COMPACTNESS VERSUS HUGENESS AT SUCCESSOR CARDINALS

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ABSTRACT. If \( \kappa \) is regular and \( 2^{<\kappa} \leq \kappa^+ \), then the existence of a weakly presaturated ideal on \( \kappa^+ \) implies \( \Box^*_\kappa \). This partially answers a question of Foreman and Magidor about the approachability ideal on \( \omega_2 \). As a corollary, we show that if there is a presaturated ideal \( I \) on \( \omega_2 \) such that \( P(\omega_2)/I \) is semiproper, then CH holds. We also show some barriers to getting the tree property and a saturated ideal simultaneously on a successor cardinal from conventional forcing methods.

The motivating question for this work is: To what extent are large cardinal properties of small cardinals mutually consistent? We focus here on a tension between versions of compactness and hugeness that make sense for successor cardinals. We show that if \( \kappa \) is regular and \( 2^{<\kappa} \leq \kappa^+ \), then we cannot have both the tree property at \( \kappa^+ \) and generic hugeness properties of \( \kappa^+ \) such as \((\kappa^+, \kappa^+)^\to (\kappa^+, \kappa)\) or the existence of a saturated ideal on \( \kappa^+ \). As a corollary, we find a tight connection between the Continuum Hypothesis and the forcing properties of the Boolean algebras associated to saturated ideals on \( \omega_2 \). We do not know whether these compactness and hugeness properties of successor cardinals are consistent with each other in the absence of the cardinal arithmetic assumption.

In Section 1, we discuss preliminaries about ideals and trees. In Section 2, we derive \( \Box^*_\kappa \) from several generic hugeness properties of \( \kappa^+ \) under cardinal arithmetic assumptions that are compatible with the tree property at \( \kappa^+ \). Proposition 2.1 and Theorem 2.4 are due to the first author, while Theorem 2.5 and Corollary 2.9 are due to the second author. Section 3 presents some barriers to combining compactness and hugeness properties at successor cardinals, showing that a general template for generically lifting huge and almost-huge embeddings, while collapsing the relevant cardinals down to successors, must force the failure of the tree property at the critical point.

1. Preliminaries

1.1. Ideals and generic embeddings. Proofs of many of the facts stated in this subsection can be found in [5]. An ideal on a set \( Z \) is a collection of subsets of \( Z \) closed under taking subsets and pairwise unions. If \( \kappa \) is a cardinal, we say that an ideal is \( \kappa \)-complete if it is closed under unions of size \( < \kappa \). If \( Z \subseteq P(X) \) and \( I \) is an ideal on \( Z \), then we say that \( I \) is normal if for all \( x \in X \), \( \{ z \in Z : x \notin z \} \in I \), and \( I \) closed under diagonal unions of the form \( \Delta_{x \in X} A_x := \{ z : \exists x \in X (z \in A_x) \} \). It is not difficult to show that if \( I \) is a normal ideal on \( Z \subseteq P(\lambda) \) and \( \kappa \leq \lambda \), then \( I \) is \( \kappa \)-complete if and only if for every \( \alpha < \kappa \), \( \{ z \in Z : \alpha \notin z \} \in I \). The smallest normal ideal on a set \( Z \) is the nonstationary ideal, \( NS_Z \), and its dual filter is called the club filter, which is generated by the collection of sets \( C_F \subseteq Z \), where \( F : X^{<\omega} \to X \).
and \( C_F = \{ z \in Z : F[z^{\omega}] \subseteq z \} \). If \( I \) is an ideal on \( Z \) and \( A \subseteq Z \), then \( I \upharpoonright A \) denotes the smallest ideal containing \( I \cup \{ Z \setminus A \} \). We say \( I \) concentrates on \( A \) if \( Z \setminus A \in I \). We refer to the collection \( \mathcal{P}(Z) \setminus I \) as \( I^+ \) and say the members are \( I \)-positive. The NS-positive sets are called \textit{stationary}.

If \( I \) is an ideal on \( Z \), \( \mathcal{P}(Z)/I \) is the quotient of the Boolean algebra \( \mathcal{P}(Z) \) by the equivalence relation \( A \sim B \iff (A \setminus B) \cup (B \setminus A) \in I \). If we take a generic filter \( G \subseteq \mathcal{P}(Z)/I \), then \( \bigcup G \) is a ultrafilter over \( \mathcal{P}(Z)^V \), and we can form the generic ultrapower embedding \( j_G : V \rightarrow V^Z/G \). The critical point of \( j_G \) is the least \( \kappa \) such that for some \( A \in \bigcup G \), \( I \upharpoonright A \) is \( \kappa \)-complete and \( A \) is the union of \( \kappa \)-many sets from \( I \). If \( Z \subseteq \mathcal{P}(X) \) and \( I \) is normal, then the well-founded part of \( V^Z/G \), which we will identify with its transitive collapse, has height at least \( |X|^{+} \), and the pointwise image \( j_G[X] \) is represented in \( V^Z/G \) by the identity function on \( Z \). Los’ Theorem is quite useful in determining properties of \( j_G[X] \) via what is satisfied by \( I \)-almost-all \( z \in Z \).

The key to showing the degree of well-foundedness of \( V^Z/G \) is the notion of a \textit{canonical function}. If \( \lambda \) is a cardinal and \( Z \subseteq \mathcal{P}_\lambda(\lambda) \), these are functions from \( Z \) to \( \lambda \) that are forced to represent particular ordinals below \( \lambda^+ \) in any generic ultrapower arising from a normal ideal on \( Z \). To define the canonical function representing \( \alpha \), we choose some surjection \( \sigma_\alpha : \lambda \rightarrow \alpha \). For any two surjections \( f, g : \lambda \rightarrow \alpha \), the set of \( z \in Z \) such that \( f[z] \neq g[z] \) is nonstationary. It is easy to check that for any normal ideal \( I \) on \( Z \) and any generic \( G \subseteq \mathcal{P}(Z)/I \), \( \alpha \) is represented in the ultrapower by the function \( z \mapsto \text{ot}(\sigma_\alpha[z]) \), where for a set of ordinals \( s \), \( \text{ot}(s) \) denotes its order-type.

We say a normal ideal \( I \) on \( Z \subseteq \mathcal{P}(X) \) is \textit{saturated} if \( \mathcal{P}(Z)/I \) has the \( |X|^{+}\)-chain condition. We say \( I \) is \textit{presaturated} when for every collection of maximal antichains \( \langle A_x : x \in X \rangle \), there is a dense set of \( B \) in \( \mathcal{P}(Z)/I \) such that for all \( x \in X \), \( \{ A \in A_x : A \cap B \notin I \} \) has size \( \leq |X| \). Obviously, saturation implies presaturation. It is well-known that if \( I \) is a presaturated normal ideal on \( Z \subseteq \mathcal{P}(\lambda) \), then the generic ultrapower \( V^Z/G \) is forced to be closed under \( \lambda \)-sequences from \( V[G] \).

An ideal is called \textit{precitous} when the generic ultrapower is forced to be well-founded. An ideal on a successor cardinal \( \kappa \) is called \textit{strong} when it is precipitous and it is forced that \( j_G(\kappa) = (\kappa^+)^V \). It is well-known that for normal ideals on successor cardinals, presaturated implies strong. The following further weakening is due to Woodin \cite{22}. We say that a normal ideal \( I \) is \textit{weakly presaturated} if it is forced that \( V^Z/G \) is well-founded up to \( (\kappa^+)^V + 1 \) and \( j_G(\kappa) = (\kappa^+)^V \). The following characterization comes from translating this into a forcing relation in \( V \):

**Proposition 1.1.** Suppose \( \kappa \) is a successor cardinal and \( I \) is a normal ideal on \( \kappa \). The following are equivalent:

1. \( I \) is weakly presaturated.
2. For all \( I \)-positive sets \( A \) and all functions \( f : \kappa \rightarrow \kappa \), \( f \) is bounded by a canonical function on an \( I \)-positive subset of \( A \).

Claverie and Schindler \cite{13} showed that a strong ideal on \( \omega_1 \) is equiconsistent with a Woodin cardinal, but it follows from a result of Silver and Lemma \cite{14} below that the consistency strength of a weakly presaturated ideal on \( \omega_1 \) is much lower.

