A NOTE ON THE FROBENIUS-EULER NUMBERS AND POLYNOMIALS ASSOCIATED WITH BERNSTEIN POLYNOMIALS

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Abstract. The present paper deals with Bernstein polynomials and Frobenius-Euler numbers and polynomials. We apply the method of generating function and fermionic $p$-adic integral representation on $\mathbb{Z}_p$, which are exploited to derive further classes of Bernstein polynomials and Frobenius-Euler numbers and polynomials. To be more precise we summarize our results as follows, we obtain some combinatorial relations between Frobenius-Euler numbers and polynomials. Furthermore, we derive an integral representation of Bernstein polynomials of degree $n$ on $\mathbb{Z}_p$. Also we deduce a fermionic $p$-adic integral representation of product Bernstein polynomials of different degrees $n_1, n_2, \cdots$ on $\mathbb{Z}_p$ and show that it can be written with Frobenius-Euler numbers which yields a deeper insight into the effectiveness of this type of generalizations. Our applications possess a number of interesting properties which we state in this paper.

1. Introduction and Notations

Let $p$ be a fixed odd prime number. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. The $p$-adic absolute value is defined by

$$|p|_p = \frac{1}{p}.$$ 

In this paper, we assume $|q - 1|_p < 1$ as an indeterminate. In [17-19], let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by T. Kim:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(\xi) \, d\mu_{-1}(\xi) = \lim_{N \to \infty} \sum_{\xi=0}^{p^N-1} f(\xi) (-1)^\xi.$$ (1.1)

From (1.1), we have well known the following equality:

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0)$$ (1.2)

here $f_1(x) := f(x + 1)$ (for details, see[3-24]).
Let \( C ([0,1]) \) be the space of continuous functions on \([0,1]\). For \( C ([0,1]) \), the Bernstein operator for \( f \) is defined by

\[
B_n (f, x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) B_k, n (x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k}
\]

where \( n, k \in \mathbb{Z}^+ := \{0, 1, 2, 3, \ldots\} \). Here \( B_k, n (x) \) is called Bernstein polynomials, which are defined by

\[
B_k, n (x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1]
\]

(for more informations on this subject, see [1-6, 11, 14, 15, 17, 21-24]).

In [7], as is well known, Frobenius-Euler polynomials are defined by means of the following generating function:

\[
\sum_{n=0}^{\infty} H_n (u, x) t^n n! = e^{H(u,x)t} - 1 - u e^t - u e^{xt}.
\]

By (1.4) and (2.1), we easily see the following applications:

\[
e^{H(u)t} = \sum_{n=0}^{\infty} H_n (u) \frac{t^n}{n!} = \frac{1-u}{e^t-u}.
\]

After these applications, we derive the following Lemma.

**Lemma 1.** For \(|u| > 1\) and \(n \in \mathbb{Z}^+ := \mathbb{N} \cup \{0\}\), we have

\[
(H (u) + 1)^n - uH_n (u) = \begin{cases} 1-u, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}
\]

In this paper, we obtained some relations between the Frobenius-Euler numbers and polynomials and the Bernstein polynomials. From these relations, we derive some interesting identities on the Frobenius-Euler numbers.

2. **On the Frobenius-Euler numbers and polynomials**

Let us take \( f (x) = u^x e^{tx} \) in (1.1), by (1.2), we see that

\[
\int_{\mathbb{Z}^+} u^n e^{\eta t} d\mu_{-1} (\eta) = \frac{2}{1+u} H_n (-u^{-1}).
\]

By (1.4) and (2.1), we have the following theorem.

