A Logic for Aristotle’s Modal Syllogistic

Clarence Lewis Protin

Independent Researcher, Montemor-o-Novo, Portugal

ABSTRACT
We propose a new modal logic endowed with a simple deductive system to interpret Aristotle’s theory of the modal syllogism. While being inspired by standard propositional modal logic, it is also a logic of terms that admits a (sound) extensional semantics involving possible states-of-affairs in a given world. Applied to the analysis of Aristotle’s modal syllogistic as found in the Prior Analytics A8-22, it sheds light on various fine-grained distinctions which when made allow us to clarify some ambiguities and obtain a completely consistent system and prove all of the modal syllogisms considered valid by Aristotle. This logic allows us also to make a connection with the axioms of modern propositional modal logic and to perceive to what extent these are implicit in Aristotle’s reasoning. Further work will involve addressing the question of the completeness of this logic (or variants thereof) together with an extension of the logic to include a calculus of relations (for instance the relational syllogistic treated in Galen’s Introduction to Logic) which Slomkowsky has argued is already found in the Topics.

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1. Introduction

Aristotle’s theory of the ‘modal’ syllogism in the Prior Analytics A8-22 has traditionally been considered one of the most problematic parts of the entire Aristotelic corpus. Opinions as to the overall consistency and merit of this theory have generally been unfavourable (for instance Kneale 1962; van Rijen 1989; Striker 1994, McCall 1963, p. 1). Interpreting the different notions of ‘contingent’, ‘possible’ or ‘necessary’ predication presents considerable difficulties. This has been seen as due to the ‘interlacing of logical and metaphysical theory’ Thom 1996, p. 2 and Patterson 1995 argued for a fundamental connection between the modal syllogistic and Aristotle’s ‘metaphysical essentialism’. It is often doubted as to whether Aristotle uses these predication types in an entirely consistent way in his proofs of the validity of the different syllogisms or in his counter-examples. While much of Aristotle’s theory seems to anticipate modern formal logic (in particular his use of concrete models to disprove the validity of certain syllogisms and his use, as the modern mathematical logician, of a metalogic ¹ to formally prove the validity of syllogisms)

¹ We use the term ‘metalogic’ to refer to the reasoning processes Aristotle employs to deduce results about syllogisms.

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there is also much that seems quite intractable to modern methods. Since the days of Bocheński and Łukasiewicz and specially more recently there has been an active interest in interpreting the modal syllogistic using modern formal methods (cf. Rini 2011; Malink 2013; Bocheński 1951; Łukasiewicz 1951). But these approaches end up being quite involved and tied up with semantic and ontological problems.

Aristotle’s concept of ‘term’ would seem to correspond to either an individual or to a predicate (in general unary). Such predicate terms have an extension, the totality of individuals which fall under them. The extensionalist interpretation of the predication all A is B is that all individuals which fall under A also fall under B: in modern predicate logic, \( \forall x. A(x) \rightarrow B(x) \). This interpretation is that of the standard Tarskian semantics of classical first-order logic. It is an interpretation which finds its most adequate use in mathematics.

It is certain that Aristotle was highly influenced by the mathematics of his day in developing the theory in the Analytics. But the goal of the Organon was to develop a general methodology for valid reasoning in philosophy and science, to find a formal logic for natural language based reasoning. It is not necessary to recall here the serious problems of extensionalism (at least in its naive form) once we leave the domain of mathematics. Frege already remarked that in All men are mortal we are not referring to the relationship between the unknown, vague and context-dependent extensions of the concepts of Man and Mortal but rather to a relationship of concepts: our concept of Man includes the concept of Mortal. It is clear that if we employ extensions then these must reflect epistemic and spatio-temporal limitations.

We point out that the ‘logic’ present in the Topics, which probably reflects to a large extent the rules of formal debate prevalent in Greek philosophical schools since the time of Socrates, is clearly much more language-based than that of the Analytics as well as having a richer ontological texture. The Topics also contain implicitly and in a more intensional form much of the reasoning in the Analytics (this is argued for in Malink 2013, chs. 8–9). The syllogistic theory of the Analytics might be said to hint at a greater importance of extension in the choice of terminology (major, minor or middle premise), in some of the arguments used to establish the validity of syllogisms and in the importance of mathematical examples. As mentioned above, Aristotle’s use of counter-examples in the treatment of the modal syllogism anticipates modern modal logic. But what is clear is that in the Analytics we have categorical and ontological neutrality (in modern terminology, terms are not ‘typed’): the category of a term, or its being a genus, species, property or accident, and the particular type of relationship between the two terms (species to genus, accident to substance, etc.) is completely immaterial. As Striker put it in Striker 1994: the assertoric syllogistic is, as it were, metaphysically neutral from the point of view of Aristotle’s ontology. However Aristotle became conscious of difficulties in this approach to the modal syllogism and invoked the elimination of considerations of time in predication (I Pr. An. 34b6–11).

In this work, we consider a type-neutral formal logic in the style of the Analytics. Although we will provide a sound extensional semantics for our logic (and be guided by the concrete intuition it provides), the neutral position we take is that terms have extensions but are not to be identified with their extension.

For reasons of both elegance and faithfulness to the structure of Aristotle’s logic, we do not employ modern predicate logic as in Łukasiewicz 1951 and Rini 2011 for instance. Modern classical propositional logic (anticipated by the Stoics but also present in portions of
the Analytics) is also clearly inadequate. Our logic, called Aristotelic Modal Logic (AML), is a fragment of minimal propositional logic built over a rudimentary modal term calculus. It is given a simple sound semantics in which each model consists of a collection of possible state-of-affairs for a given world. Connections can be made with algebraic logic, in particular Boolean algebras with operators (modal algebras). Our approach is in fact inspired by Boole’s original method of formalising the (assertoric) syllogism. As a term calculus AML incorporates term negation and both particular and universal predication as primitive notions as well as two forms of necessitation □ and ◊ which give rise to two fundamental forms of possibility/contingency which correspond to two different term-operators ◊ and ◊. Aristotle does appear to have made such a distinction as seen in his use of the terms endekhomenon and dunaton. AML also expresses several fundamental axioms of S5 modal logic which we find were implicitly used by Aristotle. Different combinations of the above operators allow us to distinguish up to eight different kinds of contingent predication of one term to another. We are also lead to distinguish between a weak and strong form of assertoric predication. Many of the difficulties in Aristotle’s theory can be traced to a confusion in certain instances between a certain type of ‘contingent’, weak assertoric, strong assertoric and ‘necessary’ predication.

Equipped with these concepts we can make a fine analysis of the entirety of the theory of the syllogism in A4–A22 and explain the ambiguities and alleged errors and inconsistencies in Aristotle’s theory by making the concepts precise and selecting adequate interpretations within AML so as to obtain a consistent interpretation with only minor alterations or clarifications.

In what follows, we will make use of the Ross edition of the text of the Prior Analytics Ross 1957.

2. Aristotelic Modal Logic

Definition 2.1: The terms and formulas of Aristotelic Modal Logic (AML) over a countable set $T$ of atomic terms $t_1, t_2, \ldots, t_n, \ldots$ are defined as follows:

- A term is either an atomic term $t$ or an expression of the form $A$, □$A$ or ◊$A$ where $A$ and $B$ are terms.
- Atomic formulas are defined as follows: if $A$ and $B$ are terms, then $A \rightarrow B$ and $A \leftrightarrow B$ are atomic formulas. The last two are called universal and particular formulas.
- Formulas are either atomic formulas or expressions of the form $\phi \& \psi$ or $\phi \Rightarrow \psi$ where $\phi$ and $\psi$ are formulas.

We use the notation $\diamond A = (\square A)^c$ and $\Theta A = (\Diamond A)^c$. $\Theta$ represents bidirectional possibility which we call bicontingency (see the following remark). By analogy, we call ◊ the binecessity operator.