The principle \( (\kappa^+, \kappa^+) \rightarrow (\kappa^+, \kappa) \), a generalized version of Chang’s Conjecture, asserts that every structure \( \mathfrak{A} \) in a countable language on \( \kappa^+ \) contains a substructure \( \mathfrak{B} \) such that \( |\mathfrak{B}| = \kappa^+ \) and \( |\mathfrak{B} \cap \kappa^+| = \kappa \). This is equivalent to saying that the
collection of \( z \subseteq \kappa^{++} \) of order-type \( \kappa^{+} \), or \([\kappa^{++}]^{\kappa^{+}}\), is stationary in \( \mathcal{P}(\kappa^{++}) \). An important fact is that if \((\kappa^{++}, \kappa^{+}) \rightarrow (\kappa^{+}, \kappa)\), then the set of \( z \in [\kappa^{++}]^{\kappa^{+}} \) such that \( z \cap \kappa^{+} \) is an ordinal is also stationary. If we force below this set in \( \mathcal{P}(\kappa^{++})/\text{NS} \), then the critical point of the embedding will be \( \kappa^{+} \).

We will use the following result from \cite{12}:

**Theorem 1.2** (Foreman-Magidor). For any infinite cardinal \( \kappa \), the set \( \{ z \in [\kappa^{++}]^{\kappa^{+}} : \text{cf}(z \cap \kappa^{+}) \neq \text{cf}(\kappa) \} \) is nonstationary.

A connection between Chang’s Conjecture and saturation properties of ideals is given by the following:

**Lemma 1.3.** Suppose \((\kappa^{++}, \kappa^{+}) \rightarrow (\kappa^{+}, \kappa)\). Then there is a weakly presaturated ideal on \( \kappa^{+} \) concentrating on \( \{ \alpha : \text{cf}(\alpha) = \text{cf}(\kappa) \} \).

**Proof.** Let \( I \) be the nonstationary ideal on \( \mathcal{P}(\kappa^{++}) \) restricted to \( Z = \{ z \subseteq \kappa^{++} : z \cap \kappa^{+} \in \kappa^{+} \land \text{ot}(z) = \kappa^{+} \} \). Let \( \pi : Z \rightarrow \kappa^{+} \) be defined by \( \pi(z) = z \cap \kappa^{+} \). Let \( J \) be the collection of \( A \subseteq \kappa^{+} \) such that \( \pi^{-1}[A] \in I \). It is easy to check that \( J \) is a normal ideal on \( \kappa^{+} \). Note that if \( A \in I^{+} \), then \( \pi[A] \in J^{+} \).

Suppose \( f : \kappa^{+} \rightarrow \kappa^{+} \) and \( A \in J^{+} \). Then \( \bar{A} = \pi^{-1}[A] \). For each \( z \in \bar{A} \), there is \( \alpha_{z} \in z \) such that \( f(z \cap \kappa^{+}) < \text{ot}(z \cap \alpha_{z}) \). By normality, there is \( \alpha < \kappa^{+} \) and \( B \in I^{+} \) such that \( \alpha_{z} = \alpha \) for all \( z \in B \). If \( \sigma_{\alpha} : \kappa^{+} \rightarrow \kappa^{+} \) is a surjection, then for \( I \)-almost-all \( z \in B \), \( \sigma_{\alpha} \upharpoonright z \) is a surjection from \( z \cap \kappa^{+} \) to \( z \cap \alpha \). We may assume this holds for all \( z \in B \).

Thus for all \( z \in \bar{B} \), \( f(z \cap \kappa^{+}) < \text{ot}(\sigma_{\alpha}[z \cap \kappa^{+}]) \). Let \( B = \pi[\bar{B}] \). Then \( B \) is a \( J \)-positive subset of \( A \), and \( f \) is bounded by a canonical function on \( B \). By Theorem 1.2, \( \{ \beta < \kappa^{+} : \text{cf}(\beta) \neq \text{cf}(\kappa) \} \) is nonstationary.

Silver showed that \((\omega_{2}, \omega_{1}) \rightarrow (\omega_{1}, \omega)\) can be forced from an \( \omega_{1} \)-Erdős cardinal (see \cite{12}), and Donder \cite{5} showed that \((\omega_{2}, \omega_{1}) \rightarrow (\omega_{1}, \omega)\) implies that there is an \( \omega_{1} \)-Erdős cardinal in an inner model. The existence of a weakly presaturated ideal on \( \kappa \) clearly implies that there is no single \( f : \kappa \rightarrow \kappa \) that dominates all canonical functions modulo \( \text{NS}_{\kappa} \). Donder and Koepke \cite{14} showed that for \( \kappa = \omega_{1} \), this statement is equiconsistent with an almost \( <\omega_{1} \)-Erdős cardinal, which implies the existence of \( \theta^{\kappa} \).

### 1.2. Trees and weak square

A *tree* is a partial order that is well-ordered below any element. For an infinite cardinal \( \kappa \), a \( \kappa \)-tree is a tree of height \( \kappa \) with levels of size \( < \kappa \). We say that \( \kappa \) has the *tree property* if every \( \kappa \)-tree has a cofinal branch. A \( \kappa^{+} \)-tree is called *special* if there is a function \( f : T \rightarrow \kappa \) such that \( x < y \) implies \( f(x) \neq f(y) \). Clearly, special \( \kappa^{+} \)-trees cannot have cofinal branches, since a branch would witness that \( \kappa^{+} \) is not a cardinal. Jensen \cite{11} showed the existence of a special \( \kappa^{+} \)-tree is equivalent to the weak square principle \( \square_{\kappa}^{\ast} \), which states that there is a sequence \( \langle C_{\alpha} : \alpha < \kappa^{+} \rangle \) such that:

1. Each \( C_{\alpha} \) is a nonempty set of size \( \leq \kappa \) consisting of closed unbounded subsets of \( \alpha \), each of order-type \( \leq \kappa \).
2. If \( \alpha < \kappa^{+} \), \( D \in C_{\alpha} \), and \( \beta < \alpha \) is a limit point of \( D \), then \( D \cap \beta \in C_{\beta} \).

Mitchell \cite{14} showed that having the tree property at the successor of a regular cardinal is equiconsistent with a weakly compact cardinal, and the failure of \( \square_{\kappa}^{\ast} \) for a regular \( \kappa \) is equiconsistent with a Mahlo cardinal. Starting with regular cardinals \( \mu < \kappa \) such that \( 2^{\mu} = \mu \) and a weakly compact \( \lambda > \kappa \), Mitchell constructed a
forcing extension preserving $\mu$ and $\kappa$ and in which $\lambda = \kappa^+$, $2^\mu = \lambda$, and every $\lambda$-tree has a cofinal branch. Starting from a Mahlo $\lambda > \kappa$, the same forcing works to produce a model in which there are no special $\kappa^+$-trees.

A strictly weaker principle than $\square^*_\kappa$ is the approachability property at $\kappa^+$. For a regular cardinal $\kappa$, we say a set $S \subseteq \kappa$ is approachable if there is a sequence $\langle a_\alpha : \alpha < \kappa \rangle$ such that for a club $C \subseteq \kappa$ and all $\alpha \in S \cap C$, there is an unbounded $A \subseteq \alpha$ of order-type $\text{cf}(\alpha)$ such that each initial segment of $A$ is in $\{a_\beta : \beta < \alpha\}$. The collection of approachable subsets of $\kappa$ generates a possibly non-proper normal ideal denoted by $I[\kappa]$. Shelah [19] showed that if $\kappa$ is regular, then $\kappa^+ \cap \text{cof}(< \kappa) \in I[\kappa^+]$, where $\text{cof}(< \kappa)$ denotes the class of ordinals of cofinality $< \kappa$.

The weak square principle $\square^*_\kappa$ is absolute to any outer model with the same $\kappa^+$, including outer models in which $\kappa$ is not a cardinal. This is because it does not matter how we bound the order-types:

**Lemma 1.4.** Suppose $\xi < \kappa^+$ and $\langle C_\alpha : \alpha < \kappa^+ \rangle$ is a sequence such that:

1. Each $C_\alpha$ is a set of $\leq \kappa$ clubs in $\alpha$, each of order-type $\leq \xi$.
2. If $\alpha < \kappa^+$, $D \in C_\alpha$, and $\beta \in \alpha \cap \text{lim} D$, then $D \cap \beta \in C_\beta$.

