On Warped Product Gradient Yamabe Soliton

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Abstract

In this paper, we provide a necessary and sufficient conditions for the warped product $M = B \times_f F$ to be a gradient Yamabe soliton when the base is conformal to an $n$-dimensional pseudo-Euclidean space, which are invariant under the action of an $(n-1)$-dimensional translation group, and the fiber $F$ is scalar-constant. As application, we obtain solutions in steady case with fiber scalar-flat. Besides, on the warped product we consider the potential function as separable variables and obtain some characterization of the base and the fiber.

1 Introduction and main statements

A Yamabe soliton is a pseudo-Riemannian manifold $(M, g)$ admitting a vector field $X \in \mathfrak{X}(M)$ such that

$$(S_g - \rho)g = \frac{1}{2} \mathfrak{L}_X g,$$

(1)

where $S_g$ denotes the scalar curvature of $M$, $\rho$ is a real number and $\mathfrak{L}_X g$ denotes the Lie derivative of the metric $g$ with respect to $X$. We say that $(M^n, g)$ is shrinking, steady or expanding, if $\rho > 0$, $\rho = 0$, $\rho < 0$, respectively. When $X = \nabla h$ for some smooth function $h \in C^\infty(M)$, we say that $(M^n, g, \nabla h)$ is a gradient Yamabe soliton with potential function $h$. In this case the equation (1) turns out

$$(S_g - \rho)g = \text{Hess}(h),$$

(2)

where $\text{Hess}(\phi)$ denote the hessian of $\phi$. When $\phi$ is constant, we call it a trivial Yamabe soliton.

Yamabe solitons are self-similar solutions for the Yamabe flow

$$\frac{\partial}{\partial t}g(t) = -R_{g(t)}g(t),$$

and are important to understand the geometric flow since they can appear as singularity models. It has been known that every compact gradient Yamabe soliton is of constant scalar curvature, hence, trivial, since $f$ is harmonic, see [14], [15], [18]. For the non-compact case many interesting results is obtained in [10], [11], [23], [24], [28].

As pointed in [9] and [22], it is important to emphasize here that although the Yamabe flow are wellposed in the Riemannian setting, they do not necessarily exist in the semi-Riemannian case, where even the existence of short-time solutions is not guaranteed in general due to the lack of parabolicity. However, the existence of self-similar solutions of the flow is equivalent to the existence of Yamabe solitons as in [11]. Semi-Riemannian Yamabe solitons have been intensively studied, showing many differences with respect to the Riemannian case, see for instance [3] and [9].
Brozos-Vázquez et al. in [8], obtain a local characterization of pseudo Riemannian manifold endowed with gradient Yamabe soliton metric, its results establish that if a pseudo Riemannian gradient Yamabe soliton \((M, g)\), with potential function \(h\) and such that \(|\nabla h| \neq 0\) is locally isometric to warped product of unidimensional base and scalar-constant fiber. In the Riemannian context a global structure result was given in [10].

In [15] Daskalopoulos and Sesum investigated gradient Yamabe soliton and proved that all complete locally conformally flat gradient Yamabe solitons with positive sectional curvature are rotationally symmetric. Proceeding in the same locally conformally flat context, Neto and Tenenblat in [4] consider the study of pseudo Riemannian manifold \(\left(\mathbb{R}^n, \frac{1}{\varphi^2}g_0\right)\), where \(g_0\) is the canonical pseudo metric, and obtain a necessary and sufficient condition to this manifold be a gradient Yamabe soliton. In the search for invariant solutions they consider the invariant action of an \((n-1)\)-dimensional translation group and exhibit a complete solution in steady case.

Recently, Pina and De Sousa in [26], consider the study of gradient Ricci solitons on warped product structure \(M = B^n \times_f F^m\), where the base is conformal to an \(n\)-dimensional pseudo-Euclidean space, invariant under the action of an \((n-1)\)-dimensional translation group, the fiber chosen to be an Einstein manifold and potential function \(h\) depending only on the base and they give a necessary an sufficient condition for \(M\) to be a gradient Ricci soliton.

As far as we know, there are no results for gradient Yamabe solitons related to its potential function in the warped products of two Riemannian manifolds of arbitrary dimensions. Thus, in this paper we consider the study of gradient Yamabe solitons with warped product structure, where we choose the fiber with dimension greater than 1. Initially we provide a sufficient condition for the potential function on warped product depends only on the base.

**Proposition 1.1.** Let \(M = B \times_f F\) be a warped product manifold with metric \(\tilde{g}\). If the metric \(\tilde{g} = g_B \oplus f^2 g_F\) is a gradient Yamabe soliton with potential function \(h : M \rightarrow \mathbb{R}\) and there exist a pair of orthogonal vectors \((X_i, X_j)\) of the base \(B\), such that \(\text{Hess}_{g_B}(f)(X_i, X_j) \neq 0\), then the potential function \(h\) depends only on the base.

**Remark 1.2.** The previous proposition extend for Yamabe solitons the result obtained in [19] where the authors studied warped product gradient Ricci solitons with one-dimensional base.

Motivated by natural extension of Ricci solitons given by Rigoli, Pigola and Setti in [27] the authors Barbosa and Ribeiro in [2], define the concept of *Almost Yamabe soliton* allowing the constant \(\rho\) in definition of Yamabe soliton \((\mathbb{1})\) to be a differentiable function on \(M\). The following example was obtained by Barbosa and Ribeiro in [2], where the manifold is endowed with a warped product metric.

**Example 1.3.** Let \(M^{n+1} = \mathbb{R} \times_{\cosh t} S^n\) with metric \(g = dt^2 + \cosh^2 t g_0\), where \(g_0\) is the canonical metric of \(S^n\). Taking \((M^{n+1}, g, \nabla h, \rho\), where \(h(t, x) = \sinh t\) and \(\rho(t, x) = \sinh t + n\). A straightforward computation gives that \(M^{n+1} = \mathbb{R} \times_{\cosh t} S^n\) is a noncompact almost gradient Yamabe soliton.

In what follows, inspired by Proposition \(\mathbb{1}\), we consider a warped product gradient Yamabe soliton \(M = B \times_f F\) with potential function \(h\) splitting of the form

\[
h(x, y) = h_1(x) + h_2(y), \text{ where } h_1 \in \mathcal{C}^\infty(B) \text{ and } h_2 \in \mathcal{C}^\infty(F),
\]

and get the following characterization theorem.
Theorem 1.4. Let $M = B \times_f F$ be a warped product manifold with metric $\tilde{g} = g_B \oplus f^2 g_F$, and gradient Yamabe soliton structure with potential function $h : B \times F \to \mathbb{R}$ given by (5), then one of the following cases occurs

(a) $M$ is the Riemannian product between a trivial gradient Yamabe soliton and a gradient Yamabe soliton.