**Theorem 1.**

\[
\int_{\mathbb{Z}^+} u^n (x + \eta)^n d\mu_{-1} (\eta) = \frac{2}{u+1} H_n (-u^{-1}, x).
\]
By applying some combinatorial techniques in (2.2), we derive the following
\[ \int_{\mathbb{Z}_p} u^n (x + \eta)^n \, d\mu_{-1} (\eta) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \left\{ \int_{\mathbb{Z}_p} u^n \eta^k \, d\mu_{-1} (\eta) \right\}. \]

So, from above, we have the well known identity
\[ (2.3) \quad H_n (-u^{-1}, x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} H_k (-u^{-1}) = (H (-u^{-1} + x)^n. \]

by using the umbral(symbolic) convention \( H^n (u) := H_n (u). \)

The Frobenius-Euler polynomials have to symmetric properties, which is shown by Choi et al. in [7], as follows:
\[ H_n (-u^{-1}, 1 - x) = (-1)^n H_n (-u^{-1}, x). \]

For \( n \in \mathbb{N} \), by (2.4), Choi et al. derived the following equality:
\[ (2.4) \quad u^2 H_n (-u^{-1}, 2) = u^2 + u + H_n (-u^{-1}). \]

From (2.2) and (2.4), we easily see that
\[ (2.5) \quad \int_{\mathbb{Z}_p} u^n (1 - \eta)^n \, d\mu_{-1} (\eta) = \begin{cases} (-1)^n \int_{\mathbb{Z}_p} u^n (\eta - 1)^n \, d\mu_{-1} (\eta) \\ \frac{2}{u+1} (-1)^n H_n (-u^{-1}, -1) \\ \frac{2}{u+1} H_n (-u^{-1}, 2). \end{cases} \]

Thus, we obtain the following Theorem.

**Theorem 2.** The following identity
\[ (2.6) \quad \int_{\mathbb{Z}_p} u^n (1 - \eta)^n \, d\mu_{-1} (\eta) = \frac{2}{u+1} H_n (-u^{-1}, 2) \]

is true.

Let \( n \in \mathbb{N} \). By expression of (2.4) and (2.6), we get
\[ (2.7) \quad \int_{\mathbb{Z}_p} u^n (1 - \eta)^n \, d\mu_{-1} (\eta) = \frac{2}{u+1} + \frac{2}{u^2 + u} + \frac{2}{u^3 + u} H_n (-u^{-1}). \]

From (2.7), we procure the following corollary.

**Corollary 1.** For \( n \in \mathbb{N} \), we have
\[ \int_{\mathbb{Z}_p} u^n (1 - \eta)^n \, d\mu_{-1} (\eta) = \frac{2}{u+1} + \frac{2}{u^2 + u} + \frac{2}{u^3 + u} H_n (-u^{-1}). \]
3. Some identities on the Frobenius-Euler numbers

In this section, we develop Frobenius-Euler numbers, that is, we derive some interesting and worthwhile relations for studying in Theory of Analytic Numbers.

Now also, for \( x \in [0,1] \), we rewrite definition of Bernstein polynomials as follows:

\[
B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{where } n, k \in \mathbb{Z}_+.
\]

By expression of (3.1), we have the properties of symmetry of Bernstein polynomials as follows:

\[
B_{k,n}(x) = B_{n-k,n}(1-x), \quad \text{(for detail, see [21]).}
\]

Thus, from Corollary 1, (3.1) and (3.2), we see that

\[
\int_{\mathbb{Z}_p} B_{k,n}(\eta) u^d \, d\mu_{-1}(\eta) = \int_{\mathbb{Z}_p} B_{n-k,n}(1-\eta) u^d \, d\mu_{-1}(\eta)
\]

\[
= \left( \sum_{l=0}^{k} \binom{k}{l} \eta^l \right) \left( \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n-l}(-u^{-1}) \right)
\]

\[
= \begin{cases} 
\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_n(-u^{-1}), & \text{if } k = 0, \\
\sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left( \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n-l}(-u^{-1}) \right), & \text{if } k > 0.
\end{cases}
\]

Let us take the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) on the Bernstein polynomials of degree \( n \) as follows:

\[
\int_{\mathbb{Z}_p} B_{k,n}(\eta) u^d \, d\mu_{-1}(\eta) = \left( \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} H_{l+k}(-u^{-1}) \right)
\]

Consequently, by expression of (3.3) and (3.4), we state the following Theorem:

**Theorem 3.** The following identity holds true:

\[
\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l} H_{l+k}(-u^{-1}) = \begin{cases} 
1 + u^{-1} + u^{-2} H_n(-u^{-1}), & \text{if } k = 0, \\
\sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} (1 + u^{-1} + u^{-2} H_{n-l}(-u^{-1})), & \text{if } k > 0.
\end{cases}
\]
Let \( n_1, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 > 2k \). Then, we derive the followings

\[
\int_{\mathbb{Z}_p} B_{k, n_1} (\eta) B_{k, n_2} (\eta) u^n d\mu_{-1} (\eta)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \int_{\mathbb{Z}_p} (1 - \eta)^{n_1+n_2-l} u^n d\mu_{-1} (\eta)
\]

\[
= \left( \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left( \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2-l} (-u^{-1}) \right) \right)
\]

\[
= \begin{cases} 
\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2} (-u^{-1}), & \text{if } k = 0, \\
\binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left( \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2-l} (-u^{-1}) \right), & \text{if } k \neq 0.
\end{cases}
\]

Therefore, we obtain the following Theorem:

**Theorem 4.** For \( n_1, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 > 2k \), we have

\[
\int_{\mathbb{Z}_p} B_{k, n_1} (\eta) B_{k, n_2} (\eta) u^n d\mu_{-1} (\eta)
\]

\[
= \begin{cases} 
\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2} (-u^{-1}), & \text{if } k = 0, \\
\binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left( \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2-l} (-u^{-1}) \right), & \text{if } k \neq 0.
\end{cases}
\]

By using the binomial theorem, we can derive the following equation.

(3.5) \[
\int_{\mathbb{Z}_p} B_{k, n_1} (\eta) B_{k, n_2} (\eta) u^n d\mu_{-1} (\eta)
\]

\[
= \prod_{i=1}^{2} \binom{n_i}{k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (n_1+n_2-2k-l)! \int_{\mathbb{Z}_p} \eta^{2k+l} u^n d\mu_{-1} (\eta)
\]

\[
= \prod_{i=1}^{2} \binom{n_i}{k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (n_1+n_2-2k-l)! H_{2k+l} (-u^{-1})
\]

Thus, we can obtain the following Corollary:

**Corollary 2.** For \( n_1, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 > 2k \), we have

\[
\sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (n_1+n_2-2k-l)! H_{2k+l} (-u^{-1})
\]

\[
= \begin{cases} 
1 + u^{-1} + u^{-2} H_{n_1+n_2} (-u^{-1}), & \text{if } k = 0, \\
\sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} (1 + u^{-1} + u^{-2} H_{n_1+n_2-l} (-u^{-1})), & \text{if } k \neq 0.
\end{cases}
\]

For \( \eta \in \mathbb{Z}_p \) and \( s \in \mathbb{N} \) with \( s \geq 2 \), let \( n_1, n_2, ..., n_s, k \in \mathbb{Z}_+ \) with \( \sum_{i=1}^{s} n_i > sk \). Then we take the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) for the Bernstein polynomials.
of degree \( n \) as follows:

\[
\int_{\mathbb{Z}_p} B_{k,n_1} (\eta) B_{k,n_2} (\eta) ... B_{k,n_s} (\eta) u^n d\mu_{-1} (\eta)
\]

\[=
\sum_{i=1}^{s} \left( \frac{n_i}{k} \right) \int_{\mathbb{Z}_p} \eta^{sk} (1 - \eta)^{n_1 + n_2 + ... + n_s - sk} u^n d\mu_{-1} (\eta)
\]

\[=
\prod_{i=1}^{s} \left( \frac{n_i}{k} \right) \sum_{l=0}^{sk} \left( \frac{sk}{l} \right) (-1)^{l+sk} \int_{\mathbb{Z}_p} (1 - \xi)^{n_1 + n_2 + ... + n_s - l} u^n d\mu_{-1} (\eta)
\]

\[=
\begin{cases}
\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2+...+n_s} (-u^{-1}), & \text{if } k = 0, \\
\prod_{i=1}^{s} \left( \frac{n_i}{k} \right) \sum_{l=0}^{sk} \left( \frac{sk}{l} \right) (-1)^{sk+l} \left( \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2+...+n_s-l} (-u^{-1}) \right), & \text{if } k \neq 0.
\end{cases}
\]

So from above, we have the following Theorem:

