PROXIMALITY AND EQUIDISTRIBUTION ON THE FURSTENBERG BOUNDARY

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Abstract. Let \( G \) be a connected semisimple Lie group with finite center and without compact factors, \( P \) a minimal parabolic subgroup of \( G \), and \( \Gamma \) a lattice in \( G \). We prove that every \( \Gamma \)-orbits in the Furstenberg boundary \( G/P \) is equidistributed for the averages over Riemannian balls. The proof is based on the proximality of the action of \( \Gamma \) on \( G/P \).

1. Introduction

Let \( G \) be a connected semisimple Lie group with finite center and without compact factor, and \( \Gamma \) a lattice in \( G \), that is, a discrete subgroup of \( G \) such that \( \Gamma \backslash G \) has finite volume. In this article we investigate the distribution of orbits of \( \Gamma \) acting on the Furstenberg boundary of \( G \). Recall that the Furstenberg boundary can be identified with the factor space \( G/P \), where \( P \) is a minimal parabolic subgroup of \( G \). It is known that every orbit of \( \Gamma \) in \( G/P \) is dense (see [Mo]). We show that orbits of \( \Gamma \) are equidistributed with respect to the averages over Riemannian balls.

Since we study the action of a nonamenable group on a space without a finite invariant measure, our result lies outside the scope of the classical ergodic theory. The published results about distribution of dense orbits of nonamenable groups are limited to a few special examples. Arnold and Krylov showed in [AK] that dense orbits of groups generated by two rotations acting on the 2-dimensional sphere are equidistributed. A similar problem was considered by Kazhdan in [Kaz] where he studied the action of a group generated by two affine isometries on the plane \( \mathbb{R}^2 \). Distribution of dense orbits of a lattice in \( SL(2, \mathbb{R}) \) acting on \( \mathbb{R}^2 \) was investigated by Ledrappier [L] and Nogueira [N].

Let \( X \) be the symmetric space of \( G \) equipped with a right invariant Riemannian metric \( d \). Note that \( X \) can be identified with \( L \backslash G \) for a maximal compact subgroup \( L \) of \( G \).

Fix \( x, \tilde{x} \in X \) and denote by \( K \) and \( \tilde{K} \) the stabilizers of \( x \) and \( \tilde{x} \) respectively. Let \( \nu \) and \( \tilde{\nu} \) be the probability Haar measures on \( K \) and \( \tilde{K} \) and \( m_{\tilde{x}} \) the harmonic measures at \( \tilde{x} \) on \( G/P \), that is, the unique \( \tilde{K} \)-invariant probability measure on \( G/P \). For \( S \subset G \)

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and $T > 0$, define

$$S_T(\tilde{x}) = \{ s \in S : d(x, \tilde{x}s) < T \},$$

$$S_T = S_T(x).$$

Our main result is the following theorem.

**Theorem 1.** For every $f \in C(G/P)$, $\tilde{x} \in X$, and $y \in G/P$,

$$\lim_{T \to \infty} \frac{1}{|\Gamma_T(\tilde{x})|} \sum_{\gamma \in \Gamma_T(\tilde{x})} f(\gamma y) = \int_{G/P} f dm_{\tilde{x}},$$

Moreover, the convergence is uniform for $y \in G/P$.

We remark that it was shown in [EM] (see also [DRS]) that

$$(1) \quad |\Gamma_T(\tilde{x})| \sim_{T \to \infty} \frac{\text{Vol}(G_T(\tilde{x}))}{\text{Vol}(\Gamma \backslash G)},$$

and the exact asymptotics of the volume $\text{Vol}(G_T(\tilde{x})) = \text{Vol}(G_T)$ as $T \to \infty$ was computed in [Kn].

The first result in the direction of Theorem 1 was established in [Ma], where the case of the real hyperbolic spaces was considered. A different proof of Theorem 1 is given in [GO]. An advantage of the approach presented here is that it shows that the convergence is uniform. While the proof in [GO] uses equidistribution of solvable flows on $\Gamma \backslash G$, our proof is based on the strong proximality of the action of $G$ on $G/P$ (see Theorem 2 below). This result is of independent interest, and it might be useful for other applications.

Recall that an action of a group $H$ on a compact metric space $(Y, d)$ is called proximal if for every $u, v \in Y$ there exists a sequence $\{h_n\} \subset H$ such that $d(h_n u, h_n v) \to 0$ as $n \to \infty$. The fact that the action of $G$ on $G/P$ is proximal plays important role in the study of random walks on $G$ (see, for example, [F]). It turns out that a typical sequence in $G$ acts on $G/P$ in proximal fashion.

**Theorem 2 (Strong proximality).** Let $O$ be neighborhood of the diagonal in $G/P \times G/P$ and $u, v \in G/P$. Then

$$\lim_{T \to \infty} \frac{\text{Vol}(\{ g \in G_T(\tilde{x}) : (gu, gv) \notin O \})}{\text{Vol}(G_T(\tilde{x}))} = 0$$

and

$$\lim_{T \to \infty} \frac{|\{ \gamma \in \Gamma_T(\tilde{x}) : (\gamma u, \gamma v) \notin O \}|}{|\Gamma_T(\tilde{x})|} = 0$$

uniformly on $u, v$.

