ON ORBITS OF ANTICHAINS OF POSITIVE ROOTS

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ABSTRACT. For any finite poset $P$, there is a natural operator $X = X_P$, acting on the set of antichains of $P$. We discuss conjectural properties of $X$ for some graded posets associated with irreducible root systems. In particular, if $\Delta^+$ is the set of positive roots and $\Pi$ is the set of simple roots in $\Delta^+$, then we consider the cases $P = \Delta^+$ and $P = \Delta^+ \setminus \Pi$. For the root system of type $A_n$, we consider an $X$-invariant integer-valued function on the set of antichains of $\Delta^+$ and establish some properties of it.

1. INTRODUCTION

Let $(P, \preceq)$ be an arbitrary finite poset. For any $S \subset P$, let $S_{\text{min}}$ and $S_{\text{max}}$ denote the set of minimal and maximal elements of $S$, respectively. An antichain in $P$ is a subset of mutually incomparable elements. In other words, $\Gamma$ is an antichain if and only if $\Gamma = \Gamma_{\text{min}}$ (or $\Gamma = \Gamma_{\text{max}}$). Write $\mathfrak{A}_n(P)$ for the set of all antichains in $P$. An upper ideal (or filter) is a subset $I \subset P$ such that if $\gamma \in I$ and $\gamma \preceq \beta$, then $\beta \in I$. If $\Gamma \in \mathfrak{A}_n(P)$, then $I(\Gamma)$ denotes the upper ideal of $P$ generated by $\Gamma$. That is,

$$I(\Gamma) = \{ \varepsilon \in P \mid \exists \gamma \in \Gamma \text{ such that } \gamma \preceq \varepsilon \}.$$

For instance, $\Gamma = \emptyset$ is an antichain and $I(\emptyset)$ is the empty upper ideal. Conversely, if $I$ is an upper ideal of $P$, then $I_{\text{min}} \in \mathfrak{A}_n(P)$. This yields a natural bijection between the upper ideals and antichains of $P$. Letting $\Gamma' \preceq \Gamma$ if $I(\Gamma') \subset I(\Gamma)$, we make $\mathfrak{A}_n(P)$ a poset.

For $\Gamma \in \mathfrak{A}_n(P)$, we set $X(\Gamma) = (P \setminus I(\Gamma))_{\text{max}}$. This defines the map $X = X_P : \mathfrak{A}_n(P) \to \mathfrak{A}_n(P)$. Clearly, $X$ is one-to-one, i.e., it is a permutation of the finite set $\mathfrak{A}_n(P)$. We say that $X$ is the reverse operator for $P$. If $\# \mathfrak{A}_n(P) = m$, then $X$ is an element of the symmetric group $\Sigma_m$. Let $\langle X \rangle$ denote the cyclic subgroup of $\Sigma_m$ generated by $X$. The order of $X$, $\text{ord}(X)$, is the order of the group $\langle X \rangle$. As the definition of $X$ is quite natural, one can expect that properties of $\langle X \rangle$-orbits in $\mathfrak{A}_n(P)$ are closely related to other properties of $P$. One of the problems is to determine the cyclic structure of $X$, i.e., possible cardinalities of $\langle X \rangle$-orbits in $\mathfrak{A}_n(P)$. In particular, one can ask about a connection between properties of $P$ and $\text{ord}(X)$. For simplicity, we will speak about $X$-orbits in what follows. If $P$ is a Boolean lattice, then $X$-orbits has been studied under the name "loops of clutters", see [2]. Some conjectures stated in [2] for that special situation are proved in [3, 4] for an arbitrary graded poset $P$.

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We say that $\mathcal{P}$ is graded (of level $r$) if there is a function $d : \mathcal{P} \to \{1, 2, \ldots, r\}$ such that both $d^{-1}(1)$ and $d^{-1}(r)$ are non-empty, and $d(y) = d(x) + 1$ whenever $y$ covers $x$. Then $d^{-1}(1) \subset \mathcal{P}_{\text{min}}$ and $d^{-1}(r) \subset \mathcal{P}_{\text{max}}$.

**Lemma 1.1.** Suppose $\mathcal{P}$ is graded of level $r$, $d^{-1}(1) = \mathcal{P}_{\text{min}}$ and $d^{-1}(r) = \mathcal{P}_{\text{max}}$. Then $\mathcal{X}$ has an orbit of cardinality $r + 1$.

**Proof.** Clearly, $\mathcal{P}(i) : = d^{-1}(i)$ is an antichain for any $i$. From our hypotheses, it follows that $\mathcal{X}(\mathcal{P}(i)) = \mathcal{P}(i-1)$ for $i = 2, \ldots, r$, $\mathcal{X}(\mathcal{P}(1)) = \emptyset$, and $\mathcal{X}(\emptyset) = \mathcal{P}(r)$. Thus, \(\emptyset, \mathcal{P}(r), \ldots, \mathcal{P}(1)\) is an $\mathcal{X}$-orbit. \(\square\)

Such an orbit of $\mathcal{X}$ is said to be **standard**.

The goal of this note is to present several observations and conjectures on orbits of reverse operators for some graded posets associated with a root system $\Delta$. In Section 2, we discuss conjectural properties of reverse operators for $\Delta^+, \Delta^+ \setminus \Pi$, and $\Delta_s^+$ (see notation below). Roughly speaking, all our conjectures are verified up to rank 5. In particular, our calculations for $F_4$ are presented in Appendix. In Section 3, we work with the root system of type $A_n$. In this particular case, we

(1) describe an $\mathcal{X}$-invariant function $y : \mathfrak{An}(\Delta^+) \to \mathbb{N}$ (this is due to O. Yakimova);

(2) prove that $\mathcal{X}$ satisfies the relation $\mathcal{X}^{-1} = * \circ \mathcal{X} \circ *$, where $* : \mathfrak{An}(\Delta^+) \to \mathfrak{An}(\Delta^+)$ is the involutory mapping (duality) constructed in [5]. In other words, for any $\Gamma \in \mathfrak{An}(\Delta^+)$, one has $\mathcal{X}^{-1}(\Gamma^*) = \mathcal{X}(\Gamma)^*$;

(3) show that $y(\Gamma) = y(\Gamma^*)$ for any $\Gamma \in \mathfrak{An}(\Delta^+)$.\[\]

This is an expanded version of my talk at the workshop “$B$-stable ideals and nilpotent orbits” (Rome, October 2007).

