Towards random uniform sampling of bipartite graphs with given degree sequence

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Abstract

In this paper we consider a simple Markov chain for bipartite graphs with given degree sequence on \( n \) vertices. We show that the mixing time of this Markov chain is bounded above by a polynomial in \( n \) in case of semi-regular degree sequence. The novelty of our approach lays in the construction of the canonical paths in Sinclair’s method.

1 Introduction

The degree sequence, \( d(G) \), of a graph \( G \) is the non-increasing sequence of its vertex degrees. A sequence \( d = (d_1, \ldots, d_n) \) is graphical iff \( d(G) = d \) for some simple graph \( G \), and \( G \) is a graphical realization of \( d \).

Already at the beginning of the systematic graph theoretical research (late fifties and early sixties) there were serious efforts to decide whether a non-increasing sequence is graphical. Erdős and Gallai (1960, [4]) gave a necessary and sufficient condition, while Havel (1955, [6]) and Hakimi (1962, [5]) independently developed a greedy algorithm to built a graphical realization if there exists any. (For more details see for example [10].)

Generating some (or all possible) graphs realizing a given degree sequence or finding a typical one among the different realizations is an ubiquitous problem in network modeling, ranging from social sciences to chemical compounds and biochemical reaction networks in the cell. (See for example the book [13] for a detailed analysis, or the paper [10] for a short explanation.)

When the number of different realizations is small, then the uniform sampling of the different realizations can be carried out by generating all possible ones and choosing among them uniformly.

However in cases where there are many different realizations this approach can not work. In these cases some stochastic processes can provide solutions.
Here we mention only one of the preceding results: Molloy and Reed (1995, [12]) applied the configuration model (Bollobás (1980, [1]) for the problem. They successfully used it to generate random graphs with given degree sequences where the degrees are (universally) bounded. Although it is well known, that this method is computationally infeasible in case of general, unbounded degree sequences.

A completely different method was proposed by Kannan, Tetali and Vempala (1999, [9]), which is based on the powerful Metropolis-Hastings algorithm: some local transformations (these are the well-known swaps) generate a random walk on the family of all realizations. They conjectured that this process is fast mixing i.e. starting from an arbitrary realization of the degree sequence the process reaches in a completely random realization in reasonable (i.e. polynomial) time. They succeeded to prove it for bipartite regular graphs. Their conjecture was proved for arbitrary regular graphs by Cooper, Dyer and Greenhill (2007, [2]).

The goal of this paper was to attack Kannan, Tetali and Vempala’s conjecture for arbitrary bipartite degree sequences. We obtained the following result, which is one further step toward the general Kannan, Tetali and Vempala’s conjecture. A bipartite graph is called semi-regular if in one class all the vertices have the same degree, while the degrees in the other class are arbitrary.

**Theorem 1.1.** The Markov process - defined by Kannan, Tetali and Vempala - is fast mixing on bipartite semi-regular degree sequences.

This paper is rather long therefore at first we give a short sketch what is going to happen later.

## 2 Outline of the paper

Assume that $G$ and $G'$ are two realizations of the same bipartite degree sequence (see formula (3.1)). The realizations are given by the mean of adjacency matrices $M_G$ of the graph $G$. As Ryser proved ([14], see Theorem 3.2) they can be transformed to each other with a sequence of swaps (for exact description see Definition 3.1). In this paper we give a short, transparent proof of this fact (using the language of graphs instead of the language of 0-1 matrices).

Since the inverse transformation of a swap is also a swap, therefore the Markov graph $G$ (see Section 4) - which consists of all possible realizations of a fixed bipartite degree sequence - is connected. One can easily prove that $G$’s diameter is at most the sum of the elements in the degree sequence (see the Proof of Theorem 3.2). Equipping with the appropriate transition probabilities (see formula (4.1)), this depicts a Markov chain on the set of all realizations. It is easy to see that this Markov process is reversible with the uniform distribution as the globally stable stationary distribution. Therefore an application of the Metropolis-Hastings algorithm will converge to some completely random element, independently from its starting point. The only open problem here is its mixing time: how fast a random sample can be reached.
In their papers, Kannan, Tetali and Vempala (1999, [9]) (and Cooper, Dyer and Greenhill, 2007, [2]) used this method to sample almost uniformly all possible realizations of bipartite regular (general regular) degree sequences, resp. Both approaches are based on the seminal Sinclair’s method (Section 5).

For the successful application of this method you have to construct a set of canonical paths, which satisfies a long list of properties (see Section 6). The key property is the following: if \( X \) and \( Y \) are two realizations of the degree sequence, then for each canonical path \( P \) (leading from \( X \) to \( Y \)) and for all vertices \( Z \) in \( P \) there exists a realization \( U \) such that these four realizations together satisfy the condition:

\[
d(M_X + M_Y - M_Z, M_U) \leq C
\]

where \( C \) is a fixed constant (here \( d(M, M') \) denotes the number of different entries in two integer matrices). The successful construction of the canonical path system provides the necessary input of the Sinclair’s method, which in turns proves the fast mixing behavior of the Markov process.

In the canonical path construction process, for all pairs \( X \) and \( Y \) of possible realizations we have to define a collection of swap sequences leading from \( X \) to \( Y \) (and, automatically, from \( Y \) to \( X \)). For that end one can take the symmetric difference of the edges of the two realizations, then it can be uniquely decomposed into cycles (with edges from \( X \) and \( Y \) alternatingly, see the list ”Outline of the construction of the path system” in Section 6). In this point both the Tetali and Vempala and the Cooper, Dyer and Greenhill methods apply a uniform ”folding” process on those cycles to determine the swap sequences.

The main contribution of our paper is the introduction of the friendly path method to construct suitable canonical path systems. Our method takes into consideration the environment of the alternating cycles. For semi-regular bipartite degree sequences one can prove that our construction satisfies condition (2.1) therefore the Markov process under review is fast mixing. However when the edge sets of the bipartite graphs \( X \) and \( Y \) differ precisely in one (alternating) cycle, then the friendly path method always provides a suitable canonical path system. (See Theorem 8.1).

To finish this Section we also want to mention that any graph can be represented as a semi-regular bipartite graph: here one class represents the vertices, while the other class describes the (labeled) edges. Using this representation for sampling randomly the original graphs faces the problem that different graphs will appear with different frequencies. An appropriate weighted Metropolis-Hastings algorithm may provide a solution.

3 Basic definitions and preliminaries

Let \( G = (U, V; E) \) be a simple bipartite graph (no parallel edges) with vertex classes \( U = \{u_1, \ldots, u_l\}, V = \{v_1, \ldots, v_k\} \). The (bipartite) degree sequence of \( G \), \( bd(G) \) is defined as follows:

\[
bd(G) = \left( (d(u_1), \ldots, d(u_l)), (d(v_1), \ldots, d(v_l)) \right)
\]
where the vertices are ordered such that both sequences are non-increasing.
From now on when we say “degree sequence” of a bipartite graph, we will always mean the bipartite degree sequence.

A pair \((a, b)\) of sequences is a \((bipartite)\) \emph{graphical sequence} (BGS for short) if \((a, b) = bd(G)\) for some simple bipartite graph \(G\), while the graph \(G\) is a \((graphical)\) realization of \((a, b)\).

Next we define the swaps, our basic operation on bipartite graphs.

**Definition 3.1.** Let \(G = (U, V, E)\) be a bipartite graph, \(u_1, u_2 \in U, v_1, v_2 \in V,\) such that induced subgraph \(G[u_1, u_2; v_1, v_2]\) is a 1-factor, (i.e. \((u_1, v_j), (u_2, v_{3-j}) \in E,\) but \((u_1, v_{3-j}), (u_2, v_j) \notin E\) for some \(j\).) Then we say that the swap on \((u_1, u_2; v_1, v_2)\) is allowed, and it transforms the graph \(G\) into a graph \(G' = (U, V, E')\) by replacing the edges \((u_1, v_j), (u_2, v_{3-j})\) by edges \((u_1, v_{3-j}), (u_2, v_j)\), i.e.

\[
E' = E \setminus \{(u_1, v_j), (u_2, v_{3-j})\} \cup \{(u_1, v_{3-j}), (u_2, v_j)\}.
\]

(3.2)

So a swap transforms one realization of the BGS to another (bipartite graph) realization of the same BGS. The following proposition is a classical result of Ryser (1957, [14]).

**Theorem 3.2 (Ryser).** Let \(G_1 = (U, V; E_1)\) and \(G_2 = (U, V; E_2)\) be two realizations of the same BGS. Then there exists a sequence of swaps which transform \(G_1\) into \(G_2\) through different realizations of the same BGS.

Ryser’s result used the language of 0 - 1 matrices. Here, to make the paper self contained, we give a short proof, using the notion of swaps. The proof is based on a well known observation of Havel and Hakimi ([3,5]):

**Lemma 3.3 (Havel and Hakimi).** Let \(G = (U, V; E)\) be a simple bipartite graph, and assume that \(d(u') \leq d(u)\), furthermore \(u'v \in E\) and \(uv \notin E\). Then there exists a vertex \(v'\) such that the swap on \((u, u'; v, v')\) is allowed, and so it produces a bipartite graph \(G'\) from \(G\) such that \(\Gamma_{G'}(v) = (\Gamma_G(v) \setminus \{u'\}) \cup \{u\}\).

**Proof:** By the pigeonhole principle there exists a vertex \(v' \neq v\) such that \(uv' \in E\) and \(u'v' \notin E\). So swap defined on vertices \((u, u'; v, v')\) is allowed. \(\square\)

We say that the previous operation is \emph{pushing up} the neighbors of vertex \(v\). Applying the pushing up operation \(d\) times we obtain the following push up lemma.

**Lemma 3.4 (Havel and Hakimi).** If \(G = (U, V; E)\) is a simple bipartite graph, \(d(u_1) \geq d(u_2) \geq \cdots \geq d(u_k)\) and \(v \in V, d = d(v)\). Then there is a sequence \(S\) of \(d\) many swaps which transforms \(G\) into a graph \(G'\) such that \(\Gamma_{G'}(v) = \{u_1, \ldots, u_d\}\).

