Adaptive efficient robust sequential analysis for autoregressive big data models *

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Abstract

In this paper we consider high dimension models based on dependent observations defined through autoregressive processes. For such models we develop an adaptive efficient estimation method via the robust sequential model selection procedures. To this end, firstly, using the Van Trees inequality, we obtain a sharp lower bound for robust risks in an explicit form given by the famous Pinsker constant (see in [21, 20] for details). It should be noted, that for such models this constant is calculated for the first time. Then, using the weighted least square method and sharp non asymptotic oracle inequalities from [4] we provide the efficiency property in the minimax sense for the proposed estimation procedure, i.e. we establish, that the upper bound for its risk coincides with the obtained lower bound. It should be emphasized that this property is obtained without using sparse conditions and in the adaptive setting when the parameter dimension and model regularity are unknown.

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1 Introduction

1.1 Problem and motivations

We study the observations model defined for $1 \leq j \leq n$ through the following difference equation

$$y_j = \left(\sum_{i=1}^{q} \beta_i \psi_i(x_j)\right) y_{j-1} + \xi_j, \quad x_j = a + \frac{(b-a)j}{n},$$

(1.1)

where the initial value $y_0$ is a non random known constant, $(\psi_i)_{i \geq 1}$ are known linearly independent functions, $a < b$ are fixed known constants and $(\xi_j)_{j \geq 1}$ are i.i.d. unobservable random variables with an unknown density distribution $p$ from some functional class which will be specified later.

The problem is to estimate the unknown parameters $(\beta_i)_{1 \leq i \leq q}$ in the high dimension setting, i.e. when the number of parameters more than the number of observations, i.e. $q > n$.

Usually, in these cases one uses one of two methods: Lasso algorithm or the Dantzig selector method (see, for example, [14] and the references therein). It should be emphasized, that in the papers devoted to big data models the number of parameters $q$ must be known and, moreover, usually it is assumed sparse conditions to provide optimality properties. Therefore, unfortunately, these methods can’t be used to estimate, for example, the number of parameters $q$. To overcome these restrictions in this paper similar to the approach proposed in [13] we study this problem in the nonparametric setting, i.e. we embed the observations (1.1) in the general model defined as

$$y_j = S(x_j)y_{j-1} + \xi_j,$$

(1.2)

where $S(\cdot) \in L_2[a, b]$ is unknown function. The nonparametric setting allows to consider the models (1.1) with unknown $q$ or even with $q = +\infty$. Note that the case when the number of parameters $q$ is unknown is one of challenging problems in the signal and image processing theory (see, for example, [7, 5]). So, now the problem is to estimate the function $S(\cdot)$ on the basis of the observations (1.2) under the condition that the noise distribution $p$ is unknown and belongs to some functional class $\mathcal{P}$. There is a number of papers which consider these models (see, for example, [6, 8, 18] and the references therein). Firstly, minimax estimation problems for the model (1.2) has been treated in [2, 19] in the nonadaptive case, i.e. for the known regularity of the function $S$. Then, in [1] and [3] it is proposed to use the sequential analysis for the adaptive pointwise estimation problem, i.e. in the case when the Hölder regularity is unknown. Moreover, it turned out that only the sequential methods can provide the adaptive estimation for autoregressive models. That is why in this paper we use the adaptive sequential procedures from [4] for the efficient estimation, which we study for the quadratic risks

$$\mathcal{R}_p(\hat{S}_n, S) = \mathbb{E}_{p,S}\|\hat{S}_n - S\|^2, \quad ||S||^2 = \int_a^b S^2(x)dx,$$

where $\hat{S}_n$ is an estimator of $S$ based on observations $(y_j)_{1 \leq j \leq n}$ and $\mathbb{E}_{p,S}$ is the expectation with respect to the distribution law $P_{p,S}$ of the process $(y_j)_{1 \leq j \leq n}$ given the distribution density $p$ and the function $S$. Moreover, taking into account that $p$ is unknown, we use the robust risk defined as

$$\mathcal{R}^*(\hat{S}_n, S) = \sup_{p \in \mathcal{P}} \mathcal{R}_p(\hat{S}_n, S),$$

(1.3)

where $\mathcal{P}$ is a family of the distributions defined in Section 2.
1.2 Main tools

To estimate the function $S$ in model (1.2) we make use of the model selection procedures proposed in [4]. These procedures are based on the family of the optimal pointwise truncated sequential estimators from [3] for which sharp oracle inequalities are obtained through the model selection method developed in [9]. In this paper, on the basis of these inequalities we show that the model selection procedures are efficient in the adaptive setting for the robust quadratic risk (1.3). To this end, first of all, we have to study the sharp lower bound for the these risks, i.e. we have to provide the best potential accuracy estimation for the model (1.2) which usually for quadratic risks is given by the Pinsker constant (see, for example, in [21, 20]). To do this we use the approach proposed in [10, 11] based on the Van-Trees inequality. It turns out that for the model (1.2) the Pinsker constant equals to the minimal quadratic risk value for the filtration signal problem studied in [21] multiplied by the coefficient obtained through the integration of the optimal pointwise estimation risk on the interval $[a,b]$. This is a new result in the nonparametric estimation theory for the statistical models with dependent observations. Then, using the oracle inequality from [4] and the weight least square estimation method we show that, for the model selection procedure the upper bound asymptotically coincides with the obtained Pinsker constant without using the regularity properties of the unknown functions, i.e. it is efficient in adaptive setting with respect to the robust risk (1.3).

1.3 Organization of the paper

The paper is organized as follows. In Section 2 we construct the sequential pointwise estimation procedures which allows us to pass from the autoregression model to the corresponding regression model, then in Section 3 we develop the model selection method. We announce the main results in Section 4. In Section 5 we present Monte-Carlo results which numerically illustrate the behavior of the proposed model selection procedures. In Section 6 we show the Van-Trees inequality for the model (1.2). We obtain the lower bound for the robust risk in Section 7 and in Section 8 we get the upper bound for the robust risk of the constructed sequential estimator. In Appendix we give the all auxiliary tools.

2 Sequential point-wise estimation method

To estimate the function $S$ in the model (1.2) on the interval $[a,b]$ we use the kernel sequential estimators proposed in [3] at the points $(z_l)_{1 \leq l \leq d}$ defined as

$$z_l = a + \frac{l}{d}(b-a),$$

(2.1)

where $d$ is an integer valued function of $n$, i.e. $d = d_n$, such that $d_n/\sqrt{n} \to 1$ as $n \to \infty$. Note that in this case the kernel estimator has the following form

$$\hat{S}(z_l) = \frac{\sum_{j=1}^{n} Q_{l,j} y_j - y_{l-1}}{\sum_{j=1}^{n} Q_{l,j} y_j^2 - Q_{l,j} y_{l-1}^2} \text{ and } Q_{l,j} = Q\left(\frac{x_j - z_l}{h}\right),$$

where $Q(\cdot)$ is a kernel function and $h$ is a bandwidth. As is shown in [2] to provide an efficient point wise estimation, the kernel function must be chosen as the indicator of the interval
\[ -1;1 \], i.e. \( Q(u) = 1_{[-1,1]}(u) \). This means that we can rewrite the estimator (2) as
\[
\hat{S}(z_l) = \frac{\sum_{j = k_{1,l}}^{k_{2,l}} y_{j-1} y_j}{\sum_{j = k_{1,l}}^{k_{2,l}} y_{j-1}^2}, \tag{2.2}
\]
where \( k_{1,l} = [n \tilde{z}_l - n\hat{h}] + 1 \) and \( k_{2,l} = [n \tilde{z}_l + n\hat{h}] \wedge n \), \([x]\) is the integer part of \( x \), \( \tilde{z}_l = l/d \) and \( \hat{h} = h/(b-a) \). To use the model selection method from [4] we need to obtain the uncorrelated stochastic terms in the kernel estimators for the function \( S \) at the points (2.1), i.e. one needs to use the disjoint observations sets \( (y_j)_{k_{1,l} \leq j \leq k_{2,l}} \). To this end it suffices to choose \( h \) for which for all \( 2 \leq l \leq d \) the bounds \( k_{2,l-1} < k_{1,l} \), i.e. we set
\[
h = \frac{b-a}{2d} \quad \text{and} \quad \hat{h} = \frac{1}{2d}. \tag{2.3}
\]
Note that the main difficulty is that the kernel estimator is the non linear function of the observations due to the random denominator. To control this denominator we need to assume conditions providing the stability properties for the model \( (1.2) \). To obtain the stable (uniformly with respect to the function \( S \) ) model \( (1.2) \), we assume that for some fixed \( 0 < \epsilon < 1 \) and \( L > 0 \) the unknown function \( S \) belongs to the \( \epsilon \)-stability set introduced in [3] as
\[
\Theta_{\epsilon,L} = \left\{ S \in C_1([a,b],\mathbb{R}) : |S|_* \leq 1 - \epsilon \quad \text{and} \quad |S|_* \leq L \right\}, \tag{2.4}
\]
where \( C_1([a,b]) \) is the Banach space of continuously differentiable \([a,b] \to \mathbb{R}\) functions and \( |S|_* = \sup_{a \leq x \leq b} |S(x)| \). As is shown in [4] \( \forall S \in \Theta_{\epsilon,L} \)
\[
\sum_{j = k_{1,l}}^{k_{2,l}} y_{j-1}^2 \approx \frac{k_{2,l} - k_{1,l}}{1 - S^2(\tilde{z}_l)} \quad \text{as} \quad k_{2,l} - k_{1,l} \to \infty. \tag{2.5}
\]
Therefore, to replace the denominator in (2.2) with its limit we need firstly preliminary estimate the function \( S(z_l) \). We estimate it as
\[
\hat{S}_l = \frac{\sum_{j = k_{1,l}}^{\mu_l} y_{j-1} y_j}{\sum_{j = k_{1,l}}^{\mu_l} y_{j-1}^2} \quad \text{and} \quad \mu_l = k_{1,l} + q, \tag{2.5}
\]
where \( q = q_n = [(n\hat{h})^\mu_0] \) for some \( 0 < \mu_0 < 1 \). Indeed, we can not use this estimator directly to replace the random denominator since in general it can be closed to one. By this reason we use its projection into the interval in \([ -1 + \tilde{\epsilon}, 1 - \tilde{\epsilon} ]\), i.e. \( \hat{S}_l = \min(\max(\hat{S}_l, 1 - \tilde{\epsilon}), 1 - \tilde{\epsilon}) \) and \( \tilde{\epsilon} = (2 + \ln n)^{-1} \). Finally, omitting some technical details, we will replace the denominator (2.2) with the threshold \( H_l \) defined as
\[
H_l = \frac{1 - \tilde{\epsilon}}{1 - \hat{S}_l^2} (k_{2,l} - \mu_l). \tag{2.6}
\]
It should be noted that \( H_l \) is a function the observations \( y_{k_{1,l}}, \ldots, y_{\mu_l} \). To replace the random denominator in (2.2) with the \( H_l \) we use the sequential estimation method through the following stopping time
\[
\tau_l = \inf\{ k > \mu_l : \sum_{j = \mu_l+1}^{k} u_{j,l} \geq H_l \}, \tag{2.7}
\]
where \( u_{j,l} = y_{j-1}^2 \mathbf{1}_{\{i_l+1 \leq j < k_{2,l}\}} + H_l \mathbf{1}_{\{j = k_{2,l}\}} \). It is clear that \( \tau_l \leq k_{2,l} \) a.s. Now we define the sequential estimator as

