z(v) - (1 + v^2)^2 T(v) - 4v^2 T(v)]
= \frac{3}{70} Z_4 + \frac{Z_2}{7} - \frac{Z_0}{10} - \frac{3}{70} T_4 + \frac{6}{7} T_2 - \frac{3}{5} T_0,
\] (6.16)

where, as usual, \( Z_\ell \) and \( T_\ell \) denote the \( \ell \)th multipoles of the corresponding functions. The results obtained in equations (6.11), (6.12) and (6.13) with the partial treatment of the tensor modes developed in [27]. As in the case of the vectors instead of working with the linear polarizations of the relic gravitons, we could as well work with the circular polarization. The results obtained so far can be easily translated to the case when the relic gravitons are circularly polarized, always assuming that \( \hat{k} \parallel \hat{e}_3 \), i.e. that the direction of propagation of the relic gravitons is parallel to the orientation of the magnetic field intensity.

As expected from the vector case, when the relic tensor propagates orthogonally to the magnetic field direction, the two tensor polarizations will obey different equations but, at the same time, there will be differences in comparison with the vector case. The two polarizations of the relic gravitons when projected along the directions of the photon propagation will lead to a quadrupole term. The \( \oplus \) and \( \otimes \) polarization of the tensor mode are illustrated in figure 3 which should be compared with the shaded area of figure 2. When the \( \oplus \) polarization
Figure 3. The ⊕ and ⊗ polarizations are illustrated, respectively, with full and dashed lines. The direction of propagation of the wave is not shown and it is orthogonal to the plane spanned by \( \hat{a} \) and \( \hat{b} \). When \( \vec{B} \parallel \hat{k} \) the magnetic field is oriented perpendicularly to the plane of the figure.

propagates orthogonally to the magnetic field direction its evolution equations are given by

\[
\partial_\tau Z^{\oplus} + (ik\mu + \epsilon')Z^{\oplus} - \frac{1}{2}\partial_\tau h^{\oplus} = \frac{\epsilon'}{2}(\chi^2 \Lambda^2_1 + \Lambda^2_2)\Sigma^{(i)} + \frac{\epsilon'}{2(1 - \mu^2)\cos 2\varphi} \\
\times \left[ \Lambda^2_1(1 + \mu^2) - \chi^2(\Lambda^2_1(1 + \mu^2) - 2f'\Lambda^2_2(\mu^2 - 1)) \right] \Sigma^{(i)},
\]

(6.17)

\[
\partial_\tau T^{\oplus} + (ik\mu + \epsilon')T^{\oplus} = -\frac{\epsilon'}{2}(\chi^2 \Lambda^2_1 + \Lambda^2_2)\Sigma^{(i)} - \frac{\epsilon'}{2(1 - \mu^2)\cos 2\varphi} \\
\times \left[ \Lambda^2_1(1 + \mu^2) - \chi^2(\Lambda^2_1(1 + \mu^2) - 2f'\Lambda^2_2(\mu^2 - 1)) \right] \Sigma^{(i)},
\]

(6.18)

\[
\partial_\tau S^{\oplus} + (ik\mu + \epsilon')S^{\oplus} = -\frac{3}{16}f'\xi \Lambda_1 \Lambda_3 \frac{1 - \mu^2}{\mu} \frac{\sin \varphi}{\cos 2\varphi} \Sigma^{(i)},
\]

(6.19)

where it has been assumed that \( \alpha = \beta = -\pi/2 \). When the ⊗ polarization propagates orthogonally to the magnetic field direction its evolution equations are given by

\[
\partial_\tau Z^{\otimes} + (ik\mu + \epsilon')Z^{\otimes} - \frac{1}{2}\partial_\tau h^{\otimes} = -\epsilon' \xi \Lambda_1 \Lambda_3 \Sigma^{(i)},
\]

(6.20)

\[
\partial_\tau T^{\otimes} + (ik\mu + \epsilon')T^{\otimes} = -\epsilon' \xi \Lambda_1 \Lambda_3 \Sigma^{(i)},
\]

(6.21)

\[
\partial_\tau S^{\otimes} + (ik\mu + \epsilon')S^{\otimes} = -\epsilon' f\xi \Lambda_1 \Lambda_3 \frac{\cos \varphi}{\sin 2\varphi} \frac{1 - \mu^2}{\mu} \Sigma^{(i)}.
\]

