Semi-supervised Learning on Graph with an Alternating Diffusion Process

Qilin Li*, Senjian An, Ling Li and Wanquan Liu

Department of Computing, Curtin University, Perth, Australia
li.qilin@postgrad.curtin.edu.au, {s.an, l.li, w.liu}@curtin.edu.au

Abstract

Graph-based semi-supervised learning usually involves two separate stages, constructing an affinity graph and then propagating labeling for transductive inference on the graph. It is suboptimal to solve them independently, as the correlation between the affinity graph and labels are not fully exploited. In this paper, we integrate the two stages into one unified framework by formulating the graph construction as a regularized function estimation problem similar to label propagation. We propose an alternating diffusion process to solve the two problems simultaneously, which allows us to learn the graph and unknown labels in an iterative fashion. With the proposed framework, we are able to adequately leverage both the given labels and estimated labels to construct a better graph, and effectively propagate labels on such a dynamic graph updated simultaneously with the newly obtained labels. Extensive experiments on various real-world datasets have demonstrated the superiority of the proposed method compared to other state-of-the-art methods.

1 Introduction

Semi-supervised learning (SSL) refers to the problem of learning from labeled and unlabeled data. SSL is of particular interest in many real-world applications since labeled data is often scarce, whereas unlabeled data is abundant. One family of SSL methods known as graph-based approaches have attracted much attention during the past decade, where a typical graph is constructed during the past decade, where a typical graph is constructed using Gaussian kernel in the Euclidean space, as we know Euclidean distance cannot approximate the geodesic distance on the manifold, especially for high-dimensional data due to the “curse of dimensionality”.

Over the last decade, many works of GSSL, such as Label Propagation [Zhu and Ghahramani, 2002], GFHF [Zhu et al., 2003] and LGC [Zhou et al., 2004], focused on how to effectively propagate labels on a predefined graph. The prototype approach is to formulate the problem as regularized function estimation, targeting at a trade-off between the accuracy of the classification function on the labeled points and the regularization that favors a function which is sufficiently smooth with respect to the intrinsic manifold structure revealed by both labeled and unlabeled points. These label propagation methods can effectively spread labels, given the underlying manifold is appropriately represented by the affinity graph. What if the graph itself is problematic? That is indeed the case during the past decade, where a typical graph is constructed using Gaussian kernel in the Euclidean space, as we know Euclidean distance cannot well approximate the geodesic distance on the manifold, especially for high-dimensional data due to the “curse of dimensionality”.

Recently, research focus of GSSL has been shifted to constructing an adequate graph [Li et al., 2015; Zhuang et al., 2017; Zhang et al., 2018; Fang et al., 2018] that better reveals the data manifold in order to facilitate the following label propagation. State-of-the-art graph construction methods are usually based on the self-expressive model which reconstructs each data point by a linear combination of all other data points. The reconstruction coefficients, regularized by sparsity or low-rankness, are then used to construct a graph. Despite the success of these methods on many application, a common drawback is that the correlation between the label and graph is not fully exploited, as graph construction and label propagation are formulated as two independent stages. Typical pipeline of current GSSL methods start with constructing a graph using Gaussian kernel or sparse representation, and then obtain results by applying label propagation, e.g., GFHF or LGC, on the static graph defined in the previous stage. Although limited works [Zhuang et al., 2017] attempted to use initial label information to guide the graph construction, it is still suboptimal as it has been shown in [Li et al., 2015] that even the estimated labels of the unlabeled points can provide “weakly” supervision for building a better graph.
In this work, we attempt to integrate graph construction and label propagation into one unified iterative framework. Specifically, we formulate graph construction as a regularized function estimation problem similar to label propagation. We show that these two function estimation problems can be solved efficiently by two iterative diffusion processes that are fundamentally similar. Most importantly, by alternately or jointly running these two diffusion processes, we are allowed to fully interact the two stages in a iterative manner, as in each iteration, labels are propagated on a dynamic graph updated simultaneously with the supervision of the estimated labels obtained in the previous iteration.

**Paper contribution.** The main contribution of this paper can be summarized as three folds:

- We formulate the graph construction as a regularized optimization problem that shares the same intuition as label propagation. They both attempt to fit the label information while regularizing the function smoothness using graph Laplacian.
- We propose an iterative diffusion process that can fully exploit the given and estimated labels to efficiently solve the optimization problem.
- We integrate graph construction and label propagation into one unified iterative diffusion framework, allowing them to be updated simultaneously.