In the case of the real hyperbolic space, Theorem 2 was proved in [Ma] using geometric methods.
2. Proof of Theorem 2

2.1. Cartan decomposition. Let $G = K_0 \exp(p)$ be the Cartan decomposition of $G$ and $A \subset \exp(p)$ a split Cartan subgroup of $G$, that is, a maximal connected abelian subgroup in $\exp(p)$. We fix a system of positive roots $\Sigma^+$ on $a = \text{Lie}(A)$, and let $A^+ = \{a \in A : \alpha(\log a) \geq 0 \text{ for all } \alpha \in \Sigma^+\}$ denote the closed positive Weyl chamber in $A$. Then $G = KA^+K$, and a Haar measure on $G$ can be given by

\begin{equation}
\int_G \psi(g)dg = \int_K \int_{A^+} \int_K \psi(k_1a_2k_2)\xi(\log a)d\nu(k_1)d\nu(k_2), \quad \psi \in C_c(G),
\end{equation}

where $da$ denotes the Lebesgue measure on $A$, $\xi(s) = \prod_{\alpha \in \Sigma^+} \sinh(\alpha(s))^{m_\alpha}$, $s \in a$, and $m_\alpha$ denotes the dimension of the root space for the root $\alpha \in \Sigma^+$.

Let $\tilde{g} \in G$ be such that $x\tilde{g} = \tilde{x}$. Then $G = \tilde{g}^{-1}KA^+K$, $G_T(\tilde{x}) = \tilde{g}^{-1}KA^+_TK$, and

\begin{equation}
\int_G \psi(g)dg = \int_K \int_{A^+} \int_K \psi(\tilde{g}^{-1}k_1ak_2)\xi(\log a)d\nu(k_1)d\nu(k_2), \quad \psi \in C_c(G).
\end{equation}

In particular, it follows that

\begin{equation}
\text{Vol}(G_T(\tilde{x})) = \text{Vol}(G_T) = \int_{A^+_T} \xi(\log a)da.
\end{equation}

2.2. Reduction to maximal parabolics. Fix a system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \Sigma^+$. Here $r = \text{dim } A$ is the $\mathbb{R}$-rank of $G$. It is well-known that the closed subgroups of $G$ that contain $P$ are in one-to-one correspondence with the subsets of $\Pi$ (see [W, Sec. 1.2]). In particular, $P_i = P_{\{\alpha_i\}}$, $i = 1, \ldots, r$, are the maximal parabolic subgroups of $G$ and

$$P = \bigcap_{i=1}^r P_i.$$ 

We consider the projection maps $\pi_i : G/P \times G/P \to G/P_i \times G/P_i$, $i = 1, \ldots, r$. Let $\Delta$ and $\Delta_i$ denote the diagonals in $G/P \times G/P$ and $G/P_i \times G/P_i$ respectively. Then

$$\Delta = \bigcap_{i=1}^r \pi_i^{-1}(\Delta_i).$$
Since
\[
\prod_{i=1}^{r} \pi_i : G/P \times G/P \to \prod_{i=1}^{r} G/P_i \times G/P_i
\]
is a continuous injective map from a compact space to a Hausdorff space, it is a homeomorphism onto its image. It follows that for any neighborhood \( \mathcal{O} \) of \( \Delta \) in \( G/P \times G/P \), there exist neighborhoods \( \mathcal{O}_i \) of \( \Delta_i \) in \( G/P_i \times G/P_i \) such that
\[
\mathcal{O} \supset \bigcap_{i=1}^{r} \pi_i^{-1}(\mathcal{O}_i).
\]
Then for every \((u,v) \in G/P \times G/P\),
\[
\{g \in G : g \cdot (u,v) \notin \mathcal{O}\} \subset \bigcup_{i=1}^{r} \{g \in G : g \cdot \pi_i(u,v) \notin \mathcal{O}_i\}.
\]
This inclusion shows that it suffices to prove Theorem 2 under the assumption that \( P \) is a maximal parabolic subgroup of \( G \). We keep this assumption until the end of this section.

2.3. Dynamics on projective space. By a result from [T], there is an irreducible representation \( G \to \text{GL}(V) \) such that the highest weight space is one-dimensional, and the stabilizer of this space is \( P \). We consider the induced action of \( G \) on the projective space \( \mathbb{P}(V) \), and let \( w^+ \in \mathbb{P}(V) \) be the direction of the highest weight space. The map \( g \mapsto gw^+ \) defines an embedding of \( G/P \) in \( \mathbb{P}(V) \). Note that if \( \lambda \) is the highest weight, the other weights of the representation are of the form \( \lambda - \sum_{\alpha \in \Sigma^+} n_\alpha \alpha \) for integers \( n_\alpha \geq 0 \). We denote by \( V^< \) the sum of all root spaces with weights other than \( \lambda \). We fix a \( K \)-invariant scalar product on \( V \), which gives rise to a metric \( d \) on \( \mathbb{P}(V) \), which is \( K \)-invariant. Put \( d(w_1,w_2) = d(\tilde{g}w_1,\tilde{g}w_2) \). Let \( V^<_\epsilon \) be the open \( \epsilon \)-neighborhood of \( V^< \) in \( \mathbb{P}(V) \) with respect to the metric \( d \).

