Constraint Dissolving Approaches for Riemannian Optimization

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Abstract

In this paper, we propose a class of constraint dissolving approaches for optimization problems over closed Riemannian manifolds. In these proposed approaches, solving a Riemannian optimization problem is transferred into the unconstrained minimization of a constraint dissolving function named CDF. Different from existing exact penalty functions, the exact gradient and Hessian of CDF are easy to compute. We study the theoretical properties of CDF and prove that the original problem and CDF have the same first-order and second-order stationary points, local minimizers, and Łojasiewicz exponents in a neighborhood of the feasible region. Remarkably, the convergence properties of our proposed constraint dissolving approaches can be directly inherited from the existing rich results in unconstrained optimization. Therefore, the proposed constraint dissolving approaches build up short cuts from unconstrained optimization to Riemannian optimization. Several illustrative examples further demonstrate the potential of our proposed constraint dissolving approaches.

1 Introduction

1.1 Problem description

In this paper, we consider the following constrained optimization problem

\[
\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad c(x) = 0.
\]

We denote the feasible region of (OCP) by \( \mathcal{M} := \{ x \in \mathbb{R}^n : c(x) = 0 \} \). In addition, the objective function \( f : \mathbb{R}^n \to \mathbb{R} \) and constraint mapping \( c : \mathbb{R}^n \to \mathbb{R}^p \) of (OCP) satisfy the following assumptions.

Assumption 1.1. Blank assumptions

1. \( \nabla f(x) \) is locally Lipschitz continuous in \( \mathbb{R}^n \);

2. The transposed Jacobian of \( c \), denoted as \( J_c(x) \in \mathbb{R}^{n \times p} \), is locally Lipschitz continuous in \( \mathbb{R}^n \);

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3. The linear independence constraint qualification (LICQ) holds for any $x \in \mathcal{M}$, i.e. $J_c(x) \in \mathbb{R}^{n \times p}$ has full column rank for any $x \in \mathcal{M}$.

Since $c$ is smooth in $\mathbb{R}^n$, the set $\mathcal{M}$ is a closed Riemannian manifold embedded in $\mathbb{R}^n$ [17]. In fact, OCP satisfying Assumption 1.1 covers almost all practically interesting smooth optimization problems over closed Riemannian manifolds.

### 1.2 Existing approaches

Due to the diffeomorphisms between the Euclidean space and the Riemannian manifold, various unconstrained optimization approaches (i.e., approaches for solving unconstrained nonconvex optimization) can be transferred to their corresponding Riemannian optimization approaches (i.e., the approaches for Riemannian optimization). In practice, [3] provides several well-recognized frameworks based on two basic materials in differential geometry: geodesics and parallel transports. The geodesics generalize the concept of straight lines from Euclidean spaces to Riemannian manifolds, but may be expensive to compute in most cases. To this end, [3] provides the concept of retractions as relaxations to geodesics, which makes it more affordable to update the iterates on a certain Riemannian manifold at the cost of introducing approximation errors. Besides, parallel transports are mappings that move tangent vectors along given curves on a Riemannian manifold “parallelly”. Computing parallel transports is essential in computing the difference of two vectors from different tangent spaces. Specifically, computing parallel transports is necessary for algorithms that utilize information in the past iterates to construct searching direction (e.g., quasi-Newton methods, nonlinear conjugate gradient methods, momentum accelerated methods). For various Riemannian manifolds, computing the parallel transports amounts to solving differential equations, which is generally unaffordable in practice [3]. To alleviate the computational cost, [3] proposes the concept of vector transports as approximations to parallel transports. As mentioned in [50], computing vector transports is usually cheaper than parallel transports. With retractions and vector transports, many unconstrained optimization approaches have been extended to their Riemannian versions, including Riemannian gradient descent with line-search [2, 3, 58], Riemannian conjugate gradient methods [1, 51], Riemannian accelerated gradient methods [70, 69, 52, 15], and Riemannian adaptive gradient methods [6]; see [3, 8] for instances.

In recent years, there emerge an increasing number of approaches for solving unconstrained optimization problems, which have superior convergence properties, marvelous numerical behaviors, or both. However, transferring an unconstrained optimization approach to its Riemannian versions requires some basic geometrical materials of the Riemannian manifold, including computing Riemannian gradients, retractions, and vector transports [5]. Determining those geometrical materials can be challenging for various Riemannian manifolds, see [15, 46, 23] for instances. Based on those geometrical materials, transferring an unconstrained optimization approach to its Riemannian versions requires profound modifications, including replacing the computation of differentials by Riemannian differentials, introducing retractions to keep the iterates feasible, and employing vector transports to move vectors on the Riemannian manifold. As a result, it is challenging to keep Riemannian optimization approaches updated with the advances in unconstrained nonconvex optimization.

Furthermore, the convergence properties of many practically useful Riemannian optimization approaches cannot directly follow from existing results for unconstrained optimization, as vector transports are only approximations to parallel transports, and the iterates are usually not updated along geodesics. Therefore, their convergence properties must be carefully revisited due to the approximation errors introduced by vector transports and retractions. Additionally, a number of existing Riemannian optimization approaches are built upon parallel transports and geodesics [4, 71, 69, 71, 4, 32]. It is challenging to verify whether their convergence properties are still valid when the approximation errors are introduced by vector transports and retractions.

Extending the existing unconstrained optimization approaches to their Riemannian manifold versions and retaining the convergence guarantee are nontrivial and sometimes intractable. Therefore, it is quite natural to ask the following question.
Could unconstrained optimization approaches, together with their convergence properties, have a straightforward implementation for the constrained optimization problem OCP?

This question drives us to propose constraint dissolving approaches for OCP i.e., transferring OCP into an unconstrained optimization problem while keeping stationary points unchanged. Therefore, constraint dissolving approaches enable direct implementation of unconstrained optimization approaches to solve OCP while the convergence properties of those unconstrained optimization approaches are retained simultaneously.

We should mention that there are two classes of existing approaches attempting to achieve a similar goal but using completely different philosophies. One of them is the well-known $\ell_1$ penalty function methods. The $\ell_1$ penalty function is known as an exact penalty function. However, its non-smooth penalty term, with the nonconvex manifold constraints inside, usually leads to difficulties in developing efficient unconstrained optimization approaches [47]. The other class of methods are based on the augmented Lagrange penalty function [29, 49]. The Lagrangian penalty function is an exact penalty function for OCP when the Lagrange multipliers $\lambda$ take their optimal values, which are certainly unknown in advance. Therefore, classical augmented Lagrange methods need to solve the unconstrained penalty function subproblem with a fixed $\lambda$ and then update the multipliers in each iteration. These two hierarchical approaches are not as efficient as the existing Riemannian optimization approaches for solving OCP. In particular, [20] proposes a class of exact penalty functions named Fletcher’s penalty function below

$$\phi(x) := f(x) - u(x)^\top c(x) + \frac{\beta}{2} \|c(x)\|^2. \quad (1.1)$$

Here $u(x)$ is defined by the following linear least squares problem

$$u(x) := \arg \min_{y \in \mathbb{R}^p} \frac{1}{2} \left\| \sum_{i=1}^p y_i \nabla c_i(x) - \nabla f(x) \right\|^2. \quad (1.2)$$

Fletcher’s penalty function and its variants [16, 67, 19] involve the first-order derivative of the original objective function $f$. Therefore, their differentiability depend on the second-order differentiability of $f$. Moreover, calculating the derivatives of these penalty functions requires the second-order derivative of $f$, which are not always available in practice. As a result, existing approaches based on Fletcher’s penalty function, such as approximated steepest descent methods [19], approximated Newton methods [55, 54, 67], and approximated quasi-Newton methods [19], are usually combined with certain approximation strategies to estimate higher-order derivatives. Hence, various existing unconstrained optimization approaches are not compatible with the Fletcher’s penalty function framework.

For optimization problems on the Stiefel manifold, i.e. $\mathcal{M} = S_{m,s} := \{ X \in \mathbb{R}^{m \times s} : X^\top X = I_s \}$, [63] presents an exact penalty model named PenC based on the explicit expression of the multipliers [22], which further yields efficient infeasible algorithms [63, 64, 31, 65]. However, PenC involves $\nabla f$ in its objective function as well. Therefore, those aforementioned limitations of Fletcher’s penalty function approaches still remain unsolved.

1.3 Constraint dissolving function

Very recently, for optimization problems on the Stiefel manifold, [62] proves that under mild conditions, all the stationary points of the following smooth penalty function are either its strict saddle points or are the first-order stationary points of the original problems:

$$\psi(X) := f \left( X \left( \frac{3}{2} I_s - \frac{1}{2} X^\top X \right) \right) + \frac{\beta}{4} \left\| X^\top X - I_s \right\|_F^2. \quad (\text{ExPen})$$

As a result, various algorithms designed for unconstrained optimization can be directly applied to optimization problems on the Stiefel manifold, while the convergence properties are straightforwardly retained.
The constraint dissolving approaches for Riemannian optimization proposed in this paper is motivated by (ExPen). To this end, we first introduce the following constraint dissolving operator $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$, which is a smooth mapping independent of $f$ and satisfies the following assumptions.

**Assumption 1.2. Blanket assumptions on $\mathcal{A}$**

- $\mathcal{A}$ is locally Lipschitz smooth in $\mathbb{R}^n$;
- $\mathcal{A}(x) = x$ holds for any $x \in \mathcal{M}$;
- The Jacobian of $c(\mathcal{A}(x))$ equals to 0 for any $x \in \mathcal{M}$. That is, $J_{\mathcal{A}}(x)J_{c}(x) = 0$ holds for any $x \in \mathcal{M}$, where $J_{\mathcal{A}}(x) \in \mathbb{R}^{n \times n}$ refers to the transposed Jacobian of $\mathcal{A}$ at $x$.

With the constraint dissolving operator, we propose the constraint dissolving function (CDF) for OCP:

$$h(x) := f(\mathcal{A}(x)) + \frac{\beta}{2} \|c(x)\|^2.$$  

(CDF)

Clearly, the first requirement in Assumption 1.2 guarantees the smoothness of $h(x)$. Meanwhile the second requirement ensures that $f(x) = h(x)$ for any $x \in \mathcal{M}$. Finally, the last requirement implies that the first-order derivative of $c(\mathcal{A}(x))$ vanishes at any feasible $x$. As a result, we can further conclude that $c(\mathcal{A}(x)) = O(\|c(x)\|^2)$ when $\|c(x)\|$ is sufficiently small, whose rigorous proof is presented later.

The practical choices of $\mathcal{A}$ are introduced in Section 4.

### 1.4 Contribution

In this paper, we propose a class of constraint dissolving approaches, which transfer OCP into minimizing the corresponding constraint dissolving function (CDF) in $\mathbb{R}^n$. We prove that OCP and CDF have the same first-order stationary points, second-order stationary points, and local minimizers in a neighborhood of $\mathcal{M}$. In addition, we show that CDF has the same Łojasiewicz exponent as OCP over $\mathcal{M}$. Furthermore, we show that the exact gradient and Hessian of CDF can be easily obtained.

As CDF requires a constraint dissolving operator $\mathcal{A}$ satisfying Assumption 1.2, we present representative formulations of $\mathcal{A}$ for many well-known Riemannian manifolds. Moreover, we discuss how to choose $\mathcal{A}$ for general cases and demonstrate that the general formulation does not involve any information on the objective function, and hence CDF is different from the Fletcher’s penalty function. More importantly, constructing CDF is completely independent of any geometrical material of $\mathcal{M}$. Since $\nabla h(x)$ is not necessarily restricted to the tangent space of $\mathcal{M}$ when $x$ is feasible, CDF waives all the calculations of geometrical materials of $\mathcal{M}$, including computing Riemannian gradients, retractions, and vector transports on $\mathcal{M}$. Therefore, we can develop various constraint dissolving approaches to solve optimization problems over a broad class of Riemannian manifolds, without prior knowledge of their geometrical properties.

The convergence properties, including the global convergence and iteration complexity of applying any unconstrained optimization approach to CDF can be guaranteed by a unified framework. We also present a representative example to demonstrate how to adopt CDF and invoke the theoretical framework. These examples further highlight the significant advantages and great potentials of CDF.

### 1.5 Organization

The rest of this paper is arranged as follows. In Section 2, we present some notations, definitions, and constants that are necessary for concise narrative in later parts of the paper. We establish the theoretical properties of CDF and illustrate how our proposed constraint dissolving approaches inherit the convergence properties from the implemented unconstrained approaches in Section 3. The proofs for the theoretical properties of CDF are presented in the appendix. In Section 4, we discuss how to choose the constraint dissolving operator $\mathcal{A}$ for CDF. We conclude the paper in the last section.
2 Notations, definitions and constants

2.1 Notations

Let range(A) be the subspace spanned by the column vectors of matrix A, and \( \| \cdot \| \) represents the \( \ell_2 \)-norm of a vector or an operator. The notations \( \text{diag}(A) \) and \( \text{Diag}(x) \) stand for the vector formed by the diagonal entries of a matrix A, and the diagonal matrix with the entries of \( x \in \mathbb{R}^n \) as its diagonal, respectively. We denote the smallest and largest eigenvalues of \( A \) by \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \), respectively. Besides, \( \sigma_{\min}(A) \) refers to the smallest singular value of matrix \( A \). Furthermore, for any matrix \( A \in \mathbb{R}^{n \times p} \), the pseudo-inverse of \( A \) is denoted by \( A^\dagger \in \mathbb{R}^{p \times n} \), which satisfies \( AA^\dagger A = A \), \( A^\dagger AA^\dagger = A^\dagger \), and both \( A^\dagger A \) and \( AA^\dagger \) are symmetric [25].

For any \( x \in \mathcal{M} \), we denote \( T_x := \{ d \in \mathbb{R}^n : d^\top J_c(x) = 0 \} = \text{Null}(J_c(x)\top) \) and \( N_x := \{ d \in \mathbb{R}^n : d^\top u = 0, \forall u \in T_x \} = \text{range}(J_c(x)) \) as the tangent and normal spaces of \( \mathcal{M} \) at \( x \), respectively.

For any \( x \in \mathbb{R}^n \), we define the projection from \( x \in \mathbb{R}^n \) to \( \mathcal{M} \) as
\[
\text{proj}(x, \mathcal{M}) := \arg\min_{y \in \mathcal{M}} \| x - y \|.
\]

It is worth mentioning that the optimality condition of the above problem leads to the fact that \( x - \text{proj}(x, \mathcal{M}) \in \text{range}(J_c(\text{proj}(x, \mathcal{M}))) \). Furthermore, \( \text{dist}(x, \mathcal{M}) \) refers to the distance between \( x \) and \( \mathcal{M} \), i.e.
\[
\text{dist}(x, \mathcal{M}) := \| x - \text{proj}(x, \mathcal{M}) \|.
\]

The transposed Jacobian of the mapping \( \mathcal{A} \) is denoted as \( J_{\mathcal{A}}(x) \in \mathbb{R}^{n \times n} \). Recall the definition of \( J_c(x) \), and let \( c_i \) and \( A_i \) be the \( i \)-th coordinate of the mapping \( c \) and \( \mathcal{A} \), respectively. Then \( J_c \) and \( J_{\mathcal{A}} \) can be expressed by
\[
J_c(x) := \begin{bmatrix}
\frac{\partial c_1(x)}{\partial x_1} & \cdots & \frac{\partial c_p(x)}{\partial x_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial c_1(x)}{\partial x_n} & \cdots & \frac{\partial c_p(x)}{\partial x_n}
\end{bmatrix} \in \mathbb{R}^{p \times n}, \quad \text{and} \quad J_{\mathcal{A}}(x) := \begin{bmatrix}
\frac{\partial A_1(x)}{\partial x_1} & \cdots & \frac{\partial A_p(x)}{\partial x_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial A_1(x)}{\partial x_n} & \cdots & \frac{\partial A_p(x)}{\partial x_n}
\end{bmatrix} \in \mathbb{R}^{p \times n}.
\]

Besides, \( \mathcal{D}_{J_c}(x) : d \mapsto \mathcal{D}_{J_c}(x)[d] \) denotes the second-order derivative of the mapping \( \mathcal{A} \), which can be regarded as a linear mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^{n \times n} \) and satisfies \( \mathcal{D}_{J_c}(x)[d] = \lim_{t \to 0} \frac{1}{t} (J_c(x + td) - J_c(x)) \in \mathbb{R}^{n \times n} \). Similarly, \( \mathcal{D}_{J_{\mathcal{A}}}(x) \) refers to the second-order derivative of the mapping \( c \), which satisfies \( \mathcal{D}_{J_{\mathcal{A}}}(x)[d] = \lim_{t \to 0} \frac{1}{t} (J_{\mathcal{A}}(x + td) - J_{\mathcal{A}}(x)) \in \mathbb{R}^{p \times p} \). Additionally, we set
\[
\mathcal{A}^k(x) := \mathcal{A}(\mathcal{A}(\cdots \mathcal{A}(x) \cdots)),
\]
for \( k \geq 1 \), and define \( \mathcal{A}^0(x) := x \), \( \mathcal{A}^\infty(x) := \lim_{k \to \infty} \mathcal{A}^k(x) \). Furthermore, we denote \( g(x) := f(\mathcal{A}(x)) \) and use \( \nabla f(\mathcal{A}(x)) \) to denote \( \nabla f(z) |_{z=\mathcal{A}(x)} \) in the rest of this paper.

2.2 Definitions

We first state the first-order optimality condition of \( \text{OCP} \) as follows.

