THE AUTOMORPHISM GROUP OF A SUPERSINGULAR $K_3$ SURFACE WITH ARTIN INVARIANT 1 IN CHARACTERISTIC 3

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Abstract. We present a finite set of generators of the automorphism group of a supersingular $K_3$ surface with Artin invariant 1 in characteristic 3.

1. Introduction

To determine the automorphism group $\text{Aut}(Y)$ of a given algebraic $K_3$ surface $Y$ is an important problem. It follows from the Torelli type theorem for $K_3$ surfaces defined over $\mathbb{C}$ (Piatetskii-Shapiro, Shafarevich [27]) that $\text{Aut}(Y)$ is isomorphic to $O(S_Y)/W(S_Y)^{(-2)}$ up to finite groups where $S_Y$ is the Picard lattice of $Y$, $O(S_Y)$ is the orthogonal group of $S_Y$ and $W(S_Y)^{(-2)}$ is the normal subgroup of $O(S_Y)$ generated by $(-2)$-reflections. In particular $\text{Aut}(Y)$ is finite if and only if $S_Y$ is $(-2)$-reflective, that is, $W(S_Y)^{(-2)}$ is of finite index in $O(S_Y)$. All $(-2)$-reflective lattices are classified (Nikulin [22], [23], Vinberg [50]). On the other hand, Shioda-Inose [43] showed that $\text{Aut}(Y)$ is infinite for all singular $K_3$ surfaces, that is, complex $K_3$ surfaces with the maximum Picard number 20. In case $\text{Aut}(Y)$ is infinite, it is, in general, very difficult to describe it explicitly. If $S_Y$ is reflective, that is, the full reflection group $W(S_Y)$ generated by not only $(-2)$-reflections but also all reflections in $O(S_Y)$ is of finite index in $O(S_Y)$, then one may find a fundamental domain of $W(S_Y)$ (The reflective lattices in rank 3 are classified in Allcock [1]). By this method, Vinberg [49] gave a concrete description of $\text{Aut}(Y)$ for two singular $K3$ surfaces. However, in case that $S_Y$ is not reflective, there were no general methods to study $\text{Aut}(Y)$. In case that $S_Y$ is non-reflective, the first author [17] gave a set of generators of the automorphism group of a generic Jacobian Kummer surface by applying the results of Conway [8] and Borcherds [5, 6] on the reflection group of the even unimodular lattice of signature $(1, 25)$.

In this paper, by using the same method, we present a set of generators of the automorphism group of a supersingular $K3$ surface $X$ in characteristic 3 with Artin invariant 1. We note that $S_X$ is not reflective. Our method is computational, and relies heavily on computer-aided calculation. It gives us generators in explicit form, and it can be easily applied to many other $K3$ surfaces by modifying computer programs.

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A $K3$ surface defined over an algebraically closed field $k$ is said to be supersingular (in the sense of Shioda) if its Picard number is 22. Supersingular $K3$ surfaces exist only when $k$ is of positive characteristic. Let $Y$ be a supersingular $K3$ surface in characteristic $p > 0$, and let $S_Y$ denote its Néron-Severi lattice. Artin [4] showed that the discriminant group of $S_Y$ is a $p$-elementary abelian group of rank $2\sigma$, where $\sigma$ is an integer such that $1 \leq \sigma \leq 10$. This integer $\sigma$ is called the Artin invariant of $Y$. Ogus [24, 25] proved that a supersingular $K3$ surface with Artin invariant 1 in characteristic $p$ is unique up to isomorphisms and the Torelli type theorem for them in characteristic $p > 2$ (see also Rudakov, Shafarevich [28]).

In the following, we consider the Fermat quartic surface $X := \{ w^4 + x^4 + y^4 + z^4 = 0 \} \subset \mathbb{P}^3$ defined over an algebraically closed field $k$ of characteristic 3, which is a supersingular $K3$ surface with Artin invariant 1. Let $h_0 := [O_X(1)] \in S_X$ denote the class of the hyperplane section of $X$. The projective automorphism group $\text{Aut}(X, h_0)$ of $X \subset \mathbb{P}^3$ is equal to the finite subgroup $\text{PGU}_4(\mathbb{F}_9)$ of $\text{PGL}_4(k)$ with order 13,063,680. Segre [33] showed that $X$ contains 112 lines. (In characteristic 0, the Fermat quartic surface contains only 48 lines and its projective automorphism group is of order 1536. See Segre [32].) Tate [47] showed that $X$ is supersingular by means of representations of $\text{PGU}_4(\mathbb{F}_9)$. Mizukami [19] showed that the classes of these 112 lines form a lattice of discriminant $-9$ (see Schütt, Shioda, van Luijk [31]). Hence these classes span $S_X$ and the Artin invariant of $X$ is 1 (see also Shioda [44] and [45]).

Let $(w, x, y)$ be the affine coordinates of $\mathbb{P}^3$ with $z = 1$, and let $F_i^j$ and $F_2^j$ be polynomials of $(w, x, y)$ with coefficients in

$$\mathbb{F}_9 = \mathbb{F}_3(i) = \{0, \pm 1, \pm i, \pm (1 + i), \pm (1 - i)\}, \quad \text{where } i := \sqrt{-1},$$
given in Table 1.1.

**Proposition 1.1.** For $\nu = 1$ and 2, the rational map

$$(w, x, y) \mapsto [F_{\nu 0} : F_{\nu 1} : F_{\nu 2}] \in \mathbb{P}^2$$

induces a morphism $\phi_\nu : X \to \mathbb{P}^2$ of degree 2.

We denote by $X \xrightarrow{\phi_\nu} Y_\nu \xrightarrow{\pi_\nu} \mathbb{P}^2$ the Stein factorization of $\phi_\nu : X \to \mathbb{P}^2$, and let $B_\nu \subset \mathbb{P}^2$ be the branch curve of the finite morphism $\pi_\nu : Y_\nu \to \mathbb{P}^2$ of degree 2. Note that $Y_\nu$ is a normal $K3$ surface, and hence $Y_\nu$ has only rational double points as its singularities (see Artin [2, 3]). Let $[x_0 : x_1 : x_2]$ be the homogeneous coordinates of $\mathbb{P}^2$. 
Our main result is as follows:

\[ F_{10} = (1 + i) + (1 + i) w + (1 - i) x - y - (1 - i) wx - x^2 + i wy \]
\[ + i xy - i y^2 + (1 + i) w^3 - i w^2 x + (1 + i) wx^2 - i x^2 + w^2 y \]
\[ + (1 + i) wxy + (1 + i) x^2 y - (1 - i) wy^2 - (1 + i) xy^2 + i y^3 \]

\[ F_{11} = (1 - i) - (1 + i) x - (1 - i) y - (1 - i) w - (1 - i) wx - (1 - i) x^2 \]
\[ - (1 + i) wy - xy - (1 + i) y^2 - w^3 + (1 - i) w^2 x + w^2 x - i x^3 \]
\[ - (1 + i) w^2 y - (1 + i) wxy + x^2 y - i wy^2 - x^2 y + (1 - i) y^3 \]

\[ F_{12} = (1 + i) w - i x - y - w^2 - wx - i x^2 - i xy + i y^2 + i w^3 \]
\[ - (1 + i) wx^2 + i x^3 - i w^2 y - wxy + (1 - i) wy^2 + (1 + i) y^3 \]

\[ F_{20} = -1 - i w + (1 + i) x - y - (1 + i) w^2 - wx - (1 - i) x^2 - i wy + (1 + i) xy \]
\[ - (1 - i) w^3 + w^3 x - wx^2 + x^3 - w^3 y + (1 - i) wxy + x^2 y + (1 - i) wy^2 \]
\[ + (1 - i) xy^2 + (1 + i) y^3 - w^3 x - i w^2 x^2 - wx^3 + w^3 y - (1 + i) w^2 xy \]
\[ - (1 - i) wxy^2 + x^2 y^2 - (1 + i) wy^3 - (1 + i) xy^2 - y^4 + (1 - i) w^3 x^2 - i x^5 \]
\[ + (1 - i) w^2 x^3 + (1 + i) wxy^2 + x^3 y - i w^3 y^2 + (1 + i) w^2 xy^2 - (1 + i) wy^3 \]
\[ + i x^3 y - w^2 y^3 - (1 + i) wxy^2 - (1 - i) x^2 y^3 + i wy^4 + (1 - i) x^4 y^4 + (1 + i) y^5 \]

\[ F_{21} = -(1 - i) + i w + (1 - i) y - (1 + i) w^2 + wx + (1 + i) x^2 + (1 + i) wy - (1 + i) xy \]
\[ - i y^2 - w^3 + i w^2 x + (1 + i) wxy - x^3 - (1 + i) w^3 y - (1 - i) x^2 y - i (1 - i) x^2 y \]
\[ - i wy^2 - (1 + i) xy^2 + y^3 - (1 - i) w^3 x - wx^3 + (1 - i) x^4 + (1 - i) w^3 y + i w^2 xy \]
\[ + (1 - i) w^2 y^2 - i x^3 y + (1 - i) w^3 y^3 + (1 - i) wxy^2 - (1 + i) x^2 y^2 + (1 - i) wy^3 \]
\[ - i xy^3 + i y^4 + w^3 x^2 + w^3 x^2 + (1 - i) w^4 x - i x^5 - i w^3 xy + w^2 x^2 y + (1 + i) w^3 y \]
\[ + x^4 y + w^3 y^2 - w^2 x y^2 - w^2 x y + i w^2 y^3 + (1 + i) wxy^3 - i wy^3 - i w^4 - i w^4 + y^5 \]

\[ F_{22} = (1 - i) - (1 + i) w - (1 + i) x - (1 - i) y + i w^2 - (1 + i) wx - (1 - i) x^2 + i wy \]
\[ - (1 + i) wy - x^3 - i w^2 x - wx^2 + x^3 - (1 - i) w^2 y + wxy + x^2 y + (1 + i) wy^2 \]
\[ - (1 + i) xy^2 - y^3 + i w^3 x - (1 - i) w^3 x - wx^2 - (1 + i) x^2 + i w^3 y + w^2 xy \]
\[ + (1 - i) w^2 y^2 - (1 - i) w^2 y^2 + (1 + i) wxy^2 + i wy^3 + x^4 + (1 - i) y^4 - i w^3 x^2 \]
\[ - (1 + i) xw^2 + x^5 - (1 - i) wxy + x^2 y + (1 + i) wx^2 y + (1 - i) x^2 y - x^3 y^2 \]
\[ - (1 + i) w^2 xy^2 + i wx^2 y^2 + i x^3 y - w^2 y^2 - (1 - i) x^2 y^3 - wy^4 - x^4 y - y^5 \]

### Table 1.1. Polynomials $F_{ij}$ and $F_{2j}$

**Proposition 1.2.** (1) The ADE-type of the singularities of $Y_1$ is $6A_1 + 4A_2$. The branch curve $B_1$ is defined by $f_1 = 0$, where

\[ f_1 := x_0^6 + x_0^5 x_1 - x_0^3 x_1^3 - x_0 x_1^5 - x_0^4 x_2^2 \]
\[ + x_0 x_1^2 x_2^2 + x_1^4 x_2^2 + x_0^2 x_2^4 + x_1^2 x_2^4 + x_2^6. \]

(2) The ADE-type of the singularities of $Y_2$ is $A_1 + A_2 + 2A_3 + 2A_4$. The branch curve $B_2$ is defined by $f_2 = 0$, where

\[ f_2 := x_0^5 x_1 + x_0^2 x_1^4 - x_0^4 x_2^2 + x_0 x_1^3 x_2^2 \]
\[ + x_1^4 x_2^2 - x_0^2 x_2^4 - x_0 x_1 x_2^4 - x_1^2 x_2^4 - x_2^6. \]

Our main result is as follows:
Theorem 1.3. Let $g_\nu \in \text{Aut}(X)$ denote the involution induced by the deck transformation of $\pi_\nu: Y_\nu \to \mathbb{P}^2$. Then $\text{Aut}(X)$ is generated by $\text{Aut}(X, h_0) = \text{PGU}_4(\mathbb{F}_9)$ and $g_1, g_2$.

