Bayesian Analysis of Bell Inequalities

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Abstract

Statistical tests are needed to determine experimentally whether a hypothetical theory based on local realism can be an acceptable alternative to quantum mechanics. It is impossible to rule out local realism by a single test, as often claimed erroneously. The “strength” of a particular Bell inequality is measured by the number of trials that are needed to invalidate local realism at a given confidence level. Various versions of Bell’s inequality are compared from this point of view. It is shown that Mermin’s inequality for Greenberger-Horne-Zeilinger states requires fewer tests than the Clauser-Horne-Shimony-Holt inequality or than its chained variants applied to a singlet state, and also than Hardy’s proof of nonlocality.

I. Formulation of the problem

Bell inequalities are upper bounds on the correlations of results of distant measurements. These inequalities are obeyed by any local realistic theory, namely a theory that uses local variables with objective values. Since Bell’s original discovery [1], many inequalities of
that type have been published, with various claims of superiority. The purpose of this article is to compare their relative strengths for various quantum states.

In actual experimental tests, there are no infinite ensembles for accurate measurements of mean values. Experimental physicists perform a finite number of tests, and then they state that their results violate the inequality at some confidence level. The problem I wish to discuss here is of a different nature. I am a theorist and I trust that quantum mechanics gives a reliable description of nature. However, I have a friend who is a local realist. We have only a finite number of trials at our disposal. How many tests are needed to make my realist friend feel uncomfortable?

The problem is not whether the validity of a Bell inequality can be salvaged by invoking clever loopholes, as some local realists try to trick us into, but whether there can be any local realistic theory that reproduces the experimental results. When these results are analyzed we have to take into account detector inefficiencies, and this should be done honestly in the same way when our analysis is based on quantum theory or on a local realistic theory. To simplify the discussion, I shall assume that there are ideal detectors, and that the rate at which particles are produced by the apparatus is perfectly known. The disagreement is only on the choice of the correct theory.

Consider a yes-no test. Quantum mechanics (QM) predicts that the probability of the “yes” result is $q$, and an alternative local realistic (LR) theory predicts a probability $r$. An experimental test is performed $n$ times and yields $m$ “yes” results. What can we infer about the likelihood of the two theories? The answer is given by Bayes’s theorem [2]. Denote by $p'_q$ and $p'_r$ the prior probabilities that we assign to the validity of the two theories. These are subjective probabilities, expressing our personal beliefs. For example, if my friend is willing to bet 100 to 1 that LR is correct and QM is wrong, then $p'_r/p'_q = 100$. The question is how many experimental tests are needed to change my friend’s opinion to $p''_r/p''_q = 0.01$ say, before he is driven to bankruptcy.

It follows from Bayes’s theorem that

$$
\frac{p''_r}{p''_q} = \frac{p'_r}{p'_q} \frac{E_r}{E_q},
$$

(1)

where $E_r$ and $E_q$ are the probabilities of the experimentally found result (namely $m$ successes in $n$ trials), according to the two theories. These are, by the binomial theorem,

$$
E_r = \frac{n!}{m!(n-m)!} r^m (1-r)^{n-m},
$$

(2)
\[ E_q = \left\lfloor \frac{n!}{m!(n-m)!} \right\rfloor q^m (1-q)^{n-m}, \]  

whence

\[ \frac{E_q}{E_r} = \left( \frac{q}{r} \right)^m \left[ (1-q)/(1-r) \right]^{n-m}. \]  

I shall call the ratio

\[ D = \frac{E_q}{E_r} \]

the confidence depressing factor for hypothesis LR with respect to hypothesis QM.

II. Greenberger-Horne-Zeilinger state

As a first example, consider the Greenberger-Horne-Zeilinger (GHZ) state \([3, 4]\) for a tripartite system, namely \((|000\rangle - |111\rangle)/\sqrt{2}\), where 0 and 1 denote two orthogonal states of each subsystem. This state is experimentally difficult to produce but its theoretical analysis is quite simple. Three distant observers examine the three subsystems. The first observer has a choice of two tests. The first test can give two different results, that we label \(a = \pm 1\), and likewise the other test yields \(a' = \pm 1\). Symbols \(b, b', c\) and \(c'\) are similarly defined for the two other observers. Any possible values of their results satisfy

\[ a'bc + ab'c + abc' - a'b'c' \equiv \pm 2, \]  

whence it follows that

\[ -2 \leq \langle a'bc + ab'c + abc' - a'b'c' \rangle \leq 2. \]  

This is Mermin’s inequality \([5]\).

