EXPLICIT STRUCTURE OF THE VANISHING VISCOSITY LIMITS FOR THE ZERO-PRESSURE GAS DYNAMICS SYSTEM INITIATED BY THE LINEAR COMBINATION OF A CHARACTERISTIC FUNCTION AND A $\delta$-DISTIBUTION

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Abstract. In this article, we consider the one-dimensional zero-pressure gas dynamics system

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad \rho_t + (\rho u)_x = 0$$

in the upper-half plane with a linear combination of a characteristic function and a $\delta$-measure

$$u|_{t=0} = u_a \chi_{(-\infty,a)} + u_b \delta_{x=b}, \quad \rho|_{t=0} = \rho_c \chi_{(-\infty,c)} + \rho_d \delta_{x=d}$$

as initial data, where $a, b, c, d$ are distinct points on the real line ordered as $a < c < b < d$, and provide a detailed analysis of the vanishing viscosity limits for the above system utilizing the corresponding modified adhesion model

$$u_\epsilon_t + \left(\frac{(u_\epsilon)^2}{2}\right)_x = \epsilon^2 u_\epsilon_{xx}, \quad \rho_\epsilon_t + (\rho_\epsilon u_\epsilon)_x = \epsilon^2 \rho_{xx}.$$ 

For this purpose, we use suitable Hopf-Cole transformations and various asymptotic properties of the function $\text{erfc}: z \mapsto \int_0^\infty e^{-s^2} ds$.

1. Introduction

We consider the initial value problem for the zero-pressure gas dynamics system

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad \rho_t + (\rho u)_x = 0, \quad (x,t) \in \mathbb{R}^1 \times (0,\infty), \quad (1)$$

under the assumption that the initial data consists of a linear combination of a characteristic function and a $\delta$-measure, more precisely,

$$u|_{t=0} = u_a \chi_{(-\infty,a)} + u_b \delta_{x=b}, \quad \rho|_{t=0} = \rho_c \chi_{(-\infty,c)} + \rho_d \delta_{x=d}. \quad (2)$$

Here $a, b, c, d$ are fixed with $a < c < b < d$ and $u_a, u_b, \rho_c, \rho_d$ are real constants.

The system (1) is an important system of partial differential equations finding applications in cosmology and having close relations with the Zeldovich approximation (9). (1) is used to describe the evolution of matter in the expansion of universe as cold dust moving only under the effect of gravity. Here the objects under study, namely $u$ and $\rho$, respectively denote the velocity and the density of the particles, $x \in \mathbb{R}^1$ denotes the space variable and $t > 0$ denotes time.

In (1), the first equation involving the velocity component $u$ alone is the Burgers equation. In general, we cannot expect to find solutions of this problem in the class of smooth functions even for smooth initial data and so we need the notion of weak solutions satisfying (1) in the sense of distributions, namely

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\[ \int_0^\infty \int_{-\infty}^\infty (u \phi_t + f(u) \phi_x) \, dx \, dt + \int_{-\infty}^\infty u(x,0) \, \phi(x,0) \, dx = 0, \quad \phi \in C^\infty_c(\mathbb{R}^1 \times [0,\infty)) \].

The initial value problem for the Burgers equation was studied by Hopf [5] using the method of vanishing viscosity, where the author has considered the problem

\[ u_t + \left( \frac{u^2}{2} \right)_x = \mu u_{xx}, \]

\[ u|_{t=0} = u_0 \]

for \( \mu > 0 \) with locally integrable initial data \( u_0 \) under the additional assumption that

\[ \lim_{|x| \to \infty} \frac{\int_0^x u_0(\xi) \, d\xi}{x^2} = 0. \]

This resulted in an explicit formula

\[ u(x, t) = \frac{\int_{-\infty}^\infty \frac{x-y}{t} e^{-\frac{F(x-y, t)}{2\mu}}}{\int_{-\infty}^\infty e^{-\frac{F(x-y, t)}{2\mu}}} \],

where \( F : (x, y, t) \mapsto \frac{(x-y)^2}{2t} + \int_0^y u_0(\xi) \, d\xi \). The subsequent asymptotic limit as \( \mu \to 0 \) was discussed in detail.

The work of Hopf [5] was generalized by Lax [7] to general conservation laws of the form

\[ u_t + [f(u)]_x = 0, \]

under bounded measurable initial data \( u|_{t=0} = u_0 \), under the assumption that the flux function \( f \in C^2 \) is strictly convex (\( f'' > 0 \)) with superlinear growth at infinity (\( \lim_{y \to \infty} \frac{f(y)}{y} = \infty \)). It was observed that for each fixed \( t > 0 \), except possibly for countably many \( x \in \mathbb{R}^1 \), there exists a unique minimizer \( y(x, t) \) for

\[ \theta(x, y, t) := f^\ast \left( \frac{x-y}{t} \right) + \int_0^y u_0(\xi) \, d\xi, \]

where \( f^\ast : z \mapsto \max_{p \in \mathbb{R}^1} \{pz - f(p)\} \) denotes the convex conjugate of \( f \).

A weak solution of the associated initial value problem could then be given by

\[ \overline{u}(x, t) = (f^\ast)^{-1} \left( \frac{x-y(x, t)}{t} \right), \]

which is defined pointwise a.e.

There are many published works for scalar conservation laws with bounded nonnegative measures as initial data (see Bertsch et. al. [1], Demengel and Serre [4], Liu and Pierre [8] and the references given there). These works focus on scalar conservation laws of the form

\[ u_t + (\phi(u))_x = 0, \]

with the initial data \( u|_{t=0} = u_0 \) at \( t = 0 \) being a bounded Borel measure. The existence and uniqueness of solutions will then depend on the following factors:

1. The initial data \( u_0 \) and its sign
Whether the flux $\phi$ is odd or convex

It is natural to expect the flux function $\phi$ will have a smoothing effect on the solution by virtue of its strong nonlinearity.

The vanishing viscosity method has been previously applied by Joseph [6] for the system (1) under Riemann-type initial data

$$(u(x, 0), \rho(x, 0)) = \begin{cases} (u_L, \rho_L), & x < 0, \\ (u_R, \rho_R), & x > 0. \end{cases}$$

The modified adhesion model corresponding to the given problem (1), that is

$$u_\varepsilon^t + \left( \frac{(u_\varepsilon^t)^2}{2} \right)_x = \varepsilon u_{\varepsilon xx}, \quad \rho_\varepsilon^t + (\rho_\varepsilon^t u_\varepsilon^t)_x = \frac{\varepsilon}{2} \rho_{\varepsilon xx},$$

was subsequently reduced to the new system

$$U_\varepsilon^t + \left( \frac{(U_\varepsilon^t)^2}{2} \right)_x = \frac{\varepsilon}{2} U_{\varepsilon xx}, \quad R_\varepsilon^t + R_\varepsilon^t U_\varepsilon^t_x = \frac{\varepsilon}{2} R_{\varepsilon xx},$$

with new initial data

$$(U_\varepsilon^t(x, 0), R_\varepsilon^t(x, 0)) = \begin{cases} (u_L x, \rho_L x), & x < 0, \\ (u_R x, \rho_R x), & x > 0. \end{cases}$$

It was observed that the distributional derivatives

$$\left( \pi, R \right) := (U_\varepsilon^t, R_\varepsilon^t)$$

with respect to the space variable $x$ solved the system (1) under the prescribed Riemann-type initial data.

The first step involved the linearization of the first equation in (1) by the Hopf-Cole transformation

$$V_\varepsilon = e^{-\frac{u_\varepsilon}{2\varepsilon}},$$

resulting in the consideration of the new linear problem

$$V_\varepsilon^t = V_\varepsilon_{xx},$$

$$V_\varepsilon^t(x, 0) = \begin{cases} e^{-\frac{u_\varepsilon}{2\varepsilon}}, & x < 0, \\ e^{-\frac{u_\varepsilon}{2\varepsilon}}, & x > 0. \end{cases}$$

The second equation in (1) involving $\rho$ was linearised by a modified Hopf-Cole transformation

$$S_\varepsilon = R_\varepsilon e^{-\frac{\rho_\varepsilon}{2\varepsilon}}.$$

The new linear problem to be considered was
\[ S_t^\varepsilon = S_{xx}^\varepsilon, \]
\[ S^\varepsilon(x,0) = \begin{cases} 
\rho_L x e^{-\frac{u_L^\varepsilon}{\varepsilon}}, & x < 0, \\
\rho_R x e^{-\frac{u_R^\varepsilon}{\varepsilon}}, & x > 0.
\end{cases} \]

These linear problems were explicitly solved and \( u^\varepsilon, \rho^\varepsilon \) were recovered through the relation
\[
(u^\varepsilon, \rho^\varepsilon) = \left( -\varepsilon \cdot \frac{V^\varepsilon}{V^\varepsilon_x}, \left( S^\varepsilon \frac{V^\varepsilon}{V^\varepsilon_x} \right)_x \right),
\]
where \( u^\varepsilon \) and \( \rho^\varepsilon \) are related to the original system (1) through the modified adhesion model
\[
u^\varepsilon_t + \left( \frac{(u^\varepsilon)_x^2}{2} \right)_x = \frac{\varepsilon}{2} u^\varepsilon_{xx}, \quad \rho^\varepsilon_t + (\rho^\varepsilon u^\varepsilon)_x = \frac{\varepsilon}{2} \rho^\varepsilon_{xx},
\]
\[
(u^\varepsilon(x,0), \rho^\varepsilon(x,0)) = \begin{cases} 
(u_L, \rho_L), & x < 0, \\
(u_R, \rho_R), & x > 0.
\end{cases}
\]

The above strategy has been previously applied in

• [2] for the problem
\[
u_t + \left( \frac{u^2}{2} \right)_x = 0, \quad \rho_t + (\rho u)_x = 0,
\]
\[
u|_{t=0} = u_a \delta_{x=a} + u_b \delta_{x=b}, \quad \rho|_{t=0} = \rho_a \delta_{x=a} + \rho_d \delta_{x=d}.
\]
• [3] in the context of the problem discussed in this article for the case \( c = a, d = b. \)

2. Computation of the vanishing viscosity limits

Let us now generalize the ideas discussed in the previous section for our problem. For each \( \varepsilon > 0, \) suppose \( (u^\varepsilon, \rho^\varepsilon) \) are approximate solutions for the problem
\[
u^\varepsilon_t + \left( \frac{(u^\varepsilon)_x^2}{2} \right)_x = \frac{\varepsilon}{2} u^\varepsilon_{xx}, \quad \rho^\varepsilon_t + (\rho^\varepsilon u^\varepsilon)_x = \frac{\varepsilon}{2} \rho^\varepsilon_{xx},
\]
\[
u^\varepsilon|_{t=0} = u_a \chi_{(-\infty,a)} + u_b \delta_{x=b}, \quad \rho^\varepsilon|_{t=0} = \rho_c \chi_{(-\infty,c)} + \rho_d \delta_{x=d}.
\]

First let us observe that if \( (U^\varepsilon, R^\varepsilon) \) is a solution to the system
\[
u^\varepsilon + \left( \frac{(U^\varepsilon)_x^2}{2} \right)_x = \frac{\varepsilon}{2} U^\varepsilon_{xx}, \quad R^\varepsilon + R^\varepsilon U^\varepsilon_x = \frac{\varepsilon}{2} R^\varepsilon_{xx}
\]
under the initial conditions

\[
U^\epsilon(x,0) = \begin{cases} 
  u_a(x-a), & x < a, \\
  0, & a < x < b, \\
  u_b, & x > b, 
\end{cases}
\]

\[
R^\epsilon(x,0) = \begin{cases} 
  \rho_c(x-c), & x < c, \\
  0, & c < x < d, \\
  \rho_d, & x > d, 
\end{cases}
\]

then the distributional derivatives \( u^\epsilon := U^\epsilon_x \) and \( \rho^\epsilon := R^\epsilon_x \) in the space variable \( x \) will solve the problem (3) under the initial conditions (4).

Theorem 1. Suppose \( a, b, c, d \) are points on the real line ordered according to the inequalities \( a < c < b < d \). Given real constants \( u_a, u_b, \rho_c \) and \( \rho_d \), let us consider the one-dimensional zero-pressure gas dynamics system

\[
\begin{align*}
  u_t + \frac{u^2}{2} x &= 0, \\
  \rho_t + (\rho u)_x &= 0,
\end{align*}
\]

under the initial data

\[
\begin{align*}
  u|_{t=0} &= u_a \chi_{(-\infty,a)} + u_b \delta_{x=b}, \\
  \rho|_{t=0} &= \rho_c \chi_{(-\infty,c)} + \rho_d \delta_{x=d}.
\end{align*}
\]

Suppose \( u^\epsilon, \rho^\epsilon \) are approximate solutions of the system

\[
\begin{align*}
  u^\epsilon_t + \left( \frac{(u^\epsilon)^2}{2} \right)_x &= \frac{\epsilon}{2} u^\epsilon_{xx}, \\
  \rho^\epsilon_t + (\rho^\epsilon u^\epsilon)_x &= \frac{\epsilon}{2} \rho^\epsilon_{xx},
\end{align*}
\]

\[
\begin{align*}
  u^\epsilon|_{t=0} &= u_a \chi_{(-\infty,a)} + u_b \delta_{x=b}, \\
  \rho^\epsilon|_{t=0} &= \rho_c \chi_{(-\infty,c)} + \rho_d \delta_{x=d}.
\end{align*}
\]

Then the structure of the vanishing viscosity limit \( (u,\rho) = \lim_{\epsilon \to 0} (u^\epsilon,\rho^\epsilon) \) can be explicitly described under various cases depending on the relative signs of \( u_a \) and \( u_b \) as follows:

**Case 1.** \( u_a < 0, u_b > 0 \)

In this case, we consider the curves

(i) \( r(s) := a + u_a s, \)

(ii) \( p(s) := b + \sqrt{2u_b s} \)

defined for every \( s \geq 0 \). The explicit structure of \( \lim_{\epsilon \to 0} (u^\epsilon,\rho^\epsilon) \) can then be described as follows:
\[ u(x,t) = \begin{cases} 
    u_a, & x < r(t), \\
    \frac{u_a}{\epsilon}, & x \in (r(t), a), \\
    0, & x \in (a, b) \cup (p(t), \infty), \\
    \frac{x-b}{\epsilon}, & x \in (b, p(t)), 
\end{cases} \]

\[ \rho = \rho_c \left( \chi_{(-\infty, a+u_at)} + \frac{4(x-a) - 2u_at}{u_at} \chi_{(a+u_at, a)} + \chi_{(a, \infty)} \right) + \rho_d \delta_{x=\gamma_d(t)}. \]

