Norm of matrix operator on Orlicz-binomial spaces and related operator ideal

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Abstract

The purpose of this article is to introduce Orlicz extension of binomial sequence spaces $b_{\psi}^r,s$ and investigate some topological and inclusion properties of the new spaces. We give an upper estimation of $\|A\|_{\ell_{\psi}^p,b_{\psi}^r,s}$, where $A$ is the Hausdorff matrix operator or Nörlund matrix operator. A Hardy type formula is established in the case of Hausdorff matrix operator. Finally we introduce operator ideal using the space $b_{\psi}^r,s$ and the sequence of s-number function and prove its completeness under certain assumptions.

Keywords: Binomial sequence space, upper bounds, Hausdorff Matrix, Nörlund Matrix, Orlicz function, s-number, operator ideal.

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1. Introduction

Throughout this paper $\mathbb{N}_0 = \{0,1,2,\ldots\}$ and $\omega$ denotes the linear space of all real sequences. Any vector subspace of $\omega$ is called a sequence space. Also by $\ell_p$, we mean the space of absolutely $p$-summable series, where $1 \leq p < \infty$. The space $\ell_p$ is a Banach space according to the $\ell_p$ norm given by

$$||x||_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{\frac{1}{p}}. \quad (1.1)$$

For some recent papers on sequence spaces, one may refer to [5, 8, 16, 20, 36, 39, 40]. Let $X$ and $Y$ be two sequence spaces and let $A = (a_{n,k})$ be an infinite matrix of real entries. We write $A_n$ to denote the sequences in the $n$th row of the matrix $A$. We recall that $A$ defines a matrix mapping from $X$ to $Y$ if for
every sequence \( x = (x_k) \), the \( A \)-transform of \( x \), i.e., \( Ax = (A_n x)_{n=0}^\infty \in Y \), where

\[
A_n x = \sum_{k=0}^\infty a_{nk} x_k, \quad n \in \mathbb{N}_0.
\]

An Orlicz function \( \varphi \) is a function from \((0, \infty)\) to \((0, \infty)\) which is continuous, increasing and convex with \( \varphi(0+) = 0 \) and so has a unique inverse \( \varphi^{-1} : (0, \infty) \to (0, \infty) \). As usual in the Orlicz theory the domain of \( \varphi \) is extended to the real line by \( \varphi(x) = \varphi(|x|) \) and \( \varphi(0) = 0 \) (for details on Orlicz functions see [12, 13, 27]).

Let \( x = (x_n) \) be a sequence of real numbers with \( x_n > 0 \) for all \( n \in \mathbb{N}_0 \). The Orlicz sequence space is defined as

\[
\ell_{\varphi} = \left\{ x \in \omega : \sum_{n=0}^\infty \varphi \left( \frac{x_n}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}.
\]

The space \( \ell_{\varphi} \) is a Banach space equipped with the Orlicz-Luxemborg norm \( \| x \|_{\varphi} \), defined by

\[
\| x \|_{\varphi} = \inf \left\{ \rho > 0 : \sum_{n=0}^\infty \varphi \left( \frac{x_n}{\rho} \right) \leq 1 \right\}.
\] (1.2)

Clearly \( |x_n| \leq |y_n| \) for all \( n \in \mathbb{N}_0 \), then \( \| x \|_{\varphi} \leq \| y \|_{\varphi} \). Further if \( 0 < \| x \|_{\varphi} < \infty \), then \( \sum_{n=0}^\infty \varphi \left( \frac{x_n}{\| x \|_{\varphi}} \right) \leq 1 \)

[25, Lemma 1].

In particular, if \( \varphi(t) = |t|^p, \ p \geq 1 \), then the space \( \ell_{\varphi} \) reduces to the \( \ell_p \) space and the norm \( \| x \|_{\varphi} \) given by (1.2) reduces to the norm \( \| x \|_p \) given by (1.1).

By supermultiplicative function, we shall mean any function \( \varphi : (0, \infty) \to (0, \infty) \) such that for all positive \( a \) and \( b \)

\[
\varphi(ab) \geq \varphi(a) \varphi(b).
\]

An immediate example of supermultiplicative function is \( \varphi(t) = t^p, \ p \geq 1 \). Throughout the article, we consider this supermultiplicative Orlicz function \( \varphi \) which satisfies \( \varphi(1) = 1 \).

We recall that an upper bound for a matrix operator \( T \) from a sequence space \( X \) into another sequence space \( Y \) is the value of \( U \) satisfying the inequality

\[
\|Tx\|_Y \leq U \|x\|_X,
\]

where \( \| - \|_X \) and \( \| - \|_Y \) are the norms on the spaces \( X \) and \( Y \), respectively. Here, \( U \) does not depend on \( x \). The best possible value of \( U \) is regarded as the operator norm of \( T \).

The Euler mean matrix \( E^r \) of order \( r \) is defined by the matrix \( E^r = (e^r_{nk}) \), where \( 0 < r < 1 \) and

\[
e^r_{nk} = \begin{cases} \binom{n}{k} (1 - r)^{n-k} r^k, & (0 \leq k \leq n), \\ 0, & (k > n). \end{cases}
\]

The Euler sequence spaces \( e^r_p \) and \( e^r_\infty \) were introduced by Altay et al. [7] as follows (also see Altay and Başar [6]):

\[
e^r_p = \left\{ x = (x_k) \in \omega : \sum_{n=0}^\infty \left| \sum_{k=0}^n \binom{n}{k} (1 - r)^{n-k} r^k x_k \right|^p < \infty \right\}
\]

and

\[
e^r_\infty = \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}_0} \left| \sum_{k=0}^n \binom{n}{k} (1 - r)^{n-k} r^k x_k \right| < \infty \right\}.
\]

Let \( r, s \in \mathbb{R} \) and \( r + s \neq 0 \), then the binomial matrix \( B^{r,s} = (b^{r,s}_{nk}) \) is defined by:

\[
b^{r,s}_{nk} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k, & (0 \leq k \leq n), \\ 0, & (k > n). \end{cases}
\]
Bisgin [10, 11] obtained another generalization of Euler sequence spaces by introducing the binomial sequence spaces $b^{r,s}_p$ and $b^{r,s}_\infty$ as follows:

$$b^{r,s}_p = \left\{ x = (x_k) \in \omega : \sum_{n=0}^{\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right\}^p < \infty$$

and

$$b^{r,s}_\infty = \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}_0} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right\} < \infty.$$

It is clear that, if we take $r + s = 1$, then the binomial matrix $B^{r,s}$ reduces to the Euler mean matrix $E^r$ of order $r$. Thus binomial matrix generalizes the Euler mean matrix.

