Parallel Batch-Dynamic Graph Connectivity*

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Abstract

With the rapid growth of graph datasets over the past decade, a new kind of dynamic algorithm, supporting the ability to ingest batches of updates and exploit parallelism is needed in order to efficiently process large streams of updates. In this paper, we study batch and parallel algorithms for the dynamic connectivity problem, a fundamental problem that has received considerable attention in sequential setting. Perhaps the best known sequential algorithm is the elegant level-set algorithm of Holm, de Lichtenberg and Thorup (HDT), which achieves $O(\lg^2 n)$ amortized time per edge insertion or deletion, and $O(\lg n)$ time per query.

In this paper, we design a parallel batch-dynamic connectivity algorithm that is work-efficient with respect to the HDT algorithm for small batch sizes, and is asymptotically faster when the average batch size is sufficiently large. Given a sequence of batched updates, where $\Delta$ is the average batch size of all deletions, our algorithm achieves $O(\lg n \lg (1 + n/\Delta))$ expected amortized work per edge insertion and deletion and $O(\lg^3 n)$ depth w.h.p. Our algorithm answers a batch of $k$ connectivity queries in $O(k \lg (1 + n/k))$ expected work and $O(\lg n)$ depth w.h.p. To the best of our knowledge, our algorithm is the first parallel batch-dynamic algorithm for connectivity.

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1 Introduction

Computing the connected components of a graph is a fundamental problem that has been studied in many different models of computation [54, 51, 6, 31]. The **connectivity problem** takes as input an undirected graph \( G \) and assigns labels to vertices so that two vertices have the same label if and only if they are in the same connected component. The **dynamic connectivity problem** requires maintaining a data structure over an \( n \) vertex undirected graph that supports operations which query whether two vertices are in the same connected component, or inserts and deletions of edges. Despite the large body of work on the dynamic connectivity problem over the past two decades [27, 20, 31, 60, 61, 28, 66, 35, 37, 42, 67], little is known about batch-dynamic connectivity algorithms that process batches of queries and updates, or on parallel algorithms for dynamic connectivity.

Traditional dynamic algorithms were motivated by applications from the early days of computing where data would undergo small changes that could be adequately handled by these algorithms. Today, however, applications operate on increasingly large datasets which change rapidly over time: for example, millions of customers can log on to a web site at the same time, make phone calls at the same time, send an email at the same time and so on. In such applications, traditional dynamic algorithms require serializing the changes made and processing them one at a time, missing an opportunity to exploit the parallelism that is afforded by processing batches of changes.

Motivated by such dynamic applications, there has been recent interest in developing parallel batch-dynamic algorithms [2, 57, 1, 62]. In the batch-dynamic framework, instead of applying one update or query at a time a whole batch is applied. A batch could be of size \( \lg n \), \( \sqrt{n} \), or \( n/\lg n \) for example. There are two advantages of applying operations in batches. The first more obvious one is that it might allow for parallelism. The second, perhaps less obvious one, is that it could reduce the cost of each update. In this paper we are interested in both these advantages. We use the term parallel batch-dynamic to be algorithms that process batches of operations instead of single ones, and for which the algorithm itself is parallel. The underlying parallel model we use in this paper is a formalization of the widely used work-depth model [9, 15, 12, 7].

Understanding the connectivity structure of graphs is of significant practical interest, for example, due to its use as a primitive for clustering the vertices of a graph [52]. Existing batch-dynamic connectivity algorithms are designed for restricted settings, e.g., in the incremental setting when all updates are edge insertions [57], or when the underlying graph is a forest [50, 1, 62]. In practice, since most real-world graphs change over time, many implementations of parallel batch-dynamic connectivity algorithms have been implemented [40, 32, 65, 33, 53, 64]. In the worst case, these algorithms may recompute the connected components of the entire graph even for very small batches. Since this requires \( O(m + n) \) work, it makes the worst-case performance of the algorithms no better than running a static parallel algorithm.

Therefore, two important questions are:

1. **Is there a batch-dynamic connectivity algorithm that is asymptotically faster than existing dynamic connectivity algorithms for large enough batches of insertions, deletions and queries?**

2. **Can the batch-dynamic connectivity algorithm be parallelized to achieve low worst-case depth?**

In this paper we give an algorithm that answers both of these questions affirmatively. To simplify our exposition and present the main ideas, we first give a less efficient version of the algorithm that runs in \( O(\lg^4 n) \) depth w.h.p.\(^1\) and performs \( O(\lg^2 n) \) expected amortized work per update, making it

\(^1\) We say that an algorithm has \( O(f(n)) \) cost **with high probability (w.h.p.)** if it has \( O(k \cdot f(n)) \) cost with probability at least \( 1 - 1/n^k \), \( k \geq 1 \).
work-efficient with respect to the classic HDT algorithm. Next, we describe the improved algorithm
which runs in $O(\log^3 n)$ depth w.h.p. and achieves an improved work bound that is asymptotically
faster than the HDT algorithm for sufficiently large batch sizes. We note that our depth bounds
hold even when processing the updates one a time, ignoring batching. Our improved work bounds
are derived by a novel analysis of the work performed by the algorithm over all batches of deletions.

Our contribution is summarized by the following theorem:

**Theorem 1.** There is a parallel batch-dynamic data structure which, given batches of edge insertions,
deletions, and connectivity queries processes all updates in $O(\lg n \lg (1 + \frac{n}{\Delta}))$ expected amortized
work per edge insertion or deletion where $\Delta$ is the average batch size of deletion. The cost of
connectivity queries is $O(k \lg (1 + n/k))$ expected work and $O(\lg n)$ depth w.h.p. for a batch of $k$
queries. The depth to process a batch of edge insertions and deletions is $O(\lg n)$ w.h.p. and $O(\log^3 n)$
w.h.p. respectively.

**Technical Overview.** The starting point of our algorithm is the classic Holm, de Lichtenberg and
Thorup (HDT) dynamic connectivity algorithm [31]. Like nearly all existing dynamic connectivity
algorithms, the HDT algorithm maintains a spanning forest certifying the connectivity of the graph.
The algorithm maintains a set of $\log n$ nested forests under two carefully designed invariants. The
forests are represented using the Euler tour tree (ET-tree) data structure [41, 27].

The main challenge in a dynamic connectivity algorithm is to efficiently find a replacement edge,
or a non-tree edge going between the two disconnected components after deleting a tree edge. The
main idea of the HDT algorithm is to organize the spanning-forest of the graph into $\log n$ levels of
trees. The top-most level of the structure stores a spanning forest of the entire graph, and each
level contains all tree-edges stored in levels below it. The algorithm ensures that the largest size
of a component at level $i$ is $2^i$. Using these invariants, the algorithm is able to cleverly search the
tree edges so that each non-tree edge is examined at most $\log n$ times as a candidate replacement
edge. The main idea is to store each non-tree edge at a single level (initially the top-most level),
and push the edge to a lower level each time it is unsuccessfully considered as a replacement edge.
Since there are $\log n$ levels, and the cost of discovering, processing, and removing an edge from each
level using ET-tree operations is $O(\log n)$, the amortized cost of the HDT algorithm is $O(\log^2 n)$ per
degree operation. We now discuss the main challenges and sequential bottlenecks that arise in the
HDT algorithm that a parallel batch-dynamic algorithm must address.

**Efficiently searching for replacements:** A challenge, and sequential bottleneck in the HDT
algorithm is the fact that it processes each non-tree edge sequentially—a property which is crucial
for achieving good amortized bounds. Aside from hindering parallelism, processing the edges one at
a time eliminates any potential for improved batch bounds, since finding the representative of the
endpoints of an edge costs $O(\log n)$ time per query. Therefore, to obtain an efficient batch or parallel
algorithm we must examine batches of *multiple* non-tree edges at a time, while also ensuring that we
do not perform extra work that cannot be charged to level-decreases on an edge. Our approach is
to use an *doubling technique*, where we examine sets of non-tree edges with geometrically increasing
sizes.

**Handling Batches of Deletions:** Another challenge is that processing a batch of deletions
can shatter a component into multiple disconnected pieces. Since the HDT algorithm deletes at
most a single tree edge per deletion operation, it handles exactly two disconnected pieces per level.
In contrast, since we delete batches of edges in our batch-dynamic algorithms, we may have many
disconnected pieces at a given level, and must find replacement edges reconnecting these pieces. In
order to achieve good parallel bounds our algorithm searches for a replacement edge from each piece
that is small enough to be pushed down to the next lower level.

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2We provide full details of the HDT algorithm in Section 2.2.
Each round of both of our algorithms can be viewed as calling an oracle which returns a set of replacement edges incident on the disconnected pieces that we are trying to reconnect. Unlike in the HDT algorithm which terminates once it finds any replacement edge, the edges returned by the oracle may not fully restore the connectivity of the original component. In particular, the replacement edges that we find may contain multiple edges going between the same pieces (like in Borůvka’s algorithm) or may contain cycles, which must be dealt with since each level of the data structure represents a forest. Our approach to handling these issues is to run a static spanning forest algorithm on the replacement edges found in this round, and insert only the spanning forest edges into the ET-tree at the current level.

