A STUDY ON FREE ROOTS OF BORCHERDS-KAC-MOODY LIE
SUPERALGEBRAS

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Abstract. Let $g$ be a Borcherds-Kac-Moody Lie superalgebra (BKM superalgebra in short) with the associated graph $G$. Any such $g$ is constructed from a free Lie superalgebra by introducing three different sets of relations on the generators: (1) Chevalley relations, (2) Serre relations, and (3) Commutation relations coming from the graph $G$. By Chevalley relations we get a triangular decomposition $g = n_+ \oplus h \oplus n_- $ and each roots space $g_\alpha$ is either contained in $n_+$ or $n_-$. In particular, each $g_\alpha$ involves only the relations (2) and (3). In this paper, we are interested in the root spaces of $g$ which are independent of the Serre relations. We call these roots free roots of $g$. Since these root spaces involve only commutation relations coming from the graph $G$ we can study them combinatorially. We use heaps of pieces to study these roots and prove many combinatorial properties. We construct two different bases for these root spaces of $g$: One by extending the Lalonde’s Lyndon heap basis of free partially commutative Lie algebras to the case of free partially commutative Lie superalgebras and the other by extending the basis given in [1] for the free root spaces of Borcherds algebras to the case of BKM superalgebras. This is done by studying the combinatorial properties of super Lyndon heaps. We also discuss a few other combinatorial properties of free roots.

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1-These roots spaces are free from Serre relations. Also, these roots spaces can be identified with certain grade spaces of free partially commutative Lie superalgebras [c.f. Lemma 1]. So we call them free roots and the associated root spaces free root spaces of $g$. 

1
1. Introduction

In this paper, we are interested in the combinatorial properties of roots of a Borcherds-Kac-Moody Lie superalgebra (BKM superalgebra in short) \( \mathfrak{g} \). In particular, we are interested in the roots of \( \mathfrak{g} \) whose associated root spaces are independent of the Serre relations. We call these roots free roots of \( \mathfrak{g} \). BKM superalgebras \([40, 41, 36, 29]\) are a natural generalization of two important classes of Lie algebras namely Borcherds algebras (Generalized Kac-Moody algebras) \([3, 22, 23, 1]\) and the Kac-Moody Lie superalgebras \([24, 7, 26, 9, 10]\). BKM superalgebras have a wide range of applications in mathematical physics \([37, 20, 16, 12, 4]\). For example, physicists applied these algebras to describe supersymmetry, chiral supergravity, and Gauge theory \([17, 19, 27, 13, 8, 21]\).

We explain our results in detail: In \([1, \text{Theorem 1}]\) the following connection between the root multiplicities of a Borcherds algebra \( \mathfrak{g} \) and the \( k \)-chromatic polynomial \( \pi_{k}^{G}(q) \) [c.f. Definition 7] of the associated quasi Dynkin diagram \( G \) [c.f. Definition 2] is proved.

**Theorem 1.** Let \( \mathfrak{g} \) be a Borcherds algebra with associated Borcherds-Cartan matrix \( A \) and the quasi Dynkin diagram \( G \). Assume that the matrix \( A \) is indexed by the countable (finite/countably infinite) set \( I \). Let \( k = (k_{i} : i \in I) \in \mathbb{Z}_{+}[I] := \oplus_{i \in I} \mathbb{Z}\alpha_{i} \) be such that \( k_{i} \leq 1 \) for \( i \in I^{re} \). Then

\[
\pi_{k}^{G}(q) = (-1)^{ht(\eta(k))} \sum_{J \in L_{G}(k)} (-1)^{|J|} \prod_{J \in J} \left( q^{\text{mult}(\beta(J))} D(J, J) \right)
\]

where \( \eta(k) := \sum_{i \in I} k_{i}\alpha_{i} \), \( ht(\eta(k)) := \sum_{i \in I} k_{i} \), and \( L_{G}(k) \) is the bond lattice of weight \( k \) of the graph \( G \) [c.f. Definition 8].

Further in \([1]\), using **Theorem 1**, the following theorem is proved in which a set of the basis for the root space \( \mathfrak{g}_{\eta(k)} \) is constructed using the combinatorial model \( C^{i}(k, G) \) [c.f. Equation (4.3)].
Theorem 2. Let \( k \in \mathbb{Z}_+[[I]] \) be as in Theorem 1. Then the set \( \{i(w) : w \in C^k(G)\} \) is a basis for the root space \( g_{\eta(k)} \). Moreover, if \( k_i = 1 \), the set \( \{e(w) : w \in \mathcal{X}_i, \text{wt}(w) = \eta(k)\} \) forms a left-normed basis of \( g_{\eta(k)} \) [c.f. Section 4]. We call these bases (one basis for each \( i \in I \)) as LLN bases (Lyndon - Left normed bases.) [c.f. Example 8].

We observe that, by Lemma 1, the root spaces considered in Theorems 1 and 2 are precisely the set of free roots of the Borcherds algebra \( g \). In particular, the proof shows that the associated root spaces are independent of the Serre relations. In this paper, we construct two different bases for the free root spaces of BKM superalgebras. First, we prove Theorem 2 for the case BKM superalgebras. In particular, This will give us the LLN basis for the root spaces of BKM superalgebras. Our proof is completely different from [1] even for Borcherds algebras.

For example, our proof of Theorem 2 is independent of Theorem 1 (for Borcherds algebras) and its super analog Theorem 3 (for BKM superalgebras). We give direct simple proof. Second, we construct a Lyndon heaps basis for the free root spaces of BKM superalgebras. We remark that this basis is not discussed in [1].

This is done in the following steps. Let \( g(A, \Psi) \) be a BKM superalgebra with the associated quasi Dynkin diagram \((G, \Psi)\). Assume that \( k \in \mathbb{Z}_+[[I]] \) satisfies \( k_i \leq 1 \) for \( i \in I^e \cup \Psi_0 \).

1. First, we introduce the notion of a supergraph \((G, \Psi)\) [c.f. Definition 1]. Using this definition, we give the definition of free partially commutative Lie superalgebra \( \mathcal{L}(\mathcal{S}(G, \Psi)) \) associated with a supergraph \((G, \Psi)\) (when \( \Psi \) is the empty set \( \mathcal{L}(\mathcal{S}(G)) \) is the free partially commutative Lie algebra \( \mathcal{L}(G) \) associated with the graph \( G \)). Then we explain the Lyndon heaps basis of \( \mathcal{L}(G) \) due to Lalonde [31]: The set \( \{\Lambda(E) : E \in \mathcal{H}(I, \zeta) \} \) is a Lyndon heap (Basics definitions and results in the theory of heaps of pieces are given in Section 3.3).

Next, we identify a free root space \( g_{\eta(k)} \) of the BKM superalgebra \( g(A, \Psi) \) with the \( k \)-grade space \( \mathcal{L}(\mathcal{S}_k(G, \Psi)) \) of the free partially commutative Lie superalgebra \( \mathcal{L}(\mathcal{S}(G, \Psi)) \) associated with the supergraph \((G, \Psi)\). The exact statement is as follows: The root space \( g_{\eta(k)} \) can be identified with the grade space \( \mathcal{L}(\mathcal{S}_k(G)) \) of the free partially commutative Lie superalgebra \( \mathcal{L}(G) \). In particular, \( \text{mult} \eta(k) = \dim \mathcal{L}(\mathcal{S}_k(G)) \). This is our first main result. This observation plays a crucial role in giving an alternate proof of Theorem 2 when \( g \) is a Borcherds algebra [c.f. Section 4].

2. Step (1) shows that to construct a basis for a free root space \( g_{\eta(k)} \) of the BKM superalgebra \( g(A, \Psi) \) it is enough to extend the Lyndon heaps basis of the free partially commutative Lie algebra \( \mathcal{L}(G) \) to the case of free partially commutative Lie superalgebra \( \mathcal{L}(G) \). We introduce the notion of super Lyndon heaps and construct a (super) Lyndon heaps basis for \( \mathcal{L}(G) \) following the proof idea of Lalonde. The precise statement is as follows [c.f. Theorem 5]: The set \( \{\Lambda(E) : E \in \mathcal{H}(I, \zeta) \} \) is super Lyndon heap (Basics definitions and results in the theory of heaps of pieces are given in Section 3.3).

This is our second main result. This gives us Lyndon heaps basis for the free root spaces of BKM superalgebras.

3. Next, we extend Theorem 2 to the case of BKM superalgebras, namely we construct LLN basis for the free root spaces of BKM superalgebras. The main step is the identification of the combinatorial model \( C^k(G) \) given in Theorem 2 with the (super) Lyndon heaps of weight \( k \) over the graph \( G \). This gives a different proof (heap theoretic proof) to Theorem 2 when \( g \) is a Borcherds algebra. We are using neither the denominator identity nor Theorem 1 in our proof. We use only the identification of the spaces \( g_{\eta(k)} \) and \( \mathcal{L}(\mathcal{S}_k(G)) \), and the super
Lyndon heaps basis for free partially commutative Lie superalgebra $\mathcal{L}(G, \Psi)$ [c.f. Theorem 5]. In this sense, our proof is simpler and transparent. This is our third main result. We remark that the LLN basis and the Lyndon heaps basis of a free roots space $\mathfrak{g}_{\eta(k)}$ are different in general. The cases when the elements of these two bases are the same are discussed in Section 4.9.

(4) Next, along with various other combinatorial results on the free roots, we prove the following super analogue of Theorem 1 and its corollary. This is our final main result.

**Theorem 3.** Let $G$ be the quasi Dynkin diagram of a BKM superalgebra $\mathfrak{g}$. Assume that $k = (k_i : i \in I) \in \mathbb{Z}_+[I]$ satisfies the assumptions of Theorem 1 and in addition $k_i \leq 1$ for $i \in \Psi_0$. Then

$$
\pi^G_k(q) = (-1)^{ht(\eta(k))} \sum_{J \in L_G(k)} (-1)^{|J|+|J_1|} \prod_{J \in J_0} \left( q \mult(\beta(J)) D(J,J) \right) \prod_{J \in J_1} \left( -q \mult(\beta(J)) D(J,J) \right).
$$

where $L_G(k)$ is the bond lattice of weight $k$ of the graph $G$.

We have the following corollary to the above theorem which gives us a recurrence formula for the root multiplicities of free roots of $\mathfrak{g}$.

**Corollary 1.** We have

$$
\mult(\eta(k)) = \sum_{\ell | k} \frac{\mu(\ell)}{\ell} \left| \pi^{G}_{k/\ell}(q)[q] \right|
$$

if $\eta(k) = \sum_{i \in I} k_i \alpha_i \in \Delta^+_0$ and

$$
\mult(\eta(k)) = \sum_{\ell | k} \frac{(-1)^{l+1} \mu(\ell)}{\ell} \left| \pi^{G}_{k/\ell}(q)[q] \right|
$$

if $\eta(k) \in \Delta^+_1$ where $|\pi^{G}_{k}(q)[q]|$ denotes the absolute value of the coefficient of $q$ in $\pi^{G}_{k}(q)$ and $\mu$ is the Möbius function. See Example 11 for an working example of this formula.

If $k_i$’s are relatively prime (In particular if $k_i = 1$ for some $i \in I$) then the above formula becomes much simpler:

$$
\mult(\eta(k)) = |\pi^{G}_{k}(q)[q]|
$$

for any $\eta(k) \in \Delta^+$.

We also discuss why the expression given in Theorem 3 exists only for free roots explaining the main assumptions made in [49] and [1]. Various examples explaining our results are provided throughout the paper.

The paper is organized as follows. In Section 2, the definition and the basic results on the Borcherds Kac Moody Lie superalgebra are given. In Section 3, we construct the Lyndon heaps basis of free root spaces of BKM superalgebras. In Section 4, we construct the LLN basis of free root spaces of BKM superalgebras. In Section 5, we study the further combinatorial properties of free roots of BKM superalgebras.

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2. Structure theory of BKM superalgebras

In this section, we recall the basic properties and the denominator identity of BKM superalgebras from [50]. The theory of BKM superalgebras can also be seen in [40]. Our base field will be complex numbers throughout the paper.

2.1. Generators and Relations. Let \( I = \{1, 2, \ldots, n\} \) or the set of natural numbers. Fix a subset \( \Psi \) of \( I \) to describe the odd simple roots. A complex matrix \( A = (a_{ij})_{i,j \in I} \) together with a choice of \( \Psi \) is said to be a Borcherds-Kac-Moody supermatrix (BKM supermatrix in short) if the following conditions are satisfied: For \( i, j \in I \) we have

1. \( a_{ii} = 2 \) or \( a_{ii} \leq 0 \).
2. \( a_{ij} \leq 0 \) if \( i \neq j \).
3. \( a_{ij} = 0 \) if and only if \( a_{ji} = 0 \).
4. \( a_{ij} \in \mathbb{Z} \) if \( a_{ii} = 2 \).
5. \( a_{ij} \in 2\mathbb{Z} \) if \( a_{ii} = 2 \) and \( i \in \Psi \).
6. \( A \) is symmetrizable, i.e., \( DA \) is symmetric for some diagonal matrix \( D = \text{diag}(d_1, \ldots, d_n) \) with positive entries.

Denote by \( I^{\text{re}} = \{i \in I : a_{ii} = 2\} \), \( I^{\text{im}} = I \setminus I^{\text{re}} \), \( \Psi^{\text{re}} = \Psi \cap I^{\text{re}} \), and \( \Psi_0 = \{i \in \Psi : a_{ii} = 0\} \). The Borcherds-Kac-Moody Lie superalgebra (BKM superalgebra in short) associated with a BKM supermatrix \((A, \Psi)\) is the Lie superalgebra \( g(A, \Psi) \) (simply \( g \) when the presence of \( A \) and \( \Psi \) are understood) generated by \( e_i, f_i, h_i, i \in I \) with the following defining relations [50, Equations (2.10)-(2.13), (2.24)-(2.26)]:

1. \( [h_i, h_j] = 0 \) for \( i, j \in I \).
2. \( [h_i, e_j] = a_{ij}e_j \), \( [h_i, f_j] = -a_{ij}f_j \) for \( i, j \in I \).
3. \( [e_i, f_j] = \delta_{ij}h_i \) for \( i, j \in I \).
4. \( \deg h_i = 0 \), \( i \in I \).
5. \( \deg e_i = 0 \) \( \deg f_i \) if \( i \notin \Psi \).
6. \( \deg e_i = 1 \) \( \deg f_i \) if \( i \in \Psi \).
7. \( (\text{ad} e_i)^{1-a_{ij}}e_j = 0 = (\text{ad} f_i)^{1-a_{ij}}f_j \) if \( i \in I^{\text{re}} \) and \( i \neq j \).
8. \( (\text{ad} e_i)^{1-\frac{a_{ij}}{2}}e_j = 0 \) \( (\text{ad} f_i)^{1-\frac{a_{ij}}{2}}f_j \) if \( i \in I^{\text{re}} \) and \( i \neq j \).
9. \( (\text{ad} e_i)^{1-\frac{a_{ij}}{2}}e_j = 0 \) \( (\text{ad} f_i)^{1-\frac{a_{ij}}{2}}f_j \) if \( i \in \Psi_0 \) and \( i = j \).
10. \( [e_i, e_j] = 0 = [f_i, f_j] \) if \( a_{ij} = 0 \).

The relations (7), (8) and (9) are called the Serre relations of \( g \). We define \( I_j = \{i \in I : \deg e_i = j \} \) for \( j = 0, 1 \) and these sets will be identified with the set of even and odd simple roots of \( g \) respectively.

Remark 1. If \( \Psi \) is the empty set then \((A, \Psi)\) becomes a Borcherds Cartan matrix and the resulting Lie algebra \( g(A) \) is a Borcherds algebra. Assume that \( a_{ii} \neq 0 \). If \( i \in I \setminus \Psi \), then the Lie sub-superalgebra \( S_i = C f_i \oplus Ch_i \oplus C e_i \) of the BKM superalgebra \( g \) is isomorphic to \( \mathfrak{sl}_2 \) and if \( i \in \Psi \), then the Lie sub-superalgebra \( S_i = C [f_i, f_i] \oplus C f_i \oplus C h_i \oplus C e_i \) is isomorphic to \( \mathfrak{sl}(0, 1) \). If \( a_{ii} = 0 \), the Lie sub-superalgebra \( S_i = C f_i \oplus C h_i \oplus C e_i \) is isomorphic to the three dimensional Heisenberg Lie algebra (resp. superalgebra) if \( i \in I \setminus \Psi \) (resp. if \( i \in \Psi \)). The conditions (5) and (6) defines a \( \mathbb{Z}_2 \) gradation on \( g \) which makes it a Lie superalgebra. The presence of \( \mathfrak{sl}(0, 1) \) explains the appearances of even integers in the definition of BKM
supermatrix. This fact is also reflected in condition (8) of the defining relations of \( \mathfrak{g} \). This is one of the main structural differences between Borcherds algebras and the BKM superalgebras.

2.2. Quasi Dynkin diagram. First, we define the notion of a supergraph.

**Definition 1.** Let \( G \) be a finite/countably infinite simple graph with vertex set \( I \). Let \( \Psi \subseteq I \) be a subset of the vertex set. Then the pair \((G, \Psi)\) is said to be a supergraph and the vertices in \( \Psi \) (resp. \( I \setminus \Psi \)) are said to be odd (resp. even) vertices of \( G \). Let \( A \) be the classical adjacency matrix of the graph \( G \). Then the pair \((A, \Psi)\) is said to be the adjacency matrix of the supergraph \((G, \Psi)\).

