THE NICA-TOEPLITZ ALGEBRA OF ABELIAN LATTICE-ORDERED GROUPS IS A FULL CORNER IN GROUP CROSSED PRODUCT

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Abstract. Consider the pair \((G, P)\) consisting of an abelian lattice-ordered discrete group \(G\) and its positive cone \(P\). Let \(\alpha\) be an action of \(P\) by extendible endomorphisms of a \(C^*\)-algebra \(A\). We show that the Nica-Toeplitz algebra \(T_{\text{cov}}(A \times_\alpha P)\) is a full corner of a group crossed product \(B \times_\beta G\), where \(B\) is a subalgebra of \(\ell^\infty(G, A)\) generated by a collection of faithful copies of \(A\), and the action \(\beta\) on \(B\) is given by shift on \(\ell^\infty(G, A)\). By using this realization, we identify (the essential) ideal \(I\) of \(T_{\text{cov}}(A \times_\alpha P)\) such that the quotient algebra \(T_{\text{cov}}(A \times_\alpha P)/I\) is the isometric crossed product \(A \times_{\alpha}^{\text{iso}} P\).

1. Introduction

Let \(P\) be the positive cone of an abelian lattice-ordered discrete group \(G\). The identity element of \(G\) is denoted by \(e\), and \(s^{-1}\) denotes the inverse of an element \(s \in G\). Note that we have \(P^{-1} \cap P = \{e\}\) and \(G = P^{-1} P\). For every \(s, t \in G\), we let \(s \vee t\) and \(s \wedge t\) denote the supremum and infimum of the elements \(s\) and \(t\), respectively. Suppose that \((A, P, \alpha)\) is a dynamical system consisting of a \(C^*\)-algebra \(A\), an action \(\alpha : P \to \text{End}(A)\) of \(P\) by endomorphisms of \(A\) such that \(\alpha_e = \text{id}\). Note that since the \(C^*\)-algebra \(A\) is not necessarily unital, we need to assume that each endomorphism \(\alpha_s\) is extendible, which means that, it extends to a strictly continuous endomorphism \(\pi_s\) of the multiplier algebra \(\mathcal{M}(A)\). Recall that an endomorphism \(\alpha\) of \(A\) is extendible if and only if there exists an approximate identity \(\{a_\lambda\}\) in \(A\) and a projection \(p \in \mathcal{M}(A)\) such that \(\alpha(a_\lambda)\) converges strictly to \(p\) in \(\mathcal{M}(A)\). However, the extendibility of \(\alpha\) does not necessarily imply \(\pi(1_{\mathcal{M}(A)}) = 1_{\mathcal{M}(A)}\). In [3], for the system \((A, P, \alpha)\), Fowler defined a covariant representation called the Nica-Toeplitz covariant representation of the system, such that the endomorphisms \(\alpha_s\) are implemented by partial isometries. He then showed that there exists a universal \(C^*\)-algebra \(T_{\text{cov}}(A \times_\alpha P)\) associated with the system \((A, P, \alpha)\) generated by a universal Nica-Toeplitz covariant representation of the system such that there is a bijection between the Nica-Toeplitz covariant representation of the system and the nondegenerate representations of \(T_{\text{cov}}(A \times_\alpha P)\). This universal algebra is called the Nica-Toeplitz algebra or Nica-Toeplitz crossed product of the system \((A, P, \alpha)\). We recall that when the group \(G\) is totally ordered and abelian, the algebra \(T_{\text{cov}}(A \times_\alpha P)\) is the partial-isometric crossed product \(A \times_{\alpha}^{\text{piso}} P\) of the system \((A, P, \alpha)\) introduced and studied in [13]. Further studies on the structure of the algebra \(A \times_{\alpha}^{\text{piso}} P\) were carried out in [1], [3], [4], [10], and [18]. In particular, it was shown in [18] that \(A \times_{\alpha}^{\text{piso}} P\) is a full corner in classical crossed product by group. This is the main inspiration of the present work, where by following the framework

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of [18], we generalize this corner realization of \( \mathcal{T}_{\text{cov}}(A \times \alpha P) \) to more general groups, namely, (abelian) lattice-ordered groups. However, compared to the totally ordered case, the discussions here are more complicated which involve quite huge computations. We believe that such efforts are very useful on understanding the structure of \( C^* \)-algebras constructed out of semigroup dynamical systems, on which we could import many information from the well-established theory of the classical crossed products by groups (for example, see [10]).

Following the idea of [18], a subalgebra \( \mathcal{B} \) of the algebra \( \ell^\infty(G,A) \) of norm bounded \( A \)-valued functions of \( G \) will be defined. Then, the shift on \( \ell^\infty(G,A) \) gives an action of \( G \) on \( \mathcal{B} \) by automorphisms. Let \( \mathcal{B} \times G \) be the associated group crossed product of \( \mathcal{B} \) by \( G \). Next, a Nica-Toeplitz covariant representation of \( (A,P,\alpha) \) in the multiplier algebra of \( \mathcal{B} \times G \) will be constructed, and we show that the corresponding homomorphism of the Nica-Toeplitz algebra is an isomorphism of \( \mathcal{T}_{\text{cov}}(A \times \alpha P) \) onto a full corner of \( \mathcal{B} \times G \). We then apply this realization to identify the kernel of the natural surjective homomorphism \( \Omega : (\mathcal{T}_{\text{cov}}(A \times \alpha P), i_A, i_P) \to (A \times^\text{iso} \alpha P, \mu_A, \mu_P) \) induced by the canonical isometric covariant pair \((\mu_A, \mu_P)\) of \((A, P, \alpha)\). Moreover, we will see that \( \ker \Omega \) is an essential ideal.

We begin with a preliminary section containing a summary on the Nica-Toeplitz algebra and the theory of isometric crossed products. In section 3 we introduce a subalgebra \( \mathcal{B} \) of \( \ell^\infty(G,A) \) and its crossed product \( \mathcal{B} \times_\beta G \) by \( G \), where the action \( \beta \) is given by the shift on \( \ell^\infty(G,A) \). In section 4 a Nica-Toeplitz covariant representation of \( (A,P,\alpha) \) in \( \mathcal{M}(\mathcal{B} \times_\beta G) \) will be constructed, from which we get an isomorphism of the Nica-Toeplitz algebra \( \mathcal{T}_{\text{cov}}(A \times_\alpha P) \) onto a full corner of \( \mathcal{B} \times_\beta G \). By using this realization, in section 5 for the system \((A,\mathbb{N}^2,\alpha)\) associated with the abelian lattice-ordered group \((\mathbb{Z}^2, \mathbb{N}^2)\), we show that the kernel of the natural surjective homomorphism of the Nica-Toeplitz algebra onto the isometric crossed product is the direct sum of two full corners in algebras of compact operators. Finally, in section 6 we show that when the action of \( P \) is given by automorphisms the Nica-Toeplitz algebra \( \mathcal{T}_{\text{cov}}(A \times_\alpha P) \) is a full corner in the classical crossed product \((B_G \otimes A) \times G\) by group.

2. Preliminaries

2.1. Morita equivalence and full corner. The \( C^* \)-algebras \( A \) and \( B \) are called Morita equivalent if there is an \( A-B \)-imprimitivity bimodule \( X \). If \( p \) is a projection in the multiplier algebra \( \mathcal{M}(A) \) of \( A \), then the \( C^* \)-subalgebra \( pAp \) of \( A \) is called a corner in \( A \). We say a corner \( pAp \) is full if \( A_{pAp} := \overline{\text{span}}\{apb : a, b \in A\} \) is \( A \). Any full corner of \( A \) is Morita equivalent to \( A \) via the imprimitivity bimodule \( Ap \) (see more in [9] or [15]).

2.2. Nica-Toeplitz algebra. A partial-isometric representation of \( P \) on a Hilbert space \( H \) is a map \( V : P \to B(H) \) such that each \( V_x := V(x) \) is a partial isometry, and \( V_xV_y = V_{xy} \) for all \( x, y \in P \).

A Toeplitz covariant representation of \( (A,P,\alpha) \) on a Hilbert space \( H \) is a pair \((\pi,V)\) consisting of a nondegenerate representation \( \pi : A \to B(H) \) and a partial-isometric representation \( V : P \to B(H) \) of \( P \) such that

\[
\pi(\alpha_x(a)) = V_x\pi(a)V_x^* \quad \text{and} \quad V_x^*V_y\pi(a) = \pi(a)V_y^*V_x
\]
for all \( a \in A \) and \( x \in P \). A Nica-Toeplitz covariant representation of \((A,P,\alpha)\) on a Hilbert space \( H \) is a Toeplitz covariant representation \((\pi,V)\) such that

\[
(2.2) \quad V_x^* V_x V_y = V_{x \vee y}^* V_{x \vee y}
\]

for all \( x, y \in P \).

Note that every Nica-Toeplitz covariant pair \((\pi, V)\) extends to a Nica-Toeplitz covariant representation \((\pi, V)\) of \((M(A), P, \overline{\alpha})\), and \((2.1)\) is equivalent to

\[
(2.3) \quad \pi(\alpha_x(a))V_x = V_x \pi(a) \quad \text{and} \quad V_x V_x^* = \overline{\pi(\alpha_x(1))}
\]

for \( a \in A \) and \( x \in P \).

**Definition 2.1.** A Nica-Toeplitz crossed product of \((A,P,\alpha)\) is a triple \((B, i_A, i_P)\) consisting of a \( C^* \)-algebra \( B \), a nondegenerate homomorphism \( i_A : A \rightarrow B \), and a map \( i_P : P \rightarrow M(B) \) such that:

(i) if \( \Lambda \) is a nondegenerate representation of \( B \), then the pair \((\Lambda \circ i_A, \overline{\Lambda} \circ i_P)\) is a Nica-Toeplitz covariant representation of \((A, P, \alpha)\);

(ii) for every Nica-Toeplitz covariant representation \((\pi, V)\) of \((A, P, \alpha)\) on a Hilbert space \( H \), there exists a nondegenerate representation \( \pi \times V : B \rightarrow B(H) \) such that \((\pi \times V) \circ i_A = \pi \) and \((\pi \times V) \circ i_P = V \); and

(iii) \( B \) is generated by \( \{ i_A(a)i_P(x) : a \in A, x \in P \} \).

Fowler in [5] showed that the Nica-Toeplitz crossed product of \((A,P,\alpha)\) exists, and it is unique up to isomorphism (see in particular [5, Proposition 9.2]). He denotes this algebra by \( T_{\text{cov}}(A \times_\alpha P) \), which is also called the Nica-Toeplitz algebra.

**Remark 2.2.** Note that in the definition \((2.1)\) as the algebra \( B \) can be embedded in some algebra \( B(H) \) by a faithful nondegenerate representation, (i) is indeed equivalent to the following statement:

\( i \)' the pair \((i_A, i_P)\) is a Nica-Toeplitz covariant representation of \((A, P, \alpha)\) in \( B \).

This means that the pair \((i_A, i_P)\) is consisting of a nondegenerate homomorphism \( i_A : A \rightarrow B \) and a partial-isometric representation \( i_P : P \rightarrow M(B) \) which satisfies the covariance equations

\[
(2.4) \quad i_A(\alpha_x(a)) = i_P(x)i_A(a)i_P(x)^* \quad \text{and} \quad i_P(x)^* i_P(x)i_A(a) = i_A(a)i_P(x)^* i_P(x),
\]

such that

\[
(2.5) \quad i_P(x)^* i_P(x)i_P(y)^* i_P(y) = i_P(x \vee y)^* i_P(x \vee y)
\]

for all \( a \in A \) and \( x, y \in P \). So, one can calculate to see that we actually have

\[
(2.6) \quad B = \overline{\text{span}}\{ i_P(x)^* i_A(a)i_P(y) : x, y \in P, a \in A \}.
\]

We recall that by [5, Theorem 9.3], a Nica-Toeplitz covariant representation \((\pi, V)\) of \((A, P, \alpha)\) on \( H \) induces a faithful representation \( \pi \times V \) of \( T_{\text{cov}}(A \times_\alpha P) \) if and only if for every finite subset \( F = \{ x_1, x_2, ... , x_n \} \) of \( P \setminus \{ e \} \), \( \pi \) is faithful on the range of

\[
\prod_{i=1}^n (1 - V_{x_i}^* V_{x_i}) = 0.
\]
2.3. **Isometric crossed products.** An **isometric covariant representation** of \((A, P, \alpha)\) on a Hilbert space \(H\) is a pair \((\pi, V)\) consisting of a nondegenerate representation \(\pi : A \to B(H)\) and an isometric representation \(V : P \to B(H)\) of \(P\) such that

\[
\pi(\alpha_x(a)) = V_x \pi(a) V_x^* \tag{2.7}
\]

for all \(a \in A\) and \(x \in P\).

**Definition 2.3.** An **isometric crossed product** of \((A, P, \alpha)\) is a triple \((C, \mu_A, \mu_P)\) consisting of a \(C^*\)-algebra \(C\), a nondegenerate homomorphism \(\mu_A : A \to C\), and an isometric representation \(\mu_P : P \to \mathcal{M}(C)\) such that:

1. \(\mu_A(\alpha_x(a)) = \mu_P(x) \mu_A(a) \mu_P(x)^*\) for all \(a \in A\) and \(x \in P\);
2. for every isometric covariant representation \((\pi, V)\) of \((A, P, \alpha)\) on a Hilbert space \(H\), there exists a nondegenerate representation \(\pi \times V : C \to B(H)\) such that \((\pi \times V) \circ \mu_A = \pi\) and \((\pi \times V) \circ \mu_P = V\); and
3. \(C\) is generated by \(\{\mu_A(a) \mu_P(x) : a \in A, x \in P\}\), indeed we have

\[
C = \text{span}\{\mu_P(x) \mu_A(a) \mu_P(y) : x, y \in P, a \in A\}.
\]

Note that the isometric crossed product of the system \((A, P, \alpha)\) exists if the system admits a nontrivial (isometric) covariant representation, and it is unique up to isomorphism. Thus, the isometric crossed product of the system \((A, P, \alpha)\) is denoted by \(A \times^{\text{iso}}_\alpha P\). We refer readers to [2, 5, 8, 11, 12, 16] for more on isometric crossed products.

Consider the dynamical system \((A, P, \alpha)\). Let \((\mathcal{T}_{\text{cov}}(A \times_\alpha P), i_A, i_P)\) and \((A \times^{\text{iso}}_\alpha P, \mu_A, \mu_P)\) be the Nica-Toeplitz algebra and the isometric crossed product associated with the system, respectively. One can see that the pair \((\mu_A, \mu_P)\) is a Nica-Toeplitz covariant representation of \((A, P, \alpha)\) in the \(C^*\)-algebra \(A \times^{\text{iso}}_\alpha P\). Thus, there exists a nondegenerate homomorphism

\[\Omega : (\mathcal{T}_{\text{cov}}(A \times_\alpha P), i_A, i_P) \to (A \times^{\text{iso}}_\alpha P, \mu_A, \mu_P)\]

such that

\[\Omega(i_P(x)^* i_A(a) i_P(y)) = \mu_P(x)^* \mu_A(a) \mu_P(y)\]

for all \(a \in A\) and \(x, y \in P\). So, it follows that \(\Omega\) is surjective, and hence, we have the following short exact sequence:

\[
0 \longrightarrow \ker \Omega \longrightarrow \mathcal{T}_{\text{cov}}(A \times_\alpha P) \xrightarrow{\Omega} A \times^{\text{iso}}_\alpha P \longrightarrow 0,
\]

Next, we want to identify spanning elements for the ideal \(\ker \Omega\). To do so, first, see that, for every \(r, s \in P\), by the covariance equation of \((i_A, i_P)\), we have

\[
i_P(r)[1 - i_P(s)^* i_P(s)] = i_P(r) - i_P(r) i_P(s)^* i_P(s) = i_P(r) - i_P(r)[i_P(r)^* i_P(r) i_P(s)^* i_P(s)] = i_P(r) - i_P(r) i_P(r \lor s)^* i_P(r \lor s) = i_P(r) - i_P(r) i_P(r)^* i_P((r \lor s)r^{-1}) i_P((r \lor s)r^{-1}) i_P(r) = i_P(r) - i_P(r)((r \lor s)r^{-1})^* i_P((r \lor s)r^{-1}) i_P(r) = i_P(r) - i_P(r)((r \lor s)r^{-1})^* i_P((r \lor s)r^{-1}) i_P(r) = i_P(r) - i_P(r)((r \lor s)r^{-1})^* i_P((r \lor s)r^{-1}) i_P(r) = 1 - i_P((r \lor s)r^{-1})^* i_P((r \lor s)r^{-1}) i_P(r).
\]
Therefore,
\[
(2.9) \quad i_P(r)[1 - i_P(s)^*i_P(s)] = [1 - i_P((r \vee s)r^{-1})^*i_P((r \vee s)r^{-1})]i_P(r)
\]
for every \( r, s \in P \). This equation will be applied in the following proposition regarding the identifying spanning elements for the ideal ker \( \Omega \).

**Proposition 2.4.** Let

\[ \mathcal{I} := \text{span}\{i_P(x)^*i_A(a)(1 - i_P(s)^*i_P(s))i_P(y) : a \in A, x, y, s \in P\}. \]

Then \( \mathcal{I} \) is an ideal of \( \mathcal{T}_{\text{cov}}(A \times_\alpha P), i_A, i_P \), and ker \( \Omega = \mathcal{I} \).

**Proof.** To see that \( \mathcal{I} \) is an ideal of \( \mathcal{T}_{\text{cov}}(A \times_\alpha P) \), it suffices to show that \( i_P(t)^*\mathcal{I}, i_P(t)\mathcal{I}, \) and \( i_A(b)\mathcal{I} \) are all contained in \( \mathcal{I} \) for every \( t \in P \) and \( b \in A \) (by only computing on the spanning elements of \( \mathcal{I} \)). The first one is trivial. For the second one, first note that, by applying the covariance equation of \( (i_A, i_P) \), we have

\[
i_P(t)i_P(x)^* = i_P(t)[i_P(t)^*i_P(t)i_P(x)^*i_P(x)i_P(x)^*]
= i_P(t)[i_P(t)^*i_P(t)i_P(x)^*i_P(x)i_P(x)^*]
= i_P(t)i_P(t)^*i_P((t \vee x)^{-1}i_P((t \vee x)x^{-1})i_P(x)i_P(x)^*)
= i_P(t)\bar{t}_A(\bar{\alpha}_{x}(1))i_P((t \vee x)^{-1}i_P((t \vee x)x^{-1})\bar{a}_x(1))
= i_P((t \vee x)^{-1}i_P(\bar{t}_A(\bar{\alpha}_{x}(1)))i_A(\alpha_{(t \vee x)x^{-1}}(1))i_P((t \vee x)x^{-1})
= i_P((t \vee x)^{-1}i_A(\alpha_{(t \vee x)x^{-1}}(1))i_A(\alpha_{(t \vee x)x^{-1}}(1)))i_P((t \vee x)x^{-1})
= i_P((t \vee x)^{-1}).
\]

Therefore,

\[
i_P(t)[i_P(x)^*i_A(a)(1 - i_P(s)^*i_P(s))i_P(y)]
= [i_P(t)^*i_P(x)^*]i_A(a)(1 - i_P(s)^*i_P(s))i_P(y)
= i_P((t \vee x)^{-1}i_A(\alpha_{(t \vee x)x^{-1}}(1)))i_A(\alpha_{(t \vee x)x^{-1}}(1))i_P((t \vee x)x^{-1})(1 - i_P(s)^*i_P(s))i_P(y)
= i_P((t \vee x)^{-1}).
\]

where \( c = \bar{\alpha}_{(t \vee x)x^{-1}}(a) \in A \) and \( r = (t \vee x)x^{-1} \in P \). Then, in the bottom line, for \( i_P(r)(1 - i_P(s)^*i_P(s)) \), we apply [2.9], which gives us

\[
i_P(t)[i_P(x)^*i_A(a)(1 - i_P(s)^*i_P(s))i_P(y)]
= i_P((t \vee x)^{-1})i_A(\alpha_{(t \vee x)x^{-1}}(a))i_P((t \vee x)x^{-1})(1 - i_P(s)^*i_P(s))i_P(y)
= i_P((t \vee x)^{-1})i_A(\alpha_{(t \vee x)x^{-1}}(a))i_P((t \vee x)x^{-1})(1 - i_P(s)^*i_P(s))i_P(y)
= i_P((t \vee x)^{-1})i_A(\alpha_{(t \vee x)x^{-1}}(a))i_P((t \vee x)x^{-1})(1 - i_P(s)^*i_P(s))i_P(y)
= i_P((t \vee x)^{-1}).
\]

which belongs to \( \mathcal{I} \). For the last one, \( i_A(b)\mathcal{I} \), again by using the covariance equation of \( (i_A, i_P) \), we see that

\[
i_A(b)[i_P(x)^*i_A(a)(1 - i_P(s)^*i_P(s))i_P(y)]
= [i_A(b)i_P(x)^*i_A(a)(1 - i_P(s)^*i_P(s))i_P(y)]
= i_P(x)^*i_A(\alpha_x(b))i_A(a)(1 - i_P(s)^*i_P(s))i_P(y)
= i_P(x)^*i_A(\alpha_x(b))a(1 - i_P(s)^*i_P(s))i_P(y) \in \mathcal{I}.
\]

Thus, \( \mathcal{I} \) is an ideal of \( \mathcal{T}_{\text{cov}}(A \times_\alpha P) \).

Now, we show that ker \( \Omega = \mathcal{I} \). The inclusion \( \mathcal{I} \subset \text{ker} \Omega \) follows immediately as \( \bar{\Omega}(1 - i_P(s)^*i_P(s)) = 1 - \mu_P(s)^*\mu_P(s) = 0 \) for every \( s \in P \). For the other inclusion, take a nondegenerate representation \( \sigma \) of \( \mathcal{T}_{\text{cov}}(A \times_\alpha P) \) on some Hilbert space \( H \) such that ker \( \sigma = \mathcal{I} \). Then, \( (\pi := \sigma \circ i_A, V := \sigma \circ i_P) \) is a Nica-Toeplitz covariant
representation of \((A, P, \alpha)\) on \(H\). However, each \(V_s\) is actually an isometry. To see this, take any approximate identity \(\{a_\lambda\}\) in \(A\). Then, we have
\[
0 = \sigma(i_A(a_\lambda)(1 - i_P(s)^*i_P(s))) = \pi(a_\lambda)(1 - V_s^*V_s)
\]
for each \(\lambda\). One can see that \(\pi(a_\lambda)(1 - V_s^*V_s)\) converges strongly to \((1 - V_s^*V_s)\), and hence, we must have \(1 - V_s^*V_s = 0\). It follows that the pair \((\pi, V)\) is indeed a covariant isometric representation of \((A, P, \alpha)\) on \(H\). Therefore, there is a nondegenerate representation \(\varphi\) of the isometric crossed product \((A \times^\text{iso} P, \mu_P, \mu_P)\) on \(H\), such that \(\varphi(\mu_P(a)) = \pi(a) = \sigma(i_A(a))\) and \(\varphi(\mu_P(x)) = V_x = \pi(i_P(x))\) for all \(a \in A\) and \(x \in P\).
This implies that \(\varphi \circ \Omega = \sigma\), from which, we conclude that \(\ker \Omega \subset \ker \sigma\).

