REAL RANK AND SQUARING MAPPING FOR UNITAL C*-ALGEBRAS

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Abstract. It is proved that if $X$ is a compact Hausdorff space of Lebesgue dimension $\dim(X)$, then the squaring mapping $\alpha_m: (C(X)_{sa})^m \to C(X)_+$, defined by $\alpha_m(f_1, \ldots, f_m) = \sum_{i=1}^m f_i^2$, is open if and only if $m - 1 \geq \dim(X)$. Hence the Lebesgue dimension of $X$ can be detected from openness of the squaring maps $\alpha_m$. In the case $m = 1$ it is proved that the map $x \mapsto x^2$, from the self-adjoint elements of a unital C*-algebra $A$ into its positive elements, is open if and only if $A$ is isomorphic to $C(X)$ for some compact Hausdorff space $X$ with $\dim(X) = 0$.

1. Introduction

A compact Hausdorff space $X$ is defined to have Lebesgue dimension $\leq m$ if for every closed subset $F$ of $X$, each continuous map $F \to S^m$ has a continuous extension $X \to S^m$.

Various types of ranks for (unital) C*-algebras have been inspired by corresponding prototypes in the classical dimension theory of (compact) spaces, such as the one given above. While the Lebesgue dimension of a compact space has numerous equivalent formulations, the extensions of these equivalent formulations to non-commutative C*-algebras most often differ. Examples of such ranks for C*-algebras are the stable rank defined by Rieffel in [7], the real rank defined by Brown and Pedersen in [1], the analytic rank defined by Murphy in [4], the completely positive rank considered by Winter in [8], and the bounded rank defined in [2]. Other ranks of C*-algebras are the tracial rank defined by Lin in [3] and the exponential rank defined by Phillips in [6].

It was shown in [2] that a unital C*-algebra $A$ has real rank at most $n$ if the squaring map $(x_1, \ldots, x_{n+1}) \mapsto \sum_{i=1}^{n+1} x_i^2$, from the set of $(n + 1)$-tuples of self-adjoint elements to the set of positive elements in $A$, is open; and it was asked if the reverse also holds, in which case openness of the squaring maps would determine the real rank of the C*-algebra.

In the present note we answer this question in the affirmative in the commutative case — and in the negative in the general (non-commutative) case.

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The latter — negative — answer follows from our result that the squaring map $x \mapsto x^2$ (from the set of self-adjoint elements to the set of positive elements) is open if and only if the $C^*$-algebra is commutative and of real rank zero. Hence the squaring map $x \mapsto x^2$ is not open for the $C^*$-algebra, $M_2$, of 2 by 2 matrices, but this $C^*$-algebra has real rank zero.

2. Results

For a $C^*$-algebra $A$ we use the standard notation $A_{sa}$ and $A_+$ to denote the set of all self-adjoint and the set of all positive elements of $A$, respectively. The real rank of a unital $C^*$-algebra, denoted by $RR(A)$, is in [1] defined as follows: For each non-negative integer $n$, $RR(A) \leq n$ if for every $(n+1)$-tuple $(x_1, \ldots, x_{n+1})$ in $A_{sa}$ and every $\varepsilon > 0$, there exists an $(n+1)$-tuple $(y_1, \ldots, y_{n+1})$ in $A_{sa}$ such that $\sum_{k=1}^{n+1} y_k^2$ is invertible and $\sum_{k=1}^{n+1} \|x_k - y_k\| < \varepsilon$.

Let us say that a unital $C^*$-algebra $A$ has an open $m$-squaring map if the map $\alpha_m: (A_{sa})^m \to A_+$, defined by $\alpha_m(x_1, \ldots, x_m) = \sum_{k=1}^m x_k^2$, is open. Observe that $\alpha_m$ is open at $(x_1, \ldots, x_m)$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $a \in A_+$ with $\|\sum_{k=1}^m x_k^2 - a\| < \delta$ there is an $m$-tuple $(y_1, \ldots, y_m)$ in $A_{sa}$ with $\sum_{k=1}^m y_k^2 = a$ and $\sum_{k=1}^m \|x_k - y_k\| < \varepsilon$.

For the readers convenience we present a shorter proof of [2, Proposition 7.1].

**Proposition 2.1.** Let $A$ be a unital $C^*$-algebra. If the $(n + 1)$-squaring map on $A$ is open, then $RR(A) \leq n$.

