Acoustic modes in He I and He II
in the presence of an alternating electric field

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Abstract

By solving the equations of ordinary and two-fluid hydrodynamics, we study the oscillatory modes in isotropic nonpolar dielectrics He I and He II in the presence of an alternating electric field \( E = E_0 \delta_z \sin (k_0 z - \omega_0 t) \). The electric field and oscillations of the density become "coupled," since the density gradient causes a spontaneous polarization \( P_s \), and the electric force contains the term \( (P_s \nabla)E \). The analysis shows that the field \( E \) changes the velocities of first and second sounds, propagating along \( E \), by the formula \( u_j \approx c_j + \chi_j E_0^2 \) (where \( j = 1, 2 \); \( c_j \) is the velocity of the \( j \)-th sound for \( E_0 = 0 \), and \( \chi_j \) is a constant). We have found that the field \( E \) jointly with a wave of the first (second) sound \((\omega, k)\) should create in He II hybrid acousto-electric (thermo-electric) density waves \((\omega + l\omega_0, k + lk_0)\), where \( l = \pm 1, \pm 2, \ldots \). The amplitudes of acousto-electric waves and the quantity \(|u_1 - c_1|\) are negligibly small, but they should increase in the resonance way at definite \( \omega \) and \( \omega_0 \). Apparently, the first resonance corresponds to the decay of a photon into two phonons with the transfer of a momentum to the whole liquid. Therefore, the spectrum of an electromagnetic signal should contain a narrow absorption line like that in the Mössbauer effect.

Keywords: first sound, second sound, spontaneous polarization, dielectric.

1 Introduction

It is well known that the external electric field \( E^\text{ext} \) polarizes an isotropic dielectric \[ 1, 2 \]. The measure of such polarization is the dielectric permittivity \( \varepsilon \). In addition, the isotropic nonpolar dielectric can polarize itself spontaneously. The spontaneous polarization related to the acceleration and the density gradient was theoretically studied, respectively, in \[ 3, 4, 5, 6, 7, 8 \] and \[ 3, 5, 9, 10, 11, 12 \]. The density gradient causes the spontaneous polarization, because two nonpolar atoms polarize each other \[ 13, 14, 15 \]. It was shown \[ 16 \] how the spontaneous polarization of an isotropic nonpolar dielectric should be taken in Maxwell equations for a medium into account.

The electric properties of such isotropic nonpolar dielectrics as He I and He II were experimentally studied in a number of works. In the experiment by A.S. Rybal’ko it was found that the standing half-wave of the second sound in He II is accompanied by an electric signal
This effect was confirmed in subsequent experiments [18, 19, 20]. A lot of theoretical explanations of the Rybalko’s effect, of various degrees of plausibility, were proposed [3, 7, 8, 10, 21, 22, 23, 24, 25]. An analogous effect for the first sound was predicted theoretically [23, 24] and then was found experimentally [26]. The attempt to explain this effect was also made in [8]. We note that, for first and second sounds, the electric signal was not observed at $T > T_\lambda$. It is natural for the second sound (that simply does not exist at $T > T_\lambda$), but it is strange for the first one. We note also that a supernarrow absorption line at the roton frequency was found in experiments with an electromagnetic resonator imbedded in He II [27, 28]. Several models were proposed to explain this line [29, 30, 31].

In work [17], Rybalko mentioned the observation of a second-sound wave induced by an alternating electric field. However, this effect was not revealed in the recent experiment [32]. In what follows, we will study theoretically the influence of an external alternating electric field $E^{ext}$ on the oscillatory modes of He I and He II and will show that the field leads to several interesting effects. In Sections 2 and 3 we present the results of calculations. The physical consequences and experiments will be considered in Section 4.

## 2 Nonsuperfluid liquid dielectric (He I)

Consider an isotropic nonpolar liquid dielectric in an alternating electric field. The motion of an ideal liquid is described by the equations [33, 34]

$$\rho \partial \mathbf{v} / \partial t + \rho (\mathbf{v} \nabla) \mathbf{v} = -\nabla p + \mathbf{F},$$

$$\partial \rho / \partial t + \mathrm{div} (\rho \mathbf{v}) = 0,$$

where $\rho$ is the density, $p$ is the pressure, $\mathbf{v}$ is the velocity, and $\mathbf{F}$ is a nonmechanical force per unit volume. In our case, the force $\mathbf{F}$ is induced by the electric field $\mathbf{E}$:

$$\mathbf{F} = \nabla \left[ \frac{E^2}{8\pi \rho} \frac{\partial \varepsilon}{\partial \rho} \right] - \frac{E^2}{8\pi} \nabla \varepsilon +$$

$$\left( \mathbf{P}_s \nabla \right) \mathbf{E} + (a - 1) \nabla (\mathbf{P}_s \mathbf{E}) + \frac{1}{2} \nabla \times (\mathbf{P}_s \times \mathbf{E}).$$

Here, two first terms were obtained by H.L. Helmholtz (see [1]), and the rest ones are related to the spontaneous polarization and are found in [16]. $a$ is the parameter from the formula $\mathbf{P}_s (r) = const \cdot \rho^* \nabla \rho$ [3, 5, 9, 10, 11, 12]. We will consider only nonpolar liquids and gases. Then the dielectric permittivity $\varepsilon$ satisfies the Clausius–Mossotti formula [2] $\frac{\varepsilon - 1}{\varepsilon + 2} = \frac{4\pi}{3} n \beta$ (here, $\beta$ is the polarizability of a molecule, and $n = \rho / m$). For the gases and some liquids, including He I and He II, $\varepsilon$ is close to 1 [35]. Then $\rho \partial \varepsilon / \partial \rho \approx \varepsilon - 1$, and formula (3) takes the form

$$\mathbf{F} = \frac{\varepsilon - 1}{8\pi} \nabla E^2 + (\mathbf{P}_s \nabla) \mathbf{E} + (a - 1) \nabla (\mathbf{P}_s \mathbf{E}) + \frac{1}{2} \nabla \times (\mathbf{P}_s \times \mathbf{E}).$$
As was mentioned above, the spontaneous polarization $P_s$ of an isotropic nonpolar dielectric can be related to the acceleration and the density gradient. We note that the available calculations of the polarization caused by the acceleration are rather crude [3, 4, 5, 6, 7, 8].

The motion of an element of the liquid dielectric volume in an acoustic wave is accompanied by the acceleration and the density gradient. If we subtract the contribution related to the density gradient from the total polarization, we get the part of the polarization which is caused only by the acceleration. It is worth noting that the available works contain no proof that this part is nonzero. According to the estimations made in [5], this part should be much less than the polarization caused by the density gradient. This is related to the fact that the electron shell of a nonpolar atom is hard to be stretched. In view of this fact we will neglect a possible polarization caused by the acceleration.

The polarization caused by the density gradient was studied theoretically in works [3, 5, 9, 10, 11, 12]. It can be accurately evaluated [5, 9, 11, 12] on the basis of the formula for a mutual polarization of two nonpolar atoms [13, 14, 15]. Additionally, we need to average over different configurations of atoms, which gives the following formulas [5]:

$$\mathbf{P_s}(r) = \xi \nabla n(r), \quad \xi \approx \zeta (7/3)d_0\vec{r}_0(n(r)/n_0)^\alpha, \quad (5)$$

where $a = 1$, $\vec{r}_0 = n_0^{-1/3}$ is the mean interatomic distance, $d_0 = -D_7|\epsilon a_B a_0^{1/3}$, $a_B = \hbar^2/4m$ is Bohr radius, $D_7$ is the atomic constant [13, 14, 15], and $\zeta \approx 7.5$ for He II [5]. For He I, the value of $\zeta$ should be almost the same. Indeed, $\zeta$ depends on the pair correlation function $g(r)$ [5], and the latter is almost independent of the temperature, if $T = 1-4.27\, K$ [36]. We note that the value of $\xi$ was determined in [5] with wrong sign. In formula (5), we took the relation $S_7 \approx 15(n/n_0)^{4/3}$ into account [see Eqs. (28) and (29) in [5]] and assumed that the deviations of the particle number density $n(r)$ from the mean value $n_0$ are small.

The spontaneous polarization (5) is caused by the interaction of atoms and, therefore, exists at any temperature: in He II, He I, and gaseous helium at high temperatures.

We now determine the influence of an external field $\mathbf{E}^{ext} = E_0i_z \sin(k_0z - \omega_0t)$ on the oscillatory modes of a liquid. The total field $\mathbf{E}$ in (4) is the sum $\mathbf{E}^{ext} + \mathbf{E}^{own}$, where $\mathbf{E}^{own}$ is the field created by the dipoles of a dielectric. We can roughly consider that $\mathbf{E}^{own} \sim \mathbf{P}$, where $\mathbf{P} = \mathbf{P}_i + \mathbf{P}_s$, and $\mathbf{P}_i = (\varepsilon - 1)\mathbf{E}/(4\pi)$ is the induced polarization [16]. Since $(\varepsilon - 1)/(4\pi) \lesssim 0.0045$ for liquid helium-4 [35], we can neglect the field created by induced dipoles as compared with $\mathbf{E}^{ext}$. Below, we will see that the field created by the spontaneous dipoles is also low. Therefore, we set $\mathbf{E} \approx \mathbf{E}^{ext}$.

