GAGA THEOREMS

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Abstract. We prove a new and unified GAGA theorem. This recovers all analytic and formal GAGA results in the literature, and is also valid in the non-noetherian setting. Our method can also be used to establish various Lefschetz theorems and comparison results for the Fargues–Fontaine curve.

1. Introduction

Let $X$ be a scheme. Frequently associated to $X$ is a morphism

$$c : X \to X,$$

where $X$ is some type of analytic space. When $X$ is proper over $\text{Spec } R$, where $R$ is a noetherian ring (usually connected to the construction of $X$), there is often an induced comparison isomorphism on cohomology of coherent sheaves:

$$H^i(X, F) \simeq H^i(X, c^* F)$$

and an equivalence of abelian categories of coherent sheaves:

$$c^* : \text{Coh}(X) \simeq \text{Coh}(X).$$

Since Serre’s famous paper [GAGA], such results have been called “GAGA theorems”. Restricting these comparison isomorphisms and equivalences to specific subcategories of sheaves (e.g., vector bundles or finite étale algebras) leads to both local and global “Lefschetz Theorems” [SGA2]. We briefly recall some of these phenomena below.

1.1. Archimedean analytification. Assume $X$ is proper over $\text{Spec } \mathbb{C}$. Naturally associated to $X$ is an analytic space $X_{\text{an}}$. This consists of endowing the $\mathbb{C}$-points of $X$ with its Euclidean topology and its sheaf of holomorphic functions. There is an induced morphism of locally ringed spaces $c : X_{\text{an}} \to X$. This is the setting of the original GAGA theorems [GAGA] and [SGA1, XII.4.4].

1.2. Non-archimedean analytification. Assume $X$ is proper over $\text{Spec } K$, where $K$ is a complete non-archimedean field. Then there are various analytifications associated to $X$: the Berkovich $X_{\text{Berk}}$, the adic $X_{\text{adic}}$, and the rigid $X_{\text{rig}}$. There is also a GAGA theorem in this context [Köp74] (see [Con06] for a more recent account).

1.3. Formal completion. Let $Z = V(I) \subseteq X$ be a closed subscheme. Then there is the formal completion $\hat{X}_{/Z}$. The locally ringed space $\hat{X}_{/Z}$ has underlying topological space $Z$ and sheaf of rings $\mathcal{O}_{\hat{X}_{/Z}} = \lim_{\leftarrow n} \mathcal{O}_X/I^n$. If $X$ is proper over a complete noetherian ring, then there is a GAGA theorem [EGA, III].
1.4. Unification. All of these results have previously been proved separately, though their general strategies are very similar. First one treats projective spaces directly via a direct computation of the cohomology of line bundles, then a dévissage is performed using Chow’s Lemma.

The main theorem of this article is that these GAGA results are true much more generally and can be put into a single framework. Essentially all existing results follow very easily from ours (see §9). We state one such result in the noetherian situation for locally ringed \(G\)-spaces.

**Theorem A.** Let \(R\) be a noetherian ring. Let \(X \to \text{Spec} R\) be a proper morphism of schemes. Let \(c \colon X \to X\) be a morphism of locally ringed \(G\)-spaces. Let \(X_{c!}\) be the set of closed points of \(X\) and let \(X_{c!} = c^{-1}(X_{cl})\). Assume that

1. \(\mathcal{O}_X\) is coherent;
2. \(F \in \text{Coh}(X)\), then \(\oplus_i H^i(X, F)\) is a finitely generated \(R\)-module;
3. \(c \colon X_{c!} \to X_{cl}\) is bijective;
4. \(F \in \text{Coh}(X)\) and \(F_x = 0\) for all \(x \in X_{c!}\), then \(F = 0\); and
5. \(x \in X_{c!}\), then \(\mathcal{O}_{X_{c},x} \to \mathcal{O}_{X,x}\) is flat and \(\kappa(x) \to \kappa(c(x)) \otimes_{\mathcal{O}_{X,c(x)}} \mathcal{O}_{X,x}\) is an isomorphism.

Then the comparison map:

\[
H^i(X, F) \to H^i(X, c^* F)
\]

is an isomorphism for all coherent sheaves \(F\) on \(X\) and

\[
c^* : \text{Coh}(X) \to \text{Coh}(\mathcal{X})
\]

is an exact equivalence of abelian categories.

In future work, we will apply it to algebraic stacks and their derived counterparts. Our approach is related to the non-noetherian GAGA results in the Stacks Project [Stacks, Tag 0DL1]. While Theorem A (and B, below) is an immediate consequence of much more general results (Theorems 6.1, 8.1, and 9.1), we give a simple and direct proof in §2.

**Remark 1.1.** Basic properties of noetherian local rings show that Theorem A(5) is implied by:

5’ \(x \in X_{c!}\), then \(\mathcal{O}_{X,x}\) is noetherian and the morphism \(\mathcal{O}_{X,c|} \to \mathcal{O}_{X,x}\) induces an isomorphism on maximal-adic completions.

Conditions (1) of (2) of Theorem A are typically established by theorems in analysis. For example, in the archimedean case, they are Oka’s Theorem [Oka50] and Cartan–Serre’s finiteness theorem [CS53]. In the non-archimedean case, it is Kiehl’s finiteness theorem [Kie67]. In the case of formal schemes, it is Grothendieck’s finiteness theorem [EGA, III.3.4.2]. Note that there is no circularity here: none of these results have anything to do with GAGA theorems or their proofs. It is even possible to use the analytic finiteness results to prove the algebraic ones [Duc15]. The recent work [Zav21] shows that the ideas in this article persist to almost mathematics. Theorem A (and its generalizations considered in this paper) are currently not well-adapted to local or henselian type affincifications (e.g., [AT19, Thm. C.1.1] or [Dev20]), however, as the boundedness condition (2) is established via induction and GAGA.

Without finiteness results, however, we can establish the following.

**Theorem B.** Let \(X\) be a quasi-compact and quasi-separated scheme. Let \(c \colon X \to X\) be a morphism of locally ringed \(G\)-spaces. Let \(x \in X\) be a closed point. Assume

1. \(U = X - \{x\}\) is quasi-affine;
2. \(c^{-1}(x)\) consists of a single point \(y\);
3. \(\mathcal{O}_{X,x} \to \mathcal{O}_{X,y}\) is flat and \(\kappa(x) \to \kappa(y) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,y}\) is an isomorphism; and
4. \(H^i(X, \mathcal{O}_X) \simeq H^i(X, c_* \mathcal{O}_X)\) and \(H^1(X, \mathcal{O}_X) \to H^1(X, c_* \mathcal{O}_X)\)

Then the comparison map:

\[
H^i(X, E) \to H^i(X, c^* E)
\]

is an isomorphism for all vector bundles \(E\) of finite rank on \(X\).
Theorem B is sufficient to establish the full-faithfulness of the analytification of vector bundles on the Fargues–Fontaine curve. Surprisingly, Theorem B is a consequence of a general Lefschetz-style result (see §6), which establishes all existing full faithfulness results in the literature (see §9).

In order to establish non-noetherian GAGA results in formal geometry (e.g., [FK18] or [Stacks, Tag 0DIA]), the non-flatness of completions complicates matters significantly. These are dealt with in §5. We work quite generally and expect these methods to be applicable to other types of GAGA and Lefschetz-style results (e.g., in o-minimal geometry). In Appendix §A, we establish functoriality results for the projection formula in symmetric monoidal categories, which turn out to be critical for results like Theorem A.

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2. Proof of Theorems A and B

We now prove Theorem A and B in complete generality. Keeping track of the residue fields, whileussy, is the crucial calculation.

We work with the unbounded derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves, $\mathcal{D}_{qc}(X)$, and the unbounded derived category of $\mathcal{O}_X$-modules, $\mathcal{D}(X)$. If $X$ is smooth and projective over a field, then we can instead work with $\mathcal{D}_{qc}^{\text{Coh}}(X)$ and $\mathcal{D}_{qc}^{\text{Coh}}(X)$. In general, the unbounded derived category—while perhaps not strictly necessary—is much more convenient. There is an unbounded derived pullback

$$
\mathcal{L}c^*: \mathcal{D}_{qc}(X) \to \mathcal{D}(X).
$$

Throughout, we let $t \in \mathcal{X}$ and let $s = c(t)$. In the case of Theorem A we assume that $t \in \mathcal{X}_{cl, s}$ and in Theorem B we take $t = y$. We now obtain a commutative diagram of ringed $G$-spaces:

$$
\begin{array}{ccc}
(X, \kappa(t)) & \xrightarrow{c^*} & (X, \kappa(s)) \\
\downarrow{c^*} & & \downarrow{i_*} \\
\mathcal{X} & \xrightarrow{c} & X.
\end{array}
$$

Note that $c^*: \text{Mod}(X, \kappa(s)) \to \text{Mod}(X, \kappa(t))$ is an exact equivalence of abelian categories because $\text{Mod}(X, \kappa(s)) \simeq \text{Mod}(\kappa(s))$, $\text{Mod}(X, \kappa(t)) \simeq \text{Mod}(\kappa(t))$, and $\kappa(s) \simeq \kappa(t)$ (see Lemma 2.7). In particular, the natural morphism $\mathcal{L}c^* \kappa(s) \to \kappa(t)$ is an isomorphism.

Now consider the unbounded derived pushforward

$$
\mathcal{R}_c*: \mathcal{D}(X) \to \mathcal{D}(X),
$$

which is right adjoint to $\mathcal{L}c^*$. Then $t$ and $s$ are closed points, so $(i'_t)_*$ and $(i_s)_*$ are exact. Also, $(c^*)_*$ is exact. In particular, $\mathcal{R}_c, \kappa(t) \simeq \mathcal{R}_c, (i'_t)_*, \kappa(t) \simeq (i'_t)_*, (c^*)_*, \kappa(t) \simeq (c^*)_*, \kappa(s)$. Putting this all together, we have now established that the relevant adjunctions induce isomorphisms:

$$
\kappa(s) \simeq \mathcal{R}_c^* \mathcal{L}c^* \kappa(s) \quad \mathcal{L}c^* \mathcal{R}_c^* \kappa(t) \simeq \kappa(t).
$$

By general adjoint functor theorems, the inclusion $\mathcal{D}_{qc}(X) \subseteq \mathcal{D}(X)$ admits a right adjoint,

$$
\mathcal{R}Q_X: \mathcal{D}(X) \to \mathcal{D}_{qc}(X).
$$

Since $\mathcal{D}_{qc}(X) \subseteq \mathcal{D}(X)$ is a fully faithful embedding, if $M \in \mathcal{D}_{qc}(X)$, then $M \simeq \mathcal{R}Q_X(M)$. Composing $\mathcal{R}Q_X$ with $\mathcal{R}_c$ above gives a right adjoint

$$
\mathcal{R}_{qc, c^*}*: \mathcal{D}(X) \to \mathcal{D}_{qc}(X),
$$

see §4 for a more in depth discussion. Thus, we can upgrade (2.1) to

$$
\kappa(s) \simeq \mathcal{R}_{qc, c^*}^* \mathcal{L}c^* \kappa(s) \quad \mathcal{L}c^* \mathcal{R}_{qc, c^*}^* \kappa(t) \simeq \kappa(t).
$$

Let $N \in \mathcal{D}_{qc}(X)$ and consider the adjunction

$$
\eta_N: N \to \mathcal{R}_{qc, c^*} \mathcal{L}c^* N.
$$
The subcategory $T$ of $\mathcal{D}_{qc}(X)$ with objects those $N$ for which $\eta_N$ is an isomorphism is certainly thick and triangulated. By (2.2), it contains $\kappa(s)$. A short inductive argument shows that $T$ contains every $N \in \mathcal{D}^b_{\mathcal{Coh}}(X)$ such that $Lj^*_s N \simeq 0$, where $j_s : X - \{s\} \subseteq X$ (see Lemma 3.4).

We now make another key observation: if $P \in \mathcal{D}_{qc}(X)$ is perfect and $N \in \mathcal{D}(X)$, then there is the projection formula:

$$P \otimes^L_{\mathcal{O}_X} (R\mathcal{C}_{qc,s} N) \simeq R\mathcal{C}_{qc,s}(Lc^*P \otimes^L_{\mathcal{O}_X} N).$$

This follows from formal properties of adjoints (see Lemma 4.3 and Appendix A).

By perfect approximation [LN07], there is a perfect complex $P_s$ supported only at $s$ and a morphism $\phi_s : P_s \to \kappa(s)$ such that $\tau_{\leq 0} \phi_s$ is an isomorphism. But $P_s \otimes^L_{\mathcal{O}_X} N$ is supported only at $s$ too, so the projection formula implies that

$$P_s \otimes^L_{\mathcal{O}_X} H_N \simeq H_{P_s \otimes^L_{\mathcal{O}_X} N} \simeq 0,$$

where $H_N$ is the cone of $\eta_N$ (see (A.8)). The proofs of Theorem A and B now diverge.

**Proof of Theorem A.** First, observe that $Lc^*$ restricts to a $t$-exact functor on $\mathcal{D}^b_{\mathcal{Coh}}(X)$. Indeed, if $M \in \mathcal{D}^b_{\mathcal{Coh}}(X)$, then $(Lc^* M)_t \simeq M_s \otimes^L_{\mathcal{O}_X} \mathcal{O}_{X,t}$. In particular, (5) implies that if $\tau_{< n} M \simeq 0$ for some $n$, then

$$\tau_{< n}(Lc^* M)_t \simeq \tau_{< n}(M_s \otimes^L_{\mathcal{O}_X} \mathcal{O}_{X,t}) \simeq (\tau_{< n} M_s) \otimes^L_{\mathcal{O}_X} \mathcal{O}_{X,t} \simeq 0.$$

Condition (4) gives $\tau_{< n}(Lc^* M) \simeq 0$ as claimed.

Condition (2) now allows us to apply a deep finiteness result, which implies that the restriction of $\mathcal{R}_{qc,s}$ to $\mathcal{D}^b_{\mathcal{Coh}}(X)$ factors through $\mathcal{D}^b_{\mathcal{Coh}}(X)$ (see Proposition 4.1 or [BZN17, Prop. 3.9.9]). If $X$ is projective over a field $k$, this step also follows from [Bal11, Rem. 4.6] (if $X$ is smooth, it is the well-known [BB03, Thm. A.1]). In summary, our functors restrict to an adjoint pair

$$Lc^* : \mathcal{D}^b_{\mathcal{Coh}}(X) \rightleftarrows \mathcal{D}^b_{\mathcal{Coh}}(X) : \mathcal{R}_{qc,s}.$$

Note that we now have $H_N \in \mathcal{D}^b_{\mathcal{Coh}}(X)$. Let $v$ be an integer such that $\tau_{> v} H_N = 0$; then $\kappa(s) \otimes^L_{\mathcal{O}_X} \mathcal{H}^v(H_N) \simeq \mathcal{H}^v(P_s \otimes^L_{\mathcal{O}_X} H_N) \simeq 0$, by (2.4). Since this is true for all closed points $s \in |X|$ and $\mathcal{H}^v(H_N) \in \mathcal{Coh}(X)$, Nakayama’s Lemma implies that $\tau_{> v-1} H_N \simeq 0$. It follows immediately that $H_N \simeq 0$. Hence, $\eta_N$ is an isomorphism for all $N \in \mathcal{D}^b_{\mathcal{Coh}}(X)$ and so $Lc^*$ is fully faithful on $\mathcal{D}^b_{\mathcal{Coh}}(X)$.

Further, we have the comparison result: if $N \in \mathcal{Coh}(X)$ and $i \geq 0$, then

$$H^i(X, N) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, N[i]) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{R}_{qc,s} c^* N[i]) \simeq \operatorname{Hom}_{\mathcal{O}_X}(c^* \mathcal{O}_X, c^* N[i]) \simeq H^i(X, c^* N).$$

For the essential surjectivity of $Lc^*$, we proceed similarly as before: let $M \in \mathcal{D}^b_{\mathcal{Coh}}(X)$; then we must prove that $\epsilon_M : Lc^* \mathcal{R}_{qc,s} M \to M$ is an isomorphism. Let $E_M$ be the cone of $\epsilon_M$; then $E_M \in \mathcal{D}^b_{\mathcal{Coh}}(X)$. The projection formula and (2.2) give $Lc^* P_s \otimes^L_{\mathcal{O}_X} E_M \simeq E_{Lc^* P_s \otimes^L_{\mathcal{O}_X} M} \simeq 0$. Condition (4) now shows that $E_M \simeq 0$.

**Remark 2.5.** Let $c : X \to X$ be a morphism of locally ringed $G$-spaces $\mathcal{O}_X$ is coherent and such that $Lc^* : \mathcal{D}^b_{\mathcal{Coh}}(X) \to \mathcal{D}^b_{\mathcal{Coh}}(X)$ is a $t$-exact equivalence. If there is a set of closed points $S$ of $X$ such that

(a) if $s \in S$, then $\mathcal{O}_{X,s}$ is noetherian;
(b) if $s \in S$, then $\kappa(s) \in \mathcal{Coh}(X)$; and
(c) if $I \in \mathcal{Coh}(\mathcal{F})$ and $I_s = 0$ for all $s \in S$, then $I = 0$.