This pushing-up lemma also suggests (and proves the correctness of) a greedy algorithm to construct a concrete realization of a BGS \((a, b)\).

**Proof of Theorem 3.2** We prove the following stronger statement:
there exists a sequence of \(2e\) swaps which transform \(G_1\) into \(G_2\), where \(e\) is the number of edges of \(G_1\), by induction on \(e\). Assume that (\(\heartsuit\)) holds for \(e' < e\). We can assume that \(d(u_1) \geq d(u_2) \geq \cdots \geq d(u_k)\). By the Havel-Hakimi Push-up Lemma there is a sequence \(T_1\) (\(T_2\) of \(d = d(v_1)\) many swaps which transforms \(G_1\) (\(G_2\) into a \(G'_1\) (\(G'_2\) such that \(\Gamma_{G'_1}(v_1) = \{u_1, \ldots, u_d\} \ (\Gamma_{G'_2}(v_1) = \{u_1, \ldots, u_d\})\).

We consider the bipartite graphs \(G'_1 = G_1 \setminus \{v_1\}\) and \(G'_2 = G_2 \setminus \{v_1\}\), i.e. we remove the vertex \(v_1\) and all the edges connected to \(v_1\). Since \(bd(G'_1) = bd(G'_2)\) and the number of edges of \(G'_i\) is \(e - d\), by the inductive assumption there is a sequence \(T\) of \(2(e - d)\) many swaps which transforms \(G'_1\) into \(G'_2\).

Now observe that if a swap transforms \(H\) into \(H'\), then the “inverse swap” (choosing the same four vertices, and changing back the edges) transforms \(H'\) into \(H\). So the swap sequence \(T_2\) has an inverse \(T'_2\) which transform \(G'_2\) into \(G_2\). Hence the sequence \(T_1T'_2\) is the required swap sequence: it transforms \(G_1\) into \(G_2\) and its length is at most \(d + 2(e - d) + d = 2e\). \(\square\)

4 The Markov chain \((\mathbb{G}, \mathcal{P})\)

For a bipartite graphical sequence \((a, b)\) - following Kannan, Tetali and Vempala’s lead - we define a Markov chain \((\mathbb{G}, \mathcal{P})\) in the following way. \(\mathbb{G}\) is a graph, the vertex set \(V(\mathbb{G})\) of the graph \(\mathbb{G}\) consists of all possible realizations of our BGS, while the edges represent the possible swap operations: two realizations are connected if there is a swap operation which transforms one realization into the other (and, recall, the inverse swap transforms the second one to the first one as well).

We use the convention that upper case letters \(X, Y\) and \(Z\) stands for vertices of \(V(\mathbb{G})\).

The graph \(\mathbb{G}\) clearly may have exponentially many vertices (that many different realizations of the degree sequence). However, by the statement (\(\heartsuit\)) (in the proof of Theorem \(3.2\)), its diameter is always relatively small:

**Corollary 4.1.** The swap distance of any two realizations is at most \(2e\), where \(e\) is the number of edges.

\(\mathcal{P}\) denotes the set of the transition probabilities, which are defined as follows: if the current realization (status of the process) is \(G\) then we choose uniformly two-two vertices \(u_1, u_2; v_1, v_2\) from classes \(U\) and \(V\) respectively and perform the swap if it is possible and move to \(G'\). Otherwise we do not perform a move. The swap moving from \(G\) to \(G'\) is unique, therefore the probability of this transformation (the jumping probability from \(G\) to \(G' \neq G\) is:

\[
\text{Prob}(G \rightarrow G') := T(G'|G) = \frac{1}{\binom{d}{2}}. \tag{4.1}
\]

The probability of transforming \(G'\) to \(G\) in any time is just the same. The transition probabilities are *time* and edge *independent* and they are also *symmetric*. Therefore \(\mathcal{P}\) can be imagined as a symmetric matrix, where all off
main-diagonal, non-zero elements are the same, while the entries in the main
diagonal are non-zero, but (probably) different values.

As we observed, the graph $G$ is connected, therefore the Markov process
is irreducible. Furthermore since in each realization (state) our random walk
can stay in that state with positive probability (except for the unique degree
sequence $((1, 1), (1, 1))$), therefore our Markov process is clearly aperiodic.

Finally, as we saw, the jumping probabilities are symmetric $T(G|G') = T(G'|G)$, therefore our Markov process is reversible with the uniform distribu-
tion as the globally stable stationary distribution.

Since our Markov process is determined by the graph $G$, sometimes $G$ will
be mentioned as the Markov graph.

5 The Sinclair’s Method

To start with we recall some definitions and notations from the literature. Let
denote $P^t$ the $t$th power of the transition probability matrix and define

$$
\Delta_x(t) := \frac{1}{2} \max_{y \in G} |P^t(y, x) - 1/N|,
$$

and

$$
\tau_x(\epsilon) := \min_{t} \{\Delta_x(t') \leq \epsilon \text{ for all } t' \geq t\}.
$$

Our Markov chain is said to be fast mixing iff

$$
\tau_x(\epsilon) \leq O\left(\text{poly(ln } \log(1/\epsilon) + \log N)\right).
$$

Consider the different eigenvalues of $P$ in decreasing order:

$$
1 = \lambda_1 > \lambda_2 \cdots > \lambda_m.
$$

The relaxation time $\tau_{rel}$ is defined as

$$
\tau_{rel} = \frac{1}{1 - \lambda_2}.
$$

The following result was proved by Diaconis and Strook in 1991:

**Theorem 5.1** (Diaconis and Strook, [3]). $\tau_x(\epsilon) \leq \tau_{rel} \cdot \text{poly(ln } \epsilon^{-1} + \log N)$. □

So one way to prove rapid convergence is to find a polynomial upper bound of
$\tau_{rel}$ and we need fast convergence of the method to the stationary distribution
otherwise the method cannot be used in practice.

Kannan, Tetali and Vempala in [9] could prove that the relaxation time of
the Markov chain $(G, P)$ is a polynomial function of the size $n := 2k$ of $(a, b)$
if it is a regular bipartite degree sequence. Here we extend their proof to show
the fast convergence of the process for the semi-regular bipartite case.

There are several different methods to prove fast convergence, here we use -
similarly to [9] - Sinclair’s canonical path method ([15]).
Theorem 5.2. Let $\mathbb{H}$ be a graph whose vertices represent the possible states of a time reversible finite state Markov chain $\mathcal{M}$, and where $uv \in E(\mathbb{H})$ iff the jumping probabilities in $\mathcal{M}$ satisfy $T(u|v)T(v|u) \neq 0$. For all $x \neq y \in V(\mathbb{H})$ denote $\Gamma_{x,y}$ a set of paths in $\mathbb{H}$ connecting $x$ and $y$ together with a probability distribution $\pi_{x,y}$ on $\Gamma_{x,y}$. Furthermore let

$$\Gamma := \bigcup_{x \neq y \in V(\mathbb{H})} \Gamma_{x,y}$$

where the elements of $\Gamma$ are called canonical paths. We also assume that there is a stationary distribution $\pi$ on the vertices $V(\mathbb{H})$. Finally let

$$\kappa_{\Gamma} := \max_{e \in E(\mathbb{H})} \sum_{x,y \in V(\mathbb{H})} \frac{\pi(x)\pi(y)\pi_{x,y}(\gamma)}{\sum_{(w,z) \in E(\mathbb{H})} T(z|w)\pi(w)} \cdot (5.1)$$

Then

$$\tau_{rel}(\mathcal{M}) \leq \kappa_{\Gamma} \quad (5.2)$$

holds.

We are going to apply Theorem 5.2 for our Markov chain $(\mathbb{G}, \mathcal{P})$. Using the notation $|V(\mathbb{G})| := N$, the (uniform) stationary distribution has the value $\pi(X) = 1/N$ for each vertex $X \in V(\mathbb{G})$. Furthermore each jumping probability has the property $T(x|y) \geq 1/n^4$. So if we can design a canonical path system such that each path is shorter then an appropriate poly($n$) function, then simplifying inequality (5.1) we can turn inequality (5.2) to the form:

$$\tau_{rel} \leq \frac{\text{poly}(n)}{N} \left( \max_{e \in E(H)} \sum_{x,y \in V(H)} \frac{\pi_{x,y}(\gamma)}{\sum_{\gamma \in \Gamma_{x,y}} : e \in \gamma} \right) \quad (5.3)$$

So our task is to find a canonical path system $\{\Gamma_{x,y} : x, y \in \mathbb{G}\}$ such that

- each path is shorter then an appropriate poly($n$) function,
- in (5.3) we have a polynomial upper bound for the right side.

6 Canonical paths - general considerations

Our construction method for canonical paths commences on the trail of Kannan, Tetali and Vempala [3], and Cooper, Dyer and Greenhill [2].

If $X, Y \in V(\mathbb{G})$ let $E(X \triangle Y)$ be the symmetric difference of $E(X)$ and $E(Y)$, set $E(X - Y) = E(X) \setminus E(Y)$, and $E(Y - X) = E(Y) \setminus E(X)$. The edges in $E(X - Y)$ are the $X$-edges and the edges in $E(Y - X)$ are the $Y$-edges. Due to the fact, that the two degree sequences are equal, for every vertex $u \in U \cup V$
the number of $X$-edges adjacent to $u$ is the same as the number of $Y$-edges adjacent to $u$.

Before we describe the construction of our canonical path system we fix some definitions:

**Definition 6.1.** In a simple graph a **circuit** is a sequence of pairwise disjoint edges $e_1, \ldots, e_\ell$, where each pair of edges $e_i, e_{i+1}$ (the summation goes by modulo $\ell$) share a common end point, furthermore the common end points of $e_{i-1}, e_i$ and $e_i, e_{i+1}$ are different. If these common end points are pairwise distinct, then this circuit is a **cycle**.

