Abstract. The purpose of this paper is to study $*$-Ricci tensor on Sasakian manifold. Here, $\varphi$-conformally flat and conformally flat $*-\eta$-Einstein Sasakian manifold are studied. Next, we consider $*$-Ricci symmetric condition on Sasakian manifold. Finally, we study a special type of metric called $*$-Ricci soliton on Sasakian manifold.

1. Introduction

The notion of contact geometry has evolved from the mathematical formalism of classical mechanics[9]. Two important classes of contact manifolds are $K$-contact manifolds and Sasakian manifolds[2]. An odd dimensional analogue of Kähler geometry is the Sasakian geometry. Sasakian manifolds were firstly studied by the famous geometer Sasaki[21] in 1960, and for long time focused on this, Sasakian manifold have been extensively studied under several points of view in [5, 8, 14, 19, 23], and references therein.

On the other hand, it is mentioned that the notion of $*$-Ricci tensor was first introduced by Tachibana[22] on almost Hermitian manifolds and further studied by Hamada and Inoguchi[12] on real hypersurfaces of non-flat complex space forms.

Motivated by these studies the present paper is organized as follows: In section 2, we recall some basic formula and result concerning Sasakian manifold and $*$-Ricci tensor which we will use in further sections. A $\varphi$-conformally flat Sasakian manifold is studied in section 3, in which we obtain some interesting result. Section 4 is devoted to the study of conformally flat $*-\eta$-Einstein Sasakian manifold. In section 5, we consider $*$-Ricci symmetric Sasakian manifold and found that $*$-Ricci symmetric Sasakian manifold is $*$-Ricci flat, moreover, it is $\eta$-Einstein manifold. In the last section, we studied a special type of metric called $*$-Ricci soliton. Here we have proved an important result on Sasakian manifold admitting $*$-Ricci soliton.

2. Preliminaries

In this section, we collect some general definition and basic formulas on contact metric manifolds and Sasakian manifolds which we will use in further sections. We may refer to [1] [3] [16] and references therein for more details and information about Sasakian geometry.

A $(2n+1)$-dimensional smooth connected manifold $M$ is called almost contact manifold if it admits a triple $(\varphi, \xi, \eta)$, where $\varphi$ is a tensor field of type $(1,1)$, $\xi$ is a global vector field and $\eta$ is a 1-form, such that

\begin{equation}
\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0,
\end{equation}

2010 Mathematics Subject Classification. 53D10 · 53C25 · 53C15 · 53B21.

Key words and phrases. Sasakian metric · $*$-Ricci tensor · Conformal curvature tensor · $\eta$-Einstein manifold.
for all $X, Y \in TM$. If an almost contact manifold $M$ admits a structure $(\varphi, \xi, \eta, g)$, $g$ being a Riemannian metric such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

then $M$ is called an almost contact metric manifold. An almost contact metric manifold $M(\varphi, \xi, \eta, g)$ with $d\eta(X, Y) = \Phi(X, Y)$, $\Phi$ being the fundamental 2-form of $M(\varphi, \xi, \eta, g)$ as defined by $\Phi(X, Y) = g(X, \varphi Y)$, is a contact metric manifold and $g$ is the associated metric. If, in addition, $\xi$ is a killing vector field (equivalently, $\mathcal{L}_\xi \varphi = 0$, where $\mathcal{L}$ denotes Lie differentiation), then the manifold is called $K$-contact manifold. It is well known that if the contact metric structure $(\varphi, \xi, \eta, g)$ is normal, that is, $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$ holds, then $(\varphi, \xi, \eta, g)$ is Sasakian. An almost contact metric manifold is Sasakian if and only if

$$\nabla_X \varphi Y = g(X, Y)\xi - \eta(X)\eta(Y),$$

for any vector fields $X, Y$ on $M$, where $\nabla$ is Levi-Civita connection of $g$. A Sasakian manifold is always a $K$-contact manifold. The converse also holds when the dimension is three, but which may not be true in higher dimensions. On Sasakian manifold, the following relations are well known;

$$\nabla_X \xi = -\varphi X$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X,$$

$$Ric(X, \xi) = 2n\eta(X) \quad (or \ Q\xi = 2n\xi),$$

for all $X, Y \in TM$, where $R, Ric$ and $Q$ denotes the curvature tensor, Ricci tensor and Ricci operator, respectively.