**Case 2.** \( u_a > 0, u_b > 0 \)

Consider the curves

\[ l(s) := a + \frac{u_b}{u_a} + \frac{u_a}{2} \cdot s, \]
\[ \tilde{l}(s) := a + \frac{u_a}{2} \cdot s, \]
\[ r(s) := a + u_a s, \]
\[ p(s) := b + \sqrt{2}u_b s, \]
\[ q(s) := b + u_a s - \sqrt{2} u_a (b-a)s, \]

\[ \gamma_a(s) := \begin{cases} 
    a + \frac{u_a}{2} \cdot s, & 0 \leq s \leq \frac{2(b-a)}{u_a}, \\
    b + u_a s - \sqrt{2} u_a (b-a)s, & \frac{2(b-a)}{u_a} \leq s \leq t_{p,l} := \left( \frac{\sqrt{2 u_b} + \sqrt{2 u_a (b-a)}}{u_a} \right)^2, \\
    a + \frac{u_a}{2} + \frac{u_a}{2} \cdot s, & s \geq t_{p,l}, 
\end{cases} \]

\[ \gamma_b(s) := \begin{cases} 
    b + \sqrt{2}u_b s, & 0 \leq s \leq t_{p,l}, \\
    a + \frac{u_a}{2} + \frac{u_a}{2} \cdot s, & s \geq t_{p,l}, 
\end{cases} \]

\[ \gamma_c(s) := \begin{cases} 
    c, & 0 \leq s \leq \frac{2(c-a)}{u_a}, \\
    \gamma_a(s), & s \geq \frac{2(c-a)}{u_a}, 
\end{cases} \]

\[ \gamma_d(s) := \begin{cases} 
    d, & 0 \leq s \leq \frac{(d-b)^2}{2u_b}, \\
    \gamma_b(s), & s \geq \frac{(d-b)^2}{2u_b}, 
\end{cases} \]

defined over \([0, \infty)\). Then the limit \( \lim_{\epsilon \to 0} (u', \rho') \) has the following explicit representation:
\[
u(x, t) = \begin{cases} 
\frac{x-a}{t}, & x \in \left((b, \infty) \setminus \{ d \}\right) \cap \left((r(t), p(t)) \cup (q(t), \min \{ p(t), r(t) \}) \right), \\
\frac{a-b}{t}, & x \in \left((a, b) \setminus \{ c \}\right) \cap \left(\tilde{l}(t), \infty \right) \setminus \{ \max \{ l(t), p(t) \}, \min \{ q(t), r(t) \} \right) \\
o, & \cup \left((b, \infty) \setminus \{ d \}\right) \cap \left((\max \{ p(t), r(t) \}, \infty \right) \\
& \cup \left(\max \{ p(t), q(t) \}, r(t) \right) \cup (\max \{ l(t), p(t) \}, \min \{ q(t), r(t) \}) \right), \\
\rho = \rho_{c} \left(\chi_{(-\infty, \gamma_{a}(t))} - (x-c) \delta_{x=\gamma_{a}(t)} - u_{a} t \delta_{x=\gamma_{a}(t)} \right) + \rho_{d} \delta_{x=\gamma_{d}(t)}.
\end{cases}
\]

**Case 3.** \( u_{a} > 0, \ u_{b} < 0 \)

Consider the curves

\[
l(t) := a + \frac{u_{a}}{2} t, \\
\tilde{l}(t) := a + \frac{u_{a}}{2} t, \\
r(t) := a + u_{a} t, \\
p(t) := b - \sqrt{-2u_{b} t}, \\
q(t) := b + u_{a} t - \sqrt{2 (u_{a} (b-a) - u_{b}) t}
\]
defined over \([0, \infty)\). As in the previous cases, we can define additional curves \( x = \gamma_{a}(t), \ x = \gamma_{c}(t) \) and \( x = \gamma_{d}(t) \) so that the limit \( \lim_{t \to 0} (u', \rho') \) has the following explicit representation:

\[
u(x, t) = \begin{cases} 
\frac{x-a}{t}, & x \in \left((-\infty, a) \cap (-\infty, q(t))\right) \\
\frac{a-b}{t}, & x \in \left((-\infty, a) \cap (q(t), \infty)\right) \\
o, & \cup \left((a, b) \setminus \{ c \}\right) \cap \left(-\infty, \min \{ q(t), r(t) \}\right) \cap \left((-\infty, \min \{ \tilde{l}(t), p(t) \}\right) \cup (p(t), \infty) \\
& \cup \left((b, \infty) \setminus \{ d \}\right) \cap \left(-\infty, \min \{ l(t), r(t) \}\right), \\
x \in \left((a, b) \setminus \{ c \}\right) \cap \left((r(t), p(t)) \cup \left((-\infty, \min \{ p(t), r(t) \}\right) \cap \left(\tilde{l}(t), q(t) \cup (q(t), \infty)\right) \right) \\
& \cup \left((b, \infty) \setminus \{ d \}\right) \cap \left(l(t), r(t) \right) \cup (r(t), \infty) \right), \\
\rho = \rho_{c} \left(\chi_{(-\infty, \gamma_{a}(t))} - (x-c) \delta_{x=\gamma_{a}(t)} - u_{a} t \delta_{x=\gamma_{a}(t)} \right) + \rho_{d} \delta_{x=\gamma_{d}(t)}.
\end{cases}
\]

**Case 4.** \( u_{a} < 0, \ u_{b} < 0 \)

In this case, introduce the curves
\( l(s) := \frac{a + b}{2} + \frac{u_b}{b - a} \cdot s, \)
\( r(s) := a + u_a s, \)
\( p(s) := b - \sqrt{-2u_b s}, \)
\( q(s) := b + u_a s - \sqrt{2(u_a (b - a) - u_b)} s \) (defined if \( u_a (b - a) > u_b \))

over \([0, \infty)\). Under this case, we have to consider two separate cases: \( u_a (b - a) > u_b \) and \( u_a (b - a) \leq u_b \). As done before, we can also define additional curves \( x = \gamma_{a_1}(t), x = \gamma_{a_2}(t), x = \gamma_c(t) \) and \( x = \gamma_a(t) \) in each case so that the limit \( \lim_{t \to 0} (u^*, \rho^* ) \) has the following explicit representation:

- \( u_a (b - a) > u_b \)

\[
\begin{align*}
u(x, t) &= \begin{cases}
u_a, & x \in (-\infty, a) \cap \left(-\infty, \min \{q(t), r(t)\}\right), \\
\frac{x-a}{t}, & x \in (-\infty, a) \cap \left(r(t), l(t)\right), \\
\frac{x-b}{t}, & x \in \left(-\infty, a\right) \cap \left(\left(\max \{l(t), r(t)\}, \infty\right) \cup \{q(t), r(t)\}\right), \\
0, & x \in \left((a, b) \setminus \{c\}\right) \cap \left((-\infty, \rho(t))\right) \cup \left((b, \infty) \setminus \{d\}\right),
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\rho &= \rho_c \left[ X_{(-\infty, \gamma_{a_1}(t))} + \left(\frac{4(x - a)}{u_a l} - 2\right) X_{(\gamma_{a_1}(t), \gamma_{a_2}(t))} + X_{(\gamma_{a_2}(t), \gamma_c(t))} \\
&+ \left(2(c-a) + u_a l + \gamma_c(t) - x - 2 - \frac{(x - a)^2}{u_a l}\right) \delta_{x=\gamma_{a_1}(t)} \\
&+ \left(2(x - a) + c - \gamma_c(t) - 2 - \frac{(x - a)^2}{u_a l}\right) \delta_{x=\gamma_{a_2}(t)} \right] + \rho_d \delta_{x=\gamma_d(t)}.
\end{align*}
\]

- \( u_a (b - a) \leq u_b \)

\[
\begin{align*}
u(x, t) &= \begin{cases}
u_a, & x \in (-\infty, a) \cap \left(-\infty, r(t)\right), \\
\frac{x-a}{t}, & x \in (-\infty, a) \cap \left(r(t), l(t)\right), \\
\frac{x-b}{t}, & x \in \left(-\infty, a\right) \cap \left(\left(\max \{l(t), r(t)\}, \infty\right) \cup \{q(t), r(t)\}\right), \\
0, & x \in \left((a, b) \setminus \{c\}\right) \cap \left((-\infty, \rho(t))\right) \cup \left((b, \infty) \setminus \{d\}\right),
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\rho &= \rho_c \left[ X_{(-\infty, \gamma_{a_1}(t))} + \left(\frac{4(x - a)}{u_a l} - 2\right) X_{(\gamma_{a_1}(t), \gamma_{a_2}(t))} + X_{(\gamma_{a_2}(t), \gamma_c(t))} \\
&+ \left(2(c-a) + u_a l + \gamma_c(t) - x - 2 - \frac{(x - a)^2}{u_a l}\right) \delta_{x=\gamma_{a_1}(t)} \\
&+ \left(2(x - a) + c - \gamma_c(t) - 2 - \frac{(x - a)^2}{u_a l}\right) \delta_{x=\gamma_{a_2}(t)} \right] + \rho_d \delta_{x=\gamma_d(t)}.
\end{align*}
\]
Proof. The first step is to consider the generalized Hopf-Cole transformations

\[ V^\epsilon := e^{-\frac{ut}{\epsilon}}, \quad S^\epsilon := R^\epsilon e^{-\frac{ut}{\epsilon}}, \tag{7} \]

leading us to the consideration of the linear problem

\[ V^\epsilon_t = \frac{\epsilon}{2} V^\epsilon_{xx}, \quad S^\epsilon_t = \frac{\epsilon}{2} S^\epsilon_{xx} \tag{8} \]

under the initial conditions

\[ V^\epsilon(x, 0) =\begin{cases} 
  e^{-\frac{u_a(x-a)}{\epsilon}}, & x < a, \\
  1, & a < x < b, \\
  e^{-\frac{u_b}{\epsilon}}, & x > b, 
\end{cases} \tag{9} \]

\[ S^\epsilon(x, 0) =\begin{cases} 
  \rho_c (x-c) e^{-\frac{u_a(x-a)}{\epsilon}}, & x < a, \\
  \rho_c (x-c), & a < x < c, \\
  0, & c < x < d, \\
  \rho_d e^{-\frac{u_b}{\epsilon}}, & x > d. 
\end{cases} \tag{10} \]

The system (8) under the initial conditions (9)-(10) can be solved explicitly by

\[
\begin{pmatrix}
V^\epsilon(x, t) \\
S^\epsilon(x, t)
\end{pmatrix} = \frac{1}{\sqrt{\pi}} \begin{pmatrix}
\text{erfc} \left( \frac{x-a-u_a t}{\sqrt{2t\epsilon}} \right) \exp \left( \frac{(x-a-u_a t)^2-(x-a)^2}{2t\epsilon} \right) \\
\left( \text{erfc} \left( \frac{x-b}{\sqrt{2t\epsilon}} \right) - \text{erfc} \left( \frac{x-a}{\sqrt{2t\epsilon}} \right) \right) + \text{erfc} \left( -\frac{x-d}{\sqrt{2t\epsilon}} \right) e^{-\frac{u_b}{\epsilon}}
\end{pmatrix}
\]

\[
+ \frac{1}{\sqrt{2\pi t\epsilon}} \begin{pmatrix}
t \rho_c e^{-\frac{(x-a)^2}{2t\epsilon}} + \sqrt{2t\epsilon} \rho_c (x-c-u_at) e^{\frac{(x-x-\sqrt{2t\epsilon}^2-(x-a)^2}{2t\epsilon}} \text{erfc} \left( \frac{x-a-u_a t}{\sqrt{2t\epsilon}} \right) \\
+ t \rho_c \left( e^{-\frac{(x-a)^2}{2t\epsilon}} - e^{-\frac{(x-c)^2}{2t\epsilon}} \right) + \sqrt{2t\epsilon} \rho_c (x-c) \left( \text{erfc} \left( \frac{x-a}{\sqrt{2t\epsilon}} \right) - \text{erfc} \left( \frac{x-c}{\sqrt{2t\epsilon}} \right) \right)
\end{pmatrix}
\]

where \( \text{erfc} : z \mapsto \int_z^\infty e^{-t^2} \, dt \) for every \( z \in \mathbb{R}^1 \). Coming back to the original problem (3)-(4), we explicitly recover \( u^\epsilon \) and \( R^\epsilon \) as follows:
$u^t = -\epsilon \cdot \sqrt{\frac{\epsilon}{\sqrt{\epsilon}}}$

$u_a \frac{\sqrt{2t\epsilon}}{\epsilon} \text{erfc}\left(\frac{x - a - u_a t}{\sqrt{2t\epsilon}}\right) e^{\frac{(x-a-u_a t)^2-(x-a)^2}{2t\epsilon}}$

$= \frac{\epsilon}{\sqrt{2t\epsilon}} \text{erfc}\left(\frac{x - a - u_a t}{\sqrt{2t\epsilon}}\right) \exp\left(\frac{(x - a - u_a t)^2 - (x - a)^2}{2t\epsilon}\right) + e^{-\frac{(x-b)^2}{2t\epsilon}} \left(1 - e^{-\frac{u_a}{2t\epsilon}}\right)$

$R^e = \text{S}_{\epsilon} \left(\frac{t\epsilon}{\sqrt{\epsilon}} \rho_c \epsilon^{-\frac{(x-a)^2}{2t\epsilon}} + \rho_c(x-c-u_a t) \epsilon^{-\frac{(x-a-u_a t)^2-(x-a)^2}{2t\epsilon}} \text{erfc}\left(\frac{x - a - u_a t}{\sqrt{2t\epsilon}}\right) \exp\left(\frac{(x - a - u_a t)^2 - (x - a)^2}{2t\epsilon}\right) + \rho_d \epsilon^{-\frac{b}{2t\epsilon}} \text{erfc}\left(\frac{x - d}{\sqrt{2t\epsilon}}\right) + \left(\text{erfc}\left(\frac{x - b}{\sqrt{2t\epsilon}}\right) - \text{erfc}\left(\frac{x - a}{\sqrt{2t\epsilon}}\right)\right) + \text{erfc}\left(\frac{x - b}{\sqrt{2t\epsilon}}\right) e^{-\frac{u_a}{2t\epsilon}} \right)$

Throughout this article, we assume that $u_a, u_b \neq 0$. For each $\epsilon > 0$ and $(x, t) \in \mathbb{R}^1 \times (0, \infty)$, define