Both of our algorithms alternate between a first phase which calls the oracle to find a set of replacement edges, and a second phase which determines a set of replacement edges that can be committed as tree edges. The difference is that our first algorithm (Section 3) requires $O(\lg^2 n)$ oracle queries per level, whereas the second algorithm (Section 4) only requires $O(\lg n)$ oracle queries due to a more careful doubling technique.

**Dynamic trees supporting batching:** Another obstacle to improving on the bounds of the HDT algorithm is that the classic ET-tree data structure performs links and cuts one at a time. To achieve good batch bounds for forest operations, we use a recently developed solution to the Batch-Parallel forest connectivity problem by Tseng et al. [62]. Their data structure, which we refer to as a batch-parallel ET-tree processes a set of $k$ links, cuts, or connectivity queries in $O(k \lg(1 + n/k))$ work and $O(\lg n)$ depth. We extend the data structure in order to perform operations such as fetching the first $l$ non-tree edges in the tree efficiently.

## 2 Preliminaries

**Model.** In this paper we analyze our algorithms in the work-depth model using fork-join style parallelism. Specifically, we use a particular work-depth model called the MT-RAM, which is closely related to the PRAM but more closely models current machines and programming paradigms that are asynchronous and support dynamic forking. The model can work-efficiently cross-simulate a CRCW PRAM, equipped with the same atomic instructions, and is therefore essentially equivalent to the classic CRCW PRAM model. We formally define the model and provide more details about the simulations in Appendix 7 and refer the interested reader to [10] for full details.

Our algorithms are designed using **nested fork-join parallelism** in which a procedure can fork off another procedure call to run in parallel and then wait for forked calls to complete with a join synchronization [9]. Our efficiency bounds are stated in terms of work and depth, where work is the total number of vertices in the thread DAG and where depth (span) is the length of the longest path in the DAG [9].

**Parallel Primitives.** The following parallel procedures are used throughout the paper. A **semisort** takes an input array of elements, where each element has an associated key and reorders the elements so that elements with equal keys are contiguous, but elements with different keys are not necessarily ordered. The purpose is to collect equal keys together, rather than sort them. Semisorting a sequence of length $n$ can be performed in $O(n)$ expected work and $O(\lg n)$ depth with high probability assuming access to a uniformly random hash function mapping keys to integers in the range $[1, n^{O(1)}]$ [49, 24].

A **parallel dictionary** data structure supports batch insertion, batch deletion, and batch lookups of elements from some universe with hashing. Gil et al. describe a parallel dictionary that uses linear space and achieves $O(k)$ work and $O(\lg^* k)$ depth w.h.p. for a batch of $k$ operations [23].
The **pack** operation takes an $n$-length sequence $A$ and an $n$-length sequence $B$ of booleans as input. The output is a sequence $A'$ of all the elements $a \in A$ such that the corresponding entry in $B$ is **true**. The elements of $A'$ appear in the same order that they appear in $A$. Packing can be easily implemented in $O(n)$ work and $O(\log n)$ depth [34].

**Useful Lemmas.** The following lemmas are useful for analyzing the work bounds of our parallel algorithms.

**Lemma 1.** Let $n_1, n_2, ..., n_c$ and $k_1, k_2, ..., k_c$ be sequences of non-negative integers such that $\sum k_i = k$, and $\sum n_i = n$. Then
\[
\sum_{i=1}^{c} k_i \log \left( 1 + \frac{n_i}{k_i} \right) \leq k \log \left( 1 + \frac{n}{k} \right).
\]

**Lemma 2.** For any non-negative integers $n$ and $r$,
\[
\sum_{w=0}^{n} 2^w \log \left( 1 + \frac{n}{2^w} \right) = O\left( 2^{r} \log \left( 1 + \frac{n}{2^r} \right) \right).
\]

**Lemma 3.** For any $n \geq 1$, the function $x \log\left( 1 + \frac{n}{x} \right)$ is strictly increasing with respect to $x$ for $x \geq 1$.

### 2.1 Batch-Dynamic Trees

The batch-dynamic trees problem is to represent a forest as it undergoes batches of links, cuts, and connectivity queries. A **link** operation inserts an edge connecting two trees in the forest. A **cut** deletes an edge from the forest, breaking one tree into two trees. A **connectivity** query returns whether two vertices are connected by a path (in the same tree) in the forest. The interface is formally defined as follows:

**Batch-Dynamic Trees Interface.**

- **BatchLink**($\{(u_1, v_1), ..., (u_k, v_k)\}$) takes an array of edges and adds them to the graph $G$. The input edges must not create a cycle in $G$.
- **BatchCut**($\{(u_1, v_1), ..., (u_k, v_k)\}$) takes an array of edges and removes them from the graph $G$.
- **BatchConnected**($\{(u_1, v_1), ..., (u_k, v_k)\}$) takes an array of tuples representing queries. The output is an array where the $i$-th entry returns whether vertices $u_i$ and $v_i$ are connected by a path in $G$.
- **BatchFindRep**($\{x_1, ..., x_k\}$) takes an array of pointers to tree elements. The output is an array where the $i$-th entry is the representative (rep) of the tree in which $x_i$ lives. The representative is defined so that $\text{rep}(u) = \text{rep}(v)$ if and only if $u$ and $v$ are in the same tree. Note that representatives are invalidated after the sequences are modified.

**Batch-Parallel Euler Tour Trees.** In this paper we make use of a recently developed, parallel solution to the batch-dynamic trees problem, called a batch-parallel Euler tour tree (batch-parallel ET-trees) [62]. The data structure represents each ET-tree sequence using a concurrent skip-list, and reduces bulk link, cut, and query operations to bulk operations on the concurrent skip-list.

Tseng et al. [62] prove the following theorem on the efficiency of the batch-parallel ET-tree:

**Theorem 2.** A batch of $k$ links, $k$ cuts, $k$ connectivity queries, or $k$ representative queries over an $n$-vertex forest can be processed in $O(k \log(1 + n/k))$ expected work and $O(\log n)$ depth with high probability.
The trees also support augmentation with an associative and commutative function \( f : D^2 \to D \) with values from \( D \) assigned to vertices and edges of the forest. The goal is to compute \( f \) over subtrees of the represented forest. The interface can be easily extended with the following batch-parallel primitives for updating and querying augmented values.

Appendix 9 contains information about additional tree operations that are needed to efficiently implement our algorithms.

### 2.2 The sequential (HDT) algorithm

Our parallel algorithm is based on the sequential algorithm of Holm, De Lichtenberg, and Thorup [31], which we refer to as the HDT algorithm. The HDT algorithm assigns to each edge in the graph, an integer level from 1 to \( \lg n \). The levels correspond to sequence of subgraphs \( G_1 \subset G_2 \subset ... \subset G_{\lg n} = G \), such that \( G_i \) contains all edges with level at most \( i \). The algorithm also maintains a spanning forest \( F_i \) of each \( G_i \) such that \( F_1 \subset F_2 \subset ... \subset F_{\lg n} \). Each forest is maintained using a set of augmented ET-trees which we describe shortly. Throughout the algorithm, the following invariants are maintained.

**Invariant 1.** \( \forall i = 1...\lg n \), the connected components of \( G_i \) have size at most \( 2^i \).

**Invariant 2.** \( F_{\lg n} \) is a minimum spanning forest where the weight of each edge is its level.

**Connectivity Queries.** To perform a connectivity query in \( G \), it suffices to query \( F_{\lg n} \), which takes \( O(\lg n) \) time by querying for the root of each Euler tour tree and returning whether the roots are equal. We note that in [31], a query time of \( O(\lg n/\lg \lg n) \) is achieved by storing the Euler tour of \( F_{\lg n} \) in a B-tree with branching factor \( \lg n \).

**Inserting an Edge.** An edge insertion is handled by assigning the edge to level \( \lg n \). If the edge connects two currently disconnected components, then it is added to \( F_{\lg n} \).

**Deleting an Edge.** Deletion is the most interesting part of the algorithm. If the deleted edge is not in the spanning forest \( F_{\lg n} \), the algorithm removes the edge and does nothing to \( F_{\lg n} \) as the connectivity structure of the graph is unchanged. Otherwise, the component containing the edge is split into two. The goal is to find a replacement edge, that is, an edge crossing the split component.

If the deleted edge had level \( i \), then the smaller of the two resulting components is searched starting at level \( i \) in order to locate a replacement edge. Before searching this component, all tree edges whose level is equal to \( i \) have their level decremented by one. As the smaller of the split components at level \( i \) has size \( \leq 2^i - 1 \), pushing the entire component to level \( i - 1 \) does not violate Invariant 1. Next, the non-tree edges at level \( i \) are considered one at a time as possible replacement edges. Each time the algorithm examines an edge that is not a replacement edge, it decreases the level of the edge by one. If no replacement is found, it moves up to the next level and repeats. Note that because the algorithm first pushes all tree edges to level \( i - 1 \), any subsequent non-tree edges that may be pushed from level \( i \) to level \( i - 1 \) will not violate Invariant 2.

