On the convergence of the Kac-Moody correction factor

Yanze Chen

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Abstract

The Kac-Moody correction factor, first studied by Macdonald in the affine case, corrects the failure of an identity found by Macdonald in finite-dimensional root systems in 1972. Subsequently this factor appeared in several formulas in the affine or Kac-Moody analogue of $p$-adic spherical theory for reductive groups. In this article we view the inverse of this correction factor as a function, prove the convergency and holomorphy of this function on the complexified Tits cone.

Keywords Kac-Moody root system · Kac-Moody algebra · Tits cone

1 Introduction

1.1 Kac-Moody correction factor

Let $(V, \Delta = \{a_1, \cdots, a_l\}, V^*, \Delta^\vee)$ be a Kac-Moody root system, $\Phi_{re}$ be the set of real roots in $V$, $\Phi_{re,+}$ be the set of positive real roots, $W$ be the Weyl group with length function $\ell$. Then

$$\Delta_{re,t} := \prod_{a \in \Phi_{re,+}} (1 - te^{-a})$$

is a unit in the ring $\mathbb{Z}[[t]][[Q_-]] := \mathbb{Z}[[t]][[e^{-a_1}, \cdots, e^{-a_l}]]$, and

$$\frac{\Delta_{re,t}}{\Delta_{re,1}} = \prod_{a \in \Phi_{re,+}} \frac{1 - te^{-a}}{1 - e^{-a}}$$

is an element in the same ring by formally expanding all the denominators in terms of negative roots. There is no reasonable $W$-action on this ring, but we can formally define $w\left(\frac{\Delta_{re,t}}{\Delta_{re,1}}\right)$ by
in \[ 12 \]. In the affine case, \( m \) possibly non-reduced affine root systems and gave explicit formulas for \( ( \Phi_1(w) \) is the Poincaré series \( \Phi_1(w) \) is the quotient of two polynomials in \( w \) and \( e^w \). It can be proved that \( \sum_{w \in W} w (\frac{\Delta_{re,t}}{\Delta_{re,1}}) \) is a well-defined element in the ring \( \mathbb{Z}[[t]][[Q_-]] \) (see \[14\]), and the “constant term” of \( \sum_{w \in W} w (\frac{\Delta_{re,t}}{\Delta_{re,1}}) \) is the Poincaré series \( W(t) := \sum_{w \in W} t^{\ell(w)} \) of the Coxeter group \( W \), i.e. if we set all the formal exponentials \( e^{-a} \) to be zero, then the summation just gives \( W(t) \). It is known that \( W(t) \in 1 + t\mathbb{Z}[[t]] \) is “rational”, namely it is the quotient of two polynomials in \( \mathbb{Z}[t] \) (c.f. \[17\]). In particular, \( \sum_{w \in W} w (\frac{\Delta_{re,t}}{\Delta_{re,1}}) \) is a unit in the ring \( \mathbb{Z}[[t]][[Q_-]] \), so there exists \( m \in \mathbb{Z}[[t]][[Q_-]] \) such that

\[
m \sum_{w \in W} w (\frac{\Delta_{re,t}}{\Delta_{re,1}}) = W(t)
\]

If the Kac-Moody root system \( (V, \Delta) \) is of finite type, then Macdonald proved that \( m = 1 \) in \[12\]. In the affine case, \( m \) is first studied also by Macdonald in \[11\], where he considered possibly non-reduced affine root systems and gave explicit formulas for \( m \) case by case. Note that the computation is equivalent to the Macdonald constant term conjecture \[13\], which is proved by Cherednik \[6\]. For example, if the affine root system \( (V, \Delta) \) is of type \( A_1^{(1)} \) in the sense of \[7\], where the simple roots are given by \( \Delta = \{a_0 = c - \alpha, a_1 = \alpha\} \), the corresponding \( m \) takes the following form

\[
m = \frac{(qt, qt^{-1}, q)_{\infty}}{(q, qt^2, q)_{\infty}} \in \mathbb{Z}[[t]][[q]]
\]

where \( q = e^c \) and

\[
(x_1, x_2, \cdots, x_r; q)_{\infty} = \prod_{i=1}^{r} \prod_{k \geq 0} (1 - x_i q^k)
\]
c.f. [11].

In the general Kac-Moody case \( m \) is called the **Kac-Moody correction factor**. The affine correction factor appears in the affine analogue of the Gindikin-Karpelevich formula [3, 4], the Macdonald formula [5], and the Casselman-Shalika formula [15] in the works towards a generalization of \( p \)-adic spherical theory to affine Kac-Moody groups. Subsequently the Kac-Moody correction factor also appeared in the generalizations of the above works to arbitrary \( p \)-adic Kac-Moody groups [1, 2, 16].

The Kac-Moody correction factor \( m \) was first studied in [14] and later in [9]. In [14] the authors considered a formal product expansion

\[
m = \prod_{\lambda \in Q^-} \prod_{n \geq 0} (1 - t^n e^{\lambda})^{-m(\lambda, n)}
\]

of the correction factor, gave a generalized Peterson algorithm to compute \( m \) recursively, and they showed that \( m_0(t) = \sum_{n \geq 0} m(\lambda, n) t^n \) are polynomials and gave a closed formula for this polynomial. In the paper [9] the authors proved that \( m^{-1} \) times a certain infinite product over imaginary roots can be expressed as an infinite \( \mathbb{Z}[t] \)-linear combination of irreducible characters of the corresponding Kac-Moody algebra, and they studied the coefficients. Note that in [9, 14], the correction factor was treated as a formal power series.

In this paper we will study the inverse \( m^{-1} \) of the correction factor from an analytic perspective. In view of the further goal of generalizing the \( p \)-adic spherical theory and even the theory of automorphic forms to Kac-Moody groups, eventually this correction factor should be “evaluated” on the parameters of a principal series representation. Our main result is that \( m^{-1} \) converges to a holomorphic function on the union of all \( W \)-translates of the domain \( V + iC \) in \( V \otimes \mathbb{C} = V + iV \) (throughout this article, for subsets \( A, B \) of \( V \), \( A + iB \) denotes the set \( \{x + iy \in V \otimes \mathbb{C} = V + iV : x \in A, y \in B \} \)), where \( C \) is the dominant chamber in \( V \), and it can be analytically continued to a Weyl group invariant holomorphic function on the domain \( \Omega = V + iI^o \), where \( I^o \) is the interior of the Tits cone. Note that the characters of irreducible highest weight modules of a symmetrizable Kac-Moody Lie algebra with dominant integral highest weights converges absolutely to a Weyl group invariant holomorphic function on the domain \( \Omega \) [8, 10].

### 1.2 The inverse of the correction factor as an analytic function

In this section we will describe the ideas of the proof of our main result. Note that we are going to change into the coroot version of the correction factor in accordance with the formulas in [2, 5, 16], etc.

By the definition (1.1), the inverse of the correction factor is

\[
m^{-1} = W(t)^{-1} \left( \sum_{w \in W} \prod_{b \in \Phi(w)} \frac{t - e^{b\vee}}{1 - te^{b\vee}} \right) \cdot \left( \prod_{a \in \Phi_{re,+}} \frac{1 - te^{-a\vee}}{1 - e^{-a\vee}} \right)
\]

We view the formal exponential \( e^{-a\vee} \) for \( a \in \Phi_{re,+} \) as a function on \( V \otimes \mathbb{C} \) whose value on \( h \in V \otimes \mathbb{C} \) is \( e^{2\pi i \langle h, a\vee \rangle} \), so we hope that

\[
\mathcal{C}(t, h) = \left( \sum_{w \in W} \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i \langle h, b\vee \rangle}}{1 - te^{2\pi i \langle h, b\vee \rangle}} \right) \cdot \left( \prod_{a \in \Phi_{re,+}} \frac{1 - te^{2\pi i \langle h, a\vee \rangle}}{1 - e^{2\pi i \langle h, a\vee \rangle}} \right) \tag{1.3}
\]

defines a function on a subset of \( \mathbb{C} \times (V + iV) \).
The main work of this article is to prove that $C(t, h)$ “defines” a complex analytic function for $t \in D_r$ and $h \in \Omega$, where $r$ is the radius of convergence of $W(t)$ and $D_r$ is the open disk $\{|t| < r\}$. More precisely, we prove that the above defined $C(t, h)$ is absolutely convergent for $(t, h) \in D_r \times (V + i W \cdot C)$, uniformly on compact subsets, where $W \cdot C \subseteq V$ is the union of all $W$-translates of $C$. It is an open dense subset of $\Omega$, so $C(t, h)$ is a holomorphic function on $D_r \times (V + i W \cdot C)$. Then we prove that $C(t, h)$ can be analytically continued to a holomorphic function on the domain $D_r \times \Omega$.