Euler weighted sequence space $e^{\theta}_{w,p}$ has been studied recently by Talebi and Dehgan [37] as follows:

$$e^{\theta}_{w,p} = \left\{ x = (x_k) \in \omega : \sum_{n=0}^{\infty} w_n \left( \frac{n}{1-\theta} \right)^{n-k} r^k x_k \right\}^p < \infty,$$

where $1 \leq p < \infty$, $0 < \theta < 1$ and $w = (w_n)$ is a decreasing non-negative sequences of real numbers with $\sum_{n=0}^{\infty} \frac{w_n}{n+1} = \infty$.

More recently, Manna [26] has studied Orlicz extension of weighted Euler sequence space and obtained norm inequalities involving generalized Hausdorff and Nörlund matrix operators which strengthen the results of Talebi and Dehgan [38]. The lower bounds of operators on different sequence spaces were studied in [18, 21–24]. Recently Roopaei and Foroutaninia [34, 35] and Ilkhan [15] discussed the norms of matrix operators on different sequence spaces. Following Bisgin [10], Manna [26], Talebi and Dehgan [37], we introduce Orlicz extension of binomial sequence spaces $b^{r,s}_\varphi$.

The paper is organized as follows. In the Section 2, we introduce Orlicz-binomial sequence space $b^{r,s}_\varphi$, investigate topological properties and inclusion relations. In the Section 3, we give an upper bound estimation for the norm of Hausdorff matrix as an operator from $\ell_\varphi$ to $b^{r,s}_\varphi$ and provide some immediate corollaries. In the Section 4, we give an upper bound estimation for $\|N\|_{\ell_{\varphi},b^{r,s}_\varphi}$ where $N = N(x_n)$ is the Nörlund matrix associated with the sequence $x = (x_n)$. In the final section, we introduce operator ideal $\mathcal{L}_{b^{r,s}_\varphi}$ using the space $b^{r,s}_\varphi$ and the sequence of s-number functions and prove its completeness under certain assumption.

2. Orlicz-binomial sequence spaces $b^{r,s}_\varphi$

Let $\varphi$ be an Orlicz function. Then the Orlicz-binomial sequence space $b^{r,s}_\varphi$ can be defined as the set of all sequences whose $B^{r,s}_\varphi$-transform is in $\ell_\varphi$. That is

$$b^{r,s}_\varphi = \left\{ x \in \omega : B^{r,s}_\varphi x \in \ell_\varphi \right\} = \left\{ x \in \omega : \sum_{n=0}^{\infty} \varphi \left( \frac{1}{\rho} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right) < \infty \text{ for some } \rho > 0 \right\}.$$

One can observe that the sequence space $b^{r,s}_\varphi$ reduces to $b^{r,s}_p$ when $\varphi(t) = t^p$, $p \geq 1$ as studied by Bisgin [11] which further reduces to the space $b^{r}_p$ as studied by Altay et al. [7] when $r + s = 1$. Also if $r + s = 1$, the sequence space $b^{r,s}_\varphi$ reduces to the Orlicz-Euler sequence space $e^{\varphi}_{w,p}$ studied by Manna [26].

Clearly the space $b^{r,s}_\varphi$ is a normed linear space equipped with the norm $\|x\|^{r,s}_\varphi = \|B^{r,s}_\varphi x\|_\varphi$. We begin with the following theorem.

**Theorem 2.1.** The sequence space $b^{r,s}_\varphi$ is a Banach space equipped with the norm $\|\cdot\|^{r,s}_\varphi$. 

Proof. Let \((x^i)\) be a Cauchy sequence in \(b^{r,s}_\varphi\). Then for any \(\varepsilon > 0\), there exist \(n_0 \in \mathbb{N}_0\) such that
\[
\|x^i - x^j\|_{\varphi} < \varepsilon \quad \text{for each} \quad i, j \geq n_0.
\]
Choose \(0 < \rho \varepsilon < \varepsilon\) such that
\[
\sum_{n=0}^{\infty} \varphi \left( \frac{1}{\rho \varepsilon} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} \left( x^i_k - x^j_k \right) \right) \leq 1 \quad \text{for each} \quad i, j \geq n_0.
\]
Using the assumption \(\varphi(1) = 1\), we obtain
\[
\frac{1}{\rho \varepsilon} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} \left( x^i_k - x^j_k \right) \leq 1 \quad \text{for each} \quad i, j \geq n_0.
\]
Thus it is clear that the sequence \((x^i_k)\) is a Cauchy sequence of real numbers and hence converges. Let \((x^i_k) \to x_k\) as \(i \to \infty\) for each \(k \geq 0\). Since \(\varphi\) is continuous, we obtain from (2.1)
\[
\sum_{n=0}^{\infty} \varphi \left( \frac{1}{\rho \varepsilon} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} \left( x^i_k - x^j_k \right) \right) \leq 1 \quad \text{for each} \quad i \geq n_0.
\]
Thus \(x \in b^{r,s}_\varphi\) and \(\|x^i\|_{\varphi} \leq \rho \varepsilon < \varepsilon\) for \(i \geq n_0\). So \((b^{r,s}_\varphi, \|\cdot\|_{\varphi}^{r,s})\) is a Banach space.

