On the structure of the adjacency matrix of the line digraph of a regular digraph

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Abstract

We show that the adjacency matrix $M$ of the line digraph of a $d$-regular digraph $D$ on $n$ vertices can be written as $M = AB$, where the matrix $A$ is the Kronecker product of the all-ones matrix of dimension $d$ with the identity matrix of dimension $n$ and the matrix $B$ is the direct sum of the adjacency matrices of the factors in a dicycle factorization of $D$.

Key words: Line digraph; adjacency matrix; de Bruijn digraph

Introduction

Line digraphs of regular digraphs and their generalizations are important in the design of point-to-point interconnection networks for parallel computers and distributed systems. For instance, de Bruijn digraphs and Reddy-Pradhan-Kuhl digraphs, which are important topologies for interconnection networks, are all examples of line digraphs of regular digraphs (see, e.g., [1],[2] and [6]). In this note, we describe a special regularity property of the adjacency matrix of the line digraph of a regular digraph. Before stating formally our main result, we recall the necessary graph-theoretic terminology.

A (finite) directed graph, for short digraph, consists of a non-empty finite set of elements called vertices and a (possibly empty) finite set of ordered pairs of vertices called arcs. The digraphs considered here are without multiple arcs. We denote by $D = (V, A)$ a digraph with vertex-set $V(D)$ and arc-set $A(D)$. A labeling of the vertices of a digraph $D$ is a function $l : V(D) \rightarrow L$, where $L$ is a set of labels. Chosen a bijective labeling, the adjacency matrix of a digraph $D$ with $n$ vertices, denoted by $M(D)$, is the $n \times n$ $(0, 1)$-matrix with $ij$-th element defined by $M(i, j)(D) = 1$ if $(v_i, v_j) \in A(D)$ and $M(i, j)(D) = 0$, otherwise. For any vertex $v_i \in V(D)$ of a digraph $D$, let $d^+_D(v_i) := | \{v_j : (v_j, v_i) \in A(D)\}$ |
and \( d_D^+(v_i) := | \{ v_j : (v_i, v_j) \in A(D) \} | \). A digraph \( D \) is said to be \( d \)-regular if, for every vertex \( v_i \in V(D) \), \( d_D^+(v_i) = d \). A digraph \( H \) is a subdigraph of a digraph \( D \) if \( V(H) \subseteq V(D) \) and \( A(H) \subseteq A(D) \). A subdigraph \( H \) of a digraph \( D \) is said to be a spanning subdigraph of \( D \), or equivalently, a factor of \( D \), if \( V(H) = V(D) \). A decomposition of a digraph \( D \) is a set \( \{ H_1, H_2, ..., H_k \} \) of subdigraphs of \( D \) whose arc-sets are exactly the classes of a partition of \( A(D) \). A factorization of a digraph \( D \), if there exists one, is a decomposition of \( D \) into factors. A dicycle factor \( H \) of a digraph \( D \) is a spanning subdigraph of \( D \) such that \( M(H) \) is a permutation matrix. The disjoint union of digraphs \( D_1, D_2, ..., D_k \), is the digraph with vertex-set \( \bigcup_{i=1}^k V(D_i) \), and arc-set \( \bigcup_{i=1}^k A(D_i) \). Then a dicycle factor \( H \) of a digraph \( D \) is a spanning subdigraph of \( D \) and it is the disjoint union of dicycles. A dicycle factorization is a factorization into dicycle factors. The line digraph of a digraph \( D \), denoted by \( \overrightarrow{L}D \), is defined as follows: the vertex-set of \( \overrightarrow{L}D \) is \( A(D) \); for \( v_h, v_i, v_j, v_k \in V(D) \), \( ((v_h, v_i), (v_j, v_k)) \in A(\overrightarrow{L}D) \) if and only if \( v_i = v_j \). Kronecker product and direct sum of matrices \( M \) and \( N \) are respectively denoted by \( M \otimes N \) and \( M \oplus N \). The identity matrix and the all-ones matrix of size \( n \) are respectively denoted by \( I_n \) and \( J_n \). In the next section, we prove the following theorem:

**Theorem** Let \( D \) be a \( d \)-regular digraph on \( n \) vertices and let \( \{ H_1, H_2, ..., H_d \} \) be a dicycle factorization of \( D \). Then there is a labeling of \( V(\overrightarrow{L}D) \) such that

\[
M(\overrightarrow{L}D) = (J_d \otimes I_n) \bigoplus_{i=1}^d M(H_i).
\]

1 Proof of the theorem

The proof of the theorem is based on two simple observations and a result proved by Hasunuma and Shibata [4] (see also Kawai et al. [5]).

**Lemma 1** Let \( D \) be a \( d \)-regular digraph. Then \( D \) has a dicycle factorization. In particular, if \( \{ H_1, H_2, ..., H_d \} \) is a dicycle factorization of \( D \) then \( M(H_1), M(H_2), ..., M(H_d) \) are permutation matrices such that

\[
M(D) = \sum_{i=1}^d M(H_i).
\]

Two digraphs \( D \) and \( D' \) are said to be isomorphic if there is a permutation matrix \( P \) such that \( P \cdot M(D) \cdot P^{-1} = M(D') \). If \( D \) and \( D' \) are isomorphic we then write \( D \cong D' \). An \( n \)-dicycle, denoted by \( \overrightarrow{C}_n \), is a digraph with vertex-set
\{v_1, v_2, \ldots, v_n\} \text{ and arc-set } \{(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}. \text{ A } d\text{-spiked } n\text{-dicycle is the digraph obtained from } \overrightarrow{C}_n \text{ as follows: for every vertex } v_i \in V(\overrightarrow{C}_n), \text{ we add } d \text{ new vertices } w_1, w_2, \ldots, w_d; \text{ we connect } v_i \in (\overrightarrow{C}_n) \text{ to the vertices } w_1, w_2, \ldots, w_d, \text{ obtaining the arcs } (v_i, w_1), (v_i, w_2), \ldots, (v_i, w_d).

