Tractability of the Function Approximation Problem in Terms of the Kernel’s Shape and Scale Parameters

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Abstract This article studies the problem of approximating functions belonging to a Hilbert space $\mathcal{H}_d$ with a reproducing kernel of the form

$$\tilde{K}_d(x,t) := \prod_{\ell=1}^{d} \left( 1 - \alpha_\ell^2 + \alpha_\ell^2 K_\gamma(x_\ell, t_\ell) \right)$$

for all $x, t \in \mathbb{R}^d$. The $\alpha_\ell \in [0,1]$ are scale parameters, and the $\gamma_\ell > 0$ are sometimes called shape parameters. The reproducing kernel $K_\gamma$ corresponds to some Hilbert space of functions defined on $\mathbb{R}$. The kernel $\tilde{K}_d$ generalizes the anisotropic Gaussian reproducing kernel, whose tractability properties have been established in the literature. We present sufficient conditions on $\{\alpha_\ell, \gamma_\ell\}_{\ell=1}^\infty$ under which polynomial tractability holds for function approximation problems on $\mathcal{H}_d$. The exponent of strong polynomial tractability arises from bounds on the eigenvalues of a positive definite linear operator.

1 Introduction

This article addresses the problem of function approximation. In a typical application we are given data of the form $y_i = f(x_i)$ or $y_i = L_i(f)$ for $i = 1, \ldots, n$. That is, a function $f$ is sampled at the locations $\{x_1, \ldots, x_n\}$, usually referred to as the data sites or the design, or more generally we know the values of $n$ linear functionals.
$L_1, \ldots, L_n$ applied to $f$. Here we assume that the domain of $f$ is a subset of $\mathbb{R}^d$. The goal is to construct $A_n(f)$, a good approximation to $f$ that is inexpensive to evaluate.

Algorithms for function approximation based on symmetric positive definite kernels have arisen in both the numerical computation literature [2, 4, 11, 16], and the statistical learning literature [1, 3, 6, 10, 12, 13, 14, 15]. These algorithms go by a variety of names, including radial basis function methods [2], scattered data approximation [16], meshfree methods [4], (smoothing) splines [15], kriging [13], Gaussian process models [10] and support vector machines [14].

Many kernels commonly used in practice are associated with a sequence of shape parameters $\gamma = \{\gamma_r\}_{r=1}^\infty$, which allows more flexibility in the function approximation problem. Examples of such kernels include the Matérn, the multiquadrics, the inverse multiquadrics, and the extensively studied Gaussian kernel (also known as the squared exponential kernel). The anisotropic stationary Gaussian kernel, is given by

$$\tilde{K}_d(x,t) := e^{-\gamma_1(x_1-t_1)^2} \cdots e^{-\gamma_d(x_d-t_d)^2} = \prod_{i=1}^d e^{-\gamma_i(x_i-t_i)^2} \text{ for all } x,t \in \mathbb{R}^d,$$  

where $\gamma_i$ is a positive shape parameter for each variable $x_i$. Choosing a small $\gamma_i$ has a beneficial effect on the rate of decay of the eigenvalues of the Gaussian kernel. The optimal choice of $\gamma_i$ is application dependent and much work has been spent on the quest for the optimal shape parameter. Note that taking $\gamma_i = \gamma$ for all $\ell$ will recover the isotropic Gaussian kernel.

For the Gaussian kernel (1), convergence rates with polynomial and tractability results are established in [5]. These rates are summarized in Table 1. Note that the error of an algorithm $A_n$ in this context is the worst case error based on the following $L_2^2$ criterion:

$$e_{wor}(A_n) := \sup_{\|f\|_{L_2^2} \leq 1} \|f - A_n(f)\|_{L_2^2}, \quad \|f\|_{L_2^2} := \left( \int_{\mathbb{R}^d} f^2(t) \rho_d(t) \, dt \right)^{1/2},$$

where $\rho_d$ is a probability density function with independent marginals, namely $\rho_d(x) = \rho_1(x_1) \cdots \rho_d(x_d)$. For real $\theta$, the notation $\sim n^\theta$ (with $n \to \infty$ implied) means that for all $\delta > 0$ the quantity is bounded above by $C_\delta n^\theta + \delta$ for all $n > 0$, where $C_\delta$ is some positive constant that is independent of the sample size, $n$, and the dimension, $d$, but may depend on $\delta$. The notation $\gtrsim n^\theta$ is defined analogously, and means that the quantity is bounded from below by $C_\delta n^{\theta-\delta}$ for all $\delta > 0$. The notation $\asymp n^\theta$ means that the quantity is both $\leq n^\theta$ and $\geq n^\theta$. The term $r(\gamma)$ appearing in Table 1

| Data Available | Absolute Error Criterion | Relative Error Criterion |
|----------------|--------------------------|--------------------------|
| Linear Functionals | $\asymp n^{\max(r(\gamma)/1/2)}$ | $\asymp n^{r(\gamma)}$, if $r(\gamma) > 0$ |
| Function values | $\lesssim n^{\max(r(\gamma)/[1+1/(2r(\gamma))],1/4)}$ | $\lesssim n^{r(\gamma)/[1+1/(2r(\gamma))]}$, if $r(\gamma) > 1/2$ |
denotes the rate of convergence to zero of the shape parameter sequence \( \gamma \) and is defined by
\[
r(\gamma) := \sup \left\{ \beta > 0 \mid \sum_{\ell=1}^{\infty} \gamma_\ell^{1/\beta} < \infty \right\}.
\] (3)

The kernel studied in this article has a more general product form
\[
\tilde{K}_d(x,t) = \tilde{K}_{\alpha,\gamma}(x,t) := \prod_{\ell=1}^{d} \hat{K}_{\alpha_\ell,\gamma}(x_{\ell},t_{\ell}) \quad \text{for all } x,t \in \mathbb{R}^d,
\] (4)
where \( 0 \leq \alpha_\ell \leq 1, \gamma_\ell > 0 \) and
\[
\hat{K}_{\alpha,\gamma}(x,t) := 1 - \alpha^2 + \alpha^2 K_\gamma(x,t), \quad x,t \in \mathbb{R}.
\] (5)

We assume that we know the eigenpair expansion of the one-dimensional kernel \( K_\gamma \) in terms of its shape parameter \( \gamma \). Many kernels in the numerical integration and approximation literature take the form of (4), where \( \alpha_\ell \) governs the vertical scale of the kernel across the \( \ell \)th dimension. In particular, taking \( \alpha_\ell = 1 \) for all \( \ell \) and \( K_\gamma(x,t) = \exp(-\gamma^2 (x-t)^2) \) recovers the anisotropic Gaussian kernel (1).

