Indeterminate moment problem associated with continuous dual $q$-Hahn polynomials

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ABSTRACT

We study a limiting case of the Askey–Wilson polynomials when one of the parameters goes to infinity, namely continuous dual $q$-Hahn polynomials when $q > 1$. Solutions to the associated indeterminate moment problem by general theory are found and an orthogonality relation is established.

1. Introduction

One of the classical questions in functional analysis, related to orthogonal polynomials, is the moment problem: for a sequence of real numbers $\{\mu_n\}_{n \geq 0}$, is there a positive Borel measure $\mu$ with $\text{supp}(\mu)$, the support of $\mu$, in $\mathbb{R}$ such that

$$\mu_n = \int x^n d\mu, \quad n \in \{0, 1, 2, \ldots\}? \quad (1)$$

When such a measure exists, if the measure is unique, then the moment problem is determine otherwise the moment problem is called an indeterminate moment problem. When the sequence $\{\mu_n\}_{n \geq 0}$ is positive, that is for all $n \in \{0, 1, 2, \ldots\}$ the Hankel determinant

$$D_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}$$
is positive, there exists a positive Borel measure \( \mu \), supported on \( \mathbb{R} \), such that (1) is satisfied (cf. [1]). Moreover, the family of polynomials

\[
P_n(x) = \frac{1}{\sqrt{D_nD_{n-1}}} \begin{vmatrix}
\mu_0 & \mu_1 & \ldots & \mu_n \\
\mu_1 & \mu_2 & \ldots & \mu_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{n-1} & \mu_n & \ldots & \mu_{2n-1} \\
1 & x & \ldots & x^n
\end{vmatrix}, \quad n = 1, 2, \ldots,
\]

with \( P_0(x) = 1 \), is orthonormal with respect to the measure \( \mu \) on its support, and the sequence \( \{P_n\}_{n=0}^\infty \) satisfies the three-term recurrence relation

\[
xP_n(x) = b_nP_{n+1}(x) + a_nP_n(x) + b_{n-1}P_{n-1}(x), \quad (2)
\]

where

\[
b_n = \frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}, \quad a_n = \int xP_n(x)^2 \, d\mu(x).
\]

The initial conditions are \( P_{-1}(x) = 0 \) and \( P_0(x) = 1 \). Observing from the above relation that the leading coefficient of \( P_n \), \( n = 0, 1, 2, \ldots \), is \( \frac{1}{b_0b_1\ldots b_{n-1}} \), the monic polynomial \( p_n(x) = b_0b_1\ldots b_{n-1}P_n(x) \) satisfies the three-term recurrence relation

\[
xp_n(x) = p_{n+1}(x) + c_np_n(x) + \lambda np_{n-1}(x), \quad (3)
\]

with \( c_n = a_n \in \mathbb{R} \) and \( \lambda_n = b_{n-1}^2 > 0 \), where \( a_n \) and \( b_{n-1} \) are the constants appearing in (2). The three-term recurrence relations (2) and (3) both provide useful information about the moment problem. For example, in [7, Thm 2], Chihara proved that, under the assumption

\[
\lim_{n \to \infty} c_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{\lambda_{n+1}}{c_n} = L < \frac{1}{4},
\]

the moment problem is indeterminate when

\[
\lim \inf_{n \to \infty} \frac{c_n^{1/2}}{1 + \sqrt{1 - 4L}} > \frac{1 + \sqrt{1 - 4L}}{1 - \sqrt{1 - 4L}} \quad (4)
\]

and determinate when the opposite (strict) inequality holds. Specifically, when \( c_n = f_nq^{-n} \) with \( 0 < q < 1 \) and where \( \{f_n\}_{n \geq 0} \) is both bounded and bounded away from 0, the moment problem is determinate when

\[
L < \frac{q}{(1 + q)^2}
\]

and indeterminate when

\[
L > \frac{q}{(1 + q)^2}.
\]

When the moment problem is indeterminate, there are infinitely many measures satisfying (1) (cf. [22]). Moreover, these measures can be described by means of the Stieltjes
\[
\int_{\text{supp}(\mu)} \frac{d\mu(t)}{z-t} = \frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \text{supp}(\mu),
\]
where \(\varphi\) belongs to the space of Pick functions augmented with the point \(\infty\). The entire functions \(A, B, C,\) and \(D\) are uniform limits, on compact subsets of \(\mathbb{C}\), as \(n \to \infty\), of

\[
A_n(z) = b_n(Q_n(0)Q_{n+1}(z) - Q_{n+1}(0)Q_n(z)),
\]
\[
B_n(z) = b_n(Q_n(0)P_{n+1}(z) - Q_{n+1}(0)P_n(z)),
\]
\[
C_n(z) = b_n(P_n(0)Q_{n+1}(z) - P_{n+1}(0)Q_n(z)),
\]
\[
D_n(z) = b_n(P_n(0)P_{n+1}(z) - P_{n+1}(0)P_n(z)),
\]

where \(\{Q_n\}_{n=0}^{\infty}\) is the first associated orthogonal polynomial system defined by the three-term recurrence relation (2) with the initial condition \(Q_0(x) = 0\) and \(Q_1(x) = 1\). In cases where it is possible to express the so-called Nevanlinna matrix

\[
\begin{pmatrix}
A & C \\
C & D
\end{pmatrix}
\]
in terms of known functions, formula (5) is a useful tool for determining explicit measures that solve the indeterminate moment problems (cf. [8–10, 16]). A useful collection of examples of indeterminate moment problems can be found in the recent publication [5, §11.3]. If \(\varphi\) is of the form \(\varphi(z) = t, \ t \in \mathbb{R} \cup \{\infty\}\), the space of polynomials \(\mathbb{C}[x]\) is dense in \(L^2(\mathbb{R}, \mu)\).

