Multicopy metrology with many-particle quantum states

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We consider quantum metrology with several copies of bipartite and multipartite quantum states. We characterize the metrological usefulness by determining how much the state outperforms separable states. We identify a large class of entangled states that become maximally useful for metrology in the limit of infinite number of copies. The maximally achievable metrological usefulness is attained exponentially fast in the number of copies. We show that, on the other hand, pure entangled states with even a small amount of white noise do not become maximally useful even in the limit of infinite number of copies. We also make general statements about the usefulness of a single copy of pure entangled states. We show that the multiqubit states presented in Hyllus et al. [Phys. Rev. A 82, 012337 (2010)], which are not useful, become useful if we embed the qubits locally in qudits. We discuss the relation of our scheme to error correction, and possible use for quantum information processing in a noisy environment.

Quantum entanglement plays a central role in quantum physics, as well as in quantum information processing applications [1–3]. In quantum metrology, it is known that quantum entanglement is needed to surpass the classical limit in the precision of parameter estimation [4]. However, there are even highly entangled pure states that are not useful for metrology [5], while weakly entangled bound entangled states can still be better than separable states [6–8]. It is natural to ask, whether all entangled states can outperform separable states, if some more general scheme is considered which is nevertheless still relevant from the practical point of view. Such a scheme is when we consider several copies of the bipartite or the multipartite state.

In this paper, we will show that for $N$ qudits of dimension $d$, there is a class of entangled states of at most rank-$d$ that become maximally useful compared to separable states in the limit of infinite number of copies. On the other hand, pure entangled multiqubit states in this class are not all useful metrologically (i.e., not more useful than separable states) in the single copy case, however, they can be made useful if we embed the qubits locally in qudits. We will also show that entangled states outside of this class might not be maximally useful even in the limit of infinite number of copies. Adding further copies does not even necessarily help after the optimal number of copies is reached. There are even highly entangled pure states that are not useful in the single copy nor in the multicopy case. We will also discuss relations to error correction, namely, that states outside of the class mentioned above can be brought back to this class by simple operations. Our approach, stemming from the idea of activation in quantum information theory [9–12] might offer a viable information processing scenario in the noisy, intermediate-scale quantum (NISQ) era [13], in which simple error mitigation techniques are needed [14–25].

Quantum metrology. Before discussing our main results, we review some of the fundamental relations of quantum metrology. A basic metrological task in a linear interferometer is estimating the small angle $\theta$ for a unitary dynamics $U_\theta = \exp(-i\theta H)$, where the Hamiltonian is the sum of local terms. In particular, for $N$-partite systems it is

$$\mathcal{H} = h_1 + h_2 + \ldots + h_N, \quad (1)$$

where $h_n$ for $n = 1, 2, \ldots, N$ are single-subsystem operators [26]. The precision is limited by the Cramér-Rao bound as [27–37]

$$\langle \Delta \theta \rangle^2 \geq \frac{1}{\nu F_Q[\rho, \mathcal{H}]}, \quad (2)$$

where $\nu$ is the number of independent repetitions, and the quantum Fisher information, a central quantity in quantum metrology is defined by the formula [27–31]

$$F_Q[\rho, \mathcal{H}] = 2 \sum_{k,l} (\lambda_k - \lambda_l)^2 \frac{1}{\lambda_k + \lambda_l} |\langle k|\mathcal{H}|l\rangle|^2. \quad (3)$$

Here, $\lambda_k$ and $|k\rangle$ are the eigenvalues and eigenvectors, respectively, of the density matrix $\rho$, which is used as a probe state for estimating $\theta$.

We are interested in the ratio of the quantum Fisher information of a state and the maximum of the quantum Fisher
information for the same Hamiltonian for separable states, which we call the metrological gain for that particular unitary dynamics [38]

\[ g_N(\rho) = \frac{\mathcal{F}_Q[\rho, \mathcal{H}]}{\mathcal{F}_Q[\text{sep}](\mathcal{H})}. \]  

(4)

The maximum for separable states is given as

\[ \mathcal{F}_Q[\text{sep}](\mathcal{H}) = \sum_{n=1}^{N} [\lambda_{\text{max}}(h_n) - \lambda_{\text{min}}(h_n)]^2, \]

(5)

where \( \lambda_{\text{max}}(X) \) and \( \lambda_{\text{min}}(X) \) denote the maximum and minimum eigenvalues of \( X \), respectively. We also define the metrological gain optimized over all local Hamiltonians as

\[ g(\rho) = \max_\mathcal{H} g_N(\rho). \]

(6)

If \( g(\rho) \geq 1 \) then we call the state metrologically useful.

