Junction conditions in quadratic gravity: thin shells and double layers

Borja Reina, José M M Senovilla and Raúl Vera

Física Teórica, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain
E-mail: josemm.senovilla@ehu.es, raul.vera@ehu.es and borja.reina@ehu.es

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Abstract
The junction conditions for the most general gravitational theory with a Lagrangian containing terms quadratic in the curvature are derived. We include the cases with a possible concentration of matter on the joining hypersurface—termed as thin shells, domain walls or braneworlds in the literature—as well as the proper matching conditions where only finite jumps of the energy-momentum tensor are allowed. In the latter case we prove that the matching conditions are more demanding than in general relativity. In the former case, we show that generically the shells/domain walls are of a new kind because they possess, in addition to the standard energy-momentum tensor, a double layer energy-momentum contribution which actually induces an external energy flux vector and an external scalar pressure/tension on the shell. We prove that all these contributions are necessary to make the entire energy-momentum tensor divergence-free, and we present the field equations satisfied by these energy-momentum quantities. The consequences of all these results are briefly analyzed.

Keywords: modified theories of gravity, extended classical solutions, cosmic strings, domain walls, texture, strings and branes

1. Introduction

Quadratic gravity refers to theories generalizing general relativity (GR) by adding terms quadratic in the curvature to the Lagrangian density. The motivations for such modifications go back several decades (see the critical paper [18]), and today there is a general consensus that modern string theory (see, for example, [1]) and other approaches to quantum gravity (see, for example, [17]) present that structure, even with higher powers of curvature tensor, in their effective actions.

1 Author to whom any correspondence should be addressed.
On the other hand, it is often convenient to have a description of concentrated sources, that is, of concentrated matter and energy in gravity theories. These concentrated sources represent, for instance, thin shells of matter (or braneworlds/domain walls) and impulsive matter or gravitational waves. They can mathematically be modeled by using distributions, such as Dirac deltas or the like, hence, one has to resort to using tensor distributions. However, one cannot simply assume that the metric is a distribution because the products of distributions are not well defined in general, and therefore the curvature (and Einstein) tensor will not be defined. Thus, one must identify the class of metrics whose curvature is defined as a distribution, and such that the field equations make sense. For sources on thin shells, the appropriate class of metrics was identified in [12, 14, 23] in GR, further discussed in [10]. Essentially, these are the metrics which are smooth except on localized hypersurfaces where the metric is only continuous.

We carry on a similar program in the most general quadratic theory of gravity, where extra care must be taken: the field equations, as well as the Lagrangian density, contain products of Riemann tensors and, moreover, their second derivatives. Therefore, the singular distributional part—such as the Dirac deltas—cannot arise in the Riemann tensor itself, which can have, at most, finite jumps except in some very exceptional situations. We identify these and then concentrate on the generic, and more relevant, situation performing a detailed calculation using the rigorous calculus of tensor distributions (see the appendices for definitions and fundamental formulas with derivations) to obtain the energy-momentum quantities on the shells. They depend on the extrinsic geometrical properties of the hypersurface supporting it, as well as on the possible discontinuities of the curvature and their derivatives.

Surprisingly, and as already demonstrated in [19–21], a contribution of ‘dipole’ type also appears in the energy-momentum content supported on the shell. This is what we call a double layer, in analogy with the terminology used in classical electrodynamics [13] for the case of electrodipole surface distributions. This analogy makes the interpretation of these double layers somewhat mysterious, as there are no negative masses—and thus no mass dipoles—in gravitation. One of our purposes is to shed some light on this new mystery. From our results and those in [19–21], these double layers seem to arise when abrupt changes in the Einstein tensor occur.

We also find the field equations obeyed by all these energy-momentum quantities, which generalize the traditional Israel equations [12], and describe the conservation of energy and momentum. Actually, we explicitly prove that the full energy-momentum tensor is divergence-free (in the distributional sense) by virtue of the mentioned field equations.

Previous works on junction conditions in quadratic gravity include [2, 5, 7, 24]—see also [6, 11] for the Gauss–Bonnet case—but none of them provided the correct full field equations with matter outside the shell, and they all missed the double-layer contributions, which are fundamental for the energy-momentum conservation. Maybe this is due to the extended use of Gaussian coordinates based on the thin shell: this prevents one making a mathematically sound analysis of the distributional part of the energy-momentum tensor, as the derivatives of the Dirac delta supported on the shell seem to be ill-defined in those coordinates. This is explained in detail in appendix E.

The paper is structured as follows. In section 2 we present a purely geometric review on spacetimes with distributional curvature constructed by joining smooth spacetimes. The quadratic gravity field equations are introduced in section 3, where the proper junction conditions for the description of thin shells (layers) are found. This is achieved by using distributional calculus, briefly reviewed in the appendices. In section 4, the matter content supported on the layer, i.e. the distributional part of the global energy-momentum tensor, is found to contain a ‘usual’ Dirac delta term $\bar{T}_{\mu\nu} \delta^3 \zeta$ together with another contribution of
double-layer type as mentioned above; the latter is denoted by $T_{\mu\nu}$. Then, both $T_{\mu\nu}$ and $I_{\mu\nu}$ are computed in terms of geometrical quantities: the curvatures at either side of the layer and the extrinsic and intrinsic geometry of the hypersurface supporting it. The tensor $T_{\mu\nu}$ is decomposed into the proper energy momentum of the shell $\tau_{\alpha\beta}$, external flux momentum $\tau_{\alpha}$ and external pressure (or tension) $\tau$ corresponding to the completely tangent, tangent-normal and normal parts respectively. The double layer energy-momentum tensor distribution is found to resemble the energy-momentum content of a dipole surface charge distribution with strength $\mu_{\alpha\beta}$. This strength depends on the jump of the Einstein—or equivalently the Ricci—tensor at the layer. The allowed jumps of the curvature (and its derivatives up to second order) at the layer are determined in section 5, again from a purely geometrical perspective.

The general quadratic gravity field equations are obtained in section 6. These are the inherited field equations on the layer, and they involve $\tau_{\alpha\beta}$, $\tau_{\alpha}$, $\tau$ and $\mu_{\alpha\beta}$ together with jumps on the layer of the spacetime energy-momentum tensor. These fundamental equations are the generalization of the Israel equations in GR to the general quadratic gravity theories. The covariant conservation of the full energy-momentum tensor with its distributional parts is explicitly demonstrated in section 7, where we discuss how the double layer term is necessary for that. The field equations on the layer are analyzed and further discussed in section 8, where a classification of the junction conditions in the following cases are presented: proper matching, thin shells with no double layers, and pure double layers. In particular we find that if there is no double layer, then no external flux momentum $\tau_{\alpha}$ nor external tension $\tau$ can exist. Finally, in section 9 some comparisons with the general GR case, and particular matchings of spacetimes, are provided. It is found that any GR solution containing a proper matching hypersurface (say, a boundary surface in GR) will contain a double layer and/or a thin shell at the matching hypersurface if the true theory is quadratic. Therefore, if any quantum regimes require, excite or switch on quadratic terms in the Lagrangian density, then GR solutions modeling two regions with different matter contents will develop thin shells and double layers on their interfaces.

In order to have a self-contained text, we devote some appendices to review distributional calculus in manifolds and to present some useful general calculations with distributions. On the other hand, we present in appendix E a (we hope clarifying) discussion about the difficulty, and in fact inconvenience, of using Gaussian coordinates for dealing with layers in quadratic Lagrangian theories, as has been often done in the literature.

### 2. Junction: spacetimes with distributional curvature

The spacetime is given by an $(n + 1)$-dimensional Lorentzian manifold $(V, g)$. Let us consider the case where $(V, g)$ possesses two different regions, say with different matter contents or different gravitational fields, separated by a border. This border will locally be a hypersurface $\Sigma \subset V$ which can have any causal character, the physically more interesting case arising when it is timelike, which we will assume throughout in this paper. $\Sigma$ divides the manifold $V$ into two regions $V^\pm$, as shown schematically in figure 1.

The metrics $g^\pm_{\mu\nu}$ are assumed to be smooth on $V^\pm$ respectively ($\mu, \nu, \ldots = 0, 1, \ldots, n$). An important observation is that one can actually deal with two different coordinate systems on $V^\pm$. In fact, this is needed in most practical problems, as one is usually given two distinct solutions of the field equations that are to be matched: for instance, one solution describing an interior with matter and another describing a vacuum; or a background solution upon which a localized perturbation, such as a wavefront or a shell of matter, propagates. Thus, we will be
presented with two sets of local coordinates $\{x^a_\pm\}$ with no relation whatsoever, each valid on the corresponding part $V^$ [12].

Two corresponding timelike hypersurfaces $\Sigma^\pm \subset V^\pm$ which bound the regions $V^\pm$ must be chosen on each $\pm$-side to be matched. Of course, these two hypersurfaces are to be identified in the final glued spacetime, so that they must be diffeomorphic. The junction of $V^+$ with $V^-$ by identifying $\Sigma^+$ with $\Sigma^-$ depends crucially on the particular diffeomorphism used for this identification, hence we assume that this has already been chosen and is known. The glued global manifold $V$ is defined as the disjoint union of $V^+$ and $V^-$ with diffeomorphically related points of $\Sigma^+$ and $\Sigma^-$ identified. This unique hypersurface is the matching hypersurface we denote simply by $\Sigma$.

Let $\{\xi^a\}$ be a set of local coordinates on $\Sigma$ ($a, b, \ldots = 1, \ldots, n$). Then, there are two parametric representations:

$$x^a_\pm = x^a_\pm(\xi^a)$$

of $\Sigma$, one for each imbedding into each of $V^\pm$. As explained in appendix B, in order to have well-defined curvature tensors in the sense of distributions we need a global metric which is at least continuous across $\Sigma$. As is known [3, 16], this happens if and only if the two first fundamental forms $h^\pm$ of $\Sigma$ inherited from both sides $V^\pm$ agree. This agreement requires the equalities on $\Sigma$:

$$h^\pm_{ab} = h_{ab}, \quad h^\pm_{ab} \equiv g^\pm_{ab}(x(\xi)) \frac{\partial x^a}{\partial \xi^a} \frac{\partial x^b}{\partial \xi^b}$$

and implies that one can build local coordinate systems in which the metric can be extended to be continuous across $\Sigma$. The unique metric defined on the entire manifold that coincides with $g^\pm$ in the respective $V^\pm$ and is continuous across $\Sigma$ is denoted simply by $g$.

Let $n^*_a$ be the unit normals to $\Sigma$ as seen from $V^\pm$ respectively. They are fixed up to a sign by the conditions

![Figure 1. Schematic diagram of the situation under consideration: $\Sigma$ is a timelike hypersurface separating two regions of the spacetime, $V^+$ and $V^-$, with corresponding smooth metrics $g^+$ and $g^-$. These two metrics also have well-defined, definite limits, when approaching $\Sigma$. If, and only if, the first fundamentals forms inherited by $\Sigma$ from $V^+$ and $V^-$ agree, one can build a local coordinate system such that the entire metric is continuous across $\Sigma$ too. In that case, one can define a unique unit normal $n^*$, which we choose to point from $V^-$ towards $V^+$, as shown.](image-url)
and one must choose one of them (say \( n^- \)) pointing outwards from \( V^- \) and the other (\( n^+ \)) pointing towards \( V^+ \). Hence, the two bases on the tangent spaces at any point of \( \Sigma \)

\[
\begin{align*}
\{ n^{\pm \mu} , \, \frac{\partial x^a}{\partial \xi^a} \} \quad &\leftrightarrow \quad \{ n^{-\mu} , \, \frac{\partial x^a}{\partial \xi^a} \}
\end{align*}
\]

give rise to bases which agree and are then identified, so we drop the ± (even though, in explicit calculations, one can still use both versions using the two coordinate systems on each side). We denote by \( \tilde{e}_a \) the vector fields tangent to \( \Sigma \) defined by the above imbeddings:

\[
\tilde{e}_a := \frac{\partial x^a}{\partial \xi^a} \frac{\partial}{\partial x^\nu} |_{\Sigma} = \frac{\partial x^a}{\partial \xi^a} \frac{\partial}{\partial x^\nu} |_{\Sigma}.
\]

Note that \( \{ \tilde{e}_a \} \) are defined only on \( \Sigma \). The basis dual to \( \{ n^\mu , \, e^\mu_a \} \) is denoted by \( \{ \eta_{\mu} , \, \omega^\mu_a \} \) where the one-forms \( \omega^\mu_a \) are characterized by

\[
n^\nu \omega^\mu_a = 0, \quad e^\mu_b \omega^\mu_a = \delta^a_b.
\]

The spacetime version of the first fundamental form, which is now unique due to (1), is given by the projector to \( \Sigma \) (defined only on \( \Sigma \))

\[
h_{\mu \nu} = g_{\mu \nu} - n_{\mu} n_{\nu}.
\]

Notice that

\[
n^\mu h_{\mu \nu} = 0, \quad h_{\mu \nu} h^{\nu} = h_{\mu \nu}, \quad h^\mu = n, \quad h_{\mu \nu} e^\mu_a e^\nu_b = h_{ab}
\]

and that

\[
e^\mu_a = h_{ab} \omega^{a}_{\nu} g^{\nu \mu}, \quad e^\mu_b \omega^{a}_{\nu} = h^{a}_{b}.
\]

Despite all of the above, the extrinsic curvatures, or second fundamental forms, inherited by \( \Sigma \) from both sides \( V^\pm \) will be, in principle, different, because the derivatives of the metric are not continuous in general. We denote them by \( K_{\mu \nu}^\pm \), and they are defined, as usual, by

\[
K_{\mu \nu}^\pm := h^\rho \partial h^\mu_{\nu} / \partial n_\rho, \quad K_{\mu \nu} = K_{\mu \nu}^+ - K_{\mu \nu}^-.
\]

where only tangent derivatives are involved. Obviously \( n^\mu K_{\mu \nu}^\pm = 0 \), thus only the \( n(n+1)/2 \) components tangent to \( \Sigma \) are non-identically vanishing. In terms of the imbeddings these components are given by

\[
K_{ab}^\pm \equiv -n^a \left( \frac{\partial^2 x^b}{\partial \xi^a \partial \xi^b} + \Gamma_{\rho \sigma}^{\mu} \frac{\partial x^a}{\partial \xi^\rho} \frac{\partial x^b}{\partial \xi^\sigma} \right),
\]

which is adapted to explicit calculations. These components correspond to the second fundamental form, defined as a tensor in \( \Sigma \) by

\[
K_{ab}^\pm = -n^a \tilde{e}^b_a \nabla^\pm e^\mu_b = e^\mu_b \tilde{e}^a_a \nabla^\pm n^\mu_a.
\]

As shown in the appendix C, the Riemann tensor can be computed in the distributional sense and acquires, in general, a singular part proportional to the distribution \( \delta^\Sigma \) supported on \( \Sigma \) (which is defined in appendix B).
Here $H^{\alpha\beta\mu\nu}$ is called the singular part of the Riemann tensor distribution and, as shown in the appendix C, reads

$$H^{\alpha\beta\mu\nu} = n_\alpha [\Gamma^\alpha_{\beta\lambda}] - n_\mu [\Gamma^\alpha_{\beta\lambda}]$$

where the square brackets always denote the jump of the enclosed object across $\Sigma$ according to the definition (137) given in appendix B. We can provide a more interesting formula for this singular part. First, note that from the general formula for the discontinuities of derivatives (158) in appendix D.2 we have

$$[\partial_\alpha g_{\mu\nu}] = n_\alpha \zeta_{\mu\nu}$$

for some symmetric tensor field $\zeta_{\mu\nu}$ defined only on $\Sigma$. This immediately gives

$$[\Gamma^\alpha_{\beta\lambda}] = \frac{1}{2}(\zeta^\alpha_{\beta\mu} n_\lambda + \zeta^\alpha_{\lambda\mu} n_\beta - n^\mu \zeta_{\beta\lambda})$$

which implies

$$H_{\alpha\beta\lambda\mu} = \frac{1}{2}(-n_\alpha \zeta_{\beta\mu} n_\lambda + n_\alpha \zeta_{\lambda\mu} n_\beta - n_\beta \zeta_{\alpha\lambda} n_\mu + n_\beta \zeta_{\alpha\mu} n_\lambda).$$

Note that this expression is invariant under the change $\zeta_{\mu\nu} \longrightarrow \zeta_{\mu\nu} + n_\mu X_\nu + n_\nu X_\mu$ for arbitrary $X_\mu$ and thus only the part of $\zeta_{\mu\nu}$ tangent to $\Sigma$ enters into the formula. Actually, one can prove the existence of $C^1$, piecewise $C^\infty$, changes of coordinates that remove any normal part of $\zeta_{\mu\nu}$ arising in (5)—see, for example, [16]. Thus, from now on we assume that such a change has been performed and we will restrict ourselves to assuming that $\zeta_{\mu\nu}$ is tangent to $\Sigma$: $n^\mu \zeta_{\mu\nu} = 0$. But using (3) together with (5) we deduce

$$K^+_{ab} - K^-_{ab} = -n_\mu [\Gamma^\mu_{ab}] e^a_b e^b_c = \frac{1}{2} \zeta_{ac} e^a_b e^b_c$$

that is to say, the tangent part of $\zeta_{\mu\nu}$ is characterized by the difference of the two $\pm$-second fundamental forms Thus, defining the jump on $\Sigma$ of the second fundamental form as usual:

$$[K_{\mu\nu}] := K^+_{\mu\nu} - K^-_{\mu\nu}, \quad n_\mu [K_{\mu\nu}] = 0$$

we can rewrite (6) as the desired formula for the singular part of the Riemann tensor distribution:

$$H_{\alpha\beta\lambda\mu} = -n_\alpha [K_{\beta\mu}] n_\lambda + n_\alpha [K_{\lambda\mu}] n_\beta - n_\beta [K_{\alpha\lambda}] n_\mu + n_\beta [K_{\alpha\mu}] n_\lambda.$$

This important formula informs us that the singular part of the Riemann tensor distribution vanishes if, and only if, the jump of the second fundamental form vanishes. If and only if that of the full Riemann tensor distribution.

By contractions on (4) we get (with obvious notations):

- The Ricci tensor distribution

$$R_{\beta\mu} = R_{\beta\mu}^\rho \theta^\rho + R_{\beta\mu}^\rho (1 - \theta) + H_{\beta\mu}^\rho e^\rho$$

where its singular part is given by

$$H_{\beta\mu} := H^\rho_{\beta\mu} = -[K^\rho_{\beta\mu}] - [K^\rho_{\mu\beta}] n_\beta n_\mu.$$
The scalar curvature distribution
\[ R = R^+ \mathcal{O} + R^- (1 - \mathcal{O}) + H^\Sigma \]
whose singular part reads
\[ H = H^\rho_\rho = -2 [K^\rho_\rho]. \]

It follows that the singular part of the scalar curvature distribution vanishes if, and only if, the jump of the \textit{trace} of second fundamental form vanishes.

