Regularity and multiplicity of toric rings of three-dimensional Ferrers diagrams

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Abstract
We investigate the Castelnuovo–Mumford regularity and the multiplicity of the toric ring associated with a three-dimensional Ferrers diagram. In particular, in the rectangular case, we provide direct formulas for these two important invariants. Then, we compare these invariants for an accompanying pair of Ferrers diagrams under some mild conditions and bound the Castelnuovo–Mumford regularity for more general cases.

Keyword Special fiber · Toric rings · Blowup algebras · Ferrers graph · Castelnuovo–Mumford regularity · Multiplicity

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1 Introduction

In this work, we will consider the Castelnuovo–Mumford regularity and the multiplicity of the toric varieties arising from the squarefree monomial ideals associated with the three-dimensional Ferrers diagrams.

The study of toric varieties is an important part of algebraic geometry. From our point of view, it suffices to notice that the image of the rational map defined by an equi-generated monomial ideal is a toric variety. Meanwhile, the special fiber of the blowup algebra defined by an equi-generated monomial ideal is the toric variety associated...
with the monomial ideal in this case. This type of approach has applications in statistics [9], geometric modeling [25], and coding theory [18]. In addition, the recent work of Cox, Lin, and Sosa [8] provides a connection of chemical reaction networks with the toric variety arising from the monomial ideal.

In this paper, we will focus on the two most important invariants of toric varieties: the Castelnuovo–Mumford regularity and the multiplicity (also known as the degree).

Throughout this paper, $\mathbb{K}$ is a field of characteristic zero. Recall that, for a finitely generated graded nonzero module $M$ over the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$, the Castelnuovo–Mumford regularity of $M$, denoted by $\text{reg}(M)$, is $\max\{j - i : \beta_{i,j} \neq 0\}$, where the $\beta_{i,j}$'s are the graded Betti numbers of $M$. The regularity can be used to bound the degree of syzygy generators of the module $M$.

Another important invariant that we investigate here is the multiplicity $e(M)$ of $M = \bigoplus_k M_k$ with respect to the graded maximal ideal. Recall that $H(M, k) := \dim_{\mathbb{K}}(M_k)$ is eventually a polynomial in $k$ of degree $\dim(M) - 1$. The leading coefficient of this polynomial is of the form $e(M)/(\dim(M) - 1)!$ for the positive integer $e(M)$. When $M$ is the homogeneous coordinate ring of a projective variety $X$, the multiplicity is just the degree of $X$.

These two invariants measure the complexity of the toric variety from different perspectives. Finding a reasonable estimate of these two invariants is still wildly open [27]. Researchers in commutative algebra, algebraic geometry, geometry modeling, coding theory, and statistics are still working restlessly on investigating these invariants.

Recently, Biermann et al. [1] found bounds on Castelnuovo–Mumford regularity of toric rings associated with edge ideals arising from complete bipartite graphs. Later, Beyarslan, Hà, and O’Keefe bound the regularity of toric rings associated with simple graphs using their induced subgraphs in [16]. Consequently, Galetto and his co-authors [12] computed the graded Betti numbers for the toric ideal of graphs constructed by adjoining cycles to complete bipartite graphs. The regularity of linearly presented toric edge ideals related to the bipartite complement of a graph is investigated by Greif and McCullough in [15]. Furthermore, graphs such that their associated toric rings have regularity $r$ and $h$-polynomials have degree $d$ are classified in [11] by Favacchio and his co-authors. From this list of recent work, we can see clearly that there are still many unknown cases deserving investigation, even with toric rings associated with simple graphs.

Meanwhile, Eto gave a method for computing the multiplicity of monoid rings in [10]. Gitler and Valencia [13] related the multiplicity of the edge subring of a simple graph with the volume of its edge polytope. At the same time, Villarreal [29, chapter 10] presented sharp upper bounds for the multiplicity of edge subrings.

In general, not much is known when the toric ring is associated with monomial ideals of degree greater than 2. One important reason is that finding the explicit defining equations of the toric rings is quite difficult in general. Even when the defining equations are known in some cases, the computation of regularity or multiplicity is still very involved; see, for example, the regularities of toric rings (fiber cones) associated with initial ideals of secant varieties of rational normal scrolls that we determined in [20].
In this work, we will continue our previous work in [21]. The toric variety that we consider here is associated with the squarefree monomial ideal $I_D$ where $D$ is a three-dimensional Ferrers diagram. The classical two-dimensional Ferrers diagram is an important object studied in combinatorics, and it has applications in permutation statistics [3] and inverse rook problems [14]. They are also known as Young diagrams and have important applications to the representation theory of symmetric and general linear groups, and to Schubert calculus; see, for example, [24].

The regularity and multiplicity problem for the classical Ferrers diagram has been solved by Corso and Nagel in [5]. As a natural generalization, in our previous work [21], we have shown that, under some mild conditions, the special fiber ideal $J_D$ associated with a three-dimensional Ferrers diagram $D$ has a squarefree quadratic initial ideal with respect to the lexicographic order. In addition, the accompanying Stanley–Reisner complex $\Delta_1(D)$ is pure vertex-decomposable. The main purpose of the current paper is then to investigate the regularity and multiplicity of the special fiber ring $F(I_D)$. Related key ingredients from our previous paper will be summarized here in Sect. 2.

By some algebraic argument, we can transfer the calculation of these two vital invariants of the special fiber ring $F(I_D)$ to those of the associated Stanley–Reisner ring $\mathbb{K}[\Delta(D)]$; see Corollary 3.4. Notice that the vertex-decomposability of the associated Stanley–Reisner complex $\Delta(D)$ provides naturally a short exact sequence, on which both invariants behave nicely; see Remark 3.6. Therefore, one can quickly extract an algorithm for calculating the reduction number of $I_D$ as well as the regularity and multiplicity of the special fiber ring $F(I_D)$ associated with the three-dimensional Ferrers diagram $D$.

When the given three-dimensional Ferrers diagram $D = [a] \times [b] \times [c]$ is in a full rectangular shape, we provide a clean formula for the reduction number, the regularity, and the multiplicity of the special fiber ring in Sect. 4. On the other hand, we do not intend to give a closed formula for them in the general case. Even for the multiplicity in the two-dimensional case, the formula given in [5, Corollary 5.6] is already highly entangled. Therefore, the main object of this paper is to give combinatorically reasonable upper bounds for these two invariants.

We exemplify two approaches. In Sect. 5, we consider an accompanying pair of Ferrers diagrams $D_1 \subseteq D_2$. To compare these invariants on these two diagrams, we fall back on the aforementioned shedding order of the associated vertex-decomposable Stanley–Reisner complexes $\Delta(D_1)$ and $\Delta(D_2)$. Since this approach requires a synchronizing comparison along the decomposing process, we need a slightly stronger condition, given in Definition 5.4. In the second approach, an identical estimate can be achieved for the reduction number and the regularity under the same (weaker) condition as assumed in [21]. No doubt, the proof, given in Sect. 6, invites a more intricate combinatorial maneuver.

2 Preliminaries

Throughout this paper, when $n$ is a non-negative integer, we will follow the common convention and denote the set $\{1, 2, \ldots, n\}$ simply as $[n]$. 
Let \( D \) be a nonempty set of finite lattice points in \( \mathbb{Z}^3_+ \). Let
\[
m := \max\{i : (i, j, k) \in D\}, \quad n := \max\{j : (i, j, k) \in D\} \quad \text{and} \quad p := \max\{k : (i, j, k) \in D\}.
\]

We associate with \( D \) the polynomial ring
\[
R = \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p]
\]
and the monomial ideal
\[
I_D := (x_i y_j z_k : (i, j, k) \in D) \subset R.
\]

This ideal will be called the \textit{defining ideal} of \( D \).

If we write \( m \) for the graded maximal ideal of \( R \), the \textit{special fiber ring} of \( I_D \) is
\[
\mathcal{F}(I_D) := \bigoplus_{l \geq 0} I_D^l / m I_D^l \cong R[I_D] \otimes_R R / m.
\]

Sometimes, we also call it the \textit{toric ring} of \( I_D \) and denote it by \( \mathbb{K}[I_D] \). Let
\[
\mathbb{K}[T_D] := \mathbb{K}[T_{i, j, k} : (i, j, k) \in D]
\]
be the polynomial ring in the variables \( T_{i, j, k} \) over the field \( \mathbb{K} \). Consider the map
\[
\varphi : \mathbb{K}[T_D] \to R, \quad \text{given by} \quad T_{i, j, k} \mapsto x_i y_j z_k,
\]
and extend algebraically. Then \( \mathcal{F}(I_D) \) is canonically isomorphic to \( \mathbb{K}[T_D] / \ker(\varphi) \). We will denote the kernel ideal \( \ker(\varphi) \) by \( J_D \) and call it the \textit{special fiber ideal} of \( I_D \). Sometimes, we also call it the \textit{toric ideal} of \( I_D \). It is well-known that \( J_D \) is a graded binomial ideal; see, for instance, [26, Corollary 4.3] or [28]. We also observe that \( \mathcal{F}(I_D) \), being isomorphic to a subring of \( R \), is a domain. Hence, \( J_D \) is a prime ideal.

\textbf{Definition 2.1} Let \( D \neq \emptyset \) be a finite set of lattice points in \( \mathbb{Z}^3_+ \).

(a) We will call
\[
\begin{align*}
a_D &:= |\{i : (i, j, k) \in D\}|, \\
b_D &:= |\{j : (i, j, k) \in D\}| \quad \text{and} \\
c_D &:= |\{k : (i, j, k) \in D\}|
\end{align*}
\]
the \textit{essential length, width, and height} of \( D \), respectively.

(b) The set \( D \) is called a \textit{three-dimensional Ferrers diagram} if for each \((i_0, j_0, k_0) \in D\), and for every positive integers \( i \leq i_0, j \leq j_0 \) and \( k \leq k_0 \), one has \((i, j, k) \in D\).

\textbf{Definition 2.2} Let \( D \) be a three-dimensional Ferrers diagram. For each \( i \in [a_D] \), let \( b_i = \max\{j : (i, j, 1) \in D\} \) and \( c_i = \max\{k : (i, 1, k) \in D\} \). Then the \textit{projection} of the \( x = i + 1 \) layer is the set
\[
\{(i, j, k) \in \mathbb{Z}_+^3 : j \leq b_{i+1} \text{ and } k \leq c_{i+1}\}
\]
when \( i < a_\mathcal{D} \). And \( \mathcal{D} \) is said to satisfy the projection property if the \( \mathcal{x} = i \) layer covers the projection of the \( \mathcal{x} = i + 1 \) layer for each \( i \in [a_\mathcal{D} - 1] \), i.e., the following equivalent conditions hold:
(a) \((i, b_{i+1}, c_{i+1}) \in \mathcal{D}\);
(b) if \((i + 1, j_1, k_1) \in \mathcal{D} \) and \((i + 1, j_2, k_2) \in \mathcal{D} \), then \((i, j_1, k_1) \in \mathcal{D} \).