2. **Reverse operators for posets associated with root systems**

Let $\Delta$ be a reduced irreducible root system in an $n$-dimensional real vector space $V$ and $W \subset GL(V)$ the corresponding Weyl group. Choose a system of positive roots $\Delta^+$ with the corresponding subset of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. The root order in $\Delta^+$ is given by letting $x \preceq y$ if $y - x$ is a non-negative integral combination of positive roots. In particular, $y$ covers $x$ if $y - x$ is a simple root. The highest root in $\Delta^+$ is denoted by $\theta$. It is the unique maximal element of $(\Delta^+, \preceq)$. If $\Delta$ has two root lengths, then $\theta_s$ is the dominant (highest) short root. Let $w_0 \in W$ be the longest element, i.e., the unique element that takes $\Delta^+$ to $-\Delta^+$. If $\gamma = \sum_{i=1}^n a_i \alpha_i \in \Delta^+$, then $\text{ht}(\gamma) := \sum a_i$ is the height of $\gamma$. For $I \subset \Pi$, $\Delta(I)$ is the root subsystem of $\Delta$ generated by $I$. If $X_n$ is one of the Cartan types, then $\Delta(X_n)$ denotes the root system of type $X_n$.\[\]
2.1. Orbits in $\Delta^+$. In this subsection, we consider antichains in $\Delta^+$ and the reverse operator $\mathcal{X} = \mathcal{X}_{\Delta^+} : \mathfrak{An}(\Delta^+) \to \mathfrak{An}(\Delta^+)$. Let $h = h(\Delta)$ be the Coxeter number and $e_1, \ldots, e_n$ the exponents of $\Delta$. It is known that $\#(\mathfrak{An}(\Delta^+)) = \prod_{i=1}^n \frac{h + e_i + 1}{e_i + 1}$. The function $\alpha \mapsto \text{ht}(\alpha)$ turns $\Delta^+$ into a graded poset of level $h-1$. Set $\Delta(i) = \{\alpha \in \Delta^+ \mid \text{ht}(\alpha) = i\}$ and $\Delta(\geq i) = \{\alpha \in \Delta^+ \mid \text{ht}(\alpha) \geq i\}$. Then $\Delta(1) = \Pi = \Delta_{\text{min}}^+$ and $\Delta(h-1) = \{\theta\} = \Delta_{\text{max}}^+$.

Let us point out two specific orbits of $\mathcal{X}$:

1) By Lemma 1.1 there is an orbit of cardinality $h$. Namely, $\{\emptyset, \Delta(h-1), \ldots, \Delta(2), \Delta(1)\}$ is the standard $\mathcal{X}$-orbit in $\mathfrak{An}(\Delta^+)$. 

2) There is an $\mathcal{X}$-orbit of order 2. Let $\mathcal{A} \subset \Pi$ be a set of mutually orthogonal roots such that $\Pi \setminus \mathcal{A}$ also has that property. (The partition $\{\mathcal{A}, \Pi \setminus \mathcal{A}\}$ is uniquely determined, since the Dynkin diagram of $\Delta$ is a tree.) Then $\mathcal{X}(\mathcal{A}) = \Pi \setminus \mathcal{A}$ and $\mathcal{X}(\Pi \setminus \mathcal{A}) = \mathcal{A}$.

If $\Delta$ is of rank 2, then these two orbits exhaust $\mathfrak{An}(\Delta^+)$. 

**Conjecture 2.1.**

(i) If $w_0 = -1$, then $\text{ord}(\mathcal{X}) = h$;

(ii) If $w_0 \neq -1$, then $\mathcal{X}^w$ is the involution of $\mathfrak{An}(\Delta^+)$ induced by $-w_0$ and $\text{ord}(\mathcal{X}) = 2h$;

(iii) Let $O$ be an arbitrary $\mathcal{X}$-orbit in $\mathfrak{An}(\Delta^+)$. Then $\frac{1}{\#O} \sum_{\Gamma \in O} \#\Gamma = \frac{\#\Delta^+}{h} = \frac{n}{2}$.

Recall that $w_0 \neq -1$ if and only if $\Delta$ is of type $\mathbf{A}_n$ ($n \geq 2$), $\mathbf{D}_{2n+1}$, $\mathbf{E}_6$. Furthermore, the posets $\Delta^+$ are isomorphic for $\mathbf{B}_n$ and $\mathbf{C}_n$ [3, Lemma 2.2]. Conjecture 2.1 has been verified for $\mathbf{A}_n$ ($n \leq 5$), $\mathbf{C}_n$ ($n \leq 4$), $\mathbf{D}_4$, $\mathbf{F}_4$. It is easily seen that $\#\Gamma$ equals the number of elements of $\mathfrak{An}(\Delta^+)$ covered by $\Gamma$. For, $\Gamma$ covers $\Gamma'$ with respect to the order ‘$<$’ described in the Introduction if and only if $\Gamma' = (J(\Gamma) \setminus \{\gamma_i\})_{\text{min}}$ for some $\gamma_i \in \Gamma$. Hence $\sum_{\Gamma \in \mathfrak{An}(\Delta^+)} \#\Gamma$ equals the total number of edges in the Hasse diagram of $(\mathfrak{An}(\Delta^+), <)$. Therefore it follows from [7 Cor. 3.4] that

$$\sum_{\Gamma \in \mathfrak{An}(\Delta^+)} \frac{\#\Gamma}{\#\mathfrak{An}(\Delta^+)} = \frac{\#\Delta^+}{h}.$$ 

Thus, part (iii) can be regarded as a refinement of the last equality.