**Theorem 2.2.** The sequence space \(b^{r,s}_\varphi\) is linearly isomorphic to \(\ell_{\varphi}\).

**Proof.** Define a mapping \(T : b^{r,s}_\varphi \to \ell_{\varphi}\) by \(x \mapsto y = T x\), where the sequence \(y = \{y_k\}\) is the \(B^{r,s}\)-transform of the sequence \(x = \{x_k\}\), i.e.,
\[
y_k = \sum_{i=0}^{k} \frac{1}{(s+r)^i} \binom{k}{i} (-s)^{k-i} r^i x^i_i.
\]
Clearly, \(T\) is linear and injective. Let \(y \in \ell_{\varphi}\) and define the sequence \(x = \{x_k\}\) by
\[
x_k = \sum_{i=0}^{k} \frac{1}{(s+r)^i} \binom{k}{i} (-s)^{k-i} r^i y^i_i.
\]
Then, one obtains
\[
\|x\|_{\varphi}^{r,s} = \|B^{r,s}_\varphi x\|_{\varphi} = \sum_{k=0}^{\infty} \varphi \left( \frac{B^{r,s}_\varphi x}{\rho} \right)
\]
\[
= \sum_{k=0}^{\infty} \varphi \left( \frac{1}{\rho} \sum_{j=0}^{k} \frac{1}{(s+r)^j} \binom{k}{j} s^{k-j} r^j x^j_j \right)
\]
\[
= \sum_{k=0}^{\infty} \varphi \left( \frac{1}{\rho} \sum_{j=0}^{k} \frac{1}{(s+r)^j} \binom{k}{j} s^{k-j} r^j \sum_{i=0}^{j} (s+r)^i \binom{j}{i} (-s)^{i-j} r^j y^j_i \right)
\]
\[
= \sum_{k=0}^{\infty} \varphi \left( \frac{\sum_{j=0}^{k} \delta_{kj} y^j_i}{\rho} \right) = \sum_{k=0}^{\infty} \varphi \left( \frac{y_k}{\rho} \right) = \|y\|_{\varphi} \leq \rho.
\]
Thus we conclude that \(x \in b^{r,s}_\varphi\) and \(T\) is norm preserving. Consequently, \(T\) is surjective. Thus \(b^{r,s}_\varphi \cong \ell_{\varphi}\).
Now we establish certain inclusion properties concerning Orlicz-binomial sequence space. We start with the following result.

**Theorem 2.3.** Let $\varphi$ be an Orlicz and supermultiplicative function. Then the inclusion $\ell_\varphi \subset b^{r,s}_\varphi$ holds.

**Proof.** Let $x = (x_k) \in \ell_\varphi$ with $x \neq 0$. Applying Jensen’s inequality, we have

$$
\sum_{n=0}^{\infty} \varphi \left( \frac{B^{r,s}x}{\rho} \right) \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k \varphi \left( \frac{x_k}{\rho} \right)
$$

$$
= \sum_{k=0}^{\infty} \varphi \left( \frac{x_k}{\rho} \right) \sum_{n=k}^{\infty} \binom{n}{k} \left( \frac{s}{s+r} \right)^{n-k} \left( \frac{r}{s+r} \right)^k
$$

$$
= \frac{s+r}{r} \sum_{k=0}^{\infty} \varphi \left( \frac{x_k}{\rho} \right)
$$

$$
= \sum_{k=0}^{\infty} \varphi \left( \frac{x_k}{\rho} \right) \varphi^{-1} \left( \frac{s+r}{r} \right) = \sum_{k=0}^{\infty} \varphi \left( \frac{x_k}{\|x\|_\varphi} \right) \leq 1.
$$

Let us put $\rho = \|x\|_\varphi \varphi^{-1} \left( \frac{s+r}{r} \right)$. Then the above inequality implies that

$$
\sum_{n=0}^{\infty} \varphi \left( \frac{B^{r,s}x}{\rho} \right) \leq \sum_{k=0}^{\infty} \varphi \left( \frac{x_k}{\rho} \right) \varphi^{-1} \left( \frac{s+r}{r} \right) = \sum_{k=0}^{\infty} \varphi \left( \frac{x_k}{\|x\|_\varphi} \right) \leq 1.
$$

Hence, by the definition of Orlicz-Luxemborg norm, we obtain that

$$
\|x\|_{\ell_\varphi} \leq \rho = \varphi^{-1} \left( \frac{s+r}{r} \right) \|x\|_\varphi.
$$

Therefore, $\ell_\varphi \subset b^{r,s}_\varphi$. To establish the strictness part, we consider $\varphi(t) = t^p$, $p \geq 1$. Then the sequence $x = (x_k) = (-1)^k \in b^{r,s}_\varphi$ but $x \notin \ell_\varphi$. This completes the proof. \qed

**Theorem 2.4.** Let $\varphi$ be an Orlicz and supermultiplicative function. Then the inclusion $e^{r,s}_\varphi \subset b^{r,s}_\varphi$ is strict.