The goal of this paper is to extend the results in Table 1 to the kernel in (4). In essence we are able to replace \( r(\gamma) \) by \( \tilde{r}(\alpha,\gamma) \) defined below in (12).

Knowing the eigenpair expansion of \( K_\gamma(x,t) \) does not give us explicit formulae for the eigenvalues and eigenfunctions of the kernel \( \hat{K}_{\alpha,\gamma} \). However, since the kernel (4) is of tensor product form and each factor is a convex combination of the constant kernel and a kernel with a known eigenpair expansion, we can derive upper and lower bounds of the eigenvalues of \( \hat{K}_{\alpha,\gamma} \) by approximating the corresponding linear operators by finite rank operators and applying some inequalities for eigenvalues of matrices. These bounds then yield tractability results for the general kernels.

2 Function Approximation

2.1 Reproducing Kernel Hilbert Spaces

Let \( \mathcal{H}_d = \mathcal{H}(\tilde{K}_d) \) denote a reproducing kernel Hilbert space of real functions defined on \( \mathbb{R}^d \). The goal is to approximate any function in \( \mathcal{H}_d \) given a finite number of data about it. The reproducing kernel \( \tilde{K}_d : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) is symmetric and positive definite. It takes the form (4), where \( K_\gamma \) satisfies the unit trace condition
\[
\int_{\mathbb{R}} K_\gamma(t,t) \rho_1(t) \, dt = 1 \quad \forall \gamma > 0.
\] (6)

This condition implies that \( \mathcal{H}_d \) is continuously embedded in the space \( \mathcal{L}_2 = \mathcal{L}_2(\mathbb{R}^d, \rho_d) \) of square Lebesgue integrable functions, where the \( \mathcal{L}_2 \) norm was de-
fined in (2). Continuous embedding means that $\|I_d f\|_{\mathcal{L}_2} = \|f\|_{\mathcal{L}_2} \leq \|I_d\| \|f\|_{\mathcal{H}_d}$ for all $f \in \mathcal{H}_d$.

Functions in $\mathcal{H}_d$ are approximated by linear algorithms of the form

$$(A_n f)(x) := \sum_{j=1}^{n} L_j(f) a_j(x) \quad \text{for all} \quad f \in \mathcal{H}_d, \; x \in \mathbb{R}^d \quad \text{for all} \quad x \in \mathbb{R}^d.$$  

for some continuous linear functionals $L_j \in \mathcal{H}_d^*$, and functions $a_j \in \mathcal{L}_2$. Note that for known functions $a_j$, the cost of computing $A_n(f)(x)$ is $O(n)$, if we do not consider the cost of generating the data samples $L_j(f)$. The linear functionals, $L_j$, used by an algorithm $A_n$ may either come from the class of arbitrary bounded linear functionals, $\Lambda_{all} = \mathcal{H}_d^*$, or from the class of function evaluations, $\Lambda_{std}$. The $n$th minimal worst case error over all possible algorithms is defined as

$e^{w_{\partial}}(n, \mathcal{H}_d) := \inf_{A_n \text{ with } L_j \in \Lambda_{\partial}} e^{w_{\partial}}(A_n) \quad \partial \in \{\text{std, all}\}$.

### 2.2 Tractability

While typical numerical analysis focuses on the rate of convergence, it does not take into consideration the effects of $d$. The study of tractability arises in information-based complexity and it considers how the error depends on the dimension, $d$, as well as the number of data, $n$.

In particular, we would like to know how $e^{w_{\partial}}(n, \mathcal{H}_d)$ depends not only on $n$ but also on $d$. Because of the focus on $d$-dependence, the absolute and normalized error criteria described in the previous section may lead to different answers. For a given positive $\varepsilon \in (0,1)$ we want to find an algorithm $A_n$ with the smallest $n$ for which the error does not exceed $\varepsilon$ for the absolute error criterion, and does not exceed $\varepsilon e^{w_{\partial}}(0, \mathcal{H}_d) = \varepsilon \|I_d\|$ for the normalized error criterion. That is,

$$n_{w_{\psi}}^{w_{\partial}}(\varepsilon, \mathcal{H}_d) = \min \left\{ n \mid e^{w_{\partial}}(n, \mathcal{H}_d) \leq \begin{cases} \varepsilon, & \psi = \text{abs}, \\ \varepsilon \|I_d\|, & \psi = \text{norm}, \end{cases} \right\}.$$

Let $\mathcal{I} = \{I_d\}_{d \in \mathbb{N}}$ denote the sequence of function approximation problems. We say that $\mathcal{I}$ is polynomially tractable if and only if there exist numbers $C, p$ and $q$ such that

$$n_{w_{\psi}}^{w_{\partial}}(\varepsilon, \mathcal{H}_d) \leq C d^q \varepsilon^{-p} \quad \text{for all} \quad d \in \mathbb{N} \quad \text{and} \quad \varepsilon \in (0,1). \quad (7)$$

If $q = 0$ above then we say that $\mathcal{I}$ is strongly polynomially tractable and the infimum of $p$ satisfying the bound above is the exponent of strong polynomial tractability.

