A HOMOGENIZATION RESULT IN FINITE PLASTICITY
AND ITS APPLICATION TO HIGH-CONTRAST MEDIA

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Abstract. We carry out a variational study for integral functionals that model the stored energy of a heterogeneous material governed by finite-strain elastoplasticity with hardening. Assuming that the composite has a periodic microscopic structure, we firstly establish the $\Gamma$-convergence of the energies in the limiting of vanishing periodicity. Then, in the second part of the paper, we use the result to derive a macroscopic description for an elastoplastic medium with high-contrast microstructure. Specifically, we consider a composite obtained by filling the voids of a periodically perforated stiff matrix by soft inclusions. Again, we study the $\Gamma$-convergence of the related energy functionals as the periodicity tends to zero. The main challenge is posed by the lack of coercivity brought about by the degeneracy of the material properties in the soft part. We prove that the $\Gamma$-limit, which we compute with respect to a suitable notion of convergence, is the sum of the contributions resulting from each of the two components separately.

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1. INTRODUCTION

This paper deals with the homogenization of some variational integrals related to the theory of finite plasticity. Precisely, we focus on those materials governed by finite-strain elastoplasticity with hardening. Starting from the microscopic level, we aim at retrieving effective macroscopic descriptions by means of a $\Gamma$-convergence approach.

First, we briefly outline the content of our results. Later, we elaborate on the motivation and the background of our research.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with Lipschitz boundary, and let us denote by $SL(3)$ the group of $3 \times 3$ real matrices with determinant equal to 1. Our main result describes the limiting behavior of the functionals

$$F_\varepsilon(y, P) := \int_{\Omega} W\left(\frac{x}{\varepsilon}, \nabla y(x) P^{-1}(x)\right) \, dx + \int_{\Omega} H\left(\frac{x}{\varepsilon}, P(x)\right) \, dx + \int_{\Omega} |\nabla P(x)|^q \, dx,$$

where $y \in W^{1,2}(\Omega; \mathbb{R}^3)$, $P \in L^{1,q}(\Omega; K)$ with $q > 3$ and $K \subset SL(3)$ compact, and $W$ and $H$ encode, respectively, the periodic nonlinear elastic energy density and the hardening of the elastoplastic material sitting in $\Omega$. The precise statement is contained in Theorem 2.2, and the exact mathematical setting of the analysis is detailed in Section 2.

In the second part of the paper we employ Theorem 2.2 to derive the effective macroscopic energy of a heterogeneous material with a high-contrast microstructure. Specifically, we consider the case in which the pores of a stiff perforated matrix $\Omega^1_\varepsilon$ are filled by soft inclusions, which form the set $\Omega^0_\varepsilon$ (see Figure 2). When the matrix and the inclusions exhibit the same plastic-hardening $H$, the functionals encoding the stored energy associated with the deformation $y \in W^{1,2}(\Omega; \mathbb{R}^3)$ and the plastic strain $P \in L^{1,q}(\Omega; SL(3))$ read

$$J_\varepsilon(y, P) := \int_{\Omega} W^0_\varepsilon \left(\varepsilon \nabla y(x) P^{-1}(x)\right) \, dx + \int_{\Omega^0_\varepsilon} W^1 \left(\nabla y(x) P^{-1}(x)\right) \, dx$$

$$+ \int_{\Omega} H(P(x)) \, dx + \int_{\Omega} |\nabla P(x)|^q \, dx,$$

where $\{W^0_\varepsilon\}_{\varepsilon > 0}$ and $W^1$ are, respectively, the elastic energy densities of the inclusions and of the matrix. The factor $\varepsilon$ multiplying the argument of $W^0_\varepsilon$, instead, encodes the high-contrast between the two components, and it results in a loss of coercivity in the problem. From a modeling perspective, this heuristically means that very large deformations of the inclusions are allowed or, in other words, that the inclusions are very soft. The asymptotics of the functionals $J_\varepsilon$ is detailed in Theorem 2.8, and we refer again to Section 2 for further details on the definitions and on the roles of the terms in $J_\varepsilon$. A discussion on alternative modeling choices for the hardening contribution is presented in Remark 2.4.

As we touched upon before, we work within the classical mathematical theory of elastoplasticity at finite strains (see, e.g., [42]). We assume that the elastic behavior of our sample $\Omega$ is independent of preexistent plastic distortions. This can be rephrased as the assumption that the deformation gradient $\nabla y$ associated with any deformation $y: \Omega \to \mathbb{R}^3$ of the body decomposes into an elastic strain and a plastic one. Within the framework of linearized elastoplasticity, the
decomposition would have an additive nature; in the setting of finite plasticity, instead, a multiplicative structure is traditionally assumed \[41, 44\]. Following this approach, as in \[46, 45\], we suppose that for any deformation \( y \in W^{1,2}(\Omega; \mathbb{R}^3) \) there exist an elastic strain \( F_{\text{el}} \in L^2(\Omega; \mathbb{R}^{3\times 3}) \) and a plastic strain \( P \in L^2(\Omega; \text{SL}(3)) \) such that

\[
\nabla y(x) = F_{\text{el}}(x)P(x) \quad \text{for a.e.} \ x \in \Omega.
\]

Such multiplicative decomposition, recently justified in the setting of dislocation systems and crystal plasticity in \[19, 20\], motivates the definitions of the energy functionals in (1.1) and (1.2). Other constitutive models in finite plasticity have been also taken into account in the mathematical literature (see, e.g., \[25, 35, 36, 47\]). Concerning the regularization via \( \nabla P \) in (1.1) and (1.2), we note that it is common in engineering models and prevents the formation of microstructures, see \[8, 12\]. We also refer to \[28\] for a discussion of its drawbacks and for alternative higher order regularizations.

In the small strain regime, the homogenization of the plasticity equations has been undertaken in the series of works \[50, 39, 40\] both in the periodic and in the aperiodic and stochastic settings. The framework of perfect plasticity in the case of a linear elastic response has been completely characterized in the seminal work \[31\]. A homogenization analysis for the Fleck and Willis model in linearized elastoplasticity has been established in \[33, 32\]. See also \[38\] for a homogenization result in the small strain regime with hardening. To the authors’ knowledge, the only available results in the large strain setting are instead confined to the framework of crystal plasticity for stratified composites. In particular, we refer to \[15, 16\] for the superlinear growth case, to \[24\] for the linear growth scenario, and to \[26\] for some first homogenization analysis in the evolutionary setting.

Theorem 2.2 fills the gap in the study of elastoplastic microstructures by providing a static homogenization result in the large strain regime. In the second part of the paper we then present an application of our main theorem to the modeling of a special class of composite materials, namely those exhibiting a strong difference - high-contrast - between their elastoplastic behaviors. The interest in these metamaterials stems from the experimental observation of an infinite number of band gaps in their mechanical behavior. In other words, high-contrast materials exhibit infinitely many interval of frequencies in which wave propagation is not allowed. This, in turn, makes them extremely interesting for possible cloaking applications. Recent applications in civil engineering, such as in seismic waves cloaking, and in the modeling of advanced sensor and actuator devices, call for advancements in the mathematical modeling of high-contrast materials in settings that have not yet been fully studied, such as those of inelastic phenomena.

The mathematical literature on high-contrast materials is vast. To keep our presentation concise, we only point out that besides the aforementioned results for stratified elastoplastic composites \[15, 16, 24, 26\], the only additional available contributions in the inelastic setting concern the study of brittle fracture problems \[6, 7, 49\]. For the modeling of nonlinear elastic high-contrast composites we single out the works \[11, 14\].

We conclude this introduction with a few words on the proofs. The existence of an integral \( \Gamma \)-limit is based on the localization technique in the context of integral representation results. The biggest hurdle is devising a version of the so-called \textit{fundamental estimate} that complies with the constraint that plastic deformations must belong to \( \text{SL}(3) \). This is done by regarding \( \text{SL}(3) \) as a manifold endowed with a Finsler metric, which establishes a connection to the mechanism of plastic dissipation. We refer to Section 3 for a review of the localization method and of the geometric properties of \( \text{SL}(3) \). The characterization of the limiting energy density is obtained by a perturbative argument that grounds on standard homogenization results, see Theorem 3.1.
Turning to our second result, a delicate point is choosing a convergence that ensures effective compactness properties. Indeed, the fact that the energy contributions in the soft inclusions are evaluated in terms of $\varepsilon \nabla y$ leads to a loss of coercivity for which compactness in classical weak Sobolev topologies is prevented. On the other hand, arguing with strong two-scale convergence of the gradients, as in [14] does not guarantee convergence of minimizers of $J_\varepsilon$ to minimizers of the limiting functional. To cope with this difficulty, we adapt the approach in [27] and introduce an ad hoc notion of convergence for deformations, to which we refer as convergence in the sense of extensions. Roughly speaking, a sequence of deformations converges in the sense of extensions if it is bounded in $L^2$ and can be extended in $W^{1,2}$ in such a way that the extensions are weakly compact in the Sobolev sense, cf. Definition 2.5 and Remarks 2.6 and 2.7 for the precise definition and some basic properties. For the plastic strains, we argue instead with the weak convergence in $W^{1,q}$. This choice is motivated by the fact that sequences of deformations and plastic strains with uniformly bounded energies are precompact with respect to the above topology. Thus our $\Gamma$-convergence analysis directly entails convergence of minimizers. We observe that this result easily extends to functionals which take into account also plastic dissipation. On this point we refer to Section 7.

The strategy relies on extension results on perforated domains, on two-scale convergence and periodic unfolding techniques, as well as on equiintegrability arguments to control the behavior of the microstructure close to the boundary of the set $\Omega$. A key-step is a splitting procedure that allows to treat the soft and the stiff parts separately.

The setup of our analysis and the main results, Theorems 2.2 and 2.8, are presented in Section 2. Section 3 contains some reminders about the technical tools to be used in the sequel. The proof of Theorem 2.2 is the subject of Section 4. With Section 5 we begin the analysis of high-contrast composites, and we discuss the equicoercivity of the related energy functionals, as well as the splitting procedure. The asymptotic behavior of the soft inclusions is characterized in Section 6. The ground is then laid for the proof of the Theorem 2.8, to whom Section 7 is devoted. We conclude the paper with an appendix in which we compare our findings with a related result in the context of elasticity.

2. Mathematical setting and results

Throughout the paper, $\Omega$ is an open, bounded, and connected set with Lipschitz boundary in $\mathbb{R}^3$ (the analysis would not change significantly if we settled the problem in $\mathbb{R}^d$ with $d = 2$ or $d > 3$). We use $\mathbb{R}^{3 \times 3}$ and $\mathbb{R}^{3 \times 3 \times 3}$ to denote real-valued $3 \times 3$ and $3 \times 3 \times 3$ tensors, respectively. We use the notation $I$ for the identity matrix. The symbol $| \cdot |$ is indiscriminately adopted for the Euclidean norms in $\mathbb{R}^3$, $\mathbb{R}^{3 \times 3}$ and $\mathbb{R}^{3 \times 3 \times 3}$. To deal with plastic strains, we recall the classical notation

$$\text{SL}(3) := \{F \in \mathbb{R}^{3 \times 3} : \det F = 1\}.$$

In Subsection 3.2 we will endow $\text{SL}(3)$ with a metric structure and regard it as a Finsler manifold.

If $A \subset \mathbb{R}^3$ is a measurable set, we will denote by $\mathcal{L}^3(A)$ its three-dimensional Lebesgue measure.

The building block of our study is the following variational notion of convergence:

Definition 2.1. Let $X$ be a set endowed with a notion of convergence. We say that the family $\{G_\varepsilon\}$, with $G_\varepsilon : X \to [0, \infty]$, $\Gamma$-converges as $\varepsilon \to 0$ to $G : X \to [0, \infty]$ if for all $x \in X$ and all infinitesimal sequences $\{\varepsilon_k\}_{k \in \mathbb{N}}$ the following holds:
(1) for every sequence \( \{x_k\}_{k \in \mathbb{N}} \subset X \) such that \( x_k \to x \), we have
\[
\mathcal{G}(x) \leq \liminf_{k \to +\infty} \mathcal{G}_{\varepsilon_k}(x_k);
\]
(2) there exists a sequence \( \{x_k\}_{k \in \mathbb{N}} \subset X \) such that \( x_k \to x \) and
\[
\limsup_{k \to +\infty} \mathcal{G}_{\varepsilon_k}(x_k) \leq \mathcal{G}(x).
\]

When \( X \) is equipped with a topology \( \tau \), we sometimes use expressions such as \( \Gamma(\tau) \)-convergence or \( \Gamma(\tau) \)-limit to emphasize what is the underlying convergence for sequences in \( X \). We elaborate on the connections between \( \Gamma \)-convergence and homogenization in Subsection 3.1. In what follows, for notational convenience, we indicate the dependence on \( \varepsilon_k \) by means of the subscript \( k \) alone, e.g. \( \mathcal{F}_k := \mathcal{F}_{\varepsilon_k} \).

2.1. Homogenization in finite plasticity. We collect here the assumptions under which the \( \Gamma \)-convergence of the functionals in (1.1) will be proved.

Let \( Q := [0,1)^3 \) be the periodicity cell.

We assume that the elastic energy density \( W : \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \to [0, +\infty] \) satisfies the following:

**E1:** It is a Carathéodory function such that \( W(\cdot, F) \) is \( Q \)-periodic for every \( F \in \mathbb{R}^{3 \times 3} \).

**E2:** It is 2-coercive and has at most quadratic growth, i.e., there exist \( 0 < c_1 \leq c_2 \) such that for a.e. \( x \in \mathbb{R}^3 \) and for all \( F \in \mathbb{R}^{3 \times 3} \)
\[
c_1 |F|^2 \leq W(x, F) \leq c_2 \left( |F|^2 + 1 \right).
\]

**E3:** It is 2-Lipschitz: there exists \( c_3 > 0 \) such that for a.e. \( x \in \mathbb{R}^3 \) and for all \( F_1, F_2 \in \mathbb{R}^{3 \times 3} \)
\[
|W(x, F_1) - W(x, F_2)| \leq c_3 (1 + |F_1| + |F_2|) |F_1 - F_2|.
\]

Let us compare E1–E3 with the common requirements for physically admissible elastic energy densities. Setting
\[
\text{GL}^+ (3) := \{ F \in \mathbb{R}^{3 \times 3} : \det F > 0 \},
\]
a couple of natural conditions are \( W \equiv +\infty \) on \( \mathbb{R}^{3 \times 3} \setminus \text{GL}^+ (3) \), which rules out interpenetrations, and \( \lim_{\det F \to 0} W(F) = +\infty \), which means that it should take infinite energy to squeeze a small block of material down to a point. Feasible elastic densities are also required to be frame indifferent and to attain their minimum, conventionally set to 0, on the identity matrix \( I \).

In our analysis, it would be particularly challenging to preserve the constraint of positive determinant for deformation gradients. Therefore, we work in a simplified setting and consider the whole space of \( 3 \times 3 \) matrices as domain for \( W \). Similarly, the quadratic growth from above in E2 rules out the blow up of the energy densities, but it is a rather common assumption in variational studies. Frame indifference, that is, \( W(x, RF) = W(x, F) \) for a.e. \( x \in \mathbb{R}^3 \), for all \( F \in \text{GL}^+ (3) \) and all rotations \( R \), does not play any role in our analysis, thus we ignore it. Finally, up to adding a constant to \( W \), the growth condition E2 is compatible with the requirement \( \min W(x, \cdot) = W(x, I) = 0 \) for a.e. \( x \in \mathbb{R}^3 \).

As for the hardening functional, we assume that \( H : \mathbb{R} \times \mathbb{R}^{3 \times 3} \to [0, +\infty] \) meets the ensuing requirements:

**H1:** \( H \) is a Carathéodory function such that \( H(\cdot, F) \) is \( Q \)-periodic for every \( F \in \mathbb{R}^{3 \times 3} \).

**H2:** Assume that a Finsler structure on \( \text{SL}(3) \) is assigned. For a.e. \( x \in \Omega \), \( H(x, F) \) is finite if and only if \( F \in K \), where \( K \subset \text{SL}(3) \) is a geodesically convex, compact neighborhood of \( I \).

**H3:** The restriction of \( H(x, \cdot) \) to \( K \) is Lipschitz continuous, uniformly in \( x \).
Let us spend some words on $H_2$. This rather strong hypothesis prescribes that the effective domain of $H(x, \cdot)$, namely the set $\{F \in SL(3) : H(x, F) < +\infty\}$ is an $x$-independent compact set containing $I$. A consequence of technical advantage is that uniform $L^\infty$-bounds on the plastic strains become available, provided the latter give rise to finite hardening. Indeed, if $K$ is as in $H_2$, then there exists $c_K > 0$ such that
\[
|F| + |F^{-1}| \leq c_K \quad \text{for every } F \in K,
\]
which is ensured by Proposition 3.3 below. and Remark 3.4 for the details). roots in the localization argument that we adopt to establish the $\Gamma$-convergence (see Subsection 3.1). We recall that a subset of a Finsler manifold is said to be geodesically convex if, for any couple of points in the set, there is a unique shortest path contained in the set that joins those two points. The existence of a compact set $K$ complying with $H_2$ is ensured by Proposition 3.3 and Remark 3.4 below.

We are now ready to state the first main result of this paper, namely the homogenization formula for the functionals in (1.1). We work in the space $W^{1,2}((\Omega; \mathbb{R}^3) \times W^{1,3}(\Omega; SL(3))$ endowed with the topology $\tau$ characterized by
\[
(y_k, P_k) \xrightarrow{\tau} (y, P) \quad \text{if and only if} \quad \begin{cases} y_k \to y & \text{strongly in } L^2(\Omega; \mathbb{R}^3), \\ P_k \to P & \text{uniformly}. \end{cases}
\]

**Theorem 2.2.** Let $F_\varepsilon$ be the functionals in (1.1), which we extend by setting
\[
F_\varepsilon(y, P) = +\infty \quad \text{on } [L^2(\Omega; \mathbb{R}^3) \times L^q(\Omega; SL(3))] \setminus [W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,3}(\Omega; K)].
\]
If $W$ and $H$ satisfy E1–E3 and H1–H3, respectively, then for all $y \in L^2(\Omega; \mathbb{R}^3)$, $P \in L^q(\Omega; SL(3))$ the $\Gamma$-limit
\[
F(y, P) := \Gamma(\tau)\lim_{\varepsilon \to 0} F_\varepsilon(y, P)
\]
exists. We also have that
\[
F(y, P) = \begin{cases} \int_{\Omega} \left( W_{\text{hom}}(\nabla y(x), P(x)) + H_{\text{hom}}(P(x)) + |\nabla P(x)|^q \right) dx & \text{if } (y, P) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,3}(\Omega; K), \\ +\infty & \text{otherwise in } L^2(\Omega; \mathbb{R}^3) \times L^q(\Omega; SL(3)), \end{cases}
\]
where $W_{\text{hom}} : \mathbb{R}^{3 \times 3} \times K \to [0, +\infty)$ and $H_{\text{hom}} : K \to [0, +\infty)$ are defined as
\[
W_{\text{hom}}(F, G) := \lim_{\lambda \to +\infty} \frac{1}{\lambda^3} \inf_{l} \int_{(0, \lambda)^3} W(x, (F + \nabla y(x))G^{-1}) dx : y \in W_0^{1,2}((0, \lambda)^3; \mathbb{R}^3),
\]
\[
H_{\text{hom}}(F) := \int_{Q} H(z, F) dz.
\]

**2.2. Homogenization of a high-contrast elastoplastic composite.** In the second part of the paper (Sections 5–7) we exploit the previous homogenization result to study an elastoplastic medium with high-contrast periodic microstructure. We consider the situation in which soft inclusions are inserted in a perforated stiff matrix. To describe the geometry in precise terms, let $Q^0 \subset Q$ be an open set such that $Q^1 := Q \setminus \overline{Q^0}$ is connected and has a Lipschitz boundary (see Figure 1). The set $\Omega$, which represents the region of space occupied by the composite, is
Figure 1. The periodicity cell $Q$ and its partition into the soft inclusion $Q^0$ (white) and the stiff matrix $Q^1$ (gray).

Figure 2. The microstructure of the composite in $\Omega$. The soft inclusions that form $\Omega^0_\varepsilon$ correspond to the white holes, while the grey region represents the matrix $\Omega^1_\varepsilon$.

then subdivided by means of the sets

$$\Omega^0_\varepsilon := \bigcup_{t \in T_\varepsilon} \varepsilon(t + Q^0), \quad \text{with} \quad T_\varepsilon := \{t \in \mathbb{Z}^3 : \varepsilon(t + Q^0) \subset \Omega\}, \quad (2.4)$$

$$\Omega^1_\varepsilon := \Omega \setminus \overline{\Omega^0_\varepsilon}, \quad (2.5)$$

which stand respectively for the collection of the inclusions and for the matrix (see Figure 2). We also define the $Q$-periodic set

$$E^1 := \bigcup_{t \in \mathbb{Z}^3} (t + Q^1), \quad (2.6)$$

where we say that a set $E \subset \mathbb{R}^3$ is $Q$-periodic if $E + t = E$ for all $t \in \mathbb{Z}^3$. Note that the set $\Omega^1_\varepsilon$ is connected and Lipschitz, because (2.4) ensures that the inclusions are well separated from $\partial \Omega$. Our assumptions allow for some flexibility on the geometry of the inclusions, which could for instance form interconnected fibers (see Figure 3).

Our second $\Gamma$-convergence result deals with the asymptotic behavior, as $\varepsilon$ tends to 0, of the family $\{J_\varepsilon\}$ defined by (1.2). Before stating the result, we collect the hypotheses we use in the following lines.
Figure 3. In the 3-dimensional space, interconnected soft fibers do not disconnect the matrix. A simple case is depicted here: the cylindrical perforation $Q^0$ runs through the periodicity cell and its complement $Q^1$ is connected.

The elastic energy density of the stiff matrix $W^1: \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \to [0, +\infty]$ meets conditions E1–E3 with $W$ replaced by $W^1$. The assumptions on the soft densities $W^0_\varepsilon: \mathbb{R}^{3 \times 3} \to [0, +\infty]$ are analogous:

**E4:** There exist $0 < c_1 \leq c_2$ such that for all $F \in \mathbb{R}^{3 \times 3}$, and all $\varepsilon > 0$,
$$c_1 |F|^2 \leq W^0_\varepsilon(F) \leq c_2 \left(||F||^2 + 1\right).$$

**E5:** There exists $c_3 > 0$ such that for all $F_1, F_2 \in \mathbb{R}^{3 \times 3}$, and all $\varepsilon > 0$,
$$|W^0_\varepsilon(F_1) - W^0_\varepsilon(F_2)| \leq c_3 \left(1 + |F_1| + |F_2|\right) |F_1 - F_2|.$$

**E6:** There exists $W^0: \mathbb{R}^{3 \times 3} \to [0, +\infty]$ such that for all $F \in \mathbb{R}^{3 \times 3}$
$$\lim_{\varepsilon \to 0} W^0_\varepsilon(F) = W^0(F).$$

**Remark 2.3.** The function $W^0$ possesses the same growth and regularity properties of $W^0_\varepsilon$.

As for the hardening term in (1.2), we assume that $H: \mathbb{R}^{3 \times 3} \to [0, +\infty]$ is finite if and only if $P$ lies in a geodesically convex, compact neighborhood of $I$, and that the restriction of $H$ to such set is Lipschitz continuous. These are essentially the same requirements introduced in the previous subsection, so, for the sake of brevity, in the sequel we will refer to the property of $H$ as $H_2$ and $H_3$ as well.

**Remark 2.4.** Note that in principle it would be reasonable to suppose that the soft and the stiff components feature different hardening behaviors. For instance, it could be imposed that the soft hardening is evaluated on an $\varepsilon$-rescaling of the plastic stress, thus replicating the structure of the elastic contribution. In order to focus our attention just on the application of Theorem 2.2, we leave such scenarios for possible future investigation and we restrain ourselves to a simpler setting, namely we choose to model both hardening terms by a single function satisfying $H_2$ and $H_3$. We point out that under these assumptions making a distinction between $H^i = H^i(P)$, $i = 0, 1$ would not require any substantial change in our approach, therefore we dispense with it. Qualitatively, keeping the soft hardening contribution of order 1 amounts to the situation in which, for small $\varepsilon$, elastic deformations of a much larger magnitude than the plastic ones are allowed.

We can now state the homogenization result for high-contrast elastoplastic media. Since we want our analysis to yield convergence of minima and minimizers of $J_\varepsilon$ to the ones of the limiting energy, we need to introduce a convergence that is compliant with the degeneracy of the soft
inclusions. For shortness, we refer to it as convergence in the sense of extensions, even though the name is not at all standard.

