NON-ABELIAN SYMMETRIES OF QUASITORIC MANIFOLDS

MICHAEL WIEMELER

Abstract. A quasitoric manifold \( M \) is a \( 2n \)-dimensional manifold which admits an action of an \( n \)-dimensional torus which has some nice properties. We determine the isomorphism type of a maximal compact connected Lie-subgroup \( G \) of \( \text{Homeo}(M) \) which contains the torus. Moreover, we show that this group is unique up to conjugation.

1. Introduction

A quasitoric manifold is a smooth connected orientable \( 2n \)-dimensional manifold \( M \) with a smooth action of an \( n \)-dimensional torus \( T \) such that:

- The \( T \)-action on \( M \) is locally standard, i.e. the \( T \)-action is locally modelled on the standard \( T \)-action on \( \mathbb{C}^n \).
- If the first property is satisfied, then \( M/T \) is naturally an \( n \)-dimensional manifold with corners. We require that the orbit space of the \( T \)-action on \( M \) is face-preserving homeomorphic to an \( n \)-dimensional simple polytope.

Quasitoric manifolds were introduced by Davis and Januszkiewicz in 1991. A symplectic \( 2n \)-dimensional manifold with a hamiltonian action of an \( n \)-dimensional torus is an example of a quasitoric manifold. We call such a manifold a symplectic toric manifold.

In this paper we construct a maximal compact connected Lie-subgroup of the homeomorphism group of \( M \) which contains the torus \( T \). To be more precise, we have the following theorem.

Theorem 1.1. Let \( M \) be a quasitoric manifold. Then there is a compact connected Lie-subgroup \( G \) of \( \text{Homeo}(M) \) which contains the torus \( T \) such that:

1. \( G \) acts smoothly on \( M \) for some smooth structure on \( M \).
2. If \( G' \subset \text{Homeo}(M) \) is another compact connected Lie-subgroup which contains the torus \( T \) and acts smoothly on \( M \) for some smooth structure on \( M \), then \( G' \) is conjugated in \( \text{Homeo}(M) \) to a subgroup of \( G \).
3. If the \( G \)- and \( G' \)-actions are smooth with respect to the same smooth structure on \( M \), then \( G' \) is conjugated in \( \text{Diff}(M) \) to a subgroup of \( G \).
4. If \( M \) is a symplectic toric manifold, then \( G \) is conjugated in \( \text{Homeo}(M) \) to a subgroup of the symplectomorphism group of \( M \).

A smooth structure on \( M \) for which the \( G \)-action from the above theorem is smooth can be described as follows. By Theorem 5.6 of \( [4] \) the \( T \)-equivariant smooth structures on \( M \) correspond one-to-one to smooth structures on the orbit space \( M/T \). The \( G \)-action is smooth for the \( T \)-equivariant smooth structure on \( M \) for which \( M/T \) is diffeomorphic to a simple polytope.

In this paper all actions of compact Lie-groups on manifolds \( M \) are smooth with respect to some smooth structure on \( M \).

2010 Mathematics Subject Classification. Primary 57S05, 57S15.
Key words and phrases. quasitoric manifolds, non-abelian Lie-groups.
Parts of the research were supported by SNF Grant No. PBFRP2-133466 and a grant from the MPG.
This article is organized as follows. In section 2, we review some basic facts about quasitoric manifolds and introduce an automorphism group for the characteristic pair corresponding to a quasitoric manifold. In section 3, we construct the group $G$ from Theorem 1.1. In section 4, we review the classification of quasitoric manifolds with $G$-action up to $G$-equivariant diffeomorphism given in $\mathbb{H}$. Moreover, we give a classification of these manifolds up to $G$-equivariant homeomorphism. In section 5, we apply the results of the previous section and show that the group $G$ has the properties described in Theorem 1.1.

2. Characteristic pairs and their automorphism groups

Let $M$ be a $2n$-dimensional quasitoric manifold, $P$ its orbit polytope and $\pi : M \to P$ the orbit map. Denote by $\mathfrak{F}(M)$ the set of facets of $P$. We write also $\mathfrak{F}$ instead of $\mathfrak{F}(M)$ if it is clear from the context which quasitoric manifold is meant. Then the preimage $M_i = \pi^{-1}(F_i)$ of $F_i \in \mathfrak{F}$ is a codimension two submanifold of $M$ which is fixed by a one dimensional subtorus $\lambda(F_i) = \lambda(M_i)$ of $T$. These $M_i$ are called characteristic submanifolds of $M$. Since the facets of $P$ correspond one-to-one to the characteristic submanifolds of $M$, we denote the set of characteristic manifolds also by $\mathfrak{F}$.

Let $IT \subset LT$ be the integral lattice in the Lie algebra of $T$. The characteristic map $\lambda : \mathfrak{F} \to \{\text{one-dimensional subtori of } T\}$ lifts to a map $\bar{\lambda} : \mathfrak{F} \to \mathbb{Z}^n$ such that, for a subset $\sigma$ of $\mathfrak{F}$ with $\bigcap_{F_i \in \sigma} F_i \neq \emptyset$, $\{\bar{\lambda}(F_i); F_i \in \sigma\}$ is part of a basis of $IT$. Note that each $\bar{\lambda}(M_i)$ is unique up to sign. We call $\bar{\lambda}$ a characteristic function for $M$.

Dual to $P$ there is a simplicial complex $K$ with vertex set $\mathfrak{F}$. A subset $\sigma \subset \mathfrak{F}$ is a simplex of $K$ if and only if $\bigcap_{F_i \in \sigma} F_i \neq \emptyset$.

Note that $M$ is determined by the combinatorial type of $P$ (or $K$) and $\bar{\lambda}$ up to equivariant homeomorphism $\mathbb{H}$ Proposition 1.8. This construction motivates the following definition.

**Definition 2.1.** Let $K$ be a simplicial complex of dimension $n - 1$ with vertex set $\mathfrak{F}$. Moreover let $T$ be an $n$-dimensional torus and $\bar{\lambda} : \mathfrak{F} \to IT \cong \mathbb{Z}^n$ a map such that for all simplices $\sigma$ of $K$ the set $\{\bar{\lambda}(F_i); F_i \in \sigma\}$ is part of a basis of $IT$. Then we call $(K, \bar{\lambda})$ a characteristic pair.

