Multimodal Oscillations in Systems with Strong Contraction

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Mixed-mode oscillations

Applications: neuroscience, chemical oscillations, combustion

Dynamical features: dynamic bifurcations, complex bifurcation structure of families of periodic solutions, windows of chaotic behavior

Medvedev, Cisternas, Physica D, 2004
Mixed-mode oscillations in a model of solid combustion

Frankel, Kovačić, Roytburd, Timofeev, Physica D, 2000

- Time vs. Variable $v$ with $\nu = 0.33$
- Time vs. Variable $v$ with $\nu = 0.325$
- Time vs. Variable $v$ with $\nu = 0.25$
- Time vs. Variable $v$ with $\nu = 0.1$

- Period of $x_2$ vs. $\mu$
- Winding Number vs. $\mu$
Analytical approaches

- Analysis of the Hopf-homoclinic bifurcation: Belyakov; Bosch, Simo (93); Deng; Guckenheimer, Willms (97)

- Asymptotic expansions: Baer, Erneux (86,92); Mischenko, Rozov

- Reduction to 1D maps: Medvedev, Cisternas (04), Milik et al (98)

- Geometric singular perturbations: Krupa, Szmolyan; Brons et al (05)
The Model

A 3D approximation of a free boundary problem modeling solid combustion (Frankel, Kovačić, Roytburd, Timofeev, Physica D, 2000)

\[
\begin{align*}
\dot{\nu} &= \frac{3(\xi + \eta - \nu - \nu k(\nu) - v^2)}{\nu k'(\nu)} \\
\dot{\eta} &= \eta - \nu \\
\dot{\xi} &= 9(\nu - \eta) - 6\xi + \nu(\nu + 1)k(\nu) + 2v^2
\end{align*}
\]

Kinetic function:

\[
k(\nu) = \frac{(1 - \nu)^p - (1 - \nu)^{-q}}{p + q}, \quad q = 1
\]

Control parameters: \(\nu, p\)

Andronov-Hopf (AH) bifurcation: \(\nu = \frac{1}{3}\)

AH bifurcation is supercritical: \(0.6 < p < 2.63\)
The Linearized System

Linearize about \((0, 0, 0)\):

\[
\begin{pmatrix}
\dot{\nu} \\
\dot{\xi} \\
\dot{\eta}
\end{pmatrix} = \begin{pmatrix}
\frac{3-\nu}{\nu} & -\frac{3}{\nu} & -\frac{3}{\nu} \\
-1 & 0 & 1 \\
9-\nu & -6 & -9
\end{pmatrix}
\begin{pmatrix}
\nu \\
\xi \\
\eta
\end{pmatrix}
\]

The eigenvalues:

\[
\lambda_1 = -1, \quad \lambda_2 = \frac{-9\nu + 3 + \sqrt{69\nu^2 - 54\nu + 9}}{2\nu}, \quad \lambda_3 = \frac{-9\nu + 3 - \sqrt{69\nu^2 - 54\nu + 9}}{2\nu}.
\]

The AH bifurcation is at \(\nu = \frac{1}{3}\)

The AH bifurcation is supercritical: \(0.6 < p < 2.63\)

The eigenvectors:

\[
v_1 = \begin{pmatrix}
\frac{2}{3-\nu} \\
\frac{\nu-1}{3-\nu} \\
1
\end{pmatrix}, \quad v_2 = \begin{pmatrix}
1 \\
-\frac{\nu}{3\nu+1} \\
\frac{2}{3\nu+1}
\end{pmatrix}, \quad v_3 = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}.
\]
Two bifurcation scenarios

Subcritical AH bifurcation: \((p = 3)\)

Supercritical AH bifurcation \((p = 2.2)\)
The Approximate System

A coordinate transformation:

\[ y = (v, \xi, \eta) \quad \Longleftrightarrow \quad x = (x_1, x_2, x_3) \]
\[ v \quad \Longleftrightarrow \quad \mu + \frac{1}{3} \]
\[ y = P(\mu)x \quad P(\mu) = (v_1(\mu)v_2(\mu)v_3(\mu)) \]