Next, the force (4) contains three terms, in which the field $\mathbf{E}$ “couples” with $\mathbf{P}_s$. Since $\mathbf{P}_s \sim \nabla n(r)$, the electric field must create some density wave in the medium. We consider it to be weak and the perturbations of parameters of the system to be small. Therefore, we seek the deviations from the unperturbed values in the linear approximation. Let the unperturbed system be characterized by the parameters $\rho_0, p_0 = const$, and $\mathbf{v}, \mathbf{E} = 0$. For the perturbed system, we take the velocity $\mathbf{v}$ to be nonzero and small. We set $\rho = \rho_0 + \rho'$,
\[ p = p_0 + p', \quad \mathbf{E} = E_0 \mathbf{i}_z \sin (k_0 z - \omega_0 t) \] and take into account that the ideal liquid moves adiabatically \((s = \text{const})\). Therefore, \(p' = \frac{\partial p}{\partial \rho} \rho' = c_1^2 \rho'\), where \(c_1\) is the sound velocity \([33, 34]\). First, we set \(P_s = 0\) in (4). Equations (1) and (4) imply that the velocity must be directed along the field \(E\): \(v = vi_z\). In (1) and (2), we remain only the terms linear in \(\rho', p', v\) and get the following equations for small perturbations:

\[
\rho_0 \frac{\partial v}{\partial t} + c_1^2 \frac{\partial \rho'}{\partial z} = \frac{\varepsilon - 1}{8\pi} E_0^2 k_0, \tag{6}
\]

\[
\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v}{\partial z} = 0. \tag{7}
\]

We set \(v = \tilde{v} \cos (kz - \omega t + \alpha)\), \(p' = \tilde{\rho} \cos (kz - \omega t + \alpha)\). For \(E = 0\) Eqs. (6) and (7) yield

\[
-kc_1^2 \tilde{\rho} + \omega \rho_0 \tilde{v} = 0, \tag{8}
\]

\[
\omega \tilde{\rho} - k \rho_0 \tilde{v} = 0. \tag{9}
\]

From whence, we obtain the sound dispersion law: \(\omega^2 = c_1^2 k^2\). For \(E = E_0 \mathbf{i}_z \sin (k_0 z - \omega_0 t)\) Eqs. (6), (7) contain the driving force. The corresponding solution takes the form \(v = \tilde{v}_0,2 \cos (2k_0 z - 2\omega_0 t)\), \(p' = \tilde{\rho}_0,2 \cos (2k_0 z - 2\omega_0 t)\). Then

\[
-2k_0 c_1^2 \tilde{\rho}_0,2 + 2\omega_0 \rho_0 \tilde{v}_0,2 = \frac{\varepsilon - 1}{8\pi} E_0^2 k_0, \tag{10}
\]

\[
2\omega_0 \tilde{\rho}_0,2 - 2k_0 \rho_0 \tilde{v}_0,2 = 0, \tag{11}
\]

and we get the amplitudes

\[
\tilde{\rho}_0,2 = \frac{\rho_0}{\omega_0} \tilde{v}_0,2, \quad \tilde{v}_0,2 = \frac{\varepsilon - 1}{16\pi \rho_0 \omega_0^2 - c_1^2 k_0^2} E_0^2 k_0. \tag{12}
\]

Now, we consider the terms with \(P_s\) in (4). In this case, (2) leads, as before, to (7), and the linearized equation (1) with force (4) takes the form

\[
\rho_0 \frac{\partial v}{\partial t} + c_1^2 \frac{\partial \rho'}{\partial z} = \frac{\varepsilon - 1}{8\pi} E_0^2 k_0 \sin (k_0 z - \omega_0 t) - \frac{\xi E_0 \tilde{\rho}}{2m} (ak k_0 + (a - 1)k^2) \sin (k_1 z - \omega_1 t + \alpha) - \frac{\xi E_0 \tilde{\rho}}{2m} (ak k_0 - (a - 1)k^2) \sin (k_1 z - \omega_1 t + \alpha). \tag{13}
\]

Any real liquid has a nonzero temperature. Therefore, it contains an ensemble of thermal phonons, including those moving along the field \(E\). The latter create the density waves of the form \(\rho' = \tilde{\rho} \cos (kz \pm \omega t + \alpha)\), where \(\omega > 0\). We will consider only the wave \(\rho' = \tilde{\rho} \cos (kz - \omega t + \alpha)\), by considering that \(\omega\) can be positive or negative. For such wave, the right-hand side of (13) reduces to the form

\[
-\frac{\varepsilon - 1}{8\pi} E_0^2 k_0 \sin (k_0 z - \omega_0,2 t) - \frac{\xi E_0 \tilde{\rho}}{2m} (ak k_0 + (a - 1)k^2) \sin (k_1 z - \omega_1,2 t + \alpha) - \frac{\xi E_0 \tilde{\rho}}{2m} (ak k_0 - (a - 1)k^2) \sin (k_1 z - \omega_1,2 t + \alpha). \tag{14}
\]

Here and below, we use the notations \(k_{l,i} = lk + ik_0, \omega_{l,i} = l\omega + i\omega_0\). It shows that \(\rho'\) should be sought in the form

\[
\rho' = \tilde{\rho}_{1,0} \cos (k_{1,0} z - \omega_0,0 t + \alpha) + \tilde{\rho}_{0,2} \cos (k_{0,2} z - \omega_0,2 t) + \tilde{\rho}_{1,1} \cos (k_{1,1} z - \omega_1,1 t + \alpha) + \tilde{\rho}_{1,-1} \cos (k_{1,-1} z - \omega_1,-1 t + \alpha).
\]
If we substitute this expansion to the right-hand side of (13), the latter will generate several new harmonics. They should be taken into account in solutions for \( \rho', v \). Then we again substitute \( \rho', v \) in (7), (13), and so on. Finally, we have found that a solution of Eqs. (7) and (13) should be sought in the form

\[
\rho' = \sum_{i=1,2,\ldots} \tilde{\rho}_{0,i} \cos(k_{0,i}z - \omega_{0,i}t) + \sum_{i=0,\pm 1,\pm 2,\ldots} \tilde{\rho}_{1,i} \cos(k_{1,i}z - \omega_{1,i}t + \alpha),
\]

where \( \alpha \) is any real number. We substitute expansions (15) and (16) in Eqs. (7), (13) and take (14) into account. Then (7) and (13) take the form

\[
\sum_{l,i} A_{l,i} \sin(k_{l,i}z - \omega_{l,i}t + q_0\alpha) = 0, \quad \sum_{l,i} B_{l,i} \sin(k_{l,i}z - \omega_{l,i}t + q_1\alpha) = 0,
\]

where \( l, i \) run the values \( l = 0, 1; i = 0, \pm 1, \pm 2, \ldots \) (except for \( l = i = 0 \)). In this case, \( q_0 = 0, q_1 = 1 \). Equations (17) are satisfied, if \( A_{l,i} = 0 \) and \( B_{l,i} = 0 \) for all \( l, i \). As a result, (7) yields the equations

\[
A_{l,i} \equiv \omega_{l,i} \tilde{\rho}_{l,i} - k_{l,i} \rho_0 \tilde{\nu}_{l,i} = 0.
\]

Moreover, (13) leads to

\[
B_{0,1} \equiv \rho_0 \omega_0 \tilde{\nu}_{0,1} - c_1^2 k_0 \tilde{\rho}_{0,1} + \frac{\xi E_0}{2m} \tilde{\rho}_{0,2}(2ak_0^2 - (a - 1)4k_0^2) = 0,
\]

\[
B_{0,2} \equiv \rho_0 2\omega_0 \tilde{\nu}_{0,2} - c_1^2 k_0 \tilde{\rho}_{0,2} = \frac{\xi E_0}{2m} \tilde{\rho}_{0,1}(a^2k_0^2 + (a - 1)k_0^2) + \frac{\xi E_0}{2m} \tilde{\rho}_{0,3}(3ak_0^2 - (a - 1)9k_0^2) = 0,
\]

\[
B_{0,j} \equiv \rho_0 \omega_{0,j} \tilde{\nu}_{0,j} - c_1^2 k_0 \tilde{\rho}_{0,j} + \frac{\xi E_0}{2m} \tilde{\rho}_{0,j-1}(ak_0k_{0,j-1} + (a - 1)k_{0,j-1}^2) + \frac{\xi E_0}{2m} \tilde{\rho}_{0,j+1}(ak_0k_{0,j+1} + (a - 1)k_{0,j+1}^2) = 0,
\]

\[
B_{1,0} \equiv \rho_0 \omega \tilde{\nu}_{1,0} - c_1^2 k \tilde{\rho}_{1,0} + \frac{\xi E_0}{2m} \tilde{\rho}_{1,-1}[a(k_0 - k_0)(k - (k_0)^2)] + \frac{\xi E_0}{2m} \tilde{\rho}_{1,1}[a(k_0 + k_0) - (a - 1)(k + k_0)^2] = 0,
\]

\[
B_{1,i} \equiv \rho_0 \omega_{1,i} \tilde{\nu}_{1,i} - c_1^2 k_{1,i} \tilde{\rho}_{1,i} + \frac{\xi E_0}{2m} \tilde{\rho}_{1,i-1}[ak_0k_{1,i-1} + (a - 1)k_{1,i-1}^2] + \frac{\xi E_0}{2m} \tilde{\rho}_{1,i+1}[ak_0k_{1,i+1} + (a - 1)k_{1,i+1}^2] = 0,
\]

where \( j = 3, 4, \ldots, i = \pm 1, \pm 2, \ldots \). For Eqs. (18)–(23) and similar equations of the following section, the small parameter is

\[
\vartheta = \frac{\xi E_0 k_0}{mc_1^2}.
\]
Even in strong fields \( E_0 \), we have \( \vartheta \ll 1 \) for characteristic \( k_0 \). The smallness of \( \vartheta \) ensures the convergence of series [15]. It is convenient to introduce the phase velocity \( u_{t,i} = \omega_{t,i}/k_{t,i} \) for each harmonic. Then [18] takes the form

\[
\tilde{v}_{t,i} = u_{t,i}\tilde{\rho}_{t,i}/\rho_0. \tag{25}
\]

The system of equations [19]–[25] is separated into two independent systems: for the harmonics \((0, i)\) and for the harmonics \((1, i)\). In Eqs. [19]–[21] we present \( \tilde{v}_{0,i} \) in terms of \( \tilde{\rho}_{0,i} \) with the help of [25]. The solutions for the harmonics \((0, 1)\), \((0, 2)\), and \((0, 3)\) are as follows:

\[
\tilde{\rho}_{0,2} \approx \frac{\varepsilon - 1}{16\pi} \frac{E_0^2}{c^2 - c_1^2}, \quad \tilde{v}_{0,2} = \frac{c\tilde{\rho}_{0,2}}{\rho_0}, \tag{26}
\]

\[
\tilde{\rho}_{0,1} \approx \frac{\xi E_0 k_0(a - 2)}{m(c^2 - c_1^2)} \tilde{\rho}_{0,2}, \quad \tilde{v}_{0,1} = \frac{c\tilde{\rho}_{0,1}}{\rho_0}, \tag{27}
\]

\[
\tilde{\rho}_{0,3} \approx \frac{\xi E_0 k_0(2/3 - a)}{m(c^2 - c_1^2)} \tilde{\rho}_{0,2}, \quad \tilde{v}_{0,3} = \frac{c\tilde{\rho}_{0,3}}{\rho_0}, \tag{28}
\]

where \( c = \omega_0/k_0 \) is the velocity of light in a dielectric. The mode \((0, 2)\) is dominant, and the remaining modes are weak: \( \tilde{\rho}_{0,1} \sim \vartheta \tilde{\rho}_{0,2}, \tilde{\rho}_{0,4} \sim \vartheta^2 \tilde{\rho}_{0,2}, \tilde{\rho}_{0,5} \sim \vartheta^3 \tilde{\rho}_{0,2}, \) and so on. Moreover, the amplitudes \( \tilde{\rho}_{0,i} \sim \vartheta^{i-2} \) with \( i \geq 4 \) and \( i \geq 6 \) must include the contributions, respectively, from the nonlinear terms \( \tilde{\rho}_{0,i_1}\tilde{v}_{0,i_2} \) and \( \tilde{\rho}_{0,i_1}\tilde{v}_{0,i_2}\tilde{v}_{0,i_3} \) [from Eqs. [11] and [2]], which were neglected. Solution [26] coincides with [12].

