Hydromagnetic Stability Analysis of a Partially Ionized Medium

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Abstract

Rayleigh-Taylor instability of a composite medium with variable density and viscosity is considered by taking into account the frictional effect of collisions of ionized with neutral atoms in the presence of a variable horizontal magnetic field. The criteria determining stability and instability are independent of the effects of viscosity and collisional effects. The magnetic field stabilizes the system which is otherwise unstable in the absence of the magnetic field. The viscosity of the medium has stabilizing as well as destabilizing effect on the growth rates. The collisional frequency has stabilizing effect on the growth rates, but has also destabilizing effect in some region.

Keywords: Rayleigh-taylor instability; Variable density; Variable horizontal magnetic field.

1. Introduction

A detailed treatment of Rayleigh-Taylor instability, together with the possible extensions in various domains of interest has been given by Chandrasekhar [1]. The medium has been considered to be fully ionized. Quite often the plasma is not fully ionized and is, instead, partially ionized. Partially ionized plasma represents a state which often exists in the Universe and there are several situations when the interaction between the ionized and neutral gas components becomes important in cosmic physics. The study of partially ionized plasmas has become a hot topic because solar structures such as spicules, prominences, as well as layers of the solar atmosphere (photosphere and chromosphere), are made of partially ionized plasmas. On the other hand, considerable developments have taken place in the study of partially ionized plasmas applied to the physics of the interstellar medium, molecular clouds, the formation of protostellar discs, planetary magnetospheres/ionospheres, exoplanets atmospheres, etc. For instance, molecular clouds are mainly made up of neutral material which does not interact with magnetic fields. However, neutrals are not the only constituent of molecular clouds since there are also several types of charged species which do interact with magnetic fields. Furthermore, the charged fraction also interacts with the neutral material through collisions. These multiple interactions produce many different physical effects which may have a strong influence on star formation and molecular cloud turbulence. A further example can be found in the formation of dense cores in molecular clouds induced by MHD waves. Because of the low ionization fraction, neutrals and charged particles are weakly coupled and ambipolar diffusion plays an important role in the formation process. Even in the primeval universe, during the recombination era, when the plasma, from which all the matter of the universe was formed, evolved from fully ionized to neutral, it went through a phase of partial ionization. Partially ionized plasmas introduce physical effects which are not considered in fully ionized plasmas, for instance, Cowling’s resistivity, isotropic thermal conduction by neutrals, heating due to ion-neutral friction, heat transfer due to collisions, charge exchange, ionization energy, etc., which are crucial to fully understand the behaviour of astrophysical plasmas in different environments. Stromgren [2], has reported that ionized hydrogen is limited to certain rather sharply bounded regions in space surrounding, for example, O-type stars and clusters of such stars and that the gas outside these regions is essentially non-ionized. Other examples of the existence of such situations are given by [3] theory on the origin of the planetary system, in which a high ionization rate is suggested to appear from collisions between a plasma and a neutral-gas cloud and by the absorption of plasma waves due to ion-neutral collisions such as in the solar photosphere and chromosphere and in cool interstellar clouds [4, 5]. Lehner [6], has found that both ion viscosity and neutral gas friction have a stabilizing influence on cosmiclal plasma interacting with a neutral gas. According to Hans [7] and Bhatia [8], the medium may be idealized as a composite mixture of a hydromagnetic (ionized) component and a neutral component, the two interacting through mutual collisional (frictional) effects. A stabilizing effect of collisionals on Rayleigh-Taylor configuration has been shown by Hans [7] and Bhatia [8]. But the collisional effects are found to be destabilizing for a sufficiently large collisional frequency on Kelvin-Helmholtz configuration by Rao and Kalra [9] and Hans [7]. Chhajlani, et al. [10], considered the hydromagnetic Rayleigh-Taylor instability of a composite medium in the presence of suspended particles for an exponentially varying density
distribution. The Rayleigh-Taylor instability of a partially ionized plasma in a porous medium in the presence of magnetic field perpendicular to gravity has been considered by Sharma and Sunil [11]. The gravitational instability of a rotating Walters B’ viscoelastic partially ionized plasma permeated by an oblique magnetic field in the presence of the effects of Hall currents, electrical resistivity and ion viscosity has been considered by El-Sayed and Mohamed [12]. Hoshoudy [13], has investigated the Rayleigh-Taylor instability in stratified plasma in the presence of combined effect of horizontal and vertical magnetic field. Sharma, et al. [14], have investigated the effect of surface tension on hydromagnetic Rayleigh-Taylor instability of two incompressible superimposed fluids in a medium with suspended dust particles in a uniform horizontal magnetic field.