Then a short argument gives $S = c^{-1}(X_G)$ and the remaining conditions of Theorem A are met.

**Proof of Theorem B.** It suffices to prove that if $F$ is a vector bundle on $X$, then $\tau_{\leq 0} H_F \simeq 0$. Indeed, there is an exact sequence:

$$0 \to H^1(X, F) \to H^0(X, F) \to H^0(X, \mathcal{R}_{qc,s} c^* F) \to H^0(X, H_F) \to 0.$$

If $\tau_{\leq 0} H_F \simeq 0$, then the terms at the ends vanish and by adjunction $H^0(X, \mathcal{R}_{qc,s} c^* F) \simeq H^0(X, c^* F)$.

Now since $F$ is a vector bundle, the projection formula (2.3) implies that $H_{\mathcal{O}_X} \otimes^L_{\mathcal{O}_X} F \simeq H_F$. Set $H = H_{\mathcal{O}_X}$; hence, it suffices to prove that $\tau_{\leq 0} H \simeq 0$. By (2.4), $P_s \otimes^L_{\mathcal{O}_X} H \simeq 0$. Now let
j: U → X be the resulting open immersion that is complementary to \{s\}; then localization theory (e.g., [HR17b, Ex. 1.4]) now implies that \(H \cong Rj_*Lj^*H\). But (1) implies that \(U\) is quasi-affine, so \(\tau^{\leq 0}H \cong 0\) if and only if \(\tau^{\leq 0}\mathcal{R}\Gamma(X, H) \cong 0\). We now have the long exact sequence:

\[
0 \to H^1(X, H) \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X) \to H^0(X, H) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X).
\]

Certainly, \(\tau^{\leq 0}H \cong 0\), so we have \(H^{-1}(X, H) \cong H^0(X, \mathcal{O}_X)\). But \(H^0(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X)\), so \(H^0(X, \mathcal{O}_X) = 0\) and \(\mathcal{H}^{-1}(H) \cong 0\). Hence, \(H^0(X, H) \cong H^0(X, \mathcal{O}_X)\). The injection \(H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)\) allows us to conclude that \(\mathcal{H}^0(H) \cong 0\). The result follows.

\[\square\]

**Remark 2.6.** A support theory for “big” objects in \(D(X)\), or some suitable subcategory, would aid in establishing the essential surjectivity of \(c^*\) in Theorem B.

**Lemma 2.7.** Let \(\mathcal{Y}\) be a \(G\)-space. Let \(B\) be a ring. Let \(\{y\} \subseteq \mathcal{Y}\) be the inclusion of a point. If \(y\) belongs to an admissible open and \(\mathcal{Y} - \{y\}\) is covered by admissible opens, then

\[
i^{-1}: \Mod(\mathcal{Y}, i_*B) \Rightarrow \Mod(B): i_*
\]

is an exact equivalence of abelian categories.

**Proof.** Let \(\mathcal{F}\) be an \(i_*B\)-module. Let \(V \subseteq \mathcal{Y}\) be admissible. If \(y \notin V\), then \((i_*B)(V) = 0\). But \(\mathcal{F}(V)\) is a \((i_*B)(V)\)-module, so \(\mathcal{F}(V) = 0\). Let \(U \subseteq V\) be another admissible open and assume that \(y \in U\). Let \(\{W_i \subseteq V \setminus \{y\}\}_{i \in I}\) be an admissible cover. The sheaf condition gives an exact sequence:

\[
0 \to \mathcal{F}(V) \to \mathcal{F}(U) \times \prod_{i \in I} \mathcal{F}(V \cap W_i) \xrightarrow{\bigoplus_{i \in I} \mathcal{F}(U \cap V \cap W_i)} \left(\prod_{i \in I} \mathcal{F}(V \cap W_i \cap W_j)\right).
\]

Since \(y \notin V \cap W_i\) for all \(i \in I\), it follows that the sequence above collapses to the restriction morphism \(\mathcal{F}(V) \to \mathcal{F}(U)\) being an isomorphism.

Since \(i^{-1}\mathcal{F} = \mathcal{F}_\alpha\), it immediately follows that the adjunction \(\mathcal{F} \to i_*i^{-1}\mathcal{F}\) is an isomorphism of abelian sheaves. Now let \(M\) be a \(B\)-module. Since it is clear that \(i^{-1}i_*M \to M\) is an isomorphism of \(B\)-modules, the result follows.

\[\square\]

3. A finiteness result

Our first task is to consider a variant of the finiteness result [BB03, Thm. 1.1] for non-noetherian algebraic spaces. This was recently established in the noetherian case in [BZNP17] and in general in [Stacks], where it is formulated in terms of pseudo-coherence [SGA6]. Since the non-noetherian situation will be important to us, we will briefly recall these ideas.

Let \(B\) be a ring. A bounded complex of finitely generated and projective \(B\)-modules is called **strictly perfect**. A complex of \(B\)-modules \(M\) is \(m\)-pseudo-coherent if there is a morphism \(\phi: P \to M\) such that \(P\) is strictly perfect and the induced morphism \(\mathcal{H}^i(\phi): \mathcal{H}^i(P) \to \mathcal{H}^i(M)\) is an isomorphism for \(i > m\) and surjective for \(i = m\). A complex of \(B\)-modules \(M\) is **pseudo-coherent** if it is \(m\)-pseudo-coherent for all integers \(m \in \mathbb{Z}\); equivalently, it is quasi-isomorphic to a bounded above complex of finitely generated and projective \(B\)-modules [Stacks, Tag 064T]. These conditions are all stable under derived base change [Stacks, Tag 0650] and are flat local [Stacks, Tag 068R].

We let \(D^{-\infty}_{pc}(B)\) denote the full triangulated subcategory of the derived category of \(B\)-modules, \(D(B)\), with objects those complexes of \(B\)-modules that are quasi-isomorphic to a pseudo-coherent complex of \(B\)-modules. We let \(D^{-\infty}_{pc}(B) \subseteq D^{-\infty}_{pc}(B)\) be the triangulated subcategory of objects with bounded cohomological support. If \(B \to C\) is a ring homomorphism, then derived base change induces \(- \otimes^L_C: D^{-\infty}_{pc}(B) \to D^{-\infty}_{pc}(C)\). If \(C\) has finite tor-dimension over \(B\) (e.g., \(C\) is \(B\)-flat), then the derived base change sends bounded pseudo-coherent complexes to bounded pseudo-coherent complexes.

The above generalizes to ringed sites [Stacks, Tag 08FS]. Let \(X\) be a ringed site. A complex of \(\mathcal{O}_X\)-modules is **strictly perfect** if it is bounded and each term is a direct summand of a finitely generated and free \(\mathcal{O}_X\)-module [Stacks, Tag 08FL]. A complex of \(\mathcal{O}_X\)-modules is **perfect** if locally on \(X\) is strictly perfect.
Example 3.1. Let $X$ be an algebraic space. If $i: D \subseteq X$ is a Cartier divisor, then $i_* \mathcal{O}_D \in \mathcal{D}_{pc}(X)$ is perfect. More generally, if $i: Z \to X$ is a regular embedding (i.e., $i$ is locally the zero locus of a regular section of a vector bundle), then $i_* \mathcal{O}_Z \in \mathcal{D}_{pc}(X)$ is perfect. Also, if $X$ is quasi-compact and quasi-separated and $j: U \subseteq X$ is a quasi-compact open immersion, then there is a perfect complex $P \in \mathcal{D}_{pc}(X)$ whose cohomological support is precisely $X \setminus U$ [HR17a, Thm. A].

Let $m \in \mathbb{Z}$. A complex of $\mathcal{O}_X$-modules $M$ is $m$-pseudo-coherent if locally on $X$ there is a morphism $\phi: \mathcal{P} \to M$ such that $\mathcal{P}$ is strictly perfect and the induced morphism $\mathcal{H}(\phi): \mathcal{H}(\mathcal{P}) \to \mathcal{H}(M)$ is an isomorphism for $i > m$ and surjective for $i = m$. A complex of $\mathcal{O}_X$-modules is pseudo-coherent if it is $m$-pseudo-coherent for every $m \in \mathbb{Z}$.

Let $\mathcal{D}_{pc}(X)$ denote the full triangulated subcategory of $\mathcal{D}(X)$, the unbounded derived category of $\mathcal{O}_X$-modules, with objects those complexes that are quasi-isomorphic to a bounded above pseudo-coherent complex. We let $\mathcal{D}_{pc}^b(X) \subseteq \mathcal{D}_{pc}(X)$ be the full triangulated subcategory of objects with bounded cohomological support. If $c: X \to Y$ is a morphism of ringed sites, then the restriction of $Lc^*: \mathcal{D}(X) \to \mathcal{D}(Y)$ to $\mathcal{D}_{pc}(X)$ factors through $\mathcal{D}_{pc}(X)$ [Stacks, Tag 08H4]. Moreover, if $c$ has finite tor-dimension (e.g., it is flat), then $Lc^*$ preserves bounded complexes.

Example 3.2. Perfect complexes are pseudo-coherent. In particular, vector bundles of finite rank are pseudo-coherent.

Example 3.3. Let $X$ be a ringed site with a coherent structure sheaf. For example, a locally noetherian algebraic space or an analytic space. Let $\ast \in \{-, b\}$. Then $\mathcal{D}_{pc}^\ast(X) = \mathcal{D}_{coh}^\ast(X)$: that is, a complex $M \in \mathcal{D}(X)$ is pseudo-coherent if and only if it is quasi-isomorphic to a bounded above complex of sheaves with coherent cohomology [SGA6, Cor. I.3.5].

The following lemma improves upon those given in Example 3.1 in the coherent setting. To state this lemma, we recall the following definition [Nee01, §2.1]. Let $\mathcal{T}$ be a triangulated category. A subcategory $S \subseteq \mathcal{T}$ is thick (or épaisse) if it is triangulated and is closed under direct summands. If $S \subseteq \mathcal{T}$ is a collection of objects, we let $\langle S \rangle \subseteq \mathcal{T}$ denote the thick closure of $S$; that is, it is the smallest thick triangulated subcategory of $\mathcal{T}$ containing $S$.

Lemma 3.4. Let $X$ be a quasi-compact and quasi-separated algebraic space. Let $i: Z \hookrightarrow X$ be a finitely presented closed immersion. Let

$$
\mathcal{D}_{pc}^b(Z)(X) = \ker(Lj^* : \mathcal{D}_{pc}^b(X) \to \mathcal{D}_{pc}^b(U)),
$$

where $j: U \to X$ is the open immersion complementary to $i: Z \to X$. If $\mathcal{O}_X$ is coherent, then $(Ri_* \mathcal{D}_{pc}^b(Z)) = \mathcal{D}_{pc}^b(Z)(X)$.

Proof. Clearly, $Ri_* \mathcal{D}_{pc}^b(Z) \subseteq \mathcal{D}_{pc}^b(Z)(X)$. Since $\mathcal{D}_{pc}^b(Z)(X)$ is a thick subcategory of $\mathcal{D}_{pc}^b(X)$, it follows that $(Ri_* \mathcal{D}_{pc}^b(Z)) \subseteq \mathcal{D}_{pc}^b(Z)(X)$. For the reverse inclusion, by induction on the length of a complex, it is sufficient to prove that if $M \in \mathcal{Coh}^b_{Z}(X) = \ker(j^* : \mathcal{Coh}(X) \to \mathcal{Coh}(U))$ then $M \in (Ri_* \mathcal{D}_{pc}^b(Z))$. Let $I = \ker(\mathcal{O}_X \to i_* \mathcal{O}_Z)$. By [HR22, Lem. 2.5(i)] it follows that there exists an integer $n > 0$ such that $I^{n+1}M = 0$. Hence, $M$ admits a finite filtration by $i_* \mathcal{O}_Z$-modules and so belongs to $(Ri_* \mathcal{D}_{pc}^b(Z))$. \hfill \Box

Let $A$ be a ring. A $B$-algebra $A$ is pseudo-coherent if it admits a surjection from a polynomial ring $\phi: A[x_1, \ldots, x_n] \twoheadrightarrow B$ such that $B$ is a pseudo-coherent $A[x_1, \ldots, x_n]$-module. Pseudo-coherence is stable under flat base change on $A$ and is étale local on $B$. See [Stacks, Tag 067X] for more background material. This definition generalizes readily to morphisms of algebraic spaces [Stacks, Tag 06BQ]. We now recall some examples that will be important to us.

Example 3.5. Let $A$ be a noetherian ring. If $X \to \text{Spec} A$ is a locally of finite type morphism of algebraic spaces, then it is pseudo-coherent [Stacks, Tag 06BX].

Example 3.6. Let $A$ be a ring. If $X \to \text{Spec} A$ is a flat and locally of finite presentation morphism of algebraic spaces, then it is pseudo-coherent [Stacks, Tag 06BV].
Example 3.7. Let $A$ be a universally cohesive ring. That is, every finitely presented $A$-algebra is a coherent ring. The standard example is an $a$-adic locally complete valuation ring: for example, $A = \mathcal{O}_{\mathbb{C}^p}$, the ring of integers in the $p$-adically completed algebraic closure of $\mathbb{Q}_p$, $\mathbb{C}_p$. If $X \to \text{Spec } A$ is a locally of finite presentation morphism of algebraic spaces, then it is pseudo-coherent. This is the setting for Fujitawa–Kato’s formalism of rigid geometry [FK18].

The main result of this section is the following small refinement of [Stacks, Tag 0CTT].

**Theorem 3.8.** Let $A$ be a ring. Let $X \to \text{Spec } A$ be a quasi-compact, separated, and pseudo-coherent morphism of algebraic spaces. Let $M \in \mathcal{D}_{qc}(X)$. If $\text{RHom}_{\mathcal{O}_X}(P, M) \in \mathcal{D}(A)$ is pseudo-coherent (pseudo-coherent and bounded) for all perfect complexes $P$, then $M$ is pseudo-coherent (pseudo-coherent and bounded).

**Proof.** It is sufficient to prove that $\text{R}(\mathcal{H}om)(X, E \otimes_{\mathcal{O}_X} M) \in \mathcal{D}(A)$ is pseudo-coherent for all pseudo-coherent $E$ on $X$. Indeed, pseudo-coherent morphisms are locally finitely presented, so $M$ is pseudo-coherent relative to $A$ [Stacks, Tag 0CTT]. Since $X \to \text{Spec } A$ is pseudo-coherent, $M$ is pseudo-coherent on $X$ [Stacks, Tag 0DHQ]. The boundedness result is [BZNP17, Lem. 3.0.14]—also see [Stacks, Tag 0GEF].

We begin by observing that if $P \in \mathcal{D}_{qc}(X)$ is perfect, then $\text{RHom}_{\mathcal{O}_X}(P^\vee, M) \simeq \text{R}(\mathcal{H}om)(X, P \otimes^\mathbb{L}_{\mathcal{O}_X} M) \in \mathcal{D}(A)$, which is pseudo-coherent by assumption. Now there exists an integer $n$ such that $H^r(X, N) = 0$ for all $r > n$ and $N \in \text{QCoh}(X)$ [Stacks, Tag 073G]. By [L008, Rem. 2.1.11],

$$
\tau_{\geq m} \text{R}(\mathcal{H}om)(X, G) \simeq \tau_{\geq m} \text{R}(\mathcal{H}om)(X, \tau_{\geq j} G) \quad \text{for all } m \in \mathbb{Z}, \quad l \leq m - n,
$$

and $G \in \mathcal{D}_{qc}(X)$. Let $E \in \mathcal{D}_{qc}(X)$ be pseudo-coherent and fix $m \in \mathbb{Z}$. Let $j \in \mathbb{Z}$ be such that $\tau_{\geq j} M \simeq 0$ and $\tau_{\geq j} E \simeq 0$. Choose a perfect complex $P \in \mathcal{D}_{qc}(X)$ and morphism $\phi: P \to E$ such that $\tau_{\geq a} \text{cone}(\phi) \simeq 0$, where $a = m - n - j$ [Stacks, Tag 08HP]. Then

$$
\tau_{\geq m} \text{R}(\mathcal{H}om)(X, E \otimes^\mathbb{L}_{\mathcal{O}_X} M) \simeq \tau_{\geq m} \text{R}(\mathcal{H}om)(X, \tau_{\geq m-n}(E \otimes^\mathbb{L}_{\mathcal{O}_X} M))
\simeq \tau_{\geq m} \text{R}(\mathcal{H}om)(X, \tau_{\geq m-n}(E \otimes^\mathbb{L}_{\mathcal{O}_X} (\tau_{\geq m-n-j} E) \otimes^\mathbb{L}_{\mathcal{O}_X} M))
\simeq \tau_{\geq m} \text{R}(\mathcal{H}om)(X, \tau_{\geq m-n}(P \otimes^\mathbb{L}_{\mathcal{O}_X} M))
\simeq \tau_{\geq m} \text{R}(\mathcal{H}om)(X, P \otimes^\mathbb{L}_{\mathcal{O}_X} M).
$$

We have already seen that $\text{R}(\mathcal{H}om)(X, P \otimes^\mathbb{L}_{\mathcal{O}_X} M) \in \mathcal{D}(A)$ is pseudo-coherent, and the claim follows. □

Remark 3.9. Theorem 3.8 has a converse if $X \to \text{Spec } A$ is proper. If $M$ is pseudo-coherent (resp. pseudo-coherent and bounded) and $P$ is perfect, then $\text{RHom}_{\mathcal{O}_X}(P^\vee, M) \simeq \text{R}(\mathcal{H}om)(X, P^\vee \otimes^\mathbb{L}_{\mathcal{O}_X} M)$. Replacing $M$ by $P^\vee \otimes^\mathbb{L}_{\mathcal{O}_X} M$, it suffices to prove that $\text{R}(\mathcal{H}om)(X, \cdot)$ sends pseudo-coherent complexes to pseudo-coherent complexes. If $A$ is noetherian, this is just the usual coherence theorem for algebraic spaces [Km71, Thm. IV.4.1]. If $X$ is a scheme and $A$ is not necessarily noetherian, this is Kiehl’s Finiteness Theorem [Kie72, Thm. 2.9]. If $X \to \text{Spec } A$ is flat, this is in the Stacks Project [Stacks, Tag 0CSC]. If $A$ is universally cohesive, then this is due to Fujitawa–Kato [FK18, Thm. I.8.1.2]. Using derived algebraic geometry, the argument given in the Stacks Project readily extends to the general (i.e., non-flat) situation; that is, a version of Kiehl’s finiteness theorem for algebraic spaces. This is done by Lurie in [SAG].

