IMPROVING THE ESTIMATES FOR A SEQUENCE INVOLVING PRIME NUMBERS

CHRISTIAN AXLER

Abstract. Based on new explicit estimates for the prime counting function, we improve the currently known estimates for the particular sequence \( C_n = np_n - \sum_{k \leq n} p_k, \ n \geq 1 \), involving the prime numbers.

1. Introduction

Let \( p_n \) denotes the \( n \)th prime number. In this paper, we establish new explicit estimates for the sequence \((C_n)_{n \geq 1}\) with
\[
C_n = np_n - \sum_{k \leq n} p_k
\]
(see [3]). In [1, Theorem 10], the present author used the identity
\[
C_n = \int_2^{p_n} \pi(x) \, dx
\]
where \( \pi(x) \) denotes the number of primes not exceeding \( x \), to derive that the asymptotic formula
\[
C_n = \sum_{k=1}^{m-1} (k - 1)! \left( 1 - \frac{1}{2k} \right) \frac{p_n^k}{\log^k p_n} + O \left( \frac{p_n^2}{\log^m p_n} \right)
\]
holds for each positive integer \( m \). By setting \( m = 9 \) in (2), we get
\[
C_n = \frac{p_n^2}{2 \log p_n} + \frac{3p_n^2}{4 \log^2 p_n} + \frac{7p_n^2}{4 \log^3 p_n} + \chi(n) + O \left( \frac{p_n^2}{\log^9 p_n} \right)
\]
where
\[
\chi(n) = \frac{45p_n^2}{8 \log^4 p_n} + \frac{93p_n^2}{8 \log^5 p_n} + \frac{945p_n^2}{8 \log^6 p_n} + \frac{5715p_n^2}{8 \log^7 p_n} + \frac{80325p_n^2}{16 \log^8 p_n}.
\]
In the direction of (3), the present author [1, Theorem 3 and Theorem 4] showed that
\[
C_n \geq \frac{p_n^2}{2 \log p_n} + \frac{3p_n^2}{4 \log^2 p_n} + \frac{7p_n^2}{4 \log^3 p_n} + \Theta(n)
\]
for every \( n \geq 52703656 \), where
\[
\Theta(n) = \frac{43.6p_n^2}{8 \log^4 p_n} + \frac{90.9p_n^2}{8 \log^5 p_n} + \frac{927.5p_n^2}{8 \log^6 p_n} + \frac{5620.5p_n^2}{8 \log^7 p_n} + \frac{79075.5p_n^2}{16 \log^8 p_n}
\]
and that the upper bound
\[
C_n \leq \frac{p_n^2}{2 \log p_n} + \frac{3p_n^2}{4 \log^2 p_n} + \frac{7p_n^2}{4 \log^3 p_n} + \Omega(n)
\]
holds for every positive integer \( n \), where
\[
\Omega(n) = \frac{46.4p_n^2}{8 \log^4 p_n} + \frac{95.1p_n^2}{8 \log^5 p_n} + \frac{962.5p_n^2}{8 \log^6 p_n} + \frac{5809.5p_n^2}{8 \log^7 p_n} + \frac{118848p_n^2}{16 \log^8 p_n}.
\]
Using new explicit estimates for the prime counting function \( \pi(x) \), which are found in [2, Proposition 3.6 and Proposition 3.12], we improve the inequalities [4] and [5] by showing the following both results.
Theorem 1.1. For every positive integer $n \geq 44200309$, we have
\[
C_n \geq \frac{p_n^2}{2 \log p_n} + \frac{3p_n^2}{4 \log^2 p_n} + \frac{7p_n^2}{4 \log^3 p_n} + L(n),
\]
where
\[
L(n) = \frac{44.4p_n^2}{8 \log^4 p_n} + \frac{92.1p_n^2}{4 \log^5 p_n} + \frac{937.5p_n^2}{8 \log^6 p_n} + \frac{5674.5p_n^2}{8 \log^7 p_n} + \frac{79789.5p_n^2}{16 \log^8 p_n}.
\]

Theorem 1.2. For every positive integer $n$, we have
\[
C_n \leq \frac{p_n^2}{2 \log p_n} + \frac{3p_n^2}{4 \log^2 p_n} + \frac{7p_n^2}{4 \log^3 p_n} + U(n),
\]
where
\[
U(n) = \frac{45.6p_n^2}{8 \log^4 p_n} + \frac{93.9p_n^2}{4 \log^5 p_n} + \frac{952.5p_n^2}{8 \log^6 p_n} + \frac{5755.5p_n^2}{8 \log^7 p_n} + \frac{116371p_n^2}{16 \log^8 p_n}.
\]

