Abstract. By introducing $k$-marked Durfee symbols, Andrews found a combinatorial interpretation of $2k$-th symmetrized moment $\eta_{2k}(n)$ of ranks of partitions of $n$. Recently, Garvan introduced the $2k$-th symmetrized moment $\mu_{2k}(n)$ of cranks of partitions of $n$ in the study of the higher-order spt-function $spt_k(n)$. In this paper, we give a combinatorial interpretation of $\mu_{2k}(n)$. We introduce $k$-marked Dyson symbols based on a representation of ordinary partitions given by Dyson, and we show that $\mu_{2k}(n)$ equals the number of $(k+1)$-marked Dyson symbols of $n$. We then introduce the full crank of a $k$-marked Dyson symbol and show that there exist an infinite family of congruences for the full crank function of $k$-marked Dyson symbols which implies that for fixed prime $p \geq 5$ and positive integers $r$ and $k \leq (p-1)/2$, there exist infinitely many non-nested arithmetic progressions $An + B$ such that $\mu_{2k}(An + B) \equiv 0 \pmod{p^{r}}$.

1 Introduction

Dyson’s rank [9] and the Andrews-Garvan-Dyson crank [2] are two fundamental statistics in the theory of partitions. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, the rank of $\lambda$, denoted $r(\lambda)$, is the largest part of $\lambda$ minus the number of parts. The crank $c(\lambda)$ is defined by

$$c(\lambda) = \begin{cases} 
\lambda_1, & \text{if } n_1(\lambda) = 0, \\
\mu(\lambda) - n_1(\lambda), & \text{if } n_1(\lambda) > 0,
\end{cases}$$

where $n_1(\lambda)$ is the number of ones in $\lambda$ and $\mu(\lambda)$ is the number of parts larger than $n_1(\lambda)$.

Andrews [3] introduced the symmetrized moments $\eta_{2k}(n)$ of ranks of partitions of $n$ given by

$$\eta_{2k}(n) = \sum_{m=-\infty}^{+\infty} \binom{m + \left\lfloor \frac{k-1}{2} \right\rfloor}{k} N(m, n), \quad (1.1)$$

1

$k$-Marked Dyson Symbols and Congruences for Moments of Cranks

William Y.C. Chen\textsuperscript{1}, Kathy Q. Ji\textsuperscript{2} and Erin Y.Y. Shen\textsuperscript{3}

\textsuperscript{1,2,3}Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

\textsuperscript{1}Center for Applied Mathematics
Tianjin University, Tianjin 300072, P. R. China

\textsuperscript{1}chen@nankai.edu.cn, \textsuperscript{2}ji@nankai.edu.cn, \textsuperscript{3}shenyijing@mail.nankai.edu.cn
where $N(m, n)$ is the number of partitions of $n$ with rank $m$.

In view of the symmetry $N(-m, n) = N(m, n)$, we have $\eta_{2k+1}(n) = 0$. As for the even symmetrized moments $\eta_{2k}(n)$, Andrews [3] showed that for fixed $k \geq 1$, $\eta_{2k}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of $n$. Kursungöz [15] and Ji [13] provided the alternative proof of this result respectively. Bringmann, Lovejoy and Osburn [7] defined two-parameter generalization of $\eta_{2k}(n)$ and $k$-marked Durfee symbols. In [3], Andrews also introduced the full rank of a $k$-marked Durfee symbol and defined the full rank function $NF_k(r; t; n)$ to be the number of $k$-marked Durfee symbols of $n$ with full rank congruent to $r$ modulo $t$.

The full rank function $NF_k(r; t; n)$ have been extensively studied and they posses many congruence properties, see for example, [5–8,14]. Recently, Bringmann, Garvan and Mahlburg [6] used the automorphic properties of the generating functions of $NF_k(r; t; n)$ to prove the existence of infinitely many congruences for $NF_k(r; t; n)$. More precisely, for given positive integers $j$, $k \geq 3$, odd positive integer $t$, and prime $Q$ not divisible by $6t$, there exist infinitely many non-nested arithmetic progressions $An + B$ such that for every $0 \leq r < t$, we have

$$NF_k(r; t; An + B) \equiv 0 \pmod{Q^j}. \quad (1.2)$$

Since

$$\eta_{2k}(n) = \sum_{r=0}^{t-1} NF_{k+1}(r; t; n),$$

by (1.2), we see that there exist an infinite family of congruences for $\eta_{2k}(n)$, namely, for given positive integers $j$, $k \geq 3$, odd positive integer $t$, and prime $Q > 3$, there exist infinitely many non-nested arithmetic progressions $An + B$ such that

$$\eta_{2k}(An + B) \equiv 0 \pmod{Q^j}.$$

Analogous to the symmetrized moments $\eta_k(n)$ of ranks, Garvan [12] introduced the $k$-th symmetrized moments $\mu_k(n)$ of cranks of partitions of $n$ in the study of the higher-order spt-function $spt_k(n)$. To be more specific,

$$\mu_k(n) = \sum_{m=-\infty}^{+\infty} \left( m + \left\lfloor \frac{k-1}{2} \right\rfloor \right) M(m, n), \quad (1.3)$$

where $M(m, n)$ denotes the number of partitions of $n$ with crank $m$ for $n > 1$. For $n = 1$ and $m \neq -1, 0, 1$, we set $M(m, 1) = 0$; otherwise, we define

$$M(-1, 1) = 1, \ M(0, 1) = -1, \ M(1, 1) = 1.$$ 

It is clear that $\mu_{2k+1}(n) = 0$, since $M(m, n) = M(-m, n)$.

In this paper, we give a combinatorial interpretation of $\mu_{2k}(n)$. We first introduce the notion of $k$-marked Dyson symbols based on a representation for ordinary partitions given
by Dyson [9]. We show that for fixed \( k \geq 1 \), \( \mu_{2k}(n) \) equals the number of \((k + 1)\)-marked Dyson symbols of \( n \). Moreover, we define the full crank of a \( k \)-marked Dyson symbol and define full crank function \( NC_k(r, t; n) \) to be the number of \( k \)-marked Dyson symbols of \( n \) with full crank congruent to \( r \) modulo \( t \). We prove that for fixed prime \( p \geq 5 \) and positive integers \( r \) and \( k \leq (p + 1)/2 \), there exists infinitely many non-nested arithmetic progressions \( An + B \) such that for every \( 0 \leq i \leq p^r - 1 \),

\[
NC_k(i, p^r; An + B) \equiv 0 \pmod{p^r}.
\]  

(1.4)

Note that

\[
\mu_{2k}(n) = \sum_{i=0}^{p^r-1} NC_{k+1}(i, p^r; n),
\]

so that from (1.4) we can deduce that there exist an infinite family of congruences for \( \mu_{2k}(n) \), that is, for fixed prime \( p \geq 5 \), positive integers \( r \) and \( k \leq (p - 1)/2 \), there exist infinitely many non-nested arithmetic progressions \( An + B \) such that

\[
\mu_{2k}(An + B) \equiv 0 \pmod{p^r}.
\]

2 Dyson symbols and \( k \)-marked Dyson symbols

In this section, we introduce the notion of \( k \)-marked Dyson symbols. A 1-marked Dyson symbol is called a Dyson symbol, which is a representation of a partition introduced by Dyson [10]. For \( 1 \leq i \leq k \), we define the \( i \)-th crank of a \( k \)-marked Dyson symbol. Moreover, we define the function \( F_k(m_1, m_2, \ldots, m_k; n) \) to be the number of \( k \)-marked Dyson symbol of \( n \) with the \( i \)-th crank equal to \( m_i \) for \( 1 \leq i \leq k \). The following theorem shows that the number of \( k \)-marked Dyson symbols of \( n \) can be expressed in terms of the number of Dyson symbols of \( n \).