On the other hand, let $M(\varphi, \xi, \eta, g)$ be an almost contact metric manifold with Ricci tensor $Ric$. The $*$-Ricci tensor and $*$-scalar curvature of $M$ respectively are defined by

$$\text{Ric}^*(X, Y) = \sum_{i=1}^{2n+1} R(X, e_i, \varphi e_i, \varphi Y), \quad r^* = \sum_{i=1}^{2n+1} \text{Ric}^*(e_i, e_i),$$

for all $X, Y \in TM$, where $e_1, ..., e_{2n+1}$ is an orthonormal basis of the tangent space $TM$. By using the first Bianchi identity and (2.8) we get

$$\text{Ric}^*(X, Y) = \frac{1}{2} \sum_{i=1}^{2n+1} g(\varphi R(X, \varphi Y)e_i, e_i).$$

An almost contact metric manifold is said to be $*$-Einstein if $\text{Ric}^*$ is a constant multiple of the metric $g$. One can see $\text{Ric}^*(X, \xi) = 0$, for all $X \in TM$. It should be remarked that $\text{Ric}^*$ is not symmetric, in general. Thus the condition $*$-Einstein automatically requires a symmetric property of the $*$-Ricci tensor.

Now we make an effort to find $*$-Ricci tensor on Sasakian manifold.

**Lemma 2.1.** In a $(2n+1)$-dimensional Sasakian manifold $M$, the $*$-Ricci tensor is given by

$$\text{Ric}^*(X, Y) = \text{Ric}(X, Y) - (2n - 1)g(X, Y) - \eta(X)\eta(Y).$$
Proof. In a \((2n+1)\)-dimensional Sasakian manifold \(M\), the Ricci tensor \(\text{Ric}\) satisfies the relation (see page 284, Lemma 5.3 in \([25]\)):

\[
\text{Ric}(X, Y) = \frac{1}{2} \sum_{i=1}^{2n+1} g(\varphi R(X, \varphi Y) e_i, e_i) + (2n - 1)g(X, Y) + \eta(X)\eta(Y).
\]  

Using the definition of \(\text{Ric}^\ast\) in (2.11), we obtain (2.10) \(\square\)

Definition 2.2. \([7]\) An almost contact metric manifold \(M\) is said to be weakly \(\varphi\)-Einstein if

\[
\text{Ric}^\varphi(X, Y) = \beta g^\varphi(X, Y), \quad X, Y \in \mathcal{T}M,
\]

for some function \(\beta\). Here \(\text{Ric}^\varphi\) denotes the symmetric part of \(\text{Ric}^\ast\), that is,

\[
\text{Ric}^\varphi(X, Y) = \frac{1}{2} \{\text{Ric}^\ast(X, Y) + \text{Ric}^\ast(Y, X)\}, \quad X, Y \in \mathcal{T}M,
\]

we call \(\text{Ric}^\varphi\), the \(\varphi\)-Ricci tensor on \(M\) and the symmetric tensor \(g^\varphi\) is defined by

\[
g^\varphi(X, Y) = g(\varphi X, \varphi Y).
\]

When \(\beta\) is constant, then \(M\) is said to be \(\varphi\)-Einstein.

Definition 2.3. If the Ricci tensor of a Sasakian manifold \(M\) is of the form

\[
\text{Ric}(X, Y) = \alpha g(X, Y) + \gamma \eta(X)\eta(Y),
\]

for any vector fields \(X, Y\) on \(M\), where \(\alpha\) and \(\gamma\) being constants, then \(M\) is called an \(\eta\)-Einstein manifold.

Let \(M(\varphi, \xi, \eta, g)\) be a Sasakian \(\eta\)-Einstein manifold with constants \((\alpha, \gamma)\). Consider a \(D\)-homothetic Sasakian structure \(S = (\varphi', \xi', \eta', g') = (\varphi, a^{-1}\xi, a\eta, ag+a(a-1)\eta\otimes\eta)\). Then \((M, S)\) is also \(\eta\)-Einstein with constants \(\alpha' = \frac{\alpha+2-2a}{a}\) and \(\gamma' = 2n - \alpha'\) (see proposition 18 in \([6]\)). Here we make a remark that the particular value: \(\alpha = -2\) remains fixed under a \(D\)-homothetic deformation\([11]\). Thus, we state the following definition.

Definition 2.4. A Sasakian \(\eta\)-Einstein manifold with \(\alpha = -2\) is said to be \(D\)-homothetically fixed.

3. \(\varphi\)-CONFORMALLY FLAT SASAKIAN MANIFOLD

The Weyl conformal curvature tensor\([25]\) is defined as a map \(C : TM \times TM \times TM \rightarrow TM\) such that

\[
C(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1}\{\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y + g(Y, Z)QX
\]
\[
- g(X, Z)QY\} + \frac{r}{2n(2n-1)}\{g(Y, Z)X - g(X, Z)Y\}, \quad X, Y \in TM.
\]

In\([3]\), Cabrerizo et al proved some necessary condition for \(K\)-contact manifold to be \(\varphi\)-conformally flat. In the following theorem we find a condition for \(\varphi\)-conformally flat Sasakian manifold.

