A STUDY OF THE NAVIER-STOKES EQUATIONS WITH THE KINEMATIC AND NAVIER BOUNDARY CONDITIONS

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Abstract. We study the initial-boundary value problem of the Navier-Stokes equations for incompressible fluids in a domain in $\mathbb{R}^3$ with compact and smooth boundary, subject to the kinematic and Navier boundary conditions. We first reformulate the Navier boundary condition in terms of the vorticity, which is motivated by the Hodge theory on manifolds with boundary from the viewpoint of differential geometry, and establish basic elliptic estimates for vector fields subject to the kinematic and Navier boundary conditions. Then we develop a spectral theory of the Stokes operator acting on divergence-free vector fields on a domain with the kinematic and Navier boundary conditions. Finally, we employ the spectral theory and the necessary estimates to construct the Galerkin approximate solutions and establish their convergence to global weak solutions, as well as local strong solutions, of the initial-boundary problem. Furthermore, we show as a corollary that, when the slip length tends to zero, the weak solutions constructed converge to a solution to the incompressible Navier-Stokes equations subject to the no-slip boundary condition for almost all time. The inviscid limit of the strong solutions to the unique solutions of the initial-boundary value problem with the slip boundary condition for the Euler equations is also established.

1. Introduction

We are concerned with solutions of the initial-boundary value problem of the Navier-Stokes equations for incompressible fluids in a general domain in $\mathbb{R}^3$ with compact and smooth boundary, subject to the kinematic boundary condition (i.e. the slip condition) and the Navier boundary condition. The incompressible fluid flows are governed by the Navier-Stokes equations:

$$\partial_t u + u \cdot \nabla u = \mu \Delta u - \nabla p, \quad \nabla \cdot u = 0, \quad x \in \Omega \subset \mathbb{R}^3,$$

where $u$ represents the Eulerian velocity vector field of the fluid flow, $p$ is a scalar pressure function (up to a function of time $t$) which maintains the incompressibility of the fluid, $\mu$ is the kinematic viscosity, and the density has been renormalized as one in this setting. As a system of partial differential equations, $u$ and $p$ are unknown functions, and a solution $u$ of (1.1) determines $p$ uniquely up to a function depending only on $t$. The initial condition for the fluid flow is

$$u|_{t=0} = u_0(x).$$
If $\Omega \subset \mathbb{R}^3$ has a non-empty boundary $\Gamma = \partial \Omega$, then system (1.1) must be supplemented with boundary conditions on $u$ in order to be well-posed. In fluid dynamics, if the rigid surface $\Gamma$ is at rest, the kinematic and no-slip conditions are often imposed. The kinematic condition means that the normal component of the velocity vanishes, that is, the velocity $u$ is tangent to the boundary $\Gamma$:

$$u^\perp|_\Gamma = 0,$$

(1.3)

while the no-slip condition demands for the coincidence of the tangent component of the fluid velocity with that of the boundary $\Gamma$. These two boundary conditions lead to the Dirichlet boundary problem associated with the Navier-Stokes equations. There has been a large literature for the Navier-Stokes equations subject to the Dirichlet boundary condition; see [14, 19, 20, 21, 25, 26, 10, 43] and the references cited therein. The fundamental problem of the global (in time) existence and uniqueness of a strong solution remains open; however, the Dirichlet boundary problem of the Navier-Stokes equations is well-posed at least for a small time, or for small data globally in time.

However, the usual no-slip assumption does not always match with the experimental results. Navier [34] first proposed the slip-with-friction boundary condition, that is, the Navier boundary condition: The tangent part of the velocity $u$ is proportional to that of the normal vector field of the stress tensor with proportional constant $\zeta > 0$, which is called the slip length (see (2.5) below). In the recent years, the Navier boundary condition has been received much attention, especially when fluids with larger Reynolds number or fluids past a rigid surface with considerable speeds for which the curvature effect becomes apparent (cf. [11, 15, 17, 18, 23, 47] and the references cited therein). Such boundary conditions can be induced by effects of free capillary boundaries, a perforated boundary, or an exterior electric field (cf. [1, 4, 5, 8, 37, 39]). In particular, this friction boundary condition was rigorously justified as the effective boundary condition for flows over rough boundary; see [16, 17]. Thus, it becomes important to analyze solutions to the equations for such fluids subject to the Navier boundary condition.

The rigorous mathematical analysis of the Navier-Stokes equations with the kinematic and Navier boundary conditions may date back the work by Solonnikov-Šcadilov [12] for the stationary linearized Navier-Stokes system under the boundary condition that the tangent part of the normal vector field of the stress tensor is zero. The existence of weak solutions and regularity for the stationary Navier-Stokes equations with the kinematic and Navier boundary conditions was only recently obtained by Beirão da Veiga [6] for the half-space. In a two-dimensional, simply connected, bounded domain, the well-posedness problem has been rigorously established by Yodovich [46]. See also Clop, Mikhailč, and Robert [9] and Lopes Filho, Nussenzveig Lopes and Planas [28] for the vanishing viscosity limit, and Mucha [33] under some geometrical constraints on the shape of the domain. These two-dimensional results are based on the fact that the vorticity is scalar and satisfies the maximum principle. However, in the three-dimensional case, the standard maximum principle for the vorticity fails, so that the techniques employed in the two-dimensional case cannot be directly extended to this case. Furthermore, the Navier boundary condition causes additional difficulties in developing apriori estimates which require to be compatible with the nonlinear convection term. The main purpose of this paper is to develop a general approach to establish the well-posedness, the no-slip limit, as well as the inviscid limit for the initial-boundary value problem for the Navier-Stokes equations in a general domain.
\[ \Omega \subset \mathbb{R}^3, \text{ subject to the Navier boundary condition, together with the kinematic condition } (1.3) \text{ and the initial condition } (1.2). \]

By careful local computations in Section 2, we first reformulate the Navier boundary condition in terms of the vorticity \( \omega = \nabla \times u \) (see Proposition 2.1 below). This is motivated by the Hodge theory on manifolds with boundary, since the kinematic and vorticity conditions are the natural boundary conditions for the Hodge theory from the viewpoint of differential geometry. Indeed, the Navier boundary condition under the kinematic condition (1.3) is equivalent to the condition that the tangent portion of the vorticity \( \omega \) is of the following form:

\[
\left. \omega^\parallel \right|_\Gamma = -\frac{1}{\zeta} (\ast u) + 2 (\ast \pi(u)),
\]

where \( \pi = (\pi_{ij}) \) is the curvature of the boundary \( \Gamma \), that is, the second fundamental form. The form \( \pi \) is identified with the self-adjoint operator which sends a tangent vector \( (\xi^1, \xi^2) \) on the boundary surface \( \Gamma \) to the tangent vector \( (\sum_j \pi_{1j} \xi^j, \sum_j \pi_{2j} \xi^j) \), the operator \( \ast \) is the Hodge star operator which rotates (towards the interior of the domain \( \Omega \)) a tangent vector \( (\xi^1, \xi^2) \) by 90\(^0\) degree. The Navier boundary condition in form (1.4) has appealing physical interpretation: The vorticity on the boundary \( \Gamma \) is created mainly from the slip of the fluid and the curvature of the boundary \( \Gamma \) (see [22] for more information and further references on the slip length for different fluid media). In particular, the effect of the curvature becomes significant when the curvature of the boundary \( \Gamma \) becomes large (comparable to the reciprocal of the slip length).

In Section 3, we establish some basic elliptic estimates for vector fields subject to the kinematic and Navier boundary conditions. In particular, we establish some \( L^2 \)-estimates which are uniform in the slip length by a careful analysis of several boundary integrals.

In Section 4, we develop a spectral theory of the Stokes operator acting on divergence-free vector fields in a general domain \( \Omega \) subject to the kinematic and Navier boundary conditions. We establish several fundamental estimates for the symmetric form defined by the Stokes operator. Besides the difficulties caused by the divergence-free condition on the vector fields, the Navier boundary condition causes additional difficulties in developing \textit{apriori} estimates for the Galerkin approximations to solutions of the Navier-Stokes equations. To overcome these difficulties, we establish an estimate for the third derivatives of the vector fields satisfying the Navier boundary condition (Theorem 4.4 and Corollary 4.2) and a uniform gradient estimate for the Galerkin approximations (Theorem 4.5). Then, in Section 5, we employ the spectral theory and all the estimates established in Sections 3–4 to construct the Galerkin approximate solutions and establish the global existence of weak solutions and the local existence of strong solutions for the initial-boundary problem (1.1)–(1.4). Furthermore, we show as a corollary that, for any weak solution \( u_\zeta(t, x) \) corresponding to the slip length \( \zeta \) to problem (1.1)–(1.4) constructed in Theorem 5.1, when \( \zeta \to 0 \), there exists a subsequence (still denoted) \( u_\zeta(t, x) \) converging to \( u(t, x) \) such that \( u(t, x) \) is a solution to (1.1) subject to the no-slip condition for almost all time \( t \). On the other hand, when \( \zeta \to \infty \), there also exists a subsequence (still denoted) \( u_\zeta(t, x) \) converging to \( u(t, x) \) such that \( u(t, x) \) is a solution to (1.1) subject to the complete slip boundary condition:

\[
\left. \omega^\parallel \right|_\Gamma = 2 (\ast \pi(u)),
\]
in the weak sense. Such a nonhomogeneous vorticity boundary problem has been carefully investigated in [7].

Finally, in Section 6, we study the inviscid limit and establish the $L^2$-convergence of the strong solutions of problem (1.1)-(1.4) to the unique smooth solution of the initial-boundary value problem with the slip boundary condition for the Euler equations for incompressible fluid flows.

2. The Navier boundary condition

In this section we reformulate the Navier boundary condition in terms of the vorticity for every slip length $\zeta > 0$ and introduce several notions and notations which are used throughout the paper.

For simplicity, we use the conventional notation that the repeated indices in a formula are understood to be summed up from 1 to 3 unless confusion may occur. Furthermore, we use a universal constant $C > 0$ that is independent of the slip length $\zeta > 0$, and a universal constant $M > 0$ that may depend on $\zeta$ among others, which may be different at each occurrence.

We use the same notation for both scalar functions and vector fields in the $L^p$-space and Sobolev spaces $W^{k,p}(\Omega)$ ($H^k(\Omega)$ if $p = 2$). Denote $\|T\|_p$ as the $L^p$-norm of the length $|T|$ of $T$ on $\Omega$ with respect to the Lebesgue measure, and $\|T\|_{L^p(\Gamma)}$ as the $L^p$-norm of the vector field $T$ on the boundary $\Gamma$ with respect to the induced surface-area measure on $\Gamma$. That is,

$$\|T\|_p = \left( \int_{\Omega} |T(x)|^p \, dx \right)^{1/p}, \quad \|T\|_{L^p(\Gamma)} = \left( \int_{\Gamma} |T(x)|^p \, d\mathcal{H}^2(x) \right)^{1/p},$$

where $dx$ is the usual Lebesgue measure on $\mathbb{R}^3$ and $\mathcal{H}^2$ is the two-dimensional Hausdorff measure (i.e. the surface area measure) on $\Gamma$. From now on, $dx$ and $d\mathcal{H}^2(x)$ in the integrals will be suppressed, unless confusion may arise. For a vector field $T$ on $\Omega$,

$$\|T\|_{W^{k,p}} = \left( \sum_{j=0}^{k} \int_{\Omega} |\nabla^j T|^p \, dx \right)^{1/p},$$

where $\nabla^j T$ is the $j$-th derivative of $T$. See [2,13] for the details.

For the bounded domain $\Omega \subset \mathbb{R}^3$ with a boundary $\Gamma = \partial \Omega$ that is smooth, compact, and oriented, unless otherwise specified, we carry out local computations on the boundary in a moving frame compatible to $\Gamma$. More precisely, if $\nu$ is the unit normal to $\Gamma$ pointing outwards with respect to $\Omega$, by a moving frame we mean any local orthonormal basis $(e_1, e_2, e_3)$ of the tangent space $T\Omega$ such that $e_3 = \nu$ when restricted to $\Gamma$. If $u = \sum_{j=1}^{3} u^j e_j$ is a vector field on $\Omega$, then, restricted to the boundary surface $\Gamma$, $u^\parallel = \sum_{j=1,2} u^j e_j$ (resp. $u^\perp = u^3 \nu$) denotes its tangent part (resp. normal part). The Christoffel symbols $\Gamma^k_{ij}$ are determined by the directional derivatives $\nabla_i e_j = \Gamma^k_{ij} e_k$, where $\nabla_i$ is the directional derivative in the direction $e_i$.

The tensor $(\pi_{ij})_{1 \leq i,j \leq 2}$, where $\pi_{ij} = -\Gamma^3_{ij}$ for $i,j = 1,2$, is a symmetric tensor on $\Gamma$, which is the second fundamental form, denoted by $\pi$. That is,

$$\pi(u^\parallel, v^\parallel) := \sum_{i,j=1,2} \pi_{ij} u^i v^j \quad \text{for any } u^\parallel, v^\parallel \in T\Gamma.$$
We say that $\pi$ is bounded above (resp. below) by a constant $\lambda$ if two eigenvalues (which are functions on $\Gamma$) are bounded above (resp. below) by $\lambda$. We will also identify $\pi$ with the linear transformation $(h_{ij})$: If $u^\parallel = \sum_{j=1,2} u^j e_j$ is tangent to $\Gamma$, then

$$\pi(u^\parallel) := \sum_{j=1,2} \pi(u^\parallel) e_j = \sum_{j=1,2} \pi_{ij} u^j e_j,$$

with $\langle \pi(u^\parallel), v^\parallel \rangle = \pi(u^\parallel) - \pi(v^\parallel)$, and $H = \sum_{j=1,2} \pi_{jj}$ is the mean curvature. We refer to [35, 38] for further facts and notations in differential geometry used in this paper.

