“Minimal geometric data” approach to Dirac algebra, spinor groups and field theories

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Abstract

The three first sections contain an updated, not-so-short account of a partly original approach to spinor geometry and field theories introduced by Jadczynk and myself [3, 4, 5]; it is based on an intrinsic treatment of 2-spinor geometry in which the needed background structures have not to be assumed, but rather arise naturally from a unique geometric datum: a vector bundle with complex 2-dimensional fibres over a real 4-dimensional manifold. The two following sections deal with Dirac algebra and 4-spinor groups in terms of two spinors, showing various aspects of spinor geometry from a different perspective. The last section examines particle momenta in 2-spinor terms and the bundle structure of 4-spinor space over momentum space.

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Introduction

The precise equivalence between the 4-spinor and 2-spinor settings for electrodynamics was exposed by Jadczyn and myself in \[2, 3, 4, 5\]. In summary one sees that, from an algebraic point of view, the only notion of a complex 2-dimensional vector space $S$ yields, naturally and without any further assumptions, all the needed algebraic structures through functorial constructions; conversely in a 4-spinor setting, provided one makes the minimum assumptions which are needed in order to formulate the standard physical theory, the 4-spinor space naturally splits (Weyl decomposition) into the direct sum of two 2-dimensional subspaces which are anti-dual to each other. In a sense, which setting one regards as fundamental is then mainly a matter of taste. The 4-spinor setting is closer to standard notations, and some formulas can be written in a more compact way, while the relations among the various objects are somewhat more involved. The 2-spinor setting turns out to give a much more direct formulation, in which all the basic objects and the relations among them naturally set into their places; just from $S$ one automatically gets exactly the needed algebraic structure, nothing more, nothing less: 4-spinor space $W$ with the ‘Dirac adjoint’ anti-isomorphism, Minkowski space $H$ and Dirac map $\gamma : H \to \text{End}(W)$ with the required properties. Further objects which are commonly considered depend on the choice of a gauge of some sort, whose nature is precisely described.

When we consider a vector bundle $S \to M$, where now the fibres are complex 2-dimensional and $M$ is a real 4-dimensional manifold, then we don’t have to assign any further background structure in order to formulate a full Einstein-Cartan-Maxwell-Dirac theory. In fact we naturally get a vector bundle $H \to M$ whose fibres are
Minkowski spaces, a 4-spinor bundle \( W \to M \) and so on. Any object which is not determined by geometric construction from the unique geometric datum \( S \to M \) is a field of the theory, namely we consider: the tetrad \( \Theta : TM \to L \otimes H \), the 2-spinor connection \( \Gamma \), the electromagnetic and Dirac fields. (Even coupling factors naturally arise as covariantly constant sections of the real line bundle \( L \) of length units, which is geometrically constructed from \( S \).) The gravitational field is described by the tetrad (which can be seen as a ‘square root’ of spacetime metric) and by the connection induced by \( F \) on \( H \), while the remaining part of the spinor connection can be viewed as the electromagnetic potential. A natural Lagrangian density for all these fields is then introduced; the relations between metric and connection and between e.m. potential and e.m. field follow from the (Euler-Lagrange) field equations. All considered, this setting has some original aspects but is not in contrast to the (by now classical) Penrose formalism \([12]\).

In §4 and §5 I’ll show how the above said algebraic setting, and in particular the natural splitting of the 4-spinor space into the direct sum of its Weyl subspaces, enables us to examine the structures of the Dirac algebra, the Clifford group and its subgroups from a different perspective.

In §6 I’ll show the strict relation existing between the two-spinor setting and the geometry of particle momenta, in particular the bundle structure of \( W \) over the space of momenta. These results are a preparation to a 2-spinor formulation of quantum electrodynamics along lines of a previous paper \([6]\), in which the classical structure underlying electron states is a 2-fibred bundle over spacetime.

1 Two-spinor geometry

In this section we’ll see how all the fundamental geometric structures needed for Dirac theory naturally arise through functorial constructions from a two-dimensional complex vector space, with no further assumptions.

1.1 Complex conjugated spaces

If \( A \) is a set and \( f : A \to \mathbb{C} \) is any map, then \( \overline{f} : A \to \mathbb{C} : a \mapsto \overline{f(a)} \) is the conjugated map. Let \( V \) be a complex vector space of finite-dimension \( n \); its dual space \( V^* \) and antidual space \( \overline{V}^* \) are defined to be the \( n \)-dimensional complex vector spaces of all maps \( V \to \mathbb{C} \) which are respectively linear and antilinear. One then has the distinguished anti-isomorphism \( V^* \to \overline{V}^* : \lambda \mapsto \overline{\lambda} \).

Set now \( \overline{V} := V^{**} \), and call this the conjugate space of \( V \). One has the natural isomorphisms

\[ V \cong V^{**} \cong \overline{V}^{**}, \quad \overline{V} := V^{**} \cong \overline{V}^{**}. \]

Summarizing, one one gets the four distinct spaces

\[ V \leftrightarrow \overline{V}, \quad V^* \leftrightarrow \overline{V}^*, \]

where the arrows indicate the conjugation anti-isomorphisms.

Accordingly, coordinate expressions have four types of indices. Let \((b_A), 1 \leq A \leq n\), be a basis of \( V \) and \((b^A)\) its dual basis of \( V^* \). The corresponding indices in the conjugate spaces are distinguished by a dot, namely one writes

\[ \bar{b}_A := \overline{b_A}, \quad \bar{b}^A := \overline{b^A}. \]
so that \( \{ b_\lambda \} \) is a basis of \( V \) and \( \{ b^A \} \) its dual basis of \( V^* \). For \( v \in V \) and \( \lambda \in V^* \) one has

\[
\begin{align*}
v &= v^A b_A, & \bar{v} &= \bar{v}^\lambda b_\lambda, \\
\lambda &= \lambda_A b^A, & \bar{\lambda} &= \bar{\lambda}_A \bar{b}^A,
\end{align*}
\]

where \( \bar{v}^A = \bar{v}^\lambda b_\lambda \), \( \bar{\lambda}_A := \bar{\lambda}_A \bar{b}^A \) and Einstein summation convention is used.

The conjugation morphism can be extended to tensors of any rank and type; if \( \tau \) is a tensor then all indices of \( \tau \) are of reversed (dotted/non-dotted) type; observe that dotted indices cannot be contracted with non-dotted indices. In particular if \( K \in \text{Aut}(V) \subset V \otimes V^* \) then \( \bar{K} \in \text{Aut}(V) \subset V^* \otimes V^* \) is the induced conjugated transformation (under a basis transformation, dotted indices transform with the conjugate matrix).

### 1.2 Hermitian tensors

The space \( V \otimes \bar{V} \) has a natural real linear (complex anti-linear) involution \( w \mapsto \bar{w} \), which on decomposable tensors reads

\[
(u \otimes \bar{v})^\dagger = v \otimes \bar{u}.
\]

Hence one has the natural decomposition of \( V \otimes \bar{V} \) into the direct sum of the real eigenspaces of the involution with eigenvalues \( \pm 1 \), respectively called the Hermitian and anti-Hermitian subspaces, namely

\[
V \otimes \bar{V} = (\bar{V} \wedge V) \oplus \mathbb{i}(V \wedge \bar{V}).
\]

In other terms, the Hermitian subspace \( V \wedge \bar{V} \) is constituted by all \( w \in V \otimes \bar{V} \) such that \( \bar{w} = w \), while an arbitrary \( w \) is uniquely decomposed into the sum of an Hermitian and an anti-Hermitian tensor as

\[
w = \frac{1}{2}(w + w^\dagger) + \frac{1}{2}(w - w^\dagger).
\]

In terms of components in any basis, \( w = w^{\alpha \beta} b_\alpha \otimes \bar{b}_\beta \) is Hermitian (anti-Hermitian) iff the matrix \( (w^{\alpha \beta}) \) of its components is of the same type, namely \( \bar{w}^{\beta \alpha} = \pm w^{\alpha \beta} \).

Obviously \( V^* \otimes \bar{V}^* \) decomposes in the same way, and one has the natural isomorphisms

\[
(V \wedge \bar{V})^* \cong V^* \wedge \bar{V}^*, \quad (i V \wedge \bar{V})^* \cong i V^* \wedge \bar{V}^*,
\]

where * denotes the real dual.

A Hermitian 2-form is defined to be a Hermitian tensor \( h \in \bar{V}^* \wedge V^* \). The associated quadratic form \( v \mapsto h(v, v) \) is real-valued. The notions of signature and non-degeneracy of Hermitian 2-forms are introduced similarly to the case of real bilinear forms. If \( h \) is non-degenerate then it yields the isomorphism \( h^\#: \bar{V} \to V^* : \bar{v} \mapsto h(\bar{v}, \cdot) \); its conjugate map is an anti-isomorphism \( \bar{V} \to \bar{V}^* \) which, via composition with the canonical conjugation, can be seen as the isomorphism \( \bar{h}^\#: \bar{V} \to \bar{V}^* : v \mapsto h(\cdot, v) \). The inverse isomorphisms \( h^\# \) and \( \bar{h}^\# \) are similarly related to a Hermitian tensor \( h^{-1} \in \bar{V} \wedge V \). One has the coordinate expressions

\[
\begin{align*}
(h^\#(\bar{v}))_B &= h_{AB} \bar{v}^A, & (\bar{h}^\#(v))_A &= h_A^B v^B, \\
(h^\#(\lambda))^A &= h^{AB} \lambda_B, & (\bar{h}^\#(\lambda))^\lambda &= h_B^A \lambda_B \lambda^A,
\end{align*}
\]

where \( h^{CA} h_{CB} = \delta^A_B \), \( h^{AC} h_{BC} = \delta^A_C \).
1.3 Two-spinor space

Let $S$ be a 2-dimensional complex vector space. Then $\wedge^2 S$ is a 1-dimensional complex vector space; its dual space $(\wedge^2 S)^\ast$ will be identified with $\wedge^2 S^\ast$ via the rule

$$\omega(s \wedge s') := \frac{1}{2} \omega(s, s') , \quad \forall \omega \in \wedge^2 S^\ast, \ s, s' \in S .$$

Any $\omega \in \wedge^2 S^\ast \setminus \{0\}$ (a ‘symplectic’ form on $S$) has a unique ‘inverse’ or ‘dual’ element $\omega^{-1}$. Denoting by $\omega^b : S \rightarrow S^\ast$ the linear map defined by $\langle \omega^b(s), t \rangle := \omega(s, t)$ and by $\omega^\# : S^\ast \rightarrow S$ the linear map defined by $\langle \mu, \omega^\#(\lambda) \rangle := \omega^{-1}(\lambda, \mu)$, one has

$$\omega^\# = -(\omega^b)^{-1} .$$

The Hermitian subspace of $(\wedge^2 S) \otimes (\wedge^2 S^\ast)$ is a 1-dimensional real vector space with a distinguished orientation, whose positively oriented semispace

$$\mathbb{L}^2 := [(\wedge^2 S) \vee (\wedge^2 S^\ast)]^+ := \{w \otimes \bar{w}, \ w \in \wedge^2 S\}$$

has the square root semi-space $\mathbb{L}$, called the space of *length units*.\(^1\)

Next, consider the complex 2-dimensional space

$$U := \mathbb{L}^{-1/2} \otimes S .$$

This is our 2-spinor space. Observe that the 1-dimensional space

$$Q := \wedge^2 U = \mathbb{L}^{-1} \otimes \wedge^2 S$$

has a distinguished Hermitian metric, defined as the unity element in

$$\overline{Q}^\ast \vee Q^\ast \equiv (\wedge^2 U^\ast) \vee (\wedge^2 S^\ast) = \mathbb{L}^{-2} \otimes (\wedge^2 S^\ast) \vee (\wedge^2 S^\ast) \cong \mathbb{R} .$$

Hence there is the distinguished set of normalized symplectic forms on $U$, any two of them differing by a phase factor.\(^2\)

Consider an arbitrary basis $(\xi_A)$ of $S$ and its dual basis $(x^A)$ of $S^\ast$. This determines the mutually dual bases

$$w := \varepsilon^{AB} \xi_A \wedge \xi_B, \quad w^{-1} := \varepsilon_{AB} x^A \wedge x^B ,$$

respectively of $\wedge^2 S$ and $\wedge^2 S^\ast$ (here $\varepsilon^{AB}$ and $\varepsilon_{AB}$ both denote the antisymmetric Ricci matrix), and the basis

$$l := \sqrt{w \otimes w} \text{ of } \mathbb{L} .$$

Then one also has the induced mutually dual, normalized bases

$$\left(\zeta_A\right) := (l^{-1/2} \otimes \xi_A) , \quad (z^A) := (l^{1/2} \otimes x^A) .$$

---

1. Here, $s \wedge s' := \frac{1}{2}(s \otimes s' - s' \otimes s)$. This contraction, defined in such a way to respect usual conventions in two-spinor literature, corresponds to $1/4$ standard exterior-algebra contraction.
2. A unit space is defined to be a 1-dimensional real semi-space, namely a positive semi-field associated with the semi-ring $\mathbb{R}^+$ (see [1, 2] for details). The square root $\mathbb{L}^{1/2}$ of a unit space $\mathbb{L}$, is defined by the condition that $\mathbb{L}^{1/2} \otimes \mathbb{L}^{1/2}$ be isomorphic to $\mathbb{U}$. More generally, any rational power of a unit space is defined up to isomorphism (negative powers correspond to dual spaces). In this article we only use the unit space $\mathbb{L}$ of lengths and its powers; essentially, this means that we take $\hbar = c = 1$.
3. One says that elements of $U$ and of its tensor algebra are ‘conformally invariant’, while tensorializing by $\mathbb{L}^r$ one obtains ‘conformal densities’ of weight $r$. 
of $\mathbf{U}$ and $\mathbf{U}^*$, and also
\[
\varepsilon := l \otimes w^{-1} = \varepsilon_{AB} z^A \wedge z^B \in \mathbf{Q}^* \equiv \Lambda^2 \mathbf{U}^* ,
\]
\[
\varepsilon^{-1} \equiv l^{-1} \otimes w = \varepsilon^{AB} \zeta_A \wedge \zeta_B \in \mathbf{Q} \equiv \Lambda^2 \mathbf{U} .
\]

**Remark.** In contrast to the usual 2-spinor formalism, no symplectic form is fixed. The 2-form $\varepsilon$ is unique up to a phase factor which depends on the chosen 2-spinor basis, and determines isomorphisms
\[
\varepsilon^b : \mathbf{U} \to \mathbf{U}^* : u \mapsto u^b , \quad \langle u^b, v \rangle := \varepsilon(u, v) \Rightarrow (u^b)_B = \varepsilon_{AB} v^A ,
\]
\[
\varepsilon^\#: \mathbf{U}^* \to \mathbf{U} : \lambda \mapsto \lambda^# , \quad \langle \lambda^#, \mu \rangle := \varepsilon^{-1}(\lambda, \mu) \Rightarrow (\lambda^#)_B = \varepsilon^{AB} \lambda_A .
\]
If no confusion arises, we’ll make the identification $\varepsilon^\# \equiv \varepsilon^{-1}$.

