ON THE GROUP OF AUTOMORPHISMS OF THE SEMIGROUP $B_2^\mathcal{F}$ WITH THE FAMILY $\mathcal{F}$ OF INDUCTIVE NONEMPTY SUBSETS OF $\omega$

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ABSTRACT. We study automorphisms of the semigroup $B_2^\mathcal{F}$ with the family $\mathcal{F}$ of inductive nonempty subsets of $\omega$ and prove that the group $\text{Aut}(B_2^\mathcal{F})$ of automorphisms of the semigroup $B_2^\mathcal{F}$ is isomorphic to the additive group of integers.

1. Introduction, motivation and main definitions

We shall follow the terminology of [1,2,11,12,15]. By $\omega$ we denote the set of all non-negative integers and by $\mathbb{Z}$ the set of all integers.

Let $\mathcal{P}(\omega)$ be the family of all subsets of $\omega$. For any $F \in \mathcal{P}(\omega)$ and $n,m \in \omega$ we put $n - m + F = \{n - m + k : k \in F\}$ if $F \neq \emptyset$ and $n - m + \emptyset = \emptyset$. A subfamily $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called $\omega$-closed if $F_1 \cap (n + F_2) \in \mathcal{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathcal{F}$. For any $a \in \omega$ we denote $[a] = \{x \in \omega : x \geq a\}$.

A subset $A$ of $\omega$ is said to be inductive, if $i \in A$ implies $i + 1 \in A$. Obviously, that $\emptyset$ is an inductive subset of $\omega$.

Remark 1.1 ([8]). (1) By Lemma 6 from [7] nonempty subset $F \subseteq \omega$ is inductive in $\omega$ if and only if $(-1 + F) \cap F = F$.

(2) Since the set $\omega$ with the usual order is well-ordered, for any nonempty inductive subset $F$ in $\omega$ there exists nonnegative integer $n_F \in \omega$ such that $[n_F] = F$.

(3) Statement (2) implies that the intersection of an arbitrary finite family of nonempty inductive subsets in $\omega$ is a nonempty inductive subset of $\omega$.

A semigroup $S$ is called inverse if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element $x^{-1}$ is called the inverse of $x \in S$. If $S$ is an inverse semigroup, then the function $\text{inv}: S \to S$ which assigns to every element $x$ of $S$ its inverse element $x^{-1}$ is called the inversion.

A partially ordered set (or shortly a poset) $(X, \leq)$ is the set $X$ with the reflexive, antisymmetric and transitive relation $\leq$. In this case relation $\leq$ is called a partial order on $X$. A partially ordered set $(X, \leq)$ is linearly ordered or is a chain if $x \leq y$ or $y \leq x$ for any $x, y \in X$. A map $f$ from a poset $(X, \leq)$ onto a poset $(Y, \leq)$ is said to be an order isomorphism if $f$ is bijective and $x \leq y$ if and only if $f(x) \leq f(y)$.

If $S$ is a semigroup, then we shall denote the subset of all idempotents in $S$ by $E(S)$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a band (or the band of $S$). Then the semigroup operation on $S$ determines the following partial order $\preceq$ on $E(S)$: $e \preceq f$ if and only if $ef = fe = e$. This order is called the natural partial order on $E(S)$. A semilattice is a commutative semigroup of idempotents.

If $S$ is an inverse semigroup then the semigroup operation on $S$ determines the following partial order $\preceq$ on $S$: $s \preceq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the natural partial order on $S$ [16].

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The bicyclic monoid \( \mathcal{C}(p, q) \) is the semigroup with the identity 1 generated by two elements \( p \) and \( q \) subjected only to the condition \( pq = 1 \). The semigroup operation on \( \mathcal{C}(p, q) \) is determined as follows:

\[
q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.
\]

It is well known that the bicyclic monoid \( \mathcal{C}(p, q) \) is a bisimple (and hence simple) combinatorial \( E \)-unitary inverse semigroup and every non-trivial congruence on \( \mathcal{C}(p, q) \) is a group congruence [1].

On the set \( B_\omega = \omega \times \omega \) we define the semigroup operation \( \cdot \) in the following way

\[
(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} (i_1 - j_1 + i_2, j_1 - i_2 + j_2) & \text{if } j_1 \leq i_2; \\ (i_1, j_1 - i_2 + j_2) & \text{if } j_1 \geq i_2. 
\end{cases}
\]

It is well known that the bicyclic monoid \( \mathcal{C}(p, q) \) to the semigroup \( B_\omega \) is isomorphic by the mapping \( h: \mathcal{C}(p, q) \rightarrow B_\omega, q^kp^l \mapsto (k, l) \) (see: [1, Section 1.12] or [14, Exercise IV.1.11(ii)]).

Next we shall describe the construction which is introduced in [7].

Let \( B_\omega \) be the bicyclic monoid and \( \mathcal{F} \) be an \( \omega \)-closed subfamily of \( \mathcal{P}(\omega) \). On the set \( B_\omega \times \mathcal{F} \) we define the semigroup operation \( \cdot \) in the following way

\[
(i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} (i_1 - j_1 + i_2, j_1 - i_2 + j_2, (j_1 - i_2 + j_2) \cap (i_2 - j_1 + F_2)) & \text{if } j_1 \leq i_2; \\ (i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)) & \text{if } j_1 \geq i_2.
\end{cases}
\]

In [7] is proved that if the family \( \mathcal{F} \subseteq \mathcal{P}(\omega) \) is \( \omega \)-closed then \( (B_\omega \times \mathcal{F}, \cdot) \) is a semigroup. Moreover, if an \( \omega \)-closed family \( \mathcal{F} \subseteq \mathcal{P}(\omega) \) contains the empty set \( \emptyset \) then the set \( I = \{(i, j, \emptyset) : i, j \in \omega\} \) is an ideal of the semigroup \( (B_\omega \times \mathcal{F}, \cdot) \). For any \( \omega \)-closed family \( \mathcal{F} \subseteq \mathcal{P}(\omega) \) the following semigroup

\[
B_\mathcal{F}^\omega = \left\{ \frac{(B_\omega \times \mathcal{F}, \cdot)}{I}, \begin{array}{ll} (B_\omega \times \mathcal{F}, \cdot) & \text{if } \emptyset \in \mathcal{F}; \\ (B_\omega \times \mathcal{F}, \cdot) & \text{if } \emptyset \notin \mathcal{F}. \end{array} \right\}
\]

is defined in [7]. The semigroup \( B_\mathcal{F}^\omega \) generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in [7] that \( B_\mathcal{F}^\omega \) is a combinatorial inverse semigroup and Green’s relations, the natural partial order on \( B_\mathcal{F}^\omega \) and its set of idempotents are described. Here, the criteria when the semigroup \( B_\mathcal{F}^\omega \) is simple, 0-simple, bisimple, 0-bisimple, is isomorphic to the bicyclic monoid if and only if \( \mathcal{F} \) consists of a non-empty inductive subset of \( \omega \).