As noted in [BZN17, Rem. 3.0.6], it is Theorem 3.8 that fails miserably for algebraic stacks with infinite stabilizers. In future work, we will describe a variant of Theorem 3.8 for a large class of algebraic stacks with infinite stabilizers that is sufficient to establish integral transform and GAGA results.

We conclude this section with a simple corollary of Theorem 3.8. Variants of this are well-known (see [Lip9, Ex. 4.3.9] and [Ryd14] in the finite type noetherian, but non-separated situation).

**Corollary 3.10.** Let $A$ be a universally cohesive ring (e.g., noetherian). Let $X \to \text{Spec } A$ be a quasi-compact and separated morphism of algebraic spaces. If $\text{R}(\mathcal{H}om)(X, \cdot)$ sends $D^b_{qc}(X)$ to $D^b_{qc}(A)$, then $X \to \text{Spec } A$ is proper and of finite presentation.
Proof. By absolute noetherian approximation [Ryd15], there is an affine morphism \( a: X \to X_0 \), where \( X_0 \) is a separated and finitely presented algebraic space over \( \text{Spec} \, A \). Using Nagata’s compactification theorem for algebraic spaces [CLO12], a blow-up, and absolute noetherian approximation again, we may further assume that \( X_0 \to \text{Spec} \, A \) is proper and finitely presented. Since \( A \) is universally cohesive, \( \mathcal{O}_{X_0} \) is coherent. Now let \( P \in \mathcal{D}_{qc}(X_0) \) be a perfect complex; then

\[
\mathbf{R}\text{Hom}_{\mathcal{O}_{X_0}}(P, \mathcal{O}_X) = \mathbf{R}\text{Hom}_{\mathcal{O}_X}(La^* P, \mathcal{O}_X) = \mathbf{R}\Gamma(X, La^* P^\vee) \in \mathcal{D}^b_{\text{Coh}}(A).
\]

Hence, Theorem \( 3.8 \) implies that \( a_* \mathcal{O}_X \in \text{Coh}(X_0) \). That is, \( a \) is finite and finitely presented. By composition, \( X \to \text{Spec} \, A \) is proper and of finite presentation. \( \square \)

4. Adjoints

Throughout we let \( X \) be an algebraic space. Consider a morphism of ringed topoi \( c: X \to X_{et} \).

There is an adjoint pair on the level of unbounded derived categories

\[
\mathcal{L}c^*: \mathcal{D}(X) \rightleftarrows \mathcal{D}(\mathcal{X}): \mathcal{R}c_*.
\]

The inclusion \( \mathcal{D}_{qc}(X) \subseteq \mathcal{D}(X) \) is fully faithful and also admits a right adjoint, the quasi-coherator \( \mathcal{R}Q_X: \mathcal{D}(X) \to \mathcal{D}_{qc}(X) \). It follows immediately that

(1) the restriction of \( \mathcal{L}c^* \) to \( \mathcal{D}_{qc}(X) \) is left adjoint to \( \mathcal{R}Q_X \mathcal{R}c_* \); and

(2) if \( M \in \mathcal{D}_{qc}(X) \), then the natural map \( M \to \mathcal{R}Q_X(M) \) is a quasi-isomorphism.

We will let

\[
\mathcal{L}c^*: \mathcal{D}_{qc}(X) \rightleftarrows \mathcal{D}(\mathcal{X}): \mathcal{R}c_*^\cdot
\]

denote the resulting adjoint pair. Let \( M \in \mathcal{D}_{qc}(X) \) and \( N \in \mathcal{D}(\mathcal{X}) \). Let

\[
\eta_M: : M \to \mathcal{R}c_{qc,*} \mathcal{L}c^* M \quad \text{and} \quad \epsilon_N: \mathcal{L}c^* \mathcal{R}c_{qc,*} N \to N
\]
denote the morphisms resulting from the adjunctions.

We now use Theorem \( 3.8 \) to show that \( \mathcal{R}c_{qc,*} \) frequently preserves pseudo-coherence.

Proposition 4.1. Let \( A \) be a ring. Let \( X \to \text{Spec} \, A \) be a quasi-compact, separated, and pseudo-coherent morphism of algebraic spaces. Let \( c: X \to X_{et} \) be a morphism of ringed topoi. Let \( * \in \{b, - \} \). If \( \mathbf{R}\Gamma(X, -) \) sends \( \mathcal{D}^b_{pc}(X) \) to \( \mathcal{D}^b_{pc}(A) \), then the restriction of \( \mathcal{R}c_{qc,*} \) to \( \mathcal{D}^b_{qc}(X) \) factors through \( \mathcal{D}^b_{pc}(X) \).

Proof. Let \( M \in \mathcal{D}^b_{pc}(X) \) and let \( P \in \mathcal{D}_{qc}(X) \) be perfect. Then \( \mathcal{L}c^* P^\vee \in \mathcal{D}_{qc}(X) \) is perfect, so \( \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}c^* P \in \mathcal{D}^b_{pc}(X) \). Hence,

\[
\mathbf{R}\text{Hom}_{\mathcal{O}_X}(P, \mathcal{R}c_{qc,*} M) \simeq \mathbf{R}\text{Hom}_{\mathcal{O}_X}(\mathcal{L}c^* P, M) \simeq \mathbf{R}\Gamma(X, \mathcal{L}c^* P^\vee \otimes^{\mathbb{L}}_{\mathcal{O}_X} M) \in \mathcal{D}^b_{pc}(A).
\]

By Theorem \( 3.8 \), \( \mathcal{R}c_{qc,*} M \in \mathcal{D}^b_{pc}(X) \). \( \square \)

Remark 4.2. A variant of Proposition 4.1 that is valid for finite type cohomological functors for proper schemes over noetherian bases, which generalizes [BB03, Thm. 1.1], appears in [Nee18].

We return to our general discussion. The categories \( \mathcal{D}_{qc}(X) \) and \( \mathcal{D}(\mathcal{X}) \) are symmetric monoidal and the derived pullback \( \mathcal{L}c^* \) is strong monoidal. This lets us apply the formalism in Appendix A to our situation. We record some consequences here.

Lemma 4.3. If \( M \in \mathcal{D}_{qc}(X) \) and \( N \in \mathcal{D}(\mathcal{X}) \), then there is a natural projection morphism

\[
\tau_{M,N}: M \otimes_{\mathcal{O}_X}^L (\mathcal{R}c_{qc,*} N) \to \mathcal{R}c_{qc,*} (\mathcal{L}c^* M \otimes_{\mathcal{O}_X}^L N).
\]

This is an isomorphism if \( M \) is perfect or \( \mathcal{O}_X \) is a compact object of \( \mathcal{D}(\mathcal{X}) \).

Proof. If \( M \) is perfect, then it is a dualizable object of \( \mathcal{D}_{qc}(X) \) [HR17a, Lem. 4.3]. Hence, the projection morphism is an isomorphism in this case by Theorem A.12. If \( \mathcal{O}_X \) is a compact object of \( \mathcal{D}(\mathcal{X}) \), then \( \mathcal{L}c^*: \mathcal{D}_{qc}(X) \to \mathcal{D}(\mathcal{X}) \) preserves compact objects [HR17b, Lem. 2.3(2)], so its right adjoint \( \mathcal{R}c_{qc,*} \) preserves small coproducts [Nee96, Thm. 5.1]. Hence, the full subcategory of \( \mathcal{D}_{qc}(X) \) consisting of those \( M \) for which \( \tau_{M,N} \) is an isomorphism is localizing and contains the perfect complexes. By Thomason’s localization theorem [Nee96, Thm. 2.1.2], the result follows. \( \square \)
Remark 4.4. The condition that \( \mathcal{O}_X \) is a compact object of \( D(X) \) is subtle, but frequently satisfied. A useful criterion is [Stacks, Tag 094D], which shows that it is sufficient for cohomology of abelian sheaves on \( X \) to commute with filtered colimits and have finite cohomological dimension. It follows that \( \mathcal{O}_X \) is a compact object of \( D(X) \) whenever \( X \) is equivalent to the topos of
1. a noetherian space of finite Krull dimension [Gro57, Thm. 3.6.5]; or
2. a spectral space of finite Krull dimension [Sch92, Thm. 4.5]; or
3. a compact Hausdorff space of finite cohomological dimension.

5. Equivalences

In this section, the following setup will feature frequently.

Setup 5.1. Let \( i: Z \to X \) be a morphism of quasi-compact and quasi-separated algebraic spaces. Consider a 2-commutative diagram of ringed topoi:

\[
\begin{array}{ccc}
Z & \xrightarrow{cz} & Z_{\text{ét}} \\
\downarrow i' & & \downarrow i \\
X & \xrightarrow{c} & X_{\text{ét}}.
\end{array}
\]

We will also assume that

5.1.1 \( \text{Lc}_Z^* : D_{pc}(Z) \to D_{pc}(Z) \) is an equivalence of categories; and

5.1.2 \( \tau^*_{N,R^i_*Q} : N \otimes_{\mathcal{O}_X} R^i_*\mathcal{O} \simeq R^i_* (\text{Lc}_Z^* N \otimes_{\mathcal{O}_Z} \mathcal{O}) \) for all \( N \in D_{pc}(X) \) and \( Q \in D_{pc}(Z) \).

Setup 5.1 is said to be tor-independent if the following condition holds:

5.1.3 \( \text{Lc}_Z^* R^i_*Q \simeq R^i_* \text{Lc}_Z^* Q \) for all \( Q \in D_{pc}(Z) \).

Example 5.2. If \( c \) is the formal completion along a closed immersion \( i: Z \to X \) and when \( Z \) and \( Z \) are points with the same residue field, then 5.1.1 holds. Indeed, here we simply take \( Z = Z_{\text{ét}} \).

Lemma 5.4 will show that 5.1.2 also holds in this setting. Further, if \( c \) and \( i \) are tor-independent (e.g., \( c \) is flat or \( X \) is noetherian and \( c \) is the formal completion along \( i \)), then Lemma 5.5 implies that 5.1.3 holds.

The next result is a sanity check for 5.1.1.

Lemma 5.3. Assume 5.1.1. Then the restriction of \( R\text{c}_{Z,\text{qc},*} \) to \( D_{pc}^- (Z) \) induces an equivalence:

\( \text{Lc}_Z^* : D_{pc}(Z) \simeq D_{pc}(Z) : R\text{c}_{Z,\text{qc},*} \).

Proof. Let \( P, Q \in D_{pc}(Z) \). Then there are isomorphisms:

\[
\text{RHom}_{O_Z}(P, Q) \simeq \text{RHom}_{O_Z}(\text{Lc}_Z^* P, \text{Lc}_Z^* Q) \simeq \text{RHom}_{O_Z}(P, R\text{c}_{Z,\text{qc},*} \text{Lc}_Z^* Q).
\]

Let \( H_Q \) be the cone of the adjunction morphism \( \eta_{Z,Q} : Q \to R\text{c}_{Z,\text{qc},*} \text{Lc}_Z^* Q \). Then \( \text{RHom}_{O_Z}(P, H_Q) \simeq 0 \) for all \( P \in D_{pc}(Z) \). But \( D_{pc}(Z_{\lambda}) \) contains the perfect complexes of \( Z \), so \( H_Q \simeq 0 \) [HK17a, Thm. A]. That is, \( \eta_{Z,Q} \) is an isomorphism for all \( Q \in D_{pc}(Z) \). Now let \( \Omega \in D_{pc}(Z) \). Then \( Q \in D_{pc}(Z) \) and there is an isomorphism \( \Omega \simeq \text{Lc}_Z^* Q \). By what we have proved so far, it follows that \( R\text{c}_{Z,\text{qc},*} \Omega \simeq R\text{c}_{Z,\text{qc},*} \text{Lc}_Z^* Q \simeq Q \). That is, \( R\text{c}_{Z,\text{qc},*} \) restricts to a functor from \( D_{pc}^- (Z) \) to \( D_{pc}^- (Z) \). It follows immediately from general nonsense that \( R\text{c}_{Z,\text{qc},*} \) is right adjoint to \( \text{Lc}_Z^* : D_{pc}^- (Z) \to D_{pc}^- (Z) \) and we have the claimed adjoint equivalence.

The following lemma shows that 5.1.2 is a very mild condition.

Lemma 5.4. Assume 5.1.1. If one of the following hold, then 5.1.2 holds.

1. The projection morphism \( \tau^*_{N,R^i_*Q} \) is an isomorphism for all \( N \in D_{pc}(X) \) and \( Q \in D_{pc}(Z) \).
2. The projection morphism \( \tau^*_{L^i,N,R^i_*Q} \) is an isomorphism for all \( N \in D_{pc}(X) \) and \( Q \in D_{pc}(Z) \).
3. There exists an integer \( K \) such that for each \( U \) of \( X \), \( \tau^{\geq K} R^i_* Q \simeq 0 \) for all \( Q \in D_{pc}^- (U)^0 (i^{\geq -1} U) \), where \( i_U^* : i^{\geq -1} (U) \to U \) is the induced morphism.
Proof. Certainly, we have (1)⇒(2). We next show (3)⇒(1): let \( X \in D^b_{qc}(\mathcal{X}) \) and \( \Omega \in D^b_{pc}(\mathcal{Z}) \) for some integer \( b \). Let \( n \in \mathbb{Z} \); then it remains to prove that the projection morphism

\[ \pi_{X,\Omega} : N \otimes_{D_X} R_i^* \Omega \to R_i^*(L^i N \otimes_{D_Z} \Omega) \]

is \( n \)-coconnected. This assertion is local on \( X \). Thus, by pseudo-coherence of \( N \), we may assume that there is an \((n - k - b)\)-coconnected morphism \( \phi : \mathcal{P} \to \mathcal{N} \), where \( \mathcal{P} \) is perfect. Equivalently, if \( \mathcal{E} \) sits in a distinguished triangle:

\[ \mathcal{P} \xrightarrow{\phi} \mathcal{N} \xrightarrow{\epsilon} \mathcal{E} \xrightarrow{[1]} \]

then \( \tau^{>n-K-b} \mathcal{E} \simeq 0 \). Now \( \mathcal{P} \) is perfect, so \( \pi_{X,\Omega}^L \mathcal{E} \) is an isomorphism (Theorem A.12). However, we certainly have \( \tau^{>K+b} \mathcal{E} \simeq 0 \), so \( \tau^{>n}(\mathcal{E} \otimes_{D_X} \Omega) \simeq 0 \). Similarly, \( \tau^{>n-K}(\mathcal{E} \otimes_{D_Z} \Omega) \simeq 0 \) and so \( \tau^{>n} \mathcal{E} \simeq 0 \). It follows immediately that the projection morphism is \( n \)-coconnected.

It remains to prove that (2) implies 5.1.2. This is essentially just the functoriality properties of the projection formula together with Lemma 5.3. Specifically, we apply Lemma A.9 and Remark A.5 with \( \mathcal{E} = D_{qc}(X) \), \( \mathcal{D} = D(\mathcal{X}) \), \( \mathcal{E}' = D_{qc}(Z) \), and \( \mathcal{D}' = D(\mathcal{Z}) \) with the natural functors and adjoints already described. In more details: there is a commutative diagram

\[ \begin{array}{ccc} N \otimes_{D_X} R_{i*} \mathcal{E} & \xrightarrow{=} & \mathcal{E} \otimes_{D_Z} \mathcal{E}' \otimes_{D_Z} \mathcal{E} \\ \downarrow & & \downarrow \\ N \otimes_{D_X} R_{i*} \mathcal{E} \otimes_{D_X} L_{j*} \mathcal{E} & \xrightarrow{=} & \mathcal{E} \otimes_{D_Z} \mathcal{E}' \otimes_{D_Z} \mathcal{E} \end{array} \]

The map along the top is what we want to show is an isomorphism. The upper morphism on the right is an isomorphism by (2). The lower morphism on the right and the upper morphism on the left are isomorphisms by functoriality. The lower morphism on the left is an isomorphism by the usual projection formula for quasi-compact and quasi-separated morphisms of algebraic spaces [HR17a, Cor. 4.12]. Finally, the bottom morphism is an isomorphism by Remark A.3 applied to the equivalence of Lemma 5.3.