In all the above studies, the magnetic field has been considered to be uniform. Generally the magnetic field has a stabilizing effect on the instability. But a few exceptions are there. For example, Kent [15] studied the effect of a horizontal magnetic filed, which varies in the vertical direction, on the stability of parallel flows and showed that the system is unstable under certain conditions, while, in the absence of magnetic field, the system is known to be stable. Therefore, in this comprehensive review, we study the collisional effects on the Rayleigh-Taylor instability of a composite medium with variable viscosity and density in the presence of a variable horizontal magnetic field.

2. Formulation of the Problem and Perturbation Equations

The model we consider consists of an incompressible composite layer of an infinitely conducting hydromagnetic fluid of density $\rho$, permeated with neutrals of density $\rho_d$, arranged in horizontal strata and acted on by the gravity force $\vec{g}(0,0,-g)$ and the variable horizontal magnetic field $\vec{H}(H_0(z),0,0)$. Assume that both the ionized fluid and the neutral gas behave like continuum fluids and that effects on the neutral component resulting from the presence of magnetic field, pressure and gravity are neglected.

Let $\vec{q}(u,v,w)$, $\vec{h}(h_x, h_y, h_z)$, $\delta \rho$ and $\delta \varphi$ denote, respectively, the perturbations in velocity, magnetic field, density $\rho$ and pressure $p$; $\vec{q}_a, \vec{v}_c, \mu_a$, and $\mu$ denote the velocity of the neutral gas, the mutual collisional (frictional) frequency between the two components of the composite medium, the magnetic permeability and the viscosity of the hydromagnetic fluid, respectively. Then the linearized perturbation equations governing the motion of the composite medium are

$$\rho \frac{\partial \vec{q}}{\partial t} = -\nabla \delta \rho + \frac{\mu_a}{4\pi} \left[ \left( \nabla \times \vec{h} \right) \times \vec{h} + \left( \nabla \cdot \vec{h} \right) \right] + \vec{g} \delta \rho + \nabla \left( \mu \nabla \vec{q} \right)$$  \hspace{1cm} (1)

$$\frac{\partial \vec{h}}{\partial t} = \left( \nabla \cdot \vec{v} \right) \vec{q} - (\vec{q} \cdot \nabla) \vec{h}$$  \hspace{1cm} (2)

$$\frac{\partial \delta \rho}{\partial t} = -w \frac{\partial \rho}{\partial z}$$  \hspace{1cm} (5)

3. Dispersion Relation

Analysing in terms of normal modes, we seek solutions of the above equations whose dependence on space-time co-ordinates is of the form

$$f(z) \exp[i k_x x + ik_y y + nt]$$  \hspace{1cm} (6)

where $f(z)$ is some function of $z$ only, $k_x$ and $k_y$ ($k^2 = k_x^2 + k_y^2$) are horizontal wave numbers and $n$ is the frequency of the harmonic disturbance.

If we eliminate $\vec{q}_a$ in equations (1) and (2) and use (6), equations (1)-(5) give

$$n^2 \rho u = -ik_x \delta \rho + \mu (D^2 - k_x^2) u + D \mu (Du + ik_x w) + \frac{\mu_a}{4\pi} h_x (D H_0),$$  \hspace{1cm} (7)

$$n^2 \rho v = -ik_y \delta \rho + \mu (D^2 - k_y^2) v + D \mu (Dv + ik_y w) + \frac{\mu_a}{4\pi} (i k_x h_y - ik_y h_x),$$  \hspace{1cm} (8)

$$n^2 \rho w = -D \delta \rho + \mu (D^2 - k_x^2) w + 2(D \mu)(Dw) + \frac{\mu_a}{4\pi} \left( i k_x h_x - D h_x - h_x \frac{D H_0}{H_0} \right)$$  \hspace{1cm} (9)