Group congruences on the semigroup \( B_\mathcal{F}^\omega \) and its homomorphic retracts in the case when an \( \omega \)-closed family \( \mathcal{F} \) consists of inductive non-empty subsets of \( \omega \) are studied in [8]. It is proven that a congruence \( \mathcal{C} \) on \( B_\mathcal{F}^\omega \) is a group congruence if and only if its restriction on a subsemigroup of \( B_\mathcal{F}^\omega \), which is isomorphic to the bicyclic semigroup, is not the identity relation. Also in [8], all non-trivial homomorphic retracts and isomorphisms of the semigroup \( B_\mathcal{F}^\omega \) are described.

In [5, 13] the algebraic structure of the semigroup \( B_\mathcal{F}^\omega \) is established in the case when \( \omega \)-closed family \( \mathcal{F} \) consists of atomic subsets of \( \omega \).

The set \( B_\mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \) with the semigroup operation defined by formula (1) is called the extended bicyclic semigroup [17]. On the set \( B_\mathbb{Z} \times \mathcal{F} \), where \( \mathcal{F} \) is an \( \omega \)-closed subfamily of \( \mathcal{P}(\omega) \), we define the semigroup operation \( \cdot \) by formula (2). In [9] it is proved that \( (B_\mathbb{Z} \times \mathcal{F}, \cdot) \) is a semigroup. Moreover, if an \( \omega \)-closed family \( \mathcal{F} \subseteq \mathcal{P}(\omega) \) contains the empty set \( \emptyset \) then the set \( I = \{(i, j, \emptyset) : i, j \in \mathbb{Z}\} \) is an ideal of the semigroup \( (B_\mathbb{Z} \times \mathcal{F}, \cdot) \). For any \( \omega \)-closed family \( \mathcal{F} \subseteq \mathcal{P}(\omega) \) the following semigroup

\[
B_\mathcal{F}^\mathbb{Z} = \left\{ \frac{(B_\mathbb{Z} \times \mathcal{F}, \cdot)}{I}, \begin{array}{ll} (B_\mathbb{Z} \times \mathcal{F}, \cdot) & \text{if } \emptyset \in \mathcal{F}; \\ (B_\mathbb{Z} \times \mathcal{F}, \cdot) & \text{if } \emptyset \notin \mathcal{F}. \end{array} \right\}
\]

is defined in [9] similarly as in [7]. In [9] it is proven that \( B_\mathcal{F}^\mathbb{Z} \) is a combinatorial inverse semigroup. Green’s relations, the natural partial order on the semigroup \( B_\mathcal{F}^\mathbb{Z} \) and its set of idempotents are described. Here, the criteria when the semigroup \( B_\mathcal{F}^\mathbb{Z} \) is simple, 0-simple, bisimple, 0-bisimple, is isomorphic to the extended bicyclic semigroup, are derived. In particularly in [9] it is proved that the semigroup \( B_\mathcal{F}^\mathbb{Z} \) is isomorphic to the semigroup of \( \omega \times \omega \)-matrix units if and only if \( \mathcal{F} \) consists of a singleton set and
the empty set, and $B_Z^\varphi$ is isomorphic to the extended bicyclic semigroup if and only if $\varphi$ consists of a non-empty inductive subset of $\omega$. Also, in [9] it is proved that in the case when the family $\varphi$ consists of all singletons of $\omega$ and the empty set, the semigroup $B_Z^\varphi$ is isomorphic to the Brandt $\lambda$-extension of the semilattice $(\omega, \min)$, where $(\omega, \min)$ is the set $\omega$ with the semilattice operation $x \cdot y = \min\{x, y\}$.

It is well-known that every automorphism of the bicyclic monoid $B_\omega$ is the identity self-map of $B_\omega$ [1], and hence the group $\text{Aut}(B_\omega)$ of automorphisms of $B_\omega$ is trivial. The group $\text{Aut}(B_Z)$ of automorphisms of the extended bicyclic semigroup $B_Z$ is established in [6] and there it is proved that $\text{Aut}(B_Z)$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$. Also in [10] the semigroups of endomorphisms of the bicyclic semigroup and the extended bicyclic semigroup are described.

Later we assume that an $\omega$-closed family $\mathcal{F}$ consists of inductive nonempty subsets of $\omega$.

In this paper we study automorphisms of the semigroup $B_Z^\varphi$ with the family $\mathcal{F}$ of inductive nonempty subsets of $\omega$ and prove that the group $\text{Aut}(B_Z^\varphi)$ of automorphisms of the semigroup $B_Z^\varphi$ is isomorphic to the additive group of integers.

2. Algebraic properties of the semigroup $B_Z^\varphi$

**Proposition 2.1.** Let $\mathcal{F}$ be an arbitrary nonempty $\omega$-closed family of subsets of $\omega$ and let $n_0 = \min\{\bigcup \mathcal{F}\}$. Then the following statements hold:

1. $\mathcal{F}_0 = \{-n_0 + F : F \in \mathcal{F}\}$ is an $\omega$-closed family of subsets of $\omega$;
2. the semigroups $B_Z^\varphi$ and $B_Z^{\varphi_0}$ are isomorphic by the mapping

$$(i, j, F) \mapsto (i, j, -n_0 + F), \quad i, j \in \mathbb{Z};$$

**Proof.** Statement (1) is proved in [8, Proposition 1(1)]. The proof of (2) is similar to the one of Proposition 1(2) from [8]. \qed

