COMPACT ORBITS OF PARABOLIC SUBGROUPS

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Abstract. We study the action of a real reductive group $G$ on a real submanifold $X$ of a Kähler manifold $Z$. We suppose that the action of a compact connected Lie group $U$ with Lie algebra $u$ extends holomorphically to an action of the complexified group $U^C$ and that the $U$-action on $Z$ is Hamiltonian. If $G \subset U^C$ is compatible, there exists a gradient map $\mu : X \rightarrow p$ where $g = \mathfrak{t} \oplus \mathfrak{p}$ is a Cartan decomposition of $g$. In this paper, we describe compact orbits of parabolic subgroups of $G$ in terms of the gradient map $\mu_g$.

§1. Introduction

In this paper, we study the actions of real reductive groups on real submanifolds of Kähler manifolds.

Let $U$ be a compact connected Lie group with Lie algebra $u$, and let $U^C$ be its complexification. We say that a subgroup $G$ of $U^C$ is compatible if $G$ is closed and the map $K \times p \rightarrow G, (k, \beta) \mapsto k \exp(\beta)$ is a diffeomorphism where $K := G \cap U$ and $p := g \cap iu; g$ is the Lie algebra of $G$. The Lie algebra $u^C$ of $U^C$ is the direct sum $u \oplus iu$. It follows that $G$ is compatible with the Cartan decomposition $U^C = U \exp(iu)$, $K$ is a maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$, and that $g = \mathfrak{t} \oplus \mathfrak{p}$. Note that $G$ has finitely number of connected components. In the sequel, we always assume that $G$ is connected.

Let $(Z, \omega)$ be a Kähler manifold with a holomorphic action of the complex reductive group $U^C$. We also assume $\omega$ is $U$-invariant, and that there is a $U$-equivariant momentum map $\mu : Z \rightarrow u^*$. By definition, for any $\xi \in u$ and $z \in Z$, $d\mu^\xi = i\xi Z \omega$, where $\mu^\xi(z) := \langle \mu(z), \xi \rangle$, and $\xi Z$ denotes the fundamental vector field induced on $Z$ by the action of $U$,

$$\xi Z(z) := \frac{d}{dt} |_{t=0} \exp(t \xi) \cdot z.$$  

The inclusion $\mathfrak{p} \hookrightarrow \mathfrak{u}$ induces, by restriction, a $K$-equivariant map $\mu_{ip} : Z \rightarrow (i \mathfrak{p})^*$. Using an $\text{Ad}(U)$-invariant inner product on $u$ to identify $(i \mathfrak{p})^*$ and $\mathfrak{p}, \mu_{ip}$ can be viewed as a map $\mu_p : Z \rightarrow \mathfrak{p}$. For $\beta \in \mathfrak{p}$, let $\beta^\mathfrak{p}$ denote $\mu^{-i \beta}$, that is, $\mu^\beta_p(z) := -(\mu(z), i\beta)$. Then, the grad $\mu_p^\beta = \beta Z$ where grad is computed with respect to the Riemannian metric induced by the Kähler structure. The map $\mu_p$ is called the gradient map associated with $\mu$. For a $G$-stable locally closed real submanifold $X$ of $Z$, we consider $\mu_p$ as a mapping $\mu_p : X \rightarrow \mathfrak{p}$ (see [9] for more details).

Let $\mathfrak{a} \subset \mathfrak{p}$ be an abelian subalgebra, and let $\pi : \mathfrak{p} \rightarrow \mathfrak{a}$ be the orthogonal projection onto $\mathfrak{a}$. Then, $\mu_\mathfrak{a} = \pi_\mathfrak{a} \circ \mu_p$ is the gradient map associated with $A = \exp(\mathfrak{a})$. 

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If $\beta \in \mathfrak{p}$, set

$$G^\beta := \{ g \in G : \lim_{t \to -\infty} \exp(t\beta)g\exp(-t\beta) \text{ exists} \},$$

$$R^\beta := \{ g \in G : \lim_{t \to -\infty} \exp(t\beta)g\exp(-t\beta) = e \},$$

$$\mathfrak{t}^\beta := \bigoplus_{\lambda > 0} V_{\lambda}(\text{ad}\beta),$$

$$G^\beta = \{ g \in G : \text{Ad}(g)(\beta) = \beta \}. $$

Note that $\mathfrak{g}^\beta = \mathfrak{g}^\beta + \mathfrak{t}^\beta$. It is well known that $G^\beta$ is a parabolic subgroup of $G$ with Lie algebra $\mathfrak{g}^\beta$, and every parabolic subgroup of $G$ arises as $G^\beta$ for some $\beta \in \mathfrak{p}$. $R^\beta$ is compact, and it is the unipotent radical of $G^\beta$. $G^\beta$ is a Levi factor of $G^\beta$ (see [5, Lem. 9], [6] and [11] for more details). Our first main result is the following.