**Definition 2.** The quasi Dynkin diagram of a BKM superalgebra is defined as follows [50, Section 2.1]. Let \((A = (a_{ij}), \Psi)\) be a BKM supermatrix and let \( \mathfrak{g} \) be the associated BKM superalgebra. The quasi Dynkin diagram of \( \mathfrak{g} \) is the supergraph \((G, \Psi)\) with vertex set \( I \) and two vertices \( i, j \in I \) are connected by an edge if and only if \( a_{ij} \neq 0 \). We often refer to \((G, \Psi)\) simply as the graph of \( \mathfrak{g} \). An example of a quasi Dynkin diagram of a BKM superalgebra \( \mathfrak{g} \) is given in Example 2.

**Remark 2.** We observe that the quasi Dynkin diagram of \( \mathfrak{g} \) is the underlying simple graph of the classical Dynkin diagram of \( \mathfrak{g} \) [50, Definition 2.4 above]. In other words, the quasi Dynkin diagram can be obtained from the Dynkin diagram of \( \mathfrak{g} \) by replacing all the multi edges with a single edge. In Section 5, we find a connection between root multiplicities of a BKM superalgebra \( \mathfrak{g}(A) \) and the chromatic polynomial of the associated supergraph \((G, \Psi)\). We will see that the chromatic polynomial of \( G \) depends on whether two vertices of \( G \) are adjacent or not but not on the actual number of edges connecting them. Therefore, we work with quasi Dynkin diagrams instead of the Dynkin diagrams.

For any subset \( S \subseteq \Pi \), we denote by \( |S| \) the number of elements in \( S \). The subgraph induced by the subset \( S \) is denoted by \( G_S \). We say a subset \( S \subseteq \Pi \) is connected if the corresponding subgraph \( G_S \) is connected. Also, we say \( S \) is independent if there is no edge between any two elements of \( S \), i.e., \( G_S \) is totally disconnected.

2.3. Root system and the Weyl group. Let \( \Delta \) be the root system of a BKM superalgebra \( \mathfrak{g} \) [50, Section 2.3]. Let \( \Pi \) the set of simple roots of \( \mathfrak{g} \). Define \( Q := \bigoplus_{\alpha \in \Pi} \mathbb{Z}\alpha, \quad Q_+ := \sum_{\alpha \in \Pi} \mathbb{Z}_+\alpha \).

**Definition 3.** An element \( \alpha = \sum_{i \in I} k_i\alpha_i \in Q_+ \) (or its weight \( k = (k_i : i \in I) \in \mathbb{Z}_+[I] \)) is said to be free if \( k_i \leq 1 \) for \( i \in I^{re} \cup \Psi_0 \).

**Remark 3.** In Lemma 1, we will show that any root space of a BKM superalgebra that corresponds to a free root is independent of (or free from) the Serre relations.

The set of positive roots is denoted by \( \Delta_+ := \Delta \cap Q_+ \). All the root spaces of \( \mathfrak{g} \) are finite dimensional and for any \( \alpha \in \mathfrak{h}^* \) either \( \mathfrak{g}_\alpha \subseteq \mathfrak{g}_0 \) or \( \mathfrak{g}_\alpha \subseteq \mathfrak{g}_1 \), i.e., every root is either even or odd. Set \( \Delta_+^l \) (resp. \( \Delta_+^l \)) to be the set of positive even (resp. odd) roots. We have a triangular decomposition \( \mathfrak{g} \cong n^- \oplus \mathfrak{h} \oplus n^+ \), where \( n^\pm = \bigoplus_{\alpha \in \pm \Delta_+} \mathfrak{g}_\alpha \). Given \( \gamma = \sum_{i \in I} k_i\alpha_i \in Q_+ \), we set \( ht(\gamma) := \sum_{i \in I} k_i \). The real vector space spanned by \( \Delta \) is denoted by \( R = \mathbb{R} \otimes \mathbb{Z}_+ Q \). For \( \alpha \in \Pi^e \), define the linear isomorphism \( s_\alpha \) of \( R \) by \( s_\alpha(\lambda) = \lambda - 2(\lambda, \alpha)\alpha, \quad \lambda \in R \). Note that the simple reflections are defined for odd real simple roots also. The Weyl group \( W \) of \( \mathfrak{g} \) is the subgroup
of GL(R) generated by the simple reflections \(s_\alpha, \alpha \in \Pi^e\). Note that the above bilinear form is \(W\)-invariant and \(W\) is a Coxeter group with canonical generators \(s_\alpha, \alpha \in \Pi^e\). Define the length of \(w \in W\) by \(\ell(w) := \min \{k \in \mathbb{N} : w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_k}}\}\) and any expression \(w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_k}}\) with \(k = \ell(w)\) is called a reduced expression. The set of real roots is denoted by \(\Delta^r = W(\Pi^e)\) and the set of imaginary roots is denoted by \(\Delta^i = \Delta \setminus \Delta^r\). Equivalently, a root \(\alpha\) is real if and only if \((\alpha, \alpha) > 0\) and else imaginary. Let \(\rho\) be any element of \(\mathfrak{h}^*\) satisfying \(2(\rho, \alpha) = (\alpha, \alpha)\) for all \(\alpha \in \Pi\).

2.4. Denominator identity of BKM superalgebras. Let \(\Omega\) be the set of all \(\gamma \in \mathbb{Q}_+\) such that

1. \(\gamma = \sum_{j=1}^r \alpha_{ij} + \sum_{k=1}^s l_{ik} \beta_{ik}\) where the \(\alpha_{ij}\) (resp. \(\beta_{ik}\)) are distinct even (resp. odd) imaginary simple roots,
2. \((\alpha_{ij}, \alpha_{ik}) = (\beta_{ij}, \beta_{ik}) = 0\) for \(j \neq k\); \((\alpha_{ij}, \beta_{ik}) = 0\) for all \(j, k\);
3. if \(l_{ik} \geq 2\), then \((\beta_{ij}, \beta_{ik}) = 0\).

The following denominator identity of BKM superalgebras is proved in [50, Section 2.6]:

\[
U := \sum_{w \in W} \sum_{\gamma \in \Omega} \epsilon(w) \epsilon(\gamma) e^{w(\rho - \gamma) - \rho} = \frac{\prod_{\alpha \in \Delta^r_0} (1 - e^{-\alpha})^{\text{mult}(\alpha)}}{\prod_{\alpha \in \Delta^r_+} (1 + e^{-\alpha})^{\text{mult}(\alpha)}}
\]  

(2.1)

where \(\text{mult}(\alpha) = \dim g_0, \epsilon(w) = (-1)^{\ell(w)}\) and \(\epsilon(\gamma) = (-1)^{|\text{ht}\gamma|}\).

Remark 4. If \(\Psi\) is the empty set then Equation (2.1) reduces to the denominator identity of the Borcherds algebras. Further, if \(I^r\) is also empty then Equation (2.1) reduces to the denominator identity of the Kac-Moody algebras.

3. Main result I: Lyndon basis of BKM superalgebras

In this section, we identify the free root spaces of a BKM superalgebra with the grade spaces of free partially commutative Lie superalgebra. Using this identification, we construct the Lyndon basis for the free root spaces of a BKM superalgebra \(g\). We start with the definition of free partially commutative Lie superalgebras.

3.1. Free partially commutative Lie superalgebra \(\mathcal{LS}(G, \Psi)\). Given a supergraph \((G, \Psi)\), the associated free partially commutative Lie superalgebra \(\mathcal{LS}(G, \Psi)\) is defined as follows. First, we define the free Lie superalgebra on a \(\mathbb{Z}_2\)-graded set.

Definition 4. Let \(I = I_0 \sqcup I_1\) be a non-empty superset (\(\mathbb{Z}_2\)-graded set). Let \(I^*\) be the free monoid generated by \(I\). A word \(w \in I^*\) is called even if the number of odd alphabets (i.e. the elements of \(I_1\)) in \(w\) is even otherwise it is called odd. This defines a \(\mathbb{Z}_2\)-gradation on \(I^*\).

Let \(V\) be the free super vector space on the set \(I\) and let \(T(V)\) be the tensor algebra on \(V\). The algebra \(T(V)\) has an induced \(\mathbb{Z}_2\)-gradation makes it an associative superalgebra for which \(I^*\) is a basis. Since \(T(V)\) is associative, it has a natural Lie superalgebra structure. Given this, the free Lie superalgebra on the superset \(I = I_0 \sqcup I_1\) is defined to be the smallest Lie subsuperalgebra of \(T(V)\) containing \(I\). We denote the free Lie superalgebra on the superset \(I\) by \(\mathcal{FLS}(I)\).
Remark 5. If $I_1$ is the empty set then $\mathcal{FLS}(I)$ is the free Lie algebra on set $I_0$. Whenever we talk about a free Lie superalgebra on a set $I = \{1, \ldots, n\}$ or $\{1, 2, \ldots\}$, we consider the elements of $I$ as $\{e_1, \ldots, e_n\}$ and $\{e_1, e_2, \ldots\}$ instead $I$. This way, we can relate the elements of $\mathcal{FLS}(I)$ with the elements of a free root spaces of a BKM superalgebra $g$.

Definition 5. Let $(G, \Psi)$ be a supergraph with the vertex set $I$ and the edge set $E$. Let $\mathcal{FLS}(I)$ be the free Lie superalgebra on the set $I = I_0 \sqcup I_1$ where we take $I_1$ to be $\Psi$. Let $J$ be the ideal in $\mathcal{FLS}(I)$ generated by the relations $\{e_i, e_j : \{i, j\} \notin E\}$ [c.f. Remark 5]. The quotient algebra $\mathcal{FLS}(I)/J$, denoted by $\mathcal{LS}(G, \Psi)$, is the free partially commutative Lie superalgebra associated with the supergraph $(G, \Psi)$. When $\Psi$ is the empty set, $\mathcal{LS}(G, \Psi)$ is the free partially commutative Lie algebra associated with the graph $G$ and is denoted by $\mathcal{L}(G)$. It is well-known that $\mathcal{FLS}(I)$ and hence $\mathcal{LS}(G, \Psi)$ is graded by $\mathbb{Z}_+[I]$.

3.2. Free partially commutative super monoid. Let $(G, \Psi)$ be a supergraph with a finite/countably infinite vertex set $I = I_0 \sqcup I_1$ (where $I_1 = \Psi$). We assume that $I$ is totally ordered. Let $I^*$ be the free monoid generated by $I$. We note that $I^*$ is totally ordered by the lexicographical order. The free partially commutative super monoid associated with a supergraph $(G, \Psi)$ is denoted by $M(I, G, \Psi) := I^*/\sim$, where $\sim$ is generated by the relations $ab \sim ba$, if $(a, b) \notin E(G)$. When $\Psi$ is empty, $M(I, G, \Psi)$ is called the free partially commutative monoid associated with the graph $G$ and denoted simply by $M(I, G)$. We observe that $M(I, G, \Psi)$ has a natural $\mathbb{Z}_2$-gradation induced from the $\mathbb{Z}_2$-gradation of $I^*$. We associate with each element $[a] \in M(I, G, \Psi)$ a unique element $\bar{a} \in I^*$ which is the maximal element in $[a]$ with respect to the lexicographical order. We call this element the standard word of the class $[a]$ and denoted by $\text{st}([a])$. A total order on $M(I, G, \Psi)$ is then given by

$$[a] < [b] := \text{st}([a]) < \text{st}([b]). \quad (3.1)$$

Next, we explain the Lyndon heaps basis of free partially commutative Lie algebras. For this reason, in the next subsection, we give the essential definitions from the theory of heaps of pieces to define pyramids and Lyndon heaps from [32].

3.3. Heaps monoid, Pyramids, and Lyndon heaps. Heaps of pieces were introduced by Xavier Viennot in [44]. He proved many combinatorial results on heaps of pieces and gave applications of heaps of pieces to a wide range of areas: directed animals, polyominoes, Motzkin paths, and orthogonal polynomials, Rogers-Ramanujan identities, fully commutative elements in Coxeter groups, Bessel functions, Lorentzian quantum gravity and many more applications in mathematical physics. In [2, 45, 34], special types of heaps namely pyramids are the important tools in proving results. Heaps of pieces have also applications in the representation theory of complex simple Lie algebras [14]. In this book, the combinatorial aspects of minuscule representations are studied using heaps of pieces. In [2], the connection between heaps of piece , chromatic polynomials and the free partially commutative Lie algebras is discussed along with many other combinatorial properties of heaps of pieces.

Let $(G, \Psi)$ be a supergraph with a (finite/countably infinite) totally ordered vertex set $I = \{\alpha_1, \ldots, \alpha_k\}$ or $I = \{\alpha_1, \alpha_2, \ldots\}$. Let $\zeta$ be the concurrency relation complement to the commuting relation $\sim$ on $I^*$ [c.f. Section 3.2]. A pre-heap $E$ over $(I, \zeta)$ is a finite subset of $I \times \{0, 1, 2, \ldots\}$ satisfying, if $(\alpha_1, m), (\alpha_2, n) \in E$ with $\alpha_1 \zeta \alpha_2$, then $m \neq n$. Each element $(\alpha, m)$ of $E$ is called a basic piece. If $(\alpha, m) \in E$, we write $\pi(\alpha, m) = \alpha$ (the position of the
piece \((\alpha, m)\) and \(h(\alpha, m) = m\) (the level of the piece \((\alpha, m)\)). A basic piece will be simply denoted by \(\alpha\) when we don’t need to emphasize the level. The set \(\pi(E)\) is defined to be the set of all positions occupied by the pieces of \(E\). A pre-heap \(E\) defines a partial order \(\leq_E\) by taking the transitive closure of the relation: \((\alpha_1, m) \leq_E (\alpha_2, n)\) if \(\alpha_1 \alpha_2 = m < n\). We say that two heaps \(E\) and \(F\) are isomorphic if there exists a position preserving order isomorphism \(\phi\) between \((E, \leq_E)\) and \((F, \leq_F)\). A heap \(E\) over \((I, \zeta)\) is a pre-heap over \((I, \zeta)\) such that: if \((\alpha, m) \in E\) with \(m > 0\) then there exists \(E\) such that \(\alpha \zeta \beta\). Every isomorphism class of pre-heaps contains exactly one heap and this is the unique pre-heap \(E\) in the class for which \(\sum_{\alpha \in E} h(\alpha)\) is minimal.

**Remark 6.** The graph \(G\) can have a countably infinite number of vertices, but each heap \(E\) over the graph \(G\) has only a finite number of pieces by definition. This fact leads to a natural \(\mathbb{Z}_+[I]\)-gradation on the set of all heaps over the graph \(G\).

Let \(\mathcal{H}(I, \zeta)\) be the set of all heaps over \((I, \zeta)\). This set can be made into a monoid with a product called the superposition of heaps. To get superposition \(E \circ F\) of \(F\) over \(E\), let the heap \(F\) ‘fall’ over \(E\). Let \(\mathcal{H}_k(I, \zeta)\) be the set of all heaps of weight \(k\) for \(k \in \mathbb{Z}_+[I]\) where the weight counts the number of pieces in each of the positions. This gives a \(\mathbb{Z}_+[I]\)-gradation on \(\mathcal{H}(I, \zeta)\). We define a map \(\psi: I^* \to \mathcal{H}(I, \zeta)\) as follows: For a word \(p_1 p_2 \cdots p_k \in I^*\) define \(\psi(p_1 p_2 \cdots p_k) = p_1 \circ p_2 \circ \cdots \circ p_k\). Note that \(\psi^{-1}(E)\) is the set of all linear orders compatible with \(\leq_E\). It is clear that \(\psi\) extends to weight and order-preserving isomorphism of the monoids \(M(I, G, \Psi)\) and \(\mathcal{H}(I, \zeta)\). This defines a total order on \(\mathcal{H}(I, \zeta)\). It also defines a \(\mathbb{Z}_2\)-gradation \(\mathcal{H}(I, \zeta) = \mathcal{H}_0(I, \zeta) \oplus \mathcal{H}_1(I, \zeta)\). The standard word of a heap \(E\) is defined to be \(\text{st}(E) = \text{st}(\psi^{-1}(E))\) [c.f. Equation (3.1)]. For a heap \(E\), \(\min E\) is the heap composed of minimal pieces of \(E\) with respect to \(\leq_E\) and \(\max E\) is defined similarly. We write \(|E|\) for the number of pieces in \(E\) and \(|E|_\alpha\) for the number of pieces of \(E\) in the position \(\alpha\). A heap \(E\) such that \(\min(E) = \{\alpha\}\) is said to be a pyramid with the basis \(\alpha\). The set of all pyramids in \(\mathcal{H}(I, \zeta)\) is denoted by \(\mathcal{P}(I, \zeta)\) and the set of all pyramids with basis \(\alpha_i\) is denoted by \(\mathcal{P}_i(I, \zeta)\).

Let \(E\) be a heap, we say that \(E\) is periodic if there exists a heap \(F \neq 0\) (empty heap) and an integer \(k \geq 2\) such that \(E = F^k\). Similarly, \(E\) is primitive if \(E = U \circ V = V \circ U\) then either \(U = 0\) or \(V = 0\). Pyramids in which the minimum piece has the lowest position (with respect to the total order on \(I\)) are known as admissible pyramids. A pyramid \(E\) with the basis \(p\) such that \(|E|_p = 1\) is said to an elementary pyramid. An admissible pyramid that is also elementary is known as a super-letter. The set of all super-letters in \(\mathcal{H}(I, \zeta)\) is denoted by \(\mathcal{A}(I, \zeta)\). A heap \(E\) in \(\mathcal{H}(I, \zeta)\) is said to be multilinear if every basic piece occurs exactly once in \(E\).