\[
S_l^* = \frac{1}{H_l} \left( \sum_{j=i_l+1}^{\tau_l-1} y_{j-1} y_j + \kappa_l y_{\tau_l-1} y_{\tau_l} \right) \mathbf{1}_{\Gamma_l},
\]

where \( \Gamma_l = \{\tau_l < k_{2,l}\} \) and the correcting coefficient \( 0 < \kappa_l \leq 1 \) is defined as

\[
\sum_{j=i_l+1}^{\tau_l-1} u_{j,l} + \kappa_l^2 u_{\tau_l,l} = H_l.
\]

To study robust properties of this sequential procedure similarly to [3] we assume that in the model (1.2) the i.i.d. random variables \( (\xi_j)_{j \geq 1} \) have a density \( p \) (with respect to the Lebesgue measure) from the functional class \( \mathcal{P} \) defined as

\[
\mathcal{P} := \left\{ \begin{array}{l} p \geq 0 : \int_{-\infty}^{+\infty} p(x) \, dx = 1, \quad \int_{-\infty}^{+\infty} x \, p(x) \, dx = 0, \\
\int_{-\infty}^{+\infty} x^2 \, p(x) \, dx = 1 \quad \text{and} \quad \sup_{l \geq 1} \frac{\int_{-\infty}^{+\infty} |x|^{2l} \, p(x) \, dx}{l! \, \varsigma_l} \leq 1 \end{array} \right\},
\]

where \( \varsigma \geq 1 \) is some fixed parameter, which may be a function of the number observation \( n \), i.e. \( \varsigma = \varsigma(n) \), such that for any \( b > 0 \)

\[
\lim_{n \to \infty} n^{-b} \varsigma(n) = 0.
\]

Proposition 2.1. For any \( b > 0 \)

\[
\lim_{n \to \infty} n^b \max_{1 \leq i \leq d} \sup_{S \in \Theta_{k,L}} \sup_{p \in \mathcal{P}} \mathbf{P}_{p,S} \left( |\tilde{S}_i - S(z_i)| > \epsilon_0 \right) = 0,
\]

where \( \epsilon_0 = \epsilon_0(n) \to 0 \) as \( n \to \infty \) such that \( \lim_{n \to \infty} n^\gamma \epsilon_0 = \infty \) for any \( \gamma > 0 \).

Now we set

\[
Y_i = S_l^* \mathbf{1}_{\Gamma_l} \quad \text{and} \quad \Gamma = \cap_{l=1}^{d} \Gamma_l.
\]

In Theorem 3.2 from [4] it is shown that the probability of \( \Gamma \) goes to zero uniformly more rapid than any power of the observations number \( n \), which is formulated in the next proposition.

Proposition 2.2. For any \( b > 0 \) the probability of the set \( \Gamma \) satisfies the following asymptotic equality

\[
\lim_{n \to \infty} n^b \sup_{S \in \Theta_{k,L}} \mathbf{P}_{p,S} (\Gamma^c) = 0.
\]
In view of this proposition we can negligible the set $\Gamma^c$. So, using the estimators (2.13) on the set $\Gamma$ we obtain the discrete time regression model
\begin{equation}
Y_l = S(z_l) + \zeta_l \quad \text{and} \quad \zeta_l = \eta_l + \varpi_l,
\end{equation}
in which
\begin{equation}
\eta_l = \frac{\sum_{j=1-l+1}^{\eta_l-1} u_{j,l} \xi_j + \varpi_j u_{\eta_j,l} \xi_{\eta_j}}{H_l} \quad \text{and} \quad \varpi_l = \varpi_{1,l} + \varpi_{2,l},
\end{equation}
where
\begin{equation}
\varpi_{1,l} = \frac{\sum_{j=1-l+1}^{\eta_l-1} u_{j,l} (S(x_j) - S(z_l)) + \varpi_j^2 u_{\eta_j,l} (S(x_{\eta_j}) - S(z_{\eta_j}))}{H_l}
\end{equation}
and $\varpi_{2,l} = (\varpi_j - \varpi_j^2) u_{\eta_j,l} S(x_{\eta_j})/H_l$. Note that the random variables $(\eta_j)_{1 \leq j \leq d}$ (see Lemma A.2 in [4]), for any $1 \leq l \leq d$ and $p \in \mathcal{P}$ are such that
\begin{equation}
\mathbb{E}_p, S (\eta_l | G_l) = 0, \quad \mathbb{E}_p, S (\eta_l^2 | G_l) = \sigma_l^2 \quad \text{and} \quad \mathbb{E}_p, S (\eta_l^4 | G_l) \leq v^* \sigma_l^4,
\end{equation}
where $\sigma_l = H_l^{-1/2}$, $G_l = \sigma \{ \eta_1, \ldots, \eta_{l-1}, \sigma_l \}$ and $v^*$ is a fixed constant. Note that
\begin{equation}
\sigma_{0,*} \leq \min_{1 \leq l \leq d} \sigma_l^2 \leq \max_{1 \leq l \leq d} \sigma_l^2 \leq \sigma_{1,*},
\end{equation}
where
\begin{equation}
\sigma_{0,*} = \frac{1 - \epsilon^2}{2(1 - \epsilon)^n h} \quad \text{and} \quad \sigma_{1,*} = \frac{1}{(1 - \epsilon)(2nh - q - 3)}.
\end{equation}

**Remark 1.** It should be summarized that we construct the sequential pointwise procedure (2.7) – (2.8) in two steps. First, we preliminary estimate the function $S(z_l)$ in (2.5) on the observations $(y_j)_{k_1 \leq j \leq k_2}$ and through this estimator we replace the random denominator in (2.8) with the threshold $H_l$ in the second step when we construct the estimation procedure on the basis of the observations $(y_j)_{k_1 < j < k_2}$. It should be noted also that in the deviation (2.14) the main term $\eta_l$ has a martingale form and the second one, as it is shown in [4], is asymptotically small. It should be emphasized that namely these properties allow us to develop effective estimation methods.

## 3 Model selection

Now to estimate the function $S$ we use the sequential model selection procedure from [4] for the regression (2.14). To this end, first we choose the trigonometric basis $(\phi_j)_{j \geq 1}$ in $\mathcal{L}_2[a, b]$, i.e.
\begin{equation}
\phi_1 = \frac{1}{\sqrt{b - a}}, \quad \phi_j(x) = \sqrt{\frac{2}{b - a}} \text{Tr}_j (2\pi j/2) l_0(x), \quad j \geq 2,
\end{equation}
where the function $\text{Tr}_j(x) = \cos(x)$ for even $j$ and $\text{Tr}_j(x) = \sin(x)$ for odd $j$, and $l_0(x) = (x - a)/(b - a)$. Moreover, we choose the odd number $d$ of regression points (2.1), for example, $d = 2[\sqrt{n}/2] + 1$. Then the functions $(\phi_j)_{1 \leq j \leq d}$ are orthonormal for the empirical inner product, i.e.
\begin{equation}
(\phi_i, \phi_j)_d = \frac{b - a}{d} \sum_{l=1}^{d} \phi_i(z_l) \phi_j(z_l) = 1_{i=j}.
\end{equation}
It is clear that the function \( S \) can be represented as
\[
S(z_l) = \sum_{j=1}^{d} \theta_{j,d} \phi_j(z_l) \quad \text{and} \quad \theta_{j,d} = (S, \phi_j)_d. \tag{3.3}
\]
We define the estimators for the coefficients \((\theta_{j,d})_{1 \leq j \leq d}\) as
\[
\hat{\theta}_{j,d} = \frac{b - a}{d} \sum_{l=1}^{d} Y_l \phi_j(z_l). \tag{3.4}
\]
From (2.14) we obtain immediately the following regression on the set \( \Gamma \)
\[
\hat{\theta}_{j,d} = \theta_{j,d} + \zeta_{j,d} \quad \text{with} \quad \zeta_{j,d} = \sqrt{b - a} \eta_{j,d} + \varpi_{j,d}, \tag{3.5}
\]
where
\[
\eta_{j,d} = \sqrt{b - a} \sum_{l=1}^{d} \eta_l \phi_j(z_l) \quad \text{and} \quad \varpi_{j,d} = \sqrt{b - a} \sum_{l=1}^{d} \varpi_l \phi_j(z_l). \]

Through the Bounyakovskii-Cauchy-Schwarz we get that
\[
|\varpi_{j,d}|^2 \leq \|\varpi\|^2_d \|\varphi_j\|^2_d \leq (b - a) \frac{\varpi_n^*}{n}, \tag{3.6}
\]
where \( \varpi_n^* = \max_{1 \leq l \leq d} \varpi_l^2 \). Note here, that as it is shown in [4] (Theorem 3.3) for any \( b > 0 \)
\[
\lim_{n \to \infty} \frac{1}{n b} \sup_{p \in \mathcal{P}} \sup_{S \in \Theta_{s,t}} E_{p,S} \varpi_n^* 1_{\Gamma} = 0 \quad \text{for any} \quad b > 0. \tag{3.7}
\]
To construct the model selection procedure we use weighted least squares estimators defined as
\[
\hat{S}_\lambda(t) = \sum_{l=1}^{d} \hat{S}_\lambda(z_l) 1_{|z_l| \leq t}, \quad \hat{S}_\lambda(z_l) = \sum_{j=1}^{d} \lambda(j) \hat{\theta}_{j,d} \phi_j(z_l) 1_{\Gamma}, \tag{3.8}
\]
where the weight vector \( \lambda = (\lambda(1), \ldots, \lambda(d))' \) belongs to some finite set \( \Lambda \subset [0, 1]^d \), the prime denotes the transposition. Denote by \( \nu \) the cardinal number of the set \( \Lambda \), for which we impose the following condition.