Remark 2.1: We could also introduce $\&$ as a primitive binary term operator in place of $\leftrightarrow$ and ◊. Then we could define $\square A$ as $(\diamond A \& \diamond A^c)^c$. But the presentation above is chosen to be more faithful to Aristotle’s approach.
We present the deductive system of AML in natural deduction style. We have the usual rules for \& and \( \Rightarrow \) for minimal propositional logic:

\[
\begin{array}{c}
\begin{array}{c}
\phi \\
\phi \Rightarrow \psi
\end{array}
\end{array}
\begin{array}{c}
\phi \\
\Rightarrow \psi \\
\end{array}
\begin{array}{c}
\text{MP} \\
\Rightarrow I \\
\& I
\end{array}
\]

Then we have rules for \( ( ) \)\(^c\) and \( \sqcup \):

\[
\begin{array}{c}
\begin{array}{c}
A \\
\to
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A' \\
\to
\end{array}
\end{array}
\begin{array}{c}
\text{cEx}
\end{array}
\]

where \( A' \) results from \( A \) by replacing any number of occurrences of \( \mathcal{C}^c \) with \( C \) or \( \mathcal{C}^c \) with \( C \) for a subterms \( C \) of \( A \).

\[
\begin{array}{c}
\begin{array}{c}
A \\
\to
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\to
\end{array}
\end{array}
\begin{array}{c}
\text{□Ex}
\end{array}
\]

Where \( A' \) results from \( A \) by replacing any number of occurrences of \( \sqcup \mathcal{C}^c \) with \( \sqcup C \) or \( \sqcup \mathcal{C}^c \) with \( \sqcup C \) for a subterm \( C \) of \( A \).

\[
\begin{array}{c}
\begin{array}{c}
A \to B \\
A \to C \\
A \to B \to C
\end{array}
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
T \\
B \to A \to C
\end{array}
\begin{array}{c}
\text{cC}
\end{array}
\]

Finally we have the axioms:

\[
\begin{array}{c}
\Box A \to \sqcup A \\
\text{(S)} \\
\Box A \to A \\
\text{(T)} \\
\Box A \to \Box \Box A \\
\text{(4)}
\end{array}
\]

We use the notation \(*A\) for \( A \to A \). \( *A \) is to be interpreted as expressing the fact that the term \( A \) has non-empty extension.

**Lemma 2.2:** In AML we can derive:

\[
\begin{array}{c}
\Theta A \to \Diamond A^c \\
\text{(i)} \\
A \to \Diamond A \\
\text{(ii)} \\
\Theta A \to \Diamond A \\
\text{(iii)} \\
\lozenge A^c \to \sqcup A \\
\text{(iv)} \\
\Diamond \Diamond A \to \Diamond A \\
\text{(v)} \\
\Theta A \to \Theta A \\
\text{(vi)} \\
\Diamond \Box A \to \Diamond A \\
\text{(vii)}
\end{array}
\]

We also get a derived rule \( \Theta Ex \) in which we can interchange \( \Theta A \) and \( \Theta A^c \) and the derived rules:

\[
\begin{array}{c}
\begin{array}{c}
* A \\
\to
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\lozenge \\
\to
\end{array}
\end{array}
\begin{array}{c}
\Box \\
\to
\end{array}
\begin{array}{c}
\text{K}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
A \to B \\
A \to B \to C
\end{array}
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
T \\
B \to A \to C
\end{array}
\begin{array}{c}
\text{cC}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
A \to B \\
A \to C \\
A \to B \to C
\end{array}
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
T \\
B \to A \to C
\end{array}
\begin{array}{c}
\text{cC}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
A \to B \\
A \to C \\
A \to B \to C
\end{array}
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
T \\
B \to A \to C
\end{array}
\begin{array}{c}
\text{cC}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
A \to B \\
A \to C \\
A \to B \to C
\end{array}
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
T \\
B \to A \to C
\end{array}
\begin{array}{c}
\text{cC}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
A \to B \\
A \to C \\
A \to B \to C
\end{array}
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
T \\
B \to A \to C
\end{array}
\begin{array}{c}
\text{cC}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
A \to B \\
A \to C \\
A \to B \to C
\end{array}
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
T \\
B \to A \to C
\end{array}
\begin{array}{c}
\text{cC}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
A \to B \\
A \to C \\
A \to B \to C
\end{array}
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
T \\
B \to A \to C
\end{array}
\begin{array}{c}
\text{cC}
\end{array}
\]
We note that the proof of (vi) makes essential use of rule □I.

The condition $\star A$ in $\rightarrow W$ (called the weakening rule) reflects the classical problem that Aristotle’s universal quantification ‘all $A$ is $B$’ seems to implicitly assume the non-empty extension of $A$. The logic of AML is minimal logic and as such has no negation. AML is not designed to capture the classical Aristolean theory of contradictories although it would be interesting to consider an extension of the deductive system to accomplish this. In any case note that $\neg(\star A & A \rightarrow B)$ could not be $A \rightsquigarrow B^c$ for standard negation $\neg$.

We denote by $AML^{S5}$ the extension of AML which results from adding the analogue of the (S5) axiom:

$$\Diamond A \rightarrow \Box \Diamond A$$  \hspace{1cm} (S5)

It does not, however, appear to be derivable from the current axioms and rules of AML. See the conclusion of this paper for a possible proof strategy.

We denote by $AML^{\Diamond}$ the extension of AML which includes formulas of the form $\Diamond \phi$ with $\phi$ an atomic formula together with the rule: from $\phi$ we can derive $\Diamond \phi$. The significance of this extension will be explained further ahead.

Given two terms $A$ and $B$ of $AML$ we use the notation $P(A, B)$ to denote any formula of the form $XA' \circ YB'$ where $\circ$ is either $\rightarrow$ or $\rightsquigarrow$ and $X$ and $Y$ are either empty or one of the symbols $\Box, \Diamond, \ominus, \odot$ and $A'$ is either $A$ or $A^c$ and $B'$ is either $B$ or $B^c$. We call such a formula a predication of $B$ to $A$.

By necessary predication of $B$ to $A$ we mean $A \rightarrow \Box B'$ where $B'$ is either $B$ or $B^c$. We distinguish eight different kinds of contingent predication of $B$ to $A$:

contingent1: $A \circ \Diamond B'$
contingent2: $A \circ \ominus B'$
contingent3: $\Diamond A \circ \Diamond B'$
contingent4: $\ominus A \circ \ominus B'$
contingent5: $\ominus A \circ \Diamond B'$
contingent6: $\Diamond A \circ \ominus B'$
contingent7: $\Diamond A \circ B'$
contingent8: $\ominus A \circ B'$

where $B'$ is either $B$ or $B^c$. However only contingent1–6 will be of interest to us and of these chiefly contingent1–4.

An assertoric predication $B$ to $A$ is $A \rightarrow B'$ where $B'$ is either $B$ or $B^c$.

Given $A$ and $B$ together with symbols $X$ and $Y$ we have four distinguished types of predication $XA' \circ YB'$:

A corresponds to $XA \rightarrow YB$
E corresponds to $XA \rightarrow YB^c$
I corresponds to $XA \rightsquigarrow YB$
O corresponds to $XA \rightsquigarrow YB^c$

A syllogism in $A$, $B$, $C$ is a formula of the form

$$P_1(X_1, X_2) \ & P_2(X_3, X_4) \Rightarrow P_3(X_5, X_6)$$
in which the $X_i$ for $i = 1, \ldots, 6$ are elements of $\{A, B, C\}$ and $X_1 \neq X_2$, $X_3 \neq X_4$ and $X_5 \neq X_6$. The first two predications are called the premises and the third is called the conclusion.

**Definition 2.3:** An extensional model for AML is a pair $M = (I, V)$ consisting of a non-empty set individuals $I$ and non-empty set of valuations $v : T \rightarrow \mathcal{P}(I)$.

We define the extension $E_v(A)$ of a term $A$ for $v \in V$ inductively:

- $E_v(A^c) = I / E_v(A)$
- $E_v(\square A) = \bigcap_{v \in V} E_v(A)$
- $E_v(\bigcup A) = \{a \in I : \forall v \in V. a \in E_v(A) \lor \forall v \in V. a \notin E_v(A)\}$

We define satisfaction of atomic formulas as follows:

- $M \models A \rightarrow B$ if $E_v(A) \subseteq E_v(B)$ for all $v \in V$.
- $M \models A \leftrightarrow B$ if $E_v(A) \cap E_v(B) \neq \emptyset$ for all $v \in V$.
- $M \models \star A$ if $E_v(A) \neq \emptyset$ for all $v \in V$.

