On Polynilpotent Covering Groups of a Polynilpotent Group

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Abstract

Let \( N_{c_1,\ldots,c_t} \) be the variety of polynilpotent groups of class row \((c_1,\ldots, c_t)\). In this paper, first, we show that a polynilpotent group \( G \) of class row \((c_1,\ldots, c_t)\) has no any \( N_{c_1,\ldots,c_t,c_{t+1}} \)-covering group if its Baer-invariant with respect to the variety \( N_{c_1,\ldots,c_t,c_{t+1}} \) is nontrivial. As an immediate consequence, we can conclude that a solvable group \( G \) of length \( c \) with nontrivial solvable multiplier, \( S_n M(G) \), has no \( S_n \)-covering group for all \( n > c \), where \( S_n \) is the variety of solvable groups of length at most \( n \). Second, we prove that if \( G \) is a polynilpotent group of class row \((c_1,\ldots, c_t, c_{t+1})\) such that \( N_{c'_1,\ldots,c'_t,c'_{t+1}} M(G) \neq 1 \), where \( c'_i \geq c_i \) for all \( 1 \leq i \leq t \) and \( c'_{t+1} > c_{t+1} \), then \( G \) has no any \( N_{c'_1,\ldots,c'_t,c'_{t+1}} \)-covering group. This is a vast generalization of the first author’s result on nilpotent covering groups (Indian J. Pure Appl. Math. 29(7) 711-713, 1998).

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1. Introduction and Motivation

Let \( G \cong F/R \) be a free presentation for \( G \) and \( \mathcal{V} \) be a variety of groups. Then, after R. Baer [1], the Baer-invariant of \( G \) with respect to \( \mathcal{V} \) is defined to be \( \mathcal{V}M(G) = R \cap V(F)/[RV^*F] \), where \( V(F) \) is the verbal subgroup of \( F \) with respect to \( \mathcal{V} \) and

\[
[RV^*F] = \langle v(f_1, \ldots, f_{i-1}, f_ir, f_{i+1}, \ldots, f_n)v(f_1, \ldots, f_n)^{-1} \mid r \in R, f_i \in F, 1 \leq i \leq n, v \in V, n \in \mathbb{N} \rangle.
\]

In special case, if \( \mathcal{V} \) is the variety of abelian groups, then the Baer-invariant of \( G \) will be the well-known notion the Schur-multiplier of \( G \), denoted by \( M(G) = R \cap F'/[R, F] \) (See [5,6] for further details).

It is easy to see that if \( \mathcal{V} = \mathcal{N}_c \), the variety of nilpotent groups of class at most \( c \geq 1 \), then

\[
\mathcal{N}_cM(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, cF]},
\]

where \( \gamma_{c+1}(F) \) is the \((c+1)\)-st term of the lower central series of \( F \) and \( [R,1]F = [R,F], [R,c]F = [[R,c-1]F,F], \) inductively. We shall also call \( \mathcal{N}_cM(G) \) the \( c \)-nilpotent multiplier of \( G \).

In a more general case, if \( \mathcal{V} = \mathcal{N}_{c_1, \ldots, c_t} \), the variety of polynilpotent groups of class row \((c_1, \ldots, c_t)\), then

\[
\mathcal{N}_{c_1, \ldots, c_t}M(G) = \frac{R \cap \gamma_{c_{t+1}} \circ \cdots \circ \gamma_{c_1+1}(F)}{[R_{c_1} F_{c_2} \gamma_{c_1+1}(F), \ldots, c_t \gamma_{c_t-1+1} \circ \cdots \circ \gamma_{c_1+1}(F)]},
\]

where \( \gamma_{c_{t+1}} \circ \cdots \circ \gamma_{c_1+1}(F) = \gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\cdots (\gamma_{c_1+1}(F)) \cdots)) \) are the terms of iterated lower central series of \( F \). See [4, corollary 6.14] for the following equality

\[
[RN^*_{c_1, \ldots, c_t}F] = [R_{c_1} F_{c_2} \gamma_{c_1+1}(F), \ldots, c_t \gamma_{c_t-1+1} \circ \cdots \circ \gamma_{c_1+1}(F)].
\]

