Solutions of the (free boundary) Reifenberg Plateau problem

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Abstract

We solve two variants of the Reifenberg problem (minimizing or not the free boundary) for all coefficient groups. We carry out the direct method of the calculus of variation and search a solution as a "weak limit" of a minimizing sequences. This strategy has been introduced by De Lellis, De Philippis, De Rosa, Ghiraldin and Maggi in \cite{DLGM, DPDRG1, DLDRG3, DPDRG3} and allowed them to solve the Reifenberg problem. We use an analogous strategy proved in \cite{Lab} which has the advantage to take into account the free boundary. Moreover, we show that the Reifenberg class is closed under weak limits without restriction on the coefficient group.

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1 Definitions and main results

1.1 Introduction

We present the Reifenberg approach to the Plateau problem (\cite{Rei}, 1960). Reifenberg works with sets of the Euclidean space which span a boundary in the sense of algebraic topology and minimizes their (spherical) Hausdorff
measure. A $d$-dimensional set $E$ spans a boundary $\Gamma$ if $E$ contains $\Gamma$ and cancel the $(d - 1)$-cycles of $\Gamma$ (or a subgroup of them).

**Definition** (Reifenberg competitors). Fix $\Gamma$ a compact subset of $\mathbb{R}^n$ and let $L$ be a subgroup of the homology group $H_{d-1}(\Gamma)$. A Reifenberg competitor is a compact subset $E \subset \mathbb{R}^n$ such that $E$ contains $\Gamma$ and the morphism induced by inclusion,

$$H_{d-1}(\Gamma) \longrightarrow H_{d-1}(E),$$

is zero on $L$.

Reifenberg proved the existence of a solution to his problem in 1960 for the Čech homology theory and compact Abelian coefficient groups ([Rei]). The continuity property of the Čech theory implies that a Hausdorff limit of competitors is a competitor. Thus, a solution can be searched as a Hausdorff limit of a minimizing sequence. However, the area is not lower semicontinuous with respect to Hausdorff limits. For instance, one can imagine a minimizing sequence which has more and more dense tentacles so that the limit set is too large. Reifenberg worked with compact coefficient groups to have the Excision Axiom and thus to be able to cut out the tentacles and patch the holes. His construction leads to an alternative minimizing sequence for which the area is lower semicontinuous.

Nakauchi ([Na]) stated and solved a free boundary variant in 1984 (for compact Abelian coefficient groups as well). We mean by this that the intersection $E \cap \Gamma$ varies among the competitors $E$.

![Figure 1: Two types of Plateau problem. The competitors have a fixed boundary on the left and a free boundary on the right.](image)

**Definition** (Nakauchi competitors). Fix a compact subset $\Gamma$ of $\mathbb{R}^n$ and a subgroup $L$ of $H_{d-1}(\Gamma)$. The Nakauchi competitors are the compacts sets $E \subset \mathbb{R}^n$ such that for all $v \in L$, there exists $u \in H_{d-1}(E \cap \Gamma)$ such that $i_*(u) = v$ and $i'_*(u) = 0$ where $i_*$ and $i'_*$ are the morphisms induced by
inclusions:
\[ H_{d-1}(\Gamma) \xrightarrow{i_*} H_{d-1}(E \cap \Gamma) \xrightarrow{i'_*} H_{d-1}(E). \]

Nakauchi minimizes \( H^d(E \setminus \Gamma) \) but it would be also interesting to minimize \( H^d(E) \) to take into account the free boundary.

In 2015, Fang ([Fn]) gave a new proof of the Reifenberg and Nakauchi problems for all coefficient groups and minimizing \( H^d(E \setminus \Gamma) \). His idea is to take advantage of the lower semicontinuity of the area on quasiminimal sets (we present them in the next paragraph). Thanks to a construction of Feuvrier ([Feuv]), he obtains an alternative minimizing sequence composed of quasiminimal sets. Fang also replaces the Hausdorff measures with more general functionals, called elliptic integrands.

In [DLGM] and [DPDRG1], De Lellis, De Philippis, De Rosa, Ghiraldin and Maggi introduced a new type of direct method. As Reifenberg, they work with sets and Hausdorff measures of the Euclidean space but they replace local Hausdorff limits by weak limits of minimizing sequences in the ambient space \( \mathbb{R}^n \setminus \Gamma \). In [DPDRG2], [DPDRG3], the authors replace the Hausdorff measure by elliptic integrands and solve the problem of Reifenberg. However, they do not minimize the free boundary and they require a compact coefficient group.

In [Lab], we adapted this notion of weak limit to quasiminimizing sequences and we were able to take into account the free boundary. Thus, we can follow the strategy of De Lellis, De Philippis, De Rosa, Ghiraldin and Maggi and minimize the part of the competitor which lies on the boundary.

1.2 Reifenberg competitors

Given a topological space \( X \) and an integer \( k \), \( H_k(X; G) \) is the \( k \)th Čech homology group of \( X \) over an Abelian coefficient group \( G \). We abbreviate this notation as \( H_k(X) \) since the coefficient group is not significant for us. Throughout this section, we fix a closed set \( \Gamma \) of \( \mathbb{R}^n \) and a subgroup \( L \) of \( H_{d-1}(\Gamma) \).