### 1.4 2-spinors and Minkowski space

Though a normalized element $\varepsilon \in \mathbf{Q}^*$ is unique only up to a phase factor, certain objects which can be expressed through it are natural geometric objects. The first example is the unity element in $\mathbf{Q}^* \otimes \overline{\mathbf{Q}^*}$, which can be written as $\varepsilon \otimes \overline{\varepsilon}$; it can also be seen as a bilinear form $g$ on $\mathbf{U} \otimes \overline{\mathbf{U}}$, given for decomposable elements by
\[
g(p \otimes \overline{q}, r \otimes \overline{s}) = \varepsilon(p, r) \overline{\varepsilon}(\overline{q}, \overline{s}) .
\]

The fact that any $\varepsilon$ is non-degenerate implies that $g$ is non-degenerate too. In a normalized 2-spinor basis $(\zeta_A)$ one writes $w = w^{A\dot{A}} \zeta_A \otimes \overline{\zeta}_{\dot{A}} \in \mathbf{U} \otimes \overline{\mathbf{U}}$, $g_{A\dot{A}B\dot{B}} = \varepsilon_{AB} \overline{\varepsilon}_{A\dot{A}B\dot{B}}$ and
\[
g(w, w) = \varepsilon_{AB} \overline{\varepsilon}_{A\dot{A}B\dot{B}} w^{A\dot{A}} w^{B\dot{B}} = 2 \det w .
\]

Next, consider the Hermitian subspace
\[
\mathbf{H} := \mathbf{U} \vee \overline{\mathbf{U}} \subset \mathbf{U} \otimes \overline{\mathbf{U}} .
\]

This is a 4-dimensional real vector space; for any given normalized basis $(\zeta_A)$ of $\mathbf{U}$ consider, in particular, the *Pauli basis* $(\tau_\lambda)$ of $\mathbf{H}$ associated with $(\zeta_A)$, namely
\[
\tau_\lambda \equiv \tau^{A\dot{A}}_\lambda \zeta_A \otimes \overline{\zeta}_{\dot{A}} \equiv \frac{1}{\sqrt{2}} \sigma^{A\dot{A}}_\lambda \zeta_A \otimes \overline{\zeta}_{\dot{A}} , \quad \lambda = 0, 1, 2, 3 ,
\]
where $(\sigma^{A\dot{A}}_\lambda)$ denotes the $\lambda$-th Pauli matrix.\(^4\)

The restriction of $g$ to the Hermitian subspace $\mathbf{H}$ turns out to be a Lorentz metric with signature $(+, -, -, -)$. Actually, a Pauli basis is readily seen to be orthonormal, namely $g_{\lambda\mu} := g(\tau_\lambda, \tau_\mu) = \eta_{\lambda\mu} := 2 \delta^0_\lambda \delta^0_\mu - \delta_{\lambda\mu}$.

It’s not difficult to prove:

**Proposition 1.1** An element $w \in \mathbf{U} \otimes \overline{\mathbf{U}} = \mathbb{C} \otimes \mathbf{H}$ is null, that is $g(w, w) = 0$, iff it is a decomposable tensor: $w = u \otimes \overline{s}$, $u, s \in \mathbf{U}$.

\(^4\) Note how $\det w \equiv \det(\overline{w^{A\dot{A}}})$ is intrinsically defined through $\varepsilon$, even if $w$ is not an endomorphism.

\(^5\) $\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. 
A null element in $U \otimes \overline{U}$ is also in $H$ iff it is of the form $\pm u \otimes \overline{u}$. Hence the null cone $N \subset H$ is constituted exactly by such elements. Note how this fact yields a way of distinguish between time orientations: by convention, one chooses the future and past null-cones in $H$ to be, respectively,

$$N^+ := \{ u \otimes \overline{u}, \ u \in U \} , \quad N^- := \{ -u \otimes \overline{u}, \ u \in U \} .$$

**Proposition 1.2** For each $g$-orthonormal positively oriented basis $(e_\lambda)$ of $H$, such that $e_0$ is timelike and future-oriented, there exists a normalized 2-spinor basis $(\zeta_A)$ whose associated Pauli basis $(\tau_\lambda)$ coincides with $(e_\lambda)$.

**Remark.** From the above proposition it follows that any future-pointing timelike vector can be written as $u \otimes \overline{u} + v \otimes \overline{v}$, for suitable $u, v \in U$.

### 1.5 From 2-spinors to 4-spinors

Next observe that an element of $U \otimes \overline{U}$ can be seen as a linear map $U^* \rightarrow U$, while an element of $U^* \otimes U^*$ can be seen as a linear map $U \rightarrow U^*$. Then one defines the linear map

$$\gamma : U \otimes \overline{U} \rightarrow \text{End}(U \oplus \overline{U}^*) : y \mapsto \gamma(y) := \sqrt{2} (y, y^\flat^*) ,$$

i.e.

$$\gamma(y)(u, \chi) = \sqrt{2}(y|\chi, u|y^\flat^*) ,$$

where $y^\flat := g^\flat(y) \in U^* \otimes \overline{U}^*$ and $y^\flat^* \in \overline{U}^* \otimes U^*$ is the transposed tensor. In particular for a decomposable $y = p \otimes \overline{q}$ one has

$$\tilde{\gamma}(p \otimes \overline{q})(u, \chi) = \sqrt{2}(\langle \chi, \overline{q} \rangle p, \langle p^\flat, u \rangle \overline{q}^\flat) .$$

**Proposition 1.3** For all $y, y' \in U \otimes \overline{U}$ one has

$$\gamma(y) \circ \gamma(y') + \gamma(y') \circ \gamma(y) = 2g(y, y') 11 .$$

**Proof:** It is sufficient to check the statement’s formula for any couple of null i.e. decomposable elements in $U \otimes \overline{U}$. Using the identity

$$\varepsilon(p, q) r^\flat + \varepsilon(q, r) p^\flat + \varepsilon(r, p) q^\flat = 0 , \quad p, q, r \in U ,$$

which is in turn easily checked, a straightforward calculation gives

$$[\gamma(p \otimes \overline{q}) \circ \gamma(r \otimes \overline{s}) + \gamma(r \otimes \overline{s}) \circ \gamma(p \otimes \overline{q})](u + \chi) =$$

$$= 2 \varepsilon(p, r) \varepsilon(\overline{q}, \overline{s}) (u, \chi) = 2g(p \otimes \overline{q}, r \otimes \overline{s})(u, \chi) .$$

Now one sees that $\gamma$ is a Clifford map relatively to $g$ (see also §4.1); thus one is led to regard

$$W := U \oplus \overline{U}^*$$

as the space of Dirac spinors, decomposed into its Weyl subspaces. Actually, the restriction of $\gamma$ to the Minkowski space $H$ turns out to be a Dirac map.
The 4-dimensional complex vector space $W$ is naturally endowed with a further structure: the obvious anti-isomorphism

$$W \to W^* : (u, \chi) \mapsto (\bar{\chi}, \bar{u}) .$$

Namely, if $\psi = (u, \chi) \in W$ then $\bar{\psi} = (\bar{u}, \bar{\chi}) \in \overline{W}$ can be identified with $(\bar{\chi}, \bar{u}) \in W^*$; this is the so-called ‘Dirac adjoint’ of $\psi$. This operation can be seen as the “index lowering anti-isomorphism” related to the Hermitian product

$$k : W \times W \to \mathbb{C} : \left((u, \chi), (u', \chi')\right) \mapsto \langle \bar{\chi}, u' \rangle + \langle \chi', \bar{u} \rangle ,$$

which is obviously non-degenerate; its signature turns out to be $(+ + - -)$, as it can be seen in a “Dirac basis” (below).

Let $(\zeta_\alpha)$ be a normalized basis of $U$; the Weyl basis of $W$ is defined to be the basis $(\zeta_\alpha)$, $\alpha = 1, 2, 3, 4$, given by

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) := (\zeta_1, \zeta_2, -\bar{z}_1, -\bar{z}_2) .$$

Above, $\zeta_1$ is a simplified notation for $(\zeta_1, 0)$, and the like. Another important basis is the Dirac basis $(\zeta_\alpha')$, $\alpha = 1, 2, 3, 4$, where

$$\zeta_1' = \frac{1}{\sqrt{2}} (\zeta_1, \bar{z}_1) \equiv \frac{1}{\sqrt{2}} (\zeta_1 - \zeta_3) , \qquad \zeta_2' = \frac{1}{\sqrt{2}} (\zeta_2, \bar{z}_2) \equiv (\zeta_2 - \zeta_4) ,$$

$$\zeta_3' = \frac{1}{\sqrt{2}} (\zeta_1, -\bar{z}_1) \equiv (\zeta_1 + \zeta_3) , \qquad \zeta_4' = \frac{1}{\sqrt{2}} (\zeta_2, -\bar{z}_2) \equiv (\zeta_2 + \zeta_4) .$$

Setting

$$\gamma_\lambda := \gamma(\tau_\lambda) \in \text{End}(W)$$

one recovers the usual Weyl and Dirac representations as the matrices $(\gamma_\lambda)$, $\lambda = 0, 1, 2, 3$, in the Weyl and Dirac bases respectively.

### 1.6 Further structures

Some other operations on 4-spinor space, commonly used in the literature, actually depend on particular choices or conventions. Similarly to the choice of a basis or of a gauge they are useful in certain arguments or calculations, but don’t need to be fixed in the theory’s foundations. I’ll describe the cases of a Hermitian form on $U$, of charge conjugation, parity and time reversal; I’ll show the relations among these objects and how they are related to the notion of observer.

A Hermitian 2-form $h$ on $U$ is an element in $\overline{U^*} \vee U^*$, hence it can be seen as an element in $H^*$; more precisely, $\bar{h} \in H^*$. One says that $h$ is normalized if it is non-degenerate, positive and $g^\#(h) = h^{-1}$; the latter condition is equivalent to $g(h, h) = 2$. If $h$ is normalized then it is necessarily a future-pointing timelike element in $H^*$. For example, consider the Pauli basis $(\tau_\lambda)$ determined by a normalized 2-spinor basis $(\zeta_\lambda)$, and let $(t^A)$ be the dual basis; then $\sqrt{2} t^0 = \bar{z}_1 \otimes z^1 + \bar{z}_2 \otimes z^2$ is normalized; conversely, every positive-definite normalized Hermitian metric $h$ can be expressed in the above form for some suitable normalized 2-spinor bases\(^6\)

---

\(^6\) Similarly, negative-definite Hermitian metrics correspond to past-pointing timelike covectors. Hermitian metrics of mixed signature $(1, -1)$ correspond to spacelike covectors; actually, such metrics can always be written as proportional to $\sqrt{2} t^0 = \bar{z}_1 \otimes z^1 - \bar{z}_2 \otimes z^2$, in appropriate normalized 2-spinor bases.
1.7 2-spinor groups

The basic observation resulting from the above discussion is that the assignments of an ‘observer’ in $H$ and of a positive-definite Hermitian metric on $U$ are equivalent; actually, the two objects are nearly the same thing. In 4-spinor terms, the above equivalence is only slightly less obvious. If $h$ is assigned, then it extends naturally to a Hermitian metric $h$ on $W$, which can be characterized by

$$h(ψ, φ) = k(γ_0ψ, φ).$$

Charge conjugation depends on the choice of a normalized 2-form $ω = e^{i t} ε ∈ \wedge^2 U^*$, and is defined as the anti-isomorphism

$$C_ω: W → W : ψ → C_ω(ψ) ≡ C(u, χ) = (ω^#(χ), −ω^♭(u)) = e^{−it} (ε^#(χ), −ε^♭(u)).$$

Thus $C_ω = e^{−it} C_ε$. One also gets

$$C_ω ◦ C_ω = 11_W,$$

$$γ_y ◦ C_ω + C_ω ◦ γ_y = 0 ⇔ C_ω ◦ γ_y ◦ C_ω = −γ_y, \quad y ∈ H.$$

Finally, parity is an isomorphism of $W$ dependent on the choice of an observer, while time-reversal is an anti-isomorphism dependent on the choice of an observer and of a normalized 2-form; they are defined by

$$P := γ_0 ≡ γ(τ_0), \quad T_ω := γ_ηγ_0C_ω,$$

where the chosen observer is expressed as $τ_0$ in a suitable Pauli basis, and $γ_η$ is the canonical element of the Dirac algebra corresponding to the $g$-normalized volume form of $H$, and expressed in a Pauli basis as $γ_η = γ_0γ_1γ_2γ_3$ (see §4.1).

Remark. An observer, seen as a Hermitian metric on $U$, also determines an isomorphism $U ⊗ U^* → U ⊗ U^{∗} ≡ End(U)$. Through it, one can view ‘world spinors’ as endomorphisms, thus recovering the algebraic structure for the Galilean treatment of spin [1].

1.7 2-spinor groups

The group $Aut(S) ≅ Aut(U) ⊂ U ⊗ U^*$ has the natural subgroups

$$\text{Sl}(U) := \{ K ∈ Aut(U) : \det K = 1\}, \quad \dim_{\mathbb{C}} \text{Sl}(U) = 3,$$

$$\text{Sl}^c(U) := \{ K ∈ Aut(U) : |\det K| = 1\}, \quad \dim_{\mathbb{R}} \text{Sl}^c(U) = 7.$$

The former is the group of all automorphisms of $S$ (of $U$) which leave any complex volume form invariant; the latter is the group of all automorphisms which leave any complex volume form invariant up to a phase factor, and thus it can be seen as the group which preserves the two-spinor structure. One has the Lie algebras

$$\mathfrak{L}\text{Sl}(U) ≅ \{ A ∈ End(U) : \text{Tr} A = 0\},$$

$$\mathfrak{L}\text{Sl}^c(U) ≅ \{ A ∈ End(U) : \Re \text{Tr} A = 0\} = i\mathbb{R} ⊕ \mathfrak{L}\text{Sl}(U).$$

7 In the traditional notation, $γ_λ^\dagger$ indicates the $h$-adjoint of $γ_λ$, and then depends on the chosen observer.
If \( h \in U^* \oplus \overline{U}^* \) is a positive Hermitian metric then one sets

\[
U(U, h) := \{ K \in \text{Aut}(U) : K^\dagger = K^{-1} \} \subset \text{Sl}(U),
\]

\[
SU(U, h) := \{ K \in \text{Aut}(U) : K^\dagger = K^{-1}, \quad \det K = 1 \} \subset \text{Sl}(U),
\]

where \( K^\dagger \) denotes the \( h \)-adjoint of \( K \). One gets the Lie algebras

\[
L U(U, h) = \{ A \in \text{End}(U) : A + A^\dagger = 0 \} = i \mathbb{R} \oplus L SU(U, h),
\]

\[
L SU(U, h) = \{ A \in \text{End}(U) : A + A^\dagger = 0, \quad \text{Tr} A = 0 \}.\]

Now observe that \( \text{End}(U) \) can be decomposed into the direct sum of the subspaces of all \( h \)-Hermitian and anti-Hermitian endomorphisms; the restriction of this decomposition to \( L \text{Sl}(U) \) gives then

\[
L \text{Sl}(U) = L SU(U, h) \oplus i L SU(U, h).
\]

When a 2-spinor basis is fixed, then one gets group isomorphisms \( \text{Sl}(U) \to \text{Sl}(2, \mathbb{C}), \text{SU}(U, h) \to \text{SU}(2) \) and the like.