An omniorientation of a quasitoric manifold $M$ is a choice of orientations for $M$ and all characteristic submanifolds of $M$. An omniorientation of $M$ induces a complex structure on all normal bundles of the characteristic submanifolds. These complex structures may be used to make the map $\bar{\lambda}$ unique by requiring that the $S^1$-action induced by $\bar{\lambda}(M_i)$ on the normal bundle of $M_i$ is given by complex multiplication.

The cohomology $H^*(M; \mathbb{Z})$ was computed by Davis and Januszkiewicz $\mathbb{H}$ Theorem 4.14. It is torsion-free and generated by the Poincaré-duals $PD(M_i)$ of the characteristic manifolds. If we choose $\lambda$ as above, then these Poincaré-duals are subject to the following relations

$$0 = \sum_{M_i \in \mathfrak{F}} \langle v, \bar{\lambda}(M_i) \rangle PD(M_i) \quad \text{for all } v \in IT^*$$

and, for $\sigma \subset \mathfrak{F}$,

$$0 = \prod_{M_i \in \sigma} PD(M_i) \iff \sigma \text{ is not a simplex of } K.$$
columns of a matrix of the form

\[ \Lambda = \begin{pmatrix} 1 & \cdots & \Lambda' \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \]

where the first \#\sigma columns of \( \Lambda \) correspond to the \( F_i \in \sigma \). We call such a matrix a characteristic matrix of \( M \). If \( v_1, \ldots, v_n \) is the basis of \( IT^* \) dual to the standard basis of \( IT \cong \mathbb{Z}^n \), then the coefficient \( \langle v_j, \lambda(M_i) \rangle \) in equation (2.1) is the \( i \)-th entry of the \( j \)-th row of \( \Lambda \). Hence, one can read off all relations in (2.1) from the matrix \( \Lambda \).

We are interested in the symmetries of \( M \). Since quasitoric manifolds are determined by their characteristic pairs, we should also study automorphisms of characteristic pairs. Therefore we define:

**Definition 2.2.** Let \((K, \tilde{\lambda})\) be a characteristic pair. Then we define the automorphism group of \((K, \tilde{\lambda})\) to be

\[ \text{aut}(K, \tilde{\lambda}) = \{ (f, g) \in \text{aut}(K) \times \text{aut}(T); \ Lg \circ \tilde{\lambda} = \tilde{\lambda} \circ f \}. \]

**Lemma 2.3.** Let \( M \) be a quasitoric manifold. Choose an omniorientation on \( M \) such that for two characteristic submanifolds \( M_1, M_2 \subset M \) we have

\[ PD(M_1) = \pm PD(M_2) \Rightarrow PD(M_1) = PD(M_2). \]

For \( \alpha \in H^2(M) \) let

\[ \mathfrak{A}_\alpha = \{ M_i \in \mathfrak{A}; PD(M_i) = \alpha \}. \]

Then there is a unique homomorphism \( \phi: \prod_{\alpha \in H^2(M)} S(\mathfrak{A}_\alpha) \to \text{aut}(K, \tilde{\lambda}) \) such that \( \psi \circ \phi : \prod_{\alpha \in H^2(M)} S(\mathfrak{A}_\alpha) \to S(\mathfrak{A}) \) is the standard inclusion. Here, \( \psi : \text{aut}(K, \tilde{\lambda}) \to S(\mathfrak{A}) \) is the natural projection.

**Proof.** Let \( \sigma \in \prod_{\alpha \in H^2(M)} S(\mathfrak{A}_\alpha) \subset S(\mathfrak{A}) \). If \((f, g)\) is an element of \( \text{aut}(K, \tilde{\lambda}) \) such that \( \psi((f, g)) = \sigma \), then we must have \( f = \sigma \) and \( Lg(\tilde{\lambda}(F_i)) = \tilde{\lambda}(\sigma(F_i)) \) for all \( F_i \in \mathfrak{A} \). Since \( IT \) is generated by the \( \tilde{\lambda}(F_i), F_i \in \mathfrak{A}, g \) is uniquely determined by \( \sigma \). Therefore a homomorphism \( \phi: \prod_{\alpha \in H^2(M)} S(\mathfrak{A}_\alpha) \to \text{aut}(K, \tilde{\lambda}) \) with the properties described in the lemma is unique, if it exists.

Now we show that \( \phi \) exists. Let \( \sigma \in \prod_{\alpha \in H^2(M)} S(\mathfrak{A}_\alpha) \subset S(\mathfrak{A}) \). Then we have, for \( I \subset \mathfrak{A} \),

\[ \bigcap_{F_i \in I} F_i = \emptyset \iff \prod_{M_i \in I} PD(M_i) = \prod_{M_i \in I} PD(\sigma(M_i)) = 0 \iff \bigcap_{F_i \in I} \sigma(F_i) = \emptyset. \]

Therefore \( \sigma \) is an automorphism of \( K \).

Now let \( F_1, \ldots, F_n \in \mathfrak{A} \) such that \( \bigcap_{i=1}^n F_i \neq \emptyset \). Then \( \tilde{\lambda}(F_1), \ldots, \tilde{\lambda}(F_n) \) is a basis of \( IT \). Moreover, since \( \sigma \) is an automorphism of \( K \), the same holds for \( \tilde{\lambda}(\sigma(F_1)), \ldots, \tilde{\lambda}(\sigma(F_n)) \).

Therefore we may define an automorphism \( g_\sigma \) of \( T \) by \( Lg_\sigma(\tilde{\lambda}(F_i)) = \tilde{\lambda}(\sigma(F_i)) \) for \( i = 1, \ldots, n \). Let \( v_1, \ldots, v_n \) be the basis of \( IT^* \) dual to \( \lambda(F_1), \ldots, \lambda(F_n) \) and \( v'_1, \ldots, v'_n \) the basis of \( IT^* \) dual to \( \lambda(\sigma(F_1)), \ldots, \lambda(\sigma(F_n)) \). Then we have, for \( i = 1, \ldots, n \),

\[
\sum_{M_j \in \mathfrak{A} - \{M_1, \ldots, M_n\}} \langle v_i, \lambda(M_j) \rangle PD(M_j) = -PD(M_i) = -PD(\sigma(M_i))
\]

\[
= \sum_{M_j \in \mathfrak{A} - \{\sigma(M_1), \ldots, \sigma(M_n)\}} \langle v'_i, \lambda(M_j) \rangle PD(M_j)
\]

\[
= \sum_{M_j \in \mathfrak{A} - \{M_1, \ldots, M_n\}} \langle v'_i, \lambda(\sigma(M_j)) \rangle PD(M_j)
\]
Since \{PD(M_i); M_i \in \mathfrak{F} = \{M_1, \ldots, M_n\}\} is a basis of \(H^2(M)\), it follows that
\[L_g \sigma(\lambda(M_i)) = \lambda(\sigma(M_i))\] for all \(M_i \in \mathfrak{F}\).