Equation (8) with the help of equations (13) and (14), can be rewritten as

$$n^2 \rho v = -ik_y \delta \rho + \frac{\mu_a H_0}{4\pi} \left( i k_x H_0 \zeta + ik_y w D H_0 \right) + \mu (D^2 - k_y^2) v$$  \hspace{1cm} (16)
where \( \zeta = i k_x v - i k_y u \) is the vorticity in the \( z \)-direction,
\[
n' = n \left( 1 + \frac{\alpha_v v_c}{n + v_c} \right) \quad \text{and} \quad \alpha_0 = \frac{\rho_a}{\rho}.
\]

Multiplying equation (7) by \(- i k_x \) and (16) by \(- i k_y \) and adding, we get
\[
n' \rho Dw = - k^2 \delta p + \mu (D^2 - k^2) Dw + \frac{k_x k_y \mu \rho_0 H_0^2}{4\pi n} \zeta + \frac{\mu_e H_0 k_z^2}{4\pi n} w (DH_0) + (D\mu)(D^2 + k^2) w - \frac{i \mu_e k_x}{4\pi} h_x (DH_0).
\]

Eliminating \( \delta p \) in equations (9) and (17) and using equations (10)-(15), we get
\[
\left( D^2 - k^2 \right) w - \frac{k_x k_y \mu \rho_0 H_0^2}{4\pi n} \zeta + \frac{\mu_e H_0 k_z^2}{4\pi n} w (DH_0) = \frac{g}{n} (Dp) w = 0.
\]

4. Discussion

(A) Inviscid Case \(( \mu = 0) \)

In many situations of astrophysical interest the viscosity of the medium is negligible. Hence the importance and we study the inviscid case. In this case equation (18), under simplification, reduces to
\[
n' D(\rho Dw) - k^2 n' \rho w + \frac{\alpha_v v_c}{n} (Dp) w = - \frac{\mu_e k_z^2}{4\pi n} \left( H_0^2 (D^2 - k^2) w + D(H_0^2) Dw \right).
\]

Let us assume
\[
\rho = \rho_a e^{\beta z}, \quad \rho_d = \rho_{ad} e^{\beta z}, \quad H_0^2 = H_0^2 e^{\beta z},
\]
where \( \rho, \rho_d, H_0, \) and \( \beta \) are constant. Equations (20) imply that the local Alfven velocity is constant everywhere. By substituting (20), equation (19) gives
\[
D^2 w + \beta Dw = \frac{k_x k_y \mu \rho_0 H_0^2}{4\pi n} \left( n' + \frac{k_x k_y \mu \rho_0 H_0^2}{4\pi n} \right)^2 \left( \frac{4\pi n^2}{k_x k_y \mu \rho_0 H_0^2} \right) w = 0,
\]
where \( \beta W = \mu_e H_0^2 / 4\pi \rho_0 \) is the square of the Alfven velocity. The solution of (21) satisfying \( w = 0 \) at \( z = 0 \) is
\[
w = A (\exp[m_1 z] - \exp[m_2 z]),
\]
where
\[
m_1,2 = -\frac{\beta}{2} \pm 1 \sqrt{\frac{\beta^2 + 4k^2 \left( \frac{4\pi n^2}{k_x k_y \mu \rho_0 H_0^2} - \frac{\left( \frac{4\pi n^2}{k_x k_y \mu \rho_0 H_0^2} \right)^2}{\beta^2 + 4k^2 + 4\pi^2 \beta^2 / d^2} \right)} {2}.
\]

Since \( w = 0 \) at \( z = d \) also, equation (22) gives
\[
\exp[(m_1 - m_2)d] = 0,
\]
i.e.
\[
(m_1 - m_2)d = 2i s \pi ,
\]
s being any integer. Squaring this and using (23), we obtain
\[
n^2 \left( 1 + \frac{\alpha_v v_c}{n + v_c} \right) = \beta^2 + 4k^2 + 4\pi^2 \beta^2 / d^2 - k_x k_y \beta^2 ,
\]
which simplifies to give
\[
n^2 + n^2 \left[ v_c (1 + \alpha_0) \right] + n \left[ k_x k_y \beta^2 - \frac{4k^2 \beta^2}{\beta^2 + 4k^2 + 4\pi^2 \beta^2 / d^2} \right] +
\]
\[
+ v_c \left[ k_x k_y \beta^2 - \frac{4k^2 \beta^2}{\beta^2 + 4k^2 + 4\pi^2 \beta^2 / d^2} \right] = 0.
\]