Suppose that $\mathcal{F}$ is an $\omega$-closed family of inductive subsets of $\omega$. Fix an arbitrary $k \in \mathbb{Z}$. If $[0) \in \mathcal{F}$ and $[p] \in \mathcal{F}$ for some $p \in \omega$ then for any $i, j \in \mathbb{Z}$ and we have that

$$(k, k, [0]) \cdot (i, j, [p]) = \begin{cases} (k - k + i, j, (k - i + [0]) \cap [p]), & \text{if } k < i; \\
(k, j, [0] \cap [p]), & \text{if } k = i; \\
(k, k - i + j, [0] \cap (i - k + [p])), & \text{if } k > i \\
\end{cases} = \begin{cases} (i, j, [p]), & \text{if } k < i; \\
(i, j, [p]), & \text{if } k = i; \\
(k, k - i + j, [0] \cap (i - k + [p])), & \text{if } k > i \\
\end{cases}$$

and

$$(i, j, [p]) \cdot (k, k, [0]) = \begin{cases} (i - j + k, (j - k + [p]) \cap [0]), & \text{if } j < k; \\
(i, k, [p] \cap [0]), & \text{if } j = k; \\
(i, j - k + k, [p] \cap (k - j + [0])), & \text{if } j > k \\
\end{cases} = \begin{cases} (i - j + k, (j - k + [p]) \cap [0]), & \text{if } j < k; \\
(i, k, [p]), & \text{if } j = k; \\
(i, j, [p]), & \text{if } j > k. \\
\end{cases}$$

Therefore the above equalities imply that

$$(k, k, [0]) \cdot B_Z^\varphi \cdot (k, k, [0]) = (k, k, [0]) \cdot B_Z^\varphi \cap B_Z^\varphi \cdot (k, k, [0]) = \{(i, j, [p]) : i, j \geq k, [p] \in \mathcal{F}\}$$

for an arbitrary $k \in \mathbb{Z}$. We define $B_Z^\varphi[k, k, 0) = (k, k, [0]) \cdot B_Z^\varphi \cap B_Z^\varphi \cdot (k, k, [0])$. It is obvious that $B_Z^\varphi[k, k, 0)$ is a subsemigroup of $B_Z^\varphi$.

**Proposition 2.2.** Let $\mathcal{F}$ be an arbitrary nonempty $\omega$-closed family of inductive nonempty subsets of $\omega$ such that $[0) \in \mathcal{F}$. Then the subsemigroup $B_Z^\varphi[k, k, 0)$ of $B_Z^\varphi$ is isomorphic to $B_\omega^\varphi$. 
Proof. Since the family $\mathcal{F}$ does not contain the empty set, $B^\mathcal{F}_Z = (B_Z \times \mathcal{F}, \cdot)$. We define a map $\mathcal{I}: B^\mathcal{F}_Z \to B^\mathcal{F}_Z[k, k, 0]$ in the following way $(i, j, |p|) \mapsto (i + k, j + k, |p|)$. It is obvious that $\mathcal{I}$ is a bijection. Then for any $i_1, i_2, j_1, j_2 \in \mathbb{Z}$ and $F_1, F_2 \in \mathcal{F}$ we have that

$$
\mathcal{I}((i_1, j_1, F_1) \cdot (i_2, j_2, F_2)) = \begin{cases} 
\mathcal{I}(i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 < i_2; \\
\mathcal{I}(i_1, j_2, F_1 \cap F_2), & \text{if } j_1 = i_2; \\
\mathcal{I}(i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 > i_2.
\end{cases}
$$

and

$$
\mathcal{I}(i_1, j_1, F_1) \cdot \mathcal{I}(i_2 + k, j_2 + k, F_2) = (i_1 + k, j_1 + k, F_1) \cdot (i_2 + k, j_2 + k, F_2) =
\begin{cases} 
(i_1 - j_1 + i_2 + k, j_2 + k, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 + k < i_2 + k; \\
(i_1 + k, j_2 + k, F_1 \cap F_2), & \text{if } j_1 + k = i_2 + k; \\
(i_1 + k, j_1 - i_2 + j_2 + k, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 + k > i_2 + k
\end{cases}
$$

and hence $\mathcal{I}$ is a homomorphism which implies the statement of the proposition. \[\square\]

By Remarks 1.1(2) and 1.1(3) every nonempty subset $F \in \mathcal{F}$ contains the least element, and hence later for every nonempty set $F \in \mathcal{F}$ we denote $n_F = \min F$.

Later we need the following lemma from [8].

Lemma 2.3 ([8]). Let $\mathcal{F}$ be an $\omega$-closed family of inductive subsets of $\omega$. Let $F_1$ and $F_2$ be elements of $\mathcal{F}$ such that $n_{F_1} < n_{F_2}$. Then for any positive integer $k \in \{n_{F_1} + 1, \ldots, n_{F_2} - 1\}$ there exists $F \in \mathcal{F}$ such that $F = [k]$.

Proposition 2.1 implies that without loss of generality later we may assume that $[0] \in \mathcal{F}$ for any $\omega$-closed family $\mathcal{F}$ of inductive subsets of $\omega$. Hence these arguments and Lemma 5 of [7] imply the following proposition.

Proposition 2.4. Let $\mathcal{F}$ be an infinite $\omega$-closed family of inductive nonempty subsets of $\omega$. Then the diagram on Fig. 1 describes the natural partial order on the band of $B^\mathcal{F}_Z$.

By the similar way for a finite $\omega$-closed family of inductive nonempty subsets of $\omega$ we obtain the following

Proposition 2.5. Let $\mathcal{F} = \{[0], \ldots, [k]\}$. Then the diagram on Fig. 1 without elements of the form $(i, j, |p|)$ and their arrows, $i, j \in \mathbb{Z}$, $p > k$, describes the natural partial order on the band of $B^\mathcal{F}_Z$.

The definition of the semigroup operation in $B^\mathcal{F}_Z$ implies that in the case when $\mathcal{F}$ is an $\omega$-closed family subsets of $\omega$ and $F \in \mathcal{F}$ is a nonempty inductive subset in $\omega$ then the set

$$
B^{(F)}_Z = \{(i, j, F) : i, j \in \mathbb{Z}\}
$$

with the induced semigroup operation from $B^\mathcal{F}_Z$ is a subsemigroup of $B^\mathcal{F}_Z$ which by Proposition 5 from [9] is isomorphic to the extended bicyclic semigroup $B_Z$.