The essence of polynomial tractability is to guarantee that a polynomial number of linear functionals is enough to satisfy the function approximation problem to
within $\varepsilon$. Obviously, polynomial tractability depends on which class, $\Lambda^{all}$ or $\Lambda^{std}$, is considered and whether the absolute or normalized error is used.

The property of strong polynomial tractability is especially challenging since then the number of linear functionals needed for an $\varepsilon$-approximation is independent of $d$. Nevertheless, we provide here positive results on strong polynomial tractability.

## 3 Eigenvalues for the General Kernel

Let us define the linear operator corresponding to any kernel $\tilde{K}_d$ as

$$W f = \int_{\mathbb{R}^d} f(t) \tilde{K}_d(\cdot, t) \rho_d(t) \, dt \quad \text{for all } f \in \mathcal{H}_d.$$

It is known that $W$ is self-adjoint and positive definite if $\tilde{K}_d$ is a positive definite kernel. Moreover (6) implies that $W$ is compact. Let us define the eigenpairs of $W$ by $(\lambda_{d,j}, \eta_{d,j})$, where the eigenvalues are ordered, $\lambda_{d,1} \geq \lambda_{d,2} \geq \cdots$, and

$$W \eta_{d,j} = \lambda_{d,j} \eta_{d,j} \quad \text{with} \quad \langle \eta_{d,j}, \eta_{d,i} \rangle_{\mathcal{H}_d} = \delta_{i,j} \quad \text{for all } i, j \in \mathbb{N}.$$

Note also that for any $f \in \mathcal{H}_d$ we have

$$\langle f, \eta_{d,i} \rangle_{L_2} = \lambda_{d,i} \langle f, \eta_{d,j} \rangle_{\mathcal{H}_d}.$$

Taking $f = \eta_{d,i}$ we see that $\{ \eta_{d,j} \}$ is a set of orthogonal functions in $L_2$. Letting

$$\phi_{d,j} = \lambda_{d,j}^{-1/2} \eta_{d,j} \quad \text{for all } j \in \mathbb{N},$$

we obtain an orthonormal sequence $\{ \phi_{d,j} \}$ in $L_2$. Since $\{ \eta_{d,j} \}$ is a complete orthonormal basis of $\mathcal{H}_d$, we have

$$\tilde{K}_d(x,t) = \sum_{j=1}^{\infty} \eta_{d,j}(x) \eta_{d,j}(t) = \sum_{j=1}^{\infty} \lambda_{d,j} \phi_{d,j}(x) \phi_{d,j}(t) \quad \text{for all } x, t \in \mathbb{R}^d.$$

To standardize the notation, we shall always write the eigenvalues of the linear operator corresponding to the kernel $\tilde{K}_{d,a,y}$ in (4) in a weakly decreasing order $\nu_{d,a,y,1} \geq \nu_{d,a,y,2} \geq \cdots$. We drop the dependency on the dimension $d$ to denote the eigenvalues of the linear operator corresponding to the one-dimensional kernel $\tilde{K}_{a,y}$ in (5) by $\nu_{a,y,1} \geq \nu_{a,y,2} \geq \cdots$. Similarly the eigenvalues of the linear operator corresponding to the one-dimensional kernel $K_y(x,t)$ are denoted by $\lambda_{y,1} \geq \lambda_{y,2} \geq \cdots$. A useful relation between the multivariate eigenvalues $\nu_{d,a,y,j}$ and the univariate eigenvalues $\tilde{\nu}_{a,y,j}$ is given by [5, Lemma 3.1]:
where the operator norm

\[ \| A \|_{\infty} = \sup_{\| x \|_2 = 1} \| Ax \|_2 \]

Let the eigenvalues of the operators be sorted in a weakly decreasing order, i.e.,

\[ \lambda_1, \lambda_2, \ldots, \lambda_s \]

values are arranged in a weakly decreasing order, thus \( \lambda_1 \geq \lambda_2 \geq \cdots \). According to Lemma 11.1 (OS2), we have

\[ \lambda_{i,j+1} = a\lambda_{A,i,j+1} + b\lambda_{B,j+1}, \quad i, j = 1, 2, \ldots \]

We are interested in the high dimensional case where \( d \) is large, and we want to establish convergence and tractability results when \( \alpha \) and/or \( \gamma \) tend to zero as \( \ell \to \infty \). According to [8], strong polynomial tractability holds if the eigenvalues are bounded. The following lemma provides us some useful inequalities on eigenvalues of the linear operators corresponding to reproducing kernels.

**Lemma 1.** Let \( \mathcal{H}(K_A), \mathcal{H}(K_B), \mathcal{H}(K_C) \subset \mathcal{L}_2(\mathbb{R}, \rho_1) \) be Hilbert spaces with symmetric positive definite reproducing kernels \( K_A, K_B \) and \( K_C \) such that

\[ \int_{\mathbb{R}} K_k(t,t)\rho_1(t)\,dt < \infty, \quad \kappa \in \{A, B, C\}, \]

and \( K_C = aK_A + bK_B \), \( a, b \geq 0 \). Define the linear operators \( W_A, W_B, \) and \( W_C \) by

\[ W_kf = \int_{\mathbb{R}} f(t)K_k(\cdot,t)\rho_1(t)\,dt, \quad \text{for all } f \in \mathcal{H}(K_k), \quad \kappa \in \{A, B, C\}. \]

Let the eigenvalues of the operators be sorted in a weakly decreasing order, i.e., \( \lambda_{1,1} \geq \lambda_{2,1} \geq \cdots \). Then these eigenvalues satisfy

\[ \lambda_{C,i,j+1} \leq a\lambda_{A,i,j+1} + b\lambda_{B,j+1}, \quad i, j = 1, 2, \ldots \]

