WHEN HAMILTON CIRCUITS GENERATE THE CYCLE SPACE OF A RANDOM GRAPH

PETER HEINIG

Abstract. If \( \varepsilon > 0 \) and \( p(n) \geq n^{-\frac{1}{2} + \varepsilon} \), in a binomial random graph \( G_{n,p} \) a.a.s. the set of cycles which can be constructed as a symmetric difference of Hamilton circuits is as large as parity by itself permits (all cycles if \( n \) is odd, all even cycles if \( n \) is even). Moreover, every \( p \) which ensures the above property a.a.s. must necessarily be such that for any constant \( c > 0 \), eventually \( p(n) \geq \frac{\log n + 2\log \log n + c}{n} \). So whatever the smallest sufficient \( p \) for an a.a.s. Hamilton-generated cycle space might be, it does not coincide with the threshold for hamiltonicity of \( G_{n,p} \).

Keywords: cycle space, binomial random graph, finite-dimensional vector spaces, Hamilton circuit, Hamilton-connected, consequences of structured spanning subgraphs

1. Introduction

In the face of the apparent intractability of efficiently describing the set of all finite hamiltonian graphs, one realistic strategy to acquire more knowledge is the study of slices of that complicated set: under the assumption that a known sufficient condition for hamiltonicity holds, one tries to find proofs that such a graph has extra properties, such as having many Hamilton circuits, or having many well-distributed Hamilton circuits; examples in the deterministic setting (with the sufficient condition being a minimum-degree-condition) are to be found in \([6]\) and \([13]\). In \([9, \text{Theorem 1}]\), another extra property, a sort of ‘richness-’ or ‘universality-’ property (the set of all Hamilton circuits generates the cycle space), previously known for certain Cayley-graphs only \([1]\), was proved to hold in graphs of minimum degree slightly above Dirac’s bound for hamiltonicity.

There are known examples for the above paradigm in a random setting as well (e.g., \([3]\) \([4]\) \([5]\) \([11]\)), and in this note we give another variation on that theme: we will combine some recent results in order to prove a new result, Theorem 1, teaching us that sufficiently dense binomial random graphs a.a.s. have the above universality-property (and for a vastly smaller edge-probability than what would a.a.s. force the minimum-degree-condition from \([9, \text{Theorem 1}]\) to hold).

We write \( Z_1(G; \mathbb{F}_2) \) for the cycle space of a graph \( G \) in the standard graph-theoretical sense, and \( \mathcal{H}(G) \) for the set of all Hamilton circuits in \( G \). The word ‘circuit’ means ‘2-regular connected graph’, while ‘cycle’ means ‘element of \( Z_1(G; \mathbb{F}_2) \)’. For any graph \( G \), the notation \( Z_1(G; \mathbb{F}_2) = \langle \mathcal{H}(G) \rangle_{\mathbb{F}_2} \), in which \( \langle \mathcal{H}(G) \rangle_{\mathbb{F}_2} \) denotes the \( \mathbb{F}_2 \)-span of the (vectors w.r.t the standard basis of the edge-space describing the) Hamilton circuits of \( G \), is equivalent to saying ‘every cycle of \( G \) is a symmetric difference of Hamilton circuits of \( G \).

We will phrase Theorem 1 in linear-algebraic language, to be able to efficiently use auxiliary results from \([9]\). However, the results can easily be translated into more constructive phrasings: (2) and (3) are equivalent to the statement that, a.a.s., the set of cycles which can be constructed as a symmetric difference of Hamilton circuits is as large as parity alone permits. I.e., all cycles if \( n \) is odd, and all even cycles (i.e., elements of the cycle space having a support of even size, of which circuits are examples) if \( n \) is even. One of these equivalences is a deterministic one: statement (3) is deterministically equivalent to saying that for odd \( n \), every cycle of \( G \) is a symmetric difference of (edge-sets of) Hamilton-circuits of \( G \). The other implication holds in an a.a.s. sense. Statement (2) is not deterministically equivalent to saying that for even \( n \), every even cycle is a symmetric difference of Hamilton circuits: if \( n \) is even and \( G \) happens to be bipartite, then every even cycle
being a symmetric difference of Hamilton circuits is equivalent to \( \dim_{F_2} (Z_1(G;F_2)/\langle \mathcal{H}(G) \rangle_{F_2}) = 0 \), not \( = 1 \). But when restricted to non-bipartite graphs only—and \( G_{n,p} \) with the present \( p \) is non-bipartite a.a.s.—statement (2) is equivalent to saying that for even \( n \), every even cycle is a symmetric difference of Hamilton-circuits: \( G \sim G_{n,p} \) with \( p(n) \geq n^{-1/2+\varepsilon} \) a.a.s. contains triangles. Since \( n \) is even, if there were an even cycle \( z \in Z_1(G;F_2) \) not in the \( F_2 \)-span of the Hamilton-circuits of \( G \), then this cycle together with some triangle would represent two linearly independent elements modulo \( \langle \mathcal{H}(G) \rangle_{F_2} \), in contradiction to \( \dim_{F_2} (Z_1(G;F_2)/\langle \mathcal{H}(G) \rangle_{F_2}) = 1 \), the latter being guaranteed to hold a.a.s. by (2).

