Bound States in NRQCD/NRQED and the Renormalization Group

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Abstract.

The application of renormalization group techniques to bound states in non-relativistic QED and QCD is discussed. For QED bound states like Hydrogen and positronium, the renormalization group allows large logarithms of the velocity, \( \ln v \) (or equivalently \( \ln \alpha' \)'s), to be predicted in a universal and simple way. The series of \( (\alpha \ln \alpha)'s \) are shown to terminate after a few terms. For QCD one can systematically sum infinite series of the form \( \alpha_s \ln \alpha_s \)^k, and answer definitively the question “\( \alpha_s \) at what scale?”.

For Coulombic bound states of two fermions, the relevant scales include the mass of the fermions, \( m \), their momentum, \( p \sim mv \), and their energy, \( E \sim mv^2 \). Since \( v \ll 1 \) it is useful to calculate properties of these bound states in a double expansion in \( v \) and \( \alpha \), with terms \( \alpha/v \sim 1 \) kept to all orders. However, the expansion may still involve large logarithms of \( v \) or \( \alpha \), which appear through factors of \( \ln(p/m) \), \( \ln(E/p) \), and \( \ln(E/m) \). In this talk I discuss how the renormalization group can be used to systematically predict and sum powers of \( \alpha \ln v \) [1–6].

At a given order in QED, terms involving \( \ln \alpha \) typically give the largest contributions, and the precision of experiments make their prediction quite important [7]. In Ref. [5] it was shown for the first time that \( \ln \alpha' \)'s in the Lamb shift, hyperfine splittings, and annihilation decay widths can be predicted with the renormalization group. This is in contrast to the usual method of computing these logarithms by evaluating matrix elements at the scale \( m \). The calculations are simple enough that we can simultaneously treat Hydrogen, muonium (\( \mu^+e^- \)), and positronium (\( e^+e^- \)).

For the \( t\bar{t} \) system near threshold, the relevant scales are \( m_t \sim 175 \) GeV, \( m_t v \sim 30 \) GeV, and \( m_t v^2 \sim 5 \) GeV, which are all \( \gg \Lambda_{\text{QCD}} \) and can be treated perturbatively. In QCD there is a strong dependence of the coupling on the scale; \( \alpha_s(m_t) \) is much different from \( \alpha_s(m_t v^2) \). The renormalization group allows us to handle this complication, or equivalently it allows us to systematically sum terms.
FIGURE 1. a) Paths in the \((\mu_U, \mu_S)\) plane for one-stage and two-stage running. b),c),d) Examples of the \(\mu_U\) and \(\mu_S\) dependence of the Feynman rules.

\((\alpha_s \ln v)^k\). Predictions for the running coefficients of the \(t\bar{t}\) potentials and the \(t\bar{t}\) production current will be discussed below [2,4].

Effective theories for non-relativistic QED and QCD (NRQED and NRQCD) allow the double expansion in \(v\) and \(\alpha\) to be performed in a simple way [8]. However, application of the renormalization group in these theories is complicated by the presence of two low energy scales, \(p\) and \(E\), which are coupled by the equations of motion, \(E = p^2/(2m)\). If one attempts to lower the cutoff on the energy to \(E \lesssim \Lambda\), then one can still excite larger momenta \(p \lesssim \sqrt{m\Lambda}\). Using dimensional regularization one can deal with this coupling of scales by using a velocity renormalization group [1], which has a subtraction velocity \(\nu\) rather than the usual subtraction momentum \(\mu\). Running in one-stage from \(\nu = 1\) to \(\nu = v\) simultaneously lowers the subtraction point for momenta, \(\mu_S \equiv m\nu\), to the scale \(mv\), and the subtraction point for energy, \(\mu_U \equiv mv^2\), to \(mv^2\). In Fig. 1a this one-stage approach is contrasted with the alternative two-stage approach where one first runs from \(\mu = m\) to \(mv\) and then runs from \(\mu = mv\) to \(mv^2\).

In Ref. [6] it was shown that for QED bound states, the most obvious method of two-stage running fails to reproduce terms involving \((\ln \alpha)^k\) with \(k \geq 2\). This occurs because in the two-stage method the coupling between the energy and momentum is ignored.

