On Pseudo-Hermitian Hamiltonians
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We investigate some questions on the construction of $\eta$ operators for pseudo-Hermitian Hamiltonians. We give a sufficient condition which can be exploited to systematically generate a sequence of $\eta$ operators starting from a known one, thereby proving the non-uniqueness of $\eta$ for a particular pseudo-Hermitian Hamiltonian. We also study perturbed Hamiltonians for which $\eta$‘s corresponding to the original Hamiltonian still work.

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I. INTRODUCTION

In recent years there has been a growing interest in the study of non-Hermitian quantum mechanics\textsuperscript{1}, primarily because a class of these Hamiltonians admit real eigenvalues despite being non-Hermitian. Among the different non-Hermitian systems the ones with $\mathcal{PT}$-symmetry\textsuperscript{2-6} and those which are $\eta$-pseudo-Hermitian\textsuperscript{7-11} have been the most widely studied. Lately non-Hermitian quantum mechanics have found applications in diverse areas of physics\textsuperscript{12-17}.

Here our objective is to examine some features of $\eta$-pseudo-Hermitian models\textsuperscript{7-11}. It may be recalled that a non-Hermitian Hamiltonian $\mathcal{H}$ is called $\eta$-pseudo-Hermitian if it satisfies the condition

$$\eta \mathcal{H} \eta^{-1} = \mathcal{H}^\dagger, \quad (1)$$

where $\eta$ is a Hermitian linear automorphism\textsuperscript{7-11}. However, there is no definite method to determine $\eta$ and for different models it has to be constructed in different ways. Another characteristic of $\eta$ is that it is not unique. Here we propose to examine the construction of $\eta$ operators for a class of $\eta$-pseudo-Hermitian matrix Hamiltonians. We shall also discuss briefly non-matrix Hamiltonians.

II. TWO THEOREMS ON $\eta$

In this section we consider a particular class of pseudo-Hermitian Hamiltonians and present a general procedure to get an infinite set of $\eta$‘s provided that we know one for a particular Hamiltonian. This will, in particular, explicitly demonstrate that $\eta$ is not unique. We shall also construct a class of non-Hermitian Hamiltonians whose members are pseudo-Hermitian with respect to the same $\eta$.

Theorem II.1. Let $\mathcal{H}$ be a non-Hermitian Hamiltonian that is pseudo-Hermitian, i.e. there exists a Hermitian linear automorphism $\eta$ such that $\eta \mathcal{H} \eta^{-1} = \mathcal{H}^\dagger$. Suppose that $\mathcal{H}^\dagger$ is invertible. Then there exists a possibly infinite no.of $\eta$‘s, with respect to which $\mathcal{H}$ is pseudo-Hermitian.

Proof. Let $\eta_0 = \eta$. Define a sequence $\{\eta_k\}_{k \geq 1}$ by $\eta_k = \mathcal{H}^\dagger \eta_{k-1} \mathcal{H}^\dagger$. We claim that each $\eta$ in $\{\eta_k\}_{k \geq 0}$ renders $\mathcal{H}$ pseudo-Hermitian.

We prove the claim by induction. Suppose that for some $m \geq 0$, $\eta_m$ satisfies the claim, i.e., $\eta_m$ is a Hermitian linear automorphism with $\eta_m \mathcal{H} \eta_m^{-1} = \mathcal{H}^\dagger$. Consider $\eta_{m+1}$. Clearly,

$$\eta_{m+1}^\dagger = (\mathcal{H}^\dagger \eta_m \mathcal{H}^\dagger)^\dagger = \eta_m^\dagger \mathcal{H} \eta_m \mathcal{H}^\dagger = \eta_{m+1}. \quad (2)$$

Also, as both $\mathcal{H}^\dagger$ and $\eta_m$ are linear automorphisms, so is $\eta_{m+1}$. It remains therefore to check that $\eta_{m+1} \mathcal{H} \eta_{m+1}^{-1} = \mathcal{H}^\dagger$. Indeed we have

$$\eta_{m+1} \mathcal{H} \eta_{m+1}^{-1} = \mathcal{H}^\dagger \eta_m \mathcal{H} \eta_m \mathcal{H} \eta_m^{-1} \mathcal{H} = \mathcal{H}^\dagger \eta_m \mathcal{H} \mathcal{H}^\dagger \eta_m^{-1} \mathcal{H} = \mathcal{H}^\dagger \mathcal{H} \mathcal{H}^\dagger = \mathcal{H}^\dagger \mathcal{H} \mathcal{H}^\dagger \mathcal{H} = \mathcal{H}^\dagger \mathcal{H} = \mathcal{H}^\dagger. \quad (3)$$

This completes the induction since $\eta_0$ satisfies the claim by definition.