**Definition 2.1 ([47]).** Given \( x \in \mathbb{R}^n \), we say \( x \) is a first-order stationary point of \( \text{OCP} \) if there exists \( \bar{\lambda} \in \mathbb{R}^p \) that satisfies
\[
\begin{align*}
\nabla f(x) - \sum_{i=1}^p \bar{\lambda}_i \nabla c_i(x) &= 0, \\
c(x) &= 0.
\end{align*}
\]
(2.1)

For any given \( x \in \mathcal{M} \), we define \( \lambda(x) \) as
\[
\lambda(x) := J_c(x)^\top \nabla f(x) \in \arg\min_{\lambda \in \mathbb{R}^p} \| \nabla f(x) - J_c(x) \lambda \|,
\]
where \( J_c(x)^\top = J_c(x) (J_c(x)^\top J_c(x))^{-1} \) since \( J_c(x) \) has full column rank when \( x \in \mathcal{M} \). Then it can be easily verified that
\[
\nabla f(x) = J_c(x) \lambda(x).
\] (2.2)

**Definition 2.2.** Given \( x \in \mathbb{R}^n \), we say \( x \) is a second-order stationary point of \( \text{OCP} \) if \( x \) is a first-order stationary point of \( \text{OCP} \) and for any \( d \in T_x \), it holds that
\[
d^\top \left( \nabla^2 f(x) - \sum_{i=1}^p \lambda_i(x) \nabla^2 c_i(x) \right) d \geq 0.
\] (2.3)

**Definition 2.3.** Given \( x \in \mathcal{M} \), we define the Riemannian gradient of \( f \) at \( x \) as
\[
\text{grad} f(x) := \nabla f(x) - J_c(x) \lambda(x).
\] (2.4)

Besides, \( \text{hess} f(x) \) refers to the Riemannian Hessian of \( f \) at \( x \) that is defined as the following self-adjoint linear transform \( \text{hess} f(x) : T_x \to T_x \) such that
\[
d^\top \text{hess} f(x)d = d^\top \left( \nabla^2 f(x) - \sum_{i=1}^p \lambda_i(x) \nabla^2 c_i(x) \right) d, \quad \text{for any} \ d \in T_x.
\]

Furthermore, the smallest eigenvalue of \( \text{hess} f(x) \) is defined as
\[
\lambda_{\min} (\text{hess} f(x)) := \min_{d \in T_x, \|d\|=1} d^\top \text{hess} f(x)d.
\]

Given \( x \in \mathcal{M} \), let \( U_x \) be a matrix whose columns forms an orthonormal basis of \( T_x \), we define the projected Hessian of \( \text{OCP} \) at \( x \) as
\[
\mathcal{H}(x) := U_x^\top \left( \nabla^2 f(x) - \sum_{i=1}^p \lambda_i(x) \nabla^2 c_i(x) \right) U_x,
\] (2.5)

The following proposition characterizes the relationship between \( \mathcal{H}(x) \) and \( \text{hess} f(x) \).

**Proposition 2.4.** Given any \( x \in \mathcal{M} \), suppose \( x \) is a first-order stationary point of \( \text{OCP} \) then \( \mathcal{H}(x) \) and \( \text{hess} f(x) \) have the same eigenvalues. Moreover, \( \lambda_{\min} (\text{hess} f(x)) = \lambda_{\min} (\mathcal{H}(x)) \) and \( x \) is a second-order stationary point of \( \text{OCP} \) if and only if \( \mathcal{H}(x) \succeq 0 \).

The proof of the above proposition directly follows from the Definition 2.3 and [3], and hence we omit it for simplicity.

**Definition 2.5.** Given \( x \in \mathbb{R}^n \), we say \( x \) is a first-order stationary point of \( \text{CDF} \) if
\[
\nabla h(x) = 0.
\] (2.6)

Besides, when \( h \) is twice-order differentiable, we say a point \( x \in \mathbb{R}^n \) is a second-order stationary point of \( h \) if \( x \) is a first-order stationary point of \( h \) and satisfies
\[
\nabla^2 h(x) \succeq 0.
\] (2.7)

Next, we present the definition of the Łojasiewicz gradient inequality [39, 40, 7], which is a powerful tool in analyzing the convergence of various unconstrained optimization approaches.

**Definition 2.6.** Given \( x \in \mathbb{R}^n \), the function \( f \) is said to satisfy the (Euclidean) Łojasiewicz gradient inequality at \( x \) if and only if there exist a neighborhood \( U \) of \( x \), and constants \( \theta \in (0, 1], C > 0 \), such that the following inequality holds for any \( y \in U \),
\[
\|\nabla f(y)\| \geq C |f(y) - f(x)|^{1-\theta}.
\]

The Łojasiewicz gradient inequality on a Riemannian manifold [30] can be similarly defined in the following definition.
2.3 Constants

Based on these constants, we further set
\[ \rho_x := \arg \max_{0 < \rho \leq 1} \rho \quad \text{s.t.} \quad \inf \{ \sigma_{\min}(f(x)) : y \in \mathbb{R}^n, \|y - x\| \leq \rho \} \geq \frac{1}{2} \sigma_{\min}(f(c)). \]

The constant \( \theta \) is usually referred as (Riemannian) Łojasiewicz exponent in the gradient inequality, which is simply abbreviated as (Riemannian) Łojasiewicz exponent.

Definition 2.7. Given \( x \in \mathcal{M} \), the function \( f \) is said to satisfy the Riemannian Łojasiewicz gradient inequality at \( x \) if and only if there exist a neighborhood \( U \subset \mathcal{M} \) of \( x \), and constants \( \theta \in (0, 1], C > 0 \), such that the following inequality holds for any \( y \in U \),
\[ \|\nabla f(x)\| \geq C|f(y) - f(x)|^{1-\theta}. \]

It is worth mentioning that Assumption 1.1 guarantees that \( \sigma_{\min} > 0 \) for any given \( x \in \mathcal{M} \), which implies that \( \epsilon_x > 0 \). On the other hand, we can conclude that \( \Omega_x \subset \Omega_x \subset \Theta_x \) holds for any given \( x \in \mathcal{M} \), and \( \mathcal{M} \) lies in the interior of \( \Omega \).

Definition 2.8. For any given \( x \in \mathcal{M} \), we set
\[ \beta_x := \max \left\{ \frac{128L_{x,g}(M_{x,A} + 1)^2}{\sigma_{x,c}^2}, \frac{64M_{x,f}(M_{x,A} + 1)L_{x,b}}{\sigma_{x,c}^3}, \frac{16M_{x,f}L_{x,A}(2M_{x,A} + 1)}{\sigma_{x,c}^2} \right\}. \]
Remark 2.9. When the manifold $\mathcal{M}$ is compact, there exists a finite set $\mathcal{I} \subset \mathcal{M}$ such that $\mathcal{M} \subseteq \bigcup_{x \in \mathcal{I}} \Theta_{x}$. Therefore, we can choose uniform positive lower bounds for $\sigma_{x,c}$ and $\varepsilon_{y}$, while find uniform upper bounds for all the other aforementioned constants. Specifically, we can choose a uniform upper bound for $\beta_{y}$, which can be marked as a threshold. Any $\beta$ greater than this threshold can determine an exact penalty function for $\text{CDF}$.

Finally, the following assumption is needed when we discuss the second-order stationarity of $\text{CDF}$.

**Assumption 2.10. Assumption on twice-order differentiability**

- $f$, $c$ and $\mathcal{A}$ are twice differentiable in $\mathbb{R}^{n}$.

## 3 Theoretical Results

In this section, we present some theoretical properties of $\text{CDF}$. We begin with the characteristics of the mapping $\mathcal{A}$ in Section 3.1. Then we investigate the stationarity of $\text{CDF}$ as well as the Łojasiewicz exponents in Section 3.2. Finally, in Section 3.3 we propose a framework showing that the convergence properties of $\text{CDF}$ can directly be inherited from those of the applied unconstrained optimization approaches.

### 3.1 Theoretical properties of $\mathcal{A}$

We start with evaluating the relationships among $\|c(x)\|$, $\|c(\mathcal{A}(x))\|$ and $\text{dist}(x, \mathcal{M})$ in the following lemmas.

#### Lemma 3.1

For any given $x \in \mathcal{M}$, the following inequalities hold for any $y \in \Omega_{x}$,

$$
\frac{1}{\sigma_{x,c}} \|c(y)\| \leq \text{dist}(y, \mathcal{M}) \leq \frac{2}{\sigma_{x,c}} \|c(y)\|. \quad (3.1)
$$

**Proof.** Let $z = \text{proj}(y, \mathcal{M})$, then from the definition of $z$ we can conclude that $\|z - y\| \leq \|y - x\|$. Thus $\|z - x\| \leq \|z - y\| + \|y - x\| \leq 2\varepsilon_{y} \leq \rho_{x}$, and hence $z \in \Theta_{x}$. By the mean-value theorem, for any fixed $v \in \mathbb{R}^{n}$ there exists a point $\tilde{c} \in \mathbb{R}^{n}$ that is a convex combination of $y$ and $z$ such that $v^\top c(y) = (y - x)^\top J_{c}(\tilde{c})v$. By the convexity of $\Theta_{x}$, $\tilde{c} \in \Theta_{x}$ holds for any $v \in \mathbb{R}^{n}$. Therefore, we get

$$
\|c(y)\| = \sup_{v \in \mathbb{R}^{n}, \|v\| = 1} v^\top c(y) = \sup_{v \in \mathbb{R}^{n}, \|v\| = 1} (y - x)^\top J_{c}(\tilde{c})v \leq \sup_{v \in \Theta_{x}} \|J_{c}(\tilde{c})\| \|y - z\| \leq \sigma_{x,c} \text{dist}(y, \mathcal{M}).
$$

Moreover, it follows from the definition of $z$ that $y - z \in \text{range}(J_{c}(z))$. As a result, let $\tilde{v} = \frac{J_{c}(z)^\top (y - z)}{\|J_{c}(z)^\top (y - z)\|}$, we have

$$
\|c(y)\| = \sup_{v \in \mathbb{R}^{n}, \|v\| = 1} (y - z)^\top J_{c}(\tilde{c})v \geq (y - z)^\top J_{c}(\tilde{c})\tilde{v} = (y - z)^\top J_{c}(z)\tilde{v} + (y - z)^\top (J_{c}(z) - J_{c}(\tilde{c}))\tilde{v} \geq \|J_{c}(z)^\top (y - z)\| - \rho_{x,c} \|y - z\|^{2} \geq (\sigma_{x,c} - \varepsilon_{y} \rho_{x,c}) \text{dist}(y, \mathcal{M}) \geq \frac{\sigma_{x,c}}{2} \text{dist}(y, \mathcal{M}).
$$

#### Lemma 3.2

For any given $x \in \mathcal{M}$, it holds that

$$
\|\mathcal{A}(y) - y\| \leq \frac{2(M_{x,A} + 1)}{\sigma_{x,c}} \|c(y)\|, \quad \text{for any } y \in \Omega_{x}. \quad (3.2)
$$
Proof. For any given $y \in \Omega_x$, we define $z = \text{proj}(y, \mathcal{M})$. Then we can conclude that $z \in \Theta_x$. Furthermore, from the Lipschitz continuity of $\mathcal{A}$ and the fact that $\mathcal{A}(z) - z = 0$, it holds that

$$
\|\mathcal{A}(y) - y\| = \|(\mathcal{A}(y) - y) - (\mathcal{A}(z) - z)\| \leq (M_{x,A} + 1)\text{dist}(y, \mathcal{M}) \leq \frac{2(M_{x,A} + 1)}{\sigma_{x,c}} \|c(y)\|, \quad (3.3)
$$

where the last inequality follows from Lemma 3.1.

**Lemma 3.3.** For any given $x \in \mathcal{M}$, it holds that

$$
\|c(\mathcal{A}(y))\| \leq \frac{4L_{x,b}}{\sigma_{x,c}^2} \|c(y)\|^2, \quad \text{for any } y \in \Omega_x. \quad (3.4)
$$

Proof. For any given $y \in \Omega_x$, we define $z = \text{proj}(y, \mathcal{M})$. It holds that $z \in \Theta_x$. By the mean-value theorem, for any $v \in \mathbb{R}_p$, there exists $t_v \in [0, 1]$ and $\xi_v = t_vy + (1 - t_v)z$ such that

$$
v^\top c(\mathcal{A}(y)) = v^\top (\mathcal{A}(\xi_v)) = v^\top (\mathcal{A}(\xi_v)) (y - z). \quad (3.5)
$$

The convexity of $\Theta_x$ ensures that $\xi_v \in \Theta_x$ holds for any $v \in \mathbb{R}_p$. Therefore, from the definition of $L_{x,b}$ and $\Omega_x$, we get

$$
\|c(\mathcal{A}(y))\| = \sup_{v \in \mathbb{R}_p, \|v\| = 1} v^\top c(\mathcal{A}(y)) = \sup_{v \in \mathbb{R}_p, \|v\| = 1} v^\top (\mathcal{A}(\xi_v)) (y - z) \leq \sup_{v \in \mathbb{R}_p, \|v\| = 1} \|\mathcal{A}(\xi_v)\| \|y - z\| \leq \sup_{v \in \mathbb{R}_p, \|v\| = 1} \|\mathcal{A}(\xi_v)\| \text{dist}(y, \mathcal{M}) \leq L_{x,b} \text{dist}(y, \mathcal{M}) \leq L_{x,b} \|y - z\| \leq \frac{4L_{x,b}}{\sigma_{x,c}^2} \|c(y)\|^2,
$$

where the last inequality follows from Lemma 3.1.

For any given $x \in \mathcal{M}$ and $y \in \Omega_x$, Lemma 3.3 illustrates that the operator $\mathcal{A}$ can reduce the feasibility violation of $y$ quadratically when $y$ is sufficiently close to $\mathcal{M}$.

Next, we present the theoretical property of $\mathcal{A}^\infty(y)$.

**Lemma 3.4.** For any given $x \in \mathcal{M}$ and any $y \in \bar{\Omega}_x$, $\mathcal{A}^\infty(y)$ exists and $\mathcal{A}^\infty(y) \in \Omega_x \cap \mathcal{M}$. Moreover, it holds that

$$
\|\mathcal{A}^\infty(y) - y\| \leq \frac{4(M_{x,A} + 1)}{\sigma_{x,c}} \|c(y)\|. \quad (3.7)
$$

The proof for Lemma 3.4 is presented in Section A.1.

In the rest of this subsection, we study the properties of $J_A(x)$. The following lemmas characterize the range space and null space of $J_A(x)$ for any given $x \in \mathcal{M}$.

**Lemma 3.5.** For any given $x \in \mathcal{M}$, the inclusion $J_A(x) \mathbb{T}_x$ holds for any $d \in \mathbb{R}^n$. Moreover, when $d \in \mathbb{T}_x$, it holds that $J_A(x) \mathbb{T}_x$ holds for any $d \in \mathbb{R}^n$. Moreover, when $d \in \mathbb{T}_x$, it holds that $J_A(x) \mathbb{T}_x$ holds for any $d \in \mathbb{R}^n$.

Proof. Firstly, for any $d \in \mathbb{R}^n$, and any $d_1 \in \mathbb{N}_x$, from the fact that $J_A(x)J_A(x) = 0$, we can conclude that $J_A(x)\mathbb{T}_x$ holds for any $d_1 \in \mathbb{N}_x$, which implies that $J_A(x)\mathbb{T}_x$ holds for any $d \in \mathbb{R}^n$. On the other hand, by the Taylor expansion of $c(x + td)$ up to the second-order term, we have

$$
\|c(x + td)\| \leq M_{x,c} \|d\|^2 \quad \text{holds for any } d \in \mathbb{T}_x.
$$

In addition, when $d \in \mathbb{T}_x$, we have that $x + td \in \Omega_x$ for $t \in (0, 1)$ sufficiently small. Then by Lemma 3.2, we get

$$
\|x + td - A(x + td)\| \leq \frac{2(M_{x,A} + 1)}{\sigma_{x,c}} \|c(x + td)\| \leq \frac{2(M_{x,A} + 1)M_{x,c}}{\sigma_{x,c}} \|d\|^2 t^2.
$$
By Assumption 1.2, \( A(x) = x \). Thus we have
\[
J_A(x)^\top d = \lim_{t \to 0} \frac{A(x + td) - A(x)}{t} = \lim_{t \to 0} \frac{x + td - x}{t} = d,
\]
which completes the proof. \( \square \)

**Lemma 3.6.** For any given \( x \in \mathcal{M} \), the equality \( J_A(x)d = 0 \) holds, if and only if \( d \in \mathcal{N}_x \).

**Proof.** For any \( d_2 \) satisfying \( J_A(x)d_2 = 0 \) and any \( d_1 \in \mathcal{T}_x \), by Lemma 3.3 we have \( 0 = d_1^\top J_A(x)d_2 = d_1^\top d_2 \), which implies that \( d_2 \in \mathcal{N}_x \).

On the other hand, let \( d_2 \in \mathcal{N}_x \). Then \( d_2 = J_c(x)\eta \) for some \( \eta \in \mathbb{R}^p \). By Assumption 1.2, \( J_A(x)d_2 = J_A(x)J_c(x)\eta = 0 \). This completes the proof. \( \square \)

Next we show the idempotence of \( J_A(x) \), i.e. \( J_A(x)^2 = J_A(x) \) holds for any \( x \in \mathcal{M} \).