**Implementation and Cost.** To efficiently search for replacement edges, the ET-trees are augmented with two additional pieces of information. The first augmentation is to maintain the number of non-tree edges whose level equals the level of the tree. The second augmentation maintains the number of tree-edges whose level is equal to the level of the tree.

Using these augmentations, each successive non-tree edge (or tree edge) whose level is equal to the level of the tree can be found in \( O(\lg n) \) time. Furthermore, checking whether the edge is a replacement edge can be done in \( O(\lg n) \) time. Lastly, the cost of pushing an edge that is not a replacement edge to the lower level is \( O(\lg n) \), since it corresponds to inserting the edge into an adjacency structure and updating the augmented values. Since each edge can be processed at most
once per level, paying a cost of \(O(\lg n)\), and there are \(\lg n\) levels, the overall amortized cost per edge is \(O(\lg^2 n)\).

3 A Parallel Algorithm

In this section, we give a simple parallel batch-dynamic connectivity algorithm based on the HDT algorithm. The underlying invariants maintained by our parallel algorithm are identical to the sequential HDT algorithm: we maintain \(\lg n\) levels of spanning forests subject to Invariants 1 and 2. The main challenge, and where our algorithm departs from the HDT algorithm is in how we search for replacement edges in parallel, and how we search multiple components in parallel. We show by a charging argument that this parallel algorithm is work-efficient with respect to the HDT algorithm—it performs \(O(\lg^2 n)\) amortized work per edge insertion or deletion. Furthermore, we show that the depth of this algorithm is \(O(\lg^4 n)\). Although these bounds are subsumed by the improved parallel algorithm we describe in Section 4, the parallel algorithm in this section is useful to illustrate the main ideas in this paper.

Data Structures. Each spanning forest, \(F_i\), is represented using a set of batch-parallel ET-trees [62]. We represent the edges of the graph in a parallel dictionary \(E_D\) for convenience (see Section 2). We also store an adjacency array, \(A_i[u]\), at each level \(i\), and for each vertex \(u\) to store the tree and non-tree edges incident on \(u\) with level \(i\). Note that tree and non-tree edges are stored separately so that they can be accessed separately. Refer to Appendix 8 for details on these data structures.

3.1 Connectivity Queries

As in the sequential algorithm, a connectivity query can be answered by simply performing a query on \(F_{\lg n}\). Algorithm 1 gives pseudocode for the batch connectivity algorithm. The bound we achieve follows from the batch bounds on batch-parallel ET-trees.

Algorithm 1 The batch query algorithm

\begin{verbatim}
1: procedure BATCHQUERY\((\{(u_1, v_1), (u_2, v_2), ..., (u_k, v_k)\}\))
2: return \(F_{\lg n}.BATCHQUERY(\{(u_1, v_1), (u_2, v_2), ..., (u_k, v_k)\})\)
\end{verbatim}

Theorem 3. A batch of \(k\) connectivity queries can be processed in \(O(k \lg (1 + \frac{n}{k}))\) expected work and \(O(\lg n)\) depth w.h.p.

\(\square\)

Proof. Follows from Theorem 2.

3.2 Inserting Batches of Edges

To perform a batch insertion, we first determine a set of edges in the batch that increase the connectivity of the graph. To do so, we treat each current connected component of the graph as a vertex, and build a spanning forest of the edges being inserted over this contracted graph. The edges in the resulting spanning forest are then inserted into the topmost level in parallel.
Algorithm 2 The batch insertion algorithm

1: procedure BatchInsert( $U = \{(u_1,v_1), \ldots, (u_k,v_k)\}$ )
2: For all $e_i \in U$, set level($e_i$) = $\lg n$ in parallel
3: Update $A_{\lg n}[u]$ for edges incident on $u$
4: $U' = \text{Apply } F_{\lg n}.\text{BatchFindRep} \text{ to the endpoints of } e \in U$
5: $T' = \text{SpanningForest}(U')$
6: $T = \text{edges in } U \text{ corresponding to } T'$
7: Promote edges in $T$ to tree edges
8: $F_{\lg n}.\text{BatchInsert}(T)$

Algorithm 2 gives pseudocode for the batch insertion algorithm. We assume that the edges given as input in $U$ are not present in the graph. Each vertex $u$ that receives an updated edge inserts its edges into $A_{\lg n}[u]$ (line 3). This step can be implemented by first running a semisort to collect all edges incident on $u$.

The last step is to insert edges that increase the connectivity of the graph as tree edges (lines 4–8). The algorithm starts by computing the representatives for each edge (line 5). The output is $U'$, which contains an array of edges equivalent to $(\text{FindRep}(u), \text{FindRep}(v))$ for each $(u,v) \in U$.

Next, it computes a spanning forest over the tree edges (line 5). Finally, the algorithm promotes the corresponding edges in $U$ to tree edges. This is done by updating the appropriate adjacency lists and inserting them into $F_{\lg n}$ (lines 7&8).

Theorem 4. A batch of $k$ edge insertions can be processed in $O(k \lg(1 + \frac{n}{k}))$ expected work and $O(\lg n)$ depth w.h.p.

Proof. Lines 2–4 cost $O(k)$ work and $O(\lg k)$ depth w.h.p. using our bounds for updating $A$ (see Lemma 9). The batch find representative query costs $O(k \lg(1 + \frac{n}{k}))$ expected work and $O(\lg n)$ depth w.h.p. by Theorem 2. Computing a spanning forest can then be done in $O(k)$ expected work and $O(\lg k)$ depth w.h.p. using Gazit’s connectivity algorithm [22]. Finally, inserting the spanning forest edges into $F_{\lg n}$ and updating the adjacency lists costs $O(k \lg(1 + \frac{n}{k}))$ expected work and $O(\lg n)$ depth w.h.p.

3.3 Deleting Batches of Edges

As in the sequential HDT algorithm, searching for replacement edges after deleting a batch of tree edges is the most interesting part of our parallel algorithm. A natural idea for parallelizing the HDT algorithm is to simply scan all non-tree edges incident on each disconnected component in parallel. Although this approach has low depth per level, it may perform work that cannot be amortized to level decreases on edges. To amortize the work properly while also searching the edges in parallel we must perform a more careful exploration of the non-tree edges. Our approach is to use a doubling technique, in which we geometrically increase the number of non-tree edges explored as long as we have not yet found a replacement edge. We show using the doubling technique, the work performed (and number of non-tree edges explored) is dominated by the work of the last phase, when we either find a replacement edge, or run out of non-tree edges. Our amortized work-bounds follow by a per-edge charging argument, as in the analysis of the HDT algorithm.

The Deletion Algorithm. Algorithm 3 shows the pseudocode for our parallel batch deletion algorithm. As with the batch insertion algorithm, we assume that each edge is present in $U$ in both directions. Given a batch of $k$ edge deletions, the algorithm first deletes the given edges from their respective adjacency lists in parallel (line 2). It then filters out the tree edges and delete each tree edge $e$ from $F_i \ldots, F_{\lg n}$, where $i$ is the level of $e$ (lines 3–4). Next, it computes $C$, a set of
components (representatives) from the deleted tree edges (lines 5–8). For each deleted tree edge, e, it includes the representatives of both endpoints in the forest at l(e), which must be in different components as e is a deleted tree edge. Finally, it computes the minimum level where a tree edge was deleted (line 10) and calls `PARALLELLEVELSEARCH` with the minimum level, the disconnected components that will be searched, and an empty set of discovered spanning forest edges.

**Algorithm 3** The batch deletion algorithm

```plaintext
1: procedure BatchDeletion( U = \{e_1, \ldots, e_k\} )
2:   Delete e \in U from A_0, \ldots, A_{\lg n}
3:   T = \{e \in U \mid e \in F_{\lg n}\}
4:   Delete e \in T from F_0, \ldots, F_{\lg n}
5:   C = \emptyset
6:   for e = (u, v) \in T in parallel do
7:     (R_u, R_v) = (F_l(e).\text{FindRep}(u), F_l(e).\text{FindRep}(u))
8:     C = C \cup \{R_u, R_v\}
9:   S = \emptyset
10:  for i \in [\text{min}_l = \min_{e \in T} l(e), \lg n] do
11:    (C, S) = PARALLELLEVELSEARCH(i, C, S)
```

The bulk of the work done by the algorithm is performed by Algorithm 4, which searches the disconnected components on a given level of the data structure in parallel. Each call to `PARALLELLEVELSEARCH` takes as input an integer i, the level to search, a set of representatives of the disconnected components, L, and the set of new spanning forest edges that were found in previous levels, S. `PARALLELLEVELSEARCH` returns the set of components that are still disconnected after considering the non-tree edges at this level, and the set of spanning forest edges found so far.