## 2 Related work

### 2.1 Graph-based semi-supervised learning

Graph construction is of great importance to the success of all GSSL methods, as the effectiveness of the label propagation stage depends heavily on having a good graph that well reveals the underlying manifold. Started with a Gaussian kernel $W_{ij} = \exp\left(-\frac{\|x_i - x_j\|^2}{2\sigma^2}\right)$, where $W$ is called a weight matrix or an affinity/similarity matrix that is equivalent to a graph, recent works [He et al., 2011; Zhuang et al., 2015] on GSSL have focused on constructing a graph using the self-expressive model, i.e., expressing each data point as linear combination of all other points, while regularizing the coefficients by sparsity or low-rankness. However, these approaches arise in subspace clustering does not fit into the SSL scenario as the given labels are totally neglect. A nature extension [Zhuang et al., 2017] is to add a hard constraint to force the affinities between points with different labels to be zero. While these label information guided graph construction methods [Zhuang et al., 2017; Fang et al., 2018] showed a reasonable performance gain, they are still suboptimal as only given labels are exploited once, which often account a small portion. It was shown in [Li et al., 2015] that the estimated unknown labels can provide additional useful supervision for graph construction, given a proper feedback mechanism.

After a graph is constructed, the next step is to propagate label on it. Many classic GSSL methods [Zhu and Ghahramani, 2002; Zhu et al., 2003; Zhou et al., 2004] formulated label propagation as a regularized function estimation problem consisting of two terms $J_{fit}$ and $J_{smooth}$, namely fitness and smoothness respectively. The target classification function $F \in \mathbb{R}^{n \times c}$ can then be obtained by finding a trade-off between these two terms:

$$\arg \min\limits_{F} (J_{fit}(F) + \mu J_{smooth}(F)).$$

(1)

A common choice of the fitness term is a quadratic loss between the predicted labels and the groundtruth labels, while imposing graph Laplacian as a smooth operator for the second term, such as GFHF [Zhu et al., 2003]:

$$\min\limits_{F} \infty \sum_{i=1}^{\ell} \|F_i - Y_i\|^2 + \frac{1}{2} \sum_{i,j=1}^{n} W_{ij} \|F_i - F_j\|^2,$$

(2)

where $\ell$ is the number of labeled points. Note that the infinity weight $\infty$ clamps the predictions on the labeled points to be the given labels. LGC [Zhu et al., 2004] relaxed the infinity weight to a trade-off parameter $\mu > 0$, and applied on all data points instead of only labeled ones. GGMC [Wang et al., 2013] followed the same cost function as LGC but reformulated it to a bivariate optimization problem with respect to not only the classification function $F$ but also the binary label matrix $Y$. Updating these two variables alternately, it can be seen as learning $F$ iteratively by feeding back the estimated labels to initialize the given labels $Y$, resulting in a robust classification function less depended on the given labels. A novel label propagation method based on KL-divergence was also proposed in [Fan et al., 2018].

### 2.2 Diffusion process

Diffusion is a mechanism which propagates information through a graph represented by an affinity matrix. It has been applied in many computer vision problems, such as clustering [Li et al., 2018], saliency detection [Lu et al., 2014], image segmentation [Yang et al., 2013], low-shot learning [Douze et al., 2018] and mainly image retrieval [Donoser and Bischof, 2013; Bai et al., 2017; Iscen et al., 2017].

The diffusion process can be interpreted as random walks on the graph, where a transition matrix $P$ obtained by normalizing the initial weight matrix $W$, i.e., $P = D^{-1}W$, defines the probability of walking from one node to another. If the goal is to learning an affinity matrix $A$ as in retrieval, the random walk can be formulated as an iterative process:

$$A^{(t+1)} = A^{(t)}P.$$

(3)

After $t$ steps, the initial affinity matrix are updated by the probability distribution in $P^t$. This model was then extended to a popular retrieval system PageRank [Page et al., 1999] by introducing an additional random jump matrix $Y$:

$$A^{(t+1)} = \alpha A^{(t)}P + (1-\alpha)Y,$$

(4)

where $0 < \alpha < 1$ controls the trade-off between walk and jump. [Yang et al., 2013] applied the diffusion on a higher-order tensor graph by the following update mechanism:

$$A^{(t+1)} = \alpha PA^{(t)}P^\top + (1-\alpha)I,$$

(5)

where $I$ is the identity matrix. It has been shown in the survey paper [Donoser and Bischof, 2013] that this type of
diffusion process Eq.(5) consistently yield better results. [Bai et al., 2017] showed that this diffusion process Eq.(5) closely related to the regularized optimization problem in GSSL.

While the diffusion process has been successfully used in unsupervised scenario, such as image retrieval, it is not clear how to cope it with label information. [Li et al., 2019] attempted to leverage labels to facilitate the affinity learning, but only with given labels, limiting its effectiveness in SSL where few labels are available.