**Lemma 3.7.** For any given \( x \in \mathcal{M} \), it holds that \( J_A(x)^2 = J_A(x) \).

**Proof.** For any \( d_1 \in \mathcal{T}_x \), let \( d_2 = J_A(x)d_1 - d_1 \). Then for any \( d_3 \in \mathcal{T}_x \), it follows from Lemma 3.5 that \( d_3^\top d_2 = d_3^\top J_A(x)d_1 - d_3^\top d_1 = 0 \), which further implies \( d_2 \in \mathcal{N}_x \). On the other hand, for any \( d_4 \in \mathcal{N}_x \), it follows from Lemma 3.6 that \( J_A(x)d_4 = 0 \).

Therefore, for any \( d_1 \in \mathcal{T}_x \) and any \( d_4 \in \mathcal{N}_x \), it holds that
\[
J_A(x)^2(d_1 + d_4) = J_A(x)^2d_1 = J_A(x)d_1 + J_A(x)d_2 = J_A(x)d_1 = J_A(x)(d_1 + d_4).
\]
Therefore, we complete the proof by recalling the arbitrariness of \( d_1 \in \mathcal{T}_x \) and \( d_4 \in \mathcal{N}_x \). \( \square \)

### 3.2 Theoretical properties of \( \text{CDF} \)

This subsection investigates the relationships between \( \text{OCP} \) and \( \text{CDF} \) on their stationary points, Łojasiewicz exponents and local minimizers. We start by presenting the explicit expression for the gradient and Hessian of \( \text{CDF} \).

**Proposition 3.8.** The gradient of \( h \) in \( \text{CDF} \) can be expressed as
\[
\nabla h(x) = J_A(x)\nabla f(A(x)) + \beta J_c(x)c(x). \quad (3.8)
\]
Furthermore, under Assumption 2.10 the Hessian of \( g(x) := f(A(x)) \) can be expressed as
\[
\nabla^2 g(x) = J_A(x)\nabla^2 f(A(x))J_A(x)^\top + \mathcal{D}J_A(x)[\nabla f(A(x))]. \quad (3.9)
\]
and the Hessian of \( h(x) \) is
\[
\nabla^2 h(x) = \nabla^2 g(x) + \beta \left( J_c(x)J_c(x)^\top + \mathcal{D}J_c(x)[c(x)] \right). \quad (3.10)
\]

The statements of Proposition 3.8 can be verified by straightforward calculations, and hence its proof is omitted.

#### 3.2.1 Relationships on stationary points

This subsection shows that \( \text{OCP} \) and \( \text{CDF} \) share the same first-order and second-order stationary points in a neighborhood of \( \mathcal{M} \). Moreover, for any given \( x \in \mathcal{M} \) and \( \beta \geq \beta_x \), we can prove that all first-order stationary points of \( \text{CDF} \) in \( \Omega_x \) are feasible. For a more concise presentation, we put all the proofs for Proposition 3.9, Theorem 3.10 and Theorem 3.11 in Appendix A.2.1.

**Proposition 3.9.** Any first-order stationary point of \( \text{OCP} \) is a first-order stationary point of \( \text{CDF} \). On the other hand, any first-order stationary point of \( \text{CDF} \) over \( \mathcal{M} \) is a first-order stationary point of \( \text{OCP} \).
Theorem 3.10. For any given $x \in \mathcal{M}$, suppose $\beta \geq \beta_x$, then the following inequality holds for any $y \in \Omega_x$,
\[
\|\nabla h(y)\| \geq \frac{\beta \sigma_x c}{8(M_{x,A} + 1) c(y)}.
\] (3.11)
Moreover, any first-order stationary point of $\mathcal{CDF}$ in $\Omega_x$ is a first-order stationary point of $\mathcal{OCP}$.

Theorem 3.11. Suppose Assumption 2.10 holds. Any second-order stationary point of $\mathcal{CDF}$ is a second-order stationary point of $\mathcal{OCP}$. On the other hand, for any given first-order stationary point $x$ of $\mathcal{OCP}$ if $\beta \geq \beta_x$, then $x$ is a second-order stationary point of $\mathcal{CDF}$.

3.2.2 Stationarity at infeasible points

From Lemma 3.3, we know that the constraint dissolving operator $\mathcal{A}$ can quadratically reduce the feasibility violation of any infeasible point $x \in \bar{\Omega}$. Together with Lemma 3.3, $\mathcal{A}^\infty(x)$ is feasible and it is in a neighborhood of $x$. The relationships between $x$ and $\mathcal{A}^\infty(x)$ in terms of the function values and derivatives are of great importance in characterizing the properties of $\mathcal{CDF}$ at those infeasible points.

The detailed proofs of Proposition 3.12 – Proposition 3.15 are presented in Appendix A.2.2.

Proposition 3.12. For any given $x \in \mathcal{M}$, suppose $\beta \geq \beta_x$, then the following inequalities hold for any $y \in \Omega_x$
\[
\begin{align*}
    h(\mathcal{A}(y)) - h(y) &\leq -\frac{\beta}{8} \|c(y)\|^2, \\
    h(\mathcal{A}^\infty(y)) - h(y) &\leq -\frac{\beta}{4} \|c(y)\|^2.
\end{align*}
\] (3.12) (3.13)

When we invoke a specific unconstrained optimization algorithm to solve $\mathcal{CDF}$ we usually terminate the algorithm once the stopping criterion reaches a certain tolerance, meanwhile the feasibility violation at the returned solution $x$ may not be sufficiently small. To pursue a solution with high feasibility accuracy, certain post-processing step should be imposed. As we have mentioned, $\mathcal{A}^\infty(x)$ is feasible if $x$ is sufficiently close to $\mathcal{M}$. Thus we can recursively compute $x \leftarrow \mathcal{A}(x)$ until the desired accuracy on feasibility is satisfied.

Proposition 3.13. For any given $x \in \mathcal{M}$, suppose $\beta \geq \beta_x$, then it holds that
\[
\|\nabla h(y)\| \geq \frac{1}{2} \|\nabla \mathcal{A}(\mathcal{A}^\infty(y))\|, \quad \text{for all } y \in \Omega_x.
\] (3.14)
Proposition 3.13 shows that $\|\nabla \mathcal{A}^\infty(y)\|$ can be controlled by $\|\nabla h(y)\|$. Next, we study the relationship between the Riemannian Hessian of $f$ at $\mathcal{A}^\infty(y)$ and the Hessian of $h$ at $y$.

Proposition 3.14. Suppose Assumption 2.10 holds, then the following inequalities hold for any $x \in \mathcal{M}$
\[
\begin{align*}
    \lambda_{\min}(\mathcal{hess} f(x)) &\geq \lambda_{\min}(\nabla^2 h(x)) - L_{x,A} \|\nabla f(x)\|, \\
    \lambda_{\max}(\mathcal{hess} f(x)) &\leq \lambda_{\max}(\nabla^2 h(x)) + L_{x,A} \|\nabla f(x)\|.
\end{align*}
\] (3.15) (3.16)

Proposition 3.15. Suppose Assumption 2.10 holds. For any given $x \in \mathcal{M}$ and $\beta \geq \beta_x$, the following inequality holds for any $y \in \Omega_x$,
\[
\begin{align*}
    \lambda_{\min}(\mathcal{hess} f(\mathcal{A}^\infty(y))) &\geq \lambda_{\min}(\nabla^2 h(y)) - \left\|\nabla^2 g(\mathcal{A}^\infty(y)) - \nabla^2 g(y)\right\| \\
    &\quad - \left(\frac{5}{2} L_{x,A} + \frac{96 L_{x,A} M_{x,c} (M_{x,A} + 1)^2}{\sigma_x^2 c}\right) \|\nabla h(y)\|.
\end{align*}
\] (3.17)
3.2.3 Łojasiewicz exponents and local minimizers

The following proposition guarantees the fact that OCP and CDF share the same Łojasiewicz exponents at any \( x \in M \).

**Proposition 3.16.** For any given \( x \in M \), suppose \( \beta \geq \beta_x \) and OCP satisfies the Riemannian Łojasiewicz gradient inequality at \( x \) with Riemannian Łojasiewicz exponent \( \theta \in (0, \frac{1}{2}] \), then CDF satisfies the (Euclidean) Łojasiewicz gradient inequality at \( x \) with Łojasiewicz exponent \( \theta \).

At the end of this subsection, we establish the relationship on the local minimizers between OCP and CDF.

**Theorem 3.17.** For any given \( x \in M \), suppose \( \beta \geq \beta_x \), then any local minimizer of OCP in \( \bar{\Omega}_x \) is a local minimizer of CDF. Moreover, any local minimizer of CDF in \( \bar{\Omega}_x \) is a local minimizer of OCP.

The detailed proofs of Proposition 3.16 and Proposition 3.17 are presented in Appendix A.2.3.

3.3 Constraint dissolving approach and theoretical analysis

In this subsection, we show how to establish the convergence properties of the proposed constraint dissolving approaches directly from existing results. In our proposed constraint dissolving approaches, we first construct the corresponding CDF for OCP, then we select a specific unconstrained optimization approach to minimize CDF. Moreover, we perform a post-processing procedure to achieve high accuracy in feasibility, if necessary. The details of the constraint dissolving approach are summarized in Algorithm 1.

**Algorithm 1** Constraint Dissolving Approach for OCP.

**Require:** Input data: manifold \( M \), objective function \( f \), penalty parameter \( \beta \), initial guess \( x_0 \), stationarity tolerance \( \epsilon_s \), feasibility tolerance \( \epsilon_f \), and a selected unconstrained optimization approach (UCO).

1. Construct the constraint dissolving function CDF.
2. Initiated from \( x_0 \), invoke the UCO to solve CDF and generate a sequence \( \{x_k\} \). Terminate when tolerance \( \epsilon_s \) is reached and obtain \( \bar{x} \).
3. if require post-processing then
4. while \( \|c(\bar{x})\| > \epsilon_f \) do
5. \( \bar{x} = A(\bar{x}) \).
6. end while
7. end if
8. Return \( \bar{x} \).

For convenience, we call the selected unconstrained optimization approach in Algorithm 1 as UCO, and let \( \{x_k\} \) be the iterates generated by Algorithm 1. We assume that there exists a compact set \( \hat{\Gamma} \subset \mathbb{R}^n \) such that \( \{x_k\} \subset \hat{\Omega} \cap \hat{\Gamma} \). Hence, \( \sup_{x \in M \cap \hat{\Gamma}} \beta_x < +\infty \) due to the compactness of \( M \cap \hat{\Gamma} \). We choose sufficiently large \( \beta \) such that \( \beta \geq \sup_{x \in M \cap \hat{\Gamma}} \beta_x \). Then we can adopt the following framework to establish the corresponding theoretical results.

- **Global convergence:** Theorem 3.10 illustrates that OCP and CDF have the same first-order stationary points in \( \bar{\Omega} \cap \hat{\Gamma} \). Therefore, if a cluster point of \( \{x_k\} \) is a first-order stationary point of CDF, then we can claim that it is a first-order stationary point of OCP. Moreover, Proposition 3.16 illustrates that OCP and CDF have the same Łojasiewicz gradient exponents. Therefore, when OCP satisfies the KL property, then the sequence convergence of \( \{x_k\} \) can be guaranteed by Proposition 3.16 and existing results, for instance [7], established for the selected unconstrained optimization approach in Algorithm 1.

- **Local convergence rate:** Theorem 3.17 shows that OCP and CDF have the same local minimizers in \( \bar{\Omega} \cap \hat{\Gamma} \). Together with Proposition 3.16, the local convergence rate of the sequence \( \{x_k\} \) can be established from prior works [48, 38, 36].
• **Worst case complexity:** Proposition 3.13 establishes the relationship between \( \| \nabla h(x_k) \| \) and \( \| \text{grad } f(\mathcal{A}^\infty(x_k)) \| \). Consequently, if Algorithm 1 produces an iterate \( x_k \) satisfying \( \| \nabla f(x_k) \| \leq \varepsilon_s \), then it holds that \( \| \text{grad } f(\mathcal{A}^\infty(x_k)) \| \leq 2\varepsilon_s \). As a result, the worst case complexity of Algorithm 1 can be obtained from prior works immediately \( \square \).

• **The ability of escaping from saddle points:** If a cluster point of \( \{ x_k \} \) is a second-order stationary point of CDP then Theorem 8.11 guarantees that this cluster point is a second-order stationary point of OCP as well. Moreover, for any sequence \( \{ x_k \} \) generated by Algorithm 1, Proposition 3.15 provides the relationship among \( \min_{x \in \mathcal{F}} \) \( \| \nabla h(x_k) \| \) and \( \lambda_{\text{min}}(\nabla^2 h(x_k)) \). Consequently, Algorithm 1 inherits the escaping-from-saddle-point properties from its UCO, while the theoretical analysis directly follows existing results in unconstrained optimization \( \square \).

Clearly, the existence of a bounded set \( \hat{\Omega} \cap \hat{\Gamma} \), which contains all iterates generated by Algorithm 1, is crucial for the establishment of the above-mentioned theoretical properties. In the rest of this subsection, we provide easy-to-verify conditions for the existence of such a compact set under a mild assumption, which covers a broad class of scenarios.

**Assumption 3.18. Assumption on the coercivity of \( f(x) \) over \( M \)**

The level set \( \Gamma^*_x := \{ y \in M : f(y) \leq f(x) \} \) is compact for any \( x \in M \).

Assumption 3.18 straightforwardly holds when \( M \) is compact, and it is commonly assumed in the literature \( \square \). Those extreme situations that Assumption 3.18 does not hold are out of the scope of this paper.

For any given \( x \in M \) and any constant \( \zeta > 0 \), we set \( \Gamma_{x, \zeta} := \{ y \in \mathbb{R}^n : \text{dist}(y, \Gamma^*_x) \leq \zeta + 1 \} \) and \( \mu_{x, \zeta} := \inf_{y \in \Gamma_{x, \zeta}} \| \nabla c(y) \|^2 \). From the definition of \( \hat{\Omega} \) and the compactness of \( \Gamma_{x, \zeta} \), we can conclude that \( \mu_{x, \zeta} \) holds for any \( x \in M \). In addition, we define \( M_{\Gamma_{x, \zeta}} := \sup_{y \in \Gamma_{x, \zeta}} g(y) - g(z) \), and \( \Xi_x := \{ z \in \hat{\Omega} \cap \Gamma_{x, \zeta} : \text{dist}(z, \Gamma^*_x) \leq 1/2 \} \). Then we introduce the following threshold value of \( \beta \).

**Definition 3.19.** For any given \( x \in M \) and any \( \zeta > 0 \), we define

\[
\beta_x := \max \left\{ \frac{4M_{\Gamma_{x, \zeta}}}{\mu_{x, \zeta}}, \sup_{w \in M \cap \Gamma_{x, \zeta}} \beta_w \right\}. \tag{3.18}
\]

**Proposition 3.20.** Suppose Assumption 3.18 holds. For any given \( x \in M \) and \( \zeta > 0 \), let \( \beta \geq \beta_x \), then it holds that

\[
\{ y \in \mathbb{R}^n : h(y) \leq h(x) \} \cap \Gamma_{x, \zeta} \subset \Xi_x. \tag{3.19}
\]

**Proof.** For any \( y \in \Gamma_{x, \zeta} \setminus \hat{\Omega} \), it holds from the definition of \( M_{\Gamma_{x, \zeta}} \) and \( \mu_{x, \zeta} \) that

\[
h(y) - h(x) = g(y) + \frac{\beta}{2} \| c(y) \|^2 - g(x) \geq -(M_{\Gamma_{x, \zeta}} + \frac{\mu_{x, \zeta} \beta}{2}) > 0. \tag{3.20}
\]

Moreover, for any \( y \in (\hat{\Omega} \cap \Gamma_{x, \zeta}) \setminus \Xi_x \), we show that \( \mathcal{A}^\infty(y) \notin \Gamma^*_x \) by contradiction. Suppose on the contrary that \( \mathcal{A}^\infty(y) \in \Gamma^*_x \). Then Lemma 3.4 demonstrates that

\[
\text{dist}(y, \Gamma^*_x) \leq \| \mathcal{A}^\infty(y) - y \| \leq \sup_{z \in \Gamma_{x, \zeta} \cap M} \varepsilon_z \leq \sup_{z \in \Gamma_{x, \zeta} \cap M} \frac{\varepsilon_z}{2} \leq \frac{1}{2}.
\]

As a result, \( y \in \Xi_x \) and this contradicts the fact that \( y \in \hat{\Omega} \setminus \Xi_x \). Therefore, \( \mathcal{A}^\infty(y) \notin M \setminus \Gamma^*_x \), and from Proposition 3.12 we obtain that

\[
h(y) \geq h(\mathcal{A}^\infty(y)) = f(\mathcal{A}^\infty(y)) > f(x) = h(x). \tag{3.21}
\]

This completes the proof. \( \square \)
The following corollary illustrates that with the help of Assumption 3.18, we can actually further relax the requirement \( \{x_k\} \subset \bar{\Omega} \cap \Gamma \) to \( x_0 \in \bar{\Omega} \cap \Gamma \) under mild conditions.

**Corollary 3.21.** Given any \( x_0 \in \mathcal{M} \) and \( \zeta > 0 \), suppose Assumption 3.18 holds, \( \beta \geq \beta_{x_0} \) and Algorithm 1 generates a sequence \( \{x_k\} \) that satisfies \( h(x_k) \leq h(x_0) \) and \( \|x_{k+1} - x_k\| \leq \zeta \) for any \( k \geq 0 \). Then it holds that \( \{x_k\} \subset \bar{\Omega} \cap \Gamma_{x_0,\zeta} \).