`PARALLELLEVELSEARCH` starts by inserting the edges in S into F_i (line 2). Next, it computes C and D, which are the components that are active and inactive at this level, respectively (lines 3–4). Active components have size at most 2^{i-1}, meaning that their tree edges can be pushed to level i – 1 without violating Invariant 1. The algorithm pushes the tree edges of components in C to level i – 1 (line 5). The main loop consists of two pieces: the first (lines 6–21) searches each remaining component for a replacement edge, and the second (lines 22–30) processes the found replacement edges, and updates the components based on the new edges.

The first piece of the algorithm (lines 6–21), searches each remaining component for a replacement edge in parallel. The found replacement edges are stored in R (line 7). The search consists of a number of phases, where the w'th phase searches the first c_{sz} = 2^w non-tree edges, or all of the edges if 2^w is larger than the number of non-tree edges in c. Initially w = 0 (line 9). On each successive phase, the algorithm retrieves the first sz many non-tree edges, E_c (line 12). It then checks whether any of the edges in E_c are a replacement edge, which can be done using `BATCHFindRep`. If any of the edges are a replacement edge, it pushes all edges before the first replacement edge in E_c to level i – 1 (lines 14–16). Otherwise, if it did not find a replacement edge in this phase and the search size is equal to the number of non-tree edges in c, then we know there is no replacement edge at this level. Therefore, the algorithm removes c from C, the set of searched components, and inserts it into D to be considered at a higher level, and pushes all searched edges to level i – 1 (lines 17–20).

The second piece of the algorithm (lines 22–30) processes the replacement edges and updates the components based on the new edges. It computes the representatives for each replacement edge, and computes a spanning forest over these edges (lines 22–23). Next, it computes the original edges corresponding to the spanning forest edges (line 24), and promotes these edges to tree edges (line 25), adding them to F_i (line 26). Note that the new tree edges are not immediately inserted into
Algorithm 4 The parallel level search algorithm

1: procedure PARALLELLEVELSEARCH($i$, $L = \{c_1, c_2, \ldots\}$, $S$)
2:     $F_i$.BatchInsert($S$)
3:     $C = c \in L$ with size $\leq 2^{i-1}$
4:     $D = c \in L$ with size $> 2^{i-1}$
5:     Push level $i$ tree edges of all components in $C$ to level $i-1$
6:     while $|C| > 0$ do
7:         $R = \{\}$ // Map from components to replacement edges
8:             for each component $c \in C$ in parallel do
9:                 $w = 0$, $c_{\text{max}} = c.\text{NumNonTreeEdges}$
10:                while $R[c] = \text{null}$ and $2^w < 2 \cdot c_{\text{max}}$ do
11:                        $c_{sz} = \min(2^w, c_{\text{max}})$
12:                        $E_c = \text{First } c_{sz} \text{ non-tree edges in } c$
13:                        $R[c] = \text{First replacement edge in } E_c$
14:                        if $R[c] \neq \text{null}$ then
15:                                $E' = \text{Edges before } R[c] \text{ in } E_c$
16:                                Push all edges in $E'$ to level $i - 1$
17:                        else if $c_{sz} = c_{\text{max}}$
18:                                $D = D \cup \{c\}$
19:                                $C = C \setminus \{c\}$
20:                                Push all edges in $E_c$ to level $i - 1$
21:                 $w = w + 1$
22:             $R' = \text{Apply } F_i.\text{BatchFindRep to } e \in R$
23:             $T' = \text{SpanningForest}(R')$
24:             $T = \text{Edges in } R \text{ corresponding to edges chosen in } T'$
25:             Promote edges in $T$ to tree edges
26:             $F_i.\text{BatchInsert}(T)$
27:             $S = S \cup T$
28:             $C = F_i.\text{BatchFindRep}(C)$, and remove duplicates
29:             $D = D \cup \text{components in } C \text{ with size } > 2^{i-1}$
30:             $C = \text{Components in } C \text{ with size } \leq 2^{i-1}$
31:     return $(D, S)$
all higher level spanning trees. Instead, the edges are buffered by adding them to \( S \) (line 27) so that they will be inserted when each level is reached in the search (line 2). Finally, the algorithm updates the set of components by computing their representatives on the updated \( F_i \), and filtering out any components which have size greater than \( 2^{i-1} \) into \( D \) (lines 28–30).

**Cost Bounds.** We now prove that our parallel algorithm has low depth, and is work-efficient with respect to the sequential HDT algorithm. For simplicity, we assume that we start with no edges in a graph on \( n \) vertices.

**Theorem 5.** A batch of \( k \) edge deletions can be processed in \( O(\lg^4 n) \) depth w.h.p.

**Proof.** The algorithm doubles the number of edges searched in each phase. Therefore, after \( \lg_2 m = O(\lg n) \) phases, all non-tree edges incident on the component will be searched.

Each round of the algorithm gets rid of some components, and finds a replacement edge for each remaining component. In the worst case, the edges found for each component pair the components off, leaving us with half as many components in the subsequent round. As we lose a constant fraction of the components per round, the algorithm takes \( O(\lg n) \) rounds.

A given level can therefore perform at most \( O(\lg^2 n) \) phases. Each phase consists of fetching, examining, and pushing down non-tree edges, and hence can be implemented in \( O(\lg n) \) depth w.h.p. by Lemma 10, Theorem 2, and Lemma 11. Therefore, the overall depth for a given level is \( O(\lg^3 n) \) w.h.p. As all \( \lg n \) levels will be processed in the worst case, the overall depth of the algorithm is \( O(\lg^4 n) \).

We now analyze the work performed by the algorithm.

**Lemma 4.** The work performed by \textsc{BatchDeletion} excluding the calls to \textsc{ParallelLevelSearch} is

\[
O\left( k \lg n \lg \left( 1 + \frac{n}{k} \right) \right),
\]

in expectation.

**Proof.** The edge deletions performed by line 2 cost \( O(k) \) work by Lemma 9. Filtering the tree edges (line 3) can be done in \( O(k) \) work. Deleting the tree edges costs at most \( O(k \lg (1 + n/k)) \) work by Lemma 3.

Lines 6–8 perform a \textsc{BatchFindRep} call for each endpoint of each deleted tree edge, which costs \( O(\lg n \lg (1 + n/k)) \) work in expectation by Theorem 2. Since in the worst case each tree edge must be deleted from \( \lg n \) levels, the overall cost of this step is \( O(k \lg n \lg (1 + n/k)) \) in expectation. Summing up the costs for each step proves the lemma.

**Theorem 6.** The expected amortized cost per edge insertion or deletion is \( O(\lg^2 n) \).

**Proof.** Algorithm 3 takes as input a batch of \( k \) edge deletions. By Lemma 4, the expected work performed by \textsc{BatchDeletion} excluding the calls to \textsc{ParallelLevelSearch} is

\[
O\left( k \lg n \lg \left( 1 + \frac{n}{k} \right) \right),
\]

which is at most \( O(k \lg^2 n) \) in expectation. We now consider the cost of the calls to \textsc{ParallelLevelSearch}. Specifically, we show that the work performed during the calls to \textsc{ParallelLevelSearch} can either be charged to level decreases on edges, or is at most \( O(k \lg n) \) per call in expectation. Since the total number of calls to \textsc{ParallelLevelSearch} is at most \( \lg n \), the bounds follow.

First, observe that the number of spanning forest edges we discover, \( |S| \), is at most \( k \), since at most \( k \) tree edges were deleted initially. Therefore, the batch insertion on line 2 costs \( O(k \lg n) \)
in expectation by Theorem 2. Similarly, \( L \), the number of components that are supplied to ParallelLevelSearch, is at most \( k \). Therefore, the cost of filtering the components in \( L \) based on their size, and checking whether their representative exists in \( F_i \) is at most \( O(k \lg n) \) in expectation (lines 3–4).

To fetch, examine, and push down \( l \) tree or non-tree edges costs

\[
O\left(l \lg \left(1 + \frac{n}{T}\right)\right),
\]

work in expectation, by Lemma 10, Theorem 2, and Lemma 11. Note that this is at most \( O(\lg n) \) per edge. In particular, the cost of retrieving and pushing the tree edges of active components to level \( i - 1 \) (line 5) is therefore at most \( O(\lg n) \) per edge in expectation, which we charge to the corresponding level decreases.

We now show that all work done while searching for replacement edges (lines 6–30) can be charged to level decreases. Consider an active component, \( c \) in some round. Suppose the algorithm performs \( q > 0 \) phases before either the component is exhausted (all incident non-tree edges have been checked), or a replacement edge is found. First consider the case where it finds a replacement edge. If \( q = 1 \), only a single edge was inspected, so then we charge the \( \lg n \) work for the round to the edge, which will become a tree edge. Otherwise, it performs \( q - 1 \) phases which do not produce any replacement edge.

Since phase \( w \) inspects \( 2^w \) edges, it costs \( O(2^w \lg n) \) work. The total work over all \( q \) phases is therefore

\[
\sum_{w=0}^{q} 2^w \lg n = O(2^q \lg n)
\]

in expectation. However, since no replacement was found during the first \( q - 1 \) phases, there are at least \( 2^{q-1} = O(2^q) \) edges that will be pushed down, so we can charge \( O(\lg n) \) work to each such edge to pay for this. In the other case, \( q \) phases run without finding a replacement edge. In this case, all edges inspected are pushed down, and hence each assumes a cost of \( O(\lg n) \) in expectation.

