Direct measurement of a bulk topological observable in topological phase of matter has been a long-standing issue. Recently, detection of bulk topology through quench dynamics has attracted growing interests. Here we propose that topological characters of a quantum quadrupole insulator can be read out by quench dynamics. Specifically, we introduce a quantity, a quadrupole moment weighted by the eigenvalues of the chiral operator, which takes zero for the trivial phase and finite for the quadrupolar topological phase. By utilizing an efficient numerical method to track the unitary time evolution, we elucidate that the quantity we propose indeed serves as an indicator of topological character for both non-interacting and interacting cases. The robustness against disorders is also demonstrated.

**Introduction.**— Understanding topological aspects of quantum matters has been one of the central issues in modern condensed matter physics [1, 2]. Discovery of topological insulators (TIs) [3–7] is highlighted as one of the most prominent steps that makes the roles of topology manifest. Specifically, it was found that topological natures of Bloch electrons characterized by topological invariants result in boundary modes robust against perturbations [8]. This relation between bulk topology and boundary modes is called bulk-boundary correspondence (BBC), and it has served as a central notion in studies on topological materials [9, 10].

BBC also ties topologically-protected boundary modes with quantized responses to external fields, which is another characteristics of TIs. A representative example is the quantum Hall effect where the number of edge modes corresponds to the Hall conductance [8–10]. Another example is the quantization of an electric dipole moment attributed to the quantized Berry’s phase of Bloch electrons in one dimension [11–20]. From the viewpoint of BBC, this is attributed to the boundary states localized at the ends of one-dimensional systems. Recently, this topological viewpoint of an electric dipole is further extended [21, 22] to higher-rank multipole moments [23–27] (e.g., quadrupole and octapole) in two or higher dimensions, that are attributed to the boundary states localized at the corners. Such a topological phase of matter hosting boundary modes with co-dimension greater than one is nowadays established as a higher-order topological phase, and large amount of theoretical [28–46] and experimental [47–60] efforts have been devoted to understanding and realizing this phase.

It had been a common belief that topological invariants themselves are not observed from featureless gapped ground states, but characteristic boundary modes enable us to observe them. However, recently, an approach to directly access bulk topological natures was proposed, that is, dynamical aspects of topological phases [61–75]. In the literature, the semi-classical approach of wave-packet dynamics has successfully illustrated the role of Berry curvature in transport phenomena [61, 62]. Another direction, which we focus on this Letter, is considering quench dynamics from completely localized initial states. At single-particle level, i.e., without the Fermi sea of the filled bands, the information of Bloch bands in an entire Brillouin zone can be embedded by setting a spatially localized initial state; this is attributed to the fact that the localized states can be expressed as a linear combination of all the states in the Fourier space. As such, topological invariants can be extracted from the time-dependent quantities. For instance, for chiral-symmetric one-dimensional systems, the quantity called the mean chiral displacement (MCD), which is the polarization weighted by the eigenvalue of the chiral operator, successfully extracts the topological winding number of the system, and thus distinguishes the topological states from the trivial states [67–69, 75]. Moreover, measurements of such topological invariants are experimentally feasible in various setups, e.g., discrete quantum walk in a photonic system based on the orbital angular momentum of a light beam [67].

Considering the findings listed above, one is naturally tempted to ask the following questions: (i) Can we apply the measurement of topological invariants through...
dynamics to higher-order topological (or quadrupolar) phases? (ii) If so, can it be also applicable to interacting systems and/or disordered ones? In this Letter, to address these issues, we investigate two-particle dynamics of the interacting Benalcazar-Bernevig-Hughes (BBH) model \([23, 24]\). We heuristically find a quantity whose long-time average can characterize the topology. This quantity is a modified bulk quadrupole moment, which is reminiscent of the MCD in one-dimensional systems. Therefore, this quantity is also experimentally measurable. By using a numerically efficient algorithm of tracing the unitary time evolution of the two-particle wave function (one may increase the number of particle in principle), we elucidate that the quantity introduced here characterizes the topological nature of the BBH model, for both non-interacting and interacting cases. Furthermore, the characteristic behavior of this quantity is robust against moderate strength of disorders, indicating the feasibility of experimental measurements in realistic setups that are not completely clean.

Hereafter, we set \(\hbar = 1\).

Model and method.— We consider the model proposed in Refs. \([23, 24]\), incorporating an interaction and a disorder. The Hamiltonian reads \(\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} + \hat{H}_{\text{rand}}\), where \(\hat{H}_0 = \sum_{(i,j)} t_{i,j} a_i^\dagger a_j + (\text{H.c.})\), \(\hat{H}_{\text{int}} = V \sum_i n_i a_i^\dagger a_i\), and \(\hat{H}_{\text{rand}} = \sum_i w_i n_i\). Here \(a\) and \(a^\dagger\) denotes, respectively, the annihilation and creation operators of spinless fermions, and \(i\) denotes the sites on a square lattice specified by a pair of indices \(r \equiv (r_x, r_y)\) and \(\alpha = 1, 2, 3, 4\), where \(r = (r_x, r_y)\) is the position of the unit cell, and \(\alpha\) labels the sublattice (Fig. 1). The symbol \(\langle \, , \rangle\) represents the nearest-neighbor pairs of sites. The transfer integral \(t_{i,j}\) is indicated in Fig. 1; there are two parameters, \(t_a\) and \(t_b\). In addition to \(\hat{H}_0\), we consider two additional terms \(\hat{H}_{\text{int}}\) and \(\hat{H}_{\text{rand}}\). Here, \(V\) denotes the strength of the intersite interaction, \(n_i := a_i^\dagger a_i\) is the density operator, and \(w_i\) is the strength of the disorder potential, chosen randomly in \([-W, W]\).

