GENERA OF TWO-BRIDGE KNOTS AND EPIMORPHISMS OF THEIR KNOT GROUPS

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ABSTRACT. Let $K, K'$ be two-bridge knots of genus $n, k$ respectively. We show the necessary and sufficient condition of $n$ in terms of $k$ that there exists an epimorphism from the knot group of $K$ onto that of $K'$.

1. INTRODUCTION

Let $K$ be a knot in $S^3$ and $G(K)$ the knot group, that is, the fundamental group of the complement of $K$ in $S^3$. We denote by $g(K)$ the genus of $K$. Recently, many papers have investigated epimorphisms between knot groups. In particular, Simon’s conjecture in [8], which states that every knot group maps onto at most finitely many knot groups, was settled affirmatively in [2]. In the same Kirby’s problem list [5], Simon also proposed another conjecture. Namely, if there exists an epimorphism from $G(K)$ onto $G(K')$, then is $g(K)$ greater than or equal to $g(K')$? This problem is also mentioned in [9]. It is known that if there exists an epimorphism from $G(K)$ onto $G(K')$, then the Alexander polynomial of $K$ is divisible by that of $K'$. Moreover, Crowell [7] showed that the genus of an alternating knot is equal to a half of the degree of the Alexander polynomial. Then the above conjecture is true for alternating knots, especially two-bridge knots.

In this paper, we give a more explicit condition on genera of two-bridge knots $K$ and $K'$ such that there exists an epimorphism between their knot groups. As a corollary, we show that if there exists an epimorphism from $G(K)$ onto $G(K')$, then $g(K) \geq 3g(K') - 1$.

A knot is called minimal if its knot group admits epimorphisms onto the knot groups of only the trivial knot and itself. Many types of minimal knots are already shown in [10], [17], [4], [11], [13], [15], and [14]. By using the main theorem of this paper, we obtain several types of minimal knots. For example, a two-bridge knot of genus 2 is minimal if and only if it is not the two-bridge knot $C[2a, 4b, 4a, 2b]$ in Conway’s notation for any non-zero integers $a, b$.

2. OHTSUKI-RILEY-SAKUMA CONSTRUCTION

In this section, we review some known facts about two-bridge knots, see [5] and [12] for example. Especially, we recall Ohtsuki-Riley-Sakuma construction of epimorphisms between two-bridge knot groups.

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It is known that a two-bridge knot corresponds to a rational number and that it can be expressed as a continued fraction

\[ [a_1, a_2, \ldots, a_{m-1}, a_m] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{m-1} + a_m}}} }, \]

where \( a_1 > 0 \). We define the length of the continued fraction to be

\[ \ell([a_1, a_2, \ldots, a_{m-1}, a_m]) = m. \]

Note that the length depends on the choice of continued fractions. For example, we can delete zeros in a continued fraction by using

\[ [a_1, a_2, \ldots, a_{i-2}, a_{i-1}, 0, a_{i+1}, a_{i+2}, \ldots, a_m] = [a_1, a_2, \ldots, a_{i-2}, a_{i-1} + a_{i+1}, a_{i+2}, \ldots, a_m]. \]

Then, we can reduce the length by 2, if the continued fraction contains a 0.

**Theorem 2.1** (Ohtsuki-Riley-Sakuma [16], Agol [1], Aimi-Lee-Sakuma [3]). Let \( K(r), K(\hat{r}) \) be 2-bridge knots, where \( r = [a_1, a_2, \ldots, a_m] \). There exists an epimorphism \( \varphi : G(K(\hat{r})) \to G(K(r)) \) if and only if \( \hat{r} \) can be written as

\[ \hat{r} = [\varepsilon_1 a, 2c_1, \varepsilon_2 a^{-1}, 2c_2, \varepsilon_3 a, 2c_3, \varepsilon_4 a^{-1}, 2c_4, \ldots, \varepsilon_{2n} a^{-1}, 2c_{2n}, \varepsilon_{2n+1} a], \]

where \( a = (a_1, a_2, \ldots, a_m), a^{-1} = (a_m, a_{m-1}, \ldots, a_1), \varepsilon_i = \pm 1 (\varepsilon_1 = 1), \) and \( c_i \in \mathbb{Z} \).

Remark that we can exclude the case where \( c_i = 0 \) and \( \varepsilon_i \cdot \varepsilon_{i+1} = -1 \) without loss of generality (see [17] for details).