Now, we argue that the work done while processing the replacement edges is \( O(k \lg n) \) in expectation over all rounds (lines 22–30). Since \( k \) edges were deleted, the algorithm discovers at most \( k \) replacement edges. We charge the work in these steps to the replacement edges that we find. Let \( k' \) be the number of replacement edges that we find. Filtering the edges, and computing a spanning forest all costs \( O(k') \) work. Promoting the edges to tree edges (inserting them into \( F_i \) and updating the adjacency lists) costs \( O(k' \lg n) \) work in expectation. Finally, updating the components (lines 27–30) costs \( O(k' \lg n) \) work in expectation, which we can charge to either the component, if it is removed from \( C \) in this round, or to the replacement edge that it finds, which is promoted to a tree edge. Since the algorithm can find at most \( k \) replacement edges, the cost per level is \( O(k \lg n) \) in expectation for these steps as necessary.

In total, on each level the algorithm performs \( O(k \lg n) \) expected work that is not charged to a level decrease. Summing over \( \lg n \) levels, this yields an amortized cost of \( O(\lg^2 n) \) expected work per edge deletion. Finally, since the level of an edge can decrease at most \( \lg n \) times, and an edge is charged \( O(\lg n) \) expected work each time its level is decreased, the expected amortized cost per edge insertion is \( O(\lg^2 n) \).

\[\square\]
4 An Improved Algorithm

In this section we design a improved version of the parallel algorithm that performs less work than our algorithm from Section 3. Furthermore, the improved algorithm runs in $O(\log^3 n)$ depth w.h.p., improving on the $O(\log^4 n)$ depth obtained by using Algorithm 4.

**Algorithm 5** The interleaved level search algorithm

1: procedure InterleavedLevelSearch($i, L = \{c_1, c_2, \ldots\}, S$)
2: $F_i$.BatchInsert($S$)
3: $C = c \in L$ with size $\leq 2^{i-1}$
4: $D = c \in L$ with size $> 2^{i-1}$
5: Push level $i$ tree edges of all components in $C$ to level $i - 1$
6: $r = 0, T = \emptyset, E_P = \emptyset$
7: $M = \{c : c\}$ \quad \triangleright Map of components to supercomponents
8: while $|C| > 0$ do
9: $R = \emptyset, sz = 2^r$
10: for each component $c \in C$ in parallel do
11: $c_{\text{max}} = c$.NumNonTreeEdges
12: $c_{sz} = \min(sz, c_{\text{max}})$
13: $E_c =$ First $c_{sz}$ non-tree edges in $c$
14: $R_c =$ All replacement edges in $E$
15: $R = R \cup R_c$
16: $R' =$ Apply $M$ to each $e \in R$ to compute representatives
17: $C_r = C \cup \{\text{components affected by } R\}$
18: $T'_r =$ SpanningForest($R'$)
19: $T_r =$ Edges in $R$ corresponding to edges chosen in $T'_r$
20: $T = T \cup T_r$
21: Update $M$, the map of supercomponents and their sizes
22: for each component $c \in C$ in parallel do
23: $T_c =$ The set of tree edges chosen from $E_c$
24: if $M[c].\text{size} \leq 2^{i-1} \text{ and } c_{sz} < \text{max}_{sz}$ then
25: Remove edges in $E_c$ from level $i$
26: $E_P = E_P \cup E_c$
27: else
28: Mark $c$ as an inactive component
29: $D_r =$ components in $C$ that are inactive
30: $D = D \cup D_r$
31: $C = C \setminus D_r$
32: $r = r + 1$
33: Promote edges in $T$ to tree edges
34: $F_i$.BatchInsert($T$)
35: Insert pushed edges in $E_P$ onto level $i - 1$
36: return $(D, S \cup T)$

4.1 The Interleaved Deletion Algorithm

**Overview.** Algorithm 5 is based on *interleaving* the phases of doubling that search for replacement edges with the spanning forest computation performed on the replacement edges. Recall that in
Algorithm 4, the number of edges examined in each round is reset, and the doubling algorithm must therefore start with an initial search size of 1 on the next round. Because the doubling resets from round to round, the number of phases per round can be $O(\lg n)$ in the worst case, making the total number of phases per level $O(\lg^2 n)$, and the depth per level $O(\lg^3 n)$. Instead, the interleaved algorithm avoids resetting the search size by maintaining a single, geometrically increasing search size over all rounds of the search.

The most important difference in Algorithm 5 compared with Algorithm 4 is that it defers inserting tree edges found on this level until the end of the search. Instead, it continues to search for replacement edges from the initial components until the component is deactivated. This property is important to show that the work done for a component across all rounds is geometrically dominated, since the number of vertices in the component is always fixed, but the number of non-tree edges examined doubles in each round. For the same reason, it also defers inserting the pushed edges onto level $i - 1$. This property is crucial for the improved work bounds that we obtain (Section 4.3).

Another difference in the modified algorithm is that if a component is still active after adding the replacement edges found in this round (i.e., the component on level $i$ still has size at most $2^i - 1$), then all of the edges found in this round can be pushed to level $i - 1$ without violating Invariant 1. Notice now that when pushing down edges, both the tree and non-tree edges that are found in this round are pushed. Pushing down all edges ensures that we have enough level decreases to which to charge the work performed on the next round. The component deactivates either once it runs out of incident edges, or when it becomes too large. Since the algorithm defers adding the new tree edges found until the end of the level, it also maintains an auxiliary data structure that dynamically tracks the size of the resulting components as new edges are found.

The Deletion Algorithm. We briefly describe the main differences between INTERLEAVEDLEVELSEARCH, the new level search procedure, and PARALLELLEVELSEARCH. We use $r$ to track the round numbers, and we use $E_P$ to store the set of edges that will be pushed to level $i - 1$ at the end of the search at this level (line 6). $T$ stores the set of tree edges that have been selected, which will be added to the spanning forest at the end of the level. Lastly, we use $M$ to maintain a mapping from all the components in $L$ to a unique representative for their contracted supercomponent (initially itself), and the size of the contracted supercomponent.

In round $r$, the algorithm first retrieves the first $2^r$ (or fewer) edges of the active components, and find replacement edges (lines 11–14). All replacement edges are added to the set $R$ (line 15).

After synchronizing, it computes a spanning forest over the edges in $R$, and compute $T_r$, which are the original replacement edges that were selected in the spanning forest (lines 18–20). The spanning forest computation returns, in addition to the tree edges, a mapping from the vertices in $R'$ to their connectivity label (line 18), which can be used on line 21 to efficiently update the representatives of all affected components ($C_r$), and the sizes of the supercomponents.

The next step in a round loops over all active components again in parallel (line 22). If a component is still active (its new size is small enough to still be searched, and the component still has some non-tree edges remaining) (line 24), all of the searched edges are removed from $G_i$ (line 25) and are added to the set of edges that will be pushed to level $i - 1$ at the end of the level (line 26).

The last steps in the round (lines 29–31) handle updating the set of components.

Finally, once all components are inactive, the tree edges found at this level are inserted into $F_i$ (line 34), and all edges added to $E_P$ in line 26 are pushed down to level $i - 1$ (line 35) (note that any tree-edges found in this set are added to $F_{i-1}$). The procedure returns the set of components and all replacement edges found at this level and levels below it (line 36).
4.2 Cost Bounds

We start by showing that the depth of this algorithm is $O(\lg^3 n)$.

**Lemma 5.** The number of rounds performed by Algorithm 5 is $O(\lg n)$ and the depth of each round is $O(\lg n)$ w.h.p. The depth of the InterleavedLevelSearch is therefore $O(\lg^2 n)$ w.h.p.

**Proof.** Each round of the algorithm increases the search size of a component by a factor of 2. Therefore, after $O(\lg n)$ rounds, every non-tree edge incident on a component will be considered and the algorithm will terminate.

To argue the depth bound, we consider the main steps performed during a round. Fetching, examining and removing the edges from level $i$ takes $O(\lg n)$ depth w.h.p. by Lemma 10, Theorem 2, and Lemma 11. Computing a spanning forest on the replacement edges and filtering the components (at most $k$ replacement edges, or components) can be done in $O(\lg k)$ depth. The depth per round is therefore $O(\lg n)$ w.h.p. and the depth of InterleavedLevelSearch is $O(\lg^2 n)$ w.h.p. \(\square\)

Combining Lemma 5 with the fact that there are $\lg n$ levels gives the following theorem.

**Theorem 7.** A batch of $k$ edge deletions can be processed in $O(\lg^3 n)$ depth w.h.p.

We now consider the work performed by the algorithm. We start with a lemma showing that the search-size for a component increases geometrically until the round where the component is deactivated.

**Lemma 6.** Consider a component, $c$, that is active at the end of round $r - 1$. If $c$ is not removed from $C$, then it examines $\geq 2^{r-1}$ edges that are pushed down to level $i - 1$ at the end of the search.