Let \(E\) be a heap. If \(E = U \circ V\) for some heaps \(U\) and \(V\), we say that \(V \circ U\) is a transpose of \(E\). The transitive closure of transposition is an equivalence relation on \(\mathcal{H}(I, \zeta)\), which we call the conjugacy relation of heaps and is denoted by \(\sim_c\). A non-empty heap \(E\) is said to be Lyndon if \(E\) is primitive and minimal in its conjugacy class. We write \(\mathcal{LH}(I, \zeta)\) for the set of all Lyndon heaps over the super graph \((G, \Psi)\).

**Remark 7.** According to Viennot, a pyramid is a heap with a unique maximal piece [44, Definition 5.9]. In this paper, we follow Lalonde’s convention on pyramids [32], i.e., A pyramid is a heap with a unique minimal piece.
Given the definition of Lyndon heaps, in the next subsection, we explain the Lyndon heaps basis of free partially commutative Lie algebras.

3.4. Lyndon heaps basis of free partially commutative Lie algebras. In this subsection, we recall the Lyndon heaps basis of Lalonde from [31]. If $E$ is a Lyndon heap then the standard factorization $\sum(E)$ of $E$ is given by $\sum(E) = (F, N)$, where

1. $F \neq 0$ (empty heap)
2. $E = F \circ N$
3. $N$ is Lyndon
4. $N$ is minimal in the total order on $H(I, \zeta)$.

To each Lyndon heap $E \in H(I, \zeta)$ we associate a Lie monomial $\Lambda(E)$ in $L(G)$ as follows. If $E \in I$, then $\Lambda(E) = E$ and otherwise $\Lambda(E) = [\Lambda(F_1), \Lambda(F_2)]$, where $\sum(E) = (F_1, F_2)$ is the standard factorization of $E$. Given these notions, we have the following theorem which gives the Lyndon basis of the free partially commutative Lie algebra $L(G)$.

Theorem 4. [31] The set $\{\Lambda(E) : E \in H(I, \zeta)\}$ forms a basis of $L(G)$.

3.5. The identification of the spaces $g_{\eta(k)}$ and $LS_k(G)$. Let $g$ be a BKM superalgebra with the associated supergraph $(G, \Psi)$. Let $LS(G, \Psi)$ be the free partially commutative Lie superalgebra associated with the supergraph $(G, \Psi)$. Let $I$ be the vertex set of $G$. Fix $k \in Z^+[I]$ such that $k_i \leq 1$ for $i \in I^{re} \cup \Psi_0$. In this subsection, we claim that there is a natural vector space isomorphism between the root space $g_{\eta(k)}$ of $g$ and the grade space $LS_k(G, \Psi)$ of $LS(G, \Psi)$.

The precise statement is the following.

Lemma 1. Fix $k \in Z^+[I]$ such that $k_i \leq 1$ for $i \in I^{re} \cup \Psi_0$. Then

1. The root space $g_{\eta(k)}$ can be identified with the grade space $LS_k(G)$ of the free partially commutative Lie superalgebra $LS(G, \Psi)$. In particular, $\mult \eta(k) = \dim LS_k(G, \psi)$.
2. The root space $g_{\eta(k)}$ is independent of the Serre relations.

Proof. The positive part $n_+$ of $g$ can be written as $\Big( \bigoplus_{\alpha \in \Delta_+} g_{\alpha} \big) \bigsqcup \Big( \bigoplus_{\alpha \in \Delta_+} g_{\alpha} \Big)$. From the defining relations (relation (9)) of $g$, there is a natural grade preserving surjection $\phi$ from $LS(G, \Psi)$ to $n_+$. Further, by the defining relations (7), (8) and (9), the kernel of this map is generated by the elements

$$(\ad e_i)^{1-a_{ij}} e_j \text{ if } i \in I^{re} \text{ and } i \neq j,$$

$$(\ad e_i)^{1-a_{ij}} e_j \text{ if } i \in \Psi^{re} \text{ and } i \neq j, \text{ and}$$

$$(\ad e_i)^{1-a_{ij}} e_j \text{ if } i \in \Psi_0 \text{ and } i = j$$

of $LS(G, \Psi)$. We observe that in all these elements some $e_i$’s (corresponding to a real simple root or an odd simple root of norm zero) are occurring at least twice. Since $\phi$ preserves the grading, by our assumption on $k$, the grade space $LS_k(G, \Psi)$ is injectively mapped onto the free root space $g_{\eta(k)}$ of $g$. This completes the proof. \[\square\]
3.6. Super Lyndon heaps and the standard factorization. Let \( g \) be a BKM superalgebra with the associated supergraph \((G, \Psi)\). In Theorem 4, we saw the Lyndon heaps of the free partially commutative Lie algebra \( \mathcal{L}(G) \). We construct a similar basis for the free root spaces of the BKM superalgebra \( g \). Given Lemma 1, to construct such a basis it is enough to extend Theorem 4 to the case of free partially commutative Lie superalgebra associated with the supergraph \((G, \Psi)\). This extension is the main result of this section [c.f. Theorem 5]. This is done by introducing and studying the combinatorial properties of super Lyndon heaps.

Definition 6. Let \( I = I_0 \cup I_1 \) be a non-empty set. Let \((G, \Psi)\) be a supergraph with vertex set \( I \) and \( I_1 = \Psi \). A heap \( E \in \mathcal{H}(I, \zeta) = \mathcal{H}_0(I, \zeta) \oplus \mathcal{H}_1(I, \zeta) \) (heap monoid over the supergraph \((G, \Psi)\)) is said to be a super Lyndon heap if \( E \) satisfies one of the following conditions:

- \( E \) is a Lyndon heap.
- \( E = F \circ F \) where \( F \in \mathcal{H}_1(I, \zeta) \) is Lyndon.

The set of all super Lyndon heaps over the supergraph \((G, \Psi)\) is denoted by \( \mathcal{SLH}(I, \zeta) \).

Example 1. A super Lyndon heap over the path graph on 4 vertices with \( I_1 = \{\alpha_1, \alpha_2, \alpha_3\} \) and \( I_0 = \{\alpha_4\} \) is the following.

\[
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{array}
\]

This is an example of a super Lyndon heap \( E = 123123 \) with \( F = 123 \) is a Lyndon heap in \( \mathcal{H}_1(I, \zeta) \).

Let \( E \) be a super Lyndon heap in \( \mathcal{H}(I, \zeta) \). Assume that \( E = F \circ F \) with \( F \) is a Lyndon heap in \( \mathcal{H}_1(I, \zeta) \). We define the standard factorization of \( E \) to be \( \Sigma(E) = (F, F) \). To each super Lyndon heap \( E \in \mathcal{H}(I, \zeta) \) we associate a Lie word \( \Lambda(E) \) in \( \mathcal{LS}(G, \Psi) \) as follows. If \( E \in I \), then \( \Lambda(E) = E \) and otherwise \( \Lambda(E) = [\Lambda(F_1), \Lambda(F_2)] \), where \( E = F_1 \circ F_2 \) is the standard factorization of \( E \). Given these notions, we have our following theorem which gives a basis for free partially commutative Lie superalgebra \( \mathcal{LS}(G, \Psi) \).

3.7. Super Lyndon heaps basis of free partially commutative Lie superalgebras. The following theorem is the main result of this subsection in which we construct the Lyndon heaps basis for free partially commutative Lie superalgebras.

Theorem 5. The set \( \{\Lambda(E) : E \in \mathcal{H}(I, \zeta) \text{ is super Lyndon}\} \) forms a basis of \( \mathcal{LS}(G, \Psi) \).

The rest of the section is dedicated to the proof of the above theorem. The proof of the following lemma is immediate.
Lemma 2. Let $E \in \mathcal{SH}_k(I, \zeta)$. Then $\Lambda(E) = \sum_{F \in \mathcal{SH}_k(I, \zeta)} \alpha_F F$ where $\alpha_F \in \mathbb{Z}$. Since there are finite number of heaps of degree $k$, the sum is a finite sum.

Proposition 1. The set $\mathcal{H}(I, \zeta)$ indexes a basis for the universal enveloping algebra of the free partially commutative Lie superalgebra $\mathcal{LS}(G, \Psi)$.

Proof. Let $\mathfrak{U}$ be the $\mathbb{C}$-span of the heaps monoid $\mathcal{H}(I, \zeta)$ associated with the supergraph $(G, \Psi)$. Then $\mathfrak{U}$ has an algebra structure induced from the multiplication in $\mathcal{H}(I, \zeta)$. This is the free partially commutative superalgebra associated with the supergraph $(G, \Psi)$. This is the smallest associative superalgebra containing $\mathcal{LS}(G)$. Therefore $\mathfrak{U}$ is the universal enveloping algebra of the Lie superalgebra $\mathcal{LS}(G, \Psi)$. $\square$

Proposition 2. Let $L$ be a super Lyndon heap of weight $k$ over the supergraph $(G, \Psi)$. Put $\Lambda(L) = \sum_{E \in \mathcal{SH}_k(I, \zeta)} \alpha_E E$. Then

(i) $\alpha_L = 1$ if $L$ is a Lyndon heap
(ii) $\alpha_L = 2$ if $L = L_1 \circ L_1$, $L_1$ is Lyndon heap in $\mathcal{H}_1(I, \zeta)$
(iii) If $\alpha_E \neq 0$ then $E \geq L$.

Proof. If $E$ is a Lyndon heap, then the proofs of (i) and (iii) are given in [31, Theorem 4.2]. So we prove (ii) and (iii) when $L$ is a super Lyndon heap. Let $L = L_1 \circ L_1$ where $L_1$ is Lyndon heap in $\mathcal{H}_1(I, \zeta)$. Now,

$$\Lambda(L) = [\Lambda(L_1), \Lambda(L_1)]$$

$$= \left[ \sum_{E \geq L_1} \alpha_E E, \sum_{E' \geq L_1} \alpha_{E'} E' \right] \quad \text{(Using part(i) and (iii)) for Lyndon heaps)$$

$$= [L_1, L_1] + \sum_{E > L_1 \ , \ E' > L_1} \alpha_E \alpha_{E'} [E, E'] + \sum_{E > L_1} \alpha_E [E, L_1] + \sum_{E' > L_1} \alpha_{E'} [L_1, E']$$

$$= 2L + \sum_{K > L} \alpha_K K.$$ 

This proves (ii). Now, $st(E \circ E') \geq st(E) \cdot st(E') > st(L_1) \cdot st(L_1) =: st(L_1 \circ L_1) \Rightarrow E \circ E' > L$. Similarly, we have $E' \circ E > L$. Also, $E \circ L_1 > L, L_1 \circ E > L$. Hence (iii) follows. $\square$

Corollary 2. The set $\mathcal{B} = \{ \Lambda(L) : L \text{ is a super Lyndon heap} \}$ is linearly independent in $\mathcal{LS}(G, \Psi)$.

Proof. Assume that

$$\sum_{L \in \mathcal{B}} \beta_L \Lambda(L) = 0, \quad \beta_L \in \mathbb{C}$$

where all but finitely many $\beta_L$ are zero. Then by the above proposition, we have

$$\sum_{L \in \mathcal{B}} \beta_L \left( \sum_{E \geq L \ , \ \text{wt}(E) = \text{wt}(L)} \alpha_E E \right) = 0$$
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\[ \Rightarrow \sum_{L \in \mathcal{B}} \beta_L \left( \alpha_L L + \sum_{E > L, \text{wt}(E) = \text{wt}(L)} \alpha_E E \right) = 0 \]

where \( \alpha_L = \begin{cases} 1 & \text{if } L \in \mathcal{LH}(I, \zeta) \\ 2 & \text{if } L = E \circ E, E \in \mathcal{H}_1(I, \zeta) \end{cases} \)

\[ \Rightarrow \sum_{L \in \mathcal{LH}(I, \zeta)} \beta_L L + 2 \sum_{L = E \circ E, E \in \mathcal{LH}_1(I, \zeta)} \beta_L L + \sum_{L \in \mathcal{B}} \sum_{E > L, \text{wt}(E) = \text{wt}(L)} \beta_L \alpha_E E = 0 \]

Taking modulo in the grade space \( \mathcal{L} \mathcal{S}_k(G, \Psi) \) we get

\[
\begin{cases}
\sum_{L \in \mathcal{LH}_k(I, \zeta)} \beta_L L + \sum_{E > L, \text{wt}(E) = \text{wt}(L), E \in \mathcal{S}\mathcal{LH}_k(I, \zeta)} \beta_L \alpha_E E = 0 & \text{if } k_i \text{ is odd for some } i \in \text{supp}(k) \\
\sum_{L \in \mathcal{LH}_k(I, \zeta)} \beta_L L + 2 \sum_{L = E \circ E, E \in \mathcal{LH}_1(I, \zeta)} \beta_L L + \sum_{E > L, \text{wt}(E) = \text{wt}(L), L \in \mathcal{S}\mathcal{LH}_k(I, \zeta)} \beta_L \alpha_E E = 0 & \text{otherwise}
\end{cases}
\]

which are finite linear combinations of heaps of weight \( k \) in the universal enveloping algebra \( \mathfrak{U} \) of \( \mathcal{L} \mathcal{S}(G, \Psi) \). Since heaps form a basis of \( \mathfrak{U} = \mathbb{C}(\mathcal{H}(I, \zeta)) \) we get all the \( \beta_L = 0 \) in the above equations. Since \( k \) is arbitrary we get \( \beta_L = 0 \) for all \( L \in \mathcal{B} \). This completes the proof.

**Proposition 3.** Let \( L \) and \( M \) be super Lyndon heaps such that \( L < M \). Then we can write

\[ [\Lambda(L), \Lambda(M)] = \sum_{N \in \mathcal{S}\mathcal{LH}(I, \zeta), N < M, \deg(N) = \deg(L \circ M)} \alpha_N \Lambda(N). \]

**Proof.** The proof is by case by case analysis.

Case (i):- Suppose \( L, M \) are Lyndon heaps satisfying \( L < M \) then result follows from [31, Theorem 4.4].

Case (ii):- Suppose exactly one of \( L, M \) is a super Lyndon heap. Without loss of generality assume that \( L = L_1 \circ L_1 \) where \( L_1 \) is Lyndon heap in \( \mathcal{H}_1(I, \zeta) \) and \( M \) is an arbitrary Lyndon heap. Now,
\[ [\Lambda(L), \Lambda(M)] = [[\Lambda(L_1), \Lambda(L_1)], \Lambda(M)] \]
\[ = 2[\Lambda(L_1), [\Lambda(L_1), \Lambda(M)]] \] (3.2)
\[ = 2[\Lambda(L_1), \sum_{N_1 \leq M \text{ such that } \deg(N_1) = \deg(L_1 \circ M)} \alpha_{N_1} \Lambda(N_1)] (\because L_1 \circ L_1 = L < M) \] (3.3)
\[ = 2 \sum_{N_1 \in \mathcal{L}(I, \zeta), N_1 < M \text{ such that } \deg(N_1) = \deg(L_1 \circ M)} \alpha_{N_1} [\Lambda(L_1), \Lambda(N_1)] \] (3.4)
\[ = 2 \left( \sum_{N_1 \in \mathcal{L}(I, \zeta), N_1 < M \text{ such that } \deg(N_1) = \deg(L_1 \circ M)} \alpha_{N_1} [\Lambda(L_1), \Lambda(N_1)] + \sum_{N_1 \in \mathcal{L}(I, \zeta), N_1 < M \text{ such that } \deg(N_1) = \deg(L_1 \circ M)} \alpha_{N_1} [\Lambda(L_1), \Lambda(N_1)] \right) \] (3.5)

Using case (i) in the first term of the above equation we get

\[ \sum_{N_1 \in \mathcal{L}(I, \zeta), N_1 < M \text{ such that } \deg(N_1) = \deg(L_1 \circ M)} \alpha_{N_1} [\Lambda(L_1), \Lambda(N_1)] = \sum_{N_1 \in \mathcal{L}(I, \zeta), N_1 < M \text{ such that } \deg(N_1) = \deg(L_1 \circ M)} \alpha_{N_1} \left( \sum_{N_2 \in \mathcal{L}(I, \zeta), N_2 < M \text{ such that } \deg(N_2) = \deg(L_1 \circ M)} \alpha_{N_2} \Lambda(N_2) \right) \]
\[ = \sum_{N_2 \in \mathcal{L}(I, \zeta), N_2 < M \text{ such that } \deg(N_2) = \deg(L_1 \circ M)} \left( \sum_{N' \in \mathcal{L}(I, \zeta), N_2 < N' < M \text{ such that } \deg(N') = \deg(L_1 \circ M)} \alpha_{N'} \right) \alpha_{N_2} \Lambda(N_2) \]
\[ = \sum_{N_2 \in \mathcal{L}(I, \zeta), N_2 < M \text{ such that } \deg(N_2) = \deg(L_1 \circ M)} \left( \sum_{N' \in \mathcal{L}(I, \zeta), N_2 < N' < M \text{ such that } \deg(N') = \deg(L_1 \circ M)} \alpha_{N'} \right) \alpha_{N_2} \Lambda(N_2) \]
\[ = \sum_{N_2 \in \mathcal{L}(I, \zeta), N_2 < M \text{ such that } \deg(N_2) = \deg(L_1 \circ M)} \left( \sum_{N' \in \mathcal{L}(I, \zeta), N_2 < N' < M \text{ such that } \deg(N') = \deg(L_1 \circ M)} \alpha_{N'} \right) \alpha_{N_2} \Lambda(N_2) \]