3 Alternating Diffusion Process

We consider the problem of learning from labeled and unlabeled points:

**Problem 1 (Semi-supervised learning).** Given a data point set \( X = \{x_1, ..., x_t, x_{t+1}, ..., x_n \} \subset \mathbb{R}^d \) and a label set \( Y = \{1, 2, ..., c\} \), the first \( \ell (\ell < n) \) points \( x_i (i \leq \ell) \) are labeled as \( y_i \in Y \) and the other points \( x_u (\ell + 1 \leq u \leq n) \) are unlabeled. The goal of SSL is to predict the label of the unlabeled points.

Let a matrix \( F \in \mathbb{R}^{n \times c} \) denote a classification on the point set \( X \), where its entry \( F_{ij} \) represents the probability of \( x_i \) belonging to \( j \)-th class. \( F \) is nonnegative and each row sum up to 1. The target label \( y_i \) can be obtained by \( y_i = \arg \max_j F_{ij} \). Let us transfer the labels to another \( n \times c \) matrix \( Y \) by one-hot-encoding, where \( Y_{ij} = 1 \) if \( y_i = j \) and \( Y_{ij} = 0 \) otherwise. We first show how to construct a graph represented by an affinity matrix \( A \in \mathbb{R}^{n \times n} \).

3.1 Given classification \( F \), update graph \( A \)

**Optimization problem**

With classification \( F \), we formulate the graph construction as a regularized function estimation problem:

\[
\min_A \frac{1}{2} \sum_{i,j,k=1}^{n} W_{ij} W_{kl} \left( \frac{A_{ki} + Z_{ki}}{\sqrt{D_{ii} D_{kk}}} - \frac{A_{lj} + Z_{lj}}{\sqrt{D_{jj} D_{ll}}} \right)^2 + \left( -2 \sum_{k,i=1}^{n} A_{ki} Z_{ki} \right) + \mu \sum_{k,i=1}^{n} (A_{ki} - I_{ki})^2 ,
\]

where \( \mu > 0 \) is a regularization parameter, \( D \) is a diagonal matrix with its element \( D_{ii} = \sum_{j=1}^{n} W_{ij} \), \( I \) is the \( n \times n \) identity matrix and \( Z = FF^\top \) is the label similarity/affinity matrix. A large element \( Z_{ij} \) indicates that the label vector \( F_i \) is similar to \( F_j \).

As shown in Eq.(6), the objective function consists of three terms. The first term is a local smoothness term, where it encourages nearby points identified by \( W \) taking large values on the learned affinity \( A \). Instead of considering pair-wise similarity independently on the original graph represented by \( W \), ADP attempts to smooth \( A \) with 4 nodes at a time by using a higher-order tensor graph [Yang et al., 2013]. Specifically, if \( W_{ij} \) is large (\( x_i \) is similar to \( x_j \)) and \( W_{kl} \) is large (\( x_k \) is similar to \( x_l \)), then \( A_{ki} \) and \( A_{lj} \) are encouraged to be similar after adding the label similarity. The second term is a fitness term. It encourages a large \( A_{ki} \) when \( Z_{ki} \) is large. The assumption is that data points with similar label (large \( Z_{ki} \)) should have a large similarity as well. The last term is a regularization term controlling the scale of \( A \).

It can be seen that the problem of constructing graph \( A \) Eq.(6) is quite similar to the problem of label propagation such as Eq.(2). They both can be seen as estimating a local smooth function on the graph \( W \), while regularizing the global fitness to the labels.

The objective function Eq.(6) can be transferred to:

\[
\mathcal{J} = \text{vec}(A + Z)^\top L \text{vec}(A + Z) - 2 \text{vec}(A)^\top \text{vec}(Z) + \mu \| \text{vec}(A) - \text{vec}(I) \|^2 ,
\]

where \( L = I - S \) is the normalized graph Laplacian of the tensor product graph. \( S^{n \times c} \) is the Kronecker product \( S \otimes \mathbb{S} \), where \( \mathbb{S} = D^{-1/2} W D^{-1/2} \), \( \text{vec} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{mn} \) is an operator that stacks the columns of a matrix into a column vector. Its inverse is denoted as \( \text{vec}^{-1} \).