Therefore \((\sigma, g\sigma) \in \text{aut}(K, \lambda)\) and we define \(\phi(\sigma) = (\sigma, g\sigma)\). \(\square\)

**Lemma 2.4.** Let \(M\) be a quasitoric manifold as in Lemma 2.3. Then for \(x \in M^T\) and \(\alpha \in H^2(M)\) we have
\[\#\mathfrak{F}_\alpha - 1 \leq \#\{M_i \in \mathfrak{F}_\alpha; x \in M_i\} \leq \#\mathfrak{F}_\alpha\]

**Proof.** It follows from the relations (2.1) that the Poincaré duals of the \(M_i \in \mathfrak{F}_\alpha\), \(x \not\in M_i\), form a basis of \(H^2(M)\). \(\square\)

### 3. Constructing group actions

In this section we construct an action of a compact connected Lie-group on a quasitoric manifold which extends the torus action.

Before we do that we explain how an action of a compact connected Lie-group acts on a quasitoric manifold as in Lemma 2.3. Then there is a compact connected Lie-group \(G\) such that there is a characteristic submanifold of \(M\), \(x \not\in M_i\), forming a basis of \(H^2(M)\). We call \(G\), the set of characteristic submanifolds of \(M\) which are permuted by \(W(G)\).

Moreover, \(W(G)\) acts on \(T\) by conjugation. The actions of \(W(G)\) on \(T\) and \(\mathfrak{F}\) induce an action of \(W(G)\) on the characteristic pair \((K, \lambda)\). The homomorphism \(W(G) \to \text{aut}(K, \lambda)\) corresponding to this action is the restriction of the homomorphism \(\phi\) constructed in Lemma 2.3 to \(W(G)\).

Now we state the main theorem of this section.

**Theorem 3.1.** Let \(M\) be a quasitoric manifold as in Lemma 2.3. Then there is a smooth structure and a smooth action of a compact connected Lie-group \(G\) on \(M\) which extends the torus action such that \(W(G) = \prod_{\alpha \in H^2(M)} S(\mathfrak{F}_\alpha)\).

**Proof.** We prove this theorem by induction on the dimension of \(M\). If \(\dim M = 0\) or \(\#\mathfrak{F}_\alpha \leq 1\) for all \(\alpha \in H^2(M)\), then there is nothing to prove. Therefore assume that there is an \(\alpha \in H^2(M)\) such that \(\#\mathfrak{F}_\alpha \geq 2\).

By Lemmas 2.3 and 2.4 there are two cases:

1. \(\bigcap_{M_i \in \mathfrak{F}_\alpha} M_i = \emptyset\) and, for all \(M_{i_0} \in \mathfrak{F}_\alpha\), \(\bigcap_{M_i \in \mathfrak{F}_\alpha - \{M_{i_0}\}} M_i \neq \emptyset\),

2. \(\bigcap_{M_i \in \mathfrak{F}_\alpha} M_i \neq \emptyset\).

We consider at first the case 1. In this case we have with \(N = \bigcap_{M_i \in \mathfrak{F}_\alpha - \{M_{i_0}\}} M_i\) that

1. \(M/T = \Delta^{\#\mathfrak{F}_\alpha - 1} \times N/T\)
Here the sums are taken over those characteristic submanifolds of $S$ belong to $F$ follows.

Let $G/H$ acts on $\bar{M}$, such that the action of $\bar{G}$ on $\bar{M}$ induces an action of $\bar{G}$. Then the action of $SU(\#\mathfrak{g}_\alpha) \times G'$ on $S^{2\#\mathfrak{g}_\alpha} \times N$ induces an action of $G = SU(\#\mathfrak{g}_\alpha) \times Z_G(\phi(S^1))$ on $M$ such that the action of $G/H$ extends the torus action. Here $Z_G(\phi(S^1))$ is the centralizer of $\phi(S^1)$ in $G'$ and $H$ is the ineffective kernel of the $G$-action on $M$.

Therefore, by Proposition 1.8 of [1]. $M$ is equivariantly homeomorphic to $S^{2\#\mathfrak{g}_\alpha} \times S^1 N$.

Here the action of $S^1$ on $N$ is induced by the homomorphism $\phi: S^1 \to T'$ to the torus, which acts on $N$, defined by $\mu = (\lambda_1, \ldots, \lambda_k)\lambda = \sum_{\alpha \in \mathfrak{g}_\alpha} \lambda(M_\alpha)$. Moreover, $S^1$ acts on $S^{2\#\mathfrak{g}_\alpha} \subset \mathbb{C}^{2\#\mathfrak{g}_\alpha}$ by multiplication.

By the induction hypothesis there is a compact connected Lie-group $G'$ which acts on $N$ by an extension of the torus action, such that $G'$ realizes the action of $\prod_{\beta \in H^2(N)} S(\mathfrak{g}_\beta)$ on the simplicial complex dual to $N/T$.

Then the action of $SU(\#\mathfrak{g}_\alpha) \times G'$ on $S^{2\#\mathfrak{g}_\alpha} \times N$ induces an action of $G = SU(\#\mathfrak{g}_\alpha) \times Z_G(\phi(S^1))$ on $M$ such that the action of $G/H$ extends the torus action. 

Here $Z_G(\phi(S^1))$ is the centralizer of $\phi(S^1)$ in $G'$ and $H$ is the ineffective kernel of the $G$-action on $M$.