**Definition 1.2.1** (Reifenberg competitor). A Reifenberg competitor is a compact subset \( E \subset \mathbb{R}^n \) such that the morphism induced by inclusion,
\[ H_{d-1}(\Gamma) \longrightarrow H_{d-1}(E \cup \Gamma), \]
is zero on \( L \).
This definition is different from the original definition of Reifenberg because the competitor $E$ may not contain $\Gamma$. In the remainder of this paragraph, we compare this definition to the definition of Nakauchi (\cite{Na}). Let $E$ be a compact subset of $\mathbb{R}^n$ and consider the following commutative diagram induced by the inclusions:

\[
\begin{array}{ccc}
H_{d-1}(E \cap \Gamma) & \xrightarrow{i_*} & H_{d-1}(E) \\
& \downarrow{j_*} & \\\nH_{d-1}(\Gamma) & \xrightarrow{i'_*} & H_{d-1}(E \cup \Gamma). \\
& \downarrow{j'_*} & \\\n\end{array}
\]

The set $E$ is a Nakauchi competitor provided that for all $v \in L$, there exists $u \in H_{d-1}(E \cap \Gamma)$ such that $i_*(u) = v$ and $i'_*(u) = 0$. Assuming that the Mayer Vietoris sequence holds for the sets $\Gamma$, $E$ in $E \cup \Gamma$, the following sequence is exact:

\[
H_{d-1}(E \cap \Gamma) \xrightarrow{(i_*,i'_*)} H_{d-1}(\Gamma) \otimes H_{d-1}(E) \xrightarrow{j_*-j'_*} H_{d-1}(E \cup \Gamma).
\]

Observe that $E$ satisfies Definition 1.2.1 if and only if all elements of the form $(v,0) \in L \otimes H_{d-1}(E)$ are in the kernel of $j_* - j'_*$. And $E$ is a Nakauchi competitor if and only if all elements of the form $(v,0) \in L \otimes H_{d-1}(E)$ are in the image of $(i_*,i'_*)$. Thus, the Mayer Vietoris sequence implies that Definition 1.2.1 is equivalent to the definition of Nakauchi. In that sense, we consider these definitions to be essentially equivalent. We favor Definition 1.2.1 because we are able to prove that it is stable under weak limits (see Lemma 2.3.1).

Our goal is to prove the two following results (we omit the regularity of the boundary here, see Theorems 3.2.1 and 3.2.2 for the full statement).

**Theorem** (Reifenberg - minimizing the free boundary). We assume that

\[
m = \inf \{ H^d(E) \mid E \text{ Reifenberg competitor} \} < \infty \tag{1}
\]

and that there exists a compact set $C \subset \mathbb{R}^n$ such that

\[
m = \inf \{ H^d(E) \mid E \text{ Reifenberg competitor}, \ E \subset C \} . \tag{2}
\]

Then there exists a Reifenberg competitor $E \subset C$ such that $H^d(E) = m$.

The next theorem is similar to Theorem 1.3 of \cite{Fm} (which is based on Feuvrier’s construction) and Theorem 3.4 of \cite{DPDRG3} (which is based on weak limits of minimizing sequences). However, we have not yet dealt with elliptic integrands.
Theorem (Reifenberg - without minimizing the free boundary). We assume that
\[ m = \inf \{ H^d(E \setminus \Gamma) \mid E \text{ Reifenberg competitor} \} < \infty \tag{3} \]
and that there exists a compact set \( C \subset \mathbb{R}^n \) such that
\[ m = \inf \{ H^d(E \setminus \Gamma) \mid E \text{ Reifenberg competitor, } E \subset C \} \tag{4} \]
Then there exists a Reifenberg competitor \( E \subset C \) such that \( H^d(E \setminus \Gamma) = m \).

Remark. If \( \Gamma \) is compact and \( H^d(\Gamma) < \infty \), this amounts to minimizing \( H^d(E) \) among Reifenberg competitors containing \( \Gamma \).

1.3 Sliding deformations

Our ambient space is an open set \( X \) of \( \mathbb{R}^n \). We fix an integer \( 1 \leq d \leq n \). The term a closed set \( S \subset X \) means that \( S \) is relatively closed in \( X \). The interval \([0, 1]\) is denoted by the capital letter \( I \). Given a set \( E \subset X \) and a function \( F: I \times E \to X \), the notation \( F_t \) means \( F(t, \cdot) \). Given two sets \( A, B \subset \mathbb{R}^n \), the notation \( A \subset\subset B \) means that there exists a compact set \( K \subset \mathbb{R}^n \) such that \( A \subset K \subset B \). For a ball \( U \) of center \( x \) and radius \( r \), for \( h \geq 0 \), the symbol \( hU \) denotes the ball of center \( x \) and radius \( hr \). We fix a closed subset \( \Gamma \) of \( X \) (the boundary).

Definition 1.3.1 (Sliding deformation along a boundary). Let \( E \) be a closed, \( H^d \) locally finite subset of \( X \). A sliding deformation of \( E \) in an open set \( U \subset X \) is a Lipschitz map \( f: E \to X \) such that there exists a continuous homotopy \( F: I \times E \to X \) satisfying the following conditions:
\[ F_0 = \text{id} \tag{5a} \]
\[ F_1 = f \tag{5b} \]
\[ \forall t \in I, \ F_t(E \cap \Gamma) \subset \Gamma \tag{5c} \]
\[ \forall t \in I, \ F_t(E \cap U) \subset U \tag{5d} \]
\[ \forall t \in I, \ F_t = \text{id} \text{ in } E \setminus K, \tag{5e} \]
where \( K \) is some compact subset of \( E \cap U \). Alternatively, the last axiom can be stated as
\[ \{ x \in E \mid \exists t \in I, \ F_t(x) \neq x \} \subset\subset E \cap U. \tag{6} \]

2 Operations on the competitors

We present three operations that preserve the Reifenberg competitors: supsets, continuous image by sliding deformations and weak limits.
2.1 Supset

Lemma 2.1.1. Let $E$ be a Reifenberg competitor. Let $F$ be a compact subset of $\mathbb{R}^n$ containing $E$. Then $F$ is a Reifenberg competitor.