**Proof.** We prove the inclusion \( \{x_k\} \subset \bar{\Omega} \cap \Gamma_{x_0,\zeta} \) by induction. Suppose \( x_j \in \bar{\Omega} \cap \Gamma_{x_0,\zeta} \) for \( 0 \leq j \leq k \). Then Proposition 3.20 and the fact that \( h(x_k) \leq h(x_0) \) implies that \( x_k \in \Xi_{x_0} \). Moreover, it follows from the definition of \( \Gamma_{x_0,\zeta} \) and \( \Xi_{x_0} \) that

\[
\text{dist}(\mathbb{R}^n \setminus \Gamma_{x_0,\zeta}, \Xi_{x_0}) \geq \zeta.
\]

Therefore, the fact that \( \|x_{k+1} - x_k\| \leq \zeta \) implies that \( x_{k+1} \in \Gamma_{x_0,\zeta} \). Furthermore, together with Proposition 3.20 and the fact that \( h(x_{k+1}) \leq h(x_0) \), we arrive at \( x_{k+1} \in \Xi_{x_0} \subset \bar{\Omega} \cap \Gamma_{x_0,\zeta} \). Namely, the inclusion \( x_j \in \bar{\Omega} \cap \Gamma_{x_0,\zeta} \) holds for \( 0 \leq j \leq k \). Then by induction, we obtain that \( \{x_k\} \subset \bar{\Omega} \cap \Gamma_{x_0,\zeta} \) holds for any \( k \geq 0 \).

**Remark 3.22.** The conditions in Corollary 3.21 are not restrictive at all. By choosing any monotone algorithm or algorithm that employ nonmonotone line search techniques [27] as UCO, it is easy to guarantee that the relationship \( h(x_k) \leq h(x_0) \) holds for any \( k \geq 0 \). Moreover, the condition that \( \|x_{k+1} - x_k\| \leq \zeta \) holds for any \( k \geq 0 \) is also priorly verifiable in most cases. For example, we can choose the UCO in Algorithm 1 as a line-search method with maximal stepsize, a trust-region method with maximal radius [66], or a cubic regularization method with an appropriate regularization parameter [43]. For the other situations, we can prefix a large \( \zeta \) as a loose upper-bound for the distance between two consecutive iterates. Additionally, when \( \{x_k\} \) has sequential convergence, the restriction \( \|x_{k+1} - x_k\| \leq \zeta \) naturally holds for any sufficiently large \( k \).

### 4 Implementation

In this section, we first show that we can construct the constraint dissolving operator \( \mathcal{A} \) directly from \( c(x) \) without any prior knowledge of the geometrical properties of \( \mathcal{M} \). In addition, we provide easy-to-compute formulations of \( \mathcal{A} \) for several well-known Riemannian manifolds, such as the Stiefel manifold, the Grassmann manifold, the symplectic Stiefel manifold, the hyperbolic manifold, etc. Moreover, we provide an illustrative example of selecting the momentum-accelerated cubic regularization method as the unconstrained optimization approach in Algorithm 1 and establish its convergence properties directly from existing works.

#### 4.1 Construction of constraint dissolving operators

When we have no prior knowledge on the constraints \( c(x) = 0 \) in \( \text{OCP} \), we can consider the following mapping

\[
\mathcal{A}_c(x) := x - J_c(x) \left( J_c(x)^\top J_c(x) + \alpha \|c(x)\|^2 I_p \right)^{-1} J_c(x) c(x).
\]

It is worth mentioning that \( J_c(x) \) may be rank-deficient for some \( x \in \mathbb{R}^n \setminus \mathcal{M} \) [21, 19], resulting in the discontinuity of \( J_c(x)^\top \). To this end, we choose to add a regularization term to \( J_c(x)^\top J_c(x) \) with a prefixed constant \( \alpha > 0 \) in (4.1). Then \( \mathcal{A}_c \) is locally Lipschitz smooth in \( \mathbb{R}^n \) and we can consider the following penalty function,

\[
h_c(x) := f(\mathcal{A}_c(x)) + \frac{\beta}{2} \|c(x)\|^2.
\]

The following lemma illustrates that \( \mathcal{A}_c \) satisfies Assumption 1.2.

**Lemma 4.1.** Suppose \( c \) is twice locally Lipschitz continuously differentiable, then the constraint dissolving operator \( \mathcal{A}_c \) satisfies Assumption 1.2.
Table 1: Implementation of $\mathcal{A}$ for several Riemannian manifolds. Here $0_{m \times m}$ denotes the $m$-th order zero matrix, and $X^H$ denotes the conjugate transpose of a complex matrix $X$.

Proof. The Lipschitz smoothness of $\mathcal{A}_c$ is guaranteed by the twice locally Lipschitz continuous differentiability of $c$. In addition, for any $x \in \mathcal{M}$, it follows from the equality $c(x) = 0$ that the equality $\mathcal{A}_c(x) = x$ holds. Moreover, according to the fact that $J_{\mathcal{A}_c}(x) = I_n - J_c(x)J_c(x)^\dagger$ holds for any $x \in \mathcal{M}$, we obtain

$$J_{\mathcal{A}_c}(x)J_c(x) = J_c(x) - J_c(x)J_c(x)^\dagger J_c(x) = 0.$$ (4.3)

Therefore, we can conclude that $\mathcal{A}_c$ satisfies Assumption [1,2].

The mapping $\mathcal{A}_c$ in (4.2) only depends on $J_c(x)$. As a result, for a wide range of Riemannian manifolds, we can develop the corresponding constraint dissolving function without any prior knowledge on the geometrical properties of $\mathcal{M}$.

On the other hand, for several Riemannian manifolds with explicit expressions, which are widely used in real life, we can choose specific constraint dissolving operators that are easy to calculate. We present the details in Table 1. It can be easily verified that all the constraint dissolving operators presented in Table 1 satisfy Assumption [1,2] and we omit the proofs for simplicity. Moreover, calculating these operators $\mathcal{A}$ and the corresponding $J_{\mathcal{A}}(x)$ only involve matrix-matrix multiplications. This implies that it is efficient to compute $\nabla h$ once $\nabla f$ is obtained. In particular, compared with the Fletcher’s penalties, $\mathcal{CDF}$ avoids the needs to solve a system of linear equations in each function evaluation by appropriately selecting the the constraint dissolving operators for a variety of Riemannian manifolds in Table 1.

4.2 Comparison with existing penalty approaches

We summarize the differences between $\mathcal{CDF}$ and Fletcher’s penalty function (1.1) in Table 2. As mentioned in the introduction, Fletcher’s penalty function involves $\nabla f$ in its function value. Therefore, the cost of computing its exact gradient is similar to computing the Hessian of $\mathcal{CDF}$ meanwhile, calculating the Hessian of Fletcher’s penalty function is usually intractable.

On the other hand, the constraint dissolving operator $\mathcal{A}$ that satisfies Assumption [1,2] its transposed Jacobian $J_{\mathcal{A}}(x)$ is not necessary symmetric. As a result, from the expression of $\nabla f(x)$ presented in Proposition 3.8, $\nabla h(x)$ is not necessarily in $T_x$ even when $x \in \mathcal{M}$. However, from the expression of Fletcher’s penalty function in (1.1), $\nabla \phi(x) = \text{grad} f(x) \in T_x$ holds for any given $x \in \mathcal{M}$.

To further illustrate the differences between $\mathcal{CDF}$ and Fletcher’s penalty function, we provide an example by considering a problem in $\mathbb{R}^2$ that minimizes $f(w) := \|w - [1, 1]^T\|^2$ over the constraint $w^T C w = 1$ with $C := \text{Diag}(1, -1)$. As illustrated in Section 4.1, the mapping $\mathcal{A} : w \mapsto$
Objective function $f$ in CDF | CDF | Fletcher’s penalty function (1.1)
---|---|---
Bounded below | Bounded below | Not bounded below
Lipschitz smooth | Lipschitz smooth | Lipschitz continuous
Twice differentiable | Twice differentiable | Differentiable
$\nabla f$ is available | $h$ and $\nabla h$ are achievable | Only $\phi$ is achievable
$\nabla f$ and $\nabla^2 f$ are available | $h$, $\nabla h$ and $\nabla^2 h$ are achievable | Only $\phi$ and $\nabla \phi$ are achievable

Table 2: Differences between CDF and Fletcher’s penalty function.

Figure 1: The contours of CDF and (1.1) with $\beta = 1$ for $w \in [0, 2] \times [-0.5, 1.5]$, where a lighter contour corresponds to a higher function value. The red lines denote the feasible set. (a) The contours of $h(w)$; (b) The contours of Fletcher’s penalty function.

$w - \frac{1}{2}w(w^\top Cw - 1)$ satisfies Assumption 1.2 and thus its constraint dissolving function $h(w) := f(w - \frac{1}{2}w(w^\top Cw - 1)) + \frac{\beta}{2}(w^\top Cw - 1)^2$ shares the same first-order stationary points with itself. We plot the contours of $h(w)$, together with the contours of the corresponding Fletcher’s penalty function in Figure 1. These figures illustrate that even when $w$ is feasible, $\nabla h(w)$ is not necessarily contained in the tangent space of the feasible set. Therefore, $\nabla h(x)$ is independent of the Riemannian gradient of $f$ even when $x$ is feasible, which further illustrates that minimizing CDF can completely waive the computation of geometrical materials of the Riemannian manifold $M$. However, the expression of the corresponding Fletcher’s penalty function forces $\nabla \phi(x) = \text{grad } f(x) \in T_x$ for any given $x \in M$. That is, for any $x \in M$, computing $\nabla \phi(x)$ is equivalent to computing the Riemannian gradient of $f$ at $x$.

Furthermore, when $f$ is bounded below, it is easy to verify that CDF is bounded below in $\mathbb{R}^n$. Meanwhile, Fletcher’s penalty function does not have this property. As illustrated in Figure 1, Fletcher’s penalty function can be unbounded below in $\mathbb{R}^n$ even if the objective function is bounded below. Moreover, Fletcher’s penalty function is not well defined in $\mathbb{R}^n$ since the Jacobian of the constraint is singular at $[0, 0]^\top$.

4.3 Example

In this subsection, we present a representative example to illustrate how to apply CDF to solve OCP by the routine in Algorithm 1 and that it inherits all the convergence properties from the selected unconstrained optimization approach. Moreover, we present several supplementary examples in Appendix B. These supplementary examples further illustrate that the proposed constraint dissolving approaches enable us to directly employ various existing efficient unconstrained solvers to CDF.

Before we start the proof, we first define several constants based on Assumption 3.18 for any given $x_0 \in M$ and any $\zeta > 0$:
Recently, there is a growing interest in designing algorithms that can escape from saddle points in unconstrained nonconvex optimization. Among these approaches, the cubic regularization Newton’s method is a popular optimization algorithm. Recently, [59] proposed a cubic regularization method with momentum (CRm), which achieves the best possible convergence rate to a second-order stationary point for nonconvex optimization. However, transferring CRm into its Riemannian version by the framework from [3] requires deep modifications to the original framework, since solving the cubic step in tangent space requires specially designed solvers, computing the momentum step involves vector transports on the Riemannian manifold, and retractions should also be introduced to enforce the feasibility of the iterates. Noting that the iterates are not updated along geodesics, and the momentum steps involve vector transports, we need great efforts in establishing the convergence properties for the Riemannian version of CRm.

Alternatively, we can directly apply CRm algorithm to solve OCP through CDF. The detailed algorithm is presented in Algorithm 2.

Algorithm 2 Cubic regularization method with momentum for CDF

Require: Input data: functions \(f, \rho < 1, \nu\).
1. Choose initial guess \(x_0 \in \mathcal{M}\), and \(\beta \geq \beta_{x_0}\) according to Definition 3.19 set \(y_0 = x_0, k := 0\).
2. while not terminated do
3. Compute cubic step:
   \[
d_k = \arg\min_{d \in \mathbb{R}^n} d^T \nabla h(x_k) + \frac{1}{2} d^T \nabla^2 h(x_k) d + \frac{\nu}{6} \|d\|^3. \tag{4.4}
   \]
4. \(y_{k+1} = x_k + d_k\).
5. \(\tau_k = \min\{\rho, \|\nabla h(y_{k+1})\|, \|y_{k+1} - x_k\|\}\).
6. Compute momentum step:
   \[
v_{k+1} = y_{k+1} + \tau_k(y_{k+1} - y_k). \tag{4.5}
   \]
7. Set \(x_{k+1} = y_{k+1}\) if \(h(y_{k+1}) \leq h(v_{k+1})\); otherwise, set \(x_{k+1} = v_{k+1}\).
8. \(k = k + 1\).
9. end while
10. Return \(x_k\).

Next, we establish the convergence results of Algorithm 2 by combining Theorem 1 in [59], Theorem 3.10 and Theorem 3.11.

**Theorem 4.2.** Suppose Assumption 2.10 and Assumption 3.18 hold, \(\nabla^2 g(x)\) is locally Lipschitz continuous in \(\mathbb{R}^n\), Algorithm 2 sets \(\zeta > 0\), chooses its parameters as
\[
\tilde{M} := \sup_{y, z \in \Gamma_{x_0}, y \neq z} \left\| \frac{\nabla^2 g(y) - \nabla^2 g(z)}{y - z} \right\|, \quad \beta \geq \max\left\{ \tilde{\beta}_{x_0}, \frac{96 \tilde{M}(\tilde{M}_{x_0} + 1)^2}{\tilde{\sigma}_{x_0}^2} \right\},
\]
\[
\nu = \max\left\{ \sup_{y, z \in \Gamma_{x_0}, y \neq z} \left\| \frac{\nabla^2 h(y) - \nabla^2 h(z)}{y - z} \right\|, \sup_{y \in \Gamma_{x_0}} \frac{2 \left\| \nabla^2 h(y) \right\|}{\zeta}, \sup_{y \in \Gamma_{x_0}} \left\| \nabla h(y) \right\| \frac{1}{\zeta^2} \right\},
\]

...
and produces iterates \( \{x_k\} \). Then for any \( \epsilon \in (0, 1) \), there exists an constant \( C \) that is dependent on \( \beta, M_{x,f}, L_{x,g} \) and \( v \), such that for any

\[
K \geq \frac{C}{\epsilon^{3/2}},
\]

there exists an \( \bar{x} \in \{x_i\}_{0 \leq i \leq K} \) such that

\[
\left\| \nabla f(A^\infty(\bar{x})) \right\| \leq \epsilon, \quad \text{and} \quad \lambda_{\min}(\text{hess } f(A^\infty(\bar{x}))) \geq -\sqrt{\epsilon}.
\]

**Proof.** We first conclude from the definition for \( \nu \) that \( \|x_{k+1} - x_k\| \leq \xi \) holds for any \( k \geq 0 \). Due to step 7 in Algorithm \( \mathcal{A} \), the sequence \( \{x_k\} \) generated by Algorithm \( \mathcal{A} \) satisfies \( h(x_{j+1}) \leq h(x_j), j = 0, 1, ... \), which leads to the fact that \( \{x_k\} \subseteq \Gamma_{x_0} \). Then Corollary 3.11 ensures that the sequence is restricted in \( \Omega \cap \Gamma_{x_0} \), which implies the validity of Assumption 1 in [59]. Together with Theorem 1 in [59], we can conclude that for any \( \epsilon \in (0, 1) \) and any \( K \geq \frac{C}{\epsilon^{3/2}} \), there exists an \( \bar{x} \in \{x_i\}_{0 \leq i \leq K} \) satisfying

\[
\left\| \nabla h(\bar{x}) \right\| \leq \frac{\epsilon}{2 + 8L_{x_0,A} + \frac{288L_{x_0,c}M_{x_0,c}(M_{x_0,A} + 1)^2}{\sigma_{x_0,c}}}, \quad \text{and} \quad \lambda_{\min}(\nabla^2 h(\bar{x})) \geq -\sqrt{\epsilon}.
\]

By Proposition 3.13, the relationship \( \left\| \nabla f(A^\infty(\bar{x})) \right\| \leq 2 \left\| \nabla h(\bar{x}) \right\| \leq \epsilon \) holds. In addition, it follows from Theorem 3.10 and Lemma 3.4 that

\[
\left\| A^\infty(\bar{x}) - \bar{x} \right\| \leq \frac{4(M_{x_0,A} + 1)}{\sigma_{x_0,c}} \|c(\bar{x})\| \leq \frac{32(M_{x_0,A} + 1)^2}{\beta\sigma_{x_0,c}^2} \left\| \nabla h(\bar{x}) \right\| \leq \frac{1}{3M}\epsilon.
\]

Finally, we recall Proposition 3.15 and arrive at

\[
\lambda_{\min}(\text{hess } f(A^\infty(\bar{x})))
\geq \lambda_{\min}(\nabla^2 h(\bar{x})) - \left\| \nabla^2 g(A^\infty(\bar{x})) - \nabla^2 g(\bar{x}) \right\| - \left( \frac{\sigma_{x_0,c}^2}{2L_{x_0,A}} + \frac{96L_{x_0,c}M_{x_0,c}(M_{x_0,A} + 1)^2}{\sigma_{x_0,c}^2} \right) \left\| \nabla h(\bar{x}) \right\|
\geq \lambda_{\min}(\nabla^2 h(\bar{x})) - \bar{M} \|A^\infty(\bar{x}) - \bar{x}\| - \left( \frac{\sigma_{x_0,c}^2}{2L_{x_0,A}} + \frac{96L_{x_0,c}M_{x_0,c}(M_{x_0,A} + 1)^2}{\sigma_{x_0,c}^2} \right) \left\| \nabla h(\bar{x}) \right\|
\geq -\frac{\sqrt{\epsilon}}{3} - \frac{\epsilon}{3} - \frac{\epsilon}{3} = -\sqrt{\epsilon},
\]

which completes the proof. \( \square \)

## 5 Conclusion

Riemannian optimization has close connections with unconstrained optimization. To extend existing unconstrained optimization approaches to solve Riemannian optimization problems and establish the corresponding theoretical properties, most existing approaches are developed based on the frameworks summarized in [3] by utilizing various geometrical materials from differential geometry. However, determining and preparing the geometrical materials are challenging for various Riemannian manifolds. In addition, incorporating these geometrical materials requires significant modifications to the original unconstrained optimization approaches. Last but not least, the approximation errors introduced by retractions and vector transports generally will lead to difficulties in establishing the theoretical convergence properties. Therefore, it is challenging to apply advanced unconstrained optimization approaches to solve Riemannian optimization problems.