For \( w \in \mathbb{P}(V) \) and \( \tau > 0 \), define
\[
K_\tau(w) = \{k \in K : kw \notin V^<_\tau\}.
\]

Lemma 3. For every \( w \in G \cdot w^+ \),
\[
\lim_{\tau \to 0^+} \nu(K - K_\tau(w)) = 0.
\]

Proof. It follows from the Iwasawa decomposition that \( G \cdot w^+ = K \cdot w^+ \). Thus, without loss of generality, we may assume that \( w = w^+ \). By the continuity of the measure, it suffices to prove that
\[
\nu(\{k \in K : kw^+ \in V^<\}) = 0.
\]
Suppose that this is false. For a subspace \( W \) of \( V \), define
\[
K_W = \{k \in K : kw^+ \in W\}.
\]
Let \( W \) be a minimal subspace of \( V^< \) such that \( \nu(K_W^+) > 0 \). We claim that \( \text{Stab}_K(W) = K \). If \( \text{Stab}_K(W) \) has infinite index in \( K \), then there exist \( k_i \in K \), \( i \geq 1 \), such that \( k_iW \neq k_jW \) for \( i \neq j \). Since all sets \( k_iK_W \subset K \), \( i \geq 1 \), have the same positive measure, it follows that for some \( i \neq j \), \( k_iK_W \cap k_jK_W \) has positive measure. Then \( k_j^{-1}k_iK_W \cap K_W \) has positive measure too, and for \( k \in k_j^{-1}k_iK_W \cap K_W \),

\[
k_w^+ \in k_j^{-1}k_iW \cap W.
\]

Since \( k_j^{-1}k_iW \cap W \) is a proper subspace of \( W \), this contradicts the choice of \( W \). Thus, \( \text{Stab}_K(W) \) is a closed subgroup of finite index in \( K \). Since \( K \) is connected, it follows that \( K = \text{Stab}_K(W) \). Then \( w^+ \in K_W^{-1}W \subset V^< \). This contradiction proves the lemma.

We consider the sets

\[
A_T^\eta = \{ a \in A_T : \alpha(\log a) \geq \eta \text{ for all } \alpha \in \Sigma^+ \},
\]

\[
G_{T,\eta}(u, v) = \{ g \in G_T(\tilde{x}) : \tilde{d}(gu, gv) > \varepsilon \},
\]

\[
\Omega_{T,\tau}^\eta(u, v) = \tilde{g}^{-1}KA_T^\eta(K_\tau(u) \cap K_\tau(v))
\]

defined for \( T, \eta, \tau, \varepsilon > 0 \) and \( u, v \in P(V) \).

**Lemma 4.** For every \( \varepsilon > 0 \) and \( \tau > 0 \), there exists \( \eta > 0 \) such that for every \( T > 0 \) and \( u, v \in G \cdot w^+ \),

\[
\Omega_{T,\tau}^\eta(u, v) \cap G_{T,\eta}(u, v) = \emptyset.
\]

**Proof.** Note that an element \( a \in A_T^\eta \) acts by diagonal transformations on \( V \) with respect to some fixed basis, and the eigenvalue associated to the vector \( w^+ \) is at least \( \exp(\eta) \) times greater than the other eigenvalues. Therefore, for all \( w \notin V^<_\varepsilon \) and sufficiently large \( \eta \) (depending only on \( \tau \) and \( \varepsilon \)), we have \( \tilde{d}(aw, w^+) < \varepsilon/2 \) when \( a \in A_T^\eta \). Thus, for

\[
\tilde{g}^{-1}k_1ak_2 \in \Omega_{T,\tau}^\eta(u, v) = \tilde{g}^{-1}KA_T^\eta(K_\tau(u) \cap K_\tau(v)),
\]

we have

\[
\tilde{d}(\tilde{g}^{-1}k_1ak_2u, \tilde{g}^{-1}k_1ak_2v) = \tilde{d}(ak_2u, ak_2v) \leq \tilde{d}(ak_2u, w^+) + \tilde{d}(ak_2v, w^+) < \varepsilon,
\]

This proves the lemma. \( \square \)

**2.4. Completion of the proof.** By \([3]\),

\[
\text{Vol}(\Omega_{T,\tau}^\eta(u, v)) = \left( \int_{A_T^\eta} \xi(\log a) da \right) \cdot \nu(K_\tau(u) \cap K_\tau(v)).
\]

Let \( \varepsilon, \delta \in (0, 1) \). Using Lemma \([3]\) we choose \( \tau > 0 \) such that

\[
\nu(K_\tau(u) \cap K_\tau(v)) > 1 - \delta.
\]
Let $\eta > 0$ be as Lemma 4. By Lemma 9(a), for sufficiently large $T$,
\[\int_{A_T^+} \xi(a)da \geq (1 - \delta) \int_{A_T^+} \xi(\log a)da.\]
Thus, it follows from (4) and (7) that
\[\text{Vol}(\Omega_{T,\tau}^{\eta}(u, v)) \geq (1 - \delta)^2 \text{Vol}(G_T(\bar{x})).\]
for sufficiently large $T > 0$. Therefore, by (6),
\[\text{Vol}(G_{T,\epsilon}(u, v)) \leq (1 - (1 - \delta)^2) \text{Vol}(G_T(\bar{x})),\]
for all $\delta \in (0, 1)$ and sufficiently large $T > 0$. Since the sets
\[\{(g_1 P, g_2 P) : \bar{d}(g_1 w^+, g_2 w^+) < \epsilon\}, \epsilon > 0,
\]
form a base of the neighborhoods of the diagonal in $G/P \times G/P$, this proves the first part of Theorem 2.