- $A_\epsilon = A_\epsilon(x, t) := \frac{|x-a|}{\sqrt{2t\epsilon}}$
- $B_\epsilon = B_\epsilon(x, t) := \frac{|x-b|}{\sqrt{2t\epsilon}}$
- $C_\epsilon = C_\epsilon(x, t) := \frac{|x-c|}{\sqrt{2t\epsilon}}$
- $D_\epsilon = D_\epsilon(x, t) := \frac{|x-d|}{\sqrt{2t\epsilon}}$
- $P_\epsilon = P_\epsilon(x, t) := \frac{|x-a-u_a t|}{\sqrt{2t\epsilon}}$

Then

- $A_\epsilon(x, t) \xrightarrow{t \to 0} \infty$ whenever $x \neq a$
- $B_\epsilon(x, t) \xrightarrow{t \to 0} \infty$ whenever $x \neq b$
- $C_\epsilon(x, t) \xrightarrow{t \to 0} \infty$ whenever $x \neq c$
- $D_\epsilon(x, t) \xrightarrow{t \to 0} \infty$ whenever $x \neq d$
- $P_\epsilon(x, t) \xrightarrow{t \to 0} \infty$ whenever $x \neq a + u_a t$

Our next objective is to obtain the corresponding explicit expressions of $u^t = u^t(x, t)$ and $R^e = R^e(x, t)$ in the regions $x < a$, $a < x < c$, $c < x < b$, $b < x < d$ and $x > d$ in terms of $A_\epsilon$, $B_\epsilon$, $C_\epsilon$, $D_\epsilon$ and $P_\epsilon$. The details of the relevant computations involved here have been provided in the appendix.
\[\begin{align*}
\bullet \ x &< a \\
u' &= \begin{cases}
u_a \frac{\text{erfc}(P_c) e^{P_c^2 - A^2} + \frac{\epsilon}{\sqrt{2\pi}} \left(1 - e^{-\frac{x^2}{2}}\right) e^{-B_c^2}}{\text{erfc}(A_c) + \text{erfc}(B_c) \left(e^{-\frac{x^2}{2}} - 1\right) + \text{erfc}(P_c) e^{P_c^2 - A^2}} , & x > a + u_a t, \\
&\nu_a \left(\sqrt{\pi} - \text{erfc}(P_c)\right) e^{P_c^2 - A^2} + \frac{\epsilon}{\sqrt{2\pi}} \left(1 - e^{-\frac{x^2}{2}}\right) e^{-B_c^2} , & x < a + u_a t,
\end{cases}
\end{align*}\]

\[\begin{align*}
\bullet \ a < x < c \\
R' &= \begin{cases}
u_a \frac{\sqrt{2\pi} \left(1 - e^{\frac{x^2}{2}}\right) + (x - c - u_a t) \text{erfc}(P_c) e^{P_c^2 - A^2} + (x - c) \left(\text{erfc}(A_c) - \text{erfc}(C_c)\right) + \rho_d \text{erfc}(D_c) e^{-\frac{x^2}{2}}}{\text{erfc}(A_c) + \text{erfc}(B_c) \left(e^{-\frac{x^2}{2}} - 1\right) + \text{erfc}(P_c) e^{P_c^2 - A^2}} , & x > a + u_a t, \\
&\nu_a \left(\sqrt{\pi} - \text{erfc}(P_c)\right) e^{P_c^2 - A^2} + \frac{\epsilon}{\sqrt{2\pi}} \left(1 - e^{-\frac{x^2}{2}}\right) e^{-B_c^2} , & x < a + u_a t,
\end{cases}
\end{align*}\]

\[\begin{align*}
\bullet \ c < x < b \\
R' &= \begin{cases}
u_a \frac{\sqrt{2\pi} \left(1 - e^{\frac{x^2}{2}}\right) + (x - c - u_a t) \text{erfc}(P_c) e^{P_c^2 - A^2} + (x - c) \left(\sqrt{\pi} - \text{erfc}(P_c)\right) e^{P_c^2 - A^2} + (x - c) \left(\sqrt{\pi} - \text{erfc}(P_c)\right) e^{P_c^2 - A^2} + (x - c) \left(\sqrt{\pi} - \text{erfc}(P_c)\right) e^{P_c^2 - A^2} + (x - c) \left(\sqrt{\pi} - \text{erfc}(P_c)\right) e^{P_c^2 - A^2} + (x - c) \left(\sqrt{\pi} - \text{erfc}(P_c)\right) e^{P_c^2 - A^2} , & x > a + u_a t, \\
&\nu_a \left(\sqrt{\pi} - \text{erfc}(P_c)\right) e^{P_c^2 - A^2} + \frac{\epsilon}{\sqrt{2\pi}} \left(1 - e^{-\frac{x^2}{2}}\right) e^{-B_c^2} , & x < a + u_a t,
\end{cases}
\end{align*}\]
\[
u' = \begin{cases} 
\frac{u_a \text{erfc} (P_c) e^{P^2 - A^2} + \frac{e}{\sqrt{2\pi e}} (1 - e^{-\frac{2A}{e^2}}) e^{-B^2}}{\sqrt{\pi - \text{erfc}(A_c) + \text{erfc}(B_c) (e^{-\frac{2A}{e^2}} - 1) + \text{erfc} (P_c) e^{P^2 - A^2}}}, & x > a + u_a t, \\
\frac{u_a \left(\sqrt{\pi - \text{erfc} (P_c)}\right) e^{P^2 - A^2} + \frac{e}{\sqrt{2\pi e}} (1 - e^{-\frac{2A}{e^2}}) e^{-B^2}}{\sqrt{\pi - \text{erfc}(A_c) + \text{erfc}(B_c) (e^{-\frac{2A}{e^2}} - 1) + \left(\sqrt{\pi - \text{erfc} (P_c)}\right) e^{P^2 - A^2}}}, & x < a + u_a t,
\end{cases}
\]

\[
R' = \begin{cases} 
\frac{\rho_c \left[\sqrt{2\pi e} \left(\frac{1}{e^{A^2}} - \frac{1}{2 e^{C^2}}\right)\right] + (x - c - u_a t) \text{erfc} (P_c) e^{P^2 - A^2}}{\sqrt{\pi - \text{erfc}(A_c) + \text{erfc}(B_c) (e^{-\frac{2A}{e^2}} - 1) + \text{erfc} (P_c) e^{P^2 - A^2}}}, & x > a + u_a t, \\
\frac{\rho_c \left[\sqrt{2\pi e} \left(\frac{1}{e^{A^2}} - \frac{1}{2 e^{C^2}}\right)\right] + (x - c - u_a t) \left(\sqrt{\pi - \text{erfc} (P_c)}\right) e^{P^2 - A^2}}{\sqrt{\pi - \text{erfc}(A_c) + \text{erfc}(B_c) (e^{-\frac{2A}{e^2}} - 1) + \left(\sqrt{\pi - \text{erfc} (P_c)}\right) e^{P^2 - A^2}}}, & x < a + u_a t,
\end{cases}
\]

- \(b < x < d\)

\[
u' = \begin{cases} 
\frac{u_a \text{erfc} (P_c) e^{P^2 - A^2} + \frac{e}{\sqrt{2\pi e}} (1 - e^{-\frac{2A}{e^2}}) e^{-B^2}}{\sqrt{\pi e^{-\frac{2A}{e^2}} - \text{erfc}(A_c) + \text{erfc}(B_c) (1 - e^{-\frac{2A}{e^2}}) + \text{erfc} (P_c) e^{P^2 - A^2}}}, & x > a + u_a t, \\
\frac{u_a \left(\sqrt{\pi - \text{erfc} (P_c)}\right) e^{P^2 - A^2} + \frac{e}{\sqrt{2\pi e}} (1 - e^{-\frac{2A}{e^2}}) e^{-B^2}}{\sqrt{\pi e^{-\frac{2A}{e^2}} - \text{erfc}(A_c) + \text{erfc}(B_c) (1 - e^{-\frac{2A}{e^2}}) + \left(\sqrt{\pi - \text{erfc} (P_c)}\right) e^{P^2 - A^2}}}, & x < a + u_a t,
\end{cases}
\]

\[
R' = \begin{cases} 
\frac{\rho_c \left[\sqrt{2\pi e} \left(\frac{1}{e^{A^2}} - \frac{1}{2 e^{C^2}}\right)\right] + (x - c - u_a t) \text{erfc} (P_c) e^{P^2 - A^2}}{\sqrt{\pi e^{-\frac{2A}{e^2}} - \text{erfc}(A_c) + \text{erfc}(B_c) (1 - e^{-\frac{2A}{e^2}}) + \text{erfc} (P_c) e^{P^2 - A^2}}}, & x > a + u_a t, \\
\frac{\rho_c \left[\sqrt{2\pi e} \left(\frac{1}{e^{A^2}} - \frac{1}{2 e^{C^2}}\right)\right] + (x - c - u_a t) \left(\sqrt{\pi - \text{erfc} (P_c)}\right) e^{P^2 - A^2}}{\sqrt{\pi e^{-\frac{2A}{e^2}} - \text{erfc}(A_c) + \text{erfc}(B_c) (1 - e^{-\frac{2A}{e^2}}) + \left(\sqrt{\pi - \text{erfc} (P_c)}\right) e^{P^2 - A^2}}}, & x < a + u_a t,
\end{cases}
\]

- \(x > d\)
asymptotic properties of the function \( \text{erf}c \):

Depending on the relative positions of \( u_a \) and \( u_b \), we study the asymptotic behavior of \((u', R')\) as \( \epsilon \to 0 \) in each of the regions mentioned above. For discussing the passage to the limit, we will extensively use the following asymptotic properties of the function \( \text{erf}c:\)

1. \( \lim_{z \to \infty} \text{erf}c(z) = 0 \)
2. \( \text{erf}c(z) = \left( \frac{1}{2z} - \frac{1}{4z^3} + o \left( \frac{1}{z^3} \right) \right) e^{-z^2} \) as \( z \to \infty \)
3. \( \lim_{z \to \infty} f(z) = \frac{1}{2} \), where \( f : z \mapsto z \text{erf}c(z) e^{z^2} \) for every \( z \in \mathbb{R}^1 \)

These properties have been proved in [2]. However, for the sake of completeness, the derivations have again been provided in the appendix.

**Case 1.** \( u_a < 0, u_b > 0 \)

1. \( x < a \)

Within this region, the limit \( \lim_{\epsilon \to 0} (u', R') \) has to be separately evaluated in the subregions \( x > a + u_a t \)

and \( x < a + u_a t \) as follows:
Subregion 1. $x > a + u_at$

$$\lim_{\epsilon \to 0} \frac{b - x}{a - x} \left( \frac{(b - x) f(A_x)}{x - a - u_at} + \frac{(b - x) \left( 1 - e^{-\frac{|x-a|}{2t}} \right)}{e^{B_x^2 - A_x^2}} \right) + \rho_c (x - c - u_at) \left( \frac{(b - x) f(P_a)}{x - a - u_at} \right)$$

$$= \left( \frac{x - a}{\epsilon}, \rho_c \frac{2(x - a - u_at)(x - a) + (a - c) u_at}{u_at} \right).$$

Subregion 2. $x < a + u_at$

$$\lim_{\epsilon \to 0} \frac{b - x}{a - x} \left( \frac{(b - x) f(A_x)}{x - a - u_at} + \frac{(b - x) \left( 1 - e^{-\frac{|x-a|}{2t}} \right)}{e^{B_x^2 + P_x^2 - A_x^2}} \right) + \rho_c (x - c - u_at) \left( \sqrt{\pi} - \text{erfc}(P_c) \right)$$

$$= (u_a, \rho_c (x - c - u_at)).$$

Here we have used the strict inequalities

$$(b - x)^2 + (a + u_at - x)^2 - (a - x)^2 = (a + u_at - x)^2 + 2 \left( b - a \right) \left| \frac{a + b}{2} - x \right| > 0,$$

$$(d - x)^2 + (a + u_at - x)^2 - (a - x)^2 = (a + u_at - x)^2 + 2 \left( d - a \right) \left| \frac{a + d}{2} - x \right| > 0.$$

For the remaining regions, the restriction $u_a < 0$ will automatically imply that $x > a + u_at$. Therefore, the limit $\lim_{\epsilon \to 0} (u', R')$ in each of these regions will be evaluated as follows:
\( b < x < d \)

\[
\begin{aligned}
\lim_{\epsilon \to 0} & \quad \left( u_a \frac{f(P_e)}{P_e e^{A_e^2}} + \epsilon \frac{1}{\sqrt{2t\epsilon}} \left( 1 - e^{-\frac{|x|}{\epsilon}} \right) e^{-B_e^2} \right)
\end{aligned}
\]

\[
\begin{aligned}
\sqrt{\pi} - \text{erfc}(A_e) + \text{erfc}(B_e) \left( e^{-\frac{|x|}{\epsilon}} - 1 \right) + \frac{f(P_e)}{P_e e^{A_e^2}}
\end{aligned}
\]

\[
\begin{aligned}
\sqrt{\pi} - \text{erfc}(A_e) + \text{erfc}(B_e) \left( e^{-\frac{|x|}{\epsilon}} - 1 \right) + \frac{f(P_e)}{P_e e^{A_e^2}}
\end{aligned}
\]

\[
\begin{aligned}
\left( 0, \rho_e (x-c) \right).
\end{aligned}
\]

\( c < x < b \)

\[
\begin{aligned}
\lim_{\epsilon \to 0} & \quad \left( u_a \frac{f(P_e)}{P_e e^{A_e^2}} + \epsilon \frac{1}{\sqrt{2t\epsilon}} \left( 1 - e^{-\frac{|x|}{\epsilon}} \right) e^{-B_e^2} \right)
\end{aligned}
\]

\[
\begin{aligned}
\sqrt{\pi} - \text{erfc}(A_e) + \text{erfc}(B_e) \left( e^{-\frac{|x|}{\epsilon}} - 1 \right) + \frac{f(P_e)}{P_e e^{A_e^2}}
\end{aligned}
\]

\[
\begin{aligned}
\sqrt{\pi} - \text{erfc}(A_e) + \text{erfc}(B_e) \left( e^{-\frac{|x|}{\epsilon}} - 1 \right) + \frac{f(P_e)}{P_e e^{A_e^2}}
\end{aligned}
\]

\[
\begin{aligned}
(0, 0).
\end{aligned}
\]