Finally, we mention the Wigner-Yanase skew information [39]

\[ I_\rho(\mathcal{H}) = \text{Tr}(\rho \mathcal{H}^2) - \text{Tr}(\sqrt{\rho} \mathcal{H} \sqrt{\rho} \mathcal{H}), \]

(7)

for which

\[ \mathcal{F}_Q[\rho, \mathcal{H}] \geq 4I_\rho(\mathcal{H}) \]

(8)

holds. It is often easier to calculate \( I_\rho(\mathcal{H}) \) than \( \mathcal{F}_Q[\rho, \mathcal{H}] \).

The setup. We consider the \( N \)-partite state \( \rho \) acting on parties \( A_n \), as shown in Fig. 1. We take \( M \) copies of this state such that the resulting state is still \( N \)-partite. The system consists of the subsystems \( A_n^{(m)} \) for \( m = 1, 2, ..., M \) and \( n = 1, 2, ..., N \). First, we calculate the \( M \)-copy quantum Fisher information \( \mathcal{F}_Q[\rho^{\otimes M}, \mathcal{H}] \). We consider local Hamiltonians of the form

\[ h_n = \otimes_{m=1}^{M} h_{n^{(m)}} \]

(9)

for \( n = 1, 2, ..., N \), as well as more general local Hamiltonians. Then, with the maximum for separable states given in Eq. (5) we calculate the metrological gain defined in Eq. (4).

We already know that any entangled bipartite pure state is maximally useful metrologically, in the limit of infinite number of copies [38]. However, what is the situation in the multiqubit case, relevant for quantum metrology with particle ensembles? What can we tell about the usefulness of mixed states?

Multi-qudit quantum states. Next, we will study a class of entangled multiqubit states that turn out to be maximally useful in the infinite copy limit.

Observation 1. Entangled states of \( N \geq 2 \) qudits of dimension \( d \) are maximally useful in the infinite copy limit if they live in the

\[ \{000, 000, 111, 111, ..., |d-1, d-1, ..., d-1\}\]

(10)

subspace. States that can be transformed to this form with local unitaries have also the same property.

Proof. Let us consider \( \rho^{\otimes M} \) with the Hamiltonian \( \mathcal{H} \), where

\[ \rho = \sum_{k, l=0}^{d-1} c_{kl} |k\rangle \langle l| \otimes N, \quad \mathcal{H} = \sum_{n=1}^{N} (D^{\otimes M}) A_n, \]

(11)

with \( c_{kl} \) being the matrix elements of \( \rho \) in the basis from Eq. (10) and \( D = \text{diag}(+1, -1, +1, -1, ...) \). To simplify the calculation, we use the mapping

\[ \rho \rightarrow \tilde{\rho} = \sum_{k, l=0}^{d-1} c_{kl} |k\rangle \langle l|, \quad \mathcal{H} \rightarrow \tilde{\mathcal{H}} = ND^{\otimes M}, \]

(12)

for which \( \mathcal{F}_Q[\tilde{\rho}^{\otimes M}, \tilde{\mathcal{H}}] = \mathcal{F}_Q[\tilde{\rho}^{\otimes M}, \tilde{\mathcal{H}}] \) holds. We can bound the quantum Fisher information as

\[ \mathcal{F}_Q[\tilde{\rho}^{\otimes M}, \tilde{\mathcal{H}}] \geq 4I_{\tilde{\rho}^{\otimes M}}(\tilde{\mathcal{H}}), \]

(13)

where the Wigner-Yanase skew information is

\[ I_{\tilde{\rho}^{\otimes M}}(\tilde{\mathcal{H}}) = N^2[1 - \text{Tr}(\sqrt{\tilde{\rho}} D \sqrt{\tilde{\rho}} D)^M]. \]

(14)

Simple algebra shows that the state is always maximally useful in the limit of infinitely many copies, if it is entangled. The skew information given in Eq. (14) converges to the maximum except if \( [\sqrt{\tilde{\rho}}, D] = 0 \), which is equivalent to \( c_{kl} = 0 \) for all \( k \neq l \). Such a state is a mixture of product states. If \( c_{01} = c_{11} = 1/2 \) and \( d = 2 \) then we can even have an explicit formula

\[ I_{c_{01}, N} = N^2[1 - (1 - 4|c_{01}|^2)^{M/2}]. \]

(15)

The details of the derivation are in the Supplemental Material [40].

The method given in Observation 1 can be used to calculate precision for the multicopy metrology with the state

\[ \rho_p = p|\text{GHZ}\rangle \langle \text{GHZ}| + (1 - p) |0\rangle \langle 0|^N + |1\rangle \langle 1|^N, \]

(16)
where the Greenberger-Horne-Zeilinger (GHZ) state is [50]

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\ldots00\rangle + |111\ldots11\rangle),$$

(17)

thus the state turns out to be maximally useful in the limit of very many copies. Such a state appears in ion trap experiments [51–53].

Observation 1 also implies that all entangled pure states of the form

$$\sum_{k=0}^{d-1} |\sigma_k|^2 = 1$$

(18)

with \(\sum_k |\sigma_k|^2 = 1\) are maximally useful in the infinite copy limit.

