AUTOMORPHISMS OF THE FINE CURVE GRAPH

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Abstract. Building on work of Farb and the second author, we prove that the group of automorphisms of the fine curve graph for a surface is isomorphic to the group of homeomorphisms of the surface. This theorem is analogous to the seminal result of Ivanov that the group of automorphisms of the (classical) curve graph is isomorphic to the extended mapping class group of the corresponding surface.

1. Introduction

The fine curve graph $C^\dagger(S)$ was recently introduced by Bowden–Hensel–Webb as a combinatorial tool for studying $\text{Homeo}(S)$, the group of homeomorphisms of a surface $S$. Its vertices are essential simple closed curves in $S$ and the edges are pairs of disjoint curves. There are two versions of $C^\dagger(S)$ in the literature, according to whether the curves are smooth or topological (the two are quasi-isometric). In this paper we take the vertices to be topological curves.

Let $S_g$ be the closed, connected, orientable surface of genus $g$. Our main theorem is that the group of simplicial automorphisms of $C^\dagger(S_g)$ is isomorphic to $\text{Homeo}(S_g)$ when $g \geq 2$. We can think of this as saying that $C^\dagger(S_g)$ is a combinatorial model for $S_g$, in that they have isomorphic groups of automorphisms. More precisely, we have the following statement.

Theorem 1.1. For $g \geq 2$ the natural map
\[ \eta : \text{Homeo}(S_g) \to \text{Aut } C^\dagger(S_g) \]
is an isomorphism.

Theorem 1.1 should be viewed as an analogue of the celebrated theorem of Ivanov [8, Theorem 1] that the group of simplicial automorphisms of the curve graph for $S_g$ is isomorphic to the mapping class group of $S_g$ when $g \geq 3$. There are some immediate complications that arise for the fine curve graph that distinguish it from the curve graph. To begin, the graph $C^\dagger(S)$ has uncountably many vertices, and is even locally uncountable. Moreover, two vertices of $C^\dagger(S)$ can bound (countably many) bigons and can intersect along (uncountably many) intervals, etc. The main difficulty in our work is to overcome these topological pathologies.

There is a precursor to Theorem 1.1 that we use in our proof. Specifically, Farb and the second author studied what we presently refer to as the extended fine curve graph $E C^\dagger(S)$. The vertices of $E C^\dagger(S)$ are all simple closed curves in $S$, including the inessential ones, and the edges again are pairs of disjoint curves.

Theorem 1.2 (Farb–Margalit). For any surface $S$ without boundary, the natural map
\[ \nu : \text{Homeo}(S) \to \text{Aut } E C^\dagger(S) \]
is an isomorphism.

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We give the original (unpublished) proof of Theorem 1.2 in Section 4. In the unpublished preprint of Farb and the second author [6], Theorem 1.2 is stated more generally, where the vertices of the graph are locally flat \((n - 1)\)-spheres in an \(n\)-manifold \(M\) and edges are for disjointness (with the automorphism group being \(\text{Homeo}(M)\)). The proof is essentially the same in this greater generality. We refer the reader to Farb’s lecture for more details [4].

**Outline of the proof of Theorem 1.1.** In order to prove Theorem 1.1, we construct an inverse map \(\text{Aut} \mathcal{C}(S_g) \to \text{Homeo}(S_g)\). Because Theorem 1.2 already gives a map \(\text{Aut} \mathcal{E} \mathcal{C}(S_g) \to \text{Homeo}(S_g)\) we can construct our inverse as a composition

\[
\text{Aut} \mathcal{C}(S_g) \to \text{Aut} \mathcal{E} \mathcal{C}(S_g) \to \text{Homeo}(S_g).
\]

So besides giving the proof of Theorem 1.2, our main task is to construct a homomorphism \(\text{Aut} \mathcal{C}(S_g) \to \text{Aut} \mathcal{E} \mathcal{C}(S_g)\). In other words, given an automorphism \(\alpha\) of \(\mathcal{C}(S_g)\) we would like to define an extension \(\hat{\alpha}\) which is an automorphism of \(\mathcal{E} \mathcal{C}(S_g)\). This means that given \(\alpha\) and an inessential curve \(e\) in \(S_g\), we need to associate another inessential curve \(e'\) in a natural way. We can then define \(\hat{\alpha}\) by the rule \(\hat{\alpha}(e) = e'\).

To this end, we associate to each such \(e\) a pair of vertices \(\{c, d\}\) of \(\mathcal{C}(S_g)\), called a bigon pair, and define \(e'\) to be the inessential curve associated to the bigon pair \(\{\alpha(c), \alpha(d)\}\). See the right-hand side of Figure 1 for an example of a bigon pair. This definition requires us to prove that bigon pairs are preserved by automorphisms of \(\mathcal{C}(S_g)\), which is the content of Proposition 2.1, the main technical result of the paper.

The proof of Theorem 1.2 mirrors the proof of Theorem 1.1: we use certain collections of vertices—convergent sequences of curves—to encode points in a surface \(S\) in order to define a map \(\text{Aut} \mathcal{E} \mathcal{C}(S) \to \text{Homeo}(S)\). As such, convergent sequences play the role in the proof of Theorem 1.2 that bigon pairs play in the proof of Theorem 1.1.

**The torus case.** As defined, the graph \(\mathcal{C}(T^2)\) is not connected since disjoint curves in \(T^2\) lie in the same homotopy class. In fact, the connected components precisely correspond to the homotopy classes of essential curves. Since all of these components are isomorphic, it follows that every permutation of the components is induced by some element of \(\text{Aut} \mathcal{C}(T^2)\). On the other hand, \(\text{Homeo}(T^2)\) preserves the geometric intersection number between components, so the set of permutations of components arising from \(\text{Homeo}(T^2)\) is countable (this set of permutations is isomorphic to \(\text{PGL}_2(\mathbb{Z})\)). Thus, \(\text{Aut} \mathcal{C}(T^2)\) properly contains \(\text{Homeo}(T^2)\) as a subgroup of uncountable index.

Bowden–Hensel–Webb give a modified definition of \(\mathcal{C}(T^2)\), where the edges connect curves that intersect at most once. It seems plausible that with this definition the natural map \(\text{Homeo}(T^2) \to \text{Aut} \mathcal{C}(T^2)\) is an isomorphism. However, our arguments for Theorem 1.1 do not apply, since they rely heavily on the fact that edges correspond to disjointness. If one
can show that an element of $\text{Aut} \mathcal{C}^\dagger(T^2)$ preserves the set of edges corresponding to disjoint curves, then it would be possible to apply many of our arguments to the torus case.

The smooth case. As mentioned at the outset, there is a smooth version of the fine curve graph, where the vertices are smooth curves in a surface. We conjecture that for most surfaces, the group of simplicial automorphisms of the smooth fine curve graph is the group of diffeomorphisms of the surface. A first step in proving this conjecture would be to promote Theorem 1.2 to the case of smooth curves and diffeomorphisms. Even given this, many of the arguments we give for Theorem 1.1 would not apply because our constructions would in general produce piecewise smooth curves (on the other hand, it also makes sense to consider a graph whose vertices are piecewise smooth curves...).

Outline of the paper. We begin in Section 2 by showing that automorphisms of $\mathcal{C}^\dagger(S_g)$ preserve certain configurations of curves in $S_g$. Specifically, these are the aforementioned bigon pairs, which are used to define the map $\epsilon$ discussed above, and sharing pairs, which are used in the proof that $\epsilon$ is well defined. In Section 3, we prove that three different fine arc graphs are connected. The last of these, the fine linked arc graph $\mathcal{A}^\dagger_{\text{lk}}(S)$, is also used in the proof that $\epsilon$ is well defined. In Section 4 we give the original proof of Theorem 1.2. Finally, in Section 5 we assemble the preceding results to prove Theorem 1.1.