\( b < x < d \)

\[
\begin{aligned}
\lim_{\epsilon \to 0} & \quad \left( u_a \frac{(x-b) f(P_e)}{(x-a - u_a t) e^{A_e^2-B_e^2}} + \frac{x-b}{2t} \left( 1 - e^{-\frac{|x|}{\epsilon}} \right) \right)
\end{aligned}
\]

\[
\begin{aligned}
\sqrt{\pi} B_e e^{B_e^2 - \frac{|x|}{\epsilon}} - \frac{(x-b) f(A_e)}{(x-a) e^{A_e^2-B_e^2}} + f(B_e) \left( 1 - e^{-\frac{|x|}{\epsilon}} \right) + \frac{(x-b) f(P_e)}{(x-a - u_a t) e^{A_e^2-B_e^2}}
\end{aligned}
\]

\[
\begin{aligned}
\sqrt{\pi} B_e e^{B_e^2 - \frac{|x|}{\epsilon}} - \frac{(x-b) f(A_e)}{(x-a) e^{A_e^2-B_e^2}} + f(B_e) \left( 1 - e^{-\frac{|x|}{\epsilon}} \right) + \frac{(x-b) f(P_e)}{(x-a - u_a t) e^{A_e^2-B_e^2}}
\end{aligned}
\]

\[
\begin{aligned}
\left( 0, 0 \right), \quad x > b + \sqrt{2u_b t},
\end{aligned}
\]

\[
\begin{aligned}
\left( \frac{x-b}{t}, 0 \right), \quad x < b + \sqrt{2u_b t}.
\end{aligned}
\]

(5) \( x > d \)
The explicit structure of \( u \) and \( R \) under the present case can then be described as follows:

\[
\begin{align*}
\lim_{t \to 0} \rho c \frac{x - b}{2 e^{C^2/B^2}} + \rho c \frac{x - c - u_a t}{(x - a - u_a t) e^{A^2/B^2}} f(A_x) \\
&+ \rho d B e^{B^2 - \frac{2}{\pi}} \frac{\rho e (x - b) f(P_x)}{(x - a - u_a t) e^{A^2/B^2}} f(B_x) \left(1 - e^{-\frac{2}{\pi}} \right)
\end{align*}
\]

\[
\begin{align*}
= \begin{cases}
(0, \rho d), & x > b + \sqrt{2u_b}, \\
\left(\frac{x - b}{t}, 0\right), & x < b + \sqrt{2u_b}.
\end{cases}
\end{align*}
\]

Now, to recover \( \rho \), set \( u = \lim_{t \to 0} u^t \) and \( R = \lim_{t \to 0} R^t \). For each \( s \geq 0 \), let us define

1. \( r(s) := a + u_a s \),
2. \( p(s) := b + \sqrt{2u_b s} \).

The next step is to consider an arbitrary test function \( \phi \in C_0^\infty (\mathbb{R}^d, \mathbb{R}^1) \) and consider the action of the distributional derivative \( R_x \) of \( R \) with respect to the space variable \( x \) on \( \phi \). We see that
\((R_x, \phi) = -(R, \phi_t)\)

\[
= -\rho c \int_0^\infty \left[ \int_a^{a+u_at} (x - c - u_at) \phi_x dx + \int_{a+u_at}^a 2\frac{(x - a - u_at)(x - a) + (a - c)u_at}{u_at} \phi_x dx \right. \\
+ \left. \int_a^c (x - c) \phi_x dx \right] dt - \rho d \left[ \int_0^\infty \int_{\gamma_d(t)}^a \phi_x dx dt \right] \\
= \rho c \int_0^\infty \int_a^{a+u_at} \phi(x, t) dx dt + (c - a) \int_0^\infty \phi(a + u_at, t) dt \\
+ \frac{4}{u_at} \int_0^\infty \int_a^{a+u_at} (x - a) \phi(x, t) dx dt - 2 \int_0^\infty \int_{a+u_at}^a \phi(x, t) dx dt \\
+ (c - a) \int_0^\infty \phi(a, t) dt - (c - a) \int_0^\infty \phi(a + u_at, t) dt \\
+ \int_0^\infty \int_a^c \phi(x, t) dx dt - (c - a) \int_0^\infty \phi(a, t) dt \right] + \rho d \int_0^\infty \phi(\gamma_a(t), t) dt \\
= \left( \rho c \left( \chi_{(-\infty, a+u_at)} + \frac{4(x - a) - 2u_at}{u_at} \chi_{(a+u_at, a)} + \chi_{(a,c)} \right) + \rho d \delta_x=\gamma_a(t), \phi \right),
\]

where the curve \(x = \gamma_a(t)\) is defined on \([0, \infty)\) by

\[
\gamma_a(t) := \begin{cases} 
  d, & 0 \leq t \leq t^* := \frac{(d-b)^2}{2u_b}, \\
  b + \sqrt{2u_b}, & t > t^*.
\end{cases}
\]

Therefore \(\rho = \rho_c \left( \chi_{(-\infty, a+u_at)} + \frac{4(x - a) - 2u_at}{u_at} \chi_{(a+u_at, a)} + \chi_{(a,c)} \right) + \rho d \delta_x=\gamma_a(t)\).

**Case 2.** \(u_a > 0, u_b > 0\)

(1) \(x < a\)

In this region, we have \(x < a + u_at\), since \(u_a > 0\). Hence \(\lim_{c \to 0} (u', R')\) equals

\[
\lim_{c \to 0} \rho_c \left( \begin{array}{c}
 u_a \left( \sqrt{\pi} - \text{erfc}(P_e) \right) + \frac{a - x}{2t} \frac{1 - e^{-\frac{|x|}{2}}} {A_e e^{P_e^2 + B_e^2 - \Lambda^2}} \\
 f(A_e) e^{P_e^2} + f(B_e) e^{P_e^2 + B_e^2 - \Lambda^2} + \left( \sqrt{\pi} - \text{erfc}(P_e) \right)
\end{array} \right) \rightarrow T
\]

\[
= \left( u_a, \rho_c (x - c - u_at) \right).
\]

Here we have used the strict inequalities

\[
(x - a - u_at)^2 + (x - b)^2 - (x - a)^2 = (x - a - u_at)^2 + 2 \left| x - \frac{a + b}{2} \right| (b - a) > 0,
\]

\[
(x - a - u_at)^2 + (x - d)^2 - (x - a)^2 = (x - a - u_at)^2 + 2 \left| x - \frac{a + d}{2} \right| (d - a) > 0.
\]

In each of the remaining regions under this case, the limit \(\lim_{c \to 0} (u', R')\) has to be evaluated separately in the subregions \(x > a + u_at\) and \(x < a + u_at\).
(2) \( a < x < c \)

Subregion 1. \( x > a + u_a t \)

\[
\lim_{\epsilon \to 0} \frac{u_a f(P_c) e^{A_c^2} + (b - x) \left( 1 - e^{-\frac{|u_a|}{2}} \right)}{2 t A_c e^{B_c^2}}
\]

\[
\frac{(\sqrt{\pi} - \text{erfc}(A_c)) + \text{erfc}(B_c) \left( e^{-\frac{|u_a|}{2}} - 1 \right) + \frac{f(P_c)}{P_c e^{A_c^2}}}{\rho_c \left[ \sqrt{2t} \left( e^{-A_c^2} - \frac{1}{2} e^{C_c^2} \right) + (x - c - u_a t) \left( \sqrt{\pi} - \text{erfc}(P_c) \right) e^{P_c^2 - A_c^2} + (x - c) \left( \sqrt{\pi} - \text{erfc}(A_c) - \text{erfc}(C_c) \right) \right] + \rho_d \text{erfc}(D_c) e^{-\frac{|u_a|}{2}}}
\]

\[
\left( \sqrt{\pi} - \text{erfc}(A_c) \right) + \text{erfc}(B_c) \left( e^{-\frac{|u_a|}{2}} - 1 \right) + \frac{f(P_c)}{P_c e^{A_c^2}}
\]

\[
= (0, \rho_c (x - c)).
\]

Subregion 2. \( x < a + u_a t \)

\[
\lim_{\epsilon \to 0} \frac{u_a \left( \sqrt{\pi} - \text{erfc}(P_c) \right) e^{P_c^2 - A_c^2} + (x - a) \left( 1 - e^{-\frac{|u_a|}{2}} \right)}{2 t A_c e^{B_c^2}}
\]

\[
\frac{(\sqrt{\pi} - \text{erfc}(A_c)) + \text{erfc}(B_c) \left( e^{-\frac{|u_a|}{2}} - 1 \right) + \left( \sqrt{\pi} - \text{erfc}(P_c) \right) e^{P_c^2 - A_c^2}}{\rho_c \left[ \sqrt{2t} \left( e^{-A_c^2} - \frac{1}{2} e^{C_c^2} \right) + (x - c - u_a t) \left( \sqrt{\pi} - \text{erfc}(P_c) \right) e^{P_c^2 - A_c^2} + (x - c) \left( \sqrt{\pi} - \text{erfc}(A_c) - \text{erfc}(C_c) \right) \right] + \rho_d \text{erfc}(D_c) e^{-\frac{|u_a|}{2}}}
\]

\[
= \begin{cases} 
(u_a, \rho_c (x - c - u_a t)), & x \in \left( a, a + \frac{u_a}{2}, t \right), \\
(0, \rho_c (x - c)), & x \in \left( a + \frac{u_a}{2}, t, a + u_a t \right).
\end{cases}
\]

(3) \( c < x < b \)

Subregion 1. \( x > a + u_a t \)

\[
\lim_{\epsilon \to 0} \frac{u_a f(P_c) e^{A_c^2} + (b - x) \left( 1 - e^{-\frac{|u_a|}{2}} \right)}{2 t A_c e^{B_c^2}}
\]

\[
\frac{(\sqrt{\pi} - \text{erfc}(A_c)) + \text{erfc}(B_c) \left( e^{-\frac{|u_a|}{2}} - 1 \right) + \frac{f(P_c)}{P_c e^{A_c^2}}}{\rho_c \left[ \sqrt{2t} \left( e^{-A_c^2} - \frac{1}{2} e^{C_c^2} \right) + (x - c - u_a t) \left( \sqrt{\pi} - \text{erfc}(P_c) \right) e^{P_c^2 - A_c^2} + (x - c) \left( \sqrt{\pi} - \text{erfc}(C_c) - \text{erfc}(A_c) \right) \right] + \rho_d \text{erfc}(D_c) e^{-\frac{|u_a|}{2}}}
\]

\[
\left( \sqrt{\pi} - \text{erfc}(A_c) \right) + \text{erfc}(B_c) \left( e^{-\frac{|u_a|}{2}} - 1 \right) + \frac{f(P_c)}{P_c e^{A_c^2}}
\]

\[
= (0, 0).
\]
Subregion 2. \( x < a + u_a t \)

\[
\lim_{\epsilon \to 0} \left( u_a \left( \sqrt{\pi} - \text{erfc} \left( P_\epsilon \right) \right) e^{P_\epsilon^2 - A^2} + \frac{(b - x) \left( 1 - e^{-\frac{|\epsilon|}{\epsilon}} \right)}{2t \, B_\epsilon e^{B_\epsilon^2}} \right) T
\]

\[
\frac{\sqrt{\pi} - \text{erfc} \left( A_\epsilon \right) + \text{erfc} \left( B_\epsilon \right) \left( e^{-\frac{|\epsilon|}{\epsilon}} - 1 \right) + \left( \sqrt{\pi} - \text{erfc} \left( P_\epsilon \right) \right) e^{P_\epsilon^2 - A^2}}{\rho_c \left[ \sqrt{2t} \left( \frac{1}{e^{A^2}} - \frac{1}{2 \, e^{B_\epsilon^2}} \right) + (x - c - u_a t) \left( \sqrt{\pi} - \text{erfc} \left( P_\epsilon \right) \right) e^{P_\epsilon^2 - A^2} \right] + \rho_d \, \text{erfc} \left( D_\epsilon \right) e^{-\frac{|\epsilon|}{\epsilon}} + (x - c) \left( \text{erfc} \left( C_\epsilon \right) - \text{erfc} \left( A_\epsilon \right) \right)}
\]

\[
= \begin{cases} (u_a, \rho_c (x - c - u_a t)), & x \in (c, a + \frac{u_a}{2} \cdot t), \\ (0, 0), & x \in (a + \frac{u_a}{2} \cdot t, a + u_a t). \end{cases}
\]

(4) \( b < x < d \)

Let us first define

(i) \( l(s) := a + \frac{u_a}{u_a} + \frac{u_a}{2} \cdot s, \)

(ii) \( r(s) := a + u_a s, \)

(iii) \( p(s) := b + \sqrt{2u_a s}, \)

(iv) \( q(s) := b + u_a s - \sqrt{2u_a(b - a) s} \)

for each \( s \geq 0 \). Then we can evaluate \( \lim_{\epsilon \to 0} (u', R') \) separately in the subregions \( x > a + u_a t \) and \( x < a + u_a t \) as follows:

Subregion 1. \( x > a + u_a t \)

\[
\lim_{\epsilon \to 0} \left( u_a \left( \sqrt{\pi} - \text{erfc} \left( P_\epsilon \right) \right) e^{P_\epsilon^2 - A^2} + \frac{(x - b) \left( 1 - e^{-\frac{|\epsilon|}{\epsilon}} \right)}{2t \, B_\epsilon e^{B_\epsilon^2}} \right) T
\]

\[
\frac{\sqrt{\pi} - \text{erfc} \left( A_\epsilon \right) + \text{erfc} \left( B_\epsilon \right) \left( e^{-\frac{|\epsilon|}{\epsilon}} - 1 \right) + \left( \sqrt{\pi} - \text{erfc} \left( P_\epsilon \right) \right) e^{P_\epsilon^2 - A^2}}{\rho_c \left[ \sqrt{2t} \left( \frac{1}{e^{A^2}} - \frac{1}{2 \, e^{B_\epsilon^2}} \right) + (x - c - u_a t) \left( \sqrt{\pi} - \text{erfc} \left( P_\epsilon \right) \right) e^{P_\epsilon^2 - A^2} \right] + \rho_d \, \text{erfc} \left( D_\epsilon \right) e^{-\frac{|\epsilon|}{\epsilon}} + (x - c) \left( \text{erfc} \left( C_\epsilon \right) - \text{erfc} \left( A_\epsilon \right) \right)}
\]

\[
= \begin{cases} (0, 0), & x \in \left( \max \{p(t), r(t)\}, d \right), \\ \left( \frac{x - b}{u_a}, 0 \right), & x \in \left( r(t), p(t) \right). \end{cases}
\]
Subregion 2. $x < a + u_at$

\[
\lim_{\epsilon \to 0} \left\{ \begin{array}{rl}
\rho_c \left[ & (x-b) \left( e^{B_2^2-A^2} \left( \frac{1}{2} e^{C_2^2-B_2^2} \right) + \left( \frac{x-a}{x-a-u_at} \right) \right) + f(B_c) \left( 1 - e^{-\frac{|u|}{2t}} \right) \right] \\
& \left( \frac{x-a}{x-a-u_at} \right) \frac{\sqrt{\pi} B_c e^{B_2^2-A_2^2} - (x-b) f(A_c) e^{B_2^2-A_2^2}}{x-a} + f(B_c) \left( 1 - e^{-\frac{|u|}{2t}} \right) + \sqrt{\pi} B_c e^{B_2^2-A_2^2} \\
& + \rho_d \left( \sqrt{\pi} - \text{erfc} (D_c) \right) B_c e^{B_2^2-A_2^2} \\
\end{array} \right.
\]

\[
\begin{cases}
(0, 0), & x \in \left( \max \{p(t), q(t)\}, \infty \right) \\
(u_a, \rho_c (x-c-u_at)), & x \in \left( -\infty, \min \{p(t), q(t)\} \right) \cup \left( \max \{l(t), p(t)\}, \min \{q(t), r(t)\} \right) \\
\left( \frac{x-a}{x-a-u_at}, 0 \right), & x \in \left( q(t), \min \{p(t), r(t)\} \right) \\
\end{cases}
\]