**Example 2.2.** We use the standard notation for roots in $\Delta^+(\mathbf{A}_n)$; e.g., $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, 2, \ldots, n$, and $\theta = \varepsilon_1 - \varepsilon_{n+1}$. If $\Gamma = \{\alpha_1\}$ for $\mathbf{A}_n$ and $n \geq 3$, then

$$\mathcal{X}^k(\{\alpha_1\}) = \{\gamma \in \Delta(\alpha_1, \ldots, \alpha_{n-1}) \mid \text{ht}(\gamma) = n + 1 - k\} \cup \{\alpha_{k+1} + \ldots + \alpha_n\}, \ 1 \leq k \leq n.$$
In particular, $\mathcal{X}^n(\{\alpha_1\}) = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and hence $\mathcal{X}^{n+1}(\{\alpha_1\}) = \{\alpha_n\}$. Therefore the $\mathcal{X}$-orbit of $\{\alpha_1\}$ is of cardinality $2h = 2n + 2$. The ratio $\frac{1}{\#O} \sum_{\Gamma \in \mathcal{O}} \#\Gamma$ equals $n/2$ for this orbit, as required.

It is an interesting problem to construct “invariants” of $\mathcal{X}$, i.e., functions on $\mathfrak{A}(\Delta^n)$ that are constant on the $\mathcal{X}$-orbits. Ideally, one could ask for a family of invariants that separates the orbits. Our achievement in this direction is rather modest. We know only one invariant in the case of type $\mathfrak{A}_n$, see Section 3.

2.2. Orbits in $\Delta^n \setminus \Pi$. We regard $\Delta^n \setminus \Pi = \Delta(...)$ as a subposet of $\Delta^n$. The theory of antichains (upper ideals) in $\Delta^n \setminus \Pi$ is quite similar to that for $\Delta^n$. In particular, $\#(\mathfrak{A}(\Delta^n \setminus \Pi)) = \prod_{i=1}^{n} \frac{h + e_i - 1}{e_i + 1}$. Let $\mathcal{X}_0 : \mathfrak{A}(\Delta^n \setminus \Pi) \to \mathfrak{A}(\Delta^n \setminus \Pi)$ be the reverse operator for $\Delta^n \setminus \Pi$. The function $\alpha \mapsto (\text{ht} \alpha) - 1$ turns $\Delta^n \setminus \Pi$ into a graded poset of level $h-2$. It follows that $\mathcal{X}_0$ has the standard orbit of cardinality $h - 1$. As the simple roots are removed, the corresponding orbit of order 2 also vanishes from $\mathfrak{A}(\Delta^n \setminus \Pi)$.

**Conjecture 2.3.**

(i) If $w_0 = -1$, then $\text{ord}(\mathcal{X}_0) = h - 1$;

(ii) If $w_0 \neq -1$, then $\mathcal{X}_0^{h-1}$ is the involution of $\mathfrak{A}(\Delta^n \setminus \Pi)$ induced by $-w_0$ and $\text{ord}(\mathcal{X}_0) = 2h - 2$;

(iii) For any $\mathcal{X}_0$-orbit $\mathcal{O} \subset \mathfrak{A}(\Delta^n \setminus \Pi)$, we have $\frac{1}{\#\mathcal{O}} \sum_{\Gamma \in \mathcal{O}} \#\Gamma = \frac{\#(\Delta^n \setminus \Pi)}{h - 1} = \frac{n}{2} \cdot \frac{h - 2}{h - 1}$.

Here are empirical evidences supporting the conjecture. The poset $\Delta^n \setminus \Pi$ for $\mathfrak{A}_{n+1}$ is isomorphic to $\Delta^n$ for $\mathfrak{A}_n$. Therefore Conjecture 2.3 holds for $\mathfrak{A}_n$ ($n \leq 6$). It has also been verified for $\mathfrak{C}_n$ ($n \leq 5$), $\mathfrak{D}_n$ ($n \leq 5$), and $\mathfrak{F}_4$. Again, $\sum_{\Gamma \in \mathfrak{A}(\Delta^n \setminus \Pi)} \#\Gamma$ equals the number of edges on the Hasse diagram of $\mathfrak{A}(\Delta^n \setminus \Pi)$, and it was verified in [8, Sect. 3] that

$$\frac{1}{\#\mathfrak{A}(\Delta^n \setminus \Pi)} \sum_{\Gamma \in \mathfrak{A}(\Delta^n \setminus \Pi)} \#\Gamma = \frac{\#(\Delta^n \setminus \Pi)}{h - 1}.$$

Hence part (iii) can be regarded as a refinement of the last equality.

If $w_0 = -1$ and $h - 1$ is prime, then Conjecture 2.3 predicts that all $\mathcal{X}_0$-orbits have the same cardinality. This is really the case for $\mathfrak{F}_4$, $\mathfrak{C}_3$, and $\mathfrak{C}_4$. Actually, this seems to be true for any $\mathfrak{C}_n$, see Conjecture 2.5.