To prove the second part of Theorem 2 we choose a neighborhood $\mathcal{P}$ of $e$ in $G$ and a neighborhood $\mathcal{Q}$ of the diagonal in $G/P \times G/P$ such that
\begin{align*}
\mathcal{P}^{-1} \mathcal{P} \cap \Gamma &= \{e\}, \\
\mathcal{P}^{-1} \cdot \mathcal{Q} &\subset \mathcal{O}, \\
\mathcal{P} \cdot G_T(\bar{x}) &\subset G_{T+c}(\bar{x}).
\end{align*}
for fixed $c > 0$ and all $T > 0$. Here we use that $\Gamma$ is discrete, the space $G/P$ is compact, and the metric on the symmetric space is uniformly continuous. By (3), for every $\gamma \in \Gamma$ such that $\gamma \cdot (u, v) \notin \mathcal{O}$, we have $\mathcal{P}\gamma \cdot (u, v) \cap \mathcal{Q} = \emptyset$. Thus, using (10), we deduce that
\[\mathcal{P} \cdot \{\gamma \in \Gamma_T(\bar{x}) : \gamma \cdot (u, v) \notin \mathcal{O}\} \subset \{g \in G_{T+c}(\bar{x}) : g \cdot (u, v) \notin \mathcal{Q}\}.
\]
Then by (3), $\mathcal{P}\gamma_1 \cap \mathcal{P}\gamma_2 = \emptyset$ for $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, and
\[|\{\gamma \in \Gamma_T(\bar{x}) : \gamma \cdot (u, v) \notin \mathcal{O}\}| \leq \frac{1}{\text{Vol}(\mathcal{P})} \text{Vol}(\{g \in G_{T+c}(\bar{x}) : g \cdot (u, v) \notin \mathcal{Q}\}) = o(\text{Vol}(G_{T+c}(\bar{x})))\]
as $T \to \infty$. Now the second statement of Theorem 2 follows from Lemma 9(d) and (1).

3. Equidistribution on $\Gamma \backslash G$

Recall that $K$ is a maximal compact subgroups of $G$, and $\nu$ is the probability Haar measure on $K$. Denote by $\varrho$ a right Haar measure on the minimal parabolic subgroup $P$. For a suitable normalization of $\varrho$, the Haar measure on $G$ is given by
\[\int_G \psi(g)dg = \int_K \int_P \psi(kp)d\varrho(p)d\nu(k), \quad \psi \in C_c(G).\]
We also define a measure \( \mu \) on \( G \) by

\[
\int_G \psi(g) d\mu(g) = \int_K \int_P \psi(kp^{-1}) d\varphi(p) d\nu(k), \quad \psi \in C_c(G).
\]

Note that \( \mu \) is left \( K \)-invariant.

The first step in the proof of Theorem 1 is the following result.

**Proposition 5.** For every \( \Psi \in C_c(\Gamma \backslash G) \) and \( z \in \Gamma \backslash G \),

\[
\lim_{T \to \infty} \frac{1}{\mu(G_T)} \int_{G_T} \Psi(zg) d\mu(g) = \frac{1}{\Vol(\Gamma \backslash G)} \int_{\Gamma \backslash G} \Psi dg
\]

where \( G_T = \{ g \in G : d(x, zg) < T \} \).

Proposition 5 is a consequence of the equidistribution of translates of \( K \) in \( \Gamma \backslash G \) proved by Eskin and McMullen in [EM] (see also [S] for a more general result). They showed that for every strongly divergent sequence \( \{ g_n \} \subset G \),

\[
\lim_{n \to \infty} \int_K \Psi(\kappa g_n) d\nu(k) = \frac{1}{\Vol(\Gamma \backslash G)} \int_{\Gamma \backslash G} \Psi dg.
\]

Recall that a sequence \( \{ g_n \} \subset G \) is strongly divergent if the projection of \( \{ g_n \} \) on every noncompact simple factor of \( G \) is divergent. Note that (13) was proved in [EM] under the condition that the lattice \( \Gamma \) is irreducible. Since the proof of (13) is based on mixing properties of the action of \( G \) on \( \Gamma \backslash G \), it is applicable to the case of a reducible lattice \( \Gamma \) provided that the sequence \( \{ g_n \} \) is strongly divergent.

Denote by \( \pi_i : G \to G_i, \ i = 1, \ldots, s \), the projections of \( G \) onto its simple factors. Let \( C_{i,j} \subset G_i, j \geq 1 \), be an increasing sequence of compact subsets such that \( G_i = \bigcup_{j \geq 1} C_{i,j} \). Define

\[
G_{T,n} = G_T - \bigcup_{1 \leq i \leq s} \pi_i^{-1}(C_{i,n}).
\]

**Lemma 6.** For every \( n \geq 1 \), \( \mu(G_{T,n}) \sim \mu(G_T) \) as \( T \to \infty \).

*Proof.* It suffices to show that for every \( i = 1, \ldots, s \) and \( n \geq 1 \),

\[
\mu(G_T \cap \pi_i^{-1}(C_{i,n})) = o(\mu(G_T)) \quad \text{as} \quad T \to \infty.
\]

Fix \( i = 1, \ldots, s \) and \( n \geq 1 \). Note that \( G = DH \), where \( D \) and \( H = \ker(\pi_i) \) are normal connected semisimple Lie subgroups with finite centers, and \( D \) and \( H \) commute. We have \( \pi_i^{-1}(C_{i,n}) = D_{i,n}H \) for some compact set \( D_{i,n} \subset D \). There is a constant \( \delta > 0 \) such that

\[
D_{i,n}H_{T-\delta} \subset (D_{i,n}H)_T \subset D_{i,n}H_{T+\delta} \quad \text{for all} \quad T > 0.
\]

We define measures \( \mu_D \) and \( \mu_H \) for the groups \( D \) and \( H \) respectively as in (12). With appropriate normalization, \( \mu = \mu_D \otimes \mu_H \). Thus, it follows from (13) that

\[
\mu(G_T \cap \pi_i^{-1}(C_{i,n})) = \mu((D_{i,n}H)_T) \ll \mu_H(H_{T+\delta}).
\]
Since \( G_T = KP_T \) and \( P_T^{-1} = P_T \), using (11) and (12), we conclude that

\[
\mu(G_T) = \varrho(P_T^{-1}) = \varrho(P_T) = \text{Vol}(G_T).
\]