And the Einstein tensor distribution
\[ \mathcal{G}_{\beta \mu} := R_{\beta \mu} - \frac{1}{2} g_{\beta \mu} R = G^+_{\beta \mu} \mathcal{O} + G^-_{\beta \mu} (1 - \mathcal{O}) + \mathcal{G}_{\beta \mu} \delta^\Sigma \]
with a singular part
\[ \mathcal{G}_{\beta \mu} = -[K_{\beta \mu}] + h_{\beta \mu} [K^\rho_\rho], \quad \eta^\nu \mathcal{G}_{\beta \mu} = 0 \]
which is tangent to \( \Sigma \).

A general result proven in [16] is that the second Bianchi identity holds in the distributional sense:
\[ \nabla_\mu R^\alpha_{\beta \mu} + \nabla_\mu R^\alpha_{\beta \nu} + \nabla_\nu R^\alpha_{\beta \mu} = 0 \]
from where one deduces by contraction
\[ \nabla^\beta \mathcal{G}_{\beta \mu} = 0 \]
for the Einstein tensor distribution. By using (14) and the general formula (138) this implies
\[ 0 = \nabla^\beta \mathcal{G}_{\beta \mu} = n^\beta [G_{\beta \mu}] \delta^\Sigma + \nabla^\beta (\mathcal{G}_{\beta \mu} \delta^\Sigma). \]

The second summand on the right-hand side is computed according to the general formula (163) in appendix D.1
\[ \nabla^\beta (\mathcal{G}_{\beta \mu} \delta^\Sigma) = g^{\beta \rho} \nabla_\rho (\mathcal{G}_{\beta \mu} \delta^\Sigma) = g^{\beta \rho} \nabla_\rho (\mathcal{G}_{\beta \mu} n^\nu \delta^\Sigma) + g^{\beta \rho} h_\rho^\lambda \nabla_\lambda \mathcal{G}_{\beta \mu} \delta^\Sigma = h_\rho^\lambda \nabla_\lambda \mathcal{G}_{\beta \mu} \delta^\Sigma \]
which, via (148) finally gives
\[ \nabla^\beta (\mathcal{G}_{\beta \mu} \delta^\Sigma) = (\nabla^\beta \mathcal{G}_{\beta \mu} - K^\rho_\rho \mathcal{G}^\alpha_\alpha n^\nu) \delta^\Sigma. \]

Introducing this into (16) we arrive at
\[ 0 = \delta^\Sigma \left( n^\beta [G_{\beta \mu}] + \nabla^\beta \mathcal{G}_{\beta \mu} - \frac{1}{2} n^\nu \mathcal{G}^\alpha_\alpha (K^\rho_\rho + K^-_\rho_\rho) \right) \]
which implies, by taking the normal and tangent components, the following relations:
\[ (K^+_\rho_\rho + K^-_\rho_\rho) \mathcal{G}^\alpha_\alpha = 2 n^\beta n^\nu [G_{\beta \mu}] = 2 n^\beta n^\nu [R_{\beta \mu}] - [R], \]
\[ \nabla^\beta \mathcal{G}_{\beta \mu} = -n^\nu h^\sigma_\mu [G_{\sigma \rho}] = -n^\nu h^\sigma_\mu [R_{\rho \sigma}]. \]

(These equations can also be obtained [12] by using part of the Gauss and Codazzi equations for \( \Sigma \) on both sides, specifically (154) and (155) in appendix D.1).
3. Quadratic gravity

We are going to concentrate on the case of quadratic theories of gravity because, apart from its own intrinsic interest, they allow for cases where gravitational double layers arise, as we are going to discuss. Let us consider a quadratic theory of gravity in $n + 1$ dimensions described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2\kappa} (R - 2\Lambda + a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}) + \mathcal{L}_{\text{matter}},$$

(19)

where $\kappa = 8\pi G/c^4$ is the gravitational coupling constant, $\Lambda$ is the cosmological constant, $a_1, a_2, a_3$ are three constants selecting the particular theory, and $\mathcal{L}_{\text{matter}}$ is the Lagrangian density describing the matter fields. $\Lambda^{-1}$ and $a_1, a_2, a_3$ have physical units of $L^2$. The field equations derived from this Lagrangian read (see, for example, [8] and references therein):

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} + G_{\alpha\beta}^{(2)} = \kappa T_{\alpha\beta},$$

(20)

where $T_{\alpha\beta}$ is the energy-momentum tensor of the matter fields derived from $\mathcal{L}_{\text{matter}}, G_{\alpha\beta}$ is the Einstein tensor and $G_{\alpha\beta}^{(2)}$ encodes the part that comes from the quadratic terms:

$$G_{\alpha\beta}^{(2)} = 2\{a_1 R R_{\alpha\beta} - 2a_3 R_{\alpha\mu} R_{\beta}^{\mu} - a_3 R_{\alpha\beta\mu\nu} R^{\mu\nu} + (a_2 + 2a_3) R_{\alpha\mu\beta\nu} R^{\mu\nu} - \left( a_1 + \frac{1}{2} a_2 + a_3 \right) \nabla_{\alpha}\nabla_{\beta} R + \left( \frac{1}{2} a_1 + 2a_3 \right) \Box R_{\alpha\beta} \}$$

$$- \frac{1}{2} a_{\alpha\beta} \left( (a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 R_{\rho\gamma\mu\nu} R^{\rho\gamma\mu\nu}) - (4a_1 + a_2) \Box R \right)$$

(21)

where $\Box = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu}$ is the notation for the D'Alembertian in $(V, g)$.

If we want to find the proper junction conditions, or a description of thin shells or braneworlds in these theories, we have to resort to the distributional calculus (see appendices) and use the formulas provided in the previous section. Then, in order to have the Lagrangian density as well as the tensor $G_{\alpha\beta}^{(2)}$ well defined in a distributional sense—so that the field equations (20) are sensible mathematically—one has to avoid any multiplication of singular distributions (such as $\delta^D \delta^{\Sigma}$). One could also hope for some cancellation of such terms between different parts of the Lagrangian, and of $G_{\alpha\beta}^{(2)}$, and this is discussed in the following subsection for completeness, but one has to bear in mind that these cancellations are probably ill-defined anyway, and thus not relevant. In order to properly deal with products of distributions we would need a more general calculus, based, for example, on Colombeau algebras [4, 22], and hope that those cancellations certainly occur and are well defined.

3.1. Dubious possible cancellation of non-linear $\delta^D \delta^{\Sigma}$ terms

Let us start by examining the Lagrangian (19) recalling that the different curvature terms possess now singular parts proportional to $\delta^{\Sigma}$, as given in (9) and its contractions (11) and (13). One could naively compute the products of these singular parts arising from the quadratic terms in (19) and collect them in a common-factor fashion. The result would be a term of type
\[ \delta^{\nu\sigma} \delta^{\alpha\beta} (2\kappa_1 [K_{\mu\nu}]^2 + 2\kappa_2 [K_{\alpha\beta}][K^{\alpha\beta}]) \]

where we have introduced the abbreviations
\[ \kappa_1 := 2a_1 + a_2/2, \quad \kappa_2 := 2a_1 + a_2/2 \]

(22)
to be used repeatedly in what follows. Then, one should require the vanishing of the term in brackets. A similar naive compilation could be performed with the non-linear distributions arising from the quadratic terms in the field equations (21). Imposing again that the full combination must vanish, and separating the resulting condition into its normal and tangent parts to \( \Sigma \), we would find
\[ \{ \kappa_1 [K_{\mu\nu}]^2 + \kappa_2 (3[K^{\mu\nu}][K_{\mu\nu}] - 2[K_{\mu\nu}]^2) \} n_\alpha n_\beta \]
\[ + \kappa_1 [K_{\mu\nu}]^2 (2[K_{\alpha\beta}] - [K_{\mu\nu}]h_{\alpha\beta}) + \kappa_2 (2[K_{\mu\nu}]^2 - [K_{\mu\nu}][K^{\mu\nu}]h_{\alpha\beta}) = 0. \]  

(23)
(24)
The normal (23) and tangent (24) parts should vanish separately. In particular the trace of the tangent part reads
\[ \kappa_1 [K_{\mu\nu}]^2 (2 - n) + \kappa_2 (2[K_{\mu\nu}]^2 - n[K_{\mu\nu}][K^{\mu\nu}]) = 0. \]

(25)
We see directly that \( \kappa_1 = \kappa_2 = 0 \) solves (23) and (24), but in order to find all solutions we compute the determinant of the system (23) and (25). This yields
\[ (3 - n)[K_{\mu\nu}]^2 ([K_{\alpha\beta}]^2 - [K^{\alpha\beta}][K_{\alpha\beta}]) = 0. \]

(26)
Take first \( [K_{\mu\nu}]^2 = 0 \). Then, (23) and (25) reduce to \( \kappa_2 [K_{\mu\nu}][K^{\mu\nu}] = 0 \). If \( [K_{\mu\nu}]^2 \neq 0 \) but \( [K_{\mu\nu}]^2 = [K^{\mu\nu}][K_{\mu\nu}] \), (23) reads \( (\kappa_1 + \kappa_2)[K_{\mu\nu}]^2 = 0 \) and (25) is redundant since it becomes \( (\kappa_1 + \kappa_2)[K_{\mu\nu}]^2 (2[K_{\alpha\beta}] - [K_{\mu\nu}]h_{\alpha\beta}) = 0 \). Thus, \( \kappa_1 + \kappa_2 = 0 \) would follow. Finally, if \( n = 3 \) (and \( [K_{\mu\nu}]^2 = 0 \)), (23), (24) and (25) yield a new possibility not considered so far, summarized in
\[ [K_{\alpha\beta}] = \frac{1}{3} h_{\alpha\beta} \Rightarrow [K_{\mu\nu}] = 1, \quad [K_{\alpha\beta}][K^{\alpha\beta}] = \frac{1}{3}, \quad \kappa_1 - \kappa_2 = 0. \]  

(27)
In short, each of the following possibilities would seem to allow for the mutual annihilation of \( \delta^{\nu\sigma} \delta^{\alpha\beta} \) terms in (21)—and in (19)—:

1. \( \kappa_1 = \kappa_2 = 0 \).
2. \( [K_{\mu\nu}]^2 = 0 \) and \( \kappa_2 = 0 \).
3. \( [K_{\mu\nu}]^2 = [K_{\mu\nu}][K^{\mu\nu}] = 0 \).
4. \( [K_{\mu\nu}]^2 = [K_{\mu\nu}][K^{\mu\nu}] \neq 0 \) and \( \kappa_1 + \kappa_2 = 0 \).
5. If the spacetime is 4-dimensional, \( \kappa_1 - \kappa_2 = 0 \) and \( [K_{\alpha\beta}] = h_{\alpha\beta}/3 \).

Despite the fact that we have included this analysis here for completeness, we should not forget that these cases are not mathematically correct, and therefore they should not be taken seriously unless a more rigorous study is performed showing their feasibility. To understand the problems behind these naive calculations, we want to emphasize that there is no known way to give a sensible meaning to \( \delta^{\nu\sigma} \delta^{\alpha\beta} \), let alone to things such as \( f \delta^{\nu\sigma} \delta^{\alpha\beta} \). Thus, taking for granted that combinations of type \( f \delta^{\nu\sigma} \delta^{\alpha\beta} \) are related to \( (f_1 + f_2) \delta^{\nu\sigma} \delta^{\alpha\beta} \), at the least, dubious. Such difficulties were, for instance, noted in [6] for the Gauss–Bonnet case, corresponding to the possibility 1 above, and one has to resort to analyzing thick shells, that is, layers with a finite width, or to a setting more general than distributions, such as the theory of nonlinear generalized functions described in [4, 22] and references therein. The thin shell formalism is simply not available. Therefore, we will abandon this route for now, and in this
paper we will concentrate on the generic and well-defined cases analyzed in the next subsection.

3.2. Well-defined possibilities: no $\delta^C \delta^C$ terms

The only mathematically well-defined possibilities in the available theory of distributions for the thin shell formalism, as just argued, are those where no $\delta^C \delta^C$ term ever arises, leading to two different possibilities if we set aside the case of GR (defined by $a_1 = a_2 = a_3 = 0$):

1. If either $a_2$ or $a_3$ is different from zero, then products of the Ricci tensor by itself, or by the Riemann tensor, appear in (21), and these are ill-defined if the singular parts (9) and (11) are non-zero. Thus, we must demand that the singular parts (9) and (11) vanish, which happens, as proven above, if and only if the jump of the second fundamental form vanishes. Thus, in this situation it is indispensable to require

\[
[K_{\mu \nu}] = 0. \tag{28}
\]

In this case, all the curvature tensors are tensor distributions associated to tensor fields—see appendix A—with possible discontinuities across $\Sigma$. Observe that then the Lagrangian density (19) is also a well-defined, locally integrable, function.

2. If, on the other hand, $a_2 = a_3 = 0$ then only products of $R$ by itself or by the Ricci tensor appear in (21), and thus it is enough to demand that $R$ is a locally integrable function without singular part. Hence, in this case it is enough to require that (13) vanishes, that is to say, that the trace of the second fundamental form has no jump: $[K'_{\mu \nu}] = 0$. Observe that, again, the Lagrangian density (19) is in this case a well-defined locally integrable function.

In any of the above two possibilities, expression (20) with (21) has a remarkable property: there are no terms quadratic in derivatives of the curvature tensors. Taking into account that tensor distributions can be covariantly differentiated according to the rules explained in the appendices, the derivatives of the curvature tensors may have singular parts and still the field equations (20) are mathematically sound. This opens the door for the existence of matching hypersurfaces which represent double layers. Case 2 above was extensively treated in [19–21], where gravitational double layers were found for the first time. Therefore, we will here concentrate on the more general case 1, and thus we will assume hereafter that (28) holds. Notice that (28) coincides precisely with the matching conditions that are needed in general relativity to avoid distributional matter contents, as follows from (15) together with the Einstein field equations.

Once (28) is enforced, the left-hand side of field equation (20) can be computed in the distributional sense. From (4) and (28) we know that the Riemann tensor distribution

\[
\bar{R}_{\alpha \beta \mu \nu} = R_{\alpha \beta \mu \nu}^+ \bar{\theta} + R_{\alpha \beta \mu \nu}^- (1 - \bar{\theta}),
\]

is actually associated to a locally integrable (and piecewise differentiable) tensor field. However, this tensor field may be discontinuous across $\Sigma$, and thus $[R_{\alpha \beta \mu \nu}]$ may be non-vanishing. This leads, when computing covariant derivatives of $R_{\alpha \beta \mu \nu}$, to singular terms proportional to $\delta^\Sigma$ and its derivatives. And these are going to arise in $G_{\alpha \beta}^{(2)}$. Thus, the energy-momentum tensor on the right-hand side of (20) must be treated as a tensor distribution and contain such terms, localized on $\Sigma$, giving the energy-matter contents of the thin shell or double layer.

In order to compute this matter content supported on $\Sigma$ we only have to calculate the singular part of $G_{\alpha \beta}^{(2)}$, because $G_{\alpha \beta}$ in (14) vanishes as follows from (28) with (15). But the
only terms in (21) that are relevant for this singular part are \( \nabla_\alpha \nabla_\beta R \) and \( \square R_{\alpha\beta} \) (and its contraction \( \square R \)). More precisely, we need to obtain the singular part of the expression
\[
- (2a_1 + a_2 + 2a_3) \nabla_\alpha \nabla_\beta R + (a_2 + 4a_3) \square R_{\alpha\beta} + \left( 2a_1 + \frac{1}{2} a_2 \right) \square R_{\alpha\beta} g_{\alpha\beta} = -(\kappa_1 + \kappa_2) \nabla_\alpha \nabla_\beta R + 2\kappa_2 \square R_{\alpha\beta} + \kappa_1 \square R_{\alpha\beta} g_{\alpha\beta}. \tag{29}
\]
This is the purpose of the next section.