(b) $M$ is the Riemannian product between two gradient Yamabe solitons.

(c) $M$ is the warped product between a Almost gradient Yamabe solitons and a trivial gradient Yamabe soliton.

This characterization theorem shows us that if we take the potential function depending only on the base then the fiber $F$ is of constant scalar curvature. In what follows we will take a warped product gradient Yamabe soliton with potential function of the form $h(x,y) = h_1(x) + \text{constant}$, the base conformal to an $n$-dimensional pseudo-Euclidean space, and the fiber chosen to be a scalar-constant space. More precisely, let $(\mathbb{R}^n, g)$ be the pseudo-Euclidean space, $n \geq 3$ with coordinates $x = (x_1, \ldots, x_n)$ and $g_{ij} = \delta_{ij}\epsilon_i$, and let $M = (\mathbb{R}^n, \tilde{g}) \times_f F^n$ be a warped product where $\tilde{g} = \frac{1}{\varphi^2} g_F$, $F$ a semi-Riemannian scalar-constant manifold with curvature $\lambda_F$, $m \geq 1$, $f, \varphi$, $h : \mathbb{R}^n \to \mathbb{R}$, smooth functions, and $f$ is a positive function. Then we obtain necessary and sufficient conditions for the warped product metric $g_B \oplus f^2 g_F$ to be a gradient Yamabe soliton.

Theorem 1.5. Let $(\mathbb{R}^n, g)$ be a pseudo-Euclidean space, $n \geq 3$ with coordinates $x = (x_1, \ldots, x_n)$ and $g_{ij} = \delta_{ij}\epsilon_i$, and let $M = (\mathbb{R}^n, \tilde{g}) \times_f F^n$ be a warped product where $\tilde{g} = \frac{1}{\varphi^2} g_F$, $F$ a semi-Riemannian scalar-constant manifold with curvature $\lambda_F$, $m \geq 1$, $f, \varphi$, $h : \mathbb{R}^n \to \mathbb{R}$, smooth functions, and $f$ is a positive function. Then the warped product metric $\tilde{g}$ is a gradient Yamabe soliton with potential function $h$ if, and only if, the functions $f, \varphi, h$ satisfy

\begin{equation}
\begin{aligned}
 h_{,x_i} + \frac{\varphi_{x_i}}{\varphi} h_{,x_i} + \frac{\varphi_{x_j}}{\varphi} h_{,x_j} = 0 \quad i \neq j,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
 \left[ (n-1) \left( 2\varphi \sum_k \varepsilon_k \varphi_{,x_k} - n \sum_k \varepsilon_k \varphi_{,x_k}^2 \right) + \lambda_F \frac{f^2}{f} - \frac{2m}{f} \left( \varphi^2 \sum_k \varepsilon_k f_{,x_k} - (n - 2) \varphi \sum_k \varepsilon_k \varphi_{,x_k} f_{,x_k} \right) + \right. \\
 - \frac{m(m-1)}{f^2} \varphi^2 \sum_k \varepsilon_k f_{,x_k}^2 - \rho \left. \right] \varepsilon_i \varphi^2 = h_{,x_i} + 2 \frac{\varphi_{x_i}}{\varphi} h_{,x_i} - \varepsilon_i \sum_k \varepsilon_k \frac{\varphi_{,x_k}^2}{\varphi} h_{,x_k} \quad i = j,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
 (n-1) \left( 2\varphi \sum_k \varepsilon_k \varphi_{,x_k} - n \sum_k \varepsilon_k \varphi_{,x_k}^2 \right) + \lambda_F \frac{f^2}{f} - \frac{2m}{f} \left( \varphi^2 \sum_k \varepsilon_k f_{,x_k} - (n - 2) \varphi \sum_k \varepsilon_k \varphi_{,x_k} f_{,x_k} \right) + \\
 - \frac{m(m-1)}{f^2} \varphi^2 \sum_k \varepsilon_k f_{,x_k}^2 - \rho = \frac{\varphi^2}{f} \sum_k \varepsilon_k f_{,x_k} h_{,x_k}.
\end{aligned}
\end{equation}

In order to obtain solutions for equations in Theorem 1.5, we consider $f$, $\varphi$ and $h$ invariant under the action of an $(n-1)$-dimensional translation group, and $\xi = \sum_{i=1}^n \alpha_i x_i, \alpha_i \in \mathbb{R}$, be a basic invariant for the $(n-1)$-dimensional translation group, then we obtain
Theorem 1.6. Let \((\mathbb{R}^n, g)\) be a pseudo-Euclidean space, \(n \geq 3\) with coordinates \(x = (x_1, \ldots, x_n)\), \(g_{ij} = \delta_{ij}\varepsilon_i\), and let \(M = (\mathbb{R}^n, \bar{g}) \times_f F^m\) be a warped product where \(\bar{g} = \frac{1}{\rho^2}g\), \(F\) a semi-Riemannian scalar-constant manifold with curvature \(\lambda_F\), \(m \geq 1\), \(f, \varphi, h : \mathbb{R}^n \rightarrow \mathbb{R}\), smooth functions and \(f > 0\). Consider the functions \(f(\xi), \varphi(\xi)\) and \(h(\xi)\), where \(\xi = \sum_{k=1}^n \alpha_k x_k, \alpha_k \in \mathbb{R}\) and \(\sum_{k=1}^n \varepsilon_k \alpha_k^2 = \varepsilon_{k_0}\) or \(\sum_{k=1}^n \varepsilon_k \alpha_k^2 = 0\). Then the warped product metric \(\bar{g}\) is a gradient Yamabe soliton with potential function \(h\) if, and only if, \(f, h\) and \(\varphi\), satisfy

\[
h'' + 2\frac{\varphi'h'}{\varphi} = 0 \tag{7}
\]

\[
\varepsilon_{k_0}[(n-1)(2\varphi\varphi'' - n(\varphi')^2) - 2\frac{m}{f}(\varphi^2f'' - (n-2)\varphi'f') - \frac{m(m-1)}{f^2}\varphi^2(f')^2 + \varphi'h'\varphi] = \rho - \lambda_F \frac{f'}{f^2} \tag{8}
\]

\[
\varepsilon_{k_0}[(n-1)(2\varphi\varphi'' - n(\varphi')^2) - 2\frac{m}{f}(\varphi^2f'' - (n-2)\varphi'f') - \frac{m(m-1)}{f^2}\varphi^2(f')^2 - \frac{\varphi^2}{f}f'h'] = \rho - \lambda_F \frac{f'}{f^2} \tag{9}
\]

when \(\sum_{k=1}^n \varepsilon_k \alpha_k^2 = \varepsilon_{k_0}\).