At the zero temperature the liquid contains no acoustic waves, and the electric field generates in a liquid only oscillations of the density of the type \((0, i)\) [see [26]–[28]] that have the phase velocity equal to the velocity of light. If an acoustic wave is generated at \( T = 0 \) artificially, hybrid modes obtained below should additionally appear in the system.

Let us consider the chain of equations [22], [23] for the harmonics \((1, i)\). Let us set \( \tilde{v}_{1,i} = u_{1,i}\tilde{\rho}_{1,i}/\rho_0 \) and \( \omega_{1,i} = u_{1,i}k_{1,i} \). Then [22] and [23] take the form

\[
\tilde{\rho}_{1,0}k(u^2 - c_1^2) = -\frac{\xi E_0}{2m} \tilde{\rho}_{1,-1}[ak_0(k - k_0) + (a - 1)(k - k_0)^2] - \frac{\xi E_0}{2m} \tilde{\rho}_{1,1}[ak_0(k + k_0) - (a - 1)(k + k_0)^2], \tag{29}
\]

\[
\tilde{\rho}_{1,i}k_{1,i}(u_{1,i}^2 - c_1^2) = -\frac{\xi E_0}{2m} \tilde{\rho}_{1,i-1}[ak_0k_{1,i-1} + (a - 1)k_{1,i-1}^2] - \frac{\xi E_0}{2m} \tilde{\rho}_{1,i+1}[ak_0k_{1,i+1} - (a - 1)k_{1,i+1}^2], \tag{30}
\]

where \( i = \pm 1, \pm 2, \ldots \). With regard for the smallness of \( \vartheta \), relation [30] gives the recurrence relations

\[
\tilde{\rho}_{1,i} \approx -\frac{\xi E_0}{2m} \frac{ak_0k_{1,i-1} + (a - 1)k_{1,i-1}^2}{(u_{1,i}^2 - c_1^2)k_{1,i}}, \tag{31}
\]

\[
\tilde{\rho}_{1,-i} \approx -\frac{\xi E_0}{2m} \frac{ak_0k_{1,-i+1} - (a - 1)k_{1,-i+1}^2}{(u_{1,-i}^2 - c_1^2)k_{1,-i}}. \tag{32}
\]
\(i = 1, 2, 3, \ldots\). Using them, we can express all \(\tilde{\rho}_{1, \pm i}\) in terms of \(\tilde{\rho}_{1, 0}\). We consider the quantity \(\tilde{\rho}_{1, 0}\) to be known. It represents small fluctuations of the density related to thermal phonons \((\omega, k)\).

Substituting \(\tilde{\rho}_{1, \pm 1}\) \[31\], \[32\] in \(29\), we get the formula for the sound velocity \(u_{1, 0} \equiv u:\)

\[
u^2 = \frac{c^2_1 + \chi \left(\frac{\xi E_0}{2m}\right)^2}{C_1^2},
\]

\[
\chi \approx \frac{k_0 + (a - 1)k[ak_0 - (a - 1)k]}{u_{1, i}^2 - c_1^2} + \frac{k_0 - (a - 1)k[ak_0 + (a - 1)k]}{u_{1, i}^2 - c_1^2}.
\]

For \(a = 1\) we get

\[
\chi \approx \frac{k_0^2}{u_{1, i}^2 - c_1^2} + \frac{k_0^2}{u_{1, i}^2 - c_1^2}.
\]

We consider the quantities \(k_0\) and \(\omega_0\) to be positive (this can always be attained in the formula \(E = E_0\hat{i}_z\sin(k_0z - \omega_0t)\) by the choice of a direction of the axis \(z\)). We also consider \(k\) of phonons in \(\tilde{\rho}_{1, 0}\cos(kz - \omega t + \alpha)\) to be positive. In this case, the angular frequency \(\omega = uk\) can be positive or negative, since the phase velocity \(u\) can have different signs. From \[33\], we get two solutions:

\[
u \approx \pm \left(c_1 + \frac{\chi}{2c_1} \left(\frac{\xi E_0}{2m}\right)^2\right).
\]

Thus, we have found the solutions for small oscillations of the density for a nonsuperfluid liquid dielectric placed in an alternating electric field \(E = E_0\hat{i}_z\sin(k_0z - \omega_0t)\).

As was mentioned above, the electric field \(E_s\) induced by spontaneous dipoles can be neglected. This is seen from the formula \(P_s \sim P_s = (\xi/m)\nabla \rho'\) and from the fact that the main contribution to \(\rho'\) is given by \(\tilde{\rho}_{1,0}\) and \(\tilde{\rho}_{0,2}\). The latter leads to \(E_s \sim (\xi/m)\nabla \tilde{\rho}_{0,2} \sim \frac{\eta (\xi - 1) c^2}{8\pi \varepsilon_0} \frac{E_{ext}}{c^2} \ll E_{ext}\), and the quantity \(\tilde{\rho}_{1,0}\) gives \(E_s \sim (\xi/m)k\tilde{\rho}_{1,0}\). Since the density \(\tilde{\rho}_{1,0}\) of thermal phonons with momentum \((k_x, k_y, k_z) = (0, 0, k)\) is very small, we have \(E_s \ll E_0\) for not too small \(E_0\).

The above solutions have interesting properties. The modes \((0, 1), (0, 2), (0, 3)\) \[26\]–\[28\] correspond to weak oscillations of the density with parameters \((\omega_0, k_0)\), \((2\omega_0, 2k_0)\), and \((3\omega_0, 3k_0)\). These waves have a phase velocity equal to the velocity of light in the medium \(c = c_v/\sqrt{\varepsilon \mu}\) (where \(c_v\) is the velocity of light in vacuum), which is larger by 6 orders than the velocity of sound. The modes \((1, \pm 1)\) are hybrid acousto-electric modes. They exist, if the phonon mode \(\tilde{\rho}_{1,0}\cos(kz - \omega t + \alpha)\) and the electric field \(E = E_0\hat{i}_z\sin(k_0z - \omega_0t)\) are present. Phonons exist always at \(T > 0\). According to the solutions, the modes \((1, \pm 1)\) should be stronger than the modes \((1, \pm 2)\). The acousto-electric modes (“acouelons”) \((1, \pm 1)\) have rather unusual properties. The mode \((1, 1)\) is a wave with frequency \(\omega + \omega_0\) and with wave vector \(k + k_0\). If \(\omega_0 \sim \omega\), then \(k_0 \ll k\) and \(k + k_0 \approx k\). Therefore, if the wave vector \(k + k_0\) is close to \(k\), the frequency \(\omega + \omega_0\) can be any one, in fact, from the interval \([\omega, 10^6 \omega]\). In particular, for \(10^2 \omega \lesssim \omega_0 \lesssim 10^5 \omega\), we have \(k + k_0 \approx k\) and \(\omega + \omega_0 \approx \omega_0\). Such acouelon has
the wave length close to that of a sound wave (phonon) and the frequency close to that of an electromagnetic wave (photon). The modes \((1, -1)\) have similar properties as well.

In addition, the solutions \(\tilde{\rho}_{1,i} \) and \(\tilde{\rho}_{1,-i} \) \((31), (32)\) are characterized by a parametric resonance, respectively, at

\[
|u_{1,i}| = c_1(1 + \delta), \quad \delta \to 0
\]

and

\[
|u_{1,-i}| = c_1(1 + \delta), \quad \delta \to 0.
\]

At \(i = 1\), if any of these conditions is satisfied, we get a resonant growth of \(\chi \) \((34)\). As is seen, the resonance arises, if the phase velocity \(u_{1,\pm i}\) of a hybrid wave coincides with the sound velocity \(c_1\) for the medium without a field \(\mathbf{E}\).

Of course, solutions \((31)\) and \((32)\) do not work near the resonance point (i.e., as \(\delta \to 0\)). In order to obtain a solution in this region, one needs to use methods of the theory of nonlinear oscillations. In this case, we need to consider the viscosity in \((1)\) and the nonlinear terms in \((1)\) and \((2)\), as well as in the chain of equations for \(\tilde{\rho}_{1,\pm i}\) following from \((1), (2)\). In such approach, the solutions \(\tilde{\rho}_{1,\pm i}\) and \(\chi\) should be finite at the resonance point. We will restrict ourselves by solutions \((31)\)–\((35)\) which are true for values of \(k\) not too close to the resonance point. Therefore, we consider \(|\delta|\) to be small \(|\delta| \ll 1\), but not too small. For \(|\delta| \ll 1\), relations \((35), (37), (38)\) yield

\[
\chi \approx \frac{k^2}{2\delta \cdot c_1^2}.
\]

We consider \(|\delta|\) to be not too small, if \(|\delta| \gg \vartheta^2/16\). In this case, \(|\lambda| \approx \frac{\vartheta^2}{2c_1} \left(\frac{\xi E_0}{2m}\right)^2 \ll c_1\) and \(|u| \approx c_1\), according to \((33)\) and \((39)\).

For the modes \((1, 1)\) and \((1, -1)\), we consider a neighborhood of the resonance corresponding to not too small \(|\delta|\). Condition \((38)\) is equivalent to two conditions: \(u_{1,-1} = -c_1(1 + \delta)\) or \(u_{1,-1} = c_1(1 + \delta)\). In the first and second cases, the phase velocity \(u_{1,-1}\) is, respectively, negative and positive. Let \(u > 0\). The first condition yields the relations

\[
k_{1,-1} \approx k \approx \frac{k_0 c}{2c_1}, \quad u \approx c_1, \quad \omega_{1,-1} \approx -\frac{\omega_0}{2}.
\]

From the second condition we get

\[
k_{1,-1} \approx k \approx \frac{k_0 c}{2c_1}, \quad \delta_I = \frac{\chi}{2c_1} \left(\frac{\xi E_0}{2m}\right)^2 - \delta, \quad \delta \approx c_1, \quad \omega_{1,-1} \approx \frac{\omega_0}{\delta_I},
\]

where \(\zeta = 1\). Since \(|\delta_I| \ll 1\), the value of \(k_{1,-1} \) \((31)\) is much larger than \(k_{1,-1} \) \((31)\). For the real electric waves, the values of \(k_{1,-1} \) \((31)\) are very large and should go beyond the phonon region of the spectrum. Therefore, we do not consider solution \((31)\).