Now we begin to sketch the proof of the main result. We start by proving that $C(t, h)$ is absolutely convergent if $|t| < r$ and $h = x + iy \in V + i C$ where $C$ is the dominant chamber

$$C = \{ y \in V : \langle y, a^\vee \rangle > 0, \forall a \in \Delta \}$$

Indeed, the second bracket of (1.3) can be bounded by a geometric progression, and the first bracket of (1.3) can be compared to $W(t)$ since as the depth of the root $b$ tends to $\infty$, we have $|e^{2\pi i (h, b^\vee)}| \to 0$ and thus $\frac{t - e^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}} \to t$. Since $|\Phi(w)| = \ell(w)$, the term corresponding to $w \in W$ in the summation of the first bracket is close to $t^{\ell(w)}$ when the positive root $b$ goes deep. Moreover, this convergence is uniform on compact sets, so $C(t, h)$ converges to a holomorphic function on the domain $D_r \times (V + i C)$.

Note that the formal computation in (1.2) implies that for $(t, h) \in D_r \times (V + i C)$ we have

$$\prod_{a \in \Phi_{r, +}} \frac{1 - te^{2\pi i (h, wa^\vee)}}{1 - e^{2\pi i (h, wa^\vee)}} = \left( \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i (h, b^\vee)}}{1 - te^{2\pi i (h, b^\vee)}} \right) \left( \prod_{a \in \Phi_{r, +}} \frac{1 - te^{2\pi i (h, a^\vee)}}{1 - e^{2\pi i (h, a^\vee)}} \right)$$

Namely, the infinite product on the left hand side is well-defined since the right hand side is absolutely convergent. Moreover, for $(t, h) \in D_r \times (V + i C)$ we also have

$$\sum_{w \in W} \prod_{a \in \Phi_{r, +}} \frac{1 - te^{2\pi i (h, wa^\vee)}}{1 - e^{2\pi i (h, wa^\vee)}} = \left( \sum_{w \in W} \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i (h, b^\vee)}}{1 - te^{2\pi i (h, b^\vee)}} \right) \left( \prod_{a \in \Phi_{r, +}} \frac{1 - te^{2\pi i (h, a^\vee)}}{1 - e^{2\pi i (h, a^\vee)}} \right)$$

namely the left hand side summation is absolutely convergent uniformly on compact sets since the right hand side is so. Thus we have

$$C(t, h) = \sum_{w \in W} \prod_{a \in \Phi_{r, +}} \frac{1 - te^{2\pi i (h, wa^\vee)}}{1 - e^{2\pi i (h, wa^\vee)}}$$

for $(t, h) \in D_r \times (V + i C)$. This is a series of summation over $W$, so by $W$-invariance $C(t, h)$ actually defines a holomorphic function on $D_r \times (V + i W \cdot C)$.

However, $V + i W \cdot C \neq \Omega$ unless $(V, \Delta)$ is of finite type. In order to analytically continue the function $C(t, h)$ to the whole $\Omega$ we need to use the detailed description of $\Omega$ given by Lootjenga in [10]. We call a subset $X \subseteq \Delta$ $W$-finite if the subgroup $W_X$ in $W$ generated by fundamental reflections corresponding to simple roots in $X$ is finite. Following [10], for such a subset $X$ we define

$$F_X = \{ x \in V : \langle x, a^\vee \rangle = 0 \text{ for } a \in X, \langle x, a^\vee \rangle > 0 \text{ for } a \in \Delta - X \}$$
Then $\Omega$ is the union of $V + iW \cdot F_X$ for all $W$-finite $X \subseteq \Delta$. Moreover we have $F_X \subseteq \tilde{F}_Y$ iff. $Y \subseteq X$, so $V + iW \cdot C$ is an open dense subset of $\Omega$. We then extend $\mathcal{C}(t, h)$ to a holomorphic function on $D_r \times \Omega$.

For simplicity, in this introduction we will only consider the situation $h \in V + iF_X$, since other cases can be treated by applying $W$-translations. The formula (1.3) can not be used to define a function $\mathcal{C}(t, h)$ for $h \in V + iF_X$ because in the second bracket of (1.3) the denominator can be zero for some $h \in V + iF_X$. But it turns out that if we apply Macdonald’s identity (see (1.5) below) for the finite-dimensional root system $(V, X)$, the poles will cancel.

Let’s return to the formal sum

$$\sum_{w \in W} w \left( \prod_{a \in \Phi_{rc,+}} \frac{1 - te^{-a^\vee}}{1 - e^{-a^\vee}} \right)$$

and consider another decomposition of it. For each $W$-finite $X \subseteq \Delta$, we first sum over $W_X$, and make a formal computation as follows:

$$\sum_{w \in W} w \left( \prod_{a \in \Phi_{rc,+}} \frac{1 - te^{-a^\vee}}{1 - e^{-a^\vee}} \right) = \sum_{w_1 \in W^X} \sum_{w_2 \in W_X} \prod_{a \in \Phi_{rc,+}} \frac{1 - te^{-w_1 w_2 a^\vee}}{1 - e^{-w_1 w_2 a^\vee}}$$

$$= \sum_{w_1 \in W^X} \sum_{w_2 \in W_X} \prod_{a \in \Phi_{rc,+} - \Phi_{X,+}} \frac{1 - te^{-w_1 w_2 a^\vee}}{1 - e^{-w_1 w_2 a^\vee}} \prod_{a \in \Phi_{rc,+} \Phi_{X,+}} \frac{1 - te^{-w_1 a^\vee}}{1 - e^{-w_1 a^\vee}}$$

$$= \sum_{w_1 \in W^X} w_1 \left( \sum_{w_2 \in W_X} \prod_{a \in \Phi_{X,+}} \frac{1 - te^{-w_1 a^\vee}}{1 - e^{-w_1 a^\vee}} \right) \prod_{a \in \Phi_{rc,+} - \Phi_{X,+}} \frac{1 - te^{-w_1 a^\vee}}{1 - e^{-w_1 a^\vee}}$$

(1.4)

where $W^X$ is the set of minimal length representatives for $W / W_X$, $\Phi_{X,+}$ is the set of positive roots for the root system $(V, X)$, and the last equality is because each $w_2 \in W_X$ permutes the positive roots in $\Phi_{rc,+} - \Phi_{X,+}$ (since the positive roots inverted by $w_2$ all lies in $\Phi_{X,+}$).

A main result in [12] is that

$$\sum_{w_2 \in W_X} \prod_{a \in \Phi_{X,+}} \frac{1 - te^{-w_2 a^\vee}}{1 - e^{-w_2 a^\vee}} = W_X(t)$$

(1.5)

where $W_X(t)$ is the Poincaré polynomial of the Coxeter group $W_X$, which is a polynomial in $t$ with integral coefficients since $W_X$ is finite, thus the the right hand side of (1.4) is equal to

$$W_X(t) \sum_{w_1 \in W^X} \prod_{a \in \Phi_{rc,+} - \Phi_{X,+}} \frac{1 - te^{-w_1 a^\vee}}{1 - e^{-w_1 a^\vee}}$$

$$= W_X(t) \left( \sum_{w \in W_X} \prod_{b \in \Phi(w)} \frac{t - e^{-b^\vee}}{1 - te^{-b^\vee}} \right) \left( \prod_{a \in \Phi_{rc,+} - \Phi_{X,+}} \frac{1 - te^{-a^\vee}}{1 - e^{-a^\vee}} \right)$$

(1.6)
In this formal expression, every term in the denominator has no zero points in \( V + i F_X \). So we define

\[
C_X(t, h) = W_X(t) \left( \sum_{w \in W_X} \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i \langle h, b^\vee \rangle}}{1 - te^{2\pi i \langle h, b^\vee \rangle}} \right) \left( \prod_{a \in \Phi_{re,+} \setminus \Phi_{X,+}} \frac{1 - te^{2\pi i \langle h, a^\vee \rangle}}{1 - e^{2\pi i \langle h, a^\vee \rangle}} \right)
\]  

(1.7)

We can prove the convergence of the right hand side of (1.7). Our proof is similar to that of the convergence of \( C(t, h) \), but is technically more involved. In the proof we need to use a relation between the root systems \((V, \Delta)\) and \((V, X)\) (see Lemma 3.2.1 below). Also the formal computation (1.6) yields

\[
C_X(t, h) := W_X(t) \sum_{w \in W_X} \prod_{a \in \Phi_{re,+} \setminus \Phi_{X,+}} \frac{1 - te^{2\pi i \langle h, wa^\vee \rangle}}{1 - e^{2\pi i \langle h, wa^\vee \rangle}}
\]  

(1.8)

Moreover, the right hand side of (1.7) is also convergent on each \( V + i F_Y \) with \( Y \subseteq X \), and the convergence is uniform on compact subsets of \( V + i \bigcup_{Y \subseteq X} F_Y \). By absolute convergence we have \( C(t, -) = C_X(t, -) \) on \( V + i C \). Thus \( C_X(t, h) \) can be viewed as an extension of the function \( C(t, h) \) to the boundary component \( V + i \bigcup_{Y \subseteq X} F_Y \) of \( I^\circ \). But this is not an open set, so we do not get an analytic continuation at this point.