**Theorem 5.** For \( s \in \mathbb{N} \) with \( s \geq 2 \), let \( n_1, n_2, ..., n_s, k \in \mathbb{Z}_+ \) with \( \sum_{l=1}^{s} n_l > sk \). Then we have

\[
\int_{\mathbb{Z}_p} u^n \prod_{i=1}^{s} B_{k,n_i} (\eta) u^n d\mu_{-1} (\eta)
\]

\[=
\begin{cases}
\frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2+...+n_s} (-u^{-1}), & \text{if } k = 0, \\
\prod_{i=1}^{s} \left( \frac{n_i}{k} \right) \sum_{l=0}^{sk} \left( \frac{sk}{l} \right) (-1)^{sk+l} \left( \frac{2}{u+1} + \frac{2}{u^2+u} + \frac{2}{u^3+u} H_{n_1+n_2+...+n_s-l} (-u^{-1}) \right), & \text{if } k \neq 0.
\end{cases}
\]

From the definition of Bernstein polynomials and the binomial theorem, we easily get

\[
\int_{\mathbb{Z}_p} B_{k,n_1} (\eta) B_{k,n_2} (\eta) ... B_{k,n_s} (\eta) u^n d\mu_{-1} (\eta)
\]

\[=
\sum_{i=1}^{s} \left( \frac{n_i}{k} \right) \sum_{l=0}^{sk} \left( \sum_{d=1}^{s} \binom{n_d - k}{l} \right) (-1)^{l+sk} u^n d\mu_{-1} (\eta)
\]

\begin{equation}
(3.6) = \frac{2}{u+1} \sum_{i=1}^{s} \left( \frac{n_i}{k} \right) \sum_{l=0}^{sk} \left( \sum_{d=1}^{s} \binom{n_d - k}{l} \right) (-1)^{sk+l} H_{sk+l} (-u^{-1}).
\end{equation}

Therefore, by (3.6), we get novel properties of Frobenius-Euler numbers with the following corollary:

**Corollary 3.** For \( s \in \mathbb{N} \) with \( s \geq 2 \), let \( n_1, n_2, ..., n_s, k \in \mathbb{Z}_+ \) with \( \sum_{l=1}^{s} n_l > sk \). Then, we have

\[
\sum_{l=0}^{sk} \left( \sum_{d=1}^{s} \binom{n_d - k}{l} \right) \left( u^2 + u + H_{n_1+n_2+...+n_s} (-u^{-1}) \right)
\]

\[=
\begin{cases}
\sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left( u^2 + u + H_{n_1+n_2+...+n_s-l} (-u^{-1}) \right), & \text{if } k \neq 0.
\end{cases}
\]
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References

[1] Açıkgoz, M. and Araci, S., A study on the integral of the product of several type Bernstein polynomials, IST Transaction of Applied Mathematics-Modeling and Simulation, vol.1, no. 1, pp. 10–14, 2010.

[2] Açıkgoz, M. and Şimşek, Y., A New generating function of \(q\)-Bernstein type polynomials and their interpolation function, Abstract and Applied Analysis, Article ID 769095, 12 pages, doi: 10.1155/2010/769095-01-313.

[3] Araci, S., Erdal, D., and Seo, J-J., A study on the Fermionic \(p\)-adic \(q\)-integral Representation on \(\mathbb{Z}_p\) Associated with Weighted \(q\)-Bernstein and \(q\)-Genocchi Polynomials, Abstract and Applied Analysis, Volume 2011, Article ID 649248, 10 pages.

[4] Araci, S. Erdal, D. and Kang, D-J., Some New Properties on the \(q\)-Genocchi numbers and Polynomials associated with \(q\)-Bernstein polynomials, Honam Mathematical J. 33 (2011) no. 2, pp. 261-270

[5] Araci, S., Acikgoz, M., Qi, F., On the \(q\)-Genocchi numbers and polynomials with weight 0 and their applications, http://arxiv.org/abs/1202.2643.

[6] A. Bayad, T. Kim, Identities involving values of Bernstein \(q\)-Bernoulli, and \(q\)-Euler polynomials, Russ. J. Math. Phys. 18 (2011), no. 2, 133-143.

