Online learning algorithms are a key tool in web search and content optimization, adaptively learning what users want to see. In a typical application, each time a user arrives, the algorithm chooses among various content presentation options (e.g., news articles to display), the chosen content is presented to the user, and an outcome (e.g., a click) is observed. Such algorithms must balance exploration (making potentially suboptimal decisions now for the sake of acquiring information that will improve decisions in the future) and exploitation (using information collected in the past to make better decisions now). Exploration could degrade the experience of a current user but improves user experience in the long run. This exploration–exploitation tradeoff is commonly studied in the online learning framework of multi-armed bandits [Bubeck and Cesa-Bianchi 2012].

Concerns have been raised about whether exploration in such scenarios could be unfair in the sense that some individuals or groups may experience too much of the downside of exploration without sufficient upside [Bird et al. 2016]. We formally study these concerns in the linear contextual bandits model [Li et al. 2010, Chu et al. 2011], a standard variant of multi-armed bandits appropriate for content personalization scenarios. We focus on externalities arising due to exploration, that is, undesirable side effects that the presence of one party may impose on another.

We first examine the effects of exploration at a group level. We introduce the notion of a group externality in an online learning system, quantifying how much the presence of one population (which we dub the majority) impacts the rewards of another (the minority). We show that this impact can be negative, and that, in
a particular precise sense, no algorithm can avoid it. This cannot be explained by the absence of suitably good policies since our adoption of the linear contextual bandits framework implies the existence of a feasible policy that is simultaneously optimal for everyone. Instead, the problem is inherent to the process of exploration. We come to a surprising conclusion that more data can sometimes lead to worse outcomes for the users of an explore-exploit-based system.

We next turn to the effect of exploration at an individual level. We interpret exploration as a potential externality imposed on the current user by future users of the system. Indeed, it is only for the sake of the future users that the algorithm would forego the action that currently looks optimal. To avoid this externality, one may use the greedy algorithm that always chooses the action that appears optimal according to current estimates of the problem parameters. While this greedy algorithm performs poorly in the worst case, it tends to work well in many applications and experiments.\(^1\)

In a new line of work, Bastani et al. [2020] and Kannan et al. [2018] analyzed conditions under which inherent diversity in the data makes explicit exploration unnecessary. Kannan et al. [2018] proved that the greedy algorithm achieves a regret rate of \(\hat{O}(\sqrt{T})\) in expectation over small perturbations of the context vectors (which ensure sufficient data diversity). This is the best rate that can be achieved in the worst case (i.e., for all problem instances, without data diversity assumptions), but it leaves open the possibilities that (i) another algorithm may perform much better than the greedy algorithm on some problem instances or (ii) the greedy algorithm may perform much better than worst case under the diversity conditions. We expand on this line of work. We prove that under the same diversity conditions the greedy algorithm almost matches the best possible Bayesian regret rate of any algorithm on the same problem instance. This could be as low as \(\text{polylog}(T)\) for some instances, and, as we prove, at most \(\hat{O}(T^{1/3})\) whenever the diversity conditions hold.

Returning to group-level effects, we show that under the same diversity conditions the negative group externalities imposed by the majority essentially vanish if one runs the greedy algorithm. Together, our results illustrate a sharp contrast between the high individual and group externalities that exist in the worst case, and the ability to remove all externalities if the data is sufficiently diverse.

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\(^{1}\) Both positive and negative findings are folklore. One way to precisely state the negative result is that the greedy algorithm incurs constant per-round regret with constant probability; while results of this form have likely been known for decades, Mansour et al. [2018, corollary A.2(b)] proved this for a wide variety of scenarios. Very recently, the good empirical performance has been confirmed by state-of-art experiments in Bietti et al. [2018].
Additional motivation. Whether and when explicit exploration is necessary is an important concern in the study of the exploration–exploitation tradeoff. Fairness considerations aside, explicit exploration is expensive. It is wasteful and risky in the short term, it adds a layer of complexity to algorithm design [Langford and Zhang 2007, Agarwal et al. 2014], and its adoption at scale tends to require substantial systems support and buy-in from management [Agarwal et al. 2016, 2017]. A system based on the greedy algorithm would typically be cheaper to design and deploy.

Further, explicit exploration can run into incentive issues in applications such as recommender systems. Essentially, when it is up to the users which products or experiences to choose and the algorithm can only issue recommendations and ratings, an explore–exploit algorithm needs to provide incentives to explore for the sake of the future users [Frazier et al. 2014, Kremer et al. 2014, Mansour et al. 2015, Che and Hörner 2018, Papanastasiou et al. 2018]. Such incentive guarantees tend to come at the cost of decreased performance and rely on assumptions about human behavior. The greedy algorithm avoids this problem as it is inherently consistent with the users’ incentives.

Additional related work. Our research draws inspiration from the growing body of work on fairness in machine learning (e.g., Dwork et al. [2012], Hardt et al. [2016b], Kleinberg et al. [2017] and Chouldechova [2017]). Several other authors have studied fairness in the context of the contextual bandits framework. Our work differs from the line of research on meritocratic fairness in online learning [Kearns et al. 2017, Liu et al. 2017], which considers the allocation of limited resources such as bank loans and requires that nobody should be passed over in favor of a less qualified applicant. We study a fundamentally different scenario in which there are no allocation constraints, and we would like to serve each user the best content possible. Our work also differs from that of Celis and Vishnoi [2017], who studied an alternative notion of fairness in the context of news recommendations. According to this notion, all users should have approximately the same probability of seeing a particular type of content (e.g., Republican-leaning articles), regardless of their individual preferences, in order to mitigate the possibility of discriminatory personalization.

The data diversity conditions in Kannan et al. [2018] and this chapter are inspired by the smoothed analysis framework of Spielman and Teng [2004], who proved that the expected running time of the simplex algorithm is polynomial for perturbations of any initial problem instance (whereas the worst-case running time has long been known to be exponential). Such disparity implies that very bad problem instances are brittle. We find a similar disparity for the greedy algorithm in our setting.
Our results on group externalities. A typical goal in online learning is to minimize regret, the (expected) difference between the cumulative reward that would have been obtained had the optimal policy been followed at every round and the cumulative reward obtained by the algorithm. We define a corresponding notion of minority regret, the portion of the regret experienced by the minority. Since online learning algorithms update their behavior based on the history of their observations, minority regret is influenced by the entire population on which an algorithm is run. If the minority regret is much higher when a particular algorithm is run on the full population than it is when the same algorithm is run on the minority alone, we can view the majority as imposing a negative externality on the minority; the minority population would achieve a higher cumulative reward if the majority were not present. Asking whether this can ever happen amounts to asking whether access to more data points can ever lead an explore-exploit algorithm to make inferior decisions. One might think that more data should always lead to better decisions and therefore better outcomes for the users. Surprisingly, we show that this is not the case, even with a standard algorithm.

Consider LinUCB [Li et al. 2010, Abbasi-Yadkori et al. 2011, Chu et al. 2011], a standard algorithm for linear contextual bandits that is based on the principle of “optimism under uncertainty.” We provide a specific problem instance on which, after observing $T$ users, LinUCB would have a minority regret of $\Omega(\sqrt{T})$ if run on the full population but only constant minority regret if run on the minority alone. While stylized, this example is motivated by the problem of providing driving directions to different populations of users, about which fairness concerns have been raised [Bird et al. 2016]. Further, the situation is reversed on a slight variation of this example: LinUCB obtains constant minority regret when run on the full population and $\Omega(\sqrt{T})$ on the minority alone. That is, group externalities can be large and positive in some cases and large and negative in others.

Although these regret rates are specific to LinUCB, we show that this phenomenon is, in some sense, unavoidable. Consider the minority regret of LinUCB when run on the full population and the minority regret that LinUCB would incur if run on the minority alone. We know that one may be much smaller or larger than the other. We ask whether any algorithm could achieve the minimum of the two on every problem instance. Using a variation of the same problem instance, we prove that this is impossible; in fact, no algorithm could simultaneously approximate both up to any $o(\sqrt{T})$ factor. In other words, an externality-free algorithm would sometimes “leave money on the table.”

In terms of techniques, we rely on the special structure of our example, which can be viewed as an instance of the sleeping bandits problem [Kleinberg et al. 2010]. This simplifies the behavior and analysis of LinUCB, allowing us to obtain
the $O(1)$ upper bounds. The lower bounds are obtained using Kullback-Leibler (KL)-divergence techniques to show that the two variants of our example are essentially indistinguishable, and an algorithm that performs well on one must obtain $\Omega(\sqrt{T})$ regret on the other.

**Our results on the greedy algorithm.** We consider a version of linear contextual bandits in which the latent weight vector $\theta$ is drawn from a known prior. In each round, an algorithm is presented several actions to choose from, each represented by a context vector. The expected reward of an action is a linear product of $\theta$ and the corresponding context vector. The tuple of context vectors is drawn independently from a fixed distribution. In the spirit of smoothed analysis, we assume that this distribution has a small amount of jitter. Formally, the tuple of context vectors is drawn from some fixed distribution, and then a small perturbation—small-variance Gaussian noise—is added independently to each coordinate of each context vector. This ensures arriving contexts are diverse. We are interested in Bayesian regret, that is, regret in expectation over the Bayesian prior. Following the literature, we are primarily interested in the dependence on the time horizon $T$.

We focus on a batched version of the greedy algorithm, in which new data arrives to the algorithm's optimization routine in small batches, rather than every round. This is well-motivated from a practical perspective—in high-volume applications data usually arrives to the “learner” only after a substantial delay [Agarwal et al. 2016, 2017]—and is essential for our analysis.

Our main result is that the greedy algorithm matches the Bayesian regret of any algorithm up to polylogarithmic factors, for each problem instance, fixing the Bayesian prior and the context distribution. We also prove that LinUCB achieves regret $O(T^{1/3})$ for each realization of $\theta$. This implies a worst-case Bayesian regret of $\tilde{O}(T^{1/3})$ for the greedy algorithm under the perturbation assumption.

Our results hold for both natural versions of the batched greedy algorithm, Bayesian and frequentist, henceforth called BatchBayesGreedy and BatchFreqGreedy. In BatchBayesGreedy, the chosen action maximizes expected reward according to the Bayesian posterior. BatchFreqGreedy estimates $\theta$ using ordinary least squares regression and chooses the best action according to this estimate. The results for BatchFreqGreedy come with additive polylogarithmic factors, but are stronger in that the algorithm does not need to know the prior. Further, the $\tilde{O}(T^{1/3})$ regret bound for BatchFreqGreedy is approximately prior-independent, in the sense that it applies even to very concentrated priors such as independent Gaussians with standard deviation on the order of $T^{-2/3}$.

The key insight in our analysis of BatchBayesGreedy is that any (perturbed) data can be used to simulate any other data, with some discount factor. The analysis of
BatchFreqGreedy requires an additional layer of complexity. We consider a hypothetical algorithm that receives the same data as BatchFreqGreedy, but chooses actions based on the Bayesian-greedy selection rule. We analyze this hypothetical algorithm using the same technique as BatchBayesGreedy, and then upper bound the difference in Bayesian regret between the hypothetical algorithm and BatchFreqGreedy.

Our analyses extend to group externalities and (Bayesian) minority regret. In particular, we circumvent the impossibility result mentioned above. We prove that both BatchBayesGreedy and BatchFreqGreedy match the Bayesian minority regret of any algorithm run on either the full population or the minority alone, up to polylogarithmic factors.

**Detailed comparison with prior work.** We substantially improve over the $\tilde{O}(\sqrt{T})$ worst-case regret bound from Kannan et al. [2018], at the cost of some additional assumptions. First, we consider Bayesian regret, whereas their regret bound is for each realization of $\theta$. Second, they allow the context vectors to be chosen by an adversary before the perturbation is applied. Third, they extend their analysis to a somewhat more general model, in which there is a separate latent weight vector for every action (which amounts to a more restrictive model of perturbations). However, this extension relies on the greedy algorithm being initialized with a substantial amount of data. The results of Kannan et al. [2018] do not appear to have implications on group externalities.

Bastani et al. [2020] show that the greedy algorithm achieves logarithmic regret in an alternative linear contextual bandits setting that is incomparable to ours in several important ways. They consider two-action instances where the actions share a common context vector in each round but are parameterized by different latent vectors. They ensure data diversity via a strong assumption on the context distribution. This assumption does not follow from our perturbation conditions; among other things, it implies that each action is the best action in a constant fraction of rounds. Further, they assume a version of Tsybakov’s margin condition, which is known to substantially reduce regret rates in bandit problems (e.g., see Rigollet and Zeevi [2010]).

### 3.1 Preliminaries

We consider the model of linear contextual bandits [Li et al. 2010, Chu et al. 2011]. Formally, there is a learner who serves a sequence of users over $T$ rounds, where $T$ is the (known) time horizon. For the user who arrives in round $t$, there are (at most) $K$ actions available, with each action $a \in \{1, \ldots, K\}$ associated with a context

---

2. Equivalently, they allow point priors, whereas our priors must have variance $T^{-\Omega(1)}$. 
vector $x_{a,t} \in \mathbb{R}^d$. Each context vector may contain a mix of features of the action, features of the user, and features of both. We assume that the tuple of context vectors for each round $t$ is drawn independently from a fixed distribution. The learner observes the set of contexts and selects an action $a_t$ for the user. The user then experiences a reward $r_t$ that is visible to the learner. We assume that the expected reward is linear in the chosen context vector. More precisely, we let $r_{a,t}$ be the reward of action $a$ if this action is chosen in round $t$ (so that $r_t = r_{a_t,t}$), and assume that there exists an unknown vector $\theta \in \mathbb{R}^d$ such that $\mathbb{E}[r_{a,t} | x_{a,t}] = \theta^\top x_{a,t}$ for any round $t$ and action $a$. Throughout most of the chapter, the realized rewards are either in $\{0, 1\}$ or are the expectation plus independent Gaussian noise of variance at most 1. We sometimes consider a Bayesian version, in which the latent vector $\theta$ is initially drawn from some known prior $\mathcal{P}$.

A standard goal for the learner is to maximize the expected total reward over $T$ rounds, or $\sum_{t=1}^T \theta^\top x_{a,t}$. This is equivalent to minimizing the learner’s regret, defined as

$$\text{Regret}(T) = \sum_{t=1}^T \theta^\top x_t^* - \theta^\top x_{a,t}$$  \hspace{1cm} (3.1)

where $x_t^* = \arg \max_{x \in \{x_1, \ldots, x_K\}} \theta^\top x$ denotes the context vector associated with the best action at round $t$. We are mainly interested in expected regret, where the expectation is taken over the context vectors, the rewards, and the algorithm's random seed, and Bayesian regret, where the expectation is taken over all of the above and the prior over $\theta$.

We introduce some notation in order to describe and analyze algorithms in this model. We write $x_t$ for $x_{a_t,t}$, the context vector chosen at time $t$. Let $X_t \in \mathbb{R}^{d \times d}$ be the context matrix at time $t$, a matrix whose rows are vectors $x_1, \ldots, x_t \in \mathbb{R}^d$. A $d \times d$ matrix $Z_t := \sum_{t=1}^t x_t x_t^\top = X_t^\top X_t$, called the empirical covariance matrix, is an important concept in some of the prior work on linear contextual bandits (e.g., Abbasi-Yadkori et al. [2011] and Kannan et al. [2018]), as well as in this chapter.