A system in new coordinate:

\[ \dot{x} = D(\mu)x + P^{-1}(\mu)F(P(\mu)x) \]

The approximate system:

\[ \dot{x} = D(\mu)x + P^{-1}(0)F(P(0)x), \]
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 & 0 \\
0 & \alpha(\mu) & \beta(\mu) \\
0 & -\beta(\mu) & \alpha(\mu)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
+ \begin{pmatrix}
g_1(x_1, x_2, x_3) \\
g_2(x_1, x_2, x_3) \\
g_3(x_1, x_2, x_3)
\end{pmatrix}
\]

\[ \alpha(\mu) = \frac{-27\mu}{2(3\mu + 1)}, \quad \beta(\mu) = \frac{\sqrt{12 + 72\mu - 621\mu^2}}{2(3\mu + 1)} \]
Bifurcations of periodic orbits of the approximate system (subcritical AH bifurcation)
Bifurcations of periodic orbits of the approximate system (supercritical AH bifurcation)
Formulation of the problem

a. Proximity to AH bifurcation

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 & 0 \\
0 & a(\mu) & b(\mu) \\
0 & -b(\mu) & a(\mu)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} +
\begin{pmatrix}
h_1(x) \\
h_2(x) \\
h_3(x)
\end{pmatrix}
\]

\[a(0) = 0, \quad b(0) = \beta, \quad a'(0) > 0\]

b. Return mechanism

c. Strong contraction
Local dynamics near the saddle-focus

\[
\begin{align*}
\dot{x}_1 &= -x_1 + \sum a_{ij} x_i x_j + \sum a_{ijk} x_i x_j x_k + O(4), \\
\dot{x}_2 &= a(\mu)x_2 - b(\mu)x_3 + \sum b_{ij} x_i x_j + \sum b_{ijk} x_i x_j x_k + O(4), \\
\dot{x}_3 &= b(\mu)x_2 + a(\mu)x_3 + \sum c_{ij} x_i x_j + \sum c_{ijk} x_i x_j x_k + O(4).
\end{align*}
\]

a. Change to cylindrical coordinates: \( x_1, x_2 = \rho \cos \theta, x_3 = \rho \sin \theta \)
b. Rescale the variables: \( x_1 = \xi \epsilon^2 \) and \( \rho = r \epsilon \)
c. Recall: \( a(\mu) = \alpha \mu \epsilon^2 + O(\epsilon^4) \) and \( b(\mu) = \beta + O(\epsilon^2) \)

In new coordinates,

\[
\begin{align*}
\dot{\xi} &= -\xi + r^2 T_1(\theta) + O(\epsilon) \\
\dot{r} &= \epsilon r^2 Q_1(\theta) + \epsilon^2 r (\alpha \mu + \xi Q_2(\theta) + r^2 Q_3(\theta)) + O(\epsilon^3) \\
\dot{\theta} &= \beta + \epsilon r L_1(\theta) + O(\epsilon^2)
\end{align*}
\]
The rescaled system

\[
\begin{align*}
\dot{\xi} &= -\xi + r^2 T_1(\theta) + O(\epsilon) \\
\dot{r} &= \epsilon r^2 Q_1(\theta) + \epsilon^2 r (\alpha \mu + \xi Q_2(\theta) + r^2 Q_3(\theta)) + O(\epsilon^3) \\
\dot{\theta} &= \beta + \epsilon r L_1(\theta) + O(\epsilon^2)
\end{align*}
\]