[7] J. Choi, D. S. Kim, T. Kim and Y. H. Kim, A note on Some identities of Frobenius-Euler Numbers and Polynomials, International Journal of Mathematics and Mathematical Sciences, Volume 2012, Article ID 861797, 9 pages.

[8] T. Kim and B. Lee, Some Identities of the Frobenius-Euler polynomials, Abstract and Applied Analysis, Volume 2009, Article ID 639439, 7 pages.

[9] T. Kim, On the multiple \(q\)-Genocchi and Euler numbers, Russian J. Math. Phys. 15 (4) (2008) 481-486. arXiv:0801.0978v1 [math.NT]

[10] T. Kim, A Note on the \(q\)-Genocchi Numbers and Polynomials, Journal of Inequalities and Applications 2007 (2007) doi:10.1155/2007/71452. Article ID 71452, 8 pages.

[11] T. Kim, A note \(q\)-Bernstein polynomials, Russ. J. Math. Phys. 18 (2011), 41-50.

[12] T. Kim, \(q\)-Volkenborn integration, Russ. J. Math. Phys. 9 (2002), 288-299.

[13] T. Kim, \(q\)-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients, Russ. J. Math. Phys. 15 (2008), 51-57.

[14] T. Kim, J. Choi, Y. H. Kim and C. S. Ryoo, On the fermionic \(p\)-adic integral representation of Bernstein polynomials associated with Euler numbers and polynomials, J. Inequal. Appl. 2010 (2010), Art ID 864247, 12pp.

[15] T. Kim, J. Choi and Y. H. Kim Some identities on the \(q\)-Bernstein polynomials, \(q\)-Stirling numbers and \(q\)-Bernoulli numbers, Adv. Stud. Contemp. Math. 20 (2010), 335-341.

[16] T. Kim, An invariant \(p\)-adic \(q\)-integrals on \(\mathbb{Z}_p\), Applied Mathematics Letters, vol. 21, pp. 105-108, 2008.

[17] T. Kim, J. Choi and Y. H. Kim \(q\)-Bernstein Polynomials Associated with \(q\)-Stirling Numbers and Carlitz’s \(q\)-Bernoulli Numbers, Abstract and Applied Analysis, Article ID 150975, 11 pages, doi:10.1155/2010/150975.

[18] T. Kim, \(q\)-Euler numbers and polynomials associated with \(p\)-adic \(q\)-integrals, J. Nonlinear Math. Phys., 14 (2007), no. 1, 15–27.

[19] T. Kim, New approach to \(q\)-Euler polynomials of higher order, Russ. J. Math. Phys., 17 (2010), no. 2, 218–225.

[20] T. Kim, Some identities on the \(q\)-Euler polynomials of higher order and \(q\)-Stirling numbers by the fermionic \(p\)-adic integral on \(\mathbb{Z}_p\), Russ. J. Math. Phys., 16 (2009), no.4, 484–491.

[21] T. Kim, A. Bayad, Y. H. Kim, A Study on the \(p\)-Adic \(q\)-Integrals Representation on \(\mathbb{Z}_p\) Associated with the weighted \(q\)-Bernstein and \(q\)-Bernoulli polynomials, Journal of Inequalities and Applications, Article ID 513821, 8 pages, doi:10.1155/2011/513821.

[22] C. S. Ryoo, A note on the weighted \(q\)-Euler numbers and polynomials, Adv. Stud. Contemp. Math. 21 (2011), 47-54.

[23] H. Y. Lee, N. S. Jung, and C. S. Ryoo, Some Identities of the Twisted \(q\)-Genocchi Numbers and Polynomials with weight \(\alpha\) and \(q\)-Bernstein Polynomials with weight \(\alpha\), Abstract and Applied Analysis, Volume 2011 (2011), Article ID 123483, 9 pages.

[24] N. S. Jung, H. Y. Lee and C. S. Ryoo, Some Relations between Twisted \((h,q)\)-Euler Numbers with Weight \(\alpha\) and \(q\)-Bernstein Polynomials with Weight \(\alpha\), Discrete Dynamics in Nature and Society, Volume 2011 (2011), Article ID 176296, 11 pages.
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