\[ H_1 : \text{Assume that the number of the weight vectors} \nu \text{as a function of} \ n, \ i.e. \ \nu = \nu_n, \text{such that for any} \ b > 0 \text{the sequence} \ n^{-b} \nu_n \to 0 \text{as} \ n \to \infty. \]

To choose a weight vector \( \lambda \in \Lambda \) in (3.8) we will use the following risk
\[
\mathrm{Err}_d(\lambda) = \|\hat{S}_\lambda - S\|^2_d = \frac{b - a}{d} \sum_{l=1}^{d} (\hat{S}_\lambda(z_l) - S(z_l))^2. \tag{3.9}
\]
Using (3.3) and (3.8) it can be represented as
\[
\mathrm{Err}_d(\lambda) = \sum_{j=1}^{d} \lambda^2(j) \hat{\varphi}_{j,d}^2 - 2 \sum_{j=1}^{d} \lambda(j) \hat{\theta}_{j,d} \theta_{j,d} + \sum_{j=1}^{d} \theta_{j,d}^2. \tag{3.10}
\]
Since the coefficients $\theta_{j,d}$ are unknown we can’t minimize this risk directly to obtain an optimal weight vector. To modify it we set

$$\tilde{\theta}_{j,d} = \hat{\theta}_{j,d}^2 - \frac{b-a}{d} s_{j,d} \quad \text{with} \quad s_{j,d} = \frac{b-a}{d} \sum_{l=1}^{d} \sigma_l^2 \phi_j^2(z_l).$$  \hspace{1cm} (3.11)

Note here that in view of (2.16) - (3.2) the last term can be estimated as

$$\sigma_{0,*} \leq s_{j,d} \leq \sigma_{1,*}. \hspace{1cm} (3.12)$$

Now, we modify the risk (3.10) as

$$J_d(\lambda) = \sum_{j=1}^{d} \lambda^2(j) \hat{\theta}_{j,d}^2 - 2 \sum_{j=1}^{d} \lambda(j) \tilde{\theta}_{j,d} + \delta P_d(\lambda),$$  \hspace{1cm} (3.13)

where the coefficient $0 < \delta < 1$ will be chosen later and the penalty term is

$$P_d(\lambda) = \frac{b-a}{d} \sum_{j=1}^{d} \lambda^2(j) s_{j,d}. \hspace{1cm} (3.14)$$

Now using (3.13) we define the sequential model selection procedure as

$$\hat{\lambda} = \arg\min_{\lambda \in \Lambda} J_d(\lambda) \quad \text{and} \quad \hat{S}_* = \hat{S}_{\hat{\lambda}}. \hspace{1cm} (3.15)$$

To study the efficiency property we specify the weight coefficients $(\lambda(j))_{1 \leq j \leq n}$ as it is proposed, for example, in [10]. First, for some $0 < \varepsilon < 1$ introduce the two dimensional grid to adapt to the unknown parameters (regularity and size) of the Sobolev ball, i.e. we set

$$A = \{1, \ldots, k^*\} \times \{\varepsilon, \ldots, m \varepsilon\}, \hspace{1cm} (3.16)$$

where $m = [1/\varepsilon^2]$. We assume that both parameters $k^* \geq 1$ and $\varepsilon$ are functions of $n$, i.e. $k^* = k^*(n)$ and $\varepsilon = \varepsilon(n)$, such that

$$\begin{cases} 
\lim_{n \to \infty} k^*(n) = +\infty, & \lim_{n \to \infty} \frac{k^*(n)}{\ln n} = 0, \\
\lim_{n \to \infty} \varepsilon(n) = 0 \quad \text{and} \quad \lim_{n \to \infty} n^b \varepsilon(n) = +\infty 
\end{cases} \hspace{1cm} (3.17)$$

for any $b > 0$. One can take, for example, for $n \geq 2$

$$\varepsilon(n) = \frac{1}{\ln n} \quad \text{and} \quad k^*(n) = k_0^* + [\sqrt{\ln n}], \hspace{1cm} (3.18)$$

where $k_0^* \geq 0$ is some fixed integer number. For each $\alpha = (k, t) \in A$, we introduce the weight sequence $\lambda_{\alpha} = (\lambda_{\alpha}(j))_{1 \leq j \leq d}$ with the elements

$$\lambda_{\alpha}(j) = 1_{\{1 \leq j < j_*\}} + \left(1 - \frac{j}{\omega_{\alpha}}\right)^k 1_{\{j_* \leq j \leq \omega_{\alpha}\}}, \hspace{1cm} (3.19)$$

and

$$\omega_{\alpha} = \omega_{\ast} + (b-a)^{2k/(2k+1)} \left(\frac{(k+1)(2k+1)}{\pi^{2k} k} t n\right)^{1/(2k+1)}. \hspace{1cm} 8$$
Here, $j_*$ and $\omega_*$ are such that $j_* \to \infty$, $j_*=o\left((n/\varepsilon)^{1/(2k+1)}\right)$ and $\omega_* = O(j_*)$ as $n \to \infty$. In this case we set $\Lambda = \{\lambda_\alpha, \alpha \in A\}$. Note, that these weight coefficients are used in [16, 17] for continuous time regression models to show the asymptotic efficiency. It will be noted that in this case the cardinal of the set $\Lambda$ is $\nu = k^* m$. It is clear that the properties (3.17) imply the condition $H_1$. In [4] we showed the following result.

**Theorem 3.1.** Assume that the conditions (2.11) and $H_1$ hold. Then for any $n \geq 3$, any $S \in \Theta_{\varepsilon,L}$ and any $0 < \delta \leq 1/12$, the procedure (3.15) with the coefficients (3.19) satisfies the following oracle inequality

$$R^*(\hat{S}_*, S) \leq \frac{(1 + 4\delta)(1 + \delta)^2}{1 - 6\delta} \min_{\lambda \in \Lambda} R^*(\hat{S}_\lambda, S) + \frac{B^*_n}{\delta n},$$

(3.20)

where the term $B^*_n$ is such that $\lim_{n \to \infty} n^{-b} B^*_n = 0$ for any $b > 0$.

**Remark 2.** In this paper we will use the inequality (3.20) to study efficiency properties for the model selection procedure (3.15) with the weight coefficients (3.19) in adaptive setting, i.e. in the case when the regularity of the function $S$ (1.2) is unknown.

**4 Main results**

First, to study the minimax properties for the estimation problem for the model (1.2) we need to introduce some functional class. To this end for any fixed $r > 0$ and $k \geq 2$ we set

$$W_{k,r} = \left\{ f \in \Theta_{\varepsilon,L}: \sum_{j=1}^{+\infty} a_j \theta_j^2 \leq r \right\},$$

(4.1)

where $a_j = \sum_{j=0}^{k} (2\pi j/2)(b - a)^j$, $(\theta_j)_{j \geq 1}$ are the trigonometric Fourier coefficients in $L_2[a,b]$, i.e. $\theta_j = (f, \phi_j) = \int_a^b f(x)\phi_j(x)dx$ and $(\phi_j)_{j \geq 1}$ is the trigonometric basis defined in (3.1). It is clear that we can represent this functional class as the Sobolev ball

$$W_{k,r} = \left\{ f \in \Theta_{\varepsilon,L}: \sum_{j=0}^{k} \|f^{(j)}\|^2 \leq r \right\}.$$

Now, for this set we define the normalizing coefficients

$$l_k(r) = l_k(r) = ((1 + 2k)r)^{1/(2k+1)} \left( \frac{k}{\pi(k+1)} \right)^{2k/(2k+1)}$$

and

$$c_* = c_*(S) = \int_a^b (1 - S^2(u))du.$$  

(4.2)

It is well known that in regression models with the functions $S \in W_{k,r}$ the minimax convergence rate is $n^{-2k/(2k+1)}$ (see, for example, [10, 15] and the references therein). Our goal in this paper is to show the same property for the non parametric auto-regressive models (1.2). First we have to obtain a lower bound for the risk (1.3) over all possible estimators $\Xi_n$, i.e. any measurable function of the observations $(y_1, \ldots, y_n)$. 

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Theorem 4.1. For the model (1.2) the robust risk (1.3) normalized by the coefficient $v(S) = ((b - a)c)^k/(2k+1)$ can be estimated from below as

$$\liminf_{n \to \infty} \inf_{\hat{S}_n \in \mathcal{X}_n} n^{2k/(2k+1)} \sup_{S \in W_{k,r}} v(S)\overline{R}^*(\hat{S}_n, S) \geq l_k(r).$$

(4.3)

Now to study the procedure (3.15) we have to add some condition on the penalty coefficient $\delta$ which provides sufficiently small penalty term in (3.13).

**H_2:** Assume that the parameter $\delta$ is a function of $n$, i.e. $\delta = \delta_n$ such that $\lim_{n \to \infty} \delta_n = 0$ and $\lim_{n \to \infty} n^{-b}\delta_n = 0$ for any $b > 0$.

Theorem 4.2. Assume that the conditions $H_1 - H_2$ hold. Then the model selection procedure $\hat{S}_*$ defined in (3.15) with the weight vectors (3.19) admits the following asymptotic upper bound

$$\limsup_{n \to \infty} n^{2k/(2k+1)} \sup_{S \in W_{k,r}} v(S)\overline{R}(\hat{S}_*, S) \leq l_k(r).$$

Now Theorems 4.1 - 4.2 imply the following efficiency property.