The boundary surface $\Gamma \subset \mathbb{R}^3$ has a natural induced metric and hence a natural notion of directional derivatives, the Lévi-Civita connection, denoted by $\nabla\Gamma$. The following formulas will be useful in treating with integrals on the boundary $\Gamma$. Let $u \in H^2(\Omega)$ be a vector field on $\Omega$. Then, on $\Gamma$,

$$\langle u \cdot \nabla u, \nu \rangle = -\pi(u^\parallel, u^\parallel) - H|u^\parallel|^2 + \langle u, \nu \rangle \nabla \cdot u + 2\langle u^\parallel, \nabla \langle u, \nu \rangle \rangle - \nabla \cdot ((\langle u, \nu \rangle u^\parallel), (2.1)$$

and

$$\frac{1}{2} \partial_\nu (|u|^2) = (u \times (\nabla \times u), \nu) + (u \cdot \nabla u, \nu). (2.2)$$

The first formula (2.1) may be verified by means of the moving frame method (see [7] for the details). The second follows from the vector identity:

$$\frac{1}{2} \nabla |u|^2 = u \times (\nabla \times u) + u \cdot \nabla u. (2.3)$$

Let $f$ be a scalar function on $\Omega$. Then, as a special case of (2.2),

$$\partial_\nu (|\nabla f|^2) = -2\pi(\nabla f^\parallel, \nabla f^\parallel) - 2H |\partial_\nu f|^2 + \partial_\nu f \Delta f + 4\langle \nabla f^\parallel, \nabla \langle \partial_\nu f \rangle \rangle - 2\nabla \cdot (\partial_\nu f \nabla \langle f^\parallel \rangle). (2.4)$$

The connecting condition over an interface $\Gamma$ of a fluid is expressed as the Navier boundary condition; see Einzel-Panzer-Liu [11] for its physical interpretation. This condition may be written in a moving frame compatible to $\Gamma$ as follows:

$$u^k = -\zeta \left( \nabla_3 u^k + \nabla_k u^3 \right) \quad \text{on } \Gamma \quad \text{for } k = 1, 2, (2.5)$$

where $\zeta$ is the slip length that is a positive scalar function on $\Gamma$ depending only on the nature of the fluid and the material of the rigid boundary. In order to write down (2.5) in a global form in terms of the vorticity and the curvature of $\Gamma$, we recall that the Hodge operator $*$ sends a vector field $(v^1, v^2)$ on the surface $\Gamma$ to $*(v^1, v^2) := (-v^2, v^1)$. The effect of the Hodge operator $*$ is to rotate a vector on $\Gamma$ by $90^\circ$ degree with respect to the normal vector pointing the interior of $\Omega$. The Hodge operator $*$ is independent of the choice of a moving frame on $\Gamma$ and may be defined via the identity:

$$\langle w \times (*u^\parallel), \nu \rangle = \langle u^\parallel, w^\parallel \rangle \quad \text{on } \Gamma \quad (2.6)$$

for any vector fields $u$ and $w$.

**Proposition 2.1.** Let $u \in C^1(\Omega)$. Then $u$ satisfies the Navier boundary condition on $\Gamma$ if and only if

$$\langle \nabla \times u \rangle_{\Gamma} = -\frac{1}{\zeta} (*u^\parallel) - 2(* \nabla \langle u, \nu \rangle) + 2(* \pi(u^\parallel)). (2.7)$$
In terms of the components in a moving frame compatible to the boundary, condition (2.7) takes the following forms:
\[
(\nabla \times u)^1 = \frac{1}{\zeta}u^2 + 2\nabla_2 \langle u, \nu \rangle - 2 \sum_{j=1,2} \pi_{j2}u^j, \\
(\nabla \times u)^2 = -\frac{1}{\zeta}u^1 - 2\nabla_1 \langle u, \nu \rangle + 2 \sum_{j=1,2} \pi_{j1}u^j.
\]

In particular, if \( u \) satisfies the kinematic condition (1.3), then the Navier condition (2.5) is equivalent to (1.4).

**Proof.** It suffices to show the results in a moving frame compatible to the boundary surface \( \Gamma \). Then, for \( k = 1, 2 \),
\[
\nabla_k u^3 = e_k(u^3) + \sum_{j=1,2} \Gamma_{kj}^3u^j = e_k(u, \nu) - \sum_{j=1,2} \pi_{jk}u^j.
\]

Therefore, along the boundary surface \( \Gamma \),
\[
\nabla_3 u^k + \nabla_k u^3 = \varepsilon_{3ka}\omega^a + 2\varepsilon_k(u, \nu) - 2 \sum_{j=1,2} \pi_{jk}u^j \quad \text{for any } k = 1, 2,
\]
so that
\[
\omega^a|_\Gamma = -\frac{1}{\zeta}\varepsilon_{3ka}u^k - 2\varepsilon_{3ka}\varepsilon_k(u, \nu) + 2\varepsilon_{3ka} \sum_{j=1,2} \pi_{jk}u^j \quad \text{for } a = 1, 2,
\]
where \( \varepsilon_{ijk} \) is the Kronecker symbols. This completes the proof. \( \square \)

**Definition 2.1.** Let \( \zeta > 0 \) be a constant. Then a vector field \( u \) on \( \Omega \) is said to satisfy the Navier’s \( \zeta \)-condition if (1.4) holds, which is equivalent to
\[
(\nabla \times u)^1 = \frac{1}{\zeta}u^2 - 2 \sum_{j=1,2} \pi_{j2}u^j, \quad (\nabla \times u)^2 = -\frac{1}{\zeta}u^1 + 2 \sum_{j=1,2} \pi_{j1}u^j \quad \text{(2.8)}
\]
in a moving frame compatible to the boundary surface \( \Gamma \).

In this paper we study the initial-boundary problem (1.1)–(1.4) for the Navier-Stokes equations with fixed constants \( \mu > 0 \) and \( \zeta > 0 \).

### 3. Elliptic estimates for vector fields

The fundamental estimates in the standard elliptic theory state that, for any function \( f \) on \( \Omega \) subject to a certain boundary condition (Dirichlet or Neumann), \( \|f\|_{H^2} \) is dominated by the \( L^2 \)-norm of its Laplacian together with its \( L^2 \)-norm:
\[
\|f\|_{H^2} \leq C(\|\Delta f\|_2 + \|f\|_2)
\]
for some constant \( C \) depending only on \( \Omega \). A correct boundary condition here plays an essential role, and the previous estimate can not be true without a proper boundary condition. A version of elliptic estimates for vector fields has been established in [3] (also see [32]) for vector fields satisfying the Dirichlet or Neumann condition. If \( u \) is a vector field in \( H^2(\Omega) \) such that \( u^1|_{\Gamma_1} = 0 \) and \( u|_{\Gamma_2} = 0 \) for \( \Gamma = \Gamma_1 \cup \Gamma_2 \), then
\[
\|u\|_{H^1(\Omega)} \leq C(\|\nabla \times u\|_2 + \|\nabla \cdot u\|_2 + \|u\|_2), \quad (3.1)
\]
which is a special case of a general result in [3].

For our problem, we need to develop the $L^2$-estimates for divergence-free vector fields that satisfy the kinematic condition (1.3) and Navier’s $ζ$-condition (1.4) in a general domain $Ω$. To our knowledge, these estimates are not covered in the previous literature, although they can be considered as a part of the standard elliptic theory.

We begin with an elementary lemma which implies (3.1) and may be verified by means of integration by parts.

Lemma 3.1. Let $u ∈ H^2(Ω)$ be a vector field on $Ω$. Then

$$\int_Ω (Δ u, u) = -\|∇ u\|_2^2 + \frac{1}{2} \int_Γ \partial_ν (|u|^2)$$

and

$$\int_Ω |∇ u|^2 = \|∇ × u\|_2^2 + \|∇ · u\|_2^2 - \int_Γ (Δ u) (u, ν) + \int_Γ (u · ∇ u, ν).$$

In particular, if $u^⊥ |_Γ = 0$, then

$$\|∇ u\|_2^2 = \|∇ × u\|_2^2 + \|∇ · u\|_2^2 - \int_Γ \pi (u, u).$$

Lemma 3.2. If $g$ is a smooth function on $Ω$ (up to the boundary $Γ$), then

$$\|∇^2 g\|_2^2 = \|Δ g\|_2^2 - \int_Γ \pi ((∇ g)\|, (∇ g)\|) - \int H |∂_ν g|^2 + 2 \int_Γ (∇^Γ g, ∇^Γ (∂_ν g)).$$

This elementary fact can be proved by using integration by parts and the Bochner identity:

$$|∇^2 g|^2 = \frac{1}{2} |Δ g|^2 - ⟨∇ Δ g, ∇ g⟩.$$  

Now we are in a position to prove our first main estimate, an elliptic estimate, for vector fields satisfying (1.3)–(1.4).

Theorem 3.1. There exists a constant $C$ depending only on $Ω$ such that

$$\|∇^2 u\|_2^2 + \frac{1}{ζ} \|∇^Γ u\|_{L^2(Γ)}^2 ≤ C (\|Δ u\|_2^2 + \|u\|_{H^1})$$

for any vector field $u$ satisfying (1.3)–(1.4).

Proof. Under an orthonormal frame, we have

$$\|∇^2 u\|_2^2 = \sum_{k=1}^3 \int_Ω (Δ u^k)^2 - \int_Γ \sum_{i,j=1,2} \pi_{ij} (∇_i u^k) (∇_j u^k)$$

$$- \int_Γ H \sum_{k=1}^3 |∂_ν u^k|^2 + 2 \int_Γ (∇^Γ u^k, ∇^Γ (∂_ν u^k))$$

$$= \|Δ u\|_2^2 - \int_Γ \pi (⟨∇ u^k⟩\|, ⟨∇ u^k⟩\|) - \int_Γ H \sum_{k=1}^3 ⟨∇ u^k, ν⟩^2$$

$$+ 2 \int_Γ (∇^Γ u^k, ∇^Γ ⟨∇ u^k⟩, ν).$$
where the second equality follows from (3.5) applying to \( g = u^k \). The second and third boundary integrals can be dominated by \( \int_{\Gamma} \| \nabla u \|^2 \). Thus, we have to handle the last boundary integral, where we use the Navier boundary condition (1.4). Working in a frame compatible to \( \Gamma \), since \( u^\perp |_{\Gamma} = 0 \) so that \( \nabla^\Gamma u^3 = 0 \), then

\[
I = \sum_{k=1}^{3} \frac{3}{\Gamma} \left( \nabla^\Gamma u^k, \nabla^\Gamma (\partial_v u^k) \right) = \sum_{k=1,2} \left( \nabla^\Gamma u^k, \nabla^\Gamma (\partial_v u^k) \right).
\]

On \( \Gamma \), \( u^1 \) and \( u^2 \) are the tangent components of \( u \), and \( u^3 = 0 \),

\[
\partial_v u^k = e_3(u^k) = \nabla_3 u^k - \sum_{j=1,2} w^j \Gamma_{3j}^j,
\]

and \( \nabla_k u^3 = -\sum_{i=1,2} u^i \pi_{ki} \), \( k = 1, 2 \). For \( \omega = \nabla \times u \), \( \nabla_3 u^k - \nabla_k u^3 = \varepsilon_{3kj} \omega^j \) so that, for \( k = 1, 2 \),

\[
\partial_v u^k = \varepsilon_{3kj} \omega^j + \nabla_k u^3 - \sum_{j=1,2} w^j \Gamma_{3j}^k \varepsilon_{3kj} \omega^j - \sum_{i=1,2} u^i \pi_{ki} - \sum_{j=1,2} u^j \Gamma_{3j}^k.
\] (3.9)

According to the Navier’s \( \zeta \)-condition (1.4) (also see the proof of Proposition 2.1):

\[
\omega^j = -\frac{1}{\zeta} \varepsilon_{3aj} u^a + 2 \varepsilon_{3aj} \sum_{b=1,2} h_{ba} u^b.
\]

Substitution it into (3.9) yields

\[
\partial_v u^k = -\frac{1}{\zeta} \varepsilon_{3aj} \varepsilon_{3kj} u^a + 2 \varepsilon_{3aj} \varepsilon_{3aj} \sum_{b=1,2} h_{ba} u^b - \sum_{i=1,2} u^i \pi_{ki} - \sum_{j=1,2} u^j \Gamma_{3j}^k.
\] (3.10)

It follows that

\[
\nabla^\Gamma (\partial_v u^k) = -\frac{1}{\zeta} \nabla^\Gamma u^k + \nabla^\Gamma \left( \sum_{i=1,2} \pi_{ik} u^i \right) - \nabla^\Gamma \left( \sum_{j=1,2} w^j \Gamma_{3j}^k \right),
\]

so that

\[
\sum_{k=1,2} \left( \nabla^\Gamma u^k, \nabla^\Gamma (\partial_v u^k) \right) = -\frac{1}{\zeta} \| \nabla^\Gamma u \|^2 + \sum_{k=1,2} \left( \nabla^\Gamma u^k, \nabla^\Gamma \left( \sum_{i=1,2} \pi_{ik} u^i \right) \right) - \sum_{k=1,2} \nabla^\Gamma \left( \sum_{j=1,2} w^j \Gamma_{3j}^k \right) \right)
\]

\[
\leq -\frac{1}{\zeta} \| \nabla^\Gamma u \|^2 + C \left( \| \nabla^\Gamma u \|^2 + \| u \|^2 \right).
\] (3.11)

Therefore, we have

\[
I \leq -\frac{1}{\zeta} \int_{\Gamma} \| \nabla^\Gamma u \|^2 + C \int_{\Gamma} \left( \| \nabla^\Gamma u \|^2 + \| u \|^2 \right),
\]

where \( C \) is a constant depending only on \( \Omega \). Combining this inequality with (3.8) yields

\[
\| \nabla^2 u \|^2 + \frac{2}{\zeta} \int_{\Gamma} \| \nabla^\Gamma u \|^2 \leq \| \Delta u \|^2 + C \int_{\Gamma} \left( \| \nabla^\Gamma u \|^2 + \| u \|^2 \right).
\]
Then the conclusion follows from the Sobolev embedding:
\[ \int_{\Gamma} |\nabla u|^2 \leq \varepsilon \|\nabla^2 u\|_2^2 + \frac{C}{\varepsilon} \|\nabla u\|_2^2 \]
for some constant \( C = C(\Omega) \), independent of \( \varepsilon > 0 \), since the boundary \( \Gamma \) has bounded geometry.