**Optimism under uncertainty.** Optimism under uncertainty is a common paradigm in problems with an explore-exploit tradeoff [Bubeck and Cesa-Bianchi 2012]. The idea is to evaluate each action “optimistically”—assuming the best-case scenario for this action—and then choose an action with the best optimistic evaluation. When applied to the basic multi-armed bandit setting, it leads to a well-known algorithm called UCB1 [Auer et al. 2002], which chooses the action with the highest upper confidence bound (henceforth, UCB) on its mean reward. The UCB is computed as the sample average of the reward for this action plus a term that captures the amount of uncertainty.
Optimism under uncertainty has been extended to linear contextual bandits in the LinUCB algorithm [Abbasi-Yadkori et al. 2011, Chu et al. 2011]. The high-level idea is to compute a confidence region \( \Theta_t \subset \mathbb{R}^d \) in each round \( t \) such that \( \theta \in \Theta_t \) with high probability, and choose an action \( a \) that maximizes the optimistic reward estimate \( \sup_{\theta \in \Theta_t} x_a^\top \theta \). Concretely, one uses regression to form an empirical estimate \( \hat{\theta}_t \) for \( \theta \). Concentration techniques lead to high-probability bounds of the form \( |x^\top (\theta - \hat{\theta}_t)| \leq f(t) \sqrt{x^\top Z_t^{-1} x} \), where the interval width function \( f(t) \) may depend on hyperparameters and features of the instance. LinUCB simply chooses an action

\[
 a_t^{\text{LinUCB}} := \arg \max_a x_a^\top \hat{\theta}_t + f(t) \sqrt{x_a^\top Z_t^{-1} x_a},
\]  

Among other results, Abbasi-Yadkori et al. [2011] use

\[
 f(t) = S + \sqrt{d c_0 \log(T + tTL^2)},
\]

where \( L \) and \( S \) are known upper bounds on \( |x_{a,t}| \) and \( |\theta| \), respectively, and \( c_0 \) is a parameter. For any \( c_0 \geq 1 \), they obtain regret \( O(dS \sqrt{c_0 K T}) \), with only a polylog dependence on \( TL/d \).

### 3.2 GroupExternality of Exploration

In this section, we study the externalities of exploration at a group level, quantifying how much the presence of one population impacts the rewards of another in an online learning system. We consider linear contextual bandits in a setting in which there are two underlying user populations, called the majority and the minority. The user who arrives at round \( t \) is assumed to come from the majority population with some fixed probability and the minority population otherwise, and the population from which the user comes is known to the learner. The tuple of context vectors at time \( t \) is then drawn independently from a fixed group-specific distribution.

We assume there is a single hidden vector \( \theta_t \), and that the distribution of rewards conditioned on the chosen context vector is the same for both groups. Only the distribution over tuples of available context vectors differs between groups. This implies that externalities cannot be explained by the absence of a good policy since there always exists a policy that is simultaneously optimal for everyone. This allows us to focus only on externalities inherent to the process of exploration.

We define the minority regret to be the regret experienced by the minority. The group externality imposed on the minority by the majority is then the difference
between the minority regret of an algorithm run on the minority alone and the minority regret of the same algorithm run on the full population. A negative group externality implies that the minority is worse off due to the presence of the majority. It is generally more meaningful to bound the multiplicative difference between the minority regret obtained with and without the majority present. Several of our results have this form.

We first ask whether large group externalities can exist. We show that on a simple toy example a large negative group externality arises under LinUCB while a slight variant of this example leads to a large positive externality. Put another way, more available data can lead to either better or worse outcomes for the users of a system. We show that this general phenomenon is unavoidable. That is, no algorithm can simultaneously approximate the minority regret of LinUCB run on the full population and LinUCB run on the minority alone, up to any $o(\sqrt{T})$ multiplicative factor.

### 3.2.1 Two-bridge Instance

We consider a toy example, motivated by a scenario in which a learner is choosing driving routes for two groups of users. Each user starts at point $A, B,$ or $C$, and wants to get to the same destination, point $D$, which requires taking one of two bridges, as shown in Figure 3.1. The travel costs for each of the two bridges are unknown. For simplicity, assume all other edges are known to have 0 cost.

Suppose that 95% of users are in the majority group. All of these users start at point $A$ and have access only to the top bridge. The other 5% are in the minority. Of these users, 95% start at point $C$, from which they have access only to the bottom bridge. The remaining 5% of the minority users start at point $B$, and have access to both bridges.

Consider the behavior of an algorithm that follows the principle of optimism under uncertainty. If run on the full user population, it will quickly collect many observations of the commute time for the top bridge since all users in the majority group must travel over the top bridge. It will collect relatively fewer observations of the commute time over the bottom bridge. Therefore, when the algorithm is faced with a member of the minority population who starts at point $B$, the algorithm

\[ A \xrightarrow{\theta_1} B \xrightarrow{\theta_2} D \]

\[ C \]

![Figure 3.1](image.png) Visual illustration of the two-bridge instance.
will likely send this user over the bottom bridge in order to collect more data and improve its estimate.

If the same algorithm is instead run on the minority alone, it will quickly collect many more observations of the commute time for the bottom bridge relative to the top. Now when the algorithm is faced with a user who starts at point $B$, it will likely send her over the top bridge.

Which is better depends on which bridge has the longer commute time. If the top bridge is the better option, then the presence of the majority imposes a negative externality on the minority. If not, then the presence of the majority helps. These two scenarios may be difficult to distinguish.

This toy example can be formalized in the linear contextual bandits framework. There are two underlying actions (the two bridges), but these actions are not always available. To capture this, we define a parameter vector $\theta$ in $[0, 1]^2$, with the two coordinates $\theta_1$ and $\theta_2$ representing the expected rewards for taking the top and bottom bridge, respectively. (Though we motivated the example in terms of costs, it can be expressed equivalently in terms of rewards.) There are two possible context vectors: $[1 0]^T$ and $[0 1]^T$. A user has available an action with context vector $[1 0]^T$ if and only if she has access to the top bridge. Similarly, she has available an action with context vector $[0 1]^T$ if and only if she has access to the bottom bridge. The instance can then be formalized as follows.

**Definition 3.1**

**Two-bridge Instance**

The two-bridge instance is an instance of linear contextual bandits. On each round $t$, the user who arrives is from the majority population with probability 0.95, in which case $x_{1,t} = x_{2,t} = [1 0]^T$. Otherwise, the user is in the minority. In this case, with probability 0.95, $x_{1,t} = x_{2,t} = [0 1]^T$ (based on Figure 3.1, we call these $C$ rounds), while with probability 0.05, $x_{1,t} = [1 0]^T$ and $x_{2,t} = [0 1]^T$ ($B$ rounds). We consider two values for the hidden parameter vector $\theta$, $\theta^{(0)} = [1/2 1/2 - \epsilon]^T$ and $\theta^{(1)} = [1/2 - \epsilon 1/2]^T$ where $\epsilon = 1/\sqrt{T}$.

**3.2.2 Performance of LinUCB**

We start by analyzing the performance of LinUCB on the two-bridge instance. Our main result formalizes the intuition above, showing that when $\theta = \theta^{(0)}$ (that is, the top bridge is better) the majority imposes a large negative externality on the minority, while the majority imposes a large positive externality when $\theta = \theta^{(1)}$. We assume rewards are 1-subgaussian.\(^3\)

---

\(^3\) A random variable $X$ is called $\sigma$-subgaussian if $E[e^{\sigma X}] < \infty$. A special case is Gaussians with variance $\sigma^2$. 

Theorem 3.1 Consider LinUCB with any interval width function $f$ satisfying $f(t) \geq 2\sqrt{\log(T)}$.\footnote{For instance, the interval width function in Equation (3.3) satisfies this condition whenever $d_{c_0} \geq 4$, so one can either set $c_0 \geq 2$ or add two more dimensions to the problem instance (and set $\theta_1 = \theta_2 = 0$).} On the two-bridge instance, assuming $1$-subgaussian noise on the rewards, when $\theta = \theta^0$, LinUCB achieves expected minority regret $O(1)$ when run on the minority alone but $\Omega(\sqrt{T})$ when run on the full population. In contrast, when $\theta = \theta^1$, LinUCB achieves expected minority regret $O(1)$ when run on the full population but $\Omega(\sqrt{T})$ when run on the minority alone.

We omit the proofs of the $\Omega(\sqrt{T})$ lower bounds, which both follow a similar structure to the one used in the proof of the general impossibility result in Section 3.2.3; in fact, both of these lower bounds could be stated as an immediate corollary of Theorem 3.2. Essentially, an argument based on KL-divergence shows that it is difficult to distinguish between the case in which $\theta = \theta^0$ and the case in which $\theta = \theta^1$, and therefore LinUCB must choose similar actions in these two cases.

To prove the $O(1)$ upper bounds, we make heavy use of the special structure of the two-bridge instance, which significantly simplifies the analysis of LinUCB. We exploit the fact that the only context vectors available to the learner are the basis vectors $[1 \; 0]^T$ and $[0 \; 1]^T$, which essentially makes this an instance of sleeping bandits [Kleinberg et al. 2010]. In this special case, the covariance matrix $Z_t$ is always diagonal, which simplifies Equation (3.2) and leads to LinUCB choosing the $i$th basis, where $i$ maximizes $(\hat{\theta}_i)_i + f(t)/\sqrt{(Z_t)_{ii}}$ and $(Z_t)_{ii}$ is simply the number of times that this basis vector was already chosen. Additionally, in this setting $(\hat{\theta}_i)_i$ is just the average reward observed for the $i$th basis vector, allowing us to bound the difference between each $(\hat{\theta}_i)_i$ and $\theta_i$ using standard concentration techniques. Using this, we show that with high probability, after a logarithmic number of rounds—during which the learner can amass at most $O(1)$ regret since the worst-case regret on any round is $\epsilon = 1/\sqrt{T}$—the probability that LinUCB chooses the wrong action on a $B$ round is small ($O(1/\sqrt{T})$). This leads to constant regret on expectation.

The proof makes use of the following concentration bound:

**Lemma 3.1** Let $C_t$ be the number of $C$ rounds observed in the first $t$ minority rounds in the two-bridge instance. For any $\delta \in (0, 1)$, with probability at least $1 - \delta$, $C_t \geq 0.9t$ for all $t \geq 760 \log(T/\delta)$.

**Proof.** We apply the following form of the Chernoff bound:

$$\Pr\left[ C_t \leq (1 - \gamma) \mathbb{E}[C_t] \right] \leq \exp\left(-\frac{\gamma^2}{2} \mathbb{E}[C_t]\right).$$

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$$\Pr\left[ C_t \leq (1 - \gamma) \mathbb{E}[C_t] \right] \leq \exp\left(-\frac{\gamma^2}{2} \mathbb{E}[C_t]\right).$$
Setting $\gamma = 1/19$, we get

$$
\Pr \left[ C_t \leq \frac{9}{10} \right] = \Pr \left[ C_t \leq \left( 1 - \frac{1}{19} \right)^{19/20} t \right] \leq \exp \left( - \frac{(1/19)^2 \cdot 19}{20} \right) \\
= \exp \left( - \frac{t}{760} \right) \leq \frac{\delta}{T}
$$

for $t \geq 760 \log(T/\delta)$. Applying a union bound over all $T$ rounds, we have $C_t \geq 0.9t$ for all $t \geq 760 \log(T/\delta)$ with probability at least $1 - \delta$.

**Proof of Theorem 3.1.** Now, consider LinUCB run on the minority population alone on the two-bridge instance with $\theta = \theta^{(0)}$. Since we are considering running LinUCB on the minority only, majority rounds are irrelevant, so throughout this proof we abuse notation and use $t \in \{1, \ldots, T_0\}$ for some $T_0 \leq T$ to index minority rounds. $T$ is still the total number of (minority plus majority) rounds.

This proof heavily exploits the special structure of the two-bridge instance to simplify the analysis of LinUCB. In particular, we exploit the fact that the only contexts ever available are the basis vectors $[1 \ 0]^T$ and $[0 \ 1]^T$. This implies that the covariance matrix $Z_t$ is always diagonal, which greatly simplifies the expression for the chosen action in Equation (3.2). The optimistic estimate of the reward for choosing the $i$th basis vector is simply

$$
UCB'_i := (\hat{\theta}_i)_i + f(t) / \sqrt{(Z_t)_{ii}}.
$$

(3.4)

Additionally, in this special case, $(Z_t)_{ii}$ is simply the number of times that the $i$th basis vector was chosen over the first $t$ minority rounds, and $(\hat{\theta}_i)_i$ is the average reward observed over the $(Z_t)_{ii}$ rounds on which it was chosen.

Using this fact, we can apply concentration bounds to bound the difference between each $(\hat{\theta}_i)_i$ and $\theta_i$. Since rewards were assumed to be 1-subgaussian, Lemma C.7 and a union bound give us that for any $\delta_1$, for any $t$, with probability at least $1 - 4\delta_1$, for all $i \in \{1, 2\}$,

$$
|\theta_i - (\hat{\theta}_i)_i| \leq \sqrt{2 \log(1/\delta_i)/(Z_t)_{ii}}
$$

(3.5)

Let $B_t$ and $C_t$ be the number of $B$ and $C$ rounds, respectively, before round $t$. By Lemma 3.1, for any $\delta_2$, with probability $1 - \delta_2$, $C_t \geq 9B_t$ when $t \geq 760 \log(T/\delta_2)$. Suppose this is the case. Since it is only possible to choose $[1 \ 0]$ on $B$ rounds, we have $(Z_t)_{11} \leq B_T$. Similarly, since the algorithm can only choose $[0 \ 1]$ on every $C$ round, $(Z_t)_{22} \geq C_T \geq 9B_T$. Fixing $\delta_1 = 1/\sqrt{T}$ and using the assumption that
\[ f(t) \geq 2\sqrt{\log(T)}, \] Equations (3.4) and (3.5) then imply that for any \( t \geq 760 \log(T/\delta_2) \), with probability at least \( 1 - 2\delta_1 = 1 - 2\sqrt{T} \),

\[
\text{UCB}_1^t \geq \theta_1 - \sqrt{\frac{2 \log(\sqrt{T})}{(Z_t)_{11}}} + \frac{f(t)}{\sqrt{(Z_t)_{11}}} \geq \frac{1}{2} + \frac{1}{2\sqrt{B_t}},
\]

and similarly,

\[
\text{UCB}_2^t \leq \theta_2 + \sqrt{\frac{2 \log(\sqrt{T})}{(Z_t)_{22}}} + \frac{f(t)}{\sqrt{(Z_t)_{22}}} \leq \frac{1}{2} - \varepsilon + \frac{1}{2} + \frac{1}{2\sqrt{B_t}} \leq \text{UCB}_1^t.
\]

This shows that with probability at least \( 1 - \delta_2 \), after the first \( 760 \log(T/\delta_2) \) rounds, LinUCB picks \([1 \ 0] \) on each \( B \) round with probability at least \( 1 - 2\delta_1 \), leading to zero regret on that round. To turn this into a bound on expected regret, first note that with at most \( \delta_2 \) probability, the argument above fails to hold, in which case the minority regret is still bounded by \( \varepsilon B_T \leq \varepsilon T \). When the argument above holds, LinUCB may suffer up to \( \varepsilon \) regret on each of the first \( 760 \log(T/\delta_2) \) majority rounds. On each additional round, there is a failure probability of \( 2\delta_1 \), and in this case LinUCB again suffers regret of at most \( \varepsilon \). Putting this together and setting \( \delta_2 = 1/\sqrt{T} \), we get that the expected regret is bounded by \( \delta_2 \varepsilon T + 760 \log(T/\delta_2) \varepsilon + 4\delta_1 \varepsilon T = O(1) \).

### 3.2.3 An Impossibility Result

It is natural to ask whether it is possible to design an algorithm that can distinguish between the two scenarios analyzed above, obtaining minority regret that is close to the best of LinUCB run on the minority alone and LinUCB run on the full population on any problem instance. In this section, we show that the answer is no. In particular, we prove that on the two-bridge instance, if \( \Pr[\theta = \theta^{(0)}] = \Pr[\theta = \theta^{(1)}] = 1/2 \), then any algorithm must suffer \( \Omega(\sqrt{T}) \) regret on expectation (and therefore \( \Omega(\sqrt{T}) \) minority regret since all regret is incurred by minority users).

To prove this result, we begin by formalizing the idea that it is hard to distinguish between the case in which \( \theta = \theta^{(0)} \) and the case in which \( \theta = \theta^{(1)} \). To do so, we bound the KL-divergence between the joint distributions over the sequences of context vectors, actions taken by the given algorithm, and the given algorithm’s rewards that are induced by the two choices of \( \theta \). By applying the high-probability Pinsker lemma [Tsybakov 2009], we show that a low KL-divergence between these distributions implies that the algorithm must be likely either to choose the top bridge on \( B \) rounds more than half the time when the bottom bridge is better or to choose the bottom bridge on \( B \) rounds more than half the time when the top bridge
is better, either of which would lead to high \(\Omega(\sqrt{T})\) regret as long as the number of 
B rounds is sufficiently large. To finish the proof, we use a simple Chernoff bound 
to show that the number of B rounds is large with high probability.

To derive the KL-divergence bound, we make use of the assumption that the 
realized rewards \(r_t\) at each round are either 0 or 1. This assumption is not strictly 
necessary. An analogous argument could be made, for instance, for real-valued 
rewards with Gaussian noise.