**Definition 2.5.** Let \( \{ \varepsilon_k \} \) be an infinitesimal sequence. We say that \( \{ y_k \} \subset W^{1,2}(\Omega;\mathbb{R}^3) \) converges to \( y \in W^{1,2}(\Omega;\mathbb{R}^3) \) in the sense of extensions with respect to the scales \( \varepsilon_k \) if the following hold:

1. \( \{ y_k \} \) is bounded in \( L^2(\Omega;\mathbb{R}^3) \);
2. there exists a sequence \( \{ \tilde{y}_k \} \subset W^{1,2}(\Omega;\mathbb{R}^3) \) such that \( y_k \to \tilde{y}_k \) in \( \Omega_{\varepsilon_k}^1 := \Omega_{\varepsilon_k}^1 \) and \( \tilde{y}_k \to y \) weakly in \( W^{1,2}(\Omega;\mathbb{R}^3) \).

**Remark 2.6.** Let \( \tilde{y}_k = \tilde{y}_k' \) a.e. in \( \Omega_{\varepsilon_k}^1 \). Let as well \( \tilde{y}_k \to y \) and \( \tilde{y}_k' \to y' \) weakly in \( W^{1,2}(\Omega;\mathbb{R}^3) \). Then, recalling (2.5)–(2.6) and observing that \( \Omega \cap \varepsilon_k E^1 \subset \Omega_{\varepsilon_k}^1 \),

\[
0 = \lim_{k \to +\infty} \int_{\Omega_{\varepsilon_k}^1} |\tilde{y}_k - \tilde{y}_k'| \, dx \geq \lim_{k \to +\infty} \int_{\Omega} \chi_{\varepsilon_k E^1}(x)|\tilde{y}_k - \tilde{y}_k'| \, dx = c \int_{\Omega} |y - y'| \, dx,
\]

for a constant \( c > 0 \). From this, we conclude that \( y = y' \) a.e. in \( \Omega \). In particular, if the limit in the sense of extensions exists, then it is unique.

**Remark 2.7.** By the definition of \( \Omega_{\varepsilon_k}^1 \), there exists a tubular neighborhood \( O \) of \( \partial\Omega \) such that \( \Omega_{\varepsilon_k}^1 \cap O \equiv \Omega \cap O \). Therefore, if \( y \) and \( \tilde{y} \) coincide in \( \Omega_{\varepsilon_k}^1 \), their traces on \( \partial\Omega \) are also equal.

The asymptotic behavior of the family \( \{ J_\varepsilon \} \) with respect to the notion of convergence that we have just introduced is described in the next theorem:

**Theorem 2.8.** Let \( \{ W^1 \} \) and \( \{ W^0 \} \) satisfy E1–E6, and let \( H \) satisfy H2–H3. For all \( y \in L^2(\Omega;\mathbb{R}^3) \) and \( P \in L^q(\Omega;SL(3)) \) there exists

\[
J(y, P) := \Gamma \cdot \lim_{\varepsilon \to 0} J_\varepsilon(y, P),
\]

where the underlying convergences are the one in the sense of extensions and the uniform one, respectively for the first and for the second argument. The \( \Gamma \)-limit is characterized as follows:

\[
J(y, P) = J^0(0, P) + J^1(y, P),
\]

where

\[
J^0(y, P) := \begin{cases} L^3(Q^0) \int_{\Omega} \left[ Q' W^0(\nabla y(x), P^{-1}(x)) + H(P(x)) \right] \, dx & \text{if } y = 0 \text{ and } P \in W^{1,q}(\Omega; K), \\
+\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3) \times L^q(\Omega; SL(3)),
\end{cases}
\]

and

\[
J^1(y, P) := \begin{cases} \int_{\Omega} \left[ \overline{W}_{\text{hom}}(\nabla y(x), P(x)) + L^3(Q^1) H(P(x)) + |\nabla P(x)|^q \right] \, dx & \text{if } (y, P) \in W^{1,2}(\Omega;\mathbb{R}^3) \times W^{1,q}(\Omega; K), \\
+\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3) \times L^q(\Omega; SL(3)).
\end{cases}
\]

Here, for \( F, G \in \mathbb{R}^{3 \times 3} \),

\[
Q' W^0(F, G) := \inf \left\{ \int_Q W^0((F + \nabla v(z))G) \, dz : v \in W^{1,2}_0(Q;\mathbb{R}^3) \right\},
\]
while

\[
\widetilde{W}_{\text{hom}}(F, G) := \lim_{\lambda \to +\infty} \frac{1}{\lambda^3} \inf \left\{ \int_{(0, \lambda)^3 \cap \Omega^1} W^1 \left( (F + \nabla y(x)) G^{-1} \right) \, dx : y \in W^{1,2}_{0}((0, \lambda)^3; \mathbb{R}^3) \right\}.
\]

The formula defining \( Q W^0 \) provides a variant of the classical quasiconvex envelope of \( W^0 \). We refer to Section 6 for further discussion on this point.

**Remark 2.9.** In principle, it cannot be excluded that some nontrivial energy densities \( W^0_\varepsilon \) do not contribute to the elastic homogenized energy, in the sense that, when finite, for the corresponding \( J^0 \) we have

\[
J^0(0, P) = \mathcal{L}^3(Q^0) \int_{\Omega} H(P(x)) \, dx.
\]

As an instance of this phenomenon, we consider the following example. For any \( \chi \in \mathbb{R}^3 \), the analysis of each single component. In view of this approach, it is useful to introduce the minimum problems associated with the energy functionals and of the related (quasi) minimizers.

**Corollary 2.10.** Let the same assumptions and notation of Theorem 2.8 hold, and let \( \{(y_k, P_k)\} \subset W^{1,2}_0(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; \text{SL}(3)) \) be a sequence of almost minimizers, that is,

\[
\lim_{k \to +\infty} \left( J_k(y_k, P_k) - \inf J_k(y, P) \right) = 0,
\]

where the infimum is taken over \( W^{1,2}_0(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; \text{SL}(3)) \). Then, there exists a minimizer \( (y, P) \in W^{1,2}_0(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; \text{SL}(3)) \) of \( J \) such that, up to subsequences, \( y_k \to y \) in the sense of extensions and \( P_k \to P \) uniformly. Moreover,

\[
\inf J_k \to \min J.
\]

The proof of Theorem 2.8 consists of three steps. First, we study the compactness properties of sequences \( \{(y_\varepsilon, P_\varepsilon)\} \) satisfying \( \sup_\varepsilon J_\varepsilon(y_\varepsilon, P_\varepsilon) \leq C \), and characterize their limits. Second, we show that the two components of the material can be studied independently. Finally, we perform the analysis of each single component. In view of this approach, it is useful to introduce the functionals that account for the two different contributions, namely

\[
E_\varepsilon^0(y, P) := \int_{\Omega} \chi_\varepsilon^0(x) \left[ W_\varepsilon^0 \left( \varepsilon \nabla y(x) P^{-1}(x) \right) \right] + H(P(x)) \, dx,
\]

\[
E_\varepsilon^1(y, P) := \int_{\Omega} \chi_\varepsilon^1(x) \left[ W^1 \left( x, \nabla y(x) P^{-1}(x) \right) \right] + H(P(x)) \, dx,
\]

where, for \( i = 0, 1 \), \( \chi_\varepsilon^i(x) \) denotes the characteristic function of \( \Omega_\varepsilon^i \), i.e. \( \chi_\varepsilon^i(x) = 1 \) if \( x \in \Omega_\varepsilon^i \) and \( \chi_\varepsilon^i(x) = 0 \) otherwise. We also decompose the functional \( J_\varepsilon \) accordingly:

\[
J_\varepsilon = J_\varepsilon^0 + J_\varepsilon^1,
\]

with

\[
J_\varepsilon^0(y, P) := \begin{cases} E_\varepsilon^0(y, P) & \text{if } (y, P) \in W^{1,2}_0(\Omega_\varepsilon^0; \mathbb{R}^3) \times W^{1,q}(\Omega; K), \\ +\infty & \text{otherwise in } L^2(\Omega; \mathbb{R}^3) \times L^q(\Omega; \text{SL}(3)), \end{cases}
\]

\[
J_\varepsilon^1(y, P) := \begin{cases} E_\varepsilon^1(y, P) + \| \nabla P \|_{L^q(\Omega; \mathbb{R}^{3 \times 3 \times 3})}^q & \text{if } (y, P) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K), \\ +\infty & \text{otherwise in } L^2(\Omega; \mathbb{R}^3) \times L^q(\Omega; \text{SL}(3)). \end{cases}
\]
In contrast to $\mathcal{J}^1_\varepsilon(y, P)$, whose asymptotic behavior is derived from Theorem 2.2, the soft part requires a dedicated treatment. This happens already in the setting of nonlinear elasticity (see [14]). Recall the topology $\tau$ in (2.3). We obtain the following:

**Proposition 2.11.** Let $(v, P) \in W^{1,2}_0(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; \mathbb{SL}(3))$. For an infinitesimal sequence $\{\varepsilon_k\}$, consider $\mathcal{J}^0_k$ and $\mathcal{J}^0$ as in (2.12) and (2.7), respectively.

1. For every sequence $\{(v_k, P_k)\} \subset W^{1,2}_0(\Omega_k; \mathbb{R}^3) \times W^{1,q}(\Omega; \mathbb{SL}(3))$ such that $\varepsilon_k v_k, P_k \overset{\mathcal{J}^0_k}{\rightarrow} (v, P)$ we have
   \[ \mathcal{J}^0(v, P) \leq \liminf_{k \to +\infty} \mathcal{J}^0_k(v_k, P_k), \]
   provided that $\{v_k\}$ is bounded in $L^2(\Omega; \mathbb{R}^3)$ and that $\{\varepsilon_k \nabla v_k\}$ is 2-equiintegrable.

2. There exists a sequence $\{v_k\} \subset W^{1,2}_0(\Omega_k; \mathbb{R}^3)$ such that $\varepsilon_k v_k \rightarrow v$ in $L^2(\Omega; \mathbb{R}^3)$ and that
   \[ \limsup_{k \to +\infty} \mathcal{J}^0_k(v_k, P_k) \leq \mathcal{J}^0(v, P), \]
   provided $P_k \rightarrow P$ uniformly.

In the statement above, the space $W^{1,2}_0(\Omega_\varepsilon; \mathbb{R}^3)$ is regarded for each $\varepsilon$ as a subset of $W^{1,2}(\Omega; \mathbb{R}^3)$ by extending its elements to 0 on $\Omega_\varepsilon^c$.

**Remark 2.12.** Let $\Omega \subset \mathbb{R}^3$ be bounded Lipschitz domain and, for $p > 1$, let us consider the local integral functionals on $W^{1,p}(\Omega; \mathbb{R}^3)$

\[ v \mapsto \int_\Omega W_k(\nabla v) \, dx. \]

If the energy densities $\{W_k\}$ satisfy standard $p$-growth conditions, as a consequence of Rellich-Kondrachov theorem, the $\Gamma$-limits with respect to the strong $L^p$-convergence and with respect to the weak $W^{1,p}$-convergence coincide (if they exist).

For the sequence of functionals

\[ v \mapsto \int_\Omega W_k(\varepsilon_k \nabla v) \, dx, \tag{2.14} \]

again under standard growth conditions for $\{W_k\}$, the analysis is more delicate. The natural bound that follows from the $p$-coercivity is $\|\varepsilon_k \nabla v_k\|_{L^p} \leq C$, and it suggests the use of weak two-scale convergence (see Subsection 3.5). However, this estimate alone is not enough to deduce convergence of the sequence $\{v_k\}$: a further control on the $\varepsilon$-difference quotients is required to guarantee that a two-scale variant of Rellich-Kondrachov theorem holds (see [52, Theorem 4.4]).

In other words, in our degenerate setting, compactness of sequences of gradients, say $\{\varepsilon_k \nabla v_k\}$, does not bring compactness of $(v_k)$. This explains why in Proposition 2.11 we need to require a bound also for $\|v_k\|_{L^2}$ in order to establish the lower limit inequality.

We note incidentally that, by means of Lemma 3.10(4) below, it can be shown that the $\Gamma$-limit of the functionals (2.14) with respect to the strong two-scale convergence in $L^p$ of $\{v_k\}$ is the same as the one computed by combining the latter convergence and the weak two-scale convergence of $\{\varepsilon_k \nabla v_k\}$. Those are not suitable choices for our goals, though, because, as we commented above, they do not match the natural compactness of the problem. This explains why in [14], where strong two-scale convergence is considered, the asymptotic behavior of minimum problems is not immediately determined by the $\Gamma$-convergence (see [14, Sec. 10]). We also refer to the Appendix for a comparison between our findings and the ones in [14].
3. Preliminaries

We gather in this section the technical tools to be employed in the sequel.

3.1. Localization and integral representation. To the aim of laying the ground for the proof of Theorem 2.2, we briefly outline the localization technique in the context of integral representation results for $\Gamma$-limits. For definiteness, we report on the most relevant case for our analysis, that is, the homogenization of connected media. The process described in the following lines will be then tailored to the energy functionals (1.1) in Section 4. More detail and a thorough treatment of $\Gamma$-convergence, which we introduced in Definition 2.1, may instead be found in the monographs [10, 23, 9].

The localization method underpins the following well-known result (see e.g. [10, Theorem 14.5]).

**Theorem 3.1.** Let $g: \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \to [0, +\infty)$ be a Borel function that is $Q$-periodic in its first argument and that satisfies standard $p$-growth conditions for some $p \in (1, +\infty)$. For $\varepsilon > 0$ and $y \in L^p(\Omega; \mathbb{R}^3)$ we define

$$G_\varepsilon(y) := \begin{cases} \int_\Omega g \left( \frac{x}{\varepsilon}, \nabla y(x) \right) \, dx & \text{if } y \in W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, we have

$$\Gamma(L^p)\text{-}\lim \varepsilon \to 0 G_\varepsilon(y) = \begin{cases} \int_\Omega g_{\text{hom}}(\nabla y(x)) \, dx & \text{if } y \in W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where the $\Gamma$-limit is taken with respect to the strong $L^p(\Omega; \mathbb{R}^3)$-topology, and $g_{\text{hom}}: \mathbb{R}^{3 \times 3} \to [0, +\infty)$ is a quasiconvex function characterized by the asymptotic homogenization formula

$$g_{\text{hom}}(F) = \lim_{\lambda \to +\infty} \frac{1}{\lambda^3} \inf \left\{ \int_{(0,\lambda)^3} g(x, F + \nabla y(x)) \, dx : y \in W^{1,p}_0((0,\lambda)^3; \mathbb{R}^3) \right\}$$

for all $F \in \mathbb{R}^{3 \times 3}$.

As starting point to establish this theorem, one resorts to a general property of $\Gamma$-convergence which ensures that if $X$ is a separable metric space, then any family $\{G_\varepsilon\}$ with $G_\varepsilon: X \to [-\infty, +\infty]$ has a $\Gamma$-convergent subsequence (see e.g. [10, Proposition 7.9]). Such $\Gamma$-compactness principle yields that, up to subsequences, an abstract $\Gamma(L^p)$-limit of $\{G_\varepsilon\}$ exists. One is then naturally led to wonder whether the limit is in turn a functional of integral type. In order to show that this is in fact the case, a localization argument is employed. This amounts to regard $G_\varepsilon$ as a function of the pair $(y, A)$, where $y \in L^p(\Omega; \mathbb{R}^3)$ and $A \in \mathcal{A}(\Omega) := \{\text{open subsets of } \Omega\}$; more precisely, setting

$$G_\varepsilon(y, A) := \int_A g \left( \frac{x}{\varepsilon}, \nabla y(x) \right) \, dx \quad \text{if } y \in W^{1,p}(A; \mathbb{R}^3)$$

and $G_\varepsilon(y, A) := +\infty$ otherwise, the idea is to focus on the properties of $G_\varepsilon(y, \cdot)$ as a set function. In this respect, we recall some terminology.

**Definition 3.2.** Let $\alpha: \mathcal{A}(\Omega) \to [0, +\infty]$ be a set function. We say that $\alpha$ is

- an increasing set function if $\alpha(\emptyset) = 0$ and $\alpha(A) \leq \alpha(B)$ if $A \subseteq B$;
- subadditive if $\alpha(A \cup B) \leq \alpha(A) + \alpha(B)$ for all $A, B$;
- superadditive if $\alpha(A \cup B) \geq \alpha(A) + \alpha(B)$ for all $A, B$ such that $A \cap B = \emptyset$.
• inner regular if $\alpha(A) = \sup \{\alpha(B) : B \in \mathcal{A}(\Omega), B \subset A\}$.

De Giorgi-Letta criterion (see e.g. [10, Theorem 10.2]) states that an increasing set function is a restriction to $\mathcal{A}(\Omega)$ of Borel measure if and only if subadditive, superadditive and inner regular. Therefore, since the $\Gamma(L^p)$-limit of $\{G_\varepsilon\}$ must be an increasing set function and we work under $p$-growth conditions, the integral representation boils down to proving that $\Gamma(L^p)$-lim $G_\varepsilon$ is subadditive, superadditive and inner regular. The subadditivity and inner regularity are the most delicate points, and they hinge in turn upon the so-called fundamental estimate (see e.g. [10, Definition 11.2]). Roughly speaking, given $A, B \in \mathcal{A}(\Omega), y \in L^p(\Omega; \mathbb{R}^3)$ and sequences $\{y_\varepsilon\}$ and $\{y_\varepsilon''\}$ converging to $y$ in $L^p$ such that

$$\Gamma(L^p)-\lim G_\varepsilon(y, A) = \lim_{\varepsilon \to 0} G_\varepsilon(y_\varepsilon', A) \quad \text{and} \quad \Gamma(L^p)-\lim G_\varepsilon(y, B) = \lim_{\varepsilon \to 0} G_\varepsilon(y_\varepsilon'', B),$$

the fundamental estimate allows to construct a third sequence $\{y_\varepsilon\}$ such that

$$G_\varepsilon(y_\varepsilon, A \cup B) \leq G_\varepsilon(y_\varepsilon', A) + G_\varepsilon(y_\varepsilon'', B) + R_\varepsilon, \quad \lim_{\varepsilon \to 0} R_\varepsilon = 0,$$

with $R_\varepsilon$ satisfying a precise estimate in terms of $\|y_\varepsilon' - y_\varepsilon''\|_{L^p(A \cap B)}$. The exact form of the fundamental estimate to be used in our analysis is contained in (4.3), while its application to establish subadditivity and inner regularity is discussed in Proposition 4.6.

### 3.2. Finsler structure on $\text{SL}(3)$.

To the purpose of devising a form of the fundamental estimate that suits the functionals under consideration, it is convenient to endow $\text{SL}(3)$ with a metric structure. In order to link the latter to the physics of the system we would like to model, we follow the approach in [45], which is grounded on the concept of plastic dissipation.

We recall some basic facts about the geometry of $\text{SL}(3)$, which is a smooth manifold with respect to the topology induced by the inclusion in $\mathbb{R}^{3 \times 3}$. For every $F \in \text{SL}(3)$ the tangent space at $F$ is characterized as

$$T_F \text{SL}(3) = F\mathfrak{sl}(3) := \{FM \in \mathbb{R}^{3 \times 3} : \text{tr} M = 0\}.$$

In particular, $T_\text{I} \text{SL}(3)$ coincides with $\mathfrak{sl}(3) := \{M \in \mathbb{R}^{3 \times 3} : \text{tr} M = 0\}$. We equip $\text{SL}(3)$ with a Finsler structure starting from a $C^2$ function $\Delta_F : \mathfrak{sl}(3) \to [0, +\infty)$, on which we make the following assumptions:

- **D1:** It is positively 1-homogeneous: $\Delta_F(cM) = c\Delta_F(M)$ for all $c \geq 0$ and $M \in \mathfrak{sl}(3)$;
- **D2:** It is 1-coercive and has at most linear growth: there exist $0 < c_4 \leq c_5$ such that for all $M \in \mathfrak{sl}(3)$

$$c_4 |M| \leq \Delta_F(M) \leq c_5 |M|.$$  

- **D3:** It is strictly convex.

We point out that we work under stronger regularity assumptions than the ones in [45]. This is mainly due to the fact that we borrow results developed within the context of differential geometry, where smoothness is customarily required. As a consequence, some models, such as single crystal plasticity, are not covered by our analysis; on the other hand, our assumptions encompass Von Mises plasticity, see the considerations in [37, 45]. We also recall that Finslerian structures are known to appear as homogenized limits of periodic Riemannian metrics [1].

Essentially, $\Delta$ is a Minkowski norm on $\mathfrak{sl}(3)$, which we can “translate” to the other tangent spaces by setting

$$\Delta : T\text{SL}(3) \to [0, +\infty)$$

$$(F, M) \mapsto \Delta_F(F^{-1}M),$$
where $\text{TSL}(3)$ is the tangent bundle to $\text{SL}(3)$. Then, it can be checked that $(\text{SL}(3), \Delta)$ is a $C^2$ Finsler manifold. We refer to the monograph [5] by Bao, Chern & Shen for an introduction to Finsler geometry.

Let now $\mathcal{C}(F_0, F_1)$ be the family of piecewise $C^2$ curves $\Phi : [0, 1] \to \text{SL}(3)$ such that $\Phi(0) = F_0$ and $\Phi(1) = F_1$. We define a non-symmetric distance on $\text{SL}(3)$ as follows:

$$D(F_0, F_1) := \inf \left\{ \int_0^1 \Delta(\Phi(t), \dot{\Phi}(t)) dt : \Phi \in \mathcal{C}(F_0, F_1) \right\}, \quad (3.1)$$

where $\dot{\Phi}$ is the velocity of the curve. The function $D$ is positive, attains 0 if and only if it is evaluated on the diagonal of $\text{SL}(3) \times \text{SL}(3)$, and fulfills the triangular inequality; in general, however, $D(F_0, F_1) \neq D(F_1, F_0)$.

From a physical viewpoint, if $P_0, P_1 : \Omega \to \text{SL}(3)$ are admissible plastic strains, the integral of $D(P_0, P_1)$ over $\Omega$ is interpreted as the minimum amount of energy that is dissipated when the system moves from a plastic configuration to another (see also Subsection 7.2).

An application of the direct method of the calculus of variations (cf. [45, Theorem 5.1]) proves that for every $F_0, F_1 \in \text{SL}(3)$ there exists a curve $\Phi \in C^{1,1}([0,1];\text{SL}(3))$ such that $\Phi(0) = F_0$, $\Phi(1) = F_1$ and

$$D(F_0, F_1) = \int_0^1 \Delta(\Phi(t), \dot{\Phi}(t)) \, dt.$$

We call such $\Phi$ a shortest path between $F_0$ and $F_1$. The following result, which summarizes the content of [5, Exercises 6.3.3], is crucial for the proof of the fundamental estimate.

**Proposition 3.3.** For any point $F$ in the Finsler manifold $\text{SL}(3)$ there exists a relatively compact neighborhood $U$ of $F$ such that for any $F_0, F_1 \in U$ there exists a unique shortest path $\Phi$ joining $F_0$ and $F_1$, and such path depends smoothly on its endpoints $F_0$ and $F_1$.

The uniqueness of the shortest path allows to characterize it in terms of the exponential map. We recall that a path between $F_0$ and $F_1$ is called a geodesic if it is a critical point of the length functional under variations that do not change the endpoints, and that, given $(F, M) \in \text{TSL}(3)$ such that $\Delta(F, M)$ is sufficiently small, there exists a unique geodesic $\gamma_{F,M} : (-2, 2) \to \text{SL}(3)$ satisfying $\gamma_{F,M}(0) = F$ and $\gamma_{F,M}(0) = M$. Therefore, for $M$ in a neighborhood of $0 \in T_F \text{SL}(3)$, the exponential map

$$\exp(F, M) := \begin{cases} F & \text{if } M = 0, \\ \gamma_{F,M}(1) & \text{otherwise} \end{cases}$$

is well defined [5, p. 126] and $\exp(F, \cdot)$ is invertible. Denoting by $\exp^{-1}$ the inverse, if $\Phi$ is the unique shortest path from $F_0$ to $F_1$, one sees that necessarily

$$\Phi(t) = \exp(F_0, t \exp^{-1}_{F_0}(F_1)). \quad (3.2)$$

**Remark 3.4.** Grounding on Proposition 3.3, we can show that there exists a compact $K$ meeting the requirements in **H2**. Let $U$ be a relatively compact neighborhood of $I \in \text{SL}(3)$ such that for any $F_0, F_1 \in U$ there is a unique shortest path $\Phi$ joining $F_0$ and $F_1$. By a Finsler variant of a theorem by Whitehead [5, Exercise 6.4.3], there exists an open neighborhood $V$ of $I$ that is compactly contained in $U$ and geodesically convex. Since $K := V \subset U$, there is a unique shortest path $\Phi$ from $F_0$ to $F_1$ for any $F_0, F_1 \in K$. By (3.2), the path depends smoothly on its endpoints. The fact that $K$ is geodesically convex as well follows then by the same argument that proves that the closure of a convex set is still convex.
3.3. A decomposition lemma. In our analysis of heterogeneous media it will be often desirable to disregard the energy contributions arising from the region close to $\partial \Omega$, for the composite fails to be periodic there (recall positions (2.4)–(2.5)). To this aim, it is natural to resort to $p$-equiintegrable arguments, because such boundary strip has small measure. We recall that a family $\mathcal{C} \subset L^p(\Omega; \mathbb{R}^3)$ is said to be $p$-equiintegrable if for all $\delta > 0$ there exists $m > 0$ such that

$$\sup_{u \in \mathcal{C}} \int_E |u|^p \, dx < \delta \quad \text{whenever } E \subset \Omega \text{ satisfies } \mathcal{L}^3(E) < m.$$  

The ensuing lemma grants that for any bounded sequence in $L^p$ we can always find another one which is $p$-equiintegrable and “does not differ too much” from the given one.