We start our construction process by defining

$$S(u; (E(X-Y), E(Y-X))) = \{ f \cup f^{-1} \mid f : E(X-Y)(u) \to E(Y-X)(u) \text{ is a bijection} \}$$

for each vertex $u$. The set $S(u; (E(X-Y), E(Y-X)))$ is the set of all possible bijections from the $X$-edges adjacent to $u$ into the $Y$-edges adjacent to $u$. We also define

$$S(X,Y) = \prod_u S(u; (E(X-Y), E(Y-X))).$$

and

$$S = \bigcup \{ S(X,Y) : X, Y \in V(G) \}.$$

For $s \in S$ let

$$\Delta_s = \bigcup \{ \text{dom}(s(w)) : w \in U \cup V \}.$$ 

If $s \in S(X,Y)$ then clearly

$$\Delta_s = E(X \triangle Y). \quad (6.1)$$

If $M$ and $M'$ are $m \times m'$ matrices then let $\mathfrak{d}(M, M')$ be the number of non-zero elements in $M - M'$. For $G \in \mathbb{G}$ let $M_G$ be the bipartite adjacency matrix of $G$. For $X, Y, Z \in \mathbb{G}$ write $M_{X,Y,Z} = M_X + M_Y - M_Z$.

**Outline of the construction of the path system.**

(A) To each $s \in S$ assign a circuit decomposition

$$W_1^s, W_2^s, \ldots, W_{k_s}^s$$

of $\Delta_s$ as follows:

(a) Consider the graph $G = (\Delta_s, \mathcal{E})$, where

$$((u,v), (u'v')) \in \mathcal{E} \text{ iff } u = u' \text{ and } s(u)(u,v) = (u, v').$$

(b) $G$ is a 2-regular graph because $(u,v) \in \Delta_s$ is adjacent to $s(u)(u,v)$ and $s(v)(u,v)$, so it is the union of vertex disjoint cycles $\{W_1^s, \ldots, W_{k_s}^s\}$. 

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(c) The cycles of $\mathcal{G}$ are circuits in $E(X \triangle Y)$, so we set $\{W_1^s, \ldots, W_k^s\}$ as our desired decomposition of $\Delta_s$ into circuits.

(B) For each $X, Y \in \mathcal{G}$ and $s \in \mathcal{S}(X, Y)$

$$\Delta_s = E(X \triangle Y),$$

by (6.1) and the circuits

$$W_1^s, \ldots, W_k^s$$

form a decomposition of $E(X \triangle Y)$ into alternating circuits.

(C) Decompose each $W_i^s$ into alternating cycles

$$C_1^{s,i}, C_2^{s,i}, \ldots, C_{\ell_i}^{s,i}.$$

(D) Let

$$C_1^s, C_2^s, \ldots, C_m^s,$$

be the short hand notation of the cycle decomposition

$$C_1^{s,1}, C_2^{s,1}, \ldots, C_{\ell_1}^{s,1}, C_1^{s,2}, C_2^{s,2}, \ldots, C_{\ell_2}^{s,2}, \ldots, C_1^{s,k_s}, C_2^{s,k_s}, \ldots, C_{\ell_{k_s}}^{s,k_s}$$

of $E(X \triangle Y)$.

(E) For each cycle $C$ in this decomposition we fix a walk-around of the cycle. If $a, b \in C$ let $[a, b]_C$ be the walk from $a$ to $b$ in $C$ according to the fixed walk-around.

(F) Define the path $\Upsilon(X, Y, s)$

$$X = G_0, G_1, \ldots, G_{n_1}, G_{n_1+1}, \ldots, G_{n_2}, \ldots, G_{n_m} = Y$$

in $\mathcal{G}$ from $X$ to $Y$ such that

(a) $n_{m_s} \leq C \cdot n^2$.

(b) $E(G_{n_i+1}) = (E(G_{n_i}) \cup (E(Y) \cap E(C_i))) \setminus (E(X) \cap E(C_i))$.

(c) if $n_i \leq j \leq n_{i+1}$ then there are two vertices $a$ and $b$ in $C_{n_i}$ such that

$$|E(G_j) \triangle F| \leq \Omega_1,$$

where

$$F = (E(G_{n_i}) \cup ([a, b]_{C_j} \cap E(Y)) \setminus ([a, b]_{C_j} \cap E(X))),$$

(d) for each $i$ there is $H_i \in \mathcal{G}$ such that

$$\varphi(\hat{M}_{X,Y,Z}, M_{H_i}) \leq \Omega_2,$$

where $\Omega_i$ are fixed “small” natural numbers.
For $\gamma \in \Gamma_{X,Y}$ let

$$
\pi_{X,Y}(\gamma) = \frac{|\{s \in \mathcal{S}(X,Y) : \Upsilon(X,Y,s) = \gamma\}|}{|\mathcal{S}(X,Y)|}
$$

(6.2)

**Key Lemma 6.2.** If we can assign paths

$$
\{\Upsilon(X,Y,s) : s \in \mathcal{S}(X,Y), X,Y \in G\}
$$

according to (A)-(F) then for each $e \in E(G)$

$$
\sum \{\pi_{X,Y}(\gamma) : e \in \gamma \in \Gamma_{X,Y} : x,y \in G\} \leq \text{poly}(n) \cdot N,
$$

(6.3)

and so our Markov chain is fast mixing.

To prove this statement we need some preparations:

**Lemma 6.3.** There is function $\Psi$ and a parameter set $\mathbb{B}$ such that $\mathbb{B}$ has $\text{poly}(n)$ elements, and for each $X,Y \in V(G)$, for each $\gamma = \Upsilon(X,Y,s) \in \Gamma(X,Y)$ and for each edge $e = (Z,Z')$ from $\gamma$ there is $B \in \mathbb{B}$ such that

$$
\Psi(Z,M_{X,Y,Z},s,B) = (X,Y,\gamma).
$$

(6.4)

**Proof:** By definition $s$ determines the circuit decomposition $(W_1^s, \ldots, W_k^s)$ of $\Delta_s$, and this circuit decomposition in turn determines a canonical alternating cycle decomposition $(C_1^s, \ldots, C_m^s)$ by (A)-(D).

Assume that $Z = G_j$ in the path $\Upsilon(X,Y,s)$ such that $n_i \leq j \leq n_{i+1}$. Put

$$
F = \bigcup_{i' < i} \{E(C_{i'}) \cap E(Z)\} \cup \left(\{a,b|_{C_i} \cap E(Z)\} \right.
$$

$$
\cup \left(\{b,a|_{C_i} \setminus E(Z)\} \cup \bigcup_{i' > i} \{E(C_{i'}) \setminus E(Z)\}\right).
$$

Then, by (F)(b)-(c)

$$
|\left(\{E(X) \setminus E(Y)\} \cup F\right| \leq \Omega.
$$

So if the parameter $B$ contains $j,a,b$ and the set $\left(\{E(X) \setminus E(Y)\} \cup F\right)$ then we can determine $E(X) \setminus E(Y)$. Clearly the size of the parameter set is polynomial. Since $Z$ and $M_{X,Y,Z}$ determine $E(X) \cap E(Y)$, we can compute $E(X)$. Similarly we can compute $E(Y)$. Then path $\gamma = \Upsilon(X,Y,s)$ is uniquely determined by $X$, $Y$ and $s$.

**Proof of the Key Lemma:** Using these observations, we can rearrange HS of (6.3) as follows:

$$
\sum \{\pi_{X,Y}(\gamma) : X,Y \in V(G) \text{ and } \gamma \in \Gamma_{X,Y} : e \in \gamma\} =
$$

$$
\sum \{\pi_{X,Y}(\gamma) : \exists M; \exists s; \exists B \exists Z \in e
$$

s.t. $\Psi(Z,M,s,B) = (X,Y,\gamma)\}$

(6.5)
(It is worth to mention that it may happen, that - as we already mentioned it - several s give rise to the same path γ, but π_{X,Y}(γ) is still taken into consideration only once in the sum of LHS.) Let

\[ \mathcal{M} = \left\{ \hat{M}_{X,Y,Z} : Z \in \mathcal{T}(X,Y,s) \text{ for some } X,Y \in \mathcal{G} \text{ and } s \in \mathcal{S}(X,Y) \right\}. \]

By (F)(d) each \( \hat{M} \) is in a small radius neighborhood of some \( U \in \mathcal{G} \), so

\[ |\mathcal{M}| \leq \text{poly}(n) \cdot |V(\mathcal{G})| = \text{poly}(n) \cdot N. \]

For \( \hat{M} \in \mathcal{M} \) and \( B \in \mathcal{B} \) write

\[ T(\hat{M}, B) = \sum \left\{ \pi_{X,Y}(\gamma) : (Z, Z') \in e \in \gamma \wedge \exists s \Psi(Z, \hat{M}, s, B) = (X, Y, \gamma) \right\}. \]

It is enough to prove that

\[ T(\hat{M}, B) \leq \text{poly}(n). \]

So fix \( \hat{M} \) and \( B \). Let

\[ \mathcal{X}(\hat{M}, B) = \left\{ (X,Y,s) : \Psi(Z, \hat{M}, s, B) = (X,Y,\gamma) \wedge e \in \gamma \right\}. \]

We can assume \( \mathcal{X} \neq \emptyset \). As we observed,

\[ E(X \triangle Y) = \Delta_s. \]

\( E(X \triangle Y) \) has a degree sequence \((2d_1, \ldots, 2d_h)\). Put

\[ t_\Delta = \prod_{1}^{h} (d_i!). \]

Since \( t_\Delta = |\mathcal{S}(X,Y)| \) for each \((X,Y,s) \in \mathcal{X}\), expression \([12]\) can be written as

\[ \pi_{X,Y}(\gamma) = \frac{|\{ s \in \mathcal{S}(X,Y) : \mathcal{Y}(X,Y,s) = \gamma \}|}{t_\Delta}. \]

Then

\[ T(\hat{M}, B) = \sum_{X,Y \in \mathcal{G}: \gamma \in \mathcal{P}_{X,Y}} \left\{ \frac{|\{ s \in \mathcal{S}(X,Y) : \mathcal{Y}(X,Y,s) = \gamma \}|}{t_\Delta} : (X,Y,s) \in \mathcal{X}(\hat{M}, B) \right\}. \]