**Proof.** Let $(x_1, \ldots, x_{n+1})$ be an $(n + 1)$-tuple of self-adjoint elements in $A$ and let $\varepsilon > 0$. By openness of the $(n + 1)$-squaring map there is $\delta > 0$ and an $(n + 1)$-tuple $(y_1, \ldots, y_{n+1})$ of self-adjoint elements in $A$ such that $\sum_{k=1}^{n+1} \|x_k - y_k\| < \varepsilon$ and $\sum_{k=1}^{n+1} y_k^2 = \sum_{k=1}^{n+1} x_k^2 + \delta \cdot 1$, and the latter element is invertible (because each $x_k^2$ is positive).

Next, we prove the reverse of Proposition 2.1 in the commutative case.

**Proposition 2.2.** If $X$ is a compact space such that $\dim X \leq n$, then $C(X)$ has open $(n + 1)$-squaring map.

**Proof.** Let $f = f_1^2 + \cdots + f_{n+1}^2$. Put $m_i = \sup f_i$, $i = 1, \ldots, n + 1$ and let $m = \max\{m_i\}$. Fix $\varepsilon > 0$ and let

$$\delta = \min \left\{ \frac{(\varepsilon/3)^4}{m^2}, \left(\frac{\varepsilon}{3}\right)^2 \right\} \text{ and } U = \left\{ x \in X : f(x) > \left(\frac{\varepsilon}{3}\right)^2 \right\}.$$

Let also $A = f^{-1}([0, (\varepsilon/3)^2])$ and $S = f^{-1}((\varepsilon/3)^2)$. Then $A$ and $S$ are closed subsets of $X$ such that $A = X \setminus U$ and $S \subseteq A$.

Now consider the diagonal product

$$F(x) = \Delta\{f_1(x), \ldots, f_{n+1}(x)\} : X \to \mathbb{R}^{n+1}$$
and note that

\[ A = F^{-1}(B^{n+1}) \text{ and } S = F^{-1}(S^n), \]

where

\[ B^{n+1} = B^{n+1}(0, \varepsilon/3) \text{ and } S^n = \partial B^{n+1}(0, \varepsilon/3). \]

Since \( \dim A \leq \dim X \leq n \), the map \( F|S: S \to S^n \) admits an extension \( H: A \to S^n \) (see, for instance, [5, Ch. 3, Theorem 2.2]). Represent \( H \) as the diagonal product \( H = \Delta\{h_i, \ldots, h_{n+1}\} \), where each \( h_i \) maps \( A \) into \( \mathbb{R} \). Since \( H(A) \subseteq S^n \) it follows that \( h_1^2 + \cdots + h_{n+1}^2 = (\varepsilon/3)^2 \). Note also that since \( H|S = F|S \) we have \( h_i|S = f_i|S \) for each \( i = 1, \ldots, n + 1 \).

The last condition allows us to define for each \( i = 1, \ldots, n + 1 \) a continuous map \( \tilde{h}_i \) on \( X \) by letting

\[ \tilde{h}_i(x) = \begin{cases} f_i(x), & \text{if } x \in U \\ h_i(x), & \text{if } x \in A \end{cases} \]

Observe that the function \( \tilde{h} = \tilde{h}_1^2 + \cdots + \tilde{h}_{n+1}^2 \) is strictly positive on \( X \). Notice also that \( \tilde{h}|_U = f|_U \) and \( \tilde{h}|_A = (\varepsilon/3)^2 \).

Next consider \( g \in O(f, \delta) \cap C_+(X) \) and define a function \( \lambda \) on \( X \) by \( \lambda(x) = (g(x)/\tilde{h}(x))^{1/2} \). Note that \( \lambda \geq 0 \).

Now define \( g_i, i = 1, \ldots, n + 1 \), on \( X \) by the formula \( g_i(x) = \tilde{h}_i(x) \cdot \lambda(x) \).