An interesting example of an indeterminate moment problem on the real line is the moment problem corresponding to continuous \(q\)-Hermite polynomials [20, (14.26.1)] for \(q > 1\). Askey was the first to give an explicit weight function for continuous \(q\)-Hermite polynomials when \(q > 1\) (cf. [2]). Ismail and Masson studied these polynomials extensively and introduced the name \(q^{-1}\)-Hermite polynomials in [17]. Berg and Ismail [6] showed that continuous \(q\)-Hermite polynomials are fundamental in the hierarchy of classical \(q\)-orthogonal polynomials and can be used to systematically build other \(q\)-orthogonal polynomials by attaching generating functions to measures. They used the solution of the \(q^{-1}\)-Hermite moment problem to derive a family of explicit solutions to a special case of the Al-Salam–Chihara moment problem for \(q > 1\). The indeterminate moment problem for Al-Salam–Chihara polynomials when \(q > 1\) was first discussed by Askey and Ismail in [3]. In [10], Christiansen and Ismail gave more solutions for the \(q^{-1}\)-Hermite moment problem and also discussed the moment problem for a symmetric case of the Al-Salam–Chihara polynomials. The indeterminant moment problems associated with polynomials in the Askey scheme were classified and investigated in [9]. Christiansen and Koelink (cf. [11]) provided an alternative derivation of the N-extremal measures for the continuous \(q^{-1}\)-Hermite and Al-Salam–Chihara polynomials. More recent contributions to the indeterminate Hamburger moment problem associated with Al-Salam–Chihara polynomials are due to Groenevelt (cf. [13]) and Ismail (cf. [14]). In [14], new infinite families of orthogonality measures were provided for \(q^{-1}\)-Hermite polynomials, \(q\)-Laguerre polynomials and Stieltjes–Wigert polynomials.
The moment problem for continuous dual $q$-Hahn polynomials [20, (14.3.1)] when $q > 1$ was first pointed out by Askey and Wilson in [4, p. 31-32] as the limiting case, letting one parameter approach infinity, of the Askey–Wilson polynomials [4, (1.15)] [20, (14.1.1)]

$$
\frac{a^n p_n(x; a, b, c, d|q)}{(ab, ac, ad; q)_n} = 4\phi_3 \left( \begin{array}{c}
q^{-n}, abcdq^{n-1}, ae^{-\theta}, ae^{\theta} \\
ab, ac, ad
\end{array} ; q, q \right), \quad x = \cos \theta. \quad (6)
$$

Here the $q$-shifted factorials are given by

$$(a; q)_0 = 1, \quad (a; q)_k = \prod_{j=0}^{k-1} (1 - aq^j) \quad \text{for } k = 1, 2, \ldots \text{ or } \infty,$$

while the multiple $q$-shifted factorials are defined by

$$(a_1, \ldots, a_i; q)_k = \prod_{j=1}^{i} (a_j; q)_k$$

and

$$s+1\phi_s \left( \begin{array}{c}
a_1, \ldots, a_{s+1} \\
b_1, \ldots, b_s
\end{array} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{s+1}; q)_k}{(b_1, \ldots, b_s; q)_k} \frac{z^k}{(q; q)_k}. \quad (7)$$

The limiting case for Askey–Wilson polynomials for $q > 0$ and $q \neq 1$, as $d$ tends to infinity, is given by (cf. [18, Proposition 3.6(i)])

$$\lim_{d \to \infty} \frac{p_n(x; a, b, c, d|q)}{(ad; q)_n} = (bc)^n q^{n(n-1)} p_n(x; a^{-1}, b^{-1}, c^{-1}|q^{-1}), \quad n = 0, 1, 2, \ldots,$$

where $p_n(x; a, b, c|q)$ denotes continuous dual $q$-Hahn polynomials (cf. [20, (14.3.1)]). The orthogonality relation for $p_n(x; a, b, c|q)$ when $0 < q < 1$ can be found in [20, (14.3.2)].

In the monograph [9], Christiansen proved that the moment problem associated with continuous dual $q$-Hahn polynomials when $q > 1$ is indeterminate, and obtained solutions for the special case when $c = 0$ and $b = -a$. The moment problem for the general case of continuous dual $q$-Hahn polynomials when $q > 1$, for parameter values $a$, $b$ and $c$ such that $ab$, $ac$ and $bc \in (-1, 0)$, was considered by Koelink and Stokman in [21] where they used spectral analysis of a $q$-difference operator to obtain an explicit measure as well as a natural orthogonal basis of the complement of the polynomials in the corresponding weighted $L^2$ space. The solution has an absolutely continuous as well as a discrete part.

In this paper, we use our recent results in [18] to derive certain orthogonality measures of the continuous dual $q$-Hahn polynomials for $q > 1$ and parameter values $a$, $b$ and $c$ with $ab$, $ac$ and $bc \notin [-1, 0]$ obtained as a limit case of the Askey–Wilson polynomials. Note that, since the polynomial in (6) is symmetric in $a$, $b$, $c$ and $d$, it suffices to study the case when $d \to \infty$. We state and prove an orthogonality relation satisfied by the continuous dual $q$-Hahn polynomials. The measures we obtained solve the indeterminate moment problem associated with continuous dual $q$-Hahn polynomials by general theory. That is, we found measures, not necessarily positive, that can be used to compute moments associated with this family of orthogonal polynomials.
2. Continuous dual $q^{-1}$-Hahn polynomials

Assume throughout that $0 < q < 1$. The monic polynomials

$$\frac{1}{\binom{n}{2}} p_n(x; a^{-1}, b^{-1}, c^{-1} | q^{-1})$$

satisfy the three-term recurrence relation (3) with (cf. [18, (3.30)])

$$c_n = \frac{(ab + ac + bc) q^n + q^{n+1} - q - 1}{2abc q^{2n}} \quad \text{and}$$

$$\lambda_n = \frac{(1 - q^n)(1 - bcq^{n-1})(1 - acq^{n-1})(1 - abq^{n-1})}{4a^2b^2c^2 q^{4n-3}}.$$

(8)

For $0 < q < 1$, $\lim_{n \to \infty} c_n = \infty$ and

$$L = \lim_{n \to \infty} \frac{\lambda_{n+1}}{c_n c_{n+1}} = \lim_{n \to \infty} \frac{q(1 - q^{n+1})(1 - bcq^n)(1 - acq^n)(1 - abq^n)}{(ab + ac + bc) q^n + q^{n+1} - q - 1)((ab + ac + bc) q^{n+1} + q^{n+2} - q - 1)} = \frac{q}{(q + 1)^2} < \frac{1}{4}.$$ 

Moreover, since $c_n = f_n q^{-2n}$ with

$$f_n = \frac{(ab + ac + bd) q^n + q^{n+1} - q - 1}{2abc}$$

bounded, it follows that

$$\lim_{n \to \infty} c_n^{\frac{1}{2}} = \frac{1}{q^2} > \frac{1}{q} = \frac{1 + \sqrt{1 - 4L}}{1 - \sqrt{1 - 4L}}.$$ 

Therefore (4) holds and the moment problem associated with the polynomials

$$\frac{1}{\binom{n}{2}} p_n(x; a^{-1}, b^{-1}, c^{-1} | q^{-1})$$

is indeterminate.