**Lemma 3.5** (Theorem 2.20 in [4]; see also Lemma 1.2 in [30]). Let $\Omega$ be as in Section 2. For any sequence $\{v_k\} \subset W^{1,2}(\Omega; \mathbb{R}^3)$ such that $v_k \rightharpoonup v$ weakly in $W^{1,2}(\Omega; \mathbb{R}^3)$ there exist a subsequence $\{k_j\}$ and a sequence $\{u_j\} \subset W^{1,2}(\Omega; \mathbb{R}^3)$ satisfying the following:

1. $u_j \rightharpoonup v$ weakly in $W^{1,2}(\Omega; \mathbb{R}^3)$;
2. $u_j = v$ in a neighborhood of $\partial \Omega$;
3. $\{\nabla u_j\}$ is 2-equiintegrable;
4. $\lim_{j \to +\infty} \mathcal{L}^3(\{x \in \Omega : v_{k_j}(x) \neq u_j(x)\}) = 0$.

Point (4) yields $\lim_{j \to +\infty} \mathcal{L}^3(\{|\nabla v_{k_j} | \neq |\nabla u_j|\}) = 0$, because by standard properties of Sobolev functions (see e.g. [34, Lemma 7.7]) the inclusion $\{v_{k_j} \neq u_j\} \supset \{|\nabla v_{k_j} | \neq |\nabla u_j|\}$ holds true.

3.4. A couple of tools to deal with periodic heterogeneous media. The periodic geometry of the composite calls for an extension result for Sobolev maps on perforated domains. Since the perforations of the matrix are well detached from the boundary, by applying [10, Lemma B.7] the following can be proved:

**Lemma 3.6** (Lemma 8 in [14]). Let $\Omega$ be open and bounded, and let $\Omega^1_{\varepsilon}$ be as in Section 1. There exists a linear and continuous extension operator

$$T_{\varepsilon} : W^{1,2}(\Omega^1_{\varepsilon}; \mathbb{R}^3) \to W^{1,2}(\Omega; \mathbb{R}^3)$$

such that for all $y \in W^{1,2}(\Omega^1_{\varepsilon}; \mathbb{R}^3)$

$$T_{\varepsilon} y = y \quad \text{a.e. in } \Omega^1_{\varepsilon},$$

$$\|T_{\varepsilon} y\|_{L^2(\Omega; \mathbb{R}^3)} \leq c \|y\|_{L^2(\Omega^1_{\varepsilon}; \mathbb{R}^3)},$$

$$\|\nabla (T_{\varepsilon} y)\|_{L^2(\Omega; \mathbb{R}^3 \times \mathbb{R}^3)} \leq c \|\nabla y\|_{L^2(\Omega^1_{\varepsilon}; \mathbb{R}^3 \times \mathbb{R}^3)},$$

where $c$ is independent of $\varepsilon$ and $\Omega$.

**Remark 3.7.** Even though the lemma above is a classical result, it is worth clarifying the way we employ it.

In the sequel, we always work with sequences which are already defined on the whole $\Omega$. When we apply Lemma 3.6 to such a sequence, say $\{y_\varepsilon\} \subset W^{1,2}(\Omega; \mathbb{R}^3)$, it is tacitly understood that the functions that are extended are the restrictions $y_\varepsilon |_{\Omega^1_{\varepsilon}}$. So, in a sense, the process modifies $y_\varepsilon$ on the region occupied by the soft inclusions rather than extending it. Note that the modification is a true one, because $T_{\varepsilon}$ cannot be the identity. The two crucial points for our analysis are that

1. if $\{y_\varepsilon \cap \Omega^1_{\varepsilon}\}$ and $\{\nabla y_\varepsilon \cap \Omega^1_{\varepsilon}\}$ are bounded in $L^2$, then $\{T_{\varepsilon} y_\varepsilon\}$ is bounded in $W^{1,2}(\Omega; \mathbb{R}^3)$;
2. if $\{y_\varepsilon\}$ is bounded in $L^2(\Omega; \mathbb{R}^3)$ and $\{\nabla y_\varepsilon\}$ is a 2-equiintegrable sequence, then $\{\nabla (T_{\varepsilon} y_\varepsilon)\}$ is 2-equiintegrable as well.
The second point follows from the construction of $T_\varepsilon$, which is modeled on the proof of [10, Lemma B.8] by patching together the extensions from $W^{1,2}(Q^1;\mathbb{R}^3)$ to $W^{1,2}(Q;\mathbb{R}^3)$ given by [10, Lemma B.7] via partitions of unity (this is also the reason why the constant $c$ above depends only on $Q^1$ and $Q$). The extensions in [10, Lemma B.7] preserve equiintegrability, because they rely on the classical reflection procedure.

The first application of the extension lemma is the following Poincaré inequality on periodic heterogeneous media (cf. formula (4.5) in [2] where, however, the proof is not provided).

**Proposition 3.8.** Let $\Omega$, $\Omega_\varepsilon^0$ and $\Omega_\varepsilon^1$ be as in Section 1. There exists a constant $c$ independent of $\varepsilon$, and such that for every $y \in W^{1,2}_0(\Omega;\mathbb{R}^3)$

$$\|y\|_{L^2(\Omega;\mathbb{R}^3)} \leq c \left( \varepsilon \|\nabla y\|_{L^2(\Omega_\varepsilon^0;\mathbb{R}^{3\times 3})} + \|\nabla y\|_{L^2(\Omega_\varepsilon^1;\mathbb{R}^{3\times 3})} \right).$$

**Proof.** For $\varepsilon$ fixed, we use the extension operator $T_\varepsilon$ from Lemma 3.6 to obtain

$$\|y\|_{L^2} \leq \|y - T_\varepsilon y\|_{L^2} + \|T_\varepsilon y\|_{L^2} = \|y - T_\varepsilon y\|_{L^2(\Omega_\varepsilon^0)} + \|T_\varepsilon y\|_{L^2}.$$  

(3.3)

Observe that $T_\varepsilon y \in W^{1,2}_0(\Omega;\mathbb{R}^3)$, as $T_\varepsilon y = y$ a.e. in $\Omega_\varepsilon^1$ and there exists a tubular neighborhood $O$ of $\partial\Omega$ such that $\Omega_\varepsilon^1 \cap O \equiv \Omega \cap O$. Then, by the standard Poincaré’s inequality,

$$\|T_\varepsilon y\|_{L^2} \leq c \|\nabla (T_\varepsilon y)\|_{L^2} \leq c \|\nabla y\|_{L^2(\Omega_\varepsilon^0)}.$$  

(3.4)

Observe that $y - T_\varepsilon y \in W^{1,2}_0(\Omega_\varepsilon^0;\mathbb{R}^3)$ as well. In view of the periodic structure of $\Omega_\varepsilon^0$ and of Poincaré inequality on each cube, we infer

$$\|y - T_\varepsilon y\|_{L^2(\Omega_\varepsilon^0)}^2 = \sum_{t \in T_\varepsilon} \|y - T_\varepsilon y\|_{L^2(\varepsilon(t+D_0))}^2 = \sum_{t \in T_\varepsilon} \varepsilon^3 \int_{D_0} |y(\varepsilon(t+z)) - T_\varepsilon y(\varepsilon(t+z))|^2 \, dz$$

$$\leq c \sum_{t \in T_\varepsilon} \varepsilon^5 \int_{D_0} |\nabla (y - T_\varepsilon y)(\varepsilon(t+z))|^2 \, dz = \varepsilon^2 \|\nabla (y - T_\varepsilon y)\|_{L^2(\Omega_\varepsilon^0)}^2,$$

where $c$ depends only on $D_0$. By applying again Lemma 3.6 we find

$$\|y - T_\varepsilon y\|_{L^2(\Omega_\varepsilon^0)} \leq c \left( \varepsilon \|\nabla y\|_{L^2(\Omega_\varepsilon^0)} + \|\nabla y\|_{L^2(\Omega_\varepsilon^1)} \right).$$

This, together with (3.3) and (3.4), yields the result. \hfill \Box

3.5. **Two-scale convergence and the unfolding method.** From a mathematical perspective, the high-contrast structure of the functional $J_\varepsilon$ results in the absence of uniform bounds in $L^2$ for sequences with equibounded energy; indeed, only bounds on $\{\varepsilon \nabla y_\varepsilon^P \varepsilon^{-1}\}$ are available. Such degenerate bounds are conveniently dealt with by means of two-scale convergence [2, 48], whose definition we recall next. Hereafter, the subscript per denotes spaces of $Q$-periodic functions, e.g.,

$$W^{1,2}_{\text{per}}(\mathbb{R}^3) := \{u \in W^{1,2}_{\text{loc}}(\mathbb{R}^3) : u(x+t) = u(x) \text{ a.e. for all } t \in \mathbb{Z}^3\}.$$

**Definition 3.9.** Let $\{\varepsilon_k\} \subset (0, +\infty)$ be infinitesimal. A sequence $\{y_k\} \subset L^2(\Omega;\mathbb{R}^3)$ weakly two-scale converges in $L^2$ to a function $y \in L^2(\Omega;L^2_{\text{per}}(\mathbb{R}^3;\mathbb{R}^3))$ if for every $v \in L^2(\Omega;C_{\text{per}}(\mathbb{R}^3;\mathbb{R}^3))$

$$\lim_{k \to +\infty} \int_{\Omega} y_k(x) \cdot v\left(x, \frac{x}{\varepsilon_k}\right) \, dx = \int_{\Omega} \int_Q y(x,z) \cdot v(x,z) \, dz \, dx.$$
A sequence \( \{y_k\} \subset L^2(\Omega; \mathbb{R}^3) \) strongly two-scale converges in \( L^2 \) to \( y \in L^2(\Omega; L^2_{\text{per}}(\mathbb{R}^3; \mathbb{R}^3)) \) if \( y_k \rightharpoonup^2 y \) in \( L^2 \) and \( \|y\|_{L^2(\Omega; \mathbb{R}^3)} \to \|y\|_{L^2(\Omega \times Q; \mathbb{R}^3)} \). We use the notations \( y_k \rightharpoonup^2 y \) and \( y_k \rightharpoonup^3 y \) for the weak and strong two-scale convergence, respectively.

Recalling that for \( i = 0, 1 \) \( \chi^i_k(x) = 1 \) if \( x \in \Omega^i_k \) and \( \chi^i_k(x) = 0 \) otherwise, an example of strong two-scale convergence is provided by the sequences \( \{\chi^0_k\} \) and \( \{\chi^1_k\} \). Indeed,

\[
\chi^i_k \overset{k}{\rightharpoonup} \chi^i \quad \text{strongly two-scale in } L^2,
\]

where \( \chi^i(x, z) := \chi_{Q^i}(z) \) for all \((x, z) \in \Omega \times Q\).

We collect in the next lemma some basic properties of two-scale convergence which we will resort to in the following. Proofs and more details can be found in [2, 51, 52].

**Lemma 3.10.** Let \( \{\varepsilon_k\} \subset (0, +\infty) \) be infinitesimal and consider \( \{y_k\} \subset L^2(\Omega; \mathbb{R}^3) \).

1. If \( \{y_k\} \) is weakly two-scale convergent, then it is bounded in \( L^2(\Omega; \mathbb{R}^3) \); conversely, if \( \{y_k\} \) is bounded in \( L^2(\Omega; \mathbb{R}^3) \), then it admits a weakly two-scale convergent subsequence.
2. If \( y_k \rightharpoonup^2 y \) weakly two-scale in \( L^2 \), then \( y_k \rightharpoonup \int_Q y(\cdot, z) \, dz \) weakly in \( L^2(\Omega; \mathbb{R}^3) \).
3. If \( y_k \rightharpoonup^2 y \) weakly two-scale in \( L^2 \) and if \( \{u_k\} \subset L^2(\Omega; \mathbb{R}^3) \) converges to \( u \) strongly two-scale in \( L^2 \), then \( y_k u_k \rightharpoonup^2 y u \) weakly two-scale in \( L^2 \).
4. Suppose that \( \{y_k\} \subset W^{1,2}(\Omega; \mathbb{R}^3) \) and that \( \{y_k\} \) and \( \{\varepsilon_k \nabla y_k\} \) are bounded in \( L^2 \). Then, there exists \( y \in L^2(\Omega; W^{1,2}_{\text{per}}(\mathbb{R}^3; \mathbb{R}^3)) \) such that, up to subsequences, \( y_k \rightharpoonup^2 y \) and \( \varepsilon_k \nabla y_k \rightharpoonup^2 \nabla_y \) weakly two-scale in \( L^2 \).

Two-scale convergence in \( L^2 \) can be related to \( L^2 \) convergence by means of unfolding operator, which, for \( \varepsilon > 0 \), is the map \( S_\varepsilon : L^2(\Omega) \to L^2(\mathbb{R}^3; L^2_{\text{per}}(\mathbb{R}^3; \mathbb{R}^3)) \) defined as

\[
S_\varepsilon y(x, z) := \hat{y}\left(\varepsilon \frac{x}{\varepsilon} + \varepsilon z\right),
\]

where \( \hat{y} \) denotes the extension of \( y \) by 0 outside \( \Omega \).

**Lemma 3.11.** If \( \{y_\varepsilon\} \subset L^2(\Omega; \mathbb{R}^3) \) is bounded, the following hold:

1. \( y_\varepsilon \rightharpoonup^2 y \) weakly two-scale in \( L^2 \) if and only if \( S_\varepsilon y_\varepsilon \rightharpoonup y \) weakly in \( L^2(\mathbb{R}^3 \times Q; \mathbb{R}^3) \);
2. \( y_\varepsilon \rightharpoonup^2 y \) strongly two-scale in \( L^2 \) if and only if \( S_\varepsilon y_\varepsilon \rightharpoonup y \) strongly in \( L^2(\mathbb{R}^3 \times Q; \mathbb{R}^3) \).

In addition, if \( \{y_\varepsilon\} \) is 2-equiintegrable, the family of unfoldings \( \{S_\varepsilon y_\varepsilon\} \) is as well 2-equiintegrable on \( \mathbb{R}^3 \times Q \). Lastly, if \( y \in W^{1,2}(\Omega; \mathbb{R}^3) \), then

\[
S_\varepsilon(\varepsilon \nabla y)(x, z) = \nabla_z(S_\varepsilon y)(x, z).
\]

For a proof of Lemma 3.11 and for further reading on the unfolding operator we refer to [51, 52, 17, 18].

4. **Homogenization in finite plasticity**

We devote this section to the proof of Theorem 2.2, that is, we exhibit the \( \Gamma(\tau) \)-limit of the functionals in (1.1). We recall that the topology \( \tau \) was introduced in (2.3).

**Remark 4.1.** The energy functionals at stake depend on the plastic strain \( P \) through its inverse \( P^{-1} \). In this respect, it is useful to notice that if \( P_k \to P \) uniformly, then \( P_k^{-1} \to P^{-1} \) uniformly as well. Indeed, recalling that for any \( k \in \mathbb{N} \) we can write

\[
P_k^{-1} = \frac{(\text{cof } P_k)^T}{\det P_k} = (\text{cof } P_k)^T,
\]
we deduce convergence for the sequence of inverses.

Since \((W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; \text{SL}(3)), \tau)\) is a separable metric space, we know from general properties of \(\Gamma\)-convergence that, up to extraction of subsequences, the functionals in (1.1) \(\Gamma(\tau)\)-converge. Our main task is therefore to show that the limit is an integral functional, and we achieve this by a localization approach as the one streamlined in Subsection 3.1. Precisely, denoting by \(\mathcal{A}(\Omega)\) the family of open subset of \(\Omega\), with a slight abuse of notation, for any triple \((y, P, A) \in L^2(A; \mathbb{R}^3) \times L^q(A; \text{SL}(3)) \times \mathcal{A}(\Omega)\) we set

\[
\mathcal{F}_k(y, P, A) := \left\{ \begin{array}{ll}
\int_A W \left( \frac{x}{\varepsilon_k}, \nabla y P^{-1} \right) \, dx + \int_A H \left( \frac{x}{\varepsilon_k}, P \right) \, dx + \int_A |\nabla P|^q \, dx & \text{if } (y, P) \in W^{1,2}(A; \mathbb{R}^3) \times W^{1,q}(A; K), \\
+\infty & \text{otherwise in } L^2(A; \mathbb{R}^3) \times L^q(A; \text{SL}(3)),
\end{array} \right.
\]

(4.1)

where \(\varepsilon_k\) is an infinitesimal sequence.

We first show that the limits of \(\Gamma\)-convergent subsequences are in turn integral functionals. Second, we characterize the limiting energy densities, proving as well that the whole family \(\{\mathcal{F}_k\}\) \(\Gamma\)-converges.

### 4.1. Integral representation

In this subsection we establish the following:

**Theorem 4.2.** Let \(\mathcal{F}_k\) be as in (4.1), where \(W\) and \(H\) satisfy E1–E3 and H1–H3, respectively. Then, up to subsequences, \(\{\mathcal{F}_k\}\ \Gamma(\tau)\)-converges and for all \(y \in L^2(A; \mathbb{R}^3)\) and \(P \in L^q(A; \text{SL}(3))\)

\[
\Gamma(\tau)\lim_{k \to +\infty} \mathcal{F}_k(y, P, A) = \left\{ \begin{array}{ll}
\int_A f(x) \, dx & \text{if } (y, P) \in W^{1,2}(A; \mathbb{R}^3) \times W^{1,q}(A; K), \\
+\infty & \text{otherwise in } L^2(A; \mathbb{R}^3) \times L^q(A; \text{SL}(3)),
\end{array} \right.
\]

(4.2)

for some \(f \in L^1_{\text{loc}}(\mathbb{R}^3)\) (which depends on the subsequence).

As a first step, we introduce a version of the fundamental estimate fit for the functionals in (4.1). We recall that, given \(A, A' \in \mathcal{A}(\Omega)\) with \(A' \subset A\) (i.e. \(A'\) is a compact set contained in \(A\)), we say that a function \(\varphi\) is a cut-off function between \(A'\) and \(A\) if \(\varphi \in C_0^\infty(A), 0 \leq \varphi \leq 1\) and \(\varphi \equiv 1\) in a neighborhood of \(A\).

**Definition 4.3.** Let \(\mathcal{C}\) be a class of functionals \(\mathcal{F}: W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K) \times \mathcal{A}(\Omega) \to [0, +\infty]\). We say that \(\mathcal{C}\) satisfies uniformly the fundamental estimate if for every \(A, A', B \in \mathcal{A}(\Omega)\) with \(A' \subset A\) and for every \(\sigma > 0\) there exists a constant \(M_\sigma > 0\) with the following property: for all \(\mathcal{F} \in \mathcal{C}\) and for every \((y_1, P_1), (y_2, P_2) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K)\) there exist a cut-off function \(\varphi\) between \(A'\) and \(A\) and a path \(\gamma: [0, 1] \times K \times K \to K\) satisfying \(\gamma(0, F, G) = F, \gamma(1, F, G) = G\) for all \(F, G \in K\) such that

\[
\mathcal{F}(\varphi y_1 + (1 - \varphi) y_2, \gamma \circ (\varphi, P_2, P_1), A' \cup B) \\
\leq (1 + \sigma) \mathcal{F}(y_1, P_1, A) + \mathcal{F}(y_2, P_2, B) \\
+ M_\sigma \int_{(A \setminus B) \setminus A'} \left| y_1 - y_2 \right|^2 + |P_1 - P_2|^q \, dx + \sigma.
\]

(4.3)

Note that, as far as the functionals in (4.1) are concerned, it is not restrictive to assume that plastic deformations range just in \(K\) and not in \(\text{SL}(3)\), because (4.3) is otherwise trivially satisfied.
The use of the path $\gamma$ in (4.3) is motivated by the simple observation that the “convex combination” $\varphi P_1 + (1 - \varphi)P_2$ does not belong to $\mathcal{SL}(3)$ in general. For example, if we let

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we see that $\det P_1 = \det P_2 = 1$, but $\det(P_1/2 + P_2/2) = 0$.

Variational problems involving manifold-valued Sobolev maps were already considered in the literature, see e.g. [3, 22]. The main difference between these references and ours is that not only we constrain our maps to take values in a manifold, but we also require them to range in a specific compact subset.

The next statement grants that the class $C = \{F_k\}$ meets the definition above.

**Proposition 4.4.** The sequence of functionals $\{F_k\}_{k \in \mathbb{N}}$ defined in (4.1) satisfies uniformly the fundamental estimate (4.3), upon choosing $\gamma$ as the map that associates to $(t, F, G) \in [0,1] \times K \times K$ the image at $t$ of the unique shortest path connecting $F$ and $G$.

**Proof.** Fix $A, A', B \in \mathcal{A}(\Omega)$ with $A' \Subset A$ and $\sigma > 0$. Fix also $(y_1, P_1), (y_2, P_2) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,2}(\Omega; K)$. We need to choose suitably the cut-off function $\varphi$.

Let $\delta := \text{dist}(A', \partial A) > 0$. For a fixed $N \in \mathbb{N}$, $N > 1$, we define

$$C_0 := A',
C_j := \left\{ x \in A : \text{dist}(x, A') < \frac{j}{N} \delta \right\}, \quad j = 1, \ldots, N,$$

$$D_j := C_j \setminus C_{j-1}.$$

Observe that for all $j$ we can construct a cut-off function $\varphi_j$ between $C_{j-1}$ and $C_j$ such that $|\nabla \varphi_j| \leq 2N/\delta$. Now, setting $\gamma^{-1}(t, F, G) := (\gamma(t, F, G))^{-1}$ for all $t \in [0, 1]$, we have

$$F_k (\varphi_j y_1 + (1 - \varphi_j) y_2, \gamma \circ (\varphi_j, P_2, P_1), A' \cup B) = I_{el} + I_{hard} + I_{reg},$$

(4.4)

where

$$I_{el} := \int_{A' \cup B} W \left( \frac{x}{\varepsilon_k} \left[ \varphi_j \nabla y_1 + (1 - \varphi_j) \nabla y_2 + \nabla \varphi_j \otimes (y_1 - y_2) \right] (\gamma^{-1} \circ (\varphi_j, P_2, P_1)) \right) dx$$

$$I_{h} := \int_{A' \cup B} H \left( \frac{x}{\varepsilon_k}, \gamma \circ (\varphi_j, P_2, P_1) \right) dx$$

$$I_{reg} := \int_{A' \cup B} |\nabla (\gamma \circ (\varphi_j, P_2, P_1))|^q dx$$

Since $P_1, P_2 \in K$, (2.1) holds. We now choose $\gamma$ as the map such that $\gamma(t, F, G)$ is the evaluation at $t$ of the unique shortest path connecting $F$ and $G$ for all $(t, F, G) \in [0, 1] \times K \times K$. By Proposition 3.3 we know that such $\gamma$ exists, that it lies completely in $K$ if $F, G \in K$ (see H2), and that it depends smoothly on its arguments.
Let us estimate the three summands above separately. For the elastic contribution we have

\[
I_{el} = \int_{(A' \cup B) \cap C_j \setminus \Omega} W \left( \frac{x}{\varepsilon_k}, \nabla y_1 P_1^{-1} \right) \, dx + \int_{(A' \cup B) \setminus C_j} W \left( \frac{x}{\varepsilon_k}, \nabla y_2 P_2^{-1} \right) \, dx \\
+ \int_{(A' \cup B) \cap D_j} W \left( \frac{x}{\varepsilon_k}, \left[ \varphi_j \nabla y_1 + (1 - \varphi_j) \nabla y_2 + \nabla \varphi_j \otimes (y_1 - y_2) \right] (\gamma^{-1} \circ (\varphi_j, P_2, P_1)) \right) \, dx
\]

\[
\leq \int_A W \left( \frac{x}{\varepsilon_k}, \nabla y_1 P_1^{-1} \right) \, dx + \int_B W \left( \frac{x}{\varepsilon_k}, \nabla y_2 P_2^{-1} \right) \, dx \\
+ c \int_{B \cap D_j} \left( 1 + \|\nabla y_1\|^2 + \|\nabla y_2\|^2 + \left( \frac{2N}{\delta} \right)^2 \|y_1 - y_2\|^2 \right) \, dx,
\]

(4.5)

where we used the fact that \((A' \cup B) \cap C_j \subseteq C_j \subseteq A\), \((A' \cup B) \setminus C_j \subseteq B\), \((A' \cup B) \cap D_j = B \cap D_j\), the growth condition \(E2\) and the uniform bound (2.1). Analogously, for the second summand we find

\[
I_h = \int_{(A' \cup B) \cap C_j \setminus \Omega} H \left( \frac{x}{\varepsilon_k}, P_1 \right) \, dx + \int_{(A' \cup B) \setminus C_j} H \left( \frac{x}{\varepsilon_k}, P_2 \right) \, dx \\
+ \int_{(A' \cup B) \cap D_j} H \left( \frac{x}{\varepsilon_k}, \gamma (\varphi_j, P_2, P_1) \right) \, dx
\]

\[
\leq \int_A H \left( \frac{x}{\varepsilon_k}, P_1 \right) \, dx + \int_B H \left( \frac{x}{\varepsilon_k}, P_2 \right) \, dx + c \mathcal{L}^3 (B \cap D_j),
\]

(4.6)

where we exploited again the geodesic convexity of \(K\) and the fact that \(H\) is bounded on \(K\), since it is Lipschitz.