However, the RHS of the previous formula is just \(|\mathcal{X}|/t_\Delta\). So it is enough to prove that

\[ |\mathcal{X}(\hat{M}, B)| \leq \text{poly}(n) \cdot t_\Delta. \]
Since $e, \hat{M}, s$ and $B$ determine $X$ and $Y$ it is enough to prove that
\[
\left| \left\{ s : \exists X, Y \in G \ (X, Y, s) \in \mathcal{X}(\hat{M}, B) \right\} \right| \leq \text{poly}(n) \cdot t_{\Delta}. \tag{6.6}
\]
Assume that $(X, Y, s) \in \mathcal{X}(\hat{M}, B)$. Assume that $Z = G_j$ and $n_i \leq j \leq n_{i+1}$
Then we can observe that $s$ is “almost” in $\mathcal{S}(\Delta \cap E(Z), \Delta \setminus E(Z))$. More precisely there is $s' \in \mathcal{S}(\Delta \cap E(Z), \Delta \setminus E(Z))$ such that
\[
\left| (\cup s) \triangle (\cup s') \right| \leq \theta \tag{6.7}
\]
for some constant $\theta$. Indeed, by (F)(c), apart from a few edges, for $i' \neq i$ the cycles $C_{i'}$ alternates between $\Delta \cap E(Z)$ and $\Delta \setminus E(Z)$ by (F)(c). Moreover both $[a, b]_{C_{i'}}$ and $[b, a]_{C_{i'}}$ are “almost alternate. This proves (6.7) which implies (6.6) because $|\mathcal{S}(\Delta \cap E(Z), \Delta \setminus E(Z))| = t_{\Delta}$.\hfill \Box

We try to carry out the plan we just described. So:

- Fix $X, Y \in G$.
- Pick $s \in \mathcal{S}(X, Y)$.
- $s$ gives an alternating cycle decomposition
  \[C_0, C_1, \ldots, C_\ell.\] \tag{6.8}
  of $E(X \triangle Y)$.

We want to define a path
\[X = G_0, \ldots, G_i, \ldots, G_m = Y\]
from $X$ into $Y$ in $G$ - denoted by $T(X, Y, s)$ - such that

(i) the length of this path is $\leq C \cdot n^2$,
(ii) for some increasing indices $0 < n_1 < n_2 < \ldots < n_\ell$ we have $G_{n_i} = H_i,$ where
\[E(H_i) = E(X) \setminus \left( \bigcup_{i' < i} E(C_{i'}) \right).\]

So we have certain “fixed points” of our path $\Upsilon(X, Y, s)$, and this observation reduces our task to the following:

- for each $i < \ell$ construct the path
  \[H_i = G'_0, G'_1, \ldots, G'_{m'} = H_{i+1}.\]
  between $G_{n_i}$ and $G_{n_{i+1}}$ such that $m'$ is $\leq |C_i|$ and (F)(d) holds, i.e. for each $j$ there is $K_j \in G$ such that $\delta(M_{X,Y,G'_j}, K_j) \leq \Omega_3.$
From now on we work in that construction. To simplify the notation we write $G = H_i$ and $G' = H_{i+1}$. We know that the symmetric difference of $G$ and $G'$ is just the cycle $C_i$. Now we are in the following situation:

**Generic situation - construction a path along a cycle**

(i) $X, Y, G, G' \in G$.

(ii) The symmetric difference of $E(G)$ and $E(G')$ is a cycle $C$.

(iii) the symmetric differences $E(X \triangle G), E(G \triangle G')$ and $E(G' \triangle Y)$ are pairwise disjoint.

Construct a path $G = G_0, \ldots, G_m = G'$ (6.9)

in the Markov graph $G$ such that

(a) $m \leq c \cdot |C|$, 

(b) for each $j$ there is $K_j \in G$ such that $\varrho(\hat{M}_{X,Y,G_j}, M_{K_j}) \leq \Omega_4$.

We will carry out this construction in the next chapters. The burden of such a construction is to meet requirement (b). In [9] and in [2] the regularity of the realizations was used.

The "friendly path method".

In the next chapters we describe a new general method based on the notion of *friendly paths* (see Definition 7.3) to construct the paths $\Upsilon(X, Y, s)$.

The novelty of our friendly path method can be summarized as follows:

- if our bipartite degree sequence is semi-regular then the paths $\Upsilon(X, Y, s)$ satisfy (b)

- if our bipartite degree sequence is arbitrary, then $\Upsilon(X, Y, s)$ satisfies (b) provided the symmetric difference of $X$ and $Y$ is a cycle. (You can not get such paths using the methods of [9] and [2].)

Originally we conjectured, that our friendly path method always produces paths which satisfy (b). However we were unable to prove it, and now we think that essentially new ideas are needed to prove the case of general bipartite degree sequences.

### 7 Canonical paths - along a cycle

In this section we describe the construction a canonical path along an alternating cycle $C$.

In what follows we will imagine our cycles as convex polygons in the plane, and we will denote the vertices of any particular cycle of $2\ell$ edges with $u_1, v_1, u_2, v_2, \ldots, u_{\ell}, v_{\ell}$ The edges of the cycle are $(u_1, v_1), (v_1, u_2), \ldots, (u_{\ell}, v_{\ell}), (v_{\ell}, u_1)$ and
they belong alternately to $X$ and $Y$. All the other (possible, but not necessarily existing) edges among vertices of a particular cycle are the chords. (With other words we will use the notion of chord if we want to emphasize that we do not know whether the two vertices form an edge or not in the current graph.) A chord is a shortest one, if in one direction there are only two vertices (that is three edges) of the cycle between its end points. The middle edge of this three is the root of the chord.

When we start the process we have realization $G$, when we finish it we have realization $G'$. These two have almost the same edge set, except that the $X$-edges in $C$ belong to $G$ while $G'$ contains the $Y$-edges. W.l.o.g. we may assume that the vertices are $u_1, v_1, u_2, v_2, \ldots, u_\ell, v_\ell$ and $(u_1, v_1)$ is an edge in $G$ while $(v_1, u_2)$ belongs to $G'$. We are going to construct now a sequence of graphical realizations between $G$ and $G'$ such that any two consecutive elements in this sequence differ from each other in one swap operation. The general element of this sequence will be denoted by $Z$.

We have to control which graphs belong to this sequence. For that purpose we assigned a matrix $\hat{M}$ to each graph $Z$. If $G$ is a vertex in $G$ then denoted $M_G$ the adjacency matrix of the bipartite realization $G$ where the columns and rows are indexed by the vertices of $U$ and $V$ resp., but the columns are numbered from left to right while the lines are numbered from bottom to the top. Then let

$$\hat{M}(X + Y - Z) = M_X + M_Y - M_Z.$$  

By definition each entry of an adjacency matrix is 0 or 1. Therefore only $-1, 0, 1, 2$ can be the entries of $\hat{M}$. An entry is $-1$ if the edge is missing from both $X$ and $Y$ but it exists in $Z$. This is 2 if the edge is missing from $Z$ but exists in both $X$ and $Y$. It is 1 if the edge exists in all three graphs $(X, Y, Z)$ or it is there only in one of $X$ and $Y$ but not in $Z$. Finally it is 0 if the edge is missing from all three graphs, or the edge exists in exactly one of $X$ and $Y$ and in $Z$. (Therefore if a chord denotes an existing edge in exactly one of $X$ and $Y$ then the entry corresponding to this chord is always 0 or 1.)

**Observation 7.1.** The raw and column sums of $\hat{M}(X + Y - Z)$ are the same as raw and column sums in $M_X$ (or $M_Y$ or $M_Z$). □

We need some more notions:

**Definition 7.2.** The type of a chord is 1 or 0 depending whether it is an edge in $G$ or - what is the same - in $G'$. A chord $f = (u_\alpha, v_\beta)$ is a cousin of a chord $e = (u_\gamma, v_\delta)$, if $0 \leq \delta - \alpha \leq 1$ and $0 \leq \epsilon - \beta \leq 1$. A chord $e$ is friendly if at least one of its cousins has the same type as $e$ itself.

Since the notion of the cousins is not a very natural one we want to illustrate it. Therefore now we introduce our main tool what we will use later in this paper to illustrate different procedures in our current realizations. This tool $F(G+G'+Z)$ (or $F_Z$ for short, and we use $F$ for “figure”) is a matrix again, similar to the matrix $M_G + M_{G'} + M_Z$. This always illustrate the subgraph in $Z$ spanned by
the vertices of one of the alternating (therefore of an even cardinality) cycle $C_l$. The positions $(1,1), \ldots, (\ell, \ell)$ form the main-diagonal while the positions right above the main-diagonal as well as the rightmost bottom one (these are $(1,2), (2,3), \ldots, (\ell-1, \ell)$ finally $(\ell,1)$) form the small-diagonal.

The matrix $F_Z$’s entries in the main-diagonal and the entries in the small-diagonal come from the adjacency matrix of $M_Z$. The other entries come from the corresponding entries of $M_G+M_G'+M_Z$. In that way in the main- and small-diagonal’s elements are 0 or 1 while the others (the off-diagonal entries) can be 0, 1, 2, 3. (There is an easy algorithm to construct $F_Z$ from the corresponding $\hat{M}(G+G'-Z)$ and vice versa (please recognize that here we use $G$ and $G'$ instead of $X$ and $Y$): In the main-diagonal and in the small-diagonal the zeros and ones must be interchanged to the opposite ones. Outside of these diagonal entries $-1,0,1,2$ of $\hat{M}(G+G'-Z)$ become 1, 0, 3, 2 in $F_Z$.) Since in this alternating cycle $G$ and $G'$ contain the same chords therefore the off-diagonal elements in $F_Z$ are odd when the edge exists in the actual $Z$ and even otherwise. When $Z = G$ then the main-diagonal entries are 1 while the small-diagonal elements are 0. This matrix $F_Z$ will be used in our illustrating figures.

Now Figure 1 illustrates the cousins of the chord $(u_6, v_2)$ in the initial $Z$. (They are $(u_1, v_6), (u_1, v_7), (u_2, v_6)$, finally $(u_2, v_7)$ and let’s recall that the word chord indicates that the definition does not depend on the actual existence or non-existence of that edge.)