As a consequence, we have the following elliptic estimate.

**Corollary 3.1.** There exists a positive constant \( C \) depending only on \( \Omega \) such that
\[ \|u\|_{H^2}^2 + \frac{1}{\zeta} \|\nabla u\|_{L_2(\Gamma)}^2 \leq C \|\nabla \times (\nabla \times u), \nabla \times u, u\|_2^2 \] (3.13)
for any vector field \( u \in H^2(\Omega) \) satisfying (1.3)–(1.4).

Therefore, for any divergence-free vector field \( u \) on \( \Omega \) satisfying (1.3)–(1.4),
\[ C \|\nabla \times (\nabla \times u, \nabla \times u, u)\|_2^2 \leq \|u\|_{H^2}^2 \leq C^{-1} \|\nabla \times (\nabla \times u, \nabla \times u, u)\|_2^2 \] (3.14)
for some constant \( C > 0 \) depending only on the domain \( \Omega \), but independent of \( \zeta > 0 \).

4. The Stokes Operator with the Navier Boundary Condition

In this section, we develop a theory of the Stokes operator acting on divergence-free vector fields in a general domain \( \Omega \) subject to the kinematic and Navier boundary conditions (1.3)–(1.4).

Note that the divergence operator \( \nabla \cdot \) defined for smooth vector fields with compact supports in \( \Omega \) is closable in \( L^2(\Omega) \). The kernel, \( \text{ker}(\nabla \cdot) \), is a closed subspace of \( L^2(\Omega) \), denoted by \( K_2(\Omega) \). Any vector field \( u \in K_2(\Omega) \cap H^1(\Omega) \) is divergence-free: \( \nabla \cdot u = 0 \), and satisfies the kinematic condition (1.3). The orthogonal complement of \( K_2(\Omega) \) is a closed subspace of \( L^2(\Omega) \), denoted by \( G_2(\Omega) \), and the decomposition
\[ L^2(\Omega) = K_2(\Omega) \oplus G_2(\Omega) \]
is called the Helmholtz decomposition. Any element in \( G_2(\Omega) \) can be identified with the gradient of a scalar function, that is,
\[ G_2(\Omega) = \{ \nabla p \in L^2(\Omega) : p \in L^2_{\text{loc}}(\Omega) \} \].

Let
\[ P_\infty : L^2(\Omega) \to K_2(\Omega) \]
be the projection from \( L^2(\Omega) \) onto \( K_2(\Omega) \). The following fact is easy but important.

**Lemma 4.1.** Let \( u \in H^1(\Omega) \), and let \( u = P_\infty(u) + \nabla q_u \) be the Helmholtz decomposition of \( u \). Then
\[ \nabla \times P_\infty(u) = \nabla \times u, \quad \nabla \cdot P_\infty(u) = 0, \quad P_\infty(u) \big|_\Gamma = 0. \]

**Proposition 4.1.** Let \( u \in H^1(\Omega) \). Then
\[ \|\nabla P_\infty(u)\|_2^2 = \|\nabla \times u\|_2^2 - \int_{\Gamma} \pi(P_\infty(u), P_\infty(u)), \] (4.1)

and
\[ \|\nabla P_\infty(u)\|_2 \leq C \|\nabla \times (u, u)\|_2 \] (4.2)
for some constant \( C > 0 \) depending only on \( \Omega \).
Remark 4.1. Therefore, \( C \) and (4.3) follows from the Navier's ζ components, we can easily see that 
\[ \nabla \cdot K \nabla \cdot \nabla u \Delta K \nabla \nabla \cdot \nabla x \]
which gives (4.2). Then (4.1) follows from integration by parts and (4.2).

The Stokes operator \( S \) can be defined to be the composition \( S = P_\infty \circ \Delta \) with domain \( H^2(\Omega) \). We often restrict the Stokes operator on the Hilbert space \( K_2(\Omega) \), hence with domain \( H^2(\Omega) \cap K_2(\Omega) \), but we will use the same notation \( S \) if no confusion may arise.

### 4.1. The Stokes operator with the Navier boundary condition \( (\Omega) \)

Let \( \zeta > 0 \) be a constant, and let \( D_{0,\zeta}(S) \) be the space of all vector fields \( u \in K_2(\Omega) \cap C^\infty(\Omega) \) (so that \( \nabla \cdot u = 0 \) and \( u^\perp |_{\Gamma} = 0 \)) which satisfy the Navier's ζ-condition \( (\Omega) \). Then \( D_{0,\zeta}(S) \) is dense in \( K_2(\Omega) \), and \( (S,D_{0,\zeta}(S)) \) is a densely defined linear operator on the Hilbert space \( K_2(\Omega) \).

**Lemma 4.2.** Let \( u \in D_{0,\zeta}(S) \), and let \( \Delta u = S(u) + \nabla p \) be the Helmholtz decomposition of \( \Delta u \). Then \( p \) is the unique solution (up to a constant) of the Neumann boundary problem:

\[ \nabla p = \nabla \cdot (\nabla u), \quad \partial_\nu p |_{\Gamma} = \frac{1}{\zeta} \nabla \cdot u - 2\nabla u \cdot \pi(u). \]  

**Proof.** Since \( \nabla u = Su + \nabla p \), then, by taking the divergence and considering the normal components, we can easily see that \( p \) satisfies the Poisson equation:

\[ \nabla p = \nabla \cdot (\nabla u), \quad \partial_\nu p |_{\Gamma} = \langle \Delta u, \nu \rangle. \]  

Since \( \nabla \cdot u = 0 \), \( \Delta u = -\nabla \times (\nabla \times u) \) so that

\[ \langle \Delta u, \nu \rangle = -\langle \nabla \times (\nabla \times u), \nu \rangle = -\nabla \times (\nabla \times u) \parallel, \]

and (4.3) follows from the Navier’s ζ-condition \( (\Omega) \).

**Remark 4.1.** For any \( p \geq 2 \), there exists a constant \( C(p) > 0 \) depending only on \( \Omega \) (e.g., \( C(2) = 1 \)) such that

\[ \| \nabla p \|_p \leq C \| \Delta u \|_p. \]

Therefore,

\[ \| S(u) \|_p \leq C(p) \| \Delta u \|_p \quad \text{for any } u \in D_{0,\zeta}(S). \]

Of course, \( \| S(u) \|_2 = \| P_\infty \Delta u \|_2 \leq \| \Delta u \|_2 \), if \( u \in H^2(\Omega) \).

**Theorem 4.1.** Consider the bilinear form \( (E, D_{0,\zeta}(S)) \) on \( K_2(\Omega) \):

\[ E(u, w) = -\int_\Omega \langle S u, w \rangle \quad \text{for any } u, w \in D_{0,\zeta}(S). \]  

Then

(i) The bilinear form \( (E, D_{0,\zeta}(S)) \) on the Hilbert space \( K_2(\Omega) \) is densely definite, symmetric, and

\[ E(u, w) = \int_\Omega \langle u, \nabla u \rangle + \frac{1}{\zeta} \int_\Gamma \langle u, w \rangle - \int_\Gamma \pi(u, w) \quad \text{for any } u, w \in D_{0,\zeta}(S). \]  

(ii) For any \( \epsilon \in (0, 1) \), there exists a constant \( C(\epsilon, \Omega) \) such that

\[ E(u, u) \geq (1 - \epsilon) \| \nabla u \|_2^2 - C(\epsilon, \Omega) \| u \|_2^2 \quad \text{for any } u \in D_{0,\zeta}(S). \]  

(4.7)
(iii) \((E, D_0, \zeta(S))\) is closable on \(K_2(\Omega)\), its closure is denoted by \((E, D_\zeta(E))\). Identity (4.6) remains true for any \(u, w \in D_\zeta(E)\).

(iv) If \(\pi \leq \frac{1}{\zeta}\), then

\[
E(u, u) \geq \|\nabla u\|^2_2 \quad \text{for any} \ u \in D_\zeta(E).
\]

(v) \(D_\zeta(E) = K_2(\Omega) \cap H^1(\Omega)\) which is thus independent of \(\zeta\) and hence denoted by \(D(E)\).

Proof. Let \(u, w \in D_0, \zeta(S)\) and write \(\Delta u = S(u) + \nabla p\). Since \(\nabla \cdot u = 0\), then

\[
S(u) = -\nabla \times (\nabla \times u) - \nabla p,
\]

(4.9)

where \(p\) solves the Neumann problem (4.3). Taking inner product on both sides of (4.9) with \(w\) and integration by parts on \(\Omega\) yields

\[
E(u, w) = \int_\Omega \langle \nabla \times (\nabla \times u), w \rangle + \int_\Omega \langle \nabla p, w \rangle + \frac{1}{\zeta} \int_\Gamma \langle u, w \rangle,
\]

where we have used (4.3)–(4.4) so that

\[
\langle (\nabla \times u) \times w, \nu \rangle = \frac{1}{\zeta} \langle u, w \rangle - 2\pi(u, w).
\]

(4.10)

Therefore, \((u, w) \to E(u, w)\) is symmetric and bilinear. Since \(u^\perp|_\Gamma = w^\perp|_\Gamma = 0\), then

\[
\int_\Omega \langle \nabla u, \nabla w \rangle = \int_\Omega \langle \nabla \times u, \nabla \times w \rangle + \int_\Gamma \pi(u, w),
\]

and hence

\[
E(u, w) = \int_\Omega \langle \nabla u, \nabla w \rangle + \frac{1}{\zeta} \int_\Gamma \langle u, w \rangle.
\]

(4.11)

If \(\pi \leq \frac{1}{\zeta}\), then

\[
E(u, u) \geq \|\nabla u\|^2_2.
\]

Let \(\lambda_1\) be a upper bound of the second fundamental form \(\pi\), i.e., \(\pi \leq \lambda_1\), then

\[
E(u, u) \geq \|\nabla u\|^2_2 - \lambda_1 \int_\Gamma |u|^2 \geq (1 - \varepsilon)\|\nabla u\|^2_2 - \frac{C}{\varepsilon}\|u\|^2_2
\]

for some \(C = C(\Omega) > 0\), where we have used the trace imbedding inequality:

\[
\int_\Gamma |u|^2 \leq \varepsilon\|\nabla u\|^2_2 + \frac{C}{\varepsilon}\|u\|^2_2.
\]

(4.12)

Next, we show that \((E, D_0, \zeta(S))\) is closable on \(K_2(\Omega)\). Indeed, if \(u_n \in D_0, \zeta(S)\) such that \(E(u_n - u_m, u_n - u_m) \to 0\) and \(\|u_n - u_m\|^2_2 \to 0\), then, since

\[
\frac{1}{2}\|\nabla(u_n - u_m)\|^2_2 \leq C\|u_n - u_m\|^2_2 + E(u_n - u_m, u_n - u_m),
\]

we have

\[
\|\nabla(u_n - u_m)\|^2_2 \to 0.
\]
That is, \( \{u_n\} \) is a Cauchy sequence in \( H^1(\Omega) \), and hence there exists a unique \( u \in H^1(\Omega) \) such that
\[
\|u_n - u\|_2^2 + \|\nabla (u_n - u)\|_2^2 \to 0.
\]
It follows by the Sobolev imbedding that
\[
\lim_{n \to \infty} \int_\Gamma \langle u_n, u_n \rangle = \int_\Gamma |u|^2, \quad \lim_{n \to \infty} \int_\Gamma \pi(u_n, u_n) = \int_\Gamma \pi(u, u)
\]
so that
\[
\lim_{n \to \infty} \mathcal{E}(u_n, u_n) = \int_\Omega |\nabla u|^2 + \frac{1}{\zeta} \int_\Omega |u|^2 - \int_\Gamma \pi(u, u),
\]
and \( u \) belongs to the closure of \( (\mathcal{E}, D_{0, \zeta}(\mathcal{S})) \).