**Proof.** The inclusion part is obvious since the sequence space $b^{r,s}_\varphi$ reduces to $e^{r,s}_\varphi$ when $r + s = 1$. To establish the strictness part, we consider $\varphi(t) = t^p$, $p \geq 1$ and a sequence $x = (x_k) = (-1)^k \varphi^{-1} \left( \frac{s+r}{r} \right)$ and let $0 < r < 1$ and $s = 4$. Then one can easily deduce that $(x_k) = (-1)^k \notin \ell_\varphi$, $E^r(x) = (-2-r)^k \notin \ell_\varphi$ and $B^{r,s}x = \left( \frac{1}{s+r} \right) \varphi^{-1} \left( \frac{s+r}{r} \right) \notin \ell_\varphi$. Thus there exists at least one sequence $x = (x_k) \in b^{r,s}_\varphi \setminus e^{r,s}_\varphi$. This establishes the result. \qed

### 3. Upper bound for Hausdorff matrix operators

In this section, we establish a Hardy type formula as an upper estimate for $\|H_{\mu}\|_{\ell_\varphi, b^{r,s}_\varphi}$, where $d\mu$ is a Borel probability measure on $[0,1]$ and $H_{\mu}$ is the generalized Hausdorff matrix associated with $d\mu$. Let $\alpha > -1$ and $c > 0$, then the generalized Hausdorff matrix, $H_{\mu} = H_{\mu}(\vartheta) = (h_{nk}(\vartheta))_{n,k \geq 0}$ is defined by (see [9, 17])

$$
h_{nk} = \begin{cases} 
(n+a) \Delta^{n-k} \mu_k, & (k \leq n), \\
0, & (k > n).
\end{cases}
$$

where $\Delta \mu_k = \mu_k - \mu_{k+1}$ and $\mu = (\mu_n)_{n \in \mathbb{N}}$ is a sequence of real numbers, normalized so that $\mu_0 = 1$ and

$$
\mu_k = \int_0^1 \theta^k d\mu(\theta), \quad k = 0, 1, 2, \ldots.
$$
An equivalent expression for the generalized Hausdorff matrix $H_\mu = (h_{nk})$ is given by

$$h_{nk} = \begin{cases} \frac{\theta^c}{1-n} \int_0^\theta \theta^{(k+\alpha)}(1-\theta^c)^{n-k} \, d\mu(\theta), & (k \leq n), \\ 0, & (k > n). \end{cases}$$

When $a = 0$ and $c = 1$, then $H_\mu$ reduces to the ordinary Hausdorff matrix (see [9]) which generalizes various classes of matrices. These classes are:

(a) taking $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1} \, d\theta$ gives the Cesàro matrix of order $\alpha$;
(b) taking $d\mu(\theta) = \text{point evaluation at } \theta = \alpha$ gives the Euler matrix of order $\alpha$;
(c) taking $d\mu(\theta) = \log(\theta)^{\alpha-1} / \Gamma(\alpha) \, d\theta$ gives the Hölder matrix of order $\alpha$;
(d) taking $d\mu(\theta) = \alpha\theta^{\alpha-1} \, d\theta$ gives the Gamma matrix of order $\alpha$.

The Cesàro, Hölder, and Gamma matrices have non-negative entries when $\alpha > 0$, and also the Euler matrices, when $0 < \alpha < 1$. Now we consider the following hypothesis related to Orlicz function and Hausdorff matrix:

‘Hypothesis OH’: Let $\varphi$ be an Orlicz and supermultiplicative function, $\varphi^{-1}$ be its inverse, and $\|\cdot\|_\varphi$ be the Orlicz-Luxemborg norm. Denote $(x)_q = \frac{r[x+q]}{x}$ for $x \geq 0$ and $H_\mu = (h_{nk})$, $h_{nk} \geq 0$. Further, let $a > -1, c > 0, q > -a -1$ and $\frac{m+a+q}{n} \geq 0$. Now we state a lemma due to Love [25] which is essential for deducing our results.

**Lemma 3.1 ([25, Theorem 2]).** Suppose that the ‘Hypothesis OH’ holds. Then for any non-negative sequence $x = (x_k)$ and $\mu = (\mu_k)$ of real numbers normalized so that $\mu_0 = 1$, the following inequality holds:

$$\|H_\mu x\|_\varphi \leq \hat{C} \|x\|_\varphi,$$

where

$$\hat{C} = \int_0^1 \varphi^{-1}(\theta^{-(q+1)c}) \, d\mu(\theta).$$

**Theorem 3.2.** Suppose that the ‘Hypothesis OH’ holds. Then the Hausdorff matrix $H_\mu$ maps $\ell_\varphi$ to $b^r_\varphi$ and

$$\|H_\mu\|_{\ell_\varphi, b^r_\varphi} \leq \hat{C} \varphi^{-1} \left(\frac{s+r}{r}\right),$$

where $\hat{C}$ is given by (3.2).

**Proof.** Let $x = (x_n)$ be a non-negative sequence of real numbers in $\ell_\varphi$. Let $\rho > 0$ be a real number, then using Jensen’s inequality, we have

$$\|H_\mu\|_{\ell_\varphi, b^r_\varphi} = \sum_{k=0}^\infty \varphi \left(\frac{1}{\rho} \sum_{i=0}^k \sum_{l=k}^n \binom{n}{k} h_{ki} x_i \right)^{-1} \left(\frac{1}{\rho} \sum_{i=0}^k h_{ki} x_i \right)^{-1},$$

$$\leq \sum_{k=0}^\infty \sum_{n=0}^\infty \sum_{k=0}^n \frac{1}{(s+r)^n} \binom{n}{k} h_{ki} x_i \sum_{n=0}^\infty \varphi \left(\frac{1}{\rho} \sum_{i=0}^k \binom{n}{k} \frac{s}{s+r}^{-n} \right)^{-1} \left(\frac{r}{s+r}\right)^k,$$

$$= \frac{s+r}{r} \sum_{k=0}^\infty \varphi \left(\frac{1}{\rho} \sum_{i=0}^k h_{ki} x_i \right) \varphi \left(\varphi^{-1} \left(\frac{s+r}{r}\right) \right).$$
Let $\rho = \|H_\mu x\|_{\varphi} \varphi^{-1}\left(\frac{s+r}{r}\right)$. Then the above inequality implies that

$$\|H_\mu\|_{\ell_\varphi,b_{r,s}^\varphi} \leq \sum_{k=0}^{\infty} \varphi\left(\frac{1}{\rho} \sum_{l=0}^{k} h_{kl} x_l \varphi^{-1}\left(\frac{s+r}{r}\right)\right) = \sum_{k=0}^{\infty} \varphi\left(\frac{H_\mu x}{\|H_\mu x\|_{\varphi}}\right) \leq 1.$$ 

