COMPUTATIONAL STUDY OF NON-UNITARY PARTITIONS

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Abstract. Following Cayley, MacMahon, and Sylvester, define a non-unitary partition to be an integer partition with no part equal to one, and let \( \nu(n) \) denote the number of non-unitary partitions of size \( n \). In a 2021 paper, the sixth author proved a formula to compute \( p(n) \) by enumerating only non-unitary partitions of size \( n \), and recorded a number of conjectures regarding the growth of \( \nu(n) \) as \( n \to \infty \). Here we refine and prove some of these conjectures. For example, we prove
\[
p(n) \sim \nu(n) \sqrt{n/\zeta(2)}
\]
as \( n \to \infty \), and give Ramanujan-like congruences between \( p(n) \) and \( \nu(n) \) such as
\[
p(5n) \equiv \nu(5n) \pmod{5}.
\]

1. Introduction and statement of results

1.1. Non-unitary partitions and \( p(n) \). Let \( \mathcal{P} \) denote the set of integer partitions, including the empty partition \( \emptyset \). Let \( \mathcal{P}_n \) denote partitions of size (sum) equal to \( n \geq 0 \), with \( \mathcal{P}_0 := \{\emptyset\} \). The partition function \( p(n) \) gives the cardinality \( \#\mathcal{P}_n \), with \( p(0) := 1 \) [2].

In a 2021 paper [14], the sixth author (Schneider) studies the class of non-unitary partitions,\(^1\) which are partitions having no part equal to one, and records a number of conjectures. Non-unitary partitions appear to have first arisen in the literature in the 1880s in the work of MacMahon [9] and Sylvester [17] in connection with seminvariants in the theory of invariants. Cayley made another early contribution with [3]. Guy [7] proved the number of non-unitary partitions of size \( n \) into odd parts, equals the number of partitions of size \( n \) into distinct parts none of which is a power of two. The present authors, using tools from number theory and computer science, made a computational study of non-unitary partitions that led us to prove some of the conjectures in [14].

Let \( \mathcal{N} \subset \mathcal{P} \) denote the set of non-unitary partitions, let \( \mathcal{N}_n \subset \mathcal{P}_n \) denote non-unitary partitions of size \( n \geq 0 \), and let \( \nu(n) \) denote the cardinality \( \#\mathcal{N}_n \), with \( \nu(0) := 1 \). It is not hard to see
\[
p(n) = p(n - 1) + \nu(n),
\]
since every partition of \( n \) that is not non-unitary, can be obtained by adjoining 1 to a partition of \( n - 1 \). Immediately this recursion implies \( p(n) = \nu(0) + \nu(1) + \nu(2) + \cdots + \nu(n) \).

As a result of (1), in [14] it is proved one can view partitions in \( \mathcal{N}_n \) as “decaying” to produce the rest of \( \mathcal{P}_n \) by a certain algorithm resembling nuclear decay, which yields a formula for \( p(n) \) requiring one to generate only non-unitary partitions of size \( n \) (see [14, Thm. 1]). To see this is useful, let us compare \( \#\mathcal{N}_n \) to \( \#\mathcal{P}_n \). In [14] it is proved using the Hardy–Ramanujan asymptotic [8, Eq. 1.41], viz.
\[
p(n) \sim \frac{e^{A\sqrt{n}}}{Bn} \quad \text{with} \quad A = \pi \sqrt{2/3}, \quad B = 4\sqrt{3},
\]
that \( \nu(n) = o(p(n)) \). What more can one deduce about the relative growths of \( p(n), \nu(n) \)?

\(^1\)Non-unitary partitions are referred to as “nuclear partitions” in [14].
1.2. Our main results. Based on numerical patterns in Table 1 below, an explicit comparison relating \( p(n) \) and \( \nu(n) \) is conjectured in [14]:

\[
As n \to \infty, we have p(2n) \approx \nu(2n)\sqrt{2n},
\]

where “\( \approx \)” means approximately equal in magnitude. However, the approximation as stated is not compatible with (2). Our more extensive computations up to \( n = 3000 \) suggested the estimate needs a multiplicative constant, and extends to odd \( n \) as well.

Recall the well-known evaluation \( \zeta(2) = \pi^2/6 \) of the Riemann zeta function.