Proposition 2.6. Let $\mathcal{F}$ be an arbitrary $\omega$-closed family of inductive subsets of $\omega$ and $S$ be a subsemigroup of $B^\mathcal{F}_Z$ which is isomorphic to the extended bicyclic semigroup $B_Z$. Then there exists a subset $F \in \mathcal{F}$ such that $S$ is a subsemigroup in $B^{(F)}_Z$.
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**Proof.** Suppose that $\mathcal{J}: B_Z \to S \subseteq B_Z^\mathcal{F}$ is an isomorphism. Proposition 21(2) of [12, Section 1.4] implies that the image $\mathcal{J}(0,0)$ is an idempotent of $B_Z^\mathcal{F}$, and hence by Lemma 1(2) from [9], $\mathcal{J}(0,0) = (i,i,F)$ for some $i \in Z$ and $F \in \mathcal{F}$. By Proposition 2.1(viii) of [3] the subset $(0,0)B_Z(0,0)$ of $B_Z$ is isomorphic to the bicyclic semigroup, and hence the image $\mathcal{J}((0,0)B_Z(0,0))$ is isomorphic to the bicyclic semigroup $B_\omega$. Then the definition of the natural partial order on $E(B_Z^\mathcal{F})$ and Corollary 1 from [9] imply that there exists an integer $k$ such that $(i,i,F) \preceq (k,k,[0])$. By Proposition 2.2 the subsemigroup

$$B_Z^\mathcal{F}[k,k,0] = (k,k,[0]) \cdot B_Z^\mathcal{F} \cdot (k,k,[0])$$

of $B_Z^\mathcal{F}$ is isomorphic to $B_\omega^\mathcal{F}$. Since $(i,i,F) \preceq (k,k,[0])$ we have that $\mathcal{J}((0,0)B_Z(0,0)) \subseteq B_Z^\mathcal{F}[k,k,0]$, and hence $\mathcal{J}((0,0)B_Z(0,0)) \subseteq B_Z^{\{F\}}$ by Proposition 4 of [8].

Next, fix any negative integer $n$. By Proposition 2.1(viii) of [3] the subset $(n,n)B_Z(n,n)$ of $B_Z$ is isomorphic to the bicyclic semigroup. Since $(0,0)B_Z(0,0)$ is an inverse subsemigroup of $(n,n)B_Z(n,n)$, the above arguments imply that $\mathcal{J}((n,n)B_Z(n,n)) \subseteq B_Z^{\{F\}}$ for any negative integer $n$. Since

$$B_Z = \bigcup \{(k,k)B_Z(k,k) : -k \in \omega\},$$

we get that $\mathcal{J}(B_Z) \subseteq B_Z^{\{F\}}$. □
3. ON AUTOMORPHISMS OF THE SEMIGROUP $B^\mathcal{F}_Z$

Recall [4] define relations $\mathcal{L}$ and $\mathcal{R}$ on an inverse semigroup $S$ by

$$(s, t) \in \mathcal{L} \iff s^{-1}s = t^{-1}t \quad \text{and} \quad (s, t) \in \mathcal{R} \iff ss^{-1} = tt^{-1}.$$ 

Both $\mathcal{L}$ and $\mathcal{R}$ are equivalence relations on $S$. The relation $\mathcal{D}$ is defined to be the smallest equivalence relation which contains both $\mathcal{L}$ and $\mathcal{R}$, which is equivalent that $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ [12].

**Remark 3.1.** It is obvious that every semigroup isomorphism $i$: $S \to T$ maps a $\mathcal{D}$-class (resp. $\mathcal{L}$-class, $\mathcal{R}$-class) of $S$ onto a $\mathcal{D}$-class (resp. $\mathcal{L}$-class, $\mathcal{R}$-class) of $T$.

In this section we assume that $[0] \in \mathcal{F}$ for any $\omega$-closed family $\mathcal{F}$ of inductive subsets of $\omega$.

An automorphism $a$ of the semigroup $B^\mathcal{F}_Z$ is called a $(0, 0, [0])$-automorphism if $a(0, 0, [0]) = (0, 0, [0])$.

**Theorem 3.2.** Let $\mathcal{F}$ be an $\omega$-closed family of inductive nonempty subsets of $\omega$. Then every $(0, 0, [0])$-automorphism of the semigroup $B^\mathcal{F}_Z$ is the identity map.

**Proof.** Let $a: B^\mathcal{F}_Z \to B^\mathcal{F}_Z$ be an arbitrary $(0, 0, [0])$-automorphism.

By Theorem 4(iv) of [9] the elements $(i_1, j_1, F_1)$ and $(i_2, j_2, F_2)$ of $B^\mathcal{F}_Z$ are $\mathcal{D}$-equivalent if and only if $F_1 = F_2$. Since every automorphism preserves $\mathcal{D}$-classes, the above argument implies that $a(B^{(F_1)}_Z) = B^{(F_2)}_Z$ if and only if $F_1 = F_2$ for $F_1, F_2 \in \mathcal{F}$. Hence we have that $a(B^{(0)}_Z) = B^{(0)}_Z$. By Proposition 21(6) of [12, Section 1.4] every automorphism preserves the natural partial order on the semilattice $E(B^\mathcal{F}_Z)$ and since $a$ is a $(0, 0, [0])$-automorphism of $B^\mathcal{F}_Z$ we get that $a(i, i, [0]) = (i, i, [0])$ for any integer $i$.

Fix arbitrary $k, l \in \mathbb{Z}$. Suppose that $a(k, l, [0]) = (p, q, [0])$ for some integers $p$ and $q$. Since the semigroup $B^\mathcal{F}_Z$ is inverse, Proposition 21(1) of [12, Section 1.4] and Lemma 1(4) of [9] imply that

$$(a(k, l, [0]))^{-1} = (p, q, [0])^{-1} = (q, p, [0]).$$

Again by Proposition 21(1) of [12, Section 1.4] we have that

$$(k, k, [0]) = a(k, k, [0]) = a((k, l, [0]) \cdot (k, k, [0])) = a(k, l, [0]) \cdot a((k, l, [0])^{-1}) = (p, q, [0]) \cdot (q, p, [0]) = (p, p, [0]),$$

and hence $p = k$. By similar way we get that $l = q$. Therefore, $a(k, l, [0]) = (k, l, [0])$ for any integers $k$ and $l$.