Similarly, \( H = LQT \) where \( L \) is a maximal compact subgroup of \( H \) contained in \( K \), and \( Q \) is a minimal parabolic subgroup of \( H \). As in (17), we deduce that \( \mu_H(H_T) = \text{Vol}_H(H_T) \). By (16), it is sufficient to show that

\[
\text{Vol}_H(H_T + \delta) = o(\text{Vol}(G_T)) \quad \text{as} \quad T \to \infty.
\]

Note that with appropriate normalization the Haar measure on \( G \) is the product of Haar measures on \( D \) and \( H \). Without loss of generality, \( \text{Vol}_D(D_{i,n}) > 0 \). Then by (15),

\[
\text{Vol}_H(H_T + \delta) \ll \text{Vol}(D_{i,n}H_T + \delta) \leq \text{Vol}((D_{i,n}H)_{T+2\delta}).
\]

Let \( G_T^n \) be defined as in (24). Since the set \( D_{i,n} \) is compact, there exists \( \eta > 0 \) such that

\[
(D_{i,n}H)_{T+2\delta} \subset G_{T+2\delta} = G_T^n.
\]

Thus, (18) follows from Lemma 9(b).

**Proof of Proposition 5.** The map \( K \times A^+ \times K \to G \) is a diffeomorphism on an open set of full measure. Since the measure \( \mu \) is left \( K \)-invariant and smooth, for some \( \sigma \in C(A^+ \times K) \),

\[
\int_G \psi(g) d\mu(g) = \int_K \int_{A^+} \int_K \psi(k_1ak_2)\sigma(a,k_2) d\nu(k_1) d\nu(k_2), \quad \psi \in C_c(G).
\]

Let \( G_{T,n} \) be defined as in (14), and it is \( K \)-bi-invariant (equivalently, all \( C_{i,j} \) are \( \pi_i(K) \)-bi-invariant). Then

\[
G_{T,n} = KA_{T,n}^+ K \quad \text{and} \quad \mu(G_{T,n}) = \int_K \int_{A_{T,n}^+} \sigma(a,k_2) d\nu(k_2),
\]

where \( A_{T,n}^+ = G_{T,n} \cap A^+ \).

Let \( \varepsilon > 0 \). By (13),

\[
\left| \int_K \Psi(zk_1ak_2) d\nu(k_1) - \frac{1}{\text{Vol}(\Gamma \setminus G)} \int_{\Gamma \setminus G} \Psi \, dg \right| < \varepsilon
\]
for $a \in A^+_T,n$ and $k_2 \in K$ when $n > n_0(\varepsilon)$. Thus, for $n > n_0(\varepsilon)$,

$$\left| \int_{G_T,n} \Psi(zg) d\mu(g) - \frac{\mu(G_T,n)}{\text{Vol}(\Gamma \setminus G)} \int_{\Gamma \setminus G} \Psi dg \right|$$

$$= \left| \int_K \int_{A^+_T,n} \int_K \Psi(zk_1ak_2) d\nu(k_1) \sigma(a, k_2) d\nu(k_2) \right|$$

$$\leq \int_K \int_{A^+_T,n} \int_K \Psi(zk_1ak_2) d\nu(k_1)$$

$$- \frac{\mu(G_T,n)}{\text{Vol}(\Gamma \setminus G)} \int_{\Gamma \setminus G} \Psi dg \leq \int_K \int_{A^+_T,n} \int_K \Psi(zk_1ak_2) d\nu(k_1)$$

$$- \frac{1}{\text{Vol}(\Gamma \setminus G)} \int_{\Gamma \setminus G} \Psi dg \sigma(a, k_2) d\nu(k_2) < \varepsilon \mu(G_T,n).$$

By Lemma 6 for every $n \geq 1$,

$$\int_{G_T} \Psi(zg) d\mu(g) = \int_{G_T} \Psi(zg) d\mu(g) + o(\mu(G_T,n))$$

as $T \to \infty$. Thus, it follows from (19) that

$$\limsup_{T \to \infty} \left| \frac{1}{\mu(G_T)} \int_{G_T} \Psi(zg) d\mu(g) - \frac{1}{\text{Vol}(\Gamma \setminus G)} \int_{\Gamma \setminus G} \Psi dg \right| < \varepsilon$$

for every $\varepsilon > 0$. This proves the proposition. □

4. Equidistribution on average

In this section we prove that Theorem 1 holds “on average”. In the case of hyperbolic spaces, the following proposition is a consequence of the work of Roblin [R].

**Proposition 7.** For every $f \in C(G/P)$ and $y \in G/P$,

$$\lim_{T \to \infty} \frac{1}{\Gamma_T(\tilde{x})} \sum_{\gamma \in \Gamma_T(\tilde{x})} \int_K f(\gamma ky) d\nu(k) = \int_{G/P} f dm_{\tilde{x}}$$

where $\Gamma_T(\tilde{x}) = \{\gamma \in \Gamma : d(x, \tilde{x} \gamma) < T\}$.