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2. Characterizations of curve configurations

The goal of this section is to prove that two specific types of configurations of curves in a surface $S$ are preserved under automorphisms of $\mathcal{C}^\dagger(S)$. The corresponding statements are Propositions 2.1 and 2.2. We restrict ourselves in this section to the case $g \geq 2$.

We begin with some preliminaries. We say that curves $c$ and $d$ are noncrossing at a component $a$ of $c \cap d$ if there is a neighborhood $U$ of $a$ and a homeomorphism $U \to \mathbb{R}^2$ so that the image of $c \cap U$ and $d \cap U$ lie in the (closed) upper and lower half-planes of $\mathbb{R}^2$, respectively. The curves $c$ and $d$ are noncrossing if they are noncrossing at each component of $c \cap d$. 
Next, we say that a pair \( \{c,d\} \) of essential simple closed curves in \( S \) (equivalently, a pair of vertices of \( \mathcal{C}^1(S) \)) is...

- a **torus pair** if \( c \cap d \) is a single interval and \( c \) and \( d \) cross at that interval,
- a **pants pair** if \( c \cap d \) is a single interval, \( c \) and \( d \) do not cross at that interval, and \( c \) and \( d \) are not homotopic, and
- a **bigon pair** if \( c \cap d \) is a nontrivial closed interval and \( c \) and \( d \) are homotopic.

If \( c \) and \( d \) form a torus pair, then there is a neighborhood of \( c \cup d \) that is a torus with boundary. If \( c \) and \( d \) form a pants pair, then there is a neighborhood of \( c \cup d \) that is a pair of pants. We say that a torus pair or pants pair \( \{c,d\} \) is **degenerate** if \( c \cap d \) is a single point.

See Figure 1 for pictures of the three types of pairs; in each case, we show the union of the two curves in the pair. Given a nondegenerate torus pair \( \{c,d\} \), there are three ways to write \( c \cup d \) as a union of two simple closed curves, and each of the three resulting pairs of curves is a torus pair. The same statement is true for pants pairs. As such, the first two pictures in Figure 1 really show three torus pairs and three pants pairs, respectively.

If \( \{c,d\} \) is a nondegenerate torus pair (or pants pair) in \( S \), then there is exactly one other essential curve \( e \) contained in \( c \cup d \); the curve \( e \) is the closure in \( S \) of the symmetric difference \( c \triangle d \). We also refer to \( \{c,d,e\} \) as a **torus triple** (or pants triple), since any two elements of the triple form a torus pair determining the third.

If \( c \) and \( d \) form a bigon pair, then \( c \) and \( d \) determine an inessential simple closed curve \( e \); specifically, \( e \) is the closure of the symmetric difference \( c \triangle d \). When the two curves in a bigon pair are nonseparating, we call the pair a **nonseparating bigon pair**. The following proposition is the first of the two main goals of the section.

**Proposition 2.1.** Let \( g \geq 2 \). Then every automorphism of \( \mathcal{C}^1(S_g) \) preserves the set of nonseparating bigon pairs.

For the second proposition, we require another definition. Suppose that the bigon pairs \( \{a,b\} \) and \( \{a',b'\} \) determine the same inessential curve \( e \). In this case, each bigon pair gives rise to a single arc in the surface obtained by deleting the interior of the disk bounded by \( e \); we identify this surface with \( S^1_g \). We say that the pair of bigon pairs \( \{\{a,b\},\{a',b'\}\} \) is a **sharing pair** if the corresponding arcs in \( S^1_g \) have disjoint interiors. We further say that the sharing pair is **linked** if these two arcs are linked at \( e \), which means that all boundary parallel curves in \( S^1_g \) sufficiently close to the boundary intersect the two arcs alternately. We note that if a sharing pair is linked, then all four of the corresponding curves must be nonseparating. See Figure 2 for pictures of linked sharing pairs; there are three configurations according to how many endpoints of the arcs agree at \( e \). The following proposition is the second of the two main goals of the section.

**Proposition 2.2.** Let \( g \geq 2 \). Then every automorphism of \( \mathcal{C}^1(S_g) \) preserves the set of linked sharing pairs.
As discussed in the introduction, Propositions 2.1 and 2.2 will be used in Section 5
to define a map \( \text{Aut} \mathcal{C}^1(S_g) \rightarrow \text{Aut} \mathcal{EC}^1(S_g) \). More specifically, given \( \alpha \in \text{Aut} \mathcal{C}^1(S_g) \) we will
define an extension \( \hat{\alpha} \in \text{Aut} \mathcal{EC}^1(S_g) \). If \( e \) is an inessential curve determined by the bigon pair
\( \{c,d\} \), then Proposition 2.1 allows us to define \( \hat{\alpha}(e) \) to be the inessential curve determined
by \( \{\alpha(c),\alpha(d)\} \), and Proposition 2.2 will be used in the proof that \( \hat{\alpha}(e) \) is well defined.

We begin in Section 2.1 by proving a preliminary result, Lemma 2.5, which states that
automorphisms preserve torus pairs. This lemma will be used in the proofs of Propositions 2.1
and 2.2, which we give in Sections 2.2 and 2.3, respectively.

Tame curves, wild pairs. Before we begin in earnest, we make some comments about the
point-set topological issues that arise in this work. First, it is a fact that every curve in a
surface is tame in the sense that it is flat at each point (see [3, Theorem A1]). It follows, for
example, that any two nonseparating curves in a surface \( S \) differ by a homeomorphism of \( S \).
This can be thought of as a version of the change of coordinates principle in the theory of
mapping class groups [5, Section 1.3].

While curves themselves are tame, pairs of curves can exhibit complicated behavior. If
\( a \) and \( b \) are curves in a surface \( S \), then \( a \cap b \) can be regarded as an open set in \( S^1 \), and hence
is a countable union of disjoint intervals. On the other hand, \( a \cap b \) is a compact set, but it
can be complicated. The components of \( a \cap b \) are (possibly degenerate) intervals, but there
is a non-empty set of torus pairs, the set of (non)degenerate torus pairs, and the set of
torus triples. Along the way, we prove two auxiliary lemmas, Lemmas 2.3 and 2.4.

2.1. Torus pairs. In this section we prove Lemma 2.5, which states that automorphisms of
\( \mathcal{C}^1(S_g) \) preserve the set of torus pairs, the set of (non)degenerate torus pairs, and the set of
torus triples. Along the way, we prove two auxiliary lemmas, Lemmas 2.3 and 2.4.

Sides. For the first lemma, a multicurve is a finite collection of pairwise disjoint essential
simple closed curves in \( S \) (a curve is an example of a multicurve). As such, multicurves are
the same as finite cliques in \( \mathcal{C}^1(S) \). We emphasize that two curves in a multicurve are allowed
to be parallel.

A multicurve in \( S \) is separating if its complement has more than one component. We
say that two curves \( a \) and \( b \) lie on the same side of a separating multicurve \( m \) if they are
disjoint from \( m \) and lie in the same complementary component.

In the proof, we say that a graph is a join if we can partition the set of vertices into
two or more nonempty sets in such a way that every vertex from one set is connected by an
de edge to every vertex in the other sets. Also, the link of a set \( A \) of vertices in a graph is the
subgraph spanned by the set of vertices that are not in \( A \) and are connected by an edge to
each vertex in \( A \).

Lemma 2.3. Let \( S = S_g \) with \( g \geq 2 \), and let \( \alpha \in \text{Aut} \mathcal{C}^1(S) \). Then \( \alpha \) preserves the set
of separating curves in \( \mathcal{C}^1(S) \) and also preserves the set of separating multicurves in \( \mathcal{C}^1(S) \).
Moreover, \( \alpha \) preserves the sides of a separating multicurve, that is, \( a \) and \( b \) lie on the same
side of \( m \) if and only if \( \alpha(a) \) and \( \alpha(b) \) lie on the same side of \( \alpha(m) \).