To calculate observables at the hard scale \(m\), the way the effective theory is formulated does not matter too much as long as it has a consistent power counting in \(v\). Below \(m\), only on-shell degrees of freedom are kept in the effective theory, i.e. degrees of freedom which fluctuate near their mass shell. For example, a potential gluon exchanged between two quarks has energy \(\sim mv^2\) but momentum \(\sim mv\), and is therefore far offshell. Instead of a potential gluon the effective theory has a four quark operator

\[
\mathcal{L}_p = - \sum_{p,p'} V(p,p') \mu_S^{2\epsilon} \bar{\psi}_{p'} \gamma_\mu \psi_p \chi_{-p'} \chi_{-p},
\]
TABLE 1. QED $\ln \alpha$’s which follow from the leading order (LO) and next-to-leading order (NLO) anomalous dimensions in Ref. [5]. $\mu^+ e^-$ predictions include $1/m_{\mu}$ dependence. (h.f.s is hyperfine splitting and $\Delta \Gamma / \Gamma$ is the $e^+ e^-$ decay width correction.)

| $\alpha^n \ln^m \alpha$ | Lamb shift | $H$ | $\mu^+ e^-, e^+ e^-$ | $\mu^+ e^-, e^+ e^-$ |
|-------------------------|------------|-----|----------------------|---------------------|
| ($\alpha^8 \ln^3 \alpha$) | (no h.f.s.) | $\alpha^7 \ln^2 \alpha$ | $\alpha^3 \ln^2 \alpha$ | $\alpha^3 \ln^2 \alpha$ |
| ($\alpha^4 \ln^3 \alpha$) | (no h.f.s.) | $\alpha^6 \ln \alpha$ | $\alpha^6 \ln \alpha$ | $\alpha^6 \ln \alpha$ |
| ($\alpha^2 \ln \alpha$) | $\alpha^4 \ln \alpha$ | $\alpha^4 \ln \alpha$ | $\alpha^4 \ln \alpha$ | $\alpha^4 \ln \alpha$ |
| ($\alpha \ln \alpha$) | $\alpha \ln \alpha$ | $\alpha \ln \alpha$ | $\alpha \ln \alpha$ | $\alpha \ln \alpha$ |

* There is a growing consensus that the value in Ref. [9] is correct [16].

where $\psi_p (\chi_p)$ destroys a quark (antiquark) with momentum $p$ and spin and color labels are suppressed. $V(p, p')$ is the potential

$$V(p, p') = \frac{U_c}{k^2} + \frac{U_k}{|k|} + U_2 + U_3 S^2 + \frac{U_6 (p^2 + p'^2)}{2 k^2} + U_t \left( \sigma_1 \cdot \sigma_2 - \frac{3 k \cdot \sigma_1 k \cdot \sigma_2}{k^2} \right)$$

$$- \frac{i U_A \cdot (p' \times p)}{k^2} + U_3 |k| + U_3 \ S^2 |k| + \frac{U_{r \ k} (p^2 + p'^2)}{2 |k|} + \ldots ,$$

(2)

where the $U_i(\nu)$’s are running coefficients. In QCD the color singlet and octet coefficients have different values. We have absorbed in the $U_i$’s the dependence on the fermion masses; only the momentum dependence is important for the power counting. For the $QQ$ and $Q\bar{Q}$ potentials in an arbitrary Lie gauge group, the matching coefficients at one-loop to order $v^2$ can be found in Ref. [3].

The effective Lagrangian has quarks with $(E, p) \sim (mv^2, mv)$ interacting with soft gluons which have $(E, p) \sim (mv, mv)$, and ultrasoft gluons which have $(E, p) \sim (mv^2, mv^2)$ (see Refs. [1–4]). To implement the velocity renormalization group we renormalize the Lagrangian and compute anomalous dimensions. This procedure is fairly simple since it can be done treating the Coulomb potential perturbatively. In Figs. 1b,c,d the $\mu$’s which appear in some typical interactions are shown. Fig. 1b shows a quark interacting with a single ultrasoft gluon. Physically, the fact that the ultrasoft mode involves the coupling $g(\mu_U)$ makes sense; due to the multipole expansion the scale $mv^2$ is the only scale it sees. Fig. 1c shows a soft gluon scattering off a quark, and Fig. 1d shows an insertion of the potential. For these interactions the parameter $\mu_S \sim mv$ appears.
TABLE 2. LO and LL values of coefficients of the \( \bar{t}t \) potential in QCD. Here \( U_\Lambda = S U_\Lambda \) and the values are in units of the top mass \( m_t \).

| \( \nu \) | \( U^{(s)}_k m_t \) | \( U^{(s)}_r m_t^2 \) | \( U^{(s)}_2 m_t^2 \) | \( U^{(s)}_s m_t^2 \) | \( U^{(s)}_\Lambda m_t^2 \) | \( U^{(s)}_\theta m_t^2 \) |
|---|---|---|---|---|---|---|
| 1 | -0.36 | -1.81 | 0 | 0.60 | 0.15 | 2.71 |
| \( \nu = \nu \) | -0.03 | -1.49 | 0.63 | 0.53 | 0.16 | 3.11 |

