Double quantization on coadjoint representations of simple Lie groups and its orbits

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Abstract

Let $M$ be a manifold with an action of a Lie group $G$, $\mathcal{A}$ the function algebra on $M$. The first problem we consider is to construct a $U_h(g)$ invariant quantization, $\mathcal{A}_h$, of $\mathcal{A}$, where $U_h(g)$ is a quantum group corresponding to $G$.

Let $s$ be a $G$ invariant Poisson bracket on $M$. The second problem we consider is to construct a $U_h(g)$ invariant two parameter (double) quantization, $\mathcal{A}_{t,h}$, of $\mathcal{A}$ such that $\mathcal{A}_{t,0}$ is a $G$ invariant quantization of $s$. We call $\mathcal{A}_{t,h}$ a $U_h(g)$ invariant quantization of the Poisson bracket $s$.

In the paper we study the cases when $G$ is a simple Lie group and $M$ is the coadjoint representation $g^*$ of $G$ or a semisimple orbit in this representation.

First of all, we describe Poisson brackets and pairs of Poisson brackets related to $U_h(g)$ invariant quantizations for arbitrary algebras. After that we construct a two parameter quantization on $g^*$ for $g = sl(n)$ and $s$ the Lie bracket and show that such a quantization does not exist for other simple Lie algebras. As the function algebra on $g^*$ we take the symmetric algebra $Sg$. In $sl(n)$ case, we also consider the problem of restriction of the family $(Sg)_{t,h}$ on orbits. In particular, we describe explicitly the Poisson bracket along the parameter $h$ of this family, which turns out to be quadratic, and prove that it can be restricted on each orbit in $g^*$. We prove also that the family $(Sg)_{t,h}$ can be restricted on the maximal semisimple orbits.

For $M$ a manifold isomorphic to a semisimple orbit in $g^*$, we describe the variety of all brackets related to the one parameter quantization. Actually, it is a variety making $M$ into a Poisson manifold with a Poisson action of $G$. It turns out that not all such brackets and not all orbits admit a double quantization with $s$ the Kirillov-Kostant-Souriau bracket. We classify the orbits and pairs of brackets admitting a double quantization and construct such a quantization for almost all admissible paires.

1 Introduction

Quantum groups can be considered as symmetry objects of certain “quantum spaces” described by noncommutative algebra of functions. This point of view was developed, for example, in [RTF] and [Ma]. Here we study the inverse problem: given the quantum group corresponding to a Lie group $G$, we want to define a “quantum space” corresponding to a given classical $G$-manifold.
Let $M$ be a manifold with an action of a Lie group $G$, $\mathfrak{g}$ the Lie algebra of $G$, and $U_h(\mathfrak{g})$ the quantized universal enveloping algebra. Let $\mathcal{A}$ be the sheaf of function algebras on $M$. It may be a sheaf of smooth, analytic, or algebraic functions. For shortness, we simply call $\mathcal{A}$ a function algebra. The algebra $\mathcal{A}$ is of course invariant under the induced action of the bialgebra $U(\mathfrak{g})$.

We consider the following two general problems.

**The first problem.** Does there exist a deformation quantization, $\mathcal{A}_h$, of $\mathcal{A}$, which is invariant under the action of the quantum group $U_h(\mathfrak{g})$?

**The second problem.** Suppose $\mathcal{A}_t$ is a $U(\mathfrak{g})$ invariant quantization of $\mathcal{A}$. Does there exist a two parameter quantization, $\mathcal{A}_{t,h}$, of $\mathcal{A}$ such that $\mathcal{A}_{t,0} = \mathcal{A}_t$, which is invariant under $U_h(\mathfrak{g})$?

In this paper, we study the first and the second problems for two cases. The first case, when $M$ is the coadjoint representation of a simple Lie group. The second case, when $M$ is a semisimple orbit in this representation. This paper is motivated by papers [Do2] and [DGS] where we started to study these problems. In this paper we develop results of [Do2] and [DGS] and present some additional results.

The paper is organized as follows.

In Section 2 we recall some facts about quantum groups and related categories, which are essential for a strict formulation of our problems and for our approach to $U_h(\mathfrak{g})$ invariant quantization of algebras. In particular, we use the Drinfeld category with non-trivial associativity constraint determined by an invariant element $\varphi_h \in U(\mathfrak{g}) \otimes \mathfrak{g}$ and show that the problem of $U_h(\mathfrak{g})$ invariant quantization is equivalent to the problem of deforming the function algebra in such a way that the deformed algebra to be $G$ invariant and $\varphi_h$ associative (see Subsection 2.3).

Subsection 2.4 is very important for the paper. In this subsection we give, for all commutative algebras, a description of Poisson brackets related to one and two parameter $U_h(\mathfrak{g})$ invariant quantizations. We show the following. If $\mathcal{A}_h$ is a $U_h(\mathfrak{g})$ invariant quantization, the corresponding Poisson bracket, $p$, on $M$ has to be a difference of two brackets, $p = f - r_M$. Here $r_M$ is the so called $r$-matrix bracket obtained from a classical $r$-matrix $r \in \wedge^2 \mathfrak{g}$ with the help of the action morphism $\mathfrak{g} \to \text{Vect}(M)$. So, the Schouten bracket $[r_M, r_M]$ is equal to the image $\varphi_M$ of the invariant element $\varphi \in \wedge^3 \mathfrak{g}$. The bracket $f$ is $U(\mathfrak{g})$ invariant and such that $[f, f] = -\varphi_M$. Of course, any invariant bracket, $f$, is compatible with $r_M$, so that $[p, p] = 0$.

We see that for existence of the family $\mathcal{A}_h$ one needs existence of an invariant bracket $f$ on $M$ such that

$$[f, f] = -\varphi_M. \quad (1.1)$$

Note that the manifold $M$ endowed with the bracket $p = f - r_M$ is a Poisson manifold with a Poisson action of $G$, where $G$ is considered to be the Poisson-Lie group with Poisson structure defined by $r$. We shall not use this fact in the paper.

Similarly, given a two parameter quantization, $\mathcal{A}_{t,h}$, a pair of compatible Poisson brackets is determined. These brackets are: the bracket $p = f - r_M$ considered above and a $U(\mathfrak{g})$ invariant Poisson bracket, $s$, the initial term of the $U(\mathfrak{g})$ invariant quantization $\mathcal{A}_t$. We may perceive the family $\mathcal{A}_{t,h}$ as a $U_h(\mathfrak{g})$ invariant quantization of the Poisson bracket $s$. 

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We assume that $s$ is given in advance and determined, for example, by a $G$ invariant simplectic structure on $M$. From the compatibility of $p$ and $s$ (this means $[p, s] = 0$) follows that
\[ [f, s] = 0. \quad (1.2) \]

So, for existence of the family $A_{t,h}$ one needs existence of an invariant bracket $f$ on $M$ such the both equations (1.1) and (1.2) hold.

Thus, our problems divide into two steps. The first step is looking for invariant brackets $f$ on $M$ satisfying either (1.1) (in case of the first problem) or both (1.1) and (1.2) (in case of the second problem). The second step is quantizing these brackets.

In Section 3 we consider the one and two parameter quantization on $M = \mathfrak{g}^*$, the coadjoint representation of a simple Lie algebra $\mathfrak{g}$. As a function algebra on $\mathfrak{g}^*$, we take the symmetric algebra $S\mathfrak{g}$. It turns out that the cases $\mathfrak{g} = \mathfrak{sl}(n)$ and $\mathfrak{g} \neq \mathfrak{sl}(n)$ are quite different.

We prove that for $\mathfrak{g} \neq \mathfrak{sl}(n)$ the two parameter family which is a $U_h(\mathfrak{g})$ invariant quantization of the Lie bracket on $S\mathfrak{g}$ does not exist. Moreover, as a conjecture we state that in this case even a one parameter $U_h(\mathfrak{g})$ invariant quantization of $S\mathfrak{g}$ does not exist.

In the case $\mathfrak{g} = \mathfrak{sl}(n)$, the two parameter quantization of $S\mathfrak{g}$ exists. Moreover, the picture looks like in the classical case. Recall that in the classical case, the natural one parameter $U(\mathfrak{g})$ invariant quantization of $S\mathfrak{g}$ is given by the family $(S\mathfrak{g})_t = T(\mathfrak{g})[[t]]/J_t$, where $J_t$ is the ideal generated by the elements of the form $x \otimes y - \sigma(x \otimes y) - t[x, y]$, $x, y \in \mathfrak{g}$, $\sigma$ is the permutation. By the PBW theorem, $(S\mathfrak{g})_t$ is a free module over $\mathbb{C}[t]$. We have $(S\mathfrak{g})_0 = S\mathfrak{g}$, so this family of quadratic-linear algebras gives a $U(\mathfrak{g})$ invariant quantization of $S\mathfrak{g}$. It is obvious that the Poisson bracket, $s$, related to this quantization is the Lie bracket on $\mathfrak{g}^*$.

We show that for $\mathfrak{g} = \mathfrak{sl}(n)$ this picture can be extended to the quantum case. Namely, there exist deformations, $\sigma_h$ and $[\cdot, \cdot]_h$, of both the mappings $\sigma$ and $[\cdot, \cdot]$ such that the two parameter family of algebras $(S\mathfrak{g})_{t,h} = T(\mathfrak{g})[[h]][[t]]/J_{t,h}$, where $J_{t,h}$ is the ideal generated by the elements of the form $x \otimes y - \sigma_h(x \otimes y) - t[x, y]_h$, $x, y \in \mathfrak{g}$, gives a $U_h(\mathfrak{g})$ invariant quantization of the Lie bracket $s$ on $\mathfrak{g}^*$. In this case, the corresponding bracket $f$ from (1.2) is a quadratic bracket which is, up to a factor, a unique nontrivial invariant map $\wedge^2 \mathfrak{g} \to S^2 \mathfrak{g}$.

Taking $t = 0$ we obtain the family $(S\mathfrak{g})_h$ which is a quadratic algebra over $\mathbb{C}[[h]]$. This algebra can be called the quantum symmetric algebra (or quantum polynomial algebra on $\mathfrak{g}^*$). We show (Subsection 3.4) that $(S\mathfrak{g})_h$ can be included in the deformed graded differential algebra (deformed de Rham complex). In Subsection 3.5 we prove that the family $(S\mathfrak{g})_{t,h}$ can be restricted on the maximal semisimple orbits in $\mathfrak{g}^*$ to give a two parameter quantization on these orbits.

In Section 4 we study the problems of one and two parameter quantization on semisimple orbits in $\mathfrak{g}^*$ for all simple Lie algebras $\mathfrak{g}$. First of all, we classify all the brackets $f$ satisfying (1.1) and both (1.1) and (1.2) for $s$ being the Kirillov-$A_{t,h}$ (KKS) bracket on the orbit. After that, we construct quantizations of these brackets.

Let $M$ be a semisimple orbit. In Subsection 4.1 we prove that the brackets $f$ satisfying (1.1) form a $\dim H^2(M)$-dimensional variety. We give a description of this variety and prove (in Subsection 4.3) that almost all these brackets can be quantized. So, we obtain for $M$ a $\dim H^2(M)$ parameter family of non-equivalent one parameter quantizations.
Note that in [DG2] we have built one of these quantizations, the quantization of the so-called Sklyanin-Drinfeld Poisson bracket.

It turns out that brackets $f$ satisfying (1.1) and (1.2) exist not for all orbits. We call an orbit $M$ good if there exists a bracket $f$ satisfying (1.1) and (1.2) for the Kirillov-Kostant-Souriau (KKS) bracket $s$.

In Subsection 4.1 we give the following classification of the semisimple good orbits for all simple $\mathfrak{g}$. [DGS].

In the case $\mathfrak{g} = sl(n)$ all semisimple orbits are good. (Actually we prove that in this case all orbits are good.)

For $\mathfrak{g} \neq sl(n)$ all symmetric orbits (which are symmetric spaces) are good. In this case $\varphi_M = 0$, so $\rho_M$ itself is a Poisson bracket compatible with $s$.

Only in the case $\mathfrak{g}$ of type $D_n$ and $E_6$ (except of $A_n$) there are good orbits different from the symmetric ones. For such orbits $\varphi_M \neq 0$.

We show that brackets $f$ on a good orbit satisfying (1.1) and (1.2), form a one parameter family.

In Subsection 4.2 we consider cohomologies of an invariant complex with the differential given by the Schouten bracket with the bivector $f$. These cohomologies are needed for our construction of quantization.

In Subsection 4.3 we construct one and two parameter quantizations for semisimple orbits. According to our approach, as a first step we construct a $G$ invariant $\Phi_h$ associative quantization, i.e., a quantization in the Drinfeld category with non-trivial associativity constraint given by $\Phi_h$. Note that the bracket $f$ from (1.1) can be considered as a “Poisson bracket” in that category. As a second step, we make a passage to the category with trivial associativity to obtain the associative $U_h(\mathfrak{g})$ invariant quantization. We applied this method earlier for quantizing the function algebra on the highest weight orbits in irreducible representations of $G$, the algebra of sections of linear vector bundles over flag manifolds, and the function algebra on symmetric spaces, [DGM], [DG1], [DS1].

I put in the text some questions which naturally appeared by exposition. They are open (for me) and seem to be important.

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2 Preliminaries

2.1 Quantum groups

We shall consider quantum groups in sense of Drinfeld, [Dr2], as deformed universal enveloping algebras. If $U(\mathfrak{g})$ is the universal enveloping algebra of a complex Lie algebra $\mathfrak{g}$, then the quantum group (or quantized universal enveloping algebra) corresponding to $U(\mathfrak{g})$ is a topological Hopf algebra, $U_h(\mathfrak{g})$, over $\mathbb{C}[[\hbar]]$, isomorphic to $U(\mathfrak{g})[[[\hbar]]]$ as a
topological \( \mathbb{C}[[h]] \) module and such that \( U_h(\mathfrak{g})/hU_h(\mathfrak{g}) = U(\mathfrak{g}) \) as a Hopf algebra over \( \mathbb{C} \).

In particular, the deformed comultiplication in \( U_h(\mathfrak{g}) \) has the form

\[
\Delta_h = \Delta + h\Delta_1 + o(h),
\]

(2.1)

where \( \Delta \) is the comultiplication in the universal enveloping algebra \( U(\mathfrak{g}) \). One can prove, \( \text{Dr}^2 \), that \( \Delta_1 : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}) \) is such a map that \( \Delta_1 - \sigma \Delta_1 = \delta \) (\( \sigma \) is the usual permutation) being restricted on \( \mathfrak{g} \) gives a map \( \delta : \mathfrak{g} \to \wedge^2 \mathfrak{g} \) which is a 1-cocycle and defines the structure of a Lie coalgebra on \( \mathfrak{g} \) (the structure of a Lie algebra on the dual space \( \mathfrak{g}^* \)). The pair \( (\mathfrak{g}, \delta) \) is considered as a quasiclassical limit of \( U_h(\mathfrak{g}) \).

In general, a pair \( (\mathfrak{g}, \delta) \), where \( \mathfrak{g} \) is a Lie algebra and \( \delta \) is such a 1-cocycle, is called a Lie bialgebra. It is proven, \( \text{EK} \), that any Lie bialgebra \( (\mathfrak{g}, \delta) \) can be quantized, i.e., there exists a quantum group \( U_h(\mathfrak{g}) \) such that the pair \( (\mathfrak{g}, \delta) \) is its quasiclassical limit.