Use \( \phi = \beta^{-1} \theta \) as an independent variable to obtain a slow-fast system

\[
\begin{align*}
\frac{d\xi}{d\phi} &= -\xi + r^2 P_1(\phi) + O(\epsilon) \\
\frac{dr}{d\phi} &= \epsilon Q_1(\phi) r^2 + \epsilon^2 r (\alpha \mu + \xi Q_2(\phi) + r^2 Q_3(\phi)) + (\epsilon^3)
\end{align*}
\]
The slow manifold

The fast-slow system

\[
\begin{align*}
\frac{d\xi}{d\phi} &= -\xi + r^2 P_1(\phi) + O(\epsilon) \\
\frac{dr}{d\phi} &= \epsilon Q_1(\phi)r^2 + \epsilon^2 r (\alpha \mu + \xi Q_2(\phi) + r^2 Q_3(\phi)) + (\epsilon^3)
\end{align*}
\]

The slow manifold:

\[
S = \{(\xi, r, \theta) : \xi = U(\theta)r^2, \ r \geq 0, \ \theta \in [0, 2\pi)\} \quad U(\theta) = a + A \cos (2\theta - \Delta_1),
\]
The slow manifold

\[
\frac{d\xi}{d\phi} = -\xi + r^2 P_1(\phi) + O(\epsilon)
\]

\[
\frac{dr}{d\phi} = \epsilon Q_1(\phi) r^2 + \epsilon^2 r (\alpha \mu + \xi Q_2(\phi) + r^2 Q_3(\phi)) + (\epsilon^3)
\]

The slow manifold:

\[
S = \{ (\xi, r, \theta) : \xi = U(\theta)r^2, \ r \geq 0, \ \theta \in [0, 2\pi) \} \quad U(\theta) = a + A \cos (2\theta - \Delta_1),
\]

\[
D' = \{ (\xi, r) : |\xi| \leq M, \ r \leq 2M \} \quad D'_0 = \{ (\xi, r) : |\xi| \leq M, \ r \leq M \}.
\]

Lemma: Let \((\xi(0), r(0)) \in D'_0\). Then for \(\phi \geq |\ln \epsilon|\) the trajectory of (1), (2) stays in an \(O(\epsilon)\) neighborhood of \(S\) as long as it remains in \(D\).
Use the slow equation:

\[ \frac{dr}{d\phi} = \epsilon Q_1 r^2 + \epsilon^2 r (\alpha \mu + \xi Q_2 + r^2 Q_3) + O(\epsilon^3) \]

and the approximation for \( \xi \),

\[ \xi(\phi) = \xi_0(\phi) r^2(\phi) + O(\epsilon), \]

to derive the reduced system

\[ \dot{I} = -2\epsilon^2 (\alpha \mu I + \gamma + Q(\phi)) + O(\epsilon^3) \]
\[ \dot{\phi} = 1 + O(\epsilon) \]

where

\[ I(\phi) = \left( \frac{1}{r(\phi)} + \epsilon q_1(\phi) \right)^2 \quad \gamma = \frac{1}{2\pi} \int_0^{2\pi} (U(\theta) Q_2(\theta) + Q_3(\theta)) d\theta, \]
The reduced system

\[
\dot{I} = -2\epsilon^2 (\alpha\mu I + \gamma + Q(\phi)) + O(\epsilon^3)
\]
\[
\dot{\phi} = 1 + O(\epsilon)
\]

The averaged equation

\[
\dot{J} = -2\epsilon^2 (\alpha\mu J + \gamma) + O(\epsilon^3)
\]

\(\gamma < 0\) (supercritical): a stable fixed point

\[
\bar{J} = \frac{-\gamma}{\alpha\mu} > 0 \quad \Rightarrow \quad \bar{r} = \sqrt{\frac{\alpha\mu}{-\gamma}} + O(\epsilon)
\]

\(\gamma > 0\) (subcritical): \(J\) decreases \(\Rightarrow\) \(r\) increases
A supercritical Andronov-Hopf bifurcation ($\gamma < 0$)