**Proof.** Pure states given in Eq. (18) are subset of states considered in Observation 1. Among such states, only states of the type \(|k\rangle^\otimes N\) have been found useless in the infinite copy limit.

Surprisingly, in the single copy case not all entangled states of the type Eq. (18) are useful. A single copy of such a state for \(d = 2\) is not useful metrologically if \(\sigma_1\) is sufficiently small and \(N \geq 3\) [5]. For a short proof of this fact, including the well-known relation [55, 56], see the Supplemental Material [40]. This is in contrast to the bipartite case where all entangled pure states are metrologically useful [38].

Next, we will examine the single-copy case for higher \(d\).

**Observation 2.** All entangled pure states of the form given in Eq. (18) with \(\sum_k |\sigma_k|^2 = 1\) are useful for \(d \geq 3\) and \(N \geq 3\).

**Proof.** Let us see first the case of odd \(d \geq 3\) and the block diagonal matrix

$$\mathcal{H}_{\text{odd}} = \text{diag}(1, X_{d-1}),$$

(19)

where \(X_{d-1}\) is a \((d-1) \times (d-1)\) matrix with 1's in the antidiagonal and all other elements being 0. Hence,

$$\mathcal{F}_Q[\rho, \mathcal{H}_{\text{odd}}] = 4N + 4N\sigma_1^2|N(1 - \sigma_1^2) - 1|,$$

(20)

where the separable limit is \(\mathcal{F}_Q^{(\text{sep})}(\mathcal{H}_{\text{odd}}) = 4N\). With the appropriate permutation of the columns and rows of \(\mathcal{H}_{\text{odd}}\) we can obtain a Hamiltonian for which \(\sigma_k\) appears in the place of \(\sigma_1\) in Eq. (20). Hence, if for any \(\sigma_k\),

$$0 < \sigma_k^2 < (N-1)/N$$

(21)

holds then the given state is useful. If the state is entangled, then at least two of the \(\sigma_k\) are nonzero, and one of them fulfills Eq. (21).

Let us now consider the case of even \(d \geq 4\), with

$$\mathcal{H}_{\text{even}} = \text{diag}(1, 1, X_{d-2}),$$

(22)

for which

$$\mathcal{F}_Q[\rho, \mathcal{H}_{\text{even}}] = 4N + 4N(\sigma_1^2 + \sigma_2^2)|N(1 - \sigma_1^2 - \sigma_2^2) - 1|$$

(23)

hold. Similary to the case of odd \(d\), with the appropriate permutation of the columns and rows of \(\mathcal{H}_{\text{even}}\) we can obtain a Hamiltonian for which \(\sigma_k\) and \(\sigma_l\) with \(l \neq k\) appear in the place of \(\sigma_1\) and \(\sigma_2\), respectively, in Eq. (23). Hence, if

$$0 < \sigma_k^2 + \sigma_l^2 < (N-1)/N$$

(24)

then the state is useful. If the state is entangled, then at least two of the \(\sigma_k\) are nonzero. We have to consider the following cases. If the number of nonzero \(\sigma_k\) is three or more, then two of the \(\sigma_k\) will fulfill Eq. (24). If only two of them are nonzero then we can consider a problem of odd \(d\) with \(d = 3\) with one \(\sigma_k\) set to zero, and the state is useful.

A surprising consequence of Observation 2 is that all entangled states of the form Eq. (18) are useful for \(d = 2\), if we embed the qubits locally in qutrits, and consider a state as in Eq. (18) for \(d = 3\) by setting \(\sigma_3 = 0\). Thus, just by increasing the local dimension of the system, the states that have been found useless in Ref. [5] can be made useful.

Next, we present highly entangled pure states that remain useless even for multi-copy metrology.

**Observation 3.** The ring cluster states for \(N \geq 5\) are not useful even in the infinite copy limit.

**Proof.** Since the state is pure, the quantum Fisher information equals the variance times four. Moreover, many cluster states have strong three-body correlations, while most two-body reduced density matrices are completely mixed [54]. Ref. [5] found that for ring cluster states for \(N \geq 5\) all two-qubit reduced density matrices are the completely mixed state. Hence, for the reduced two-qubit states \(\langle \sigma_k \otimes \sigma_l \rangle = 0\) holds for all \(k, l = x, y, z\), while for all reduced single-qubit states \(\langle \sigma_k \rangle = 0\) holds for \(k = x, y, z\). It is easy to see that this is also true for the multicopy case. Due to that, the variance is the same as the variance of the completely mixed state and can be obtained as

$$\mathcal{F}_Q[\rho, \mathcal{H}] = 4(\Delta \mathcal{H})_{\text{disc}}^2 = 4 \sum_{n=1}^{N} \text{Tr}(h_n^2)/2^M,$$

(25)

where we assumed \(\text{Tr}(h_n) = 0\), and \(h_n\) are now \(2^M \times 2^M\) matrices representing operators acting on \(A_n\). Equation (25) is never larger than the separable limit in Eq. (5) based on the well-known relation [55, 56]

$$\sum_{k=1}^{2^M} \lambda_k^2(h_n)/2^M \leq |\lambda_{\text{max}}(h_n) - \lambda_{\text{min}}(h_n)|^2/4,$$

(26)

where \(\lambda_k(h_n)\) denote the eigenvalues of \(h_n\).