**Theorem 3.2** On the two-bridge instance with realized rewards \(r_t \in \{0, 1\}\), any algorithm must 
incur \(\Omega(\sqrt{T})\) minority regret in expectation when \(\Pr[\theta = \theta^{(0)}] = \Pr[\theta = \theta^{(1)}] = \frac{1}{2}\).

Note that “any algorithm” here includes algorithms run on the minority alone, 
essentially ignoring data from the majority. Theorems 3.1 and 3.2 immediately 
implicate the following corollary.

**Corollary 3.1** No algorithm can simultaneously approximate the minority regret of both Lin-
UCB run on the minority and LinUCB run on the full population up to any \(o(\sqrt{T})\) 
multiplicative factor.

**Proof of Theorem 3.2.** Fix any algorithm \(A\). We will first derive an \(\Omega(\sqrt{T})\) lower 
bound on the expected regret of \(A\) conditioned on the number of \(B\) rounds, \(B_T\), 
being large. To complete the proof, we then show that \(B_T\) is large with high 
probability.

Let \(h_t = \{(x_{1,t}, x_{2,t}, a_t, r_t)\}_{t=1}^T\) be a history of all context vectors, chosen actions, 
and rewards up to round \(t\), with \(h_0 = \emptyset\). Running \(A\) on the two-bridge instance 
with \(\theta = \theta^{(0)}\) induces a distribution over histories \(h_T\). Let \(P\) denote the conditional 
distribution of these histories, conditioned on the event that \(B_T \geq T/800\). That is, 
we define

\[
P(h_T) := \Pr[h_T \mid \theta = \theta^{(0)}, B_T \geq T/800].
\]

Similarly, we define

\[
Q(h_T) := \Pr[h_T \mid \theta = \theta^{(1)}, B_T \geq T/800].
\]

We first show that \(\text{KL}(P(h_T) \mid Q(h_T))\) is upper bounded a constant that does not 
depend on \(T\). By the chain rule for KL divergences, since \(r_t\) is independent of any 
previous contexts, actions, or rewards conditioned on \(x_t\),

\[
\text{KL}(P(h_T) \mid Q(h_T))
\]

\[
= \sum_{t=1}^T \mathbb{E}_{h_{t-1}} \text{KL}(P((x_{1,t}, x_{2,t}, a_t) \mid h_{t-1}) \mid Q((x_{1,t}, x_{2,t}, a_t) \mid h_{t-1}))
\]

\[
+ \sum_{t=1}^T \mathbb{E}_{x_{1,t}, x_{2,t}, a_t} \text{KL}(P(r_t \mid x_{1,t}, x_{2,t}, a_t) \mid Q(r_t \mid (x_{1,t}, x_{2,t}, a_t))).
\]
Since the choice of context vectors available at time $t$ is independent of the value of the parameter $\theta$ and $\mathcal{A}$ may only base its choices on the observed history and current choice of contexts, it is always the case that $P((x_{1,t}, x_{2,t}, a_t) \mid h_{t-1}) = Q((x_{1,t}, x_{2,t}, a_t) \mid h_{t-1})$, so the first sum in this expression is equal to 0.

To bound the second sum, we make use of the assumption that $r_t \in \{0, 1\}$ for all $t$. Lemma C.9 then tells us that for any round $t$, $\text{KL}(P(r_t \mid x_{1,t}, x_{2,t}, a_t))$ is at most $\frac{7\epsilon^2}{2}$ since the probability of getting reward 1 conditioned on a chosen context is always either $1/2$ or $1/2 - \epsilon$. Putting this together, we get that

$$\text{KL}(P(h_T) \mid Q(h_T)) \leq \frac{7\epsilon^2 T}{2} = \frac{7}{2}.$$

Now, let $E$ be the event that the algorithm $\mathcal{A}$ chooses arm 2 on at least half of the $B$ rounds, conditioned on $B_T \geq T/800$. If $E$ occurs when $\theta = \theta^{(1)}$, the regret of $\mathcal{A}$ is at least $B_T h_T / 2$, which is on the order of $\sqrt{T}$ when $B_T \geq T/800$. If $E$ does not occur (i.e., $\overline{E}$ occurs) when $\theta = \theta^{(1)}$, $\mathcal{A}$ again has regret at least $B_T e / 2$. We will use the bound on KL divergence to show that one of these bad cases happens with high probability.

By Lemma C.8,

$$P(E) + Q(\overline{E}) \geq \frac{1}{2} e^{-\text{KL}(P(h_T) \mid Q(h_T))} \geq \frac{1}{2} e^{-7/2}.$$

Let $R$ be the regret of $\mathcal{A}$. We then have that

$$\mathbb{E} \left[ R \mid B_T \geq \frac{T}{800} \right] = \frac{1}{2} \mathbb{E} \left[ R \mid \theta = \theta^{(0)}, B_T \geq \frac{T}{800} \right] + \frac{1}{2} \mathbb{E} \left[ R \mid \theta = \theta^{(1)}, B_T \geq \frac{T}{800} \right]$$

$$\geq \frac{1}{2} \mathbb{E} \left[ R \mid \theta = \theta^{(0)}, B_T \geq \frac{T}{800} \right] + \frac{1}{2} \mathbb{E} \left[ R \mid \theta = \theta^{(1)}, B_T \geq \frac{T}{800} \right]$$

$$\geq \frac{1}{2} (P(E) + Q(\overline{E})) \frac{\sqrt{T}}{1600}$$

$$\geq \frac{\sqrt{T} e^{-7/2}}{6400}.$$

It remains to bound the probability that $B_T \geq T/800$. By a Chernoff bound,

$$\Pr \left[ B_T \leq \frac{T}{800} \right] = \Pr \left[ B_T \leq \frac{\mathbb{E}[B_T]}{2} \right] \leq \exp \left( - \frac{\mathbb{E}[B_T]}{8} \right) = \exp \left( - \frac{T}{3200} \right).$$

5. If we instead assumed rewards had Gaussian noise with variance $\sigma^2$, we would have $\text{KL}(P(r_t \mid x_{1,t}, x_{2,t}, a_t)) = \epsilon^2 / (2\sigma^2)$, and the proof would still go through.
Thus, for any $\delta \in (0, 1)$, if $T \geq 3200 \log(1/\delta)$, then with probability at least $1 - \delta$, $B_T \geq T/800$. In particular, let $\delta = 1/2$. Then if $T \geq 3200 \log 2$, we have

$$
\mathbb{E}[R] \geq \Pr\left[B_T \geq \frac{T}{800}\right] \mathbb{E}\left[R \mid B_T \geq \frac{T}{800}\right] \geq \left(\frac{1}{2}\right) \left(\frac{\sqrt{T}e^{-\frac{1}{2}}}{6400}\right).
$$

This completes the proof that the regret of $A$ is $\Omega(\sqrt{T})$ on this problem instance.

### 3.3 Greedy Algorithms and LinUCB with Perturbed Contexts

We now turn our attention to externalities at an individual level. We interpret exploration as a potential externality imposed on the current user of a system by future users since the current user would prefer the learner to take the action that appears optimal. One could avoid such externalities by running the greedy algorithm, which does just that, but it is well known that the greedy algorithm performs poorly in the worst case. In this section, we build on a recent line of work analyzing the conditions under which inherent data diversity leads the greedy algorithm to perform well.

We analyze the expected performance of the greedy algorithm under small random perturbations of the context vectors. We focus on greedy algorithms that consume new data in batches rather than every round. We consider both Bayesian and frequentist versions, BatchBayesGreedy and BatchFreqGreedy. Our main result is that for any specific problem instance both algorithms match the Bayesian regret of any algorithm on that particular instance up to polylogarithmic factors. We also prove that under the same perturbation assumptions, LinUCB achieves regret $\mathcal{O}(T^{1/3})$ for each realization of $\theta$, which implies a worst-case Bayesian regret of $\mathcal{O}(T^{1/3})$ for the greedy algorithms. Finally, we repurpose our analysis to derive a positive result in the group setting, implying that the impossibility result of Section 3.2.3 breaks down when the data is sufficiently diverse.

**Setting and notation.** We consider a Bayesian version of linear contextual bandits, with $\theta$ drawn from a known multivariate Gaussian prior $\mathcal{P} = \mathcal{N}(\bar{\theta}, \Sigma)$, with $\bar{\theta} \in \mathbb{R}^d$ and invertible $\Sigma \in \mathbb{R}^{d \times d}$.

To capture the idea of data diversity, we assume the context vectors on each round $t$ are generated using the following *perturbed context generation* process: First, a tuple $(\mu_{1,t}, \ldots, \mu_{K,t})$ of *mean context vectors* is drawn independently from some fixed distribution $D_{\mu}$ over $\left(\mathbb{R} \cup \{\bot\}\right)^K$, where $\mu_{a,t} = \bot$ means action $a$ is not available. For each available action $a$, the context vector is then $x_{a,t} = \mu_{a,t} + \epsilon_{a,t}$, where $\epsilon_{a,t}$ is a vector of random noise. Each component of $\epsilon_{a,t}$ is drawn independently from a zero-mean Gaussian with standard deviation $\rho$. We refer to $\rho$ as the *perturbation*
size. In general, our regret bounds deteriorate if $\rho$ is very small. Together we refer to a distribution $D_\rho$, prior $\mathcal{P}$, and perturbation size $\rho$ as a problem instance.

We make several technical assumptions. First, the distribution $D_\rho$ is such that each context vector has bounded 2-norm, that is, $|\mu_{a,t}|_2 \leq 1$. It can be arbitrary otherwise. Second, the perturbation size needs to be sufficiently small, $\rho \leq 1/\sqrt{d}$. Third, the realized reward $r_{a,t}$ for each action $a$ and round $t$ is $r_{a,t} = x_{a,t}^\top \theta + \eta_{a,t}$, the mean reward $x_{a,t}^\top \theta$ plus standard Gaussian noise $\eta_{a,t} \sim \mathcal{N}(0,1)$. The history up to round $t$ is denoted by the tuple $h_t = ((x_1, r_1), \ldots, (x_t, r_t))$.

The greedy algorithms. For the batch version of the greedy algorithm, time is divided in batches of $Y$ consecutive rounds each. When forming its estimate of the optimal action at round $t$, the algorithm may only use the history up to the last round of the previous batch, denoted $t_0$.

BatchBayesGreedy forms a posterior over $\theta$ using prior $\mathcal{P}$ and history $h_{t_0}$. In round $t$, it chooses the action that maximizes reward in expectation over this posterior. This is equivalent to choosing

$$a_t = \arg \max_a x_{a,t}^\top \hat{\theta}_t^{bay}, \quad \text{where } \hat{\theta}_t^{bay} := \mathbb{E}[\theta \mid h_{t_0}].$$

BatchFreqGreedy does not rely on any knowledge of the prior. It chooses the best action according to the least squares estimate of $\theta$, denoted $\hat{\theta}_t^{fre}$, computed with respect to history $h_{t_0}$:

$$a_t = \arg \max_a x_{a,t}^\top \hat{\theta}_t^{fre}, \quad \text{where } \hat{\theta}_t^{fre} = \arg \min_{\theta'} \sum_{t=1}^{t_0} (\theta')^\top x_t - r_t)^2.$$  

3.3.1 Main Results

We first state our main results before describing the intuition behind them. We state each theorem in terms of the main relevant parameters $T$, $K$, $d$, $Y$, and $\rho$. First, we prove that in expectation over the random perturbations both greedy algorithms favorably compare to any other algorithm.

Theorem 3.3

With perturbed context generation, there is some $Y_0 = \text{polylog}(d,T)/\rho^2$ such that with batch duration $Y \geq Y_0$, the following holds. Fix any bandit algorithm, and let $R_0(T)$ be its Bayesian regret on a particular problem instance. Then on that same instance,

(a) BatchBayesGreedy has Bayesian regret at most $Y \cdot R_0(T/Y) + \tilde{O}(1/T)$.

(b) BatchFreqGreedy has Bayesian regret at most $Y \cdot R_0(T/Y) + \tilde{O}(\sqrt{d}/\rho^2)$.

---

6. Our analysis can be easily extended to handle reward noise of fixed variance, that is, $\eta_{a,t} \sim \mathcal{N}(0,\sigma^2)$. BatchFreqGreedy would not need to know $\sigma$. BatchBayesGreedy would need to know either $\Sigma$ and $\sigma$ or just $\Sigma/\sigma^2$. 

Our next result asserts that the Bayesian regret for LinUCB and both greedy algorithms is on the order of (at most) $T^{1/3}$. This result requires additional technical assumptions.

**Theorem 3.4** Assume that the maximal eigenvalue of the covariance matrix $\Sigma$ of the prior $\mathcal{P}$ is at most $1,^7$ and the mean vector satisfies $|\theta|_2 \geq 1 + \sqrt{3 \log T}$. With perturbed context generation,

(a) With appropriate parameter settings, LinUCB has Bayesian regret $\tilde{O}(d^2 K^{2/3} T^{1/3}/\rho^2)$.

(b) If $Y \geq Y_0$ as in Theorem 3.3, then both BatchBayesGreedy and BatchFreqGreedy have Bayesian regret at most $\tilde{O}(d^2 K^{2/3} T^{1/3}/\rho^2)$.

The assumption $|\theta|_2 \geq 1 + \sqrt{3 \log T}$ in Theorem 3.4 can be replaced with $d \geq \log T / \log \log T$. We use Theorem 3.4(b) to derive an “approximately prior-independent” result for BatchFreqGreedy. (For clarity, we state it for independent priors.) The bound in Theorem 3.4(b) deteriorates if $\mathcal{P}$ gets very sharp, but it suffices if $\mathcal{P}$ has standard deviation on the order of (at least) $T^{-2/3}$.

**Corollary 3.2** Assume that the prior $\mathcal{P}$ is independent over the components of $\theta$, with variance $\kappa^2 \leq 1$ in each component. Suppose the mean vector satisfies $|\theta|_2 \geq 1 + \sqrt{3 \log T}$. With perturbed context generation, if $Y \geq Y_0$ as in Theorem 3.3, then BatchFreqGreedy has Bayesian regret at most $\tilde{O}(d^2 K^{2/3} T^{1/3}/\rho^2)$ as long as $\kappa \geq T^{-2/3}$.

Finally, we derive a positive result on group externalities. We find that with perturbed context generation, the minority Bayesian regret of the greedy algorithms (i.e., the Bayesian regret incurred on minority rounds) is small compared to the minority Bayesian regret of any algorithm, whether run on the full population or on the minority alone. This sidesteps the impossibility result of Section 3.2.3.

**Theorem 3.5** Assume $Y \geq Y_0$ as in Theorem 3.3 and perturbed context generation. Fix any bandit algorithm and instance, and let $R_{\text{min}}(T)$ be the minimum of its minority Bayesian regrets when it is only run over minority rounds or when it is run over the full population. Both greedy algorithms run on the full population achieve minority Bayesian regret at most $Y \cdot R_{\text{min}}(T) + \tilde{O}(\sqrt{d}/\rho^2)$.

### 3.3.2 Key Techniques

The key idea behind our approach is to show that, with perturbed context generation, BatchBayesGreedy collects data that is informative enough to “simulate” the history of contexts and rewards from the run of any other algorithm ALG over fewer

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7. In particular, if $\mathcal{P}$ is independent across the coordinates of $\theta$, then the variance in each coordinate is at most 1.
3.3 Greedy Algorithms and LinUCB with Perturbed Contexts

rounds. This implies that it remains competitive with ALG since it has at least as much information and makes myopically optimal decisions.

We use the same technique to prove a similar simulation result for BatchFreq-Greedy. To treat both algorithms at once, we define a template that unifies them. A bandit algorithm is called batch-greedy-style if it divides the timeline in batches of \( Y \) consecutive rounds each, in each round \( t \) chooses some estimate \( \hat{\theta}_t \) of \( \theta \), based only on the data from the previous batches, and then chooses the best action according to this estimate, so that \( a_t = \arg \max_{x} \hat{\theta}_t^\top x \). For a batch that starts at round \( t_0 + 1 \), the batch history is the tuple \((x_{t_0+1}, \ldots, x_{t_0+r}) : r \in [Y]\) and the batch context matrix is the matrix \( X \) whose rows are vectors \((x_{t_0+t} : r \in [Y])\); here \([Y] = \{1, \ldots, Y\}\). Similarly to the “empirical covariance matrix,” we define the batch covariance matrix as \( X^\top X \).