Tor-independence is more subtle to arrange, however.

Lemma 5.5. Assume 5.1.1. If

1. \( c \) and \( i \) are tor-independent morphisms;
2. \( i \) factors as \( Z \xrightarrow{i} \tilde{X} \xrightarrow{j} X \), where \( \tilde{i} \) is affine and \( j \) is étale; and
3. \( D_{pc}(\mathcal{Z}) \simeq D_{pc}(c^{-1}(\tilde{X}), \tilde{i}_* \mathcal{O}_\mathcal{Z}) \),

then 5.1.3 holds.

Proof. Let \( Z \xrightarrow{i} c^{-1}(\tilde{X}) \xrightarrow{j} X \) and \( \tilde{c} : c^{-1}(\tilde{X}) \to \tilde{X}_{ét} \) be the induced morphisms. We may assume that \( j \) is a cover, so \( j' \) is a cover. Let \( Q \in D_{pc}(\mathcal{Z}) \); then we must prove that the base change morphism \( L^i R_{i*} \mathcal{Q} \to R_{i*} L^i \mathcal{Q} \) is an isomorphism. Since \( j' \) is covering, it suffices to prove that \( L^i R_{i*} \mathcal{Q} \to L^{i'} R_{i'j'} \mathcal{Q} \) is an isomorphism. Since the following diagram commutes:

\[ \begin{array}{ccc} L^{i'} R_{i'j'} \mathcal{Q} & \xrightarrow{=} & L^{i'} R_{i'j'} \mathcal{Q} \\ \downarrow & & \downarrow \\ L^{i'} R_{i'j'} \mathcal{Q} & \xrightarrow{=} & L^{i'} R_{i'j'} \mathcal{Q} \\ \downarrow & & \downarrow \\ L^{i} R_{i} \mathcal{Q} & \xrightarrow{=} & R_{i} \mathcal{Q} \end{array} \]
where the vertical morphisms are all isomorphisms, and the bottom morphism is the base change map, we are reduced to the situation where \( X = \bar{X} \). The result now follows from Lemma B.2. \( \square \)

The following lemma collects some of the technicalities of this section.

**Lemma 5.6.** Assume Setup 5.1. Let \( N \in D^-(pc)(X) \) and define
\[
\mathcal{V}_N = \{ \Omega \in D(X) : \pi_{N,\Omega} \text{ is an isomorphism} \}, \\
\mathcal{T} = \{ \Omega \in D(X) : \epsilon_\Omega \text{ is an isomorphism} \}, \text{ and} \\
\mathcal{S} = \{ P \in D_{qc}(X) : \eta_P \text{ is an isomorphism} \}.
\]
Then \( (R_i D^{-}_{pc}(\mathbb{Z})) \subseteq \mathcal{V}_N \). If tor-independent, then \( (R_i D^{-}_{pc}(\mathbb{Z})) \subseteq \mathcal{T} \) and \( (R_i D^{-}_{pc}(\mathbb{Z})) \subseteq \mathcal{S} \).

**Proof.** We may view \( \mathcal{V}_N \), \( \mathcal{T} \), and \( \mathcal{S} \) as full subcategories of \( D(X) \), \( D(X) \), and \( D_{qc}(X) \), respectively. They are obviously triangulated and thick subcategories. It remains to prove the following.

1. If \( \Omega_0 \in D^-(pc)(\mathbb{Z}) \), then \( R'_i \Omega_0 \in \mathcal{V}_N \). This is 5.1.2.
2. If tor-independent and \( \Omega_0 \in D^-(pc)(\mathbb{Z}) \), then \( R'_i \Omega_0 \in \mathcal{T} \). To see this: the diagram
\[
\begin{array}{c}
\text{Le}^* R_{qc,0} R'_i \Omega_0 \\
\downarrow \epsilon_{R'_i} \downarrow \\
R'_i \Omega_0
\end{array}
\]
commutes. The claim now follows from functoriality (the top morphism) Lemmas 5.3 (the bottom morphism) and 5.1.3 (the right morphism).
3. If tor-independent and \( \Omega_0 \in D^-(pc)(\mathbb{Z}) \), then \( R_i \Omega_0 \in \mathcal{S} \): this similar to the previous step, so is omitted. \( \square \)

We now introduce a key definition. We appreciate that it is difficult to parse. Such a definition appears necessary, however, to treat the lack of tor-independence that appears in the non-noetherian situation as well as the subtleness of the projection morphism. When tor-independence is available, Proposition 5.8 provides a useful criterion for \( Z \)-equivalence.

**Definition 5.7.** Assume Setup 5.1. Let \( M \in D(X), N \in D^{-}_{pc}(X) \). We say that \( c \) is

(a) **faithful along** \( M \) at \( N \) if
\[
\nu_{M,N} : R_{qc,0} M \otimes_{\mathbb{O}_X} c R_{qc,0} N \to R_{qc,0} (M \otimes_{\mathbb{O}_X} N)
\]
is an isomorphism; and

(b) an **equivalence** along \( M \) at \( N \) if (a) holds and
\[
M \otimes \epsilon_N : M \otimes_{\mathbb{O}_X} \text{Le}^* R_{qc,0} N \to M \otimes_{\mathbb{O}_X} N
\]
is an isomorphism.

If these hold for all \( N \), then we omit the “at \( N \)”. If \( M = i'_* \mathbb{O}_Z \), then we will replace “\( M \)” with “\( Z \)”.

The simplest method to produce the above is to use the following.

**Proposition 5.8.** Assume Setup 5.1. Let \( M \in D_{qc}(X) \) and \( N \in D^{-}_{pc}(X) \).

1. If \( \eta_M \) is an isomorphism, then \( c \) is faithful along \( \text{Le}^* M \) at \( N \) if and only if \( \pi_{M,N} \) is an isomorphism.

2. If tor-independent and \( M \in (R_i D^{-}_{pc}(\mathbb{Z})) \), then \( c \) is an equivalence along \( M \) at \( N \) if and only if \( \pi_{M,N} \) is an isomorphism.

**Proof.** By definition of the projection morphism (A.2), the following diagram commutes:
\[
\begin{array}{c}
M \otimes_{\mathbb{O}_X} (R_{qc,0} N) \\
\downarrow \pi_{M,N} \\
R_{qc,0} (\text{Le}^* M \otimes_{\mathbb{O}_X} N).
\end{array}
\]
This proves (1). If \( M \in (R_iD^\bullet_{pc}(Z)) \) and \( c \) and \( i \) are tor-independent, then \( \eta_M \) is an isomorphism (Lemma 5.6). By Lemma A.4, the following diagram commutes:

\[
\begin{array}{ccc}
Le^*(M \otimes_{O_X}^L R_{qc,*}N) & \xrightarrow{Le^*\pi_M,N} & Le^*M \otimes_{O_X}^L Le^*R_{qc,*}N \\
\downarrow & & \downarrow Le^*M \otimes_{\epsilon_N} N \\
Le^*R_{qc,*}(Le^*M \otimes_{O_X}^L N) & \xrightarrow{\tau^* M \otimes_{\delta_N} N} & Le^*M \otimes_{\delta_N} N.
\end{array}
\]

The top morphism is an isomorphism, as is the bottom (Lemmas 5.6 and B.1). The stated equivalence follows.

Many examples are provided by the following two results.

**Corollary 5.9.** Assume tor-independent Setup 5.1. If

1. \( i \) is a Cartier divisor; or
2. \( i_*\mathcal{O}_Z \) is perfect; or
3. \( \mathcal{O}_X \) is a compact object of \( D(X) \) (see Remark 4.4);

then \( c \) is an equivalence along \( Z \).

**Proof.** We use the criterion of Proposition 5.8. Case (1) is a special case of (2). In cases (2) and (3) the projection morphism is an isomorphism by Lemma 4.3.

**Corollary 5.10.** Assume tor-independent Setup 5.1 and \( i \) is a closed immersion. If \( \mathcal{O}_X \) is coherent, then there is a perfect complex \( M \in \langle R_iD^\bullet_{pc}(Z) \rangle \) with \( \tau^{=0}M \simeq \mathcal{O}_Z \) such that \( c \) is an equivalence along \( Le^*M \).

**Proof.** By perfect approximation [Stacks, Tag 08HP], there exists a perfect complex \( M \in D^\bullet_{Coh,b}(X) \) with \( \tau^{=0}M \simeq \mathcal{O}_Z \). By Lemma 3.4, \( M \in (R_iD^\bullet_{pc}(Z)) \), so \( Le^*M \in (R_i^\bullet D^\bullet_{pc}(Z)) \) (Lemma B.2). Now apply Lemma 4.3 and Proposition 5.8.

Lacking tor-independence, these notions can be quite subtle. We will give some interesting examples at the end of this section, however. In the meantime, we content ourselves with the following useful lemma.

**Lemma 5.11.** Assume Setup 5.1. Let \( \mathcal{M}_0 \in D^-_{pc}(Z) \), \( N \in D^-_{pc}(X) \). If \( R_{qc,*}N \in D^-_{pc}(X) \) and \( c \) is faithful along \( R_i^\bullet \mathcal{M}_0 \) at \( N \), then \( c \) is an equivalence along \( R_i^\bullet \mathcal{M}_0 \) at \( N \).

**Proof.** We have the following commutative diagram:

\[
\begin{array}{ccc}
R_{qc,*}(Le^*R_{qc,*}N \otimes_{O_X}^L R_i^\bullet \mathcal{M}_0) & \xrightarrow{\tau_{R_{qc,*},N,R_i^\bullet \mathcal{M}_0}} & R_{qc,*}(N) \otimes_{O_X}^L R_{qc,*}R_i^\bullet \mathcal{M}_0 \\
\downarrow R_{qc,*}(\epsilon_N \otimes_{O_X}^L \epsilon_{i_*^\bullet \mathcal{M}_0}) & & \downarrow R_{qc,*}(N \otimes_{O_X}^L R_i^\bullet \mathcal{M}_0) \\
R_{qc,*}(N \otimes_{O_X}^L R_i^\bullet \mathcal{M}_0).
\end{array}
\]

Since \( c \) is faithful along \( R_i^\bullet \mathcal{M}_0 \) at \( N \), the vertical map is an isomorphism. By Lemma 5.6, the horizontal map is an isomorphism. It follows that the diagonal map is an isomorphism. Let \( Q \) be a cone for \( \epsilon_N \); then \( Q \in D^-_{pc}(X) \). Hence, \( Q \otimes_{O_X}^L R_i^\bullet \mathcal{M}_0 \simeq R_i^\bullet (L^i\epsilon^* Q \otimes_{O_Z}^L \mathcal{M}_0) \) (Lemma B.1). Let \( Q_0 = Li^*Q \otimes_{O_Z}^L \mathcal{M}_0 \); then \( Q_0 \in D^-_{pc}(Z) \) and

\[
0 \simeq R_{qc,*}(Q \otimes_{O_X}^L R_i^\bullet \mathcal{M}_0) \simeq R_{qc,*}R_i^\bullet Q_0 \simeq R_i^\bullet R_{CZ,qc,*}Q_0.
\]

It follows immediately that \( Q_0 \simeq 0 \) and the claim follows.

The whole reason for introducing these notions is the following key result.

**Proposition 5.12.** Assume Setup 5.1. Let \( \mathcal{M} \in D(X) \) and \( N = Le^*N, \) where \( N \in D^-_{pc}(X) \). If \( \pi_{N,M} \) is an isomorphism, then the following conditions are equivalent:

1. \( c \) is faithful along \( M \) at \( N \);
2. \( R_{qc,*}M \otimes \eta_N \) is an isomorphism.
**Proof.** This is immediate from the commutativity of the following diagram (A.2):

\[
\begin{array}{c}
N \otimes_{O_X} (Rc_{qc,*}M) \xrightarrow{\eta N \otimes Rc_{qc,*}} Rc_{qc,*}Lc^*N \otimes_{O_X} (Rc_{qc,*}M) \\
\pi_{N,M} \downarrow \downarrow \downarrow
\end{array}
\]

We conclude this section with more methods to produce examples.

**Lemma 5.13.** Assume Setup 5.1.

1. \(c\) is faithful along \(M\) if and only if
   \[
   \Gamma(X, Rc_{qc,*}M \otimes_{O_X} Rc_{qc,*}N) \to \Gamma(X, M \otimes_{O_X} N)
   \]
   is an isomorphism for all \(N \in \mathcal{D}_{pc}^-(X)\).

2. If \(f: X \to \text{Spec} A\) is flat; \(Z = X \otimes_A B\), where \(B\) is an \(A\)-algebra; and
   \[
   B \otimes_A^L \Gamma(\chi, N) \to \Gamma(\chi, Li^{**}N)
   \]
   is an isomorphism for all \(N \in \mathcal{D}_{pc}^-(X)\); then \(c\) is faithful along \(Z\).

**Proof.** For (1): the necessity is clear. For the sufficiency, the perfect complexes compactly generate \(\mathcal{D}_{qc}(X)\) [HR17a, Thm. A], so it suffices to prove that

\[
\text{RHom}_{O_X}(P, Rc_{qc,*}M \otimes_{O_X} Rc_{qc,*}N) \to \text{RHom}_{O_X}(P, Rc_{qc,*}(M \otimes_{O_X} N))
\]

is an isomorphism for all perfect \(P\). Since perfects are dualizable, the morphism above is an isomorphism if and only if the following morphism is an isomorphism:

\[
\Gamma(X, P^\vee \otimes_{O_X} Rc_{qc,*}M \otimes_{O_X} Rc_{qc,*}N) \to \Gamma(X, P^\vee \otimes_{O_X} Rc_{qc,*}(M \otimes_{O_X} N)).
\]

The projection formula (Lemma 4.3) and adjunction says that this morphism is an isomorphism if and only if the following is an isomorphism:

\[
\Gamma(X, Rc_{qc,*}(Lc^*P^\vee \otimes_{O_X} M) \otimes_{O_X} Rc_{qc,*}N)) \to \Gamma(X, Lc^*P^\vee \otimes_{O_X} M \otimes_{O_X} N).
\]

The claim follows.

For (2), we take \(M = c^*f^*(B) = i'_L O_Z\); then \(Rc_{qc,*}M \simeq R_i^* Rc_{Z,qc,*}O_Z \simeq f^*(B) \simeq Li^{**}N\). We next observe that the usual projection formula [HR17a, Cor. 4.12] implies that

\[
\Gamma(X, Lf^*(B) \otimes_{O_X} Rc_{qc,*}N) \simeq B \otimes_A^L \Gamma(X, Rc_{qc,*}N) \simeq B \otimes_A \Gamma(\chi, N).
\]

The claim now follows from (1). \(\square\)

**Remark 5.14.** Lemma 5.13 can easily be refined when \(X\) is quasi-affine:

1. \(c\) is faithful along \(M\) at \(N\) if and only if the following is an isomorphism:
   \[
   \Gamma(X, Rc_{qc,*}M \otimes_{O_X} Rc_{qc,*}N) \to \Gamma(X, M \otimes_{O_X} N).
   \]

2. If \(f: X \to \text{Spec} A\) is flat; \(B = A/I\), where \(I \subseteq A\) is an ideal; and
   \[
   A/I \otimes_A \Gamma(\chi, N) \to \Gamma(\chi, Li^{**}N)
   \]
   is an isomorphism; then \(c\) is a faithful along \(Z\) at \(N\).

We have the following non-noetherian and non-tor-independent example that comes from [Stacks, Tag 0DIA].

**Example 5.15.** Let \(\{A_n\}_{n \geq 0}\) be an inverse system of rings with surjective transition maps and locally nilpotent kernel. Let \(A = \varinjlim A_n\). Let \(X \to \text{Spec} A\) be a proper, flat and finitely presented morphism of algebraic spaces. Let \(\tilde{I}_n = \ker(A \to A_n)\) and let \(O_{X_n} = O_X/\tilde{I}_n O_{X_n}\). Let \(X\) be the ringed topos with underlying topos \(X_0\) and sheaf of rings \(O_X = \varprojlim O_{X_n}\) in \(\text{Mod}(X)\). There is a morphism of ringed topoi \(c: X \to X_{et}\) corresponding to \(O_X \to O_X\). Let \(i_n: X_n \to X\) and \(i'_n: X_n \to X\) be the resulting morphisms; note that \(c \circ i'_n = i_n\).
We claim that if \( M \in D_{\text{pc}}(X) \), then \( \Gamma(X, M) \in D_{\text{pc}}(A) \). For each \( n \geq 0 \) let \( M_n = \text{Li}^n_n M \in D_{\text{pc}}(X_n) \). By [Stacks, Tag 0CQF] and a local calculation, \( M \simeq \text{holim}_n \text{Li}^n_n M_n \) in \( D(X) \). Also \( \Gamma(X, \cdot) \) preserves homotopy limits, so

\[
M = \Gamma(X, M) \simeq \text{holim}_n \Gamma(X, \text{Li}^n_n M_n) \simeq \text{holim}_n \Gamma(X_n, M_n).
\]

Let \( M_n = \Gamma(X_n, M_n) \). Then \( M_n \) is a pseudo-coherent complex of \( A_n \)-modules (a special case of Kiehl’s Finiteness Theorem, see [Stacks, Tag 0CQF]) and the projection formula \([HR17a, \text{Cor. 4.12}]\) implies that:

\[
M_{n+1} \otimes_{X_{n+1}} A_n \simeq \Gamma(X_{n+1}, M_{n+1} \otimes_{X_{n+1}} \mathcal{O}_{X_n}) \simeq M_n.
\]

Thus, \( M \) is \( A \)-pseudo-coherent and \( \Gamma(X, M) \otimes_X A_n \simeq \Gamma(X_n, M_n) \) [Stacks, Tag 0CQF]. By Proposition 4.1, \( \text{Rec}_{qC, n} M \in D_{\text{pc}}(X) \).