Condition \((38)\) with \(i = 2, 3, \ldots\) leads to solution \((31)\) with \(\zeta = 1\) and with the changes \((1, -1) \to (1, -i)\) and \(k_0 \to i k_0\), as well as to solution \((30)\) with the changes \((1, -1) \to (1, -i)\) and \(k_0 \to i k_0\):

\[
k_{1,-i} \approx k \approx \frac{i k_0 c}{2c_1}, \quad u \approx c_1, \quad \omega_{1,-i} \approx -\frac{i \omega_0}{2}.
\]
Formulas (42) describe a near-resonance solution for the modes \((1, -i)\) with \(i = 1, 2, \ldots\).

For \(u < 0\), condition (38) gives solutions with \(k < 0\) (what is unphysical) and \(k > 0\) (but \(k\) are too large and go beyond the phonon region).

Let us turn to condition (37). It can be written in the form \(u_{1,i} = c_i(1 + \delta)\) or \(u_{1,i} = -c_i(1 + \delta)\). In the first case for \(u > 0\) and \(i = 1\), we get solution (41) with \(\zeta = -1\) and the change \((1, -1) \rightarrow (1, 1)\). Here, we go outside the phonon region of the spectrum; we have the analogous situation for \(i = 2, 3, \ldots\). For \(u < 0\) the solution reads

\[
k_{1,i} \approx k \approx \frac{i k_0 c}{2 c_1}, \quad u \approx -c_1, \quad \omega_{1,i} \approx \frac{i \omega_0}{2}.
\]

It differs from (42) by signs of the phase velocities \(u\) and \(u_{1,i}\). In this case, the value of \(\chi\) is set by formula (39), like for solution (42). The second case, \(u_{1,i} = -c_i(1 + \delta)\), is possible for \(u < 0\). But here, the solutions are characterized by \(k\) outside the phonon region.

Thus, we have found two near-resonance solutions: (42) and (43). For clarity, let us consider their behavior, as the phonon wave vector \(k\) increases. For \(k\) ranging from the smallest value \(k = \pi/L\) (\(L\) is the resonator length) to \(k = 10^5 k_0\) (suppose that \(10^5 k_0 > \pi/L\)), we have \(|u_{1,\pm i}| \gg c_1\) at any \(i\). Therefore, the values of \(\tilde{p}_{1,\pm i}\) and \(\chi\) are small, and the sound velocity \(u \approx c_1\). However, at \(k \approx \frac{k_0 c}{2 c_1} \sim 10^6 k_0\), the relation \(|u_{1,\pm i}| \approx c_1\) holds, and the quantities \(\tilde{p}_{1,\pm i}\) and \(\chi\) increase in the resonance way. In this case for solutions (42) and (43), we have \(\delta, \chi > 0\) at \(k < \frac{k_0 c}{2 c_1}\) and \(\delta, \chi < 0\) at \(k > \frac{k_0 c}{2 c_1}\). Therefore, by (36), the phonon energy \(|\omega(k)|\) must be somewhat higher than \(c_1k\) at \(k < \frac{k_0 c}{2 c_1}\) and somewhat lower than \(c_1k\) at \(k > \frac{k_0 c}{2 c_1}\). Near the point \(k \approx \frac{k_0 c}{2 c_1}\), these deviations can be large. And at the very point \(k \approx \frac{k_0 c}{2 c_1}\), the phonon dispersion curve \(|\omega(k)|\) should be discontinuous, and the amplitude \(|\tilde{p}_{1,-1}|\) (or \(|\tilde{p}_{1,1}|\), depending on the sign of \(u\)) should sharply increase. In this case, the velocity \(u_{1,-1}\) (or \(u_{1,1}\)) becomes equal to the sound velocity \(c_1\). In other words, at the resonance point the hybrid mode becomes similar to a phonon, and vice versa. At \(k > \frac{k_0 c}{2 c_1}\), we leave the resonance region, as \(k\) increases. Near the points \(k \approx \frac{k_0 c}{2 c_1}\) (\(i = 2, 3, \ldots\)), the amplitudes \(\tilde{p}_{1,\pm i}\) have resonances.

3 Superfluid liquid dielectric (He II)

We now consider the analogous problem for superfluid He II. The equations of hydrodynamics for He II describe the motion of the normal and superfluid components [34] [37]:

\[
\partial J_i/\partial t + \sum_{j=1,2,3} \frac{\partial}{\partial r_j} (p \delta_{ij} + \rho_n v_{n,i} v_{n,j} + \rho_s v_{s,i} v_{s,j}) = F_i, \tag{44}
\]

\[
\partial \rho/\partial t + \text{div} \mathbf{J} = 0, \tag{45}
\]

\[
\partial (\rho s)/\partial t + \text{div} (\rho s \mathbf{v}_n) = 0, \tag{46}
\]

\[
\partial \mathbf{v}_s/\partial t + (\mathbf{v}_s \nabla) \mathbf{v}_s = -\nabla (\mu + \Omega), \tag{47}
\]
where $\rho = \rho_s + \rho_n$, $\mathbf{J} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$, and $\mathbf{F}/\rho = -\nabla \Omega$ is a nonmechanical force per unit mass. Such force acting on the superfluid component must be the gradient of some function (according to (47), this ensures the potentiality of the motion of the superfluid component, $\text{rot} \mathbf{v}_s = 0$). The microscopic substantiation of Eqs. (44)–(47) was proposed in [38].

Let the equilibrium system be characterized by the parameters $\rho_0, p_0, s_0, T_0 = \text{const}, \mathbf{v}_s = \mathbf{v}_n = 0$, and $\Omega = 0$. We now find the oscillatory modes of the system in the presence of a force $\mathbf{F} = -\rho \nabla \Omega$. As usual, the sound and thermal waves are considered as small deviations from the equilibrium. Therefore, we consider $\mathbf{v}_s$ and $\mathbf{v}_n$ to be small and $\rho, p, s, T$ to be close to the equilibrium values. Then, from (44)–(47) we can pass to the linearized system

$$\partial \mathbf{J}/\partial t + \nabla p = -\rho \nabla \Omega, \quad (48)$$
$$\partial \rho/\partial t + \text{div} \mathbf{J} = 0, \quad (49)$$
$$\partial(\rho s)/\partial t + \rho s \cdot \text{div} \mathbf{v}_n = 0, \quad (50)$$
$$\partial \mathbf{v}_s/\partial t = -\nabla (\mu + \Omega). \quad (51)$$

Equations (49)–(51) and the thermodynamic relation (52)

$$dp = \rho d\mu + \rho s dT + (\rho_n/2) d(\mathbf{v}_n - \mathbf{v}_s)^2$$

(we neglect the last term) yield the equation (53)

$$\frac{\partial^2 s}{\partial t^2} = \frac{s^2 \rho_s}{\rho_n} \Delta T. \quad (53)$$

In addition, Eqs. (48) and (49) lead to the equation (54)

$$\frac{\partial^2 \rho}{\partial t^2} = \Delta p + \rho \Delta \Omega. \quad (54)$$

We set $\rho = \rho_0 + \rho', p = p_0 + p', s = s_0 + s', T = T_0 + T'$, where $\rho', p', s'$, and $T'$ are small. Then it is convenient to write Eqs. (53), (54) in the form (57)

$$\frac{\partial s}{\partial p}|_T \frac{\partial^2 p'}{\partial t'^2} + \frac{\partial s}{\partial T}|_p \frac{\partial^2 T'}{\partial t'^2} - \frac{s^2 \rho_s}{\rho_n} \Delta T' = 0, \quad (55)$$

$$\frac{\partial \rho}{\partial p}|_T \frac{\partial^2 p'}{\partial t'^2} + \frac{\partial \rho}{\partial T}|_p \frac{\partial^2 T'}{\partial t'^2} - \Delta p' = \rho_0 \Delta \Omega. \quad (56)$$

These are the basic equations which will be analyzed in what follows. They differ from the Landau’s equations [37] by the additional term $\rho_0 \Delta \Omega$ characterizing the influence of the electric field on the oscillatory modes.

We write the perturbations $p'$ and $T'$ in (55), (56) as $p' = \tilde{p} \cos(kz - \omega t + \alpha)$ and $T' = \tilde{T} \cos(kz - \omega t + \alpha)$ and use the relations (57)

$$\frac{\partial \rho}{\partial p}|_T = \frac{C_p}{C_V C_l^2}, \quad \frac{\partial s}{\partial T}|_p = \frac{C_p}{T}. \quad (57)$$
Then, instead of (55) and (56) we get
\[
\left[ \dot{p} \left( -u^2 \frac{\partial s}{\partial p} \right) + \ddot{T} \left( \frac{s^2 \rho_n - u^2 C_p}{C_V} \right) \right] k^2 \cos (kz - \omega t + \alpha) = 0, \tag{58}
\]
\[
\left[ \dot{p} \left( 1 - \frac{u^2 C_p}{c_1^2 C_V} \right) + \ddot{T} \left( -u^2 \frac{\partial p}{\partial T} \right) \right] k^2 \cos (kz - \omega t + \alpha) = \rho_0 \Delta \Omega, \tag{59}
\]
where \( u = \omega/k \). For \( \Omega = 0 \), Eqs. (58) and (59) and formula [34]
\[
\frac{\partial \rho}{\partial T} \bigg|_p = \frac{C_p}{T c_1^2} \left( \frac{C_p}{C_V} - 1 \right) \tag{60}
\]
yield the well-known equation for the velocities of first and second sounds [34]:
\[
\left( \frac{u^2}{c_1^2} - 1 \right) \left( \frac{u^2}{c_2^2} - 1 \right) + \frac{C_V}{C_p} - 1 = 0, \tag{61}
\]
where
\[
c_1^2 = \frac{\partial p}{\partial s} \bigg|_s, \quad c_2^2 = \frac{\rho s^2 T}{\rho_n C_V}. \tag{62}
\]
We note that the electric field \( \mathbf{E} = E_0 \mathbf{i}_z \sin (k_0 z - \omega_0 t) \) depends only on the coordinate \( z \) and the time. It is clear from the symmetry of the problem that, for the infinite system, \( \rho', p', s', \) and \( T' \) should depend only on \( z \) and \( t \) as well. Let us set \( \rho' = \bar{\rho} \cos (kz - \omega t + \alpha) \). Then the force [11] can be represented in the form \( \mathbf{F} = -\rho \nabla \Omega \), where
\[
\Omega = -\frac{\varepsilon - 1}{16 \pi \rho} E_0^2 (1 - \cos (2k_0 z - 2\omega_0 t)) - \frac{\xi E_0 k_0 \bar{\rho}}{2 \mu \rho_0}
\cdot \left\{ \left( \frac{k_0}{k_{1,1}} + a - 1 \right) \cos (k_{1,1} z - \omega_{1,1} t + \alpha) + \right. \\
\left. + \left( \frac{k_0}{k_{1,-1}} - a + 1 \right) \cos (k_{1,-1} z - \omega_{1,-1} t + \alpha) \right\} \tag{63}
\]
and \( k_{1,-1} \neq 0 \). For \( k_{1,-1} = 0 \) the term \( \frac{k_0}{k_{1,-1}} \cos (k_{1,-1} z - \omega_{1,-1} t + \alpha) \) should be replaced by \( k_0 z \sin (\omega_{1,-1} t - \alpha) \).

