AKSZ CONSTRUCTION OF TOPOLOGICAL OPEN P-BRANE ACTION AND NAMBU BRACKETS

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Abstract. We review the AKSZ construction as applied to the topological open membranes and Poisson sigma models. We describe a generalization to open topological p-branes and Nambu-Poisson sigma models.

1. Introduction

The purpose of this paper is two-fold. First, we review the Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) formulation of topological open membranes and Poisson sigma models, following [41], [25] and [11]. Second, we propose a generalization to the case of topological open p-branes and Nambu-Poisson sigma models.

The Poisson sigma model, introduced in [50] and [28], and its twisted version introduced in [35], play an important role in modern mathematical physics. The most striking application of the open Poisson sigma model is the path integral derivation of Kontsevich’s formality, in particular of his celebrated star product [36], in [9]. For this, it was necessary to use BV formalism. Within this formalism Kontsevich’s formality appears as a consequence of the Ward identities for the BV quantized open Poisson sigma model. Another example of the use of Poisson sigma models is the integration of Poisson manifolds to symplectic groupoids in [10].

The AKSZ formalism [2] is a geometric formalization of the BV formalism. It leads to a powerful method for constructing BV actions starting from geometric data, the super worldvolume and the superspace. The super worldvolume is a differential graded manifold, equipped with a measure invariant under the cohomological vector field and the super spacetime a differential graded symplectic manifold, for which the cohomological vector filed is a Hamiltonian one. The AKSZ formalism is a prescription how to construct, starting with these data, a solution to master equation on the space of the corresponding superfields. Examples of the construction comprise the BF model in any dimension, the 2 dimensional A- and B-model, 3 dimensional
Chern-Simons [2], [11] and 3 dimensional Rozansky-Witten theory [43]. Further applications of the AKSZ construction can be found, e.g., in [30], [31] and [16]. The AKSZ formulation of the Poisson sigma model was given in [11]. Papers [41] and [25] describe an AKSZ formulation of open topological membranes in the presence of a closed 3-form and a twisted Poisson tensor, which leads on shell to the twisted Poisson sigma model. The BV quantization of such topological membranes is described in [26]. In [25] the authors also describe the relation of topological open membranes to Lie and Courant algebroids and Dirac structures, cf. also [45], [29], for some related results about 3 dimensional Courant sigma model.

Nambu-Poisson structures are the most natural generalizations of Poisson structures. The original definition of the Nambu bracket of order 3 goes back to the Nambu’s seminal paper [39]. The generalization and the modern geometric formulation is due to Takhtajan [54]. An order $p$ Nambu-Poisson bracket on a manifold $M$ determines the structure of a Filippov $p$-algebra on $C^\infty(M)$ [17]. The recent interest in such generalized structures was motivated by the Bagger-Lambert-Gustavsson model for M2-branes [6], [4], [5], [21]. More recently, the relevance of Nambu-Poisson brackets of order 3 for the description of M5-branes was noticed in [24].

From the above discussion, it seems to be natural (and hopefully useful) to search for a proper generalization of the above mentioned works on topological open membranes and Poisson sigma models to the case of topological open $p$-branes and Nambu-Poisson sigma models. We also hope that the topological open $p$-branes can be quantized using the BV quantization scheme, similarly to topological open membranes [26]. This could possibly lead to a kind of deformation quantization of Nambu-Poisson structures.

This paper is organized as follows. In Section 2 we summarize, for the reader’s convenience, the relevant facts regarding the Dorfman (Courant) bracket on sections of $TM \oplus \wedge^{p-1}T^*M$. The discussion includes the twistings of a $p$ Dorfman bracket by a $p+1$-form and a $p$-vector field, which leads to a generalization of the so called Roytenberg bracket. Most of the material of this section can be found in [22] and [8].

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1For deformation quantization of Nambu-Poisson structures see [13]. Interested reader can consult, e.g., the review [3] for a short discussion of other approaches to the quantization of Nambu-Poisson structures.
In Section 3, we recall the basic facts concerning Nambu-Poisson structures. In particular, we describe different, but equivalent, forms of the fundamental identity (FI) and recall the relation between the Dorfman and Poisson-Nambu brackets following, e.g. [22] and [8]. We discuss the possibility of twisting a Nambu-Poisson bracket of order $p$ with a closed $p + 1$ form, which, as it turns out, is not possible for $p \geq 3$ (this is probably known to the experts, but we could not find it anywhere in the existing literature). Further, we describe gauge equivalence of Nambu-Poisson structures and relate it to the higher (semiclassical) version of the Seiberg-Witten map, well-known from non-commutative gauge theory. Our discussion of the Seiberg-Witten map is a generalization of the low dimensional cases $p = 2$ in [33, 34], and $p = 3$ in [14].

In Section 4, we collect the relevant material about differential graded (dg) symplectic manifolds, which are the natural framework for the Batalin-Vilkovisky formalism. Among various relevant examples, we describe twisted Dorfman brackets within the framework of dg symplectic manifolds. Here we closely follow the review [12], which is based on [45].

The AKSZ construction of BV actions is reviewed in Section 5 following, with small modifications, [12] and [25]. As already mentioned above, the AKSZ formalism [2] provides for a solution of the classical master equation on the space of maps between a super worldvolume and a super spacetime under rather mild assumptions. Since we are interested in open $p$-branes, we describe in detail the effect of canonical transformations on the boundary conditions and the boundary terms too. This is an important point in understanding the construction of actions described in this paper.

In Section 6, we provide examples of the AKSZ construction. The first one is the Poisson sigma model, following [11], the second one is the open topological membrane and the twisted Poisson sigma model, following [41] and [25]. The open topological membrane action has a WZW-type bulk term originating from a closed 3-form $c$ and a boundary term originating from a Poisson tensor $\pi$ twisted by $c$. There is a gauge symmetry which changes the 3-form $c$ by an exact piece and the twisted Poisson structure to an equivalent one, hence relating the bulk and boundary interactions. This gauge symmetry is closely related to the semiclassical Seiberg-Witten map.
The twisted Poisson sigma model is obtained from the open topological membrane on shell.

The open topological membrane and the twisted Poisson model are generalized to an open topological $p$-brane and a Nambu-Poisson sigma model (recall, there the twisting of a Nambu-Poisson structure doesn’t work for $p > 2$) in Section 7. Again, the construction is based on AKSZ. However, the generalization is not a straightforward one. For instance, it does include the case $p = 2$ in a non-trivial way. Namely, for $p = 2$, we have twice as many fields compared to the action of the topological open membrane and the latter is obtained only after imposing further constrains on the fields. Nevertheless, also for general $p$, the resulting topological open $p$-brane action contains a $p + 1$-form $c$ coupled to the bulk through the WZW term and a Nambu-Poisson tensor coupled to the boundary. Also here, we have a gauge symmetry which changes the $p + 1$-form $c$ by an exact piece and Nambu-Poisson structure to a gauge equivalent one (in the sense of Section 3), hence relating the bulk and boundary interactions. This gauge symmetry is closely related to the higher semiclassical Seiberg-Witten map as described in Section 3. On shell we have a generalization of the Poisson sigma model, which we call the Nambu-Poisson sigma model. This model is also obtained quite naturally as the topological limit of models arising in the study of p-brane actions with background fields [32]. Further generalizations, properties and applications of the Nambu-Poisson sigma model will be discussed in this forthcoming paper.

2. Dorfman brackets

Let $M$ be a smooth finite-dimensional manifold, and let $E$ denote a vector bundle over $M$. The set of sections of $E$ will be denoted by $\Gamma E$. In particular we have the tangent bundle $TM$, whose sections $\mathfrak{X}(M) = \Gamma(TM)$ are vector fields, and the cotangent bundle $T^*M$ whose sections $\Gamma(T^*M)$ are 1-forms. We also denote by $\mathfrak{X}^p(M) = \Gamma(\wedge^p TM)$, and $\Omega^p(M) = \Gamma(\wedge^p T^*M)$ the set of $p$-vector fields and $p$-forms, respectively.

In this section we collect some basic facts concerning the Dorfman bracket on $\Gamma(TM \oplus \wedge^{p-1} T^*M) = \mathfrak{X}(M) \oplus \Omega^{p-1}(M)$. We have the following definition [22] (see
also [23, 20, 8, 56]. Everything in this section, with an exception of maybe Remark 2.6 and Proposition 2.7, can be found, e.g., in [8].

**Definition 2.1.** The *Dorfman bracket* of order $p \geq 2$ on sections of $E = TM \oplus \wedge^{p-1}T^*M$ is defined by

$$[(X, \alpha), (Y, \beta)] = ([X, Y], \mathcal{L}_X \beta - \iota_Y d\alpha),$$

where $X, Y \in \mathfrak{X}(M)$, and $\alpha, \beta \in \Omega^{p-1}(M)$.

Let $\rho$ denote the projection $\rho : E \to TM$, the so-called *anchor map*, and let $\langle \cdot, \cdot \rangle$ denote the $\Omega^{p-2}(M)$-valued non-degenerate bilinear pairing between $\mathfrak{X}(M)$ and $\Omega^{p-1}(M)$ given by

$$\langle (X, \alpha), (Y, \beta) \rangle = \frac{1}{2}(\iota_X \beta + \iota_Y \alpha).$$

The important properties of the Dorfman bracket $[\cdot, \cdot]$ on $\Gamma E = \Gamma(TM \oplus \wedge^{p-1}T^*M)$ are summarized in the following theorem.

**Theorem 2.2.** For $e_1, e_2, e_3 \in \Gamma E$ and $f \in C^\infty(M)$

$$\rho[e_1, e_2] = [\rho e_1, \rho e_2],$$

$$[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2,$$

and the bracket is a Leibniz bracket, i.e. the Jacobi identity

$$[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]],$$

is satisfied. The pairing and the bracket are compatible, i.e.,

$$\mathcal{L}_{\rho(e_1)}\langle e_2, e_3 \rangle = \langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle.$$

Let $c$ be a $(p+1)$-form on $M$. Then the Dorfman bracket $[\cdot, \cdot]$ can be twisted by $c$.

**Definition 2.3.** The twisted Dorfman bracket is defined as

$$[(X, \alpha), (Y, \beta)]_c := ([X, Y], \mathcal{L}_X \beta - \iota_Y d\alpha + \iota_X \wedge \iota_Y c).$$

We have the following proposition

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2Let us note, that almost everything in this and the subsequent sections can be formulated more generally, by replacing the tangent bundle $TM$ and cotangent bundle $T^*M$ by a Lie algebroid $A$ over $M$ and its dual $A^*$, respectively [57].

3We will use the same notation, $\rho$, for the mapping induced on the sections, $\rho : \Gamma(E) \to \Gamma(TM)$. 
Proposition 2.4. The twisted Dorfman bracket is a Leibniz bracket iff $c$ is closed; $dc = 0$. Further, for any $b \in \Omega^p(M)$ define $e^b : E \to E$ as

$$e^b(X, \alpha) := (X, \alpha + \iota_X b).$$

We have

$$e^b[e_1, e_2]_{c+db} = [e^b e_1, e^b e_2]_c.$$

Remark 2.5. The Dorfman bracket (twisted or not) is not antisymmetric; its antisymmetrization is called the (twisted) Courant bracket. However, the Courant bracket (twisted or not) does not obey the Jacobi identity. The triple $(E, \cdot, \cdot, \rho)$ has the structure of a Leibniz algebroid (properties (2.1)-(2.3)) \cite{22, 8}. Moreover, taking into account the differential $d : \Omega^{p-2}(M) \to \Omega^{p-1}(M) \oplus \mathfrak{X}(M)$, defined by $d : \alpha \mapsto (d\alpha, 0)$, the quadruple $(E, \cdot, \cdot, \langle \cdot, \cdot \rangle, d)$ has a structure of a (Courant-) Dorfman algebra \cite{17, 15}.