**Proof.** Let \( \{u_j\}_{j\in\mathbb{N}} \) be any orthonormal basis in \( \mathcal{L}_2(\mathbb{R}, \rho_1) \). We assign the orthogonal projections \( P_n \) given by

\[ P_n x = \sum_{j=1}^{n} \langle x, u_j \rangle u_j, \quad x \in \mathcal{L}_2(\mathbb{R}, \rho_1). \]

Since \( W_A \) is compact due to (8), it can be shown that \( \|(I - P_n)W_A\| \to 0 \) as \( n \to \infty \), where the operator norm

\[ \|(I - P_n)W_A\| := \sup_{\|x\|_2 \leq 1} \|(I - P_n)W_Ax\|_{\mathcal{L}_2(\mathbb{R}, \rho_1)}. \]

Furthermore [9, Lemma 11.1 (OS2)] states that for every pair \( T_1, T_2 : X \to Y \) of compact operators we have \( |s_j(T_1) - s_j(T_2)| \leq \|T_1 - T_2\|, \ j \in \mathbb{N} \), where the singular values \( s_j(T_k), k = 1, 2 \) are the square roots of the eigenvalues \( \lambda_j(T_k) \) arranged in a weakly decreasing order, thus \( s_j(T_k) = \sqrt{\lambda_j(T_k^* T_k)} \). Now we can bound

\[ |s_j(W_A) - s_j(P_n W_A P_n)| \leq |s_j(W_A) - s_j(P_n W_A)| + |s_j(P_n W_A) - s_j(P_n W_A P_n)| \leq \|W_A - P_n W_A\| + \|P_n W_A - P_n W_A P_n\| \leq \|(I - P_n)W_A\| + \|W_A (I - P_n)\| \to 0. \]
as $n \to \infty$. Thus the eigenvalues $\lambda_{P_nW_A^P_n,j} \to \lambda_{W_A,j}$ for all $j$ as $n \to \infty$. Similarly this applies to the operators $W_B$ and $W_C$. Note that we have

$$P_nW_CP_n = aP_nW_AP_n + bP_nW_BP_n$$

and these finite rank operators correspond to self-adjoint matrices. These matrices are symmetric and positive definite because the kernels are symmetric and positive definite. Since the inequalities (9) and (10) hold for the eigenvalues of symmetric positive definite matrices, they also hold for the operators corresponding to symmetric and positive definite kernels.

We are now ready to present the main results of this article in the following two sections.

4 Tractability for the Absolute Error Criterion

We now consider the function approximation problem for Hilbert spaces $\mathcal{H}_d = \mathcal{H} (\tilde{K}_d)$ with a general kernel using the absolute error criterion. From the discussion of eigenvalues in the previous section and from (6) it follows that

$$\sum_{j=1}^{\infty} \tilde{\lambda}_{\gamma,j} = \int_{\mathbb{R}} K_{\gamma}(t,t) \rho_1(t) dt = 1, \quad \forall \gamma > 0. \quad (11)$$

We want to verify whether polynomial tractability holds, namely whether (7) holds.

4.1 Arbitrary Linear Functionals

We first analyze the class $\Lambda^\text{all}$ and polynomial tractability. Similar to (3), let us define the rate of decay of scale and shape parameters $\tilde{r}(\alpha, \gamma)$ as

$$\tilde{r}(\alpha, \gamma) = \sup \left\{ \beta > 0 \left| \sum_{\ell=1}^{\infty} (\alpha_{\gamma})_{\ell}^{1/\beta} \right| < \infty \right\} \quad (12)$$

with the convention that the supremum of the empty set is taken to be zero.

Theorem 1. Consider the function approximation problem $\mathcal{F} = \{ I_d \}_{d \in \mathbb{N}}$ for Hilbert spaces for the class $\Lambda^\text{all}$ and the absolute error criterion with the kernels (4) satisfying (11). Let $\tilde{r}(\alpha, \gamma)$ be given by (12). If there exist constants $C_1, C_2, C_3 > 0$, which are independent of $\gamma$ but may depend on $\tilde{r}(\alpha, \gamma)$ and $\sup \{ \gamma | \ell \in \mathbb{N} \}$, such that
\[
\int_{\mathbb{R}^2} K_\gamma(x,t) \rho_1(x) \rho_1(t) dx dt \geq 1 - C_1 \gamma^2,
\]
\[
C_2 \leq \sum_{j=2}^{\infty} \left( \frac{\tilde{\lambda}_j}{\gamma^j} \right) \leq C_3
\]
hold for all \( 0 < \gamma < \sup \{ \gamma | \ell \in \mathbb{N} \} \), then it follows that

- \( \mathcal{I} \) is strongly polynomially tractable with exponent
  \[
p^{\text{all}} = \min \left( 2, \frac{1}{\bar{r} (\alpha, \gamma)} \right).
\]
- For all \( d \in \mathbb{N} \) we have
  \[
e^{\text{wor-all}} (n, \mathcal{K}_d) \leq n^{-1/p^{\text{all}}} = n^{-\max(\bar{r}(\alpha, \gamma), 1/2)} \quad n \to \infty,
\]
  \[
n^{\text{wor-abs-all}} (\varepsilon, \mathcal{K}_d) \leq \varepsilon^{-p^{\text{all}}} \quad \varepsilon \to 0,
\]
  where \( \lesssim n^q \) with \( n \to \infty \) was defined in Section 1, and \( \lesssim \varepsilon^q \) with \( \varepsilon \to 0 \) is analogous to \( \lesssim (1/\varepsilon)^{-q} \) with \( 1/\varepsilon \to \infty \).
- For the isotropic kernel with \( \alpha_\ell = \alpha \) and \( \gamma_\ell = \gamma \) for all \( \ell \), the exponent of strong tractability is 2. Furthermore strong polynomial tractability is equivalent to polynomial tractability.