Proof. This follows from the following commutative diagram

$$
\begin{array}{ccc}
H_{d-1}(\Gamma) & \longrightarrow & H_{d-1}(E \cup \Gamma) \\
\downarrow f_* & & \downarrow f_* \\
H_{d-1}(F \cup \Gamma) & \longrightarrow & H_{d-1}(f(E) \cup \Gamma)
\end{array}
$$

where the arrows are the morphisms induced by inclusion. \qed

2.2 Continuous image

Lemma 2.2.1. Let $E$ be a Reifenberg competitor. Let $f: E \cup \Gamma \to \mathbb{R}^n$ be a continuous map such that there exists a continuous map $F: I \times \Gamma \to \Gamma$ satisfying $F_0 = \text{id}$ and $F_1 = f$. Then $f(E)$ is a Reifenberg competitor.

Proof. Consider the following commutative diagram

$$
\begin{array}{ccc}
H_{d-1}(\Gamma) & \longrightarrow & H_{d-1}(E \cup \Gamma) \\
\downarrow f_* & & \downarrow f_* \\
H_{d-1}(\Gamma) & \longrightarrow & H_{d-1}(f(E) \cup \Gamma)
\end{array}
$$

where the unlabeled arrows are the morphisms induced by inclusion. As $f: \Gamma \to \Gamma$ is homotopic to $\text{id}$, $f_* = \text{id}$ on $H_{d-1}(\Gamma)$. \qed

The lemma assumed $f$ to be defined on $E \cup \Gamma$ but the image $f(E)$ depends only on the values of $f$ on $E$. In the two following remarks, we are going to see that it is generally enough for $f$ to be defined on $E$. In particular, the second remark applies to sliding deformations.

Remark 2.2.1. Let $f: E \to \mathbb{R}^n$ be a continuous map such that $f = \text{id}$ on $E \cap \Gamma$. As $E$ and $\Gamma$ are closed sets of $\mathbb{R}^n$, the gluing

$$
g = \begin{cases} 
 f & \text{in } E \\
 \text{id} & \text{in } \Gamma
\end{cases}
$$

(7)

is continuous. Then $G_t = (1-t)\text{id} + tg$ is a continuous homotopy from $\text{id}$ to $g$ and $G_t = \text{id}$ on $\Gamma$. We deduce that $f(E)$ is a Reifenberg competitor.

Remark 2.2.2. Let $f: E \to \mathbb{R}^n$ be a continuous map such that there exists a continuous map $F: I \times (E \cap \Gamma) \to \Gamma$ satisfying $F_0 = \text{id}$ and $F_1 = f$. Let us assume that $\Gamma$ is a neighborhood retract i.e. there exists an open set $O \subset \mathbb{R}^n$ and a continuous map $r: O \to \Gamma$ such that $r = \text{id}$ on $\Gamma$. According to the
Homotopy Extension Lemma, $F$ extends as a continuous map $F: I \times \Gamma \to \Gamma$. Moreover, the gluing

$$g = \begin{cases} f & \text{in } E \\ F_1 & \text{in } \Gamma \end{cases}$$

is continuous because $E$ and $\Gamma$ are closed sets of $\mathbb{R}^n$. We deduce that $f(E)$ is a Reifenberg competitor.

### 2.3 Weak limit

We finally present our lemma about weak limits of Reifenberg competitors.

**Lemma 2.3.1.** Let $(E_k) \subset \mathbb{R}^n$ be a sequence of Reifenberg competitors. Let $E$ be a compact subset of $\mathbb{R}^n$. We assume that

1. there exists a compact set $C \subset \mathbb{R}^n$ such that for all $k$, $E_k \subset C$;
2. for all open sets $V$ containing $E \cup \Gamma$,

$$\lim_{k} H^d(E_k \setminus V) = 0.$$ (9)

Then $E$ is a Reifenberg competitor.

The proof requires a preliminary lemma about the general position of spheres. For $x \in \mathbb{R}^n$ and $r > 0$, let $S(x, r)$ denote the Euclidean sphere of center $x$ and radius $r$ of $\mathbb{R}^n$. Given an integer $k$, a $k$-sphere is an Euclidean sphere of positive radius relative to a $(k + 1)$-affine plane. We extend this definition to $k < 0$, by calling $k$-sphere the empty set.

**Lemma 2.3.2.** Let $S^k$ be a $k$-sphere in $\mathbb{R}^n$ and let $x$ be a point in $\mathbb{R}^n$. Then for all $r > 0$ (except for at most one value), $S^k \cap S(x, r)$ is a subset of a $(k - 1)$-sphere.