**Definition 2.3** Let \( \mathcal{D} \neq \emptyset \) be a finite set of lattice points in \( \mathbb{Z}_+^3 \). For \( \mathbf{u} = (i_1, j_1, k_1) \) and \( \mathbf{v} = (i_2, j_2, k_2) \) in \( \mathcal{D} \), define

\[
I_{2,x}(\mathbf{u}, \mathbf{v}) := \begin{cases} 
T_uT_v - T_{i_2,j_2,k_2}T_{i_1,j_2,k_2} & \text{if } (i_1, j_1, k_1), (i_2, j_2, k_2) \in \mathcal{D}, \\
0 & \text{otherwise}.
\end{cases}
\]

When this is a nonzero binomial, we will say switching the \( x \)-coordinates is allowed between \( \mathbf{u} \) and \( \mathbf{v} \). We can similarly define \( I_{2,y}(\mathbf{u}, \mathbf{v}) \) and \( I_{2,z}(\mathbf{u}, \mathbf{v}) \). Now, let

\[
I_2(\mathcal{D}) := \left( I_{2,x}(\mathbf{u}, \mathbf{v}), I_{2,y}(\mathbf{u}, \mathbf{v}), I_{2,z}(\mathbf{u}, \mathbf{v}) : \mathbf{u}, \mathbf{v} \in \mathcal{D} \right) \subseteq \mathbb{K}[T_\mathcal{D}],
\]

and call it the 2-minors ideal of \( \mathcal{D} \).

**Example 2.4** Let \( \mathcal{D} \) be a three-dimensional Ferrers diagram consisting the following lattice points

\[
(1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 2, 1), (1, 2, 2), (1, 2, 3), (1, 3, 1), (1, 3, 2), (1, 3, 3),
(2, 1, 1), (2, 1, 2), (2, 1, 3), (2, 2, 1), (2, 2, 2), (2, 2, 3), (3, 1, 1), (3, 1, 2), (3, 2, 1).
\]

This diagram satisfies the projection property by easy verification. We can use the base ring

\[
S = \mathbb{K}[T_{1,1,1}, T_{1,1,2}, T_{1,1,3}, T_{1,2,1}, T_{1,2,2}, T_{1,2,3}, T_{1,3,1}, T_{1,3,2}, T_{1,3,3},
T_{2,1,1}, T_{2,1,2}, T_{2,1,3}, T_{2,2,1}, T_{2,2,2}, T_{2,2,3}, T_{3,1,1}, T_{3,1,2}, T_{3,2,1}],
\]

and apply the canonical epimorphism. The toric ideal \( J_\mathcal{D} \) in \( S \) contains, among others, the binomials

\[
T_{2,1,2}T_{3,3,2} - T_{2,2,1}, \quad \text{(an } x \text{-coordinate switch)},
T_{1,2,1}T_{3,3,2} - T_{1,3,1}T_{2,2,1}, \quad \text{(a } y \text{-coordinate switch)},
T_{1,1,1}T_{2,2,2} - T_{1,1,2}T_{2,1,1}, \quad \text{(a } z \text{-coordinate switch}).
\]

If \( \mathcal{D} \) is a three-dimensional Ferrers diagram that satisfies the projection property, it is proved in [21, Corollary 2.14 and Theorem 6.1] that the toric ideal \( J_\mathcal{D} \) coincides with the 2-minors ideal \( I_2(\mathcal{D}) \subseteq \mathbb{K}[T_\mathcal{D}] \). Furthermore, if we order the \( T \)-variables lexicographically with respect to their subscripts and apply the lexicographic monomial order to \( \mathbb{K}[T_\mathcal{D}] \) (we will call it the lexicographic order for short), the 2-minors in \( I_2(\mathcal{D}) \) provide a minimal Gröbner basis of \( J_\mathcal{D} \). On the other hand, for arbitrary integer \( p \geq 4 \), we can consider the three-dimensional Ferrers diagram \( \mathcal{D} \) minimally generated (governed) by the following extremal lattices points:

\[
(1, 2, p - 1), (2, 3, p - 2), (3, 4, p - 3), \ldots, (p - 1, p, 1), (p, 1, 2),
(2, 1, p - 1), (3, 2, p - 2), (4, 3, p - 3), \ldots, (p, p - 1, 1), (1, p, 2).
\]
Then, we can actually prove that the minimal generating set of the special fiber ideal \( J_D \)
contains a degree \( p \) binomial, generalizing the phenomenon observed in [21, Example 2.4] that
the binomials generating the ideal need not form a Gröbner basis and that the ring need not be Koszul.

As the main result of [21], we proved that the special fiber ring \( \mathcal{F}(I_D) \) is a Koszul
Cohen–Macaulay normal domain, provided that \( D \) is a three-dimensional Ferrers diagram
that satisfies the projection property. Its proof is quite involved and depends on
the following two key ingredients.

(a) For both the Cohen–Macaulayness and the primeness of the ideal \( I_2(D) \), we had
to use a common elaborate induction process, which will be explained afterward.
(b) To establish the Cohen–Macaulayness of \( I_2(D) \), we actually proved that the
Stanley–Reisner complex \( \Delta(D) \) associated with the squarefree initial ideal
in \( (I_2(D)) \) is pure vertex-decomposable. Furthermore, the above induction pro-
cess induces a shedding-decomposable for that purpose.

To introduce the aforementioned induction process, we still need some preparation.

**Definition 2.5** For each \( u = (i_0, j_0, k_0) \in D \), let
\[
\alpha_D(u) := \max\{i : (i, j_0, k_0) \in D\}.
\]
In a similar vein, we can define \( \beta_D(u) \) and \( \gamma_D(u) \). Meanwhile, we use the superscript
to denote the corresponding \( x \) layers. For instance,
\[
D^1 := \{(1, j, k) \in D\} \quad \text{and} \quad D^{d^1} := \{(i, j, k) \in D : i \geq d\}.
\]
It is clear that when \( u = (1, 1, 1) \) and \( D \) is a three-dimensional Ferrers diagram, then
\( \alpha_D(u), \beta_D(u), \) and \( \gamma_D(u) \) are \( a_D, b_D, \) and \( c_D, \) respectively; see also Definition 2.1.

**Definition 2.6** Let \( D \) be a three-dimensional Ferrers diagram. Take arbitrary \( u = (i_0, j_0, k_0) \in D \) and for simplicity, write
\[
\alpha = \alpha_D(u), \quad \beta = \beta_D(u) \quad \text{and} \quad \gamma = \gamma_D(u).
\]
Then we can divide \( D^{i_0} \) into the following six zones:
\[
\begin{align*}
Z_1(D, u) & := \{(i, j, k) \in D^{i_0} : 1 \leq j \leq j_0 \text{ and } k > \gamma\}, \\
Z_2(D, u) & := \{(i, j, k) \in D^{i_0} : 1 \leq j \leq j_0 \text{ and } k_0 < k \leq \gamma\}, \\
Z_3(D, u) & := \{(i, j, k) \in D^{i_0} : 1 \leq j \leq j_0 \text{ and } 1 \leq k \leq k_0\}, \\
Z_4(D, u) & := \{(i, j, k) \in D^{i_0} : j_0 < j \leq \beta \text{ and } k_0 < k \leq \gamma\}, \\
Z_5(D, u) & := \{(i, j, k) \in D^{i_0} : j_0 < j \leq \beta \text{ and } 1 \leq k \leq k_0\}, \\
Z_6(D, u) & := \{(i, j, k) \in D^{i_0} : j > \beta \text{ and } 1 \leq k < k_0\}.
\end{align*}
\]
It is clear that \( D^{i_0} \) is the disjoint union of the above six zones. In the subsequent
discussion, they will be called the \( Z \)-zones with respect to \( D \) and \( u \). We will omit
some of the parameters if they are clear from the context. Figure 1 gives the idea of
the division of \( D^{i_0} \) with respect to these zones.
Definition 2.7 Let $\mathcal{D}$ be a nonempty subset of $\mathbb{Z}_3^+$. We say that a total order $\prec$ on $\mathcal{D}$ is a quasi-lexicographic order if it satisfies the following two conditions.

(a) The points in $\mathcal{D}^i$ precede the points in $\mathcal{D}^{i+1}$ with respect to $\prec$ for each $i \in [a_D - 1]$.
(b) For distinct $u = (i, j_1, k_1)$ and $v = (i, j_2, k_2)$ in $\mathcal{D}^i$, if $j_1 \leq j_2$ and $k_1 \leq k_2$, then $u$ precedes $v$ with respect to $\prec$.

Obviously, the common lexicographic order is a quasi-lexicographic order.

Definition 2.8 Let $\prec$ be a quasi-lexicographic order on a nonempty subset $\mathcal{D}$ of $\mathbb{Z}_3^+$. Given a lattice point $u \in \mathcal{D}$, let $A_u$ be the subset obtained from $\mathcal{D}$ by removing the points before $u$ with respect to $\prec$. We also write $A_u^+ := A_u \setminus u$.

Setting 2.9 (Induction Order) Suppose that $\mathcal{D}$ is a three-dimensional Ferrers diagram with $a_D \geq 2$. We adopted in [21] the following total order $\prec_D$ on $\mathcal{D}^1$ for considering both the vertex-decomposability of the associated Stanley–Reisner complex $\Delta(\mathcal{D})$ and the primeness of $I_2(\mathcal{D})$. By abuse of terminology, we call this total order the induction order with respect to $\mathcal{D}$.

Let $C_1 = C_1(\mathcal{D}) := \{(1, j, k) \in \mathcal{D} : k \leq c_{D^{\geq 2}}\}$ be the points in the first stage, and $C_2 = C_2(\mathcal{D}) := \mathcal{D}^1 \setminus C_1$ be the points in the second stage. For $\prec_D$, we require the following.

(a) The points in the first stage precede the points in the second stage with respect to $\prec_D$.
(b) The restriction of $\prec_D$ to $C_1$ is the lexicographic order.
(c) Consider the symmetry operation $S : \mathbb{Z}_3^+ \to \mathbb{Z}_3^+$ by sending $(i, j, k)$ to $(i, k, j)$.

We will also call this operation as a flip. Then the restriction of $\prec_D$ to $C_2$ corresponds to the lexicographic order on $S(C_2)$.

The total order $\prec_D$ on $\mathcal{D}^1$ is completely determined by the three requirements just stated. We may also extend this order to the whole $\mathcal{D}$ by arranging the points in $\mathcal{D}^{\geq 2}$ in a similar fashion and putting them after the points in $\mathcal{D}^1$. The induction order $\prec_D$ constructed by this approach is a quasi-lexicographic order.
Figure 2 gives an idea of how this proceeds on the $x = 1$ layer. The first point is $\circ = (1, 1, 1)$. When $c_{D^2} < c_D$, the last point is $\bullet = (1, \beta_D((1, 1, \gamma_D)), c_D)$ of $D$, using the notation we introduced in Definition 2.5. Otherwise, $c_{D^2} = c_D$ with the second stage disappears and the last point is $(1, b_D, \gamma_D((1, b_D, 1)))$ of $D$. Meanwhile, in Fig. 2, $\triangle$ denotes the last point in the first stage while $\square$ denotes the first point in the second stage.

Definition 2.10 Let $D$ be a nonempty subset of $\mathbb{Z}_+^3$.