We seek the solutions for \( \rho', p', s', \) and \( T' \) in the form of expansions analogous to (15), (16). In this case, \( \Omega \) acquires a rather awkward form, but it can be easily found with the help of (63). We substitute the formula for \( \Omega \) and the expansions for \( p' \) and \( T' \) in (55), (56). With regard for formula (57), Eqs. (55) and (56) take, respectively, the forms
\[
\sum_{l,i} A_{l,i} \cos (k_{l,i} z - \omega_{l,i} t + q_{l} \alpha) = 0, \quad \sum_{l,i} B_{l,i} \cos (k_{l,i} z - \omega_{l,i} t + q_{l} \alpha) = 0. \tag{64}
\]
Here, analogously to the previous section, \( l = 0, 1; i = 0, \pm 1, \pm 2, \ldots \) (the case \( l = i = 0 \) is excluded), and \( q_0 = 0, q_1 = 1 \). Equations (64) are valid for \( A_{l,i} = 0, B_{l,i} = 0 \) for all \( l, i \). In such a way, relation (55) yields
\[
\tilde{T}_{l,i} = \tilde{p}_{l,i} \frac{u_{l,i}^2 \frac{\partial s}{\partial p} \bigg|_T}{s^2 \rho_n - u_{l,i}^2 C_p \bigg|_T}, \tag{65}
\]
where \( u_{t,i} = \omega_{t,i}/k_{t,i} \). Moreover, Eq. 56 yields the following chain of equations:

\[
\tilde{p}_{0,1} \left( 1 - \frac{u_{0,1}^2 C_p}{c_1^2 C_V} \right) - \tilde{T}_{0,1} \left( \frac{c_1^2 \partial \rho}{\partial T} \right) = \frac{\xi E_0}{2m} \tilde{p}_{0,2} 2k_0 \left( \frac{1}{2} - a + 1 \right),
\]

\[
\tilde{p}_{0,2} \left( 1 - \frac{u_{0,2}^2 C_p}{c_1^2 C_V} \right) - \tilde{T}_{0,2} \left( \frac{c_1^2 \partial \rho}{\partial T} \right) = -\frac{\varepsilon - 1}{16\pi} E_0^2 + \frac{\xi E_0}{2m} \left[ \tilde{p}_{0,1} k_0 \left( \frac{1}{2} + a - 1 \right) + \tilde{p}_{0,3} 3k_0 \left( \frac{1}{2} - a + 1 \right) \right],
\]

\[
\tilde{p}_{0,j} \left( 1 - \frac{u_{0,j}^2 C_p}{c_1^2 C_V} \right) - \tilde{T}_{0,j} \left( \frac{c_1^2 \partial \rho}{\partial T} \right) = \frac{\xi E_0}{2m} \left[ \tilde{p}_{0,j-1} k_{0,j-1} \left( \frac{1}{j} + a - 1 \right) + \tilde{p}_{0,j} k_{0,j+1} \left( \frac{1}{j} - a + 1 \right) \right],
\]

where \( j = 3, 4, \ldots, i = \pm 1, \pm 2, \ldots, \) and \( u_{0,1} = u_{0,2} = \ldots = u_{0,j} = c. \) We solve Eqs. 66–70 similarly to Sect. 2. We substitute

\[
\tilde{p}_{l,i} = \frac{\partial \rho}{\partial T} T \cdot \tilde{p}_{l,i} + \frac{\partial \rho}{\partial T} \tilde{T}_{l,i} = \frac{C_p}{c_1^2 C_V} \tilde{p}_{l,i} + \frac{\partial \rho}{\partial T} \tilde{T}_{l,i}
\]

into the right-hand sides of those equations and then present \( \tilde{T}_{l,i} \) in terms of \( \tilde{p}_{l,i} \) with the help of formula 65.

At small \( \vartheta \) Eqs. 66–68 yield

\[
\tilde{p}_{0,2} \approx \frac{\varepsilon - 1}{16\pi} \frac{E_0^2 C_V c_1^2}{C_p c^2},
\]

\[
\tilde{p}_{0,1} \approx \frac{\xi E_0 k_0 (a - 2)}{mc^2} \frac{C_V}{C_p} \tilde{p}_{0,2},
\]

\[
\tilde{p}_{0,3} \approx \frac{\xi E_0 k_0 (2/3 - a)}{mc^2} \frac{C_V}{C_p} \tilde{p}_{0,2}.
\]

The remaining \( \tilde{p}_{0,j} \) are very small: \( \tilde{p}_{0,4} \sim \vartheta^2 \tilde{p}_{0,2}, \tilde{p}_{0,5} \sim \vartheta^3 \tilde{p}_{0,2}, \) etc.
With the help of formulas (65) and (71), we write (69), (70) as the equations for \( \tilde{p}_{1,i} \):

\[
\tilde{p}_{1,0} \frac{G_{1,0}}{c_p} - u_{1,0}^2 \frac{c_p}{c_p^2} = \frac{\xi E_0}{2m} \left[ \tilde{p}_{1,-1}(k - k_0) \left( \frac{k_0}{k} + a - 1 \right) \frac{C_p (c_1^2 - u_{1,-1}^2)}{c_1^2 (c_1^2 C_V - u_{1,-1}^2 C_p)} + \right. \\
+ \left. \tilde{p}_{1,1}(k + k_0) \left( \frac{k_0}{k} - a + 1 \right) \frac{C_p (c_2^2 - u_{1,1}^2)}{c_2^2 (c_2^2 C_V - u_{1,1}^2 C_p)} \right],
\]

(75)

\[
\tilde{p}_{1,i} \frac{G_{1,i}}{c_p} - u_{1,i}^2 \frac{c_p}{c_p^2} = \frac{\xi E_0}{2m} \left[ \tilde{p}_{1,i-1}k_{1,i-1} \left( \frac{k_0}{k_{1,i}} + a - 1 \right) \frac{c_2^2 C_V - u_{1,i-1}^2 C_p}{c_2^2 C_V - u_{1,i-1}^2 C_p} \frac{c_2^2 - u_{1,i-1}^2}{c_2^2 G_{1,i}} \right. \\
+ \left. \tilde{p}_{1,i+1}k_{1,i+1} \left( \frac{k_0}{k_{1,i}} - a + 1 \right) \frac{c_2^2 C_V - u_{1,i+1}^2 C_p}{c_2^2 C_V - u_{1,i+1}^2 C_p} \frac{c_2^2 - u_{1,i+1}^2}{c_2^2 G_{1,i}} \right],
\]

(76)

where \( i = \pm 1, \pm 2, \ldots \), and we denoted

\[
G_{1,i} = \left( \frac{u_{1,i}^2}{c_1^2} - 1 \right) \left( \frac{u_{1,i}^2}{c_2^2} - 1 \right) + \frac{C_V}{C_p} - 1.
\]

(77)

At small \( \vartheta \), formula (76) leads to the recurrence relations

\[
\tilde{p}_{1,i} \approx \frac{\xi E_0}{2m} \tilde{p}_{1,i-1}k_{1,i-1} \left( \frac{k_0}{k_{1,i}} + a - 1 \right) \frac{c_2^2 C_V - u_{1,i-1}^2 C_p}{c_2^2 C_V - u_{1,i-1}^2 C_p} \frac{c_2^2 - u_{1,i-1}^2}{c_2^2 G_{1,i}},
\]

(78)

\[
\tilde{p}_{1,i} \approx \frac{\xi E_0}{2m} \tilde{p}_{1,i+1}k_{1,i+1} \left( \frac{k_0}{k_{1,i+1}} - a + 1 \right) \frac{c_2^2 C_V - u_{1,i+1}^2 C_p}{c_2^2 C_V - u_{1,i+1}^2 C_p} \frac{c_2^2 - u_{1,i+1}^2}{c_2^2 G_{1,i}}
\]

(79)

\((i = 1, 2, \ldots)\). These relations allow us to present \( \tilde{p}_{1,\pm i} \) in terms of \( \tilde{p}_{1,0} \). Like in Section 2, we consider the quantity \( \tilde{p}_{1,0} \) to be known. We now substitute \( \tilde{p}_{1,1} \) (78) and \( \tilde{p}_{1,-1} \) (79) in Eq. (75), reduce both sides of the equation by \( \tilde{p}_{1,0} \), and determine the dispersion relation

\[
G_{1,0} \equiv \left( \frac{u^2}{c_1^2} - 1 \right) \left( \frac{u^2}{c_2^2} - 1 \right) + \frac{C_V}{C_p} - 1 = \frac{\chi}{c_2^2} \left( \frac{\xi E_0}{2m} \right)^2,
\]

(80)

where \( u \equiv u_{1,0} \) and

\[
\chi \approx \frac{c_2^2 - u^2}{c_1^4} \left\{ [k_0 + (a - 1)k][ak_0 - (a - 1)k] \frac{c_2^2 - u_{1,-1}^2}{c_2^2 G_{1,-1}} + \right. \\
+ \left. [k_0 - (a - 1)k][ak_0 + (a - 1)k] \frac{c_2^2 - u_{1,1}^2}{c_2^2 G_{1,1}} \right\}.
\]

(81)

For \( a = 1 \) we get

\[
\chi \approx \frac{k_0}{c_1^4} \left( c_2^2 - u^2 \right) \left\{ \frac{c_2^2 - u_{1,-1}^2}{c_2^2 G_{1,-1}} + \frac{c_2^2 - u_{1,1}^2}{c_2^2 G_{1,1}} \right\}.
\]

(82)

For liquid \(^4\)He the quantity \( C_p/C_V - 1 \) is very small: \( 0 < C_p/C_V - 1 \lesssim 0.0005 \) for the He II temperatures and pressures \( \lesssim 0.1 \text{ atm} \). Therefore, it is convenient to write (80) in the form

\[
\left( \frac{u^2}{c_1^2} - 1 \right) \left( \frac{u^2}{c_2^2} - 1 \right) = 1 - \frac{C_V}{C_p} + \frac{\chi}{c_2^2} \left( \frac{\xi E_0}{2m} \right)^2 \equiv 2\delta_u,
\]

(83)
where $\delta_u$ is small ($0 < \delta_u \ll 1$) at sufficiently small $\vartheta$. From (83) we get the solutions for the velocities of first and second sounds:

$$|u| \approx c_1 \left( 1 + \frac{\delta_u}{c_1^2/c_2^2 - 1} \right), \quad (84)$$

$$|u| \approx c_2 \left( 1 + \frac{\delta_u}{c_2^2/c_1^2 - 1} \right). \quad (85)$$

We have found the solutions for small oscillations of the pressure in a superfluid dielectric placed in the electric field $E = E_0 i_z \sin (k_0 z - \omega_0 t)$.