**Proof.** We assume $k \geq 1$. The proof is based on the observation that the intersection of a sphere with a $k$-affine plane is either empty, a point, or a $(k - 1)$-sphere. In all cases, this intersection is part of a $(k - 1)$-sphere. Let $P_0$ be the $(k+1)$-affine plane associated to $S^k$, let $x_0 \in P_0$ be the center of $S^k$ and $r_0 > 0$ be its radius. For $r > 0$, a point $y \in S^k \cap S(x, r)$ is characterized by the system

$$y \in P_0$$

$$|y - x| = r$$

$$|y - x_0| = r_0$$

or equivalently

$$y \in P_0$$

$$|y - x| = r$$

$$|y - x|^2 - |y - x_0|^2 = r^2 - r_0^2.$$ (11c)
Assume \( x = x_0 \). If \( r \neq r_0 \) (this removes one value of \( r \)), equation (11c) has no solutions. Then, \( S^k \cap S(x, r) \) is empty and it is part of a \((k - 1)\)-sphere. Assume \( x \neq x_0 \). Equation (11c) defines a hyperplane and, if \( |x - x_0|^2 \neq r^2 - r_0^2 \) (this removes at most one value of \( r \)), this hyperplane does not contain \( x_0 \). Then, the intersection of the two planes (11a) and (11c) is included in a \( k \)-affine plane. The intersection of this plane with the sphere (11b) is part of a \((k - 1)\)-sphere as seen in introduction.

The following proof makes use of the notion of complex (Subsection 3.1 of [Lab]) and of the Federer-Fleming projection (Proposition 3.1 of [DS] or Proposition 3.3.1 of [Lab]).

Proof of Lemma 2.3.1. Observe that the sequence \((E_k \cup E)_k\) also satisfies the Lemma assumptions. So without loss of generality, we assume that for all \( k \), \( E \subset (E_k \cup \Gamma) \). We define a general covering as an open family \( \gamma = (\gamma_j)_{j \in V_\gamma} \) of \( \mathbb{R}^n \) satisfying the following properties:

1. there exists \( k \) such that \( E_k \cup \Gamma \subset \bigcup_{j \in V_\gamma} \gamma_j \);
2. for every subset \( S \subset V_\gamma \) of cardinal \( d + 1 \),
   \[ \bigcap_S \gamma_j \neq \emptyset \implies (E \cup \Gamma) \cap \bigcap_S \gamma_j \neq \emptyset. \] (12)

The main goal of the proof is to show that for any open covering \( \alpha = (\alpha_i)_i \) of \( E \cup \Gamma \), there exists a general covering \( \gamma = (\gamma_j)_{j \in V_\gamma} \) such that \(((E \cup \Gamma) \cap \gamma_j)_j\) is a refinement of \( \alpha \). Let us explain how to conclude from there. A general covering \( \gamma \) induces simplicial complexes:

\[
K(\Gamma) = \{ S \subset V_\gamma \text{ finite} \mid \Gamma \cap \bigcap_S \gamma_j \neq \emptyset \},
\]
(13)

\[
K(E \cup \Gamma) = \{ S \subset V_\gamma \text{ finite} \mid (E \cup \Gamma) \cap \bigcap_S \gamma_j \neq \emptyset \},
\]
(14)

\[
K(E_k \cup \Gamma) = \{ S \subset V_\gamma \text{ finite} \mid (E_k \cup \Gamma) \cap \bigcap_S \gamma_j \neq \emptyset \}.
\]
(15)

The inclusions \( K(\Gamma) \subset K(E \cup \Gamma) \subset K(E_k \cup \Gamma) \) induce morphisms \( i_* \) and \( j_* \):

\[
H_{d-1}(K(\Gamma)) \xrightarrow{i} H_{d-1}(K(E \cup \Gamma)) \xrightarrow{j} H_{d-1}(K(E_k \cup \Gamma)).
\]

As \( E_k \) is a Reifenberg set, we have \( j_* \circ i_* = 0 \) on \( L \). However, the second axiom of general coverings says that the simplicial complexes \( K(E \cup \Gamma) \) and \( K(E_k \cup \Gamma) \) have the same \( d \)-simplexes. Hence the \( d \)-chains of \( K(E \cup \Gamma) \) and \( K(E_k \cup \Gamma) \) are identical and they induce the same boundaries. We deduce that \( j_* \) is injective and then, \( i_* = 0 \) on \( L \). Since every open covering \( \alpha \) of \( E \cup \Gamma \) is refined by such general covering \( \gamma \), we conclude that the morphism
induced by inclusion $H_{d-1}(\Gamma) \to H_{d-1}(E \cup \Gamma)$ is nul on $L$. Step 1. We fix a relative open covering $\alpha = (\alpha_i)_i$ of $E \cup \Gamma$ and we build a locally finite open sequence $\beta = (\beta_j)_{j \in \mathbb{N}}$ in $\mathbb{R}^n$ such that

1. $\beta$ cover $E \cup \Gamma$ and $((E \cup \Gamma) \cap \beta_j)_j$ is a refinement of $\alpha$;

2. for every finite subset $S \subset \mathbb{N}$, the intersection of boundaries $\bigcap S \partial \beta_i$ is included in a finite union of $(n-m)$-spheres, where $m$ is the cardinal of $S$;

3. for every finite subset $S \subset \mathbb{N}$,

$$\bigcap_S \beta_j \neq \emptyset \implies (E \cup \Gamma) \cap \bigcap_S \beta_j \neq \emptyset. \quad (16)$$

We work with the closed set $F := E \cup \Gamma$. For all $x \in F$, there exists $i$ such that $x \in \alpha_i$ so there exists an open ball $B$ centred at $x$ such that $F \cap 2B \subset \alpha_i$. We extract a sequence of open ball $(B_j)_{j \in \mathbb{N}}$ covering $F$ such that $(2B_j)_j$ is locally finite in $\mathbb{R}^n$ and $(F \cap 2B_j)_j$ is a refinement of $\alpha$. Next, we build by induction an open sequence $(\beta_j)_{j \in \mathbb{N}}$ such that for all $j$,

1. $F \cap \overline{B_j} \subset \beta_j$ and there exists $i$ such that $F \cap \beta_j \subset \alpha_i$.