(a) Suppose that $D^{i_0} = \emptyset$ for some $i_0 \in \mathbb{Z}_+$. Then, we can remove the whole $x = i_0$ layer from the ambient space, and consider a new set of lattice points

$$D' := \{(i, j, k) \in D : i < i_0\} \cup \{(i - 1, j, k) : (i, j, k) \in D \text{ with } i > i_0\}.$$

Deriving $D'$ from $D$ above will be called a reduction along the $x$-direction. It is not difficult to see that $F(I_D) \cong F(I_{D'})$.

(b) One can similarly define reductions along the $y$-direction and $z$-direction. In this paper, when we say a set $D$ essentially satisfies some property $(P)$, we mean that after a finite sequence of reductions, the derived new set satisfies the property $(P)$.

Example 2.11 For example, we may have

$$D = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (3, 1, 1), (3, 1, 2)\}.$$

Since $D^2 = \emptyset$, we can apply a reduction along the $x$-direction to get

$$D' = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1), (2, 1, 2)\}$$

with respect to $i_0 = 2$. The essential length, width and height do not change: $a_D = a_{D'}$, $b_D = b_{D'}$ and $c_D = c_{D'}$. But visually, this $D'$ is more concise than the original $D$. 
3 Considering the Hilbert polynomials

Let \( M \neq 0 \) be a finitely generated graded module over the polynomial ring \( S = \mathbb{K}[x_1, \ldots, x_n] \). The field \( \mathbb{K} \) has characteristic zero. Let \( \text{Hilb}_M(t) := \sum_{k \in \mathbb{Z}} H(M, k)t^k \) be the Hilbert series of \( M \). It is well-known that

\[
\text{Hilb}_M(t) = \frac{P_M(t)}{1 - t^d} \quad \text{with} \quad P_M(1) \neq 0
\]

for some \( h \)-polynomial \( P_M(t) \in \mathbb{Z}[t, t^{-1}] \). The number \( d \) is the Krull dimension of \( M \), while \( P_M(1) = e(M) > 0 \) is the multiplicity of \( M \); see [19, Section 6.1.1].

Lemma 3.1 [2, Proposition 7.43] Let \( M \) be a graded Cohen–Macaulay module of dimension \( d \) over the polynomial ring \( S \). Let \( P_M(t) \) be the \( h \)-polynomial of the Hilbert series of \( M \). Then \( \text{reg}(M) = \deg(P_M(t)). \)

Lemma 3.2 Let \( J \) be a homogeneous ideal of \( S \). If with respect to some monomial order of \( S \), the initial ideal \( \text{in}(J) \) is Cohen–Macaulay, then \( \text{reg}(S/J) = \text{reg}(S/\text{in}(J)) \).

Proof It is well-known that \( \text{Hilb}_{S/\text{in}(J)}(t) \) coincides with \( \text{Hilb}_{S/\text{in}(J)}(t) \); see [19, Corollary 6.1.5]. Furthermore, since \( \text{in}(J) \) is Cohen–Macaulay, so is \( J \) of the same dimension by [19, Theorem 3.3.4 and Corollary 3.3.5]. Therefore,

\[
\text{reg}(S/\text{in}(J)) = \deg(P_{S/\text{in}(J)}(t)) = \deg(P_{S/J}(t)) = \text{reg}(S/J).
\]

Let \( I \) be an ideal. A subideal \( J \subseteq I \) is called a reduction of \( I \) if there is a number \( n \) such that \( I^{n+1} = J \cdot I^n \). The least number \( n \) with the above property is the reduction number of \( I \) with respect to \( J \) and denoted by \( r_J(I) \). A reduction \( J \) is minimal if no proper subideal of \( J \) is a reduction of \( I \). The (absolute) reduction number of \( I \) is defined as

\[
r(I) := \min\{r_J(I)|J \text{is a minimal reduction of } I\}.
\]

Lemma 3.3 [6, Proposition 6.6] Let \( I \subset S \) be a homogeneous ideal that is generated in one degree, say \( \delta \). Assume that the special fiber ring \( \mathcal{F}(I) \) is Cohen-Macaulay. Then each minimal reduction of \( I \) is generated by \( \dim(\mathcal{F}(I)) \) homogeneous polynomials of degree \( \delta \), and \( I \) has reduction number \( r(I) = \text{reg}(\mathcal{F}(I)) \).

Let \( D \) be a nonempty finite subset of \( \mathbb{Z}_+^3 \) such that for the 2-minors ideal \( I_2(D) \), all its minimal Gröbner basis elements with respect to the lexicographic order are quadratic (this requirement is satisfied when \( D \) is a three-dimensional Ferrers diagram that satisfies the projection property, by [21, Corollary 2.14]). In particular, \( \text{in}(I_2(D)) \) is squarefree by the well-known Buchberger’s criterion and the explicit description of the generating set of \( I_2(D) \) in Definition 2.3. Whence, we will write \( \Delta(D) \) for the associated Stanley–Reisner complex.

Once we have a simplicial complex \( \Delta \), we will consider the regularity

\[
\text{reg}(\Delta) := \text{reg}(\mathbb{K}[\Delta]) = \text{reg}(S/I_\Delta)
\]
for the Stanley–Reisner ideal \( I_\Delta \) of \( \Delta \) in appropriate polynomial ring \( S \) over the field \( \mathbb{K} \). We will similarly consider the multiplicity

\[
e(\Delta) := e(\mathbb{K}[\Delta]) = e(S/I_\Delta).
\]

**Corollary 3.4** Let \( D \) be a three-dimensional Ferrers diagram that satisfies the projection property. Then,

\[
\text{reg}(\Delta(D)) = \text{reg}(\mathbb{K}[T_D]/J_D) = \text{reg}(\mathcal{F}(I_D)) = r(I_D)
\]

(1) and

\[
e(\Delta(D)) = e(\mathbb{K}[T_D]/J_D) = e(\mathcal{F}(I_D)).
\]

(2)

**Proof** With respect to (1), the first equality follows from Lemma 3.2, and the last equality is by Lemma 3.3. With respect to (2), the first equality follows from the fact that the Hilbert series of \( \mathbb{K}[T_D]/J_D \) and \( \mathbb{K}[T_D]/\text{in}(J_D) \) coincide, as already mentioned in the proof of Lemma 3.2. \( \square \)

**Remark 3.5** It is recently proved in [4, Corollary 2.7] that if \( I \) is a homogeneous ideal of \( S \) with arbitrary term order such that the initial ideal \( \text{in}(I) \) is squarefree, then depth \( (S/I) \) = depth \( (S/\text{in}(I)) \) and \( \text{reg}(S/I) = \text{reg}(S/\text{in}(I)) \). In particular, the first equality of (1) also follows.

In [21, Theorem 4.1], we have shown that if the three-dimensional Ferrers diagram satisfies the projection property, then the associated Stanley–Reisner complex is pure vertex-decomposable. To investigate the regularity and the multiplicity of the corresponding Stanley–Reisner ring, we will fall back on the following critical observation.

**Remark 3.6** [See also [23, Remark 2.4]] Let \( \Delta \) be a pure vertex-decomposable simplicial complex on the finite set \([n]\) and assume that \( n \) is a shedding vertex. Let \( I_\Delta \) be the Stanley–Reisner ideal of \( \Delta \) considered as a complex on \([n]\) in \( S = \mathbb{K}[x_1, \ldots, x_n] \). Then the cone over \( \text{link}_\Delta(n) \) with apex \( n \) considered as a complex on \([n]\) has Stanley–Reisner ideal \( J_{\text{link}_\Delta(n)} = I_\Delta : x_n \). And the Stanley–Reisner ideal of \( \Delta \setminus n \) considered as a complex on \([n]\) is \( (x_n, I_{\Delta\setminus n}) \) where \( I_{\Delta\setminus n} \subset \mathbb{K}[x_1, \ldots, x_{n-1}] \) is the Stanley–Reisner ideal of \( \Delta \setminus n \) considered as a complex on \([n-1]\). Furthermore, we have a short exact sequence of graded \( S \)-modules of the same positive dimension:

\[
0 \rightarrow S/J_{\text{link}_\Delta(n)}(-1) \rightarrow S/I_\Delta \rightarrow S/I_{\Delta\setminus n}S \rightarrow 0.
\]

As multiplicity is additive on such a sequence, we have

\[
e(\Delta) = e(\Delta \setminus n) + e(\text{link}_\Delta(n)).
\]

(3)

Meanwhile, by [17, Theorem 4.2], we have

\[
\text{reg}(\Delta) = \max\{\text{reg}(\Delta \setminus n), \text{reg}(\text{link}_\Delta(n)) + 1\}.
\]

(4)
4 Full rectangular case

In this section, we will focus on the special case when \( D = [a_D] \times [b_D] \times [c_D] \) is a full three-dimensional Ferrers diagram. In this situation, the most convenient tool will be the Segre product of graded modules. Say, that \( R = \mathbb{K}[x_1, \ldots, x_m] \) and \( S = \mathbb{K}[y_1, \ldots, y_n] \) are two standard graded polynomial rings over \( \mathbb{K} \). Then the Segre product of \( R \) and \( S \) is \( \bigotimes_{\ell \in \mathbb{Z}} (R_{\ell} \otimes_{\mathbb{K}} S_{\ell}) \), which is a graded ring. For a graded \( R \)-module \( M \) and a graded \( S \)-module \( N \), the Segre product of \( M \) and \( N \) is defined as \( M \otimes N = \bigoplus_{\ell \in \mathbb{Z}} (M_{\ell} \otimes_{\mathbb{K}} N_{\ell}) \), which is a graded \( (R \otimes S) \)-module.

Now, we study the special fiber ring in the full rectangular case. Firstly, we consider multiplicity.

**Lemma 4.1** If \( M \) and \( N \) above are finitely generated and have positive dimensions, then

\[
\dim(M \otimes N) = \dim(M) + \dim(N) - 1
\]

and

\[
e(M \otimes N) = \left( \frac{\dim(M) + \dim(N) - 2}{\dim(M) - 1} \right) e(M) e(N).
\]

**Proof** By definition, the Hilbert functions satisfy

\[
H(M, t) = \frac{e(M)}{(\dim(M) - 1)!} t^{\dim(M) - 1} + \text{lower degrees},
\]

and

\[
H(N, t) = \frac{e(N)}{(\dim(N) - 1)!} t^{\dim(N) - 1} + \text{lower degrees}
\]

for \( t \gg 0 \). Thus,

\[
H(M \otimes N, t) = H(M, t) H(N, t)
\]

\[
= \frac{e(M)}{(\dim(M) - 1)!} \frac{e(N)}{(\dim(N) - 1)!} t^{\dim(M) + \dim(N) - 2} + \text{lower degrees}
\]

for \( t \gg 0 \). The expected dimension and the multiplicity formula can be read off from the last equation. \( \square \)

**Proposition 4.2** Suppose that \( D \) is the full three-dimensional Ferrers diagram \( [a_D] \times [b_D] \times [c_D] \). Then the multiplicity of the special fiber ring is given by the trinomial:

\[
e(\mathcal{F}(I_D)) = \left( \frac{a_D + b_D + c_D - 3}{a_D - 1, b_D - 1, c_D - 1} \right) \equiv \frac{(a_D + b_D + c_D - 3)!}{(a_D - 1)!(b_D - 1)!(c_D - 1)!}.
\]
**Proof** Notice that

\[ F(I_D) \cong \mathbb{K}[x_1, \ldots, x_{a_D}] \otimes (\mathbb{K}[y_1, \ldots, y_{b_D}] \otimes \mathbb{K}[z_1, \ldots, z_{c_D}]). \]

Thus, by Lemma 4.1,

\[ e(F(I_D)) = \left( \frac{a_D + b_D + c_D - 3}{a_D - 1} \right) \left( \frac{b_D + c_D - 2}{b_D - 1} \right) = \left( \frac{a_D + b_D + c_D - 3}{a_D - 1, b_D - 1, c_D - 1} \right), \]

as

\[ e(\mathbb{K}[x_1, \ldots, x_{a_D}]) = e(\mathbb{K}[y_1, \ldots, y_{b_D}]) = e(\mathbb{K}[z_1, \ldots, z_{c_D}]) = 1. \]

\[ \square \]

Secondly, we consider the regularity. The following is known.