The main contribution of this paper is to propose a class of constraint dissolving approaches, based on the so-called constraint dissolving functions \([CD]\). We prove that under mild assumptions, \(\text{OCP} \) and \(\text{CDF} \) have the same first-order stationary points, second-order stationary points, local minimizers, and Łojasiewicz exponents in a neighborhood of the feasible region. In addition, the exact gradient and Hessian of \(\text{CDF} \) can directly be calculated based on the same order of differentials of \( f \).
We summarize our proposed constraint dissolving approaches in Algorithm 1. We provide a framework to establish the global convergence, worst-case complexity, and escaping-from-saddle-point properties of Algorithm 1 under mild assumptions directly based on existing results for the selected unconstrained optimization approach.

Moreover, we discuss how to choose the constraint dissolving operator $A$ for CDF and present an easy-to-compute form of $A$ for several well-known manifolds, including the generalized Stiefel manifold, the symplectic Stiefel manifold, and the hyperbolic manifold. The construction of CDF is independent of the geometrical properties of $M$, which avoids the difficulties in analyzing the geometrical properties of the underlying manifold and hence it enables us to design constraint dissolving approaches for a number of well-known Riemannian manifolds. Finally, we use the cubic regularization method with momentum as an example to illustrate how to directly apply unconstrained optimization approaches to OCP and inherit existing theoretical results.

References

[1] Traian Abrudan, Jan Eriksson, and Visa Koivunen. Conjugate gradient algorithm for optimization under unitary matrix constraint. *Signal Processing*, 89(9):1704–1714, 2009.

[2] Traian Abrudan, Jan Eriksson, and Visa Koivunen. Steepest descent algorithms for optimization under unitary matrix constraint. *IEEE Transactions on Signal Processing*, 56(3):1134–1147, 2008.

[3] P-A Absil, Robert Mahony, and Rodolphe Sepulchre. *Optimization algorithms on matrix manifolds*. Princeton University Press, 2009.

[4] Kwangjun Ahn and Suvrit Sra. From Nesterov’s estimate sequence to Riemannian acceleration. In *Conference on Learning Theory*, pages 84–118. PMLR, 2020.

[5] Zhaojun Bai and Ren-Cang Li. Minimization principles and computation for the generalized linear response eigenvalue problem. *BIT Numerical Mathematics*, 54(1):31–54, 2014.

[6] Gary Be´ cigneul and Octavian-Eugen Ganea. Riemannian adaptive optimization methods. *arXiv preprint arXiv:1810.00760*, 2018.

[7] J´ erˆ ome Bolte, Shoham Sabach, and Marc Teboulle. Proximal alternating linearized minimization for nonconvex and nonsmooth problems. *Mathematical Programming*, 146(1):459–494, 2014.

[8] Nicolas Boumal. An introduction to optimization on smooth manifolds. *Available at http://sma.epfl.ch/nboumal/book/index.html*, 2020.

[9] James Bradbury, Roy Frostig, Peter Hawkins, Matthew James Johnson, Chris Leary, Dougal Maclaurin, George Nacula, Adam Paszke, Jake VanderPlas, Skye Wanderman-Milne, and Qiao Zhang. JAX: composiable transformations of Python+NumPy programs, 2018.

[10] Richard H Byrd, Peihuang Lu, Jorge Nocedal, and Ciyou Zhu. A limited memory algorithm for bound constrained optimization. *SIAM Journal on scientific computing*, 16(5):1190–1208, 1995.

[11] Coralia Cartis, Nicholas IM Gould, and Ph L Toint. An adaptive cubic regularization algorithm for nonconvex optimization with convex constraints and its function-evaluation complexity. *IMA Journal of Numerical Analysis*, 32(4):1662–1695, 2012.

[12] Coralia Cartis, Nicholas IM Gould, and Ph L Toint. Complexity bounds for second-order optimality in unconstrained optimization. *Journal of Complexity*, 28(1):93–108, 2012.

[13] Coralia Cartis, Nicholas IM Gould, and Philippe L Toint. Worst-case evaluation complexity and optimality of second-order methods for nonconvex smooth optimization. In *Proceedings of the International Congress of Mathematicians: Rio de Janeiro 2018*, pages 3711–3750. World Scientific, 2018.
[14] Andrew R Conn, Nicholas IM Gould, and Philippe L Toint. Trust region methods. SIAM, 2000.

[15] Chris Criscitiello and Nicolas Boumal. An accelerated first-order method for non-convex optimization on manifolds. arXiv preprint arXiv:2008.02252, 2020.

[16] Gianni Di Pillo and Luigi Grippo. An exact penalty function method with global convergence properties for nonlinear optimization problems. Mathematical Programming, 36(1):1–18, 1986.

[17] David W Dreisigmeyer. Equality constraints, Riemannian manifolds and direct search methods. Available at http://www.optimization-online.org/DB_FILE/2007/08/1743.pdf, 2007.

[18] Alan Edelman, Tomás A Arias, and Steven T Smith. The geometry of algorithms with orthogonality constraints. SIAM journal on Matrix Analysis and Applications, 20(2):303–353, 1998.

[19] Ron Estrin, Michael P Friedlander, Dominique Orban, and Michael A Saunders. Implementing a smooth exact penalty function for equality-constrained nonlinear optimization. SIAM Journal on Scientific Computing, 42(3):A1809–A1835, 2020.

[20] Roger Fletcher. A class of methods for nonlinear programming with termination and convergence properties. Integer and nonlinear programming, pages 157–173, 1970.

[21] Roger Fletcher and Sven Leyffer. Nonlinear programming without a penalty function. Mathematical Programming, 91(2):239–269, 2002.

[22] Bin Gao, Xin Liu, and Ya-xiang Yuan. Parallelizable algorithms for optimization problems with orthogonality constraints. SIAM Journal on Scientific Computing, 41(3):A1949–A1983, 2019.

[23] Bin Gao, Nguyen Thanh Son, P-A Absil, and Tatjana Stykel. Riemannian optimization on the symplectic Stiefel manifold. SIAM Journal on Optimization, 31(2):1546–1575, 2021.

[24] Rong Ge, Furong Huang, Chi Jin, and Yang Yuan. Escaping from saddle points–online stochastic gradient for tensor decomposition. In Conference on learning theory, pages 797–842. PMLR, 2015.

[25] Gene H Golub and Charles F Van Loan. Matrix computations. JHU press, 2013.

[26] Nicholas IM Gould, Stefano Lucidi, Massimo Roma, and Philippe L Toint. Solving the trust-region subproblem using the Lanczos method. SIAM Journal on Optimization, 9(2):504–525, 1999.

[27] Luigi Grippo, Francesco Lampariello, and Stephano Lucidi. A nonmonotone line search technique for Newton’s method. SIAM Journal on Numerical Analysis, 23(4):707–716, 1986.

[28] Charles R Harris, K Jarrod Millman, Stéfan J van der Walt, Ralf Gommers, Pauli Virtanen, David Cournapeau, Eric Wieser, Julian Taylor, Sebastian Berg, Nathaniel J Smith, et al. Array programming with numpy. Nature, 585(7825):357–362, 2020.

[29] Magnus R Hestenes and Eduard Stiefel. Methods of conjugate gradients for solving linear systems. Journal of research of the National Bureau of Standards, 49(6):409, 1952.

[30] S Hosseini. Convergence of nonsmooth descent methods via Kurdyka–Lojasiewicz inequality on Riemannian manifolds. Hausdorff Center for Mathematics and Institute for Numerical Simulation, University of Bonn (2015,(INS Preprint No. 1523)), 2015.

[31] Xiaoyin Hu and Xin Liu. An efficient orthonormalization-free approach for sparse dictionary learning and dual principal component pursuit. Sensors, 20(3041), 2020.

[32] Wen Huang and Ke Wei. Riemannian proximal gradient methods. arXiv preprint arXiv:1909.06065, 2019.

[33] Chi Jin, Rong Ge, Praneeth Netrapalli, Sham M Kakade, and Michael I Jordan. How to escape saddle points efficiently. In International Conference on Machine Learning, pages 1724–1732. PMLR, 2017.
[34] Chi Jin, Praneeth Netrapalli, and Michael I Jordan. Accelerated gradient descent escapes saddle points faster than gradient descent. In Conference On Learning Theory, pages 1042–1085. PMLR, 2018.

[35] Max Kochurov, Rasul Karimov, and Serge Kozlukov. Geoopt: Riemannian optimization in pytorch. Technical report, arXiv preprint arXiv:2005.02819, 2020.

[36] Yunwen Lei, Ting Hu, Guiying Li, and Ke Tang. Stochastic gradient descent for nonconvex learning without bounded gradient assumptions. IEEE transactions on neural networks and learning systems, 31(10):4394–4400, 2019.

[37] Felix Lenders, Christian Kirches, and Andreas Potschka. trlib: A vector-free implementation of the gltr method for iterative solution of the trust region problem. Optimization Methods and Software, 33(3):420–449, 2018.

[38] Huan Li and Zhouchen Lin. Accelerated proximal gradient methods for nonconvex programming. In Advances in neural information processing systems, pages 379–387, 2015.

[39] Stanislaw Łojasiewicz. Sur le probleme de la division. 1961.

[40] Stanislaw Lojasiewicz. Une propri´ et´ e topologique des sous-ensembles analytiques r´ eels. Les ´equations aux d´ eriv´ ees partielles, 117:87–89, 1963.

[41] Dougal Maclaurin, David Duvenaud, and Ryan P Adams. Autograd: Effortless gradients in numpy. In ICML 2015 AutoML workshop, volume 238, page 5, 2015.

[42] Mayank Meghwanshi, Pratik Jawanpuria, Anoop Kunchukuttan, Hiroyuki Kasai, and Bamdev Mishra. Mctorch, a manifold optimization library for deep learning. Technical report, arXiv preprint arXiv:1810.01811, 2018.

[43] José Luis Morales and Jorge Nocedal. Remark on “algorithm 778: L-BFGS-B: Fortran subroutines for large-scale bound constrained optimization”. ACM Transactions on Mathematical Software (TOMS), 38(1):1–4, 2011.

[44] Stephen G Nash. Newton-type minimization via the Lanczos method. SIAM Journal on Numerical Analysis, 21(4):770–788, 1984.

[45] Yuriii Nesterov and Boris T Polyak. Cubic regularization of Newton method and its global performance. Mathematical Programming, 108(1):177–205, 2006.

[46] Maximillian Nickel and Douwe Kiela. Learning continuous hierarchies in the Lorentz model of hyperbolic geometry. In International Conference on Machine Learning, pages 3779–3788. PMLR, 2018.

[47] Jorge Nocedal and Stephen Wright. Numerical optimization. Springer Science & Business Media, 2006.

[48] Peter Ochs, Yunjin Chen, Thomas Brox, and Thomas Pock. ipiano: Inertial proximal algorithm for nonconvex optimization. SIAM Journal on Imaging Sciences, 7(2):1388–1419, 2014.

[49] Michael JD Powell. A method for nonlinear constraints in minimization problems. Optimization, pages 283–298, 1969.

[50] Chunhong Qi, Kyle A Gallivan, and P-A Absil. Riemannian BFGS algorithm with applications. In Recent advances in optimization and its applications in engineering, pages 183–192. Springer, 2010.

[51] Hiroyuki Sato. A Dai–Yuan-type Riemannian conjugate gradient method with the weak Wolfe conditions. Computational optimization and Applications, 64(1):101–118, 2016.

[52] Jonathan W Siegel. Accelerated optimization with orthogonality constraints. arXiv preprint arXiv:1903.05204, 2019.
[53] Nguyen Thanh Son, P-A Absil, Bin Gao, and Tatjana Stykel. Symplectic eigenvalue problem via trace minimization and Riemannian optimization. arXiv preprint arXiv:2101.02618, 2021.

[54] Trond Steihaug. The conjugate gradient method and trust regions in large scale optimization. SIAM Journal on Numerical Analysis, 20(3):626–637, 1983.

[55] Philippe Toint. Towards an efficient sparsity exploiting Newton method for minimization. In Sparse matrices and their uses, pages 57–88. Academic Press, 1981.

[56] James Townsend, Niklas Koep, and Sebastian Weichwald. Pymanopt: A python toolbox for optimization on manifolds using automatic differentiation. arXiv preprint arXiv:1603.03236, 2016.

[57] Pauli Virtanen, Ralf Gommers, Travis E Oliphant, Matt Haberland, Tyler Reddy, David Cournapeau, Evgeni Burovski, Pearu Peterson, Warren Weckesser, Jonathan Bright, et al. Scipy 1.0: fundamental algorithms for scientific computing in python. Nature methods, 17(3):261–272, 2020.

[58] Lei Wang, Bin Gao, and Xin Liu. Multipliers correction methods for optimization problems over the Stiefel manifold. arXiv preprint arXiv:2011.14781, 2020.

[59] Zhe Wang, Yi Zhou, Yingbin Liang, and Guanghui Lan. Cubic regularization with momentum for nonconvex optimization. In Uncertainty in Artificial Intelligence, pages 313–322. PMLR, 2020.

[60] Zaiwen Wen and Wotao Yin. A feasible method for optimization with orthogonality constraints. Mathematical Programming, 142(1-2):397–434, 2013.

[61] R-B Wu, Raj Chakrabarti, and Herschel Rabitz. Critical landscape topology for optimization on the symplectic group. Journal of optimization theory and applications, 145(2):387–406, 2010.

[62] Nachuan Xiao and Xin Liu. Solving optimization problems over the Stiefel manifold by smooth exact penalty function. arXiv preprint arXiv:2110.08986, 2021.

[63] Nachuan Xiao, Xin Liu, and Ya-xiang Yuan. A class of smooth exact penalty function methods for optimization problems with orthogonality constraints. Optimization Methods and Software, pages 1–37, 2020.

[64] Nachuan Xiao, Xin Liu, and Ya-xiang Yuan. Exact penalty function for \ell_{2,1} norm minimization over the Stiefel manifold. SIAM Journal on Optimization, 31(4):3097–3126, 2021.

[65] Nachuan Xiao, Xin Liu, and Ya-xiang Yuan. A penalty-free infeasible approach for a class of nonsmooth optimization problems over the Stiefel manifold. arXiv preprint arXiv:2103.03514, 2021.

[66] Ya-xiang Yuan. Recent advances in trust region algorithms. Mathematical Programming, 151(1):249–281, 2015.

[67] Victor M Zavala and Mihai Anitescu. Scalable nonlinear programming via exact differentiable penalty functions and trust-region Newton methods. SIAM Journal on Optimization, 24(1):528–558, 2014.

[68] Erchuan Zhang and Lyle Noakes. Riemannian cubics in quadratic matrix Lie groups. Applied Mathematics and Computation, 375:125082, 2020.

[69] Hongyi Zhang and Suvrit Sra. Towards Riemannian accelerated gradient methods. arXiv preprint arXiv:1806.02812, 2018.

[70] Jingzhao Zhang, Hongyi Zhang, and Suvrit Sra. R-spider: A fast Riemannian stochastic optimization algorithm with curvature independent rate. arXiv preprint arXiv:1811.04194, 2018.

[71] Pan Zhou, Xiao-Tong Yuan, and Jiashi Feng. Faster first-order methods for stochastic non-convex optimization on Riemannian manifolds. In The 22nd International Conference on Artificial Intelligence and Statistics, pages 138–147. PMLR, 2019.
[72] Ciyou Zhu, Richard H Byrd, Peihuang Lu, and Jorge Nocedal. Algorithm 778: L-BFGS-B: Fortran subroutines for large-scale bound-constrained optimization. ACM Transactions on mathematical software (TOMS), 23(4):550–560, 1997.