(5) $x > d$

As in the preceding case, we need to consider the following curves defined in $[0, \infty)$:

(i) $l(s) = a + \frac{u_a}{u_a} + \frac{u_a}{2} s$

(ii) $r(s) = a + u_at$

(iii) $p(s) = b + \sqrt{2u_a}s$

(iv) $q(s) = b + u_as - \sqrt{2u_a} (b-a) s$

We can now describe the explicit structure of $\lim_{\epsilon \to 0} (u^t, R^t)$ in different subregions as follows:

\[
\text{Subregion 1. } x > a + u_at
\]

\[
\lim_{\epsilon \to 0} \left\{ \begin{array}{rl}
\rho_c \left[ & (x-b) f(P_c) e^{B_2^2-A_2^2} \left( \frac{1}{2} e^{C_2^2-B_2^2} \right) + \left( \frac{x-a}{x-a-u_at} \right) \right) + f(P_c) \left( 1 - e^{-\frac{|u|}{2t}} \right) \right] \\
& \left( \frac{x-a}{x-a-u_at} \right) \frac{\sqrt{\pi} B_c e^{B_2^2-A_2^2} - (x-b) f(A_c) e^{B_2^2-A_2^2}}{x-a} + f(P_c) \left( 1 - e^{-\frac{|u|}{2t}} \right) + \sqrt{\pi} B_c e^{B_2^2-A_2^2} \\
& - \rho_d \left( \sqrt{\pi} - \text{erfc} (D_c) \right) B_c e^{B_2^2-A_2^2} \\
\end{array} \right.
\]

\[
\begin{cases}
(0, \rho_d), & x \in \left( \max \{p(t), r(t)\}, \infty \right), \\
\left( \frac{x-a}{x-a-u_at}, 0 \right), & x \in \left( r(t), p(t) \right). \\
\end{cases}
\]
Subregion 2. $x < a + u_a t$

\[
\lim_{\epsilon \to 0} \rho_c \begin{cases}
(0, \rho_d) , & x \in \left( \max \{p(t), q(t)\} , r(t) \right) \cup \left( \max \{l(t), p(t)\} , \min \{q(t), r(t)\} \right), \\
(u_a, \rho_c (x - c - u_a t)) , & x \in \left( -\infty , \min \{p(t), q(t), r(t)\} \right) \cup \left( p(t) , \min \{l(t), q(t), r(t)\} \right), \\
\left( \frac{x-b}{\sqrt{\pi}} , 0 \right) , & x \in \left( q(t) , \min \{p(t), r(t)\} \right). 
\end{cases}
\]

Now we have to recover the $\rho$ component. For this purpose, set $R = \lim_{\epsilon \to 0} R'$. There are three cases to consider, namely

- $2u_b < u_a (b - a)$
- $u_b < u_a (b - a) \leq 2u_b$
- $u_a (b - a) \leq u_b$

Here we provide the details only for the case $2u_b < u_a (b - a)$. The other cases can be studied similarly.

For further simplification, let us also assume that $x_{p,r} < d < x_{p,l}$, where $x_{p,r}$ and $x_{p,l}$ denote the respective $x$ co-ordinates of the points of intersection of $x = p(t)$ with the curves $x = r(t)$ and $x = l(t)$.

In addition to the curves $x = l(t)$, $x = r(t)$, $x = p(t)$ and $x = q(t)$ defined above, let us also introduce the following curves on $[0, \infty)$:
\[\tilde{l}(t) := a + \frac{u_2}{2} \cdot t,\]

\[\gamma_a(t) := \begin{cases} 
    a + \frac{u_2}{2} \cdot t, & 0 \leq t \leq \frac{2(b-a)}{u_a}, \\
    b + u_2 t - \sqrt{2u_a (b-a) t}, & \frac{2(b-a)}{u_a} \leq t \leq t_{p,1}, \\
    a + \frac{u_2}{u_a} + \frac{u_2}{2} \cdot t, & t \geq t_{p,1},
\end{cases}\]

\[\gamma_b(t) := \begin{cases} 
    b + \sqrt{2u_2 t}, & 0 \leq t \leq t_{p,1}, \\
    a + \frac{u_2}{u_a} + \frac{u_2}{2} \cdot t, & t \geq t_{p,1},
\end{cases}\]

\[\gamma_c(t) := \begin{cases} 
    c, & 0 \leq t \leq \frac{2(c-a)}{u_a}, \\
    \gamma_a(t), & t \geq \frac{2(c-a)}{u_a},
\end{cases}\]

\[\gamma_d(t) := \begin{cases} 
    d, & 0 \leq t \leq \frac{(d-b)^2}{2u_a}, \\
    \gamma_b(t), & t \geq \frac{(d-b)^2}{2u_a}.
\end{cases}\]

Here we have used the notation \(t_{p,1}\) to denote the positive \(t\) co-ordinate for the intersection of \(x = p(t)\) and \(x = l(t)\).

These curves can be used to describe the explicit structure of \((u, R) = \lim_{x \to 0} (u^*, R^*)\) as follows:

\[u(x, t) = \begin{cases} 
    u_a, & x \in \left( -\infty, a \right) \cup \left( (a, b) \setminus \{c\} \right) \cap \left( -\infty, \tilde{l}(t) \right) \\
    \cup \left( (b, \infty) \setminus \{d\} \right) \cap \left( \left( -\infty, \min \{p(t), q(t), r(t)\} \right) \cup \{p(t), \min \{l(t), q(t), r(t)\}\} \right), \\
    \frac{x-b}{t}, & x \in \left( (b, \infty) \setminus \{d\} \right) \cap \left( \{r(t), p(t)\} \cup \{q(t), \min \{p(t), r(t)\}\} \right), \\
    \left( (a, b) \setminus \{c\} \right) \cap \left( \tilde{l}(t), \infty \right) \setminus \{r(t)\}, & 0, \\
    \cup \left( (b, \infty) \setminus \{d\} \right) \cap \left( \{\max \{p(t), r(t)\}, \infty\} \\
    \cup \{\max \{p(t), q(t)\}, r(t)\} \cup \{\max \{l(t), p(t)\}, \min \{q(t), r(t)\}\} \right),
\end{cases}\]

\[R(x, t) = \begin{cases} 
    \rho_c (x - c - u_a t), & x \in \left( -\infty, \gamma_c(t) \right), \\
    \rho_c (x - c), & x \in \left( \gamma_c(t), \gamma_c(t) \right), \\
    0, & x \in \left( \gamma_c(t), \gamma_c(t) \right) \cup \left( \gamma_b(t), \gamma_d(t) \right), \\
    \rho_d, & x > \gamma_d(t).
\end{cases}\]

For any test function \(\phi \in C^\infty_c \left( \mathbb{R}^1 \times [0, \infty); \mathbb{R}^1 \right),\) we observe that
\[ \langle R_x, \phi \rangle = - \langle R, \phi_x \rangle \]
\[ = - \int_0^\infty \left[ \int_{-\infty}^{\gamma_a(t)} \rho_c(x - c - u_at) \phi_x \, dx + \int_{\gamma_a(t)}^{\gamma_c(t)} \rho_c(x - c) \phi_x \, dx + \int_{\gamma_c(t)}^\infty \rho_d \phi_x \, dx \right] \, dt \]
\[ = - \int_0^\infty \rho_c \left[ - \int_{-\infty}^{\gamma_a(t)} \phi \, dx + (\gamma_a(t) - c - u_at) \phi(\gamma_a(t), t) \right. \]
\[ - \left. \int_{\gamma_a(t)}^{\gamma_c(t)} \phi \, dx + (\gamma_c(t) - c) \phi(\gamma_c(t), t) - (\gamma_a(t) - c) \phi(\gamma_a(t), t) \right] \, dt \]
\[ + \rho_d \int_0^\infty \phi(\gamma_d(t), t) \, dt \]
\[ = \left( \rho_c \left( \chi_{(-\infty, \gamma_a(t))} - (x - c) \delta_{x=\gamma_a(t)} - u_at \delta_{x=\gamma_a(t)} \right) + \rho_d \delta_{x=\gamma_d(t)} \phi \right). \]

Therefore \( \rho = \rho_c \left( \chi_{(-\infty, \gamma_a(t))} - (x - c) \delta_{x=\gamma_a(t)} - u_at \delta_{x=\gamma_a(t)} \right) + \rho_d \delta_{x=\gamma_d(t)} \).

**Case 3.** \( u_a > 0, \, u_b < 0 \)

(1) \( x < a \)

In this region, we have \( x < a + u_at \) and hence, utilizing the inequality

\[ (a + u_at - x)^2 + (c - x)^2 - (a - x)^2 = (a + u_at - x)^2 + 2(c - a) \frac{a + c}{2} - x > 0, \]

the required limit \( \lim_{e \to 0} (u^e, R^e) \) equals

\[
\lim_{e \to 0} \left( \begin{array}{c}
\frac{u_a(\sqrt{e} - \text{erfc}(P_e)) + (b - x) \left( e^{-\frac{\|w\|}{\sqrt{e}}} - 1 \right)}{2t B_e e^{P_e^2 + B_e^2 - A_e^2 + \frac{2b}{c}}} \\
\frac{f(A_e)}{A_e e^{P_e^2}} + \frac{f(B_e) \left( 1 - e^{-\frac{\|w\|}{\sqrt{e}}} \right)}{B_e e^{P_e^2 + B_e^2 - A_e^2 + \frac{2b}{c}}} + (\sqrt{\pi} - \text{erfc}(P_e))
\end{array} \right)^T
\]

\[ = \left\{ \begin{array}{ll}
(u_a, \rho_c (x - c - u_at)), & x < b + u_at - \sqrt{2(u_a (b - a) - u_b)} t, \\
\left( \frac{x - b}{2}, 0 \right), & x > b + u_at - \sqrt{2(u_a (b - a) - u_b)} t.
\end{array} \right. \]

For discussing the passage to the limit in the remaining regions, we introduce the curves

(i) \( \tilde{l}(s) := a + \frac{\mu}{\sqrt{e}} \cdot s, \)
(ii) \( r(s) := a + u_as, \)
(iii) \( p(s) := b - \sqrt{-2u_b}s, \)
(iv) \( q(s) := b + u_as - \sqrt{2(u_a (b - a) - u_b)} s \)

defined for each \( s \geq 0 \). The limit \( \lim_{e \to 0} (u^e, R^e) \) can then be evaluated as follows:

(2) \( a < x < c \)
Subregion 1. $x > a + u_a t$

$$\begin{aligned}
\left(\begin{array}{c}
\frac{u_a f(P_c)}{P_c e^{A^2}} + \frac{(b - x) \left(e^{-\frac{|x|}{2}} - 1\right)}{2t B_c e^{B^2 + \frac{A^2}{2}}} \\
\sqrt{\pi} - \operatorname{erfc}(A_c) + \frac{f(B_c) \left(1 - e^{-\frac{|x|}{2}}\right)}{B_c e^{B^2 + \frac{A^2}{2}}} + \frac{f(P_c)}{P_c e^{A^2}} \\
\sqrt{\pi} - \operatorname{erfc}(A_c) + \frac{f(B_c) \left(1 - e^{-\frac{|x|}{2}}\right)}{B_c e^{B^2 + \frac{A^2}{2}}} + \frac{f(P_c)}{P_c e^{A^2}}
\end{array}\right)^T
\end{aligned}$$

$$\begin{aligned}
= \left\{ (0, \rho_c (x - c)), \quad x \in \left( r(t), p(t) \right), \\
\left(\frac{b - b}{t}, 0\right), \quad x \in \left( \max \{ r(t), p(t) \}, \infty \right). \right. 
\end{aligned}$$