Taking the partial derivative of \( \mathcal{J} \) with respect to \( \text{vec}(A) \), we have

\[
\frac{\partial \mathcal{J}}{\partial \text{vec}(A)} = 2 (I - S) (\text{vec}(A) + \text{vec}(Z)) - 2 \text{vec}(Z) + 2 \mu (\text{vec}(A) - \text{vec}(I)) .
\]

Setting this derivative Eq.(8) to zero, we obtain

\[
\text{vec}(A) = \frac{1}{\mu + 1} S \left( I - \frac{1}{\mu + 1} S \right)^{-1} \text{vec}(Z) + \frac{\mu}{\mu + 1} \left( I - \frac{1}{\mu + 1} S \right)^{-1} \text{vec}(I) .
\]

After setting \( \alpha = \frac{1}{\mu + 1} \) and applying \( \text{vec}^{-1} \) on both sides of Eq.(9), the closed-form solution can be obtained as

\[
\mathcal{A}^* = \text{vec}^{-1} \left( \left( I - \alpha S \right)^{-1} - I \right) \text{vec}(Z) + (1 - \alpha) \left( I - \alpha S \right)^{-1} \text{vec}(I) .
\]

Note that \((I - \alpha S)^{-1}\) is a diffusion kernel [Zhou et al., 2004]. Though we can obtain the closed-form solution, it is impractical to use due to the heavy computation of the tensor inverse. Next, we introduce an efficient iterative diffusion process which converges to the same solution as Eq.(10).

**Iterative solver**

The closed-form solution can be efficiently obtained by running the following iteration:

\[
\mathcal{A}^{(t+1)} = \alpha S (\mathcal{A}^{(t)} + Z) S^\top + (1 - \alpha) I .
\]

As shown in Eq.(11), the affinity matrix \( \mathcal{A} \) can be learned by iteratively spreading the previous affinity values with the label similarity, while continuously drawing information from the prior affinity \( I \).

Next, we prove the convergence of the iteration. By applying the operator \( \text{vec} \) on both sides of Eq.(11), we obtain

\[
\text{vec} \left( \mathcal{A}^{(t+1)} \right) = \text{vec} \left( \alpha S \mathcal{A}^{(t)} S^\top + (1 - \alpha) \text{vec}(I) \right) .
\]
As \( \text{vec}(ABC^T) = (C \otimes A) \text{vec}(B) \), Eq.(12) can be rewritten to
\[
\text{vec}(A^{(t+1)}) = \alpha S \text{vec}(A^{(t)}) + (1 - \alpha) \text{vec}(Z) + (1 - \alpha) \text{vec}(I).
\]
By running the iteration for \( t \) times, we obtain
\[
\text{vec}(A^{(t+1)}) = (\alpha S)^t \text{vec}(A^{(1)}) + \frac{t}{\alpha} \sum_{i=1}^{t} (\alpha S)^i \text{vec}(Z) + (1 - \alpha) \sum_{i=0}^{t-1} (\alpha S)^i \text{vec}(I).
\]
Since the eigenvalues of \( S \) are bounded in \([-1,1]\) and \( 0 < \alpha < 1 \), we have
\[
\lim_{t \to \infty} (\alpha S)^t = 0, \quad \lim_{t \to \infty} \sum_{i=0}^{t-1} (\alpha S)^i = (I - \alpha S)^{-1}.
\]
Hence, Eq.(14) converges to
\[
\lim_{t \to \infty} \text{vec}(A^{(t+1)}) = \alpha S(I - \alpha S)^{-1} \text{vec}(Z) + (1 - \alpha) (I - \alpha S)^{-1} \text{vec}(I).
\]
After applying \( \text{vec}^{-1} \) on both sides of Eq.(16), we obtain the solution
\[
A^* = \text{vec}^{-1}\left( (I - \alpha S)^{-1} - I \right) \text{vec}(Z)
+ (1 - \alpha) (I - \alpha S)^{-1} \text{vec}(I),
\]
which is exactly the same as Eq.(10) obtained by solving the optimization problem Eq.(6). Note that the solution is independent with the initialization of \( A \). In practice, it is initialized as \( S \) for a faster convergence speed.

3.2 Given graph \( A \), update classification \( F \)

Once a graph is constructed, the next step is to propagate label on it, where the standard label propagation methods can be directly used, such as GFHF [Zhu et al., 2003], LGC [Zhou et al., 2004], and GGMC [Wang et al., 2013]. We take LGC as an example. Given the affinity matrix \( A \) and the initial one-hot label matrix \( Y \), LGC attempts to propagate labels by solving the following optimization problem:
\[
\min_{F} \frac{1}{2} \sum_{i,j=1}^{n} A_{ij} \left\| \frac{F_i}{\sqrt{D_{ii}}} - \frac{F_j}{\sqrt{D_{jj}}} \right\|^2 + \frac{\mu}{2} \sum_{i=1}^{n} \| F_i - Y_i \|^2,
\]
where \( \mu > 0 \) is a regularization parameter, and \( D_{ii} = \sum_{j=1}^{n} A_{ij} \). Note that Eq.(18) and Eq.(6) share the same intuition. They both attempt to smooth the target function with normalized graph Laplacian while being regularized by the label information. It is not surprisingly that Eq.(18) has an equivalent iterative form:
\[
F^{(t+1)} = \alpha S_A F^{(t)} + (1 - \alpha) Y,
\]
where \( S_A = D^{-1/2} A D^{-1/2} \). Equ.(18) and Eq.(19) converge the same closed-form solution, \( F^* = (I - \alpha S_A)^{-1} Y \), where \( (I - \alpha S_A)^{-1} \) is another diffusion kernel [Zhou et al., 2004].