From Lemma 2.5 of [5] we know that $W(SU(\#\mathfrak{g}_\alpha)) = S(\mathfrak{g}_\alpha)$. Therefore we have to show that

$$W(Z_G(\phi(S^1))) = \prod_{\beta \in H^2(M) - \{\alpha\}} S(\mathfrak{g}_\beta(M)) \subset \prod_{\beta \in H^2(N)} S(\mathfrak{g}_\beta(N)) = W(G').$$

It follows from the remarks at the beginning of this section that $W(Z_G(\phi(S^1)))$ is a subgroup of $\prod_{\beta \in H^2(M) - \{\alpha\}} S(\mathfrak{g}_\beta(M))$. Therefore let $w \in \prod_{\beta \in H^2(M) - \{\alpha\}} S(\mathfrak{g}_\beta(M))$. Since $\prod_{\beta \in H^2(M) - \{\alpha\}} S(\mathfrak{g}_\beta(M))$ is generated by transpositions, we may assume that $w$ is a transposition.

Let $M_1, \ldots, M_m \in \mathfrak{g}(M) - \mathfrak{g}_\alpha$ such that $\cap_{i=1}^m M_i \cap N$ is a single point. Then $\lambda(M_1), \ldots, \lambda(M_m)$ form a basis of $IT'$. Let $v_1, \ldots, v_m$ be the dual basis of $IT''$. Let $i \in \{1, \ldots, m\}$. At first assume that $w(M_i) = M_i$ with $i' \in \{1, \ldots, m\}$. Then we have

$$\langle v_i, \mu \rangle + \sum_{i'=1}^m \langle v_i, \lambda(M_{i'}) \rangle PD(M_{j'}) = -PD(M_i) = -PD(M_{i'})$$

$$= \langle v_{i'}, \mu \rangle + \sum_{i'=1}^m \langle v_{i'}, \lambda(M_{j'}) \rangle PD(M_{j'}).$$

Here the sums are taken over those characteristic submanifolds of $M$ which do not belong to $\mathfrak{g}_\alpha \cup \{M_1, \ldots, M_m\}$. Their Poincaré duals together with $\alpha$ form a basis of $H^2(M)$. Therefore we have

$$\langle v_i, \mu \rangle = \langle v_{i'}, \mu \rangle = \langle w^* v_i, \mu \rangle = \langle v_i, w^* \mu \rangle.$$
Now assume that \( w(M_i) \neq M_1, \ldots, M_{m'} \). Then, because \( PD(M_i) = PD(w(M_j)) \) we must have \( \langle v_i, \tilde{\lambda}(M_j) \rangle = 0 \) for all \( M_j \in \mathfrak{F}(M) - \{ M_i, w(M_i) \} \). This implies \( \langle v_i, \mu \rangle = 0 \). Moreover, we have

\[
\langle v_i, w_* \mu \rangle = \sum_{j=1}^{m'} \langle v_i, w_* \tilde{\lambda}(M_j) \rangle \langle v_j, \mu \rangle = 0.
\]

Therefore we have \( \mu = w_* \mu \). This implies \( w \in W(Z_{G'}(\phi(S^1))) \). Hence, the claim follows in this case.

Now assume that \( \bigcap_{M_i \in \mathfrak{F}_\alpha} M_i \) is non-empty. Then let \( \tilde{M} \) be the blow-up of \( M \) along \( \bigcap_{M_i \in \mathfrak{F}_\alpha} M_i \) (see Section 4 of [5] for details). If we write the characteristic matrix of \( \tilde{M} \) in the form

\[
\begin{pmatrix}
1 & \cdots & \Lambda' \\
\vdots & \ddots & \\
1 & \cdots & \Lambda' \\
1 & \cdots & 0
\end{pmatrix},
\]

such that the first \#\( \mathfrak{F}_\alpha \) columns correspond to the \( F_i \in \mathfrak{F}_\alpha \), then the characteristic matrix of \( \tilde{M} \) is given by

\[
\begin{pmatrix}
1 & \cdots & \Lambda' \\
\vdots & \ddots & \\
1 & \cdots & \Lambda' \\
1 & \cdots & 0
\end{pmatrix} - 
\begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \\
1 & \cdots & 0
\end{pmatrix},
\]

where the proper transforms of the characteristic submanifolds of \( M \) are ordered as in [5], the last column corresponds to the exceptional submanifold and the first \#\( \mathfrak{F}_\alpha \) entries in this column are equal to \(-1\).

Hence two characteristic submanifolds of \( M \) have the same Poincaré-duals if and only if their proper transforms have the same Poincaré-duals. Moreover, the Poincaré-dual of the exceptional submanifold is distinct from the Poincaré-duals of the other characteristic submanifolds of \( \tilde{M} \).

By the first case there is a \( G \)-action on \( \tilde{M} \) which extends the torus action. Since the exceptional submanifold is fixed by \( \phi(S^1) \), we can \( G \)-equivariantly blow down \( \tilde{M} \) along the exceptional manifold to get a \( G \)-action on \( M \) (see [5] Section 4 for details).

**Corollary 3.2.** The group \( G \) constructed in Theorem 3.1 has a covering group of the form \( \prod_{\alpha \in H^2(M,\mathbb{Z})} SU(\#\mathfrak{F}_\alpha) \times T^{l_0} \).

**Proof.** This follows from the results of Section 2 of [5] and the description of the \( W(G) \)-action on \( \mathfrak{g} \) given in Theorem 3.1.

In [4] we proved that the equivariant smooth structures on a quasitoric manifold correspond one-to-one to the smooth structures on its orbit space. We will show that the \( G \)-action constructed in Theorem 3.1 is smooth with respect to the smooth structure on \( M \) which corresponds to the natural smooth structure on the simple polytope \( P \). This will follow from the proof of Theorem 3.1 Corollary 5.3 of [4] and the following lemma.
Lemma 3.3. In the situation of Theorem 5.16 of [5] we have: $N/T$ is diffeomorphic to a simple polytope if and only if $M/T$ is diffeomorphic to a simple polytope.

Proof. If $M/T$ is diffeomorphic to a simple polytope, then $N/T$ is diffeomorphic to a simple polytope because $N/T$ is a face of $M/T$.

So we only have to prove the other implication. If $N/T$ is diffeomorphic to a simple polytope, then all face-preserving homeomorphisms constructed in the proof of the cited theorem may be replaced by diffeomorphisms. So if we follow the proof of this theorem we end-up with a diffeomorphism $g : F_1 \times \Delta_{l_1} \to F_1 \times \Delta_{l_1}$, which we want to extend to a diffeomorphism $F_1 \times \Delta_{l_1} \to F_1 \times \Delta_{l_1}$. That this is possible follows from Theorem 5.1 of [4] because every facet of $F_1 \times \Delta_{l_1}$ of the form $F_1 \times F$, where $F$ is a facet of $\Delta_{l_1}$, is mapped my $g$ to a facet of the same form.