Remark 2.6. The Dorfman bracket can also be twisted by a $p$ multi-vector field $\zeta \in \mathfrak{X}^p(M)$. Define $e^\zeta : E \to E$ as

$$e^\zeta(X, \alpha) := (X + \zeta^\sharp \alpha, \alpha),$$

and introduce the $(c, \zeta)$-twisted Dorfman bracket

$$e^\zeta[e_1, e_2]_{(c, \zeta)} = [e^\zeta e_1, e^\zeta e_2]_c.$$

In the case of $p = 2$, it is due to Roytenberg \cite{41}.

Proposition 2.7. Assume that the map $(1 + b_2 \circ \zeta^\sharp) : \Omega^{p-1}(M) \to \Omega^{p-1}(M)$ is invertible and denote $\zeta'^\sharp = \zeta^\sharp \circ (1 + b_2 \circ \zeta^\sharp)^{-1}$. We have

$$e^b e^\zeta = e^{\zeta'} e^{(b, \zeta)},$$

where $e^{(b, \zeta)}(X, \alpha) = ((1 - \zeta'^\sharp \circ b_2)(X), (1 + b_2 \circ \zeta^\sharp)(\alpha) + b_2(X)).$

\footnote{Henceforth, for any $\zeta \in \mathfrak{X}^p(M)$, and $b \in \Omega^p(M)$, we define maps $\zeta^\sharp : \Omega^{p-1}(M) \to \mathfrak{X}(M)$, and $b_\sharp : \mathfrak{X}(M) \to \Omega^{p-1}(M)$ by $\zeta^\sharp(\alpha) = \iota_\alpha \zeta$, and $b_\sharp(X) = \iota_X b$, for all $\alpha \in \Omega^{p-1}(M)$, and $X \in \mathfrak{X}(M)$, respectively.}
3. Nambu-Poisson structures

The original definition of the Nambu bracket of order 3 goes back to the Nambu’s seminal paper [39]. The generalization and the modern geometric formulation is due to Takhtajan [54] (for a review on n-ary algebras, see [3]).

Definition 3.1. A Nambu-Poisson bracket of order \( p \geq 2 \) on \( M \) is a skew-symmetric \( p \)-linear map \( \{\cdot, \ldots, \cdot\} : C^\infty(M) \times \ldots \times C^\infty(M) \rightarrow C^\infty(M) \) having the Leibniz property

\[
\{fg, f_2, \ldots, f_p\} = f \{g, f_2, \ldots, f_p\} + g \{f, f_2, \ldots, f_p\},
\]

and satisfying the so called fundamental identity (FI)

\[
\{f_1, \ldots, f_{p-1}, \{g_1, \ldots, g_p\}\} = \sum_{i=1}^{p} \{g_1, \ldots, \{f_1, \ldots, g_i, \ldots, f_{p-1}\}, \ldots, g_p\}. \tag{3.1}
\]

In terms of the corresponding \( p \)-vector field \( \pi \in \mathfrak{X}^p(M) \), defined by

\[
\pi(df_1 \wedge \ldots \wedge df_p) := \{f_1, \ldots, f_p\},
\]

the fundamental identity can be expressed as in the following proposition, which can be read off from, e.g., [22], [8] (cf. also [27]).

Proposition 3.2. A Nambu-Poisson bracket of order \( p \) on \( M \) is uniquely determined by a multi-vector field \( \pi \) of order \( p \), satisfying either of the following:

(i) for all \( \alpha, \beta \in \Omega^{p-1}(M) \)

\[
[\pi^\sharp \alpha, \pi^\sharp \beta]_S = (\mathcal{L}_{\pi^\sharp \alpha} \pi)^\sharp \beta = -\pi^\sharp (\iota_{\pi^\sharp \beta} d\alpha).
\]

where \([\cdot, \cdot]_S\) denotes the Schouten bracket of multi-vector fields.

(ii) for \( f_1, \ldots, f_{p-1} \in C^\infty(M) \),

\[
[\pi^\sharp (df_1 \wedge \ldots \wedge df_{p-1}), \pi]_S = \mathcal{L}_{\pi^\sharp (df_1 \wedge \ldots \wedge df_{p-1})} \pi = 0,
\]

(iii) for \( (p-1) \)-forms \( \alpha \) and \( \beta \),

\[
[\pi^\sharp \alpha, \pi^\sharp \beta] = \pi^\sharp [\alpha, \beta]_\pi, \tag{3.2}
\]

where

\[
[\alpha, \beta]_\pi := \mathcal{L}_{\pi^\sharp \alpha} \beta - \iota_{\pi^\sharp \beta} d\alpha.
\]

\(^5\)Equivalently, as an equality of maps from \( \Omega^{p-1}(M) \) to \( \mathfrak{X}(M) \) in the form \([\pi^\sharp \alpha, \pi]_S^\sharp = (\mathcal{L}_{\pi^\sharp \alpha} \pi)^\sharp = -\pi^\sharp \circ (d\alpha) \circ \pi^\sharp\).
Corollary 3.3. In some local coordinates $x^i$ on $M$, the fundamental identity gives two conditions, an algebraic and a differential one. The algebraic one is

$$
\Sigma_{i_1...i_p} \pi_{j_1...j_p} = \Sigma_{j_1...j_p} \pi_{i_1...i_p},
$$

where

$$
\Sigma_{i_1...i_p} = \frac{1}{p!} \epsilon_{i_1...i_p} \epsilon_{j_1...j_p} \pi_{l_1...l_{p-1}} \pi_{l_2...l_{p+1}},
$$

and the differential one is

$$
\pi^{i_1...i_p-1} \partial_k \pi_{j_1...j_p} = \frac{1}{(p-1)!} \epsilon_{j_1...j_p} \pi^{l_1...l_{p-1}} \partial_k (\pi_{i_1...i_p-1} l). (3.4)
$$

Remark 3.4. The algebraic condition (3.3) assures that the second order derivative terms in the fundamental identity (3.1) vanish, which is a nontrivial statement for $p \geq 3$. This condition is equivalent, for $p \geq 3$, to the decomposability of the Nambu-Poisson tensor [18], [1], (cf. also [40], [37], [38]). More precisely: Let us fix a point $x \in M$ for which $\pi(x) \neq 0$, then, locally around $x$, $\pi = v_1 \wedge ... \wedge v_p$, with some local vector fields $v_1, ..., v_p$. Let us also note that any $p$-vector field of such a form trivially fulfills the fundamental identity. If $V_x \subset T_x M$ denotes the $p$-dimensional subspace generated at the point $x \in M$ by $(v_1, ..., v_p)$, then the fundamental identity in the form (3.2) assures that the field of subspaces $V_x$ is integrable.

Corollary 3.5. Due to the decomposability, for $p \geq 3$, the fundamental identity in the form of footnote 5 can be rewritten as

$$
[\pi^{i_1...i_p} \pi, \pi^{j_1...j_p}]_{S^2} = -\pi^{i_1...i_p} (d\alpha)_{i_1} \pi^{j_1...j_p} = (-1)^p (t_{\alpha, \pi}) \pi^{i_1...i_p}.
$$

The following characterization of a decomposability of a $p$-vector, $p \geq 3$, will be useful later.

Lemma 3.6. A $p$-vector $\pi$ is decomposable iff

$$
\pi^{[i_1...i_p] j_1...j_p} = 0,
$$

where the square brackets denote antisymmetrization.

The relation between the Dorfman bracket on $\Gamma(TM \oplus \wedge^{p-1}T^*M)$ and Nambu-Poisson structures is as follows, see, e.g., [22], [8].

6In [18], an observation that the decomposability follows from the so-called Weitzenböck condition is attributed to L. Takhtajan.

7See [22], where also the equivalence to the Weitzenböck condition is shown.
Theorem 3.7. Let $\pi$ be a $p$-vector field. Its graph, $\text{graph}(\pi) = \{(\pi^\sharp \alpha, \alpha), \ \alpha \in \Omega^{p-1}(M)\} \subset \mathfrak{X}(M) \oplus \Omega^{p-1}(M)$, is closed under the Dorfman bracket iff $\pi$ is a Nambu-Poisson vector of order $p$.

Remark 3.8. Let $c$ be a closed $(p+1)$-form. One can try to introduce a Nambu-Poisson structure twisted by $c$, in analogy with the twisted Poisson bracket for $p = 2$.

Again, let $\pi$ be a $p$-vector field and let us determine when its graph, $\text{graph}(\pi) \subset \mathfrak{X}(M) \oplus \Omega^{p-1}(M)$, is closed under the twisted Dorfman bracket. We will find, similarly to (3.2), the following (necessary and sufficient) condition. For $(p-1)$-forms $\alpha$ and $\beta$,

$$[\pi^\sharp \alpha, \pi^\sharp \beta] = \pi^\sharp [\alpha, \beta]_{\pi,c},$$

where now

$$[\alpha, \beta]_{\pi,c} := \mathcal{L}_{\pi^\sharp \alpha} \beta - \iota_{\pi^\sharp \beta} d\alpha + \iota_{\pi^\sharp \alpha} \wedge \pi^\sharp \beta c.$$

Equivalently, we have for $f_1, \ldots, f_{p-1}, g_1, \ldots, g_p \in C^\infty(M)$

$$\{f_1, \ldots, f_{p-1}, \{g_1, \ldots, g_p\}\} = \sum_{i=1}^p \{f_1, \ldots, g_i, \ldots, f_{p-1}\}, \ldots, g_p\} + c(X_{f_1,\ldots,f_{p-1}} \wedge X_{g_1} \wedge \ldots \wedge X_{g_{p-1}} \wedge X_{g_p}), \quad (3.6)$$

where $X_{f_1,\ldots,f_{p-1}} \in \mathfrak{X}(M)$ denotes the Hamiltonian vector field associated to functions $f_1, \ldots, f_{p-1} \in C^\infty(M)$, defined by

$$X_{f_1,\ldots,f_{p-1}}(h) = X_h(f_1, \ldots, f_{p-1}) := \{f_1, \ldots, f_{p-1}, h\}.$$

Now, following Remark 3.4. In some local coordinates $x^i$ on $M$, the twisted fundamental identity again gives two conditions, an algebraic and a differential one. The algebraic one is identical to (3.3) and the differential one (3.4) gets a contribution coming from the closed $p+1$-form $c$, which is proportional to $c_{i_1, \ldots, i_{p+1}} \pi^{i_1 \ldots i_{p-1} i_{p} \pi^{i_{p+1} i_{p+1} \ldots i_{p+1}}} \pi^{j_1 \ldots j_{p-1} j_{p} \pi^{i_{p+1} i_{p+1} \ldots i_{p+1}}} \pi^{j_1 \ldots j_{p+1} j_{p+1}}$.