Proof. First of all, it follows from the definition of \( \mathcal{C}^1(S) \) that \( \alpha \) preserves multicurves and also
preserves curves. Therefore, for the first statement it suffices to distinguish the separating
multicurves from the nonseparating ones.

We claim that a multicurve \( m = \{c_1, \ldots, c_k\} \) is separating if and only if the link of
\( m \) is a join. Indeed if \( m \) is a separating multicurve then the sets for the join decomposition
are the curves that lie in the various complementary components of \( m \) (each of these sets is
nonempty because they contain curves parallel to the \( c_i \)). For the other direction, we observe
that if $m$ is a nonseparating multicurve, and $a$ and $b$ lie in the link of $m$, then there is a curve $d$ that intersects both $a$ and $b$. It follows from this that the link of $m$ cannot be a join, as desired.

It follows from the argument in the previous paragraph that the two sets used to define the join decomposition for a separating multicurve are uniquely defined. From this the second statement follows. \hfill \Box

**Hulls.** We define the *hull* of a collection of curves in a surface to be the union of the curves along with any embedded disks bounded by the curves.

**Lemma 2.4.** Let $S = S_g$ with $g \geq 2$, and let $\alpha \in \text{Aut} \mathcal{C}^\uparrow(S)$. If $X$ is a finite set of vertices of $\mathcal{C}^\uparrow(S)$ and a vertex $d$ lies in the hull of $X$, then $\alpha(d)$ lies in the hull of $\alpha(X)$.

**Proof.** It suffices to prove that $d$ lies in the hull of $X$ if and only if the link of $d$ contains the link of $X$. To this end, suppose that $d$ is a vertex of $\mathcal{C}^\uparrow(S)$ that does not lie in the hull of $X$. This means that there is a component of $d \setminus X$ that lies in a component $R$ of $S \setminus X$ that is not a disk. Because $R$ is not a disk, it contains simple closed curves that are essential in $S$, and in particular it contains one that intersects the arc of $d$ in $R$, as desired.

Suppose on the other hand that $d$ is a vertex of $\mathcal{C}^\uparrow(S)$ that lies in the hull of $X$. Suppose also that $e$ is a simple closed curve in $S$ that intersects $d$ but not $X$. Since $e$ is disjoint from $X$ it must lie in the complement of $X$. And since $d$ is contained in the hull of $X$ it must then be that $e$ lies in one of the components of $S \setminus X$ that is a disk. It follows that $e$ is inessential, and the lemma is proven. \hfill \Box

The statement of Lemma 2.4 is specifically geared towards closed surfaces. For surfaces with punctures we would, among other things, need to define the hull to include all once-punctured disks.

**Torus pairs.** We now prove the main result of this subsection.

**Lemma 2.5.** Let $S = S_g$ with $g \geq 2$, and let $\alpha \in \text{Aut} \mathcal{C}^\uparrow(S)$. Then $\alpha$ preserves the set of torus pairs, the set of degenerate torus pairs, the set of nondegenerate torus pairs, and the set of torus triples.

**Proof.** We proceed in four steps. First we show that $\alpha$ preserves the union of the torus pairs and the pants pairs. Then we show that $\alpha$ preserves the set of torus pairs. Next, we show that $\alpha$ preserves the set of degenerate torus pairs, hence it also preserves the nondegenerate torus pairs. Finally, we prove that $\alpha$ preserves the set of torus triples.

**Step 1.** For the first step, it suffices to show that the following three statements are equivalent for a pair of intersecting vertices $\{c, d\}$ of $\mathcal{C}^\uparrow(S)$:

1. The pair $\{c, d\}$ is a torus pair or a pants pair.
2. There is at most one other vertex of $\mathcal{C}^\uparrow(S)$ that lies in the hull of $\{c, d\}$.
3. There is at most one other vertex of $\mathcal{C}^\uparrow(S)$ whose link contains the link of $\{c, d\}$.

The second and third statements are equivalent by Lemma 2.4, so it suffices to prove the equivalence of the first two statements.

The first statement implies the second because if $\{c, d\}$ is a torus pair or pants pair, then the hull of $\{c, d\}$ is $c \cup d$, and in this case there are either no other simple curves or one other simple curve in $c \cup d$, depending on whether or not $\{c, d\}$ is degenerate.

We prove that the second statement implies the first. Let $\{c, d\}$ be a pair of vertices that are intersecting curves and assume that there is at most one other vertex in the hull of
\{c, d\}. We claim that no complementary region of \(c \cup d\) is a disk. Indeed, suppose to the contrary that one such region were a disk. Any curve that agrees with \(c\) away from this disk and disagrees with \(c\) inside the disk lies in the hull of \(\{c, d\}\). There are infinitely many such curves, a contradiction.

If \(\{c, d\}\) is not a torus pair or a pants pair, then it must be that \(c \cap d\) has more than one connected component. Let \(a_1\) and \(a_2\) be two such components. Let \(c_1, c_2, d_1,\) and \(d_2\) be the closures of the complementary components of \(a_1 \cup a_2\) in \(c\) and \(d\). In \(c \cup d\) there are four distinct simple closed curves \(e_1, \ldots, e_4\) that contain \(c_1 \cup d_1, c_1 \cup d_2, c_2 \cup d_1,\) and \(c_2 \cup d_2\), respectively. These curves intersect each \(a_i\) in the empty set, an endpoint, or all of \(a_i\). The \(e_i\) are all distinct from \(c\) and \(d\). If some \(e_i\) were inessential, then it would be the boundary of a disk in \(S\). It would follow that \(c \cup d\) bounds a (possibly smaller) disk, contradicting the claim in the previous paragraph.

**Step 2.** For the second step, we assume that \(\{c, d\}\) is a torus pair or pants pair. We will show that, under this assumption, the pair \(\{c, d\}\) is a torus pair if and only if there is a separating curve \(e\) disjoint from \(c\) and \(d\) and with the following property: all nonseparating simple closed curves in \(S_g\) lying on the same side of \(e\) as \(\{c, d\}\) fail to be disjoint from \(c \cup d\). The proposition then follows from the definition of \(C^\dag(S_g)\) and Lemma 2.3.

We begin with the forward direction. Let \(\{c, d\}\) be a torus pair, let \(R\) be a neighborhood of \(c \cup d\) homeomorphic to a torus with one boundary component \(e\). The surface obtained by cutting \(R\) along \(c \cup d\) is an annulus. Any nonseparating curve in \(S_g\) that lies in \(R\) is not parallel to the boundary (otherwise it is parallel to \(e\), hence separating), and hence this nonseparating curve intersects either \(c\) or \(d\), as desired.

For the reverse direction, we assume that \(\{c, d\}\) is a pants pair, and we let \(e\) be any separating curve disjoint from \(c \cup d\). Let \(R\) be the subsurface of \(S_g\) that contains \(c \cup d\) and has boundary \(e\). It must be that \(R\) has positive genus. There is a closed neighborhood of \(c \cup d\) that is a pair of pants \(P\) contained in \(R\); we denote its interior by \(P^\circ\). Since \(P\) has genus 0, there must exist a curve in \(R \setminus P^\circ\) that is nonseparating in \(R\), hence in \(S_g\). This completes the proof of the second step.

**Step 3.** The following three statements are equivalent for a torus pair \(\{c, d\}\) of \(C^\dag(S)\):

1. The torus pair \(\{c, d\}\) is nondegenerate.
2. There is exactly one other vertex of \(C^\dag(S)\) in the hull of \(\{c, d\}\).
3. There is exactly one other vertex of \(C^\dag(S)\) whose link contains the link of \(\{c, d\}\).