Letting $a$, $b$ and $c$ go to infinity in (8), we obtain $c_n = 0$ and $\lambda_n = \frac{1}{4}(1 - \frac{1}{q^n})$, and therefore (3) reads as

$$p_{n+1}(x) = xp_n(x) - \frac{q^{-n}(q^n - 1)}{4} p_{n-1}(x).$$

The corresponding monic polynomial system (cf. [18, Prop. 3.6 (iv)]), known as $q^{-1}$-Hermite polynomials (cf. [17], [16, p. 533]), is orthogonal on the imaginary axis (cf. [2]).

To keep the orthogonality on the real line when one or more parameters tends to infinity in (6), we follow Askey [2] in using the change of variable $x \to ix$, as well as the change of parameters, $(a, b, c) \to (-ia, -ib, -ic)$, or, equivalently, $(a^{-1}, b^{-1}, c^{-1}) \to$
\[(a^{-1}, b^{-1}, c^{-1}),\]
to obtain
\[
q_{n+1}(x; a, b, c|q) = (x - \tilde{c}_n)q_n(x; a, b, c|q) - \tilde{\lambda}_n q_{n-1}(x; a, b, c|q)
\] (9a)

with
\[
\tilde{c}_n = \frac{(ab + ac + bc)q^n - q^{n+1} + q + 1}{2abcq^{2n}},
\] (9b)

\[
\tilde{\lambda}_n = -\frac{1}{4} \left(1 - \frac{1}{q^n}\right) \left(1 + \frac{1}{bcq^{n-1}}\right) \left(1 + \frac{1}{acq^{n-1}}\right) \left(1 + \frac{1}{abq^{n-1}}\right),
\] (9c)

where continuous dual \(q^{-1}\)-Hahn polynomials \(q_n(x; a, b, c|q)\) are defined as
\[
q_n(x; a, b, c|q) = \frac{(-i)^n}{2^n} p_n(ix; ia^{-1}, ib^{-1}, ic^{-1}|q^{-1})
\]
\[
= \left(-\frac{a}{2}\right)^n \left(-\frac{1}{ab}, -\frac{1}{ac}; \frac{1}{q}\right)_n \phi_2 \left(\frac{q^n, -\frac{e^u}{a}, \frac{e^{-u}}{a}}{-\frac{1}{ab}, -\frac{1}{ac}}; \frac{1}{q}, \frac{1}{q}\right),
\] \(x = \sinh(u)\). (10)

The analogue of the Askey-Wilson operator for the parametrization \(x = \sinh(u)\), introduced by Ismail (cf. [15]), is given by
\[
D_q f(x) = \tilde{f}(q^{1/2} e^u) - \tilde{f}(q^{-1/2} e^u) \quad \text{cosh } u
\] (11)

with
\[
\tilde{f}(e^u) = f \left(\frac{e^u - e^{-u}}{2}\right) = f(x).
\]

The divided-difference operator (11), as well as the averaging operator [16, (21.6.3)]
\[
S_q f(x) = \frac{f(q^{1/2} e^u) + f(q^{-1/2} e^u)}{2},
\]
will play a fundamental role in the sequel.

**Proposition 2.1:** The action of the divided-difference operator \(D_q\) defined in (11) on continuous dual \(q^{-1}\)-Hahn polynomial \(q_n(x; a, b, c|q), n = 1, 2, 3, \ldots\) is given by
\[
D_q q_n(x; a, b, c|q) = \gamma_n q_{n-1} \left(x; aq^{1/2}, bq^{1/2}, cq^{1/2}|q\right), \quad \gamma_n = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.
\] (12)
Proof: By definition, the action of the operator $\mathcal{D}_q$ on $v_k(u, a; q) = \left(-\frac{e^u}{a}, \frac{e^{-u}}{a}; \frac{1}{q}\right)_k$, $k = 1, 2 \cdots$ is

\[
\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) \left(\frac{e^u + e^{-u}}{2}\right) \mathcal{D}_q v_k(u, a; q)
\]

\[
= \left(-\frac{e^u q^{\frac{1}{2}}}{a}, \frac{e^{-u}}{aq^{\frac{1}{2}}}; \frac{1}{q}\right)_k - \left(-\frac{e^u}{a}, \frac{e^{-u} q^{\frac{1}{2}}}{aq}; \frac{1}{q}\right)_k
\]

\[
= \left(1 + \frac{e^u q^{\frac{1}{2}}}{a}\right) \cdots \left(1 + \frac{e^u}{aq^{\frac{1}{2}}}ight) \left(1 - \frac{e^{-u} q^{\frac{1}{2}}}{aq^2 - 1}\right) \cdots \left(1 - \frac{e^{-u}}{aq^{\frac{1}{2}} - 1}\right)
\]

\[
= \left(1 + \frac{e^u q^{\frac{1}{2}}}{a}\right) \cdots \left(1 + \frac{e^u}{aq^{\frac{1}{2}}}ight) \left(1 - \frac{e^{-u}}{aq}ight) \cdots \left(1 - \frac{e^{-u} q^{\frac{1}{2}}}{aq^{\frac{1}{2}} - 1}\right)
\]