For stable density stratification \( \beta < 0 \), equation (26) does not have any change of sign and so no positive root occurs. The system is, therefore, stable for stable stratification. For unstable density stratification \( \beta > 0 \), the system is unstable or unstable according as
\[
k_x k_y \beta^2 \geq \frac{4k^2 \beta^2}{\beta^2 + 4k^2 + 4\pi^2 \beta^2 / d^2} .
\]

The system is clearly unstable for \( \beta > 0 \) in the absence of a magnetic field. However, the system can be completely stabilized by a large enough magnetic field as can be seen from equation (26) if
\[
V_A^2 > \frac{4k^2 \beta^2}{\beta^2 + 4k^2 + 4\pi^2 \beta^2 / d^2} .
\]

Thus, if
\[
\beta > 0 \quad \text{and} \quad k_x k_y \beta^2 < \frac{4k^2 \beta^2}{\beta^2 + 4k^2 + 4\pi^2 \beta^2 / d^2} ,
\]
Equation (26) has one positive root. Let \( n_0 \) denote the positive root of equation (26), then
\[
n_0^2 + n_0^2 \left( v_c (1 + \alpha_0) \right) + P n_0 + v_c P = 0,
\]
where
To find the role of collisions on the growth rate of unstable modes, we examine the nature of \( \frac{dn_0}{dv_c} \).

Equation (28) gives
\[
\frac{dn_0}{dv_c} = \frac{(1 + \alpha_0)n_0^2 + P}{3n_0^2 + 2n_0[v_c(1 + \alpha_0)] + P}.
\]

Therefore, if, in addition to \( \left| P \right| > 3n_0^2 + 2v_c(1 + \alpha_0)n_0 \),
\[
\frac{dn_0}{dv_c} = \frac{4k^2g\beta}{\beta^2 + 4k^2 + 4\pi^2s^2/d^2}.
\]

which is a sufficient condition for instability, we have either of the conditions
\[
\left| P \right| > 3n_0^2 + 2v_c(1 + \alpha_0)n_0,
\]
or
\[
\frac{dn_0}{dv_c} \text{ is always negative, as } \alpha_0(=\rho_d/\rho) \text{ can almost be equal to 1. The growth rates, therefore, decrease}
\]

with the increase in collisional frequency.

However, the growth rates increase with the increase in collisional frequency if
\[
(1 + \alpha_0)n_0^2 < |P| < 3n_0^2 + 2v_c(1 + \alpha_0)n_0,
\]

for then \( \frac{dn_0}{dv_c} \) is positive.

**B) Viscous Case**

Equation (18) after simplification can be rewritten as
\[
n'\rho(D\rho) - k^2n'p\omega - \frac{gk^2}{n}(D\rho)w = -\frac{\mu k^2}{4\pi n}(H_2^2(D^2 - k^2)w + D(H_2^2 Dw) +
\]

\[
+\mu(D^2 - k^2)w + 2(D\mu)(D^2 - k^2) Dw + (D^2\mu)(D^2 + k^2)w.
\]

Let us assume
\[
\rho = \rho_0 e^{\text{exp}[\beta z]}, \rho_d = \rho_0 e^{\text{exp}[\beta z]}, \mu = \mu_0 e^{\text{exp}[\beta z]}, H_2^2(z) = H_2^2 e^{\text{exp}[\beta z]},
\]

which imply that the coefficient of kinematic viscosity \( \nu \) and the Alfvén velocity are constant everywhere.

Substituting (33) in (32) and neglecting the effect of heterogeneity on inertia, we obtain
\[
(D^2 - k^2)w = \frac{\nu_v}{n_0} (D^2 - k^2)w - \frac{g\beta k^2}{\nu_0} w = 0,
\]

where \( \nu_v = \mu_0/\rho_0 \). Considering the case of two free boundaries, we must have
\[
w = D^2w = 0 \text{ at } z = 0 \text{ and } z = d.
\]

The appropriate solution of (34) satisfying (35) is
\[
w = A \sin \frac{m\pi z}{d}.
\]

Substituting (36) in equation (34), we obtain the dispersion relation
\[
\left[ \frac{mn^2}{d^2} + k^2 \right] + \left[ \frac{\nu_v^2}{n_0} + \frac{n^4}{\nu_0} \right] \left( \frac{mn^2}{d^2} + k^2 \right) - \frac{g\beta k^2}{\nu_0} = 0.
\]