**Lemma 4.3** [22, Theorem 5.3] Let \( S_1, \ldots, S_s \) be graded polynomial rings on disjoint sets of variables over \( \mathbb{K} \). For \( i = 1, \ldots, s \), let \( M_i \) be a graded finitely generalized Cohen–Macaulay \( S_i \)-module of positive dimension.

(a) If \( \dim(M_i) = 1 \) for all \( i \), then \( M_1 \otimes \cdots \otimes M_s \) is a Cohen–Macaulay \( S_1 \otimes \cdots \otimes S_s \)-module, and

\[ \text{reg}(M_1 \otimes \cdots \otimes M_s) = \max\{\text{reg}(M_1), \ldots, \text{reg}(M_s)\} . \]

(b) Assume that at least for one \( j \), \( \dim(M_j) \geq 2 \), and for all \( i = 1, \ldots, s \), \( M_i \) is an \( \mathbb{N} \)-graded \( S_i \)-module with \( \text{reg}(M_i) < \dim(M_i) \). Then, \( M_1 \otimes \cdots \otimes M_s \) is a Cohen–Macaulay \( S_1 \otimes \cdots \otimes S_s \)-module, and

\[ \text{reg}(M_1 \otimes \cdots \otimes M_s) = (\dim(M_1) + \cdots + \dim(M_s) - s + 1) - \max\{\dim(M_i) - \text{reg}(M_i) : 1 \leq i \leq s\} . \]

**Proposition 4.4** Suppose that \( D \) is the full three-dimensional Ferrers diagram \([a_D] \times [b_D] \times [c_D]\). If \( a_D \leq b_D \leq c_D \), then

\[ \text{reg}(F(I_D)) = a_D + b_D - 2 = r(I_D). \]

**Proof** The regularity formula follows from Lemma 4.3 by proceeding as in the proof of 4.2. The reduction number part then follows from (1) in Corollary 3.4. Alternatively, [7, Theorem 5.2] gives the reduction number. \( \square \)

**5 A uniform treatment in the strong projection case**

In this section, we want to provide reasonable estimates of the regularity and the multiplicity of the toric ring associated with the three-dimensional Ferrers diagram.
The approach we take here is to consider simultaneously a pair of such diagrams under some conditions. It allows us to investigate these two invariants together.

Recall that for a given lattice point \( u \in D \), \( A_u(D) \) defined in Definition 2.8 is the subset obtained from \( D \) by removing the points before \( u \) with respect to a given quasi-lexicographic order \( \prec \). Meanwhile, we also write \( A_u^+(D) := A_u(D) \setminus u \). For simplicity, when the diagram \( D \) is clear from the context, we will write directly \( A_u \) and \( A_u^+ \), respectively.

**Definition 5.1** For the diagram \( D \) above, we define

\[
N(D) := \{ u \in D^1 : \text{in}(I_2(A_u)) \supset \text{in}(I_2(A_u^+)) \} \subseteq K[A_u]
\]

and \( \text{Phan}(D) := D^1 \setminus N(D) \) to be the set of normal points and phantom points (with respect to a chosen quasi-lexicographic order \( \prec \)), respectively. Note that the initial ideals are with respect to the lexicographic monomial order on \( K[T_D] \).

These notions were introduced in our previous paper [21]. Pertinent properties regarding them are summarized in Observation 5.2 and Remark 5.3.

**Proposition 5.2** [21, Corollary 2.14, Remark 3.5, and Observation 3.8] Let \( D \) be a three-dimensional Ferrers diagram that satisfies the projection property. For any \( u \in D^1 \), we have the following facts.

(a) Let \( K[T_{A_u}] \) be the subring of \( K[T_D] \) induced by the containment \( A_u \subseteq D \). Then the 2-minors ideal \( I_2(A_u) \) is precisely \( I_2(D) \cap K[T_{A_u}] \), and the minimal monomial generating set \( \text{gens}(\text{in}(I_2(A_u))) \) is precisely \( \text{gens}(\text{in}(I_2(D))) \cap K[T_{A_u}] \).

In particular, the restriction complex \( \Delta(D, A_u) \) is \( \Delta(A_u) \).

(b) If \( u \) is a phantom point, then trivially \( \text{codim} I_2(A_u) = \text{codim} I_2(A_u^+) \).

(c) If \( u \) is a normal point, then \( \dim \Delta(A_u) = \dim \Delta(A_u^+) \) and \( \text{codim} I_2(A_u) = \text{codim} I_2(A_u^+) + 1 \).

**Remark 5.3** Let \( D \) be a three-dimensional Ferrers diagram that satisfies the projection property. Suppose that a quasi-lexicographic order \( \prec \) induces a shedding order on \( \Delta(D) \) (the induction order \( \prec_D \) in Setting 2.9 satisfies this requirement by [21, Theorem 4.1]). Now, take an arbitrary \( u \in D^1 \). When \( u \) is a phantom point, we observed in [21, Remark 3.6] that \( \Delta(A_u) \) is a cone over \( \Delta(A_u) \setminus T_u = \Delta(A_u^+) \) with the apex \( T_u \). Trivially we have

\[
\text{reg}(\Delta(A_u)) = \text{reg}(\Delta(A_u^+)) \quad \text{and} \quad e(\Delta(A_u)) = e(\Delta(A_u^+)) \quad (5)
\]

in this case. On the other hand, if \( u \) is a normal point with respect to \( D \), then by our assumption, \( T_u \) is a shedding vertex of \( \Delta(A_u) \). Whence, it follows from (3) and (4) in Remark 3.6 that

\[
\text{reg}(\Delta(A_u)) = \max\{\text{reg}(\Delta(A_u^+)), \text{reg}(\text{link}_{\Delta(A_u)}(T_u)) + 1\} \quad (6),
\]

\[
e(\Delta(A_u)) = e(\Delta(A_u^+)) + e(\text{link}_{\Delta(A_u)}(T_u)) \quad (7).
\]
In particular,
\[
\text{reg}(\Delta(A_u)) \geq \text{reg}(\Delta(A_u^+)) \quad \text{and} \quad e(\Delta(A_u)) \geq e(\Delta(A_u^+)). \quad (8)
\]

The synchronizing treatment applied in this section requires the introduction of a stronger condition. The necessity for introducing this condition is discussed in Remark 5.11.

**Definition 5.4** Let \( \mathcal{D} \) be a three-dimensional Ferrers diagram. Then \( \mathcal{D} \) is said to satisfy the **strong projection property** if the following equivalent conditions hold for each \( i \) in \([a_D - 1]\):

(a) for each \( u \in \mathcal{D}' \), both \( Z_i^{\geq i+1}(\mathcal{D}, u) \) and \( Z_6^{\geq i+1}(\mathcal{D}, u) \) are empty;

(b) for each \( u \in \mathcal{D}' \), both \( b_{\mathcal{D}^i+1} \leq \beta_{\mathcal{D}}(u) \) and \( c_{\mathcal{D}^i+1} \leq \gamma_{\mathcal{D}}(u) \) hold;

(c) \((i, b_{\mathcal{D}^i+1}, c_{\mathcal{D}^i}) \in \mathcal{D} \) and \((i, b_{\mathcal{D}^i}, c_{\mathcal{D}^i+1}) \in \mathcal{D} \).

**Example 5.5** Consider the diagram \( \mathcal{D} \) in Example 2.4. Since \((2, b_{\mathcal{D}^2}, c_{\mathcal{D}^3}) = (2, 3, 2) \notin \mathcal{D} \), this diagram does not satisfy the strong projection property. The minimal three-dimensional Ferrers diagram that contains \( \mathcal{D} \) and satisfies the strong projection property, is the diagram \( \mathcal{D}' \), consisting of the following lattice points

\[
(1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 2, 1), (1, 2, 2), (1, 2, 3), (1, 3, 1), (1, 3, 2), (1, 3, 3),
(2, 1, 1), (2, 1, 2), (2, 1, 3), (2, 2, 1), (2, 2, 2), (2, 2, 3), (2, 3, 1), (2, 3, 2),
(3, 1, 1), (3, 1, 2), (3, 2, 1).
\]

The following facts are clear from the definition.

**Observation 5.6** (a) If \( \mathcal{D} \) is a full three-dimensional Ferrers diagram, then it satisfies the strong projection property.

(b) If \( \mathcal{D} \) is a three-dimensional Ferrers diagram and one of \( a_{\mathcal{D}}, b_{\mathcal{D}} \) and \( c_{\mathcal{D}} \) is 1, then \( \mathcal{D} \) is practically a two-dimensional Ferrers diagram and satisfies the strong projection property.

(c) If \( \mathcal{D} \) is a three-dimensional Ferrers diagram that satisfies the strong projection property, then it satisfies the projection property.

(d) Suppose that \( \mathcal{D} \) is a three-dimensional Ferrers diagram satisfying the strong projection property. Then all the three truncated subdiagrams

\[
\{(i, j, k) \in \mathcal{D} : i \neq i_0\}, \quad \{(i, j, k) \in \mathcal{D} : j \neq j_0\} \quad \text{and} \quad \{(i, j, k) \in \mathcal{D} : k \neq k_0\}
\]

are essentially three-dimensional Ferrers diagrams that still satisfy the strong projection property; see also Definition 2.10.

Let \( \mathcal{D} \) be a three-dimensional Ferrers diagram that satisfies the projection property. In [21, Theorem 4.1], we proved that the complex \( \Delta(\mathcal{D}) \) is pure vertex-decomposable of dimension \( a_{\mathcal{D}} + b_{\mathcal{D}} + c_{\mathcal{D}} - 3 \), and the induction order in Setting 2.9 gives a shedding order. As a matter of fact, in the early draft of that paper, we showed that the usual
lexicographic order gives a shedding order. However, a proof for the primeness of the 2-minors ideals is hard to achieve if we use the lexicographic order.

In the following, we will give a direct proof that when \( D \) satisfies the strong projection property, then the lexicographic order gives a shedding order. Unlike the proof of [21, Theorem 4.1], we don’t have a second stage to deal with here. Thus, the proof here is relatively shorter than the previous one. In the proof, we will use backward induction on the lattice points of \( D \) with respect to the lexicographic order. Undoubtedly from the definition of vertex decomposable complexes, we need to check both the link complexes and the deletion complexes are pure of expected dimensions. Counting the related phantom points is our tool for achieving this goal. Of course, we need to know where to find the phantom points. The following observation is summarized from [21, Discussion 3.7].