The topological properties of the hopping term \(\hat{H}_0\) has been well-investigated in the literatures. For \(|t_a| \neq |t_b|\), the system is gapped at the half-filling. The half-filled ground state is topologically trivial (nontrivial) when \(|t_a| > |t_b|\) \((|t_a| < |t_b|)\); the topological nature can be captured by topological invariants such as the nested Wilson loop \([23, 24]\), the quadrupole moment \([25–27]\), the entanglement-related quantities \([35, 46, 76]\), and the Berry phase \([44]\). Furthermore, nontrivial topology results in the emergence of the corner states, which is characteristics of the higher-order topological phases.

The quench dynamics of the system can be dictated by the unitary time evolution of the many-body wave function, \(\langle \Psi(t)\rangle = e^{-i\hat{H}t} \langle 0\rangle\). To obtain \(\langle \Psi(t)\rangle\) numerically, we approximate \(e^{-i\hat{H}t}\) as follows. First, we discretize the time as \(t_l = l\Delta\tau\), with \(\Delta\tau\) being small time step; we set \(\Delta\tau = 0.01\) in the present work. Then, we have \(e^{-i\hat{H}t} \sim (e^{-i\hat{H}\Delta\tau})^l\). The remaining task is to approximate \(e^{-i\hat{H}\Delta\tau}\). To this end, we employ the fourth-order Suzuki-Trotter decomposition \([77, 78]\), namely, \(e^{-i\hat{H}\Delta\tau} = S(-i\Delta\tau)S(-i(1 - 2p)\Delta\tau)S(-i p\Delta\tau)\), where \(p := (2 - 2^1/3)^{-1}\) and \(S(x) = e^{x H_1} \cdots e^{x H_{2p}} e^{x H_{2p+1}} \cdots e^{x H_p}\). Note that, in defining \(S(x)\), we divide the Hamiltonian \(\hat{H}\) into \(q\) pieces, \(\hat{H} = H_1 + \cdots + H_q\), which do not necessarily commute each other. Here we set \(q = 5\), and the explicit forms of \(H_1\)–\(H_5\) are presented in Supplemental Materials \([79]\). The Suzuki-Trotter decomposition of \(e^{-i\hat{H}t}\) largely reduces computational costs. Hence, we can access long-time dynamics with relatively large system size in short computational time, compared with other methods such as exact diagonalization.

Mean chiral quadrupole moment.— The main proposal of this Letter is the introduction of a quantity characterizing the topological nature of the quadrupolar phase, which may be termed the mean chiral quadrupole moment (MCQM): \(C_q(t) = \langle \Psi(t)\rangle \langle Q |\Psi(t)\rangle\),

\[
C_q(t) = \sum_{r,\alpha} r_x r_y \Gamma_\alpha n_{r,\alpha}.
\]
the initial state such that two particles are localized at two diagonal sites on the inter-unit-cell plaquette located at the middle of the system, as schematically depicted as green circles in Fig. 1.

How the MCQM extracts the topological nature of the quadrupole insulators? To see this, we rely on the argument of the decoupled four-site cluster limit. In the prior works [21, 42, 44, 80–82], it was found that this argument is essential for understanding the ground-state properties of the insulating state, since the ground state is adiabatically connected to this limit and topological properties of gapped ground states are unchanged under the change of model parameters unless the excitation gap is closed. Regarding the dynamical properties, for which the information of all the eigenstates matters, the notion of adiabatic connection does not work straightforwardly, but it still gives a useful insight, as we will see below. In fact, such an argument works in one-dimensional systems as well [79].

For the decoupled limits, the particles are confined in the plaquette on which the particle is initially located, thus unitary time evolution can exactly be tracked by solving the four-site problem. Thus, in these limits, the exact form of \( C_q \) is accessible. For the details of the calculations, see Supplemental Materials [79]. For the present choice of the initial state, we find that, for the trivial limit, i.e., \( t_b = 0 \) [Fig. 2(a)], one has

\[
\bar{C}_q(t) = 0,
\]

where \( \bar{C}_q \) stands for the long-time average of \( C_q(t) \). Meanwhile, for the nontrivial limit i.e., \( t_a = 0 \) [Fig. 2(b)], one has

\[
\bar{C}_q(t) = \frac{1}{2}.
\]

Equation (4) that non-vanishing value of \( \bar{C}_q \) under the proper choice of the initial state reflects the presence of the nontrivial topology in the bulk.

We briefly remark the role of \( \Gamma_\alpha \). In fact, the similar factor is included in the MCD for one-dimensional systems [68, 69]. In that case, its role is to make the contributions from the negative-energy bands and those from the positive-energy bands additive; otherwise they cancel each other. This fact also implies that the MCD is adaptable to chiral symmetric systems with an even number of topological bands. In fact, \( \Gamma_\alpha \) in the MCQM is incorporated in the same spirit, but in a rather heuristic manner. Nevertheless, it is indeed essential so that the MCQM serves as a topological observable, as clarified in the decoupled cluster argument [79]. Moreover, the topological characterization is valid even in the presence of the chiral-symmetry-breaking term, \( H_{\text{rand}} \), as we will show later.