A Proofs for Main Results

A.1 Proofs for Section 3.1

Proof for Lemma 3.4

Proof. We first show that the inclusion \( A^i(y) \in \Omega_x \) holds for any \( j \in \mathbb{N} \cup \{0\} \) by mathematical induction. It is clear that this statement holds at \( j = 0 \). Suppose that the statement holds for any \( 0 \leq j \leq K - 1 \) with certain \( K \in \mathbb{N} \). Then for any \( 1 \leq k \leq K - 1 \), Lemma 3.1, Lemma 3.3 and the definition of \( \varepsilon_x \) imply that

\[
\left\| c(A^k(y)) \right\| \leq \frac{4L_{x,b}}{\sigma_{x,c}} \left\| c(A^{k-1}(y)) \right\|^2 \leq \frac{1}{2} \left\| c(A^{k-1}(y)) \right\|. 
\] (A.1)

By simple calculations, we obtain

\[
\sum_{i=0}^{K} \left\| c(A^i(y)) \right\| < 2 \left\| c(y) \right\|. 
\] (A.2)

On the other hand, it follows from Lemma 3.2 that

\[
\left\| A^k(y) - A^{k-1}(y) \right\| \leq \frac{2(M_{x,A} + 1)}{\sigma_{x,c}} \left\| c(A^{k-1}(y)) \right\|, 
\]

which implies that

\[
\sum_{i=1}^{K} \left\| A^i(y) - A^{i-1}(y) \right\| \leq \sum_{i=1}^{K} \frac{2(M_{x,A} + 1)}{\sigma_{x,c}} \left\| c(A^{i-1}(y)) \right\| 
\leq \frac{4(M_{x,A} + 1)}{\sigma_{x,c}} \left\| c(y) \right\| \leq \frac{4M_{x,c}(M_{x,A} + 1)}{\sigma_{x,c}} \left\| y - x \right\|. 
\] (A.3)

Together with the fact that \( y \in \bar{\Omega}_x \), it holds that

\[
\left\| A^K(y) - x \right\| \leq \left\| y - x \right\| + \sum_{i=1}^{K} \left\| A^i(y) - A^{i-1}(y) \right\| 
\leq \frac{4M_{x,c}(M_{x,A} + 1) + \sigma_{x,c}}{\sigma_{x,c}} \left\| y - x \right\| \leq \varepsilon_x.
\]

Namely, the inclusion \( A^i(y) \in \Omega_x \) holds at \( j = K \). By mathematical induction, we can conclude that this statement holds for any \( j \in \mathbb{N} \cup \{0\} \).

Now, it is easy to extend inequalities (A.2) and (A.3) to the following infinite case, leading to

\[
\sum_{i=0}^{\infty} \left\| c(A^i(y)) \right\| \leq \sum_{i=0}^{\infty} \frac{1}{2^i} \left\| c(y) \right\| = 2 \left\| c(y) \right\|,
\]

and

\[
\sum_{i=1}^{\infty} \left\| A^i(y) - A^{i-1}(y) \right\| \leq \frac{4(M_{x,A} + 1)}{\sigma_{x,c}} \left\| c(y) \right\|. 
\] (A.4)
Then by the dominated convergence theorem, we have that the sequence \( \{A^k(x)\} \) is convergent, i.e. \( A^\infty(y) \) exists. Moreover,

\[
\|A^\infty(y) - y\| \leq \sum_{k=1}^{\infty} \|A^k(y) - A^{k-1}(y)\| \leq \frac{4(M_{x,A} + 1)}{\sigma_{x,c}} \|c(y)\|. \quad (A.5)
\]

Finally, we show that \( A^\infty(y) \in \mathcal{M} \). From (A.1), \( \|c(A^k(y))\| \leq \frac{1}{2} \|c(A^{k-1}(y))\| \) holds for any \( k \geq 1 \). Then

\[
\|c(A^\infty(y))\| = \lim_{k \to +\infty} \|c(A^k(y))\| = 0.
\]

This implies that \( A^\infty(y) \in \mathcal{M} \) and the proof is completed. \( \square \)

### A.2 Proofs for Section 3.2

#### A.2.1 Proofs for Section 3.2.1

**Proof for Proposition 3.9**

**Proof.** For any first-order stationary point, \( x \in \mathcal{M} \) of \( \text{[OCP]} \) it follows from the equality (2.2) that

\[
\nabla h(x) = J_A(x)\nabla f(x) + \beta J_c(x) c(x) = J_A(x)J_c(x)\lambda(x) = 0,
\]

which implies that \( x \) is a first-order stationary point of \( \text{[CDF]} \).

On the other hand, for any \( x \in \mathcal{M} \) that is a first-order stationary point of \( \text{[CDF]} \) it holds that

\[
J_A(x)\nabla f(x) = 0,
\]

which results in the inclusion \( \nabla f(x) \in \mathcal{N}_x = \text{range}(J_c(x)) \) from Lemma 3.6. By Definition 2.1 we conclude that \( x \) is a first-order stationary point of \( \text{[OCP]} \). \( \square \)

**Proof for Theorem 3.10**

**Proof.** For any \( y \in \Omega_x \), it follows from the definition of \( M_{x,A} \) that

\[
\|(J_A(y) - I_n)\nabla h(y)\| \leq (M_{x,A} + 1) \|\nabla h(y)\|.
\]

Moreover, we can obtain

\[
\|(J_A(y) - I_n)\nabla g(y)\| = \|(J_A(y) - I_n)J_A(y)\nabla f(A(y))\| \leq \|J_A(y)^2 - J_A(y)\| \|\nabla f(A(y))\| \leq L_{x,A}(2M_{x,A} + 1) \text{dist}(y, \mathcal{M}) \|\nabla f(A(y))\| \leq \frac{2M_{x,f}L_{x,A}(2M_{x,A} + 1)}{\sigma_{x,c}} \|c(y)\|.
\]

Here, the second inequality results from the Lipschitz continuity of \( J_A(y) \) and Lemma 3.7. The last inequality is implied by Lemma 3.1. On the other hand, it holds that

\[
\|(J_A(y) - I_n)J_c(y)c(y)\| \geq \|J_c(y)c(y)\| - \|J_A(y)J_c(y)c(y)\| \\
\geq \frac{\sigma_{x,c}}{2} \|c(y)\| \geq \|c(y)\| - \|J_A(y)J_c(y)c(y)\| \\
\geq \frac{\sigma_{x,c}}{2} \|c(y)\| - \|J_A(y)J_c(y)c(y)\| \geq \frac{\sigma_{x,c}}{4} \|c(y)\|.
\]

Combining the above two inequalities, we have

\[
\|\nabla h(y)\| \geq \frac{1}{M_{x,A} + 1} \|(J_A(y) - I_n)\nabla h(y)\| \geq \frac{\beta}{M_{x,A} + 1} \frac{\|J_A(y) - I_n\|J_c(y)c(y)\| - \|(J_A(y) - I_n)\nabla g(y)\|}{\sigma_{x,c}} \|c(y)\| \geq \frac{\beta\sigma_{x,c}}{8(M_{x,A} + 1)} \|c(y)\| \quad (A.6)
\]

\[
\frac{\beta\sigma_{x,c}}{8(M_{x,A} + 1)} \|c(y)\| \geq \frac{\beta\sigma_{x,c}}{8(M_{x,A} + 1)} \|c(y)\|.
\]

\[
\frac{\beta\sigma_{x,c}}{8(M_{x,A} + 1)} \|c(y)\| \geq \frac{\beta\sigma_{x,c}}{8(M_{x,A} + 1)} \|c(y)\|.
\]

\[
\frac{\beta\sigma_{x,c}}{8(M_{x,A} + 1)} \|c(y)\| \geq \frac{\beta\sigma_{x,c}}{8(M_{x,A} + 1)} \|c(y)\|.
\]
holds for any $y \in \hat{\Omega}_c$. Here, the last inequality results from Definition 2.8.

Finally, for any first-order stationary point $x^*$ of $\text{CDF}$ that satisfies $x^* \in \Omega_c$, it follows from the inequality $(A.6)$ that the feasibility $\|c(x^*)\| = 0$ is implied by the stationarity $\nabla h(x^*) = 0$. Together with Proposition 5.9, we conclude that $x^*$ is a first-order stationary point of $\text{OCP}$.

In the rest of this part, we aim to prove Theorem 3.11. We start with two auxiliary lemmas.

**Lemma A.1.** Suppose Assumption 2.10 holds. For any given $x \in \mathcal{M}$, the following equation holds for any $d \in \mathcal{T}_x$,

$$(\mathcal{D}_{I_A}(x)[\nabla f(x)])d = (\mathcal{D}_{I_A}(x)[\nabla f(x)])d + \sum_{i=1}^{p} \lambda_i(x)I_A(x)\nabla^2 c_i(x)J_A(x)^\top d.$$ 

Moreover, if $x$ is a first-order stationary point of $\text{OCP}$, then for any $d \in \mathcal{T}_x$, it holds that

$$(\mathcal{D}_{I_A}(x)[\nabla f(x)])d = -\sum_{i=1}^{p} \lambda_i(x)I_A(x)\nabla^2 c_i(x)J_A(x)^\top d.$$ 

**Proof.** Firstly, for any given $x \in \mathcal{M}$, it follows from Assumption 1.2 that the identity $J_A(x)I_c(A(x)) = I_A(x)J_c(A(x)) = 0$ holds. Then for any $d \in \mathcal{T}_x$, we denote $\hat{x} = \text{proj}(x + d, \mathcal{M})$ and obtain

$$\|J_A(x+d)J_c(A(x+d))\| = \|J_A(x+d)\| - \|J_A(\hat{x})\| \\ \leq 2L_{x,b}\|x + d - \hat{x}\| = 2L_{x,b}\text{dist}(x + d, \mathcal{M}) \leq \frac{2L_{x,b}}{\sigma_{x,c}} \|c(x + d)\| \leq \frac{L_{x,b}L_{x,c}}{\sigma_{x,c}} \|d\|^2.$$ 

Here, the inequality $(i)$ directly follows from Lemma 3.1. Meanwhile, combining the second-order Taylor expansion of $c(x + d)$ at $x$ with the facts that $x \in \mathcal{M}$ and $d \in \mathcal{T}_x$, we can obtain the inequality $(ii)$.

Therefore, it follows from the fact $\nabla (c \circ A)(x) = J_A(x)J_c(A(x))$ that

$$\lim_{t \to 0} \frac{I_A(x + td)\nabla c_i(A(x+td)) - I_A(x)\nabla c_i(A(x))}{t} = \lim_{t \to 0} \frac{I_A(x + td)\nabla c_i(A(x+td))}{t} = 0$$

holds for any $i = 1, \ldots, p$ and $d \in \mathcal{T}_x$, which further implies

$$\nabla^2((c \circ A)(x))d = 0.$$ 

Here $\nabla^2((c \circ A)(x))$ denotes the Hessian of $c_i(A(x))$ with respect to $x$.

On the other hand, we notice that

$$\nabla^2((c \circ A)(x)) = I_A(x)\nabla^2 c_i(A(x))I_A(x)^\top + \mathcal{D}_{I_A}(x)[\nabla c_i(A(x))].$$

Combining Definition 2.1 with the equality (2.4) and $A(x) = x$, we have

$$0 = \sum_{i=1}^{p} \lambda_i(x)\nabla^2((c \circ A)(x))d$$

$$= (\mathcal{D}_{I_A}(x)[\sum_{i=1}^{p} \lambda_i(x)\nabla c_i(x)])d + \sum_{i=1}^{p} \lambda_i(x)I_A(x)\nabla^2 c_i(x)J_A(x)^\top d$$

$$= (\mathcal{D}_{I_A}(x)[\nabla f(x) - \nabla^2 f(x)])d + \sum_{i=1}^{p} \lambda_i(x)I_A(x)\nabla^2 c_i(x)J_A(x)^\top d.$$ 

From here, the first required result follows directly. The second required equality holds because $\nabla f(x) = 0$ when $x$ is a stationary point of $\text{OCP}$.
Lemma A.2. Suppose Assumption 2.10 holds, then for any \( x \in \mathcal{M} \) and \( d \in \mathcal{T}_x \), it holds that

\[
\nabla^2 h(x)d - J_A(x) \left( \nabla^2 f(x) - \sum_{i=1}^{p} \lambda_i(x) \nabla^2 c_i(x) \right) J_A(x)^\top d = (\mathcal{D}_{J_A}(x)[\nabla f(x)]) d.
\]

Moreover, if \( x \) is a first-order stationary point of OCP, then for any \( d \in \mathcal{T}_x \), it holds that

\[
\nabla^2 h(x)d = J_A(x) \left( \nabla^2 f(x) - \sum_{i=1}^{p} \lambda_i(x) \nabla^2 c_i(x) \right) J_A(x)^\top d.
\]

Proof. Firstly, it follows from Proposition 3.8 that

\[
\nabla^2 h(x) = J_A(x) \nabla^2 f(x) J_A(x)^\top + \mathcal{D}_{J_A}(x)[\nabla f(x)] + \beta J_c(x) J_c(x)^\top.
\]

Together with Lemma A.1 and the fact that \( J_c(x)^\top d = 0 \), we conclude that

\[
\nabla^2 h(x)d = J_A(x) \nabla^2 f(x) J_A(x)^\top d + \mathcal{D}_{J_A}(x)[\nabla f(x)]d
\]

\[
= J_A(x) \left( \nabla^2 f(x) - \sum_{i=1}^{p} \lambda_i(x) \nabla^2 c_i(x) \right) J_A(x)^\top d + (\mathcal{D}_{J_A}(x)[\nabla f(x)]) d,
\]

and complete the proof. \(\square\)

Now, we are ready to prove the main theorem.

Proof for Theorem 3.11

Proof. Firstly, if \( x \) is a second-order stationary point of 2.10 then Theorem 3.10 implies that \( x \in \mathcal{M} \). Therefore, we conclude that \( \nabla f(x) = 0 \) and \( \lambda_{\min}(\nabla^2 h(x)) \geq 0 \). It follows from Lemma 3.5 that \( J_A(x)^\top d_1 = d_1 \) holds for any \( d_1 \in \mathcal{T}_x \). Then together with Lemma A.2, we obtain that

\[
d_1^\top \nabla^2 h(x)d_1 = d_1^\top \left( \nabla^2 f(x) - \sum_{i=1}^{p} \lambda_i(x) \nabla^2 c_i(x) \right) d_1.
\]

Together with Lemma A.2 and Definition 2.3, we arrive at

\[
\lambda_{\min}(\text{hess } f(x)) \geq \min_{d \in \mathcal{T}_x, ||d|| = 1} d^\top \nabla^2 h(x)d \geq 0,
\]

which implies that \( x \) is a second-order stationary point of OCP.

On the other hand, suppose \( x \) is a second-order stationary point of OCP which implies that \( \nabla f(x) = 0 \) and \( \text{hess } f(x) \geq 0 \). As a result, we have \( \mathcal{H}(x) \geq 0 \), where \( \mathcal{H}(x) \) is defined in (7.3). Then it follows from Lemma A.2 that for any \( d_1 \in \mathcal{T}_x \), we have

\[
d_1^\top \nabla^2 h(x)d_1 = d_1^\top J_A(x) \left( \nabla^2 f(x) - \sum_{i=1}^{p} \lambda_i(x) \nabla^2 c_i(x) \right) J_A(x)^\top d_1
\]

\[
= d_1^\top \left( \nabla^2 f(x) - \sum_{i=1}^{p} \lambda_i(x) \nabla^2 c_i(x) \right) d_1.
\]

Moreover, by (5.9) in Proposition 3.8, Lemma 3.5 and Lemma A.1, we have that for \( d_1 \in \mathcal{T}_x \),

\[
d_1^\top \nabla^2 g(x)d_1 = d_1^\top J_A(x) \left( \nabla^2 f(x) - \sum_{i=1}^{p} \lambda_i(x) \nabla^2 c_i(x) \right) J_A(x)^\top d_1
\]

\[
= d_1^\top \left( \nabla^2 f(x) - \sum_{i=1}^{p} \lambda_i(x) \nabla^2 c_i(x) \right) d_1.
\]
Thus,
\[
\lambda_{\text{max}}(\mathcal{H}(x)) = \sup_{d_1 \in \mathcal{T}_x, \|d_1\|_1 = 1} d_1^T \left( \nabla^2 f(x) - \sum_{i=1}^p \lambda_i(x) \nabla^2 c_i(x) \right) d_1
\]
(A.8)
\[
= \sup_{d_1 \in \mathcal{T}_x, \|d_1\|_1 = 1} d_1^T \nabla^2 g(x) d_1 \leq L_{x,g}.
\]
As a result, for any \( \beta \geq \beta_x \), we obtain that for any \( d_1 \in \mathcal{N}_x \),
\[
d_1^T \nabla^2 h(x) d_2 = \beta d_1^T J_c(x) J_c(x)^\top d_2 + d_1^T \nabla^2 g(x) d_2
\]
\[
\geq \beta \sigma^2_{x,c} \|d_2\|^2 - L_{x,g} \|d_2\|^2 \geq L_{x,g} M_{x,A}^2 \|d_2\|^2
\]
\[
\geq \lambda_{\text{max}}(\mathcal{H}(x)) \|J_A(x)\|^2 \|d_2\|^2
\]
\[
\geq d_2^T J_A(x) \left( \nabla^2 f(x) - \sum_{i=1}^p \lambda_i(x) \nabla^2 c_i(x) \right) J_A(x)^\top d_2.
\]
Note that the last inequality holds because \( J_A(x)^\top d_2 \in \mathcal{T}_x \).