Below the electron mass the electromagnetic coupling in NRQED does not run, but coefficients in the potential do. For fermions with mass and charge \((m_1, -e)\) and \((m_2, Ze)\) we find the anomalous dimensions:

\[
\begin{align*}
\nu \frac{dU_2}{d\nu} \bigg|_{\text{LO}} &= \frac{2\alpha}{3\pi} \left( \frac{1}{m_1} + \frac{Z}{m_2} \right)^2 U_c + \frac{14Z^2\alpha^2}{3m_1m_2}, \\
\nu \frac{dU_{2+s}}{d\nu} \bigg|_{\text{NLO}} &= \rho_{c22} U_c \left( U_{2+s}^2 + 2U_{2+s}U_r + \frac{3}{4}U_r^2 - 9U_t^2S^2 \right) + \rho_{ccc} U_c^3 \\
&\quad + \rho_{cc2} U_c \left( U_{2+s} + U_r \right) + \rho_{ck} U_c U_k + \rho_{k2} U_k \left( U_{2+s} + U_r/2 \right) \\
&\quad + \rho_{c3} U_c \left( U_3 + U_{3a}S^2 + \frac{U_{ck}}{2} \right) + \rho_s \frac{Z^3\alpha^3}{m_1m_2}, \tag{3}
\end{align*}
\]

where \( U_{2+s} = U_2 + U_sS^2 \) and the \( \rho_i \)'s are mass dependent numbers [5]. Solving these equations gives the results summarized in Table 1.\(^2\) Taking the matrix element of the leading log (LL) value of \( U_2(\nu) \) gives the \( \alpha^5 \ln \alpha \) Lamb shifts for Hydrogen, muonium, and positronium. Furthermore, the LO anomalous dimension is independent of \( \nu \) so there are no higher terms, \( \alpha^{k+4} \ln^k \alpha \) for \( k \geq 2 \). At next-to-leading log (NLL) order the most logarithms are generated by the \( \rho_{c22} U_c U_2(\nu)^2 \) term which gives the \( \alpha^8 \ln^3 \alpha \) Lamb shifts. Thus, there are no terms \( \alpha^{k+4} \ln^k \alpha \) for \( k \geq 4 \). Hyperfine splittings are generated by the \( S^2 \) terms in Eq. (3), and the ortho and para-positronium widths are generated by imaginary terms which enter through the matching condition for \( U_{2+s}(1) \).

NRQCD is better at generating logarithms than NRQED since the running of \( \alpha_s \) causes all potential coefficients to run. For \( \bar{t}t \) the change in the color singlet couplings from \( \nu = 1 \) to \( \nu = \nu = 0.15 \) are shown in Table 2.\(^3\) The largest changes occur in the spin independent couplings \( U_k^{(s)}(\nu) \), \( U_r^{(s)}(\nu) \), and \( U_2^{(s)}(\nu) \). It is these couplings which depend on \( \alpha_s(m_t\nu^2) \) since their anomalous dimensions have contributions from ultrasoft diagrams. In Fig. 2a we plot the two-loop running of \( U_k^{(s)}(\nu) \) whose value changes by an order of magnitude between \( \nu = 1 \) and \( \nu = \nu \). The full theory \( \bar{t}t \) production current gets matched onto a current in the effective theory \( \bar{t}\gamma^\mu t = c_1\gamma^\mu_1\sigma^\dagger\chi^{*}\gamma_\mu + \ldots \). At NLL order the running potentials mix into the running of the production current coefficient \( c_1(\nu) \) [1]. Fig. 2b plots the running of \( c_1(\nu) \) at

\(^2\) Note that the \( \alpha^7 \ln^2 \alpha \) Lamb shift requires the LL running of \( V^{(1)} \) since this potential mixes into \( U_2 \), and the \( \alpha^6 \ln \alpha \) Lamb shift requires the NLL running of \( U_k \). These will be discussed in a future publication.

\(^3\) The renormalization group improved static potential is considered in Ref. [17].
FIGURE 2. For $t\bar{t}$ production near threshold the running of the color singlet and octet $1/|k|$ potentials are shown in a), and the NLL value of the production current is given in b) (solid line) [4]. In b) the large, medium, and small dashes are the LO, NLO, and NNLO [18] matching results. In a) and b), $\nu = 1$ is the scale $\mu = m_t$ and the solid vertical line is the Coulombic velocity.

NLL from Ref. [4]. Summing the logarithms improves the convergence by reducing the size of the NLO matching coefficient by a factor of 2. It would be interesting to see if a similar improvement in the convergence takes place for the rather large NNLO matching correction found in Ref. [18].

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