The equivalence of the first two statements can be proved by inspection of the two possible configurations for a torus pair (degenerate and nondegenerate). The last two statements are equivalent by Lemma 2.4. This completes the third step.

**Step 4.** Suppose \(\{c, d, e\}\) is a torus triple. Then \(e\) is the unique vertex (other than \(c\) and \(d\)) contained in the hull of \(\{c, d\}\). By Lemma 2.4, \(e\) is the unique curve whose link contains the link of \(\{c, d\}\). Since torus pairs are preserved, it now follows that torus triples are preserved. \(\square\)

### 2.2. Bigon pairs

In this subsection we prove Proposition 2.1. We begin by defining annulus sets and describing their basic properties. We prove in Lemma 2.6 that these properties are preserved under automorphisms of \(C^\dag(S)\). With that in hand, we proceed to the proof of Proposition 2.1.

**Annulus sets.** Suppose that \((a, b)\) is an ordered pair of vertices of \(C^\dag(S_g)\) that are disjoint, homotopic curves. Assuming \(g \geq 2\), there is a unique annulus \(A\) in \(S_g\) whose boundary is
Let \( C^\dagger(a, b) \) be the set of vertices of \( C^\dagger(S_g) \) given by curves contained in the interior of \( A \). We refer to \( C^\dagger(a, b) \) as an **annulus set**. We say that a pair of vertices of \( C^\dagger(S_g) \) is an **annulus pair** if they lie in some \( C^\dagger(a, b) \). A **nonseparating noncrossing annulus pair** is an annulus pair where both curves are nonseparating and the pair is noncrossing.

There is a natural partial ordering on the annulus set \( C^\dagger(a, b) \): we say that \( c \preceq d \) if \( c \) and \( d \) are noncrossing and each component of \( c \setminus d \) lies in the component of \( A \setminus d \) bounded by \( a \).

**Lemma 2.6.** Let \( S = S_g \) with \( g \geq 2 \), let \( \alpha \) be an automorphism of \( C^\dagger(S_g) \), let \( a \) and \( b \) be disjoint, homotopic nonseparating curves.

1. The curves \( \alpha(a) \) and \( \alpha(b) \) are disjoint, homotopic nonseparating curves.
2. The image of \( C^\dagger(a, b) \) under \( \alpha \) is \( C^\dagger(\alpha(a), \alpha(b)) \).
3. If \( c, d \in C^\dagger(a, b) \) are noncrossing then \( \alpha(c) \) and \( \alpha(d) \) are noncrossing.
4. If \( c \preceq d \) in \( C^\dagger(a, b) \) then \( \alpha(c) \preceq \alpha(d) \) in \( C^\dagger(\alpha(a), \alpha(b)) \).

**Proof.** We prove the four statements in turn. The first statement is a consequence of Lemma 2.3 and the fact that two disjoint nonseparating curves \( a \) and \( b \) in \( S \) are homotopic if and only if the following conditions hold: \( a \) and \( b \) form a separating multicurve and all separating curves disjoint from both \( a \) and \( b \) lie on the same side of the multicurve \( a \cup b \).

The second statement is an immediate consequence of the first statement and Lemma 2.3.

We proceed to the third statement. By the first statement, \( \alpha(a) \) and \( \alpha(b) \) are disjoint, homotopic nonseparating curves. Any two such ordered pairs differ by a homeomorphism of \( S \), and hence by an automorphism of \( C^\dagger(S_g) \). Thus, we may assume without loss of generality that \( \alpha \) preserves \( a \) and \( b \) (that is, we may postcompose \( \alpha \) with the automorphism from the previous sentence to make this so). By the second statement, \( \alpha \) preserves the annulus set \( C^\dagger(a, b) \).

We claim that two curves \( c \) and \( d \) in \( C^\dagger(a, b) \) are noncrossing if and only if there is a different curve \( e \in C^\dagger(a, b) \) with the property that every curve in \( C^\dagger(a, b) \) that intersects \( c \) and \( d \) must also intersect \( e \). Indeed, when \( c \) and \( d \) are noncrossing the curve \( e \) is any curve that contains \( c \cap d \), is contained in the hull of \( \{c, d\} \), and that passes through the interior of at least one such bigon (the last condition ensures that \( e \) is not equal to \( c \) or \( d \)). For the other direction of the claim, we assume that there is a curve \( e \) with the property that every curve in \( C^\dagger(a, b) \) that intersects \( c \) and \( d \) must also intersect \( e \). The curve \( e \) divides the annulus into two smaller annuli \( A_- \) and \( A_+ \), with the former being bounded by \( a \) and \( e \) and the latter being bounded by \( b \) and \( e \). The defining property of \( e \) implies that \( c \) and \( d \) are each contained in one of these smaller annuli. It follows that \( c \) and \( d \) are noncrossing, as desired. This completes the proof of the third statement.

The fourth statement holds by the previous three statements and the fact that noncrossing curves \( c, d \in C^\dagger(a, b) \) satisfy \( c \preceq d \) if and only if there is a vertex of \( C^\dagger(S_g) \) that intersects \( a \) and \( c \) but not \( b \) or \( d \). This completes the proof of the lemma.

**Type 1 and type 2 curves.** Suppose that \( \{c, d\} \) is a nonseparating noncrossing annulus pair, and suppose that \( e \) is a curve so that \( \{c, e\} \) and \( \{d, e\} \) are degenerate torus pairs. If \( c \cap e \) and \( d \cap e \) are the same point, then we say that \( e \) is a **type 1 curve** for \( \{c, d\} \). Otherwise we say that \( e \) is a **type 2 curve** for \( \{c, d\} \). The reason for the terminology is that type 1 and type 2 curves intersect \( c \cup d \) in one point and two points, respectively.

**Lemma 2.7.** Let \( S = S_g \) with \( g \geq 2 \), and let \( \alpha \in \text{Aut} C^\dagger(S) \). Then \( \alpha \) preserves type 1 and type 2 curves for nonseparating noncrossing annulus pairs. More precisely, if \( \{c, d\} \) is a
nonseparating noncrossing annulus pair and \( e \) is a type 1 curve for \( \{ c, d \} \), then \( \alpha(e) \) is a type 1 curve for the nonseparating noncrossing annulus pair \( \{ \alpha(c), \alpha(d) \} \), and similarly for type 2 curves.

**Proof.** Since \( \alpha \) preserves degenerate torus pairs (Lemma 2.5), we may assume that \( \alpha \) preserves the union of the type 1 and type 2 curves for \( \{ c, d \} \). So it remains to show that \( \alpha \) preserves the two types.

Say that \( c, d \in C^\dagger(a, b) \). By parts (1) and (2) of Lemma 2.6, we may assume without loss of generality that \( \alpha \) preserves \( C^\dagger(a, b) \) (as in the proof of Lemma 2.6 we may postcompose \( \alpha \) with an automorphism induced by an element of \( \text{Homeo}(S) \) to make this so).

Let \( e \) be a curve with the property that \( \{ c, e \} \) and \( \{ d, e \} \) are degenerate torus pairs, so \( e \) is either a type 1 or type 2 curve for \( \{ c, d \} \). We claim that \( e \) is a type 2 curve if and only if there is a curve \( f \) with the following properties:

- \( f \) is contained in the hull of \( \{ c, d, e \} \),
- \( f \) is not contained in \( C^\dagger(a, b) \), and
- \( f \) is not equal to \( e \).

The forward direction is proved by construction, as follows. If \( e \) is a type 2 curve, it passes through the interior of a bigon \( B \) bounded by arcs of \( c \) and \( d \). By replacing the arc of \( e \) that passes through the bigon with a different arc, we obtain the desired curve \( f \). For the other direction, we suppose that \( e \) is a type 1 curve for \( \{ c, d \} \). Any curve that satisfies the first two given properties would have to contain all of \( e \), hence would fail the third property. The claim follows and the lemma thus follows from Lemma 2.4.