**Remark.** One might have thought that posets $\Delta(\geq j)$ enjoy similar good properties for any $j$. However, this is not the case. For $\mathfrak{F}_4$ and $\Delta(\geq 3)$, the reverse operator has orbits of cardinality 10 and 8. Hence its order equals 40, while $h - 2 = 10$. Furthermore, the mean value of the size of antichains along the orbits is not constant.
2.3. Orbits in $\Delta^+_s$. Suppose $\Delta$ has two root lengths. Then $\Delta^+_s$ denotes the set of short positive roots in $\Delta^+$. We regard $\Delta^+_s$ as subposet of $\Delta^+$. Then $\theta_s$ is the unique maximal element of $\Delta^+_s$ and $(\Delta^+_s)_{min} = \Pi \cap \Delta^+_s =: \Pi_s$. General results on antichains in $\Delta^+_s$ are obtained in [6, Sect. 5]. Suppose $m = \#\Pi_s$ and the exponents $\{e_i\}$ are increasingly ordered. Then $\#(\mathcal{A}_{\Pi}(\Delta^+_s)) = \prod_{i=1}^{m} \frac{h + e_i + 1}{e_i + 1}$. Let $\mathcal{X}_s : \mathcal{A}_{\Pi}(\Delta^+_s) \rightarrow \mathcal{A}_{\Pi}(\Delta^+_s)$ be the reverse operator for $\mathcal{A}_{\Pi}(\Delta^+_s)$. Let $h^*(\Delta)$ denote the dual Coxeter number of $\Delta$. Recall that $h^*(\Delta^\vee) - 1 = \text{ht}(\theta_s)$, where $\Delta^\vee = \{ \frac{2\alpha}{(\alpha,\alpha)} \mid \alpha \in \Delta \}$ is the dual root system. The function $\text{ht}(\cdot)$ turns $\Delta^+_s$ into a graded poset of level $h^*(\Delta^\vee) - 1$. It follows that $\mathcal{X}_s$ has the standard orbit of cardinality $h^*(\Delta^\vee)$.

**Conjecture 2.4.**

1. $\text{ord}(\mathcal{X}_s) = h^*(\Delta^\vee)$;
2. Let $\mathcal{O}$ be an arbitrary $\mathcal{X}_s$-orbit in $\mathcal{A}_{\Pi}(\Delta^+_s)$. Then $\frac{1}{\#\mathcal{O}} \sum_{\Gamma \in \mathcal{O}} \#\Gamma = \frac{\#(\Delta^+_s)}{h^*(\Delta^\vee)}$.

The conjecture is easily verified for $\mathcal{B}_n$, $\mathcal{F}_4$, and $\mathcal{G}_2$, where the number of $\mathcal{X}_s$-orbits equals 1, 3, and 1, respectively. We have also verified it for $\mathcal{C}_n$ with $n \leq 5$.

For $\mathcal{C}_n$, the posets $\Delta^+ \setminus \Pi$ and $\Delta^+_s$ (hence $\mathcal{A}_{\Pi}(\Delta^+ \setminus \Pi)$ and $\mathcal{A}_{\Pi}(\Delta^+_s)$) are isomorphic. We also have a more precise conjecture in this case.

**Conjecture 2.5.** For $\Delta^+_s(\mathcal{C}_n)$, every $\mathcal{X}_s$-orbit is of cardinality $2n - 1 = h^*(\mathcal{B}_n)$. Each $\mathcal{X}_s$-orbit contains a unique antichain lying in $\Delta^+ (\alpha_1, \ldots, \alpha_{n-2}) \simeq \Delta^+(\mathcal{A}_{n-2})$.

Since $\#(\mathcal{A}_{\Pi}(\Delta^+_s)) = \binom{2n-1}{n}$ for $\mathcal{C}_n$ [6, Theorem 5.5], Conjecture 2.5 would imply that the number of $\mathcal{X}_s$-orbits equals $\frac{1}{2n-1} \binom{2n-1}{n}$, the $(n-1)$-th Catalan number. Note that this conjecture also provides a canonical representative in each $\mathcal{X}_s$-orbit in $\mathcal{A}_{\Pi}(\Delta^+_s(\mathcal{C}_n))$.

2.4. Orbits in $\Delta^+_s \setminus \Pi_s$. We regard $\Delta^+_s \setminus \Pi_s$ as subposet of $\Delta^+_s$. For the reverse operator $\mathcal{X}_{s,0} : \mathcal{A}_{\Pi}(\Delta^+_s \setminus \Pi_s) \rightarrow \mathcal{A}_{\Pi}(\Delta^+_s \setminus \Pi_s)$, one can state a similar conjecture, where $h^*(\Delta^\vee)$ is replaced with $h^*(\Delta^\vee) - 1$. However, this does not make much sense. The case of $\mathcal{B}_n$ and $\mathcal{G}_2$ is trivial. For $\mathcal{C}_n$, the poset $\Delta^+_s \setminus \Pi_s$ is isomorphic to $\Delta^+(\mathcal{C}_{n-1})$. Hence this case is covered by previous conjectures. The only new phenomenon occurs for $\mathcal{F}_4$, where everything is easily verified. Here $\#\mathcal{A}_{\Pi}(\Delta^+_s \setminus \Pi_s) = 16$ and $\mathcal{X}_{s,0}$ has two orbits, both of cardinality $8 = h^*(\mathcal{F}_4) - 1$.

**Example 2.6.** A slight modification of a poset can drastically change properties of reverse operators. Consider two graded posets of level 3, with Hasse diagrams.
The reverse operator for $P_1$ has three orbits of cardinality 8, 4, and 2 (and the properties stated in Conjecture 2.1). For $P_2$, there are two orbits of cardinality 16 and 7. Thus, $\text{ord}(X_1) = 8$, while $\text{ord}(X_2) = 16 \cdot 7$. Furthermore, the mean values of the size of antichains for two $X_2$-orbits are different.

3. Results for $\Delta^+(A_n)$

In this section, $\Delta = \Delta(A_n) = \Delta(sl_{n+1})$.