See Theorem 7.1 for a more explicit description of the involutions $g_1$ and $g_2$.

Let $L$ denote an even unimodular lattice of rank 26 with signature $(1, 25)$, which is unique up to isomorphisms by Eichler’s theorem. Conway [8] determined the fundamental domain in a positive cone of $L \otimes \mathbb{R}$ under the action of $W(L)(-2)$. Borcherds [5, 6] applied Conway theory to the investigation of the orthogonal groups of even hyperbolic lattices $S$ primitively embedded in $L$ with the orthogonal complement $T$ being a (negative definite) root lattice. He obtained a chamber decomposition of a positive cone of $S \otimes \mathbb{R}$ by restricting Conway’s chamber decomposition of a positive cone of $L \otimes \mathbb{R}$.

We employ Borcherds’ method in our case. We take the root lattice $A_2 \oplus A_2$ as $T$. Then the orthogonal complement of $T$ in $L$ is isomorphic to the Néron-Severi lattice $S_X$. Let $\mathcal{P}_{S_X}$ denote the connected component of $\{x \in S_X \otimes \mathbb{R} \mid x^2 > 0\}$ that contains $h_0$. We prove Theorem 1.3 by calculating a closed chamber $D_{S_0}$ in the cone $\mathcal{P}_{S_X}$ with the following properties (see Section 6):

(1) The chamber $D_{S_0}$ is invariant under the action of $\text{Aut}(X, h_0)$.
(2) For any nef class $v \in S_X$, there exists $\gamma \in \text{Aut}(X)$ such that $v^\gamma \in D_{S_0}$.
(3) For nef classes $v, v'$ in the interior of $D_{S_0}$, there exists $\gamma \in \text{Aut}(X)$ such that $v' = v^\gamma$ if and only if there exists $\tau \in \text{Aut}(X, h_0)$ such that $v' = v^\tau$.

This chamber $D_{S_0}$ is bounded by $112 + 648 + 5184$ hyperplanes in $\mathcal{P}_{S_X}$. See Proposition 4.5 for the explicit description of these walls. Using $D_{S_0}$ and these walls, we can also present a finite set of generators of $O^+(S_X)$ (see Theorem 8.2). This chamber $D_{S_0}$ is not only important for the calculation of $\text{Aut}(X)$, but also useful for the classification of geometric objects on $X$ modulo $\text{Aut}(X)$. See Corollary 6.2 for an example.

As mentioned above, the first author [17] determined the automorphism group of a generic Jacobian Kummer surface by embedding its Néron-Severi lattice into $L$ primitively with the orthogonal complement being a root lattice of type $6A_1 + A_3$ (The description of the automorphism group was applied by Kumar [18] to obtain all elliptic fibrations with a section up to automorphisms).

Keum and the first author [16] applied the same method to the Kummer surface of the product of two elliptic curves, Dolgachev and Keum [13] applied it to quartic Hessian surfaces, and Dolgachev and the first author [12] applied it to the supersingular $K3$ surface in characteristic 2 with Artin invariant 1. We remark that the Picard lattice of the last $K3$ surface is reflective.

There are many $K3$ surfaces $Y$ such that $S_Y$ is isomorphic to the orthogonal complement to a primitive root sublattice of $L$; for example, a complex $K3$ surface
whose Picard lattice is isomorphic to $U \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle$, which is obtained as the orthogonal complement of the root lattice $D_7$ in $L$. We hope that, for these $K3$ surfaces, we can obtain a set of generators of $\text{Aut}(Y)$ by modifying our computer programs.

If the orthogonal complement of $S_Y$ embedded in $L$ is not a root lattice, then this method does not always work. Consider, for example, the Fermat quartic surface in characteristic 0 or a supersingular $K3$ surface with Artin invariant 1 in characteristic 5. We can embed $S_Y$ into $L$ primitively, but the orthogonal complement is \textit{not} a root lattice. An experimental computation shows that many isomorphism classes of chambers in $S_Y \otimes \mathbb{R}$ are obtained as the intersections with the Conway chambers in $L \otimes \mathbb{R}$. However, for example, in case of Fermat quartic surface in characteristic 0, we obtain 48 faces of a chamber corresponding to 48 lines on the Fermat quartic. On the other hand, a supersingular $K3$ surface with Artin invariant 1 in characteristic 5 is obtained as the double cover of $\mathbb{P}^2$ branched along the Fermat sextic curve. The pullback of 126 lines on $\mathbb{P}^2(F_{25})$ are 252 smooth rational curves on the $K3$ surface. In this case, we have these 252 smooth rational curves as 252 faces of a chamber. Moreover the group $\text{PGU}(3, F_5)$ appears as the group of automorphisms of the chamber. By a similar way, we can see that the sporadic simple groups $M_{11}$ and $M_{22}$ called the Mathieu groups act on a supersingular $K3$ surface in characteristic 11 with Artin invariant 1. Note that the last $K3$ surface is isomorphic to the Fermat quartic surface in characteristic 11 (see Shioda [44] and [45]).

The new idea introduced in this paper is that, in order to find automorphisms of $X$ necessary to generate $\text{Aut}(X)$, we search for polarizations of degree 2 whose classes are located on the walls of the chamber decomposition of the cone $P_{S_X}$. Since $S_X$ is generated by the classes of lines, and the defining ideals of these lines are easily calculated, we can write the automorphisms of $X$ as a list of rational functions on $X$. The computational tools used in this paper have been developed by the second author for the study [39] of various double plane models of a supersingular $K3$ surface in characteristic 5 with Artin invariant 1. The computational data for this paper is available from the second author’s webpage [40].

In [36] and [38], the second author showed that every supersingular $K3$ surface in any characteristic with arbitrary Artin invariant is birational to a double cover of the projective plane. In [37], [41] and [26, 39], projective models of supersingular $K3$ surfaces in characteristic 2, 3 and 5 were investigated, respectively.

The study of the automorphism group of the Fermat quartic surface goes back to Segre [32], in which it was proved that the automorphism group is infinite in characteristic 0. Then Shioda [42] showed that $\text{Aut}(X)$ contains $\mathbb{Z}^2$ by showing the existence of an elliptic fibration of rank 2 on $X$. Recently, elliptic fibrations on $X$ was classified in Sengupta [34] by embedding $S_X$ into $L$. On the other hand,
configurations of smooth rational curves on $X$ was studied in Katsura and the second author \cite{Katsura-Shimada} with respect to an embedding of $S_X$ into $L$.

This paper is organized as follows. In Section 2, we give a review of the theory of Conway and Borcherds, and investigate chamber decomposition induced on a positive cone of a primitive hyperbolic sublattice $S$ of $L$. In Section 3, we give explicitly a basis of the Néron-Severi lattice $S_X$ of $X$, and describe a method to compute the action of $\text{Aut}(X, h_0)$ on $S_X$. In Section 4, we embed $S_X$ into $L$, and study the obtained chamber decomposition in detail. In particular, we investigate the walls of the chamber $D_{S_0}$ that contains the class $h_0$. In Section 5, we prove Propositions 1.1 and 1.2, and show that the involutions $g_1$ and $g_2$ map $h_0$ to its mirror images into walls of the chamber $D_{S_0}$. Then we can prove Theorem 1.3 in Section 6. In Section 7, we give another description of the involutions $g_\nu$. In the last section, we give a set of generators of $O^+(S_X)$.

2. Leech Roots

2.1. Terminologies and notation. We fix some terminologies and notation about lattices. A lattice $M$ is a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric bilinear form $(\ , )_M : M \times M \to \mathbb{Z}$.

A submodule $N$ of $M$ is said to be primitive if $M/N$ is torsion free. For a submodule $N$ of $M$, we denote by $N^\perp \subset M$ the submodule defined by

$$ N^\perp := \{ u \in M \mid (u,v)_M = 0 \text{ for all } v \in N \}, $$

which is primitive by definition. We denote by $O(M)$ the orthogonal group of $M$. Throughout this paper, we let $O(M)$ act on $M$ from right. Suppose that $M$ is of rank $r$. We say that $M$ is hyperbolic (resp. negative-definite) if the signature of the symmetric bilinear form $(\ , )_M$ on $M \otimes \mathbb{R}$ is $(1, r-1)$ (resp. $(0, r)$). We define the dual lattice $M^\vee$ of $M$ by

$$ M^\vee := \{ u \in M \otimes \mathbb{Q} \mid (u,v)_M \in \mathbb{Z} \text{ for all } v \in M \}. $$

Then $M$ is contained in $M^\vee$ as a submodule of finite index. The finite abelian group $M^\vee/M$ is called the discriminant group of $M$. We say that $M$ is unimodular if $M = M^\vee$.

A lattice $M$ is said to be even if $(v,v)_M \in 2\mathbb{Z}$ holds for any $v \in M$. The discriminant group $M^\vee/M$ of an even lattice $M$ is naturally equipped with the quadratic form

$$ q_M : M^\vee/M \to \mathbb{Q}/2\mathbb{Z} $$

defined by $q_M(u \mod M) := (u,u)_M \mod 2\mathbb{Z}$. We call $q_M$ the discriminant form of $M$. The automorphism group of $q_M$ is denoted by $O(q_M)$. There exists a natural homomorphism $O(M) \to O(q_M)$. 
Suppose that $M$ is hyperbolic. Then the open subset
\[
\{ x \in M \otimes \mathbb{R} \mid (x, x)_M > 0 \}
\]
of $M \otimes \mathbb{R}$ has two connected components. A *positive cone* of $M$ is one of them. We fix a positive cone $\mathcal{P}$. The *autochronous orthogonal group* $O^+(M)$ of $M$ is the group of isometries of $M$ that preserve $\mathcal{P}$. Then $O^+(M)$ is a subgroup of $O(M)$ with index 2. Note that $O^+(M)$ acts on $\mathcal{P}$. For a nonzero vector $u \in M \otimes \mathbb{R}$, we denote by $(u)^+_M$ the hyperplane of $M \otimes \mathbb{R}$ defined by
\[
(u)^+_M := \{ x \in M \otimes \mathbb{R} \mid (x, u)_M = 0 \}.
\]
Let $\mathcal{R}$ be a set of non-zero vectors of $M \otimes \mathbb{R}$, and let
\[
\mathcal{H} := \{ (u)^+_M \mid u \in \mathcal{R} \}
\]
be the family of hyperplanes defined by $\mathcal{R}$. Suppose that $\mathcal{H}$ is locally finite in $\mathcal{P}$. Then the closure in $\mathcal{P}$ of each connected component of
\[
\mathcal{P} \setminus \left( \mathcal{P} \cap \bigcup_{u \in \mathcal{R}} (u)^+_M \right)
\]
is called an $\mathcal{R}$-chamber. Let $D$ be an $\mathcal{R}$-chamber. We denote by $D^0$ the interior of $D$. We say that a hyperplane $(u)^+_M \in \mathcal{H}$ bounds $D$, or that $(u)^+_M$ is a wall of $D$, if $(u)^+_M \cap D$ contains a non-empty open subset of $(u)^+_M$. We denote the set of walls of $D$ by
\[
\mathcal{W}(D) := \{ (u)^+_M \in \mathcal{H} \mid (u)^+_M \text{ bounds } D \}.
\]
Suppose that $\mathcal{R}$ is invariant under $u \mapsto -u$. We choose a point $p \in D^0$, and put
\[
\tilde{\mathcal{W}}(D) := \{ u \in \mathcal{R} \mid (u)^+_M \text{ bounds } D \text{ and } (u, p)_M > 0 \},
\]
which is independent of the choice of $p$. It is obvious that $D$ is equal to
\[
\{ x \in \mathcal{P} \mid (x, u)_M \geq 0 \text{ for all } u \in \tilde{\mathcal{W}}(D) \}.
\]