**Noise and imperfections prohibiting to reach the maximum metrological gain for infinite number of copies.** First, it is clear that states that are biseparable with respect to a bipartition remain biseparable even in the infinite copy limit [57]. Thus, such states cannot reach the maximum metrological performance [58, 59].

Next, we consider noisy quantum states.
Let us consider now the case when \( h_n \) is the tensor product of single-particle operators as in Eq. (9). Extensive numerics show that such \( h_n \) can often reach the maximal metrological performance, which makes the implementation easier [38, 40]. For this case, the condition in Eq. (29) can be reformulated as

\[
\langle h_n \otimes h_{n'} \rangle = \prod_{m=1}^{M} \langle h_{A_n^{(m)}} \otimes h_{A_{n'}^{(m)}} \rangle_\varrho = w_n w_{n'},
\]

for \( n \neq n' \), where \( \varrho \) is a single copy of the \( N \)-partite state and we took into account that the density matrix is of the form \( \varrho \otimes M \). To satisfy Eq. (29), we need that \( \langle h_{A_n^{(m)}} \otimes h_{A_{n'}^{(m)}} \rangle_\varrho \) is maximal for all \( n \neq n' \).

Let us consider a concrete example of a two-qubit isotropic state with \( h_{A_n^{(m)}} = \sigma_z \) for all \( n \). Then, based on Eq. (30) we have

\[
\langle h_n \otimes h_{n'} \rangle = \langle \sigma_z^{(n)} \otimes \sigma_z^{(n')} \rangle_\varrho^M.
\]

It is clear that if \( \langle \sigma_z^{(n)} \otimes \sigma_z^{(n')} \rangle_\varrho < 1 \) then the upper bound on the quantum Fisher information given in Eq. (27) is going to the separable limit for large \( M \). Hence, there is an optimal number of copies above which the metrological performance does not improve, which is illustrated in Fig. 2. One can see that with lower noise the state is useful, but the gain starts to decrease from \( M = 4 \) copies. In the example with higher noise, the quantum Fisher information is increasing with \( M \) but still does not overcome the separable limit. Apart from the quantum Fisher information, we also plotted the variance and the Wigner-Yanase skew information, as they constitute bounds that are easy to compute.

Scaling with the number of particles. For a constant \( M \), Eqs. (13) and (14) indicate a Heisenberg scaling with \( N \) for all entangled states given by the same coefficients \( c_{kl} \). We can also see that the difference from the maximal quantum Fisher information is decreasing exponentially with \( M \).

Let us consider the realistic scenario \( c_{01} = \frac{1}{2} e^{-t/T} \), where \( T \) is a time constant of the decay of \( c_{01} \). Then, from Eq. (15) we obtain

\[
\mathcal{I}(c_{01}, N) \approx N^2 [1 - (2t/T)^{M/2}]
\]

for \( t \ll T \), using the first two terms for the Taylor expansion of the exponential. Thus, using several copies slows down considerably the decay of the metrological abilities with \( t \).

Let us now consider a family of states for which the coefficients \( c_{kl} \) appearing in Eq. (11) depend on \( N \). In particular, we examine how the metrological gain scales with \( N \) for a given...
number of copies $M$ for the multiqubit pure state of the form Eq. (18) for which $g = 1$, analyzed in the Supplement in more detail [40]. In Fig. 3, it can be seen that for $N \ll M$ we reach the Heisenberg limit, while for $N \gg M$ we have shot-noise scaling.

**Relation to the bit flip code.** The states mentioned in Observation 1 appear in the context of error correction, in particular, in bit flip codes [63–66]. Using such an error correcting code, a single qubit is stored in many qubits in the subspace mentioned in Observation 1. If, due to some error, the state leaves the subspace, then with error correcting steps one can force the state back to the subspace. In the multiqubit case this would mean carrying out the syndrome measurements $\sigma_z \otimes \sigma_z$ on the qubit pairs. If some of the results are $-1$ then some of the spins must be flipped.

Such a step can be used to transform the initial state of the metrology and move it to the desired subspace. Note that we do not need to restore or protect a given quantum state, as usually the case in error correction. We need only to keep a quantum state that is useful for metrology. This method can be used also throughout the dynamics. Moreover, we do not even have to carry out the error correcting step. It is sufficient to employ a different Hamiltonian to states with different syndrome measurement results.