### 1.8 2-spinor groups and Lorentz group

Up to an obvious transposition we can make the identification

\[
\text{End}(U) \otimes \text{End}(\overline{U}) \cong \text{End}(U \otimes \overline{U}).
\]

We then write

\[
(K \otimes \overline{H})^\lambda_A \tau^\mu_B = K^A_B \overline{H}^\lambda_B, \quad K \in \text{End}(U),
\]

\[
(K \otimes \overline{H}) \tau^\lambda_A \tau^\mu_B = K^A_B \overline{H}^\lambda_B \tau^\lambda_A \tau^\mu_B.
\]

The group \( \text{Aut}(U) \times \text{Aut}(\overline{U}) \) can be identified with the subgroup of \( \text{Aut}(U \otimes \overline{U}) \) constituted of all elements of the type \( K \otimes \overline{H} \) with \( K, H \in \text{Aut} U \). This subgroup is sometimes written as \( \text{Aut}(U) \otimes \text{Aut}(\overline{U}) \), which of course must not be intended as a true tensor product. It has the proper subgroup \( \text{Aut}(U) \culos \text{Aut}(\overline{U}), \) constituted of all automorphisms of the type \( K \otimes \overline{K} = K \in \text{Aut}(U) \).

**Proposition 1.4** \( \text{Aut}(U) \culos \text{Aut}(\overline{U}) \) preserves the splitting \( U \otimes \overline{U} = H \oplus i H \) and the causal structure of \( H \).

**Proof:** There exist bases of \( H \) composed of isotropic elements; these are also complex bases of isotropic elements of \( U \otimes \overline{U} \). Then \( A \in \text{Aut}(U \otimes \overline{U}) \) preserves the splitting and the causal structure iff it sends any element of the form \( u \otimes \overline{u} \) in an element of the form \( v \otimes \overline{v} \). \( \square \)

\( ^8 \) The elements of the dual Pauli basis can be written as \( t^\lambda = \tau^\lambda_A \overline{z}^A \otimes \overline{z}^\lambda B \) with \( \tau^\lambda_A = g_{\lambda \mu} \epsilon^{AB} \epsilon^{\lambda \rho} \tau^\rho_{\mu A B} \).
Accordingly, on sets
\[ \text{Sl}^r(U) \vee \text{Sl}^r(U) = \text{Sl}(U) \vee \text{Sl}(U) := \{ K \otimes \bar{K} : K \in \text{Sl}(U) \} . \]
Since \( K \) preserves \( \varepsilon \) up to a phase factor, \( K \otimes \bar{K} \) preserves \( \varepsilon \otimes \bar{\varepsilon} \equiv g \); moreover it is immediate to check that any Pauli basis is transformed to another Pauli basis. From proposition \[ \text{Lor}^r(U) \leftrightarrow \text{Lor}(U) \], it then follows that \( \text{Sl}(U) \vee \text{Sl}(U) \) restricted to \( H \) coincides with the special orthochronous Lorentz group \( \text{Lor}^r_+(H, g) \). Actually, the epimorphism \( \text{Sl}(U) \to \text{Lor}^r_+(H, g) \) turns out to be 2-to-1.

The Lie algebra of \( \text{Sl}(U) \vee \text{Sl}(U) \) is the Lie subalgebra of \( \text{End}(U) \otimes \text{End}(U) \) constituted by all elements which can be written in the form
\[ A \otimes \mathbb{1}_U + \mathbb{1}_U \otimes \tilde{A}, \quad A \in \mathfrak{L} \text{Sl}(U). \]

One easily checks that these restrict to endomorphisms of \( H \), actually they constitute the vector space of all \( g \)-antisymmetric endomorphisms of \( H \) namely the Lie algebra \( \mathfrak{L} \text{Lor}(H, g) \). Let a normalized 2-spinor basis be fixed; then the isomorphism \( \mathfrak{L} \text{Sl}(U) \leftrightarrow \mathfrak{L} \text{Lor}(H, g) \), taking into account the isomorphism \( \mathfrak{L} \text{Lor}(H, g) \leftrightarrow \wedge^2 H^\ast \) induced by the Lorentz metric \( g \), associates the basis \( (\nu_i; \tilde{\nu}_i) \) with the basis \( (\rho_i; \tilde{\rho}_i) \), \( i = 1, 2, 3 \), where \[ \nu_i := -i \tilde{\nu}_i, \quad \tilde{\nu}_i := \frac{1}{2} \sigma_i \equiv \frac{1}{2} \sigma^A \zeta_A \otimes \eta^0, \quad \rho_i := -i \tilde{\rho}_i, \quad \tilde{\rho}_i := 2 \epsilon^0 \wedge \epsilon^i. \]

A Hermitian metric \( h \) on \( U \), besides the above said \( (\mathfrak{I}, \mathfrak{L}) \) splitting of \( \mathfrak{L} \text{Sl}(U) \), also determines an “observer” \( \tau_0 := \frac{1}{\sqrt{2}} \tilde{h}^\# \), hence also the splitting of \( \mathfrak{L} \text{Lor}(H, g) \) into “infinitesimal rotations” and “infinitesimal boosts” as
\[ \mathfrak{L} \text{Lor}(H, g) = \mathfrak{L} \text{Lor}_r(H, g, \tau_0) \oplus \mathfrak{L} \text{Lor}_h(H, g, \tau_0). \]

If one chooses a normalized 2-spinor basis such that the element \( \tau_0 \) of the corresponding Pauli basis of \( H \) coincides with the given observer, then the bases \( (\nu_i; \tilde{\nu}_i) \) and \( (\rho_i; \tilde{\rho}_i) \) turn out to be adapted to the respective splittings.

**Remark.** On \( \mathfrak{L} \text{Lor}(H, g) \) one has the pseudo-metric induced by \( g \); moreover, consider the real symmetric 2-form
\[ K_{\mathfrak{L} \text{Sl}} : \mathfrak{L} \text{Sl}(U) \times \mathfrak{L} \text{Sl}(U) \to \mathbb{R} : (A, B) \mapsto 2 \Re \text{Tr}(A \circ B). \]

Then it turns out that the bases \( (\nu_i; \tilde{\nu}_i) \) and \( (\rho_i; \tilde{\rho}_i) \) are orthonormal, and that the signature of both metrics is \( (-, -, -), (+, +, +) \). So, the splittings of the two algebras determined by the choice of an “observer” can’t be into arbitrary subspaces: the two components must be mutually orthogonal subspaces of opposite signature.

## 2 Two-spinor bundles

### 2.1 Two-spinor connections

Consider any real manifold \( M \) and a vector bundle \( S \to M \) with complex 2-dimensional fibres. Denote base manifold coordinates as \( (x^\alpha) \); choose a local frame

---

9 Here again \( (\sigma^A_B) \) denotes the \( i \)-th Pauli matrix. \( (\epsilon^1) \) is the dual Pauli basis. Also note that the Hodge isomorphism restricts to a complex structure on \( \wedge^2 H^\ast \).
(ξ_A) of S, determining linear fibre coordinates (x^A). According to the constructions of the previous sections, one now has the bundles Q, L, U, H over M, with smooth natural structures; the frame (ξ_A) yields the frames ε, l, (ξ_A) and (τ_A), respectively. Moreover for any rational number r ∈ Q one has the semi-vector bundle L^r.

Consider an arbitrary C-linear connection Γ on S → M, called a 2-spinor connection. In the fibred coordinates (x^A, x^A) Γ is expressed by the coefficients Γ^A_B : M → C, namely the covariant derivative of a section s : M → S is expressed as

\[ \nabla s = (\partial a s^A - \Gamma^A_B s^B) dx^a \otimes \xi_A. \]

The rule \( \nabla s = \nabla s \) yields a connection \( \bar{\Gamma} \) on \( S \to M \), whose coefficients are given by

\[ \bar{\Gamma}^A_{\dot{B}} = \Gamma^A_B. \]

Actually, Γ determines linear connections on each of the above said induced vector bundles over M (in particular, it is easy to see that any C-linear connection on a complex vector bundle determines a R-linear connection on the induced Hermitian tensor bundle). Denote by 2G and 2Y the connections induced on L and Q (this notation makes sense because the fibres are 1-dimensional), namely

\[ \nabla l = -2G_a dx^a \otimes l, \quad \nabla \varepsilon = 2iY_a dx^a \otimes \varepsilon, \]
\[ \nabla w^{-1} \equiv \nabla (l^{-1} \otimes \varepsilon) = 2(G_a + iY_a) dx^a \otimes l^{-1} \otimes \varepsilon \]

and the like. By direct calculation we find

\[ G_a = \Re \left( \frac{1}{2} \Gamma^A_{\dot{A}} \right) = \frac{1}{4} (\Gamma^A_A + \bar{\Gamma}^A_{\dot{A}}), \]
\[ Y_a = \Im \left( \frac{1}{2} \Gamma^A_{\dot{A}} \right) = \frac{1}{4i} (\Gamma^A_A - \bar{\Gamma}^A_{\dot{A}}). \]

Note that since Y_a are real the induced linear connection on Q is Hermitian (preserves its natural Hermitian structure).

The coefficients of the connection \( \tilde{\Gamma} \) induced on U are given by

\[ \tilde{\Gamma}^A_{\dot{B}} = \Gamma^A_B - G_a \delta^A_B. \]

Let \( \tilde{\Gamma} \) be the connection induced on U ⊗ \overline{U}, and \( \Gamma' \) the connection induced on S ⊗ \overline{S}. Then

\[ \Gamma'_{A \dot{A}}_{B \dot{B}} = \Gamma^A_B \delta^A_B + \delta^A_B \bar{\Gamma}^A_{\dot{B}}, \]
\[ \tilde{\Gamma}'_{A \dot{A}}_{B \dot{B}} = \Gamma^A_B \delta^A_B + \delta^A_B \bar{\Gamma}^A_{\dot{B}} - 2G_a \delta^A_B \delta^A_B. \]

Since the above coefficients are real, Γ' and \( \tilde{\Gamma} \) turn out to be reducible to real connections on \( S \vee \overline{S} \) and \( H \equiv U \vee \overline{U} \), respectively. Moreover

**Proposition 2.1** The connection \( \tilde{\Gamma} \) induced on H by any 2-spinor connection is metric, namely \( \nabla [\tilde{\Gamma}] g = 0 \).

**Proof:** The Lorentz metric g of H can be identified with the identity of the bundle \( \mathbb{L}^{-2} \), namely it is the canonical section \( 1 \equiv \varepsilon^{-1} \otimes \varepsilon : M \to \mathbb{L}^{-2} \otimes \mathbb{L}^2 \equiv M \times \mathbb{R}^+ \), which obviously has vanishing covariant derivative. \( \square \)
Because of metricity the coefficients $\tilde{\Gamma}^\lambda_{a \mu}$ of $\tilde{\Gamma}$ in the frame $(\tau_\lambda)$ are antisymmetric and traceless, namely
\[
\tilde{\Gamma}^\lambda_a + \tilde{\Gamma}^\lambda_a = 0, \quad \tilde{\Gamma}^\lambda_a = 0
\]
(the second formula says $\nabla \eta = 0$, where $\eta$ is the $g$-normalized volume form of $H$).

The above relations between $\Gamma$ and the induced connections can be inverted as follows:

**Proposition 2.2** One has

\[
\Gamma^A_a b = (-G_a + i Y_a) \delta^A_B + \frac{1}{2} \Gamma'_{ab}^{A\dot{A}} = (G_a + i Y_a) \delta^A_B + \frac{1}{2} \tilde{\Gamma}^\lambda_{a \mu} \delta^\lambda_B.
\]

In 4-spinor formalism the above relation reads

\[
\Gamma^\alpha_a \beta = (G_a + i Y_a) \delta^\alpha_{\beta} + \frac{1}{4} \tilde{\Gamma}^\lambda_{a \mu} (\gamma^\lambda \gamma_{\mu})_{\alpha \beta},
\]

where now $\Gamma^\alpha_a \beta$ stands for the coefficients of the naturally induced connection $(\tilde{\Gamma}, \bar{\tilde{\Gamma}})$ on $W \equiv U \oplus M \bar{U}^*$ in any 4-spinor frame, $\alpha, \beta = 1, \ldots, 4$.

A similar relation holds among the curvature tensors, namely

\[
R_a^{AB} = 2 (dG - i dY)_{ab} \delta^A_B + \frac{1}{2} R'_{ab}^{A\dot{A}} =
\]

\[
= -2 (dG + i dY)_{ab} \delta^A_B + \frac{1}{2} \tilde{R}_a^{A\dot{A}} + \dot{R}_a^{A\dot{A}},
\]

where $R, R'$ and $\tilde{R}$ are the curvature tensors of $\Gamma, \Gamma'$ and $\tilde{\Gamma}$, respectively.

**Remark.** Under a local gauge transformation $K : M \to \text{Gl}(2, \mathbb{C})$ the above coefficients transform as

\[
\Gamma^A_a b \mapsto (K^{-1})^A_C K^C_B \Gamma^C_a D - (K^{-1})^A_C \partial_a K^C_B,
\]

\[
G_a \mapsto G_a - \frac{1}{2} \partial_a \log |\det K|, \quad Y_a \mapsto Y_a - \frac{1}{2} \partial_a \arg \det K,
\]

\[
\tilde{\Gamma}^\lambda_a \mu \mapsto (K^{-1})^\lambda_\nu K^\nu_\mu \tilde{\Gamma}_a^\nu - (K^{-1})^\lambda_\nu \partial_\mu K^\nu,
\]

2.2 **Two-spinor tetrad**

Henceforth I'll assume that $M$ is a real 4-dimensional manifold. Consider a linear morphism

\[
\Theta : TM \to S \otimes \bar{S} = \mathbb{C} \otimes L \otimes H,
\]

namely a section

\[
\Theta : M \to \mathbb{C} \otimes L \otimes H \otimes T^* M
\]

(all tensor products are over $M$). Its coordinate expression is

\[
\Theta = \Theta^\lambda_a \tau_\lambda \otimes dx^a = \Theta^A_{\dot{A}} \zeta_A \otimes \dot{\zeta}_{\dot{A}} \otimes dx^a,
\]

with $\Theta^A_{\dot{A}} : M \to \mathbb{C} \otimes L$.

We'll assume that $\Theta$ is non-degenerate and valued in the Hermitian subspace $L \otimes H \subset S \otimes \bar{S}$; then $\Theta$ can be viewed as a 'scaled' tetrad (or soldering form, or vierbein); the coefficients $\Theta^\lambda_a$ are real (i.e. valued in $\mathbb{R} \otimes L$) while the coefficients $\Theta^A_{\dot{A}}$ are Hermitian, i.e. $\Theta^A_{\dot{A}} = \Theta^A_{\dot{A}}$. 