**Theorem 1.** As \( n \to \infty \), we have the asymptotic relation

\[
p(n) \sim \nu(n)\sqrt{\frac{n}{\zeta(2)}}.
\]

**Proof.** We use the asymptotic estimate \( 1 - e^{-x} \sim x \) as \( x \to 0 \), that arises from the Maclaurin series expansion for \( f(x) = e^{-x} \). Note by (1) that as \( n \to \infty \),

\[
\frac{p(n)}{\nu(n)} = \frac{p(n)}{p(n) - p(n-1)} = \left(1 - \frac{p(n-1)}{p(n)}\right)^{-1} \sim \left(1 - e^{A(\sqrt{n} - \sqrt{n-1})}\right)^{-1}.
\]

with \( 2A^{-1} = \sqrt{6/\pi^2} = \sqrt{\zeta(2)^{-1}} \), which leads to the statement of the theorem. We use (2) for the first asymptotic above, use \( 1 - e^{-x} \sim x \) for the second asymptotic, and use \( \sqrt{n} + \sqrt{n-1} \sim 2\sqrt{n} \) for the third asymptotic after rationalizing the denominator.

**Remark.** We note that Theorem 1 also follows from (2) together with [16, Eq. (1.2)].

In fact, an even smaller subset of \( P \) can be shown to generate \( N \), and to control the growth of \( \nu(n) \) in the same sense that \( \nu(n) \) controls \( p(n) \). Let a *ground state non-unitary partition* denote a partition in \( N \) such that the largest part appears two or more times. Let \( G \) denote the set of all ground state non-unitary partitions, let \( G_n \) denote the ground state non-unitary partitions of size \( n \), and let \( \gamma(n) \) denote the cardinality \#\( G_n \) with \( \gamma(0) := 0 \). One can compute \( p(n) \) as a linear combination of the values \( \gamma(k), k \leq n \) [14, Eq. 5]. Table 1 gives a comparison of the values of \( \gamma(n), \nu(n), p(n) \) for small \( n \).

**Remark.** Consideration of Young diagrams shows the sets \( G \) and \( G_n \) are both closed under partition conjugation, which is also true for \( P \) and \( P_n \), but is not true for \( N \) or \( N_n \).

Note that a partition in \( G_n \) is a non-unitary partition of size \( n \), and each element of \( N_n \) that is not an element of \( G_n \) can be constructed by adding 1 to the largest part of a partition in \( N_{n-1} \). Then we also have the recursion

\[
\nu(n) = \nu(n-1) + \gamma(n).
\]

This recursion together with the given initial values yields \( \nu(n) = 1 + \gamma(1) + \gamma(2) + \gamma(3) + \gamma(4) + \cdots + \gamma(n) \). One can deduce by repeated application of (2) that \( \gamma(n) = o(\nu(n)) \).

Another approximation is conjectured in [14] based on numerical evidence in Table 1:

\[
As n \to \infty, we have p(2n) \approx 2n \cdot \gamma(2n).
\]

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2 We note the \( n = 0 \) row is omitted from Table 1. In [14], the initial values \( \gamma(0) = \nu(0) = 0 \) are defined; alternatively, the definitions \( \gamma(0) = \nu(0) = 1 \) are consistent with the convention \( p(0) = 1 \).

3 These proofs for Theorems 1 and 2 were suggested by G. E. Andrews, and simplify our original proofs.
Again, the stated approximation is incompatible with (2). However, with a multiplicative constant it turns out to be true, and extends to odd \( n \), as a corollary of the next theorem.

**Theorem 2.** As \( n \to \infty \), we have the asymptotic relation

\[
\nu(n) \sim \gamma(n) \sqrt{\frac{n}{\zeta(2)}}.
\]

**Proof.** Note by (4) that as \( n \to \infty \), we have

\[
\begin{aligned}
\frac{\nu(n)}{\gamma(n)} &= \frac{\nu(n)}{\nu(n) - \nu(n-1)} = \left(1 - \frac{\nu(n-1)}{\nu(n)}\right)^{-1} \\
&\sim \left(1 - \frac{p(n-1)\sqrt{n}}{p(n)\sqrt{n-1}}\right)^{-1} \sim \left(1 - \frac{p(n-1)}{p(n)}\right)^{-1} \sim \sqrt{n/\zeta(2)},
\end{aligned}
\]

where we use Theorem 1 for the first asymptotic, use \( \sqrt{n/(n-1)} \sim 1 \) for the second asymptotic, and refer back to (3) for the third asymptotic. \( \square \)