Since, for $p \geq 3$, the algebraic condition is the same as in the untwisted case, it is again equivalent to the decomposability of the tensor $\pi$. Let us also note that for any $p$-vector field of such a form, the above mentioned contribution to the differential condition $c_{i_1, \ldots, i_{p+1}} \pi^{i_1 \ldots i_{p-1} i_{p} \pi^{j_1 \ldots j_{p-1} j_{p} \pi^{i_{p+1} i_{p+1} \ldots i_{p+1}}} \pi^{j_1 \ldots j_{p+1} j_{p+1}}$ vanishes identically. Hence, $\pi$ also fulfills the untwisted fundamental identity. We can conclude that for a $p$-vector field, $p \geq 3$, from the twisted Dorfman bracket we only get an “ordinary” Nambu-Poisson
tensor. For \( p = 2 \), however, we get a twisted Poisson tensor. We have the following corollary.

**Corollary 3.9.** Let \( \pi \) be a \( p \)-tensor and \( b \) a \( p \)-form. Then \( e^b(\text{graph}(\pi)) \) corresponds to a graph of a \( p \)-vector \( \pi^b \) iff \( (1 + b \circ \pi^z) \) is invertible on \( \Omega^{p-1}(M) \). On \( \Omega^{p-1}(M) \), we have

\[
\pi^b = \pi^z \circ (1 + b \circ \pi^z)^{-1}.
\]

For \( p = 2 \), if \( \pi \) is a Poisson bracket twisted by \( c \), then \( \pi^b \) is a Poisson tensor twisted by \( c - db \). If \( \pi \) is a Nambu-Poisson tensor, for \( p \geq 3 \), due to the decomposability, we have

\[
\pi^b = (1 + (-1)^{p-1}b(\pi))^{-1} \pi,
\]

in which case \( \pi^b \) is again a Nambu-Poisson tensor. We say that \( \pi^b \) and \( \pi \) are gauge equivalent.

**Remark 3.10.** Let us note that, if it makes sense, \( \pi^b_t := \pi^z_t \circ (1 + tb \circ \pi^z_t)\) is a solution to the differential equation

\[
\dot{\pi}^z_t = -\pi^z_t \circ b \circ \pi^z_t,
\]

interpolating between \( \pi \) and \( \pi^b \).

**Seiberg-Witten map.** In case of an exact \( b, b = da \), we have the so called Seiberg-Witten map (see, e.g., [33], [34] for the case of a Poisson structure and [14] for the case \( p = 3 \)). The Seiberg-Witten map is a (formal) diffeomorphism relating the Nambu-Poisson tensors \( \pi \) and \( \pi^{da} \). We have the following general proposition and its corollary, due to the decomposability, valid for \( p \geq 3 \).

**Proposition 3.11.** Suppose that two Nambu-Poisson \( p \)-tensors \( \pi \) and \( \pi^{da} \) on \( M \) are gauge equivalent, the gauge equivalence being given by an exact \( p \)-form \( b = da \), such that, for \( t \in [0, 1] \), the map \( (1 + tb \circ \pi^z) \) is invertible on \( \Omega^{p-1}(M) \) and the vector field \( \pi^t_a \), where \( \pi^t \) is defined by \( \dot{\pi}^z_t = -\pi^z_t \circ b_t \circ \pi^z_t \), \( \pi^z_0 = \pi^z \), is complete. Then there exists a Nambu-Poisson map relating \( \pi \) and \( \pi^{da} \).

**Proof.** Let us consider a one-parameter family of \( p \)-tensors \( \pi_t \) defined by

\[
\pi^z_t = \pi^z \circ (1 + tb \circ \pi^z)^{-1}.
\]

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\(^8\)Recall that, due to the decomposability, multiplying a Nambu-Poisson tensor by a smooth function gives again a Nambu-Poisson tensor.
We have $\pi_0 = \pi$ and $\pi_1 = \pi^{da}$. Moreover, it is straightforward to check that
\[
\dot{\pi}_t^z = -\pi_t^z \circ b_z \circ \pi_t^z = (\mathcal{L}_{\pi_t^z a} \pi_t^z)^z.
\]
Denote the corresponding flow by $\phi_t$. The map $\phi := \phi_1$ is the sought Nambu-Poisson map. \hfill $\square$

**Corollary 3.12.** Suppose that two Nambu-Poisson $p$-tensors $\pi$ and $\pi^{da}$, $p \geq 3$ on $M$ are gauge equivalent, the gauge equivalence being given by an exact $p$-form $b = da$, such that, for $t \in [0, 1]$, the function $(1 + (-1)^{p-1}tb(\pi))$ is invertible and the vector field $\pi_t^z a$, where $\pi_t^z$ is defined by $\dot{\pi}_t^z = (-1)^p b(\pi_t)\pi_t^z$, $\pi_0^z = \pi^z$, is complete. Then there exists a Nambu-Poisson map relating $\pi$ and $\pi^{da}$.

**Remark 3.13.** Although the Nambu-Poisson tensor $\pi_t$ does not depend on the choice of the primitive $a$, the flow $\phi_t$ does. In order to indicate the dependence of the flow $\phi_t$ on $a$ explicitly, we will use the notation $\phi_t^a$ for it.

**Proposition 3.14.** For a $(p - 2)$-form $\lambda$, the flow $\phi_t^{\lambda + d\lambda} (\phi_t^a)^{-1}$ is generated by a Hamiltonian vector field $X_{\lambda,a}$, i.e., there exists a $(p - 2)$-form $\mu_{\lambda,a}$ such that $X_{\lambda,a} = \pi^z d\mu_{\lambda,a}$.

The explicit formal series formula for $\mu_{\lambda,a}$ can be worked out using the BCH formula. Up to the first order in $\lambda$ we have
\[
\mu_{\lambda,a} = \sum_k \left. \frac{(-\mathcal{L}_{\pi_t^z a} + \partial_t)^k(\lambda)}{(k + 1)!} \right|_{t=0} + o(\lambda^2). \tag{3.7}
\]

Applications of the SW map will be discussed in [32].

4. **Differential graded symplectic manifolds**

Here we closely follow [12]. Another nice discussion of the relevant material can be found in [45], on which [12] is based, and in the original paper on the AKSZ formalism [2].

**Definition 4.1.** A differential graded (dg-) manifold $M$ is a graded manifold equipped with a cohomological vector field $Q$, i.e., a graded vector field of degree $+1$ such that $Q^2 = 0$. 
**Example 4.2.** The basic example of a dg-manifold is $T[1]\Sigma$, where $\Sigma$ is an ordinary manifold. The algebra of smooth functions on $X = T[1]\Sigma$ is isomorphic to the algebra $(\Omega(\Sigma), \wedge)$ of differential forms. For the cohomological vector field on $X = T[1]\Sigma$ we take the vector field $Q_X$ corresponding, under this isomorphism, to the de Rham differential $d$ on $(\Omega(\Sigma), \wedge)$. If we choose some local coordinates $x^\mu$ on $\Sigma$ and denote the corresponding induced odd coordinates on the fibre of $X$ by $\theta^\mu$, we will have $Q_X = \theta^\mu \partial_\mu$.

**Definition 4.3.** A symplectic form $\omega$ of degree $k$ on a graded manifold $M$ is a closed, non-degenerate 2-form, which is homogeneous of degree $k$. The corresponding graded Poisson bracket $\{\cdot, \cdot\}$ is of degree $-k$. It is defined similarly to the non-graded case by $\{f, g\} := X_f g$, where $\iota_{X_f} \omega = df$, i.e., $\{f, g\} = \iota_{X_f} \iota_{X_g} \omega = \omega^{-1}(df, dg)$.

**Example 4.4.** For $V$ a smooth manifold we take $T^*[1]V$. The canonical symplectic structure $\omega$ on $T^*[1]V$ is of degree $|\omega| = 1$. We will denote the degree 0 local coordinates on $V$ as $X^i$ and the induced degree 1 fibre coordinates on $T^*[1]V$ by $\chi_i$. The canonical symplectic form in these coordinates is $\omega = d\chi_i \wedge dX^i$. The potential one-form $\vartheta$, such that $\omega = d\vartheta$, can be taken as $\vartheta = \chi_i dX^i$.

**Example 4.5.** Let $V$ be a smooth manifold. Consider $T^*[p]T[1]V$, with $p$ an integer, $p \geq 2$. The canonical symplectic structure $\omega$ on $T^*[p]T[1]V$ is of degree $p$. We will denote the degree 0 local coordinates on $V$ as $X^i$ and the induced degree 1 fibre coordinates on $T[1]V$ by $\psi^j$. Dual fibre coordinates on $T^*[p]T[1]V \rightarrow T[1]M$, of respective degrees $p$ and $p-1$, will be denoted by $F_i$ and $\chi_i$. The canonical symplectic form in these coordinates is $\omega = dF_i \wedge dX^i + d\psi^j \wedge d\chi_i$. The potential one-form $\vartheta$ can be taken as $\vartheta = F_i dX^i + \psi^i d\chi_i$.

**Example 4.6.** Again, let $V$ be a smooth manifold. Consider $T^*[p](\Lambda^{p-1})(T[1]V)$, with $p$ an integer, $p \geq 2$. The canonical symplectic structure $\omega$ on $T^*[p](\Lambda^{p-1})(T[1]V)$ is of degree $p$. We will denote the degree 0 local coordinates on $V$ as $X^i$, the induced degree 1 fibre coordinates on $T[1]V \rightarrow V$ by $\psi^j$, the induced degree $p-2$ and degree $p-1$ fibre coordinates on $(\Lambda^{p-1} T)[p-1](T[1]V) \rightarrow T[1]V$ as $H^I := H^{i_1 \ldots i_{p-1}}$, with $i_1 < \ldots < i_{p-1}$ and $\eta^I := \eta^{i_1 \ldots i_{p-1}}$, $i_1 < \ldots < i_{p-1}$, respectively. Further, the dual fibre coordinates on $T^*[p](\Lambda^{p-1} T)[p-1](T[1]V) \rightarrow (\Lambda^{p-1} T)[p-1](T[1]V)$ of the respective degrees $p-1$, $p$, 2 and 1 will be denoted by $\chi_i$, $F_i$, $G_I := $
The canonical symplectic form in these coordinates is \( \omega = dF_i \wedge dX^i + d\psi^i \wedge d\chi_i + dG_I \wedge dH^I + d\eta^I \wedge dA_I. \)

The potential one-form \( \vartheta \) can be taken as \( \vartheta = F_i dX^i + \psi^i d\chi_i + G_I dH^I + \eta^I dA_I. \)

**Remark 4.7.** If \( V \) is a graded vector space, which has only finitely many non-zero homogeneous components (all of them finite-dimensional) then a (formal) cohomological vector field is the same thing as an \( L_\infty \)-structure on \( V \). [2]

**Remark 4.8.** If \( A \) is a vector bundle over a manifold \( V \), then a cohomological vector field on \( A[1] \) is the same thing as a Lie algebroid structure on \( A \). [55]

**Remark 4.9.** A graded symplectic form \( \omega \) of a non-zero degree \( k \) is exact [46]. The symplectic potential \( \theta \) is given by the contraction \( \iota_E \omega \), where \( E \) is the graded Euler vector field, i.e., the vector field acting on a homogeneous function \( f \) of degree \( |f| \) as \( Ef = |f|f \).

In some homogeneous coordinates \( x^i, E = |x^i|x^i \partial_i \).

**Definition 4.10.** A vector field \( \chi \) on a graded symplectic manifold \( M \) is called symplectic, if it preserves the symplectic structure, i.e. \( \mathcal{L}_\chi \omega = 0 \), where \( \mathcal{L}_\chi \) is the Lie derivative with respect to \( \chi \). It is Hamiltonian, if the 1-form \( \iota_\chi \omega \) is exact, i.e., \( \iota_\chi \omega = dh \) for some smooth function \( h \).