**Bigon pairs.** We are now ready for the proof of Proposition 2.1.

**Proof of Proposition 2.1.** By Lemma 2.6(3), we have that \( \alpha \) preserves nonseparating noncrossing annulus pairs. If \( \{ c, d \} \) is such a pair, then \( c \cup d \) is a union of (potentially infinitely many) inessential curves with (possibly degenerate) arcs connecting these inessential curves cyclically. If there are \( k \) (or more) inessential curves, then we may find curves \( e_1, f_1, \ldots, e_k, f_k \) with the following properties:

- any two of the \( 2k \) curves are homotopic and pairwise disjoint,
- each \( e_i \) is a type 1 curve for \( \{ c, d \} \),
- each \( f_i \) is a type 2 curve for \( \{ c, d \} \),
- and the \( 2k \) curves lie in the given order in the annulus bounded by \( e_1 \) and \( f_k \).

Conversely, if we can find \( 2k \) curves with the above properties, then \( c \) and \( d \) form at least \( k \) inessential curves. Thus, by Lemmas 2.6 and 2.7, \( \alpha \) preserves the number of inessential curves formed by \( \{ c, d \} \) (this number may be infinite). In particular, it preserves the set of noncrossing annulus pairs that form exactly one inessential curve.

A bigon pair is a nonseparating noncrossing annulus pair \( \{ c, d \} \) that forms exactly one inessential curve and has the additional property that \( c \cap d \) is a nondegenerate interval. Among the nonseparating noncrossing annulus pairs forming exactly one inessential curve, the bigon pairs are exactly those for which there exists two curves \( e_1 \) and \( e_2 \) that are disjoint and are both type 1 curves for \( \{ c, d \} \). The proposition follows.

### 2.3. Sharing pairs.

The goal of this subsection is to prove Proposition 2.2, which states that automorphisms of \( C^\dagger(S_g) \) preserve sharing pairs.

In the following proof we write \( A \equiv B \) if \( A \) and \( B \) are two sets with \( A \Delta B \) a finite set. The relation \( \equiv \) is an equivalence relation.
We claim that bigon pairs follow we will use the fact that if $e$ is a bigon pair, then the inessential curve and only if the following conditions hold:

$$ e \cong c \Delta d. $$

In what follows we will use the fact that if $e$ and $e'$ are curves in a surface with $e \cong e'$ then $e = e'$.

Proof of Proposition 2.2. We claim that bigon pairs $\{c, d\}$ and $\{c', d'\}$ form a sharing pair if and only if the following conditions hold:

1. each of $\{c, d'\}$ and $\{c', d\}$ is an nondegenerate torus pair, and
2. there is a curve that forms a torus triple with both $\{c, d\}$ and $\{c', d\}$.

The proposition follows from the claim, Proposition 2.1, and Lemma 2.5. The first direction of the claim can be verified by explicitly constructing the curve $e$; in Figure 3 we indicate the curve $e$ that forms a torus triple with both $\{c, d\}$ and $\{c', d\}$.

For the other direction, say that $e$ is the inessential curve determined by $\{c, d\}$, and that $e'$ is the inessential curve determined by $\{c', d'\}$. Since there is a curve that forms a torus triple with both $\{c, d\}$ and $\{c', d\}$, this means that $c \Delta d' \cong c' \Delta d$. Using the basic fact about symmetric differences that $A \Delta B = (A \Delta C) \Delta (B \Delta C)$, we have:

$$ e \cong e' = c \Delta d = (c \Delta c') \Delta (c' \Delta d) \cong (c \Delta c') \Delta (c \Delta d') = e' \cong e'. $$

Thus $e = e'$, which is to say that $\{c, d\}$ and $\{c', d'\}$ determine the same inessential curve $e$. We identify the complement of the interior of $e$ with $S_g^1$. The pairs $\{c, d\}$ and $\{c', d'\}$ determine arcs $a$ and $a'$ in $S_g^1$. If $a$ and $a'$ were not disjoint and linked, then this would violate the condition that $\{c, d\}$ (and also $\{c', d\}$) is a nondegenerate torus pair. This completes the proof. 

\[\square\]

3. Connectedness of fine arc graphs

Let $S_g^b$ denote the surface obtained from $S_g$ by deleting the interiors of $b$ disjoint disks. Let $S = S_g^b$ with $b > 0$. The goal of this section is to prove that three fine arcs graphs are connected: the fine arc graph $A^f(S)$, the fine nonseparating arc graph $N A^f(S)$, and the fine linked arc graph $A^f_{Lk}(S)$. We begin by proving that $A^f(S)$ is connected (Proposition 3.1), and then derive the connectivity of the other two graphs as corollaries (Corollaries 3.2 and 3.3). For the proof of Theorem 1.1, we will only use the connectivity of $A^f_{Lk}(S)$.

The fine arc graph. We begin with the basic definitions. An arc in $S = S_g^b$ is the image of a map $a : [0, 1] \to S$. We say that the arc is simple if the map $a$ is injective, we say that the arc is proper if $a^{-1}(\partial S) = \{0, 1\}$, and we say that the arc is essential if it is not homotopic into $\partial S$. We say that two arcs have disjoint interiors if they are disjoint away from $\partial S$. When two arcs have no intersections at all (including at the boundary), we say that the arcs are completely disjoint.
The fine arc graph $A^\dagger(S)$ is the graph whose vertices are essential simple proper arcs in $S$ and whose edges connect vertices with disjoint interiors. The next proposition states that $A^\dagger(S)$ is connected; we note that for $S = S^3_0$, the graph $A^\dagger(S)$ is empty, hence vacuously connected.

**Proposition 3.1.** For any $S = S^b_g$ with $b > 0$, the graph $A^\dagger(S)$ is connected.

Our proof of Proposition 3.1 is based on the proof of Bowden–Hensel–Webb that the fine (smooth) curve graph is connected [2, Section 3]. As that proof relies on the connectivity of the classical curve graph, our proof relies in the connectivity of the classical arc graph $A(S)$. The vertices of $A(S)$ are isotopy classes of essential simple proper arcs in $S$, where isotopies are allowed to move endpoints of arcs along the boundary of $S$. The edges are pairs of vertices with disjoint representatives. The arc complex is the flag complex associated to $A(S)$.

**Proof of Proposition 3.1.** There is a natural simplicial map $A^\dagger(S) \to A(S)$ given by taking isotopy classes. The arc complex is contractible [7], so in particular its 1-skeleton $A(S)$ is connected. Thus, it suffices to show that for any vertex of $A(S)$, the subgraph of $A^\dagger(S)$ spanned by its preimage is connected. In other words, it suffices to show that between any two isotopic essential simple proper arcs in $S$ there is a path in $A^\dagger(S)$ connecting the two.

Let $a$ and $b$ be vertices of $A(S)$ that are isotopic arcs and let $H : S^1 \times [0,1] \to S$ be an isotopy from $a$ to $b$. For $t \in [0,1]$, let $a_t$ be the image of $S^1 \times \{t\}$, so $a_0 = a$ and $a_1 = b$. For each vertex $c$ of $A^\dagger(S)$, we define

$$I_c = \{ t \in [0,1] \mid a_t \text{ is completely disjoint from } c \}$$

Each $I_c$ is open. Thus we may find a sequence of open intervals $I_0, \ldots, I_k$ that cover $[0,1]$ and so each $I_i$ is some $I_{c_i}$. We may further assume that $0 \in I_0$ and that $I_i \cap I_j$ is nonempty if and only if $|i - j| = 1$. Let $t_0 = 0$, let $t_{k+1} = 1$, and for $i \in \{1, \ldots, k\}$ let $t_i$ be an element of $I_i \cap I_{i+1}$. For $i \in \{0, \ldots, k+1\}$ let $a_i = a_{t_i}$.