Quantum mechanics happens to make a very simple prediction for the GHZ state: there are well chosen tests that give with certainty

\[ a'bc = ab'c = abc' = -a'b'c' = 1. \]  

Naturally, performing any such test can verify the value 1 for only one of these products, since each product corresponds to a different experimental setup. Yet, if we take all these results together they manifestly conflict with Eq. (3), and many authors \([6]\) have stated
that a single experiment is sufficient to invalidate local realism. This is sheer nonsense: a single experiment can only verify one occurrence of one of terms in (8).

Let us return to our realist friend. He believes that, in each experimental run, each term in Eq. (8) has a definite value even if that term is not actually measured in that run. Let us therefore ask him to propose just a rule giving the average values of the products $a'bc$, etc., that appear in Eq. (7). How many tests are needed for depressing his confidence in that rule by a factor $10^4$, say?

The most successful LR theory, namely the one that gives the least depressing factor, is to assume that

$$\langle a'bc \rangle = \langle ab'c \rangle = \langle abc' \rangle = \langle -a'b'c' \rangle = 0.5.$$ (9)

This obviously attains the right hand side of Mermin’s inequality (7). The LR prediction thus is that if we measure $a'bc$, we shall find the result 1 (i.e., “yes”) in 75% of cases, and the opposite result in 25%; and likewise for the other tests. We thus have, with the notations introduced above, $q = 1$ and $r = 0.75$. For $n$ tests, with ideal detectors, we have $m = n$ (I am assuming here that quantum theory is correct), and the depressing factor in Eq. (4) is $0.75^n$. For example, 32 tests give $D \simeq 10^4$, as required.

III. The singlet state

The second example involves just two correlated quantum systems far away from each other. An observer, located near one of the systems, has a choice of several yes-no tests, labelled $A_1$, $A_3$, $A_5$, etc. Likewise, another observer, near the second system, has a choice of several yes-no tests, $B_2$, $B_4$, $B_6$ . . . Let $p(A_iB_j)$ denote the probability that tests $A_i$ and $B_j$ give the same result (both “yes” or both “no”). It was shown long ago by Clauser, Horne, Shimony, and Holt (CHSH) [7] that local realism implies

$$p(A_1B_2) + p(B_2A_3) + p(A_3B_4) \geq p(A_1B_4).$$ (10)

(In the original paper [7], this equation was written in terms of correlations, namely $2p - 1$, but it is much simpler to use probabilities, as here.) More generally, Braunstein and Caves [8] derived chained Bell inequalities that can be written

$$p(A_1B_2) + p(B_2A_3) + \cdots + p(A_{2k-1}B_{2k}) \geq p(A_1B_{2k}).$$ (11)
There are \((k!)^2\) independent inequalities of that type, obtainable by relabelling the various \(A_i\) and \(B_j\). Local realism guarantees that all these inequalities are satisfied.

Consider a pair of spin-\(\frac{1}{2}\) particles in the singlet state (similar results hold for maximally entangled pairs of polarized photons, except that all the angles mentioned below should be halved). Each observer can measure a spin component along one of \(k\) possible directions, as illustrated in Fig. 1, where the angle between consecutive directions is \(\theta = \pi/2k\). Quantum theory predicts that each one of the probabilities on the left hand side of Eq. (11) is \(q = (1 - \cos \theta)/2\), and the probability on the right hand side is \(1 - q\). These predictions manifestly violate Eq. (11).

What could be the predictions of an alternative theory, based on local realism? These predictions have to satisfy Eq. (11). The closest they can approach quantum theory is when equality holds in the latter equation. Moreover, it is reasonable to assume that all the terms on the left hand side are equal (this follows from rotational symmetry, and it can also be shown that any deviation from this symmetry would only increase the depression factor \(D\)). Let \(r\) be the common value of all these terms. Then the right hand side of Eq. (11) has to be \(1 - r\) (again, because of rotational symmetry and because the spin projection along \(\beta_{2k}\) is opposite to that along \(\beta = 0\)). It follows that in a local realistic theory which mimics as closely as possible quantum mechanics and saturates the inequality (11), we have \((2k - 1)r = 1 - r\), whence \(r = 1/2k = \theta/\pi\), where \(\theta\) is the angle between consecutive rays. This is indeed the result obtainable from a crude semi-classical model, where a spinless system splits into two fragments with opposite angular momenta [9]. Quantum theory, on the other hand, predicts for the same angle \(\theta\) a probability \(q = (1 - \cos \theta)/2\) that both observers obtain the same result.