A Lie bialgebra \( (\mathfrak{g}, \delta) \) is said to be a coboundary one if there exists an element \( r \in \wedge^2 \mathfrak{g} \), called the classical \( r \)-matrix, such that \( \delta(x) = [r, \Delta(x)] \) for \( x \in \mathfrak{g} \). Since \( \delta \) defines a Lie coalgebra structure, \( r \) has to satisfy the so-called classical Yang-Baxter equation which can by written in the form

\[
[r, r] = \varphi,
\]

(2.2)

where \( [\cdot, \cdot] \) stands for the Schouten bracket and \( \varphi \in \wedge^3 \mathfrak{g} \) is an invariant element. We denote the coboundary Lie bialgebra by \( (\mathfrak{g}, r) \).

In case \( \mathfrak{g} \) is a simple Lie algebra, the most known is the Sklyanin-Drinfeld \( r \)-matrix:

\[
r = \sum_\alpha X_\alpha \wedge X_{-\alpha},
\]

where the sum runs over all positive roots; the root vectors \( X_\alpha \) are chosen is such a way that \( (X_\alpha, X_{-\alpha}) = 1 \) for the Killing form \( (\cdot, \cdot) \). This is the only \( r \)-matrix of weight zero, \( \text{SS} \), and its quantization is the Drinfeld-Jimbo quantum group. A classification of all \( r \)-matrices for simple Lie algebras was given in \( \text{BL} \).

We are interested in the case when \( \mathfrak{g} \) is a semisimple finite dimensional Lie algebra. In this case, from results of Drinfeld and Etingof and Kazhdan one can derive the following

**Proposition 2.1.** Let \( \mathfrak{g} \) be a semisimple Lie algebra. Then

a) any Lie bialgebra \( (\mathfrak{g}, \delta) \) is a coboundary one;

b) the quantization, \( U_h(\mathfrak{g}) \), of any coboundary Lie bialgebra \( (\mathfrak{g}, r) \) exists and is isomorphic to \( U(\mathfrak{g})[[h]] \) as a topological \( \mathbb{C}[[h]] \) algebra;

c) the comultiplication in \( U_h(\mathfrak{g}) \) has the form

\[
\Delta_h(x) = F_h \Delta(x) F_h^{-1}, \quad x \in U(\mathfrak{g}),
\]

(2.3)

where \( F_h \in U(\mathfrak{g})^\otimes[[h]] \) and can be chosen in the form

\[
F_h = 1 \otimes 1 + \frac{h}{2} r + o(h).
\]

(2.4)
Proof. a) follows from the fact that $H^1(\mathfrak{g}, \wedge^2 \mathfrak{g}) = 0$. From the fact that $H^2(\mathfrak{g}, U(\mathfrak{g})) = 0$ follows that $U(\mathfrak{g})$ does not admit any nontrivial deformations as an algebra, (see [Dr1]), which proves b). From the fact that $H^1(\mathfrak{g}, U(\mathfrak{g})^\otimes 2) = 0$ follows that any deformation of the algebra morphism $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ appears as a conjugation of $\Delta$. In particular, the comultiplication in $U_h(\mathfrak{g})$ looks like (2.3) with some $F_h$ such that $F_0 = 1 \otimes 1$.

From the coassociativity of $\Delta_h$ follows that $F_h$ satisfies the equation

$$(F_h \otimes 1) \cdot (\Delta \otimes id)(F_h) = (1 \otimes F_h) \cdot (id \otimes \Delta)(F_h) \cdot \Phi_h$$

(2.5)

for some invariant element $\Phi_h \in U(\mathfrak{g})^\otimes 3[[h]]$.

The element $F_h$ satisfying (2.3) and (2.4) can be obtained by correction of some $F_h$ only obeying (2.3), [Dr2]. This procedure also makes use simple cohomological arguments and essentially (2.7). This proves c). \hfill \Box

From (2.3) follows that if $F_h$ has the form (2.4), then the coefficient by $h$ for $\Phi_h$ vanishes. Moreover, as a coefficient by $h^2$ one can take the element $\varphi$ from (2.2), i.e.,

$$\Phi_h = 1 \otimes 1 \otimes 1 + h^2 \varphi + o(h^2).$$

(2.6)

In addition, from (2.3) follows that $\Phi_h$ satisfies the pentagon identity

$$(id^\otimes 2 \otimes \Delta)(\Phi_h) \cdot (\Delta \otimes id^\otimes 2)(\Phi_h) = (1 \otimes \Phi_h) \cdot (id \otimes \Delta \otimes id)(\Phi_h) \cdot (\Phi_h \otimes 1).$$

(2.7)

**Question 2.1.** Let $(\mathfrak{g}, r)$ be a coboundary Lie bialgebra. Does there exist a quantization of it, $U_h(\mathfrak{g})$, such that $U_h(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})[[h]]$ as a topological $\mathbb{C}[[h]]$ algebra and the comultiplication has the form (2.3)?

From [Dr4] follows that if $[r, r] = 0$, the answer to this question is positive.

### 2.2 Categorical interpretation

It is known that the elements constructed above have a nice categorical interpretation. First, recall some facts about the Drinfeld algebras and the monoidal categories determined by them.

Let $A$ be a commutative algebra with unit, $B$ a unitary $A$-algebra. The category of representations of $B$ in $A$-modules, i.e. the category of $B$-modules, will be a monoidal category if the algebra $B$ is equipped with an algebra morphism, $\Delta : B \to B \otimes_A B$, called comultiplication, and an invertible element $\Phi \in B^\otimes 3$ such that $\Delta$ and $\Phi$ satisfy the conditions (see [Dr2])

$$(id \otimes \Delta)(\Delta(b)) \cdot \Phi = \Phi \cdot (\Delta \otimes id)(\Delta(b)), \quad b \in B,$$

(2.8)

$$(id^\otimes 2 \otimes \Delta)(\Phi) \cdot (\Delta \otimes id^\otimes 2)(\Phi) = (1 \otimes \Phi) \cdot (id \otimes \Delta \otimes id)(\Phi) \cdot (\Phi \otimes 1).$$

(2.9)

Define a tensor product functor for the category of $B$ modules $\mathcal{C}$, denoted $\otimes_C$ or simply $\otimes$ when there can be no confusion, in the following way: given $B$-modules $M, N$, $M \otimes_C N = M \otimes_A N$ as an $A$-module. The action of $B$ is defined by

$$b(m \otimes n) = (\Delta b)(m \otimes n) = b_1 m \otimes b_2 n,$$
where \( \Delta b = b_1 \otimes b_2 \) (we use the Sweedler convention of an implicit summation over an index). The element \( \Phi = \Phi_1 \otimes \Phi_2 \otimes \Phi_3 \) defines the associativity constraint,

\[
a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P), \quad a_{M,N,P}((m \otimes n) \otimes p) = \Phi_1 m \otimes (\Phi_2 n \otimes \Phi_3 p).
\]

Again the summation in the expression for \( \Phi \) is understood. By virtue of (2.8) \( \Phi \) induces an isomorphism of \( B \)-modules, and by virtue of (2.4) the pentagon identity for monoidal categories holds. We call the triple \((B, \Delta, \Phi)\) a Drinfeld algebra. The definition is somewhat non-standard in that we do not require the existence of an antipode. The category \( C \) of \( B \)-modules for \( B \) a Drinfeld algebra becomes a monoidal category. When it becomes necessary to be more explicit we shall denote \( C(B, \Delta, \Phi) \).

Let \((B, \Delta, \Phi)\) be a Drinfeld algebra and \( F \in B^{\otimes 2} \) an invertible element. Put

\[
\tilde{\Delta}(b) = F\Delta(b)F^{-1}, \quad b \in B,
\]

\[
\tilde{\Phi} = (1 \otimes F) \cdot (id \otimes \Delta)(F) \cdot \Phi \cdot (\Delta \otimes id)(F^{-1}) \cdot (F \otimes 1)^{-1}.
\]

Then \( \tilde{\Delta} \) and \( \tilde{\Phi} \) satisfy (2.3) and (2.9), therefore the triple \((B, \tilde{\Delta}, \tilde{\Phi})\) also becomes a Drinfeld algebra. We say that it is obtained by twisting from \((B, \Delta, \Phi)\). It has an equivalent monoidal category of modules, \( \tilde{C}(B, \tilde{\Delta}, \tilde{\Phi}) \). Note that the equivalent categories \( C \) and \( \tilde{C} \) consist of the same objects as \( B \)-modules, and the tensor products of two objects are isomorphic as \( A \)-modules. The equivalence \( C \to \tilde{C} \) is given by the pair \((\text{Id}, F)\), where \( \text{Id} : C \to \tilde{C} \) is the identity functor of the categories (considered without the monoidal structures, but only as categories of \( B \)-modules), and \( F : M \otimes_C N \to M \otimes_{\tilde{C}} N \) is defined by \( m \otimes n \mapsto F_1 m \otimes F_2 n \) where \( F_1 \otimes F_2 = F \).

We are interested in the case when \( A = \mathbb{C}[[h]], \quad B = U(\mathfrak{g})[[h]] \) where \( \mathfrak{g} \) is a complex semisimple Lie algebra. In this case, all tensor products over \( \mathbb{C}[[h]] \) are completed in \( h \)-adic topology.

We have two nontrivial Drinfeld algebras. The first is \( (U(\mathfrak{g})[[h]], \Delta, \Phi_h) \), with the usual comultiplication and \( \Phi_h \) from (2.3). The condition (2.8) means the invariance of \( \Phi_h \), while (2.9) coincides with (2.7). The second Drinfeld algebra is \( (U(\mathfrak{g})[[h]], \Delta_h, 1) \). It obtains by twisting of the first by the element \( F_h \) from (2.3). The equation (2.11) follows from (2.7). The pair \((\text{Id}, F_h)\) defines an equivalence between the corresponding monoidal categories \( C(U(\mathfrak{g})[[h]], \Delta, \Phi_h) \) and \( C(U(\mathfrak{g})[[h]], \Delta_h, 1) \). The last is the category of representations of the quantum group \( U_h(\mathfrak{g}) \).

It is clear that reduction modulo \( h \) defines a functor from either of these categories to the category of representations of \( U(\mathfrak{g}) \) and the equivalence just described reduces to the identity modulo \( h \). In fact, both categories are \( \mathbb{C}[[h]] \)-linear extensions (or deformations) of the \( \mathbb{C} \)-linear category of representations of \( \mathfrak{g} \). Ignoring the monoidal structure the extension is a trivial one, but the associator \( \Phi_h \) in the first case and the comultiplication \( \Delta_h \) in the second case make the extension non-trivial from the point of view of monoidal categories.

### 2.3 \( U_h(\mathfrak{g}) \) invariant quantizations of algebras

Let \((B, \Delta, \Phi)\) be a Drinfeld algebra. Assume \( A \) is a \( B \)-module with a multiplication \( \mu : A \otimes_A A \to A \) which is a homomorphism of \( A \)-modules. We say that \( \mu \) is \( \Delta \) invariant
if
\[ b\mu(x \otimes y) = \mu\Delta(b)(x \otimes y) \quad \text{for } b \in B, \, x, y \in \mathcal{A}, \] (2.12)
and \( \mu \) is \( \Phi \) associative, if
\[ \mu(\Phi_1 x \otimes \mu(\Phi_2 y \otimes \Phi_3 z)) = \mu(\mu(x \otimes y) \otimes z) \quad \text{for } x, y, z \in \mathcal{A}. \] (2.13)

Note, that a \( B \)-module \( \mathcal{A} \) equipped with \( \Delta \) invariant and \( \Phi \) associative multiplication
is an associative algebra in the monoidal category \( \mathcal{C}(B, \Delta, \Phi) \). If \((B, \tilde{\Delta}, \tilde{\Phi})\) is a Drinfeld
algebra twisted by \((2.10) \) and \((2.11)\), then the algebra \( \mathcal{A} \) may be transfered into the
equivalent category \( \tilde{\mathcal{C}}(B, \tilde{\Delta}, \tilde{\Phi}) \): the multiplication \( \tilde{\mu} = \mu_{F^{-1}} : M \otimes_A M \to M \) is \( \tilde{\Phi} \)-
associative and invariant in the category \( \tilde{\mathcal{C}} \).

Let \( \mathcal{A} \) be a \( U(\mathfrak{g}) \) invariant associative algebra, i.e., an algebra with \( U(\mathfrak{g}) \) invariant
multiplication \( \mu \) in sense of (2.12). A deformation (or quantization) of \( \mathcal{A} \) is an associative
algebra, \( \mathcal{A}_h \), which is isomorphic to \( \mathcal{A}[[h]] = \mathcal{A} \otimes \mathbb{C}[[h]] \) (completed tensor product) as
a \( \mathbb{C}[[h]] \)-module, with multiplication in \( \mathcal{A}_h \) having the form \( \mu_h = \mu + h\mu_1 + o(h) \). The
algebra \( U(\mathfrak{g})[[h]] \) is clearly acts on the \( \mathbb{C}[[h]] \) module \( \mathcal{A}_h \).

We will study quantizations of \( \mathcal{A} \) which will be invariant under the comultiplication \( \Delta_h \).
In other words, \( \mathcal{A}_h \) will be an algebra in the category of representations of the quantum
group \( U_h(\mathfrak{g}) \). It is clear from the previous Subsection that if \( \mathcal{A}_h \) is such a quantization,
then the multiplication \( \mu_h F_h \) makes the module \( \mathcal{A}[[h]] \) into an algebra in the category
\( \mathcal{C}(U(\mathfrak{g})[[h]], \Delta, \Phi_h) \), i.e., this multiplication is \( U(\mathfrak{g}) \) invariant and \( \Phi_h \) associative.

We shall see that often it is easier to constrcut \( U(\mathfrak{g}) \) invariant and \( \Phi_h \) associative
quantization of \( \mathcal{A} \). After that, the invariant quantization with respect to any quantum
group from Proposition 2.1 can be obtained by twisting by the appropriate \( F_h \).

As an algebra \( \mathcal{A} \) we may take an algebra \( \mathcal{A}_t \) that is itself a \( U(\mathfrak{g}) \) invariant quantization
of a commutative algebra \( \mathcal{A} \). In this case, a \( U_h(\mathfrak{g}) \) invariant quantization of \( \mathcal{A}_t \) is an algebra
\( \mathcal{A}_{t,h} \) over \( \mathbb{C}[[t, h]] \).

### 2.4 Poisson brackets associated with the \( U_h(\mathfrak{g}) \) invariant quantization

Let \( \mathcal{A} \) be a \( U(\mathfrak{g}) \) invariant commutative algebra with multiplication \( \mu \) and \( \mathcal{A}_h \) its quantization
with multiplication \( \mu_h = \mu + h\mu_1 + o(h) \). The Poisson bracket corresponding to
the quantization is given by \( \{a, b\} = \mu_1(a, b) - \mu_1(b, a), \, a, b \in \mathcal{A} \).

In general, we call a skew-symmetric bilinear form \( \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \) a bracket, if it satisfies
the Leibniz rule in either argument when the other is fixed. The term Poisson bracket
indicates that the Jacobi identity is also true.

A bracket of the form
\[ \{a, b\}_r = (r_1 a)(r_2 b) = \mu r(a \otimes b) \quad a, b \in \mathcal{A}, \] (2.14)
where \( r = r_1 \otimes r_2 \) (summation implicit) is the representation of \( r \)-matrix \( r \), will be called
an \( r \)-matrix bracket.
Assume $\mathcal{A}_h$ is a $U_h(\mathfrak{g})$ invariant quantization, i.e., the multiplicity $\mu_h$ is $\Delta_h$ invariant. We shall show that in this case the Poisson bracket $\{\cdot, \cdot\}$ has a special form. Suppose $f$ and $g$ are two brackets on $\mathcal{A}$. Define their Schouten bracket $[[f, g]]$ as

$$[[f, g]](a, b, c) = f(g(a, b), c) + g(f(a, b), c) + \text{cyclic permutations of } a, b, c. \quad (2.15)$$

Then $[[f, g]]$ is a skew-symmetric map $\mathcal{A}^\otimes 3 \to \mathcal{A}$. We call $f$ and $g$ compatible if $[[f, g]] = 0$.