2.2. **Conway theory.** We review the theory of Conway [8]. Let $L$ be an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphisms by Eichler’s theorem (see, for example, [7, Chapter 11, Theorem 1.4]). We choose and fix a positive cone $\mathcal{P}_L$ once and for all. A vector $r \in L$ is called a root if the reflection $s_r : L \otimes \mathbb{R} \to L \otimes \mathbb{R}$ defined by
\[
x \mapsto x - \frac{2(x, r)_L}{(r, r)_L} \cdot r
\]
preserves $L$ and $\mathcal{P}_L$, or equivalently, if $(r, r)_L = -2$. We denote by $\mathcal{R}_L$ the set of roots of $L$, which is invariant under $r \mapsto -r$. Let $W(L)$ denote the subgroup of $O^+(L)$ generated by the reflections $s_r$ associated with all the roots $r \in \mathcal{R}_L$. Then $W(L)$ is a normal subgroup of $O^+(L)$. The family of hyperplanes
\[
\mathcal{H}_L := \{ (r)^+_L \mid r \in \mathcal{R}_L \}
\]
is locally finite in $\mathcal{P}_L$. Hence we can consider $\mathcal{R}_L$-chambers. By definition, each $\mathcal{R}_L$-chamber is a fundamental domain of the action of $W(L)$ on $\mathcal{P}_L$.

A non-zero primitive vector $w \in L$ is called a Weyl vector if $(w,w)_L = 0$, $w$ is contained in the closure of $\mathcal{P}_L$ in $L \otimes \mathbb{R}$, and the negative-definite even unimodular lattice $\langle w \rangle^\perp/\langle w \rangle$ of rank 24 has no vectors of square norm $-2$. Let $w \in L$ be a Weyl vector. We put

$$LR(w) := \{ r \in \mathcal{R}_L \mid (w,r)_L = 1 \}.$$ 

A root in $LR(w)$ is called a Leech root with respect to $w$.

Suppose that $w$ is a non-zero primitive vector of norm 0 contained in the closure of $\mathcal{P}_L$. Then there exists a vector $w' \in L$ such that $(w,w')_L = 1$ and $(w',w')_L = 0$. Let $U \subset L$ denote the hyperbolic sublattice of rank 2 generated by $w$ and $w'$. By Niemeier’s classification [20] of even definite unimodular lattices of rank 24 (see also [11, Chapter 18]), we see that the condition that $\langle w \rangle^\perp/\langle w \rangle$ have no vectors of square norm $-2$ is equivalent to the condition that the orthogonal complement $U^\perp$ of $U$ in $L$ be isomorphic to the (negative-definite) Leech lattice $\Lambda$. From this fact, we can deduce the following:

**Proposition 2.1.** The group $O^+(L)$ acts on the set of Weyl vectors transitively.

**Proposition 2.2.** Suppose that $w$ is a Weyl vector and that $w' \in L$ satisfies $(w,w')_L = 1$ and $(w',w')_L = 0$. Via an isomorphism $\rho: \Lambda \cong U^\perp$, the map

$$\lambda \mapsto -\frac{2 + (\lambda,\lambda)}{2}w + w' + \rho(\lambda)$$

induces a bijection from the Leech lattice $\Lambda$ to the set $LR(w)$.

Using Vinberg’s algorithm [48] and the result on the covering radius of the Leech lattice [10], Conway [8] proved the following:

**Theorem 2.3.** Let $w \in L$ be a Weyl vector. Then

$$D_L(w) := \{ x \in \mathcal{P}_L \mid (x,r)_L \geq 0 \text{ for all } r \in LR(w) \}$$

is an $\mathcal{R}_L$-chamber, and $\mathcal{W}(D_L(w))$ is equal to $LR(w)$; that is, $(r)_L^+$ bounds $D_L(w)$ for any $r \in LR(w)$. The map $w \mapsto D_L(w)$ is a bijection from the set of Weyl vectors to the set of $\mathcal{R}_L$-chambers.

**Remark 2.4.** Using Proposition 2.2, Conway [8] also showed that the automorphism group $\text{Aut}(D_L(w)) \subset O^+(L)$ of an $\mathcal{R}_L$-chamber $D_L(w)$ is isomorphic to the group $\cdot \infty$ of affine automorphisms of the Leech lattice $\Lambda$. Hence $O^+(L)$ is isomorphic to the split extension of $\cdot \infty$ by $W(L)$. 
2.3. Restriction of $\mathcal{R}_L$-chambers to a primitive sublattice. Let $S$ be an even hyperbolic lattice of rank $r < 26$ primitively embedded in $L$. Following Borcherds [5, 6], we explain how the Leech roots of $L$ induce a chamber decomposition on the positive cone $\mathcal{P}_S := \mathcal{P}_L \cap (S \otimes \mathbb{R})$ of $S \otimes \mathbb{R}$.

The orthogonal complement $T := S^\perp$ of $S$ in $L$ is negative-definite of rank $26 - r$, and we have

$$S \oplus T \subset L \subset S^\vee \oplus T^\vee$$

with $[L : S \oplus T] = [S^\vee \oplus T^\vee : L]$. The projections $L \otimes \mathbb{R} \rightarrow S \otimes \mathbb{R}$ and $L \otimes \mathbb{R} \rightarrow T \otimes \mathbb{R}$ are denoted by $x \mapsto x_S$ and $x \mapsto x_T$, respectively. Note that, if $v \in L$, then $v_S \in S^\vee$ and $v_T \in T^\vee$.

Let $r \in L$ be a root. Then the hyperplane $(r)_L^\perp$ contains $S \otimes \mathbb{R}$ if and only if $r_S = 0$, or equivalently, if and only if $r \in T$. Since $T$ is negative-definite, the set

$$\mathcal{R}_T := \{ v \in T \mid (v, v)_T = -2 \}$$

is finite, and therefore there exist only finite number of hyperplanes $(r)_L^\perp$ that contain $S \otimes \mathbb{R}$. Suppose that $r_S \neq 0$. If $(r_S, r_S)_S \geq 0$, then either $\mathcal{P}_S$ is entirely contained in the interior of the halfspace

$$\{ x \in L \otimes \mathbb{R} \mid (x, r)_L \geq 0 \}$$

or is disjoint from this halfspace. Hence the hyperplane

$$(r_S)_S^\perp = (r)_L^\perp \cap (S \otimes \mathbb{R})$$

of $S \otimes \mathbb{R}$ intersects $\mathcal{P}_S$ if and only if $(r_S, r_S)_S < 0$. We put

$$\mathcal{R}_S := \{ r_S \mid r \in \mathcal{R}_L \text{ and } (r_S, r_S)_S < 0 \} = \{ r_S \mid r \in \mathcal{R}_L \text{ and } (r_S)_S^\perp \cap \mathcal{P}_S \neq \emptyset \}.$$  

Then the associated family of hyperplanes

$$\mathcal{H}_S := \{ (r_S)_S^\perp \mid r_S \in \mathcal{R}_S \}$$

is locally finite in $\mathcal{P}_S$, and hence we can consider $\mathcal{R}_S$-chambers in $\mathcal{P}_S$. Note that $\mathcal{R}_S$ is invariant under $r_S \mapsto -r_S$. We investigate the relation between $\mathcal{R}_S$-chambers and $\mathcal{R}_L$-chambers.

If $D_S \subset \mathcal{P}_S$ is an $\mathcal{R}_S$-chamber, then there exists an $\mathcal{R}_L$-chamber $D_L(w) \subset \mathcal{P}_L$ such that $D_S = D_L(w) \cap (S \otimes \mathbb{R})$ holds. For a given $\mathcal{R}_S$-chamber $D_S$, the set of $\mathcal{R}_L$-chambers $D_L(w)$ satisfying $D_S = D_L(w) \cap (S \otimes \mathbb{R})$ is in one-to-one correspondence with the set of connected components of

$$(T \otimes \mathbb{R}) \setminus \bigcup_{r \in \mathcal{R}_T} (r)_T^\perp.$$
Conversely, suppose that an $\mathcal{R}_L$-chamber $D_L(w)$ is given.

**Definition 2.5.** We say that $D_L(w)$ is $S$-nondegenerate if $D_L(w) \cap (S \otimes \mathbb{R})$ is an $\mathcal{R}_S$-chamber.

By definition, $D_L(w)$ is $S$-nondegenerate if and only if $w$ satisfies the following two conditions:

(i) There exists $v \in \mathcal{P}_S$ such that $(v, r)_L \geq 0$ holds for any $r \in LR(w)$.

(ii) There exists $v' \in \mathcal{P}_S$ such that $(v', r)_L > 0$ holds for any $r \in LR(w)$ with $\langle r_S, r_S \rangle_S < 0$.

If $D_S = D_L(w) \cap (S \otimes \mathbb{R})$ is an $\mathcal{R}_S$-chamber, then $\tilde{W}(D_S)$ is contained in the image of the set

$$LR(w, S) := \{ r \in LR(w) \mid r_S \notin \mathcal{R}_S \} = \{ r \in LR(w) \mid \langle r_S, r_S \rangle_S < 0 \}$$

by the projection $L \to S^\vee$. The following proposition shows that $D_S$ is bounded by a finite number of walls if $w_T \neq 0$, and its proof indicates an effective procedure to calculate $LR(w, S)$. (See [39, Section 3] for the details of the necessary algorithms.)

**Proposition 2.6.** Let $w \in L$ be a Weyl vector such that $w_T \neq 0$. Then $LR(w, S)$ is a finite set.

**Proof.** Since $T$ is negative-definite and $w_T \neq 0$, we have

$$(w_S, w_S)_S = -(w_T, w_T)_T > 0.$$ 

Suppose that $r \in LR(w)$. Then we have

$$(w_S, r_S)_S + (w_T, r_T)_T = 1, \quad (r_S, r_S)_S + (r_T, r_T)_T = -2.$$ 

We have $(r_S, r_S)_S < 0$ if and only if $(r_T, r_T)_T > -2$. Since $T$ is negative-definite, the set

$$V_T := \{ v \in T^\vee \mid (v, v)_T > -2 \}$$

is finite. For $v \in V_T$, we put

$$a_v := 1 - (w_T, v)_T, \quad n_v := -2 - (v, v)_T \quad \text{and} \quad A := \{ (a_v, n_v) \mid v \in V_T \}.$$ 

For each $(a, n) \in A$, we put

$$V_S(a, n) := \{ u \in S^\vee \mid (w_S, u)_S = a, (u, u)_S = n \}.$$ 

Since $S$ is hyperbolic and $(w_S, w_S)_S > 0$, the set $V_S(a, n)$ is finite, because $(\ , \ )_S$ induces on the affine hyperplane

$$\{ x \in S \otimes \mathbb{R} \mid (x, w_S)_S = a \}$$

of $S \otimes \mathbb{R}$ an inhomogeneous quadratic function whose quadratic part is negative-definite. Then the set $LR(w, S)$ is equal to

$$L \cap \{ u + v \mid v \in V_T, u \in V_S(a_v, n_v) \},$$

where the intersection is taken in $S^\vee \oplus T^\vee$. \qed
The notion of $R_S$-chamber is useful in the study on $O^+(S)$ because of the following:

**Proposition 2.7.** Suppose that the natural homomorphism $O(T) \to O(q_T)$ is surjective. Then the action of $O^+(S)$ preserves $R_S$. In particular, for an $R_S$-chamber $D_S$ and an isometry $\gamma \in O^+(S)$, the image $D_S^\gamma$ of $D_S$ by $\gamma$ is also an $R_S$-chamber. Moreover, if the interior of $D_S^\gamma$ has a common point with $D_S$, then $D_S^\gamma = D_S$ holds and $\gamma$ preserves $\tilde{W}(D_S)$.