Corollary 4.1. Assume that the conditions $H_1 - H_2$ hold. The model selection procedure $\hat{S}_*$ defined in (3.15) and (3.19) is efficient, i.e.

$$\lim_{n \to \infty} \inf_{\hat{S}_n \in \mathcal{Z}_n} \sup_{S \in W_{k,r}} v(S)\overline{R}^*(\hat{S}_n, S) \overline{R}(\hat{S}_*, S) = 1.$$  

(4.4)

Moreover,

$$\lim_{n \to \infty} n^{2k/(2k+1)} \sup_{S \in W_{k,r}} v(S)\overline{R}^*(\hat{S}_*, S) = l_k(r).$$

(4.5)

Remark 3. Note that the limit equalities (4.4) and (4.5) imply that the function $l_k(r)/v(S)$ is the minimal value of the normalized asymptotic quadratic robust risk, i.e. the Pinsker constant in this case. We remind that the coefficient $l_k(r)$ is the well known Pinsker constant for the "signal+standard white noise" model obtained in [21]. Therefore, the Pinsker constant for the model (1.2) is represented by the Pinsker constant for the "signal+white noise" model in which the noise intensity is given by the function (4.2).

Now we assume that in the model (1.2) the functions $(\psi_i)_{i \geq 1}$ are orthonormal in $L_2[a, b]$, i.e. $(\psi_i, \psi_j) = I_{i=j}$. We use the estimators (3.8) to estimate the parameters $\beta = (\beta_i)_{i \geq 1}$ as $\hat{\beta}_\lambda = (\hat{\beta}_{\lambda,i})_{i \geq 1}$ and $\beta_{\lambda,i} = (\psi_i, \hat{S}_\lambda)$. Then, similarly we use the selection model procedure (3.15) as

$$\hat{\beta}_s = (\hat{\beta}_{s,i})_{i \geq 1} \quad \text{and} \quad \beta_{s,i} = (\psi_i, \hat{S}_s).$$

(4.6)

It is clear, that in this case $|\hat{\beta}_\lambda - \beta|^2 = \sum_{i=1} (\hat{\beta}_{\lambda,i} - \beta_i)^2 = ||\hat{S}_\lambda - S||^2$ and $||\hat{S}_s - S||^2 = ||\hat{S}_s - S||^2$.

Note, that Theorem 3.1 implies that the estimator (4.6) is optimal in the sharp oracle inequality sense which is established in the following theorem.

Theorem 4.3. For any $S \in \Theta_{x,L}$, $n \geq 3$ and $0 < \delta \leq 1/12$,

$$R^*(\hat{\beta}_s, \beta) \leq \frac{(1 + 4\delta)(1 + \delta)^2}{1 - 6\delta} \min_{\lambda \in \Lambda} R^*(\hat{\beta}_\lambda, \beta) + \frac{B_n^*}{\delta n},$$

where $R^*(\hat{\beta}, \beta) = \sup_{p \in \mathcal{P}} E_{p,S} |\hat{\beta} - \beta|^2$ and $B_n^*$ satisfies the limit property mentioned in Theorem 3.1.
Note now, that Theorems 4.1 and 4.2 imply the efficiency property for the estimate (4.6) based on the model selection procedure (3.15) constructed with the penalty threshold $\delta$ satisfying the condition $H_2$.

**Theorem 4.4.** Then the estimate (4.6) is asymptotically efficient, i.e.

$$\lim_{n \to \infty} n^{2k/(2k+1)} \sup_{S \in W_{k,r}} v(S)R^* (\hat{\beta}_n, \beta) = l_k(r)$$

and

$$\lim_{n \to \infty} \inf_{\hat{\beta}_n \in \Xi_n} \sup_{S \in W_{k,r}} v(S)R^* (\hat{\beta}_n, \beta) = 1,$$

where $\Xi_n$ is the set estimators for $\beta$ based on the observations $(y_j)_{1 \leq j \leq n}$.

**Remark 4.** It should be noted that we obtained the efficiency property (4.7) for the big data autoregressive model (1.2) without using the parameter dimension $q$ or sparse conditions usually used for such models (see, for example, in [14]).

## 5 Monte - Carlo simulations

In this section we present the numeric results obtained through the Python soft for the model (1.2) in which $(\xi_j)_{1 \leq j \leq n}$ are i.i.d. $\mathcal{N}(0,1)$ random variables and $0 \leq x \leq 1$, i.e. $a = 0$ and $b = 1$. In this case we simulate the model selection procedure (3.15) with the weights (3.19) in which $k^* = 150 + \sqrt{\ln n}$, $m = [\ln^2 n]$, $\varepsilon = 1/\ln n$. Moreover, the parameters $j_*$ and $\omega_*$ are chosen as

$$j_* = \frac{\omega}{200 + \ln \omega}, \quad \omega_* = \ln n + \left( \frac{(k + 1)(2k + 1)}{n^{2k}} \right)^{1/(2k+1)}$$

and $\omega_* = j_* + \ln n$. First we study the model (1.2) with $S_1(x) = 0, 5 \cos(2\pi x)$ and then for the function

$$S_2(x) = 0, 1 + \sum_{j=1}^{q} \frac{\cos(2\pi jx)}{(j + 3)^2} \quad \text{and} \quad q = 100000.$$

In the model selection procedures we use $d = 2[\sqrt{n}/2] + 1$ points in (2.1).
Figure 1: Model selection for $n = 200$

(a) Observations for $n = 200$  
(b) Estimator of $S_1$ for $n = 200$

Figure 2: Model selection for $n = 500$

(a) Observations for $n = 500$  
(b) Estimator of $S_1$ for $n = 500$
Figure 3: Model selection for $n = 10000$

Figure 4: Model selection for $n = 70000$
Figures 1–4 show the behavior of the function $S_1$ and its estimators by the model selection procedure (3.15) depending on the observations number $n$. In the figures (a) the observations are given and in (b) the red dotted is the regression function and the black full line is its estimator at the points (2.1). Then we calculate the empiric risks as

$$\overline{R} = \frac{1}{d} \sum_{j=1}^{d} \hat{E} \left( \hat{S}_n(z_j) - S(z_j) \right)^2,$$

where the expectation is taken as an average over $M = 50$ replications, i.e.

$$\hat{E} \left( \hat{S}_n(.) - S(.) \right)^2 = \frac{1}{M} \sum_{l=1}^{M} \left( \hat{S}_{ln}(.) - S(.) \right)^2.$$

We use also the relative risk

$$\overline{R}_* = \frac{\overline{R}}{\|S\|_n^2} \quad \text{and} \quad \|S\|_n^2 = \frac{1}{n} \sum_{j=1}^{n} S^2(x_j).$$

The tables below give the values for the sample risks (5.1) and (5.2) for different numbers of observations $n$.

**Table 1: Empirical risks for $S_1$**

| $n$   | $\overline{R}$ | $\overline{R}_*$ |
|-------|-----------------|-------------------|
| 200   | 0.135           | 0.98              |
| 500   | 0.0893          | 0.624             |
| 10000 | 0.043           | 0.362             |
| 70000 | 0.03523         | 0.281             |

**Table 2: Empirical risks for $S_2$**

| $n$   | $\overline{R}$ | $\overline{R}_*$ |
|-------|-----------------|-------------------|
| 200   | 0.0821          | 5.685             |
| 500   | 0.0386          | 2.623             |
| 10000 | 0.0071          | 0.516             |
| 70000 | 0.0067          | 0.419             |

**Remark 5.** From numerical simulations of the procedure (3.15) with various observation durations $n$ and for the different functions $S$ we may conclude that the quality of the estimation improves as the number of observations increases.
6 The van Trees inequality

In this section we consider the nonparametric autoregressive model (1.2) with the \((0, 1)\) gaussian i.i.d. random variable \((\xi_l)_{1 \leq l \leq n}\) and the parametric linear function \(S\), i.e.

\[
S_{\theta}(x) = \sum_{j=1}^{d} \theta_j \psi_j(x), \quad \theta = (\theta_1, \ldots, \theta_d)' \in \mathbb{R}^d. \tag{6.1}
\]

We assume that the functions \((\psi_j)_{1 \leq j \leq d}\) are orthogonal with respect to the scalar product (3.2). Let now \(P_n^{(\theta)}\) be the distribution in \(\mathbb{R}^n\) of the observations \(y = (y_1, \ldots, y_n)\) in the model (1.2) with the function (6.1) and \(\nu_n^{(\xi)}\) be the distribution in \(\mathbb{R}^n\) of the gaussian vector \((\xi_1, \ldots, \xi_n)\). In this case the Radon - Nykodim density is given as

\[
f_n(y, \theta) = \frac{dP_n^{(\theta)}}{d\nu_n^{(\xi)}} = \exp \left\{ \sum_{l=1}^{n} S_{\theta}(x_l)y_{l-1} - \frac{1}{2} \sum_{j=1}^{n} S_{\theta}^2(x_l)y_{l-1}^2 \right\}. \tag{6.2}
\]

Let \(\rho\) be a prior distribution density on \(\mathbb{R}^d\) for the parameter \(\theta\) of the following form

\[
\rho(\theta) = \prod_{j=1}^{d} \rho_j(\theta_j), \quad \text{where} \quad \rho_j(\theta_j) \text{ is some probability density in} \ \mathbb{R} \ \text{with continuously derivative} \ \dot{\rho}_j \ \text{for which the Fisher information is finite, i.e.}
\]

\[
I_j = \int_{\mathbb{R}^d} \frac{\dot{\rho}_j^2(z)}{\rho_j(z)} \, dz < \infty. \tag{6.3}
\]

Let \(g(\theta)\) be a continuously differentiable \(\mathbb{R}^d \rightarrow \mathbb{R}\) function such that

\[
\lim_{|\theta_j| \rightarrow \infty} g(\theta) \rho_j(\theta_j) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |g_j'(\theta)| \, u(\theta) \, d\theta < \infty, \tag{6.4}
\]

where \(g_j'(\theta) = \partial g(\theta)/\partial \theta_j\).

For any \(Q(\mathbb{R}^{n+d})\)-measurable integrable function \(H = H(y, \theta)\) we denote

\[
\tilde{E} H = \int_{\mathbb{R}^{n+d}} H(y, \theta) f_n(y, \theta) \rho(\theta) d\nu_n^{(\xi)} d\theta. \tag{6.5}
\]

Let \(\mathcal{F}_n^y\) be the field generated by the observations (1.2), i.e. \(\mathcal{F}_n^y = \sigma\{y_1, \ldots, y_n\}\). Now we study the Gaussian the model (1.2) with the function (6.1).