We now estimate \(I_{reg}\). The chain rule yields

\[
\nabla (\gamma \circ (\varphi_j, P_2, P_1)) = [\hat{\gamma} \circ (\varphi_j, P_2, P_1)] \otimes \nabla \varphi_j + [\partial_F \gamma \circ (\varphi_j, P_2, P_1)] \nabla P_2 + [\partial_G \gamma \circ (\varphi_j, P_2, P_1)] \nabla P_1,
\]

whence, observing that by Proposition 3.3, the two differentials \(\partial_F \gamma\) and \(\partial_G \gamma\) are continuous functions restricted to compact sets,

\[
I_{reg} = \int_{(A' \cup B) \cap C_j \setminus \Omega} |\nabla P_1|^q \, dx + \int_{(A' \cup B) \setminus C_j} |\nabla P_2|^q \, dx + \int_{(A' \cup B) \cap D_j} |\nabla (\gamma \circ (\varphi_j, P_2, P_1))|^q \, dx
\]

\[
\leq \int_A |\nabla P_1|^q \, dx + \int_B |\nabla P_2|^q \, dx \\
+ \left( \frac{2N}{\delta} \right)^q \int_{B \cap D_j} |\hat{\gamma} \circ (\varphi_j, P_2, P_1)|^q \, dx + c \int_{B \cap D_j} (|\nabla P_1|^q + |\nabla P_2|^q) \, dx.
\]

We now resort to the explicit expression of \(\gamma\) in terms of the exponential map, see (3.2). From (2.1), \(D2\) and \(D3\) it follows

\[
|\hat{\gamma}(t, F, G)| = |\exp (F, t \exp_F^{-1}(G)) \exp_F^{-1}(G)| \leq c \Delta_1 (F^{-1} \exp_F^{-1}(G)) \leq c |G - F|,
\]

so that

\[
I_{reg} \leq \int_A |\nabla P_1|^q \, dx + \int_B |\nabla P_2|^q \, dx \\
+ c \left[ \left( \frac{2N}{\delta} \right)^q \int_{B \cap D_j} |P_1 - P_2|^q \, dx + \int_{B \cap D_j} (|\nabla P_1|^q + |\nabla P_2|^q) \, dx \right].
\]

(4.7)
By gathering (4.4)–(4.7) we obtain
\[ F_k(\varphi_j y_1 + (1 - \varphi_j) y_2, \gamma \circ (\varphi_j, P_2, P_1), A' \cup B) \]
\[ \leq F_k(y_1, P_1, A) + F_k(y_2, P_2, B) \]
\[ + c \int_{B \cap D_j} \left(1 + |\nabla y_1|^2 + |\nabla y_2|^2 + \left(\frac{2N}{\delta}\right)^2 |y_1 - y_2|^2\right) dx \]
\[ + c \left[ L^3(B \cap D_j) + \left(\frac{2N}{\delta}\right)^q \int_{B \cap D_j} |P_1 - P_2|^q dx + \int_{B \cap D_j} (|\nabla P_1|^q + |\nabla P_2|^q) dx \right]. \]

Now, note that
\[ \sum_{j=1}^N \int_{B \cap D_j} \left(1 + |\nabla y_1|^2 + |\nabla y_2|^2 + |\nabla P_1|^q + |\nabla P_2|^q\right) dx \]
\[ \leq \int_{(A \setminus A') \cap B} \left(1 + |\nabla y_1|^2 + |\nabla y_2|^2 + |\nabla P_1|^q + |\nabla P_2|^q\right) dx, \]
thus there certainly exists \( \ell \in 1, \ldots, N \) such that
\[ \int_{B \cap D_\ell} \left(1 + |\nabla y_1|^2 + |\nabla y_2|^2 + |\nabla P_1|^q + |\nabla P_2|^q\right) dx \]
\[ \leq \frac{1}{N} \int_{(A \setminus A') \cap B} \left(1 + |\nabla y_1|^2 + |\nabla y_2|^2 + |\nabla P_1|^q + |\nabla P_2|^q\right) dx \]
\[ \leq \frac{1}{N} L^3((A \setminus A') \cap B) + c \left( F_k(y_1, P_1, A) + F_k(y_2, P_2, B) \right) \]
where in the last inequality we exploited the fact that the set \((A \setminus A') \cap B\) is contained in both \( A \) and \( B \), together with the growth condition from below in E2. Therefore, we obtain
\[ F_k(\varphi_\ell y_1 + (1 - \varphi_\ell) y_2, \gamma \circ (\varphi_\ell, P_2, P_1), A' \cup B) \]
\[ \leq \left(1 + \frac{c}{N}\right) \left( F_k(y_1, P_1, A) + F_k(y_2, P_2, B) \right) \]
\[ + c \int_{(A \setminus B) \setminus A'} \left[ \left(\frac{2N}{\delta}\right)^2 |y_1 - y_2|^2 + \left(\frac{2N}{\delta}\right)^q |P_1 - P_2|^q \right] dx + \frac{2c}{N} L^3((A \setminus A') \cap B). \]

Finally, choosing \( N \) such that
\[ \frac{c}{N} < \sigma, \]
\[ \frac{2c L^3((A \setminus A') \cap B)}{N} < \sigma, \]
and letting
\[ M_\sigma := c \left[ \left(\frac{2N}{\delta}\right)^2 + \left(\frac{2N}{\delta}\right)^q \right], \]
we see (4.3) is satisfied, and the proof is complete. \( \square \)

Still following the strategy outlined in Subsection 3.1, we analyze the properties of the \( \Gamma(\tau) \)-lower and \( \Gamma(\tau) \)-upper limit of our sequence \( F_k \) when regarded as set functions. We recall that
(see e.g. [10, Chapter 7]), if \( \{\varepsilon_k\}_{k \in \mathbb{N}} \) is such that \( \varepsilon_k \to 0^+ \) and if \( (y, P, A) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,2}(\Omega; SL(3)) \times \mathcal{A}(\Omega) \), we have
\[
\mathcal{F}'(y, P, A) := \Gamma(\tau) \cdot \liminf_{k \to +\infty} \mathcal{F}_k(y, P, A)
= \inf \left\{ \liminf_{k \to +\infty} \mathcal{F}_k(y_k, P_k, A) : (y_k, P_k) \xrightarrow[k \to +\infty]{\tau} (y, P) \right\},
\]
\[
\mathcal{F}''(y, P, A) := \Gamma(\tau) \cdot \limsup_{k \to +\infty} \mathcal{F}_k(y, P, A)
= \inf \left\{ \limsup_{k \to +\infty} \mathcal{F}_k(y_k, P_k, A) : (y_k, P_k) \xrightarrow[k \to +\infty]{\tau} (y, P) \right\},
\]
and that \( \{\mathcal{F}_k\} \) \( \Gamma(\tau) \)-converges to \( \mathcal{F} \) if and only if \( \mathcal{F} = \mathcal{F}' = \mathcal{F}'' \). A key-step is the following “almost subadditivity” result, which is a modification of [10, Proposition 11.5] or [23, Proposition 18.3]. It is a consequence of the fundamental estimate, but it does not depend on the explicit form of the functionals at stake.

**Proposition 4.5.** Let \( \{\mathcal{F}_k\}_{k \in \mathbb{N}}, \mathcal{F}' \) and \( \mathcal{F}'' \) be as above. Then for all \( A, A', B \in \mathcal{A}(\Omega) \) with \( A' \Subset A \) and for all \( (y, P) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K) \)
\[
\mathcal{F}'(y, P, A' \cup B) \leq \mathcal{F}'(y, P, A) + \mathcal{F}''(y, P, B),
\]
\[
\mathcal{F}''(y, P, A' \cup B) \leq \mathcal{F}'(y, P, A) + \mathcal{F}''(y, P, B).
\]

**Proof.** We need to prove the inequalities only when \( \mathcal{F}'(y, P, A), \mathcal{F}''(y, P, A) \) and \( \mathcal{F}''(y, P, B) \) are finite. By the definitions in (4.8), there exist sequences \( \{y'_k\}_k, \{P'_k\}_k, \{y''_k\}_k, \{P''_k\}_k \) such that
\[
(y'_k, P'_k) \xrightarrow[k \to +\infty]{\tau} (y, P), \quad (y''_k, P''_k) \xrightarrow[k \to +\infty]{\tau} (y, P),
\]
\[
\mathcal{F}'(y, P, A) = \liminf_{k \to +\infty} \mathcal{F}_k(y'_k, P'_k, A), \quad \mathcal{F}''(y, P, B) = \limsup_{k \to +\infty} \mathcal{F}_k(y''_k, P''_k, B).
\]

Let us fix \( \sigma > 0 \). The fundamental estimate (4.3) gives a constant \( M_\sigma \), a sequence \( \{\varphi_k\}_k \) of cut-off functions between \( A' \) and \( A \), and a sequence \( \{\gamma_k\}_k \) of shortest paths from \( P''_k \) to \( P'_k \) such that
\[
\mathcal{F}_k \left( \varphi_k y'_k + (1 - \varphi_k) y''_k, \gamma_k \circ (\varphi_k, P''_k, P'_k), A' \cup B \right)
\leq (1 + \sigma) \left( \mathcal{F}_k(y'_k, P'_k, A) + \mathcal{F}_k(y''_k, P''_k, B) \right)
\]
\[
+ M_\sigma \int_{(A \cap B) \setminus A'} \left( |y'_k - y''_k|^2 + |P'_k - P''_k|^q \right) dx + \sigma.
\]

Recalling (3.2), we find
\[
\varphi_k y'_k + (1 - \varphi_k) y''_k \to y \quad \text{strongly in } L^2(A' \cup B; \mathbb{R}^3),
\]
\[
\gamma_k \circ (\varphi_k, P''_k, P'_k) \to P \quad \text{uniformly.}
\]

Taking the lower limit in (4.9) we obtain
\[
\mathcal{F}'(y, P, A' \cup B) \leq \liminf_{k \to +\infty} \mathcal{F}_k \left( \varphi_k y'_k + (1 - \varphi_k) y''_k, \gamma_k \circ (\varphi_k, P''_k, P'_k), A' \cup B \right)
\]
\[
\leq (1 + \sigma) \left( \liminf_{k \to +\infty} \mathcal{F}_k(y'_k, P'_k, A) + \limsup_{k \to +\infty} \mathcal{F}_k(y''_k, P''_k, B) \right) + \sigma
\]
\[
= (1 + \sigma) \left( \mathcal{F}'(y, P, A) + \mathcal{F}''(y, P, B) \right) + \sigma,
\]
and the first inequality in the statement follows by letting \( \sigma \to 0 \). The second one is obtained by taking the upper limit in (4.9) and arguing in a similar way. \( \square \)
Building on Proposition 4.5, we next establish subadditivity and inner regularity. We exploit the fact that $F'(y, P, \cdot)$ and $F''(y, P, \cdot)$ are increasing set functions, as well as the growth condition
\[ F''(y, P, A) \leq c \int_A \left(1 + |\nabla y P^{-1}|^2 + |\nabla P|^q \right) \, dx. \] (4.10)

**Proposition 4.6** (Subadditivity and inner regularity). Let $\{F_k\}_{k \in \mathbb{N}}$, $F'$ and $F''$ be as before. For all $(y, P) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,\varphi}(\Omega; K)$

1. $F'(y, P, \cdot)$ and $F''(y, P, \cdot)$ are inner regular,
2. $F''(y, P, \cdot)$ is subadditive.

**Proof.** We prove the two statements separately.

1. Let us focus on $F'$. According to Definition 3.2 we need to prove that
   \[ F'(y, P, A) = \sup \{ F'(y, P, B) : B \in \mathcal{A}(\Omega), B \subseteq A \}. \]
   Since $F'(y, P, \cdot)$ is an increasing set function, it is trivial that $F'(y, P, A)$ is larger than the supremum in the formula above. As for the reverse inequality, let us fix $A \in \mathcal{A}(\Omega)$ and let $C \subset A$ be compact. There exist $B, B' \in \mathcal{A}(\Omega)$ such that $C \subset B' \subseteq B \subseteq A$. Then $B' \cup (A \setminus C) = A$. We now apply Proposition 4.5:
   \[ F'(y, P, A) = F'(y, P, B' \cup (A \setminus C)) \leq F'(y, P, B) + F''(y, P, A \setminus C) \leq \sup \{ F'(y, P, B) : B \in \mathcal{A}(\Omega), B \subseteq A \} + c \int_{A \setminus C} \left(1 + |\nabla y P^{-1}|^2 + |\nabla P|^q \right) \, dx. \]
   By the arbitrariness of $C$, we can let $C$ invade $A$ and we obtain
   \[ F'(y, P, A) \leq \sup \{ F'(y, P, B) : B \in \mathcal{A}(\Omega), B \subseteq A \}. \]
   The same argument applies to $F''$.

2. Fix $A, B \in \mathcal{A}(\Omega)$ and let $C \in \mathcal{A}(\Omega)$ satisfy $C \subseteq A \cup B$. We find $A' \in \mathcal{A}(\Omega)$ such that $A' \Subset A$ and that $C \subseteq A' \cup B$. Note that $\overline{C} \setminus B$ is compact and $\overline{C} \setminus B \subset A$. Now, by the monotonicity of $F''(y, P, \cdot)$ and Proposition 4.5,
   \[ F''(y, P, C) \leq F''(y, P, A' \cup B) \leq F''(y, P, A) + F''(y, P, B). \]
   By taking the supremum over $C$, inner regularity yields
   \[ F''(y, P, A \cup B) \leq F''(y, P, A) + F''(y, P, B). \]

By gathering the previous results, we infer the existence of a $\Gamma$-limit admitting an integral representation.

**Proof of Theorem 4.2.** Let $\{\varepsilon_k\}_{k \in \mathbb{N}}$ be such that $\varepsilon_k \to 0^+$, and let $F'$ and $F''$ be as in (4.8). It follows from the definitions that they are increasing set functions, and Proposition 4.6 yields inner regularity for both and subadditivity for $F''$. By standard arguments (see e.g. [10, Theorem 10.3]), we deduce that, upon extraction of a subsequence,
\[ F(y, P, A) := \Gamma(\tau) - \lim_{k \to +\infty} F_k(y, P, A) \]
exists for all triple $(y, P, A)$. From the definition of $\Gamma$-limit and from the fact that $F_k(y, P, \cdot)$ is a measure for each $k$, it follows that $F(y, P, \cdot)$ is superadditive. We are then in a position to apply De Giorgi-Letta criterion (see Subsection 3.1), which ensures that $F(y, P, \cdot)$ is the
restriction of a Borel measure to \( \mathcal{A}(\Omega) \) for \( (y, p) \in W^{1,2}(A; \mathbb{R}^3) \times W^{1,q}(A; K) \). Thanks to (4.10) we infer also that it is absolutely continuous with respect to the Lebesgue measure.

Thus, by Radon-Nikodym theorem there exists \( f \in L^1_{\text{loc}}(\mathbb{R}^3) \) such that
\[
\mathcal{F}(y, P, A) = \int_A f(x) \, dx.
\]

\[\square\]

4.2. Characterization of the limiting energy density. So far, we have proved a \( \Gamma(\tau) \)-compactness result for the functionals (4.1), and we have shown that the limit is actually an integral functional. We proceed by providing an explicit formula for the limiting energy density.

**Proof of Theorem 2.2.** Let \( y \in W^{1,2}(\Omega; \mathbb{R}^3) \), \( P \in W^{1,q}(\Omega; K) \). Our aim is to identify \( f \) in (4.2) as
\[
f(x) = W_{\text{hom}}(\nabla y(x), P(x)) + H_{\text{hom}}(P(x)) + |\nabla P(x)|^q \quad \text{for a.e. } x \in \Omega. \tag{4.11}
\]

This also entails (see [10, Proposition 7.11]) that the whole family \( \{\mathcal{F}_\varepsilon\} \) \( \Gamma(\tau) \)-converges to \( \mathcal{F} \) as in the statement of the theorem.

For all \( x \in \mathbb{R}^3 \), \( F \in \mathbb{R}^{3 \times 3} \) and \( G \in K \) let us define
\[
\tilde{W}(x, F, G) := W(x, FG^{-1}).
\]

From (E1) and (E2) we deduce that

1. \( \tilde{W} \) is a Borel function and \( \tilde{W}(\cdot, F, G) \) is \( Q \)-periodic;
2. \( \tilde{W}(x, \cdot, G) \) is 2-coercive and has at most quadratic growth, uniformly in \( x \) and \( G \): there exist \( 0 < \tilde{c}_1 \leq \tilde{c}_2 \) such that for a.e. \( x \in \mathbb{R}^3 \), for all \( F \in \mathbb{R}^{3 \times 3} \) and for all \( G \in K \)
\[
\tilde{c}_1 |F|^2 \leq \tilde{W}(x, F, G) \leq \tilde{c}_2 \left(|F|^2 + 1\right).
\]

Note that the independence of \( \tilde{c}_1 \) and \( \tilde{c}_2 \) from \( G \) is a consequence of (2.2) and (2.1), respectively.

For \( A \in \mathcal{A}(\Omega) \), we now introduce the functional
\[
\mathcal{G}_k(y, G, A) := \int_A \left[ \tilde{W} \left( \frac{x}{\varepsilon_k}, \nabla y(x), G \right) + H \left( \frac{x}{\varepsilon_k}, G \right) \right] \, dx \tag{4.12}
\]
where \( G \in K \) is fixed, and we observe that
\[
\mathcal{F}_k(y, P, A) = \int_A \left[ \tilde{W} \left( \frac{x}{\varepsilon_k}, \nabla y(x), P(x) \right) + H \left( \frac{x}{\varepsilon_k}, P(x) \right) + |\nabla P(x)|^q \right] \, dx.
\]

From Theorem 4.2 we know that
\[
\mathcal{F}(y, P, A) := \Gamma(\tau)- \lim_{k \to +\infty} \mathcal{F}_k(y, P, A) = \int_A f(x) \, dx,
\]
while, if \( x_0 \in A \) and \( P_0 := P(x_0) \), Theorem 3.1 and Riemann-Lebesgue lemma on rapidly oscillating functions yield
\[
\mathcal{G}(y, P_0, A) := \Gamma(L^2) - \lim_{k \to +\infty} \mathcal{G}_k(y, P_0, A)
\]
\[
= \int_A \left( W_{\text{hom}}(\nabla y(x), P(x_0)) + H_{\text{hom}}(P(x_0)) \right) \, dx. \tag{4.13}
\]

We next show that (4.11) holds by exploiting these two convergence results.

From now on, let \( x_0 \in \Omega \) be a Lebesgue point of the functions \( f, \nabla y \) and \( \nabla P \). Notice that almost every \( x_0 \in \Omega \) has such property. Let \( A \in \mathcal{A}(\Omega) \) contain \( x_0 \), and let \( \{(y_k, P_k)\} \subset \)
$W^{1,2}(A; \mathbb{R}^3) \times W^{1,q}(A; K)$ be a generic sequence. We use $E3$ and (2.1) to estimate the difference between the elastic contribution in $\mathcal{F}_k(y_k, P_k, A)$ and $\mathcal{G}_k(y_k, P_0, A)$:

$$\int_A \left| \tilde{W} \left( \frac{x}{\varepsilon_k}, \nabla y_k, P_k \right) - \tilde{W} \left( \frac{x}{\varepsilon_k}, \nabla y_k, P_0 \right) \right| \, dx$$

$$\leq c_3 \int_A \left( 1 + |\nabla y_k P_k^{-1}| + |\nabla y_k P_0^{-1}| \right) |\nabla y_k| \left| P_k^{-1} - P_0^{-1} \right| \, dx$$

$$\leq c_3 \int_A \left( 1 + 2cK|\nabla y_k| |\nabla y_k| \left| P_k^{-1} - P^{-1} \right| + |P^{-1} - P_0^{-1}| \right) \, dx$$

$$\leq c \left( \mathcal{L}^3(A) + \|\nabla y_k\|_{L^2(A)}^2 \right) \left( \|P_k^{-1} - P^{-1}\|_{L^\infty(A)} + \|P^{-1} - P_0^{-1}\|_{L^\infty(A)} \right),$$

where $c$ is a constant independent of $k$ and $A$. We now use $H3$ to estimate the difference between the hardenings in $\mathcal{F}_k(y_k, P_k, A)$ and $\mathcal{G}_k(y_k, P_0, A)$:

$$\int_A \left| H \left( \frac{x}{\varepsilon_k}, P_k \right) - H \left( \frac{x}{\varepsilon_k}, P_0 \right) \right| \, dx \leq c \mathcal{L}^3(A) \left( \|P_k - P\|_{L^\infty(A)} + \|P - P_0\|_{L^\infty(A)} \right).$$

By the definitions of $\mathcal{F}_k$ and $\mathcal{G}_k$ we obtain

$$\left| \mathcal{F}_k(y_k, P_k, A) - \int_A |\nabla P_k|^q \, dx - \mathcal{G}_k(y_k, P_0, A) \right|$$

$$\leq c \left( \mathcal{L}^3(A) + \|\nabla y_k\|_{L^2(A)}^2 \right) \left( \|P_k^{-1} - P^{-1}\|_{L^\infty(A)} + \|P^{-1} - P_0^{-1}\|_{L^\infty(A)} \right)$$

$$+ c \mathcal{L}^3(A) \left( \|P_k - P\|_{L^\infty(A)} + \|P - P_0\|_{L^\infty(A)} \right).$$

We now prove that

$$f(x) \geq W_{\text{hom}}(\nabla y(x), P(x)) + H_{\text{hom}}(P(x)) + |\nabla P(x)|^q \quad \text{for a.e. } x \in \Omega. \quad (4.15)$$

To this aim, we select a recovery sequence $\{(y_k, P_k)\} \subset W^{1,2}(A; \mathbb{R}^3) \times W^{1,q}(A; K)$ for $\mathcal{F}(y, P, A)$, namely

$$(y_k, P_k) \rightharpoonup (y, P), \quad \lim_{k \to +\infty} \mathcal{F}_k(y_k, P_k, A) = \mathcal{F}(y, P, A). \quad (4.16)$$

Owing to the growth assumptions on the energy densities, we can without loss of generality suppose that

$$y_k \to y \quad \text{weakly in } W^{1,2}(A; \mathbb{R}^3), \quad P_k \to P \quad \text{weakly in } W^{1,q}(A; K).$$

From (4.14) coupled with the 2-coercivity of $\tilde{W}$, recalling Remark 4.1 about the convergence of inverses, (4.13) and (4.16), we infer

$$\mathcal{F}(y, P, A) = \lim_{k \to +\infty} \mathcal{F}_k(y_k, P_k, A)$$

$$\geq \liminf_{k \to +\infty} \mathcal{G}_k(y_k, P_0, A) + \liminf_{k \to +\infty} \int_A |\nabla P_k|^q \, dx$$

$$- c \|P^{-1} - P_0^{-1}\|_{L^\infty(A)} \left( \mathcal{L}^3(A) + \lim_{k \to +\infty} \mathcal{F}_k(y_k, P_k, A) \right) - c \mathcal{L}^3(A) \|P - P_0\|_{L^\infty(A)}$$

$$\geq \mathcal{G}(y, P_0, A) + \int_A |\nabla P|^q \, dx$$

$$- c \|P^{-1} - P_0^{-1}\|_{L^\infty(A)} \left( \mathcal{L}^3(A) + \mathcal{F}(y, P, A) \right) - c \mathcal{L}^3(A) \|P - P_0\|_{L^\infty(A)}.$$
estimates by $L^3(B_r(x_0))$. By letting $r \to 0$, we apply Lebesgue differentiation theorem to deduce (4.15) for $x = x_0$. Indeed,

\[
\lim_{r \to 0} \|P^{-1} - P_0^{-1}\|_{L^\infty(B_r(x_0))} \left(1 + \frac{1}{L^3(B_r(x_0))} \mathcal{F}(y, P, B_r(x_0)) \right) = 0,
\]

because (4.10) grants

\[
\mathcal{F}(y, P, A) \leq c \int_A \left(1 + |\nabla y P^{-1}|^2 + |\nabla P|^q \right) dx.
\]

Eventually, we are only left to prove that

\[
f(x) \leq W_{\text{hom}}(\nabla y(x), P(x)) + H_{\text{hom}}(P(x)) + |\nabla P(x)|^q \quad \text{for a.e. } x \in \Omega. \tag{4.17}
\]

To establish the estimate, we let $\{y_k\} \subset W^{1,2}(A; \mathbb{R}^3)$ be a recovery sequence for $G(y, P_0, A)$, that is,

\[
\lim_{k \to +\infty} G_k(y_k, P_0, A) = G(y, P_0, A),
\]

with $y_k \rightharpoonup y$ weakly in $W^{1,2}(A; \mathbb{R}^3)$. We consider estimate (4.14) for this particular $\{y_k\}$ and the constant sequence given by $P_k = P$ for all $k$. Exploiting this time the 2-coercivity of $G_k$ we obtain

\[
\mathcal{F}(y, P, A) \leq \limsup_{k \to +\infty} \mathcal{F}_k(y_k, P, A)
\]

\[
\leq \lim_{k \to +\infty} G_k(y_k, P_0, A) + \int_A |\nabla P|^q \, dx + c \|P^{-1} - P_0^{-1}\|_{L^\infty(A)} \left(\mathcal{L}^3(A) + \lim_{k \to +\infty} G_k(y_k, P_0, A)\right) + c \mathcal{L}^3(A) \|P - P_0\|_{L^\infty(A)}
\]

\[
\leq G(y, P_0, A) + \int_A |\nabla P|^q \, dx + c \|P^{-1} - P_0^{-1}\|_{L^\infty(A)} \left(\mathcal{L}^3(A) + G(y, P_0, A)\right) + c \mathcal{L}^3(A) \|P - P_0\|_{L^\infty(A)}.
\]

Arguing as above with $A = B_r(x_0)$ and exploiting the 2-growth conditions of $W_{\text{hom}}$, we conclude that (4.17) holds for $x = x_0$.