**Figure 1: An edge and its cousins**

![Figure 1](image)

**Definition 7.3.** A sequence of pairwise distinct chords $e_1, \ldots, e_j$ is a friendly path if

(i) each chord is friendly,

(ii) each $e_h$ has previously one common endpoint with $e_{h+1}$ while the other end points are in distance 2,

(iii) finally $e_1$ and $e_j$ are shortest chords and their roots have different types.
A friendly path can be quite complicated, and it is important to remark that such a friendly path is NOT a path in a particular graph. The name is justified by the image of the friendly path in the illustration of $F_Z$ what we are going to show next. (It shows the path itself, but it does not show why the individual elements of the path are friendly - see Figure 2). This figure is not for illustration only: whenever we consider a friendly path we always work on the matrix itself.

Figure 2: A friendly path

![Image of a friendly path]

7.1 There exists a friendly path in our cycle

In this subsection we describe the construction of the canonical path along this cycle in the case if a friendly path (going from the main-diagonal to the small-diagonal) exists. Let the chords of the existing friendly path correspond to the positions $A_1, \ldots, A_N$ where $A_j = (a_{1j}, a_{2j})$. As we have already mentioned we now consider (“comfortably ordered”) $\ell \times \ell$ submatrices of our adjacency matrices. It is important to remark, that all descriptions of this subsection use the language of the matrices $F_Z$ (that is the entries can be 0, 1, 2, 3), unless it is announced otherwise.

Before we start we introduce a metric on pairs of positions of this matrix: $\|A, \bar{A}\|$ says how many steps are necessary to go from $A$ to $\bar{A}$ if in every step we can move to a (horizontally or vertically) neighboring position, we cannot cross the main-diagonal, finally the position $(i,1)$ is neighboring to $(i, \ell)$ and analogously $(\ell, i)$ is neighboring to $(1, i)$.

By definition our friendly path has the following properties: (i) $a_{1j} \neq a_{2j}$ and $a_{1j} + 1 \neq a_{2j}$. (ii) $\|A_j, A_{j+1}\| = 1$. Finally (iii) $A_1$ is in distance 1 from the main-diagonal, while $A_N$ is in distance 1 from the small-diagonal.

First we introduce two new structures:

Definition 7.4. Let $1 \leq \alpha, \beta \leq \ell$, $\alpha + 1 < \beta$. We say an $\ell \times \ell$-matrix $M = (m_{i,j})$ is $(\alpha, \beta)$-OK matrix iff
(i) $m_{\alpha, \beta} = 2$,
(ii) $m_{i,i} = 0$ for $\alpha < i < \beta$, and $m_{ii} = 1$ for $i \leq \alpha$ and $\beta \leq i$,
(iii) $m_{i,i+1} = 1$ for $\alpha \leq i < \beta$, and $m_{i,i+1} = 0$ for $i < \alpha$ and $\beta \leq i$

(See Figure 3) Please recall that the entry 2 in $F_Z$ is an edge which is missing from $Z$ but exists in both $G$ and $G'$ (the off-diagonal entries are the same in $M_G$ and $M_{G'}$).

**Definition 7.5.** Let $1 \leq \alpha, \beta \leq \ell$, $\alpha + 1 < \beta$. We say an $\ell \times \ell$-matrix $M = (m_{i,j})$ is $(\beta, \alpha)$-KO matrix iff

(i) $m_{\alpha, \beta} = 1$,
(ii) $m_{i,i} = 0$ for $\alpha \leq i \leq \beta$, and $m_{ii} = 1$ for $i < \alpha$ and $\beta < i$,
(iii) $m_{i,i+1} = 1$ for $\alpha \leq i < \beta$, and $m_{i,i+1} = 0$ for $i < \alpha$ and $\beta \leq i$

(See Figure 3) Please recall that the entry 1 in $F_Z$ is an edge which exists in $Z$ but missing from both $G$ and $G'$.

**Lemma 7.6.** Let $M = (m_{i,j})$ be an $(\alpha, \beta)$-OK matrix and $m_{\alpha-1, \beta+2} = 3$. Assume that $M' = (m'_{i,j})$ is an $(\alpha-1, \beta+2)$-OK matrix such that

(1) $m'_{\alpha, \beta} = 3$
(2) $m'_{i,j} = m_{i,j}$ if $i \neq j$, $i+1 \neq j$, and $(i,j) \neq (\alpha, \beta), (\alpha-1, \beta+2)$.

Then there exists an absolute constant $\Theta$ such that one can transform $M$ into $M'$ with at most $\Theta$ swaps.
Proof: It is enough to observe that the symmetric difference of \(M\) and \(M'\) is a cycle which alternates between \(M\) and \(M'\). Indeed, in the next figure (1) indicates edges in \(E(M' - M)\) and (0) indicates edges in \(E(M - M')\). (The non-empty positions of this figure are the circled positions in the previous matrix \(M')\).

Therefore
\[(\alpha - 1, \alpha), (\alpha, \alpha), (\alpha, \beta),\]
\[(\beta, \beta), (\beta, \beta + 1), (\beta + 1, \beta + 1),\]
\[(\beta + 1, \beta + 2), (\alpha, \beta + 2)\]

is an alternating cycle of length 8.

So the difference of the realizations lay within in a subgraph with 8 vertices \(V\). The subgraphs in the two original graphs spanned by \(V\) have the same (bipartite) degree sequence and they contain alternately the edges of the cycle. By Theorem 3.2 we know one can be transformed by swaps into the other one. Since the cycle contains four vertices from both classes, and there at most 12 edges, therefore the number of the necessary swaps (by Corollary 4.1) is at most \(2 \times 12\) therefore \(\Theta = 24\) is an upper bound of the number of the necessary swaps.

Clearly the same argument gives the following more general lemma.
Lemma 7.7. For each natural number $u$ there is a natural number $\Theta_u$ with the following property: assume that $M = (m_{i,j})$ is an $(\alpha, \beta)$-OK matrix and $m_{\alpha', \beta'} = 3$ where $\|(\alpha, \beta); (\alpha', \beta')\| < u$, furthermore $M' = (m'_{i,j})$ is an $(\alpha', \beta')$-OK matrix such that

1. $m'_{\alpha, \beta} = 3$
2. $m'_{i,j} = m_{i,j}$ if $i \neq j$, $i + 1 \neq j$, and $(i, j) \neq (\alpha, \beta), (\alpha', \beta')$.

Then at most $\Theta_u$ swap transforms $M$ into $M'$.

Proof: The only difference is that here the symmetric difference of $M$ and $M'$ is a cycle of length at most $2 + 2u$ which alternates between $M$ and $M'$.

We also have the analogous general result for KO matrices.

Lemma 7.8. For each natural number $u$ there is a natural number $\Theta'_u$ with the following property: assume that $M = (m_{i,j})$ is an $(\beta, \alpha)$-KO matrix and $m_{\beta', \alpha'} = 0$ where $\|(\beta, \alpha); (\beta', \alpha')\| < u$, furthermore $M' = (m'_{i,j})$ is an $(\beta', \alpha')$-KO matrix such that

1. $m'_{\beta, \alpha} = 0$
2. $m'_{i,j} = m_{i,j}$ if $i \neq j$, $i + 1 \neq j$, and $(i, j) \neq (\beta, \alpha), (\beta', \alpha')$.

Then at most $\Theta'_u$ swap transforms $M$ into $M'$.

Proof: The proof is very similar to the proof of Lemma 7.7 which is left to the diligent reader.

Lemma 7.9. Assume that $M = (m_{i,j})$ is $(\alpha, \beta)$-OK matrix and $m_{\beta+2, \alpha-1} = 0$. Assume that $M' = (m'_{i,j})$ is a $(\beta + 2, \alpha - 1)$-KO matrix such that

1. $m'_{\alpha, \beta} = 3$
2. $m'_{i,j} = m_{i,j}$ if $i \neq j$, $i + 1 \neq j$, and $(i, j) \neq (\alpha, l), (\beta + 2, \alpha - 1)$.

Then there exists a natural number $\Omega$ such that one can transform $M$ into $M'$ with at most $\Omega$ swaps.
Proof: It is enough to observe that the symmetric difference of \( M \) and \( M' \) is a cycle which alternates between \( M \) and \( M' \). Indeed, in the next figure values 1 indicate edges in \( E(M' - M) \) and values 0 indicate edges in \( E(M - M') \).

\[
\begin{array}{c|c|c}
\beta & 1 & 0 \\
\hline
1 & 1 & 0 \\
\hline
1 & 0 & 0 \\
\hline
0 & 0 & 1 \\
\end{array}
\]

Therefore

\((\alpha - 1, \alpha - 1), (\beta + 2, \alpha - 1), (\beta + 2, \beta + 2), (\beta + 1, \beta + 2), (\beta + 1, \beta + 1), (\beta, \beta), (\alpha, \beta), (\alpha, \alpha), (\alpha - 1, \alpha)\)

is an alternating cycle of length 10.

The proof goes like the proof of Lemma 7.6. The difference of the realizations lay within in a subgraph with 10 vertices \( \bar{V} \). The subgraphs in the two original graphs spanned by \( \bar{V} \) have the same (bipartite) degree sequence and they contain alternately the edges of the cycle. By Theorem 3.2 we know one can be transformed by swaps into the other one. Since the cycle contains five vertices from both classes, and there are at most 20 edges, the number of the necessary swaps (by Corollary 4.1) is at most \( 2 \times 20 \) therefore there exists a constant upper bound \( \Omega \leq 40 \) of the number of the necessary swaps.

Lemma 7.10. For each natural number \( u \) there is a natural number \( \Omega_u \) with the following property: assume that \( M = (m_{i,j}) \) is \((\alpha, \beta)\)-OK and \( m_{\beta', \alpha'} = 0 \) where

\[ \|[(\alpha, \beta); (\alpha', \beta')]\| < u, \]

and \( M' = (m'_{i,j}) \) is a \((\beta', \alpha')\)-KO matrix such that

1. \( m'_{\alpha, \beta} = 3 \)
2. \( m'_{i,j} = m_{i,j} \) if \( i \neq j \), \( i + 1 \neq j \), and \((i, j) \neq (\alpha, \ell), (\beta', \alpha')\).