If $\mathcal{F} \neq \{[0]\}$ then by Lemma 2.3, $[1] \in \mathcal{F}$. The definition of the natural partial order on the semilattice $E(B^\mathcal{F}_Z)$ (also, see Proposition 2.4) and Corollary 5 of [9] imply that $(0, 0, [1])$ is the unique idempotent $\varepsilon$ of the semigroup $B^\mathcal{F}_Z$ with the property

$$(1, 1, [0]) \leq \varepsilon \leq (0, 0, [0]).$$

Since by Proposition 21(6) of [12, Section 1.4] the automorphism $a$ preserves the natural partial order on the semilattice $E(B^\mathcal{F}_Z)$, we get that $a(0, 0, [1]) = (0, 0, [1])$. Similar arguments as in the above paragraph imply that $a(k, l, [1]) = (k, l, [1])$ for any integers $k$ and $l$.

Next, by induction we obtain that $a(k, l, [p]) = (k, l, [p])$ for any $k, l \in \mathbb{Z}$ and $[p] \in \mathcal{F}$. $\square$

**Proposition 3.3.** Let $\mathcal{F}$ be an $\omega$-closed family of inductive nonempty subsets of $\omega$. Then for every integer $k$ the map $h_k: B^\mathcal{F}_Z \to B^\mathcal{F}_Z$, $(i, j, [p]) \mapsto (i + k, j + k, [p])$ is an automorphism of the semigroup $B^\mathcal{F}_Z$. 

The proof of Proposition 3.3 is similar to Proposition 2.2.

For a partially ordered set \((P, \leq)\), a subset \(X\) of \(P\) is called order-convex, if \(x \leq z \leq y\) and \(x, y \in X\) implies that \(z \in X\), for all \(x, y, z \in P\) [11].

**Lemma 3.4.** If \(\mathcal{F}\) is an infinite \(\omega\)-closed family of inductive nonempty subsets of \(\omega\) then

\[
\{(0,0,[k]) : k \in \omega\}
\]

is an order-convex linearly ordered subset of \((E(B^\mathcal{F}_Z), \preceq)\).

**Proof.** Fix arbitrary \((0,0,[m]), (0,0,[n]), (0,0,[p]) \in E(B^\mathcal{F}_Z)\). If \((0,0,[m]) \preceq (0,0,[n]) \preceq (0,0,[p])\) then Corollary 1 of [9] implies that \([m] \subseteq [n] \subseteq [p]\). Hence we have that \(m \geq n \geq p\), which implies the statement of the lemma.

**Proposition 3.5.** Let \(\mathcal{F}\) be an infinite \(\omega\)-closed family of inductive nonempty subsets of \(\omega\). Then

\[
a(0,0,[0]) \in B^\mathcal{F}_Z
\]

for any automorphism \(a\) of the semigroup \(B^\mathcal{F}_Z\).

**Proof.** Suppose to the contrary that there exists an automorphism \(a\) of the semigroup \(B^\mathcal{F}_Z\) such that

\[
a(0,0,[0]) \notin B^\mathcal{F}_Z.
\]

Then \(a(0,0,[0])\) is an idempotent of the semigroup \(B^\mathcal{F}_Z\). Lemma 1(2) of [9] implies that \(a(0,0,[0]) = (i,i,[p])\) for some integer \(i\) and some positive integer \(p\). Since the automorphism \(a\) maps a \(\mathcal{D}\)-class of the semigroup \(B^\mathcal{F}_Z\) onto its \(\mathcal{D}\)-class there exists an element \((0,0,[s])\) of the chain

\[
\cdots \preceq (0,0,[k]) \preceq (0,0,[k-1]) \preceq \cdots \preceq (0,0,[2]) \preceq (0,0,[1]) \preceq (0,0,[0])
\]

such that \(a(0,0,[s]) = (n,n,[0]) \in B^\mathcal{F}_Z\) for some integer \(m\). By Proposition 21(6) of [12, Section 1.4] every automorphism preserves the natural partial order on the semilattice \(E(B^\mathcal{F}_Z)\), and hence the inequality \(0,0,[s]) \preceq (0,0,[0])\) implies that

\[
a(0,0,[s]) = (m,m,[0]) \preceq (i,i,[p]) = a(0,0,[0]).
\]

By Corollary 1 of [9] we have that \(m \geq i\) and \((0) \subseteq i - m + [p]\). The last inclusion implies that \(m \geq i + p\). Since the chain (3) is infinite and any its two distinct elements belong to distinct two \(\mathcal{D}\)-classes of the semigroup \(B^\mathcal{F}_Z\), Proposition 21(6) of [12, Section 1.4] and Remark 3.1 imply that there exists a positive integer \(q > s\) such that \(a(0,0,[q]) = (t,t,[x])\) for some positive integer \(x > p\) and some integer \(t\). Then

\[
a(0,0,[q]) = (t,t,[x]) \preceq (m,m,[0]) = a(0,0,[s])
\]

and by Corollary 1 of [9] we have that \(t \geq m\) and \([x] \subseteq t - m + [0]\), and hence \(x \geq t - m\).

Next we consider the idempotent \((i + 1,i + 1,[p])\) of the semigroup \(B^\mathcal{F}_Z\). By Corollary 1 of [9] we get that \((i + 1,i + 1,[p]) \preceq (i,i,[p])\) in \(E(B^\mathcal{F}_Z)\). Since \(x > p\) we have that \(x \geq p + 1\). The inequalities \(t \geq m \geq i + p\) and \(p \geq 1\) imply that \(t \geq i + 1\). Also, the inequalities \(t \geq m \geq i\) and \(x \geq p + 1\) imply that \(t + x \geq i + 1 + p\), and hence we obtain the inclusion \([x] \subseteq i + 1 + t + [p]\). By Corollary 1 of [9] we have that \((t,t,[x]) \preceq (i + 1,i + 1,[p])\). Since \(a\) is an automorphism of the semigroup \(B^\mathcal{F}_Z\), its restriction \(a|_{E(B^\mathcal{F}_Z)} : E(B^\mathcal{F}_Z) \rightarrow E(B^\mathcal{F}_Z)\) onto the band \(E(B^\mathcal{F}_Z)\) is an order automorphism of the partially ordered set \((E(B^\mathcal{F}_Z), \preceq)\), and hence the map \(a|_{E(B^\mathcal{F}_Z)}\) preserves order-convex subsets of \((E(B^\mathcal{F}_Z), \preceq)\).