For the second summation, \[ [\Lambda(L_1), \Lambda(N_1)] = -(-1)^{a_{N_1} b_{L_1}} [\Lambda(N_1), \Lambda(L_1)] \text{ where } a_{N_1}, b_{L_1} \in \{0,1\} \text{ according to } N_1, L_1 \in \mathcal{H}_i(I, \zeta) \text{ for } i \in \{0,1\}. \] Therefore,
\[ [\Lambda(L_1), \Lambda(N_1)] = -(-1)^{a_{N_1} b_{L_1}} \sum_{K \in \mathcal{H}(I, \zeta), K < L_1 < M \text{ such that } \deg(K) = \deg(N_1 \circ L_1) = \deg(L_1 \circ M)} \alpha_K \Lambda(K) \]
\[
\sum_{N_1 \in \mathcal{LH}(I, \zeta), \deg(N_1) = \deg(L_1 \circ M), L_1 > N_1} \alpha_{N_1} \left[ \Lambda(L_1), \Lambda(N_1) \right]
\]

\[
\sum_{N_1 \in \mathcal{LH}(I, \zeta), \deg(N_1) = \deg(L_1 \circ M)} \alpha_{N_1} \left[ \Lambda(L_1), \Lambda(N_1) \right]
\]

\[
- \sum_{N_1 \in \mathcal{LH}(I, \zeta), \deg(N_1) = \deg(L_1 \circ M)} \alpha_{N_1} \left( (-1)^{a_{N_1} b_{L_1}} \sum_{K \in \mathcal{LH}(I, \zeta), \deg(K) = \deg(N_1 \circ L_1) = \deg(L \circ M)} \alpha_K \Lambda(K) \right)
\]

\[
- \sum_{K \in \mathcal{LH}(I, \zeta), \deg(K) = \deg(L \circ M)} \left( \sum_{N' \in \mathcal{LH}(I, \zeta), \deg(N') = \deg(L \circ M)} (-1)^{a_{N'} b_{L_1} a_{N'}} \right) \alpha_K \Lambda(K)
\]

\[
\sum_{K \in \mathcal{LH}(I, \zeta), \deg(K) = \deg(L \circ M)} \alpha'_K \Lambda(K) \quad \text{where} \quad \alpha'_K = -c_K \alpha_K
\]

\[
\Rightarrow [\Lambda(L), \Lambda(M)] = 2 \left( \sum_{N_2 \in \mathcal{LH}(I, \zeta), \deg(N_2) = \deg(L \circ M)} (c_{N_2} a_{N_2}) \Lambda(N_2) + \sum_{K \in \mathcal{LH}(I, \zeta), \deg(K) = \deg(L \circ M)} \alpha'_K \Lambda(K) \right)
\]

Case (iii): Suppose \( L = L_1 \circ L_1, M = M_1 \circ M_1 \) where \( L_1, M_1 \) are Lyndon heaps in \( \mathcal{H}_1(I, \zeta) \) satisfying \( L < M \). Then

\[
\left[ \Lambda(L), \Lambda(M) \right] = \left[ \left[ \Lambda(L_1), \Lambda(L_1) \right], \Lambda(M) \right]
\]

\[
= 2 \left[ \Lambda(L_1), \left[ \Lambda(L_1), \Lambda(M) \right] \right]
\]

\[
= 2 \left[ \Lambda(L_1), \sum_{N \in \mathcal{LH}(I, \zeta), \deg(N) = \deg(L_1 \circ M)} \alpha_N \Lambda(N) \right] \quad \text{(by the previous case)}
\]

\[
= 2 \sum_{N \in \mathcal{LH}(I, \zeta), \deg(N) = \deg(L_1 \circ M)} \alpha_N \left[ \Lambda(L_1), \Lambda(N) \right]
\]
\[
= 2 \left( \sum_{N \in \mathcal{LH}(I, \zeta)} \alpha_N[\Lambda(L_1), \Lambda(N)] + \sum_{N \in \mathcal{LH}(I, \zeta)} \alpha_N[\Lambda(L_1), \Lambda(N)] \right)
\]

For those \( N \in \mathcal{LH}(I, \zeta) \) such that \( L_1 < N \) then by first case

\[
[\Lambda(L_1), \Lambda(N)] = \sum_{K \in \mathcal{LH}(I, \zeta)} \beta_K \Lambda(K)
\]

\[
\Rightarrow \sum_{N \in \mathcal{LH}(I, \zeta)} \alpha_N[\Lambda(L_1), \Lambda(N)] = \alpha_N \left( \sum_{N \in \mathcal{LH}(I, \zeta)} \beta_K \Lambda(K) \right)
\]

\[
= \sum_{K \in \mathcal{LH}(I, \zeta)} \alpha'_K \Lambda(K)
\]

For those \( N \in \mathcal{LH}(I, \zeta) \) such that \( L_1 > N \) then

\[
[\Lambda(L_1), \Lambda(N)] = -[\Lambda(N), \Lambda(L_1)]
\]

\[
= - \sum_{N \in \mathcal{LH}(I, \zeta)} \beta_{K_2} \Lambda(K_2)
\]

\[
\Rightarrow \sum_{N \in \mathcal{LH}(I, \zeta)} \alpha_N[\Lambda(L_1), \Lambda(N)] = \alpha_N \left( \sum_{K \in \mathcal{LH}(I, \zeta)} \sum_{K_2 \in \mathcal{LH}(I, \zeta)} \beta_{K_2} \Lambda(K_2) \right)
\]

\[
= \sum_{K_2 \in \mathcal{LH}(I, \zeta)} \alpha'_{K_2} \Lambda(K_2)
\]

\[
\Rightarrow [\Lambda(L), \Lambda(M)] = 2 \left( \sum_{K \in \mathcal{LH}(I, \zeta)} \alpha'_K \Lambda(K) + \sum_{K_2 \in \mathcal{LH}(I, \zeta)} \alpha'_{K_2} \Lambda(K_2) \right)
\]

This completes the proof.
By the above proposition, the Lie subsuperalgebra generated by $B = \{A(L) : L \text{ is super Lyndon heap}\}$ in $\mathcal{LS}(G, \Psi)$ contains $I$. So this subalgebra is equal to $\mathcal{LS}(G, \Psi)$. This completes the proof of Theorem 5 and in turn, gives the Lyndon basis for the free roots spaces of BKM superalgebra $g$ whose associated supergraph is $(G, \Psi)$ [c.f Lemma 1].

Example 2. Consider the BKM superalgebra $g$ associated with the BKM supermatrix

$$A = \begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & -3 & -4 & -1 & 0 & 0 \\
0 & -4 & -4 & 0 & 0 & -1 \\
0 & -1 & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & -2 & 0 \\
0 & 0 & -1 & 0 & 0 & -3
\end{bmatrix}.$$

The quasi-Dynkin diagram $G$ of $g$ is as follows:

We have $I = \{1, 2, 3, 4, 5, 6\}, \Psi = \{3, 5\}$ and $I^{re} = \{1, 4\}$. Assume the natural total order on $I$. Let $k = (0, 0, 3, 0, 0, 3) \in \mathbb{Z}_+^I$. Then $\eta(k) = 3\alpha_3 + 3\alpha_6 \in \Delta^+_+$. Fix $i = 3$ (minimal element in the support of $k$), then the super Lyndon heaps of weight $\eta(k)$ are $\{336636, 333666, 336366\}$ with standard factorization $3366|36, 3|33666$ and $3|36366$. The associated Lie monomials

$$\{(3, [3, 6], 6), [3, 6], [3, 3, [3, 6], 6], [3, [3, 6], [3, 6], 6]\}$$

spans $g_{\eta(k)}$. We have $\text{mult}(\eta(k)) = 3$ [c.f. Example 10]. So these Lie monomials form a basis for $g_{\eta(k)}$.

Example 3. Consider the BKM superalgebra $g$ associated with the BKM supermatrix

$$A = \begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & -3 & -4 & -1 & 0 & 0 \\
0 & -2 & -4 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & -2 & -1 \\
0 & 0 & 0 & 0 & -1 & -3
\end{bmatrix}.$$

The quasi-Dynkin diagram $G$ of $g$ is as follows:

We have $I = \{1, 2, 3, 4, 5, 6\}, \Psi = \{3, 5\}, I^{re} = \{1, 4\}, \eta(k) = 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$. Assume the natural total order on $I$. Fix $i = 3$, observe that the Lyndon heaps of weight $\eta(k)$ are
\{33456, 334565\} with standard factorizations \{3\{34565, 3\{34556\} respectively. The associated Lie monomials are
\[[3, [3, [4, [[5, 6], 5]]]], [3, [3, [4, [5, 6]]]]\}
which form a spanning set of \( g_{\eta(k)} \). Since \( \text{mult}(\eta(k)) = 2 \) [c.f. Example 11]. These lie monomials form a basis of \( g_{\eta(k)} \).

4. Main result II: LLN basis of BKM superalgebras

Let \( \mathfrak{g} \) be the BKM Lie superalgebra associated with a Borcherds–Cartan matrix \((A, \Psi)\). Let \((G, \Psi)\) be the quasi-Dynkin diagram of \( \mathfrak{g} \) with the vertex set \( I \) [c.f. Definition 2]. In this section, we extend Theorem 2 to the case of free root spaces of \( \mathfrak{g} \).

In what follows in this section, we use the super-Jacobi identity (up to sign) to prove our results. Also, whenever we fix an \( i \in I \), it is assumed that \( i \) is the least element in the total order of \( I \).

4.1. Initial alphabet and Left normed Lie word associated with a word. Let \( \mathfrak{w} \in M(I, G) \) [c.f. Section 3.2] and \( \mathfrak{w} = a_1 \cdots a_r \) be an element in the class \( \mathfrak{w} \). We define the length of the word \( |\mathfrak{w}| = r \). We define, \( i(\mathfrak{w}) = |\{j : a_j = i\}| \) and \( \text{supp}(\mathfrak{w}) = \{i \in I : i(\mathfrak{w}) \neq 0\} \).

We define the weight of \( \mathfrak{w} \) to be \( \text{wt}(\mathfrak{w}) = \sum_{i \in I} i(\mathfrak{w}) \alpha_i \). The following definition of the initial alphabet is different from the one given in [1, Section 4]. Our definition is compatible with the definition of pyramids given in Section 3.3 and pyramids will be the main tool in this section.

For \( i \in I \), its initial multiplicity in \( \mathfrak{w} \) is defined to be the largest \( k \geq 0 \) for which there exists \( \mathfrak{u} \in M(I, G) \) such that \( \mathfrak{w} = i^k \mathfrak{u} \). We define the initial alphabet \( \text{IA}_m(\mathfrak{w}) \) of \( \mathfrak{w} \) to be the multiset in which each \( i \in I \) occurs as many times as its initial multiplicity in \( \mathfrak{w} \). The underlying set is denoted by \( \text{IA}(\mathfrak{w}) \). The left normed Lie word associated with \( \mathfrak{w} \) is defined by
\[
\epsilon(\mathfrak{w}) = [[a_1, a_2], a_3] \cdots , a_{r-1}]a_r \in \mathfrak{g}.
\]

Using the Jacobi identity, it is easy to see that the association \( \mathfrak{w} \mapsto \epsilon(\mathfrak{w}) \) is well-defined and preserves the \( \mathbb{Z}_2 \)-grading.

4.2. Lyndon words and their Standard factorization. For a fixed \( i \in I \) (which is assumed to be minimal in the total order on \( I \)), consider the set
\[
\mathcal{X}_i = \{ \mathfrak{w} \in M(I, G) : \text{IA}_m(\mathfrak{w}) = \{i\} \text{ and } i(\mathfrak{w}) = 1\}.
\]

Observe that \( \mathcal{X}_i \) (and hence \( \mathcal{X}_i^* \)) is \( \mathbb{Z}_2 \)-graded and also totally ordered using (3.1). We denote by \( \text{FLS}(\mathcal{X}_i) \) the free Lie superalgebra generated by \( \mathcal{X}_i = \mathcal{X}_{i,0} \sqcup \mathcal{X}_{i,1} \) where \( \mathcal{X}_{i,0} \) (resp. \( \mathcal{X}_{i,1} \)) is the set of even (resp. odd) elements in \( \mathcal{X}_i \). A non-empty word \( \mathfrak{w} \in \mathcal{X}_i^* \) is called a Lyndon word if it satisfies one of the following equivalent definitions:

- \( \mathfrak{w} \) is strictly smaller than any of its proper cyclic rotations.
- \( \mathfrak{w} \in \mathcal{X}_i \) or \( \mathfrak{w} = uv \) for Lyndon words \( \mathfrak{u} \) and \( \mathfrak{v} \) with \( \mathfrak{u} < \mathfrak{v} \).

There may be more than one choice of \( \mathfrak{u} \) and \( \mathfrak{v} \) with \( \mathfrak{w} = uv \) and \( \mathfrak{u} < \mathfrak{v} \) but if \( \mathfrak{v} \) is of maximal possible length we call it the standard factorization. Equivalently, we can define \( \mathfrak{v} \) to be the minimal Lyndon word in the lexicographic order satisfying \( \mathfrak{w} = uv \). This is called the standard factorization of \( \mathfrak{w} \) and denoted by \( \sigma(\mathfrak{w}) = (\mathfrak{u}, \mathfrak{v}) \). Note that, when \( G \) is a complete graph, the
heap monoid $\mathcal{H}(I, \zeta)$ is isomorphic to the free monoid $I^*$. Further, the standard factorization $\Sigma(E)$ of a Lyndon heap $E \in \mathcal{H}(I, \zeta)$ coincides with the factorization $\sigma(E)$.

4.3. Super Lyndon words and their associated Lie word. Next, we recall the definition and some properties of super Lyndon words and state the main theorem of this section, we define a word $w \in \mathcal{X}_i^*$ to be super Lyndon if $w$ satisfies one of the following conditions [5]:

- $w$ is a Lyndon word.
- $w = uu$ where $u \in \mathcal{X}_{i,1}^*$ is Lyndon. In this case, we define $w = uu$ is the standard factorization of $w$.

In [35], super Lyndon words are known as $s$–regular words. We will use super Lyndon words (resp. super Lyndon words) to construct a basis for the Borcherds algebras (resp. BKM Lie superalgebras). To each super Lyndon word $w \in \mathcal{X}_i^*$ we associate a Lie word $L(w)$ in $\mathcal{F}_i$ as follows. If $w \in \mathcal{X}_i$, then $L(w) = w$ and otherwise $L(w) = [L(u), L(v)]$ where $w = uv$ is the standard factorization of $w$. If $\mathcal{X}_{i,1}$ is empty then the map $L$ assigns Lie monomials in free Lie algebras to Lyndon words. For more details about Lyndon words and super Lyndon words, we refer the readers to [6, 35, 42]. The following result can be seen in [6, 35] and the basis constructed is known as the Lyndon basis for free Lie superalgebras.

**Proposition 4.** The set $\{L(w) : w \in \mathcal{X}_i^* \text{ is super Lyndon}\}$ forms a basis of $\mathcal{F}(\mathcal{X}_i)$.

**Corollary 3.** If the set $\mathcal{X}_{i,1}$ is empty then $\mathcal{F}(\mathcal{X}_i)$ becomes the free Lie algebra $\mathcal{F}(\mathcal{X}_i)$. In this case, $\{L(w) : w \in \mathcal{X}_i^* \text{ is Lyndon}\}$ forms a basis of $\mathcal{F}(\mathcal{X}_i)$.

Universal property of the free Lie superalgebra $\mathcal{F}(\mathcal{X}_i)$: Let $I$ be a Lie superalgebra and let $\Phi : \mathcal{X}_i \to I$ be a set map that preserves the $\mathbb{Z}_2$ grading. Then $\Phi$ can be extended to a Lie superalgebra homomorphism $\Phi : \mathcal{F}(\mathcal{X}_i) \to I$.

4.4. Idea of the proof. Let $g$ be a BKM superalgebra with the associated supergraph $(G, \Psi)$. Define a map $\Phi : \mathcal{X}_i \to g$ by $\Phi(w) = e(w)$, the left normed Lie word associated with $w$. The map $\Phi$ preserves the $\mathbb{Z}_2$ grading. By the universal property, we have a Lie superalgebra homomorphism

$$\Phi : \mathcal{F}(\mathcal{X}_i) \to g, \quad w \mapsto e(w) \quad \forall \ w \in \mathcal{X}_i. \quad (4.2)$$

Since $\Phi$ preserves the $Q_+ = \mathbb{Z}_+[I]$–grading and $g$ can be finite-dimensional the map $\Phi$ need not be surjective. Let $g^i$ be the image of the homomorphism $\Phi$ in $g$. Then $g^i$ is the Lie sub-superalgebra of $g$ generated by $\{e(w) : w \in \mathcal{X}_i\}$. Note that $g^i$ is again $Q_+$–graded. Any basis of the free Lie superalgebra $\mathcal{F}(\mathcal{X}_i)$ can be pushed forward through the map $\Phi$ to $g^i$ and the image will be a spanning set of $g^i$. We construct a basis for the root space $\mathfrak{h}(k)$ from this spanning set; This is the main theorem of this section. This is done by identifying the following combinatorial model from [1] with the set of super Lyndon heaps of weight $k$.

$$C^i(k, G) = \{w \in \mathcal{X}_i^* : w \text{ is a super Lyndon word}, \ \text{wt}(w) = \eta(k)\}, \quad \iota(w) = \Phi \circ L(w). \quad (4.3)$$

The precise statement is given in the next subsection.

4.5. Theorem 6: LLN basis of BKM superalgebras. In this subsection, we state Theorem 6 and two main lemmas which are essential to prove this theorem. The proofs of these lemmas are postponed to the subsequent subsection.
The set $\{\iota(w) : w \in C^i(k, G)\}$ is a basis of the root space $g_{\eta(k)}$. Moreover, if $k_i = 1$, the set $\{e(w) : w \in X_i, \, \text{wt}(w) = \eta(k)\}$ forms a left-normed basis of $g_{\eta(k)}$.