Then $\square^*_\kappa$ holds.

**Proof.** It is easy to show by induction that for each $\delta < \kappa^+$, there is a sequence $\langle E_\alpha : \alpha < \delta \rangle$ such that:

1. Each $E_\alpha$ is a club in $\alpha$ of order-type $\leq \kappa$.
2. If $\alpha < \delta$ and $\beta \in \alpha \cap \text{lim} E_\alpha$, then $E_\beta = E_\alpha \cap \beta$.

Fix such a sequence $\langle E_\alpha : \alpha < \xi \rangle$. For each $\alpha < \kappa^+$, and $D \in C_\alpha$ define $D' = \{\beta \in D : \text{ot}(D \cap \beta) \in E_{\text{ot}(D)}\}$. Clearly, for each $\alpha < \kappa^+$, $|C_\alpha| \leq |C_\alpha| \leq \kappa$ and if $D \in C_\alpha$, then $\text{ot}(D') \leq \text{ot}(E_{\text{ot}(D)}) \leq \kappa$, and $D'$ is a club in $\alpha$.

Let $C'_\alpha = \{D' : D \in C_\alpha\}$. Furthermore, $\text{ot}(D \cap \beta) \in \text{lim} E_{\text{ot}(D)}$, so $E_{\text{ot}(D \cap \beta)} = E_{\text{ot}(D)} \cap \text{ot}(D \cap \beta)$, and $D \cap \beta = \{\gamma \in D \cap \beta : \text{ot}(D \cap \gamma) \in E_{\text{ot}(D)}\} = D' \cap \beta$.

**2. Weak square from strong ideals**

In this section, we deduce $\square^*_\kappa$ from strong hypotheses about ideals plus cardinal arithmetic assumptions that are compatible with the tree property at $\kappa^+$. We start with some deductions from $(\kappa^{++}, \kappa^+) \rightarrow (\kappa^+, \kappa)$ and related principles that are easier and serve to illustrate the main idea. While these are surpassed by Theorem 2.3 in the case where $\kappa$ is regular, the arguments from Chang’s Conjecture also work in the case of singular $\kappa$.

**Proposition 2.1.** Suppose $2^\kappa = \kappa^+$ and either $(\kappa^{++}, \kappa^+) \rightarrow (\kappa^+, \kappa)$ holds, or there is a weakly presaturated ideal on $\kappa^+$ with the property that $\mathcal{P}(\kappa^+)^V$ is forced to be a member of the generic ultrapower. Then $\square^*_\kappa$ holds.

**Proof.** First assume that $(\kappa^{++}, \kappa^+) \rightarrow (\kappa^+, \kappa)$ holds. Let $N < H_{\kappa^{++}}$ contain $\kappa^{++}$ and have size $\kappa^{++}$. Let $Z = \{z \subseteq \kappa^{++} : z \cap \kappa^+ \in \kappa^+ \wedge \text{ot}(z) = \kappa^+\}$. Let $G \subseteq \mathcal{P}(Z)/\text{NS}$ be generic, and let $j : V \rightarrow M$ be the ultrapower embedding. Since $[\text{id}]_G = j[\kappa^{++}]$, Los’ Theorem implies that $\text{ot}(j[\kappa^{++}]) = (\kappa^{++})^V = j(\kappa^+)$. Since $j[\kappa^{++}] \in M$ and $N$ is coded by a subset of $\kappa^{++}$, $N \in M$. 

Under the second hypothesis, since in general $H_{\beta^+}$ is constructible from $\mathcal{P}(\theta)$, we have that $H_{\beta^+}^+ \in M$. Thus under both assumptions, some forcing introduces a generic elementary embedding $j: V \to M$ with critical point $\kappa^+$ such that $j(\kappa^+) = (\kappa^+)^V$, and there is a transitive structure $N \in V \cap M$ of height $(\kappa^+)^V$, with $\mathcal{P}(\kappa)^V \in N$, in which $(\kappa^+)^V$ is the largest cardinal, and such that $N \models ZFC - \{\text{Powerset]\}$. Working in $M$, we construct a $\Box^*_\kappa$ sequence as follows. For $\alpha < (\kappa^+)^V$ such that $N \models \text{cf}(\alpha) = \kappa^+$, let $D_\alpha \in N$ be a club in $\alpha$ of order-type $\kappa^+$, and let $C_\alpha = \{D_\alpha\}$. For $\alpha < (\kappa^+)^V$ such that $N \models \text{cf}(\alpha) \leq \kappa$, let $C_\alpha$ be the collection of all clubs in $\alpha$ of order-type $< \kappa^+$ that are members of $N$. Since $N \models 2^\kappa = \kappa^+$, and $M \models |(\kappa^+)^N| = \kappa$, we have $|C_\alpha| \leq \kappa$ for all $\alpha$. Since $N$ satisfies enough set theory, we have that if $D \in C_\alpha$ and $\beta < \alpha$ is a limit point of $D$, then $D \cap \beta \in C_\beta$. Lemma 1.3 implies that $M \models \Box^*_\kappa$. Since $\kappa < \text{crit}(j)$, $V \models \Box^*_\kappa$ by the elementarity of $j$. \hfill \Box

Remark 2.2. It is shown in \cite{7} that if $\text{cf}(\kappa) = \omega$ and $(\kappa^+, \kappa) \rightarrow (\kappa^+, \kappa)$, then $\Box^*_\kappa$ holds, regardless of cardinal arithmetic.

Proposition 2.3. Suppose $\kappa^\kappa \leq \kappa^+$, $(\kappa^+, \kappa^+) \rightarrow (\kappa^+, \kappa)$, and the set $\{\alpha < \kappa^+ : \exists z \in [\kappa^+]^\kappa^+ (\alpha = z \cap \kappa^+)\}$ is approachable. Then $\Box^*_\kappa$ holds.

Proof. In $V$, let $\langle a_\alpha : \alpha < \kappa^+ \rangle$ list all elements of $\mathcal{P}_\kappa(\kappa^+)$. Let $Z = \{z \subseteq \kappa^+ : z \cap \kappa^+ \in \kappa^+ \land \text{ot}(z) = \kappa^+\}$, and let $j: V \rightarrow M$ be a generic ultrapower embedding obtained by forcing with $\mathcal{P}(Z)/\text{NS}$. By the approachability assumption and Theorem 1.2, for all but nonstationary-many $z \in Z$, there is an unbounded $A \subseteq z \cap \kappa^+$ of order-type $\text{cf}(\kappa)$ such that all initial segments are in $\{a_\beta : \beta < z \cap \kappa^+\}$. By Loś’s Theorem, there is in $M$ a club $E \subseteq (\kappa^+)^V$ of order-type $\text{cf}(\kappa)$ such that every initial segment of $E$ is in $V$.

In $V$, choose a sequence $\langle b_\alpha : \alpha < \kappa^+ \rangle$ such that each $b_\alpha$ is a club in $\alpha$ of order-type $\text{cf}(\alpha)$. Since $j(\kappa^+)^V \in M$, this sequence is in $M$, and the $V$-cofinalities of ordinals $< j(\kappa^+)$ are definable in $M$ in terms of this parameter. For the same reasons, $\langle \mathcal{P}_\kappa(\kappa)^V : \alpha < (\kappa^+)^V \rangle \in M$. Note that each element of this sequence has size $\kappa$ in $M$.