3.1. The OY-invariant. Here we describe an $X$-invariant function $Y : \mathfrak{An}(\Delta^+) \rightarrow \mathbb{N}$, which is found by Oksana Yakimova.

Let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be an arbitrary antichain in $\Delta^+$ and $J = J(\Gamma)$ the corresponding upper ideal, so that $\Gamma = J_{\text{min}}$. To each $\gamma_s$, we attach certain integer as follows. Clearly, $J \setminus \{\gamma_s\}$ is again an upper ideal. Set

$$r_{\Gamma}(\gamma_s) := \#(J \setminus \{\gamma_s\})_{\text{min}} - \#J_{\text{min}} + 1.$$  

For $sl_{n+1}$, the difference between the numbers of minimal elements of $J$ and $J \setminus \{\gamma_s\}$ always belongs to $\{-1, 0, 1\}$. Therefore $r_{\Gamma}(\gamma_s) \in \{0, 1, 2\}$. The OY-number of $\Gamma$ is defined by

$$Y(\Gamma) := \sum_{s=1}^{k} r_{\Gamma}(\gamma_s).$$

This definition only applies to non-empty $\Gamma$, and we specially set $Y(\emptyset) = 0$.

Example 3.1. a) For $\Gamma = \Pi = \{\alpha_1, \ldots, \alpha_n\}$, we have $Y(\Pi) = 0$. More generally, the same is true for $\Gamma = \Delta(i)$. b) For $\Gamma = \{\alpha_1, \alpha_3, \ldots\}$ (all simple roots with odd numbers) or $\Gamma = \{\alpha_2, \alpha_4, \ldots\}$ (all simple roots with even numbers), we have $Y(\Gamma) = n - 1$.

Theorem 3.2 (O. Yakimova). The OY-number is $X$-invariant, i.e., $Y(\Gamma) = Y(X(\Gamma))$ for all $\Gamma \in \mathfrak{An}(\Delta^+)$. 

Proof. Let us begin with an equivalent definition of $Y(\Gamma)$, which is better for the proof. Recall that $\Delta^+(A_n) = \{\varepsilon_i - \varepsilon_{j+1} \mid 1 \leq i \leq j \leq n\}$. The positive root $\varepsilon_i - \varepsilon_{j+1} = \alpha_i + \ldots + \alpha_j$ will be denoted by $(i, j)$. Suppose $\gamma_s = (i_s, j_s)$. Without loss of generality, we may assume that the $i$-components of all roots in $\Gamma$ form an increasing sequence. Then the fact that $\Gamma = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ is an antichain is equivalent to that $1 \leq i_1 < \ldots < i_k$, $j_1 < \ldots < j_k \leq n$, and $i_s \leq j_s$ for each $s$. Obviously, $\Gamma \setminus \{\gamma_s\} \subset (J \setminus \{\gamma_s\})_{\text{min}}$. Furthermore, if
$i_s - i_{s-1} \geq 2$, then $(i_s - 1, j_s) \in (J \setminus \{\gamma_s\})_{\min}$; and if $j_{s+1} - j_s \geq 2$, then $(i_s, j_s + 1) \in (J \setminus \{\gamma_s\})_{\min}$ as well. This observation shows that $r_{\Gamma}(\gamma_s) = \chi(i_s - i_{s-1}) + \chi(j_{s+1} - j_s)$, where $i_0 := 0$, $j_{k+1} := n + 1$, and the function $\chi$ on $\{1, 2, \ldots\}$ is defined by

$$\chi(a) = \begin{cases} 1, & a \geq 2 \\ 0, & a = 1 \end{cases}.$$  

Hence

$$Y(\Gamma) = \sum_{s=1}^{k} \chi(i_s - i_{s-1}) + \sum_{s=1}^{k} \chi(j_{s+1} - j_s).$$

We say that the difference $b - a$ is essential if $b - a \geq 2$. Thus, $Y(\Gamma)$ counts the total number of consecutive essential differences in the sequences $(0, i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k, n + 1)$. For this reason, we will think of $\Gamma$ as a two-row array:

$$\Gamma = \begin{pmatrix} 0 & i_1 & \cdots & i_k \\ j_1 & \cdots & j_k & n + 1 \end{pmatrix},$$

where each 2-element column represents a positive root.

Let us describe the operator $X$ using this notation. The first step is to replace $\Gamma$ in Eq. (3.3) with

$$\tilde{\Gamma} = \begin{pmatrix} 0 & 1 & i_1 + 1 & \cdots & i_{k-1} + 1 & i_k + 1 \\ j_1 - 1 & j_2 - 1 & \cdots & j_{k-1} & n & n + 1 \end{pmatrix}.$$  

It may happen, however, that some 2-element columns of $\tilde{\Gamma}$ are “bad”, i.e., they do not represent positive roots; e.g., if $j_1 = 1$ or $i_{s-1} + 1 > j_s - 1$. The second step is to remove all bad columns. The remaining array is exactly $X(\Gamma)$, cf. Figure 1.
Thus, our task is to check that such a procedure does not change the total number of essential differences.

(a) If $\mathcal{X}(\Gamma) = \tilde{\Gamma}$, then the essential differences themselves for $\Gamma$ and $\mathcal{X}(\Gamma)$ are the same.

(b) Let us realise what happens with essential differences if $\tilde{\Gamma}$ contains bad columns. Assume the column $\kappa_s = (i_s - 1 + 1, j_s - 1)$ is bad for $2 \leq s \leq k - 1$. It is easily seen that in this case $i_{s-1} + 1 = j_s$ and $\gamma_{s-1}, \gamma_s$ are adjacent simple roots. If both the surrounding columns for $\kappa_s$ are good and, say, $\gamma_s = \alpha_t = (t, t)$, then the array $\tilde{\Gamma}$ contains a fragment of the form

\[
\begin{pmatrix}
\cdots & x & t+1 & t+2 & \cdots \\
\cdots & t-1 & t & y & \cdots 
\end{pmatrix},
\]

where $x \leq t - 1$ and $y \geq t + 2$. It follows that removing the bad column changes the value of essential differences, but does not change their number.