In Setup 5.1, we take \( Z = X_0 \). By the above and Lemma 5.13(2), \( c \) is faithful along \( Z \). By Lemma 5.11, \( c \) is even an equivalence along \( Z \).

**Example 5.16.** Let \( X \) be a quasi-compact and quasi-separated algebraic space. Let \( L \) be an \( \mathcal{O}_X \)-line bundle on \( X \) and let \( s \in \Gamma(X, L) \). Let \( s^\vee : L^\vee \to \mathcal{O}_X \) be the dual morphism and let \( I = \text{im}(s^\vee) \) and \( K = \ker(s^\vee) \). Let \( i : Z \hookrightarrow X \) be the closed immersion defined by \( I \). That is, \( i \) is the vanishing locus of \( s \). Let \( \mathcal{O}_X \) be a sheaf of \( \mathcal{O}_X \)-algebras (not necessarily quasi-coherent) such that

1. \( \mathcal{O}_X/I \to \mathcal{O}_X/I \mathcal{O}_X \) is an isomorphism; and
2. \( K \cong \ker(s^\vee \otimes_{\mathcal{O}_X} \mathcal{O}_X) \).

This holds, for example, when \( X \) is locally noetherian, \( s \) is a regular section of \( L \) (i.e., \( K = 0 \)), and \( \mathcal{O}_X \) is the formal completion of \( X \) along \( I \). This also holds when \( X \) is non-noetherian [Stacks, Tag 0BNG]. More generally, it holds when \( s \) is a regular section of \( L \) and of \( c^*L \). Note that if \( s \) is a regular section, then \( s \) remaining a regular section of \( c^*L \) is easily seen to be equivalent to the tor-independence of \( c \) and \( i \).

Let \( Z = Z \) in Setup 5.1. Let \( \mathcal{X} = (X_{et}, \mathcal{O}_X) \) and \( C = [L^\vee \to \mathcal{O}_X] \), which is perfect. We will establish the following:

1. \( \mathcal{X} \) is an isomorphism;
2. \( c \) is faithful along \( c^*C \); and
3. if \( N \in D_{\text{pc}}(X) \) is such that \( N \otimes_{\mathcal{O}_X}^L C \) or \( N \otimes_{\mathcal{O}_X}^L K \) belongs to \( D_{qC}(X) \), then \( c \) is an equivalence along \( c^*C \) at \( N \).

Condition (3) is of course trivially satisfied when \( s \) is a regular section of \( L \).

We first prove (1). Consider the morphism of distinguished triangles:

\[
\begin{array}{ccc}
\mathcal{H}^{-1}(C)[1] & \longrightarrow & C \\
\downarrow & & \downarrow \\
\mathcal{H}^{-1}(\mathcal{R}_{qC}L^*C)[1] & \longrightarrow & \mathcal{R}_{qC}L^*C \\
\downarrow & & \downarrow \\
\mathcal{H}^{-1}(\mathcal{H}^0(\mathcal{R}_{qC}L^*C))[1] & \longrightarrow & \mathcal{H}^0(\mathcal{R}_{qC}L^*C)[0] \\
\downarrow & & \downarrow \\
\mathcal{H}^{-1}(C)[2] & \longrightarrow & \mathcal{H}^{-1}(C)[2].
\end{array}
\]

Now \( \mathcal{H}^{-0}(C) = \mathcal{O}_X/I \) and \( \mathcal{H}^{-0}(L^*C) = \mathcal{O}_X/I \mathcal{O}_X \), which are isomorphic by (a). Similarly, \( \mathcal{H}^{-1}(C) = \ker(L^\vee \to \mathcal{O}_X) \) and \( \mathcal{H}^{-1}(L^*C) = \ker((L^\vee \otimes_{\mathcal{O}_X} \mathcal{O}_X) \to \mathcal{O}_X) \) are isomorphic by (b). In particular, both \( \mathcal{H}^0(L^*C) \) and \( \mathcal{H}^{-1}(L^*C) \) are quasi-coherent \( \mathcal{O}_X \)-modules. This follows immediately that \( L^*C \) is a quasi-coherent \( \mathcal{O}_X \)-module and so \( \mathcal{R}_{qC}L^*C \simeq \mathcal{R}_cL^*C \simeq C \otimes_{\mathcal{O}_X}^L \mathcal{O}_X \) as \( \mathcal{O}_X \)-modules. The claim follows.

We next prove (2). Now condition (a) implies that \( \text{Mod}(Z) \simeq \text{Mod}(Z) \), so \( D_{\text{pc}}(Z) \simeq D_{\text{pc}}(Z) \). Since \( C \) is perfect, Lemma 4.3 and Proposition 5.8(1) together with (1) imply \( c \) is faithful along \( c^*C \). Claim (3) is similar, so its proof is omitted.

The previous example can be generalized to the vanishing locus of a section of a vector bundle.

**Example 5.17.** Let \( X \) be a quasi-compact and quasi-separated algebraic space. Let \( F \) be a vector bundle on \( X \) and let \( s \in \Gamma(X, F) \). Let \( i : Z \hookrightarrow X \) be the vanishing locus of \( s \). Let \( \mathcal{O}_X \) be a sheaf of \( \mathcal{O}_X \)-algebras (not necessarily quasi-coherent) such that \( K(s^\vee) \to K(s^\vee) \otimes_{\mathcal{O}_X} \mathcal{O}_X \) is a quasi-isomorphism of \( \mathcal{O}_X \)-modules, where \( K(s^\vee) \) is the Koszul complex associated to \( s^\vee : F^\vee \to \mathcal{O}_X \).
[FL85, §IV.2]. If $s$ is a regular section of $F$, then this condition is equivalent to $s$ remaining a regular section of $F \otimes_{\mathcal{O}_X} \mathcal{O}_X$. As before, this condition is satisfied when $s$ is a regular section and $\mathcal{O}_X$ is the formal completion of $\mathcal{O}_X$ along $f = \text{im}(s')$. If $F$ is a line bundle, then $K(s') = [F^\vee \xrightarrow{\sim} \mathcal{O}_X]$. Hence, we see that this condition is equivalent to those in Example 5.16. Arguing as in Example 5.16, one can establish the following:

1. $c$ is faithful along $\text{Le}^* K(s')$; and
2. if $N \in D^-_{pc}(X)$ is such that $N \otimes_{\mathcal{O}_X} L(s') \in D_{qc}(X)$, then $c$ is an equivalence along $\text{Le}^* K(s')$ at $N$.

6. **Lefschetz Theorems**

To illustrate the strength of our reformulation, we can give a brief proof of the following Lefschetz theorem.

**Theorem 6.1.** Let $X$ be a quasi-compact and quasi-separated algebraic space. Let $c: \mathcal{X} \to X_{\text{ét}}$ be a morphism of ringed topos. Let $i: Z \to X$ be a closed immersion and let $r \geq 0$ be an integer. If

1. $U = X - Z$ is quasi-affine;
2. $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$ is $r$-connected; and
3. there exists $\mathcal{M} \in D(X)$ such that
   
   - $R_{\mathcal{O}_X, *} \mathcal{M}$ is perfect with cohomological support $|Z|$, and
   - $c$ is faithful along $\mathcal{O}_X$;

then $\mathcal{O}_X \to R_{\mathcal{O}_X, *} \mathcal{O}_X$ is $r$-connected. In particular, if $N \in D_{qc}(X)$ is perfect of tor-amplitude $\geq a$, then $\Gamma(X, N) \to \Gamma(X, \text{Le}^* N)$ is $r + a$-connected.

**Proof.** By the projection formula (Lemma 4.3), $\pi_{\mathcal{O}_X, \mathcal{M}}$ is an isomorphism. Thus, Proposition 5.12 implies that $R_{\mathcal{O}_X, *} \mathcal{M} \otimes \eta_{\mathcal{O}_X}$ is an isomorphism. Let $H$ be a cone for $\eta_{\mathcal{O}_X}: \mathcal{O}_X \to R_{\mathcal{O}_X, *} \mathcal{O}_X$; then we have just proved that $H \otimes_{\mathcal{O}_X} R_{\mathcal{O}_X, *} \mathcal{M} \simeq 0$ (see (A.8)). It remains to prove that $\tau^{\leq r} H \simeq 0$. Let $j: U \to X$ be the resulting open immersion. The theory of smashing Bousfield localization implies immediately that $H \simeq R_{j_*} j^* H$ (e.g., [HR17b, Ex. 1.4]). Now $\tau^{\leq -2} H \simeq 0$ and so

$\Gamma(U, \mathcal{H}^{-1}(j^* H)) \simeq H^{-1}(U, j^* H) \simeq \mathcal{H}^{-1}(X, H) \simeq 0$.

Since $U$ is quasi-affine, $\mathcal{H}^{-1}(j^* H) = 0$ and so $\mathcal{H}^{-1}(H) \simeq j_* \mathcal{H}^{-1}(j^* H) \simeq 0$. That is, $\tau^{\leq -1} H \simeq 0$. Repeating this argument, we obtain that $\tau^{\leq r} H \simeq 0$; that is, $\mathcal{O}_X \to R_{\mathcal{O}_X, *} \mathcal{O}_X$ is $r$-connected. \hfill $\square$

In the tor-independent case, we have the following variant of Theorem 6.1.

**Corollary 6.2.** Let $X$ be a quasi-compact and quasi-separated algebraic space. Let $c: \mathcal{X} \to X_{\text{ét}}$ be a morphism of ringed topos. Let $i: Z \hookrightarrow X$ be a closed immersion and let $r \geq 0$ be an integer. If

1. $U = X - Z$ is quasi-affine;
2. the morphism $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$ is $r$-connected;
3. $c$ and $i$ are tor-independent;
4. $\mathcal{O}_X$ is coherent or $i_* \mathcal{O}_Z$ is perfect, and
5. $D_{pc}(X, \mathcal{O}_X) \simeq D_{pc}(X, c^* i_* \mathcal{O}_Z)$;

then $\mathcal{O}_X \to R_{\mathcal{O}_X, *} \mathcal{O}_X$ is $r$-connected. In particular, if $N \in D_{qc}(X)$ is perfect of tor-amplitude $\geq a$, $\Gamma(X, N) \to \Gamma(X, c^* E)$ is $r + a$-connected.

**Proof.** If $i_* \mathcal{O}_Z$ is perfect, then set $\mathcal{M} = \text{Le}^* i_* \mathcal{O}_Z$. If $\mathcal{O}_X$ is coherent, then set $\mathcal{M} = \text{Le}^* M$, where $M$ is as in Lemma 5.10. Since $c$ and $i$ are tor-independent, Corollary 5.9(2) (in the perfect case) and Corollary 5.10 (in the coherent case) imply that $c$ is faithful along $\mathcal{M}$. Moreover, Lemma 5.6 implies that $R_{\mathcal{O}_X, *} \mathcal{M} = R_{\mathcal{O}_X, *} \text{Le}^* M \simeq M$, which is perfect with cohomological support $|Z|$. The result now follows from Theorem 6.1. \hfill $\square$
In the following theorem, we can optimize the above results substantially in the case of a Cartier divisor, making them amenable to an inductive process.

**Theorem 6.3.** Let $X$ be a quasi-compact and quasi-separated algebraic space. Let $c : X \to X_{ét}$ be a morphism of ringed topoi. Let $L$ be a line bundle on $X$, $s \in \Gamma(X, L)$, and $i : Z \hookrightarrow X$ its vanishing locus. Let $r \geq 0$ be an integer. If

1. $U = X - Z$ is quasi-affine;
2. $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$ is $r$-connected; and either
3. (a) $C \to \mathcal{Rc}_{qc}* C$ is $r$-connected, where $C = [L^s \to \mathcal{O}_X]$; or
   (b) $c$ and $i$ are tor-independent and either
   (i) $\mathcal{O}_Z \to \mathcal{Rc}_{Z, qc} \mathcal{O}_Z$ is $r$-connected; or
   (ii) $Z$ is quasi-affine and $\Gamma(Z, \mathcal{O}_Z) \to \Gamma(Z, \mathcal{O}_Z)$ is $r$-connected;
then $\mathcal{O}_X \to \mathcal{Rc}_{qc} \mathcal{O}_X$ is $r$-connected. In particular, if $N \in D_{qc}(X)$ is perfect of tor-amplitude $\geq a$, $\Gamma(X, E) \to \Gamma(X, e^*E)$ is $r + a$-connected.

**Proof.** Let $H_N$ be the cone of $\eta_N$ and set $H = H_{0_X}$; it suffices to show that $\tau^{\leq r} H \simeq 0$. Since $C$ is perfect, $\tau_{C,L^s \mathcal{O}_X}$ is an isomorphism (Lemma 4.3). In particular, $H \otimes_{\mathcal{O}_X} C \simeq H_C$. By condition (3), we conclude that $\tau^{\leq r} (H \otimes_{\mathcal{O}_X} C) \simeq 0$. Now consider the distinguished triangle:

$$L^s \otimes_{\mathcal{O}_X} H \rightarrow H \rightarrow C \otimes_{\mathcal{O}_X} \mathcal{O}_X \rightarrow L^s \otimes_{\mathcal{O}_X} H[1].$$

But $L^s$ is a line bundle, so $\tau^{\leq r} (L^s \otimes_{\mathcal{O}_X} H) \simeq L^s \otimes_{\mathcal{O}_X} H$. It follows immediately from the distinguished triangle above that $C \otimes_{\mathcal{O}_X} \tau^{\leq r} H \simeq 0$. Let $j : U = X - Z \hookrightarrow X$ be the resulting open immersion, which is affine; then the theory of smashing Bousfield localizations implies immediately that $\tau^{\leq r} H \simeq Rj_* L^s \tau^{\leq r} H$ (e.g., [HR17b, Ex. 1.4]). Noting that $\tau^{\geq r} \Gamma(X, H) \simeq \tau^{\geq r} \mathcal{R} \Gamma(X, \tau^{\geq r} H)$ for all $j \in J$, now argue as in Theorem 6.1. \hfill \Box

### 7. Pseudo-conservation

Let $X$ be a ringed space or quasi-compact and quasi-separated algebraic space. Let $|X|_cl$ be the set of closed points of $X$. The collection $\{\kappa(x)\}_{x \in |X|_cl}$ is pseudo-conservative. This is immediate from Nakayama’s Lemma.

**Example 7.2.** Let $A$ be a ring. Let $X \to \text{Spec } A$ be a quasi-compact and closed morphism of algebraic spaces. Let $I \subseteq A$ be an ideal contained in the Jacobson radical of $A$. Let $X_0 = X \times_{\text{Spec } A} \text{Spec } (A/I)$ and take $i : X_0 \to X$ be the resulting closed immersion. Then $\{\mathcal{O}_{X_0}\}$ is pseudo-conservative. Indeed, if $M \in D_{qc}(X)$ is non-zero, then its top cohomology group $H^{top}(M)$ is finitely generated. It follows that its support $W$ is a non-empty closed subset of $X$. Hence, the image of $W$ in $\text{Spec } A$ is closed and non-empty. Since $I$ is contained in the Jacobson radical of $A$, $W$ meets $\text{Spec } (A/I)$.

We have the following useful lemma.

**Lemma 7.3.** Let $X$ be a ringed topos. Let $S \subseteq D^-(X)$ be a collection of objects. Consider

$$S' = \{t^t(\mathcal{O}) : \mathcal{O} \in S\} \subseteq \text{Mod}(X),$$

where $t(\mathcal{O})$ denotes the top cohomological degree of $\mathcal{O}$. If $\mathcal{O}_X$ is coherent, then the following are equivalent:

1. $S$ is pseudo-conservative;
2. $t(\mathcal{O})$ and $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O} \simeq 0$ for all $\mathcal{O} \in S'$, then $\mathcal{M} \equiv 0$.

**Proof.** This is immediate from the following: if $M, N \in D^-(X)$, then

$$\text{tt}^{t(\mathcal{M}) + t(\mathcal{N})} (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) \equiv \text{tt}^{t(\mathcal{M})} (M) \otimes_{\mathcal{O}_X} \text{tt}^{t(\mathcal{N})} (N),$$

\hfill \Box
8. GAGA

In this section, we prove our general GAGA theorem. We will see in §9 that this implies all existing results in the literature for algebraic spaces. Given what we have already established, its proof is straightforward.