4. Energy-momentum on the layer \( \Sigma \)

From (10) and the assumption (28) we know that

\[
R_{\alpha\beta} = R_{\alpha\beta}^+ \theta + R_{\alpha\beta}^- (1 - \theta)
\]

where, from using the general formula (138) twice, we deduce
\[
\nabla_\alpha R_{\alpha\beta} = \nabla_\alpha R_{\alpha\beta}^+ \theta + \nabla_\alpha R_{\alpha\beta}^- (1 - \theta) + [R_{\alpha\beta}]_{\mu\nu} \delta^\Sigma, \\
\nabla_\mu R_{\alpha\beta} = \nabla_\mu R_{\alpha\beta}^+ \theta + \nabla_\mu R_{\alpha\beta}^- (1 - \theta) + [\nabla_\mu R]_{\mu\nu} \delta^\Sigma + \nabla_\mu ([R_{\alpha\beta}] n_{\nu} \delta^\Sigma) + [\nabla_\mu R_{\alpha\beta}] \delta^\Sigma
\]

as well as
\[
\square R_{\alpha\beta} = \square R_{\alpha\beta}^+ \theta + \square R_{\alpha\beta}^- (1 - \theta) + n^\rho [\nabla_\rho R_{\alpha\beta}] \delta^\Sigma + g^{\mu\nu} [\nabla_\mu R]_{\mu\nu} \delta^\Sigma + [\nabla_\mu R_{\alpha\beta}] \delta^\Sigma, \tag{32}
\]

\[
\square R = \square R^+ \theta + \square R^- (1 - \theta) + n^\rho [\nabla_\rho R] \delta^\Sigma + g^{\mu\nu} [\nabla_\mu R]_{\mu\nu} \delta^\Sigma. \tag{33}
\]

Thus, we need to control the discontinuities of the Ricci tensor and the scalar curvature, and also to provide an expression for the singular distribution \( \nabla_\mu ([R_{\alpha\beta}] n_{\nu} \delta^\Sigma) \) supported on \( \Sigma \). The general formula (163) provides
\[
\nabla_\mu (n_{\nu} [R_{\alpha\beta}] \delta^\Sigma) = \nabla_\mu ([R_{\alpha\beta}]_{\mu\nu} n_{\nu} \delta^\Sigma) + \{ h^\rho_{\mu} \nabla_\rho (n_{\nu} [R_{\alpha\beta}]) \} - K^\rho_{\mu} [R_{\alpha\beta}] n_{\nu} n_{\nu} \delta^\Sigma.
\]

At this point we introduce a 4-covariant tensor distribution \( \Delta_{\mu\nu\rho\sigma} \) with support on \( \Sigma \), which takes care of the first summand here and is defined by
\[
\Delta_{\mu\nu\rho\sigma} \nabla_\rho ([R_{\alpha\beta}]_{\mu\nu} n_{\nu} n^\rho \delta^\Sigma)
\]
or equivalently by
\[
\langle \Delta_{\mu\nu\rho\sigma}, Y_{\mu\nu\rho\sigma} \rangle = -\int_{\Sigma} [R_{\alpha\beta}]_{\mu\nu} n_{\nu} n^\rho \nabla_\rho Y_{\mu\nu\rho\sigma} \mathrm{d}\sigma.
\]

Note that \( \Delta_{\mu\nu\rho\sigma} = \Delta_{\rho\mu\nu\sigma} = \Delta_{\mu\rho\nu\sigma} \). In summary, we have
\[
\nabla_\mu (n_{\nu} [R_{\alpha\beta}] \delta^\Sigma) = \Delta_{\mu\nu\rho\sigma} + \{ n_{\nu} h^\rho_{\mu} \nabla_\rho [R_{\alpha\beta}] + [R_{\alpha\beta}] (K_{\mu\nu} - K^\rho_{\mu} n_{\rho} n_{\nu}) \} \delta^\Sigma
\]
and therefore (30) becomes

\[
\nabla_{\mu} \nabla_{\nu} R_{\alpha \beta} = \nabla_{\mu} \nabla_{\nu} R_{\alpha \beta}^+ (1 - \theta) + \Delta_{\mu \nu} \delta^\Sigma + \{[\nabla_{\nu} R_{\alpha \beta}]n_{\mu} + n_{\nu} h^{\mu}_{\rho} \nabla_{\rho} [R_{\alpha \beta}] + [R_{\alpha \beta}](K_{\rho \nu} - K^\nu_{\rho} n_{\nu} n_{\rho}) \} \delta^\Sigma.
\]

From the general formula (161), conveniently generalized, we have

\[
[\nabla_{\nu} R_{\beta \alpha}] = n_{\nu} r_{\beta \alpha} + h^{\mu}_{\nu} \nabla_{\mu} [R_{\beta \alpha}],
\]

where

\[
r_{\beta \alpha} = n^\nu [\nabla_{\nu} R_{\beta \alpha}], \quad r_{\beta \alpha} = r_{\alpha \beta}
\]

are the discontinuities of the normal derivatives of the Ricci tensor. Thus, we finally get

\[
\nabla_{\nu} \nabla_{\mu} R_{\alpha \beta} = \nabla_{\nu} \nabla_{\mu} R_{\alpha \beta}^+ (1 - \theta) + \Delta_{\mu \nu} \delta^\Sigma + \{r_{\alpha \beta} n_{\nu} + n_{\nu} h^{\mu}_{\nu} \nabla_{\rho} [R_{\alpha \beta}] + n_{\nu} h^{\mu}_{\rho} \nabla_{\rho} [R_{\alpha \beta}] + [R_{\alpha \beta}](K_{\rho \nu} - K^\nu_{\rho} n_{\nu} n_{\rho}) \} \delta^\Sigma.
\]

Observe that the entire singular part is symmetric in \((\alpha \beta)\) and in \((\mu \nu)\).

From (36) we immediately get all the sought terms. First, by contracting with \(g^{\nu \beta}\) we find [19–21]

\[
\nabla_{\nu} \nabla_{\mu} R = \nabla_{\nu} \nabla_{\mu} R^+ (1 - \theta) + \Delta_{\mu \nu} \delta^\Sigma + \{bn_{\nu} n_{\mu} + n_{\nu} \nabla_{\mu} [R] + n_{\nu} \nabla_{\mu} [R] + [R](K_{\rho \nu} - K^\nu_{\rho} n_{\nu} n_{\rho}) \} \delta^\Sigma
\]

where [20, 21]

\[
b := r^\rho_{\rho} = n^\nu \nabla_{\rho} [R]
\]

measures the discontinuity on the normal derivative of the scalar curvature, and [20]

\[
\Delta_{\mu \nu} := g^{\alpha \beta} \Delta_{\alpha \beta \mu \nu}
\]

is a 2-covariant symmetric tensor distribution with support on \(\Sigma\) acting as follows\(^2\):

\[
\langle \Delta_{\mu \nu}, Y^{\mu \nu} \rangle = -\int_\Sigma [R] n_{\nu} n_{\mu} n^\rho \nabla_{\rho} Y^{\mu \nu} d\sigma; \quad \Delta_{\mu \nu} = \nabla_{\mu} ([R] n_{\nu} n_{\mu} n^\rho \delta^\Sigma).
\]

Similarly, contracting (36) with \(\gamma^{\mu \nu}\) we readily get

\[
\square R_{\alpha \beta} = \square R_{\alpha \beta}^+ (1 - \theta) + \Delta_{\alpha \beta} \delta^\Sigma + \gamma^{\mu \nu} \Delta_{\alpha \beta \mu \nu}
\]

where the last distribution acts as follows:

\[
\langle \gamma^{\mu \nu} \Delta_{\alpha \beta \mu \nu}, Y^{\alpha \beta} \rangle = \langle \Delta_{\alpha \beta \mu \nu}, Y^{\mu \nu} \rangle = -\int_\Sigma [R_{\alpha \beta}] n_{\mu} n_{\nu} n^\sigma \nabla_{\rho} (Y^{\alpha \beta}) \gamma^{\mu \nu} d\sigma; \quad \gamma^{\mu \nu} \Delta_{\alpha \beta \mu \nu} = \nabla_{\nu} ([R_{\alpha \beta}] n^\rho \delta^\Sigma).
\]

Finally, by tracing either of (37) or (40) we easily derive

\[
\square R = \square R^+ (1 - \theta) + b \delta^\Sigma + \Delta,
\]

where we have introduced the notation \(\Delta := g^{\mu \nu} \Delta_{\mu \nu}\). Note that [19]

\[
\langle \Delta, Y \rangle = \langle g^{\mu \nu} \Delta_{\mu \nu}, Y \rangle = -\int_\Sigma [R] n^\rho \nabla_{\rho} Y d\sigma; \quad \Delta = \nabla_{\rho} ([R] n^\rho \delta^\Sigma)
\]

\(^2\) There are some errata in the formulae for \(\Delta_{\mu \nu}\) and \(\Omega_{\mu \nu}\) in [19], and for \(J_{\mu \nu}\) in [20, 21]; in all cases \(Y\) must be replaced by \(Y^{\mu \nu}\).
What we have proven is that the distribution $G^{(2)}_{\alpha \beta}$ takes the following form:

$$G^{(2)}_{\alpha \beta} = G^{(2)+} + G^{(2)-}(1 - \theta) + \bar{G}^{(2)}_{\alpha \beta} \delta^{\Sigma} + G_{\alpha \beta}$$

(42)

where

$$\bar{G}^{(2)}_{\alpha \beta} = 2\kappa_2 \rho_{\alpha \beta} + \kappa_1 b_{\rho_{\alpha \beta}} - (\kappa_1 + \kappa_2) \{ b_{\alpha \beta} n_{\gamma} + n^\gamma \nabla_{\gamma}[R] + n_{\beta} \nabla_{\alpha}[R] + [R](K_{\alpha \beta} - K^{\rho}_{\alpha \beta} n_{\alpha \beta})$$

(43)

and after a trivial rearrangement

$$G_{\alpha \beta} = (\kappa_1 + \kappa_2) \Omega_{\alpha \beta} - \Delta_{\alpha \beta} + 2\kappa_2 (2\rho^{\mu \nu} \Delta_{\mu \alpha \beta} - \Delta_{\alpha \beta}).$$

(44)

From (44) we define two new 2-covariant tensor distributions with support on $\Sigma$ [20]:

$$\Omega_{\alpha \beta} = g_{\alpha \beta} - \Delta_{\alpha \beta} = \nabla_{\mu}[R] h_{\alpha \beta}(\nu \delta^{\Sigma}); \quad \langle \Omega_{\alpha \beta}, Y^{\alpha \beta} \rangle = -\int_{\Sigma} [R] h_{\alpha \beta} \nu \nabla_{\gamma} Y^{\alpha \beta} d\sigma$$

(45)

and

$$\Phi_{\alpha \beta} = g^{\mu \nu} \Delta_{\mu \alpha \beta} - \frac{1}{2} \Delta_{\alpha \beta} = \nabla_{\mu}[R] g_{\alpha \beta}(\nu \delta^{\Sigma}); \quad \langle \Phi_{\alpha \beta}, Y^{\alpha \beta} \rangle = -\int_{\Sigma} [G_{\alpha \beta}] \nu \nabla_{\gamma} Y^{\alpha \beta} d\sigma$$

(46)

(recall that $[G_{\alpha \beta}]$ is tangent to $\Sigma$, $n^\nu [G_{\alpha \beta}] = 0$, due to (17) and (18) together with the vanishing of $G_{\alpha \beta}$ as follows from (15) and (28)). With these definitions, (44) is rewritten simply as

$$G_{\alpha \beta} = (\kappa_1 + \kappa_2) \Omega_{\alpha \beta} + 2\kappa_2 \Phi_{\alpha \beta}; \quad G_{\alpha \beta} = \nabla_{\mu}[R] (\nu \delta^{\Sigma}) + \nu \nabla_{\gamma} Y^{\alpha \beta}.$$

(47)

Given the structure (42), the field equations (20) can only be satisfied if the energy-momentum tensor on the right-hand side is a tensor distribution with the following terms:

$$\mathcal{I}_{\mu \nu} = T_{\mu \nu} + \tau_{\mu \nu} \epsilon_{\Sigma} + \bar{T}_{\mu \nu} \delta^{\Sigma} + I_{\mu \nu}$$

(48)

where $\bar{T}_{\mu \nu}$ is a symmetric tensor field defined only on $\Sigma$ and $I_{\mu \nu}$ is by definition the singular part of $\bar{T}_{\mu \nu}$ with support on $\Sigma$ not proportional to $\delta^{\Sigma}$. We perform an orthogonal decomposition of $\bar{T}_{\mu \nu}$ into tangent, normal-tangent and normal parts with respect to $\Sigma$:

$$\bar{T}_{\mu \nu} = \tau_{\mu \nu} + \tau_{\mu} n_{\nu} + \tau_{\nu} n_{\mu} + \tau n_{\mu} n_{\nu}$$

(49)

with

$$\tau_{\mu \nu} = h_{\mu}^{\rho} h_{\nu}^{\sigma} \bar{T}_{\rho \sigma}, \quad \tau_{\mu 
abla} = \nu \tau_{\mu 
abla}, \quad n^{\mu} \tau_{\mu 
abla} = 0; \quad \tau = n^{\mu} n^{\nu} T_{\mu \nu}$$



so that

$$\mathcal{I}_{\mu \nu} = T_{\mu \nu} \theta + T_{\mu \nu}(1 - \theta) + (\tau_{\mu \nu} + \tau_{\mu} n_{\nu} + \tau_{\nu} n_{\mu} + \tau n_{\mu} n_{\nu}) \delta^{\Sigma} + I_{\mu \nu}$$

(50)

Following [20, 21] the proposed names for the objects in (50) supported on $\Sigma$, with their respective explicit expressions, are:
1. the energy-momentum tensor $\tau_{\alpha\beta}$ on $\Sigma$, given by
\[ \kappa \tau_{\alpha\beta} = -(\kappa_1 + \kappa_2)[R]h_{\alpha\beta} + 2\kappa_2 f_{\mu\nu}h^{\mu}_\alpha h^{\nu}_\beta. \] (51)
$\tau_{\alpha\beta}$ is the only quantity usually defined in standard shells.

2. the external flux momentum $\tau_\alpha$, defined by
\[ \kappa \tau_\alpha = -(\kappa_1 + \kappa_2)\nabla_\alpha [R] + 2\kappa_2 f_{\mu\nu}n^\mu h^{\nu}_\alpha. \] (52)
This momentum vector describes normal-tangent components of $\tau_{\mu\nu}$ supported on $\Sigma$. Nothing like that exists in GR. Let us stress this ‘external’ flux momentum should not be confused with the ‘flux momentum’ defined in thin shells in GR (see, for example, [9]).

3. the external pressure or tension $\tau$
\[ \kappa \tau = (\kappa_1 + \kappa_2)[R]K^\alpha_\alpha + \kappa_2 (2r_{\mu\nu}n^\mu n^\nu - b). \] (53)

Taking the trace of (51) one obtains a relation between $b$, $\tau$ and the trace of $\tau_{\mu\nu}$:
\[ \kappa(\tau^\mu_\mu + \tau) = (\kappa_1 n + \kappa_2)\bar{b}. \] (54)
The scalar $\tau$ measures the total normal pressure/tension supported on $\Sigma$. Again, such a scalar does not exist in GR.

4. the double-layer energy-momentum tensor distribution $I_{\alpha\beta}$, which is defined by
\[ \kappa I_{\alpha\beta} = G_{\alpha\beta} \equiv \nabla_\rho (\{ (\kappa_1 + \kappa_2)[R]h_{\alpha\beta} + 2\kappa_2 [G_{\alpha\beta}] \} n^\rho \delta^3 \Sigma) \] (55)
or, equivalently, by acting on any test tensor field $Y^{\alpha\beta}$ as
\[ \kappa \langle I_{\alpha\beta}, Y^{\alpha\beta} \rangle = -\int_{\Sigma} \{ (\kappa_1 + \kappa_2)[R]h_{\alpha\beta} + 2\kappa_2 [G_{\alpha\beta}] \} n^\mu \nabla_\mu Y^{\alpha\beta} d\sigma. \] (56)
$I_{\alpha\beta}$ is a symmetric tensor distribution of ‘delta-prime’ type: it has support on $\Sigma$, but its product with objects intrinsic to $\Sigma$ is not defined unless their extensions off $\Sigma$ are known. As argued in [20, 21], $I_{\alpha\beta}$ resembles the energy-momentum content of double-layer surface charge distributions, or ‘dipole distributions’, with strength
\[ \kappa \mu_{\alpha\beta} \equiv (\kappa_1 + \kappa_2)[R]h_{\alpha\beta} + 2\kappa_2 [G_{\alpha\beta}], \]
\[ \rho_{\alpha\beta} = \rho_{3\alpha\beta}, \quad n^\mu \rho_{\alpha\beta} = 0. \] (57)
We note in passing that
\[ \kappa \mu^\mu_\rho = (\kappa_1 n + \kappa_2)[R], \quad \kappa T^\rho_\rho = (\kappa_1 n + \kappa_2) \Delta \] (58)
The appearance of such double layers is remarkable, as ‘massive dipoles’ do not exist. However, in quadratic theories of gravity they arise, as we have just shown, in the generic situation when thin shells are considered. In this case, $I_{\alpha\beta}$ seems to represent the idealization of abrupt changes, or jumps, in the curvature of the spacetime.

5. Curvature discontinuities
In the next section, we are going to derive the field equations satisfied by the energy-momentum quantities (51), (52), (53) and (57) supported on $\Sigma$. To that end, we have to perform a detailed calculation of the discontinuities of the field equations (20): they obviously include the discontinuities of the energy-momentum tensor $T_{\mu\nu}$ which must be related to the energy-momentum content concentrated on $\Sigma$. 

The discontinuity of the left-hand side of (20) contains \( G_{0j}^2 \) (actually, we will only need \( n^a[G_{0j}^2] \)) and this involves discontinuities of quadratic terms in the Riemann tensor, such as \([R^2], [R_{\alpha\beta\nu\rho}R^\alpha\gamma^\rho\nu\rho], [RR_{\alpha\beta}], [R_{\alpha\beta\nu\rho}R^\nu^\rho_{\beta\rho}], [R_{\alpha\beta\nu\rho}R^\nu^\rho_{\beta\rho}]\) and \([R_{\alpha\beta\nu\rho}R^\nu^\rho_{\beta\rho}]\), as well as discontinuities of derivatives of the curvature tensors, such as \([\nabla_\alpha\nabla_\beta R], [\Box R_{\alpha\beta}]\) or \([\Box R]\). Thus, we have to use systematically the rules (157) and either of (161) or (162) supplemented with (28), and we also need to have some knowledge on the discontinuities of the Riemann tensor (and its derivatives).

5.1. Discontinuities of the curvature tensors

Thus, let us start by controlling the allowed discontinuities of the Riemann tensor across \( \Sigma \). From the requirement (28) we know that \( \zeta_{\mu\nu} = 0 \) and thus

\[
[\Gamma_{0j}^a] = 0.
\]

Then, the general formula (158) gives

\[
[\partial_j \Gamma_{0j}^a] = n_\alpha \gamma_{0j}^a \beta
\]

for some functions \( \gamma_{0j}^a \beta \), such that \( \gamma_{0j}^a \beta = \gamma_{0j}^a \beta \), and therefore

\[
[R_{\alpha\beta\nu\rho}^\rho] = n_\nu \nabla_{\alpha\beta} R_{\nu\rho}.
\]

The antisymmetry of \( R_{\alpha\beta\nu\rho} \) in \([\alpha\beta]\) implies then that \( \gamma_{\alpha\beta\nu}^a = -B_{\alpha\beta} n_\nu \) for some symmetric tensor \( B_{\alpha\beta} \) defined only on \( \Sigma \). Hence

\[
\gamma_{0j}^a = \tilde{\gamma}_{0j}^a - B_{0j}^a n_\mu, \quad \tilde{\gamma}_{0j}^a = -\tilde{\tilde{\gamma}}_{0j}^a
\]

and

\[
[R_{0j\alpha\beta}] = n_\nu \tilde{\gamma}_{0j}^a - n_\beta \tilde{\tilde{\gamma}}_{0j}^a.
\]

However, the symmetry of \( \tilde{\gamma}_{0j}^a \) in \([\alpha\beta]\) implies \( \tilde{\gamma}_{\alpha\beta\nu}^a = \tilde{\gamma}_{\beta\alpha\nu}^a = 0 \) as well as \( \tilde{\tilde{\gamma}}_{\alpha\beta\nu}^a = 0 \) from where one easily derives

\[
\tilde{\gamma}_{0j}^a = 2\tilde{\gamma}_{0j}^a = n_\nu B_{\nu\mu} - n_\beta B_{\beta\mu}
\]

and we recover the standard formula [16]

\[
[R_{0j\alpha\beta}] = n_\nu n_\lambda B_{\nu\mu} - n_\nu n_\mu B_{\lambda\mu} + n_\lambda n_\mu B_{\mu\lambda}.
\]

As argued after formula (9), there is no loss of generality by assuming that \( B_{0j} \) is tangent to \( \Sigma \), i.e. such that

\[
B_{0j} n^a = 0 \quad \implies \quad B_{0j} = B_{ab} n^a \omega_a^b.
\]

Given this, plus the symmetry of \( B_{0j} \) there are \((n^2 + n)/2\) independent allowed discontinuities for the curvature tensor, all encoded in \( B_{ab} \).