\textbf{Numerical demonstration.}— We now demonstrate the validity of the MCQM for topological characterization. In Figs. 3(a) and 3(b), we plot \( C_q(t) \) to \( t = 50 \) for topological and trivial cases respectively, for the clean systems (i.e., \( W = 0 \)). Here the numerical computations are carried out for 20 × 20-site (i.e., 10 × 10-unit cell) systems under the open boundary condition. At the initial state, two particles are localized at the sublattice 1 at the unit cell (0, 0) and the sublattice 3 at the unit cell (−1, −1). We see in these figures that, for the topological case with weak interaction (\( V = 0.3 \)), \( C_q(t) \) oscillates around 1/2 as expected, whereas it oscillates around 0 for the trivial case. Therefore, the long-time average of \( \bar{C}_q(t) \) indeed can be used to extract the topological character of this model.

We also remark the boundary effects. In fact, the particles are initially located near the center of the system, and they reach the boundary at \( t \sim 20 \). Although the amplitude of the oscillation of the MCQM increases after reaching the boundary, the center of the oscillation is still unchanged, manifesting the robustness of \( \bar{C}_q \) against the boundary effects. Also, the fact that the finite value of MCQM in the topological case is obtained before reaching the boundary indicates that the finite MCQM is not attributed to the corner states, and thus this is indeed the bulk property.

It can also be found in Fig. 3(a) that the role of the interaction becomes manifest even for moderate strength of the interaction (\( V = 1.0 \)). In the topological case, the MCQM deviates from the non-interacting case, namely, the MCQM exhibits gradual increase (decrease) to \( t \lesssim 30 \).
FIG. 4. The MCQM for the disordered system with (a) $t_a = -0.3$, $t_b = -1.0$, $W = 0.2$ and (b) $t_a = -0.3$, $t_b = -1.0$, $W = 1.0$. The error bars are represented by the shades.

$(t \gtrsim 30)$. This value of $V$ is smaller than the band gap at the half-filling. This result indicates the essential difference between the dynamical properties and the ground-state properties at the half-filling, because the latter is stable against interactions as far as the excitation gap is not closed [44]. We also note that, in the trivial case, the MCQM seems to be rather insensitive to the interaction strength.

It should also be noted that the “topological transition” of the dynamical properties is not as sharp as that for the ground state. To show this, we plot $\mathcal{C}_q(t)$ as a function of $|t_a|/|t_b|$ in Fig. 3(c). We see that the dependence on $|t_a|/|t_b|$ of $\mathcal{C}_q$ is smooth, rather than a steep jump; this is another indication of the difference between the dynamical properties and the ground-state properties, as the latter is characterized by the jump of the quantized topological number.

Robustness against disorders.— We further study the effects of disorder potentials, to test the robustness of the MCQM. In Fig. 4, we plot $\mathcal{C}_q(t)$ for the topological case with weak ($W = 0.2$) and moderate ($W = 1.0$) disorders. Here the average is taken over 432 configurations of the random disorder potential. We see that $\mathcal{C}_q(t)$ remains to oscillate around 1/2 for a weak disorder case. In particular, for $t \lesssim 20$ where the particles do not reach the boundary, the error bars due to the disorder average are very small. Even for the moderate disorder case, the short-time behavior (i.e., $t \lesssim 5$) is almost unaffected by the disorders. However, after the long time ($t \gtrsim 20$), the $\mathcal{C}_q(t)$ starts to decrease gradually and deviates from 1/2. These results manifest the robustness of the MCQM against disorders, which indicates that this is a measurable quantity in experiments for moderately clean samples.

Summary.— We have proposed how to extract the topological character of the quadrupolar phase by the quench dynamics. The key quantity to measure is the MCQM, the quadrupole moment weighted by the eigenvalue of the chiral operator. Although the initial state in the present setup is localized, the system is translationally invariant without any boundaries. The results of our numerical calculations on the two-particle BBH model indicate that the MCQM indeed captures the topological nature for weakly-interacting and moderately clean systems. It has also been clarified that there are essential differences from the ground-state topological properties at the half-filling and the MCQM, with respect to the stability against the interactions and the sharpness of the topological transition. This is attributed to the fact that the former is protected by the finite excitation gap, whereas the information of all the eigenvalues and eigenvectors affects the latter.

It is worth pointing out that our method of tracking two-particle dynamics is also applicable to bosonic systems. We find that the results are qualitatively the same as those for fermions [79]. Furthermore, it is also straightforward to use this quantity to extract topology of the single-particle dynamics. Considering this fact, the present method can cover a wide range of experimental setups, including fermionic and bosonic ultracold atoms under the optical lattice, photonic crystals, and discrete quantum walks. We hope our proposal opens up a way to understanding novel aspects of the quadrupolar phase.

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* mizoguchi@rhodia.ph.tsukuba.ac.jp

[1] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982).
[2] F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (1988).
[3] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005).
[4] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005).
[5] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, 314, 1757 (2006).
[6] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
[7] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
[8] B. I. Halperin, Phys. Rev. B 25, 2185 (1982).
[9] Y. Hatsugai, Phys. Rev. Lett. 71, 3697 (1993).
[68] E. J. Meier, F. A. An, A. Dauphin, M. Maffei, P. Massignan, T. L. Hughes, and B. Gadway, (2018), 10.1126/science.aat3406.