In particular, one has

\[ \wedge \Theta : M \to \mathbb{C} \otimes \mathbb{L}^2 \otimes T^*M \otimes T^*M, \]
\[ \eta := \Theta^* \tilde{g} : M \to \mathbb{C} \otimes \mathbb{L}^4 \otimes ^4T^*M, \]
\[ \gamma := \tilde{g} \circ \Theta : TM \to \mathbb{L} \otimes \text{End}(W), \]

which have the coordinate expressions

\[ g = \eta_{\lambda \mu} \Theta^\lambda_b \Theta^\mu_c \, dx^a \otimes dx^b = \varepsilon_{AB} \varepsilon_{\lambda \mu} \Theta^A_b \Theta^\mu_c \, dx^a \otimes dx^b, \]
\[ \eta = \text{det}(\Theta) \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \]
\[ \gamma = \sqrt{2} \Theta^A_c (\zeta_A \otimes \tilde{\zeta}_A + \varepsilon_{AB} \varepsilon_{\lambda \mu} \tilde{z}^\mu \otimes z^\lambda) \otimes dx^a. \]

The above objects turn out to be a Lorentz metric, the corresponding volume form and a Clifford map. Moreover

\[ \Theta^b_{\mu} := \Theta^\lambda_a \eta_{\mu \lambda} g^{ab} = (\Theta^{-1})^b_{\mu} : M \to \mathbb{C} \otimes \mathbb{L}^{-1}, \]
\[ g^{ab} : M \to \mathbb{C} \otimes \mathbb{L}^{-2}. \]

A non-degenerate tetrad, together with a two-spinor frame, yields mutually dual orthonormal frames \((\Theta^\lambda)\) of \(\mathbb{L}^{-1} \otimes TM\) and \((\tilde{\Theta}^\lambda)\) of \(\mathbb{L} \otimes T^*M\), given by

\[ \Theta^\lambda := \Theta^{-1}(\tau^\lambda) = \Theta^\lambda_a \, dx_a, \quad \tilde{\Theta}^\lambda := \Theta^* (\tau^\lambda) = \Theta^\lambda_a \, dx^a. \]

We also write

\[ \gamma = \gamma^\lambda \otimes \tilde{\Theta}^\lambda = \gamma^a \otimes dx^a, \quad \gamma^\lambda := \gamma(\Theta^\lambda) : M \to \text{End}(W), \]
\[ \gamma^a := \gamma(\partial x_a) = \Theta^\lambda_a \gamma^\lambda : M \to \mathbb{L} \otimes \text{End}(W). \]

### 2.3 Cotetrad

One defines a natural ‘exterior’ product of elements in the fibres of \(H \otimes_M T^*M\) by requiring that, for decomposable tensors, it is given by

\[ (y_1 \otimes \alpha_1) \wedge (y_2 \otimes \alpha_2) = (y_1 \wedge y_2) \otimes (\alpha_1 \wedge \alpha_2), \quad \alpha_1, \alpha_2 \in T^*M, \; u_1, u_2 \in H. \]

We’ll consider the exterior products

\[ \wedge^q \Theta : M \to \mathbb{C} \otimes \mathbb{L}^q \otimes \wedge^q H \otimes \wedge^q T^*M, \quad q = 1, 2, 3, 4. \]

In particular, one has \(\wedge^2 \Theta = \Theta \wedge \Theta\), that is

\[ \wedge^2 \Theta(u \wedge v) = \Theta(u) \wedge \Theta(v) \Rightarrow \wedge^2 \Theta = \Theta^\lambda_a \Theta^\mu_b (\tau^\lambda \wedge \tau^\mu) \otimes (dx^a \wedge dx^b). \]
Next, consider the linear map over $M$

$$\tilde{\Theta} : (S \otimes \bar{S}) \otimes T^*M \rightarrow \mathbb{C} \otimes L^4 \otimes \wedge^4 T^*M$$

defined by

$$\tilde{\Theta}(\xi) := \frac{1}{3!} \tilde{\eta} \mid \langle \xi \wedge \Theta \wedge \Theta \wedge \Theta \rangle = \frac{1}{3!} \tilde{\eta} \mid [\xi \wedge (\wedge^3 \Theta)] .$$

Its coordinate expression is

$$\tilde{\Theta}(\xi) = \tilde{\Theta}^a_\lambda \xi^\lambda_a d^4x := \frac{1}{3!} \varepsilon^{abcd} \varepsilon_{\lambda\mu\nu\rho} \Theta^a_b \Theta^\rho_c \Theta^\mu_d \xi^\lambda_a d^4x ,$$

$$\xi = \xi^\lambda_a \tau_\lambda \otimes dx^a , \quad \xi^\lambda_a : M \rightarrow \mathbb{C} \otimes L .$$

Now $\tilde{\Theta}$ can be seen as a bilinear map $(S \otimes \bar{S}) \times T^*M \rightarrow \mathbb{C} \otimes L^4 \otimes \wedge^4 T^*M$ over $M$, or also as a linear map $S \otimes S \rightarrow \mathbb{C} \otimes L^4 \otimes \wedge^4 T^*M$ over $M$. Using the latter point of view, if $\Theta$ is non-degenerate then one has

$$\tilde{\Theta} = \Theta^{-1} \otimes \eta .$$

Namely, in general one may regard $\tilde{\Theta}$, which is called the co-tetrad, as a kind of ‘pseudo-inverse’ of $\Theta$, defined even if $\Theta$ is degenerate.

The above construction can be easily generalized, for $p = 0, 1, 2, 3, 4$, to a map

$$\tilde{\Theta}^{(p)} : \wedge^p (S \otimes \bar{S}) \otimes (\wedge^p T^*M) \rightarrow \mathbb{C} \otimes L^4 \otimes \wedge^4 T^*M .$$

We’ll be concerned with $\tilde{\Theta}^{(1)} = \tilde{\Theta}$ and $\tilde{\Theta}^{(2)}$. Note that $\tilde{\Theta}^{(0)} = \eta$.

### 2.4 Tetrad and connections

If $\Gamma$ is a complex-linear connection on $S$, and $G$ and $\tilde{\Gamma}$ are the induced connections on $L$ and $H$, then a non-degenerate tetrad $\Theta : TM \rightarrow L \otimes H$ yields a unique connection $\Gamma$ on $TM$, characterized by the condition

$$\nabla|\Gamma \otimes \tilde{\Gamma}|\Theta = 0 .$$

Moreover $\Gamma$ is metric, i.e. $\nabla|\Gamma|g = 0$. Denoting by $\Gamma^\lambda_{a\mu}$ the coefficients of $\Gamma$ in the frame $\Theta^\lambda_\mu \equiv \Theta^{-1} (l \otimes \tau_\lambda)$ one obtains

$$\Gamma^\lambda_{a\mu} = \tilde{\Gamma}^\lambda_{a\mu} + 2 G_a \delta^\lambda_{\mu} .$$

The curvature tensors of $\Gamma$ and $\tilde{\Gamma}$ are related by $R_{ab\mu}^\lambda = \tilde{R}_{ab\mu}^\lambda$, or

$$R_{ab\mu}^c = \tilde{R}_{ab\mu}^\lambda \Theta^\lambda_c \Theta^\mu_d .$$

Hence the Ricci tensor and the scalar curvature are given by

$$R_{ad} = \tilde{R}_{ad}^b = \tilde{R}_{ab\mu}^\lambda \Theta^b \Theta^\mu_d ,$$

$$R_a^a = \tilde{R}_{ab\mu}^\lambda \Theta^b \Theta^\mu .$$
In general, the connection $\Gamma$ will have non-vanishing torsion\footnote{This is the tensor field $T : M \to TM \otimes \Lambda^2 T^*M$ defined by $T(u, v) = \nabla_u v - \nabla_v u - [u, v]$, where $u, v : M \to TM$ are any two vector fields, and has the coordinate expression $T^c_{ab} = -\Gamma^c_{ab} + \Gamma^c_{ba}$.} which can be expressed as

$$\Theta^\lambda T^c_{ab} = \partial^a \Theta^\lambda_{b} + \Theta^\mu_{[a} \tilde{\Gamma}^\lambda_{b] \mu} + 2 \Theta^\lambda_{[a} G_{b]}.$$ 

Remark. The torsion can be seen as the Frölicher-Nijenhuis bracket

$$\tilde{T} := T_j \Theta = [\Gamma', \Theta] : M \to \Lambda^2 T^*M \otimes H',$$

where $H' = \mathbb{L} \otimes H$, $\Gamma' : H' \to T^*M \otimes_{H'} T H'$ is the induced connection on $H' \to M$, and $\Theta$ is seen as a vertical-valued form $\Theta : H' \to T^*M \otimes_{H'} \mathbb{V}$. 

2.5 The Dirac operator

Given a tetrad and a two-spinor connection, one introduces the Dirac operator acting on sections $\psi : M \to \Lambda^{-3/2} \otimes W$.

Writing $\tilde{\gamma}^\# : M \to H \otimes \text{End}(W)$, $\nabla \psi : M \to \Lambda^{-3/2} \otimes T^*M \otimes_M W$, one has

$$\tilde{\gamma}^\# \nabla \psi : M \to \Lambda^{-3/2} \otimes H \otimes T^*M \otimes W,$$

where contraction in $W$ is understood. Next, one contracts the factors $H$ and $T^*M$ above via

$$\tilde{\Theta} : M \to \mathbb{C} \otimes \Lambda^3 \otimes H^* \otimes TM \otimes \Lambda^4 T^*M,$$

obtaining

$$\tilde{\nabla} \psi := \langle \tilde{\Theta}, \tilde{\gamma}^\# \nabla \psi \rangle : M \to \Lambda^{3/2} \otimes W \otimes \Lambda^4 T^*M,$$

which has the coordinate expression

$$\tilde{\nabla} \psi = \tilde{\Theta}^\lambda_{\mu} \left( \sigma^\lambda_{\mu\nu} \nabla_a \chi_{\nu} \zeta_{\lambda}, \sigma^\lambda_{\mu\nu} \nabla_a \dot{u} \bar{z}^\nu \right) \otimes d^4x.$$

This definition works even if $\Theta$ were degenerate; in the non-degenerate case one simply has $\tilde{\nabla} \psi = \nabla \psi \otimes \eta$.

3 Two-spinors and field theories

3.1 The fields

In this section I’ll present a “minimal geometric data” field theory: actually, the unique “geometric datum” is a vector bundle $S \to M$ with complex 2-dimensional fibres and real 4-dimensional base manifold. All other bundles and fixed geometric objects are determined just by this datum through functorial constructions, as we saw in the previous sections; no further background structure is assumed. Any considered bundle section which is not functorially fixed by our geometric datum is a field. In this way one obtains a field theory which turns out to be essentially equivalent to a classical theory of Einstein-Cartan-Maxwell-Dirac fields.

The fields are taken to be the tetrad $\Theta$, the 2-spinor connection $\Gamma$, the electromagnetic field $F$ and the electron field $\psi$. The gravitational field is represented by...
3.2 Gravitational Lagrangian

Θ (which can be viewed as a ‘square root’ of the metric) and the traceless part of \( \Gamma \), namely \( \tilde{\Gamma} \), seen as the gravitational part of the connection. If \( \Theta \) is non-degenerate one obtains, as in the standard metric-affine approach \([10, 11, 13, 18]\), essentially the Einstein equation and the equation for torsion; the metricity of the spacetime connection is a further consequence. But note that the theory is non-singular also in the degenerate case. The connection \( G \) induced on \( L \) will be assumed to have vanishing curvature, \( dG = 0 \), so that one can always find local charts such that \( G_a = 0 \); this amounts to gauging away the conformal (‘dilaton’) symmetry. Coupling constants will arise as covariantly constant sections of \( L \), which now becomes just a vector space.

The Dirac field is a section

\[
\psi : M \to \mathbb{L}^{-3/2} \otimes W := \mathbb{L}^{-3/2} \otimes (U \oplus U^*) ,
\]

assumed to represent a semiclassical particle with one-half spin, mass \( m \in \mathbb{L}^{-1} \) and charge \( q \in \mathbb{R} \).

The electromagnetic potential can be thought of as the Hermitian connection \( Y \) on \( \wedge^2 U \) determined by \( \tilde{\Gamma} \), whose coefficients are indicated as \( i Y_a \); locally one writes

\[
Y_a = q A_a ,
\]

where \( A : M \to T^* M \) is a local 1-form.

The electromagnetic field is represented by a spinor field

\[
\tilde{F} : M \to \mathbb{L}^{-2} \otimes \wedge^2 H^* \]

which, via \( \Theta \), determines the 2-form \( F := \Theta^* \tilde{F} : M \to \wedge^2 T^* M \). The relation between \( Y \) and \( F \) will follow as one of the field equations; note how this setting allows a first-order linear Lagrangian and non-singularity in the degenerate case also for the electromagnetic sector.

The total Lagrangian and the Euler-Lagrange operator will be the sum of a gravitational, an electromagnetic and a Dirac term

\[
\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{\text{em}} + \mathcal{L}_D , \quad \mathcal{E} = \mathcal{E}_g + \mathcal{E}_{\text{em}} + \mathcal{E}_D .
\]

Observe that all Lagrangian 4-forms are defined in terms of the cotetrad \( \tilde{\Theta} \), while a direct translation of the standard formulation in terms of our fields would force one to use \( \Theta^{-1} \), resulting in a less simple and natural theory.

3.2 Gravitational Lagrangian

The tetrad \( \Theta \) and the curvature tensor \( \tilde{R} \) of \( \tilde{\Gamma} \) can be assembled into a 4-form \( \mathcal{L}_g \) which, in the non-degenerate case, turns out to be the usual gravitational Lagrangian density:

\[
\mathcal{L}_g := \frac{1}{4k} \tilde{\Theta}^{(2)}(\tilde{R}^\#) = \frac{1}{8k} \tilde{\eta} | (\tilde{R}^\# \wedge \Theta \wedge \Theta) : M \to \wedge^4 T^* M ,
\]

where \( \tilde{R}^\# : M \to \wedge^2 T^* M \otimes \wedge^2 H \) is the curvature tensor of \( \tilde{\Gamma} \) with one index raised via \( \tilde{g} \), and \( k \in \mathbb{L}^2 \) is Newton’s gravitational constant. Note how this is necessary in
order to obtain a true (non-scaled) 4-form on $M$ and the correct coupling with the spinor field. One has the coordinate expression $L_g = \ell_g \, d^4x$ with

$$\ell_g = \frac{1}{8k} \varepsilon_{\lambda\mu\nu\rho} \varepsilon^{abcd} R_{ab}^{\lambda \mu} \Theta_c^\nu \Theta_d^\rho = \frac{1}{2k} R \det \Theta,$$

where $R$ is the scalar curvature and the last equality holds if $\Theta$ is non-degenerate.