Let us formulate what we mean by “simulation.” We want to use the data collected from a single batch in order to simulate the reward for any one context \( x \). More formally, we are interested in the randomized function that takes a context \( x \) and outputs an independent random sample from \( N(\theta^\top x, 1) \). We denote it \( \text{Rew}_\theta(\cdot) \); this is the realized reward for an action with context vector \( x \).

**Definition 3.2** Consider batch \( B \) in the execution of a batch-greedy-style algorithm. Batch history \( h_B \) can simulate \( \text{Rew}_\theta(\cdot) \) up to radius \( R > 0 \) if there exists a function \( g : \{\text{context vectors}\} \times \{\text{batch histories } h_B\} \to \mathbb{R} \) such that \( g(x, h_B) \) is identically distributed to \( \text{Rew}_\theta(x) \) conditional on the batch context matrix, for all \( \theta \) and all context vectors \( x \in \mathbb{R}^d \) with \(|x|_2 \leq R \).

Let us comment on how it may be possible to simulate \( \text{Rew}_\theta(x) \). For intuition, suppose that \( x = \frac{1}{2} x_1 + \frac{1}{2} x_2 \). Then \( \frac{1}{2} r_1 + \frac{1}{2} r_2 + \xi \) is drawn independently from \( N(0, \frac{1}{2}) \). Thus, we can define \( g(x, h) = \frac{1}{2} r_1 + \frac{1}{2} r_2 + \xi \) in Definition 3.2. We generalize this idea and show that a batch history can simulate \( \text{Rew}_\theta \) as long as the batch covariance matrix has a sufficiently large minimum eigenvalue, which holds with high probability when the batch size is large.

**Lemma 3.2** With perturbed context generation, there is some \( Y_0 = \text{polylog}(d, T)/\rho^2 \) and \( R = O(\sqrt{d \log(TKd)}) \) such that with probability at least \( 1 - T^{-2/3} \) any batch history from a batch-greedy-style algorithm can simulate \( \text{Rew}_\theta(\cdot) \) up to radius \( R \), as long as \( Y \geq Y_0 \).

If the batch history of an algorithm can simulate \( \text{Rew}_\theta \), the algorithm has enough information to simulate the outcome of a fresh round of any other algorithm \( \text{ALG} \). Eventually, this allows us to use a coupling argument in which we couple a run of BatchBayesGreedy with a slowed-down run of \( \text{ALG} \), and prove that the former accumulates at least as much information as the latter, and therefore the Bayesian-greedy action choice is, in expectation, at least as good as that of \( \text{ALG} \). This leads to Theorem 3.3(a). We extend this argument to a scenario in which both the greedy
algorithm and ALG measure regret over a randomly chosen subset of the rounds, which leads to Theorem 3.5.

To extend these results to BatchFreqGreedy, we consider a hypothetical algorithm that receives the same data as BatchFreqGreedy but chooses actions based on the (batched) Bayesian-greedy selection rule. We analyze this hypothetical algorithm using the same technique as above, and then argue that its Bayesian regret cannot be much smaller than that of BatchFreqGreedy. Intuitively, this is because the two algorithms form almost identical estimates of \( \theta \), differing only in the fact that the hypothetical algorithm uses the \( P \) as well as the data. We show that this difference amounts to effects on the order of \( 1/t \), which add up to a maximal difference of \( O(\log T) \) in Bayesian regret.

### 3.4 Analysis: LinUCB with Perturbed Contexts

In this section, we prove Theorem 3.4(a), a Bayesian regret bound for the LinUCB algorithm under perturbed context generation. We focus on a version of LinUCB from Abbasi-Yadkori et al. [2011], as defined in (3.3).

Recall that the interval width function in (3.3) is parameterized by numbers \( L, S, c_0 \). We use

\[
L \geq 1 + \rho \sqrt{2d \log(2T^3Kd)}, \\
S \geq \|\bar{\theta}\|_2 + \sqrt{3d \log T} \quad \text{(and } S < T) \\
c_0 = 1.
\]  

(3.8)

Recall that \( \rho \) denotes perturbation size and \( \bar{\theta} = \mathbb{E}[\theta] \), the prior means of the latent vector \( \theta \).

**Remark 3.1**

Ideally, we would like to set \( L, S \) according to (3.8) with equalities. We consider a more permissive version with inequalities so as to not require the exact knowledge of \( \rho \) and \( \|\bar{\theta}\|_2 \).

While the original result in Abbasi-Yadkori et al. [2011] requires \( |x_{a,t}|_2 \leq L \) and \( |\theta|_2 \leq S \), in our setting this only happens with high probability.

We prove the following theorem (which implies Theorem 3.4(a)):

**Theorem 3.6**

Assume perturbed context generation. Further, suppose that the maximal eigenvalue of the covariance matrix \( \Sigma \) of the prior \( P \) is at most 1, and the mean vector satisfies \( \|\bar{\theta}\|_2 \geq 1 + \sqrt{3 \log T} \). The version of LinUCB with interval width function (3.3) and parameters given by (3.8) has Bayesian regret at most

\[
T^{1/3} \left( d^2 S (K^2/\rho)^{1/3} \right) \cdot \text{polylog}(TKLd).
\]

(3.9)
Remark 3.2  
The theorem also holds if the assumption on $|\bar{\theta}|_2$ is replaced with $d \geq \frac{\log T}{\log \log T}$. The only change in the analysis is that in the concluding steps (Section 3.4.2) we use Lemma 3.5(b) instead of Lemma 3.5(a).

On a high level, our analysis proceeds as follows. We massage algorithm's regret so as to elucidate the dependence on the number of rounds with small “gap” between the best and second-best action, call it $N$. This step does not rely on perturbed context generation and makes use of the analysis from Abbasi-Yadkori et al. [2011]. The crux is that we derive a much stronger upper-bound on $E[N]$ under perturbed context generation. The analysis relies on some non-trivial technicalities on bounding the deviations from the “high-probability” behavior, which are gathered in Section 3.4.1.

We reuse the analysis in Abbasi-Yadkori et al. [2011] via the following lemma. To state this lemma, define the instantaneous regret at time $t$ as $R_t = \theta^T x_{t} - \theta^T x_{a,t}$, and let

$$\beta_T = \left( \sqrt{d \log \left( T (1 + TL) \right)} + S \right)^2.$$ 

Lemma 3.3  
Abbasi-Yadkori et al. [2011]

Consider a problem instance with reward noise $\mathcal{N}(0, 1)$ and a specific realization of latent vector $\theta$ and contexts $x_{a,t}$. Consider LinUCB with parameters $L, S, c_0$ that satisfy $|x_{a,t}|_2 \leq L, |\theta|_2 \leq S$, and $c_0 = 1$. Then

(a) with probability at least $1 - \frac{1}{T}$ (over the randomness in the rewards) it holds that

$$\sum_{t=1}^{T} R_t^2 \leq 16 \beta_T \log(\det(Z_t + I)),$$

where $Z_t = \sum_{t=1}^{T} x_{a,t} x_{a,t}^\top \in \mathbb{R}^{d \times d}$ is the “empirical covariance matrix” at time $t$. (b) $\det(Z_t + I) \leq (1 + TL^2/d)^d$.

The following lemma captures the essence of the proof of Theorem 3.6. From here on, we assume perturbed context generation. In particular, reward noise is $\mathcal{N}(0, 1)$.

Lemma 3.4  
Suppose parameter $L$ is set as in (3.8). Consider a problem instance with a specific realization of $\theta$ such that $|\theta|_2 \leq S$. Then for any $\gamma > 0$,

$$E[\text{Regret}(T)] \leq |\theta|_2^{-1/3} \left( \frac{1}{2\sqrt{\pi}} + 16 \beta_T d \log(1 + TL^2/d) \right) \left( \frac{TK^2}{\rho} \right)^{1/3} + \tilde{O}(1).$$

---

8. Lemma 3.3(a) is implicit in the proof of theorem 3 from Abbasi-Yadkori et al. [2011], and Lemma 3.3(b) is asserted by Abbasi-Yadkori et al. [2011, lemma 10].
Proof. We will prove that for any \( \gamma > 0 \),

\[
\mathbb{E} [\text{Regret}(T)] \leq T \cdot \frac{\gamma^2 K^2}{2\rho |\theta|_2 \sqrt{T}} + \frac{16 \beta_T}{\gamma} d \log(1 + TL^2/d) + \tilde{O}(1). \tag{3.10}
\]

The Lemma easily follows by setting \( \gamma = (TK^2/(\rho|\theta|_2))^{-1/3} \).

Fix some \( \gamma > 0 \). We distinguish between rounds \( t \) with \( R_t < \gamma \) and those with \( R_t \geq \gamma \):

\[
\text{Regret}(T) = \sum_{t=1}^{T} R_t \leq \sum_{t \in T_t} R_t + \sum_{t = 1}^{T} \frac{R_t^2}{\gamma} \leq \gamma |T_t| + \frac{1}{\gamma} \sum_{t=1}^{T} R_t^2, \tag{3.11}
\]

where \( T_t = \{ t : R_t \in (0, \gamma) \} \).

We use Lemma 3.3 to upper-bound the second summand in (3.11). To this end, we condition on the event that every component of every perturbation \( \epsilon_a \), has absolute value at most \( \sqrt{2 \log 2T^3Kd} \); denote this event by \( U \). This implies \( |x_a|_2 \leq L \) for all actions \( a \) and all rounds \( t \). By Lemma C.3, \( U \) is a high-probability event: \( \Pr[U] \geq 1 - \frac{1}{T^2} \). Now we are ready to apply Lemma 3.3:

\[
\mathbb{E} \left[ \sum_{t=1}^{T} R_t^2 \mid U \right] \leq 16 d \beta_T \log(1 + tl^2/d). \tag{3.12}
\]

To plug this into (3.11), we need to account for the low-probability event \( \bar{U} \). We need to be careful because \( R_t \) could, with low probability, be arbitrarily large. By Lemma 3.6 with \( \ell = 0 \),

\[
\mathbb{E} [R_t \mid \bar{U}] \leq 2 \left( |\theta|_2 \left( 1 + \rho (1 + \sqrt{2 \log K}) + \sqrt{2 \log (2T^3Kd)} \right) \right).
\]

\[
\mathbb{E} [\text{Regret}(T) \mid \bar{U}] \Pr[\bar{U}] = \sum_{t=1}^{T} \mathbb{E} [R_t \mid \bar{U}] / T^2 < \tilde{O}(1).
\]

\[
\mathbb{E} [\text{Regret}(T) \mid U] \Pr[U] \leq \gamma \mathbb{E} [\left| T_t \right| ] + \frac{1}{\gamma} \mathbb{E} \left[ \sum_{t=1}^{T} R_t^2 \mid U \right] \quad \text{(by (3.11))}
\]

Putting this together and using (3.12), we obtain:

\[
\mathbb{E} [\text{Regret}(T)] \leq \gamma \mathbb{E} [\left| T_t \right| ] + \frac{16}{\gamma} d \beta_T \log(1 + tl^2/d) + \tilde{O}(1). \tag{3.13}
\]

To obtain (3.10), we analyze the first summand in (3.13). Let \( \Lambda_t \) be the “gap” at time \( t \): the difference in expected rewards between the best and second-best actions at time \( t \) (where “best” and “second-best” is according to expected rewards). Here, we’re taking expectations after the perturbations are applied, so the only randomness comes from the noisy rewards. Consider the set of rounds with small gap, \( \mathcal{G}_t := \{ t : \Lambda_t < \gamma \} \).

Notice that \( r_t \in (0, \gamma) \) implies \( \Lambda_t < \gamma \), so \( |T_t| \leq |\mathcal{G}_t| \).
3.4 Analysis: LinUCB with Perturbed Contexts

In what follows we prove an upper bound on $E[|G_t|]$. This is the step where perturbed context generation is truly used. For any two arms $a_1$ and $a_2$, the gap between their expected rewards is

$$\theta^T(x_{a_1,t} - x_{a_2,t}) = \theta^T(\mu_{a_1,t} - \mu_{a_2,t}) + \theta^T(\epsilon_{a_1,t} - \epsilon_{a_2,t}).$$

Therefore, the probability that the gap between those arms is smaller than $\gamma$ is

$$\Pr[|\theta^T(\mu_{a_1,t} - \mu_{a_2,t}) + \theta^T(\epsilon_{a_1,t} - \epsilon_{a_2,t})| \leq \gamma] = \Pr[-\gamma - \theta^T(\mu_{a_1,t} - \mu_{a_2,t}) \leq \theta^T(\epsilon_{a_1,t} - \epsilon_{a_2,t}) \leq \gamma - \theta^T(\mu_{a_1,t} - \mu_{a_2,t})].$$

Since $\theta^T\epsilon_{a_1,t}$ and $\theta^T\epsilon_{a_2,t}$ are both distributed as $\mathcal{N}(0, \rho^2|\theta|_2^2)$, their difference is $\mathcal{N}(0, 2\rho^2|\theta|_2^2)$. The maximum value that the Gaussian measure takes is $\frac{1}{2\rho|\theta|_2\sqrt{\pi}}$, and the measure in any interval of width $2\gamma$ is therefore at most $\frac{\gamma}{\rho|\theta|_2\sqrt{\pi}}$. This gives us the bound

$$\Pr[|\theta^T(\mu_{a_1,t} - \mu_{a_2,t}) + \theta^T(\epsilon_{a_1,t} - \epsilon_{a_2,t})| \leq \gamma] \leq \frac{\gamma}{\rho|\theta|_2\sqrt{\pi}}.$$

Union-bounding over all $K \choose 2$ pairs of actions, we have

$$\Pr[\Delta_t \leq \gamma] \leq \Pr\left[\bigcup_{a_1, a_2 \in [K]} |\theta^T(x_{a_1,t} - x_{a_2,t})| \leq \gamma\right] \leq \frac{K^2}{2} \frac{\gamma}{\rho|\theta|_2\sqrt{\pi}}. $$

$$\mathbb{E}[|G_t|] = \sum_{t=1}^{T} \Pr[\Delta_t \leq \gamma] \leq T \cdot \frac{K^2}{2} \frac{\gamma}{\rho|\theta|_2\sqrt{\pi}}.$$

Plugging this into (3.13) (recalling that $|T_t| \leq |G_t|$) completes the proof.

### 3.4.1 Bounding the Deviations

We make use of two results that bound deviations from the “high-probability” behavior, one on $|\theta|_2$ and another on instantaneous regret. First, we prove high-probability upper and lower bounds on $|\theta|_2$ under the conditions in Theorem 3.6. Essentially, these bounds allow us to use Lemma 3.4.

**Lemma 3.5** Assume the latent vector $\theta$ comes from a multivariate Gaussian, $\theta \sim \mathcal{N}(\bar{\theta}, \Sigma)$, here the covariate matrix $\Sigma$ satisfies $\lambda_{\text{max}}(\Sigma) \leq 1$.

(a) If $|\theta|_2 \geq 1 + \sqrt{3\log T}$, then for sufficiently large $T$, with probability at least $1 - \frac{2}{T}$,

$$\frac{1}{2\log T} \leq |\theta|_2 \leq |\bar{\theta}|_2 + \sqrt{3d \log T}. \quad (3.14)$$
(b) Same conclusion if \( d \geq \frac{\log T}{\log \log T} \).

*Proof.* We consider two cases, based on whether \( d \geq \log T / \log \log T \). We need both cases to prove part (a), and we obtain part (b) as an interesting by-product. We repeatedly use Lemma C.6, a concentration inequality for \( \chi^2 \) random variables, to show concentration on the Gaussian norm.