**Theorem 2.1.** For fixed integers \( m_1, m_2, \ldots, m_k \), we have

\[
F_k(m_1, \ldots, m_k; n) = \sum_{t_1, \ldots, t_k = 0}^{+\infty} F_1 \left( \sum_{i=1}^{k} |m_i| + 2 \sum_{i=1}^{k-1} t_i + k - 1; n \right). \tag{2.1}
\]

For a partition \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \), let \( \ell(\lambda) \) denote the number of parts of \( \lambda \) and \( |\lambda| \) denote the sum of parts of \( \lambda \). A Dyson symbol of \( n \) is a pair of restricted partitions \((\alpha, \beta)\) satisfying the following conditions:

(1) If \( \ell(\alpha) = 0 \), then \( \beta_1 = \beta_2 \);

(2) If \( \ell(\alpha) = 1 \), then \( \alpha_1 = 1 \);

(3) If \( \ell(\alpha) > 1 \), then \( \alpha_1 = \alpha_2 \);
When we display a Dyson symbol, we shall put $\alpha$ on the top of $\beta$ in the form of a Durfee symbol [3] or a Frobenius partition [1].

For example, there are 5 Dyson symbols of 4:

\[
\begin{pmatrix} 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}.
\]

**Theorem 2.2 (Dyson).** There is a bijection $\Omega$ between the set of partitions of $n$ and the set of Dyson symbols of $n$.

For completeness, we give a proof of the above theorem.

**Proof of Theorem 2.2:** Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of $n$. A Dyson symbol $(\alpha, \beta)$ of $n$ can be constructed via the following procedure. There are two cases.

Case 1: One is not a part of $\lambda$. We set $\alpha = \emptyset$ and $\beta = \lambda'$.

Case 2: One is a part of $\lambda$. Assume that one occurs $M$ times in $\lambda$. We decompose the Ferrers diagram of $\lambda$ into three blocks as illustrated in Figure 2.1, where $N$ is the number of parts of $\lambda$ that are greater than $M$. In this case, we see that $\lambda = (\lambda_1, \ldots, \lambda_N, \lambda_N+1, \ldots, \lambda_s, 1^M)$, where $\lambda_N > M$, $\lambda_N+1 \leq M$ and $1^M$ means $M$ occurrences of 1. Then remove all parts equal to one from $\lambda$ and insert a new part $M$, so that we get a partition $\mu = (\lambda_1, \ldots, \lambda_N, M, \lambda_N+1, \ldots, \lambda_s)$ as shown in Figure 2.2.
Now the partitions $\alpha$ and $\beta$ can be obtained from $\mu$. First, let $\beta = (\lambda_1 - M, \lambda_2 - M, \ldots, \lambda_N - M)$, and let $\nu = (M, \lambda_{N+1}, \ldots, \lambda_s)$. Then we get $\alpha = (\nu'_1, \nu'_2, \ldots, \nu'_M)$, where $\nu'$ the conjugate of $\nu$, see Figure 2.2.

It is easy to verify that $(\alpha, \beta)$ is a Dyson symbol of $n$ and the above procedure is reversible, and hence the proof is complete.

For a Dyson symbol $(\alpha, \beta)$, Dyson [10] considered the difference between the number of parts of $\alpha$ and $\beta$, which we call the crank of $(\alpha, \beta)$. Let $F_1(m; n)$ denote the number of Dyson symbols of $n$ with crank $m$. Dyson [10] observed the following relation based on the construction in Theorem 2.2.

**Corollary 2.3 (Dyson).** For $n \geq 2$ and integer $m$,

$$M(-m, n) = F_1(m; n). \quad (2.2)$$

A $k$-marked Dyson symbol is defined as the following array

$$\eta = \left( \begin{array}{cccc} \alpha^{(k)}, & \alpha^{(k-1)}, & \ldots, & \alpha^{(1)} \\ p_{k-1}, & p_{k-2}, & \ldots, & p_1 \\ \beta^{(k)}, & \beta^{(k-1)}, & \ldots, & \beta^{(1)} \end{array} \right),$$

consisting of $k$ pairs of partitions $(\alpha^{(i)}, \beta^{(i)})$ and a partition $p = (p_{k-1}, p_{k-2}, \ldots, p_0)$ subject to the following conditions:

1. The smallest part of $p$ equals 1, that is, $p_{k-1} \geq \cdots \geq p_1 \geq p_0 = 1$. 

Figure 2.2: The Dyson symbol $(\alpha, \beta)$. 

Now the partitions $\alpha$ and $\beta$ can be obtained from $\mu$. First, let $\beta = (\lambda_1 - M, \lambda_2 - M, \ldots, \lambda_N - M)$, and let $\nu = (M, \lambda_{N+1}, \ldots, \lambda_s)$. Then we get $\alpha = (\nu'_1, \nu'_2, \ldots, \nu'_M)$, where $\nu'$ the conjugate of $\nu$, see Figure 2.2.

It is easy to verify that $(\alpha, \beta)$ is a Dyson symbol of $n$ and the above procedure is reversible, and hence the proof is complete.

For a Dyson symbol $(\alpha, \beta)$, Dyson [10] considered the difference between the number of parts of $\alpha$ and $\beta$, which we call the crank of $(\alpha, \beta)$. Let $F_1(m; n)$ denote the number of Dyson symbols of $n$ with crank $m$. Dyson [10] observed the following relation based on the construction in Theorem 2.2.

**Corollary 2.3 (Dyson).** For $n \geq 2$ and integer $m$,

$$M(-m, n) = F_1(m; n). \quad (2.2)$$

A $k$-marked Dyson symbol is defined as the following array

$$\eta = \left( \begin{array}{cccc} \alpha^{(k)}, & \alpha^{(k-1)}, & \ldots, & \alpha^{(1)} \\ p_{k-1}, & p_{k-2}, & \ldots, & p_1 \\ \beta^{(k)}, & \beta^{(k-1)}, & \ldots, & \beta^{(1)} \end{array} \right),$$

consisting of $k$ pairs of partitions $(\alpha^{(i)}, \beta^{(i)})$ and a partition $p = (p_{k-1}, p_{k-2}, \ldots, p_0)$ subject to the following conditions:

1. The smallest part of $p$ equals 1, that is, $p_{k-1} \geq \cdots \geq p_1 \geq p_0 = 1$. 

Figure 2.2: The Dyson symbol $(\alpha, \beta)$. 