**Lemma 6.1.** For any \(\mathcal{F}_n^y\)-measurable square integrable \(\mathbb{R}^n \rightarrow \mathbb{R}\) function \(\hat{g}_n\) and for any \(1 \leq j \leq d\), the mean square accuracy of the function \(g(\cdot)\) with respect to the distribution (6.5) can be estimated from below as

\[
\tilde{E} (\hat{g}_n - g(\theta))^2 \geq \frac{\bar{g}_j^2}{E \Psi_{n,j} + I_j}, \tag{6.6}
\]

where \(\Psi_{n,j} = \sum_{l=1}^{n} \psi_j^2(x_l) y_{l-1}^2\) and \(\bar{g}_j = \int_{\mathbb{R}^d} g_j'(\theta) \rho(\theta) \, d\theta\).
Proof. First, for any $\theta \in \mathbb{R}^d$ we set

$$\tilde{U}_j = \frac{1}{f_n(y, \theta) u(\theta)} \frac{\partial (f_n(y, \theta) u(\theta))}{\partial \theta_j}.$$ 

Taking into account the condition (6.4) and integrating by parts we get

$$\tilde{E} \left( \hat{g}_n - g(\theta) \tilde{U}_j \right) = \int_{\mathbb{R}^{n+d-1}} \left( \int_{-\infty}^{+\infty} g_j'(\theta) f_n(y, \theta) u(\theta) d\theta \right) \left( \prod_{i \neq j} d\theta_i \right) d\nu^{(n)}_\xi = \mathcal{I}_j.$$ 

Now by the Bouniakovskii-Cauchy-Schwarz inequality we obtain the following lower bound for the quadratic risk

$$\tilde{E} (\hat{g}_n - g(\theta))^2 \geq \frac{\mathcal{I}_j}{\tilde{E} \tilde{U}_j^2}.$$ 

To study the denominator in this inequality note that in view of the representation (6.2)

$$\int_{\mathbb{R}^{n+d}} \left( \int_{-\infty}^{+\infty} g_j'(\theta) f_n(y, \theta) u(\theta) d\theta \right) \left( \prod_{i \neq j} d\theta_i \right) d\nu^{(n)}_\xi$$

we get

$$\tilde{E} \tilde{U}_j^2 = \tilde{E} \Psi_{n,j} + I_j.$$ 

Hence Lemma 6.1. □

7 Lower bound

First, taking into account that the (0, 1) gaussian density $p_0$ belongs to the class (2.10), we get $\mathcal{R}^*(\hat{S}_n, S) \geq \mathcal{R}_{p_0}(\hat{S}_n, S)$. Now, according to the general lower bounds methods (see, for example, in [10]) one needs to estimate this risk from below by some bayesian risk and, then to apply the van Trees inequality. To define the bayesian risk we need to choose a prior distribution on $W_k^r$. To this end, first for any vector $\kappa = (\kappa_j)_{1 \leq j \leq d} \in \mathbb{R}^d$, we set

$$S_\kappa(x) = \sum_{j=1}^d \kappa_j \phi_j(x),$$

(7.1)

where $(\phi_j)_{1 \leq j \leq d_n}$ is the trigonometric basis defined in (3.1). We will choose a prior distribution on the basis of the optimal coefficients in (7.1) (see, for example, in [20, 21]) providing
asymptotically the maximal value for the risk \( R_{p_0}(\hat{S}_n, S) \) over \( S \). So, we choose it as the distribution \( \mu_\kappa \) in \( \mathbb{R}^d \) of the random vector \( \kappa = (\kappa_j)_{1 \leq j \leq d_n} \) defined by its components as

\[
\kappa_j = s_j \eta_j^* ,
\]

where \( \eta_j^* \) are i.i.d. random variables with the continuously differentiable density \( \rho_n(\cdot) \) defined in Lemma A.1 with \( \mathbf{N} = \ln n \) and \( n > e^2 \),

\[
s_j = \sqrt{\frac{(b - a)s_j^*}{n}} \quad \text{and} \quad s_j^* = \left( \frac{d_n}{j} \right)^k - 1 .
\]

To choose the number of the terms \( d \) in (7.1) one needs to keep this function in \( \mathcal{W}_{k,r} \), i.e. one needs to provide for arbitrary fixed \( 0 < \rho < 1 \) the following property

\[
\lim_{n \to \infty} \sum_{j=1}^d a_j s_j^2 = \rho r := r_\rho.
\]

To do this we set

\[
d = d_n = \left[ g_k n^{1/(2k+1)} \right] ,
\]

where \( g_k = (b - a)^{(2k-1)/(2k+1)} l_k(r_\rho)(k + 1)/k \) and

\[
l_k(r_\rho) = \left( (1 + 2k)r_\rho \right)^{1/(2k+1)} \left( \frac{k}{\pi(k + 1)} \right)^{2k/(2k+1)} = \rho^{1/(2k+1)} l_k(r) .
\]

It is clear that almost sure the function (7.1) can be bounded as

\[
\max_{a \leq x \leq b} \left( |S_\kappa(x)| + |\hat{S}_\kappa(x)| \right) \leq c_* \frac{\ln n}{\sqrt{n}} \sum_{j=1}^d j \left( \frac{d_n}{j} \right)^{k/2} = \delta_n^* ,
\]

where \( c_* = \sqrt{2(b-a+\pi)/(b-a)^{3/2}} \). Note, that \( \delta_n^* \to 0 \) as \( n \to \infty \) for \( k \geq 2 \). Therefore, for sufficiently large \( n \) the function (7.1) belongs to the class (2.4). Now, \( \forall f \in \mathcal{L}_2[a,b] \), we denote by \( h(f) \) its projection in \( \mathcal{L}_2[a,b] \) onto the ball \( \mathcal{W}_{r'} = \{ f \in \mathcal{L}_2[a,b] : ||f|| \leq r \} \), i.e. \( h(f) = (r/\max((r,||f||))f \). Since \( \mathcal{W}_{k,r} \subset \mathcal{W}_{r'} \), then for \( S \in \mathcal{W}_{k,r} \) we have \( ||\hat{S}_n - S||^2 \geq ||\hat{h}_n - S||^2 \), where \( \hat{h}_n = h(\hat{S}_n) \). Therefore, for large \( n \)

\[
\sup_{S \in \mathcal{W}_{k,r}} u(S) \mathcal{R}_{p_0}(\hat{S}_n, S) \geq \int_{D_n} u(S) \mathbf{E}_{p_0} |S|_z \int_{D_n} \mathbf{E}_{p_0} |\hat{h}_n - S||^2 \mu_\kappa(dz) \geq v_* \int_{D_n} \mathbf{E}_{p_0} |\hat{h}_n - S||^2 \mu_\kappa(dz) ,
\]

where \( D_n = \{ z \in \mathbb{R}^d : \sum_{j=1}^d a_j z_j^2 \leq r \} \) and \( v_* = \inf_{|S| \leq \delta_n^*} u(S) \). Note that \( v_* \to (b - a)^{-4k/(2k+1)} \) as \( n \to \infty \). Using the distribution \( \mu_\kappa \) we introduce the following Bayes risk as

\[
\bar{R}_0(\hat{S}_n) = \mathbf{E}_0 ||\hat{S}_n - S||^2 = \int_{\mathbb{R}^d} \mathbf{E}_{p_0} ||\hat{S}_n - S||^2 \mu_\kappa(dz) .
\]
Now taking into account that \( \|\hat{h}_n\|^2 \leq r \), we get
\[
\sup_{S \in \mathcal{W}_{k,r}} v(S) \mathcal{R}_{p_0}(\hat{S}_n, S) \geq v_n \tilde{\mathcal{R}}_0(\hat{h}_n) - 2v_n R_{0,n} \tag{7.8}
\]
and \( R_{0,n} = \int_{D_n} (r + \|S_z\|^2) \mu_\nu(dz) = \int_{D_n} (r + |z|^2) \mu_\nu(dz) \). Note here that this term is studied in Lemma A.3. Moreover, note also that for any \( z \in \mathbb{R}^d \) we get \( \|\hat{h}_n - S_z\|^2 \geq \sum_{j=1}^d (\hat{z}_j - z_j)^2 \) and \( \hat{z}_j = (\hat{h}_n, \phi_j) \). Therefore, from Lemma 6.1 with \( g(\theta) = \theta_j \) it follows, that for any \( 1 \leq j \leq d \) and any \( F_n \) measurable random variable \( \hat{r}_j \)
\[
\tilde{E}_0(\hat{r}_j - \kappa_j)^2 \geq \frac{1}{\tilde{E}_0 \Psi_{n,j} + s_j^{-2} J_n},
\]
where \( \Psi_{n,j} = \sum_{j=1}^n \phi_j^2(x_i) y_{i-1}^2 \) and \( J_n = \int_{-n}^{n} (\rho_n(t))^2 / \rho_n(t) dt \). Therefore, the Bayes risk can be estimated from below as
\[
\tilde{\mathcal{R}}_0(\hat{h}_n) \geq \frac{1}{n} \sum_{j=1}^n \tilde{E}_0 \Psi_{n,j} + s_j^{-2} J_n = b - a \frac{1}{n} \sum_{j=1}^n \Psi_{n,j} + s_j^2 \tilde{E}_0 y_{i-1}^2 \tag{7.9}
\]
where
\[
\bar{\Psi}_{n,j} = \frac{(b-a)}{n} \tilde{E}_0 \Psi_{n,j} \frac{(b-a)}{n} \sum_{j=1}^n \phi_j^2(x_i) \tilde{E}_0 y_{i-1}^2.
\]
Note here, that in view of Lemmas A.1 and A.2 for any \( 0 < \rho_1 < 1 \) and sufficiently large \( n \) we have \( J_n \leq 1 + \rho_1 \) and \( \max_{1 \leq j \leq a} \bar{\Psi}_{n,j} \leq 1 + \rho_1 \), therefore,
\[
\tilde{\mathcal{R}}_0(\hat{h}_n) \geq \frac{b - a}{(1 + \rho_1) n} \sum_{j=1}^n s_j^2 \frac{1}{1 + s_j^2} = b - a \frac{1}{(1 + \rho_1) n} \sum_{j=1}^n \left( 1 - \frac{j^2}{d_k} \right).
\]
Using here that \( \lim_{d \to \infty} d^{-1} \sum_{j=1}^d (1 - j^2 d^{-k}) = \int_0^1 (1 - t^2) dt = k/(k + 1) \), we obtain for sufficiently large \( n \)
\[
\tilde{\mathcal{R}}_0(\hat{h}_n) \geq \frac{(b-a)(1-\rho_1)d_n}{(1+\rho_1)n} \frac{k}{k+1}.
\]
Therefore, taking into account this in (7.8) we conclude through the definition (7.4) and Lemma A.3, that for any \( 0 < \rho \) and \( \rho_1 < 1 \)
\[
\liminf_{n \to \infty} \inf_{S_n \in S_n} n^\frac{2k}{2k+1} \sup_{S \in \mathcal{W}_{k,r}} v(S) \mathcal{R}(\hat{S}_n, S) \geq \frac{(b-a)(1-\rho_1)}{1+\rho_1} l_k(r).
\]
Taking here limit as \( \rho \to 1 \) and \( \rho_1 \to 0 \) we come to the Theorem 4.1. \( \Box \)