**Case 1:** \( d \geq \log T / \log \log T \).

Since the Gaussian measure is decreasing in distance from 0, the \( \Pr [ \| \theta \|_2 \leq c ] \leq \Pr [ \| \hat{\theta} - \theta \|_2 \leq c ] \) for any \( c \). In other words, the norm of a Gaussian is most likely to be small when its mean is 0. Let \( X = \Sigma^{-1/2}(\hat{\theta} - \theta) \). Note that \( X \) has distribution \( N(0, I) \), and therefore \( |X|_2^2 \) has \( \chi^2 \) distribution with \( d \) degrees of freedom. We can bound this as

\[
\Pr [ \| \hat{\theta} - \theta \|_2 \leq \frac{1}{2 \log T} ] = \Pr [ \| \Sigma^{-1/2}X \|_2 \leq \frac{1}{2 \log T} ] \\
\leq \Pr [ \| \sqrt{\lambda_{\max}(\Sigma)}X \|_2 \leq \frac{1}{2 \log T} ] \\
\leq \Pr [ \| X \|_2 \leq \frac{1}{2 \log T} ] \\
= \Pr [ \| X \|^2_2 \leq \frac{1}{4(\log T)^2} ] \\
\leq \left( \frac{1}{4d(\log T)^2} e^{1-1/(4(\log T)^2 d)} \right)^{d/2} \quad \text{(By Lemma C.6)} \\
\leq \frac{\log \log T}{(\log T)^3} \log T / (2 \log \log T) \quad \text{(}d \geq \log T / \log \log T\text{)} \\
= \frac{T^{\log \log T / (2 \log \log T)}}{T^{3/2}} \\
\leq T^{-1}
\]

Similarly, we can show

\[
\Pr [ \| \hat{\theta} - \theta \|_2 \geq \sqrt{d \log T} ] = \Pr [ \| \Sigma^{-1/2}X \|_2 \geq \sqrt{d \log T} ] \\
\leq \Pr [ \| \sqrt{\lambda_{\max}(\Sigma)}X \|_2 \geq \sqrt{d \log T} ] \\
\leq \Pr [ \| X \|_2 \geq \sqrt{d \log T} ] \\
= \Pr [ \| X \|^2_2 \geq d \log T ] \\
\leq \left( \log T e^{1-\log T} \right)^{d/2} \quad \text{(By Lemma C.6)} \\
\leq (\exp (1 + \log \log T - \log T))^{\log T / (2 \log \log T)} \quad \text{(}d \geq \log T / \log \log T\text{)}
\]
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\[ = T^{(1+\log \log T - \log T)/(2 \log \log T)} \]
\[ \leq T^{-1} \] (For \( T > 1 + 3 \log \log T \))

By the triangle inequality,
\[ |\bar{\theta}_2 - |\bar{\theta} - \theta|_2 \leq |\bar{\theta}_2| \leq |\bar{\theta}_2| + |\bar{\theta} - \theta|_2. \]

Thus, in this case, \( \frac{1}{2 \log T} \leq |\theta|_2 \leq |\bar{\theta}_2| + \sqrt{d \log T} \) with probability at least \( 1 - 2T^{-1} \).

**Case 2:** \( |\bar{\theta}_2| \geq 1 + \sqrt{3 \log T} \) and \( d < \log T / \log \log T \).

For this part of the proof, we just need that \( d < \log T \), which it is by assumption. Using the triangle inequality, if \( |\bar{\theta}_2| \) is large, it suffices to show that \( |\bar{\theta} - \theta|_2 \) is small with high probability. Again, let \( X = \sum_{1}^{-1/2}(\bar{\theta} - \theta) \). Then,
\[
\Pr \left[ |\bar{\theta} - \theta|_2 \geq \sqrt{3 \log T} \right] = \Pr \left[ \sum_{1}^{1/2}X |X|_2 \geq \sqrt{3 \log T} \right]
\[
\geq \Pr \left[ \lambda_{\max}(\Sigma) |X|_2 \geq \sqrt{3 \log T} \right]
\[
= \Pr \left[ |X|_2 \geq \sqrt{3 \log T} \right]
\[
\geq \Pr \left[ |X|_2 \geq \sqrt{3 \log T} \right]
\]

By Lemma C.6,
\[
\Pr \left[ |X|_2 \geq \sqrt{3 \log T} \right] \leq \left( \frac{3 \log T}{e^{1-3 \log T}} \right)^{d/2}
\[
= \left( T^{-3/d} e^{3 \log T} \right)^{d/2}
\[
= T^{-1} \left( T^{-1/d} e^{3 \log T} \right)^{d/2}
\[
\leq T^{-1} \] (for sufficiently large \( T \))

Because \( |\bar{\theta}_2| \geq 1 + \sqrt{3 \log T} \), \( 1 \leq |\theta|_2 \leq |\bar{\theta}_2| + \sqrt{3 \log T} \) with probability at least \( 1 - T^{-1} \).

Next, we show how to upper-bound expected instantaneous regret in the worst case.\(^9\)

---

\(^9\) We state and prove this result in a slightly more general version test we use to support Section 3.3. For the sake of this section, a special case of \( f = 0 \) suffices.
Lemma 3.6  Fix round $t$ and parameter $t > 0$. For any $\theta$, conditioned on any history $h_{t-1}$ and the event that $|e_{a,t}|_\infty \geq t$ for each arm $a$, the expected instantaneous regret of any algorithm at round $t$ is at most

$$2 |\theta|_2 \left( 1 + \rho(2 + \sqrt{2 \log K} + t) \right).$$

Proof. The expected regret at round $t$ is upper-bounded by the reward difference between the best arm $x^*_t$ and the worst arm $x^t$, which is

$$\theta^\top(x^*_t - x^t).$$

Note that $x^*_t = \mu^*_t + \epsilon^*_t$ and $x^t = \mu^t + \epsilon^t$. Then, this is

$$\theta^\top(x^*_t - x^t) = \theta^\top(\mu^*_t - \mu^t) + \theta^\top(\epsilon^*_t - \epsilon^t)$$

$$\leq 2 |\theta|_2 + \theta^\top(\epsilon^*_t - \epsilon^t)$$

since $|\mu_{a,t}|_2 \leq 1$. Next, note that

$$\theta^\top \epsilon^*_t \leq \max_a \theta^\top e_{a,t}$$

and

$$\theta^\top \epsilon^t \geq \min_a \theta^\top e_{a,t}.$$  

Since $e_{a,t}$ has symmetry about the origin conditioned on the event that at least one component of one of the perturbations has absolute value at least $t$, that is, $v$ and $-v$ have equal likelihood, $\max_a \theta^\top e_{a,t}$ and $\min_a \theta^\top e_{a,t}$ are identically distributed. Let $E_{t,t}$ be the event that at least one of the components of one of the perturbations has absolute value at least $t$. This means for any choice $\mu_{a,t}$ for all $a$,

$$\mathbb{E} \left[ \theta^\top(x^*_t - x^t) \mid E_{t,t} \right] \leq 2 |\theta|_2 + 2 \mathbb{E} \left[ \max_a \theta^\top e_{a,t} \mid E_{t,t} \right]$$

where the expectation is taken over the perturbations at time $t$.

Without loss of generality, let $(e_{a',t})_j$ be the component such that $|(e_{a',t})_j| \geq t$. Then, all other components have distribution $\mathcal{N}(0, \rho^2)$. Then,

$$\mathbb{E} \left[ \max_a \theta^\top e_{a,t} \mid E_{t,t} \right]$$

$$= \mathbb{E} \left[ \max_a \theta^\top e_{a,t} \mid (e_{a',t})_j \geq t \right]$$

$$= \mathbb{E} \left[ \max(\theta^\top e_{a',t}, \max_{a \neq a'} \theta^\top e_{a,t}) \mid (e_{a',t})_j \geq t \right]$$

$$\leq \mathbb{E} \left[ \max \left( |(e_{a',t})_j| \mid \sum_{i \neq j} \theta_i (e_{a',t})_i, \max_{a \neq a'} \theta^\top e_{a,t} \right) \mid (e_{a',t})_j \geq t \right]$$
Let $(\epsilon_{a,t})_i = 0$ if $a = a'$ and $i = j$, and $(\epsilon_{a,t})_i$ otherwise. In other words, we simply zero out the component $(\epsilon_{a',t})_j$. Then, this is

$$
\mathbb{E} \left[ \max_a \left( |\theta_j(\epsilon_{a',t})| + \theta^T \tilde{\epsilon}_{a,t} \right) \right] \leq \mathbb{E} \left[ \max_a \left( |\theta_j(\epsilon_{a',t})| + \theta^T \tilde{\epsilon}_{a,t} \right) \right] 
$$

because by Lemma C.4

$$
\mathbb{E} \left[ \max_a \theta^T \tilde{\epsilon}_{a,t} \right] \leq \mathbb{E} \left[ \max_a \theta^T e_{a,t} \right] \leq \rho |\theta|_2 \sqrt{2 \log K}.
$$

Next, note that by symmetry and since $\theta_j \leq |\theta|_2$,

$$
\mathbb{E} \left[ |\theta_j(\epsilon_{a',t})| \right] \leq |\theta|_2 \mathbb{E} \left[ (\epsilon_{a',t})_j \right]
$$

By Lemma C.1,

$$
\mathbb{E} \left[ (\epsilon_{a',t})_j \right] \leq \max(2\rho, t + \rho) \leq 2\rho + t.
$$

Putting this all together, the expected instantaneous regret is bounded by

$$
2 \left( |\theta|_2 \left( 1 + \rho(2 + \sqrt{2 \log K} + t) \right) \right),
$$

proving the lemma.

### 3.4.2 Finishing the Proof of Theorem 3.6

We focus on the “nice event” that $(3.14)$ holds, denote it $\mathcal{E}$ for brevity. In particular, note that it implies $|\theta|_2 \leq S$. Lemma 3.4 guarantees that expected regret under this event, $\mathbb{E} [\text{Regret}(T) | \mathcal{E}]$, is upper-bounded by the expression (3.9) in the theorem statement.

In what follows, we use Lemma 3.5(a) and Lemma 3.6 guarantee that if $\mathcal{E}$ fails, then the corresponding contribution to expected regret is small. Indeed, Lemma 3.6 with $t = 0$ implies that

$$
\mathbb{E} \left[ R_t | \mathcal{E} \right] \leq BT |\theta|_2 \quad \text{for each round } t,
$$
where \( B = 1 + \rho(2 + \sqrt{2 \log K}) \) is the “blow-up factor.” Since (3.14) fails with probability at most \( \frac{2}{T} \) by Lemma 3.5(a), we have

\[
\mathbb{E}[\text{Regret}(T) | \hat{E}] \leq \frac{2B}{T} \mathbb{E}[\|\theta\|_2 | \hat{E}]
\]

\[
\leq \frac{2B}{T} \mathbb{E}[\|\theta\|_2 | \|\theta\|_2 \geq \frac{1}{2 \log T}]
\]

\[
\leq O\left(\frac{B}{T}\right) \left(\|\theta\|_2 + d \log T\right)
\]

\[
\leq O(1).
\]

The antecedent inequality follows by Lemma C.2 with \( \alpha = \frac{1}{2 \log T} \), using the assumption that \( \lambda_{\max}(\Sigma) \leq 1 \). The theorem follows.

### 3.5 Analysis: Greedy Algorithms with Perturbed Contexts

We present the proofs for our results on greedy algorithms in Section 3.3.10 This section is structured as follows. In Section 3.5.1, we quantify the diversity of data collected by batch-greedy-style algorithms, assuming perturbed context generation. In Section 3.5.2, we show that a sufficiently “diverse” batch history suffices to simulate the reward for any given context vector, in the sense of Definition 3.2. Jointly, these two subsections imply that batch history generated by a batch-greedy-style algorithm can simulate rewards with high probability, as long as the batch size is sufficiently large. Section 3.5.3 builds on this foundation to derive regret bounds for BatchBayesGreedy. The crux is that the history collected by BatchBayesGreedy suffices to simulate a “slowed-down” run of any other algorithm. This analysis extends to a version of BatchFreqGreedy equipped with a Bayesian-greedy prediction rule (and tracks the performance of the prediction rule). Finally, Section 3.5.4 derives the regret bounds for BatchFreqGreedy by comparing the prediction-rule version of BatchFreqGreedy with BatchFreqGreedy itself. To derive the results on group externalities, we present all our analysis in Sections 3.5.3 and 3.5.4 in a more general framework in which only the minority rounds are counted for regret.

### Preliminaries

We assume perturbed context generation in this section, without further mention.

Throughout, we will use the following parameters as a shorthand:

\[
\delta_R = T^{-2}
\]

\[
\hat{R} = \rho \sqrt{2 \log(2TKd/\delta_R)}
\]

\[
R = 1 + \hat{R} \sqrt{d}.
\]

10. That is, all results in Section 3.3 except the regret bound for LinUCB (Theorem 3.4(a)), which is proved in Section 3.4.
Recall that $\rho$ denotes perturbation size and $d$ is the dimension. The meaning of $\hat{R}$ and $R$ is that they are high-probability upper bounds on the perturbations and the contexts, respectively. More formally, by Lemma C.4 we have:

$$\Pr[|e_{a,t}|_\infty \leq \hat{R}] \text{ for all arms } a \text{ and all rounds } t \leq \delta_{\hat{R}}. \quad (3.15)$$

$$\Pr[|x_{a,t}|_2 \leq R] \text{ for all arms } a \text{ and all rounds } t \leq \delta_R. \quad (3.16)$$

Let us recap some of the key definitions from Section 3.3.2. We consider batch-greedy-style algorithms, a template that unifies BatchBayesGreedy and BatchFreqGreedy. A bandit algorithm is called batch-greedy-style if it divides the timeline in batches of $Y$ consecutive rounds each, in each round $t$ chooses some estimate $\theta_t$ of $\theta$, based only on the data from the previous batches, and then chooses the best action according to this estimate, so that $a_t = \arg\max_a \theta_t^\top x_{a,t}$.

For a batch $B$ that starts at round $t_0 + 1$, the batch history $h_B$ is the tuple $((x_{t_0+t}, r_{t_0+t}) : \tau \in \{Y\})$, and the batch context matrix $X_B$ is the matrix whose rows are vectors $(x_{t_0+t} : \tau \in \{Y\})$. Here and elsewhere, $|Y| = \{1, \ldots, Y\}$. The batch covariance matrix is defined as

$$Z_B := X_B^\top X_B = \sum_{t=t_0+1}^{t_0+Y} x_t x_t^\top. \quad (3.17)$$

### 3.5.1 Data Diversity under Perturbations

We are interested in the diversity of data collected by batch-greedy-style algorithms, assuming perturbed context generation. Informally, the observed contexts $x_1, x_2, \ldots$ should cover all directions in order to enable good estimation of the latent vector $\theta$. Following Kannan et al. [2018], we quantify data diversity via the minimal eigenvalue of the empirical covariance matrix $Z_t$. More precisely, we are interested in proving that $\lambda_{\min}(Z_t)$ is sufficiently large. We adapt some tools from Kannan et al. [2018], and then derive some improvements for batch-greedy-style algorithms.

#### 3.5.1.1 Tools from Kannan et al. [2018]

Kannan et al. [2018] prove that $\lambda_{\min}(Z_t)$ grows linearly in time $t$, assuming $t$ is sufficiently large.

**Lemma 3.7** Kannan et al. [2018] \[\]

Fix any batch-greedy-style algorithm. Consider round $t \geq t_0$, where $t_0 = 160 \frac{\rho^2}{\rho^3} \log \frac{2d}{\delta} \cdot \log T$. Then for any realization of $\theta$, with probability $1 - \delta$

$$\lambda_{\min}(Z_t) \geq \frac{\rho^2 t}{32 \log T}.$$
The claimed conclusion follows from an argument inside the proof of lemma B.1 from Kannan et al. [2018], plugging in \( \lambda_0 = \frac{\rho^2}{2 \log T} \). This argument applies for any \( t \geq \tau_0^\prime \), where \( \tau_0^\prime = \max\left(32 \log \frac{2}{\delta}, 160 \frac{d^2}{\rho^2} \log \frac{2d}{\delta} \cdot \log T \right) \). We observe that \( \tau_0^\prime = \tau_0 \) since \( R \geq \rho \).

Recall that \( Z_t \) is the sum \( Z_t := \sum_{i=1}^t x_i x_i^\top \). A key step in the proof of Lemma 3.7 zeroes in on the expected contribution of a single round. We use this tool separately in the proof of Lemma 3.10.