**Remark 4.11.** Let \( \omega \) be a symplectic form of degree \( k \) and \( \chi \) a symplectic vector field of degree \( l \) such that \( k + l \neq 0 \). Then \( \chi \) is Hamiltonian \( \iota_\chi \omega = d \left( \frac{\iota_\chi \omega}{k+l} \right) \) [46].

**Definition 4.12.** A graded manifold equipped with a graded symplectic form and a symplectic cohomological vector field \( Q \) is called a symplectic dg manifold. It follows from the above Remark [4.11] that if the symplectic form \( \omega \) on a symplectic dg manifold \( M \) is of degree \( k \neq -1 \), then the cohomological vector field \( Q \) on \( M \) is Hamiltonian. Let \( S \) be the corresponding Hamiltonian function, \( Q = \{ S, . \} \). It can be chosen to have degree \( k+1 \). Then \( Q^2 = 0 \) and \( k \neq -2 \) implies that \( S \) is a solution to the classical master equation

\[ \{ S, S \} = 0. \] (4.1)

**Example 4.13.** A symplectic cohomological vector field \( Q_M \) on \( M = T^*[1]V \) (equipped with the canonical symplectic structure) can be determined by a Poisson 2-vector on \( V \), i.e. a 2-vector field \( \pi = \frac{1}{2} \pi^{ij} \partial_i \wedge \partial_j \) with vanishing Schouten bracket \( [\pi, \pi]_S \). More
explicitly,
\[ Q = \pi^{ij} \chi_j \partial_{x_i} + \frac{1}{2} \partial_k \pi^{ij} \chi_i \chi_j \partial_{\chi_k}. \]

The corresponding Hamiltonian function is
\[ \gamma = \frac{1}{2} \pi^{ij} \chi_i \chi_j. \]

This last example is an illustration of the following general fact [49].

**Theorem 4.14.** Let \((M, \omega)\) be a degree 1 graded symplectic structure. Then it is symplectomorphic to to \(T^*[1]V\) with the canonical symplectic form, where \(V\) can be chosen to be an ordinary manifold. Under the symplectomorphism, the degree -1 Poisson bracket \(\{\cdot, \cdot\}\) on \(C^\infty\) is mapped to the Schouten bracket \([\cdot, \cdot]_S\) on \(T^*[1]V = \Gamma(\bigwedge TV)\). A smooth vector function \(S\) of degree 2 is mapped to a bivector field \(\pi\) on \(V\). Hence, \(S\) solves the classical master equation is translated into \(\pi\) being Poisson.

**Remark 4.15.** The above Theorem can be restated as follows. A graded symplectic manifold of non-negative degree is called a \(N\)-manifold if all its coordinates are of non-negative degree. Isomorphism classes of dg symplectic \(N\)-manifolds of degree 1 are one-to-one with isomorphism classes of Poisson manifolds.

**Remark 4.16.** It is also known (letter 7 of [51] and [46]) that isomorphism classes of dg symplectic \(N\)-manifolds of degree 2 are one-to-one with isomorphism classes of Courant algebroids.

**Example 4.17.** In the example of \(T^*[p]T[1]V\), we have the canonical symplectic cohomological vector field \(\psi^i \partial_{x^i}\), corresponding to the de Rham differential on \(V\). The corresponding degree \((p + 1)\) Hamiltonian function is \(-\psi^i F_i\). Hence, if \(c\) is a closed \((p + 1)\)-form, the sum
\[ S = -\psi^i F_i + \frac{1}{(p + 1)!} c_{i_1 \ldots i_{p+1}} \psi^{i_1} \ldots \psi^{i_{p+1}} \]

is also a solution to the master equation and the corresponding Hamiltonian vector field \(Q = \{S, \cdot\}\) is another example of a symplectic cohomological vector field on \(T^*[p]T[1]V\).
Twisted Dorfman bracket. Using the above cohomological vector field $Q$ on $M = T^*[p]T[1]V$, the twisted Dorfman bracket can be identified as a derived bracket on $T^*[p]T[1]V$. Let $C_n$ for $n \leq p - 1$ denote the subspace of all degree $n$ functions on $T^*[p]T[1]V$. For $n \leq p - 2$, a degree $n$ function $\tilde{\alpha} \in C_n$ corresponds to an $n$-form $\alpha \in \Omega^n(V)$ via $\tilde{\alpha} = \frac{1}{(n)!} \alpha_{i_1\ldots i_n} \psi^{i_1} \ldots \psi^{i_n}$. Also, a degree $p - 1$ function $\tilde{e} = \frac{1}{(p-1)!} \omega_{i_1\ldots i_{p-1}} \psi^{i_1} \ldots \psi^{i_{p-1}} + v^i \chi_i \in C_{p-1}$ on $T^*[p]T[1]V$ corresponds to a pair $e = (\omega, v) \in \Omega^{p-1}(V) \oplus \mathfrak{X}(V)$ consisting of a $(p-1)$-form and a vector field. We have the following relation between the dg symplectic manifold structure of $T^*[p]T[1]V$ and the Courant algebroid structure on $\mathfrak{X}(V) \oplus \Omega^{p-1}(V)$ (given by the pairing, the anchor, the twisted Dorfman bracket and the differential as described in Section 2).

$$\{\tilde{e}_1, \tilde{e}_2\} = \langle e_1, e_2\rangle^\sim,$$

$$\{\{S, \tilde{e}\}, f\} = \rho(e)f,$$

$$\{\{S, \tilde{e}_1\}, \tilde{e}_2\} = [e_1, e_2]^\sim,$$

$$\{S, \tilde{\alpha}\} = (d\alpha)^\sim,$$

where, $\alpha$ is $(p - 2)$-form and $f$ is a function on $M$.

In particular, the subspace of degree $p - 1$ functions on $T^*[p]T[1]V$ is closed under the derived bracket $\{\{S, \cdot\}, \cdot\}$, which can be identified with the twisted Dorfman bracket. Actually, as a consequence of a theorem in [19], the complex $(C_n[p - 1], \psi_i \partial \chi_i)_{n=0}^{p-1}$ can be equipped with a Lie $p$-algebra structure, the twisted Dorfman bracket being one of the binary brackets.

Let us note that in the case of $p = 2$ the canonical transformation $e^\delta \zeta$ generated by a degree 2 function $\zeta = \zeta_{ij} \chi_i \chi_j$ gives $e^\delta \tilde{e} = (e^c e)^\sim$.

Twisted Dorfman bracket – continuation. In the example of $M = T^*[p](\wedge^p T)[p-1](T[1]V)$, we can take as the Hamiltonian vector field $Q = \{S, \cdot\}$ corresponding to the Hamiltonian

$$S = -\psi^i F_i + G_I \eta^I + \frac{1}{(p + 1)!} c_{i_1\ldots i_{p+1}} \psi^{i_1} \ldots \psi^{i_{p+1}}.$$

\[9\text{See [58] for the explicit formulas. For the original work on case } p = 2, \text{ see [48]. Also, see [2] for another related (generalized) } L_\infty\text{-structure.}\]

\[10\text{We use the following convention. Only when using the upper case notation for a multi-index, we assume it ordered, otherwise not.}\]
We can embed $T^*[p]T[1]V$ into $T^*[p]((\wedge^m T)[p-1](T[1]V))$ as the zero section of the vector bundle $T^*[p]((\wedge^m T)[p-1](T[1]V)) \to T^*[p]T[1]V$. Obviously, this embedding is a Poisson map. Hence, the Dorfman bracket can be identified as (a part of) the restriction to $T^*[p]T[1]V$ of the derived bracket on $T^*[p]((\wedge^m T)[p-1](T[1]V))$ too. For the further reference, let us also note that under the canonical transformation generated by the degree $p$-function $-\frac{1}{(p-1)!} \psi^{i_1} \ldots \psi^{i_{p-1}} A_{i_1 \ldots i_{p-1}}$ the Hamiltonian function $S$ changes to

$$S' = -\psi^{i} F_i + \frac{1}{(p-1)!} G_{i_1 \ldots i_{p-1}} (\eta^{i_1} \ldots \psi^{i_{p-1}} - \psi^{i_1} \ldots \psi^{i_{p-1}}) + \frac{1}{(p+1)!} c_{i_1 \ldots i_{p+1}} \psi^{i_1} \ldots \psi^{i_{p+1}}.$$

This gives another way of identifying the Dorfman bracket as (a part of) the restriction of the derived bracket on $T^*[p]((\wedge^m T)[p-1](T[1]V))$.

5. AKSZ construction

In this section we review the AKSZ construction. We follow mainly [12] and [25], where the interested reader can find missing details. See also [45], [11] and, of course, [2]. The AKSZ formalism [2] provides for a solution of the classical master equation (4.1) on the space of maps $\mathcal{M} := C^\infty(X, M) = M^X$.

Here,

(i) the source $(X, Q_X, \mu)$ (a super worldvolume) is a differential graded manifold $X$, equipped with a measure $\mu$ which is invariant under the cohomological vector field $Q_X$, and

(ii) the target $(M, Q_M, \omega)$ (a super spacetime) is a differential graded symplectic manifold $M$ with the graded symplectic form $\omega$, such that the cohomological vector field $Q_M$ is a Hamiltonian vector field.

Using the above structures on $X$ and $M$ the AKSZ construction produces:

(i) a graded symplectic structure $\tilde{\omega}$ on the space of maps $\mathcal{M}$, and

(ii) a symplectic cohomological vector field $Q$ on $\mathcal{M}$.

The cohomological vector field on the space of maps. We describe very briefly the construction. The tangent space $T_f\mathcal{M}$ to $\mathcal{M}$, at some function $f \in \mathcal{M}$, is identified with with the space of sections $\Gamma(X, f^*TM)$. Then a vector field on $\mathcal{M}$ is an assignment of an element in $T_{f(x)}M$ to each $x \in X$ and $f \in \mathcal{M}$. In particular, the vector
fields \( Q_0 \) and \( \check{Q} \) on \( M \), associated to the cohomological vector fields \( Q_X \) and \( Q_M \), respectively, are defined as

\[
(Q_0 f)(x) = Q_0(x, f) = df(x)Q_X(x),
\]

and

\[
(\check{Q} f)(x) = \check{Q}(x, f) = Q_M(f(x)).
\]

We note that \( Q_0 \) and \( \check{Q} \) are of the degree 1, square to zero and graded commute with each other.

**Proposition 5.1.** Let \( X \) and \( M \) be differential graded manifolds. Then the space of smooth maps \( M = M^X \) is a differential graded manifold, with cohomological vector field \( Q = Q_0 + \check{Q} \).

**The source.** For our purposes it will be sufficient to consider the case \( X = T[1] \Sigma \), where \( \Sigma \) is an ordinary manifold of dimension \( p+1 \) with boundary \( \partial \Sigma \). The algebra of smooth functions on \( X = T[1] \Sigma \) is isomorphic to the algebra \((\Omega(\Sigma), \wedge)\) of differential forms. We denote the isomorphism \( j \). For the cohomological vector field on \( X = T[1] \Sigma \) we take the vector field \( Q_X \) corresponding, under this isomorphism, to the de Rham differential \( d \) on \((\Omega(\Sigma), \wedge)\). If we choose some local coordinates \( x^\mu \) on \( \Sigma \) and denote the corresponding induced odd coordinates on the fibre by \( \theta^\mu \), we will have \( Q_X = \theta^\mu \partial_\mu \). Take the canonical measure \( \mu \) on \( X = T[1] \Sigma \), which maps a function \( f \) on \( X \) to \( \int_X f := \int_\Sigma j(f) \), where \( j \) is the isomorphism between smooth functions on \( T[1] \Sigma \) and smooth forms \( \Omega(\Sigma) \) on \( \Sigma \). In local coordinates \( \mu = d^{p+1}x d^{p+1}\theta \).