All graphs considered, the property of graphs \( G \) with \( Z_1(G;F_2) = \langle \mathcal{H}(G) \rangle_{F_2} \) (no Hamilton-connectedness required) is not monotone, so a priori one should be cautious of speaking of a threshold for that property in \( G_{n,p} \). It seems likely, though, that for binomial random graphs \( G_{n,p} \) the property can only arise ‘within a monotone property’ and that there is a threshold. Should it be possible to improve Theorem 8 in Section 3 to the threshold for Hamilton-connectedness of \( G_{n,p} \), then one would know that for any \( p \) which a.a.s. ensures \( Z_1(G;F_2) = \langle \mathcal{H}(G) \rangle_{F_2} \), the property necessarily comes together with Hamilton-connectedness, i.e. then one would know that, if a.a.s. \( G_{n,p} \in \{ G: Z_1(G;F_2) = \langle \mathcal{H}(G) \rangle_{F_2} \} \), then a.a.s. \( G \in \{ G: Z_1(G;F_2) = \langle \mathcal{H}(G) \rangle_{F_2} \} \cap \{ G: \) Hamilton-connected \} , a monotone graph property. If this is the case, then by [7, Theorem 1.1] we would know that \( Z_1(G;F_2) = \langle \mathcal{H}(G) \rangle_{F_2} \) has a sharp threshold. Although it is clear that a.a.s. \( Z_1(G;F_2) = \langle \mathcal{H}(G) \rangle_{F_2} \) implies that a.a.s. any two adjacent vertices are connected by a Hamilton-path, it is not clear that it also implies that a.a.s. any two non-adjacent vertices are.

2. AN UPPER BOUND FOR THE SMALLEST SUFFICIENT \( p \)

In this section we prove an upper bound on the smallest \( p \) sufficient for \( Z_1(G;F_2) = \langle \mathcal{H}(G) \rangle_{F_2} \) to a.a.s. hold in \( G_{n,p} \):

**Theorem 1.** If \( \varepsilon > 0 \), \( p \in [0,1]^N \) with \( p(n) \geq n^{-1/2+\varepsilon} \), \( \mathcal{H}(G) \) denotes the set of all Hamilton circuits of a graph \( G \) and \( Z_1(G;F_2) \) its cycle space, then for \( n \to \infty \) a random graph \( G \sim G_{n,p} \) asymptotically almost surely has the following properties:

1. the \( F_2 \)-span of all circuits having length \( n \) or \( n-1 \) in \( G \) is \( Z_1(G;F_2) \),
2. if \( n \) is even, then \( \dim_{F_2} (Z_1(G;F_2)/\langle \mathcal{H}(G) \rangle_{F_2}) = 1 \),
3. if \( n \) is odd, the \( F_2 \)-span of \( \mathcal{H}(G) \) is \( Z_1(G;F_2) \).