**Proof.** From [8, Theorem 5.1] it follows that \( \mathcal{I} \) is strongly polynomially tractable if and only if there exist two positive numbers \( c_1 \) and \( \tau \) such that

\[
c_2 := \sup_{d \in \mathbb{N}} \left( \sum_{j=1,|c_1]}^{\infty} V_{d,\alpha,\gamma,j} \right)^{1/\tau} < \infty.
\]

Furthermore, the exponent \( p^{\text{all}} \) of strong polynomial tractability is the infimum of 2\( \tau \) for which this condition holds. Obviously (15) holds for \( c_1 = 1 \) and \( \tau = 1 \) because

\[
\sum_{j=1}^{\infty} V_{d,\alpha,\gamma,j} = \prod_{\ell=1}^{d} \left( \sum_{j=1}^{\infty} \tilde{V}_{\alpha,\gamma,j} \right) = \prod_{\ell=1}^{d} \left( \int_{\mathbb{R}} \left( 1 - \alpha_\ell^2 + \alpha_\ell^2 K_\gamma(t,t) \right) \rho_1(t) dt \right) = \prod_{\ell=1}^{d} (1 - \alpha_\ell^2 + \alpha_\ell^2) = 1.
\]

This shows that \( p^{\text{all}} \leq 2 \).

Take now \( \bar{r}(\alpha, \gamma) > 0 \). Consider first the case \( d = 1 \) for simplicity. Then kernel \( \tilde{K}_{d,\alpha,\gamma} \) in (4) becomes \( \tilde{K}_{\alpha,\gamma} \). We will show that for \( \tau = 1/(2\bar{r}(\alpha, \gamma)) \), the eigenvalues of \( \tilde{K}_{\alpha,\gamma} \) satisfy

\[
\sum_{j=1}^{\infty} V_{\alpha,\gamma,j}^{\tau} \leq 1 + C_U (\alpha \gamma)^{2\tau},
\]

where \( C_U \) depends on the kernel parameters.
where the constant $C_U$ does not depend $\alpha$ or $\gamma$. Since all the eigenvalues of $K_\gamma$ are non-negative, we clearly have for the first eigenvalue of $K_\gamma$,

$$\hat{\nu}_{\alpha, \gamma, 1} \leq 1.$$  \hfill (17)

On the other hand, (13) gives the lower bound of the first eigenvalue of $\hat{K}_{\alpha, \gamma}$

$$\hat{\nu}_{\alpha, \gamma, 1} \geq \int_{\mathbb{R}^2} \hat{K}_{\alpha, \gamma}(x, t) \rho_1(x) \rho_1(t) \, dx = \int_{\mathbb{R}^2} \left(1 - \alpha^2 + \alpha^2 K_\gamma(x, t)\right) \rho_1(x) \rho_1(t) \, dx = 1 - \alpha^2 + \alpha^2 \int_{\mathbb{R}^2} K_\gamma(x, t) \rho_1(x) \rho_1(t) \, dx \geq 1 - C_1 (\alpha \gamma)^2. \tag{18}$$

It follows from (11) that

$$\hat{\nu}_{\alpha, \gamma, 2} \leq C_1 (\alpha \gamma)^2. \tag{19}$$

For $j \geq 3$, the upper bound of $\hat{\nu}_{\alpha, \gamma, j}$ is given by (9) with $i = 1$:

$$\hat{\nu}_{\alpha, \gamma, j} \leq \alpha^2 \hat{\lambda}_{\gamma, j-1}, \tag{20}$$

which in turn yields

$$\sum_{j=3}^{\infty} \hat{\nu}_{\alpha, \gamma, j} \alpha^2 \sum_{j=3}^{\infty} \hat{\lambda}_{\gamma, j-1} \leq C_3 (\alpha \gamma)^{2\tau} \tag{21}$$

by (14). Combining (17), (19) and (21) gives (16), where the constant $C_U = C_1^2 + C_3$.

The lower bound we want to establish is that for $\tau < 1/(2 \hat{r}(\alpha, \gamma))$,

$$\sum_{j=1}^{\infty} \hat{\nu}_{\alpha, \gamma, j} \geq 1 + C_L (\alpha \gamma)^{2\tau} \text{ if } \alpha \gamma < \left(\frac{C_2}{2C_1}\right)^{1/[2(1-\tau)]}, \tag{22}$$

where $C_L := C_2/2$. It follows from (18) that

$$\hat{\nu}_{\alpha, \gamma, 1} \geq \hat{\nu}_{\alpha, \gamma, 1} \geq 1 - C_1 (\alpha \gamma)^2. \tag{23}$$

In addition we apply the eigenvalue inequality (9) to obtain

$$\hat{\nu}_{\alpha, \gamma, j} \geq \alpha^2 \hat{\lambda}_{\gamma, j}, \quad j = 2, 3, \ldots$$

which in turn gives

$$\sum_{j=2}^{\infty} \hat{\nu}_{\alpha, \gamma, j} \alpha^2 \sum_{j=2}^{\infty} \hat{\lambda}_{\gamma, j} \geq C_2 (\alpha \gamma)^{2\tau}, \tag{24}$$

where the last inequality follows from (14). Inequalities (23) and (24) together give

$$\sum_{j=1}^{\infty} \hat{\nu}_{\alpha, \gamma, j} \geq 1 - C_1 (\alpha \gamma)^2 + C_2 (\alpha \gamma)^{2\tau} \geq 1 + (C_2/2)(\alpha \gamma)^{2\tau}$$
under the condition in (22) on small enough $\alpha \gamma$. Thus we obtain (22).

For the multivariate case, the sum of the $\tau$-th power of the eigenvalues is bounded from above for $\tau = 1/(2\tilde{r}(\alpha, \gamma))$ because

$$\sum_{j=1}^{\infty} v_{d,\alpha,\gamma,j}^\tau = \prod_{\ell=1}^d \left( \sum_{j=1}^{\infty} \tilde{v}_{d,\alpha,\gamma,j}^\tau \right) \leq \prod_{\ell=1}^d \left( 1 + C_U(\alpha \gamma)^{2\tau} \right) = \exp \left( \sum_{j=1}^{\infty} \ln \left( 1 + C_U(\alpha \gamma)^{2\tau} \right) \right) \leq \exp \left( C_U \sum_{\ell=1}^{\infty}(\alpha \gamma)^{2\tau} \right) < \infty. \quad (25)$$

This shows that $p^{\text{all}} \leq 1/\tilde{r}(\alpha, \gamma)$.