**The graded symplectic structure on \( M \).** For any \( n \)-form \( \alpha \in \Omega^n(M) \), we obtain an \( n \)-form \( \check{\alpha} \in \Omega^n(M) \) by

\[
\check{\alpha} = \int_X \text{ev}^*(\alpha),
\]

where \( \text{ev}^* \) is the pullback under the evaluation map \( \text{ev} : X \times M \to M \), defined by \( \text{ev}(x, f) = f(x) \). Integrating over \( X = T[1] \Sigma \) we obtain an \( n \)-form of degree \( |\alpha| - (p + 1) \). In particular, from the symplectic form \( \omega \) on \( M \), we get the symplectic form \( \check{\omega} \) on \( M \) and if there exists a symplectic potential \( \vartheta \) on \( M \), we obtain a corresponding symplectic potential \( \check{\vartheta} \) on \( M \) as well. For example, from a degree \( p + 1 \) function

\[1\]Hence, a point \( x \in X \) is locally parametrized by \((x^\mu, \theta^\mu)\).
$f \in \Omega^0(M) = C^\infty(M)$, we obtain a function $\tilde{f} \in \Omega^0(M) = C^\infty(M)$, defined on $\phi \in \mathcal{M}$ by

$$\tilde{f}[\phi] = \int_X \phi^*(f).$$

Furthermore, we can use the coordinates on $M$, say $X^i$, to parametrize a general the superfield $\phi : X \to M$, hence introduce the “coordinate” superfields

$$\phi^i(x^\mu) = \phi^*(X^i)(x^\mu).$$

In particular, in the case $|\omega| = p$ corresponding to the BV formalism, which we will consider from now, we have

**Proposition 5.2.** For a degree $p$ symplectic form $\omega$ on $M$, the 2-form $\tilde{\omega}$ is a degree -1 symplectic form on $\mathcal{M}$. Further, if $\partial$ is a symplectic potential for $\omega$, then $\tilde{\partial}$ is a symplectic potential for $\tilde{\omega}$. Moreover, since $\iota_\tilde{Q} \tilde{\omega} = \int_X \ev^* \iota_Q \omega$, we also have $\mathcal{L}_Q \tilde{\omega} = 0$. In particular, if $\gamma$ is the a degree $p + 1$ Hamiltonian function on $M$ corresponding to $Q$ then $\tilde{\gamma} := \tilde{\gamma} = \int_X \ev^* \gamma$ is the degree 0 Hamiltonian function on $\mathcal{M}$ corresponding to $\tilde{Q}$.

Let $\{\cdot, \cdot\}$ be the degree 1 Poisson bracket, the BV bracket, on $\mathcal{M}$ corresponding to the degree $-p$ Poisson bracket $\{\cdot, \cdot\}$ on $M$.

**Proposition 5.3.** The map $\int_X \ev^*$ is a degree $-(p + 1)$ Lie algebra map from $(M, \{\cdot, \cdot\})$ to $(\mathcal{M}, \{\cdot, \cdot\})$, i.e., $\int_X \ev^* \{f, g\} = \{\int_X \ev^* f, \int_X \ev^* g\}$, for any two functions $f, g$ on $M$.

**Notation** We will use the following notation. For a function $f$ on $M$, we will omit the $\ev^*$ symbol under the integral sign and simply write $\int_X f$ instead of $\int_X \ev^* f$, etc.

**Solution to the master equation.** Since $p \geq 0$, we will assume that we have chosen a symplectic potential $\partial$ on $M$ and that $Q_M$ is Hamiltonian with the degree $(p + 1)$ Hamiltonian function $\gamma$. To proceed further, we should be careful about the boundary conditions. This is discussed in great detail in [25], [26] and [11]. We will need a

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12In general, we may expand a superfield $\phi : X \to M$ as a polynomial in the odd variables $\theta$ with coefficients being $M$-valued functions on $\Sigma$, i.e., $\phi(x^\mu, \theta^\mu) = \phi^{(0)}(x^\mu) + \phi^{(1)}(x^\mu)\theta^\nu + \ldots + \phi^{(p+1)}(x^\mu)\theta^{\nu_1} \ldots \theta^{\nu_{p+1}}$. Then if $\phi^{(0)}$ is degree $k$ $\phi^{(i)}$ will be degree $k - i$. Assume that $M$ is non-negatively graded. Factoring out the ideal generated by coefficients with negative degrees will give the space of the fields (including the ghosts) and factoring out the ideal generated by coefficients with nonzero degrees (ghost and antifields) will give the space of classical fields.
slight modification of that discussion. Let $L$ denote the Lagrangian submanifold of $M$ which is the zero locus of $\vartheta$, and $L' \subset L$ some submanifold. We will consider only a subspace $\mathcal{M}_{L'}$ of $\mathcal{M}$, which consists of maps that map the boundary $\partial X = T[1]\partial \Sigma$ into $L'$. Also, we will assume that $\gamma$ when restricted to $L'$ vanishes. Hence, we assume

$$\{\gamma, \gamma\} = 0,$$

and

$$\gamma|_{L'} = 0.$$

**Theorem 5.4.** On $\mathcal{M}_{L}$, the vector field $Q_0$ is Hamiltonian, with the corresponding Hamiltonian function

$$S_0 = -\iota_{Q_0} \check{\vartheta}.$$ 

For $\gamma$ satisfying $\{\gamma, \gamma\} = 0$, and $\gamma|_{L'} = 0$ the sum

$$S = S_0 + \check{\gamma},$$

is a solution of the master equation on $\mathcal{M}_{L'}$, i.e., $S$ is a BV action.\(^{14}\)

We will not give a formal proof, since it is a only a slight modification of the discussion, in the case $L' = L$ in [11] and [25]. We sketch as an example the case when $X = T[1]\Sigma$ and $M = T^*[p]N$, with $N$ some graded manifold $N$, with coordinates of degrees from 0 to $p$. Choose homogeneous local coordinates $X^i$ on $N$ and the dual fibre coordinates $P_i$ on $T^*[p]N \rightarrow N$. For the symplectic potential on $M$ we take $\vartheta = P_i dX^i$.

Hence, $L$ is given by $P = 0$.

We will use the notation $X^i, P_i$ for the superfields associated with the local coordinates $X^i, P_i$ on $M$ respectively, i.e., $X^i = \Phi^*X^i, P_i = \Phi^*P_i$.

So in the local coordinates

$$\check{\omega} = \int_X \delta P_i \wedge \delta X^i,$$

\(^{13}\)To be more precise, $\mathcal{M}_{L'}$ has to be properly regularized, see [11] for the detailed discussion for $L' = L$. We will not discuss the subtleties related to this and refer to [11] for the general discussion, cf. also [26].

\(^{14}\)I.e., for $\phi \in \mathcal{M} = M^X$, $S[\phi] = S_0[\phi] + \check{\gamma}[\phi] = \int_X (-\iota_{Q_0} \phi^* \alpha + \phi^*(\gamma))$. 

and the BV bracket is determined by the bivector\footnote{Here the superscripts $R$ and $L$ refer the right and left (functional) derivative. We will omit them in the sequel.}

\[
\int_X \partial^R X^i \wedge \partial^L P_i.
\]

Hence, for the $S_0$ part of the BV action we have\footnote{More precisely, we should have written $S_0[\phi]$ instead of $S_0$. We will continue, hopefully without causing confusion, with this shorthand notation in the sequel.}

\[
S_0 = \int_X P_i D X^i.
\]

In the above formulas and in the sequel we use the notation $D$ for $\theta^\mu \partial_\mu$. Now we can explicitly check that $S_0$ is the Hamiltonian for $Q_0$ iff

\[
\int_X D(P_i D X^i) = \int_{\partial X} P_i D X^i = 0.
\]

By assumption, $(P_i(x), X^i(x)) \in L' \subset L$ on the boundary and it follows that for $x \in \partial X$ we have $P^i(x) = 0$. Therefore, $S_0$ is indeed the Hamiltonian for $Q_0$.

Furthermore, we have

\[
Q_0 \gamma = \int_X D \gamma = \int_{\partial X} \gamma,
\]

where we used the symbol $D$ also for the lift of $\theta^\mu \partial_\mu$ to $M \times X$. For $x \in \partial X$, by assumption, $(P_i(x), X^i(x)) \in L'$ and we see that if $\gamma|_{L'}$ vanishes then $Q_0 \gamma = \{S_0, \gamma\} = 0$. We conclude that $S_0 + \gamma$ is a BV action.

Remark 5.5. Integrating in $S$ over the odd variables $\theta$ and restricting it to the degree zero fields, we obtain the “classical” action $S_{cl}$. Then the solutions of the classical field equations are dg maps from $(T[1]\Sigma, D)$ to $(M, (-1)^{p+1}Q)$ \footnote{Remark 5.5. Integrating in $S$ over the odd variables $\theta$ and restricting it to the degree zero fields, we obtain the “classical” action $S_{cl}$. Then the solutions of the classical field equations are dg maps from $(T[1]\Sigma, D)$ to $(M, (-1)^{p+1}Q)$}. This follows from the fact that the critical points of $S$ are the fixed points of $Q$.

**Canonical transformations.** From the above discussion, it follows that a canonical transformation on $M$, generated by a function $\alpha$ of degree $p$, induces a canonical transformation on $M$ generated by the function $\bar{\alpha}$. We will use the notation $\delta_\alpha$ for the corresponding Hamiltonian vector field and $e^{\delta_\alpha}$ and $e^{\bar{\delta}_\alpha}$ for the respective canonical transformations. From $e^{-\delta_\alpha}\{e^{\delta_\alpha}\gamma, e^{\delta_\alpha}\gamma\} = \{\gamma, \gamma\}$ we see that $\{e^{\delta_\alpha}\gamma, e^{\delta_\alpha}\gamma\} = 0$, provided $\{\gamma, \gamma\} = 0$. We can write

\[
e^{\bar{\delta}_\alpha}(S_0 + \int_X \gamma) = S_0 + \int_X e^{\delta_\alpha}\gamma + \sum_{n \geq 1} \frac{1}{n!}\delta_\alpha^{n-1} \int_{\partial X} \alpha.
\]
Hence, when $\alpha|_{L'} = 0$, the BV actions $S_0 + \int_X \gamma$ and $S_0 + \int_X e^{\delta \alpha} \gamma$ are equivalent, i.e., related by a canonical transformation.

In general, the symplectic potential $\vartheta$ (and hence also the the Lagrangian submanifold $L$ defined by its locus) or the submanifold $L'$ may have changed due to the canonical transformation. Hence, for degree $p$ generating function $\beta$ on $M$ it may happen that $L'_{\beta} := e^{\delta \beta} (L') \neq L'$.