Remark II.1. By the definition in the above theorem we have $\eta_k = (\mathcal{H}^\dagger)^k \eta$.

Example II.1. Consider now a quantum particle on a coordinate axis consisting of two points. The Hamiltonian for this system is given by a $2 \times 2$ matrix of the form\textsuperscript{18}

$$\mathcal{H} = \begin{pmatrix} x & y \\ \bar{y} & \bar{x} \end{pmatrix} \quad (4)$$

where $\Im(x) \neq 0$. This Hamiltonian is $\mathcal{PT}$-symmetric\textsuperscript{18}. However it is also pseudo Hermitian w.r.t. the following

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Consider now a two level system associated with the classical motion of a simple harmonic oscillator with frequency \( \omega(t) \). Then the relevant Hamiltonian is of the form\(^7\)-\(^\text{11}\)

\[
\mathcal{H}(t) = \begin{pmatrix} 0 & i \omega(t) \cr -i \omega(t)^2 & 0 \end{pmatrix}.
\]

\(^{(3)}\)

It can be easily verified that in this case one can choose

\[
\eta = \begin{cases} 0 & \text{if } y \neq 0; \\ 1 & \text{otherwise}. \end{cases}
\]

Therefore, in the case \( y = 0 \), we have

\[
\eta_k = (\mathcal{H}^\dagger)^k \eta = \begin{pmatrix} 0 & \hat{x}^k \\ \hat{x}^k & 0 \end{pmatrix}.
\]

Although the operator can be also be obtained when \( y \neq 0 \), the expression for \((\mathcal{H}^\dagger)^k \eta\) is cumbersome and so, we omit it.

**Example II.2.** Consider now a two level system associated with the classical motion of a simple harmonic oscillator with frequency \( \omega(t) \). Then the relevant Hamiltonian is of the form\(^7\)-\(^\text{11}\)

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\mathcal{H}(t) = \begin{pmatrix} 0 & i \omega(t) \cr -i \omega(t)^2 & 0 \end{pmatrix}.
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Although the operator can be also be obtained when \( y \neq 0 \), the expression for \((\mathcal{H}^\dagger)^k \eta\) is cumbersome and so, we omit it.

**Theorem II.2.** Consider a non-Hermitian Hamiltonian \( \mathcal{H} \). Suppose \( \eta \) is a Hermitian linear automorphism such that \( \eta \mathcal{H} \eta^{-1} = \mathcal{H}^\dagger \). Further suppose \( K \) is Hermitian and commutes with \( \eta \). Let \( \mathcal{H} = \mathcal{H} + K \). Then \( \mathcal{H} \) is also non-Hermitian and pseudo Hermitian with respect to \( \eta \).

**Proof.** Clearly, \( \mathcal{H}^\dagger = \mathcal{H} + K \). Let \( \mathcal{H} = \mathcal{H} + K \). Then \( \mathcal{H} \) is also non-Hermitian and pseudo Hermitian with respect to \( \eta \).

**Remark II.3.** Under the hypotheses of Theorem II.2, \( f(K) \) is also Hermitian for any \( f(x) \in \mathbb{R}[x] \), the ring of all real polynomials, and commutes with \( \eta \). Thus, the same \( \eta \) works for \( \mathcal{H} = \mathcal{H} + f(K) \) also.

Below we give two automatic choices for \( K \).

1. Take \( K = \alpha \eta \), where \( \alpha \in \mathbb{R} \). Then \( K \) satisfies the hypotheses of the above theorem and hence \( \eta \) works for the class of Hamiltonians defined by \( \mathcal{H}_\alpha = \mathcal{H} + \alpha \eta \).

2. Take \( K = \alpha I \), where \( \alpha \in \mathbb{R} \). Then clearly \( K \) is Hermitian and commutes with every other operator, in particular \( \eta \). Thus \( K \) satisfies the hypotheses of the above theorem and hence for \( \mathcal{H} = \mathcal{H} + \alpha I \), the same \( \eta \) works.