More generally, $m$ consecutive bad columns occur in $\tilde{\Gamma}$ if and only if $\Gamma$ contains $m+1$ consecutive simple roots. Here the argument is practically the same.

(c) Assume the column $\kappa_1 = (1, j_1 - 1)$ is bad. Then $j_1 = 1$ and $i_1 = 1$, i.e., $\gamma_1 = \alpha_1$. If the next-to-right column is good, then $\tilde{\Gamma}$ contains a fragment of the form

\[
\begin{pmatrix}
0 & 1 & 2 & i_2 + 1 & \cdots \\
0 & j_2 - 1 & j_3 - 1 & \cdots 
\end{pmatrix},
\]

where $j_2 \geq 3$. Having removed the bad column $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we gain the essential difference ‘2’ in the first row instead of the essential difference $(j_2 - 1)$ in the second row. However, the total number of essential differences remains intact. The similar argument applies if there are several consecutive bad columns including $\kappa_1$ or if $\kappa_{k+1} = \begin{pmatrix} i_{k+1} \\ n \end{pmatrix}$ is bad. \hfill \square

In what follows, the function $\mathcal{Y} : \mathfrak{A}_n(\Delta^+) \to \mathbb{N}$ is said to be the $\mathcal{OY}$--invariant. Here are further properties of $\mathcal{Y}$.

**Proposition 3.3.** The minimal (resp. maximal) value of $\mathcal{Y}$ is 0 (resp. $n-1$). Each of them is attained on a unique $\mathcal{X}$-orbit. Namely, $\mathcal{Y}(\Gamma) = 0$ if and only if $\Gamma$ lies in the standard $\mathcal{X}$-orbit; $\mathcal{Y}(\Gamma) = n-1$ if and only if $\Gamma = \{\alpha_1, \alpha_3, \ldots\}$ or $\{\alpha_2, \alpha_4, \ldots\}$.

**Proof.** This is easily verified using Eq. (3.2). \hfill \square

**Remark 3.4.** The definition of $\mathcal{Y}(\Gamma)$ given in Eq. (3.1) can be repeated verbatim for any other root system. However, such a function will not be $\mathcal{X}$-invariant. To save $\mathcal{X}$-invariance, one might attempt to endow summands in Eq. (3.1) with certain coefficients. This works in the symplectic case. Namely, one has to put coefficient ‘2’ in front of $r_{\Gamma}(\gamma_s)$ if $\gamma_s$ is short. The explanation stems from the fact that there is an unfolding procedure that takes $C_n$ to $A_{2n-1}$. This procedure allows us to identify an antichain (upper ideal) in $\Delta^+(C_n)$ with a
“self-conjugate” antichain (upper ideal) in \( \Delta^+ (A_{2n-1}) \), see [5, 5.1] for details. Under this procedure, each short root in \( \Delta^+ (C_n) \) is replaced with two roots in \( \Delta^+ (A_{2n-1}) \). Therefore, the modified sum for an antichain in \( \Delta^+ (C_n) \) actually represents the OY-invariant for the corresponding “self-conjugate” antichain in \( \Delta^+ (A_{2n-1}) \). Since \( \Delta^+ (C_n) \cong \Delta^+ (B_n) \), the modified formula can also be transferred to the \( B_n \)-setting. But the last isomorphism does not respect root lengths. Therefore the definition becomes quite unnatural for \( B_n \).

Also, it is not clear how to construct an \( X \)-invariant in case of \( D_4 \).

3.2. \( X \)-orbits and duality. For \( \Delta \) of type \( A_n \), we introduced in [5, § 4] a certain involutory map (“duality”) \( \ast : \mathfrak{An}(\Delta^+) \rightarrow \mathfrak{An}(\Delta^+) \). It has the following properties:

1. \( \# \Gamma + \# (\Gamma^*) = n \);
2. If \( \Gamma \subset \Pi \), then \( \Gamma^* = \Pi \setminus \Gamma \);
3. \( \Delta(i)^* = \Delta(n + 2 - i) \).

Say that \( \Gamma^* \) is the dual antichain for \( \Gamma \). Our aim is to establish a relationship between \( X \) and \( \ast \). To this end, recall the explicit definition of the duality \( \Gamma \mapsto \Gamma^* \).

Suppose \( \Gamma = \{(i_1, j_1), \ldots, (i_k, j_k)\} \) as above. In this subsection, we represent \( \Gamma \) as the usual two-row array:

\[
\Gamma = \begin{pmatrix}
i_1 & \cdots & i_k \\
\vdots & \ddots & \vdots \\
\end{pmatrix}
\begin{pmatrix}
\vdots & \ddots & \vdots \\
j_1 & \cdots & j_k
\end{pmatrix}
\]

Set \( I = I(\Gamma) = (i_1, \ldots, i_k) \) and \( J = J(\Gamma) = (j_1, \ldots, j_k) \). That is, \( \Gamma = (I, J) \) is determined by two strictly increasing sequences of equal cardinalities lying in \( [n] := \{1, \ldots, n\} \) such that \( I \leq J \) (componentwise). Then \( \Gamma^* = (I^*, J^*) \) is defined by

\[
I^* := [n] \setminus J \text{ and } J^* := [n] \setminus I.
\]

It is not hard to verify that \( \Gamma^* \) is an antichain, see [5, Theorem 4.2] (Our notation for the roots of \( \mathfrak{sl}_{n+1} \) is slightly different from that in [5], therefore the definition of \( \Gamma^* \) has become a bit simpler.)