Now using the definition of Orlicz-Luxemborg norm and equation (3.1), we get

$$\|H_\mu x\|_{\varphi}^{r,s} \leq \rho = \|H_\mu x\|_{\varphi} \varphi^{-1}\left(\frac{s+r}{r}\right) \leq C \varphi^{-1}\left(\frac{s+r}{r}\right) \|x\|_{\varphi}.$$ 

This gives

$$\|H_\mu\|_{\ell_\varphi,b_{r,s}^\varphi} \leq C \varphi^{-1}\left(\frac{s+r}{r}\right).$$

\[\square\]

**Corollary 3.3.** Choose $c = 1$ and $a = 0$. Then $\mathcal{C} = \int_{0}^{1} \varphi^{-1}(\theta^{-(q+1)}) \, d\mu(\theta)$ and Cesàro, Hölder, Euler, and Gamma matrices map $\ell_\varphi$ into $b_{r,s}^\varphi$ and

$$\|C(\alpha)\|_{\ell_\varphi,b_{r,s}^\varphi} \leq \alpha \varphi^{-1}\left(\frac{s+r}{r}\right) \int_{0}^{1} \varphi^{-1}\left(\theta^{-(q+1)}(1-\theta)^{\alpha-1}\right) \, d\theta, \quad \alpha > 0;$$

$$\|H(\alpha)\|_{\ell_\varphi,b_{r,s}^\varphi} \leq \frac{1}{\Gamma(\alpha)} \varphi^{-1}\left(\frac{s+r}{r}\right) \int_{0}^{1} \varphi^{-1}\left(\theta^{-(q+1)} \log \theta\right)^{\alpha-1} \, d\theta, \quad \alpha > 0;$$

$$\|E(\alpha)\|_{\ell_\varphi,b_{r,s}^\varphi} \leq \varphi^{-1}\left(\frac{s+r}{r}\right) \varphi^{-1}\left(\alpha^{-}q+1\right), \quad 0 < \alpha < 1;$$

$$\|\Gamma(\alpha)\|_{\ell_\varphi,b_{r,s}^\varphi} \leq \alpha \varphi^{-1}\left(\frac{s+r}{r}\right) \int_{0}^{1} \varphi^{-1}\left(\theta^{-(q+1)}\right) \theta^{\alpha-1} \, d\theta.$$

**Corollary 3.4.** Choose $c = 1$, $a = 0$, $q = 0$ and $\varphi(t) = t^p$, $p \geq 1$, and denote $p^* = \frac{p}{p-1}$. Then Cesàro, Hölder, Euler and Gamma matrices map $\ell_\varphi$ to $b_{r,s}^\varphi$ and

$$\|C(\alpha)\|_{\ell_\varphi,b_{r,s}^\varphi} \leq \frac{\varphi^{-1}\left(\frac{s+r}{r}\right)}{\Gamma(\alpha + \frac{1}{p^*})}, \quad \alpha > 0;$$

$$\|H(\alpha)\|_{\ell_\varphi,b_{r,s}^\varphi} \leq \frac{1}{\Gamma(\alpha)} \varphi^{-1}\left(\frac{s+r}{r}\right) \int_{0}^{1} \theta^{\frac{1}{p^*}} \log \theta^{\alpha-1} \, d\theta, \quad \alpha > 0;$$

$$\|E(\alpha)\|_{\ell_\varphi,b_{r,s}^\varphi} \leq \frac{\varphi^{-1}\left(\frac{s+r}{r}\right)}{\alpha^{\frac{1}{p^*}}}, \quad 0 < \alpha < 1;$$

$$\|\Gamma(\alpha)\|_{\ell_\varphi,b_{r,s}^\varphi} \leq \frac{\varphi^{-1}\left(\frac{s+r}{r}\right)}{\alpha^{\frac{1}{p^*}}}, \quad \alpha > 1.$$ 

4. Upper bound for Nörlund matrix operator

In this section, we give an upper bound estimation for the norm of Nörlund matrix as an operator
from $\ell_p$ to $b_{qs}^r$ . Let $u = (u_n)$ be a sequence of non-negative numbers with $u_0 > 0$ . We write $U_n = \sum_{k=0}^{n} u_k, \ n \geq 0$ . Then the Nörlund mean with respect to the sequence $u = (u_n)$ is defined by the matrix $N = N(u_n) = (a^u_{nk})$ given by

$$a^u_{nk} = \begin{cases} \frac{u_n - u_k}{U_n}, & (0 \leq k \leq n) \\ 0, & k > n. \end{cases}$$

In the case when $u_n = e$, Nörlund matrix reduces to Cesàro matrix. Note that one can assume $u_0 = 1$ because $N(u_n) = N(cu_n)$ for any $c > 0$.

**Theorem 4.1.** Let $u = (u_n)$ be a sequence of non-negative real numbers with $u_0 = 1$ . Then

$$||N||_{\ell_p,b_{qs}^r} \leq \varphi^{-1} \left( \frac{s + r}{r} \sum_{n=0}^{\infty} \frac{u_n}{U_n} \right).$$