Now we construct our $\Box^*_\kappa$ sequence in $M$ as follows. If $V \models \text{cf}(\alpha) = \kappa^+$, then enumerate $b_\alpha$ in increasing order as $\langle \gamma_\beta : \beta < \kappa^+ \rangle$, and let $C_\alpha = \{\gamma_\beta : \beta \in E\}$. Every initial segment of $\{\gamma_\beta : \beta \in E\}$ is in $V$. If $V \models \text{cf}(\alpha) = \kappa$, let $C_\alpha = \{b_\alpha\}$. If $V \models \text{cf}(\alpha) < \kappa$, let $C_\alpha$ be the collection of all club subsets of $\alpha$ from $V$ of order-type $< \kappa$. For each $\alpha$, $M \models |C_\alpha| \leq \kappa$. If $D \in C_\alpha$ and $\beta \in \alpha \cap \text{lim} D$, then $D \cap \beta \in V$. Since $\text{ot}(D) \leq \kappa$, $V \models \text{cf}(\beta) < \kappa$, and so $D \cap \beta \in C_\beta$. By elementarity, $V \models \Box^*_\kappa$. \hfill \Box

In order to derive $\Box^*_\kappa$ from weaker hypotheses about ideals on $\kappa^+$, we have to contend with the fact there is no guarantee that a given object of size $> \kappa^+$ from $V$ is a member of the generic ultrapower $M$. The key to our arguments will be a refinement of the approximation property introduced by Hamkins \cite{10}. Suppose $A$ is a set, $X \subseteq A$, and $\mathcal{F} \subseteq \mathcal{P}(A)$. We will say that $X$ is approximated by $\mathcal{F}$ when for all $a \in A$, $a \cap X \in \mathcal{F}$. The $\kappa$-approximation property can be stated in these terms as follows: For models of set theory $M \subseteq N$ and an $M$-cardinal $\kappa$, the pair $(M, N)$ satisfies the $\kappa$-approximation property when for all ordinals $\lambda \in M$, if $X \in \mathcal{P}(\lambda)^N$ is approximated by $\mathcal{P}_\kappa(\lambda)^M$, then $X \in M$. We say that a forcing $\mathbb{P}$ has the $\kappa$-approximation property when it forces that the pair $(V, V[G])$ has this property.
Theorem 2.4. Suppose $\kappa$ is regular, $2^{<\kappa} \leq \kappa^+$, and $I$ is a normal ideal on $\kappa^+$ such that $\mathcal{P}(\kappa^+)/I$ is a forcing with the $\kappa$-approximation property. Then $I$ is not weakly presaturated.

Proof. Recall that a weak $\kappa$-Kurepa tree is a tree of height $\kappa$, with levels of size $\leq \kappa$, and with more than $\kappa$-many cofinal branches. If $I$ is a normal ideal on $\kappa^+$ such that $\mathcal{P}(\kappa^+)/I$ has the $\kappa$-approximation property, then there are no weak $\kappa$-Kurepa trees $T$, since the generic embedding would necessarily add branches to $T$, whereas any branch is approximated by $\mathcal{P}_\kappa(T)^V$. Baumgartner [11] showed that if $\kappa$ is regular, $2^{<\kappa} \leq \kappa^+$, and there are no weak $\kappa$-Kurepa trees, then $2^\kappa = \kappa^+$.

Suppose $j : V \rightarrow M$ is a generic embedding arising from forcing with a normal ideal $I$ as above. Since $2^{<\kappa} \leq \kappa^+$, $\mathcal{P}_\kappa(\kappa^+)^V \in M$. By the $\kappa$-approximation property, every $X \subseteq \kappa^+$ in $M$ that is approximated by $\mathcal{P}_\kappa(\kappa^+)^V$ is in $V$. But this means that $\mathcal{P}(\kappa^+)^V$ is definable in $M$ as the collection of all such $X$, since for any $X \in \mathcal{P}(\kappa^+)^V$, $X = j(X) \cap \kappa^+ \in M$. Since $V \models 2^\kappa = \kappa^+$, Proposition 2.1 implies that if $I$ is weakly presaturated, then $V \models \square^*_\kappa$.

On the other hand, $\mathcal{P}(\kappa^+)/I$ having the $\kappa$-approximation property implies the tree property at $\kappa^+$, since any $\kappa^+$-tree $T$ acquires a branch in $V[G]$ by looking below a node of $j(T)$ at level $(\kappa^+)^V$. But this branch is approximated by $\mathcal{P}_\kappa(T)^V$ and is thus in $V$. \hfill \Box

We note that it is consistent relative to a measurable cardinal that $2^\kappa = \omega_2$ and there is a precipitous normal ideal $I$ on $\omega_2$ such that $\mathcal{P}(\omega_2)/I$ has the $\omega_1$-approximation property. For example, if $\kappa$ is measurable, this is forced by (a variation of) the Mitchell forcing up to $\kappa$, countable support iteration of Sacks forcing [2], and by the pure side conditions forcings of Krueger [13] and Neeman [15].

Theorem 2.5. Suppose $\kappa$ is a regular cardinal, $2^{<\kappa} \leq \kappa^+$, and there is a weakly presaturated ideal on $\kappa^+$ concentrating on $\text{cof} (\kappa)$. Then $\square^*_\kappa$ holds.

Proof. We may assume that $\kappa > \omega$, since $\square^*_\kappa$ always holds. Let $\delta = (\kappa^+)^V$. A forcing introduces an elementary embedding $j : V \rightarrow M$ with critical point $\delta$, such that $M$ is well-founded up to $(\kappa^+)^V + 1$, $j(\delta) = (\kappa^+)^V$, and $M \models \text{cf}(\delta) = \kappa$. Since $V \models \delta^{<\kappa} = \delta$, $\mathcal{P}_\kappa(\delta)^V \in M$. Define in $M$ the set $\mathcal{A}$ of subsets of $\delta$ that are approximated by $\mathcal{P}_\kappa(\delta)^V$.

Fix in $M$ a club $C^* \subseteq \delta$ of order-type $\kappa$, and let $\langle \xi_\alpha : \alpha < \kappa \rangle$ be its increasing enumeration. In $V$, let $\vec{\sigma} = \langle \sigma_\alpha : \alpha < \delta \rangle$ be a sequence such that $\sigma_\alpha : \kappa \rightarrow \alpha$ is a surjection, and note that $\vec{\sigma} \in M$. We can write $\delta$ as the union of a continuous increasing sequence of sets of size $< \kappa$, $\langle z_\alpha : \alpha < \kappa \rangle$, by putting $z_\alpha = \bigcup_{\beta < \alpha} \sigma_\beta [\alpha]$. Take $N < M^j_{\delta(\delta)}$ such that $\{C^*, \mathcal{P}_\kappa(\delta)^V, \vec{\sigma} \} \cup \delta \subseteq N$ and $M \models |N| = \kappa$. Let $Q = \mathcal{P}_\kappa(\delta)^V \cap N$.

Recall that a prewellordering is a transitive reflexive binary relation in which every two elements are comparable, such that the quotient by the equivalence relation, $x \sim y \iff x \leq y \wedge y \leq x$, is a wellorder. An ordinal $\alpha$ has cardinality $\leq \beta$ if and only if there is a prewellordering on $\beta$ whose quotient has order-type $\alpha$. There is a natural correspondence between surjections from sets onto ordinals and prewellorderings of those sets. For a set of ordinals $Z$ closed under the Gödel pairing function, a set $X \subseteq Z$ codes a relation on $Z$ via this function. If $X \subseteq Z$ codes a prewellordering whose quotient has order-type $\alpha$, let $f_X : Z \rightarrow \alpha$ be the corresponding surjection.
Claim 2.6. Suppose \( X, Y \in \mathcal{A} \) code prewellorderings of \( \delta \) of the same length. Then \( \{ f_X[z] : z \in Q \} = \{ f_Y[z] : z \in Q \} \).

Proof of claim. Let \( r \in Q \). We need to show that there is some \( s \in Q \) such that \( f_X[r] = f_Y[s] \). There is a club \( C \subseteq \kappa \) such that for all \( \alpha \in C, f_X[z_\alpha] = f_Y[z_\alpha] \). We may assume that for all \( \alpha \in C, z_\alpha \) is closed under Gödel pairing.

Let \( \alpha \in C \) be such that \( r \subseteq z_\alpha \). By definition, \( z_\alpha \subseteq \xi_\alpha < \delta \). Since \( |z_\alpha| < \kappa \), there is \( \beta < \kappa \) such that \( z = \sigma_{\xi_\alpha}[^{\beta}z_\alpha] \). Since \( X, Y \in \mathcal{A}, X \cap z \) and \( Y \cap z \) are in \( V \). Thus \( X \cap z_\alpha \) and \( Y \cap z_\alpha \) are in \( N \). These sets code prewellorderings of \( z_\alpha \) of order-type \( \eta = \text{ot}(f_X[z_\alpha]) = \text{ot}(f_Y[z_\alpha]) \).