Finally, we prove that \( D_\zeta(\mathcal{E}) = K_2(\Omega) \cap H^1(\Omega) \), which is not surprising, since the Navier’s \( \zeta \)-condition \[1.4\] that has to be satisfied for any \( u \in D_{0, \zeta}(\mathcal{S}) \) will be “forgot” when passing to the limit in \( H^1(\Omega) \) (in which the boundary values of the first derivative can not be retained). Therefore, \( D_{0, \zeta}(\mathcal{S}) \) is dense in \( K_2(\Omega) \cap H^1(\Omega) \) in the \( H^1 \)-norm.

To see this, consider the case that \( \Omega = \{(x_1, x_2, x_3) : x_3 > 0\} \), and \( u = (u^1, u^2, u^3) \in \mathcal{D}(\mathcal{S}) \cap C^0(\mathbb{R}^3) \) such that \( \nabla \cdot u = 0 \) and \( u^3(x_1, x_2, 0) = 0 \). Then, for every \( \varepsilon > 0 \), choose
\[
(u_{\varepsilon}^1, u_{\varepsilon}^2, u_{\varepsilon}^3) = (u^1, u^2, u^3) + x_3 \mathcal{X}_{\{x_3 < \varepsilon\}}(h_1, h_2, h_3).
\]

Then
\[
|\nabla \times u_{\varepsilon}|_\Gamma = |\partial_{x_3} u_{\varepsilon}^2|_\Gamma = \partial_{x_3} u^2(x_1, x_2, 0) + h_2(x_1, x_2),
\]
\[
|\nabla \times u_{\varepsilon}|_\Gamma = - |\partial_{x_3} u_{\varepsilon}^1|_\Gamma = - \partial_{x_3} u^1(x_1, x_2, 0) - h_1(x_1, x_2),
\]
\[
\nabla \cdot u_{\varepsilon} = \mathcal{X}_{\{x_3 < \varepsilon\}}(x_3 \nabla \cdot h + h_3).
\]

To match the Navier’s \( \zeta \)-condition, we set
\[
h_j(x_1, x_2) = \frac{1}{\zeta} u^j(x_1, x_2, 0) - \varepsilon x^j(x_1, x_2, 0), \quad j = 1, 2,
\]
\[
h_3(x_1, x_2) = - x_3 (\partial_{x_1} h_1 + \partial_{x_2} h_2).
\]

Hence, \( u_{\varepsilon} \in D_{\zeta}(\mathcal{E}) \) and \( u_{\varepsilon} \to u \) in \( H^1 \), which concludes the proof.

\[\square\]

**Corollary 4.1.** \( (\mathcal{E}, D(\mathcal{E})) \) is a densely defined, bounded below, and closed symmetric form on the Hilbert space \( K_2(\Omega) \). Moreover,
\[
\mathcal{E}(u, u) = \|\nabla u\|_2^2 + \frac{1}{\zeta} \|u\|_{L^2(\Gamma)}^2 - \int_\Gamma \pi(u, u)
\]
for any \( u \in D_\zeta(\mathcal{E}) \),
and there exists \( M(\varepsilon, \zeta) > 0 \) such that
\[
\|\nabla u\|_2^2 \leq (\mathcal{E} + M)(u, u) \leq (1 + \varepsilon)\|\nabla u\|_2^2 + M(\varepsilon, \zeta)\|u\|_2^2
\]
for all \( u \in D_\zeta(\mathcal{E}) \). \quad (4.13)

**Proof.** Suppose that \( \pi \geq -C_0 \) for some \( C_0 \geq 0 \). Then
\[
\mathcal{E}(u, u) \leq \|\nabla u\|_2^2 + \frac{1}{\zeta} \|u\|_{L^2(\Gamma)}^2.
\]
The Sobolev imbedding yields that, for every \( \varepsilon \in (0, 1) \), there exists \( C_0 > 0 \) such that
\[
\left(\frac{1}{\zeta} + C_0\right)\|u\|_{L^2(\Gamma)}^2 \leq \varepsilon \|\nabla u\|_2^2 + M(\varepsilon, \zeta)\|u\|_2^2
\]
so that (4.13) follows.

\[\square\]
**Definition 4.1.** Let \( \zeta > 0 \). Then the unique self-adjoint operator on \( K_2(\Omega) \) associated with the closed symmetric form \((\mathcal{E}, D_\zeta(\mathcal{E}))\) is denoted again by \( S \), with its domain \( D_\zeta(S) \), called the Stokes operator with Navier’s \( \zeta \)-condition, or simply the Stokes operator if no confusion may arise.

According to Definition 4.1, \((S, D_\zeta(S))\) is the unique self-adjoint operator on \( K_2(\Omega) \) such that
\[
\mathcal{E}(u, w) = -\int_\Omega \langle Su, w \rangle \quad \text{for any } u \in D_\zeta(S), \ w \in D_\zeta(S),
\]
and
\[
D_{0,\zeta}(S) \subset D_\zeta(S) \subset H^1(\Omega) \cap K_2(\Omega).
\]
Moreover, if \( u \in D_\zeta(S) \), then \( u \in H^1(\Omega) \) with \( \nabla \cdot u = 0 \) and \( u^+|_\Gamma = 0 \). In particular, there exists \( \Lambda \geq 0 \) such that \(-S + \Lambda I\) is positive definite (when \( \pi \leq \frac{1}{\zeta}, \Lambda = 0 \)).

To end this section, we establish an \( L^2 \)-estimate for the total derivative of \( S(u) \).

**Lemma 4.3.** Let \( p_u \) be the unique solution (up to a constant) of the Neumann problem:
\[
\Delta p_u = 0, \quad \partial_\nu p_u|_\Gamma = \frac{1}{\zeta} \nabla \cdot u - 2 \nabla \cdot \pi(u) \tag{4.14}
\]
for \( u \in H^2(\Omega) \) satisfying \( u^+|_\Gamma = 0 \). Then, for every \( \epsilon > 0 \), there exists a constant \( M(\epsilon, \zeta) \) depending only on \( \epsilon, \zeta \), and the domain \( \Omega \) such that
\[
\|\nabla p_u\|_2 \leq \epsilon \|
abla \times \nabla \times u\|_2 + M(\epsilon, \zeta)\|u\|_2. \tag{4.15}
\]

**Proof.** By integration by parts, one obtains
\[
\|\nabla p_u\|_2^2 = -\int_\Omega p_u \Delta p_u + \int_\Gamma p_u \partial_\nu p_u
\]
\[
= \frac{1}{\zeta} \int_\Gamma p_u \nabla \cdot u - 2 \int_\Gamma p_u \nabla \cdot \pi(u)
\]
\[
\leq \|p_u\|_{L^2(\Gamma)} M(\epsilon) \|\nabla^2 u\|_2 + \|u\|_{H^1}\]
\[
\leq \|\nabla p_u\|_{L^2(\Gamma)} (\epsilon \|\nabla^2 u\|_2 + \|u\|_{H^1})
\]
for some constant \( M > 0 \) which may depend on \( \zeta \) and \( \epsilon \). Then (4.15) follows from (3.1). \( \square \)

**Proposition 4.2.** Let \( u \in D_{0,\zeta}(S) \), \( \omega = \nabla \times u \), and \( \psi = \nabla \times \omega = -\Delta u \). Then
\[
\|S(u)\|_{H^1} \leq M(\|\nabla \times \psi\|_2 + \|\psi, u\|_2), \tag{4.16}
\]
where \( M(\epsilon, \zeta) > 0 \) depend only on \( \epsilon, \zeta \), and \( \Omega \).

**Proof.** Since \( \nabla \cdot S(u) = 0 \) and \( S(u)^+|_\Gamma = 0 \), according to (3.1),
\[
\|S(u)\|_{H^1} \leq C \|(S(u), \nabla \times S(u))\|_2^2,
\]
where \( C > 0 \) depends only on \( \Omega \).

Let \( S(u) = -\psi - \nabla p \), where \( p \) solves (4.14). Then \( \nabla \times S(u) = -\nabla \times \psi \) so that
\[
\|S(u)\|_{H^1}^2 = C \|(\nabla \times \psi, S(u))\|_2^2.
\]
Together with (4.15), (4.16) follows immediately. \( \square \)
4.2. Spectral theory of the Stokes operator subject to the kinematic and Navier boundary conditions (1.3)–(1.4). To establish other important properties of the Stokes operator \((S, D_{\zeta}(S))\), we study the boundary problem of the Stokes equation:

\[
\lambda u + \Delta u - \nabla p = f, \quad \nabla \cdot u = 0
\]

subject to the boundary conditions (1.3)–(1.4), where \(f \in K_2(\Omega) \cap C^\infty(\Omega)\) and \(\lambda \in \mathbb{R}\) is a constant.

Taking the divergence to both sides of equation (4.17) yields that the scalar function \(p\) satisfies the Neumann problem of the Laplace equation:

\[
\Delta p = 0, \quad \partial_{\nu} p \big|_\Gamma = \frac{1}{\zeta} \nabla^\Gamma \cdot u - 2\nabla^\Gamma \cdot \pi(u).
\]

Let \(\Lambda > 0\) be the constant such that \(-S + \Lambda I \geq 0\). Then, for \(\lambda > \Lambda\), let \(R_\lambda\) denote the resolvent, i.e.,

\[
R_\lambda = (\lambda I - S)^{-1},
\]

which is a bounded linear operator on \(K_2(\Omega)\).

**Theorem 4.2.** For any \(\lambda > \Lambda\), \(R_\lambda\) is a compact operator on \(K_2(\Omega)\).

**Proof.** Let \(f \in K_2(\Omega)\) and \(u = R_\lambda f\). Then \(u \in D_{\zeta}(S)\) and

\[
(\lambda I - S) u = f.
\]

Suppose in addition that \(u \in D_{0,\zeta}(S)\), so that \(\nabla \cdot u = 0\), and \(u\) satisfies (1.3)–(1.4). Let \(\Delta u = Su + \nabla p\). Then

\[
\lambda u - \Delta u + \nabla p = f, \quad \nabla \cdot u = 0,
\]

and \(p\) solves the Neumann problem (4.18). Hence, for every \(\varepsilon > 0\),

\[
\|\nabla p\|_2 \leq \varepsilon \|\nabla^2 u\|_2 + M(\varepsilon, \zeta) \|u\|_{H^1}
\]

for some constant \(M(\varepsilon, \zeta)\). It is easy to devise the energy estimate for \(u\). Indeed, since

\[
\lambda \langle u, u \rangle - \langle u, Su \rangle = \langle u, f \rangle,
\]

then

\[
\lambda \|u\|_2^2 - \int_\Omega \langle u, Su \rangle = \int_\Omega \langle u, f \rangle \leq \|u\|_2 \|f\|_2.
\]

Furthermore, since \(-S + \Lambda \geq 0\), i.e., \(- \int_\Omega \langle u, Su \rangle \geq \|\nabla u\|_2^2 - \Lambda \|u\|_{L^2}\), we have

\[
(\lambda - \Lambda) \|u\|_2^2 + \|\nabla u\|_2^2 \leq \|u\|_2 \|f\|_2.
\]

Therefore, we have

\[
\frac{3(\lambda - \Lambda)}{4} \|u\|_2^2 + \|\nabla u\|_2^2 \leq \frac{1}{\lambda - \Lambda} \|f\|_2^2.
\]

Next, we estimate the second-order derivative of \(u\). Since \(u\) satisfies (1.3)–(1.4), according to the elliptic estimate (Theorem 3.1):

\[
\|\nabla^2 u\|_2^2 + \frac{2}{\zeta} \|\nabla^\Gamma u\|_{L^2(\Gamma)}^2 \leq C \left( \|\Delta u\|_2^2 + \|u\|_{H^1}^2 \right),
\]

then

\[
\|\nabla^2 u\|_2^2 + \frac{2}{\zeta} \|\nabla^\Gamma u\|_{L^2(\Gamma)}^2 \leq C \left( \|\Delta u\|_2^2 + \|u\|_{H^1}^2 \right)
\]

\[
\leq C \left( \|\nabla p\|_2^2 + \|f\|_2^2 + \|u\|_{H^1}^2 \right)
\]

\[
\leq \varepsilon \|\nabla^2 u\|_2^2 + M(\varepsilon, \zeta) \left( \|u\|_{H^1}^2 + \|f\|_2^2 \right).
\]
It follows that
\[ \|\nabla^2 u\|_2^2 \leq M(\varepsilon, \zeta, \lambda) \left( \|u\|_{H^1}^2 + \|f\|_2^2 \right) \leq M(\varepsilon, \zeta, \lambda) \|f\|_2^2, \]
so that
\[ \|u\|_{H^2} \leq M(\zeta, \lambda, \Lambda) \|f\|_2 \]  
(4.22)
for some constant $M > 0$ depending only on $\zeta, \lambda$, and $\Lambda$. Therefore, $R_{\lambda}$ is compact. \qed

**Theorem 4.3.** The spectrum of the Stokes operator $(S, D_\zeta(S))$ with Navier's $\zeta$-condition \[1.4] is discrete and belongs to $(-\infty, \Lambda]$ for some constant $\Lambda = \Lambda(\Omega, \zeta)$. The eigenvalues $\lambda_j \leq \Lambda$ can be ordered as
\[ \Lambda \geq \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq \cdots, \quad \lambda_n \downarrow -\infty. \]
Moreover, there are eigenfunctions $a_n$: $S a_n = \lambda_n a_n$, where $a_n \in D_{0,\zeta}(S)$ (so that $a_n$ satisfy \[1.3]-\[1.4]) such that $\{a_n : n \geq 0\}$ is a complete orthonormal basis of $K_2(\Omega)$. In particular, when $\pi \leq \frac{1}{2}$, $\Lambda = 0$, i.e., $\lambda_j \leq 0$ for $j = 0, 1, 2, \ldots$.