3. The \(C^*\)-algebra \(B\) and its crossed product by \(G\)

Let \((G, P)\) be an abelian lattice-ordered group, and \((A, P, \alpha)\) a dynamical system in which \(\alpha\) is an action of \(P\) by extendible endomorphisms of a \(C^*\)-algebra \(A\). Consider the algebra \(\ell^\infty(G, A)\) of all norm bounded \(A\)-valued functions of \(G\). For every \(s \in G\), we define a map \(\phi_s : A \to \ell^\infty(G, A)\) by
\[
\phi_s(a)(x) = \begin{cases} 
\alpha_{xs^{-1}}(a) & \text{if } s \leq x \\
0 & \text{otherwise.}
\end{cases}
\]

It is not difficult to see that each map \(\phi_s\) is actually an injective \(\ast\)-homomorphism (embedding). Now, let \(B\) be the \(C^*\)-subalgebra of \(\ell^\infty(G, A)\) generated by \(\{\phi_s(a) : s \in G, a \in A\}\). Note that, since \(\phi_s(a)^* = \phi_s(a^*)\), and
\[
\phi_s(a)\phi_t(b) = \phi_{svt}(\alpha_{svt}s^{-1}(a)\alpha_{svt}t^{-1}(b)),
\]
we actually have
\[
B = \overline{\text{span}}\{\phi_s(a) : s \in G, a \in A\}.
\]

**Lemma 3.1.** Each homomorphism \(\phi_s : A \to B\) extends to a strictly continuous homomorphism \(\overline{\phi_s} : \mathcal{M}(A) \to \mathcal{M}(B)\) of multiplier algebras.

**Proof.** Let \(\{a_\lambda\}\) be an approximate identity in \(A\). We show that there exists a projection \(p_s \in \mathcal{M}(B)\) such that \(\phi_s(a_\lambda) \to p_s\) strictly in \(\mathcal{M}(B)\). It suffices to see that \(\phi_s(a_\lambda)\phi_t(a) \to p_s\phi_t(a)\) and \(\phi_t(a)\phi_s(a_\lambda) \to \phi_t(a)p_s\) in the norm topology of \(B\) for every \(a \in A\) and \(t \in G\). Consider the algebra \(\ell^\infty(G, \mathcal{M}(A))\) which contains \(\ell^\infty(G, A)\) as an essential ideal. Then, similar to \(B\), define \(\overline{B}\) to be the \(C^*\)-subalgebra of \(\ell^\infty(G, \mathcal{M}(A))\) spanned by \(\{\chi_s(m) : s \in G, m \in \mathcal{M}(A)\}\), where \(\chi_s : \mathcal{M}(A) \to \ell^\infty(G, \mathcal{M}(A))\) is a map defined by
\[
\chi_s(m)(x) = \begin{cases} 
\pi_{xs^{-1}}(m) & \text{if } s \leq x \\
0 & \text{otherwise.}
\end{cases}
\]

Each \(\chi_s\) is then an embedding such that \(\chi_s|_A = \phi_s\). Now, since \(B\) sits in \(\overline{B}\) as an essential ideal, \(\overline{B}\) sits in \(\mathcal{M}(B)\) as a \(C^*\)-subalgebra. Let \(p_s = \chi_s(1)\) for every \(s \in G\), which is a projection in \(\mathcal{M}(B)\). Then, by (3.1), we have
\[
\phi_s(a_\lambda)\phi_t(a) = \phi_{svt}(\alpha_{svt}s^{-1}(a_\lambda)\alpha_{svt}t^{-1}(a)),
\]
which is convergent to
\[
\phi_{svt}(\overline{\alpha_{svt}s^{-1}(1)\alpha_{svt}t^{-1}(a))}
\]
in the norm topology of \( \mathcal{B} \) (This is due to the facts that each \( \alpha_x \) is extendible, and each \( \phi_s \) is an isometry). On the other hand, again by a similar equation to (3.3) for the spanning elements \( \chi_s \) of \( \mathcal{B} \),

\[
p_s \phi_t(a) = \chi_s(1)\phi_t(a) = \chi_s(1)\chi_t(a) = \chi_{s\sqrt{t}}(\chi_{s\sqrt{t}})_{s-1}(1\chi_{s\sqrt{t}}t-1(a)) = \chi_{s\sqrt{t}}(\alpha_{s\sqrt{t}}s-1(1\alpha_{s\sqrt{t}}t-1(a))) = \phi_{s\sqrt{t}}(\alpha_{s\sqrt{t}}s-1(1\alpha_{s\sqrt{t}}t-1(a)))
\]

Thus, it follows that \( \phi_s(\alpha_x)\phi_t(a) \) is indeed convergent to \( p_s\phi_t(a) \) in \( \mathcal{B} \). We also have \( \phi_t(a)\phi_s(\alpha_x) \rightarrow \phi_t(a)p_s \) by a similar argument, and therefore each \( \phi_s \) is extendible. □

**Remark 3.2.** Note that, therefore, by Lemma 3.1 we have \( \overline{\phi_s} = \chi_s \) for every \( s \in G \). Also, we would like to recall that

\[
(3.2) \quad \overline{\phi_s}(m)\overline{\phi_t}(n) = \overline{\phi_{s\sqrt{t}}(\alpha_{s\sqrt{t}}s-1(m\alpha_{s\sqrt{t}}t-1(n))}
\]

for all \( s, t \in G \) and \( m, n \in \mathcal{M}(A) \). So, in particular, if \( s \leq t \), then, since \( s \vee t = t \),

\[
(3.3) \quad \overline{\phi_s}(m)\overline{\phi_t}(n) = \overline{\phi_t(\chi_{ts-1}(m)n)},
\]

and similarly,

\[
(3.4) \quad \overline{\phi_t}(m)\overline{\phi_s}(n) = \overline{\phi_s(m\alpha_{st-1}(n))},
\]

if \( t \leq s \). These equations have key roles in some computations in section 4.

Next, let \( \mathcal{J} \) be the \( C^* \)-subalgebra of \( \mathcal{B} \) generated by \( \{\phi_s(a) - \phi_t(\alpha_{ts^{-1}}(a)) : s < t \in G, a \in A\} \). Then:

**Proposition 3.3.** We have

\[
(3.5) \quad \mathcal{J} = \text{span}\{\phi_s(a) - \phi_t(\alpha_{ts^{-1}}(a)) : s < t \in G, a \in A\},
\]

which is in fact an essential ideal of \( \mathcal{B} \).

**Proof.** Calculations show that the product

\[
[\phi_s(a) - \phi_t(\alpha_{ts^{-1}}(a))] [\phi_s(b) - \phi_y(\alpha_{yx^{-1}}(b))]
\]

equals the sum of two elements of the same form, and

\[
[\phi_s(a) - \phi_t(\alpha_{ts^{-1}}(a))]^* = \phi_s(a^*) - \phi_t(\alpha_{ts^{-1}}(a))^* = \phi_s(a^*) - \phi_t(\alpha_{ts^{-1}}(a^*))
\]

for all \( a, b \in A \) and \( s, t, x, y \in G \) with \( s < t \) and \( x < y \). Therefore, (3.5) holds. Then, by the following calculation on spanning elements, one can see that \( \mathcal{J} \) is actually an ideal of \( \mathcal{B} \). For all \( a, b \in A \) and \( r, s, t \in G \) with \( s < t \), we have

\[
\phi_r(b)[\phi_s(a) - \phi_t(\alpha_{ts^{-1}}(a))]
\]
\[
= \phi_r(b)\phi_s(a) - \phi_r(b)\phi_t(\alpha_{ts^{-1}}(a))
\]
\[
= \phi_r(\alpha_{r(s)}s^{-1}(b))\alpha_{r(s)}s^{-1}(a)) - \phi_{r(t)}(\alpha_{r(t)}t^{-1}(b))\alpha_{r(t)}t^{-1}(\alpha_{ts^{-1}}(a))
\]
\[
= \phi_r(\alpha_{r(s)}s^{-1}(b))\alpha_{r(s)}s^{-1}(a)) - \phi_{r(t)}(\alpha_{r(t)}t^{-1}(b))\alpha_{r(t)}t^{-1}(\alpha_{ts^{-1}}(a))
\]
\[
= \phi_s(c) - \phi_y(\alpha_{yx^{-1}}(c)) \in \mathcal{J},
\]

where \( c = \alpha_{r(s)}s^{-1}(b)\alpha_{r(s)}s^{-1}(a), x = r \vee s, \) and \( y = r \vee t \) (note that as \( s < t, x \leq y \)).

Finally we show that \( \mathcal{J} \) is an essential ideal of \( \mathcal{B} \). If \( \xi \mathcal{J} = 0 \) for some \( \xi \in \mathcal{B} \), then for each \( s \in G \),

\[
\xi[\phi_s(\xi(s)^*) - \phi_t(\alpha_{ts^{-1}}(\xi(s)^*))] = 0,
\]
where }t\in G\text{ with }s<t.\text{ So, we must have }\xi(s)^*\xi(s) = 0\text{ in }A\text{ for each }s\in G,\text{ which implies that }\xi(s) = 0.\text{ Thus }\xi = 0,\text{ and therefore }\mathcal{J}\text{ is essential.}\]

Note that a simple calculation shows that
\[
(3.6)\quad [\phi_s(a) - \phi_t(\alpha_{ts^{-1}}(a))][\phi_s(b) - \phi_t(\alpha_{ts^{-1}}(b))] = \phi_s(ab) - \phi_t(\alpha_{ts^{-1}}(ab))
\]
is valid for all }a,b\in A\text{ and }s<t\in G.\text{ This equation will be applied later in Lemma 4.2.}

There is an action }\beta\text{ of }G\text{ by automorphisms on }\mathcal{B}\text{ induced by the shift on }\ell^\infty(G,A),\text{ such that }\beta_t \circ \phi_s = \phi_{ts}\text{ for all }s,t\in G.\text{ Thus we obtain a dynamical system }\mathcal{(B,G,\beta)}\text{. Define a map }\rho : \mathcal{B} \to \mathcal{L}(\ell^2(G,A))\text{ by }\rho(\xi)(x) = \xi(x)f(x),\text{ and }U : G \to \mathcal{L}(\ell^2(G,A))\text{ by }U_t f(x) = f(t^{-1}x),\text{ where }\xi \in \mathcal{B}\text{ and }f \in \ell^2(G,A).\text{ Then }\rho\text{ is a nondegenerate representation, and }U\text{ is a unitary representation such that we have }\rho(\beta_t(\xi)) = U_t \rho(\xi) U_t^*.\text{ Therefore the pair }\rho,U\text{ is a covariant representation of }\mathcal{(B,G,\beta)}\text{ on }\ell^2(G,A).\text{ Moreover, let }\mathcal{(B \times_\beta G, j_B,j_G)}\text{ be the group crossed product associated to the system }\mathcal{(B,G,\beta)}\text{. Since }\mathcal{J}\text{ is a }\beta\text{-invariant essential ideal of }\mathcal{B},\mathcal{J \times_\beta G}\text{ sits in }\mathcal{B \times_\beta G}\text{ as an essential ideal [7, Proposition 2.4].}

### 4. The Nica-Toeplitz Algebra }\mathcal{T}_{\text{cov}}(A \times_\alpha P)\text{ as a Full Corner of }\mathcal{B \times_\beta G}

**Theorem 4.1.** Suppose that }\mathcal{(A,P,\alpha)}\text{ is a dynamical system consisting of a }C^*-\text{ algebra }A\text{ and an action }\alpha\text{ of }P\text{ by extendible endomorphisms of }A.\text{ Let }p = j_B \circ \phi_e(1),\text{ and }\]
\[
k_A : A \to p(\mathcal{B \times_\beta G})p \text{ and } k_P : P \to \mathcal{M}(p(\mathcal{B \times_\beta G})p)
\]
be the maps defined by }k_A(a) = (j_B \circ \phi_e)(a)\text{ and }k_P(x) = p j_G(x)^* p\text{ for all }a \in A\text{ and }x \in P.\text{ Then the triple }\mathcal{(p(\mathcal{B \times_\beta G})p,k_A,k_P)}\text{ is a Nica-Toeplitz crossed product for }\mathcal{(A,P,\alpha)},\text{ and hence }\mathcal{(\mathcal{T}_{\text{cov}}(A \times_\alpha P),i_A,i_P)} \simeq \mathcal{(p(\mathcal{B \times_\beta G})p,k_A,k_P)}.\text{ Moreover, }\mathcal{T}_{\text{cov}}(A \times_\alpha P)\text{ is a full corner in }\mathcal{B \times_\beta G}.

**Proof.** First of all, since }j_B\text{ and }\phi_e\text{ are injective, so is }k_A.\text{ Let }\Lambda\text{ be a nondegenerate representation of }p(\mathcal{B \times_\beta G})p\text{ on a Hilbert space }H.\text{ We show that }\pi := \Lambda \circ k_A, W := \overline{\mathcal{N} \circ k_P}\text{ is a Nica-Toeplitz covariant representation of }\mathcal{(A,P,\alpha)}\text{ on }H.\text{ Take an approximate identity }\{a_\lambda\}\text{ in }A.\text{ Since }\phi_e(a_\lambda) \to \phi_e(1)\text{ strictly in }\mathcal{M}(\mathcal{B}),\text{ and }j_B\text{ is nondegenerate, we get }k_A(a_\lambda) \to j_B(\phi_e(1))\text{ strictly in }\mathcal{M}(\mathcal{B \times_\beta G}),\text{ where }\overline{j_B(\phi_e(1))} = j_B \circ \phi_e(1) = p.\text{ Therefore, as }\Lambda\text{ is nondegenerate, }\pi(a_\lambda) = \Lambda(k_A(a_\lambda))\text{ converges strictly to }\overline{\mathcal{N}(p)} = 1\text{ strictly in }B(H) = \mathcal{M}(\mathcal{K}(H)),\text{ which implies that }\pi\text{ is nondegenerate. Next, we show that }W : P \to B(H)\text{ is a partial-isometric representation of }P\text{ on }H\text{ which satisfies the equation }
\]
\[
W_x^* W_x W_y^* W_y = W_{x\cdot y} W_{x\cdot y}
\]
for all }x,y \in P.\text{ To see that each }W_x\text{ is a partial-isometry, note that, by applying the covariance equation of the pair }\mathcal{(j_B,j_G)},\text{ we have }
\[
(4.1)\quad k_P(x) k_P(x)^* k_P(x) = p j_G(x)^* p j_G(x) j_G(x)^* p = p j_G(x)^* p j_B(\phi_e(1)) j_G(x)^* p = p j_G(x)^* j_B(\phi_e(1)) j_B(\phi_e(1)) p = p j_G(x)^* j_B(\phi_e(1)) \overline{j_B(\phi_e(1))} p
\]
Then, in the bottom line, for \( \phi_e(1) \phi_x(1) \), since \( e \leq x \), by (3.3), we have
\[
\phi_e(1) \phi_x(1) = \phi_x(\alpha_x(1)),
\]
and therefore
\[
k_P(x) k_P(x)^* k_P(x) = p j_T(x)^* \overline{\phi_x(\alpha_x(1))} p
\]
Now, again, by the covariance equation of \((j_B, j_T)\),
\[
k_P(x) k_P(x)^* k_P(x) = p j_B(\phi_e(1))^* j_T(x)^* p
\]
\[
= q j_B(\phi_e(1) \phi_x^{-1}(1))^* j_T(x)^* p \quad \text{[by (3.4), as } x^{-1} \leq e\]
\[
= p j_B(\phi_e(1))^* j_B(\phi_x^{-1}(1))^* j_T(x)^* p
\]
\[
= p j_B(\phi_e(1))^* \overline{j_B(\phi_x^{-1}(1))} p
\]
\[
= p j_T(x)^* \overline{\phi_e(1)} p = p j_T(x)^* p = k_P(x).
\]
Therefore, it follows that
\[
W_x W_x^* W_x = \overline{\Lambda}(k_P(x) k_P(x)^* k_P(x)) = \overline{\Lambda}(k_P(x)) = W_x,
\]
which means that each \( W_x \) is a partial-isometry. Moreover,
\[
k_P(x) k_P(y) = p j_T(x)^* j_B(\overline{\phi_e(1)}) j_T(y)^* p
\]
\[
= p j_T(x)^* j_B(\overline{\phi_e(1)}) j_B(\overline{\phi_y(1)}) p
\]
\[
= p j_T(x)^* j_B(\overline{\phi_y(1)}) j_B(\overline{\phi_y(1)}) p
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= p j_T(x)^* j_B(\overline{\phi_y(1)}) j_B(\overline{\phi_y(1)}) p
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= p j_T(x)^* j_B(\overline{\phi_y(1)}) j_B(\overline{\phi_y(1)}) p
\]
\[
= p j_T(x)^* j_B(\overline{\phi_y(1)}) j_B(\overline{\phi_y(1)}) p
\]
Now, for the bottom line, if we continue the computation similar to (4.2) (see that \( p j_B(\phi_e(1))^* j_T(x)^* p = k_P(x) \)), then we get
\[
k_P(x) k_P(y) = p j_B(\overline{\phi_x(\alpha_x(1))}) j_T(x)^* p = k_P(x).\]
Thus,
\[
W_x W_y = \overline{\Lambda}(k_P(x) k_P(y)) = \overline{\Lambda}(k_P(x)) = W_{xy}.
\]
To see that \( W_x^* W_x W_y^* W_y = W_{x \vee y}^* W_{x \vee y} \), note that, by a similar computation done in (4.1), we have
\[
k_P(x)^* k_P(x) = p j_B(\overline{\phi_x(1)}) p = p j_B(\overline{\phi_x(\alpha_x(1))}) p.
\]
So, it follows that
\[
 k_P(x)^*k_P(x)k_P(y)^*k_P(y) = p j_B(\phi_x(1))p j_B(\phi_y(1))p = p j_B(\phi_x(1))j_B(\phi_e(a))j_B(\phi_y(1))p = p j_B(\phi_x(1))p \phi_e(1)\phi_y(1)p = p j_B(\phi_x(1))\phi_y(\phi_y(1))p \quad \text{[by (3.3), as } e \leq y]\]
\[
 = p j_B(\phi_{x\vee y}(\phi_y(1))^p) \quad \text{[by (3.2)]}
\]
\[
 = p j_B(\phi_{x\vee y}(\phi_y(1)))p \quad \text{[since } (x \vee y)x^{-1} \leq (x \vee y)]
\]
\[
 = k_P(x \vee y)^*k_P(x \vee y). \quad \text{[by (4.3)]}
\]

Therefore, we have
\[
 W_x^* W_y^* W_x^* W_y = \overline{\Lambda}(k_P(x)^*k_P(x)k_P(y)^*k_P(y))
\]
\[
 = \overline{\Lambda}(k_P(x \vee y)^*k_P(x \vee y)) = W_{x\vee y}^* W_{x\vee y}
\]

Now, we show that the pair \((\pi, W)\) satisfies the covariance equations
\[
 \pi(\alpha_x(a)) = W_x \pi_A(a) W_x^* \quad \text{and} \quad W_x^* W_x \pi(a) = \pi(a) W_x^* W_x
\]
for every \(a \in A\) and \(x \in P\). We have
\[
 k_P(x)k_A(a)k_P(x)^* = p j_B(\phi_x(1))p j_B(\phi_e(a))j_B(\phi_y(1))p
\]
\[
 = p j_B(\phi_x(1))\phi_x(-1)(\phi_e(a))p \quad \text{[by the covariance of } (j_B, j_\Gamma)\]
\[
 = p j_B(\phi_x(1))\phi_x(-1)(a)p
\]
\[
 = p j_B(\phi_x(1))j_B(\phi_x^{-1}(a))p = p j_B(\phi_x^{-1}(a))p
\]
\[
 = p j_B(\phi_x(1))p \phi_x^{-1}(a)p \quad \text{[by (3.4), as } x^{-1} \leq e]\]
\[
 = (j_B \circ \phi_e)(\alpha_x(a)) = k_A(\alpha_x(a)).
\]

Thus, it follows that
\[
 \pi(\alpha_x(a)) = \Lambda(k_A(\alpha_x(a))) = \Lambda(k_P(x)k_A(a)k_P(x)^*) = W_x \pi(a) W_x^*
\]

To see that \(W_x^* W_x \pi(a) = \pi(a) W_x^* W_x\), first, by using (1.3), we have
\[
 k_P(x)^*k_P(x)k_A(a) = p j_B(\phi_x(1))p j_B(\phi_e(a))
\]
\[
 = p j_B(\phi_x(1))j_B(\phi_e(a))p = p j_B(\phi_x(1))\phi_x(a)p
\]
\[
 = p j_B(\phi_x(1)\alpha_x(a))p \quad \text{[by (3.4), as } e \leq x]\]
\[
 = p j_B(\phi_x(1)\alpha_x(a))p
\]
\[
 = p j_B(\phi_x(a))j_B(\phi_x(1))p = p j_B(\phi_x(a))p \phi_x(1)p
\]
\[
 = j_B(\phi_e(a))j_B(\phi_x(1))p = k_A(\alpha_x(a))k_P(x)^*k_P(x).
\]

Therefore, we get
\[
 W_x^* W_x \pi(a) = \Lambda(k_P(x)^*k_P(x)k_A(a))
\]
\[
 = \Lambda(k_A(\alpha_x(a))k_P(x)^*k_P(x)) = \pi(a) W_x^* W_x
\]

So, the pair \((\pi, W)\) is a Nica-Toeplitz covariant representation of \((A, P, \alpha)\) on \(H\).