**Theorem 1.** Let $\beta \in \mathfrak{p}$. Then:

- If $G^\beta \cdot x$ is compact, then $\mathcal{O} = G \cdot x$ is compact and $G^\beta \cdot x$ is a finite union of connected components of $\max_{\mathcal{O}}(\beta) = \left\{ p \in \mathcal{O} : \max_{x \in \mathcal{O}} \mu_x^\beta = \mu_x^\beta(p) \right\}$.
- If $\mathcal{O}$ is a compact $G$-orbit, then $\max_{\mathcal{O}}(\beta)$ is a finite union of compact $G^\beta$-orbits.

In particular, the number of compact $G^\beta$-orbits is equal or bigger than the number of compact $G$-orbits.

Observe that $\mu_\mathcal{O}(\mathcal{O})$ is a $K$-orbit, but it is not true, in general, that $\mu_\mathcal{O}$ defines a diffeomorphism between $\mathcal{O}$ and $\mu_\mathcal{O}(\mathcal{O})$, without the assumption that $G$ is a complex reductive group. Therefore, Theorem 1.2 in [5, p. 582] does not apply in our context.

Let $\xi \in \mathfrak{u}$. The standard notation for parabolic subgroups of complex reductive groups (see, e.g., [12]) is given by

$$U^{\mathbb{C}}(\xi) = \{ g \in U^{\mathbb{C}} : \lim_{t \to -\infty} \exp(it\xi)g\exp(-it\xi) \text{ exists} \}.$$

It is well known that $U^{\mathbb{C}}(\xi)$ is connected, and it contains a Borel subgroup, that is, a maximal solvable subgroup of $U^{\mathbb{C}}$ (see [1]). Hence, if $\beta \in i\mathfrak{u}$, then $U^{\mathbb{C}}(-i\beta)$ corresponds to $(U^{\mathbb{C}})^{\beta^+}$ in our notation. If $\mathcal{O}$ is a compact orbit of $U^{\mathbb{C}}$, then it is a complex $U$-orbit (see [9]), and so it is a flag manifold (see [8]). Since

$$\max_{\mathcal{O}}(\beta) = \left\{ p \in \mathcal{O} : \max_{x \in \mathcal{O}} \mu^{-i\beta} = \mu^{-i\beta}(p) \right\},$$

it follows that $\max_{\mathcal{O}}(\beta)$ is connected (see [2], [7]). Hence, the following result (see also [5]) holds.

**Corollary 1.0.1.** The number of compact $(U^{\mathbb{C}})^{\beta^+}$-orbits is equal to the number of compact $U^{\mathbb{C}}$-orbits. Moreover, any closed $(U^{\mathbb{C}})^{\beta^+}$-orbit arises as $\max_{\mathcal{O}}(\beta)$, where $\mathcal{O}$ is a compact $U^{\mathbb{C}}$-orbit.

Assume that $G$ is a real form of $U^{\mathbb{C}}$. Assume there exists $p \in Z$ such that $X = U^{\mathbb{C}} \cdot p$ is compact. If $Z$ is compact, then the $U^{\mathbb{C}}$-orbit throughout the maximum of the norm square function $\|\mu\|^2$ is a compact orbit, and so it is a flag manifold (see [9]). It is well known that $G$ has a unique closed orbit $\mathcal{O}$ in $X$. This is an old result of Wolf [16] (see also [9]). In this setting, we prove the following result.
Theorem 2. The set $\max_{O}(\beta)$ is the unique closed orbit of $G^{\beta+}$ acting on $X$. This orbit is connected, and it is a $(K^{\beta}o)$-orbit.

As a consequence of the proof, we obtain the following result.

Proposition 1.1. Let $a \subset p$ be an abelian subalgebra. Let $\mu_a : X \rightarrow a$ be the corresponding $A = \exp(a)$-gradient map. Then, $\mu_a(X) = \mu_a(O)$.