**Proof.** The standard spectral theory for self-adjoint operators yields that the spectrum of $(S, D_\zeta(S))$ belongs to $(-\infty, \Lambda]$ and is discrete, and there exists an orthonormal basis $\{a_n : n \geq 0\}$, where each $a_n \in D_\zeta(S)$ and $S a_n = \lambda_n a_n$ for the corresponding eigenvalues $\lambda_n$. The standard elliptic theory then implies that $a_n \in D_{0,\zeta}(S)$. \qed

### 4.3. Several facts about the Navier's $\zeta$-condition \[1.4]

In what follows, we will assume that $\{a_n : n \geq 0\}$ is the orthonormal basis of $K_2(\Omega)$ constructed in Theorem 4.3. Let $N$ be an integer, which may be infinity. Let $X_N$ be the Hilbert space spanned by $\{a_k : k \leq N\}$, and let $P_N : L^2(\Omega) \to X_N$ be the projection. That is, for every $u \in L^2(\Omega),$
\[ P_N u = \sum_{k=0}^{N} a_k \int_\Omega \langle a_k, u \rangle. \]  
(4.23)

Of course, $P_\infty u = \sum_{k=0}^{\infty} a_k \int_\Omega \langle a_k, u \rangle$ is the projection from $L^2(\Omega)$ onto $K_2(\Omega)$. If $N < \infty$, $P_N(u) \in D_{0,\zeta}(S)$ for any $u \in L^2(\Omega)$.

**Proposition 4.3.** Let $u \in \cup_{N \in \mathbb{N}} X_N$ the vector space spanned by elements in $X_N$ where $N$ runs over all natural numbers, $\omega = \nabla \times u$, and $\psi = \nabla \times \omega = -\Delta u$. Then
\[ \langle \psi, \nu \rangle_{\Gamma} = -\frac{1}{\zeta} \nabla^\Gamma \cdot u + 2\nabla^\Gamma \cdot \pi(u), \]  
(4.24)

and
\[ (\nabla \times \psi)_{\Gamma} = \frac{1}{\zeta} \left( \ast S(u) \right) - 2 \left( \ast \pi(S(u)) \right). \]  
(4.25)

**Proof.** Let $u \in X_N$ for some $N$. By definition, $S(u) = \Delta u - \nabla p$ so that $\psi = -S(u) - \nabla p$. Since $\nabla \cdot S(u) = 0$ and $S(u)_{\Gamma} = 0$, then $p$ is the solution of the Neumann problem:
\[ \Delta p = 0, \quad \partial_{\nu} p |_{\Gamma} = \frac{1}{\zeta} \nabla^\Gamma \cdot u + 2\nabla^\Gamma \cdot \pi(u), \]  
(4.26)
and $\langle \psi, \nu \rangle_{\Gamma} = -\partial_{\nu} p |_{\Gamma}$, which yields (4.24). To see the tangent component of $\nabla \times \psi$, we note that $\nabla \times \psi = -\nabla \times S(u)$. Since $u = \sum_{k=0}^{N} a_k \int_\Omega \langle a_k, u \rangle,$ then
\[ S(u) = \sum_{k=0}^{N} S(a_k) \int_\Omega \langle a_k, u \rangle = -\sum_{k=0}^{N} \lambda_k a_k \int_\Omega \langle a_k, u \rangle. \]
Therefore, 

\[ \nabla \times \psi = -\nabla \times S(u) = \sum_{k=0}^{N} \lambda_k (\nabla \times a_k) \int_{\Omega} \langle a_k, u \rangle, \]

and it follows that

\[ (\nabla \times \psi)^\| \big|_\Gamma = - \sum_{k=0}^{N} \lambda_k (\nabla \times a_k)^\| \int_{\Omega} \langle a_k, u \rangle \]

\[ = \sum_{k=0}^{N} \lambda_k \left( \frac{1}{\zeta} (\ast a_k) - 2(\ast \pi(a_k)) \right) \int_{\Omega} \langle a_k, u \rangle \]

\[ = \frac{1}{\zeta} \left( \ast \left( \sum_{k=0}^{N} \lambda_k a_k \int_{\Omega} \langle a_k, u \rangle \right) \right) - 2(\ast \pi(\sum_{k=0}^{N} \lambda_k a_k \int_{\Omega} \langle a_k, u \rangle)) \]

\[ = \frac{1}{\zeta} \left( \ast S(u) \right) - 2(\ast \pi(S(u))), \]

which yields the claim. □

**Lemma 4.4.** For every \( \varepsilon > 0 \), there exists \( M(\varepsilon, \zeta) > 0 \) such that

\[ \frac{1}{2} \int_{\Gamma} \partial_\nu(|\psi|^2) \leq \varepsilon \|\nabla^3 u\|_2^2 + M\|\psi, u\|_2^2 \] (4.27)

for any \( u \in \cup_{N \in \mathbb{N}} X_N \), \( \omega = \nabla \times u \), and \( \psi = \nabla \times \omega = -\Delta u \).

**Proof.** Recall (cf. (2.11)) that

\[ \frac{1}{2} \partial_\nu(|\psi|^2) = -\nabla^\Gamma \cdot (\langle \psi, \nu \rangle \psi^\|) - \pi(\psi^\|, \psi^\|) - H|\psi^\perp|^2 \]

\[ + 2\langle \psi^\|, \nabla^\Gamma \langle \psi, \nu \rangle \rangle + \langle \psi^\| \times (\nabla \times \psi)^\|, \nu \rangle. \]

Integrating the equation over \( \Omega \) and using the boundary data in Lemma 4.3 yield

\[ \frac{1}{2} \int_{\Gamma} \partial_\nu(|\psi|^2) = - \int_{\Gamma} \nabla^\Gamma \cdot (\langle \psi, \nu \rangle \psi^\|) - \int_{\Gamma} \pi(\psi^\|, \psi^\|) - \int_{\Gamma} H|\psi^\perp|^2 \]

\[ + \frac{1}{\zeta} \int_{\Gamma} \langle \psi^\|, S(u) \rangle - 2 \int_{\Gamma} \pi(\psi^\|, S(u)) \]

\[ - \frac{2}{\zeta} \int_{\Gamma} \langle \psi^\|, \nabla^\Gamma (\nabla^\Gamma \cdot u) \rangle + 4 \int_{\Gamma} \langle \psi^\|, \nabla^\Gamma (\nabla^\Gamma \cdot \pi(u)) \rangle. \]

The first integral vanishes by Stokes' theorem applying to the surface \( \Gamma \). Using the Hölder inequality, it follows that

\[ \frac{1}{2} \int_{\Gamma} \partial_\nu(|\psi|^2) \leq M\|\psi^\|_2^2 \left( 1 + \|S(u)\|_{L^2(\Gamma)} + \|\nabla^2 u, \nabla u, u\|_{L^2(\Gamma)} \right). \] (4.28)

All the boundary integrals on the right-hand side can be estimated via the Sobolev imbedding:

\[ \|\nabla^2 u, \nabla u, u\|_{L^2(\Gamma)} \leq \varepsilon \|\nabla^3 u\|_2 + M\|\psi, u\|_2 \]

and

\[ \|S(u)\|_{L^2(\Gamma)} \leq \varepsilon \|\nabla S(u)\|_2 + M\|S(u)\|_2 \leq \varepsilon \|\nabla \times \psi\|_2 + M\|\psi, u\|_2, \]
where the second inequality follows from (4.16). Again, by the Sobolev imbedding,
\[ \|\psi\|_{L^2(\Gamma)}^2 \leq \varepsilon \|\nabla \psi\|_2^2 + M \|\psi\|_2^2. \]
Plugging these estimates into (4.28), we conclude
\[ \frac{1}{2} \int_\Gamma \partial_\nu(|\psi|^2) \leq \left( \varepsilon \|\nabla \psi\|_2 + C_2 \|\psi\|_2 \right) (\varepsilon \|\nabla^3 u\|_2 + C \|(\psi, u)\|_2 + 1), \]
and (4.27) follows. \[ \square \]

**Theorem 4.4.** Let \( u \in \cup_{N \in \mathbb{N}} X_N, \omega = \nabla \times u, \) and \( \psi = -\Delta u. \) Then
\[ \|\nabla^3 u\|_2 \leq M (\|\nabla \psi\|_2 + \|u\|_{H^2}) \]
so that
\[ \|u\|_{H^3} \leq M \|(\nabla \psi, \psi, u)\|_2, \] \hspace{1cm} (4.29) where \( M > 0 \) is a constant depending only on \( \zeta \) and the domain \( \Omega, \) which may be different in each occurrence.

**Proof.** First, we need to apply integration by parts to reduce the \( L^2 \)-norm of the third total derivative \( \nabla^3 u \) into the \( L^2 \)-norm of the total derivative \( \nabla \psi, \) plus some boundary integrals, which can be dominated by the use of boundary conditions on \( u \) and the facts that \( \nabla \cdot u = 0 \) and \( \nabla \cdot \psi = 0. \) Let us carry out this step in the ordinary coordinate system, and thus \( \partial_j = \frac{\partial}{\partial x_j} \) are the coordinate differentiations. Then
\[ |\nabla^3 u|^2 = \left( \partial_i \partial_j \partial_k u^i \right) \left( \partial_i \partial_j \partial_k u^j \right), \]
where, as usual, the repeated indices are summed up from 1 to 3. Therefore, by using integration by parts twice,
\[
\|\nabla^3 u\|_2^2 = \int_{\Omega} (\partial_k \psi^i) (\partial_k \psi^j) + \int_{\Gamma} (\partial_k \psi^i) (\partial_j \partial_k u^i) (\partial_j \nu) + \int_{\Gamma} (\partial_j \partial_k u^i) (\partial_i \partial_j \partial_k u^j) (\partial_i \nu) \\
= \int_{\Omega} |\nabla \psi|^2 + \int_{\Gamma} (\partial_k \psi^i) (\partial_j \partial_k u^i) (\partial_j \nu) + \frac{1}{2} \int_{\Gamma} \partial_\nu(|\nabla^2 u|^2), \]
(4.30)
where \( \psi^i = -\Delta u^i \) has been used. We first handle the last boundary integral \( \frac{1}{2} \int_{\Gamma} \partial_\nu(|\nabla^2 u|^2). \)

To this end, we use a moving frame \((\nabla_1, \nabla_2, \nabla_3)\) so that \( \nabla_3 \) coincides with the unit normal \( \nu \) and \( \nabla_i, \ i = 1, 2, \) are (local) tangent vector fields, so that we may perform integration by parts on the surface \( \Gamma \) for \( \nabla_i, \ i = 1, 2. \) Then
\[ J \equiv \frac{1}{2} \partial_\nu(|\nabla^2 u|^2) \sim (\nabla_j \nabla_k u^j) (\nabla_j \nabla_k \nabla_3 u^j), \]
where \( \sim \) means that the difference between the two sides contains only the quadratic terms involving \( u \) and its 1st and 2nd order derivatives, the second fundamental form \( \pi \) and its derivatives. These terms come from the exchange of orders by applying the derivatives \( \nabla_i. \) According to the Ricci identity, the difference after the exchange of \( \nabla_i \nabla_j \) to \( \nabla_j \nabla_i \) is a term which has order 0. More precisely,
\[
\nabla_i \nabla_j T^1 - \nabla_j \nabla_i T^1 = \sum_{l=1}^{3} C_{ij}^l T^l,
\]
where $T$ is a vector field and $I$ can be a multi-index, depending on the order of the vector field $T$. The boundary integrals of these lower-order terms can be controlled by

$$C\|(\nabla^2 u, \nabla u, u)\|_{L^2(\Gamma)}^2,$$

which, in turn, can be dominated by

$$\varepsilon \|\nabla^3 u\|_2^2 + M\|(\psi, u)\|_2^2.$$

Note that, in this step, no boundary condition on $u$ which, in turn, can be dominated by $u$ without using any boundary condition on the field integral on the right-hand side of (4.30) can be estimated similarly.