2. Preliminaries

In 1793, Gauß [4] stated a conjecture concerning an asymptotic magnitude of $\pi(x)$, namely
\[
\pi(x) \sim \text{li}(x) \quad (x \to \infty),
\]
where the logarithmic integral $\text{li}(x)$ defined for every real $x \geq 0$ as
\[
\text{li}(x) = \int_0^x \frac{dt}{\log t} = \lim_{\varepsilon \to 0} \left\{ \int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t} \right\} \approx \int_2^x \frac{dt}{\log t} + 1.04516 \ldots.
\]
Using the method of integration of parts, [7] implies that
\[
\text{li}(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \frac{24x}{\log^5 x} + \ldots + \frac{(m-1)!x}{\log^m x} + O\left( \frac{x}{\log^{m+1} x} \right),
\]
for every positive integer $m$. The asymptotic formula [6] was proved independently by Hadamard [5] and by de la Vallée-Poussin [7] in 1896, and is known as the Prime Number Theorem. By proving the existence of a zero-free region for the Riemann zeta-function $\zeta(s)$ to the left of the line $\text{Re}(s) = 1$, de la Vallée-Poussin [8] was able to estimate the error term in the Prime Number Theorem by
\[
\pi(x) = \text{li}(x) + O(x \exp(-a \sqrt{\log x})),
\]
where $a$ is a positive absolute constant. Together with [8], we obtain that the asymptotic formula
\[
\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \frac{24x}{\log^5 x} + \ldots + \frac{(m-1)!x}{\log^m x} + O\left( \frac{x}{\log^{m+1} x} \right),
\]
holds for every positive integer $m$.

3. A Proof of Theorem 1.1

Now, we use some recent obtained lower bound for the prime counting function $\pi(x)$ to give a proof of Theorem 1.1.

Proof of Theorem 1.1 First, let $m$ be a positive integer with $m \geq 2$, and let $a_2, \ldots, a_m, x_0$, and $y_0$ be real numbers, so that
\[
\pi(x) \geq \frac{x}{\log x} + \sum_{k=2}^{m} \frac{a_k x}{\log^k x}
\]
for every $x \geq x_0$ and
\[
\text{li}(x) \geq \sum_{j=1}^{m-1} \frac{(j-1)!x}{\log^j x}
\]
for every $x \geq y_0$. The asymptotic formulae [9] and [8] guarantee the existence of such parameters. In [4] Theorem 13], the present author showed that
\[
C_n \geq d_0 + \sum_{k=1}^{m-1} \left( \frac{(k-1)!}{2k} \left( 1 + 2t_{k-1,1} \right) \right) \frac{p_n^2}{\log^k p_n}
\]
for every \( n \geq \max \{ \pi(x_0) + 1, \pi(\sqrt{y_0}) + 1 \} \), where \( t_{i,j} \) is defined by

\[
t_{i,j} = (j - 1)! \sum_{l=j}^{m} \frac{2^{l-j}a_{l+1}}{l!}.
\]

and \( d_0 \) is given by

\[
d_0 = d_0(m, a_2, \ldots, a_m, x_0) = \int_{2}^{x_0} \pi(x) \, dx - (1 + 2t_{m-1,1}) \lambda(x_0) + \sum_{k=1}^{m-1} t_{m-1,k} \frac{x_0^k}{\log^k x_0}.
\]

Now, we choose \( x = 841160647 \) for every \( d \geq 0 \), \( m = 9 \), \( a_2 = 1 \), \( a_3 = 2 \), \( a_4 = 5.85 \), \( a_5 = 23.85 \), \( a_6 = 119.25 \), \( a_7 = 715.5 \), \( a_8 = 5008.5 \), \( a_9 = 0 \), \( x_0 = 19027490297 \) and \( y_0 = 4171 \). By [2, Proposition 3.12], we obtain that the inequality (10) holds for every \( x \geq x_0 \) and (11) holds for every \( x \geq y_0 \) by [1, Lemma 15]. Substituting these values in (12), we get

\[
C_n \geq d_0 + \frac{4p_n^2}{\log^2 p_n} + \frac{3p_n^2}{4\log^3 p_n} + \frac{7p_n^2}{4\log^4 p_n} + L(n)
\]

for every \( n \geq 841160647 = \pi(x_0) \), where \( d_0 = d_0(9, 1, 2, 5.85, 23.85, 119.25, 715.5, 5008.5, 0, x_0) \) is given by

\[
d_0 = \int_{2}^{x_0} \pi(x) \, dx - 253.3\log(x_0^2) + \frac{126.15}{\log x_0} + \frac{62.57}{\log^2 x_0} + \frac{61.75}{\log^3 x_0}
\]

\[
+ \frac{89.4375}{\log^4 x_0} + \frac{165.95}{\log^5 x_0} + \frac{357.75}{\log^6 x_0} + \frac{715.5}{\log^7 x_0}.
\]

The present author [1, Lemma 16] found that

\[
\lambda(x) \leq \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \frac{24x}{\log^5 x} + \frac{120x}{\log^6 x} + \frac{900x}{\log^7 x}
\]

for every \( x \geq 10^{16} \). Applying this inequality to (14), we get

\[
d_0 \geq \int_{2}^{x_0} \pi(x) \, dx - \frac{x_0^2}{2\log x_0} - \frac{3x_0^2}{4\log^2 x_0} - \frac{7x_0^2}{4\log^3 x_0} - \frac{5.55x_0^2}{\log^4 x_0} - \frac{23.025x_0^2}{\log^5 x_0}.
\]