And

\[
h'' + 2\frac{\varphi'h'}{\varphi} = 0 \tag{10}
\]

\[
\rho - \frac{\lambda_F}{f^2} = 0 \tag{11}
\]

when \(\sum_{k=1}^n \varepsilon_k \alpha_k^2 = 0\).

It is interesting to know how geometry of the fiber manifold \(F\) affects the geometry of the warped product \(M = (\mathbb{R}^n, \bar{g}) \times_f F^m\). He in [16] has shown that any complete steady gradient Yamabe soliton on \(\mathbb{R} \times_f F\) is necessarily isometric to the Riemannian product with constant \(f\) and \(F\) being of zero scalar curvature. Moreover, he showed that there is no complete steady gradient Yamabe soliton on \(\mathbb{R} \times_f F^n\) with \(n \geq 2\) and \(F\) positive scalar constant manifold.

As consequence of Theorem 1.6 in the context of lightlike vector invariance and scalar-constant fiber, we prove that if \(F\) has positive scalar constant curvature then there is no shrinking or steady gradient Yamabe soliton \(M = (\mathbb{R}^n, \bar{g}) \times_f F^m\) and when \(F\) has negative constant scalar curvature, there is no expanding or steady gradient Yamabe soliton \(M = (\mathbb{R}^n, \bar{g}) \times_f F^m\), this is translated into the following corollary.

Corollary 1.7. In the context of Theorem 1.6, if \(X = \sum_k \alpha_k \frac{\partial}{\partial x_k}\) is a lightlike vector, assume that \(\lambda_F > 0\), then there is no expanding or steady gradient Yamabe soliton with warped metric \(\bar{g}\) and potential function \(h\). Similarly, if we assume that \(\lambda_F < 0\), then there is no shrinking or steady gradient Yamabe soliton with warped metric \(\bar{g}\) and potential function \(h\).
Now, by equations (7) and (10) in Theorem 1.6 we easily see that a necessary condition for \( M = (\mathbb{R}^n, \tilde{g}) \times_f F^m \) be a gradient Yamabe soliton with invariant solution \( f(\xi), \varphi(\xi) \) and \( h(\xi) \), where \( \xi = \sum_{k=1}^n \alpha_k x_k, \alpha_k \in \mathbb{R} \) is that \( h \) is a monotone function. That is,

\[
h'(\xi) = \frac{\alpha}{\varphi^2(\xi)},
\]

for some \( \alpha \in \mathbb{R} \).

We provide solutions for ODE in Theorem 1.6 in two cases: \( h' = 0 \) and \( h' \neq 0 \), with metric \( \tilde{g} = g_B \oplus f^2 g_F \) be a steady gradient Yamabe soliton, i.e. \( \rho = 0 \), and \( F \) is a scalar-flat pseudo-Riemannian manifold.

**Theorem 1.8.** In the context of Theorem 1.6, if \( \sum_{k=1}^n \varepsilon_k \alpha_k^2 = \varepsilon_{k_0} \neq 0 \) and \( F \) scalar-flat fiber, then the warped product metric \( \tilde{g} \) is a steady gradient Yamabe soliton with potential function \( h \) and \( h' \neq 0 \) if, and only if, \( f, h \) and \( \varphi \), satisfies

\[
f(\xi) = \frac{e^c}{\varphi(\xi)} ,
\]

\[
h(\xi) = \alpha \int \frac{1}{\varphi^2(\xi)} d\xi ,
\]

\[
(n + m - 1)(n + m + 2) \int \frac{\varphi d\varphi}{\alpha - \frac{23(\alpha + m - 1)(n + m + 2)}{n + m - 2} - \frac{2n + m + 1}{2}} = \xi + \nu , \quad c \in \mathbb{R}
\]

where \( c, \nu, \alpha, \beta \in \mathbb{R} \) with \( \alpha \neq 0 \).

**Theorem 1.9.** In the context of Theorem 1.6, if \( \sum_{k=1}^n \varepsilon_k \alpha_k^2 = \varepsilon_{k_0} \neq 0 \) and \( F \) scalar-flat fiber, then, given a smooth function \( \varphi > 0 \), the warped product metric \( \tilde{g} \) is a steady gradient Yamabe soliton with potential function \( h \) and \( h' = 0 \) if, and only if, \( f, h \) and \( \varphi \), satisfies

\[
h(\xi) = \text{constant},
\]

\[
f = \varphi^{\frac{n-m+2}{m+2}} e^{\int z_p d\xi} \left( \int e^{-(m+1) \int z_p d\xi} + \frac{2}{m+1} \right)^{\frac{1}{m+1}}
\]

for an appropriate function \( z_p \).

In the null case \( \sum_{k=1}^n \varepsilon_k \alpha_k^2 = 0 \) we obtain

**Theorem 1.10.** In the context of Theorem 1.6, if \( \sum_{k=1}^n \varepsilon_k \alpha_k^2 = 0 \) and \( F \) scalar-flat fiber, then, given smooth functions \( \varphi(\xi) \) and \( f(\xi) \), the warped product metric \( \tilde{g} \) is a steady gradient Yamabe soliton with potential function \( h \) if, and only if

\[
h(\xi) = \alpha \int \frac{1}{\varphi^2(\xi)} d\xi .
\]

**Remark 1.11.** As pointed in [12], see Theorem 3.6, a necessary and sufficient condition to warped product \( B \times_f F \) be a conformally flat is that the function \( f \) defines a global conformal deformation such that \( (B, \frac{1}{2} g_B) \) is a space of constant curvature \( c \) and \( F \) has constant curvature \( -c \). With this observation, we see that the solutions of theorems 1.8, 1.9 and 1.10 defines a non locally conformally flat metric if the warping function \( f \) is not constant.
Remark 1.12. As we can see in the proof of Theorem 1.6, if $\rho$ is a function defined only on the base, then we can easily extend Theorem 1.6 into context of almost gradient Yamabe solitons. In the particular case of lightlike vectors there are infinitely many solutions, that is, given $\varphi$ and $f$

$$\rho(\xi) = \frac{\lambda_F}{f(\xi)^2}$$

$$h(\xi) = \alpha \int \frac{1}{\varphi^2(\xi)} d\xi$$

provide a family of almost gradient Yamabe soliton with warped product structure.

Before proving our main results, we present some examples illustrating the above theorems.