(6.22)

Equations (6.17), (6.18) and (6.19) can then be compared to equations (6.20), (6.21) and (6.22) recalling that, now the magnetic field is directed along \( \hat{z} \). The two tensor polarizations read \( \tilde{\epsilon}^{\oplus}_{ij} = \langle \hat{a}_i \hat{a}_j - \hat{b}_i \hat{b}_j \rangle \) and \( \tilde{\epsilon}^{\otimes}_{ij} = \langle \hat{a}_i \hat{b}_j + \hat{a}_j \hat{b}_i \rangle \). But since it has been assumed that \( \hat{k} = \hat{z} \)
we shall also have $\hat{a} = (1, 0, 0) = \hat{x}$ and $\hat{b} = (0, 1, 0) = \hat{y}$. The addition of a magnetic field either along $\hat{a}$ or along $\hat{b}$ (i.e. orthogonally to $\hat{k}$) is illustrated in figure 3 for the two tensor polarizations. The polarization $\otimes$ spans the shaded area bounded by the full lines. The polarization $\oplus$ spans the shaded area bounded by the dashed lines. The effect of having an extra source of circular dichroism along the $\hat{a}$ (or along the $\hat{b}$ axis) will be felt by both tensor polarizations as quantitatively established in equations (6.17)–(6.19) and in equations (6.20)–(6.22).

The last step is to compute the evolution equations by averaging the source functions over the directions of the magnetic field:

\[
\mathcal{L}_i^{(0)}(\mu, \nu, \phi, \tau) = \frac{3e'}{128\pi^2} \int_{-1}^{1} d\nu' \int_0^{2\pi} d\phi' \int_0^\pi \sin \beta \, d\beta \int_0^{2\pi} d\alpha \mathcal{F}_i^{(0)}(\mu, \nu, \phi', \alpha, \beta),
\]

(6.23)

\[
\mathcal{L}_Q^{(0)}(\mu, \nu, \phi, \tau) = \frac{3e'}{128\pi^2} \int_{-1}^{1} d\nu' \int_0^{2\pi} d\phi' \int_0^\pi \sin \beta \, d\beta \int_0^{2\pi} d\alpha \mathcal{F}_Q^{(0)}(\mu, \nu, \phi', \alpha, \beta),
\]

(6.24)

\[
\mathcal{L}_U^{(0)}(\mu, \nu, \phi, \tau) = \frac{3e'}{128\pi^2} \int_{-1}^{1} d\nu' \int_0^{2\pi} d\phi' \int_0^\pi \sin \beta \, d\beta \int_0^{2\pi} d\alpha \mathcal{F}_U^{(0)}(\mu, \nu, \phi', \alpha, \beta),
\]

(6.25)

\[
\mathcal{L}_V^{(0)}(\mu, \nu, \phi, \tau) = \frac{3e'}{128\pi^2} \int_{-1}^{1} d\nu' \int_0^{2\pi} d\phi' \int_0^\pi \sin \beta \, d\beta \int_0^{2\pi} d\alpha \mathcal{F}_V^{(0)}(\mu, \nu, \phi', \alpha, \beta).
\]

(6.26)

The result for the evolution equations of the tensor polarizations with averaged sources is given by

\[
\partial_\tau \mathcal{Z} + (ik\mu + e')\mathcal{Z} - \frac{1}{2}\partial_\tau h = \frac{e'}{15} \left[ \xi^2 (7\Lambda_1^2 - 5f_c^2\Lambda_3^2) - 6\xi\Lambda_1\Lambda_3 + 2\Lambda_3^2 \right] \Sigma^{(0)},
\]

(6.27)

\[
\partial_\tau \mathcal{T} + (ik\mu + e')\mathcal{T} = -\frac{e'}{15} \left[ \xi^2 (7\Lambda_1^2 - 5f_c^2\Lambda_3^2) - 6\xi\Lambda_1\Lambda_3 + 2\Lambda_3^2 \right] \Sigma^{(0)},
\]

(6.28)

\[
\partial_\tau \mathcal{S} + (ik\mu + e')\mathcal{S} = 0.
\]

(6.29)

