THE DEGREE OF BIHOLOMORPHISMS OF QUASI-REINHARDT DOMAINS FIXING THE ORIGIN

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Abstract. We give a description of biholomorphisms of quasi-Reinhardt domains fixing the origin via Bergman representative coordinates, which are shown to be polynomial mappings with a degree bound given by the so-called “resonance order”.

1. Introduction

The study of “special domains” invariant under a compact Lie group action is of classical interest (see e.g. [4]), with Cartan’s Linearity Theorem for circular domains being one of the most well-known results. While it is known that biholomorphisms between these special domains fixing the origin are all polynomials with uniform degree upper bound (see e.g. [7]), such a uniform upper bound can be explicitly given in the case of quasi-circular domains and quasi-Reinhardt domains, thanks to the notion of resonance order and quasi-resonance order introduced in [8, 2].

Let $T^r$ be the torus group of dimension $r \geq 1$. Let $\rho : T^r \to \text{GL}(n, \mathbb{C})$ be a holomorphic linear action of $T^r$ on $\mathbb{C}^n$ such that the only $\rho$-invariant holomorphic functions on $\mathbb{C}^n$ are constant. Let $\lambda = (\lambda_1, \cdots, \lambda_r) \in T^r$, $m = (m_{i1}, \cdots, m_{ir}) \in \mathbb{Z}^r$, $1 \leq i \leq n$, and $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$. Then an action $\rho$ of $T^r$ on $\mathbb{C}^n$ can be written as

\begin{equation}
\rho(\lambda)z = (\lambda^{m_1}z_1, \cdots, \lambda^{m_n}z_n),
\end{equation}

where $\lambda^{m_i} = \prod_{j=1}^r \lambda_j^{m_{ij}}$ for each $i$. We call $m_i$, $1 \leq i \leq n$, the weight of the action $\rho$.

Let $D$ be a bounded domain in $\mathbb{C}^n$ containing the origin. We say that $D$ is quasi-Reinhardt (of rank $r$) if it is $\rho$-invariant. When $r = 1$, we say that $D$ is quasi-circular.

Denote by $\mathbb{N}$ the set of non-negative integers. Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ and $m^j = (m_{i1}, \cdots, m_{nj})$, $1 \leq j \leq r$. For $1 \leq i \leq n$, we define the $i$-th resonance set as

$E_i := \{ \alpha; \alpha \cdot m^j = m_{ij}, 1 \leq j \leq r \}$,

where $\alpha \cdot m^j = \sum_{i=1}^n \alpha_im_{ij}$, and the $i$-th resonance order as

$\mu_i := \max \{|\alpha|; \alpha \in E_i\}$,

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where \( |\alpha| := \sum_{i=1}^{n} \alpha_i \). Then, the resonance order is defined as
\[
\mu := \max_{1 \leq i \leq n} \mu_i.
\]

Let \( K_{D}(z, w) \) be the Bergman kernel, i.e. the reproducing kernel of the space of square integrable holomorphic functions on \( D \). Since \( D \) is bounded, we have \( K_{D}(z, z) > 0 \) for \( z \in D \). The Bergman metric tensor \( T_{D}(z, w) \) is defined as the \( n \times n \) matrix with entries \( t_{ik}^{D}(z, w) = \frac{\partial}{\partial w_{i} \partial z_{k}} \log K_{D}(z, w), 1 \leq i, k \leq n \). For \( z \in D \), we know that \( T_{D}(z, z) \) is a positive definite Hermitian matrix (see e.g. [1]).

The Bergman representative coordinates at \( \xi \in D \) is defined as (see e.g. [3])
\[
\sigma_{\xi}^{D}(z) := T_{D}(\xi, \xi)^{-1} \text{grad}_{w} \log \frac{K_{D}(z, w)}{K_{D}(w, w)} \bigg|_{w=\xi}.
\]
As shown in [5], one has \( K_{D}(z, 0) \equiv K_{D}(0, 0) \) when \( D \) is a quasi-Reinhardt domain. Thus the Bergman representative coordinates \( \sigma_{D}(z) \) is defined for all \( z \in D \).