We shall also call \( \mathcal{N}_{c_1, \ldots, c_t}M(G) \), the \((c_1, \ldots, c_t)\)-polynilpotent multiplier of \( G \).
Let \( \mathcal{V} \) be a variety of groups and \( G \) be an arbitrary group, then a \( \mathcal{V} \)-covering group of \( G \) (a generalized covering group of \( G \) with respect to the variety \( \mathcal{V} \)) is a group \( G^* \) with a normal subgroup \( A \) such that \( G^*/A \cong G \), \( A \subseteq V(G^*) \cap V^*(G^*) \), and \( A \cong \mathcal{V}M(G) \), where \( V^*(G^*) \) is the marginal subgroup of \( G^* \) with respect to \( \mathcal{V} \) (see [6]).

Note that if \( \mathcal{V} \) is the variety of abelian groups, then the \( \mathcal{V} \)-covering group of \( G \) will be ordinary covering group (sometimes it is called representing group) of \( G \). Also if \( \mathcal{V} = N_{c_1, \ldots, c_t} \), then an \( N_{c_1, \ldots, c_t} \)-covering group of \( G \) is a group \( G^* \) with a normal subgroup \( A \) such that

\[
G \cong G^*/A, \\
A \cong N_{c_1, \ldots, c_t} M(G^*) \text{ and} \\
A \subseteq N_{c_1, \ldots, c_t}^*(G^*) \cap \gamma_{c_{t+1}}(\cdots(\gamma_{c_1}(G^*))\cdots).
\]

We shall also call \( G^* \) a \((c_1, \ldots, c_t)\)-polynilpotent covering group of \( G \).

It is a well-known fact that every group has at least a covering group (see [5,13]). Also, the first author proved that every group has a \( \mathcal{V} \)-covering group if \( \mathcal{V} \) is the variety of all groups, \( \mathcal{G} \), or the variety of all abelian groups, \( \mathcal{A} \), or the variety of all abelian groups of exponent \( m \), \( \mathcal{A}_m \), where \( m \) is square free (see [7,9]).

Moreover, C. R. Leedham-Green and S. Mckay [6] proved, by a homological method, that a sufficient condition for existence of a \( \mathcal{V} \)-covering group of \( G \) is that \( G/V(G) \) should be a \( \mathcal{V} \)-splitting group.

Some people have tried to construct a covering group for some well-known structures of groups. For example, the generalized quaternion group \( Q_{4n} = \langle a, b | a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle \) is a covering group of the dihedral group \( D_{2n} = \langle a, b | a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle \) (see [5]).

Also J. Wiegold [12] presented a covering group for a direct product of two finite groups. W. Haebich [2, 3] generalized the Wiegold’s result and gave a covering group for a regular product of a family of groups and also
for a verbal wreath product of two groups. Moreover, the first author [10] recently proved the existence and presented a structure of an $N_c$-covering group for a nilpotent product of a family of cyclic groups.

It is interesting to mention that there are some groups which have no any $V$-covering group, for some variety $V$. The first author [8] gave an example of the group $G \simeq \mathbb{Z}_r \oplus \mathbb{Z}_s$, where $(r, s) \neq 1$, which has no $N_c$-covering group for all $c \geq 2$. Moreover, the first author [7, 9] proved that a nilpotent group $G$ of class $n$ with nontrivial $c$-nilpotent multiplier $N_c M(G)$, has no $N_c$-covering group for all $c > n$.

Now, in this paper, we concentrate on nonexistence of polynilpotent covering groups of a polynilpotent group. More precisely, we show that if $G$ is a polynilpotent group of class row $(c_1, \ldots, c_t)$ such that $N_{c_1 \ldots, c_t, c_{t+1}} M(G) \neq 1$, then $G$ has no $(c_1, \ldots, c_t, c_{t+1})$-polynilpotent covering group of $G$. Also, if $N_{c'_1 \ldots, c'_t} M(G) \neq 1$ and $c'_i \geq c_i$ for all $1 \leq i \leq t - 1$ and $c'_t > c_t$, then $G$ has no $(c'_1, \ldots, c'_t)$-polynilpotent covering group of $G$.

2. The Main Results

Let $G$ be a group and $V$ be a variety of groups. It is clear, by definition, that if $V M(G) = 1$, then $G$ is the only $V$-covering group of itself. So it is natural to put the condition $V M(G) \neq 1$ for nonexistence of $V$-covering group of $G$.