Successive contractions on (60) provide

\[
[R_{\beta\mu}] = B_{\beta\mu} + \frac{1}{2} n_{0j} n_\mu [R], \quad [R] = 2 B_{\beta\mu},
\]

or equivalently

\[
B_{\beta\mu} = [R_{\beta\mu}] - \frac{1}{2} [R] n_{\beta\mu} = [G_{\beta\mu}] + \frac{1}{2} h_{\beta\mu} [R].
\]
In other words, the \( n(n+1)/2 \) allowed independent discontinuities of the Riemann tensor can be chosen to be the discontinuities of the \( \Sigma \)-tangent part of the Einstein tensor (or equivalently, of the Ricci tensor).

5.2. Discontinuities of the curvature tensors derivative

Concerning the covariant derivative of the Riemann tensor, the general formula (161) leads to

\[
[\nabla, R_{\alpha\beta\lambda\mu}] = n_{\mu} r_{\alpha\beta\lambda\mu} + h_{\mu}^{\nu} \nabla_{\nu}[R_{\alpha\beta\lambda\mu}],
\]

(63)

where \( r_{\alpha\beta\lambda\mu} \) is a tensor field defined only on \( \Sigma \) and with the symmetries of a Riemann tensor. Using the second Bianchi identity for the Riemann tensor the previous formula implies

\[
n_{\mu} r_{\alpha\beta\lambda\mu} + h_{\mu}^{\nu} \nabla_{\nu}[R_{\alpha\beta\lambda\mu}] = 0
\]

which, on using (60) and after some calculations, implies the following structure for \( r_{\alpha\beta\lambda\mu} \):

\[
r_{\alpha\beta\lambda\mu} = K_{\alpha\beta} B_{\lambda\mu} - K_{\alpha\lambda} B_{\beta\mu} + K_{\alpha\mu} B_{\beta\lambda} - K_{\beta\lambda} B_{\mu\alpha}
\]

\[
+ (\nabla_{\beta} B_{\alpha\mu} - \nabla_{\alpha} B_{\beta\mu})(n_{\lambda} h_{\mu}^{\rho} - n_{\mu} h_{\lambda}^{\rho}) + (\nabla_{\lambda} B_{\alpha\beta} - \nabla_{\alpha} B_{\lambda\beta})(n_{\mu} h_{\beta}^{\rho} - n_{\beta} h_{\mu}^{\rho})
\]

\[
+ n_{\lambda} n_{\mu} \rho_{\lambda\mu} - n_{\alpha} n_{\nu} \rho_{\alpha\nu} - n_{\lambda} n_{\mu} \rho_{\lambda\nu} + n_{\beta} n_{\mu} \rho_{\beta\mu},
\]

(64)

where \( \rho_{\beta\mu} \) is a new symmetric tensor field, defined only on \( \Sigma \) and tangent to \( \Sigma \). Using the second Bianchi identity for the Riemann tensor the previous formula implies

\[
[r_{\alpha\beta\lambda\mu} + h_{\alpha}^{\nu} \nabla_{\nu}[R_{\alpha\beta\lambda\mu}] = 0
\]

which encodes the allowed new independent discontinuities of the covariant derivative of the Riemann tensor. There are \( n(n+1)/2 \) of those again. As far as we know, relation (64) has only been derived in [16].

Contraction of (64) leads to equation (34), but now with an explicit expression for the discontinuity of the normal derivative of the Ricci tensor which reads, on using (62)

\[
r_{\beta\mu} = \rho_{\beta\mu} + K_{\mu}^{\nu} B_{\nu\beta} + \frac{1}{2}[R] K_{\beta\nu} - K_{\nu\beta} B_{\nu\mu} - B_{\nu\beta} K_{\nu}^{\rho}
\]

\[
+ n_{\nu} \nabla_{\nu}[G_{\beta}^{\rho}] - n_{\rho} \nabla_{\rho}[G_{\beta}^{\rho}]
\]

\[
+ n_{\lambda} n_{\mu} \rho_{\lambda\mu},
\]

(65)

where a natural orthogonal decomposition of \( r_{\beta\mu} \) appears: the first line is its complete tangent part which, given that \( \rho_{\beta\mu} \) entails the allowed new independent discontinuities, is in itself a symmetric tensor field tangent to \( \Sigma \) codifying those discontinuities. We are going to denote it by

\[
R_{\beta\mu} := h_{\mu}^{\nu} h_{\nu}^{\rho} r_{\rho\beta} = h_{\mu}^{\nu} h_{\nu}^{\rho} n^{\lambda}[\nabla_{\lambda} R_{\rho\beta}] = 0
\]

(66)

the second line is its tangent-normal part, which is completely determined by the covariant derivative within \( \Sigma \) of the discontinuity of the Einstein tensor

\[
n^{\lambda} h_{\mu}^{\nu} r_{\rho\beta} = - \nabla^{\lambda}[G_{\rho\beta}] = 0
\]

(67)

and finally, the third line gives the total normal component of \( r_{\beta\mu} \), which can be related to the discontinuity (38) of the normal derivative of \( R \) by simply taking the trace \( r_{\mu}^{\mu} = b \) leading to

\[
r_{\beta\mu} n^{\beta} n^{\mu} = \frac{b}{2} + K^{\alpha\mu}[G_{\beta\alpha}],
\]

(68)

Using this we get a useful relation for the trace of \( R_{\alpha\beta} \), that does not depend on \( \rho_{\alpha\beta} \)

\[
R_{\alpha}^{\alpha} = \frac{b}{2} - K^{\alpha\mu}[G_{\beta\alpha}].
\]

(69)
5.3. Second-order derivative discontinuities

Let us now consider the jumps in the second derivatives of the Ricci tensor. The starting point is equation (34). We can find an expression for the second summand there by differentiating (61) along $\Sigma$ and using the general rule (148) (see appendix D.1)

$$h^\rho_\gamma \nabla_\rho [R_{\gamma \rho}] = \frac{1}{2} n_\rho n_\mu \nabla_\mu [R] + n_\rho (K_{\beta \gamma \rho} [R] = 2B_{\beta \gamma \rho}K^\rho_{\beta \gamma}) + \nabla_\rho B_{\gamma \rho}. \quad (70)$$

The jumps of the second-order derivatives of the Ricci tensor, due to the general formula (161), can be written as

$$[\nabla_\rho \nabla_\sigma R_{\gamma \rho}] = n_\rho A_{\rho \beta \gamma} + h^\nu_\phi \nabla_\phi [\nabla_\rho R_{\gamma \rho}] \quad (71)$$

where $A_{\rho \beta \gamma} = A_{\rho (\beta \gamma)}$ is a shorthand for

$$A_{\rho \beta \gamma} = n^\lambda [\nabla_\lambda \nabla_\rho R_{\gamma \rho}]$$

The last term $h^\nu_\phi \nabla_\phi [\nabla_\rho R_{\gamma \rho}]$ can be further expanded by first using (34) to obtain

$$h^\nu_\phi \nabla_\phi [\nabla_\rho R_{\gamma \rho}] = K_{\rho \phi \beta \gamma} + n_\rho h^\nu_\phi \nabla_\phi r_{\beta \gamma} + h^\nu_\phi \nabla_\phi (h^\rho_\gamma \nabla_\gamma [R_{\gamma \rho}]).$$

and then computing the last summand here, which leads to

$$h^\nu_\phi \nabla_\phi [\nabla_\rho R_{\gamma \rho}] = K_{\rho \phi \beta \gamma} + 2n_\rho K_{\phi \beta \gamma} + [R]K_{\phi \beta \gamma} - 4K_{\rho \phi \beta \gamma}B_{\gamma \beta \rho \lambda} - \frac{1}{2} n_\rho n_\mu n_\lambda K^\rho_{\phi \beta \gamma \lambda} \nabla_\gamma [R]

+ \left(\nabla_\gamma K^\rho_{\phi \beta \gamma} - K_{\phi \beta \gamma} \nabla^\rho [R] - [R]K_{\phi \beta \gamma} + 2K^\rho_{\phi \beta \gamma}B_{\gamma \beta \rho \lambda}

+ \nabla_\gamma \nabla_\rho B_{\gamma \rho} - 2K_{\phi \beta \gamma} [K_{\gamma \beta \rho \lambda} + n_\rho K_{\gamma \phi \beta \lambda} + n_\rho h^\nu_\phi \nabla_\phi r_{\beta \gamma}] \right). \quad (72)$$

Let us stress the fact that all the terms in the first two lines in the above expression are symmetric in $(\lambda \rho)$. Concerning $A_{\rho \beta \gamma}$, let us first decompose it into normal and tangential parts by

$$A_{\rho \beta \gamma} = n_\rho A_{\rho \beta \gamma} + h^\nu_\phi A_{\rho \beta \gamma}, \quad A_{\beta \gamma} := n^\rho A_{\rho \beta \gamma}, \quad A_{\beta \gamma} = A_{\beta \gamma}$$

In order to obtain an expression for $h^\rho_\gamma A_{\beta \gamma}$, we take the antisymmetric part of (71) with respect to $[\lambda \rho]$, and contract with $n^\lambda$. For the left-hand side of (71) we use the Ricci identity applied to the Ricci tensor at both sides $V^\pm$, and take the difference of the limits on $\Sigma$, so that

$$[(\nabla_\rho \nabla_\nu - \nabla_\nu \nabla_\rho)R_{\gamma \rho}] = [R_{\gamma \beta \rho \lambda} - [R]_{\gamma \beta \rho \lambda}] + [R_{\rho \lambda \beta \gamma} - [R]_{\rho \lambda \beta \gamma}].$$

For the right-hand side of (71), after the contraction with $n^\lambda$, we get

$$A_{\rho \beta \gamma} = n^\rho A_{\rho \beta \gamma} - n^\nu h^\rho_\gamma \nabla_\gamma [\nabla_\rho R_{\gamma \rho}] = h^\rho_\gamma A_{\gamma \beta \rho} - n^\lambda h^\rho_\gamma \nabla_\gamma [\nabla_\rho R_{\gamma \rho}].$$

Isolating $h^\rho_\gamma A_{\gamma \beta \rho}$ and using (72) for the last term of the above equation, it is then straightforward to obtain

$$A_{\rho \beta \gamma} = n^\rho A_{\rho \beta \gamma} + n^\nu [R_{\beta \rho \gamma \nu}] + n^\nu [R_{\rho \lambda \beta \gamma}] + h^\rho_\gamma \nabla_\gamma r_{\beta \gamma} - \frac{1}{2} n_\rho n_\mu n_\lambda K^\rho_{\phi \beta \gamma \lambda} \nabla_\gamma [R] - K^\rho_{\phi \beta \gamma} \nabla_\phi B_{\gamma \beta \rho \lambda} - [R]K^\rho_{\phi \beta \gamma}K_{\gamma \beta \rho \lambda} + 2K^\rho_{\phi \beta \gamma}K^\rho_{\phi \beta \gamma}B_{\gamma \beta \rho \lambda}. \quad (73)$$

The expression for $[\nabla_\rho \nabla_\nu R_{\gamma \rho}]$ now follows by combining (71) with (72) and (73). After little rearrangements, that reads
\[ [\nabla_\alpha \nabla_\beta R_{\gamma\delta}] = n_\alpha n_\beta A_{\gamma\delta} + n_\gamma n_\delta A_{\alpha\beta} + n_\gamma n_\mu [R^\gamma_{\delta\beta\rho} R_{\gamma\mu}] + [R^\gamma_{\rho\mu\delta} R_{\gamma\beta}] + 2n_\gamma h_{\mu\delta} \nabla_\gamma r_{\mu\delta} - n_\gamma h_{\mu\delta} n_\rho n_\sigma \nabla_\gamma [R] - 2n_\gamma h_{\mu\delta} n_\sigma n_\beta \nabla_\gamma B_{\gamma\mu\rho} + 2n_\gamma h_{\mu\delta} K_{\gamma\mu\rho} n_\sigma + 4n_\gamma h_{\mu\delta} K_{\gamma\mu\rho} B_{\gamma(\beta\gamma)} + 2n_\gamma h_{\mu\delta} K_{\gamma\mu\rho} n_{\beta\gamma} - 4K_{\gamma\beta} \nabla_\alpha B_{\gamma(\beta\gamma)} n_{\mu\delta} + 2n_{\beta\gamma} \nabla_\gamma B_{\gamma(\beta\gamma)} - 2K_{\gamma\beta} B_{\gamma(\beta\gamma)} n_{\mu\delta}, \]

(74)

We must stress the fact that there are still terms in \( R_{\alpha\beta} \) and \( r_{\beta\mu} \) that are not completely independent.

The contraction of (74) with \( g^{\alpha\lambda} \) yields

\[ [\Box R_{\beta\mu}] = A_{\beta\mu} + K_{\beta\mu} + n_\beta n_\mu \left( \frac{1}{2} \Box [R] - [R] K^{\alpha\sigma} K_{\alpha\sigma} + 2K^{\alpha\sigma} K_{\beta\mu} \right) + 2n_\beta h_{\mu\gamma} \left( \nabla_\gamma [R] K_{\gamma\mu} - 2K^{\alpha\sigma} \nabla_\alpha B_{\gamma(\beta)} + \nabla_\gamma K_{\gamma\mu} - \nabla_\alpha K^{\alpha\sigma} B_{\gamma(\beta)} \right) + [R] K_{\mu\beta} K_{\rho\sigma} + \Box B_{\mu\beta} - 2K_{\mu\beta} K_{\rho\sigma} B_{\gamma(\beta)} \gamma, \]

(75)

while contracting with \( g^{\beta\mu} \) we obtain \[ [\nabla_\gamma \nabla_\mu R] = A_{\beta\mu} n_\beta n_\mu + 2n_\gamma \nabla_\mu b - 2n_\gamma h_{\mu\gamma} \nabla_\gamma [R] + b K_{\beta\mu} + \nabla_\gamma \nabla_\mu [R]. \]

(76)

From any of the previous we readily have

\[ [\Box R] = A_{\beta\mu} + b K + \Box [R]. \]

(77)

The energy-momentum quantities (51–53) will arise from the discontinuities of the normal components of the left-hand side of (20). In other words, we will only need to consider \( n^\alpha [G_{\alpha\beta}^{(2)}] \). Observe then that \( A_{\beta\mu} \) only appears in \([\Box R_{\beta\mu}]\), and since we only need the terms contracted with the normal once, in particular \([\Box R_{\beta\mu}]\), we are only interested in controlling \( n^\alpha A_{\beta\mu} \). This can be done by using the identities \( 2 \nabla_\gamma [R] = \nabla_\mu R^\mu = \nabla_\mu R^\mu \) at both sides of \( \Sigma \), and taking the difference after one further differentiation:

\[ n^\alpha [\nabla_\gamma \nabla_\mu R_{\beta\mu}] = \frac{1}{2} n^\alpha [\nabla_\gamma \nabla_\mu R]. \]

(78)

The left-hand side here comes from (71) combined with (73) after one contraction, whereas for the right-hand side we simply have to contract (76) with \( n^\alpha \). Equation (78) is thus found to be equivalent to

\[ n^\alpha A_{\beta\mu} + n^\alpha [-R^\alpha_{\gamma\beta} R_{\gamma\mu}] + [R_{\gamma\mu\rho\delta} R_{\gamma\delta}] + h^{\beta\delta} \nabla_\delta r_{\beta\mu} - K^{\alpha\beta} \nabla_\delta B_{\beta\mu} - n_\mu \left( \frac{1}{2} [R] K_{\gamma\mu} K^{\alpha\sigma} - K^{\alpha\sigma} K_{\gamma\mu} B_{\gamma(\beta)} \right) = \frac{1}{2} \left( A_{\beta\mu} n_\beta + \nabla_\mu b - K_{\gamma\beta} \nabla_\gamma [R] \right). \]

(79)

5.4. Discontinuities of terms quadratic in the curvature

Now, let us concern ourselves with the many terms in (21) quadratic in curvature tensors. To start with, using (61) with (157) we readily obtain

\[ [R_{\alpha\gamma}^\beta R_{\delta\beta}^\gamma] = 2[R_{\alpha\gamma}^\beta] R_{\delta\beta}^\gamma = \left( B_{\alpha\beta} + \frac{1}{2} [R] n^\alpha n^\beta \right) R_{\alpha\gamma}^\beta. \]

(80)
\[ [R_{\alpha\beta}] = R^\Sigma_{\alpha\beta}[R_{\alpha\beta}] + [R]R^\Sigma_{\alpha\beta} = R^\Sigma_{\alpha\beta} \left( B_{\alpha\beta} + \frac{1}{2} n_{\alpha} n_{\beta} [R] \right) + RR^\Sigma_{\alpha\beta}, \quad (81) \]

\[ [R^2] = 2[R]R^\Sigma. \quad (82) \]

Regarding \( n^\alpha [R_{\alpha\beta} R^\mu_\beta] \), let us first consider the contraction \( n^\alpha n^\beta [R^\Sigma_{\alpha\beta} R_{\mu\beta}] \). The chain of equalities

\[ n^\alpha n^\beta [R^\Sigma_{\alpha\beta} R_{\mu\beta}] = 2n^\alpha n^\beta R^\Sigma_{\alpha\beta} [R^\mu_\beta] = [R]n^\alpha n^\beta R^\Sigma_{\mu\beta} \]

follows from (61) and (157). Half-adding the two \( \pm \) equations (154) and using the result in (83) we derive

\[ n^\alpha n^\beta [R^\Sigma_{\alpha\beta} R_{\mu\beta}] = \frac{1}{2} [R] \left( R^{\Sigma}_{\gamma\gamma} - [R] + (K^\rho_\mu)^2 - K^\rho_\mu K^{\rho\alpha
\mu} \right). \quad (84) \]

Analogous procedures using the Gauss equation (149) accordingly yield

\[ n^\alpha n^\beta [R^\Sigma_{\alpha\beta} R_{\mu\beta}] = B_{\mu\nu} \left( \nabla^\alpha K^{\lambda\alpha}_\nu - \nabla^\nu K^{\rho}_\mu \right) + \frac{1}{2} [R] \left( \nabla^\alpha K^{\rho}_\nu - \nabla^\nu K^{\rho}_\mu \right), \quad (85) \]

\[ n^\alpha n^\beta [R_{\alpha\beta\mu} R^{\mu\nu}] = (2R^{\Sigma}_{\mu\nu} - R^{\mu\nu} + K^{\rho}_\mu K^{\rho}_{\mu\nu} - K^{\rho}_\mu K^{\rho}_{\nu\mu}) B^{\mu\nu}, \quad (86) \]

\[ n^\alpha n^\beta [R_{\alpha\beta\mu\nu} R^{\mu\nu}] = \left( \nabla^\beta K^{\lambda\alpha}_{\nu\mu} - \nabla^\nu K^{\rho}_{\mu\nu} \right) B^{\mu\nu} - \left( \nabla^\nu K^{\rho}_{\mu\nu} - \nabla^\beta K^{\rho}_{\nu\mu} \right) B^{\mu\nu}. \quad (87) \]

\[ n^\alpha n^\beta [R_{\alpha\beta\mu\nu} R^{\mu\nu}] = 4n^\alpha n^\beta [R_{\alpha\beta\mu\nu} R^{\mu\nu}] - 4R^{\Sigma}_{\alpha\beta} B^{\mu\nu}, \quad (88) \]

\[ n^\alpha n^\beta [R_{\alpha\beta\mu\nu} R^{\mu\nu}] = 2B^{\alpha\beta} \left( \nabla_{\alpha} K_{\beta\mu\nu} - \nabla_{\beta} K_{\alpha\mu\nu} \right) - \left( \nabla_{\alpha} K_{\beta\mu\nu} - \nabla_{\beta} K_{\alpha\mu\nu} \right) B^{\mu\nu}. \quad (89) \]

\[ [R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}] = 2n^\alpha n^\beta [R_{\alpha\beta\mu\nu} R^{\mu\nu}] \]

\[ = 8B^{\alpha\beta} \left( R^{\Sigma}_{\alpha\beta} - R^{\alpha\beta} + K^{\rho}_\mu K^{\rho\alpha}_{\mu\beta} - K^{\rho}_\mu K^{\rho\beta}_{\mu\alpha} \right). \quad (90) \]

### 5.5. Discontinuities of the quadratic part \([G^{(2)}_{\alpha\beta}]\)

We are now ready to compute the full \( n^\alpha [G^{(2)}_{\alpha\beta}] \). To keep track of the different terms, we split the compilation of terms into three parts, corresponding to the terms multiplied by either of the three constants \( a_1, a_2, a_3 \) in (21).