**Proof.** There exists $\tilde{p} \in P$ such that $\tilde{x} = x \tilde{p}$. Then $\tilde{K} = \tilde{p}^{-1} K \tilde{p}$, and it follows from (11) that

$$\int_K \psi(g) dg = \int_{\tilde{K}} \int_P \psi(k \tilde{p}^{-1} p) d\varphi(p) d\tilde{\nu}(k), \quad \psi \in C_c(G).$$

Without loss of generality, $f \geq 0$, and since $G = K P$, we may assume that $y = eP$. Let $\varepsilon > 0$, $O_{\varepsilon} = \{z \in X : d(x, z) < \varepsilon\}$, and $\phi_{\varepsilon} \in C_c(X)$ such that

$$\phi_{\varepsilon} \geq 0, \quad \text{supp}(\phi_{\varepsilon}) \subset O_{\varepsilon}, \quad \int_P \phi_{\varepsilon}(xp^{-1}) d\varphi(p) = 1.$$
Since \( X = \tilde{x}P \) and \( g \) is right invariant, it follows that
\[
(21) \quad \int_P \phi_\varepsilon(zp^{-1})dg(p) = 1 \quad \text{for every } z \in X.
\]
Let
\[
\psi_\varepsilon(g) = f(gP)\phi_\varepsilon(\tilde{x}g), \quad g \in G.
\]
Clearly, \( \psi_\varepsilon \in C_c(G) \) and
\[
\Psi_\varepsilon(\Gamma g) \overset{\text{def}}{=} \sum_{\gamma \in \Gamma} \psi_\varepsilon(\gamma g) \in C_c(\Gamma\backslash G).
\]
By Proposition 5,
\[
(22) \quad \lim_{T \to \infty} \frac{1}{\mu(G_T)} \sum_{\gamma \in \Gamma} \int_{G_T} \psi_\varepsilon(\gamma g)dg = \frac{1}{\text{Vol}(\Gamma\backslash G)} \int_{\Gamma \backslash G} \Psi_\varepsilon(\Gamma g)dg
\]
and by (20),
\[
\text{Vol}(\Gamma \backslash G) \int_{\Gamma \backslash G} \Psi_\varepsilon(\Gamma g)dg = \int_G \psi_\varepsilon(g)dg = \int_K f(kP) d\nu(k) \cdot \int_P \phi_\varepsilon(\tilde{x}p^{-1})dg(p)
\]
\[
= \int_{G/P} f dm_{\tilde{x}} \cdot \int_P \phi_\varepsilon(xp)dg(p).
\]
Denote by \( \delta \) the modular function of \( P \). By (21),
\[
\left| \int_P \phi_\varepsilon(xp)dg(p) - 1 \right| = \left| \int_P \phi_\varepsilon(xp^{-1})\delta(p) - 1)dg(p) \right|
\]
\[
\leq \max\{|\delta(p) - 1| : xp^{-1} \in O_\varepsilon\}.
\]
The sets \( \{p \in P : xp^{-1} \in O_\varepsilon\}, \varepsilon > 0 \), form a base of neighborhoods of \( P \cap K \) in \( P \). Since \( \delta|_{P \cap K} = 1 \) and \( P \cap K \) is compact,
\[
\max\{|\delta(p) - 1| : xp^{-1} \in O_\varepsilon\} \to 0 \quad \text{as } \varepsilon \to 0^+.
\]
Thus, it follows from (22) that
\[
(23) \quad \lim_{\varepsilon \to 0^+} \lim_{T \to \infty} \frac{1}{\mu(G_T)} \sum_{\gamma \in \Gamma} \int_{G_T} \psi_\varepsilon(\gamma g)dg = \int_{G/P} f dm_{\tilde{x}}.
\]
Since \( G_T = KP_T \),
\[
\sum_{\gamma \in \Gamma} \int_{G_T} \psi_\varepsilon(\gamma g)dg
\]
\[
\overset{\text{12}}{=} \sum_{\gamma \in \Gamma} \int_{K \times P_T^{-1}} \psi_\varepsilon(\gamma kp^{-1})d\nu(k)dg(p)
\]
\[
\overset{\text{12}}{=} \sum_{\gamma \in \Gamma} \int_K f(\gamma kP) \left( \int_{P_T^{-1}} \phi_\varepsilon(\tilde{x}\gamma kp^{-1})dg(p) \right) d\nu(k).
\]
For $\gamma \in \Gamma - \Gamma_{T+\epsilon}(\bar{x})$, $k \in K$, and $p \in P_{T}^{-1}$,
\[d(x, \bar{x}\gamma k p^{-1}) = d(xp^{-1}, \bar{x}\gamma) \geq d(x, \bar{x}\gamma) - d(x, xp^{-1}) \geq \epsilon.\]
This implies that $\int_{P_{T}^{-1}} \phi_{\epsilon}(\bar{x}\gamma k p^{-1}) d\varrho(p) = 0$ for $\gamma \in \Gamma - \Gamma_{T+\epsilon}(\bar{x})$. Thus,
\[
\sum_{\gamma \in \Gamma} \int_{G_{T}} \psi_{\epsilon}(\gamma g) d\mu(g) = \sum_{\gamma \in \Gamma_{T+\epsilon}(\bar{x})} \int_{K} f(\gamma k P) \left( \int_{P_{T}^{-1}} \phi_{\epsilon}(\bar{x}\gamma k p^{-1}) d\varrho(p) \right) d\nu(k) \leq \sum_{\gamma \in \Gamma_{T+\epsilon}(\bar{x})} \int_{K} f(\gamma k P) d\nu(k).
\]
Combining (23), (17), (1), and Lemma 9(c), we deduce that
\[
\liminf_{T \to \infty} \frac{1}{|\Gamma_{T}(\bar{x})|} \sum_{\gamma \in \Gamma_{T}(\bar{x})} \int_{K} f(\gamma k P) d\nu(k) \geq \int_{G/P} f dm_{\bar{x}}.
\]
On the other hand, for $\gamma \in \Gamma_{T-\epsilon}(\bar{x})$, $k \in K$, and $p \in P$ such that $d(x, \bar{x}\gamma k p^{-1}) < \epsilon$,
\[d(x, xp^{-1}) \leq d(x, \bar{x}\gamma k p^{-1}) + d(xp^{-1}, \bar{x}\gamma k p^{-1}) < T.\]
This shows that for $\gamma \in \Gamma_{T-\epsilon}(\bar{x})$,
\[
\int_{P_{T}^{-1}} \phi_{\epsilon}(\bar{x}\gamma k p^{-1}) d\varrho(p) = \int_{P} \phi_{\epsilon}(\bar{x}\gamma k p^{-1}) d\varrho(p) \quad \text{21} = 1.
\]
Hence,
\[
\sum_{\gamma \in \Gamma} \int_{G_{T}} \psi_{\epsilon}(\gamma g) d\mu(g) \geq \sum_{\gamma \in \Gamma_{T-\epsilon}(\bar{x})} \int_{K} f(\gamma k P) \left( \int_{P_{T}^{-1}} \phi_{\epsilon}(\bar{x}\gamma k p^{-1}) d\varrho(p) \right) d\nu(k) = \sum_{\gamma \in \Gamma_{T-\epsilon}(\bar{x})} \int_{K} f(\gamma k P) d\nu(k).
\]
By (23), (17), (1), and Lemma 9(c),
\[
\limsup_{T \to \infty} \frac{1}{|\Gamma_{T}(\bar{x})|} \sum_{\gamma \in \Gamma_{T}(\bar{x})} \int_{K} f(\gamma k P) d\nu(k) \leq \int_{G/P} f dm_{\bar{x}}.
\]
This proves the proposition. $\square$
5. Proof of Theorem 1