By definition, each pair $\{a_i, a_{i+1}\}$ is completely disjoint from $c_i$. Thus the sequence

$$a = a_0, c_0, a_1, c_1, a_2, \ldots, a_{k-1}, c_{k-1}, a_k = b$$

is the desired path from $a$ to $b$ in $A^\dagger(S)$. \qed

We remark that it is possible to define $A^\dagger(S)$ in a similar manner when $S$ has punctures instead of boundary. However, in some cases, this graph is not connected. For instance, if $S = S_{g,1}$ and an arc $a$ spirals around the puncture infinitely many times relative to $b$, then $a$ and $b$ are not connected by a path in $A^\dagger(S)$. The part of the proof of Proposition 3.1 that fails for surfaces with punctures is that the sets $I_c$ are not always open.

The fine nonseparating arc graph. We say that an arc in a surface $S$ is nonseparating if its complement in $S$ is connected. The fine nonseparating arc graph is the subgraph of $A^\dagger(S)$ spanned by the nonseparating arcs. The proof of the following corollary is modeled on the proof of the corresponding statement for curve graphs [5, Theorem 4.4].

**Corollary 3.2.** For any $S = S^b_g$ with $b > 0$, the graph $NA^\dagger(S)$ is connected.

**Proof.** Let $a$ and $b$ be vertices of $NA^\dagger(S)$. By Proposition 3.1 there is a path

$$a = a_0, \ldots, a_k = b$$

Such a path has $a_1$ and $a_k$ as disjoint arcs. Thus we only need to show that $a_1$ and $a_k$ are connected in $NA^\dagger(S)$. Since $NA^\dagger(S)$ is a subgraph of $A^\dagger(S)$, it is connected. \qed
A graph

Corollary 3.3. Let

Proof. NA

d are linked at

entirely in

S

arcs in

NA

the above sequence of vertices.

□
The lemma follows.

The fine linked arc graph. Let

S = S^b_g

be a surface with
g \geq 1

and

b > 0.

Let

d_0

be a distinguished component of \partial S. Let

a

and

b

be two vertices of \mathcal{A}^\dagger(S) that are arcs with disjoint interiors. We say that

a

and

b

are linked at

d_0

if all four endpoints lie on

d_0

and the boundary curve for any sufficiently small neighborhood of

d_0

alternates between intersections with

a

and

b.

Some examples of linked arcs in

S^1_g

with disjoint interiors are shown in Figure 4.

We define

\mathcal{A}^\dagger_{Lk}(S, d_0)

to be the graph whose vertices are nonseparating simple proper arcs in

S

with both endpoints at

d_0

and whose edges connect arcs with disjoint interiors that are linked at

d_0.

When convenient we suppress

d_0

in the notation in what follows and write

\mathcal{A}^\dagger_{Lk}(S).

Corollary 3.3. Let

S = S^b_g

with
g \geq 1

and

b > 0,

and let

d_0

be a component of \partial S. The graph

\mathcal{A}^\dagger_{Lk}(S, d_0)

is connected.

Proof. Let

a

and

b

be vertices of \mathcal{A}^\dagger_{Lk}(S, d_0). By Corollary 3.2 there is a path

a = a_0, \ldots, a_k = b

in \mathcal{N}\mathcal{A}^\dagger(S).

For a given edge \{a_i, a_{i+1}\} in this path where

a_i

and

a_{i+1}

are not linked, we would like to show there is an arc

b_i

that is linked with both

a_i

and

a_{i+1}

and disjoint from their interiors. For then we may obtain a path from

a

to

b

in \mathcal{A}^\dagger_{Lk}(S) by inserting all such

b_i

into the above sequence of vertices.

So let \{x, y\} be an arbitrary edge in \mathcal{N}\mathcal{A}^\dagger(S) where

x

and

y

are unlinked. Since each of

x

and

y

is nonseparating, it follows that

x \cup y

separates

S

into at most two components. We consider a small annular neighborhood

A

of

d_0.

The intersections of

x

and

y

with

A

it into 4 components. These components come in a cyclic order; call them $A_0$, $A_1$, $A_2$, and $A_3$. Exactly two of the 4 components of $A$ have the property that they are bounded by one arc of $x$ and one arc of $y$. These components are not adjacent; say they are $A_0$ and $A_2$. Since $x \cup y$ separates $S$ into at most two components, $A_1$ and $A_3$ lie in the same component of $S$ cut along $x \cup y$. Thus there is an arc $z$ in $S$ that is disjoint from $x$ and $y$ away from $d_0$ and connects $A_1$ to $A_3$. This arc $z$ is thus linked with both $x$ and $y$ by definition. By virtue of being linked with other arcs, $z$ is nonseparating, hence a vertex of $A_{lk}^+(S,d_0)$. The corollary follows. 

4. AUTOMORPHISMS OF THE EXTENDED FINE CURVE GRAPH

In this section we prove Theorem 1.2, which states that the natural map $\nu : \text{Homeo}(S) \to EC^+(S)$ is an isomorphism. As discussed in the introduction, the proof we give is the original one, due to Farb and the second author. We emphasize that the proof applies to all surfaces without boundary, including those whose fundamental group is not finitely generated.

In Section 4.1 we introduce convergent sequences of vertices and prove several related lemmas about them. Then in Section 4.2 we use convergent sequences to prove Theorem 1.2.

4.1. Convergent sequences. Let $S$ be a surface without boundary. We say that a sequence of vertices $(c_i)$ of $EC^+(S)$ converges to a point $x \in S$ if the corresponding curves converge to $x$ in the Hausdorff metric. In other words, every neighborhood of $x$ contains all but finitely many of the $c_i$. In this case we write $\lim(c_i) = x$. If $(c_i)$ is convergent, it must be that there exists $M > 0$ so that each $c_i$ with $i > M$ is inessential.

The main goal of this section is to prove Lemma 4.2, which states that automorphisms of $EC^+(S)$ preserve convergent sequences. We also prove several related statements, Corollaries 4.3, 4.4, and 4.5. Before proving these we introduce a technical tool, the connect-the-dots lemma, Lemma 4.1.

Connect-the-dots lemma. We will use the following technical lemma to prove that automorphisms of $EC^+(S)$ preserve convergent sequences.

Lemma 4.1. Let $S$ be a surface, and let $(x_i)$ be a sequence of points in $S$ that converges to a point $x$ in the interior of $S$. Then there is a simple closed curve in $S$ that contains infinitely many of the $x_i$.

Proof. Let $U$ be an open disk in $S$ that contains $x$. By the classification of surfaces of infinite type [9, Theorem 1], the surface $U \setminus (\{x_i\} \cup \{x\})$ is homeomorphic to $F = \mathbb{C} \setminus (\{1/n\} \cup \{0\})$ (this surface is sometimes called the flute surface). Regarding the punctures in $F$ as marked points, there is clearly a simple closed curve containing infinitely many marked points of $F$ (any curve containing a nontrivial interval $[0,\epsilon]$). Any such curve corresponds to a curve in $U$ containing infinitely many $x_i$. □

Convergent sequences are preserved. For the statement of Lemma 4.2, we require one more definition. We say that a vertex $a$ of $EC^+(S)$ intersects the tail of a sequence of vertices $(c_i)$ if there are infinitely many $i$ so that the curve $a$ intersects the curve $c_i$.

Lemma 4.2. Let $S$ be a surface without boundary. Automorphisms of $EC^+(S)$ preserve convergent sequences.