Theorem 3.1. If a \((2n+1)\)-dimensional Sasakian manifold \(M\) is \(\varphi\)-conformally flat, then \(M\) is \(*\text{-}\eta\text{-Einstein manifold. Moreover, }M\text{ is weakly }\varphi\text{-Einstein.}\)
Proof. It is well known that (see in [3]), if a $K$-contact manifold is $\varphi$-conformally flat then we get the following relation:

\begin{equation}
R(\varphi X, \varphi Y, \varphi Z, \varphi W) = \frac{r - 4n}{2n(2n - 1)} \{ g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W) \}.
\end{equation}

In a Sasakian manifold, in view of (2.5) and (2.6) we can verify that

\begin{equation}
R(\varphi^2 X, \varphi^2 Y, \varphi^2 Z, \varphi^2 W) = R(X, Y, Z, W) - g(Y, Z)\eta(X)\eta(Y) + g(X, Z)\eta(Y)\eta(W) + g(Y, W)\eta(X)\eta(Z) - g(X, W)\eta(Y)\eta(Z),
\end{equation}

for all $X, Y, Z, W \in T M$. Replacing $X, Y, Z, W$ by $\varphi X, \varphi Y, \varphi Z, \varphi W$ respectively in (3.2) and making use of (2.2) and (3.3) we get

\begin{equation}
R(X, Y, Z, W) = \frac{r - 4n}{2n(2n - 1)} \{ g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \} - \frac{r - 2n(2n + 1)}{2n(2n - 1)} \{ g(Y, Z)\eta(X)\eta(W) - g(X, Z)\eta(Y)\eta(W) \} + g(X, W)\eta(Y)\eta(Z) - g(Y, W)\eta(X)\eta(Z).
\end{equation}

By the definition of $\text{Ric}^*\varphi$, direct computation yields

\begin{equation}
\text{Ric}^*\varphi(X, Y) = \sum_{i=1}^{2n+1} R(X, e_i, \varphi e_i, Y) = \beta g(X, Y) - \beta \eta(X)\eta(Y),
\end{equation}

where $\beta = \frac{r - 4n}{2n(2n - 1)}$, showing that $M$ is $\eta$-$\varphi$-Einstein. Next, in view of (2.2) we have

\begin{equation}
\text{Ric}^*\varphi(X, Y) = \frac{r - 4n}{2n(2n - 1)} g^\varphi(X, Y),
\end{equation}

for all $X, Y \in T M$. Hence $\text{Ric}^*\varphi = \text{Ric}^\varphi$ and hence it is weakly $\varphi$-Einstein. This completes the proof.

Suppose the scalar curvature of the manifold is constant. Then in view of (3.6), we have

**Corollary 3.2.** If a $\varphi$-conformally flat Sasakian manifold has constant scalar curvature, then it is $\varphi$-Einstein.

In a Sasakian manifold, the $\ast$-Ricci tensor is given by (2.10) and so in view of (3.5), we state the following;

**Corollary 3.3.** A $\varphi$-conformally flat Sasakian manifold is $\eta$-Einstein.

The notion of $\eta$-parallel Ricci tensor was introduced in the context of Sasakian manifold by Kon [17] and is defined by $(\nabla_{\varphi X})\text{Ric}(\varphi X, \varphi Y) = 0$, for all $X, Y \in T M$. From this definition, we define a $\eta$-parallel $\ast$-Ricci tensor by $(\nabla_{\varphi X})\text{Ric}^*(\varphi X, \varphi Y) = 0$.

Replacing $X$ by $\varphi X$ and $Y$ by $\varphi Y$ in (3.5), we obtain $\text{Ric}^*(\varphi X, \varphi Y) = \beta g(\varphi X, \varphi Y)$. Now taking covariant differentiation with respect to $W$, we get $(\nabla_W \text{Ric}^*)(\varphi X, \varphi Y) = dr(W)g(\varphi X, \varphi Y)$. Therefore we have the following;

**Corollary 3.4.** A $(2n+1)$-dimensional $\varphi$-conformally flat Sasakian manifold has $\eta$-parallel $\ast$-Ricci tensor if and only if the scalar curvature of the manifold is constant.
4. CONFORMALLY FLAT $\ast$-$\eta$-EINSTEIN SASAKIAN MANIFOLD

