Homoclinic orbit bifurcations in a pattern formation system

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Abstract. An extensive numerical exploration of delicate global dynamics of the pulse self-replication and the stability of singular homoclinic stationary solutions and its bifurcations in the one-dimensional Gray-Scott model are carried out. A careful analysis of the scenario of the global bifurcation diagram suggests that the dynamics of self-replicating system is related to a hierarchy structure of folding bifurcation branches in parameter regions. The numerics suggests the Bogdanov-Takens points together with a presence of critical points emanating from the particular codimension-two homoclinic orbit play a central role for global bifurcation of periodic orbits and the homoclinic solutions and the complex chaotic dynamics. Numerical simulation also reveals the existence of the modulating two-pulse or multi-pulse, which accompanies the procedure of pulse self-replicating in reaction-diffusion systems

1. Introduction

Global bifurcation or Homoclinic bifurcation of dynamical systems is important in application for a number of reasons [1-28]. If the dynamical system arises as the travelling wave equation for a partial differential equation or system, then homoclinic solutions of it describe solitary waves, which are of importance in many fields. One example of this type is the reaction-diffusion systems which are multi-component models involving diffusion and non-linear interaction among the components. Such systems are commonplace in many areas of physics, chemistry and biology. They are used as a model for such diverse phenomena as cell differentiation, reaction of chemical, propagations of flame fronts, laser interference and sea shell patterns. The Gray-Scott model is referred to as an activator-substrate system that exhibits a variety of new patterns including spots that self replicate and develop into a variety of asymptotic states in two dimensions as well as pulses that self replicate in one dimension [2-7]. Here, the chemical $u$ can be interpreted as substrate depleted by $v$. The long-range inhibition here is due to depletion of $u$. The activation is due to the presence of the source term in the equation for $u$. There are some cornerstones in the development of the Gray-Scott model. The numerical and laboratory experiments by Pearson and Swinney et al. have spurred the development of new analytical techniques to understand these patterns [8].

Some analytical results on spike type solution were derived by Doelman and his co-workers in a series of papers starting with [2]. Their theory explains analytically certain stability properties of spike-type solutions in one dimensional Gray-Scott model. Several interesting analytical works have also appeared: for instance, construction of single-spot solution to the Gray-Scott model and its stability has been done by W. N. Reynolds et al. [9] with the aid of formal matched asymptotic analysis, which is closely related to the replicating phenomenon; a rigorous analysis concerning the existence and stability of steady single pulse as well as nonexistence of traveling pulses has been done

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by A. Doelman [10-14]. These works are very suggestive, nevertheless, very little is known about the mechanism that derives the replication dynamics itself. The aim of this paper is to present a key mechanism for the self-replicating dynamics for the Gray-Scott model.

In this paper, we confine ourselves to the one-dimensional case and mainly investigate the apparent loss of stability of one-pulse and multi-pulse solutions and the transition to shift dynamics in the excitable regime. The rest of the paper is outlined as follows. Section 2 gives the derived scaled Gray-Scott model and briefly analyzes the stability of the stationary solutions. Section 3 presents the numerical results. The final section (Section 4) provides insight into the central role of the global bifurcation and also provides a qualitative comparison between the numerical simulation results and those obtained from analytical results and it concludes with some conclusion remarks.

2. Model equations

In this paper, we carry out an extensively numerical global bifurcations study on the one-dimensional Gray-Scott model [7] (see also the reference in [8-15] for the derivations and early studies of this model):

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u v^2 + A (1 - u) \tag{1}
\]

\[
\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial t^2} + u v^2 - B v
\]

For all positive values of the parameters \(A\), \(B\) and \(D\), the background homogeneous state \(u = 1, v = 0\) is stable. The homoclinic orbits or pulse solutions considered in this paper are asymptotic to this basic state as \(x \to \infty\). The pulses correspond to excursions of the \(v\)-components away from the \(v = 0\) background state on narrow intervals; the \(u\)-component varies significantly outside these regions and is in general not close to \(u = 1\) in between the \(v\)-pulses. For stationary solutions of eq.(1) we further reduce the number of parameters by introducing new independent and dependent variables and constants. This yields the coupled ODE's

\[
\frac{\partial^2 u}{\partial x^2} - u v^2 + \lambda (1 - u) = 0 \tag{2}
\]

\[
\gamma \frac{\partial^2 v}{\partial x^2} + u v^2 - v = 0
\]