8 Upper bound

We start with the estimation problem for the functions \( S \) from \( \mathcal{W}_{k,r} \) with known parameters \( k, r \) and \( \zeta_* \) defined in (4.2). In this case we use the estimator from family (3.19)
\[
\tilde{S}_n = \tilde{S}_n \quad \text{and} \quad \tilde{\alpha} = (k, \tilde{t}_n), \tag{8.1}
\]
where \( \tilde{\tau}_n = [\tau(S)/\varepsilon] \varepsilon, \tau(S) = r/\varsigma_s \) and \( \varepsilon = 1/Ln. \) Note that for sufficiently large \( n \), the parameter \( \tilde{\alpha} \) belongs to the set (3.16). In this section we study the risk for the estimator (8.1). To this end we need firstly to analyse the asymptotic behavior of the sequence

\[
Y_n(S) = \sum_{j=1}^{d} (1 - \tilde{\lambda}(j))^2 \theta_{j,d}^2 + \frac{\varsigma_n}{n} \sum_{j=1}^{d} \tilde{\chi}(j). \tag{8.2}
\]

**Proposition 8.1.** The sequence \( Y_n(S) \) is bounded from above

\[
\limsup_{n \to \infty} \sup_{S \in W_{k,r}} n^{k_1} v(S) Y_n(S) \leq I_k(r) \quad \text{and} \quad k_1 = 2k/(2k + 1).
\]

**Proof.** First, note that \( 0 < c^2(b - a) \leq \inf_{S \in \Theta_{k,L}} \varsigma_s \leq \sup_{S \in \Theta_{k,L}} \varsigma_s \leq b - a. \) This implies directly that

\[
\limsup_{n \to \infty} \sup_{S \in \Theta_{k,L}} \left| \tilde{\tau}_n/\tau(S) - 1 \right| = 0,
\]

where \( \tilde{\tau}_n = [\tau(S)/\varepsilon] \varepsilon \) and \( \tau(S) = r/\varsigma_s. \) Moreover, note that

\[
n^{k_1} v(S) Y_n(S) \leq n^{k_1} v(S) G_n + \frac{(\varsigma_s)^{1/(2k+1)}}{(b - a)^{k_1} n^{1/(2k+1)}} \sum_{j=1}^{d} \tilde{\chi}(j)
\]

and \( G_n = \sum_{j=1}^{d} (1 - \tilde{\lambda}(j))^2 \theta_{j,d}^2 = G_{1,n} + G_{2,n}, \) where

\[
G_{1,n} = \sum_{j=j_*}^{[\hat{\omega}]} (1 - \tilde{\lambda}(j))^2 \theta_{j,d}^2 \quad \text{and} \quad G_{2,n} = \sum_{j=[\hat{\omega}] + 1}^{d} \theta_{j,d}^2.
\]

Remind, that \( \tilde{\omega} = \tilde{\omega}_s + (b - a)^{k_1} \left( \frac{\tilde{\tau}_n}{\pi_k n} \right)^{1/(2k+1)} + \pi_k \) \( = (k + 1)(2k + 1)/(\pi 2k) \). Note now, that Lemma A.5 and Lemma A.6 yield

\[
G_{1,n} \leq (1 + \tilde{\varepsilon}) \sum_{j=j_*}^{[\hat{\omega}]} (1 - \tilde{\lambda}(j))^2 \theta_{j}^2 + 4r(1 + \tilde{\varepsilon}^{-1}) \frac{(b - a)^{2k\tilde{\omega}}}{d^{2k}}
\]

and

\[
G_{2,n} \leq (1 + \tilde{\varepsilon}) \sum_{j=[\hat{\omega}] + 1}^{d} \theta_{j}^2 + r(1 + \tilde{\varepsilon}^{-1}) \frac{(b - a)^{2k}}{d^{2k\omega^2(k-1)}},
\]

i.e. \( G_n \leq (1 + \tilde{\varepsilon}) G_n^* + 4r(b - a)^{2k}(1 + \tilde{\varepsilon}^{-1}) \tilde{G}_n, \) where

\[
G_n^* = \sum_{j=1}^{d} (1 - \tilde{\lambda}(j))^2 \theta_{j}^2 = \sum_{j \leq \hat{\omega}} (1 - \tilde{\lambda}(j))^2 \theta_{j}^2 + \sum_{j > \hat{\omega}} \theta_{j}^2 := G_{1,n}^* + G_{2,n}^*
\]

and \( \tilde{G}_n = \tilde{\omega} d^{-2k} + d^{-2\omega^{-2(k-1)}}. \) Note, that

\[
n^{k_1} v(S) G_{1,n}^* = \frac{v(S)}{(b - a)^{2k_1} (\pi_k n)^{k_1}} \sum_{j=j_*}^{[\hat{\omega}]} j^{2k} \theta_{j}^2 \leq \frac{u_{k_1}^{n}}{(\pi_k n)^{k_1}} \sum_{j=j_*}^{[\hat{\omega}]} a_{j} \theta_{j}^2,
\]
where \( \mathbf{u}_n^* = \sup_{j \geq 2k} j^{2k}/((b-a)^2\mathbf{a}_j) \). It is clear that \( \lim_{n \to \infty} \mathbf{u}_n^* = \pi^{-2k} \). Therefore, from (8.3) it follows that

\[
\limsup_{n \to \infty} \sup_{S \in \Theta_{s,t}} \frac{n^{k_1} v(S) G_{1,n}^*}{\sum_{|j| = j} a_j \theta_j^2} \leq \pi^{-2k(\sqrt{k}_r)} - k_1.
\]

Next, note that for any \( \tilde{\varepsilon} < 1 \) and for sufficiently large \( n \)

\[
G_{2,n}^* \leq \frac{1}{\alpha^{|\omega|} + 1} \sum_{j \geq |\omega| + 1} a_j \theta_j^2 \leq \frac{(1 + \tilde{\varepsilon})((b-a)\varsigma)^{k_1}}{\pi^{2k}(\sqrt{k}_r)^{k_1}} \sum_{j \geq |\omega| + 1} a_j \theta_j^2,
\]

i.e.

\[
\limsup_{n \to \infty} \sup_{S \in \Theta_{s,t}} \frac{n^{k_1} v(S) \sum_{j \geq |\omega| + 1} \theta_j^2}{\sum_{|\omega| = j} a_j \theta_j^2} \leq \pi^{-2k(\sqrt{k}_r)} - k_1.
\]

Next, since \( \sum_{j=1}^{d} \tilde{\lambda}^2(j)/\tilde{\omega} = k_{k}/(k + 1) \) as \( n \to \infty \), we get directly that

\[
\lim_{T \to \infty} \sup_{S \in \Theta_{s,t}} \frac{(\varsigma_j)^{1/(2k + 1)} \sum_{j=1}^{d} \tilde{\lambda}^2(j)}{(b-a)^{k_{k}/(2k + 1)} \tilde{\omega}^2} \leq \frac{(\sqrt{k}_r)^{1/(2k + 1)} k_{k}}{(k + 1)} = 0.
\]

Finally, taking into account, that \( \lim_{n \to \infty} \sup_{S \in W_{k,r}} n^{k_1} \mathbf{G}_n = 0 \), we obtain Proposition 8.1.

\[\Box\]

**Theorem 8.1.** The estimator \( \tilde{S} \) constructed on the trigonometric basis satisfies the following asymptotic upper bound

\[
\limsup_{n \to \infty} n^{2k/(2k + 1)} \sup_{S \in W_{k,r}} v(S) E_{p,S} \| \tilde{S} - S \|_d \leq I_k(r). \tag{8.4}
\]

**Proof.** We denote \( \tilde{\lambda} = \lambda_{\tilde{\omega}} \) and \( \tilde{\omega} = \omega_{\tilde{\omega}} \). Now we recall that the Fourier coefficients on the set \( \Gamma \)

\[
\hat{\theta}_{j,d} = \theta_{j,d} + \varsigma_{j,d} \quad \text{with} \quad \varsigma_{j,d} = \sqrt{\frac{b-a}{d}} \eta_{j,d} + w_{j,d}. \]