Suppose $M$ is conformally flat Sasakian manifold, then from (3.1) we have

\[ R(X,Y)Z = \frac{1}{2n-1} \{ Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \} - \frac{r}{2n(2n-1)} \{ g(Y,Z)X - g(X,Z)Y \}. \]

(4.1)

If we set $Y = Z = \xi \perp X$, we find $QX = \frac{r-2n}{2n}X$. From this equation, (4.1) becomes

\[ R(X,Y)Z = \frac{r-4n}{2n(2n-1)} \{ g(Y,Z)X - g(X,Z)Y \}. \]

(4.2)

By definition of $\ast$-Ricci tensor, direct computation yields

\[ Ric^\ast(X,Y) = \frac{r-4n}{2n(2n-1)} g(\varphi X, \varphi Y). \]

(4.3)

Since $M$ is a conformally flat Sasakian manifold, we have the following equations from the definition of $Ric^\ast$ and equation (4.2):

\[
\begin{align*}
Ric^\ast(\varphi Y, \varphi X) &= \sum_{i=1}^{2n+1} R(\varphi Y, e_i, \varphi e_i, \varphi^2 X) \\
&= \sum_{i=1}^{2n+1} \{ -R(X, \varphi e_i, e_i, \varphi Y) + \eta(X)R(\varphi Y, e_i, \varphi e_i, \xi) \} \\
&= Ric^\ast(X, Y).
\end{align*}
\]

If we set $Y = \varphi X$ such that $X$ is unit, we obtain $Ric^\ast(\varphi^2 X, \varphi X) = Ric^\ast(X, \varphi X)$ which implies that $Ric^\ast(X, \varphi X) = 0$. Thus, from the definition of $\ast$-$\eta$-Einstein and (4.3), we obtain $ag(X,Y) + b\eta(X)\eta(Y) = \frac{r-4n}{2n(2n-1)} g(\varphi X, \varphi Y)$. If we choose $X = Y = \xi$, we find $a + b = 0$. If we set $Y = X \perp \xi$ such that $X$ and $Y$ are units, we get

\[ a = \frac{r-4n}{2n(2n-1)} = K(X, \varphi X). \]

(4.4)

In [18], the author proved that every conformally flat Sasakian manifold has a constant curvature +1, that is, $R(X,Y)Z = g(Y,Z)X - g(X,Z)Y$. From this result and (4.2), we find $r = 2n(2n-1) + 4n$. In view of (4.4), we obtain $a = 1$. Therefore we have the following:

**Theorem 4.1.** Let $M$ be a $(2n+1)$-dimensional conformally flat Sasakian manifold. If $M$ is $\ast$-$\eta$-Einstein, then it is of constant curvature +1.

We know that every Riemannian manifold of constant sectional curvature is locally symmetric. From theorem (4.1), we have

**Corollary 4.2.** A conformally flat $\ast$-$\eta$-Einstein Sasakian manifold is locally symmetric.

**Theorem 4.3.** A $(2n+1)$-dimensional conformally flat Sasakian manifold is $\varphi$-Einstein.

**Proof.** The theorem follows from (4.3) and definition (2.2).
5. ∗-Ricci Semi-symmetric Sasakian Manifold

A contact metric manifold is called Ricci semi-symmetric if \( R(X,Y) \cdot Ric = 0 \), for all \( X,Y \in TM \). Analogous to this definition, we define ∗-Ricci semi-symmetric by \( R(X,Y) \cdot Ric^* = 0 \).

**Theorem 5.1.** If a \((2n+1)\)-dimensional Sasakian manifold \( M \) is ∗-Ricci semi-symmetric, then \( M \) is ∗-Ricci flat. Moreover, it is η-Einstein manifold and the Ricci tensor can be expressed as

\[
Ric(X,Y) = (2n - 1)g(X,Y) + \eta(X)\eta(Y).
\]

**Proof.** Let us consider \((2n+1)\)-dimensional Sasakian manifold which satisfies the condition \( R(X,Y) \cdot Ric^* = 0 \). Then we have

\[
Ric^*(R(X,Y)Z, W) + Ric^*(Z, R(X,Y)W) = 0.
\]

Putting \( X = Z = \xi \) in (5.1), we have

\[
Ric^*(R(\xi, Y)\xi, W) + Ric^*(\xi, R(\xi, Y)W) = 0.
\]

It is well known that \( Ric^*(X, \xi) = 0 \). Making use of (2.5) in (5.2) and by virtue of last equation, we find

\[
Ric^*(Y, W) = 0, \quad Y, W \in TM,
\]

showing that \( M \) is ∗-Ricci flat. Moreover, in view of (5.3) and (2.10), we have the required result. \( \square \)