Clearly

\[ g_1^2(x) + \cdots + g_{n+1}^2(x) = \left( \tilde{h}_1(x) \cdot \lambda(x) \right)^2 + \cdots + \left( \tilde{h}_{n+1}(x) \cdot \lambda(x) \right)^2 = \]

\[ \lambda^2(x) \left( \tilde{h}_1^2(x) + \cdots + \tilde{h}_{n+1}^2(x) \right) = \frac{g(x)}{\tilde{h}(x)} \cdot \left( \tilde{h}_1^2(x) + \cdots + \tilde{h}_{n+1}^2(x) \right) = g(x). \]

Next let us show that \( g_i \) is sufficiently close to \( f_i \) for each \( i = 1, \ldots, n + 1 \). Indeed, since \( g \in O(f, \delta) \) we have for each \( x \in A \) and therefore

\[ g(x) < f(x) + \delta < \left( \frac{\varepsilon}{3} \right)^2 + \left( \frac{\varepsilon}{3} \right)^2. \]

Since \( g_i^2 \leq g \) and \( f_i^2 \leq f \), the last inequality implies that

\[ |g_i(x)| < \sqrt{2 \left( \frac{\varepsilon}{3} \right)^2} < 2\frac{\varepsilon}{3} \] and \( |f_i(x)| \leq \frac{\varepsilon}{3} \)

for all \( x \in A \). Hence

\[ |f_i(x) - g_i(x)| \leq |f_i(x)| + |g_i(x)| < \frac{\varepsilon}{3} + 2\frac{\varepsilon}{3} = \varepsilon \]
as required.

Further, if $x \in U$, then $\tilde{h}(x) = f(x)$ and consequently

$$\tilde{h}(x) - \delta < g(x) < \tilde{h}(x) + \delta.$$ 

Hence

$$1 - \frac{\delta}{h(x)} < g(x) < 1 + \frac{\delta}{h(x)}$$

for $x \in U$. Since

$$\tilde{h}(x) = f(x) > \left(\frac{\varepsilon}{3}\right)^2 \text{ for } x \in U \text{ and } \delta \leq \frac{\varepsilon/3}{m^2}$$

we have (for $x \in U$)

$$1 - \left(\frac{\varepsilon}{3m}\right)^2 = 1 - \frac{1}{m^2} \cdot \left(\frac{\varepsilon}{3}\right)^4 < \frac{g(x)}{h(x)} < 1 + \frac{1}{m^2} \cdot \left(\frac{\varepsilon}{3}\right)^4 = 1 + \left(\frac{\varepsilon}{3m}\right)^2$$

and

$$1 - \left(\frac{\varepsilon}{3m}\right)^2 < \lambda^2(x) < 1 + \left(\frac{\varepsilon}{3m}\right)^2.$$ 

Consequently,

$$|1 - \lambda^2(x)| < \left(\frac{\varepsilon}{3m}\right)^2.$$ 

Since $\lambda(x) \geq 0$, this implies

$$[1 - \lambda(x)]^2 \leq |1 - \lambda(x)| \cdot |1 + \lambda(x)| = |1 - \lambda^2(x)| < \left(\frac{\varepsilon}{3m}\right)^2.$$ 

Therefore

$$|1 - \lambda(x)| \leq \frac{\varepsilon}{3m} \text{ for any } x \in U.$$ 

Finally we have

$$|f_i(x) - g_i(x)| = |1 - \lambda(x)| \cdot |f_i(x)| < \frac{\varepsilon}{3m} \cdot m < \frac{\varepsilon}{3} \text{ for any } x \in U.$$ 

This completes the verification of the fact that $|f_i(x) - g_i(x)| < \varepsilon$ for each $x \in X$ and any $i = 1, \ldots, n + 1$. \hfill \qed

**Corollary 2.3.** Let $A$ be a unital $C^*$-algebra. Then the following conditions are equivalent:

(i) The squaring map $x \mapsto x^2$ from $A_{sa}$ to $A_+$ is open.

(ii) $A$ is commutative and $\text{RR}(A) = 0$. 


(iii) \( A \) is isomorphic to a \( C^* \)-algebra of the form \( C(X) \) for a compact Hausdorff space \( X \) with \( \dim X = 0 \).

**Proof.** The equivalence of (ii) and (iii) follows from Gelfand’s duality and [1, Proposition 1.1].

The implication (iii) \( \Rightarrow \) (i) follows from Proposition 2.2.

(i) \( \Rightarrow \) (ii). Assume that (i) holds. Then \( \text{RR}(A) = 0 \) by Proposition 2.1. It remains to show that \( A \) is commutative. Since \( A \) is of real rank zero it suffices to show that any two projections \( p, q \) in \( A \) commute.