Proof. It suffices to prove the following statement. Suppose that $(c_i)$ is a sequence of vertices of $EC^+(S)$. Then $(c_i)$ is convergent if and only if the following two conditions hold:
(1) there exists a vertex $a$ that intersects the tail of $(c_i)$, and
(2) if $a$ and $b$ are distinct vertices that intersect the tail of $(c_i)$, then $a$ and $b$ intersect.

The forward direction follows immediately from the definition of a convergent sequence and the fact that vertices of $\mathcal{EC}^\dagger(S)$ correspond to closed subsets of $S$. Indeed, if $(c_i)$ converges to the point $x$ then any two vertices that intersect the tail of $(c_i)$ must be curves that intersect at the point $x$.

For the reverse direction, suppose that $(c_i)$ is not convergent. In the case where the sequence $(c_i)$ leaves every compact subsurface of $S$, the first condition fails: there is no vertex $a$ intersecting the tail of $(c_i)$. Thus, we may assume that there is a compact subsurface $R \subseteq S$ with the property that $c_i \cap R$ is nonempty for infinitely many $i$.

Since $R$ is compact, we may choose a subsequence $(c_{i_j})$ of $(c_i)$ and a sequence of points $x_{i_j} \in c_{i_j}$ with the property that $x_{i_j}$ converges to a point $x$ in $R$. And since $(c_i)$ is not convergent, we may choose another subsequence $(c_{i_k})$ of $(c_i)$ and a sequence of points $y_{i_k} \in c_{i_k}$ with the property that $y_{i_k}$ converges to a point $y$ in $R$. By replacing $R$ with a neighborhood of $R$, we may assume that $x$ and $y$ both lie in the interior of $R$.

Let $U$ and $V$ be disjoint open neighborhoods of $x$ and $y$. These neighborhoods contain infinitely many $x_{i_j}$ and infinitely many $y_{i_k}$, respectively. By Lemma 4.1 there are simple closed curves in $U$ and $V$ containing infinitely many of the $x_{i_j}$ and $y_{i_k}$, respectively. The curves $a$ and $b$ are disjoint since $U$ and $V$ are. Both curves intersect the tail of $(c_i)$ by construction. This completes the proof.

\[\square\]

**Coincidence of convergent sequences is preserved.** For the statement of the following corollary, we say that two convergent sequence of vertices of $\mathcal{EC}^\dagger(S)$ are coincident if they converge to the same point of $S$. We also define the interleave of two sequences $(c_i)$ and $(d_i)$ to be the sequence $c_1, d_1, c_2, d_2, \ldots$.

We have the following corollary of Lemma 4.2. The first statement follows from the definition of convergence in point set topology, and the second statement follows from Lemma 4.2.

**Corollary 4.3.** Let $S$ be a surface without boundary. Let $(c_i)$ and $(d_i)$ be two convergent sequences of vertices of $\mathcal{EC}^\dagger(S)$. Then $(c_i)$ and $(d_i)$ are coincident if and only if the interleave of $(c_i)$ and $(d_i)$ is convergent. In particular, automorphisms of $\mathcal{EC}^\dagger(S)$ preserve coincidence of convergent sequences.

**Convergence of convergent sequences is preserved.** For the next corollary to Lemma 4.2, we say that a sequence of convergent sequences

$$(c_i^1), (c_i^2), (c_i^3), \ldots$$

in $\mathcal{EC}^\dagger(S)$ converges if the sequence of limit points

$$\lim(c_i^1), \lim(c_i^2), \lim(c_i^3), \ldots$$

converges to a point $x \in S$. In this case we say that the sequence converges to $x$.

A diagonal sequence for a sequence of sequences as above is a sequence $(d_j)$ with each $d_j$ equal to some $c_i^j$. In other words, there is a function $D : \mathbb{N} \to \mathbb{N}$ so that $d_j = c_{D(j)}^j$. We impose a partial order on diagonal sequences for convergent sequences as follows: $(d_j) \preceq (e_j)$ if the corresponding functions satisfy $D(j) \leq E(j)$ for all $j$. In the statement of the next corollary, we say that a diagonal subsequence is sufficiently large if it is sufficiently large with respect to this ordering.
Let \((c_i^1), (c_i^2), (c_i^3), \ldots\) be a sequence of convergent sequences of vertices of \(\mathcal{EC}^\dagger(S)\) and let \(x \in S\). Then this sequence converges to \(x \in S\) if and only if all sufficiently large diagonal subsequences converge to \(x\). We thus have the following consequence of Lemma 4.2.

**Corollary 4.4.** Let \(S\) be a surface without boundary. Automorphisms of \(\mathcal{EC}^\dagger(S)\) preserve convergent sequences of convergent sequences of vertices of \(\mathcal{EC}^\dagger(S)\).

Convergence of a sequence to a curve is preserved. For the next corollary, we say that a vertex \(c\) is a *limit curve* for a sequence of vertices \((c_i)\) of \(\mathcal{EC}^\dagger(S)\) if
\[
\lim(c_i) \in c
\]
(here we regard the \(c_i\) as subsets of \(S\) as opposed to vertices of \(\mathcal{EC}^\dagger(S)\)). We have that \(c\) is a limit curve for \((c_i)\) if and only if the following condition holds: if \(a\) is any vertex of \(\mathcal{EC}^\dagger(S)\) that intersects the tail of \((c_i)\) then \(a\) intersects \(c\). In particular we have the following corollary of Lemma 4.2, which follows by an argument similar to the one used for Lemma 4.2.

**Corollary 4.5.** Let \(S\) be a surface without boundary. Automorphisms of \(\mathcal{EC}^\dagger(S)\) respect the relationship between convergent sequences and limit curves. More precisely, \(c\) is a limit curve for a sequence of vertices \((c_i)\) and \(\alpha\) is an element of \(\text{Aut} \mathcal{EC}^\dagger(S)\), then \(\alpha(c)\) is a limit curve for \((\alpha(c_i))\).

### 4.2. Finishing the proof.

We require one more lemma for the proof of Theorem 1.2.

**Lemma 4.6.** For any surface \(S\) without boundary, the natural map
\[
\nu : \text{Homeo}(S) \to \text{Aut} \mathcal{EC}^\dagger(S)
\]
is injective.

**Proof.** Suppose that \(f \in \text{Homeo}(S)\) lies in \(\ker \nu\), and let \(x \in S\). Let \(c\) and \(d\) be two vertices of \(\mathcal{EC}^\dagger(S)\) with \(c \cap d = \{x\}\) (regarding \(c\) and \(d\) as subsets of \(S\)). Since \(f(c) = c\) and \(f(d) = d\), it follows that \(f(x) = f(c \cap d) = c \cap d = x\). Since \(x\) was arbitrary, \(f\) is the identity, as desired. \(\square\)

**Proof of Theorem 1.2.** As in the statement of the theorem, let \(\nu : \text{Homeo}(S) \to \text{Aut} \mathcal{EC}^\dagger(S)\) be the natural map. As per the statement, we would like to show that \(\nu\) is an isomorphism. By Lemma 4.6, the map \(\nu\) is injective.

We wish to construct a left inverse \(\xi : \text{Aut} \mathcal{EC}^\dagger(S) \to \text{Homeo}(S)\) for \(\nu\), as this will imply that \(\nu\) is surjective. For \(\alpha\) an arbitrary element of \(\text{Aut} \mathcal{EC}^\dagger(S)\) let \(f_\alpha : S \to S\) be the map given by the following rule: for \(x \in S\) we choose a convergent sequence \((c_i)\) of vertices of \(\mathcal{EC}^\dagger(S)\) with \(\lim(c_i) = x\) and define
\[
f_\alpha(x) = \lim(\alpha(c_i)).
\]
The right hand side is well defined because \(\alpha\) preserves convergent sequences (Lemma 4.2). The function \(f_\alpha\) is well defined and bijective by Corollary 4.3.