**Proof.** By the assumption $O(T) \to O(q_T)$, every element $\gamma \in O^+(S)$ lifts to an element $\tilde{\gamma} \in O(L)$ that satisfies $\tilde{\gamma}(S) = S$ and $\tilde{\gamma}|_S = \gamma$ (see [21, Proposition 1.6.1]). Since $\tilde{\gamma}$ preserves $R_L$ and $\gamma$ preserves $P_S$, $\gamma$ preserves $R_S$. $\square$

3. A Basis of the Néron-Severi Lattice of $X$

Recall that $X \subset \mathbb{P}^3$ is the Fermat quartic surface in characteristic 3. From now on, we put

$$S := S_X,$$

which is an even hyperbolic lattice of rank 22 such that $S^\vee/S \cong (\mathbb{Z}/3\mathbb{Z})^2$. We use the affine coordinates $w, x, y$ of $\mathbb{P}^3$ with $z = 1$.

Note that $X$ is the Hermitian surface over $\mathbb{F}_9$ (see [14, Chapter 23]). Hence the number of lines contained in $X$ is 112 (see [33, n. 32] or [35, Corollary 2.22]). Since the indices of these lines are important throughout this paper, we present defining equations of these lines in Table 3.1. (Note that $\ell_i \subset X$ implies that $\ell_i$ is not contained in the plane $z = 0$ at infinity.) From these 112 lines, we choose the following:

\begin{equation}
(3.1) \quad \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_9, \ell_{10}, \ell_{11}, \ell_{17}, \ell_{18}, \ell_{19}, \ell_{21}, \ell_{22}, \ell_{23}, \ell_{25}, \ell_{26}, \ell_{27}, \ell_{33}, \ell_{35}, \ell_{49}.
\end{equation}

The intersection matrix $N$ of these 22 lines is given in Table 3.2. Since $\det N = -9$, the classes $[\ell_i] \in S$ of the lines $\ell_i$ in (3.1) form a basis of $S$. Throughout this paper, we fix this basis, and write elements of $S \otimes \mathbb{R}$ as row vectors

$$[x_1, \ldots, x_{22}]_S.$$

When we use its dual basis, we write

$$[\xi_1, \ldots, \xi_{22}]_S^\vee.$$

Since the hyperplane $w + (1 + i) = 0$ cuts out from $X$ the divisor $\ell_1 + \ell_2 + \ell_3 + \ell_4$, the class $h_0 = [O_X(1)] \in S$ of the hyperplane section is equal to

$$h_0 = [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S$$

$$= [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]_S^\vee.$$

As a positive cone $P_S$ of $S$, we choose the connected component containing $h_0$. 
From the intersection numbers of the 112 lines, we can calculate their classes $[\ell_i] \in S$.

Remark 3.1. Any other choice of the 22 lines that span $S$ will do for the calculation of $\text{Aut}(X)$. We have chosen and fixed (3.1) only in order to fix the notation of vectors in $S^\vee$.

Remark 3.2. Since these 112 lines are all defined over $\mathbb{F}_9$, every class $v \in S$ is represented by a divisor defined over $\mathbb{F}_9$. More generally, Schütt [30] showed that a supersingular $K3$ surface with Artin invariant 1 in characteristic $p$ has a projective model defined over $\mathbb{F}_p$, and its Néron-Severi lattice is generated by the classes of divisors defined over $\mathbb{F}_{p^2}$.

\[
\begin{pmatrix}
-2 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & -2 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & -2 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -2 & 1
\end{pmatrix}
\]

Table 3.2. Gram matrix $N$ of $S$
Proposition 3.3. We have \( h_0 = \frac{1}{28} \sum_{i=1}^{112} \ell_i \).

Proof. The number of \( \mathbb{F}_9 \)-rational points on \( X \) is 280. For each \( \mathbb{F}_9 \)-rational point \( P \) of \( X \), the tangent plane \( T_{X,P} \subset \mathbb{P}^3 \) to \( X \) at \( P \) cuts out a union of four lines from \( X \). Since each line contains ten \( \mathbb{F}_9 \)-rational points, we have \( 280 h_0 = 10 \sum \ell_i \). \( \square \)

As before, we let \( O(S) \) act on \( S \) from right, so that

\[
O(S) = \{ T \in \text{GL}_{22}(\mathbb{Z}) \mid T N^t T = N \}.
\]

We also let the projective automorphism group \( \text{Aut}(X, h_0) = \text{PGU}_4(\mathbb{F}_9) \) act on \( X \) from right. For each \( \tau \in \text{PGU}_4(\mathbb{F}_9) \), we can calculate its action \( \tau_* \) on \( S \) by looking at the permutation of the 112 lines induced by \( \tau \).

Example 3.4. Consider the projective automorphism

\[
\tau : \begin{bmatrix} w : x : y : z \end{bmatrix} \mapsto \begin{bmatrix} i & 0 & i & -1+i \\ 1 & 1-i & -1 & 0 \\ 1 & i & i & -i \\ 1 & -1 & -i & -1 \end{bmatrix}
\]

of \( X \). Then the images \( \ell_i^\tau \) of the lines \( \ell_i \) in (3.1) are

\[
\begin{align*}
\ell_1^\tau &= \ell_{60}, & \ell_2^\tau &= \ell_{31}, & \ell_3^\tau &= \ell_{105}, & \ell_4^\tau &= \ell_{95}, & \ell_5^\tau &= \ell_{92}, & \ell_6^\tau &= \ell_{30}, & \ell_7^\tau &= \ell_{76}, \\
\ell_8^\tau &= \ell_{110}, & \ell_9^\tau &= \ell_{29}, & \ell_{10}^\tau &= \ell_{6}, & \ell_{11}^\tau &= \ell_{6}, & \ell_{12}^\tau &= \ell_{20}, & \ell_{13}^\tau &= \ell_{96}, & \ell_{14}^\tau &= \ell_{102}, \\
\ell_{15}^\tau &= \ell_{13}, & \ell_{16}^\tau &= \ell_{87}, & \ell_{17}^\tau &= \ell_{91}, & \ell_{18}^\tau &= \ell_{108}, & \ell_{19}^\tau &= \ell_{26}, & \ell_{20}^\tau &= \ell_{10}, \\
\ell_{21}^\tau &= \ell_{57}, & \ell_{22}^\tau &= \ell_{52}, & \ell_{23}^\tau &= \ell_{51}, & \ell_{24}^\tau &= \ell_{9}, & \ell_{25}^\tau &= \ell_{59}.
\end{align*}
\]

Therefore the action \( \tau_* \) on \( S \) is given by \( v \mapsto vT_\tau \), where \( T_\tau \) is the matrix whose row vectors are given in Table 3.3

We put the representation

\[
(3.2) \quad \tau \mapsto T_\tau
\]

of \( \text{Aut}(X, h_0) = \text{PGU}_4(\mathbb{F}_9) \) to \( O^+(S) \) in the computer memory. It turns out to be faithful. On the other hand, \( \text{Aut}(X, h_0) \) is just the stabilizer subgroup in \( \text{Aut}(X) \) of \( h_0 \in S \). Therefore we confirm the following fact ([28, Section 8, Proposition 3]):

Proposition 3.5. The action of \( \text{Aut}(X) \) on \( S \) is faithful.

From now on, we regard \( \text{Aut}(X) \) as a subgroup of \( O^+(S) \), and write \( v \mapsto v_\gamma \) instead of \( v \mapsto v_\gamma^* \) for the action \( \gamma_* \) of \( \gamma \in \text{Aut}(X) \) on \( S \).
\[ [\ell_{60}] = [1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1]_S, \\
[\ell_{31}] = [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 1]_S, \\
[\ell_{105}] = [2, 2, 2, 3, -1, -1, -1, -1, -1, -1, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1, 0, 0, 0]_S, \\
[\ell_{95}] = [-3, -2, -2, -3, 1, 1, 2, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1]_S, \\
[\ell_{92}] = [-1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, -1, -1, -1]_S, \\
[\ell_{30}] = [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0]_S, \\
[\ell_{76}] = [0, -1, -1, -1, 0, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S, \\
[\ell_{110}] = [-1, 0, 0, -1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0]_S, \\
[\ell_{29}] = [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0]_S, \\
[\ell_{6}] = [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S, \\
[\ell_{20}] = [1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S, \\
[\ell_{96}] = [4, 2, 3, 4, -2, -3, -2, -1, -2, 0, 1, 0, -1, 0, -1, 0, 0, -1, -1, -1, 2, 0, 1]_S, \\
[\ell_{102}] = [-1, -1, -1, -1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, -1, 0, 0]_S, \\
[\ell_{13}] = [0, 1, 1, 1, 1, 1, 1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S, \\
[\ell_{47}] = [-1, -2, -1, -1, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S, \\
[\ell_{91}] = [4, 2, 3, 3, -1, -2, -1, 0, -1, 0, -1, 0, -1, -1, -1, -1, -1, -1, 1, 0, 1]_S, \\
[\ell_{108}] = [-1, -2, -2, -3, 1, 0, 1, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0]_S, \\
[\ell_{10}] = [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S, \\
[\ell_{57}] = [1, 2, 1, 2, -1, 0, -1, 0, -1, 0, 1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S, \\
[\ell_{52}] = [-1, 0, -1, -1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S, \\
[\ell_{51}] = [1, 1, 1, 2, 0, 0, -1, -1, 0, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S, \\
[\ell_{59}] = [2, 1, 2, 2, -1, -2, 0, 0, -1, 1, 0, -1, -1, 0, -1, 0, -1, -1, -1, 1, 1, 1]_S.

Table 3.3. Row vectors of $T_\tau$ in Example 3.4

4. Embedding of $S$ into $L$

Next we embed the Néron-Severi lattice $S$ of $X$ into the even unimodular hyperbolic lattice of rank 26, and calculate the walls of an $R_S$-chamber.

Let $T$ be the negative-definite root lattice of type $2A_2$. We fix a basis of $T$ in such a way that the Gram matrix is equal to

\[
\begin{bmatrix}
-2 & 1 & 0 & 0 \\
1 & -2 & 0 & 0 \\
0 & 0 & -2 & 1 \\
0 & 0 & 1 & -2
\end{bmatrix}.
\]

When we use this basis, we write elements of $T \otimes \mathbb{R}$ as $[y_1, y_2, y_3, y_4]_T$, while when we use its dual basis, we write as $[\eta_1, \eta_2, \eta_3, \eta_4]^\vee_T$. Elements of $(S \oplus T) \otimes \mathbb{R}$ are written as

\[ [x_1, \ldots, x_{22} \mid y_1, \ldots, y_4] \]
using the bases of $S$ and $T$, or as
\[ [\xi_1, \ldots, \xi_22 \mid \eta_1, \ldots, \eta_4] \]
using the dual bases of $S^\vee$ and $T^\vee$.