(2) For $1 \leq i \leq k - 1$, each part of $\alpha^{(i)}$ and $\beta^{(i)}$ is between $p_{i-1}$ and $p_i$, namely,
\[ p_i \geq \alpha^{(i)}_1 \geq \alpha^{(i)}_2 \geq \cdots \geq \alpha^{(i)}_t \geq p_{i-1} \quad \text{and} \quad p_i \geq \beta^{(i)}_1 \geq \beta^{(i)}_2 \geq \cdots \geq \beta^{(i)}_t \geq p_{i-1}. \]

(3) Each part of $\alpha^{(k)}$ and $\beta^{(k)}$ is no less than $p_{k-1}$, namely,
\[ \alpha^{(k)}_1 \geq \alpha^{(k)}_2 \geq \cdots \geq \alpha^{(k)}_t \geq p_{k-1} \quad \text{and} \quad \beta^{(k)}_1 \geq \beta^{(k)}_2 \geq \cdots \geq \beta^{(k)}_t \geq p_{k-1}. \]

(4) If $\ell(\alpha^{(k)}) = 1$, then $\alpha^{(k)}_1 = p_{k-1}$;
If $\ell(\alpha^{(k)}) > 1$, then $\alpha^{(k)}_1 = \alpha^{(k)}_2$;
If $\ell(\alpha^{(k)}) = 0$ and $\ell(\beta^{(k)}) = 1$, then $\beta^{(k)}_1 = p_{k-1}$;
If $\ell(\alpha^{(k)}) = 0$ and $\ell(\beta^{(k)}) \geq 2$, then $\beta^{(k)}_1 = \beta^{(k)}_2$;
If $\ell(\alpha^{(k)}) = 0$ and $\ell(\beta^{(k)}) = 0$, then $p_{k-1} = \max\{\alpha^{(k-1)}_1, \beta^{(k-1)}_1\}$.

For example, the array below
\[ \eta = \begin{pmatrix} (5, 5, 4) & (3, 3, 2) & (1, 1) \\ 4 & 2 & \\ (4) & (3, 2, 2) & (2, 1, 1) \end{pmatrix} \tag{2.3} \]
is a 3-marked Dyson symbol.

We next define the weight of a $k$-marked Dyson symbol. Recall that for a pair of partitions $(\alpha, \beta)$ with $\ell(\alpha) \geq \ell(\beta)$, a balanced part $\beta_i$ of $\beta$ is defined recursively as follow. If the number of parts greater than $\beta_i$ in $\alpha$ is equal to the number of unbalanced parts before $\beta_i$ in $\beta$, that is, the number of unbalanced parts $\beta_j$ with $1 \leq j < i$; otherwise, we call $\beta_i$ is an unbalanced part, see [13, p.992]. We use $b(\alpha, \beta)$ to denote the number of balanced parts of $(\alpha, \beta)$.

For example, for the pair of partitions
\[ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3 & 3 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix}, \]
the first part 3 of $\beta$ is balanced, and the second part 2 and the third part 2 are unbalanced. Therefore, $b(\alpha, \beta) = 1$.

We now define the $i$-th crank and the $i$-th balanced number of a $k$-marked Dyson symbol. Let
\[ \eta = \begin{pmatrix} \alpha^{(k)}_1, & \cdots, & \alpha^{(1)}_1, \\ p_{k-1}, & \cdots, & p_1, \\ \beta^{(k)}_1, & \cdots, & \beta^{(1)}_1 \end{pmatrix} \]
be a $k$-marked Dyson symbol. The pair of partitions $(\alpha^{(i)}, \beta^{(i)})$ is called the $i$-th vector of $\eta$. For $1 \leq i \leq k$, we define $c_i(\eta)$, the $i$-th crank of $\eta$, to be the difference between the number of parts of $\alpha^{(i)}$ and $\beta^{(i)}$, that is, $c_i(\eta) = \ell(\alpha^{(i)}) - \ell(\beta^{(i)})$. 
For $1 \leq i < k$, we define $b_i(\eta)$, the $i$-th balanced number of $\eta$ by

$$
b_i(\eta) = \begin{cases} 
b(\alpha^{(i)}, \beta^{(i)}), & \text{if } \ell(\alpha^{(i)}) \geq \ell(\beta^{(i)}), \\
b(\beta^{(i)}, \alpha^{(i)}), & \text{if } \ell(\alpha^{(i)}) < \ell(\beta^{(i)}). 
\end{cases}
$$

For $i = k$, we set $b_k(\eta) = 0$.

For the 3-marked Dyson symbol $\eta$ in (2.3), we have $c_1(\eta) = -1$, $c_2(\eta) = 0$, $c_3(\eta) = 2$ and $b_1(\eta) = 1$, $b_2(\eta) = 1$, $b_3(\eta) = 0$.

For $1 \leq i \leq k$, we define $l_i(\eta)$, the $i$-th large length of $\eta$ by

$$
l_i(\eta) = \begin{cases} 
\ell(\alpha^{(i)}), & \text{if } \ell(\alpha^{(i)}) \geq \ell(\beta^{(i)}), \\
\ell(\beta^{(i)}), & \text{if } \ell(\alpha^{(i)}) < \ell(\beta^{(i)}). 
\end{cases}
$$

Similarly, we define the $i$-th small length $s_i(\eta)$ of $\eta$ by

$$
s_i(\eta) = \begin{cases} 
\ell(\beta^{(i)}), & \text{if } \ell(\alpha^{(i)}) \geq \ell(\beta^{(i)}), \\
\ell(\alpha^{(i)}), & \text{if } \ell(\alpha^{(i)}) < \ell(\beta^{(i)}). 
\end{cases}
$$

The weight of $k$-marked Dyson symbol is defined by

$$
|\eta| = \sum_{i=1}^{k} (|\alpha^{(i)}| + |\beta^{(i)}|) + \sum_{i=1}^{k-1} p_i + (l(\eta) + D + k - 1)(s(\eta) - D),
$$

where

$$
l(\eta) = \sum_{i=1}^{k} l_i(\eta), \quad s(\eta) = \sum_{i=1}^{k} s_i(\eta), \quad \text{and} \quad D = \sum_{i=1}^{k} b_i(\eta). \quad (2.5)
$$

For example, the weight of the 3-marked Dyson symbol $\eta$ in (2.3) equals 97.

For a $k$-marked Dyson symbol $\eta$, if the weight of $\eta$ equals $n$, we call $\eta$ a $k$-marked Dyson symbol of $n$. We can now define the function $F_k(m_1, \ldots, m_k; n)$ as the number of $k$-marked Dyson symbols of $n$ with the $i$-th crank equal to $m_i$ for $1 \leq i \leq k$. Note that a 1-marked Dyson symbol is a Dyson symbol and $F_1(m; n) = M(-m, n)$. The following theorem shows the function $F_k(m_1, \ldots, m_k; n)$ has the mirror symmetry with respect to each $m_j$.

**Theorem 2.4.** For $n \geq 2$, $k \geq 1$ and $1 \leq j \leq k$, we have

$$
F_k(m_1, \ldots, m_j, \ldots, m_k; n) = F_k(m_1, \ldots, -m_j, \ldots, m_k; n). \quad (2.6)
$$

**Proof.** The above identity is trivial for $m_j = 0$. We now assume that $m_j > 0$. Let $H_k(m_1, \ldots, m_k; n)$ denote the set of $k$-marked Dyson symbols of $n$ counted by $F_k(m_1, \ldots,$
We aim to build a bijection $\Lambda$ between the set $H_k(m_1, \ldots, m_j, \ldots, m_k; n)$ and the set $H_k(m_1, \ldots, -m_j, \ldots, m_k; n)$.