6. Sasakian Manifold Admitting ∗-Ricci Soliton

Ricci flows are intrinsic geometric flows on a Riemannian manifold, whose fixed points are solitons and it was introduced by Hamilton[13]. Ricci solitons also correspond to self-similar solutions of Hamilton’s Ricci flow. They are natural generalization of Einstein metrics and is defined by

\[
(L_V g)(X,Y) + 2Ric(X,Y) + 2\lambda g(X,Y) = 0,
\]

for some constant \( \lambda \), a potential vector field \( V \). The Ricci soliton is said to be shrinking, steady, and expanding according as \( \lambda \) is negative, zero, and positive respectively.

The notion of ∗-Ricci soliton was introduced by George and Konstantina[10], in 2014, where they essentially modified the definition of Ricci soliton by replacing the Ricci tensor in (6.1) with the ∗-Ricci tensor. Recently, the authors studied ∗-Ricci soliton on para-Sasakian manifold in the paper [20] and obtain several interesting result. A Riemannian metric \( g \) on \( M \) is called ∗-Ricci soliton, if there is vector field \( V \), such that

\[
(L_V g)(X,Y) + 2Ric^*(X,Y) + 2\lambda g(X,Y) = 0,
\]

for all vector fields \( X,Y \) on \( M \). In this section we study a special type of metric called ∗-Ricci soliton on Sasakian manifold. Now we prove the following result:

**Theorem 6.1.** If the metric \( g \) of a \((2n+1)\)-dimensional Sasakian manifold \( M(\varphi, \xi, \eta, g) \) is a ∗-Ricci soliton with potential vector field \( V \), then (i) \( V \) is Jacobi along geodesic of \( \xi \), (ii) \( M \) is an η-Einstein manifold and the Ricci tensor can be expressed as

\[
Ric(X,Y) = \left[ 2n - 1 - \frac{\lambda}{2} \right] g(X,Y) + \left[ 1 + \frac{\lambda}{2} \right] \eta(X)\eta(Y).
\]
Next, taking covariant differentiation of equation (6.6):
\[
(L_V \nabla_X g - \nabla_X L_V g - \nabla_{[V,X]} g)(Y, Z) = -g((L_V \nabla)(X, Y), Z) - g((L_V \nabla)(X, Z), Y),
\]
and is well known for all vector fields \(X, Y, Z\) on \(M\). Since \(g\) is parallel with respect to Levi-Civita connection \(\nabla\), then the above relation becomes
\[
(\nabla_X L_V g)(Y, Z) = g((L_V \nabla)(X, Y), Z) + g((L_V \nabla)(X, Y), Z).
\]
Since \(L_V \nabla\) is a symmetric tensor of type \((1, 2)\), i.e., \((L_V \nabla)(X, Y) = (L_V \nabla)(Y, X)\), it follows from (6.5) that
\[
g((L_V \nabla)(X, Y), Z) = \frac{1}{2}\{(\nabla_X L_V g)(Y, Z) + (\nabla_Y L_V g)(Z, X) - (\nabla_Z L_V g)(X, Y)\}.
\]
Next, taking covariant differentiation of \(-Ricci\) soliton equation (6.2) along a vector field \(X\), we obtain \((\nabla_X L_V g)(X, Y) = -2(\nabla_X Ric^*)(X, Y)\). Substituting this relation into (6.6) we have
\[
g((L_V \nabla)(X, Y), Z) = (\nabla_Z Ric^*)(X, Y) - (\nabla_X Ric^*)(Y, Z) - (\nabla_Y Ric^*)(X, Z).
\]
Again, taking covariant differentiation of (2.10) with respect to \(Z\), we get
\[
(\nabla_Z Ric^*)(X, Y) = (\nabla_Z Ric)(X, Y) - \{g(Z, \varphi X)\eta(Y) + g(Z, \varphi Y)\eta(X)\}.
\]
Combining (6.8) with (6.7), we find
\[
g((L_V \nabla)(X, Y), Z) = (\nabla_Z Ric)(X, Y) - (\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(X, Z) + 2g(X, \varphi Z)\eta(Y) + 2g(Y, \varphi Z)\eta(X).
\]
In Sasakian manifold we know the following relation[11]:
\[
\nabla_\xi Q = Q\varphi - \varphi Q = 0, \quad (\nabla_X Q)\xi = Q\varphi X - 2n\varphi X.
\]
Replacing \(Y\) by \(\xi\) in (6.9) and then using (6.10) we obtain
\[
(L_V \nabla)(X, \xi) = -2Q\varphi X + 2(2n - 1)\varphi X.
\]
From the above equation, we have
\[
(L_V \nabla)(\xi, \xi) = 0.
\]
Now, substituting \(X = Y = \xi\) in the well known formula [24]:
\[
(L_V \nabla)(X, Y) = \nabla_X \nabla_Y V - \nabla_{\nabla_Y V} V + R(V, X)Y,
\]
and then making use of equation (6.12) we obtain
\[
\nabla_\xi \nabla_\xi V + R(V, \xi)\xi = 0,
\]
which proves part (i). Further, differentiating (6.11) covariantly along an arbitrary vector field \(Y\) on \(M\) and then using (2.3) and last equation of (2.7), we obtain
\[
(\nabla_Y L_V \nabla)(X, \xi) - (L_V \nabla)(X, \varphi Y) = 2\{(\nabla_Y Q)\varphi X - g(X, Y)\xi + \eta(X)QY - (2n - 1)\eta(X)Y\}.
\]
According to Yano[24], we have the following commutation formula:
\[
(L_V R)(X, Y)Z = (\nabla_X L_V \nabla)(Y, Z) - (\nabla_Y L_V \nabla)(X, Z).
\]
Substituting $\xi$ for $Z$ in the foregoing equation and in view of (6.13), we obtain