2. for every subset $S \subset \{1, \ldots, j\}$, the intersection of boundaries $\bigcap S \partial \beta_i$ is included in a finite union of $(n-m)$-spheres, where $m$ is the cardinal of $S$;

3. for every subset $S \subset \{1, \ldots, j\}$,

$$\bigcap_S \beta_i \neq \emptyset \implies F \cap \bigcap_S \beta_i \neq \emptyset. \quad (17)$$

Assume that $\beta_0, \ldots, \beta_{j-1}$ has been built and let us built $\beta_j$. For all $x \in F \cap \overline{B_j}$, there exists an open ball $B$ centered at $x$ such that

1. $B \subset 2B_j$;

2. for all finite subset $S \subset \{1, \ldots, j-1\}$, the intersection of boundaries $\partial B \cap \bigcap S \partial \beta_i$ is included in a finite union of $(n-m-1)$-spheres, where $m$ is the cardinal of $S$;

3. for all finite subset $S \subset \{1, \ldots, j-1\}$,

$$(F \cap \overline{B_j}) \subset \mathbb{R}^n \setminus \bigcap_S \overline{\beta_j} \implies \overline{B} \subset \mathbb{R}^n \setminus \bigcap_S \overline{\beta_j} \quad (18)$$

or, equivalently

$$\overline{B} \cap \bigcap_S \overline{\beta_j} \neq \emptyset \implies (F \cap \overline{B_j}) \cap \bigcap_S \overline{\beta_j} \neq \emptyset. \quad (19)$$
Extract a finite covering of \( F \cap B_j \) by such balls \( B \) and denote \( \beta_j \) their union. Then \( \beta_j \) solves the next step of the induction.

**Step 2.** We complete the family \( \beta \) with an open set \( \beta_\infty \) to obtain a covering of one of the \( E_k \). We take care not to introduce new \( d \)-simplexes on \( E \cup \Gamma \).

We want to reduce the problem to the case where for some \( k \), \( E_k \setminus \bigcup_j \beta_j \) is a \((d-1)\)-dimensional grid. Using a Federer-Fleming projection, we are going to project \( E_k \) in a \((d-1)\)-dimensional grid away from \( E \cup \Gamma \). Let \( \ell > 0 \) and consider a complex \( K \) describing a uniform grid of sidelength \( \ell \) in \( \mathbb{R}^n \). In particular, \( \mathbb{R}^n = |K| = U(K) \) and the cells of \( K \) have a diameter \( \leq \sqrt{n} \ell \). We select the cells in which we want to perform the Federer-Fleming projection. Let \( B_0 \) be an open ball such that for all \( k \), \( E_k \subset \overline{B_0} \). Let \( L \) be the subcomplex of \( K \) defined by

\[
L = \{ A \in K \mid \exists x \in A, x \in \overline{2B_0} \text{ and } d(x, E \cup \Gamma) \geq 2\sqrt{n} \ell \}. \tag{20}
\]

Consider \( x \in \overline{2B_0} \) such that \( d(x, E \cup \Gamma) \geq 2\sqrt{n} \ell \). As \( \mathbb{R}^n = U(K) \), there exists a cell \( A \in K \) such that \( x \in \text{int}(A) \) and, in particular, \( A \in L \). We deduce that

\[
\{ x \in 2B_0 \mid d(x, E \cup \Gamma) \geq 2\sqrt{n} \ell \} \subset U(L) \tag{21}
\]

As \( E \cup \Gamma \) is a closed set included in \( \bigcup_j \beta_j \), the function \( x \mapsto d(x, E \cup \Gamma) \) is positive on \( \mathbb{R}^n \setminus \bigcup_j \beta_j \). Moreover, \( 2B_0 \setminus \bigcup_j \beta_j \) is compact so the function \( x \mapsto d(x, E \cup \Gamma) \) has a positive minimum on \( 2B_0 \setminus \bigcup_j \beta_j \). This minimum does not depend on \( \ell \) so we can assume \( \ell \) small enough so that for all \( x \in 2B_0 \setminus \bigcup_j \beta_j \), \( d(x, E \cup \Gamma) > 4\sqrt{n} \ell \). By contraposition,

\[
\{ x \in 2B_0 \mid d(x, E \cup \Gamma) \leq 4\sqrt{n} \ell \} \subset \bigcup_j \beta_j. \tag{22}
\]

Next, we introduce the Federer-Fleming projection of \( E_k \cap |L| \) in \( L \). First, we justify that \( H^d(E_k \cap |L|) < \infty \). In fact, we are going to have much better. By local finitness of \( K \), \( |L| \) is a closed subset of \( \mathbb{R}^n \). Since the cells of \( K \) have a diameter \( \leq \sqrt{n} \ell \), the definition of \( L \) implies that the cells of \( L \) cannot meet \( E \cup \Gamma \). Thus, the set \( V = \mathbb{R}^n \setminus |L| \) is open and contains \( E \cup \Gamma \). According to the Lemma assumptions,

\[
\lim_k H^d(E_k \cap |L|) = 0. \tag{23}
\]

Now, we apply Proposition 3.3.1 of [Lab] and we obtain a continuous map \( \phi : |L| \to |L| \) such that