Then at most \( \Omega_u \) swap transforms \( M \) into \( M' \).

Proof: Similar to Lemma 7.7.

Now using our friendly path we are going to define a sequence of OK- and KO-matrices, such that we can achieve the required edge changes in \( G \) getting \( G' \) along this sequence, using operations described in the previous Lemmas. At
first we define a new sequence $A'_1, \ldots, A'_\Lambda$ from $A_1, \ldots, A_\Lambda$ with the following way:

$$A'_i = \begin{cases} A_i, & \text{if } M_G(A_i) = 0; \\ \text{Cousin}(A_i), & \text{if } M_G(A_i) = 3, \end{cases} \quad (7.1)$$

where Cousin($A$) denote one position where $A$ has a cousin. (So $M_G($Cousin($A$)) = $M_G(A)$.)

**Observation 7.11.** By definitions,

(i) if $M_G(A_i) = M_G(A_{i+1})$ then $\|A'_i, A'_{i+1}\| \leq 3,$

(ii) if $M_G(A_i) \neq M_G(A_{i+1})$ then $\|\text{Cousin}(A'_i), A'_{i+1}\| \leq 3.$

Next we define the matrix sequence $M_G = L_0, L_1, \ldots, L_\Lambda, L_{\Lambda+1} = M_G'$ as follows:

**Definition 7.12.** The matrix $L_i$ ($i = 1, \ldots, \Lambda$) is defined from the matrix $L_{i-1}$ by the formulae:

$$L_i = \begin{cases} \text{the } (A'_i)-\text{OK matrix}, & \text{if } L_i(A_i) = 3; \\ \text{the } (A'_i)-\text{KO matrix}, & \text{if } L_i(A_i) = 0. \end{cases}$$

Here all positions $(u, v)$ which are NOT determined by the definitions of the OK- and KO-matrices satisfy $L_i(u,v) = L_{i-1}(u,v).$ □

It is quite clear that $(\Lambda-1)$ consecutive applications of (the appropriate) Lemmas 7.6 - 7.10 will take care the definition of the required swap sub-sequences between $L_1$ and $L_\Lambda.$ However, the swap-sequence transforming $L_0$ into $L_1$ furthermore the one transforming $L_\Lambda$ into $L_{\Lambda+1}$ require special considerations:

- If $L_1(A_1) = 3$ then there are two possibilities - depending on the position of the Cousin($A_1$). (The squares denoted with dashed lines contain the possible positions of friendly cousins.)

**Case I:**

\[
\begin{array}{c|c|c}
0 & 0 & 1 \\
\cline{3-3}
0 & 1 & 3 \\
\cline{3-3}
\end{array} \quad \Rightarrow \quad \begin{array}{c|c|c}
2 & 1 & 0 \\
\cline{3-3}
1 & 0 & 3 \\
\cline{3-3}
\end{array}
\]

and **Case II.**

\[
\begin{array}{c|c|c}
0 & 0 & 1 \\
\cline{3-3}
0 & 1 & 3 \\
\cline{3-3}
\end{array} \quad \Rightarrow \quad \begin{array}{c|c|c}
2 & 1 & 0 \\
\cline{3-3}
1 & 1 & 2 \\
\cline{3-3}
\end{array} \quad \Rightarrow \quad \begin{array}{c|c|c}
2 & 1 & 0 \\
\cline{3-3}
1 & 0 & 3 \\
\cline{3-3}
\end{array}
\]
• If, however, \( M(A_1) = 0 \) then there is only one case:

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 0 \\
\end{bmatrix} \implies 
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

The connecting swap-sequence from the matrix \( L_\Lambda \) to \( L_{\Lambda+1} \) (what is \( M_G' \)) can be defined analogously to the previous one.

Next we will analyze the behavior of the current matrices \( \hat{M}(G, G', Z) \) along this sub-sequences. At first we consider those \( Z \)'s which correspond to matrices \( L_i \).

Let \( M \) be an integer matrix and let \( M' \) be a \( 2 \times 2 \) submatrix of it. If we add 1’s to the values of the positions of one diagonal in \( M' \) and -1’s to the values of the positions of the other diagonal, then the acquired matrix has the same row and column sums as \( M \) had. Such an operation is called a **switch**. When our matrix \( M \) is the adjacency matrix of a degree sequence realization, then any swap clearly corresponds to a switch of that matrix. The two matrices are in **switch-distance** 1 from each other.

**Lemma 7.13.** All adjacency matrices \( L_1, \ldots, L_\Lambda \) are in switch-distance 1 from the adjacency matrix of some appropriate realizations.

**Proof:** We show here the statement for such an \( L_i \) where \( M(A_i) = 0 \) therefore \( L_i \) itself is an \( (A'_i) \)-KO-matrix, and where - by definition - \( A_i = A'_i \) (the other case is similar). Due to the definitions \( A_i \) originally is not an edge either in \( G \) or in \( G' \). It belongs to the friendly path, therefore we also know that Cousin(\( A_i \)) = 0 also holds in \( M_G, M'_G \). In \( L_i \) this value is 1, so \( A_i \) is an edge in \( Z \). Therefore \( L_i \) which is \( = F_Z \) looks like the matrix to the left in the following figure (the circled element is the cousin of \( A_i \)). The corresponding \( \hat{M}(G + G' - Z) \) is shown on the right hand side:

\[
\begin{array}{cccccc}
6 & 0 & 0 & 0 & 1 & \\
5 & 0 & 0 & 1 & 0 & \\
4 & 1 & 0 & & & \\
3 & 0 & 1 & & & \\
2 & 0 & 0 & 1 & & \\
1 & 0 & & & & \\
\end{array}
\quad
\begin{array}{cccccc}
6 & 0 & 0 & 0 & 1 & \\
5 & 0 & 0 & 1 & 0 & \\
4 & 0 & 1 & & & \\
3 & 0 & 1 & & & \\
2 & 1 & 1 & -1 & & \\
1 & 0 & & & & \\
\end{array}
\]

It is clear that adding 1 to the values of the positions \( A' \) and Cousin(\( A' \)) of \( \hat{M}(G + G' - Z) \) and subtracting 1 from the other two corners of the spanned submatrix constitutes the required switch. \( \square \)
Lemma 7.14. The realization $G$ can be transformed into the realization $G'$ through realizations represented by the adjacency matrices $L_i$ ($i = 1, \ldots, \Lambda$) in such a way that the lengths of the swap sub-sequences leading from each $L_i$ to $L_{i+1}$ (where $0 = 1, \ldots, \Lambda$) can be bounded from above with the absolute constant \(\max\{\Theta_3, \Theta'_3, \Omega_3\}\). In the process each arisen matrix $\hat{M}(G + G' - Z)$ is within a constant switch-distance from some vertices in $V(G)$ (that is some realizations of the bipartite degree sequence).

Proof: By Observation 7.11 any two $A'_i$ and $A'_{i+1}$ are at most distance 3. Therefore for each $i$ (where $i = 2, \ldots, \Lambda$) the corresponding process chosen among Lemma 7.7, Lemma 7.8 and Lemma 7.10 will describe the desired swap sub-sequences. The length of any such swap-subsequence is bounded from above by \(\max\{\Theta_3, \Theta'_3, \Omega_3\}\).

Furthermore, when in the process the current realization $Z$ corresponds to an $L_i$, then Lemma 7.9 applies, and matrix $\hat{M}(G + G' - Z)$ has switch-distance 1 from the adjacency matrix of some realization $\in V(G)$.

Let now $Z$ be an intermediate realization in the process, say, between the matrices $L_i$ and $L_{i+1}$: then $\hat{M}(G + G' - Z)$ can be transform through swaps into $\hat{M}(G + G' - L_{i+1})$ (assume, this end is the closer one to $Z$). As we know all swaps are specialized switches, and they keep the row and column sums. Combining this with the previous paragraph, we have for every $Z$ that $\hat{M}(G + G' - Z)$ is at most \(\frac{1}{2} \max\{\Theta_3, \Theta'_3, \Omega_3\} + 1\) switch distance from some realization $\in V(G)$.

\[\square\]

Key problem

One can say that we are very close to proving the fast mixing property of our Markov process on all bipartite degree sequences: we should prove, that in the case when there exists a friendly path from $G$ to $G'$ then for each intermediate $Z$ the matrix $\hat{M}(X + Y - Z)$ is in a constant distance from some realization $\in V(G)$. If we can manage this then we must handle the cases when there are no friendly paths. It is somewhat surprising that this second requirement can be satisfied successfully (as it will be shown in Subsection 7.2).

However, we cannot manage to prove the first requirement. The problem is the following: we can try to repeat the proof of Lemma 7.14 but, unfortunately, it is not true anymore that for each graph $Z$, corresponding to a particular matrix $L_i$, the matrix $\hat{M}(X + Y - Z)$ is also in distance 1 from some realization $\in V(G)$.

In the realizations $G$ and $G'$ all chord have the same values, but this is not the case for realizations $X$ and $Y$. The edges in $E(X - Y) \cup E(Y - X)$ belong to only one of them. Therefore if a swap turns an entry to 2 in $\hat{M}(G + G' - Z)$ then this entry originally was 1: the edge belonged to $G$ and $G'$ and $Z$ as well. Therefore its cousin bears the entry 1 (also belonged to $G$ and $G'$ and $Z$ as well). So this entry was appropriate to perform a switch to turn the matrix under investigation into the adjacency matrix of a realization.
However, if the cousin entry is 0 in $\hat{M}(X + Y - Z)$ (this edge belongs only to one of realizations $X$ and $Y$, say, it belongs to $X$ only), then the required switch cannot be performed. (The value $-1$ can be cause a similar problem and can be handled similarly as this case.)

A good solution for this particular problem would probably ends up in a complete proof of the fast mixing property.

The following observation is enough to handle the switch-distance problem for $\hat{M}(X + Y - Z)$ in semi-regular bipartite degree sequences: recall, that $(a, b)$ is semi-regular if in $a$ all degrees are the same, while in $b$ can be anything.