**Proof.** We prove the contrapositive. Suppose that $< 2^{r-1}$ edges are pushed down in total by $c$ in the last round. Then, we will show that $c$ cannot be active in the next round (i.e., it is removed from $C$ in round $r - 1$).

Notice that $c$ must be active at the start of round $r - 1$. Consider the check on line 24, which checks whether $c_{sz} \leq 2^{r-1}$ and $c_{sz} < c_{max}$ on this round. Suppose for the same of contradiction that both conditions are true. Then, by the fact that $c_{sz} < c_{max}$, it must be the case that $c_{sz} = sz = 2^{r-1}$ by line 12. If the condition is true, then on line 26 the algorithm adds $2^{r-1}$ edges to be pushed to level $i - 1$, contradicting our assumption that $< 2^{r-1}$ edges are pushed.

Therefore the check on line 24 must be false, giving that either $c_{sz} > 2^{i-1}$, or $c_{sz} = c_{max}$. This means that $c$ will be marked as inactive on line 28, and then become deactivated on line 29. Therefore, if $< 2^{r-1}$ edges are pushed down by $c$ in round $r - 1$, it is deactivated at the end of the round, concluding the proof. \(\square\)

**Lemma 7.** Consider the work done by some component $c$ over the course of InterleavedLevelSearch at a given level. Let $R$ be the total number rounds that $c$ is active. Then, $c$ pushes down $p_c = 2^R - 1$ edges in total. Furthermore, the total cost of searching for and pushing down replacement edges performed by $c$ is

$$O\left(p_c \lg \left(1 + \frac{n_c}{p_c}\right)\right)$$

in expectation, where $n_c$ is the number of vertices in $c$.

**Proof.** By Lemma 6, for each round $r < R$, $c$ adds $2^r$ edges to be pushed down. Summing over all rounds shows that the total number of edges added to be pushed down is $2^R - 1$. The cost of
pushing down these edges at the end of the search at this level is exactly

\[ O\left(p_c \log \left(1 + \frac{n_c}{p_c}\right)\right). \]

by Lemma 11, since the size of the tree that is affected is \( n_c \).

We now consider the cost of fetching and examining the edges over all rounds. The cost of fetching and examining \( 2^r \) edges is

\[ O\left(2^r \log \left(1 + \frac{n_c}{2^r}\right)\right), \]

in expectation by Theorem 2 and Lemma 10. Summing over all rounds \( r < R \), the work is

\[ \sum_{r=1}^{R-1} O\left(2^r \log \left(1 + \frac{n_c}{2^r}\right)\right) \]

in expectation to fetch and examine edges in the first \( R - 1 \) rounds, which is equal to

\[ O\left(2^R \log \left(1 + \frac{n_c}{2R}\right)\right), \]

by Lemma 2. Since on round \( R \), the algorithm searches at most \( 2^R \) edges, the total cost of searching for replacement edges over all rounds is at most

\[ O\left(2^R \log \left(1 + \frac{n_c}{2R}\right)\right) = O\left(p_c \log \left(1 + \frac{n_c}{p_c}\right)\right). \]

Lemma 8. The cost of \textsc{InterleavedLevelSearch} is at most

\[ O\left(k \log \left(1 + \frac{n}{k}\right) + p \log \left(1 + \frac{n}{p}\right)\right) \]

in expectation where \( p \) is the total number of edges pushed down.

Proof. First consider lines 2–5. Since we are deleting a batch of \( k \) edges, we can find at most \( k \) replacement edges to reconnect these components. Therefore line 2 performs \( O(k \log (1 + \frac{n}{k})) \) expected work by Theorem 2. Pushing \( t \) spanning tree edges to the next level (line 5) can be done in \( O(t \log (\frac{n}{t} + 1)) \) expected work by Lemmas 10, 11, and 1, and Theorem 2. Hence in total, lines 2–5 perform at most \( O(k \log (1 + \frac{n}{k}) + t \log (1 + \frac{n}{t})) \) work in expectation.

Now, consider the cost of the steps which scan or update the components that are active in each round. On the first round, this cost is \( O(k) \). In every subsequent round, \( r \), by Lemma 6 each currently active component must have added \( 2^{r-1} \) edges to be pushed down on the previous round. Therefore, we can charge the \( O(1) \) work per component performed in this round to these edge pushes.

Next, we analyze the work done while searching for and pushing replacement edges. Consider some component \( c \in C \) that is searched on this level. By Lemma 7, the cost of searching for and pushing down the replacement edges incident on this component is

\[ O\left(p_c \log \left(1 + \frac{n_c}{p_c}\right)\right) \]
in expectation, where \( n_c \) is the number of vertices in \( c \) and \( p_c \) is the total number of edges pushed down by \( c \).

The total work done over all components to search for replacement edges and push down both the original tree edges, and the edges in each round is therefore

\[
O\left( t \lg \left( 1 + \frac{n}{\tau} \right) + \sum_{c \in C} p_c \lg \left( 1 + \frac{n_c}{p_c} \right) \right).
\]

\[
\sum n_c = n,
\]

by Lemma 1 this costs

\[
O\left( p \lg \left( 1 + \frac{2n}{p} \right) \right) = O\left( p \lg \left( 1 + \frac{n}{p} \right) \right)
\]

work in expectation, where \( p = t + \sum p_c \) is the total number of edges pushed, including tree and non-tree edges. Therefore, the total cost is

\[
O\left( k \lg \left( 1 + \frac{n}{k} \right) + p \lg \left( 1 + \frac{n}{p} \right) \right)
\]

in expectation.

**Theorem 8.** The expected amortized cost per edge insertion or deletion is \( O(\lg^2 n) \).

**Proof.** The proof follows from the same argument as Theorem 6, by using Lemma 8. \( \square \)

### 4.3 A Better Work Bound

We now show that by a more careful analysis, we can obtain a tighter bound on the amount of work performed by the interleaved algorithm. In particular, we show in this section that the algorithm performs

\[
O\left( \lg n \lg \left( 1 + \frac{n}{\Delta} \right) \right)
\]

amortized work per edge in expectation, where \( \Delta \) is the average batch size of all batches of deletions. Therefore, if we process batches of deletions of size \( O(n/\text{polylog}(n)) \) on average, our algorithm performs \( O(\lg n \lg \lg n) \) expected amortized work per edge, rather than \( O(\lg^2 n) \). Furthermore, if we have batches of size \( O(n) \), the cost is just \( O(\lg n) \) per edge.

At a high level, our proof formalizes the intuition that in the worst case, all edges are pushed down at every level, and that performing fewer deletion operations results in larger batches of pushes which take advantage of work bounds of the ET-tree. Our proof crucially relies on the fact that although the deletion algorithm at a level can perform \( O(\lg n) \) ET-tree operations per component, since the batch sizes are geometrically increasing, these operations have the cost of a single ET-tree operation per component. Furthermore, Lemma 8 shows that the costs per component can be combined so that the total cost is equivalent to the cost of a single ET-tree operation on all the vertices. Therefore, the number of deletion operations can be exactly related to the effective number of ET-tree operations at a level. We relate the number of deletions to the average batch size, which lets us obtain a single unified bound for both insertions and deletions.

**Theorem 9.** Using the interleaved deletion algorithm, the amortized work performed by \textsc{BatchDeletion} and \textsc{BatchInsertion} on a batch of \( k \) edges is

\[
O\left( k \lg n \lg \left( 1 + \frac{n}{\Delta} \right) \right),
\]

in expectation where \( \Delta \) is the average batch size of all batch deletions.
Proof. Batch insertions perform only $O(k \lg(1 + \frac{n}{k}))$ work by Theorem 4, so we focus on the cost of deletion since it dominates. Consider the total amount of work performed by all batch deletion operations at any given point in the lifetime of the data structure. We will denote by $k_b$, the size of batch $b$, and by $p_{b,i}$, the number of edges pushed down on level $i$ during batch $b$. Combining Lemmas 4, and 8, the total work is bounded above by

$$O\left(\sum_{\text{batch} b} \sum_{\text{level} i} k_b \lg \left(1 + \frac{n}{k_b}\right) + p_{b,i} \lg \left(1 + \frac{n}{p_{b,i}}\right)\right).$$

We begin by analyzing the first term, which is paid for by the deletion algorithm. Let

$$K = \sum_{\text{batch} b} k_b$$

denote the total number of deleted edges. Applying Lemma 1, and using the fact that there are $\lg n$ levels, we have

$$O\left(\sum_{\text{batch} b} \sum_{\text{level} i} k_b \lg \left(1 + \frac{n}{k_b}\right)\right) = O\left(K \lg n \lg \left(1 + \frac{n \cdot d}{K}\right)\right),$$

where $d$ is the number of batches of deletions. Since $K/d = \Delta$, this is equal to

$$O\left(K \lg n \lg \left(1 + \frac{n}{\Delta}\right)\right),$$

work in expectation. Each batch can therefore be charged a cost of $\lg n \lg(1 + n/\Delta)$ per edge, and hence the amortized cost of batch deletion is

$$O\left(k \lg n \lg \left(1 + \frac{n}{\Delta}\right)\right)$$

in expectation.