A calculation gives the $\Theta$- and $\tilde{\Gamma}$-components of the gravitational part $E_g$ of the Euler-Lagrange operator:

$$(E_g)^c\nu = \frac{1}{4k} \varepsilon_{\lambda\mu\nu\rho} \varepsilon^{abcd} R_{ab}^{\lambda \mu} \Theta_c^\nu \Theta_d^\rho,$$

$$(E_g)^a_{\lambda\mu} = \frac{1}{2k} \varepsilon_{\lambda\mu\nu\rho} \varepsilon^{abcd} (\partial_b \Theta_c^\nu + \Theta_e^\nu \tilde{\Gamma}_{c \sigma}^\nu) \Theta_d^\rho.$$ 

In the non-degenerate case these are essentially the Einstein tensor and the torsion of the spacetime connection, respectively. The first, in particular, can be written

$$(E_g)^c\nu = \frac{1}{4k} \Theta^a_{\lambda\mu} \Theta^b_{\nu} \Theta^c_{\rho} \det \Theta = \frac{1}{k} (R_{bc}^{\lambda \mu} - \frac{1}{2} R_{bd}^{\lambda \mu} \delta_{\lambda \sigma}) \Theta^a_{\nu} \det \Theta.$$ 

The $\tilde{\Gamma}$-component of $E_g$ can be expressed in terms of the torsion as

$$(E_g)^a_{\lambda\mu} = \frac{1}{4k} \varepsilon_{\lambda\mu\nu\rho} \varepsilon^{abcd} \hat{T}^e_{bc} \Theta_e^\nu \Theta_d^\rho.$$ 

### 3.3 Electromagnetic Lagrangian

The electromagnetic potential and the Maxwell field will be considered independent fields. The former is represented by a local section $A : M \rightarrow T^*M$, related to the connection $Y$ induced by $F$ on $\wedge^2 U$ by the relation $Y = q A$. The Maxwell field is a section $\tilde{F} : M \rightarrow \Lambda^{-2} \otimes \wedge^2 H^*$, written in coordinates as $\tilde{F} = \tilde{F}^{\lambda \mu} t^\lambda \otimes t^\mu$. The e.m. Lagrangian density is defined to be

$$L_{em} = \ell_{em} \, d^4x = \left[ -\frac{1}{2} \Theta^{(2)} (dA \otimes \tilde{F}) + \frac{1}{4} (\tilde{F} \cdot \tilde{F}) \right] \eta,$$

with coordinate expression

$$\ell_{em} = -\frac{1}{4} \varepsilon^{abcd} \varepsilon_{\lambda\mu\rho\sigma} \partial_a A_b \tilde{F}^{\lambda \mu} \Theta_c^\nu \Theta_d^\rho + \frac{1}{4} \tilde{F}^{\alpha \beta} \tilde{F}_{\alpha \beta} \det \Theta.$$ 

In the non-degenerate case, this turns out to be essentially the Lagrangian used in the ADM formalism.

Since $\tilde{F}$ does not appear in the other terms of the total Lagrangian, the $\tilde{F}$-component of the field equations is immediately seen to yield

$$-\frac{1}{2} \varepsilon^{abcd} \varepsilon_{\lambda\mu\rho\sigma} \partial_a A_b \Theta_c^\nu \Theta_d^\rho + \tilde{F}_{\lambda \mu} \det \Theta = 0,$$

which in the non-degenerate case gives

$$F := \Theta^\nu \tilde{F} = 2 dA \quad \Rightarrow \quad L_{em} = -\frac{1}{4} F^2 \eta.$$ 

The $A$-component of the Euler-Lagrange operator is

$$(E_{em})^a = \frac{1}{2} \varepsilon^{abcd} \varepsilon_{\lambda\mu\rho\sigma} (\partial_b \tilde{F}^{\lambda \mu} \Theta_c^\nu \Theta_d^\rho + 2 \tilde{F}^{\lambda \mu} \partial_b \Theta_c^\nu \Theta_d^\rho) = \frac{1}{2} \varepsilon^{abcd} (d \ast F)_{bcd}.$$
The Θ-component is

\[(E_{em})^c_\nu = -\frac{1}{2} \epsilon^{abcd} \epsilon_{\mu\nu\rho} \partial_a A_b \tilde{F}^\lambda \Theta^d_\rho + \frac{1}{4} \tilde{F}^2 \tilde{\Theta}^c_\nu,\]

which in the non-degenerate case becomes essentially the usual Maxwell stress-energy tensor

\[(E_{em})^c_\nu = (F_{ab}F^{ac} - \frac{1}{4} F^2 \delta^c_\nu) \tilde{\Theta}^b_\rho.\]

### 3.4 Dirac Lagrangian

The Dirac spinor field and its ‘Dirac adjoint’ are sections

\[\psi = (u, \chi) : M \to \mathbb{L}^{-3/2} \otimes W = \mathbb{L}^{-3/2} \otimes (U \oplus U^*) ,\]

\[\bar{\psi} = (\bar{\chi}, \bar{u}) : M \to \mathbb{L}^{-3/2} \otimes (U^* \oplus U) = \mathbb{L}^{-3/2} \otimes W^*.\]

In coordinates:

\[u^A \zeta_A \quad, \quad \chi^A \bar{z}_A \quad, \quad u^A, \chi_A : M \to \mathbb{C} \otimes \mathbb{L}^{-3/2} \]

\[\langle \bar{\psi}, \psi \rangle = (\bar{u}^A \chi_A + \bar{\chi}_A u^A) : M \to \mathbb{C} \otimes \mathbb{L}^{-3} \]

The Dirac operator (§2.5) yields a section

\[\tilde{\nabla} \psi : M \to \mathbb{L}^{3/2} \otimes W \otimes \Lambda^4 T^* M ,\]

so that

\[\langle \bar{\psi}, \tilde{\nabla} \psi \rangle : M \to \mathbb{C} \otimes \Lambda^4 T^* M .\]

Now we introduce the scalar density

\[L_D = \frac{1}{2} \left( \langle \bar{\psi}, \tilde{\nabla} \psi \rangle - \langle \tilde{\nabla} \bar{\psi}, \psi \rangle \right) - m \langle \bar{\psi}, \psi \rangle \eta : M \to \Lambda^4 T^* M ,\]

where \(\tilde{\nabla} \psi := \overline{\nabla} \psi\), and \(m \in \mathbb{L}^{-1}\) is the described particle’s mass. This is a version of the Dirac Lagrangian which remains non-singular when Θ is degenerate. In the non-degenerate case one also has

\[L_D = \left[ \frac{1}{2} \left( \langle \bar{\psi}, \nabla \psi \rangle - \langle \nabla \bar{\psi}, \psi \rangle \right) - m \langle \bar{\psi}, \psi \rangle \right] \eta ;\]

in 2-spinor terms this reads

\[L_D = \frac{1}{\sqrt{2}} \overline{\Theta}^a_A \left( \nabla a u^A \tilde{u}^A - u^A \nabla a \tilde{u}^A + \epsilon^{AB} \epsilon^{AD} \left( \tilde{\chi}_B \nabla a \chi^B - \nabla a \tilde{\chi}_B \chi^B \right) \right) - m \left( \langle \chi, \tilde{u} \rangle + \langle \tilde{\chi}, u \rangle \right) \eta ,\]

with the coordinate expression

\[L_D = \frac{1}{\sqrt{2}} \overline{\Theta}^a_A \left( \nabla a u^A \tilde{u}^A - u^A \nabla a \tilde{u}^A + \epsilon^{AB} \epsilon^{AD} \left( \tilde{\chi}_B \nabla a \chi^B - \nabla a \tilde{\chi}_B \chi^B \right) \right) - m \left( \langle \chi, \tilde{u} \rangle + \langle \tilde{\chi}, u \rangle \right) \det \Theta .\]

Next we compute the Euler-Lagrange operator \(E_D\). The \(\bar{u}\)-component is

\[(E_D)^{\bar{u}}_\nu = \sqrt{2} i \overline{\Theta}^a_A \nabla a u^A \det \Theta + \frac{1}{\sqrt{2}} T_{\bar{A}A} u^A ,\]
where $T_{AA} := \tilde{\Theta}^a_{AA} T^b_{ab}$ is used for replacing the term with $\partial_a \Theta^\mu_b$ (see §2.4).

The $\bar{\chi}$-component is

$$ (E_D)^A = \sqrt{2} i \tilde{\Theta}^\mu_{AA} \nabla_a \chi_a^\mu - m u^A \det \Theta + \frac{i}{\sqrt{2}} T^A_{AA} \chi^\mu_A, $$

with $\tilde{\Theta}^\mu_{AA} := \tilde{\Theta}^a_{AA} \varepsilon^{AB} \varepsilon^{B'A'}$ and $T^A_{AA} := \varepsilon^{BA} \varepsilon^{B'A'} T_{BB'}$.

The $\tilde{\Gamma}$-component is

$$ (E_D)^q_{\lambda \mu} = \frac{1}{4 \sqrt{2}} \left[ (\tilde{\Theta}^a_{AC} \tau_{[\lambda} D_{\nu]} A - \tilde{\Theta}^a_{CA} \tau_{[\lambda} D_{\mu]} A) u^A \tilde{u}^A + (\tilde{\Theta}^a_{AC} \tau_{[\lambda} D_{\mu]} A - \tilde{\Theta}^a_{CA} \tau_{[\lambda} D_{\nu]} A) u^A \tilde{u}^A \right] =$$

$$ = \frac{1}{4} \varepsilon^{abcd} \varepsilon_{\lambda \mu \rho} \Theta^a_b \Theta^b_d \left( \tilde{\psi} \gamma^\lambda \gamma^\rho \psi - \tilde{\gamma}^\lambda \gamma^\rho \psi \right) - m \tilde{\psi} \psi \Theta^c_{\nu}. $$

The $A$-component is simply

$$ (E_D)'^a = \sqrt{2} q \tilde{\Theta}^a_{A'A} \left( u^A \tilde{u}^A + \varepsilon^{BA} \varepsilon^{B'A'} \bar{\chi}_B \chi_B' \right) = q \tilde{\Theta}^a_{A'A} \left( \tilde{\psi} \gamma^\lambda \psi \right). $$

### 3.5 Field equations

Having calculated the various pieces of $E = E_\xi + E_{em} + E_D$, writing down the field equations $E = 0$ is a simple matter. These equations are non-singular also when $\Theta$ is degenerate; in the non-degenerate case one expects this approach to reproduce essentially the usual Einstein-Cartan-Maxwell-Dirac field equations.

The $\Theta$-component

$$ (E_\Theta)^e_{\nu} = - (E_{em} + E_D)^e_{\nu}, $$

corresponds to the Einstein equation; actually, as already discussed, in the non-degenerate case the left-hand side is essentially the Einstein tensor, while the right-hand side can be viewed as the sum of the energy-momentum tensors of the electromagnetic field and of the Dirac field.

The $\tilde{\Gamma}$-component gives the equation for torsion

$$ (E_{\tilde{\Gamma}})^a_{\lambda \mu} = - (E_D)^a_{\lambda \mu}. $$

From this one sees that the spinor field is a source for torsion, and that in this context one cannot formulate a torsion-free theory.

It was already seen (§3.3) that the $\tilde{F}$-component reads $F = 2 dA$ in the non-degenerate case, and of course this yields the first Maxwell equation $dF = 0$. The $A$-component is

$$ - \frac{1}{2} \varepsilon^{abcd} (d*F)_{bcd} + q \tilde{\Theta}^a_{\lambda \mu} (\tilde{\psi} \gamma^\lambda \psi) = 0 \quad \text{i.e.} \quad \frac{1}{2} c \varepsilon^{abcd} (d*F)_{bcd} = q \tilde{\Theta}^a_{\lambda \mu} (\tilde{\psi} \gamma^\lambda \psi). $$
In the non-degenerate case this gives the second Maxwell equation
\[ \frac{1}{2} * d * F = j, \]
where \( j : M \to \otimes T^* M \) is the Dirac current, with coordinate expression
\[ j := \frac{q}{c} \Theta_a^\lambda (\bar{\psi} \gamma_\lambda \psi) dx^a. \]

The \( \bar{u} \)- and \( \bar{\chi} \)-components \((\mathcal{E}_0)_A = 0 \) and \((\mathcal{E}_0)^B = 0 \) give the following generalized form of the standard Dirac equation:
\[
\begin{cases}
\sqrt{2} i \Theta_{Aa} \nabla_a u^A - m \chi^A \det \Theta + \frac{i}{\sqrt{2}} T_A u^A = 0 \\
\sqrt{2} i \Theta^{aA} \nabla_a \chi^A - m u^A \det \Theta + \frac{i}{\sqrt{2}} T^A \chi^A = 0
\end{cases}
\]

Denoting by \( \bar{T}^a \) the 1-form obtained from the torsion by contraction, with coordinate expression \( \bar{T}^a \equiv T_{ab} \), the above equation can be written in coordinate-free form as
\[ \left( i \nabla - m + \frac{i}{2} \gamma^\# (\bar{T}) \right) \psi = 0. \]

4 Dirac algebra in two-spinor terms

4.1 Dirac algebra

If \( V \) is a finite-dimensional real vector space endowed with a non-degenerate scalar product, then its Clifford algebra \( C(V) \) is the associative algebra generated by \( V \) where the product of any \( u, v \in V \) is subjected to the condition
\[ u v + v u = 2 u \cdot v, \quad u, v \in V. \]

The Clifford algebra fulfills the following universal property: if \( A \) is an associative algebra with unity and \( \gamma : V \to A \) is a linear map such that \( \gamma(v) \gamma(v) = v \cdot v \forall v \in V \), then \( \gamma \) extends to a unique homomorphism \( \tilde{\gamma} : C(V) \to A \). It turns out that \( C(V) \) is isomorphic, as a vector space, to the vector space underlying the exterior algebra \( \wedge V \); through this isomorphism one identifies \( v_1 \wedge \ldots \wedge v_p \) with the antisymmetrized Clifford product
\[ \frac{1}{p!} (v_1 v_2 \cdots v_p - v_2 v_1 \cdots v_p + \cdots) \]
where the sum is extended to all permutations of the set \( \{1, \ldots, p\} \), with the appropriate signs. In other terms, one has two distinct algebras on the same underlying vector space: any element of \( C(V) \) can be uniquely expressed as a sum of terms, each of well-defined exterior degree. For example, one has \( u v = u \wedge v + u \cdot v \); from this one sees that the Clifford algebra product does not preserve the exterior algebra degree, but only its parity: \( C(V) \) is \( \mathbb{Z}_2 \)-graded. If \( \phi \in \wedge^r V, \theta \in \wedge^s V \), then the Clifford product \( \phi \theta \) turns out to be a sum of terms of exterior degree \( r+s, r+s-2, \ldots, |r-s| \).

The Clifford algebra \( D := C(H) \) of Minkowski space \( H \) (§1.4) is called the Dirac algebra. The Dirac map \( \gamma : H \to \text{End}(W) \) is a Clifford map, hence by virtue of the above said universal property one can see the Dirac algebra as a real vector
subspace $D \subset \text{End}(W)$ of dimension $2^4 = 16$. Since this coincides with the complex dimension of $\text{End}(W) \equiv W \otimes W^*$, one gets $\text{End}(W) = \mathbb{C} \otimes D$.