In order to get an analytic continuation, we recall from [10] that the star of \( F_X \) is the set

\[
S_X = \bigcup_{w \in W_X} \bigcup_{Y \subseteq X} w F_Y
\]

This is open [10], and we consider the open domain \( V + i W \cdot S_X \) in \( I^\circ \). By the \( W \)-invariance of \( C_X(t, -) \), we can prove that \( C_X(t, h) \) actually defines a \( W \)-invariant holomorphic function on \( V + i W \cdot S_X \), hence is an analytic continuation of \( C(t, -) \). By putting all the \( C_X \) together, we have defined a holomorphic function on \( \Omega \).

This article will be organized as follows: in sect. 2 we will recall Looijenga’s study of the structure of the interior of the Tits cone, and in sect. 3 we will prove the main theorem following the above recipe.

**2 Kac-Moody root systems and interior of the tits cone**

In this section we briefly review Kac-Moody root systems, and then review the description of the interior of the Tits cone following [10]. We also recall some properties of Coxeter groups which will be used in sect. 3.

**2.1 Kac-Moody root systems**

**Definition 2.1.1** A Kac-Moody root system is a quadruple \((V, \Delta, V^*, \Delta^\vee)\) where

- \( V \) is a finite-dimensional real vector space.
- \( \Delta \) is a finite linearly independent subset of \( V \).
- \( V^* \) is the linear dual of \( V \).
\begin{itemize}
  \item $\Delta^\vee$ is a finite linearly independent subset of $V$ with a bijection $\vee : \Delta \to \Delta^\vee$. We will denote the image of $a \in \Delta$ under $\vee$ by $a^\vee$, and denote the image of a subset $X \subseteq \Delta$ under $\vee$ by $X^\vee$.
\end{itemize}

satisfying the following axioms:

(R1) $\langle a, a^\vee \rangle = 2$ for $a \in \Delta$, where $\langle -, - \rangle$ is the natural pairing between $V$ and $V^*$.

(R2) $\langle a, b^\vee \rangle$ is a non-positive integer for $a \neq b \in \Delta$.

(R3) $\langle a, b^\vee \rangle = 0$ implies $\langle b, a^\vee \rangle = 0$.

If $\Delta \neq 0$, we usually suppose $\Delta = \{a_1, \cdots, a_l\}$. Elements in $\Delta$ are sometimes called **simple roots**. Note that $(V^*, \Delta^\vee, V, \Delta)$ is also a Kac-Moody root system, called the dual of $(V, \Delta, V^*, \Delta^\vee)$. From now on, we will loosely refer to a Kac-Moody system only by the pair $(V, \Delta)$.

**Definition 2.1.2** For $a \in \Delta$, define the corresponding **fundamental reflection** $s_a : V \to V$ by $s_a(v) = v - \langle v, a^\vee \rangle a$. The **Weyl group** $W$ of the Kac-Moody root system $(V, \Delta)$ is the subgroup of $\text{Aut}(V)$ generated by the fundamental reflections corresponding to simple roots. We let $W$ acts on $V^*$ by $\langle v, wI \rangle = \langle w^{-1}v, I \rangle$ for $v \in V, I \in V^*, w \in W$.

Let $\Phi_{re} \subseteq V$ be the set of real roots of the root system $(V, \Delta)$, $\Phi_{re,+}$ be the set of positive real roots in $\Phi_{re}$, namely the subset of real roots that can be written as a $\mathbb{Z}_{\geq 0}$-linear combination of simple roots.

**Remark 2.1.3** In [10], Looijenga used the word “root” for what is called real root here, because the imaginary roots do not play a role here.

### 2.2 The interior of the Tits cone

Let

$$C = \{x \in V : \langle x, a^\vee \rangle > 0 \text{ for all simple roots } a \in \Delta\}$$

be the dominant chamber. Recall that the Tits cone is defined to be the subset $I = \bigcup_{w \in W} wC$ of $V$. Let $I^\circ$ be the interior of the Tits cone. We will give a concrete description of $I^\circ$ in this section following [10].

For each subset $X \subseteq \Delta$, define

$$FX := \{x \in V : \langle x, a^\vee \rangle = 0 \text{ for } a \in X, \langle x, a^\vee \rangle > 0 \text{ for } a \in \Delta - X\}$$

then clearly we have a disjoint decomposition $C = \bigcup_{X \subseteq \Delta} FX$ and $FX \subseteq FC$ iff $Y \subseteq X$. In particular we have $F_{\emptyset} = C$. A **facet** is defined to be a $W$-translate of $FX$ for some $X \subseteq \Delta$. For $X \subseteq \Delta$, the **star** of $FX$, denoted $SX$, is defined to be the union of all facets which contains $FX$ in the closure. We recall some results from [10]:

**Lemma 2.2.1** ([10] (1.3)) If $w \in W$, $X \subseteq \Delta$ are such that $wFX \cap FC \neq \emptyset$, then $w \in WX$ and hence $w$ leaves $FX$ pointwise fixed. In particular, $C$ is a fundamental domain of the action of $W$ on $I$.

**Definition 2.2.2** A subset $X$ of $\Delta$ is called **$W$-finite** if $|WX| < \infty$.

**Lemma 2.2.3** For a $W$-finite subset $X \subseteq \Delta$, we have

$$SX = \bigcup_{w \in WX} \bigcup_{Y \subseteq X} wFY$$
Proof If $F_X \subseteq \overline{wF_Y} = \overline{wY}$, then $w^{-1}F_X \cap \overline{C} \neq \phi$, so by the above lemma, $w \in W_X$ and $w^{-1}F_X = F_X$, namely $F_X \subseteq \overline{F_Y}$, which implies $Y \subseteq X$. \hfill\Box

In view of this lemma, we define the set $\Gamma_X := \bigcup_{Y \subseteq X} F_Y$, then from definition we have

$$\Gamma_X = \{x \in V : \langle x, a \rangle \geq 0 \text{ for } a \in X, \langle x, a \rangle > 0 \text{ for } a \in \Delta - X\}$$

And Lemma 2.2.2 says that if $X \subseteq \Delta$ is $W$-finite, then $S_X = W_X \cdot \Gamma_X$ is the union of all $W_X$-translates of $\Gamma_X$, so

$$S_X = \{x \in V : \langle x, a \rangle > 0 \text{ for } a \in \Delta - X\}$$

In particular, $S_X$ is an open subset of $I^\circ$.

Lemma 2.2.4 ([10] (1.14)) $I^\circ$ is a disjoint union of facets with finite stabilizers in $W$. Namely, we have

$$I^\circ = \bigcup_{w \in W} \bigcup_{X \subseteq \Delta : |W_X| < \infty} wF_X$$

where $W_X$ is the subgroup of $W$ generated by simple reflections corresponding to elements in $X$. Moreover, $w_1F_X = w_2F_X$ iff. $w_1W_X = w_2W_X$.

Proof This follows from the above lemma. \hfill\Box

Corollary 2.2.5 Every point in $\Omega$ is a $W$-translate of a point in $V + iF_X$ for some $W$-finite subset $X \subseteq \Delta$.

2.3 Poincaré series of a Coxeter group

Let $(W, S)$ be a Coxeter system with length function $\ell$, with the set $S$ of simple reflections in $W$ finite. For any subset $V \subseteq W$, we define

$$V(t) := \sum_{w \in V} t^{\ell(w)}$$

the Poincaré series of $V$. We recall the following fundamental result in the theory of Coxeter groups:

Proposition 2.3.1 (Minimal Length Representatives) For any $X \subseteq S$, let $W_X$ be the corresponding parabolic subgroup generated by simple reflections in $X$, define

$$W^X := \{w \in W : \ell(ws) > \ell(w), \forall s \in X\}$$

Then $W^X$ is a set of coset representatives of $W/W_X$. Each element $w' \in W^X$ is the unique element of minimal length in the coset $w'W_X$. For arbitrary $w' \in W^X$, $w \in W_X$, we have $\ell(w'w) = \ell(w') + \ell(w)$.