Consider the following vectors of $S^\vee \oplus T^\vee$:
\[
\begin{aligned}
o_1 &:= \frac{1}{3} [2,2,0,0,0,1,2,1,1,2,2,1,1,2,0,0,1,1,0,0,0 | 1,2,0,0],  \\
o_2 &:= \frac{1}{3} [2,0,2,0,2,1,1,0,2,1,2,1,0,2,2,1,0,1,0,0,0 | 0,0,1,2].
\end{aligned}
\]
We define $\alpha_1, \alpha_2 \in (S \oplus T)^\vee/(S \oplus T)$ by
\[
\alpha_1 := a_1 \mbox{ mod } (S \oplus T), \quad \alpha_2 := a_2 \mbox{ mod } (S \oplus T).
\]
Then $\alpha_1$ and $\alpha_2$ are linearly independent in $(S \oplus T)^\vee/(S \oplus T) \cong \mathbb{F}_3^4$. Since
\[ q_{S \oplus T}(\alpha_1) = q_{S \oplus T}(\alpha_2) = q_{S \oplus T}(\alpha_1 + \alpha_2) = 0, \]
the vectors $\alpha_1$ and $\alpha_2$ generate a maximal isotropic subgroup of $q_{S \oplus T}$. Therefore, by [21, Proposition 1.4.1], the submodule
\[ L := (S \oplus T) + \langle a_1 \rangle + \langle a_2 \rangle \]
of $S^\vee \oplus T^\vee$ is an even unimodular overlattice of $S \oplus T$ into which $S$ and $T$ are primitively embedded.

By construction, $L$ is hyperbolic of rank 26. We choose $\mathcal{P}_L$ to be the connected component that contains $\mathcal{P}_S$. Then, by means of the roots of $L$, we obtain a decomposition of $\mathcal{P}_S$ into $\mathcal{R}_S$-chambers.

The order of $O(T)$ is 288, while the order of $O(q_T)$ is 8. It is easy to check that the natural homomorphism $O(T) \to O(q_T)$ is surjective. Therefore we obtain the following from Proposition 2.7:

**Proposition 4.1.** The action of $O^+(S)$ on $S \otimes \mathbb{R}$ preserves $\mathcal{R}_S$.

We put
\[
\begin{aligned}
w_0 &:= [1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] \mbox{ mod } \mathbb{Z}^{26}  \\
&= [1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1] \vee.
\end{aligned}
\]

Note that the projection $w_{0S} \in S^\vee$ of $w_0$ to $S^\vee$ is equal to $h_0$.

Since $(w_0,w_0)_L = 0$ and $(w_0,h_0)_L > 0$, we see that $w_0$ is on the boundary of the closure of $\mathcal{P}_L$ in $L \otimes \mathbb{R}$.

**Proposition 4.2.** The vector $w_0$ is a Weyl vector, and the $\mathcal{R}_L$-chamber $D_L(w_0)$ is $S$-nondegenerate. The $\mathcal{R}_S$-chamber
\[ D_{S0} := D_L(w_0) \cap (S \otimes \mathbb{R}) \]
contains $w_{0S} = h_0$ in its interior.
The only non-trivial part of the first assertion is that \( \langle w_0 \rangle^\perp / \langle w_0 \rangle \) has no vectors of square norm \(-2\). We put
\[
w'_0 := [7, 6, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 5, 5, 5, 7, 7, 7]^\vee.
\]
Then we have \( (w'_0, w'_0)_L = 0 \) and \( (w_0, w'_0)_L = 1 \). Let \( U \subset L \) be the sublattice generated by \( w_0 \) and \( w'_0 \). Calculating a basis \( \lambda_1, \ldots, \lambda_{24} \) of \( U^\perp \subset L \), we obtain a Gram matrix of \( U^\perp \), which is negative-definite of determinant 1. By the algorithm described in [39, Section 3.1], we verify that there are no vectors of square norm \(-2\) in \( U^\perp \).

We show that \( w_0 \) satisfies the conditions (i) and (ii) given after Definition 2.5. By Proposition 2.2, in order to verify the condition (i), it is enough to show that the function \( Q : U^\perp \to \mathbb{Z} \) given by
\[
Q(\lambda) := (h_0, -\frac{2 + (\lambda, \lambda)_L}{2} w_0 + w'_0 + \lambda)_L
\]
does not take negative values. Using the basis \( \lambda_1, \ldots, \lambda_{24} \) of \( U^\perp \), we can write \( Q \) as an inhomogeneous quadratic function of 24 variables. Its quadratic part turns out to be positive-definite. By the algorithm described in [39, Section 3.1], we verify that there exist no vectors \( \lambda \in U^\perp \) such that \( Q(\lambda) < 0 \). Next we show that \( w_0S = h_0 \in P_S \) has the property required for \( v' \) in the condition (ii), and hence \( h_0 \) is contained in the interior of \( D_{S_0} \). Note that \( w_{0T} = [-1, -1, -1, -1]^T \) is non-zero. Hence we can calculate
\[
LR(w_0, S) = \{ r \in LR(w_0) \mid (r_S, r_S)_S < 0 \}
\]
by the method described in the proof of Proposition 2.6. Then we can easily show that \( h_0 \) satisfies \( (h_0, r)_L > 0 \) for any \( r \in LR(w_0, S) \). \( \square \)

Remark 4.3. There exist exactly four vectors \( \lambda \in U^\perp \) such that \( Q(\lambda) = 0 \). They correspond to the Leech roots \( r \in LR(w_0) \) such that \( r = r_T \).

From the surjectivity of \( O(T) \to O(q_T) \) and Proposition 2.7, we obtain the following:

**Corollary 4.4.** The action of \( \text{Aut}(X, h_0) \) on \( S \otimes \mathbb{R} \) preserves \( D_{S_0} \) and \( \tilde{W}(D_{S_0}) \).

**Proposition 4.5.** The maps \( r \mapsto r_S \) and \( r_S \mapsto (r_S)_S^\perp \) induce bijections
\[
LR(w_0, S) \cong \tilde{W}(D_{S_0}) \cong W(D_{S_0}).
\]
The action of \( \text{Aut}(X, h_0) \) decomposes \( \tilde{W}(D_{S_0}) \) into the three orbits
\[
\tilde{W}_{112} := \tilde{W}(D_{S_0})[1,-2], \quad \tilde{W}_{648} := \tilde{W}(D_{S_0})[2,-4/3] \quad \text{and} \quad \tilde{W}_{5184} := \tilde{W}(D_{S_0})[3,-2/3]
\]
of cardinalities 112, 648 and 5184, respectively, where
\[
\tilde{W}(D_{S_0})[a,n] := \{ r_S \in \tilde{W}(D_{S_0}) \mid (r_S, h_0)_S = a, \ (r_S, r_S)_S = n \}.
\]
The set \( \mathring{W}_{112} \) coincides with the set of the classes \([\ell_i]\) of lines contained in \( X \):

\[
\mathring{W}_{112} = \{[\ell_1], [\ell_2], \ldots, [\ell_{112}]\}.
\]

The sets \( \mathring{W}_{648} \) and \( \mathring{W}_{5184} \) are the orbits of

\[
b_1 = \frac{1}{3}[1, 0, -1, 0, 2, 1, 0, 2, 1, -1, 1, 0, -1, -1, 1, 0, 0, 0]_S \in \mathring{W}_{648}
\]

and

\[
b_2 := \frac{1}{3}[0, 1, -1, 0, 2, 0, 2, 1, 0, 0, -1, 2, 1, 0, 1, -1, 0, 0, 0]_S \in \mathring{W}_{5184}
\]

by the action of \( \text{Aut}(X, h_0) \), respectively.

**Proof.** We have calculated the finite set \( LR(w_0, S) \) in the proof of Proposition 4.2. We have also stored the classes \([\ell_i]\) of the 112 lines and the action of \( \text{Aut}(X, h_0) \) on \( S \) in the computer memory. Thus the assertions of Proposition 4.5 are verified by a direct computation, except for the fact that, for any \( r \in LR(w_0, S) \), the hyperplane \((rs)^1_2\) actually bounds \( D_{S_0} \). This is proved by showing that the point

\[
p := h_0 - \frac{(h_0, rs)_S}{(r, r)_S} r_S
\]

on \((rs)^1_2\) satisfies \( (p, r')_L > 0 \) for any \( r' \in LR(w_0, S) \setminus \{r\} \). \( \square \)

Since Proposition 3.3 implies that the interior point \( h_0 \) of \( D_{S_0} \) is determined by \( \mathring{W}_{112} \) and since \( O(T) \to O(q_T) \) is surjective, we obtain the following from Proposition 2.7:

**Corollary 4.6.** For \( \gamma \in O^+(S) \), the following are equivalent: (i) the interior of \( D_{S_0}^\gamma \) has a common point with \( D_{S_0} \), (ii) \( D_{S_0}^\gamma = D_{S_0} \), (iii) \( \mathring{W}_{112}^{\gamma} = \mathring{W}_{112} \), (iv) \( h_0^\gamma = h_0 \), and (v) \( h_0^0 \in D_{S_0} \).

In particular, we obtain the following:

**Corollary 4.7.** If \( \gamma \in \text{Aut}(X) \) satisfies \( h_0^\gamma \in D_{S_0} \), then \( \gamma \) is in \( \text{Aut}(X, h_0) \).

5. The Automorphisms \( g_1 \) and \( g_2 \)

In order to find automorphisms \( \gamma \in \text{Aut}(X) \) such that \( h_0^\gamma \notin D_{S_0} \), we search for polarizations of degree 2 that are located on the walls \((b_1)_S^1\) and \((b_2)_S^1\).

We fix terminologies and notation. For a vector \( v \in S \), we denote by \( L_v \to X \) a line bundle defined over \( F_0 \) whose class is \( v \) (see Remark 3.2). We say that a vector \( h \in S \) is a polarization of degree \( d \) if \((h, h)_S = d\) and the complete linear system \(|L_h|\) is nonempty and has no fixed components. If \( h \) is a polarization, then \(|L_h|\) has no base-points by [29, Corollary 3.2] and hence defines a morphism

\[
\Phi_h : X \to \mathbb{P}^N,
\]

where \( N = \dim |L_h| \).
A polynomial in \( \mathbb{F}_9[w, x, y] \) is said to be of normal form if its degree with respect to \( w \) is \( \leq 3 \). For each polynomial \( G \in \mathbb{F}_9[w, x, y] \), there exists a unique polynomial \( \overline{G} \) of normal form such that

\[
G \equiv \overline{G} \mod (w^4 + x^4 + y^4 + 1).
\]

We say that \( \overline{G} \) is the normal form of \( G \). For any \( d \in \mathbb{Z} \), the vector space \( H^0(X, L_{dh_0}) \) over \( \mathbb{F}_9 \) is naturally identified with the vector subspace

\[
\Gamma(d) := \{ G \in \mathbb{F}_9[w, x, y] \mid G \text{ is of normal form with total degree } \leq d \}
\]

of \( \mathbb{F}_9[w, x, y] \). For an ideal \( J \) of \( \mathbb{F}_9[w, x, y] \), we put

\[
\Gamma(d, J) := \Gamma(d) \cap J.
\]

A basis of \( \Gamma(d, J) \) is easily obtained by a Gröbner basis of \( J \). Let \( \ell_i \) be a line contained in \( X \). We denote by \( I_i \subset \mathbb{F}_9[w, x, y] \) the affine defining ideal of \( \ell_i \) in \( \mathbb{P}^3 \) (see Table 3.1), and put

\[
I_i^{(\nu)} := I_i^{\prime} + (w^4 + x^4 + y^4 + 1) \subset \mathbb{F}_9[w, x, y]
\]

for nonnegative integers \( \nu \). Suppose that \( v \in S \) is written as

\[
v = d h_0 - \sum_{i=1}^{112} a_i [\ell_i],
\]

where \( a_i \) are nonnegative integers. Then there exists a natural isomorphism

\[
H^0(X, \mathcal{L}_v) \cong \Gamma(d, \bigcap_{i=1}^{112} I_i^{(a_i)})
\]

with the property that, for another vector \( v' = d' h_0 - \sum_{i=1}^{a'_i} a_i' [\ell_i] \) with \( a_i' \in \mathbb{Z}_{\geq 0} \), the multiplication homomorphism

\[
H^0(X, \mathcal{L}_v) \times H^0(X, \mathcal{L}_{v'}) \to H^0(X, \mathcal{L}_{v+v'})
\]

is identified with

\[
\Gamma(d, \bigcap_{i=1}^{a_i} I_i^{(a_i)}) \times \Gamma(d', \bigcap_{i=1}^{a_i'} I_i^{(a_i')}) \to \Gamma(d + d', \bigcap_{i=1}^{a_i + a_i'} I_i^{(a_i + a_i')})
\]

given by \( (\mathcal{G}, \mathcal{G}') \mapsto \mathcal{G} \mathcal{G}' \).