\[
\times \left[\left(1 + \frac{e^u q^{\frac{1}{2}}}{a}\right) \left(1 - \frac{e^{-u}}{aq^{\frac{1}{2}} - 1}\right) - \left(1 + \frac{e^u}{aq^{\frac{1}{2}} - 1}\right) \left(1 - \frac{e^{-u} q^{\frac{1}{2}}}{aq^2 - 1}\right)\right]
\]

\[
v_{k-1}(u, a q^{\frac{1}{2}}; q) \frac{1}{a} \left[\frac{e^u q^{\frac{1}{2}} - e^{-u}}{q - 1} - \frac{e^u}{q^{\frac{1}{2}} - 1} + e^{-u} q^{\frac{1}{2}}\right]
\]

\[
v_{k-1}(u, a q^{\frac{1}{2}}; q) \frac{1}{a} \left[\left(e^u + e^{-u}\right)\left(q^\frac{1}{2} - q^{-\frac{1}{2}}\right)\right],
\]

which yields the relation

\[
\mathcal{D}_q v_k(u, a; q) = \frac{2}{aq^{k-1}} \left(q^k - 1\right) v_{k-1}(u, a q^{\frac{1}{2}}; q), \quad k \geq 1.
\]

Use (10) as well as (7) to expand $q_n(x; a, b, c|q)$ in terms of $v_k(u, a; q)$ and apply the operator $\mathcal{D}_q$ to obtain, after some computation, (12). \hfill \blacksquare

Lemma 2.2: Let $x(s) = \frac{1}{2}(q^s - q^{-s})$, $q^s = e^u$, $\alpha_n = \frac{1}{2}(q^{\frac{n}{2}} + q^{-\frac{n}{2}})$ and

\[
\gamma_n = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \quad \text{for} \quad n = 0, 1, \ldots
\]

Then polynomial solutions $P_n(x)$ of degree exactly $n$ of the Sturm–Liouville type equation

\[
\phi(x) \mathcal{D}_q^2 y(x) + \psi(x) \mathcal{S}_q \mathcal{D}_q y(x) + \lambda y(x) = 0,
\]

where $\phi(x) = \phi_2 x^2 + \phi_1 x + \phi_0$ and $\psi(x) = \psi_1 x + \psi_0$ are polynomials of degree at most two and one, can be expanded as

\[
P_n(x) = \sum_{k=0}^{n} d_k \prod_{j=0}^{k-1} \left[x(s) - x(\mu + j)\right],
\]
where $\mu$ is a complex number such that $\sigma(x(\mu)) = 0$ with

$$\sigma(x(s)) = \phi(x(s)) - \frac{x(s + \frac{1}{2}) - x(s - \frac{1}{2})}{2} \psi(x(s))$$

and $d_k$ is solution to the first order recurrence relation

$$
\left( \gamma_k \gamma_{k+1} \left( \phi_2(x(\mu + k) + x(\mu)) + \phi_1 - \frac{\psi_1(x(\mu + \frac{1}{2}) - x(\mu - \frac{1}{2}))}{2} \right) 
+ \alpha_k \gamma_{k+1} \psi(x(\mu + k)) \right) d_{k+1} 
+ (\lambda + \gamma_k \gamma_{k-1} \phi_2 + \gamma_k \alpha_{k-1} \psi_1) d_k = 0
$$

with $\lambda = -\gamma_n \gamma_{n-1} \phi_2 - \gamma_n \alpha_{n-1} \psi_1$.

**Proof:** See [19, Lemma 3.1] for a proof, observing that

$$
D_qf(x) = \frac{f(x(s + \frac{1}{2})) - f(x(s - \frac{1}{2}))}{x(s + \frac{1}{2}) - x(s - \frac{1}{2})} = \mathbb{D}_xf(x(s)),
$$

(14a)

$$
S_qf(x) = \frac{f(x(s + \frac{1}{2})) + f(x(s - \frac{1}{2}))}{2} = S_xf(x(s)),
$$

(14b)

when $x(s) = \frac{1}{2}(q^s - q^{-s})$ with $q^s = e^{\alpha}$.

**Proposition 2.3:** Continuous dual $q^{-1}$-Hahn polynomials $q_n(x; a, b, c|q)$, $n = 0, 1, 2, \ldots$ solve the Sturm–Liouville type equation

$$
\phi(x) D^2_{q,y}(x) + \psi(x) S_{q,y}(x) + \lambda y(x) = 0,
$$

(15)

with

$$
\phi(x) = 2 x^2 + \left( \frac{1}{abc} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} \right) x + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} + 1,
$$

(16a)

$$
\psi(x) = \frac{4\sqrt{q}}{q-1} x - 2 \sqrt{q} \frac{(bc + ac + ab + 1)}{abc (q-1)},
$$

(16b)

$$
\lambda = -4 \frac{\sqrt{q}}{q-1} \frac{(q^n - 1)}{(q-1)^2}.
$$

(16c)
**Proof:** Observing that, for $x(s) = \frac{1}{2} (q^s - q^{-s})$, $q^s = e^u$ and $q^n = -a$,

$$v_k(u, a; q) = \binom{2}{a} q^{-k(k+1)/2} \prod_{j=0}^{k-1} [x(s) - x(\eta + j)]$$

it follows that

$$q_n(x; a, b, c|q) = \sum_{k=0}^{n} d_k \prod_{j=0}^{k-1} [x(s) - x(\eta + j)],$$

where

$$d_k = \frac{(-a)^n}{2^n} \left( -\frac{1}{ab}, -\frac{1}{ac}, \frac{1}{q} \right) \frac{(q^n; \frac{1}{q})_k (\frac{2}{a} q^{-\frac{k+1}{2}})_k}{n \left( \frac{1}{ab}, -\frac{1}{ac}, \frac{1}{q} \right) \left( \frac{1}{q}, \frac{1}{q} \right)_k}, \quad k = 0, 1, 2, \ldots \quad (17)$$