Since we can select almost every point in $\Omega$ as $x_0$, the conclusion follows from (4.15) and (4.17). \hfill \Box

Thanks to the decomposition Lemma 3.5, we are able to refine the choice of recovery sequences for $\mathcal{F}$. This will come in handy in the proof of Corollary 2.10.

**Corollary 4.7.** Under the same assumptions of Theorem 2.2, for any $(y, P) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,2}(\Omega; K)$ there exists a recovery sequence $(y_k, P_k)$ for $\mathcal{F}(y, P)$ satisfying the following:

1. $y_k \rightharpoonup y$ weakly in $W^{1,2}(\Omega; \mathbb{R}^3)$;
2. $y_k = y$ in a neighborhood of $\partial \Omega$;
3. $\{\nabla y_k\}$ is 2-equiintegrable.

**Proof.** Let $\{(w_k, P_k)\}$ be a recovery sequence for $\mathcal{F}(y, P)$ as provided by Theorem 2.2. We apply Lemma 3.5 to $\{w_k\}$. We deduce the existence of sequences $\{k_j\}$ and $\{u_j\} \subset W^{1,2}(\Omega; \mathbb{R}^3)$ such that the sequence defined by

\[
y_k := \begin{cases} u_j & \text{if } k = k_j \text{ for some } j \in \mathbb{N}, \\ y & \text{otherwise} \end{cases}
\]
satisfies properties (1)-(3) and \((y_k, P_k) \xrightarrow{j} (y, P)\). Moreover
\[
\lim_{j \to +\infty} \mathcal{L}^3(N_j) = 0,
\]
where \(N_j := \{x \in \Omega : w_{k_j}(x) \neq u_j(x)\}\).

We are left to prove that \(\{(y_k, P_k)\}\) satisfies the upper limit inequality. Loosely speaking, this is a consequence of the fact that passing to a 2-equintegrable sequence “does not increase the energy”. Upon passing to a subsequence, which we do not relabel, we can assume that \(\{\mathcal{F}_k(y_k, P_k)\}\) is convergent. We provisionally focus just on the elastic and hardening parts of the energy \(\mathcal{F}_k\). It holds
\[
\int_\Omega \left[ W \left( \frac{x}{\varepsilon_{k_j}}, \nabla w_{k_j} P_{k_j}^{-1} \right) + H \left( \frac{x}{\varepsilon_{k_j}}, P_{k_j} \right) \right] dx
= \int_{N_j} \left[ W \left( \frac{x}{\varepsilon_{k_j}}, \nabla w_{k_j} P_{k_j}^{-1} \right) + H \left( \frac{x}{\varepsilon_{k_j}}, P_{k_j} \right) \right] dx
+ \int_{\Omega \setminus N_j} \left[ W \left( \frac{x}{\varepsilon_{k_j}}, \nabla u_j P_{k_j}^{-1} \right) + H \left( \frac{x}{\varepsilon_{k_j}}, P_{k_j} \right) \right] dx
\geq \int_{\Omega \setminus N_j} \left[ W \left( \frac{x}{\varepsilon_{k_j}}, \nabla u_j P_{k_j}^{-1} \right) + H \left( \frac{x}{\varepsilon_{k_j}}, P_{k_j} \right) \right] dx,
\]
so that
\[
\limsup_{j \to +\infty} \int_\Omega \left[ W \left( \frac{x}{\varepsilon_{k_j}}, \nabla w_{k_j} P_{k_j}^{-1} \right) + H \left( \frac{x}{\varepsilon_{k_j}}, P_{k_j} \right) \right] dx
\geq \limsup_{j \to +\infty} \int_{\Omega \setminus N_j} \left[ W \left( \frac{x}{\varepsilon_{k_j}}, \nabla u_j P_{k_j}^{-1} \right) + H \left( \frac{x}{\varepsilon_{k_j}}, P_{k_j} \right) \right] dx
= \limsup_{j \to +\infty} \int_{\Omega \setminus N_j} \left[ W \left( \frac{x}{\varepsilon_{k_j}}, \nabla u_j P_{k_j}^{-1} \right) + H \left( \frac{x}{\varepsilon_{k_j}}, P_{k_j} \right) \right] dx,
\]
where the equality follows from the growth condition \(\text{E2}\) and from the 2-equintegrability of \(\{\nabla u_j\}\) (recall that \(\sup_{k \in \mathbb{N}} \|P_k^{-1}\|_{\infty} \leq C\), together with the boundedness of \(H\). Therefore, coming back to the full functional \(\mathcal{F}_{k_j}\),
\[
\lim_{j \to +\infty} \mathcal{F}_{k_j}(w_{k_j}, P_{k_j})
\geq \limsup_{j \to +\infty} \int_{\Omega} \left[ W \left( \frac{x}{\varepsilon_{k_j}}, \nabla w_{k_j} P_{k_j}^{-1} \right) + H \left( \frac{x}{\varepsilon_{k_j}}, P_{k_j} \right) \right] dx + \liminf_{j \to +\infty} \int_{\Omega} |\nabla P_{k_j}|^q dx
\geq \limsup_{j \to +\infty} \int_{\Omega} \left[ W \left( \frac{x}{\varepsilon_{k_j}}, \nabla u_j P_{k_j}^{-1} \right) + H \left( \frac{x}{\varepsilon_{k_j}}, P_{k_j} \right) \right] dx + \liminf_{j \to +\infty} \int_{\Omega} |\nabla P_{k_j}|^q dx
\geq \lim_{j \to +\infty} \mathcal{F}_{k_j}(u_j, P_{k_j}).
\]

Recalling that \(\{(u_j, P_k)\}\) is a recovery sequence, we find
\[
\lim_{k \to +\infty} \mathcal{F}_k(y_k, P_k) = \lim_{j \to +\infty} \mathcal{F}_{k_j}(u_j, P_{k_j}) \leq \lim_{j \to +\infty} \mathcal{F}_{k_j}(w_{k_j}, P_{k_j}) = \mathcal{F}(y, P),
\]
which in turn yields that \(\{(y_k, P_k)\}\) is also a recovery sequence. \(\square\)

We conclude this section with a variant of Theorem 2.2 that is instrumental to Theorem 2.8. We only state it, for its proof is an adaptation of that of Theorem 2.2, the most substantial difference being the use of [10, Theorem 19.1] instead of Theorem 3.1.
Theorem 4.8. Let $E$ be an open and connected set that is $Q$-periodic and that has Lipschitz boundary. For every $(y, P) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,\tilde{q}}(\Omega; K)$, let

$$
\tilde{F}_\varepsilon(y, P) := \int_{\Omega \cap \varepsilon E} W \left( \frac{x}{\varepsilon}, \nabla y(x) P^{-1}(x) \right) \, dx + \int_{\Omega \cap \varepsilon E} H \left( \frac{x}{\varepsilon}, P(x) \right) \, dx + \int_{\Omega} |\nabla P(x)|^q \, dx,
$$

(4.19)

which we extend by setting

$$
\tilde{F}_\varepsilon(y, P) = +\infty \text{ on } [L^2(\Omega; \mathbb{R}^3) \times L^q(\Omega; \text{SL}(3))] \setminus [W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,\tilde{q}}(\Omega; K)].
$$

If $W$ and $H$ satisfy E1–E3 and H1–H3, respectively, then for all $(y, P) \in L^2(\Omega; \mathbb{R}^3) \times L^q(\Omega; \text{SL}(3))$ the $\Gamma$-limit

$$
\tilde{F}(y, P) := \Gamma(\tau) - \lim_{\varepsilon \to 0} \tilde{F}_\varepsilon(y, P)
$$

exists and we have that

$$
\tilde{F}(y, P) = \begin{cases} 
\int_{\Omega} \left( \tilde{W}_\text{hom}(\nabla y(x), P(x)) + \tilde{H}_\text{hom}(P(x)) + |\nabla P(x)|^q \right) \, dx & \text{if } (y, P) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,\tilde{q}}(\Omega; K), \\
+\infty & \text{otherwise in } L^2(\Omega; \mathbb{R}^3) \times L^q(\Omega; \text{SL}(3)),
\end{cases}
$$

where $\tilde{W}_\text{hom} : \mathbb{R}^{3 \times 3} \times K \to [0, +\infty)$ and $\tilde{H}_\text{hom} : K \to [0, +\infty)$ are defined as

$$
\tilde{W}_\text{hom}(F, G) := \lim_{\lambda \to +\infty} \frac{1}{\lambda^3} \inf \left\{ \int_{(0, \lambda)^3 \cap E} W(x, (F + \nabla y(x)) G^{-1}) \, dx : y \in W_0^{1,2}((0, \lambda)^3; \mathbb{R}^3) \right\},
$$

$$
\tilde{H}_\text{hom}(F) := \int_{Q \cap E} H(z, F) \, dz.
$$

We observe that the theorem above is similar in spirit to homogenization results for perforated domains. The case at stake is however different, in that later we will deal with functions defined on the nonperforated domain $\Omega$. This makes the analysis simpler because it spares us the need of extending $\text{SL}(3)$-valued Sobolev maps.

5. High-contrast energy: compactness and splitting

From now on we turn to the analysis of the high-contrast energy in (1.2). We investigate in this section the compactness properties of sequences with equibounded energy. We will see that, as a consequence of the behavior of the hardening functional $H$, we can reduce the problem to the case of pure elasticity addressed by K. Cherdantsev & M. Cherednichenko [14], and we adapt their approach.

Lemma 5.1 (Compactness). Let $\{\varepsilon_k\}$ be an infinitesimal sequence. We suppose that $\{(y_k, P_k)\}_{k \in \mathbb{N}} \subset W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,\tilde{q}}(\Omega; \text{SL}(3))$ satisfies

$$
\|y_k\|_{L^2(\Omega; \mathbb{R}^3)} \leq C, \quad J_k(y_k, P_k) \leq C
$$

for some $C \geq 0$, uniformly in $k$. Let us denote by $\tilde{y}_k$ the extension of $y_k$ in the sense of Remark 3.7 above. Then, there exist subsequences of $\{\varepsilon_k\}$, $\{y_k\}$, and $\{P_k\}$, which we do not relabel, as well as $y \in L^2(\Omega; W^{1,2}_{\text{per}}(\mathbb{R}^3; \mathbb{R}^3))$, $y^1 \in W^{1,2}(\Omega; \mathbb{R}^3)$, $v \in L^2(\Omega; W^{1,2}_0(Q^0; \mathbb{R}^3))$, and
$P \in W^{1,q}(\Omega; SL(3))$ such that the following hold:

\begin{align}
\chi_k \frac{\varepsilon}{\varepsilon_k} = y_k & \text{ weakly two-scale in } L^2, \\
\varepsilon_k \nabla y_k \frac{\varepsilon}{\varepsilon_k} \nabla_z & \text{ weakly two-scale in } L^2, \\
\chi_k \frac{\varepsilon}{\varepsilon_k} & \text{ weakly in } W^{1,2}(\Omega; \mathbb{R}^3),
\end{align}

\begin{align}
P_k & \rightarrow P, \quad P_k^{-1} \rightarrow P^{-1} \text{ weakly in } W^{1,q}(\Omega; SL(3)) \text{ and uniformly in } C(\bar{\Omega}; SL(3)), \\
\nabla \bar{y}_k P_k^{-1} & \rightarrow \nabla y_1 P^{-1} \text{ weakly in } L^2(\Omega; \mathbb{R}^{3 \times 3}).
\end{align}

**Proof.** From the definition of $\mathcal{J}_k$, for all $k \in \mathbb{N}$

\begin{equation}
\|\nabla P_k\|_{L^q} \leq C.
\end{equation}

Besides, for all $k$, hypothesis E4, the definition of $H$ and the bound (2.1) imply

\begin{equation}
\left\|\varepsilon_k \chi_k^0 \nabla y_k P_k^{-1}\right\|_{L^2} + \left\|\chi_k^1 \nabla y_k P_k^{-1}\right\|_{L^2} \leq C,
\end{equation}

\begin{equation}
\|P_k\|_{L^\infty} + \|P_k^{-1}\|_{L^\infty} \leq C.
\end{equation}

Thanks to (2.2), from the first estimate we deduce

\begin{equation}
\left\|\varepsilon_k \chi_k^0 \nabla y_k\right\|_{L^2} + \left\|\chi_k^1 \nabla y_k\right\|_{L^2} \leq C,
\end{equation}

which is precisely formula (21) in [14]. Thus, for what concerns the sequence of deformations, the same bounds as the purely elastic case are retrieved. While referring to [14] for details, here we limit ourselves to sketch how (5.9) entails two-scale compactness.

The boundedness of $\{y_k\}$ in $L^2$ and Lemma 3.10(4) yield the existence of a function $y \in L^2(\Omega; W^{1,2}_{per}(\mathbb{R}^3; \mathbb{R}^3))$ such that, up to subsequences, (5.2) holds and

\begin{equation}
\varepsilon_k \nabla y_k \frac{\varepsilon}{\varepsilon_k} \nabla_z y \text{ weakly two-scale in } L^2.
\end{equation}

Thanks to (3.5) and Lemma 3.10(3), we also infer that

\begin{equation}
\chi_k^1 y_k \frac{\varepsilon}{\varepsilon_k} \chi_k, \quad \varepsilon_k \chi_k^1 \nabla y_k \frac{\varepsilon}{\varepsilon_k} \nabla_1 \chi_k \text{ weakly two-scale in } L^2.
\end{equation}

Moreover, there exist $y^1 \in W^{1,2}(\Omega; \mathbb{R}^3)$ and $v \in L^2(\Omega; W^{1,2}_0(Q^0; \mathbb{R}^3))$ such that the decomposition (5.1) and the convergence (5.4) hold. By combining (5.1) and (5.10), (5.3) follows.

We now turn to the sequence of plastic strains. By (5.6) and (5.8), we see that $\{P_k\}$ is bounded in $W^{1,q}(\Omega; SL(3))$. Since $q > 3$, Morrey’s embedding yields the uniform convergence of (a subsequence of) $\{P_k\}$ to some $P \in W^{1,q}(\Omega; SL(3))$, whence, by Remark 4.1, we also deduce that $P_k^{-1} \rightarrow P^{-1}$ uniformly.

Finally, we observe that, thanks to (5.4) and the uniform convergence of $\{P_k^{-1}\}$, (5.5) is also inferred.

It is well-known that $\Gamma$-limits are not additive. In our case, however, we are able to show that the asymptotic behavior of the functionals $\mathcal{J}_\varepsilon$ is given exactly by the sum of the $\Gamma$-limits of the soft and of the stiff contributions. Such splitting will enable us to treat the $\Gamma$-limits of $\mathcal{J}_\varepsilon^0$ and of $\mathcal{J}_\varepsilon^1$ separately. We premise a simple lemma, which deals with the hardening part of the energy. We recall that, for $i = 0, 1$, $\chi_i$ is the characteristic function of $\Omega_i^\varepsilon$.\hfill \Box
Lemma 5.2. Under assumptions H2–H3, for any sequence \( \{P_k\} \subset W^{1,q}(\Omega; K) \) converging uniformly to \( P \in W^{1,q}(\Omega; K) \) it holds
\[
\lim_{k \to +\infty} \int_{\Omega} \chi_k^i(x) H(P_k(x)) \, dx = \mathcal{L}^3(Q^i) \int_{\Omega} H(P(x)) \, dx \quad \text{for } i = 0, 1.
\]

Proof. Let us focus on the case \( i = 0 \) first. We set
\[
E^0 := \bigcup_{t \in \mathbb{Z}^3} (t + Q^0) = \mathbb{R}^3 \setminus \overline{E}, \quad \hat{\Omega}_k^0 := \bigcup_{t \in T_k} \varepsilon_k(t + Q^0),
\]
where
\[
\hat{T}_k := \{t \in \mathbb{Z}^3 : \varepsilon_k(t + Q) \subset \Omega\} \subset T_k.
\]
By definition of \( \Omega_k^0 \) (see (2.4)), we have
\[
\varepsilon_k E^0 \setminus \Omega_k^0 \subset \varepsilon_k E^0 \setminus \hat{\Omega}_k^0.
\]
Note that \( \Omega \cap (\varepsilon_k E^0 \setminus \hat{\Omega}_k^0) \) is contained in the strip \( \{x \in \Omega : \text{dist}(x, \partial \Omega) < \sqrt{3}\varepsilon_k\} \). Since \( \{H(P_k)\} \) is uniformly bounded by H2 and H3, we see that
\[
\lim_{k \to +\infty} \int_{\Omega} \chi_k^0(x) H(P_k(x)) \, dx = \lim_{k \to +\infty} \int_{\Omega} \chi_{\varepsilon_k E^0}(x) H(P_k(x)) \, dx - \lim_{k \to +\infty} \int_{\Omega} (\chi_{\varepsilon_k E^0}(x) - \chi_k^0(x)) H(P_k(x)) \, dx
\]
Then, by the Lipschitz continuity of \( H \) on its domain,
\[
\lim_{k \to +\infty} \int_{\Omega} \chi_{\varepsilon_k E^0}(x) H(P_k(x)) \, dx = \lim_{k \to +\infty} \int_{\Omega} \chi_{\varepsilon_k E^0}(x) H(P(x)) \, dx
\]
Then the case \( i = 1 \) follows from the previous one by the identities \( \chi_k^1 = \chi_k^0 \) and \( \mathcal{L}^3(Q^1) = 1 - \mathcal{L}^3(Q^0) \).

The splitting process is explained by the ensuing proposition.

Proposition 5.3 (Splitting). Let \( \{\varepsilon_k\} \) be an infinitesimal sequence, and let \( \{(y_k, P_k)\}_{k \in \mathbb{N}} \subset W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; \text{SL}(3)) \) be a sequence satisfying
\[
\|y_k\|_{L^2(\Omega; \mathbb{R}^3)} \leq C, \quad \mathcal{J}_k(y_k, P_k) \leq C
\]
for some \( C \geq 0 \), uniformly in \( k \). Let \( \tilde{y}_k \) be the extension of \( y_k \) in the sense of Remark 3.7, and let \( v \in L^2(\Omega; W^{1,2}_0(Q^0; \mathbb{R}^3)) \) be as in Lemma 5.1. Then, defining \( v_k := y_k - \tilde{y}_k \), the following hold:
\[
\{v_k\} \subset W^{1,2}_0(\Omega^0_k; \mathbb{R}^3),
\]
\[
\|v_k\|_{L^2(\Omega; \mathbb{R}^3)} \leq C,
\]
\[
\varepsilon_k \nabla v_k \xrightarrow{k \to +\infty} \nabla v \quad \text{weakly two-scale in } L^2,
\]
\[
\liminf_{k \to +\infty} \mathcal{J}_k^0(v_k, P_k) + \liminf_{k \to +\infty} \mathcal{J}_k^1(\tilde{y}_k, P_k) \leq \liminf_{k \to +\infty} \mathcal{J}_k(y_k, P_k),
\]
\[
\limsup_{k \to +\infty} \mathcal{J}_k(y_k, P_k) \leq \limsup_{k \to +\infty} \mathcal{J}_k^0(v_k, P_k) + \limsup_{k \to +\infty} \mathcal{J}_k^1(\tilde{y}_k, P_k).
\]
Moreover, in (5.13), \( \{v_k\} \) may be replaced with another sequence \( \{w_k\} \subset W^{1,2}_0(\Omega_k;\mathbb{R}^3) \) such that \( \{\varepsilon_k \nabla w_k\} \) is 2-equiintegrable and \( \varepsilon_k \nabla w_k \rightharpoonup 0 \) weakly in \( L^2(\Omega;\mathbb{R}^{3\times 3}) \).

**Proof.** We first prove that (5.12) – (5.14) hold for the sequence \( \{v_k\} \). Afterwards, we will show how to recover the equiintegrability for the sequence of gradients.

We split the functional \( J_k \) evaluated on \( (y_k, P_k) \) as follows:

\[
J_k(y_k, P_k) = J_k^0(y_k, P_k) + J_k^1(y_k, P_k) = J_k^0(v_k, P_k) + J_k^1(\bar{y}_k, P_k) + R_k,
\]

(5.15)

where \( J_k^0 \) and \( J_k^1 \) are as in (2.12) and (2.13), and

\[
R_k := J_k^0(y_k, P_k) - J_k^0(v_k, P_k) = \int_\Omega \chi_k^0 \left[ W_\varepsilon^0 \left( \varepsilon_k \nabla y_k P_k^{-1} \right) - W_\varepsilon^0 \left( \varepsilon_k \nabla v_k P_k^{-1} \right) \right] \, dx.
\]

We next show that \( R_k \) is asymptotically negligible.

Hypothesis E5 yields

\[
|R_k| \leq c_3 \int_\Omega \chi_k^0 \left( 1 + \left| \varepsilon_k \nabla y_k P_k^{-1} \right| + \left| \varepsilon_k \nabla v_k P_k^{-1} \right| \right) \left| \varepsilon_k \nabla \bar{y}_k P_k^{-1} \right| \, dx.
\]

(5.16)

Since \( \{(y_k, P_k)\} \) is equibounded in energy, the sequences \( \{\varepsilon_k \chi_k^0 \nabla y_k P_k^{-1}\}, \{\chi_k^1 \nabla y_k P_k^{-1}\} \), and \( \{P_k^{-1}\} \) are bounded in suitable Lebesgue spaces (see (5.7) and (5.8)). By the properties of the extension operator \( T_\varepsilon \) in Lemma 3.6, we deduce that

\[
\int_\Omega \left| \nabla \bar{y}_k P_k^{-1} \right|^2 \, dx \leq c \int_\Omega \left| \nabla \bar{y}_k \right|^2 \, dx \leq c \int_\Omega \left| \chi_k^1 \nabla y_k \right|^2 \, dx \leq c \int_\Omega \left| \chi_k^1 \nabla y_k P_k^{-1} \right|^2 \, dx \leq C
\]

(recall estimate (2.2)). So, thanks to (5.3), we deduce that

\[
\varepsilon_k \nabla v_k = \varepsilon_k \nabla y_k - \varepsilon_k \nabla \bar{y}_k \overset{\text{A}}{\rightarrow} \nabla z v \quad \text{weakly two-scale in } L^2,
\]

In particular, by Lemma 3.10(1), \( \{\varepsilon_k \chi_k^0 \nabla y_k P_k^{-1}\} \) is bounded in \( L^2(\Omega;\mathbb{R}^{3\times 3}) \). By applying Hölder’s inequality to the right-hand side of (5.16), we then find \( R_k = O(\varepsilon_k) \). Owing to (5.15) we conclude that (5.13) and (5.14) hold.