For later use, we also record here the following well-known transformation formula for the Bergman metric tensor
\[
T_{D}(z, w) = \text{Jac}_{f}^{2}(w) T_{D}(f(z), f(w)) \text{Jac}_{f}(z), \quad f \in \text{Aut}(D).
\]

The main purpose of this paper is to prove the following

**Theorem 1.1.** Let \( f \) be a biholomorphism of quasi-Reinhardt domains \( D_{1} \) and \( D_{2} \), fixing the origin. Set \( \sigma^{i}(z) = \sigma_{D_{1}}^{i}(z), i = 1, 2, \) and \( J_{f} \) the linear part of \( f \). Then,

(i) \( f = (\sigma^{2})^{-1} \circ J_{f} \circ \sigma^{1}; \)

(ii) The degrees of \( \sigma^{i} \) and \( (\sigma^{i})^{-1}, i = 1, 2, \) are bounded by the resonance order of \( D_{i} \).

The proof of Theorem 1.1 is given in section 2. The corresponding result for automorphisms of quasi-circular domains fixing the origin was given by [9, Lemma 3.1; Corollary 3.2; Lemma 3.3], although the stronger statement [9, Theorem 1.1] is not correct. We give a clarification of the situation in section 3.

## 2. Quasi-Reinhardt Domains

Let \( D_{1} \) and \( D_{2} \) be bounded quasi-Reinhardt domains containing the origin. Set \( \sigma^{i}(z) = \sigma_{D_{1}}^{i}(z), i = 1, 2, \) Then, it is well-known (see e.g. [3]) that for any biholomorphism \( f \) between \( D_{1} \) and \( D_{2} \) with \( f(0) = 0 \) there exists a linear map \( L_{f} \) between \( \sigma^{1}(D_{1}) \) and \( \sigma^{2}(D_{2}) \) such that
\[
\sigma^{2} \circ f = L_{f} \circ \sigma^{1}.
\]
From the definition (1.2), one readily checks that \( \sigma^{i}(0) = 0 \) and
\[
\text{Jac}_{\sigma^{i}}(z) = T_{D_{i}}(0, 0)^{-1} T_{D_{i}}(z, 0).
\]
In particular, one has \( J_{\sigma^{i}}(z) := \text{Jac}_{\sigma^{i}}(0) \cdot z^{i} = z, \) which also implies that \( \sigma^{i} \)'s are invertible. Combining these facts with (2.1), one sees that the linear map in (2.1) is in fact just the linear part \( J_{f} \) of \( f \), and
\[
f = (\sigma^{2})^{-1} \circ J_{f} \circ \sigma^{1}.
\]
This proves Theorem 1.1 (i).

To study the degree of the Bergman representative coordinates of a quasi-Reinhardt \( D \), we first need to order the weight \( m_{i}, 1 \leq i \leq n \), in a proper way.
Without loss of generality and for simplicity, we will assume that all \( m_i \)'s are distinct. (In the general case, whenever \( m_i = m_j \) one can then treat \( z_i \) and \( z_j \) as the same, resulting in “Jordan blocks”, which do not affect the degree.)

We say that \( m_i < m_j \) if there exists \( \alpha \in E_j \) with \( \alpha_i \neq 0 \).