**Remark 5.7** Let \( D \) be a three-dimensional Ferrers diagram satisfying the projection property and assume that \( d_D \geq 2 \). The point \( u = (1, j_1, k_1) \in D^1 \) is called a **border point** if \((1, j_1 + 1, k_1 + 1) \notin D\). Let \( B \subseteq D^1 \) denote the set of border points of \( D \). It has the following two special subsets:

- \( B_y \) : the border points on the \( y = 1, 2, \ldots, b_{D \geq 2} - 1 \) lines with minimal \( z \)-coordinates;
- \( B_z \) : the border points on the \( z = 1, 2, \ldots, c_{D \geq 2} - 1 \) lines with minimal \( y \)-coordinates.

Then, \( B_y \) and \( B_z \) are disjoint and \( B \setminus (B_y \cup B_z) \) is precisely \( \text{Phan}(D) \). Consequently, the normal points and phantom points are independent of the concrete choice of quasi-lexicographic order.

**Example 5.8** Let \( D \) be a typical three-dimensional Ferrers diagram that satisfies the projection property. In Fig. 3, the union of the cross-hatch cells \( \square \) provides \( B_y \) and the union of the grid cells \( \□ \) provides \( B_z \). The remaining shaded cells \( \blacklozenge \) give the phantom points. The set \( B \) of border points is the disjoint union of these three groups.

**Proposition 5.9** Let \( D \) be a three-dimensional Ferrers diagram that satisfies the strong projection property. Then the lexicographic order on \( D \) gives a shedding order on the pure vertex-decomposable complex \( \Delta(D) \).

Fig. 3  Border points
Proof  We prove by induction on \( a_D \); this will be called the outer induction process in the proof. The base case of the induction is when \( a_D = 0 \) and \( D = \emptyset \). The claimed result holds trivially in this case.

For the induction step of the outer induction process, in the following, we will assume that \( a_D \geq 1 \). It suffices to prove that \( \Delta(A_u(D)) \) is pure vertex-decomposable, for each \( u = (1, j_0, k_0) \in D^1 \). As a reminder, in this proof, both \( A \) and \( A^+ \) are with respect to the lexicographic order. We will prove this by backward induction with respect to the lexicographic order; this will be called the inner induction process in the proof. The base case of the inner induction process is when we remove the whole \( x = 1 \) layer \( D^1 \) and get \( D^{\geq 2} \); whence, \( u \) is indeed \((2, 1, 1)\). As \( a_{D^{\geq 2}} = a_D - 1 \) and \( D^{\geq 2} \) essentially still satisfies the strong projection property, by induction on \( a_D \), \( \Delta(D^{\geq 2}) \) is pure vertex-decomposable. This establishes the validity in the base case for the inner induction process.

For the induction step of the inner induction process, in the following, we take a general point \( u = (1, j_0, k_0) \in D^1 \). Without loss of generality, we may assume that \( u \) is a normal point with respect to \( D \). As the lexicographic order is automatically a quasi-lexicographic order, the restriction complex \( \Delta(D, A_u(D)) \) is \( \Delta(A_u(D)) \) by Proposition 5.2 (a). Similarly, we have \( \Delta(D, A_u^+(D)) = \Delta(A_u^+(D)) \).

Firstly, we deal with the deletion complex \( \Delta(A_u(D)) \setminus T_u = \Delta(A_u^+(D)) \). Note that for any \( v \in D^1 \cap A_u^+(D) \), the restriction complex \( \Delta(D, A_v(D)) = \Delta(A_v(D)) \) is pure vertex decomposable with respect to the lexicographic order, by induction and by Proposition 5.2 (a). Thus, by applying Proposition 5.2 (b) and (c) repeatedly, we also have

\[
\dim(\Delta(A_u^+(D))) = \dim(D^{\geq 2}) + \#(\text{Phan}(D) \cap A_u^+(D));
\]

note that only phantom points provide dimensional change. Here, we use \# to denote the cardinality of the corresponding set. Notice that the phantom points do not depend on the concrete choice of quasi-lexicographic order by Remark 5.7. It follows from both Proposition 5.2 and [21, Theorem 4.1] that

\[
\dim(\Delta(D)) = \dim(D^{\geq 2}) + \#(\text{Phan}(D)).
\]

Consequently,

\[
\dim(\Delta(A_u^+(D))) = \dim(\Delta(D)) - \#(\text{Phan}(D) \setminus A_u^+(D)). \tag{9}
\]

Since \( u \) is not a phantom point, we actually have

\[
\dim(\Delta(A_u^+(D))) = \dim(\Delta(D)) - \#(\text{Phan}(D) \setminus A_u(D)). \tag{10}
\]

Next, we consider the link complex \( L_u(D) := \text{link}_{\Delta(A_u(D))}(T_u) \). It suffices to show that \( L_u(D) \) is pure vertex decomposable of dimension \( \dim(\Delta(A_u^+(D))) - 1 \), and the lexicographic order gives a shedding order. Notice that \( D \) satisfies the strong projection property. Now, we define
\[ \mathcal{H} := \mathcal{Z}_3^\perp(D, u) \cup \mathcal{Z}_5^1(D, u) \cup \mathcal{Z}_6^1(D, u). \] (11)

Then, \( \mathcal{L}_u(D) \) is the join of the restriction complex \( \Delta(D, \mathcal{H}) \) with a simplex of dimension \( \gamma - k_0 - 1 \) for \( \gamma = \gamma_D(u) \); cf. the detailed calculation in the first stage proof of [21, Theorem 4.1]. The simplex here corresponds to the set \( \{(1, j, k) : k_0 < k \leq \gamma \} \). Since this \( \mathcal{H} \) satisfies the detaching condition in [21, Proposition 2.17], we can use it to deduce that \( \Delta(\mathcal{H}) \) agrees with the restriction complex \( \Delta(D, \mathcal{H}) \).

To study \( \Delta(\mathcal{H}) \), we turn to consider \( \mathcal{D}' := \mathcal{Z}_3(D, u) \cup \mathcal{Z}_5^1(D, u) \cup \mathcal{Z}_6^1(D, u) \). Notice that \( \mathcal{D}' \) is a three-dimensional Ferrers diagram that still satisfies the strong projection property on \( \mathcal{D} \). Furthermore, \( \mathcal{H} = \mathcal{A}_u^+(\mathcal{D}') \). Since \( \mathcal{H} \) has fewer cells than \( \mathcal{A}_u(D) \), by induction, \( \Delta(\mathcal{H}) \) is pure vertex-decomposable and the lexicographic order gives a shedding order. Now, it remains to show that

\[ \dim(\Delta(\mathcal{H})) + (\gamma - k_0) = \dim(\Delta(\mathcal{A}_u^+(\mathcal{D}))) - 1, \] (12)

since the left-hand side gives the dimension of \( \mathcal{L}_u(D) \). Similar to (9), we have

\[ \dim(\Delta(\mathcal{H})) = \dim(\Delta(\mathcal{D}')) - \#(\text{Phan}(\mathcal{D}') \setminus \mathcal{A}_u^+(\mathcal{D}')). \] (13)

As \( a_{\mathcal{D}'} = a_{\mathcal{D}}, b_{\mathcal{D}'} = b_{\mathcal{D}} \) and \( c_{\mathcal{D}'} = k_0 \), we have

\[ \dim(\Delta(\mathcal{D})) - \dim(\Delta(\mathcal{D}')) = c_D - k_0. \]

By Remark 5.7, we have

\[ \text{Phan}(\mathcal{D}') \setminus \mathcal{A}_u^+(\mathcal{D}') = \{(1, j, k_0) : b_{(\mathcal{D}')^2} \leq j \leq j_0 \}. \]

However, \( b_{(\mathcal{D}')^2} = \min(j_0, b_{\mathcal{D}^2}) \). Thus,

\[ \#(\text{Phan}(\mathcal{D}') \setminus \mathcal{A}_u^+(\mathcal{D}')) = j_0 - \min(j_0, b_{\mathcal{D}^2}) + 1. \] (14)

By combining Eqs. (10), (12), (13), and (14) together, it is clear that we have to show

\[ \#(\text{Phan}(\mathcal{D}) \setminus \mathcal{A}_u(\mathcal{D})) = c_D + j_0 - \min(j_0, b_{\mathcal{D}^2}) - \gamma. \] (15)

We will write \( Q := \{(1, j, k) \in \mathcal{B} : j < j_0 \} \). Obviously, to show (15), we have two cases.

(a) Suppose that \( b_{\mathcal{D}^2} \geq j_0 \). As \( \mathcal{D} \) satisfies the strong projection property, we have \( \gamma = c_D \). Therefore, we deduce immediately from Remark 5.7 that

\[ \text{Phan}(\mathcal{D}) \setminus \mathcal{A}_u(\mathcal{D}) \subseteq Q \setminus \mathcal{B}_y = \{(1, j, \gamma) \in \mathcal{B} : j < j_0 \} \setminus \mathcal{B}_y = \emptyset, \]

which gives the desired formula (15).
Suppose instead that $b_{D^{\geq 2}} < j_0$. If we picture the set of border points $B$ as in Fig. 3, then the southeast extremal point of $Q$ is $(1, j_0 - 1, \gamma)$. Thus, it is not difficult to check that $\# Q = c_D - \gamma + j_0 - 1$. Notice that $B_y \subset Q$ with $\#B_y = b_{D^{\geq 2}} - 1$ by Remark 5.7. Now, as $D$ satisfies the strong projection property, $\gamma \geq c_{D^{\geq 2}}$. Since for any $(1, j, k) \in Q$, one has $k \geq \gamma$. Thus, $B_z \cap Q = \emptyset$. Consequently, $Q \setminus B_y = \text{Phan}(D) \setminus \mathcal{A}_u(D)$, giving the desired formula (15). This completes our proof of Proposition 5.9.

We are now ready to estimate the regularity and multiplicity of three-dimensional Ferrers diagrams that satisfy the strong projection property. In particular, we find an upper bound of those invariants with the help of Propositions 4.2 and 4.4; see Example 5.13.

To provide such a reasonable estimate, we consider simultaneously two diagrams of this type. We argue by induction with respect to the lexicographic order and consider the estimation problems for the accompanied subdiagrams of these two. For that purpose, we need to analyze and compare different zones defined in Definition 2.6 of the given subdiagrams.

**Theorem 5.10** Let $D_1 \subset D_2$ be two three-dimensional Ferrers diagrams that satisfy the strong projection property. Then, we have

$$r(I_{D_1}) = \text{reg}(\Delta(D_1)) \leq r(I_{D_2}) = \text{reg}(\Delta(D_2)) \quad \text{and} \quad e(\Delta(D_1)) \leq e(\Delta(D_2)).$$

(16)

**Proof** Once we have the regularity relationship, the reduction-number part follows from Corollary 3.4. Thus, in the following, we will focus on establishing (16) for regularity and multiplicity. We prove this by induction on $a_D$; this is the outer induction process of our proof. Its base case is when $a_D = 0$ and $D = \emptyset$. Whence, these inequalities hold trivially. In the following, we assume that $a_D \geq 1$.