**Proof.** Let $x \in \ell_p$ be any non-negative sequence of real numbers and $\rho > 0$ . Applying Jensen’s inequality, we obtain

$$\sum_{n=0}^{\infty} \varphi\left( \frac{1}{\rho} \sum_{k=0}^{n} \frac{1}{(s + r)^n} \binom{n}{k} s^{n-k} r^k \sum_{i=0}^{k} \frac{u_{k-i}}{U_k} x_i \right) \leq \sum_{n=0}^{\infty} \frac{n}{(s + r)^n} \binom{n}{k} s^{n-k} r^k \varphi\left( \frac{1}{\rho} \sum_{i=0}^{k} \frac{u_{k-i}}{U_k} x_i \right)$$

$$= \sum_{k=0}^{\infty} \varphi\left( \frac{1}{\rho} \sum_{i=0}^{k} \frac{u_{k-i}}{U_k} x_i \right) \sum_{n=k}^{\infty} \binom{n}{k} \left( \frac{s}{s + r} \right)^{n-k} \left( \frac{r}{s + r} \right)^k$$

$$\leq \frac{s + r}{r} \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{u_{k-i}}{U_k} \varphi\left( \frac{x_i}{\rho} \right)$$

$$= \frac{s + r}{r} \sum_{k=0}^{\infty} \frac{u_k}{U_k} \sum_{i=0}^{\infty} \varphi\left( \frac{x_i}{\rho} \right)$$

$$= \sum_{i=0}^{\infty} \varphi\left( \frac{x_i}{\rho} \right) \varphi^{-1} \left( \frac{s + r}{r} \sum_{k=0}^{\infty} \frac{u_k}{U_k} \right)$$

$$\leq \sum_{i=0}^{\infty} \varphi\left( \frac{x_i}{\rho} \right) \varphi^{-1} \left( \frac{s + r}{r} \sum_{k=0}^{\infty} \frac{u_k}{U_k} \right).$$

Put $\rho = \|x\|_{\ell_p} \varphi^{-1} \left( \frac{s + r}{r} \sum_{k=0}^{\infty} \frac{u_k}{U_k} \right)$ , then the above inequality becomes

$$||Nx||_{\ell_p,b_{qs}^r} \leq \sum_{i=0}^{\infty} \varphi\left( \frac{x_i}{\|x\|_{\ell_p}} \right) \leq 1.$$

Now using the definition of Orlicz-Luxemborg norm, we get

$$||Nx||_{\ell_p,b_{qs}^r} \leq \rho = \varphi^{-1} \left( \frac{s + r}{r} \sum_{k=0}^{\infty} \frac{u_k}{U_k} \right) \|x\|_{\ell_p}.$$

This establishes the result. □
Corollary 4.2. Let \( u = (u_n) \) be a non-negative sequence of real numbers such that \( \frac{u_n}{n} = \frac{1}{(n+1)^p}, n = 0, 1, 2, \ldots \). Then the Nörlund matrix maps \( \ell_p \) into \( b^{r,s}_p \) and

\[
\|N\|_{\ell_p, b^{r,s}_p} \leq \varphi^{-1} \left( \frac{(s+r)^2}{6r} \right).
\]

Corollary 4.3. Let \( \varphi(t) = t^p p \geq 1 \) and \( u = (u_n) \) be a non-negative sequence of real numbers such that \( \frac{u_n}{n} = \frac{1}{(n+1)^p}, n = 0, 1, 2, \ldots \). Then the Nörlund matrix maps \( \ell_p \) into \( b^{r,s}_p \) and

\[
\|N\|_{\ell_p, b^{r,s}_p} \leq \left( \frac{(s+r)^2}{6r} \right)^{\frac{1}{p}}.
\]

5. The operator ideals \( \mathcal{A}^{(s)}_{b^{r,s}_p} \)

Throughout this section, we denote by \( X \) and \( Y \), the Banach spaces over the complex field \( \mathbb{C} \) and by \( L(X, Y) \), the class of all bounded linear maps from \( X \) to \( Y \). Let \( L \) be the class of all bounded linear operators between any pair of Banach spaces.

A map \( s : L \rightarrow \omega^+ \), where \( \omega^+ \) is the class of sequences of non-negative real numbers, is called an \( s \)-number function if it satisfies the following conditions:

(i) \( \|s\| = s_0(S) \geq s_1(S) \geq \cdots \geq 0 \), \( s(S) = \{s_n(S)\}, S \in L; \)
(ii) \( s_n(S+T) \leq s_n(S) + \|T\| \) for \( S, T \in L(X, Y) \) and \( n \in \mathbb{N}_0; \)
(iii) \( s_n(RST) \leq \|R\| s_n(S) \|T\| \) for \( R, T \in L(X_0, X), S \in L(Y, Z) \), \( R \in L(Y_0, Y), \) and \( n \in \mathbb{N}_0; \)
(iv) if \( \text{rank}(S) < n \), then \( s_n(S) = 0; \)
(v) if \( \dim X \geq n \), then \( s_n(S_X) = 1 \), where \( S_X \) denotes the identity map of \( X \).

An \( s \)-number function is called additive if the condition (ii) is replaced by

(ii) \( s_{m+n-1}(S+T) \leq s_m(S) + s_n(T) \) for \( S, T \in L(X, Y) \) and \( m, n \in \mathbb{N}_0. \)

If the condition (iii) is replaced by

(iii) \( s_{m+n-1}(RT) \leq s_m(R)s_n(T) \) for \( R \in L(Y_0, Y), T \in L(X, Y), \) and \( m, n \in \mathbb{N}_0, \)
then the \( s \)-number function is called multiplicative, where the negative subscript is consider to be naught.

For a subset \( A \) of \( L \), we write \( A(X, Y) = A \cap L(X, Y) \) where \( X \) and \( Y \) are Banach spaces. The collection \( A \) is said to be an operator ideal if it satisfies the following conditions:

(i) \( A \) contains all finite rank operators;
(ii) \( T + S \in A(X, Y) \) for \( S, T \in A(X, Y); \)
(iii) if \( T \in A(X, Y) \) and \( S \in L(Y, Z) \), then \( ST \in A(X, Z) \) and also if \( T \in L(X, Y) \) and \( S \in A(Y, Z) \), then \( ST \in A(X, Z). \)

The collection \( A(X, Y) \), for a given pair of Banach spaces \( X \) and \( Y \), is called a component of \( A \). For more details on \( s \)-number and operator ideal, we strictly refer to [1–4, 14, 19, 28–33] and the references cited therein.