Moreover, Lemma 3.5 implies that \( J_A(x)^\top (d_1 + d_2) \in \mathcal{T}_x \) holds for any \( d_1 \in \mathcal{T}_x \) and \( d_2 \in \mathcal{N}_x \). Together with Definition 2.2 and the fact that \( \text{hess} f(x) \succeq 0 \), we arrive at
\[
(d_1 + d_2)^\top J_A(x) \left( \nabla^2 f(x) - \sum_{i=1}^p \lambda_i(x) \nabla^2 c_i(x) \right) J_A(x)^\top (d_1 + d_2) \geq 0.
\]
Additionally, it follows from Lemma A.2 that
\[
d_2^T \nabla^2 h(x) d_1 = d_2^T J_A(x) \left( \nabla^2 f(x) - \sum_{i=1}^p \lambda_i(x) \nabla^2 c_i(x) \right) J_A(x)^\top d_1.
\]
Finally, we can conclude that
\[
0 \leq (d_1 + d_2)^\top J_A(x) \left( \nabla^2 f(x) - \sum_{i=1}^p \lambda_i(x) \nabla^2 c_i(x) \right) J_A(x)^\top (d_1 + d_2)
\]
\[
= d_1^T \nabla^2 h(x) d_1 + 2d_1^T \nabla^2 h(x) d_2 + d_2^T J_A(x) \left( \nabla^2 f(x) - \sum_{i=1}^p \lambda_i(x) \nabla^2 c_i(x) \right) J_A(x)^\top d_2
\]
\[
\leq d_1^T \nabla^2 h(x) d_1 + 2d_1^T \nabla^2 h(x) d_2 + d_2^T \nabla^2 h(x) d_2
\]
\[
= (d_1 + d_2)^\top \nabla^2 h(x) (d_1 + d_2).
\]
From the arbitrariness of \( d_1 \in \mathcal{T}_x \) and \( d_2 \in \mathcal{N}_x \), we get \( \nabla^2 h(x) \succeq 0 \). This complete the proof.

A.2.2 Proofs for Section 3.2.2

Proof for Proposition 3.12

Proof. It follows from Lemma 3.2 and Lemma 3.3 that
\[
|f(A^2(y)) - f(A(y))| \leq M_{x,f} \left| A^2(y) - A(y) \right| \leq \frac{2M_{x,f}(M_{x,A} + 1)}{\sigma_{x,c}} \|c(A(y))\|^2
\]
\[
\leq \frac{8M_{x,f}(M_{x,A} + 1)L_{x,b}}{\sigma_{x,c}^2} \|c(y)\|^2.
\]
By Lemma 3.3 we know that the inequality \( \|c(A(y))\| \leq \frac{1}{2} \|c(y)\| \) holds for any \( y \in \Omega_x \). As a result,
\[
h(A(y)) - h(y) \leq \left| f(A^2(y)) - f(A(y)) \right| + \frac{\beta}{2} \left( \|c(A(y))\|^2 - \|c(y)\|^2 \right)
\]
\[
\leq - \left( \frac{\beta}{4} - \frac{8M_{x,f}(M_{x,A} + 1)L_{x,b}}{\sigma_{x,c}^3} \right) \|c(y)\|^2 \leq - \frac{\beta}{8} \|c(y)\|^2.
\]
Here the last inequality uses the fact that \( \beta \geq \beta_x \).
Together with Lemma \ref{lem:lowerbound_derivative}
and Definition \ref{def:gradient}, further implies that
\[
\begin{align*}
h(A^\infty(y)) - h(y) &= \sum_{i=0}^{+\infty} h(A^{i+1}(y)) - h(A^i(y)) \\
&\leq - \frac{\beta}{8} \sum_{i=0}^{+\infty} \|c(A^i(y))\|^2 \leq - \frac{\beta}{4} \|c(y)\|^2,
\end{align*}
\]
and we complete the proof. \( \square \)

\textbf{Proof for Proposition \ref{prop:gradient}}

\begin{proof}
Firstly, it follows from Lemma \ref{lem:lowerbound_derivative}
that \( A^\infty(y) \) exists and \( A^\infty(y) \in \Omega_x \cap M \). Let \( U \) be a matrix
whose columns form an orthonormal basis of the tangent space at \( A^\infty(y) \), from the definition of \( \mathcal{T}_{A^\infty(y)} = \text{Null}(J_c(A^\infty(y))^\top) \),
it holds that \( U^\top J_c(A^\infty(y)) = 0 \). Then we can conclude that
\[
\begin{align*}
\left\| U^\top J_c(y)c(y) \right\| &\leq \left\| U^\top J_c(y) \right\| \|c(y)\| \leq \left\| U^\top (J_c(y) - J_c(A^\infty(y))) \right\| \|c(y)\| \\
&\leq \| J_c(y) - J_c(A^\infty(y)) \| \|c(y)\| \leq \frac{4L_{x,y}(M_{x,y} + 1)}{\sigma_{x,y}} \|c(y)\|^2.
\end{align*}
\]
\tag{A.10}

Here the last inequality results from Lemma \ref{lem:lowerbound_derivative}.
Because \( U \) be a matrix whose columns form an orthonormal basis of the tangent space at \( A^\infty(y) \),
Lemma \ref{lem:lowerbound_derivative} and Definition \ref{def:gradient} imply that
\[
UU^\top \nabla g(A^\infty(y)) = \text{grad} f(A^\infty(y)).
\]
Therefore, we can obtain
\[
\| \nabla h(y) \| \geq \left\| U^\top \nabla h(y) \right\| = \left\| U^\top (\nabla g(y) + \beta J_c(y)c(y)) \right\|
\geq \left\| U^\top \nabla g(y) \right\| - \beta \left\| U^\top J_c(y)c(y) \right\|
\geq \left\| U^\top \nabla g(A^\infty(y)) \right\| - L_{x,y} \|A^\infty(y) - y\| - \beta \left\| U^\top J_c(y)c(y) \right\|
\geq \| \text{grad} f(A^\infty(y)) \| - \frac{4L_{x,y}(M_{x,y} + 1)}{\sigma_{x,y}} \|c(y)\| - \frac{4\beta L_{x,y}(M_{x,y} + 1)}{\sigma_{x,y}} \|c(y)\|^2.
\]

Here the inequality \((i)\) results from \tag{A.10} and Lemma \ref{lem:lowerbound_derivative} As a result, we further have
\[
\| \nabla h(y) \| \geq \frac{1}{2} \| \nabla h(y) \| + \frac{1}{2} \| \nabla h(y) \|
\geq \frac{1}{2} \| \nabla h(y) \| + \frac{\beta \sigma_{x,y}}{16(M_{x,y} + 1)} \|c(y)\|
\geq \frac{1}{2} \| \text{grad} f(A^\infty(y)) \| + \frac{\beta \sigma_{x,y}}{16(M_{x,y} + 1)} \|c(y)\| - \frac{2L_{x,y}(M_{x,y} + 1)}{\sigma_{x,y}} \|c(y)\| \\
- \frac{2\beta L_{x,y}(M_{x,y} + 1)}{\sigma_{x,y}} \|c(y)\|^2
\geq \frac{1}{2} \| \text{grad} f(A^\infty(y)) \| + \frac{\beta \sigma_{x,y}}{16(M_{x,y} + 1)} \|c(y)\| - \frac{2L_{x,y}(M_{x,y} + 1)}{\sigma_{x,y}} \|c(y)\| \\
- \frac{2\beta L_{x,y}(M_{x,y} + 1)}{\sigma_{x,y}} \cdot \frac{M_{x,y}\sigma_{x,y} \varepsilon_{x}}{4M_{x,y}(M_{x,y} + 1) + \sigma_{x,y}} \|c(y)\|
\geq \frac{1}{2} \| \text{grad} f(A^\infty(y)) \|.
\]

Here the inequality \((ii)\) is implied by Theorem \ref{thm:gradient} and the inequality \((iii)\) follows from the definition of \( \hat{\Omega}_x \) and Lemma \ref{lem:lowerbound_derivative} \( \square \)

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Proof for Proposition 3.14

Proof. It follows from Lemma 3.4 that $J(x)^{\top}d = d$ holds for any $d \in T_x$, which implies the equality

$$d^{\top}\nabla^2 h(x)d = d^{\top}\nabla^2 g(x)d = d^{\top}\left(\nabla^2 f(x) - \sum_{i=1}^{p}\lambda_i(x)\nabla^2 c_i(x)\right)d + d^{\top}(D_{f_A(x)}\nabla f(x))d.$$

Together with Definition 2.3 we obtain the follow inequalities

$$\lambda_{\min}(hess f(x)) \geq \min_{d \in T_x, ||d||=1} d^{\top}\nabla^2 h(x)d - L_{x,A}||\nabla f(x)|| \geq \lambda_{\min}(\nabla^2 g(x)) - L_{x,A}||\nabla f(x)||,$$

$$\lambda_{\max}(hess f(x)) \leq \max_{d \in T_x, ||d||=1} d^{\top}\nabla^2 h(x)d + L_{x,A}||\nabla f(x)|| \leq \lambda_{\max}(\nabla^2 g(x)) + L_{x,A}||\nabla f(x)||,$$

which complete the proof. □

Proof for Proposition 3.15

Proof. Firstly, it follows from Lemma 3.4 and Theorem 3.10 that

$$||y - A^{\infty}(y)|| \leq \frac{4(M_{x \lambda} + 1)}{\sigma_{x,c}} ||c(y)|| \leq \frac{32(M_{x \lambda} + 1)^2}{\beta \sigma_{x,c}^2} \||\nabla h(y)||. \quad (A.11)$$

Let $U_y$ be a matrix whose columns form an orthonormal basis of $T_{A^{\infty}(y)}$. Lemma A.2 implies that

$$\lambda_{\min}(hess (A^{\infty}(y))) \geq \lambda_{\min}(U_y^{\top}\nabla^2 g(A^{\infty}(y))U_y) - L_{x,A}||\nabla f(A^{\infty}(y))||$$

$$\geq \lambda_{\min}(U_y^{\top}\nabla^2 g(A^{\infty}(y))U_y) - 2L_{x,A}||\nabla h(y)||$$

$$\geq \lambda_{\min}(U_y^{\top}\nabla^2 g(A^{\infty}(y))U_y) - 2L_{x,A}||\nabla h(y)|| - 2L_{x,A}L_{x,g}||y - A^{\infty}(y)||$$

$$\geq \lambda_{\min}(U_y^{\top}\nabla^2 g(A^{\infty}(y))U_y) - 2L_{x,A} \left(1 + \frac{32L_{x,g}(M_{x \lambda} + 1)^2}{\beta \sigma_{x,c}^2}\right)||\nabla h(y)||$$

$$\geq \lambda_{\min}(U_y^{\top}\nabla^2 g(A^{\infty}(y))U_y) - \frac{5}{2}L_{x,A}||\nabla h(y)||.$$

Here the last inequality follows from the fact that $\beta \geq \beta_x$. Then, we can obtain

$$\lambda_{\min}(U_y^{\top}\nabla^2 g(A^{\infty}(y))U_y) \geq \lambda_{\min}(U_y^{\top}\nabla^2 g(y)U_y) - \left\|U_y^{\top}\left(\nabla^2 g(A^{\infty}(y)) - \nabla^2 g(y)\right)U_y\right\|$$

$$\geq \lambda_{\min}(U_y^{\top}\nabla^2 g(y)U_y) - \left\|\nabla^2 g(A^{\infty}(y)) - \nabla^2 g(y)\right\|.$$
Finally, we conclude that
\[
\lambda_{\text{min}}(\text{hess } f(A^\infty(y))) \geq \lambda_{\text{min}}(U_y^T \nabla^2 g(A^\infty(y)) U_y) - L_{x,A} \|\nabla f(A^\infty(y))\| \\
\geq \lambda_{\text{min}}(U_y^T \nabla^2 g(y) U_y) - \left\| \nabla^2 g(A^\infty(y)) - \nabla^2 g(y) \right\| - L_{x,A} \|\nabla f(A^\infty(y))\| \\
\geq \lambda_{\text{min}}(\nabla^2 h(y)) - \left\| \nabla^2 g(A^\infty(y)) - \nabla^2 g(y) \right\| - \left(\frac{5}{2} L_{x,A} + \frac{96 L_{x,c} M_{x,A} (M_{x,A} + 1)^2}{\sigma_{x,c}^2} \right) \|\nabla h(y)\|,
\]
and complete the proof.

A.2.3 Proofs for Section 3.2.3

Proof for Proposition 3.16

Proof. Since \( f(x) \) satisfies the Riemannian Łojasiewicz gradient inequality at \( x \in \mathcal{M} \) with exponent \( \theta \), there exists a neighborhood \( \mathcal{U} \subset \mathcal{O}_x \) and a constant \( C > 0 \) such that for any \( y \in \mathcal{M} \cap \mathcal{U} \), \( A^\infty(y) \in \mathcal{U} \) and

\[
\|\nabla f(y)\| \geq C |f(y) - f(x)|^{1-\theta}.
\]

For any \( z \in \mathcal{U} \subset \mathcal{O}_x \), we denote \( w := A^\infty(z) \in \mathcal{M} \cap \mathcal{U} \), it then follows from Theorem 3.11 and Proposition 3.13 that

\[
\|\nabla h(z)\| \geq \frac{1}{2} \|\nabla f(w)\| + \frac{\beta \sigma_{x,c}}{16(M_{x,A} + 1)} \|c(z)\| \\
\geq \frac{C}{2} \|f(w) - f(x)\|^{1-\theta} + \frac{\beta \sigma_{x,c}}{16(M_{x,A} + 1)} \|c(z)\|,
\]

which further implies

\[
|h(z) - h(w)|^{1-\theta} = \left| f(A(z)) - f(A^\infty(z)) + \frac{\beta}{2} \|c(z)\| \right|^{1-\theta} \\
\leq \left| f(A(z)) - f(A^\infty(z)) \right|^{1-\theta} + \beta^{1-\theta} \|c(z)\|^{2-2\theta} \\
\leq \left( M_{x,f} \|A(z) - A^\infty(z)\|^{1-\theta} + \beta^{1-\theta} \|c(z)\|^{2-2\theta} \right) \\
\leq \left( \frac{16 M_{x,f} (M_{x,A} + 1) L_{x,b}}{\sigma_{x,c}^2} \right)^{1-\theta} + \beta^{1-\theta} \|c(z)\|^{2-2\theta} \leq \frac{5 \beta^{1-\theta}}{4} \|c(z)\|^{2-2\theta}.
\]

Here, the inequality \((i)\) results from the Lipschitz continuity of \( f \). Meanwhile, the inequality \((ii)\) is concluded from Lemma 3.3 and Lemma 3.4. As a result, it follows from the monotonicity of \( t^{1-\theta} \) that

\[
|h(z) - h(x)|^{1-\theta} \leq |h(w) - h(x)|^{1-\theta} + |h(z) - h(w)|^{1-\theta} \leq |f(w) - f(x)|^{1-\theta} + \frac{5 \beta^{1-\theta}}{4} \|c(z)\|^{2-2\theta}.
\]

Substituting \((A.13)\) into \((A.12)\), we obtain

\[
\|\nabla h(z)\| \geq \frac{C}{2} \|f(w) - f(x)\|^{1-\theta} + \frac{\beta \sigma_{x,c}}{16(M_{x,A} + 1)} \|c(z)\| \\
\geq \frac{C}{2} |h(z) - h(x)|^{1-\theta} - \frac{5 \beta^{1-\theta} \|c(z)\|^{2-2\theta}}{8} + \frac{\beta \sigma_{x,c}}{16(M_{x,A} + 1)} \|c(z)\|.
\]

Denote \( \hat{O}_x := \left\{ \tilde{x} \in \mathcal{O}_x : \|c(\tilde{x})\|^{1-2\theta} \leq \left( \frac{\beta \sigma_{x,c}}{16(M_{x,A} + 1)} \right) \right\} \), then \( \hat{O}_x \) is a neighborhood of \( x \). Resulting from the fact that \( \theta \in (0, \frac{1}{2}) \), the following inequality holds for any \( z \in \hat{O}_x \),

\[
\frac{\beta \sigma_{x,c}}{16(M_{x,A} + 1)} \|c(z)\| \geq \frac{5 \beta^{1-\theta}}{8} \|c(z)\|^{2-2\theta},
\]
which further implies the inequality

\[ \| \nabla h(z) \| \geq \frac{C}{2} |h(z) - h(x)|^{1-\theta}. \]

Therefore, we complete the proof.

\[ \square \]

**Proof for Theorem 3.17**

**Proof.** Suppose \( \tilde{x} \) is a local minimizer of \( \text{OCP} \), then there exists a neighborhood \( U_1 \) of \( \tilde{x} \) such that \( f(y) \geq f(\tilde{x}) \) holds for any \( y \in \mathcal{M} \cap U_1 \). We denote \( \tilde{U}_1 := \{ x \in \Omega_{\tilde{x}} : A^\infty(x) \in U_1 \} \) which is a neighborhood of \( \tilde{x} \) from Lemma 3.4. Then it follows from Proposition 3.12 that the inequality

\[ h(y) \geq h(A^\infty(y)) + \frac{\beta}{16} \| c(y) \|^2 \geq h(\tilde{x}), \]

holds for any \( y \in \Omega_{\tilde{x}} \cap \tilde{U}_1 \), which further implies that \( \tilde{x} \) is a local minimizer of \( \text{CDF} \).