[69] M. Maffei, A. Dauphin, F. Cardano, M. Lewenstein, and P. Massignan, New Journal of Physics 20, 013023 (2018).

[70] C. Yang, L. Li, and S. Chen, Phys. Rev. B 97, 060304 (2018).

[71] Z. Gong and M. Ueda, Phys. Rev. Lett. 121, 250601 (2018).

[72] L. Zhang, L. Zhang, and X.-J. Liu, Phys. Rev. A 99, 053606 (2019).

[73] H. Hu and E. Zhao, Phys. Rev. Lett. 124, 160402 (2020).

[74] S. Longhi, Opt. Lett. 44, 2530 (2019).

[75] A. Halle, P. Massignan, and M. Rizzi, arXiv:2001.09074 [quant-gas].

[76] Q. Wang, D. Wang, and Q.-H. Wang, EPL (Europhysics Letters) 124, 50005 (2018).

[77] Y. Hatsugai and A. Sugi, International Journal of Modern Physics B 15, 2045 (2001).

[78] M. Suzuki, Physics Letters A 146, 319 (1990).

[79] See Supplemental Material.

[80] Y. Hatsugai, Journal of the Physical Society of Japan 75, 123601 (2006).

[81] Y. Hatsugai, Journal of Physics: Condensed Matter 19, 145209 (2007).

[82] T. Mizoguchi, H. Araki, and Y. Hatsugai, Journal of the Physical Society of Japan 88, 104703 (2019).
Supplemental Material for “Detecting Bulk Topology of Quadrupolar Phase from Quench Dynamics”

Tomonari Mizoguchi, Yoshihito Kuno, and Yasuhiro Hatsugai
Department of Physics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan
S1. DIVISION OF THE HAMILTONIAN

To implement the Suzuki-Trotter decomposition described in the main text, we decompose the Hamiltonian into five parts:

\[ H = H^{(1)} + H^{(2)} + H^{(3)} + H^{(4)} + H^{(5)}, \]  

(S1)

Note that \( H^{(m)} \)'s do not necessarily commute each other. Each portion consists of the sum of the local Hamiltonian:

\[ H^{(m)} = \sum_i h^{(m)}_i, \]  

(S2)

where \( h^{(m)}_i \)'s with the same superscript \( (m) \) commute each other, i.e., \([h^{(m)}_i, h^{(m)}_j] = 0 \) for \( i \neq j \).

In the present work, we use the following decomposition:

\[ H^{(1)} = t_a \sum_r \left( a^\dagger_{r,1} a_{r,2} + a^\dagger_{r,4} a_{r,3} \right) + \text{(H.c.)}, \]  

(S3a)

\[ H^{(2)} = t_b \sum_r \left( a^\dagger_{r,2} a_{r+e_x,1} + a^\dagger_{r,3} a_{r+e_y,4} \right) + \text{(H.c.)}, \]  

(S3b)

\[ H^{(3)} = t_a \sum_r \left( a^\dagger_{r,1} a_{r,4} - a^\dagger_{r,2} a_{r,3} \right) + \text{(H.c.)}, \]  

(S3c)

\[ H^{(4)} = t_b \sum_r \left( a^\dagger_{r,4} a_{r+e_y,1} - a^\dagger_{r,3} a_{r+e_x,2} \right) + \text{(H.c.)}, \]  

(S3d)

and

\[ H^{(5)} = H_{\text{int}} + H_{\text{rand}}. \]  

(S3e)

Note that \( e_x \) and \( e_y \) represent unit vectors in \( x \) and \( y \) directions, respectively, where the length of the edge of the unit cell is set to be unity.

S2. MCQM IN THE DECOUPLED CLUSTER LIMIT

In this section, we derive the exact form of the MCQM the decoupled cluster limit for non-interacting case.
A. Warm-up: MCD in the SSH model

Before discussing the MCQM in the BBH model, we first discuss the mean chiral displacement (MCD)\textsuperscript{1,2} in the Su-Schrieffer-Heeger (SSH) model\textsuperscript{3} in the decoupled dimer limit, in order to grasp how the decoupled cluster argument works. Through this argument, the connection between the MCD and topology is clarified without relying on the momentum-space picture.

The Hamiltonian of the SSH model reads

\[ H = t_a \sum_r c_r^\dagger_{r,A} c_{r,B} + \text{(H.c.)} + t_b \sum_r c_r^\dagger_{r,B} c_{r+1,A} + \text{(H.c.)}, \tag{S4} \]

where \( r \) denotes the position of the unit cell and A and B label the sublattices. The schematic figure of the model is shown in Fig. S1(a). When |\( t_a | > |t_b | \) (|\( t_a | < |t_b | \)), the half-filled ground state is topologically trivial (non-trivial). This can be evidenced by calculating the topological invariant, i.e., the winding number, which takes 0 for the trivial case and 1 for the topological case.