Proposition 1.1 in Introduction is an immediate consequence of the following:

**Proposition 5.1.** Consider the vectors

\[
m_1 := [-1, 0, -1, -1, 2, 2, 1, 1, 2, 0, -1, 1, 1, -1, 0, 0, 1, -1, 0, 0]_S \quad \text{and}
\]

\[
m_2 := [2, 2, 1, 2, 1, -1, 1, 1, 1, 0, -1, 0, 0, 0, 0, 0, -1, -1, 0, 0]_S
\]

of \( S \). Then each \( m_\nu \) is a polarization of degree 2. If we choose a basis of the vector space \( H^0(X, \mathcal{L}_{m_\nu}) \) appropriately, the morphism \( \Phi_{m_\nu} : X \to \mathbb{P}^2 \) associated with \( \mathcal{L}_{m_\nu} \) coincides with the morphism \( \phi_\nu : X \to \mathbb{P}^2 \) given in the statement of Proposition 1.1.
Proof. We have \((m_{\nu}, m_{\nu})_S = 2\). By the method described in [39, Section 4.1], we see that \(m_{\nu}\) is a polarization; namely, we verify that the sets
\[
\{ v \in S \mid (v, v)_S = -2, \ (v, m_{\nu})_S < 0, \ (v, h_0)_S > 0 \}
\]
and
\[
\{ v \in S \mid (v, v)_S = 0, \ (v, m_{\nu})_S = 1 \}
\]
are both empty. Since
\begin{align}
\{ v \in S \mid (v, v)_S = -2, \ (v, m_{\nu})_S < 0, \ (v, h_0)_S > 0 \} &= \emptyset \\
\{ v \in S \mid (v, v)_S = 0, \ (v, m_{\nu})_S = 1 \} &= \emptyset
\end{align}
are both empty. Since
\[
m_1 = 3h_0 - ([\ell_{21}] + [\ell_{22}] + [\ell_{50}] + [\ell_{65}] + [\ell_{88}])
\]
and
\[
m_2 = 5h_0 - ([\ell_1] + [\ell_3] + [\ell_6] + [\ell_{18}] + [\ell_{35}] + [\ell_{74}] + [\ell_{90}] + [\ell_{92}] + [\ell_{110}] + [\ell_{111}])
\]
the vector spaces \(H^0(X, \mathcal{L}_{m_1})\) and \(H^0(X, \mathcal{L}_{m_2})\) are identified with the subspaces
\[
\Gamma_1 := \Gamma(3, I_{21} \cap I_{22} \cap I_{50} \cap I_{63} \cap I_{65} \cap I_{88}) \quad \text{and} \\
\Gamma_2 := \Gamma(5, I_1 \cap I_3 \cap I_6 \cap I_{18} \cap I_{35} \cap I_{74} \cap I_{90} \cap I_{92} \cap I_{110} \cap I_{111})
\]
of \(F_3[w, x, y]\), respectively. We calculate a basis of \(\Gamma_{\nu}\) by means of Gröbner bases of the ideals \(I_{\nu}\). The set \(\{F_{\nu 0}, F_{\nu 1}, F_{\nu 2}\}\) of polynomials in Table 1.1 is just a basis of \(\Gamma_{\nu}\) thus calculated.

\[\square\]

Remark 5.2. The polarizations \(m_1\) and \(m_2\) in Proposition 5.1 are located on the hyperplanes \((b_1)_{\frac{2}{3}}\) and \((b_2)_{\frac{2}{3}}\) bounding \(D_{S_0}\), respectively, where \(b_1 \in \tilde{W}_{648}\) and \(b_2 \in \tilde{W}_{5184}\) are given in Proposition 4.5.

We now prove Proposition 1.2.

Proof. The set \(\text{Exc}(\phi_{\nu})\) of the classes of \((-2)\)-curves contracted by \(\phi_{\nu} : X \to \mathbb{P}^2\) is calculated by the method described in [39, Section 4.2]. We first calculate the set
\[
R^+_\nu := \{ v \in S \mid (v, v)_S = -2, \ (v, m_{\nu})_S = 0, \ (v, h_0)_S > 0 \}.
\]
It turns out that every element of \(R^+_\nu\) is written as a linear combination with coefficients in \(\mathbb{Z}_{\geq 0}\) of elements \(l \in R^+_\nu\) such that \((l, h_0)_S = 1\). Hence we have
\[
\text{Exc}(\phi_{\nu}) = \{ l \in R^+_\nu \mid (l, h_0)_S = 1 \}.
\]
The \(ADE\)-type of the root system \(\text{Exc}(\phi_{\nu})\) is equal to \(6A_1 + 4A_2\) for \(\nu = 1\) and \(A_1 + A_2 + 2A_3 + 2A_4\) for \(\nu = 2\). Thus the assertion on the \(ADE\)-type of the singularities of \(Y_{\nu}\) is proved. Moreover we have proved that all \((-2)\)-curves contracted by \(\phi_{\nu} : X \to \mathbb{P}^2\) are lines. See Tables 5.1 and 5.2, in which the lines \(\ell_{k_1}, \ldots, \ell_{k_r}\) contracted by \(\phi_{\nu}\) to a singular point \(P\) of type \(A_r\) are indicated in such an order that \((\ell_{k_j}, \ell_{k_{j+1}})S = 1\) holds for \(j = 1, \ldots, r - 1\).

The defining equation \(f_{\nu} = 0\) of the branch curve \(B_{\nu} \subset \mathbb{P}^2\) is calculated by the method given in [39, Section 5]. We calculate a basis of the vector space \(H^0(X, \mathcal{L}_{3m_{\nu}})\) of dimension 11 using (5.2), (5.3) and Gröbner bases of \(I^3_i\). Note that
the ten normal forms $M_{\nu,1}, \ldots, M_{\nu,10}$ of the cubic monomials of $F_{\nu 0}, F_{\nu 1}, F_{\nu 2}$ are contained in $H^0(X, L_{3m_{\nu}})$. We choose a polynomial $G_{\nu} \in H^0(X, L_{3m_{\nu}})$ that is not contained in the linear span of $M_{\nu,1}, \ldots, M_{\nu,10}$. In the vector space $H^0(X, L_{6m_{\nu}})$ of dimension 38, the 39 normal forms of the monomials of $G_{\nu}, F_{\nu 0}, F_{\nu 1}, F_{\nu 2}$ of weighted degree 6 with weight $\deg G_{\nu} = 3$ and $\deg F_{\nu j} = 1$ have a non-trivial linear relation. Note that this linear relation is quadratic with respect to $G_{\nu}$. Completing the square and re-choosing $G_{\nu}$ appropriately, we confirm that

$$G_{\nu}^2 + f_{\nu}(F_{\nu 0}, F_{\nu 1}, F_{\nu 2}) = 0$$

holds. Hence $Y_{\nu}$ is defined by $y^2 + f_{\nu}(x_0, x_1, x_2) = 0$. \hfill $\square$

Remark 5.3. In order to obtain a defining equation of $B_{\nu}$ with coefficients in $\mathbb{F}_3$, we have to choose the basis $F_{\nu 0}, F_{\nu 1}, F_{\nu 2}$ of $\Gamma_{\nu} = H^0(X, L_{m_{\nu}})$ carefully. See [39, Section 6.10] for the method.

Remark 5.4. The polynomial

$$G_1 = G_{1(0)}(x,y) + G_{1(1)}(x,y) w + G_{1(2)}(x,y) w^2 + G_{1(3)}(x,y) w^3$$

is given in Table 5.3. The polynomial $G_2$ is too large to be presented in the paper (see [40]).
Then the action $\gamma_0$ such a way that $(\nu_0)$ are contracted to a singular point

$\gamma_0 = (1 + i) x + (1 + i) y + i x^2 - (1 + i) y^2 - xy^2$

$+ (1 + i) y^3 - (1 - i) x^4 - (1 + i) x^3 y - xy^3 - (1 - i) y^4 - (1 - i) x^5 - x^3 y^2$

$- i x y^2 - (1 - i) x y^3 + (1 + i) y^5 - (1 - i) x^6 + x^2 y + i x^4 y^2 - (1 - i) x^2 y^3$

$+ (1 + i) x^2 y^4 - i x y^5 + (1 + i) y^6 + (1 + i) x^7 + x^4 y^3 + (1 + i) x y^4$

$+ i x y^6 - i y^7 + i x^8 + (1 + i) x^2 y - i x^6 y^2 - (1 + i) y^8 + (1 + i) x y^4$

$+ (1 - i) x y^5 - (1 + i) x^3 y^6 - (1 - i) x^2 y^7 - (1 - i) x y^8 - (1 + i) y^9$

$G_1(0) = (1 - i) + (1 - i) x - (1 + i) x^2 - xy + iy^2 + x^3 - (1 - i) x^2 y - xy^2$

$G_1(1) = (1 - i) + (1 - i) x - (1 - i) x^2 - xy + iy^2 + x^3 - (1 - i) x^2 y - xy^2$

$G_1(2) = (1 - i) + (1 - i) x - (1 + i) xy + iy^2 - (1 - i) x^3 - (1 + i) x^2 y - i x y^2$

$G_1(3) = (1 + i) x - (1 - i) y - (1 - i) x^2 - (1 - i) xy + (1 + i) y^2 - x^3 - x^2 y$

$- xy^2 + y^3 - (1 - i) x^2 - i x^3 y + (1 - i) x y^3 - iy^4 + i x^3 + x^4 y$

$+ (1 + i) x^3 y^2 + (1 + i) x^2 y^3 + (1 + i) x y^4 + (1 + i) y^5 - x^6 - (1 - i) x^5 y$

$+ (1 + i) x^2 y^4 + i x^3 y^3 - (1 - i) x^2 y^4 - (1 + i) xy^5 + (1 + i) y^6$

| Table 5.3. Polynomial $G_1$ |

Proposition 5.5. Let $g_1$ and $g_2$ be the involutions of $X$ defined in Theorem 1.3. Then the action $g_{r\nu}$ on $S$ is given by $v \mapsto vA_{\nu}$, where $A_{\nu}$ is the matrix given in Tables 5.4 and 5.5.