An ideal quasi norm is a real valued function \( f \) defined on an operator ideal \( A \), which satisfies the following properties:

(i) \( 0 \leq f(\mathcal{I}) < \infty \), for each \( \mathcal{I} \in A \) and \( f(\emptyset) = 0 \) if and only if \( \mathcal{I} = \emptyset; \)
(ii) there exists a constant \( N \geq 1 \) such that \( f(S+T) \leq N[f(S) + f(T)] \) for \( S, T \in A(X, Y) \), where \( A(X, Y) \) is any component of \( A; \)
(iii) (a) \( f(RS) \leq \|R\| f(S) \) for \( S \in A(X, Z), R \in L(Z, Y); \) and
(b) \( f(RS) \leq \|S\| f(R) \) for \( S \in L(X, Z), R \in A(Z, Y). \)
We start with the following definition.

**Definition 5.1.** An operator \( T \in \mathcal{L}(X, Y) \) is said to be of type \( b^{r,s}_\varphi \) if \( \{ s_n(T) \} \in b^{r,s}_\varphi \).

Let \( \mathcal{L}^{(s)}_{b^{r,s}_\varphi} \) denotes the collection of all such mappings, i.e.,

\[
\mathcal{L}^{(s)}_{b^{r,s}_\varphi} = \{ T \in \mathcal{L}(X, Y) : \{ s_n(T) \} \in b^{r,s}_\varphi \}.
\]

For \( T \in \mathcal{L}^{(s)}_{b^{r,s}_\varphi} \), we define

\[
\| T \|^{(s)}_{b^{r,s}_\varphi} = \inf \left\{ \rho > 0 : \sum_{n=0}^{\infty} \varphi \left( \frac{1}{\rho_1} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k}r^k s_k(T) \right) \leq 1 \right\}.
\]

**Theorem 5.2.** The class \( \mathcal{L}^{(s)}_{b^{r,s}_\varphi} \) is an operator ideal equipped with the norm \( \| - \|^{(s)}_{b^{r,s}_\varphi} \).

**Proof.** Note that all the finite rank operators are contained in \( \mathcal{L}^{(s)}_{b^{r,s}_\varphi} \), since \( s_n(T) = 0 \) for \( n \geq n_0 \) if \( \text{rank}(T) < n_0 \). Let \( T_1, T_2 \in \mathcal{L}^{(s)}_{b^{r,s}_\varphi} \), then

\[
\begin{align*}
\sum_{n=0}^{\infty} \varphi \left( \frac{1}{\rho_1 + \rho_2} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k}r^k s_k(T_1 + T_2) \right) &< \infty \text{ for some } \rho_1 > 0, \text{ and} \\
\sum_{n=0}^{\infty} \varphi \left( \frac{1}{\rho_2} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k}r^k s_k(T_1 + T_2) \right) &< \infty \text{ for some } \rho_2 > 0.
\end{align*}
\]

Now

\[
\begin{align*}
\sum_{n=0}^{\infty} \varphi \left( \frac{1}{\rho_1 + \rho_2} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k}r^k s_k(T_1 + T_2) \right) &\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k}r^k \varphi \left( \frac{1}{\rho_1 + \rho_2} s_k(T_1 + T_2) \right) (\text{using Jensen's inequality}) \\
&= \sum_{k=0}^{\infty} \varphi \left( \frac{1}{\rho_1 + \rho_2} s_k(T_1 + T_2) \right) \sum_{n=k}^{\infty} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k}r^k \\
&= \frac{s+r}{r} \sum_{k=0}^{\infty} \varphi \left( \frac{s_k(T_1 + T_2)}{\rho_1 + \rho_2} \right) + \sum_{k=0}^{\infty} \frac{\varphi(s_k(T_2))}{\rho_2} < \infty.
\end{align*}
\]

Thus \( T_1 + T_2 \in \mathcal{L}^{(s)}_{b^{r,s}_\varphi} \). Let \( T \in \mathcal{L}^{(s)}_{b^{r,s}_\varphi}(X_0, Y_0), R \in \mathcal{L}^{(s)}_{b^{r,s}_\varphi}(X, X_0), S \in \mathcal{L}^{(s)}_{b^{r,s}_\varphi}(Y_0, Y) \). Using the property (iii) of s-number function, we obtain

\[
\begin{align*}
\sum_{n=0}^{\infty} \varphi \left( \frac{1}{\rho} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k}r^k s_k(RTS) \right) &\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k}r^k \varphi \left( \frac{1}{\rho} s_k(RTS) \right) (\text{using Jensen's inequality}) \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k}r^k \varphi \left( \frac{\| R \| \| S \|}{\rho} s_k(T) \right) (\text{using property (iii) of s-number function})
\end{align*}
\]
\[
\frac{s + r}{\rho} \sum_{k=0}^{\infty} \phi \left( \frac{\|R\| s_k(T) \|S\|}{\rho} \right) < \infty.
\]

Thus \( RTS \in \mathcal{L}_{b_p^{(s)}}^{(s)}. \) Thus \( \mathcal{L}_{b_{q}^{(s)}}^{(s)} \) is an operator ideal. \( \Box \)

**Theorem 5.3.** The operator ideal \( \mathcal{L}_{b_{q}^{(s)}}^{(s)} \) is complete under the quasi-norm \( \| \cdot \|^{(s)}_{b_{q}^{(s)}}. \)

**Proof.** First we shall show that \( \| \cdot \|^{(s)}_{b_{q}^{(s)}} \) is a quasi-norm on \( \mathcal{L}_{b_{q}^{(s)}}^{(s)}. \) Note that \( \|T\|^{(s)}_{b_{q}^{(s)}} \geq 0 \) for each \( T \in \mathcal{L}_{b_{q}^{(s)}}^{(s)} \) and \( \|T\|^{(s)}_{b_{q}^{(s)}} = 0 \) for \( T = 0. \) Now, let \( T \in \mathcal{L}_{b_{q}^{(s)}}^{(s)} \) such that \( \|T\|^{(s)}_{b_{q}^{(s)}} = 0. \) Then for \( \varepsilon > 0, \) we can find \( 0 < \rho < \varepsilon \) and

\[
\sum_{n=0}^{\infty} \phi \left( \frac{1}{\rho} \sum_{k=0}^{n} \frac{1}{(s + r)^n} \left( \frac{n}{k} \right) s^{n-k} r^{k} s_k(T) \right) \leq 1.
\]