**Proposition 2.2.** Let $\mathcal{A}$ be a $U(\mathfrak{g})$ invariant commutative algebra and $\mathcal{A}_h$ a $U_h(\mathfrak{g})$ invariant quantization. Then the corresponding Poisson bracket has the form

$$\{a, b\} = f(a, b) - \{a, b\}_r \quad (2.16)$$

where $f(a, b)$ is a $U(\mathfrak{g})$ invariant bracket.

The brackets $f$ and $\{\cdot, \cdot\}_r$ are compatible and $[[f, f]] = -\varphi_\mathcal{A}$, where $\varphi_\mathcal{A}(a, b, c) = (\varphi_1)(\varphi_2)(\varphi_3)$ and $\varphi_1 \otimes \varphi_2 \otimes \varphi_3 = \varphi \in \wedge^3 \mathfrak{g}$ is the invariant element from (2.2).

**Proof.** Let the comultiplication for $U_h(\mathfrak{g})$ have the form (2.1). Let $\mathcal{A}$ be a commutative algebra with the $U(\mathfrak{g})$ invariant multiplication $\mu$. Suppose $\mathcal{A}_h$ is a $U_h(\mathfrak{g})$ invariant quantization of $\mathcal{A}$. This means that the deformed multiplication has the form

$$\mu_h = \mu + h\mu_1 + o(h) \quad (2.17)$$

and satisfies the relation

$$x\mu_h(a \otimes b) = \mu_h\Delta_h(x)(a \otimes b) \quad \text{ for } \quad x \in U(\mathfrak{g}), \ a, b \in \mathcal{A}. \quad (2.18)$$

Substituting (2.1) and (2.17) in (2.18) and collecting the terms by $h$ we obtain

$$\mu_1(a \otimes b) = \mu\Delta(x)(a \otimes b) + m\Delta_1(x)(a \otimes b).$$

Subtracting from this equation the similar one with permuting $a$ and $b$ and making use that $\Delta$ is commutative and $\delta = \Delta_1 - \sigma\Delta_1$ is skew-commutative, we derive that the Poisson bracket $p = \{\cdot, \cdot\}$ has to satisfy the property

$$xp(a \otimes b) = p\Delta(x)(a \otimes b) + \mu\delta(x)(a \otimes b), \quad x \in U(\mathfrak{g}). \quad (2.19)$$

Let us prove that the bracket $f(a, b) = \{a, b\} + \{a, b\}_r$ is $U(\mathfrak{g})$ invariant. Indeed, from (2.14) we have for $x \in U(\mathfrak{g}), \ a, b \in \mathcal{A}$

$$x\mu r(a \otimes b) = \mu\Delta(x)r(a \otimes b) = \mu r\Delta(x)(a \otimes b) - \mu[r, \Delta(x)](a \otimes b).$$

Using this expression, (2.19), and the fact that $\delta(x) = [r, \Delta(x)]$, we obtain

$$xf = xp + x\mu r = (p\Delta(x) + \mu[r, \Delta(x)]) + (\mu r\Delta(x) - \mu[r, \Delta(x)]) =$$

$$= p\Delta(x) + \mu r\Delta(x) = f\Delta(x),$$

which proves the invariantness of $f$.

So, we have $\{a, b\} = f(a, b) - \{a, b\}_r$, as required.

It is easy to check that any bracket of the form $\{a, b\} = (X_1a)(X_2b) = \mu(X_1a, X_2b)$, for $X_1 \otimes X_2 \in \mathfrak{g} \wedge \mathfrak{g}$, is compatible with any invariant bracket. In particular, an $r$-matrix bracket is compatible with $f$. In addition, $\{\cdot, \cdot\}$ is a Poisson bracket, so its Schouten bracket with itself is equal to zero. Using this and the fact that the Schouten bracket of $r$-matrix bracket with itself is equal to $\varphi_\mathcal{A}$, we obtain from (2.16) that $[[f, f]] = -\varphi_\mathcal{A}$. \qed
Remark 2.1. Let $\mathcal{A}$ be the function algebra on a $G$-manifold $M$, where the Lie group $G$ corresponds to the Lie algebra $\mathfrak{g}$. It is easy to see that condition (2.19) with $\delta(x) = [r, \Delta(x)]$ is equivalent to the condition that the pair $(M, p)$ becomes a $(G, \tilde{r})$-Poisson manifold, where $\tilde{r}$ is the Poisson structure on $G$ defined by the $r$-matrix:

$$\tilde{r} = r' - r''.$$

It is known that $\tilde{r}$ makes $G$ into a Poisson-Lie group. So Proposition 2.2 gives a description of Poisson structures $p$ on $M$ making $(M, p)$ into a $(G, \tilde{r})$-Poisson manifold.

We shall also consider two parameter quantizations of algebras. A two parameter quantization of an algebra $\mathcal{A}$ is an algebra $\mathcal{A}_{t,h}$ isomorphic to $\mathcal{A}[[t, h]]$ as a $\mathbb{C}[[t, h]]$ module and having a multiplication in the form:

$$\mu_{t,h} = \mu + t\mu_1' + h\mu_1'' + o(t, h).$$

With such a quantization, one associates two Poisson brackets: the bracket $s(a,b) = \mu_1'(a,b) - \mu_1'(b,a)$ along $t$, and the bracket $p(a,b) = \mu_1''(a,b) - \mu_1''(b,a)$ along $h$. It is easy to check that $p$ and $s$ are compatible Poisson brackets, i.e., the Schouten bracket $[p, s] = 0$.

A pair of compatible Poisson brackets we call a Poisson pencil.

Corollary 2.1. Let $\mathcal{A}_{t,h}$ be a two parameter $U_h(\mathfrak{g})$ invariant quantization of a commutative algebra $\mathcal{A}$ such that $\mathcal{A}_{t,0}$ is a one parameter $U(\mathfrak{g})$ invariant quantization of $\mathcal{A}$ with Poisson bracket $s$. Then the $U_h(\mathfrak{g})$ invariant quantization $\mathcal{A}_{0,h}$ has a Poisson bracket $p$ of the form (2.16): $p = f - \{\cdot, \cdot\}_r$, where $f$ is an invariant bracket such that $[f, f] = -\varphi_A$ and compatible with $s$, i.e.,

$$[f, s] = 0. \quad (2.20)$$

Proof. For the two parameter quantization, the Poisson brackets $p$ and $s$ form a Poisson pencil, hence must be compatible. Also, $s$ is a $U(\mathfrak{g})$ invariant bracket, so that $s$ is compatible with the $r$-matrix bracket $\{\cdot, \cdot\}_r$. It follows from (2.16) that $s$ has to be compatible with $f$. \qed

In what follows, we shall often call $\mathcal{A}_{t,h}$ a $U_h(\mathfrak{g})$ invariant quantization (or double quantization) of the invariant Poisson bracket $s$, or of the Poisson pencil $s$ and $p$.

Remark 2.2. As we have seen in Subsection 2.3, to construct a $U_h(\mathfrak{g})$ invariant quantization of $\mathcal{A}$ is the same that to construct a $U(\mathfrak{g})$ invariant $\Phi_h$ associative quantization of $\mathcal{A}$. We shall see that the last problem often turns out to be simpler (see Subsection 4.3). We observe that if $p = f - \{\cdot, \cdot\}_r$ is an admissible Poisson bracket for $U_h(\mathfrak{g})$ invariant quantization, then the invariant bracket $f$ with the property $[f, f] = -\varphi_A$ may be considered as a “Poisson bracket” of quantization in the category with $\Phi_h$ defining the associativity constraint. Also, the pair $f, s$ is a Poisson pencil in that category.

3 Double quantization on coadjoint representations

In this section we study a two parameter (or double) quantization on coadjoint representations of simple Lie algebras.
Let \( \mathfrak{g} \) be a complex Lie algebra. Then, the symmetric algebra \( S\mathfrak{g} \) can be considered as a function algebra on \( \mathfrak{g}^* \). The algebra \( U(\mathfrak{g}) \) is included in the family of algebras \( (S\mathfrak{g})_t = T(\mathfrak{g})[t]/J_t \), where \( J_t \) is the ideal generated by the elements of the form \( x \otimes y - \sigma(x \otimes y) - t[x, y] \), \( x, y \in \mathfrak{g} \), \( \sigma \) is the permutation. By the PBW theorem, \( (S\mathfrak{g})_t \) is a free module over \( \mathbb{C}[t] \). We have \( (S\mathfrak{g})_0 = S\mathfrak{g} \), so this family of quadratic-linear algebras gives a \( U(\mathfrak{g}) \) invariant quantization of \( S\mathfrak{g} \) by the Lie bracket \( s \).

It turns out that for \( \mathfrak{g} = sl(n) \) this picture can be extended to the quantum case, \cite{Do2}. Namely, there exist deformations, \( \sigma_h \) and \( [\cdot, \cdot]_h \), of both the mappings \( \sigma \) and \( [\cdot, \cdot] \) such that the two parameter family of algebras \( (S\mathfrak{g})_{t, h} = T(\mathfrak{g})[[h]][t]/J_{t, h} \), where \( J_{t, h} \) is the ideal generated by the elements of the form \( x \otimes y - \sigma_h(x \otimes y) - t[x, y]_h \), \( x, y \in \mathfrak{g} \), gives a \( U_h(\mathfrak{g}) \) invariant quantization of the Lie bracket \( s \) on \( \mathfrak{g}^* \). In this case, the corresponding bracket \( f \) from \( \mathfrak{g}^* \) is a quadratic bracket which is, up to a factor, a unique nontrivial invariant map \( \wedge^2 \mathfrak{g} \to S^2 \mathfrak{g} \).

We shall show that for other simple Lie algebras, double quantizations of the Lie brackets do not exist.

We give two constructions of the algebra \( (S\mathfrak{g})_{t, h} \). The first construction uses an idea from the paper \cite{LS} on a quantum analog of Lie algebra for \( sl(n) \). The second construction using the so called reflection equations (RE), \cite{KS}, \cite{Ma}, is presented in Remark 3.4.

### 3.1 Quantum Lie algebra for \( U_h(sl(n)) \)

Let \( U_h(\mathfrak{g}) \) be a quantized universal enveloping algebra for a Lie algebra \( \mathfrak{g} \). We consider \( U_h(\mathfrak{g}) \) as a \( U_h(\mathfrak{g}) \) module with respect to the left adjoint action: \( \text{ad}(x)y = x_1 y \gamma(x_2) \), where \( x, y \in U_h(\mathfrak{g}) \), \( \Delta_h(x) = x_1 \otimes x_2 \) (summation implicit).

There were attempts to define quantum Lie algebras as deformed standard classical embeddings of \( \mathfrak{g} \) into \( U_h(\mathfrak{g}) \) obeying some additional properties, \cite{DG}, \cite{LS}.

In the classical case, there is probably the following way (not using comultiplication) to distinguish the standard embedding \( \mathfrak{g} \to U(\mathfrak{g}) \) from other invariant embeddings: with respect to this embedding \( U(\mathfrak{g}) \) is a quadratic-linear algebra. So, we give the following (working) definition of quantum Lie algebras.

**Definition 3.1.** Let \( \mathfrak{g}_h \) be a subrepresentation of \( U_h(\mathfrak{g}) \), which is a deformation of the standard embedding of \( \mathfrak{g} \) in \( U(\mathfrak{g}) \). We call \( \mathfrak{g}_h \) a quantum Lie algebra, if the kernel of the induced homomorphism \( T(\mathfrak{g}_h) \to U_h(\mathfrak{g}) \) is defined by (deformed) quadratic-linear relations.

We are going to show that the quantum Lie algebra exists in case \( \mathfrak{g} = sl(n) \). On the other hand, if such an algebra exists for some Lie algebra \( \mathfrak{g} \), then a double quantization of the Lie bracket on \( \mathfrak{g}^* \) also exists. But, as we shall see, no double quantization exists for simple \( \mathfrak{g} \neq sl(n) \). So, among simple finite dimensional Lie algebras, only \( sl(n) \) has a quantum Lie algebra in our sense.

Our construction is the following. Let \( R = R_i' \otimes R_i'' \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}) \) (completed tensor product) be the R-matrix (summation by \( i \) is assumed). It satisfies the properties \cite{Dr2}

\[
\Delta'_h(x) = R \Delta_h(x) R^{-1}, \quad x \in U_h(\mathfrak{g}),
\]

(3.1)
where $\Delta_h$ is the comultiplication in $U_h(\mathfrak{g})$ and $\Delta'_h$ is the opposite one,

\[
(\Delta_h \otimes 1)R = R^{13}R^{23} = R'_i \otimes R'_j \otimes R''_k R''_l,
\]

\[
(1 \otimes \Delta_h)R = R^{13}R^{12} = R'_i R'_j \otimes R''_k \otimes R''_l,
\]

and

\[
(1 \otimes \varepsilon)R = (\varepsilon \otimes 1)R = 1 \otimes 1,
\]

where $\varepsilon$ is the counit in $U_h(\mathfrak{g})$.

Consider the element $Q = Q'_i \otimes Q''_l = R^{21}R$. It follows from (3.2) that $Q$ commutes with elements from $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ of the form $\Delta_h(x)$. This is equivalent for $Q$ to be invariant under the adjoint action of $U_h(\mathfrak{g})$ on $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$.

Let $V$ be an irreducible finite dimensional representation of $U_h(\mathfrak{g})$ and $\rho : U_h(\mathfrak{g}) \to \text{End}(V)$ the corresponding map of algebras. Consider the dual space $\text{End}(V)^*$ as a left $U_h(\mathfrak{g})$ module setting

\[
(x\varphi)(a) = \varphi(\gamma(x_{1(1)})ax_{2(2)}),
\]

where $\varphi \in \text{End}(V)^*$, $a \in \text{End}(V)$, $\Delta_h(x) = x_{1(1)} \otimes x_{2(2)}$ in Sweedler notions, and $\gamma$ denotes the antipode in $U_h(\mathfrak{g})$.

Consider the map

\[
f : \text{End}(V)^* \to U_h(\mathfrak{g})
\]

defined as $\varphi \mapsto \varphi(\rho(Q'_i)Q''_l)$. From the invariance of $Q$ it follows that $f$ is a $U_h(\mathfrak{g})$ equivariant map, so $\mathcal{L} = \text{Im}(f)$ is a $U_h(\mathfrak{g})$ submodule.

It follows from (3.2) that $\mathcal{L}$ is a left coideal in $U_h(\mathfrak{g})$, i.e., $\Delta(x) \in U_h(\mathfrak{g}) \otimes \mathcal{L}$ for any $x \in \mathcal{L}$. Indeed, $Q = R''_i R''_j \otimes R''_l R''_k$. Applying (3.2) we obtain

\[
(1 \otimes \Delta_h)R^{21}R = R''_i R''_j R''_l R''_k R'_i \otimes R''_i R''_j R''_k R'_l R''_l R''_l R''_l R''_l
\]

Let $\varphi \in \text{End}(V)^*$. Define $\psi_{il} \in \text{End}(V)^*$ setting $\psi_{il}(a) = \varphi(R''_i a R'_l)$ for $a \in \text{End}(V)$. Then $\Delta\varphi(R''_i R''_j) R''_l R''_k R''_l = R''_i R''_j R''_l R''_k R''_l R''_l R''_l R''_l$ which obviously belongs to $U_h(\mathfrak{g}) \otimes \mathcal{L}$.