Let us assume that $(e^{\delta \beta} \gamma)|_{L'} = 0$ and, therefore, $S_0 + (e^{\delta \beta} \gamma)$ is a BV action. Also assume, for simplicity, that $\{\beta, \beta\} = 0$. We have

$$
S_0 + \int_X e^{\delta \beta} \gamma \sim e^{-\delta \beta} (S_0 + \int_X e^{\delta \beta} \gamma) = e^{-\delta \beta} S_0 + \int_X e^{\delta \beta} e^{\delta \beta} \gamma = S_0 - \int_X D\beta + \int_X \gamma = S_0 + \int_X \gamma - \int_{\partial X} \beta,
$$

where in the first equality we have used Proposition 5.3 and in the second equality the fact that only first two terms in the expansion of $e^{-\delta \beta}$ will survive due to the (graded) Jacobi identity and the assumption $\{\beta, \beta\} = 0$.

Hence, we have a slight modification of the corresponding statement of [25].

**Theorem 5.6.**

(i) Assume that $\{\gamma, \gamma\} = 0$, $\gamma|_{L'} = 0$ and $\alpha|_{L'} = 0$. The BV actions $S_0 + \int_X \gamma$ and $S_0 + \int_X e^{\delta \alpha} \gamma$ are equivalent, i.e., related by a canonical transformation.

(ii) Assume that $\{\gamma, \gamma\} = 0$ and $\{\beta, \beta\} = 0$. Also, assume that $(e^{\delta \beta} \gamma)|_{L'} = 0$.

Then the BV action $S_0 + \int_X e^{\delta \beta} \gamma$ on $M_{L'}$ and the BV action $S_0^\beta + \int_X \gamma$ on $M_{i L'}$, where $S_0^\beta$ corresponds to the symplectic potential $\vartheta - d\beta$, are equivalent.

The latter BV action can be written as a bulk/boundary action $S_0 + \int_X \gamma - \int_{\partial X} \beta$.

Related to this we have the following corollary [25].

**Corollary 5.7.** Let $\alpha$ and $\beta$ be degree $p$ functions on $M$, $\alpha|_{L'} = 0$ and $\beta \in \text{Im } i_{L'}$. Here $i_{L'}$ is some embedding of functions on $L'$ into functions on $M$ such that $P_{L'} i_{L'} = \text{id}$ with $P_{L'}$ being the restriction to $L'$. Assume that we have found $\alpha'|_{L'} = 0$, and $\beta' \in \text{Im } i_{L'}$ such that $e^{\delta \alpha} e^{\delta \beta} = e^{\delta \beta'} e^{\delta \alpha'}$. Then the BV actions $S_0 + \int_X \gamma + \int_{\partial X} \beta$ and $S_0 + \int_X e^{\delta \alpha} \gamma + \int_{\partial X} \beta'$ are equivalent.\[17\]

\[17\] Of course, we assume that $\{\gamma, \gamma\} = 0$, $\gamma|_{L'_\beta} = 0$, $\{\beta, \beta\} = 0$ and $\{\beta', \beta'\} = 0$. 


The statement of the corollary follows from the chain of equivalences and equalities

\[ S_0 + \int_X \gamma + \int_{\partial X} \beta \sim S_0 + \int_X e^{-\delta_{\beta}} \gamma \sim S_0 + \int_X e^{\delta_{\alpha}} e^{-\delta_{\beta}} \gamma \]

\[ = S_0 + \int_X e^{-\delta_{\beta'}} \delta_{\alpha} \gamma \sim S_0 + \int_X e^{\delta_{\alpha}} \gamma + \int_{\partial X} \beta' , \]

where we have used that, according to the assumptions,

\[ e^{\delta_{\alpha}} e^{-\delta_{\beta'}} \gamma |_{L'} = e^{\delta_{\alpha}} e^{-\delta_{\beta'}} \gamma |_{L'} = e^{-\delta_{\beta}} \gamma |_{L'} = 0. \]

6. Examples of the AKSZ construction

6.1. Poisson sigma model. Here we follow \[\text{[11]}\]. For \( V \) a smooth manifold, we take \( M = T^* [1] V \). The canonical symplectic structure \( \omega \) on the target \( T^* [1] V \) is of degree \( |\omega| = 1 \). Hence, we take a 2-dimensional \( \Sigma \). We will denote the degree 0 local coordinates on \( V \) as \( X^i \) and the induced degree 1 fibre coordinates on \( T^* [1] V \) by \( \chi_i \). The canonical symplectic form in these coordinates is \( \omega = d\chi_i \wedge dX^i \). The potential one-form \( \vartheta \) can be taken as \( \vartheta = \chi_i dX^i \). Its zero locus \( L \) is given by \( \chi_i = 0 \). A symplectic cohomological vector field \( Q_M \) on \( M = T^* [1] V \) is necessarily determined by a Poisson 2-vector on \( V \), i.e. a 2-vector field \( \pi = \frac{1}{2} \pi^{ij} \partial_{X^i} \wedge \partial_{X^j} \) with vanishing Schouten bracket \( [\pi, \pi]_S \). More explicitly,

\[ Q_M = \pi^{ij} \chi_j \partial_{X^i} + \frac{1}{2} (\partial_{k} \pi^{ij}) \chi_i \chi_j \partial_{\chi_k}. \]

The corresponding Hamiltonian function is

\[ \gamma = \frac{1}{2} \pi^{ij} \chi_i \chi_j. \]

Furthermore, we may use the superfields \( X^i, \chi^i \) to write the BV bracket in the form

\[ \int_X \partial_{X^i} \wedge \partial_{X^i}. \]

For the \( S_0 \) part of the BV action we have

\[ S_0 = \int_X \chi_i D X^i. \]
Hence, in this case, the AKSZ construction gives the Poisson sigma model of \[25\], \[50\].

\[ S = \int_X (\chi_i D X^i + \frac{1}{2} \pi^{ij} \chi_i \chi_j). \]

It follows from Remarks \[5.5\] and \[4.8\] that the solutions to the classical field equations are Lie algebroid maps from \( T[1] \Sigma \) to \( T^* [1] V \), cf. \[53\].

If the boundary \( \partial \Sigma \) is nonzero, we take the Dirichlet boundary conditions for the superfields \( \chi_i \).

6.2. **Open topological membrane and twisted Poisson sigma model.** This subsection is based on \[25\]. Let \( V \) be a smooth manifold and put \( M = T^*[p]T[1]V \), with \( p \) an integer \( p \geq 2 \). The case \( p = 2 \), corresponds to the open topological membrane. The canonical symplectic structure \( \omega \) on the target \( T^*[p]T[1]V \) is of degree \( p \). Hence, we take a \( p + 1 \) dimensional \( \Sigma \). We will denote the degree 0 local coordinates on \( V \) as \( X^i \) and the induced degree 1 fibre coordinates on \( T[1]V \) by \( \psi^i \). Dual fibre coordinates on \( T^*[p]T[1]V \to T[1]V \) of the respective degrees \( p - 1 \) and \( p \) will be denoted by \( \chi_i \) and \( F_i \). The canonical symplectic form in these coordinates is \( \omega = dF_i \wedge dX^i + d\psi^i \wedge d\chi_i \). The potential one-form \( \vartheta \) can be taken as \( \vartheta = F_i dX^i + \psi^i d\chi_i \). Its zero locus \( L \) is given by \( F_i = 0 \) and \( \psi^i = 0 \).

We may use the superfields corresponding to the local coordinates on \( T^*[p]T[1]V \) to write the BV bracket in the form

\[
\int_X (\partial_{X^i} \wedge \partial_{F_i} + \partial_{\chi_i} \wedge \partial_{\psi^i}).
\]

For the \( S_0 \) part of the BV action we have

\[
S_0 = \int_X (F_i D X^i + \psi^i D \chi_i).
\]

Furthermore, the canonical symplectic cohomological vector field corresponding to the de Rham differential on \( V \) is \( Q = \psi_i \partial_{X^i} \) and the corresponding degree \( (p + 1) \) Hamiltonian \( \gamma_0 = \psi^i F_i \) is a solution to the classical master equation on \( T^*[p]T[1]V \) satisfying \( \gamma_0 |_L = 0 \). From now on, we will use notation \( \gamma_0 \) for \( \gamma_0 \). Hence we have the “free part” of the BV action

\[
S_0 + \Gamma_0 = \int_X (F_i D X^i + \psi^i D \chi_i - \psi^i F_i).
\]

\[18\] Recall footnote \[16\].
Bulk interaction. Let $c$ be a $(p+1)$-form on $V$. In local coordinates,

$$c = \frac{1}{(p+1)!} c_{i_1...i_{p+1}} dX^{i_1} \wedge \ldots \wedge dX^{i_{p+1}}.$$ 

We associate with it the degree $p+1$ function $C = \frac{1}{(p+1)!} c_{i_1...i_{p+1}} \psi^{i_1} \ldots \psi^{i_{p+1}}$ on $M$ and the corresponding degree 0 function $\Gamma_1 = \int_X C$ on $M$. By construction,

$$\{ S_0 + \Gamma_0 + \Gamma_1, S_0 + \Gamma_0 + \Gamma_1 \} = 2 \{ \Gamma_0, \Gamma_1 \} = 2 \int_X \{ -\psi^i F_i, C \}.$$

Since the function $-\psi^i F_i$ on $T^*[p]V$ corresponds to the de Rham differential on $V$, we see that the sum

$$S_0 + \Gamma_0 + \Gamma_1 = \int_X (F_i D X^i + \psi^i D \chi_i - \psi^i F_i + \frac{1}{(p+1)!} c_{i_1...i_{p+1}} \psi^{i_1} \ldots \psi^{i_{p+1}})$$

is a solution to the master equation iff the $(p+1)$-form $c$ is closed. Let us also note, that the canonical transformation on $M$ generated by the degree $p$ function $B$, where $B = \frac{1}{p!} b_{i_1...i_p} \psi^{i_1} \ldots \psi^{i_p}$ corresponds to a $p$-form $b$ on $V$, amounts to the gauge transformation $c \mapsto c - db$.

Boundary interaction. One can also consider $\Sigma$ to have a nonempty boundary $\partial \Sigma$. For our discussion it is relevant that the boundary conditions can be chosen so that the superfields $X^i, F_i, \psi^i$ and $\chi_i$ restrict on the boundary to maps to the zero locus of $\theta$. This means that we take the Dirichlet boundary conditions for the superfields $F_i$ and $\psi^i$.\footnote{Other boundary conditions and the related effects are discussed in \cite{25}. Also, see \cite{26} and \cite{11} for discussion of boundary conditions for the remaining superfields.}

The case $p = 2$ was thoroughly discussed in \cite{25}. If $p = 2$, we can consider a boundary term associated to a 2-vector field $\pi$ on $V$. This can be done considering the canonical transformation on $M$ generated by the degree 2 function $-\frac{1}{2} \pi^{ij} \chi_i \chi_j$.