Using the assumption \( \phi(1) = 1, \) one obtains

\[
\frac{1}{\varepsilon} \sum_{k=0}^{n} \frac{1}{(s + r)^n} \left( \frac{n}{k} \right) s^{n-k} r^{k} s_k(T) \leq \frac{1}{\varepsilon} \sum_{k=0}^{n} \frac{1}{(s + r)^n} \left( \frac{n}{k} \right) s^{n-k} r^{k} s_k(T) \leq 1.
\]

Now

\[
\frac{1}{\varepsilon} \left( \frac{s}{s + r} \right)^n s_0(T) \leq \frac{1}{\varepsilon} \sum_{k=0}^{n} \frac{1}{(s + r)^n} \left( \frac{n}{k} \right) s^{n-k} r^{k} s_k(T) \leq 1.
\]

Since \( \varepsilon \) is arbitrary, we get

\[
\|T\| = s_0(T) = 0 \implies T = 0.
\]

Next we establish the triangular inequality. Let \( T_1, T_2 \in \mathcal{L}_{b_{q}^{(s)}}^{(s)} \) and \( \varepsilon > 0 \) arbitrary. Choose \( \rho_1 > 0, \rho_2 > 0 \) such that

\[
\frac{1}{\rho_1} \sum_{k=0}^{n} \frac{1}{(s + r)^n} \left( \frac{n}{k} \right) s^{n-k} r^{k} s_k(T_1) \leq 1, \quad \rho_1 \leq \|T_1\|^{(s)}_{b_{q}^{(s)}} + \frac{\varepsilon}{2}, \quad \text{and}
\]

\[
\frac{1}{\rho_2} \sum_{k=0}^{n} \frac{1}{(s + r)^n} \left( \frac{n}{k} \right) s^{n-k} r^{k} s_k(T_2) \leq 1, \quad \rho_2 \leq \|T_2\|^{(s)}_{b_{q}^{(s)}} + \frac{\varepsilon}{2}.
\]

We choose \( N > 1. \) Then

\[
\sum_{n=0}^{\infty} \phi \left( \frac{1}{N (\rho_1 + \rho_2)} \sum_{k=0}^{n} \frac{1}{(s + r)^n} \left( \frac{n}{k} \right) s^{n-k} r^{k} s_k(T_1 + T_2) \right) \leq 1.
\]

From which one can deduce that

\[
\|T_1 + T_2\|^{(s)}_{b_{q}^{(s)}} \leq N (\rho_1 + \rho_2) \leq N \left( \|T_1\|^{(s)}_{b_{q}^{(s)}} + \|T_2\|^{(s)}_{b_{q}^{(s)}} + \varepsilon \right).
\]

Since \( \varepsilon > 0 \) is arbitrary, therefore

\[
\|T_1 + T_2\|^{(s)}_{b_{q}^{(s)}} \leq N \left( \|T_1\|^{(s)}_{b_{q}^{(s)}} + \|T_2\|^{(s)}_{b_{q}^{(s)}} \right).
\]

Now we shall establish the completeness of \( \mathcal{L}_{b_{q}^{(s)}}^{(s)}. \) Let \( (T^{(i)}) \) be a Cauchy sequence in \( \mathcal{L}_{b_{q}^{(s)}}^{(s)}, \) then for \( \varepsilon > 0 \) there exists a positive integer \( n_0 \) such that \( \|T^{(i)} - T^{(j)}\|^{(s)}_{b_{q}^{(s)}} < \varepsilon \) for each \( i, j \geq n_0. \) We choose \( 0 < \rho < \varepsilon \) and

\[
\sum_{n=0}^{\infty} \phi \left( \frac{1}{\rho} \sum_{k=0}^{n} \frac{1}{(s + r)^n} \left( \frac{n}{k} \right) s^{n-k} r^{k} s_k(T^{(i)} - T^{(j)}) \right) \leq 1 \quad (5.1)
\]
for $i, j \geq n_0$. Using the assumption $\varphi(1) = 1$ and the same argument above, one can deduce that $\|T^{(i)} - T^{(j)}\| \to 0$ as $i, j \to \infty$. Hence $(T^{(i)})$ is a Cauchy sequence in $L(X, Y)$ and hence converges, say to $T$, i.e., $\|T^{(i)} - T\| \to 0$ as $i \to \infty$. Since $\varphi$ is continuous, therefore using equation (5.1) as $i \to \infty$,

$$
\sum_{n=0}^{\infty} \varphi \left( \frac{1}{\rho} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \left( \begin{array}{c} n \\ k \end{array} \right) s^{n-k} r^k \right) \leq 1.
$$

Thus $T \in L_{b_\varphi^s}$ and $\|T^{(i)} - T\|_{b_\varphi^s} \leq \rho < \varepsilon$ as $i \to \infty$. This establishes the result. 

6. Conclusion

In this article, we give an upper bound estimation for the norms of Hausdorff matrix and Nörlund matrix as operators from $\ell_\varphi$ to $b_\varphi^s$, thereby obtaining a Hardy type formulae in the case of Hausdorff matrix. We have used Jensen’s inequality to prove all the results. Note that by ignoring the weighted version, i.e., by taking $\lambda_n = 1$ and $v_n = 1$ for all $n \in \mathbb{N}_0$ in the results of Manna [26] and Talebi and Dehgan [37], respectively, then our investigated results in this paper intend to generalize the results obtained by the authors in [26, 37]. We also defined operator ideal for Orlicz-binomial sequence space and proved its completeness. We expect that the results obtained in this paper might be a reference for further studies in this field.

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