Now the proof can be completed using the argument from [Ma]. Let \( \varepsilon > 0 \). Since the space \( G/P \times G/P \) is compact, there exists a neighborhood \( \mathcal{O} \) of the diagonal in \( G/P \times G/P \) such that for every \( (z_1, z_2) \in \mathcal{O} \), we have \(|f(z_1) - f(z_2)| < \varepsilon\). Then for every \( k \in K \),

\[
\left| \sum_{\gamma \in \Gamma_T(\tilde{x})} f(\gamma y) - \sum_{\gamma \in \Gamma_T(\tilde{x})} f(\gamma ky) \right| \\
\leq \sum_{\gamma \in \Gamma_T(\tilde{x})} |f(\gamma y) - f(\gamma ky)| + \sum_{\gamma \in \Gamma_T(\tilde{x})} |f(\gamma y) - f(\gamma ky)| \\
\leq \varepsilon |\Gamma_T(\tilde{x})| + 2 \sup |f| \cdot |\{ \gamma \in \Gamma_T(\tilde{x}) : (\gamma y, \gamma ky) \notin \mathcal{O} \}|.
\]

Thus, it follows from Theorem 2 that

\[
\lim_{T \to \infty} \frac{1}{|\Gamma_T(\tilde{x})|} \left| \sum_{\gamma \in \Gamma_T(\tilde{x})} f(\gamma y) - \sum_{\gamma \in \Gamma_T(\tilde{x})} f(\gamma ky) \right| = 0
\]

for all \( k \in K \). Hence, by the dominated convergence theorem,

\[
\lim_{T \to \infty} \frac{1}{|\Gamma_T(\tilde{x})|} \left| \sum_{\gamma \in \Gamma_T(\tilde{x})} f(\gamma y) - \frac{1}{|\Gamma_T(\tilde{x})|} \sum_{\gamma \in \Gamma_T(\tilde{x})} f(\gamma ky) \right| d\nu(k) = 0.
\]

Finally, Theorem 1 follows from Proposition 7.

6. Appendix: Volume estimates

In this section, we give proofs of volume estimates, which are used in Theorems 1 and 2. There are other ways to establish these volume estimates. For example, one can use the exact asymptotics of the volume of Riemannian balls from [Kn] (see also [GO]). We present a straightforward proof that does not use asymptotics.

Let \( a \) be the Lie algebra of the Cartan subgroup \( A \) and \( a^+ \) the positive Weyl chamber with respect to the root system \( \Sigma^+ \). The Riemannian metric defines a scalar product on \( a \) and, by duality, on the dual space of \( a \). For \( \alpha \in \Sigma^+ \), we denote by \( m_\alpha \) the dimension of the corresponding root space and put \( \rho = \frac{1}{2} \sum_{\beta \in \Sigma^+} m_\beta \beta \).

Lemma 8. The maximum of \( \rho \) on \( \{ a \in a : \|a\| \leq 1 \} \) is achieved at a unique point in the interior of \( a^+ \).

Proof. Since the set \( \{ a \in a : \|a\| = 1 \} \) is strictly convex, it is clear that the point of maximum is unique. It is sufficient to show that \( (\rho, \alpha) > 0 \) for every \( \alpha \in \Sigma^+ \). Denote by \( \sigma_\alpha \) the reflection with respect to the hyperplane \( \{ \alpha = 0 \} \). The map \( \sigma_\alpha \) permutes the elements of the set \( \Sigma^+ - \{ \alpha, 2\alpha \} \) and \( \sigma_\alpha(\alpha) = -\alpha \). Since \( m_\sigma(\beta) = m_\beta \), we have

\[
\sigma_\alpha(\rho) = \rho - 2m_\alpha\alpha - 4m_{2\alpha} \alpha.
\]
Thus,
\[(\rho, \alpha) = (\sigma_\alpha(\rho), \sigma_\alpha(\alpha)) = 2m_\alpha(\alpha, \alpha) + 4m_{2\alpha}(\alpha, \alpha) - (\rho, \alpha)\]
and \((\rho, \alpha) = (m_\alpha + 2m_{2\alpha})(\alpha, \alpha)\) is positive.