Let

$$\eta = \left( \begin{array}{ccccccc} \alpha^{(k)}, & \alpha^{(k-1)}, & \ldots, & \alpha^{(j)}, & \ldots, & \alpha^{(1)} \\ p_{k-1}, & \beta^{(k-1)}, & \ldots, & p_j, & \ldots, & p_1 \\ \beta^{(k)}, & \beta^{(k-1)}, & \ldots, & \beta^{(j)}, & \ldots, & \beta^{(1)} \end{array} \right)$$

be a $k$-marked Dyson symbol in $H_k(m_1, \ldots, m_j, \ldots, m_k; n)$. To define the map $\Lambda$, we need to construct a new $j$-th vector $(\alpha^{(j)}, \beta^{(j)})$ from $(\alpha^{(k)}, \beta^{(k)})$. There are four cases.

Case 1: $1 \leq j \leq k - 1$. Set $\bar{\alpha}^{(j)} = \beta^{(j)}$ and $\bar{\beta}^{(j)} = \alpha^{(j)}$.

Case 2: $j = k$ and $\ell(\alpha^{(k)}) = 1$. In this case, we have $\alpha_1^{(k)} = p_{k-1}$ and $\beta^{(k)} = \emptyset$. Set $\bar{\alpha}^{(k)} = \emptyset$ and $\bar{\beta}^{(k)} = \alpha^{(k)}$.

Case 3: $j = k$, $\ell(\alpha^{(k)}) \geq 2$ and $\ell(\beta^{(k)}) \neq 1$. Let $t = \beta_1^{(k)} - \beta_2^{(k)}$. Set

$$\bar{\alpha}^{(k)} = (\beta_1^{(k)} - t, \beta_2^{(k)}, \ldots, \beta_\ell^{(k)}) \quad \text{and} \quad \bar{\beta}^{(k)} = (\alpha_1^{(k)} + t, \alpha_2^{(k)}, \ldots, \alpha_\ell^{(k)})$$

Case 4: $j = k$, $\ell(\alpha^{(k)}) \geq 2$ and $\ell(\beta^{(k)}) = 1$. Let $t = \beta_1^{(k)} - p_{k-1}$. Set

$$\bar{\alpha}^{(k)} = (\beta_1^{(k)} - t) \quad \text{and} \quad \bar{\beta}^{(k)} = (\alpha_1^{(k)} + t, \alpha_2^{(k)}, \ldots, \alpha_\ell^{(k)})$$

From the above construction, it can be checked that

$$\ell(\bar{\alpha}^{(j)}) - \ell(\bar{\beta}^{(j)}) = -(\ell(\alpha^{(j)}) - \ell(\beta^{(j)}))$$

Then $\Lambda(\eta)$ is defined as

$$\left( \begin{array}{ccccccc} \alpha^{(k)}, & \alpha^{(k-1)}, & \ldots, & \bar{\alpha}^{(j)}, & \ldots, & \alpha^{(1)} \\ p_{k-1}, & \beta^{(k-1)}, & \ldots, & p_j, & \ldots, & p_1 \\ \beta^{(k)}, & \beta^{(k-1)}, & \ldots, & \bar{\beta}^{(j)}, & \ldots, & \beta^{(1)} \end{array} \right).$$

Hence $\Lambda(\eta)$ is a $k$-marked Dyson symbol in $H_k(m_1, \ldots, -m_j, \ldots, m_k; n)$. Furthermore, it can be seen that the above process is reversible. Thus $\Lambda$ is a bijection.

We are now ready to prove Theorem 2.1, which says that the number of $k$-marked Dyson symbols of $n$ can be expressed in terms of the number of Dyson symbols of $n$. This theorem is needed in the combinatorial interpretation of $\mu_{2k}(n)$ given in Theorem 3.1. By Theorem 2.4, we see that Theorem 2.1 can be deduced from the following formula.

**Theorem 2.5.** For $n \geq 2$ and $m_1, m_2, \ldots, m_k \geq 0$, we have

$$F_k(m_1, \ldots, m_k; n) = \sum_{t_1, \ldots, t_{k-1}=0}^{+\infty} F_1 \left( \sum_{i=1}^k m_i + 2 \sum_{i=1}^{k-1} t_i + k - 1; n \right).$$

(2.7)
To prove the above theorem, we introduce the structure of strict \( k \)-marked Dyson symbols. Recall that a strict bipartition of \( n \) is a pair of partitions \((\alpha, \beta)\) such that \( \alpha_i > \beta_i \) for \( i = 1, 2, \ldots, \ell(\beta) \) and \(|\alpha| + |\beta| = n\). Note that for a strict bipartition \((\alpha, \beta)\) we have \( \ell(\alpha) \geq \ell(\beta) \). For example,

\[
\begin{pmatrix}
3 & 3 & 2 & 2 & 1 \\
2 & 1 & 1 & 1
\end{pmatrix}
\]

is a strict bipartition.

Strict bipartitions are the building blocks of strict \( k \)-marked Dyson symbols. For \( k \geq 2 \), let

\[
\eta = \begin{pmatrix}
\alpha^{(k)}, & \alpha^{(k-1)}, & \ldots, & \alpha^{(1)} \\
p_{k-1}, & p_{k-2}, & \ldots, & p_1 \\
\beta^{(k)}, & \beta^{(k-1)}, & \ldots, & \beta^{(1)}
\end{pmatrix}
\]

be a \( k \)-marked Dyson symbols of \( n \). If \((\alpha^{(i)}, \beta^{(i)})\) is a strict bipartition for any \( 1 \leq i < k \), we say that \( \eta \) a strict \( k \)-marked Dyson symbol of \( n \).

Notice that there is no balanced part in a strict bipartition. Consequently, if \( \eta \) is a strict \( k \)-marked Dyson symbol, then the \( i \)-th balanced number \( b_i(\eta) \) of \( \eta \) equals zero for \( 1 \leq i < k \). To prove Theorem 2.5, we define a function \( F_k^s(m_1, \ldots, m_k; n) \) as the number of \( k \)-marked Dyson symbols of \( n \) with the \( i \)-th crank equal to \( m_i \) for \( 1 \leq i \leq k \) and define a function \( F_k(m_1, \ldots, m_k, t_1, \ldots, t_{k-1}; n) \) as the number of \( k \)-marked Dyson symbols of \( n \) with the \( i \)-th crank equal to \( m_i \) for \( 1 \leq i \leq k \) and the \( i \)-th balance number equal to \( t_i \) for \( 1 \leq i \leq k - 1 \). The relation stated in Theorem 2.5 can be established via two steps as stated in the following two theorems.