Corollary 3.4. The action of the group $G$ constructed in Theorem 5.7 is smooth with respect to the smooth structure on $M$ corresponding to the natural smooth structure on $P$.

Proof. We use the same induction and notations as in the proof of Theorem 5.1. If the $G$-action on $N$ is smooth with respect to the smooth structure on $N$ for which $N/T$ is diffeomorphic to a simple polytope, then by Lemma 3 the $G$-action on $M$ is smooth with respect to the smooth structure on $M$. Since two simple polytopes are combinatorially equivalent if and only if they are diffeomorphic [4 Corollary 5.3], it follows that the $G$-action on $M$ is smooth with respect to the smooth structure for which $M/T$ is diffeomorphic to $P$.

4. Classification

Let $G$ be a compact Lie-group with maximal torus $T$. A quasitoric manifold with $G$-action is a smooth $G$-manifold $M$ such that $M$ together with the action of the maximal torus of $G/H$ is a quasitoric manifold. Here $H$ is a finite subgroup of $G$ which acts trivially on $M$. In [5] we classified quasitoric manifolds with $G$-action. As a first step towards this classification we showed that $G$ has a covering group of the form $\prod_{i=1}^k SU(l_i + 1) \times T^{n_0}$.

The classification was given in terms of admissible triples $(\psi, N, (A_1, \ldots, A_k))$, where

- $\psi$ is an homomorphism $\prod_{i=1}^k SU(U_i) \times U(1) \to T^{n_0}$.
- $N$ is a 2n-dimensional quasitoric manifold.
- The $A_i$ are characteristic submanifolds of $N$ or empty. If $A_i$ is non-empty then $im(\psi|_{SU(U_i)\times U(1)})$ acts trivially on $A_i$ and $ker(\psi|_{SU(U_i)\times U(1)}) = SU(U_i)$.

Two such triples $(\psi, N, (A_1, \ldots, A_k))$ and $(\psi', N', (A'_1, \ldots, A'_k))$ are called equivalent or diffeomorphic if

- $\psi|_{SU(U_i)\times U(1)} = \psi'|_{SU(U_i)\times U(1)}$ if $l_i > 1$.
- $\psi|_{SU(U_i)\times U(1)} = \psi'|_{SU(U_i)\times U(1)}$ if $l_i = 1$.
- There is an $T^{n_0}$-equivariant diffeomorphism $f : N \to N'$ such that $f(A_i) = A_i'$ for all $i$.

The main theorem of [5] may be formulated in the following way:

**Theorem 4.1** ([5] Theorem 8.6). Let $G = \prod_{i=1}^k SU(l_i + 1) \times T^{n_0}$. Then the $G$-equivariant diffeomorphism classes of quasitoric manifolds with $G$-action are in one-to-one correspondence with the diffeomorphism classes of admissible triples.
We call two admissible triples \((\psi, N, (A_1, \ldots, A_k))\) and \((\psi', N', (A_1', \ldots, A_k'))\) homeomorphic if

- \(\psi_i|_{SU(l_i) \times U(1)} = \psi'_i|_{SU(l_i) \times U(1)}\) if \(l_i > 1\).
- \(\psi_i|_{SU(l_i) \times U(1)} = \psi'^{l_i-1}_i|_{SU(l_i) \times U(1)}\) if \(l_i = 1\).
- There is a \(T^{l_i}\)-equivariant homeomorphism \(f : N \to N'\) such that \(f(A_i) = A'_i\) for all \(i\).

With this notation we have the following classification of quasitoric manifolds with \(G\)-action up to \(G\)-equivariant homeomorphism.

**Theorem 4.2.** Let \(G = \prod_{i=1}^k SU(l_i + 1) \times T^{l_i}\). Then the \(G\)-equivariant homeomorphism classes of quasitoric manifolds with \(G\)-action are in one-to-one correspondence with the homeomorphism classes of admissible triples.

**Proof.** At first note that the homeomorphism type of the admissible triple corresponding to a quasitoric manifold with \(G\)-action depends only on the \(G\)-equivariant homeomorphism type of \(M\) because \(N\) and the \(A_i\) may be identified with intersections of characteristic submanifolds of \(M\) and \(\psi\) depends only on the isotropy groups of points in these intersections (see Sections 5 and 8 of [5] for details).

Because by the proof of Corollary 8.8 of [5] \(M/G = N/T^{l_0}\), it follows from Lemma 4.3 below that the homeomorphism type of the admissible triple determines the \(G\)-equivariant homeomorphism type of \(M\).

**Lemma 4.3.** Let \(M\) be a quasitoric manifold with \(G\)-action, \(G = \prod_{i=1}^k SU(l_i + 1) \times T^{l_i}\). Then \(M\) is equivariantly homeomorphic to \(M/G \times G/\sim\), where \((x, g) \sim (x', g')\) if and only if \(x = x'\) and \(gg'^{-1} \in H_x\). Here the groups \(H_x\) depend only on \(x\) and the admissible triple corresponding to \(M\).

**Proof.** We prove this lemma by induction on the number of simple factors of \(G\). If \(G\) is a torus then this lemma is due to Davis and Januszkiewicz [1] Proposition 1.8.

Therefore we may assume that there is at least one simple factor. There are two cases:

1. \(M^{SU(l_i+1)} = \emptyset\)
2. \(M^{SU(l_i+1)} \neq \emptyset\).