Therefore, on the set \( \Gamma \) we can represent the empiric squared error as

\[
\| \tilde{S} - S \|_d^2 = \sum_{j=1}^{d} (1 - \tilde{\lambda}(j))^2 \theta_{j,d}^2 - 2M_n
\]

\[
- 2 \sum_{j=1}^{d} (1 - \tilde{\lambda}(j)) \tilde{\lambda}(j) \theta_{j,d} \tilde{\omega}_{j,d} + \sum_{j=1}^{d} \tilde{\lambda}^2(j) \varsigma_{j,d}^2,
\]

where \( M_n = \sqrt{b-a} \sum_{j=1}^{d} (1 - \tilde{\lambda}(j)) \tilde{\lambda}(j) \eta_{j,d} / \sqrt{d} \). Now for any \( 0 < \varepsilon_1 < 1 \)

\[
2 \sum_{j=1}^{d} (1 - \tilde{\lambda}(j)) \tilde{\lambda}(j) \hat{\theta}_{j,d} \tilde{\omega}_{j,d} \leq \varepsilon_1 \sum_{j=1}^{d} (1 - \tilde{\lambda}(j))^2 \theta_{j,d}^2 + \varepsilon^{-1} \sum_{j=1}^{d} \tilde{\omega}_{j,d}^2
\]

\[
\leq x \varepsilon_1 \sum_{j=1}^{d} (1 - \tilde{\lambda}(j))^2 \theta_{j,d}^2 + \frac{\tilde{\omega}_{j,d}^2}{\varepsilon_1 n}.
\]
where \( \varpi_n^* \) is defined in (3.6). Therefore,
\[
\| \tilde{S} - S \|_d^2 \leq (1 + \varepsilon_1) \sum_{j=1}^{d} (1 - \tilde{\lambda}(j))^2 \theta_{j,d}^2 - 2M_n + \frac{\varpi_n^*}{\varepsilon_1 n} + \sum_{j=1}^{d} \tilde{\lambda}^2(j) \zeta_{j,d}^2 .
\]

By the same way we get
\[
\sum_{j=1}^{d} \tilde{\lambda}^2(j) \zeta_{j,d}^2 \leq (1 + \varepsilon_1) (b - a) \eta_{j,d}^2 + (1 + \varepsilon_1^{-1}) \frac{\varpi_n^*}{n} .
\]
Thus, on the set \( \Gamma \) we find that for any \( 0 < \varepsilon_1 < 1 \)
\[
\| \tilde{S}_n - S \|_d^2 \leq (1 + \varepsilon_1) \Upsilon_n(S) - 2M_n + (1 + \varepsilon_1) \Upsilon_n + \frac{3 \varpi_n^*}{\varepsilon_1 n} ,
\]
where \( \Upsilon_n(S) \) is defined in (8.2) and
\[
\Upsilon_n = \frac{1}{d^2} \sum_{j=1}^{d} \tilde{\lambda}^2(j) \left( (b - a) \eta_{j,d}^2 - \zeta \right) .
\]

We recall that the variance \( \zeta \) is defined in (4.2). In view of Lemma A.7
\[
\mathbb{E}_{p,S} M_n^2 \leq \frac{\sigma_{1,\ast} (b - a)}{d} \sum_{j=1}^{d} \theta_{j,d}^2 = \frac{\sigma_{1,\ast} (b - a)}{d} \| S \|_d^2 \leq \frac{\sigma_{1,\ast} (b - a)^2}{d} ,
\]
where \( \sigma_{1,\ast} \) is given in (2.16). Moreover, using that \( \mathbb{E}_{p,S} M_n = 0 \), we get
\[
| \mathbb{E}_{p,S} M_n \mathbf{1}_\Gamma | = | \mathbb{E}_{p,S} M_n \mathbf{1}_{\Gamma^c} | \leq (b - a) \sqrt{\frac{\sigma_{1,\ast} \mathbb{P}_{p,S}(\Gamma^c)}{d}} .
\]
Therefore, Proposition 2.2 yields
\[
\lim_{n \to \infty} n^{2k/(2k+1)} \sup_{S \in \Theta_{s,L}} | \mathbb{E}_{p,S} M_n \mathbf{1}_\Gamma | = 0 .
\]

Now, the property (3.7) Proposition 8.1 and Lemma A.8 imply the inequality (8.4). Hence Theorem 8.1. \( \Box \)

It is clear, that Theorem 3.1 and Theorem 8.1 imply Theorem 4.2.

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A Appendix

A.1 Properties of the prior distribution 7.2

In this section we study properties of the distribution used in (7.7).
Lemma A.1. For any $N > 2$ there exists a continuously differentiable probability density $\rho_N(\cdot)$ on $\mathbb{R}$ with the support on the interval $[-N, N]$, i.e. $\rho_N(z) > 0$ for $-N < z < N$ and $\rho_N(z) = 0$ for $|z| \geq N$, such that for any $N > 2$ the integral $\int_{-N}^{N} z^2 \rho_N(z) \, dz = 0$ and, moreover, $\int_{-N}^{N} z^2 \rho_N(z) \, dz \rightarrow 1$ and $J_N = \int_{-N}^{N} (\rho_N(z))^2/\rho_N(z) \, dz \rightarrow 1$ as $N \rightarrow \infty$.

Proof. First we set $V(z) = \left( \int_{z}^{1} e^{-(t-1)^2} \, dt \right)^{-1} e^{-\frac{(z-1)^2}{2}} 1_{|z| \leq 1}$. It is clear that this function is infinitely times continuously differentiable, such that $V(z) > 0$ for $|z| < 1$, $V(z) = 0$ for $|z| \geq 1$ and $\int_{-1}^{1} V(z) \, dz = 1$. Now for $N \geq 2$ we set $\chi_N(z) = \int_{1}^{N} 1_{|z+u| \leq N-1} V(u) \, du = \int_{R} 1_{(|t| \leq N-1)} V(t-z) \, dt$. Using here the properties of the function $V$ we can obtain directly that $\chi_N(z) = \chi_N(z)$ for $z \in \mathbb{R}$, $\chi_N(z) = 1$ for $|z| \leq N - 2$, $\chi_N(z) > 0$ for $N - 2 < |z| < N$ and $\chi_N(z) = 0$ for $|z| \geq N$. Moreover, it is clear that the derivative $\chi_N'(z) = -\int_{R} 1_{|u| \leq N-1} V(t-z) \, dt - \int_{|u+z| \leq N-1} V(u) \, du$. Note here that $|\dot{V}(z)| \leq c_0 \sqrt{V(z)}$ for some $c_0 > 0$. Now through the Bunyakovsky - Cauchy - Schwartz inequality we get that $\chi_N^2(z) \leq 2c_0 \chi_N(z)$ for $|z| < N$. Now we set $\rho_N(z) = \left( \int_{-N}^{N} \varphi(t) \chi_N(t) \, dt \right)^{-1} \varphi(z) \chi_N(z)$, where $\varphi(z)$ is the (0,1) Gaussian density. It is clear that $\rho_N(z)$ is the the continuously differentiable probability density with the support $[-N, N]$ such that for any $N$ the integral $\int_{-N}^{N} z^2 \rho_N(z) \, dz = 0$ and $\int_{-N}^{N} \varphi(t) \chi_N(t) \, dt \rightarrow 1$, $\int_{-N}^{N} \rho_N(z) \, dz \rightarrow 1$ as $N \rightarrow \infty$. Moreover, the Fisher information can be represented as

$$J_N = \left( \int_{-N}^{N} \varphi(t) \chi_N(t) \, dt \right)^{-1} \left( \int_{\mathbb{R}} \varphi^2(z) \chi_N(z) \, dz + \Delta_N \right),$$

where, taking into account that $\dot{\chi}_N(z) = 0$ for $|z| \leq N - 2$,

$$\Delta_N = 2 \int_{|z| \geq N-2} \varphi(z) \chi_N(z) \, dz + \int_{|z| \geq N-2} \varphi(z) \chi_N^2(z) \, dz.$$

Therefore, $\Delta_N \rightarrow 0$ as $N \rightarrow \infty$. Hence Lemma A.1.

Lemma A.2. The term (7.9) is such that $\lim_{n \rightarrow \infty} \max_{1 \leq j \leq d} |\bar{y}_{n,j} - 1| = 0$.

Proof. First, note that $y_l = y_0 \prod_{i=1}^{l} S_{\kappa}(x_i) + \sum_{l=1}^{l} \prod_{i=l+1}^{l} S_{\kappa}(x_i) \xi_i$ for $l \geq 1$. Therefore, $E_0 y^2 = y_0^2 \hat{E}_0 \prod_{i=1}^{l} S_{\kappa}^2(x_i) + \sum_{l=1}^{l-1} \hat{E}_0 \prod_{i=l+1}^{l} S_{\kappa}^2(x_i) + 1$ and due to (7.5) we obtain that for any $n \geq 1$ for which $\delta_n < 1$

$$\sup_{l \geq 1} \left| E_0 y^2 - 1 \right| \leq (\delta_n^*)^2 \left( y_0^2 + 1 \right) \frac{1}{1 - (\delta_n^*)^2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since, $(b-a) \sum_{l=1}^{n} \phi^2_j(x_i) = n$, we get Lemma A.2.

Lemma A.3. The term $R_{0,n}$ in (7.8) is such that $\lim_{n \rightarrow \infty} n^b R_{0,n} = 0$ for any $b > 0$ and $0 < \rho < 1$.

Proof. First note, that taking into account in the definition of term $R_{0,n}$ in (7.8), that $|\eta_j^s| \leq \ln n$, we get that $R_{0,n} \leq \left( r + \ln^2 n \sum_{j=1}^{d} s_j^2 \right) \mu_\kappa(D_{n})$. Therefore, to show this lemma
it suffices to check that \( \lim_{n \to \infty} n^b \mu_n(D_n^c) = 0 \) for any \( b > 0 \). To do this note, that the definition of \( D_n \) in (7.6) implies \( \mu_n(D_n^c) \leq P(\zeta_n > r) \) and \( \zeta_n = \sum_{j=1}^{d_n} a_j \kappa_j^2 \). So, it suffices to show that

\[
\lim_{n \to \infty} n^b P(\zeta_n > r) = 0 \quad \text{for any} \quad b > 0.
\] (A.1)

Indeed, first note, that the definition (7.2) through Lemma A.1 and the property (7.3) imply directly

\[
\lim_{n \to \infty} \mathbb{E} \zeta_n = \lim_{n \to \infty} \sum_{j=1}^{d} a_j \delta_j^2 \mathbb{E}(\eta_1^*)^2 = \lim_{n \to \infty} \sum_{j=1}^{d} a_j \delta_j^2 = \rho r.
\]

Setting now \( \tilde{\zeta}_n = \zeta_n - \mathbb{E} \zeta_n = (b-a) \sum_{j=1}^{d_n} a_j \tilde{\eta}_j/n \) and \( \tilde{\eta}_j = (\eta_j^*)^2 - \mathbb{E}(\eta_j^*)^2 \), we get for large \( n \) that \( \{\zeta_n > r\} \subset \{\tilde{\zeta}_n > r\} \) for \( r_1 = r (1 - \rho)/2 \). Now the correlation inequality from [12] and the bound \( |\tilde{\eta}_j| \leq 2 \ln n \) imply that for any \( p \geq 2 \) there exists some constant \( C_p > 0 \) for which

\[
\mathbb{E} \tilde{\zeta}_n^p \leq C_p (\ln n)^{2p}/n^p \left( \sum_{j=1}^{d} (s_j^*)^2 a_j^2 \right)^{p/2} \leq C_p n^{-2/p} \ln n)^{2p},
\]

i.e. the expectation \( \mathbb{E} \tilde{\zeta}_n^p \to 0 \) as \( n \to \infty \) and, therefore, \( n^b P(\tilde{\zeta}_n > r_1) \to 0 \) as \( n \to \infty \) for any \( b > 0 \). This implies (A.1) and, hence Lemma A.3. \( \square \)

**A.2 Properties of the trigonometric basis.**

First we need the following lemma from [17].