- **Terms with \( a_1 \):** The terms in (21) that go with \( a_1 \) are

  \[ G^{(2)}_{\alpha\beta} := 2RR_{\alpha\beta} - 2\nabla_{\beta} \nabla_{\alpha} R - \frac{1}{2} g_{\alpha\beta} R^2 + 2g_{\alpha\beta} \Box R, \]

  and we can compute their jump using (81), (76) and (77) to obtain

  \[ n^\alpha n^\beta [G^{(2)}_{\alpha\beta}] = 2[R] R^{\Sigma}_{\alpha\beta} n^\alpha n^\beta + 2b K^\rho_\mu + 2\Box [R] \]

  and

  \[ n^\alpha h^\beta_\mu [G^{(2)}_{\alpha\beta}] = 2[R] R^{\Sigma}_{\alpha\beta} n^\alpha h^\beta_\mu - 2\nabla^\beta b + 2K^\rho_\mu \nabla^\alpha [R]. \]

- **Terms with \( a_2 \):** The terms in (21) relative to \( a_2 \) are

  \[ G^{(2)}_{\alpha\beta} := 2R_{\alpha\beta\mu\nu} R^{\mu\nu} - \nabla_{\beta} \nabla_{\alpha} R + \Box R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} (R_{\mu\nu} R^{\mu\nu} - \Box R). \]
Before using (86) and (87) it is convenient to write down \( n^\alpha [\Box R_{\alpha \beta}] \) using (75) combined with (79), since some terms simplify. With the help of (86), (87), (76), (81) and (66–68) it is then easy to get

\[
n^\alpha n^\beta [G_{\alpha \beta}^{(2)\alpha \beta}] = \frac{b}{2} K^\rho_\rho + \Box [R] + \frac{1}{4} [R](R^\Sigma - R) + (K^\mu_\mu)^2 \left( -\frac{3}{4} [R]K_{\mu \rho}K^{\rho \sigma} + \nabla_\rho K^{\rho \sigma} + \nabla_\mu K^{\mu \sigma} \nabla_\nu G^{\nu \mu \sigma} + B^{\mu \nu}(R^\Sigma_{\mu \nu} - R_{\mu \nu} + K^\rho_{\mu \nu}K_{\rho \nu}), \right) \tag{93}
\]

\[
n^\alpha h^\beta_\mu [G_{\alpha \beta}^{(2)\alpha \beta}] = \frac{1}{2} \nabla_\mu b + \frac{3}{2} K^\alpha_\mu \nabla_\alpha [R] + [R]\left( \nabla_\alpha K^\alpha_\mu - \frac{1}{2} \nabla_\mu K \right) - \nabla_\alpha R_{\alpha \beta} + K^\mu_\mu \nabla_\nu [G_{\nu \beta}] + B^{\alpha \beta}(\nabla_\mu K_{\alpha \beta} - \nabla_\beta K_{\alpha \mu}) - B_{\alpha \beta \mu \nu} \nabla_\nu K^{\alpha \beta} - K^{\alpha \beta} \nabla_\mu B_{\alpha \beta}. \tag{94}
\]

• Terms with \( a_3 \): Regarding \( a_3 \) we have

\[
G_{\alpha \beta}^{(2)\alpha \beta} := -4 R_{\alpha \beta} R^\beta_\beta + 2 R_{\alpha \mu \beta \rho} R^{\rho \mu \nu \beta} + 4 R_{\alpha \beta \mu \nu} R^{\mu \nu \beta}_\beta - 2 \nabla_\beta R_{\alpha \beta} + 4 \nabla_\beta R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} R_{\rho \mu \nu} R^{\rho \mu \nu}. \tag{90}
\]

All terms have already appeared except for the last one, for which we use (90). Straightforward calculations lead to

\[
n^\alpha n^\beta [G_{\alpha \beta}^{(2)\alpha \beta}] = 4 R_{\alpha \beta} K^{\alpha \beta} + 4 \nabla_\alpha \nabla_\beta [G^{\alpha \beta}] + 4 [G_{\alpha \beta}] K^{\alpha \beta} K^\mu_\mu + 2 \Box [R] + 4 B^{\alpha \beta}(R_{\alpha \beta} - R_{\alpha \beta} + K_{\alpha \beta} K^\rho_\rho - K_{\alpha \beta} K^\rho_\rho), \tag{95}
\]

\[
n^\alpha h^\beta_\mu [G_{\alpha \beta}^{(2)\alpha \beta}] = 4 K^\mu_\alpha \nabla_\beta [G_{\beta \alpha}] - 4 \nabla_\beta R_{\alpha \beta} + 4 K^\alpha_\mu \nabla_\alpha [R] - 4 \nabla_\beta (B_{\alpha \beta} K^{\alpha \beta}) + 2 [R] \nabla_\beta K^\alpha_\mu - 4 B_{\alpha \beta \mu \nu} \nabla_\nu K^{\alpha \beta} + 4 B^{\alpha \beta}(\nabla_\mu K_{\alpha \beta} - \nabla_\beta K_{\alpha \mu}). \tag{96}
\]

Collecting all of the above, we finally obtain

\[
n^\alpha n^\beta [G_{\alpha \beta}^{(2)}] = \kappa \left\{ b K^\rho_\rho + \Box [R] + \frac{1}{2} (R^\Sigma - R + (K^\mu_\mu)^2 - K_{\mu \rho}K^{\rho \sigma}) \right\} \\
+ \kappa \left\{ 2 R_{\alpha \beta} K^{\alpha \beta} + 2 \nabla_\alpha \nabla_\beta [G^{\alpha \beta}] + 2 B^{\alpha \beta}(R_{\alpha \beta} - R_{\alpha \beta} + K_{\alpha \beta} K^\rho_\rho - K_{\alpha \beta} K^\rho_\rho) \right\} \\
+ 2 [G_{\alpha \beta}] K^{\alpha \beta} K^\mu_\mu + \Box [R]. \tag{97}
\]

\[
n^\alpha h^\beta_\mu [G_{\alpha \beta}^{(2)}] = \kappa \left\{ [R](\nabla_\mu K^\alpha_\alpha - \nabla_\alpha K^\alpha_\mu) - \nabla_\mu b + K^\rho_\alpha \nabla_\alpha [R] \right\} \\
+ \kappa \left\{ -2 \nabla_\beta R_{\alpha \beta} + 2 K^\alpha_\mu \nabla_\beta [G_{\mu \alpha}] + 2 B^{\alpha \beta}(\nabla_\mu K_{\alpha \beta} - \nabla_\beta K_{\alpha \mu}) + 2 K^{\alpha \beta} \nabla_\mu [R] \right\} \\
+ [R] \nabla_\mu K^\alpha_\alpha - 2 B_{\alpha \beta \mu \nu} \nabla_\nu K^{\alpha \beta} + 2 K^{\alpha \beta} \nabla_\mu B_{\alpha \beta}. \tag{98}
\]
As a final remark, we would like to stress that all the discontinuities computed in this section 5 are purely geometrical, and therefore valid in any theory based on a Lorentzian manifold whenever (28) holds.

6. Field equations on the layer Σ

Relations (97) and (98) are the equations we were looking for, but we wish to rewrite them in terms of (derivatives of) the energy-momentum quantities supported on Σ given in (51–53) and (57). Observe, first of all, that the three relations (66–68) allow us to rewrite the energy-momentum contents supported on Σ given in (51–53) as follows:

\[ \kappa \tau_{\alpha \beta} = -(\kappa_1 + \kappa_2) [R] K_{\alpha \beta} + \kappa_1 b h_{\alpha \beta} + 2 \kappa_2 R_{\alpha \beta}, \]

\[ \kappa \tau_\alpha = -(\kappa_1 + \kappa_2) \nabla_\alpha [R] - 2 \kappa_2 \nabla^\beta [G_{\alpha \beta}], \]

\[ \kappa \tau = (\kappa_1 + \kappa_2) [R] K^\alpha + 2 \kappa_2 K^{\alpha \nu} [G_{\alpha \nu}], \]

and using the definition of the double-layer strength (57) the last two here can be rewritten as

\[ \tau_\alpha = -\nabla^\alpha \mu_{\alpha \beta}, \]

\[ \tau = K^{\alpha \nu} \mu_{\alpha \nu}. \]

Now, a direct computation provides the following expressions for some combinations of derivatives of these objects:

\[ \kappa (\nabla^3 \tau_{\alpha \beta} + K^\mu_{\alpha \beta} \tau_\mu + \nabla_\alpha \tau_\beta) = -(\kappa_1 + \kappa_2) (K_{\alpha \beta} \nabla_\alpha [R] + [R] \nabla^3 K_{\alpha \beta} - \nabla_\alpha K^\mu_\rho) + \kappa_1 \nabla_\alpha b + 2 \kappa_2 (\nabla^3 R_{\alpha \beta} + \nabla_\alpha (K^{\alpha \nu} [G_{\alpha \nu}]) + K^\mu_\nu \nabla^\nu [G_{\alpha \nu}]), \]

\[ \kappa (\tau_{\alpha \beta} K^{\alpha \beta} - \nabla^3 \tau_\alpha) = (\kappa_1 + \kappa_2) (\Delta [R] - [R] K^{\alpha \nu} K^{\nu \alpha}) + \kappa_1 b K^\mu_{\alpha \beta} + 2 \kappa_2 (K^{\alpha \nu} R_{\alpha \nu} + \nabla^3 K^{\alpha \beta} [G_{\alpha \beta}]). \]

Using these, equations (98) and (97) become, respectively (after some rewriting using (150) and (151) and (20))

\[ \kappa (n^\alpha h^\beta_\nu [T_{\alpha \beta}] + \nabla^\beta \tau_{\alpha \beta} + K^\mu_\nu \tau_\beta + \nabla_\beta \tau) = 2 \kappa_2 [K^{\alpha \nu} \nabla_\beta [G_{\alpha \nu}] - K^\mu_\nu \nabla^\alpha [G_{\alpha \nu}] + \nabla_\mu ([G^{\alpha \nu}] K_{\alpha \beta}) - \nabla_\beta ([G_{\nu \beta}] K^{\alpha \mu})], \]

\[ \kappa (n^\alpha h^\beta_\nu [T_{\alpha \beta}] + \nabla^\beta \tau_{\alpha \beta} - \tau_{\alpha \beta} K^{\alpha \beta}) = (\kappa_1 + \kappa_2) [R] (n^\alpha n^\beta R^\gamma_{\alpha \beta} + K_{\nu \beta} K^{\alpha \gamma}) + 2 \kappa_2 [G^{\mu \nu}] (n^\alpha n^\beta R^\gamma_{\alpha \beta} R_{\nu \gamma} + K^\mu_\nu K_{\mu \nu}). \]

Now, using the definition of the strength (57) these become

\[ n^\alpha h^\beta_\nu [T_{\alpha \beta}] + \nabla^\beta \tau_{\alpha \beta} + K^\mu_\nu \tau_\beta + \nabla_\beta \tau = K^{\alpha \nu} \nabla_\beta \mu_{\alpha \rho} - K^\mu_\nu \nabla^\alpha \mu_{\alpha \beta} + \nabla_\nu (\mu^{\alpha \nu} K_{\alpha \beta}) - \nabla_\beta (\mu_{\alpha \beta} K^{\nu \gamma}), \]

\[ n^\alpha n^\beta [T_{\alpha \beta}] + \nabla^\alpha \tau_\alpha - \tau_{\alpha \beta} K^{\alpha \beta} = \mu^{\mu \nu} (n^\alpha n^\beta R^\gamma_{\alpha \beta} R_{\nu \gamma} + K^\mu_\nu K_{\mu \nu}). \]

Recalling here the relations (102) and (103) between \( \tau_\alpha \) and \( \tau \) with the double-layer strength \( \mu_{\alpha \beta} \), we finally obtain the following field equations:
\[ n^\alpha h_{\beta}^\rho [T_{\rho\sigma}] + \nabla^\alpha \tau_{\alpha\beta} = -\mu_{\alpha\rho} \nabla_\beta K^{\alpha\rho} + \nabla_\rho (\mu^{\alpha\rho} K_{\alpha\beta}) - \nabla_\rho (\mu_{\alpha\beta} K^{\alpha\rho}), \]  
(107)

\[ n^\alpha n_{\beta} [T_{\rho\sigma}] - \tau_{\alpha\beta} K^{\alpha\beta} = \nabla^\alpha \nabla_\beta \mu_{\alpha\beta} + \mu^{\alpha\beta} (n^\alpha n_{\beta} R^{\Sigma}_{\mu\nu\rho\sigma} + K_{\alpha\beta} K^{\alpha\beta}). \]
(108)

These are the fundamental field equations satisfied by the energy-momentum quantities (51) and (57) within \( \Sigma \). They generalize the classical Israel equations of GR [12] and they are very satisfactory from a physical point of view. They possess an obvious structure with a clear interpretation of energy-momentum conservation relations. There are three type of terms in these relations. The first type is given by the corresponding first summands on the left-hand side. They simply describe the jump of the normal components of the energy-momentum tensor across \( \Sigma \). Therefore, they are somehow the main source for the energy-momentum contents in \( \Sigma \). The second type of terms are those on the left-hand side involving \( \tau_{\alpha\beta} \), the energy-momentum tensor in the shell/layer \( \Sigma \). We want to remark that the first equation (107) provides the divergence of \( \tau_{\alpha\beta} \). Finally, the third type of terms are those on the right-hand side, involving the strength \( \mu_{\alpha\beta} \) of a double layer. These terms also act as sources of the energy-momentum contents within \( \Sigma \), combined with extrinsic geometric properties of \( \Sigma \) and curvature components in the spacetime.

An alternative version of (107), after use of the Codazzi equation (152), reads

\[ n^\alpha h_{\beta}^\rho [T_{\rho\sigma}] + \nabla^\alpha \tau_{\alpha\beta} = \mu^{\alpha\beta} n^\rho \nabla_\rho \nabla^{\Sigma}_{\alpha\beta} \tau_{\alpha\beta} + K^{\alpha\beta} \nabla_\rho (\mu_{\alpha\beta} K^{\rho\sigma}) \]
(109)

Note that the allowed jumps in the Riemann tensor (60) lead to \( n^\rho \nabla_{\rho} h^\alpha_{\beta} = 0 \) and therefore the term \( \tau_{\alpha\beta} K^{\alpha\beta} \) in the last formula can be written simply as \( \mu^{\alpha\beta} n^\rho \nabla^{\Sigma}_{\alpha\beta} \tau_{\alpha\beta} \).

7. Energy-momentum conservation

The divergence of the left-hand side of the field equations (20) vanishes identically due to the Ricci and Bianchi identities and, therefore, the conservation equation for the energy-momentum tensor \( \nabla_{\alpha} T^{\mu\nu} = 0 \) follows. In our situation, however, we are dealing with tensor distributions, and with (20) considered in a distributional sense. The question arises whether or not the energy-momentum tensor distribution (50) is covariantly conserved. We know that the Bianchi and Ricci identities hold for distributions (see appendices), hence it is expected that the divergence of the \( T_{\mu\nu} \) vanishes when distributions are considered. In this section we prove that this is the case, when taking into account the fundamental field equations (107) and (108). The following calculation can alternatively be seen, therefore, as an independent derivation of (107) and (108)—from the covariant conservation of \( T_{\mu\nu} \).