**Lemma 3.8** Kannan et al. [2018]
Fix any batch-greedy-style algorithm, and the latent vector \( \theta \). Assume \( T \geq 4K \). Condition on the event that all perturbations \( \epsilon_{a,t} \) are upper-bounded by \( \hat{R} \), denote it with \( \mathcal{E} \). Then with probability at least \( \frac{1}{4} \),

\[
\lambda_{\min}\left( \mathbb{E}\left[ x_i x_i^\top \mid h_{t-1}, \mathcal{E} \right] \right) \geq \frac{\rho^2}{2 \log T}.
\]

**Proof.** The proof is easily assembled from several pieces in the analysis in Kannan et al. [2018]. Let \( \hat{\theta}_t \) be the algorithm’s estimate for \( \theta \) at time \( t \). As in Kannan et al. [2018], define

\[ \hat{c}_{a,t} = \max_{a' \neq a} \hat{\theta}_{t}^\top x_{a',t}, \]

where \( \hat{c}_{a,t} \) depends on all perturbations other than the perturbation for \( x_{a,t} \). Let us say that \( \hat{c}_{a,t} \) is “good” for arm \( a \) if

\[ \hat{c}_{a,t} \leq \hat{\theta}_{t}^\top \mu_{a,t} + \rho \sqrt{2 \log T} |\hat{\theta}_t|_2. \]

First, we argue that

\[ \Pr\left[ \hat{c}_{a,t} \text{ is good for } a \mid a_t = t, \mathcal{E} \right] \geq \frac{1}{4}. \quad (3.18) \]

Indeed, in the proof of their lemma 3.4, Kannan et al. [2018] show that for any round, conditioned on \( \mathcal{E} \), if the probability that arm \( a \) was chosen over the randomness of the perturbation is at least \( 2/T \), then the round is good for \( a \) with probability at least \( \frac{1}{2} \). Let \( B_t \) be the set of arms at round \( t \) with probability at most \( 2/T \) of being chosen over the randomness of the perturbation. Then,

\[
\Pr_{\epsilon \sim \mathcal{N}(0,\rho^2 I)}[a_t \in B_t] \leq \sum_{a \in B_t} \Pr_{\epsilon \sim \mathcal{N}(0,\rho^2 I)}[a_t = a] \leq \frac{2}{T} |B_t| \leq \frac{2K}{T} \leq \frac{1}{2}.
\]

Since by assumption \( T \geq 4K \), (3.18) follows.
Second, we argue that

\[ \lambda_{\min} \left( \mathbb{E} \left[ x_{a,t} x_{a,t}^T \mid a_t = a, \hat{c}_{a,t} \text{ is good} \right] \right) \geq \frac{\rho^2}{2 \log T}. \]  

(3.19)

This is where we use conditioning on the event \( \{c_{a,t} \leq \hat{R} \} \). We plug in \( r = \rho \sqrt{2 \log T} \) and \( \lambda_0 = \frac{\rho^2}{2 \log T} \) into Lemma 3.2 of Kannan et al. [2018]. This lemma applies because with these parameters the perturbed distribution of context arrivals satisfies the \((\rho \sqrt{2 \log T}, \rho^2/(2 \log T))\)-diversity condition from Kannan et al. [2018]. The latter is by Lemma 3.6 of Kannan et al. [2018]. This completes the proof of (3.19). The lemma follows from (3.18) and (3.19).

Let \( \theta_t^\text{greedy} \) be the BatchFreqGreedy estimate for \( \theta \) at time \( t \), as defined in (3.7). We are interested in quantifying how the quality of this estimate improves over time. Kannan et al. [2018] prove, essentially, that the distance between \( \theta_t^\text{greedy} \) and \( \theta \) scales as \( \sqrt{t/\lambda_{\min}(Z_t)} \).

**Lemma 3.9** Kannan et al. [2018]

Consider any round \( t \) in the execution of BatchFreqGreedy. Let \( t_0 \) be the last round of the previous batch. For any \( \theta \) and any \( \delta > 0 \), with probability \( 1 - \delta \),

\[ |\theta - \theta_t^\text{greedy}|_2 \leq \frac{C_{\text{batch}}^\delta}{\lambda_{\min}(Z_{t_0})}. \]

**Some improvements.** We focus on the batch covariance matrix \( Z_B \) of a given batch in a batch-greedy-style algorithm. We would like to prove that \( \lambda_{\min}(Z_B) \) is sufficiently large with high probability, as long as the batch size \( Y \) is large enough. The analysis from Kannan et al. [2018] (a version of Lemma 3.7) would apply, but only as long as the batch size is least as large as the \( r_0 \) from the statement of Lemma 3.7. We derive a more efficient version, essentially shoving off a factor of 8.\(^{11}\)

**Lemma 3.10** Fix a batch-greedy-style algorithm and any batch \( B \) in the execution of this algorithm. Fix \( \delta > 0 \) and assume that the batch size \( Y \) is at least

\[ Y_0 := \left( \frac{R}{\rho} \right)^2 \frac{8e^2}{(e-1)^2} \left( 1 + \log \frac{2\delta}{\delta} \right) \log(T) + \frac{4e}{e-1} \log \frac{2}{\delta}. \]  

(3.20)

Condition on the event that all perturbations in this batch are upper-bounded by \( \hat{R} \), more formally:

\[ \mathcal{E}_B = \{|e_{a,t}| \leq \hat{R} : \text{for all arms } a \text{ and all rounds } t \in B \}. \]

\(^{11}\) Essentially, the factor of 160 in Lemma 3.7 is replaced with factor \( \frac{8e^2}{(e-1)^2} < 20.022 \) in (3.20).
Further, condition on the latent vector \( \theta \) and the history \( h \) before batch \( B \). Then

\[
\text{Pr}\left[ \lambda_{\text{min}}(Z_B) \geq R^2 \mid E_B, h, \theta \right] \geq 1 - \delta. \tag{3.21}
\]

The probability in (3.21) is over the randomness in context arrivals and rewards in batch \( B \).

The improvement over Lemma 3.7 comes from two sources: we use a tail bound on the sum of geometric random variables instead of a Chernoff bound on a binomial random variable, and we derive a tighter application of the eigenvalue concentration inequality of Tropp [2012].

**Proof.** Let \( t_0 \) be the last round before batch \( B \). Recalling (3.17), let

\[
W_B = \sum_{t = t_0 + 1}^{t_0 + Y} \mathbb{E}[x_t x_t^\top \mid h_{t-1}]
\]

be a similar sum over the expected per-round covariance matrices. Assume \( Y \geq Y_0 \).

The proof proceeds in two steps: first we lower-bound \( \lambda_{\text{min}}(Z_B) \), and then we show that it implies (3.21). Denoting \( m = R^2 \frac{\delta}{\epsilon - 1} (1 + \log \frac{2d}{\delta}) \), we claim that

\[
\text{Pr}\left[ \lambda_{\text{min}}(W_B) < m \mid E_B, h \right] \leq \frac{\delta}{2}. \tag{3.22}
\]

To prove this, observe that \( W_B \)'s minimum eigenvalue increases by at least \( \lambda_0 = \rho^2/(2 \log T) \) with probability at least \( 1/4 \) each round by Lemma 3.8, where the randomness is over the history, that is, the sequence of (context, reward) pairs. If we want it to go up to \( m \), this should take \( 4m/\lambda_0 \) rounds in expectation. However, we need it to go to \( m \) with high probability. Notice that this is dominated by the sum of \( m/\lambda_0 \) geometric random variables with parameter \( 1/4 \). We'll use the following bound from Janson [2018]: for \( X = \sum_{i=1}^n X_i \), where \( X_i \sim \text{Geom}(p) \) and any \( c \geq 1 \),

\[
\text{Pr}[X \geq c\mathbb{E}[X]] \leq \exp\left(-n(c - 1 - \log c)\right).
\]

Because we want the minimum eigenvalue of \( W_B \) to be \( m \), we need \( n = m/\lambda_0 \), so \( \mathbb{E}[X] = 4m/\lambda_0 \). Choose \( c = \left(1 + \frac{\lambda_0}{m} \log \frac{2}{\delta} \right) \frac{\epsilon}{\epsilon - 1} \). By Corollary C.1,

\[
c - 1 - \log c \geq \frac{\epsilon - 1}{\epsilon} \cdot c - 1 = \frac{\lambda_0}{m} \log \frac{2}{\delta}.
\]

Therefore,

\[
\text{Pr}[X \geq c\mathbb{E}[X]] \leq \exp\left(-n \cdot \frac{\lambda_0}{m} \log \frac{2}{\delta}\right) = \left(\frac{\delta}{2}\right)^{\frac{n\lambda_0/m}{2}} = \frac{\delta}{2}.
\]
Thus, with probability $1 - \frac{\delta}{2}$, $\lambda_{\min}(W_B) \geq m$ as long as the batch size $Y$ is at least

$$\frac{e}{e - 1} \left( 1 + \frac{\lambda_0}{m} \log \frac{2}{\delta} \right) \cdot \mathbb{E}[X] = \frac{4e}{e - 1} \left( \frac{m}{\lambda_0} + \log \frac{2}{\delta} \right) = Y_0.$$ 

This completes the proof of (3.22).

To derive (3.21) from (3.22), we proceed as follows. Consider the event

$$\mathcal{E} = \{ \lambda_{\min}(Z_B) \leq R^2 \text{ and } \lambda_{\min}(W_B) \geq m \}.$$ 

Letting $\alpha = 1 - R^2/m$ and rewriting $R^2$ as $(1 - \alpha)m$, we use a concentration inequality from Tropp [2012] to guarantee that

$$\Pr[\mathcal{E} | \mathcal{E}_B, h] \leq d \left( e^{(1 - \alpha)^{1 - \alpha} m/R^2} \right).$$

Then, using the fact that $x^x \geq e^{-1/e}$ for all $x > 0$, we have

$$\Pr[\mathcal{E} | \mathcal{E}_B, h] \leq d \left( e^{1 - R^2/m - 1/e} \right)^{-m/R^2} = d e^{-m - R^2 - m/e} \leq \frac{\delta}{2},$$

since $m \geq \frac{e}{e - 1} R^2 \left( 1 + \log \frac{2d}{\delta} \right)$. Finally, observe that, omitting the conditioning on $\mathcal{E}_B, h$, we have:

$$\Pr[\lambda_{\min}(Z_B) \leq R^2] \leq \Pr[\mathcal{E}] + \Pr[\lambda_{\min}(W_B) < m] \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$ 

### 3.5.2 Reward Simulation with a Diverse Batch History

We consider reward simulation with a batch history, in the sense of Definition 3.2. We show that a sufficiently “diverse” batch history suffices to simulate the reward for any given context vector. Coupled with the results of Section 3.5.1, it follows that batch history generated by a batch-greedy-style algorithm can simulate rewards as long as the batch size is sufficiently large.

Let us recap the definition of reward simulation (Definition 3.2). Let $\text{Rew}_d(\cdot)$ be a randomized function that takes a context $x$ and outputs an independent random sample from $\mathcal{N}(\theta^T x, 1)$. In other words, this is the realized reward for an action with context vector $x$. 

Consider batch $B$ in the execution of a batch-greedy-style algorithm. Batch history $h_B$ can simulate $\text{Rew}_\theta(x)$ up to radius $R > 0$ if there exists a function $g : \{\text{context vectors}\} \times \{\text{batch histories } h_B\} \to \mathbb{R}$ such that $g(x, h_B)$ is identically distributed to $\text{Rew}_\theta(x)$ conditional on the batch context matrix, for all $\theta$ and all context vectors $x \in \mathbb{R}^d$ with $|x|_2 \leq R$.

Note that we do not require the function $g$ to be efficiently computable. We do not require algorithms to compute $g$; a mere existence of such function suffices for our analysis.

The result in this subsection does not rely on the “greedy” property. Instead, it applies to all “batch-style” algorithms, defined as follows: time is divided in batches of $Y$ consecutive rounds each, and the action at each round $t$ only depends on the history up to the previous batch. The data diversity condition is formalized as $\lambda_{\min}(Z_B) \geq R^2$; recall that it is a high-probability event, in a precise sense defined in Lemma 3.10. The result is stated as follows:

**Lemma 3.11** Fix a batch-style algorithm and any batch $B$ in the execution of this algorithm. Assume the batch covariance matrix $Z_B$ satisfies $\lambda_{\min}(Z_B) \geq R^2$. Then batch history $h_B$ can simulate $\text{Rew}_\theta$ up to radius $R$.

**Proof.** Let us construct a suitable function $g$ for Definition 3.3. Fix a context vector $x \in \mathbb{R}^d$ with $|x|_2 \leq R$. Let $r_B$ be the vector of realized rewards in batch $B$, that is, $r_B = (r_t : \text{rounds } t \text{ in } B) \in \mathbb{R}^Y$. Define

$$g(x, h_B) = w_B^\top r_B + \mathcal{N}(0, 1 - |w_B|_2^2), \text{where } w_B = X_B Z_B^{-1} x \in \mathbb{R}^Y.$$ (3.23)

Recall that the variance of the reward noise is 1. (We can also handle a more general version in which the variance of the reward noise is $\sigma^2$. Then the noise variance in (3.23) should be $\sigma^2(1 - |w_B|_2^2)$, with essentially no modifications throughout the rest of the proof.)

Note that $w_B$ is well-defined: indeed, $Z_B$ is invertible since $\lambda_{\min}(Z_B) \geq R^2 > 0$. In the rest of the proof, we show that $g$ is as needed for Definition 3.3.

First, we will show that for any $x \in \mathbb{R}^d$ such that $|x|_2 \leq R$, the weights $w_B \in \mathbb{R}^t$ as defined above satisfy $X_B^\top w_B = x$ and $|w_B|_2 \leq 1$. Then, we’ll show that if each $r_t \sim \mathcal{N}(\theta^\top x_t, 1)$, then $r_B^\top w_B + \mathcal{N}(0, 1 - |w_B|_2^2) \sim \mathcal{N}(\theta^\top x, 1)$.

Trivially, we have

$$X_B^\top w_B = X_B^\top X_B (X_B^\top X_B)^{-1} x = x$$

as desired. We must now show that $|w_B|_2 \leq 1$. Note that

$$|w_B|_2^2 = w_B^\top w_B = \|w_B Z_B^{-1} x\|^2 = x^\top Z_B^{-1} Z_B^{-1} x = |x|_Z^2,$$
where \( |v|_M^2 \) simply denotes \( v^T M v \). Thus, it is sufficient to show that \( |x|_{z_B}^2 \leq 1 \). Since \( |x|_2 \leq R \) and \( \lambda_{\text{min}}(z_B) \geq R^2 \), we have by Lemma C.10

\[
Z_B \succeq R^2 I \succeq xx^T.
\]

By Lemma C.11, we have

\[
I \succeq Z_B^{-1/2} xx^T Z_B^{-1/2}.
\]

Let \( z = Z_B^{-1/2} x \), so \( I \succeq zz^T \). Again by Lemma C.10, \( \lambda_{\text{max}}(zz^T) = z^T z \). This means that

\[
1 \geq z^T z = (Z_B^{-1/2} x)^T Z_B^{-1/2} x = x^T Z_B^{-1} x = |x|_{z_B}^2 = |w_B|^2
\]

as desired. Finally, observe that

\[
r_B^T w_B = (X_B \theta + \eta)^T w_B = \theta^T X_B^T w_B + \eta^T w_B = \theta^T x + \eta^T w_B,
\]

where \( \eta \sim \mathcal{N}(0, I) \) is the noise vector. Notice that \( \eta^T w_B \sim \mathcal{N}(0, |w_B|_2^2) \), and therefore, \( \eta^T w_B + \mathcal{N}(0, 1 - |w_B|_2^2) \sim \mathcal{N}(0, 1) \). Putting this all together, we have

\[
r_B^T w_B + \mathcal{N}(0, 1 - |w_B|_2^2) \sim \mathcal{N}(\theta^T x, 1),
\]

and therefore \( D \) can simulate \( E \) for any \( x \) up to radius \( R \).

### 3.5.3 Regret Bounds for BatchBayesGreedy

We apply the tools from Sections 3.5.1 and 3.5.2 to derive regret bounds for BatchBayesGreedy. On a high level, we prove that the history collected by BatchBayesGreedy suffices to simulate a “slowed-down” run of any other algorithm \( \text{ALG}_0 \). Therefore, when it comes to choosing the next action, BatchBayesGreedy has at least as much information as \( \text{ALG}_0 \), so its Bayesian-greedy choice cannot be worse than the choice made by \( \text{ALG}_0 \).