**Lemma A.4.** Let \( f \) be an absolutely continuous function, \( f : [a, b] \to \mathbb{R} \), with \( \|f\| < \infty \) and \( g \) be a piecewise constant function \( [a, b] \to \mathbb{R} \) of a form \( g(x) = \sum_{j=1}^{d} c_j \chi_{(z_{j-1}, z_j]}(x) \) where \( c_j \) are some constants. Then for any \( \varepsilon > 0 \), the function \( \Delta = f - g \) satisfies the following inequalities

\[
\|\Delta\|_{d}^2 \leq (1 + \varepsilon)\|\Delta\|^2 + (1 + \varepsilon^{-1})(b-a)^2 \|\hat{f}\|^2 d^2.
\]

**Lemma A.5.** For any \( 1 \leq j \leq d \) the trigonometric Fourier coefficients \( (\theta_{j,d})_{1 \leq j \leq d} \) for the functions \( S \) from the class \( W_{k,r} \) with \( k \geq 1 \) satisfy, for any \( \varepsilon > 0 \), the following inequality \( \theta_{j,d}^2 \leq (1 + \varepsilon) \theta_{j}^2 + 4r (1 + \varepsilon)(b-a)^2k_d^{-2k} \).

**Proof.** First we represent the function \( S \) as \( S(x) = \sum_{l=d}^{d} \phi_l(x) + \Delta_d(x) \) and \( \Delta_d(x) = \sum_{l>d} \phi_l(x) \), i.e. \( \theta_{j,d} = (S, \phi_l)_d = \theta_{j} + (\Delta_d, \phi_l)_d \) and, therefore, \( \forall \varepsilon > 0 \) we get \( \theta_{j,d}^2 \leq (1 + \varepsilon) \theta_{j}^2 + 4r (1 + \varepsilon)(b-a)^2k_d^{-2k} \).

Using here that \( 2\pi [l/2] \geq l \) for \( l \geq 2 \), we obtain that \( \|\Delta_d\|^2 = \sum_{l>d} \theta_{l}^2 \leq r/a_d \leq (b-a)^2k_d^{-2k} \) and

\[
\|\Delta_d\|^2 = (2\pi/(b-a))^2 \sum_{l>d} \theta_{l}^2 \leq r/a_d \leq (b-a)^2k_d^{-2k}.
\]

Hence Lemma A.5 \( \square \)

**Lemma A.6.** For any \( d \geq 2 \) and \( 1 \leq N \leq d \) the coefficients \( (\theta_{j,d})_{1 \leq j \leq d} \) of functions \( S \) from the class \( W_{r,k} \) with \( k \geq 1 \) satisfy, for any \( \varepsilon > 0 \), the inequality \( \sum_{j=N}^{d} \theta_{j,d}^2 \leq (1 + \varepsilon) \sum_{j=N}^{d} \theta_{j}^2 + (1 + \varepsilon^{-1})(b-a)^2k_d^{-2k} \).

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Proof. Note that \( \sum_{j=N}^{d} \theta_{j,d}^2 = \min_{x_1,\ldots,x_{N-1}} \| S - \sum_{j=1}^{N-1} x_j \phi_j \|^2 \leq \| \Delta_N \|^2 \) and \( \Delta_N(t) = \sum_{j \geq N} \theta_j \phi_j(t) \). Lemma A.4 and (A.2), imply Lemma A.6

### A.3 Technical lemmas

**Lemma A.7.** For any non random coefficients \( (u_{j,l})_{1 \leq j \leq d} \)

\[
E \left( \sum_{j=1}^{d} u_{j,l} \eta_{j,d} \right)^2 \leq \sigma_{1,s} \sum_{j=1}^{d} u_{j,l}^2 ,
\]

where the coefficient \( \sigma_{1,s} \) is given in (2.16).

**Proof.** Using the definition of \( \eta_{j,d} \) in (3.5) and the bounds (2.16), we get

\[
E \left( \sum_{j=1}^{d} u_{j,l} \eta_{j,d} \right)^2 = \frac{b-a}{d} E \sum_{l=1}^{d} \sigma_l^2 \left( \sum_{j=1}^{d} u_{j,l} \phi_j(z_l) \right)^2 \leq \sigma_{1,s} \frac{b-a}{d} \sum_{l=1}^{d} \left( \sum_{j=1}^{d} u_{j,l} \phi_j(z_l) \right)^2.
\]

Now, the orthonormality property (3.2) implies this lemma.

**Lemma A.8.** For the sequence (8.5) the following limit property holds true

\[
\lim_{n \to \infty} n^{2k/(2k+1)} \sup_{S \in \Theta_{k,r}} \sup_{p \in \mathcal{P}} \left| E_{p,S} U_n 1_{\Gamma} \right| = 0.
\]

**Proof.** First of all, note that, using the definition of \( s_{j,d} \) in (3.11), we obtain

\[
E_{p,S} \eta_{j,d}^2 = E_{p,S} s_{j,d} = \frac{1}{d} \sum_{l=1}^{d} E_{p,S} \frac{1}{H_l} + \frac{1}{d} E_{p,S} s_{j,d},
\]

where \( s_{j,d} = \sum_{l=1}^{d} \sigma_l^2 \phi_j(x_l) \) and \( \phi_j(z) = (b-a) \phi_j^2(z) - 1 \). Therefore, we can represent the expectation of \( U_n \) as

\[
E_{p,S} U_n = \frac{||\tilde{\lambda}||^2}{d^2} E_{p,S} U_{1,n} + \frac{b-a}{d} E_{p,S} U_{2,n},
\]

where \( ||\tilde{\lambda}||^2 = \sum_{j=1}^{d} \tilde{\lambda}^2(j) \),

\[
U_{1,n} = \frac{b-a}{d} \sum_{l=1}^{d} \frac{1}{H_l} - \varsigma_\ast \quad \text{and} \quad U_{2,n} = \sum_{j=1}^{d} \tilde{\lambda}^2(j) s_{j,d}.
\]

Note now, that using Proposition 2.12 and the dominated convergence theorem in the definition (2.6) we obtain that

\[
\lim_{n \to \infty} \max_{1 \leq l \leq d} \sup_{S \in \Theta_{\ast,l}} \sup_{p \in \mathcal{P}} \left| \frac{d}{H_l} - (1 - S^2(z_l)) \right| = 0.
\]

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Taking into account that for the functions from the class (2.4) their derivatives are uniformly bounded, we can deduce that

$$\lim_{n \to \infty} \sup_{S \in \Theta_{\epsilon, L}} \left| \frac{b - a}{d} \sum_{l=1}^{d} (1 - S^2(z_l)) - \varsigma \right| = 0,$$

i.e. $\lim_{n \to \infty} \sup_{S \in \Theta_{\epsilon, L}} \sup_{p \in P} |E_{p, S} U_{1, n}| = 0$. Therefore, taking into account that

$$\lim_{n \to \infty} \sup_{S \in \Theta_{\epsilon, L}} \sup_{p \in P} \left| E_{p, S} U_{1, n} \right| = 0,$$

we obtain that

$$\lim_{n \to \infty} \sup_{S \in \Theta_{\epsilon, L}} \sup_{p \in P} \left| E_{p, S} U_{1, n} \right| = 0,$$

(i.e. $\lim_{n \to \infty} \sup_{S \in \Theta_{\epsilon, L}} \left| E_{p, S} U_{1, n} \right| = 0$).

Now, using Lemma A.2 from [9] we obtain that

$$\left| E_{p, S} U_{2, n} \right| = \left| \sum_{l=1}^{d} E_{p, S} \sigma_l^2 \sum_{j=1}^{d} \tilde{\lambda}_l^2(z_l) \phi_j(z_l) \right| \leq d \sigma_1 \eta^2 \left( 2^{2k+1} + 2^{k+2} + 1 \right) \leq 5 d \sigma_1 \eta^{2k}.$$

The definition of $\sigma_1 \eta$ in (2.16) implies $\lim \sup_{n \to \infty} d \sigma_1 \eta < \infty$, i.e.

$$\lim \sup_{n \to \infty} \sup_{S \in \Theta_{\epsilon, L}} \sup_{p \in P} \left| E_{p, S} U_{2, n} \right| < \infty.$$

Therefore, the using this bound in (A.3) implies

$$\lim_{n \to \infty} \sup_{S \in \Theta_{\epsilon, L}} \sup_{p \in P} |E_{p, S} U_{2, n}| = 0.$$

Using the inequality (A.4) from [4] we get $E_{p, S} \eta_{j, d} \leq 64 \nu^* \sigma_1 \eta^2$, where the coefficient $\nu^*$ is given in (2.15). From this we obtain, that

$$E_{p, S} |U_n L^c \leq \frac{(b - a)}{d} \sum_{j=1}^{d} E_{p, S} \eta_{j, d}^2 \eta_{j, d} L^c \varsigma + \varsigma \varsigma P_{p, S}(L^c) \leq \frac{8 \sigma_1 (b - a) \sqrt{\nu^*}}{d} \varsigma P_{p, S}(L^c) \varsigma + \varsigma \varsigma P_{p, S}(L^c).$$

So, Proposition 2.2 implies $\lim_{n \to \infty} n^{2k/(2k+1)} \sup_{S \in \Omega_{k, r}} E_{p, S} |U_n L^c \leq 0$. Hence Lemma A.8.
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