Proposition 2.3.2 ([17]) $W(t) \in 1 + t\mathbb{Z}[t]$ is rational, namely it is a quotient of two polynomials in $\mathbb{Z}[t]$. 

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3 Correction factor as complex analytic functions

3.1 Convergence of the \( W \)-summation with imaginary part in the dominant cone

We first prove locally uniform convergence of the function

\[
\mathcal{C}(t, h) = \left( \sum_{w \in W} \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}} \right) \left( \prod_{b \in \Phi_{re,+}} \frac{1 - t e^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}} \right) \quad (3.1)
\]

for \( t \in D_r \) and \( h \) in the region \( V + iC \). Then we prove that it also converges on \( W \)-translates of \( V + iC \) by using another expression of \( \mathcal{C}(t, h) \) in terms of a summation over \( W \).

**Proposition 3.1.1** Let \( r \) be the radius of convergence of \( W(t) \). For \( t \in D_r \) and \( h = x + iy \in V + iC \),

\[
\mathcal{C}(t, h) = \left( \sum_{w \in W} \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}} \right) \left( \prod_{b \in \Phi_{re,+}} \frac{1 - t e^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}} \right)
\]

is absolutely convergent uniformly on compact subsets of \( D_r \times (V + iC) \), defining a holomorphic function on the domain \( D_r \times (V + iC) \).

**Proof** First note that \( r \leq 1 \) since the series for \( W(t) \) diverges at \( t = 1 \). By \( h = x + iy \in V + iC \) we have

\[
|e^{2\pi i (h, b^\vee)}| = e^{-2\pi (y, b^\vee)} < 1 \quad \text{for } b \in \Phi_{re,+}
\]

so the denominators are all non-zero. We first prove the convergence of the infinite product

\[
\prod_{b \in \Phi_{re,+}} \frac{1 - e^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}}
\]

Recall that an infinite product \( \prod_{i \in I} (1 - a_i) \) is called **absolutely convergent** if the summation \( \sum_{i \in I} |a_i| \) is convergent. If an infinite product is absolutely convergent, then we can freely change the order of the product without affecting the limit of the product. So we consider the infinite product

\[
\prod_{b \in \Phi_{re,+}} (1 - t e^{2\pi i (h, b^\vee)})
\]

whose absolute convergence is guaranteed as long as the summation

\[
\sum_{b \in \Phi_{re,+}} |te^{2\pi i (h, b^\vee)}| = t \sum_{b \in \Phi_{re,+}} e^{-2\pi (y, b^\vee)}
\]

is convergent. Since every positive real coroot is a finite sum of the simple coroots in \( \Delta^\vee = \{a_1^\vee, \cdots, a_l^\vee\} \), we have

\[
\sum_{b \in \Phi_{re,+}} e^{-2\pi (y, b^\vee)} \leq \sum_{n_1, \cdots, n_l \in \mathbb{Z}_{\geq 0}} e^{-2\pi (y, n_1 a_1^\vee + \cdots + n_l a_l^\vee)}
\]

\[
= \prod_{i=1}^l \left( \sum_{n_i = 0}^{\infty} e^{-2\pi (y, a_i^\vee) n_i} \right) = \prod_{i=1}^l \frac{1}{1 - e^{-2\pi (y, a_i^\vee)}}
\]
Hence the infinite product \( \prod_{b \in \Phi_{re,+}} (1 - te^{2\pi i (h,b^\vee)}) \) is absolutely convergent for arbitrary \( t \in \mathbb{C} \) and \( h \in V + iC \). For \( h \) varying in compact subsets of \( V + iC \), the imaginary part \( y \) of \( h \) will vary in a compact subset of \( C \), so the linear functions \( \langle y, a_i^\vee \rangle \) have maximum and minimum and the summation \( \sum_{b \in \Phi_{re,+}} |te^{2\pi i (h,b^\vee)}| \) is thus uniformly absolutely convergent on compact subsets of \( Dr \times (V + iC) \). By taking \( t = 1 \), we see the product

\[
\prod_{b \in \Phi_{re,+}} \frac{1 - te^{2\pi i (h,b^\vee)}}{1 - e^{2\pi i (h,b^\vee)}}
\]

is absolutely convergent uniformly on compact subsets because the denominators are non-zero.

Next we prove the absolute convergence of the summation

\[
\sum_{w \in W} \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i (h,b^\vee)}}{1 - te^{2\pi i (h,b^\vee)}}
\]

The idea of the proof is, as the depth of the positive root \( b \) grow large, \( |e^{2\pi i (h,b^\vee)}| \) becomes small, then \( \frac{t - e^{2\pi i (h,b^\vee)}}{1 - te^{2\pi i (h,b^\vee)}} \) is close to \( t \), so we can try to compare the above sum with \( W(t) = \sum_{w \in W} t^\ell(w) \). So we fix \( 0 < r_0 < r_1 < r \), let \( \overline{D}_{r_0} \) be the closed disk \( \overline{D}_{r_0} = \{ t \in \mathbb{C} : |t| \leq r_0 \} \), let \( K \) be a compact subset of \( C \), we will prove the above summation is uniformly absolutely convergent for \( t \in \overline{D}_{r_0} \) and \( h \in V + iR \). Since \( \lim_{z \to 0} \frac{z - r}{1 - rz} = t \), there exists \( c > 0 \) such that for all \( |z| < c \) and \( t \in \overline{D}_{r_0} \), we have

\[
\frac{t - z}{1 - tz} < r_1
\]

Take \( N = -\frac{\log c}{2\pi} \). Note that each positive real coroot is a finite \( \mathbb{Z}_{\geq 0} \)-linear combination of simple coroots, and for each simple coroot \( a_i^\vee \) we have \( \min_{y \in R} \langle y, a_i^\vee \rangle > 0 \), there exists a finite subset \( S \subseteq \Phi_{re,+} \) of positive roots such that \( \langle y, b^\vee \rangle > N \) for all \( b \in \Phi_{re,+} \) and \( y \in R \). Then for \( b \in \Phi_{re,+} \) and \( y \in R \) we have \( |e^{2\pi i (x,b^\vee)} - 2\pi (y,b^\vee)| < c \) and

\[
\left| \frac{t - e^{2\pi i (x,b^\vee)}}{1 - te^{2\pi i (x,b^\vee)}} \right| = \left| \frac{t - e^{2\pi i (x,b^\vee)} - 2\pi (y,b^\vee)}{1 - te^{2\pi i (x,b^\vee)} - 2\pi (y,b^\vee)} \right| < r_1
\]

Since \( S \) is finite, there exists \( A > 0 \) such that for all \( b \in S \) we have

\[
\left| \frac{t - e^{2\pi i (h,b^\vee)}}{1 - te^{2\pi i (h,b^\vee)}} \right| \leq A
\]

So when \( \ell(w) > S \), the term

\[
\prod_{b \in \Phi(w)} \left| \frac{t - e^{2\pi i (x,b^\vee)} - 2\pi (y,b^\vee)}{1 - te^{2\pi i (x,b^\vee)} - 2\pi (y,b^\vee)} \right|
\]

is bounded by

\[
r^\ell(w) - |S| A^{|S|} = r^\ell(w) - |S| A^{|S|} r^\ell(w)
\]

uniformly for \( |t| \leq r_0 \) and \( y \in R \), since each term is a product of \( \ell(w) \) fractions, all but \( |S| \) of them are smaller than \( r_0 \), the rest are smaller than 1. The summation

\[
\sum_{w \in W} r^\ell(w)
\]
is convergent since \(0 < r_1 < r\). Thus the summation

\[
\sum_{w \in W} \prod_{a \in \Phi_{r, +}} \frac{1 - t e^{2\pi i (h, w a^\vee)}}{1 - e^{2\pi i (h, w a^\vee)}}
\]

is uniformly convergent on \(\mathbb{D}_{r_0} \times (V + i \mathbb{R})\) by Weierstrass test. So we proved that the summation \(\mathcal{C}(t, h)\) is absolutely convergent uniformly on compact subsets on \(D_r \times (V + i \mathbb{C})\), hence defining a holomorphic function on this domain.

\[\square\]

**Lemma 3.1.2** For \(t \in D_r\) and \(h = x + iy \in V + i \mathbb{C}\), the sum

\[
\sum_{w \in W} \prod_{a \in \Phi_{r, +}} \frac{1 - t e^{2\pi i (h, w a^\vee)}}{1 - e^{2\pi i (h, w a^\vee)}}
\]

is absolutely convergent uniformly on compact subsets of \(D_r \times (V + i \mathbb{C})\), and is equal to \(\mathcal{C}(t, h)\).