Assuming that $q_n(x, a, b, c|q), n = 0, 1, 2, \ldots$ satisfies (15) with $\phi(x) = \phi_2 x^2 + \phi_1 x + \phi_0$ and $\psi(x) = \psi_1 x + \psi_0$, the use of (13) with $d_k$ given by (17), leads to

$$\sum_{j=1}^{4} H_j(\phi_2, \phi_1, \phi_0, \psi_1, \psi_0, \lambda, q^n)(q^k)^j = 0,$$

where $H_j, j = 1, 2, 3, 4$, is a linear combination of $\phi_2, \phi_1, \phi_0, \psi_1, \psi_0, \text{and } \lambda$. Solving the system of linear equations $H_j(\phi_2, \phi_1, \phi_0, \psi_1, \psi_0, \lambda, q^n) = 0$ in terms of the coefficients $\phi_2, \phi_1, \psi_1$ and $\psi_0$, we obtain

$$\phi_2 = -\frac{(q - 1)^2 \lambda}{2 \sqrt{q} (q^n - 1)}, \quad \phi_1 = \frac{(q - 1)^2 (ab + ac + bc - 1) \lambda}{4abc (q^n - 1) \sqrt{q}},$$

$$\psi_1 = -\frac{(q - 1) \lambda}{q^n - 1}, \quad \psi_0 = \frac{(q - 1) (ab + ac + bc + 1) \lambda}{2abc (q^n - 1)}.$$

Since these coefficients do not depend on $n$, taking

$$\lambda = -\frac{4 \sqrt{q} (q^n - 1)}{(q - 1)^2},$$

we get $\phi_2, \phi_1, \psi_1$ and $\psi_0$ as given in the theorem. Substituting $\phi(x)$ and $\psi(x)$ into (15) with $y(x) = q_n(x, a, b, c|q)$, and taking $n = 2$ we derive $\phi_0$. 

**Proposition 2.4:** If $\{q_n(x; a, b, c|q)\}_{n=0}^{\infty}$ is orthogonal with respect to a weight function $w(x)$ then $\{D_q^2 q_n(x; a, b, c|q)\}_{n=2}^{\infty}$ is orthogonal with respect to $\pi(x)w(x)$, where $\pi(x)$ is the polynomial defined by

$$\pi(x) = \frac{8}{abc} \left( x + \frac{a - a^{-1}}{2} \right) \left( x + \frac{b - b^{-1}}{2} \right) \left( x + \frac{c - c^{-1}}{2} \right).$$
Proof: Since $q_n(x; a, b, c|q)$, $n = 0, 1, 2, \ldots$, satisfies (15), it follows from (14a) and [19, Thm 4.2] that $\{D^2_qq_n(x; a, b, c|q)\}_{n \geq 2}$ is orthogonal with respect to $\pi(x)w(x)$, where (cf. [19, Remark 4.7])

$$\pi(x) = \phi(x)^2 - U_2(x)\psi(x)^2$$

(18)

and

$$U_2(x) = \left(\frac{x(s + \frac{1}{2}) - x(s - \frac{1}{2})}{2}\right)^2, \quad x(s) = \frac{q^s - q^{-s}}{2}, q^s = e^u$$

is the polynomial given in [19, p.5]. Observing that $e^u = x + \sqrt{1 + x^2}$ and $e^{-u} = \sqrt{1 + x^2} - x$, we see that

$$U_2(x) = \frac{1}{16}(e^u + e^{-u})^2(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2$$

$$= \frac{1}{4q}(1 + x^2)(q - 1)^2.$$  

(19)

Substituting (19) and the expressions for $\phi$ and $\psi$, given in (16a), into (18), we obtain the result.  

3. Orthogonality measures for continuous dual $q^{-1}$-Hahn polynomials

Theorem 3.1: Let $a$, $b$ and $c$ be real numbers such that $ab, ac, bc \notin \{-q^{-k}, k = 1, 2, \ldots\}$ and $ab, ac, bc \notin [-1, 0]$. Then the sequence of continuous dual $q^{-1}$-Hahn polynomials $\{q_n(x; a, b, c|q)\}_{n}$ defined in (10) is orthogonal with respect to the distribution $d\mu(x) = w(x; a, b, c)dx$, where

$$w(x; a, b, c) = N(x; a, b, c)w(x), \quad x = \sinh(u)$$

(20)

with

$$N(x; a, b, c) = \left(-\frac{q}{a}e^u, \frac{q}{a}e^{-u}, -\frac{q}{b}e^u, \frac{q}{b}e^{-u}, -\frac{q}{c}e^u, \frac{q}{c}e^{-u}, q\right)_{\infty}$$

and $w(x)$ a $q^{-1}$-Hermite weight.

Proof: The continuous dual $q^{-1}$-Hahn polynomials satisfy the three-term recurrence relation (9a). Since $ab, ac, bc \notin \{-q^{-k}, k = 1, 2, \ldots\}$ and $ab, ac, bc \notin [-1, 0]$, $\lambda_n > 0$ for all $n \in \mathbb{N}$, it follows from the spectral theorem for monic orthogonal polynomials that there exists a positive measure $\mu$ on the real line such that these polynomials are monic orthogonal polynomials satisfying

$$\int_{\mathbb{R}} q_n(x; a, b, c|q) q_m(x; a, b, c|q) \, d\mu(x) = k_n\delta_{m,n}, \quad m, n = 0, 1, 2, \ldots$$

Since $\lim_{n \to \infty} \lambda_n = \infty$, the measure $\mu$ has infinite, unbounded support. We look for $d\mu(x)$ of the form $d\mu(x) = w(x; a, b, c)dx$, for a function depending continuously on the
parameters $a$, $b$ and $c$. It follows from (12) that we have
\[ D_q^2 q_n(x; a, b, c|q) = \gamma_n \gamma_{n-1} q_{n-2} (x; aq, bq, cq|q). \] (21)

Therefore, the sequence \( \{D_q^2 q_n(x; a, b, c|q)\}_{n=2}^\infty \) is orthogonal with respect to \( w(x; aq, bq, cq) \). Let us prove that
\[ w(x; aq, bq, cq) = (1 + \frac{e^u}{a})(1 - \frac{e^{-u}}{a})(1 + \frac{e^u}{b})(1 - \frac{e^{-u}}{b})(1 + \frac{e^u}{c})(1 - \frac{e^{-u}}{c})w(x; a, b, c). \] (22)