$$(LV R)(X, Y)\xi - (LV \nabla)(Y, \varphi X) + (LV \nabla)(X, \varphi Y)$$

(6.15)

$$= 2\{(\nabla_X Q)\varphi X - (\nabla_X Q)\varphi Y + \eta(Y)QX - \eta(X)QY + (2n - 1)(\eta(X)Y - \eta(Y)X)\}. \tag{6.15}$$

Replacing $Y$ by $\xi$ in (6.15) and then using last equation of (2.7), (6.10) and (6.11), we have

$$(LV R)(X, \xi)\xi = 4\{QX - \eta(X)\xi - (2n - 1)X\}. \tag{6.16}$$

Since $Ric^*(X, \xi) = 0$, then $*$-Ricci soliton equation (6.2) gives $(LV g)(X, \xi) + 2\lambda\eta(X) = 0$, which gives

$$(LV \eta)(X) - g(X, LV \xi) + 2\lambda\eta(X) = 0. \tag{6.17}$$

Lie-derivative of $g(\xi, \xi) = 1$ along $V$ gives $\eta(LV \xi) = \lambda$. Next, Lie-differentiating the formula $R(X, \xi)\xi = X - \eta(X)\xi$ along $V$ and then by virtue of last equation, we obtain

$$(LV R)(X, \xi)\xi - g(X, LV \xi)\xi + 2\lambda X = -((LV \eta)X)\xi. \tag{6.18}$$

Combining (6.18) with (6.16), and making use of (6.17), we obtain part (ii). This completes the proof. \[\square\]

By virtue of (2.10) and (6.3), the $*$-Ricci soliton equation (6.2) takes the form

$$(LV g)(X, Y) = -\lambda\{g(X, Y) + \eta(X)\eta(Y)\}. \tag{6.19}$$

Now, differentiating this equation covariantly along an arbitrary vector field $Z$ on $M$, we have

$$(\nabla_Z LV g)(X, Y) = -\lambda\{g(Z, \varphi X)\eta(Y) + g(Z, \varphi Y)\eta(X)\}. \tag{6.20}$$

Substitute this equation in commutation formula (6.6), we find

$$(LV \nabla)(X, Y) = \lambda\{\eta(Y)\varphi X + \eta(X)\varphi Y\}. \tag{6.21}$$

Taking covariant differentiation of (6.21) along a vector field $Z$ and then using (2.3), we obtain

$$(\nabla_Z LV \nabla)(X, Y) = \lambda\{g(Z, \varphi Y)\varphi X + g(Z, \varphi X)\varphi Y + g(X, Z)\eta(Y)\xi \tag{6.22}$$

$$+ g(Y, Z)\eta(X)\xi - 2\eta(X)\eta(Y)Z\}. \tag{6.22}$$

Making use of (6.22) in commutation formula (6.14), we have

$$(LV R)(X, Y)Z = \lambda\{g(X, \varphi Z)\varphi Y + 2g(X, \varphi Y)\varphi Z - g(Y, \varphi Z)\varphi X + g(X, Z)\eta(Y)\xi \tag{6.23}$$