**Proof of Theorem 1.** We use some notation from [9]: we denote by \( C_n^2 \) the square of an \( n \)-vertex circuit. The symbol \( \mathcal{M}_{i:1,0} \) denotes the set of all finite graphs which are simultaneously Hamilton-connected and have cycle equal to a symmetric difference of Hamilton circuits, while \( \mathcal{M}_{i:1,1} \) denotes the set of all finite graphs \( G \) which are simultaneously Hamilton-connected and have \( \dim_{F_2} (Z_1(G;F_2)/\langle \mathcal{H}(G) \rangle_{F_2}) = 1 \). The symbol \( \mathcal{M}_{i:1,1} \) denotes the set of all graphs \( G \) which simultaneously have the property that any two vertices are connected by a path of length at least \( |G| - 2 \) and that the \( F_2 \)-span of the set of all circuits of length at least \( |G| - 1 \) is equal to \( Z_1(G;F_2) \). By a recent theorem of Kühm and Osthus [14, Theorem 1.2 specialized to \( k = 2 \)], we know that, asymptotically almost surely, \( G_{n,p} \) contains \( C_n^2 \) as a subgraph. By (a3), (a4) and (a5) in [9, Lemma 17], \( C_n^2 \in \mathcal{M}_{i:1,0} \) if \( n \) is odd, while both \( C_n^2 \in \mathcal{M}_{i:1,1} \) if \( n \) is even. By [9, Lemma 18.(1)], each of the graph properties \( \mathcal{M}_{i:1,0} \), \( \mathcal{M}_{i:1,1} \) and \( \mathcal{M}_{i:1,1} \) is monotone. Therefore, a.a.s. \( G_{n,p} \) itself is in \( \mathcal{M}_{i:1,0} \) for odd \( n \) (proving (3)), and in both \( \mathcal{M}_{i:1,1} \) (proving (2)) and \( \mathcal{M}_{i:1,1} \) for even \( n \). The latter completes the proof of (1), for although (1) is formulated without a parity condition, in the case of odd \( n \), (1) is implied by (3) anyway. \( \square \)
3. A LOWER BOUND FOR THE SMALLEST SUFFICIENT $p$

In this section we will derive a lower bound (larger than the hamiltonicity threshold) that any $p$ which a.a.s. ensures $Z_1(G_n,p;F_2) = \langle H(G_n,p) \rangle_{F_2}$ must satisfy. We prepare with a few lemmas.

**Lemma 2.** If $G = (V,E)$ is a graph such that
(1) $G$ is neither a forest nor a circuit ,
(2) every cycle in $G$ is a symmetric difference of Hamilton circuits ,
(3) $G$ contains a vertex of degree 2 ,
then for every vertex $v$ as in (3), the graph $G - v$ obtained by deleting $v$ is bipartite.

**Proof.** Let $G$ be any such graph. Being a non-forest, the cycle space of $G$ is non-trivial. Since otherwise (2) fails (for trivial reasons), we may assume that $G$ contains a Hamilton circuit, in particular, $G$ is 2-connected. We now choose any $v \in V$ with $\deg(v) = 2$. If all vertices of $G$ had degree 2, then $G$ would be one single circuit, contradicting hypothesis (1). We may thus assume that there exists $w' \in V$ with $\deg(w') \geq 3$. By connectedness, there exists a $v-w'$-path $P$, and by finiteness there exists a vertex $w$ on this path such that $\deg(w) \geq 3$ and all vertices between $v$ and $w$ have degree 2 in $G$. (Possibly, $w$ is a neighbour of $v$ and there are no vertices between $v$ and $w$ at all.) Let $v^-$ and $v^+$ denote the two neighbours of $v$, with $v^+$ the one in the direction of $w$ along $P$ (possibly, $v^+ = w$). Then all vertices from $v$ up to and including the predecessor $v^-$ of $w$ on $P$ have degree 2, hence

every circuit in $G$ either contains all or none of the edges $v^-v$, $vv^+$, ..., $w^-w$.  \hspace{1cm} (1)

Now consider some circuit $C$ in $G$ which does not contain any of the edges $v^-v$, $vv^+$, ..., $w^-w$. Such circuits exist, since for example any two neighbours $w'$, $w''$ of the $\geq 2$ neighbours of $w$ other than $w^-$ are connected by a path $\bar{P}$ which neither contains $w$ (by 2-connectedness of $G$ and Menger’s theorem) nor any of the vertices $v$, $v^+$, ..., $w^-$ (since these all have degree 2), so the circuit $w'\bar{P}w''w$ is an example. By hypothesis (2), there exist Hamilton circuits $H_1, \ldots, H_t$ of $G$ such that $C$ equals their symmetric difference. Each $H_t$ contains $v$, hence $vv^+$, hence in view of (1) must contain all edges $v^-v$, $vv^+$, ..., $w^-w$. Since $C$ itself does not contain any of these edges, $t$ is even. We have shown that every circuit $C$ in $G$ which does not contain any of the edges $v^-v$, $vv^+$, ..., $w^-w$ is the symmetric difference of an even number of Hamilton circuits; since every such circuit has an even number of edges (no matter what the parity of $|G|$ is), and since every circuit in $G - v$ is a circuit not containing any of the edges $v^-v$, $vv^+$, ..., $w^-w$, this proves that $G - v$ is bipartite. \hfill $\Box$