We now verify whether the solutions for He II pass into solutions for He I as $\rho_s \to 0$ ($c_2 \to 0$). At $C_p = C_V$ and $c_2 \to 0$, (80) yields (83). Turning $c_2 \to 0$ and $u \to c_1$ (81), it is easy to see that formula (81) passes in (34). In reality, we have $C_p \neq C_V$. Therefore, the solutions for He II do not pass exactly into solutions for He I, which is related to the fact that the first sound in He II is not quite identical to the ordinary sound in He I.

Formulas (78), (79), (81) imply that the quantities $\tilde{p}_{1,\pm 1}$ and $\chi$ should resonantly increase as $G_{1,\pm 1} \to 0$. To find solutions in a neighborhood of the resonance, we set $G_{1,-1} = 2\delta$ (or $G_{1,1} = 2\delta$). Like in the previous section, we consider $|\delta|$ to be small, but not too small $(1 - C_V/C_p \ll |\delta| \ll 1)$. Then the condition $G_{1,-1} = 2\delta$ is equivalent to four possible solutions for $u_{1,-1}$:

$$u_{1,-1} \approx \pm c_1 \left( 1 + \frac{\delta_{1,-1}}{c_1^2/c_2^2 - 1} \right), \quad (86)$$

$$u_{1,-1} \approx \pm c_2 \left( 1 + \frac{\delta_{1,-1}}{c_2^2/c_1^2 - 1} \right), \quad (87)$$

where $2\delta_{1,-1} = 2\delta + 1 - C_V/C_p$. The situation is analogous for the condition $G_{1,1} = 2\delta$. By analyzing these solutions, we should take into account that the phase velocity $u$ in (84) and (85) can be positive or negative. In such a way, we find the following near-resonance solutions with positive and not too large (phonon) values of $k$ for the modes $\tilde{p}_{1,-1}$:

$$k_{1,-1} \approx k \approx \frac{ck_0}{c_1 + c_2}, \quad \omega_{1,-1} \approx -\frac{c_1\omega_0}{c_1 + c_2}, \quad u_{1,-1} \approx -c_1, \quad u \approx c_2, \quad (88)$$

$$k_{1,-1} \approx k \approx \frac{ck_0}{c_1 - c_2}, \quad \omega_{1,-1} \approx -\frac{c_1\omega_0}{c_1 - c_2}, \quad u_{1,-1} \approx -c_1, \quad u \approx -c_2, \quad (89)$$

$$k_{1,-1} \approx k \approx \frac{ck_0}{c_1 + c_2}, \quad \omega_{1,-1} \approx -\frac{c_2\omega_0}{c_1 + c_2}, \quad u_{1,-1} \approx -c_2, \quad u \approx c_1, \quad (90)$$

$$k_{1,-1} \approx k \approx \frac{ck_0}{2c_2}, \quad \omega_{1,-1} \approx -\frac{\omega_0}{2}, \quad u_{1,-1} \approx -c_2, \quad u \approx c_2, \quad (91)$$

$$k_{1,-1} \approx k \approx \frac{ck_0}{c_1 - c_2}, \quad \omega_{1,-1} \approx -\frac{c_2\omega_0}{c_1 - c_2}, \quad u_{1,-1} \approx c_2, \quad u \approx c_1, \quad (92)$$

$$k_{1,-1} \approx k \approx \frac{ck_0}{2c_1}, \quad \omega_{1,-1} \approx -\frac{\omega_0}{2}, \quad u_{1,-1} \approx -c_1, \quad u \approx c_1. \quad (93)$$
Resonance (93) is characterized by the resonance-like increase in the value of \( \chi \approx \frac{k_1^2 (c_1^2 - c_2^2)^2}{28 c_1^2 c_2^2} \) as \( \delta \to 0 \). In this case, according to (88), (89), (93), the velocity of the first sound resonantly varies as \( \delta \to 0 \). For solutions (88) and (89), we find \( \chi \approx -\frac{k_1^2}{c_1^4} (1 - C_V/C_p) [2\delta + \vartheta^2 c_1^2/(4c_2^2)]^{-1} \).

Here, the resonances are possible for \( \chi \) and the velocity of the second sound. For solutions (90)–(92), the value of \( \chi \) is close to a constant as \( \delta \to 0 \) (\( \chi \approx -k_0^2/c_1^2 \) for (90) and (92), and \( \chi \approx \frac{k_1^2 c_1^2}{(c_1^2 - c_2^2)^2} (1 - C_V/C_p) [1 - \vartheta^2 c_1^2/(4c_2^2)]^{-1} \) for (91)); that is, there is no resonance for \( \chi \).

For the mode \( \tilde{p}_{1,1} \) we get the following near-resonance solutions:

\[
\begin{align*}
 k_{1,1} & \approx k \approx \frac{ck_0}{c_1 + c_2}, \quad \omega_{1,1} \approx \frac{c_1 \omega_0}{c_1 + c_2}, \quad u_{1,1} \approx c_1, \quad u \approx -c_2, \\
 k_{1,1} & \approx k \approx \frac{ck_0}{c_1 + c_2}, \quad \omega_{1,1} \approx \frac{c_2 \omega_0}{c_1 + c_2}, \quad u_{1,1} \approx c_2, \quad u \approx -c_1, \\
 k_{1,1} & \approx k \approx \frac{ck_0}{c_1 - c_2}, \quad \omega_{1,1} \approx \frac{-c_2 \omega_0}{c_1 - c_2}, \quad u_{1,1} \approx -c_2, \quad u \approx -c_1, \\
 k_{1,1} & \approx k \approx \frac{ck_0}{2c_2}, \quad \omega_{1,1} \approx \frac{\omega_0}{2}, \quad u_{1,1} \approx c_2, \quad u \approx -c_2, \\
 k_{1,1} & \approx k \approx \frac{ck_0}{2c_1}, \quad \omega_{1,1} \approx \frac{\omega_0}{2}, \quad u_{1,1} \approx c_1, \quad u \approx -c_1, \\
 k_{1,1} & \approx k \approx \frac{ck_0}{c_1 - c_2}, \quad \omega_{1,1} \approx \frac{c_1 \omega_0}{c_1 - c_2}, \quad u_{1,1} \approx c_1, \quad u \approx c_2. 
\end{align*}
\]

The function \( \chi(\delta) \) is not constant as \( \delta \to 0 \) for solutions (98) (\( \chi \approx \frac{k_0^2 (c_1^2 - c_2^2)^2}{28 c_1^2 c_2^2} \)) and (99), (94) (\( \chi \approx -\frac{k_0^2}{c_1^2} (1 - C_V/C_p) [2\delta + \vartheta^2 c_1^2/(4c_2^2)]^{-1} \)). In the last case due to the smallness of \( 1 - C_V/C_p \) it will be apparently difficult to observe \( \chi \) near the resonance (\( \delta \to 0 \)).

If we replace \( k_0 \to ik_0 \) and \( \omega_0 \to i\omega_0 \) in formulas (88)–(93) and (94)–(99), we get the near-resonance solutions for \( \tilde{p}_{1,-i} \) and \( \tilde{p}_{1,i} \), respectively (\( i \geq 2 \)).

We note that, according to (78) and (79), the quantities \( \tilde{p}_{1,i} \) and \( \tilde{p}_{1,-i} \) (\( i \geq 1 \)) should sharply increase also as \( c_2^2 C_V - u_{1,i-1}^2 C_p \to 0 \) and \( c_2^2 C_V - u_{1,i+1}^2 C_p \to 0 \), respectively (in this case, there is no resonance for \( \chi \) and other \( \tilde{p}_{1,\pm i} \)). For \( i = 2 \), this leads to solutions (90)–(92) and (95)–(97) (if we replace \( k_0 \to (i - 1)k_0 \) and \( \omega_0 \to (i - 1)\omega_0 \) in them, we get solutions for \( i > 2 \)). Thus, each of solutions (90)–(92) and (95)–(97) corresponds to two closely located resonances. In order to distinguish them theoretically, we should find solutions directly at resonance points.

### 4 Main physical consequences

We will try to understand the physical nature of solutions and discuss which of the above-found peculiarities can be observed.

Let us estimate the intensity of the above-obtained modes for He I (for He II, the results are analogous). At small \( \vartheta \), the mode \((2\omega_0, 2k_0)\) is the most intense from the modes \((i\omega_0, ik_0)\)
(both for He I and He II). It is a density wave whose phase velocity is equal to the velocity of light. For this mode, the frequency and the wave vector are two times larger than for the field $E$. Let us consider critical the field $E_0 = E_0^c$ for which $\rho_{0,2} = 0.01 \rho_0$ (for $E_0 > E_0^c$ the density perturbation $\rho_{0,2}$ becomes sufficiently high so that our approximation of small perturbations fails). With the use of the parameters of He II $\bar{\alpha}_0 = 3.58 \text{ Å}$, $d_0 \approx -1.88 \cdot 10^{-5} |e|\text{Å}^{[6]}$ and formula (26), we find $E_0^c \approx 3.4 \cdot 10^{10} \text{g/cm/seg} \approx 10^{15} \text{V/m}$. This is a very strong field. For comparison, in experiments $^{[27,28]}$ the field $E_0$ near a resonator was 11 orders of magnitude weaker. For $E_0 = E_0^c$, $a = 1$, and $L = 1 \text{ cm}$, formulas (26), (28) yield $\bar{\rho}_{0,1} \sim \bar{\rho}_{0,3} \sim 10^{-21} \bar{\rho}_{0,2}$ (in this case, we used the resonance relations $k_0 \approx 2c_1 k/c$ and $k = \pi/L$, see below).