Example 1.13. In Theorem 1.8 consider $\beta = 0$, then we have

$$f(\xi) = e^{c\sqrt{(n-1)(n+m+2)}}$$

$$h(\xi) = \frac{\alpha}{2}(n-1)(n+m+2) \ln |\xi + \nu|$$

$$\varphi(\xi) = \sqrt{\frac{2\alpha(\xi + \nu)}{(n-1)(n+m+2)}}$$

where $\alpha(\xi + \nu) > 0$ and $c \in \mathbb{R}$. Thus, the metric $\tilde{g} = \frac{1}{\varphi}g_0 \oplus f^2g_F$ is a steady gradient Yamabe soliton defined in the semi-space of Euclidean space $\mathbb{R}^n$ with potential function $h$.

Example 1.14. In Theorem 1.9 consider the warped product $M = (\mathbb{R}^2, g) \times F^3$. Given the function $\varphi(\xi) = e^{\frac{3\xi^2}{4}}$, we have that

$$\tilde{g} = e^{-\frac{3\xi^2}{4}}g_0, \quad h(\xi) = \text{constant}, \quad f(\xi) = e^{\xi}$$

defines a steady gradient Yamabe soliton in warped metric.

Example 1.15. In Theorem 1.10 consider the Lorentzian space $(\mathbb{R}^n, g)$ with coordinates $(x_1, \ldots, x_n)$ and signature $\epsilon_1 = -1$, $\epsilon_k = 1$ for all $k \geq 2$, and $F^m$ a complete scalar flat manifold. Let $\xi = x_1 + x_2$ and choose $\varphi(\xi) = \frac{1}{1+\xi^2}$. Then, for $\alpha \neq 0$

$$\tilde{g} = (1 + \xi^2)^2 g, \quad h(\xi) = \alpha \left( \xi + \frac{2}{3}\xi^3 + \frac{1}{5}\xi^5 \right), \quad f \in C^\infty$$

defines a steady gradient Yamabe soliton $(\mathbb{R}^n, \tilde{g}) \times F^m, g_{\text{flat}}$ with potential function $h$ and warping function $f$. Observe that, since the conformal function $\varphi$ is bounded, we have that $\tilde{g}$ is complete, and consequently $\tilde{g} = \frac{1}{\varphi^2}g_0 \oplus f^2g_F$ is complete.
2  Proofs of the Main Results

Proof of Proposition 1.1. Let $M = B \times_f F$ be a gradient Yamabe soliton with potential function $h : M \to \mathbb{R}$, then by equation (2), we obtain

$$ (S_g - \rho)g = Hess(h) $$  \hspace{1cm} (17) 

Now, it is well known that on warped metric $\tilde{g}$ the scalar curvature of base $B$, fiber $F$ and $M$ is related by (see chapter 7 of [25])

$$ S_g = S_{g_B} + \frac{S_{g_F}}{f^2} - \frac{2d}{f} \Delta^B f - d(d-1) \frac{\langle grad_B f, grad_B f \rangle}{f^2} $$  \hspace{1cm} (18) 

where $\Delta^B$ denote the laplacian defined on $B$. Then considering $X_1, X_2, \ldots, X_n \in \mathcal{L}(B)$ and $Y_1, Y_2, \ldots, Y_m \in \mathcal{L}(F)$, where $\mathcal{L}(B)$ and $\mathcal{L}(F)$ are respectively the space of lifts of vector fields on $B$ and $F$ to $B \times F$, we obtain substituting equation (18) in (17) that

\[
\begin{align*}
\left\{ \left( S_{g_B} + \frac{S_{g_F}}{f^2} - \frac{2d}{f} \Delta^B f - d(d-1) \frac{\langle grad_B f, grad_B f \rangle}{f^2} - \rho \right) g_B(X_i, X_j) = Hess g_B(X_i, X_j) \right. \\
\left. \left( S_{g_B} + \frac{S_{g_F}}{f^2} - \frac{2d}{f} \Delta^B f - d(d-1) \frac{\langle grad_B f, grad_B f \rangle}{f^2} - \rho \right) g_f(Y_i, Y_j) = Hess g_f(Y_i, Y_j) \right. \\
\left. \left( S_{g_B} + \frac{S_{g_F}}{f^2} - \frac{2d}{f} \Delta^B f - d(d-1) \frac{\langle grad_B f, grad_B f \rangle}{f^2} - \rho \right) f^2 g_f(Y_i, Y_j) = Hess g_f(Y_i, Y_j). \right.
\end{align*}
\]

Thus, using the fact $\tilde{g}(X_i, Y_j) = 0$, we obtain by expression (ii) that

$$ Hess(h)(X_i, Y_j) = 0. $$

Now, using Lemma 2.1 of [17], we obtain

$$ h(x, y) = z(x) + f(x)v(y) $$  \hspace{1cm} (19) 

where $z : B \to \mathbb{R}$ and $v : F \to \mathbb{R}$. Then since

$$ Hess g_B(X_i, X_j) = Hess g_B h(X_i, X_j), $$

we have by expression (i) that

$$ \left( S_{g_B} + \frac{S_{g_F}}{f^2} - \frac{2d}{f} \Delta^B f - d(d-1) \frac{\langle grad_B f, grad_B f \rangle}{f^2} - \rho \right) g_B(X_i, X_j) = Hess g_B h(X_i, X_j). $$  \hspace{1cm} (20) 

We have from (19) that the right size of (20) is given by

$$ Hess g_B z + v Hess g_B f. $$  \hspace{1cm} (21) 

By hypothesis, there are two orthogonal vector fields $X_i, X_j$ such that $Hess g_B f(X_i, X_j) \neq 0$. Then combining this fact with equations (20) and (21), we obtain

$$ v = \frac{Hess g_B z(X_i, X_j)}{Hess g_B f(X_i, X_j)}. $$  \hspace{1cm} (22) 

This show that $v(y)$ is constant, and then by expression (19) we have that $h$ depends only on the base. 

\[ \square \]
Proof of Theorem 1.4 Let $M = B \times_f F$ be a warped product with gradient Yamabe soliton structure and potential function $h(x, y) = h_1(x) + h_2(y)$. In the same way as in the proof of Proposition 1.1 for $X_1, X_2, \ldots, X_n \in \mathcal{L}(B)$ and $Y_1, Y_2, \ldots, Y_m \in \mathcal{L}(F)$ we obtain

$$\text{Hess}(h)(X_i, Y_j) = 0.$$ 

As we know, the connection of warped product is particularly simple, that is, for $X \in \mathcal{L}(B)$ and $Y \in \mathcal{L}(F)$, we have

$$\nabla_X Y = \nabla_Y X = \frac{X(f)}{f} Y.$$ 

Thus,

$$\text{Hess}(h)(X_i, Y_j) = X_i(Y_j(h)) - (\nabla_X Y_j)h = X_i(Y_j(h)) - \frac{X_i(f)}{f} Y_j = 0.$$ 