**Theorem 2.6.** For \( n \geq 2 \), \( k \geq 2 \), \( m_1, m_2, \ldots, m_k \geq 0 \) and \( t_1, t_2, \ldots, t_{k-1} \geq 0 \), we have

\[
F_k(m_1, \ldots, m_k, t_1, \ldots, t_{k-1}; n) = F_k^s(m_1 + 2t_1, \ldots, m_{k-1} + 2t_{k-1}, m_k; n). \tag{2.8}
\]

**Theorem 2.7.** For \( n \geq 2 \), \( k \geq 2 \) and \( m_1, m_2, \ldots, m_k \geq 0 \), we have

\[
F_k^s(m_1, \ldots, m_k; n) = F_1 \left( \sum_{i=1}^{k} m_i + k - 1; n \right). \tag{2.9}
\]

To prove Theorem 2.6, we need a bijection in [13, Theorem 2.4]. Let \( P(r; n) \) denote the set of pairs of partitions \((\alpha, \beta)\) of \( n \) where there are \( r \) balanced parts and \( \ell(\alpha) - \ell(\beta) \geq 0 \), and let \( Q(r; n) \) denote the set of strict bipartitions \((\alpha, \beta)\) of \( n \) with \( \ell(\alpha) - \ell(\beta) \geq r \). Given two positive integers \( n \) and \( r \), there is a bijection \( \psi \) between \( P(r; n) \) and \( Q(2r; n) \). Furthermore, the bijection \( \psi \) possesses the following properties. For \((\alpha, \beta) \in P(r; n)\), let \((\bar{\alpha}, \bar{\beta}) = \psi(\alpha, \beta)\). Then we have

\[
\bar{\alpha}_1 = \max\{\alpha_1, \beta_1\}, \quad \bar{\alpha}_\ell = \alpha_\ell, \quad \text{and} \quad \bar{\beta}_\ell \geq \beta_\ell. \tag{2.10}
\]

\[
\ell(\bar{\alpha}) = \ell(\alpha) + r \quad \text{and} \quad \ell(\bar{\beta}) = \ell(\beta) - r. \tag{2.11}
\]
We next give a proof of Theorem 2.6 by using the bijection $\psi$.

**Proof of Theorem 2.6.** Let $P_k(m_1, \ldots, m_k, t_1, t_2, \ldots, t_{k-1}; n)$ denote the set of $k$-marked Dyson symbols of $n$ with the $i$-th crank equal to $m_i$ and the $i$-th balanced number equal to $t_i$, and let $Q_k(m_1, \ldots, m_k; n)$ denote the set of strict $k$-marked Dyson symbols of $n$ with the $i$-th crank equal to $m_i$. We proceed to define a bijection $\Omega$ between $P_k(m_1, \ldots, m_k, t_1, t_2, \ldots, t_{k-1}; n)$ and $Q_k(m_1 + 2t_1, \ldots, m_{k-1} + 2t_{k-1}, m_k; n)$.

Let

$$\eta = \left( \alpha^{(k)}, \alpha^{(k-1)}, \ldots, \alpha^{(1)} \right)$$

be a $k$-marked Dyson symbol in $P_k(m_1, \ldots, m_k, t_1, t_2, \ldots, t_{k-1}; n)$. For $1 \leq i < k$, we apply the bijection $\psi$ described above to $(\alpha^{(i)}, \beta^{(i)})$ to get a pair of partitions $(\tilde{\alpha}^{(i)}, \tilde{\beta}^{(i)})$. From the properties of the bijection $\psi$, we see that $(\tilde{\alpha}^{(i)}, \tilde{\beta}^{(i)})$ is a strict bipartition and

$$\tilde{\alpha}^{(i)} = \max\{\alpha^{(i)}, \beta^{(i)}\}, \quad \tilde{\beta}^{(i)} = \alpha^{(i)}, \quad \tilde{\tilde{\beta}}^{(i)} = \beta^{(i)}$$

and

$$\ell(\tilde{\alpha}^{(i)}) = \ell(\alpha^{(i)}) + t_i, \quad \ell(\tilde{\tilde{\beta}}^{(i)}) = \ell(\beta^{(i)}) - t_i.$$  \quad (2.13)

Then $\Omega(\eta)$ is defined to be

$$\left( \alpha^{(k)}, \tilde{\alpha}^{(k-1)}, \ldots, \tilde{\alpha}^{(1)} \right)$$

By (2.12), we see that that for $1 \leq i < k - 1$, each part of $\tilde{\alpha}^{(i)}$ and $\tilde{\tilde{\beta}}^{(i)}$ is between $p_{i-1}$ and $p_i$, namely,

$$p_i \geq \tilde{\alpha}^{(i)} \geq \tilde{\alpha}^{(i)} \geq \cdots \geq \tilde{\alpha}^{(i)} \geq p_{i-1} \quad \text{and} \quad p_i \geq \tilde{\beta}^{(i)} \geq \tilde{\tilde{\beta}}^{(i)} \geq \cdots \geq \tilde{\tilde{\beta}}^{(i)} \geq p_{i-1}.$$

It is also clear from (2.13) that the $i$-th crank of $\Omega(\eta)$ is equal to $m_i + 2t_i$ for $1 \leq i < k$ and the $k$-th crank of $\Omega(\eta)$ is equal to $m_k$. Using (2.13) again, we get

$$l(\Omega(\eta)) = \sum_{i=1}^{k-1} \ell(\tilde{\alpha}^{(i)}) + \ell(\alpha^k) = \sum_{i=1}^{k} \ell(\alpha^{(i)}) + t_i = \sum_{i=1}^{k} \ell(\alpha^{(i)}) + D = l(\eta) + D$$

and

$$s(\Omega(\eta)) = \sum_{i=1}^{k-1} \ell(\tilde{\beta}^{(i)}) + \ell(\beta^k) = \sum_{i=1}^{k} \ell(\beta^{(i)}) - t_i = \sum_{i=1}^{k} \ell(\alpha^{(i)}) - D = s(\eta) - D.$$  \quad (2.14)

Thus the weight of $\Omega(\eta)$ is equal to

$$\sum_{i=1}^{k} (|\alpha^{(i)}| + |\beta^{(i)}|) + \sum_{i=1}^{k-1} p_i + (l(\Omega(\eta)) + k - 1) \cdot s(\Omega(\eta))$$

$$= \sum_{i=1}^{k} (|\alpha^{(i)}| + |\beta^{(i)}|) + \sum_{i=1}^{k-1} p_i + (l(\eta) + k - 1 + D) \cdot (s(\eta) - D),$$  \quad (2.15)
which is in accordance with the definition of $|\eta|$. So $\Omega(\eta)$ is in $Q_k(m_1 + 2t_1, \ldots, m_{k-1} + 2t_{k-1}, m_k; n)$. Since $\psi$ is a bijection, it is readily verified that $\Omega$ is also a bijection, and hence the proof is complete.

We now turn to the proof of Theorem 2.7.