$$- g(Y, Z)\eta(X)\xi + 2\eta(X)\eta(Z)Y - 2\eta(Y)\eta(Z)X\}. \tag{6.23}$$

Contracting (6.23) over $Z$, we have

$$(LV Ric)(Y, Z) = 2\lambda\{g(Y, Z) - (2n + 1)\eta(Y)\eta(Z)\}. \tag{6.24}$$

On other hand, taking Lie-differentiation of (6.3) along the vector field $V$ and using (6.19), we obtain

$$(LV Ric)(Y, Z) = \left[1 + \frac{\lambda}{2}\right]\{(LV \eta)(Y)\eta(Z) + \eta(Y)(LV \eta)(Z)\} \tag{6.25}$$

$$- \lambda\left[2n - 1 - \frac{\lambda}{2}\right]\{g(Y, Z) + \eta(Y)\eta(Z)\}. \tag{6.25}$$
Comparison of (6.24) and (6.25) gives

$$\lambda \left[ 1 + \frac{\lambda}{2} \{ (L_{\nu} \eta)(Y) \eta(Z) + \eta(Y)(L_{\nu} \eta)(Z) \} - \lambda \left[ 2n - 1 - \frac{\lambda}{2} \right] \{ g(Y, Z) + \eta(Y) \eta(Z) \} \right]$$

(6.26)

$$= 2 \lambda \{ g(Y, Z) - (2n + 1) \eta(Y) \eta(Z) \}.$$  

Taking \( Y \) by \( \xi \) in the foregoing equation, we get

$$\left[ 1 + \frac{\lambda}{2} \right] (L_{\nu} \eta)(Y) = - \left[ \lambda + \frac{\lambda^2}{2} \right] \eta(Y).$$

Substitute (6.27) in (6.26) and then replacing \( Z \) by \( \varphi Z \), we obtain

$$\lambda \left[ 2n + 1 - \frac{\lambda}{2} \right] g(Y, \varphi Z) = 0.$$  

Since \( g(Y, \varphi Z) \) is non-vanishing everywhere on \( M \), thus we have either \( \lambda = 0 \), or \( \lambda = 2(2n + 1) \).

**Case I:** If \( \lambda = 0 \), then from (6.19) we can see that \( L_{\nu} g = 0 \), i.e., \( V \) is Killing. From (6.3), we have

$$\text{Ric}(X, Y) = (2n - 1) g(X, Y) + \eta(X) \eta(Y).$$

This shows that \( M \) is \( \eta \)-Einstein manifold with scalar curvature \( r = 2n(\alpha + 1) = 4n^2 \).

**Case II:** If \( \lambda = 2(2n + 1) \), then plugging \( Y \) by \( \varphi Y \) in (6.27) we have the relation

$$\left[ 1 + \frac{\lambda}{2} \right] (L_{\nu} \eta)(\varphi Y) = 0.$$  

Since \( \lambda = 2(2n + 1) \), by virtue of last equation we have \( \lambda \neq -2 \), thus we must have \( (L_{\nu} \eta)(\varphi Y) = 0 \). Replacing \( Y \) by \( \varphi Y \) in the foregoing equation and then using (2.1), we have

$$\left( L_{\nu} \eta \right)(Y) = -2(2n + 1) \eta(Y).$$

This shows that \( V \) is a non-strict infinitesimal contact transformation. Now, substituting \( Z \) by \( \xi \) in (6.19) and using (6.29) we immediately get \( L_{\nu} \xi = 2(2n + 1) \xi \). Using this in the commutation formula (see [24], page 23)

$$L_{\nu} \nabla_X \xi - \nabla_X L_{\nu} \xi - \nabla_{[V, X]} \xi = (L_{\nu} \nabla)(X, \xi),$$

for an arbitrary vector field \( X \) on \( M \) and in view of (2.4) and (6.21) gives \( L_{\nu} \varphi = 0 \). Thus, the vector field \( V \) leaves the structure tensor \( \varphi \) invariant.

Other hand, using \( \lambda = 2(2n + 1) \) in (5.3) if follows that

$$\text{Ric}(X, Y) = -2g(X, Y) + 2(n + 1) \eta(X) \eta(Y),$$

showing that \( M \) is \( \eta \)-Einstein with \( \alpha = -2 \). Thus \( M \) is a \( D \)-homothetically fixed.

In [6], the authors give a wonderful information on \( \eta \)-Einstein Sasakian geometry; in this paper authors says, when \( M \) is null (transverse Calabi-Yau) then always \((\alpha, \gamma) = (-2, 2n + 2) \) (see page 189). By this we conclude that \( M \) is a \( D \)-homothetically fixed null \( \eta \)-Einstein manifold. Therefore, we have the following;

**Theorem 6.2.** Let \( M \) be a \((2n+1)\)-dimensional Sasakian manifold. If \( M \) admits *-Ricci soliton, then either \( V \) is Killing, or \( M \) is \( D \)-homothetically fixed null \( \eta \)-Einstein manifold. In the first case, \( M \) is \( \eta \)-Einstein manifold of constant scalar curvature \( r = 2n(\alpha + 1) = 4n^2 \) and in second case, \( V \) is a non-strict infinitesimal contact transformation and leaves the structure tensor \( \varphi \) invariant.