Establishing the notation $h_{,x_i} = X_i(h), h_{,x_i,x_j} = X_j(X_i(h))$, we have that

$$h_{,y_jx_i} - \frac{f_{x_i}}{f} h_{,y_j} = 0 - \frac{f_{x_i}}{f}(h_2)_{,y_j} = 0 \quad \forall i, j.$$ 

Then, $f$ is constant or $h(x, y) = h_1(x) + \text{constant}$. We separate the proof in three cases:

Case(I) : ($f$ is constant and $h(x, y) = h_1(x) + \text{constant}$). In this case, $M = B \times_f F$ is a Riemannian product and we have

$$\begin{cases}
(S_{g_B} + \frac{S_{g_F}}{f} - \rho) g_B(X_i, X_j) = \text{Hess}_{g_B} h_1 (X_i, X_j) \\
(S_{g_B} + \frac{S_{g_F}}{f} - \rho) \tilde{g}(X_i, Y_j) = \text{Hess}_{\tilde{g}} h_1 (X_i, Y_j) = 0 \\
(S_{g_B} + \frac{S_{g_F}}{f} - \rho) f^2 g_F(Y_i, Y_j) = \text{Hess}_{g_F} h_2 (Y_i, Y_j) + f \nabla f (h_1) g_F(Y_i, Y_j)
\end{cases}$$

where in equation $(iii)$ we use Proposition 35 of [25] and the Hessian definition to get

$$\text{Hess}_\tilde{g} h(Y_i, Y_j) = Y_i(Y_j(h)) - (\nabla_{Y_i} Y_j)^M h = Y_i(Y_j(h)) - (\mathcal{H}(\nabla_{Y_i} Y_j) + \nabla(Y_i Y_j))(h) = Y_i(Y_j(h)) + \frac{\langle Y_i, Y_j \rangle}{f} \text{grad}_\tilde{g} f(h) - \nabla_{Y_i} Y_j(h) = Y_i(Y_j(h)) + f g_F(Y_i, Y_j) \text{grad}_\tilde{g} f(h) - \nabla_{Y_i} Y_j(h) = \text{Hess}_{g_F} h_2 (Y_i, Y_j) + f \nabla f (h_1) g_F(Y_i, Y_j).$$

Since $S_{g_F}$ is constant on $B$, we have from $(i)$ that $B$ is a gradient Yamabe soliton of the form $(B, g_B, \nabla h_1, -\frac{S_{g_B}}{f} + \rho)$. Furthermore, since $h(x, y) = h_1(x) + \text{cte}$ we have by $(iii)$ that $F$ is a trivial gradient Yamabe soliton of the form $(F, g_F, \nabla 0, f^2 \rho - f^2 S_{g_B})$. This proves the item $(a)$.

Case(II) : ($f$ is constant and $h(x, y) = h_1(x) + h_2, h_2$ not necessarily constant). In this case, $M = B \times_f F$ is a Riemannian product and we have
\[ \begin{align*}
\left( S_{gb} + \frac{s_{gf}}{f^2} - \rho \right) g_B(X_i, X_j) &= \text{Hess}_{g_B} h_1(X_i, X_j) \quad (i) \\
\left( S_{gb} + \frac{s_{gf}}{f^2} - \rho \right) \tilde{g}(X_i, Y_j) &= \text{Hess}_{\tilde{g}} h(X_i, Y_j) = 0 \\
\left( S_{gb} + \frac{s_{gf}}{f^2} - \rho \right) f^2 g_F(Y_i, Y_j) &= \text{Hess}_{g_F} h_2(Y_i, Y_j) + f \nabla f(h_1) g_F(Y_i, Y_j). 
\end{align*} \]

Since \( S_{gb} \) is constant on \( B \), we have that \( B \) is a gradient Yamabe soliton of the form \((B, g_B, \nabla h_1, -\frac{s_{gf}}{f^2} + \rho)\). Furthermore, by equation \((iii)\) we have that \( F \) is a gradient Yamabe soliton of the form \((F, g_F, \nabla h_2, f^2 \rho - f^2 S_{gb})\). This proves the item \((b)\).

**Case(III)\( : (f \) is non constant and \( h(x, y) = h_1(x) + \text{constant} \).** In this case we have:

\[ \begin{align*}
\left( S_{gb} + \frac{s_{gf}}{f^2} - \frac{2d}{f} \Delta B f - d(d - 1) \frac{\langle \text{grad}_B f, \text{grad}_B f \rangle}{f^2} - \rho \right) g_B(X_i, X_j) &= \text{Hess}_{g_B} h_1(X_i, X_j) \\
\left( S_{gb} + \frac{s_{gf}}{f^2} - \frac{2d}{f} \Delta B f - d(d - 1) \frac{\langle \text{grad}_B f, \text{grad}_B f \rangle}{f^2} - \rho \right) \tilde{g}(X_i, Y_j) &= \text{Hess}_{\tilde{g}} h_1(X_i, Y_j) = 0 \\
\left( S_{gb} + \frac{s_{gf}}{f^2} - \frac{2d}{f} \Delta B f - d(d - 1) \frac{\langle \text{grad}_B f, \text{grad}_B f \rangle}{f^2} - \rho \right) f^2 g_F(Y_i, Y_j) &= f \nabla f(h_1) g_F(Y_i, Y_j).
\end{align*} \]

Since \( f > 0 \), by equation \((iii)\) we have that

\[ (S_{gf} - \psi) g_F(Y_i, Y_j) = 0 \quad (24) \]

where \( \psi = -f^2 S_{gb} + 2d f \Delta B f + d(d - 1) \frac{\langle \text{grad}_B f, \text{grad}_B f \rangle}{f^2} + f^2 \rho + f \nabla f(h). \)

Now, since \( \psi \) depend only on \( B \), we have that \( \psi \) is constant on \( F \), then by equation \((iii)\), we have that \( F \) is a trivial gradient Yamabe soliton. Furthermore, by equation \((i)\) we have that \((B, g_B)\) is a gradient almost Yamabe soliton of the form

\[ (B, g_B, \nabla h_1, -\frac{s_{gf}}{f^2} - \frac{2d}{f} \Delta B f - d(d - 1) \frac{\langle \text{grad}_B f, \text{grad}_B f \rangle}{f^2} - \rho). \]

This proves the item \((c)\). \( \square \)

**Proof of Theorem 1.5.** Let \( M \) be a warped product with gradient Yamabe soliton structure and potential function \( h \), that is,

\[ (S_{\tilde{g}} - \rho) \tilde{g} = \text{Hess}_{\tilde{g}}(h). \quad (25) \]