Let \( h_X : z_\alpha \rightarrow \eta \) and \( h_Y : z_\alpha \rightarrow \eta \) be the corresponding surjections. Let \( r' = h_X[r] \). Note that if \( \pi : \eta \rightarrow f_X[z_\alpha] \) is the unique order-preserving map, then \( \pi[r'] = f_X[r] \). Let \( s = f_{h_Y^{-1}}[r'] \). Then \( s \in N \), and \( h_Y[s] = r' \). Furthermore, \( \pi \circ h_Y[s] = f_Y[s] = f_X[r] \), as desired.

If \( f \) is a function from \( Z \) to an ordinal \( \alpha \) and \( \beta < \alpha \), let \( f \upharpoonright \beta \) be the function \( g \) such that \( g(\gamma) = f(\gamma) \) when \( f(\gamma) < \beta \) and \( g(\gamma) = 0 \) otherwise. If \( R \) is a prewellordering on a set \( Z \) of order-type \( \alpha \), \( f_R : Z \rightarrow \alpha \) is the corresponding surjection, and \( \beta < \alpha \), then let \( R \upharpoonright \beta \) denote the canonical alteration of \( R \) to represent \( f_R \upharpoonright \beta \), where we make \( x \) equivalent to the \( R \)-least element of \( Z \) if \( f_R(x) \geq \beta \), and leave the ordering between the elements of rank \( < \beta \) the same.

Claim 2.7. If \( X \in \mathcal{A} \) codes a prewellordering of order-type \( \alpha \) and \( \beta < \alpha \), then \( X \upharpoonright \beta \in \mathcal{A} \). Furthermore, if \( r \in Q \), then \( f_X[r] \cap \beta = f_{X \upharpoonright \beta}[s] \) for some \( s \in Q \).

Proof of claim. Suppose \( y \in \mathcal{P}(\kappa) \) and \( r \in Q \). Let \( \zeta_0 \) be any ordinal such that \( f_X(\zeta_0) = 0 \). Let \( \xi < \delta \) and \( \beta < \kappa \) be such that \( y \cup r \cup \{ \zeta_0 \} \subseteq \sigma_{\xi}[\beta] \). Let \( z \) be the closure of \( \sigma_{\xi}[\beta] \) under Gödel pairing, which is in \( V \). Let \( h : z \rightarrow \eta \) be the surjection coded by \( X \cap z \). There is some \( \xi \leq \eta \) such that \( h(\gamma) < \xi \Leftrightarrow f_X(\gamma) < \beta \) for \( \gamma \in z \). The function \( h \upharpoonright \xi \) and its code \( x \subseteq z \) are in \( V \). We have that \( (X \upharpoonright \beta) \cap z = x \). Thus \( (X \upharpoonright \beta) \cap y = (X \upharpoonright \beta) \cap z \cap y = x \cap y \in V \). This shows that \( X \upharpoonright \beta \in \mathcal{A} \).

For the second part, let \( s = \{ \gamma \in r : h(\gamma) < \xi \} \). Then \( s \in N \), and \( f_X[s] = f_{X \upharpoonright \beta}[s] = f_X[r] \cap \beta \).

To define a \( \square \) sequence in \( M \), first consider ordinals \( \alpha < j(\delta) \) of cofinality \( < \kappa \). Let \( C_\alpha \) be the set of all clubs \( D \in \alpha \) of order-type \( < \kappa \), such that for some \( X \in \mathcal{A} \) that codes a prewellordering of \( \delta \) of order-type \( \alpha \), \( D = f_X[s] \) for some \( s \in Q \). By Claim 2.6, the choice of \( X \) does not matter, so the cardinality of this set is at most \( |N| = \kappa \). By Claim 2.7, if \( C \subseteq C_\alpha \) and \( \beta \) is a limit point of \( C \), then \( C \cap \beta \in C_\beta \). Furthermore, each such \( C_\alpha \) is nonempty, since the cofinality of \( \alpha \) cannot change between \( V \) and \( M \). For suppose \( V \models \text{cf}(\alpha) = \mu \) and \( M \models \text{cf}(\alpha) = \mu' < \kappa \). Let \( Y \in \mathcal{P}(\delta)^V \) code a witness to \( V \models \text{cf}(\alpha) = \mu \). Then \( Y \in M \), so \( M \models \text{cf}(\mu) = \mu' \). We cannot have \( \mu = \delta \) because \( M \models \text{cf}(\delta) = \kappa \), so \( \mu < \delta \). By elementarity, \( M \models \text{cf}(\mu) = \mu \). Thus there is \( X \in \mathcal{P}(\delta)^V \) that codes a prewellordering of \( \delta \) of order-type \( \alpha \) and a set \( s \in \mathcal{P}(\kappa)^V \) such that \( f_X[s] \) is club in \( \alpha \).

Now suppose \( V \models \text{cf}(\alpha) = \kappa \). Let \( D \in \mathcal{V} \) be a club in \( \alpha \) of order-type \( \kappa \). Let \( f : \delta \rightarrow \alpha \) be a surjection in \( V \). If \( s \) is an initial segment of \( D \) of limit order-type, then \( r = f^{-1}[s] \in V \). If \( \beta = \sup(s) \), then \( s = (f \upharpoonright \beta)[r] \), so \( s \in C_\beta \). Thus in \( M \), there is a club \( C \subseteq \alpha \) of order-type \( \kappa \) such that all initial segments of limit length are in \( C_\beta \) for some \( \beta < \alpha \).
Finally, suppose \( V \models \text{cf}(\alpha) = \delta \). Let \( D \in V \) be a club in \( \alpha \) of order-type \( \delta \), and let \( \langle \gamma_\beta : \beta < \delta \rangle \) be its increasing enumeration. Let \( f : \delta \to \alpha \) be a surjection in \( V \). Let \( g : \delta \to \delta \) be a function in \( V \) such that for all \( \beta < \delta \), \( f \circ g(\beta) = \gamma_\beta \). In \( M \), let \( D' = \{ \gamma_\beta : \beta \in C^* \} \). Let \( s \) be an initial segment of \( C^* \). Let \( \gamma = \sup(s) \), and let \( \beta < \kappa \) be such that \( s \subseteq z = \sigma_{\gamma}[\beta] \). Then \( g \upharpoonright z \) is coded by an element of \( \mathcal{P}_\kappa(\delta)^V \), and so \( g \upharpoonright s \in N \). Thus \( \{ \gamma_\beta : \beta \in s \} = f[r] \) for some \( r \in N \). In particular, there exists \( C \in M \) that is club in \( \alpha \), of order-type \( \kappa \), and such that all initial segments of limit length are in \( C_\beta \) for some \( \beta < \alpha \).

Although \( M \) may not know which ordinals \( \alpha \) of cofinality \( \kappa \) have cofinality \( \delta \) in \( V \), we can just choose in either case some club \( C \subseteq \alpha \) of order-type \( \kappa \) such that all initial segments are in \( C_\beta \) for some \( \beta < \alpha \). Let \( C_\alpha = \{ C \} \) for any such \( C \). This completes the construction of a \( {\square}^*_\kappa \)-sequence in \( M \). By elementarity, \( V \models {\square}^*_\kappa. \)

Shelah \[18\] showed that if \( I \) is a normal presaturated ideal on \( \kappa^+ \), then \( I \) concentrates on \( \{ \alpha : \text{cf}(\alpha) = \text{cf}(\kappa) \} \). Thus the hypothesis of Theorem \[22\] can be simplified if we assume the ideal is presaturated. On the other hand, a combination of theorems of Sargsyan \[17\] and Woodin \[22\] shows that we cannot drop the assumption that the ideal concentrates on the highest possible cofinality:

**Theorem 2.8** (Sargsyan–Woodin). Assume the consistency of a Woodin limit of Woodin cardinals. Then there is a model of ZFC satisfying:

1. Bounded Martin’s Maximum, which implies \( 2^\omega = \omega_2 \) and the tree property at \( \omega_2 \).
2. \( \text{NS}_{\omega_2} \upharpoonright \text{cof}(\omega) \) is strong.