From (48) and (138) we directly get

\[ \nabla^0 T_{\alpha\beta} = n^\alpha [T_{\alpha\beta}] \delta^\Sigma + \nabla^0 (\nabla_{\alpha} \delta^\Sigma) + \nabla^0 T_{\alpha\beta}. \]
(110)

Let us first compute the middle term on the right-hand side. From the orthogonal decomposition (49)

\[ \nabla^0 (\nabla_{\alpha} \delta^\Sigma) = \nabla^0 (\{ \tau_{\alpha\beta} + \tau_{\beta\alpha} \} n^\alpha \delta^\Sigma) + \nabla^0 (\{ \tau_{\alpha\beta} + \tau_{\beta\alpha} \} \delta^\Sigma) \]

and using the general formula (163) the second summand can be expanded to get

\[ \nabla^0 (\nabla_{\alpha} \delta^\Sigma) = \nabla^0 (\{ \tau_{\alpha\beta} + \tau_{\beta\alpha} \} n^\alpha \delta^\Sigma) + (\nabla^0 \tau_{\alpha\beta} - \tau_{\alpha\beta} K^{\alpha\rho} \tau_{\rho\beta} + \tau_{\alpha\beta} K_{\alpha\beta} + \tau_{\alpha\beta} \nabla^0 \tau_{\alpha} \delta^\Sigma) \]
so that with the help of (102) we get
\[
\nabla^{a}(\mathcal{T}_{a,b}^{\Sigma}) = \nabla^{a}(\{\tau_{b} + \tau n_{a}\}n_{b}^{\Sigma}) \\
+ (\nabla^{a}r_{a,b}^{\Sigma} - \tau_{a,b}K_{a}^{\nu}n_{b} - K_{a,b}^{\nu}\nabla^{\rho}\mu_{p,a} - n_{b}^{a}\nabla^{\rho}\mu_{p,b})^{\Sigma}.
\]
(111)

With respect to the last term in (110), on using definitions (56) and (57) we can write for any test vector field \(Y^{b}\) (and using the Ricci identity)
\[
\langle \nabla^{a}l_{a,b}, Y^{b} \rangle = -\langle l_{a,b}^{n}, \nabla^{a}Y^{b} \rangle = \int_{\Sigma} \mu_{a,b}n^{\rho} \nabla_{\rho}Y^{b} \, d\sigma \\
= \int_{\Sigma} (\mu_{a,b}n^{\rho} \nabla_{\rho}Y^{b} + R^{\rho}_{\sigma}n^{\sigma}Y^{b}) \, d\sigma \\
= \int_{\Sigma} \mu_{a,b}n^{\rho} \nabla_{\rho}Y^{b} \, d\sigma = \langle n^{a\rho\sigma}R_{\rho\sigma}^{\Sigma}, Y^{b} \rangle.
\]
The first integral here can be expanded as
\[
\int_{\Sigma} \mu_{a,b}n^{\rho} \nabla_{\rho}Y^{b} \, d\sigma = \int_{\Sigma} \mu_{a,b} \{ \nabla^{a}(n^{\rho}Y^{b}) - K_{a}^{\nu}n_{b}^{\rho}Y^{b} \} \, d\sigma \\
= \int_{\Sigma} n^{\rho} \nabla_{\rho}Y^{b}(\mu_{a,b}K_{a}^{\nu}n_{b} - \nabla^{a}\mu_{a,b}) \, d\sigma - \int_{\Sigma} \mu_{a,b}K^{\nu}(\nabla_{\rho}Y^{b} + (n_{a}^{\rho}Y^{b})K_{a}^{\rho}) \, d\sigma \\
= \int_{\Sigma} (\tau_{a} + \tau_{b})n^{\rho} \nabla_{\rho}Y^{b} \, d\sigma + \int_{\Sigma} Y^{b}(\mu_{a,b}K^{\nu} - n_{b}^{a}\mu_{a}K_{b}^{\nu}) \, d\sigma \\
= -\langle \nabla^{a}(\{\tau_{a} + \tau_{b}\}n_{a}^{\Sigma}), Y^{b} \rangle + \langle \nabla^{a}(\mu_{a,b}K^{\nu} - n_{b}^{a}\mu_{a}K_{b}^{\nu})^{\Sigma}, Y^{b} \rangle.
\]
so that we arrive at
\[
\nabla^{a}l_{a,b}^{n} = -\nabla^{a}(\{\tau_{a} + \tau_{b}\}n_{a}^{\Sigma}) + (\nabla_{\rho}(\mu_{a,b}K^{\nu} - n_{b}^{a}\mu_{a}K_{b}^{\nu}) - n^{a\rho\sigma}R_{\rho\sigma}^{\Sigma})^{\Sigma}.
\]
(112)

Adding up (111) and (112) to (110) we finally obtain
\[
\nabla^{a}l_{a,b} = \{n^{a}[l_{a,b}^{n}] + \nabla^{a}\tau_{a,b}^{\Sigma} - \tau_{a,b}K_{a}^{\nu}n_{b}^{\Sigma} + \nabla_{\rho}(\mu_{a,b}K^{\nu}) \\
- n_{b}^{a}\mu_{a}K_{b}^{\nu} - n^{a\rho\sigma}R_{\rho\sigma}^{\Sigma} - K_{a,b}^{\nu}\nabla^{\rho}\mu_{p,a} - n_{b}^{a}\nabla^{\rho}\mu_{p,b}\}^{\Sigma}.
\]

The fundamental equations (108) and (109) prove the vanishing of this expression leading to
\[
\nabla^{a}l_{a,b}^{n} = 0
\]
as claimed. As remarked in [20, 21], this calculation shows that the double-layer energy-momentum distribution \(l_{a,b}^{n}\) is essential to keep energy-momentum conservation. Without the double-layer contribution the total energy-momentum tensor distribution \(l_{a,b}^{n}\) would not be covariantly conserved.

8. Matching hypersurfaces, thin shells and double layers

Once we have discussed the junction in the case of gravity theories with quadratic terms, and have obtained the corresponding field equations on \(\Sigma\), we are in a disposition to analyze their consequences. Before entering into this discussion, it is convenient to note the following important result.
Result 8.1. If there is no double layer (that is $\mu_{\alpha \beta} = 0$), then there can be neither external flux momentum $\tau_\alpha$ nor external pressure/tension $\tau$.

This follows directly from expressions (102) and (103). In other words, there exist non-vanishing external flux momentum and/or external pressure/tension only if there is a double layer.

Thus, there are three levels of junction depending on whether or not thin shells and/or double layers are allowed. We will term them as:

- **Proper matching:** this is the case where the matching hypersurface $\Sigma$ does not support any distributional matter content, describing simply an interface with jumps in the energy-momentum tensor, so that there are neither thin shells nor double layers. This situation models, for instance, the gravitational field of stars (non-empty interior) with a vacuum exterior. Or the case of vacuoles in cosmological surroundings.

- **Thin shells, but no double layer:** this is an idealized situation where an enormous quantity of matter is concentrated on a very thin region mathematically described by $\Sigma$, but where no double layer is permitted to exist. Thus, delta-type terms proportional to $\delta^\Sigma$ are allowed, and the expression (51) provides the energy-momentum tensor of the thin shell. However, from result 8.1 the other possible quantities (52) and (53) accompanying $\delta^\Sigma$ vanish identically. This situation is analogous to that in GR where only (51) appears. The main difference with a generic quadratic gravity arises in the explicit expression for (51), as the field equations turn out to adopt the same form.

- **Double layers:** this is the general case with no further assumptions, which describes a large concentration of matter on $\Sigma$, as in the previous case, but accompanied with a brusque jump in the curvature of the spacetime. Still, there are several sub-possibilities depending on the vanishing or not of any of (51–53). There is also an extreme possibility, that we term a pure double layer, where the thin shell is not present, but the double layer is: this is the case with all (51–53) vanishing, but a non-vanishing (56). Nothing like any of these different possibilities can be described in GR.

We classify the junction condition for these different cases in turn.

8.1. Thin shells without double layer

From (56), it follows that the strength of the double layer $\mu_{\alpha \beta}$ must be set to zero, and thus from (57) we have

$$ (\kappa_1 + \kappa_2)[R]h_{\alpha \beta} + 2\kappa_2[G_{\alpha \beta}] = 0 \quad \implies \quad (\kappa_2 + n\kappa_1)[R] = 0, $$

which implies that $\tau$ and $\tau_\alpha$ both vanish (see result 8.1). Hence, only the tangential part of the distributional energy-momentum tensor on $\Sigma$ survives, given explicitly by (99). Its trace, upon using (69), reads

$$ \kappa\tau^\alpha = (n\kappa_1 + \kappa_2)b - K^{\alpha \beta}\mu_{\alpha \beta} = (n\kappa_1 + \kappa_2)b. $$

The equations (107) and (108) in this case read

$$ n^\alpha h^\beta_{\alpha \beta}[T_{\alpha \beta}] = -\nabla^\alpha \tau_{\alpha \beta}, \quad n^\alpha n^\beta[T_{\alpha \beta}] = \tau_{\alpha \beta} K^{\alpha \beta}. $$

Observe that, remarkably, these are identical with the Israel conditions derived in GR.

We have to distinguish whether $\kappa_2 = 0$ or not.
\* $\kappa_2 \neq 0$. If $(n \kappa_1 + \kappa_2) \neq 0$ relations (113) imply that $[R] = 0$ and $[G_{\alpha\beta}] = 0$ in full. The direct consequences are $[R_{\alpha\beta}] = [R_{\alpha\beta\mu\nu}] = 0$, and the discontinuities in the derivatives are given by

$$\left[ \nabla_\mu R_{\alpha\beta\lambda\nu} \right] = (n_\alpha n_\beta R_{\gamma\nu} - n_\alpha n_\gamma R_{\beta\nu} - n_\beta n_\gamma R_{\alpha\nu} + n_\gamma n_\beta R_{\alpha\mu}) n_\mu,$$

(116)

for some symmetric tensor $R_{\alpha\beta}$ tangent to $\Sigma$. From (54) we get $b = 2R'_{\mu}$ and therefore the energy-momentum tensor (51) on $\Sigma$ just reads

$$\kappa \tau_{\alpha\beta} = \kappa_1 b h_{\alpha\beta} + 2\kappa_2 R_{\alpha\beta}.$$ 

With regard to the exceptional possibility $n \kappa_1 + \kappa_2 = 0$, equation (113) implies, in particular, that the tensor $B_{\alpha\beta}$ is proportional to the first fundamental form. The explicit relation reads

$$B_{\alpha\beta} = \frac{1}{2n} [R] h_{\alpha\beta},$$

which for the discontinuity of the Riemann tensor produces

$$[R_{\alpha\beta\gamma\lambda}] = \frac{[R]}{2n} (n_\alpha n_\beta h_{\gamma\lambda} - n_\alpha n_\gamma h_{\beta\lambda} - n_\beta n_\gamma h_{\alpha\lambda} + n_\gamma n_\beta h_{\alpha\lambda}).$$

(117)

Taking contractions in this last expression we find the allowed jumps in the Ricci and Einstein tensor

$$[R_{\alpha\beta}] = \frac{[R]}{2} \left( \frac{1}{n} h_{\alpha\beta} + n_\alpha n_\beta \right) \Rightarrow [G_{\alpha\beta}] = \frac{1 - n}{2n} [R] h_{\alpha\beta}.$$ 

(18)

Note, $[R]$ is the only degree of freedom allowed for the discontinuities of the curvature tensors.

The remaining allowed discontinuities of the derivative of the Ricci tensor are encoded in $r_{\alpha\beta} = n^\mu [\nabla_\mu R_{\alpha\beta}]$, so that

$$[\nabla_\mu R_{\alpha\beta}] = r_{\alpha\beta} n_\mu + \frac{1}{2} \left( n_\alpha n_\beta + \frac{1}{n} h_{\alpha\beta} \right) \nabla_\mu [R]$$

$$+ \left( \frac{1 - n}{2n} \right) [R] (n_\alpha K_{\beta\mu} + n_\beta K_{\alpha\mu}).$$

(19)

Recalling that $b = r'_{\alpha\beta} = n^\mu [\nabla_\mu R]$ the explicit form of the energy-momentum tensor on $\Sigma$ can be obtained from (99). Due to (114), $\tau_{\alpha\beta}$ is traceless. Nevertheless, the relevance of this exceptional case is probably marginal, as the coupling constants satisfy a dimensionally dependent condition.

\* $\kappa_2 = 0$. We have to assume then that $\kappa_1 \neq 0$, as otherwise all the terms (51–53) vanish identically and thus there are no thin shells. Let us also recall that $a_2$ and $a_3$ are assumed not to vanish simultaneously, as that case was fully analyzed in [19–21], so it would be more precise to label this case as $a_2 = -4a_3$ with $a_1 = a_3$. This case reduces to the condition $[R] = 0$ (see (113)). All the remaining jumps on the curvature tensor and its derivatives are allowed. The energy-momentum tensor on $\Sigma$ is simply given by

$$\kappa \tau_{\alpha\beta} = \kappa_1 b h_{\alpha\beta},$$

(20)

with $b = n^\mu [\nabla_\mu R]$, and therefore the thin shell $\Sigma$ only contains, at most, a ‘cosmological constant’ type of matter content.
8.2. Proper matching hypersurface

In addition to the requirement imposed in the previous case of thin shells, we demand now that the full $\mathcal{F}_{\alpha\beta}$ vanishes. Thus we have to add $\tau_{\alpha\beta} = 0$ to the conditions discussed in the previous section 8.1. In general, from (115) we have

$$n^\nu [T_{\alpha\beta}] = 0$$

which adopt exactly the same form as in GR and we call the generalized Israel conditions. They imply the continuity of the normal components of the energy-momentum tensor across $\Sigma$.

Again, we have to distinguish two cases depending on whether $\kappa_2$ vanishes or not.

- $\kappa_2 \equiv 0$. If $(n\kappa_1 + \kappa_2) = 0$, we already know from the previous section that $[R] = 0$ and $[G_{\alpha\beta}] = 0$. The trace relation (114) provides $b = 0$ and moreover $\tau_{\alpha\beta} = 0$ implies, via (99), $\mathcal{R}_{\alpha\beta} = 0$. Plugging this information into (116) it follows that the derivatives of the curvature tensors do not present discontinuities.

**Result 8.2.** In the generic case with $4a_1 + a_2 \neq 0$ and $4a_3 + (1 + n)a_2 + 4n\kappa_1 \neq 0$, the full set of matching conditions amount to those of GR (agreement of the first and second fundamental forms on $\Sigma$) plus the agreement of the Ricci tensor and its first derivative on $\Sigma$:

$$[R_{\alpha\beta}] = 0, \quad [\nabla_\rho R_{\alpha\beta}] = 0.$$ 

This actually implies that the full Riemann tensor and its first derivatives have no jumps across $\Sigma$:

$$[R_{\alpha\beta\mu\rho}] = 0, \quad [\nabla_\rho R_{\alpha\beta\mu\rho}] = 0.$$ 

With regard to the exceptional possibility $\kappa_2 + n\kappa_1 = 0$, the curvature tensors satisfy (117) and (118). Now $\tau_{\alpha\beta} = 0$ provides

$$\mathcal{R}_{\alpha\beta} = \frac{1}{2n} ((n - 1)[R] K_{\alpha\beta} + b h_{\alpha\beta}),$$

and thus $r_{\alpha\beta} = n^\nu [\nabla_\rho R_{\alpha\beta}]$ gets determined in terms of $[R]$ and $b$, so that (119) for $[\nabla_\rho R_{\alpha\beta}]$ reads

$$[\nabla_\rho R_{\alpha\beta}] = \left( \frac{1}{2n} ((n - 1)[R] K_{\alpha\beta} + b h_{\alpha\beta}) - 2n\kappa_1 [R] + \frac{b}{2} + \frac{1 - n}{2n} [R] K_{\alpha\beta} n_{\alpha} n_{\beta} \right) n_{\rho} + \frac{1}{2} \left( n_{\rho} n_{\beta} + \frac{1}{n} h_{\alpha\beta} \right) \nabla_\rho [R] + \left( \frac{1 - n}{2n} \right) [R] (n_{\alpha} K_{\beta\mu} + n_{\beta} K_{\alpha\mu}).$$

Hence, the entire set of discontinuities of the Riemann tensor and its first derivative can be written just in terms of $[R]$ and $b = n^\nu [\nabla_\rho R]$, which remain as two free degrees of freedom. As mentioned before, this case is probably irrelevant due to its defining condition depending on the dimension $n$.

- $\kappa_2 = 0$ but $\kappa_1 \equiv 0$. From the results from the previous section we know that $[R] = 0$ and the energy momentum on $\Sigma$ is given by (120). Thus, for a proper matching we find $b = 0$.

The discontinuity in the derivative is

$$[\nabla_\rho R_{\alpha\beta}] = n_{\rho} \left( [R]_{\rho\beta} [K^{\nu\sigma} n_{\beta} - 2 \nabla_\nu [R_{\rho(\beta\sigma)}] n_{\alpha}) + \mathcal{R}_{\alpha\beta} \right) + \nabla_\rho [R_{\alpha\beta}] - 2 K^\rho_{\mu} [R_{\rho(\alpha\beta)} n_{\beta}],$$

where also $\mathcal{R}^\rho_{\mu} = - K^\nu_{\rho(\nu} [R_{\nu\mu}]$. 

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**8.3. The double layer fauna: pure double layers**

The generic occurrence in quadratic gravity, as shown above, is that any thin shell comes accompanied by a double layer, which in turn generically implies the existence of non-zero external pressure/tension and external flux momentum. However, there are several special possibilities in which one of these quantities, or all, disappear. This gives rise to a fauna of different kinds of double layers. There is also the possibility that the double layer term (56) is non-zero while the remaining distributional part in the energy-momentum tensor, that is \( \tilde{T}_{\alpha \beta} \delta^{\Sigma} \), vanishes. In other words, a double layer without a classical thin shell. We call such a case a **pure double layer**. In the rest of this section we explore this novel possibility.

For pure double layers, the vanishing of the external pressure \( \tau \) plus the energy flux \( \tau_{\alpha} \) first imply, by virtue of (52) and (102)

\[
\mu_{\alpha \beta} K^{\alpha \beta} = 0, \quad \nabla^\mu \mu_{\rho \alpha} = 0. \tag{123}
\]

In particular, then, the double layer strength is conserved.

The first equation in (123) yields

\[
(\kappa_1 + \kappa_2)[R] K^{\sigma}_{\sigma} + 2 \kappa_2 K^{\alpha \rho} [G_{\rho \sigma}] = 0 \tag{124}
\]

while the second gives

\[
(\kappa_1 + \kappa_2) \nabla_\alpha [R] + 2 \kappa_2 \nabla^{\rho} [G_{\rho \alpha}] = 0. \tag{125}
\]

Equation (124) combined with the vanishing of the trace of \( \tau_{\alpha \beta} \) provides

\[
(\kappa_1 n + \kappa_2) b = 0 \tag{126}
\]

so that, generically \((n\kappa_1 + \kappa_2 = 0)\) one has \(b = 0\). A first consequence is that the jump in the derivative of the Ricci scalar is now tangent to \( \Sigma \) and fully determined by the tangent derivative of \([R]\):

\[
[\nabla_\alpha [R]] = \nabla_\alpha [R]. \tag{127}
\]

The vanishing of \( \tau_{\alpha \beta} \), using (51), is now equivalent to

\[
\kappa_2 \mathcal{R}_{\alpha \beta} = (\kappa_1 + \kappa_2) [R] K_{\alpha \beta}. \tag{128}
\]

The expression for the discontinuity of the normal derivative of the Ricci tensor has to be studied depending on \( \kappa_2 \) vanishing or not.