**Corollary 3.** As \( n \to \infty \), we have the asymptotic relation

\[
p(n) \sim \frac{n \cdot \gamma(n)}{\zeta(2)}.
\]

**Proof.** Substitute the right side of the formula in Theorem 2, for \( \nu(n) \) in Theorem 1. \( \square \)

2. **Discussion of other conjectures and observations**

2.1. **Questions of monotonicity of \( \gamma(n) \).** In the previous section, we addressed conjectures raised in [14] related to the asymptotic behaviors of \( \nu(n) \) and \( \gamma(n) \) as \( n \) increases. There are other questions brought up in that work based on inspection of Table 1, related to arithmetic properties of the sequences \( \nu(n) \), \( \gamma(n) \), which we address in this section.

As noted in [14], an immediate contrast between the \( \gamma(n) \) column of Table 1 and the other two columns, is that while \( p(n), \nu(n) \) both grow (weakly) monotonically,\(^4\) the values of \( \gamma(n) \) oscillate as they increase, with both \( \gamma(2k-1) < \gamma(2k) \) and \( \gamma(2k) > \gamma(2k+1) \) holding for the entries in the table. On the other hand, inspection of the table up to \( n = 20 \) suggests \( \gamma(n) \) does grow monotonically if \( n \to \infty \) through only odd or even \( n \)-values; this trend continues in our computations up to \( n = 3000 \), and is readily proved.

**Proposition 4.** For all \( n \geq 3 \), we have that

\[
\gamma(n) \geq \gamma(n-2).
\]

**Proof.** For \( n \geq 3 \), adjoin 2 as a part to every partition in \( \mathcal{G}_{n-2} \), to produce the partitions in \( \mathcal{G}_n \) having smallest part 2; thus \( \gamma(n) \) is at least equal to \( \gamma(n-2) \). Noting for \( n \geq 6 \) that \( \mathcal{G}_n \) might also contain partitions with smallest part \( \geq 3 \), gives the inequality. \( \square \)

On the other hand, our computations for \( 20 < n \leq 3000 \) display another trend, contradicting our comments above about oscillating behavior.

**Proposition 5.** For all \( n \geq 26 \), we have that

\[
\gamma(n) \geq \gamma(n-1).
\]

\(^4\)That both \( p(n) - p(n-1) = \nu(n) \) and \( \nu(n) - \nu(n-1) = \gamma(n) \) are non-negative is clear.
Table 1. Comparison of $\gamma(n)$, $\nu(n)$, $p(n)$ reproduced from [14].

| $n$ | $\gamma(n)$ | $\nu(n)$ | $p(n)$ |
|-----|--------------|----------|--------|
| 1   | 0            | 0        | 1      |
| 2   | 0            | 1        | 2      |
| 3   | 0            | 1        | 3      |
| 4   | 1            | 2        | 5      |
| 5   | 0            | 2        | 7      |
| 6   | 2            | 4        | 11     |
| 7   | 0            | 4        | 15     |
| 8   | 3            | 7        | 22     |
| 9   | 1            | 8        | 30     |
| 10  | 4            | 12       | 42     |
| 11  | 2            | 14       | 56     |
| 12  | 7            | 21       | 77     |
| 13  | 3            | 24       | 101    |
| 14  | 10           | 34       | 135    |
| 15  | 7            | 41       | 176    |
| 16  | 14           | 55       | 231    |
| 17  | 11           | 66       | 297    |
| 18  | 22           | 88       | 385    |
| 19  | 17           | 105      | 490    |
| 20  | 32           | 137      | 627    |
| ... | ...          | ...      | ...    |
| 100 | 2,307,678    | 21,339,417 | 190,569,292 |