By the same arguments used in proof of Proposition 1.1 for \( X_1, X_2, \ldots, X_n \in \mathcal{L}(B) \) and \( Y_1, Y_2, \ldots, Y_m \in \mathcal{L}(F) \) we obtain

\[ \begin{align*}
\left( S_{\tilde{g}} + \frac{s_{gf}}{f^2} - \frac{2m}{f} \Delta \tilde{g} f - m(m - 1) \frac{\langle \text{grad}_f \text{grad}_f \rangle}{f^2} - \rho \right) \tilde{g}(X_i, X_j) &= \text{Hess}_{\tilde{g}} h(X_i, X_j) \\
\left( S_{\tilde{g}} + \frac{s_{gf}}{f^2} - \frac{2m}{f} \Delta \tilde{g} f - m(m - 1) \frac{\langle \text{grad}_f \text{grad}_f \rangle}{f^2} - \rho \right) \tilde{g}(X_i, Y_j) &= \text{Hess}_{\tilde{g}} h(X_i, Y_j) = 0 \\
\left( S_{\tilde{g}} + \frac{s_{gf}}{f^2} - \frac{2m}{f} \Delta \tilde{g} f - m(m - 1) \frac{\langle \text{grad}_f \text{grad}_f \rangle}{f^2} - \rho \right) f^2 g_F(Y_i, Y_j) &= \text{Hess}_{g_F} h(Y_i, Y_j). 
\end{align*} \]
It is well known that for the conformal metric $\tilde{g} = \frac{1}{\varphi^2} g_0$, the Christofel symbol is given by

$$\tilde{\Gamma}_{ij}^k = 0, \quad \tilde{\Gamma}_{ij}^k = -\frac{\varphi_x x_i x_j}{\varphi}, \quad \tilde{\Gamma}_{ii}^k = \xi_i \xi_k \frac{\varphi_x x_k}{\varphi} \quad \text{and} \quad \tilde{\Gamma}_{ii}^i = -\frac{\varphi_x x_i}{\varphi}.$$  

Then, we obtain by Hessian definiton that

$$\begin{cases} \operatorname{Hess}_{\tilde{g}}(h)_{ij} = h_{x_i x_j} + \frac{\varphi_x x_i h_{x_j}}{\varphi} + \frac{\varphi_x x_j h_{x_i}}{\varphi} \quad &i \neq j \\ \operatorname{Hess}_{\tilde{g}}(h)_{ii} = h_{x_i x_i} + 2 \frac{\varphi_x x_i h_{x_i}}{\varphi} - \varepsilon_i \sum_{k=1}^n \varepsilon_k \frac{\varphi_x x_k}{\varphi} h_{x_k} \quad &i = j. \end{cases} \quad (26)$$

The Ricci curvature is given by

$$\operatorname{Ric}_{\tilde{g}} = \frac{1}{\varphi^2} \left( (n-2) \varphi \operatorname{Hess}_{g}(\varphi) + [\varphi \Delta_{\tilde{g}} \varphi - (n-1) |\nabla_{\tilde{g}} \varphi|^2] g \right)$$

and then we easily see that the scalar curvature on conformal metric $\tilde{g}$ is given by

$$S_{\tilde{g}} = (n-1)(2\varphi \Delta_{\tilde{g}} \varphi - n |\nabla_{\tilde{g}} \varphi|^2) = (n-1)(2\varphi \sum_{k=1}^n \varepsilon_k \varphi_{,x_k x_k} - n \sum_{k=1}^n \varepsilon_k \varphi^2_{,x_k}). \quad (27)$$

Since $h : \mathbb{R}^n \rightarrow \mathbb{R}$, we obtain

$$\operatorname{Hess}_g h(X_i, X_j) = \operatorname{Hess}_{\tilde{g}} h(X_i, X_j), \quad \forall i, j. \quad (28)$$

On the other hand

$$\begin{cases} S_F g_F = \lambda_F g_F \\ \tilde{g}(Y_i, Y_j) = f^2 g_f(Y_i, Y_j) \\ \Delta_f f = \varphi^2 \sum_k \varepsilon_k f_{,x_k x_k} - (n-2) \varphi \sum_k \varepsilon_k \varphi_{,x_k} f_{,x_k} \\ (\varphi f_{,x_k})_{,x_k} = \varphi f_{,x_k} = \varphi^2 \sum_k \varepsilon_k f_{,x_k}^2. \end{cases} \quad (29)$$

Now, substituting the second equation of (20), the equations of (29) and equation (27) in (i), we have

$$\left[(n-1)(2\varphi \sum_k \varepsilon_k \varphi_{,x_k x_k} - n \sum_k \varepsilon_k \varphi^2_{,x_k}) + \frac{\lambda_F}{f^2} - \frac{2m}{f^2} (\varphi^2 \sum_k \varepsilon_k f_{,x_k x_k} - (n-2) \varphi \sum_k \varepsilon_k \varphi_{,x_k} f_{,x_k}) + \frac{m(m-1)}{f^2} \varphi^2 \sum_k \varepsilon_k f_{,x_k}^2 \right] \frac{\varepsilon_i}{\varphi} = h_{,x_i x_i} + 2 \frac{\varphi_x x_i}{\varphi} h_{,x_i} - \varepsilon_i \sum_k \varepsilon_k \frac{\varphi_x x_k}{\varphi} h_{,x_k}$$

which is the equation (5).

Analogously, substituting the first equation of (20) and equation (29) in (i), we obtain

$$h_{,x_i x_j} + \frac{\varphi_x x_i}{\varphi} h_{,x_i} + \frac{\varphi_x x_j}{\varphi} h_{,x_j} = 0 \quad (30)$$

which is the equation (4). In the similar way that equation (23), we have that
\[ \text{Hess}_g h(Y_i, Y_j) = Y_i(Y_j(h)) - (\nabla_i Y_j)^T h \]
\[ = \text{Hess}_g h(Y_i, Y_j) + f g_F(Y_i, Y_j) \text{grad}_f(h) \]
\[ = f \varphi^2 \sum_k \varepsilon_k f_{x_k} h_{x_k} g_F(Y_i, Y_j). \] (31)

Then, substituting equation (31), (29) and equation (27) in (iii), we obtain equation (6). A direct calculation shows us the converse implication. This concludes the proof of Theorem 1.5.

\[ \square \]

**Proof of Theorem 1.6.** Since we are assuming that \( \varphi(\xi) \), \( h(\xi) \) and \( f(\xi) \) are functions of \( \xi \), where \( \xi = \sum_k \alpha_k x_k \), \( \alpha_k \in \mathbb{R}^n \) and \( \sum_{k=1}^{n} \varepsilon_k \alpha_k^2 = \varepsilon_{k_0} \) or \( \sum_{k=1}^{n} \varepsilon_k \alpha_k^2 = 0 \), then we have

\[ \varphi, = \varphi' \alpha_i; \varphi_{x_i} = \varphi'' \alpha_i \alpha_j; f, = f' \alpha_i; f_{x_i} = f'' \alpha_i \alpha_j; h, = h'' \alpha_i; h_{x_i} = h'' \alpha_i \alpha_j. \]