Satisfaction for complex formulas are defined as usual. A formula $\phi$ of AML is valid if it is satisfied in all extensional models. If this is the case we write $\models \phi$.

Note that it follows that $E_v(\Diamond A) = \{x \in I : \exists w \in V. x \in E_w(A)\}$ and $E_v(\Box A) = \{x \in I : \exists u \in V. x \in E_u(A) \land \exists w \in V. x \notin E_w(A)\}$.

The elements of $V$ can be seen as different possible states-of-affairs for a world. Extensional models can be visualised nicely as modified Venn diagrams in which circles are given different colours according to the $v$ they express. Thus two circles may intersect for one colour and be disjoint for another.

Our models also interpret AML$\Diamond$: for an extensional model $M$ we have $M \models \Diamond \phi$ according to the cases. If $\phi$ is $A \rightarrow B$ then the condition holds if there is a $v$ such that $E_v(A) \subseteq E_v(B)$. Similarly if $\phi$ is $A \leftrightarrow B$ then the condition holds if there is a $v$ such that $E_v(A) \cap E_v(B) \neq \emptyset$.

The following can be easily checked:

**Lemma 2.4:** The rules and axioms of AML, AMLS$^5$ and AML$\Diamond$ are sound for extensional models.

It is an open question if S5 is independent from AML and whether we have completeness for extensional models.

The following is easy (if somewhat tedious) to show:

**Lemma 2.5:** There are no syllogisms in which the premises are particular predications.

**Remark 2.2:** Note that our models do not have, as usual, a distinguished actual world $v_0 \in V$. The interpretation of $A \rightarrow B$ and $A \leftrightarrow B$ is very strong, as suggested by Pr. An.
34b6–11. It is in fact a hidden necessitation at the ‘predicate’ level rather than the ‘term’ level (cf. the rule →K). A weaker interpretation of the assertoric in terms of an actual world would cause a great deal of problems. Our approach also saves us from having to deal with restrictions on the hypotheses in our natural deduction presentation of rule K. Rule K can be seen as the analogue of using classical necessitation followed by K in the classical sense. One can perhaps think of our atomic formulas being prefixed by an invisible □A → B. In Aristotle there is no great difference between syllogisms with only assertoric premises and those with only necessary premises. The term anangkê is also used at the metalogical level, in particular in the sense that the conclusion of a syllogism follows by necessity. In the case of contingent predication, it is quite arguable that Aristotle’s concept was not a hypothetical predicate-level diamond ◻A → B but contingent predication as we have defined. Just consider the syntax of frequently used expressions of the type A tó(i) de B panti endekhesthai: A may apply to all B, which should be interpreted for instance as A → ⊢B. Also it is remarkable that we find rule ◻T directly expressed in I Pr. An. 36a6–10.

2.1. Preliminaries on Aristotle’s Theory of the Syllogism

Let us consider syllogisms in AML with only assertoric predications. Assertoric predications of B to A can be of type A,E,I or O. The classical theory of the three figures of the assertoric syllogism is found in the Prior Analytics A4–8 and is codified as

- First figure: AAA, EAE, AII, EIO
- Second figure: EAE, AEE, EIO, AOO
- Third figure: *AAI, IAI, AII, *EAO, EIO, OAO

in which the first figure is of the form $P_1(B, A) \land P_2(C, B) \Rightarrow P_3(C, A)$, the second of the form $P_1(B, A) \land P_2(C, A) \Rightarrow P_3(B, C)$ and the third of the form $P_1(C, A) \land P_2(C, B) \Rightarrow P_3(A, B)$. We will allow freedom on the order of the terms in the conclusion in the second and third figure when it is obvious that one conclusion is inter-derivable from another. The star * means that we must add a third premise ◻C to the syllogism. This is so we can apply the weakening rule.

Lemma 2.6: All the purely assertoric syllogisms in the three figures A4–A8 are derivable in AML.

Proof: As an example of a proof in AML consider Aristotle’s derivation of the assertoric syllogism ◻AAI of the third figure

\[
\begin{array}{c|cc}
\text{Lemma 2.6: All the purely assertoric syllogisms in the three figures A4–A8 are derivable in AML.}
\end{array}
\]

$\star A \quad C \rightarrow A \quad C \rightarrow B$

It could be expressed:

\[
\begin{array}{c|c|c}
\text{Lemma 2.6: All the purely assertoric syllogisms in the three figures A4–A8 are derivable in AML.}
\end{array}
\]

$\star C \quad C \rightarrow A$

\[
\begin{array}{c|c|c|c}
\text{Lemma 2.6: All the purely assertoric syllogisms in the three figures A4–A8 are derivable in AML.}
\end{array}
\]

$\frac{\begin{array}{c}
A \rightsquigarrow B
\end{array}}{C \rightsquigarrow A}$

\[
\begin{array}{c|c|c|c}
\text{Lemma 2.6: All the purely assertoric syllogisms in the three figures A4–A8 are derivable in AML.}
\end{array}
\]

$\frac{\begin{array}{c}
C \rightsquigarrow A
\end{array}}{A \rightsquigarrow C}$

\[
\begin{array}{c|c|c|c}
\text{Lemma 2.6: All the purely assertoric syllogisms in the three figures A4–A8 are derivable in AML.}
\end{array}
\]

$\frac{\begin{array}{c}
C \rightarrow B
\end{array}}{A \rightarrow B}$
The syllogism \(\star\) EAO of the third figure could be proven:

\[
\begin{array}{c}
\text{\(\star\) } C \\
C \rightarrow A^c \\
\hline
\hline
C \rightarrow B \\
A^c \rightarrow A \\
\hline
\hline
A^c \rightarrow B \\
B \rightarrow A^c \\
\hline
\end{array}
\]

The syllogism OAO in the third figure is obtained:

\[
\begin{array}{c}
C \rightarrow B \\
C \rightarrow A^c \\
\hline
\hline
A^c \rightarrow B \\
B \rightarrow A \\
\hline
\hline
\end{array}
\]

and that of AOO in the second figure:

\[
\begin{array}{c}
C \rightarrow A^c \\
\hline
\hline
A^c \rightarrow B^c \\
B \rightarrow B^c \\
\hline
\end{array}
\]

A well-known statement in the *Prior Analytics* 36b35–37a9 is that universal contingent negatives do not convert. If we interpret ‘contingent’ as contingent2 then this is true in AML:

**Lemma 2.7:** \( \not\vdash A \rightarrow \ominus B \Rightarrow B \rightarrow \ominus A \)

**Proof:** We can use Aristotle’s own example from 36a4–10. Consider a model \( M = (I, V) \) with \( I = \{m, s\} \) and an AML language with atomic terms *Man*, *Snow* and *White*. \( V \) consists of two valuations \( v_0 \) and \( v_1 \) with:

\[
\begin{align*}
v_0(\text{Man}) &= v_1(\text{Man}) = \{m\} \\
v_1(\text{Snow}) &= v_1(\text{Snow}) = \{s\} \\
v_0(\text{White}) &= \{s\} \\
v_1(\text{White}) &= \{s, m\}
\end{align*}
\]

That is to say, we have world with two individuals \( m \) and \( s \) with two possible state of affairs \( v_0 \) and \( v_1 \). The man \( m \) is always a *Man* and the portion of snow \( s \) is always *Snow*. However, there is a situation in which \( m \) is *White* and a situation in which \( m \) is not *White*. \( s \) is always *White*. We can check that

\[
M \not\models \text{Man} \rightarrow \ominus \text{White}^c
\]

However, we do not have

\[
M \not\models \text{White} \rightarrow \ominus \text{Man}^c
\]

for clearly \( s \in E_{v_0}(\text{White}) \) and \( s \) cannot be both *Man* and not a *Man*. \( s \) is always not a *Man* and hence \( s \notin E_{v_0}(\ominus \text{Man}^c) \)

In a famous passage, Aristotle argues for the conversion of the contingent particular (and this is used in the proofs of the modal syllogisms in the third figure). For contingent 3 and contingent 4 predication, this is immediate. We note the following fact concerning contingent1 and contingent 2:

**Lemma 2.8:** \( \not\vdash A \rightarrow \Diamond B \Rightarrow B \rightarrow \Diamond A \) and \( \not\vdash A \rightarrow \ominus B \Rightarrow B \rightarrow \ominus A \)
Aristotle’s arguments sometimes hinge on the conversion of the necessary negative universal. Also arguments which make this move have also been argued by certain scholars to be fallacious (cf. Aristotle 1996, p. 301).