Proof. Recall that $\text{Exc}(\phi_0)$ is the set of the classes of $(-2)$-curves contracted by $\phi_0 : X \to \mathbb{P}^2$. Suppose that $\gamma_1, \ldots, \gamma_r \in \text{Exc}(\phi_0)$ are the classes of $(-2)$-curves that are contracted to a singular point $P \in \text{Sing}(B_\nu)$ of type $A_r$. We index them in such a way that $(\gamma_{\nu}, \gamma_{\nu+1})_S = 1$ holds for $\nu = 1, \ldots, r - 1$. Then $g_{\nu\nu}$ interchanges $\gamma_\nu$ and $\gamma_{\nu+1}$. Let $V(P) \subset S \otimes \mathbb{Q}$ denote the linear span of the invariant vectors $\gamma_\nu + \gamma_{\nu+1}$. Then the eigenspace of $g_{\nu\nu}$ on $S \otimes \mathbb{Q}$ with eigenvalue 1 is equal to

$$\langle m_\nu \rangle \oplus \bigoplus_{P \in \text{Sing}(B_{\nu})} V(P),$$

and the eigenspace with eigenvalue $-1$ is its orthogonal complement. \hfill \Box

Using the matrix representations $A_{\nu}$ of $g_{\nu\nu}$, we verify the following facts:

1. The eigenspace of $g_{\nu\nu}$ with eigenvalue 1 is contained in $(b_{\nu})_{(2)}$. In particular, we have $b_{\nu} = -b_{\nu}$. 


(2) The vector $h_0^{b_0}$ is equal to the image of $h_0$ by the reflection into the wall $(b_0)_S$, that is $h_0^{b_0} = h_0 + 3b_1$ and $h_0^{b_0} = h_0 + 9b_2$ hold.

Since $\text{Aut}(X, h_0)$ acts on each of $\widetilde{W}_{648}$ and $\widetilde{W}_{5184}$ transitively, we obtain the following:

**Corollary 5.6.** For any $r_S \in \widetilde{W}_{648} \cup \widetilde{W}_{5184}$, there exists $\tau \in \text{Aut}(X, h_0)$ such that

$$h_0^{b_0 \tau} = h_0 + c_\nu r_S$$

holds, where $\nu = 1$ and $c_1 = 3$ if $r_S \in \widetilde{W}_{648}$ while $\nu = 2$ and $c_2 = 9$ if $r_S \in \widetilde{W}_{5184}$.

![Table 5.4. The matrix $A_1$](image-url)
6. Proof of Theorem 1.3

We denote by

\[ G := \langle \text{Aut}(X, h_0), g_1, g_2 \rangle \]

the subgroup of \(\text{Aut}(X)\) generated by \(\text{Aut}(X, h_0), g_1\) and \(g_2\). Note that the action of \(\text{Aut}(X)\) on \(S\) preserves the set of nef classes.

**Theorem 6.1.** If \(v \in S\) is nef, there exists \(\gamma \in G\) such that \(v^\gamma \in D_{S_0}\).

**Proof.** Let \(\gamma \in G\) be an element such that \((v^\gamma, h_0)_S\) attains

\[ \min\{(v^\gamma', h_0)_S \mid \gamma' \in G\}. \]
We show that \((v^\gamma, r_S)_S \geq 0\) holds for any \(r_S \in \tilde{W}(D_{S_0})\). If \(r_S \in \tilde{W}_{112}\), then \(r_S = [\ell_i]\) for some line \(\ell_i \subset X\), and hence \((v^\gamma, r_S)_S \geq 0\) holds because \(v^\gamma\) is nef. Suppose that \(r_S \in \tilde{W}_{648} \cup \tilde{W}_{5184}\). By Corollary 5.6, there exists \(\tau \in \text{Aut}(X, h_0)\) such that \(h_{0,\tau}^{\gamma}\tau = h_0 + c_r r_S\) holds, where \(\nu = 1\) and \(c_1 = 3\) if \(r_S \in \tilde{W}_{648}\) while \(\nu = 2\) and \(c_2 = 9\) if \(r_S \in \tilde{W}_{5184}\). Since \(\gamma^{\tau^{-1}} g_{\nu} \in G\), we have
\[
(v^\gamma, h_0)_S \leq (v^{\gamma\tau^{-1} g_{\nu}}, h_0)_S = (v^\gamma, h_{0,\tau}^{\gamma})_S = (v^\gamma, h_0)_S + c_{\nu} (v^\gamma, r_S)_S.
\]
Therefore \((v^\gamma, r_S)_S \geq 0\) holds. \(\square\)

The properties (1), (2), (3) of \(D_{S_0}\) stated in Introduction follow from Corollaries 4.4, 4.6, 4.7 and Theorem 6.1. We now prove Theorem 1.3.

**Proof.** By Corollary 4.7, it is enough to show that, for any \(\gamma \in \text{Aut}(X)\), there exists \(\gamma' \in G\) such that \(h_0^{\gamma \gamma'} \in D_{S_0}\) holds. Since \(h_0^{\gamma}\) is nef, this follows from Theorem 6.1. \(\square\)

As a geometric consequence, we present the following:

**Corollary 6.2.** The group \(\text{Aut}(X)\) acts on the set of \((-2)\)-curves on \(X\) transitively with the stabilizer group of order 13063680/112 = 116640.

**Remark 6.3.** See Sterk [46] for a general result on the action of the automorphism group on the set of \((-2)\)-curves.

### 7. The Fermat Quartic Polarizations for \(g_1\) and \(g_2\)

A polarization \(h \in S\) of degree 4 is said to be a Fermat quartic polarization if, by choosing an appropriate basis of \(H^0(X, L_h)\), the morphism \(\Phi_h : X \to \mathbb{P}^3\) associated with \(|L_h|\) induces an automorphism of \(X \subset \mathbb{P}^3\). It is obvious that \(h_0^\gamma\) is a Fermat quartic polarization for any \(\gamma \in \text{Aut}(X)\). Conversely, if \(h\) is a Fermat quartic polarization, then the pull-back of \(h_0\) by the automorphism \(\Phi_h\) of \(X\) is \(h\).

Therefore the set of Fermat quartic polarizations is the orbit of \(h_0\) by the action of \(\text{Aut}(X)\) on \(S\). Consider the Fermat quartic polarizations
\[
\begin{align*}
\gamma_1 & := h_{0,1}^{g_1} = h_0 A_1 = [0, 1, 0, 1, 2, 1, 1, 0, 2, 1, -1, 1, 0, -1, -1, 1, 1, 0, 1, 0, 0, 0]_S, \\
\gamma_2 & := h_{0,2}^{g_2} = h_0 A_2 = [1, 4, -2, 1, 6, 0, 6, 3, 3, 0, 0, -3, 6, 3, 0, 3, 3, -3, 0, 0, 0, 0]_S.
\end{align*}
\]

Using the equalities
\[
\begin{align*}
\gamma_1 &= 6h_0 - (\lfloor \ell_3 \rfloor + \lfloor \ell_6 \rfloor + \lfloor \ell_s \rfloor + \lfloor \ell_{14} \rfloor + \lfloor \ell_{15} \rfloor + \lfloor \ell_{17} \rfloor + \lfloor \ell_{19} \rfloor + \\
&+ \lfloor \ell_{22} \rfloor + \lfloor \ell_{31} \rfloor + \lfloor \ell_{34} \rfloor + \lfloor \ell_{70} \rfloor + \lfloor \ell_{79} \rfloor + \lfloor \ell_{92} \rfloor), \\
\gamma_2 &= 15h_0 - (3\lfloor \ell_3 \rfloor + 4 \lfloor \ell_6 \rfloor + \lfloor \ell_{13} \rfloor + \lfloor \ell_{14} \rfloor + 3 \lfloor \ell_{18} \rfloor + \\
&+ \lfloor \ell_{22} \rfloor + + \lfloor \ell_{26} \rfloor + \lfloor \ell_{27} \rfloor + 2 \lfloor \ell_{35} \rfloor + \lfloor \ell_{44} \rfloor + 2 \lfloor \ell_{50} \rfloor + \\
&+ 3 \lfloor \ell_{52} \rfloor + \lfloor \ell_{93} \rfloor + \lfloor \ell_{106} \rfloor + \lfloor \ell_{108} \rfloor + 3 \lfloor \ell_{111} \rfloor),
\end{align*}
\]
we obtain another description of the involutions \(g_1\) and \(g_2\).
Theorem 7.1. Let \((w, x, y)\) be the affine coordinates of \(\mathbb{P}^3\) with \(z = 1\), and let
\[
H_{1j}(w, x, y) = H_{1j0}(x, y) + H_{1j1}(x, y)w + H_{1j2}(x, y)w^2 + H_{1j3}(x, y)w^3
\]
be polynomials given in Table 7.1. Then the rational map
\[
(w, x, y) \mapsto [H_{10} : H_{11} : H_{12} : H_{13}] \in \mathbb{P}^3
\]
gives the involution \(g_1\) of \(X\).

Proof. We put
\[
Z := \{3, 6, 8, 14, 15, 17, 19, 22, 31, 34, 63, 70, 79, 92\},
\]
which is the set of indices of lines on \(X\) that appear in the right-hand side of (7.1). The polynomials \(H_{10}, H_{11}, H_{12}, H_{13}\) form a basis of the vector space
\[
H^0(X, \mathcal{L}_{h_1}) \cong \Gamma(6, \bigcap_{i \in Z} \mathcal{I}_i).
\]
(See Section 5 for the notation.) We can easily verify that
\[
H_{10}^4 + H_{11}^4 + H_{12}^4 + H_{13}^4 \equiv 0 \mod (w^4 + x^4 + y^4 + 1)
\]
holds. Hence the rational map (7.3) induces an automorphism \(g'\) of \(X\). We prove \(g' = g_1\) by showing that the action \(g'_v\) of \(g'\) on \(S\) is equal to the action \(v \mapsto vA_1\) of \(g_1\). We homogenize the polynomials \(H_{1j}\) to \(\tilde{H}_{1j}(w, x, y, z)\) so that \(g'\) is given by
\[
[w : x : y : z] \mapsto [\tilde{H}_{10} : \tilde{H}_{11} : \tilde{H}_{12} : \tilde{H}_{13}].
\]
Let \(\ell_k\) be a line on \(X\) whose index \(k\) is not in \(Z\). We calculate a parametric representation
\[
[u : v] \mapsto [l_{k0} : l_{k1} : l_{k2} : l_{k3}]
\]
of \(\ell_k\) in \(\mathbb{P}^3\), where \(u, v\) are homogeneous coordinates of \(\mathbb{P}^3\) and \(l_{kw}\) are homogeneous linear polynomials of \(u, v\). We put
\[
L_{1j}^{(k)} := \tilde{H}_{1j}(l_{k0}, l_{k1}, l_{k2}, l_{k3})
\]
for \(j = 0, \ldots, 3\), which are homogeneous polynomials of \(u, v\). Let \(M^{(k)}\) be the greatest common divisor of \(L_{10}^{(k)}, L_{11}^{(k)}, L_{12}^{(k)}, L_{13}^{(k)}\) in \(\mathbb{F}_9[u, v]\). Then
\[
\rho_k : [u : v] \mapsto [L_{10}^{(k)} / M^{(k)} : L_{11}^{(k)} / M^{(k)} : L_{12}^{(k)} / M^{(k)} : L_{13}^{(k)} / M^{(k)}]
\]
is a parametric representation of the image of \(\ell_k\) by \(g'\). (If \(k \in Z\), then \(L_{1j}^{(k)}\) are constantly equal to 0.) Pulling back the defining homogeneous ideal of \(\ell_k'\) by \(\rho_k\), we can calculate the intersection number \(\langle [\ell_k']^g, [\ell_k'] \rangle_S\). Since the classes \([\ell_k']\) with \(k \notin Z\) span \(S \otimes \mathbb{Q}\), we can calculate the action \(g'_v\) of \(g'\) on \(S\), which turns out to be equal to \(v \mapsto vA_1\). \(\square\)