We now consider the lower bound in the multivariate case and define the set $A$ by

$$A = \left\{ \ell \mid \alpha \gamma < \left( \frac{C_2}{2C_1} \right)^{1/[2(1-\tau)]} \right\}.$$

Then

$$\sup_{d \in \mathbb{N}} \left( \sum_{j=1}^{\infty} \tilde{v}_{d,\alpha,\gamma,j}^\tau \right) = \prod_{\ell=1}^d \left( \sum_{j=1}^{\infty} \tilde{v}_{d,\alpha,\gamma,j}^\tau \right) \geq \prod_{\ell \in A} \left( \sum_{j=1}^{\infty} \tilde{v}_{d,\alpha,\gamma,j}^\tau \right) \prod_{\ell \not\in A} \left( \sum_{j=1}^{\infty} \tilde{v}_{d,\alpha,\gamma,j}^\tau \right).$$

We want to show that this supremum is infinite for $\tau < 1/(2\tilde{r}(\alpha, \gamma))$. We do this by proving that the first product on the right is infinite. Indeed for $\tau < 1/(2\tilde{r}(\alpha, \gamma))$,

$$\prod_{\ell \in A} \left( \sum_{j=1}^{\infty} \tilde{v}_{d,\alpha,\gamma,j}^\tau \right) \geq \prod_{\ell \in A} \left[ 1 + C_L(\alpha \gamma)^{2\tau} \right] \geq 1 + C_L \sum_{\ell \in A} (\alpha \gamma)^{2\tau} = \infty.$$

Therefore, $p^{\text{all}} \geq 1/\tilde{r}(\alpha, \gamma)$, which establishes the formula for $p^{\text{all}}$. The estimates on $e^{\text{wor-all}}(n, \mathcal{H}_d)$ and $n^{\text{wor-abs-all}}(\varepsilon, \mathcal{H}_d)$ follow from the definition of strong tractability.

Finally, the exponent of strong tractability is 2 for the isotropic kernel because $\tilde{r}(\alpha, \gamma) = 0$ in this case. To prove that strong polynomial tractability is equivalent to polynomial tractability, it is enough to show that polynomial tractability implies strong polynomial tractability. From [8, Theorem 5.1] we know that polynomial tractability holds if and only if there exist numbers $c_1 > 0$, $q_1 \geq 0$, $q_2 \geq 0$ and $\tau > 0$ such that

$$c_2 := \sup_{d \in \mathbb{N}} \left\{ d^{-q_2} \left( \sum_{j=1}^{\infty} \lambda_{d,j}^\tau \right)^{1/\tau} \right\} < \infty.$$

If so, then

$$n^{\text{wor-abs-all}}(\varepsilon, \mathcal{H}_d) \leq (c_1 + c_2^\tau) d^{\max(q_1, q_2 \tau)} \varepsilon^{-2\tau}$$

for all $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$. Note that for all $d$ we have
This implies that $\tau \geq 1$. On the other hand, for $\tau = 1$ we can take $q_1 = q_2 = 0$ and arbitrarily small $C_1$, and obtain strong tractability. This completes the proof. \hfill $\Box$

Theorem 1 states that the exponent of strong polynomial tractability is at most 2, while for all shape parameters for which $\tilde{r}(\gamma) > 1/2$ the exponent is smaller than 2. Again, although the rate of convergence of $e^{\text{wor-std}}(n, \mathcal{K}_d)$ is always excellent, the dependence on $d$ is eliminated only at the expense of the exponent which must be roughly $1/p^{\text{all}}$. Of course, if we take an exponentially decaying sequence of the products of scale parameters and shape parameters, say, $\alpha^\ell \gamma^\ell = q^\ell$ for some $q \in (0, 1)$, then $\tilde{r}(\gamma) = \infty$ and $p^{\text{all}} = 0$. In this case, we have an excellent rate of convergence without any dependence on $d$.

### 4.2 Only Function Values

The tractability results for the class $\Lambda^{\text{std}}$ are stated in the following theorem.

**Theorem 2.** Consider the function approximation problem $\mathcal{J} = \{I_d\}_{d \in \mathbb{N}}$ for Hilbert spaces for the class $\Lambda^{\text{std}}$ and the absolute error criterion with the kernels (4) satisfying (11). Let $\tilde{r}(\alpha, \gamma)$ be given by (12). If there exist constants $C_1, C_2, C_3 > 0$, which are independent of $\gamma$ but may depend on $\tilde{r}(\alpha, \gamma)$ and $\sup\{\gamma^\ell | \ell \in \mathbb{N}\}$, such that (13) and (14) are satisfied for all $0 < \gamma < \sup\{\gamma^\ell | \ell \in \mathbb{N}\}$, then

- $\mathcal{J}$ is strongly polynomially tractable with exponent of strong polynomial tractability at most 4. For all $d \in \mathbb{N}$ and $\varepsilon \in (0, 1)$ we have
  
  $$e^{\text{wor-std}}(n, \mathcal{K}_d) \leq \sqrt{2} \frac{\varepsilon}{n^{1/4}} \left(1 + \frac{1}{2\sqrt{n}}\right)^{1/2},$$
  
  $$n^{\text{wor-abs-std}}(\varepsilon, \mathcal{K}_d) \leq \left[\frac{(1 + \sqrt{1 + \varepsilon^2})^2}{\varepsilon^4}\right].$$

- For the isotropic kernel with $\alpha^\ell = \alpha$ and $\gamma^\ell = \gamma$ for all $\ell$, the exponent of strong tractability is at least 2 and strong polynomial tractability is equivalent to polynomial tractability.