Computing the right-hand side of the last inequality, we get

\[
d_0 \geq \int_{2}^{x_0} \pi(x) \, dx - 8.188366 \cdot 10^{18}.
\]

Since \( x_0 = p_{841160647} \), we use [1] and a computer to obtain

\[
\int_{2}^{x_0} \pi(x) \, dx = C_{841160647} = 8188378036394419009.
\]

Hence, by (15), we get \( d_0 \geq 1.12 \cdot 10^{13} > 0 \). So we obtain the desired inequality for every \( n \geq 841160647 \). For every \( 440200309 \leq n \leq 841160646 \) we check the inequality with a computer. \( \square \)

4. A PROOF OF THEOREM 1.2

Next, we use a recent result concerning an upper bound for the prime counting function \( \pi(x) \) to establish the required inequality stated in Theorem 1.2.

**Proof of Theorem** [1,2]. Let \( m \) be a positive integer with \( m \geq 2 \), let \( a_2, \ldots, a_m, x_1 \) be real numbers so that

\[
\pi(x) \leq \frac{x}{\log x} + \sum_{k=2}^{m} \frac{a_k x}{\log^k x}
\]

for every \( x \geq x_1 \) and let \( \lambda, y_1 \) be real numbers so that

\[
\lambda(x) \leq \sum_{j=1}^{m-2} \frac{(j - 1)! x}{\log^j x} + \frac{\lambda x}{\log^{m-1} x}
\]
for every $x \geq y_1$. Again, the asymptotic formulae \((9)\) and \((8)\) guarantee the existence of such parameters. The present author \([1, \text{Theorem } 14]\) found that the inequality

\[
C_n \leq d_1 + \sum_{k=1}^{m-2} \frac{(k-1)!}{2^k} \left(1 + 2t_{k-1,1}\right) \frac{p_n^2}{\log^k p_n} + \left(\frac{1 + 2t_{m-1,1}}{2^{m-1}} - \frac{a_m}{m-1}\right) \frac{p_n^2}{\log^{m-1} p_n}
\]

(18)
holds for every $n \geq \max\{\pi(x_1) + 1, \pi(\sqrt{y_1}) + 1\}$, where $t_{i,j}$ is defined by \((13)\), and

\[
d_1 = d_1(m, a_2, \ldots, a_m, x_1) = \int_2^{x_1} \pi(x) \, dx - \left(1 + 2t_{m-1,1}\right) \text{li}(x_1^2) + \sum_{k=1}^{m-1} t_{m-1,k} \frac{x_1^2}{\log^k x_1}.
\]

Next, we choose $m = 9, a_2 = 1, a_3 = 2, a_4 = 6.15, a_5 = 24.15, a_6 = 120.75, a_7 = 724.5, a_8 = 6601, a_9 = 0, \lambda = 6300, x_1 = 13$ and $y_1 = 10^{18}$. By \([2, \text{Proposition } 3.6]\), we get that the inequality \((16)\) holds for every $x \geq x_1$ and by \([1, \text{Lemma } 19]\), that \((17)\) holds for every $y \geq y_1$. By substituting these values \((18)\), we get

\[
C_n \leq d_1 + \frac{p_n^2}{4 \log^5 p_n} + \frac{3p_n^2}{4 \log^3 p_n} + \frac{7p_n^2}{4 \log^4 p_n} + \frac{U(n)}{16 \log^8 p_n} + 0.375p_n^2
\]

(19)
for every $n \geq 50847535$, where $d_1 = d_1(9, 1, 2, 6.15, 24.15, 120.75, 724.5, 6601, 0, x_1)$ is given by

\[
d_1 = \int_2^{x_1} \pi(x) \, dx = \frac{26599}{90} \text{li}(x_1^2) + \frac{26509 x_1^2}{180 \log x_1} + \frac{26329 x_1^2}{360 \log^2 x_1} + \frac{25969 x_1^2}{360 \log^3 x_1} + \frac{25231 x_1^2}{240 \log^4 x_1} + \frac{11891 x_1^2}{60 \log^5 x_1} + \frac{5221 x_1^2}{12 \log^6 x_1} + \frac{943 x_1^2}{\log^7 x_1}.
\]

A computation shows that $d_1 \leq 453$. We define

\[
f(x) = \frac{0.375x^2}{16 \log^5 x} - 453.
\]

Since $f(9187322) > 0$ and $f'(x) \geq 0$ for every $x \geq e^4$, we get $f(p_n) \geq 0$ for every $n \geq \pi(9187322) + 1 = 614124$. Now we can use \((19)\) to obtain the desired inequality for every positive integer $n \geq 50847535$.

Finally, we check the remaining cases with a computer. \(\square\)

References

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