**Lemma 7.15.** Assume that our bipartite degree sequence $(a, b)$ is semi-regular. Then the statement of Lemma 7.14 applies for the matrices $\hat{M}(X + Y - Z)$ as well.

**Proof:** We follow the proof of Lemma 7.14. To do so the only requirement is to show (somewhat loosely) that the matrices $\hat{M}(X + Y - L_i)$ are in a constant switch-distance from the adjacency matrix of some realizations. As we know any of these matrices contains exactly one entry of value different from 1 and 0. So consider a particular $L_i$ and assume that this “extra” value in this case is a 2. If the switch, described in the proof of Lemma 7.13 is also a possible switch in $\hat{M}(X + Y - Z)$ then we are ready. If this not the case then the entry (with value 1 in matrix $\hat{M}(G + G' - L_i)$) has value 0 in $\hat{M}(X + Y - L_i)$. (In this case, as we discussed it previously, the corresponding edge is missing from $Y$.) Let this corresponding edge be $(u, v)$, then this entry in $\hat{M}(X + Y - L_i)$ is 0. Since the column sums are fixed in these matrices, they are the same (and equal to entries in $a$).

Now vertex $v$ has degree at least 2 (it is a vertex on cycle $C$ and it also end point of at least one chord of $C$ in $X$). Therefore the raw $v$ contains some 1s. One of them is $(w, v)$ (this $w$ cannot be the column of the 2, since the entry there is 0 due that it belongs to the originally intended switch). Now by the pigeonhole principle there is a raw $z$ such that $\hat{M}(w, z) = 0$ and $\hat{M}(u, z) = 1$. Therefore the $u, w; v, z$ switch (actually this is a swap) will change $\hat{M}(u, v)$ into 1, and now the original switch finishes the job. The matrix $\hat{M}(X + Y - L_i)$ is in switch-distance at most 2 from the adjacency matrix of some realization. □

### 7.2 Their is no friendly path in our cycle

In this subsection we discuss the case where there is no friendly path in $M_G$. Our plan is this: at first we show that the non-existence of the friendly paths yields a strong structural property of the matrix $M_G$. Using this property we can divide our problem into two smaller ones, where one of the smaller matrices possesses a suitable friendly path. So we can solve our original problem in a recursive manner.

This recursive approach must be carried out with caution: a careless ”greedy” algorithm can increase the switch-distances very fast. We will deal with this
problem using a simple "fine tuning" (which is described at the end of this subsection).

We start with some further notions and notations.

**Definition 7.16.** Let \( M \) be an \( \ell \times \ell \) matrix, then the positions \((i, i), (i + 1, i - 1), (i + 2, i - 2), \ldots, (j, j + 1)\) form the \( i \)th **down-line** of the matrix. (The arithmetic operations are thought to be considered modulo \( \ell \), that is, for example, \( \ell + 1 = 1 \) while \( 1 - 3 = \ell - 2 \). The positions \((i, i), (i - 1, i + 1), (i - 2, i + 2), \ldots, (j', j' - 1)\) form the \( i \)th **up-line** of the matrix.

The lines have orientations: the down-lines connect the main-diagonal to the small-diagonal, while an up-line connects the small-diagonal to the main-diagonal.

**Definition 7.17.** A set \( T \) of positions of an \( \ell \times \ell \) matrix called **connected** if a chess king, staying inside \( T \), can visit all elements of \( T \).

The following lemma is a well-known version of the classical Steinhaus lemma (see [16]).

**Lemma 7.18.** Assume that the positions of an \( \ell \times \ell \) matrix \( M \) are colored for white and black. Then either the king has a white path from the main diagonal to the small diagonal, or there is a connected set \( T \) of black positions which intersects all rook’s path from the main diagonal to the small diagonal. \( \square \)

We use the previous result without proof. The set \( T \), which was identified in the previous lemma, will be called a **Steinhaus set**.

**Definition 7.19.** The **cousin-set** \( \mathcal{C}(u, v) \) is the set of the off diagonal cousins of the position \((u, v)\). If \( T \) is a set of positions, then the cousin set \( \mathcal{C}(T) \) is defined as \( \bigcup \{ \mathcal{C}(e) : e \in T \} \).

**Lemma 7.20.** Assume that in the matrix \( M_G \) there is a connected set \( T \) of non-friendly positions. Then the type of all positions in the cousin-set \( \mathcal{C}(T) \) are the same (all chords have the same type). All edges in \( T \) have the opposite type.

**Proof:** W.l.o.g. we may assume that a position \( P \) in \( T \) has type 0, then all positions in its cousin-set must have type 1. However, for each other position \( P' \) in \( T \), which can be reached from \( P \) in one king step, the cousin sets \( \mathcal{C}(P) \) and \( \mathcal{C}(P') \) have common position(s). Therefore all type in those two cousin sets must be the same (1), therefore both positions \( P \) and \( P' \) have the same type (0) as well. \( \square \)

**Lemma 7.21.** Let \( T \) be a Steinhaus set of unfriendly positions in \( M_G \), then its cousin-set \( \mathcal{C}(T) \) intersects all down-lines and up-lines.

**Proof:** Actually we can prove more: namely that any king-path from the main-diagonal to the small-diagonal intersects the cousin-set \( \mathcal{C}(T) \). We will argue by contradiction, assume that there exists a king-path which does not cross the cousin-set. For position \( P \) let denote \( I(P) \) be those positions whose cousin-sets
contain \( P \). Then if \( P \in \text{king path} \), then \( I(P) \) is disjoint from \( C(T) \) (otherwise the king-path intersects \( T \)). But any two neighboring positions in the king-path have overlapping \( I(P) \) sets. Therefore we have an overlapping sequence of \( I(P) \) sets, connecting the main-diagonal to the small-diagonal. And such an overlapping sequence clearly contains a rock-path, connecting the main-diagonal to the small-diagonal. A contradiction. Finally it is clear, that every down- and up-line forms a required king-path. \( \square \)

Lemma 7.22. We assume that in the matrix \( M_G \) there is no friendly path. Then for each \( i \) \((i = 1, \ldots, \ell)\) there exists a pair of \( j, j' \) s.t. \( 0 \leq j' - j \leq 1 \) and all entries \((i + 1, i - 1), \ldots, (i + j, i - j)\) have the same type \( t \), furthermore the entries \((i - 1, i + 1), \ldots, (i - j' + 1, i + j' - 1)\) have type \( 1 - t \) while \((i + j', i - j')\) has type \( t \) again.

Proof: Assume indirectly, there is no such \( j \) for a particular \( i \). W.O.L.G. we may assume, that \( M(i + 1, i - 1) = 0 \). Then, by the assumption, \( M(i - 1, i + 1) = M(i - 2, i + 2) = 1 \) must hold. Then, again by our assumption, \( M(i + 2, i - 2) = 0 \) must hold, etc. All entries along the down-line are 0, while all entries along the up-line must be 1. However both lines intersect (see Lemma 7.21) the cousin-set \( C(T) \) of the Steinhaus set \( T \). But, by Lemma 7.20 all its entries have the same type. A contradiction.

Corollary 7.23. If conditions of Lemma 7.22 hold, and \( j' \geq 2 \) for a particular \( i \) (in which case \( j' = j + 1 \)), the submatrix spanned by \((i + j, i - j)\) and \((i - j, i + j)\) contains friendly path(s).

Proof: We argue by contradiction: assume that the submatrix does not contain a friendly path. Then - due to Lemma 7.18 - it contains a Steinhaus set. Due to Lemma 7.20 in its cousin-set \( C(T) \) - which intersects all down- and up-lines - all positions have the same type. But it contradicts to the fact, that in the \( i \)-th down-line all positions have type \( t \), while in the \( i \)-th up-line all positions have type \( 1 - t \). A contradiction, again.

That finishes the preliminaries what are needed to describe our recursive algorithm, which is essentially a divide and conquer approach. Due to the previous fact here we should handle separately two possibilities: when \( j' = 1 \) and when \( j' \geq 2 \). We start with the

First possibility: assume that for a particular \( i \) our \( j' = 1 \). We should take care for two cases:

Case 1: If \( M_G = (i + 1, i - 1) = M_G(i - 1, i + 1) = 1 \) then we are in an easily handleable situation: at first we swap the quartet \( u_i, u_{i+1}, v_i, v_{i+1} \). (The dash-ed square in our illustration. Here we use the adjacency matrix \( M_G \).) The entries \((i - 1, i), (i, i)\) and \((i, i + 1)\) have the required types. However, entry \( M(i - 1, i + 1) = 0 \) therefore along the procedure we should take care to change it back to its original value.
The remaining subproblem, indicated by thick black lines (for a formal description see case $j' \geq 2$), fortunately is already in the required form. Indeed: its main-diagonal contains only 1s, while its small-diagonal is full with 0s. Denote the alternating cycle of this smaller problem by $C'$. The (recursive) solution of the subproblem $C'$ will switch the value of $M(i-1, i+1)$ automatically back to 1. Since the recursive procedure can use any down- and up-lines (see Corollary 7.23), therefore we can take care, that this switch-back will happen in the next recursion.

It is important to recognize that matrices $\hat{M}(G + G' - Z)$ and $\hat{M}(X + Y - Z)$ may have contain 2 at the position $(i-1, i+1)$. Fortunately this “problematic” entry will be present only along one recursive step. Furthermore this entry will increase the switch-distance of the current $\hat{M}$ with at most one: the positions $(i-1, i), (i, i)$ and $(i, i+1)$ (outside of our subproblem), provides a suitable switch to handle the entry 2 at position $(i-1, i+1)$.

**Case 2:** Now we have $M_G = (i+1, i-1) = M_G(i-1, i+1) = 0$. Here we perform two swaps, as it is shown below (the places of the swaps are denoted with dash-ed squares):
If $\hat{M}(X + Y - Z)(i + 1, i - 1) = -1$ holds, then it increases the switch-distance of the current $M$ with at most one (since it can be directly back-swapped). The result of the second swap (after which the previous problem is just solved automatically), together with our further strategy is shown below:

Here we distinguish between two cases, according to the value $M_G(i - 2, i + 2) = x$.