* \( \kappa_2 \neq 0 \). The relations above allow us to write the discontinuity of the normal derivative of the Ricci tensor as

\[
r_{\alpha \beta} = \frac{1}{2} \left( 1 + \frac{\kappa_1}{\kappa_2} \right) [(R) K_{\alpha \beta} + n_\beta \nabla_\alpha [R] + n_\alpha \nabla_\beta [R] - K [R] n_\alpha n_\beta],
\]

whereas the tangent part of the derivative keeps its original form given in (70).

* \( \kappa_2 = 0 \) (and \( \kappa_1 \neq 0 \)). Equations (125) and (128) read

\[
\nabla_\alpha [R] = 0, \quad [R] K_{\alpha \beta} = 0, \tag{129}
\]
and (124) is automatically satisfied. Thus, (127) implies \( \nabla_a R = 0 \). Observe that since \( \kappa_2 = 0 \), (57) establishes that the strength of the double layer is proportional to \( [R] \). Hence, in order to have a non-zero \( \rho_{a\beta} [R] \) cannot vanish. Then, \( K_{a\beta} = 0 \) necessarily, and the allowed jumps are encoded in \( [R_{a\beta}] \) and \( r_{a\beta} \).

For completeness, we provide, finally, the formulas for the exceptional case \( n\kappa_1 = n\kappa_2 = 0 \) — discarding \( \kappa_1 = \kappa_2 = 0 \) for which the double layer simply disappears. The equations \( \tau = 0 \), \( \tau_a = 0 \) and \( \tau_{a\beta} = 0 \) result, respectively, in

\[
(1 - n)[R]K_{\beta}^{\alpha} - 2nK^{\alpha\beta}[G_{\alpha\beta}] = 0,
\]

\[
(1 - n)\nabla_a [R] - 2n\nabla^\alpha [G_{\alpha\beta}] = 0,
\]

\[
(1 - n)[R]K_{a\beta} - bh_{a\beta} + 2n\mathcal{R}_{a\beta} = 0.
\]

While the third equation provides \( \mathcal{R}_{a\beta} \), the first two constitute constraints on the allowed jumps of the Ricci tensor that should be analyzed in each particular situation. In all cases, the allowed discontinuity in the derivative of the Ricci tensor can be written as

\[
r_{a\beta} = \frac{1}{2n} \left( (1 - n)[R]K_{a\beta} - bh_{a\beta} \right) - \frac{1 - n}{2n} (n\beta \nabla_a [R] + n\alpha \nabla_\beta [R])
\]

\[
+ \frac{1}{2} \left( b + \frac{1 - n}{n} [R]K^\beta_\rho \right) n_\alpha n_\beta.
\]

Observe that now the strength of the double layer is traceless, \( \rho^\rho = 0 \) (see, for example, (58)).

9. Consequences

The proper matching conditions in GR are the agreement of the first and second fundamental forms on \( \Sigma \). Therefore, any matching hypersurface in GR satisfies (28), and the allowed jumps in the energy-momentum tensor are equivalent to non-vanishing discontinuities of the Ricci (and Riemann) tensor. Thus, in GR, properly matched spacetimes will generally have \( [R_{a\beta}] = 0 \).

This simple known fact implies that any GR solution containing a proper matching hypersurface will contain a double layer and/or a thin shell at the matching hypersurface if the true theory is quadratic.

At least two relevant consequences follow from this fact: (i) generically, matched solutions in GR are no longer solutions in quadratic theories; and (ii) if any quantum regimes require, excite or switch on quadratic terms in the Lagrangian density, then GR solutions modeling two regions with different matter contents will develop thin shells and double layers on their interfaces. Let us elaborate.

Consider, for instance, the case of a perfect fluid matched to a vacuum in GR. As is well known, the GR matching hypersurface is determined by the condition that

\[
p^{GR}_\Sigma = 0
\]

where \( p^{GR} \) is the isotropic pressure of the fluid in GR. It follows that the Ricci tensor has a discontinuity of the following type:

\[
[G_{a\beta}] = \kappa^{GR}_a \kappa^{GR}_\Sigma, \quad [R_{a\beta}] = \kappa^{GR}_a \kappa^{GR}_\Sigma + \frac{1}{n - 1} g_{a\beta}\]
\( u^\alpha \) being the unit velocity vector of the perfect fluid. Therefore, using (99–101) and (56) we see that the very same spacetime has, in any quadratic theory of gravity, an energymomentum tensor distribution with all types of thin-shell and double-layer terms.

Imagine the situation of a collapsing perfect fluid (to form a black hole, say) with a vacuum exterior. Then, one can use any of the known solutions in GR to describe this situation—the reader may have in mind, for instance, the Oppenheimer-Snyder model. The GR solution describes this process accurately in the initial and intermediate stages, when the curvature of the spacetime is moderate and the values of \( a_1 R^2, a_2 R_{\alpha \beta} R^{\alpha \beta} \) and \( a_3 R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} \) for instance, or other similar quantities, are small enough to render any quadratic terms in the Lagrangian totally negligible. However, as the collapse proceeds and one approaches the black hole regions—and later the classical singularity—regimes with very high curvatures are reached. Then, the quadratic terms coming from any quantum corrections (be they from string theory counter-terms, or any other) to the Einstein–Hilbert Lagrangian start to be important, and actually to dominate (the curvature being enormous). In this regime, the original matching hypersurface actually becomes a thin double layer.

Of course, the description of a global spacetime via a matching is an approximation, and the use of tensor distributions is also just another approximation to a real situation where a gigantic quantity of matter can be concentrated around a very thin region of the spacetime. Nevertheless, both approximations are satisfactory in the sense that they are believed to actually mimic a realistic situation where the layer is thick and the jumps in the energy variables are extremely big, but finite. If this is the case, then the above reasoning seems to imply that, if quadratic theories of gravity are correct, at least in some extreme regimes, then a huge concentration of matter will develop around the interface of the interior and the exterior of the collapsing star. And this huge concentration will generically manifest as a shell with double-layer properties.

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**Appendix A. Tensor distributions**

Let \( \mathcal{D}(V) \) be the set of sufficiently differentiable tensor fields of any order with *compact support* on the (oriented) Lorentzian manifold \((V, g)\) and denote by \( \mathcal{D}_p^q \) the subset of \( p \)-covariant \( q \)-contravariant such tensor fields. The elements of these sets are called *test tensor fields*. The definition of tensor distributions is as follows (see, for instance [14, 16, 23]).

**Definition A.1 (Tensor distribution)** A \( p \)-covariant \( q \)-contravariant tensor distribution \( \chi^q_p \) is a linear and continuous functional:

\[
\chi^q_p : \mathcal{D}_p^q \to \mathbb{R},
\]

\[
Y^p_q \mapsto \chi^q_p(Y^p_q) = \langle \chi^q_p, Y^p_q \rangle.
\]

The set of tensor distributions of the same type constitutes a \( \mathbb{R} \)-vector space with the natural
definitions. This set also contains all locally integrable fields: any $p$-covariant $q$-contravariant tensor field $T^{p}_{q}$ defines a unique tensor distribution $\mathcal{D}^{p}_{q}$ by means of

$$\mathcal{I}^{q}_{p} : \mathcal{D}^{p}_{q} \rightarrow \mathbb{R}$$

$$Y^{p}_{q} \rightarrow \langle \mathcal{I}^{q}_{p}, Y^{p}_{q} \rangle \equiv \int_{V} (T^{q}_{p}, Y^{p}_{q}) \eta$$

where $\eta$ is the volume element $(n + 1)$-form and $(T, Y)$ is a shorthand for the scalar

$$(T^{q}_{p}, Y^{p}_{q}) \equiv T^{q}_{\mu_{1} \cdots \mu_{p}} Y^{\mu_{1}, \ldots, \mu_{p}}_{\mu_{1} \cdots \mu_{p}}$$

Observe that, following [16], we use the convention of distinguishing between a tensor field and its associated tensor distribution by using an underline. We will also put an underline on some other tensor distributions, not associated to tensor fields, to emphasize their distributional character.

**Definition A.2** (Tensor distribution components) The components of a $(p, q)$-tensor distribution $\chi$ relative to dual bases $\{\theta^{\alpha}\}, \{\bar{e}_{\alpha}\}$ are scalar distributions $\chi^{\alpha_{1}, \ldots, \alpha_{q}}_{\beta_{1}, \ldots, \beta_{p}}$ defined by

$$\langle \chi^{\alpha_{1}, \ldots, \alpha_{q}}_{\beta_{1}, \ldots, \beta_{p}}, Y \rangle \equiv \langle \chi^{q}_{p}, Y \theta^{\alpha_{1}} \otimes \cdots \otimes \theta^{\alpha_{q}} \otimes \bar{e}_{\beta_{1}} \otimes \cdots \otimes \bar{e}_{\beta_{p}} \rangle$$

where $Y$ is any test function.

It easily follows:

$$\langle \chi^{q}_{p}, Y^{q}_{p} \rangle = \langle \chi^{q}_{p}, Y^{q}_{p} \rangle_{\alpha_{1}, \ldots, \alpha_{q}}^{\alpha_{1}, \ldots, \alpha_{q}}$$

(130)

Contraction of indexes are then well defined and independent of the basis for tensor distributions.

A tensor distribution $\chi^{q}_{p}$ is said to vanish on an open set $U \subset V$ if

$$\langle \chi^{q}_{p}, Y^{p}_{q} \rangle = 0$$

for all test tensor fields with support contained in $U$.

**Definition A.3** (Support of tensor distributions) The support of a tensor distribution $\chi^{q}_{p}$ is the complement in $V$ of the union of all open sets where $\chi^{q}_{p}$ vanishes.

The support of a tensor distribution is always a closed set.

**Definition A.4** (Tensor product by tensor fields) The tensor product of a tensor distribution $\chi^{q}_{p}$ by a tensor field $T^{r}_{s}$—defined on a neighborhood of the support of $\chi^{q}_{p}$—is the $(p + s)$-covariant $(q + r)$-contravariant tensor distribution acting as

$$\langle T^{r}_{s} \otimes \chi^{q}_{p}, Y^{r+p}_{s+q} \rangle \equiv \langle \chi^{q}_{p}, (T, Y)^{r+p}_{s+q} \rangle$$

where $(T, Y)^{r+p}_{s+q} \in \mathcal{D}^{r+p}_{s+q}$ is given, in any basis, by

$$(T, Y)^{r+p}_{s+q} = T^{\mu_{1}, \ldots, \mu_{p}}_{\alpha_{1}, \ldots, \alpha_{q}} Y^{\mu_{1}, \ldots, \mu_{p}}_{\alpha_{1}, \ldots, \alpha_{q}}$$

Sometimes, for this product to make sense it is enough that the tensor field $T^{r}_{s}$ is defined just on the support of $\chi^{q}_{p}$, however this will not be the case when derivatives are involved. Care must be taken with this problem.
Tensor distributions can be differentiated (by acting on differentiable elements of $\mathcal{D}(V)$). The main definition is as follows.

**Definition A.5 (Covariant derivative of tensor distributions)** The covariant derivative $\nabla \chi^q_p$ of a $(p, q)$-tensor distribution $\chi^q_p$ is the $(p + 1, q)$-tensor distribution defined by

$$\langle \nabla \chi^q_p, Y^{p+1} \rangle = -\langle \chi^q_p, (DY)^q_p \rangle$$

where $(DY)_{\beta_1 \ldots \beta_q}^{\alpha_1 \ldots \alpha_p} = \nabla_{\mu} Y_{\beta_1 \ldots \beta_q}^{\alpha_1 \ldots \alpha_p}$.

This definition is well posed in the sense that it recovers the usual covariant derivative for differentiable tensor fields:

$$\nabla T = \nabla^T.$$

The components of $\nabla \chi^q_p$ in any basis are denoted by $\nabla_{\mu} \chi^{q}_{\beta_1 \ldots \beta_q}$. 

**Appendix B. Matching hypersurfaces and thin shells**

Consider the case where the spacetime $(V, g)$ contains a hypersurface $\Sigma$ such that the metric tensor (or its first derivatives) may be not differentiable across $\Sigma$. In that case, its derivatives and the Christoffel symbols may not exist at points of $\Sigma$—but they do outside this hypersurface. Similarly, the curvature tensors and their derivatives will not be defined on $\Sigma$ as tensor fields. However, given that the metric tensor $g$ is continuous, it defines a tensor distribution $g$ which can be differentiated in the distributional sense. Thus, one can define the curvature and its derivatives as tensor distributions.

To that end, we can define the $\Sigma$-step function $\theta : V \to \mathbb{R}$ as

$$\theta = \begin{cases} 
1 & \text{on } V^+ \\
1/2 & \text{on } \Sigma \\
0 & \text{on } V^- 
\end{cases} \quad (131)$$

This function is locally integrable so that it defines a scalar distribution $\varrho$:

$$\langle \varrho, Y \rangle = \int_{V^+} Y \eta$$

whose covariant derivative is a one-form distribution with support on $\Sigma$ acting as

$$\langle \nabla \varrho, \bar{Y} \rangle = -\langle \varrho, D\bar{Y} \rangle = -\int_{V^+} \nabla_{\mu} Y^\mu \eta = \int_{\Sigma} Y^\mu d\sigma_\mu = \int_{\Sigma} Y^\mu n_\mu d\sigma$$

where $d\sigma_\mu = n_\mu d\sigma$ is oriented from $V^-$ towards $V^+$, $n_\mu$ is the unit normal to $\Sigma$, and thus $d\sigma$ is its canonical volume element $n$-form. It gives rise to a natural scalar distribution $\delta^\Sigma$ with support on $\Sigma$ defined by

$$\langle \delta^\Sigma, Y \rangle := \int_{\Sigma} Y d\sigma. \quad (132)$$

This distribution can act on every locally integrable test function defined at least at the points of $\Sigma$. Similarly, it can be multiplied by any locally integrable tensor field defined at least on $\Sigma$. Observe in particular, from the above, that

$$\nabla_{\mu} \varrho = n_\mu \delta^\Sigma. \quad (133)$$

Let $T$ be any $(p, q)$-tensor field which (i) may be discontinuous across $\Sigma$, (ii) is differentiable on $V^+$ and $V^-$, and (iii) such that $T$ and its covariant derivative have definite limits on
Σ coming from both $V^+$ and $V^-$. We will use the notation $T^\pm$ for the restriction of $T$ to $V^\pm$ respectively. The tensor distribution associated to such a $T$, which exists because the tensor is locally integrable, is

$$T = T^+\theta + T^- (1 - \theta).$$

By definition, we also take

$$T := T^+\theta + T^- (1 - \theta)$$
as the tensor field defined everywhere, and thus at each point of $\Sigma$:

$$T^\Sigma := T|_\Sigma = \frac{1}{2} \left( \lim_{x \in \Sigma} T^+(x) + \lim_{x \rightarrow \Sigma} T^-(x) \right).$$

The covariant derivative of $T$ can be shown to be [16]

$$\nabla T = \nabla T^+\theta + \nabla T^-(1 - \theta) + \mathbf{n} \otimes [T] \delta^\Sigma$$

where $[T]$ is a $(p, q)$-tensor field defined only on $\Sigma$, that we call the ‘jump’ or the ‘discontinuity’ of $T$ at $\Sigma$ and is defined as

$$\forall q \in \Sigma, \quad [T](q) \equiv \lim_{x \rightarrow q} T^+(x) - \lim_{x \rightarrow q} T^-(x).$$

The index version of (136) reads

$$\nabla_\mu T^{\alpha_1 \ldots \alpha_q}_{\beta_1 \ldots \beta_p} = \nabla_\mu T^{+\alpha_1 \ldots \alpha_q}_{\beta_1 \ldots \beta_p} \theta + \nabla_\mu T^{-\alpha_1 \ldots \alpha_q}_{\beta_1 \ldots \beta_p} (1 - \theta) + [T^{\alpha_1 \ldots \alpha_q}_{\beta_1 \ldots \beta_p}] n_\mu \delta^\Sigma.$$ (138)

Formula (133) is just a particular case of this general formula.

It must be noted that formula (136), or (138), is precisely the formula one would derive by using a naive calculation starting from (134), applying Leibniz’s rule and using (133). However, such a naive approach cannot be used when the tensor distribution to be differentiated involves non-tensorial distributions—such as $\delta^\Sigma$. For instance, one may be tempted to compute the second covariant derivative of $\theta$ starting from (133) and write

$$\nabla_\mu \nabla_\nu \theta \propto \nabla_\mu n_\nu \delta^\Sigma + n_\mu \nabla_\nu \delta^\Sigma \quad \text{wrong!!}$$

Neither term on the right-hand side is well defined due the the fact that $n_\mu$ exists only on $\Sigma$ and therefore its derivatives non-tangent to $\Sigma$ are not defined at all. Nevertheless, $\nabla_\mu \nabla_\nu \theta$ is certainly well defined as a distribution, and one can compute an explicit formula by strictly following the rules of tensor-distribution derivation and multiplication. This is done in full generality in appendix D.3, formula (163), which provides the desired result sought in (164) as well as later in the formula (168).