Next, we want to prove that
\[
 (4.4) \quad p(B \times G)p = \overline{\text{span}}\{k_P(x)^*k_A(a)k_P(y) : a \in A, x, y \in P\}.
\]
We only need to show that \( p(B \times_\beta G)p \) is a subset of the right hand side of (1.4), as the other inclusion is obvious. So, recall that, since elements of the form \( j_B(\phi_r(a))j_C(s) \) span \( B \times_\beta G \), where \( a \in A \) and \( r, s \in G \), \( p(B \times_\beta G)p \) is spanned by the elements \( p j_B(\phi_r(a))j_C(s)p \). However,

\[
\begin{align*}
    pj_B(\phi_r(a))j_C(s)p &= \overline{pj_B(\phi_e(1))} j_B(\phi_r(a))j_C(s)p \\
    &= pj_B(\phi_e(1))j_B(\phi_r(a))j_C(s)p \\
    &= pj_B(\phi_e(1))j_B(\phi_r(a))j_C(s)p, \quad \text{by (3.2)}
\end{align*}
\]

where \( (e \lor r) \in P \), as \( e \leq e \lor r \). So, for the spanning elements \( pj_B(\phi_r(a))j_C(s)p \) of \( p(B \times_\beta G)p \), we can assume that \( r \in P \) without loss of generality. Furthermore,

\[
\begin{align*}
    pj_B(\phi_r(a))j_C(s)p &= pj_C(s)j_B(\beta_{s^{-1}}(\phi_r(a)))p \\
    &= pj_C(s)j_B(\beta_{s^{-1}}(\phi_r(a)))j_B(\phi_e(1))p \\
    &= pj_C(s)j_B(\phi_{s^{-1}r}(a))j_B(\phi_e(1))p \\
    &= pj_C(s)j_B(\phi_{s^{-1}r}(a))j_B(\phi_e(b))p \quad [b = \alpha_{s^{-1}r \lor e}^{-1}(a)\overline{\alpha_{s^{-1}r \lor e}(1)}] \\
    &= pj_C(s)j_B(\phi_{s^{-1}r \lor e}(b))j_B(\phi_{s^{-1}r \lor e}^{-1}(b))p \\
    &= pj_C(s)j_B(\phi_{s^{-1}r \lor e}(b))j_B(\phi_{s^{-1}r \lor e}^{-1}(b))j_B(\phi_{s^{-1}r \lor e}^{-1}(b))p \\
    &= pj_C(s)j_B(\phi_{s^{-1}r \lor e}(b))j_B(\phi_{s^{-1}r \lor e}^{-1}(b))j_B(\phi_{s^{-1}r \lor e}^{-1}(b))p \\
    &= pj_C(s)j_B(\phi_{s^{-1}r \lor e}(b))j_B(\phi_{s^{-1}r \lor e}^{-1}(b))j_B(\phi_{s^{-1}r \lor e}^{-1}(b))p,
\end{align*}
\]

where \( x = s(s^{-1}r \lor e) \) and \( y = (s^{-1}r) \lor e \). Then, \( y \in P \), obviously, and since \( s^{-1}r \leq s^{-1}r \lor e \), we have

\[
e \leq r = s(s^{-1}r) \leq s(s^{-1}r \lor e) = x.
\]

It follows that \( e \leq x \), and hence \( x \in P \), too. Therefore, we have

\[
    pj_B(\phi_r(a))j_C(s)p = pj_C(x)j_B(\phi_{e(b)})j_B(\phi_{s^{-1}r \lor e})p = k_P(x)^*k_A(b)k_P(y)
\]

for some \( b \in A \) and \( x, y \in P \). This implies that \( p(B \times_\beta G)p \) is a subset of the right hand side of (1.4), and therefore, (1.4) is indeed valid.

Now, suppose that \( (\sigma, V) \) is a Nica-Toeplitz covariant representation of \( (A, P, \alpha) \) on a Hilbert space \( K \). We show that there is a nondegenerate representation \( \Phi \) of \( p(B \times_\beta G)p \) on \( K \) such that \( \Phi \circ k_A = \sigma \) and \( \Phi \circ k_P = V \). To do so, first, we take a faithful and nondegenerate representation \( \Delta \) of \( p(B \times_\beta G)p \) on a Hilbert space \( H \). Then, similar to the earlier argument, one can see that the pair \( (\pi := \Delta \circ k_A, W := \Delta \circ k_P) \) is a Nica-Toeplitz covariant representation of \( (A, P, \alpha) \) on \( H \). Let \( \pi \times W \) be the associated nondegenerate representation (integrated form) of the Nica-Toeplitz algebra \( (T_{\text{cov}}(A \times_\alpha P), i_A, i_P) \) on \( H \), such that \( (\pi \times W) \circ i_A = \pi \) and \( (\pi \times W) \circ i_P = W \). We claim that \( \pi \times W \) is faithful. To prove our claim, we apply [5, Theorem 9.3]. So, take any finite subset \( F = \{x_1, x_2, ..., x_n\} \) of \( P \setminus \{e\} \), and assume that

\[
\pi(a) \prod_{i=1}^{n} (1 - W_{x_i}^*W_{x_i}) = 0,
\]

where \( a \in A \). It follows that

\[
0 = \pi(a) \prod_{i=1}^{n} (1 - W_{x_i}^*W_{x_i}) = \Delta(k_A(a) \prod_{i=1}^{n} (p - k_P(x_i)^*k_P(x_i))),
\]
and since $\Delta$ is faithful, we must have

$$k_A(a) \prod_{i=1}^{n} (p - k_P(x_i)^*k_P(x_i)) = 0.$$ 

However,

$$k_A(a) \prod_{i=1}^{n} (p - k_P(x_i)^*k_P(x_i)) = j_B(\phi_e(a)) \prod_{i=1}^{n} [p - p\overline{j_B(\phi_e)}(\overline{x}_i(1))] \tag{4.3}$$

and since

$$\overline{j_B(\phi_e(a))} = j_B(\phi_e(a)) \prod_{i=1}^{n} [p - p\overline{j_B(\phi_e)}(\overline{x}_i(1))] \tag{4.3}$$

we have

$$\overline{j_B(\phi_e(a))} = j_B(\phi_e(a)) \prod_{i=1}^{n} [p - p\overline{j_B(\phi_e)}(\overline{x}_i(1))] \tag{4.3}$$

By some calculation, it is not difficult to see that, in fact,

$$j_B(\phi_e(a)) - j_B(\phi_e(a)) \prod_{i=1}^{n} [p - p\overline{j_B(\phi_e)}(\overline{x}_i(1))] \prod_{i=2}^{n} [p - p\overline{j_B(\phi_e)}(\overline{x}_i(1))]$$

Thus, it follows from (4.5) that

$$(\phi_e(a) - \phi_{x_1}(\alpha_{x_1}(a))) \prod_{i=2}^{n} (\phi_e(1) - \phi_{x_i}(\overline{x}_i(1))) = 0$$

Also, note that, by some calculation, it is not difficult to see that, in fact,

$$(\phi_e(a) - \phi_{x_1}(\alpha_{x_1}(a))) \prod_{i=2}^{n} (\phi_e(1) - \phi_{x_i}(\overline{x}_i(1))) = \phi_e(a) + \sum_{k=1}^{m} \phi_{y_k}(a_k),$$

where $a_k \in A$ and $y_k \in P \setminus \{e\}$ for every $1 \leq k \leq m$. Thus, it follows from (4.5) that

$$j_B(\phi_e(a) + \sum_{k=1}^{m} \phi_{y_k}(a_k)) = 0.$$ 

Now, if $\rho \times U$ is the nondegenerate representation of $B \times G$ corresponding to the covariant pair $(\rho, U)$ of $(B, G)$ in $L(\ell^2(G, A))$ (see (3)), then, by (4.6), we get

$$\rho \times U[j_B(\phi_e(a) + \sum_{k=1}^{m} \phi_{y_k}(a_k))] = \rho(\phi_e(a) + \sum_{k=1}^{m} \phi_{y_k}(a_k)) = 0$$

in $L(\ell^2(G, A))$. It follows that, for $\varepsilon_e \otimes a^* \in \ell^2(G) \otimes A \simeq \ell^2(G, A)$, we must have

$$\rho(\phi_e(a) + \sum_{k=1}^{m} \phi_{y_k}(a_k)) (\varepsilon_e \otimes a^*) = 0.$$ 

But, since each $y_k$ satisfies $e < y_k$,

$$0 = \rho(\phi_e(a) + \sum_{k=1}^{m} \phi_{y_k}(a_k)) (\varepsilon_e \otimes a^*) = (\rho(\phi_e(a)) + \sum_{k=1}^{m} \rho(\phi_{y_k}(a_k))) (\varepsilon_e \otimes a^*) = \rho(\phi_e(a)) (\varepsilon_e \otimes a^*) + \sum_{k=1}^{m} \rho(\phi_{y_k}(a_k)) (\varepsilon_e \otimes a^*) = \varepsilon_e \otimes a^* + 0 = \varepsilon_e \otimes a^*.$$
Therefore, $\varepsilon_e \otimes aa^* = 0$, and hence, $aa^* = 0$, which implies that $a = 0$. Thus, by [5, Theorem 9.3], $\pi \times W$ is faithful. Then, define a map

$$\Psi_0 : \text{span}\{i_P(x)^*i_A(a)i_P(y) : a \in A, x, y \in P\} \to p(B \times_\beta G)p$$

by

$$\Psi_0(\sum i_P(x_m)^*i_A(a_{m,n})i_P(y_n)) = \sum k_P(x_m)^*k_A(a_{m,n})k_P(y_n).$$

One can see that $\Psi_0$ is linear. Also, the following computation

$$\|\sum k_P(x_m)^*k_A(a_{m,n})k_P(y_n)\| = \|\Delta(\sum k_P(x_m)^*k_A(a_{m,n})k_P(y_n))\| = \|\sum \pi W_{a_{m,n}}W_{y_n}\| = \|\pi \times W(\sum i_P(x_m)^*i_A(a_{m,n})i_P(y_n))\| = \|\sum i_P(x_m)^*i_A(a_{m,n})i_P(y_n)\|$$

shows that $\Psi_0$ preserves the norm. It therefore follows that it is a well-defined linear map on the dense subspace $\text{span}\{i_P(x)^*i_A(a)i_P(y) : a \in A, x, y \in P\}$ of $T_{\text{cov}}(A \times_\alpha P)$. Hence, it extends to a norm-preserving linear map $\Psi$ of $T_{\text{cov}}(A \times_\alpha P)$ into $p(B \times_\beta G)p$. Then, it is not difficult to see that $\Psi$ preserves the involution, too. Moreover, one can calculate to verify that we have

$$\Psi((i_P(x)^*i_A(a)i_P(y))[i_P(s)^*i_A(b)i_P(t)])$$

$$= \Psi(i_P(x(y \vee s))y^{-1}i_A(\alpha_{(y\vee s)s^{-1}}(a)\alpha_{(y\vee s)s^{-1}}(b))i_P((y \vee s)s^{-1}t))$$

$$= k_P(x(y \vee s))y^{-1}k_A(\alpha_{(y\vee s)s^{-1}}(a)\alpha_{(y\vee s)s^{-1}}(b))k_P((y \vee s)s^{-1}t)$$

$$= [k_P(x)^*k_A(a)k_P(y)][k_P(s)^*k_A(b)k_P(t)] = \Psi(i_P(x)^*i_A(a)i_P(y))\Psi(i_P(s)^*i_A(b)i_P(t))$$

on the spanning elements of $T_{\text{cov}}(A \times_\alpha P)$. So, this implies that $\Psi$ also preserves the multiplication, and thus, it is indeed an injective $*$-homomorphism of $T_{\text{cov}}(A \times_\alpha P)$ into $p(B \times_\beta G)p$. Moreover, since

$$k_P(x)^*k_A(a)k_P(y) = \Psi(i_P(x)^*i_A(a)i_P(y)) \in T_{\text{cov}}(A \times_\alpha P),$$

it follows by [4.4] that $\Psi$ is also onto. Therefore, $\Psi$ is actually an isomorphism of $T_{\text{cov}}(A \times_\alpha P)$ onto $p(B \times_\beta G)p$. Finally, if $\sigma \times V$ is the associated nondegenerate representation (integrated form) of $T_{\text{cov}}(A \times_\alpha P)$ on $K$ such that $(\sigma \times V) \circ i_A = \sigma$ and $(\sigma \times V) \circ i_B = V$, then $\Phi := (\sigma \times V) \circ \Psi^{-1}$ is the desired nondegenerate representation of $p(B \times_\beta G)p$ on $K$ which satisfies $\Phi \circ k_A = \sigma$ and $\Phi \circ k_B = V$.

In order to see that $T_{\text{cov}}(A \times_\alpha P)$ is a full corner in $B \times_\beta G$, we have to show that $(B \times_\beta G)p(B \times_\beta G)$ is dense in $B \times_\beta G$. If $\{a_n\}$ is an approximate identity in $A$, then for any spanning element $j_B(\phi_r(a))j_G(s)$ of $B \times_\beta G$, where $a \in A$ and $r, s \in G$, we have $j_B(\phi_r(\alpha_Aa))j_G(s) \to j_B(\phi_r(a))j_G(s)$ in the norm topology of $B \times_\beta G$. Moreover,

$$j_B(\phi_r(\alpha_Aa))j_G(s) = j_B((\beta_r \circ \phi_e(\alpha_Aa))j_G(s) = j_G(r)j_B(\phi_e(\alpha_Aa))j_G(r)^*j_G(s) = j_G(r)j_B(\phi_e(\alpha_Aa)\overline{\phi_e(1)}j_B(\phi_e(a))j_G(r^{-1})j_G(s) = j_G(r)j_B(\phi_e(\alpha_Aa))j_B(\phi_e(1))j_B(\phi_e(a))j_G(r^{-1})s) = [j_G(r)j_B(\phi_e(\alpha_Aa))]^*j_B(\phi_e(a))j_G(r^{-1}s)]$$

As the bottom line belongs to $(B \times_\beta G)p(B \times_\beta G)$, so does $j_B(\phi_r(\alpha_Aa))j_G(s)$. It therefore follows that $j_B(\phi_r(a))j_G(s) \in (B \times_\beta G)p(B \times_\beta G)$, which implies that $(B \times_\beta G)p(B \times_\beta G) = B \times_\beta G$. Thus, $T_{\text{cov}}(A \times_\alpha P)$ is a full corner in $B \times_\beta G$. This completes the proof. □

Lemma 4.2. The ideal $I$ of $T_{\text{cov}}(A \times_\alpha P)$ is isomorphic to $p(J \times_\beta G)p$, which is a full corner in $J \times_\beta G$. 
Proof. We show that the isomorphism $\Psi$ in Theorem 4.1 maps the ideal $\mathcal{I}$ onto $p(\mathcal{J} \times \beta G)p$. Firstly, note that, as $\mathcal{J} \times \beta G$ sits in $\mathcal{B} \times \beta G$ as an essential ideal, $\mathcal{B} \times \beta G$ is embedded in $\mathcal{M}(\mathcal{J} \times \beta G)$ as a $C^*$-subalgebra. So does $\mathcal{M}(\mathcal{B} \times \beta G)$, and therefore, $p \in \mathcal{M}(\mathcal{J} \times \beta G)$. Also recall that $\mathcal{J} \times \beta G$ is spanned by elements of the form $j_B(\phi_r(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z)$, where $a \in A$ and $r, s, z \in G$ such that $r < s$. Now, in order to see that $\Psi(\mathcal{I}) = p(\mathcal{J} \times \beta G)p$, let $a \in A$ and $x, y, t \in P$. We have

$$\Psi(i_P(x)^*i_A(a)(1 - i_P(t)i_P(t))i_P(y)) = k_P(x)^*k_A(a)(p - k_P(t)^*k_P(t))k_P(y).$$

Then, by a similar calculation that implies (4.5), we have

$$k_A(a)(p - k_P(t)^*k_P(t)) = j_B(\phi_e(a) - \phi_t(\alpha_t(a))),$$

and hence, we obtain

$$(4.7) \quad k_P(x)^*k_A(a)(p - k_P(t)^*k_P(t))k_P(y) = p[j_B(x)j_B(\phi_e(a) - \phi_t(\alpha_t(a)))j_G(y)^*]p$$

which belongs to $p(\mathcal{J} \times \beta G)p$. So it follows that $\Psi(\mathcal{I}) \subset p(\mathcal{J} \times \beta G)p$. For the other inclusion, we show that each spanning element $p[j_B(\phi_r(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z)]p$ of $p(\mathcal{J} \times \beta G)p$ belongs to $\Psi(\mathcal{I})$. In order to do so, first, we can assume that $r, s \in P$ with $r < s$ without loss of generality. This is due to the fact that

$$p[j_B(\phi_r(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z)]p = p[j_B(\phi_r(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z)]p = p[j_B(\phi_r(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z)]p,$$

where

$$\overline{\phi_e(1)\phi_r(a) - \phi_e(1)\phi_s(\alpha_{sr^{-1}}(a))} = \phi_{e\vee r}(\overline{\alpha_{e\vee r}^{-1}(\alpha_{sr^{-1}}(a))}) - \phi_{e\vee s}(\overline{\alpha_{e\vee s}^{-1}(\alpha_{sr^{-1}}(a))}) = \phi_{e\vee r}(\overline{\alpha_{e\vee r}^{-1}(\alpha_{sr^{-1}}(a))}) - \phi_{e\vee s}(\overline{\alpha_{e\vee s}^{-1}(\alpha_{sr^{-1}}(a))}).$$

Thus, since $r < s$, $e \leq e \vee r \leq e \vee s$, from which, for $\overline{\alpha_{e\vee s}^{-1}(\alpha_{sr^{-1}}(a))}$, we have

$$\overline{\alpha_{e\vee r}^{-1}(\alpha_{sr^{-1}}(a))} = \alpha_{e\vee s}(\overline{\alpha_{e\vee s}^{-1}(\alpha_{sr^{-1}}(a))}).$$

Consequently, we see that each spanning element $p[j_B(\phi_r(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z)]p$ of $p(\mathcal{J} \times \beta G)p$ can actually be written of the form

$$p[j_B(\phi_r(b) - \phi_y(\alpha_{y^{-1}}(b)))j_G(z)]p$$

for some $b \in A$ and $x, y, t \in P$ with $x < y$. So, take any spanning element $p[j_B(\phi_r(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z)]p$ of $p(\mathcal{J} \times \beta G)p$, where $a \in A, z \in G$, and $r, s \in P$ with $r < s$. It follows by the covariance equation of $(j_B, j_G)$ that

$$p[j_B(\phi_r(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z)]p = p[j_G(z)(j_B \circ \beta_{z^{-1}})(\phi_r(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z)]p = p[j_G(z)(j_B(\phi_{z^{-1}}(a) - \phi_{z^{-1}}(\alpha_{sr^{-1}}(a))))j_B(\overline{\phi_e(1)})]p$$

where

$$\phi_{z^{-1}}(a)\overline{\phi_e(1)} - \phi_{z^{-1}}(\alpha_{sr^{-1}}(a))\overline{\phi_e(1)} = \phi_{z^{-1}r\vee e}(\alpha_{z^{-1}r\vee e}^{-1}(\alpha_{sr^{-1}}(a)))\overline{\phi_{z^{-1}r\vee e}(1)} - \phi_{z^{-1}s\vee e}(\alpha_{z^{-1}s\vee e}^{-1}(\alpha_{sr^{-1}}(a)))\overline{\phi_{z^{-1}s\vee e}(1)}.$$


Therefore, it follows that

\[
p[j_B(\phi_r(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z)]p = p[j_G(z)j_B(\phi_e(b) - \phi_y(\alpha_{yz^{-1}}(b)))]p,
\]

where \( x = (z^{-1}r) \lor e, y = (z^{-1}s) \lor e, \) and \( b = \alpha_{(z^{-1}r)e} \). Then,

\[
\begin{align*}
p[j_G(z)j_B(\phi_e(b) - \phi_y(\alpha_{yz^{-1}}(b)))]p &= p[j_G(z)(j_B \circ \beta_x(\phi_e(b) - \phi_y(\alpha_{yz^{-1}}(b)))])p \quad \text{[see (??)]} \\
&= \Psi(i_p(x)^*e_p(b)(1 - i_p(yx^{-1})^*i_p(yx^{-1}))i_p(x)) \in \Psi(I).
\end{align*}
\]

Now, \( x = (z^{-1}r) \lor e \in P, \) clearly, and for \( zx, \) we have

\[
e \leq r = z(z^{-1}r) \leq z((z^{-1}r) \lor e) = zx.
\]

Thus, \( zx \in P, \) too, and therefore,

\[
p[j_G(z)j_B(\phi_e(b) - \phi_y(\alpha_{yz^{-1}}(b)))]j_G(x^*)]p
\]

So, \( p[j_B(\phi_e(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z)]p \) belongs to \( \Psi(I) \) which implies that \( p(J \times _\beta G) \subset \Psi(I). \) Therefore, we indeed have

\[
I \simeq \Psi(I) = p(J \times _\beta G)p.
\]

In order to show that \( I \) is a full corner in \( J \times _\beta G, \) we take an approximate identity in \( \{a_\lambda\} \) in \( A. \) Then, for any spanning element spanned \( j_B(\phi_r(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z) \) of \( J \times _\beta G, \) where \( a \in A \) and \( r, s, z \in G \) with \( r < s, \) we have

\[
j_B(\phi_r(aa_\lambda) - \phi_s(\alpha_{sr^{-1}}(aa_\lambda)))j_G(z) \rightarrow j_B(\phi_r(a) - \phi_s(\alpha_{sr^{-1}}(a)))j_G(z)
\]
in \( J \times _\beta G \) with norm topology. Now, for \( j_B(\phi_r(aa_\lambda) - \phi_s(\alpha_{sr^{-1}}(aa_\lambda)))j_G(z) \), by applying the covariance equation of \( (j_B, j_G) \), we have

\[
j_B(\phi_r(aa_\lambda) - \phi_s(\alpha_{sr^{-1}}(aa_\lambda)))j_G(z)
\]

Then, in the bottom line, for \( \phi_e(aa_\lambda) - \phi_{sr^{-1}}(\alpha_{sr^{-1}}(aa_\lambda)), \) it follows by the equation \( [3.6] \) that

\[
j_B(\phi_r(aa_\lambda) - \phi_s(\alpha_{sr^{-1}}(aa_\lambda)))j_G(r^{-1}z)
\]

\[
= j_G(r)j_B(\phi_e(aa_\lambda) - \phi_{sr^{-1}}(\alpha_{sr^{-1}}(aa_\lambda)))j_G(r^{-1}z)
\]

\[
= j_G(r)j_B(\phi_e(aa_\lambda) - \phi_{sr^{-1}}(\alpha_{sr^{-1}}(aa_\lambda)))j_B(\phi_e(aa_\lambda) - \phi_{sr^{-1}}(\alpha_{sr^{-1}}(aa_\lambda)))j_G(r^{-1}z)
\]

\[
= j_G(r)j_B(\phi_e(aa_\lambda) - \phi_{sr^{-1}}(\alpha_{sr^{-1}}(aa_\lambda)))j_B(\phi_e(aa_\lambda) - \phi_{sr^{-1}}(\alpha_{sr^{-1}}(aa_\lambda)))j_G(r^{-1}z)
\]

\[
= [j_G(r)j_B(\phi_e(aa_\lambda) - \phi_{sr^{-1}}(\alpha_{sr^{-1}}(aa_\lambda)))j_B(\phi_e(aa_\lambda) - \phi_{sr^{-1}}(\alpha_{sr^{-1}}(aa_\lambda)))j_G(r^{-1}z),
\]
which is an element of \((\mathcal{J} \times_\beta G)p(\mathcal{J} \times_\beta G)\). So, it follows that \(j_B(\phi_r(a) - \phi_s(\alpha_{r-1}(a)))j_G(z) \in (\mathcal{J} \times_\beta G)p(\mathcal{J} \times_\beta G)\), and hence \((\mathcal{J} \times_\beta G)p(\mathcal{J} \times_\beta G) = \mathcal{J} \times_\beta G\). Therefore, \(\mathcal{I} \cong p(\mathcal{J} \times_\beta G)p\) is a full corner in \(\mathcal{J} \times_\beta G\).