Subregion 2. $x < a + u_a t$

$$\begin{aligned}
\lim_{c \to 0} \left(\begin{array}{c}
\frac{u_a \sqrt{\pi} - \operatorname{erfc}(P_c)}{B_c e^{P^2 + B^2 - A^2 + \frac{A^2}{2}}} + \frac{b - x \left(e^{-\frac{|x|}{2}} - 1\right)}{2t B_c e^{B^2 + \frac{A^2}{2}}} \\
\left(\sqrt{\pi} - \operatorname{erfc}(A_c)\right) B_c e^{B^2 + \frac{A^2}{2}} + f(B_c) \left(1 - e^{-\frac{|x|}{2}}\right) \\
\left(\sqrt{\pi} - \operatorname{erfc}(A_c)\right) B_c e^{B^2 + \frac{A^2}{2}} + f(B_c) \left(1 - e^{-\frac{|x|}{2}}\right)
\end{array}\right)^T
\end{aligned}$$

$$\begin{aligned}
= \left\{ (0, \rho_c (x - c)), \quad x \in \left( -\infty, \min \{ r(t), p(t) \} \right) \cap \left( (\tilde{r}(t), q(t)) \cup (q(t), \infty) \right), \\
\left(\frac{x - b}{t}, 0\right), \quad x \in \left( \max \{ p(t), q(t) \}, r(t) \right), \\
(u_a, \rho_c (x - c - u_a t)), \quad x \in \left( -\infty, \min \{ r(t), q(t) \} \right) \cap \left( (-\infty, \min \{ \tilde{r}(t), p(t) \}) \cup (p(t), \infty) \right). \right. 
\end{aligned}$$
Subregion 1. $x > a + u_a t$

$$
\begin{align*}
\lim_{e \to 0} \rho_e \left[ \sqrt{2t e} \left( \frac{1}{e^{A^2}} - \frac{1}{2 e^{C^2}} \right) + (x - c - u_a t) \frac{f(P_c)}{P_c e^{A^2}} \right. \\
&+ \left. \frac{u_a}{P_c e^{A^2}} \frac{(b - x) \left( e^{-\frac{ib_2}{b}} - 1 \right)}{2t B_e e^{B_2 + \frac{2x}{b}}} \right] e^{-\frac{ib_2}{b}} \\
&\quad - \left( \sqrt{\pi} - \text{erfc}(A_c) \right) \frac{f(B_c) \left( 1 - e^{-\frac{ib_2}{b}} \right)}{B_c e^{B_2 + \frac{2x}{b}}} \\
&\quad \left( \sqrt{\pi} - \text{erfc}(A_c) \right) \frac{f(B_c) \left( 1 - e^{-\frac{ib_2}{b}} \right)}{B_c e^{B_2 + \frac{2x}{b}}}
\end{align*}
$$

$$
\begin{align*}
\lim_{e \to 0} \rho_e \left[ \sqrt{2t e} \left( \frac{1}{e^{A^2}} - \frac{1}{2 e^{C^2}} \right) + (x - c - u_a t) \frac{f(P_c)}{P_c e^{A^2}} \right. \\
&+ \left. \frac{u_a}{P_c e^{A^2}} \frac{(b - x) \left( e^{-\frac{ib_2}{b}} - 1 \right)}{2t B_e e^{B_2 + \frac{2x}{b}}} \right] e^{-\frac{ib_2}{b}} \\
&\quad - \left( \sqrt{\pi} - \text{erfc}(A_c) \right) \frac{f(B_c) \left( 1 - e^{-\frac{ib_2}{b}} \right)}{B_c e^{B_2 + \frac{2x}{b}}} \\
&\quad \left( \sqrt{\pi} - \text{erfc}(A_c) \right) \frac{f(B_c) \left( 1 - e^{-\frac{ib_2}{b}} \right)}{B_c e^{B_2 + \frac{2x}{b}}}
\end{align*}
$$

$$
\begin{align*}
\begin{cases}
(0, 0), & x \in \left( r(t), p(t) \right), \\
\left( \frac{x - b}{t}, 0 \right), & x \in \left( \max \{ r(t), p(t) \}, \infty \right).
\end{cases}
\end{align*}
$$

Subregion 2. $x < a + u_a t$

$$
\begin{align*}
\lim_{e \to 0} \rho_e \left[ \sqrt{2t e} \left( \frac{1}{e^{A^2}} - \frac{1}{2 e^{C^2}} \right) + (x - c - u_a t) \left( \sqrt{\pi} - \text{erfc}(P_c) \right) \frac{f(B_c) \left( 1 - e^{-\frac{ib_2}{b}} \right)}{B_c e^{B_2 + \frac{2x}{b}}} \\
+ (x - c) \left( \text{erfc}(C_c) - \text{erfc}(A_c) \right) \frac{f(B_c)}{B_c e^{B_2 + \frac{2x}{b}}} \right. \\
\left. \left( \sqrt{\pi} - \text{erfc}(A_c) \right) \frac{f(B_c) \left( 1 - e^{-\frac{ib_2}{b}} \right)}{B_c e^{B_2 + \frac{2x}{b}}} \\
+ \left( \sqrt{\pi} - \text{erfc}(P_c) \right) \frac{f(B_c) \left( 1 - e^{-\frac{ib_2}{b}} \right)}{B_c e^{B_2 + \frac{2x}{b}}} \right) e^{-\frac{ib_2}{b}}
\end{align*}
$$

$$
\begin{align*}
\begin{cases}
(0, 0), & x \in \left( -\infty, \min \{ r(t), p(t) \} \right) \cap \left( \bar{t}, q(t) \cup (q(t), \infty) \right), \\
\left( \frac{x - b}{t}, 0 \right), & x \in \left( \max \{ p(t), q(t) \}, r(t) \right), \\
(\rho_a, \rho_c (x - c - u_a t)), & x \in \left( -\infty, \min \{ r(t), q(t) \} \right) \cap \left( (\infty, \min \{ \bar{t}, p(t) \}) \cup (p(t), \infty) \right).
\end{cases}
\end{align*}
$$

Before considering the remaining two regions, we introduce the curve $l : s \mapsto a + \frac{\alpha_s}{\alpha} + \frac{\beta_s}{2} \cdot s$ defined over $[0, \infty)$. Then we are able to study the behaviour of $\lim_{s \to 0} \left( u', R' \right)$ in these regions as follows:

(4) $b < x < d$

Subregion 1. $x > a + u_a t$

$$
\begin{align*}
\lim_{e \to 0} \rho_e \left[ \sqrt{2t e} \left( \frac{1}{e^{A^2}} - \frac{1}{2 e^{C^2}} \right) + (x - c - u_a t) \frac{f(P_c)}{P_c e^{A^2}} \right. \\
\left. \left( \sqrt{\pi} - \text{erfc}(A_c) \right) e^{-\frac{ib_2}{b}} + \text{erfc}(B_c) \left( e^{-\frac{ib_2}{b}} - 1 \right) \right. \\
\left. \left( \sqrt{\pi} - \text{erfc}(A_c) \right) e^{-\frac{ib_2}{b}} + \text{erfc}(B_c) \left( e^{-\frac{ib_2}{b}} - 1 \right) \right. \\
\left. \left( \sqrt{\pi} - \text{erfc}(A_c) \right) e^{-\frac{ib_2}{b}} + \text{erfc}(B_c) \left( e^{-\frac{ib_2}{b}} - 1 \right) \right)
\end{align*}
$$

$$
\begin{align*}
\begin{cases}
(0, 0), & x \in \left( -\infty, \min \{ r(t), p(t) \} \right) \cap \left( \bar{t}, q(t) \cup (q(t), \infty) \right), \\
\left( \frac{x - b}{t}, 0 \right), & x \in \left( \max \{ p(t), q(t) \}, r(t) \right), \\
(\rho_a, \rho_c (x - c - u_a t)), & x \in \left( -\infty, \min \{ r(t), q(t) \} \right) \cap \left( (\infty, \min \{ \bar{t}, p(t) \}) \cup (p(t), \infty) \right).
\end{cases}
\end{align*}
$$
We provide the computations only for the case \( c > x \).

\[ u_0 \left( e^{-\frac{\pi t}{2}} \right) + \frac{(x - b) \left( e^{-\frac{\pi t}{2}} - 1 \right)}{2t B_{e} e^{\frac{b}{2}}} \]

\[ \lim_{t \to 0} \rho_c \left[ \frac{1}{eA_x^2 - eC_x^2} \right] e^{-\frac{t}{C_x^2}} - \frac{(x - c - u_0 t) \left( e^{-\frac{t}{C_x^2}} - 1 \right)}{\rho_d e^{\frac{C_x}{2}}} \]
under this case along with the curves \( x = c \) and \( x = d \). For example, \((x_{p,c}, t_{p,c})\) will denote the point of intersection of \( x = p(t) \) and \( x = c \) in the upper-half plane.

The restriction imposed above will ensure that \( t_{p,c} < t_{p,q} \). In addition to the curves \( x = l(t) \), \( x = \tilde{l}(t) \), \( x = p(t) \) and \( x = q(t) \), let us now introduce the curves

\[
\gamma_a(t) := \begin{cases} 
  a + \frac{u_a}{2} \cdot t, & 0 \leq t \leq t_{q,l} := \left( \frac{\sqrt{2(ua(b-a)-ua)} - \sqrt{2ua}}{2ua} \right)^2, \\
  b + ua - \sqrt{2(ua(b-a)-ua)}t, & t_{q,l} \leq t \leq t_{q,t}, \\
  a + \frac{u_a}{u_a} + \frac{u_a}{2} \cdot t, & t \geq t_{q,t}, 
\end{cases}
\]

\[
\gamma_b(t) := \begin{cases} 
  b - \sqrt{2}u_{b}t, & 0 \leq t \leq t_{p,q} := \left( \frac{\sqrt{2(u_a(b-a)-u_b)} - \sqrt{2u_b}}{u_a} \right)^2, \\
  \gamma_a(t), & t \geq t_{p,q}, 
\end{cases}
\]

\[
\gamma_c(t) := \begin{cases} 
  c, & 0 \leq t \leq t_{p,c} := \frac{(b-c)^2}{2|u_a|}, \\
  \gamma_b(t), & t \geq t_{p,c}, 
\end{cases}
\]

\[
\gamma_d(t) := \begin{cases} 
  d, & 0 \leq t \leq t_{1,d} := \frac{2(d-a + |u_a|)}{u_a}, \\
  l(l), & t \geq t_{1,d}, 
\end{cases}
\]

Using these newly introduced curves, we may describe \((u, R) = \lim_{\epsilon \to 0} (u^*, R^*)\) as follows:

\[
u(x, t) = \begin{cases} 
  x \in \left( \{ -\infty, a \} \cap \{ -\infty, q(t) \} \right) \\
  u_a, & x \in \left( \{ -\infty, a \} \cap \{ q(t), r(t) \} \right) \\
  \left( (a, b) \setminus \{ c \} \right) \cap \left( -\infty, \min \{ q(t), r(t) \} \right) \cap \left( \{ -\infty, \min \{ l(t), p(t) \} \cup (p(t), \infty) \} \right), \\
  \left( (b, \infty) \setminus \{ d \} \right) \cap \left( -\infty, \min \{ l(t), r(t) \} \right), \\
  x \in \left( \{ a, b \} \setminus \{ c \} \right) \cap \left( \max \{ p(t), q(t) \}, r(t) \right) \cup \left( \max \{ p(t), r(t) \}, \infty \right), \\
  x \in \left( \{ a, b \} \setminus \{ c \} \right) \cap \left( \max \{ p(t), q(t) \}, r(t) \right) \cup \left( \{ -\infty, \min \{ p(t), r(t) \} \right) \cap \left( (l(t), q(t)) \cup (q(t), \infty) \right) \cup \left( (b, \infty) \setminus \{ d \} \right) \cap (l(t), r(t)) \cup (r(t), \infty), \\
  0, & x \in \left( \{ a, b \} \setminus \{ c \} \right) \cap \left( \max \{ p(t), q(t) \}, r(t) \right) \cup \left( \{ -\infty, \min \{ p(t), r(t) \} \right) \cap \left( (l(t), q(t)) \cup (q(t), \infty) \right) \cup \left( (b, \infty) \setminus \{ d \} \right) \cap (l(t), r(t)) \cup (r(t), \infty), \\
  \frac{x-b}{t}, & x \in \left( \{ -\infty, a \} \cap \{ q(t), \infty) \right) \\
  \left( (a, b) \setminus \{ c \} \right) \cap \left( \max \{ p(t), q(t) \}, r(t) \right) \cup \left( \max \{ p(t), r(t) \}, \infty \right), \\
  0, & x \in \left( \{ a, b \} \setminus \{ c \} \right) \cap \left( \max \{ p(t), q(t) \}, r(t) \right) \cup \left( \{ -\infty, \min \{ p(t), r(t) \} \right) \cap \left( (l(t), q(t)) \cup (q(t), \infty) \right) \cup \left( (b, \infty) \setminus \{ d \} \right) \cap (l(t), r(t)) \cup (r(t), \infty), \\
  \rho_c(x - c - u_at), & x \in \left( -\infty, \gamma_a(t) \right), \\
  \rho_c(x - c), & x \in \left( \gamma_a(t), \gamma_c(t) \right), \\
  0, & x \in \left( \gamma_c(t), \gamma_a(t) \right) \cup \left( \gamma_c(t), \gamma_d(t) \right), \\
  \rho_d, & x > \gamma_d(t).
\end{cases}
\]

We conclude our discussion for this case with the observation that any \( \phi \in C^\infty_c (\mathbb{R}^1 \times [0, \infty); \mathbb{R}^1) \) satisfies
\( \langle R_a, \phi \rangle = - \langle R_x, \phi_x \rangle \)

\[
= - \int_0^\infty \left[ \int_{-\infty}^{\gamma_a(t)} \rho_x (x - c - u_a t) \phi_x \, dx + \int_{\gamma_a(t)}^{\gamma_d(t)} \rho_x (x - c) \phi_x \, dx + \int_{\gamma_d(t)}^{\infty} \rho_x \phi_x \, dx \right] dt 
- \int_0^\infty \rho_x \left[ - \int_0^{\gamma_a(t)} \phi \, dx + (\gamma_a(t) - c - u_a t) \phi (\gamma_a(t), t) \right.
- \int_{\gamma_a(t)}^{\gamma_d(t)} \phi \, dx + (\gamma_a(t) - c) \phi (\gamma_a(t), t) - (\gamma_a(t) - c) \phi (\gamma_a(t), t) \left. \right] dt 
+ \rho_d \int_0^\infty \phi (\gamma_d(t), t) \, dt 
= \langle \rho_x (x < a), q \rangle - \left( (x - c) \delta_{x=\gamma_a(t)} - u_a t \delta_{x=\gamma_a(t)} \right) + \rho_d \delta_{x=\gamma_d(t)}.
\]

which implies that the density component \( \rho \) is given by

\[
\rho = \rho_x \left( \chi_{(-\infty, \gamma_a(t))} - (x - c) \delta_{x=\gamma_a(t)} - u_a t \delta_{x=\gamma_a(t)} \right) + \rho_d \delta_{x=\gamma_d(t)}.
\]

**Case 4.** \( u_A < 0, u_B < 0 \)

1. \( x < a \)

In this region, let us first introduce the curves

(i) \( l(s) := \frac{a + b}{2} + \frac{u_A}{b - a} \cdot s, \)

(ii) \( r(s) := a + u_A s, \)

(iii) \( p(s) := b - \sqrt{2 u_B s}, \)

(iv) \( q(s) := b + u_A s - \sqrt{2 (u_A (b - a) - u_B s) \quad \text{(defined if} \quad u_A (b - a) > u_B) \}

defined for each \( s \geq 0. \) The subsequent evaluations of the limit \( \lim_{t \to 0} (u_t', R_t') \) in the subregions \( x > a + u_A t \)
and \( x < a + u_A t \) are shown separately as follows:

**Subregion 1.** \( x > a + u_A t \)

\[
\lim_{t \to 0} \left( \begin{array}{c}
\frac{f(A_x) + \left( x - a - u_A t \right) \left( e^{-\frac{|u_A|}{a}} - 1 \right)}{2t \left( e^{B_t^2 - A_t^2 + \frac{u_A}{B} \cdot s} \right)} \\
\frac{x - a - u_A t}{a - x} \cdot f(A_e) + \frac{x - a - u_A t}{b - x} \left( 1 - e^{-\frac{|u_A|}{b - a}} \right) f(B_e) + f(P_e) \\
\end{array} \right)^T
= \left\{ \begin{array}{ll}
\frac{x - a}{a - x} \cdot f(A_x) + \left( x - a - u_A t \right) \left( e^{-\frac{|u_A|}{a}} - 1 \right)
& \frac{b - x}{(b - x) e^{B_t^2 - A_t^2 + \frac{u_A}{B} \cdot s}} f(B_t) + f(P_t) \\
\frac{x - a - u_A t}{a - x} \cdot f(A_e) + \left( x - a - u_A t \right) \left( 1 - e^{-\frac{|u_A|}{b - a}} \right)
& \frac{b - x}{(b - x) e^{B_t^2 - A_t^2 + \frac{u_A}{B} \cdot s}} f(B_t) + f(P_t) \\
\end{array} \right\}
\]

\[
\frac{x - a - u_A t}{a - x} \cdot f(A_x) + \left( x - a - u_A t \right) \left( e^{-\frac{|u_A|}{a}} - 1 \right)
\]

\[
\frac{b - x}{(b - x) e^{B_t^2 - A_t^2 + \frac{u_A}{B} \cdot s}} f(B_t) + f(P_t)
\]
Subregion 2. \( x < a + u_a t \)

\[
\begin{align*}
\lim_{\epsilon \to 0} \left( u_a \left( \sqrt{\pi} - \text{erfc}(P_c) \right) + \frac{b - x}{2t} \frac{e^{-\frac{|x|}{2\epsilon}} - 1}{2t B_e e^{p^2 + b^2 - A_e^2 + \frac{t^2}{4}}} \right) & \\
\frac{f(A_c)}{A_e e^{p^2}} + \frac{f(B_c) \left( 1 - e^{-\frac{|x|}{2\epsilon}} \right)}{B_e e^{p^2 + b^2 - A_e^2 + \frac{t^2}{4}}} + \left( \sqrt{\pi} - \text{erfc}(P_i) \right) \left( \sqrt{\pi} - \text{erfc}(P_i) \right)
\end{align*}
\]

\[
\begin{align*}
\rho_c \left[ \sqrt{2\epsilon} \left( \frac{1}{2 e^{c^2} - A_e^2} \right) e^{-\frac{t^2}{4}} + (x - c - u_a t) \left( \sqrt{\pi} - \text{erfc}(P_i) \right) \right] & \\
+ (x - c) \left( \frac{f(A_c)}{A_e} - \frac{f(C_c)}{C_e e^{c^2 - A_e^2}} \right) e^{-\frac{t^2}{4}} + \rho_d \frac{f(D_c) \left( b - x \right) e^{b^2 - D^2}}{(d - x) B_e e^{p^2 + b^2 - A_e^2 + \frac{t^2}{4}}} \left( \sqrt{\pi} - \text{erfc}(P_i) \right)
\end{align*}
\]

\[
\begin{align*}
&= \begin{cases}
(u_a, \rho_c (x - c - u_a t)), & u_a (b - a) > u_b, \ x \in \left( -\infty, \min \{q(t), r(t)\} \right), \\
\left( \frac{x - b}{0} \right), & u_a (b - a) > u_b, \ x \in \left( q(t), r(t) \right), \\
(u_a, \rho_c (x - c - u_a t)), & u_a (b - a) \leq u_b, \ x \in \left( -\infty, r(t) \right).
\end{cases}
\end{align*}
\]

In the remaining regions, we have \( x > a + u_a t \) because of the restriction \( u_a < 0 \). Therefore the required limit \( \lim_{\epsilon \to 0} (u^\epsilon, R^\epsilon) \) will be evaluated as follows:

(2) \( a < x < c \)

\[
\begin{align*}
\lim_{\epsilon \to 0} \left( u_a \left( \sqrt{\pi} - \text{erfc}(P_c) \right) + \frac{b - x}{2t} \frac{e^{-\frac{|x|}{2\epsilon}} - 1}{2t B_e e^{p^2 + b^2 - A_e^2 + \frac{t^2}{4}}} \right) & \\
\frac{f(A_c)}{A_e e^{p^2}} + \frac{f(B_c) \left( 1 - e^{-\frac{|x|}{2\epsilon}} \right)}{B_e e^{p^2 + b^2 - A_e^2 + \frac{t^2}{4}}} + \left( \sqrt{\pi} - \text{erfc}(P_i) \right) \left( \sqrt{\pi} - \text{erfc}(P_i) \right)
\end{align*}
\]

\[
\begin{align*}
\rho_c \left[ \sqrt{2\epsilon} \left( \frac{1}{2 e^{c^2} - A_e^2} \right) e^{-\frac{t^2}{4}} + (x - c - u_a t) \left( \sqrt{\pi} - \text{erfc}(P_i) \right) \right] & \\
+ (x - c) \left( \sqrt{\pi} - \text{erfc}(A_c) \right) & \\
+ \rho_d \frac{f(D_c) \left( b - x \right) e^{b^2 - D^2}}{(d - x) B_e e^{p^2 + b^2 - A_e^2 + \frac{t^2}{4}}} \left( \sqrt{\pi} - \text{erfc}(P_i) \right)
\end{align*}
\]

\[
\begin{align*}
&= \begin{cases}
(0, \rho_c (x - c)), & x < b + \sqrt{-2u_b t}, \\
\left( \frac{x - b}{0} \right), & x > b - \sqrt{-2u_b t}.
\end{cases}
\end{align*}
\]

(3) \( c < x < b \)
\[
\begin{align*}
\lim_{\epsilon \to 0} & \left( u_n \frac{f(P_\epsilon)}{P_\epsilon \, e^{\frac{A^2}{2}}} + \frac{(b - x) \left( e^{-\frac{|u|}{\epsilon}} - 1 \right)}{2t \, B_\epsilon \, e^{B^2/\epsilon}} \right) \\
& - \left( \sqrt{\pi} - \text{erfc}(A_\epsilon) \right) \left( 1 - e^{-\frac{|u|}{\epsilon}} \right) \frac{f(B_\epsilon)}{B_\epsilon \, e^{B^2/\epsilon}} + f(P_\epsilon) \frac{P_\epsilon \, e^{A^2/\epsilon}}{P_\epsilon \, e^{A^2/\epsilon}} \\
& = \begin{cases} 
(0,0), & x < b - \sqrt{-2u_\epsilon t} \\
\left( \frac{x - b}{\pi}, 0 \right), & x > b - \sqrt{-2u_\epsilon t}.
\end{cases}
\end{align*}
\]

\( (4) \quad b < x < d \)

\[
\begin{align*}
\lim_{\epsilon \to 0} & \left( u_n \frac{f(P_\epsilon)}{P_\epsilon \, e^{\frac{A^2}{2} + \frac{u_{\epsilon a}}{\epsilon}}} + \frac{(x - b) \left( e^{-\frac{|u|}{\epsilon}} - 1 \right)}{2t \, B_\epsilon \, e^{B^2/\epsilon}} \right) \\
& - \left( \sqrt{\pi} - \text{erfc}(A_\epsilon) \right) e^{-\frac{|u|}{\epsilon}} \frac{\text{erfc}(B_\epsilon)}{P_\epsilon \, e^{A^2/\epsilon}} + e^{-\frac{|u|}{\epsilon}} - 1 \frac{f(P_\epsilon)}{P_\epsilon \, e^{A^2/\epsilon}} \\
& = (0,0).
\end{align*}
\]

\( (5) \quad x > d \)

\[
\begin{align*}
\lim_{\epsilon \to 0} & \left( u_n \frac{f(P_\epsilon)}{P_\epsilon \, e^{\frac{A^2}{2} + \frac{u_{\epsilon a}}{\epsilon}}} + \frac{(x - b) \left( e^{-\frac{|u|}{\epsilon}} - 1 \right)}{2t \, B_\epsilon \, e^{B^2/\epsilon}} \right) \\
& - \left( \sqrt{\pi} - \text{erfc}(A_\epsilon) \right) e^{-\frac{|u|}{\epsilon}} \frac{\text{erfc}(B_\epsilon)}{P_\epsilon \, e^{A^2/\epsilon}} + e^{-\frac{|u|}{\epsilon}} - 1 \frac{f(P_\epsilon)}{P_\epsilon \, e^{A^2/\epsilon}} \\
& = (0, \rho_d).
\end{align*}
\]

For recovering \( \rho_d \), set \( R = \lim_{\epsilon \to 0} R^\epsilon \). The explicit structure of \( R \) under this case is given by
where \( \gamma_d(s) := d \) for every \( s \geq 0 \) and the curves \( x = \gamma_{a,1}(t) \), \( x = \gamma_{a,2}(t) \) and \( x = \gamma_c(t) \) are defined as follows:

- If \( u_a (b - a) \leq u_b \), then

\[
\gamma_{a,1}(s) := a + u_a s, \quad s \geq 0,
\]

\[
\gamma_{a,2}(s) := \begin{cases} 
  a, & 0 \leq s \leq t_{a,1} := \frac{(b-a)^2}{2|u_a|}, \\
  \frac{a+b}{2} + \frac{u_a}{b-a} s, & s \geq t_{a,1},
\end{cases}
\]

\[
\gamma_c(s) := \begin{cases} 
  c, & 0 \leq s \leq t_{c,p} := \frac{(b-c)^2}{2|u_a|}, \\
  \frac{a+b}{2} + \frac{u_a}{b-a} s, & s \geq t_{c,p},
\end{cases}
\]

- If \( u_a (b - a) > u_b \), then

\[
\gamma_{a,1}(s) := \begin{cases} 
  a + u_a s, & 0 \leq s \leq t_{c,p} := \frac{(b-a)^2}{2(u_a(b-a)-u_b)}, \\
  b + u_a s - \sqrt{2(u_a(b-a) - u_b)} s, & s \geq t_{c,p},
\end{cases}
\]

\[
\gamma_{a,2}(s) := \begin{cases} 
  a, & 0 \leq s \leq t_{a,1}, \\
  \frac{a+b}{2} + \frac{u_a}{b-a} s, & t_{a,1} \leq s \leq t_{a,q} := \frac{(b-a)^2}{2(u_a(b-a)-u_b)}, \\
  b + u_a s - \sqrt{2(u_a(b-a) - u_b)} s, & s \geq t_{a,q},
\end{cases}
\]

\[
\gamma_c(s) := \begin{cases} 
  c, & 0 \leq s \leq t_{c,p} := \frac{(b-c)^2}{2|u_a|}, \\
  \frac{a+b}{2} + \frac{u_a}{b-a} s, & s \geq t_{c,p}.
\end{cases}
\]

In either case, we see that any test function \( \phi \in C^\infty_c \left( \mathbb{R}^1 \times [0, \infty); \mathbb{R}^1 \right) \) will satisfy
\[ (R_x, \phi) = - (R, \phi_x) \]

\[
= - \rho \int_{0}^{\infty} \left[ \int_{-\infty}^{\gamma_a,1(t)} (x - c - u_a t) \phi_x \, dx + \int_{\gamma_a,1(t)}^{\gamma_a,2(t)} \frac{2(x - a - u_a t)(x - a) + (a - c) u_a t}{u_a t} \phi_x \, dx \right. \\
+ \left. \int_{\gamma_a,2(t)}^{\infty} (x - c) \phi_x \, dx \right] \, dt - \rho \delta \int_{0}^{\infty} \phi \, dx \]

\[
= \rho \int_{0}^{\infty} \int_{-\infty}^{\gamma_a,1(t)} (4(x - a) - 2) \phi(x, t) \, dx \, dt \\
+ \rho \int_{0}^{\infty} \left( \gamma_a,2(t) - 2a + c - 2 \frac{2(\gamma_a,2(t) - a)^2}{u_a t} \right) \phi(\gamma_a,2(t), t) \, dt \\
- \rho \int_{0}^{\infty} \left( \gamma_a,1(t) - 2a + c - 2 \frac{2(\gamma_a,1(t) - a)^2}{u_a t} \right) \phi(\gamma_a,1(t), t) \, dt \\
+ \rho \int_{0}^{\infty} \int_{\gamma_a,2(t)}^{\gamma_a,1(t)} \phi(x, t) \, dx \, dt + \rho \int_{0}^{\infty} (\gamma_a,2(t) - \gamma_a,1(t)) \phi(\gamma_a,2(t), t) \, dt \\
- \rho \int_{0}^{\infty} (\gamma_a,1(t) - \gamma_a,2(t)) \phi(\gamma_a,1(t), t) \, dt \right] + \rho \int_{0}^{\infty} \phi(d, t) \, dt \\
= \rho \left[ \chi_{(-\infty, \gamma_a,1(t))} + \frac{4(x - a)}{u_a t} - 2 \right] \chi_{(\gamma_a,1(t), \gamma_a,2(t))} + \chi_{(\gamma_a,2(t), \gamma_a,1(t))} \\
+ \left( 2(c - a) + u_a t + \gamma_a(t) - x - 2 \frac{2(x - a)^2}{u_a t} \right) \delta_{x = \gamma_a,1(t)} \\
+ \left( 2(x - a) + c - \gamma_a(t) - 2 - \frac{2(x - a)^2}{u_a t} \right) \delta_{x = \gamma_a,2(t)} \right] + \rho \delta_{x = \gamma_a(t)}.
\]

Therefore \( \rho \) is given by

\[
\rho = \rho \left[ \chi_{(-\infty, \gamma_a,1(t))} + \frac{4(x - a)}{u_a t} - 2 \right] \chi_{(\gamma_a,1(t), \gamma_a,2(t))} + \chi_{(\gamma_a,2(t), \gamma_a,1(t))} \\
+ \left( 2(c - a) + u_a t + \gamma_a(t) - x - 2 \frac{2(x - a)^2}{u_a t} \right) \delta_{x = \gamma_a,1(t)} \\
+ \left( 2(x - a) + c - \gamma_a(t) - 2 - \frac{2(x - a)^2}{u_a t} \right) \delta_{x = \gamma_a,2(t)} \right] + \rho \delta_{x = \gamma_a(t)}.
\]

\[
\\square
\]