By Lemma 3.4 chain (3) is order-convex in the partially ordered set \((E(B^\mathcal{F}_Z), \preceq)\). The inequalities \((t,t,[x]) \preceq (i + 1,i + 1,[p]) \preceq (i,i,[p])\) in \(E(B^\mathcal{F}_Z)\) imply that the image of order-convex chain (3) under the order automorphism \(a|_{E(B^\mathcal{F}_Z)}\) is not an order-convex subset of \((E(B^\mathcal{F}_Z), \preceq)\), a contradiction. The obtained contradiction implies the statement of the proposition.

Later for any integer \(k\) we assume that \(h_k : B^\mathcal{F}_Z \rightarrow B^\mathcal{F}_Z\) is an automorphism of the semigroup \(B^\mathcal{F}_Z\) defined in Proposition 3.3.

**Theorem 3.6.** Let \(\mathcal{F}\) be an infinite \(\omega\)-closed family of inductive nonempty subsets of \(\omega\). Then for any automorphism \(a\) of the semigroup \(B^\mathcal{F}_Z\) there exists an integer \(p\) such that \(a = h_p\).
Proof. By Proposition 3.5 there exists an integer $p$ such that $a(0,0,0) = (-p,-p,0))$. Then the composition $h_p \circ a$ is a $(0,0,0)$-automorphism of the semigroup $B^\mathbb{F}_Z$, i.e., $(h_p \circ a)(0,0,0) = (0,0,0)$, and hence by Theorem 3.2 the composition $h_p \circ a$ is the identity map of $B^\mathbb{F}_Z$. Since $h_p$ and $a$ are bijections of $B^\mathbb{F}_Z$, the above arguments imply that $a = h_p$. □

Since $h_{k_1} \circ h_{k_2} = h_{k_1+k_2}$ and $h_{k_1}^{-1} = h_{-k_1}$, $k_1, k_2 \in \mathbb{Z}$, for any automorphisms $h_{k_1}$ and $h_{k_2}$ of the semigroup $B^\mathbb{F}_Z$, Theorem 3.6 implies the following corollary.

**Corollary 3.7.** Let $\mathcal{F}$ be an infinite $\omega$-closed family of inductive nonempty subsets of $\omega$. Then the group of automorphisms $\text{Aut}(B^\mathbb{F}_Z)$ of the semigroup $B^\mathbb{F}_Z$ is isomorphic to the additive group of integers $(\mathbb{Z}, +)$.

The following example shows that for an arbitrary nonnegative integer $k$ and the finite family $\mathcal{F} = \{(0), (1), \ldots, (k)\}$ there exists an automorphism $\tilde{a} : B^\mathbb{F}_Z \to B^\mathbb{F}_Z$ which is distinct from the form $h_p$.

**Example 3.8.** Fix an arbitrary nonnegative integer $k$. Put
\[
\tilde{a}(i,j,[s]) = (i+s,j+s,[k-s])
\]
for any $s = 0, 1, \ldots, k$ and all $i,j \in \mathbb{Z}$.

**Lemma 3.9.** Let $k$ be an arbitrary nonnegative integer and $\mathcal{F} = \{(0), (1), \ldots, (k)\}$. Then $\tilde{a} : B^\mathbb{F}_Z \to B^\mathbb{F}_Z$ is an automorphism.

Proof. Fix arbitrary $i,j,m,n \in \mathbb{Z}$. Without loss of generality we may assume that $s,t \in \{0,1,\ldots,k\}$ with $s < t$. Then we have that
\[
\tilde{a}((i,j,[s]) \cdot (m,n,[t])) = \begin{cases} 
\tilde{a}(i,j,m,n,(j-m+[s]) \cap [t]), & \text{if } j < m; \\
\tilde{a}(i,n,[s] \cap [t]), & \text{if } j = m; \\
\tilde{a}(i,j,m,n,[s] \cap (m-j+[t])), & \text{if } j > m
\end{cases}
\]
\[
= \begin{cases} 
\tilde{a}(i-j+m,n,[t]), & \text{if } j < m; \\
\tilde{a}(i,n,1), & \text{if } j = m; \\
\tilde{a}(i,j-m+n,[s]), & \text{if } j > m \text{ and } m+t < j+s; \\
\tilde{a}(i,j-m+n,m-j+[t]), & \text{if } j > m \text{ and } m+t > j+s
\end{cases}
\]
\[
\tilde{a}(i,j,[s]) \cdot \tilde{a}(m,n,[t]) = (i+s,j+s,[k-s]) \cdot (m+t,n+t,[k-t]) = 
\begin{cases} 
(i-j+m+t+n,t+(j-s-m-t+[k-s]) \cap [k-t]), & \text{if } j+s < m+t; \\
(i+s,n+t,[k-s] \cap [k-t]), & \text{if } j+s = m+t; \\
(i+s,j+m+n+s,[k-s] \cap (m+t-s-j+[k-t])), & \text{if } j > m+t
\end{cases}
\]
\[
= \begin{cases} 
(i-j+m+t,n+t,[k-t+j-m]), & \text{if } j < m \text{ and } j+s < m+t; \\
(i+t,n+t,[k-t] \cap [k-t]), & \text{if } j = m \text{ and } j+s < m+t; \\
(i-j+m+t,n+t,[k-t+j-m] \cap [k-t]), & \text{if } j > m \text{ and } j+s < m+t;
\end{cases}
\]
\[
\text{vagueness, if } j < m \text{ and } j+s = m+t; \\
\text{vagueness, if } j = m \text{ and } j+s = m+t; \\
\text{vagueness, if } j > m \text{ and } j+s > m+t; \\
\text{vagueness, if } j = m \text{ and } j+s > m+t; \\
(i+s,j-m+n+s,[k-s] \cap [k-s-j+m]), & \text{if } j > m \text{ and } j+s > m+t
\end{cases}
\]
\[
\tilde{a}(m, n, [t]) \cdot (i, j, [s]) = \begin{cases}
\tilde{a}(m - n + i, j, (n - i + [t]) \cap [s]), & \text{if } n < i; \\
\tilde{a}(m, j, [t]) \cap [s]), & \text{if } n = i; \\
\tilde{a}(m, n - i + j, [t] \cap (i - n + [s])), & \text{if } n > i \\
\end{cases}
\]