These results extend and partially correct the results derived in [27]. The correction has to do with the source term of equation (6.29) which vanishes exactly unlike stated in [27] because of an error in the azimuthal integrations. Finally the same comments made in connection with equations (5.46), (5.47) and (5.48) also apply to equations (6.27), (6.28) and (6.29). Both sets of equations are exact in the sense that the averaging procedure has been carried out over the original equations without making use of the expansion (equation (4.5)) which has been instead implemented, for the reason of space, when presenting equations (4.28), (4.29), (4.30) and (4.31). Equations (6.27), (6.28) and (6.29) can also be expanded according to equation (4.5); the result is

\[
\partial_\tau \mathcal{Z} + (ik\mu + e')\mathcal{Z} - \frac{1}{2}\partial_\tau h = e' \left( \frac{3}{5} - \frac{4}{3} f_c^2 \right) \Sigma^{(0)},
\]

(6.30)

\[
\partial_\tau \mathcal{T} + (ik\mu + e')\mathcal{T} = -e' \left( \frac{3}{5} - \frac{4}{3} f_c^2 \right) \Sigma^{(0)},
\]

(6.31)

\[
\partial_\tau \mathcal{S} + (ik\mu + e')\mathcal{S} = 0.
\]

(6.32)

Equations (6.30), (6.31) and (6.32) are the tensor counterpart of equation (5.49), (5.50) and (5.51).
7. Concluding remarks

An arbitrarily oriented magnetic field has been incorporated in the Stokes matrix of last electron–photon scattering. The transport equations for the scalar, vector and tensor components of the brightness perturbations have been derived and studied in various physical situations. In particular, the main results can be summarized as follows:

- in the scalar case the transport equations for arbitrary magnetic field directions are now available in their exact form, in their approximate form and in their averaged form; the results concerning the scalar modes have been mainly presented in section 4;
- the same analysis performed in the scalar case has also been carried out for the vector modes of the geometry (section 5) as well as for the tensor modes (section 6);
- the explicit expressions of all the relevant matrix elements describing the magnetized electron–photon scattering for arbitrary magnetic field direction can be found in the appendix and have been heavily used in the explicit derivations.

In this study the presence of the magnetic field in the scattering processes has been included by using a hybrid approach which is somehow intermediate between the Jones and the Mueller calculus. The components of the Stokes matrix reported in appendix A can be used to compute, if needed, the Mueller matrix for arbitrary magnetic field direction.

In discussing the effects of the magnetic field direction on the transport equations for the vector and for the tensor modes it is practical to investigate, separately, the propagation along different directions. If the relic vector propagates along the same magnetic field direction, the two vector polarizations satisfy the same equation. The same kind of phenomenon occurs for the tensor modes even if the vector and tensor equations are clearly different. Conversely, if the propagation of the tensor and vector modes of the geometry follows an orientation which is orthogonal to the magnetic field direction, the two polarizations obey, in each of the two cases, different equations. As explained in bulk of the paper, the parallel and orthogonal propagations are only two illustrative examples: the general situation which is fully described by the general set of equations derived here.

Many ideas on the nature of primordial magnetogenesis have been put forward in the last 60 years (see [37] for a recent discussion on inflationary magnetogenesis and see [11] for a general review). The true problem before us today is to use CMB physics for gaining physical insights on the possibility that magnetic fields were present prior to matter-radiation equality. The results obtained in this paper can impact on two complementary aspects of the physics of CMB anisotropies which can be summarized, in short, as follows:

- a consistent improvement of the available analytical and numerical tools used for the calculation of magnetized CMB anisotropies;
- the careful assessment of the level of circular polarization induced at last scattering by the presence of a small-scale component of the pre-decoupling magnetic field.

The available codes led so far to the following results:

- a first attempt of estimation of the parameters of the magnetized background either in the context of the $\Lambda$CDM paradigm [15] or in the $w$CDM scenario where the dark energy fluctuations are dynamical [16] insofar as $w \neq -1$ (where $w$ denotes the barotropic index of the dark energy component);
- a first analysis of the induced $V$-mode polarization from the adiabatic mode in the presence of a magnetic field [24, 25];
- a set of constraints on the magnetic field intensity from the measured $V$-mode correlations [28].
The distortions induced by gravitating magnetic fields on the temperature autocorrelations (i.e. the TT correlations) and on the cross-correlations between temperature and polarization (i.e. the TE correlations) can be confronted with the angular power spectra measured by the WMAP satellite and by other CMB experiments\(^6\). It has been shown that comoving magnetic fields exceeding \(3.5 \text{ nG}\) with a typical correlation scale of the order of \(1–10 \text{ Mpc}\) are excluded to \(95\% \text{ CL}\) and for magnetic spectral indices \(n_B = 1.6^{+0.3}_{-0.1}\) [16]. The latter result did assume that the magnetic fields were gravitating but not contributing to the scattering term. The derivations presented here offer then the opportunity of relaxing those assumptions. Another portion of studies touched by the present considerations, as mentioned above, involve the direct determinations of V-mode polarizations by using appropriate instruments with sensitivity to circular polarizations [24, 25]. The current bounds on the V-mode polarization from direct searches are rather old [38, 39] and imply that the normalized V-mode autocorrelation must not exceed a fraction of the mK depending upon the angular scale. These bounds have been jointly analyzed and discussed in [28] where further detail can be found. Improved direct bounds on the V-mode polarization will also provide independent bounds on the small-scale component of the pre-decoupling magnetic field according to the results discussed in [28].

Appendix A. Stokes and Mueller matrices

The explicit form of the Stokes and Mueller matrices for arbitrary orientation of the magnetic field will now be reported. The four distinct entries of the Stokes matrix \(M(\Omega, \Omega', \alpha, \beta)\) appearing in equation (2.14) will now be written in explicit terms:

\[
M_{11}(\Omega, \Omega', \alpha, \beta) = \frac{\zeta \Lambda_1 - \Lambda_3}{2} [\sqrt{1 - \mu^2} \sqrt{1 - v^2} + \mu \nu \cos (\varphi - \alpha) \cos (\varphi' - \alpha)] \\
+ \frac{\zeta \Lambda_1 + \Lambda_3}{2} [\cos 2\beta [\mu \nu \cos (\varphi - \alpha) \cos (\varphi' - \alpha) - \sqrt{1 - \mu^2} \sqrt{1 - v^2}]
+ \sin 2\beta [\mu \nu \sqrt{1 - v^2} \cos (\varphi - \alpha) + \nu \sqrt{1 - \mu^2} \cos (\varphi' - \alpha)]]
+ \zeta \Lambda_1 \mu \nu \sin (\varphi - \alpha) \sin (\varphi' - \alpha) + i f_c \zeta \Lambda_2 [\sin \beta [\mu \nu \sqrt{1 - v^2} \sin (\varphi - \alpha) - \nu \sqrt{1 - \mu^2} \sin (\varphi' - \alpha)]
- \nu \sqrt{1 - \mu^2} \sin (\varphi' - \alpha)] + \mu \nu \cos \beta \sin (\varphi - \varphi')], \tag{A.1}
\]

\[
M_{12}(\Omega, \Omega', \alpha, \beta) = \frac{\Lambda_3 - \Lambda_1 \zeta}{2} \mu \sin (\varphi' - \alpha) \cos (\varphi - \alpha)
- \frac{\Lambda_3 + \Lambda_1 \zeta}{2} [\mu \sin (\varphi' - \alpha) \cos (\varphi - \alpha) \cos 2\beta]
+ \sqrt{1 - \mu^2} \sin (\varphi' - \alpha) \sin 2\beta)] + \zeta \Lambda_1 \mu \sin (\varphi - \alpha) \cos (\varphi' - \alpha)
- i f_c \zeta \Lambda_2 [\mu \cos \beta \cos (\varphi' - \varphi) + \sqrt{1 - \mu^2} \sin \beta \cos (\varphi' - \alpha)]], \tag{A.2}
\]

\[
M_{21}(\Omega, \Omega', \alpha, \beta) = - \frac{\zeta \Lambda_1 + \Lambda_3}{2} \sqrt{1 - v^2} \sin 2\beta \sin (\varphi - \alpha)
+ \frac{1}{4} [\zeta \Lambda_1 + \Lambda_3] [\sin (\varphi + \varphi' - 2\alpha)] - \sin (\varphi' - \varphi)] \]