From the hybrid modes $(\omega \pm i\omega_0, k \pm ik_0)$, the modes $(\omega \pm \omega_0, k \pm k_0)$ are the most intense. From formula (31) for $E_0 = E_0^c$, $a = 1$, $k_0 \approx 2c_1 k/c$, $k = \pi/L$, $L = 1 \text{ cm}$, and $|u_{1,1}| = c_1 (1 + \delta)$ (resonance for the $(1,1)$-mode), we get $\bar{\rho}_{1,1} \approx (\delta/4\delta)\bar{\rho}_{1,0} \approx (6 \cdot 10^{-10}/\delta)\bar{\rho}_{1,0}$. Here we also see the smallness of $|\vartheta|$: $\vartheta \approx -2.5 \cdot 10^{-9}$. For the resonance of the $(1, -1)$-mode, the estimate is analogous. That is, all hybrid waves are much weaker than the bare acoustic wave $\bar{\rho}_{1,0}$. These estimates show that, in real fields $E_0 \ll E_0^c$, all waves-satellites of the $(0, i)$- and $(1, i)$-modes are extremely weak. In this case, the $(1, i)$-modes can in principle become observable, if the frequency $\omega_0$ is very close to the resonance one or if the amplitude $\bar{\rho}_{1,0}$ of a bare acoustic (or a thermal one, for He II) wave $(\omega, k)$ is artificially made very high.

For the phase velocity $u$ of an acoustic wave in He I, we have found that $u^2 = c_1^2 + (\delta u)^2$, where $\delta u = \sqrt{\chi E_0}/2m$. The estimate with the use of the parameters above gives $\delta u \approx c_1 \vartheta/\sqrt{8\delta} \approx 10^{-9} c_1/\sqrt{\delta}$ (near resonance). Such value can be observable (i.e., $\sim c_1$) only in a small vicinity of the resonance. For the second sound, $\chi \sim 1 - C_V/C_p$, which suppresses $\delta u$. As is known from the theory of oscillations $^{[40]}$, the amplitude of oscillations at the resonance point should increase with the time until the growth is terminated by the nonlinear viscosity and nonlinear corrections which were neglected in the solution of $^{[11]}$, (2). In addition, the linear viscosity leads to that the resonance exists only in a field $E_0$ higher than some threshold one.

Near the resonances, the hybrid modes $(\omega \pm \omega_0, k \pm k_0)$ sharply increase. We note that the hybrid modes at resonances are characterized by the phase velocity equal to the velocity of the first or second sound. It is natural that the energy of an electric wave easily transits into the energy of a hybrid mode, if this mode is similar to an eigenmode of the system. Thus, the resonance point corresponds to the intersection of dispersion curves of the sound and hybrid modes (see Fig. 1). In this case, apparently, the reconnection (hybridization) of two curves should occur. To clarify this point, one needs to accurately find a solution near the resonance.

The absorption of the energy of an electromagnetic wave that occurs at the resonance amplification of a hybrid mode has the quantum origin. We can try to establish the character of the process from the conditions for a resonance. For example, for resonances $^{[42]}$ and $^{[93]}$, we have $u_{1,-i} \approx -c_1$ and $u \approx c_1$. Therefore, $\omega_{1,-i} \equiv \omega - i\omega_0 = uk - ick_0 = u_{1,-i} k_{1,-i} \approx u_{1,-i} k \approx -c_1 k$ (here, we took into account that $k_0 \ll k$), which yields $ichk_0 \approx 2c_1 hk$. The
Fig. 1: [Color online] The dispersion laws $\omega_{1,\pm 1}(k_{1,\pm 1})$ of the hybrid modes $(1, -1)$ and $(1, 1)$ for He I at different $\omega_0$. 1) The mode $(1, -1)$ for $\omega_0 = 2\pi c_1/L$ (open circles), $\omega_0 = 6\pi c_1/L$ (open triangles), and $\omega_0 = 20\pi c_1/L$ (open diamonds); 2) the mode $(1, 1)$ for $\omega_0 = 2\pi c_1/L$ (filled circles), $\omega_0 = 6\pi c_1/L$ (filled triangles), and $\omega_0 = 20\pi c_1/L$ (filled diamonds). Resonance points (stars) correspond to the intersection of the curves of the modes $(1, \pm 1)$ with the acoustic dispersion law $\omega = \pm c_1 k$ (solid lines), which is characteristic of the medium without a field $E$; $\omega_{1,\pm 1}$ and $k_{1,\pm 1}$ are given in dimensionless units, for which $c_1 = 2\pi/L = 1$; values of $\omega_0$ correspond to formula (104) with $i = 1$ and $j = 1, 3, 10$. In the formula $\omega_{1,\pm 1} = uk \pm ck_0$, we set $u = c_1$ for the mode $(1, -1)$ and $u = -c_1$ for the mode $(1, 1)$; in this case, we omit the fact that the values of $|u|$ at the resonance points should significantly differ from the values of $c_1$.

same equation can be obtained for resonances (43) and (98). For $i = 1$, we get $chk_0 \approx 2c_1\hbar k$. This indicates that such resonance corresponds to the decay of a photon with energy $chk_0$ into two phonons, each possessing the energy $c_1\hbar k$. Since $k_0 \sim 10^{-6}k$, the momentum conservation law can hold for low $k \ll 10\pi/L$ only if the phonons have the momenta $\hbar k$ and $-\hbar k$, and the photon momentum is transferred to the whole liquid. Therefore, we assume that resonances (42) and (93) for $i = 1$ and low $k$ correspond to the following exact equations:

$$chk_0 = u\hbar k + u\hbar k + P_{liq}^2/(2M),$$  

(100)

$$\hbar k_0 = \hbar k - \hbar k + P_{liq} = P_{liq},$$

(101)

where $P_{liq}^2/(2M)$ and $P_{liq}$ are, respectively, the energy and momentum of the liquid as a whole, $M$ is the liquid mass, and $u$ is the velocity of the first sound in the medium with field $E$. Such process is similar to the Mössbauer effect [41].

The Mössbauer effect is observed in crystals. In this effect, the momentum is transferred to the whole crystal due to its stiffness. In our case, the momentum should be transferred to the liquid which has no stiffness. Apparently, this is possible due to that the process has the quantum origin and involves the whole system (because the wavelength $\lambda$ of a phonon is of the order of system size $L$, and $\lambda$ of a photon is much larger than $L$).

Resonances (42), (43), (93), (98) with any $i$ correspond to the conditions

$$ichk_0 = u\hbar k + u\hbar k + (P_{liq})^2/(2M),$$

(102)
Here, $i$ photons with the same momentum are transformed into two phonons with opposite momenta, and the recoil momentum $ihk_0$ is transferred to a liquid as a whole. It is clear that the process with $i \geq 2$ and the reverse process with any $i$ should be unlikely. We note that since the field $E$ has the form of a running wave, the momenta of all photons must be directed to the same side.

The conditions for resonances (88) and (91) for any $i$ lead to the relation $i\hbar k_0 \approx c_1 \hbar k + c_2 \hbar k$. This can be interpreted as the coalescence of $i$ photons with the formation of a phonon and a “quantum” of the thermal wave with the momenta $\hbar k$ and $-\hbar k$. In a similar way, resonances (89) and (92) give the relation $i\hbar k_0 + c_2 \hbar k \approx c_1 \hbar k$ which can be interpreted as the coalescence of $i$ photons and a “quantum” of the thermal wave with the creation of a phonon. Such interpretations are questionable, because a thermal wave is a classical structure, namely, a wave in the gas of quasiparticles. Nevertheless, it is worth to verify in experiments whether the spectrum of electromagnetic waves has the absorption lines at the corresponding $\omega_0$.

The other resonances correspond to the second sound: $u_{1,\pm i} \approx \pm c_2$. Consider the resonances for the modes $(\omega \pm \omega_0, k \pm k_0)$ for He II. These are solutions (90)–(92) and (95)–(97). They can be joined in pairs: (90), (95); (91), (96); and (92), (97). In each pair, the waves have the same wave vector, and the phase velocities of waves differ from one another only by a sign. The sum of such waves forms a standing wave (we assume that the constants $\alpha$ for both waves are close; this is possible, if the system contains many phonons or second-sound waves). Such a standing wave is stable, if its wavelength is $\lambda = 2L/j$, where $j = 1, 2, 3, \ldots$, and $L$ is the resonator length. Therefore, the following equalities must be valid: $k_{1,\pm 1} = 2\pi/\lambda = \pi j/L$, $|\omega_{1,\pm 1}| = c_2 \pi j/L$. It follows from (90)–(92) that $\omega_0 = 2\pi j c_2/L$ or $\omega_0 = \pi j (c_1 \pm c_2)/L$. The account for the resonances for the modes $(\omega \pm i\omega_0, k \pm ik_0)$ with $i > 1$ leads to the more general formulas: $\omega_0 = 2\pi j c_2/(iL)$ or $\omega_0 = \pi j (c_1 \pm c_2)/(iL)$, $i = 1, 2, \ldots$. However, all these resonances are suppressed by the factors $c_2^2 C_V - u_{1,i}^2 C_p$ in (78) and $c_2^2 C_V - u_{1,i}^2 C_p$ in (79) which are close to zero at $u_{1,\pm i} \approx c_2$ (the condition which should be satisfied for the second sound). In the experiment [32], the frequency band $\omega_0 = 2\pi j c_2/4L$, $j = 1 \div 8$, including our theoretical frequencies $2\pi j c_2/(iL)$ with $j = 1, i = 1, 2, 3, 4$, was measured. In this case, no second sound was registered. This result agrees with our analysis in view of the above-indicated suppression and the fact that the second-sound wave can exist only at resonances $\omega_0$ (note that the search for such narrow bands at $\omega_0$ was not performed in [32]).