Writing eq.(2) as a system of four first-order differential equations

\[
\begin{align*}
    u' &= p \\
    p' &= u v^2 - \lambda (1 - u) \\
    v' &= q \\
    q' &= \frac{1}{\gamma} (v - u v^2)
\end{align*} \tag{3}
\]

Where prime denotes \(d/dx\). Constant stationary solutions or fixed points satisfy

\[
\begin{align*}
    p &= 0 \\
    u v^2 - \lambda (1 - u) &= 0 \\
    q &= 0 \\
    \frac{1}{\gamma} (v - u v^2) &= 0 \tag{4}
\end{align*}
\]

We obtain three fixed points \(S_i\) \((i = 1, 2, 3)\), i.e.
Numerical experiments confirm that the fixed point $S_3$ is of saddle type, and depending on the parameter region, $S_1$ and $S_2$ might become saddle type, saddle-center type, focus type, saddle-focus type, double-Hopf type, Bogdanov-Takens type or Bogdanov-Takens-Hopf type points. In next section, we focus on critical point $S_1$ to study the global bifurcation dynamics at the onset of pulse self-replication.

3. Numerical simulation results

The mainly numerical simulation results are presented in Figures 1-3. Figure 1 depicts the global bifurcation diagram for parameter values, which were computed using the standard local bifurcation features of AUTO and HOMCONT [16]. The vertical axis denotes the $L^2$ norm of solutions and the horizontal one is the bifurcation parameter $\gamma$. The norm used in Figure 1 is the $L^2$ norm of the solution $(u(t), p(t), v(t), q(t))$. We do not mark the BT-point exactly but can infer their existence from the bifurcation diagrams in their neighborhood. Curves labeled $E_1$, $E_2$, and $E_7$ are loci of one-pulse solution. Curves labeled $E_6$, $E_3$, and $E_4$ are loci of two-pulse solution. Curve labeled $E_2$, Curve $C_2C_4$ and curve $AC_1$ are loci of spatial period-two solutions.

FIG. 1. Hierarchy structure of the global bifurcation
Figure 2 is obtained by following two-homoclinic orbit in parameter space, while, in this situation, the continuation is carried out in codimension-two starting off at fold bifurcation point. Figure 3 is the phase diagram of four-homoclinic orbit corresponding to Figure 2. Also there are two spatially periodic orbits included in Figure 3 for visual comparison between homoclinic orbit and periodic orbit. It is observed that the so-called resonant homoclinic orbits bifurcation is accompanied by periodic orbits doubling. The similarity between periodic and homoclinic bifurcation will in fact be more than a superficial analogy. Viewing homoclinic orbits as limits of nearby periodic orbits will serve as in many cases the basic technical tool for global continuation. The right three small-amplitude excursions in Figure 3 are located so closely in phase space that they are hardly to be distinguished visually from each other. Magnification of the top part of these small-amplitude excursions in Figure 3 shows clearly three pulses corresponding these excursions. Figure 3 also shows obviously how the double homoclinic orbits emanated from doubling periodic orbits.

4. Conclusion remarks

We discussed in this work about the basic process of self-replication, typically from a one-pulse. The mechanism proposed here seems rather universal and has a potential to become a general framework to explain self-replication pattern. We confirmed that the Gray-Scott model has a rich variety of global bifurcation phenomena or patterns. This type of patterns only exists in a restricted “horn” area in a parameter space, and the whole global bifurcation diagram is an intricate hierarchical structure. Such a skeleton of coherent structure demonstrate its organizing center from which emanates an infinite number of fold and flip (period-doubling) bifurcations involving periodic orbits corresponding to the
existence of a double homoclinic orbit. Numerics also confirmed a finite number of modulated \(n\)-homoclinic orbits existed in the process of global bifurcation. Full numerical simulations indicate that the Hopf bifurcation corresponding to oscillatory profile instability is sub-critical, as such, leads to the eventual collapse of the spike. In the process of present numerical simulation, It is confirmed that a kind of replicating from \(2n\) pulses to \(2n+1\) pulses is extremely unstable which does not persist longer as parameter changes its values. This numerical result is in accordance with the analytical conclusion from Arjen Doelman [10].

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