It is interesting to discuss the relation of our scheme to error correction in quantum metrology [67–69]. In the usual error correction scheme correcting bitflip errors, a pure state $\sum_{m,n=0,1} d_{mn}|m\rangle \otimes |n\rangle$ is stored as $\sum_{m,n=0,1} d_{mn}|m\rangle \otimes |n\rangle \otimes M$, where $M$ physical qubits are used to encode a logical qubit. One can see that even the two-qubit state $(|00\rangle + |11\rangle)/\sqrt{2}$ is encoded as a highly entangled GHZ-state. In contrast, in our case, we have $M$ copies of the state that are not entangled to each other and will become slightly entangled during the metrology, which makes our scheme more appropriate for a realization with present day or near future technology. Thus, our scheme is promising from the point of view of the noisy, intermediate-scale quantum (NISQ) era [13].

**Realization.** GHZ states have been created in trapped cold ions [51, 53, 70], as well as error correction has also been realized [71, 72]. Superconducting circuits have also been used to implemented error correction [73, 74]. Thus, such systems might be used to test our approach.

**Conclusions.** We considered how to use many copies of a bipartite or multipartite quantum state for quantum metrology. If the state is in a certain subspace, corresponding to the case of correlated noise, then, even if it was weakly entangled, it becomes maximally useful compared to separable states in the limit of infinite number of copies. States that are not in this subspace, on the other hand do not become more useful any more, if too many copies are added. In particular, if the state is full rank, then it cannot be maximally useful with any local Hamiltonian acting within the parties. This is similar to the case of noisy metrology, when uncorrelated noise hinders reaching the Heisenberg scaling above a certain particle number [75]. Steps similar to the ones applied in error correction can be used to force the state into the desirable space. Our method can immediately be tested in present day quantum devices requiring very moderate resources.

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It is sufficient to consider the case when $h_n^2 = w_n^2$ for $n = 1, 2, \ldots, N$. Otherwise, there are always Hamiltonians $h'_n$ with eigenvalues $\pm \lambda_{\text{max}}(h_n)$ and eigenvectors identical to those of $h_n$, with a metrological gain that is not smaller [38].
Supplemental Material for
“Multicopy metrology with many-particle quantum states”

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The Supplemental Material contains some additional results. We give a concise proof for the metrological usefulness of a single copy of multiqubit states. We present a simple example about scaling of the metrological gain with the particle number and the number of copies. We consider GHZ states mixed with white noise, as well as W states. We present a simple algorithm that can be used to calculate the quantum Fisher information for the multicopy case for large systems. We present a quantum state that is maximally useful and lives in the two-copy space. Finally, we examine whether two-body interaction could be used instead of many-body interactions between the copies.

DETAILS OF THE DERIVATION OF OBSERVATION 1

In this section, we give further details of the derivation of Observation 1 for the $d = 2$ case. The expression for the Wigner-Yanase skew information given in Eq. (14) can be written out

$$ I = 1 - \left[ 8c_{01}^2 \sqrt{-c_{00}^2 + c_{01}^2 + 4(c_{00} - 1)c_{00} + 1} \right]^M $$

(S1)

if $c_{01} \neq 0$, otherwise $I = 0$. If $c_{00} = 1/2$ then from Eq. (S1) we obtain Eq. (15).

GHZ-LIKE STATES

The condition of the usefulness of multipartite states of the type Eq. (18) for $d = 2$ and for $N \geq 3$ has been given already in Ref. [5], considering Hamiltonians of the type

$$ h_n = \sum_{l=x,y,z} c_{l,n} \sigma_l, $$

(S2)

where $c_{l,n}$ are real numbers, and $|\tilde{c}| = 1$. For completeness, we present a very short proof, which also includes the case $|\tilde{c}| \neq 1$ as discussed in Eq. (2).

Observation S1. A single copy of the state given in Eq. (18) for $d = 2$ is useful metrologically if and only if

$$ 1/N < E := 4|\sigma_0\sigma_1|^2. $$

(S3)

Proof. Let us consider local Hamiltonians of the type Eq. (S2), where $|\tilde{c}_n| = L_n$. Then, we can consider the series of inequalities

$$ g_H = 1 + \frac{E (\sum_n c_{z,n})^2 - \sum_n c_{z,n}^2}{\sum_n L_n^2} $$

$$ \leq 1 + \frac{(E - 1/N)(\sum_n c_{z,n})^2}{\sum_n L_n^2} $$

$$ \leq 1 + \frac{(E - 1/N)(\sum_n L_n)^2}{\sum_n L_n^2} $$

$$ \leq N \times E. $$

(S4)

In the first inequality, we used the inequality between the arithmetic and quadratic means for the $c_{z,n}$, $(\sum_n c_{z,n})^2 / N \leq \sum_n c_{z,n}^2 / n$. In the third inequality, we used the same relation for $L_n$. All inequalities are saturated if $c_n = L_n$ for all $n$ and they are all equal to each other. Hence, we obtain that the state is useful if and only if Eq. (S3) is fulfilled.