To complete the proof, we are only left to establish the existence of the sequence \( \{w_k\} \). Upon extraction of a subsequence, which we do not relabel, we may assume that in (5.13) the lower limit involving \( J_k^0 \) is a limit. From the equiboundedness of the energy, by arguing as in the lines before (5.9), we get

\[
\|\varepsilon_k \nabla y_k\|_{L^2} \leq C, \quad \|\chi_k^1 \nabla y_k\|_{L^2} \leq C.
\]

(5.17)

Then, (5.3) holds and, by Lemma 3.10(2), we obtain

\[
\varepsilon_k \nabla y_k \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}).
\]

Lemma 3.5 applied to the sequence \( \{\varepsilon_k \nabla y_k\} \) yields two sequences, \( \{k_j\} \) and \( \{u_j\} \subset W^{1,2}(\Omega;\mathbb{R}^3) \), such that \( \{\varepsilon_{k_j} \nabla u_j\} \) is 2-equiintegrable,

\[
\varepsilon_{k_j} \nabla u_j \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}),
\]

(5.18)

\[
\lim_{j \to +\infty} L^3(N_j) = 0, \quad \text{with } N_j := \{x \in \Omega : y_{k_j}(x) \neq u_j(x)\}.
\]

Besides, we have

\[
\varepsilon_{k_j} \chi_{k_j}^1 \nabla u_j \rightharpoonup 0 \quad \text{strongly in } L^2(\Omega;\mathbb{R}^{3\times 3}).
\]

(5.19)
Indeed, it holds
\[ \|\varepsilon_{k_j} \chi_{k_j} \nabla u_j\|_2^2 = \|\varepsilon_{k_j} \chi_{k_j} \nabla u_j\|_{L^2(N_j)}^2 + \|\varepsilon_{k_j} \chi_{k_j} \nabla y_{k_j}\|_{L^2(\Omega \setminus N_j)}^2 \]
\[ \leq \|\varepsilon_{k_j} \nabla u_j\|_{L^2(N_j)} + \varepsilon_{k_j} \|\chi_{k_j} \nabla y_{k_j}\|_2, \]
and the conclusion follows by the 2-equiintegrability of \{\varepsilon_{k_j} \nabla u_j\} and from (5.17).

We now define \( \tilde{u}_j := T_{k_j} u_j \), with \( T_{k_j} \) as in Lemma 3.6. From Remark 3.7 it follows that \{\varepsilon_{k_j} \nabla \tilde{u}_j\} is 2-equiintegrable as well. Thus, the sequence defined by
\[ w_k := \begin{cases} u_j - \tilde{u}_j & \text{if } k = k_j \text{ for some } j \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases} \]
has the properties that \( w_k \in W^{1,2}_0(\Omega_j; \mathbb{R}^3) \) and \{\varepsilon_{k} \nabla w_k\} is 2-equiintegrable. Moreover,
\[ \varepsilon_{k} \nabla w_k \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega; \mathbb{R}^{3 \times 3}). \]
To see this, we write
\[ \varepsilon_{k_j} \nabla w_{k_j} = \varepsilon_{k_j} \nabla u_j - \varepsilon_{k_j} \nabla \tilde{u}_j. \]
The first term converges to 0 weakly in \( L^2(\Omega; \mathbb{R}^{3 \times 3}) \), as stated in (5.18). Additionally, Lemma 3.6 entails
\[ \|\varepsilon_{k_j} \nabla \tilde{u}_j\|_2 \leq c \|\varepsilon_{k_j} \chi_{k_j} \nabla u_j\|_2, \]
and the weak convergence of \{\varepsilon_{k} \nabla w_k\} follows from (5.19).

We are now ready to prove the validity of (5.13) when \{\varepsilon_{k} \nabla v_k\} is replaced by the 2-equiintegrable sequence \{\varepsilon_{k} \nabla w_k\}. By the definition of the sequence at stake, we have
\[ \varepsilon_{k_j}(\nabla v_{k_j} - \nabla w_{k_j}) = \varepsilon_{k_j}(\nabla \tilde{y}_{k_j} - \nabla \tilde{u}_j) - \varepsilon_{k_j}(\nabla y_{k_j} - \nabla u_j) \quad \text{a. e. in } \Omega. \quad (5.20) \]
Lemma 3.6 yields
\[ \varepsilon_{k_j}\|\nabla \tilde{y}_{k_j} - \nabla \tilde{u}_j\|_{L^2}^2 = \varepsilon_{k_j}\|\nabla (T_{k_j} (y_{k_j} - u_j))\|_{L^2}^2 \]
\[ \leq c \varepsilon_{k_j}\|\chi_{k_j} \nabla (y_{k_j} - u_j)\|_{L^2}^2 \]
\[ = c \varepsilon_{k_j}\|\chi_{k_j} (\nabla y_{k_j} - \nabla u_j)\|_{L^2(N_j)} \]
\[ \leq c \left( \varepsilon_{k_j}\|\chi_{k_j} \nabla y_{k_j}\|_{L^2} + \|\varepsilon_{k_j} \nabla u_j\|_{L^2(N_j)} \right). \]
Thus, (5.17) and the 2-equiintegrability of \{\varepsilon_{k_j} \nabla u_j\} entail
\[ \varepsilon_{k_j} (\nabla \tilde{y}_{k_j} - \nabla \tilde{u}_j) \rightharpoonup 0 \quad \text{strongly in } L^2(\Omega; \mathbb{R}^{3 \times 3}). \quad (5.21) \]
Therefore, using (5.20) and the fact that the densities \( W^0_{k_j} \) are bounded from below, we have
\[ \int_{\Omega} \chi_{k_j}(x) W^0_{k_j}(\varepsilon_{k_j} \nabla v_{k_j}(x) P^{-1}_{k_j}(x)) \, dx \]
\[ = \int_{N_j} \chi_{k_j}(x) W^0_{k_j}(\varepsilon_{k_j} \nabla v_{k_j}(x) P^{-1}_{k_j}(x)) \, dx \]
\[ + \int_{\Omega \setminus N_j} \chi_{k_j}(x) W^0_{k_j} \left( (\varepsilon_{k_j} \nabla w_{k_j}(x) - \varepsilon_{k_j} (\nabla \tilde{y}_{k_j}(x) - \nabla \tilde{u}_j(x))) P^{-1}_{k_j}(x) \right) \, dx \]
\[ - \int_{\Omega \setminus N_j} \chi_{k_j}(x) W^0_{k_j} (\varepsilon_{k_j} \nabla w_{k_j}(x) P^{-1}_{k_j}(x)) \, dx + \int_{\Omega \setminus N_j} \chi_{k_j}(x) W^0_{k_j} (\varepsilon_{k_j} \nabla w_{k_j}(x) P^{-1}_{k_j}(x)) \, dx \]
\[ \geq -c \left( \int_{\Omega \setminus N_j} |\varepsilon_{k_j} (\nabla y_{k_j}(x) - \nabla \tilde{u}_j(x))|^2 \, dx \right)^{1/2} + \int_{\Omega \setminus N_j} \chi_{k_j}(x) W^0_{k_j} (\varepsilon_{k_j} \nabla w_{k_j}(x) P^{-1}_{k_j}(x)) \, dx, \]
where the Lipschitz regularity $E_5$ and Hölder’s inequality were employed to derive the last bound (recall that $\sup_{k \in \mathbb{N}} \| P_k^{-1} \|_\infty \leq C$). We now take the limit in the inequality above. According to Lemma 5.2, the hardening term has a limit. Therefore, also the elastic contribution is convergent, and it satisfies
\[
\lim_{k \to +\infty} \mathcal{J}^0_k(v_k, P_k) = \lim_{j \to +\infty} \int_{\Omega} \chi_{k_j}(x) W_0^0 \left( \varepsilon_{k_j} \nabla v_{k_j}(x) P_{k_j}^{-1}(x) \right) \, dx + \mathcal{L}^3(Q^0) \int_{\Omega} H(P(x)) \, dx.
\]
The strong converge (5.21) implies
\[
\lim_{j \to +\infty} \int_{\Omega} \chi_{k_j}(x) W_0^0 \left( \varepsilon_{k_j} \nabla v_{k_j}(x) P_{k_j}^{-1}(x) \right) \, dx \geq \liminf_{j \to +\infty} \int_{\Omega \setminus N_j} \chi_{k_j}(x) W_0^0 \left( \varepsilon_{k_j} \nabla w_{k_j}(x) P_{k_j}^{-1}(x) \right) \, dx
\]
\[
= \liminf_{j \to +\infty} \int_{\Omega} \chi_{k_j}(x) W_0^0 \left( \varepsilon_{k_j} \nabla w_{k_j}(x) P_{k_j}^{-1}(x) \right) \, dx,
\]
where the equality follows from the growth condition $E_4$ and from the equiintegrability of $\{ \varepsilon_{k_j} \nabla w_{k_j} \}$. We thereby infer
\[
\liminf_{k \to +\infty} \mathcal{J}^0_k(w_k, P_k) \leq \liminf_{j \to +\infty} \mathcal{J}^0_k(w_{k_j}, P_{k_j}) \leq \lim_{k \to +\infty} \mathcal{J}^0_k(v_k, P_k),
\]
and this concludes the proof. \(\square\)

6. $\Gamma$-LIMIT OF THE SOFT COMPONENT

We devote this section to the study of the asymptotics of the functional $\mathcal{J}^0_\varepsilon$ in (2.12), which encodes the energy of the inclusions. After some observations on the limiting functional $\mathcal{J}^0_\varepsilon$ in (2.7), in the second and third subsections we deal respectively with the lower and with the upper limit inequality for the elastic part of the energy. The other contributions will be taken into account in Subsection 6.4, where we prove Proposition 2.11.

6.1. The limiting functional. The definition of $Q'W^0$ in (2.9), which encodes the limiting elastic contribution of the soft inclusions, may be regarded as a variant of the well known Dacorogna’s formula for the quasiconvex envelope [21, Theorem 6.9]. As such, the infimum in (2.9) does not depend on $Q$, and we may rewrite $Q'W^0$ as follows:
\[
Q'W^0(F, G) = \inf \left\{ \int_{Q^0} W^0 \left( (F + \nabla v(z)) G \right) \, dz : v \in W^{1,2}(Q^0; \mathbb{R}^3) \right\}.
\]

Note that here quasiconvexification occurs just with respect to the first argument, since a very strong convergence is considered for the second one. The fact that different variables in a problem may call for different relaxation procedures has been already observed. As an example, we mention the concept of cross-quasiconvexity introduced by Le Dret & Raoult [43] to deal with dimension reduction problems in elasticity.

For the sake of completeness, we explicitly mention some basic properties of $Q'W^0$.

**Lemma 6.1.** Let $W^0 : \mathbb{R}^{3 \times 3} \to \mathbb{R}$, and assume there exist $0 < c_1 \leq c_2$ such that for all $F \in \mathbb{R}^{3 \times 3}$
\[
c_1 |F|^2 \leq W^0(F) \leq c_2 \left( |F|^2 + 1 \right).
\]
Let $Q'W^0$ be as in (2.9).
Suppose further that there exists \( c > 0 \) such that for all \( F_1, F_2 \in \mathbb{R}^{3 \times 3} \)
\[
\left| Q'W^0(F_1, G) - Q'W^0(F_2, G) \right| \leq c (1 + |F_1| + |F_2|) |F_1 - F_2|.
\]
(6.2)

Proof. The growth conditions on \( Q'W^0 \) are an immediate consequence of the ones on \( W^0 \) and of the definition of \( Q'W^0 \).

For what concerns the 2-Lipschitz property, let us set \( W^0_G(F) := W^0(FG) \) for any fixed \( G \in \mathbb{R}^{3 \times 3} \). Then, \( Q'W^0(\cdot, G) \) coincides with the quasiconvex envelope of \( W^0_G \). By [21, Remark 5.3(iii)] it follows that \( Q'W^0(\cdot, G) \) is separately convex, and hence, in view of the growth assumptions on \( W^0 \), the proof of item (1) is concluded by [21, Proposition 2.32].

As for point (2), let \( G_k \to G \) in \( \mathbb{R}^{3 \times 3} \). In view of (6.2), for every \( \delta > 0 \) there exists \( c_\delta > 0 \) such that
\[
Q'W^0(F; G_k) - Q'W^0(F; G) \leq c_\delta |G_k - G| + \delta.
\]

Similarly, for any \( k \in \mathbb{N} \) there exists \( v_k \in W^{1,p}_0(Q; \mathbb{R}^{3 \times 3}) \) such that
\[
Q'W^0(F, G_k) - Q'W^0(F, G) \geq -c_3 |G_k - G| \int_Q (1 + |(F + \nabla v_k)G_k| + |(F + \nabla v_k)G|)|F + \nabla v_k| dx - \frac{1}{k}.
\]

Thanks to the coercivity of the integrand, it follows that \( \{\nabla v_k\} \) is bounded in \( L^2 \), whence
\[
Q'W^0(F, G_k) - Q'W^0(F, G) \geq -c |G_k - G| - \frac{1}{k}
\]
for a constant \( c \) independent of \( k \). The continuity of \( Q'W^0(F, \cdot) \) is then proved by letting first \( k \to +\infty \) and then \( \delta \to 0 \).

Eventually, taking into account points (1) and (2), as well as the compact embedding of \( W^{1,q}(\Omega) \) into \( C(\Omega) \), we can employ the dominated convergence theorem to obtain the continuity property in (3). \( \square \)

We now exhibit an alternative expression for the soft limiting elastic energy, which is to be exploited in the proof of Proposition 6.7.

**Lemma 6.2.** For every couple \( (V, P) \in L^2(\Omega; \mathbb{R}^{3 \times 3}) \times W^{1,q}(\Omega; \text{SL}(3)) \) we have
\[
\int_{\Omega} Q'W^0(V(x), P^{-1}(x)) dx = \inf \left\{ \int_{\Omega} \int_{Q^0} W^0((V(x) + \nabla_z w(x, z)) P^{-1}(x)) dz : w \in L^2(\Omega; W^{1,2}_0(Q^0; \mathbb{R}^3)) \right\}.
\]
(6.3)
The identity above rests on a measurable selection criterion that we recall next.

**Lemma 6.3** (Lemma 3.10 in [29]). Let $S$ be a multifunction defined on the measurable space $X$ and taking values in the collection of subsets of the separable metric space $Y$. If $S(x)$ is nonempty and open in $Y$ for every $x \in X$, and if the set $\{ x \in X : y \in S(x) \}$ is measurable for every $y \in Y$, then $S$ admits a measurable selection, that is, there exists a measurable function $s: X \to Y$ such that $s(x) \in S(x)$ for all $x \in X$.

The previous lemma is a variant of [13, Theorem III.6], and we refer to that monograph for a comprehensive treatment of measurable selection principles.

**Proof of Lemma 6.2.** The argument follows the one proposed in [29, Corollary 3.2].

Let us fix $w \in L^2(\Omega; W^{1,2}_0(Q^0; \mathbb{R}^3))$, so that, for almost every $x \in \Omega$, $w(x, \cdot) \in W^{1,2}_0(Q^0; \mathbb{R}^3)$. Hence, according to (6.1), we have

$$\mathcal{Q} W^0(V(x), P^{-1}(x)) \leq \int_{Q^0} W^0 \left( (V(x) + \nabla v(x, z)) P^{-1}(x) \right) dz \quad \text{for a.e. } x \in \Omega,$$

whence, after integration over $\Omega$, we deduce that in (6.3) the left-hand side is smaller that the right-hand one.

In order to establish the opposite inequality, we first observe that, by (6.1), for every $x \in \Omega$ and every $\delta > 0$ there exists $v_{x,\delta} \in W^{1,2}_0(Q^0; \mathbb{R}^3)$ such that

$$\int_{Q^0} W^0 \left( (V(x) + \nabla v_{x,\delta}(z)) P^{-1}(x) \right) dz - \mathcal{Q} W^0(V(x), P^{-1}(x)) < \delta. \quad (6.4)$$

We introduce the multifunction $S$ defined for $x \in \Omega$ by

$$S(x) := \left\{ v \in W^{1,2}_0(Q^0; \mathbb{R}^3) : (6.4) \text{ holds for } v_{x,\delta} = v \right\}.$$

We show that it admits a measurable selection. To this purpose, observe that, as a consequence of the growth assumptions on $W^0$ and of the dominated convergence theorem, $S(x)$ is a nonempty, open subset of $W^{1,2}_0(Q^0; \mathbb{R}^3)$ for every $x \in \Omega$. Second, for every $v \in W^{1,2}_0(Q^0; \mathbb{R}^3)$ the set $\{x \in \Omega : v \in S(x)\}$ is measurable, because it is the sublevel set of a measurable function.

Thanks to Lemma 6.3, for every $\delta > 0$ we retrieve a measurable function $w_\delta: \Omega \to W^{1,2}_0(Q^0; \mathbb{R}^3)$ that satisfies

$$\int_{\Omega} \int_{Q^0} W^0 \left( (V(x) + \nabla w_\delta(x, z)) P^{-1}(x) \right) dz dx \leq \int_{\Omega} \mathcal{Q} W^0(V(x), P^{-1}(x)) + O(\delta).$$

In particular, by the growth conditions on $W^0$, $w_\delta$ must belong to $L^2(\Omega; W^{1,2}_0(Q^0; \mathbb{R}^3))$. Therefore, since $\delta$ is arbitrary, we conclude that the left-hand side in (6.3) bounds from above the right-hand one. \hfill \Box

### 6.2. Lower bound for the elastic energy.

The goal of this subsection is to prove the ensuing:

**Proposition 6.4.** Let $\{ W^0_k \}_{k}$ satisfy assumptions E4–E6, and let $P \in W^{1,q}(\Omega; \text{SL}(3))$. For every sequence $\{(\ve_k, P_k)\} \subset W^{1,2}_0(\Omega_k; \mathbb{R}^3) \times W^{1,q}(\Omega; \text{SL}(3))$ such that $\{ \ve_k \nabla v_k \}$ is $2$-equiintegrable and $P_k \to P$ uniformly, it holds

$$L^3(Q^0) \int_{\Omega} \mathcal{Q} W^0(0, P^{-1}(x)) dx \leq \liminf_{k \to +\infty} \int_{\Omega} \chi_{k}^0(x) W_k^0(\ve_k \nabla v_k(x) P_k^{-1}(x)) dx. \quad (6.5)$$
At a first glance, it may look bizarre that no convergence for the sequence \( \{ \varepsilon_k \nabla v_k \} \) is prescribed. The statement becomes clearer once we recall that if \( Qf \) is the quasiconvex envelope of \( f : \mathbb{R}^{3 \times 3} \to \mathbb{R} \), then

\[
Qf(0) \leq \int_{\Omega} f(\nabla v(x)) \, dx
\]

for any \( v \in W^{1,\infty}_0(\Omega; \mathbb{R}^3) \).

In order to establish (6.5), it is convenient to unfold the elastic energy.

**Lemma 6.5.** Let \( \{ W_k^0 \}_k \) satisfy assumptions E4–E6. For any \( (v, P) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; \text{SL}(3)) \) it holds

\[
\int_{\Omega} \chi_k(x) W_k^0(\varepsilon_k \nabla v(x) P^{-1}(x)) \, dx = \sum_{t \in T_k} \int_{\varepsilon_k(t+Q)} W_k^0(\nabla z \hat{v}(x, z) \hat{P}^{-1}(x, z)) \, dz \, dx \tag{6.6}
\]

where \( \hat{v} := S_k v, \hat{P} := S_k P \) and \( S_k := S_{\varepsilon_k} \) is the unfolding operator introduced in Lemma 3.11.

**Proof.** According to the definition of \( \Omega_k^0 \) in (2.4), the left-hand side of (6.6) equals

\[
\varepsilon_k^3 \sum_{t \in T_k} \int_{Q_0^0} W_k^0(\varepsilon_k \nabla v(\varepsilon_k(t + z)) P^{-1}(\varepsilon_k(t + z))) \, dz.
\]

We use the unfolding operator to rewrite this quantity as a double integral. Recalling Lemma 3.11, we firstly observe that for every \( t \in T_k \) and \( z \in Q_0^0 \) we have the identities

\[
S_k(\varepsilon_k \nabla v)(\varepsilon_k t, z) = \varepsilon_k \nabla v(\varepsilon_k(t + z)), \quad S_k P^{-1}(\varepsilon_k t, z) = P^{-1}(\varepsilon_k(t + z)).
\]

Then, we also have

\[
S_k(\varepsilon_k \nabla v) = \nabla \hat{v}(S_k v) = \nabla \nabla \hat{v}, \quad S_k P^{-1} = (S_k P)^{-1} = \hat{P}^{-1}.
\]

We obtain

\[
\int_{\Omega} \chi_k(x) W_k^0(\varepsilon_k \nabla v(x) P^{-1}(x)) \, dx
\]

\[
= \varepsilon_k^3 \sum_{t \in T_k} \int_{Q_0^0} W_k^0(\varepsilon_k \nabla v)(S_k(\varepsilon_k \nabla v) P^{-1}(\varepsilon_k(t, z))) \, dz
\]

\[
= \sum_{t \in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q_0^0} W_k^0(\nabla z \hat{v}(S_k \varepsilon_k(t, z), z) P^{-1}(S_k \varepsilon_k(t, z), z)) \, dz \, dx,
\]

because \( x/\varepsilon_k = t \) for all \( x \in \varepsilon_k(t + Q) \). Since, in general, it holds

\[
S_k u \left( \varepsilon_k \left( S_k \varepsilon_k \left( \frac{x}{\varepsilon_k} \right), z \right) = u \left( \frac{x}{\varepsilon_k} + \varepsilon_k z \right) = S_k u(x, z),
\]

(6.6) follows.  \( \square \)

A crucial ingredient in the proof of Proposition 6.4 is a sort of lower semicontinuity result for the elastic contribution to the energy.

**Lemma 6.6.** Let \( \{ W_k^0 \}_k \) satisfy assumptions E4–E6. Let also \( \{ w_k \} \subset L^2(\Omega; W^{1,2}_0(Q_0; \mathbb{R}^3)) \) be such that \( \{ \nabla_z w_k \} \) is 2-equiintegrable. Then, for all \( P \in W^{1,q}(\Omega; \text{SL}(3)) \),

\[
L^2(Q_0^0) \int_{\Omega} Q' W^0(0, P^{-1}(x)) \, dx \leq \liminf_{k \to +\infty} \int_{\Omega} \int_{Q_0^0} W_k^0(\nabla z w_k(x, z) P_k^{-1}(x)) \, dz \, dx,
\]

whenever \( P_k \to P \) uniformly.
Proof. From (6.1) it follows that for all $k \in \mathbb{N}$

$$
\mathcal{L}^3(Q^0) \int_{\Omega} \mathcal{Q}'W^0(0, P_k^{-1}(x)) \, dx \leq \int_{\Omega} \int_{Q^0} W^0(\nabla_z w_k(x, z) P_k^{-1}(x)) \, dz \, dx. \tag{6.7}
$$

Next, relying on the pointwise convergence of $\{W_k^0\}$ to $W^0$, we adapt the argument in the proof of [23, Theorem 5.14] to pass from $W^0$ to $W_k^0$ on the right-hand side (see also [27, Lemma 5.2] for a similar result in the context of $\mathcal{A}$-quasiconvexity). Fix $\delta > 0$. If $\{\nabla_z w_k\}$ is 2-equiintegrable, then so is $\{\nabla_z w_k P_k^{-1}\}$. Therefore, since the 2-growth assumptions on $\{W_k^0\}$ transfer to the pointwise limit $W^0$, there exists $r > 0$ such that

$$
\sup_{k \in \mathbb{N}} \int_{\{(x, z) \in \Omega \times Q^0 : |\nabla_z w_k(x, z) P_k^{-1}(x)| > r\}} W^0(\nabla_z w_k(x, z) P_k^{-1}(x)) \, dz \, dx \leq \delta. \tag{6.8}
$$

Owing to assumption E5 and Remark 2.3, we can find $\rho > 0$ such that for every $F, G \in \mathbb{R}^{3 \times 3}$ contained in the open ball $B(0, \rho)$

$$
\sup_{k \in \mathbb{N}} |W_k^0(F) - W_k^0(G)| + |W^0(F) - W^0(G)| \leq \delta. \tag{6.9}
$$

Let now $F_1, \ldots, F_n \in B(0, r)$ be such that

$$
\overline{B(0, r)} \subset \bigcup_{i=1}^{n} B(F_i, \rho). \tag{6.10}
$$

Due to the pointwise convergence of $W_k^0$ to $W^0$, for any such $F_i$ there exist $\tilde{k}_i \in \mathbb{N}$ such that $|W_k^0(F_i) - W^0(F_i)| \leq \delta$ if $k \geq \tilde{k}_i$. Letting $k := \max\{\tilde{k}_1, \ldots, \tilde{k}_n\}$, it follows that for any $i = 1, \ldots, n$

$$
|W_k^0(F_i) - W^0(F_i)| \leq \delta \quad \text{if } k > \tilde{k}. \tag{6.11}
$$

By (6.10), for every $G \in \overline{B(0, r)}$ there exists $i \in \{1, \ldots, n\}$ such that $G \in B(F_i, \rho)$. For this particular $i$, the combination of the triangle inequality, (6.9) and (6.11) yields

$$
|W_k^0(G) - W^0(G)| \leq |W_k^0(G) - W_k^0(F_i)| + |W_k^0(F_i) - W^0(F_i)| + |W^0(G) - W^0(F_i)| \leq 3\delta, \tag{6.12}
$$

for every $G \in \overline{B(0, r)}$ and every $k > \tilde{k}$.