A continued fraction \([a_1, a_2, \ldots, a_m] \) is called even if all \( a_i \)'s are even integers. Moreover, it is called reduced if all \( a_i \)'s are non-zero.

### 3. Main Theorem

First, we define a set \( S_k \) as follows:

\[ S_k = \mathbb{N} \cap \left( \bigcup_{r=1}^{k-2} [(2r+1)k + r + 1, (2r+3)k - r - 2] \right). \]

For \( j \in \mathbb{Z} \), we let \( \mathbb{Z}_{\geq j} \) denote the set of all integers greater than or equal to \( j \).

In this section, we show the following theorem.

**Theorem 3.1**. Let \( K' \) be a two-bridge knot of genus \( k \). There exists a two-bridge knot \( K \) of genus \( n \) such that the knot group \( G(K) \) admits an epimorphism onto \( G(K') \) if and only if

\[ n \in \mathbb{Z}_{\geq 3k-1} \setminus S_k. \]

**Proof.** Recall that the length of the reduced even continued fraction corresponding to a two-bridge knot is twice the genus of the knot, see [3] for example. A continued fraction of a rational number corresponding to \( K' \) can be written as \([a_1, a_2, \ldots, a_{2k}] \) where all \( a_i \)'s are even and non-zero, since the genus of \( K' \) is \( k \). Suppose that there exists an epimorphism from \( G(K) \) onto \( G(K') \). By Theorem 2.1, a rational number corresponding to \( K \) admits a continued fraction in the form

\[ [\varepsilon_1 a, 2c_1, \varepsilon_2 a^{-1}, 2c_2, \varepsilon_3 a, 2c_3, \varepsilon_4 a^{-1}, 2c_4, \ldots, \varepsilon_{2r} a^{-1}, 2c_{2r}, \varepsilon_{2r+1} a] \]
where $a = (a_1, a_2, \ldots, a_{2k})$. As mentioned in Section 2, if $c_i = 0$, then we can reduce the length of the continued fraction by 2. After deleting 0, the length of the continued fraction of $K$ is

$$2n = (2r + 1)\ell([a_1, a_2, \ldots, a_{2k}]) + \sum_{i=1}^{2r} w_i = 2(2r + 1)k + \sum_{i=1}^{2r} w_i$$

where

$$w_i = \begin{cases} 1 & \text{if } c_i \neq 0 \\ -1 & \text{if } c_i = 0. \end{cases}$$

We define $\ell$ as

$$\ell = \frac{1}{2} \sum_{i=1}^{2r} w_i = r - \# \{i \mid c_i = 0, 1 \leq i \leq 2r\}.$$ 

Then $-r \leq \ell \leq r$ and $n = (2r + 1)k + \ell$. Namely,

$$n \in \mathbb{N} \cap \left( \bigcup_{r \in \mathbb{N}} \left( [(2r + 1)k - r, (2r + 1)k + r] \right) \right).$$

Here if $r \geq k - 1$, each interval does not have a gap with the next interval. Therefore the complement of the set to which $n$ belongs is

(3.1) \hspace{1cm} \mathbb{N} \cap \left( [1, 3k - 2] \cup \bigcup_{r=1}^{k-2} [(2r + 1)k - r, (2r + 1)k + r - 2] \right).$

Conversely, if $n$ belongs to $\mathbb{Z}_{\geq (3k-1)} \setminus S_k$, we can construct a two-bridge knot $K$ of genus $n$ whose knot group admits an epimorphism onto $G(K')$ as above. \hfill \Box

**Corollary 3.2.** Let $K$ be a two-bridge knot and $K'$ a knot. If there exists an epimorphism $\varphi : G(K) \rightarrow G(K')$, then

(3.2) \hspace{1cm} g(K) \geq 3g(K') - 1.$$

**Remark 3.3.** We denote by $c(K)$ the crossing number of a knot $K$. Let $K$ be a two-bridge knot and suppose that there exists an epimorphism from $G(K)$ onto the knot group $G(K')$ of another knot $K'$. By the previous paper [17], the following inequality holds

(3.3) \hspace{1cm} c(K) \geq 3c(K').$

Moreover, for a given two-bridge knot $K'$, we can construct a two-bridge knot $K$ with any crossing number satisfying the inequality (3.3) such that $G(K)$ admits an epimorphism onto $G(K')$. However, Theorem 3.1 implies we can not always construct a two-bridge knot $K$ of any genus even if it satisfies the inequality (3.2).