Recall, [Dr2], that $R = F_h e^{2t}F_h^{-1}$. Here $t = \sum_i t_i \otimes t_i$ is the split Casimir, where $t_i$ form an orthonormal basis in $\mathfrak{g}$ with respect to the Killing form, $F = 1 \otimes 1 + \frac{1}{2}r + o(h)$ (see (2.4)), and $r$ is a classical $r$-matrix. Therefore,

\[
Q = R^{21}R = Fe^{ht}F^{-1} = 1 \otimes 1 + ht + \frac{h^2}{2}(t^2 + [r, t]) + o(h^2).
\]

Denote by $\text{Tr}$ the unique (up to a factor) invariant element in $\text{End}(V)^*$. Let $Z_0 = \rho_0(\mathfrak{g})$, and denote by $Z_h$ some $U_h(\mathfrak{g})$ invariant deformation of $Z_0$ in $\text{End}(V)$. Then we have a decomposition $\text{End}(V) = I \oplus Z_h \oplus W$, where $I$ is the one dimensional invariant subspace generated by the identity map, $W$ is a complement to $I \oplus Z_h$ invariant subspace. This gives a decomposition $\text{End}(V)^* = I^* \oplus Z_h^* \oplus W^*$ where $W^*$ consists of all the elements which are equal to zero on $I \oplus Z_h$. The space $I^*$ is generated by $\text{Tr}$, and after normalizing in such a way that $\text{Tr}(\text{id}) = 1$, we obtain that $C_V = f(\text{Tr})$ is of the form

\[
C_V = \text{Tr}(\rho(Q'_i))Q''_l = 1 + h^2c + o(h^2),
\]

(3.7)
where \( c \) is an invariant element of \( U(\mathfrak{g}) \). It follows from (3.3) that \( \varepsilon(C) = 1 \).

From (3.6) follows that the elements of \( f(Z^*_h) \) have the form

\[
z = hx + o(h), \quad x \in \mathfrak{g},
\]

hence the subspace \( L_1 = h^{-1}f(Z^*_h) \) forms a subrepresentation of \( U_h(\mathfrak{g}) \) with respect to the left adjoint action of \( U_h(\mathfrak{g}) \) on itself, which is a deformation of the standard embedding of \( \mathfrak{g} \) into \( U(\mathfrak{g}) \). It follows from (3.3) that \( \varepsilon(L_1) = 0 \).

The elements from \( f(W^*) \) have the form \( w = h^2b + o(h^2) \) and \( \varepsilon(W^*) = 0 \). Denote \( L_2 = h^{-2}f(W^*) \).

So, \( \mathcal{T} = \mathbb{C}C_V \oplus hL_1 \oplus h^2L_2 = \mathbb{C}C_V + hL \), where \( L = L_1 \oplus hL_2 \). Since \( \mathcal{T} \) is a left coideal in \( U_h(\mathfrak{g}) \), for any \( x \in \mathcal{T} \) we have

\[
\Delta_h(x) = x(1) \otimes x(2) = z \otimes C_V + v \otimes x',
\]

where \( z, v \in U_h(\mathfrak{g}) \), \( x' \in L \). Applying to the both hand sides \( (1 \otimes \varepsilon) \) and multiplying we obtain \( x = x(1)\varepsilon(x(2)) = z\varepsilon(C_V) + v\varepsilon(x') = z \). So, \( z \) has to be equal to \( x \). and we obtain

\[
\Delta_h(x) = x(1) \otimes x(2) = x \otimes C_V + v \otimes x', \quad x, x' \in L.
\]

From (3.9) we have for any \( y \in L \)

\[
xy = x(1)y \gamma(x(2))x(3) = x(1)y \gamma(x(2))C_V + v(1)y \gamma(v(2))x'.
\]

Introduce the following maps:

\[
\sigma'_h : \quad L \otimes L \to L \otimes L, \quad x \otimes y \mapsto v(1)y \gamma(v(2)) \otimes x',
\]

\[
[\cdot, \cdot]'_h : \quad L \otimes L \to L, \quad x \otimes y \mapsto x(1)y \gamma(x(2)).
\]

We may rewrite (3.10) in the form

\[
m(x \otimes y - \sigma'_h(x \otimes y)) - [x, y]'_h C_V = 0.
\]

Observe now that, as follows from (3.7), \( C_V \) is an invertible element in \( U_h(\mathfrak{g}) \). Put \( P = C_V^{-1} \). Transfer the maps (3.11) to the space \( P \cdot L \), i.e., define

\[
\sigma_h(Px, Py) = (P \otimes P)\sigma'_h(x, y),
\]

\[
[Px, Py]'_h = P[x, y]'_h.
\]

From (3.9) we obtain

\[
P(1)x(1) \otimes P(2)x(2) = P(1)x \otimes P(2)C_V + P(1)v \otimes P(2)x'.
\]

Using this relation and taking into account that \( P \) commutes with all elements from \( U_h(\mathfrak{g}) \), we obtain as in (3.11)

\[
P x P y = P(1)x(1)P y \gamma(x(2)) \gamma(P(2))P(3)x(3) =
\]

\[
P(1)x(1)P y \gamma(x(2)) \gamma(P(2))P(3)C_V + P(1)v(1)P y \gamma(v(2)) \gamma(P(2))P(3)x' =
\]

\[
P[x, y]'_h + P^2m\sigma'_h(x \otimes y) = [Px, Py]'_h + m\sigma_h(Px \otimes Py).
\]
This equality may be written as
\[ m(x \otimes y - \sigma_h(x \otimes y)) - [x, y]_h = 0, \quad x, y \in C_V^{-1}L. \] (3.17)

Define \( L_V = C_V^{-1}L \). Let \( T(L_V) = \bigoplus_{k=0}^{\infty} L_V^k \) be the tensor algebra over \( L_V \). Notice, that \( T(L_V) \) is not supposed to be completed in \( h \)-adic topology. Let \( J \) be the ideal in \( T(L_V) \) generated by the relations
\[ (x \otimes y - \sigma_h(x \otimes y)) - [x, y]_h, \quad x, y \in L_V. \] (3.18)

Due to (3.17) we have a homomorphism of algebras over \( \mathbb{C}[[h]] \)
\[ \psi_h : T(L_V)/J \to U_h(\mathfrak{g}), \] (3.19)

extending the natural embedding \( L_V \to U_h(\mathfrak{g}) \) of \( U_h(\mathfrak{g}) \) modules.

Now we can prove

**Proposition 3.1.** For \( \mathfrak{g} = sl(n) \) the quantum Lie algebra exists.

**Proof.** Apply the above construction to \( V = \mathbb{C}^n[[h]] \), the deformed basic representation of \( \mathfrak{g} \). In this case \( \text{End}(V) = I \oplus \mathbb{Z}_h \), where \( \mathbb{Z}_h \) is a deformed adjoint representation. So, \( \mathfrak{g}_h = L_V = h^{-1} C_V^{-1} f(Z_h^*) \) is a deformation of the standard embedding of \( \mathfrak{g} \) in \( U(\mathfrak{g}) \). It is easy to see that in this case \( \sigma_h \) is a deformation of the usual permutation: \( \sigma_0(x \otimes y) = y \otimes x \), and \( [\cdot, \cdot]_h \) is a deformation of the Lie bracket on \( \mathfrak{g} : [x, y]_0 = [x, y], \quad x, y \in \mathfrak{g} \subset U(\mathfrak{g}) \). Hence, at \( h = 0 \), the quadratic-linear relations (3.18) are exactly the defining relations for \( U(\mathfrak{g}) \), therefore the map (3.19) is an isomorphism at \( h = 0 \). It follows that (3.19) is an embedding. (Actually, (3.19) is essentially an isomorphism, i.e., it is an isomorphism after completion of \( T(L_V) \) in \( h \)-adic topology.) So, the kernel of the map \( T(L_h) \to U_h(\mathfrak{g}) \) is defined by the quadratic-linear relations (3.18). \( \square \)

**Remark 3.1.** Quadratic-linear relations (3.18) can be obtained in another way. Note that equation (3.5) may be rewritten as
\[ (1 \otimes \Delta_h)Q = R_{21}Q_{13}R_{12}. \] (3.20)
Since \( Q \) commutes with all elements of the form \( \Delta_h(x), \quad x \in U_h(\mathfrak{g}) \), one derives from (3.20):
\[ Q_{23}R_{21}Q_{13}R_{12} = R_{21}Q_{13}R_{12}Q_{23}. \] (3.21)
Consider the element \( Q_\rho = \rho(Q_1) \otimes Q_2 \) as a \( \dim(V) \times \dim(V) \) matrix with the entries from \( U_h(\mathfrak{g}) \). Applying to (3.21) operator \( \rho \otimes \rho \otimes 1 \), we obtain the following relation for \( Q_\rho \):
\[ (Q_\rho)_{23} \overline{R}_{21}(Q_\rho)_{12} = \overline{R}_{21}(Q_\rho)_{12}R_{23}(Q_\rho)_{23}. \] (3.22)
where \( \overline{R} = (\rho \otimes \rho) R \) is a number matrix, the Yang-Baxter operator in \( V \otimes V \). Replacing in this equation \( \overline{R} \) by \( S = \sigma \overline{R} \), we obtain that the matrix \( Q_\rho \) satisfies the following reflection equation (RE):
\[ (Q_\rho)_{23}S(Q_\rho)_{23} = S(Q_\rho)_{23}S(Q_\rho)_{23}. \] (3.23)
It is clear that the entries of the matrix $Q_\rho$ generate the image of the map (3.4). From (3.7) follows that $Q_\rho$ has the form

$$Q_\rho = \text{Id}_V C_V + hB',$$  

(3.24)

where $B'$ has the form $B' = \sum D_i \otimes b_i, D_i$ belong to the complement to $\mathbb{C} \text{Id}_V$ submodule in $\text{End}(V)$ and $b_i \in U_h(\mathfrak{g})$. Note that the entries of the matrix $B'$ form the subspace $L$, whereas the entries of $B = C^{-1}_V B'$ form the subspace $L_V$ from (3.17). From (3.24) we obtain

$$C^{-1}_V Q_\rho = \text{Id} + hB.$$  

(3.25)

Since the element $C^{-1}_V$ belongs to the center of $U_h(\mathfrak{g})$, the matrix $C^{-1}_V Q_\rho$ obeys the RE (3.23) as well. So, $B$ satisfies the relation

$$(\text{Id} + hB)_2 S (\text{Id} + hB)_2 S = S (\text{Id} + hB)_2 S (\text{Id} + hB)_2.$$  

(3.26)

One checks that (3.26), considered as a quadratic-linear relations for indetermined entries of $B$, is equivalent to (3.18) in the case $\mathfrak{g} = sl(n)$.

### 3.2 Double quantization on $sl(n)^*$

Introduce a new variable, $t$, and consider a homomorphism of algebras, $T(L_V)[t] \to U_h(\mathfrak{g})[t]$, which extends the embedding $t \cdot \iota : L_V[t] \to U_h(\mathfrak{g})[t]$, where $\iota$ stands for the standard embedding $L_V \to U_h(\mathfrak{g})$. From (3.17) follows that $t \cdot \iota$ factors through the homomorphism of algebras over $\mathbb{C}[\hbar][t]$

$$\phi_{t,h} : T(L_V)[t]/J_t \to U_h(\mathfrak{g})[t],$$  

(3.27)

where $J_t$ is the ideal generated by the relations

$$(x \otimes y - \sigma_h(x \otimes y)) - t[x, y], \quad x, y \in L_V.$$  

(3.28)

**Proposition 3.2.** For $\mathfrak{g} = sl(n)$ the algebra $(S\mathfrak{g})_{t,h} = T(L_V)[t]/J_t$ is a double quantization of the Lie bracket on $S\mathfrak{g}$.

**Proof.** Since in this case $L_V = \mathfrak{g}_h$, from Proposition 3.1 follows that (3.27) is a monomorphism at $t = 1$. Due to the PBW theorem the algebra $\text{Im}(\phi_{t,h})$ at the point $h = 0$ is a free $\mathbb{C}[t]$-module and is equal to

$$(S\mathfrak{g})_t = T(\mathfrak{g})/\{x \otimes y - y \otimes x - t[x, y]\}. $$  

(3.29)

For $t = 0$ this algebra is the symmetric algebra $S\mathfrak{g}$, the algebra of algebraic functions on $\mathfrak{g}^*$. For $t \neq 0$ this algebra is isomorphic to $U(\mathfrak{g})$. Since $U_h(\mathfrak{g})$ is a free $\mathbb{C}[\hbar][t]$-module, it follows that $\phi_{t,h}$ in (3.27) is a monomorphism of algebras over $\mathbb{C}[\hbar][t]$ and $\text{Im}(\phi_{t,h})$ is a free $\mathbb{C}[\hbar][t]$-module isomorphic to

$$(S\mathfrak{g})_{t,h} = T(\mathfrak{g}_h)[t]/\{x \otimes y - \sigma_h(x \otimes y) - t[x, y]h\}. $$  

(3.30)

It is clear that $(S\mathfrak{g})_t = (S\mathfrak{g})_{t,0}$ is the standard quantization of the Lie bracket on $\mathfrak{g}^*$. □
Call the algebra
\[(S\mathfrak{g})_h = (S\mathfrak{g})_{0,h} = T(\mathfrak{g}_h)/\{x \otimes y - \sigma_h(x \otimes y)\}\]
a quantum symmetric algebra (or quantum polynomial algebra on $\mathfrak{g}^*$). It is a free $\mathbb{C}[[h]]$ module and a quadratic algebra equal to $S\mathfrak{g}$ at $h = 0$.

**Remark 3.2.** Up to now, all our constructions were considered for the quantum group in sense of Drinfeld, $U_h(\mathfrak{g})$, defined over $\mathbb{C}[[h]]$. But one can deduce the results above for the quantum group in sense of Lusztig, $U_q(\mathfrak{g})$, defined over the algebra $\mathbb{C}[q, q^{-1}]$. We show, for example, how to obtain the quantum symmetric algebra over $\mathfrak{g}$. Let $E$ be a Grassmannian consisting of subspaces in $\mathfrak{g} \otimes \mathfrak{g}$ of dimension equal to $\dim(\wedge^2 \mathfrak{g})$, and $Z$ the closed algebraic subset of $E$ consisting of subspaces $J$ such that $\dim(E \otimes J \cap J \otimes E) \geq \dim(\wedge^3 \mathfrak{g})$. Let $\mathcal{X}$ be the algebraic subset in $Z \times (\mathbb{C} \setminus 0)$ consisting of points $(J, q)$ such that $J$ is invariant under the action of $U_q(\mathfrak{g})$. The projection $\pi : \mathcal{X} \to \mathbb{C} \setminus 0$ is a proper map. It is clear that the fiber of this projection over $q = 1$ contains the point corresponding to the symmetric algebra $S\mathfrak{g}$ as an isolated point, because there are no quadratic $U(\mathfrak{g})$ invariant Poisson brackets on $S\mathfrak{g}$.

As follows from the existence of $(S\mathfrak{g})_h$ (completed situation at $q = 1$), the dimension of $\mathcal{X}$ is equal to $1$. Hence, the projection $\pi : \mathcal{X} \to \mathbb{C} \setminus 0$ is a covering. For $x \in \mathcal{X}$ let $J_x$ be the corresponding subspace in $\mathfrak{g} \otimes \mathfrak{g}$ and $(S\mathfrak{g})_x = T(\mathfrak{g})/\{J_x\}$ the corresponding quadratic algebra. Due to the projection $\pi$, the family $(S\mathfrak{g})_x$, $x \in \mathcal{X}$, is a module over $\mathbb{C}[q, q^{-1}]$. Since $J_x$ is $U_{p(x)}(\mathfrak{g})$ invariant, $(S\mathfrak{g})_x$ is a $U_{p(x)}(\mathfrak{g})$ invariant algebra. Hence, after possible deleting from $\mathcal{X}$ some countable set of points, we obtain a family of quadratic algebras with the same dimensions of graded components as $S\mathfrak{g}$. So, the family $(S\mathfrak{g})_x$, $x \in \mathcal{X}$ can be considered as a quantum symmetric algebra over $U_q(\mathfrak{g})$.