Thanks to Lemma 6.1(3) and (6.7) we deduce

$$
\mathcal{L}^3(Q^0) \int_{\Omega} \mathcal{Q}'W^0(0, P^{-1}(x)) \, dx
= \mathcal{L}^3(Q^0) \lim_{k \to +\infty} \int_{\Omega} \mathcal{Q}'W^0(0, P_k^{-1}(x)) \, dx
\leq \lim \inf_{k \to +\infty} \int_{\Omega} \int_{Q^0} W^0(\nabla_z w_k(x, z) P_k^{-1}(x)) \, dz \, dx
\leq \lim \inf_{k \to +\infty} \int_{\{(x, z) \in \Omega \times Q^0 : |\nabla_z w_k(x, z) P_k^{-1}(x)| \leq r\}} W^0(\nabla_z w_k(x, z) P_k^{-1}(x)) \, dz \, dx + \delta
\leq \lim \inf_{k \to +\infty} \int_{\Omega} \int_{Q^0} W_k^0(\nabla_z w_k(x, z) P_k^{-1}(x)) \, dz \, dx + 3\delta \mathcal{L}^3(\Omega \times Q^0) + \delta,
$$

where the second inequality is due to (6.8), and the last one to (6.12). The arbitrariness of $\delta > 0$ yields the conclusion. \hfill \Box

We are now ready to prove the lower bound for the elastic contribution of the soft part.
Proof of Proposition 6.4. Let \( \hat{v}_k := S_k v_k \). In view of the 2-equiintegrability of the sequence \( \{\varepsilon_k \nabla v_k\} \) and of Lemma 3.11, \( \{\nabla_z \hat{v}_k\} \) is 2-equiintegrable as well. Hence it is a fortiori bounded in \( L^2 \). From Lemma 6.5, restricting the summation in (6.6) to the set of translations in (5.11), we deduce

\[
\liminf_{k \to +\infty} \int_{\Omega} \chi_k^0(x) W_k^0(\varepsilon_k \nabla v_k(x) P_k^{-1}(x)) \, dx \geq \liminf_{k \to +\infty} \int_{\Omega} \int_{Q^0} W_k^0(\nabla_z \hat{v}_k(x, z) P_k^{-1}(x)) \, dz \, dx,
\]

where

\[
\Omega_k^Q := \bigcup_{t \in T_k} \varepsilon_k(t + Q). \tag{6.13}
\]

We rewrite the right-hand side of the previous inequality as the difference between the integrals

\[
I_k' := \int_{\Omega} \int_{Q^0} W_k^0(\nabla_z \hat{v}_k(x, z) P_k^{-1}(x)) \, dz \, dx,
\]

\[
I_k'' := \int_{\Omega \setminus \Omega_k^Q} \int_{Q^0} W_k^0(\nabla_z \hat{v}_k(x, z) P_k^{-1}(x)) \, dz \, dx.
\]

Being \( \{\nabla_z \hat{v}_k\} \) 2-equiintegrable, the sequence \( \{\nabla_z \hat{v}_k P_k^{-1}\} \) is still 2-equiintegrable. Thus, by the growth condition \( E_4 \), we obtain

\[
\lim_{k \to +\infty} I_k'' = 0.
\]

Taking into account Lemma 6.6 we conclude

\[
\liminf_{k \to +\infty} \int_{\Omega} \chi_k^0(x) W_k^0(\varepsilon_k \nabla v_k(x) P_k^{-1}(x)) \, dx \geq \liminf_{k \to +\infty} I_k' \geq \mathcal{L}^3(Q^0) \int_{\Omega} Q' W^0(0, P^{-1}(x)) \, dx.
\]

\[\square\]

6.3. Upper bound for the elastic energy. In this subsection we address the proof of \( \Gamma \)-upper limit inequality for the elastic contribution of the soft component. Differently from the previous subsection, in order to establish the desired inequality we perform an analysis that is genuinely two-scale, in the sense that we interpret 0 as the average with respect to the periodic variable of the two-scale limit of the sequence \( \{\varepsilon_k \nabla v_k\} \).

Proposition 6.7. Let \( \{W^0_k\}_k \) satisfy assumptions \( E_4 - E_6 \), and let \( P \in W^{1,q}(\Omega; SL(3)) \). For all \( \delta > 0 \) there exists a sequence \( \{v_k\} \subset W^{1,2}_0(\Omega_k^0; \mathbb{R}^3) \) such that \( \varepsilon_k \nabla v_k \to 0 \) weakly in \( L^2(\Omega; \mathbb{R}^{3 \times 3}) \) and that

\[
\limsup_{k \to +\infty} \int_{\Omega} \chi_k^0(x) W_k^0(\varepsilon_k \nabla v_k(x) P_k^{-1}(x)) \, dx < \mathcal{L}^3(Q^0) \int_{\Omega} Q' W^0(0, P^{-1}(x)) \, dx + \delta, \tag{6.14}
\]

whenever \( P_k \to P \) uniformly.

We begin with a lemma that provides a strong two-scale approximation of any sufficiently regular function. The result has already appeared in [14], where, however, the proof is just sketched. In order to keep the exposition self-contained, we include it in the Appendix, where we also compare our result with the one in [14].

Lemma 6.8. Let \( w \in L^2(\Omega; W^{1,2}_0(Q^0; \mathbb{R}^3)) \cap C^2(\Omega \times Q^0; \mathbb{R}^3) \). Then, there exists a sequence \( \{v_k\} \subset L^2(\Omega; \mathbb{R}^3) \) such that, letting \( \hat{v}_k := S_k v_k \), it holds

\[
\nabla_z \hat{v}_k \to \nabla_z w \quad \text{strongly in } L^2(\Omega \times Q; \mathbb{R}^{3 \times 3}). \tag{6.15}
\]

We are now ready to prove the \( \Gamma \)-lmsup inequality for the soft inclusions functional.
Proof of Proposition 6.7. According to Lemma 6.2, for every δ > 0 there exists \( w_δ \in L^2(Ω; W^{1,2}_0(Q_0; \mathbb{R}^3)) \) satisfying

\[
\int_Ω \int_{Q_0} W^0(\nabla_z w_δ(x, z) P^{-1}(x)) \, dz \, dx < \mathcal{L}(Q_0) \int_Ω W^0(0, P^{-1}(x)) \, dx + \delta \tag{6.16}
\]

We would like to apply Lemma 6.8 which, however, requires \( w_δ \in L^2(Ω; W^{1,2}_0(Q_0; \mathbb{R}^3)) \cap C^2(Ω \times Q_0; \mathbb{R}^3) \). We therefore establish the inequality first for a sufficiently regular \( w_δ \), and then we extend the result by a density argument.

**Case 1: \( w_δ \) regular**

Let \( w_δ \in L^2(Ω; W^{1,2}_0(Q_0; \mathbb{R}^3)) \cap C^2(Ω \times Q_0; \mathbb{R}^3) \). We consider the recovery sequence \( \{v_k\} \) coming from Lemma 6.8. Lemmas 3.11 and 3.10(2) yield \( \varepsilon_k \nabla v_k \rightharpoonup 0 \) weakly in \( L^2(Ω; \mathbb{R}^{3×3}) \).

We consider the recovery sequence \( \{v_k\} \) coming from Lemma 6.8. Lemmas 3.11 and 3.10(2) yield \( \varepsilon_k \nabla v_k \rightharpoonup 0 \) weakly in \( L^2(Ω; \mathbb{R}^{3×3}) \).

Assumption E5 and Hölder’s inequality entail

\[
\sum_{t \in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q_0} \left| W^0_k \left( \nabla_z \hat{v}_k(x, z) P^{-1}(x) \right) - W^0_k \left( \nabla_z w_δ(x, z) P^{-1}(x) \right) \right| \, dz \, dx 
\]

\[
\leq c \sum_{t \in T_k} \left( \int_{\varepsilon_k(t+Q)} \int_{Q_0} |\nabla_z \hat{v}_k(x, z) - \nabla_z w_δ(x, z)|^2 \, dz \, dx \right)^{1/2},
\]

where the constant \( c \) bounds \( \|P^{-1}\|_{L^∞} \). Thanks to the strong convergence of \( \{\nabla_z \hat{v}_k\} \), we obtain that the term above is infinitesimal when \( k \to +∞ \). From Lemma 6.5 we then deduce

\[
\limsup_{k \to +∞} \int_Ω \rho_k^0(x) W^0_k \left( \varepsilon_k \nabla v_k(x) P^{-1}(x) \right) \, dx 
\]

\[
= \limsup_{k \to +∞} \sum_{t \in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q_0} W^0_k \left( \nabla_z w_δ(x, z) P^{-1}(x) \right) \, dz \, dx 
\]

\[
= \limsup_{k \to +∞} \sum_{t \in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q_0} W^0_k \left( \nabla_z w_δ(x, z) P^{-1}(x) \right) \, dz \, dx 
\]

\[
= \limsup_{k \to +∞} \sum_{t \in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q_0} W^0_k \left( \nabla_z w_δ(x, z) P^{-1}(x) \right) \, dz \, dx,
\]

where the second identity follows from E5 and the last one from E6. Note also that, by absolute continuity of the Lebesgue integral,

\[
= \int_Ω \int_{Q_0} W^0 \left( \nabla_z w_δ(x, z) P^{-1}(x) \right) \, dz \, dx.
\]

Therefore, by combining the equalities that we have just found with (6.16), we achieve the conclusion in the case under consideration.

**Case 2: \( w_δ \) generic**

Let now \( w_δ \in L^2(Ω; W^{1,2}_0(Q_0; \mathbb{R}^3)) \). By mollification, we retrieve a function \( \tilde{w}_δ \in L^2(Ω; W^{1,2}_0(Q_0; \mathbb{R}^3)) \cap C^2(Ω \times Q_0; \mathbb{R}^3) \) such that

\[
\int_Ω \int_{Q_0} W^0 \left( \nabla_z \tilde{w}_δ(x, z) P^{-1}(x) \right) \, dz \leq \int_Ω \int_{Q_0} W^0 \left( \nabla_z w_δ(x, z) P^{-1}(x) \right) \, dz + \delta.
\]

To achieve the conclusion, it only suffices to repeat the argument in Case 1 for \( \tilde{w}_δ \). □
6.4. **Proof of Proposition 2.11.** We are eventually in a position to reap the fruits of the previous subsections.

**Proof of Proposition 2.11.** Let us start with the lower limit inequality. If the lower limit of \( J^0_k(v_k, P_k) \) is not finite, there is nothing to prove. Otherwise, recalling Lemma 3.10(4), we deduce that \( \varepsilon_k \nabla v_k \overset{2}{\rightharpoonup} \nabla_z \tilde{v} \) weakly two-scale in \( L^2 \) for some \( \tilde{v} \in L^2(\Omega; W^{1,2}_{\text{per}}(\mathbb{R}^3; \mathbb{R}^3)) \). In particular, by Lemma 3.10(2), it must be

\[
\nabla v(x) = \int_Q \nabla_z \tilde{v}(x, z) \, dz = 0 \quad \text{for a.e. } x \in \Omega,
\]

whence, being \( \Omega \) connected, must \( v \) be identically zero. Statement (1) in Proposition 2.11 then follows by combining Proposition 6.4 and Lemma 5.2.

We now turn to the upper bound. The only nontrivial case corresponds to \( v = 0 \). Proposition 6.7 provides for all \( \delta > 0 \) a sequence \( \{v_k\} \subset W^{1,2}_0(\Omega_k; \mathbb{R}^3) \) such that \( \varepsilon_k \nabla v_k \rightharpoonup 0 = \nabla v \) weakly in \( L^2(\Omega; \mathbb{R}^{4 \times 3}) \) and (6.14) holds. By the Rellich-Kondrachov theorem in \( W^{1,2}_0(\Omega; \mathbb{R}^3) \), it follows that \( \varepsilon_k v_k \to 0 \) strongly in \( L^2(\Omega; \mathbb{R}^3) \) (up to subsequences). We employ again Lemma 5.2 to deduce that

\[
\limsup_{k \to +\infty} J^0_k(v_k, P_k) < J^0(v, P) + \delta.
\]

This inequality is actually equivalent to the desired one (cf. [9, Section 1.2]), and the proof is therefore concluded. \( \square \)

7. **Conclusions and a variant**

We devote this final section to the proof of the homogenization result for high-contrast composites and to the discussion of a variant of the problem featuring plastic dissipation.

7.1. **Proof of Theorem 2.8 and convergence of minimum problems.** As we outlined before, the proof of Theorem 2.8 is achieved by combining the splitting procedure in Proposition 5.3 with Theorem 4.8 and Proposition 2.11, which account for the asymptotics of the stiff and the soft components, respectively. Once the homogenization theorem is on hand, the convergence of the minimum problems and of their minimizers will follow thanks to the compactness result in Lemma 5.1.

**Proof of Theorem 2.8.** Let \( \{\varepsilon_k\} \) be an infinitesimal sequence and let us fix \( y \in L^2(\Omega; \mathbb{R}^3) \) and \( P \in L^q(\Omega; SL(3)) \). We separate the proof of the lower and of the upper limit inequalities.

**Lower bound**

We consider a sequence \( \{(y_k, P_k)\} \subset L^2(\Omega; \mathbb{R}^3) \times L^q(\Omega; SL(3)) \) such that \( y_k \to y \) in the sense of extensions and that \( P_k \to P \) uniformly. The only case to discuss is the one in which the lower limit of \( J_k(y_k, P_k) \) is finite, and we may thus assume that \( \{J_k(y_k, P_k)\} \) is bounded. Keeping in force the notation of Definition 2.5, we let \( \{\tilde{y}_k\} \subset W^{1,2}(\Omega; \mathbb{R}^3) \) be a sequence such that \( y_k = \tilde{y}_k \) in \( \Omega^1_k \) and \( \tilde{y}_k \rightharpoonup y \) weakly in \( W^{1,2}(\Omega; \mathbb{R}^3) \). In the light of (5.4) and Remark 2.6, we may without loss of generality assume that \( \tilde{y}_k := T_k y_k \), with \( T_k \) as in Lemma 3.6.

We now apply Proposition 5.3, which yields \( \{v_k\} \subset W^{1,2}_0(\Omega^1_k; \mathbb{R}^3) \) satisfying (5.13) and such that \( \{v_k\} \) is bounded in \( L^2 \), \( \{\varepsilon_k \nabla v_k\} \) is 2-equintegrable and \( \varepsilon_k v_k \to 0 \) strongly in \( L^2 \). In particular, \( (\varepsilon_k v_k, P_k) \rightharpoonup (0, P) \) and Proposition 2.11 yields

\[
J^0(0, P) \leq \liminf_{k \to +\infty} J^0_k(v_k, P_k).
\]
At this stage, recalling (5.13), the proof of the lower bound is concluded as soon as we show that
\[
\mathcal{J}^1(y, P) \leq \liminf_{k \to +\infty} \mathcal{J}^1_k(\tilde{y}_k, P_k) = \liminf_{k \to +\infty} \mathcal{J}^1_k(y_k, P_k)
\] (7.1)
with \(\mathcal{J}(y, P)\) given by (2.8). This is what we prove next.

Let us set
\[
\tilde{W}^1(x, F) := \chi_{E^1}(x)W^1(F), \quad \tilde{H}(x, P) := \chi_{E^1}(x)H(P),
\]
\[
\tilde{J}^1_k(y, P) := \int_\Omega \left[ \tilde{W}^1 \left( \frac{x}{\varepsilon_k}, \nabla \tilde{y} P^{-1} \right) + \tilde{H} \left( \frac{x}{\varepsilon_k}, P \right) + |\nabla P|^q \right] \, dx.
\] (7.2)
It holds
\[
\liminf_{k \to +\infty} \tilde{J}^1_k(\tilde{y}_k, P_k) \leq \liminf_{k \to +\infty} \tilde{J}^1_k(y_k, P_k).
\]

Since \((\tilde{y}_k, P_k) \rightharpoonup (y, P)\), by applying Theorem 4.8 to the left-hand side of the previous inequality, (7.1) is deduced.

**Upper bound**

If \(P \notin W^{1,q}(\Omega; K)\) there is nothing to prove; let us then assume that \(P \in W^{1,q}(\Omega; K)\).

As we have already observed, \(\{\tilde{J}^1_k\}\) satisfies the requirements of Theorem 4.8. In view of Corollary 4.7, for any \((y, P) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K)\) there exists a sequence \(\{(u_k, P_k)\} \subset W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K)\) such that \(\nabla u_k\) is 2-equiintegrable, \((u_k, P_k) \rightharpoonup (y, P)\), and
\[
\limsup_{k \to +\infty} \tilde{J}^1_k(u_k, P_k) \leq \mathcal{J}^1(y, P).
\] Note that
\[
0 \leq \mathcal{J}^1_k(u_k, P_k) - \tilde{J}^1_k(u_k, P_k)
\]
\[
= \int_{\Omega} \left( \chi_k(x) - \chi_{E^1}(x) \right) (W^1(\nabla u_k P^{-1}) + H(P_k)) \, dx
\]
\[
\leq c \int_{\Omega} \left( \chi_k(x) - \chi_{E^1}(x) \right) (|\nabla u_k|^2 + 1) \, dx
\]
for all \(k \in \mathbb{N}\). Thanks to the 2-equiintegrability of \(\{\nabla u_k\}\), we deduce
\[
\limsup_{k \to +\infty} \mathcal{J}^1_k(u_k, P_k) = \limsup_{k \to +\infty} \tilde{J}^1_k(u_k, P_k) \leq \mathcal{J}^1(y, P).
\] (7.3)

We focus now on the soft part. Proposition 2.11 grants the existence of a sequence \(\{v_k\} \subset W^{1,2}_0(\Omega^0_k; \mathbb{R}^3)\) such that \(\varepsilon_k v_k \to 0\) strongly in \(L^2(\Omega; \mathbb{R}^{3 \times 3})\) and that
\[
\limsup_{k \to +\infty} \mathcal{J}^0_k(v_k, P_k) \leq \mathcal{J}^0(0, P),
\] (7.4)
where \(\{P_k\}\) is as in (7.3). Notice that if \(y_k := u_k + v_k\), then \(\{\tilde{J}_k(y_k, P_k)\}\) is bounded and \(\{y_k\}\) converges to \(y\) in the sense of extensions. Letting \(\bar{y}_k := \mathcal{T}_k y_k\), thanks to (5.14) we conclude the proof of the upper limit inequality:
\[
\limsup_{k \to +\infty} \tilde{J}_k(\bar{y}_k, P_k) \leq \limsup_{k \to +\infty} \mathcal{J}^0_k(y_k - \bar{y}_k, P_k) + \limsup_{k \to +\infty} \mathcal{J}^1_k(\bar{y}_k, P_k)
\]
\[
= \limsup_{k \to +\infty} \mathcal{J}^0_k(v_k, P_k) + \limsup_{k \to +\infty} \mathcal{J}^1_k(u_k, P_k)
\]
\[
\leq \mathcal{J}(y, P).
\]
In the previous lines, the equality is a consequence of the facts that \(\{\nabla u_k\}\) and \(\{\nabla \bar{y}_k\}\) are bounded and that \(u_k = y_k\) on \(\Omega^1_k\), whereas the last bound accounts for (7.3) and (7.4). \(\square\)
Finally, we are only left to establish the convergence of the minimum problems associated with the energy functionals $\mathcal{J}_\varepsilon$. What we need is an adaptation of the $\Gamma$-convergence statement that we have just proved so as to make it comply with Dirichlet boundary conditions. To this aim, as it is customary (see e.g. [10, Proposition 11.7]), we could employ the fundamental estimate (4.3) on the functionals $\{\hat{J}_k^1\}$ in (7.2); indeed, boundary data concern only the stiff part, cf. Remark 2.7. In the light of Corollary 4.7 we can adopt an alternative strategy.

**Proof of Corollary 2.10.** Since $\{(y_k, P_k)\}$ is a sequence of almost-minimizers, there exists $C$ such that $\mathcal{J}_k(y_k, P_k) \leq C$. The 2-growth condition from below, together with Lemma 3.8, provides a bound on $\|y_k\|_{L^2}$. By Proposition 5.1, there exists $(y, P) \in W^{1,2}_0(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K)$ such that, up to subsequences, $y_k \to y$ in the sense of extensions and $P_k \to P$ uniformly. Theorem 2.8 ensures that

$$\mathcal{J}(y, P) \leq \liminf_{k \to +\infty} \mathcal{J}_k(y_k, P_k).$$

We now prove the existence of a recovery sequence meeting the boundary conditions. As suggested by Remark 2.7, we focus on the stiff part. Let us consider again the functional $\hat{J}_k^1$ in (7.2). Since the sequence $\{\hat{J}_k^1\}$ falls within the scopes of Theorem 4.8, for any $(\hat{y}, \hat{P}) \in W^{1,2}_0(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K)$ Corollary 4.7 provides a sequence $\{(u_k, \hat{P}_k)\} \subset W^{1,2}_0(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K)$ such that $\{\nabla u_k\}$ is 2-equiintegrable, $(u_k, \hat{P}_k) \rightharpoonup (\hat{y}, \hat{P})$ and

$$\limsup_{k \to +\infty} \hat{J}_k^1(u_k, \hat{P}_k) \leq \mathcal{J}^1(y, P).$$

By reasoning as in the proof of the upper bound in Theorem 2.8 we retrieve a sequence $\{\hat{y}_k, \hat{P}_k\} \in W^{1,2}_0(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K)$ such that $\hat{y}_k \to \hat{y}$ in the sense of extensions, $\hat{P}_k \to \hat{P}$ uniformly and

$$\limsup_{k \to +\infty} \mathcal{J}_k(\hat{y}_k, \hat{P}_k) \leq \mathcal{J}(\hat{y}, \hat{P}),$$

whence

$$\limsup_{k \to +\infty} (\inf \mathcal{J}_k) \leq \inf \mathcal{J}.$$

Recalling that $\{(y_k, P_k)\}$ is a sequence of almost minimizers, we conclude

$$\inf \mathcal{J} \leq \mathcal{J}(y, P) \leq \liminf_{k \to +\infty} \mathcal{J}_k(y_k, P_k) = \liminf_{k \to +\infty} \mathcal{J}_k \leq \inf \mathcal{J},$$

as desired. \hfill \Box

**7.2. A variant with plastic dissipation.** With a view to applying Theorem 2.8 to time-dependent problems, it is useful to modify the functionals $\mathcal{J}_\varepsilon$ by adding a term that encodes the plastic dissipation mechanism of the system. Precisely, we take into account the non-symmetric distance $D: \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \to [0, +\infty]$ in (3.1) and we define the dissipation between $P_0, P_1 : \Omega \to \text{SL}(3)$ as

$$D(P_0; P_1) := \int_\Omega D(P_0, P_1) \, dx.$$ 

Then, assuming that $\tilde{P} \in W^{1,q}(\Omega; \text{SL}(3))$ represents a pre-existent plastic strain of the body, we set

$$\mathcal{J}^\text{diss}_\varepsilon(y, P) := E_\varepsilon(y, P) + D(\tilde{P}; P) + \|\nabla P\|_{L^q(\Omega, \mathbb{R}^{3 \times 3})}. \quad (7.5)$$

In the same spirit of (2.10) and (2.11), we distinguish between the dissipation of the inclusions and the one of the matrix, respectively

$$D^0_\varepsilon(\tilde{P}; P) := \int_\Omega \chi^0_\varepsilon(x) D(\tilde{P}, P) \, dx, \quad D^1_\varepsilon(\tilde{P}; P) := \int_\Omega \chi^1_\varepsilon(x) D(\tilde{P}, P) \, dx.$$
For what concerns the compactness of sequences with equibounded energy, we notice that the presence of the dissipation $D$ does not affect Lemma 5.1: the same conclusions hold if the bound on $\mathcal{J}_k(y_k, P_k)$ is replaced by a bound on $\mathcal{J}_k^{\text{diss}}(y_k, P_k)$.