**Theorem 2.1**

Let $G$ be a polynilpotent group of class row $(c_1, \ldots, c_t)$ and $N_{c_1 \ldots, c_t, c_{t+1}} M(G) \neq 1$, for some $c_{t+1} \geq 1$. Then $G$ has no any $N_{c_1 \ldots, c_t, c_{t+1}}$-covering group.

**Proof.**

Let $G^*$ be a $(c_1, \ldots, c_t, c_{t+1})$-polynilpotent covering group of $G$ with the normal subgroup $A$ of $G^*$ such that

\[ G \cong G^*/A, \]
\[ A \cong N_{c_1 \ldots, c_t, c_{t+1}} M(G^*) \]

and
By hypothesis, $\rho$ is an element of $N_{t_{i+1}} A$.

Since $\rho$ is an element of $N_{t_{i+1}} A$, we have $\rho(M) = \gamma_{t_{i+1}}(\rho_{i-1}(M))$, for $i > 1$.

We define $\rho_t$ inductively, for any group $M$ and $t \geq 0$, as follows:

$$\rho_0(M) = M$$

and

$$\rho_i(M) = \gamma_{t_{i+1}}(\rho_{i-1}(M)),$$

for $i > 1$.

By hypothesis, $\rho_t(G) = 1$ and so $\rho_t(G^*/A) = 1$. Hence $A \subseteq \rho_{t+1}(G^*)$, then $\rho_t(G^*) \subseteq \rho_{t+1}(G^*)$. Clearly $\rho_{t+1}(G^*) \subseteq \rho_t(G^*)$, so $\rho_{t+1}(G^*) = \rho_t(G^*)$.

In particular,

$$\rho_t(G^*) = \gamma_2(\rho_t(G^*)) = \cdots = \gamma_{t_{i+1}}(\rho_t(G^*)) = \gamma_{t_{i+1}}(\rho_t(G^*)) = \rho_{t+1}(G^*)$$

(1).

Since $A \subseteq N_{t_{i+1}} A$ and $\rho_t(G^*) \subseteq A$, we have

$$\cdots [\rho_t(G^*)_{t_i} G^*]_{t_2} \gamma_{t_{i+1}}(G^*)], \cdots \gamma_{t_{i+1}}(\cdots (\gamma_{t_{i+1}}(G^*) \cdots)] = 1,$$

or by the above notation,

$$\cdots [\rho_t(G^*)_{t_i} G^*]_{t_2} \rho_1(G^*)], \cdots \gamma_{t_{i+1}}(\cdots (\gamma_{t_{i+1}}(G^*) \cdots)] = 1.$$

First, we show that $[M, i, N] \supseteq [\gamma_i(N), M]$ for each natural number $i$ and normal subgroups $M$ and $N$ of any group. By Three Subgroups Lemma, we have

$$[M, i, N] = [M, i, 3, 2, N], [M, i, 2, N] = [[[M, i, 2, N], [N, N]] = [[M, i, 3, N], [N, N]]$$

(II)

$$[[M, i, 2, N], [N, N]] = [[[M, i, 2, N], [N, N]] = [[[M, i, 3, N], [N, N]] = [[M, i, 4, N], [N, N]]$$

(III)

$$[[M, i, 2, N], [N, N]] = [[[M, i, 2, N], [N, N]] = [[[M, i, 3, N], [N, N]] = [[M, i, 4, N], [N, N]]$$

(III)

for all $1 \leq i \leq t + 1$.

Clearly the equality is valid for $i = 1$. Now for $i = 2$, we can write

$$[[\rho_t(G^*)_{t_i} G^*]_{t_2} \rho_1(G^*)] \supseteq [\gamma_{t_2}(G^*)], \rho_1(G^*)]$$

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$$= [[\rho_1(G^*), \rho_1(G^*)]$$

for all $1 \leq i \leq t + 1$.
Suppose the inclusion (III) holds for \( i = j \). Now, we prove it for \( i = j + 1 \).