Proof. For $f : \mathbb{Z} \to \mathbb{C}$, let $\Delta f(n) := f(n) - f(n - 1)$ denote the first difference of $f$, let $\Delta^2 f(n) := \Delta(\Delta f(n))$ denote the second difference, and for $r \geq 1$, let $\Delta^r f(n) := \Delta(\Delta^{r-1} f(n))$ denote the $r$th difference of the function $f$. Noting for $n \geq 2$ that $\nu(n) = \Delta p(n)$ and $\gamma(n) = \Delta^2 p(n)$, then the proposition is equivalent to the statement that $\Delta^3 p(n) \geq 0$ for all $n \geq 26$. In [6], Gupta establishes that for each $r \geq 1$, there exists $n_0 = n_0(r) \geq 1$ such that $\Delta^r p(n) \geq 0$ for all $n \geq n_0$. Gupta’s proof exploits the Hardy-Ramanujan asymptotic (2) extended to $\Delta^r p(n)$, using an analytic argument about the error term; for $r = 3$, Gupta gives $n_0(3) = 26$.5 We note that Odlyzko proves in [10] that $\Delta^r p(n)$ oscillates for $n < n_0$, $r \geq 1$, and that $n_0(r) \sim \frac{6}{\pi^2} n^2 (\log n)^2$ as $n \to \infty$. □

Table 2 gives the values of $\gamma(n)$ and $\gamma(n) - \gamma(n - 1)$ for $20 < n \leq 40$. Gupta’s analytic proof [6] of Proposition 5, while elegant, does not provide combinatorial insight into the behavior of $\gamma(n)$. Let us briefly analyze the problem further, by decomposing $G_n$ as a union of two disjoint subsets. Let $G_n^{(1)} \subseteq G_n$ denote the subset of partitions in $G_n$ such that 1 can be subtracted from some part with the resulting partition being in $G_{n-1}$. Let $G_n^{(2)} \subseteq G_n$ denote size-$n$ partitions of either shape $(c, c)$ or $(c, c, 2, 2, \ldots, 2)$, with $c \geq 2$, noting $G_n^{(2)}$ is empty if $n$ is odd, and $G_n^{(1)} \cap G_n^{(2)}$ is empty. If a partition $\lambda \in G_n$ does not lie

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5 The authors are grateful to F. Zanello for pointing out Gupta’s proof to us.
in \( G_n^{(2)} \), then \( \lambda \) will have at least one part from which 1 can be subtracted with the result being in \( G_{n-1} \), and lies in \( G_n^{(1)} \). Thus \( G_n = G_n^{(1)} \cup G_n^{(2)} \), and

\[ \gamma(n) = \#G_n^{(1)} + \#G_n^{(2)}. \]

We derive formulas for \( \#G_n^{(1)}, \#G_n^{(2)} \). For \( x \in \mathbb{R} \), define a modified floor function \([x]^{*} := 0\) if \( x \) is an integer, and \([x]^{*} := [x]\) otherwise. Each partition \( \pi \in G_n^{(2)}, k > 1 \), has either the form \( \pi = (k-j, k-j, k, 2, 2, \ldots, 2) \) with \( k-j > 2 \) and \( j \) copies of 2, \( 0 \leq j < k-2 \), or \( \pi = (2, 2, \ldots, 2) \) with \( k \) copies of 2. Then \( \#G_n^{(2)} = [\frac{n-1}{2}]^{*} \).

To address \( \#G_n^{(1)} \), we need to define another, closely-related subset. For \( k \geq 1 \), let \( G_k^{(0)} \subseteq G_k \) denote the subset of partitions in \( G_k \) that are not of the form \((d, d, \ldots, d)\) with \( d \in \mathbb{N} \) a nontrivial divisor of \( k \), i.e., those \( \sigma_0(k)-2 \) partitions in \( G_k \) whose Young diagrams are not rectangles, with \( \sigma_0(k) \) the number of divisors of \( k \). It follows that

\[ \#G_k^{(0)} = \gamma(k) - \sigma_0(k) + 2. \]

Now, adding 1 to an allowable part of a partition \( \lambda \) in \( G_{n-1}^{(0)} \) (allowable, in that adding 1 results in another ground state non-unitary partition) yields a partition in \( G_n^{(1)} \), and this can be done in one or more ways for each \( \lambda \in G_n^{(0)} \). This resembles the construction of the partitions \( \mathcal{P}_n \) from the partitions \( \mathcal{P}_{n-1} \) in Young’s lattice in representation theory \cite{18}.