**Lemma 2.1.** For \( 1 \leq i \neq j \leq n \), \( m_i < m_j \) and \( m_j < m_i \) can not hold at the same time.

**Proof.** Assume that \( m_i < m_j \) and \( m_j < m_i \) hold at the same time. Then, there exists \( \alpha \in \mathbb{N}^n \) with \( \alpha_i \geq 1 \), \( \alpha_j = 0 \) and \( |\alpha| \geq 2 \) such that

\[
(2.4) \quad \alpha \cdot m^k = m_{jk}, \quad \forall \ 1 \leq k \leq r,
\]

and there exists \( \beta \in \mathbb{N}^n \) with \( \beta_j \geq 1 \), \( \beta_i = 0 \) and \( |\beta| \geq 2 \) such that

\[
(2.5) \quad \beta \cdot m^k = m_{ik}, \quad \forall \ 1 \leq k \leq r.
\]

Set \( \gamma = (\gamma_1, \ldots, \gamma_n) \) with \( \gamma_i = \beta_j \alpha_i - 1 \) and \( \gamma_l = \beta_j \alpha_l + \beta_i, \ l \neq i \). Then one has \( \gamma \in \mathbb{N}^n \) and

\[
|\gamma| = \beta_j |\alpha| + \sum_{l \neq i} \beta_l - 1 \geq |\beta| + (2\beta_j - 1) \geq |\beta| + 1 \geq 3.
\]

From \( (2.4) \) and \( (2.5) \), one gets

\[
\gamma \cdot m^k = 0, \quad \forall \ 1 \leq k \leq r.
\]

This implies that \( z^\gamma \) is invariant under the \( \rho \)-action \((1.1)\), which contradicts with the assumption that the only \( \rho \)-invariant holomorphic functions on \( \mathbb{C}^n \) are constant. \( \square \)

For a quasi-Reinhardt domain \( D \) with weight \( m_i \), we say that \( m_i \)'s are **properly ordered** if \( m_i < m_j \) only for \( i < j, 1 \leq i \neq j \leq n \). (Note that such an ordering is only partial and not unique, and for definitiveness we can assign the larger indices to those \( m_i \)'s without any resonance relations.)

A monomial \( z^n \) is called an \( i \)-th resonant monomial if \( \alpha \in E_i \).

**Proposition 2.2.** Let \( D \) be a bounded quasi-Reinhardt domain containing the origin with weight \( m_i \) properly ordered. Set \( \sigma(z) = \sigma_0^D(z) = (\sigma_1(z), \ldots, \sigma_n(z)) \). Then for each \( 1 \leq i \leq n \), \( \sigma_i(z) = z_i + g_i(z) \), where \( g_i(z) \) contains only nonlinear \( i \)-th resonant monomials. The same is true for \( \sigma^{-1}(z) \).

**Proof.** By \((1.3)\), we have

\[
T_D(z, w) = \text{Jac}_{\rho(\lambda)}(w)T_D(\rho(\lambda)(z), \rho(\lambda)(w))\text{Jac}_{\rho(\lambda)}(z).
\]

Setting \( w = 0 \) in \((2.6)\), we obtain

\[
(2.7) \quad T_D(z, 0) = \text{diag}(\lambda^{-m_1}, \ldots, \lambda^{-m_n})T_D(\rho(\lambda)(z), 0)\text{diag}(\lambda^m_1, \ldots, \lambda^m_n).
\]

Write \( t_{ik}^{D}(z, 0) = \sum_{|\alpha| \geq 0} a_{ik}^{\alpha}z^\alpha = \sum_{|\alpha| \geq 0} a_{ik}^{\alpha}z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \ 1 \leq i, k \leq n \). Set \( \beta = (\beta_1, \ldots, \beta_r) := (\alpha \cdot m^1, \ldots, \alpha \cdot m^r) \). Then from \((2.7)\), we get

\[
(2.8) \quad a_{ik}^{\alpha} = \lambda^{-m_i + m_k + \beta}a_{ik}^{\alpha}.
\]

Since \((2.8)\) holds for any \( \lambda \in T^r \), \( a_{ik}^{\alpha} \) can be nonzero only when

\[
-m_i + m_k + \beta = 0,
\]

which is satisfied if and only if

\[
(2.9) \quad \alpha + e_k \in E_i.
\]
Here \( \epsilon_k \) denotes the \( k \)-th unit multi-index.