The Dirac algebra $D$ is multiplicatively generated by $\gamma(H) \subset \text{End}(W)$, simply identified with $H$. One has the natural decompositions

$$D = D^{(+)} \oplus D^{(-)} = (\mathbb{R} \oplus \wedge^2 H \oplus \wedge^3 H) \oplus (H \oplus \wedge^3 H),$$

where $D^{(+)}$ and $D^{(-)}$ denote the even-degree and odd-degree subspaces, respectively (the former is a subalgebra). Also, one has the distinguished elements

$$1 \equiv 1_W \subset \mathbb{R} \subset D^{(+)} , \quad \eta^\# \subset \wedge^4 H \subset D^{(+)} ,$$

where $\eta^\# \equiv g^\#(\eta)$ is the contravariant tensor corresponding to the unimodular volume form $\eta$. One gets

$$\eta^\# \eta^\# = -1 , \quad \forall \theta \eta^\# = \ast \theta \quad \forall \theta \in \wedge H ,$$

where $\ast$ is the Hodge isomorphism.

### 4.2 Decomposition of $\text{End} W$ and $\varepsilon$-transposition

One has the natural decomposition

$$\text{End}(W) \equiv \text{End}(U \oplus U^*) = (U \otimes U^*) \oplus (U \otimes U) \oplus (U^* \otimes U^*) \oplus (U^* \otimes U) .$$

Accordingly, any $\Phi \in \text{End}(W)$ is a 4-uple of tensors, which will be conveniently written in matricial form as

$$\Phi = \begin{pmatrix} K & P \\ Q & J \end{pmatrix} , \quad K \in U \otimes U^* , \quad P \in U \otimes U , \quad Q \in U^* \otimes U^* , \quad J \in U^* \otimes U .$$

We now introduce an operation which acts on each of the above 4 types of tensors in a similar way. This operation, called $\varepsilon$-transposition, is actually independent of the particular normalized $\varepsilon \in \wedge^2 U^*$ chosen; it is defined by

$$U \otimes U^* \rightarrow U^* \otimes U : K \mapsto \tilde{K} := \langle \varepsilon^b \otimes \varepsilon^\#, K \rangle = \varepsilon_{CA} K^{C}_{D} \varepsilon^{DB} z^A \otimes \zeta_B ,$$

$$U \otimes \bar{U} \rightarrow U^* \otimes U^* : P \mapsto \tilde{P} := \langle \varepsilon^b \otimes \bar{\varepsilon}^\#, P \rangle = \varepsilon_{CA} P^{C}_{D} \bar{\varepsilon}_{DB} z^A \otimes \bar{\zeta}_B ,$$

$$U^* \otimes U^* \rightarrow U \otimes U : Q \mapsto \tilde{Q} := \langle \varepsilon^\# \otimes \varepsilon^\#, Q \rangle = \varepsilon^{\#CA} Q_{CD} \varepsilon^{DB} \zeta_A \otimes \bar{\zeta}_B ,$$

$$U^* \otimes \bar{U} \rightarrow U \otimes U^* : J \mapsto \tilde{J} := \langle \varepsilon^\# \otimes \varepsilon^\#, J \rangle = \varepsilon^{\#CA} J_{C}^{D} \varepsilon_{DB} \bar{\zeta}_A \otimes \bar{\zeta}_B .$$

Namely, $\varepsilon$-transposition changes the position (either high or low) of both indices of the tensor it acts on. For elements in $U \otimes \bar{U}$ or $U^* \otimes U^*$ it essentially amounts to index lowering (resp. raising) by the Lorentz metric $g$ in complexified Minkowski space; for invertible elements in $U \otimes U^* \equiv \text{End}(U)$ or $U^* \otimes \bar{U} \equiv \text{End}(U^*)$, $\varepsilon$-transposition amounts to

$$\tilde{X} = (\det X) (X^{-1})^\ast ,$$

where the superscript $\ast$ denotes standard transposition.
4.3 \( \varepsilon \)-adjoint and characterization of \( D \)

It is clear that \( \varepsilon \)-transposition can be similarly defined\(^{12}\) on \( U^* \otimes U, U^* \otimes U^* \), \( \overline{U} \otimes U \) and \( U \otimes \overline{U}^* \), and in all cases one gets

\[
\tilde{X} = X, \quad (\tilde{X})^\ast = (X^*)^\sim, \quad \tilde{X} X^\ast = X^\ast \tilde{X} = (\det X) \mathbb{I}, \quad \det X = \det \tilde{X}. 
\]

**Remark.** The determinant is uniquely defined, via any \( \varepsilon \), also for elements in \( U \otimes \overline{U}, U^* \otimes \overline{U}^*, \overline{U} \otimes U \) and \( U^* \otimes U^* \). In these cases, the determinant of a tensor equals one-half its Lorentz pseudo-norm.

Moreover, whenever the composition of tensors \( X \) and \( Y \) is defined, one has

\[
(X Y)^\sim = \tilde{X} \tilde{Y}, \quad \text{Tr}(\tilde{X} \tilde{Y}) = \text{Tr}(XY).
\]

Whenever \( A \) and \( B \) are tensors of the same type, one has

\[
\det(A + B) = \det(A) + \det(B) + \text{Tr}(A^\ast B),
\]

where the *scalar product* \( (A, B) \mapsto \text{Tr}(A^\ast B) \) is *symmetric*\(^{13}\).

**Proposition 4.1** Let \( \Phi = \begin{pmatrix} K & P \\ Q & J \end{pmatrix} \in W \otimes W^* \) be non-singular. Then

\[
\det \Phi = (\det K)(\det J) + (\det P)(\det Q) - \text{Tr}(K^\ast \tilde{P} J^\ast \tilde{Q}),
\]

\[
(\det \Phi) \Phi^{-1} = \begin{pmatrix} (\det J) \tilde{K}^\ast - \tilde{Q}^\ast J \tilde{P}^\ast & (\det P) \tilde{Q}^\ast - \tilde{K}^\ast P \tilde{J}^\ast \\ (\det Q) \tilde{P}^\ast - \tilde{J}^\ast Q \tilde{K}^\ast & (\det K) \tilde{J}^\ast - \tilde{P}^\ast K \tilde{Q}^\ast \end{pmatrix}.
\]

**Proof:** It can be checked by a direct calculation, taking into account the above identities. \( \square \)

### 4.3 \( \varepsilon \)-adjoint and characterization of \( D \)

If \( X \) is a tensor of any of the above types, then its \( \varepsilon \)-adjoint is the tensor

\[
X^\dagger := \tilde{X}.
\]

Using this operation one defines the real involution

\[
\dagger : W \otimes W^* \to W \otimes W^* : \begin{pmatrix} K & P \\ Q & J \end{pmatrix} \mapsto \begin{pmatrix} J^\dagger & Q^\dagger \\ P^\dagger & K^\dagger \end{pmatrix}.
\]

**Proposition 4.2** \( D \) and \( iD \) are the eigenspaces of \( \dagger \) corresponding to eigenvalues \(+1\) and \(-1\), respectively. Namely, \( D \) is the real subspace of \( W \otimes W^* \) constituted by all endomorphisms which can be written in the form

\[
\begin{pmatrix} K & P^\dagger \\ P & K^\dagger \end{pmatrix}, \quad K \in U \otimes U^*, \quad P \in U \otimes \overline{U}.
\]

Moreover one has the following characterisations

\[
D^0 \equiv \mathbb{R} = \left\{ r \begin{pmatrix} \mathbb{I}_U & 0 \\ 0 & \mathbb{I}_{U^*} \end{pmatrix}, \quad r \in \mathbb{R} \right\},
\]

\(^{12}\) One could introduce \( \varepsilon \)-transposition on further spaces such as \( U \otimes U, U^* \otimes U^* \) and so on. These extensions however would depend from the chosen normalized \( \varepsilon \); phase factors cancel out only in the considered cases.

\(^{13}\) On \( U \otimes \overline{U} \) and \( \overline{U} \otimes U \) (resp. \( U^* \otimes U^* \) and \( \overline{U}^* \otimes U^* \)) this coincides with \( 2g \) (resp. \( 2g^\# \)).
\[ D^1 \equiv H = \left\{ \begin{pmatrix} 0 & P \\ P^\dagger & 0 \end{pmatrix} , \quad P \in H \right\}, \]

\[ D^2 \equiv \lambda^2 H = \left\{ \begin{pmatrix} K & 0 \\ 0 & K^\dagger \end{pmatrix} , \quad K \in U \otimes U^* , \quad \text{Tr } K = 0 \right\}, \]

\[ D^3 \equiv \lambda^3 H = \left\{ \begin{pmatrix} 0 & P \\ P^\dagger & 0 \end{pmatrix} , \quad P \in iH \right\}, \]

\[ D^4 \equiv \lambda^4 H = \left\{ \begin{pmatrix} 1r \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ U^* \end{pmatrix} \right) \right) , \quad r \in \mathbb{R} \right\}, \]

\[ D^{(\pm)} = D^0 \oplus D^2 \oplus D^4 = \left\{ \begin{pmatrix} K & 0 \\ 0 & K^\dagger \end{pmatrix} , \quad K \in U \otimes U^* \right\}, \]

\[ D^{(-)} = D^1 \oplus D^3 = \left\{ \begin{pmatrix} 0 & P \\ P^\dagger & 0 \end{pmatrix} , \quad P \in U \otimes \mathbb{U} \right\}. \]

**Proof:** The Dirac map \( \gamma : H \rightarrow \text{End } W \) can be written as

\[ \gamma : v \mapsto \begin{pmatrix} 0 & \sqrt{2}v \\ \sqrt{2}v^\dagger & 0 \end{pmatrix}, \]

whence the characterization of \( D^1 \). It immediately follows that \( D^{(\pm)} \) is constituted by diagonal-block elements, while \( D^{(-)} \) is constituted by off-diagonal-block elements. The other characterizations can be checked by matrix calculations. \( \square \)

5 Clifford group and its subgroups

5.1 Clifford group

Let \( D^* := D \cap \text{Aut } W \) be the group of all invertible elements in \( D \). The Clifford group \( \text{Cl} \equiv \text{Cl}(W) \) is defined to be \([7, 9]\) the subgroup of \( D^* \) under whose adjoint action \( H \) is stable. In other terms, \( \Phi \in D^* \) is an element of \( \text{Cl} \) if

\[ \text{Ad}[\Phi]v \equiv \Phi \gamma(v) \Phi^{-1} \in \gamma(H) , \quad \forall v \in H. \]

Using proposition 4.1 we write the adjoint action as

\[
\left( \det \Phi \right) \text{Ad}[\Phi]v = \begin{pmatrix} K & P \\ P^\dagger & K^\dagger \end{pmatrix} \begin{pmatrix} 0 & V \\ V^\dagger & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Y^\dagger & X^\dagger \end{pmatrix} = \begin{pmatrix} PV^\dagger X + KV Y^\dagger & PV^\dagger Y + KV X^\dagger \\ K^\dagger V^\dagger X + P^\dagger V Y^\dagger & K^\dagger V^\dagger Y + P^\dagger V X^\dagger \end{pmatrix},
\]

where \( V \equiv \sqrt{2}v \) and

\[ X \equiv (\det K) K^* - \vec{P}^* \vec{K} \vec{P}^* , \quad Y \equiv (\det P) \vec{P}^* - \vec{K}^* P \vec{K}^* , \]

\[ X^\dagger = (\det K) K^* - \vec{P}^* K \vec{P}^* , \quad Y^\dagger = (\det P) \vec{P}^* - \vec{K}^* P \vec{K}^* . \]
5.1 Clifford group

Lemma 5.1 An element of $D^*$ which belongs to the Clifford group is necessarily either odd or even, so that the Clifford group is the disjoint union $\text{Cl} = \text{Cl}^{(+)} \cup \text{Cl}^{(-)}$ where $\text{Cl}^{(+)} = \text{Cl} \cap D^{(+)}$, $\text{Cl}^{(-)} = \text{Cl} \cap D^{(-)}$.

Proof: If $\Phi$ is in $\text{Cl}$ then the $U \otimes U^*$-component of $\text{Ad}[\Phi]v$ vanishes for all $v \in H$, namely

$$KV\tilde{Y} = -P\tilde{V}X, \quad \forall V \in H.$$  

Composing both sides with $\tilde{V}^*\tilde{K}^*$ on the left and with $\tilde{X}^*$ on the right one finds

$$(\det K)(\det V)\tilde{Y}\tilde{X}^* = -(\det \Phi)(\det \tilde{K})\tilde{V}^*\tilde{K}^*P\tilde{V}.$$  

Now the above equality is certainly fulfilled in the particular case when $\det \tilde{K} = 0$. Suppose $\det K \neq 0$ for the moment (the other case will be considered later). The left-hand side vanishes for all null elements $V \in H$, thus also $\tilde{V}^*\tilde{K}^*P\tilde{V}$ vanishes for all null vectors $V$; it's not difficult to see that this implies $\tilde{K}^*P = 0$, which on turn implies $P = 0$. Summarizing, if $\Phi \in \text{Cl}$ and $\det K \neq 0$ then $P = 0$. By a similar argument, composing the equation $KV\tilde{Y} = -P\tilde{V}X$ on the left by $\tilde{V}^*\tilde{P}^*$ and on the right by $\tilde{Y}^*$, one finds that if $\Phi \in \text{Cl}$ and $\det P \neq 0$ then $K = 0$.