We know that the result is equivalent to the boundary/bulk BV action

$$S = \int_X (F_i D X^i + \psi^i D \chi_i - \psi^i F_i + \frac{1}{6} c_{i j k} \psi^i \psi^j \psi^k) + \frac{1}{2} \int_{\partial X} \pi^{ij} \chi_i \chi_j.$$

For $\gamma = \psi^i F_i + \frac{1}{6} c_{i j k} \psi^i \psi^j \psi^k$, the condition $\{ \gamma, \gamma \} = 0$ gives, as before, $dc = 0$. Regarding the condition $\gamma|_{L_\pi} = 0$, we notice that the Lagrangian submanifold $L_\pi$ is given by equations

$$\psi^i = \frac{1}{2} \pi^{ij} \chi_j, \psi^i \psi^j = \pi^{ij} \chi_j,$$
and
\[ F_i = \{ \frac{1}{2} \pi^{ij} \chi_i \chi_j, F_i \} = \frac{1}{2} \partial_i \pi^{jk} \chi_j \chi_k. \]

Let us note that the first equation gives the graph of the map \( \pi \), whereas the second equation is the integrability condition for this graph. These two conditions determine a Dirac structure in \( T^*V \oplus TV \).\(^{20}\)

From here it follows that
\[ [\pi, \pi]_S = - \wedge^3 \pi^\sharp c, \]
where the bracket subscript \( S \) stands for the Schouten bracket. Hence, \( \pi \) defines a twisted Poisson bracket on \( V \). This is, up to equivalence, the most general BV action of the AKSZ form in our case \(^{25}\).\(^{21}\)

**Gauge transformations.** Recall Corollary \( \text{5.7} \). Following \(^{25}\), we consider a degree 2 function \( \alpha = \frac{1}{2} \alpha_{ij} \psi^i \psi^j \), corresponding to a 2-form \( \tilde{\alpha} = \frac{1}{2} \alpha_{ij} dX^i \wedge dX^j \) on \( V \) and a degree 2 function \( \beta = \frac{1}{2} \pi^{ij} \chi_i \chi_j \) corresponding to a twisted Poisson tensor \( \pi = \frac{1}{2} \pi^{ij} \partial_i \wedge \partial_j, [\pi, \pi]_S = - \wedge^3 \pi^\sharp c \) on \( V \). From Proposition \( \text{2.7} \) it follows that in Corollary \( \text{5.7} \) we have to choose \( \beta' = \frac{1}{2} \pi^{ij} \chi_i \chi_j \), with \( \pi'^\sharp = \pi^\sharp \circ (1 + \alpha^\sharp \circ \pi^\sharp)^{-1}, [\pi', \pi']_S = - \wedge^3 \pi'^\sharp (c - d\alpha) \).

This means that we have the equivalence of the two bulk/boundary actions given by the respective gauge transformations \( c \mapsto c - d\alpha \) and \( \pi^\sharp \mapsto \pi^\sharp \circ (1 + \alpha^\sharp \circ \pi^\sharp)^{-1} \).

**Poisson sigma model.** On shell, using the equations of motion for \( F \)'s, we obtain the closed twisted Poisson sigma model \( \text{[41, 35]} \)
\[ S(\pi, c) = \int_{\partial X} (\chi^i DX^i + \frac{1}{2} \pi^{ij} \chi_i \chi_j) + \frac{1}{6} \int_X c_{ijk} DX^i DX^j DX^k. \]

**Remark 6.1.** In order to obtain the open (twisted) Poisson sigma model, in \(^{25}\) it is proposed to include boundaries with corners and allow for different boundary conditions on various regions of the boundary. If, for example, the boundary \( \partial X \) is divided in two regions, on the first one takes the same boundary conditions as before and on the second one restricts the superfields to only \( V \). The interface of the two regions can then be viewed as the boundary of the first region. On the interface live

\(^{20}\) The paper \(^{25}\) discusses also the more general case of \( M = T^* [2] T[1] A \), where \( A \) is a Lie algebroid, and describes the correspondence between Lagrangian submanifolds on \( M \) and Dirac structures in \( A^* \oplus A \) in this more general case.

\(^{21}\) We will describe the original argument of \(^{25}\) later in relation with the higher dimensional case.
only superfields corresponding to functions on $V$, as on the boundary of the Poisson sigma model.

7. OPEN TOPOLOGICAL $p$-BRANES AND NAMBU-POISSON SIGMA MODELS

Let, again, $V$ be a smooth manifold. We put $M = T^*[p](\Lambda^{p-1}T)[p-1](T[1]V))$, with $p \geq 2$ an integer. The canonical symplectic structure $\omega$ on the target $M$ is of degree $p$. Hence, we take a $p + 1$ dimensional $\Sigma$. We will denote the degree 0 local coordinates on $V$ as $X^i$, the induced degree 1 fibre coordinates on $T[1]V \to V$ by $\psi^i$, the induced degree $p - 2$ and degree $p - 1$ fibre coordinates on $(\Lambda^{p-1}T)[p-1](T[1]V) \to T[1]V$ as $H^i := H^{i_1...i_{p-1}}$, with $i_1 < \ldots < i_{p-1}$ and $\eta^I := \eta^{i_1...i_{p-1}}$, $i_1 < \ldots < i_{p-1}$, respectively. Further, the dual fibre coordinates on $T^*[p](\Lambda^{p-1}T)[p-1](T[1]V)) \to (\Lambda^{p-1}T)[p-1](T[1]V)$ of the respective degrees $p - 1$, $p$, 2 and 1 will be denoted by $\chi_i$, $F_i$, $G_I := G_{i_1...i_{p-1}}$, $i_1 < \ldots < i_{p-1}$, and $A_I := A_{i_1...i_{p-1}}$, $i_1 < \ldots < i_{p-1}$. The canonical symplectic form in these coordinates is

$$\omega = dF_i \wedge dX^i + d\psi^i \wedge d\chi_i + dG_I \wedge dH^I + d\eta^I \wedge dA_I.$$ 

The potential one-form $\vartheta$ can be taken as $\vartheta = F_i dX^i + \psi^i d\chi_i + G_I dH^I + \eta^I dA_I$. Its zero locus $L$ is given by $F_i = 0$, $\psi^i = 0$, $G_I = 0$ and $\eta^I = 0$. We choose the submanifold $L' \subset L$ of $L$ by letting $H^I = 0$ and requiring that all the products $\chi_i A_{i_1...i_{p-1}}$ are totally antisymmetric in all their indices.

We may use the superfields $X^i$, $F_i$, $\psi^i$, $\chi_i$, $H^I$, $G_I$, $\eta^I$ and $A_I$ to write the BV bracket in the form

$$\int_X (\partial X^i \wedge \partial F_i + \partial \psi^i \wedge \partial \chi_i + \partial H^I \wedge \partial G_I + \partial A_I \wedge \partial \eta^I).$$

For the $S_0$ part of the BV action we have

$$S_0 = \int_X (F_i dX^i + \psi^i d\chi_i + G_I dH^I + \eta^I dA_I).$$

Furthermore, we can now add the Hamiltonian function $-\psi^i F_i$ corresponding to the de Rham differential on $V$ and further terms not depending on a background, so that the corresponding degree $(p + 1)$ function

$$\gamma_0 = -\psi^i F_i + \frac{1}{(p - 1)!} G_{i_1...i_{p-1}} (\eta^{i_1...i_{p-1}} - \psi_1^{i_1} ... \psi_1^{i_{p-1}}),$$
is still a solution to the classical master equation on \( M = T^*[p](\wedge^{p-1} T)[p-1](T[1]V) \) (cf. the remark at the end of Section 4), satisfying \( \gamma_0|_L = 0 \). Hence, we have the following BV action

\[
S_0 + \Gamma_0 = \int_X (F_i D X^i + \psi^i D \chi_i + G_I D H^I + \eta^I D A_I) \\
+ \int_X (-\psi^i F_i + \frac{1}{(p-1)!} G_{i_1...i_p-1} (\eta^{i_1...i_p-1} - \psi^{i_1} ... \psi^{i_{p-1}})).
\]

**Bulk interaction.** Let \( c \) be a \((p+1)\)-form on \( V \). In local coordinates,

\[
c = \frac{1}{(p+1)!} c_{i_1...i_{p+1}} dX^{i_1} \wedge ... \wedge dX^{i_{p+1}}.
\]

We associate with it the degree \( p+1 \) function

\[
C = \frac{1}{(p+1)!} c_{i_1...i_{p+1}} \psi^{i_1} ... \psi^{i_{p+1}}
\]

on \( M \) and the corresponding degree 0 function \( \Gamma_1 \) on \( M \). By construction,

\[
\{S_0 + \Gamma_0 + \Gamma_1, S_0 + \Gamma_0 + \Gamma_1\} = 2\{\Gamma_0, \Gamma_1\}.
\]

Obviously, the sum

\[
S_0 + \Gamma_0 + \Gamma_1 = \int_X (F_i D X^i + \psi^i D \chi_i + G_I D H^I + \eta^I D A_I) \\
+ \int_X (-\psi^i F_i + \frac{1}{(p-1)!} G_{i_1...i_p-1} (\eta^{i_1...i_p-1} - \psi^{i_1} ... \psi^{i_{p-1}})) \\
+ \frac{1}{(p+1)!} \int_X c_{i_1...i_{p+1}} \psi^{i_1} ... \psi^{i_{p+1}}
\]

is a solution to the master equation iff the \((p+1)\)-form \( c \) is closed. Let us also note that the canonical transformation on \( M \), generated by the degree \( p \) function \( \alpha \), \( \alpha|_L' = 0 \), \( \{\alpha, \alpha\} = 0 \), where \( \alpha = \frac{1}{p!} (b_{i_1...i_p} \eta^{i_2...i_p} - \partial_i b_{i_2...i_{p+1}} \psi^{i_1} ... \psi^{i_{p+1}}) \), with \( b \) being a \( p \)-form on \( V \), amounts into the gauge transformation \( c \mapsto c - db \). Such a canonical transformation preserves \( L \) as well as \( L' \).

**Boundary interaction.** Again, we can allow for a nonempty boundary \( \partial \Sigma \neq \emptyset \) of \( \Sigma \) and try to add a boundary interaction. The boundary conditions can be chosen so that the superfields restrict on the boundary to maps to the submanifold \( L' \subset L \) of the zero locus \( L \) of \( \vartheta \). This means that for the superfields \( F_i, \psi^i, G_I \) and \( \eta^I \) we take Dirichlet boundary conditions, as well as for \( H^I \) and \( (\chi_{i_1} A_{i_2...i_p} + \chi_{i_1} A_{i_2...i_{k-1}i_{k+1}...i_p}) \).