Let us consider two different limits, i.e., trivial [Fig. S1(b)] and non-trivial [Fig. S1(c)] ones. In the following, we assume that the particle is initially localized at A sublattice in the unit cell \( r \). However, the essence of the following argument holds even if the initial position is B sublattice. Throughout the unitary time evolution, the particle is completely confined in the dimer to which the particle initially belongs; it is the red bond in the unit cell \( r \) for the trivial case, and the blue bond between the unit cell \( r \) and \( r - 1 \) for the non-trivial case. By solving a two-site problem, one finds \( n_{r,A}(t) = \cos^2 t_a t \) and \( n_{r-1,B}(t) = \sin^2 t_b t \) for the trivial case, whereas \( n_{r,A}(t) = \cos^2 t_b t \) and \( n_{r,B}(t) = \sin^2 t_a t \) for the topological case. Then, the MCD is obtained as \( C_{\text{triv}}(t) = r \cos^2 t_a t \) while \( C_{\text{topo}}(t) = r \cos^2 t_b t + \sin^2 t_b t \). Clearly, the long-time average of \( C_{\text{triv}}(t) \) is zero, while that of \( C_{\text{topo}}(t) \) is \( \frac{1}{2} \), meaning that the MCD obtained by the momentum-space argument (i.e., the MCD equals to the half of the winding number) is successfully reproduced.

From the decoupled dimer argument, we obtain the intuitive understanding of the MCD, namely, the strong inter-unit-cell bond is vital to obtain the finite value. This intuition is useful for the MCQM in the BBH model, as we will elucidate below.

B. Four-site problem

FIG. S2. Schematic figure of the four-site plaquette.

Let us now move on to the BBH model. To calculate the MCQM for the decoupled cases, we first solve the four-site problem (Fig. S2):

\[ H_{4\text{-site}} = a^\dagger \mathcal{H} a, \tag{S5} \]

where \( a = (a_{(i)}, a_{(ii)}, a_{(iii)}, a_{(iv)})^T \) and

\[ \mathcal{H} = \begin{pmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}. \tag{S6} \]

Note that we set the hopping of the solid (dashed) bonds in Fig. S2 as \(-1 \) (+1) for simplicity.
The eigenstate of $H_{4\text{-site}}$ is written as
\[ \gamma^\dagger \xi = a^\dagger \cdot \phi \xi, \] (S7)
with $\xi = 1, 2, 3, 4$, which satisfies $[\gamma^\dagger \xi, H_{4\text{-site}}] = \varepsilon \gamma^\dagger \xi$. The eigenenergy $\varepsilon \xi$ is $-\sqrt{2}$ and $+\sqrt{2}$, for $\xi = 1, 2$ and $\xi = 3, 4$, respectively. The wave functions $\phi \xi$ are
\[\begin{align*}
\phi_1 &= \left( \frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}, 0 \right)^T, \\
\phi_2 &= \left( \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{\sqrt{2}} \right)^T, \\
\phi_3 &= \left( \frac{1}{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}, 0 \right)^T, \\
\phi_4 &= \left( \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{\sqrt{2}} \right)^T.
\end{align*}\] (S8a) (S8b) (S8c) (S8d)

C. Trivial case

For the trivial case [Fig. 2(a) in the main text], two particles are placed on different plaquettes from each other; one is at the sublattice 3 in the unit cell $(r_x, r_y)$, and the other is at the sublattice 1 in $(r_x + 1, r_y + 1)$. Therefore, we can calculate the contributions from two particles separately, within the single-particle level.

To proceed, we first derive the explicit form of the time-dependent particle density of each site for a generic choice of the initial state. Let $|\psi(0)\rangle$ be the single-particle initial state:
\[|\psi(0)\rangle = a^\dagger \cdot \psi(0) |0\rangle,\] (S9)
with $|0\rangle$ being the vacuum state, and $\psi(0)$ being an generic initial state whose explicit form is
\[\psi(0) = (\psi_1, \psi_2, \psi_3, \psi_4)^T.\] (S10)

Note that $\psi(0)$ is normalized such that $\sum_{\alpha=1}^{4} |\psi_{\alpha}|^2 = 1$. Then, one can explicitly calculate the state at time $t$, $|\psi(t)\rangle$, as
\[|\psi(t)\rangle = e^{-iH_{4\text{-site}}t} |\psi(0)\rangle = \sum_{\xi=1}^{4} e^{-i\varepsilon_{\xi}t} [\phi_{\xi} \cdot \psi(0)] \gamma^\dagger_{\xi} |0\rangle.\] (S11)

Writing
\[|\psi(t)\rangle = a^\dagger \cdot \psi(t) |0\rangle,\] (S12)
and using Eqs. (S8a)-(S8d), one has
\[\psi(t) = \begin{pmatrix}
\psi_1 \cos \sqrt{2}t + \frac{i}{\sqrt{2}} \sin \sqrt{2}t (\psi_2 + \psi_4) \\
\psi_2 \cos \sqrt{2}t + \frac{i}{\sqrt{2}} \sin \sqrt{2}t (\psi_1 - \psi_3) \\
\psi_3 \cos \sqrt{2}t + \frac{i}{\sqrt{2}} \sin \sqrt{2}t (\psi_4 - \psi_2) \\
\psi_4 \cos \sqrt{2}t + \frac{i}{\sqrt{2}} \sin \sqrt{2}t (\psi_1 + \psi_3)
\end{pmatrix}.\] (S13)