More precisely, if $g(K) \geq 3g(K') - 1$ but $g(K) \in S_{g(K')}$, then there does not exist an epimorphism from $G(K)$ onto $G(K')$. Note that the cardinality of $S_k$ is

$$\sum_{r=1}^{k-2} ((2r + 3)k - r - 2) - ((2r + 1)k + r + 1) + 1 = (k - 1)(k - 2)$$

and that of the set (3.1) is

$$3k - 2 + \# S_k = 3k - 2 + (k - 1)(k - 2) = k^2.$$
4. SMALL GENUS

In this section, we see some examples of small genus. Namely, for a small given $n$, we describe the continued fractions of two-bridge knots $K$ of genus $n$ which admit epimorphisms onto another knot $K'$ of genus $k$. Note that by the argument of the proof of Theorem 3.1, we have the following

\[(2r + 1)k - r \leq n \leq (2r + 1)k + r,\]
\[\mathbb{Z}\{i \mid c_i = 0, 1 \leq i \leq 2r\} = (2r + 1)k + r - n.\]

**Case:** $n = 2$. By Corollary 3.2, the genus of $K'$ is 1. Then we can take $[2a, 2b]$ for a continued fraction of $K'$, where $a, b \in \mathbb{Z} \setminus \{0\}$. The inequality (4.1) implies $r = 1$ or 2. When $r = 1$, one $c_i$ is 0 and the other $c_i$ is not 0 by the equation (4.2). Then the continued fraction of $K$ is

\[\lfloor 2a, 2b, 0, 2b, 2a, 0, 2a, 2b \rfloor = \lfloor 2a, 4b, 4a, 2b \rfloor.\]

Furthermore, if a continued fraction of a two-bridge knot of genus 2 cannot be expressed in this form, then this knot is minimal.

**Case:** $n = 3$. Similarly, the genus of $K'$ is 1 and $\lfloor 2a, 2b \rfloor$ can be taken as a continued fraction of $K'$. The inequality (4.1) implies $r = 1$ or 2. When $r = 1$, one $c_i$ is 0 and the other $c_i$ is not 0 by the equation (4.2). Then the continued fraction of $K$ is

\[\lfloor 2a, 2b, 0, 2b, 2a, 0, 2a, 2b, 0, 2a, 0, 2a, 2b \rfloor = \lfloor 2a, 4b, 4a, 4b, 4a, 2b \rfloor.\]

**Case:** $n = 4$. The genus of $K'$ is 1 and a continued fraction of $K'$ is $\lfloor 2a, 2b \rfloor$. The inequality (4.1) implies $r = 1, 2, 3$. When $r = 1$, all $c_i$’s are not 0. Then the continued fraction of $K$ is

\[\lfloor 2a, 2b, 0, 2b, 2a, 0, 2a, 2b, 0, 2b, 2a, 2c_4, 2\varepsilon_5a, 2\varepsilon_5b \rfloor = \lfloor 2a, 4b, 4a, 4b, 2a, 2c_4, 2\varepsilon_5a, 2\varepsilon_5b \rfloor\]

or

\[\lfloor 2a, 2b, 0, 2b, 2a, 0, 2a, 2b, 0, 2b, 2a, 0, 2a, 2b, 0, 2b, 2a, 0, 2a, 2b \rfloor = \lfloor 2a, 4b, 4a, 4b, 4a, 2b \rfloor.\]

up to mirror image, where $\varepsilon_2, \varepsilon_3 = \pm 1$. When $r = 2$, three $c_i$’s are 0 and one $c_i$ is not 0. Then the continued fraction of $K$ is

\[\lfloor 2a, 2b, 0, 2b, 2a, 0, 2a, 2b, 0, 2b, 2a, 2c_4, 2\varepsilon_5a, 2\varepsilon_5b \rfloor = \lfloor 2a, 4b, 4a, 4b, 2a, 2c_4, 2\varepsilon_5a, 2\varepsilon_5b \rfloor\]

**Case:** $n = 5$. In this case, the genus of $K'$ is 1 or 2 by Corollary 3.2. First, we consider the case that the genus of $K'$ is 1 and that a continued fraction of $K'$ is
The continued fraction of $K$ continues fraction of $\varepsilon$ knot of genus up to 5 is minimal.