It is well known that both $\mu_a(X)$ and $\mu_a(O)$ are polytopes (see [9]). The above result tells us that $O$ captures much of the information of $\mu_a$. Note that if $a \subset p$ is a maximal abelian subalgebra, then by a beautiful theorem of Kostant [14], keeping in mind that $\mu_a = \pi_a \circ \mu_p$ and $\mu_p(O)$ is a $K$-orbit in $p$, the set $\mu_a(X)$ is the convex hull of an orbit of the Weyl group $W(g,a) = \{\text{Ad}(k) : k \in K, \text{Ad}(k)(a) = a\}$ (see [13] for more details on the Weyl group).

§ 2. Preliminaries

2.1 Convex geometry

In this section, some definitions and results in convex geometry are recalled. The reader can see, for example, [5], [15] for further details on the topic.

Let $V$ be a real vector space with a scalar product $\langle , \rangle$, and let $E \subset V$ be a compact convex subset. The relative interior of $E$, denoted by $\text{relint } E$, is the interior of $E$ in its affine hull. For $a, b \in E$, denote the close segment joining $a$ and $b$ by $[a,b]$. Then, a face of $E$ is a convex subset $F$ of $E$ such that if $a, b \in E$ and $\text{relint}[a,b] \cap F \neq \emptyset$, then $[a,b] \subset F$. The extreme points of $E$, denoted by $\text{ext } E$, are the points $a \in E$ such that $\{a\}$ is a face. Since $E$ is compact, the faces are closed (see [15, p. 62]). The empty set and $E$ are faces of $E$: the other faces are called proper.

Definition 2.1. The support function of $E$ is defined by the function $h_E : V \rightarrow \mathbb{R}$, $h_E(u) = \max \{\langle x,u \rangle : x \in E\}$. If $u \neq 0$, the hyperplane $H(E,u) := \{x \in E : \langle x,u \rangle = h_E(u)\}$ is called the supporting hyperplane of $E$ for $u$. The set

$$F_u(E) := E \cap H(E,u)$$

is a face, and it is called the exposed face of $E$ defined by $u$.

In general, not all faces of a convex subset are exposed. For instance, consider the convex hull of a closed disc and a point outside the disc: the resulting convex set is the union of the disc and a triangle. The two vertices of the triangle that lie on the boundary of the disc are nonexposed 0-faces.

The following result is probably well known. A proof is given in [4]. For the sake of completeness, we give a proof.

Proposition 2.1. Let $C_1 \subseteq C_2$ be two compact convex set of $V$. Assume that for any $\beta \in V$, we have

$$\max_{y \in C_1} \langle y,\beta \rangle = \max_{y \in C_2} \langle y,\beta \rangle.$$

Then, $C_1 = C_2$.

Proof. We may assume without loss of generality that the affine hull of $C_2$ is $V$. Assume by contradiction that $C_1 \subsetneq C_2$. Since $C_1$ and $C_2$ are both compact, it follows that there exists $p \in \partial C_1$ such that $p \in \partial C_2$. Since every face of a convex compact set is contained in an
exposed face (see [15]), there exists $\beta \in V$ such that
\[
\max_{y \in C_1} \langle y, \beta \rangle = \langle p, \beta \rangle.
\]
This means the linear function $x \mapsto \langle x, \beta \rangle$ restricted on $C_2$ achieves its maximum at an interior point which is a contradiction.

\[\Box\]

2.2 Gradient map
Let $(Z, \omega)$ be a Kähler manifold. Let $U^C$ acts holomorphically on $Z$, that is, the action $U^C \times Z \to Z$ is holomorphic. Assume that $U$ preserves $\omega$ and that there is a $U$-equivariant momentum map $\mu : Z \to u$. If $\xi \in u$, we denote by $\xi_Z$ the induced vector field on $Z$, and we let $\mu^\xi \in C^\infty(Z)$ be the function $\mu^\xi(z) := \langle \mu(z), \xi \rangle$. By definition, we have
\[
d\mu^\xi = i_{\xi_Z} \omega.
\]