Note also that the family $(S\mathfrak{g})_h$ can be thought of as a formal section of the map $\pi : \mathcal{X} \to (\mathbb{C} \setminus 0)$ over the formal neighborhood of point $q = 1$. It follows that there is also an analytic section of $\pi$ over some neighborhood, $U$, of the point $q = 1$. If $(S\mathfrak{g})_h$ is a quantization with Poisson bracket $f - \{\cdot, \cdot\}_r$ (see Proposition 2.2), then a quantization with Poisson bracket $-f - \{\cdot, \cdot\}_r$ gives another section of $\pi$ over $U$. Hence, in a neighborhood of the “classical” point $x_0 \in \mathcal{X}$, $\pi(x_0) = 1$, the space $\mathcal{X}$ has a singularity of type “cross”.

### 3.3 Poisson pencil corresponding to $(S\mathfrak{g})_{t,h}$

Let $\mathfrak{g} = sl(n)$ and $(S\mathfrak{g})_{t,h}$ be the double quantization from Proposition 3.2.

**Proposition 3.3.** The Poisson pencil corresponding to the quantization $(S\mathfrak{g})_{t,h}$ consists of two compatible Poisson brackets:

- $s$ (along $t$) is the Lie bracket;
- $p$ (along $h$) is a quadratic Poisson bracket of the form $p = f - \{\cdot, \cdot\}_r$, where $f$ is an invariant quadratic bracket which is a unique up to a factor invariant map $f : \wedge^2 \mathfrak{g} \to S^2 \mathfrak{g}$, and $\{\cdot, \cdot\}_r$ is the $r$-matrix bracket. Moreover, $[s, f] = 0$ and $[f, f] = -\varphi$, where $\varphi$ has the form $\varphi(a, b, c) = [\varphi_1, a][\varphi_2, b][\varphi_3, c]$, and $\varphi = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 = \[r, r\]$. Recall that $\varphi$ is a unique up to a factor invariant element of $\wedge^3 \mathfrak{g}$. 

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Proof. That \( s \) coincides with the Lie bracket is obvious from (3.29). From Corollary 2.1, we have \( p = f - \{ \cdot, \cdot \}_r \). Since \((S\mathfrak{g})_h\) is a quadratic algebra over \( \mathbb{C}[[h]] \), \( p \) must be a quadratic bracket. But the \( r \)-matrix bracket \( \{ \cdot, \cdot \}_r \) is quadratic, too. Hence, \( f \) must be a quadratic invariant bracket. There is only one possibility for such a bracket: it must be a unique (up to a factor) nontrivial invariant map \( f : \wedge^2 \mathfrak{g} \to S^2 \mathfrak{g} \). Now apply Proposition 2.2 and Corollary 2.1.

Consider now the quadratic bracket \( f \) in more detail.

We say that a \( k \)-vector field, \( g \), on a manifold \( M \) is strongly restricted on a submanifold \( N \subset M \) if at any point of \( N \) the polyvector \( g \) can be presented as an exterior power of tangent vectors to \( N \).

Consider the coadjoint action of the Lie group \( G = SL(n) \) on \( \mathfrak{g}^* = sl(n)^* \). We want to prove that the bracket \( f \) is strongly restricted on any orbit of \( G \) in \( \mathfrak{g} \). It turns out that there is the following general fact.

**Proposition 3.4.** Let \( G \) be a semisimple Lie group with its Lie algebra \( \mathfrak{g} \), \( s = [\cdot, \cdot] \) the Lie bracket on \( \mathfrak{g}^* \). Let \( f = \{ \cdot, \cdot \} \) be an invariant bracket on \( \mathfrak{g}^* \) such that the Schouten bracket \( [s, f] \) is a three-vector field, \( \psi \), strongly restricted on an orbit \( O \) of \( G \). Then \( f \) is strongly restricted on \( O \).

**Proof.** Let \( x, y, z \in \mathfrak{g} \). The invariance condition for \( \{ \cdot, \cdot \} \) means:

\[
[x, \{ y, z \}] = \{ [x, y], z \} + \{ y, [x, z] \}. \tag{3.32}
\]

The Schouten bracket \( [s, f] \) is:

\[
[x, \{ y, z \}] + [y, \{ z, x \}] + [z, \{ x, y \}] + \{ x, [y, z] \} + \{ y, [z, x] \} + \{ z, [x, y] \} = \psi(x, y, z).
\]

In the left hand side of this expression, the 1-st, 5-th, and 6-th terms are canceled due to (3.32), and we have

\[
[y, \{ z, x \}] + [z, \{ x, y \}] + \{ x, [y, z] \} = \psi(x, y, z).
\]

Putting in this equation instead of \([y, \{ z, x \}]\) its expression from (3.32), i.e., \([y, \{ z, x \}] + \{ z, [y, x] \}\), we obtain, since the term \([x, [y, z]]\) is canceled:

\[
\{ z, [x, y] \} = [z, \{ x, y \}] + \psi(x, y, z). \tag{3.33}
\]

Now observe that, due to the Leibniz rule, equation (3.33) is valid for any \( z \in S\mathfrak{g} \). To prove the proposition, it is sufficient to show that if \( z \) belongs to the ideal \( I_O \) defining the orbit \( O \), then \( \{ z, u \} \) also belongs to this ideal. Again, due to the Leibniz rule, it is sufficient to show this for \( u \in \mathfrak{g} \). Since \( \mathfrak{g} \) is semisimple, there are elements \( x, y \in \mathfrak{g} \) such that \([x, y] = u\). We have from (3.33)

\[
\{ z, u \} = \{ z, [x, y] \} = [z, \{ x, y \}] + \psi(x, y, z).
\]

But \([z, \{ x, y \}] \in I_O \), since the Lie bracket is restricted on any orbit, \( \psi(x, y, z) \in I_O \) by hypothesis of the proposition. So, \( \{ z, u \} \in I_O \). \qed
As a consequence we obtain

**Proposition 3.5.** Let $\mathfrak{g} = \mathfrak{sl}(n)$. Then the bracket $f$ from Proposition 3.3 is strongly restricted on any orbit of $\text{SL}(n)$.

**Proof.** Follows from Propositions 3.3 and 3.4. \qed

**Remark 3.3.** According to Remark 2.1, this Proposition shows that in case $G = \text{SL}(n)$ any orbit in coadjoint representation has a Poisson bracket $p = f - r_M$ such that the pair $(M, p)$ becomes a $(G, \tilde{r})$-Poisson manifold.

**Remark 3.4.** Recall that in case $\mathfrak{g} = \mathfrak{sl}(n)$ the tensor square $\mathfrak{g} \otimes \mathfrak{g}$, considered as a representation of $\mathfrak{g}$, has a decomposition into irreducible components which are contained in $\mathfrak{g} \otimes \mathfrak{g}$ with multiplicity one, except of the component isomorphic to $\mathfrak{g}$ having multiplicity two. Moreover, both the symmetric and skew-symmetric parts of $\mathfrak{g} \otimes \mathfrak{g}$ contain components, $\mathfrak{g}^1$ and $\mathfrak{g}^2$, isomorphic to $\mathfrak{g}$. Hence, the bracket $f$ takes $\mathfrak{g}^2$ onto $\mathfrak{g}^1$ and all the other components to zero.

For $\mathfrak{g}$ simple not equal to $\mathfrak{sl}(n)$, the decomposition of $\mathfrak{g} \otimes \mathfrak{g}$ is multiplicity free, hence non-trivial invariant maps $\wedge^2 \mathfrak{g} \to S^2 \mathfrak{g}$ do not exist at all. It follows that for $\mathfrak{g} \neq \mathfrak{sl}(n)$, there do not exist quadratic algebras $(S\mathfrak{g})_h$ which are $U_h(\mathfrak{g})$ invariant quantizations of $S\mathfrak{g}$.

**Question 3.1.** Prove that there exist no one parameter $U_h(\mathfrak{g})$ invariant quantizations of $S\mathfrak{g}$ (not necessarily in the class of quadratic algebras) for all simple Lie algebras $\mathfrak{g} \neq \mathfrak{sl}(n)$.

Now we prove that for simple $\mathfrak{g} \neq \mathfrak{sl}(n)$, the double quantization does not exist (not necessarily in the class of quadratic-linear algebras).

**Proposition 3.6.** Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra not equal to $\mathfrak{sl}(n)$. Then a $U_h(\mathfrak{g})$ invariant quantization of the Lie bracket on $\mathfrak{g}^*$ does not exist.

**Proof.** If such a quantization exists, then from Corollary 2.1 follows that there exists an invariant bracket $f$ on $\mathfrak{g}^*$ such that $[s, f] = 0$ and $[f, f] = -\varphi$. Here $s$ is the Lie bracket and $\varphi$ is the three-vector field induced by $\varphi$ (see Proposition 3.3). We show that such $f$ does not exist. Observe that $\varphi$ has type $(3, 3)$, i.e., is a sum of terms of the view $b \partial_x \wedge \partial_y \wedge \partial_z$, where $b$ is a homogeneous polynomial of degree 3. Observe also that the Schouten bracket of two polyvector fields of degrees $(i, j)$ and $(k, l)$ is a polyvector field of degree $(i + k - 1, j + l - 1)$. We shall write $i$ for degree $(i, j)$ when the second number, $j$, is clear from context.

It is obvious that on $\mathfrak{g}$ there are no invariant bivector fields of degree 0 and, up to a factor, there is a unique invariant bivector field of degree 1, the Lie bracket $s$ itself. Since $\mathfrak{g} \neq \mathfrak{sl}(n)$, there are no bivector fields of degree 2 (see Remark 3.4). Therefore, $f$ must be of the form: $f = s + f_1$, where $f_1$ is a bracket of degree $\geq 3$. Since $f$ is compatible with $s$ and $[f, f] = -\varphi$, it must be $[f_1, f_1] = -\varphi$. But it is impossible, because $[f_1, f_1]$ has at least degree 5. \qed

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3.4 Quantum de Rham complex on \((\mathfrak{sl}(n))^*\)

Consider the algebra \( \Omega^* \) of differential forms on \( \mathfrak{g}^* \) with polynomial coefficients. This is a graded differential algebra with differential \( d \) of degree 1 which forms the de Rham complex

\[
\Omega^*: (S\mathfrak{g})_h \xrightarrow{d_h} \Omega^1_h \xrightarrow{d_h} \Omega^2_h \xrightarrow{d_h} \cdots \tag{3.34}
\]

where \( \Omega^k \) is the space of \( k \)-forms with polynomial coefficients.

We call a complex over \( \mathbb{C}[[h]] \)

\[
\Omega^*_h: (S\mathfrak{g})_h \xrightarrow{d_h} \Omega^1_h \xrightarrow{d_h} \Omega^2_h \xrightarrow{d_h} \cdots \tag{3.35}
\]

a quantum (deformed) de Rham complex if it consists of \( U_h(\mathfrak{g}) \) invariant topologically free modules over \( \mathbb{C}[[h]] \) and coincides with (3.34) at \( h = 0 \).

**Proposition 3.7.** Let \( \mathfrak{g} = sl(n) \). Then the quantized polynomial algebra \((S\mathfrak{g})_h\) from (3.34) can be included in a \( U_h(\mathfrak{g}) \) invariant graded differential algebra, \( \Omega^*_h \), which form a quantum de Rham complex (3.33).

**Proof.** First of all, define a quantum exterior algebra, \((\Lambda\mathfrak{g})_h\), an algebra of differential forms with constant coefficients. Let us modify the operator \( \sigma_h \) from (3.31). Since the representation \( \mathfrak{g}^* \) is isomorphic to \( \mathfrak{g} \), there exists a \( U_h(\mathfrak{g}) \) invariant bilinear form on \( \mathfrak{g} \), deformed Killing form. This form can be naturally extended to all tensor degrees \( \mathfrak{g}_{\otimes k} \).

Let \( \mathfrak{W}_h \) be the \( \mathbb{C}[[h]] \) submodule in \( \mathfrak{g}_h \otimes \mathfrak{g}_h \) orthogonal to \( \mathfrak{W}_h^2 = \text{Im}(\text{id} \otimes \text{id} - \sigma_h) \). Define an operator \( \bar{\sigma}_h \) on \( \mathfrak{g}_h \otimes \mathfrak{g}_h \) in such a way that it has the eigenvalues \(-1\) on \( \mathfrak{W}_h^2 \) and \(1\) on \( \mathfrak{W}_h^2 \). It is clear that \( \mathfrak{W}_h^2 \) and \( \mathfrak{W}_h^2 \) are deformed skew symmetric and symmetric subspaces of \( \mathfrak{g} \otimes \mathfrak{g} \).

Now observe that the third graded component in the quadratic algebra \((S\mathfrak{g})_h\) is the quotient of \( \mathfrak{g}_{\otimes 3} \) by the submodule \( \mathfrak{V}_h^2 \otimes \mathfrak{g}_h + \mathfrak{g}_h \otimes \mathfrak{V}_h^2 \), hence this submodule and, therefore, the submodule \( \mathfrak{V}_h^2 \otimes \mathfrak{g}_h \cap \mathfrak{g}_h \otimes \mathfrak{V}_h^2 \) are direct submodules in \( \mathfrak{g}_{\otimes 3} \), i.e., they have complement submodules. As the complement submodules one can choose the submodules \( \mathfrak{V}_h^2 \otimes \mathfrak{g}_h \cap \mathfrak{g}_h \otimes \mathfrak{W}_h^2 \) and \( \mathfrak{V}_h^2 \otimes \mathfrak{g}_h + \mathfrak{g}_h \otimes \mathfrak{W}_h^2 \), respectively, since they are complement at the point \( h = 0 \) and \( \mathfrak{W}_h^2 \) is orthogonal to \( \mathfrak{V}_h^2 \) with respect to the Killing form extended to \( \mathfrak{g}_h \otimes \mathfrak{g}_h \). Hence, \( \mathfrak{W}_h^2 \otimes \mathfrak{g}_h + \mathfrak{g}_h \otimes \mathfrak{W}_h^2 \) is a direct submodule. Also, the symmetric algebra \( S\mathfrak{g} \) is Koszul. From a result of Drinfeld, \([\text{Dr3}](\text{see also } [\text{DM}])\), follows that the quadratic algebra \((\Lambda\mathfrak{g})_h = T(\mathfrak{g}_h)/\{T(\mathfrak{V}_h^2)\}\) is a free \( \mathbb{C}[[h]] \) module, i.e., is a \( U_h(\mathfrak{g}) \)-invariant deformation of the exterior algebra \( \Lambda\mathfrak{g} \).

Call \((\Lambda\mathfrak{g})_h \) a quantum exterior algebra over \( \mathfrak{g} \).

Define a quantum algebra of differential forms over \( \mathfrak{g}^* \) as the tensor product \( \Omega^*_h = (S\mathfrak{g})_h \otimes (\Lambda\mathfrak{g})_h \) in the tensor category of representations of the quantum group \( U_h(\mathfrak{g}) \). The multiplication of two elements \( a \otimes \alpha \) and \( b \otimes \beta \) looks like \( ab_1 \otimes \alpha_1 \beta \), where \( b_1 \otimes \alpha_1 = S(\alpha \otimes b) \) for \( S = \sigma R \) being the permutation in that category. So, \( \Omega^*_h = (S\mathfrak{g})_h \otimes (\Lambda^k\mathfrak{g})_h \).