It follows form Eq. (S13) that the particle density at each site is given as
\[n_{(i)}(t) = |\psi_{(i)}|^2 \cos^2 \sqrt{2}t + \frac{|\psi_2 + \psi_4|^2}{2} \sin^2 \sqrt{2}t + \frac{1}{\sqrt{2}} \sin 2\sqrt{2}t \Im \left[ \psi_1 (\psi_2^* + \psi_4^*) \right],\] (S14a)
\[ n_{(ii)}(t) = |\psi_2|^2 \cos^2 \sqrt{2}t + \frac{|\psi_1 - \psi_3|^2}{2} \sin^2 \sqrt{2}t \\
+ \frac{1}{\sqrt{2}} \sin 2\sqrt{2}t \text{Im} \left[ \psi_2(\psi^\dagger_1 - \psi^\dagger_3) \right], \quad (S14b) \]

\[ n_{(iii)}(t) = |\psi_3|^2 \cos^2 \sqrt{2}t + \frac{|\psi_1 - \psi_3|^2}{2} \sin^2 \sqrt{2}t \\
+ \frac{1}{\sqrt{2}} \sin 2\sqrt{2}t \text{Im} \left[ \psi_3(\psi^\dagger_1 - \psi^\dagger_3) \right], \quad (S14c) \]

\[ n_{(iv)}(t) = |\psi_4|^2 \cos^2 \sqrt{2}t + \frac{|\psi_1 + \psi_3|^2}{2} \sin^2 \sqrt{2}t \\
+ \frac{1}{\sqrt{2}} \sin 2\sqrt{2}t \text{Im} \left[ \psi_4(\psi^\dagger_1 + \psi^\dagger_3) \right], \quad (S14d) \]

Using (S14a)-(S14d), one can calculate the MCQM of each particle. For the particle at \((r_x, r_y)\), we replace the labels of sites in Fig. S2 in the following manner:

- (i) \(\rightarrow (r_x, r_y), 1\)
- (ii) \(\rightarrow (r_x, r_y), 2\)
- (iii) \(\rightarrow (r_x, r_y), 3\)
- (iv) \(\rightarrow (r_x, r_y), 4\)

The corresponding initial state of this plaquette is \(\psi(0) = (0, 0, 1, 0)\), thus the MCQM is given as

\[ C_{q}^{\text{triv}(1)}(t) = r_x r_y (n_{(i)}(t) - n_{(ii)}(t)) + n_{(iii)}(t) - n_{(iv)}(t)) \]
\[ = r_x r_y \cos 2\sqrt{2}t. \quad (S15) \]

The same calculation can be performed for the particle at \((r_x + 1, r_y + 1)\), where the corresponding initial state is \(\psi(0) = (1, 0, 0, 0)\), and we have

\[ C_{q}^{\text{triv}(2)}(t) = (r_x + 1)(r_y + 1) (n_{(i)}(t) - n_{(ii)}(t)) + n_{(iii)}(t) - n_{(iv)}(t) \]
\[ = (r_x + 1)(r_y + 1) \cos 2\sqrt{2}t. \quad (S16) \]

Clearly, its long-time average of \(C_{q}^{\text{triv}(1)}(t) + C_{q}^{\text{triv}(2)}(t)\) is zero, since both of these are proportional to \(\cos 2\sqrt{2}t\).

### D. Topological case

For the topological case [Fig. 2(b) in the main text], the plaquette traverses the unit cells. The correspondence between Fig. S2 and Fig. 2(b) in the main text is as follows:

- (i) \(\rightarrow (r_x + 1, r_y + 1), 1\)
- (ii) \(\rightarrow (r_x, r_y + 1), 2\)
- (iii) \(\rightarrow (r_x, r_y), 3\)
- (iv) \(\rightarrow (r_x + 1, r_y), 4\)

In this case, two particles are confined in the same cluster, so we need to investigate the two-particle dynamics explicitly. In the absence of the interaction, the two-particle eigenstates can simply be written as \(|(\xi, \xi')\rangle \equiv \gamma_{\xi}^\dagger \gamma_{\xi'} |0\rangle\) with \(\xi \prec \xi'\), whose eigenenergy is \(\varepsilon_\xi + \varepsilon_\xi'\). Writing the local basis as \(|\Phi_1\rangle := a_{(i)}^\dagger a_{(ii)}^\dagger |0\rangle, |\Phi_2\rangle := a_{(i)}^\dagger a_{(iii)}^\dagger |0\rangle, |\Phi_3\rangle := a_{(i)}^\dagger a_{(iv)}^\dagger |0\rangle, |\Phi_4\rangle := a_{(ii)}^\dagger a_{(iii)}^\dagger |0\rangle, |\Phi_5\rangle := a_{(ii)}^\dagger a_{(iv)}^\dagger |0\rangle, \) and \(|\Phi_6\rangle := a_{(iii)}^\dagger a_{(iv)}^\dagger |0\rangle\), the explicit forms of the eigenstates are

\[ |(1, 2)\rangle = \frac{1}{\sqrt{2}} \left( |\Phi_1\rangle - |\Phi_3\rangle - |\Phi_4\rangle + |\Phi_6\rangle \right) - \frac{1}{\sqrt{2}} \left( |\Phi_2\rangle + |\Phi_5\rangle \right), \quad (S17a) \]
\[ |(1, 3)\rangle = \frac{1}{\sqrt{2}} (|\Phi_1\rangle + |\Phi_4\rangle), \]  
(S17b)

\[ |(1, 4)\rangle = \frac{1}{2\sqrt{2}} \left( |\Phi_1\rangle + |\Phi_3\rangle - |\Phi_4\rangle - |\Phi_6\rangle \right) - \frac{1}{2} \left( |\Phi_2\rangle - |\Phi_5\rangle \right), \]  
(S17c)

\[ |(2, 3)\rangle = \frac{1}{2\sqrt{2}} \left( -|\Phi_1\rangle - |\Phi_3\rangle + |\Phi_4\rangle + |\Phi_6\rangle \right) - \frac{1}{2} \left( |\Phi_2\rangle - |\Phi_5\rangle \right), \]  
(S17d)

\[ |(2, 4)\rangle = -\frac{1}{\sqrt{2}} (|\Phi_3\rangle + |\Phi_6\rangle), \]  
(S17e)

and

\[ |(3, 4)\rangle = \frac{1}{2\sqrt{2}} \left( |\Phi_1\rangle - |\Phi_3\rangle - |\Phi_4\rangle + |\Phi_6\rangle \right) + \frac{1}{2} \left( |\Phi_2\rangle + |\Phi_5\rangle \right). \]  
(S17f)