\(^6\) In [15] the parameter dependence of magnetized CMB observables has been investigated and it has been demonstrated that the cleanest framework for the study of pre-decoupling magnetism is provided, in principle, by the polarization autocorrelations (i.e. the EE correlations) which are, at the moment, the least accurate. The \(B\)-mode polarization is also affected by the magnetic fields through the Faraday effect (see [15, 16] and references therein).
\[
-\frac{v}{4}(\cos \alpha + \Lambda_3)[\sin (\varphi + \varphi') - \sin (\varphi' - \varphi)] \cos 2\beta
\]
\[
if \Lambda_2 \zeta [v \cos \beta \cos (\varphi' - \varphi) + \sqrt{1 - v^2} \sin \beta \cos (\varphi' - \varphi)],
\]
(A.3)

\[
M_{22}(\Omega, \Omega', \alpha, \beta) = \frac{\Lambda_1 + \Lambda_3}{4} \sin 2\alpha(1 - \cos 2\beta) \sin (\varphi' + \varphi) - \Lambda_3 \cos (\varphi' - \varphi)
\]
\[
+ \frac{\Lambda_1 + \Lambda_3}{4}(1 + \cos 2\beta)[\cos (\varphi' - \varphi) - \cos 2\alpha \cos (\varphi' + \varphi)]
\]
\[
+ \frac{\Lambda_1 + \Lambda_3}{2}[\cos (\varphi' - \varphi) + \cos 2\alpha \cos (\varphi' + \varphi)]
\]
\[- if \Lambda_2 \cos \beta \sin (\varphi' - \varphi).
\]
(A.4)

Equations (A.1), (A.2), (A.3) and (A.4) have various interesting limits which depend upon the specific orientation of the magnetic field intensity. In the absence of a magnetic field we have that
\[
\Lambda_1 \rightarrow 1, \quad \Lambda_2 \rightarrow 1, \quad \Lambda_3 \rightarrow 1, \quad \zeta \rightarrow -1, \quad f_e \rightarrow 0.
\]
(A.5)

In the limit defined by equation (A.5), equations (A.1)–(A.4) reduce to
\[
M_{11}(\mu, \varphi, v, \varphi') = -\sqrt{1 - \mu^2}\sqrt{1 - v^2} - \mu v \cos (\varphi' - \varphi),
\]
\[
M_{12}(\mu, \varphi, v, \varphi') = \mu \sin (\varphi' - \varphi), \quad M_{21}(\mu, \varphi, v, \varphi') = -\nu \sin (\varphi' - \varphi),
\]
\[
M_{22}(\mu, \varphi, v, \varphi') = -\cos (\varphi' - \varphi),
\]
(A.6)

where, as already pointed out in section 2, \( \mu = \cos \theta \) and \( v = \cos \theta' \). The integrands appearing in the source terms of equations (2.17), (2.18), (2.19) and (2.20) are given by
\[
\mathcal{F}_J(\Omega, \Omega', \alpha, \beta) = T_{IJ}(\Omega, \Omega', \alpha, \beta) I(\Omega') + T_{IU}(\Omega, \Omega', \alpha, \beta) U(\Omega') + T_{IV}(\Omega, \Omega', \alpha, \beta) V(\Omega'),
\]
(A.7)
\[
\mathcal{F}_Q(\Omega, \Omega', \alpha, \beta) = T_{QI}(\Omega, \Omega', \alpha, \beta) I(\Omega') + T_{QU}(\Omega, \Omega', \alpha, \beta) U(\Omega') + T_{QV}(\Omega, \Omega', \alpha, \beta) V(\Omega'),
\]
(A.8)
\[
\mathcal{F}_U(\Omega, \Omega', \alpha, \beta) = T_{UI}(\Omega, \Omega', \alpha, \beta) I(\Omega') + T_{UV}(\Omega, \Omega', \alpha, \beta) U(\Omega') + T_{UU}(\Omega, \Omega', \alpha, \beta) V(\Omega'),
\]
(A.9)
\[
\mathcal{F}_V(\Omega, \Omega', \alpha, \beta) = T_{VI}(\Omega, \Omega', \alpha, \beta) I(\Omega') + T_{VV}(\Omega, \Omega', \alpha, \beta) U(\Omega') + T_{UV}(\Omega, \Omega', \alpha, \beta) V(\Omega'),
\]
(A.10)