Such suppression is absent for the hybrid modes with the velocity of the first sound ($u_{1,\pm 1} \approx \pm c_1$). Therefore, such modes should be intense near a resonance. From formulas (88), (89), (93), (94), (95), and (92), we get that the standing wave of the first sound with $\lambda = 2L/j$, corresponding to the hybrid mode, is possible for $k_{1,\pm 1} = \pi j/L \approx k$, $|\omega_{1,\pm 1}| = c_1 \pi j/L$, $\omega_0 = 2\pi j c_1/L$ (or $\omega_0 = \pi j (c_1 \pm c_2)/L$). The account for resonances for the modes $(\omega \pm i\omega_0, k \pm ik_0)$
with $i > 1$ leads to the following formulas for He II:

$$\omega_0 = 2\pi j c_1 / (i L) \approx 2\omega / i,$$

(104)

or

$$\omega_0 = \pi j (c_1 \pm c_2) / (i L) \approx (\omega_1 \pm \omega_2) / i,$$

(105)

where $i, j = 1, 2, \ldots$. For He I, we have only relation (104). Relations (104) and (105) for frequencies are characteristic of the parametric resonance [40]. Thus, if the frequency of the electric field is close to (104) or (105), then an acoustic gage should register the weak first sound with the frequency $|\omega_1, \pm i| = c_1 \pi j / L$ equal to $i\omega_0 / 2$ or $i\omega_0 c_1 / (c_1 \pm c_2)$. As $i$ increases, the width $\Delta \omega$ of the resonance strongly decreases, as usual [40] (for our solutions, $\Delta \omega_{i,j} \sim |\theta| \omega_{i,j}^{res}$; though the account for small nonlinear corrections, including the friction, strongly affects $\Delta \omega_{i,j}$ and can increase it by many orders of magnitude). Therefore, only the resonances with $i = 1$ can be apparently observed. For $j$ such limitation is absent. We believe that one needs to seek in experiments firstly the modes with small $j$ ($\lesssim 10$), because the standing waves with such $j$ are more stable. For example, the frequency $\omega_0 = 2\pi c_1 / L$ corresponds to formula (104) with $i, j = 1$, or $i, j = 2$, and so on. In this case, a first-sound wave with $\omega = c_1 \pi / L$ and very weak waves with $\omega = j c_1 \pi / L$ ($j = 2, 3, \ldots$) should arise. The hybrid mode corresponding to the very resonance point should get the highest amplification. We did not find the amplitudes $\tilde{\rho}_{1,\pm i}$ at the resonance point. If $\tilde{\rho}_{1,\pm 1} \gg \tilde{\rho}_{1,0}$ at the resonance point, then the hybrid wave with $\omega = c_1 \pi / L$ is intense enough and can be observed.

Note that since the hybrid modes accompany always the acoustic (or thermal) one $(1, 0)$, they can be considered as a “coat” of the acoustic (thermal) wave arising in the presence of an electric field $E = E_0 z \sin (k_0 z - \omega_0 t)$. We have considered only the acoustic modes running along the field $E$. It is clear that the hybrid waves-satellites should arise also for the modes running not in the line of $E$. Most likely, the resonances also exist for them. The neutron passing through a liquid should create namely a “dressed” phonon. Therefore, based on the spectrum of scattered neutrons, we will find the dispersion law of dressed phonons. In this case, $k$ of a phonon should be quantized as usual, because the law of quantization is defined by boundary conditions. But the energy of a dressed phonon should differ from the energy of a “bare” one according to the formula $E_{n}^{\text{phon}}(k) = E_{n}^{\text{phon}}(k) + \alpha_1(k) E_0 + \alpha_2(k) E_0^2 + \ldots$, where $E_{n}^{\text{phon}} = c_1 k$ is the energy of a bare phonon, and the constants $\alpha_j$ can be determined from a microscopic calculation. Such constants should be negligibly small (except for the cases where $k$ and $k_0$ are resonance quantities) and have no influence on the heat capacity of the system.

Formulas (104) and (105) are obtained on the basis of the classical approach in Sections 2 and 3. The quantum formulas (102) and (103) yield the condition

$$\omega_0 = \frac{2\pi j u}{i L} \left[ 1 + \frac{uhk}{Mc^2} \right],$$

(106)

where $k = \pi j / L$ and it is assumed that $k$ and $k_0$ are co-directional. The distinction of formulas
and (106) is mainly related to the difference of the values of \( u \) and \( c_1 \) (velocities of dressed and bare phonons, respectively), since the correction \( u \hbar k/(Me^2) \) is negligible for a macroscopic body. According to our analysis, the value of \( u \) near resonances should significantly differ from \( c_1 \). Nevertheless, we think that formulas (104) and (106) describe the same resonance. The difference of these formulas can be due to the fact that the classical approach in Sections 2 and 3 somewhat distorts the exact quantum solutions.

In the general case, when in Eqs. (102) and (103) \( k \) and \( k_0 \) are not co-directional, instead of (106) we get

\[
\omega_0 = \frac{2k_{jx,jy,jz}u}{i} \left[ 1 + \frac{\hbar k_{jx,jy,jz}u}{Mc^2} \right],
\]

(107)

where \( k_{jx,jy,jz} = \pi \sqrt{\frac{j_x^2}{L_x^2} + \frac{j_y^2}{L_y^2} + \frac{j_z^2}{L_z^2}} \), \( j_x, j_y, j_z = 0, 1, 2, 3, \ldots \), \( L_x, L_y, L_z \) are the system sizes (above we assumed \( L_z = L \)). The probabilities of the corresponding processes can be very small. Observing such absorption lines would mean observing the discrete energy spectrum of a quantum liquid, \( E = \hbar jx,jy,jz \). So far, this spectrum has not been observed, although discrete spectra of individual atoms were recorded more than 100 years ago.

These resonances can be observed by means of the measurement of an electric signal as well, since the mode \((\omega \pm i\omega_0, k \pm ik_0)\) must generate the electric field satisfying the equations

\[
div \mathbf{D} = 0, \quad \mathbf{D} = \varepsilon \mathbf{E} + 4\pi \mathbf{P}_s
\]

(108)

(the last equation was obtained in [16]). For an infinite system whose properties depend only on the coordinate \( z \), Eqs. (108) have the solution \( \mathbf{D} = D_0 \mathbf{i}_z, \varepsilon E \mathbf{i}_z = D_0 \mathbf{i}_z - 4\pi P_s \mathbf{i}_z \), where \( D_0 = \text{const} \). In view of [3], it is clear that the density wave must induce a wave of the field \( \mathbf{E} \) with the same \( \omega, k \). Therefore, the resonance for a density wave \((\omega \pm i\omega_0, k \pm ik_0)\) must be accompanied by the electric field \( \mathbf{E} \) with the same frequency \( \omega_1, \pm k \). In order to get solutions with \( i \geq 2 \) for He II, we should change \( k_0 \to ik_0 \) and \( \omega_0 \to i\omega_0 \) in the formulas; in this case, \( \omega_0 \) is given by formula (104) or (105). One can try to register mode \((0, 2)\) in the same way (according to the estimates in Section 2, this mode creates the field \( \mathbf{E}_s \sim \hat{\varphi}(\varepsilon - 1)^2 E_0 \mathbf{i}_z \) with frequency \( 2\omega_0 \)). These properties also show that a phonon can possess a very weak coat without an external field \( \mathbf{E} \) as well, because a phonon creates oscillations of the density, which induces the electric field and the spontaneous polarization of a medium.

It is of importance that the wave vector of a phonon at the zero boundary conditions is quantized by the law \( k = \pi j/L \) which coincides with the above condition \( \lambda = 2L/j \) for a standing wave of the first sound. Therefore, this condition should be satisfied.

We wrote no solutions for the resonances with large values of \(|k_{1,i}|\) [except for (111)]. However, it is possible that they can be experimentally realized for very large \( \lambda \) of an electric wave.

Above we have found the solutions for oscillatory modes of an infinite system, i.e.,
without consideration of the boundaries. We note that the one-dimensional field $E = E_0 i_z \sin (k_0 z - \omega_0 t)$ can be created only in a resonator with sizes $L_z \ll L_x, L_y$. Otherwise, the field $E$ should depend on three coordinates [16] [44], and the solutions should differ from the above-presented ones. Since the boundaries change the frequency of the second sound only by 2–10% [20] [32], we expect that the above-presented solutions for frequencies will not be strongly changed, if the boundaries are taken into account. If $L_z \ll L_x, L_y$ is satisfied, then our solutions should be true with good accuracy.

If the solutions will be experimentally confirmed, it will be interesting to elucidate whether the relation $a = 1$ is satisfied.

5 Conclusion

According to our study, the external field $E = E_0 i_z \sin (k_0 z - \omega_0 t)$ in the presence of phonons (or temperature waves, for He II) should create a set of hybrid acousto-electric or thermoelectric waves (acouelons/“thermoelons”) in a nonpolar liquid dielectric. Such a set accompanies each wave of the first (second) sound. Such waves-satellites are very weak and unobservable. However, at certain frequencies of a phonon ($\omega$) and a field $E$ ($\omega_0$), one of the waves-satellites should be amplified in the resonance way and can become observable. According to our solutions, the hybrid wave should be intense in the case where $\omega_0$ is very close to the resonance frequency $\omega^\text{res}_{i,j}$: $|\omega_0 - \omega^\text{res}_{i,j}| \equiv \Delta \omega_{i,j} \sim |\hat{\omega}| \omega^\text{res}_{i,j}$. In this case, the field $E$ should be sufficiently strong in order that many quanta of the field with the frequencies in the interval $[\omega^\text{res}_{i,j} - \Delta \omega_{i,j}, \omega^\text{res}_{i,j} + \Delta \omega_{i,j}]$ exist.

The resonances are of the parametric nature. Apparently, the absorption lines should be observed in the electromagnetic spectrum at the energies $\hbar \omega^\text{res}_{i,j}$ equal to the double energies of lowest levels of a system (see (104), (107)). Detecting such lines would mean observing of a discrete structure of the energy spectrum of a liquid. This is of particular interest, since no such structure was directly observed, though its existence raises no doubts. In this case, the resonances should correspond to the absorption of one or several quanta of an electromagnetic field with the momentum recoil to the whole liquid, like the Mössbauer effect. The Mössbauer effect was earlier observed only in crystals, to our knowledge.

We have found the resonance frequencies and the values of amplitudes far from resonances. It is important to find the amplitudes at the resonance points with regard for small nonlinear corrections in the equations, including the friction. Such solutions would correspond to real systems. The main questions are the following: Will the above-obtained singularities be preserved, and will they be observable?
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