**Proof of Theorem 2.7.** Recall that $Q_k(m_1, \ldots, m_k; n)$ denotes the set of strict $k$-marked Dyson symbols of $n$ with the $i$-th crank equal to $m_i$ and $H_1(m; n)$ denotes the set of Dyson symbols of $n$ with crank $m$. To establish a bijection $\Phi$ between $Q_k(m_1, \ldots, m_k; n)$ and $H_1(m_1 + \cdots + m_k + k - 1; n)$, let

$$
\eta = \left(\begin{array}{cccc}
\alpha^{(k)}, & \alpha^{(k-1)}, & \ldots, & \alpha^{(1)} \\
p_{k-1}, & p_{k-2}, & \ldots, & p_1 \\
\beta^{(k)}, & \beta^{(k-1)}, & \ldots, & \beta^{(1)}
\end{array}\right)
$$

be a strict $k$-marked Dyson symbol in $Q_k(m_1, \ldots, m_k; n)$. Let $\alpha$ be the partition consisting of all parts of $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}$ together with $p_1, \ldots, p_{k-1}$, and let $\beta$ be the partition consisting of all parts of $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(k)}$. Then $\Phi(\eta)$ is defined to be $(\alpha, \beta)$. From the definition of $k$-marked Dyson symbols, we see that $(\alpha, \beta)$ is a Dyson symbol. It is also easily seen that

$$
\ell(\alpha) = l(\eta) + k - 1, \quad \ell(\beta) = s(\eta) \tag{2.14}
$$

and

$$
|\alpha| = \sum_{i=1}^{k} |\alpha^{(i)}| + \sum_{i=1}^{k-1} p_i, \quad |\beta| = \sum_{i=1}^{k} |\beta^{(i)}|. \tag{2.15}
$$

It follows from (2.14) that

$$
\ell(\alpha) - \ell(\beta) = \sum_{i=1}^{k} m_i + k - 1.
$$

Combining (2.14) and (2.15), we deduce that the weight of $(\alpha, \beta)$ equals

$$
|\alpha| + |\beta| + \ell(\alpha)\ell(\beta) = \sum_{i=1}^{k} |\alpha^{(i)}| + \sum_{i=1}^{k-1} p_i + \sum_{i=1}^{k} |\beta^{(i)}| + (l(\eta) + k - 1)s(\eta) = |\eta|.
$$

This proves that $(\alpha, \beta)$ is a Dyson symbol in $H_1(m_1 + \cdots + m_k + k - 1; n)$.

We next describe the reverse map of $\Phi$. Let

$$
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
= \begin{pmatrix}
\alpha_1 & \alpha_2 & \ldots & \alpha_{\ell} \\
\beta_1 & \beta_2 & \ldots & \beta_{\ell}
\end{pmatrix}
$$

be a Dyson symbol in $H_1(m_1 + \cdots + m_k + k - 1; n)$. We proceed to show that a strict $k$-marked Dyson symbol $\eta$ can be recovered from the Dyson symbol $(\alpha, \beta)$.
First, we see that the $k$-th vector $(\alpha^{(k)}, \beta^{(k)})$ of $\eta$ and $p_{k-1}$ can be recovered from $(\alpha, \beta)$. Let $j_k$ be largest nonnegative integer such that $\beta_{j_k} \geq \alpha_{m_k + j_k + 1}$, that is, for any $i \geq j_k + 1$, we have $\beta_i < \alpha_{m_k + i + 1}$. Define

$$
\begin{pmatrix}
\alpha^{(k)} \\
\beta^{(k)}
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_1 & \alpha_2 & \ldots & \alpha_{m_k + j_k} \\
\beta_1 & \beta_2 & \ldots & \beta_{j_k}
\end{pmatrix}
\quad \text{and} \quad
p_{k-1} = \alpha_{m_k + j_k + 1}.
$$

Obviously, $\ell(\alpha^{(k)}) - \ell(\beta^{(k)}) = m_k$.

To recover $(\alpha^{(k-1)}, \beta^{(k-1)})$ and $p_{k-1}$, we let

$$
\begin{pmatrix}
\alpha' \\
\beta'
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_{m_k + j_k + 2} & \alpha_{m_k + j_k + 3} & \ldots & \alpha_{e} \\
\beta_{j_k + 1} & \beta_{j_k + 2} & \ldots & \beta_{e}
\end{pmatrix}.
$$

By the choice of $j_k$, we find that $\alpha_{m_k + j_k + i + 1} > \beta_{j_k + i}$ for any $i$, in other words, $\alpha'_i > \beta'_i$. Consequently, $(\alpha', \beta')$ is a strict bipartition. Then $(\alpha^{(k-1)}, \beta^{(k-1)})$ and $p_{k-1}$ can be constructed from $(\alpha', \beta')$. Let $j_{k-1}$ be the largest nonnegative integer such that $\beta'_{j_{k-1}} \geq \alpha'_{m_{k-1} + j_{k-1} + 1}$. Define

$$
\begin{pmatrix}
\alpha^{(k-1)} \\
\beta^{(k-1)}
\end{pmatrix}
= 
\begin{pmatrix}
\alpha'_1 & \alpha'_2 & \ldots & \alpha'_{m_{k-1} + j_{k-1}} \\
\beta'_1 & \beta'_2 & \ldots & \beta'_{j_{k-1}}
\end{pmatrix}
\quad \text{and} \quad
p_{k-2} = \alpha'_{m_{k-1} + j_{k-1} + 1}.
$$

Now we have $\ell(\alpha^{(k-1)}) - \ell(\beta^{(k-1)}) = m_{k-1}$. Since $(\alpha', \beta')$ is a strict bipartition, we deduce that $(\alpha^{(k-1)}, \beta^{(k-1)})$ is a strict bipartition.

The above procedure can be repeatedly used to determine $(\alpha^{(k-2)}, \beta^{(k-2)}, p_{k-3}, \ldots, p_1, (\alpha^{(1)}, \beta^{(1)})$. The $k$-marked Dyson symbol $\eta$ can be defined as

$$
\begin{pmatrix}
\alpha^{(k)} \\
\beta^{(k)}
\end{pmatrix}
= 
\begin{pmatrix}
\alpha^{(k-1)} & \ldots & \alpha^{(1)} \\
p_{k-1} & \ldots & p_1 \\
\beta^{(k-1)} & \ldots & \beta^{(1)}
\end{pmatrix}.
$$

It can be checked that $\eta$ is a strict $k$-marked Dyson symbol in $Q_k(m_1, \ldots, m_k; n)$. Moreover, it can be seen that $\Phi(\eta) = (\alpha, \beta)$, that is, $\Phi$ is indeed a bijection. This completes the proof.

Here is an example to illustrate the reverse map $\Phi^{-1}$. Assume that $m_1 = 1, m_2 = 1, m_3 = 0$, and

$$
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
= 
\begin{pmatrix}
6 & 6 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 & 1 \\
5 & 5 & 4 & 2 & 1 & 1 & 1
\end{pmatrix},
$$

which a Dyson symbol of 127, that is, $(\alpha, \beta) \in H_1(4; 127)$. From $(\alpha, \beta)$, we get

$$
\begin{pmatrix}
\alpha^{(3)} \\
\beta^{(3)}
\end{pmatrix}
= 
\begin{pmatrix}
6 & 6 & 3 \\
5 & 5 & 4
\end{pmatrix},
$$

$p_2 = 3$, 

$$
\begin{pmatrix}
\alpha' \\
\beta'
\end{pmatrix}
= 
\begin{pmatrix}
3 & 3 & 2 & 2 & 1 & 1 & 1 \\
2 & 1 & 1 & 1
\end{pmatrix}.
$$

12
Based on \((\alpha', \beta')\), we get
\[
\begin{pmatrix}
\alpha^{(2)} \\
\beta^{(2)}
\end{pmatrix} = \begin{pmatrix}
3 & 3 & 2 & 2 & 1 \\
2 & 1 & 1 & 1
\end{pmatrix}, \quad p_2 = 1, \quad \begin{pmatrix}
\alpha^{(1)} \\
\beta^{(1)}
\end{pmatrix} = \begin{pmatrix}
1 \\
1
\end{pmatrix}.
\]

Finally, we obtain
\[
\eta = \begin{pmatrix}
6 & 6 & 3 \\
3 & 3 & 2 & 2 & 1 \\
5 & 5 & 4 \\
3 & 2 & 1 & 1 & 1
\end{pmatrix}.
\]

It can be checked that \(\eta \in Q_3(1,1,0;127)\).