Our analysis extends to a more general scenario that is useful for the analysis of BatchFreqGreedy. We formulate and prove our results for this scenario directly. We consider an extended bandit model that separates data collection and reward collection. Each round \( t \) proceeds as follows: the algorithm observes available actions and the context vectors for these actions, then it chooses two actions, \( a_t \) and \( a'_t \), and observes the reward for the former but not the latter. We refer to \( a'_t \) as the “prediction” at round \( t \). We will refer to an algorithm in this model as a bandit algorithm (which chooses actions \( a_t \)) with “prediction rule” that chooses the predictions \( a'_t \). More specifically, we will be interested in an arbitrary batch-greedy-style algorithm.
with prediction rule given by BatchBayesGreedy, as per (3.6). We assume this prediction rule henceforth. We are interested in prediction regret: a version of regret (3.1) if actions \( a_t \) are replaced with predictions \( a_t^* \):

\[
P\text{Reg}(T) = \sum_{t=1}^{T} \theta^\top x_t^* - \theta^\top x_{a_{t,t}}.
\]

where \( x_t^* \) is the context vector of the best action at round \( t \), as in (3.1). More precisely, we are interested in Bayesian prediction regret, the expectation of (3.24) over everything: the context vectors, the rewards, the algorithm’s random seed, and the prior over \( \theta \).

We use essentially the same analysis to derive implications on group externalities. For this purpose, we consider a further generalization in which regret is restricted to rounds that correspond to a particular population. Formally, let \( T \subseteq \mathbb{N} \) be a randomly chosen subset of the rounds where \( \Pr[t \in T] \) is a constant and rounds are chosen to be in \( T \) independently of one another. We allow for the possibility that the underlying context distribution differs for rounds in \( T \) compared to rounds in \( [T] \setminus T \). More precisely, we allow the event \( \{ t \in T \} \) to be correlated with the context tuple at round \( t \). Similar to the definition of minority regret, we define \( T \)-restricted regret (respectively, prediction regret) in \( T \) rounds to be the portion of regret (respectively, prediction regret) that corresponds to \( T \)-rounds:

\[
R^T(T) = \sum_{t \in T} \theta^\top x_t^* - \theta^\top x_{a_{t,t}}.
\]

\[
P\text{Reg}^T(T) = \sum_{t \in T} \theta^\top x_t^* - \theta^\top x_{a_{t,t}}.
\]

\( T \)-restricted Bayesian (prediction) regret is defined as an expectation over everything.

Thus, the main theorem of this subsection is formulated as follows:

**Theorem 3.7** Consider perturbed context generation. Let \( \text{ALG} \) be an arbitrary batch-greedy-style algorithm whose batch size is at least \( Y_0 \) from (3.20). Fix any bandit algorithm \( \text{ALG}_0 \), and let \( R^T_0(T) \) be the \( T \)-restricted regret of this algorithm on a particular problem instance \( \mathcal{I} \). Then on the same instance, \( \text{ALG} \) has \( T \)-restricted Bayesian prediction regret

\[
\mathbb{E} \left[ P\text{Reg}^T(T) \right] \leq Y \cdot \mathbb{E} \left[ R^T_0(T/Y) \right] + \tilde{O}(1/T).
\]

**Proof sketch.** We use a \( t \)-round history of \( \text{ALG} \) to simulate a \( (t/Y) \)-round history of \( \text{ALG}_0 \). More specifically, we use each batch in the history of \( \text{ALG} \) to simulate one round of \( \text{ALG}_0 \). We prove that the simulated history of \( \text{ALG}_0 \) has exactly the same distribution as the actual history, for any \( \bar{\theta} \). Since \( \text{ALG} \) predicts the Bayesian-optimal
Corollary action given the history (up to the previous batch), this action is at least as good (in expectation over the prior) as the one chosen by $\text{ALG}_0$ after $t/Y$ rounds. The detailed proof is deferred to Section 3.5.3.1.

**Implications.** As a corollary of this theorem, we obtain regret bounds for BatchBayesGreedy in Theorems 3.3 and 3.4. We take $\mathcal{T}$ to be the set of all rounds, that is, $\Pr[t \in \mathcal{T}] = 1$, and $\text{ALG}$ to be BatchBayesGreedy. For Theorem 3.4(b), we take $\text{ALG}_0$ to be LinUCB. Thus:

**Corollary 3.3** In the setting of Theorem 3.7, BatchBayesGreedy has Bayesian regret at most

$$Y \cdot \mathbb{E}[R_0(T/Y)] + \tilde{O}(1/T)$$

on problem instance $\mathcal{I}$. Further, under the assumptions of Theorem 3.4, BatchBayesGreedy has Bayesian regret at most $\tilde{O}(d^2 K^{2/3} T^{-1/3} \rho^2)$ on all instances.

We also obtain a similar regret bound on the Bayesian prediction regret of BatchFreqGreedy, which is essential for Section 3.5.4.

**Corollary 3.4** In the setting of Theorem 3.7, BatchFreqGreedy has Bayesian prediction regret

$$(3.27).$$

To derive Theorem 3.5 for BatchBayesGreedy, we take $\mathcal{T}$ to be the set of all minority rounds, and apply Theorem 3.7 twice: first when $\text{ALG}_0$ is run over the minority rounds only (and can behave arbitrarily on the rest), and then when $\text{ALG}_0$ is run over the full population.

3.5.3.1 **Proof of Theorem 3.7**

We condition on the event that all perturbations are bounded by $\hat{R}$, more precisely, on the event

$$\mathcal{E}_1 = \{|l_{a,t}| \leq \hat{R} : \text{for all arms } a \text{ and all rounds } t\}.$$  \hfill (3.28)

Recall that $\mathcal{E}_1$ is a high-probability event, by (3.15). We also condition on the event

$$\mathcal{E}_2 = \{|\lambda_{\text{min}}(Z_B) | \geq R^2 : \text{for each batch } B\},$$

where $Z_B$ is the batch covariance matrix, as usual. Conditioned on $\mathcal{E}_1$, this too is a high-probability event by Lemma 3.10 plugging in $\delta/T$ and taking a union bound over all batches.

We will prove that $\text{ALG}$ satisfies

$$\mathbb{E}[\text{PReg}^{\mathcal{T}}(T) | \mathcal{E}_1, \mathcal{E}_2] \leq Y \cdot \mathbb{E}[R_0(T/Y) | \mathcal{E}_1, \mathcal{E}_2],$$

where the expectation is taken over everything: the context vectors, the rewards, the algorithm’s random seed, and the prior over $\theta$. Then we take care of the “failure event” $\bar{\mathcal{E}}_1 \cap \bar{\mathcal{E}}_2$. **
**History simulation.** Before we prove (3.29), let us argue about using the history of ALG to simulate a (shorter) run of ALG\(_0\). Fix round \(t\). We use a \(t\)-round history of ALG to simulate a \(\lfloor t/Y \rfloor\)-round run of ALG\(_0\), where \(Y\) is the batch size in ALG. Stating this formally requires some notation. Let \(A_t\) be the set of actions available in round \(t\), and let \(\text{con}_t = (x_{a,t} : a \in A_t)\) be the corresponding tuple of contexts. Let \(\text{CON}\) be the set of all possible context tuples, more precisely, the set of all finite subsets of \(\mathbb{R}^d\). Let \(h_t\) and \(h_0^t\) denote, respectively, the \(t\)-round history of ALG and ALG\(_0\). Let \(\mathcal{H}_t\) denote the set of all possible \(t\)-round histories. Note that \(h_t\) and \(h_0^t\) are random variables that take values on \(\mathcal{H}_t\). We want to use history \(h_t\) to simulate history \(h_0^t\). Thus, the simulation result is stated as follows:

**Lemma 3.12** Fix round \(t\) and let \(\sigma = (\text{con}_1, \ldots, \text{con}_{\lfloor t/Y \rfloor})\) be the sequence of context arrivals up to and including round \(\lfloor t/Y \rfloor\). Then there exists a “simulation function”

\[
\text{sim} = \text{sim}_t : \mathcal{H}_t \times \text{CON}_{\lfloor t/Y \rfloor} \to \mathcal{H}_{\lfloor t/Y \rfloor}
\]

such that the simulated history \(\text{sim}(h_t, \sigma)\) is distributed identically to \(h_0^t\), conditional on sequence \(\sigma\), latent vector \(\theta\), and events \(E_1, E_2\).

**Proof.** Throughout this proof, condition on events \(E_1\) and \(E_2\). Generically, \(\text{sim}(h_t, \sigma)\) outputs a sequence of pairs \(\{(x_r, r_r)_{r=1}^{\lfloor t/Y \rfloor}\}\), where \(x_r\) is a context vector and \(r_r\) is a simulated reward for this context vector. We define \(\text{sim}(h_t, \sigma)\) by induction on \(r\) with base case \(r = 0\). Throughout, we maintain a run of algorithm ALG\(_0\). For each step \(\tau \geq 1\), suppose ALG\(_0\) is simulated up to round \(\tau - 1\), and the corresponding history is recorded as \(((x_1, r_1), \ldots, (x_{\tau-1}, r_{\tau-1}))\). Simulate the next round in the execution of ALG\(_0\) by presenting it with the action set \(A_\tau\) and the corresponding context tuple \(\text{con}_\tau\). Let \(x_\tau\) be the context vector chosen by ALG\(_0\). The corresponding reward \(r_\tau\) is constructed using the \(\tau\)-th batch in \(h_t\), denote it with \(B\). By Lemmas 3.10 and 3.11, the batch history \(h_B\) can simulate a single reward, in the sense of Definition 3.3. In particular, there exists a function \(g(x, h_B)\) with the required properties (recall that it is explicitly defined in (3.23)). Thus, we define \(r_\tau = g(x_\tau, h_B)\), and return \(r_\tau\) as a reward to ALG\(_0\). This completes the construction of \(\text{sim}(h_t, \sigma)\). The distribution property of \(\text{sim}(h_t, \sigma)\) is immediate from the construction.

**Proof of Equation (3.29).** We argue for each batch separately, and then aggregate over all batches in the very end. Fix batch \(B\), and let \(t_0 = t_0(B)\) be the last round in this batch. Let \(\tau = 1 + t_0/Y\), and consider the context vector \(x_0^\tau\) chosen by ALG\(_0\) in round \(\tau\). This context vector is a randomized function \(f\) of the current context tuple \(\text{con}_\tau\) and the history \(h_0^{\tau-1}\):

\[
x_0^\tau = f(\text{con}_\tau; h_0^{\tau-1}).
\]
By Lemma 3.12, letting \( \sigma = (\mathbf{c}_{t1}, \ldots, \mathbf{c}_{t|Y|}) \), it holds that
\[
\mathbb{E}[x_t^0 \cdot \theta \mid \sigma, \theta, \mathcal{E}_1, \mathcal{E}_2] = \mathbb{E}[f(\mathbf{c}_{ti}; \text{sim}(h_{t0}, \sigma)) \cdot \theta \mid \sigma, \theta, \mathcal{E}_1, \mathcal{E}_2].
\] (3.30)

Let \( t \) be some round in the next batch after \( B \), and let \( x_t^0 = x_{\mathcal{C}_t} \) be the context vector predicted by \( \text{ALG} \) in round \( t \). Recall that \( x_t^0 \) is a Bayesian-greedy choice from the context tuple \( \mathbf{c}_{t1} \), based on history \( h_{t0} \). Observe that the Bayesian-greedy action choice from a given context tuple based on history \( h_{t0} \) cannot be worse, in terms of the Bayesian-expected reward, than any other choice from the same context tuple and based on the same history. Using (3.30), we obtain:
\[
\mathbb{E}[x_t^0 \cdot \theta \mid \mathbf{c}_{t1} = \mathbf{c}, \mathcal{E}_1, \mathcal{E}_2] \geq \mathbb{E}[x_t^0 \cdot \theta \mid \mathbf{c}, \mathcal{E}_1, \mathcal{E}_2],
\] (3.31)
for any given context tuple \( \mathbf{c} \in \mathcal{C} \) that has a non-zero arrival probability given \( \mathcal{E}_1 \cap \mathcal{E}_2 \).

Given \( \mathbf{c}_{t1} = \mathbf{c} \), the event \( t \in \mathcal{T} \) is independent of everything else. Likewise, given \( \mathbf{c}_{t1} = \mathbf{c} \), the event \( \tau \in \mathcal{T} \) is independent of everything else. It follows that
\[
\mathbb{E}[x_t^0 \cdot \theta \mid \mathbf{c}_{t1} = \mathbf{c}, t \in \mathcal{T}, \mathcal{E}_1, \mathcal{E}_2] \geq \mathbb{E}[x_t^0 \cdot \theta \mid \mathbf{c}_{t1} = \mathbf{c}, \tau \in \mathcal{T}, \mathcal{E}_1, \mathcal{E}_2],
\] (3.32)
for any given context tuple \( \mathbf{c} \in \mathcal{C} \) that has a non-zero arrival probability given \( \mathcal{E}_1 \cap \mathcal{E}_2 \).

Observe that \( \mathbf{c}_{t1} \) and \( \mathbf{c}_{t1} \) have the same distribution, even conditioned on event \( \mathcal{E}_1 \cap \mathcal{E}_2 \). (This is because the definitions of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) treat all rounds in the same batch in exactly the same way.) Therefore, we can integrate (3.32) over the context tuples \( \mathbf{c} \):
\[
\mathbb{E}[x_t^0 \cdot \theta \mid t \in \mathcal{T}, \mathcal{E}_1, \mathcal{E}_2] \geq \mathbb{E}[x_t^0 \cdot \theta \mid \tau \in \mathcal{T}, \mathcal{E}_1, \mathcal{E}_2].
\] (3.33)

Now, let us sum up (3.33) over all rounds \( t \) in the next batch after \( B \), denote it \( \text{next}(B) \).
\[
\sum_{t \in \text{next}(B)} \mathbb{E}[x_t^0 \cdot \theta \mid t \in \mathcal{T}, \mathcal{E}_1, \mathcal{E}_2] \geq Y \cdot \mathbb{E}[x_t^0 \cdot \theta \mid \tau \in \mathcal{T}, \mathcal{E}_1, \mathcal{E}_2].
\] (3.34)

Note that the right-hand side of (3.33) stays the same for all \( t \), hence the factor of \( Y \) on the right-hand side of (3.34). This completes our analysis of a single batch \( B \).

We obtain (3.29) by summing over all batches \( B \). Here it is essential that the expectation \( \mathbb{E} [\mathbf{1}_{t \in \mathcal{T}} \theta^\top x_t^0] \) does not depend on round \( t \), and therefore the “regret benchmark” \( \theta^\top x_t^0 \) cancels out from (3.29). In particular, it is essential that the context tuples \( \mathbf{c}_{t1} \) are identically distributed across rounds.

\[ \blacksquare \]
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Proof of Theorem 3.7 given Equation (3.29). We must take care of the low-probability failure events \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \). Specifically, we need to upper-bound the expression

\[
\mathbb{E}_{\theta, p} \left[ \text{PReg}_T(T) \ | \ \mathcal{E}_1 \cup \mathcal{E}_2 \right] \cdot \Pr[\mathcal{E}_1 \cup \mathcal{E}_2].
\]

For ease of exposition, we focus on the special case \( \Pr[t \in \mathcal{T}] = 1 \); the general case is treated similarly. We know that \( \Pr[\mathcal{E}_1 \cup \mathcal{E}_2] \leq \delta + \delta_R \). Lemma 3.6 with \( t = \tilde{R} \) gives us that the instantaneous regret of every round is at most

\[
2 \mathbb{E}_{\theta \sim \mathcal{P} | h_{t-1}} \left[ \|\theta\|_2 \left(1 + \rho(2 + \sqrt{2 \log K} + \tilde{R})\right) \right]
\leq 2 \left[ \|\theta\|_2 \left(1 + \rho(2 + \sqrt{2 \log K} + \tilde{R})\right) \right]
\]

by Lemma C.5. Letting \( \delta = \delta_R = \frac{1}{T^2} \), we verify that our definition of \( Y \) means that Lemma 3.10 indeed holds with probability at least \( 1 - T^{-2} \). Using (3.29), the Bayesian prediction regret of ALG is

\[
\mathbb{E}_{\theta \sim \mathcal{P}} \left[ \text{PReg}_T(T) \right]
\leq Y \mathbb{E}_{\theta \sim \mathcal{P}} \left[ R^{\tilde{R}}_0 \left( \frac{T}{T} \right) \right]
\leq Y \mathbb{E}_{\theta \sim \mathcal{P}} \left[ R^{\tilde{R}}_0 \left( \frac{T}{T} \right) \right] + \tilde{O} \left( \frac{1}{T} \right).
\]

This completes the proof of Theorem 3.7.