The remainder of the cost, which comes entirely from searching for replacement edges, is charged to the insertions. Consider this cost and let

$$P = \sum_{\text{batch} b} \sum_{\text{level} i} p_{b,i}$$

denote the total such number of edge pushes. Since the total number of terms in the double sum is $d \lg n$, Lemma 1 allows us to bound the total work of all pushes by

$$\sum_{\text{batch} b} \sum_{\text{level} i} p_{b,i} \lg \left(1 + \frac{n}{p_{b,i}}\right) = O\left(P \lg \left(1 + \frac{nd \lg n}{P}\right)\right),$$

in expectation. Since every edge can only be pushed down once per level, we have

$$P \leq m \lg n,$$

where $m$ is the total number of edges ever inserted. Therefore by Lemma 3, the total work is at most

$$O\left(m \lg n \lg \left(1 + \frac{nd \lg n}{m \lg n}\right)\right) = O\left(m \lg n \lg \left(1 + \frac{nd}{m}\right)\right)$$

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in expectation. Since \( d = K/\Delta \), this is equal to

\[
O\left( m \log n \log \left( 1 + \frac{nK}{m\Delta} \right) \right)
\]

in expectation. Since each edge can be deleted only once, we have \( K \leq m \), and hence we obtain that the total work to push all tree edges down is at most

\[
O\left( m \log n \log \left( 1 + \frac{n}{\Delta} \right) \right).
\]

in expectation. We can therefore charge \( O(\log n \log(1 + n/k)) \) per edge to each batch insertion. Since this dominates the cost of the insertion algorithm itself, the amortized cost of batch insertion is therefore

\[
O\left( k \log n \log \left( 1 + \frac{n}{\Delta} \right) \right),
\]

in expectation as desired, concluding the proof.

5 Related Work

Parallel Dynamic Algorithms. There are only a few results on parallel dynamic algorithms. Earlier results [21, 18] are not work-efficient with respect to the fastest sequential dynamic algorithms, do not support batch updates, and perform polynomial work per update. Some more recent results such as parallel dynamic depth-first search [38] and minimum spanning forest [39] process updates one at a time, and are therefore not batch-dynamic algorithms. Work efficient parallel batch-dynamic algorithms include those for the well-spaced point sets problem [2] and those for the dynamic trees problem [50, 1, 62].

Parallel Connectivity. Parallel algorithms for connectivity have a long history [30, 54, 63, 8, 48, 45, 17, 36], and there are many existing algorithms that solve the problem work-efficiently and in low-depth [22, 16, 25, 26, 44, 46, 56], some of which are also practical [56, 19]. However, there is no obvious way to adapt existing parallel connectivity algorithms to the dynamic setting, particularly for batch updates.

Parallel Dictionaries and Trees. There are many results on parallel dictionaries and trees supporting batch updates [23, 13, 14, 55, 11, 5, 58]. The dictionary data structures in the literature culminated in dictionaries supporting batch insertions, deletions and lookups in linear work and \( O(\log^* n) \) depth w.h.p. [23]. Early work on batch insertions into trees focused on optimizing the depth, but was not work-efficient. Paul et al. design batch search, insertion and deletion algorithms for 2-3 trees on the EREW PRAM [43]. These results were later extended to B-trees by Higham et al. [29]. The algorithms of both Paul et al. and Higham et al. perform \( O(m \log n) \) work for \( m \) tree operations.

Recent work on parallel tree data structures has focused on how to parallelize batch operations for various balancing schemes in binary search trees [11], and also how to improve the depth of these operations [5]. There is also some very recent work on extending these tree data structures to support range and segment queries [58] as well as practical implementations of parallel trees supporting batch insertions, deletions and lookups [59].

Other Related Work. There is also recent work on parallel working-set structures that supports batching by Agrawal et al. [4]. Earlier work by Agrawal et al. [3] introduces the idea of implicit batching which uses scheduler support to convert dynamically multithreaded programs using an abstract data type to programs that perform batch accesses to an underlying parallel data structure.
6 Discussion

In this paper, we present a novel batch-dynamic algorithm for the connectivity problem. Our algorithm is always work-efficient with respect to the Holm, de Lichtenberg and Thorup dynamic connectivity algorithm, and is asymptotically faster than their algorithm when the average batch size is sufficiently large. A parallel implementation of our algorithm achieves $O(\lg^3 n)$ depth w.h.p., and is, to the best of our knowledge, the first parallel algorithm for the dynamic connectivity problem performing $O(T \text{polylog}(n))$ total expected work, where $T$ is the total number of edge operations.

There are several natural questions to address in future work. First, can the depth of our algorithm be improved to $O(\lg^2 n)$ without increasing the work? Investigating lower bounds in the batch setting would also be very interesting—are there non-trivial lower-bounds for batch-dynamic connectivity? Lastly, in this paper we show expected amortized bounds. One approach to strengthen these bounds is to show that our tree operations hold w.h.p. and argue that our amortized bounds hold w.h.p. Another is to design a deterministic batch-dynamic forest connectivity data structure with the same asymptotic complexity as the batch-parallel ET-tree, which would make the randomized bounds in this paper deterministic.

Two additional questions are whether we can extend our results to give parallel work-efficient batch-dynamic MST, 2-edge connectivity and biconnectivity algorithms. MST seems solvable using the techniques presented in this paper, although our dynamic tree structure would need to be extended with additional primitives. Existing sequential 2-edge connectivity and biconnectivity algorithms require a dynamic tree data structure supporting path queries which are not supported by ET-trees. However, RC-trees [1] can be extended to support path queries, which makes them a possible candidate for this line of work. Finally, it seems likely that ideas from our work can be extended to give a parallel batch-dynamic Monte-Carlo connectivity algorithm based on the Kapron-King-Mountjoy algorithm [35].

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7 Model

The Multi-Threaded Random-Access Machine (MT-RAM) consists of a set of threads that share an unbounded memory. Each thread is essentially a Random Access Machine—it runs a program stored in memory, has a constant number of registers, and uses standard RAM instructions (including an end to finish the computation). The only difference between the MT-RAM and a RAM is the fork instruction that takes a positive integer $k$ and forks $k$ new child threads. Each child thread receives a unique identifier in the range $[1, \ldots, k]$ in its first register and otherwise has the same state as the parent, which has a 0 in that register. All children start by running the next instruction. When a thread performs a fork, it is suspended until all the children terminate (execute an end instruction). A computation starts with a single root thread and finishes when that root thread terminates. This model supports what is often referred to as nested parallelism. Note that if root thread never forks, it is a standard sequential program.

We note that we can simulate an MT-RAM algorithm on the CRCW PRAM equipped with the same operations with an additional $O(\log^* n)$ factor in the depth due to load-balancing. Furthermore, a PRAM algorithm using $P$ processors and $T$ time can be simulated in our model with $PT$ work and $T$ depth. We equip the model with a compare-and-swap operation (see Section 2) in this paper.

Lastly, we define the cost-bounds for this model. A computation can be viewed as a series-parallel DAG in which each instruction is a vertex, sequential instructions are composed in series, and the forked subthreads are composed in parallel. The work of a computation is the number of vertices
and the depth (span) is the length of the longest path in the DAG. We refer the interested reader to [10] for more details about the model.

8 Data Structures

In this section we describe a simple adjacency-list like data structure that efficiently supports insertion and deletion of arbitrary edges, and quickly fetching a batch of $l$ edges. This is the data structure that we use to store adjacency lists of vertices at each level. Note that we actually store two adjacency lists, one for tree edges, and one for non-tree edges. The adjacency list data structure supports the following operations:

- **InsertEdges**($\{e_1, \ldots, e_l\}$): Insert a batch of edges adjacent to this vertex.
- **DeleteEdges**($\{e_1, \ldots, e_l\}$): Delete a batch of edges adjacent to this vertex.
- **FetchEdges**($l$): Return a set of $l$ arbitrary edges adjacent to this vertex.

We now show how to implement a data structure that gives us the following bounds:

**Lemma 9.** **InsertEdges**, **DeleteEdges**, and **FetchEdges** can be implemented in $O(1)$ amortized work per edge and in $O(\lg n)$ depth.

**Proof.** For a given vertex, the data structure stores a list of pointers to each adjacent edge in a resizable array. Each edge correspondingly stores its positions in the adjacency arrays of its two endpoints. Since each vertex can have at most $O(n)$ edges adjacent to it, the adjacency arrays are of size at most $O(n)$.

Insertions are easily handled by inserting the batch onto the end of the array, and resizing if necessary. This costs $O(1)$ amortized work per edge and $O(\lg n)$ depth. To fetch $l$ elements, we simply return the first $l$ elements of the array, which takes $O(1)$ work per edge and $O(\lg n)$ depth.