In the first case we have by Corollary 5.6 of [5]

\[ M = SU(l_1 + 1) \times_{SU(U(1) \times U(1))} M', \]

where \(M'\) is a quasitoric manifold with \(G'\)-action, \(G' = \prod_{i=2}^k SU(l_i + 1) \times T^{l_i}\). The action of \(S(U(l_1) \times U(1))\) on \(N\) is induced by the homomorphism

\[ \phi = (|_{SU(U(1) \times U(1))}^{-1} : S(U(l_1) \times U(1)) \to T^{l_1}, \]

where \(\psi\) is the homomorphism from the admissible triple of \(M\). Moreover, by the proof of Corollary 8.8 of [5], we have \(M/G = M'/G'\). Therefore by the induction hypothesis we have

\[ M = SU(l_1 + 1) \times_{SU(U(1) \times U(1))} (M/G \times G'/\sim'). \]

For a subgroup \(H'_x\) of \(G'\) we have

\[ SU(l_1 + 1) \times_{SU(U(1) \times U(1))} (G'/H'_x) = (SU(l_1 + 1) \times G')/|_{H'_x}. \]

Therefore the statement follows in this case with \(H_x = (|_{G'_{H_x}})^{-1}(H'_x)\).

If \(M^{SU(l_i+1)}\) is non-empty. Then \(M\) is the blow-down of some quasitoric manifold \(\tilde{M}\) with \(G\)-action but without \(SU(l_i + 1)\) fixed points. Let \(F : \tilde{M} \to M\) be the projection. Then we have \(M = \tilde{M}/\sim',\) where \(y \sim' y'\) if and only if there is a
$g \in SU(l_i + 1)$ such that $gy = y'$ in the case $y, y' \in F^{-1}(M^{SU(l_i + 1)})$ or $y = y'$ otherwise.

Since under the identification of $M/G$ with $N/T^o$ given in the proof of Corollary 8.8 of [5], $M^{SU(l_i + 1)}/G$ is identified with $A_1/T^o$. For $x \in M^{SU(l_i + 1)}/G$, we have $\im \phi = \im \psi \subset H_x'$.

Therefore we have $S(U(l_i) \times U(1)) \times H_x' = (\phi \Id_G)^{-1}(H'_x)$ if $x \in M^{SU(l_i + 1)}/G$.

Hence the statement follows with

$$H_x = \begin{cases} SU(l_i + 1) \times H_x' & \text{if } x \in M^{SU(l_i + 1)}/G \\ (\phi \Id_G)^{-1}(H'_x) & \text{if } x \not\in M^{SU(l_i + 1)}/G. \end{cases}$$

\[\square\]

**Theorem 4.4.** Let $G = \prod_{i=1}^{k} SU(l_i + 1) \times T^o$ and $M, M'$ be two quasitoric manifolds with $G$-action. Let $T$ be a maximal torus of $G$. Then $M$ and $M'$ are $G$-equivariantly homeomorphic (diffeomorphic), if and only if they are $T$-equivariantly homeomorphic (diffeomorphic).

**Proof.** Without loss of generality we may assume that $T$ is the standard maximal torus of $G$. Let $(\psi, N, (A_1, \ldots, A_k))$ be the admissible triple corresponding to $M$. We show that $(\psi, N, (A_1, \ldots, A_k))$ is determined up to homeomorphism (diffeomorphism) by the homeomorphism (diffeomorphism) type of the $T$-action on $M$.

At first, by Lemmas 2.7 and 2.10 of [5], the characteristic submanifolds which are permuted by the Weyl-group $W(SU(l_i + 1))$ are exactly those characteristic manifolds $M_i$ for which $\pi_i \circ \lambda(M_j)$ is non-trivial. Here $\pi_i : G \to SU(l_i + 1)$ is the projection. Denote by $\mathfrak{F}_i$ the set of the characteristic submanifolds which are permuted by $W(SU(l_i + 1))$.

If $l_i > 1$, there is exactly one $M_{j_i} \in \mathfrak{F}_i$ such that $\lambda(M_{j_i})$ is fixed by $W(S(U(l_i) \times U(1)))$. Because for each $M_{j_0} \in \mathfrak{F}_i$, we have $\dim \pi_i \circ \lambda(M_j) = 1$, $M_{j_0}$ is the only characteristic submanifold such that $\pi_i \circ \lambda(M_{j_0})$ is contained in the center of $S(U(l_i) \times U(1))$.

If $l_i = 1$, then $\mathfrak{F}_i$ has exactly two elements and any choose of a $M_{j_i} \in \mathfrak{F}_i$ leads to the same equivalence class of admissible triples (see [5], Section 5) for details).

Then we have

$$N = \bigcap_{i=1}^{k} M_{j_i} \in \mathfrak{F}_i \setminus \{M_{j_i}\} \quad \text{and} \quad A_i = N \cap M_{j_i}.$$

By its construction in the proof of Lemma 5.3 of [5], the homomorphism $\psi$ depends only on the $\prod_{i=1}^{k} S(U(l_i) \times U(1)) \times T^o$-representation $T_x M$ with $x \in N/T^o$.

Since $T$ is a maximal torus of $\prod_{i=1}^{k} S(U(l_i) \times U(1)) \times T^o$, this representation depends only on the $T$-equivariant homeomorphism type of $M$.

Now the statement follows from Theorem 4.4. \[\square\]

5. **Uniqueness**

In this section we prove that the group constructed in section 5.3 is a maximal compact connected Lie-subgroup of the homeomorphism group of $M$ which contains the torus and that it is unique up to conjugation.

**Lemma 5.1.** Let $M$ be a quasitoric manifold with $G$-action, $G = \prod_{i=1}^{k} SU(l_i + 1) \times T^o$ and $T$ a maximal torus of $G$, such that $T^o$ acts effectively on $M$. Denote by $\mathfrak{F}_i$, $i = 1, \ldots, k$, the set of characteristic submanifolds of $M$ which are permuted by $W(SU(l_i + 1))$. Moreover, let $\mathfrak{F}_0 = \mathfrak{F} - \bigcup_{i=1}^{k} \mathfrak{F}_i$. Then we have:
(1) The subgroup of $G$ which acts trivially on $M$ is given by
\[ H = \{(g, \psi(g)) \in G; \ g \in \bigcap_{i=1}^{k} SU(l_i + 1)\}. \]

(2) Let $M_1$ be a characteristic submanifold of $M$ which belongs to $\mathfrak{S}$, and $x \in M_1$ a generic point. Then $T_x$ is connected and
\[ H \cap T_x = \{(g, \psi(g)) \in G; \ g \in Z(SU(l_i + 1))\} \]
if $i > 0$. If $i = 0$ then $T_x$ is connected and $H \cap T_x = 1$.