Therefore, one can write \( T_D(z,0) \) as
\[
T_D(z,0) = T_D(0,0) + M(z),
\]
where \( T_D(0,0) = \text{Diag}(\tau_1, \cdots, \tau_n) \) and \( M(z) = [M_{ij}(z)]_{1 \leq i, j \leq n} \) with \( M_{ij} = 0 \) for \( i \leq j \).

By (2.10) and (2.2), one has
\[
\text{Jac}_\sigma(z) = T_D(0,0)^{-1}T_D(z,0) = I_n + T_D(0,0)^{-1}M(z) =: I_n + N(z),
\]
where \( N(z) = [N_{ij}(z)]_{1 \leq i, j \leq n} \) with \( N_{ij}(z) = 0 \) for \( i \leq j \) and \( N_{ij}(z) = \tau_i^{-1}M_{ij}(z) \) for \( i > j \).

Since \( \sigma(0) = 0 \), from (2.11) and (2.9), one sees that \( \sigma(z) \) is of the desired form.

Since \( J_\sigma = Id \) and each \( g_i(z) \) only contains terms involving \( z_j \)’s with \( j < i \) and of weighted degree equal to \( m_i \), a routine induction shows that each component of \( \sigma^{-1}(z) \) also only contains resonant monomials (cf. [9, Corollary 3.2]).

Obviously, Proposition 2.2 implies Theorem 1.1 (ii).

**Remark 2.3.** Theorem 1.1 gives a complete description of all possible forms of biholomorphisms of quasi-Reinhardt domains fixing the origin. It also gives a more transparent description of the “quasi-resonance order” (cf. [8, 2]).

### 3. Quasi-circular domains

Let \( D \) be a bounded quasi-circular domain containing the origin. The weight in the quasi-circular case is given by a set of \( n \) positive integers \( m_i, 1 \leq i \leq n \), with \( \gcd(m_1, \cdots, m_n) = 1 \). A standard proper ordering of \( m_i \)'s is requiring that \( m_i \leq m_j \) for \( i < j \).

First of all, the weight of a quasi-circular domain \( D \) is not unique, if \( D \) is in fact a quasi-Reinhardt domain of rank greater than one. For instance for \( D = B^2 \), the unit ball in \( \mathbb{C}^2 \), any \( (m_1, m_2) \in (\mathbb{Z}^+)^2 \) is a weight. The classical Cartan’s Linearity Theorem applies to \( B^2 \) and says that an automorphism of \( B^2 \) fixing the origin is linear. From the point of view of the resonance order, the weight which dictates the linearity is \((1, 1)\) or \((m_1, m_2)\) with \( \gcd(m_1, m_2) = 1 \). Therefore, if one considers \( B^2 \) as a quasi-circular domain with weight \((1, m)\) with \( m \geq 2 \), then one can not get the desired information on the degree of its automorphisms.

Secondly, the weight of a quasi-circular domain is not a biholomorphic invariant in general. For instance, if one considers \( D = \phi_k(B^2) \) with \( \phi_k(z_1, z_2) = (z_1, z_2 + z_1^k) \), then \( D \) is quasi-circular with weight \((1, k)\), but as we just noted above \( B^2 \) can be considered as a quasi-circular domain with any weight. And as noted in [2] Example 5.1, the rank of a quasi-Reinhardt domain is also not invariant under biholomorphisms, and one can consider the *maximal rank* of a quasi-Reinhardt domain.

We can define a *genuine* quasi-circular domain to be a quasi-Reinhardt domain with maximal rank one. For the study of the degree of automorphisms and biholomorphisms of quasi-circular domains, the genuine case and the “fake” case are vastly different. In dimension two, a complete study of the degree of origin-preserving automorphisms of quasi-circular domains was carried out in [10]. In higher dimensions, a similar study would be very complicated, especially for “fake” quasi-circular domains.
THE DEGREE OF BIHOLOMORPHISMS OF QUASI-REINHARDT DOMAINS

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