**Definition 3** ($K^{3,s-1}$). For every $s \geq 3$ we define $K^{3,s-1}$ to be the graph obtained from the complete bipartite graph with classes $\{1, 3, 5, \ldots, 2s - 1\}$ and $\{2, 4, \ldots, 2s - 2\}$ by adding the vertex 0 and the two edges $\{0, 1\}$ and $\{0, 2s - 1\}$.

**Proposition 4.** If $G = (V,E)$ is a graph such that
(1) $G$ is neither a forest nor a circuit ,
(2) every cycle in $G$ is a symmetric difference of Hamilton circuits ,
then it does not follow that $G$ has minimum degree $\geq 3$.

**Proof.** We prove that the graph $K^{3,3}$ from Definition 3 is an example for this non-implication. Evidently, it is neither a forest nor a circuit and it does not have minimum degree $\geq 3$. So all we have to show is that (2) holds for $G = K^{3,3}$.

The cycle space of $G$ has dimension $||K^{3,3}|| - ||K^{3,3}|| + 1 = 14 - 8 + 1 = 7$. Since a vector space does not contain proper subspaces of its own dimension, to prove (2) it suffices to exhibit seven Hamilton circuits linearly independent over $F_2$: the circuits
For $s = 3$, the graph $K^{3,3}$ is not an example for the non-implication in Proposition 4: but for every $s \geq 4$ it seems to be: while we do not need this here, let us note in passing that in view of the example $K^{3,3}$ used for proving Proposition 4, and also in view of some calculations for the case $s = 5$, it seems that for every $s \geq 4$ the graph $K^{5,s-1}$ has every cycle a symmetric difference of Hamilton circuits, despite its degree-2-vertex. I.e., it seems that the $K^{5,s-1}$ for $s \geq 4$ are an infinite set of examples for the non-implication in Proposition 4:

**Conjecture 5.** $Z_1(K^{5,s-1}; \mathbb{F}_2) = \langle H(K^{5,s-1}) \rangle_{\mathbb{F}_2}$ for every $s \geq 4$.

We will now derive a probabilistic analogue of Proposition 4. We will use:

**Lemma 6** ([2, Exercise 3.2]). For any $k \geq 0$ and any $\omega \in \mathbb{R}_{>0}$ with $\omega_n \xrightarrow{n \to \infty} \infty$,

1. if $p(n) \geq \log n + k \log \log n + \omega_n$, then $P_{G_{n,p}}[\delta \geq k + 1] \xrightarrow{n \to \infty} 1$,
2. if $p(n) = \log n + k \log \log n + c + h(n)$ with some constant $c \in \mathbb{R}$ and some function $h \in o(1)$, then $P_{G_{n,p}}[\delta = k] \xrightarrow{n \to \infty} 1 - \exp(-\exp(-c/k!))$ and $P_{G_{n,p}}[\delta = k + 1] \xrightarrow{n \to \infty} \exp(-\exp(-c/k!))$.
3. if $p(n) \leq \log n + k \log \log n - \omega_n$, then $P_{G_{n,p}}[\delta \leq k] \xrightarrow{n \to \infty} 1$.
If \( G \) is a forest, the property \( Z_1(G; \mathcal{F}_2) = \langle \mathcal{H}(G) \rangle_{\mathcal{F}_2} \) vacuously holds, and then no conclusions can be drawn from it. This is the reason for (1) of the following lemma. As to the second hypothesis in (1), of course, for any \( p \) a.a.s. \( G_{n,p} \) is not a circuit, so we could just leave out ‘nor a circuit’; but we leave it in for better analogy with the deterministic Proposition 4:

**Lemma 7** (the non-implication from Proposition 4 holds for \( G_{n,p} \)). If \( p \in [0, 1]^N \) is such that with \( G \sim G_{n,p} \) a.a.s. for odd \( n \)

1. \( G \) is neither a forest nor a circuit
2. every cycle in \( G \) is a symmetric difference of Hamilton circuits, then a.a.s. \( G \) has minimum degree \( \geq 3 \).