Note also that for odd $N$, states given in Eq. (18) for $d = 2$ do not violate Bell inequalities with full correlation terms and two two-outcome observables per party if $E < 1/2^{N-1}$ [41] [cf. Eq. (S3)].

EXAMPLES OF SCALING

It is instructive to look at the behavior of the multicopy quantum states in the limit of large $N$. Let us consider the
obtain $M$ of particles and set $E$ when the number of copies increases slower with the number which is the Heisenberg limit. Finally, let us consider a case $N$. If we set $N \ll M$, we obtain
\[ (1 - 1/N)^M \approx 1. \]
Hence, we have for the quantum Fisher information and the gain
\[ F_Q(\rho^{\otimes M}, \mathcal{H}) = 4N^2, \quad g = N, \]
which corresponds to the Heisenberg limit. Then, for $N \gg M$ we obtain
\[ (1 - 1/N)^M \approx 1 - M/N. \]
Hence, for the $N \to \infty$ limit we have for the quantum Fisher information and the gain
\[ F_Q(\rho^{\otimes M}, \mathcal{H}) = 4NM, \quad g = M. \]
This can be see in Fig. 3.

It is instructive to look at the case that $M$ depends on $N$. If we set $M = N$ then in the $N \to \infty$ limit from Eq. (S5) we obtain
\[ F_Q(\rho^{\otimes M}, \mathcal{H}) = 4N^2(1-1/e), \quad g = N(1-1/e) \approx 0.63N, \]
where $e \approx 2.7183$ is the basis of the natural logarithm. We used the well known relation
\[ \lim_{N \to \infty} (1 - 1/N)^N = 1/e. \]
If we set $N \ll M$ then in the $N \to \infty$ limit from Eq. (S5) we obtain
\[ F_Q(\rho^{\otimes M}, \mathcal{H}) = 4N^2, \quad g = N, \]
which is the Heisenberg limit. Finally, let us consider a case when the number of copies increases slower with the number of particles and set $M = \sqrt{N}$. Then, for the $N \to \infty$ limit we obtain
\[ F_Q(\rho^{\otimes M}, \mathcal{H}) = 4N\sqrt{N}, \quad g = \sqrt{N}. \]
\[ \lim_{N \to \infty} N^2[1 - (1 - 1/N)^{\sqrt{N}}]/\sqrt{N} = 1. \]
Finally, from Observation 2 we obtain that if we take the state mentioned and embed it into $d = 3$ dimensional systems, then with a single copy we obtain for $N \to \infty$ $F_Q = 5N, \quad g = 5/4$,  
while $F_Q(\text{sep}) = 4N$.

FIG. S1. Metrological gain for noisy GHZ states given in Eq. (S17) embedded into a multiqudit state with a qudit dimension (a) $d = 3$ and (b) $d = 4$ and $d = 5$. (dashed) Original three-qubit state and (solid) state with qubits embedded locally into qudits of a higher dimension.

GHZ STATES MIXED WITH WHITE NOISE

We have seen that GHZ states with noise in the $\{|000\ldots00\rangle, |111\ldots11\rangle\}$-space given in Eq. (16) can be made maximally useful in the limit of infinite number of copies, if the state is entangled. It is instructive to examine what the situation is with white noise.

We tested three-qubit GHZ states with various levels of white noise and found that two copies are not more useful than a single copy. We also tested what happens if a GHZ state mixed with white noise
\[ \rho_p(\text{noisy GHZ}) = p|\text{GHZ}\rangle\langle\text{GHZ}| + (1-p)\mathbb{I}/2^3, \]
is embedded into a multiqudit state with a qudit dimension larger than two. We used for numerics the methods of Ref. [38], which could easily be generalized from the bipartite case to the multipartite one to maximize the metrological gain over local Hamiltonians. The metrological gain is shown in Fig. S1. After embedding the state into a larger system, the metrological gain increased for small $p$. On the other hand, states that have been useless, did not become useful after the embedding. In all the cases, the state becomes useful if $p > 0.439576$.
\[ F_Q(\text{sep}) = 4N. \]

It is instructive to note that the state is genuine multipartite entangled if [42, 43]
\[ p > 0.428571 \]
holds.

W STATE

In this section, we consider states that are very different from the GHZ state studied in the main text. W states are defined as
\[ |W\rangle = \frac{1}{\sqrt{N}}(|100\ldots00\rangle + |010\ldots00\rangle + \ldots + |00\ldots00\rangle). \]
For three-qubit pure states, genuine multipartite entangled states can be equivalent to GHZ states or W states under local operations and classical communication (LOCC) [44].

Using the methods of Ref. [38], we find that for \( N = 3 \), the optimal Hamiltonian for the \(|W\rangle\) state is

\[
H = \sigma_z^{(1)} + \sigma_z^{(2)} - \sigma_z^{(3)},
\]

which is translationally not invariant. Thus, even if the state is symmetric, the optimal Hamiltonian is not. This finding is in a sense complementary to the findings of Ref. [45], where examples have been shown in which the Hamiltonian was symmetric but the optimal state was not symmetric.