\[
\begin{align*}
\lbrack \ldots [ \lbrack \rho_t(G^*), c_1 \rho_0(G^*) ] & , \, c_2 \rho_1(G^*) ] , \ldots , c_{j+1} \rho_j(G^*) ] \rbrack \\
\supseteq & \lbrack \gamma_{c_{j+1}}(\rho_j(G^*)) , \rho_{j+1}(G^*) ] \\
\supseteq & \lbrack \gamma_{c_{j+1}}(\rho_j(G^*)) , \rho_t(G^*) ] \\
\supseteq & \lbrack \rho_j(G^*) , \rho_t(G^*) ] \\
\supseteq & \lbrack \rho_{j+1}(G^*) , \rho_t(G^*) ] \\
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= & \lbrack \rho_{j+1}(G^*) , \rho_t(G^*) ] .
\end{align*}
\]

Now, we have

\[
1 = \lbrack \ldots [ \lbrack \rho_t(G^*), c_1 \rho_0(G^*) ] & , \, c_2 \rho_1(G^*) ] , \ldots , c_{j+1} \rho_j(G^*) ] \supseteq \lbrack \rho_{j+1}(G^*) , \rho_t(G^*) ] .
\]

Hence \( \lbrack \rho_{j+1}(G^*) , \rho_t(G^*) ] = 1 \). Since \( \rho_{j+1}(G^*) = \rho_t(G^*) \), we can conclude

\[
\lbrack \rho_t(G^*) , \rho_t(G^*) ] = 1 \text{ i.e. } \gamma_2(\rho_t(G^*)) = 1 .
\]

Hence by (I), we have \( \rho_{j+1}(G^*) = 1 \).

Therefore \( A = 1 \), which is a contradiction. \( \Box \)

Now we can state the following interesting corollary about nonexistence of solvable covering groups.

**Corollary 2.2**

Let \( G \) be a solvable group with derived length at most \( n \). If the \( l \)-solvable multiplier of \( G \), \( S_l M(G) \), is nontrivial, then \( G \) has no any \( S_l \)-covering group, for all \( l > n \).

**Proof.**

Note that, \( S_l \), the variety of solvable groups of derived length at most \( l \) is in fact the variety of polynilpotent groups of class row \( (1, \ldots, 1) \). Hence the result is a consequence of Theorem 2.1. \( \Box \)
In a different view, the following theorem is also about nonexistence of polynilpotent covering groups which is a vast generalization of a result of the first author (see [7, Theorem 3.1.6], [8, Theorem 2] and [9, Theorem 2.1]).

**Theorem 2.3**

Let $G$ be a polynilpotent group of class row $(c_1, \ldots, c_t, c_{t+1})$ such that $N_{c_1, \ldots, c_t, c_{t+1}} M(G) \neq 1$ where $c_i \geq c_i$ for all $1 \leq i \leq t$ and $c_{t+1} > c_{t+1}$. Then $G$ has no any $N_{c_1, \ldots, c_t, c_{t+1}}$-covering group.

**Proof.**

Let $G^*$ be a $(c_1', \ldots, c_t', c_{t+1}')$-polynilpotent covering group of $G$ with the normal subgroup $A$ of $G^*$ such that

$G \simeq G^*/A,$

$A \simeq N_{c_1', \ldots, c_t', c_{t+1}'} M^*(G)$ and

$A \subseteq N_{c_1', \ldots, c_t', c_{t+1}'} (G^*) \cap \gamma_{c_{t+1}'} (\cdots (\gamma_{c_{t+1}} (G^*) ) \cdots ).$

We consider the following notations, inductively:

$\rho_0 (G^*) = G^*$ and $\rho_i (G^*) = \gamma_{c_{t+1}} (\rho_{i-1} (G^*))$, for all $i \geq 1$,

$\rho_0' (G^*) = G^*$ and $\rho_i' (G^*) = \gamma_{c_{t+1}'} (\rho_{i-1}' (G^*))$, for all $i \geq 1$.

Since, $\rho_{t+1} (G) = 1$, so we have $\rho_{t+1} (G^*/A) = 1$, and hence $\rho_{t+1} (G^*) \subseteq A$.

Also $A \subseteq \rho_{t+1} (G^*)$, then $\rho_{t+1} (G^*) \subseteq \rho_{t+1}' (G^*)$. On the other hand, by $c_i' \geq c_i$ for all $1 \leq i \leq t$ and $c_{t+1} > c_{t+1}$ we can imply that $\rho_j' (G^*) \subseteq \rho_j (G^*)$, for all $1 \leq j \leq t + 1$. Therefore

$\rho_{t+1}' (G^*) = \rho_{t+1} (G^*)$ (I).