We will consider simultaneously the lexicographic order on both $D_1$ and $D_2$. For each $u \in D_1^1$, we will prove by backward induction with respect to the lexicographic order that

$$\text{reg}(\Delta(A_u(D_1))) \leq \text{reg}(\Delta(A_u(D_2))) \quad \text{and} \quad e(\Delta(A_u(D_1))) \leq e(\Delta(A_u(D_2))).$$

(17)

This will be the inner induction process of our proof. Note that when $u = (1, 1, 1)$, we obtain the expected inequalities in (16). As a reminder, in this proof, both $A_u$ and $A_u^+$ are with respect to the lexicographic order. Now, we carry out the inner induction argument.

**Base case**

The base case of this inner induction process is when we remove the whole $x = 1$ layer so that indeed $u = (2, 1, 1) \in D_1^1$; without loss of generality, we assume that $D^{\geq 2} \neq \emptyset$. By induction on $a_D$, we surely will have

$$\text{reg}(\Delta(D^{\geq 2}_1)) \leq \text{reg}(\Delta(D^{\geq 2}_2)) \quad \text{and} \quad e(\Delta(D^{\geq 2}_1)) \leq e(\Delta(D^{\geq 2}_2)),$$
establishing the validity in the base case. Note that when \( D^{\geq 2} = \emptyset \), these inequalities hold trivially.

**Induction step**

Now, consider a general point \( u \in D_1^1 \). Let \( v \in D_1 \) such that \( A_u^+(D_1) = A_v(D_1) \). Obviously, \( u \) precedes \( v \) lexicographically. Since Proposition 5.9 confirms that the lexicographic order gives a shedding order to both \( \Delta(D_1) \) and \( \Delta(D_2) \), we can apply the calculations in Remark 5.3.

(a) Suppose that \( u \) is a phantom point of \( D_1 \). Now, by induction, equalities (5) and (8), we have

\[
\text{reg}(A_u(D_1)) = \text{reg}(A_u^+(D_1)) = \text{reg}(A_v(D_1)) \leq \text{reg}(A_v(D_2)) \leq \text{reg}(A_u(D_2)),
\]

establishing (17) for the regularity in this case. Similarly, we can deduce the inequality for multiplicity.

(b) If \( u \) is a normal point with respect to \( D_1 \), then it is also a normal point with respect to \( D_2 \) by definition. Hence, by induction and (8), we have

\[
\text{reg}(\Delta(A_u^+(D_1))) = \text{reg}(\Delta(A_v(D_1))) \leq \text{reg}(\Delta(A_v(D_2))) \leq \text{reg}(\Delta(A_u^+(D_2))).
\]

Similarly, we have

\[
\text{e}(\Delta(A_u^+(D_1))) \leq \text{e}(\Delta(A_u^+(D_2))).
\]

For \( s = 1, 2 \), write \( L_u(D_s) := \text{link}_{\Delta(A_u(D_s))}(T_u) \). As in the proof of Proposition 5.9, we know that each \( L_u(D_s) \) is a join of a simplex with \( \Delta(H(D_s)) \), where

\[
H(D_s) := Z_{\geq 2}^s(D_s, u) \cup Z_1^s(D_s, u) \cup Z_0^1(D_s, u). \tag{18}
\]

If we write

\[
D'_s := Z_3(D_s, u) \cup Z_2^1(D_s, u) \cup Z_1^1(D_s, u),
\]

then \( H(D_s) = A_u^+(D'_s) \). Note that for each \( s \), the set \( D'_s \) is a three-dimensional Ferrers diagram that satisfies the strong projection property by Observation 5.6 (d). And obviously, we have the corresponding containment \( D'_1 \subseteq D'_2 \). Now, for each \( s \), we have \( H(D_s) = A_u^+(D'_s) = A_{w_s}(D'_s) \) for some \( w_s \in D'_s \). As \( H(D_1) = \Delta(D_2) \cap D_1 \), it is clear that \( w_2 \) precedes \( w_1 \) with respect to the lexicographic order. Thus, by induction and (8), we have

\[
\text{reg}(L_u(D_1)) = \text{reg}(\Delta(H(D_1))) = \text{reg}(\Delta(A_{w_1}(D'_1))) \leq \text{reg}(\Delta(A_{w_1}(D'_2))) \leq \text{reg}(\Delta(A_{w_2}(D'_2))) = \text{reg}(\Delta(H(D_2))) = \text{reg}(L_u(D_2)).
\]

And similarly, we have

\[
\text{e}(L_u(D_1)) \leq \text{e}(L_u(D_2)).
\]
However, by (6) and (7), we have
\[
\begin{align*}
\text{reg}(\Delta (A_u(D_s))) &= \max \{\text{reg}(\Delta (A^+_u(D_s))), \text{reg}(\mathcal{L}_u(D_s)) + 1\}, \\
\text{e}(\Delta (A_u(D_s))) &= \text{e}(\Delta (A^+_u(D_s))) + \text{e}(\mathcal{L}_u(D_s)),
\end{align*}
\]
for \( s = 1, 2 \). Putting the above data together, we can get the desired inequalities in (17). This completes our proof of Theorem 5.10. 

\[\Box\]

**Remark 5.11** In the proof above, the strong projection property condition, instead of the projection property condition, is used to get the equality \( \mathcal{H}(D_1) = \mathcal{H}(D_2) \cap D_1 \). Otherwise, the subsets \( \mathcal{H}(D_s) \) for the link complexes won’t be as simple as in (18). Instead, one has to use the formula (22) in the next section. Whence, it might be possible that \( \mathcal{H}(D_1) \not\subseteq \mathcal{H}(D_2) \) and \( \text{e}((\mathcal{L}_u(D_1)) > \text{e}(\mathcal{L}_u(D_2)) \). For instance, one can take \( D_1 \) to be the minimal three-dimensional Ferrers diagram containing \((1, 3, 2)\) and \((2, 2, 3)\), and take \( D_2 \) to be the full diagram \([2] \times [3] \times [3]\). The diagram \( D_1 \) satisfies the projection property. But the strong projection property is not satisfied, since \((1, b_{D_1}, c_{D_1}) = (1, 3, 3) \not\in D_1 \). For \( u = (1, 3, 1) \in D_1 \), we will have
\[
\text{e}(\mathcal{L}_u(D_1)) = 2 > \text{e}(\mathcal{L}_u(D_2)) = 1.
\]

In the following, we consider an easy application of Theorem 5.10.

**Definition 5.12** Let \( D \) be a three-dimensional Ferrers diagram. Its \((x, y)\)-profile is the two-dimensional Ferrers diagram \( \mathcal{P}_{x, y}(D) := \{(i, j) : (i, j, k) \in D\} \). In a similar vein, one can define the \((x, z)\)-profile \( \mathcal{P}_{x, z}(D) \).

**Example 5.13** Let \( D \) be a three-dimensional Ferrers diagram that satisfies the strong projection property.

(a) Let \( \overline{\mathcal{P}}_{x, y} := \mathcal{P}_{x, y}(D) \times [c_D] \). Then the pair \( D \subseteq \overline{\mathcal{P}}_{x, y} \) satisfies the assumptions in Theorem 5.10. Thus, by Lemmas 4.3 and 4.1, we have
\[
\begin{align*}
\text{reg}(\mathcal{F}(I_D)) &\leq \text{reg}(\mathcal{F}(I_{\overline{\mathcal{P}}_{x, y}})) \\
&= a_D + b_D + c_D - 2 - \max\{a_D + b_D - 1 - \text{reg}(\mathcal{F}(I_{\mathcal{P}_{x, y}(D)})), c_D\},
\end{align*}
\]
and
\[
\begin{align*}
\text{e}(\mathcal{F}(I_D)) &\leq \text{e}(\mathcal{F}(I_{\overline{\mathcal{P}}_{x, y}})) = \left(\frac{a_D + b_D + c_D - 3}{c_D - 1}\right) \text{e}(\mathcal{F}(I_{\mathcal{P}_{x, y}(D)})).
\end{align*}
\]
Notice that \( \mathcal{P}_{x, y}(D) \) is a two-dimensional Ferrers diagram. The associated regularity and multiplicity have been computed in Proposition 5.7 and Corollary 5.6 of [5], respectively.

(b) The pair \( D \subseteq \mathcal{X} := [a_D] \times [b_D] \times [c_D] \) satisfies the assumptions in Theorem 5.10. Thus, we have
\[
\begin{align*}
\text{reg}(\mathcal{F}(I_D)) &\leq \text{reg}(\mathcal{F}(I_{\mathcal{X}})) = \min\{a_D + b_D, a_D + c_D, b_D + c_D\} - 2,
\end{align*}
\]
where
and
\[
e(\mathcal{F}(I_D)) \leq e(\mathcal{F}(I_{\chi})) = \left( \begin{array}{c} a_D + b_D + c_D - 3 \\ a_D - 1, b_D - 1, c_D - 1 \end{array} \right),
\]
by the multiplicity and regularity calculated in Propositions 4.2 and 4.4, respectively.

### 6 General case

The main purpose of this section is to show that the inequality (19) can be generalized to the case when \(D\) only satisfies the projection property. Since we drop the stronger assumption in Theorem 5.10, the proof here will be inevitably more involved.

We will use the induction order outlined in Setting 2.9. It gives us a shedding order of the vertex decomposable complex \(\Delta(D)\) by [21, Theorem 4.1]. We will apply the standard techniques summarized in Remark 5.3. As we don’t assume the strong projection property here, the induction order requires a two-stage treatment.

Throughout this section, for a three-dimensional Ferrers diagram \(D\), we write \(\mu_D := \min(a_D + b_D, a_D + c_D, b_D + c_D)\).

**Theorem 6.1** Let \(D\) be a three-dimensional Ferrers diagram that satisfies the projection property. Then,
\[
\text{reg}(\Delta(D)) \leq \mu_D - 2.
\]
In particular, \(r(I_D) = \text{reg}(\mathcal{F}(I_D)) \leq \mu_D - 2\).

**Proof** The “in particular” part follows from Corollary 3.4 and the inequality (20). Thus, in the following, we will focus on proving (20). For this purpose, we prove by induction on \(a_D\); this will be the outer induction process in the proof.

The base case of the outer induction process is when \(a_D = 1\). In this situation, the diagram \(D\) trivially satisfies the strong projection property. Thus, the inequality (20) follows from (19) and Proposition 4.4. Therefore, in the following, we will assume that \(a_D \geq 2\). By symmetry, we will also assume that \(b_D\) and \(c_D\) are at least 2.

To achieve the inequality (20), we use backward induction with respect to the order outlined in Setting 2.9 and prove that
\[
\text{reg}(\Delta(A_u(D))) \leq \mu_D - 2
\]
for each \(u \in D^1\) in the first stage with respect to \(D\). Then, the claimed estimate (20) follows when we choose \(u = (1, 1, 1)\). This will be the inner induction process of this proof.