Likewise, subtracting 1 from an allowable part of a partition \( \lambda' \) in \( G_n^{(1)} \) (allowable, in that subtracting 1 produces a ground state non-unitary partition) yields a partition in \( G_{n-1}^{(0)} \), and this also might be done in multiple ways for each \( \lambda' \in G_n^{(1)} \).

Due to multiple copies of some partitions being produced in both directions, leading to potential over-counting, it is a difficult combinatorial task to compare the cardinalities of \( G_n^{(0)} \) and \( G_n^{(1)} \) by keeping track of these “add-one” and “subtract-one” mappings. Define a statistic \( \varepsilon(n) \in \mathbb{Z} \) to track the difference between these cardinalities:

\[ \#G_n^{(1)} = \#G_{n-1}^{(0)} + \varepsilon(n). \]

Rewriting (6) in light of (7) and (8), we arrive at an identity

\[ \gamma(n) = \gamma(n-1) + \varepsilon(n) - \sigma_0(n-1) + \left[ \frac{n-1}{2} \right]^{*} + 2, \]

somewhat analogous to (1) and (4), that identifies the difference between \( \gamma(n) \) and \( \gamma(n-1) \). From (9), we see \( \varepsilon(n) \) captures the “wild card” component in the growth of \( \gamma(n) \). Then it follows from Proposition 5 that, for \( n \geq 26 \), we have

\[ \varepsilon(n) \geq \sigma_0(n-1) - \left[ \frac{n-1}{2} \right]^{*} - 2. \]

2.2. Congruences and common divisors. For \( n \geq 0 \), recall the Ramanujan congruences for the partition function \cite{11}:

\[ p(5n+4) \equiv 0 \pmod{5}, \quad p(7n+5) \equiv 0 \pmod{7}, \quad p(11n+6) \equiv 0 \pmod{11}. \]
In [14], the sixth author made a conjecture based on numerical evidence in Table 1:
*Congruences similar to (10) exist for \( \nu(n) \) and \( \gamma(n) \).* However, potential patterns suggestive of this conjecture were likely coincidences based on the small data set in Table 1. In our computations up to \( n = 3000 \), we did not spot Ramanujan congruences for \( \nu(5n + 4), \gamma(7n + 5) \), etc.;\(^7\) but we spotted other Ramanujan-like congruences involving \( \nu(n), \gamma(n) \), that we then proved. We record these observed congruences and their proofs.

**Proposition 6.** For \( n \geq 1 \) we have the following:
\[
p(5n) \equiv \nu(5n) \pmod{5},
\]
\[
p(7n + 6) \equiv \nu(7n + 6) \pmod{7},
\]
\[
p(11n + 7) \equiv \nu(11n + 7) \pmod{11}.
\]

*Proof.* Recall \( a \mid b \) means \( a \in \mathbb{N} \) divides \( b \in \mathbb{N} \). The claimed relations follow immediately from the Ramanujan congruences (10) combined with (1), since the statement \( p(m) \equiv 0 \pmod{c} \) is equivalent to the statement \( c \mid (p(m + 1) - \nu(m + 1)) \).

The proof above depends on known congruences for \( p(n) \); without comparable results for \( \nu(n) \), we cannot prove analogous results involving \( \gamma(n) \). Other congruences \( p(an + b) \equiv 0 \pmod{c} \) (see [1]) will yield further congruences \( \nu(an + b + 1) \equiv \nu(an + b + 1) \pmod{c} \).

\(^7\)We did not check for more general types of congruences such as in [1], for \( \nu(an + b), \gamma(an + b) \).

\[\begin{array}{|c|c|c|}
\hline
n & \gamma(n) & \gamma(n) - \gamma(n - 1) \\
\hline
21 & 28 & -4 \\
22 & 45 & 17 \\
23 & 43 & -2 \\
24 & 67 & 24 \\
25 & 63 & -4 \\
26 & 95 & 32 \\
27 & 96 & 1 \\
28 & 134 & 38 \\
29 & 139 & 5 \\
30 & 192 & 53 \\
31 & 199 & 7 \\
32 & 269 & 70 \\
33 & 287 & 18 \\
34 & 373 & 86 \\
35 & 406 & 33 \\
36 & 521 & 115 \\
37 & 566 & 45 \\
38 & 718 & 152 \\
39 & 792 & 74 \\
40 & 983 & 191 \\
\hline
\end{array}\]