Finally, to delete a batch of $l$ edges, the algorithm first determines which of the edges to be deleted are contained within the final $l$ elements of the array. It then compacts the final $l$ elements of the array, removing those edges. Compaction costs $O(l)$ work and $O(\lg n)$ depth. The algorithm then considers the remaining $l'$ edges to be deleted, and in parallel, swaps these elements with the final $l'$ elements of the array. The final $l'$ elements in the array can then be safely removed. Note that any operation that moves an element in the array also updates the corresponding position value stored in the edge. Swapping and deleting can be implemented in $O(l')$ work and $(\lg n)$ depth, and hence all operations cost $O(1)$ amortized work per edge and $O(\lg n)$ depth.

9 Additional Tree Operations

**Retrieving and Pushing Down Edges.** The batch-parallel ET-trees used in this paper augment each node in the tree with two values indicating the number of tree and non-tree edges whose level is equal to the level of the forest currently stored in that subtree. The augmentation is necessary for efficiently fetching the tree edges that need to be pushed down before searching the data structure, and for fetching a subset of non-tree edges in a tree.

We extend the batch-dynamic trees interface described earlier with operations which enable efficiently retrieving, removing and pushing down batches of tree or non-tree edges.

These primitives are all similar and can be implemented as follows. We first describe the primitives which fetch and remove a set of $l$ tree (or non-tree) edges. The algorithm starts by finding
a set of vertices containing \( l \) edges. To do this we perform a binary search on the skip-list in order to find the first node that has augmented value greater than \( l \). The idea is to sequentially walk at the highest level, summing the augmented values of nodes we encounter and marking them, until the first node that we hit whose augmented value makes the counter larger than \( l \), or we return to \( v \). In the former case, we descend a level using this node’s downwards pointer, and repeat, until we reach a level 0 node. We also keep a counter, \( ctr \), indicating the number of tree (non-tree) edges to take from the rightmost marked node at level 0. In the latter case, we descend a level using this node’s downwards pointer, and repeat, until we reach a level 0 node. We also keep a counter, \( ctr \), indicating the number of tree (non-tree) edges to take from the rightmost marked node at level 0. Otherwise, all nodes at the topmost level are marked.

The last step of the algorithm is to find all descendants of marked nodes that have a non-zero number of tree (non-tree) edges, and return all tree (non-tree) edges incident on them. The only exception is the rightmost marked node, from which we only take \( ctr \) many tree (non-tree) edges.

Insertions are handled by first inserting the edges into the adjacency list data structure. We then update the augmented values in the ET-tree using the primitive from Tseng et al. \[62\]. We now argue that these implementations achieves good work and depth bounds.

**Lemma 10.** Given some vertex, \( v \) in a batch-parallel ET-tree, we can fetch the first \( l \) tree (or non-tree) edges referenced by the augmented values in the tree in \( O(l \log (1 + \frac{n_c}{l})) \) work and \( O(\log n) \) depth w.h.p. where \( n_c \) is the number of vertices in the ET-tree at the current level. Furthermore, removing the edges can be done in the same bounds.

**Proof.** Standard proofs about skip-lists shows that the number of nodes traversed in the binary search is \( O(\log n) \) w.h.p. \[47, 62\]. We can fetch \( l \) edges from each vertex’s adjacency list data structure in \( O(l) \) amortized work and \( O(\log n) \) depth by Lemma 9. The total work is therefore \( O(l \log (1 + \frac{n_c}{l})) \) in expectation, and the depth is \( O(\log n) \) w.h.p. since the depth of the adjacency list access is an additive increase of \( O(\log n) \). Observe that removing the edges can be done in the same bounds since updating the augmented values after deleting the edges costs \( O(l \log (1 + \frac{n_c}{l})) \) expected work.

**Lemma 11.** Decreasing the level of \( l \) tree (or non-tree) edges in a batch-parallel ET-tree can be performed in \( O(l \log (1 + \frac{n_c}{l})) \) expected work and \( O(\log n) \) depth w.h.p. where \( n_c \) is the number of nodes in the ET-tree at the current level.

**Proof.** The proof is identical to the proof of Lemma 10. The only difference is that the augmented values of the nodes that receive an edge must be updated after insertion which costs at most \( O(l \log (1 + \frac{n_c}{l})) \) in expectation. Note that since the forest on the lower level is a subgraph of the tree at the current level, it has size at most \( n_c \), proving the bounds.

### 10 Additional Proofs

We now state and give proofs for some of the technical lemmas used in our proofs of the improved batch bounds.

**Lemma 1.** Let \( n_1, n_2, \ldots, n_c \) and \( k_1, k_2, \ldots, k_c \) be sequences of non-negative integers such that \( \sum k_i = k \), and \( \sum n_i = n \). Then

\[
\sum_{i=1}^{c} k_i \log \left( 1 + \frac{n_i}{k_i} \right) \leq k \log \left( 1 + \frac{n}{k} \right).
\]

**Proof.** We proceed by induction on \( c \). When \( c = 1 \), the quantities are equal. For \( c > 1 \), we can write

\[
\sum_{i=1}^{c} k_i \log \left( 1 + \frac{n_i}{k_i} \right) = \sum_{i=1}^{c-1} k_i \log \left( 1 + \frac{n_i}{k_i} \right) + k_c \log \left( 1 + \frac{n_c}{k_c} \right),
\]

\[
\leq (k - k_c) \log \left( 1 + \frac{n - n_c}{k - k_c} \right) + k_c \log \left( 1 + \frac{n_c}{k_c} \right).
\]
Then, using the concavity of the logarithm function, we have
\[
\sum_{i=1}^{c} k_i \log \left( 1 + \frac{n_i}{k_i} \right) \leq k \log \left( \frac{k - k_c}{k} \left( 1 + \frac{n_i}{k - k_c} \right) + \frac{k_c}{k} \left( 1 + \frac{n_c}{k_c} \right) \right),
\]
\[
= k \log \left( \frac{k - k_c}{k} + \frac{n - n_c}{k} + \frac{k_c}{k} + \frac{n_c}{k_c} \right),
\]
\[
= k \log \left( 1 + \frac{n}{k} \right),
\]
which concludes the proof. \(\square\)

**Lemma 2.** For any non-negative integers \(n\) and \(r\),
\[
\sum_{w=0}^{r} 2^w \log \left( 1 + \frac{n}{2^w} \right) = O \left( 2^r \log \left( 1 + \frac{n}{2^r} \right) \right).
\]

**Proof.** First, write
\[
\log \left( 1 + \frac{n}{2^w} \right) = \log \left( 1 + 2^{r-w} \frac{n}{2^r} \right),
\]
\[
\leq \log \left( 2^{r-w} \left( 1 + \frac{n}{2^r} \right) \right),
\]
\[
= \log \left( 2^{r-w} \right) + \log \left( 1 + \frac{n}{2^r} \right),
\]
\[
= (r - w) + \log \left( 1 + \frac{n}{2^r} \right).
\]
Now substitute this into the sum to obtain
\[
\sum_{w=0}^{r} 2^w \log \left( 1 + \frac{n}{2^w} \right) \leq \sum_{w=0}^{r} (r - w) 2^w + \log \left( 1 + \frac{n}{2^r} \right) \sum_{w=0}^{r} 2^w,
\]
\[
= \sum_{w=0}^{r} (r - w) 2^w + O \left( 2^r \log \left( 1 + \frac{n}{2^r} \right) \right),
\]
We evaluate the remaining sum by writing
\[
\sum_{w=0}^{r} (r - w) 2^w = \sum_{w=0}^{r} \frac{r - w}{2^{r-w}} \cdot 2^r,
\]
and then use the fact that
\[
\sum_{w=0}^{r} \frac{r - w}{2^{r-w}} = O(1)
\]
to conclude that
\[
\sum_{w=0}^{r} 2^w \log \left( 1 + \frac{n}{2^w} \right) = O \left( 2^r \right) + O \left( 2^r \log \left( 1 + \frac{n}{2^r} \right) \right),
\]
\[
= O \left( 2^r \log \left( 1 + \frac{n}{2^r} \right) \right),
\]
as desired. \(\square\)
Lemma 3. For any $n \geq 1$, the function $x \log\left(1 + \frac{n}{x}\right)$ is strictly increasing with respect to $x$ for $x \geq 1$.

Proof. The derivative of the function with respect to $x$ is

$$\log\left(1 + \frac{n}{x}\right) - \frac{n}{n+x}.$$

We must show that this quantity is strictly positive for all $x \geq 1$. First, we use a well-known inequality that states

$$a^y \leq 1 + (a - 1)y,$$

for $a \geq 1$ and $y \in [0, 1]$. Using $a = 2$ and $y = n/(n + x)$, we obtain

$$2^{\frac{n}{n+x}} \leq 1 + \frac{n}{n+x},$$

Since $n \geq 1$ and $x \geq 1$, we have

$$1 + \frac{n}{n+x} < 1 + \frac{n}{x},$$

and hence by transitivity,

$$2^{\frac{n}{n+x}} < 1 + \frac{n}{x},$$

Taking logarithms on both sides yields

$$\frac{n}{n+x} < \log\left(1 + \frac{n}{x}\right),$$

which implies the desired result. \qed