Lemma 2 Let \(D\) be a \(d\)-spiked \(n\)-dicycle. Then \(D \cong \overrightarrow{L}D\).

Let \(D\) be a digraph and let \(H\) be a spanning subdigraph of \(D\). The growth of \(D\) derived by \(H\) is the digraph denoted by \(\Upsilon_D(H)\) and defined as follows: for every pair of vertices \(v_i, v_j \in V(D)\), if \((v_i, v_j) \in A(H)\) then \((v_i, v_j) \in A(\Upsilon_D(H))\); for every vertex \(v_i \in V(D)\), we add new vertices \(w_1, w_2, \ldots, w_l\), where \(l = d^+_D(v_i) - d^-_D(v_i)\); we connect \(v_i \in V(D)\) to the vertices \(w_1, w_2, \ldots, w_l\), obtaining the arcs \((v_i, w_1), (v_i, w_2), \ldots, (v_i, w_l)\).

Lemma 3 ([4]) If \(\{H_1, H_2, \ldots, H_k\}\) is a decomposition of a digraph \(D\) then

\[\{\overrightarrow{L}\Upsilon_D(H_1), \overrightarrow{L}\Upsilon_D(H_2), \ldots, \overrightarrow{L}\Upsilon_D(H_k)\}\]

is a decomposition of a digraph \(D' \cong \overrightarrow{L}D\).

Proof. [Proof of the theorem] Let \(D\) be a \(d\)-regular digraph on \(n\) vertices \(v_1, v_2, \ldots, v_n\). Let \(\{H_1, H_2, \ldots, H_d\}\) be a dicycle factorization of \(D\). The vertices of \(H_j \in \{H_1, H_2, \ldots, H_d\}\) are denoted as \((H_j, v_1), (H_j, v_2), \ldots, (H_j, v_n)\). Let us construct \(\Upsilon_D(H_j)\). For every vertex \((H_j, v_i) \in V(H_j)\), we add \(d - 1\) new vertices to \(H_j\). We label these new vertices by pairs of the form \((H_j, v_i)\). Let us label the row number \((j - 1)n + i\) of \(M(\Upsilon_D(H_j))\) by the vertex \((H_j, v_i)\), the adjacency matrix of \(\Upsilon_D(H_j)\) is the \((d \cdot n) \times (d \cdot n)\) block-matrix

\[
M(\Upsilon_D(H_j)) = \begin{pmatrix}
0 \\
X_j \\
0
\end{pmatrix},
\]

where

\[
X_j = \begin{pmatrix}
M(H_1) & M(H_2) & \cdots & M(H_j) & \cdots & M(H_{d-1}) & M(H_d)
\end{pmatrix}.
\]

Notice that \(M(H_j)\) is the \(jj\)-th block of \(M(\Upsilon_D(H_j))\). Thus, we have
\[ N = \sum_{i=j}^{d} M(\Upsilon_D(H_i)) = \\
= (J_d \otimes I_n) \bigoplus_{i=j}^{d} M(H_j). \]

Observe that, for every \( 1 \leq j \leq d \), \( \Upsilon_D(H_j) \) is the disjoint union of the \( d \)-spiked cycles corresponding to the orbits of the permutation associated to \( H_j \).

It follows from Lemma 2 that, for every \( 1 \leq j \leq d \),
\[ \Upsilon_D(H_j) \cong \overrightarrow{L} \Upsilon_D(H_j). \]

Then, for the chosen labeling,
\[ M(\Upsilon_D(H_j)) = M(\overrightarrow{L} \Upsilon_D(H_j)) \]
and
\[ N = \sum_{j=1}^{d} M(\Upsilon_D(H_j)) = \sum_{j=1}^{d} M(\overrightarrow{L} \Upsilon_D(H_j)). \]

Now, by Lemma 3, \( N = M(\overrightarrow{L} D) \).

**Remark** The graph operation transforming a digraph \( D \) in its line digraph can be naturally iterated: \( \overrightarrow{L}^k D := \overrightarrow{L}^{k-1} \overrightarrow{L} D \). Let \( \Sigma \) be an alphabet of cardinality \( d \) and let \( \Sigma^k \) be the set of all the words of length \( k \) over \( \Sigma \). The \( d \)-ary \( k \)-dimensional de Bruijn digraph, denoted by \( B(d,k) \), is defined as follows: the vertex-set of \( B(d,k) \) is \( V(B(d,k)) = \Sigma^k \); for every pair of vertices \( v_i, v_j \), we have \( (v_i, v_j) \in A(B(d,k)) \) if and only if the last \( k - 1 \) letters of \( v_i \) are the same as the first \( k - 1 \) letters of \( v_j \). Let \( K_d^+ \) be the complete digraph on \( d \) vertices with a loop at each vertex. Fiol, Yebra and Alegre [3] proved that \( B(d,k) \cong \overrightarrow{L}^{k-1} K_d^+ \). This result, together with the theorem, gives
\[ M(B(d,2)) \cong (J_d \otimes I_d) \bigoplus_{i=1}^{d} M(H_i), \]
where \( \{H_1, H_2, \ldots, H_d\} \) is any dicycle factorization of \( K_d^+ \).

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