Lemma 3. The root space $g_{\eta(k)} = \mathfrak{g}_{\eta(k)}^i$ for $k \in \mathbb{Z}_+[I]$ satisfying $k_i \leq 1$ for $i \in I^r \sqcup \Psi_0$.

Lemma 4. With the notations as above we have

$$|C^i(k, G)| = \dim FLS_k(A') = \dim LS_k(G, \Psi).$$

From the above lemmas Theorem 6 can be deduced as follows. Since $g_{\eta(k)} = \mathfrak{g}_{\eta(k)}^i$ we get $\{\iota(w) : w \in C^i(k, G)\}$ is a spanning set for $g_{\eta(k)}$ of cardinality equal to $|C^i(k, G)|$. Now, Lemmas 4 and 1 show that $\{\iota(w) : w \in C^i(k, G)\}$ is in fact a basis.

4.6. Examples to Theorem 6. First, we explain Theorem 6 by an example before giving the proofs of the Lemmas 3 and 4.

Example 4. Consider the root space $g_{\eta(k)}$ where $\eta(k) = 3\alpha_3 + 3\alpha_6$ from Example 2. Fix $i = 3$. Super Lyndon words of weight $\eta(k)$ in $C^3(k, G) = \{w \in \chi^*_3 : \text{wt}(w) = \eta(k), \, w \text{ is super Lyndon}\}$ are $\{336636, 333666, 336366\}$ and $\{[[3, [3, 6], 6], [3, 5], 6], [3, [3, [3, 6], 6]], [3, [3, [3, 6], 6]], [3, [3, 6], 334565]\}$ spans $g_{\eta(k)}$. We have mult($\eta(k)$) = 3 [c.f. Example 10]. So these Lie monomials form a basis for $g_{\eta(k)}$.

Example 5. Consider the root space $g_{\eta(k)}$ where $\eta(k) = 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$ from Example 3. Fix $i = 3$, observe that $\chi^*_3 = \{3, 34, 345, 3456, 3444, 34545, \ldots\}$. Only Lyndon words on $\chi^*_3$ of weight $\eta(k)$ are $\{334556, 334565\}$ with standard factorization $334556, 334565$ respectively. So the corresponding Lie monomials $\{[3, [[[3, 4, 5], 6], 5]], [3, [[[3, 4, 5], 5], 6]]\}$ spans $g_{\eta(k)}$. We have mult($\eta(k)$) = 2 [c.f. Example 11]. So these Lie monomials form a basis for $g_{\eta(k)}$.

4.7. Proof of Lemma 3. We claim that the root space $g_{\eta(k)} = \mathfrak{g}_{\eta(k)}^i$. This is done in multiple steps. First, we claim that the left normed Lie words of weight $k$ starting with a fixed $i \in I$ spans $g_{\eta(k)}$. More precisely,

Lemma 5. Fix an index $i \in I$. Then the root space $g_{\eta(k)}$ is spanned by the set of left normed lie words $\{e(w) : w \in \chi^*_i, \text{wt}(w) = \eta(k)\}$.

Proof. We observe that $w \in \chi^*_i$ if, and only if, $\text{IA}(w) = \{i\}$. It is well-known that, the set $B = \{e(w) : w \in M_k(I, G)\}$ forms a spanning set for $g_{\eta(k)}$. We will prove that each element of $B$ can be written as linear combination of left normed Lie words $e(w)$ satisfying $\text{wt}(w) = \eta(k)$ and $\text{IA}(w) = \{i\}$. Let $w = a_1a_2 \cdots a_p \in M_k(I, G)$. Assume that $a_1 = i$. If $|\text{IA}(w)| > 1$ then $e(w) = 0$ and nothing to prove. If $|\text{IA}(w)| = 1$ then we have $\text{IA}(w) = \{i\}$ and the proof follows in this case. Assume $a_1 \neq i$ and consider the set $i(w) = \{j : a_j = i\}$. Assume $\min\{i(w)\} = p + 1$ and set $w' = a_1a_2 \cdots a_p i$. 

First, we claim that
\[
e(w') = e(ia_1 a_2 \cdots a_p) + \sum_{j_1=2}^{p} e(ia_j a_1 a_2 \cdots \hat{a}_{j_1} \cdots a_p) + \sum_{1<j_2<j_1 \leq p} e(ia_{j_1} a_{j_2} a_1 a_2 \cdots \hat{a}_{j_1} \cdots a_{j_2} \cdots a_p)
\]
\[
+ \sum_{1<j_3<j_2<j_1 \leq p} e(ia_{j_1} a_{j_2} a_{j_3} a_1 a_2 \cdots \hat{a}_{j_1} \cdots a_{j_3} \cdots a_{j_2} \cdots a_p) + \cdots
\]
\[
+ \sum_{1<j_1<j_2<j_3 \leq p} e(ia_{j_1} a_{j_2} a_{j_3} a_1 a_2 \cdots \hat{a}_{j_1} \cdots a_{j_3} \cdots a_{j_2} \cdots a_p) + \cdots
\]
\[
+ e(ia_1 a_{p-1} \cdots a_2 a_1)
\]
where \(\hat{a}\) means the omission of the alphabet \(a\) in the expression.

We explain the above equation with an example for better understanding. Consider the root space \(g_{\eta(k)}\) where \(\eta(k) = 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6\) from Example 3. Fix \(i = 3\) and consider the word \(w = 456353\). Then \(p = 3\) and \(w' = 4563\). Now,
\[
e(w') = [[[4, 5], 6], 3] = [3, [4, 5], 6] - [[[4, 5], 3], 6]
\]
\[
= [[[3, 4], 5], 6] + [[[4, 3], 5], 6] - [[[4, 5], 3], 6]
\]
\[
= [[[3, 4], 5], 6] - [[[3, 5], 4], 6] + [[[3, 6], 4], 5] + [4, [3, 6], 5]
\]
\[
= [[[3, 4], 5], 6] - [[[3, 5], 4], 6] + [[[3, 6], 4], 5] - [[[3, 6], 5], 4]
\]
\[
= e(3456) - e(3546) + e(3645) - e(3654)
\]

We do induction on \(p\). For \(p = 1\), \(w' = a_1 i \Rightarrow e(w') = [i, a_1] = e(ia_1)\). Assume that the result is true for \(p = k\). Now consider \(p = k + 1\)
\[
[[[[[a_1, a_2], a_3], a_4] \cdots , a_k], a_{k+1}], i] = [[[a_1', a_3[a_4], \cdots , a_k], a_{k+1}], i]
\]
by taking \([a_1, a_2] = a_1'\). Using the induction hypothesis on right hand side of the above equation we get
\[
[[[[[a_1', a_3], a_4], \cdots , a_k], a_{k+1}], i] =
\]
\[
e(ia'_1 a_3 \cdots a_{k+1}) + \sum_{3 \leq j \leq k+1} e(ia_j a'_1 a_3 \cdots \hat{a}_j \cdots a_{k+1})
\]
\[
+ \sum_{3 \leq j_2 < j_1 \leq k+1} e(ia_j a_{j_2} a'_1 a_3 \cdots \hat{a}_{j_1} \cdots a_{k+1}) + \cdots
\]
\[
+ \sum_{3 \leq j_{k-1} < \cdots < j_2 < j_1 \leq k+1} e(ia_{j_1} a_{j_2} a_{j_3} \cdots \hat{a}_{j_{k-1}} a'_1 a_3 \cdots \hat{a}_{j_1} \cdots a_{k+1})
\]
\[
+ e(ia_{k+1} a_k \cdots a_1')
\]
Now,

\[ e(ia_{j_1}a_{j_2} \cdots a_{j_l}a_{k_1}) = [[[i, a_{j_1}], a_{j_2}, \cdots, a_{j_l}], [a_{k_1}]] \]
\[ = [[[i, a_{j_1}], a_{j_2}, \cdots, a_{j_l}], a_{k_1}]] + [[[i, a_{j_1}], a_{j_2}, \cdots, a_{j_l}], a_{k_1}]] \]
\[ = e(ia_{j_1}a_{j_2} \cdots a_{j_l}a_{k_1}) + e(ia_{j_1}a_{j_2} \cdots a_{j_l}a_{2a_{k_1}}) \]

\[ \Rightarrow e(ia_{j_1}a_{j_2} \cdots a_{j_l}a_{k_1}) = [[[i, a_{j_1}], a_{j_2}, \cdots, a_{j_l}], a_{k_1}]] \]
\[ = e(ia_{j_1}a_{j_2} \cdots a_{j_l}a_{k_1}) \]
\[ = m \cdot e(ia_{j_1}a_{j_2} \cdots a_{j_l}a_{k_1}) \]
\[ = m \cdot e(ia_{j_1}a_{j_2} \cdots a_{j_l}a_{k_1}) \]
\[ = m \cdot e(ia_{j_1}a_{j_2} \cdots a_{j_l}a_{k_1}) \]
\[ = m \cdot e(ia_{j_1}a_{j_2} \cdots a_{j_l}a_{k_1}) \]
\[ = m \cdot e(ia_{j_1}a_{j_2} \cdots a_{j_l}a_{k_1}) \]

Using this in Equation (4.5) we get

\[ [[[a_{k_1}], a_{k_1}], \cdots, a_{k_1}, a_{k_1}], i] = e(ia_{1}a_{2}a_{3} \cdots a_{k_1}) + e(ia_{2}a_{1}a_{3} \cdots a_{k_1}) \]
\[ + \sum_{3 \leq j \leq k_1} (e(ia_{j_1}a_{j_2}a_{j_3} \cdots a_{j_{k_1}}) + e(ia_{j_1}a_{j_2}a_{j_3} \cdots a_{j_{k_1}})) \]
\[ + \sum_{3 \leq j_1 < j_2 < k_1} (e(ia_{j_1}a_{j_2}a_{j_3} \cdots a_{j_{k_1}}a_{j_{k_1}+1}) + e(ia_{j_1}a_{j_2}a_{j_3} \cdots a_{j_{k_1}+1})) + \cdots \]
\[ + \sum_{3 \leq j_{k_1} < \cdots < j_{k_1+1}} (e(ia_{j_1}a_{j_2}a_{j_3} \cdots a_{j_{k_1}+1}) + e(ia_{j_1}a_{j_2}a_{j_3} \cdots a_{j_{k_1}+1})) \]

\[ = \left( e(ia_{1}a_{2}a_{3} \cdots a_{k_1}) + e(ia_{2}a_{1}a_{3} \cdots a_{k_1}) \right) + \sum_{3 \leq j \leq k_1} e(ia_{j_1}a_{j_2}a_{j_3} \cdots a_{j_{k_1}}a_{j_{k_1}+1}) \]
\[ + \sum_{3 \leq j_1 < j_2 < k_1} e(ia_{j_1}a_{j_2}a_{j_3} \cdots a_{j_{k_1}+1}) + \sum_{3 \leq j_2 < j_1 < k_1} e(ia_{j_2}a_{j_1}a_{j_3} \cdots a_{j_{k_1}+1}) \]
Thus the result is true for \( p = k + 1 \). This proves our claim. From this claim proof of the lemma follows from the following steps.

\[
\begin{align*}
&+ \sum_{3 \leq j_2 < j_1 \leq k+1} e(i^a_j a_{j_2} a_1 a_3 \cdots a_{j_2} \cdots a_{j_1} \cdots a_{k+1}) + \\
&+ \left( \sum_{3 \leq j_3 < j_2 < j_1 \leq k+1} e(i^a_j a_{j_2} a_1 a_3 \cdots a_{j_2} \cdots a_{j_1} \cdots a_{k+1}) \right) + \cdots + \\
&+ \left( \sum_{3 \leq j_{k-1} < \cdots < j_2 < j_1 \leq k+1} e(i^a_j a_{j_2} a_1 a_3 \cdots a_{j_2} \cdots a_{j_1} \cdots a_{k+1}) + e(i^a_{k+1} a_k \cdots a_2 a_1) \right)
\end{align*}
\]

\[
= \sum_{j=1}^{k+1} e(i^a_j a_1 a_2 \cdots a_j \cdots a_{k+1}) + \sum_{1 < j_2 < j_1 \leq k+1} e(i^a_j a_{j_2} a_1 a_2 \cdots a_{j_2} \cdots a_{j_1} \cdots a_{k+1}) + \cdots + \\
+ \sum_{1 < j_3 < j_2 < j_1 \leq k+1} e(i^a_j a_{j_2} a_1 a_2 \cdots a_{j_2} \cdots a_{j_1} \cdots a_{k+1}) + \cdots + \\
+ \sum_{1 < j_{k-1} < \cdots < j_2 < j_1 \leq k+1} e(i^a_j a_{j_2} a_1 a_2 \cdots a_{j_2} \cdots a_{j_1} \cdots a_{k+1}) + e(i^a_{k+1} a_k \cdots a_2 a_1)
\]

Thus the result is true for \( p = k + 1 \). This proves our claim. From this claim proof of the lemma follows from the following steps.

\[
e(w') = e(i^a_1 a_2 \cdots a_p) + \sum_{j_1=2}^{p} e(w_{j_1}) + \sum_{1 < j_2 < j_1 \leq p} e(w_{j_1, j_2}) + \sum_{1 < j_3 < j_2 < j_1 \leq p} e(w_{j_1, j_2, j_3}) + \cdots + \\
+ e(w_{p(p-1)\cdots 21})
\]

where \( w_{j_1, j_2, \ldots, j_l} = (i^a_1 a_{j_2} \cdots a_{j_l} a_{j_2} \cdots a_{j_l} \cdots a_{j_2} \cdots a_{j_1} \cdots a_p) \). We observe that all the words \( w_{j_1, \ldots, j_m} \) have the same weight and belongs to \( \chi_i \). This is because if some \( a_{j_p} \) commutes with \( i, a_{j_1}, a_{j_2}, \ldots, a_{j_{p-1}} \) then \( e(w_{j_1, j_2, \ldots, j_l}) = 0 \). In our example, \( e(w') = e(w'_{4}) + e(w'_{42}) + e(w'_{43}) + e(w'_{432}) \) where \( w' = a_1 a_2 a_3 a_4 = 4563 \). Now, By the linearity property of the brackets we have

\[
e(w' \cdot a_{p+2}) = \sum_{j_1=1}^{p} e(w_{j_1} \cdot a_{p+2}) + \sum_{1 < j_2 < j_1 \leq p} e(w_{j_1, j_2} \cdot a_{p+2}) + \sum_{1 < j_3 < j_2 < j_1 \leq p} e(w_{j_1, j_2, j_3} \cdot a_{p+2}) + \cdots + e(w_{p(p-1)\cdots 21} \cdot a_{p+2})
\]

Similarly, we can add all the remaining alphabets \( a_{p+3}, \ldots, a_r \) to the above expression. This will give us

\[
e(w) = \sum_{u \in X^* \cap \mathfrak{a}^*} \alpha(u) e(u) \text{ for some scalars } \alpha(u).
\]
Lemma 6. If \( u \neq v \in \chi_i^* \) are Lyndon words then exactly one element of the set \( \{uv, vu\} \) is Lyndon.

Proof. We observe that if \( u < v \) then \( uv \) is Lyndon otherwise \( vu \) is Lyndon. 

Lemma 7. If \( w, \tilde{w} \in \chi_i^* \) are Lyndon words with standard factorization \( w = u_1u_2, \tilde{w} = v_1v_2 \). Assume that \( w\tilde{w} \) is a Lyndon word. Then

\[
[L(w), L(\tilde{w})] \in \text{span}\{L(C^i(\text{wt}(w\tilde{w})), G)\}
\]

Proof. We have two possibilities: either \( u_2 \geq \tilde{w} \) or \( u_2 < \tilde{w} \).

If \( u_2 \geq \tilde{w} \) then \( u_1u_2v_1v_2 \) is the standard factorization of \( w\tilde{w} \). Observe that \( u_1u_21u_22v_1v_2 \) can’t be standard factorization for some standard factorization \( u_2 = u_21u_22 \) as \( u_22 < \tilde{w} \leq u_2 = u_21u_22 \) means \( u_22 < u_21 \) i.e. \( u_2 = u_21u_22 \) can’t be a Lyndon word. From this observation, the proof is immediate in this case: We have \( w\tilde{w} = u_1u_2v_1v_2 \) is the standard factorization and \( [L(w), L(\tilde{w})] = L(w\tilde{w}) \in \text{span}\{L(C^i(\text{wt}(w\tilde{w})), G)\} \).

If \( u_2 < \tilde{w} \) then \( u_1u_2v_1v_2 \) is the standard factorization of \( w\tilde{w} \). Observe that \( u_11u_12u_2v_1v_2 \) can’t be the standard factorization for some standard factorization of \( u_1 = u_11 \) bold \( u_12 \) as \( u_12 < u_2 \) means \( u_12u_2 \) is the longest right factor of \( w = u_11u_12u_2 \) which contradicts the statement: \( w = u_1u_2 \) is the standard factorization.

If \( w\tilde{w} = u_1u_2v_1v_2 \) is standard factorization then

\[
[L(w), L(\tilde{w})] = [L(u_1u_2), L(\tilde{w})]
\]

\[
= [[L(u_1), L(u_2)], L(\tilde{w})]
\]

\[
= [L(u_1), [L(u_2), L(\tilde{w})]] + [[L(u_1), L(\tilde{w})], L(u_2)]
\]

**subcase(i):** If \( u_2 \tilde{w} \) is Lyndon word with standard factorization \( u_2|\tilde{w} \) and \( u_1 \tilde{w} \) is a Lyndon word with standard factorization \( u_1|\tilde{w} \) then

\[
[L(w), L(\tilde{w})] = [L(u_1), L(u_2 \tilde{w})] + [L(u_1 \tilde{w}), L(u_2)]
\]

\[
= [L(u_1), L(u_2 \tilde{w})] + L(u_1 \tilde{w}u_2)
\]

as \( u_2 < \tilde{w} \) so \( u_1 \tilde{w}u_2 \) is standard factorization.
Now repeat the above procedure again on \([L(u_1), L(u_2 \tilde{w})]\). Continue this procedure on the subsequent terms till we get the term like \([L(v_1), L(v_2)]\) where \(v_1, v_2\) are Lyndon words with \(v_1 \in \chi_i\). This is possible since \(wt(u_1) < wt(w)\).