**Proof** For each \(w \in W\) we first prove that the term

\[
\prod_{a \in \Phi_{r, +}} \frac{1 - t e^{2\pi i (h, w a^\vee)}}{1 - e^{2\pi i (h, w a^\vee)}}
\]

is absolutely convergent uniformly on compact subsets of \(D_r \times (V + i \mathbb{C})\), hence defines a holomorphic function on this domain. By the formal computation (1.2) we have

\[
\prod_{a \in \Phi_{r, +}} \frac{1 - t e^{2\pi i (h, w a^\vee)}}{1 - e^{2\pi i (h, w a^\vee)}} = \left(\prod_{a \in \Phi_{r, +}, wa < 0} \frac{1 - t e^{2\pi i (h, w a^\vee)}}{1 - e^{2\pi i (h, w a^\vee)}}\right) \left(\prod_{a \in \Phi_{r, +}, wa > 0} \frac{1 - t e^{2\pi i (h, w a^\vee)}}{1 - e^{2\pi i (h, w a^\vee)}}\right)
\]

\[= \left(\prod_{b \in \Phi_{r, +}, w^{-1}b < 0} \frac{1 - t e^{-2\pi i (h, b^\vee)}}{1 - e^{-2\pi i (h, b^\vee)}}\right) \left(\prod_{b \in \Phi_{r, +}, w^{-1}b > 0} \frac{1 - t e^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}}\right)
\]

\[= \left(\prod_{b \in \Phi_{r, +}} \frac{1 - t e^{-2\pi i (h, b^\vee)}}{1 - e^{-2\pi i (h, b^\vee)}} \cdot \frac{1 - e^{2\pi i (h, b^\vee)}}{1 - t e^{2\pi i (h, b^\vee)}}\right) \left(\prod_{b \in \Phi_{r, +}} \frac{1 - t e^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}}\right)
\]

\[= \left(\prod_{b \in \Phi_{r, +}} \frac{1 - t e^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}}\right) \left(\prod_{b \in \Phi_{r, +}} \frac{1 - t e^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}}\right)
\]
Since the right hand side is absolutely convergent uniformly on compact subsets of $D_r \times (V + iC)$ by Proposition 3.1.1, so is the left hand side. Moreover, we have

$$
\sum_{w \in W} \prod_{a \in \Phi_{re,+}} \frac{1 - te^{2\pi i (h, wa^\vee)}}{1 - e^{2\pi i (h, wa^\vee)}} = \left( \sum_{w \in W} \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i (h, b^\vee)}}{1 - te^{2\pi i (h, b^\vee)}} \right) \left( \prod_{b \in \Phi_{re,+}} \frac{1 - te^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}} \right)
$$

also because the absolutely convergency of the right hand side, so the left hand side is equal to $C(t, h)$ and has the same convergency properties. \qed

**Corollary 3.1.3** Let $r$ be the radius of convergence of $W(t)$. For $t \in D_r$ and $h = x + iy$ with $y \in W \cdot C$, the sum

$$
C(t, h) = \sum_{w \in W} \prod_{a \in \Phi_{re,+}} \frac{1 - te^{2\pi i (h, wa^\vee)}}{1 - e^{2\pi i (h, wa^\vee)}}
$$

is absolutely convergent uniformly on compact subsets of $D_r \times (V + iW \cdot C)$, defining a holomorphic function on the domain $D_r \times (V + iW \cdot C)$. 

**Proof** Write $h = w'^{-1}h_0$ with $h_0 = x_0 + iy_0$ for $x_0 \in V$, $y_0 \in C$. Then we have

$$
\sum_{w \in W} \prod_{a \in \Phi_{re,+}} \frac{1 - te^{2\pi i (h, wa^\vee)}}{1 - e^{2\pi i (h, wa^\vee)}} = \sum_{w \in W} \prod_{a \in \Phi_{re,+}} \frac{1 - te^{2\pi i (h_0, w'wa^\vee)}}{1 - e^{2\pi i (h_0, w'wa^\vee)}}
$$

which is equal to a change of order of the summation

$$
\sum_{w \in W} \prod_{a \in \Phi_{re,+}} \frac{1 - te^{2\pi i (h_0, wa^\vee)}}{1 - e^{2\pi i (h_0, wa^\vee)}}
$$

which is absolutely convergent by Lemma 3.1.2. So one can change the order of summation. The rest is clear. \qed

### 3.2 A lemma on Kac-Moody root systems

We have defined a holomorphic function $C(t, h)$ on $D_r \times (V + iW \cdot C)$. The next task is to find an analytic continuation of $C(t, h)$ to $D_r \times \Omega$. We will use a factorization similar to (3.1) to define the candidate for the analytic continuation on the “boundary components”, and prove the convergence of it. Our method is similar to the proof of Proposition 3.1.1, but is more technical. The following lemma is crucial for the proof of convergence.

**Lemma 3.2.1** 1. For $W$-finite $X \subseteq \Delta$, assume $\Delta = \{a_1, \cdots, a_l\}$ and $X = \{a_1, \cdots, a_k\}$, then for each tuple $(p_{k+1}, \cdots, p_l)$ of non-negative integers, there are only finitely many choices of non-negative integers $n_1, \cdots, n_k$ such that $n_1a_1 + \cdots + n_ka_k + p_{k+1}a_{k+1} + \cdots + p_la_l$ is a root in $\Phi_{re,+}$. Moreover, this number is of polynomial growth in $p_{k+1}, \cdots, p_l$.

2. For any compact subset $R \subseteq \Gamma_X$ and $N > 0$ there exists a finite subset $S \subseteq \Phi_{re,+}$ of positive coroots such that $\langle y, b^\vee \rangle > N$ for all $b^\vee \in \Phi_{re,+}^\vee - S$ and $y \in R$. 

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Proof (1). Let \( \lambda = p_{k+1}a_{k+1} + \cdots + p_la_l \),

\[
S = \left\{ \alpha \in \Phi_{re,+} : \alpha = \sum_{i=1}^{k} n_i a_i + \lambda \text{ with } n_i \in \mathbb{Z}_{\geq 0}, \ i = 1, \cdots, k \right\}
\] (3.2)

We are going to prove that \( S \) is a finite set. If \( \lambda = 0 \), this is obvious, since \( S \) is exactly the set \( \Phi_{X,+} \) of positive roots of the root subsystem \((X, V)\), which is finite because \( X \) is \( W \)-finite. So we assume that \( \lambda \neq 0 \) from now on. Since \( W_X \) is a finite Coxeter group, it has a long element \( w_X \), which is the unique element in \( W_X \) with maximal length. For any \( \alpha \in S \), we have \( w_X(\alpha) \in \Phi_{re,+} \) since the positive roots that are sent to negative by \( w_X \) are exactly the positive roots in \( \Phi_{X,+} \), which are \( \mathbb{Z}_{\geq 0} \)-linear combinations of \( a_1, \cdots, a_k \). So

\[
w_X(\alpha) = n_1 w_X(a_1) + \cdots + n_k w_X(a_k) + p_{k+1} w_X(a_{k+1}) + \cdots + p_l w_X(a_l)
\]
is a \( \mathbb{Z}_{\geq 0} \)-linear combination of \( a_1, \cdots, a_l \). If for \( i = 1, \cdots, l \) we suppose

\[
w_X(a_i) = \sum_{j=1}^{l} c_{ij} a_j
\]
the \( c_{ij} \)'s are integers determined by the \( W \)-finite subset \( X \). Since \( w_X(a_i) \) are negative for \( i = 1, \cdots, k \) and positive for \( i = k+1, \cdots, l \), we have \( c_{ij} \leq 0 \) for \( i = 1, \cdots, k \) and \( c_{ij} \geq 0 \) for \( i = k+1, \cdots, l \). Then

\[
w_X(\alpha) = \left( \sum_{i=1}^{k} n_i c_{i1} + \sum_{j=k+1}^{l} p_j c_{ji} \right) a_1 + \cdots + \left( \sum_{i=1}^{k} n_i c_{ik} + \sum_{j=k+1}^{l} p_j c_{jk} \right) a_k
\]

\[
+ \left( \sum_{j=k+1}^{l} p_j c_{j,k+1} \right) a_{k+1} + \cdots + \left( \sum_{j=k+1}^{l} p_j c_{j,l} \right) a_l
\]
is a positive root, so we have

\[
\sum_{i=1}^{k} (-c_{is}) n_i \leq \sum_{j=k+1}^{l} p_j c_{js}
\]
for \( s = 1, \cdots, k \). Clearly, the number of non-negative integer solutions of such a linear system of inequalities can be bounded by

\[
\left( \max \left\{ \sum_{j=k+1}^{l} p_j c_{j1}, \cdots, \sum_{j=k+1}^{l} p_j c_{jk} \right\} + 2 \right)^k
\]
which can be bounded by a polynomial in \( p_{k+1}, \cdots, p_l \) of degree \( k \).