We thus have now definite predictions, from quantum theory and from an alternative local realistic theory. To distinguish experimentally between these two claims, we test \(n\) pairs of particles prepared in the singlet state. Let \(m\) be the number of “yes” answers. If \(m \simeq qn\) (that is, if quantum theory is experimentally correct), it follows from Eq. (11) that

\[
D = \left[ \left( \frac{q}{r} \right)^q \left( \frac{1 - q}{1 - r} \right)^{1-q} \right]^n. \tag{12}
\]

For example, if we wish to have \(D \simeq 10^4\) as before, we obtain \(n \simeq 287\) for \(k = 2\) (the case that was investigated by CHSH), and \(n \simeq 200\) for the more efficient configuration of
Braunstein and Caves with \( k = 4 \) (higher values of \( k \) would require a higher number of tests for giving the same depression factor \( D \)).

IV. Hardy’s proof of nonlocality

Finally, let us examine Hardy’s proof of nonlocality “without inequalities” [10, 11], which was called by Mermin “the best version of Bell’s inequality” [12]. It will be shown that this version is not stronger than the preceding ones. Stripped of all its technical details, Hardy’s paradox can be formulated as follows. There are four alternative setups as in the CHSH case, but each setup requires only one detector. The first observer has a choice of using detectors \( A \) or \( A' \), the second observer may use \( B \) or \( B' \). Detector coincidences will be labelled \( C_j \), with \( j = 1, \ldots, 4 \). Explicitly,

\[
C_1 = A \land B, \quad C_2 = A \land B', \quad C_3 = A' \land B, \quad C_4 = 0
\]

and \( C_4 \) means that in the fourth setup neither \( A' \) nor \( B' \) is excited. Other types of coincidences are not relevant in the following discussion.

Local realism implies that the probabilities \( p(C_j) \) satisfy the Clauser-Horne (CH) inequality [13],

\[
P(C_1) \leq p(C_2) + p(C_3) + p(C_4).
\]

On the other hand, quantum mechanics predicts that, for well chosen states and tests, these probabilities are \( p(C_2) = p(C_3) = p(C_4) = 0 \) and

\[
P(C_1) \equiv q = [(\sqrt{5} - 1/2)]^5 = 0.09017,
\]

so that the CH inequality is violated.

As in the preceding cases, let our LR friend propose a new set of probabilities \( r_j \) that satisfy the CH inequality. For example, a simple possibility is to postulate that all the \( r_j \) vanish (this assumption is implicit in Hardy’s proof). Then LR and QM agree for setups 2, 3, and 4, and we only have to test experimentally setup 1. According to QM, the probability of finding \( n \) consecutive “no” results (in agreement with the LR prediction) is

\[
(1 - q)^n. \quad \text{This is less than 50\% after only 8 trials. The hypothesis that all the } r_j \text{ vanish is obviously untenable, and this is why Hardy’s proof is usually considered as quite strong.}
\]
However, there is a more sophisticated way to defend local realism. Let us assume that $r_2 = r_3 = r_4 = r_1/3$, so that the CH inequality \[14\] is saturated, and let us optimize the value of $r_1 \equiv r$. There are now two types of experimental tests. Those with setup 1 lead to a value of $D$ given by Eq. \[12\]. On the other hand, setups 2, 3, and 4 have $q = 0$ and then Eq. \[14\] gives, with $r$ replaced by $r/3$, the result $D = (1 - r)^{-n}$. To invalidate local realism, we shall obviously choose the setup that minimizes $n$, the number of required tests. Therefore the best that a LR theorist can do is to choose $r$ so as to equate these two values of $D$. A straightforward calculation then gives $r = 0.03358$ and $n \simeq 270$.

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Fig. 1. The Braunstein-Caves configuration for chained Bell inequalities: there are $k$ alternative directions along which each observer can measure a spin projection.
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