Furthermore if $\tilde{r}(\alpha, \gamma) > 1/2$, then

- $\mathcal{J}$ is strongly polynomially tractable with exponent of strong polynomial tractability at most

  $$p^{\text{std}} = \frac{1}{\tilde{r}(\alpha, \gamma)} + \frac{1}{2p^2(\alpha, \gamma)} = p^{\text{all}} + \frac{1}{2}(p^{\text{all}})^2 < 4.$$
For all \( d \in \mathbb{N} \) we have
\[
e^{\text{wor-std}}(n, \mathcal{F}_d) \leq n^{-1/p^\text{std}} = \left[ \frac{-\ell(\alpha, \gamma)/[1+1/(2\ell(\alpha, \gamma))] \right] n \rightarrow \infty,
\]
\[
n^{\text{wor-std}}(\varepsilon, \mathcal{F}_d) \leq \varepsilon^{-\rho^\text{std}} \quad \varepsilon \to 0.
\]

Proof. The same proofs as for [5, Theorem 5.3 and 5.4] can be used. We only need to show that the assumption of [7, Theorem 5], which is used in [5, Theorem 5.4], is satisfied. It is enough to show that there exists \( p > 1 \) and \( B > 0 \) such that for any \( n \in \mathbb{N} \),
\[
\nu_{d, \alpha, \gamma, n} \leq \frac{B}{n^p}, \quad (26)
\]

Take \( \tau = 1/(2\ell(\alpha, \gamma)) \). Since the eigenvalues \( \tilde{\lambda}_{n, \alpha, \gamma} \) are ordered, we have for \( n \geq 2 \),
\[
\tilde{\lambda}_{n, \alpha, \gamma} \leq \frac{1}{n-1} \sum_{j=2}^{n} \tilde{\lambda}_{n, j} \leq \frac{1}{n-1} \sum_{j=2}^{n} \tilde{\lambda}_{j, \alpha, \gamma} \leq \frac{C_3 \gamma^{2\tau}}{n-1},
\]
where the last inequality follows from (14). Raising to the power \( 1/\tau_0 \) gives
\[
\tilde{\lambda}_{n, \alpha, \gamma} \leq \gamma^{2\tau} \left( \frac{C_3}{n-1} \right)^{1/\tau}.
\]

Furthermore (20) implies that for \( n \geq 3 \),
\[
\nu_{d, \alpha, \gamma, n} \leq \alpha_2^2 \gamma^{2\tau} \left( \frac{C_3}{n-2} \right)^{1/\tau} = \alpha_2^2 \gamma^{2\tau} C_3^{1/\tau} \left( \frac{n}{n-2} \right)^{1/\tau} \left( \frac{1}{n^{1/\tau}} \right) \\
\leq \frac{\alpha_2^2 \gamma^{2\tau} (3C_3)^{1/\tau}}{n^{1/\tau}}.
\]

Since \( \nu_{d, \alpha, \gamma, n} \leq 1 \) for all \( n \in \mathbb{N} \), we have that for all \( 1 \leq \ell \leq d \) and \( n \geq 3 \),
\[
\nu_{d, \alpha, \gamma, n} \leq \nu_{d, \alpha, \gamma, n} \leq \frac{C_4}{n^p},
\]
where \( C_4 = \alpha_2^2 \gamma^{2\tau} (3C_3)^{1/\tau} \) and \( p = 1/\tau > 1 \). For \( n = 1 \) and \( n = 2 \), we can always find \( C_5 \) large enough such that \( \nu_{d, \alpha, \gamma, n} \leq C_5/n^p \). Therefore (26) holds for \( B = \max\{C_4, C_5\} \). \( \square \)

Note that (26) can be easily satisfied for many kernels used in practice. This theorem implies that for large \( \ell(\alpha, \gamma) \), the exponents of strong polynomial tractability are nearly the same for both classes \( \Lambda^\text{all} \) and \( \Lambda^\text{std} \). For an exponentially decaying sequence of shape parameters, say, \( \alpha_2 \gamma^\ell = q^\ell \) for some \( q \in (0, 1) \), we have \( p^\text{all} = p^\text{std} = 0 \), and the rates of convergence are excellent and independent of \( d \).
5 Tractability for the Normalized Error Criterion

We now consider the function approximation problem for Hilbert spaces $H_d(\tilde{K}_d)$ with a general kernel for the normalized error criterion. That is, we want to find the smallest $n$ for which

$$e_{\text{wor-}\varnothing}(n, H_d) \leq \varepsilon \|I_d\|, \quad \varnothing \in \{\text{std}, \text{all}\}.$$  

Note that $\|I_d\| = \sqrt{\nu_d} \alpha \alpha \gamma_1 \leq 1$ and it can be exponentially small in $d$. Therefore the normalized error criterion may be much harder than the absolute error criterion. It follows from [5, Theorem 6.1] that for the normalized error criterion, lack of polynomial tractability holds for the isotropic kernel for the class $\Lambda_{\text{all}}$ and hence for the class $\Lambda_{\text{std}}$.

5.1 Arbitrary Linear Functionals

We do not know polynomial tractability holds for kernels with $0 < \tilde{r}(\alpha, \gamma) < 1/2$. For $\tilde{r}(\alpha, \gamma) \geq 1/2$, we have the following theorem.