As a reminder, within this proof, both \(A\) and \(A^+\) are with respect to this induction order, unless explicitly stated otherwise. Notice that this total order gives us a shedding order of the vertex decomposable complex \(\Delta(D)\) by [21, Theorem 4.1]. Hence, we can apply the standard techniques summarized in Remark 5.3. Furthermore, still from the
proof of [21, Theorem 4.1], we can see that \( \Delta(A_v(D)) \) is pure vertex-decomposable and coincides with the restriction of \( \Delta(D) \) to \( A_v(D) \) for each \( v \in D \). Now, we start the induction argument of the inner induction process.

**Base case**
We have two subcases here.

(a) Suppose that the points of the second stage exist, i.e., \( C_2(D) \neq \emptyset \). Then, we need to prove (21) for \( u = (1, 1, c_{D \geq 2} + 1) \), the initial element in the second stage. For that purpose, we flip \( D \) to get \( S(D) \) and write it as \( D' \) for simplicity; see also Setting 2.9. Then, \( S(A_v(D)) \) is precisely \( A_{S(v)}(D') \) for each \( v \in C_2(D) \) in the first stage; the \( A \) in \( A_{S(v)}(D') \) is with respect to the lexicographic order on \( D' \). Consequently, \( \Delta(D', A_{S(v)}(D')) = \Delta(A_{S(v)}(D')) \) is vertex-decomposable and the lexicographic order gives a shedding order (at least for the points in the \( x = 1 \) layer). Since \( b(D)_{\geq 2} < b(D') \) while \( \mu(D) = \mu(D') \), we can apply subsequent Lemma 6.3 to complete the proof.

(b) Suppose instead that \( C_2(D) = \emptyset \). Then, we need to prove (21) for \( u = (2, 1, 1) \), the initial element of the second layer; without loss of generality, we assume that \( D_{\geq 2} \neq \emptyset \). By induction on \( a_D \) in the outer induction process, we surely will have

\[
\text{reg}(\Delta(D_{\geq 2})) \leq \mu(D_{\geq 2}) - 2
\]

from (20). Notice that \( a_{D_{\geq 2}} = a_D - 1 \) while \( b(D_{\geq 2}) \leq b_D \) and \( c(D_{\geq 2}) \leq c_D \). Consequently, \( \mu(D_{\geq 2}) \leq \mu(D) \), establishing the validity of (21) in the base case.

**Induction step**
Now, take a general \( u = (1, j_0, k_0) \in D^1 \) in the first stage. Without loss of generality, we assume that \( u \) is a normal point with respect to \( D \). By induction, we have \( \text{reg}(\Delta(A_u(D))) \leq \mu(D) - 2 \). Thus, in view of Eq. (6) in Remark 5.3, it suffices to show that

\[
\text{reg}(\text{link}_{\Delta(A_u(D))}(T_u)) \leq \mu(D) - 3.
\]

As explicitly shown in the proof of [21, Theorem 4.1], the link complex \( \text{link}_{\Delta(A_u(D))}(T_u) \) is the join of a simplex with \( \Delta(H) \), where the subset

\[
\mathcal{H} := Z_{1}^{2}(D, u) \cup Z_{2}^{2}(D, u) \cup Z_{1}^{1}(D, u) \\
\cup Z_{6}^{1}(D, u) \cup \{(1, j, k) \in D : j \leq j_0 \text{ and } k > c_{D_{\geq 2}}\};
\]

see also Example 6.2. As in the proof of [21, Lemma 4.3], we turn to consider the diagram

\[
\tilde{D} := Z_{3}(D, u) \cup Z_{1}^{1}(D, u) \cup Z_{6}^{1}(D, u) \\
\cup \{(i, j, k) \in D : j \leq j_0 \text{ and } k > \min(\gamma, c_{D_{\geq 2}})\}
\]

for \( \gamma = \gamma_D(u) \). The new concise diagram \( \tilde{D} \) is essentially a three-dimensional Ferrers diagram satisfying the projection property. In addition, \( \tilde{D} \) contains \( \mathcal{H} \) and \( \mathcal{H} = A_u^+(\tilde{D}) \).
The inequality that we want to show is then translated into

$$\text{reg}(\Delta(\mathcal{H})) \leq \mu_D - 3.$$  \hspace{1cm} (24)

(a) Suppose that $\mathcal{H}^1 = \emptyset$. Since we have assumed earlier that $b_D \geq 2$, this happens precisely when $j_0 = b_D > 1$ and $c_{D,2} = c_D$. Whence, $\mathcal{H} = \tilde{D}^z = Z_{1}^{z,2}(D, u) \cup Z_{2}^{z,2}(D, u)$. Note that we always have $k_0 \leq \gamma$.

(i) If $k_0 = \gamma$, then $u$ is a phantom point. But we have assumed that $u$ is a normal point. This cannot happen.

(ii) If $k_0 < \gamma$, then $a_{D,2} = a_D - 1, b_{D,2} \leq b_D$ and $c_{D,2} \leq c_D - (\gamma - k_0) < c_D$. Since $\mathcal{H}$ has fewer points than $A_u(D)$, by induction,

$$\text{reg}(\text{link}_{\Delta(A_u(D))}(T_u)) = \text{reg}(\Delta(\mathcal{H})) = \text{reg}(\Delta(\tilde{D}^{z,2})) \leq \mu_{\tilde{D}^{z,2}} - 2 < \mu_D - 2,$$

establishing the expected inequality (24) in this case.

(b) Suppose instead that $\mathcal{H}^1 \neq \emptyset$. We will prove in Lemma 6.4 that $\text{reg}(\Delta(\mathcal{H})) \leq \mu_{\tilde{D}^{z,2}} - 3$. As obviously $\mu_{\tilde{D}^{z,2}} \leq \mu_D$, we still get the expected inequality (24) in this case. And this completes our proof of Theorem 6.1. \hfill \Box

Before fully proving Theorem 6.1 by establishing the accompanied lemmas, we provide an example with all regions and sub-diagrams involved in the proof of the theorem.

**Example 6.2** Let $D$ be a typical three-dimensional Ferrers diagram that satisfies the projection property. Let $u$ be a point in the first stage. When the subset $\mathcal{H}$ introduced in (22) is non-empty, in Fig. 4, we present the $\mathcal{H}^1$ by the shaded regions with two different cases: $\gamma \leq c_{D,2}$ (left-hand side) and $\gamma > c_{D,2}$ (right-hand side). The six zones are with respect to $D$ and $u$. Meanwhile, Fig. 5 shows the corresponding regions of $\tilde{D}^{z,2}$ for the diagram $\tilde{D}$ introduced in (23). Note that $\mathcal{H}^2 = \tilde{D}^{z,2}$, which is not pictured here.
The following lemma is used for proving Theorem 6.1 when we deal with the points in the second stage of diagram \( D \). It is also needed by Lemma 6.4 in a special case.

**Lemma 6.3** Let \( D \) be a three-dimensional Ferrers diagram that satisfies the projection property. Suppose that \( b_D > b_{D;2} \). Let \( L = \mathcal{A}_w(D) \) with respect to the lexicographic order for the point \( w = (1, b_{D;2} + 1, 1) \in D \). Suppose that for each \( v \in \mathcal{L}_1 \), the restriction complex \( \Delta(D, \mathcal{A}_v(D)) \) is \( \Delta(L, \mathcal{L}) \) is pure vertex-decomposable and the lexicographic order gives a shedding order (at least for the points in the first layer). Then, the regularity of \( \Delta(L) \) is bounded above by \( \mu_D - 3 \).

**Proof** It suffices to prove by backward induction with respect to the lexicographic order that

\[
\reg(\Delta(\mathcal{A}_u(D))) \leq \mu_D - 3 \tag{25}
\]

for each \( u = (1, j_0, k_0) \in \mathcal{L}_1 \).

**Base case**

The base case for this induction process is when we remove \( \mathcal{L}_1 \) and get \( \mathcal{L}_2 = \mathcal{D}_2 \).

Since \( a_D > a_{D;2} \) and \( b_D > b_{D;2} \), we have \( \mu_D > \mu_{D;2} \). Again, as \( a_{D;2} < a_D \), by induction, it is legal to apply (20) to get

\[
\reg(\Delta(D_{\geq 2})) \leq \mu_{D;2} - 2 < \mu_D - 2,
\]

confirming (25) in this base case.

**Induction step**

Consider a general \( u \in \mathcal{L}_1 \). As in the proof of Proposition 5.9, we may assume that \( u \) is a normal point with respect to \( D \) and try to prove that

\[
\reg(\text{link}_{\Delta(\mathcal{A}_u(D))}(T_u)) \leq \mu_D - 4. \tag{26}
\]

Similarly, one can introduce \( \mathcal{H} \) with respect to \( D \) and \( u \). To be more precise, here

\[
\mathcal{H} := \mathcal{Z}_{1;2}^2(D, u) \cup \mathcal{Z}_{3;2}^2(D, u) \cup \mathcal{Z}_{2}^1(D, u) \cup \mathcal{Z}_6^1(D, u);
\]
unlike in (11), we have an additional $Z_1^{\geq 2}(D, u)$, since we don’t assume strong projection property here. Whence, link$_{A_{u}(D)}(T_u)$ is the join of a simplex with $\Delta(D, H) = \Delta(H)$. The expected inequality (26) is then equivalent to

$$\text{reg}(\Delta(H)) \leq \mu_D - 4. \quad (27)$$

Notice that for each lattice point $(i, j, k) \in Z_1^{\geq 2}(D, u) \cup Z_3^{\geq 2}(D, u)$, we have indeed that $j \leq b_{D\geq 2} < j_0$. Thus, an ambient concise diagram for $H$ can be chosen as

$$D' := \{(i, j, k) \in Z_1(D, u) \cup Z_3(D, u) : j < j_0\} \cup Z_5(D, u) \cup Z_6(D, u).$$

This set is essentially a three-dimensional Ferrers diagram that still satisfies the projection property. Notice that

$$H = \begin{cases} A_{1,1,1}(D'), & \text{if } Z_5(D, u) \cup Z_6(D, u) \neq \emptyset, \\ A_{2,1,1}(D'), & \text{otherwise.} \end{cases}$$

Furthermore, as $u$ is not a phantom point and $j_0 > b_{D\geq 2}$, we have $u \in B_2 \cup (D' \setminus B)$ by Remark 5.7. Consequently, $k_0 < \gamma_D(u)$. Thus, $a_{D'} = a_D$, $b_{D'} = b_D - 1$ and $c_{D'} = c_D - (\gamma_D(u) - k_0) < c_D$. Now, by induction, we can apply (25) to get

$$\text{reg}(\Delta(H)) \leq \mu_{D'} - 3 < \mu_D - 3,$$

confirming the expected inequality (27) in this case. This completes our proof of Lemma 6.3. \hfill \Box

In the following lemma, we estimate the regularity associated with the set $H$ given by (22). The proof is similar to that for Theorem 6.1; we still use backward induction with respect to the order introduced in Setting 2.9. Along the proof, we have to reduce the ambient diagram $\tilde{D}$ to a smaller diagram called $\tilde{D}$. We invite the readers to use the figures regarding $\tilde{D}$ in Example 6.5 for reference when reading the proof of Lemma 6.4.