3.3 The complete algorithm and its variants

Until now, we have presented a widely-adopted two-stage GSSL algorithm, where the graph and label are independently optimized by the diffusion process. One of the drawbacks of these two-stage methods is that they do not fully exploit the correlation between the graph and the labels, as it has been shown by [Li et al., 2015] that the estimated labels can provide “weakly” supervised information for building a better graph and facilitate label propagation. Unlike the variants of LGC [Zhou et al., 2004] where the estimated labels are used as an re-initialization to apply another round of LGC, we feed it back a bit further to the graph construction stage to supply addition supervision for capturing the underlying manifold structure. Hence, the proposed algorithm can be seen as an alternating optimization between the affinity matrix \( A \) and the classification function \( F \). The complete algorithm is presented in Algorithm 1.

**Algorithm 1** Alternating Diffusion Process

**Input**: weight matrix \( W \in \mathbb{R}^{n \times n} \), label matrix \( Y \in \mathbb{R}^{n \times k} \)

**Parameter**: regularizer \( \alpha \), threshold \( \beta \)

**Output**: classification \( F \), affinity \( A \)

1: Normalize \( W \) symmetrically \( S \leftarrow D^{-1/2} W D^{-1/2} \)
2: Initialize \( F^{(0)} \leftarrow Y, \ A^{(0)} \leftarrow S, t \leftarrow 0 \)
3: while \( \| F^{(t+1)} - F^{(t)} \| < \beta \) do
4: Obtain \( F^{(t+1)} \) by iterating Eq.(19) until converge
5: Obtain \( A^{(t+1)} \) by iterating Eq.(11) until converge
6: end while
7: return solution

The core of the proposed ADP is that the graph and the classification are optimized alternately so that the estimated labels can be fed back to construct a better graph in order to facilitate the label propagation. Motivated by the iterative diffusion processes of these two subproblems, one possible variants of ADP is \( F \) and \( A \) can be optimized jointly by running only a single iteration for one variable before updating another, resulting the following updating strategy:
\[
\left\{ \begin{array}{l}
F^{(t+1)} = \alpha S(t) F^{(t)} + (1 - \alpha) Y \\
A^{(t+1)} = \alpha S \left( A^{(t)} + F^{(t+1)} F^{(t+1)^T} \right) S^T + (1 - \alpha) I
\end{array} \right.
\]
In this formula, instead of running two diffusion processes alternately, one diffusion process consisting of two steps in each iteration is used. It can be seen as propagating labels on a dynamic graph smoothed by the initial Gaussian weight matrix under the supervision of both given and estimated labels. We show empirically this variant, namely ADP1, also works well in the experiment.

4 Experiments

**Experimental setup.** For all GSSL methods, the graph is constructed by adaptive Gaussian sparsed by KNN, where the bandwidth \( \sigma \) is set to be the mean distance of 27 nearest neighbors and \( k \) is set to 10, unless stated otherwise. Other hyper-parameters are set according to the corresponding authors. For the proposed ADP, \( \alpha \) and \( \beta \) are set to 0.99 and 1e-2.
of all subjects. The images are cropped and resized to 32×32, and the raw pixel is used as feature.

Texture25 [Lazebnik et al., 2005] includes 1,000 texture images from 25 classes. The images are pre-processed by pre-trained VGG-net [Simonyan and Zisserman, 2014], resulting in a 4,096-dimensional feature vector per image.

MPEG7 [Latecki et al., 2000] contains 1,400 silhouette images from 70 classes. The IDSC shape descriptor [Ling and Jacobs, 2007] is used for distance calculation.

It can be seen from Figure 2 that the proposed ADP clearly outperforms other label propagation methods, given different numbers of labeled points on all three datasets. Like the observation in Table 1, ADP produces a large winning margin when the labeled points are limited. GGMC has similar effects but it does not gain much improvement when labeled points increase, as shown by [Wang et al., 2013] as well. All these methods but ADP continuously propagate labels on the initial static graph. This is problematic given a suboptimal graph, and the initial graph is usually suboptimal as few or even no label information is used to construct it. On the contrast, ADP iteratively propagates labels on a dynamic graph updated by the label information obtained in the previous iteration. It guarantees that the labels are propagated with respect to the local manifold structure and the global label information in every iteration.