**Theorem 3.5.** For any \( \Gamma \in \mathfrak{An}(\Delta^+) \), we have \( X(\Gamma)^* = X^{-1}(\Gamma^*) \).

**Proof.** We prove that the \( I \)- and \( J \)-sequences for \( X(\Gamma)^* \) and \( X^{-1}(\Gamma^*) \) coincide.

Below, we use the description of \( X \) given in the proof of Theorem 3.2. We have

\[
i \in I(X(\Gamma)^*) \iff i \not\in J(X(\Gamma)) \iff \begin{cases}i + 1 \notin J(\Gamma) \text{ or } \\
i + 1 \in J(\Gamma) \text{ and } \alpha_i, \alpha_{i+1} \in \Gamma.
\end{cases}
\]

The last possibility means that \( i + 1 \) occurs in the \( J \)-sequence of \( \Gamma \) and hence \( i \) occurs in the \( J \)-sequence of \( \Gamma \); however, it occurs in a bad column and therefore disappears after removing the bad columns.
On the other hand, consider \( I(\mathcal{X}^{-1}(\Gamma^*)) \). To this end, one needs an explicit description of \( \mathcal{X}^{-1} \) in terms of two-row arrays. As the description of \( \mathcal{X} \) includes deletion of some columns, the description of \( \mathcal{X}^{-1}(\Gamma^*) \) should include a creation of columns. More precisely, for

\[
\Gamma^* = \begin{pmatrix}
i_1^* & \cdots & i_{n-k}^* \\
j_1^* & \cdots & j_{n-k}^*
\end{pmatrix},
\]

we perform the following. First, if \( i_1^* \geq 2 \), then we put columns \( \begin{pmatrix} 1 & \cdots & i_1^*-1 \\ 1 & \cdots & i_1^*-1 \end{pmatrix} \) at the beginning. Then each pair of consecutive columns of \( \Gamma^* \) is transformed as follows:

\[
\begin{pmatrix}
i^*_s & i^*_{s+1} \\
j^*_s & j^*_{s+1}
\end{pmatrix} \mapsto \begin{cases}
\begin{pmatrix}
i^*_{s+1} & \cdots & i^*_{s+1} \\
j^*_{s+1} & \cdots & j^*_{s+1}
\end{pmatrix} & \text{if } i^*_{s+1} \leq j^*_{s+1}, \\
\begin{pmatrix}
j^*_{s+1} & \cdots & i^*_{s+1} \\
j^*_{s+1} & \cdots & j^*_{s+1}
\end{pmatrix} & \text{if } i^*_{s+1} > j^*_{s+1}.
\end{cases}
\]

Finally, if \( j_{n-k}^* < n \), then we put columns \( \begin{pmatrix} j_{n-k}^*+1 & \cdots & n \\ j_{n-k}^*+1 & \cdots & n \end{pmatrix} \) at the end. The resulting two-row array represents \( \mathcal{X}^{-1}(\Gamma^*) \). From this description, it follows that

\[
i \in I(\mathcal{X}(\Gamma)^*) \iff \left\{ \begin{array}{l}
i + 1 \in I(\Gamma^*) \text{ or} \\
i + 1 \not\in I(\Gamma^*) \text{ but} \\
i \leq i^*_1 - 1 \text{ or} \\
i \geq j^*_{n-k} + 1 \text{ or} \\
j^*_s + 1 \leq i \leq i^*_{s+1} - 1 \text{ for some } s \in [n-k-1] \end{array} \right\}
\]

\[
\iff \left\{ \begin{array}{l}
i + 1 \not\in J(\Gamma) \text{ or} \\
i + 1 \in J(\Gamma) \text{ and } \alpha_i, \alpha_{i+1} \in \Gamma.
\end{array} \right\}
\]

Thus, we have proved that \( I(\mathcal{X}(\Gamma)^*) = I(\mathcal{X}^{-1}(\Gamma^*)) \). The argument for \( J \)-sequences is similar.

There is also a connection between the duality and OY-invariant:

**Proposition 3.6.** \( \mathcal{Y}(\Gamma) = \mathcal{Y}(\Gamma^*) \).

**Proof.** As above, we think of \( \Gamma \) as union of sequences \( I = (i_1, \ldots, i_k) \) and \( J = (j_1, \ldots, j_k) \). Using Eq. (3.2), we write

\[
\mathcal{Y}(\Gamma) = r_\bullet(I) + r^*(J),
\]
where \( r_\bullet(I) = \sum_{s=1}^{k} \chi(i_s-i_{s-1}) \) and \( r_\bullet(J) = \sum_{s=1}^{k} \chi(j_s+1-j_s) \). Recall that \( i_0 = 0 \) and \( j_{k+1} = n + 1 \). Then the assertion will follow from the definition of \( \Gamma^* \) and the equalities \( r_\bullet(I) = r_\bullet([n] \setminus I) \) and \( r_\bullet(J) = r_\bullet([n] \setminus J) \). Clearly, it suffices to prove one of them.

Let us say that \( C_i = \{c_i, c_i+1, \ldots, c+mi\} \) is a connected component of \( I \cup \{0\} \), if \( C_i \subset I \cup \{0\} \) and \( c_i - 1, c + mi + 1 \not\in I \cup \{0\} \). One similarly defines the connected components of \( J \cup \{n+1\} \). Since the consecutive differences inside a connected component are unessential, we obtain

\[
\begin{align*}
\ r_\bullet(I) &= (\text{the number of connected components of } I \cup \{0\}) - 1, \\
\ r_\bullet(J) &= (\text{the number of connected components of } J \cup \{n+1\}) - 1.
\end{align*}
\]

Now, the equality \( r_\bullet(I) = r_\bullet([n] \setminus I) \) can be proved using a simple verification. One has to consider four cases depending on whether \( 1 \) and \( n \) belong to \( I \). As a sample, we consider one case.