The case which remains to be considered is that when $\det K = \det P = 0$. Since $\det P = \frac{1}{2}g(P, P)$, $P$ is an isotropic element of $U \otimes \overline{U}$, and as such it is decomposable. Similarly, $K$ is decomposable. Namely one can write

$$K = k \otimes \lambda, \quad P = p \otimes \tilde{q}, \quad V = s \otimes \tilde{s}, \quad k, p, q, s \in U, \quad \lambda \in U^*.$$

A little two-spinor algebra then yields

$$P\tilde{V}X + KV\tilde{Y} = \varepsilon(k, \tilde{p}) \left[ \langle \lambda, q \rangle |\langle \lambda, s \rangle|^2 k \otimes k^b - \langle \lambda, \tilde{q} \rangle |\varepsilon(s, q)|^2 p \otimes p^b \right],$$

$$\det \Phi = -\text{Tr}(K\tilde{P}^*\tilde{K}^*\tilde{P}^*) = |\varepsilon(k, p)|^2 |\langle \lambda, q \rangle|^2.$$  

Now one sees that in order that $\det \Phi \neq 0$ one must have $\langle \lambda, q \rangle \neq 0$ and $\varepsilon(k, p) \neq 0$. Thus $k \otimes k^b$ and $p \otimes p^b$ are linearly independent elements of $U \otimes U^*$ and, in order that $P\tilde{V}X + KV\tilde{Y}$ vanishes for all $V$, one must have $\langle \lambda, s \rangle = \varepsilon(q, s)$ for all $s \in U$, which implies $\lambda = 0$ and $q = 0$ that is $K = 0$ and $P = 0$, a contradiction. Thus the case $\det K = \det P = 0$ cannot yield an element $\Phi \in \text{Cl}$. 

\[ \square \]

Proposition 5.1

a) $\text{Cl}^{(+)}$ is the 7-dimensional real submanifold of $D^{(+)}$ constituted of all elements in $W \otimes W^*$ which are of the type

$$\begin{pmatrix} K & 0 \\ 0 & K^* \end{pmatrix}, \quad K \in U \otimes U^*, \quad \det K \in \mathbb{R} \setminus \{0\}.$$  

b) $\text{Cl}^{(-)}$ is the 7-dimensional real submanifold of $D^{(-)}$ constituted of all elements in $W \otimes W^*$ which are of the type

$$\begin{pmatrix} 0 & P \\ p^t & 0 \end{pmatrix}, \quad P \in U \otimes \overline{U}, \quad \det P \in \mathbb{R} \setminus \{0\}.$$
PROOF:
a) Let $\Phi = \left( \begin{array}{cc} K & 0 \\ 0 & K^* \end{array} \right)$, $K \in U \otimes U^*$, $\det K \neq 0$. Then

$$(\det \Phi)\ Ad[\Phi]v = \left( \begin{array}{cc} 0 & (\det K)K V K^* \\ (\det K)\bar{V} & 0 \end{array} \right), \quad V \equiv \sqrt{2} v \in H.$$ 

For $Ad[\Phi]v$ to be in $H$, the two non-zero entries of the above matrix must be in $H \equiv U \lor U$ and in $U^* \lor U^*$, respectively. Consider the $U \otimes U$-entry. Since $\bar{V} = V^*$ because $V$ is Hermitian, one finds

$$[(\det K)K V K^*]^{-\mathbf{T}} = (\det K)K V K^*,$$

and $(\det K)K V K^*$ is Hermitian for all $V \in H$ iff $\det K = \det \bar{K}$ (this argument gives the same result for the other non-zero entry).

b) Let $\Phi = \left( \begin{array}{cc} 0 & P \\ P^* & 0 \end{array} \right)$, $P \in U \otimes U^*$, $\det P = \frac{1}{2} g(P,P) \neq 0$. Then

$$(\det \Phi)\ Ad[\Phi]v = \left( \begin{array}{cc} 0 & (\det P)P \tilde{V} \tilde{P}^* \\ (\det P)\tilde{P} V & 0 \end{array} \right).$$

By the same argument as before, $\Phi \in \text{Cl}$ iff $\det P = \det \tilde{P}$. $\square$

Now it is not difficult to show that any complex $2 \times 2$-matrix with real determinant can be written as a product of Hermitian matrices. Using this, one recovers a well-known result:

**Proposition 5.2** $\text{Cl}$ is multiplicatively generated by $H^* \subset H$, the subset of all elements in $H$ with non-vanishing Lorentz pseudo-norm.

Namely any element of Cl can be written as

$$\Phi = v_1 v_2 \ldots v_n, \quad v_j \in H, \quad g(v_j,v_j) \neq 0;$$

its inverse is

$$\Phi^{-1} = \frac{1}{\nu(\Phi)} v_n \ldots v_2 v_1, \quad \nu(\Phi) := g(v_1,v_1) g(v_2,v_2) \ldots g(v_n,v_n).$$

Setting now $V_i \equiv \sqrt{2} v_i$ one has $\det V_i = \det \tilde{V}_i = g(v_i,v_i)$, hence

$$\nu(\Phi) = \det(V_1 \tilde{V}_2 V_3 \tilde{V}_4 \ldots) = \prod_{i=1}^{n} \det(V_i).$$

Namely, if $\Phi = \left( \begin{array}{cc} K & 0 \\ 0 & K^* \end{array} \right) \in \text{Cl}^{(+)}$ then $\nu(\Phi) = \det K = \det K^*$; if $\Phi = \left( \begin{array}{cc} 0 & P \\ P^* & 0 \end{array} \right) \in \text{Cl}^{(-)}$ then $\nu(\Phi) = \det P = \det P^*$.

**Remark.** Actually, it can be seen that any complex $2 \times 2$-matrix with real determinant can be written as a product of just three Hermitian matrices (but not, in general, of two matrices). This implies that an element in $\text{Cl}^{(-)}$ can be written as $\left( \begin{array}{cc} 0 & P \\ P^* & 0 \end{array} \right)$ with $P = V_1 V_2^* V_3$, and an element in $\text{Cl}^{(+)}$ can be written as $\left( \begin{array}{cc} K & 0 \\ 0 & K^* \end{array} \right)$ with $K = V_1 V_2^* V_3 V_4^*$, $V_i \in H^*$. 

5.2 Pin and Spin

The adjoint action of any \( w \in H \) on \( H \) is easily checked to be the negative of the reflection through the hyperplane orthogonal to \( w \). It follows that \( \text{Cl}^{(\pm)} \) is the subgroup of all elements in \( \text{Cl} \) whose adjoint action preserves the orientation of \( H \). Moreover, the subgroup

\[
\text{Cl}^\dagger := \{ \Phi \in \text{Cl} : \nu(\Phi) > 0 \}
\]

is constituted of all elements of \( \text{Cl} \) whose adjoint action preserves the time-orientation of \( H \). Its representation as \( \Phi = v_1 v_2 \ldots v_n \) has an even number of spacelike factors and any number of timelike factors.

The unit element of \( \text{Cl} \) is \( 1 1 \in D(+) \subset D \). Thus the Lie algebra of \( \text{Cl} \) is a 7-dimensional vector subspace

\[
\mathfrak{L} \text{Cl} \subset D^{(+)} = \mathbb{R} \oplus \wedge^2 H \oplus \wedge^4 H \cong \mathbb{R} \oplus \hat{\gamma}(\wedge^2 H) \oplus \hat{\gamma}(\wedge^4 H).
\]

Now observe that \( \wedge^4 H \) is not contained in \( \mathfrak{L} \text{Cl} \) since \( t \in \mathbb{R} \Rightarrow \exp(t \eta^\#) = \exp \begin{pmatrix} -it & 0 & 0 \\ 0 & 0 & e^{it} \\ it & 0 & e^{-it} \end{pmatrix} \) is not in \( \text{Cl} \) because the two component endomorphisms \( e^{-it} 1 1 U \in U \otimes U^* \) and \( e^{it} 1 1 U^* \in U^* \otimes U \) have non-real determinant. Hence, just by a dimension argument, one finds

\[
\mathfrak{L} \text{Cl} = \mathbb{R} \oplus \wedge^2 H.
\]

5.2 Pin and Spin

If \( \Phi \in \text{Cl} \) and \( a \in \mathbb{R} \setminus \{0\} \) then \( \text{Ad}[a \Phi] = \text{Ad}[\Phi] : H \to H \). It is then natural to consider the subgroup

\[
\text{Pin} := \{ \Phi \in \text{Cl} : \nu(\Phi) = \pm 1 \},
\]

which is multiplicatively generated by all elements in \( H \) whose Lorentz pseudo-norm is \( \pm 1 \). It has the subgroups

\[
\text{Spin} := \text{Pin}^{(+)} \equiv \text{Pin} \cap \text{Cl}^{(+)} = \{ \Phi \in \text{Cl}^{(+)} : \nu(\Phi) = \pm 1 \},
\]

\[
\text{Pin}^\dagger := \text{Pin} \cap \text{Cl}^\dagger = \{ \Phi \in \text{Cl} : \nu(\Phi) = 1 \},
\]

\[
\text{Spin}^\dagger := \text{Spin} \cap \text{Cl}^\dagger = \{ \Phi \in \text{Cl}^{(+)} : \nu(\Phi) = 1 \}.
\]

These share the same Lie algebra

\[
\wedge^2 H = \mathfrak{L} \text{Pin} = \mathfrak{L} \text{Spin} = \mathfrak{L} \text{Pin}^\dagger = \mathfrak{L} \text{Spin}^\dagger.
\]

The automorphisms of \( U \) which have unit determinant constitute the group \( \text{Sl} \equiv \text{Sl}(U) \); thus

\[
\text{Cl}^{(+)} \cap \text{Cl}^\dagger = \{ \begin{pmatrix} K & 0 \\ 0 & K^\dagger \end{pmatrix} \in \text{End} W : K \in \mathbb{R}^+ \times \text{Sl} \},
\]

\[
\text{Spin}^\dagger = \{ \begin{pmatrix} K & 0 \\ 0 & K^\dagger \end{pmatrix} \in \text{End} W : K \in \text{Sl} \}.
\]
In particular, one has the isomorphism
\[ \text{Spin}^\dagger \leftrightarrow \text{Sl} : \left( \begin{array}{cc} K & 0 \\ 0 & K^\dagger \end{array} \right) \leftrightarrow K. \]

Now remember that
\[ \hat{\gamma}(\kappa^2 H) = \left\{ \left( \begin{array}{cc} A & 0 \\ 0 & A^\dagger \end{array} \right) \in \text{End} W : \text{Tr} A = 0 \right\}, \]
\[ \hat{\gamma}(\mathbb{R} \oplus \kappa^2 H) = \left\{ \left( \begin{array}{cc} A & 0 \\ 0 & A^\dagger \end{array} \right) \in \text{End} W : \Im \text{Tr} A = 0 \right\}; \]
moreover \text{End} U can be decomposed into the direct sum of the subspace of all traceless endomorphisms, which is just \( \mathcal{L} \text{Sl} \), and the subspace \( \mathbb{C} \mathbb{I} \) generated by the identity. Then one has the Lie algebra isomorphisms
\[ \mathcal{L} \text{Cl} = \mathcal{L} \text{Cl}^{(+)} = \mathbb{R} \oplus \kappa^2 H \rightarrow (\mathbb{R} \mathbb{I}) \oplus \mathcal{L} \text{Sl}, \]
\[ \mathcal{L} \text{Pin} = \mathcal{L} \text{Spin}^\dagger = \kappa^2 H \rightarrow \mathcal{L} \text{Sl}. \]

**Proposition 5.3** Let
\[ \Phi = \left( \begin{array}{cc} K & 0 \\ 0 & K^\dagger \end{array} \right) \in \text{Spin}, \ v \in H, \ \gamma(v) = \left( \begin{array}{cc} V & 0 \\ 0 & V^\dagger \end{array} \right) \equiv \left( \begin{array}{cc} \sqrt{2} v & 0 \\ 0 & \sqrt{2} v^\dagger \end{array} \right). \]
Then
\[ \text{Ad}[\Phi] \gamma(v) = \pm \begin{pmatrix} 0 & [K \otimes K](V) \\ ([K \otimes K](V))^\dagger & 0 \end{pmatrix}, \]
where the + sign holds iff \( \Phi \in \text{Spin}^\dagger \).

**PROOF:** Remembering the previous results one finds
\[ \text{Ad}[\Phi] \gamma(v) = \frac{1}{\det K} \begin{pmatrix} 0 & KV \bar{K}^\ast \\ (KV \bar{K}^\ast)^\dagger & 0 \end{pmatrix}. \]
Moreover
\[ (KV \bar{K}^\ast)^{\alpha \lambda} = K^A_B V^{\beta B \dagger} (\bar{K}^\ast)_B^{\dagger} = K^A_B V^{\beta B \dagger} \bar{K}^{\lambda} = (K \otimes \bar{K})^{\alpha \lambda}_{BB'} V^{B B'}. \]

Now remember (§1.8) that the group \( \{ K \otimes \bar{K} : K \in \text{Aut}(U) \} \) is constituted of automorphisms of \( U \otimes \bar{U} \) which preserve the splitting \( U \otimes \bar{U} = H \oplus i H \) and the causal structure of \( H \). Its subgroup \( \{ K \otimes \bar{K} : K \in \text{Sl}(U) \} \) coincides with \( \text{Lor}^+_+(H) \). Thus one sees that the group isomorphism \( \text{Sl} \rightarrow \text{Spin}^\dagger \) determines the 2-to-1 epimorphism \( \text{Spin}^\dagger \rightarrow \text{Lor}^+_+(H) \).

One also finds that \( \text{Spin}^\dagger \) is the subgroup of \( \text{End} W \) preserving \( (\gamma, k, g, \eta, \epsilon) \) as well as time-orientation. Let’s review these properties in terms of two-spinors.
• Obviously, Spin$^+$ preserves the splitting $W = U \oplus \overline{U}^*$. If $\Phi = \begin{pmatrix} K & 0 \\ 0 & K^\dagger \end{pmatrix}$, $K \in \text{Sl}(U)$, then $\tilde{K} = K^{-1}$, so for $\psi \equiv (u, \chi), \psi' \equiv (u', \chi') \in W$ one gets
\begin{align*}
k(\Phi \psi, \Phi \psi') &= k((Ku, \chi \tilde{K}^{-1}), (Ku', \chi' \tilde{K}^{-1})) = \langle \chi \tilde{K}^{-1}, Ku \rangle + \langle \chi' \tilde{K}^{-1}, Ku' \rangle = \\
&= \langle \chi, u' \rangle + \langle \chi', \tilde{u} \rangle = k(\psi, \psi').
\end{align*}

• Since $K \otimes \tilde{K} : U \otimes \overline{U} \to U \otimes \overline{U}$ sends Hermitian tensors to Hermitian tensors and anti-Hermitian tensors to anti-Hermitian tensors, it preserves the splitting $U \otimes \overline{U} = H \oplus iH$. Also, remember that $K \otimes \tilde{K} = \text{Ad}[\Phi]$.

• $K \otimes \tilde{K} = \text{Ad}[\Phi] \in \text{Lor}^+_\uparrow (H)$, the subgroup of the Lorentz group which preserves orientation and time-orientation.

• $\Phi$ preserves the Dirac map $\gamma$. In fact if $y \in H$ then
\begin{align*}
\gamma[y] &= \begin{pmatrix} 0 & \sqrt{2} y^\dagger \\ \sqrt{2} y & 0 \end{pmatrix}, \quad y^\dagger \equiv \tilde{y} = \tilde{y}^* ,
\end{align*}

\begin{align*}
\text{Ad}[\Phi] \gamma[y] &= \begin{pmatrix} 0 & \sqrt{2} [K \otimes \tilde{K}] y \\ \sqrt{2} ([K \otimes \tilde{K}] y) & 0 \end{pmatrix} = \gamma[[K \otimes \tilde{K}] y].
\end{align*}

• If $K \in \text{Sl}$ then $K$ preserves any symplectic form $\varepsilon \in \wedge^2 U^*$. Hence $\Phi \equiv \begin{pmatrix} K & 0 \\ 0 & K^\dagger \end{pmatrix} \in \text{Spin}^+$ preserves the corresponding symplectic form $(\varepsilon, \tilde{\varepsilon}^\#) \in \wedge^2 W^*$ and charge conjugation.