Proof. At first we prove (1). We prove this statement by induction on $k$. If $k = 0$, then there is nothing to prove. Therefore assume that $k > 0$ and that the statement is proved for all quasitoric manifolds with $G' = \prod_{i=2}^{k} SU(l_i + 1) \times T^{l_0}$-action. With the notation from the proof of Lemma 4.3 the subgroup of $G$ which acts trivially on $M$ is given by
\[
H = \bigcap_{x \in M/G,g \in G} gH_xg^{-1} = \bigcap_{x \in M/G,g \in G} (g(\phi \Id_{G'})^{-1}(H_x')g^{-1}) = \bigcap_{g \in SU(l+1)} \{\{(h, \psi(h)); h \in SU(l_1) \times U(1)\}, \bigcap_{x \in M/G,g' \in G'} g'H'_xg^{-1}\} = \{(g, \psi(g)) \in G; \ g \in Z(\prod_{i=1}^{k} SU(l_i + 1))\}.
\]
Here $H'$ denote the subgroup of $G'$ which acts trivially on $N$.

Now we prove the second statement. At first assume $i > 0$. After blowing up $M$ along the fixed points of $SU(l_i + 1)$, we may assume that
\[ M = SU(l_i + 1) \times_{SU(l_i) \times U(1)} N. \]
Then there is an $SU(l_i + 1)$-equivariant projection $p: M \to \mathbb{C}P^{l_i}$. The characteristic submanifold $M_1$ of $M$ is given by a preimage of a characteristic submanifold $\mathbb{C}P^{l_i}$ of $\mathbb{C}P^{l_i}$. Now we have
\[ T_x = \{(t, \psi(t)) \in T; \ t \in T_{p(x)}^{l_i}\}, \]
where $T^{l_i}$ denotes $T \cap SU(l_i + 1)$. Since $T_{p(x)}^{l_i}$ contains the center of $SU(l_i + 1)$ the statement follows in this case.

If $i = 0$, then it follows from Lemma 2.10 of [5] that $T_x$ is contained in $T^{l_0}$. Therefore the statement follows in this case. \hfill \Box

Lemma 5.2. Let $M$ be a quasitoric manifold and $T$ the torus which acts on $M$ by $\phi: T \to \Homeo(M)$. Let $G_j$, $j = 1,2$, be compact connected Lie-groups and $t_j : T \to G$, $j = 1,2$, embeddings of $T$ as maximal tori of $G_j$. Assume that there are effective actions $\phi_j : G_j \to \Homeo(M)$, $j = 1,2$, such that $\phi = \phi_1 \circ t_1$ for $j = 1,2$. Moreover, assume that the natural actions of $W(G_j)$, $j = 1,2$, on $\mathfrak{S}$ induce identifications of $W(G_1)$ and $W(G_2)$ with a given subgroup $H$ of $S(\mathfrak{S})$. Then $\phi_1(G_1)$ and $\phi_2(G_2)$ are conjugated in $\Homeo(M)$.

Proof. Since the Weyl-groups of $G_1$ and $G_2$ are isomorphic. There is a group of the form $\tilde{G} = \prod_{i=1}^{k} SU(l_i + 1) \times T^{l_0}$ and coverings $\varphi_j : \tilde{G} \to G_j$, $j = 1,2$. \hfill \Box
Because all maximal tori in $\tilde{G}$ are conjugated, we may assume that there is a maximal torus $\tilde{T}$ of $\tilde{G}$ such that $\tilde{T} = \varphi_j^{-1}(T_j)$, $j = 1, 2$. Let $\psi$ be the automorphism of $L\tilde{T}$ given by $(L\varphi_1)^{-1} \circ L\tilde{t}_1 \circ (L\varphi_2)^{-1} \circ L\varphi_2$.

Choose an omniorientation for $M$ such that, for all characteristic submanifolds $M_i$, $M_j$ of $M$, we have:

$$PD(M_i) = \pm PD(M_j) \Rightarrow PD(M_i) = PD(M_j).$$

This omniorientation is preserved by the actions of $G_1$ and $G_2$. Denote by $\lambda$ the characteristic function for the $T$-action on $M$. Moreover, denote by $\tilde{\lambda}_j(M_i)$, $j = 1, 2$, $M_i \in \mathfrak{g}$, a primitive vector in $T\tilde{T}$ which generates the isotropy group of a generic point in $M$ with respect to the $\tilde{T}$-action $\phi_j \circ \varphi_j$. We choose this primitive vector in such a way that it is compatible with the omniorientation chosen above.

Then by Lemma 5.10 we have, for all $M_k \in \mathfrak{g}$, $i > 0$, and $j = 1, 2$,

$$L\varphi_j^{-1} \circ L\varphi_j(\tilde{\lambda}_j(M_k)) = (l_i + 1)\tilde{\lambda}(M_k).$$

For $M_k \in \mathfrak{g}_0$, we have

$$L\varphi_j^{-1} \circ L\varphi_j(\tilde{\lambda}_j(M_k)) = \tilde{\lambda}(M_k).$$

This implies that $\psi(\tilde{\lambda}_2(M_k)) = \tilde{\lambda}_1(M_k)$. Therefore we have $w \in W(\tilde{G})$ by the Lemma 2.10 of [5]

$$\psi(\tilde{\lambda}_2(wM_k)) = \tilde{\lambda}_1(wM_k) = w\tilde{\lambda}_1(M_k)w^{-1} = w\psi(\tilde{\lambda}_2(M_k))w^{-1}$$

(5.1) $\tilde{\lambda}_2(wM_k) = w\lambda_2(M_k)w^{-1}$.

It follows that $\psi$ is an automorphism of the $W(\tilde{G})$-representation $L\tilde{T}$. Because each irreducible non-trivial summand of $L\tilde{T}$ appears only once in a decomposition of $L\tilde{T}$ in irreducible representations, it follows from Schur’s Lemma that the restriction of $\psi$ to the Lie-algebra of the maximal torus $\tilde{T}_i$ of a simple factor $SU(l_i + 1)$ of $\tilde{G}$ is multiplication with a constant $a_i \in \mathbb{R}$.