Our next goal is to show that each such \(f_\alpha\) is a homeomorphism of \(S\). Since \(f_\alpha^{-1} = f_{\alpha^{-1}}\), it suffices to show that \(f_\alpha\) is continuous. And since surfaces are first countable, the continuity of \(f_\alpha\) can be verified by showing that it preserves limits of convergent sequences. But this is precisely the content of Corollary 4.4.

Now that we have shown that \(\xi : \text{Aut} \mathcal{EC}^\dagger(S) \to \text{Homeo}(S)\) is a well-defined homomorphism, it remains to show that \(\xi \circ \nu\) is the identity. To this end we require the following.
Claim 1. If \( \alpha \) is an element of \( \text{Aut} \mathcal{C}^\dagger(S) \) and \( c \) is a vertex of \( \mathcal{E}C^\dagger(S) \), then
\[
\xi(\alpha)(c) = \alpha(c).
\]

Claim 2. If \( f \) is an element of \( \text{Homeo}(S) \) and \( c \) is a vertex of \( \mathcal{E}C^\dagger(S) \), then
\[
\nu(f)(c) = f(c).
\]

The first claim follows from Corollary 4.5, and the second follows from the definition of the natural map \( \nu \).

We may now prove that \( \xi \circ \nu \) is the identity. Since \( \nu \) is injective, it suffices to show that \( \nu \circ \xi \circ \nu(f) = \nu(f) \) for each \( f \in \text{Homeo}(S) \). This is to say that \( \xi \circ \nu(f) \) and \( f \) have the same action on the set of vertices of \( \mathcal{E}C^\dagger(S) \). Let \( c \) be an arbitrary vertex of \( \mathcal{E}C^\dagger(S) \). Applying the two claims in the previous paragraph in succession we have
\[
\xi \circ \nu(f)(c) = \nu(f)(c) = f(c).
\]

This completes the proof of the theorem. \( \square \)

5. Automorphisms of the Fine Curve Graph

For the proof of Theorem 1.1, we require one additional lemma. The proof is the same as the proof of Lemma 4.6

\[\text{Lemma 5.1. For } g \geq 2, \text{ the natural map } \eta : \text{Homeo}(S_g) \to \text{Aut} C^\dagger(S_g) \text{ is injective.}\]

The proof of Theorem 1.1 also requires a definition. For a graph \( \Gamma \) and a subgraph \( \Delta \), we say that a map \( \text{Aut} \Delta \to \text{Aut} \Gamma \) is an extension map if each element of the image preserves \( \Delta \) and further that each element of \( \text{Aut} \Delta \) is equal to the restriction of its image.

\[\text{Proof of Theorem 1.1. The proof has two steps. The first step is to show that there exists an extension homomorphism } \varepsilon : \text{Aut} C^\dagger(S_g) \to \text{Aut} \mathcal{E}C^\dagger(S_g) \text{. The second step is to use } \varepsilon \text{ to complete the proof of the theorem.}\]

\[\text{Step 1. Let } \alpha \in \text{Aut} C^\dagger(S_g) \text{. We would like to define an element } \hat{\alpha} \in \text{Aut} \mathcal{E}C^\dagger(S_g) \text{. We will then define } \varepsilon(\alpha) \text{ to be } \hat{\alpha} \text{. For any essential simple closed curve } c \text{ in } S_g \text{ we define } \hat{\alpha}(c) \text{ to be } \alpha(c) \text{. For an inessential curve } e \text{ in } S_g \text{, we take any bigon pair } \{c, d\} \text{ determining } e \text{ and define } \hat{\alpha}(c) \text{ to be the inessential curve determined by } \{\alpha(c), \alpha(d)\} \text{; this makes sense because of Proposition 2.1.}\]

\[\text{We would like to show that } \hat{\alpha} \text{ is a well defined bijection of the set of vertices of } \mathcal{E}C^\dagger(S_g) \text{. Suppose that } \{c', d'\} \text{ is another bigon pair that determines } e \text{. It follows from Corollary 3.3 that there is a sequence of bigon pairs}\]
\[
\{c, d\} = \{c_0, d_0\} = \cdots = \{c_n, d_n\} = \{c', d'\}
\]
where each pair \( \{c_i, d_i\}, \{c_{i+1}, d_{i+1}\} \) is a linked sharing pair for \( e \). It follows then from Proposition 2.2 that \( \{\alpha(c'), \alpha(d')\} \) also determines the curve \( e \), and so \( \hat{\alpha} \) is well defined.

\[\text{To complete the first step, we must show that } \hat{\alpha} \text{ is indeed an automorphism of } \mathcal{C}^\dagger(S_g) \text{, that is, it takes edges to edges. For an edge spanned by two essential curves, this is automatic from the definition. For an edge spanned by one essential curve } c \text{ and one inessential curve } e \text{, this follows from the fact that we can find a bigon pair that determines } e \text{ and is disjoint from } c \text{. It remains to deal with the case of two inessential curves.}\]

\[\text{We claim that two inessential curves } e \text{ and } f \text{ are disjoint if and only if the following holds, up to relabeling } e \text{ and } f \text{: for every bigon pair } \{c, d\} \text{ that determines } e \text{, there is a bigon pair } \{c', d'\} \text{ that determines } f \text{ and is disjoint from } c \text{. The forward direction of the claim is}\]
proved by direct construction; see Figure 5. In the case where \( e \) and \( f \) are nested, we must take \( e \) to be the outer curve. For the reverse direction, we assume that \( e \) and \( f \) intersect, and that \( x \in e \cap f \). We may choose a bigon pair \( \{c,d\} \) for \( e \) where \( x \) is one of the vertices of the bigon. If \( \{c',d'\} \) is any bigon pair for \( f \), then \( c' \cup d' \) contains \( f \), hence \( x \). In particular, \( c' \cup d' \) intersects \( c \), which completes the proof of the claim. It follows from the claim that \( \hat{\alpha} \) preserves edges between inessential curves, and so \( \hat{\alpha} \) is indeed an automorphism of \( C_{\dagger}(S_g) \).

By definition the map \( \varepsilon : \text{Aut}C_{\dagger}(S_g) \to \text{Aut}\mathcal{E}C_{\dagger}(S_g) \) given by \( \varepsilon(\alpha) = \hat{\alpha} \) is the desired extension map.

**Step 2.** Recall that \( \eta : \text{Homeo}(S_g) \to \text{Aut}C_{\dagger}(S_g) \) and \( \nu : \text{Homeo}(S_g) \to \text{Aut}\mathcal{E}C_{\dagger}(S_g) \) are the natural homomorphisms. By Theorem 1.2, the map \( \nu \) is an isomorphism. Let \( \varepsilon \) be the extension homomorphism guaranteed by the first step. We consider the composition:

\[
\text{Homeo}(S_g) \xrightarrow{\eta} \text{Aut}C_{\dagger}(S_g) \xrightarrow{\varepsilon} \text{Aut}\mathcal{E}C_{\dagger}(S_g) \xrightarrow{\nu^{-1}} \text{Homeo}(S_g).
\]

We claim that this composition is the identity. Indeed, since \( \nu \) is the natural map, and since \( \varepsilon \) is an extension homomorphism, it follows that for any \( f \) we have

\[
\eta \circ \nu^{-1} \circ \varepsilon \circ \eta(f) = \eta(f).
\]

Since \( \eta \) is injective (Lemma 5.1) it follows that

\[
\nu^{-1} \circ \varepsilon \circ \eta(f) = f,
\]

which is to say that \( \nu^{-1} \circ \varepsilon \) is a left inverse to \( \eta \). The theorem follows. \( \square \)

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