Assume \( 1, n \not\in I \). Then \( \{0\} \) is a connected component of \( I \cup \{0\} \). If \( I \) itself has \( m \) connected components, then the total number of components is \( m + 1 \). Hence \( r_\bullet(I) = m \). On the other hand, the assumption shows that \( [n] \setminus I \) has \( m + 1 \) connected components. Furthermore, \( n \in ([n] \setminus I) \). Therefore \( \{n+1\} \) does not form a connected component. Thus, \( ([n] \setminus I) \cup \{n+1\} \) still has \( m + 1 \) components, and \( r_\bullet([n] \setminus I) = m \).

\[
\square
\]

APPENDIX A. COMPUTATIONS FOR \( F_4 \)

We use the numbering of simple roots from [10]. The positive root \( \beta = \sum_{i=1}^{4} n_i \alpha_i \) is denoted by \( (n_1n_2n_3n_4) \). For instance, \( \theta = (2432) \) and \( \theta_s = (2321) \).

I. \#\( \text{Aut}(\Delta^+) \) = 105 and \( h = 12 \). There are eleven \( \mathfrak{X} \)-orbits: eight orbits of cardinality 12 and orbits of cardinality 2, 3, and 4. We indicate representatives and cardinalities for all orbits:

\[
\begin{align*}
\{1000\} - 12; &\quad \{0100\} - 12; \quad \{0010\} - 12; \quad \{0001\} - 12; \quad \{0011\} - 12; \\
\{1100\} - 12; &\quad \{1111\} - 12; \quad \{2432\} - 12 (\text{the standard orbit}); \quad \{1000, 0010\} - 2; \\
\{0110\} - 3; &\quad \{0001, 1110\} - 4.
\end{align*}
\]

II. \#\( \text{Aut}(\Delta^+ \setminus \Pi) \) = 66 and \( h - 1 = 11 \). The notation \( \Gamma \sim \Gamma' \) means \( \Gamma' = \mathfrak{X}_0(\Gamma) \). The \( \mathfrak{X}_0 \)-orbits are:

1) The standard one: \( \Delta(11) = \{2432\} \sim \{2431\} \sim \cdots \sim \Delta(2) \sim \emptyset \sim \Delta(11) \);
2) \( \{1321\} \sim \{2221\} \sim \{1321, 2211\} \sim \{1221, 2210\} \sim \{0221, 1211\} \sim \{0211, 1111, 2210\} \sim \{0111, 1210\} \sim \{0011, 0210, 1110\} \sim \{0110, 1100\} \sim \{0011\} \sim \{2210\} \sim \{1321\} \);
3) \( \{1221\} \sim \{0221, 2211\} \sim \{1211, 2210\} \sim \{0221, 1111, 1210\} \sim \{0211, 1110\} \sim \{0111, 0210, 1100\} \sim \{0011, 0110\} \sim \{1100\} \sim \{0221\} \sim \{2211\} \sim \{1321, 2210\} \sim \{1221\};
\]

\[
\square
\]
III. \( \#\text{antichains}(\Delta^+_s) = 21 \) and \( h^* = 9 \). The \( \chi_s \)-orbits are:

1) standard: \( \Delta_s(8) = \{2321\} \sim \{1321\} \sim \cdots \sim \Delta_s(1) = \{1000, 0100\} \sim \emptyset \sim \Delta_s(8) \);

2) \{0100\} \sim \{1000\} \sim \{0111\} \sim \{1210\} \sim \{1111\} \sim \{0111, 1210\} \sim \{1110\} \sim \{0111, 1210\} \sim \{0110, 1000\} \sim \{0100\};

3) \{1100\} \sim \{0111, 1000\} \sim \{0110\} \sim \{1100\}.

References

[1] P. Cellini and P. Papi. ad-nilpotent ideals of a Borel subalgebra II, J. Algebra, 258(2002), 112–121.

[2] M. Deza and K. Fukuda. Loops of clutters, in: “Coding theory and design theory”, Part 1, pp. 72–92 (The IMA volumes in Mathematics and its Appl., 20), Springer-Verlag, 1990.

[3] D.G. Fon-Der-Flaass. Orbits of antichains in ranked posets, Europ. J. Combinatorics, 14(1993), 17–22.

[4] P.J. Cameron and D.G. Fon-Der-Flaass. Orbits of antichains revisited, Europ. J. Combinatorics, 16(1995), 545–554.

[5] D. Panyushev. ad-nilpotent ideals of a Borel subalgebra: generators and duality, J. Algebra, 274(2004), 822–846.

[6] D. Panyushev. Short antichains in root systems, semi-Catalan arrangements, and \( B \)-stable subspaces, Europ. J. Combinatorics, 25(2004), 93–112.

[7] D. Panyushev. The poset of positive roots and its relatives, J. Alg. Combinatorics, 23(2006), 79–101.

[8] D. Panyushev. Two covering polynomials of a finite poset, with applications to root systems and ad-nilpotent ideals, Preprint MPIM 2005–9 = math.CO/0502386, 20 pp.

[9] E. Sommers. \( B \)-stable ideals in the nilradical of a Borel subalgebra, Canad. Math. Bull. 48(2005), 460–472.

[10] Э.Б. Виньберг, А.Л. Ошичнік. Семінар по групам Ли і алгебраїчним групам. Москва: “Наука” 1988 (Russian). English translation: A.L. Onishchik and E.B. Vinberg. “Lie groups and algebraic groups”, Berlin: Springer, 1990.

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