Theorem \[24\] can be derived from Theorem \[22\]. This is because for regular \( \kappa \), if \( I \) is normal ideal on \( \kappa^+ \) such that \( \mathcal{P}(\kappa^+)/I \) has the \( \kappa \)-approximation property, then \( I \) must concentrate on \( \text{cof}(\kappa) \). This follows from Shelah’s result that \( \kappa^+ \cap \text{cof}(<\kappa) \) is approachable. Indeed, any such ideal must contain the approachability ideal. For suppose \( S \in I^+ \) and \( \langle a_\alpha : \alpha < \kappa^+ \rangle \) witnesses that \( S \) is approachable. Let \( G \subseteq \mathcal{P}(\kappa^+)/I \) be generic with \( S \in G \), and let \( j : V \to M \) be the ultrapower embedding. By Lós’ Theorem, there is \( A \in M \), an unbounded subset of \( (\kappa^+)^V \) of order-type \( \leq \kappa \), such that all initial segments are in \( \{ a_\alpha : \alpha < \kappa^+ \} \subseteq V \). But this violates the \( \kappa \)-approximation property.

Foreman and Magidor \[9\] asked whether a saturated normal ideal on \( \omega_2 \) can contain the approachability ideal. This question appeared again in \[8\]. Since \( {\square}^*_\kappa \) implies that all subsets of \( \kappa^+ \) are approachable, Theorem \[22\] shows that the answer is “no” under the assumption that \( 2^\omega \leq \omega_2 \).

If \( I \) is a saturated ideal on \( \omega_2 \) and \( 2^\omega = \omega_1 \), then forcing with \( \mathcal{P}(\omega_2)/I \) does not add reals. It is consistent relative to an almost-huge cardinal that there is a saturated ideal on \( \omega_2 \) whose associated Boolean algebra has a countably closed dense set, and in particular is a proper forcing (see \[8\]). Remarkably, this is only possible under CH:

**Corollary 2.9.** Suppose \( I \) is a normal ideal on \( \omega_2 \). Suppose either \( I \) is weakly presaturated and \( \mathcal{P}(\omega_2)/I \) is a proper forcing, or \( I \) is presaturated and \( \mathcal{P}(\omega_2)/I \) is a semiproper forcing. Then the continuum hypothesis holds.

Before giving the proof, let us define the “cofinal Strong Chang’s Conjecture,” abbreviated by \( \text{SCC}^{\text{cof}} \) in \[4\]. This states that for every large enough cardinal \( \theta \),
every countable $M < H_\beta$, and every $\alpha < \omega_2$, there is a countable $N < H_\beta$ such that $M \subseteq N$, $M \cap \omega_1 = N \cap \omega_1$, and $\sup(N \cap \omega_2) > \alpha$.

**Proof.** Let $I$ be a normal ideal on $\omega_2$. If $\mathcal{P}(\omega_2)/I$ is semiproper, then by Sakai [10], SCC$_{\text{cof}}$ holds. By Todorčević [20], SCC$_{\text{cof}}$ implies $2^\omega \leq \omega_2$. By Torres-Perez and Wu [21], if SCC$_{\text{cof}}$ holds, then the failure of CH is equivalent to the tree property at $\omega_2$. If $\mathcal{P}(\omega_2)/I$ is a proper forcing, then it cannot change the cofinality $\omega_2$ to $\omega$, so $I$ must concentrate on cof$(\omega_1)$. If $I$ is presaturated, then it concentrates on cof$(\omega_1)$ by Shelah’s Theorem. In either case, the hypotheses imply that $2^\omega \leq \omega_2$ and there is a weakly presaturated ideal concentrating on cof$(\omega_1)$, which by Theorem 2.5 implies that the tree property at $\omega_2$ fails. Therefore, $2^\omega = \omega_1$. \hfill $\square$

### 3. Weak square from lifted embeddings

The known methods for forcing either $(\mu^+, \mu^+) \rightarrow (\mu^+, \mu)$ or the existence of a saturated ideal on $\mu^+$, where $\mu$ is uncountable, start with a huge or almost-huge cardinal $\kappa > \mu$ with witnessing embedding $j : V \rightarrow M$, and collapse $\kappa$ to $\mu^+$ and $j(\kappa)$ to $\mu^{++}$ in a way that allows the embedding to be generically lifted. A variety of such constructions are described in [8]. These constructions typically force $\mu^{<\mu} = \mu$, and thus $\square^*_\mu$. However, we argue here that $\square^*_\mu$ is already guaranteed by certain abstract features of these forcings, without any *prima facie* assumptions about the effect on cardinal arithmetic. The arguments will apply to embeddings coming from hypotheses weaker than almost-hugeness.

In typical situations, we lift an almost-huge embedding $j : V \rightarrow M$ through a forcing $\mathbb{P} \ast \dot{\mathbb{Q}}$, obtaining $j' : V[G \ast H] \rightarrow M[G' \ast H']$, where $\mathbb{P}$ is crit($j$)-c.c. and $\dot{\mathbb{Q}}$ is not. In order to lift through $H$, it is usually key to the argument that at least small pieces of $G \ast H$ are members of $M[G']$. The next observation shows that this is enough to guarantee that hypothesis 3 of Theorem 5.2 holds.

**Proposition 3.1.** Suppose $\kappa \leq \delta$ are regular cardinals, $M \subseteq V$ is a $<\delta$-closed inner model, and $\mathbb{P}$ is a $\delta$-c.c. partial order. Let $G \subseteq \mathbb{P}$ be generic over $V$, and suppose $N$ is an outer model of $M$. Then the following are equivalent:

1. $\mathcal{P}_\kappa(\text{Ord})^{V[G]} \subseteq N$.
2. $\mathcal{P}_\kappa(\mathbb{P})^{V[G]} \subseteq N$.

**Proof.** We may assume that the elements of $\mathbb{P}$ are ordinals. Then $(1) \Rightarrow (2)$ is trivial. For the other direction, suppose $\xi < \kappa$ and $f : \xi \rightarrow \text{Ord}$ is in $V[G]$. Let $\dot{f}$ be a name for $f$, and let $p_0 \forces \text{dom} \dot{f} = \dot{\xi}$. For $\alpha < \xi$, let $A_\alpha$ be a maximal antichain below $p_0$ deciding $\dot{f}(\dot{\alpha})$. For $\alpha < \xi$ and $p \in A_\alpha$, let $\beta_{\alpha,p}$ be the value of $\dot{f}(\dot{\alpha})$ decided by $p$. Define a $\mathbb{P}$-name:

$$\tau := \{ (p, (\dot{\alpha}, \dot{\beta}_{\alpha,p})) : \alpha < \xi \text{ and } p \in A_\alpha \}.$$  

Then $p_0 \forces \dot{f} = \tau$. By the $\delta$-c.c. and the $<\delta$-closure of $M$, $\tau \in M$. Now for all $\alpha < \xi$ there is a unique $q_\alpha \in G \cap A_\alpha$, and $\tau^G$ can be computed from $\tau$ and $\{ q_\alpha : \alpha < \xi \}$. By hypothesis, $\{ q_\alpha : \alpha < \xi \} \in N$, and therefore $f \in N$. \hfill $\square$

**Theorem 3.2.** Suppose $j : V \rightarrow M$ is an elementary embedding with critical point $\kappa$ definable from parameters in $V$. Suppose $\mathbb{P} \ast \dot{\mathbb{Q}}$ is a two-step iteration such that:

1. $M$ is $[\mathbb{P}]$-closed, and $|\mathbb{P}| < j(\kappa)$.
2. $\mathbb{P} \ast \dot{\mathbb{Q}}$ collapses all ordinals in the open interval $(\kappa, j(\kappa))$. 


Then \( \mathbb{P} \) forces that \( \kappa = \mu^+ \) for some \( \mu < \kappa \), and \( \square^*_\mu \) holds.