3.5.4 Regret Bounds for BatchFreqGreedy

To analyze BatchFreqGreedy, we show that its Bayesian regret is not too different from its Bayesian prediction regret, and use Corollary 3.4 to bound the latter. As in the previous subsection, we state this result in more generality for the sake of group externality implications: we consider \( T \)-restricted (prediction) regret, exactly as before.

Theorem 3.8 Assuming perturbed context generation, BatchFreqGreedy satisfies

\[
\left| \mathbb{E} \left[ R^T(T) - \text{PReg}_T(T) \right] \right| \leq \tilde{O} \left( \frac{\sqrt{d}}{\rho^2} \right) \left( \sqrt{\lambda_{\max}(\Sigma)} + \frac{1}{\sqrt{\lambda_{\min}(\Sigma)}} \right),
\]

where \( \Sigma \) is the covariance matrix of the prior and \( \rho \) is the perturbation size.

Taking \( \mathcal{T} \) to be the set of all contexts, and using Corollary 3.4, we obtain Bayesian regret bounds for BatchFreqGreedy in Theorems 3.3 and 3.4. To derive Theorem 3.5 for BatchFreqGreedy, we take \( \mathcal{T} \) to be the set of all minority rounds.
The remainder of this section is dedicated to proving Theorem 3.8. On a high level, the idea is as follows. As in the proof of Theorem 3.7, we condition on the high-probability event (3.28) that perturbations are bounded. Specifically, we prove that

\[
\left| \mathbb{E} \left[ R^T(T) - \text{PReg}^T(T) \mid \mathcal{E}_1 \right] \right| \leq \tilde{O} \left( \frac{\sqrt{d}}{\rho^2} \right) \left( \frac{1}{\sqrt{\lambda_{\max}(\Sigma)}} + \frac{1}{\sqrt{\lambda_{\min}(\Sigma)}} \right).
\]

(3.35)

To prove this statement, we fix round \( t \) and compare the action \( a_t \) taken by BatchFreqGreedy and the predicted action \( a'_t \). We observe that the difference in rewards between these two actions can be upper-bounded in terms of \( \theta_{t}^{\text{bay}} - \theta_{t}^{\text{fre}} \), the difference in the \( \theta \) estimates with and without knowledge of the prior. (Recall (3.6) and (3.7) for definitions.) Specifically, we show that

\[
\mathbb{E} \left[ \theta^T(x_{a_t,t} - x_{a'_t,t}) \mid \mathcal{E}_1 \right] \leq 2R \mathbb{E}_{\theta - \mathcal{P}} \left[ |\theta_{t}^{\text{bay}} - \theta_{t}^{\text{fre}}|_2 \right].
\]

(3.36)

The crux of the proof is to show that the difference \( |\theta_{t}^{\text{bay}} - \theta_{t}^{\text{fre}}|_2 \) is small, namely

\[
\mathbb{E} \left[ |\theta_{t}^{\text{bay}} - \theta_{t}^{\text{fre}}|_2 \mid \mathcal{E}_1 \right] = \tilde{O}(1/\ell),
\]

(3.37)

ignoring other parameters. Thus, summing over all rounds, we get

\[
\mathbb{E} \left[ R^T(T) - \text{PReg}^T(T) \mid \mathcal{E}_1 \right] \leq O(\log T) = \tilde{O}(1).
\]

**Proof of (3.35).** Let \( R' \) and \( \text{PReg}' \) be, respectively, instantaneous regret and instantaneous prediction regret at time \( t \). Then

\[
\mathbb{E}_{\theta - \mathcal{P}} \left[ R^T(T) - \text{PReg}^T(T) \right] = \sum_{t \in T} \mathbb{E}_{\theta - \mathcal{P}} \left[ R' - \text{PReg}' \right].
\]

(3.38)

Thus, it suffices to bound the differences in instantaneous regret.

Recall that at time \( t \), the chosen action for BatchFreqGreedy and the predicted action are, respectively,

\[
a_t = \arg \max_{a \in A} x_{a,t}^T \theta_{t}^{\text{fre}}
\]

and

\[
a'_t = \arg \max_{a \in A} x_{a,t}^T \theta_{t}^{\text{bay}}.
\]

Letting \( t_0 = \lfloor t/Y \rfloor \) be the last round in the previous batch, we can formulate \( \theta_{t}^{\text{fre}} \) and \( \theta_{t}^{\text{bay}} \) as

\[
\theta_{t}^{\text{fre}} = (Z_{t_0-1})^{-1} X_{t_0-1}^T r_{1:t_0-1}
\]

and

\[
\theta_{t}^{\text{bay}} = (Z_{t_0-1} + \Sigma^{-1})^{-1}(X_{t_0-1}^T r_{1:t_0-1} + \Sigma^{-1} \tilde{r}).
\]
Therefore, we have
\[
E_{\theta \sim p} \left[ R' - P_{\text{Reg}}' \right] = E_{\theta \sim p} \left[ h_{t-1} \left( (x_{a^*_t} - x_{a_t})^\top \theta_t^{\text{bay}} \right) = (x_{a^*_t} - x_{a_t})^\top \theta_t^{\text{bay}}, \right.
\]
since the mean of the posterior distribution is exactly \( \theta_t^{\text{bay}} \), and \( \theta_t^{\text{bay}} \) is deterministic given \( h_{t-1} \). Taking expectation over \( h_{t-1} \), we have
\[
E_{\theta \sim p} \left[ R' - P_{\text{Reg}}' \right] = E_{\theta \sim p} \left[ (x_{a^*_t} - x_{a_t})^\top \theta_t^{\text{bay}} \right] .
\]
For any fixed \( \theta_t^{\text{bay}} \) and \( \theta_t^{\text{fre}} \), since BatchFreqGreedy chose \( a_t \) over \( a_t' \), it must be the case that
\[
\frac{\theta_t^{\text{bay}} - \theta_t^{\text{fre}}}{X_{a_t}^\top \theta_t^{\text{bay}} = (x_{a^*_t} - x_{a_t})^\top \theta_t^{\text{fre}} \leq (x_{a^*_t} - x_{a_t})^\top \theta_t^{\text{bay}} - \theta_t^{\text{fre}} \leq \frac{\theta_t^{\text{bay}} - \theta_t^{\text{fre}}}{2R|\theta_t^{\text{bay}} - \theta_t^{\text{fre}}|_2}
\]
(3.36) follows.

The crux is to prove (3.37): to bound the expected distance between the frequentist and Bayesian estimates for \( \theta \). By expanding their definitions, we have
\[
\theta_t^{\text{bay}} - \theta_t^{\text{fre}} = (Z_{t_0}-1 + \Sigma^{-1})^{-1}(X_{t_0}^\top r_{1:t_0} + \Sigma^{-1} \bar{\theta}) - Z_{t_0}^{-1}X_{t_0}^\top r_{1:t_0} - \Sigma^{-1} \bar{\theta}
\]
\[
= (Z_{t_0}-1 + \Sigma^{-1})^{-1}[X_{t_0}^\top r_{1:t_0} + \Sigma^{-1} \bar{\theta} - (Z_{t_0}-1 + \Sigma^{-1})Z_{t_0}^{-1}X_{t_0}^\top r_{1:t_0} - \Sigma^{-1} \bar{\theta}
\]
\[
= (Z_{t_0}-1 + \Sigma^{-1})^{-1}[X_{t_0}^\top r_{1:t_0} + \Sigma^{-1} \bar{\theta} - X_{t_0}^\top r_{1:t_0} - \Sigma^{-1} Z_{t_0}^{-1}X_{t_0}^\top r_{1:t_0} - \Sigma^{-1} \bar{\theta}
\]
\[
= (Z_{t_0}-1 + \Sigma^{-1})^{-1}[\Sigma^{-1} \bar{\theta} - \Sigma^{-1} Z_{t_0}^{-1}X_{t_0}^\top r_{1:t_0} - \Sigma^{-1} \bar{\theta}
\]
Next, note that
\[
\left| (Z_{t_0}-1 + \Sigma^{-1})^{-1}\Sigma^{-1} (\bar{\theta} - \theta_t^{\text{fre}}) \right|_2
\]
\[
\leq \left| (Z_{t_0}-1 + \Sigma^{-1})^{-1}\Sigma^{-1} (\bar{\theta} - \theta_t^{\text{fre}}) \right|_2
\]
\[
\leq \left| (Z_{t_0}-1 + \Sigma^{-1})^{-1}\left( |\Sigma^{-1} \bar{\theta} - \theta_t^{\text{fre}}|_2 + |\Sigma^{-1} |_2 \right) \right|_2 .
\]
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By Lemma C.12, \( \lambda_{\text{min}}(Z_{t_0-1} + \Sigma) \geq \lambda_{\text{min}}(Z_{t_0-1}) \). Therefore,

\[
|(Z_{t_0-1} + \Sigma)^{-1}\|_2 \leq \frac{1}{\lambda_{\text{min}}(Z_{t_0-1})},
\]

giving us

\[
|\theta_t^{\text{bay}} - \theta_t^{\text{fre}}|_2 \leq \frac{|\Sigma^{-1}(\theta - \theta)|_2 + |\Sigma^{-1}|_2 |\theta - \theta_t^{\text{fre}}|_2}{\lambda_{\text{min}}(Z_{t_0-1})} \leq \frac{|\Sigma^{-1/2}(\theta - \theta)|_2 + |\Sigma^{-1/2}|_2 |\theta - \theta_t^{\text{fre}}|_2}{\lambda_{\text{min}}(Z_{t_0-1})} = \frac{|\Sigma^{-1/2}(\theta - \theta)|_2 + \sqrt{\lambda_{\text{min}}(\Sigma)}|\theta - \theta_t^{\text{fre}}|_2}{\sqrt{\lambda_{\text{min}}(\Sigma)}\lambda_{\text{min}}(Z_{t_0-1})}.
\]

Next, recall that for

\[
t_0 - 1 \geq t_{\text{min}}(\delta) := 160 \frac{K^2}{\rho^2} \log \frac{2d}{\delta} \cdot \log T
\]

the following bounds hold, each with probability at least \( 1 - \delta \):

\[
\frac{1}{\lambda_{\text{min}}(Z_{t_0-1})} \leq 32 \log T \tag{Lemma 3.7}
\]

\[
|\theta - \theta_t^{\text{fre}}|_2 \leq \frac{\sqrt{2dR(t_0 - 1) \log(d/\delta)}}{\lambda_{\text{min}}(Z_{t_0-1})} \tag{Lemma 3.9}
\]

Therefore, fixing \( t_0 \geq 1 + t_{\text{min}}(\delta/2) \), with probability at least \( 1 - \delta \), we have

\[
|\theta_t^{\text{bay}} - \theta_t^{\text{fre}}|_2 \leq \frac{32 \log T}{\rho^2(t_0 - 1)\sqrt{\lambda_{\text{min}}(\Sigma)}} \left( |\Sigma^{-1/2}(\theta - \theta)|_2 + \frac{64 \sqrt{dR \log(2d/\delta)} \cdot \log T}{\rho^2 \sqrt{t_0 - 1}} \right). \tag{3.40}
\]

Note that the high-probability events we need are deterministic given \( h_{t_0-1} \), and therefore are independent of the perturbations at time \( t \). This means that Lemma 3.6 applies, with \( t = 0 \): conditioned on any \( h_{t_0-1} \), the expected regret for round \( t \) is upper-bounded by \( 2|\theta|_2(1 + \rho(1 + \sqrt{2 \log K})) \). In particular, this holds for any \( h_{t_0-1} \) not satisfying the high probability events from Lemmas 3.7 and 3.9. Therefore, for all \( t \geq t_{\text{min}}(\delta) \),

\[
\mathbb{E}_{\theta_{-t}} \left[ |\theta_t^{\text{bay}} - \theta_t^{\text{fre}}|_2 \right] \leq \mathbb{E}_{\theta_{-t}} \left[ (1 - \delta) \frac{32 \log T}{\rho^2(t_0 - 1)\sqrt{\lambda_{\text{min}}(\Sigma)}} \right]
\]
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\[
\frac{\sqrt{dR \log(2d/\delta) \cdot \log T}}{\rho^2 \sqrt{t_0 - 1}} + \delta \cdot 2|\theta|_2 (1 + \rho(2 + \sqrt{2 \log K}))
\]

Because \( \theta \sim N(\bar{\theta}, \Sigma) \), we have \( \Sigma^{-1/2}(\theta - \bar{\theta}) \sim N(0, I) \). By Lemma C.5,

\[
E_{\theta \sim P}[|\Sigma^{-1/2}(\theta - \bar{\theta})|_2^2] \leq \sqrt{d} \quad \text{and} \quad E_{\theta \sim P}[|\theta - \bar{\theta}|_2] \leq \sqrt{d\lambda_{\max}(\Sigma)}.
\]

This means

\[
E_{\theta \sim P}[|\theta_{\text{bay}} - \theta_{\text{fre}}|_2] \leq 32\sqrt{d} \log T \frac{R \log(2d/\delta) \cdot \log T}{\rho^2(t_0 - 1)\sqrt{\lambda_{\min}(\Sigma)}} \left(1 + \frac{64\sqrt{dR \log(2d/\delta) \cdot \log T}}{\rho^2 \sqrt{t_0 - 1}} \right)
\]

Choosing \( \delta = T^{-2} \), we find that

\[
\sum_{t=t_{\text{min}}(T^{-2})}^{T} \frac{\delta \cdot 2(|\bar{\theta}|_2 + \sqrt{d\lambda_{\max}(\Sigma)})(1 + \rho(2 + \sqrt{2 \log K})) = \tilde{O}(1),
\]

so this term vanishes. Furthermore,

\[
\sum_{t=t_{\text{min}}(T^{-2})}^{T} 2R \frac{32\sqrt{d} \log T}{\rho^2(t_0 - 1)\sqrt{\lambda_{\min}(\Sigma)}} \left(1 + \frac{64\sqrt{dR \log(2d/\delta) \cdot \log T}}{\rho^2 \sqrt{t_0 - 1}} \right)
\]

since \( t_0 \geq t - Y \) and \( \sum_{t=1}^{T} 1/t = O(\log T) \). Using the fact that \( R = \tilde{O}(1) \) (since by assumption \( \rho \leq d^{-1/2} \)), this is simply

\[
\tilde{O}\left(\frac{\sqrt{d}}{\rho^2 \sqrt{\lambda_{\min}(\Sigma)}}\right).
\]
Finally, we note that on the first $t_{\min}(T^{-2}) = \tilde{O}(1/\rho^2)$ rounds, the regret bound from Lemma 3.6 with $\ell = 0$ applies, so the total regret difference is at most

$$E_{\theta-P} \left[ R^T(T) - P\text{Reg}^T(T) \right] \leq \sum_{t=1}^{t_{\min}(T^{-2})} E_{\theta-P} \left[ R^t - P\text{Reg}^t \right] + 2R E_{\theta-P} \left[ \|\theta^t_{\text{bay}} - \theta^t_{\text{freq}}\|_2 \right],$$

$$\leq t_{\min}(T^{-2}) \cdot 2(\|\bar{\Theta}\|_2 + \sqrt{d\lambda_{\max}(\Sigma)})(1 + \rho(2 + \sqrt{2\log K})) + \tilde{O} \left( \frac{\sqrt{d}}{\rho^2 / \sqrt{\lambda_{\min}(\Sigma)}} \right)$$

$$= \tilde{O} \left( \frac{\sqrt{d\lambda_{\max}(\Sigma)}}{\rho^2} \right) + \tilde{O} \left( \frac{\sqrt{d}}{\rho^2 / \sqrt{\lambda_{\min}(\Sigma)}} \right),$$

which implies (3.35).

**Completing the proof of Theorem 3.8 given (3.35).** By Theorem 3.8, this holds whenever all perturbations are bounded by $\bar{R}$, which happens with probability at least $1 - \delta_R$. When the bound fail, the total regret is at most

$$2 \left[ (\|\bar{\Theta}\|_2 + \sqrt{d\lambda_{\max}(\Sigma)}) \left( 1 + \rho(2 + \sqrt{2\log K}) + \bar{R} \right) \right]$$

by Lemma 3.6 (with $\ell = \bar{R}$) and Lemma C.5. Since $\delta_R = T^{-2}$, the contribution of regret when the high-probability bound fails is $\tilde{O}(1/T) \leq \tilde{O}(1)$. 