1. for all \( A \in L \), \( \phi(A) \subset A \);
2. \( \phi(E_k \cap |L|) \subset |L| \setminus \bigcup \{ \text{int}(A) \mid A \in L, \dim(A) > d \} \);
3. for all \( A \in L^d \),

\[
H^d(\phi(E_k \cap |L|) \cap A) \leq CH^d(E_k \cap |L|) \tag{24}
\]
where \( C \) is a positive constant that depends only on \( n \). When \( k \) is big enough (depending on \( \ell \)), \( H^d(E_k \cap |L|) \) becomes sufficiently small so that one can perform additional projections in the \( d \)-dimensional cells of \( L \). Thus, the second axiom becomes

\[
\phi(E_k \cap |L|) \subset |L| \setminus \bigcup \{ \text{int}(A) \mid A \in L, \ dim(A) \geq d \}; \quad (25)
\]

In particular,

\[
\phi(E_k \cap |L|) \cap U(L) \subset \bigcup \{ \text{int}(A) \mid A \in L, \ dim(A) \leq d - 1 \}. \quad (26)
\]

The sets \( E \cup \Gamma \) and \( |L| \) are disjoint and closed so we can extend \( \phi \) continuously on \( E \cup \Gamma \) by \( \phi = \text{id} \). Observe that \( |\phi - \text{id}| \leq \sqrt{n\ell} \) because \( \phi \) preserves the cells of \( L \). We can extend \( \phi \) continuously on \( \mathbb{R}^n \) in such that way that \( |\phi - \text{id}| \leq \sqrt{n\ell} \). Let us show that

\[
\phi(E_k) \subset |L^{d-1}| \cup \bigcup_{j \in \mathbb{N}} \beta_j. \quad (27)
\]

Remember that \( E_k \subset \overline{B_0} \). We assume \( \ell \) small enough so that \( \sqrt{n\ell} \leq 1 \) whence \( \phi(E_k) \subset 2\overline{B_0} \). For \( x \in E_k \), we distinguish two cases. If \( d(x, E \cup \Gamma) \leq 3\sqrt{n\ell} \), then \( d(\phi(x), E \cup \Gamma) \leq 4\sqrt{n\ell} \) so \( \phi(x) \in \bigcup_j \beta_j \) by (22). If \( d(x, E \cup \Gamma) \geq 3\sqrt{n\ell} \), then we have both \( d(x, E \cup \Gamma) \geq 2\sqrt{n\ell} \) and \( d(\phi(x), E \cup \Gamma) \geq 2\sqrt{n\ell} \) so (22) shows that \( x \in U(L) \) and \( \phi(x) \in U(L) \). By (26), we have thus \( \phi(x) \in \bigcup \beta_j \).

Now, we are all set to introduce

\[
\beta_\infty = \mathbb{R}^n \setminus (E \cup \Gamma \cup \bigcup_{|S|=d} \bigcap_{S} \beta_j). \quad (28)
\]

First, we justify that \( \beta_\infty \) is open. It suffices to show that the family

\[
\left( \bigcap_{S \cup \beta_j} \right)_{|S|=d} \quad (29)
\]

is locally finite in \( \mathbb{R}^n \). In step 1, we have built the family \(( \beta_j )_{j \in \mathbb{N}}\) such that it is locally finite: for all \( x \in \mathbb{R}^n \), there exists an open set \( U \) containing \( x \) such that the set

\[
S_0 = \{ j \in \mathbb{N} \mid U \cap \beta_j \neq \emptyset \} \quad (30)
\]

is finite. Let \( S \) be a subset of \( \mathbb{N} \) with cardinal \( d \) such that \( U \) meets \( \bigcap_S \beta_j \). Then for all \( j \in S \), \( U \cap \beta_j \neq \emptyset \) and thus \( U \cap \beta_j \neq \emptyset \) because \( U \) is open. This means that \( S \subset S_0 \). We deduce that there exists only a finite number of subsets \( S \subset \mathbb{N} \) of cardinal \( d \) such that \( U \) meets \( \bigcap_S \beta_j \neq \emptyset \). We conclude that \( \beta_\infty \) is open. Observe that \( \beta_\infty \) is disjoint from \( E \cup \Gamma \) and that for all \( S \subset \mathbb{N} \) of cardinal \( d \),

\[
\beta_\infty \cap \bigcap_S \beta_j = \emptyset. \quad (31)
\]
In other words, for all $S \subset \mathbb{N}$ of cardinal $d + 1$, the condition $\bigcap_S \beta_j \neq \emptyset$ implies $S \subset \mathbb{N}$. This means that the family $(\beta_j)_{j \in \mathbb{N} \cup \{\infty\}}$ does not induce additional $d$-simplexes. Finally, we would like
\[ \phi(E_k) \subset \beta_\infty \cup \bigcup_{j \in \mathbb{N}} \beta_j. \] (32)
This is where $(d - 1)$-dimensional grid helps us a lot. According to (27), the condition (32) holds if $|L^{d-1}| \setminus \beta_\infty \subset \bigcup_j \beta_j$, that is, for all $S \subset \mathbb{N}$ of cardinal $d$,
\[ |L^{d-1}| \cap \bigcap_S \beta_j \subset \bigcup_j \beta_j. \] (33)

We are going to see that a suitable translation of $K$ allows to assume that $|K^{d-1}|$ is disjoint from the intersection of boundaries $\bigcap_S \partial \beta_j$. Fix $S \subset \mathbb{N}$ of cardinal $d$. As $\bigcap_S \partial \beta_j$ is included in a finite union of $(n - d)$-spheres, we deduce that for all $(d - 1)$-linear plane $P$,
\[ H^n\left( \bigcap_S \partial \beta_j + P \right) = 0. \] (34)
In particular,
\[ H^n\left( \bigcap_S \partial \beta_j + (-|K^{d-1}|) \right) = 0. \] (35)
This means that for almost every $x \in \mathbb{R}^n$, $x + |K^{d-1}|$ is disjoint from $\bigcap_S \partial \beta_j$. There are only a countable number of subsets $S \subset \mathbb{N}$ of cardinal $d$ so we can find $x$ such that this is true for all of them. To simplify the notation, we assume that $x = 0$ and that (33) holds.