Let $G \subset U^C$ be a compatible subgroup of $U^C$. For $x \in Z$, let $\mu^\beta_p(x)$ denote $-i$ times the component of $\mu(x)$ along $\beta$ in the direction of $ip$, that is,
\[
\mu^\beta_p(x) := \langle \mu_p(x), \beta \rangle = -\langle \mu(x), i\beta \rangle = \mu^{-i\beta},
\]
for any $\beta \in p$. Here, $\langle \cdot, \cdot \rangle$ denotes $K$-invariant inner product on $p \subset iu$. Then, the map defined by $\mu_p : Z \to p$ is called the gradient map. Let $\mu^\beta_p \in C^\infty(Z)$ be the function $\mu^\beta_p(z) = \langle \mu_p(z), \beta \rangle = \mu^{-i\beta}(z)$. Let $(\cdot, \cdot)$ be the Kähler metric associated with $\omega$, that is, $(v, w) = \omega(v, Jw)$, for all $z \in Z$ and $v, w \in T_z Z$, where $J$ denotes the complex structure on $TZ$. Then, $\beta_Z$ is the gradient of $\mu^\beta_p$.

For the rest of this paper, we assume that $G$ is connected, and we fix a $G$-invariant locally closed submanifold $X$ of $Z$. From now on, we denote the restriction of $\mu_p$ to $X$ by $\mu_p$. Then,
\[
\text{grad} \mu^\beta_p = \beta_X,
\]
where grad is computed with respect to the induced Riemannian metric on $X$.

Let $\beta \in p$. It is well known that $G^\beta$ is compatible and
\[
G^\beta = K^\beta \exp(p^\beta),
\]
where $K^\beta = K \cap G^\beta = \{ g \in K : \text{Ad}(g)(\beta) = \beta \}$ and $p^\beta = \{ v \in p : [v, \beta] = 0 \}$ (see [13]).

Corollary 2.1.1. If $x \in X$ and $\mu_p(x) = \beta$, there are a $G^\beta$-invariant decomposition $T_x X = g^\beta \cdot x \oplus W$, open $G^\beta$-invariant subsets $S \subset W$, $\Omega \subset X$, and a $G^\beta$-equivariant diffeomorphism $\Psi : G^\beta \times G^\beta S \to \Omega$, such that $0 \in S, x \in \Omega$, and $\Psi([e, 0]) = x$.

Proof. See [9, p. 169].

$\beta_X$ is a vector field on $X$, that is, a section of $TX$. For $x \in X$, the differential is a map $T_x X \to T_{\beta_X(x)}(TX)$. If $\beta_X(x) = 0$, there is a canonical splitting $T_{\beta_X(x)}(TX) = T_x X \oplus T_x X$. Accordingly, $d\beta_X(x)$ splits into a horizontal part and a vertical part. The horizontal part is the identity map. We denote the vertical part by $d\beta_X(x)$. It belongs to $\text{End}(T_x X)$. Let $\{ \varphi_t = \exp(t \beta_X) \}$ be the flow of $\beta_X$. There is a corresponding flow on $TX$. Since $\varphi_t(x) = x$, the flow on $TX$ preserves $T_x X$, and there it is given by $d\varphi_t(x) \in GL(T_x X)$. Thus, we get a linear $\mathbb{R}$-action on $T_x X$ with infinitesimal generator $d\beta_X(x)$.
Corollary 2.1.2. If $\beta \in \mathfrak{p}$ and $x \in X$ is a critical point of $\mu_{\mathfrak{p}}^\beta$, then there are open invariant neighbourhoods $S \subset T_x X$ and $\Omega \subset X$ and an $\mathbb{R}$-equivariant diffeomorphism $\Psi : S \rightarrow \Omega$, such that $0 \in S, x \in \Omega$, and $\Psi(0) = x$. (Here, $t \in \mathbb{R}$ acts as $d\varphi_t(x)$ on $S$ and as $\varphi_t$ on $\Omega$.)

Proof. See [9].

Let $x \in \text{Crit} (\mu_{\mathfrak{p}}^\beta)$. Let $D^2 \mu_{\mathfrak{p}}^\beta (x)$ denote the Hessian, which is a symmetric operator on $T_x X$ such that

$$
(D^2 \mu_{\mathfrak{p}}^\beta (x)v, v) = \frac{d^2}{dt^2} (\mu_{\mathfrak{p}}^\beta \circ \gamma)(0),
$$

where $\gamma$ is a smooth curve, $\gamma(0) = x$, and $\dot{\gamma}(0) = v$. Denote by $V_-$ (resp. $V_+$) the sum of the eigenspaces of the Hessian of $\mu_{\mathfrak{p}}^\beta$ corresponding to negative (resp. positive) eigenvalues. Denote by $V_0$ the kernel. Since the Hessian is symmetric, we get an orthogonal decomposition

$$
T_x X = V_- \oplus V_0 \oplus V_+.
$$

(3)

Let $\alpha : G \rightarrow X$ be the orbit map: $\alpha(g) := gx$. The differential $d\alpha_e$ is the map $\xi \mapsto \xi_X (x)$.