\[
\tilde{a}(m, n, [t]) \cdot \tilde{a}(i, j, [s]) = (m + t, n + t, [k - t]) \cdot (i + s, j + s, [k - s]) =
\begin{cases}
(m + t, j + s, [k - t] \cap [k - s]), & \text{if } n + t < i + s; \\
(m + t, j + s, (n + t - i - s + [k - t]) \cap [k - s]), & \text{if } n + t = i + s; \\
(m + t, n - i + j + t, [k - t] \cap (i + s - n + t + [k - s])), & \text{if } n + t > i + s
\end{cases}
\]
\[ \begin{align*}
&= \begin{cases}
(m - n + i + s, j + s, [k - s]), & \text{if } n < i \text{ and } n + t < i + s; \\
(m + t, j + s, [k - s]), & \text{if } n < i \text{ and } n + t = i + s; \\
(m + t, n - i + j + t, [k - t + i - n]), & \text{if } n < i \text{ and } n + t > i + s; \\
vagueness, & \text{if } n = i \text{ and } n + t < i + s; \\
vagueness, & \text{if } n = i \text{ and } n + t = i + s; \\
vagueness, & \text{if } n > i \text{ and } n + t = i + s; \\
vagueness, & \text{if } n > i \text{ and } n + t > i + s; \\
vagueness, & \text{if } n > i \text{ and } n + t > i + s; \\
(m + t, n - i + j + t, [k - t]), & \text{if } n > i \text{ and } n + t > i + s;
\end{cases}
\end{align*} \]

\[ \tilde{a}(i, j, [s]) \cdot (m, n, [s]) = \begin{cases}
\tilde{a}(i - j + m, n, (j - m + [s]) \cap [s]), & \text{if } j < m; \\
\tilde{a}(i, n, [s] \cap [s]), & \text{if } j = m; \\
\tilde{a}(i, j - m + n, [s] \cap (m - j + [s])), & \text{if } j > m
\end{cases} = \begin{cases}
(i - j + m + s, n + s, [k - s]), & \text{if } j < m; \\
(i + s, n + s, [k - s]), & \text{if } j = m; \\
(i + s, j - m + n + s, [k - s]), & \text{if } j > m.
\end{cases} \]
partial order on $E(B^\mathcal{F}_m)$ (see: Proposition 2.4) we get that
\[
\{a(0,0,[0]), a(0,0,[1]), \ldots, a(0,0,[n_0]), a(0,0,[n_0+1])\}
\]
is not an order convex subset of $(E(B^\mathcal{F}_m), \lessgtr)$, a contradiction. The obtained contradiction implies that $a(0,0,[1]) \neq (p,p,[m+1])$.

In the case $a(0,0,[1]) = (p+1,p+1,[m-1])$ by similar way we get a contradiction. □

Later we assume that $h_p$ and $\tilde{a}$ are automorphisms of the semigroup $B^\mathcal{F}_m$ defined in Proposition 3.3 and Example 3.8, respectively.

**Proposition 3.11.** Let $k$ be any positive integer and $\mathcal{F} = \{0, \ldots, [k]\}$. Let $a: B^\mathcal{F}_m \to B^\mathcal{F}_m$ be an automorphisms such that $a(0,0,[0]) \in B^\mathcal{F}_{[k]}$. Then there exists an integer $p$ such that $a = h_p \circ \tilde{a} = \tilde{a} \circ h_p$.

**Proof.** First we remark that for any integer $p$ the automorphisms $h_p$ and $\tilde{a}$ commute, i.e., $h_p \circ \tilde{a} = \tilde{a} \circ h_p$.

Suppose that $a(0,0,[0]) = (p,p,[k])$ for some integer $p$. Then $b = a \circ h_{-p}$ is an automorphism of the semigroup $B^\mathcal{F}_m$ such that $b(0,0,[0]) = (0,0,[k])$. Then the order convexity of the linearly ordered set $L_1 = \{(0,0,[0]), (0,0,[1])\}$ implies that the image $a(L_1)$ is an order convex chain in $E(B^\mathcal{F}_m)$ with the respect to the natural partial order. Remark 3.1 and the description of the natural partial order on $E(B^\mathcal{F}_m)$ (see: Proposition 2.5) imply that $b(0,0,[1]) = (1,1,[k-1])$. This completes the proof of the base of induction. Fix an arbitrary $s = 2, \ldots, k$ and suppose that $b(0,0,[j]) = (j,j,[k-j])$ for any $j < s$, which is the assumption of induction. Next, since the linearly ordered set $L_s = \{(0,0,[s-1]), (0,0,[s])\}$ is order convex in $E(B^\mathcal{F}_m)$, the image $a(L_s)$ is an order convex chain in $E(B^\mathcal{F}_m)$, as well. Then the equality $b(0,0,[s-1]) = (s-1,s-1,[k-s+1])$, Remark 3.1 and the description of the natural partial order on $E(B^\mathcal{F}_m)$ (Proposition 2.5) imply that $b(0,0,[s]) = (s,s,[k-s])$ for all $s = 2, \ldots, k$.

Fix an arbitrary $s \in \{0,1,\ldots,k\}$. Since $(1,1,[s])$ is the biggest element of the set of idempotents of $B^\mathcal{F}_{[s]}$ which are less then $(0,0,[s])$, Remark 3.1 and the description of the natural partial order on $E(B^\mathcal{F}_m)$ (see: Proposition 2.5) imply that $b(1,1,[s]) = (1+s,1+s,[k-s])$. Then by induction and presented above arguments we get that $b(i,i,[s]) = (i+s,i+s,[k-s])$ for any positive integer $i$. Also, since $(1,-1,[s])$ is the smallest element of the set of idempotents of $B^\mathcal{F}_{[s]}$ which are greater then $(0,0,[s])$, Remark 3.1 and the description of the natural partial order on $E(B^\mathcal{F}_m)$ imply that $b(-1,-1,[s]) = (-1+s,-1+s,[k-s])$. Similar, by induction and presented above arguments we get that $b(-i,-i,[s]) = (-i+s,-i+s,[k-s])$ for any positive integer $i$. This implies that $b(i,i,[s]) = (i+s,i+s,[k-s])$ for any integer $i$.