**Example II.3.** Consider the Hamiltonian in Example II.2. Here

\[
\mathcal{H}_{\alpha k}(t) = \mathcal{H}(t) + \alpha \eta_k(t),
\]

and \( \mathcal{H}_{\alpha k}(t) \) is pseudo-Hermitian with respect to \( \eta_k(t) \).

We shall now consider a class of non-matrix Hamiltonians. It was shown by Bender and Boettcher\(^1\) that each member of the class of Hamiltonians \( \mathcal{H} = p^2 + m^2 x^2 - (i x)^N \), \( N \) real, has a real spectrum and they conjectured that this is due to \( PT \)-symmetry. Although several \( PT \)-symmetric Hamiltonians were found to possess real discrete spectra, it was also found that the non-\( PT \)-symmetric complex potential of the form\(^2\),

\[
V(x) = \alpha V(x - \beta - i \gamma)
\]

always yields a real spectrum and Hamiltonians with these potentials are pseudo-Hermitian with respect to

\[
\eta = e^{-\theta \gamma},
\]

where the parameter \( \theta = 2 \gamma \) takes different forms for different potentials.

It may be noted that \( e^{-\theta \gamma} \) has the following two nice properties:
1. $e^{-\theta p}c e^{\theta p} = c$, for any constant $c \in \mathbb{C}$, and
2. $e^{-\theta p}pe^{\theta p} = p$.

In fact, it follows from the above two properties that $e^{-\theta p}f(p)e^{\theta p} = f(p)$, where $f(x) \in \mathbb{C}[x]$, the ring of all complex polynomials. Now, since $p$ is Hermitian, we can take $K = p$ or more generally, in view of Remark II.3,

$$K = f(p), \text{ where } f(x) \in \mathbb{R}[x].$$

So, $\eta = e^{-\theta p}$ renders $\tilde{H}$ pseudo-Hermitian with an appropriate choice of $\theta$, where

$$\tilde{H} = p^2 + f(p) + V(x),$$

and $V(x)$ is given by (4). It may be noted that the above Hamiltonian represents momentum dependent interaction\textsuperscript{23}.

**Example II.4.** In view of the discussion in the above paragraph, let us look at a concrete example. Take $f(x) = \alpha x$ where $\alpha \in \mathbb{R}$. Then

$$\tilde{H} = p^2 + \alpha p + V(x).$$

**Remark II.4.** The assumption in Theorem II.1 that $H^\dagger$ is invertible seems to be a bit restrictive. We can, however, relax this assumption to some extent by combining Theorems II.1 and II.2 in the following manner. Note that invertibility of $H^\dagger$ is equivalent to 0 not being in the spectrum of $\mathcal{H}$. Suppose that $\mathcal{H}$ has a discrete spectrum $\text{spec}(\mathcal{H})$. (By definition, for an operator $H$ on a complex Hilbert space, $\text{spec}(\mathcal{H}) = \{ \lambda \in \mathbb{C} : \lambda I - \mathcal{H} \text{ is singular} \}$). If $0 \notin \text{spec}(\mathcal{H})$, Theorem II.1 is readily applicable. If not, then we can find $\alpha \in \mathbb{R}$, such that $0 \notin \text{spec}(\mathcal{H} + \alpha I)$. Defining $\tilde{H} = \mathcal{H} + \alpha I$, it follows that $\tilde{H}^\dagger$ is invertible. Now Theorem II.2 applies to $\mathcal{H} = \tilde{H} + (-\alpha)I$. So, the class of $\eta$’s generated for $\tilde{H}$ using Theorem II.1 works for $\mathcal{H}$.

**III. SUMMARY**

Let us now summarize the main results of this paper. We have given sufficient conditions using which one can systematically construct a sequence of $\eta$ operators for a pseudo-Hermitian Hamiltonian starting from a known one. We have also investigated conditions which ensure that a single $\eta$ operator works for two different pseudo-Hermitian (or quasi-Hermitian) Hamiltonians. As the construction of an $\eta$ operator is not always straightforward for a general non-Hermitian Hamiltonian, these conditions are useful to enlarge the class of non-Hermitian Hamiltonians to which a known $\eta$ operator applies. This ideas are illustrated through a few useful examples.

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