**Theorem 3.** Consider the function approximation problem $\mathcal{I} = \{I_d\}_{d \in \mathbb{N}}$ for Hilbert spaces for the class $\Lambda_{\text{std}}$ and the relative error criterion with the kernels (11) satisfying (13). Let $\tilde{r}(\alpha, \gamma)$ be given by (12) and $\tilde{r}(\alpha, \gamma) \geq 1/2$. If there exist constants $C_1, C_2, C_3 > 0$, which are independent of $\gamma$ but may depend on $\tilde{r}(\alpha, \gamma)$ and $\sup\{\gamma | \ell \in \mathbb{N}\}$, such that (13) and (14) are satisfied for all $0 < \gamma < \sup\{\gamma | \ell \in \mathbb{N}\}$, then

- $\mathcal{I}$ is strongly polynomially tractable with exponent of strong polynomial tractability

$$p_{\text{all}} = \frac{1}{\tilde{r}(\alpha, \gamma)}.$$ 

- For all $d \in \mathbb{N}$ we have

$$e_{\text{wor-all}}(n, \mathcal{H}_d) \leq \|I_d\| n^{-1/p_{\text{all}}} = n^{-\tilde{r}(\gamma)} \quad n \to \infty,$$

$$e_{\text{wor-abs-all}}(\varepsilon, \mathcal{H}_d) \leq \varepsilon^{-p_{\text{all}}} \quad \varepsilon \to 0.$$

**Proof.** From [8, Theorem 5.2] we know that strong polynomial tractability holds if and only if there exists a positive number $\tau$ such that

$$c_2 := \sup_d \sum_{j=1}^{\infty} \left( \frac{V_d, \alpha, \gamma_j}{V_d, \alpha, \gamma_1} \right)^{\tau} = \sup_d \left( \frac{1}{V_d, \alpha, \gamma_1} \sum_{j=1}^{\infty} V_d, \alpha, \gamma_j \right) < \infty.$$  

If so, then $n_{\text{wor-nor-all}}(\varepsilon, \mathcal{H}_d) \leq c_2 \varepsilon^{-2\tau}$ for all $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$, and the exponent of strong polynomial tractability is the infimum of $2\tau$ for which $c_2 < \infty$. 

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For all \( d \in \mathbb{N} \), we have \( \sum_{j=1}^{\infty} \nu_{d, \alpha, \gamma}^r < \infty \) for \( \tau = 1/(2 \bar{\eta}(\alpha, \gamma)) \) from (25). It remains to note that \( \sup_d \{1/\nu_{d, \alpha, 1}^r\} < \infty \) if and only if \( \sup_d \{1/\nu_{d, \alpha, 1}^r\} < \infty \). Furthermore note that (18) implies that

\[
\sup_d \left\{ \frac{1}{\nu_{d, \alpha, 1}^r} \right\} \leq \prod_{\ell=1}^{\infty} \left( 1 - C_1(\alpha_\ell \gamma_\ell)^2 \right) \leq \prod_{\ell=1}^{\infty} \left( 1 - C_1(\alpha_\ell)^2 \right).
\]

Clearly, \( \bar{\eta}(\alpha, \gamma) \geq 1/2 \) implies that \( \sum_{\ell=1}^{\infty} (\alpha_\ell \gamma_\ell)^2 < \infty \), which yields \( c_2 < \infty \). This also proves that \( p^{\text{all}} \leq 1/\bar{\eta}(\alpha, \gamma) \). The estimates on \( e^{\text{wor-all}}(n, \mathcal{H}_d) \) and \( n^{\text{wor-nor-all}}(\varepsilon, \mathcal{H}_d) \) follow from the definition of strong tractability. \( \square \)

### 5.2 Only Function Values

We now turn to the class \( \Lambda^{\text{std}} \). We do not know if polynomial tractability holds for the class \( \Lambda^{\text{std}} \) for \( 0 < \bar{\eta}(\alpha, \gamma) \leq 1/2 \). For \( \bar{\eta}(\alpha, \gamma) > 1/2 \), we have the following theorem.

**Theorem 4.** Consider the function approximation problem \( \mathcal{F} = \{I_d\}_{d \in \mathbb{N}} \) for Hilbert spaces with the kernel (4) for the class \( \Lambda^{\text{std}} \) and the normalized error criterion. Let \( \bar{\eta}(\alpha, \gamma) \) be given by (12) and \( \bar{\eta}(\alpha, \gamma) > 1/2 \). If there exist constants \( C_1, C_2, C_3 > 0 \), which are independent of \( \gamma \) but may depend on \( \bar{\eta}(\alpha, \gamma) \) and \( \sup \{ \gamma_\ell \in \mathbb{N} \} \), such that (13) and (14) are satisfied for all \( 0 < \gamma < \sup \{ \gamma_\ell \in \mathbb{N} \} \), then

- \( \mathcal{F} \) is strongly polynomially tractable with exponent of strong polynomial tractability at most

\[
p^{\text{std}} = \frac{1}{\bar{\eta}(\alpha, \gamma)} + \frac{1}{2\bar{\eta}^2(\alpha, \gamma)} = p^{\text{all}} + \frac{1}{2}(p^{\text{all}})^2 < 4.
\]

For all \( d \in \mathbb{N} \) we have

\[
e^{\text{wor-std}}(n, \mathcal{H}_d) \leq n^{-1/p^{\text{std}}} \quad n \to \infty,
\]

\[
n^{\text{wor-nor-std}}(\varepsilon, \mathcal{H}_d) \leq \varepsilon^{-p^{\text{std}}} \quad \varepsilon \to 0.
\]

**Proof.** The initial error is

\[
\|I_d\| \geq \prod_{\ell=1}^{d} (1 - C_1(\alpha_\ell \gamma_\ell)^2)^{1/2} = \exp \left( \Theta(1) - \frac{1}{2} \sum_{\ell=1}^{d} (\alpha_\ell \gamma_\ell)^2 \right).
\]

\( \bar{\eta}(\alpha, \gamma) > 1/2 \) implies that \( \|I_d\| \) is uniformly bounded from below by a positive number. This shows that there is no difference between the absolute and normalized error criteria. This means that we can apply Theorem 2 for the class \( \Lambda^{\text{std}} \) with \( \varepsilon \) replaced by \( \varepsilon \|I_d\| = \Theta(\varepsilon) \). This completes the proof. \( \square \)
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