where the matrix elements \( T_{ij} \) are computed in terms of equations (A.1), (A.2), (A.3) and (A.4):
\[
T_{IJ} = 2[|M_{11}|^2 + |M_{12}|^2 + |M_{21}|^2 + |M_{22}|^2],
\]
(A.11)
\[
T_{IQ} = 2[|M_{11}|^2 - |M_{12}|^2 + |M_{21}|^2 - |M_{22}|^2],
\]
(A.12)
\[
T_{IU} = 2[M_{11}M_{12}^* + M_{12}M_{11}^* + M_{21}M_{22}^* + M_{22}M_{21}^*],
\]
(A.13)
\[
T_{IV} = 2[M_{11}M_{21}^* + M_{12}^*M_{11} + M_{22}M_{21}^* - M_{21}M_{22}^*],
\]
(A.14)
\[
T_{QI} = 2[|M_{11}|^2 + |M_{12}|^2 - |M_{21}|^2 - |M_{22}|^2],
\]
(A.15)
\[
T_{QQ} = 2[|M_{11}|^2 - |M_{12}|^2 - |M_{21}|^2 + |M_{22}|^2],
\]
(A.16)
\[ T_{QU} = 2[M_{11}M_{12}^* + M_{12}M_{11}^* - M_{21}M_{22}^* - M_{22}M_{21}^*], \]  
\[ T_{QV} = 2[i(M_{12}M_{11}^* - M_{11}M_{12}^* + M_{21}M_{22}^* + M_{22}M_{21}^*)], \]  
\[ T_{UI} = 2[i(M_{11}M_{12}^* + M_{12}M_{11}^* + M_{21}M_{22}^* + M_{22}M_{21}^*), \]  
\[ T_{UV} = 2[i(M_{12}M_{11}^* - M_{11}M_{12}^* + M_{21}M_{22}^* - M_{22}M_{21}^*), \]  
\[ T_{UV} = 2[M_{11}M_{12}^* + M_{12}M_{11}^* + M_{21}M_{22}^* + M_{22}M_{21}^*]. \]  

In equations (A.11)–(A.26) the explicit dependence upon the six angles has been suppressed only for the sake of simplicity. The components of the incident electric fields in the local frame \( \hat{e}_1, \hat{e}_2 \) and \( \hat{e}_3 \) can be related to the components off the electric field in the three Cartesian directions as

\[
E_1 = \cos \alpha \cos \beta E'_x + \sin \alpha \cos \beta E'_y - \sin \beta E'_z, \\
E_2 = -\sin \alpha E'_x + \cos \alpha E'_y, \\
E_3 = \cos \alpha \sin \beta E'_x + \sin \alpha \sin \beta E'_y + \cos \beta E'_z.
\]

The incident electric fields \( E'_x, E'_y \) and \( E'_z \) can be related, in turn, to their polar components as

\[
E'_x = \cos \theta' \cos \varphi' E'_\rho - \sin \varphi' E'_\varphi, \\
E'_y = \cos \theta' \sin \varphi' E'_\rho + \cos \varphi' E'_\varphi, \\
E'_z = -\sin \theta' E'_\rho,
\]

where, as already spelled out in section 2 the direction of propagation of the incident radiation \( \vec{h} \) coincides with \( \vec{r}' \) and \( (E'_\rho, E'_\varphi) \) are the components of the incident electric field in the spherical basis. The relation between the outgoing and the ingoing electric fields is given by

\[
E_{\rho}(\vartheta, \varphi, \vartheta', \varphi', \alpha, \beta) = \frac{r_c}{r} [M_{11}E_{\rho}(\vartheta, \varphi') + M_{12}E_{\rho}(\vartheta', \varphi')], \\
E_{\varphi}(\vartheta, \varphi, \vartheta', \varphi', \alpha, \beta) = \frac{r_c}{r} [M_{21}E_{\varphi}(\vartheta, \varphi') + M_{22}E_{\varphi}(\vartheta', \varphi')],
\]

where \( M_{ij} \equiv M_{ij}(\vartheta, \varphi, \vartheta', \varphi', \alpha, \beta) \) are given by equations (A.1)–(A.3).