The interpretation of necessary predication we shall be using in the analysis of the modal syllogism is $A \circ \Box B$. It is remarkable that we have the following:

Lemma 2.9: In AML$_S$ we have

$$C \rightarrow \Box B^c \Rightarrow B \rightarrow \Box C^c$$

Proof: Assume $C \rightarrow \Box B^c$. Then it is easy to show that by definition of $\diamond$ we have $\diamond B \rightarrow C^c$. Apply K to obtain $\Box \diamond B \rightarrow \Box C^c$. Since we have $B \rightarrow \diamond B$ by lemma 2.2 (ii) the result follows from the S5 axiom.

This solves the problem in McCall 1963, p. 20.

2.2. First Figure

A syllogism is of the first figure if it is a formula of the form

$$P_1(B, A) \land P_2(C, B) \Rightarrow P_3(C, A)$$

We call the first atomic formula the major premise and the second formula the minor premise.

The following is straightforward:

Lemma 2.10: There is no syllogism in the first figure in which the major premise has a particular arrow.

The conclusion will be particular or universal accordingly as the minor is particular or universal.

We now proceed to give a detailed analysis of Aristotle’s treatment of the first figure in the Prior Analytics. For this and the remaining figures, we make use of Ross’s commentary Ross 1957, pp. 287–369 and notation for syllogisms. Patterson 1995 also has a nice table listing all the syllogisms of the three figures. We also follow Ross in distinguishing between contingent and problematic premises which are denoted by superscripts $c$ (not to be confused with the $c$-operation in AML) and $p$ respectively. In what follows and in the next two sections we will interpret ‘apodeictic’ as right necessary $A \rightarrow \Box B$ but will consider different interpretations of ‘contingent’ in AML, in our terminology contingent1–6.

A8 deals with the case of two apodeictic premises. It is stated that the assertoric syllogisms remain valid if the premises are replaced by their corresponding apodeictic versions. Consider the validities

$$B \rightarrow \Box A \land C \circ \Box B \Rightarrow C \circ \Box A$$

where $\circ$ represents either a universal or particular arrow. Also $A$ may be replaced by $A^c$.

Then it is easy to see that we obtain all the first figure syllogisms:
A⁹ deals with the case in which one premise is apodeictic and another assertoric. Here we find the most important instance in which Aristotle’s conclusions differ from AML. It seems that Aristotle’s use of ‘assertoric’ hovers between a ‘strong’ assertoric (the invisible □(A → B), in which the inclusion is valid in all states-of-affairs) and a ‘weak’ notion ♦(A → B), in which the inclusion is valid one in one state-of-affairs, for instance, at one time or place, and even apodeictic notion in the sense of A → □B. Thus in Aristotle 1996, p. 190 it is shown that the relation between ‘man’ and ‘animal’ is used by Aristotle as an instance of both an ‘assertoric’ and ‘apodeictic’ predication thus demonstrating that the reading of ‘assertoric’ as ‘strong assertoric’ is legitimate. That the weak notion is inadequate is shown by the fact that we cannot even derive A♦A♦A♦ from it. If there is a situation in which all dogs are sleeping and a situation in which all sleeping things are cats it does not follow that there is a situation in which all dogs are cats (this is related to the problem of ‘joint possibility’ McCall 1963, p. 88). We can however obtain syllogisms by mixing strong and weak assertoric premises. For another observation regarding AML and ‘the problem of the two Barbaras’ see the concluding section.

In A⁹ the following syllogisms are considered valid: A⁹A⁹A⁹, E⁹A⁹E⁹, A⁹I⁹I⁹, E⁹O⁹O⁹. These follow from the validities:

\[ B \rightarrow □A \land C \circ B \Rightarrow C \circ □A \quad (\text{A2}) \]

Aristotle argues that the following are invalid: AA⁹A⁹, EA⁹E⁹, AI⁹I⁹, EI⁹O⁹.

These are, however, syllogisms in AML for we have:

\[ B \rightarrow A \land C \rightarrow □B \Rightarrow C \rightarrow □A \quad (1) \]

which follows from applying K to the first premise.

To show that AA⁹A⁹ is not valid Aristotle produces a counter-example. In this counter-example, the ‘assertoric’ major is not assertoric in the strong sense but in the weak sense. Indeed the counter-example involves the terms ‘moving’ (K), ‘animal’(Z) and ‘man’(A). We have indeed A → □Z but Z → K cannot be considered as assertoric in the strong sense for there are states-of-affairs in nature in which not all animals are in motion. Thus A♦A♦A♦, which corresponds to

\[ B \rightarrow □A \land ♦(C \rightarrow B) \Rightarrow C \rightarrow □A \]

is invalid in AML. This can be easily shown by an extensional model counter-example. In the same way, we can show that E♦A⁹E⁹, A♦I⁹I⁹ and E♦I⁹O⁹ are invalid.²

A¹⁴ deals with the case in which both premises are contingent. The following syllogisms are argued to be valid:

\[ A^c A^c A^c, E^c A^c E^c, A^c E^c A^c, E^c E^c A^c, A^c I^c I^c, E^c I^c O^c, A^c O^c I^c \]

The proofs employ transition from E^c to A^c and O^c to I^c.

We have the following validities in AML:

\[ B \rightarrow ⊖A \land C \circ ⊖B \Rightarrow C \circ ⊖A \quad (\text{A3}) \]

If we interpret ‘contingent’ as contingent 2, then the validity of all the above syllogisms follow from (A3) (in which we may need to employ the Ex⊖ rule). If we use contingent

² We note that if we discard rule K then we do indeed solve the ‘problem of the two Barbaras’ for then we have (A2) but not I. However, we cannot derive (A1).
4, then the same follows immediately. We can also validate those syllogisms which do not employ transition in the proof by interpreting ‘contingent’ as contingent 1 and using the validity

\[ B \rightarrow \Diamond A \land C \circ \Diamond B \Rightarrow C \circ \Diamond A \]  \hspace{1cm} (A4)

In A14 Aristotle then shows that there is no syllogism with contingent premises in which the major is particular. This is clearly true in AML no matter which interpretation of ‘contingent’ we take.

In A15 Aristotle deals with the case in which one premise is ‘contingent’ and the other ‘assertoric’. He establishes the following cases as valid:

\[ A^c A^c, E^c A^cE, A^c A^n A^c, E^c E^cE, A^c I^c, E^c I^cE, A^c I^cO^c, E^c O^cE, A^c I^cP, E^c O^cP. \]

Let us take again ‘contingent’ as contingent 2 and ‘problematic’ as contingent1. This is justified because in Aristotle 1996, p. 275 Aristotle is aware that the conclusion is not contingent2 (endekhomenon) but simply C not applying necessarily to A, that is, the weaker notion of contingency1 expressed by \( A \rightarrow \Diamond B \) in AML. We have the following validities:

\[ B \rightarrow A \land C \circ \Theta B \Rightarrow C \circ \Theta A \]  \hspace{1cm} (A5)

\[ B \rightarrow \Theta A \land C \circ B \Rightarrow C \circ \Theta A \]  \hspace{1cm} (A6)

and it is easy to see that we can validate all the syllogisms above using (A5) and (A6).

Although Aristotle announces rule \( \Diamond T \) in this chapter\(^3\) (which is necessary in AML to derive the syllogisms we are considering) the proof Aristotle gives, which proceeds by \( ad\impossible \), has been found to be insufficient (cf. Aristotle 1996, pp. 270–271).

In A15 Aristotle also states that \( A^c E \) and \( E^c E \) do not yield a syllogism which is immediately verified using extensional model counter-examples. It is also clear that we do not obtain anything from two particular premises.