We are about to finish the proof. We define the domain $V_\gamma = \mathbb{N} \cup \{\infty\}$ and for $j \in V_\gamma$, we define the open set $\gamma_j = \phi^{-1}(\beta_j)$. Remember that $\phi = \text{id}$ on $E \cup \Gamma$ so for all $j \in \mathbb{N} \cup \{\infty\}$,
\[ (E \cup \Gamma) \cap \gamma_j = (E \cup \Gamma) \cap \beta_j. \] (36)
The family $\gamma$ covers $E_k \cup \Gamma$ because $(\beta_j)_{j \in \mathbb{N} \cup \{\infty\}}$ covers $E \cup \Gamma$ and $\phi(E_k)$. The family $((E \cup \Gamma) \cap \gamma_j)_{j \in V_\gamma}$ is a refinement of $\alpha$ because $(E \cup \Gamma) \cap \beta_\infty = \emptyset$ and because for all $j \in \mathbb{N}$, $\gamma_j$ coincides with $\beta_j$ on $E \cup \Gamma$. Finally, for all $S \subset V_\gamma$ of cardinal $d + 1$, the condition
\[ \bigcap_S \gamma_j \neq \emptyset \] (37)
implies $\bigcap_S \beta_j \neq \emptyset$ and then by (31), $S \subset \mathbb{N}$. By construction of $(\beta_j)_{j \in \mathbb{N}}$, we have $(E \cup \Gamma) \cap \bigcap_S \beta_j \neq \emptyset$ or equivalently,
\[ (E \cup \Gamma) \cap \bigcap_S \gamma_j \neq \emptyset \] (38)
since $\gamma_j$ coincides with $\beta_j$ on $E \cup \Gamma$. \qed
3 Existence of Plateau solutions

We solve two formulations of the Reifenberg Plateau problem. In the first one, we work in $X = \mathbb{R}^n$ and minimize $H^d(E)$ among Reifenberg competitors $E$. In the second one, we work in $X = \mathbb{R}^n \setminus \Gamma$ (that is, away from the boundary) and minimize $H^d(E \setminus \Gamma)$ among Reifenberg competitors $E$. In this second case, we do not require regularity on the boundary.

3.1 Direct method

We recall Corollary 3.4.1 of [Lab]. This is the same direct method as [DLGM], [DPDRG1] but this version allows to minimize the competitors on the boundary. Our working space is an open set $X$ of $\mathbb{R}^n$. We need a few assumptions on the boundary; see Definition 2.2.2 ($H^d$ regular sets) and 3.1.9 (Lipschitz subset in [Lab]).

**Corollary 3.1.1 (Direct method).** Fix a Lipschitz subset $\Gamma$ of $X$ which is $H^d$ regular. Let $C$ be a class of closed subsets of $X$ such that

$$m = \inf \{ H^d(E) \mid E \in C \} < \infty$$

and assume that for all $E \in C$, for all sliding deformations $f$ of $E$ in $X$,

$$m \leq H^d(f(E)).$$

Let $(E_k)$ be a minimizing sequence for $H^d$ in $C$. Up to a subsequence, there exists a coral minimal set $E_\infty$ in $X$ such that

$$H^d \llcorner E_k \rightharpoonup H^d \llcorner E_\infty.$$  

where the arrow $\rightharpoonup$ denotes the weak convergence of Radon measures in $X$. In particular, $H^d(E_\infty) \leq m$.

**Remark 3.1.1.** In the works of Reifenberg, the Hausdorff limit of a minimizing sequence is a competitor but the area is not lower semicontinuous. Reifenberg worked with a compact coefficient group to build an alternative minimizing sequence. With weak limits, the lower semicontinuity follows from the previous proposition. Moreover, we proved in the last section that the limit is a competitor. Thus, there not much work to do.

3.2 Applications

**Theorem 3.2.1 (Reifenberg - minimizing the free boundary).** Fix a Lipschitz subset $\Gamma$ of $\mathbb{R}^n$ which is $H^d$ regular and fix a subgroup $L$ of $H_{d-1}(\Gamma)$. We assume that

$$m = \inf \{ H^d(E) \mid E \text{ Reifenberg competitor} \} < \infty$$

A set $E \subset X$ is coral in $X$ if $E$ is the support of $H^d \llcorner E$ in $X$. Equivalently, $E$ is closed in $X$ and for all $x \in E$ and for all $\tau > 0$, $H^d(E \cap B(x, \tau)) > 0$. 

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and that there exists a compact set $C \subset \mathbb{R}^n$ such that

$$m = \inf \{ H^d(E) \mid E \text{ Reifenberg competitor, } E \subset C \}. \quad (43)$$

Then there exists a Reifenberg competitor $E \subset C$ such that $H^d(E) = m$.