Substituting these expressions into (4) of Theorem 1.5, we obtain

\[ \left( h'' + \frac{2 \varphi' h'}{\varphi} \right) \alpha_i \alpha_j = 0, \quad \forall i \neq j. \] (32)

Similarly, considering equations (5) and (6) of Theorem 1.5, we obtain

\[ \left( n - 1 \right) \sum_k \varepsilon_k \alpha_k^2 - n \varphi^2 \sum_k \varepsilon_k \alpha_k^2 \right) + \frac{\lambda_f}{f^2} - \frac{2m}{f} \left( \varphi^2 f'' \sum_k \varepsilon_k \alpha_k^2 \right)
\[ - \frac{m(m-1)}{f^2} \varphi^2 (f')^2 \sum_k \varepsilon_k \alpha_k^2 - \rho \right) \varepsilon_i \varphi^2 = h'' \alpha_i^2 + 2 \alpha_i \frac{\varphi'}{\varphi} h' - \varepsilon_i \frac{\varphi'}{\varphi} \sum_k \varepsilon_k \alpha_k^2 \] (33)

for \( i \in \{1, 2, \ldots, n\} \), and

\[ \left( n - 1 \right) \sum_k \varepsilon_k \alpha_k^2 - n \varphi^2 \sum_k \varepsilon_k \alpha_k^2 \right) + \frac{\lambda_f}{f^2} - \frac{2m}{f} \left( \varphi^2 f'' \sum_k \varepsilon_k \alpha_k^2 \right)
\[ - \frac{m(m-1)}{f^2} \varphi^2 (f')^2 \sum_k \varepsilon_k \alpha_k^2 - \rho \right) \varepsilon_i \varphi^2 = \frac{\varphi^2 f'}{f} h' \sum_k \varepsilon_k \alpha_k^2 \] (34)

If there exist \( i, j, i \neq j \) such that \( \alpha_i \alpha_j \neq 0 \), then we get by equation (32) that

\[ \left( h'' + \frac{2 \varphi' h'}{\varphi} \right) = 0. \] (35)

It follows from (35) that the equation (33) is summed to

\[ \left( n - 1 \right) \sum_k \varepsilon_k \alpha_k^2 - n \varphi^2 \sum_k \varepsilon_k \alpha_k^2 \right) + \frac{\lambda_f}{f^2} - \frac{2m}{f} \left( \varphi^2 f'' \sum_k \varepsilon_k \alpha_k^2 \right)
\[ - \frac{m(m-1)}{f^2} \varphi^2 (f')^2 \sum_k \varepsilon_k \alpha_k^2 - \rho \right) \varepsilon_i \varphi^2 = - \varepsilon_i \varphi' \sum_k \varepsilon_k \alpha_k^2. \] (36)
Thus, isolating the term $\sum_k \varepsilon_k \alpha_k^2 = \varepsilon_{k_0}$, we obtain the equation (8).

In the same way, isolating the term $\sum_k \varepsilon_k \alpha_k^2 = \varepsilon_{k_0}$ in (34) we obtain the equation (9).

Thus, if $\sum_k \varepsilon_k \alpha_k^2 = \varepsilon_{k_0}$, then we obtain equations (7), (8) and (9). In the case $\sum_k \varepsilon_k \alpha_k^2 = 0$, we easily see that the equation (7) is summed to

\[
\begin{cases}
  \rho - \frac{\lambda_f}{f^2} = 0.
\end{cases}
\] (37)

Now, we need to consider the case $\alpha_{k_0} = 1$ and $\alpha_k = 0 \forall k \neq k_0$. In this case, equation (32) is trivially satisfied, and since equation (34) does not depend on the index $i$, we have that equation (34) is equivalent to equation (9).

Finally, we need to show the validity of equation (7) and (8). Observe that taking $i = k_0$, that is, $\alpha_{k_0} = 1$, in (33), we get

\[
\begin{align*}
\left( n - 1 \right) (2\varphi \varphi'' \varepsilon_{k_0} - n(\varphi')^2 \varepsilon_{k_0}) + \frac{\lambda_f}{f^2} (\varphi^2 f'' \varepsilon_{k_0} - (n - 2) \varphi \varphi' \varphi' \varepsilon_{k_0}) + \\
-m \left( \frac{m - 1}{f^2} \right) \varphi^2 (f')^2 \varepsilon_{k_0} - \rho \right] \varepsilon_{k_0} \varphi^2 = h'' + 2 \frac{\varphi' h'}{\varphi} - \frac{h'}{\varphi} = h'' + \frac{\varphi' h'}{\varphi} + \frac{\varphi''}{\varphi}
\end{align*}
\] (38)

and for $i \neq k_0$, that is, $\alpha_i = 0$, we have

\[
\begin{align*}
\left( n - 1 \right) (2\varphi \varphi'' \varepsilon_{k_0} - n(\varphi')^2 \varepsilon_{k_0}) + \frac{\lambda_f}{f^2} (\varphi^2 f'' \varepsilon_{k_0} - (n - 2) \varphi \varphi' \varepsilon_{k_0}) + \\
-m \left( \frac{m - 1}{f^2} \right) \varphi^2 (f')^2 \varepsilon_{k_0} - \rho \right] \varepsilon_i \varphi^2 = -\varepsilon_{k_0} \frac{\varphi'}{\varphi} h' + \frac{\varphi'}{\varphi} \varepsilon_{k_0} \frac{\varphi'}{\varphi}.\end{align*}
\] (39)

However, this equations are equivalent to equations (7) and (8). This complete the proof of Theorem 1.6.

**Proof of Corollary 1.7.** By Theorem 1.6, we have that $M$ is a gradient Yamabe soliton with potential function $h$ if, and only if,

\[
\begin{cases}
  h'' + 2 \frac{\varphi' h'}{\varphi} = 0 \\
  \rho - \frac{\lambda_f}{f^2} = 0
\end{cases}
\] (40)

Thus, we have that $\lambda_f$ and $\rho$ always have the same signal. Therefore, there is no gradient Yamabe soliton $M$ expanding/shrinking with fiber trivial gradient Yamabe soliton shrinking/expanding.