Proposition 2.2. If $\beta \in \mathfrak{p}$ and $x \in \text{Crit} (\mu_{\mathfrak{p}}^\beta)$, then

$$
D^2 \mu_{\mathfrak{p}}^\beta (x) = d\beta_X (x).
$$

Moreover, $d\alpha_e (v^{\beta\pm}) \subset V_{\pm}$ and $d\alpha_e (g^\beta) \subset V_0$. If $X$ is $G$-homogeneous, these are equalities.

Proof. See [5], [9, Prop. 2.5].

Corollary 2.2.1. For every $\beta \in \mathfrak{p}$, $\mu_{\mathfrak{p}}^\beta$ is a Morse–Bott function.

Using an $\text{Ad}(K)$-invariant inner product of $\mathfrak{p}$, we define $\nu_{\mathfrak{p}}(z) := \frac{1}{2} \| \mu_{\mathfrak{p}}(z) \|^2$. The function $\nu_{\mathfrak{p}}$ is $K$-invariant, and it is called the norm square function. The following result is proved in [9, Corollaries 6.11 and 6.12, p. 21].

Proposition 2.3. Let $x \in M$. Then:

- If $\nu_{\mathfrak{p}}$ restricted to $G \cdot x$ has a local maximum at $x$, then $G \cdot x = K \cdot x$.
- If $G \cdot x$ is compact, then $G \cdot x = K \cdot x$.

A strategy to analyzing the $G$ action on $M$ is to view $\nu_{\mathfrak{p}}$ as a generalized Morse function. In [9], the authors proved the existence of a smooth $G$-invariant stratification of $M$, and they studied its properties.

§3. Closed orbit of parabolic subgroups

Let $(Z, \omega)$ be a Kähler manifold, and $U^\mathbb{C}$ acts holomorphically on $Z$ with a momentum map $\mu : Z \rightarrow \mathfrak{u}$. Let $G \subset U^\mathbb{C}$ be a closed and compatible subgroup. Hence, $G = K \exp (\mathfrak{p})$, where $K := G \cap U$ is a maximal compact subgroup of $G$ and $\mathfrak{p} := \mathfrak{g} \cap \mathfrak{iu}; \mathfrak{g}$ is the Lie algebra of $G$. Let $X$ be a connected $G$-stable submanifold of $Z$, and let $\mu_{\mathfrak{p}} : X \rightarrow \mathfrak{p}$ the corresponding $G$-gradient map.
REMARKS. Let $Q \subset G$ be a parabolic subgroup. The following facts are easy to check:

a) If $Q \cdot p$ is compact, then $G \cdot p$ is closed since $G = KQ$.

b) Let $O$ be a compact $G$-orbit. By Proposition 2.2, it follows that $O = G \cdot p = K \cdot p$.

Since $\mu_p$ is $K$-equivariant, the restricted gradient map $\mu_p : K \cdot p \to K \cdot \mu_p(x)$ is a smooth $K$-equivariant submersion.

Let $\beta \in p$, and let

$$Y = \{ z \in X : \max_{x \in X} \mu^p_\beta(x) = \mu^p_\beta(z) \}.$$

Assume that $Y$ is not empty. By Corollary 2.1.2, $Y$ is a smooth, possibly disconnected, submanifold of $X$.