6 Spinors and particle momenta

6.1 Particle momentum in two-spinor terms

It has already been observed (§1.4) that any future-pointing non-spacelike element in $H$ can be written in the form
\begin{align*}
&u \otimes \tilde{u} + v \otimes \tilde{v} , \quad u, v \in U .
\end{align*}

If $u$ and $v$ are not proportional to each other, that is $\varepsilon(u, v) \neq 0$, then the above expression is a timelike future-pointing vector; if $\varepsilon(u, v) = 0$, then it is a null vector. Future-pointing elements in $H$ are a contravariant, “conformally invariant” version of classical particle momenta (translation to a scaled and/or covariant version, when needed, will be effortless).

Let $K$ and $N$ be the subsets of $H$ constituted of all future-pointing timelike vectors and of all future-pointing null vectors, respectively; moreover, set $J := K \cup N$ (all these sets do not contain the zero element). Consider now the real quadratic maps
\begin{align*}
\tilde{p} : U \times U &\to J : (u, v) \mapsto \frac{1}{\sqrt{2}} (u \otimes \tilde{u} + v \otimes \tilde{v}) , \\
p : W \cong U \times \overline{U}^* &\to J : (u, \chi) \mapsto \frac{1}{\sqrt{2}} (u \otimes \tilde{u} + \tilde{\chi}^\# \otimes \chi^\#) .
\end{align*}

When a normalized symplectic form $\varepsilon \in \wedge^2 U^*$ is fixed, $\tilde{p}$ and $p$ are essentially the same objects, as one can represent a given element $\frac{1}{\sqrt{2}} (u \otimes \tilde{u} + v \otimes \tilde{v})$ of $J$ by writing
$v \otimes \bar{v}$ as $(\bar{\chi} \otimes \chi)^\#$; here, $u, v \in U, \chi \in \overline{U}^*$. In such case I'll set

\[
v := -\bar{\chi}^\# \iff \chi = \bar{v}^\#, \\
\Rightarrow \langle \bar{\chi}, u \rangle = \langle v^\#, u \rangle = \varepsilon(v, u), \quad \langle \chi, \bar{u} \rangle = \langle \bar{v}^\#, \bar{u} \rangle = \bar{\varepsilon}(\bar{v}, \bar{u}).
\]

If $p = p(u, \chi) \equiv \bar{p}(u, v)$ then we'll use the shorthands

\[
\mu^2 := g(p, p) = 2|\varepsilon(u, v)|^2 = 2|\langle \bar{\chi}, u \rangle|^2,
\]

\[
h := \sqrt{\frac{2}{\mu}} p^\# = \frac{1}{|\langle \bar{\chi}, u \rangle|} (u^\# \otimes u^\# + \chi \otimes \bar{\chi}).
\]

Then, $h$ can be seen as an $\varepsilon$-normalized Hermitian metric on $U$.

**Proposition 6.1** Let $(u, \chi) \equiv (u, \bar{v}^\#) \in W, \langle \bar{\chi}, u \rangle \neq 0$; let $p \in K$. Then, the following conditions are equivalent:

i) $p = u \otimes \bar{u} + (\bar{\chi} \otimes \chi)^\#$,

ii) $\gamma(p)(u, \chi) = \mu (e^{-i\theta} u, e^{i\theta} \chi), \theta \in \mathbb{R}$,

iii) $h^\#(u) = e^{i\theta} \chi$,

iv) $h^\#(\chi) = e^{-i\theta} u$,

v) $h(\bar{u}, v) = 0$ and $|\langle \bar{\chi}, u \rangle| = h(\bar{u}, u)$,

v') $h(\bar{u}, v) = 0$ and $|\langle \bar{\chi}, u \rangle| = h(\bar{v}, v),$

where $\mu$ and $h$ are defined in terms of $(u, \chi)$ as above.

**Proof:** By straightforward calculations one sees that condition i implies conditions ii, iii, iv and v. Moreover:

( ii $\iff$ iii): It follows from $\gamma(\tau)(u, \chi) = \frac{1}{\sqrt{\varepsilon}} \gamma[h^\#](u, \chi) = (h^\#(\chi), h^\#(u))$.

( iii $\iff$ iv): If $h^\#(u) = e^{i\theta} \chi$ then $u = h^\#(h^\#(u)) = h^\#(e^{i\theta} \chi) = e^{i\theta} h^\#(\chi)$. Similarly, if $h^\#(\chi) = e^{-i\theta} u$ then $\chi = h^\#(h^\#(\chi)) = h^\#(e^{-i\theta} u) = e^{-i\theta} h^\#(u)$.

( iv $\Rightarrow$ v): $h(\bar{u}, v) = \langle h^\#(\bar{u}), -\bar{\chi}^\# \rangle = -\langle e^{-i\theta} \bar{\chi}, h^\#(\bar{u}) \rangle = e^{-i\theta} e^\#(\bar{\chi}, \bar{u}) = 0$.

Moreover $h(\bar{u}, u) = \langle \bar{h}^\#(\bar{u}), \bar{u} \rangle = \langle e^{i\theta} \chi, \bar{u} \rangle = \langle \bar{\chi}, u \rangle \langle \bar{\chi}, u \rangle / |\langle \bar{\chi}, u \rangle| = |\langle \bar{\chi}, u \rangle|$.

( v $\Rightarrow$ iv): From $0 = h(\bar{u}, v) = \langle h^\#(\bar{u}), -\bar{\chi}^\# \rangle = -\varepsilon^\#(\bar{\chi}, h^\#(\bar{u}))$ one has $\bar{\chi} = ch^\#(\bar{u})$, $c \in \mathbb{C}$. Then $\langle \bar{\chi}, u \rangle = c h(\bar{u}, u) = c |\langle \bar{\chi}, u \rangle| \Rightarrow c = e^{i\theta}$.

( v $\Rightarrow$ v'): From iv (equivalent to v) one has $h(\bar{v}, v) = \langle h, \chi^\# \otimes \bar{\chi}^\# \rangle = \langle h^\#(\chi), \bar{\chi} \rangle = e^{-i\theta} (\langle \chi, u \rangle) = |\langle \chi, u \rangle|$, hence also $h(\bar{v}, v) = |\langle \bar{\chi}, u \rangle|$

( v' $\Rightarrow$ iv): As in v $\Rightarrow$ iv one has $\bar{\chi} = ch^\#(\bar{u}), c \in \mathbb{C}$, or $u = \frac{1}{h^\#(\chi)}$. Then, from $\langle \bar{\chi}, u \rangle = \langle \bar{\chi}, \frac{1}{h^\#(\chi)} \rangle = \frac{1}{2} h^\#(\chi, \bar{\chi}) = \frac{1}{2} h^\#(\chi, \bar{\chi}) = \frac{1}{2} \bar{h}^\#(\bar{v}, v)$ one has $c = e^{-i\theta}$ i.e. $c = e^{i\theta}$.

( v $\Rightarrow$ i): Using also v' (already seen to be equivalent to v) one sees that the couple $(\zeta_u, \zeta_v) \equiv \langle u, v \rangle / \sqrt{|\langle \bar{\chi}, u \rangle|}$ is an $h$-orthonormal basis of $U$; hence $h^\# = \zeta_u \otimes \zeta_u + \zeta_v \otimes \zeta_v = \frac{1}{|\langle \bar{\chi}, u \rangle|} (u \otimes u + v \otimes v)$. Condition i then follows. \qed
6.2 Bundle structure of 4-spinor space over momentum space

The previous results show that the restriction \( p : W \setminus \{0\} \rightarrow J \) is surjective. Since the Lorentz "length" of \( p(u, \chi) \) is \( \sqrt{2} \left| \langle \chi, u \rangle \right| \) one sees that the subset of all elements in \( W \) which project onto \( N \) is the 6-dimensional real submanifold

\[
W^0 := p^{-1}(N) = \left\{ (u, \chi) \in W \setminus \{0\} : \langle \chi, u \rangle = 0 \right\} \subset W.
\]

The subset of all elements in \( W \) which project onto \( K \) is the open submanifold

\[
W^\alpha := p^{-1}(K) = \left\{ (u, \chi) \in W : \langle \chi, u \rangle \neq 0 \right\},
\]

and one has

\[
W \setminus \{0\} = W^0 \cup W^\alpha.
\]

Moreover, consider the subsets \( W^+, W^- \subset W^\alpha \) defined to be

\[
W^\pm := \left\{ (u, \chi) \in W \setminus \{0\} : \langle \chi, u \rangle \in \mathbb{R}^\pm \right\}.
\]

Recalling condition ii of proposition 5.1 one has

\[
\gamma[\psi] = \mu \left( e^{-i\theta} u, e^{i\theta} \chi \right),
\]

which holds for every \( \psi \equiv (u, \chi) \in W \) (if \( \psi \in W^0 \) then \( \mu = 0 \)). In particular

\[
W^\pm = \left\{ \psi \equiv (u, \chi) \in W \setminus \{0\} : \gamma[\psi] = \pm \mu \psi, \mu \equiv |\langle \chi, u \rangle| \right\}.
\]

Next, consider the subset

\[
\tilde{W}^\alpha := \left\{ (u, v) : \varepsilon(u, v) \neq 0 \right\} \subset U \times U,
\]

and note that when a normalized symplectic form \( \varepsilon \in \wedge^2 U^* \) is fixed, \( \tilde{W}^\alpha \) can be identified with \( W^\alpha \) via the correspondence \( \tilde{v}^\alpha \leftrightarrow \chi \). \( \tilde{W}^\alpha \) is a fibred set over \( K \); for each \( p \in K \), the fibre of \( \tilde{W}^\alpha \) over \( p \) is the subset

\[
\tilde{W}_p^\alpha := \tilde{p}^{-1}(p) = \left\{ (u, v) \in \tilde{W}^\alpha : \frac{1}{\sqrt{2}}(u \otimes \bar{u} + v \otimes \bar{v}) = p \right\}.
\]

**Proposition 6.2** \( \tilde{p} : \tilde{W}^\alpha \rightarrow K \) is a trivializable principal bundle with structure group \( U(2) \).

**Proof:** Let \( p = \tilde{p}(u, v) = \tilde{p}(u', v') \). From proposition 6.1 one then sees that \( (u, v) \) and \( (u', v') \) are orthonormal bases of \( U \) relatively to the Hermitian metric \( h \equiv \sqrt{2} \tilde{p}/\mu \). Then there exists a unique transformation \( K \in U(U, h) \) such that

\[
u' = K(u), \quad v' = K(v);
\]

hence, \( \tilde{W}_p^\alpha \) is a group-affine space, with derived group \( U(2) \).

Let now \( (\zeta_\lambda) \) be an \( \varepsilon \)-normalized basis of \( U \) and \( (\tau_\lambda) \) the associated Pauli frame. For each \( p \in K \) let \( L_p \in \text{Lor}_+^1(H) \) be the boost such that \( L_p \tau_0 = p/\mu \), where \( \mu^2 \equiv g(p, p) \); up to sign there is a unique \( B_p \in \text{Sl}(U) \) such that \( L_p = B_p \otimes B_p \), and a consistent smooth way of choosing one such \( B_p \) for each \( p \) can be fixed. It turns out that the basis \( (\sqrt{\mu} B_p \zeta_\lambda) \) is orthonormal relatively to \( \sqrt{2} \tilde{p}/\mu \) seen as a Hermitian metric on \( U \), hence \( \tilde{p}(\sqrt{\mu} B_p \zeta_1, \sqrt{\mu} B_p \zeta_2) = p \). In this way one selects an “origin” element in each fibre of \( \tilde{p} \), so getting a trivialization \( \tilde{W}^\alpha \rightarrow K \times U(2) \). \( \square \)
Using a little two-spinor algebra it is not difficult to prove:

**Proposition 6.3** Let \( \psi, \psi' \in W^\alpha \), \( \psi \equiv (u, \chi) \), \( \psi' \equiv (u', \chi') \); let \( K \in \text{Aut} U \) be the unique automorphism of \( U \) such that

\[
K u = u, \quad K \bar{\chi}^\# = \bar{\chi}'^\#.
\]

Then

\[
K = \frac{1}{\langle \bar{\chi}, u \rangle^2} \left[ \langle \bar{\chi}, u' \rangle \bar{\chi} - \varepsilon^\#(\bar{\chi}, \chi') u \otimes u^b + \varepsilon(u, u') \bar{\chi}^\# \otimes \bar{\chi} + \langle \bar{\chi}', u \rangle \bar{\chi}^\# \otimes u^b \right].
\]

Moreover, one has

\[
\chi' = K^\dagger \chi.
\]

Conversely, the conditions \( u' = Ku \) and \( \chi' = K^\dagger \chi \) determine \( K \) uniquely.

The above expression for \( K \) is invariant relatively to the transformation \( \varepsilon \mapsto e^{i\theta} \varepsilon \); hence, \( K \) is independent of the particular normalized symplectic form \( \varepsilon \) chosen.

When a normalized \( \varepsilon \in \wedge^2 U^* \) is given, one has the real vector bundle isomorphism \( W^\alpha \leftrightarrow \tilde{W}^\alpha : (u, v) \leftrightarrow (u, \bar{v}) \). Through this correspondence, \( W^\alpha \rightarrow K \) turns out to be a trivializable principal bundle with structure group \( U(2) \). If \( \psi, \psi' \in W^\alpha_p \), let

\[
(K) = c \begin{pmatrix} a & \bar{b} \\ -b & \bar{a} \end{pmatrix} \in U(2), \quad a, b, c \in \mathbb{C} : |a|^2 + |b|^2 = |c|^2 = 1,
\]

be the matrix of \( K \in \text{Aut} U \) sending \( \psi \) to \( \psi' \) (according to proposition [6.3]) relatively to the basis \((u, v)\). Then

\[
\begin{align*}
u' &= c(au - bv), \\
v' &= c(bu + au).
\end{align*}
\]

\[
\begin{align*}
u' &= c(a \bar{u} + b \bar{\chi}^\#), \\
\chi' &= \bar{c}(a \chi + b \bar{\chi}^\#).
\end{align*}
\]

If you take a different normalized symplectic form \( \varepsilon \rightarrow e^{i\theta} \varepsilon \), then \( K \) does not change, while the corresponding matrix \((K) \in U(2)\) changes according to \( c \rightarrow c, \ a \rightarrow a, \ b \rightarrow e^{i\theta}b \).

The above \( U(2) \)-action does not preserve \( W^\pm \subset W^\alpha \). In fact it’s straightforward to prove:

**Proposition 6.4** Let \( \psi, \psi' \in W^+_p \) (resp. \( \psi, \psi' \in W^-_p \)), \( \psi \equiv (u, \chi) \), \( \psi' \equiv (u', \chi') \); let \( K \) be the unique automorphism of \( U \) such that \( Ku = u \), \( K^\dagger \chi = \chi' \). Then \( K \in \text{SU}(U, h) \), where \( h \equiv \sqrt{-2} p^b / \mu \).

Hence, \( W^+ \rightarrow K \) and \( W^- \rightarrow K \) turn out to be trivializable principal bundles, with structure group \( \text{SU}(2) \).
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