**Subcase (ii):** If \(u_2 \tilde{w}\) is Lyndon word with standard factorization \(u_2 = u_{21}u_{22}\) and \(u_1 \tilde{w}\) is Lyndon word with standard factorization \(u_1 \tilde{w}\) then

\[
\begin{align*}
= [L(u_1), [[L(u_{21}), L(u_{22})], L(\tilde{w})]] + [L(u_1 \tilde{w}), L(u_2)] \\
= [L(u_1), [[L(u_{21}), L(\tilde{w})], L(u_{22})]] + [L(u_1), [L(u_{21}), [L(u_{22}), L(\tilde{w})]]] + L(u_1 u_2 \tilde{w})
\end{align*}
\]

Repeat the above procedure firstly for \([L(u_{21}), L(\tilde{w})], [L(u_{22}), L(\tilde{w})]\), then using this in the above equation and repeat the procedure for subsequent terms and so on. This process will end when we got terms like \([L(v_1), L(v_2)]\) where \(v_1, v_2\) are Lyndon words with \(v_1 \in \chi_i\).

**Subcase (iii):** If \(u_2 \tilde{w}\) is Lyndon word with standard factorization \(u_2 = u_{21}u_{22}\) and \(u_1 \tilde{w}\) is Lyndon word with standard factorization \(u_1 = u_{11}u_{12}\) then

\[
\begin{align*}
= [L(u_1), [[L(u_{21}), L(u_{22})], L(\tilde{w})]] + [[[L(u_{11}), L(u_{12})], L(\tilde{w})], L(u_2)] \\
= [L(u_1), [[L(u_{21}), L(\tilde{w})], L(u_{22})]] + [L(u_1), [L(u_{21}), [L(u_{22}), L(\tilde{w})]]] + [[[L(u_{11}), L(\tilde{w})], L(u_{12})], L(u_1)]
\end{align*}
\]

Repeat the above procedure firstly for \([L(u_{12}), L(\tilde{w})], [L(u_{11}), L(\tilde{w})]\), then using this in the above equation and repeat the procedure for subsequent terms and so on. This process will end when we got terms like \([L(v_1), L(v_2)]\) where \(v_1, v_2\) are super Lyndon words with \(v_1 \in \chi_i\).

\(\square\)

*The following example explains the above lemma.*

**Example 6.** Consider the root space \(\mathfrak{g}_0(k)\) where \(\eta(k) = 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6\) from Example 3. Fix \(i = 3\). Let \(w = 334345, w' = 34635364 \in \chi^*_3\) then \(w = 3|34345 = u_1u_2, w' = 346|35364 = v_1v_2\) are the standard factorizations. Thus \(L(w) = [L(u_1), L(u_2)] = [3, 34, 345]\) and \(L(w') = [L(v_1), L(v_2)] = [346, 35, 364]\). Since \(ww' = 334345 \underbrace{34635364}_w = 3\underbrace{3434534635364}_{w'}\) is the standard factorization, we have

\[L(w), L(w')\]

\[
= [[L(3), L(34345), L(34635364)] \underbrace{L(34635364)}_{[L(u_1), L(u_2)]}, L(w')] \\
= [L(3), [L(34345), L(34635364)] + [L(3), L(34635364)], L(34345)] \\
= [L(3), [L(34), L(345)], [L(34635364), L(34345)] + [L(34), L(345), L(34635364)], L(34345)] \\
= [L(3), [L(34), L(34635364), L(345)] + L(34), [L(345), L(34635364)], L(34345)] + [L(3463536434345), L(34), L(345), L(34635364)], L(34345)] + [L(3), L(3463536434345), L(u_1 u_2)]
\]
Case (ii):

Proof.

Let

\[ [L(u_1 w'), L(u_2')] + [L(u_1 u_2), L(w')] \]

\[ = [L(3), L(3434635364), L(3434534635364)] + L(3346356434345) \]

\[ = [L(3), L(343463536434345)] + [L(3434534635364), L(3346356434345)] \]

\[ = L(3343653643445) + L(33434536435364) + L(3343563463445) \]

Example 7. Let \( I = \{1, 2, 3, 4, 5, 6\}, \Psi = \{3, 5\}, I_1 = \{3, 5\}, I_0 = \{1, 2, 4, 6\}, I^c = \{1, 4\}, \eta(k) = 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6 \). Fix \( i = 3 \). Let \( w = 334365, w' = 3463564 \in \chi_3^+ \) then \( w = 3|34365 = u_1 u_2, w' = 3463564 = v_1 v_2 \) be standard factorization of these words. Since \( \sum_{i = 1}^{2} \) \[
\sum_{i = 1}^{2} \] is the standard factorization,

\[
[L(w), L(w')] = \left[ \frac{[L(3), L(34365)]}{L(u_1 u_2)}, \frac{L(3463564)}{L(u_1 w')} \right]
\]

\[
= \left[ \frac{[L(3), L(34365), L(3463564)]}{[L(u_1), L(u_2), L(w')]} + \frac{[L(3), L(3463564), L(34345)]}{L(u_2)} \right]
\]

\[
= \left[ \frac{L(3), L(343654363564)}{L(u_1 u_2 w')}, \frac{L(33465346564345)}{L(u_1 w') L(u_2)} \right]
\]

Lemma 8. If \( w_a \) and \( w_b \) are super Lyndon words then \( [L(w_a), L(w_b)] \in \text{span}\{L(C^i(\text{wt}(w_a w_b)), G)\} \).

Proof. Since \( w_a \) and \( w_b \) are super Lyndon words, we have the following cases:-

(i) \( w_a = uu, w_b = v_1 v_2 \)

(ii) \( w_a = u_1 u_2, w_b = v_2 \)

(iii) \( w_a = uu, w_b = vv \)

(iv) \( w_a = u_1 u_2, w_b = v_1 v_2 \) where \( u_1 \neq u_2, v_1 \neq v_2 \)

Case (i):- Since \( w_a = uu, w_b = v_1 v_2 \Rightarrow w_a w_b = uu v_1 v_2 \)

\[
[L(w_a), L(w_b)] = [[L(u), L(u)], L(w_b)]
\]

\[
= 2[L(u), [L(u), L(w_b)]]
\]

\[
= 2[L(u), L(uw_b)]
\]

\[
= L(uuw_b)
\]

Case (ii):- \( w_a = u_1 u_2, w_b = vv \). If \( u_2 < v \) then \( w_a w_b = u_1 | u_2 vv \) is the standard factorization.

\[
[L(w_a), L(w_b)] = [[L(u_1), L(u_2)], L(w_b)]
\]

\[
= [[L(u_1), L(w_b)], L(u_2)] + [L(u_1), [L(u_2), L(w_b)]]
\]

\[
= [L(u_1 w_b), L(u_2)] + [L(u_1), L(u_2 w_b)]
\]
Similarly, again using the Jacobi identity, we can write

\[ e(\omega) = e(i, e(i\alpha_1 \cdots \alpha_{p+1} \cdots \alpha_r)) + \sum_{t=p+2}^k [e(i\alpha_1 \cdots \alpha_{p+1} \cdots \alpha_t \cdots \alpha_r), e(i\alpha_t)] + \]

\[ + \sum_{p+2 \leq t_1 < t_2 \leq r} [e(i\alpha_1 \cdots \alpha_{p+1} \cdots \alpha_{t_1} \cdots \alpha_{t_2} \cdots \alpha_r), e(i\alpha_{t_1} \alpha_{t_2})] + \cdots \]

\[ + \sum_{p+2 \leq t_1 < t_2 < t_3 \leq r} [e(i\alpha_1 \cdots \alpha_{p+1} \cdots \alpha_{t_1} \cdots \alpha_{t_2} \cdots \alpha_{t_3} \cdots \alpha_r), e(i\alpha_{t_1} \alpha_{t_2} \alpha_{t_3})] + \cdots \]

\[ + [e(i\alpha_1 \cdots \alpha_p), e(i\alpha_{p+1} \cdots \alpha_r)] \]

Case(iii):- \( w_a = uu, w_b = vv \). Since \( u < v, w_a w_b = uvuv \) is the standard factorization.

\[ [L(w_a), L(w_b)] = [[L(u), L(u)], L(w_b)] = [L(u), L(uw_b)] \]

Case(iv):- \( w_a = u_1 u_2, w_b = v_1 v_2 \). This case follows from Lemma 7. \( \square \)

**Lemma 9.** The root space \( g_{\eta(k)} \) is contained in the span \( \{e(L(C^i(k, G))\} \).

**Proof.** Let \( e(w) \in g_{\eta(k)} \) for some \( w \in M_k(I, \eta) \). By Lemma 5, we can assume that \( \IA(w) = \{i\} \).

We will do the proof by induction on \( \ht(\eta(k)) \). If \( \ht(\eta(k)) = 1 \) then \( w = i \) and nothing to prove. Assume that the result is true for any \( \tilde{w} \) such that \( \ht(w) < \ht(\eta(k)) \). Let \( w = ia_{1} a_{2} \cdots a_{r} = i \cdot u \).

If \( i(u) = \phi \) then \( w \in X \Rightarrow L(w) = w \Rightarrow e(w) = e(L(w)) \in \text{span}\{e(L(C^i(k, G))\} \).

If \( i(u) \neq \phi \) then let \( \min\{i(u)\} = p + 1 \) (say). Set \( w' = ia_{1} a_{2} \cdots a_{p} \). Now,

\[ e(w') = [[[i, a_1], a_2], \cdots, a_p, i] = -[i, [[[i, a_1], a_2], \cdots, a_p]] = e(L(ia_{1} a_{2} \cdots a_{p})) \]

\[ \Rightarrow e(w') \in \text{span}\{e(L(C^i(w'), G))\} \] as \( (ia_{1} a_{2} \cdots a_{p}) \) is a super Lyndon word.

Now,

\[ e(w' \cdot a_{p+2}) = [[[i, a_1], a_2], \cdots, a_{p}, i, a_{p+2}] = [e(w'), a_{p+2}] = [[e(i), e(ia_{1} a_{2} \cdots a_{p})], a_{p+2}] = [e(i), [e(ia_{1} a_{2} \cdots a_{p})], a_{p+2}] + [e(i), a_{p+2}], e(ia_{1} a_{2} \cdots a_{p})] = [e(i), e(ia_{1} a_{2} \cdots a_{p} a_{p+2})] + [e(ia_{p+2}), e(ia_{1} a_{2} \cdots a_{p})] \]

Similarly, again using the Jacobi identity, we can write

\[ e(w) = e(i, e(ia_{1} a_{2} \cdots a_{p+1} \cdots a_{r})) + \sum_{t=p+2}^k [e(ia_{1} a_{2} \cdots a_{p+1} \cdots \alpha_t \cdots a_{r}), e(i\alpha_t)] + \]

\[ + \sum_{p+2 \leq t_1 < t_2 \leq r} [e(ia_{1} a_{2} \cdots a_{p+1} \cdots \alpha_{t_1} \cdots \alpha_{t_2} \cdots a_{r}), e(i\alpha_{t_1} \alpha_{t_2})] + \cdots \]

\[ + \sum_{p+2 \leq t_1 < t_2 < t_3 \leq r} [e(ia_{1} a_{2} \cdots a_{p+1} \cdots \alpha_{t_1} \cdots \alpha_{t_2} \cdots \alpha_{t_3} \cdots a_{r}), e(i\alpha_{t_1} \alpha_{t_2} \alpha_{t_3})] + \cdots \]

\[ + [e(ia_1 \cdots a_p), e(i\alpha_{p+1} \cdots a_r)] \]
Using the induction hypothesis, we can check that each term on the right-hand side is of the form

\[
[e(i a_1 a_2 \cdots \hat{a}_p a_{p+1} \cdots \hat{a}_t \cdots \hat{a}_j \cdots a_r), e(i a_{t_1} a_{t_2} \cdots a_{t_j})] = \left[ \sum_a e(L(w_a)), \sum_b e(L(w_b)) \right]
\]

as \( \left( wt(i a_1 a_2 \cdots \hat{a}_p a_{p+1} \cdots \hat{a}_t \cdots \hat{a}_j \cdots a_r) < wt(w) \right) \). So

\[
e(i a_1 a_2 \cdots \hat{a}_p a_{p+1} \cdots \hat{a}_t \cdots \hat{a}_j \cdots a_r) = \sum_a e(L(w_a)) \text{ where } w_a \text{ is super Lyndon word,}
\]

\[
wt(w_a) = wt(i a_1 a_2 \cdots \hat{a}_p a_{p+1} \cdots \hat{a}_t \cdots \hat{a}_j \cdots a_r) \text{ and } w_t(i a_{t_1} a_{t_2} \cdots a_{t_j}) < wt(w_b). \text{ So}
\]

\[
e(i a_{t_1} a_{t_2} \cdots a_{t_j}) = \sum_b e(L(w_b)) \text{ where } w_b \text{ is super Lyndon word and } wt(w_b) = wt(i a_{t_1} a_{t_2} \cdots a_{t_j}).
\]

\[
\Rightarrow \left[ \sum_a e(L(w_a)), \sum_b e(L(w_b)) \right] = \sum_{a,b} [e(L(w_a)), e(L(w_b))]
\]

\[
= \sum_{a,b} e([L(w_a), L(w_b)])
\]

By Lemma 8, we can write \([L(w_a), L(w_b)] \in \text{span}\{L(C^i(k,G))\}\).

Thus

\[
\sum_{a,b} e([L(w_a), L(w_b)]) \in \text{span}\{L(C^i(k,G))\},
\]

\[
\Rightarrow e(w) \in \text{span}\{e(L(C^i(k,G))\})..
\]

Hence \(g_\eta(k)\) is contained in the span of \(\{e(L(C^i(k,G))\}) \subseteq g^i\).

\[\square\]

### 4.8. Proof of Lemma 4 (Identification of \(C^i(k,G)\) and super Lyndon heaps).

Let \(g\) be the BKM superalgebra associated with the Borcherds-Kac-Moody supermatrix \((A, \Psi)\). Let \(G\) be the associated quasi-Dynkin diagram of \(g\) with the vertex set \(I\). Fix \(k \in \mathbb{Z}_+[I]\) such that \(k_i \leq 1 \text{ for } i \in I^e \sqcup \Psi_0\).

Fix \(i \in I\) and assume that \(i\) is the minimum element in the total order of \(I\). Consider

\[
X_i = \{w \in M(I,G,\Psi) : \text{IA}_m(w) = \{i\} \text{ and } i \text{ occurs only once in } w\}.
\]

Now, Let \(w \in X_i\) and \(E = \psi(w)\) be the corresponding heap. Then

1. \(\text{IA}_m(w) = \{i\}\) implies that \(E\) is a pyramid.
2. \(i\) occurs exactly once in \(w\) implies that \(E\) is elementary
3. \(i\) is the minimum element in the total order of \(I\) implies that \(E\) is an admissible pyramid.

Therefore \(w \in X_i\) if and only if \(E = \psi(w)\) is a super-letter. (4.6)

Let \(A_i(I,\zeta)\) be the set of all super-letters with basis \(i\) in \(H(I,\zeta)\). Let \(A_i^*(I,\zeta)\) be the monoid generated by \(A_i(I,\zeta)\) in \(H(I,\zeta)\). Then \(A_i^*(I,\zeta) = A_{i,0}^*(I,\zeta) \oplus A_{i,1}^*(I,\zeta)\) is also \(\mathbb{Z}_2\)-graded. This monoid is free by the discussion below [32, Definition 2.1.4], and by [32, Proposition 1.3.5 and Proposition 2.1.5]. We have \(H(I,\zeta)\) is totally ordered and hence \(A_i(I,\zeta)\) is totally ordered. This implies that \(A_i^*(I,\zeta)\) is totally ordered by the lexicographic order induced from the order in \(A_i(I,\zeta)\) (call it \(<^*\)). The following proposition from [32, Proposition 2.1.6] illustrates the
relation between the total order \( \leq \) on the heaps monoid \( \mathcal{H}(I, \zeta) \) and the total order \( \leq^* \) on the monoid \( A^*_i(I, \zeta) \).

**Proposition 5.** Let \( E, F \in A^*_i(I, \zeta) \). Then \( E \leq^* F \) if, and only if, \( E \leq F \).

Given this, we can talk about the Lyndon words over the alphabets \( A_i(I, \zeta) \). The following proposition from [32, Proposition 2.1.7] illustrates the relationship between the Lyndon words in \( A^*_i(I, \zeta) \) and the Lyndon heaps in \( \mathcal{H}(I, \zeta) \).

**Proposition 6.** Let \( E \in A^*_i(I, \zeta) \) then \( E \) is a Lyndon word in \( A^*_i(I, \zeta) \) if and only if \( E \) is a Lyndon heap as an element of \( \mathcal{H}(I, \zeta) \).

Next, we prove the following generalization of Proposition 6 for the case of super Lyndon words and super Lyndon heaps.

**Proposition 7.** Let \( E \in A^*_i(I, \zeta) \) then \( E \) is a super Lyndon word in \( A^*_i(I, \zeta) \) if and only if \( E \) is a super Lyndon heap as an element of \( \mathcal{H}(I, \zeta) \).