**Proof.** First we claim that \( \hat{Q} \) is forced to be \( \kappa \)-distributive. Let \( G \ast H \subseteq \mathbb{P} \ast \hat{Q} \) be generic. Suppose \( r \in \mathcal{P}_\kappa(\text{Ord}) \cap V[G \ast H] \setminus V[G] \). Let \( j' : V[G \ast H] \to M[G' \ast H'] \) be a lifting of \( j \) as in (3). Note that \( j'(r) = j[r] \), and there is an \( s \in \mathcal{P}_\kappa(\text{Ord}) \) that codes \( j \cap r \). Since \( s \in M[G'] \), we have that \( j'(r) \in M[G'] \). But by elementarity, \( j'(r) \in M[G' \ast H'] \setminus M[G'] \), a contradiction.

Next we claim that \( \mathcal{P}(\kappa)^{V[G \ast H]} \subseteq M[G'] \). Suppose \( X \in \mathcal{P}(\kappa)^{V[G \ast H]} \). Then \( X = j'(X) \cap \kappa \) is in \( M[G' \ast H'] \). But since \( M[G'] \models j(\hat{Q})' \) is \( j(\kappa) \)-distributive, \( X \in M[G'] \). Note that since \( \mathbb{P} \) is collapsed to \( \kappa \), \( G \) is coded by a subset of \( \kappa \) in \( V[G \ast H] \), and thus \( G \in M[G'] \).

By hypothesis (1), \( \mathcal{P}_\kappa(\text{Ord})^{V[G]} = \mathcal{P}_\kappa(\text{Ord})^{M[G]} \). In \( M[G'] \), define a sequence \( \langle C_\alpha : \alpha < j(\kappa) \rangle \) as follows. If \( M[G'] \models \text{cf}(\alpha) < \kappa \), let \( C_\alpha \) be the set of all clubs in \( \alpha \) of order-type less than \( \kappa \) that live in \( M[G] \). Since \( \mathbb{P} \prec j \ast \hat{Q} \) and \( j \) is inaccessible in \( M \), each such set has size \( j(\kappa) \) in \( M[G] \), and thus size \( \leq \kappa \) in \( M[G'] \). If \( M[G'] \models \text{cf}(\alpha) \geq \kappa \), then this is true in \( V[G] \), and it remains true in \( V[G \ast H] \) by distributivity. In \( V[G \ast H] \), \( |\alpha| = \kappa \), so there is a club \( C \subseteq \alpha \) of order-type \( \kappa \). All its initial segments are in \( \mathcal{P}_\kappa(\text{Ord})^{M[G]} \). \( C \) is coded by some \( X \subseteq \kappa \), which is in \( M[G'] \). Working in \( M[G'] \) we let \( C_\alpha = \{ D \} \), where \( D \) is any club in \( \alpha \) of order-type \( \kappa \) such that all initial segments are in \( M[G] \).

Since \( j(\kappa) \) is not a limit cardinal in \( M[G'] \), it follows by elementarity that \( \kappa = \mu^+ \) in \( V[G'] \) for some \( \mu < \kappa \), and thus \( j(\kappa) = \mu^+ \) in \( M[G'] \). We conclude using Lemma 14 that \( \square^*_\mu \) holds in \( M[G'] \). By elementarity, \( \square^*_\mu \) holds in \( V[G] \).

□

If we weaken hypothesis (2) of Theorem 5.2 to say just that all ordinals in some final segment of \( j(\kappa) \) are collapsed, then the argument does not go through:

**Proposition 3.3.** Suppose \( \kappa \) is measurable, and \( j : V \rightarrow M \) is derived from a normal measure on \( \kappa \). Then there is a two-step iteration \( \mathbb{P} \ast \hat{Q} \) such that:

1. \( \mathbb{P} \ast \hat{Q} \) forces that \( \kappa = \omega_2 \) and \( \omega_2 \) has the tree property.
2. \( \mathbb{P} \ast \hat{Q} \) collapses all ordinals in the open interval \( (\kappa^+, j(\kappa)) \).
3. Whenever \( G \ast H \) is \( \mathbb{P} \ast \hat{Q} \)-generic over \( V \), then in some outer model, \( j \) can be lifted to \( j' : V[G \ast H] \rightarrow M[G' \ast H'] \), such that \( \mathcal{P}_\kappa(\text{Ord})^{V[G \ast H]} \subseteq M[G'] \).

**Proof.** Let \( j : V \rightarrow M \) be as hypothesized, and note that \( M^c \subseteq M \). Let \( \mathbb{P} \) be any \( \kappa \)-c.c. forcing of size \( \kappa \) that forces \( \kappa = \omega_2 \) and the tree property holds at \( \omega_2 \), such as Mitchell’s forcing. In \( V^\mathbb{P} \), let \( Q = \text{Col}(\kappa^+, < j(\kappa)) \). Let \( G \ast H \subseteq \mathbb{P} \ast \hat{Q} \) be generic. \( \hat{Q} \) preserves the tree property since it does not add subsets of \( \kappa \). Thus (1), (1), and (2) hold.

Since \( \mathbb{P} \) is \( \kappa \)-c.c. and \( \kappa = \text{crit}(j) \), \( \mathbb{P} \) is a regular suborder of \( j(\mathbb{P}) \), and thus a further forcing yields \( G' \subseteq j(\mathbb{P}) \) such that \( G = G' \cap V_\kappa \). Thus we can extend the embedding to \( j : V[G] \rightarrow M[G'] \). Furthermore, \( j[H] \) generates a filter \( H' \) that is \( j(\hat{Q}) \)-generic over \( M[G'] \). This is because for each dense open \( D \subseteq j(Q) \) in \( M[G'] \), there is a function \( f : \kappa \rightarrow \mathcal{P}(Q) \) in \( V[G] \) such that \( f(\alpha) \) is a dense open subset of \( Q \) for all \( \alpha < \kappa \), and \( D = j(f)(\kappa) \). If \( E = \bigcap_{\alpha<\kappa} f(\alpha) \), then \( E \) is a dense open subset of \( Q \), and \( j(E) \subseteq D \). Since \( H \) is generic, there is \( q \in E \cap H \), so \( j(q) \in D \cap H' \).
Since \( \mathbb{Q} \) is \( \kappa^+ \)-closed, \( \mathcal{P}_{\kappa^+}(\text{Ord})^{V[G'H]} = \mathcal{P}_{\kappa^+}(\text{Ord})^{V[G]} = \mathcal{P}_{\kappa^+}(\text{Ord})^{M[G]} \subseteq M[G'] \), establishing (3).

Remark 3.4. The above argument also applies to embeddings derived from short extenders.

Nonetheless, we can carry out a similar argument as for Theorem 3.2 under weaker collapsing conditions, by adding more assumptions about the forcing. The idea is that \( \Box^\mu \) will be forced whenever \( P \ast \mathcal{Q} \) forces that \( j(\kappa) \) is the successor of a cardinal \( \lambda \) satisfying \( \lambda^\kappa = \lambda \), and \( M^{j(P)} \) can see enough of this structure.

Proposition 3.5. Suppose that \( j : V \to M \) is an elementary embedding with critical point \( \kappa \) definable from parameters in \( V \), and \( \mu, \lambda \) are regular such that \( \mu < \kappa \leq \lambda < j(\kappa) \). Suppose \( P \ast \mathcal{Q} \) is such that:

1. \( P \ast \mathcal{Q} \subseteq M \).
2. \( M \models |P| < j(\kappa), P \ast \mathcal{Q} \text{ is } j(\kappa)-c.c., \text{ and } P \ast \mathcal{Q} \subseteq V_{j(\kappa)}. \)
3. \( P \ast \mathcal{Q} \) forces over \( M \) that \( \kappa = \mu^+ \) and \( j(\kappa) = \lambda^+ \).
4. \( M_{P^\mathcal{Q}} \) is \( \lambda \)-distributive.
5. Whenever \( G \ast H \) is \( P \ast \mathcal{Q} \)-generic over \( V \), then in some outer model, \( j \) can be lifted to \( j' : V[G \ast H] \to M[G' \ast H'] \), such that for all \( \alpha < j(\kappa) \), \( G \ast H_{\alpha} \in M[G'] \), where \( H_{\alpha} = H \cap V_{\alpha} \).

Then \( P \) forces \( \Box^\mu \).