**Table 2.** Growth of \( \gamma(n) \) is monotonic for \( n \geq 25 \).
In our computer searches up to \( n = 3000 \), we also noticed interesting gcd relations; we were initially surprised to find that \( p(n), \nu(n) \) have nontrivial greatest common divisor in 94.6\% of cases. Moreover, in 91.8\% of cases, all three of \( p(n), \nu(n), \gamma(n) \) have \( \gcd > 1 \). Following up on these empirical observations, we proved by elementary means the divisibility properties of \( p(n), \nu(n), \gamma(n) \) are not independent.

**Proposition 7.** For \( n \geq 1 \), we have the following:

\[
\begin{align*}
(i) \quad \gcd (p(n), \nu(n)) &= \gcd (\nu(n), p(n-1)) = \gcd (p(n), p(n-1)) ; \\
(ii) \quad \gcd (\nu(n), \gamma(n)) &= \gcd (\gamma(n), \nu(n-1)) = \gcd (\nu(n), \nu(n-1)) .
\end{align*}
\]

**Proof.** Both of the congruence relations in the theorem are instances of the following fact: if \( x = y + z \) for \( x, y, z \in \mathbb{N} \), then \( \gcd (x, y) = \gcd (y, z) = \gcd (x, z) \). To see this, set \( d := \gcd (x, y) \); then \( d \) is also a divisor of \( z = x - y \). If \( d \geq 1 \) does not equal \( \gcd (y, z) \), then there must exist \( d' > d \) such that \( d' \) divides both \( y, z \). But then \( d' \) also divides \( x = y + z \), contradicting that \( d = \gcd (x, y) \). This proves that \( d = \gcd (y, z) \). By a similar argument, if there exists \( d'' > d \) that divides both \( x, z \), then \( d'' \) divides \( y = x - z \), again contradicting that \( d = \gcd (x, y) \). This proves \( d = \gcd (x, z) \). Setting \( (x, y, z) = (p(n), \nu(n), p(n-1)) \) gives (i). Setting \( (x, y, z) = (\nu(n), \gamma(n), \nu(n-1)) \) gives (ii). \( \square \)

It is interesting that congruence and divisibility properties of \( p(n) \) depend to some extent on the more primitive functions \( \nu(n), \gamma(n) \). In [14], it is conjectured these phenomena could be used to reverse-engineer a proof of the Ramanujan congruences (10) by induction, which would be a useful application, if it is possible. We note that, while the proofs of Propositions 6 and 7 explain our observations, in those cases they do not indicate new partition-theoretic phenomena. On the other hand, our group found gcd relations we could not prove, connecting \( p(n), \nu(n) \), and \( \gamma(n) \), to be ubiquitous in our numerical data. Further study of arithmetic connections between these functions seems warranted.

### 3. Non-unitary partition connections in the authors’ other works

Beyond the earlier usages [3, 7, 9, 17] of non-unitary partitions in the literature, and the study in [14], non-unitary partitions have distinguished themselves in the present authors’ works as being of interest. In [16], non-unitary partitions arise in a statistical application, in the context of estimating a population’s standard deviation using a linear combination of ranges of subsamples of a sample of size \( n \), which are indexed by partitions of size \( n \); only non-unitary partitions can be used (they are called “admissible” partitions there). In the theory of partition zeta functions [12], the partition-theoretic zeta series diverge over subsets of \( \mathcal{P} \) where 1’s can appear as parts with unbounded multiplicity, and non-unitary partitions are a necessary condition for the existence of partition Euler products. In [15, Thm. 29], the Dirichlet series generating function for the number of non-unitary partitions with norm equal to \( n \) is evaluated. In [13], only partitions with no 1’s have nonzero partition phi function \( \varphi_{\mathcal{P}}(\lambda) \). In [4], under the supernorm isomorphism, the set \( \mathcal{N} \) maps bijectively to the odd natural numbers, while partitions with 1’s map to the even numbers. Do non-unitary partitions distinguish themselves naturally in other contexts in the mathematical sciences?
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