\(\square\)

**Proposition 4.3.** The ideal \(\mathcal{I}\) is an essential ideal of the Nica-Toeplitz algebra \(\mathcal{T}_{\text{cov}}(A \times_\alpha P)\).

**Proof.** This is due to the facts that the ideal \(\mathcal{I}\) and \(\mathcal{T}_{\text{cov}}(A \times_\alpha P)\) are full corners in \(\mathcal{J} \times_\beta G\) and \(\mathcal{B} \times_\beta G\), respectively, and \(\mathcal{J} \times_\beta G\) is essential ideal of \(\mathcal{B} \times_\beta G\). \(\square\)

## 5. Examples

In this section, as an example, we consider the abelian lattice-ordered group \((\mathbb{Z}^2, \mathbb{N}^2)\). Thus, let \((A, \mathbb{N}^2, \alpha)\) be a dynamical system consisting of a \(C^*\)-algebra \(A\) and an action \(\alpha\) of \(\mathbb{N}^2\) by extendible endomorphisms of \(A\). Then, \(\alpha\) induces two actions \(\delta\) and \(\gamma\) of \(\mathbb{N}\) on \(A\) by extendible endomorphisms, such that

\[
\delta_n := \alpha_{(0,n)} \quad \text{and} \quad \gamma_n := \alpha_{(n,0)}
\]

for every \(n \in \mathbb{N}\). Hence, we have the dynamical systems \((A, \mathbb{N}, \delta)\) and \((A, \mathbb{N}, \gamma)\) generated by the single endomorphisms \(\delta := \delta_1\) and \(\gamma := \gamma_1\), respectively. One can easily see that

\[
\alpha_{(m,n)} = \delta_n \gamma_m = \gamma_m \delta_n
\]

for all \(m, n \in \mathbb{N}\). Now, we want to define two subalgebras \(D_\delta\) and \(D_\gamma\) of \(\ell^\infty(\mathbb{Z}, A)\), which are actually the corresponding algebra \(B\) to the totally ordered abelian group \((\mathbb{Z}, \mathbb{N})\) and the systems \((A, \mathbb{N}, \delta)\) and \((A, \mathbb{N}, \gamma)\), respectively (see also [18, §6]). Thus, we have

\[
D_\delta = \text{span}\{\phi_n^\delta(a) : n \in \mathbb{Z}, a \in A\} \quad \text{and} \quad D_\gamma = \text{span}\{\phi_n^\gamma(a) : n \in \mathbb{Z}, a \in A\},
\]

where the maps \(\phi_n^\delta : A \rightarrow \ell^\infty(\mathbb{Z}, A)\) and \(\phi_n^\gamma : A \rightarrow \ell^\infty(\mathbb{Z}, A)\) are the (extendible) isometries defined by

\[
\phi_n^\delta(a)(m) = \begin{cases} 
\delta_{m-n}(a) & \text{if } n \leq m \\
0 & \text{otherwise,}
\end{cases}
\]

and

\[
\phi_n^\gamma(a)(m) = \begin{cases} 
\gamma_{m-n}(a) & \text{if } n \leq m \\
0 & \text{otherwise,}
\end{cases}
\]

for all \(m, n \in \mathbb{Z}\) and \(a \in A\), respectively. Note that, the algebras \(D_\delta\) and \(D_\gamma\) both contain the algebra \(C_0(\mathbb{Z}) \otimes A\) as an essential ideal, such that

\[
C_0(\mathbb{Z}) \otimes A = \text{span}\{\phi_n^\delta(a) - \phi_m^\delta(\delta_{m-n}(a)) : n \leq m \in \mathbb{Z}, a \in A\}
\]

\[
= \text{span}\{\phi_n^\gamma(a) - \phi_m^\gamma(\gamma_{m-n}(a)) : n \leq m \in \mathbb{Z}, a \in A\}.
\]

In fact, \(C_0(\mathbb{Z}) \otimes A\) is the corresponding (essential) ideal \(\mathcal{J}\) for the algebras \(D_\delta\) and \(D_\gamma\).

Next, we see that the algebra \(B\) associated with the system \((A, \mathbb{N}^2, \alpha)\) seats in the algebras \(\ell^\infty(\mathbb{Z}, D_\delta)\) and \(\ell^\infty(\mathbb{Z}, D_\gamma)\) as a \(C^*\)-subalgebra, which leads us to identify its
(essential) ideal \( J \) with a familiar term. For every \( m, n \in \mathbb{Z} \) and \( a \in A \), define a map \( \psi^\delta_{(m, n)} : A \to \ell^\infty(\mathbb{Z}, D_\delta) \) by

\[
\psi^\delta_{(m, n)}(a)(r) = \begin{cases} 
\phi^\delta_m(\gamma_{r-m}(a)) & \text{if } m \leq r \\
0 & \text{otherwise}.
\end{cases}
\]

Calculations show that each map \( \psi^\delta_{(m, n)} \) is a norm-preserving \(*\)-homomorphism. Now, let \( B_\delta \) be the \( C^* \)-subalgebra of \( \ell^\infty(\mathbb{Z}, D_\delta) \) generated by \( \{ \psi^\delta_{(m, n)}(a) : m, n \in \mathbb{Z}, a \in A \} \). Note that, by calculation, we have \( \psi^\delta_{(m, n)}(a)^* = \psi^\delta_{(m, n)}(a^*) \), and \( \psi^\delta_{(m, n)}(a)\psi^\delta_{(t, u)}(b) = \psi^\delta_{(x, y)}(c) \), where \( (x, y) = (m, n) \vee (t, u) \) \( (x = \max\{m, t\} \) and \( y = \max\{n, u\} \) and

\[
c = \delta_{y-n}(\gamma_{x-m}(a))\delta_{y-u}(\gamma_{x-t}(b)).
\]

Thus, it follows that

\[
B_\delta = \overline{\text{span}}\{ \psi^\delta_{(m, n)}(a) : m, n \in \mathbb{Z}, a \in A \}.
\]

Similar to \( B_\delta \), for every \( m, n \in \mathbb{Z} \) and \( a \in A \), we define a map \( \psi^\gamma_{(m, n)} : A \to \ell^\infty(\mathbb{Z}, D_\gamma) \) by

\[
\psi^\gamma_{(m, n)}(a)(s) = \begin{cases} 
\phi^\gamma_m(\delta_{s-n}(a)) & \text{if } n \leq s \\
0 & \text{otherwise},
\end{cases}
\]

which is an injective \(*\)-homomorphism. Then, define \( B_\gamma \) to be the \( C^* \)-subalgebra of \( \ell^\infty(\mathbb{Z}, D_\gamma) \) generated by \( \{ \psi^\gamma_{(m, n)}(a) : m, n \in \mathbb{Z}, a \in A \} \). We similarly have

\[
B_\gamma = \overline{\text{span}}\{ \psi^\gamma_{(m, n)}(a) : m, n \in \mathbb{Z}, a \in A \},
\]

as \( \psi^\gamma_{(m, n)}(a)^* = \psi^\gamma_{(m, n)}(a^*) \), and \( \psi^\gamma_{(m, n)}(a)\psi^\gamma_{(t, u)}(b) = \psi^\gamma_{(x, y)}(c) \) for some \( x, y \in \mathbb{Z} \) and \( c \in A \).

**Lemma 5.1.** Each homomorphism \( \psi^\delta_{(m, n)} : A \to B_\delta \) and \( \psi^\gamma_{(m, n)} : A \to B_\gamma \) is extendible to a strictly continuous homomorphism \( \overline{\psi^\delta_{(m, n)}} : \mathcal{M}(A) \to \mathcal{M}(B_\delta) \) and \( \overline{\psi^\gamma_{(m, n)}} : \mathcal{M}(A) \to \mathcal{M}(B_\gamma) \) of multiplier algebras, respectively.

**Proof.** The discussion of proof is similar to the proof of Lemma 3.1. So, we skip it here. However, we would like to mention quickly that this is due to the extendibility of endomorphisms \( \delta_n \) and \( \gamma_n \), and homomorphisms \( \phi^\delta_n \) and \( \phi^\gamma_n \). \( \square \)

Thus, it follows by Lemma 5.1 that

\[
\overline{\psi^\delta_{(m, n)}}(c)(r) = \begin{cases} 
\phi^\delta_m(\gamma_{r-m}(c)) & \text{if } m \leq r \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
\overline{\psi^\gamma_{(m, n)}}(c)(s) = \begin{cases} 
\phi^\gamma_m(\delta_{s-n}(c)) & \text{if } n \leq s \\
0 & \text{otherwise},
\end{cases}
\]

for all \( m, n \in \mathbb{Z} \) and \( c \in \mathcal{M}(A) \).
Lemma 5.2. There is an isomorphism $\Lambda_\delta$ of $\mathcal{B}$ onto $\mathcal{B}_\delta$ such that

$$\Lambda_\delta(\phi_{(m,n)}(a)) = \psi_{(m,n)}^\delta(a)$$

for all $m, n \in \mathbb{Z}$ and $a \in A$. Similarly, there is an isomorphism $\Lambda_\gamma$ of $\mathcal{B}$ onto $\mathcal{B}_\gamma$ such that

$$\Lambda_\gamma(\phi_{(m,n)}(a)) = \psi_{(m,n)}^\gamma(a)$$

for all $m, n \in \mathbb{Z}$ and $a \in A$.

Proof. We only prove the existence of the isomorphism $\Lambda_\delta$, as the existence of the isomorphism $\Lambda_\gamma$ follows similarly. Define a map

$$\Lambda_\delta : \text{span}\{\phi_{(m,n)}(a) : (m, n) \in \mathbb{Z}^2, a \in A\} \to \mathcal{B}_\delta$$

by

$$\Lambda_\delta\left(\sum_i \phi_{(m_i,n_i)}(a_i)\right) = \sum_i \psi_{(m_i,n_i)}^\delta(a_i).$$

Obviously, $\Lambda_\delta$ is linear. We show that it preserves the norm, and therefore, it follows that it is a well-defined linear isometry. We first need to recall that, for any Hilbert space $H$, there is an isomorphism $U$ of the Hilbert space

$$\ell^2(\mathbb{Z}^2, H) \simeq \ell^2(\mathbb{Z}^2) \otimes H \simeq (\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})) \otimes H$$

onto the Hilbert space

$$\ell^2(\mathbb{Z}) \otimes (\ell^2(\mathbb{Z}) \otimes H) \simeq \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}, H),$$

such that

$$U(e_{(m,n)} \otimes h) = e_m \otimes (e_n \otimes h) = (\ldots, 0, 0, 0, (e_n \otimes h), 0, 0, 0, \ldots),$$

where $\{e_{(m,n)} : (m, n) \in \mathbb{Z}^2\}$ is the usual orthonormal basis for $\ell^2(\mathbb{Z}^2)$ ($\{e_m : m \in \mathbb{Z}\}$ is the one for $\ell^2(\mathbb{Z})$, accordingly) and $h \in H$. Note that in the equation (5.3), $(e_n \otimes h)$ is the $m$th slot. Moreover, the isomorphism $U$ induces the following isomorphism

$$B(\ell^2(\mathbb{Z}^2) \otimes H) \to B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}, H))$$

(5.4)

$$T \mapsto UTU^{-1}$$

of $C^*$-algebras. Now, let $\pi : A \to B(H)$ be a faithful and nondegenerate representation of $A$ on some Hilbert space $H$. Define the map

$$\Pi : \mathcal{B} \to B(\ell^2(\mathbb{Z}^2) \otimes H)$$

by

$$(\Pi(\xi)f)(r, s) = \pi(\xi(r,s))f(r, s)$$

for all $\xi \in \mathcal{B}$, $f \in \ell^2(\mathbb{Z}^2) \otimes H$, and $(r, s) \in \mathbb{Z}^2$. One can see that $\Pi$ is indeed a nondegenerate and faithful representation of $\mathcal{B}$ on the Hilbert space $\ell^2(\mathbb{Z}^2) \otimes H$. On the other hand, let $M : D_\delta \to B(\ell^2(\mathbb{Z}, H))$ be the nondegenerate and faithful representation defined by

$$(M(\xi)f)(s) = \pi(\xi(s))f(s)$$

for all $\xi \in D_\delta$ and $f \in \ell^2(\mathbb{Z}, H)$. Then, $M$ itself induces a map

$$\tilde{\Pi} : \mathcal{B}_\delta \to B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}, H))$$

defined by

$$(\tilde{\Pi}(\eta)g)(r) = M(\eta(r))g(r)$$
for all $\eta \in \mathcal{B}_d$ and $g \in \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}, H)$ (note that, $[M(\eta(r))g(r)](s) = \pi(\eta(r)(s))g(r)(s)$). It is not difficult to see that $\tilde{\Pi}$ is actually a nondegenerate and faithful representation of the algebra $\mathcal{B}_d$ on the Hilbert space $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}, H)$. Next, we want to show that

\begin{equation}
\Pi\left(\sum_i \phi_{(m_i,n_i)}(a_i)\right) = \tilde{\Pi}\left(\sum_i \psi_{(m_i,n_i)}^\delta(a_i)\right) U.
\end{equation}

It is enough to see that

$$\Pi(\phi_{(m,n)}(a)) = \tilde{\Pi}(\psi_{(m,n)}^\delta(a)) U$$

for all $(m, n) \in \mathbb{Z}^2$ and $a \in A$ on the spanning elements $(e_{(r,s)} \otimes h)$ of $\ell^2(\mathbb{Z}^2) \otimes H$. We have

\[
U[\Pi(\phi_{(m,n)}(a))(e_{(r,s)} \otimes h)] = U(e_{(r,s)} \otimes \pi(\phi_{(m,n)}(a))(r, s)h)
\]

\[
= \begin{cases}
U(e_{(r,s)} \otimes \pi(\alpha_{(r-m,s-n)}(a))h) & \text{if } (m, n) \leq (r, s) \\
U(e_{(r,s)} \otimes \pi(0)h) & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases}
U(e_{(r,s)} \otimes \pi(\delta_{s-n}(\gamma_{r-m}(a)))h) & \text{if } (m, n) \leq (r, s) \\
U(0) & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases}
U(e_{(r,s)} \otimes \tilde{h}) & \text{if } (m, n) \leq (r, s) \\
0 & \text{otherwise}
\end{cases}
\]

\[
e_r \otimes (e_s \otimes \tilde{h}) & \text{if } (m, n) \leq (r, s) \\
0 & \text{otherwise},
\]

where $\tilde{h} = \pi(\delta_{s-n}(\gamma_{r-m}(a)))h \in H$. On the other hand, we have

\[
\tilde{\Pi}(\psi_{(m,n)}^\delta(a))[U(e_{(r,s)} \otimes h)] = \tilde{\Pi}(\psi_{(m,n)}^\delta(a))[e_r \otimes (e_s \otimes h)]
\]

\[
= e_r \otimes \left[M(\psi_{(m,n)}^\delta(a))(r)\right] (e_s \otimes h)
\]

\[
= \begin{cases}
e_r \otimes \left[M(\phi_{\eta}^\delta(\gamma_{r-m}(a)))(e_s \otimes h)\right] & \text{if } m \leq r \\
e_r \otimes \left[M(0)(e_s \otimes h)\right] & \text{otherwise}
\end{cases}
\]

\[
e_r \otimes \left[e_s \otimes \pi(\phi_{\eta}^\delta(\gamma_{r-m}(a)))(s)\right]h] & \text{if } m \leq r \\
e_r \otimes 0 & \text{otherwise}
\]

\[
e_r \otimes \left[e_s \otimes \pi(\delta_{s-n}(\gamma_{r-m}(a)))h\right] & \text{if } m \leq r \text{ and } n \leq s \\
0 & \text{otherwise}
\]

\[
e_r \otimes (e_s \otimes \tilde{h}) & \text{if } (m, n) \leq (r, s) \\
0 & \text{otherwise},
\]
where $\tilde{h} = \pi(\delta_{s-n}(\gamma_{r-m}(a)))h \in H$. This implies that $U\Pi(\phi_{(m,n)}(a)) = \tilde{\Pi}(\psi_{(m,n)}^\delta)(a))U$

for all $(m,n) \in \mathbb{Z}^2$ and $a \in A$, and consequently, the equation (5.5) holds. So, we have

$$U\Pi(\sum_i \phi_{(m,n_i)}(a_i))U^{-1} = \tilde{\Pi}(\sum_i \psi_{(m,n_i)}^\delta)(a_i),$$

from which, we get (see the isomorphism (5.4))

$$\|\Lambda_\delta(\sum_i \phi_{(m,n_i)}(a_i))\| = \|\sum_i \psi_{(m,n_i)}^\delta)(a_i)\|

= \|\tilde{\Pi}(\sum_i \psi_{(m,n_i)}^\delta)(a_i)\|

= \|U\Pi(\sum_i \phi_{(m,n_i)}(a_i))U^{-1}\|

= \|\Pi(\sum_i \phi_{(m,n_i)}(a_i))\|

= \|\sum_i \phi_{(m,n_i)}(a_i)\|. $$

It therefore follows that $\Lambda_\delta$ is a well-defined linear map which is also norm-preserving. So, it extends to a linear isometry of the algebra $B$ into $B_\delta$. We use the same notation $\Lambda_\delta$ for the extension. We want to show that $\Lambda_\delta$ is actually a $*$-homomorphism. Since

$$\Lambda_\delta(\phi_{(m,n)}(a)^*) = \Lambda_\delta(\phi_{(m,n)}(a)^*) = \psi_{(m,n)}^\delta(a^*) = \psi_{(m,n)}^\delta(a)^* = \Lambda_\delta(\phi_{(m,n)}(a))^*$$

for all $m,n \in \mathbb{Z}$ and $a \in A$, it indeed preserves the involution. Moreover, if $m = (m_1,m_2), n = (n_1,n_2) \in \mathbb{Z}^2$ and $a,b \in A$, the following calculation suffices to see that $\Lambda_\delta$ preserves the multiplication, too:

$$\Lambda_\delta(\phi_m(a))\Lambda_\delta(\phi_n(b)) = \psi_{(m,n)}^\delta(a)\psi_{(m,n)}^\delta(b) = \psi_{(m,n)}^\delta(\alpha_{(m,n)}(a)\alpha_{(m,n)}^{-1}(b))

= \Lambda_\delta(\phi_{(m,n)}(a)\phi_{(m,n)}(b)).$$

Finally, one can see that $\Lambda_\delta$ is also surjective, as $\Lambda_\delta(\phi_{(m,n)}(a)) = \psi_{(m,n)}^\delta(a)$ for all $m,n \in \mathbb{Z}$ and $a \in A$. Thus, $\Lambda_\delta$ is an isomorphism of $B$ onto $B_\delta$. \hfill $\Box$

**Proposition 5.3.** The algebras $B_\delta$ and $B_\gamma$ contain the algebras $C_0(\mathbb{Z}) \otimes D_\delta$ and $C_0(\mathbb{Z}) \otimes D_\gamma$, respectively, as essential ideals, such that

(5.6) $C_0(\mathbb{Z}) \otimes D_\delta = \text{span}\{\psi_{(m,n)}^\delta(a) - \psi_{(t,n)}^\delta(\gamma_{t-m}(a)) : m,t,n \in \mathbb{Z} \text{ with } m < t, a \in A\}$,

and

(5.7) $C_0(\mathbb{Z}) \otimes D_\gamma = \text{span}\{\psi_{(m,n)}^\gamma(a) - \psi_{(m,u)}^\gamma(\delta_{u-n}(a)) : m,u,n \in \mathbb{Z} \text{ with } n < u, a \in A\}$.

**Proof.** We only prove for $C_0(\mathbb{Z}) \otimes D_\delta$, and proof for $C_0(\mathbb{Z}) \otimes D_\gamma$ follows by a similar discussion. Firstly, the right hand side of (5.6) is in fact equal to

(5.8) $\text{span}\{\psi_{(m,n)}^\delta(b) - \psi_{(m+1,n)}^\delta(\gamma(b)) : m,n \in \mathbb{Z}, b \in A\}$.
This is due to the fact that
\[ \psi^\delta_{(m,n)}(a) - \psi^\delta_{(t,n)}(\gamma_{t-m}(a)) = \left[ \psi^\delta_{(m,n)}(a) - \psi^\delta_{(m+1,n)}(\gamma(a)) \right] + \left[ \psi^\delta_{(m+1,n)}(\gamma(a)) - \psi^\delta_{(m+2,n)}(\gamma_{2m}(a)) \right] \\
+ \ldots + \left[ \psi^\delta_{(t-1,n)}(\gamma_{t-m-1}(a)) - \psi^\delta_{(t,n)}(\gamma_{t-m}(a)) \right] \\
= \sum_{r=1}^{t-m} \left[ \psi^\delta_{(m+r-1,n)}(\gamma_{r-1}(a)) - \psi^\delta_{(m+r,n)}(\gamma_r(a)) \right]. \]

Thus, we only need to show that
\[ (5.10) \quad C_0(\mathbb{Z}) \otimes D_\delta = \text{span} \left\{ \psi^\delta_{(m,n)}(a) - \psi^\delta_{(m+1,n)}(\gamma(a)) : m, n \in \mathbb{Z}, a \in A \right\}. \]

Since
\[ \psi^\delta_{(m,n)}(a) - \psi^\delta_{(m+1,n)}(\gamma(a)) = (\ldots, 0, 0, \phi^\delta_n(a), 0, 0, 0, \ldots), \]
where \( \phi^\delta_n(a) \) is the \( m \)th slot, is clearly an element of the algebra \( C_0(\mathbb{Z}) \otimes D_\delta \), the right hand side of (5.10) is contained in \( C_0(\mathbb{Z}) \otimes D_\delta \). The other inclusion holds, too. This is due to the fact that the algebra \( C_0(\mathbb{Z}) \otimes D_\delta \) is spanned by the elements of the form
\[ (\ldots, 0, 0, 0, \xi, 0, 0, 0, \ldots) \]
with \( \xi \in D_\delta \) as the \( m \)th slot, and \( \xi \) itself is spanned by the elements \( \phi^\delta_n(a) \) such that \( a \in A \) and \( n \in \mathbb{Z} \). This more precisely means that the algebra \( C_0(\mathbb{Z}) \otimes D_\delta \) is certainly spanned by the elements
\[ (\ldots, 0, 0, 0, \phi^\delta_n(a), 0, 0, 0, \ldots) = \psi^\delta_{(m,n)}(a) - \psi^\delta_{(m+1,n)}(\gamma(a)) \]
with \( \phi^\delta_n(a) \) as the \( m \)th slot.