As in the classical case, the algebras \((S\mathfrak{g})_h\) and \((\Lambda\mathfrak{g})_h\) can be embedded in \( T(\mathfrak{g}_h) \) as a graded submodules in the following way. Call the submodule \( W^k_h = (W^2_h \otimes \mathfrak{g}_h \otimes \cdots \otimes \mathfrak{g}_h) \cap (\mathfrak{g}_h \otimes W^2_h \otimes \mathfrak{g}_h \otimes \cdots \otimes \mathfrak{g}_h) \cap \cdots \cap (\mathfrak{g}_h \otimes \mathfrak{g}_h \otimes \cdots \otimes W^2_h) \) of \( T^k(\mathfrak{g}_h) \) a \( k \)-th symmetric part of \( T(\mathfrak{g}_h) \). It is clear that the natural map \( \pi_W: T(\mathfrak{g}_h) \to (S\mathfrak{g})_h \) restricted to \( W^k_h \) is
a bijection onto the $k$-degree component $(S^k\mathfrak{g})_h$ of $(S\mathfrak{g})_h$. Denote by $\pi'_V:(S^k\mathfrak{g})_h \to W^k_h$ the inverse bijection. Similarly we define $V^k_h$, the $k$-th skew symmetric part of $T(\mathfrak{g}_h)$, and the bijection $\pi'_V: (\Lambda^k\mathfrak{g})_h \to V^k_h$.

Now, define a differential $d_h$ in $\Omega^\bullet$ as a homogeneous operator of degree $(-1,1)$. It acts on an element, $a \otimes \omega$, of degree $(k,m)$ in the following way. Let $a \otimes \omega = (a_1 \otimes \cdots \otimes a_k) \otimes (\omega_1 \otimes \cdots \otimes \omega_m)$ be its realization as an element from $W^k_h \otimes V^m_h$. Then the formula

$$d_h(a \otimes \omega) = (a_1 \otimes \cdots \otimes a_{k-1} \otimes \pi'_V \pi_V(a_k \otimes \omega_1 \otimes \cdots \otimes \omega_m)$$

(3.36)

presents the element $d_h(a \otimes \omega)$ through its realization in $W^{k-1}_h \otimes V^{m+1}_h$. It is obvious that $d^2_h = 0$.

So, the graded differential algebra $\Omega^\bullet$ is constructed. It is easy to see that at the point $h = 0$ this algebra coincides with $\Omega^\bullet$.

Note that the quantum de Rham complex is exact, because it is exact at $h = 0$.

### 3.5 Restriction of $(S\mathfrak{g})_{t,h}$ on orbits

In this section $G = SL(n)$, $\mathfrak{g} = sl(n)$.

Let $M$ be an invariant closed algebraic subset in $\mathfrak{g}^*$ and $A$ the algebra of algebraic functions on $M$. The algebra $A$ can be presented as a quotient of $S\mathfrak{g}$ by some ideal, $S\mathfrak{g} \to A \to 0$.

We say that the quantization $(S\mathfrak{g})_{t,h}$ can be restricted on $M$ if there exists a $U_h(\mathfrak{g})$ invariant quantization, $A_{t,h}$, of $A$, which can be presented as a quotient of $(S\mathfrak{g})_{t,h}$ by some ideal, $(S\mathfrak{g})_{t,h} \to A_{t,h} \to 0$.

Note that, on the infinitesimal level, there are no obstructions for $(S\mathfrak{g})_{t,h}$ to be restricted on $M$. Indeed, the Lie bracket on $\mathfrak{g}^*$ is strongly restricted on any orbit of $G$ and induces the Kirillov-Kostant-Souriau bracket on $M$. Also, by Proposition 3.5, the bracket $f$ involved in the quantization along $h$ is also strongly restricted on any orbit.

From [DSI], one can derive that the problem of restriction of $(S\mathfrak{g})_{t,h}$ is solved positively in case $M$ is a minimal semisimple orbit, i.e., $M$ is a hermitian symmetric space.

We are going to show here that the problem also has a positive solution for $M$ being a maximal semisimple orbit, i.e., if $M$ can be defined as a set of zeros of invariant functions from $S\mathfrak{g}$. Such orbits are the orbits of diagonal matrices with distinct elements on diagonal.

**Proposition 3.8.** Let $\mathfrak{g} = sl(n)$. Then the family $(S\mathfrak{g})_{t,h}$ can be restricted on any maximal semisimple orbit in $\mathfrak{g}^*$.

**Proof.** There exists an isomorphism of $U_h(\mathfrak{g})$ modules $(S\mathfrak{g})_h \to W_h$, where $W_h = \oplus_k W^k_h$, the direct sum of the $k$-th symmetric parts of $T(\mathfrak{g}_h)$ (see previous subsection). Consider the composition $W_h[t] \to T(\mathfrak{g}_h)[t] \to (S\mathfrak{g})_{t,h}$, where the last map appears from (3.30). It is an isomorphism, since it is an isomorphism at the point $h = 0$. It follows that $(S\mathfrak{g})_{t,h}$ is isomorphic to $W_h[t]$ as a $U_h(\mathfrak{g})$-module.

Denote by $\mathcal{I}_{t,h}$ the submodule of $U_h(\mathfrak{g})$ invariant elements in $(S\mathfrak{g})_{t,h}$. It is obvious that $\mathcal{I}_{t,h}$ is isomorphic to $\oplus_k \mathcal{I}^k_h[t]$, where $\mathcal{I}^k_h$ is the invariant submodule in $W^k_h$. Hence, $\mathcal{I}_{t,h}$ is a direct free $\mathbb{C}[[h]][t]$ submodule in $(S\mathfrak{g})_{t,h}$. Moreover, $\mathcal{I}_{t,h}$ is a central subalgebra in $(S\mathfrak{g})_{t,h}$. Indeed, for a generic $t$ the algebra $(S\mathfrak{g})_{t,h}$ can be invariantly embedded in $U_h(\mathfrak{g})$. But
ad\((U_h(\mathfrak{g}))\) invariant elements in \(U_h(\mathfrak{g})\) form the center of \(U_h(\mathfrak{g})\). Also, \(\mathcal{I}_{t,h}\) as an algebra is isomorphic to \(\mathcal{I}[[h]][t]\) with the trivial action of \(U_h(\mathfrak{g})\), where \(\mathcal{I} = \mathcal{I}_{0,0}\), the algebra of invariant elements in \(S_\mathfrak{g}\). This follows from the fact that \(\mathcal{I}\) is a polynomial algebra, \([\text{Dix}]\), and, therefore, admits no nontrivial commutative deformations.

By the Kostant theorem, \([\text{Dix}]\), \(U(\mathfrak{g})\) is a free module over its center. It follows that at the point \(h = 0\) the module \((S_\mathfrak{g})_{t,0}\) is a free module over the algebra \(\mathcal{I}_{t,0}\). One can easily derive from this that \((S_\mathfrak{g})_{t,h}\) is a free module over \(\mathcal{I}_{t,h}\). In a next paper we shall prove that the quantization \((S_\mathfrak{g})_{t,h}\) can be restricted on all semisimple orbits.

**Question 3.2.** Can be the quantization \((S_\mathfrak{g})_{t,h}\) restricted on all orbits (not necessarily semisimple)?

As we have seen, the corresponding Poisson brackets are strongly restricted on all the orbits.

In next Section we consider the \(U_h(\mathfrak{g})\) invariant quantizations on semisimple orbits in \(\mathfrak{g}^*\) for all simple Lie algebras \(\mathfrak{g}\). It turns out that in general, on a given orbit there are many nonequivalent quantizations which are not restrictions from a quantization on \(\mathfrak{g}^*\). From this point of view, the quantization on maximal orbits described by Proposition (3.8) is a distinguished one.

## 4 The one and two parameter quantization on semisimple orbits in \(\mathfrak{g}^*\)

### 4.1 Pairs of brackets on semisimple orbits

Let \(\mathfrak{g}\) be a simple complex Lie algebra, \(\mathfrak{h}\) a fixed Cartan subalgebra. Let \(\Omega \subset \mathfrak{h}^*\) be the system of roots corresponding to \(\mathfrak{h}\). Select a system of positive roots, \(\Omega^+\), and denote by \(\Pi \subset \Omega\) the subset of simple roots. Fix an element \(E_\alpha \in \mathfrak{g}\) of weight \(\alpha\) for each \(\alpha \in \Omega^+\) and choose \(E_{-\alpha}\) such that

\[
(E_\alpha, E_{-\alpha}) = 1
\]

for the Killing form \((\cdot, \cdot)\) on \(\mathfrak{g}\).
Let $\Gamma$ be a subset of $\Pi$. Denote by $h^*_\Gamma$ the subspace in $h^*$ generated by $\Gamma$. Note, that $h^* = h^*_{\Pi \setminus \Gamma} \oplus h^*_\Gamma$, and one can identify $h^*_{\Pi \setminus \Gamma}$ and $h^*/h^*_\Gamma$ via the projection $h^* \to h^*/h^*_\Gamma$.

Let $\Omega^*_{\Gamma} \subset h^*_\Gamma$ be the subsystem of roots in $\Omega$ generated by $\Gamma$, i.e., $\Omega^*_{\Gamma} = \Omega \cap h^*_\Gamma$. Denote by $g_\Gamma$ the subalgebra of $g$ generated by the elements $\{E_\alpha, E_{-\alpha}\}$, $\alpha \in \Gamma$, and $h$. Such a subalgebra is called the Levi subalgebra.

Let $G$ be a complex connected Lie group with Lie algebra $g$ and $G_\Gamma$ a subgroup with Lie algebra $g_\Gamma$. Such a subgroup is called the Levi subgroup. It is known that $G_\Gamma$ is a connected subgroup. Let $M$ be a homogeneous space of $G$ and $G_\Gamma$ be the stabilizer of a point $o \in M$. We can identify $M$ and the coset space $G/G_\Gamma$. It is known, that such $M$ is isomorphic to a semisimple orbit in $g^*$. This orbit goes through an element $\lambda \in g^*$ which is just the trivial extension to all of $g^*$ (identifying $g$ and $g^*$ via the Killing form) of a map $\lambda : h^*_{\Pi \setminus \Gamma} \to \mathbb{C}$ such that $\lambda(\alpha) \neq 0$ for all $\alpha \in \Pi \setminus \Gamma$. Conversely, it is easy to show that any semisimple orbit in $g^*$ is isomorphic to the quotient of $G$ by a Levi subgroup.

The projection $\pi : G \to M$ induces the map $\pi_* : g \to T_o$, where $T_o$ is the tangent space to $M$ at the point $o$. Since the $\text{ad}$-action of $g_\Gamma$ on $g$ is semisimple, there exists an $\text{ad}(g_\Gamma)$-invariant subspace, $m = m_\Gamma$, of $g$ complementary to $g_\Gamma$, and one can identify $T_o$ and $m$ by means of $\pi_*$. It is easy to see that subspace $m$ is uniquely defined and has a basis formed by the elements $E_\gamma, E_{-\gamma}$, $\gamma \in \Omega^+ \setminus \Omega^*_{\Gamma}$.

Let $v \in g^\otimes m$ be a tensor over $g$. Using the right and the left actions of $G$ on itself, one can associate with $v$ right and left invariant tensor fields on $G$ denoted by $v^r$ and $v^l$.

We say that a tensor field, $t$, on $G$ is right $G_\Gamma$ invariant, if $t$ is invariant under the right action of $G_\Gamma$. The $G$ equivariant diffeomorphism between $M$ and $G/G_\Gamma$ implies that any right $G_\Gamma$ invariant tensor field $t$ on $G$ induces tensor field $\pi_* (t)$ on $M$. The field $\pi_* (t)$ will be invariant on $M$ if, in addition, $t$ is left invariant on $G$, and any invariant tensor field on $M$ can be obtained in such a way. Let $v \in g^\otimes m$. For $v^l$ to be right $G_\Gamma$ invariant it is necessary and sufficient that $v$ to be $\text{ad}(g_\Gamma)$ invariant. Denote $\pi^r (v) = \pi_* (v^r)$ for any $\text{ad}(g_\Gamma)$ invariant tensor $v$ on $g$. Note, that tensor $\pi^r (v)$ coincides with the image of $v$ by the map $g^\otimes m \to \text{Vect}(M)^\otimes m$ induced by the action map $g \to \text{Vect}(M)$. Any $G$ invariant tensor on $M$ has the form $\pi^l (v)$. Moreover, $v$ clearly can be uniquely chosen from $m^\otimes m$.

Denote by $[v, w] \in \wedge^{k+l-1} g$ the Schouten bracket of the polyvectors $v \in \wedge^k g$, $w \in \wedge^l g$, defined by the formula

$$[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l] = \sum (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \hat{X}_i \cdots \hat{Y}_j \cdots \wedge Y_l,$$

where $[\cdot, \cdot]$ is the bracket in $g$. The Schouten bracket is defined in the same way for polyvector fields on a manifold, but instead of $[\cdot, \cdot]$ one uses the Lie bracket of vector fields. We will use the same notation for the Schouten bracket on manifolds. It is easy to see that $\pi^r ([v, w]) = [\pi^r (v), \pi^r (w)]$, and the same relation is valid for $\pi^l$.

Denote by $\Omega^*_{\Gamma}$ the image of $\Omega$ in $h^*_{\Pi \setminus \Gamma}$ without zero. It is clear that $\Omega^*_{\Pi \setminus \Gamma}$ can be identified with a subset of $\Omega^*_{\Gamma}$ and each element from $\Omega^*_{\Gamma}$ is a linear combination of elements from $\Pi \setminus \Gamma$ with integer coefficients which all are either positive or negative. Thus, the subset $\Omega^+_{\Gamma} \subset \Omega^*_{\Gamma}$ of the elements with positive coefficients is exactly the image of $\Omega^+$. We call elements of $\Omega^+_{\Gamma}$ quasiroots and the images of $\Pi \setminus \Gamma$ simple quasiroots.
**Proposition 4.1.** The space $\mathfrak{m}$ considered as a $\mathfrak{g}_\Gamma$ representation space decomposes into the direct sum of subrepresentations $\mathfrak{m}_{\bar{\beta}}$, $\bar{\beta} \in \bar{\Omega}_\Gamma$, where $\mathfrak{m}_{\bar{\beta}}$ is generated by all the elements $E_\beta$, $\beta \in \Omega$, such that the projection of $\beta$ is equal to $\bar{\beta}$. This decomposition have the following properties:

a) all $\mathfrak{m}_{\bar{\beta}}$ are irreducible;

b) for $\bar{\beta}_1, \bar{\beta}_2 \in \bar{\Omega}_\Gamma$ such that $\bar{\beta}_1 + \bar{\beta}_2 \in \bar{\Omega}_\Gamma$ one has $[\mathfrak{m}_{\bar{\beta}_1}, \mathfrak{m}_{\bar{\beta}_2}] = \mathfrak{m}_{\bar{\beta}_1 + \bar{\beta}_2}$;

c) for any pair $\bar{\beta}_1, \bar{\beta}_2 \in \bar{\Omega}_\Gamma$ the representation $\mathfrak{m}_{\bar{\beta}_1} \otimes \mathfrak{m}_{\bar{\beta}_2}$ is multiplicity free.

**Proof.** Statements a) and b) are proven in [DGS]. Statement c) follows from the fact that all the weight subspaces for all $\mathfrak{m}_{\bar{\beta}}$ have the dimension one (see N. Bourbaki, Groupes et algèbres de Lie, Chap. 8.9, Ex. 14).

Since $\mathfrak{g}_\Gamma$ contains the Cartan subalgebra $\mathfrak{h}$, each $\mathfrak{g}_\Gamma$ invariant tensor over $\mathfrak{m}$ has to be of weight zero. It follows that there are no invariant vectors in $\mathfrak{m}$. Hence, there are no invariant vector fields on $M$.

Consider the invariant bivector fields on $M$. From the above, such fields correspond to the $\mathfrak{g}_\Gamma$ invariant bivectors from $\wedge^2 \mathfrak{m}$. Note, that any $\mathfrak{h}$ invariant bivector from $\wedge^2 \mathfrak{m}$ has to be of the form $\sum c(\alpha)E_\alpha \wedge E_{-\alpha}$.