**Proof.** Let \( E \in A^*_i(I, \zeta) \) be a super Lyndon word. Then either \( E \) is a Lyndon word in \( A^*_i(I, \zeta) \) or \( E = F \circ F \) for some Lyndon word \( F \in A^*_i(I, \zeta) \). Suppose the former case holds, then by Proposition 6, \( E \) is a Lyndon heap and hence is a super Lyndon heap in \( \mathcal{H}(I, \zeta) \). Suppose the latter case holds, then again by Proposition 6, \( F \) is a Lyndon heap in \( \mathcal{H}(I, \zeta) \) and hence \( E = F \circ F \) is a super Lyndon heap in \( \mathcal{H}(I, \zeta) \).

Conversely, suppose \( E \) is a super Lyndon heap in \( \mathcal{H}(I, \zeta) \). Then \( E \) is a Lyndon heap in \( \mathcal{H}(I, \zeta) \) or \( E = F \circ F \) for some Lyndon heap \( F \in \mathcal{H}(I, \zeta) \). Suppose the former case holds, then by Proposition 6, \( E \) is a Lyndon word and hence is a super Lyndon word in \( A^*_i(I, \zeta) \). Suppose the latter case holds, then again by Proposition 6, \( F \) is a Lyndon word in \( A^*_i(I, \zeta) \) and hence \( E = F \circ F \) is a super Lyndon word in \( A^*_i(I, \zeta) \). \( \square \)

By Equation (4.6), we can identify \( X^*_i \) with \( A^*_i(I, \zeta) \). This implies that
\[
|C^i(k, G)| = \left| \{ \text{super Lyndon words in } X^*_i \text{ of weight } k \} \right|
= \left| \{ \text{super Lyndon words of weight } k \text{ in } A^*_i(I, \zeta) \} \right|
= \left| \text{super Lyndon heaps of weight } k \text{ in } \mathcal{H}(I, \zeta) \right|
= \dim \mathcal{L}\mathcal{S}_k(G)
= \dim \mathfrak{g}_\eta(k) \quad \text{(By Theorem 5)}.
\]

This shows that the elements of \( C^i(k, G) \) are precisely the Lyndon heaps of weight \( k \) and completes the proof of Lemma 4.

**Remark 8.** The important step in the proof of Theorem 2 given in [1] is the proof of the equality \( \text{mult } \eta(k) = |C^i(k, G)| \) ([1, Proposition 4.5 (iii)]). The key idea is to reduce the proof to \( k = 1 = (1, 1, 1, \ldots) \)-case and then to use a result of Greene and Zaslavsky [15] for acyclic orientations of \( G \) to complete the proof. Also, Theorem 1 is used in the proof (in other words the denominator identity of \( \mathfrak{g} \) is used in the proof). Above, we have given a simpler and direct proof of this equality by identifying the set \( C^i(k, G) \) with the set of Lyndon heaps over graph \( G \). Our proof doesn’t use Theorem 1 and hence independent of the denominator identity of \( \mathfrak{g} \). Similarly, we don’t need the result of Greene and Zaslavsky. Also, we have extended this equality to the super case using super Lyndon heaps.
Example 8. The following diagram explains the connection between the Lyndon basis and the LLN basis of the free root spaces of a BKM superalgebra $\mathfrak{g}$.

To observe the above diagram through an example, consider the root space $\mathfrak{g}_{\eta(k)}$ where $\eta(k) = 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$ from Example 3. Fix $i = 3$, set of super-letters with basis 3 is $\mathcal{A}_3(I, \zeta) = \{3, 34, 345, 3456, 344, 34545, \cdots \}$.

Lyndon heaps on $\mathcal{A}_3^*(I, \zeta)$ of weight $\eta(k) = \{334556, 334565\}$

*Lyndon words over Super letters*

Lyndon words on super-letters of weight $\eta(k) = \{334556, 334565\}$

**4.9. Comparison of the Lyndon basis with the LLN basis.** In this section, we will give necessary and sufficient condition in which certain elements in both the bases are equal. We start with the following proposition whose proof is immediate from Propositions 5 and 6.
Proposition 8. Let \( w \in X_1^\ast \) be a Lyndon word with standard factorization \( \sigma(w) = (w_1, w_2) \). Then \( w, w_1 \) and \( w_2 \) can be thought of as Lyndon heaps [c.f. Proposition 7] and the Standard factorization of the Lyndon heap \( w \) is \( \Sigma(w) = (w_1, w_2) \). Further \( L(w) = \Lambda(w) \).

The following lemma will be helpful.

Lemma 10. [32, Lemma 2.3.5] Let \( E \in \mathcal{H}(I, P) \) with \( |E| \geq 2 \). Then \( \Sigma(E) = (F, N) \) iff \( \sigma(St(E)) = (St(F), St(N)) \).

From Example 8, to compare elements in the Lyndon basis and LLN basis we have to compare the action of the maps \( \Lambda \) and \( e \) on the super-letters. The following proposition gives a necessary and sufficient condition for the Lyndon basis element and LLN basis element associated with a super-letter to be equal.

Proposition 9. Let \( E \) be a super-letter with the associated standard word \( w = a_1a_2\cdots a_r \). Then \( \Lambda(E) = e(E) \) if, and only if, \( a_1 < a_r \leq a_{r-1} \leq \cdots \leq a_2 \).

Proof. Let \( E \) be a super-letter with the associated standard word \( w = a_1a_2\cdots a_r \). Assume that \( a_1 < a_r \leq a_{r-1} \leq \cdots \leq a_2 \). Now, the standard factorization of \( w \) is given by \( \sigma(w) = (a_1a_2\cdots a_{r-1}, a_r) \). This implies that, by Lemma 10, the standard factorization of \( E \) is \( \Sigma(E) = (F, G) \) where \( F \) and \( G \) are Lyndon heaps satisfying \( st(F) = a_1a_2\cdots a_{r-1} \) and \( st(G) = a_r \).

Therefore,

\[
\Lambda(E) = \left[ \Lambda(F), \Lambda(G) \right] \\
= \left[ \Lambda(F), a_r \right] \\
= \left[ \left[ \Lambda(F_1), \Lambda(a_{r-1}) \right], a_r \right] \text{ since } \sigma(st(F)) = (a_1a_2\cdots a_{r-1}, a_r) \\
= \left[ \left[ \Lambda(F_1), a_{r-1} \right], a_r \right] \\
= \cdots \\
= \left[ \left[ \left[ [a_1, a_2], a_3 \right], \cdots \right], a_r \right] \\
= e(E).
\]

Conversely, assume that \( \Lambda(E) = e(E) \) for a super-letter \( E \). Let \( w = a_1 \cdots a_r \) be the standard word of \( E \). We will prove \( a_1 < a_r \leq a_{r-1} \leq \cdots \leq a_2 \) by using induction on \( r \).

The base cases \( r = 1 \) and \( r = 2 \) are straightforward.

Assume that the result is true for \( r - 1 \). Write \( w = a_1a_2\cdots a_r = w' \cdot a_r \) where \( w' = a_1a_2\cdots a_{r-1} \). We have \( E = F \circ G \) with \( st(F) = a_1a_2\cdots a_{r-1} \) and \( st(G) = a_r \). We observe that \( w' \in \chi_i \) and \( F \) is a super-letter [c.f Equation (4.6)]. We claim that \( \Lambda(F) = e(F) \), i.e., \( \Lambda(F) = \left[ \left[ \left[ [a_1, a_2], a_3 \right], \cdots \right], a_{r-1} \right] \). Suppose not, then \( \Lambda(F) = \left[ \Lambda(F_1), \Lambda(F_2) \right] \) such that \( st(F_2) \neq a_{r-1} \) and

\[
\Lambda(E) = \Lambda(F_1 \circ F_2 \circ a_r) = \begin{cases} 
\left[ \Lambda(F_1), \Lambda(F_2 \circ a_r) \right] & \text{if } IA(st(F)) < a_r \\
\left[ \Lambda(F_1 \circ F_2), \Lambda(a_r) \right] & \text{otherwise}.
\end{cases}
\]

There is no other standard factorization of \( F_1 \circ F_2 \circ a_r \) is possible as if \( \sum(F_1 \circ F_2 \circ a_r) = (F_1 \circ F_{21}, F_{22} \circ a_r) \) for some standard factorization of \( F_2 = F_{21} \circ F_{22} \) then \( F_{22} < F_{21} \) which contradicts that \( F_2 = F_{21} \circ F_{22} \) is Lyndon heap.
This shows that \( \Lambda(E) \neq [[[a_1, a_2], [a_3], \ldots, [a_r]], a_r] \), a contradiction. Therefore \( \Lambda(F) = e(F) \). Thus by using the induction hypothesis, we have \( a_1 < a_{r-1} \leq \cdots \leq a_2 \). It remains to prove that \( a_r \leq a_{r-1} \). Suppose \( a_r > a_{r-1} \). We have \( a_{r-1} \leq a_r \) and \( \sigma(w) = (a_1a_2 \cdots a_{r-2}, a_{r-1}a_r) \). Therefore \( \Sigma(E) = (L, K) \) (by Lemma 10) where \( st(L) = a_1a_2 \cdots a_{r-2} \) and \( st(K) = a_{r-1}a_r \). This implies that \( \Lambda(E) = [\Lambda(L), \Lambda(K)] = [\Lambda(E), [a_{r-1}, a_r]] = [[[a_1, a_2], [a_3], \ldots, [a_{r-2}], [a_{r-1}, a_r]] \) which is not equal to \( e(E) \), a contradiction to our hypothesis. Hence \( a_r \leq a_{r-1} \) and the proof is complete. \( \square \)

Remark 9. Example 2 and Example 4 have same basis as super-letters 36, 366 and 3666 satisfy condition given in the above proposition. We remark that all these super-letters satisfying the condition given in the above proposition because there are only two elements in the support of the root. If support has more than two elements then some super-letter among these might satisfy the condition and the others may not. In particular, the Lyndon basis and the LLN basis will share only some common elements. Example 3 and Example 5 have different basis as super-letters 34565 and 34556 do not satisfy the condition given in the above proposition.

5. Combinatorial properties of free roots of BKM superalgebras

In this section, we explore the further combinatorial properties of free roots of BKM superalgebras. Also, we explain why the proof works in the papers [49, 1]: Chromatic polynomial cannot distinguish multi edges. So we lose multi edges in the Dynkin diagram when we consider the chromatic polynomial of the graph of a BKM superalgebra. In particular, we lose the Cartan integers and consequently Serre relations. The root spaces which are independent of the Serre relations are precisely the free root spaces.

5.1. Free roots of BKM superalgebras. Let \((G, \Psi)\) be a finite simple supergraph with vertex set \(I\), edge set \(E\), and the set of odd vertices \(\Psi \subseteq I\) [c.f. Definition 1]. Let \((A = (a_{ij}), \Psi)\) be the adjacency matrix of \(G\). We construct a class of BKM supermatrices from \((A, \Psi)\) as follows: Replace the diagonal zeros of \(A\) by arbitrary real numbers. If one such number is positive then replace all the non-zero entries in the corresponding row of \(A\) by arbitrary non-positive integers (resp. non-positive even integers) provided \(i \notin \Psi\) (resp. \(i \in \Psi\)). Otherwise, replace the non-zero entries in the associated row of \(A\) with arbitrary non-positive real numbers. Let \(M_{\Psi}(G)\) be the set of BKM supermatrices associated with the supergraph \((G, \Psi)\) constructed in this way. Let \(M(G) = \bigcup_{\Psi \subseteq I} M_{\Psi}(G)\). Let \(\mathcal{C}(G)\) be the set of all BKM superalgebras whose quasi-Dynkin diagram is \((G, \Psi)\) for some \(\Psi \subseteq I\). We observe that the set \(\mathcal{C}(G)\) consists of BKM superalgebras whose associated BKM supermatrices are in \(M(G)\). In the following proposition, we will prove that all the BKM superalgebras belong to \(\mathcal{C}(G)\) share the same set of free roots and have equal respective multiplicities.

Proposition 10. Let \(G\) be a graph. Let \(\mathfrak{g}\) be a BKM superalgebra which is an element of \(\mathcal{C}(G)\). Then

1. A \(\alpha \in Q_+\) is a free root in \(\mathfrak{g}\) if and only if \(\text{supp}\alpha\) is connected in \(G\). This gives a one-one correspondence between the connected subgraphs of \(G\) and the free roots of any \(\mathfrak{g} \in \mathcal{C}(G)\). In particular, \(\Delta^m(\mathfrak{g}_1) = \Delta^m(\mathfrak{g}_2)\) for \(\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{C}(G)\).
(2) For any $g \in \mathcal{C}(G)$, the multiplicity of a free root $\alpha$ depends only on the graph $G$ and this multiplicity is equal to the number of super Lyndon heaps of weight $k = (k_i : i \in I)$ where $\alpha = \sum_{i \in I} k_i \alpha_i$.

Proof. The necessity part of (1) is straightforward and we prove the sufficiency part. Assume $\alpha \in Q_+$ is free and $\text{supp} \alpha$ is connected in $G$. We claim that $\alpha$ is a root of $g$. We use induction on height of $\alpha$. The case in which $\text{ht}(\alpha) = 1$ is clear. Suppose $\text{ht}(\alpha) = 2$ then $\alpha = \alpha_i + \alpha_j$ and $a_{ij} < 0$. Suppose one of $\alpha_i$ or $\alpha_j$ is real. Without loss of generality we assume $\alpha_i$ is real. Then $S_{\alpha_i}(\alpha_j) = \alpha_j - a_{ij} \alpha_i$ is a root of $g$. This implies that $\alpha_i + \alpha_j$ is a root as the root chain of $\alpha_j$ through $\alpha_i$ contains $\alpha_j + m \alpha_i$ for all $0 \leq m \leq k$ for some $k \in \mathbb{N}$. Suppose both $\alpha_i$ and $\alpha_j$ are imaginary then Lemma 11 completes the proof. Assume that the result is true for all connected free $\alpha \in Q_+$ of height $r - 1$. Let $\beta$ be a connected free element of height $r$ in $Q_+$. Let $\alpha_i \in \text{supp} \beta$ be such that $\text{supp} \beta \setminus \{\alpha_i\}$ is connected in $G$. Since $\text{supp} \beta$ is connected such a vertex exit. Now, $\alpha = \sum_{j \neq i} \alpha_j$ is connected, free, and has height $r - 1$. By the induction hypothesis $\alpha$ is a root in $g$. If $\alpha_i$ is real then $S_{\alpha_i}(\alpha)$ is a root in turn $\beta = \alpha + \alpha_i$ is also root. If $\alpha_i$ is imaginary then, again by Lemma 11, $\beta$ is a root. This completes the proof of (1). Now, the proof of (2) follows from Lemma 1 and Theorem 5. $\square$

Example 9. Let $l_1 \geq 1, l_2 \geq 2$ and $l_3 \geq 3$ be positive integers satisfying $l_1 = l_2 = l_3$. Then the complex finite dimensional simple Lie algebras $A_{l_1}, B_{l_2}$ and $C_{l_3}$ have the same quasi-Dynkin diagram the path graph on $l_1$ vertices with $\Psi = \emptyset$. In particular, these algebras have the same set of free roots by the above proposition. In Table 9, using the following Proposition 11, we have listed the BKM superalgebras for which the path graph on 4 vertices is the quasi-Dynkin diagram along with its free roots.

Proposition 11. [41, Corollary 2.1.23] A simple finite dimensional Lie superalgebra $g$ is a BKM superalgebra if and only if $g$ is contragredient of type $A(m, 0) = \mathfrak{sl}(m + 1, 1), A(m, 1) = \mathfrak{sl}(m + 1, 2), B(0, n) = \mathfrak{osp}(1, 2n), B(m, 1) = \mathfrak{osp}(2m + 1, 2), C(n) = \mathfrak{osp}(2, 2n - 2), D(m, 1) = \mathfrak{osp}(2m, 2), D(2, 1, \alpha)$ for $\alpha = 0, -1, F(4)$, and $G(3)$.

Remark 10. We observe that the number of free roots of a BKM superalgebra is equal to the number of connected subgraphs $\mathcal{C}(G)$ of $G$. In particular, when $G$ is a tree, this number is equal to the number of subtrees of $G$. This number is well-studied in the literature. For example, see [38] and the references therein.

The rest of this section is dedicated to the proof of Theorem 3 and Corollary 1 which relates the $k$-chromatic polynomial with root multiplicities of BKM superalgebras. Hence we obtain a Lie superalgebras theoretic interpretation of $k$-chromatic polynomials.

5.2. Multicoloring and the $k$-chromatic polynomial of $G$. For any finite set $S$, let $\mathcal{P}(S)$ be the power set of $S$. For a tuple of non–negative integers $k = (k_i : i \in I)$, we have $\text{supp}(k) = \{i \in I : k_i \neq 0\}$.