Let \( n \) be an integer. The formal Fourier expansion of
\[ \frac{w(x; aq, bq, cq)}{w(x; a, b, c)} D_q^2 q_n(x; a, b, c|q) \]

in the system \( \{q_k\}_{k=0}^\infty \) is
\[ \frac{w(x; aq, bq, cq)}{w(x; a, b, c)} D_q^2 q_n(x; a, b, c|q) = \sum_{k=0}^\infty a_{n,k} q_k(x; a, b, c|q) \]

with
\[ a_{n,k} \int_{-\infty}^{\infty} w(x; a, b, c) (q_k(x; a, b, c|q))^2 \, dx \]
\[ = \int_{-\infty}^{\infty} w(x; aq, bq, cq) D_q^2 q_n(x; a, b, c|q) q_k(x; a, b, c|q) \, dx. \]

Since \( \{D_q^2 q_n(x; a, b, c|q)\}_{n=2}^\infty \) is orthogonal with respect to \( \pi(x) w(x; a, b, c) \) by Proposition 2.4, and \( \pi \) is a polynomial of degree at most four, there exist constants \( b_{n,n+j}, j \in \{-2, -1, 0, 1, 2\} \) such that (cf. [19, Prop 4.4])
\[ q_k(x; a, b, c|q) = \sum_{j=-2}^{2} b_{k,k+j} D_q^2 q_{k+j}(x; a, b, c|q). \]

Therefore
\[ a_{n,k} \int_{-\infty}^{\infty} w(x; a, b, c) q_k(x; a, b, c|q)^2 \, dx \]
\[ = \sum_{j=-2}^{2} b_{k,k+j} \int_{-\infty}^{\infty} w(x; aq, bq, cq) D_q^2 q_n(x; a, b, c|q) D_q^2 q_{k+j}(x; a, b, c|q) \, dx \]
\[ = 0, \text{ when } k + j \neq n, j = -2, -1, 0, 1, 2. \]

Hence \( a_{n,k} = 0 \) for \( k \notin \{n - 2, n - 1, n, n + 1, n + 2\} \) and
\[ \frac{w(x; aq, bq, cq)}{w(x; a, b, c)} D_q^2 q_n(x; a, b, c|q) = \sum_{k=n-2}^{n+2} a_{n,k} q_k(x; a, b, c|q). \]
From (21), for \( n = 2 \), we have
\[
\frac{w(x; aq, bq, cq)}{w(x; a, b, c)} \gamma_1 = \sum_{k=0}^{4} a_{2,k} q_k(x; a, b, c|q).
\]

This implies that \( \frac{w(x; aq, bq, cq)}{w(x; a, b, c)} \) is a polynomial of degree at most four and hence there exist a polynomial, \( Q(x) \), of degree at most four, such that \( w(x; aq, bq, cq) = Q(x)w(x; a, b, c) \). Combining with [19, Thm 4.2] and [19, Remark 4.7], we have
\[
Q(x) = \phi(x)^2 - U_2(x)\psi(x)^2. \tag{23}
\]

Substituting (19) and the expressions for \( \phi \) and \( \psi \), given in (16a), into (23), we obtain the result in (22).

Finally, substituting \((a, b, c)\) by \((aq^{-1}, bq^{-1}, cq^{-1})\) in (22) and then iterating the relation obtained, yields
\[
w(x; a, b, c)
= \prod_{j=1}^{n} \left(1 + \frac{q^j e^u}{a}(1 - \frac{q^j e^{-u}}{a})(1 + \frac{q^j e^{-u}}{b}(1 - \frac{q^j e^u}{b})(1 + \frac{q^j e^u}{c}(1 - \frac{q^j e^{-u}}{c})
\right)
\]
\[
w(x; aq^{-n}, bq^{-n}, cq^{-n}).
\]

Now letting \( n \) tend to \( \infty \) and using the definition of \( q \)-shifted factorial, the latter relation becomes
\[
w(x; a, b, c) = \left(-\frac{q}{a} e^u, \frac{q}{a} e^{-u}, -\frac{q}{b} e^u, \frac{q}{b} e^{-u}, -\frac{q}{c} e^u, \frac{q}{c} e^{-u}, q\right)_\infty
\times \lim_{n \to \infty} w(x; aq^{-n}, bq^{-n}, cq^{-n}).
\]

Since \( q_n(x; a, b, c|q) \) tends to \( q^{-1} \)-Hermite as all the parameters \( a, b \) and \( c \) go to \( \infty \),
\[
\lim_{n \to \infty} w(x; aq^{-n}, bq^{-n}, cq^{-n}), \quad 0 < q < 1
\]
is a continuous \( q^{-1} \)-Hermite weight.

We are now ready to prove the orthogonality relation for continuous dual \( q^{-1} \)-Hahn polynomials.

**Theorem 3.2:** Let \( a, b \) and \( c \) be real numbers such that \( ab, ac, bc \not\in \{-q^{-k}, k = 1, 2, \ldots\} \) and \( ab, ac, bc \not\in [-1, 0] \). Then the sequence of continuous dual \( q^{-1} \)-Hahn polynomials \( \{q_n(x; a, b, c|q)\}_n \) satisfies the orthogonality relation
\[
\int_{-\infty}^{\infty} q_n(x; a, b, c|q) q_m(x; a, b, c|q) w(x; a, b, c) \, dx = k_n \delta_{m,n}, \tag{24}
\]
where
\[
k_n = 4^{-n} q^{-\frac{n(n+1)}{2}} \left(-\frac{q}{abq^n}, -\frac{q}{acq^n}, -\frac{q}{bcq^n}; q\right)_\infty \left(q^n; \frac{1}{q}\right)_n > 0.
\]
**Proof:** Let be \( m \) and \( n \) two non-negative integers such that \( m \leq n \). We first evaluate the integral

\[
\int_{-\infty}^{\infty} q_n(x; a, b, c|q) \left( \frac{-e^u}{b}, \frac{e^{-u}}{b}; \frac{1}{q} \right)_m w(x; a, b, c) \, dx
\]