3 A combinatorial interpretation of \(\mu_{2k}(n)\)

In this section, we use Theorem 2.1 to give a combinatorial interpretation of \(\mu_{2k}(n)\) in terms of \(k\)-marked Dyson symbols.

**Theorem 3.1.** For \(k \geq 1\) and \(n \geq 2\), \(\mu_{2k}(n)\) is equal to the number of \((k + 1)\)-marked Dyson symbols of \(n\).

**Proof.** By definition of \(F_k(m_1, \ldots, m_k; n)\), the assertion of the theorem can be stated as follows
\[
\sum_{m_1, \ldots, m_{k+1} = -\infty}^{\infty} F_{k+1}(m_1, \ldots, m_{k+1}; n) = \mu_{2k}(n). \tag{3.1}
\]

Using Theorem 2.1, we see that the left-hand side of (3.1) equals
\[
\sum_{m_1, m_2, \ldots, m_{k+1} = -\infty}^{\infty} F_{k+1}(m_1, \ldots, m_{k+1}; n)
= \sum_{m_1, m_2, \ldots, m_{k+1} = -\infty}^{\infty} \sum_{t_1, \ldots, t_k = 0}^{\infty} F_1 \left( \sum_{i=1}^{k+1} |m_i| + 2 \sum_{i=1}^{k} t_i + k; n \right). \tag{3.2}
\]

Given \(k\) and \(n\), let \(c_k(j)\) denote the number of integer solutions to the equation
\[
|m_1| + \cdots + |m_{k+1}| + 2t_1 + \cdots + 2t_k = j
\]
in \(m_1, m_2, \ldots, m_{k+1}\) and \(t_1, t_2, \ldots, t_k\) subject to the further condition that \(t_1, t_2, \ldots, t_k\) are nonnegative. It can be shown that generating function of \(c_k(j)\) is equal to
\[
\sum_{j=0}^{\infty} c_k(j)q^j = \frac{1 + q}{(1 - q^{2k+1})}.
\]
so that
\[ c_k(j) = \binom{2k+j}{2k} + \binom{2k+j-1}{2k}. \]

Substituting \( j \) by \( m - k \), we get
\[ c_k(m - k) = \binom{m+k-1}{2k} + \binom{m+k}{2k}. \]

Thus (3.2) simplifies to
\[
\sum_{m_1, m_2, \ldots, m_{k+1} = -\infty}^{\infty} F_{k+1}(m_1, \ldots, m_{k+1}; n)
= \sum_{m=1}^{\infty} \left[ \binom{m+k-1}{2k} + \binom{m+k}{2k} \right] F_1(m; n).
\]

Using Corollary 2.3 and noting that \( M(-m, n) = M(m, n) \), we conclude that
\[
\sum_{m_1, m_2, \ldots, m_{k+1} = -\infty}^{\infty} F_{k+1}(m_1, \ldots, m_{k+1}; n)
= \sum_{m=1}^{\infty} \left[ \binom{m+k-1}{2k} + \binom{m+k}{2k} \right] M(m, n),
\]
which equals \( \mu_{2k}(n) \), as claimed. \( \blacksquare \)

For example, for \( n = 5 \) and \( k = 1 \), we have \( \mu_2(5) = 35 \), and there are 35 2-marked Dyson symbols of 5 as listed in the following table.

\[
\begin{align*}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
(1) & (1) & (1) & (1)
\end{pmatrix} & \\
\begin{pmatrix}
2 & (1) \\
(2) & (1)
\end{pmatrix} & \\
\begin{pmatrix}
(1) & 1 \\
(1) & (1)
\end{pmatrix} & \\
\begin{pmatrix}
(1) & 1 \\
(1) & (1)
\end{pmatrix} & \\
\begin{pmatrix}
(1) & 1 \\
(1) & (1)
\end{pmatrix} & \\
\begin{pmatrix}
(1) & 1 \\
(1) & (1)
\end{pmatrix} & \\
\begin{pmatrix}
1 & 1 & 1 & 1 \\
(1) & (1) & (1) & (1)
\end{pmatrix} & \\
\begin{pmatrix}
1 & (1) & (1) \\
(1) & (1)
\end{pmatrix} & \\
\begin{pmatrix}
1 & (1) & (1) \\
(1) & (1)
\end{pmatrix} & \\
\begin{pmatrix}
1 & 1 & 1 & 1 \\
(1) & (1) & (1) & (1)
\end{pmatrix}
\end{align*}
\]
4 Congruences for $\mu_{2k}(n)$

In this section, we introduce the full crank of a $k$-marked Dyson symbol. We show that there exist an infinite family of congruences for the full crank function of $k$-marked Dyson symbols.

To define the full crank of a $k$-marked Dyson symbol $\eta$, denoted $FC(\eta)$, we recall that $c_k(\eta)$ denotes the $k$-th crank of $\eta$, $l(\eta)$ denotes the large length of $\eta$ and $s(\eta)$ denotes the
small length of $\eta$ and $D$ denotes the balanced number of $\eta$. Then $FC(\eta)$ is given by

$$FC(\eta) = \begin{cases} 
  l(\eta) - s(\eta) + 2D + k - 1, & \text{if } c_k(\eta) > 0, \\
  -(l(\eta) - s(\eta) + 2D + k - 1), & \text{if } c_k(\eta) \leq 0.
\end{cases}$$

It is clear that for $k = 1$, the full crank of a 1-marked Dyson symbol reduces to the crank of a Dyson symbol.

Analogous to the full rank function for a $k$-marked Durfee symbol defined by Andrews [3], we define the full crank function $NC_k(i, t; n)$ as the number of $k$-marked Dyson symbols of $n$ with the full crank congruent to $i$ modulo $t$. The following theorem gives an infinite family of congruences of the full crank function.

**Theorem 4.1.** For fixed prime $p \geq 5$ and positive integers $r$ and $k \leq (p + 1)/2$. Then there exist infinitely many non-nested arithmetic progressions $An + B$ such that for each $0 \leq i \leq p^r - 1$,

$$NC_k(i, p^r; An + B) \equiv 0 \pmod{p^r}.$$

Since

$$\mu_{2k}(n) = \sum_{i=0}^{p^r-1} NC_{k+1}(i, p^r; n),$$

Theorem 4.1 implies the following congruences for $\mu_{2k}(n)$.

**Theorem 4.2.** For fixed prime $p \geq 5$, positive integers $r$ and $k \leq (p - 1)/2$. Then there exists infinitely many non-nested arithmetic progressions $An + B$ such that

$$\mu_{2k}(An + B) \equiv 0 \pmod{p^r}.$$

To prove Theorem 4.1, let $NC_k(m; n)$ denote the number of $k$-marked Dyson symbols of $n$ with the full crank equal to $m$. In this notation, we have the following relation.