For the first two terms, we perform integration by parts on $\Gamma$. Then the second boundary terms in (4.32) can be re-grouped into the following items: $(\nabla_j \nabla_k u^3)(\nabla_j \nabla_k \nabla_3 u^3)$, $(\nabla_k \nabla_3 u^3)(\nabla_k \nabla_3^2 u^3)$, $(\nabla_3^2 u^3)(\nabla_3^3 u^3)$, $(\nabla_j \nabla_k u^3)(\nabla_j \nabla_k \nabla_3 u^3)$, $(\nabla_j \nabla_3 u^3)(\nabla_j \nabla_3^2 u^3)$, and $(\nabla_3^2 u^3)(\nabla_3^3 u^3)$, where all the other repeated indices run through from 1 to 2. Indeed,

$$J \sim \sum_{j,k=1,2} (\nabla_j \nabla_k u^3)(\nabla_j \nabla_k \nabla_3 u^3) + \sum_{l,j,k=1,2} (\nabla_j \nabla_k u^3)(\nabla_j \nabla_k \nabla_3 u^3)$$

$$+ 2\sum_{l,j=1,2} (\nabla_j \nabla_3 u^3)(\nabla_j \nabla_3^2 u^3) + 2\sum_{k=1,2} (\nabla_k \nabla_3 u^3)(\nabla_k \nabla_3^3 u^3)$$

$$+ (\nabla_3^2 u^3)(\nabla_3^2 \nabla_3 u^3) + \sum_{l=1,2} (\nabla_3^2 u^3)(\nabla_3^3 u^3).$$

To estimate the boundary integrals of the right-hand side, we use the kinematic condition (4.31), the divergence-free condition:

$$\nabla_3 u^3 = -\sum_{a=1,2} \nabla_a u^a, \quad \nabla_3 \psi^3 = -\sum_{a=1,2} \nabla_a \psi^a,$$

and the definition of $\psi$:

$$\nabla_3^2 u^b = \Delta u^b - \sum_{a=1,2} \nabla_a^2 u^b = -\psi^b - \sum_{a=1,2} \nabla_a^2 u^a. \quad (4.32)$$

Both (4.31) and (4.32) hold on $\Omega$. By using these relations, we can rewrite $J$ as follows:

$$J \sim -2(\nabla_j \nabla_a^2 u + \nabla_j \psi^a)\nabla_j(\nabla_3 u^3 - \nabla_1 u^3) - (\nabla_a^2 u^b + \nabla_a \psi^a)[\psi^3]$$

$$+ (2\nabla_l \nabla_a u^a + \nabla_a^2 u^a + \psi^a)(\nabla_l \psi^3) + (\nabla_b^2 u^l + \psi^l)(\nabla_3 \psi^3 - \nabla_1 \psi^3)$$

$$+ \psi^l \nabla_a^2 (\nabla_3 u^3 - \nabla_1 u^3) + \sum_{l,j,k=1,2} (\nabla_j \nabla_k u^l)(\nabla_j \nabla_k (\nabla_3 u^3 - \nabla_1 u^3))$$

$$+ \sum_{l,b,a=1,2} (\nabla_b^2 u^l)(\nabla_a^2 (\nabla_3 u^3 - \nabla_1 u^3),$$

where the other repeated indices are added up through 1 to 2. The terms in the square brackets are to be replaced via the corresponding boundary conditions, so that the orders of taking derivatives for these terms are reduced by 1. Therefore, all the terms, except the first two, are quadratic forms of $u$ and its 1st and 2nd order derivatives, and those of $\pi$, so that these terms are dominated by

$$C\|(\nabla^2 u, \nabla u, u)\|^2.$$

For the first two terms, we perform integration by parts on $\Gamma$. The second boundary integral on the right-hand side of (4.30) can be estimated similarly.
Let $I = \frac{1}{2} \int_{\Gamma} \partial_{\nu} (|\nabla^2 u|^2)$. Integrating the above equation and using integration by parts on the surface $\Gamma$ yield

$$I \leq \int_{\Gamma} (2\psi^k + 3\nabla_k \nabla_u u^a + \nabla^2 u^k) \nabla_k \psi^3 + \int_{\Gamma} (\psi^l + \nabla^2 u^l) (\nabla_3 \psi^l - \nabla_l \psi^3)$$

$$+ 3 \int_{\Gamma} (\psi^l + \nabla^2 u^l) \nabla_j (\nabla_3 u^l - \nabla_l u^3) + \int_{\Gamma} (\nabla_j \nabla_k u^l) (\nabla_j \nabla_k (\nabla_3 u^l - \nabla_l u^3))$$

$$+ \varepsilon \|\nabla^3 u\|^2 + M \|(\psi, u)\|_2^2.$$ 

Therefore, using the boundary conditions for $\psi^3 = \langle \psi, \nu \rangle$, $\nabla^3 \psi^l - \nabla_l \psi^3$ (which is $(\nabla \psi)^\parallel$), $\nabla_3 u^l - \nabla_l u^3$ (which is $(\nabla \times u)^\parallel$), together with the Sobolev imbedding, we have

$$I \leq \varepsilon \|\nabla^3 u\|^2 + M(\varepsilon, \zeta) \|(\psi, u)\|_2^2.$$ 

Corollary 4.2. There exists $M(\zeta) > 0$ depending only on $\zeta$ and $\Omega$ such that

$$M \|(\nabla \psi, \psi, u)\|_2 \leq \|u\|_{H^3} \leq M^{-1} \|(\nabla \psi, \psi, u)\|_2$$  \hspace{1cm} (4.33)

for any $u \in \cup_{N \in \mathbb{N}} \mathbb{X}_N$, where $\psi = -\Delta u$.

4.4. The Stokes semigroup. The self-adjoint operator $S$ on $K_2(\Omega)$ has a spectral decomposition

$$S = \int_{-\infty}^{\Lambda} \lambda dE_\lambda,$$

where $\{E_\lambda : \lambda < \Lambda\}$ is the left-continuous family of the projection operator $E_\lambda$ on the space spanned by $\{a_k : \lambda_k > \lambda\}$. According to the spectral theory of self-adjoint operators, $u \in K_2(\Omega)$ belongs to $D_\zeta(S)$ if and only if

$$\int_{-\infty}^{\Lambda} \lambda^2 d\langle E\lambda u, u \rangle = \sum_{k=0}^{\infty} \lambda_k^2 \left( \int_{\Omega} \langle a_k, u \rangle \right)^2 < \infty,$$

and $D(\mathcal{E} + \Lambda I) = D(\sqrt{-S + \Lambda I}) = K_2(\Omega) \cap H^1(\Omega)$; and $u \in K_2(\Omega)$ is in $D(S)$ if and only if

$$\int_{-\infty}^{\Lambda} \lambda d\langle E\lambda u, u \rangle = \sum_{k=0}^{\infty} \lambda_k \left( \int_{\Omega} \langle a_k, u \rangle \right)^2 < \infty.$$

In this case,

$$\mathcal{E}(u, u) = -\sum_{k=0}^{\infty} \lambda_k \left( \int_{\Omega} \langle a_k, u \rangle \right)^2. \hspace{1cm} (4.34)$$

We are going to show the following estimate that plays an important role in the proof of the existence of strong solutions to the Navier-Stokes equations.

Theorem 4.5. For any $\varepsilon > 0$, there exists $M(\varepsilon, \zeta)$ such that

$$\|\nabla \times P_N(u)\|_2^2 \leq (1 + \varepsilon)\mathcal{E}(P_\infty(u), P_\infty(u)) + M \|u\|^2_2$$ \hspace{1cm} (4.35)

for any $u \in L^2(\Omega)$ and any integer $N$. 

Proof. Recall that
\[ P_N(u) = \sum_{k=0}^N a_k \int_{\Omega} \langle a_k, u \rangle = \sum_{k=0}^N a_k \int_{\Omega} \langle a_k, P_\infty(u) \rangle \]
so that \( P_N(u) \in D_{0, \zeta}(S). \) According to (3.4),
\[ \|\nabla \times P_N(u)\|_2^2 = \|\nabla P_N(u)\|_2^2 + \int_\Gamma \pi(P_N(u), P_N(u)) \]
\[ \leq (1 + \varepsilon)\|\nabla P_N(u)\|_2^2 + M\|u\|_2^2 \]
\[ = -(1 + \varepsilon) \int_\Omega \langle S(P_N(u)), P_N(u) \rangle + M\|u\|_2^2, \]
where the second inequality follows from (4.13). However,
\[ S(P_N(u)) = \sum_{k=0}^N \lambda_k a_k \int_{\Omega} \langle a_k, u \rangle. \]
Denote integer \( N_0 > 0 \) such that \( \Lambda \geq \lambda_0 \geq \cdots \geq \lambda_{N_0} > 0 \geq \lambda_{N_0+1} \geq \cdots. \) Then we find that, when \( N \geq N_0, \)
\[ - \int_\Omega \langle S(P_N(u)), P_N(u) \rangle = - \sum_{k=0}^N \lambda_k \int_{\Omega} \langle a_k, P_N(u) \rangle \int_{\Omega} \langle a_k, u \rangle \]
\[ = - \sum_{k=0}^N \lambda_k \left( \int_{\Omega} \langle a_k, P_\infty(u) \rangle \right)^2 \]
\[ \leq - \sum_{k=0}^N \lambda_k \left( \int_{\Omega} \langle a_k, P_\infty(u) \rangle \right)^2 \]
\[ = \mathcal{E}(P_\infty(u), P_\infty(u)), \]
and, while \( N \leq N_0 - 1, \)
\[ - \int_\Omega \langle S(P_N(u)), P_N(u) \rangle = - \sum_{k=0}^N \lambda_k \left( \int_{\Omega} \langle a_k, P_\infty(u) \rangle \right)^2 \]
\[ \leq - \sum_{k=0}^\infty \lambda_k \left( \int_{\Omega} \langle a_k, P_\infty(u) \rangle \right)^2 + \sum_{k=N+1}^{N_0} \lambda_k \left( \int_{\Omega} \langle a_k, P_\infty(u) \rangle \right)^2 \]
\[ = \mathcal{E}(P_\infty(u), P_\infty(u)) + M\|u\|_2^2. \]
This arrives the result expected. \( \square \)

Corollary 4.3. For any \( \varepsilon > 0, \) there exists \( M(\varepsilon, \zeta) > 0 \) depending only on \( \varepsilon, \zeta, \) and \( \Omega \) such that
\[ \|\nabla \times P_N(u), \nabla P_N(u)\|_2^2 \leq M\|\nabla \times u, u\|_2^2 \]
(4.36)
for any \( u \in H^2(\Omega) \) and any integer \( N. \)
Proof. Note that $P_\infty(u) \in K_2(\Omega) \cap H^2(\Omega)$ so that
\[ \nabla \times P_\infty(u) = \nabla \times P_\infty(u) = \nabla \times u \]
and the estimate follows from
\[ \mathcal{E}(P_\infty(u), P_\infty(u)) \leq M \| (\nabla P_\infty(u), u) \|^2, \]
which yields (4.36). Similarly, the estimate for $\nabla P_N(u)$ follows from (4.36) and (3.4). □

5. Existence of Weak and Strong Solutions

In this section, we consider the initial-boundary value problem (1.1)–(1.4), where $\zeta > 0$ is a constant. The minimal requirement on the initial data is that $u_0 \in K_2(\Omega)$.

First we introduce the notion of weak solutions to the initial-boundary problem (1.1)–(1.4). We say that a vector field $u(t, x)$ is a weak solution of (1.1)–(1.4) with slip length $\zeta$, provided that $u(t, x)$ satisfies the following conditions:

(i) For each $t > 0$, $u(t, \cdot) \in K_2(\Omega)$, and $u \in L^2([0, T]; H^1(\Omega))$ for any $T > 0$;

(ii) For any smooth vector field $\varphi(t, x)$ with $\varphi(t, \cdot) \in K_2(\Omega)$,
\[ \int_\Omega \langle u(T, \cdot), \varphi(T, \cdot) \rangle = \langle u_0, \varphi_0 \rangle + \int_0^T \int_\Omega \langle u(t, \cdot), \partial_t \varphi(t, \cdot) \rangle \]
\[ - \int_0^T \int_\Omega \langle \nabla \times u, (u \times \varphi + \mu \nabla \times \varphi) \rangle - \frac{\mu}{\zeta} \int_0^T \int_\Gamma \langle u, \varphi \rangle + 2\mu \int_0^T \int_\Gamma \pi(u, \varphi); \quad (5.1) \]

(iii) The energy inequality:
\[ \| u(T, \cdot) \|^2_2 + 2\mu \int_0^T \| \nabla u \|^2 + 2\mu \int_0^T \int_\Gamma \left( \frac{1}{\zeta} |u|^2 - \pi(u, u) \right) \leq \| u_0 \|^2_2. \quad (5.2) \]

Equation (5.1) is obtained by integrating (1.1) and performing formally integration by parts.

5.1. Construction of global weak solutions. Notice that $(S, D_\zeta(S))$ has the eigenvalues $\Lambda \geq \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n \to -\infty$ and the eigenvector functions $\{a_n : n = 1, 2, \cdots \}$, which form an orthonormal basis of $K_2(\Omega)$. Then $Sa_n = \lambda_n a_n$, and $a_n$ solves the Stokes equation:
\[ \Delta a_n - \nabla p_n = \lambda_n a_n, \quad \nabla \cdot a_n = 0, \quad (5.3) \]
subject to the kinematic condition (1.3) and the Navier’s $\zeta$-condition (1.4).