**Proposition 4.2.** A bivector $v \in \wedge^2 \mathfrak{m}$ is $\mathfrak{g}_\Gamma$ invariant if and only if it has the form $v = \sum c(\alpha)E_\alpha \wedge E_{-\alpha}$ where the sum runs over $\alpha \in \Omega^+ \setminus \Omega_{\Gamma}$, and for two roots $\alpha, \beta$ which give the same element in $\mathfrak{h}^*/\mathfrak{h}_\Gamma^*$ one has $c(\alpha) = c(\beta)$.

**Proof.** Follows from Proposition [4.1] and condition (4.1)
**Proposition 4.3.** Let \((\bar{\alpha}_1, \ldots, \bar{\alpha}_k)\) be the \(k\)-tuple of all simple quasiroots. Given a \(k\)-tuple of complex numbers \((c_1, \ldots, c_k)\), assign to each \(\bar{\alpha}_i\) the number \(c_i\). Then

a) for almost all \(k\)-tuples of complex numbers (except an algebraic subset in \(\mathbb{C}^k\) of lesser dimension) equations (4.3) uniquely define numbers \(c(\bar{\alpha})\) for all quasiroots \(\bar{\alpha} = \sum \bar{\alpha}_i\) such that the bivector \(f = \sum c(\bar{\alpha})E_{\alpha} \wedge E_{-\alpha}\) satisfies the condition

\[
[f, f] = K^2 \varphi_M;
\]

b) when \(K = 0\), the solution described in part a) defines a Poisson bracket on \(M\). Numbers \(c(\bar{\alpha})\) give a solution of (4.2) if and only if there exists a linear form \(\lambda \in \mathfrak{h}^*_\Pi_\Gamma\) such that

\[
c(\bar{\alpha}) = \frac{1}{\lambda(\bar{\alpha})}\]

(4.4)

for all quasiroots \(\bar{\alpha}\).

**Proof.** See [DGS].

**Remark 4.1.** This proposition shows that invariant brackets \(f\) on \(M\) defined by part a) of the proposition form a \(k\)-dimensional variety, \(X_k\), where \(k\) is the number of simple quasiroots. On the other hand, \(k = \dim H^2(M)\). [Bo]. If \(K\) is regarded as indeterminate, then \(f\) forms a \(k + 1\) dimensional variety, \(X \subset \mathbb{C}^k \times \mathbb{C}\), (component \(\mathbb{C}\) corresponds to \(K\)). Subvariety \(X_0\) corresponds to \(K = 0\), i.e., consists of Poisson brackets. It is easy to see that all the Poisson brackets with \(c(\bar{\alpha}) = 1/\lambda(\bar{\alpha}) \neq 0\) are nondegenerate. Since \(X\) is connected, it follows that almost all brackets \(f\) (except an algebraic subset in \(X\) of lesser dimension) are nondegenerate as well.

**Remark 4.2.** Equations (4.3) show that when \(c(\bar{\alpha}) + c(\bar{\beta}) = 0\), there appears a harm for determining \(c(\bar{\alpha} + \bar{\beta})\) from given \(c(\bar{\alpha})\) and \(c(\bar{\beta})\). Nevertheless, it is easy to derive from equations (1.2) that

\[
(*) \text{ If } c(\bar{\alpha}) + c(\bar{\beta}) = 0 \text{ then necessarily } c(\bar{\alpha}) = \pm K, \ c(\bar{\beta}) = \mp K.\]

So it is naturally to consider the quasiroots \(\bar{\alpha}\) where \(c(\bar{\alpha})\) are equal to \(\pm K\) or not separately.

Let \(c(\bar{\alpha}), \bar{\alpha} \in \overline{\Omega}_\Gamma\), be a solution of equations (1.2) (we assume \(c(-\bar{\alpha}) = -c(\bar{\alpha})\)). It is easy to derive from equations (4.2) the following properties.

\[
(**) \text{ If } c(\bar{\alpha}) = \pm K \text{ and } c(\bar{\beta}) \neq \pm K, \text{ then } c(\bar{\alpha} + \bar{\beta}) = \pm K \text{ and } c(\bar{\alpha} - \bar{\beta}) = \pm K;\]

\[
(***) \text{ If } c(\bar{\alpha}) = \pm K \text{ and } c(\bar{\beta}) = \pm K, \text{ then } c(\bar{\alpha} + \bar{\beta}) = \pm K.\]

Let \(\overline{\Omega}_\Gamma \subset \overline{\Omega}_\Gamma\) be the subset of quasiroots \(\bar{\alpha}\) such that \(c(\bar{\alpha}) \neq \pm K\). From (**) follows that \(\overline{\Omega}_\Gamma\) is a linear subset, i.e., \(\overline{\Omega}_\Gamma = \overline{\Omega}_\Gamma \cap \text{span}(\overline{\Omega}_\Gamma)\), where \(\text{span}(\overline{\Omega}_\Gamma)\) is the vector subspace of \(\mathfrak{h}^*/h^*_\Gamma\) generated by \(\overline{\Omega}_\Gamma\). Let \((\bar{\alpha}_1, \ldots, \bar{\alpha}_k)\) be a \(k\)-tuple of elements from \(\overline{\Omega}_\Gamma\) that form a basis of \(\text{span}(\overline{\Omega}_\Gamma)\). Since by (*) \(c(\bar{\alpha}) + c(\bar{\beta}) \neq 0\) for any \(\bar{\alpha}, \bar{\beta} \in \overline{\Omega}_\Gamma\), all \(c(\bar{\alpha}), \bar{\alpha} \in \overline{\Omega}_\Gamma\), can be found from (4.3) using the initial values \(c_i = c(\bar{\alpha}_i)\), as in Proposition 4.3.

Note that since \(c_i \neq \pm K\), there are uniquely defined complex numbers \(\lambda_i \neq 0, 1\) such that \(c(\bar{\alpha}_i) = c_i = K\psi(\lambda_j)\), where

\[
\psi(x) = \frac{x + 1}{x - 1}.\]
Using the formula

\[
\psi(xy) = \frac{\psi(x)\psi(y) + 1}{\psi(x) + \psi(y)},
\]

it is easy to derive that if \( \lambda : \overline{\Omega}_r \rightarrow \mathbb{C}^* \) is the multiplicative map (such that if \( \bar{\alpha}, \bar{\beta}, \bar{\alpha} + \bar{\beta} \in \overline{\Omega}_r \) then \( \lambda(\bar{\alpha} + \bar{\beta}) = \lambda(\bar{\alpha})\lambda(\bar{\beta}) \)) defined by \( c(\bar{\alpha}) = \lambda \), then the solution of (4.3) is given by the formula

\[
c(\bar{\alpha}) = K\psi(\lambda(\bar{\alpha})), \quad \bar{\alpha} \in \overline{\Omega}_r.
\] (4.5)

For correctness of this formula, one needs that the map \( \lambda \) to be regular, i.e., that \( \lambda \) to satisfy the condition: if \( \bar{\alpha}, \bar{\beta}, \bar{\alpha} + \bar{\beta} \in \overline{\Omega}_r \) then \( \lambda(\bar{\alpha} + \bar{\beta}) = \lambda(\bar{\alpha})\lambda(\bar{\beta}) \).

From property (***) follows that the numbers \( c(\bar{\alpha}) \) define a function on the set \( \pi(\overline{\Omega}_r) \), where \( \pi \) is the natural map \( h^*/h_r^* \rightarrow (h^*/h_r^*)/\text{span}(\overline{\Omega}_r) \). This function has values \( \pm K \).

Let \( X \subset \pi(\overline{\Omega}_r) \) be the subset where this function has value \( K \). From property (***') follows that \( X \) is a semilinear subset. It means that if \( x_1, x_2 \in X \) and \( x_1 + x_2 \in \pi(\overline{\Omega}_r) \) then \( x_1 + x_2 \in X \), and \( X \cap (-X) = \emptyset \).

The arguments above lead to the following description of the variety \( Z_K \) of all solutions of (4.2) (or, what is the same, the variety of invariant brackets \( f \) on \( M \) such that \[ [f, f] = K^2\varphi_M \]).

**Proposition 4.4.** Variety \( Z_K \) splits into stratas. Each strata is defined by choosing a linear subset \( \overline{\Omega}_r \) of \( \overline{\Omega}_r \) and a semi-linear subset \( X \) of \( \pi(\overline{\Omega}_r) \). Points of this strata are parameterized by the multiplicative regular maps \( \lambda : \overline{\Omega}_r \rightarrow \mathbb{C}^* \).

Let the data \((\overline{\Omega}_r, X, \lambda)\) corresponds to a point of \( Z_K \). Then the coefficients \( c(\bar{\alpha}) \) of \( f \) are determined in the following way. If \( \bar{\alpha} \in \overline{\Omega}_r \) then \( c(\bar{\alpha}) \) is found by (4.3). If \( \pi(\bar{\alpha}) \in X \) then \( c(\bar{\alpha}) = K \). If \( \pi(\bar{\alpha}) \in -X \) then \( c(\bar{\alpha}) = -K \).

Of course, in case \( K = 0 \) the choose of \( X \) does not matter: a strata of \( Z_0 \) is determined only by choosing \( \overline{\Omega}_r \).

Note also that the description of \( Z_K \) given in the proposition does not depend on choosing a basis in \( \overline{\Omega}_r \). The variety \( X_K \) from the previous remark forms an open everywhere dense subset of \( Z_K \) and does depend on choosing a basis. According to Remark 2.1 this proposition describes all the \((G, \tilde{r})\)-Poisson structures on semisimple orbits.

Now we fix a Poisson bracket \( s = \sum (1/\lambda(\bar{\alpha}))E_\alpha \wedge E_{-\alpha} \), where \( \lambda \) is a fixed linear form, and describe the invariant brackets \( f = \sum c(\bar{\alpha})E_\alpha \wedge E_{-\alpha} \) which satisfy the conditions

\[
[f, f] = K^2\varphi_M \quad \text{for} \quad K \neq 0,
\]

\[
[f, s] = 0.
\] (4.6)

Direct computation shows that the condition \([f, s] = 0\) is equivalent to the system of equations for the coefficients \( c(\bar{\alpha}) \) of \( f \)

\[
c(\bar{\alpha})\lambda(\bar{\alpha})^2 + c(\bar{\beta})\lambda(\bar{\beta})^2 = c(\bar{\alpha} + \bar{\beta})\lambda(\bar{\alpha} + \bar{\beta})^2
\] (4.7)

for all the pairs of positive quasiroots \( \bar{\alpha}, \bar{\beta} \) such that \( \bar{\alpha} + \bar{\beta} \) is a quasiroot.
Definition 4.1. Let $M$ be an orbit in $\mathfrak{g}^*$ (not necessarily semisimple). We call $M$ a good orbit, if there exists an invariant bracket, $f$, on $M$ satisfying the conditions (4.6) for $s$ the Kirillov-Kostant-Souriau (KKS) Poisson bracket on $M$.

So, a semisimple orbit $M$ is a good orbit if and only if equations (4.2) and (4.7) are compatible, i.e., have a common solution.

Proposition 4.5. The good semisimple orbits are the following:

a) For $\mathfrak{g}$ of type $A_n$ all semisimple orbits are good.

b) For all other $\mathfrak{g}$, the orbit $M$ is good if and only if the set $\Pi \setminus \Gamma$ consists of one or two roots which appear in representation of the maximal root with coefficient 1.

c) The brackets $f$ on good orbits form a one-dimensional variety: all such brackets have the form

$$\pm f_0 + ts,$$

where $t \in \mathbb{C}$ and $f_0$ is a fixed bracket satisfying (4.9).

Proof. See [DGS].

Remark 4.3. From Proposition (3.5) follows that for $\mathfrak{g} = \mathfrak{sl}(n)$ all orbits (not only semisimple) are good ones. In addition, if an orbit, $M$, is such that $\varphi_M = 0$, then $M$ is good: one can take $f = 0$. In [CP] there is a classification of orbits for all simple $\mathfrak{g}$, for which $\varphi_M = 0$.

Question 4.1. Let $\mathfrak{g}$ be a simple Lie algebra. Are all orbits in $\mathfrak{g}^*$ good? If not, what is a classification of good orbits?

4.2 Cohomologies defined by invariant brackets

In the next subsection we prove the existence of a $U_h(\mathfrak{g})$ invariant quantization of the Poisson brackets described above using the methods of [DS1]. This requires us to consider the 3-cohomology of the complex $(\Lambda^\bullet(\mathfrak{g}/\mathfrak{g}_\Gamma))^\mathbb{R} = (\Lambda^\bullet \mathfrak{m})^\mathbb{R}$ of $\mathfrak{g}_\Gamma$ invariants with differential given by the Schouten bracket with the bivector $f \in (\Lambda^2 \mathfrak{m})^\mathbb{R}$ from Proposition 4.3 a),

$$\delta_f : u \mapsto [f, u] \quad \text{for} \quad u \in (\Lambda^\bullet \mathfrak{m})^\mathbb{R}.$$

The condition $\delta_f^2 = 0$ follows from the Jacobi identity for the Schouten bracket together with the fact that $[f, f] = K^2 \varphi_M$. Denote these cohomologies by $H^k(M, \delta_f)$, whereas the usual de Rham cohomologies are denoted by $H^k(M)$.

Recall (see Remark 1.1) that the brackets $f$ satisfying $[f, f] = K^2 \varphi$ form a connected variety $\mathcal{X}$ which contains a submanifold $\mathcal{X}_0$ of Poisson brackets.

Proposition 4.6. For almost all $f \in \mathcal{X}$ (except an algebraic subset of lesser dimension) one has

$$H^k(M, \delta_f) = H^k(M)$$

for all $k$. In particular, $H^k(M, \delta_f) = 0$ for odd $k$. 

Proof. First, let \( v \) be a Poisson bracket, i.e., \( v \in \mathcal{X}_0 \). Then the complex of polyvector fields on \( M, \Theta^* \), with the differential \( \delta_v \) is well defined. Denote by \( \Omega^* \) the de Rham complex on \( M \). Since none of the coefficients \( c(\bar{\alpha}) \) of \( v \) are zero, \( v \) is a nondegenerate bivector field, and therefore it defines an \( \mathcal{A} \)-linear isomorphism \( \bar{v} : \Omega^1 \to \Theta^1, \omega \mapsto v(\omega, \cdot) \), which can be extended up to the isomorphism \( \bar{v} : \Omega^k \to \Theta^k \) of \( k \)-forms onto \( k \)-vector fields for all \( k \). Using Jacobi identity for \( v \) and invariance of \( v \), one can show that \( \bar{v} \) gives a \( G \) invariant isomorphism of these complexes, so their cohomologies are the same.

Since \( g \) is simple, the subcomplex of \( g \) invariants, \((\Omega^*)^g\), splits off as a subcomplex of \( \Omega^* \). In addition, \( g \) acts trivially on cohomologies, since for any \( g \in G \) the map \( M \to M, x \mapsto gx \), is homotopic to the identity map, \( (G \) is a connected Lie group corresponding to \( g \). It follows that cohomologies of complexes \( (\Omega^*)^g \) and \( \Omega^* \) coincide.

But \( \bar{v} \) gives an isomorphism of complexes \( (\Omega^*)^g \) and \((\Lambda^* m)^{gr}, \delta_v \). So, cohomologies of the latter complex coincide with de Rham cohomologies, which proves the proposition for \( v \) being Poisson brackets.