(2). We write each positive coroot \( b^{\vee} = n_1 a_1^{\vee} + \cdots + n_l a_l^{\vee} \) with \( n_1, \cdots, n_l \in \mathbb{Z}_{\geq 0} \), then for \( y \in \Gamma_X \) we have

\[
\langle y, b^{\vee} \rangle = \sum_{i=1}^{l} n_i \langle y, a_i^{\vee} \rangle \geq \sum_{i=k+1}^{l} n_i \langle y, a_i^{\vee} \rangle
\]
with each \( \langle y, a_i^{\vee} \rangle > 0 \). Since \( R \) is compact, the functions \( y \mapsto \langle y, a_i^{\vee} \rangle \) has maximum and minimum for \( y \in R \), let \( m_i(R) \) be its minimum, then \( m_i(R) > 0 \) for \( i = k + 1, \cdots, l \). There exists only finitely many choices of non-negative integers \( p_{k+1}, \cdots, p_l \) such that
\[ \sum_{i=k+1}^{l} n_i m_i(R) \leq N, \text{ let } S \text{ be the set of all positive coroots } a^\vee = n_1 a_1^\vee + \cdots + n_k a_k^\vee + p_{k+1} a_{k+1}^\vee + \cdots + p_l a_l^\vee \text{ such that } p_{k+1}, \cdots, p_l \text{ is one of the previous choices. By part (1) of this lemma, } S \text{ is finite, and for any positive coroot } b^\vee = n_1 a_1^\vee + \cdots + n_k a_k^\vee + p_{k+1} a_{k+1}^\vee + \cdots + n_l a_l^\vee \text{ which does not belong to } S \text{ and } y \in R \text{ we have} \]

\[ \langle y, b^\vee \rangle = \sum_{i=1}^{l} n_i \langle y, a_i^\vee \rangle \geq \sum_{i=k+1}^{l} n_i \langle y, a_i^\vee \rangle \geq \sum_{i=k+1}^{l} n_i m_i(R) > N \]

Hence the result. \hfill \Box

### 3.3 Analytic continuation to \( \Omega \)

In this section we first define an analytic continuation \( C_X(t, -) \) of \( C(t, -) \) to the open sets \( V + iW \cdot S_X \) for the \( W \)-finite subsets \( X \subseteq \Delta \). We prove that \( C_X(t, -) \) is \( W \)-invariant, so by putting the functions \( C_X \) together, we get a holomorphic function on \( \Omega \).

**Proposition 3.3.1** Let \( X \subseteq \Delta \) be a \( W \)-finite subset. For \( t \in \mathbb{C}, h = x + iy \in V + i\Gamma_X \), the infinite product

\[ \prod_{a \in \Phi_{re,+}} (1 - i e^{2\pi i (h, a^\vee)}) \]

as a function in \( t \) and \( h \) is absolutely convergent uniformly on compact subsets in \( \mathbb{C} \times (V + i\Gamma_X) \).

**Proof** We still assume \( X = \{ a_1, \cdots, a_k \} \subseteq \Delta = \{ a_1, \cdots, a_l \} \). We need to prove the absolute convergence of the summation

\[ \sum_{a \in \Phi_{re,+}} t e^{2\pi i (x, a^\vee) - 2\pi \langle y, a^\vee \rangle} \]

namely we need to prove

\[ \sum_{a \in \Phi_{re,+}} e^{-2\pi \langle y, a^\vee \rangle} \]

is convergent, uniformly for \( y \) in compact subsets of \( \Gamma_X \).

So let \( R \) be a compact subset of \( \Gamma_X \). Each positive root is a \( \mathbb{Z}_{\geq 0} \)-linear combination of simple roots, so for \( y \in \Gamma_X \) we have

\[ \sum_{a \in \Phi_{re,+}} e^{-2\pi \langle y, a^\vee \rangle} = \sum_{n_1, \cdots, n_k, p_{k+1}, \cdots, p_l \in \mathbb{Z}_{\geq 0}} e^{-2\pi \langle y, n_1 a_1^\vee + \cdots + n_k a_k^\vee + p_{k+1} a_{k+1}^\vee + \cdots + p_l a_l^\vee \rangle} \]

\[ \leq \sum_{p_{k+1}, \cdots, p_l \in \mathbb{Z}_{\geq 0}} c(p_{k+1}, \cdots, p_l) e^{-2\pi \langle y, a^{\vee}_{k+1} + \cdots + p_l a_l^\vee \rangle} \]

where \( c(p_{k+1}, \cdots, p_l) \) is the cardinality of the set

\[ \{(n_1, \cdots, n_k) \in \mathbb{Z}_{\geq 0}^k : n_1 a_1^\vee + \cdots + n_k a_k^\vee + p_{k+1} a_{k+1}^\vee + \cdots + p_l a_l^\vee \in \Phi_{re,+} \} \]

\( \Box \)
which is of polynomial growth in \( p_{k+1}, \ldots, p_l \) by Lemma 3.2.1 (1). So there exists \( \delta > 0, N > 0 \) such that for \( p_{k+1}, \ldots, p_l > N \), we have
\[
  c(p_{k+1}, \ldots, p_l)e^{2\pi(y, p_{k+1}a^{(1)}_{k+1} + \cdots + p_l a^{(1)})} \leq e^{2\pi(1-\delta)(y, p_{k+1}a^{(1)}_{k+1} + \cdots + p_l a^{(1)})}
\]
for all \( y \in R \) and the summation of the right hand side is convergent because it is a product of some geometric progressions. Hence the result. \( \square \)

**Corollary 3.3.2** Let \( X \subseteq \Delta \) be a \( W \)-finite subset. For \( t \in \mathbb{C}, h = x + i y \in V + i \Gamma_X \), the infinite product
\[
  \prod_{a \in \Phi_{re,+}-\Phi_{X,+}} \frac{1 - te^{2\pi i (h,a^\vee)}}{1 - e^{2\pi i (h,a^\vee)}}
\]
as a function in \( t \) and \( h \) is absolutely convergent uniformly on compact subsets of \( \mathbb{C} \times (V + i \Gamma_X) \).

**Proof** If \( 1 - e^{2\pi i (h,a^\vee)} = 0 \), then \( (h,a^\vee) = 0 \), so \( a^\vee \in \Phi_{X,+} \). After excluding all the roots in \( \Phi_{X,+} \), the denominator will never be zero. By Proposition 3.3.1, the infinite products
\[
  \prod_{a \in \Phi_{re,+}} (1 - te^{2\pi i (h,a^\vee)}) \quad \text{of numerators and} \quad \prod_{a \in \Phi_{re,+}} (1 - e^{2\pi i (h,a^\vee)}) \quad \text{of denominators are both absolutely convergent as infinite products (namely the second product of the denominators does not diverge to 0). Thus the original infinite product is absolutely convergent.}
\]
\( \square \)

**Lemma 3.3.3** Let \( X \subseteq \Delta \) be a \( W \)-finite subset. Let \( r \) be the radius of convergence of \( W(t), D_r := \{ z \in \mathbb{C} : |z| < r \} \). For \( t \in D_r \) and \( h = x + iy \in V + i \Gamma_X \), the sum
\[
  \sum_{w \in W} \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i (h,b^\vee)}}{1 - te^{2\pi i (h,b^\vee)}}
\]
as a function in \( t \) and \( h \) is absolutely convergent uniformly on compact subsets of \( D_r \times (V + i \Gamma_X) \).

**Proof** Any compact subset of \( D_r \times (V + i \Gamma_X) \) is contained in a set of the form \( \overline{D}_{r_0} \times (V + i R) \) for \( 0 < r_0 < r \) and \( R \subseteq \Gamma_X \) compact where \( \overline{D}_{r_0} = \{ t \in \mathbb{C} : |t| \leq r_0 \} \). We will prove uniform absolute convergence on this set.