By using the above arguments, we obtain a criterion whether a given two-bridge knot of genus up to 5 is minimal.

**Theorem 4.1.** A two-bridge knot $K$ of genus up to 5 is not minimal if and only if a continued fraction of a rational number corresponding to $K$ can be expressed as

$$[2a, 2b, 0, 2b, 2a, 0, 2a, 2b, 2c_3, 2\varepsilon_4 b, 2\varepsilon_4 a, 2c_4, 2\varepsilon_5 a, 2\varepsilon_5 b]$$

up to mirror image, where $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_7 = \pm1$. Next, the genus of $K'$ is 2 and the continued fraction of $K'$ is $[2a, 2b, 2c, 2d]$, where $a, b, c, d \in \mathbb{Z} \setminus \{0\}$. Then the continued fraction of $K$ is

$$[2a, 2b, 0, 2b, 2a, 0, 2a, 2b, 2c_2, 2\varepsilon_3 a, 2\varepsilon_3 b, 0, 2\varepsilon_3 a, 2\varepsilon_3 b]$$

up to mirror image, where $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_7 = \pm1$. Next, the genus of $K'$ is 2 and the continued fraction of $K'$ is $[2a, 2b, 2c, 2d]$, where $a, b, c, d \in \mathbb{Z} \setminus \{0\}$. Then the continued fraction of $K$ is

$$[2a, 2b, 2c, 2d, 0, 2d, 2c, 2b, 2c, 0, 2c, 2b, 0, 2b, 2a, 0, 2a, 2b, 0, 2b, 2a, 0, 2a, 2b]$$

up to mirror image, where $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_7 = \pm1$. Next, the genus of $K'$ is 2 and the continued fraction of $K'$ is $[2a, 2b, 2c, 2d]$, where $a, b, c, d \in \mathbb{Z} \setminus \{0\}$. Then the continued fraction of $K$ is

$$[2a, 2b, 2c, 2d] = [2a, 2b, 2c, 4d, 2c, 2b, 4a, 2b, 2c, 2d].$$

By using the above arguments, we obtain a criterion whether a given two-bridge knot of genus up to 5 is minimal.
one of the following:

\[ [2a, 4b, 4a, 2b], \\
[2a, 4b, 2a, 2c_2, 2e_3a, 2e_3b], [2a, 4b, 4a, 4b, 4a, 2b], \\
[2a, 2b, 2c_1, 2e_2b, 2e_2a, 2c_2, 2e_3a, 2e_3b], [2a, 4b, 4a, 4b, 2a, 2c_4, 2e_5a, 2e_5b], \\
[2a, 4b, 4a, 2b, 2c_3, 2e_4b, 2e_4a, 2e_4b], [2a, 4b, 4a, 4b, 4a, 4b, 4a, 2b], \\
[2a, 4b, 4a, 2b, 2c_3, 2e_4b, 2e_4a, 2e_4b], [2a, 4b, 4a, 4b, 2a, 2c_4, 2e_5a, 2e_5b], \\
[2a, 4b, 4a, 2b, 2c_3, 2e_4b, 2e_4a, 2e_4b], [2a, 4b, 4a, 4b, 2a, 2c_4, 2e_5a, 2e_5b], \\
[2a, 4b, 4a, 2b, 2c_3, 2e_4b, 2e_4a, 2e_4b], [2a, 4b, 4a, 4b, 2a, 2c_4, 2e_5a, 2e_5b], \\
[2a, 4b, 4a, 2b, 2c_3, 2e_4b, 2e_4a, 2e_4b], [2a, 4b, 4a, 4b, 2a, 2c_4, 2e_5a, 2e_5b], \\
[2a, 4b, 4a, 2b, 2c_3, 2e_4b, 2e_4a, 2e_4b], [2a, 4b, 4a, 4b, 2a, 2c_4, 2e_5a, 2e_5b], \\
[2a, 4b, 4a, 2b, 2c_3, 2e_4b, 2e_4a, 2e_4b], [2a, 4b, 4a, 4b, 2a, 2c_4, 2e_5a, 2e_5b], \\
[2a, 4b, 4a, 2b, 2c_3, 2e_4b, 2e_4a, 2e_4b], [2a, 4b, 4a, 4b, 2a, 2c_4, 2e_5a, 2e_5b]. \\
\]

where \( a, b, c, d, e_i \neq 0 \) and \( \varepsilon_i = \pm 1 \).

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