Theorem 8.1. Let $X$ be a quasi-compact and quasi-separated algebraic space. Let $c: \mathcal{X} \to X_{\text{ét}}$ be a morphism of ringed topoi. Let $\Lambda$ index a family of 2-commutative diagrams of ringed topoi:

$$
\begin{array}{ccc}
Z_\Lambda & \xrightarrow{c_\Lambda} & Z_{\Lambda,\text{ét}} \\
\downarrow{i_\Lambda} & & \downarrow{i_{\Lambda,\text{ét}}} \\
\mathcal{X} & \xrightarrow{c} & X_{\text{ét}},
\end{array}
$$

where $i_\Lambda$ is quasi-affine for all $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, let $M_\lambda \in \langle \text{Ri}_{\Lambda,\text{qc}}^{-}(\mathcal{Z}_\Lambda) \rangle$.

(i) Let $N \in \mathcal{D}_{\text{qc}}(X)$. Assume that $\text{Rc}_{\text{qc},\ast}N \in \mathcal{D}_{\text{qc}}^{-}(X)$ and $\{\text{Rc}_{\text{qc},\ast}M_\lambda\}_{\lambda \in \Lambda}$ is pseudo-conservative. If $c$ is faithful along $M_\lambda$ at $\text{Le}^\ast N$ for all $\lambda \in \Lambda$, then

$$
\eta_M: M \to \text{Rc}_{\text{qc},\ast}\text{Le}^\ast M
$$

is an isomorphism.

(ii) Let $N \in \mathcal{D}_{\text{qc}}(X)$. Assume that $\text{Rc}_{\text{qc},\ast}N \in \mathcal{D}_{\text{qc}}^{-}(X)$ and $\{M_\lambda\}_{\lambda \in \Lambda}$ is pseudo-conservative. If $c$ is an equivalence along $M_\lambda$ at $N$ for all $\lambda \in \Lambda$, then

$$
\epsilon_M: \text{Le}^\ast\text{Rc}_{\text{qc},\ast}M \to M
$$

is an isomorphism.

In addition assume that $\ast \in \{b, -\}$ and

1. $X$ is proper and pseudo-coherent over an affine scheme $\text{Spec} A$;
2. $\text{Ri}(X, -)$ sends $\mathcal{D}_{\text{qc}}^\ast(X)$ to $\mathcal{D}_{\text{qc}}^\ast(A)$; and
3. if $\ast = b$, then $\text{Le}^\ast$ sends $\mathcal{D}_{\text{qc}}^b(X)$ to $\mathcal{D}_{\text{qc}}^b(Y)$.

If $\{\text{Rc}_{\text{qc},\ast}M_\lambda\}_{\lambda \in \Lambda}$ (resp. $\{M_\lambda\}_{\lambda \in \Lambda}$) is pseudo-conservative and $c$ is faithful (resp. an equivalence) by $M_\lambda$ for all $\lambda \in \Lambda$, then

$$
\text{Le}^\ast: \mathcal{D}_{\text{qc}}^\ast(X) \to \mathcal{D}_{\text{qc}}^\ast(Y)
$$

is fully faithful (resp. essentially surjective).

Proof. For (i), by Lemma 5.6 and Proposition 5.12, we have that $\text{Rc}_{\text{qc},\ast}M_\lambda \otimes \eta_N$ is an isomorphism for all $\lambda \in \Lambda$. But $N$ and $\text{Rc}_{\text{qc},\ast}N \in \mathcal{D}_{\text{qc}}^\ast(X)$, so $\text{cone}(\eta_N) \in \mathcal{D}_{\text{qc}}^{-}(X)$. Since $\{\text{Rc}_{\text{qc},\ast}M_\lambda\}_{\lambda \in \Lambda}$ is pseudo-conservative, the claim follows.

For (ii), $M_\lambda \otimes \epsilon_N$ is an isomorphism for all $\lambda \in \Lambda$. But $N$, $\text{Le}^\ast\text{Rc}_{\text{qc},\ast}N \in \mathcal{D}_{\text{qc}}^{-}(X)$, so $\text{cone}(\epsilon_N) \in \mathcal{D}_{\text{qc}}^\ast(X)$. Since $\{M_\lambda\}_{\lambda \in \Lambda}$ is pseudo-conservative, the claim follows.

The last claim is immediate from the above and Proposition 4.1. □

Remark 8.2. In Theorem 8.1(i), if we do not assume that $\text{Rc}_{\text{qc},\ast}N$ belongs to $\mathcal{D}_{\text{qc}}^{-}(X)$, but the $\{\text{Rc}_{\text{qc},\ast}M_\lambda\}_{\lambda \in \Lambda}$ are conservative (i.e., if $Q \in \mathcal{D}_{\text{qc}}(X)$ and $Q \otimes \mathcal{D}_{\text{qc}}\text{Rc}_{\text{qc},\ast}M_\lambda \simeq 0$ for all $\lambda \in \Lambda$ implies that $Q \simeq 0$), then we get the same conclusion.

9. Applications

We begin with the following tor-independent refinement of Theorem 8.1.

Theorem 9.1. Let $A$ be a ring. Let $\pi: X \to \text{Spec} A$ be a proper and pseudo-coherent morphism of algebraic spaces. Let $c: \mathcal{X} \to X_{\text{ét}}$ be a morphism of ringed topoi. Let $\Lambda$ index a family of 2-commutative diagrams of ringed topoi:

$$
\begin{array}{ccc}
Z_\Lambda & \xrightarrow{c_\Lambda} & Z_{\Lambda,\text{ét}} \\
\downarrow{i_\Lambda} & & \downarrow{i_{\Lambda,\text{ét}}} \\
\mathcal{X} & \xrightarrow{c} & X_{\text{ét}},
\end{array}
$$
where $i_\lambda$ is affine for all $\lambda \in \Lambda$. Let $* \in \{b, -\}$. Assume that

(a) $R^*({\mathcal F}, -)$ sends $D^+_pc(X)$ to $D^+_pc(A)$;
(b) for all $\lambda \in \Lambda$, $c$ and $i_\lambda$ are tor-independent;
(c) for all $\lambda \in \Lambda$, $D^+_pc({\mathcal O}_Z) \cong D^+_pc({\mathcal O}_X)$;
(d) for all $\lambda \in \Lambda$, $L^c_\lambda: D^+_pc({\mathcal O}_Z) \to D^+_pc({\mathcal O}_X)$ is an equivalence;
(e) if $* = b$, then $L^c_\lambda$ sends $D^+_pc(X)$ to $D^+_pc(A)$;
(f) $\mathcal O_X$ is coherent; or $\mathcal O_X$ is a compact object of $D(X)$; or for all $\lambda \in \Lambda$, $i_\lambda, {\mathcal O}_Z$ is perfect.

If $\{\mathcal O_{Z_\lambda}\}_{\lambda \in \Lambda}$ (resp. $\{\mathcal O_{Z_\lambda}\}_{\lambda \in \Lambda}$) is pseudo-conservative, then

$L^c_*: D^+_pc(X) \to D^+_pc(X)$

is fully faithful (resp. essentially surjective).

Proof. In the case where $\mathcal O_X$ is a compact object of $D(X)$ or $i_\lambda, {\mathcal O}_Z$ is perfect for all $\lambda \in \Lambda$, the result is immediate from Theorem 8.1 and Corollary 5.9. In the case where $\mathcal O_X$ is coherent, we apply Corollary 5.10 to produce a perfect complex $M_\lambda \in (R_\zeta D^+_pc({\mathcal O}_X))$ with $c$ an equivalence along $L^c_\lambda M_\lambda$ for all $\lambda \in \Lambda$. But if $\{\mathcal O_{Z_\lambda}\}_{\lambda \in \Lambda}$ (resp. $\{\mathcal O_{Z_\lambda}\}_{\lambda \in \Lambda}$) is pseudo-conservative, then $\{M_\lambda\}_{\lambda \in \Lambda}$ (resp. $\{L^c_\lambda M_\lambda\}_{\lambda \in \Lambda}$) is pseudo-conservative (Lemma 7.3). Now apply Theorem 8.1.

It is easy to use Theorems A, 8.1, and 9.1 to prove existing GAGA results.

Example 9.2 (Analytic spaces). Let $X \to \text{Spec } C$ be a proper scheme. Let $c: X_\text{an} \to X$ be its complex analytification. Now $X_\text{an}$ is a Hausdorff topological space and $c$ is bijective on closed points; indeed $|X_\text{an}| = X(\text{Spec } C)$. Also, the local rings of $\mathcal O_{X_\text{an}}$ are noetherian and the induced morphism $\mathcal O_{X, c(x)} \to \mathcal O_{X_\text{an}, x}$ is an isomorphism on maximal-adic completions [SGA1, XII.1.1]. By Remark 1.1, we see that conditions (3), (4), and (5) are satisfied. Cartan–Serre [CS53] (also see [GR84, 10.5.6]) gives condition (2). Condition (1) is Oka’s Coherence Theorem (see [Oka50] and [GR84, 2.5.3]). By Theorem A we may conclude that if $F \in \text{Coh}(X)$, then

$H^*(X, F) \cong H^*(X_\text{an}, F_\text{an})$

and $c^*: \text{Coh}(X) \to \text{Coh}(X_\text{an})$ is an equivalence.

Example 9.3 (Formal GAGA). Let $X \to \text{Spec } R$ be a proper morphism of schemes. Assume that $R$ is noetherian. Let $I \subseteq R$ be an ideal and assume that $R$ is complete with respect to the $I$-adic topology. Let $c: \hat{X} \to X$ be the formal completion of $X$ along the closed subscheme $X_0 = X \otimes_R (R/I)$. It is easily verified using the results of [EGA, III.1] that $c$ satisfies the hypotheses of Theorem A. Hence, we have the cohomological comparison result and the equivalence on categories of sheaves. It is also easy to use these arguments and Theorem 9.1 to prove formal GAGA for proper algebraic spaces. We again leave this as an exercise to the reader. One can also use these arguments to prove the formal GAGA statements of [FK18], which hold for certain non-noetherian base rings $A$ (e.g., $A$ is the $a$-adic completion of a finitely presented $V$-algebra, where $V$ is an $a$-adically complete valuation ring.).

Example 9.4 (Rigid GAGA). Let $X \to \text{Spec } R$ be a proper morphism of schemes. Let $k$ be a complete nonarchimedean field. Assume that $R$ is an affinoid $k$-algebra; that is, it is a Banach $k$-algebra that is a quotient of some Tate algebra $T_n = k(\langle Y_1, \ldots, Y_n \rangle)$, where $Y_i$ is the subalgebra of $k[[Y_1, \ldots, Y_n]]$ consisting of power series that are convergent with respect to the Gauss norm (i.e., supernum of coefficients). Associated to $X$ is a natural morphism of locally ringed $G$-spaces $c: X_{\text{rig}} \to X$, where $X_{\text{rig}}$ is a rigid analytic space. The underlying topological space of $X_{\text{rig}}$ is Hausdorff and its points correspond to closed points of $X$. Moreover, $\mathcal O_{X_{\text{rig}}}$ is a coherent sheaf with noetherian local rings. Also, $R$ is noetherian. Kiehl’s Finiteness Theorem [Kie67] implies that the cohomology of coherent sheaves on $\mathcal O_{X_{\text{rig}}}$ satisfies the condition (2) of Theorem A. Again, we get the cohomological comparison result and equivalence on categories of coherent sheaves. Using [CT09], one can make sense of rigid analytifications of separated algebraic spaces. This allows one to prove rigid GAGA in this context too. One can also prove adic and Berkovich GAGA statements using this method.
Example 9.5 (Non-noetherian formal GAGA). Here we will use Theorem 8.1 to prove the GAGA result in the Stacks Project [Stacks, Tag 0DIA]. The situation is as in Example 5.15, and it is immediate from 8.1 that we obtain an equivalence

\[ \text{Lc}^* : \text{D}^{\text{pc}}_\text{pro}(X) \to \text{D}^{\text{pc}}_\text{pro}(\hat{X}). \]

The following is a variant of the results established in [BJ14, §1]. It is a simple consequence of Theorem 6.3.

**Theorem 9.6.** Let \( X \) be a quasi-compact and quasi-separated algebraic space. Let \( i : D \subseteq X \) be a Cartier divisor. Let \( c : \hat{X} \to X \) be the \( D \)-adic completion. If

1. \( X - D \) is quasi-affine;
2. \( H^0(X, \mathcal{O}_X) \cong \varprojlim_{r \geq 0} H^0(X, \mathcal{O}_{D_r}); \) and
3. \( H^i(X, \mathcal{O}_X(-r\hat{D})) = 0 \) for all \( r \gg 0; \)

then the comparison morphism:

\[ H^i(X, E) \to H^i(\hat{X}, \hat{E}) \]

is an isomorphism for all vector bundles \( E \) on \( X \).

**Proof.** We simplify verify that the hypotheses of Theorem 6.3 are satisfied. General properties of completion say that \( c \) and \( i \) are tor-independent [Stacks, Tag 0BNG]. By Example 5.2, it remains to prove that \( H^i(X, \mathcal{O}_X) \to H^i(\hat{X}, \mathcal{O}_{\hat{X}}) \) is an isomorphism when \( i = 0 \) and is injective when \( i = 1 \). The \( i = 0 \) condition is immediate from (2). For the \( i = 1 \) case, we have the Mittag–Leffler exact sequence:

\[ 0 \longrightarrow \varprojlim_{n} H^1(X, \mathcal{O}_{D_n}) \longrightarrow H^1(\hat{X}, \mathcal{O}_{\hat{X}}) \longrightarrow \lim_{n} H^1(X, \mathcal{O}_{D_n}) \longrightarrow 0. \]

It follows from (3) that \( \{H^0(X, \mathcal{O}_{D_r})\}_{n \geq 0} \) is eventually a surjective system, so is Mittag–Leffler. In particular, \( \varprojlim_{n} \) vanishes. Hence, \( H^1(\hat{X}, \mathcal{O}_{\hat{X}}) \cong \varprojlim_{n} H^1(X, \mathcal{O}_{D_n}). \) But (3) also gives \( H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_{\hat{X}}) \) for \( n \gg 0 \). Since inverse limits are left exact, this gives \( H^1(X, \mathcal{O}_X) \to \varprojlim_{n} H^1(X, \mathcal{O}_{D_n}); \) the result follows.

**Remark 9.7.** Theorem 9.6(2) is implied by \( H^0(X, \mathcal{O}_X(-r\hat{D})) = 0 \) for all \( r \gg 0 \).

**Example 9.8.** Let \( X \) be a quasi-affine scheme. Let \( A = \Gamma(X, \mathcal{O}_X) \) and \( a \in A \). Let \( D = \text{Spec}(A/a) \cap X \subseteq X \) and let \( c : \hat{X} \to X \) be the formal completion of \( X \) along \( D \). Assume

1. \( A \) is \( a \)-adically complete;
2. \( a \) is not a zero divisor of \( A \); and
3. \( a^n H^1(X, \mathcal{O}_X) \cong 0 \) for some \( N \geq 1 \).

Then \( H^0(X, E) \to H^0(\hat{X}, \hat{E}) \) is an isomorphism for all vector bundles \( E \). This follows immediately from Theorem 9.6. For example, let \( (A, \mathfrak{m}) \) be an \( \mathfrak{m} \)-adically complete noetherian local ring and \( X = \text{Spec} A - \{ \mathfrak{m} \}. \) Let \( a \in \mathfrak{m} \) be a non-zero-divisor and let \( c : \hat{X} \to X \) be the \( a \)-adic completion of \( X \) (as a formal scheme). If 

\[ \text{depth}_a(A/a) \geq 2 \quad (\text{e.g., } A/a \text{ is } S_2 \text{ or normal or Cohen–Macaulay}), \]

then the conditions are satisfied.

**Example 9.9.** Let \( X \) be an \( S_2 \)-variety over a field \( k \) (e.g. normal or smooth). Assume \( X \) is projective and \( D \subseteq X \) is an ample divisor. Let \( c : \hat{X} \to X \) be the \( D \)-adic completion. Then \( c^* : \text{Vect}(X) \to \text{Vect}(\hat{X}) \) is fully faithful. Since \( D \) is ample and \( X \) is projective, \( X - D \) is affine. Theorem 9.6 and Remark 9.7 now show it is sufficient to prove that \( H^i(X, \mathcal{O}_X(-r\hat{D})) = 0 \) for \( r \gg 0 \) and \( i = 0, 1 \). The \( i = 0 \) case is [Stacks, Tag 0FD7] and the \( i = 1 \) case is Enrique–Severi–Zariski vanishing [Stacks, Tag 0FD8]. This is closely related to the Lefschetz Hyperplane Theorem.

**Appendix A. The projection formula**

We recall some results on the projection formula. An excellent source is [FHM03]. Let \( (\mathcal{C}, \otimes, \alpha, \sigma, 1, \lambda, \rho) \) be a symmetric monoidal category. That is,

- \( \mathcal{C} \) is a category;
- \( - \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) is a functor;
Typically, we will just denote this data by \( C \). For background material on symmetric monoidal categories, we refer the interested reader to \([SR72, ML98, ML63, Eps66]\).