**Proof of Theorem 1.8.** Since $\lambda_f = \rho = 0$ we have by equation (8) and (9) of Theorem 1.6 that

$$\varphi' h' \varphi = -\frac{\varphi^2}{f} f' f'$$

and by condition $h' \neq 0$, we obtain

$$\frac{\varphi'}{\varphi} = -\frac{f'}{f}.\] (41)

Integrating this equation we have
\[ f(\xi) = \frac{e^c}{\varphi(\xi)} \]

for some \( c \in \mathbb{R} \), which is equation (12) of Theorem 1.8.

Integrating the equation (7), we have that

\[ h'(\xi) = \frac{\alpha}{\varphi^2(\xi)} \]

(42)

for some \( \alpha \neq 0 \), and

\[ h(\xi) = \alpha \int \frac{1}{\varphi^2(\xi)} d\xi \]

which is equation (13) of Theorem 1.8.

Substituting equation (42) into (8) we have

\[
(n - 1)(2 \varphi \varphi'' - n(\varphi')^2) - 2\int f(\varphi^2) - (n - 2)\varphi \varphi' f' - \frac{m(m - 1)}{f^2} \varphi^2(f')^2 + \alpha \frac{\varphi'}{\varphi} = 0.
\]

(43)

Inserting equation (41) into (43) we obtain

\[
\varphi \varphi'' - \frac{(n + m)}{2}(\varphi')^2 + \frac{\alpha}{2(n + m - 1)} \frac{\varphi'}{\varphi} = 0.
\]

(44)

Consider \( \varphi(\xi)^{1 - \frac{n + m}{2}} = \omega(\xi) \), then

\[
\omega'(\xi) = \left(1 - \frac{n + m}{2}\right)\varphi^{-\frac{n + m}{2}} \varphi', \quad \omega''(\xi) = \left(1 - \frac{m + n}{2}\right) \left(\varphi^{-\frac{n + m}{2} - 1}(\varphi \varphi'' - \frac{(n + m)}{2}(\varphi')^2)\right)
\]

(45)

and we obtain that the differential equation (44) is equivalent to

\[
\omega''(\xi) + \frac{\alpha}{2(n + m - 1)} \omega'(\xi)\omega(\xi)\frac{\varphi^4}{n + m - 2} = 0.
\]

(46)

Integrating equation (46) we have

\[
\omega'(\xi) + \frac{\alpha(n + m - 2)}{2(n + m + 2)(n + m - 1)} \omega(\xi)\frac{\varphi^{n + m + 2}}{n + m - 2} = \beta, \quad \beta \in \mathbb{R}.
\]

Thus,

\[
-\int \frac{1}{\alpha(n + m - 2)} \omega(\xi)\frac{\varphi^{n + m + 2}}{n + m - 2} d\omega = \xi + \nu
\]

and then

\[
(n + m - 1)(n + m + 2) \int \frac{\varphi d\varphi}{\alpha - \frac{2\beta(n + m - 1)(n + m + 2)}{n + m - 2}} = \xi + \nu
\]

which is equation (14) of Theorem 1.8. Then we prove the necessary condition. Now a direct calculation shows us the converse implication. This concludes the proof of Theorem 1.8.
Proof of Theorem 1.10. Since $h' = 0$ and $\lambda_F = \rho = 0$ we have by equation (8) and (9) of Theorem 1.6 that

$$(n-1)(2\varphi'' - n(\varphi')^2) - 2\frac{m}{f}(\varphi'^2 - (n-2)\varphi\varphi'f') - \frac{m(m-1)}{f^2}\varphi^2(f')^2 = 0$$

which is equivalent to

$$\left(\frac{f'}{f} - \frac{(n-2)\varphi'}{(m+1)\varphi}\right)^2 + \frac{2}{m+1}\left(\frac{f'}{f} - \frac{(n-2)\varphi'}{(m+1)\varphi}\right)' + \frac{n+m-1}{m(m+1)^2}\left(n\frac{\varphi'}{\varphi}^2 - 2\frac{\varphi''}{\varphi}\right) = 0.$$

Consider $z = \frac{f'}{f} - \frac{(n-2)\varphi'}{(m+1)\varphi}$, then

$$z^2 + \frac{2}{m+1}z' + \frac{n+m-1}{m(m+1)^2}\left(n\frac{\varphi'}{\varphi}^2 - 2\frac{\varphi''}{\varphi}\right) = 0.$$  \hspace{1cm} (47)

Now, recall that the Riccati differential equation is a differential equation of the form

$$z(\xi)' = p(\xi) + q(\xi)z(\xi) + r(\xi)z(\xi)^2$$

where $p$, $q$, and $r$ are smooth functions on $\mathbb{R}$, and by Picard theorem we have that the solutions of (48) is given by

$$z(\xi) = z_p(\xi) + \frac{e^{\int P(\xi)d\xi}d\xi}{-\int r(\xi)e^{\int P(\xi)d\xi}d\xi + c}$$

where $P(\xi) = q(\xi) + 2z_p(\xi)r(\xi)$, $z_p(\xi)$ is a particular solution of (48) and $c$ is a constant.

Observe that (47) is a Riccati differential equation with

$$q(\xi) = 0, \quad r(\xi) = -\frac{m+1}{2} \quad \text{and} \quad p(\xi) = -\frac{(n+m-1)}{2m(m+1)}\left(n\frac{\varphi'}{\varphi}^2 - 2\frac{\varphi''}{\varphi}\right).$$

Then we obtain

$$\frac{f'(\xi)}{f(\xi)} = \frac{(n-2)\varphi'(\xi)}{(m+1)\varphi(\xi)} + z_p(\xi) + \frac{e^{-(m+1)}\int z_p(\xi)d\xi}{\int e^{-(m+1)}\int z_p(\xi)d\xi d\xi + c}$$

and thus,

$$f = \varphi^{\frac{n-2}{m+1}}e^{\int z_p d\xi}\left(\int e^{-(m+1)}\int z_p d\xi + \frac{2}{m+1}C\right)^{\frac{2}{m+1}}$$

where $z_p(\xi)$ is a particular solution of (47). This expression is equation (10) of Theorem 1.6.

Now, since $h' = 0$, we have that $h(\xi) = constant$, which is equation (15) of theorem 1.5. Then we prove the necessary condition. Now a direct calculation shows us the converse implication. This concludes the proof of Theorem 1.5 \quad \Box

Proof of Theorem 1.10. In this case, since $\lambda_F = \rho = 0$, we have by differential equation (10) and (11) that

$$h(\xi) = \alpha \int \frac{1}{\varphi^2(\xi)}d\xi$$

for some $\alpha \neq 0$ and $f, \varphi$ are arbitrary.

Then we prove the necessary condition. Now a direct calculation shows us the converse implication. This concludes the proof of Theorem 1.10 \quad \Box
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