Let
\[ u(t, \cdot) = \sum_{k=1}^\infty c_k(t) a_k \quad \text{with} \quad c_k(t) = \int_\Omega \langle a_k, u \rangle, \]
be a solution of the Navier-Stokes equation (1.1) with initial data $u_0$. Multiplying by $a_k$ and integrating over $\Omega$, we obtain
\[ \partial_t c_k = \mu \int_\Omega \langle a_k, \Delta u \rangle - \int_\Omega \langle a_k, u \cdot \nabla u \rangle = \mu \lambda_k c_k - \sum_{i,j=0}^\infty \frac{c_i c_j}{\zeta} \int_\Omega \langle a_k, a_i \cdot \nabla a_j \rangle. \]
Thus, for each integer \( N \), we solve the Cauchy problem for the system of differential equations:

\[
\frac{d}{dt} c_k = \mu \lambda c_k - \sum_{i,j=0}^{N} c_i c_j \int_{\Omega} \langle a_k, a_i \cdot \nabla a_j \rangle
\]  

(5.4)

\[ c_k|_{t=0} = \int_{\Omega} \langle a_k, u_0 \rangle. \]  

(5.5)

Define

\[ u^N(t, \cdot) = \sum_{k=0}^{N} c_k(t) a_k. \]

Then \( u^N(t, \cdot) \in D_0(\zeta(S)) \) for \( t > 0 \) and satisfies the evolution equation:

\[
\partial_t u^N = \mu S(u^N) - \sum_{k=1}^{N} a_k \int_{\Omega} \langle a_k, u^N \cdot \nabla u^N \rangle. \]  

(5.6)

Therefore, \( \nabla \cdot u^N = 0 \), \( (u^N)^\perp|_{\Gamma} = 0 \), and \( (\nabla \times u^N)\|_{\Gamma} = -\frac{1}{\zeta}(u^N) + 2(\pi(u^N)) \) for \( t > 0 \).

Now we make the energy estimates for \( u^N \) and \( u^N_t \). First, it is easy to see that

\[
\frac{d}{dt} \| u^N \|_2^2 = 2 \sum_{k=1}^{N} c_k \frac{d}{dt} c_k = 2 \mu \sum_{k=1}^{N} \lambda_k c_k^2 - 2 \int_{\Omega} \langle u^N, u^N \cdot \nabla u^N \rangle.
\]

Since

\[
\sum_{k=1}^{N} \lambda_k c_k^2 = \int_{\Omega} \langle S(u^N), u^N \rangle = -\int_{\Omega} |\nabla u^N|^2 - \frac{1}{\zeta} \int_{\Gamma} |u^N|^2 + \int_{\Gamma} \pi(u^N, u^N),
\]

we have

\[
\frac{d}{dt} \| u^N \|_2^2 = -2\mu \int_{\Omega} |\nabla u^N|^2 - \int_{\Omega} u^N \cdot \nabla (|u^N|^2) - \frac{2\mu}{\zeta} \int_{\Gamma} |u^N|^2 + 2\mu \int_{\Gamma} \pi(u^N, u^N).
\]

Since \( \nabla \cdot u^N = 0 \) and \( (u^N)^\perp = 0 \), which implies \( \int_{\Omega} u^N \cdot \nabla (|u^N|^2) = 0 \), we obtain the energy balance identity:

\[
\frac{d}{dt} \| u^N \|_2^2 + 2\mu \| \nabla u^N \|_2^2 = -\frac{2\mu}{\zeta} \| u^N \|_{L^2(\Gamma)}^2 + 2\mu \int_{\Gamma} \pi(u^N, u^N). \]  

(5.7)

Therefore, we have

\[
\| u^N(T, \cdot) \|_2^2 + 2\mu \int_0^T \| \nabla u^N \|_2^2 + 2\mu \int_0^T \int_{\Gamma} \left( \frac{1}{\zeta} |u^N|^2 - \pi(u^N, u^N) \right) = \| u^N_0 \|_2^2 \leq \| u_0 \|_2^2. \]  

(5.8)

Using the Sobolev embedding inequality:

\[
\int_{\Gamma} |u|^2 \leq \epsilon \| \nabla u \|_2^2 + C(\epsilon) \| u \|_2^2,
\]

we have

\[
\int_{\Gamma} \pi(u^N, u^N) \leq \frac{1}{2} \| \nabla u^N \|_2^2 + C(\epsilon) \| u^N \|_2^2.
\]
Then, from (5.8), we have
\[ \|u^N(T, \cdot)\|_2^2 + \mu \int_0^T \int_\Omega \|\nabla u^N\|^2 + \frac{2\mu}{\zeta} \int_0^T \int_\Gamma |u^N|^2 \leq \|u_0\|_2^2 + C \int_0^T \|u^N(s, \cdot)\|_2^2. \] (5.9)

The Gronwall inequality and (5.9) imply that \( \|u^N(t, \cdot)\|_2^2 \) and \( \int_0^T \|\nabla u^N\|^2 \) are uniformly bounded in \( t, \zeta, \) and \( N \). The apriori estimate (5.9) also ensures that, for each integer \( N \), system (5.4) has a unique solution for all \( t > 0 \). Then we conclude

**Theorem 5.1.** Let \( u_0 \in K_2(\Omega) \). Then, for any \( T > 0 \), the family \( \{u^N(t, x)\}, 0 \leq t \leq T \), is weakly compact in the space \( L^2([0, T]; K_2(\Omega)) \) so that it has a convergent subsequence that converges to a vector field \( u \in L^2([0, T]; K_2(\Omega)) \), and the limit function \( u(t, x) \) is a weak solution to problem (1.1) - (1.4).

Furthermore, we have
\[ \|u_\zeta(T, \cdot)\|_2^2 + \mu \int_0^T \int_\Omega \|\nabla u_\zeta(t, \cdot)\|^2 + \frac{2\mu}{\zeta} \int_0^T \int_\Gamma |u_\zeta|^2 \leq \|u_0\|_2^2, \] (5.10)
where we have used \( u_\zeta \) to indicate the dependence on the slip length \( \zeta > 0 \). The uniform energy estimate (5.10) implies that the family \( u_\zeta(t, x) \) is pre-compact in \( L^2([0, T]; K_2(\Omega)) \) and hence there exists a convergent subsequence (still denoted) \( u_\zeta \to u \), when \( \zeta \to 0 \). From (5.10), we have
\[ \int_0^T \int_\Gamma |u_\zeta|^2 \leq \frac{\zeta}{2\mu} \|u_0\|_2^2, \]
which leads to
\[ \int_0^T \int_\Gamma |u|^2 = 0. \]
This implies that the limit \( u(t, \cdot) \) is subject to the no-slip condition for almost all time \( t \).

On the other hand, when \( \zeta \to \infty \), we obtain that there also exists a subsequence (still denoted) \( u_\zeta(t, x) \) converging to \( u(t, x) \) in \( L^2([0, T]; K_2(\Omega)) \) such that \( u(t, x) \) is a solution to (1.1) subject to the complete slip boundary condition:
\[ \omega \big|_{\Gamma} = 2 \left( * \pi(u) \right) \] (5.11)
in the weak sense.

**Theorem 5.2.** Let \( u_\zeta(t, x), 0 \leq t \leq T \), be a weak solution to problem (1.1) - (1.4) constructed in Theorem 5.1. Then
(i) when \( \zeta \to 0 \), there exists a subsequence (still denoted) \( u_\zeta(t, x) \) converging to \( u(t, x) \) in \( L^2([0, T]; K_2(\Omega)) \) such that \( u(t, x) \) is a solution to (1.1) subject to the no-slip condition for almost all time \( t \);
(ii) when \( \zeta \to \infty \), there exists a subsequence (still denoted) \( u_\zeta(t, x) \) converging to \( u(t, x) \) in \( L^2([0, T]; K_2(\Omega)) \) such that \( u(t, x) \) is a solution to (1.1) subject to the complete slip boundary condition (5.11).

The nonhomogeneous vorticity boundary problem related to (5.11) has been investigated in [7].
5.2. Strong solutions. In this section, we prove that there exists a strong solution to problem (1.1)–(1.4) for small time. To this end, we develop the $L^2$-estimates for $u^N$ up to second-order derivatives, uniformly in $N$. More precisely, we prove the following:

**Theorem 5.3.** Let $u_0 \in K_2(\Omega) \cap H^2(\Omega)$. Then there exist $T^* > 0$ and $M > 0$ depending only on $\zeta$, $\varepsilon$, $\mu$, $\Omega$, and $\|u_0\|_{H^2}$ (but independent of $N$) such that

$$
\|u^N(t, \cdot)\|_{H^2}^2 + \|\partial_t u^N(t, \cdot)\|_{H^2}^2 \leq M.
$$

**(5.12)**

**Proof.** For simplicity, we write $u = u^N$ given by (5.4). Let $\omega = \nabla \times u$ and $\psi = \nabla \times \omega = -\Delta u$ as usual. Recall that $u$ fulfills the evolution equation:

$$
\partial_t u = \mu S(u) - \sum_{k=1}^{N} a_k \int_{\Omega} \langle a_k, u \cdot \nabla u \rangle,
$$

where, for $t > 0$, $u(t, \cdot) \in D_{0,\zeta}(\mathcal{L})$. Taking the $t$-derivative yields the evolution equation:

$$
\partial_t u_t = \mu S(u_t) - \sum_{k=1}^{N} a_k \int_{\Omega} \langle a_k, u_t \cdot \nabla u \rangle - \sum_{k=1}^{N} a_k \int_{\Omega} \langle \partial_t a_k, u \cdot \nabla u_t \rangle,
$$

and $u_t(t, \cdot) \in D_{0,\zeta}(\mathcal{L})$. Therefore, $\nabla \cdot u_t = 0$ and $u_t$ again satisfies the same boundary conditions as those of $u$. Using the evolution equation (5.13), we have

$$
\frac{d}{dt} \|u_t\|_2^2 = 2\mu \int_{\Omega} \langle S(u_t), u_t \rangle - 2 \int_{\Omega} \langle u_t, u_t \cdot \nabla u \rangle - 2 \int_{\Omega} \langle u_t, u \cdot \nabla u_t \rangle.
$$

**(5.14)**

Integration by parts yields

$$
\int_{\Omega} \langle \Delta u_t, u_t \rangle = -\|\nabla u_t\|_2^2 + \int_{\Omega} \langle u_t \times (\nabla \times u_t), \nu \rangle + \int_{\Gamma} \langle u_t \cdot \nabla u_t, \nu \rangle
$$

$$
= -\|\nabla u_t\|_2^2 - \frac{1}{\zeta} \int_{\Gamma} \langle u_t \times (\ast u_t), \nu \rangle + 2 \int_{\Gamma} \langle u_t \times (\ast \pi u_t), \nu \rangle - \int_{\Gamma} \pi(u_t, u_t)
$$

$$
= -\|\nabla u_t\|_2^2 - \frac{1}{\zeta} \int_{\Gamma} |u_t|^2 + \int_{\Gamma} \pi(u_t, u_t).
$$

Substitution this into (5.14) leads to

$$
\frac{d}{dt} \|u_t\|_2^2 = -2\mu \|\nabla u_t\|_2^2 - 4 \int_{\Omega} \langle u_t, u_t \cdot \nabla u \rangle - \frac{2\mu}{\zeta} \int_{\Gamma} |u_t|^2 + 2\nu \int_{\Gamma} \pi(u_t, u_t).
$$

**(5.15)**

It follows that

$$
\frac{d}{dt} \|u_t\|_2^2 \leq -\frac{2\mu}{\zeta} \|u_t\|_{L^2(\Gamma)}^2 - 2\mu \|\nabla u_t\|_2^2 + 4\|u_t\|_2^2 \|\nabla u\|_\infty^2 + 2\mu \int_{\Gamma} \pi(u_t, u_t)
$$

$$
\leq -\frac{2\mu}{\zeta} \|u_t\|_{L^2(\Gamma)}^2 - \mu \|\nabla u_t\|_2^2 + \varepsilon \|\nabla^2 u\|^2 + C \left( \|\langle \psi, u, u_t \rangle\|_2^2 + \|u_t\|_2^2 \right),
$$

**(5.16)**

where the Sobolev’s imbedding has been used and $C > 0$ is a constant depending only on the domain $\Omega$.