Take \( r_1 \in (r_0, r) \). Since \( \lim_{z \to 0} \frac{t-z}{1-tz} = t \), there exists \( c > 0 \) such that for all \( |z| < c \) and \( t \in \overline{D}_{r_0} \), we have
\[
  \left| \frac{t - z}{1 - tz} \right| < r_1
\]
Take \( N = \frac{-\log c}{2\pi} \), by Lemma 3.2.1 (2), let \( S \subseteq \Phi_{re,+} \) be a finite set of positive roots such that \( (y, b^\vee) > N \) for all \( b \in \Phi_{re,+} - S \) and \( y \in R \). Then for \( b \in \Phi_{re,+} - S \) and \( y \in R \) we have \( |e^{2\pi i (x,b^\vee)} - 2\pi (y,b^\vee)| < c \) and
\[
  \left| \frac{t - e^{2\pi i (h,b^\vee)}}{1 - te^{2\pi i (h,b^\vee)}} - \frac{t - e^{2\pi i (x,b^\vee)} - 2\pi (y,b^\vee)}{1 - te^{2\pi i (x,b^\vee)} - 2\pi (y,b^\vee)} \right| < r_1
\]
Since \( S \) is finite, there exists \( A > 0 \) such that for all \( b \in S \) we have
\[
  \left| \frac{t - e^{2\pi i (h,b^\vee)}}{1 - te^{2\pi i (h,b^\vee)}} \right| \leq A
\]
\( \square \)
So when $\ell(w) > |S|$, the term
\[
\prod_{b \in \Phi(w)} \frac{t - e^{2\pi i (x,b') - 2\pi (y,b')}}{1 - te^{2\pi i (x,b') - 2\pi (y,b')}}
\]
is bounded by
\[
r_1^{\ell(w)-|S|} A^{|S|} = r_1^{-|S|} A^{|S|} r_1^{\ell(w)}
\]
uniformly for $|t| \leq r_0$ and $y \in R$, since each term is a product of $\ell(w)$ fractions, all but $|S|$ of them are smaller than $r_1$, the rest are smaller than $A$. The summation
\[
\sum_{w \in W} r_1^{\ell(w)}
\]
is convergent since $0 < r_1 < r$. Thus the summation
\[
\sum_{w \in W, \ell(w) > |S|} \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i (x,b') - 2\pi (y,b')}}{1 - te^{2\pi i (x,b') - 2\pi (y,b')}}
\]
is uniformly convergent on $\overline{D_{r_0}} \times (V + iR)$ by Weierstrass test. \hfill \Box

For each $W$-finite subset $X \subseteq \Delta$, we introduce the following function
\[
C_X(t,h) = W_X(t) \left( \sum_{w \in W^X} \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i (h,b')}}{1 - te^{2\pi i (h,b')}} \right) \left( \prod_{b \in \Phi_{r_0} - \Phi_X^+} \frac{1 - e^{2\pi i (h,b')}}{1 - e^{2\pi i (h,b')}} \right)
\]
By corollary 3.3.2 and Lemma 3.3.3 this is a well-defined function on $D_r \times (V + i\Gamma_X)$. Note that for $X = \phi$, we have $C_X(t,h) = C(t,h)$.

**Proposition 3.3.4** Let $X$ be a $W$-finite subset of $\Delta$. For $t \in D_r$ and $h \in V + i\Gamma_X$, the summation
\[
\sum_{w \in W^X} \prod_{a \in \Phi_{r_0} - \Phi_X^+} \frac{1 - te^{2\pi i (h,wa')}}{1 - e^{2\pi i (h,wa')}}
\]
is absolutely convergent uniformly on compact subsets of $D_r \times (V + i\Gamma_X)$, and is equal to
\[
\left( \sum_{w \in W^X} \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i (h,b')}}{1 - te^{2\pi i (h,b')}} \right) \left( \prod_{b \in \Phi_{r_0} - \Phi_X^+} \frac{1 - e^{2\pi i (h,b')}}{1 - e^{2\pi i (h,b')}} \right)
\]
**Proof** By the definition of $W^X$ (see Proposition 2.3.1), for any $w \in W^X, a \in X$ we have $wa > 0$, so we have $w \Phi_X^+ = \Phi_X^+$, namely $\Phi(w) \cap \Phi_X^+ = \phi$, then by similar formal computation as in (1.2), we have
\[
\sum_{w \in W^X} \prod_{a \in \Phi_{r_0} - \Phi_X^+} \frac{1 - te^{2\pi i (h,wa')}}{1 - e^{2\pi i (h,wa')}}
\]
\[
= \sum_{w \in W^X} \left( \prod_{a \in \Phi_{r_0} - \Phi_X^+, wa < 0} \frac{1 - te^{2\pi i (h,wa')}}{1 - e^{2\pi i (h,wa')}} \right) \left( \prod_{a \in \Phi_{r_0} - \Phi_X^+, wa > 0} \frac{1 - te^{2\pi i (h,wa')}}{1 - e^{2\pi i (h,wa')}} \right)
\]

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= \sum_{w \in W} \left( \prod_{b \in \Phi_{re,+} - \Phi_{rX,+}, w^{-1} b > 0} \frac{1 - te^{-2\pi i (h, b^\vee)}}{1 - e^{-2\pi i (h, b^\vee)}} \right) \left( \prod_{b \in \Phi_{re,+} - \Phi_{rX,+}, w^{-1} b > 0} \frac{1 - te^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}} \right)

= \sum_{w \in W} \left( \prod_{b \in \Phi(w)} \frac{1 - te^{-2\pi i (h, b^\vee)}}{1 - e^{-2\pi i (h, b^\vee)}} \right) \left( \prod_{b \in \Phi_{re,+} - \Phi_{rX,+}, w^{-1} b > 0} \frac{1 - te^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}} \right)

= \left( \sum_{w \in W} \prod_{b \in \Phi(w)} \frac{t - e^{2\pi i (h, b^\vee)}}{1 - te^{2\pi i (h, b^\vee)}} \right) \left( \prod_{b \in \Phi_{re,+} - \Phi_{rX,+}} \frac{1 - te^{2\pi i (h, b^\vee)}}{1 - e^{2\pi i (h, b^\vee)}} \right)

By Proposition 3.3.4, we have

\[ C_X(t, h) = W_X(t) \sum_{w \in W} \prod_{a \in \Phi_{re,+} - \Phi_{rX,+}} \frac{1 - te^{2\pi i (h, wa^\vee)}}{1 - e^{2\pi i (h, wa^\vee)}} \]

**Proposition 3.3.5** For \( t \in D_r, h \in V + iC \) we have \( C_X(t, h) = C(t, h) \).

**Proof** Because \( C(t, h) \) is absolute convergence for \( h \in V + iC \), we can change the order of summation and change the order in each infinite product, namely we have

\[
\sum_{w \in W} \prod_{a \in \Phi_{re,+}} \frac{1 - te^{2\pi i (h, wa^\vee)}}{1 - e^{2\pi i (h, wa^\vee)}} = \sum_{w_1 \in W} \sum_{w_2 \in W} \prod_{a \in \Phi_{re,+}} \frac{1 - te^{2\pi i (h, w_1 w_2 a^\vee)}}{1 - e^{2\pi i (h, w_1 w_2 a^\vee)}}
\]

\[
= \sum_{w_1 \in W} \sum_{w_2 \in W} \left( \prod_{a \in \Phi_{rX,+}} \frac{1 - te^{2\pi i (h, w_1 w_2 a^\vee)}}{1 - e^{2\pi i (h, w_1 w_2 a^\vee)}} \right) \left( \prod_{a \in \Phi_{re,+} - \Phi_{rX,+}} \frac{1 - te^{2\pi i (h, w_1 w_2 a^\vee)}}{1 - e^{2\pi i (h, w_1 w_2 a^\vee)}} \right)
\]

Since \( w_2 \in W_X, \Phi(w_2) \subseteq \Phi_X \), namely \( w_2 \) permutes the positive roots in \( \Phi_{re,+} - \Phi_{rX,+} \). Thus the above is equal to

\[
\sum_{w_1 \in W} \left( \prod_{a \in \Phi_{re,+} - \Phi_{rX,+}} \frac{1 - te^{2\pi i (h, w_1 a^\vee)}}{1 - e^{2\pi i (h, w_1 a^\vee)}} \right) \left( \sum_{w_2 \in W} \prod_{a \in \Phi_{rX,+}} \frac{1 - te^{2\pi i (h, w_1 w_2 a^\vee)}}{1 - e^{2\pi i (h, w_1 w_2 a^\vee)}} \right)
\]

By the identity in [12], we have

\[
\sum_{w' \in W} w' \left( \prod_{a \in \Phi_{rX,+}} \frac{1 - te^{-a^\vee}}{1 - e^{-a^\vee}} \right) = W_X(t)
\]