Fix any $i,j \in \mathbb{Z}$ and an arbitrary $s = 0,1,\ldots,k$. Remark 3.1 implies that $b(i,j,[s]) = (m,n,[k-s])$ for some $m,n \in \mathbb{Z}$. By Proposition 21(1) of [12, Section 1.4] and Lemma 1(4) of [9] we get that $b(j,i,[s]) = (n,m,[k-s])$. This implies that $b(i,i,[s]) = b((i,j,[s]) \cdot (j,i,[s])) = b(i,j,[s]) \cdot b(j,i,[s]) = (m,n,[k-s]) \cdot (n,m,[k-s]) = (m,m,[k-s])$ and $b(j,j,[s]) = b((j,i,[s]) \cdot (i,j,[s])) = b(j,i,[s]) \cdot b(i,j,[s]) = (n,m,[k-s]) \cdot (m,n,[k-s]) = (n,n,[k-s])$, and hence we have that $m = i+s$ and $n = j+s$. 
Therefore we obtain \( b(i, j, [s]) = (i + s, j + s, [k - s]) \) for any \( i, j \in \mathbb{Z} \) and an arbitrary \( s = 0, 1, \ldots, k \), which implies that \( b = \tilde{a} \). Then
\[
a = a \circ h_{-p} \circ h_p = b \circ h_p = \tilde{a} \circ h_p,
\]
which completes the proof of the proposition.

The following lemma describes the relation between automorphisms \( \tilde{a} \) and \( h_1 \) of the semigroup \( B_{\mathbb{Z}}^\mathcal{F} \) in the case when \( \mathcal{F} = \{[0], \ldots, [k]\} \).

**Lemma 3.12.** Let \( k \) be any positive integer and \( \mathcal{F} = \{[0], \ldots, [k]\} \). Then
\[
\tilde{a} \circ \tilde{a} = \underbrace{h_1 \circ \cdots \circ h_1}_{k \text{-times}} = h_k \quad \text{and} \quad \tilde{a}^{-1} = \underbrace{h_1^{-1} \circ \cdots \circ h_1^{-1}}_{k \text{-times}} \circ \tilde{a} = h_{-k} \circ \tilde{a}.
\]

**Proof.** For any \( i, j \in \mathbb{Z} \) and an arbitrary \( s = 0, 1, \ldots, k \) we have that
\[
(a \circ \tilde{a})(i, j, [s]) = \tilde{a}(i + s, j + s, [k - s]) = \tilde{a}(i + s + k - s, j + s + k - s, [k - (k - s)]) = (i + k, j + s, [s]) = h_k(i, j, [s]),
\]
Also, by the equality \( \tilde{a} \circ \tilde{a} = h_k \) we get that \( \tilde{a} = h_1^k \circ \tilde{a}^{-1} \), and hence
\[
\tilde{a}^{-1} = (h_1^k)^{-1} \circ \tilde{a} = \underbrace{h_1^{-1} \circ \cdots \circ h_1^{-1}}_{k \text{-times}} \circ \tilde{a} = h_{-k} \circ \tilde{a},
\]
which completes the proof. \( \square \)

For any positive integer \( k \) we denote the following group \( G_k = \langle x, y \mid xy = yx, y^2 = x^k \rangle \).

**Lemma 3.13.** For any positive integer \( k \) the group \( G_k = \langle x, y \mid xy = yx, y^2 = x^k \rangle \) is isomorphic to the additive groups of integers \( \mathbb{Z}(+) \).

**Proof.** In the case when \( k = 2p \) for some positive integer \( p \) we have that \( y^2 = x^{2p} \), and hence \( x \) is a generator of \( G_k \) such that \( y = x^p \).

In the case when \( k = 2p + 1 \) for some \( p \in \omega \) we have that \( z = y \cdot x^{-k} \) is a generator of \( G_k \) such that \( x = z^2 \) and \( y = z^{-2p + 1} \). \( \square \)

**Theorem 3.14.** Let \( k \) be any positive integer and \( \mathcal{F} = \{[0], \ldots, [k]\} \). Then the group \( \text{Aut}(B_{\mathbb{Z}}^\mathcal{F}) \) of automorphisms of the semigroup \( B_{\mathbb{Z}}^\mathcal{F} \) isomorphic to the group \( G_k \), and hence to the additive groups of integers \( \mathbb{Z}(+) \).

**Proof.** By Proposition 3.10 for any automorphism \( a \) of \( B_{\mathbb{Z}}^\mathcal{F} \) we have that either \( a(0, 0, [0]) \in B_{\mathbb{Z}}^{\{0\}} \) or \( a(0, 0, [0]) \in B_{\mathbb{Z}}^{\{k\}} \).

Suppose that \( a(0, 0, [0]) \in B_{\mathbb{Z}}^{\{0\}} \). Then \( a(0, 0, [0]) \) is an idempotent and hence by Lemma 1(2) of [9], \( a(0, 0, [0]) = (-p, -p, [0]) \) for some integer \( p \). Similar arguments as in the proof of Theorem 3.6 imply that \( a = h_p = \underbrace{h_1 \circ \cdots \circ h_1}_{p \text{-times}} \).

Suppose that \( a(0, 0, [0]) \in B_{\mathbb{Z}}^{\{k\}} \). Then by Proposition 3.11 there exists an integer \( p \) such that \( a = h_p \circ \tilde{a} = \tilde{a} \circ h_p \).

Since \( \tilde{a} \) and \( h_p \) commute, the above arguments imply that any automorphism \( a \) of \( B_{\mathbb{Z}}^\mathcal{F} \) is a one of the following forms:
\[
\begin{align*}
&\bullet \; a = h_p = (h_1)^p \text{ for some integer } p; \quad \text{or} \\
&\bullet \; a = h_p \circ \tilde{a} = \tilde{a} \circ h_p = \tilde{a} \circ (h_1)^p \text{ for some integer } p.
\end{align*}
\]
This implies that the map \( \mathfrak{A} : \text{Aut}(B_{\mathbb{Z}}^\mathcal{F}) \to G_k \) defined by the formulæ \( \mathfrak{A}( (h_1)^p ) = x^p \) and \( \mathfrak{A}( \tilde{a} \circ (h_1)^p ) = yx^p, p \in \mathbb{Z} \), is a group isomorphism. Next we apply Lemma 3.12. \( \square \)
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