Proof. Let \( G \ast H \subseteq P \ast \mathcal{Q} \) be generic, and let \( j' : V[G \ast H] \to M[G' \ast H'] \) be a lifting of \( j \) as in (5). In \( M[G'] \), we can define the set \( A \subseteq j(\kappa) \) of ordinals that have cofinality \( < \lambda \) in \( M[G] \), which is the same as the set of ordinals \( < j(\kappa) \) that have cofinality \( < \lambda \) in \( M[G \ast H] \) by \( \lambda \)-distributivity.

In \( M[G'] \), define a sequence \( \{C_\alpha : \alpha < j(\kappa)\} \) as follows. If \( \alpha \in A \), let \( C_\alpha \) be the set of all clubs in \( \alpha \) of order-type \( < \lambda \) that live in \( M[G] \). Since \( |P| < j(\kappa) \), this set has size \( < j(\kappa) \) in \( M[G'] \). If \( \alpha \in j(\kappa) \setminus A \), let \( C_\alpha = \{D\} \), where \( D \) is any club in \( \alpha \) of order-type \( \lambda \) such that all initial segments are in \( M[G] \). Such a club exists in \( M[G'] \) because there is one in \( M[G \ast H_\alpha] \) for some \( \alpha < j(\kappa) \). Conclude using Lemma 1.4 that \( \Box^\mu \) holds in \( M[G'] \) and thus in \( V[G] \) by elementarity.

We finish by giving an example to clear up a possible misconception. In the situation of Proposition 2.1 and Theorem 2.3, we derive \( \Box^\kappa \) from the fact that \( \Box^\kappa \) holds in \( V \), a generic ultrapower \( M \) can see enough information about \( V \) to know this, and the ultrapower embedding allows us to reflect this downward. But this does not characterize all of the situations we have discussed. In fact, we can force a saturated ideal on \( \omega_2 \) along with the tree property at \( \omega_3 \) using conventional methods. The weak square sequence of length \( j(\kappa) \) as constructed in Theorem 2.2 may only exist in a generic extension that we do not actually wish to take, but its virtual existence is enough to ensure the failure of the tree property at \( \kappa \) in the universe of interest.

Proposition 3.6. If there is a huge cardinal, then there is a generic extension in which there is a saturated ideal on \( \omega_2 \) and the tree property holds at \( \omega_3 \).

Proof. Suppose \( \kappa \) is huge. In particular, there is an almost-hugeness embedding \( j : V \to M \) with critical point \( \kappa \) such that \( \delta = j(\kappa) \) is weakly compact. By Magidor’s modification of Kunen’s construction (see 8), there is a countably closed forcing
If \( X \) is a huge cardinal. Then it is well-known that Mitchell’s forcing \( \mathcal{M} \) for the tree property at \( \delta \) is a projection of \( \text{Add}(\nu, \delta) \times \text{Col}(\kappa, \delta) \). Let \( Q \) be the projection of \( \mathcal{M} \times H \), which is an \( \mathcal{M} \)-generic filter over \( V[G] \). There is a \( \delta \)-c.c. forcing \( \mathbb{R} \) in \( V[G * Q] \) that produces the lifted embedding \( j'' \). In \( V[G * Q] \), we define a normal ideal \( I \) on \( \omega_2 \) by

\[
I = \{ X \subseteq \kappa : 1 \forces \kappa \notin j''(X) \}.
\]

If \( X_0, X_1 \in I^+ \), then there are \( r_0, r_1 \in \mathbb{R} \) such that \( r_i \forces \kappa \notin j''(X_i) \). If \( X_0 \cap X_1 \in I \), then \( r_0, r_1 \) are incompatible. Thus \( \mathcal{P}(\kappa)/I \) is \( \delta \)-c.c. In summary, \( V[G * Q] \) has a saturated ideal on \( \kappa = \omega_2 \) and satisfies the tree property at \( \delta = \omega_3 \).

**Remark 3.7.** Using a suitable modification of Mitchell’s forcing, we can similarly obtain a model of \( (\omega_3, \omega_2) \rightarrow (\omega_2, \omega_1) \) plus the tree property at \( \omega_3 \), starting from a huge cardinal.

**References**

[1] James E. Baumgartner, *Almost-disjoint sets, the dense set problem and the partition calculus*, Ann. Math. Logic 9 (1976), no. 4, 401–439. MR 401472

[2] James E. Baumgartner and Richard Laver, *Iterated perfect-set forcing*, Ann. Math. Logic 17 (1979), no. 3, 271–288. MR 556894

[3] Benjamin Claverie and Ralf Schindler, *Woodin’s axiom (∗), bounded forcing axioms, and precipitous ideals on \( \omega_1 \)*, J. Symbolic Logic 77 (2012), no. 2, 475–498. MR 2963017

[4] Sean D. Cox, *Chang’s conjecture and semiproperness of nonreasonable posets*, Monatsh. Math. 187 (2018), no. 4, 617–633. MR 3861321

[5] D. Donder, R. B. Jensen, and B. J. Koppelberg, *Some applications of the core model, Set theory and model theory* (Bonn, 1979), Lecture Notes in Math., vol. 872, Springer, Berlin-New York, 1981, pp. 55–97. MR 649007

[6] Hans-Dieter Donder and Peter Koepke, *On the consistency strength of “accessible” Jónsson cardinals and of the weak Chang conjecture*, Ann. Pure Appl. Logic 25 (1983), no. 3, 233–261. MR 730856

[7] Monroe Eskew and Yair Hayut, *Global Chang’s Conjecture and singular cardinals*, arXiv e-prints (2018), arXiv:1812.11768.

[8] Matthew Foreman, *Ideals and generic elementary embeddings*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 885–1147. MR 2768692

[9] Matthew Foreman and Menachem Magidor, *Large cardinals and definable counterexamples to the continuum hypothesis*, Ann. Pure Appl. Logic 76 (1995), no. 1, 47–97. MR 1359154

[10] Joel David Hamkins, *Extensions with the approximation and cover properties have no new large cardinals*, Fund. Math. 180 (2003), no. 3, 257–277. MR 2063629

[11] R. Björn Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic 4 (1972), 229–308; erratum, ibid. 4 (1972), 443, With a section by Jack Silver. MR 309729

[12] A. Kanamori and M. Magidor, *The evolution of large cardinal axioms in set theory*, Higher set theory (Proc. Conf., Math. Forschungsinstit., Oberwolfach, 1977), Lecture Notes in Math., vol. 669, Springer, Berlin, 1978, pp. 99–275. MR 520190

[13] John Krueger, *Forcing with adequate sets of models as side conditions*, MLQ Math. Log. Q. 63 (2017), no. 1-2, 124–149. MR 3647840

[14] William Mitchell, *Aronszajn trees and the independence of the transfer property*, Ann. Math. Logic 5 (1972/73), 21–46. MR 313057
[15] Itay Neeman, *Forcing with sequences of models of two types*, Notre Dame J. Form. Log. **55** (2014), no. 2, 265–298. MR 3201836

[16] Hiroshi Sakai, *Semiproper ideals*, Fund. Math. **186** (2005), no. 3, 251–267. MR 2191239

[17] Grigor Sargsyan, *A tale of hybrid mice*, ProQuest LLC, Ann Arbor, MI, 2009, Thesis (Ph.D.)–University of California, Berkeley. MR 2714009

[18] Saharon Shelah, *Proper forcing*, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin-New York, 1982. MR 675955

[19] __________, *Cardinal arithmetic*, Oxford Logic Guides, vol. 29, The Clarendon Press, Oxford University Press, New York, 1994, Oxford Science Publications. MR 1318912

[20] Stevo Todorčević, *Conjectures of Rado and Chang and cardinal arithmetic*, Finite and infinite combinatorics in sets and logic (Banff, AB, 1991), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 411, Kluwer Acad. Publ., Dordrecht, 1993, pp. 385–398. MR 1261218

[21] Víctor Torres-Pérez and Liuzhen Wu, *Strong Chang’s conjecture and the tree property at \(\omega_2\)*, Topology Appl. **196** (2015), no. part B, 999–1004. MR 3431031

[22] W. Hugh Woodin, *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, revised ed., De Gruyter Series in Logic and its Applications, vol. 1, Walter de Gruyter GmbH & Co. KG, Berlin, 2010. MR 2723878