A16 deals with the case in which one premise is apodeictic and the other contingent. Aristotle begins by stating that all the cases of A15 yield syllogisms if ‘assertoric’ is replaced by ‘apodeictic’. Let us consider the cases when both premises are universal. Aristotle considers the following valid: \( A^n A^c A^p, A^c A^n A^c, E^n A^c E(E^n A^c E^p), E^c A^n E^c, A^n E^c A^p. \)

In AML, we have the validities:

\[ B \rightarrow \Box A \land C \circ \Theta B \Rightarrow C \circ \Diamond A \]  \hspace{1cm} (A7)

\[ B \rightarrow \Theta A \land C \circ \Box B \Rightarrow C \circ \Theta A \]  \hspace{1cm} (A8)

Interpreting ‘contingent’ and ‘problematic’ as in A15, we validate all the syllogisms in the list except \( E^n A^c E \). It appears that we can only derive this syllogism in AML\(^5\) using \( \Diamond \Box A \rightarrow \Box A \). It is interesting that Aristotle in 36a15 seemingly endorses \( A \rightarrow \Diamond A \) from lemma 2.2 (for the present interpretation of ‘problematic’). Aristotle’s statement that \( A^n E^n \) and \( E^c E^n \) yield no syllogism can be easily verified using extensional models.

In A16 Aristotle gives the following list of valid syllogisms in which one premise is particular: \( E^n I^c O, E^p I^c O^c, A^n I^c I^c P \). The last two follow from (A8) and (A7) and the first is validated in AML\(^5\) as for \( E^n A^c E \).

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\(^3\) See remark 2.7.
2.3. Second Figure

The second figure is a syllogism of the form

\[ P_1(B, A) \land P_2(C, A) \Rightarrow P_3(B, C) \]

In proving the purely assertoric form of the second figure in AML rule cC plays a key role. In AML, we can generalise this rule in the form of the following easily provable equivalences:

\begin{align*}
\text{c1. } C \rightarrow \ominus B^c & \iff \sqcap B^c \rightarrow C^c \\
\text{c2. } C \rightarrow \lozenge B^c & \iff \square B \rightarrow C^c \\
\text{c3. } C \rightarrow \square B^c & \iff \lozenge B \rightarrow C^c \\
\text{c4. } C \rightarrow \sqcap B^c & \iff \ominus B^c \rightarrow C^c
\end{align*}

In (A8), Aristotle deals with the case of two apodeictic premises. We must check the claim that all the purely assertoric syllogisms of the second figure remain valid if we replace each premise and the conclusion by its corresponding apodeictic version. This follows the validities

\[ B \rightarrow \square A \land C \rightarrow \square A^c \Rightarrow B \rightarrow \square C^c \]  \hspace{1cm} (B1)

proven by using c3 on the second premise and

\[ B \rightarrow \square A^c \land C \rightarrow \square A \Rightarrow B \rightarrow \square C^c \]  \hspace{1cm} (B2)

which is proven by weakening the second premise to \( C \rightarrow A \) and applying cC and then K.

A10 deals with the case of one assertoric and one apodeictic premise. The following are considered valid: \( E^nAE^n, AE^nE^n, E^nIO^n \). These follow from the validities

\[ B \rightarrow \square A^c \land C \rightarrow A \Rightarrow B \rightarrow \square C^c \]  \hspace{1cm} (B3)

\[ B \rightarrow A \land C \rightarrow \square A^c \Rightarrow B \rightarrow \square C^c \]  \hspace{1cm} (B4)

and

\[ B \rightarrow \square A^c \land C \rightarrow \square A \Rightarrow C \rightarrow \square B^c \]  \hspace{1cm} (B3)

In order to obtain the apodeictic conclusion in (B3) (rather than a purely assertoric one) we have used the S5 axiom \( \lozenge A \rightarrow \square \lozenge A \).

These syllogisms are invalid if we take ‘assertoric’ to be weak assertoric. For instance for the first syllogism take B as ‘sleeping’, A as ‘man’ and C as ‘horse’. Then we have \( C \rightarrow \square A^c \) and \( \lozenge(B \rightarrow A) \) for there can be a situation in which the men are asleep and the horses are awake. But clearly we do not have \( B \rightarrow \square C^c \) because there can be a situation in which a horse is sleeping. We do however get \( \lozenge(B \rightarrow \square C^c) \).

The following are considered invalid: \( A^nEE^n, A^nOO^n, AO^nO^n \).

In AML, these are in fact valid. However if we take the assertoric premises to be weak assertoric (as is the case in Aristotle’s own counter-example) their invalidity can be easily shown by concrete extensional models.

A17 treats of the case of two contingent premises. It is stated that there are no syllogisms wherein both premises are contingent (36b25–30). This is related to the fact that we cannot
convert a universal contingent negative (at least for contingent1 and contingent2) but only have results like c1 and c2.

Let us consider the contingent2 premises $B \rightarrow \Theta A$ & $C \rightarrow \Theta A$. We can show that we do not obtain a conclusion $B \rightarrow \Theta C$ by the following model based on Aristotle's own example (37a1–10). Let $M = (I, V)$ with $I = \{m, h\}$ and consider an AML language with atomic terms Man, Horse and White and let $V$ consist of $\{v_1, v_2, v_3, v_4\}$ in which $v_i(\text{Man}) = \{m\}$ for $i = 1, \ldots, 4$ and such that $v_1(\text{White}) = \{m, h\}, v_2(\text{White}) = \{m\}, v_3(\text{White}) = \{h\}, v_4(\text{White}) = \emptyset$. Here we are considering that the individual man and horse $m$ and $h$ can in fact change their colour in different situations.

Then we have that $M \models \text{Man} \rightarrow \Theta \text{White} & \text{Horse} \rightarrow \Theta \text{White}$ but $M \not\models \text{Man} \rightarrow \Theta \text{Horse}$.

A18 treats of the case of one problematic and assertoric premise. The following are considered valid: $EAcEp, AcEcP, EEcEp, EcEcP, EIcOp, EOcOp$. For contingent1 and contingent2 premises we have the following validities:

\[ B \circ \Diamond A & C \rightarrow A^c \Rightarrow B \circ \Diamond C^c \quad (D1) \]
\[ B \circ \Theta A & C \rightarrow A^c \Rightarrow B \circ \Diamond C^c \quad (D2) \]

where $\circ$ is to be replaced either by a universal or particular arrow. These validate $A^cEEp$. $EA^cEp$ and $EI^cOp$ are validated by

\[ B \rightarrow A^c & C \circ \Diamond A \Rightarrow C \circ \Diamond B^c \quad (D4) \]

(D4) for a contingent 2 premise validate $EE^cEp$ and $EO^cOp$ by the transition (rule Ex$\Theta$). In the same way, (D2) validates $E^cEEp$.

Aristotle considers that $A^cA, AA^c, E^cA$ and $AE^c$ do not yield syllogisms. This can be checked in AML using extensional model counter-examples.

A19 deals with the case of one apodeictic and one contingent premise. Valid syllogisms are $EnA^cE, E^n A^c E, A^c E^n E, E^n A^c E, E^n E^n E, E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E^n E

Now Aristotle considers that the following premises do not yield a syllogism:

\[ A^nE^n, E^nA^n, A^nA^n, A^cA^n. \]

But we can obtain syllogisms from such premises in AML. The reason for this discrepancy is that in the examples Aristotle adduces the ‘apodeictic’ premise is not in fact right necessary (the interpretation we have been using) but merely (strong assertoric): it is necessary that everything awake(A) be in movement(M). This evidently cannot be read as $A \rightarrow \Box M$ but merely as strong assertoric $A \rightarrow M$.

For particular premises Aristotle considers the following as valid:

\[ E^n T^n O^n, E^n F^n O^n, E^n O^n O^n, E^n O^n O^n. \]
The last two syllogisms clearly follow from the first two by transition. The first two syllogisms follow from (22). The invalid cases considered by Aristotle are obtained by weakening the invalid cases discussed above for two universal premises.