Remark 7.2. We have a similar list of polynomials \(H_{20}, H_{21}, H_{22}, H_{23}\) that gives the involution \(g_2\). They are, however, too large to be presented in the paper (see [40]).
\[ H_{100} = -1 - (1 - i) x - (1 - i) x^2 - (1 + i) y^2 - i x^3 - (1 - i) xy^2 + x^4 - (1 - i) x^3 y \\
  + (1 + i) x^2 y^2 + (1 + i) y^3 - (1 - i) y^4 - (1 - i) x^5 + (1 + i) x^4 y - (1 - i) x^3 y^2 - i x^2 y^3 \\
  - i xy^4 + i y^5 + i x^6 - x^7 y - (1 + i) x^4 y^2 - (1 + i) x^2 y^3 + (1 + i) xy^5 + (1 - i) y^6 \]

\[ H_{101} = (1 + i) + x + (1 - i) y - i x^2 - (1 + i) xy + iy^2 + i x^3 - x^2 y \\
  - i xy^2 + y^3 + x^3 y + (1 - i) x^2 y^2 - (1 - i) xy^3 + (1 - i) y^4 \\
  + x^5 + i x^4 y + x^3 y^2 - i x^2 y^3 + (1 - i) xy^4 - (1 - i) y^5 \]

\[ H_{102} = i + x + (1 + i) y + x^2 + (1 + i) y^2 + (1 + i) x^3 - x^2 y \\
  - (1 + i) xy^2 + i y^3 + (1 + i) x^3 - (1 - i) x^2 y^2 + x^3 y + (1 + i) y^4 \]

\[ H_{103} = (1 - i) - (1 - i) x + (1 - i) y - (1 + i) x^2 - (1 - i) xy + (1 + i) x^3 - (1 + i) y^3 \]

\[ H_{110} = -i + i x + y - (1 + i) x^2 + xy - (1 - i) y^2 - x^3 - (1 - i) x^2 y + (1 + i) xy^2 - y^3 \\
  + (1 + i) x^4 - ix^2 y - (1 - i) x^2 y + x^3 y + (1 - i) y^4 - (1 + i) x^5 + (1 - i) x^3 y + i x y^3 \\
  - (1 + i) xy^4 - (1 + i) y^5 + i x^6 + (1 + i) x^4 y^2 + (1 + i) x^3 y^3 + (1 - i) xy^5 - (1 - i) y^6 \]

\[ H_{111} = -(1 - i) + x + (1 + i) y - (1 + i) x^2 - i xy - iy^2 + (1 - i) x^3 \\
  - i xy^2 - y^3 + (1 + i) x^4 + (1 - i) x^2 y - x^2 y^2 + (1 + i) y^3 \\
  - iy^4 + (1 - i) x^5 + i x^4 y + (1 - i) x^2 y^3 + (1 - i) xy^4 - y^5 \]

\[ H_{112} = -1 + (1 + i) y + x^2 - (1 - i) xy - (1 + i) y^2 - x^2 y + (1 + i) x^3 y \\
  - (1 + i) y^3 + (1 - i) x^4 + (1 + i) x^3 y - (1 + i) x^2 y^2 - xy^3 - (1 + i) y^4 \]

\[ H_{113} = (1 + i) - x + y + x^2 - i y^2 - (1 - i) x^3 + i x^2 y - (1 - i) x y^2 - i y^3 \]

\[ H_{120} = (1 + i) + (1 + i) x + (1 + i) y + (1 - i) x^2 + y^2 + (1 + i) x^3 + (1 + i) x^2 y \\
  - i xy^2 - y^3 - (1 - i) x^3 y + (1 - i) x^2 y^2 - (1 + i) xy^3 + (1 - i) y^4 \\
  + (1 - i) x^5 - i x^4 y + (1 - i) x^2 y^2 - (1 + i) x^2 y^3 + i xy^4 - y^5 + x^6 \\
  - (1 + i) x^5 y - (1 - i) x^4 y^2 + x^3 y^3 - i x^2 y^4 - (1 - i) xy^5 + (1 - i) y^6 \]

\[ H_{121} = i + x + xy - (1 + i) y^2 + x^3 - (1 + i) x^2 y - (1 - i) xy^2 + (1 + i) y^3 + x^3 - (1 - i) x^3 y \\
  - (-1 - i) x^3 y^2 + (1 + i) xy^3 - (1 - i) x^3 y + (1 + i) x^2 y^3 + (1 + i) x^3 y^3 + (1 - i) y^5 \]

\[ H_{122} = (1 - i) - x - (1 + i) y + ix^2 - (1 - i) xy - (1 + i) y^2 - x^3 - (1 - i) xy^2 \\
  - iy^3 - (1 + i) x^4 - (1 - i) x^3 y - (1 + i) x^2 y^2 - xy^3 + (1 + i) y^4 \]

\[ H_{123} = 1 - (1 + i) x + (1 - i) y + x^2 + i xy + y^2 - (1 + i) x^3 + (1 - i) x^2 y - (1 + i) xy^2 + (1 + i) y^3 \]

\[ H_{130} = -(1 - i) + i x + (1 + i) y - (1 + i) x^2 + (1 - i) xy + (1 - i) y^2 + x^3 - (1 - i) x^2 y + i y^3 \\
  - (1 - i) x^4 + i x^3 y + x^2 y^2 - (1 + i) y^4 + (1 + i) x^5 - (1 - i) x^4 y + (1 - i) x^3 y^2 - x^2 y^3 \\
  - (1 + i) y^5 - (1 + i) x^6 - (1 - i) x^5 y - (1 + i) x^4 y^2 + i x^3 y^3 + i x^2 y^4 + i xy^5 + (1 + i) y^6 \]

\[ H_{131} = 1 - x + 1 + i y - y - (1 - i) x^2 + (1 + i) xy - iy^2 - (1 + i) x^3 - i x^2 y - x^2 y^2 + i y^3 \\
  - x^3 - x^2 y^2 + xy - (1 + i) y^4 - (1 - i) x^5 + x^4 y + (1 - i) x^3 y^2 - i x^2 y^3 + (1 + i) xy^4 \]

\[ H_{132} = (1 + i) + ix + y - x^2 + xy + y^2 + i x^3 - (1 - i) x^2 y \\
  - (1 + i) x^2 y - (1 - i) x^4 - x^2 y^2 - i x^3 - (1 - i) y^4 \]

\[ H_{133} = i y + x^2 + (1 + i) xy - (1 - i) y^2 \]

**Table 7.1. Polynomials $H_{ij}$**
Remark 7.3. The polynomials $H_{10}, H_{11}, H_{12}, H_{13}$ are found by the following method. Let $H_0', H_1', H_2', H_3'$ be an arbitrary basis of $\Gamma(6, \cap_{i \in Z} I_i) \cong H^0(X, \mathcal{L}_h)$. Then the normal forms of the quartic monomials of $H_0', H_1', H_2', H_3'$ are subject to a linear relation of the following form (see [33, n. 3] or [39, Theorem 6.11]):

$$\sum_{i,j=0}^3 a_{ij} H_i^* H_j^* = 0,$$

where the coefficients $a_{ij} \in \mathbb{F}_9$ satisfy $a_{ij} = a_{ij}^3$ and $\det(a_{ij}) \neq 0$; that is, the matrix $(a_{ij})$ is non-singular Hermitian. We search for $B \in \text{GL}_3(\mathbb{F}_9)$ such that

$$(a_{ij}) = B^{(3)} B$$

holds, where $B^{(3)}$ is obtained from $B$ by applying $x \mapsto x^3$ to the entries, and put

$$(H_0'', H_1'', H_2'', H_3'') = (H_0', H_1', H_2', H_3') B.$$

Then $H_0'', H_1'', H_2'', H_3''$ satisfy

$$H_0'' + H_1'' + H_2'' + H_3'' \equiv 0 \mod (w^4 + x^4 + y^4 + 1).$$

Therefore $(w, x, y) \mapsto [H_0'': H_1'': H_2'': H_3'']$ induces an automorphism $g''$ of $X$. Using the method described in the proof of Theorem 7.1, we calculate the matrix $A''$ such that the action $g''_v$ of $g''$ on $S$ is given by $v \mapsto vA''$. Next we search for $\tau \in \text{PGU}_4(\mathbb{F}_9)$ such that $A'' T_{\tau}$ is equal to $A_1$, where $T_{\tau} \in O^+(S)$ is the matrix representation of $\tau$. Then the polynomials

$$(H_{10}, H_{11}, H_{12}, H_{13}) := (H_0'', H_1'', H_2'', H_3'') \tau$$

have the required property.

Remark 7.4. We have calculated the images of the $\mathbb{F}_9$-rational points of $X$ by the morphisms $\psi_\nu : X \to Y_\nu$ and $g_\nu : X \to X$, and confirmed that they are compatible (see [40]).

8. Generators of $O^+(S)$

Let $F \in O^+(S)$ denote the isometry of $S$ obtained from the Frobenius action $\phi$ of $\mathbb{F}_9$ over $\mathbb{F}_3$ on $X$. Calculating the action of $\phi$ on the lines ($\ell^0_1 = \ell_6, \ell^0_2 = \ell_5, \ell^0_3 = \ell_8, \ell^0_4 = \ell_7, \ldots$), we see that $F$ is given $v \mapsto v A_F$, where $A_F$ is the matrix presented in Table 8.1. Since $h_0^F = h_0$, we have $D_{S_0}^F = D_{S_0}$ by Corollary 4.6.

Proposition 8.1. The automorphism group $\text{Aut}(D_{S_0}) \subset O^+(S)$ of $D_{S_0}$ is the split extension of $(F) \cong \mathbb{Z}/2\mathbb{Z}$ by $\text{Aut}(X, h_0)$.

Proof. Since we have calculated the representation (3.2) of $\text{Aut}(X, h_0)$ into $O^+(S)$, we can verify that $F \notin \text{Aut}(X, h_0)$. Therefore it is enough to show that the order of $\text{Aut}(D_{S_0})$ is equal to 2 times $|\text{Aut}(X, h_0)|$. Since $|\text{PGU}_4(\mathbb{F}_9)|$ is equal to 4 times $|\text{PSU}_4(\mathbb{F}_9)|$, this follows from [15, Lemma 2.1] (see also [9, p. 52]). \qed
Since $(\ell_1, [\ell_1])_S = -2$, the reflection $s_1 : S \otimes \mathbb{R} \rightarrow S \otimes \mathbb{R}$ into the hyperplane $(\ell_1)^{\perp}$ is contained in $O^+(S)$. In the same way as the proof of Theorem 1.3, we obtain the following:

**Theorem 8.2.** The automorphism orthogonal group $O^+(S)$ of the Néron-Severi lattice $S$ of $X$ is generated by $\text{Aut}(X, h_0) = \text{PGU}_4(\mathbb{F}_9)$, $g_1$, $g_2$, $F$ and $s_1$.

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