Let \( L: \mathcal{C} \to \mathcal{D} \) be a functor, where \( \mathcal{C} \) and \( \mathcal{D} \) are symmetric monoidal categories. We say \( L \) is **lax monoidal** if for each \( c_1, c_2 \in \mathcal{C} \) there is a natural morphism:

\[
\mu_{c_1, c_2}: L(c_1) \otimes L(c_2) \to L(c_1 \otimes c_2)
\]

and a morphism

\[
\iota: 1 \to L(I)
\]

that is all compatible with the monoidal structures in the obvious way. We denote this package of data by \( (L, \mu, \iota) \). If \( \mu_{c_1, c_2} \) and \( \iota \) are always isomorphisms, then we say that \( L \) is **strong monoidal**. We have the following trivial lemma.

**Lemma A.1.** Consider a sequence of lax monoidal functors between symmetric monoidal categories:

\[
\mathcal{C}_1 \xrightarrow{(L_1, \mu_{L_1} \cdot \iota_{L_1}^{\cdot i})} \mathcal{C}_2 \xrightarrow{(L_2, \mu_{L_2} \cdot \iota_{L_2}^{\cdot i})} \cdots \xrightarrow{(L_n, \mu_{L_n} \cdot \iota_{L_n}^{\cdot i})} \mathcal{C}_{n+1}.
\]

(1) Then \( L_n \cdots L_1: \mathcal{C}_1 \to \mathcal{C}_{n+1} \) is lax monoidal via the inductively defined:

\[
\mu_{c_1, c_2} \circ L_n \mu_{L_{n-1} \cdots L_1} = L_n (\mu_{L_{n-1} \cdots L_1 \mathcal{C}_1(c_1), L_{n-1} \cdots L_1(c_2)}),
\]

\[
\iota_{L_n \cdots L_1} = L_n (\iota_{L_{n-1} \cdots L_1}).
\]

If the \( L_i \) are all strong monoidal, then so too is the composition \( L_n \cdots L_1 \).

(2) For \( j \geq i \) consider another sequence of lax monoidal functors:

\[
\mathcal{C}_i \xrightarrow{(L'_i, \mu_{L'_i} \cdot \iota_{L'_i}^{\cdot i})} \mathcal{C}_{i+1} \xrightarrow{(L'_{i+1}, \mu_{L'_{i+1}} \cdot \iota_{L'_{i+1}}^{\cdot i})} \cdots \xrightarrow{(L'_{j}, \mu_{L'_{j}} \cdot \iota_{L'_{j}}^{\cdot i})} \mathcal{C}_{j+1}
\]

together with a natural transformation

\[
\kappa: (L_i \cdots L_i, \mu_{L_i \cdots L_i} \cdot \iota_{L_i \cdots L_i}) \Rightarrow (L'_i \cdots L'_i, \mu_{L'_i \cdots L'_i} \cdot \iota_{L'_i \cdots L'_i}).
\]

Then there is a natural extension of \( \kappa \) to a natural transformation:

\[
(L_n \cdots L_1, \mu_{L_n \cdots L_1} \cdot \iota_{L_n \cdots L_1}) \Rightarrow (L'_n \cdots L'_1, \mu_{L'_n \cdots L'_1} \cdot \iota_{L'_n \cdots L'_1}).
\]

Now assume that \( L: \mathcal{C} \to \mathcal{D} \) is strong monoidal and consider a right adjoint

\[
R: \mathcal{D} \to \mathcal{C}.
\]

If \( c \in \mathcal{C} \) and \( d \in \mathcal{D} \), then we have the resulting unit/counit morphisms

\[
\eta_c: c \to RL(c) \quad \text{and} \quad \epsilon_d: LR(d) \to d.
\]

If \( d_1, d_2 \in \mathcal{D} \), then there is a natural conjugate of \( \mu_{c_1, c_2} \),

\[
\nu_{d_1, d_2}: R(d_1) \otimes R(d_2) \to R(d_1 \otimes d_2).
\]
It is obtained as the adjoint to the composition:

\[ L(R(d_1) \otimes_C R(d_2)) \xrightarrow{\mu_{R(d_1),R(d_2)}^{-1}} LR(d_1) \otimes_D LR(d_2) \xrightarrow{\epsilon_d \otimes \epsilon_d} d_1 \otimes_D d_2. \]

Similarly, there is a conjugate to \( c \):

\[ j: 1_c \to R(1_D). \]

It is obtained as the adjoint to \( \iota^{-1} : L(1_E) \to 1_D \). It is easily verified that \((R, \nu, j)\) is lax monoidal.

Now if \( c \in C \) and \( d \in D \), then there is a natural projection morphism

\[ \pi_{c,d}: c \otimes_C R(d) \to R(L(c) \otimes_D d). \]

Indeed, it is given as the composition:

\[ c \otimes_C R(d) \xrightarrow{\eta_c \otimes \text{Id}(R(d))} RL(c) \otimes_C R(d) \xrightarrow{\nu_{L(c),R(d)}} R(L(c) \otimes_D d). \]

**Remark A.3.** Note that if \( L \) is an equivalence, then \( \pi_{c,d} \) is an isomorphism.

There is another way to produce a projection morphism

\[ \tilde{\pi}_{c,d}: c \otimes_C R(d) \to R(L(c) \otimes_D d). \]

It can be given as the adjoint to the composition:

\[ L(c \otimes_C R(d)) \xrightarrow{\mu_{c,R(d)}^{-1}} L(c) \otimes_D LR(d) \xrightarrow{\text{Id}_{L(c)} \otimes \epsilon_d} L(c) \otimes_D d. \]

We wish to point out that \( \nu \) (and so consequently \( \pi \)) depend on the choice of the right adjoint \( R \). Occasionally, it will be useful to observe this, and we do so by using a suitable superscript (e.g., \( \nu^{L,R} \)).

**Lemma A.4.** \( \pi_{c,d} = \tilde{\pi}_{c,d} \).

**Proof.** The adjoint to \( \pi_{c,d} \) factors as:

\[ L(c \otimes_C R(d)) \xrightarrow{L(\eta_c \otimes \text{Id})} L(RL(c) \otimes_C R(d)) \xrightarrow{L(\nu_{RL(c),R(d)}) \mu_{L(c),R(d)}^{-1}} LRL(c) \otimes_D LR(d) \xrightarrow{\epsilon_{L(c)} \otimes \epsilon_d} L(c) \otimes_D d. \]

The following square also commutes, by naturality:

\[ \begin{array}{c}
L(c \otimes_C R(d)) \\
\xrightarrow{L(\eta_c \otimes \text{Id})} \\
\xrightarrow{L(\mu_{RL(c),R(d)}^{-1})} \\
L(c) \otimes_D LR(d) \\
\xrightarrow{L(\eta_c) \otimes \text{Id}} \\
\xrightarrow{L(\nu_{L(c),R(d)})} \\
LRL(c) \otimes_D LR(d). \\
\end{array} \]

Hence, the adjoint to \( \pi_{c,d} \) factors as:

\[ L(c \otimes_C R(d)) \xrightarrow{\mu_{c,R(d)}^{-1}} L(c) \otimes_D LR(d) \xrightarrow{L(\eta_c) \otimes \text{Id}} LRL(c) \otimes_D LR(d) \xrightarrow{\epsilon_{L(c)} \otimes \epsilon_d} L(c) \otimes_D d. \]

By the unit/ counit equations for adjunction, the composition of the last two morphisms results in \( \text{Id} \otimes \epsilon_d \). The result now follows.

**Remark A.5.** If \( \kappa: (L, \mu, \iota) \Rightarrow (L', \mu', \iota') \) is a natural transformation of strong monoidal functors, then there is a canonically induced natural transformation \( \kappa^\vee: R' \Rightarrow R \) between chosen right adjoints. If \( c \in C \) and \( d \in D \), then it is easily verified from Lemma A.4 that the following diagram commutes:

\[ \begin{array}{c}
c \otimes_C R'(d) \\
\xrightarrow{\pi_{c,d}', \nu'} \\
\xrightarrow{\kappa^\vee} \\
R'(L'(c) \otimes_D d) \\
\end{array} \quad \begin{array}{c}
c \otimes_C R(d) \\
\xrightarrow{\pi_{c,d} \nu} \\
\xrightarrow{R(\kappa)} \\
R(L(c) \otimes_D d) \\
\end{array}. \]

In the following lemma we record some useful commutative diagrams.
Lemma A.6. Let $c, x \in \mathcal{C}$ and $d \in \mathcal{D}$. The following diagrams commute.

\begin{align}
L(c \otimes_c R(d)) \xrightarrow{\mu_{c,R(d)}} L(c) \otimes_{\mathcal{D}} L R(d)
\end{align}

\begin{align}
R L (L(c) \otimes d) \xrightarrow{\epsilon_{L,c \otimes d}} L(c) \otimes_{\mathcal{D}} d.
\end{align}

Proof. The diagram (A.7) is just a restatement of Lemma A.4. For the commutativity of (A.8), we observe that the adjoint of the composition going right and then down is simply $\mu_{c,x}^{-1}$. Working the other way, we see that the adjoint map is the composition:

\begin{align}
L(c \otimes_c x) \xrightarrow{L(Id \otimes \eta_c)} L(c \otimes_c RL(x)) \xrightarrow{\mu_{c,RL(x)}} L(c) \otimes_{\mathcal{D}} LRL(x) \xrightarrow{Id \otimes L \epsilon_{x}} L(c) \otimes_{\mathcal{D}} L(x).
\end{align}

By functoriality and naturality, the following diagram commutes:

\begin{align}
L(c \otimes_c x) \xrightarrow{L(Id \otimes \eta_c)} L(c) \otimes_{\mathcal{D}} L(x) \xrightarrow{\mu_{c,x}^{-1}} L(c) \otimes_{\mathcal{D}} LRL(x) \xrightarrow{Id \otimes L \epsilon_{x}} L(c) \otimes_{\mathcal{D}} L(x).
\end{align}

It follows that the map we are interested in is actually the composition:

\begin{align}
L(c \otimes_c x) \xrightarrow{\mu_{c,x}^{-1}} L(c) \otimes_{\mathcal{D}} L(x) \xrightarrow{Id \otimes L \eta_c} L(c) \otimes_{\mathcal{D}} LRL(x) \xrightarrow{Id \otimes L \epsilon_{x}} L(c) \otimes_{\mathcal{D}} L(x).
\end{align}

By the unit/counit equations for the adjunction, the final two morphisms compose to give the identity. The result follows.

The following two lemmas establish the functoriality properties of the projection morphism.

Lemma A.9. Consider strong monoidal functors:

\[ \mathcal{C} \xrightarrow{(L, \mu, \eta)} \mathcal{D} \xrightarrow{(S, \nu, \epsilon)} \mathcal{D}' \]

Assume that $L$ and $S$ admit right adjoints $R$ and $T$, respectively.

(1) The conjugate (via $RT$) to $\nu_{c_1, c_2}^{S, L, RT}$ is the composition:

\[ \nu_{d_1', d_2'}^{S, L, RT} : RT(d_1') \otimes_c RT(d_2') \xrightarrow{\nu_{T(d_1'), T(d_2')}^{L,R}} R(T(d_1') \otimes_{\mathcal{D}} T(d_2')) \xrightarrow{R(\nu_{d_1', d_2'}^{S, T})} RT(d_1' \otimes_{\mathcal{D}'} d_2'). \]

(2) If $c \in \mathcal{C}$ and $d' \in \mathcal{D}'$, then the following diagram commutes:

\[ c \otimes_c RT(d') \xrightarrow{\pi_{L,R}(d')} R(L(c) \otimes_{\mathcal{D}} T(d')) \xrightarrow{R(\pi_{S,T}(d'))} RT(SL(c) \otimes_{\mathcal{D}'} d'). \]

Proof. Claim (1) follows from Lemma A.1. Claim (2) follows from the definition of the projection morphism (A.2) and (1).
The following lemma contains key compatibilities between composition, base change, and the projection formula. For the monoidal categories and their functors arising between morphisms of ringed spaces or topoi, such results can usually be checked by hand (e.g., [Stacks, Tag 0B6B]). For unbounded derived categories of quasi-coherent sheaves, these arguments can fail, because the relevant functors are obtained from adjoint functor theorems.

**Lemma A.10.** Consider a 2-commutative diagram of symmetric monoidal categories:

\[
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{L'} & \mathcal{D}' \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
& \xrightarrow{\kappa} & \mathcal{S}
\end{array}
\]

Assume that \(L, L', F,\) and \(S\) are strong monoidal and admit respective right adjoints \(R, R', G,\) and \(T.\)

1. If \(c \in \mathcal{C}\) and \(d' \in \mathcal{D}',\) then the following diagram commutes:

\[
\begin{array}{ccc}
c \otimes c' & \xrightarrow{\text{Id} \otimes \kappa'} & c \otimes c' R'(d') \\
\pi^{L,R}_{c,c'} & & \pi^{F,G}_{c,c',d'} \pi^{F,G}_{c,F(R(c) \otimes S(d))} \\
R(L(c) \otimes \mathcal{T}(d')) & \xrightarrow{\kappa' \otimes \text{Id}} & G(L'(F(c) \otimes \mathcal{T}(d'))
\end{array}
\]

2. If \(d \in \mathcal{D},\) there are natural base change morphisms:

\[
\beta^R_d, \tilde{\beta}^R_d : F R(d) \to R' S(d),
\]

which are adjoint to the compositions:

\[
\begin{array}{ccc}
R(d) & \xrightarrow{\kappa'(R(d))} & G R'(S(d)) \\
L' F R(d) & \xrightarrow{\kappa(R(d))} & S \left( \frac{\sigma_{L'} F(R(c) \otimes S(d))}{d} \right)
\end{array}
\]

respectively, and \(\beta^R_d = \tilde{\beta}^R_d.\)

3. If \(c \in \mathcal{C}\) and \(d \in \mathcal{D},\) then the following diagram commutes:

\[
\begin{array}{ccc}
F(c) \otimes R(d) & \xrightarrow{\mu^F_{c,R(d)}} & FR(L(c) \otimes d) \\
& \xrightarrow{\beta^F_{L(c) \otimes d}} & R'(S(L(c) \otimes d)) \\
& \xrightarrow{\beta^R_d} & R'(L'(F(c) \otimes S(d)) \otimes S(d)) \\
\end{array}
\]
Proof. For (1), combine Lemma A.9(2) with Remark A.5. Claim (2) follows from the commutativity of the following diagram:

\[
\begin{array}{cccccccc}
FR(d) & \xrightarrow{FR(\eta_d^R)} & FR(d) & \xrightarrow{F(\kappa')} & FGR'(d) & \xrightarrow{FGR'(d)} & R'(d) & \xrightarrow{R'(d)} \\
R'(\kappa) & \xrightarrow{R'(\kappa)} & R'(\kappa) & \xrightarrow{R'(\kappa)} & R'(\kappa) & \xrightarrow{R'(\kappa)} & R'(\kappa) & \xrightarrow{R'(\kappa)} \\
R'SL(d) & \xrightarrow{R'SL(d)} & R'SL(d) & \xrightarrow{R'SL(d)} & R'SL(d) & \xrightarrow{R'SL(d)} & R'SL(d) & \xrightarrow{R'SL(d)} \\
R'S(d) & \xrightarrow{R'S(d)} & R'S(d) & \xrightarrow{R'S(d)} & R'S(d) & \xrightarrow{R'S(d)} & R'S(d) & \xrightarrow{R'S(d)}
\end{array}
\]

together with the observations that the morphism along the top is \(\beta^c\), the morphism down the left is \(\beta^c\), and the morphisms on the bottom and right are the identity.

Claim (3) follows from the commutativity of the following diagram:

\[
\begin{array}{cccccccc}
F(c \otimes R(d)) & \xrightarrow{F(\eta_{c,R} \otimes \Id)} & F(RL(c) \otimes R(d)) & \xrightarrow{F(\kappa')_L} & F(c \otimes \Id) & \xrightarrow{F(c \otimes \Id)} & F(c \otimes R(d)) & \xrightarrow{F(c \otimes R(d))} \\
\mu_{RL(c),R(d)} & \xrightarrow{\mu_{RL(c),R(d)}} & \mu_{RL(c),R(d)} & \xrightarrow{\mu_{RL(c),R(d)}} & \mu_{RL(c),R(d)} & \xrightarrow{\mu_{RL(c),R(d)}} & \mu_{RL(c),R(d)} & \xrightarrow{\mu_{RL(c),R(d)}} \\
F(c) \otimes FR(d) & \xrightarrow{F(\eta_{c,R} \otimes \Id)} & FR(c) \otimes FR(d) & \xrightarrow{F(c) \otimes FR(d)} & FR(c) \otimes FR(d) & \xrightarrow{F(c) \otimes FR(d)} & FR(c) \otimes FR(d) & \xrightarrow{F(c) \otimes FR(d)} \\
\Id \otimes FR(d) & \xrightarrow{\Id \otimes FR(d)} & \Id \otimes FR(d) & \xrightarrow{\Id \otimes FR(d)} & \Id \otimes FR(d) & \xrightarrow{\Id \otimes FR(d)} & \Id \otimes FR(d) & \xrightarrow{\Id \otimes FR(d)} \\
\end{array}
\]

The commutativity of the squares on the right is just Lemmas A.1 and (1).