By setting \( \frac{mn^2}{d^2} + k^2 = L \), the above equation can be written as
\[
n^2 + n^2[v_c(1 + \alpha_0) + L\nu_0] + n \left( \frac{k^2\nu_v^2 - \frac{g\beta k^2}{L}}{L} + L\nu_v n_0 \right) + v_c \left( k^2\nu_A^2 - \frac{g\beta k^2}{L} \right) = 0.
\]

For the stable density stratification \( \beta < 0 \), equation (38) does not have any positive root, implying thereby that the system is stable. For unstable density stratification \( \beta > 0 \), the system is stable or unstable according as
\[
k^2\nu_A^2 \geq \frac{g\beta k^2}{L}.
\]

The system is clearly unstable for \( \beta > 0 \) in the absence of a magnetic field. However, the system can be completely stabilized by a large enough magnetic field as can be seen from equation (38), if
\[
V_A^2 > \frac{g\beta k^2}{L}.
\]

Thus, if \( \beta > 0 \) and \( k^2\nu_A^2 < \frac{g\beta k^2}{L} \), equation (38) has one positive root. Let \( n_0 \) denote the positive root of equation (38). Then
\[
n_0^3 + n_0^2[v_c(1 + \alpha_0) + L\nu_0] + n_0 \left( \frac{k^2\nu_v^2 - \frac{g\beta k^2}{L}}{L} + L\nu_v n_0 \right) + v_c \left( k^2\nu_A^2 - \frac{g\beta k^2}{L} \right) = 0.
\]

To find the role of viscosity and collisions on the growth rate of unstable modes, we examine the nature of \( \frac{dn_0}{dv_c} \) and \( \frac{dn_0}{dv_c} \). Equation (40) gives
\[
\frac{dn_0}{dv_c} = \frac{n_0^2(\alpha_0 + 1) + n_0 L\nu_0 + (k^2\nu_v^2 - g\beta k^2/L)}{3n_0^2 + 2n_0[v_c(1 + \alpha_0) + L\nu_0] + (k^2\nu_v^2 - g\beta k^2/L + L\nu_v n_0)}.
\]

Therefore, if, in addition to \( k^2 > \frac{k^2\nu_A^2 L}{g\beta} \), which is a sufficient condition for instability, we have either of the conditions
\[
\frac{k^2\nu_v^2 - \frac{g\beta k^2}{L}}{L} > 3n_0^2 + 2n_0[v_c(1 + \alpha_0) + L\nu_0] + L\nu_v n_0,
\]
or
\[
\left| k^2 \nu_A^2 - \frac{g \beta k^2}{L} \right| < (1 + \alpha_0) n_0^2 + L \nu_0 n_0 , \tag{42b}
\]
\[
dn_0/d\nu_c \text{ is always negative. The growth rates, therefore, decrease with the increase in collisional frequency. However, the growth rates increase with the increase in collisional frequency if}
\]
\[
(1 + \alpha_0) n_0^2 + \nu_0 Ln_0 < \left( k^2 \nu_A^2 - \frac{g \beta k^2}{L} \right) < 3n_0^2 + 2n_0 [\nu_c (1 + \alpha_0) + \nu_0 L] + L \nu_0 \nu_c , \tag{43}
\]
For then \( dn_0/d\nu_c \) is positive.
Equation (40) yields
\[
\frac{dn_0}{d\nu_0} = - \frac{Ln_0 (n_0 + \nu_c)}{3n_0^2 + 2[\nu_c (1 + \alpha_0) + \nu_0]n_0 + [k^2 \nu_A^2 - g \beta k^2/L + L \nu_c \nu_0]}. \tag{44}
\]
It is evident from equation (44) that \( dn_0/d\nu_0 \) is positive or negative depending on whether the denominator in equation (44) is positive or negative. The growth rates, therefore, decrease as well as increase with the increase in kinematic viscosity of the fluid.

We thus conclude the whole analysis with the following statements. The criteria determining stability or instability are independent of the effects of viscosity and collisional effects. The magnetic field stabilizes the system which is otherwise unstable unstable in the absence of the magnetic field. The viscosity of the medium has stabilizing as well as destabilizing effect on the growth rates. The collisional frequency has stabilizing effect on the growth rates, but has destabilizing effect also in the region (31) in the non-viscous case and (43) in the viscous case.

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