### 2.4. Third Figure

The third figure is a formula of the form

\[ P_1(C, A) \land P_2(C, B) \Rightarrow P_3(A, B) \]

It appears that Aristotle is lax about the order of the premises and thus considers in the third figure also formulas of the form

\[ P_1(C, A) \land P_2(C, B) \Rightarrow P_3(B, A) \]

This follows immediately if \( P_3(B, A) \) can be exchanged with \( P_3(A, B) \). If both figures are universal it is crucial to be able to pass to a particular form of one of the premises. This is in fact Aristotle’s main strategy. Hence in this situation we assume tacitly that we have an extra premise \( \star C \) in all syllogisms of the third figure with two universal premises. This is so we can make use of the weakening rule. What formalisation of contingency should we use in this figure? Aristotle makes frequent use of the conversion of the contingent particular as well as transition. Hence an obvious choice would be to use contingent4 (\( \ominus \ominus \)). This allows us to follow Aristotle’s own proofs very closely. However it turns out that we can validate all the syllogisms in the third figure for other interpretations of contingency, in particular allowing different interpretations of contingency in the premise(s) and in the conclusion. We also follow Ross in considering the distinction between contingent (c) and problematic (p) as legitimately corresponding to two different interpretations of contingent.

We cannot examine here the syllogisms for all the possible combinations of the six notions of contingency in the premises. In the notation such as \( AA^cI^p \) the first A refers to the first premise and \( A^c \) to the second.

In A8, we have the case of two apodeictic premises in the third figure. For our usual interpretation of ‘apodeictic’ these syllogisms are validated by the following syllogisms:

\[ C \rightarrow \Box A \land C \rightarrow \Box B \Rightarrow A \sim \Box B \]

\[ C \sim \Box A \land C \rightarrow \Box B \Rightarrow A \sim \Box B \]

Using these it is easy to justify Aristotle’s conclusion that if we add ‘it is necessary’ to the premises of all the purely assertoric syllogisms of the third figure we obtain valid syllogisms with apodeictic conclusions.

A11 deals with the case of one contingent and one apodeictic premise. For two universal premises valid syllogisms are \( A^n AI^n, AA^n I \) and \( E^n AO^n \). The first and the third are validated by

\[ C \rightarrow \Box A \land C \land B \Rightarrow B \land \Box A \]

To show \( AA^n I \) we use

\[ C \rightarrow A \land C \rightarrow \Box B \Rightarrow A \sim \Box B \]

It also follows immediately that \( A^n II^n, IA^n I \) and \( E^n IO^n \) are valid.
Aristotle’s argues that $EA^nO^n$ is invalid. In fact from such premises we can only derive $A^c \leadsto \square B$ or an assertoric conclusion $B \leadsto A^c$. That we do not get $B \leadsto \square A^c$ can be shown by a counter-example.

Consider a language with three atomic terms $A$, $B$, $C$ and a model $M = (I, V)$ with $I = \{x, y\}$, $V = \{v_0, v_1\}$ with $v_0(A) = v_1(C) = x$ and $v_0(C) = v_1(A) = y$ and let $v_1(B) = I$ for $i = 0, 1$. Then clearly $M \models C \leadsto A^c$ and $M \models C \leadsto \square B$. But clearly we $M \not\models B \leadsto \square A^c$ since $v_i(\square A^c) = \emptyset$ for $i = 0, 1$. The same goes for the invalidity of the conclusion $A \leadsto \square B^c$. The same example can be used to show the invalidity of $OA^nO^n$ and $E^nO^n$.

The other syllogisms with particular premises considered invalid by Aristotle are: $AInIn$, $InAIn$, $OnAOn$.

In AML, however, we can derive these syllogisms. But consider Aristotle’s example to invalidate $InAIn$: $A$ is ‘waking’, $B$ is ‘biped’ and $C$ is ‘animal’. Certainly we have $C \leadsto A$ in the AML sense of ‘assertoric’. Rather we have weak assertoric predication $\diamond(C \leadsto A)$. The text in fact reads: to de $A$ to(j) $\Gamma$ endekhetai (31b30). Indeed if we take $I^nAIn$ as $I^nA^cI^n$ then this can be shown to invalid in AML using contingency 1 or contingency 2 for instance.

$A20$ deals with the case in which both premises are contingent. Aristotle establishes as valid: $AcAcIc$, $EcAcOc$, $EcEcIc$, $IcAcIc$, $EcIcOc$, $OcAcIc$, $EcOcIc$, $OcEcIc$.

If ‘c’ in the premises is interpreted as contingent 1, then we obtain $A^cA^cIc$ easily

\[ C \leadsto \diamond A \& C \leadsto \diamond B \Rightarrow \diamond A \leadsto \diamond B \tag{C1} \]

with contingent 3 conclusion and similarly we can validate $E^cA^cOc$ but with $\diamond B \leadsto \diamond A^c$ as a conclusion. To get $E^cE^cIc$ we clearly must have a contingent 2 or contingent 4 for premise of instance. For example, we have

\[ C \leadsto \diamond A^c \& C \leadsto \ominus B^c \Rightarrow \ominus B \leadsto \diamond A^c \tag{C2} \]

In the proofs of (C1) and (C2) and similar validities, we apply weakening to one of the premises (it does not matter which one). Hence it follows immediately that we validate the syllogisms which result from weakening one of the premises in $A^cA^cIc$, $E^cA^cOc$, $E^cE^cIc$ and hence we can validate in an analogous way $I^cA^cIc$, $E^cE^cIc$, $O^cA^cIc$, $E^cO^cIc$, $O^cE^cIc$.

$A21$ deals with the case in which one premise is contingent and another assertoric. According to Aristotle the valid syllogisms with two universal premises are:

$AA^cI^p$, $A^cAI^c$, $EA^cO^p$, $E^cAO^c$, $AE^cI^p$, $EE^cO^p$

Consider the first syllogism $AA^cI^p$.

Let us take $c$ as contingent 4. Then in AML, we can prove a contingent 5 conclusion

\[ C \leadsto A \& \ominus C \leadsto \ominus B \Rightarrow \diamond A \leadsto \ominus B \tag{S1} \]

and hence also a contingent 4 conclusion

\[ C \leadsto A \& \ominus C \leadsto \ominus B \Rightarrow \diamond A \leadsto \diamond B \tag{S2} \]

This is interesting because it follows Aristotle’s own proof by converting the second premise. However we can also obtain the same conclusions with weaker contingent 1 and
contingent 2 premises such as
\[ C \to A \quad \& \quad C \to \lozenge B \Rightarrow A \not\rightarrow \lozenge B \]  
(S3)

The second syllogism \( A^c AI^c \) can be treated in the same way. It is not clear why Ross writes 'c' here but 'p' in the conclusion of the former syllogism. If in (S1) and (S2) and (S3) we replace \( A \) by \( A^c \) then we get the same validations for \( EA^c O^c \). According to Aristotle \( E^c AO^c, AE^c Ip, EE^c O^p \) follow from the first three syllogisms by conversion. For \( E^c AO^c \) we have the valid syllogisms
\[ \lozenge C \to \lozenge A^c \quad \& \quad C \to B \Rightarrow \lozenge B \not\rightarrow \lozenge A^c \]  
(S4)
\[ C \to \lozenge A^c \quad \& \quad C \to B \Rightarrow B \not\rightarrow \lozenge A^c \]  
(S5)

Note the order of the terms in the conclusion.
We can validate \( AE^c Ip \) with
\[ C \to A \quad \& \quad C \to \lozenge B^c \Rightarrow A \not\rightarrow \lozenge B \]  
(S6)

Here the use of the bicontingent seems to be essential in order to apply 'transition' (the Ex\( \Theta \) rule). We can of course replace the contingent 2 premise with its contingent 4 version. \( EE^c O^p \) is validated in a similar way to the previous syllogisms.

In the case in which one premise is particular in A21 Aristotle considers the following valid:
\( AI^c Ip, A^c II^c, IA^c I^c, IA^c Ip, EI^c O^p, E^c IO^c, IE^c I^c, O^c AOp \).
\( AI^c Ip \) can be validated in the same way as \( AA^c Ip \). \( A^c II^c \) cannot be validated with a contingent 4 premise:
\[ \lozenge C \to \lozenge A \quad \& \quad C \sim B \]

But it can with a contingent 1, contingent 2, contingent 3 or contingent 5 (\( \lozenge \Theta \)) premise. The same goes for \( IA^c I^c \) and \( IE^c I^c \). \( IA^c Ip \) and \( EI^c O^p \) are validated like \( A^c AI^p \) and \( EA^c O^p \).