An object \(c \in \mathcal{C}\) is dualizable if there is a triple \((c^*, s, t)\), where \(c^* \in \mathcal{C}\) and \(s: 1 \rightarrow c \otimes c^*\) and \(t: c^* \otimes c \rightarrow 1\) are morphisms such that the two compositions

\[
\begin{align*}
&c \xrightarrow{\lambda c} 1 \otimes c \xrightarrow{s \otimes \Id} (c \otimes c^*) \otimes c \xrightarrow{\alpha_{c,c,c}} c \otimes (c^* \otimes c) \xrightarrow{\Id \otimes t} c \otimes 1 \xrightarrow{\rho c} c, \\
&c^* \xrightarrow{\rho c} c^* \otimes 1 \xrightarrow{\Id \otimes s} c^* \otimes (c \otimes c^*) \xrightarrow{\alpha_{c,c,c}} (c^* \otimes c) \otimes c^* \xrightarrow{\Id \otimes t} 1 \otimes c^* \xrightarrow{\lambda c} c^*
\end{align*}
\]

are the identity morphism. Another way of expressing this is that the functor \(c^* \otimes -\) is left adjoint to \(c \otimes -\). In the following standard lemma, we do not require the existence of a right adjoint \(R\) to \(L\).
Lemma A.11. If $(L, \mu, \iota)$ is strong monoidal and $c$ is dualizable, then $L(c)$ is dualizable. More precisely: let $(c^*, s, t)$ be a dual of $c$. Then $(L(c^*), s_L, t_L)$, where $s_L$ is the composition:

$$1_D \xrightarrow{\iota} L(1_c) \xrightarrow{L(s)} L(c \otimes c^*) \xrightarrow{\mu_{c^*, c}^{-1}} L(c) \otimes_D L(c^*),$$

and $t_L$ is the composition:

$$L(c^*) \otimes_D L(c) \xrightarrow{\mu_{c^*, c}^{-1}} L(c^* \otimes c) \xrightarrow{L(t)} L(1_c) \xrightarrow{\iota^{-1}} 1_D.$$

is dual to $L(c)$.

Proof. This is a routine diagram chase. \hfill $\square$

We now come to the main result of this appendix.

Theorem A.12. If $c \in \mathcal{C}$ is dualizable, then

$$\pi_{c, d}: c \otimes_D R(d) \rightarrow R(L(c) \otimes_D d)$$

is an isomorphism.

It is not difficult to prove that $c \otimes_D R(d)$ and $R(L(c) \otimes_D d)$ are isomorphic when $c$ is dualizable. The subtlety is showing that this isomorphism can be witnessed by the projection morphism $\pi_{c, d}$. In applications, this is critical.

The standard reference for Theorem A.12 (in the context of closed symmetric monoidal categories) is [FHM03, Prop. 3.12]. Note that Theorem A.12 is not actually proved in [loc. cit.]—there is an extra coherence condition for strong monoidal functors specified in [FHM03, Eq. 3.7]. It is shown in [MS06, Rem. 2.2.10], however, that this coherence condition is implied by the other conditions. Because of its importance to this article, we give a self-contained proof here using dualizables.

Proof of Theorem A.12. Let $(c^*, s, t)$ be a dual of $c$. Let $x \in \mathcal{C}$. Then observe that we have the following natural sequence of bijections:

$$\text{Hom}_\mathcal{C}(x, c \otimes_D R(d)) \cong \text{Hom}_\mathcal{C}(c^* \otimes_D x, R(d))$$

$$\cong \text{Hom}_D(L(c^* \otimes_D x), d)$$

$$\cong \text{Hom}_D(L(c^*) \otimes_D L(x), d)$$

$$\cong \text{Hom}_D(L(x), L(c) \otimes_D d) \quad \text{(Lemma A.11)}$$

$$\cong \text{Hom}_\mathcal{C}(x, R(L(c) \otimes_D d)).$$

By the Yoneda lemma, it follows that there is a unique isomorphism

$$\pi'_{c, d}: c \otimes_D R(d) \simeq R(L(c) \otimes_D d)$$

inducing the above. By Lemma A.4, it remains to prove that $\pi_{c, d} = \pi'_{c, d}$. We will do this using the Yoneda lemma. Fix $f: x \rightarrow c \otimes_D R(d)$. By definition, the $L-R$ adjoint to the composition $\tilde{\pi}_{c, d} \circ f$ we can express as the composition:

$$L(x) \xrightarrow{L(f)} L(c \otimes c \otimes_D R(d)) \xrightarrow{\mu_{c^*, c}^{-1} \otimes \Id_R(d)} L(c^* \otimes_D L(c)) \otimes_D R(d) \xrightarrow{\Id \otimes_{LR} \Id_D} L(c) \otimes_D d.$$

The adjoint to this morphism (afforded by $L(c^*) \otimes -$ and $L(c) \otimes -$) is thus the composition:

$$L(c^*) \otimes_D L(x) \xrightarrow{\Id \otimes_{LR} \Id_D} L(c^*) \otimes_D L(c \otimes_D c^* \otimes_D R(d))$$

$$\xrightarrow{\Id \otimes \mu_{c^*, c}^{-1} \otimes \Id_D} L(c^*) \otimes_D (L(c) \otimes_D LR(d))$$

$$\xrightarrow{\alpha_{L(c^*), L(c), LR(d)}} (L(c^*) \otimes_D L(c)) \otimes_D LR(d)$$

$$\xrightarrow{\Id \otimes_{LR} \Id_D} (L(c^*) \otimes_D L(c)) \otimes_D d \xrightarrow{t_L \otimes \Id} 1_D \otimes_D d \xrightarrow{\lambda_{D, d}} d.$$
We now look at the image of \( f \) under the compositions defining \( \pi_{c,d} \) via the Yoneda lemma. What we see is that:

\[
\begin{align*}
& f \mapsto (c^* \otimes c \xrightarrow{\text{Id} \otimes f} c^* \otimes (c \otimes R(d)) \xrightarrow{\alpha_{c^*,c,R(d)}} (c^* \otimes c) \otimes \mathcal{D} R(d)) \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \xrightarrow{\text{Id} \otimes \text{Id}} 1_c \otimes R(d) \xrightarrow{\lambda_{R(d)}} R(d)) \\
& \mapsto (L(c^* \otimes c \xrightarrow{\text{Id} \otimes f} L(c^* \otimes (c \otimes R(d)))) \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \xrightarrow{L(\alpha_{c^*,c,R(d)})} L((c^* \otimes c) \otimes R(d)) \xrightarrow{L(\text{Id} \otimes \text{Id})} L(1_c \otimes R(d)) \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \xrightarrow{L(\lambda_{R(d)})} L(R(d) \xrightarrow{\text{Id}} d)
\end{align*}
\]

It remains to show that precomposing the above morphism with \( \mu_{c^*,c} \) coincides with the other morphism described above. This follows from the commutativity of the following diagram, and that all of the vertical arrows are isomorphisms:

\[
\begin{array}{c}
\xymatrix{ L(c^*) \ar[r]^{\text{Id} \otimes f \ar[d]_{\mu_{c^*,c}}} & L(c^* \otimes c \otimes R(d) \ar[d]_{\mu_{c^*,c} \otimes \text{Id}}} \\
L(c^*) \otimes L(c) \ar[r]_{\text{Id} \otimes \text{Id} \otimes \text{Id}} & L(c^*) \otimes L(c^* \otimes c) \otimes R(d) \ar[r]_{\alpha_{c^*,c,R(d)}} & L(1_c) \otimes R(d) \ar[r]_{\text{Id}} & L(1_c) \otimes R(d) \\
\end{array}
\]

\[\square\]

**Appendix B. Two Lemmas for Ringed Topoi**

We include in this appendix two simple lemmas, which we expect to be well-known to experts.

**Lemma B.1.** Let \( \mathcal{W} \) be a ringed topos. Let \( \mathcal{B} \) be a sheaf of \( \mathcal{O}_{\mathcal{W}} \)-algebras. Let \( \mathcal{W}' \) be the ringed topos \( (\mathcal{W}, \mathcal{B}) \). There is an induced morphism of ringed topoi \( j : \mathcal{W}' \rightarrow \mathcal{W} \). Let \( \mathcal{M} \in \mathcal{D}(\mathcal{W}) \) and \( N \in \mathcal{D}(\mathcal{W}') \). Then the projection morphism

\[
\pi_{\mathcal{M},N} : \mathcal{M} \otimes \mathcal{O}_{\mathcal{W}} j_! N \rightarrow j_! (\mathcal{M} \otimes \mathcal{L} \mathcal{O}_{\mathcal{W}} N)
\]

is an isomorphism. In particular, if \( \mathcal{Q} \in \mathcal{D}_{\mathcal{pc}}(\mathcal{W}) \) and \( \mathcal{P} \in \langle j_! j^* \mathcal{P} \rangle \), then \( \mathcal{Q} \otimes \mathcal{L} \mathcal{O}_{\mathcal{W}} \mathcal{P} \in \langle j_! j^* \mathcal{P} \rangle \).

**Proof.** Let \( \mathcal{F} \) be a K-flat complex of \( \mathcal{O}_{\mathcal{W}} \)-modules quasi-isomorphic to \( \mathcal{M} \) and \( \mathcal{P} \) a K-flat complex of \( \mathcal{O}_{\mathcal{W}} \)-modules quasi-isomorphic to \( N \). The exactness of \( j_! \) implies that \( \mathcal{M} \otimes \mathcal{L} \mathcal{O}_{\mathcal{W}} \mathcal{P} \) is the total complex of \( (\mathcal{F} \otimes \mathcal{O}_{\mathcal{W}} j_! \mathcal{P})_r \). Clearly,

\[
\mathcal{F} \otimes \mathcal{L} \mathcal{O}_{\mathcal{W}} j_! \mathcal{P} = j_! (j^* \mathcal{F} \otimes \mathcal{L} \mathcal{O}_{\mathcal{W}} \mathcal{P}).
\]

Moreover, \( j_! \) commutes with the formation of total complexes (it commutes with small coproducts). The result is now immediate. For the latter claim, we set

\[
\mathcal{D}_0 = \{ \mathcal{R} \in \langle j_! j^* \mathcal{P} \rangle : \mathcal{Q} \otimes \mathcal{L} \mathcal{R} \in \langle j_! j^* \mathcal{P} \rangle \}.
\]

Clearly, \( \mathcal{D}_0 \) is a thick triangulated subcategory of \( \langle j_! j^* \mathcal{P} \rangle \). Thus, it suffices to prove that \( N \in \mathcal{D}_{\mathcal{pc}}(\mathcal{W}) \) implies \( \mathcal{Q} \otimes \mathcal{L} \mathcal{R} j_! N \in \langle j_! j^* \mathcal{P} \rangle \). This is obvious from the projection formula. \[\square\]

**Lemma B.2.** Let \( \pi : \mathcal{Y} \rightarrow \mathcal{W} \) be a morphism of ringed topoi. Let \( \mathcal{B} \) be a sheaf of \( \mathcal{O}_{\mathcal{W}} \)-algebras. Let \( \mathcal{W}' \) and \( \mathcal{Y}' \) be the ringed topoi \( (\mathcal{W}, \mathcal{B}) \) and \( (\mathcal{Y}, \pi^* \mathcal{B}) \), respectively. There is an induced 2-commutative diagram of ringed topoi:

\[
\begin{array}{c}
\xymatrix{ \mathcal{Y}' \ar[r]^\pi & \mathcal{W}' \\
\mathcal{Y} \ar[r]^j \ar[u]_{\pi'} \ar[d]^j & \mathcal{W} \ar[u]^\pi \ar[d]^j }
\end{array}
\]

If \( \pi \) and \( j \) are tor-independent and \( N \in \mathcal{D}(\mathcal{W}') \), then there is a natural isomorphism:

\[
L\pi^* j_! N \simeq j_! L\pi'^* N.
\]
In particular, if \( \mathcal{O} \in \langle R_j^*D_{pe}^{-1}(W) \rangle \), then \( \mathcal{L}^n*\mathcal{O} \in \langle R_j^*D_{pe}^{-1}(\mathcal{Y}') \rangle \).

Proof. Now \( j_* \) and \( j'_* \) are exact and \( \pi^{-1}j_* = j'_*\pi^{-1} \). By tor-independence of \( \pi \) and \( j \):

\[
\mathcal{L}\pi^*\mathcal{R}j_*\mathcal{N} = 0_j \otimes_{\pi^{-1}\mathcal{O}_W} \mathcal{L}\pi^{-1}j_*\mathcal{N} \cong (\mathcal{O}_W \otimes_{\mathcal{O}_Y} \mathcal{L}\pi^{-1}j_*\mathcal{O}_Y') \otimes_{\mathcal{O}_{-1,j_*\mathcal{O}_Y'}} \pi^{-1}j_*\mathcal{N} \\
\cong j'_*(\mathcal{O}_Y' \otimes_{\mathcal{O}_{-1}\mathcal{O}_Y'} \mathcal{L}\pi^{-1}j_*\mathcal{O}_Y') \cong j'_*\pi^{-1}\mathcal{N} \\
\cong j'_*(\mathcal{O}_Y' \otimes_{\mathcal{O}_{-1}\mathcal{O}_Y'} \mathcal{L}\pi^{-1}j_*\mathcal{N}) = Rj'_*\mathcal{L}^n*\mathcal{N}.
\]

The latter claim follows from a similar argument to that in Lemma B.1. \( \square \)

References

[AT19] D. Abramovich and M. Temkin, Functorial factorization of birational maps for qe schemes in characteristic 0, Algebra Number Theory 13 (2019), no. 2, 379–424.

[Bal11] M. Ballard, Derived categories of sheaves on singular schemes with an application to reconstruction, Adv. Math. 227 (2011), no. 2, 895–919.

[BBD80] A. Bondal and M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Mosc. Math. J. 3 (2003), no. 1, 1–36, 258.

[BJ14] B. Bhatt and A. J. de Jong, Lefschetz for local Picard groups, J. Inst. Math. Jussieu 11 (2012), no. 4, 747–814.

[BZN17] D. Ben-Zvi, D. Nadler, and A. Preygel, Integral transforms for coherent sheaves, J. Eur. Math. Soc. (JEMS) 19 (2017), no. 12, 3763–3812.

[CLO12] B. Conrad, M. Lieblich, and M. Olsson, Nagata compactification for algebraic spaces, J. Inst. Math. Jussieu 11 (2012), no. 4, 747–814.

[Con06] B. Conrad, Relative ampleness in rigid geometry, Ann. Inst. Fourier (Grenoble) 56 (2006), no. 4, 1049–1126.

[CS53] H. Cartan and J.-P. Serre, Un théorème de finitude concernant les variétés analytiques compactes, C. R. Acad. Sci. Paris 237 (1953), 128–130.

[DCT09] S. Devadas, Morphisms and Cohomological Comparison for Henselian Schemes, ProQuest LLC, Ann Arbor, MI, 2020, Thesis (Ph.D.)–Stanford University.

[Duc15] A. Ducros, Cohomological finiteness of proper morphisms in algebraic geometry: a purely transcendental proof, without projective tools, Berkovich spaces and applications, Lecture Notes in Math., vol. 2119, Springer, Cham, 2015, pp. 135–140.

[EGA] A. Grothendieck, Éléments de géométrie algébrique, I.H.E.S. Publ. Math. 4, 8, 11, 17, 20, 24, 28, 32 (1960, 1961, 1961, 1964, 1965, 1966, 1967).

[Eps66] D. B. A. Epstein, Functors between tensored categories, Invent. Math. 1 (1966), 221–228.

[FK18] K. Fujiwara and F. Kato, Foundations of rigid geometry. I, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2018.

[FL85] W. Fulton and R. Lang, Riemann-Roch algebra, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 277, Springer-Verlag, New York, 1985.

[GAGA] J. P. Serre, Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier, Grenoble 6 (1955–1956), 1–42.

[GR84] H. Grauert and R. Remmert, Coherent analytic sheaves, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 265, Springer-Verlag, Berlin, 1984.

[Gro55] A. Grothendieck, Sur quelques points d’algèbre homologique, Tôhoku Math. J. (2) 9 (1957), 119–221.

[HHR17a] J. Hall and D. Rydh, Perfect complexes on algebraic stacks, Compositio Math. 153 (2017), no. 11, 2318–2367.

[HHR17b] J. Hall and D. Rydh, The telescope conjecture for algebraic stacks, J. Topol. 10 (2017), no. 3, 776–794.

[HR22] J. Hall and D. Rydh, Mayer–Vietoris squares in algebraic geometry, J. London Math. Soc. (2) (2022), to appear.

[Kie67] R. Kiehl, Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedischen Funktionentheorie, Invent. Math. 2 (1967), 191–214.

[Kie72] R. Kiehl, Ein “Descente”-Lemma und Grothendiecks Projektionssatz für nichtnoethersche Schemata, Math. Ann. 198 (1972), 287–316.

[Knu74] D. Knutson, Algebraic spaces, Lecture Notes in Mathematics, Vol. 203, Springer-Verlag, Berlin, 1971.

[Köp74] U. Köpf, Über eigentliche Familien algebraischer Varietäten über affinoxen Räumen., Ph.D. thesis, Schr. Math. Inst. Univ. Münster, 1974.