**Theorem**

If $\gamma < 0$ and $\mu > 0$, the trajectories converge to a limit cycle of period $\frac{2\pi}{\beta} + O(\epsilon)$, whose leading order approximation is given by

$$x_1 = \bar{\rho}^2 U(\theta) + O(\epsilon^3), \quad x_2 = \bar{\rho} \cos \theta + O(\epsilon^2), \quad \text{and} \quad x_2 = \bar{\rho} \sin \theta + O(\epsilon^2),$$

where

$$U(\theta) = a + A \cos (2\theta - \Delta_1),$$

$$\bar{\rho} = \sqrt{\frac{\text{Re}(\lambda)}{-\gamma}} = \epsilon \sqrt{\frac{\alpha \mu}{-\gamma}} + O(\epsilon^2) \quad \text{and} \quad \dot{\theta} = \beta.$$
If $\gamma > 0$ and $\mu > 0$, the evolution of the trajectories in the neighborhood of $S$ is given by

$$x_1(t) = \rho^2 U(\theta) + O(\epsilon^3), \quad x_2(t) = \rho \cos \theta + O(\epsilon^2), \quad \text{and} \quad x_3(t) = \rho \sin \theta + O(\epsilon^2),$$

where

$$\rho(t) = \frac{\epsilon}{\sqrt{I_0 e^{-2\alpha \epsilon^2 (t-t_0)} - \frac{\gamma}{\alpha \mu} - \epsilon q(\theta)}} + O(\epsilon^2), \quad q(\theta) = \ldots,$$

and

$$\theta = \theta_0 + \beta(t - t_0) + O(\epsilon).$$

These asymptotic formulae are valid uniformly on any finite interval of time $[t_0, t_1]$ provided

$$\theta_0 = \arctan\left(\frac{x_3(t_0)}{x_2(t_0)}\right) \quad \text{and} \quad I_0 = \epsilon^2 (x_2(t_0)^2 + x_3(t_0)^2)^{-1}.$$
A subcritical Andronov-Hopf bifurcation ($\gamma > 0$)

Additional assumptions:

- Return mechanism
- Strong contraction property
**Strong contraction property**

Assume that \( W^s(O) \) can be straightened in an \( O(1) \) neighborhood of the origin:

\[
h_{2,3}(x, 0, 0) = 0, \quad x \in [d_1, 0].
\]

Rewrite the system in a tubular neighborhood of \( W^s \)

\[
\begin{align*}
\dot{x} &= w(x) + \phi_1(x)y_1 + \phi_2(x)y_2, \\
\dot{y} &= A(x)y + \psi(x, y),
\end{align*}
\]

Let \( \lambda_{1,2}(x) \) denote the eigenvalues of \( A(x) \)

Rescale the control parameter: \( \mu := \mu \epsilon^2 \)

Strong contraction

\[
\nu_1 = \max_{x \in [d_1, d_2]} \frac{Re \lambda(x)}{w(x)} \leq C_2 \ln K \epsilon,
\]
Interspike intervals ($\gamma > 0$)

\[ T = T_0 + \frac{C}{\alpha \mu \epsilon^2} \ln \frac{\gamma \epsilon}{\alpha \mu} + \text{h.o.t.}, \]

\[ T = T_0 + \frac{C}{\alpha \mu \epsilon^2} \ln \frac{\gamma \epsilon}{\alpha \mu} + \text{h.o.t.}, \]
Oscillatory patterns near the border between regions of sub- and supercritical AHB ($\gamma \to 0 + 0$)
Chaotic dynamics near the border between regions of sub- and supercritical AHB \((\gamma \sim 0 + 0)\)

\[
I_{n+1} = P(I_n), \quad P(I) = \left(1 - 2\epsilon^2\alpha\mu\omega\right)I - 2\epsilon^2\omega\left(\gamma + \frac{\epsilon^2 c}{I^2}\right)
\]
Lyapunov coefficients
Conclusions

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