Aristotle validates \( O^c AO^p \) by \textit{reductio ad impossibile}. We cannot of course do this in AML but it is interesting to note that if we take the interpretation of the premises as
\[ C \sim \lozenge A^c \quad \& \quad C \to B \]

and assume with Aristotle that \( B \to \Box A \) then we can prove in AML that \( C \sim C^c \). Informally that \( B \to \Box A \) is false means that there is a \( B \) such that \( (\Box A)^c \) holds, that is, \( B \Rightarrow \lozenge A^c \). But in AML we can prove directly that
\[ C \sim \lozenge A^c \quad \& \quad C \to B \Rightarrow B \sim \lozenge A^c \]  
(S7)

and analogously for a contingent 2 premise. We can also validate versions with contingent 3 and contingent 4 premises.

Although according to Ross 1957, p. 365 Aristotle held that there is no conclusion from \( I^c E \) we however can validate in AML various syllogisms from such premises (for instance, with a contingent 1 premise). For instance \( C \sim \lozenge A \quad \& \quad C \to B^c \Rightarrow \lozenge A \sim \lozenge B^c \).

In A22, Aristotle deals with one contingent and one apodeictic premise. The syllogisms with two universal premises claimed as valid are \( A^n A^c Ip, A^c A^n I^c, E^c A^n O^c, E^n A^c O^p, A^n E^c Ip \).
Using axiom T it is not difficult to reduce these syllogisms to the case of one contingent and one assertoric premise. For instance from (S1) and (S3) we validate $A^nA^cI^p$ by

\[
C \rightarrow \Box A \quad & \quad \Box C \rightarrow \Box B \Rightarrow \Diamond A \sim \Box B \tag{N1}
\]

\[
C \rightarrow \Box A \quad & \quad C \rightarrow \Diamond B \Rightarrow A \sim \Diamond B \tag{N2}
\]

and analogously for $A^nA^cI^c$. Replacing $B$ with $B^c$ in (N1) and using Ex$\Box$ yields a validation of $A^nE^nI^p$.

It is easy to derive

\[
C \rightarrow \Box A^c \quad & \quad C \rightarrow \Diamond B \Rightarrow \Diamond C \sim \Diamond A^c \tag{N3}
\]

which validates $E^nA^cO^p$ with contingent 3 conclusion. With a contingent 4 or contingent 2 premise we obtain a contingent 5 $\Box B \Rightarrow \Diamond A^c$ conclusion. For $E^nA^nO^n$ we easily obtain

\[
C \rightarrow \Diamond A^c \quad & \quad C \rightarrow \Box B \Rightarrow B \sim \Diamond A^c \tag{N4}
\]

and analogously for contingent 2, contingent 3 and contingent 4 premises. For a contingent 4 premise we obtain a contingent 5 conclusion

\[
\Box C \rightarrow \Box A^c \quad & \quad C \rightarrow \Box B \Rightarrow \Diamond B \sim \Box A^c \tag{N5}
\]

Aristotle considers $A^cE^n$ as not yielding a syllogisms and uses examples to prove this. However in AML, we can prove for instance

\[
C \rightarrow \Diamond A \quad & \quad C \rightarrow \Box B^c \Rightarrow \Diamond A \rightarrow \Box B^c
\]

But in Aristotle's example (sleep-sleeping horse-man, sleep-waking horse-man) the 'contingent' universal does not in fact to correspond to contingent 1–6 right but to the weak assertoric contingent $\Diamond(A \rightarrow B)$ we discussed earlier. That is, this contingent only expresses that there is a state of affairs in which everything sleeping is a sleeping horse. This is different from stating that in all possible states-of-affairs what is sleeping is possibly a horse or what can sleep is possibly a horse, etc. With this reading we can confirm that $\Diamond(C \rightarrow A) \quad & \quad C \rightarrow \Box B^c$ does not yield a syllogism in AML using Aristotle's own example.

When one premise is particular in A22 Aristotle considers as valid: $A^nI^nI^p$, $I^cA^nI^p$, $A^nI^nI^c$, $I^nA^nI^c$, $E^nI^nO^p$, $O^nA^nO^p$, $E^nI^nO^c$, $O^nA^nO^c$.

It is clear that if in a proof a syllogism in AML we apply weakening to a premise then we obtain a syllogism also if we start with the weakened form of that premise. Thus $A^nI^nI^p$, $I^cA^nI^p$ and $O^nA^nO^p$ follow directly from $A^nA^cI^p$, $A^nA^cI^c$, and $E^nA^nO^c$. But what about when the apodeictic premise is weakened as in $A^cI^nI^c$, $I^nA^cI^c$ and $E^nI^nO^c$? For $I^nA^cI^c$ we can derive

\[
C \sim \Box A \quad & \quad \Box C \rightarrow \Box B \Rightarrow \Diamond A \sim \Box B \tag{N6}
\]

\[
C \sim \Box A \quad & \quad C \rightarrow \Diamond B \Rightarrow A \sim \Diamond B \tag{N7}
\]

Note that in (N6) the premise must now be contingent 5. The syllogisms $A^nI^nI^c$ and $E^nI^nO^c$ are validated in a similar way.
Consider now the surprising syllogisms $E^n I^n O$, $O^n A^n O$ which yield assertoric conclusions. In AML where we can in fact derive

$$C \rightarrow \Box A^c \land C \rightarrow \Diamond B \Rightarrow \Diamond B \rightarrow A^c$$

that is, from a contingent premise we get a contingent conclusion.

In $AML^{55}$ where we can show that $\Diamond \Box A \rightarrow \Box A$ we have

$$C \rightarrow \Box A^c \land \Diamond C \rightarrow \Diamond B \Rightarrow \Diamond B \rightarrow A^c$$

and for a contingent 4 premise we obtain a contingent 8 conclusion. For $O^n A^n O$ we have in AML

$$C \rightarrow \Box A^c \land C \rightarrow \Diamond B \Rightarrow \Diamond B \rightarrow A^c$$

and similarly for contingent 2 and contingent 3 premises. It does not seem that we obtain a valid syllogism for a contingent 4 premise but we do for a contingent 5 one. The situation for $F^n E^n$ is the same as for the discussion of $A^n E^n$. This can be shown not to yield a syllogism if we interpret contingent as weak assertoric.

3. Conclusion

We faced with the task of finding an axiomatic-deductive system (preferably with a sound semantics) which meets the following desiderata:

(i) It should be homogenous in nature with Aristotle’s logic. That is, have a similar syntactic style (distinct from quantifier logic and closer to natural language) that could be easily grasped by Aristotle himself.

(ii) The axiomatic-deductive system should economical and elegant and be closer to Aristotle’s own proofs than to modern first-order logic.

(iii) It should be more flexible and refined than Aristotle’s logic in the sense that we can embed Aristotle’s system into it and resolve ambiguities and seeming inconsistencies by having at our disposal a more refined spectrum of different notions of contingency, apodeictic and assertoric predication to choose from in different contexts. It should allow us look at the problems under a microscope so as to be able to pin-point the problems and easily rectify or slightly and naturally modify it into a completely consistent system.

(iv) It should still have a connection to modern mathematical logic, specially modern S5 modal propositional and quantifier logic as well as algebraic logic. The deductive system should ideally be related to natural deduction\(^4\) and have a sound semantics similar (but not necessarily identical) to Kripke frames.

(v) It should (if possible) furnish us with an interpretation which validates all the syllogisms in A8–22 and it should not derive conclusions from pairs of premises which Aristotle considers inconclusive.

(vi) It not only validates but extends Aristotle’s system and is an interesting system in its own right.