Next, to show that \( C_0(\mathbb{Z}) \otimes D_\delta \) is an ideal of \( B_\delta \), it is enough to calculate the product
\[ \psi^\delta_{(r,s)}(b) \left[ \psi^\delta_{(m,n)}(a) - \psi^\delta_{(m+1,n)}(\gamma(a)) \right] \]
on the spanning elements to see that it belongs to \( C_0(\mathbb{Z}) \otimes D_\delta \). However, we skip the calculation as it is similar to the one in the proof of Proposition 3.3 where we proved that \( J \) is an ideal of \( B_\delta \). To see that the ideal \( C_0(\mathbb{Z}) \otimes D_\delta \) of \( B_\delta \) is essential, suppose that \( \xi[C_0(\mathbb{Z}) \otimes D_\delta] = 0 \) for some \( \xi \in B_\delta \subset \ell^\infty(\mathbb{Z}, D_\delta) \). So, \( \xi \eta = 0 \) in \( \ell^\infty(\mathbb{Z}, D_\delta) \) for every \( \eta \in C_0(\mathbb{Z}) \otimes D_\delta \), which means that \( (\xi \eta)(m) = \xi(m)\eta(m) = 0 \) in \( D_\delta \subset \ell^\infty(\mathbb{Z}, A) \) for every \( m \in \mathbb{Z} \), and therefore, \( [\xi(m)\eta(m)](n) = [\xi(m)(n)][\eta(m)(n)] = 0 \) in \( A \) for every \( n \in \mathbb{Z} \). In particular, for every \( m, n \in \mathbb{Z} \), take \( \eta \) to be
\[ (\ldots, 0, 0, 0, \phi^\delta_n(a^*_n, n), 0, 0, 0, \ldots) \]
with
\[ \phi^\delta_n(a^*_n, n) - \phi^\delta_{n+1}(\delta(a^*_n, n)) = (\ldots, 0, 0, 0, a^*_n, 0, 0, 0, \ldots) \in (C_0(\mathbb{Z}) \otimes A) \subset D_\delta \]
as the \( m \)th slot, where \( a^*_n = \xi(m)(n)^* \in A \) is the \( n \)th slot. Thus, we have
\[ 0 = \xi \eta = (\ldots, 0, 0, 0, \xi(m)[\phi^\delta_n(a^*_n, n) - \phi^\delta_{n+1}(\delta(a^*_n, n))], 0, 0, 0, \ldots), \]
from which, it follows that
\[ 0 = \xi(m)[\phi^\delta_n(a^*_n, n) - \phi^\delta_{n+1}(\delta(a^*_n, n))]
= (\ldots, 0, 0, 0, [\xi(m)(n)]a^*_n, 0, 0, 0, \ldots)
= (\ldots, 0, 0, 0, a_n a^*_n, 0, 0, 0, \ldots). \]
This implies that we must have \(a_{m,n}a_{m,n}^* = 0\) in \(A\), and hence, \(\xi(m)(n) = a_{m,n} = 0\). Since this is true for every \(m, n \in \mathbb{Z}\), we have \(\xi = 0\), and this completes the proof. \(\Box\)

**Theorem 5.4.** Let \((G, P) = (\mathbb{Z}^2, \mathbb{N}^2)\). Consider the dynamical system \((A, \mathbb{N}^2, \alpha)\), and the (essential) ideal \(J\) of the associated algebra \(B\) with the system. Let

\[
J_{\delta} := \text{span} \{ \phi_{(m,n)}(a) - \phi_{(t,n)}(\gamma_{t-m}(a)) : m, t, n \in \mathbb{Z} \text{ with } m < t, a \in A \},
\]

and

\[
J_{\gamma} := \text{span} \{ \phi_{(m,n)}(a) - \phi_{(m,u)}(\delta_{u-n}(a)) : m, n, u \in \mathbb{Z} \text{ with } n < u, a \in A \}.
\]

Then, \(J_{\delta}\) and \(J_{\gamma}\) are essential ideals of \(B\) such that \(J = J_{\delta} + J_{\gamma}\). Moreover, \(J\) is isomorphic to

\[
(C_0(\mathbb{Z}) \otimes D_{\delta}) \oplus (C_0(\mathbb{Z}) \otimes D_{\gamma}),
\]

and the algebra \(C_0(\mathbb{Z}^2) \otimes A \simeq C_0(\mathbb{Z}) \otimes C_0(\mathbb{Z}) \otimes A\) is contained in \(J\) as an essential ideal.

**Proof.** Firstly, we actually have

\[
J_{\delta} = \text{span} \{ \phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a)) : m, n \in \mathbb{Z}, a \in A \},
\]

and

\[
J_{\gamma} = \text{span} \{ \phi_{(m,n)}(a) - \phi_{(m,n+1)}(\delta(a)) : m, n \in \mathbb{Z}, a \in A \}.
\]

These can be seen by similar calculations to (5.9) in the proof of Proposition (5.3). So, we skip them here. Then, the following calculations

\[
\Lambda_{\delta}^{-1}(\psi_{(m,n)}(a) - \psi_{(m+1,n)}(\gamma(a))) = \phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a)),
\]

and

\[
\Lambda_{\gamma}^{-1}(\psi_{(m,n)}(a) - \psi_{(m,n+1)}(\delta(a))) = \phi_{(m,n)}(a) - \phi_{(m,n+1)}(\delta(a))
\]

on spanning elements imply that

\[
C_0(\mathbb{Z}) \otimes D_{\delta} \simeq \Lambda_{\delta}^{-1}(C_0(\mathbb{Z}) \otimes D_{\delta}) = J_{\delta},
\]

and

\[
C_0(\mathbb{Z}) \otimes D_{\gamma} \simeq \Lambda_{\gamma}^{-1}(C_0(\mathbb{Z}) \otimes D_{\gamma}) = J_{\gamma}.
\]

Thus, by using the isomorphisms \(\Lambda_{\delta}^{-1}\) and \(\Lambda_{\gamma}^{-1}\), as \(C_0(\mathbb{Z}) \otimes D_{\delta}\) and \(C_0(\mathbb{Z}) \otimes D_{\gamma}\) are the essential ideals of \(B_{\delta}\) and \(B_{\gamma}\), respectively, \(J_{\delta}\) and \(J_{\gamma}\) must be the essential ideals of \(B\). Note that, one may see these by using the similar discussion to the one in the proof of Proposition 3.3.

Next, we show that \(J = J_{\delta} + J_{\gamma}\). The inclusion \(J_{\delta} + J_{\gamma} \subset J\) is immediate as \(J\) obviously contains the spanning elements of \(J_{\delta}\) and \(J_{\gamma}\). For the other inclusion, take any spanning element \(\phi_{(m,n)}(a) - \phi_{(t,u)}(\alpha(t-m, u-n)(a))\) of \(J\), where \(a \in A\) and \((m, n), (t, u) \in \mathbb{Z}^2\) with \((m, n) \leq (t, u)\). We have

\[
\phi_{(m,n)}(a) - \phi_{(t,u)}(\alpha(t-m, u-n)(a))
\]

\[
= [\phi_{(m,n)}(a) - \phi_{(t,n)}(\gamma_{t-m}(a))] + [\phi_{(t,n)}(\gamma_{t-m}(a)) - \phi_{(t,u)}(\delta_{u-n}(\gamma_{t-m}(a)))]
\]

\[
= [\phi_{(m,n)}(a) - \phi_{(t,n)}(\gamma_{t-m}(a))] + [\phi_{(t,n)}(b) - \phi_{(t,u)}(\delta_{u-n}(b))],
\]
where $b = \gamma_{t-n}(a) \in A$. The summands in above belong to $\mathcal{J}_\delta$ and $\mathcal{J}_\gamma$, respectively, which implies that $\mathcal{J}_\delta + \mathcal{J}_\gamma$ contains each spanning element of $\mathcal{J}$. Therefore, $\mathcal{J} = \mathcal{J}_\delta + \mathcal{J}_\gamma$, from which, it follows that the ideal $\mathcal{J}$ is actually spanned by the elements of the form

\begin{equation}
(5.11) \quad [\phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a))] + [\phi_{(t,u)}(b) - \phi_{(t,u+1)}(\delta(b))],
\end{equation}

where $m, n, t, u \in \mathbb{Z}$ and $a, b \in A$. Now, as we showed earlier, $\mathcal{J}_\delta$ and $\mathcal{J}_\gamma$ are isomorphic to $C_0(\mathbb{Z}) \otimes D_\delta$ and $C_0(\mathbb{Z}) \otimes D_\gamma$, respectively. Thus, there is an isomorphism of $\mathcal{J} = \mathcal{J}_\delta + \mathcal{J}_\gamma$ onto the direct sum $(C_0(\mathbb{Z}) \otimes D_\delta) \oplus (C_0(\mathbb{Z}) \otimes D_\gamma)$ such that it maps the spanning element \((5.11)\) of $\mathcal{J}$ to

\begin{equation}
[\psi_{(m,n)}^\delta(a) - \psi_{(m+1,n)}^\delta(\gamma(a))] + [\psi_{(t,u)}^\gamma(b) - \psi_{(t,u+1)}^\gamma(\delta(b))].
\end{equation}

Finally, we show that $\mathcal{J}$ contains the algebra $C_0(\mathbb{Z}^2) \otimes A$ as an essential ideal. First recall that $C_0(\mathbb{Z}^2) \otimes A$ is spanned by the elements (functions) $f_{(m,n)}^a : \mathbb{Z}^2 \to A$ defined by

\begin{equation}
(5.12) \quad f_{(m,n)}^a(r, s) = \begin{cases} 
  a & \text{if } (r, s) = (m, n) \\
  0 & \text{otherwise}
\end{cases}
\end{equation}

for every $a \in A$ and $(m, n) \in \mathbb{Z}^2$. Now, one can calculate to see that each function $f_{(m,n)}^a$ is actually qual to the element

\begin{equation}
(5.13) \quad [\phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a))] - [\phi_{(m,n+1)}(\delta(a)) - \phi_{(m+1,n+1)}(\alpha_{(1,1)}(a))] \in \mathcal{J}_\delta \subset \mathcal{J},
\end{equation}

which equals

\begin{equation}
(5.14) \quad [\phi_{(m,n)}(a) - \phi_{(m,n+1)}(\delta(a))] - [\phi_{(m,n+1)}(\gamma(a)) - \phi_{(m+1,n+1)}(\alpha_{(1,1)}(a))] \in \mathcal{J}_\gamma \subset \mathcal{J},
\end{equation}

where $\alpha_{(1,1)}(a) = \gamma(\delta(a)) = \delta(\gamma(a))$. Thus, $\mathcal{J}$ contains each spanning element of $C_0(\mathbb{Z}^2) \otimes A$, which implies that $C_0(\mathbb{Z}^2) \otimes A$ is a closed (vector) subspace of $\mathcal{J}$. Moreover, since $C_0(\mathbb{Z}^2) \otimes A \cong C_0(\mathbb{Z}^2, A)$ is contained in the algebra $\ell^\infty(\mathbb{Z}^2, A)$ as an essential ideal, $\ell^\infty(\mathbb{Z}^2, A)$ is embedded in the multiplier algebra $\mathcal{M}(C_0(\mathbb{Z}^2) \otimes A)$ as a $C^*$-subalgebra. Therefore, as

\begin{equation}
(5.15) \quad C_0(\mathbb{Z}^2) \otimes A \subset \mathcal{J} \subset B \subset \ell^\infty(\mathbb{Z}^2, A) \subset \mathcal{M}(C_0(\mathbb{Z}^2) \otimes A),
\end{equation}

$C_0(\mathbb{Z}^2) \otimes A$ is actually an essential ideal of $\mathcal{J}$. \hfill \Box

\begin{remark}
Note that, we have $\mathcal{J}_\delta \cap \mathcal{J}_\gamma = C_0(\mathbb{Z}^2) \otimes A$, which is an essential ideal of $\mathcal{J}_\delta$ and $\mathcal{J}_\gamma$, both. The inclusion $C_0(\mathbb{Z}^2) \otimes A \subset \mathcal{J}_\delta \cap \mathcal{J}_\gamma$ holds as by the similar discussion to the one in the proof Theorem 5.4. $C_0(\mathbb{Z}^2) \otimes A$ is contained in both $\mathcal{J}_\delta$ and $\mathcal{J}_\gamma$ as an essential ideal (see \((5.13)-(5.15)\)). For the other inclusion, as $\mathcal{J}_\delta \cap \mathcal{J}_\gamma = \mathcal{J}_\delta \mathcal{J}_\gamma$, it is enough to show that each product

\begin{equation}
(5.16) \quad [\phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a))][\phi_{(t,u)}(b) - \phi_{(t,u+1)}(\delta(b))]
\end{equation}

of the spanning elements of $\mathcal{J}_\delta$ and $\mathcal{J}_\gamma$ belongs to $C_0(\mathbb{Z}^2) \otimes A$. To calculate the product \((5.16)\) (of two functions in $\ell^\infty(\mathbb{Z}^2, A)$), think of the intersection point of two (discrete) half-lines (rays) in $\mathbb{R}^2$. One is vertical corresponding to $[\phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a))]$ with the initial point $(m, n)$, and the other one is horizontal corresponding to $[\phi_{(t,u)}(b) - \phi_{(t,u+1)}(\delta(b))]$ with the initial point $(t, u)$. These two rays have
only one intersection at the point \((m, u)\) if \(m \geq t\) and \(n \leq u\). Otherwise, there is no intersection point. This is equivalent to saying that the product \((5.16)\), as a function in \(\ell^\infty(\mathbb{Z}^2, A)\), has a nonzero value only at \((m, u)\) if \(m \geq t\) and \(n \leq u\). So, in this case, we have

\[
\begin{align*}
&\lbrack \phi(m,n)(a) - \phi(m+1,n)(\gamma(a)) \rbrack [\phi(t,u)(b) - \phi(t,u+1)(\delta(b))] \\
&= \phi(m,n)(a)\phi(t,u)(b) - \phi(m,n)(a)\phi(t,u+1)(\delta(b)) - \phi(m+1,n)(\gamma(a))\phi(t,u)(b) \\
&+ \phi(m+1,n)(\gamma(a))\phi(t,u+1)(\delta(b)) \\
&= \phi(m,u)(\delta_{u-n}(a)\gamma_{m-t}(b)) - \phi(m,u+1)(\delta_{u-n+1}(a)\gamma_{m-t}(b)) - \phi(m+1,u)(\delta_{u-n}(a)\gamma_{m-t+1}(b)) \\
&+ \phi(m+1,u+1)(\delta_{u-n+1}(a)\gamma_{m-t+1}(b)) \\
&= \lbrack \phi(m,u)(\delta_{u-n}(a)\gamma_{m-t}(b)) - \phi(m,u+1)(\delta_{u-n+1}(a)\gamma_{m-t+1}(b)) \rbrack \\
&- \lbrack \phi(m+1,u+1)(\delta_{u-n+1}(a)\gamma_{m-t+1}(b)) - \phi(m+1,u)(\delta_{u-n}(a)\gamma_{m-t}(b)) \rbrack \\
&= \lbrack \phi(m,u)(c) - \phi(m+1,u)(\delta(c)) \rbrack - \lbrack \phi(m+1,u)(\gamma(c)) - \phi(m+1,u+1)(\alpha(1,1)(c)) \rbrack,
\end{align*}
\]

where \(c = \delta_{u-n}(a)\gamma_{m-t}(b) \in A\). Thus,

\[
\begin{align*}
&\lbrack \phi(m,n)(a) - \phi(m+1,n)(\gamma(a)) \rbrack [\phi(t,u)(b) - \phi(t,u+1)(\delta(b))] \\
&= \lbrack \phi(m,u)(c) - \phi(m+1,u)(\delta(c)) \rbrack - \lbrack \phi(m+1,u)(\gamma(c)) - \phi(m+1,u+1)(\alpha(1,1)(c)) \rbrack,
\end{align*}
\]

which is equal to the spanning element \(f_{m,u}^c\) of \(C_0(\mathbb{Z}^2) \otimes A\) (see \((5.12)-(5.14))\). So, the product \((5.16)\) belongs to \(C_0(\mathbb{Z}^2) \otimes A\), and therefore, we have \(\mathcal{J}_\delta \cap \mathcal{J}_\gamma = C_0(\mathbb{Z}^2) \otimes A\).

The following Lemma is required for Theorem 5.7.

**Lemma 5.6.** Suppose that \((A \times_\alpha \Gamma, j_A, j_H)\) and \((B \times_\beta G, j_B, j_G)\) are the group crossed products of the classical dynamical systems \((A, \Gamma, \alpha)\) and \((B, G, \beta)\) by discrete groups, respectively. Then,

\[
(A \otimes_{\text{max}} B) \times_{\alpha \otimes \beta} (\Gamma \times G) \simeq (A \times_\alpha \Gamma) \otimes_{\text{max}} (B \times_\beta G).
\]

**Proof.** For convenience, let \(\mathbb{A} = A \times_\alpha \Gamma\) and \(\mathbb{B} = B \times_\beta G\). If \((i_\mathbb{A}, i_\mathbb{B})\) is the canonical pair of the \(C^*\)-algebras \(\mathbb{A}\) and \(\mathbb{B}\) into the multiplier algebra \(\mathcal{M}(\mathbb{A} \otimes_{\text{max}} \mathbb{B})\), then \(k_A := i_\mathbb{A} \circ j_A\) and \(k_B := i_\mathbb{B} \circ j_B\) are homomorphisms of the \(C^*\)-algebras \(A\) and \(B\) into \(\mathcal{M}(\mathbb{A} \otimes_{\text{max}} \mathbb{B})\), respectively, with commuting ranges. Therefore, there is a homomorphism

\[
k_A \otimes_{\text{max}} k_B : A \otimes_{\text{max}} B \to \mathcal{M}(\mathbb{A} \otimes_{\text{max}} \mathbb{B})
\]

such that

\[
k_A \otimes_{\text{max}} k_B(a \otimes b) = k_A(a)k_B(b) = i_\mathbb{A}(j_A(a))i_\mathbb{B}(j_B(b)) = j_A(a) \otimes j_B(b)
\]

for all \(a \in A\) and \(b \in B\). Let us denote the homomorphism \(k_A \otimes_{\text{max}} k_B\) by \(k_{A \otimes_{\text{max}} B}\). One can see that the extensions \(i_\mathbb{A}\) and \(i_\mathbb{B}\) of the nondegenerate homomorphisms \(i_A\) and \(i_B\), respectively, have commuting ranges. Thus, the unitary-valued representations \(k_{\Gamma} := i_\mathbb{A} \circ j_{\Gamma}\) and \(k_{G} := i_\mathbb{B} \circ j_{G}\) of the groups \(\Gamma\) and \(G\) into \(\mathcal{M}(\mathbb{A} \otimes_{\text{max}} \mathbb{B})\), respectively, have also commuting ranges. So, define the map

\[
k_{\Gamma \times G} : \Gamma \times G \to \mathcal{M}(\mathbb{A} \otimes_{\text{max}} \mathbb{B})
\]

by

\[
k_{\Gamma \times G}(s, t) = k_{\Gamma}(s)k_{G}(t)
\]

for all \((s, t) \in \Gamma \times G\).
for every \( s \in \Gamma \) and \( t \in G \). Note that, in fact, we have
\[
k_{\Gamma}(s)k_G(t) = i_A(j_{\Gamma}(s))i_B(j_G(t)) = i_{\max}(j_{\Gamma}(s) \otimes j_G(t)),
\]
where
\[
\tilde{i}_A \otimes_{\max} \tilde{i}_B: \mathcal{M}(A) \otimes_{\max} \mathcal{M}(B) \to \mathcal{M}(A \otimes_{\max} B)
\]
is the homomorphism induced by the pair \((\tilde{i}_A, \tilde{i}_B)\) with commuting ranges (see [12, Remark 2.2]). Therefore,
\[
k_{\Gamma \times G}(s, t) = i_{\max}(j_{\Gamma}(s) \otimes j_G(t)),
\]
and
\[
k_{A \otimes_{\max} B}(a \otimes b) = i_{\max}(j_A(a) \otimes j_B(b)).
\]
We claim that the triple \((A \otimes_{\max} B, k_{A \otimes_{\max} B}, k_{\Gamma \times G})\) is a crossed product of the system \((A \otimes_{\max} B, \Gamma \times G, \alpha \otimes \beta)\). Firstly, as the homomorphisms \(j_A\) and \(j_B\) are nondegenerate, so is the homomorphism \(k_{A \otimes_{\max} B}\). In can easily be seen that each \(k_{\Gamma \times G}(s, t)\) is a unitary, and
\[
k_{\Gamma \times G}((s_1, t_1)(s_2, t_2)) = k_{\Gamma \times G}((s_1s_2, t_1t_2))
= i_{\max}(j_{\Gamma}(s_1s_2) \otimes j_G(t_1t_2))
= i_{\max}(j_{\Gamma}(s_1)j_G(t_1) \otimes j_G(t_2))
= i_{\max}(j_{\Gamma}(s_1) \otimes j_G(t_1)) \otimes (i_{\max}(j_G(t_2))),
= k_{\Gamma \times G}(s_1, t_1)k_{\Gamma \times G}(s_2, t_2).
\]
Thus, the map \(k_{\Gamma \times G}\) is a unitary representation of the group \(\Gamma \times G\) in \(\mathcal{M}(A \otimes_{\max} B)\).
We now show that the pair \((k_{A \otimes_{\max} B}, k_{\Gamma \times G})\) satisfies the covariance equation. We have
\[
k_{\Gamma \times G}((s, t))k_{A \otimes_{\max} B}(a \otimes b)k_{\Gamma \times G}((s, t))^*
= i_{\max}(j_{\Gamma}(s) \otimes j_G(t))^* i_{\max}(j_A(a) \otimes j_B(b))i_{\max}(j_{\Gamma}(s) \otimes j_G(t))^*
= i_{\max}(j_{\Gamma}(s) \otimes j_G(t))^* i_{\max}(j_A(a) \otimes j_B(b))i_{\max}(j_{\Gamma}(s) \otimes j_G(t))^*
= i_{\max}(j_{\Gamma}(s) \otimes j_G(t))^* i_{\max}(j_A(a) \otimes j_B(b))i_{\max}(j_{\Gamma}(s) \otimes j_G(t))^*
= i_{\max}(\alpha_s(\otimes \beta_t)(a \otimes b)) = k_{A \otimes_{\max} B}((\alpha \otimes \beta)(s, t))(a \otimes b)
\]
This suffices to see that the pair \((k_{A \otimes_{\max} B}, k_{\Gamma \times G})\) is indeed covariant. Next, suppose that the pair \((\pi, U)\) is a covariant representation of \((A \otimes_{\max} B, \Gamma \times G, \alpha \otimes \beta)\) on some Hilbert space \(H\). We want to show that there is a non-degenerate representation \(\rho\) of \((A \times_{\alpha} \Gamma) \otimes_{\max} (B \times_{\beta} G)\) on \(H\) such that
\[
\rho \circ k_{A \otimes_{\max} B} = \pi \quad \text{and} \quad \rho \circ k_{\Gamma \times G} = U.
\]
Let \((i_A, i_B)\) be the canonical pair of the \(C^*\)-algebras \(A\) and \(B\) into the multiplier algebra \(\mathcal{M}(A \otimes_{\max} B)\). The following compositions
\[
A \xrightarrow{i_A} \mathcal{M}(A \otimes_{\max} B) \xrightarrow{\pi} B(H),
\]
and
\[
B \xrightarrow{i_B} \mathcal{M}(A \otimes_{\max} B) \xrightarrow{\pi} B(H)
\]
gives the non-degenerate representations \( \pi_A \) and \( \pi_B \) of the \( C^* \)-algebras \( A \) and \( B \) on \( H \), respectively. Note that, we have

\[
(5.17) \quad \pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)
\]

for every \( a \in A \) and \( b \in B \) (see also [15, Corollary B.22]). Let \( e \) and \( \bar{e} \) denote the identity elements of the groups \( \Gamma \) and \( G \), respectively. Then, one can see that the maps \( V \) and \( W \) defined by

\[
\Gamma \to B(H); \quad s \mapsto U_{(s,\bar{e})},
\]

and

\[
G \to B(H); \quad t \mapsto U_{(e,t)},
\]

are unitary representations of the groups \( \Gamma \) and \( G \) on \( H \), respectively. We claim that the pairs \( (\pi_A, V) \) and \( (\pi_B, W) \) are covariant representations of the systems \( (A, \Gamma, \alpha) \) and \( (B, G, \beta) \) on \( H \), respectively. In order to see these, we only need to show that they satisfy the covariance equation. We only do this for \( (\pi_A, V) \) as the discussion on \( (\pi_B, W) \) follows similarly. Let \( \{b_\lambda\} \) be an approximate identity for \( B \). We have

\[
V_s\pi(a \otimes b_\lambda)V_s^* = U_{(s,\bar{e})}\pi(a \otimes b_\lambda)U_{(s,\bar{e})}^* = \pi((\alpha \otimes \beta)(s,\bar{e})(a \otimes b_\lambda)) = \pi((\alpha_s \otimes \beta_\bar{e})(a \otimes b_\lambda)) = \pi((\alpha_s \otimes \text{id})(a \otimes b_\lambda)) = \pi(\alpha_s(a) \otimes b_\lambda) = \pi_A(\alpha_s(a))\pi_B(b_\lambda), \quad \text{[by (5.17)]}
\]

which is convergent strongly to \( \pi_A(\alpha_s(a)) \). On the other hand,

\[
V_s\pi(a \otimes b_\lambda)V_s^* = V_s\pi_A(a)\pi_B(b_\lambda)V_s^*, \quad \text{[by (5.17)]}
\]

which is convergent strongly to \( V_s\pi_A(a)V_s^* \). So, we must have

\[
\pi_A(\alpha_s(a)) = V_s\pi_A(a)V_s^*.
\]