Definition 7. Let $G$ be a graph with vertex set $I$ and the edge set $E(G)$. Let $k \in \mathbb{Z_+}[I]$. We call a map $\tau : I \to \mathcal{P}\{1, \ldots, q\}$ a proper vertex $k$-multicoloring of $G$ if the following conditions are satisfied:

(i) For all $i \in I$ we have $|\tau(i)| = k_i$,
Table 1. BKM superalgebras with equal set of free roots

| BKM superalgebras | Simple roots [24, Section 2.5.4] | Free roots |
|-------------------|----------------------------------|------------|
| $A_4$             | $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3,$  
|                   | $\alpha_3 = \varepsilon_3 - \varepsilon_4, \alpha_4 = \varepsilon_3 - \varepsilon_5.$ | $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{12}, \alpha_{13},$  
|                   | $\alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}.$ | |
| $B_4$             | $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3,$  
|                   | $\alpha_3 = \varepsilon_3 - \varepsilon_4, \alpha_4 = \varepsilon_4.$ | $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{12}, \alpha_{13},$  
|                   | $\alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}.$ | |
| $C_4$             | $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3,$  
|                   | $\alpha_3 = \varepsilon_3 - \varepsilon_4, \alpha_4 = 2\varepsilon_4.$ | $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{12}, \alpha_{13},$  
|                   | $\alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}.$ | |
| $A(3, 0)$         | $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3,$  
|                   | $\alpha_3 = \varepsilon_3 - \varepsilon_4, \alpha_4 = \varepsilon_4 - \delta_1.$ | $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{12}, \alpha_{13},$  
|                   | $\alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}.$ | |
| $A(2, 1)$         | $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3,$  
|                   | $\alpha_3 = \varepsilon_3 - \delta_1, \alpha_4 = \delta_1 - \delta_2.$ | $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{12}, \alpha_{13},$  
|                   | $\alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}.$ | |
| $B(0, 4)$         | $\alpha_1 = \delta_1 - \varepsilon_1, \alpha_2 = \delta_2 - \varepsilon_2,$  
|                   | $\alpha_3 = \delta_3 - \delta_4, \alpha_4 = \delta_1 - \delta_2.$ | $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{12}, \alpha_{13},$  
|                   | $\alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}.$ | |
| $B(3, 1)$         | $\alpha_1 = \delta_1 - \varepsilon_1, \alpha_2 = \varepsilon_2 - \varepsilon_3,$  
|                   | $\alpha_3 = \varepsilon_2 - \varepsilon_3, \alpha_4 = \varepsilon_3.$ | $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{12}, \alpha_{13},$  
|                   | $\alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}.$ | |
| $C(4)$            | $\alpha_1 = \delta_1 - \delta_1, \alpha_2 = \delta_1 - \delta_2,$  
|                   | $\alpha_3 = \delta_2 - \delta_3, \alpha_4 = \delta_3 - \delta_4.$ | $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{12}, \alpha_{13},$  
|                   | $\alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}.$ | |
| $F(4)$            | $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 + \varepsilon_3 + \delta,$  
|                   | $\alpha_3 = \varepsilon_3 - \varepsilon_4, \alpha_4 = \varepsilon_2 - \varepsilon_3.$ | $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{12}, \alpha_{13},$  
|                   | $\alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}.$ | |

(ii) For all $i, j \in I$ such that $(i, j) \in E(G)$ we have $\tau(i) \cap \tau(j) = \emptyset$.

The case $k_i = 1$ for $i \in I$ corresponds to the classical graph coloring of graph $G$. For more details and examples we refer to [18]. The number of ways a graph $G$ can be $k$-multicolored using $q$ colors is a polynomial in $q$, called the generalized $k$-chromatic polynomial ($k$-chromatic polynomial in short) and denoted by $\pi_k^G(q)$. The $k$-chromatic polynomial has the following well-known description. We denote by $P_k(k, G)$ the set of all ordered $k$-tuples $(P_1, \ldots, P_k)$ such that:

(1) each $P_i$ is a non-empty independent subset of $I$, i.e. no two vertices have an edge between them; and

(2) For all $i \in I$, $\alpha_i$ occurs exactly $k_i$ times in total in the disjoint union $P_1 \cup \cdots \cup P_k$.

Then we have

$$\pi_k^G(q) = \sum_{k \geq 0} |P_k(k, G)| \binom{q}{k}. \quad (5.1)$$

We have the following relation between the ordinary chromatic polynomials and the $k$-chromatic polynomials. We have

$$\pi_k^G(q) = \frac{1}{k!} \pi_1^{G(k)}(q) \quad (5.2)$$

where $\pi_1^{G(k)}(q)$ is the chromatic polynomial of the graph $G(k)$ and $k! = \prod_{i \in I} k_i!$. The graph $G(k)$ (the join of $G$ with respect to $k$) is constructed as follows: For each $j \in \text{supp}(k)$, take a clique (complete graph) of size $k_j$ with vertex set $\{j^1, \ldots, j^{k_j}\}$ and join all vertices of the $r$-th and $s$-th cliques if $(r, s) \in E(G)$.
For the rest of this paper, we fix an element \( k \in \mathbb{Z}_+[I] \) satisfying \( k_i \leq 1 \) for \( i \in I^e \sqcup \Psi_0 \), where \( \Psi_0 \) is the set of odd roots of zero norm.

5.3. Bond lattice and an isomorphism of lattices. In this subsection, we prove a lemma which will be useful in the proof of Theorem 3.

**Definition 8.** Let \( L_G(k) \) be the weighted bond lattice of \( G \), which is the set of \( J = \{J_1, \ldots, J_k\} \) satisfying the following properties:

(i) \( J \) is a multiset, i.e. we allow \( J_i = J_j \) for \( i \neq j \)

(ii) each \( J_i \) is a multiset and the subgraph spanned by the underlying set of \( J_i \) is a connected subgraph of \( G \) for each \( 1 \leq i \leq k \) and

(iii) For all \( i \in I \), \( \alpha_i \) occurs exactly \( k_i \) times in total in the disjoint union \( J_1 \sqcup \cdots \sqcup J_k \).

For \( J \in L_G(k) \) we denote by \( D(J, J) \) the multiplicity of \( J_i \) in \( J \) and set \( \text{mult}(\beta(J_i)) = \dim \mathfrak{g}_{\beta(J_i)} \), where \( \beta(J_i) = \sum_{\alpha \in J_i} \alpha \). We define \( J_0 = \{J_i \in J : \beta(J_i) \in \Delta_0^k\} \) and \( J_1 = J \setminus J_0 \).

**Lemma 11.** [50, Proposition 2.40] Let \( i \in I^\text{im} \) and \( \alpha \in \Delta_+ \setminus \{\alpha_i\} \) such that \( \alpha(h_i) < 0 \). Then \( \alpha + j\alpha_i \in \Delta_+ \) for all \( j \in \mathbb{Z}_+ \).

**Lemma 12.** [1, Lemma 3.4] Let \( P \) be the collection of multisets \( \gamma = \{\beta_1, \ldots, \beta_r\} \) (we allow \( \beta_i = \beta_j \) for \( i \neq j \)) such that each \( \beta_i \in \Delta_+ \) and \( \beta_1 + \cdots + \beta_r = \eta(k) \). The map \( \psi : L_G(k) \to P \) defined by \( \{J_1, \ldots, J_k\} \mapsto \{\beta(J_1), \ldots, \beta(J_k)\} \) is a bijection.

5.4. Proof of Theorem 3: (Chromatic polynomial and root multiplicities). For a Weyl group element \( w \in W \), we fix a reduced word \( w = s_{i_1} \cdots s_{i_t} \) and let \( I(w) = \{\alpha_{i_1}, \ldots, \alpha_{i_t}\} \). Note that \( I(w) \) is independent of the choice of the reduced expression of \( w \). For \( \gamma = \sum_{i \in I} m_i \alpha_i \in \Omega \), we set \( I_m(\gamma) \) is the multiset \( \{\alpha_{i_1}, \ldots, \alpha_{i_r} : i \in I\} \) and \( I(\gamma) \) is the underlying set of \( I_m(\gamma) \). We define \( \Psi_0(\gamma) = I(\gamma) \cap \Psi_0 \). Also, we define \( J(\gamma) = \{w \in W \setminus \{e\} : I(w) \sqcup I(\gamma) \text{ is an independent set}\} \).

The following lemma is a generalization of [49, Lemma 2.3] (for Kac-Moody Lie algebras) and [1, Lemma 3.6] (for Borcherds algebras) to the setting of BKM superalgebras. Since the proof of this lemma is similar to the proof of the Borcherds algebras case, we omit the proof here. Recall that \( k = (k_i : i \in I) \) satisfies \( k_i \leq 1 \) for \( i \in I^e \sqcup \Psi_0 \).

**Lemma 13.** Let \( w \in W \) and \( \gamma = \sum_{i \in I} \alpha_i + \sum_{i \in \Psi_0} m_i \alpha_i \in \Omega \). We write \( \rho - w(\rho) + w(\gamma) = \sum_{\alpha \in \Pi} b_\alpha(w, \gamma) \alpha \). Then we have

(i) \( b_\alpha(w, \gamma) \in \mathbb{Z}_+ \) for all \( \alpha \in \Pi \) and \( b_\alpha(w, \gamma) = 0 \) if \( \alpha \notin I(w) \sqcup I(\gamma) \).

(ii) \( b_\alpha(w, \gamma) \geq 1 \) for all \( \alpha \in I(w) \).

(iii) \( b_\alpha(w, \gamma) = 1 \) if \( \alpha \in I(\gamma) \setminus \Psi_0(\gamma) \) and \( b_\alpha(w, \gamma) = m_\alpha \) if \( \alpha \in \Psi_0(\gamma) \).

(iv) If \( w \in J(\gamma) \), then \( b_\alpha(w, \gamma) = 1 \) for all \( \alpha \in I(\gamma) \cup (I(\gamma) \setminus \Psi_0(\gamma)) \), \( b_\alpha(w, \gamma) = m_\alpha \) for all \( \alpha \in \Psi_0(\gamma) \).

(v) If \( w \notin J(\gamma) \cup \{e\} \), then there exists \( \alpha \in I(w) \subseteq \Pi^e \) such that \( b_\alpha(w, \gamma) > 1 \).

The following proposition is an easy consequence of the above lemma and essential to prove Theorem 3. Let \( U \) be the sum-side of the denominator identity (Equation (2.1)).

**Proposition 12.** Let \( q \in \mathbb{Z} \). We have

\[
U^q[e^{\eta(k)}] = (-1)^{ht(\eta(k))} \pi_k^G(q),
\]
where \( U^q[e^{-\eta(k)}] \) denotes the coefficient of \( e^{-\eta(k)} \) in \( U^q \).

**Proof.** Write \( U^q = \sum_{k \geq 0} \binom{q}{k} (U - 1)^k \). From Lemma 13 we get

\[
w(\rho) - \rho - w(\gamma) = -\gamma - \sum_{\alpha \in I(w)} \alpha, \quad \text{for } w \in J(\gamma) \cup \{e\}.
\]

Since \( k_i \leq 1 \) for \( i \in I^e \cup \Psi_0 \), the coefficient of \( e^{-\eta(k)} \) in \( (U - 1)^k \) is equal to

\[
\left( \sum_{\gamma \in \Omega} \sum_{w \in J(\gamma)} \epsilon(\gamma) \epsilon(w) e^{-\gamma - \sum_{\alpha \in I(w)} \alpha} \right)^k [e^{-\eta(k)}]. \tag{5.3}
\]

Hence the coefficient is given by

\[
\sum_{(\gamma_1, \ldots, \gamma_k) \, (w_1, \ldots, w_k)} \epsilon(\gamma_1) \cdots \epsilon(\gamma_k) \epsilon(w_1) \cdots \epsilon(w_k) \tag{5.4}
\]

where the sum ranges over all \( k \)-tuples \((\gamma_1, \ldots, \gamma_k) \in \Omega^k \) and \((w_1, \ldots, w_k) \in W^k \) such that

- \( w_i \in J(\gamma_i) \cup \{e\}, \ 1 \leq i \leq k \),
- \( I(w_1) \cup \cdots \cup I(w_k) = \{\alpha_i : i \in I^e, k_i = 1\} \),
- \( I(w_i) \cup I(\gamma_i) \neq \emptyset \) for each \( 1 \leq i \leq k \),
- \( \gamma_1 + \cdots + \gamma_k = \sum_{i \in P_m} k_i \alpha_i \).

It follows that \((I(w_1) \cup I(\gamma_1), \ldots, I(w_k) \cup I(\gamma_k)) \in P_k(k, G) \) and each element is obtained in this way. So the sum ranges over all elements in \( P_k(k, G) \). Hence \( (U - 1)^k[e^{-\eta(k)}] \) is equal to \((-1)^{ht(\eta(k))} |P_k(k, G)|\). Now, Equation (5.1) completes the proof. \( \square \)

**Remark 11.** We need the extra assumption \( k_i \leq 1 \) for \( i \in \Psi_0 \) when we extend Theorem 1 to the case of BKM superalgebras. Suppose \( k_i > 1 \) for some \( i \in \Psi_0 \). We observe that each \( \gamma_i \) contributes \( I_m(\gamma_i) \) to the required coefficient in Equation (5.3) and \( I_m(\gamma_i) \) can be a multiset. This implies that \( I(w_i) \cup I_m(\gamma_i) \) can be a multiset. The independent set considered in Equation (5.1) are sets. By assuming \( k_i \leq 1 \) for \( i \in \Psi_0 \), we have avoided the possibility of \( I_m(\gamma_i) \) being a multiset.

Now, we can prove Theorem 3 using the product side of the denominator identity (2.1). Proposition 12 and Equation (2.1) together imply that the \( k \)-chromatic polynomial \( \pi_k^G(q) \) is given by the coefficient of \( e^{-\eta(k)} \) in

\[
(-1)^{ht(\eta(k))} \frac{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha}) q^{\text{mult}(\alpha)}}{\prod_{\alpha \in \Delta_+^0} (1 + e^{-\alpha}) q^{\text{mult}(\alpha)}} = (-1)^{ht(\eta(k))} \prod_{\alpha \in \Delta_+} (1 - \epsilon(\alpha) e^{-\alpha}) \epsilon(\alpha) q^{\text{mult}(\alpha)}. \tag{5.5}
\]

where \( \epsilon(\alpha) = 1 \) if \( \alpha \in \Delta_0^+ \) and \( -1 \) if \( \alpha \in \Delta_+^1 \). Now,

\[
\prod_{\alpha \in \Delta_+} (1 - \epsilon(\alpha) e^{-\alpha}) \epsilon(\alpha) q^{\text{mult}(\alpha)} = \prod_{\alpha \in \Delta_+} \left( \sum_{k \geq 0} (-\epsilon(\alpha))^k \frac{(\epsilon(\alpha) q^{\text{mult}(\alpha)})^k}{k!} \right) e^{-ka}.
\]
A direct calculation of the coefficient of $e^{-\eta(k)}$ in the right-hand side of the above equation completes the proof of Theorem 3.

5.5. Proof of Corollary 1: (Formula for multiplicities of free roots). In this subsection, we prove Corollary 1 which gives a combinatorial formula for the multiplicities of free roots. We consider the algebra of formal power series $A := \mathbb{C}[[X_i : i \in I]]$. For a formal power series $\zeta \in A$ with constant term 1, its logarithm $\log(\zeta) = -\sum_{k \geq 1} \frac{(1-\zeta)^k}{k}$ is well-defined.

Proof. We consider $U$ as an element of $\mathbb{C}[[e^{-\alpha_i} : i \in I]]$ where $X_i = e^{-\alpha_i}$ [c.f. Lemma 13]. From the proof of Proposition 12 we obtain that the coefficient of $e^{-\eta(k)}$ in $-\log U$ is equal to $(-1)^{ht(\eta(k))} \sum_{k \geq 1} \frac{(-1)^k}{k} |P_k(k, G)|$ which by Equation (5.1) is equal to $|\pi_k^G(q)[q]|$. Now applying $-\log$ to the right hand side of the denominator identity (2.1) gives

$$\sum_{\ell \in N} \frac{1}{\ell} \text{mult}(\eta(k/\ell)) = |\pi_k^G(q)[q]| \quad (5.6)$$

if $\beta(k) \in \Delta_0^+$ and

$$\sum_{\ell \in N} \frac{(-1)^{l+1}}{\ell} \text{mult}(\eta(k/\ell)) = |\pi_k^G(q)[q]| \quad (5.7)$$

if $\beta(k) \in \Delta_1^+.$

The statement of the corollary is now an easy consequence of the following Möbius inversion formula: $g(d) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} \mu(d) g(d)$ where $\mu$ is the Möbius function. \qed

Example 10. Consider the BKM superalgebra $g$ and the root space $\eta(k) = 3\alpha_3 + 3\alpha_6 \in \Delta_1^+$ from Example 2. The $k$-chromatic polynomial of the quasi Dynkin diagram $G$ of $g$ is equal to

$$\pi_k^G(q) = \binom{q}{3} \binom{q-3}{3} = \frac{1}{3!3!} q(q-1)(q-2)(q-3)(q-4)(q-5).$$

By Corollary 1, since $\eta(k)$ is odd,

$$\text{mult}(\eta(k)) = \sum_{\ell \mid k} \frac{(-1)^{l+1} \mu(\ell)}{\ell} |\pi_{k/\ell}^G(q)[q]| = |\pi_k^G(q)[q]| + \frac{\mu(3)}{3} |\pi_{k'}^G(q)[q]|$$

where $k' = (0,0,1,0,0,1)$

$$= \frac{10}{3} - \frac{1}{3}$$

$$= 3$$
Example 11. Consider the BKM superalgebra $\mathfrak{g}$ from the previous example. Let $k = (2, 1, 0, 1, 2, 0) \in \mathbb{Z}_+^{|I|}$. Then $\eta(k) = 2\alpha_1 + \alpha_2 + \alpha_4 + 2\alpha_5 \in \Delta_0^+$. We have

$$\text{mult}(\eta(k)) = \sum_{\ell | k} \frac{\mu(\ell)}{\ell} |\pi_{k/\ell}^\mathfrak{g}(q)[q]|$$

This implies that $\text{mult}(\eta(k)) = |\pi_k^\mathfrak{g}(q)[q]|$. We have $\pi_k^\mathfrak{g}(q) = \frac{1}{4}q(q-1)^3(q-2)^2$. Therefore $\text{mult}(\eta(k)) = 1$.

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