For \(T, \eta > 0\), define
\[
A^\eta_T = \{ a \in A_T : \alpha(\log a) \geq \eta \text{ for all } \alpha \in \Sigma^+ \}
\]
\[
G^\eta_T = KA^\eta_T K.
\]

**Lemma 9.** For every \(\eta > 0\),

(a) \[
\int_{A^\eta_T} \xi(\log a) da \sim_{T \to \infty} \int_{A^+_T} \xi(\log a) da,
\]

(b) \[
\Vol(G^\eta_T) \sim_{T \to \infty} \Vol(G_T),
\]

(c) \[
\liminf_{\varepsilon \to 0^+} \left( \limsup_{T \to \infty} \frac{\Vol(G_{T+\varepsilon})}{\Vol(G_T)} \right) = 1,
\]

(d) \[
\Vol(G_{T+\eta}) \ll \Vol(G_T).
\]

**Proof.** We have
\[
\int_{a^+_T} \xi(a) da = 2^{-|\Sigma^+|} \sum_{i \in I} \int_{a^+_T} e^{\lambda_i(a)} da
\]
where \(\lambda_i\)'s the characters of the form \(2\rho - \sum_{\alpha \in \Sigma^+} n_\alpha \alpha\) for some \(n_\alpha \geq 0\). Let
\[
\delta = \max\{2\rho(a) : a \in a^+_T\},
\]
\[
\delta_i = \max\{\lambda_i(a) : a \in a^+_T\}, \quad i \in I,
\]
\[
\delta_\alpha = \max\{2\rho(a) : a \in a^+_T, \alpha(a) = 0\}, \quad \alpha \in \Sigma^+.
\]
It follows from Lemma \[\] that for \(\lambda_i \neq 2\rho\) and \(\alpha \in \Sigma^+, \delta > \max\{\delta_i, \delta_\alpha\}\). Thus,
\[
\int_{a^+_T} e^{\lambda_i(a)} da \leq \Vol(a^+_T) e^{\delta T} \ll T^r e^{\delta_i T}
\]
where \(r = \dim a\). Let \(\varepsilon > 0\) be such that
\[
\delta - \varepsilon > \max\{\delta_i, \delta_\alpha : i \in I, \alpha \in \Sigma^+\}.
\]
Then
\[
\int_{a^+_T} e^{2\rho(a)} da = T^r \int_{a^+_T} e^{2\rho(a)} da
\]
\[
\geq T^r e^{(\delta - \varepsilon)T} \Vol(\{a \in a^+_T : 2\rho(a) \geq \delta - \varepsilon\}) \gg T^r e^{(\delta - \varepsilon)T}.
\]
Combining (25), (26), and (27), we deduce that

\begin{equation}
\int_{a_T^+} \xi(a) da \gg T^r e^{(\delta-\varepsilon)T}.
\end{equation}

On the other hand, for \( \alpha \in \Sigma^+ \),

\begin{equation}
\int_{a_T^+ \cap \{ \alpha < \eta \}} \xi(a) da \leq \int_{a_T^+ \cap \{ \alpha = 0 \}} e^{2\rho(a)} da \ll \int_{a_T^+ \cap \{ \alpha = 0 \}} e^{2T\rho(a)} da \ll T^{r-1} e^{\delta_\alpha T} = o(e^{(\delta-\varepsilon)T}).
\end{equation}

Since

\[ a_T^+ - a_T^+ \subset \bigcup_{\alpha \in \Sigma^+} a_T^+ \cap \{ \alpha < \eta \}. \]

This proves part (a) of the lemma. Part (b) follows from (2).

To prove part (c), we note that

\[ \mathrm{Vol}(G_{T+\varepsilon}) = \int_{a_T^+} \xi(a) da = (T + \varepsilon)^r \int_{a_T^+} \xi((T + \varepsilon)a) da \]

It is easy to check that there exist \( b > 0 \) such that \( \sinh(t + \varepsilon) \leq e^\varepsilon \sinh(t) + b \) for every \( \varepsilon \in (0, 1) \) and \( t \geq 0 \). Thus, for \( a \in a_T^+ \) and sufficiently small \( \varepsilon > 0 \),

\[ \xi((T + \varepsilon)a) \leq \prod_{\alpha \in \Sigma^+} (a_\varepsilon \sinh(\alpha(Ta)) + b)^{m_\varepsilon} \leq d_\varepsilon \xi(Ta) + C \sum_{i \in I} e^{\lambda_i(Ta)} \]

where \( d_\varepsilon \to 1 \) as \( \varepsilon \to 0^+ \), \( C > 0 \), and \( \lambda_i \)'s are characters such that \( 2\rho - \lambda_i < 0 \) in the interior of \( a^+ \). Thus, it follows from (26) that

\[ \int_{a_T^+} \xi((T + \varepsilon)a) da \leq d_\varepsilon \int_{a_T^+} \xi(Ta) da + o(e^{(\delta-\varepsilon)T}). \]

Using (4) and (28), we deduce that

\begin{equation}
\limsup_{T \to \infty} \frac{\mathrm{Vol}(G_{T+\varepsilon})}{\mathrm{Vol}(G_T)} \leq d_\varepsilon,
\end{equation}

and part (c) of the lemma follows. The last part of lemma can be proved similarly. \( \square \)

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