Figure 2: Classification accuracy (%) of different label propagation methods with δ labeled points per class on PIE, Texture, and MPEG7 datasets.

### 4.3 Compared to other SSL methods

As shown previously ADP can achieve state-of-the-art performance in GSSL. It is also interesting to see how it compares to other non graph-based approaches, e.g., inductive methods. We select one classic inductive SSL method LapRLS [Belkin et al., 2006] as baseline and two state-of-the-art inductive SSL methods, DLSR [Luo et al., 2017] and GD [Gong et al., 2018]. Follow the setup of [Gong et al., 2018], we repeat two experiments.
5 Conclusion

In this work, we integrated the two separated stages of GSSL, graph construction and label propagation, into one unified framework. We show that learning a graph and a classification can both be formulated as a regularized function estimation problem which attempts to find a trade-off between a global fitness to the labeled points and a local smoothness with respect to the intrinsic manifold structure revealed mainly by the large portion of unlabeled points. An alternating diffusion process was also proposed to efficiently solve the function estimation problems, allowing us to update the graph and unknown labels simultaneously. It is a more plausible solution as we can fully leverage the given labels and unknown labels to construct a local-and-global consistent graph, and also propagate labels in a more effective way thanks to the dynamic graph updated iteratively. Extensive experiments have demonstrated the effectiveness of the proposed method.

Acknowledgements

This work is supported by an Australian Government Research Training Program (RTP) Scholarship.

| Dataset (δ) | KNN* | l₁-graph* | SPG* | NNLRs* | SSNNLRs* | RDP | SRD | FAML* | ADP |
|-------------|------|-----------|------|--------|----------|-----|-----|-------|-----|
| COIL20 (1)  | 84.76±1.75 | 78.01±3.56 | 77.15±5.40 | 83.46±2.62 | 85.02±3.36 | 87.00±3.54 | 89.72±2.80 | 90.54±2.14 | 98.54±1.14 | 96.49±1.71 |
| COIL20 (2)  | 91.86±1.23 | 87.07±2.32 | 88.62±4.80 | 91.20±1.89 | 92.51±2.11 | 95.00±1.32 | 96.12±1.52 | 94.20±2.18 | 99.04±0.37 | 99.08±0.55 |
| COIL20 (3)  | 93.83±1.26 | 91.11±1.73 | 91.88±3.02 | 93.60±2.21 | 93.91±2.05 | 96.81±1.04 | 97.40±1.16 | 96.80±1.41 | 99.72±0.25 | 99.44±0.99 |
| COIL20 (4)  | 94.88±1.38 | 93.50±1.03 | 93.65±2.50 | 94.16±2.00 | 94.73±1.94 | 98.55±0.37 | 99.01±0.49 | 97.35±1.07 | 99.65±0.22 | 99.55±0.28 |
| COIL20 (9)  | 96.13±0.69 | 94.46±0.88 | 94.79±2.44 | 95.32±0.76 | 95.89±1.76 | 98.41±0.49 | 98.90±0.52 | 98.34±0.66 | 99.52±0.13 | 99.58±0.37 |
| COIL20 (11)| 96.67±0.44 | 96.00±0.92 | 95.93±1.81 | 95.50±1.75 | 96.21±1.58 | 98.84±0.46 | 99.23±0.47 | 98.49±0.48 | 99.71±0.09 | 99.80±0.10 |
| Avg.        | 93.02±1.12 | 90.02±1.73 | 90.32±3.34 | 92.20±2.08 | 93.05±2.34 | 95.77±1.20 | 96.73±1.16 | 95.95±1.32 | 99.36±0.37 | 99.00±0.69 |

Table 2: Comparison with other state-of-the-art SSL methods on ORL and Wikipedia datasets. The results marked with (*) are cited from [Gong et al., 2018].

Table 1: Classification accuracy (%) of different graph construction methods with δ labeled points per class on COIL20 and YaleB datasets. The results marked with (*) are cited from [Fan et al., 2018].

References

[Bai et al., 2017] Song Bai, Xiang Bai, Qi Tian, and Long Jan Latecki. Regularized diffusion process for visual retrieval. In AAAI, pages 3967–3973, 2017.

[Belkin et al., 2006] Mikhail Belkin, Partha Niyogi, and Vikas Sindhwani. Manifold regularization: A geometric framework for learning from labeled and unlabeled examples. Journal of Machine Learning Research, 7(Nov):2399–2434, 2006.

[Donoser and Bischof, 2013] Michael Donoser and Horst Bischof. Diffusion processes for retrieval revisited. In CVPR, pages 1320–1327, 2013.