Thus, it follows that there are non-degenerate representations

\[
\pi_A \times V : A \times_\alpha \Gamma \to B(H)
\]

and

\[
\pi_B \times W : B \times_\beta G \to B(H)
\]

which take the pairs \( (j_A, j_\Gamma) \) and \( (j_B, j_G) \) to the pairs \( (\pi_A, V) \) and \( (\pi_B, W) \), respectively. We show that \( \pi_A \times V \) and \( \pi_B \times W \) have commuting ranges. In order to do so, it is enough to see that the pairs \( (\pi_A, \pi_B), (V, W), (V^*, W), (\pi_A, W), \) and \( (\pi_B, V) \) have commuting ranges. This is indeed true for \( (\pi_A, \pi_B) \) by (5.17). Also, it is easy to see that \( V(\Gamma) \) and \( W(G) \) commute, from which, as \( V_s^* = V_{s-1} \) for all \( s \in \Gamma \), it follows that \( V^* \) and \( W \) commute. So, we only show that \( \pi_A \) and \( W \) commute, and then, similar calculations show that \( \pi_B \) and \( V \) commute, too. Take any approximate identity \( \{b_\lambda\} \) in \( B \). Then, we have

\[
W_t\pi(a \otimes b_\lambda) = W_t\pi_A(a)\pi_B(b_\lambda),
\]
which converges strongly to $W_t\pi_A(a)$. On the other hand,

$$W_t\pi(a \otimes b) = U_{(e,t)}\pi(a \otimes b)$$

$$= \pi((\alpha \otimes \beta)(a \otimes b))U_{(e,t)} \quad [\text{by the covariance of } (\pi, U)]$$

$$= \pi((-\alpha \otimes \beta)(a \otimes b))U_{(e,t)}$$

$$= \pi((\text{id} \otimes \beta)(a \otimes b))U_{(e,t)}$$

$$= \pi(a \otimes \beta_t(b))U_{(e,t)}$$

$$= \pi_A(a)\pi_B(\beta_t(b))W_t$$

$$= \pi_A(a)W_t\pi_B(b), \quad [\text{by the covariance of } (\pi_B, W)]$$

which converges strongly to $\pi_A(a)W_t$. Thus, we have

$$W_t\pi_A(a) = \pi_A(a)W_t.$$
which converges strongly to $\overline{\nu}(k_{\Gamma \times G}(s, t))$. Therefore, we must have $U(s, t) = \overline{\nu}(k_{\Gamma \times G}(s, t))$, which implies that $\overline{\nu} \circ k_{\Gamma \times G} = U$.

Finally, we show that the $C^*$-algebra $(A \times_\alpha \Gamma) \otimes_{\max} (B \times_\beta G)$ is spanned by the elements $k_{A \otimes_{\max} B}(\xi)k_{\Gamma \times G}(s, t)$, where $\xi \in (A \otimes_{\max} B)$ and $(s, t) \in (\Gamma \times G)$. Since $C^*$-algebras $A \times_\alpha \Gamma$ and $B \times_\beta G$ are spanned by the elements $j_A(a)j_G(t)$ and $j_B(b)$, respectively, $(A \times_\alpha \Gamma) \otimes_{\max} (B \times_\beta G)$ is spanned by the elements

$$j_A(a)j_G(t) \otimes j_B(b)j_G(t).$$

But, we have

$$j_A(a)j_G(t) \otimes j_B(b)j_G(t) = i_A(j_A(a)j_G(t))i_B(j_B(b)j_G(t)) = i_A(j_A(a)i_B(j_B(b)))i_G(j_G(t)) = [i_A(j_A(a))i_B(j_B(b))][i_G(j_G(t))] \quad \text{[as $i_A$ and $i_B$ commute]}
$$

$$= [j_A(a) \otimes j_B(b)][i_B \otimes_{\max} i_G(j_G(t))] = k_{A \otimes_{\max} B}(a \otimes b)k_{\Gamma \times G}(s, t).$$

This completes the proof. \hfill \Box

**Theorem 5.7.** Let $(G, P) = (Z^2, N^2)$. Consider the dynamical system $(A, N^2, \alpha)$, and the (essential) ideal $J$ of the associated algebra $B$ with the system. Then, the (essential) ideal $J \times_\beta Z^2$ of $B \times_\beta Z^2$ is isomorphic to

$$[K(\ell^2(Z)) \otimes (D_\delta \times_\tau Z)] \oplus [K(\ell^2(Z)) \otimes (D_\tau \times_\tau Z)],$$

where $\tau$ denotes the action of $Z$ on $D_\delta$ and $D_\tau$ by the left translation. Moreover, $J \times_\beta Z^2$ contains the algebra $K(\ell^2(Z^2)) \otimes A \simeq K(\ell^2(Z)) \otimes K(\ell^2(Z)) \otimes A$ as an essential ideal.

**Proof.** Consider the (essential) ideals $J_\delta$ and $J_\gamma$ of $B$ in Theorem 5.4, which are $\beta$-invariant. First, we want to show that $J \times_\beta Z^2 = J_\delta \times_\beta Z^2 \cup J_\gamma \times_\beta Z^2$. The inclusion $J_\delta \times_\beta Z^2 \cup J_\gamma \times_\beta Z^2 \subset J \times_\beta Z^2$ is easy to see, as $J_\delta$ and $J_\gamma$ are, in fact, two $\beta$-invariant ideals of $J$, too. Thus, $J_\delta \times_\beta Z^2$ and $J_\gamma \times_\beta Z^2$ are indeed two ideals of $J \times_\beta Z^2$. For the other inclusion, firstly, it follows by Theorem 5.4 that, $J \times_\beta Z^2$ is spanned by the elements of the form

$$j_B([\phi(m,n)(a) - \phi(m+1,n)(\gamma(a))] + [\phi(t,u)(b) - \phi(t,u+1)(\delta(b))])j_{Z^2}(x, y),$$

where $a, b \in A$ and $(m, n), (t, u), (x, y) \in Z^2$. Then,

$$j_B([\phi(m,n)(a) - \phi(m,n+1)(\gamma(a))] + [\phi(t,u)(b) - \phi(t,u+1)(\delta(b))])j_{Z^2}(x, y)
$$

$$= [j_B(\phi(m,n)(a) - \phi(m+1,n)(\gamma(a)))] + [j_B(\phi(t,u)(b) - \phi(t,u+1)(\delta(b)))]j_{Z^2}(x, y)
$$

$$= [j_B(\phi(m,n)(a) - \phi(m+1,n)(\gamma(a)))]j_{Z^2}(x, y) + [j_B(\phi(t,u)(b) - \phi(t,u+1)(\delta(b)))]j_{Z^2}(x, y),$$

in which, the summands belong to $J_\delta \times_\beta Z^2$ and $J_\gamma \times_\beta Z^2$, respectively. Thus, each spanning element of $J \times_\beta Z^2$ belong to $(J_\delta \times_\beta Z^2 + J_\gamma \times_\beta Z^2)$, and hence $J \times_\beta Z^2 \subset J_\delta \times_\beta Z^2 + J_\gamma \times_\beta Z^2$. Moreover, note that $J_\delta \times_\beta Z^2 + J_\gamma \times_\beta Z^2 = J \times_\beta Z^2$ is actually spanned by the elements of the form

$$[j_B(\phi(m,n)(a) - \phi(m+1,n)(\gamma(a)))]j_{Z^2}(x, y) + [j_B(\phi(t,u)(b) - \phi(t,u+1)(\delta(b)))]j_{Z^2}(r, s),$$

where $a, b \in A$ and $(m, n), (x, y), (t, u), (r, s) \in Z^2$. Now, by Theorem 5.4, $J_\delta$ and $J_\gamma$ are isomorphic to $C_0(Z) \otimes D_\delta$ and $C_0(Z) \otimes D_\gamma$ by the isomorphisms $\Lambda_\delta$ and $\Lambda_\gamma$, respectively.
respectively. Let $\text{lt}$ denote the action of $\mathbb{Z}$ on $C_0(\mathbb{Z})$ by the left translation. Consider the action $\text{lt} \otimes \tau$ of $\mathbb{Z}^2$ on $C_0(\mathbb{Z}) \otimes D_\delta$ and $C_0(\mathbb{Z}) \otimes D_\gamma$. Then, we have

$$(\text{lt} \otimes \tau) \circ \Lambda_\delta = \Lambda_\delta \circ \beta \text{ and } (\text{lt} \otimes \tau) \circ \Lambda_\gamma = \Lambda_\gamma \circ \beta.$$ 

Thus, the systems $(\mathcal{J}_\delta, \mathbb{Z}^2, \beta)$ and $(C_0(\mathbb{Z}) \otimes D_\delta, \mathbb{Z}^2, \text{lt} \otimes \tau)$ are equivariant, as well as $(\mathcal{J}_\gamma, \mathbb{Z}^2, \beta)$ and $(C_0(\mathbb{Z}) \otimes D_\gamma, \mathbb{Z}^2, \text{lt} \otimes \tau)$. So, it follows by [17, Lemma 2.65] that the crossed products $\mathcal{J}_\delta \times_\beta \mathbb{Z}^2$ and $\mathcal{J}_\gamma \times_\beta \mathbb{Z}^2$ are isomorphic to $(C_0(\mathbb{Z}) \otimes D_\delta) \times_\text{lt} \otimes \tau \mathbb{Z}^2$ and $(C_0(\mathbb{Z}) \otimes D_\gamma) \times_\text{lt} \otimes \tau \mathbb{Z}^2$, respectively. More precisely, there are isomorphisms

$$\Psi_1 : \mathcal{J}_\delta \times_\beta \mathbb{Z}^2 \to ((C_0(\mathbb{Z}) \otimes D_\delta) \times_\text{lt} \otimes \tau \mathbb{Z}^2, i_{C_0(\mathbb{Z}) \otimes D_\delta}, i_{\mathbb{Z}^2})$$

and

$$\Psi_2 : \mathcal{J}_\gamma \times_\beta \mathbb{Z}^2 \to ((C_0(\mathbb{Z}) \otimes D_\gamma) \times_\text{lt} \otimes \tau \mathbb{Z}^2, k_{C_0(\mathbb{Z}) \otimes D_\gamma}, k_{\mathbb{Z}^2})$$

such that

$$\Psi_1(j_B(\xi)j_{\mathbb{Z}^2}(x, y)) = i_{C_0(\mathbb{Z}) \otimes D_\delta}(\Lambda_\delta(\xi))i_{\mathbb{Z}^2}(x, y)$$

and

$$\Psi_2(j_B(\eta)j_{\mathbb{Z}^2}(x, y)) = k_{C_0(\mathbb{Z}) \otimes D_\gamma}(\Lambda_\gamma(\eta))k_{\mathbb{Z}^2}(x, y)$$

for all $\xi \in \mathcal{J}_\delta$, $\eta \in \mathcal{J}_\gamma$, and $(x, y) \in \mathbb{Z}^2$. On the other hand, by Lemma 5.6 the crossed products $(C_0(\mathbb{Z}) \otimes D_\delta) \times_\text{lt} \otimes \tau \mathbb{Z}^2$ and $(C_0(\mathbb{Z}) \otimes D_\gamma) \times_\text{lt} \otimes \tau \mathbb{Z}^2$ are isomorphic to the (maximum) tensor products $(C_0(\mathbb{Z}) \times_\text{lt} \mathbb{Z}) \otimes (D_\delta \times_\tau \mathbb{Z})$ and $(C_0(\mathbb{Z}) \times_\text{lt} \mathbb{Z}) \otimes (D_\gamma \times_\tau \mathbb{Z})$, respectively, via the isomorphisms

$$\Psi_3 : ((C_0(\mathbb{Z}) \otimes D_\delta) \times_\text{lt} \otimes \tau \mathbb{Z}^2, i_{C_0(\mathbb{Z}) \otimes D_\delta}, i_{\mathbb{Z}^2}) \to (C_0(\mathbb{Z}) \times_\text{lt} \mathbb{Z}) \otimes (D_\delta \times_\tau \mathbb{Z})$$

and

$$\Psi_4 : ((C_0(\mathbb{Z}) \otimes D_\gamma) \times_\text{lt} \otimes \tau \mathbb{Z}^2, k_{C_0(\mathbb{Z}) \otimes D_\gamma}, k_{\mathbb{Z}^2}) \to (C_0(\mathbb{Z}) \times_\text{lt} \mathbb{Z}) \otimes (D_\gamma \times_\tau \mathbb{Z})$$

such that

$$\Psi_3(i_{C_0(\mathbb{Z}) \otimes D_\delta}(f \otimes \xi)i_{\mathbb{Z}^2}(x, y)) = [i_{C_0(\mathbb{Z})}(f)i_{\mathbb{Z}}(x)] \otimes [i_{D_\delta}(\xi)i_{\mathbb{Z}}(y)]$$

and

$$\Psi_4(k_{C_0(\mathbb{Z}) \otimes D_\gamma}(f \otimes \eta)k_{\mathbb{Z}^2}(x, y)) = [i_{C_0(\mathbb{Z})}(f)i_{\mathbb{Z}}(x)] \otimes [k_{D_\gamma}(\eta)k_{\mathbb{Z}}(y)]$$

for all $f \in C_0(\mathbb{Z})$, $\xi \in D_\delta$, $\eta \in D_\gamma$, and $x, y \in \mathbb{Z}$. Moreover, it is well-known by the Stone-von Neumann Theorem that the crossed product $C_0(\mathbb{Z}) \times_\text{lt} \mathbb{Z}$ is isomorphic to the algebra $\mathcal{K}(\ell^2(\mathbb{Z}))$ of compact operators on $\ell^2(\mathbb{Z})$ (see [15, Theorem C.34] or [17, Theorem 4.24]). The isomorphism is given by

$$\Psi_5 : (C_0(\mathbb{Z}) \times_\text{lt} \mathbb{Z}, i_{C_0(\mathbb{Z})}, i_{\mathbb{Z}}) \to \mathcal{K}(\ell^2(\mathbb{Z}))$$

such that

$$\Psi_5(i_{C_0(\mathbb{Z})}(f)i_{\mathbb{Z}}(x)) = e_r \otimes \overline{e_{r-x}},$$

where $f = (\ldots, 0, 0, 1, 0, 0, \ldots) \in C_0(\mathbb{Z})$ with 1 in $r$th slot, and $e_r \otimes \overline{e_{r-x}}$ is the rank-one operator on $\ell^2(\mathbb{Z})$ defined by $h \mapsto \langle h \mid e_{r-x} \rangle e_r$ for all $h \in \ell^2(\mathbb{Z})$ (see [15, Proposition 1.1]). Therefore, knowing that $\mathcal{J} \times_\beta \mathbb{Z}^2 = \mathcal{J}_\delta \times_\beta \mathbb{Z}^2 + \mathcal{J}_\gamma \times_\beta \mathbb{Z}^2$, we compose all the isomorphisms in above to get an isomorphism

$$\Psi_6 : \mathcal{J} \times_\beta \mathbb{Z}^2 \to [\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\delta \times_\tau \mathbb{Z})] \oplus [\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\gamma \times_\tau \mathbb{Z})],$$

such that it maps each spanning element

$$[j_B(\phi_{(m,n)}(a) - \phi_{(m+1,n)}(\gamma(a)))j_{\mathbb{Z}^2}(x, y)] + [j_B(\phi_{(t,u)}(b) - \phi_{(t,u+1)}(\delta(b)))j_{\mathbb{Z}^2}(r, s)],$$
of \( J \times _{\beta} \mathbb{Z}^2 = J_\delta \times _{\beta} \mathbb{Z}^2 + J_\gamma \times _{\beta} \mathbb{Z}^2 \) to the element
\[
[(e_m \otimes \overline{e}_{m-x}) \otimes (i_{D_\delta}(\phi_i^b(a))\bar{i}(y))] \oplus [(e_u \otimes \overline{e}_{u-r}) \otimes (k_{D_\gamma}(\phi_t^b(b))k_{\mathbb{Z}}(s))].
\]

Lemma 5.8. Let
\[
I_\delta = \text{span}\{i_{\mathbb{N}^2}(m,n)\ast i_A(a)[1 - i_{\mathbb{N}^2}(1,0)\ast i_{\mathbb{N}^2}(1,0)]i_{\mathbb{N}^2}(x,y) : m, n, x, y \in \mathbb{N}, a \in A\},
\]
and
\[
I_\gamma = \text{span}\{i_{\mathbb{N}^2}(r,s)\ast i_A(c)[1 - i_{\mathbb{N}^2}(0,1)\ast i_{\mathbb{N}^2}(0,1)]i_{\mathbb{N}^2}(t,u) : r, s, t, u \in \mathbb{N}, c \in A\}.
\]
Then, \( I_\delta \) and \( I_\gamma \) are two ideals of \( \mathcal{T}_{\text{cov}}(A \times _{\alpha} \mathbb{N}^2) \) (and thus, two ideals of \( I \)), such that \( I = I_\delta + I_\gamma \).