**Proof.** Suppose that we do not have

\[
P_{G_{n,p}}[\delta(G) \geq 3] \xrightarrow{\text{odd } n \to \infty} 1.
\]

Then by the Bolzano–Weierstrass-theorem there exists \( 0 < \xi < 1 \) and a subsequence \( (n_i)_{i \in \mathbb{N}} \) of odd numbers such that

\[
P_{G_{n_i,p}}[[G \subseteq K^{n_i}: \delta(G) \geq 3]] \xrightarrow{i \to \infty} \xi.
\]

equivalently

\[
P_{G_{n_i,p}}[[G \subseteq K^{n_i}: \delta(G) \leq 2]] \xrightarrow{i \to \infty} 1 - \xi \in (0, 1).
\]

Since with our \( p \), a.a.s. \( G \) contains some circuit and has property (2), in particular we a.a.s. have at least one Hamilton circuit in \( G \), hence in particular we a.a.s. have \( \delta(G) \geq 2 \), i.e., so

\[
P_{G_{n_i,p}}[[G \subseteq K^{n_i}: \delta(G) \geq 2]] \xrightarrow{i \to \infty} 1.
\]

Twice using the fact that intersecting with an a.a.s. property does not change an asymptotic probability, (5) and (6) together imply

\[
P_{G_{n_i,p}}[[G \subseteq K^{n_i}: \delta(G) = 2]] \xrightarrow{i \to \infty} 1 - \xi \in (0, 1),
\]

and now (7) together with the a.a.s. property \( Z_1(G; \mathcal{F}_2) = \langle \mathcal{H}(G) \rangle_{\mathcal{F}_2} \) from hypothesis (2) implies

\[
P_{G_{n_i,p}}[[G \subseteq K^{n_i}: \delta(G) = 2 \quad \text{and} \quad Z_1(G; \mathcal{F}_2) = \langle \mathcal{H}(G) \rangle_{\mathcal{F}_2}]] \xrightarrow{i \to \infty} 1 - \xi \in (0, 1).
\]

All in all we now know that with our \( p \),

\[
P_{G_{n_i,p}}\left[\left\{ G \subseteq K^{n_i}: \begin{array}{l}
G \text{ is neither a forest nor a circuit} \\
\text{and every cycle in } G \text{ is a symmetric difference} \\
of \text{Hamilton circuits} \\
\text{and } G \text{ contains a vertex of degree } 2
\end{array}\right\}\right] \xrightarrow{i \to \infty} 1 - \xi \in (0, 1).
\]

From Lemma 2 we know the deterministic implication that, for any \( n \in \mathbb{N} \),

\[
\{ G \subseteq K^n: \begin{array}{l}
G \text{ is neither a forest nor a circuit} \\
\text{and every cycle in } G \text{ is a symmetric difference} \\
of \text{Hamilton circuits} \\
\text{and } G \text{ contains a vertex of degree } 2
\end{array}\} \subseteq \{ G \subseteq K^n: \begin{array}{l}
\text{for every vertex } v \text{ of } G \text{ with } \deg(v) = 2 \\
\text{the graph } G - v \text{ is bipartite}
\end{array}\}.
\]

We abbreviate

1. \( \mathcal{B}_{G, v, n} := \{ G \subseteq K^n: \text{ for every } v \in V(G) \text{ with } \deg(v) = 2 \text{ the graph } G - v \text{ is bipartite} \} \),
2. \( \mathcal{E}_{\delta = 2, n} := \{ G \subseteq K^n: \text{ there exists in } G \text{ a vertex } v \text{ with } \deg(v) = 2 \} \).
Applying $P_{G_{n,p}}$ to (10), taking $\lim \sup$ on both sides and using (9), it follows that
\[ 0 < 1 - \xi \leq \lim \sup_{i \to \infty} P_{G_{n,p}}[B_{G-v,n_1}] . \] (11)
We now claim that from what we know about $p$, it follows that
\[ \lim \sup_{i \to \infty} P_{G_{n,p}}[B_{G-v,n}] = 0 , \] (12)
contradicting (11) and completing the proof of Lemma 7.