[Douze et al., 2018] Matthijs Douze, Arthur Szlam, Bharat Hariharan, and Hervé Jégou. Low-shot learning with large-scale diffusion. In CVPR, pages 3349–3358, 2018.

[Elhamifar and Vidal, 2013] Ehsan Elhamifar and René Vidal. Sparse subspace clustering: Algorithm, theory, and applications. IEEE Transactions on Pattern Analysis and Machine Intelligence, 35(11):2765–2781, 2013.

[Fan et al., 2018] Mingyu Fan, Xiaolin Zhang, Liang Du, Liang Chen, and Dacheng Tao. Semi-supervised learning through label propagation on geodesics. IEEE Transactions on Cybernetics, 48(5):1486–1499, 2018.

[Fang et al., 2018] Xiaozhao Fang, Na Han, Wai Keung Wong, Shaohua Teng, Jigang Wu, Shengli Xie, and Xuelong Li. Flexible affinity matrix learning for unsupervised and semisupervised classification. IEEE Transactions on Neural Networks and Learning Systems, (99):1–17, 2018.

[Georghiades et al., 2001] Athinodoros S. Georghiades, Peter N. Belhumeur, and David J. Kriegman. From few to Many: Illumination Cone Models for Face Recognition. IEEE Transac-
sions on Pattern Analysis and Machine Intelligence, 23(6):643–660, 2001.

[Gong et al., 2018] Chen Gong, Xiaojun Chang, Feng Fang, and Jian Yang. Teaching semi-supervised classifier via generalized distillation. In IJCAI, pages 2156–2162, 2018.

[He et al., 2011] Ran He, Wei-Shi Zheng, Bao-Gang Hu, and Xiang-Wei Kong. Nonnegative sparse coding for discriminative subspace learning. IEEE Transac-
sions on Pattern Analysis and Machine Intelligence, 33(8):1548–1560, 2011.

[Donoser and Bischof, 2013] Michael Donoser and Horst Bischof. Diffusion processes for retrieval revisited. In CVPR, pages 1320–1327, 2013.

[Douze et al., 2018] Matthijs Douze, Arthur Szlam, Bharat Hariharan, and Hervé Jégou. Low-shot learning with large-scale diffusion. In CVPR, pages 3349–3358, 2018.

[Elhamifar and Vidal, 2013] Ehsan Elhamifar and René Vidal. Sparse subspace clustering: Algorithm, theory, and applications. IEEE Transactions on Pattern Analysis and Machine Intelligence, 35(11):2765–2781, 2013.

[Fan et al., 2018] Mingyu Fan, Xiaolin Zhang, Liang Du, Liang Chen, and Dacheng Tao. Semi-supervised learning through label propagation on geodesics. IEEE Transactions on Cybernetics, 48(5):1486–1499, 2018.

[Fang et al., 2018] Xiaozhao Fang, Na Han, Wai Keung Wong, Shaohua Teng, Jigang Wu, Shengli Xie, and Xuelong Li. Flexible affinity matrix learning for unsupervised and semisupervised classification. IEEE Transactions on Neural Networks and Learning Systems, (99):1–17, 2018.

[Georghiades et al., 2001] Athinodoros S. Georghiades, Peter N. Belhumeur, and David J. Kriegman. From few to many: Illumination cone models for face recognition under variable lighting and pose. IEEE Transactions on Pattern Analysis and Machine Intelligence, 23(6):643–660, 2001.

[Gong et al., 2018] Chen Gong, Xiaojun Chang, Feng Fang, and Jian Yang. Teaching semi-supervised classifier via generalized distillation. In IJCAI, pages 2156–2162, 2018.

[He et al., 2011] Ran He, Wei-Shi Zheng, Bao-Gang Hu, and Xiang-Wei Kong. Nonnegative sparse coding for discriminative subspace learning. IEEE Transactions on Pattern Analysis and Machine Intelligence, 33(8):1548–1560, 2011.
inative semi-supervised learning. In CVPR, pages 2849–2856, 2011.

[Ischen et al., 2017] Ahmet Ischen, Giorgos Tolias, Yannis Avrithis, Teddy Furon, and Ondrej Chum. Efficient diffusion on region manifolds: Recovering small objects with compact cnn representations. In CVPR, pages 2077–2086, 2017.

[Latecki et al., 2000] Longin Jan Latecki, Rolf Lakamper, and T Eckhardt. Shape descriptors for non-rigid shapes with a single closed contour. In CVPR, volume 1, pages 424–429, 2000.

[Lazebnik et al., 2005] Svetlana Lazebnik, Cordelia Schmid, and Jean Ponce. A sparse texture representation using local affine regions. IEEE Transactions on Pattern Analysis and Machine Intelligence, 27(8):1265–1278, 2005.