Let us now consider the mixture

\[
\varrho_p^{(W-W\text{ mixture})} = p|W\rangle\langle W| + (1 - p)|\bar{W}\rangle\langle \bar{W}|,
\]

where

\[
|\bar{W}\rangle = \frac{1}{\sqrt{N}} (|011\ldots11\rangle + |101\ldots11\rangle + \ldots + |111\ldots10\rangle).
\]

Such states have been studied for odd \( N \), since they are genuine multipartite entangled, still possess no multipartite correlations for \( p = 1/2 \) [46–48]. We find that the metrological gain for states given in Eq. (S22) for \( N = 3 \) is minimal for \( p = 1/2 \). For \( p \approx 1 \) and for \( p \approx 0 \) the maximal gain for two copies is the same as for a single copy, while for intermediate \( p \) values the gain for two copies is larger, as can be seen in Fig. S2. We used the numerical optimization over local Hamiltonians described in Ref. [38].

**EFFICIENT NUMERICS FOR MANY COPIES OF A QUANTUM STATE**

We present efficient numerical methods to obtain the quantum Fisher information for large number of copies of a bipartite state. The methods work even if a direct calculation would not be feasible.

In order to obtain the quantum Fisher information given in Eq. (3) for many copies, we have to compute the quantity

\[
\langle k_1|\langle k_2|\langle k_3|\ldots h_{n-1}|l_{n-1}|l_{n}|l_2|l_1\rangle
\]

for \( n = 1, 2, \ldots, N \) for large systems. \(|l_m\rangle\) and \(|k_m\rangle\) are all bipartite states, the eigenstates of the state we consider. This computation is possible if we note that if \(|\Psi\rangle\) is a tensor product of two-particle states, then

\[
h_n|\Psi\rangle
\]

is also a tensor product of two-particle states.

Let us use now Eq. (S27) to calculate the quantum Fisher information. In the following, we assume that \( \lambda_k > 0 \) only for \( k = 1, 2, \ldots, r \), and the rank of the state is \( g \).
Based on Eq. (S27), we can write that
\[
\mathcal{F}_Q[\rho^\otimes M, \mathcal{H}] = 4\langle \mathcal{H}^2 \rangle - 8 \sum_{\vec{k},\vec{l}} \frac{\lambda_{\vec{k}}\lambda_{\vec{l}}}{\lambda_{\vec{k}} + \lambda_{\vec{l}}} |(k_1|k_2|k_3|\ldots|l_2)|^2,
\]
where we use the notation
\[
\lambda_{\vec{k}} = \lambda_k \lambda_{k_2} \lambda_{k_3} \ldots \lambda_{k_M}.
\]
(S31)

Here, states with a zero eigenvalue do not contribute. It is sufficient to look at \( \vec{k} \) and \( \vec{l} \) that contain only 1, 2, ..., \( r \). There are \( r^{2M} \) terms in the sum in Eq. (S31).

We can simplify further our calculations if \( h_{A^{(n)}} \) are all equal to each other
\[
h_{A^{(m)}} = h_{A_n},
\]
(S33)

for \( n = 1, 2, ..., N \) and \( m = 1, 2, ..., M \). Due to permutational invariance, only the number of different eigenvalues matter, the order does not matter. Hence, we can write that
\[
\mathcal{F}_Q[\rho^\otimes M, \mathcal{H}] = 4\langle \mathcal{H}^2 \rangle - 8 \sum_{\vec{k} \in I(r)} \mathcal{P}_{\vec{k}} \times \frac{\lambda_{\vec{k}}\lambda_{\vec{k}}}{\lambda_{\vec{k}} + \lambda_{\vec{k}}} |(k_1|k_2|k_3|\ldots|l_2)|^2,
\]
(S34)

where \( I(r) \) is the set of \( M \)-element vectors of 1, 2, ..., \( r \) with nondecreasing elements, the coefficients based on the number of appearance of a certain term in the sum in Eq. (S31) are given by the multinomial distribution
\[
\mathcal{P}_{\vec{k}} = \binom{M}{m_1(\vec{k}), m_2(\vec{k}), \ldots, m_r(\vec{k})}.
\]
(S35)

Here \( m_l(\vec{k}) \) is number of \( l \)'s in \( \vec{k} \).

Finally, we note that Eq. (S34) is a sum of positive terms that are also bounded from above. The average of positive bounded quantities can be approximated by an average obtained for a random sample (see, e.g., Ref. [49]).

**STATE LIVING IN THE TWO-COPY SPACE**

In this section, we present a quantum state living in the two-copy space of a bipartite system. We denote the two copies \( AB \) and \( A'B' \).