We can consider a boundary term associated to a \( p \)-vector field \( \pi \) on \( V \). This can be done considering the canonical transformation on \( M \), now generated by the
degree \( p \) function
\[
S_{c,\pi} = S_0 + \Gamma_0 + \Gamma_1 = \int_X (F_i D X^i + \psi^i D \chi_i + G_i D H^I + \eta^I D A_i)
\]
\[+ \int_X (-\psi^i F_i + \frac{1}{(p-1)!} G_{i_1...i_{p-1}} (\eta^{i_1...i_{p-1}} - \psi^{i_1}...\psi^{i_{p-1}}))
\]
\[+ \frac{1}{(p+1)!} \int_X c_{i_1...i_{p+1}} \psi^{i_1}...\psi^{i_{p+1}} + \int_{\partial X} \frac{1}{(p-1)!} \pi^{i_1...i_p} A_{i_1...i_{p-1}} \chi_{i_p}. \tag{7.1}
\]
For \( \gamma = -\psi^i F_i + \frac{1}{(p-1)!} G_{i_1...i_{p-1}} (\eta^{i_1...i_{p-1}} - \psi^{i_1}...\psi^{i_{p-1}}) + \frac{1}{(p+1)!} c_{i_1...i_{p+1}} \psi^{i_1}...\psi^{i_{p+1}}, \) the condition \( \{\gamma, \gamma\} = 0 \) gives, as before, \( dc = 0. \) Regarding the condition \( \gamma|_{L^\pi} = 0, \) we notice that the Lagrangian submanifold \( L^\pi \) is given by equations
\[
\psi^j = \{\pi^{i_j} A_{j} \chi_j, \psi^i\} = \pi^{i_j} A_j, \tag{7.2}
\]
\[
\eta^l = \{\pi^{i_l} A_{l} \chi_l, \eta^i\} = \pi^{i_l} \chi_j, \tag{7.3}
\]
\[
G_i = 0, \tag{7.4}
\]
and
\[
F_i = \{\pi^{i_j} A_{j} \chi_j, F_i\} = \partial_i \pi^{Kj} A_K \chi_j. \tag{7.5}
\]
Also, the conditions \( H^I = 0 \) and the products \( \chi_{i_1} A_{i_2...i_p} \) being totally antisymmetric in its indices are not affected by this canonical transformation. Hence, from \( \gamma|_{L^\pi} = 0, \) it follows that
\[
- \pi^{i_j} \partial_i (\pi^{Kj} A_{K} A_J \chi_J) + \frac{1}{(p+1)!} c_{i_1...i_{p+1}} \pi^{i_{p+1} i_{p+1}} A_{i_1}...A_{i_{p+1}} = 0. \tag{7.6}
\]
If we now assume a (locally) decomposable \( \pi, \) the second term will vanish automatically. Taking into account that \( \chi_{i_1} A_{i_2...i_p} \) are totally antisymmetric in all their indices, for a (locally) decomposable \( \pi, \) this equation gives the differential condition
\[
\text{(3.4).}
\]
Before we summarize the above discussion, let us note that the decomposability assumption of \( \pi \) is quite natural as the following remarks show.

**Remark 7.1.** With a decomposable \( \pi, \) on \( L^\pi = L_{\pi} \) also the products \( \psi^{i_1} \eta^{i_2...i_{p-1}} \) are antisymmetric in all their indices. Using the decomposability of \( \pi \) in the form of Lemma \textbf{(3.6)} and the antisymmetry of the products \( \chi_{i_1} A_{i_2...i_p} \) it is easy to see that
\[
p \psi^{i_1} \eta^{i_2...i_{p-1}} = \pi^{i_1...i_p} A_{i_1...i_{p-1}} \chi_{i_p} \pi^{i_1...i_p}, \text{ from where the claim follows.}
\]
Remark 7.2. Also, the decomposability of $\pi$ – at a point of $V$ where $\pi$ is nonzero – is a necessary condition for the second term to vanish independently of $c$. From the condition $\pi^1 \cdots \pi^{p+1} A_1 \cdots A_{p+1} = 0$ it follows that (locally) the rank of the map $\pi^2 : \Omega^{p-1}(V) \to \mathfrak{X}(V)$ has to be smaller than $(p + 1)$. On the other hand, since $\pi$ is of order $p$ the rank of this map has to be at least $p$. Hence, the rank of $\pi^2 : \Omega^{p-1}(V) \to \mathfrak{X}(V)$ is $p$. This is equivalent to the decomposability of $\pi$.

Now we can summarize the above discussion.

**Theorem 7.3.** Let $L_\pi$ be the Lagrangian submanifold given by $F_i = \psi^i = \eta^I = G_I = 0$ and let $L'_\pi \subset L_\pi$ be its submanifold given by the conditions $H^I = 0$ and $\chi_i A_{i_2 \cdots i_p} + \chi_{i_k} A_{i_2 \cdots i_{k-1} i_{k+1} \cdots i_p} = 0$. Also, let the superfields restrict at the boundary to maps into a submanifold $L'_\pi \subset L_\pi$. Let, further, $c$ be a closed $p+1$-form and $\pi$ a Nambu-Poisson tensor of order $p$, for $p \geq 2$. Then the bulk/boundary action $S_{c,\pi}$ of Eqn. (7.1) is a BV action.\(^{22}\)

For the converse statement, see Remark 7.2 above. We finish this section with the following remark.

**Remark 7.4.** Let us note that the equation (7.2) gives the graph of $\pi^2 : \Omega^{p-1}(V) \to \mathfrak{X}(V)$ in $\mathfrak{X}(V) \oplus \Omega^{p-1}(V)$ and the equation (7.3) the graph of the map dual to $\pi^2$. Let us also note that the condition (7.3) was not used at all in order to derive the integrability condition (7.6). In [22] and [57] higher Dirac structures (Nambu-Dirac structures) on $\mathfrak{X}(V) \oplus \Omega^{p-1}(V)$, or more generally on $A \oplus \wedge^{p-1} A^*$, $A$ being a Lie algebroid, were defined. The graph of the map $\pi^2$ corresponding to a $p$-tensor $\pi$ is an example of a higher Dirac structure iff $\pi$ is a Nambu-Poisson tensor. It would be interesting to further explore, similarly to [25], the relation between boundary conditions and higher Dirac structures.

**Gauge transformations.** Recall Corollary 5.7. Consider the degree $p$ function $\alpha$, $\alpha|_{L'} = 0$, $\{\alpha, \alpha\} = 0$, where $\alpha = \frac{1}{p!}(b_{i_1 \cdots i_p} \psi^{i_1} \eta^{i_2 \cdots i_p} - \partial_{i_1} b_{i_2 \cdots i_{p+1}} \psi^{i_1} \psi^{i_2} H^{i_3 \cdots i_{p+1}})$, with $b$ being a $p$-form on $V$. Let us recall that the canonical transformation generated by $\alpha$, results in the gauge transformation $c \mapsto c - dB$. Such a canonical transformation preserves both $L$ as well as $L'$.

\(^{22}\)For $p = 2$, we can also assume a not necessarily decomposable $\pi$ and choose consistently the submanifold $L'_\pi$ of $L_\pi$ given by additional conditions $A_i = \chi_i$. Then the condition (7.6) means that $\pi$ is a Poisson structure twisted by $c$. 
Also, consider the degree $p$ function $\beta$, $\{\beta, \beta\} = 0$, given as
\[
\beta = \frac{1}{(p-1)!} \pi^{i_1i_2...i_p} \chi_{i_1} A_{i_2...i_p},
\]
with $\pi$ being a Nambu-Poisson tensor. We are looking for the solution to the factorization problem $e^{\delta_{\alpha}e^{\delta_{\beta}}} = e^{\delta_{\beta'}e^{\delta_{\alpha'}}}$ (cf. Corollary 5.7). In order to solve it, we can follow the strategy of [25] in the case of the topological open membrane. For this we replace $\alpha$ by $t\alpha$. Then $\alpha'$ and $\beta'$ will depend on $t$ and therefore will be denoted as $\alpha'_t$ and $\beta'_t$, respectively. Obviously, $\beta'_0 = \beta$. Inspired by Corollary 3.9, we take $\beta'_t$ again of the form $\beta'_t = \frac{1}{(p-1)!} \pi^t_{ij_1j_2...j_p} \chi'_{i_1} A_{j_2...j_p}$, with $\pi^t_{ij}$ being an antisymmetric $p$-tensor field. We have
\[
\frac{d}{dt}(e^{-\delta_{\alpha'}e^{\delta_{\beta'}e^{\delta_{\beta'}}}}) = e^{-\delta_{\alpha'}e^{\delta_{\beta'}e^{\delta_{\beta'}}}} - \delta_{\alpha'}e^{-\delta_{\alpha'}e^{\delta_{\beta'}}} e^{\delta_{\beta'}} = \delta_{\alpha'}e^{-\delta_{\alpha'}e^{\delta_{\beta'}}} e^{\delta_{\beta'}} ,
\]
with
\[
\epsilon_t = e^{-\delta_{\beta'}\alpha} - \beta'_t
\]
\[
= \frac{1}{p!}(b_{i_1...i_p} \psi_t^{i_1} \eta_t^{i_2...i_p} - \partial_i b_{i_2...i_{p+1}} \psi_t^{i_1} \psi_t^{i_2} H_{i_3...i_{p+1}}) - \frac{1}{(p-1)!} \pi^t_{ij_1i_2...i_p} A_{i_1...i_p} \chi_{i_p},
\]
where
\[
\psi_t^i = \psi^i - \pi^t_i A_j ,
\]
and
\[
\eta_t^I = \eta^I + (-1)^p \pi^t_I j \chi_j .
\]
Since we ask $\epsilon_t$ to vanish on $L'$, the terms proportional to the products $\chi A$ have to be zero. The resulting differential equation for $\pi^t_{ij}$, with the initial condition $\pi^t_0 = \pi^t$ has the solution
\[
\pi^t_{ij} = \pi^t \circ (1 + b_t \circ \pi^t)^{-1} = (1 + (-1)^{p-1} b(\pi))^{-1} \pi^t .
\]
Hence, $\pi' = \pi'_1$ is a Nambu-Poisson tensor, gauge equivalent to $\pi$. Fortunately, we do not need the the explicit form of $\alpha'$, which is the solution to $\frac{d}{dt} e^{\delta_{\alpha'}} = \delta_{\alpha'} e^{\delta_{\alpha'}}$, $\alpha'_0 = 0$. It is enough to know that $\alpha'|_{L'} = 0$ which guaranteed by $\epsilon_t$ vanishing on $L'$.

Let us finish this subsection by recalling that in the case of an exact $b$ the gauge transformation between $\pi$ and $\pi'$ can be identified as the Seiberg-Witten map, see Proposition 3.11 ff. 

\[23\] See Remark 3.10.
Nambu-Poisson sigma model. On shell, using the equations of motion for \( F \)’s and \( G \)’s, we obtain the closed (twisted) Nambu-Poisson sigma model, cf. also [32]

\[
S_{(\pi,c)} = \int_{\partial X} \chi_i D X^i
+ \int_{\partial X} \left( \frac{1}{(p-1)!} A_{i_1 \ldots i_{p-1}} D X^{i_1} \ldots D X^{i_{p-1}} + \frac{1}{(p-1)!} \pi^{i_1 i_2 \ldots i_p} A_{i_1 \ldots i_{p-1}} \chi_{i_p} \right)
+ \int_X \frac{1}{(p+1)!} c_{i_1 \ldots i_{p+1}} D X^{i_1} \ldots D X^{i_{p+1}}.
\] (7.7)

Remark 7.5. Let us note that from equations of motion for \( \chi \)’s and \( A \)’s

\[
D X^i = \frac{1}{(p-1)!} \pi^{iJ} A_J,
\]

and

\[
D X^{i_1} \ldots D X^{i_{p-1}} = \pi^{i_1 \ldots i_{p-1} i_p} \chi_{i_p}.
\]

it follows that the products

\[
\pi^{iJ} A_J \pi^{i_1 \ldots i_{p-1} i_p} \chi_{i_p}
\]

must be antisymmetric in indices \( i, i_1 \ldots, i_{p-1} \), which is consistent due to the decomposability of \( \pi \) and the antisymmetry of the products \( \chi_i A_{i_1 \ldots i_{p-1}} \) (cf. Remark 7.1).

Remark 7.6. In order to obtain the open Nambu-Poisson sigma model, we can follow the idea of [25], cf. Remark 6.1 and include boundaries with corners and allow for different boundary conditions on various regions of the boundary. If, for example, the boundary \( \partial X \) is divided in two regions, on the first one takes the same boundary conditions as before and on the second one restricts the superfields to only \( V \). The interface of the two regions can then be viewed as the boundary of the first region. On the interface live only superfields corresponding to functions on \( V \).

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