Take the symmetry \( s = p - (1 - p) \). Then \( s \) is self-adjoint and \( s^2 = 1 \). By openness of the squaring map there are self-adjoint elements \( s_n \) in \( A \) such that \( \|s_n - s\| \to 0 \) and \( s_n^2 = 1 + n^{-1}q \). Define \( \varphi: \mathbb{R} \to \mathbb{R} \) by \( \varphi(t) = \max\{0, t\} \). For each \( n \), the element \( \varphi(s_n) \) commutes with \( s_n \), hence with \( s^2_n \), and hence with \( q \). Since \( \varphi(s) = p \) we obtain

\[
pq - qp = \lim_{n \to \infty} (\varphi(s_n)q - q\varphi(s_n)) = 0,
\]
as desired. \( \square \)

### 3. Related comments and open problems

**Existence of square roots:** Suppose that \( A \) is a unital \( C^* \)-algebra and that \( x \) is a self-adjoint element in \( A \). Does there exist a continuous square root \( \rho_x = \rho: \Omega \to A_{\text{sa}} \) (i.e., \( \rho(a)^2 = a \) for all \( a \in \Omega \)) defined on an open neighborhood \( \Omega \subseteq A_+ \) of \( x^2 \) such that \( \rho(x^2) = x \)? If this is true for all self-adjoint elements \( a \) in \( A \), then the equivalent conditions of Corollary 2.3 are satisfied.

Suppose that \( A = C(X) \) for some 0-dimensional compact Hausdorff space \( X \) (i.e., that the conditions of Corollary 2.3 are satisfied). Take a self-adjoint (i.e., real valued) \( f \in C(X) \), and suppose that there is a clopen set \( U \) such that \( f(x) \geq 0 \) for all \( x \in U \) and \( f(x) \leq 0 \) for all \( x \in X \setminus U \). Then the function \( \rho_U: C(X)_{+} \to C(X)_{\text{sa}} \) defined by

\[
\rho_U(g) = \begin{cases} 
\sqrt{g(x)}, & x \in U, \\
-\sqrt{g(x)}, & x \in X \setminus U,
\end{cases}
\]
is a continuous square root with \( \rho_U(f^2) = f \). It is not clear to the authors if there are continuous square roots at arbitrary real valued functions \( f \) in \( C(X) \).

In the case where \( A = M_n \), the \( C^* \)-algebra of \( n \) by \( n \) matrices, if \( x \) is a self-adjoint element and if \( x^2 \) has \( n \) distinct eigenvalues, then there is a continuous square root \( \rho \) with \( \rho(x^2) = x \) defined on some neighborhood of \( x^2 \).

In the case where \( A = M_2 \), it follows from Corollary 2.3 (and its proof) that there is no continuous square root \( \rho \) defined on a neighborhood of \( I \) such that \( \rho(I) = \text{diag}(1, -1) \). It is easily checked explicitly that if \( r \) is a (small) non-zero real number, then any square root of \( \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \) is of the form \( \begin{pmatrix} a & s \\ s & a \end{pmatrix} \), where \( a \) and \( s \)
are real numbers satisfying $a^2 + s^2 = 1$ and $2as = r$, and any such square root has distance at least 1 to diag(1, -1).

We end this note by listing some open problems related to openness of the squaring maps:

**Question 1.** Let $A$ be a unital $C^*$-algebra, let $m$ be a positive integer, and suppose that the squaring map $\alpha_m$ (defined above Proposition 2.1) is open. Does it follow that $\alpha_n$ is open for all $n \geq m$?

The answer to Question 1 is affirmative for commutative $C^*$-algebras by Propositions 2.1 and 2.2. The difficulty in this question lies in the fact that if $\Omega$ is an open subset of $A_+$ and if $a \in A_+$, then $a + \Omega$ need not be open in $A_+$. (For instance, $1 + A_+$ is not open in $A_+$.)

**Question 2.** Are Propositions 2.1 and 2.2 valid also in the non-unital case? (For Proposition 2.2, this means that we will be talking about locally compact Hausdorff spaces rather than compact Hausdorff spaces.) What is the relationship between openness of $\alpha_n$ on a non-unital $C^*$-algebra $A$ and openness of $\alpha_n$ on its unitization?

**Question 3.** Are the squaring maps $\alpha_m$ open for all $m \geq 2$ when $A$ is a unital $C^*$-algebra of real rank zero?

**Question 4.** Does the class of $C^*$-algebras, for which the squaring map $\alpha_2$ is open, have any nice properties? More generally, are there any justifications for considering the rank of a $C^*$-algebra defined by openness of the squaring maps; and will this rank reflect any “dimension like” properties of the $C^*$-algebra?

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