**Appendix**

(1) Let us prove the following properties of the function erfc:

(a) For every \( z \in \mathbb{R}^1 \), \( \text{erfc}(z) + \text{erfc}(-z) = \sqrt{\pi} \)

(b) \( \lim_{z \to \infty} \text{erfc}(z) = 0 \)

(c) \( \text{erfc}(z) = \left( \frac{1}{z^2} - \frac{4z^2}{12z^2 + o\left(\frac{1}{z}ight)} \right) e^{-z^2} \) as \( z \to \infty \)

(d) \( \lim_{z \to \infty} z \text{erfc}(z) e^{z^2} = \frac{1}{2} \)

For proving the first property, we fix \( z \in \mathbb{R}^1 \) and proceed as follows:
\[
\text{erfc}(-z) = \int_{-z}^{\infty} e^{-y^2} \, dy = \int_{-\infty}^{\infty} e^{-y^2} \, dy - \int_{-\infty}^{-z} e^{-y^2} \, dy
\]

\[
= \sqrt{\pi} - \int_{z}^{\infty} e^{-y^2} \, dy
\]

\[
= \sqrt{\pi} - \text{erfc}(z).
\]

The second property follows from the definition of \( \text{erfc} \):

\[
\lim_{z \to \infty} \text{erfc}(z) = \lim_{z \to \infty} \int_{z}^{\infty} e^{-x^2} \, dx = 0.
\]

We now verify the third and fourth properties. For any \( z > 1 \), we can integrate by parts to get

\[
\text{erfc}(z) = \int_{z}^{\infty} \left( -\frac{1}{2t} \right) \frac{d}{dt} \left( e^{-t^2} \right) \, dt
\]

\[
= \frac{1}{2z} e^{-z^2} + \int_{z}^{\infty} \frac{1}{4t^3} \frac{d}{dt} \left( e^{-t^2} \right) \, dt
\]

\[
= \left( \frac{1}{2z} - \frac{1}{4z^3} \right) e^{-z^2} + \int_{z}^{\infty} \frac{3}{4t^4} e^{-t^2} \, dt,
\]

so that

\[
\left| z^3 \left[ e^{z^2} \text{erfc}(z) - \left( \frac{1}{2z} - \frac{1}{4z^3} \right) \right] \right| \leq \frac{3}{8z^2} \int_{z}^{\infty} \frac{d}{dt} \left( e^{t^2} \right) \, dt \leq \frac{3}{8z^2},
\]

and since \( \lim_{z \to \infty} \frac{3}{8z^2} = 0 \), this proves our claim. This last inequality also implies that

\[
z^2 \left| e^{z^2} \text{erfc}(z) - \frac{1}{2z} \right| \to \frac{1}{4} \text{ as } z \to \infty,
\]

and hence we conclude the proof of the fourth property \( \lim_{z \to \infty} z \text{erfc}(z) e^{z^2} = \frac{1}{4} \).

(2) Let us justify the underlying computations for the case \( u_a < 0, u_b > 0 \) in the region \( x < a \):

- \( \lim_{\epsilon \to 0} e^{-\frac{|x|}{\epsilon}} = \lim_{\epsilon \to 0} e^{-\frac{|x|}{\epsilon}} = 0 \)
- **Simplification of** \( \lim_{\epsilon \to 0} A_{\epsilon} e^{A_{\epsilon}^2 + \frac{\epsilon}{A_{\epsilon}}} \):

\[
\lim_{\epsilon \to 0} A_{\epsilon} e^{A_{\epsilon}^2 + \frac{\epsilon}{A_{\epsilon}}} = \lim_{\epsilon \to 0} \frac{a - x}{\sqrt{2}\epsilon} e^{\frac{(a-x)^2 + 2u_{a1}}{2\epsilon}} = \begin{cases} 
\infty, & x < a - \sqrt{-2u_{a1}}, \\
0, & x > a - \sqrt{-2u_{a1}};
\end{cases}
\]

- **Simplification of** \( \lim_{\epsilon \to 0} \text{erfc} (A_{\epsilon}) e^{\frac{\epsilon}{A_{\epsilon}}} \):

\[
\lim_{\epsilon \to 0} \text{erfc} (A_{\epsilon}) e^{\frac{\epsilon}{A_{\epsilon}}} = \lim_{\epsilon \to 0} f(A_{\epsilon}) = \begin{cases} 
0, & x < a - \sqrt{-2u_{a1}}, \\
\infty, & x > a - \sqrt{-2u_{a1}};
\end{cases}
\]

\[
\lim_{\epsilon \to 0} A_{\epsilon} e^{A_{\epsilon}^2 + \frac{\epsilon}{A_{\epsilon}}} \cdot \text{erfc} (A_{\epsilon}) e^{\frac{\epsilon}{A_{\epsilon}}} = \lim_{\epsilon \to 0} f(A_{\epsilon}) = \frac{1}{2}
\]

\[
\lim_{\epsilon \to 0} A_{\epsilon} e^{A_{\epsilon}^2 + \frac{\epsilon}{A_{\epsilon}}} \cdot B_{\epsilon} e^{B_{\epsilon}^2 + \frac{\epsilon}{B_{\epsilon}}} = \lim_{\epsilon \to 0} \frac{a-x}{\epsilon} e^{\frac{(a-x)^2 + 2u_{a1}}{2\epsilon}} = 0
\]

\[
\lim_{\epsilon \to 0} A_{\epsilon} e^{A_{\epsilon}^2 + \frac{\epsilon}{A_{\epsilon}}} \cdot B_{\epsilon} e^{B_{\epsilon}^2 + \frac{\epsilon}{B_{\epsilon}}} = \lim_{\epsilon \to 0} \frac{a-x}{\epsilon} f(B_{\epsilon}) e^{\frac{(a-x)^2 + 2u_{a1}}{2\epsilon}} = 0
\]

The last computation combined with the inequalities \( 0 < a - x < c - x < d - x \) implies that
Here we provide the details of the computations involved in getting the explicit expressions for \( u^\varepsilon \) and \( R^\varepsilon \).

Let us recall that \( u^\varepsilon = \frac{V^\varepsilon}{V^\varepsilon} \) and \( R^\varepsilon = \frac{R^\varepsilon}{V^\varepsilon} \), where

\[
V^\varepsilon = \frac{\varepsilon}{2} V_x^\varepsilon, \quad S^\varepsilon = \frac{\varepsilon}{2} S_x^\varepsilon,
\]

\[
V^\varepsilon(x,0) = \begin{cases} 
\frac{\varepsilon}{2} \cdot e^{-\frac{u_0(x-a)}{\varepsilon}}, & x < a, \\
1, & a < x < b, \\
\frac{\varepsilon}{2} \cdot e^{-\frac{u_0}{\varepsilon}}, & x > b,
\end{cases}
\]

\[
S^\varepsilon(x,0) = \begin{cases} 
\rho_c(x-c) \cdot e^{-\frac{u_0(x-a)}{\varepsilon}}, & x < a, \\
\rho_c(x-c), & a < x < c, \\
0, & c < x < d, \\
\rho_d \cdot e^{-\frac{u_0}{\varepsilon}}, & x > d.
\end{cases}
\]

Therefore

\[
V^\varepsilon(x,t) = \frac{1}{\sqrt{2\pi t \varepsilon}} \left[ \int_{-\infty}^{a} e^{-\frac{u_0(y-a)}{\varepsilon}} \cdot e^{-\frac{(y-x)^2}{2t \varepsilon}} \, dy + \int_{a}^{b} e^{-\frac{(y-x)^2}{2t \varepsilon}} \, dy \right] + \int_{b}^{\infty} e^{-\frac{u_0}{\varepsilon}} \cdot e^{-\frac{(y-x)^2}{2t \varepsilon}} \, dy,
\]

\[
S^\varepsilon(x,t) = \frac{1}{\sqrt{2\pi t \varepsilon}} \left[ \int_{-\infty}^{a} \rho_c(y-c) \cdot e^{-\frac{u_0(y-a)}{\varepsilon}} \cdot e^{-\frac{(y-x)^2}{2t \varepsilon}} \, dy + \int_{a}^{c} \rho_c(y-c) \cdot e^{-\frac{u_0(y-a)}{\varepsilon}} \cdot e^{-\frac{(y-x)^2}{2t \varepsilon}} \, dy \right] + \int_{c}^{d} \rho_d \cdot e^{-\frac{u_0}{\varepsilon}} \cdot e^{-\frac{(y-x)^2}{2t \varepsilon}} \, dy.
\]

These integrals can be further simplified as follows:

- **Simplification of** \( \int_{-\infty}^{a} e^{-\frac{u_0(y-a)}{\varepsilon}} \cdot e^{-\frac{(y-x)^2}{2t \varepsilon}} \, dy; \)

\[
\int_{-\infty}^{a} e^{-\frac{u_0(y-a)}{\varepsilon}} \cdot e^{-\frac{(y-x)^2}{2t \varepsilon}} \, dy = e^{-\frac{(u_0)^2}{2t \varepsilon} + \frac{u_0(x-a)}{t \varepsilon}} \int_{-\infty}^{a} e^{-\frac{(y-x+u_0 t)}{2t \varepsilon}} \, dy = \sqrt{2t \varepsilon} \cdot e^{-\frac{(u_0)^2}{2t \varepsilon} + \frac{u_0(x-a)}{t \varepsilon}} \frac{e^{-\frac{(y-x+u_0 t)}{2t \varepsilon}}}{\sqrt{2t \varepsilon}} \text{erfc} \left( \frac{x-a-u_0 t}{\sqrt{2t \varepsilon}} \right)
\]

- **\( \int_{a}^{b} e^{-\frac{(y-a)^2}{2t \varepsilon}} \, dy = \sqrt{2t \varepsilon} \int_{\frac{a-b}{\sqrt{2t \varepsilon}}}^{\frac{b-a}{\sqrt{2t \varepsilon}}} e^{-z^2} \, dz = \sqrt{2t \varepsilon} \left( \text{erfc} \left( \frac{a-b}{\sqrt{2t \varepsilon}} \right) - \text{erfc} \left( \frac{b-a}{\sqrt{2t \varepsilon}} \right) \right) \)**

- **\( \int_{b}^{\infty} e^{-\frac{(y-b)^2}{2t \varepsilon}} \, dy = \sqrt{2t \varepsilon} \int_{\frac{b-a}{\sqrt{2t \varepsilon}}}^{\frac{\infty}{\sqrt{2t \varepsilon}}} e^{-z^2} \, dz = \sqrt{2t \varepsilon} \text{erfc} \left( \frac{b-a}{\sqrt{2t \varepsilon}} \right) \)**
• Simplification of $\int_{-\infty}^{a} (y-c) \ e^{-\frac{(y-a)^2}{2\epsilon}} \ dy$:

$$
\int_{-\infty}^{a} (y-c) \ e^{-\frac{(y-a)^2}{2\epsilon}} \ dy
= \int_{-\infty}^{a} (y-x+u_{a}t+x-c-u_{a}t) \ e^{-\frac{(y-x+u_{a}t)^2}{2\epsilon}} \ dy
= e^{\frac{(x-a-u_{a}t)^2-(x-a)^2}{2\epsilon^2}} \left[ \int_{-\infty}^{a} (y-x+u_{a}t) \ e^{-\frac{(y-x+u_{a}t)^2}{2\epsilon^2}} \ dy \right]
+ \int_{-\infty}^{a} (x-c-u_{a}t) \ e^{-\frac{(x-c-u_{a}t)^2}{2\epsilon^2}} \ dy

= e^{\frac{(x-a-u_{a}t)^2-(x-a)^2}{2\epsilon^2}} \left[ \sqrt{2\epsilon} \int_{-\infty}^{a} \frac{y-x+u_{a}t}{\sqrt{2\epsilon^2}} \ e^{-\frac{(y-x+u_{a}t)^2}{2\epsilon^2}} \ dy \right]
+ (x-c-u_{a}t) \int_{-\infty}^{a} e^{-\frac{(x-c-u_{a}t)^2}{2\epsilon^2}} \ dy

= e^{\frac{(x-a-u_{a}t)^2-(x-a)^2}{2\epsilon^2}} \left[ \sqrt{2\epsilon} \left( \int_{-\infty}^{a} \frac{d}{dz} \left( -e^{-z^2} \right) \ dz \right) \right.
+ \left. \sqrt{2\epsilon} \ e^{-\frac{(x-a+u_{a}t)^2}{2\epsilon^2}} + \sqrt{2\epsilon} \ (x-c-u_{a}t) \ \text{erfc} \left( \frac{x-a-u_{a}t}{\sqrt{2\epsilon}} \right) \right]
$$

• Simplification of $\int_{a}^{c} (y-c) \ e^{-\frac{(y-a)^2}{2\epsilon}} \ dy$:

$$
\int_{a}^{c} (y-c) \ e^{-\frac{(y-a)^2}{2\epsilon}} \ dy
= \int_{a}^{c} (y-x+c) \ e^{-\frac{(y-a)^2}{2\epsilon}} \ dy
= \sqrt{2\epsilon} \int_{a}^{c} \frac{y-x}{\sqrt{2\epsilon}} \ e^{-\frac{(y-a)^2}{2\epsilon}} \ dy + (x-c) \int_{a}^{c} e^{-\frac{(y-a)^2}{2\epsilon}} \ dy
= \sqrt{2\epsilon} \int_{a}^{c} \frac{d}{dz} \left( -e^{-z^2} \right) \ dz + \sqrt{2\epsilon} \ (x-c) \int_{a}^{c} e^{-z^2} \ dz
= \sqrt{2\epsilon} \left( e^{-\frac{(x-a)^2}{2\epsilon}} - e^{-\frac{(x-c)^2}{2\epsilon}} \right) + \sqrt{2\epsilon} \ (x-c) \left( \text{erfc} \left( \frac{x-a}{\sqrt{2\epsilon}} \right) - \text{erfc} \left( \frac{x-c}{\sqrt{2\epsilon}} \right) \right)
$$

• $\int_{d}^{\infty} e^{-\frac{(y-a)^2}{2\epsilon}} \ dy = \sqrt{2\epsilon} \int_{\frac{x-a}{\sqrt{2\epsilon}}}^{\infty} e^{-z^2} \ dz = \sqrt{2\epsilon} \ \text{erfc} \left( \frac{x-d}{\sqrt{2\epsilon}} \right)$

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