(25)

then derive the result by using the fact that \( q_n(x; a, b, c|q)_n \) is symmetric in \( (a, b, c) \) (i.e. permutations of \( a, b \) and \( c \) leave \( q_n(x; a, b, c|q) \) unchanged). Let us evaluate (25): Observe that \( e^u = x + \sqrt{1 + x^2} \) and \( e^{-u} = \sqrt{1 + x^2} - x \) and take \( t_1 = \frac{a}{b}, t_2 = \frac{q}{b}, t_3 = \frac{q}{c} \) and \( t_4 = 0 \) into [17, (3.8)] to obtain

\[
\int_{-\infty}^{\infty} \left( -\frac{q}{a}, \frac{q}{a}; \frac{1}{q} \right)_k \left( -\frac{q}{b}, \frac{q}{b}; \frac{1}{q} \right)_k w(x; a, b, c) = N \left( x; aq^k, bq^m, c \right) w(x),
\]

where \( k \) and \( m \) are non-negative integers. Therefore, expanding \( q_n(x; a, b, c|q) \) by use of (7) and (10), we obtain

\[
\int_{-\infty}^{\infty} q_n(x; a, b, c|q) \left( \frac{-e^u}{b}, \frac{e^{-u}}{b}; \frac{1}{q} \right)_m w(x; a, b, c) \, dx
\]

\[
= \left( -\frac{a}{2} \right)^n \left( -\frac{1}{ab}, \frac{1}{ac}; \frac{1}{q} \right) \sum_{k=0}^{n} \frac{(q^n; \frac{1}{q})}{(\frac{1}{q})} \int_{-\infty}^{\infty} N \left( x; aq^k, bq^m, c \right) w(x) \, dx.
\]

Applying (26) with \( (a, b, c) \) taken for \( (aq^k, bq^m, c) \) and using (27), first with \( (\alpha, l) \) taken as \( (-\frac{1}{ac}, k) \) and then \( (\alpha, l) \) taken as \( (-\frac{1}{abq^m}, k) \), we obtain

\[
\int_{-\infty}^{\infty} N \left( x; aq^k, bq^m, c \right) w(x) \, dx = \left( -\frac{1}{abq^m}, -\frac{1}{ac}; \frac{1}{q} \right) \left( -\frac{q}{abq^m}, -\frac{q}{bcq^m}; \frac{1}{q} \right).
\]

Therefore

\[
\int_{-\infty}^{\infty} q_n(x; a, b, c|q) \left( \frac{-e^u}{b}, \frac{e^{-u}}{b}; \frac{1}{q} \right)_m w(x; a, b, c) \, dx
\]

\[
= \left( -\frac{a}{2} \right)^n \left( -\frac{1}{ab}, -\frac{1}{ac}; \frac{1}{q} \right) \frac{(q^n; \frac{1}{q})}{(\frac{1}{q})} \left( -\frac{q}{abq^m}, -\frac{q}{bcq^m}; \frac{1}{q} \right)_\infty
\]

\[
2\phi_1 \left( \frac{q^n}{-\frac{1}{abq^m}; \frac{1}{q}} \right).
\]
Using the $q$-analogues of Vandermonde’s formula [12, (1.5.3)] with $(b, c, q)$ substituted by $(\frac{-1}{abq^m}, \frac{-1}{ab}, \frac{1}{q})$ we obtain

\[
2\phi_1 \left( \begin{array}{c} q^n, -\frac{1}{abq^m} \\ -\frac{1}{ab} & \frac{1}{q} \end{array} ; \frac{1}{q} \right)_n = \left( \frac{q^n; \frac{1}{q}}{(-\frac{1}{ab}; \frac{1}{q})_n} \frac{1}{abq^m} \right)^n.
\]

Therefore taking into account (27) with $\alpha = -\frac{1}{ac}$ and $l = n$, we obtain

\[
\int_{-\infty}^{\infty} q_n(x; a, b, c|q) \left( \frac{-e^u}{b}, \frac{e^{-u}}{b}; \frac{1}{q} \right)_m w(x; a, b, c) \, dx = \left( \frac{1}{2bq^m} \right)^n \left( -\frac{q}{abq^m}, -\frac{q}{bcq^m}, -\frac{q}{acq^m}; q \right)_{\infty} \left( q^n; \frac{1}{q} \right)_n. \tag{28}
\]

Since $q_m$ is symmetric in $(a, b, c)$, interchanging $a$ and $b$ into $q_m(x; a, b, c|q)$ and expanding using (7) we obtain

\[
q_m(x; a, b, c|q) = \left( \frac{-b}{2} \right)^m \left( \frac{q^n; \frac{1}{q}}{(-\frac{1}{q}; \frac{1}{q})_m} \frac{q^{-m}}{m} \right) \left( -\frac{e^u}{b}, \frac{e^{-u}}{b}; \frac{1}{q} \right)_m + \ldots.
\]

Taking into account the relation $(q^m; q^{-1})_m = (-1)^m q^{-\frac{m(m+1)}{2}} \left( \frac{1}{q}; \frac{1}{q} \right)_m$, $m = 0, 1, 2, \ldots$, we obtain after simplification

\[
q_m(x; a, b, c|q) = \left( \frac{b}{2} \right)^m q^{-\frac{m(m-1)}{2}} \left( -\frac{e^u}{b}, \frac{e^{-u}}{b}; \frac{1}{q} \right)_m + \ldots.
\]

Combining the previous relation with (28) we obtain (24). Finally, since $ab, ac, bc \not\in [-1, 0]$ implies that

\[
\left( -\frac{q}{abq^m}, -\frac{q}{acq^m}, -\frac{q}{bcq^m}; q \right)_{\infty} > 0,
\]

we have that $k_n > 0$. \[■\]

**Remark 3.3:** (1) Since a continuous $q^{-1}$-Hermite weight is [16, (21.6.13)]

\[
w(x) = \frac{e^u}{(-e^{2u}, -qe^{-2u}; q)_{\infty}}, x = \sinh(u), \tag{29}
\]

the orthogonality measure $d\mu = w(x; a, b, c) \, dx$ with

\[
w(x; a, b, c) = \frac{e^u \left( -\frac{q}{a} e^{iu}, \frac{q}{a} e^{-iu}, -\frac{q}{b} e^{iu}, \frac{q}{b} e^{-iu}, -\frac{q}{c} e^{iu}, \frac{q}{c} e^{-iu}; q \right)_{\infty}}{(-e^{2u}, -qe^{-2u}; q)_{\infty}}, \tag{30}
\]