Using these, we can write down the two-particle wave function at time \( t \):

\[ |\Psi(t)\rangle = \sum_{\xi < \xi'} e^{-i(\epsilon_\xi + \epsilon_{\xi'})t} [\langle \xi, \xi'|\Psi(0)\rangle] |\xi, \xi'\rangle, \]  
(S18)

with \( |\Psi(0)\rangle \) being the initial state. Setting \( |\Psi(0)\rangle = |\Phi_2\rangle \) [corresponding to Fig. 2(b) in the main text] and using the site labeling listed in the beginning of this subsection, we obtain the MCQM:

\[ C_q(t) = \langle \Psi(t)|Q|\Psi(t)\rangle = \frac{1}{2} + \left( \frac{1}{2} + r_x + r_y + 2r_xr_y \right) \cos \sqrt{2}t. \]  
(S19)

Taking the long-time average, the second term of Eq. (S19) is vanishing and we obtain \( \bar{C}_q(t) = \frac{1}{2} \).

S3. BOSONIC SYSTEMS

In this section, we show the results for the bosonic systems. For bosons, the on-site interaction of the form,

\[ H_{\text{on-site}} = U \sum_i n_i(n_i - 1), \]  
(S20)

is allowed in addition to the terms of Eqs. (1)-(3) in the main text. Therefore we also consider \( H_{\text{on-site}} \) in this section.

A. Numerical results

In Fig. S3, we plot \( C_q(t) \) for the clean system. We see its behavior is very similar to the fermionic systems; it oscillates around \( 1/2 \) (0) for the topological (trivial) case. One also see that this behavior is unchanged even if we incorporate small \( U \).

In Fig. S4, we plot \( C_q(t) \) for the disordered system. The number of the disorder-potential configurations used to take the average is the same as that for the main text. We again see the qualitatively the same behavior as the fermionic case discussed in the main text.
B. Decoupled cluster argument for bosons

To account for the numerical results shown in the previous subsection, we again analyze the decoupled cluster limit without interactions. Here we focus on the topological case, since the trivial case can be accounted for the single-particle picture regardless of the particle statistics. We note that, even at the non-interacting case, the MCQM for bosons is in general different from that for fermions, because of the difference of two-particle Hilbert space.

For the four-site problem, the two particle states of bosons can be spanned by the following orthonormalized states: $|\Phi^{(B)}_1\rangle := \frac{1}{\sqrt{2}} a_{(i)}^\dagger a_{(i)}^\dagger |0\rangle$, $|\Phi^{(B)}_2\rangle := a_{(i)}^\dagger a_{(i)}^\dagger |0\rangle$, $|\Phi^{(B)}_3\rangle := a_{(i)}^\dagger a_{(i)}^\dagger |0\rangle$, $|\Phi^{(B)}_4\rangle := a_{(i)}^\dagger a_{(iv)}^\dagger |0\rangle$, $|\Phi^{(B)}_5\rangle := \frac{1}{\sqrt{2}} a_{(i)}^\dagger a_{(i)}^\dagger |0\rangle$, $|\Phi^{(B)}_6\rangle := a_{(i)}^\dagger a_{(i)}^\dagger |0\rangle$, $|\Phi^{(B)}_7\rangle := a_{(i)}^\dagger a_{(iv)}^\dagger |0\rangle$, $|\Phi^{(B)}_8\rangle := \frac{1}{\sqrt{2}} a_{(i)}^\dagger a_{(i)}^\dagger |0\rangle$, and $|\Phi^{(B)}_9\rangle := \frac{1}{\sqrt{2}} a_{(i)}^\dagger a_{(i)}^\dagger |0\rangle$. The eigenstates can be obtained as $|\xi_\xi\rangle := \mathcal{N}_{\xi_\xi} |\xi_\xi\rangle |0\rangle$, where $\xi_\xi \leq \xi_\xi'$ and $\mathcal{N}_{\xi_\xi} = \frac{1}{\sqrt{2}}$ for $\xi = \xi'$ and $\mathcal{N}_{\xi_\xi} = 1$ for $\xi \neq \xi'$. Clearly, the eigenenergy of $|\xi_\xi\rangle$ is $\varepsilon_\xi + \varepsilon_\xi'$. For concreteness, we write down all the eigenstates using the basis $|\Phi^{(B)}\rangle$:

\begin{align}
|\langle 1, 1 |\rangle &= \frac{1}{4} \left( |\Phi^{(B)}_1\rangle + |\Phi^{(B)}_8\rangle \right) + \frac{1}{2} \left( |\Phi^{(B)}_2\rangle + |\Phi^{(B)}_5\rangle - |\Phi^{(B)}_6\rangle \right) - \frac{1}{2\sqrt{2}} |\Phi^{(B)}_3\rangle, \\
|\langle 1, 2 |\rangle &= -\frac{1}{2} |\Phi^{(B)}_7\rangle - \frac{1}{2\sqrt{2}} \left( |\Phi^{(B)}_1\rangle + |\Phi^{(B)}_2\rangle + |\Phi^{(B)}_4\rangle + |\Phi^{(B)}_5\rangle - |\Phi^{(B)}_6\rangle - |\Phi^{(B)}_9\rangle \right), \\
|\langle 1, 3 |\rangle &= \frac{1}{2} |\Phi^{(B)}_3\rangle - \frac{1}{2\sqrt{2}} \left( |\Phi^{(B)}_1\rangle + |\Phi^{(B)}_4\rangle \right) + \frac{1}{\sqrt{2}} |\Phi^{(B)}_5\rangle, \\
|\langle 1, 4 |\rangle &= \frac{1}{2} |\Phi^{(B)}_7\rangle - \frac{1}{2\sqrt{2}} \left( |\Phi^{(B)}_1\rangle + |\Phi^{(B)}_2\rangle - |\Phi^{(B)}_4\rangle + |\Phi^{(B)}_5\rangle - |\Phi^{(B)}_6\rangle - |\Phi^{(B)}_9\rangle \right), \\
|\langle 2, 2 |\rangle &= \frac{1}{4} \left( |\Phi^{(B)}_1\rangle + |\Phi^{(B)}_8\rangle \right) + \frac{1}{2} \left( |\Phi^{(B)}_4\rangle + |\Phi^{(B)}_5\rangle + |\Phi^{(B)}_10\rangle \right) + \frac{1}{2\sqrt{2}} |\Phi^{(B)}_3\rangle, \\
|\langle 2, 3 |\rangle &= -\frac{1}{2} |\Phi^{(B)}_7\rangle + \frac{1}{2\sqrt{2}} \left( |\Phi^{(B)}_1\rangle - |\Phi^{(B)}_2\rangle + |\Phi^{(B)}_4\rangle - |\Phi^{(B)}_5\rangle - |\Phi^{(B)}_6\rangle - |\Phi^{(B)}_9\rangle \right), \\
|\langle 2, 4 |\rangle &= \frac{1}{2} |\Phi^{(B)}_3\rangle + \frac{1}{2\sqrt{2}} \left( |\Phi^{(B)}_1\rangle + |\Phi^{(B)}_4\rangle \right) - \frac{1}{\sqrt{2}} |\Phi^{(B)}_10\rangle, \\
|\langle 3, 3 |\rangle &= \frac{1}{4} \left( |\Phi^{(B)}_1\rangle + |\Phi^{(B)}_8\rangle \right) - \frac{1}{2} \left( |\Phi^{(B)}_2\rangle - |\Phi^{(B)}_5\rangle - |\Phi^{(B)}_6\rangle \right) - \frac{1}{2\sqrt{2}} |\Phi^{(B)}_3\rangle, \\
|\langle 3, 4 |\rangle &= \frac{1}{2} |\Phi^{(B)}_7\rangle + \frac{1}{2\sqrt{2}} \left( |\Phi^{(B)}_1\rangle - |\Phi^{(B)}_2\rangle - |\Phi^{(B)}_4\rangle - |\Phi^{(B)}_5\rangle + |\Phi^{(B)}_6\rangle + |\Phi^{(B)}_9\rangle \right), \\
|\langle 4, 4 |\rangle &= \frac{1}{2} |\Phi^{(B)}_3\rangle + \frac{1}{2\sqrt{2}} \left( |\Phi^{(B)}_1\rangle - |\Phi^{(B)}_2\rangle - |\Phi^{(B)}_4\rangle - |\Phi^{(B)}_5\rangle + |\Phi^{(B)}_6\rangle + |\Phi^{(B)}_9\rangle \right), \quad \text{(S21i)}
\end{align}
and
\begin{equation}
|\langle 4, 4 \rangle\rangle = \frac{1}{4} \left( |\Phi_1^{(B)}\rangle + |\Phi_8^{(B)}\rangle \right) - \frac{1}{2} \left( |\Phi_4^{(B)}\rangle + |\Phi_9^{(B)}\rangle - |\Phi_{10}^{(B)}\rangle \right) + \frac{1}{2\sqrt{2}} |\Phi_3^{(B)}\rangle. \tag{S21j}
\end{equation}

Using these eigenstates, we can obtain the exact forms of $|\Psi(t)\rangle$ and the MCQM. For the present choice of the initial state $|\Psi(0)\rangle = |\Phi_3^{(B)}\rangle$, we have
\begin{equation}
C_q = \frac{1}{2} + \left( \frac{1}{2} + r_x + r_y + 2r_xr_y \right) \cos 2\sqrt{2}t, \tag{S22}
\end{equation}
whose long-time average is $\frac{1}{4}$. This is exactly the same expression we have for the fermionic case [Eq. (S19)], and also is consistent with the numerical result of Fig. S3(a).

1 F. Cardano, A. D’Errico, A. Dauphin, M. Maffei, B. Piccirillo, C. de Lisio, G. De Filippis, V. Cataudella, E. Santamato, L. Marrucci, M. Lewenstein, and P. Massignan, Nature Communications 8, 15516 (2017).
2 M. Maffei, A. Dauphin, F. Cardano, M. Lewenstein, and P. Massignan, New Journal of Physics 20, 013023 (2018).
3 W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. 42, 1698 (1979).