Also our $\Gamma$-convergence results easily extend to the family $\{\mathcal{J}_k^{\text{diss}}\}$. The dissipation is indeed a continuous perturbation:

**Lemma 7.1.** Let $P, \bar{P} \in C(\overline{\Omega}; K)$ be given. If $\{P_k\} \subset C(\overline{\Omega}; K)$ converges uniformly to $P$, then

$$\lim_{k \to +\infty} D_k^i(\bar{P}; P_k) = \mathcal{L}^3(Q^i)D(\bar{P}; P) \quad \text{for } i = 0, 1.$$  

**Proof.** We firstly observe that if $P_k \to P$ pointwise, then

$$D(P_k(x), P(x)) \to 0, \quad D(P(x), P_k(x)) \to 0. \quad (7.6)$$

To see this, let $\gamma$ be as in Proposition 4.4, that is, for all $(t, F, G) \in [0,1] \times SL(3) \times SL(3)$ $\gamma(t, F, G)$ is the evaluation at $t$ of the unique minimizing geodesic connecting $F$ and $G$. Then, by (3.1) and the definition of $\gamma$,

$$D(P_k(x), P(x)) = \int_0^1 \Delta \left( \gamma(t, P_k(x), P(x)), \dot{\gamma}(t, P_k(x), P(x)) \right) \, dt$$

$$\leq c \int_0^1 |\dot{\gamma}(t, P_k(x), P(x))| \, dt,$$

where the inequality follows from $D2$ and (2.1). Since $\dot{\gamma}$ is continuous and bounded, by dominated convergence we deduce that the last term vanishes as $k \to +\infty$. In a similar fashion, we show that $D(P, P_k) \to 0$ as well.

As second step, we notice that

$$D(\bar{P}(x), P_k(x)) \to D(\bar{P}(x), P(x)). \quad (7.7)$$

Indeed, the triangular inequality yields

$$D(\bar{P}(x), P(x)) - D(P_k(x), P(x)) \leq D(\bar{P}(x), P_k(x)) \leq D(\bar{P}(x), P(x)) + D(P(x), P_k(x)),$$

and the assertion follows as a consequence of (7.6).

Finally, we observe that (7.7) grants that

$$\lim_{k \to +\infty} D_k^i(\bar{P}; P_k) = \lim_{k \to +\infty} \int_\Omega \chi^i(x)D(\bar{P}(x), P(x)) \, dx,$$

and the conclusion is achieved by arguing as in Lemma 5.2. \hfill \Box

**APPENDIX: A NON DEGENERATE UPPER BOUND FOR THE SOFT COMPONENT**

We proved in Section 6 that the limiting behavior of the soft inclusions is described by a degenerate functional. However, under our assumptions, a non-degenerate upper bound may still be established, as we prove in the remainder. The argument follows [14], where CHERDANTSEV & CHEREDNICHENKO derived the effective energy of high-contrast nonlinear elastic materials. Differently from us, the $\Gamma$-limit that they retrieve keeps track of both the macro- and the microscopic variable, and this roots in the choice of a stronger notion of convergence. The drawback of such an approach is the lack of compactness for sequences with equibounded energy. It was shown in [27, Example 2.12] that, when weaker topologies are considered, the quasiconvex envelope does not provide the correct limiting energy density for the $\Gamma$-lower limit.

We start by proving a more detailed version of Lemma 6.8.
Lemma A.1 (cf. Lemma 22 in [14]). Let \( w \in L^2(\Omega; W^{1,2}_0(Q^0; \mathbb{R}^3)) \cap C^2(\Omega \times Q^0; \mathbb{R}^3) \). Then, there exists a sequence \( \{w_k\} \subset L^2(\Omega; W^{1,2}_{\text{per}}(\mathbb{R}^3; \mathbb{R}^3)) \) such that \( \nabla_z w_k \to \nabla_z w \) strongly in \( L^2(\Omega \times Q; \mathbb{R}^{3\times 3}) \). Besides, setting for \( x \in \Omega \)

\[
v_k(x) := w_k\left(x, \frac{x}{\epsilon_k}\right), \tag{A.1}\]

\( \{v_k\} \) converges strongly two-scale to \( w \) in \( L^2 \) and \( (6.15) \) holds.

Proof. We extend \( w \) by setting it equal to 0 on \( Q \backslash Q^0 \), so as to obtain a function in \( L^2(\Omega; W^{1,2}_{\text{per}}(\mathbb{R}^3; \mathbb{R}^3)) \) which, by a slight abuse of notation, we denote again by \( w \).

Keeping in mind the definition of \( \Omega_k^Q \) (see (6.13)), for \((\bar{x}, \bar{z}) \in \Omega \times \mathbb{R}^3\) we define \( w_k(\bar{x}, \bar{z}) \) in terms of the averages of \( w(\cdot, \cdot) \) on the cubes that form \( \Omega_k^Q \):

\[
w_k(\bar{x}, \bar{z}) := \begin{cases} 
\int_{\varepsilon_k(t+Q)} w(x, \bar{z}) \, dx & \text{if } \bar{x} \in \varepsilon_k(t+Q) \text{ for some } t \in T_k, \\
0 & \text{for any other } \bar{x} \in \Omega.
\end{cases} \tag{A.2}
\]

By definition, \( w_k(\cdot, z) \) is piecewise constant for all \( z \in \bar{Q} \). Moreover, for almost every \( x \in \Omega \), \( w_k(x, \cdot) \) is \( Q \)-periodic as well as weakly differentiable, and \( \nabla_z w_k \to \nabla_z w \) strongly in \( L^2(\Omega \times Q; \mathbb{R}^{3\times 3}) \). Indeed, from (A.2) and Jensen’s inequality, we have that

\[
\int_{\Omega} \int_{Q} |\nabla_z w_k(x, z) - \nabla_z w(x, z)|^2 \, dz \, dx
\]

\[
= \int_{\Omega} \int_{Q} |\nabla_z w_k(x, z) - \nabla_z w(x, z)|^2 \, dz \, dx + \int_{\Omega} \int_{Q} |\nabla_z w(x, z)|^2 \, dz \, dx
\]

\[
= \sum_{t \in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q} |\nabla_z w_k(x, z) - \nabla_z w(x, z)|^2 \, dz \, dx + o(1)
\]

\[
\leq \sum_{t \in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q} \int_{\varepsilon_k(t+Q)} |\nabla_z w(\xi, z) - \nabla_z w(x, z)|^2 \, d\xi \, dz \, dx + o(1),
\]

and the last term is infinitesimal for \( k \to +\infty \) (recall that \( w \in C^2 \) and the mean value theorem applies).

We now turn to the functions \( v_k \) given by (A.1). First of all, we point out that, thanks to the regularity of \( w \), \( v_k \) is measurable and vanishes on \( \Omega_k^1 \). Besides, it belongs to \( W^{1,2}_0(\Omega_k^1; \mathbb{R}^3) \). Secondly, we show that \( \{v_k\} \) converges weakly two-scale to \( w \) in \( L^2 \). To this aim, let us fix
\( \phi \in C(\Omega; C_{per}(\mathbb{R}^3, \mathbb{R}^3)) \). We find
\[
\int_{\Omega} v_k(x) \cdot \phi \left( x, \frac{x}{\varepsilon_k} \right) \, dx = \int_{\Omega_k^0} w_k \left( x, \frac{x}{\varepsilon_k} \right) \cdot \phi \left( x, \frac{x}{\varepsilon_k} \right) \, dx
\]
\[
= \sum_{t \in T_k} \int_{\varepsilon_k(t+Q^0)} w_k \left( x, \frac{x}{\varepsilon_k} \right) \cdot \phi \left( x, \frac{x}{\varepsilon_k} \right) \, dx
\]
\[
= \varepsilon_k^3 \sum_{t \in T_k} \int_{Q^0} w_k(\varepsilon_k(t+z), z) \cdot \phi(\varepsilon_k(t+z), z) \, dz
\]
\[
= \sum_{t \in T_k} \int_{Q^0} \int_{\varepsilon_k(t+Q)} w(x, z) \cdot \phi(\varepsilon_k(t+z), z) \, dx \, dz
\]
\[
= \int_{\Omega_k^0} \int_{Q^0} w(x, z) \cdot \phi_k(x, z) \, dz \, dx,
\]
where \( \phi_k(x, z) := \phi(\varepsilon_k(t+z), z) \) if \( x \in \varepsilon_k(t+Q) \) with \( t \in \hat{T}_k \). By the dominated convergence theorem, we infer
\[
\lim_{k \to +\infty} \int_{\Omega} v_k(x) \cdot \phi \left( x, \frac{x}{\varepsilon_k} \right) \, dx = \int_{\Omega} \int_{Q^0} w(x, z) \cdot \phi(x, z) \, dz \, dx,
\]
that is, \( v_k \rightharpoonup w \) weakly two-scale in \( L^2 \) (recall that \( w(x, z) = 0 \) if \( z \in Q^1 \)).

In order to prove that strong two-scale convergence actually holds, we study the limiting behavior of the \( L^2 \) norm of \( \{v_k\} \). On one hand, the weak two-scale convergence yields
\[
\|w\|_{L^2(\Omega \times Q)} \leq \liminf_{k \to +\infty} \|v_k\|_{L^2(\Omega)}.
\]  \hspace{1cm} (A.3)

On the other hand, from the properties of \( \{w_k\} \) and a change of variables we have the identities
\[
\int_{\Omega} |v_k(x)|^2 \, dx = \int_{\Omega_k^0} \left| w_k \left( x, \frac{x}{\varepsilon_k} \right) \right|^2 \, dx = \sum_{t \in T_k} \int_{\varepsilon_k(t+Q^0)} \left| w_k \left( x, \frac{x}{\varepsilon_k} \right) \right|^2 \, dx
\]
\[
= \sum_{t \in T_k} \varepsilon_k^3 \int_{Q^0} \left| w_k(\varepsilon_k(t+z), z) \right|^2 \, dz = \sum_{t \in T_k} \varepsilon_k^3 \int_{Q^0} \int_{\varepsilon_k(t+Q)} \left| w(x, z) \right|^2 \, dx \, dz.
\]
Thanks to Jensen’s inequality we deduce
\[
\int_{\Omega} |v_k(x)|^2 \, dx \leq \sum_{t \in T_k} \varepsilon_k^3 \int_{Q^0} \int_{\varepsilon_k(t+Q)} |w(x, z)|^2 \, dx \, dz = \int_{Q^0} \int_{\Omega_k^0} |w(x, z)|^2 \, dx \, dz
\]
This, combined with (A.3), ensures that
\[
\lim_{k \to +\infty} \|v_k\|_{L^2(\Omega)} = \|w\|_{L^2(\Omega \times Q)}.
\]
In view of Definition 3.9 the conclusion is achieved.

Finally, the strong convergence (6.15) follows by observing that, if \( x \in \varepsilon_k(t+Q) \), it holds
\[
\nabla_z \hat{v}_k(x, z) = \nabla_z w_k(\varepsilon_k(t+z), z).
\]

We are now in a position to prove a non-degenerate \( \Gamma \)-upper limit inequality that is the counterpart of the one in Proposition 6.7 under the current stronger convergence assumptions.
Proposition A.2. Let \( \{W_0^0\}_k \) satisfy assumptions E4–E6. For any \((w, P) \in L^2(\Omega; W_0^{1,2}(Q^0; \mathbb{R}^3)) \times W^{1,2}(\Omega; S(\mathbb{R}^3))\), there exists a sequence \( \{v_k\} \subset W_0^{1,2}(\Omega^k; \mathbb{R}^3) \) such that:

1. \( v_k \overset{2}{\rightharpoonup} w \) strongly two-scale in \( L^2 \);
2. \( \varepsilon_k \nabla v_k \overset{2}{\rightharpoonup} \nabla_z w \) weakly two-scale in \( L^2 \);
3. whenever \( P_k \to P \) uniformly, it holds

\[
\limsup_{k \to +\infty} \int_{\Omega} \chi_k^0(x) W_k^0(\varepsilon_k \nabla v_k(x) P_k^{-1}(x)) \, dx \leq \int_{\Omega} \int_{Q^0} Q^0 w(z, x) P^{-1}(x) \, dz \, dx,
\]

where \( Q^0 w \) is given by (2.9).

The conclusion is not a straightforward consequence of Lemma A.1, because along the sequence \( \{v_k\} \) in (A.1) we would not end up with the correct limiting energy density. Therefore, the actual recovery sequence is obtained by adding a “correction” to \( v_k \).

Proof of Proposition A.2. The proof consists of several steps. At first, to circumvent measurability issues, it is convenient to consider a sufficiently regular \( w \). Under such assumption, we are able to construct a recovery sequence of the form \( v_k = \tilde{v}_k + \tilde{w}_k \), where \( \{\tilde{v}_k\} \) is provided by Lemma A.1 and \( \{\tilde{w}_k\} \) allows to pass from the densities \( W_k \) to \( Q^0 W_k \). The definition of \( \tilde{w}_k \) is given in Step 1, while Step 2 deals with the upper limit inequality in the regular case. The general statement is eventually retrieved by approximation.

Step 1: Construction of \( \tilde{w}_k \) for a Regular \( w \)
Let us assume that \( w \in L^2(\Omega; W_0^{1,2}(Q^0; \mathbb{R}^3)) \cap C^2(\Omega \times Q^0; \mathbb{R}^3) \). We consider a cover of \( Q^0 \) made of cubes whose edge length is \( \varepsilon_k \). We set \( \hat{\Sigma}_k := \{s \in \mathbb{Z}^3 : \varepsilon_k(s + Q) \subseteq \bar{Q}^0 \} \) and, for all \( (t, s) \in \hat{T}_k \times \hat{\Sigma}_k \), we introduce the averages

\[
A_k(t, s) := \int_{\varepsilon_k(t+Q)} \int_{\varepsilon_k(s+Q)} \nabla_z w(x, z) \, dz \, dx
\]

and the piecewise constant functions

\[
A_k(x, z) := \begin{cases} A_k(t, s) & \text{if } (x, z) \in \varepsilon_k(t+Q) \times \varepsilon_k(s+Q), (t, s) \in \hat{T}_k \times \hat{\Sigma}_k, \\ 0 & \text{otherwise.} \end{cases}
\]

We record here for later use that, by means of Lebesgue differentiation and dominated convergence theorems, it follows

\[
\lim_{k \to +\infty} \|A_k - \nabla_z w\|_{L^2(\Omega \times Q)}^2 = \lim_{k \to +\infty} \sum_{t \in \hat{T}_k} \sum_{s \in \hat{\Sigma}_k} \int_{\varepsilon_k(t+Q)} \int_{\varepsilon_k(s+Q)} |A_k(t, s) - \nabla_z w(x, z)|^2 \, dz \, dx = 0.
\]

By the definition of \( Q^0 W_k \), for all \( k \in \mathbb{N} \) there exists \( \psi_k \in W_0^{1,2}(Q; \mathbb{R}^3) \) such that

\[
\int_Q \chi^0(z) W_k^0( (A_k(t, s) + \nabla \psi_k(z)) P_k^{-1}(x) \, dx \leq Q^0 W_k^0(A_k(t, s), P^{-1}(x)) + \frac{1}{k}.
\]

Note that, due to the smoothness of \( w \), the averages \( A_k \) are bounded uniformly in \( k, t \) and \( s \). In the light of Lemma 6.1, the values \( Q^0 W_k^0(A_k(t, s), P^{-1}(x)) \) are uniformly bounded as well. Therefore, by combining (A.6) with assumption E4, we deduce that \( \{\psi_k\} \) is bounded in \( W_0^{1,2}(Q; \mathbb{R}^3) \).
A change of variables in (A.6) yields
\[
\int_{\varepsilon_k(s+Q)} \chi^0 \left( \frac{z}{\varepsilon_k} - s \right) W_k^0 \left( \left( A_k(t, s) + \nabla \psi_k \left( \frac{z}{\varepsilon_k} - s \right) \right) P^{-1}(x) \right) \, dz \\
\leq \varepsilon_k^2 \left( Q' W_k^0 (A_k(t, s), P^{-1}(x)) + \frac{1}{k} \right), \tag{A.7}
\]
and that suggests us to introduce the functions
\[
\tilde{\psi}_k(x, z) := \begin{cases} 
\varepsilon_k \psi_k \left( \frac{z}{\varepsilon_k} - s \right) & \text{if } (x, z) \in \varepsilon_k(t + Q) \times \varepsilon_k(s + Q), \ (t, s) \in \hat{T}_k \times \hat{\Sigma}_k, \\
0 & \text{otherwise.}
\end{cases}
\]
Note that, for each \( k \) and \( x \in \Omega, \tilde{\psi}_k(x, \cdot) \) admits a weak derivative with respect to \( z \); thus, by summing over \( (t, s) \in \hat{T}_k \times \hat{\Sigma}_k \), from (A.7) we may write
\[
\sum_{(t, s) \in \hat{T}_k \times \hat{\Sigma}_k} \int_{\varepsilon_k(t+Q)} \int_{\varepsilon_k(s+Q)} \chi^0 \left( \frac{z}{\varepsilon_k} - s \right) W_k^0 \left( \left( A_k(x, z) + \nabla \tilde{\psi}_k(x, z) \right) P^{-1}(x) \right) \, dz \, dx \\
\leq \sum_{(t, s) \in \hat{T}_k \times \hat{\Sigma}_k} \int_{\varepsilon_k(t+Q)} \varepsilon_k^3 \left( Q' W_k^0 (A_k(t, s), P_k^{-1}(x)) + \frac{1}{k} \right) \, dx. \tag{A.8}
\]
We also observe that, since \( \{ \tilde{\psi}_k \} \) is bounded, \( \tilde{\psi}_k \to 0 \) strongly in \( L^2(\Omega \times Q; \mathbb{R}^3) \). Then, given that \( \{ \nabla \tilde{\psi} \} \) is bounded \( L^2(\Omega \times Q; \mathbb{R}^{3 \times 3}) \), it must converge weakly in \( L^2 \) to 0. It follows that, if \( w_k \) is as in Lemma A.1 and if \( (x, z) \in \varepsilon_k(t + Q) \times \varepsilon_k(s + Q) \) with \( (t, s) \in \hat{T}_k \times \hat{\Sigma}_k \),
\[
\nabla_z (w_k + \tilde{\psi}_k) \rightharpoonup \nabla_z w \quad \text{weakly in } L^2(\Omega \times Q; \mathbb{R}^{3 \times 3}). \tag{A.9}
\]
We further notice that
\[
\bar{w}_k(x) := \tilde{\psi}_k \left( x, \frac{x}{\varepsilon_k} \right) = \sum_{(t, s) \in \hat{T}_k \times \hat{\Sigma}_k} \varepsilon_k \psi \left( \frac{x}{\varepsilon_k} - s \right) \chi_{\varepsilon_k(t+Q)}(x) \chi_{\varepsilon_k(s+Q)} \left( \frac{x}{\varepsilon_k} \right)
\]
is a measurable function. A quick application of the definition of weak derivative proves also that \( \bar{w}_k \) belongs to \( W^{1,2}_0(\Omega^0, \mathbb{R}^3) \).

**Step 2: w regular**

We now turn to the proof of the limsup inequality along the sequence \( \{ v_k \} \) defined as
\[
v_k := \tilde{v}_k + \bar{w}_k, \tag{A.10}
\]
where
\[
\tilde{v}_k(x) := w_k \left( x, \frac{x}{\varepsilon_k} \right)
\]
with \( w_k \) as in Lemma A.1, and where \( \{ \tilde{w}_k \} \) was introduced in Step 1. We have
\[
\hat{v}_k(x, z) := S_k v_k(x, z) = w_k \left( \varepsilon_k \left[ \frac{x}{\varepsilon_k} \right] + \varepsilon_k z, z \right) + \bar{v}_k \left( \varepsilon_k \left[ \frac{x}{\varepsilon_k} \right] + \varepsilon_k z, z \right),
\]
so that if \( (x, z) \in \varepsilon_k(t + Q) \times \varepsilon_k(s + Q) \)
\[
\nabla_z \hat{v}_k(x, z) = \nabla_z w_k \left( \varepsilon_k(t + z), z \right) + \nabla \bar{v}_k \left( \frac{z}{\varepsilon_k} - s \right). \tag{A.11}
\]
Taking into account (A.9), (A.11) and Lemma 3.11(1), it follows that
\[ \varepsilon_k \nabla v_k \overset{2}{\rightharpoonup} \nabla z \] weakly two-scale in \( L^2 \).

Recalling Lemma 6.5, we have that
\[
\limsup_{k \to +\infty} \int_\hat{\Omega} \chi_0(x) W_k^0 (\varepsilon_k \nabla v_k(x) P_k^{-1}(x)) \, dx
= \limsup_{k \to +\infty} \sum_{t \in \hat{T}_k} \int_{\varepsilon_k(t+Q)} \int_{\varepsilon_k(s+Q)} W_k^0 (\nabla z \hat{v}_k(x,z) P_k^{-1}(x)) \, dz \, dx
= \limsup_{k \to +\infty} I_k,
\]
where
\[
I_k := \sum_{(t,s) \in \hat{T}_k \times \hat{\Sigma}_k} \int_{\varepsilon_k(t+Q)} \int_{\varepsilon_k(s+Q)} W_k^0 (\nabla z \hat{v}_k(x,z) P_k^{-1}(x)) \, dz \, dx.
\]

Indeed, \( \hat{v}_k \) vanishes if \( x \in \Omega \setminus \Omega_k^Q \) or if \( z \in Q^0 \cup \{ \varepsilon_k(s+Q) : s \in \hat{\Sigma}_k \} \), and the sequence \( \{ W_k^0(0) \} \) is bounded by virtue of E4. Therefore, since the measure of \( \Omega \setminus \Omega_k^Q \) and of \( Q^0 \cup \{ \varepsilon_k(s+Q) : s \in \hat{\Sigma}_k \} \) vanish for \( k \to +\infty \), the second equality holds.

Being the value of \( \nabla z \hat{v}_k(x,z) \) expressed by formula (A.11), we introduce
\[
I'_k := \sum_{t,s} \int_{\varepsilon_k(t+Q)} \int_{\varepsilon_k(s+Q)} W_k^0 \left( \left( A_k(t,s) + \nabla \psi_k \left( \frac{z}{\varepsilon_k} - s \right) \right) P_k^{-1}(x) \right) \, dz \, dx,
\]
where the summation runs over \( \hat{T}_k \times \hat{\Sigma}_k \). By exploiting assumption E5 and Hölder’s inequality, we obtain the estimate
\[
|I_k - I'_k| \leq c \sum_{t,s} \int_{\varepsilon_k(t+Q)} \int_{\varepsilon_k(s+Q)} \left| \left( \nabla z w_k(\varepsilon_k(t+z),z) - A_k(t,s) \right) P_k^{-1}(x) \right|^2 \, dz \, dx.
\]

In view of Lemma A.1 and (A.5) we deduce
\[
\lim_{k \to +\infty} |I_k - I'_k| = 0. \tag{A.12}
\]

Next, let us set
\[
I''_k := \int_{\Omega_k^Q} \int_{Q^0} Q' W_k^0 (A_k(x,z), P_k^{-1}(x)) \, dz \, dx.
\]

According to (A.8), the difference between the integrands of \( I'_k \) and \( I''_k \) is of order \( k^{-1} \):
\[
\lim_{k \to +\infty} |I'_k - I''_k| = 0. \tag{A.13}
\]
Finally, we compare $I_k'$ and the limiting functional. We have

$$
\left| I_k' - \int_{\Omega} \int_{Q^0} Q'W^0_\kappa(\nabla_z w(x,z), P^{-1}(x)) \, dz \, dx \right|
\leq \int_{\Omega} \int_{Q^0} \left| Q'W^0_\kappa(A_k(x,z), P^{-1}(x)) - Q'W^0_\kappa(\nabla_z w(x,z), P^{-1}(x)) \right| \, dz \, dx
+ \int_{\Omega} \int_{Q^0} \left| Q'W^0_\kappa(\nabla_z w(x,z), P^{-1}(x)) - Q'W^0(\nabla_z w(x,z), P^{-1}(x)) \right| \, dz \, dx
+ \int_{\Omega} \int_{Q^0} \left| Q'W^0(\nabla_z w(x,z), P^{-1}(x)) - Q'W^0(\nabla_z w(x,z), P^{-1}(x)) \right| \, dz \, dx.
$$

All the terms on the right-hand side vanish as $k \to +\infty$. Indeed, by using the Lipschitz continuity of $Q'W^0_\kappa$ (see Lemma 6.1(1)) and the uniform bound on $\{P_k\}$, the first summand is controlled by the norm of $A_k - \nabla_z v$, which, according to (A.5), is infinitesimal. For what concerns the second term, Lemma 6.1(2) and the uniform convergence of $\{P_k\}$ imply that the integrand is infinitesimal for $k \to +\infty$. The third quantity vanishes because $\{Q'W^0\}$ pointwise converges to $Q'W^0$ (recall that they are just variants of the quasiconvex envelopes). Lastly, the fourth summand is negligible since $L^3(\Omega \setminus \Omega^0_\kappa)$ tends to 0.

On the whole, taking into account (A.12) and (A.13), we conclude

$$
\lim_{k \to +\infty} I_k = \int_{\Omega} \int_{Q^0} Q'W^0(\nabla_z w(x,z), P^{-1}(x)) \, dz \, dx.
$$

**STEP 3: $w$ GENERIC**

The argument follows the one of Case 2 in the proof of Proposition 6.7.

\[ \square \]

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