**Theorem 4.3.** For $n \geq 2, k \geq 1$ and integer $m$,

$$NC_k(m; n) = \binom{m + k - 2}{2k - 2} M(m, n).$$

**Proof.** Recall that $F_k(m_1, \ldots, m_k, t_1, \ldots, t_{k-1}; n)$ is the number of $k$-marked Dyson symbols of $n$ such that for $1 \leq i \leq k$, the $i$-th crank equal to $m_i$ and the $i$-th balance number equal to $t_i$. By the definition of $NC_k(m; n)$, we see that if $m \geq 1$, then we have

$$NC_k(m; n) = \sum F_k(m_1, m_2, \ldots, m_k, t_1, t_2, \ldots, t_{k-1}; n),$$

where the summation ranges over all integer solutions to the equation

$$|m_1| + \cdots + |m_k| + 2t_1 + \cdots + 2t_{k-1} = m - k + 1$$

(4.3)
in $m_1, m_2, \ldots, m_k$ and $t_1, t_2, \ldots, t_{k-1}$ subject to the further condition that $m_k$ is positive and $t_1, t_2, \ldots, t_{k-1}$ are nonnegative.

Combining Theorem 2.6 and Theorem 2.7, we find that

$$F_k(m_1, m_2, \ldots, m_k, t_1, t_2, \ldots, t_{k-1}; n) = F_1 \left( \sum_{i=1}^{k} |m_i| + 2 \sum_{i=1}^{k-1} t_i + k - 1; n \right).$$

(4.4)

Substituting (4.4) into (4.2), we get

$$NC_k(m; n) = \sum F_1 \left( \sum_{i=1}^{k} |m_i| + 2 \sum_{i=1}^{k-1} t_i + k - 1; n \right),$$

(4.5)

where the summation ranges over all solutions to the equation (4.3). Let $\tilde{c}_k(m - k + 1)$ denote the number of integer solutions to the equation (4.3). It is not difficult to verify that

$$\tilde{c}_k(m - k + 1) = \binom{m + k - 2}{2k - 2}.$$

Thus, (4.5) simplifies to

$$NC_k(m; n) = \left( \frac{m + k - 2}{2k - 2} \right) F_1(m; n).$$

Using Corollary 2.3 and noting that $M(-m, n) = M(m, n)$, we conclude that

$$NC_k(m; n) = \left( \frac{m + k - 2}{2k - 2} \right) M(m, n),$$

as required. Similarly, it can be shown that relation (4.1) also holds for $m \leq 0$. 

Let $M(i, t; n)$ denote the number of partitions of $n$ with the crank congruent to $i$ modulo $t$. The following congruences for $M(i, t; n)$ given by Mahlburg [16] will be used in the proof of Theorem 4.1.

**Theorem 4.4 (Mahlburg).** For fixed prime $p \geq 5$ and positive integers $\tau$ and $r$, there are infinitely many non-nested arithmetic progressions $An + B$ such that for each $0 \leq m \leq p^r - 1$,

$$M(m, p^r; An + B) \equiv 0 \pmod{p^\tau}.$$

We are now ready to complete the proof of Theorem 4.1 by using Theorems 4.3 and 4.4.

**Proof of Theorem 4.1.** For $0 \leq i \leq p^r - 1$, by the definition of $NC_k(i, p^r; n)$, we have

$$NC_k(i, p^r; n) = \sum_{t=-\infty}^{+\infty} NC_k(p^\tau t + i; n).$$

(4.6)
Replacing $m$ by $p^r t + i$ in (4.1), we get

$$NC_k(p^r t + i; n) = \left( \frac{p^r t + i + k - 2}{2k - 2} \right) M(p^r t + i, n). \tag{4.7}$$

Substituting (4.7) into (4.6), we find that

$$NC_k(i, p^r; n) = \sum_{t=-\infty}^{+\infty} \left( \frac{p^r t + i + k - 2}{2k - 2} \right) M(p^r t + i, n). \tag{4.8}$$

Since $p$ is a prime and $k \leq (p + 1)/2$, we see that $(2k - 2)!$ is not divisible by $p$. It follows that

$$\left( \frac{p^r t + i + k - 2}{2k - 2} \right) \equiv \left( \frac{i + k - 2}{2k - 2} \right) \pmod{p^r}.$$

Thus (4.8) implies that

$$NC_k(i, p^r; n) \equiv \sum_{t=-\infty}^{+\infty} \left( \frac{i + k - 2}{2k - 2} \right) M(p^r t + i, n) \pmod{p^r}$$

$$= \left( \frac{i + k - 2}{2k - 2} \right) M(i, p^r; n).$$

Setting $\tau = r$ in Theorem 4.4, we deduce that there are infinitely many non-nested arithmetic progressions $An + B$ such that for every $0 \leq i \leq p^r - 1$

$$M(i, p^r; An + B) \equiv 0 \pmod{p^r}.$$ 

Consequently, there are infinitely many non-nested arithmetic progressions $An + B$ such that for every $0 \leq m \leq p^r - 1$

$$NC_k(i, p^r; An + B) \equiv 0 \pmod{p^r},$$

and hence the proof is complete. \hspace{1cm} \blacksquare

**Acknowledgments.** This work was supported by the 973 Project and the National Science Foundation of China.

**References**

[1] G. E. Andrews, Generalized Frobenius partitions, Mem. Amer. Math. Soc. 49 (1984) No. 301 iv+ 44 pp.

[2] G. E. Andrews and F. G. Garvan, Dyson’s crank of a partition, Bull. Amer. Math. Soc. 18 (1988) 167–171.
[3] G. E. Andrews, Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks, Invent. Math. 169 (2007) 37–73.

[4] A. O. L. Atkin and F. G. Garvan, Relations between the ranks and cranks of partitions, Ramanujan J. 7 (2003) 343–366.

[5] K. Bringmann, On the explicit construction of higher deformations of partition statistics, Duke Math. J. 114 (2008) 195–233.

[6] K. Bringmann, F. G. Garvan and K. Mahlburg, Partition statistics and quasiweak Maass forms, Int. Math. Res. Not. IMRN (2009) 63–97.

[7] K. Bringmann, J. Lovejoy and R. Osburn, Automorphic properties of generating functions for generalized rank moments and Durfee symbols, Int. Math. Res. Not. IMRN (2010) 238–260.

[8] K. Bringmann and B. Kane, Inequalities for full rank differences of 2-marked Durfee symbols, J. Combin. Theory A 119 (2012) 483–501.

[9] F. J. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944) 10–15.

[10] F. J. Dyson, Mappings and symmetries of partitions, J. Combin. Theory A 51 (1989) 169–180.

[11] F. G. Garvan, New combinatorial interpretations of Ramanujan’s partition congruences mod 5, 7, 11, Trans. Amer. Math. Soc. 305 (1988) 47–77.

[12] F. G. Garvan, Higher order spt-functions, Adv. Math. 228 (2011) 241–265.

[13] K. Q. Ji, The combinatorics of k-marked Durfee symbols, Trans. Amer. Math. Soc. 363 (2011) 987–1005.

[14] W. J. Keith, Distribution of the full rank in residue classes for odd moduli, Discrete Math. 309 (2009) 4960–4968.

[15] K. Kursungoz, Counting k-marked Durfee symbols, Electron. J. Combin. 18 (2011) #P41.

[16] K. Mahlburg, Partition congruences and the Andrews-Garvan-Dyson crank, Proc. Natl. Acad. Sci. USA 102 (2005) 15373–15376.