**Proof.** We work in $X = \mathbb{R}^n$ and we consider the class

$$C = \{ E \mid E \text{ is a Reifenberg competitor} \}. \quad (44)$$

By Lemma 2.2.1, the class $C$ is preserved by sliding deformations in $\mathbb{R}^n$ so it satisfies the requirement of Corollary 3.1.1. Let $(E_k)$ be a minimizing sequence of $C$ such that for all $k$, $E_k \subset C$. According to Corollary 3.1.1, there exists a coral set $E_\infty$ of $\mathbb{R}^n$ such that

$$H^d(E_k) \rightharpoonup H^d(E_\infty). \quad (45)$$

We prove that $E_\infty$ is a Reifenberg competitor. First, we show that $E_\infty$ is a compact subset of $C$. Observe that $\mathbb{R}^n \setminus C$ is an open set and that by lower semicontinuity,

$$H^d(E_\infty \setminus C) \leq \liminf_k H^d(E_k \setminus C) = 0. \quad (46)$$

This proves that the support of $H^d \downarrow E_\infty$ is included in $C$. As $E$ is coral, $E$ is a subset of $C$ and therefore compact. Next, we apply Lemma 2.3.1 to the set $E_\infty$. For all open set $V$ containing $E_\infty \cup \Gamma$,

$$\limsup_k H^d(E_k \setminus V) = \limsup_k H^d(E_k \cap C \setminus V) \leq H^d(E_\infty \cap C \setminus V) \leq 0. \quad (47)$$

We conclude that $E_\infty$ is a Reifenberg competitor. Finally, we show that $H^d(E_\infty) = m$. As $E_\infty$ is a Reifenberg competitor, we have of course $H^d(E_\infty) \geq m$. The fact that $H^d(E_\infty) \leq m$ was already observed in Corollary 3.1.1.

The next theorem is similar to Theorem 1.3 of [Fn] (which is based on Feuvrier’s construction) and Theorem 3.4 of [DPDRG3] (which is based on weak limits of minimizing sequences). However, we have not dealt with elliptic integrands yet.

**Theorem 3.2.2** (Reifenberg - without minimizing the free boundary). Fix a closed set $\Gamma$ of $\mathbb{R}^n$ and a subgroup $L$ of $H_{d-1}(\Gamma)$. We assume that

$$m = \inf \{ H^d(E \setminus \Gamma) \mid E \text{ Reifenberg competitor} \} < \infty \quad (50)$$

and that there exists a compact set $C \subset \mathbb{R}^n$ such that

$$m = \inf \{ H^d(E \setminus \Gamma) \mid E \text{ Reifenberg competitor, } E \subset C \}. \quad (51)$$

Then there exists a Reifenberg competitor $E \subset C$ such that $H^d(E \setminus \Gamma) = m$.  

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Remark 3.2.1. If \( \Gamma \) is compact and \( H^d(\Gamma) < \infty \), this amounts to minimizing \( H^d(E) \) among Reifenberg competitors containing \( \Gamma \).

Proof. We work in \( X = \mathbb{R}^n \setminus \Gamma \) (away from the boundary) and we consider the class
\[ C = \{ E \setminus \Gamma \mid E \text{ is a Reifenberg competitor} \}. \tag{52} \]
By Lemma 2.2.1, the class \( C \) is preserved by sliding deformations in \( X \) so it satisfies the requirement of Corollary 3.1.1. Let \( (E_k) \) be a sequence of Reifenberg competitor such that \( (E_k \setminus \Gamma) \) is a minimizing sequence of \( C \) and for all \( k \), \( E_k \subset C \). According to Corollary 3.1.1, there exists a coral set \( S_\infty \) of \( X \) such that
\[ H^d( (E_k \setminus \Gamma) \rightarrow H^d S_\infty \text{ in } X. \tag{53} \]
We prove that there exists a Reifenberg competitor \( E_\infty \subset C \) such that \( S_\infty = E_\infty \setminus \Gamma \). First, we justify that \( S_\infty \subset C \). Observe that \( X \setminus C \) is an open set of \( X \) and that by lower semicontinuity,
\[ H^d( S_\infty \setminus C) \leq \liminf_k H^d ( (E_k \setminus \Gamma) \setminus C) = 0. \tag{54} \]
As a consequence, the support of \( H^d S_\infty \) in \( X \) is included in \( C \). As \( S_\infty \) is coral in \( X \), \( S_\infty \) is a subset of \( C \). Now, let
\[ E_\infty = (S_\infty \cup \Gamma) \cap C \tag{55} \]
\[ = S_\infty \cup (\Gamma \cap C). \tag{56} \]
The set \( S_\infty \) is closed in \( X \) so \( S_\infty \cup \Gamma \) is closed in \( \mathbb{R}^n \) and \( E_\infty \) is compact. We apply Lemma 2.3.1 to the set \( E_\infty \). For all open set \( V \) containing \( E_\infty \cup \Gamma \), \( C \setminus V \) is a compact subset of \( X \) so
\[ \limsup_k H^d(E_k \setminus V) = \limsup_k H^d(E_k \cap C \setminus V) \tag{57} \]
\[ = \limsup_k H^d((E_k \setminus \Gamma) \cap C \setminus V) \tag{58} \]
\[ \leq H^d(E_\infty \cap C \setminus V) \tag{59} \]
\[ \leq 0. \tag{60} \]
In conclusion, \( E_\infty \) is a Reifenberg competitor and \( S_\infty = E_\infty \setminus \Gamma \in C \). Finally, we show that \( H^d(S_\infty) = m \). As \( S_\infty \in C \), we have of course \( H^d(S_\infty) \geq m \).

The fact that \( H^d(S) \leq m \) has already been observed in Corollary 3.1.1. \( \square \)

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