**Lemma 3.1.** $Y$ is $G^{\beta+}$ invariant.

**Proof.** Let $g \in G$, and let $\xi \in p$. It is easy to check that

$$(dg)_p(\xi x) = (\text{Ad}(g)(\xi))x(gp),$$

and so $G^\beta$ preserves $X^\beta$. We claim that $Y$ is $G^{\beta}$-stable. In fact, $G^\beta = K^\beta \exp(p^\beta)$, and $Y$ is $K^\beta$-invariant by $K$-invariant property of the gradient map. For each $y \in Y$, let $\xi \in p^\beta$, and let $\gamma(t) = \exp(t\xi) \cdot y$. Since $\beta_X(\gamma(t)) = 0$, it follows that $\mu^\beta_p(\gamma(t))$ is constant, and so $\exp(t\xi) \cdot y \in Y$. Now, $G^{\beta+} = G^\beta R^{\beta+}$ where $R^{\beta+}$ is connected and the unipotent radical of $G^{\beta+}$. By Proposition 2.2, $\mathfrak{t}^{\beta+} \subset V_+$, and so $\mathfrak{t}^{\beta+} \cdot z \subset G_z$, for all $z \in Y$. Since $R^{\beta+}$ is connected, this implies $R^{\beta+}$ does not act on $Y$, and the result follows.

**Lemma 3.2.** Let $O$ be a compact $G$-orbit. Let $\beta \in p$. If $x \in O$ is a local maximum of $\mu^\beta_p : O \to \mathbb{R}$, then $x$ is a global maximum of $\mu^\beta_p : O \to \mathbb{R}$.

**Proof.** If $x \in O$ is a local maximum of $\mu^\beta_p$, then $\mu_p(x)$ is a local maximum of the height function

$$K \cdot \mu_p(x) \to \mathbb{R}, \quad z \mapsto \langle z, \beta \rangle.$$

But it was noted in the proof of Proposition 3.9 in [5] that $\langle \cdot, \beta \rangle$ has only global maximum when restricted to $K \cdot \mu_p(x)$. Then, a local maximum is a global maximum, and this implies that $\mu_p(x)$ is a global maximum of the height function $\langle \cdot, \beta \rangle$. Since

$$\max_O(\beta) = \max \{ \langle z, \beta \rangle, z \in K \cdot \mu_p(x) \},$$

$x$ is a global maximum of $\mu^\beta_p$.

**Proposition 3.3.** Let $p \in O$ be such that $G^{\beta+} \cdot p$ is closed. Then, $G^{\beta+} \cdot p$ is a finite union of connected components of $\max_O(\beta)$.

**Proof.** Since $G^{\beta+} \cdot p$ is compact, $\mu^\beta_p|_{G^{\beta+} \cdot p}$ has a maximum. Let $q \in G^{\beta+} \cdot p$ denote a maximum of $\mu^\beta_p|_{G^{\beta+} \cdot p}$. By Proposition 2.2, $q$ is a $R^{\beta+}$ fixed point. Applying again, Proposition 2.2, keeping in mind that $O$ is $G$ homogeneous, $q$ is a local maximum of $\mu_p : O \to \mathbb{R}$. By Lemma 3.2, $q$ is a global maximum of $\mu^\beta_p$. By Lemma 3.1, the unipotent group $R^{\beta+}$ acts trivially on $\max_O(\beta)$, and $G^{\beta+} \cdot p \subset \max_O(\beta)$.

Let $x \in \max_O(\beta)$. By Proposition 2.2 and Corollary 2.2.1, keeping in mind that $O$ is $G$ homogeneous and $R^{\beta+}$ acts trivially on $\max_O(\beta)$, it follows that $T_x \max_O(\beta) = T_x G^{\beta} \cdot x$. By Lemma 3.1, $(G^\beta)^o$ preserves any connected component of $\max_O(\beta)$. Moreover, the
restriction of \( \mu_p \) to any connected component of \( \max_\mathcal{O}(\beta) \) defines the gradient map of \( (G^\beta)^o \) (see [9]). By Proposition 2.3, it follows that \( (G^\beta)^o \) has a closed orbit on any connected component of \( \max_\mathcal{O}(\beta) \). Since any \( (G^\beta)^o \) orbit is open in \( \max_\mathcal{O}(\beta) \), it follows that the connected component of \( \max_\mathcal{O}(\beta) \) containing \( x \) is \((G^\beta)^o \) homogeneous. The connected components of \( G^\beta \) are finite and intersect the connected components of \( K^\beta \). Therefore, keeping in mind that \( R^{\beta^+} \) acts trivially on \( \max_\mathcal{O}(\beta) \), \( G^{\beta^+} \cdot x \) is a finite union of connected components of \( \max_\mathcal{O}(\beta) \). The same result holds for \( G^{\beta^+} \cdot p \), concluding the proof. \[ \square \]

**Corollary 3.3.1.** Let \( x \in \max_\mathcal{O}(\beta) \). Then, \( G^{\beta^+} \cdot x \) is closed and is a finite union of connected components of \( \max_\mathcal{O}(\beta) \).