To prove (12), we first note that (15) can also be written
\[ P_{G_{n,p}}[E_{\delta=2,n_1}] \xrightarrow{i \to \infty} 1 - \xi > 0 , \] (13)
so we may condition on the event $E_{\delta=2,n_1}$ and write
\[ P_{G_{n,p}}[B_{G-v,n}] = P_{G_{n,p}}[B_{G-v,n_1}] \cdot P_{G_{n,p}}[E_{\delta=2,n_1}] . \] (14)

We now claim that from what we know about $p$,
\[ \lim \sup_{i \to \infty} P_{G_{n,p}}[B_{G-v,n_1} \mid E_{\delta=2,n_1}] = 0 . \] (15)
To prove (15), we prove $\lim \sup_{i \to \infty} P_{G_{n,p}}[B_{G-v,n_1} \cap E_{\delta=2,n_1}] \xrightarrow{i \to \infty} 0$; we first define $E_{\delta,v} : \Delta \in G-v,n := \{ G \subseteq K_n^\omega : \text{for every vertex } v \in G \text{ the graph } G-v \text{ contains a triangle } \}$ and note that, directly from the definitions
\[ B_{G-v,n} \cap E_{\delta=2,n_1} \cap E_{\delta,v} = \emptyset . \] (16)
We now recall that (cf. [10, Theorem 3.4], for example) if $p(n-1) \gg (n-1)^{-1}$, then $G_{n-1,p}$ a.a.s. contains a triangle. Because of (6) and Lemma (3) we must have (arbitrarily choosing $\omega_n := \log \log \log n$, $p(n) > \frac{\log n + \log \log n + \omega_n}{n}$ for sufficiently large $n$, hence $p(n) \gg (n-1)^{-1}$. Since for our $G \sim G_{n,p}$ we have $G-n \sim G_{n-1,p}$ for every $v \in V(G)$, it follows that with our $p(n) \gg (n-1)^{-1}$ we have the limit
\[ P_{G_{n,p}}[E_{\delta,v} : \Delta \in G-v,n] \xrightarrow{i \to \infty} 0 . \] (17)
Again using that intersecting with an a.a.s. property does not change an asymptotic probability,
\[ \lim \sup_{i \to \infty} P_{G_{n,p}}[B_{G-v,n_1} \cap E_{\delta=2,n_1}] \overset{(17)}{=} \lim \sup_{i \to \infty} P_{G_{n,p}}[B_{G-v,n_1} \cap E_{\delta=2,n_1} \cap E_{\delta,v} : \Delta \in G-v,n_1] \overset{(16)}{=} 0 . \] (18)

From (18), (14) and (13) now indeed follows (12). As already mentioned, this completes the proof of Lemma 7.

In [8], Glebov and Krivelevich found the asymptotic behaviour of the total number of Hamilton circuits in $G_{n,p}$, starting right from the hamiltonicity-threshold $p(n) = \frac{\log n + \log \log n + \omega(1)}{n}$. An obvious necessary condition to keep in mind when seeking to improve Theorem 1 is that the total number of Hamilton circuits a.a.s. be at least as large as $\dim Z_1(G; \mathbb{F}_2) = \| G \| - | G | + 1$. On the one hand, if $p(n) = \frac{\log n + \log \log n + \omega_n}{n}$ with $\omega \in \omega(1)$, we a.a.s. have $\| G \| \sim \frac{1}{2} n \log n$, hence a.a.s. $\dim Z_1(G; \mathbb{F}_2) \sim \frac{1}{2} n \log n$; on the other hand, by [8, Theorem 1] and Stirling’s approximation, $| \mathcal{H}(G_{n,p}) | \sim (2\pi n)^{\frac{1}{2}} ((\log n + \log \log n + \omega_n)/e)^n (1 - o(1))^n$ a.a.s., outgrowing $\frac{1}{2} n \log n$. So the above necessary condition is satisfied as soon as $G_{n,p}$ becomes hamiltonian at all. This prompts the question whether already right from the threshold for hamiltonicity, $G_{n,p}$ a.a.s. has each parity-permitted cycle a symmetric difference of Hamilton circuits. We now answer that particular question in the negative: although in $G_{n,p}$ a.a.s. hamiltonicity necessarily comes with extras (e.g., $[4] \ [8]$) into the bargain, in particular it comes with far more Hamilton circuits than the dimension of $Z_1(G_{n,p}; \mathbb{F}_2)$, the property $Z_1(G_{n,p}; \mathbb{F}_2) = \langle \mathcal{H}(G_{n,p}) \rangle_{\mathbb{F}_2}$ is not guaranteed right from the start
of a.a.s. hamiltonicity (we recall that by [12, Theorem 1] a.a.s. hamiltonicity already begins for $p \geq \log n + \log \log n + \omega_n$):