**Lemma 6.4** Under the assumptions in Theorem 6.1, suppose that $u = (1, j_0, k_0) \in D^1$ is a normal point with $k_0 \leq c_{D\geq 2}$. Let $H$ and $\tilde{D}$ be the sets given by (22) and (23), respectively. If $H^1 \neq \emptyset$, then

$$\text{reg}(\Delta(H)) \leq \mu_{\tilde{D}} - 3. \quad (28)$$

**Proof** For notational simplicity, we may assume that $k_0 = \min(\gamma_D(u), c_{D\geq 2})$. Hence, the ambient diagram $\tilde{D}$ is indeed a three-dimensional Ferrers diagram and $c_{\tilde{D}\geq 2} = c_{D\geq 2} \geq k_0$. Let $\prec_{\tilde{D}}$ be the induction order with respect to $\tilde{D}$; see also Setting 2.9. The subset of $H^1$ in the first stage, namely $C_1(\tilde{D}) \cap H$, is precisely $Z_5(D, u) \cup Z_6(D, u)$. Certainly, the subset of $H^1$ in the second stage, namely $C_2(\tilde{D}) \cap H$, is given by $\{(1, j, k) \in \tilde{D} : j \leq j_0$ and $k > c_{\tilde{D}\geq 2}\}$. Notice that $H = A_u^+(\tilde{D}) = A_{v_0}(\tilde{D})$ for some $v_0$. The following observations are clear.
(a) If $\mathcal{H}^1 \cap C_1(\widetilde{D}) \neq \emptyset$, then $v_0 = (1, j_0 + 1, 1)$ is the initial point of $\mathcal{H}^1 \cap C_1(\widetilde{D})$.
(b) If $\mathcal{H}^1 \cap C_1(\widetilde{D}) = \emptyset$ while $\mathcal{H}^1 \cap C_2(\widetilde{D}) \neq \emptyset$, then $v_0 = (1, 1, c_{\bar{D}^2z} + 1)$ is the initial point of $\mathcal{H}^1 \cap C_2(\widetilde{D})$.
(c) If $\mathcal{H}^1 = \emptyset$, then $v_0 = (2, 1, 1)$ is the initial point of $\bar{D}^{z2}$.

Hence, to achieve (28), it suffices to prove

$$\text{reg}(\Delta(A_v(\bar{D}))) \leq \mu_{\bar{D}} - 3$$

(29)

for each $v \in \mathcal{H}^1$ in the first stage with respect to $\bar{D}$. We will prove by backward induction with respect to the induction order $\prec_{\bar{D}}$.

Base case
We have two subcases to check.

(a) Suppose that the points of the second stage exist, i.e., $C_2(\bar{D}) \neq \emptyset$. Then, we need to prove (29) for $v = (1, 1, c_{\bar{D}^2z} + 1)$, the initial element in the second stage.
For that purpose, we flip $\bar{D}$ to get $S(\bar{D})$ and write it as $\bar{D}'$ for simplicity. Then, $S(A_v(\bar{D}))$ is precisely $A_{(1,b_{\bar{D}'^{z2}}+1,1)}(\bar{D}')$ (the $A$ in $A_{(1,b_{\bar{D}'^{z2}}+1,1)}(\bar{D}')$ is with respect to the lexicographic order on $\bar{D}'$). Since $b(\bar{D}'^{z2}) < b_{\bar{D}}$, while $\mu_{\bar{D}} = \mu_{\bar{D}'}$, we can apply Lemma 6.3 to complete the proof (the conditions of Lemma 6.3 are satisfied, as mentioned earlier in the proof of Theorem 6.1).

(b) Suppose instead that $C_2(\bar{D}) = \emptyset$. Then, we need to prove (29) for $v = (2, 1, 1)$, the initial element of the second layer. Since $\mathcal{H}^1 \neq \emptyset$, either $Z_2^1(\bar{D}, u) \cup Z_0^1(\bar{D}, u) \neq \emptyset$ (hence $b_{\bar{D}} > b_{\bar{D}^2z}$) or

$$c_{\bar{D}} > \min(\gamma, c_{\bar{D}^2z}) = k_0.$$  

(30)

for $\gamma = \gamma_{\bar{D}}(u)$. Notice that since $b_{\bar{D}} \geq j_0 \geq b_{\bar{D}^2z}$, if $b_{\bar{D}} = b_{\bar{D}^2z}$, then they coincide with $j_0$. Whence, by the projection property of $\bar{D}$, one must have $\gamma \geq c_{\bar{D}^2z} = c_{\bar{D}^2z}$ (the last equality holds since we have assumed that $k_0 = \min(\gamma, c_{\bar{D}^2z})$ for simplicity). Thus, the inequality in (30) will then imply that $c_{\bar{D}} > c_{\bar{D}^2z}$. In short, we have either $b_{\bar{D}} > b_{\bar{D}^2z}$ and $c_{\bar{D}} \geq c_{\bar{D}^2z}$, or $b_{\bar{D}} \geq b_{\bar{D}^2z}$ and $c_{\bar{D}} > c_{\bar{D}^2z}$. Since $a_{\bar{D}^2z} = a_{\bar{D}} - 1$, in either case, one has $\mu_{\bar{D}^2z} < \mu_{\bar{D}}$, as long as $\mathcal{H}^1 \neq \emptyset$.

Again, since $a_{\bar{D}^2z} = a_{\bar{D}} - 1$, we can assume Theorem 6.1 for $\bar{D}^{z2}$ to get

$$\text{reg}(\Delta(\bar{D}^{z2})) \leq \mu_{\bar{D}^2z} - 2 \leq \mu_{\bar{D}} - 3.$$

Hence, the expected inequality (29) holds in this subcase.

Induction step
Consider a general $v = (1, j_1, k_1) \in \mathcal{H} \cap C_1(\bar{D})$. As in the proof of Theorem 6.1, by induction, the proof is reduced to the case when $v$ is a normal point and we only need to show

$$\text{reg}(\text{link}_{\Delta(A_{s}(\bar{D}))}(T_v)) \leq \mu_{\bar{D}} - 4.$$
Just as $\mathcal{H}$ for $D$ and $u$, we introduce similarly $\tilde{\mathcal{H}}$ for $\tilde{D}$ and $v$. Then the previously expected inequality is simply
\[
\text{reg}(\Delta(\tilde{\mathcal{H}})) \leq \mu_{\tilde{D}} - 4. \tag{31}
\]
More explicitly,
\[
\tilde{\mathcal{H}} := Z_1(\tilde{D}, v)^\geq 2 \cup Z_3(\tilde{D}, v)^\geq 2 \cup Z_5(\tilde{D}, v)^1 \cup Z_6(\tilde{D}, v)^1
\]
\[
\cup \{(1, j, k) \in \tilde{D} : j \leq j_1 \text{ and } k > c_{\tilde{D}^2}\}
\]
as in (22). Write $\gamma(v)$ for $\gamma_{\tilde{D}}(v)$. As $k_1 \leq \gamma(v) \leq k_0 \leq c_{\tilde{D}^2} = c_{D^2}$, we have
\[
\{(1, j, k) \in \tilde{D} : j \leq j_1 \text{ and } k > c_{\tilde{D}^2}\} \subseteq \{(1, j, k) \in \tilde{D} : j \leq j_1 \text{ and } k > \gamma(v)\}
\]
\[
= \{(1, j, k) \in \tilde{D} : j < j_1 \text{ and } k > \gamma(v)\}.
\]
Furthermore, notice that from (23), one can say safely that for any $(i, j, k) \in Z_1(\tilde{D}, v)^\geq 2 \cup Z_3(\tilde{D}, v)^\geq 2$, we have $j \leq j_0 < j_1$. Thus, the ambient diagram for $\tilde{\mathcal{H}}$ can be chosen as the more concise subset
\[
\mathcal{D} := Z_1(\tilde{D}, (1, j_1 - 1, k_1)) \cup Z_3(\tilde{D}, (1, j_1 - 1, k_1)) \cup Z_5(\tilde{D}, v)^1 \cup Z_6(\tilde{D}, v)^1
\]
\[
\cup \{(i, j, k) \in \tilde{D} : j < j_1 \text{ and } k > \gamma(v)\}. \tag{32}
\]
Again, this diagram is essentially a three-dimensional Ferrers diagram that still satisfies the projection property.

If $k_1 = \gamma(v)$, as $j_1 > j_0 \geq b_{\tilde{D}^2}$, $v$ is a phantom point of $\tilde{D}$ by Remark 5.7. But this is against our assumption on $v$. If $k_1 < \gamma(v)$, then $c_{\tilde{D}} = c_{\tilde{D}^2} - (\gamma(v) - k_1) < c_{\tilde{D}}$. Meanwhile, $b_{\tilde{D}} = b_{\tilde{D}} - 1$. Thus, $\mu_{\tilde{D}} \leq \mu_{\tilde{D}} - 1$. Since $\tilde{D}$ is a smaller diagram than $\tilde{D}$, by induction, we can apply the inequality (28) to $\tilde{\mathcal{H}}$ with respect to $\mathcal{D}$ to get
\[
\text{reg}(\Delta(\tilde{\mathcal{H}})) \leq \mu_{\tilde{D}} - 3 \leq \mu_{\tilde{D}} - 4,
\]
confirming the expected inequality (31). And this completes our proof of Lemma 6.4.

\[\Box\]

**Example 6.5** In the proof of Lemma 6.4, we introduced a concise subdiagram $\mathcal{D}$ in (32). As a continuation to Example 6.2, in Fig. 6 we present typical $\mathcal{D}$ by the shaded regions with two different cases: $\gamma \leq c_{D^2}$ (left-hand side) and $\gamma > c_{D^2}$ (right-hand side). These diagrams are deduced from the $\tilde{D}$ in Example 6.2. Unlike in the proof Lemma 6.4, we don’t assume that $k_0 = \min(\gamma_D(u), c_{D^2})$ here.

Finally, we wrap up this paper with some quick observations. Recall that a simplicial complex $\Delta$ is called acyclic (over $\mathbb{K}$) if all the reduced simplicial homology groups $\tilde{H}_i(\Delta)$ are trivial. Cones are known to be acyclic.

**Corollary 6.6** Let $D$ be a three-dimensional Ferrers diagram that satisfies the projection property. Then the associated Stanley–Reisner complex $\Delta(D)$ is acyclic.
Proof Notice that if the Stanley–Reisner ring $\mathbb{K}[\Delta]$ of a simplicial complex $\Delta$ of dimension $d$ is Cohen–Macaulay, then by the Reisner’s criterion [19, Theorem 8.1.6], $\hat{H}_i(\Delta) = 0$ for all $i < d$. Meanwhile, it follows from Hochster’s formula on the local cohomology modules [19, Theorem A.7.3] that $\mathrm{reg}(\mathbb{K}[\Delta]) = \max\{j : \hat{H}_{j-1}(\text{link}_F(\Delta) ; \mathbb{K}) \neq 0 \text{ for some } F \in \Delta\}$; see also the proof of [19, Proposition 8.1.10]. Whence, $\mathrm{reg}(\mathbb{K}[\Delta]) \leq d = \dim(\mathbb{K}[\Delta]) - 1$ if and only if $\hat{H}_d(\Delta) = 0$.

We have shown in [21, Theorem 4.1] that $\Delta(D)$ is Cohen–Macaulay of dimension $a_D + b_D + c_D - 3$. Hence, the claimed result follows from Theorem 6.1. \(\square\)

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