Summing up, we have proved our first main result.

**Theorem 3.4.** Let \( \beta \in p \). Then:

- If \( G^{\beta^+} \cdot x \) is compact, then \( \mathcal{O} = G \cdot x \) is compact, and \( G^{\beta^+} \cdot x \) is a finite union of connected components of \( \max_\mathcal{O}(\beta) \).
- If \( \mathcal{O} \) is a compact \( G \)-orbit, then \( \max_\mathcal{O}(\beta) \) is a finite union of compact \( G^{\beta^+} \)-orbits.

In particular, the number of compact \( G^{\beta^+} \)-orbits is equal or bigger than the number of compact \( G \)-orbits.

Let \( Q \subset U^C \) be a parabolic subgroup. There exists \( \beta \in iu \) such that \( Q = (U^C)^\beta \). If \( \tilde{\mathcal{O}} \) is a compact \( U^C \)-orbit, then it is a complex \( U \)-orbit, and so it is a flag manifold (see [8]). By definition of the gradient map,

\[
\max_\tilde{\mathcal{O}}(\beta) = \max \{ p \in \tilde{\mathcal{O}} : \langle \mu(p), -i\beta \rangle = \max_{p \in \tilde{\mathcal{O}}} \mu^{-i\beta} \},
\]

and so it is connected (see [2], [7]). This means that \((U^C)^{\beta^+}\) has a unique closed orbit in \( \tilde{\mathcal{O}} \).

**Corollary 3.4.1.** The number of compact \((U^C)^{-i\beta^+}\)-orbits is equal to the number of compact \( U^C \)-orbits. Any closed \((U^C)^{-i\beta^+}\)-orbit arises as \( \max_\tilde{\mathcal{O}}(\beta) \), where \( \tilde{\mathcal{O}} \) is a compact \( U^C \)-orbit.

Assume that \( G \) is a real form of \( U^C \). Then, \( g = \mathfrak{g} \oplus \mathfrak{p}, u = \mathfrak{g} \oplus \mathfrak{i} \mathfrak{p} \), and so \( g^C = u^C \) (see [10], [13]).

Assume that there exists \( x \in Z \) such that \( U^C \cdot x \) is compact. Then, \( U^C \cdot x = U \cdot x \) and so a flag manifold. The following result is essentially an old theorem of Wolf [16] (see also [9]).

**Theorem 3.5 (Wolf).** There exists a unique closed \( G \)-orbit in \( U^C \cdot x \).

**Proof.** Let \( G_{ss} \) denote the connected subgroup of \( G \) with Lie algebra \([g,g]\). Then, \( G_{ss} \) is closed and compatible, and \( G = Z(G)^o \cdot G_{ss} \), where \( Z(G)^o \) is the connected component of the center (see [13, p. 442]). By Proposition 2.3, \( G \) has a closed orbit in \( U^C \cdot x \). The center of \( U \) does not act on \( U \cdot x \), and \( G_{ss} \) is a real form of \( (U^C)^{ss} = (U_{ss})^C \). By a theorem of Wolf [16], \( G_{ss} \) has a unique closed orbit in \( U^C \cdot x \). On the other hand, by Proposition 2.3, it follows that \( G_{ss} \) has a closed orbit on any closed orbit of \( G \). Therefore, \( G \) has a unique closed orbit in \( U^C \cdot x \). \[ \square \]

Let \( \mathcal{O} \) denote the unique compact \( G \)-orbit in \( U^C \cdot x \). Let \( \beta \in p \). We denote by

\[
\max_{U^C \cdot x}(\beta) = \left\{ p \in U^C \cdot x : \mu^\beta_p(p) = \max_{p \in U^C \cdot x} \mu^\beta_p \right\}.
\]
**Lemma 3.6.** For any $\beta \in \mathfrak{p}$, $\max_{U^C \cdot x}(\beta) \cap \mathcal{O} \neq \emptyset$. Hence, 
\[ \max_{U^C \cdot x}(\beta) \cap \mathcal{O} = \max_{\mathcal{O}}(\beta). \]