**Theorem 8** (a Hamilton-generated cycle space does not appear right from the onset of hamiltonicity of $G_{n,p}$). If $p \in [0,1]^\mathbb{N}$ is such that with $G \sim G_{n,p}$ a.a.s. for odd $n$

1. $G$ is neither a forest nor a circuit,
2. every cycle in $G$ is a symmetric difference of Hamilton circuits,
then for every $c > 0$ we have $p(n) > \frac{\log n + 2\log \log n + c}{n}$ for all sufficiently large odd $n$.

**Proof.** Assume otherwise, i.e. there are $c > 0$ and infinitely many odd $n$ with

$$p(n) \leq \frac{\log n + 2\log \log n + c}{n}. \tag{19}$$

Since we could pass to an infinite subsequence of such $n$ and still argue as we do below, we may assume that (19) holds for every odd $n$. We now argue that it is not true that a.a.s. $\delta(G) \geq 3$, contradicting Lemma 7 and completing the proof of Theorem 8.

We define $p^+(n) := \frac{\log n + 2\log \log n + c}{n}$ for every $n$. By assumption $p(n) \leq p^+(n)$ for every $n$. Since the property $\delta(G) \geq 3$ is monotone increasing, $P_{G_{n,p}}[\delta \geq 3] \leq P_{G(n,p^+)}[\delta \geq 3]$ for every $n$, so

$$\lim_{n \to \infty} P_{G_{n,p}}[\delta \geq 3] = \lim_{n \to \infty} P_{G(n,p^+)}[\delta \geq 3]. \tag{20}$$

By Lemma 6.(3) with $k := 3$ and $\omega_n := \log n - c$ it follows from the definition of $p^+$ that

$$P_{G(n,p^+)}[\delta(G) \leq 3] \xrightarrow{n \to \infty} 1. \tag{21}$$

Since intersecting with an a.a.s. property does not change the lim sup of the probabilities of the property, it follows from (21) that

$$\lim_{n \to \infty} P_{G(n,p^+)}[\delta \geq 3] = \lim_{n \to \infty} P_{G(n,p^+)}[\delta \geq 3 \text{ and } \delta \leq 3] = \lim_{n \to \infty} P_{G(n,p^+)}[\delta = 3]. \tag{22}$$

The definition of $p^+$ together with Lemma 6.(2) with $k := 2$ and $h := 0$ ensures that we have the limit $P_{G(n,p^+)}[\delta = 3] \xrightarrow{n \to \infty} \exp(-\exp(-c/2))$ which when substituted into (22) yields

$$\lim_{n \to \infty} P_{G(n,p^+)}[\delta \geq 3] = \exp(-\exp(-c/2)). \tag{23}$$

From (23) and (20) we finally get a conclusion for $p$ proper:

$$\lim_{n \to \infty} P_{G_{n,p}}[\delta \geq 3] \leq \exp(-\exp(-c/2)) < 1, \tag{24}$$

for any $c \in \mathbb{R}$. The bound (24) shows that with our $p$ it is not the case that $P_{G_{n,p}}[\delta \geq 3] \xrightarrow{n \to \infty} 1$, the contradiction to Lemma 7 already mentioned.