Proof. This can be seen by some inspection on spanning elements. Next, to see that \( I \subset I_\delta + I_\gamma \), take any spanning element
\[
i_{\mathbb{N}^2}(m,n)\ast i_A(b)(1 - i_{\mathbb{N}^2}(t,u)\ast i_{\mathbb{N}^2}(t,u))i_{\mathbb{N}^2}(x,y)
\]
of \( I \), where \( b \in A \) and \((m,n), (t,u), (x,y) \in \mathbb{N}^2\). We show that it can be written as a (finite) sum of the spanning elements of \( I_\delta \) and \( I_\gamma \). Firstly, we have
\[
1 - i_{\mathbb{N}^2}(t,u)\ast i_{\mathbb{N}^2}(t,u)
= \sum_{r=0}^{t-1} i_{\mathbb{N}^2}(r,0)\ast i_{\mathbb{N}^2}(r,0) - i_{\mathbb{N}^2}(r+1,0)\ast i_{\mathbb{N}^2}(r+1,0)
+ \sum_{s=0}^{t-1} i_{\mathbb{N}^2}(t,s)\ast i_{\mathbb{N}^2}(t,s) - i_{\mathbb{N}^2}(t,s+1)\ast i_{\mathbb{N}^2}(t,s+1)
= \sum_{r=0}^{t-1} i_{\mathbb{N}^2}(r,0)[1 - i_{\mathbb{N}^2}(1,0)\ast i_{\mathbb{N}^2}(1,0)]i_{\mathbb{N}^2}(r,0)
+ \sum_{s=0}^{t-1} i_{\mathbb{N}^2}(t,s)[1 - i_{\mathbb{N}^2}(0,1)\ast i_{\mathbb{N}^2}(0,1)]i_{\mathbb{N}^2}(t,s)
\]

Thus, for \( i_A(b)(1 - i_{\mathbb{N}^2}(t,u)\ast i_{\mathbb{N}^2}(t,u)) \), by applying the covariance equation of the pair \((i_A, i_{\mathbb{N}^2})\), we get
\[
\sum_{r=0}^{t-1} i_A(b)i_{\mathbb{N}^2}(r,0)[1 - i_{\mathbb{N}^2}(1,0)\ast i_{\mathbb{N}^2}(1,0)]i_{\mathbb{N}^2}(r,0)
+ \sum_{s=0}^{t-1} i_A(b)i_{\mathbb{N}^2}(t,s)[1 - i_{\mathbb{N}^2}(0,1)\ast i_{\mathbb{N}^2}(0,1)]i_{\mathbb{N}^2}(t,s)
= \sum_{r=0}^{t-1} i_{\mathbb{N}^2}(r,0)i_A(\gamma_r(b))[1 - i_{\mathbb{N}^2}(1,0)\ast i_{\mathbb{N}^2}(1,0)]i_{\mathbb{N}^2}(r,0)
+ \sum_{s=0}^{t-1} i_{\mathbb{N}^2}(t,s)i_A(\alpha_{(t,s)}(b))[1 - i_{\mathbb{N}^2}(0,1)\ast i_{\mathbb{N}^2}(0,1)]i_{\mathbb{N}^2}(t,s).
\]
Consequently,
\begin{align*}
i_{N^2}(m, n) i_{A}(b) (1 - i_{N^2}(t, u) i_{N^2}(t, u)) i_{N^2}(x, y) \\
= \sum_{r = 0}^{t-1} i_{N^2}(m, n) i_{N^2}(r, 0) i_{A}(\gamma_{r}(b)) [1 - i_{N^2}(1, 0) i_{N^2}(1, 0)] i_{N^2}(r, 0) i_{N^2}(x, y) \\
+ \sum_{s = 0}^{t-1} i_{N^2}(m, n) i_{N^2}(t, s) i_{A}(\alpha_{(t, s)}(b)) [1 - i_{N^2}(0, 1) i_{N^2}(0, 1)] i_{N^2}(t, s) i_{N^2}(x, y) \\
= \sum_{r = 0}^{t-1} i_{N^2}(m + r, n) i_{A}(\gamma_{r}(b)) [1 - i_{N^2}(1, 0) i_{N^2}(1, 0)] i_{N^2}(r + x, y) \\
+ \sum_{s = 0}^{t-1} i_{N^2}(m + t, n + s) i_{A}(\alpha_{(t, s)}(b)) [1 - i_{N^2}(0, 1) i_{N^2}(0, 1)] i_{N^2}(t + x, s + y).
\end{align*}

This implies that \( I \subset I_{\delta} + I_{\gamma} \). The other inclusion follows immediately as \( I \) clearly contains the spanning elements of \( I_{\delta} \) and \( I_{\gamma} \) both. Consequently, the ideal \( I \) is indeed spanned by the elements of the form
\begin{align*}
i_{N^2}(m, n) i_{A}(a) [1 - i_{N^2}(1, 0) i_{N^2}(1, 0)] i_{N^2}(x, y) \\
+ i_{N^2}(r, s) i_{A}(c) [1 - i_{N^2}(0, 1) i_{N^2}(0, 1)] i_{N^2}(t, u)
\end{align*}

\( \square \)

**Theorem 5.9.** Suppose that \((A \times_{\delta}^p N, j_{A}, v)\) and \((A \times_{\gamma}^p N, \iota_{A}, w)\) are the partial-isometric crossed products of the dynamical systems \((A, N, \delta)\) and \((A, N, \gamma)\), respectively. Let \(p^\delta\) and \(p^\gamma\) be the projections \(i_{D_{\delta}} \circ \phi_{0}^\delta(1)\) and \(k_{D_{\gamma}} \circ \phi_{0}^\gamma(1)\) in \(\mathcal{M}(D_{\delta} \times_{\tau} \mathbb{Z})\) and \(\mathcal{M}(D_{\gamma} \times_{\tau} \mathbb{Z})\), respectively. Then, the ideal \( I = I_{\delta} \oplus I_{\gamma} \) of the Nica-Toeplitz algebra \(\mathcal{T}_{\text{cov}}(A \times_{\alpha} N^2)\) is isomorphic to \((\text{the direct sum)} \ A_{\delta} \oplus A_{\gamma}\), where

\begin{equation}
A_{\delta} = Q_{\delta} [\mathcal{K}(\ell^2(N)) \otimes (A \times_{\delta}^p N)] Q_{\delta} \quad \text{and} \quad A_{\gamma} = Q_{\gamma} [\mathcal{K}(\ell^2(N)) \otimes (A \times_{\gamma}^p N)] Q_{\gamma}
\end{equation}

are full corners, in which, \(Q_{\delta}\) and \(Q_{\gamma}\) are projections in
\[\mathcal{M}(\mathcal{K}(\ell^2(N)) \otimes (A \times_{\delta}^p N)) \simeq \mathcal{L}(\ell^2(N) \otimes (A \times_{\delta}^p N))\]

and
\[\mathcal{M}(\mathcal{K}(\ell^2(N)) \otimes (A \times_{\gamma}^p N)) \simeq \mathcal{L}(\ell^2(N) \otimes (A \times_{\gamma}^p N)),\]

defined by
\[(Q_{\delta} f)(n) = \overline{J_{\delta}(\gamma_n(1))} f(n), \quad f \in \ell^2(N) \otimes (A \times_{\delta}^p N)\]

and
\[(Q_{\gamma} h)(n) = \overline{J_{\gamma}(\gamma_n(1))} h(n), \quad h \in \ell^2(N) \otimes (A \times_{\gamma}^p N),\]

respectively. Also, \(I\) contains a full corner of \(\mathcal{K}(\ell^2(N^2)) \otimes A \simeq \mathcal{K}(\ell^2(N)) \otimes \mathcal{K}(\ell^2(N)) \otimes A\) as an essential ideal. Thus, we have the exact sequence

\begin{equation}
0 \rightarrow (A_{\delta} \oplus A_{\gamma}) \xrightarrow{\sigma} \mathcal{T}_{\text{cov}}(A \times_{\alpha} N^2) \xrightarrow{\Omega} A \times_{\alpha} N^2 \rightarrow 0,
\end{equation}

such that the embedding \(\sigma\) maps the spanning element
\[Q_{\delta} [(e_m \otimes e_x) \otimes (v^*_n j_{A}(a) v_y)] Q_{\delta} \oplus Q_{\gamma} [(e_r \otimes e_t) \otimes (w^*_n \iota_{A}(c) w_u)] Q_{\gamma},\]
of $A_\delta \oplus A_\gamma$ to the element
\[ i_{\mathbb{N}^2}(m, n) i_A(a) [1 - i_{\mathbb{N}^2}(1, 0) i_{\mathbb{N}^2}(1, 0)] i_{\mathbb{N}^2}(x, y) 
+ i_{\mathbb{N}^2}(r, s) i_A(c) [1 - i_{\mathbb{N}^2}(0, 1) i_{\mathbb{N}^2}(0, 1)] i_{\mathbb{N}^2}(t, u) \]
of $T_{co}(A \times_\alpha \mathbb{N}^2)$ for all $a, c \in A$ and $(m, n), (x, y), (r, s), (t, u) \in \mathbb{N}^2$.

**Proof.** Firstly recall that, by [18] Theorem 4.1] (or even by Theorem 4.1 in §4), we have
\[ A \times_\delta^{piso} \mathbb{N} \cong p^\delta(D_\delta \times_\tau \mathbb{Z})p^\delta, \]
and
\[ A \times_\gamma^{piso} \mathbb{N} \cong p^\gamma(D_\gamma \times_\tau \mathbb{Z})p^\gamma. \]

Then, by Lemma 4.2 we have
\[ I \cong \Psi(I) = p(J \times_\beta \mathbb{Z}^2)p = p(J_\delta \times_\beta \mathbb{Z}^2 + J_\gamma \times_\beta \mathbb{Z}^2)p = p(J_\delta \times_\beta \mathbb{Z}^2)p + p(J_\gamma \times_\beta \mathbb{Z}^2)p. \]

Then, note that, as the homomorphisms $i_{C_0(\mathbb{Z}) \otimes D_\delta}$ is nondegenerate, so is the isomorphism $\Psi_1$. If $\{a_\lambda\}$ is an approximate identity in $A$, then we have
\[ \overline{\Psi_1}(j_B(\phi_{(0,0)}(a_\lambda))) = \overline{i_{C_0(\mathbb{Z}) \otimes D_\delta}(A_\delta(\phi_{(0,0)}(a_\lambda))), \]
and hence,
\[ \overline{\Psi_1}(j_B(\phi_{(0,0)}(a_\lambda))) = \overline{i_{C_0(\mathbb{Z}) \otimes D_\delta}(\psi_{(0,0)}^\delta(a_\lambda)). \]

Now, the left hand side is convergent strictly to $\overline{\Psi_1}(p)$, while the right hand side is convergent strictly to $\overline{i_{C_0(\mathbb{Z}) \otimes D_\delta}(\psi_{(0,0)}^\delta)(1))$, where $p = j_B(\phi_{(0,0)}(1))$. Therefore, we get
\[ \overline{\Psi_1}(p) = \overline{i_{C_0(\mathbb{Z}) \otimes D_\delta}(\psi_{(0,0)}^\delta)(1)) \in \mathcal{M}((C_0(\mathbb{Z}) \otimes D_\delta) \rtimes_\tau \mathbb{Z}^2). \]

Moreover, since the homomorphisms $i_{C_0(\mathbb{Z})}$ and $\tilde{i}_{D_\delta}$ are nondegenerate, so is the isomorphism $\Psi_3$. It follows that $P_\delta = \overline{\Psi_3}(\overline{\Psi_1}(p)) = \overline{\Psi_3 \circ \Psi_1}(p)$ is a projection in
\[ \mathcal{M}((C_0(\mathbb{Z}) \rtimes_\tau \mathbb{Z}) \otimes (D_\delta \rtimes_\tau \mathbb{Z}^2)) \cong \mathcal{M}(\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\delta \rtimes_\tau \mathbb{Z}^2)) \cong \mathcal{L}(\ell^2(\mathbb{Z}) \otimes (D_\delta \rtimes_\tau \mathbb{Z}^2)) \]
such that
\[ (P_\delta f)(n) = \begin{cases} \overline{\delta_\delta \circ \phi_{0}(\overline{\tau_n}(1))} f(n) & \text{if } n \geq 0 \\ 0 & \text{if } n < 0, \end{cases} \]
for all $f \in \ell^2(\mathbb{Z}) \otimes (D_\delta \rtimes_\tau \mathbb{Z})$.

Thus, by applying the isomorphisms $\Psi_1$, $\Psi_3$, and $\Psi_5$ in Theorem 5.7 we have
\[ p(J_\delta \times_\beta \mathbb{Z}^2)p \cong P_\delta[\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\delta \rtimes_\tau \mathbb{Z})]P_\delta. \]

However, $P_\delta[\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\delta \rtimes_\tau \mathbb{Z})]P_\delta$ actually equals the corner
\[ Q_\delta[\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_\delta^{piso} \mathbb{N})]Q_\delta \]
of the compact operators $\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_\delta^{piso} \mathbb{N}) \cong \mathcal{K}(\ell^2(\mathbb{N}) \otimes (A \times_\delta^{piso} \mathbb{N}))$, where $Q_\delta$ is the projection in
\[ \mathcal{M}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_\delta^{piso} \mathbb{N})) \cong \mathcal{M}(\mathcal{K}(\ell^2(\mathbb{N}) \otimes (A \times_\delta^{piso} \mathbb{N}))) \cong \mathcal{L}(\ell^2(\mathbb{N}) \otimes (A \times_\delta^{piso} \mathbb{N})) \]
defined by
\[ (Q_\delta f)(n) = \overline{j_A(\tau_n}(1)) f(n) \]
for all \( f \in \ell^2(N) \otimes (A \times^{piso} N) \). To see this, let \( \{e_n\}_{n \in \mathbb{Z}} \) be the usual orthonormal basis for \( \ell^2(\mathbb{Z}) \). Since \( \mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\delta \times \tau) \simeq \mathcal{K}(\ell^2(\mathbb{Z}) \otimes (D_\delta \times \tau)) \) (recall that the isomorphism is given by \( (e_n \otimes e_m) \otimes \xi \eta^* \mapsto \Theta_{e_n \otimes e_m \otimes \xi \eta^*} \) for all \( m, n \in \mathbb{Z} \) and \( \xi, \eta \in (D_\delta \times \tau) \)), spanned by elements (compact operators) \( \{\Theta_{e_n \otimes e_m \otimes \xi \eta^*} : m, n \in \mathbb{Z}, \xi, \eta \in (D_\delta \times \tau)\} \), we have

\[
P_\delta[\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\delta \times \tau)]P_\delta = \overline{\text{span}}\{P_\delta(\Theta_{e_n \otimes e_m \otimes \xi \eta^*})P_\delta : m, n \in \mathbb{N}, \xi, \eta \in (D_\delta \times \tau)\}
\]

However, if \( n < 0 \) or \( m < 0 \), then \( P_\delta(e_n \otimes \xi) = 0 \) or \( P_\delta(e_m \otimes \eta) = 0 \), and hence \( \Theta_{P_\delta(e_n \otimes \xi), P_\delta(e_m \otimes \eta)} = 0 \). Thus, it follows that

\[
P_\delta[\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\delta \times \tau)]P_\delta = \overline{\text{span}}\{\Theta_{P_\delta(e_n \otimes \xi), P_\delta(e_m \otimes \eta)} : m, n \in \mathbb{N}, \xi, \eta \in (D_\delta \times \tau)\}.
\]

Moreover,

\[
P_\delta(e_n \otimes \xi) = e_n \otimes (iD_\delta \otimes \phi_\delta^*)_n(1)\xi
\]

and similarly, \( P_\delta(e_m \otimes \eta) = P_\delta(e_m \otimes \eta) \). Therefore, we have

\[
\Theta_{P_\delta(e_n \otimes \xi), P_\delta(e_m \otimes \eta)} = \Theta_{P_\delta(e_n \otimes \eta), P_\delta(e_m \otimes \eta)} = P_\delta(e_n \otimes \eta, e_m \otimes \eta)P_\delta,
\]

where \( \Theta_{e_n \otimes \eta, e_m \otimes \eta} \) actually belongs to \( \mathcal{K}(\ell^2(\mathbb{Z})) \otimes (A \times^{piso} N) \), because \( \Theta_{e_n \otimes \eta, e_m \otimes \eta} \) indeed corresponds the compact operator

\[
(e_n \otimes \overline{e_m}) \otimes (p^\delta \eta^* \xi) = (e_n \otimes \overline{e_m}) \otimes (p^\delta \eta^* \xi),
\]

in which \( (p^\delta \eta^* \xi) \) is in \( p^\delta(D_\delta \times \tau) \). So, it follows that

\[
P_\delta[\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\delta \times \tau)]P_\delta = \overline{\text{span}}\{P_\delta((e_n \otimes \overline{e_m}) \otimes \xi)P_\delta : m, n \in \mathbb{N}, \xi \in (A \times^{piso} N)\}
\]

Now, since

\[
P_\delta(e_n \otimes \xi) = e_n \otimes (iD_\delta \otimes \phi_\delta^*)_n(1)\xi = e_n \otimes \overline{J_A}(\gamma_n(1))\xi = \delta_{\delta}(e_n \otimes \xi)
\]

for every \( n \in \mathbb{N} \) and \( \xi \in (A \times^{piso} N) \), we actually have

\[
P_\delta[\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\delta \times \tau)]P_\delta = \overline{\text{span}}\{Q_\delta(e_n \otimes \xi, Q_\delta(e_m \otimes \eta) : m, n \in \mathbb{N}, \xi, \eta \in (A \times^{piso} N)\}
\]

By a similar discussion, we have

\[
p(\mathcal{J}_\gamma \times \beta Z) p \simeq P_\gamma[\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\gamma \times \tau)]P_\gamma = Q_\delta[\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (A \times^{piso} N)]Q_\delta.
\]
where $P_\gamma = \Psi_3 \circ \Psi_2(p)$ is a projection in 
\[ \mathcal{M}(\mathcal{K}(\ell^2(\mathbb{Z}))) \otimes (D_\gamma \times_{\tau} \mathbb{Z})) \cong \mathcal{L}(\ell^2(\mathbb{Z}) \otimes (D_\gamma \times_{\tau} \mathbb{Z})), \]
and similarly, 
\[ \Psi_2(p) = \kappa_{\mathcal{C}_0(\mathbb{Z}) \otimes D_\gamma}(\psi_1^{\gamma}_0(1)) \in \mathcal{M}((\mathcal{C}_0(\mathbb{Z}) \otimes D_\gamma) \times_{\tau} \mathbb{Z}^2), \]
and $Q_\gamma$ is the projection in 
\[ \mathcal{M}(\mathcal{K}(\ell^2(\mathbb{N}))) \otimes (A \times_{\gamma} \mathbb{N}) \cong \mathcal{M}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\gamma} \mathbb{N})) \cong \mathcal{L}(\ell^2(\mathbb{N}) \otimes (A \times_{\gamma} \mathbb{N})) \]
defined by 
\[ (Q_\gamma h)(n) = t_A(t_\gamma(1))h(n) \]
for all $h \in \ell^2(\mathbb{N}) \otimes (A \times_{\gamma} \mathbb{N})$. Therefore, $\Psi_7 := \Psi_6 \circ \Psi$ is an isomorphism of the ideal $\mathcal{I}$ onto 
\[ Q_\delta[\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\gamma} \mathbb{N})]Q_\delta \oplus Q_\gamma[\mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times_{\gamma} \mathbb{N})]Q_\gamma. \]

We want to show that precisely how $\Psi_7$ acts on each spanning element of $\mathcal{I}$. Recall that, by Lemma 5.3, the ideal $\mathcal{I}$ is actually spanned by the elements of the form 
\[ i_{\mathbb{N}^2}(m,n) \ast i_A(ab\ast)[1 - i_{\mathbb{N}^2}(1,0) \ast i_{\mathbb{N}^2}(1,0)]i_{\mathbb{N}^2}(x,y) + i_{\mathbb{N}^2}(r,s) \ast i_A(cd\ast)[1 - i_{\mathbb{N}^2}(0,1) \ast i_{\mathbb{N}^2}(0,1)]i_{\mathbb{N}^2}(t,u), \]
where $a, b, c, d \in A$ and $(m, n), (x, y), (r, s), (t, u) \in \mathbb{N}^2$. So, it is enough to see that how actually the isomorphism $\Psi_7$ acts on such elements. First, we have (see Lemma 4.2) 
\[ \Psi(i_{\mathbb{N}^2}(m,n) \ast i_A(ab\ast)[1 - i_{\mathbb{N}^2}(1,0) \ast i_{\mathbb{N}^2}(1,0)]i_{\mathbb{N}^2}(x,y) + i_{\mathbb{N}^2}(r,s) \ast i_A(cd\ast)[1 - i_{\mathbb{N}^2}(0,1) \ast i_{\mathbb{N}^2}(0,1)]i_{\mathbb{N}^2}(t,u)) \]
\[ = \Psi(i_{\mathbb{N}^2}(m,n) \ast i_A(ab\ast)[1 - i_{\mathbb{N}^2}(1,0) \ast i_{\mathbb{N}^2}(1,0)]i_{\mathbb{N}^2}(x,y)) \]
\[ + \Psi(i_{\mathbb{N}^2}(r,s) \ast i_A(cd\ast)[1 - i_{\mathbb{N}^2}(0,1) \ast i_{\mathbb{N}^2}(0,1)]i_{\mathbb{N}^2}(t,u)) \]
\[ = p(j_{\mathbb{Z}^2}(m,n)j_B(\phi_{(0,0)}(ab\ast) - \phi_{(0,0)}(\gamma(ab\ast)))j_{\mathbb{Z}^2}(x,y) \ast p \]
\[ + p(j_{\mathbb{Z}^2}(r,s)j_B(\phi_{(0,0)}(cd\ast) - \phi_{(0,1)}(\delta(cd\ast)))j_{\mathbb{Z}^2}(t,u) \ast p, \]
which belongs to $p(\mathcal{J} \times_{\gamma} \mathbb{Z}^2)p = p(\mathcal{J}_\gamma \times_{\gamma} \mathbb{Z}^2)p + p(\mathcal{J}_\gamma \times_{\gamma} \mathbb{Z}^2)p$. For convenience, for every $a \in A$ and $m, n \in \mathbb{N}$, let $\xi_{m,a,n}$ and $\xi_{m,a,n}$ denote the elements of the form 
\[ i_{\mathbb{Z}^2}(m)(i_{D_\gamma} \circ \phi_{d}^\gamma)(a)i_{\mathbb{Z}^2}(n)^* \in D_\delta \times_{\tau} \mathbb{Z} \]
and 
\[ k_{\mathbb{Z}^2}(m)(k_{D_\gamma} \circ \phi_{d}^\gamma)(a)k_{\mathbb{Z}^2}(n)^* \in D_\gamma \times_{\tau} \mathbb{Z}, \]
respectively. Then, the isomorphism $\Psi_6$ in Theorem 5.7 maps the (spanning) element in (5.21) to 
\[ P_\delta[(e_m \otimes e_x) \otimes (\xi_{n,a,b\ast,y}^\delta)]P_\delta \oplus P_\gamma[(e_r \otimes e_t) \otimes (\xi_{s,c,d\ast,u}^\gamma)]P_\gamma. \]
However, it is not difficult to see that 
\[ \xi_{n,a,b\ast,y}^\delta = (\xi_{n,a,0}^\delta)(\xi_{0,b\ast,y}^\gamma) = (\xi_{n,a,0}^\delta)(\xi_{y,b,0}^\delta)^*, \]
and similarly, 
\[ \xi_{s,c,d\ast,u}^\gamma = (\xi_{s,c,0}^\gamma)(\xi_{0,d\ast,u}^\gamma) = (\xi_{s,c,0}^\gamma)(\xi_{u,d,0}^\gamma)^*. \]

Thus, (5.22) is indeed equal to 
\[ P_\delta[(e_m \otimes e_x) \otimes (\xi_{n,a,0}^\delta)(\xi_{y,b,0}^\delta)^*]P_\delta \oplus P_\gamma[(e_r \otimes e_t) \otimes (\xi_{s,c,0}^\gamma)(\xi_{u,d,0}^\gamma)^*]P_\gamma, \]
where, the latter one (up to isomorphism) equals

\[(5.23) \quad P_δ(Θ_εν⊗ξδ_{a,0,ε}⊗ξδ_{y,b,0})P_δ \oplus P_γ(Θ_εν⊗ξγ_{c,0,ε}⊗ξγ_{u,d,0})P_γ.\]

Moreover, by a similar computation to the one earlier, we have

\[
P_δ(Θ_εν⊗ξδ_{a,0,ε}⊗ξδ_{y,b,0})P_δ \oplus P_γ(Θ_εν⊗p^γξ_{c,0,ε}⊗p^γξ_{u,d,0})P_γ
\]

\[
= P_δ(Θ_εν⊗p^δξ_{a,0,ε}⊗p^δξ_{y,b,0})Q_δ \oplus Q_γ(Θ_εν⊗p^γξ_{c,0,ε}⊗p^γξ_{u,d,0})Q_γ
\]

\[
= Q_δ[(e_ε \otimes x_ε) \otimes (p^δξ_{a,0})(p^δξ_{y,b,0})^*]Q_δ \oplus Q_γ[(e_ε \otimes t_ε) \otimes (p^γξ_{c,0})(p^γξ_{u,d,0})^*]Q_γ
\]

Finally, since

\[
p^δξ_{a,b,c,y}P^δ = p^δζZ(n)(i_D \circ φ^δ_0(ab^*)v_y)P^δ = v^*_n j_A(ab^*)v_y \in (A \times piso N),
\]

and

\[
p^γξ_{s,c,d,u}P^γ = p^γkZ(s)(k_D \circ φ^γ_0)(cd^*)kZ(u)P^γ = w^*_u t_A(cd^*)w_u \in (A \times piso N),
\]

we get

\[
Q_δ[(e_ε \otimes x_ε) \otimes (p^δξ_{a,b,c,y}p^δ)]Q_δ \oplus Q_γ[(e_ε \otimes t_ε) \otimes (p^γξ_{s,c,d,u}p^γ)]Q_γ
\]

\[
= Q_δ[(e_ε \otimes x_ε) \otimes (v^*_n j_A(ab^*)v_y)]Q_δ \oplus Q_γ[(e_ε \otimes t_ε) \otimes (w^*_u t_A(cd^*)w_u)]Q_γ.
\]

\[\square\]

Remark 5.10. As another example, it is natural to consider the abelian lattice-ordered group \((Q_1^*, N^*)\). At the time being, the author does not know about this case. Thus, results regarding this case are left to work on.