[Li et al., 2015] Chun-Guang Li, Zhouchen Lin, Honggang Zhang, and Jun Guo. Learning semi-supervised representation towards a unified optimization framework for semi-supervised learning. In ICCV, pages 2767–2775, 2015.

[Li et al., 2018] Qilin Li, Wanquan Liu, and Ling Li. Affinity learning via a diffusion process for subspace clustering. Pattern Recognition, 84:39–50, 2018.

[Li et al., 2019] Qilin Li, Wanquan Liu, and Ling Li. Self-reinforced diffusion for graph-based semi-supervised learning. Manuscript submitted for publication, 2019.

[Ling and Jacobs, 2007] Haibin Ling and David W Jacobs. Shape classification using the inner-distance. IEEE Transactions on Pattern Analysis and Machine Intelligence, 29(2):286–299, 2007.

[Liu et al., 2013] Guangcan Liu, Zhouchen Lin, Shuicheng Yan, Ju Sun, Yong Yu, and Yi Ma. Robust recovery of subspace structures by low-rank representation. IEEE Transactions on Pattern Analysis and Machine Intelligence, 35(1):171–184, 2013.

[Lu et al., 2014] Song Lu, Vijay Mahadevan, and Nuno Vasconcelos. Learning optimal seeds for diffusion-based salient object detection. In CVPR, pages 2790–2797, 2014.

[Luo et al., 2017] Minnan Luo, Lingling Zhang, Feiping Nie, Xiaojun Chang, Buyue Qian, and Qinghua Zheng. Adaptive semi-supervised learning with discriminative least squares regression. In IJCAI, pages 2421–2427. AAAI Press, 2017.

[Nene et al., 1996] Sameer A Nene, Shree K Nayar, Hiroshi Murase, et al. Columbia object image library (COIL-20). Technical report, 1996.

[Page et al., 1999] Lawrence Page, Sergey Brin, Rajeev Motwani, and Terry Winograd. The pagerank citation ranking: Bringing order to the web. Technical report, Stanford InfoLab, 1999.

[Roweis and Saul, 2000] Sam T Roweis and Lawrence K Saul. Non-linear dimensionality reduction by locally linear embedding. Science, 290(5500):2323–2326, 2000.

[Sim et al., 2002] Terence Sim, Simon Baker, and Maan Bsate. The cmu pose, illumination, and expression (PIE) database. In IEEE Conference on Automatic Face and Gesture Recognition, pages 53–58, 2002.

[Simonyan and Zisserman, 2014] Karen Simonyan and Andrew Zisserman. Very deep convolutional networks for large-scale image recognition. arXiv preprint arXiv:1409.1556, 2014.

[Wang et al., 2013] Jun Wang, Tony Jebara, and Shih-Fu Chang. Semi-supervised learning using greedy max-cut. Journal of Machine Learning Research, 14(Mar):771–800, 2013.

[Yang et al., 2013] Xingwei Yang, Lakshman Prasad, and Longin Jan Latecki. Affinity learning with diffusion on tensor product graph. IEEE Transactions on Pattern Analysis and Machine Intelligence, 35(1):28–38, 2013.

[Zhang et al., 2018] Rui Zhang, Feiping Nie, and Xuelong Li. Semi-supervised learning with parameter-free similarity of label and side information. IEEE Transactions on Neural Networks and Learning Systems, (99):1–10, 2018.

[Zhou et al., 2004] Dengyong Zhou, Olivier Bousquet, Thomas N Lal, Jason Weston, and Bernhard Schölkopf. Learning with local and global consistency. In NIPS, pages 321–328, 2004.

[Zhu and Ghahramani, 2002] Xiaojin Zhu and Zoubin Ghahramani. Learning from labeled and unlabeled data with label propagation. 2002.

[Zhu et al., 2003] Xiaojin Zhu, Zoubin Ghahramani, and John D Lafferty. Semi-supervised learning using gaussian fields and harmonic functions. In ICML, pages 912–919, 2003.

[Zhuang et al., 2015] Liansheng Zhuang, Shenghua Gao, Jinhui Tang, Jingjing Wang, Zhouchen Lin, Yi Ma, and Nenghai Yu. Constructing a nonnegative low-rank and sparse graph with data-adaptive features. IEEE Transactions on Image Processing, 24(11):3717–3728, 2015.

[Zhuang et al., 2017] Liansheng Zhuang, Zihan Zhou, Shenghua Gao, Jingwen Yin, Zhouchen Lin, and Yi Ma. Label information guided graph construction for semi-supervised learning. IEEE Transactions on Image Processing, 26(9):4182–4192, 2017.