4. Concluding remarks

It seems very likely that the condition $p(n) \geq n^{-1/2+\varepsilon}$ in Theorem 1 can be significantly improved; apparently (work in progress), by adapting available embedding technology to the present purposes one can improve $-1/2$ to $-2/3$ in Theorem 1. It also seems as if $p(n) \geq n^{-2/3+\varepsilon}$ is the utmost of what can be achieved with the technique of proving the existence of some spanning subgraph pre-selected from the monotone property $\mathcal{M}_{[1]_0}^2$: the graph $G_n^2$ is a non-minimal element of the monotone property $\mathcal{M}_{[1]_0}$ (resp., for even $n$, of $\mathcal{M}_{[1]_1}$), and it is possible to descend quite far by deleting edges to construct sparser ‘rebar’, and yet $p(n) \geq n^{-2/3+\varepsilon}$ seems to be required even when using a minimal element of $\mathcal{M}_{[1]_0}$.

It nevertheless still seems plausible that one may even significantly improve the conjectured bound $n^{-2/3+\varepsilon}$, but this will require several new ideas: some customary arguments for showing that a set of vectors is a generating system carry with them an ‘instability’ unusual for arguments
about $G_{n_p}$. For example, an argument which were to randomly choose a dimension-sized subset of distinct Hamilton-circuits and then go on to prove this subset to be a.a.s. linearly independent over $\mathbb{F}_2$ would have to be so sensitive to the size of the subset so as to break down if only $1 + \dim_{\mathbb{F}_2} Z_1(G; \mathbb{F}_2)$ circuits would have been selected instead. Nevertheless, it is not inconceivable that with the right ideas one can improve the threshold $n^{-1/2+\varepsilon}$ all the way down to what Theorem 8 allows, i.e. $p(n) \geq \frac{\log n + 2 \log \log n + \omega}{n}$, the threshold for minimum degree $\geq 3$.

References

1. Brian Alspach, Stephen C. Locke, and Dave Witte, The Hamilton spaces of Cayley graphs on abelian groups, Discrete Math. 82 (1990), no. 2, 113–126. [1]
2. Béla Bollobás, Random graphs, second ed., Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001. [4]
3. Béla Bollobás and Alan M. Frieze, On matchings and Hamiltonian cycles in random graphs, Random graphs ’83 (Poznań, 1983), North-Holland Math. Stud., vol. 118, North-Holland, Amsterdam, 1985, pp. 23–46. [1]
4. Colin Cooper, 1-pancyclic Hamilton cycles in random graphs, Random Structures Algorithms 3 (1992), no. 3, 277–287. [1, 6]
5. Colin Cooper and Alan M. Frieze, Pancyclic random graphs, Random graphs ’87 (Poznań, 1987), Wiley, Chichester, 1990, pp. 29–39. [1]
6. Bill Cuckler and Jeff Kahn, Hamiltonian cycles in Dirac graphs, Combinatorica 29 (2009), no. 3, 299–326. [1]
7. Ehud Friedgut and Gil Kalai, Every monotone graph property has a sharp threshold, Proc. Amer. Math. Soc. 124 (1996), no. 10, 2993–3002. [2]
8. Roman Glebov and Michael Krivelevich, On the number of Hamilton cycles in sparse random graphs, SIAM J. Discrete Math. 27 (2013), no. 1, 27–42. [6]
9. Peter Heinig, On prisms, Möbius ladders and the cycle space of dense graphs, (2011), arXiv:1112.5101v1. [1, 2]
10. Svante Janson, Tomasz Łuczak, and Andrzej Rucinski, Random graphs, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000. [6]
11. Fiachra Knox, Daniela Kühn, and Deryk Osthus, Approximate Hamilton decompositions of random graphs, Random Structures Algorithms 40 (2012), no. 2, 133–149. [1]
12. János Komlós and Endre Szemerédi, Limit distribution for the existence of Hamiltonian cycles in a random graph, Discrete Math. 43 (1983), no. 1, 55–63. [7]
13. Daniela Kühn, John Lapinskas, and Deryk Osthus, Optimal packings of Hamilton cycles in graphs of high minimum degree, Combin. Probab. Comput. 22 (2013), no. 3, 394–416. [1]
14. Daniela Kühn and Deryk Osthus, On Pósa’s Conjecture for Random Graphs, SIAM J. Discrete Math. 26 (2012), no. 3, 1440–1457. [2]

ZENTRUM MATHEMATIK, M9, TECHNISCHE UNIVERSITÄT MÜNCHEN,
BOLTZMANNSTRASSE 3, D-85747 GARCHING BEI MÜNCHEN, GERMANY
E-mail address: heinig@ma.tum.de