This value can be $x = 1$ and $x = 0$.

In the case of $x = 1$ we perform a second swap, which results in a subproblem with a friendly path (the swap shown on the left side of Figure 4 while the right hand side indicates the two new subproblems):

Figure 4: The case of $x = 1$

On the RHS of Figure 4 the dash-ed submatrix is in the right form: the main diagonal contains all 1’s, while each entry in the small-diagonal is 0. The second subproblem (indicated with the thick black lines, the four pieces fit together to a square matrix, again in the right form. We will process the second ("dash-ed") subproblem along the up-line, containing position $(i - 2, i + 2)$, so the only currently improper entry will have the right value at the end of the next recursion step (that is it will be swapped back to its original value). (Here we used again Corollary 7.23)
As it happened before $\hat{M}(X + Y - Z)$ may contain 2 at the position $(i - 2, i + 2)$. Again this increases the switch-distance with at most one, since the positions $(i - 2, i - 1), (i + 1, i - 1)$ and $(i + 1, i + 2)$ are not in our subproblem.

Finally it can happen, that $x = 0$. Then we can define the following subproblem:

This figure shows the new subproblem (indicating with thick black lines) is in the right form again. We will process the subproblem along the up-line, containing position $(i - 2, i + 2)$ (so the only currently improper entry will have the right value at the end of the next recursion step).

Here, again, we may confront the fact, that $\hat{M}(X + Y - Z)(i + 1, i - 1) = -1$. Then we should consider the alternating cycle shown in the figure. All elements of the cycle, except $(i + 1, i - 1)$, is in the main- and small-diagonal, therefore along this cycle we can swap that entry into range within a small number (say $\delta$) of steps. This will increase the switch-distance of $\hat{M}$ with at most $\delta$.

We run the first recursion on the subproblem along the 1th up-line, therefore the sub-subproblem with friendly path will contain the position $(i - j', i + j')$. Therefore when we finish the first recursion, our adjacency matrix $M_G$ will be in the following form: (the figure on the left):

We have seen how one can handle the switch-distance of our matrix, if position $(i + 1, i - 1)$ is problematic (in $\hat{M}$ it is a $-1$ there) but position $(i - 2, i + 2)$ is O.K. On the other, if $M(X + Y - Z))(i - 2, i + 2) = -1$ then the swap on
the positions \((i - 2, i - 1), (i - 1, i + 2); (i + 1, i - 1), (i - 2, i + 2)\) change both \((i + 1, i - 1)\) and \((i - 2, i + 2)\) into 0. For \((i + 1, i - 1)\) that was the original type - so it cannot be wrong in \(M\).

After that we perform the swap on the positions \((i - 2, i - 1), (i + 1, i + 2); (i + 1, i - 1), (i - 2, i + 2)\) (these are the corners of the dash-ed square in the figure on the upper right). The result is shown to the right:

This completes our handling on the First Possibility, that is when for our \(i\) we have the value \(j' = 1\). Now we turn to the other (and probably more common) configuration:

**Second Possibility:** We have \(j' \geq 2\). Unfortunately, the situation can be more complicated in this case due to the switch-distance of \(\hat{M}\). We overcome this problem by showing at first the general structure of the process, and later we give the necessary fine-tuning to ensure the low switch-distance.

In our current alternating cycle (lying in the symmetric difference of \(G\) and \(G'\)) there is no friendly path, therefore there is a \(T\) Steinhaus set in the subgraph of \(Z\), spanned by the vertices of the cycle. Now fix a particular \(i\) and assume that the \(j'\) belongs to this \(i\) is \(\geq 2\). We should distinguish between two cases: where the down-line start with the value \(t = 1\) or with \(t = 0\).

**Case 1:** \(t = 1\) The first figure below shows the structure of matrix \(M_{G'}\). The dash-ed square is the first subproblem to deal with, while the thick black lines indicate the second subproblem. However, before we start the processing the subproblems, we have to perform a swap. The corners of the thin black square shows the positions of the swap. After that, the first subproblem (indicated by the dash-ed square) is in the right form. (See the figure on the right.)

Finishing the first subproblem, we have the following \(M_{G'}\) adjacency matrix:
As it can be seen, after the first phase, all entries in the dash-ed are in their required types: the small-diagonal consists of 1s (including position \((i+2, i-2)\) which in that way is back to its original type), while the main-diagonal consists of only 0s.

The second subproblem (indicated by the thick black lines) in the right form now (including position \((i-3, i+3)\) which is sitting on the small-diagonal).

After completing the solution of the black subproblem, all entries in the matrix will be in exactly the required type. We start processing the black subproblem on the \(i\)th up-line, therefore the actual types of positions \((i-3, i+3)\) and \((i+2, i-2)\) can be described as follows: Position \((i+2, i-2)\) has opposite type after the very first swap, then while processing the dash-ed subproblem it may changes between 0 and 1. Finishing the dash-ed subproblem, it will be in the type as it starts.

Position \((i-3, i+3)\) will be in type 0 all the way in the dash-ed phase, while within the black phase it will change between 1 and 0. At the end, as we already mentioned, it 1.

**Case 2: \(t = 0\)** The first figure below shows the structure of matrix \(M_G\). The dash-ed square is the first subproblem to deal with, while the thick black lines indicate the second subproblem. They can process without any preprocessing.

At the end everything will be in the right type, except the four position, showed by the thin black square (below, left side). We can finish the process with that swap.
While the overall structure of our plan is clear, we may meet problems along this procedure. Its reason is that we must be able to control the switch-distance of our $\hat{M}(X + Y - Z)$ (we will use here simply $\hat{M}$) from the adjacency matrix of some realization. There are two neuralgic points: both the positions $(i + j, i - j)$ and $(i - j', i + j')$ may contain $-1$, or both may contain $2$. When we start a new subproblem, then their types always provide suitable switch for the control (as it was seen before). However, when we proceed along our subproblem, then it can happen that one of the problematic position changes its value, while the other does not. But in this case the switching which was previously available are not useable anymore. Next we describe how we can fine tuning our procedure to avoid this trap.

As we know the first subproblem contains a friendly path, and for easier reference let call its problematic position $P_1$. We also know that second subproblem contains a problematic position, $P_2$, and probably we have to divide this subproblem into two smaller ones. If so, then the first of them becomes the new second subproblem, which contains $P_2$ and possesses a friendly path, while the third subproblem contains another problematic position, $P_3$.

**Fine tuning:**

1. We begin our swap sequence along the first subproblem but we stop just before we face the swap which changes the value of $P_1$.

2. Next we continue with the swap sequence of the second problem and we stop before we should perform a swap on $P_2$.

3. Now we finish the swap sequence of the first subproblem.

4. After that we focus on the second subproblem. Dealing effectively with this, we need to prepare the third subproblem similarly as we did with the second one, when we were working on the first one. Therefore we begin the swap sequence of the third subproblem but we stop it before the first swap would be carried out on $P_3$. 

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5. And if now we just rename our two active subproblems as first and second subproblem, we are back to a situation, which is equivalent to the beginning of the third stage.

Doing this refined algorithm we have the opportunity that at every particular moment we can control with a small number of swaps the $\hat{M}$ values at positions $P_i$ and $P_{i+1}$.

The size of the parameter set $\mathbb{B}$: In this paper we proposed the friendly path method to construct suitable canonical path systems for the application of the Sinclair’s method. Lemma 6.3 postulated several assumptions which were managed throughout the construction. However we did not check the assumption about the size of the parameter set $\mathbb{B}$ yet. Here we supply this last missing part.

Lemma 7.24. The size of the parameter set $\mathbb{B}$ is $\text{poly}(n)$.

Proof: The parameter set contains the following parameters:

1. The integer $j$ describing the ordinal number of the current step. Since no path is longer than $Cn^2$ therefore $1 \leq j \leq n^2$ also holds.

2. The edge pair $a, b$ describing the segment $[a, b]$ of the current alternating cycle which is already switched from $E(G)$ to $E(G')$. Since $n$ is a trivial upper bound of the length of any cycle, therefore there are at most $O(n^2)$ possible edge pairs.

3. We should list those chords of the current alternate cycle which have different types in $X$ and $Y$ and touched by the construction. At the very first and last steps of the construction of the swap sequence along a friendly path there may be at most one "bad" chord. While along the steps described in Lemmas 7.6 7.7 7.8 there are at most $\max\{\Theta, \Theta_u, \Theta'\}$ bad chords. (In each application of those sub-procedures at most 2 bad chords can come into existence and the same amount can be omitted. At the end of those applications there are always at most one bad chord.

4. Finally at the fine tuning a constant number $C$ of bad chords may come into existence, there at most $\binom{n^2}{C} = O(n^{2C})$ choices for them.

All together this gives an $\text{poly}(n)$ upper bound on the size of $\mathbb{B}$. (Here we gave only very rough upper bounds but they were sufficient for our purposes.)

8 Conclusions

In this paper we introduced a new way to generate canonical path systems for the Sinclair’s method to be applied for the Markov chain determined by a given bipartite degree sequence (defined by Kannan, Tetali and Vempala [9]). In the process no specific assumptions on the degree sequence are adopted, but showing
the validity of the argument requires the assumption of semi-regularity (at least for now). See Lemma 7.15 and the Key Problem. However we think that this technical limitation can be eased by the mean some new, but not necessarily breath-taking ideas. Therefore here we summarize the most important feature of our "friendly path system" method:

As Lemma 7.14 and the procedure described in Subsection 7.2 proves the following statement holds:

**Theorem 8.1.** Let $G$ and $G'$ be two realizations of an arbitrary bipartite degree sequence, such that the edges $E(G \triangle G')$ form one alternating cycle. Then realization $G$ can be transformed into the realization $G'$ through realizations in such a way that the length of the swap sequence leading from $G$ to $G'$ can be bounded from above with some absolute constant. In the process each arisen matrix $\hat{M}(G + G' - Z)$ is within a constant switch-distance from some vertices in $V(G)$ (that is some realizations of the bipartite degree sequence).

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