**Proof.** Since 
\[ \max_{U^C \cdot x}(\beta) = \max \left\{ p \in U^C \cdot x : \langle \mu(p), -i\beta \rangle = \max_{p \in U^C \cdot x} \mu^{-i\beta} \right\}, \]
by Corollary 3.4.1, it follows that $\max_{U^C \cdot x}(\beta)$ is the unique closed orbit of $(U^C)^{\beta+}$. Since 
\[ G^{\beta+} = G \cap (U^C)^{\beta+}, \]
we get $R^{\beta+} \subset R((U^C)^{-i\beta+})$, and so $R^{\beta+}$ acts trivially on $\max_{U^C \cdot x} \mu_{\mathfrak{p}}^{\beta}$. Applying the same arguments of Proposition 3.3, it follows that $G^{\beta+}$ has a closed orbit in $\max_{U^C \cdot x} \mu_{\mathfrak{p}}^{\beta}$. By Theorem 3.5, we get $\max_{U^C \cdot x}(\beta) \cap \mathcal{O} \neq \emptyset$, and the result follows.

As a consequence, we obtain the following result.

**Proposition 3.7.** Let $\mathfrak{a} \subset \mathfrak{p}$ be an abelian subalgebra. Then, 
\[ \mu_{\mathfrak{a}}(U^C \cdot x) = \mu_{\mathfrak{a}}(\mathcal{O}). \]

**Proof.** It is well known that both $\mu_{\mathfrak{a}}(U^C \cdot x)$ and $\mu_{\mathfrak{a}}(\mathcal{O})$ are polytopes (see [9], [14]). Applying the above lemma and Proposition 2.1, we get $\mu_{\mathfrak{a}}(U^C \cdot x) = \mu_{\mathfrak{a}}(\mathcal{O})$.

Now, we are ready to prove our second main result.

**Theorem 3.8.** The set $\max_{\mathcal{O}}(\beta)$ is the unique closed orbit of $G^{\beta+}$ in $\mathcal{O}$. Moreover, it is connected and is a $(K^\beta)^{\mathfrak{o}}$-orbit.

**Proof.** Let $(G^{\beta})_{ss}^o$ denote the connected subgroup whose Lie algebra is $[\mathfrak{g}^\beta : \mathfrak{g}^\beta]$. It is closed, semisimple, and compatible (see [13]). By Lemma 3.1, it preserves any connected components of $\max_{\mathcal{O}}(\beta)$. By Proposition 2.3, $(G^{\beta})_{ss}^o$ has a closed orbit on any connected component $\max_{\mathcal{O}}(\beta)$. On the other hand, $\max_{U^C \cdot x}(\beta)$ is connected and, by Proposition 2.2, is a closed orbit of $(U^C)^{\beta}$. Note that $(G^{\beta})_{ss}^o$ is a real form of $(U^C)^{\beta}_{ss}$, and $\max_{U^C \cdot x}(\beta)$, keeping in mind that it is a flag manifold and the center of $(U^C)^{\beta}$ does not act on it, is a compact $(U^C)^{ss}$ orbit. Applying a theorem of Wolf [16], it follows that $(G^{\beta})_{ss}^o$ has a unique closed orbit in $\max_{U^C \cdot x}(\beta)$. Since both $(G^{\beta})_{ss}^o$ act trivially on $\max_{U^C \cdot x}(\beta)$, the unique closed orbit of $(G^{\beta})_{ss}^o$ is contained in a closed $(G^{\beta})_{ss}^o$ is contained in a closed orbit of $G^{\beta+}$, and so it is contained in $\mathcal{O}$. By Theorem 3.4, this orbit is contained in $\max_{\mathcal{O}}(\beta)$. Since $(G^{\beta})_{ss}^o$ preserves $\max_{\mathcal{O}}(\beta)$ and it has a closed orbit on any connected component of $\max_{\mathcal{O}}(\beta)$, it follows that $\max_{\mathcal{O}}(\beta)$ is connected. This means $\max_{\mathcal{O}}(\beta)$ is the unique closed orbit of $G^{\beta+}$. In particular, keeping in mind $\max_{\mathcal{O}}(\beta)$ is $(G^{\beta})_{ss}^o$ homogeneous, applying Proposition 2.3, we get $\max_{\mathcal{O}}(\beta)$ is a $(K^\beta)^{\mathfrak{o}}$-orbit, concluding the proof.

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