Consider the following trivial inclusions:

$\rho_{t+1}' (G^*) = \gamma_{c_{t+1}'} (\rho_t' (G^*)) \subseteq \gamma_{c_{t+1}'} (\rho_t' (G^*)) \subseteq \gamma_{c_{t+1}'} (\rho_t' (G^*)) \subseteq \cdots \subseteq \gamma_{c_{t+1}'} (\rho_t' (G^*)) \subseteq \gamma_{c_{t+1}'} (\rho_t (G^*)) = \rho_{t+1} (G^*).$

Thus by the equality $(I)$, we can conclude that

$\gamma_{c_{t+1}'} (\rho_t' (G^*)) = \gamma_{c_{t+1}'} (\rho_t' (G^*))$ (II).
Since \( \rho_{t+1}(G^*) \subseteq A \subseteq N_{c_{t+1},c_{t+1}^1,...,c_{t+1}^r}^2(G^*) \), we have
\[
\cdots \left( \rho_{t+1}(G^*), \rho_0'(G^*), \rho_1'(G^*), \cdots, \rho_t'(G^*) \right) = 1.
\]
Clearly \( \rho_i'(G^*) \subseteq \rho_i'(G^*) \) for all \( 0 \leq i \leq t \), so by (II), we can conclude that
\[
\cdots \left( \gamma_{c_{t+1}+1}(\rho_t'(G^*)), \rho_0'(G^*), \rho_1'(G^*), \cdots, \rho_t'(G^*) \right) = 1.
\]
and then \( \gamma_{c_{t+1}+1+c_{t+1}^1+\cdots+c_{t+1}^r}(\rho_t'(G^*)) = 1 \). Put \( c = c_{t+1} + 1 + c_{t+1}^1 + \cdots + c_{t+1}^r \)
and \( k = c_{t+1} - c_{t+1} \). By division algorithm, there are \( q, r \in \mathbb{Z} \) such that\( c = kq + r, \) where \( r < k \). Put \( j = \min \{i \in \mathbb{N} | ki + r \geq c_{t+1} + 1 \} \). Then
\[
kj + r \geq c_{t+1} + 1 \quad \text{and} \quad k(j - 1) + r < c_{t+1} + 1.
\]
Now, using (II) we have
\[
1 = \gamma_c(\rho_t'(G^*)) = \left[ \gamma_{c_{t+1}+1}(\rho_t'(G^*)), c - c_{t+1} - 1 \right] \rho_t'(G^*)
\]
\[
= \gamma_{c_{t+1}+1}(\rho_t'(G^*), c - c_{t+1} - 1 \rho_t'(G^*)
\]
\[
= \gamma_{c-1}(\rho_t'(G^*))
\]
\[
\vdots
\]
\[
= \gamma_{c-1}(q-j)(\rho_t'(G^*))
\]
\[
= \gamma_{kq+r}(\rho_t'(G^*))
\]
\[
= \gamma_{k(j-1)+r}(\rho_t'(G^*))
\]
\[
\geq \gamma_{c_{t+1}+1}(\rho_t'(G^*))
\]
\[
= \rho_{t+1}(G^*). \quad \text{Hence} \quad \rho_{t+1}(G^*) = 1 \quad \text{and} \quad A = 1, \quad \text{which is a contradiction.} \quad \square
\]

Notes

(i) The condition \( c_{t+1} > c_{t+1} \) in the theorem 2.3 is essential, since the first author [10] showed that for any natural number \( n \), there exists a nilpotent group \( G \) of class \( n \) such that \( N_c M(G) \neq 1 \) and \( G \) has at least one \( N_c \)-covering group for all \( c \leq n \).

(ii) In a joint paper with the first author [11], it is shown that a finitely generated abelian group \( G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k} \), where \( n_{i+1}|n_i \) for all \( 1 \leq i \leq k - 1 \), has a nontrivial polynilpotent multiplier, \( N_{c_1,\ldots,c_k}(G) \), if \( k \geq 3 \). Hence we can find many groups satisfying in conditions of Theorems 2.1 and 2.3.
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