Remark 5.11. Note that, when \(G = (\mathbb{Z}, \mathbb{N})\) we completely fall into the context of \([18]\), as \(\mathcal{T}_{cov}(A \times α N) \simeq A \times α N\) (see \([13]\)).

6. The Nica-Toeplitz algebra \(\mathcal{T}_{cov}(A \times α P)\) of dynamical systems by semigroups of automorphisms

Let \((G, P)\) be an abelian lattice-ordered group. In this section we suppose that \((A, P, α)\) is a system consisting of a \(C^*\)-algebra \(A\) and an action \(α : P \rightarrow \text{Aut}(A)\) of \(P\) by automorphisms of \(A\). First, we show that the automorphic action \(α\) of \(P\) can be extended to an action of the group \(G\) on \(A\), by using the fact that \(G = P^{-1}\).

One can easily see this. For every \(s \in G\), \(s \vee e \in P\), and hence \((s \vee e)^{-1} \in P^{-1}\). Thus, we have \(s = (s(s \vee e)^{-1})(s \vee e) \in P^{-1}\). Now, if \(s \in P^{-1}\), then \(s^{-1} \in P\), and therefore, we can define \(α_s := α_{s^{-1}}\). Thus, \(α_s\) is defined for every \(s \in G\) to be an automorphism, which gives us an action of \(G\) on \(A\) by automorphisms, and hence, we obtain a classical system \((A, G, α)\).

Lemma 6.1. Suppose that \((A, P, α)\) is a system consisting of a \(C^*\)-algebra \(A\) and an action \(α : P \rightarrow \text{Aut}(A)\) of \(P\) by automorphisms of \(A\). Then, the action \(α\) extends to an action of \(G\) on \(A\) by automorphisms.

Proof. It is not difficult to see this. So, we skip the proof. \(\square\)
Now let $B_G$ be the $C^*$-subalgebra of $\ell^\infty(G)$ spanned by the characteristic functions \( \{1_s \in \ell^\infty(G) : s \in G \} \), such that
\[
1_s(t) = \begin{cases} 
1 & \text{if } s \leq t \\
0 & \text{otherwise}.
\end{cases}
\]
It can easily be seen that $1_s1_t = 1_{st}$, and $1_s^* = 1_s$ for every $s, t \in G$. Thus, we have
\[
B_G = \text{span}\{1_s : s \in G\}.
\]
Let $\tau : G \to \text{Aut}(B_G)$ be the action of $G$ on $B_G$ by automorphisms given by the translation, such that $\tau(t)1_s = 1_{st}$ for all $s, t \in G$. Hence, we have the classical system $(B_G, G, \tau)$. Let $B_{G,\infty}$ denote the $C^*$-subalgebra of $B_G$ generated by the elements \( \{1_s - 1_t : s < t \in G\} \). It is not difficult to see that
\[
B_{G,\infty} = \text{span}\{1_s - 1_t : s < t \in G\},
\]
which is an essential ideal of $B_G$. It is $\tau$-invariant, too. Moreover, $s \mapsto \tau_s \otimes \alpha_s^{-1}$ defines an action of $G$ on the algebra $B_G \otimes A$ by automorphisms (note that as the algebra $B_G$ is abelian, $B_G \otimes A = B_G \otimes_{\text{min}} A = B_G \otimes_{\text{max}} A$). Therefore, we obtain the classical dynamical system $(B_G \otimes A, G, \tau \otimes \alpha^{-1})$. Also, $B_{G,\infty} \otimes A$ is a $(\tau \otimes \alpha^{-1})$-invariant ideal of $B_G \otimes A$. Next, we see that the algebra $B$ and its ideal $\mathcal{J}$ associated with the system $(A, P, \alpha)$ can be identified with tensor product algebras.

**Proposition 6.2.** There is an isomorphism $\mu : B_G \otimes A \to B$ such that $\beta_t(\mu(\xi)) = \mu((\tau \otimes \alpha^{-1})_t(\xi))$ for all $\xi \in (B_G \otimes A)$ and $t \in G$, and it maps the ideal $B_{G,\infty} \otimes A$ onto $\mathcal{J}$. Moreover, $\mu$ induces an isomorphism $\Gamma : ((B_G \otimes A) \times_{\tau \otimes \alpha^{-1}} G, i)$ onto $(\mathcal{B} \times_{\beta} G, j)$ such that
\[
(6.1) \quad \Gamma(i_{B_G \otimes A}(\xi)i_G(s)) = j_B(\mu(\xi))j_G(s) \quad \text{for all } \xi \in (B_G \otimes A), s \in G,
\]
and it maps the ideal $(B_{G,\infty} \otimes A) \times_{\tau \otimes \alpha^{-1}} G$ onto $\mathcal{J} \times_{\beta} G$.

**Proof.** Firstly, since $\ell^\infty(G, A)$ sits in $\ell^\infty(G, \mathcal{M}(A))$ as an essential ideal, $\ell^\infty(G, \mathcal{M}(A))$ is embedded in $\mathcal{M}(\ell^\infty(G, A))$ as a $C^*$-subalgebra. Now, define the maps
\[
\varphi : B_G \to \mathcal{M}(\ell^\infty(G, A)) \quad \text{and} \quad \psi : A \to \mathcal{M}(\ell^\infty(G, A))
\]
by
\[
\varphi(f)(s) = f(s)1_{\mathcal{M}(A)} \quad \text{and} \quad \psi(a)(s) = \alpha_s(a)
\]
for every $f \in B_G$, $a \in A$, and $s \in G$. One can see that $\varphi$ and $\psi$ are *-homomorphisms with commuting ranges, which means that $\varphi(f)\psi(a) = \psi(a)\varphi(f)$ for all $f \in B_G$ and $a \in A$. Therefore, there exists a homomorphism $\mu := \varphi \otimes \psi : B_G \otimes A \to \mathcal{M}(\ell^\infty(G, A))$ such that $\mu(f \otimes a) = \varphi(f)\psi(a) = \psi(a)\varphi(f)$ for every $f \in B_G$ and $a \in A$. We prove that $\mu$ is actually an isomorphism of $B_G \otimes A$ onto the algebra $\mathcal{B}$. To do so, we first show that $\mu(B_G \otimes A) = \mathcal{B}$. For each spanning element $1_s \otimes a$ of $B_G \otimes A$, we have
\[
\mu(1_s \otimes a)(t) = (\varphi(1_s)\psi(a))(t) = \varphi(1_s)(t)\psi(a)(t) = 1_s(t)\alpha_t(a) = \begin{cases} 
\alpha_t(a) & \text{if } s \leq t \\
0 & \text{otherwise},
\end{cases}
\]
which is equal to
\[
\phi_s(\alpha_s(a))(t) = \begin{cases} 
\alpha_{ts^{-1}}(\alpha_s(a)) = \alpha_t(a) & \text{if } s \leq t \\
0 & \text{otherwise},
\end{cases}
\]
for every \( t \in G \). So, we have \( \mu(1_s \otimes a) = \phi_s(\alpha_s(a)) \in \mathcal{B} \), and therefore \( \mu(B_G \otimes A) \subset \mathcal{B} \).

To see the other inclusion, for any spanning element \( \phi_s(a) \) of \( \mathcal{B} \), we apply the equation \( \mu(1_s \otimes a) = \phi_s(\alpha_s(a)) \) to see that
\[
\mu(1_s \otimes \alpha_s^{-1}(a)) = \phi_s(\alpha_s(\alpha_s^{-1}(a))) = \phi_s(a).
\]

Therefore, \( \phi_s(a) = \mu(1_s \otimes \alpha_s^{-1}(a)) \in \mu(B_G \otimes A) \), which implies that \( \mathcal{B} \subset \mu(B_G \otimes A) \).

Next, we show that \( \mu \) is injective. Define the map \( M : B_G \to B(\ell^2(G)) \) by \( (M(f)\lambda)(s) = f(s)\lambda(s) \) for every \( f \in B_G \) and \( \lambda \in \ell^2(G) \), which is a faithful (non-degenerate) representation. Let \( \pi : A \to B(H) \) be a faithful (non-degenerate) representation of \( A \) on some Hilbert space \( H \). Then, it follows by \([15\text{ Corollary B.11}]\) that there is a faithful representation \( M \otimes \pi : B_G \otimes A \to B(\ell^2(G) \otimes H) \) such that \( M \otimes \pi(f \otimes a) = M(f) \otimes \pi(a) \). On the other hand, we have a faithful representation \( \tilde{\pi} : \mathcal{B} \to B(\ell^2(G, H)) \) of \( \mathcal{B} \) on the Hilbert space \( \ell^2(G, H) \) defined by \( (\tilde{\pi}(\xi)\eta)(s) = \pi(\alpha_s^{-1}(\xi(s)))\eta(s) \) for every \( \xi \in \mathcal{B} \) and \( \eta \in \ell^2(G, H) \). Now, take \( U \) to be the isomorphism (unitary) of \( \ell^2(G) \otimes H \) onto \( \ell^2(G, H) \) which satisfies \( U(\lambda \otimes h)(s) = \lambda(s)h \) for all \( \lambda \in \ell^2(G) \) and \( h \in H \). So, we have
\[
\tilde{\pi}(\mu(f \otimes a))U(\lambda \otimes h)(s) = \pi(\alpha_s^{-1}(\mu(f \otimes a)(s)))U(\lambda \otimes h)(s)
= \pi(\alpha_s^{-1}(f(s)\alpha_s(a)))(\lambda(s)h)
= \pi(f(s)a)(\lambda(s)h)
= f(s)\lambda(s)\pi(a)h,
\]
and
\[
U((M \otimes \pi(f \otimes a))(\lambda \otimes h))(s) = U((M(f) \otimes \pi(a))(\lambda \otimes h))(s)
= U(M(f)\lambda \otimes \pi(a)h)(s)
= (M(f)\lambda)(s)\pi(a)h = f(s)\lambda(s)\pi(a)h.
\]

Therefore, we have
\[
\tilde{\pi}(\mu(f \otimes a))U(\lambda \otimes h) = U((M \otimes \pi(f \otimes a))(\lambda \otimes h)),
\]
which implies that
\[
U^*\tilde{\pi}(\mu(\xi))U = (M \otimes \pi)(\xi)
\]
for all \( \xi \in B_G \otimes A \). So, it follows that \( \mu \) must be injective. This is due to the facts that \( \tilde{\pi} \) and \( M \otimes \pi \) are injective, and \( U \) is a unitary. Consequently, \( B_G \otimes A \cong \mu(B_G \otimes A) = \mathcal{B} \).

Moreover, \( B_{G,\infty} \otimes A \) is isomorphic to \( \mathcal{J} \) via \( \mu \). To see this, take \( a \in A \) and \( s < t \in G \). Then,
\[
\mu((1_s - 1_t) \otimes a) = \mu((1_s \otimes a) - (1_t \otimes a))
= \mu(1_s \otimes a) - \mu(1_t \otimes a)
= \phi_s(\alpha_s(a)) - \phi_t(\alpha_t(a))
= \phi_s(\alpha_s(a)) - \phi_t(\alpha_{ts^{-1}}(\alpha_s(a))) \in \mathcal{J}.
\]

Therefore, \( \mu(B_{G,\infty} \otimes A) \subset \mathcal{J} \). For the other inclusion, by the above computation in (6.2), we have
\[
\mu((1_s - 1_t) \otimes \alpha_{s^{-1}}(a)) = \phi_s(\alpha_s(\alpha_{s^{-1}}(a))) - \phi_t(\alpha_t(\alpha_{s^{-1}}(a)))
= \phi_s(a) - \phi_t(\alpha_{ts^{-1}}(a)).
\]

So, each spanning element \( \phi_s(a) - \phi_t(\alpha_{ts^{-1}}(a)) \) of \( \mathcal{J} \) is equal to \( \mu((1_s - 1_t) \otimes \alpha_{s^{-1}}(a)) \), which belongs to \( \mu(B_{G,\infty} \otimes A) \). Therefore, \( \mathcal{J} \subset \mu(B_{G,\infty} \otimes A) \), and hence \( \mu(B_{G,\infty} \otimes A) = \mathcal{J} \). This means that \( B_{G,\infty} \otimes A \cong \mathcal{J} \) via \( \mu \).
At last, we show that the isomorphism $\mu$ satisfies $\beta_t \circ \mu = \mu \circ (\tau \otimes \alpha^{-1})_t$. Therefore, by [17] Lemma 2.65, there is an isomorphism $\Gamma : ((B_G \otimes A) \times_{\tau \otimes \alpha^{-1}} G, i) \rightarrow (B \times_{\beta} G, j)$ such that
\[
\Gamma(i_{B_G \otimes A}(\xi)i_G(s)) = j_B(\mu(\xi))j_G(s) \quad \text{for all } \xi \in (B_G \otimes A), s \in G.
\]
For each spanning element $1_s \otimes a$ of $B_G \otimes A$, we have
\[
\beta_t(\mu(1_s \otimes a)) = \beta_t(\phi_s(\alpha_s(a))) = \phi_{ts}(\alpha_s(a)).
\]
On the other hand,
\[
\mu((\tau \otimes \alpha^{-1})_t(1_s \otimes a)) = \mu(\tau_t(1_s \otimes a)) = \mu(\tau_t(1_s) \otimes \alpha_t^{-1}(a)) = \mu(1_{ts} \otimes \alpha_t^{-1}(a)) = \phi_{ts}(\alpha_t^{-1}(a)) \quad \text{[by applying } \mu(1_s \otimes a) = \phi_s(\alpha_s(a))] = \phi_{ts}(\alpha_s(a)).
\]
Thus, $\beta_t \circ \mu = \mu \circ (\tau \otimes \alpha^{-1})_t$ is valid. Note that, by some routine computation on spanning elements using the equation (6.1), it follows that
\[
(B_{G,\infty} \otimes A) \times_{\tau \otimes \alpha^{-1}} G \simeq \Gamma((B_{G,\infty} \otimes A) \times_{\tau \otimes \alpha^{-1}} G) = J \times_{\beta} G.
\]
We skip it here.

**Corollary 6.3.** If $q = i_{B_G \otimes A}(1_e \otimes 1_M(A)) \in \mathcal{M}((B_G \otimes A) \times_{\tau \otimes \alpha^{-1}} G)$, then $\overline{\Gamma}(q) = p$. Thus, it follows that $\mathcal{T}_{cov}(A \times_{\alpha} P)$ and the ideal $\mathcal{I}$ are isomorphic to the full corners $q((B_G \otimes A) \times_{\tau \otimes \alpha^{-1}} G)q$ and $q((B_{G,\infty} \otimes A) \times_{\tau \otimes \alpha^{-1}} G)q$, respectively.

**Proof.** First of all, as the homomorphism $j_B$ is nondegenerate, so is the isomorphism $\Gamma$. Therefore, $\Gamma$ extends to an isometry of multiplier algebras. Now, take any approximate identity $\{a_\lambda\}$ in $A$. Then, it follows by the equation (6.1) that
\[
\Gamma(i_{B_G \otimes A}(1_e \otimes a_\lambda)) = j_B(\mu(1_e \otimes a_\lambda)) = j_B(\phi(\alpha(\lambda))).
\]
Thus, since $1_e \otimes a_\lambda \rightarrow 1_e \otimes 1_M(A)$ strictly in $\mathcal{M}(B_G \otimes A)$, in the equation above, the left hand side tends to $\overline{\Gamma}(i_{B_G \otimes A}(1_e \otimes 1_M(A))) = \overline{\Gamma}(q)$, while the right hand side tends to $j_B(\phi(1)) = p$ strictly in $\mathcal{M}(B \times_{\beta} G)$. Hence, we have $\overline{\Gamma}(q) = p$. Therefore, it follows by Proposition 6.2 that
\[
q((B_G \otimes A) \times_{\tau \otimes \alpha^{-1}} G)q \simeq \Gamma(q((B_G \otimes A) \times_{\tau \otimes \alpha^{-1}} G)q) = p(B \times_{\beta} G)p,
\]
and
\[
q((B_{G,\infty} \otimes A) \times_{\tau \otimes \alpha^{-1}} G)q \simeq \Gamma(q((B_{G,\infty} \otimes A) \times_{\tau \otimes \alpha^{-1}} G)q) = p(J \times_{\beta} G)p,
\]
where by Theorem 4.11 and Lemma 4.2, $p(B \times_{\beta} G)p$ and $p(J \times_{\beta} G)p$ are isomorphic to $\mathcal{T}_{cov}(A \times_{\alpha} P)$ and the ideal $\mathcal{I}$, respectively. Consequently,
\[
\mathcal{T}_{cov}(A \times_{\alpha} P) \simeq q((B_G \otimes A) \times_{\tau \otimes \alpha^{-1}} G)q \quad \text{and} \quad \mathcal{I} \simeq q((B_{G,\infty} \otimes A) \times_{\tau \otimes \alpha^{-1}} G)q
\]
via the isomorphism $\Gamma^{-1} \circ \Psi$. \hfill \Box

**Remark 6.4.** Let $(A, N^2, \alpha)$ be a system in which the action $\alpha$ on $A$ is given by automorphisms. It follows that each $\delta_t = \alpha_{(0,t)}$ is an automorphism of $A$ as well as each $\gamma_t = \alpha_{(t,0)}$. Therefore, the systems $(A, \tilde{N}, \delta)$ and $(A, N, \gamma)$ in §5 are actually generated by single automorphisms $\delta = \delta_1$ and $\gamma = \gamma_1$, respectively. Also, the algebras
\( D_\delta \) and \( D_\gamma \) are isomorphic to \( B_Z \otimes A \), where \( B_Z = \text{span}\{1_n : n \in \mathbb{Z}\} \) as a subalgebra of \( \ell^\infty(\mathbb{Z}, A) \), from which, we have

\[
D_\delta \times_\tau Z \simeq (B_Z \otimes A) \times_{\text{It } \otimes \delta^{-1}} Z,
\]

and

\[
D_\gamma \times_\tau Z \simeq (B_Z \otimes A) \times_{\text{It } \otimes \gamma^{-1}} Z.
\]

One can see these by Proposition 6.2 or [18, Proposition 5.1].

**Corollary 6.5.** Let \((A, \mathbb{N}^2, \alpha)\) be a system in which the action \(\alpha\) on \(A\) is given by automorphisms. Then, the full corners \(A_\delta\) and \(A_\gamma\) in Theorem 5.9 are equal to the compact operators

\[
K_\delta := \mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times^{\text{piso}} \mathbb{N}) \quad \text{and} \quad K_\gamma := \mathcal{K}(\ell^2(\mathbb{N})) \otimes (A \times^{\text{piso}} \mathbb{N}),
\]

respectively. Therefore, the ideal \(\mathcal{I} = \mathcal{I}_\delta + \mathcal{I}_\gamma\) is actually isomorphic to the direct sum \(K_\delta \oplus K_\gamma\), which contains the algebra \(\mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes A \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes \mathcal{K}(\ell^2(\mathbb{N})) \otimes A\) as an essential ideal. Consequently, we obtain the following exact sequence

\[
0 \longrightarrow K_\delta \oplus K_\gamma \xrightarrow{\sigma} T_{\text{cov}}(A \times_\alpha \mathbb{N}^2) \xrightarrow{\Omega} A \times_\alpha \mathbb{Z}^2 \longrightarrow 0,
\]

in which the embedding \(\sigma\) maps the spanning element

\[
[(e_m \otimes \overline{e}_x) \otimes (v^*_n j_A(a) v_y)] \oplus [(e_r \otimes \overline{e}_t) \otimes (w^*_s \ell_A(c) w_u)]
\]

of \(K_\delta \oplus K_\gamma\) to the element

\[
i_{\mathbb{N}^2}(m, n) \ast i_A(a)[1 - i_{\mathbb{N}^2}(1, 0) \ast i_{\mathbb{N}^2}(1, 0)]i_{\mathbb{N}^2}(x, y) + i_{\mathbb{N}^2}(r, s) \ast i_A(c)[1 - i_{\mathbb{N}^2}(0, 1) \ast i_{\mathbb{N}^2}(0, 1)]i_{\mathbb{N}^2}(t, u)
\]

of \(T_{\text{cov}}(A \times_\alpha \mathbb{N}^2)\) for all \(a, c \in A\) and \((m, n), (x, y), (r, s), (t, u) \in \mathbb{N}^2\).

**Proof.** Since each \(\gamma_n\) is an automorphism, \(\overline{\gamma_n}(1) = 1\). Therefore, as the homomorphism \(j_A\) is nondegenerate, \(\overline{j_A(\gamma_n(1))} = \overline{j_A(1)} = 1\) for every \(n \in \mathbb{N}\), and hence

\[
(Q_\delta f)(n) = \overline{j_A(\gamma_n(1))} f(n) = f(n)
\]

for all \(f \in \ell^2(\mathbb{N}) \otimes (A \times^{\text{piso}} \mathbb{N})\). This implies that the projection \(Q_\delta\) in \(\mathcal{L}(\ell^2(\mathbb{N}) \otimes (A \times^{\text{piso}} \mathbb{N}))\) is the identity operator, namely, \(Q_\delta = 1\). A similar discussion shows that the projection \(Q_\gamma\) in \(\mathcal{L}(\ell^2(\mathbb{N}) \otimes (A \times^{\text{piso}} \mathbb{N}))\) is the identity operator, too. Thus, it follows that

\[
A_\delta = Q_\delta K_\delta Q_\delta = K_\delta \quad \text{and} \quad A_\gamma = Q_\gamma K_\gamma Q_\gamma = K_\gamma.
\]

The rest follows from Theorem 5.9. \(\square\)

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