Now, consider the family of complexes \((\Lambda^* m)^{gr}, \delta_v \), \( v \in \mathcal{X} \). It is clear that \( \delta_v \) depends algebraically on \( v \). It follows from the uppersemicontinuity of \( \dim H^k(M, \delta_v) \) and the fact that \( H^k(M) = 0 \) for odd \( k \), \([E]_c \), that \( H^k(M, \delta_v) = 0 \) for odd \( k \) and almost all \( v \in \mathcal{X} \). Using the uppersemicontinuity again and the fact that the number \( \sum_k(-1)^k \dim H^k(M, \delta_v) \) is the same for all \( v \in \mathcal{X} \), we conclude that \( \dim H^k(M, \delta_v) = \dim H^k(M) \) for even \( k \) and almost all \( v \).

\( \square \)

Remark 4.4. Call \( f \in \mathcal{X} \) admissible, if it satisfies Proposition \([L.3] \). From the proof of the proposition follows that the subset \( \mathcal{D} \) such that \( \mathcal{X} \setminus \mathcal{D} \) consists of admissible brackets does not intersect with the subset \( \mathcal{X}_0 \) consisting of Poisson brackets. It follows from this fact that for each good orbit there are admissible \( f \) compatible with the KKS bracket. Indeed, let \( M \) be a good orbit and \( f_0 + ts \) the family from Proposition \([L.3] \) (c) satisfying \([L.9] \) for a fixed \( K \). Then for almost all numbers \( t \) this bracket is admissible. In fact, this family is contained in the two parameter family \( uf_0 + ts \). By \( u = 0 \), \( t \neq 0 \) we obtain admissible brackets. So, there exist \( u_0 \neq 0 \) and \( t_0 \) such that the bracket \( u_0f_0 + t_0s \) is admissible. It follows that the bracket \( f_0 + (t_0/u_0)w \) is admissible, too. So, in the family \( f_0 + ts \) there is an admissible bracket, and we conclude that almost all brackets in this family (except a finitely many) are admissible.

For the proof of existence of two parameter quantization for the cases \( D_n \) and \( E_6 \) in the next subsection, we will use the following result on invariant three-vector fields.

Denote by \( \theta \) the Cartan automorphism of \( g \).

Lemma 4.1. For either \( D_n \) or \( E_6 \) and one of the subsets, \( \Gamma \), of simple roots such that \( G_\Gamma \) defines a good orbit, any \( g_{\Gamma} \) and \( \theta \) invariant element \( v \) in \( \Lambda^3 m \) is a multiple of \( \varphi_M \), that is,

\[
(\Lambda^3 m)^{gr} \cong \langle \varphi_M \rangle.
\]

Proof. In this case the system of positive quasiroots consists of \( \bar{\alpha}, \bar{\beta}, \) and \( \bar{\alpha} + \bar{\beta} \), where \( \bar{\alpha}, \bar{\beta} \) are the simple quasiroots. From Proposition \([L.3] \) follows that invariant elements in \( m_\alpha \otimes m_\beta \otimes m_{-\alpha - \beta} \) and \( m_{-\alpha} \otimes m_{-\beta} \otimes m_{\bar{\alpha} + \bar{\beta}} \) form subspaces of dimension one, \( I_1 \) and \( I_2 \). Moreover, all the invariant elements of \( \Lambda^3 m \) are lying in \( I_1 + I_2 \). Since \( \theta \) takes \( I_1 \) onto \( I_2 \),
there is only one-dimensionl $\theta$ invariant subspace in $I_{1} + I_{2}$, which is necessarily generated by $\varphi_{M}$. 

4.3 $U_{h}(g)$ invariant quantizations in one and two parameters

In this subsection we prove the existence of one and two parameter $U_{h}(g)$ invariant quantization of the function algebras $A$ on semisimple orbits, $M$, in $g^\ast$. By Proposition 2.2, the one parameter quantization has the Poisson bracket of the form

$$f(a, b) - \{a, b\}_{r}, \quad [f, f] = -\varphi_{M}. \quad (4.8)$$

We show that the one parameter quantization exists for all semisimple orbits and all $f$ constructed in Proposition 4.3 a) and satisfying Proposition 4.6.

For two parameter quantization, there are two compatible Poisson brackets: the KKS bracket $s$ and the bracket of the form (4.8) with the additional condition

$$[f, s] = 0. \quad (4.9)$$

We show that the two parameter quantization exists for good orbits in cases $D_{n}$ and $E_{6}$ and for almost all $f$ satisfying (4.8) and (4.9).

Note that in subsection 3.5 we have proven that in case $A_{n}$ the two parameter quantization exists for maximal semisimple orbits. In a next paper we shall prove the same for all semisimple orbits.

We remind the reader of the method in [DS1]. The first step is to construct a $U(g)$ invariant quantization in the category $C(U(g)[[h]], \Delta, \Phi_{h})$. Then we use the equivalence given by the pair $(\text{Id}, F_{h})$ between the monoidal categories $C(U(g)[[h]], \Delta, \Phi_{h})$ and $C(U(g)[[h]], \Delta_{h}, 1)$ to define a $U_{h}(g)$ invariant quantization, either $\mu_{h}F_{h}^{-1}$ in the one parameter case or $\mu_{t,h}F_{h}^{-1}$ in the two parameter case (see Subsections 2.2 and 2.3). In the following we often write $\Phi$ for $\Phi_{h}$. 

**Proposition 4.7.** Let $g$ be a simple Lie algebra, $M$ a semisimple orbit in $g^\ast$. Then, for almost all (in sense of Proposition 4.4) $g$ invariant brackets $f$ satisfying $[f, f] = -\varphi_{M}$, there exists a multiplication $\mu_{h}$ on $A$

$$\mu_{h}(a, b) = ab + (h/2)f(a, b) + \sum_{n \geq 2} h^{n} \mu_{n}(a, b)$$

which is $U(g)$ invariant (equation (2.12)) and $\Phi$ associative (equation (2.13)).

**Proof.** To begin, consider the multiplication $\mu^{(1)}(a, b) = ab + (h/2)f(a, b)$. The corresponding obstruction cocycle is given by

$$\text{obs}_{2} = \frac{1}{h^2} (\mu^{(1)}(\mu^{(1)} \otimes id) - \mu^{(1)}(id \otimes \mu^{(1)}) \Phi)$$

considered modulo terms of order $h$. No $\frac{1}{h}$ terms appear because $f$ is a biderivation and, therefore, a Hochschild cocycle. The fact that the presence of $\Phi$ does not interfere with the cocycle condition and that this equation defines a Hochschild 3-cocycle was proven in [DS1].
It is well known that if we restrict to the subcomplex of cochains given by differential operators, the differential Hochschild cohomology of $\mathcal{A}$ in dimension $p$ is the space of $p$-polyvector fields on $M$. Since $\mathfrak{g}$ is reductive, the subspace of $\mathfrak{g}$ invariants splits off as a subcomplex and has cohomology given by $(\Lambda^p \mathfrak{m})^{gr}$. The complete antisymmetrization of a $p$-tensor projects the space of invariant differential $p$-cocycles onto the subspace $(\Lambda^p \mathfrak{m})^{gr}$ representing the cohomology. The equation $[f, f] + \varphi_M = 0$ implies that obstruction cocycle is a coboundary, and we can find a 2-cochain $\mu_2$, so that $\mu^{(2)} = \mu^{(1)} + h^2 \mu_2$ satisfies
\[
\mu^{(2)}(\mu^{(2)} \otimes \text{id}) - \mu^{(2)}(\text{id} \otimes \mu^{(2)}) \Phi = 0 \mod h^2.
\]
Assume we have defined the deformation $\mu^{(n)}$ to order $h^n$ such that $\Phi$ associativity holds modulo $h^n$, then we define the $(n+1)^{st}$ obstruction cocycle by
\[
\text{obs}_{n+1} = \frac{1}{h^{n+1}}(\mu^{(n)}(\mu^{(n)} \otimes \text{id}) - \mu^{(n)}(\text{id} \otimes \mu^{(n)}) \Phi) \mod h.
\]

In [DS1] (Proposition 4) we showed that the usual proof that the obstruction cocycle satisfies the cocycle condition carries through to the $\Phi$ associative case. The coboundary of $\text{obs}_{n+1}$ appears as the $h^{n+1}$ coefficient of the signed sum of the compositions of $\mu^{(n+1)}$ with $\text{obs}_{n+1}$. The fact that $\Phi = 1 \mod h^2$ together with the pentagon identity implies that the sum vanishes identically, and thus all coefficients vanish, including the coboundary in question. Let $\text{obs}'_{n+1} \in (\Lambda^3 \mathfrak{m})^{gr}$ be the projection of $\text{obs}_{n+1}$ on the totally skew symmetric part, which represents the cohomology class of the obstruction cocycle. The coefficient of $h^{n+2}$ in the same signed sum, when projected on the skew symmetric part, is $[f, \text{obs}'_{n+1}]$ which is the coboundary of $\text{obs}'_{n+1}$ in the complex $(\Lambda^3 \mathfrak{m})^{gr}, \delta f = \left[ f, . \right]$). Thus $\text{obs}'_{n+1}$ is a $\delta f$ cocycle. By Proposition 4.7, this complex has zero cohomology. Now we modify $\mu^{(n+1)}$ by adding a term $h^n \mu_n$ with $\mu_n \in (\Lambda^2 \mathfrak{m})^{gr}$ and consider the $(n+1)^{st}$ obstruction cocycle for $\mu^{(n+1)} = \mu^{(n+1)} + h^n \mu_n$. Since the term we added at degree $h^n$ is a Hochschild cocycle, we do not introduce a $h^n$ term in the calculation of $\mu^{(n)}(\mu^{(n)} \otimes \text{id}) - \mu^{(n)}(\text{id} \otimes \mu^{(n)}) \Phi$ and the totally skew symmetric projection $h^{n+1}$ term has been modified by $[f, \mu_n]$. By choosing $\mu_n$ appropriately, we can make the $(n+1)^{st}$ obstruction cocycle represent the zero cohomology class, and we are able to continue the recursive construction of the desired deformation.

Now we prove the existence of a two parameter deformation for good orbits in the cases $D_n$ and $E_6$.

**Proposition 4.8.** Given a pair of $\mathfrak{g}$ invariant brackets, $f, v$, on a good orbit in $D_n$ or $E_6$ satisfying $[f, f] = -\varphi_M$, $[f, v] = [v, v] = 0$, there exists a multiplication $\mu_{h,t}$ on $\mathcal{A}$
\[
\mu_{t,h}(a, b) = ab + (h/2)f(a, b) + (t/2)v(a, b) + \sum_{k,l \geq 1} h^k t^l \mu_{k,l}(a, b)
\]
which is $U(\mathfrak{g})$ invariant and $\Phi$ associative.

**Proof.** The existence of a multiplication which is $\Phi$ associative up to and including $h^2$ terms is nearly identical to the previous proof. Both $f$ and $v$ are anti-invariant under the Cartan involution $\theta$. We shall look for a multiplication $\mu_{t,h}$ such that $\mu_{k,l}$ is $\theta$ anti-invariant and skew-symmetric for odd $k + l$ and $\theta$ invariant and symmetric for even $k + l$. 

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So, suppose we have a multiplication defined to order \( n \),

\[
\mu_{t,h}(a, b) = ab + h\mu_1(a,b) + t\mu'_1(a,b) + \sum_{k+l \leq n} h^k t^l \mu_{k,l}(a,b),
\]

with mentioned above invariance properties and \( \Phi \) associative to order \( h^n \).

Further we shall suppose that \( \Phi \) has the properties: It is invariant under the Cartan involution \( \theta \) and \( \Phi^{-1} = \Phi_{221} \). Such \( \Phi \) always can be chosen, \( \text{[DS2]} \). Using these properties for \( \Phi \), direct computation shows that the obstruction cochain,

\[
\text{obs}_{n+1} = \sum_{k=0,...,n+1} h^k t^{n+1-k} \beta_k,
\]

has the following invariance properties: For odd \( n \), \( \text{obs}_{n+1} \) is \( \theta \) invariant and \( \text{obs}_{n+1}(a, b, c) = -\text{obs}_{n+1}(c, b, a) \), and for even \( n \), and \( \text{obs}_{n+1} \) is \( \theta \) anti-invariant and \( \text{obs}_{n+1}(a, b, c) = \text{obs}_{n+1}(c, b, a) \).

Hence, the projection of \( \text{obs}_{n+1} \) on \( (\Lambda^3 m)^{gr} \) is equal to zero for even \( n \). It follows that all the \( \beta_k \) are Hochschild coboundaries, and the standard argument implies that the multiplication can be extended up to order \( n+1 \) with the required properties.

For odd \( n \), Lemma \( 4.1 \) shows that the projection on \( (\Lambda^3 m)^{gr} \) has the form

\[
\text{obs}_{n+1} = \left( \sum_{k=0,...,n+1} a_k h^k t^{n+1-k} \right) \varphi_M.
\]

The KKS bracket is given by the two-vector

\[
v = \sum_{a \in \Omega^+ \setminus \Omega_T} \frac{1}{\lambda(\bar{a})} E_\alpha \wedge E_{-\alpha}.
\]

Setting

\[
w = \sum_{a \in \Omega^+ \setminus \Omega_T} \lambda(\bar{a}) E_\alpha \wedge E_{-\alpha},
\]

gives

\[
[v, w] = -3 \varphi_M.
\]

Defining

\[
\mu^{(n)} = \mu^{(n)} + \frac{a_0}{3} t^n w,
\]

the new obstruction cohomology class is

\[
\text{obs}'_{n+1} = \left( \sum_{k=1,...,n+1} a_k h^k t^{n+1-k} \right) \varphi_M.
\]

Finally we define

\[
\mu^{(n)} = \mu^{(n)} + \sum_{k=1,...,n+1} a_k h^{k-1} t^{n+1-k} f
\]

and get an obstruction cocycle which is zero in cohomology. Now the standard argument implies that the deformation can be extended to give a \( \Phi \) associative invariant multiplication with the required properties of order \( n+1 \).

So, we are able to continue the recursive construction of the desired multiplication. \( \square \)
Using the $\Phi_h$ associative multiplications $\mu_h$ and $\mu_{t,h}$ from Propositions 4.7 and 4.8 and the equivalence between the monoidal categories $\mathcal{C}(U(\mathfrak{g})[[h]], \Delta, \Phi_h)$ and $\mathcal{C}(U(\mathfrak{g})[[h]], \Delta, 1)$ given by the pair $(\text{Id}, F_h)$ (see Section 2), one can define $U_h(\mathfrak{g})$ invariant multiplications, either $\mu_h F_h^{-1}$ in the one parameter case or $\mu_{t,h} F_h^{-1}$ in the two parameter case.

**Remark 4.5.** After [Ko], the philosophy is that there are no obstructions for quantizations of Poisson brackets on manifolds. In this connection, the following question arises:

**Question 4.2.** Let $M$ be a $G$-manifold on which there exists an invariant connection. Given a $G$ invariant Poisson bracket, $v$, on $M$, does there exist a $G$ invariant quantization of $v$?

In case $M$ is a homogeneous manifold the bracket $v$ has a constant rank, and such a quantization can be obtained by Fedosov’s method, [Fed], [Do1].

Another question which relates to the topic of this paper is the following.

**Question 4.3.** Let $M$ be a $G$-manifold on which there exists an invariant connection, $U(\mathfrak{g})$ the corresponding to $G$ universal enveloping algebra, and $\Phi_h \in (U(\mathfrak{g}))^{\otimes 3}[[h]]$ an invariant element of the form (2.6) obeying the pentagon identity (2.7). Let $f$ be an invariant bracket on $M$ satisfying $[f, f] = -\varphi_M$. Does there exist a $U(\mathfrak{g})$ (or $G$) invariant and $\Phi_h$ associative quantization of $f$ (as in Proposition 4.7)?

Note that if the answer to this Question is positive, then the answer to Question 2.1 is also positive: we take for $M$ the group $G$ itself and consider it as a $G$-manifold by left multiplication.

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