\(^4\) Note that we can see natural deduction quantifier rules at work for instance in Aristotle’s proof in 24a14–20.
The systems of *McCall 1963* and *Malink 2013* build upon the pioneering work of *Bocheński 1951, Becker 1933* and *Łukasiewicz 1951*. Malink’s system is one of the most interesting to date as a sound first-order semantics (called *predicable semantics*) is considered which establishes as inconcluent all the syllogisms held to be inconcluent by Aristotle. Both McCall’s and Malink’s system do not meet desiderata (i) and (ii). McCall’s system is syntactically close to Aristotle but is based on a very large number of syllogisms which must be taken as axioms and also a fairly large number of deduction rules. Malink’s system is based on full first-order quantifier logic (without equality) and six axioms involving three primitive binary predicates $A$, $N$ and $\overline{N}$. In order to express the modal syllogisms Malink must build up nine complex first-order notions from these predicates (including introducing ‘ontological’ distinctions between elements of the model such as by means of the unary predicate $\Sigma$). The proofs of the validity of the moods of the modal syllogistic given in Appendix B of *Malink 2013* are often of a certain complexity. Thus it is arguable that AML is much closer to meeting desiderata (i) and (ii) than Malink’s or McCall’s systems. We note that AML could also be recast in first-order form with three sorts (individuals, concepts, states-of-affairs) and a single primitive trinary predicate $\Delta(x, t, v)$. Then we could define different modes of predications in AML such as $a \rightarrow \diamond b$ as a predicate $Q(a, b) \equiv \forall v. \forall x. \Delta(x, a, v) \rightarrow \exists v'. \Delta(x, b, v')$. We then would obtain by first-order logic alone for instance that $Q(a, b) \land Q(b, c) \rightarrow Q(a, c)$ and many other moods in the same way. There seems to be a candidate for an embedding of AML into first-order modal $\mathcal{S}_5$ logic but showing the proof-theoretic faithfulness of such an embedding presents certain difficulties.

Malink’s system does meet (iii). For instance, it is based on two kinds of necessitation just as AML we can have both ordinary $\diamond$ and strong contingency $\Theta$. However Malink’s system is based on binary predicates so has a more *dedicto* flavour while the *dere* approach of AML allows us to consider using $\diamond, \Theta, \Box, \mathcal{U}$ (and no symbol at all) up to 25 types of predication between two terms. For (iii) with AML we draw the following conclusions:

(a) Although it is clear that Aristotle distinguished clearly two types of contingency at least six (if not eight) inter-related types of contingency are desirable in order to obtain a clear and fully consistent theory of the modal syllogism. AML greatly clarifies the problem of the conversion of particular contingents. Also AML proves the conversion of the negative necessary universals.

(b) We find in places an ambiguity and confusion between weak and strong assertoric predication and between necessary and strong assertoric predication (see *Aristotle 1996*, p. 190) as well as between contingent and weak assertoric predication. This agrees with the take of *Becker 1933* on the classical problem of the oscillation between *de dicto* and *de re*.

Point (iv) is clearly met by AML. The analogues of modern modal axioms appear as essential to the structure of AML proofs of Aristotle’s theory of the modal syllogism. $\mathcal{S}_5$ makes a surprising appearance in a few places. Aristotle seems to have explicitly acknowledged $K$. Neither McCall’s nor Malink’s system seem to have much connection to modern modal logic although axiom 5, $Nab \rightarrow Aab$, in *Malink 2013*, p. 287 might be read as an analogue of $T$. 


As for (v) McCall’s only validates a certain limited amount of Aristotle’s syllogisms. Malink’s system on the hand not only validates Aristotle’s conclusions but even furnishes refutations of all the moods held to be inconcluent. There is however still the question of the legitimacy of the interpretations given to the various form of predication, as for instance the interpretation of the negative necessary predication $\neg a b$ which seems rather forced since it introduces a disjunction $\Sigma a \lor \Sigma b$ in the definition. Also Malink’s six axioms can be seen as solving the ‘problem of the two Barbaras’ by positing these two Barbara moods ad hoc as axioms. But it is generally considered that axioms should express intuitively self-evident truths or propositions generally agreed upon.

AML (or more precisely AML$^S5$ for a few syllogisms) allows us to give interpretations to ‘contingent’, ‘problematic’, ‘assertoric’ and ‘apodeictic’ which validate all the syllogisms in A8–22. AML, however, proves conclusions for a fair number of combinations of premises considered inconclusive by Aristotle (though agreeing with Aristotle about the inconclusivity of some non-obvious combinations). In AML, we can explain these discrepancies by examining the counter-examples Aristotle offers. We note that Malink’s system also has difficulties agreeing with Aristotle’s concrete counter-examples. As already mentioned, according to the AML analysis in many of Aristotle’s examples we find an ambiguity or confusion between weak and strong assertoric predication and between necessary and strong assertoric predication as well as between contingent and weak assertoric predication.

Finally it seems that both AML and Malink’s system fulfill desideratum (vi). Also we observe that while AML is connected to extensional models (based on sets of individuals) Malink’s predicative semantics is intensional in flavour (indeed heavily inspired by the ontology and logic of the Topics) and based on quantification over terms rather than individuals (cf. Rini 2011, p. 236), that is, we are arguably in the presence of disguised second-order logic. Whereas AML required the $\star \phi$ condition to deal with empty extensions and the passage from A to I predication, Malink solves this problem by the first axiom $Aa$. Obviously we cannot read this to say: unicorns are unicorns so the term unicorn has non-empty extension.

There remain the following open problems regarding AML and its extensions:

1. Obtain a complete list of all valid syllogisms in AML and its extensions (including all types of predication) perhaps employing an automatic theorem prover.
2. Investigate the complexity of the simplest model required to refute a non-valid syllogism.
3. Investigate whether there is a faithful embedding of AML$^S5$ into first-order S5 logic and whether we can use such a result to prove the completeness of AML$^S5$ for extensional models.
4. Investigate more complex types of models for AML with an accessibility relation between states-of-affairs. This seems a good strategy to prove the independence of S5. Find such a model in which S5 fails.
5. Investigate modal sorites and analyse the rest of the theory of the Prior and Posterior Analytics.
6. Extend the logic of AML to include the intuitionistic and classical negation rules in order to formalise Aristotle’s reductio ad impossibile arguments and his theory of contraries and oppositions. The metalogic of AML is in fact weaker than the metalogic used by Aristotle but we saw an example of Aristotle’s use of the classical
negation rule suggesting AML could easily be extended with negation to consistently capture a larger fragment of Aristotle’s metalogic

(7) A further topic would be to explore the alternative presentation of AML in which a binary term operation \( \land \) is introduced. In this case the semantics looks like a collection of representations of a Boolean algebra with operators (in this case a single operator \( \square \)).

(8) Another approach which comes to mind is a purely algebraic treatment of AML, to consider AML as a kind of generalised lattice with operators, independently of its semantics (representation) in terms of the set-theoretic class of models defined. This might be seen as taking an intensional approach in which we focus on the relationship between concepts.

(9) If we drop rule K from AML then we can derive \( A^n A^n A^n \), \( A^n A A^n \) but not \( A A^n A^n \) in the first figure (problem of the two Barabras). If we extend AML to include disjunction then we might devise other interpretations of contingency such as \( A \rightarrow \Diamond B \lor \Diamond A \rightarrow B \) so that many other moods remain valid. We could thus investigate weaker versions of AML for modelling the modal syllogistic and attempt to find a suitable semantics for which rule K is no longer sound.

(10) Extend this approach to a calculus of relations. Slomkowski has argued in Slomkowski 1997 that the Topics includes the rudiments of a calculus of relations which is given fuller treatment in Galen’s Introduction to Logic (see Barnes 1993). The approach in this paper might be extended to binary predicates (relations). Let \( \Delta \) be the diagonal relation. Then unary predicates are represented by terms of the form \( R \land \Delta \). We use \( \rightarrow \) and \( \cdot \) as before. We have furthermore a product operation on terms \( R \circ S \) and an inversion operator \( R^* \). We could thus obtain a straightforward extension of AML in which could express the analogue of more complex sentences in first-order logic but with the added benefit of modalities. Algebraically this gives rise to structure similar to that of a quantale but with operators.

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**ORCID**

Clarence Lewis Protin http://orcid.org/0000-0001-5854-6928

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