Therefore we have

$$\iota_1^{-1} \circ \varphi_1(\tilde{T}_i) = \iota_2^{-1} \circ \varphi_2(\tilde{T}_i).$$

Denote this subtorus of $T$ by $T_i$. By Lemma 5.1, we have that

$$IT_i/L\iota_1^{-1} \circ L\varphi_1(\tilde{T}_i) \cong \ker \iota_1^{-1} \circ \varphi_1 \cap \tilde{T}_i \cong \ker \iota_1^{-1} \circ \varphi_2 \cap \tilde{T}_i \cong IT_i/L\iota_2^{-1} \circ L\varphi_2(\tilde{T}_i).$$

Note that $I_{ij} = (\tilde{\lambda}_j(M_k); M_k \in \mathfrak{g}) \cap IT_i$ is a lattice of maximal rank in $IT_i$. Then we have

$$|IT_i/I_{11}| = \frac{|IT_i/L\iota_1^{-1} \circ L\varphi_1(\tilde{T}_i)|}{|IT_i/L\iota_2^{-1} \circ L\varphi_2(\tilde{T}_i)|} = \frac{|IT_i/L\iota_2^{-1} \circ L\varphi_2(\tilde{T}_i)|}{|IT_i/I_{12}|} = \frac{1}{|a_i|} |IT_i/I_{11}|,$$

because $\psi(I_{12}) = I_{11}$. Therefore we must have $a_i = 1$. Therefore there is an automorphism $\Psi$ of $\tilde{G}$ with $L\Psi = \psi$. Now the statement follows from Theorem 4.4 applied to the $\tilde{G}$-actions $\phi_1 \circ \varphi_1 \circ \Psi$ and $\phi_2 \circ \varphi_2$.

$\square$

**Remark 5.3.** If, in the situation of Lemma 5.2, both $G_j$-actions are smooth with respect to the same smooth structure, then it follows from Theorem 4.4 that $\phi_1(G_1)$ and $\phi_2(G_2)$ are conjugate in $\text{Diff}(M)$. 


Theorem 5.4. Let $M$ be a quasitoric manifold and $G \subset \text{Homeo}(M)$, the group constructed in section 3. If $G'$ is an other compact connected Lie-group which acts by an extension of the torus action on $M$. Then $G'$ is conjugated in $\text{Homeo}(M)$ to a subgroup of $G$. If the $G$ and $G'$-actions are smooth for the same smooth structure on $M$, then $G'$ is conjugated in $\text{Diff}(M)$ to a subgroup of $G$.

Proof. Let $\mathcal{F} = \mathcal{F}_1 \cdots \mathcal{F}_k$ be a partition in $W(G')$-orbits. Then we have $W(G') = \prod_{i=1}^k S(\mathcal{F}_k)$. Moreover since the $G'$-action on $H^*(M)$ is trivial it follows that the sets $\mathcal{F}_k$, $\alpha \in H^2(M)$, are $W(G')$-invariant.

This gives as an homomorphism $W(G') \to W(G) = \prod_{\alpha \in H^2(M)} S(\mathcal{F}_\alpha)$. There is a subgroup of maximal rank of $G$ whose Weyl-group is given by the image of this homomorphism. Therefore the statement follows from Lemma 5.2. □

Now we have proven all parts of Theorem 5.1 besides the statement about the symplectic toric manifolds. To prove this part we first recall the construction of a maximal compact Lie-subgroup of the symplectomorphism group of a symplectic toric manifold due to Masuda [2]. An alternative construction of this group was given by McDuff and Tolman [3].

He showed that there is a root system $R(M)$ such that the root system of every compact connected Lie-subgroup of the symplectomorphism group which contains the torus is a subroot system of $R(M)$. Moreover, he constructed a compact Lie-subgroup $G'$ of the symplectomorphism group which contains the torus and has a root system isomorphic to $R(M)$.

The proof of the first part of Masuda’s results is also valid for any compact connected Lie-subgroup of the homeomorphism group of $M$ which contains the torus and preserves the omniorientation induced by the symplectic form on $M$. Therefore $G$ and $G'$ are conjugated, if the $G$-action preserves this omniorientation.

Hence, it is sufficient to prove that if $M_1$ and $M_2$ are characteristic submanifolds of $M$ with $PD(M_1) = \pm PD(M_2)$, then we have $PD(M_1) = PD(M_2)$. We consider two cases $M_1 \cap M_2 = \emptyset$ and $M_1 \cap M_2 \neq \emptyset$. We should note here that if $M$ is a symplectic toric manifold then $\lambda(F_i)$ is the outward normal vector of the facet $F_i$ of $P$.

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. In the first case we may assume that $\{0\} = F_1 \cap \bigcap_{i=1}^{n+1} F_i \subset \mathbb{R}^n$ and $\lambda(F_1) = e_1, \lambda(F_i) = e_{i-1}$ for $i = 3, \ldots, n+1$. It follows from $PD(M_1) = \pm PD(M_2)$ that $\lambda(F_2) = \pm e_1 + \sum_{i=2}^n \mu_{i2} e_i$ and $\lambda(F_j) = \sum_{i=2}^n \mu_{ij} e_i$ for $j > n+1$ with $\mu_{ij} \in \mathbb{Z}$. Therefore $P \cap \{e_1\}$ is an interval with boundary $(e_1) \cup (F_1 \cup F_2)$. Hence we must have $\lambda(F_2) = -e_1 + \sum_{i=2}^n \mu_{i2} e_i$. This implies $PD(M_1) = PD(M_2)$.

Now consider the case $M_1 \cap M_2 \neq \emptyset$.

Without loss of generality we may assume that $\{0\} = \bigcap_{i=1}^n F_i \subset \mathbb{R}^n$ and $\lambda(F_i) = e_i$ for $i = 1, \ldots, n$.

Assume that $PD(M_1) = -PD(M_2)$. Then for all $F_j \in \mathcal{F}$, $j > n$, there are $\mu_{0j}, \mu_{3j}, \ldots, \mu_{nj} \in \mathbb{Z}$ such that

$$\lambda(F_j) = \mu_{0j} e_1 - e_2 + \sum_{i=3}^n \mu_{ij} e_i.$$  

Because the $\lambda(F_j)$ are the outward normal vectors of the facets of $P$ it follows that $P \cap \{e_1, e_2\}$ is non-compact. But this is impossible because $P$ is a convex polytope. Therefore we must have $PD(M_1) = PD(M_2)$.

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Max-Planck-Institute for Mathematics, Vivatsgasse 7, D-53111 Bonn, Germany

E-mail address: wiemeler@mpim-bonn.mpg.de