Let us consider the quantum state
\[
\rho_p = p|\Psi^+\rangle \langle \Psi^+| \otimes^2 + (1-p)|\Phi^+\rangle \langle \Phi^+| \otimes^2,
\]
(S36)

where the maximally entangled two-qubit states are defined as
\[
|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),
\]
\[
|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).
\]
(S37)

For state \( \rho_p \) and the Hamiltonian
\[
H = \sigma_z^A \sigma_z^{A'} + \sigma_z^B \sigma_z^{B'}
\]
(S38)

for any \( p \) the metrological gain is obtained as
\[
g = 2.
\]
(S39)

The state \( \rho_p \) given in Eq. (S36) is equivalent to the state that is the mixture of the maximally entangled state
\[
\frac{1}{2} \sum_{n=0}^3 |n\rangle_{AA'} |3-n\rangle_{BB'},
\]
(S40)

and state obtained from it after a basis transformation
\[
\frac{1}{2} \sum_{n=0}^3 |n\rangle_{AA'} |3-n\rangle_{BB'}.
\]
(S41)

It is instructive to compare the state given in Eq. (S36) to
\[
\rho_p = p|\Psi^+\rangle \langle \Psi^+| + (1-p)|\Phi^+\rangle \langle \Phi^+|,
\]
(S42)

which is nonentangled and metrologically not useful for \( p = 1/2 \).

**INTERACTION BETWEEN THE COPIES VIA TWO-PARTICLE TERMS**

In this section, we consider \( M \) copies of a state of \( N \) qubits. Instead of \( M \)-particle correlations, two-particle correlations act between the copies. We show that for large number of copies, the metrological gain will be below 1 and the state is not useful.

Let us look at the correlations
\[
h_n = \sum_{m>m'} (\sigma_z)_A^{(m)} \otimes (\sigma_z)_A^{(m')}
\]
(S43)

for \( n = 1, 2, ..., N \). Then we can make a mapping for the Hamiltonian
\[
\mathcal{H} \rightarrow \tilde{\mathcal{H}} = N \sum_{n>m} \sigma_z^{(n)} \otimes \sigma_z^{(m)}
\]
(S44)

and for the state we use the mapping given in Eq. (12) for \( d = 2 \). This way, we carry out our calculations with smaller matrices using
\[
\mathcal{F}_Q[\tilde{\rho}^\otimes M, \tilde{\mathcal{H}}] = \mathcal{F}_Q[\rho^\otimes M, \tilde{\mathcal{H}}].
\]
(S45)

Then, one can prove the following series of inequalities
\[
\mathcal{F}_Q[\tilde{\rho}^\otimes M, \tilde{\mathcal{H}}] \leq 4N^2(\Delta M^2)^2_{\tilde{\rho}^\otimes M}
\]
\[
\leq 4N^2\text{Tr}(\tilde{\rho}^\otimes M \tilde{M}^2)
\]
\[
\leq 4N^2\text{Tr}(0 \otimes M \tilde{M}^2)
\]
\[
= N^2(12M^2 - 8M).
\]
(S46)
In Eq. (S46), the first inequality is based on that for any $A$ the inequality $F_{Q}[g, A] \leq 4(\Delta A)^2$ holds, and that we write the sum of two-body correlations as
\[ \sum_{m > m'} \sigma_z^{(m)} \sigma_z^{(m')} = (M_z^2 - M 1)/2, \] 
(S47)

where the $z$-component of the collective angular momentum is given as
\[ M_z = \sum_m \sigma_z^{(m)}. \] 
(S48)

In Eq. (S46), the second inequality is based on that for any $A$ the inequality $(\Delta A)^2 \leq \langle A^2 \rangle$ holds. In Eq. (S46), the third inequality is based on that the product state $\tilde{\varrho} \otimes M_{\text{max}}$ maximizing $M_z^2$ is $\ket{0} \bra{0} \otimes M_{\text{max}}$. In Eq. (S46), the equality is based on simple analytical calculations.

For the best metrological performance for separable states we have
\[ F_{Q}^{(\text{sep})}(\tilde{H}) = N^2 [\lambda_{\text{max}}(M_z^2/2) - \lambda_{\text{min}}(M_z^2/2)]^2. \] 
(S49)

The largest eigenvalue is
\[ \lambda_{\text{max}}(M_z^2) = M^2. \] 
(S50)

The smallest eigenvalue is
\[ \lambda_{\text{min}}(M_z^2) = \begin{cases} 0, & \text{if } M \text{ is even}, \\ 1, & \text{if } M \text{ is odd}. \end{cases} \] 
(S51)

Based on these, for the metrological gain
\[ g \leq \begin{cases} (48M^2 - 32M)/M^4, & \text{if } M \text{ is even}, \\ (48M^2 - 32M)/(M^2 - 1)^2, & \text{if } M \text{ is odd}, \end{cases} \] 
(S52)

holds. The upper bound is decreasing with $M$ rapidly. For $M \geq 7$, the metrological gain will be below 1 and the state is not useful metrologically.