So

\[
\sum_{w_2 \in W \setminus \Phi_{rX,+}} \prod_{a \in \Phi_{rX,+}} \frac{1 - te^{2\pi i (h, w_1 w_2 a^\vee)}}{1 - e^{2\pi i (h, w_1 w_2 a^\vee)}} = W_X(t)
\]

is constant in \( h \). So we have

\[
= \sum_{w_1 \in W} \left( \prod_{a \in \Phi_{re,+} - \Phi_{rX,+}} \frac{1 - te^{2\pi i (h, w_1 a^\vee)}}{1 - e^{2\pi i (h, w_1 a^\vee)}} \right) \left( \sum_{w_2 \in W} \prod_{a \in \Phi_{rX,+}} \frac{1 - te^{2\pi i (h, w_1 w_2 a^\vee)}}{1 - e^{2\pi i (h, w_1 w_2 a^\vee)}} \right)
\]
Proposition 3.3.6 The summation

\[ \sum_{w_1 \in \mathcal{W}} \prod_{a \in \Phi_{r,+} - \Phi_{X,+}} \frac{1 - te^{2\pi i(h, wa^\vee)}}{1 - e^{2\pi i(h, wa^\vee)}} \cdot W_X(t) = C_X(t, h) \]

Recall that \( V + iS_X \) is an open set because \( S_X \) is open in \( V \). By Lemma 2.2.2, we have

\[ V + iS_X = V + i \bigcup \bigcup_{w \in \mathcal{W}_X \subseteq X} wF_Y = V + i \bigcup_{w \in \mathcal{W}_X} w\Gamma_X \]

is the union of all the \( \mathcal{W}_X \)-translates of the set \( V + i\Gamma_X \). Also we have \( V + iW \cdot S_X = V + iW \cdot \Gamma_X = W \cdot (V + i\Gamma_X) \) is an open set in \( \Omega \).

Proposition 3.3.6 The summation

\[ \sum_{w \in \mathcal{W}_X} \prod_{a \in \Phi_{r,+} - \Phi_{X,+}} \frac{1 - te^{2\pi i(h, wa^\vee)}}{1 - e^{2\pi i(h, wa^\vee)}} \quad (3.3) \]

is absolutely convergent on \( \mathcal{D}_r \times (V + iW \cdot S_X) \). So we can extend the domain of \( C_X(t, h) \) by defining

\[ C_X(t, h) = W_X(t) \sum_{w \in \mathcal{W}_X} \prod_{a \in \Phi_{r,+} - \Phi_{X,+}} \frac{1 - te^{2\pi i(h, wa^\vee)}}{1 - e^{2\pi i(h, wa^\vee)}} \]

for \((t, h) \in \mathcal{D}_r \times (V + iW \cdot S_X)\). Then \( C_X(t, h) \) is a \( \mathcal{W}_X \)-invariant function on \( \mathcal{D}_r \times (V + iW \cdot S_X) \).

Proof Fix \( t \in \mathcal{D}_r, h \in V + iF_Y \) for some \( Y \subseteq X \). First of all, for \( w_X \in \mathcal{W}_X \) we have

\[ \prod_{a \in \Phi_{r,+} - \Phi_{X,+}} \frac{1 - te^{2\pi i(h, wa^\vee)}}{1 - e^{2\pi i(h, wa^\vee)}} = \prod_{a \in \Phi_{r,+} - \Phi_{X,+}} \frac{1 - te^{2\pi i(h, wwa^\vee)}}{1 - e^{2\pi i(h, wwa^\vee)}} \]

since \( w_X \) permutes the positive roots in \( \Phi_{r,+} - \Phi_{X,+} \) as in the proof of Proposition 3.3.3. So for \( w_1, w_2 \in W \) lying in the same left \( \mathcal{W}_X \)-coset, we have

\[ \prod_{a \in \Phi_{r,+} - \Phi_{X,+}} \frac{1 - te^{2\pi i(h, w_1a^\vee)}}{1 - e^{2\pi i(h, w_1a^\vee)}} = \prod_{a \in \Phi_{r,+} - \Phi_{X,+}} \frac{1 - te^{2\pi i(h, w_2a^\vee)}}{1 - e^{2\pi i(h, w_2a^\vee)}} \]

Note that \( \mathcal{W}_X \) is a set of coset representatives of \( W/\mathcal{W}_X \), and for any \( w' \in W \), \( w'\mathcal{W}_X \) is also a set of coset representatives of \( W/\mathcal{W}_X \), so

\[ C_X(t, h) = W_X(t) \sum_{w \in \mathcal{W}_X} \prod_{a \in \Phi_{r,+} - \Phi_{X,+}} \frac{1 - te^{2\pi i(h, wa^\vee)}}{1 - e^{2\pi i(h, wa^\vee)}} \]

\[ = W_X(t) \sum_{w \in w'\mathcal{W}_X} \prod_{a \in \Phi_{r,+} - \Phi_{X,+}} \frac{1 - te^{2\pi i(h, wa^\vee)}}{1 - e^{2\pi i(h, wa^\vee)}} \]

\[ = W_X(t) \sum_{w \in \mathcal{W}_X} \prod_{a \in \Phi_{r,+} - \Phi_{X,+}} \frac{1 - te^{2\pi i(h, wa^\vee)}}{1 - e^{2\pi i(h, wa^\vee)}} \]

\[ = W_X(t) \sum_{w \in \mathcal{W}_X} \prod_{a \in \Phi_{r,+} - \Phi_{X,+}} \frac{1 - te^{2\pi i(h, w^{-1}h, wa^\vee)}}{1 - e^{2\pi i(h, w^{-1}h, wa^\vee)}} \]
so the right hand side is absolutely convergent and is equal to $C_X(t, w^{-1}h)$. By the description of $V+iW \cdot S_X$ before this Proposition, every element in $V+iW \cdot S_X$ is of the form $w^{-1}h$ for some $w' \in W$, $h \in V+iF_Y$ for some $Y \subseteq X$. Thus (3.3) is convergent on $D_r \times (V+iW \cdot S_X)$ and $C_X(t, h)$ is $W$-invariant.

**Proposition 3.3.7** Let $X$ be a $W$-finite subset of $\Delta$. The absolute convergence of the summation

$$
\sum_{w \in W^X} \prod_{a \in \Phi_{re,+}\Phi_X} \frac{1-te^{2\pi i(h, wa')}}{1-e^{2\pi i(h, wa')}}
$$

is uniform on compact subsets of $D_r \times (V+iS_X)$, thus $C_X$ is a holomorphic function on $D_r \times (V+iW \cdot S_X)$.

**Proof** We recall that $V+iS_X = V+i \bigcup_{w \in W_X} w\Gamma_X$. A compact set of $D_r \times (V+iS_X)$ is contained in a compact set of the form $\overline{D_{r_0}} \times E$ where $0 < r_0 < r$ and $E \subseteq V+iS_X$ is compact. Since $W_X$ is finite, the set $W_X \cdot E$ is also compact. Let $E_0 = (W_X \cdot E) \cap (V+i\Gamma_X)$, then $W_X \cdot E = W_X \cdot E_0$ and $E_0 \subseteq V+i\Gamma_X$ is compact by the concrete description of $\Gamma_X$ and $S_X$ after Lemma 2.2.2. So the convergence of this summation is uniform on $E_0$ by Proposition 3.3.4, and the convergence on $W_X \cdot E_0$ is uniform since $W_X$ is finite and this summation is $W$-invariant.

The uniformness of convergence on compact subsets of $V+iS_X$ implies that $C_X$ is holomorphic on $V+iS_X$. Since $C_X$ is $W$-invariant, it is also holomorphic on each $W$-translation of $V+iS_X$, so it is holomorphic on $W \cdot (V+iS_X) = V+iW \cdot S_X$.

So each $C_X(t, h)$ is a holomorphic continuation of $C(t, h)$ from $V+iW \cdot C$ to $V+iW \cdot S_X$. This implies that for any $W$-finite $X, X' \subseteq \Delta$ we have

$$
C_X(t, h) = C_{X'}(t, h)
$$

for $t \in D_r, h \in V+iW \cdot (S_X \cap S_{X'})$, and all these functions together give a holomorphic continuation of $C(t, h)$ to the open set

$$
D_r \times \bigcup_{X \subseteq \Delta: |W_X| < \infty} (V+iW \cdot S_X)
$$

Which is equal to the whole $D_r \times \Omega$ by corollary 2.2.4. So we have proved

**Theorem 3.3.8** $C(t, h)$ can be analytically continued to a holomorphic function on $D_r \times \Omega$.

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