On the other hand, suppose there exists \( x \in \mathcal{M} \) such that \( \tilde{x} \in \Omega_{x} \) is a local minimizer of \( \text{CDF} \), i.e. there exists a neighborhood \( U_2 \subset \Omega_{x} \) of \( \tilde{x} \) such that \( h(y) \geq h(\tilde{x}) \) holds for any \( y \in U_2 \). Firstly, we have \( \tilde{x} \in \mathcal{M} \) from Theorem 3.10. Recalling the fact that \( h(y) = f(y) \) holds for any \( y \in \mathcal{M} \), we immediately arrive at

\[ f(y) \geq f(\tilde{x}), \quad \forall y \in U_2 \cap \mathcal{M}. \]

Namely, \( \tilde{x} \) is a local minimizer of \( \text{OCP} \).

\[ \square \]

**B Supplementary Examples**

Developing Riemannian solvers by the frameworks introduced in [3] requires several basic geometrical materials of the manifold. Currently, the generalized Stiefel manifold, the hyperbolic manifold, and the symplectic Stiefel manifold are not supported by the existing Riemannian optimization packages programmed in Python [56, 42, 35]. On the other hand, even when the geometrical materials of the above-mentioned manifolds are available, it is still unclear whether various existing efficient unconstrained solvers can be easily implemented and added into the existing Python-based Riemannian optimization packages.

In this section, we present several supplementary examples to illustrate that \( \text{CDF} \) can be directly embedded in various existing unconstrained solvers to solve \( \text{OCP} \). All the numerical experiments in this section are run in serial on a platform with Intel(R) Xeon(R) Gold 6242R CPU @ 3.10GHz under Ubuntu 20.04 running Python 3.7.0 and Numpy 1.20.0 [28].

We choose the solvers from SciPy package [57], which provides various highly efficient solvers for unconstrained optimization. The detailed descriptions of the selected solvers are presented in Table 3. We select the symplectic manifold as an example since there is no existing symplectic manifold solver available in any Python package. However, we can easily solve the problem by the SciPy package with our \( \text{CDF} \) approach.

In this section, we consider the following optimization problem over the symplectic Stiefel manifold (i.e., \( \mathcal{M} = \{ X \in \mathbb{R}^{2m \times 2s} : X^\top Q_m X = Q_s \} \) with \( n = 4ms \), as described in Table 1),

\[ \min_{x \in \mathcal{M}} \frac{1}{2} \| x - \tilde{w} \|^2. \tag{B.1} \]

Problem (B.1) is usually referred as the nearest symplectic matrix problem [23, 61], which aims to calculate the nearest symplectic matrix on the symplectic Stiefel manifold to a target point \( \tilde{w} \in \mathbb{R}^n \) with respect to the \( \ell_2 \)-norm. In our numerical examples, we follow the settings in [23] to randomly generate \( \tilde{w} \) in \( \mathbb{R}^n \) and scale it by \( \tilde{w} = \tilde{w} / \text{norm}(\tilde{w}) \). We set the penalty parameter \( \beta = 2 \) in \( \text{CDF} \) and initiate all the selected solvers in Table 3 at the same initial point, which is randomly generated.
| Name       | Descriptions                                                                                                                                                                                                 | Riemannian version in Python? |
|------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------|
| CG         | The nonlinear conjugate gradient method by Polak and Ribiere, which is a variant of the Fletcher-Reeves method [47]. This solver is written in Python.                                                                 | Y                             |
| BFGS       | The quasi-Newton method proposed by Broyden, Fletcher, Goldfarb, and Shanno (BFGS) [47] and programmed in Python.                                                                                            | N                             |
| L-BFGS-B   | The limit-memory BFGS method [10]. SciPy package provides a python wrapper for the original FORTRAN solver developed by [72, 43].                                                                               | N                             |
| TNC        | The truncated Newton method [47]. SciPy package provides a python wrapper for its C implementation by [44].                                                                                                  | N                             |
| Newton-CG  | The Newton-CG method [47] that uses conjugate gradient method to compute the search direction. This solver is programmed in Python.                                                                               | N                             |
| Trust-krylov| The Newton GLTR trust-region method [28]. The trust-region subproblems are solved by trilb [57], which is a C programmed Krylov solver.                                                                  | N                             |
| Trust-ncg  | The Newton conjugate gradient trust-region method [47], programmed in Python.                                                                                                                                  | Y                             |
| Trust-exact| The trust-region method for unconstrained minimization, where the trust-region subproblems are solved exactly by factorizing the Hessian matrix [14]. Therefore, it requires the explicit expression of the Hessian matrix of the objective function. | N                             |

Table 3: Detailed descriptions for selected solvers from the SciPy package.

over the symplectic Stiefel manifold. Moreover, we adopt the auto-differentiation packages to automatically generate the gradient and Hessian from the expression of $h(x)$. Specifically, we generate the gradient $\nabla h(x)$ and explicit expression of $\nabla^2 h(x)$ by the autograd package [41]. Furthermore, for the solvers “Trust-krylov” and “Trust-ncg”, the Hessian-vector product for $\nabla^2 h(x)$ is automatically generated by the JAX package [9] from the expression of $\nabla h(x)$. We terminate the solvers when $\|\nabla h(x_k)\| \leq 10^{-5}$, or the number of iterations exceeds 10,000, while keeping all the other parameters as the default values.

Table 4 and Table 5 illustrate the performance of all the solvers from Table 3 in solving problem (B.1), under different combinations of problem parameters $m$ and $s$. The terms “Fval”, “Iter”, “Obj eval” “Grad”, “Feas”, and “CPU time” stand for the function value, the number of iterations, the number of function value evaluations, $\|\nabla h(x^*)\|$, $\|c(x^*)\|$, and the wall-clock running time, respectively. Here $x^*$ is the final solution returned by these solvers. We can learn from these tables that all the solvers can directly minimize CDF and yield solutions with similar function values and high accuracy in feasibility. This shows that solving OCP via our CDF formulation is not sensitive to the choice of the unconstrained optimization solver. Of course, the running times for various solvers may differ. But for our example, CG, L-BFGS-B, Newton-CG, Trust-krylov, Trust-ncg, and to a lesser extent TNC, can all solve the CDF problem highly efficiently.

Finally, we remark that while fixing the penalty parameter $\beta = 2$ in the above numerical experiments is sufficient for the corresponding CDF to be an exact penalty function, for other application examples, we may need to dynamically increase the penalty parameter in order to make the corresponding CDF $h(\cdot)$ an exact penalty function. We leave the strategy to adjust $\beta$ for future investigation.
| Test instance | Solver | Fval | Iter | Obj.eval | Grad | Feas | CPU time (s) |
|---------------|--------|------|------|----------|------|------|--------------|
|               | CG     | 1.64e+00 | 52  | 86       | 4.44e-06 | 8.95e-07 | 0.03         |
|               | BFGS   | 1.64e+00 | 71  | 90       | 4.61e-06 | 7.12e-07 | 0.60         |
| 2m = 50       | L-BFGS-B | 1.64e+00 | 54  | 59       | 3.98e-06 | 5.56e-07 | 0.03         |
|               | TNC    | 1.64e+00 | 40  | 410      | 9.75e-06 | 1.08e-06 | 0.10         |
|               | Newton-CG | 1.64e+00 | 16  | 20       | 1.03e-06 | 1.82e-07 | 0.09         |
|               | Trust-krylov | 1.64e+00 | 16  | 17       | 2.84e-06 | 5.15e-07 | 0.14         |
|               | Trust-neg | 1.64e+00 | 20  | 21       | 4.65e-07 | 8.05e-08 | 0.14         |
|               | Trust-exact | 1.64e+00 | 39  | 40       | 9.70e-07 | 3.37e-07 | 14.86        |
|               | CG     | 1.59e+00 | 45  | 76       | 5.77e-06 | 1.19e-06 | 0.03         |
| 2m = 100      | BFGS   | 1.59e+00 | 71  | 90       | 6.24e-06 | 9.47e-07 | 1.55         |
|               | L-BFGS-B | 1.59e+00 | 49  | 56       | 3.81e-06 | 6.68e-07 | 0.03         |
|               | TNC    | 1.59e+00 | 41  | 489      | 5.38e-06 | 5.34e-07 | 0.13         |
|               | Newton-CG | 1.59e+00 | 16  | 24       | 1.10e-06 | 1.93e-07 | 0.09         |
|               | Trust-krylov | 1.59e+00 | 15  | 16       | 7.57e-06 | 1.32e-06 | 0.12         |
|               | Trust-neg | 1.59e+00 | 19  | 20       | 5.50e-07 | 9.40e-08 | 0.12         |
|               | Trust-exact | 1.59e+00 | 37  | 38       | 6.45e-07 | 2.05e-07 | 29.10        |
|               | CG     | 1.23e+00 | 35  | 64       | 9.41e-06 | 1.92e-06 | 0.03         |
| 2m = 500      | BFGS   | 1.23e+00 | 51  | 75       | 9.21e-06 | 1.73e-06 | 258.38       |
|               | L-BFGS-B | 1.23e+00 | 39  | 45       | 6.48e-06 | 1.38e-06 | 0.04         |
|               | TNC    | 1.23e+00 | 45  | 620      | 7.36e-07 | 8.79e-08 | 0.29         |
|               | Newton-CG | 1.23e+00 | 15  | 19       | 9.18e-07 | 1.63e-07 | 0.09         |
|               | Trust-krylov | 1.23e+00 | 13  | 14       | 9.68e-06 | 1.71e-06 | 0.13         |
|               | Trust-neg | 1.23e+00 | 14  | 15       | 2.67e-07 | 4.58e-08 | 0.10         |
|               | Trust-exact | 1.23e+00 | 33  | 34       | 9.28e-07 | 2.52e-07 | 482.73       |
|               | CG     | 1.27e+00 | 27  | 52       | 9.08e-06 | 3.32e-06 | 0.05         |
| 2m = 1000     | BFGS   | 1.27e+00 | 33  | 61       | 4.51e-06 | 6.13e-07 | 0.12         |
|               | L-BFGS-B | 1.27e+00 | 35  | 40       | 9.80e-06 | 3.63e-06 | 0.06         |
|               | TNC    | 1.27e+00 | 35  | 507      | 6.12e-07 | 6.49e-08 | 0.37         |
|               | Newton-CG | 1.27e+00 | 14  | 18       | 1.77e-06 | 3.11e-07 | 0.09         |
|               | Trust-krylov | 1.27e+00 | 12  | 13       | 5.45e-06 | 9.57e-07 | 0.14         |
|               | Trust-neg | 1.27e+00 | 14  | 15       | 2.27e-07 | 3.93e-08 | 0.11         |
|               | Trust-exact | 1.27e+00 | 33  | 34       | 9.28e-07 | 2.52e-07 | > 1200       |
|               | CG     | 1.18e+00 | 33  | 61       | 4.51e-06 | 6.13e-07 | 0.12         |
| 2m = 2000     | BFGS   | - - - - | - - - | - - - | - - - | > 1200 |
|               | L-BFGS-B | 1.18e+00 | 39  | 47       | 2.28e-06 | 3.27e-07 | 0.17         |
|               | TNC    | 1.18e+00 | 33  | 553      | 3.70e-07 | 4.54e-08 | 0.72         |
|               | Newton-CG | 1.18e+00 | 14  | 17       | 2.40e-07 | 4.31e-08 | 0.15         |
|               | Trust-krylov | 1.18e+00 | 12  | 13       | 1.42e-06 | 2.64e-07 | 0.20         |
|               | Trust-neg | 1.18e+00 | 18  | 19       | 3.50e-08 | 6.05e-09 | 0.26         |
|               | Trust-exact | 1.18e+00 | 18  | 19       | 3.50e-08 | 6.05e-09 | > 1200       |
|               | CG     | 1.10e+00 | 30  | 60       | 8.85e-06 | 1.55e-06 | 0.42         |
| 2m = 10000    | BFGS   | - - - - | - - - | - - - | - - - | > 1200 |
|               | L-BFGS-B | 1.10e+00 | 33  | 38       | 5.52e-06 | 1.00e-06 | 0.66         |
|               | TNC    | 1.10e+00 | 20  | 156      | 2.84e-07 | 3.33e-08 | 1.38         |
|               | Newton-CG | 1.10e+00 | 13  | 16       | 5.83e-06 | 6.78e-07 | 0.31         |
|               | Trust-krylov | 1.10e+00 | 11  | 12       | 9.26e-06 | 1.85e-06 | 0.47         |
|               | Trust-neg | 1.10e+00 | 16  | 17       | 1.62e-07 | 2.79e-08 | 0.56         |
|               | Trust-exact | 1.10e+00 | 16  | 17       | 1.62e-07 | 2.79e-08 | > 1200       |

Table 4: Numerical results of the nearest symplectic matrix problem with fixed $2s = 10$. 
| Test instance | Solver     | Fval   | Iter | Obj eval | Grad   | Feas     | CPU time (s) |
|--------------|------------|--------|------|----------|--------|----------|--------------|
| 2s = 2       | CG         | 2.15e-01 | 16   | 33       | 6.32e-06 | 1.65e-06 | 0.02         |
|              | BFGS       | 2.15e-01 | 13   | 16       | 8.20e-06 | 8.28e-07 | 1.18         |
|              | L-BFGS-B   | 2.15e-01 | 14   | 16       | 1.34e-06 | 2.09e-07 | 0.02         |
|              | TNC        | 2.15e-01 | 35   | 333      | 2.90e-07 | 5.52e-08 | 0.12         |
|              | Newton-CG  | 2.15e-01 | 10   | 13       | 5.92e-07 | 1.49e-07 | 0.04         |
|              | Trust-krylov| 2.15e-01 | 10   | 11       | 2.25e-06 | 5.67e-07 | 0.05         |
|              | Trust-neg  | 2.15e-01 | 14   | 15       | 5.56e-07 | 1.40e-07 | 0.06         |
|              | Trust-exact| 2.15e-01 | 27   | 28       | 4.42e-06 | 1.11e-06 | 124.31       |
| 2s = 10      | CG         | 1.17e+00 | 33   | 59       | 2.71e-06 | 3.13e-07 | 0.04         |
|              | BFGS       | 1.17e+00 | 48   | 72       | 9.34e-06 | 9.89e-07 | 234.10       |
|              | L-BFGS-B   | 1.17e+00 | 43   | 49       | 1.64e-06 | 2.27e-07 | 0.13         |
|              | TNC        | 1.17e+00 | 32   | 538      | 6.86e-07 | 8.59e-08 | 0.39         |
|              | Newton-CG  | 1.17e+00 | 15   | 18       | 1.50e-06 | 2.56e-07 | 0.08         |
|              | Trust-krylov| 1.17e+00 | 12   | 13       | 3.03e-06 | 5.38e-07 | 0.14         |
|              | Trust-neg  | 1.17e+00 | 14   | 15       | 2.85e-07 | 4.87e-08 | 0.11         |
|              | Trust-exact| 1.17e+00 | 27   | 28       | 4.42e-06 | 1.11e-06 | > 1200       |
| 2s = 50      | CG         | 7.19e+00 | 47   | 80       | 1.34e-06 | 1.51e-07 | 0.41         |
|              | BFGS       | 7.19e+00 | 53   | 57       | 6.83e-07 | 1.02e-07 | 0.62         |
|              | L-BFGS-B   | 7.19e+00 | 41   | 748      | 2.64e-07 | 2.93e-08 | 4.41         |
|              | TNC        | 7.19e+00 | 17   | 20       | 2.05e-07 | 3.59e-08 | 0.63         |
|              | Newton-CG  | 7.19e+00 | 15   | 16       | 4.11e-06 | 7.06e-07 | 0.93         |
|              | Trust-neg  | 7.19e+00 | 17   | 18       | 1.91e-07 | 3.26e-08 | 0.86         |
|              | Trust-exact| 7.19e+00 | 17   | 18       | 1.91e-07 | 3.26e-08 | > 1200       |
| 2s = 100     | CG         | 1.65e+01 | 58   | 100      | 6.64e-07 | 1.51e-07 | 0.84         |
|              | BFGS       | 1.65e+01 | 67   | 73       | 2.39e-07 | 3.81e-08 | 1.21         |
|              | L-BFGS-B   | 1.65e+01 | 43   | 710      | 4.09e-07 | 4.27e-08 | 8.10         |
|              | TNC        | 1.65e+01 | 18   | 21       | 4.71e-06 | 8.52e-07 | 1.06         |
|              | Newton-CG  | 1.65e+01 | 19   | 20       | 2.54e-06 | 4.33e-07 | 1.85         |
|              | Trust-neg  | 1.65e+01 | 23   | 24       | 8.83e-08 | 1.53e-08 | 1.99         |
|              | Trust-exact| 1.65e+01 | 23   | 24       | 8.83e-08 | 1.53e-08 | > 1200       |
| 2s = 500     | CG         | 1.21e+02 | 135  | 224      | 2.65e-07 | 5.28e-08 | 10.38        |
|              | BFGS       | 1.21e+02 | 133  | 200      | 3.52e-07 | 5.78e-08 | 21.90        |
|              | L-BFGS-B   | 1.21e+02 | 67   | 977      | 5.81e-07 | 9.86e-08 | 64.28        |
|              | TNC        | 1.21e+02 | 31   | 36       | 1.26e-06 | 2.20e-07 | 12.96        |
|              | Newton-CG  | 1.21e+02 | 28   | 29       | 4.40e-06 | 7.55e-07 | 21.64        |
|              | Trust-krylov| 1.21e+02 | 47   | 37       | 2.09e-07 | 3.50e-08 | 54.90        |
|              | Trust-neg  | 1.21e+02 | 47   | 37       | 2.09e-07 | 3.50e-08 | > 1200       |
|              | Trust-exact| 1.21e+02 | 47   | 37       | 2.09e-07 | 3.50e-08 | > 1200       |

Table 5: Numerical results of the nearest symplectic matrix problem with fixed $2m = 1000$. 

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