$x = \frac{1}{2}(e^u - e^{-u})$, $u \in \mathbb{R}$ is a solution to the indeterminate moment problem for continuous dual $q^{-1}$-Hahn polynomials in a general sense. However, letting $c$ tend to $+\infty$
in (30) and substituting \((a^{-1}, b^{-1})\) by \((a, b)\) we obtain

\[
w(x, a, b) = \frac{e^u \left(-qae^u, qae^{-u}, -qbe^u, qbe^{-u}; q\right)}{(-e^{2u}, -qe^{-2u}; q)_{\infty}} = \frac{1}{2} w_{ab}(x, a, b), \quad x = \sinh(u),
\]

where \(w_{ab}(x, a, b)\), for \(b = \bar{a}, \Im a \neq 0\) is the weight function \([14, (3.9)]\) for a solution to the indeterminate moment problem associated with Al-Salam Chihara polynomials.

(2) Letting \((ia, ib, ic, e^u) = (t_0^{-1}, t_1^{-1}, t_2^{-1}, iy),\) with \(i^2 = -1,\) in (10), we obtain up to a multiplicative factor the polynomial in \([21, (9.8)]\). The condition in Theorem 3.1 that \(ab, ac, bc \notin [-1, 0]\) therefore corresponds to \(t_it_j < 1\) in the notation used in \([21]\). Since the results in \([21]\) hold for parameters \(t_0, t_1, t_2\) with \(t_i > 0\) and \(t_it_j > 1\) when \(i \neq j\), our work contributes to the general theory for the indeterminate moment problem associated with continuous dual \(q^{-1}\)-Hahn polynomials for parameter values that have not been considered in \([21]\).

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**References**

[1] N.I. Akhiezer The Classical Moment Problem and Some Related Questions in Analysis, Oliver and Boyd, Edinburgh, 1965. English translation.

[2] R.A. Askey, Continuous \(q\)-Hermite polynomials when \(q > 1\), in \(q\)-Series and Partitions, D. Stanton, ed., IMA Vol. Math. Appl., Springer-Verlag, New York, 1989, pp. 151–158.

[3] R.A. Askey and M.E.H. Ismail, Recurrence relations, continued fractions and orthogonal polynomials, Mem. Amer. Math. Soc. 300 (1984), pp. 114.

[4] R.A. Askey and J.A. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Mem. Amer. Math. Soc. 319 (1985), pp. 55.

[5] C. Berg and J.S. Christiansen, The Moment Problem, in Encyclopedia of Special Functions. The Askey–Bateman Project Volume I. Univariate Orthogonal Polynomials. M. E. H. Ismail, ed., Cambridge University Press, Cambridge, UK, 2020, pp. 269–306.

[6] C. Berg and M.E.H. Ismail, \(q\)-Hermite polynomials and classical orthogonal polynomials, Can. J. Math. 48(1) (1996), pp. 43–63.

[7] T.S. Chihara, Hamburger moment problems and orthogonal polynomials, Trans. Amer. Math. Soc. 315(1) (1989), pp. 189–203.

[8] T.S. Chihara and M.E.H. Ismail, Extremal measures for a system of orthogonal polynomials, Constr. Approx. 9(1) (1993), pp. 111–119.

[9] J.S. Christiansen, Indeterminate Moment Problems within the Askey-scheme, University of Copenhagen, Denmark, 2004.

[10] J.S. Christiansen and M.E.H. Ismail, A moment problem and family of integral evaluations, Trans. Amer. Math. Soc. 358 (2006), pp. 4071–4097.

[11] J.S. Christiansen and E. Koelink, Self-Adjoint difference operators and symmetric al-Salam-Chihara polynomials, Constr. Approx. 28 (2008), pp. 199–218.
[12] G. Gasper and M. Rahman, Basic hypergeometric series, 2nd ed., Encyclopedia of Mathematics and Its Applications, Vol. 96, Cambridge University Press, Cambridge, 2004.
[13] W. Groenvelt, A solution to the Al-Salam-Chihara Moment Problem, in Positivity and Non-commutative Analysis, G. Buskes et al. eds., Trends in Mathematics, Birkhäuser, Cham, 2019, pp. 223–248.
[14] M.E.H. Ismail, Solutions of the Al-Salam-Chihara and allied moment problems, Anal. Appl. 18(02) (2020), pp. 185–210.
[15] M.E.H. Ismail, Ladder operators for \( q^{-1} \)-Hermite polynomials, C.R. Math. Rep. Acad. Sci. Canada. 15(6) (1993), pp. 261–266.
[16] M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Encyclopedia of Mathematics and its Applications; Vol. 98, Cambridge University Press, Cambridge, 2005.
[17] M.E.H. Ismail and D. Masson, \( q \)-Hermite polynomials, biorthogonal rational functions, and \( q \)-beta integrals, Trans. Amer. Math. Soc. 346 (1994), pp. 63–116.
[18] M. Kenfack-Nangho and K. Jordaan, A characterization of Askey–Wilson polynomials, Proc. Amer. Math. Soc. 147 (2019), pp. 2465–2480.
[19] M. Kenfack-Nangho and K. Jordaan, Structure relations of classical orthogonal polynomials in the quadratic and \( q \)-Quadratic variable, SIGMA. 14 (2018), pp. 126. 26 pp.
[20] R. Koekoek, P.A. Lesky, and R.F. Swarttouw, Hypergeometric Orthogonal Polynomials and Their \( q \)-analogues, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010.
[21] E. Koelink and J.V. Stokman, The big \( q \)-Jacobi function transform, Constr. Approx. 19(2) (2003), pp. 191–235.
[22] R.H. Nevanlinna, Asymptotische entwicklungen beschränkter funktionen und das stieltjessche momentenproblem, Ann. Acad. Sci. Fenn. (A). 18(5) (1922), pp. 1–52.