Now, we prove the following result, which gives some remark on *-Ricci soliton.
**Theorem 6.3.** Let $M$ be a $(2n+1)$-dimensional Sasakian manifold admitting $*$-Ricci soliton with $Q^*\varphi = \varphi Q^*$. Then the soliton vector field $V$ leaves the structure tensor $\varphi$ invariant if and only if if $g(\varphi(\nabla_V \varphi)X, Y) = (dv)(X, Y) - (dv)(\varphi X, \varphi Y) = (dv)(X, \xi)\eta(Y)$. 

**Proof.** The $*$-Ricci soliton equation can be written as

$$
(6.30) \quad g(\nabla_X V, Y) + g(\nabla_Y V, X) + 2\text{Ric}^*(X, Y) + 2\lambda g(X, Y) = 0.
$$

Suppose $v$ is 1-form, metrically equivalent to $V$ and is given by $v(X) = g(X, V)$, for any arbitrary vector field $X$, then the exterior derivative $dv$ of $v$ is given by

$$
(6.31) \quad 2(dv)(X, Y) = g(\nabla_X V, Y) - g(\nabla_Y V, X)
$$

As $dv$ is a skew-symmetric, if we define a tensor field $F$ of type $(1, 1)$ by

$$
(6.32) \quad (dv)(X, Y) = g(X, FY),
$$

then $F$ is skew self-adjoint i.e., $g(X, FY) = -g(FX, Y)$. The equation (6.31) takes the form $2g(X, FY) = g(\nabla_X V, Y) - g(\nabla_Y V, X)$. Adding it to equation (6.30) side by side and factoring out $Y$ gives

$$
(6.33) \quad \nabla_X V = -Q^*X - \lambda X - FX,
$$

where $Q^*$ is $*$-Ricci operator. Applying $\varphi$ on (6.33), we have

$$
(6.34) \quad \varphi \nabla_X V = -\varphi Q^*X - \varphi \lambda X - \varphi FX.
$$

Next, Replacing $X$ by $\varphi X$ in (6.33), we obtain

$$
(6.35) \quad \nabla_{\varphi X} V = -Q^*\varphi X - \lambda \varphi X - F\varphi X.
$$

Subtracting (6.34) and (6.35), we have

$$
(6.36) \quad \varphi \nabla_X V - \nabla_{\varphi X} V = (Q^*\varphi - \varphi Q^*)X + (F\varphi - \varphi F)X.
$$

By our hypothesis, noting that $\varphi$ commutes with the $*$-Ricci operator $Q^*$ for Sasakian manifold, we have

$$
(6.37) \quad \varphi \nabla_X V - \nabla_{\varphi X} V = (F\varphi - \varphi F)X.
$$

Now, we note that

$$
(L_V \varphi)X = L_V \varphi X - \varphi L_V X
\quad = \nabla_V \varphi X - \nabla_{\varphi X} V - \varphi \nabla_V X + \varphi \nabla_X V
\quad = (\nabla_V \varphi)X - \nabla_{\varphi X} V + \varphi \nabla_X V.
$$

The use of foregoing equation in (6.37) gives

$$
(6.38) \quad (L_V \varphi)X - (\nabla_V \varphi)X = (F\varphi - \varphi F)X.
$$

Operating $\varphi$ on both sides of the equation (6.37) and then making use of (2.1), (6.33) and (6.35), we find

$$
(dv)(\varphi X, \varphi Y) - (dv)(X, Y) + (dv)(X, \xi)\eta(Y) = g(\varphi(F\varphi - \varphi F)X, Y).
$$

Using (6.38) in the above equation provides

$$
(dv)(\varphi X, \varphi Y) - (dv)(X, Y) + (dv)(X, \xi)\eta(Y) = g(\varphi(L_V \varphi)X - \varphi(\nabla_V \varphi)X, Y).
$$

This shows that $L_V \varphi = 0$ if and only if $g(\varphi(\nabla_V \varphi)X, Y) = (dv)(X, Y) - (dv)(\varphi X, \varphi Y) - (dv)(X, \xi)\eta(Y)$, completing the proof. \qed
SASAKIAN MANIFOLD AND ∗-RICCI TENSOR

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