Next we deal with $\Delta u$. The evolution equation for $u(t, \cdot)$ may be written as

$$
\partial_t u = \mu S(u) - P_N (u \cdot \nabla u).
$$
Together with the vector identity \( u \cdot \nabla u = \frac{1}{2} \nabla |u|^2 - u \times \omega \), we have
\[
\partial_t u = \mu S(u) + P_N(u \times \omega). \tag{5.17}
\]
Since \( \nabla \times S(u) = \nabla \times (\Delta u) \), taking curl (twice) to both sides of equation \( \text{(5.17)} \) yields
\[
\partial_t \omega = \mu \Delta \omega + \nabla \times P_N(u \times \omega), \tag{5.18}
\]
and
\[
\partial_t \psi = \mu \Delta \psi + \nabla \times \nabla \times P_N(u \times \omega). \tag{5.19}
\]
It follows from \( \text{(5.19)} \) that
\[
\frac{d}{dt} \| \psi \|_2^2 = 2\mu \int_\Omega \langle \Delta \psi, \psi \rangle + 2 \int_\Omega \langle \nabla \times \nabla \times P_N(u \times \omega), \psi \rangle. \tag{5.20}
\]
Integration by parts leads to
\[
2\mu \int_\Omega \langle \Delta \psi, \psi \rangle = -2\mu \| \nabla \psi \|_2^2 + \mu \int_\Gamma \partial_\nu (|\psi|^2),
\]
and
\[
\int_\Omega \langle \nabla \times \nabla \times P_N(u \times \omega), \psi \rangle = \int_\Omega \langle \nabla \times P_N(u \times \omega), \nabla \times \psi \rangle - \int_\Gamma \langle \psi \times (\nabla \times P_N(u \times \omega)), \nu \rangle
\]
\[
= \int_\Omega \langle \nabla \times P_N(u \times \omega), \nabla \times \psi \rangle - \int_\Gamma \langle \psi \times (-\frac{1}{\zeta} (\pi P_N(u \times \omega)) - 2\pi P_N(u \times \omega)), \nu \rangle
\]
\[
= \int_\Omega \langle \nabla \times P_N(u \times \omega), \nabla \times \psi \rangle + \frac{1}{\zeta} \int_\Gamma \langle \psi, P_N(u \times \omega) \rangle - 2 \int_\Gamma \pi (\psi, P_N(u \times \omega)).
\]
Therefore, we obtain
\[
\frac{d}{dt} \| \psi \|_2^2 = -2\mu \| \nabla \psi \|_2^2 + 2 \int_\Omega \langle \nabla \times P_N(u \times \omega), \nabla \times \psi \rangle + \mu \int_\Gamma \partial_\nu (|\psi|^2)
\]
\[
+ \frac{2}{\zeta} \int_\Gamma \langle \psi, P_N(u \times \omega) \rangle - 4 \int_\Gamma \pi (\psi, P_N(u \times \omega)). \tag{5.21}
\]
Using the H"older inequality, one obtains
\[
\frac{d}{dt} \| \psi \|_2^2 \leq -2\mu \| \nabla \psi \|_2^2 + 2 \| \nabla \times P_N(u \times \omega) \|_2 \| \nabla \times \psi \|_2
\]
\[
+ \mu \int_\Gamma \partial_\nu (|\psi|^2) + M \| \psi \|_{L^2(\Gamma)} \| P_N(u \times \omega) \|_{L^2(\Gamma)}. \tag{5.22}
\]
The first boundary integral \( \int_\Gamma \partial_\nu (|\psi|^2) \) can be estimated by using Lemma \( \text{[4.3]} \) to obtain
\[
\frac{1}{2} \int_\Gamma \partial_\nu (|\psi|^2) \leq \varepsilon \| \nabla^2 u \|_2^2 + M \| (\psi, u) \|_2^2.
\]
The product of last two boundary integrals in \( \text{(5.22)} \) can be estimated via the Sobolev imbedding to yield
\[
M \| \psi \|_{L^2(\Gamma)} \| P_N(u \times \omega) \|_{L^2(\Gamma)} \leq (\varepsilon \| \nabla \psi \|_2 + M \| \psi \|_2) (\varepsilon \| \nabla P_N(u \times \omega) \|_2 + M \| u \times \omega \|_2).
\]
Plugging these estimates into \( \text{(5.22)} \), using Corollary \( \text{[4.3]} \) and the estimate
\[
\| u \times \omega \|_2 \leq M \| u \|_{H^1} = M \| (\psi, u) \|_2^2,
\]
and rearranging the inequality, we obtain
\[
\frac{d}{dt} \|\psi\|_2^2 \leq -2\mu \|\nabla \psi\|_2^2 + \varepsilon \|\psi\|_{H^1}^2 + \varepsilon \|\nabla^3 u\|_2^2 + M \|\nabla P_N(u \times \omega)\|_2 \|\psi\|_{H^1} + M \left(\|\psi\|_2^2 + \|\psi, u\|_2^2\right).
\] (5.23)

Finally, we use Corollary 4.3 to obtain
\[
\|\nabla P_N(u \times \omega)\|_2 \leq M \|\nabla \times (u \times \omega)\|_2 + M \|u \times \omega\|_2 \leq M \|\omega \cdot \nabla u, u \cdot \nabla \omega, u \times \omega\|_2 \leq M \|u\|_{H^2}^2.
\]

Then we conclude
\[
\frac{d}{dt} \|\psi\|_2^2 \leq -\frac{3}{2} \mu \|\nabla \psi\|_2^2 + M \left(\|\psi, u\|_2^2 + \|\psi, u\|_2^2\right)
\] (5.24)

for some constant $M$ depending only on $\zeta$, $\mu$, and $\Omega$.

Let $F = \|\psi, u, u_t\|_2^2$. Combining (5.17) and (5.16) with (5.24), we obtain the following differential inequality:
\[
\frac{d}{dt} F \leq -\mu \|\nabla (\psi, u, u_t)\|_2^2 - \frac{2\mu}{\zeta} \|(u, u_t)\|_{L^2(\Gamma)}^2 + M_1 F + M_2 F^2
\] (5.25)
for some constants $M_1$ and $M_2$ depending only on $\zeta$ and $\Omega$. In particular, we have
\[
\frac{d}{dt} F \leq M_1 F + M_2 F^2.
\] (5.26)

Since
\[
\|u_t\|_2 \leq \mu \|S(u)\|_2 + \|P_N(u \times \omega)\|_2 \leq 2\mu \|\Delta u\|_2 + \|u \times \omega\|_2 \leq 2\mu \|\Delta u\|_2 + \|u\|_2 \|\omega\|_2,
\]
then
\[
F(0) \leq C(\mu, \Omega) \|u_0\|_{H^2}^2
\]
for some constant $C(\mu, \Omega)$ depending only on $\mu$ and $\Omega$.

Let $\rho$ be the solution on $[0, T^*)$ to the ordinary differential equation:
\[
\rho' = M_1 \rho + M_2 \rho^2, \quad \rho(0) = C(\mu, \Omega) \|u_0\|_{H^2}^2,
\] (5.27)
where $T^* > 0$ is the blowup time of $\rho$.

Then inequality (5.26) together with the fact that $F(0) \leq \rho(0)$ implies that $F(t) \leq \rho(t)$ on $[0, T^*)$. This completes the proof of Theorem 5.3.

\textbf{Theorem 5.4.} Let $u_0 \in K_2(\Omega) \cap H^2(\Omega)$. Then there exists $T^* > 0$ depending only on $\zeta$, $\mu$, $\Omega$, and $\|u_0\|_{H^2}$ such that there is a strong solution $u(t, x)$ of the initial-boundary value problem (1.1)–(1.4) up to $T^* > 0$.

6. Inviscid Limit as $\mu \to 0$

In this section, we analyze the inviscid limit of the solutions $u^\mu(t, x)$ of the initial-boundary value problem (1.1)–(1.4).

Let $u(t, x)$ be the unique smooth solution of the initial-boundary value problem of the Euler equations:
\[
\begin{aligned}
\partial_t u + u \cdot \nabla u &= -\nabla p, \\
\nabla \cdot u &= 0, \\
u(0, \cdot) &= u(0), \\
u^\perp \big|_{\Gamma} &= 0,
\end{aligned}
\] (6.1)
up to time $T^* > 0$. Notice that all solutions $u^\mu$ to problem (1.1)-(1.4), $\mu \in (0, \mu_0]$ for some $\mu_0 > 0$, subject to the same boundary conditions: $(u^\mu)_\Gamma = 0$ and

$$
(\nabla \times u^\mu)|_\Gamma = -\frac{1}{\zeta}(*u^\mu) + 2(\ast \pi(u^\mu)),
$$

while the solution $u$ of (6.1) satisfies only the kinematic boundary condition and is independent of the viscosity constant $\mu$.

**Theorem 6.1.** Suppose that, for all $\mu \in (0, \mu_0]$, a unique strong solution $u^\mu$ of the initial-boundary value problem (1.1)-(1.4) and a unique strong solution $u \in H^2(\Omega)$ to the initial-boundary value problem (6.1) both exist up to time $T^* > 0$. Then there exists $C = C(\mu_0, T, \|u\|_{L^2([0,T];H^2 \cap H^{1,\infty}(\Omega))}$, independent of $\mu$, such that, for any $T \in [0, T^*]$,

$$
\sup_{0 \leq t \leq T} \|u^\mu(t, \cdot) - u(t, \cdot)\|_2 \leq C \mu \rightarrow 0 \quad \text{as } \mu \downarrow 0,
$$

and

$$
\int_0^T \|\nabla (u^\mu - u)(s, \cdot)\|^2 ds \leq C.
$$

It follows that the whole solution sequence $u^\mu$ of (1.1)-(1.4) converges to the unique solution $u(t, x)$ of the initial-boundary value problem (6.1) in $L^2$ as $\nu \rightarrow 0$.

**Proof.** Let $v^\mu = u^\mu - u$. Then $v^\mu$ satisfies the following equations:

$$
\begin{align*}
\partial_t v^\mu &= \mu \Delta v^\mu - (v^\mu + u) \cdot \nabla v^\mu - \nabla P^\mu - v^\mu \cdot \nabla u + \mu \Delta u, \\
\nabla \cdot v^\mu &= 0,
\end{align*}
$$

and the initial condition:

$$
v^\mu(0, \cdot) = 0,
$$

where $P^\mu = p^\mu - p$. Since both $w^\mu$ and $u$ satisfy the kinematic condition (1.3), so does $v^\mu$. Thus, by means of the energy method, we obtain

$$
\frac{d}{dt} \|v^\mu\|^2_2 = 2\mu \int_{\Omega} \langle v^\mu, \Delta v^\mu \rangle - \int_{\Omega} \langle v^\mu + u, \nabla (\|v^\mu\|^2) \rangle - 2 \int_{\Omega} \langle \nabla P^\mu, v^\mu \rangle \\
- 2 \int_{\Omega} \langle v^\mu \cdot \nabla u, v^\mu \rangle + 2\mu \int_{\Omega} \langle \Delta u, v^\mu \rangle.
$$

Integration by parts in the first three integrals leads to

$$
\frac{d}{dt} \|v^\mu\|^2_2 = -2\mu \int_{\Omega} |\nabla \times v^\mu|^2 + 2\mu \int_{\Gamma} \langle v^\mu \times (\nabla \times v^\mu) , \nu \rangle \\
- 2 \int_{\Omega} \langle v^\mu \cdot \nabla u, v^\mu \rangle + 2\mu \int_{\Omega} \langle \Delta u, v^\mu \rangle \\
= -2\mu \|\nabla v^\mu\|^2_2 - 2\mu \int_{\Gamma} \pi(v^\mu, v^\mu) + 2\mu \int_{\Gamma} \langle v^\mu \times b, \nu \rangle \\
- 2 \int_{\Omega} \langle v^\mu \cdot \nabla u, v^\mu \rangle + 2\mu \int_{\Omega} \langle \Delta u, v^\mu \rangle,
$$

where $b = -\frac{1}{\zeta}(*u^\mu + u) + 2(\ast \pi(u^\mu + u) - (\nabla \times u)$.
Furthermore, we use the following estimate:
\[
\int_{\Gamma} \langle v_\mu \times b, \nu \rangle \leq \| b \|_{L^2(\Gamma)} \| v_\mu \|_{L^2(\Gamma)} \leq C \left( \| u \|_{H^1(\Gamma)}^2 + \| v_\mu \|_{L^2(\Gamma)}^2 \right)
\]
to obtain
\[
\frac{d}{dt} \| v_\mu \|_2^2 \leq -2\mu \| \nabla v_\mu \|_2^2 + \mu C \left( \| v_\mu \|_{L^2(\Gamma)}^2 + \| u \|_{H^1(\Gamma)}^2 \right)
\]
+ 2\| \nabla u \|_\infty \| v_\mu \|_2^2 + 2\mu \| \Delta u \|_2 \| v_\mu \|_2.
\] (6.5)

Finally, we use the Sobolev imbeddings:
\[
C \| v_\mu \|_{L^2(\Gamma)}^2 \leq \| \nabla v_\mu \|_2^2 + \tilde{C} \| v_\mu \|_2^2; \quad \| u \|_{H^1(\Gamma)}^2 \leq C \| u \|_{H^2(\Omega)}^2
\]
to establish the differential inequality:
\[
\frac{d}{dt} \| v_\mu \|_2^2 + \mu \| \nabla v_\mu \|_2^2 \leq C \left( \| \nabla u \|_\infty + \mu_0 \right) \| v_\mu \|_2^2 + \mu \| u \|_{H^2(\Omega)}^2.
\] (6.6)

Gronwall’s inequality implies that
\[
\| v_\mu(t, \cdot) \|_2^2 \leq \mu \int_0^t e^{C \int_0^\tau \| \nabla u(\tau, \cdot) \|_{\infty} d\tau + \mu_0 (t-\tau)} \| u(s, \cdot) \|_{H^2(\Omega)}^2 ds =: C\mu,
\] (6.7)
where \( C > 0 \) depends only on \( \mu_0, T^* \), and \( \| u \|_{L^2([0,T]; H^2(\Omega) \cap W^{1,\infty}(\Omega))} \)
Using this and (6.6), we further have
\[
\int_0^t \| \nabla v_\mu(s, \cdot) \|_2^2 ds \leq C\mu.
\]
This completes the proof. \( \square \)

In order to ensure the convergence of \( \mu \) to \( u \) in the strong sense (say, \( H^2(\Omega) \)), a necessary condition is that \( u \) must match the Navier’s \( \zeta \)-condition (1.4).

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