Appendix B. Angular integrations

The angular integrations of the collision term can always be performed either directly or with the help of the appropriate Rayleigh expansion. The results are however rather lengthy. Furthermore, in the case of the vector and of the tensor modes (i.e. sections 5 and 6), the integrations over \( \varphi' \) change depending upon the polarization of the vector (or of the tensor) wave. In the scalar case the results of the angular integration over \( \varphi' \) are given hereunder
following the notation of equation (4.6):

\[
\mathcal{T}_{II} = 2\pi v^2[(3\mu^2 - 1) + f_\nu^2(1 + \mu^2) - 2(1 + \mu^2) \sin^2 (\varphi - \alpha) + \mu \sqrt{1 - \mu^2} \cos (\varphi - \alpha) \sin 2\beta] + 2\pi(3 - \mu^2 + f_\nu^2(1 + \mu^2) - 2 \sin^2 (1 - \mu^2) \cos^2 (\varphi - \alpha) + \mu \sqrt{1 - \mu^2} \cos (\varphi - \alpha) \sin 2\beta)] \\
\] (B.1)

\[
\mathcal{T}_{IQ} = \frac{\pi}{2}(v^2 - 1)[4(3\mu^2 - 1) + f_\nu^2[4(1 - \mu^2) \sin^2 (\varphi - \alpha) + (1 - \mu^2)\cos 2(\varphi - \alpha - \beta) + \cos 2(\varphi - \alpha + \beta)] + 2 \cos 2\beta(1 + 3\mu^2) - 2\mu \sqrt{1 - \mu^2}(\sin (\varphi - \alpha - 2\beta) - \sin (\varphi - \alpha - 2\beta))], \\
\] (B.2)

\[
\mathcal{T}_{IV} = 8\pi f_\nu[2(1 + \mu^2) \cos \beta + \mu \sqrt{1 - \mu^2} \cos (\varphi - \alpha) \sin \beta]. \\
\] (B.3)

\[
\mathcal{T}_{OO} = \frac{\pi}{2}(1 - v^2)[12(1 - \mu^2) + f_\nu^2[2(\mu^2 - 1) - 2(1 + \mu^2) \cos 2(\varphi - \alpha) + (1 + \mu^2)\cos 2(\varphi - \alpha - \beta) + \cos 2(\varphi - \alpha + \beta)) + 6 \cos 2\beta(1 - \mu^2) + 2\mu \sqrt{1 - \mu^2}(\sin (\varphi - \alpha - 2\beta) - \sin (\varphi - \alpha + 2\beta))]. \\
\] (B.5)

\[
\mathcal{T}_{OU} = 0, \\
\] (B.6)

\[
\mathcal{T}_{OV} = 8\pi f_\nu((\mu^2 - 1) \cos \beta + \mu \sqrt{1 - \mu^2} \cos (\varphi - \alpha) \sin \beta], \\
\] (B.8)

\[
\mathcal{T}_{UI} = -2\pi f_\nu^2 \sin (\varphi - \alpha)[v^2[4\mu \cos (\varphi - \alpha) \sin^2 \beta + \sqrt{1 - \mu^2} \sin 2\beta] + \sqrt{1 - \mu^2} \sin 2\beta - 4 \mu \cos (\varphi - \alpha) \sin^2 \beta]] \\
\] (B.9)

\[
\mathcal{T}_{UO} = -4\pi f_\nu^2(v^2 - 1) \sin \beta \sin (\varphi - \alpha)[\sqrt{1 - \mu^2} \cos \beta + 2\mu \cos (\varphi - \alpha) \sin \beta], \\
\] (B.10)

\[
\mathcal{T}_{UV} = 0, \\
\] (B.11)

\[
\mathcal{T}_{VV} = -8\pi f_\nu \sqrt{1 - \mu^2} \sin (\varphi - \alpha) \sin \beta \\
\] (B.12)

\[
\mathcal{T}_{VI} = 4\pi f_\nu(v^2[2\mu \cos \beta - \sqrt{1 - \mu^2} \cos (\varphi - \alpha) \sin \beta] + [2\mu \cos \beta + 3\sqrt{1 - \mu^2} \cos (\varphi - \alpha) \sin \beta]). \\
\] (B.13)

\[
\mathcal{T}_{VO} = 2\pi f_\nu(v^2 - 1)[4\mu \cos \beta - 2\sqrt{1 - \mu^2} \cos (\varphi - \alpha) \sin \beta], \\
\] (B.14)

\[
\mathcal{T}_{VV} = 0, \\
\] (B.15)

\[
\mathcal{T}_{VV} = 8\pi v[\mu + \cos \beta f_\nu^2(\mu \cos \beta + \sqrt{1 - \mu^2} \cos (\varphi - \alpha) \sin \beta)]. \\
\] (B.16)
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