Appendix C. Curvature tensor distributions

We are now prepared to compute the connection and the Riemann tensor in the distributional sense. Recall that the metric $g$ is continuous across $\Sigma$ and can be written as a function or as a distribution:

$$g = g^+\theta + g^- (1 - \theta), \quad \bar{g} = g^+\bar{\theta} + g^- (1 - \bar{\theta}).$$

As in the general rule (138) one deduces

$$\partial_\mu \bar{g}_{\mu\nu} = \partial_\mu g^+_{\mu\nu} \theta + \partial_\mu g^-_{\mu\nu} (1 - \theta)$$

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from which, using the standard formula, one can construct the Christoffel symbols as distributions:

\[
\Gamma^\alpha_{\beta\gamma} = \Gamma^+\alpha_{\beta\gamma} + \Gamma^-\alpha_{\beta\gamma} (1 - \theta)
\]  

where \(\Gamma^\pm\alpha_{\beta\gamma}\) denote the corresponding symbols on \(V^\pm\), respectively. These scalar distributions are associated to locally integrable functions given by

\[
\Gamma^\alpha_{\beta\gamma}|_{\Sigma} = \frac{1}{2}(\Gamma^+\alpha_{\beta\gamma}|_{\Sigma} + \Gamma^-\alpha_{\beta\gamma}|_{\Sigma}).
\]

Now, using the standard formula (in a local coordinate basis) and treating the objects appearing as distributions, we define the Riemann tensor distribution by

\[
R^\alpha_{\beta\gamma\lambda} = \partial_\lambda \Gamma^\alpha_{\beta\gamma} - \partial_\gamma \Gamma^\alpha_{\beta\lambda} + \Gamma^\rho_{\beta\lambda} \Gamma^\alpha_{\rho\gamma} - \Gamma^\rho_{\beta\gamma} \Gamma^\alpha_{\rho\lambda}.
\]

First of all, we observe that the products of \(\Gamma\)'s are well defined because \(\Gamma^\pm_{\alpha\beta\gamma}\) are distributions associated to functions and, actually, they become (upon using \(\theta \cdot \theta = \theta\))

\[
\Gamma^\pm_{\alpha\beta\gamma}|_{\Sigma} = \frac{1}{2}(\Gamma^+\pm_{\alpha\beta\gamma}|_{\Sigma} + \Gamma^-\pm_{\alpha\beta\gamma}|_{\Sigma}).
\]

On the other hand, we have from (139), as in (138):

\[
\partial_\lambda \Gamma^\pm_{\alpha\beta\gamma} = \partial_\gamma \Gamma^\pm_{\beta\lambda\alpha} + \partial_\gamma \Gamma^\pm_{\alpha\lambda\beta} (1 - \theta) + [\Gamma^\pm_{\beta\lambda\alpha}]_n \delta^\Sigma
\]

so that the final expression for the Riemann tensor distribution reads

\[
R^\alpha_{\beta\gamma\lambda} = R^+\alpha_{\beta\gamma\lambda} + R^-\alpha_{\beta\gamma\lambda} (1 - \theta) + \{\partial_\lambda [\Gamma^\pm_{\alpha\beta\gamma}]_n]\}_n \delta^\Sigma
\]

where \(R^\pm\alpha_{\beta\gamma\lambda}\) are the Riemann tensors fields on \(V^\pm\) respectively. The part proportional to \(\delta^\Sigma\) is called the singular part of the Riemann tensor distribution, and it has support only on \(\Sigma\). An explicit formula for this part in terms of the jump of the second fundamental form of \(\Sigma\) is provided in the main text, expression (9).

Appendix D. Useful formulas

D.1. Concerning \(\Sigma\) and its objects

Consider a hypersurface \((\Sigma, h_{\alpha\beta})\) imbedded in a \((n+1)\)-dimensional spacetime \((V, g_{\alpha\beta})\). We will later use this construction for the + and − sides. Using the dual bases \(\{n^\alpha\}, e^\alpha\) and \(\{n_{\mu}\}, \omega^\mu\) introduced in section 2, we have

\[
e^\alpha_a \nabla_b e^\alpha_b = -K^a_{\alpha} n^\alpha + \Gamma^\alpha_{ab} e^\alpha_b,
\]

\[
e^\alpha_a \nabla_b \omega^\alpha_b = -K^b_{\alpha} n^\alpha - \Gamma^b_{ac} \omega^c_a,
\]

\[
e^a_b \nabla_b n^\alpha = K^a_{\alpha\mu} \omega^\mu
\]

where \(K_{\alpha\beta}\) is the second fundamental form introduced in (3) and

\[
\Gamma^\alpha_{ab} = \omega^\alpha_a e^a_c \nabla_c e^\alpha_b
\]

represent the Christoffel symbols of the Levi-Civita connection associated to the first fundamental form \(h_{\alpha\beta}\) of \(\Sigma\). In general—unless the jump of the second fundamental form (8) vanishes—there will be two versions, one + and one − of all these equations except for the
last one, the connection, which is uniquely defined given that the first fundamental form agrees on both sides (1).

The covariant derivative defined by $\Gamma$ is denoted by $\nabla$. The relationship between $\nabla$ and $\nabla$ on $\Sigma$ is ruled by the following formula (given here for a (1, 1)-tensor field $S^{ab}$, but generalizable in the obvious way to arbitrary ranks [16]):

$$\omega^a_{\nu} e^\nu_b \nabla_{\rho} S^b_{\alpha} = \nabla_{\nu} S^\rho_{\alpha} + (e^\beta_b S^\rho_{\alpha \beta}) K^\nu_a + (\omega^a_{\alpha \nu} n^\nu) K_{\nu b}$$  \hspace{1cm} (146)

where, for any tensor field $S$, we denote by $\tilde{S}$ its projection to $\Sigma$:

$$\tilde{S}^a_b := \omega^a_{\nu} e^\nu_b S^\nu.$$  \hspace{1cm} (147)

The equivalent ‘spacetime’ version of (146) is

$$h^\nu_b h^\mu_a \nabla_{\rho} S^b_{\alpha} = \nabla_{\nu} S^\rho_{\alpha} + (h^\beta_b S^\rho_{\alpha \beta}) K^\nu_a + (h^\nu_b S^\nu_{\alpha \beta}) K_{\nu b},$$  \hspace{1cm} (148)

where $\tilde{S}^a_b$ is the spacetime version of $S^a_b$, i.e. $\tilde{S}^a_b := \omega^a_{\nu} e^\nu_b S^\nu$.

Denoting by $\tilde{R}^a_{\alpha \beta \gamma \delta}$ the Riemann tensor of $(\Sigma, h_{ab})$, the classical Gauss equation reads

$$\omega^a_{\alpha \beta \gamma} e^\beta_b e^\gamma_c = \tilde{R}^a_{\alpha \beta \gamma} - K^d_a K^a_{\beta} - K^a_{ab} K^{a \beta},$$  \hspace{1cm} (149)

whose contractions are

$$e^\alpha_a e^\beta_c R_{\alpha \beta \gamma} - n^\alpha n^\beta R_{\alpha \beta \gamma \delta} e^\gamma_d e^\delta_b = \tilde{R}_{ac} - K^d_a K_{ac} + K^d_{ac} K^b_c,$$  \hspace{1cm} (150)

$$R - 2n^\alpha n^\beta R_{\alpha \beta} = \tilde{R} - (K_{\alpha \beta})^2 + K_{ab} K^{ab}$$  \hspace{1cm} (151)

where $\tilde{R}_{ac}$ and $\tilde{R}$ denote the Ricci tensor and scalar curvature of $(\Sigma, h_{ab})$.

Similarly, the classical Codazzi equation reads

$$n^\alpha R_{\alpha \beta \gamma} e^\beta_b e^\gamma_c = \nabla_b K_{ca} - \nabla_b K_{ca},$$  \hspace{1cm} (152)

with contraction

$$n^\alpha R_{\alpha \beta \gamma} e^\beta_b = \nabla_b K^a_c - \nabla_b K^{a c}.$$  \hspace{1cm} (153)

As mentioned before, generically there will be two versions of each of the previous equations, one for each ± side of $\Sigma$ if this is a matching hypersurface. Thus, for instance (and using spacetime notation), (151) and (153) must have the two versions:

$$R^\pm = 2R^\mu_{\mu \nu \rho} n^\nu n^\rho = \tilde{R} - (K^\pm_{\nu \rho})^2 + K_{\mu \nu} K^{\pm \mu \rho},$$  \hspace{1cm} (154)

$$n^\mu R^\pm_{\mu \nu} h^\nu_\rho = \tilde{R}^\pm_{\mu \nu} - \tilde{R}^\pm_{\mu \rho}.$$  \hspace{1cm} (155)

On the other hand, equation (146) at points of the matching hypersurface $(\Sigma, h_{ab})$ of the already glued spacetime $V = V^+ \cup V^-$ reads

$$h^\nu_b h^\mu_a \nabla_{\rho} S^b_{\alpha} = \nabla_{\nu} S^\rho_{\alpha} + (h^\beta_b S^\rho_{\alpha \beta}) K^\nu_a + (h^\nu_b S^\nu_{\alpha \beta}) K_{\nu b},$$  \hspace{1cm} (156)

where here we use $h \nabla_{\nu}$ just to make explicit that $h \nabla_{\nu}$ is being restricted to points at $\Sigma$ using the connection given by (140), whose restriction to $\Sigma$ is (141). Note, that whenever both second fundamental forms coincide $[K_{\alpha \beta}] = 0$ on a matching hypersurface $\Sigma$, equation (156) reads just as (148).
D.2. Discontinuities

In the computations we need the discontinuities of objects, such as functions and tensor fields, across $\Sigma$. This also implies that we need to control such discontinuities for the derivatives of those objects, and for their products. Here we provide the general rules.

Let $A$ and $B$ be any two functions possibly discontinuous across $\Sigma$. Then

$$[AB] = A^+ B^+ |_{\Sigma} - A^+ B^- |_{\Sigma} = A^+ B^+ |_{\Sigma} - A^+ B^- |_{\Sigma}$$

and an equivalent expression interchanging $A \leftrightarrow B$. Adding these two expressions and using (135) we get

$$[AB] = A^\Sigma [B] + [A]B^\Sigma. \quad (157)$$

Concerning derivatives, let us start with any function $f$ that may be discontinuous across $\Sigma$. If we compute the tangent derivatives on both sides of $\Sigma$ we obtain

$$e^\mu [\partial_\mu f] = [e^\mu \partial_\mu f] = \frac{\partial f^+}{\partial x^+} - \frac{\partial f^-}{\partial x^-} = \frac{\partial f^+}{\partial x^+} - \partial_\mu f^- |_{\Sigma} = \partial_\mu [f] = e^\mu \partial_\mu [f]$$

and thus, by orthogonal decomposition:

$$[\partial_\mu f] = F \n_\mu + \omega^\mu_\nu e^\mu_\nu \partial_\nu [f] = F \n_\mu + h_\mu^\nu \partial_\nu [f] \quad (158)$$

where $F$ is a function defined only on $\Sigma$ that measures the discontinuity of the normal derivatives of $f$ across $\Sigma$:

$$F \equiv n^\mu [\partial_\mu f].$$

Consider now the case of a one-form $\tilde{t}_\mu$, again possibly discontinuous across $\Sigma$. A direct computation using (158) and (157) produces

$$e^\mu [\n_\mu t_\alpha] = e^\mu_\alpha [\partial_\mu t_\alpha] = e^\mu_\alpha (\partial_\mu [t_\alpha] - \Gamma_\mu^\alpha \n_\alpha)$$

$$= e^\mu_\alpha (\partial_\mu [t_\alpha] - \Gamma_\mu^\alpha \n_\alpha)$$

Let us remark that the derivative $\n_\mu$ is restricted to points on $\Sigma$, so that the connection (141) must be used. Therefore:

$$[\n_\mu t_\alpha] = n_\mu T_\alpha + h_\mu^\nu \n_\nu t_\alpha - \Gamma_\mu^\alpha \n_\alpha h_\mu^\nu \quad (159)$$

where $T_\alpha$ is a one-form defined only on $\Sigma$ giving the discontinuity of the normal derivatives of $t_\alpha$ across $\Sigma$

$$T_\alpha := n^\mu [\n_\mu t_\alpha].$$

and the tangential derivative $h_\mu \n_\nu$ is restricted to $\Sigma$, although it is not explicitly indicated so as not to overwhelm the expressions. The right-hand side of (159) can be further computed. First, due to (156):

$$h_\mu^\nu \n_\nu [t_\alpha] = \n_\alpha [t_\alpha] + n^\mu [t_\mu] K^\Sigma_{\mu \alpha} + n_\alpha n^\rho h_\mu^\nu \n_\nu [t_\rho]$$

$$= \n_\alpha [t_\alpha] + n^\mu [t_\mu] K^\Sigma_{\mu \alpha} + n_\alpha \n_\rho (t_\rho n^\rho) - n_\alpha [t_\rho] K^\Sigma_{\mu \alpha}$$

while, for the last summand in (159) we use (5) and (7):

$$-\Gamma_\mu^\alpha \n_\alpha h_\mu^\nu = \Gamma_\mu^\alpha n^\nu [K_{\mu \alpha}] - n_\alpha \Gamma_\alpha^\Sigma_{\mu \alpha}.$$
Introducing both results into (159) we get

\[
[\nabla \mu t_\alpha] = n_\mu T_\alpha + \nabla_\mu [t_\alpha] + n^\nu [t_\rho] K^{\nu\alpha}_{\mu\rho} + n_\alpha (\nabla_\mu [t_\rho] n^\rho) - [t_\rho] K^{\nu\alpha}_{\mu\rho} + t_\rho^{\nu\alpha} [K^{\nu\alpha}_{\mu\rho}] - n_\alpha t_\rho^{\nu\alpha} [K^{\nu\alpha}_{\mu\rho}]
\]

\[= n_\mu T_\alpha + \nabla_\mu [t_\alpha] + n^\nu [t_\rho] K^{\nu\alpha}_{\mu\rho} + n_\alpha (\nabla_\mu [t_\rho] n^\rho) - [t_\rho] K^{\nu\alpha}_{\mu\rho}. \tag{160}\]

Observe that when there is no jump of the second fundamental form, 
\[K_{\alpha\beta} = 0 \Rightarrow [\Gamma^{\mu\alpha\beta}_\rho] = 0,\]
equations (159) and (160) read, respectively

\[
[\nabla \mu t_\alpha] = n_\mu T_\alpha + \nabla_\mu [t_\alpha],
\]

\[
[\nabla \mu t_\alpha] = n_\mu T_\alpha + \nabla_\mu [t_\alpha] + n^\nu [t_\rho] K^{\nu\alpha}_{\mu\rho} + n_\alpha (\nabla_\mu [t_\rho] n^\rho) - [t_\rho] K^{\nu\alpha}_{\mu\rho}. \tag{162}\]

These formulas can be generalized to arbitrary \((p, q)\)-tensor fields \(T_p^q\) in an obvious way. In that case, the term replacing \(a T_p^q\) is simply a tensor field of the same type \((p, q)\) defined only on \(\Sigma\) and measuring the discontinuities of the normal derivatives of \(T_p^q\).

### D.3. Derivatives of tensor distributions proportional to \(\delta^\Sigma\)

Let us consider tensor distributions of type

\[t_{\alpha_1...\alpha_p} \delta^\Sigma,\]

where \(t_{\alpha_1...\alpha_p}\) is any tensor field defined at least on \(\Sigma\), but not necessarily off \(\Sigma\) (for instance \(h_{\mu\nu}\) or \(n_\mu\) are not defined outside \(\Sigma\)). We want to compute the covariant derivative of such tensor distributions. Then we have

\[
\langle \nabla_\lambda (t_{\alpha_1...\alpha_p} \delta^\Sigma), Y^{\lambda\alpha_1...\alpha_p} \rangle = -\langle t_{\alpha_1...\alpha_p} \delta^\Sigma, \nabla_\lambda Y^{\lambda\alpha_1...\alpha_p} \rangle = -\langle \delta^\Sigma, t_{\alpha_1...\alpha_p} \nabla_\lambda Y^{\lambda\alpha_1...\alpha_p} \rangle
\]

\[= -\int_\Sigma t_{\alpha_1...\alpha_p} \nabla_\lambda Y^{\lambda\alpha_1...\alpha_p} \, d\sigma = -\int_\Sigma t_{\alpha_1...\alpha_p} (n_\lambda n^\rho + h^\rho_\lambda) \nabla_\rho Y^{\lambda\alpha_1...\alpha_p} \, d\sigma.\]

The first summand here is

\[-\langle t_{\alpha_1...\alpha_p} n_\rho n^\rho \delta^\Sigma, \nabla_\lambda Y^{\lambda\alpha_1...\alpha_p} \rangle = \langle \nabla_\rho (t_{\alpha_1...\alpha_p} n_\rho n^\rho \delta^\Sigma), Y^{\lambda\alpha_1...\alpha_p} \rangle\]

while the second one has derivatives tangent to \(\Sigma\) and thus

\[-\int_\Sigma t_{\alpha_1...\alpha_p} h^\rho_\lambda \nabla_\rho Y^{\lambda\alpha_1...\alpha_p} \, d\sigma = -\int_\Sigma h^\rho_\lambda \nabla_\rho (t_{\alpha_1...\alpha_p} Y^{\lambda\alpha_1...\alpha_p}) \, d\sigma + \int_\Sigma Y^{\lambda\alpha_1...\alpha_p} h^\rho_\lambda \nabla_\rho t_{\alpha_1...\alpha_p} \, d\sigma\]

and using (148) for the first integral here:

\[-\int_\Sigma \nabla_\rho [(t_{\alpha_1...\alpha_p} Y^{\lambda\alpha_1...\alpha_p})] \, d\sigma = -\int_\Sigma K^{\rho\alpha_1...\alpha_p} n_\lambda t_{\alpha_1...\alpha_p} Y^{\lambda\alpha_1...\alpha_p} \, d\sigma + \int_\Sigma Y^{\lambda\alpha_1...\alpha_p} h^\rho_\lambda \nabla_\rho t_{\alpha_1...\alpha_p} \, d\sigma\]

where we have used that, as \(Y^{\lambda\alpha_1...\alpha_p}\) has compact support, the first total divergence term integrates to zero. Summing up, we have the following basic formula:

\[
\nabla_\lambda (t_{\alpha_1...\alpha_p} \delta^\Sigma) = \nabla_\rho (t_{\alpha_1...\alpha_p} n_\rho \delta^\Sigma) + (h^\rho_\lambda \nabla_\rho t_{\alpha_1...\alpha_p} - K^{\rho\alpha_1...\alpha_p} n_\lambda t_{\alpha_1...\alpha_p}) \delta^\Sigma. \tag{163}\]
In particular, for the second derivative of $\nabla_q$ one gets
\[
\nabla_q \nabla_q q = \nabla_q (n_\mu \delta^\Sigma) = \nabla_q (n_\mu n_\rho \delta^\Sigma) + (K_{\mu \rho} - K^\Sigma_{\mu \rho} n_\rho) \delta^\Sigma.
\] (164)

\subsection*{D.4. Ricci and Bianchi identities}

The Bianchi identity holds in the distributional sense (for a proof consult [16]):
\[
\nabla_q R_{\alpha \beta \gamma \mu} + \nabla_{\alpha} R_{\beta \gamma \mu} + \nabla_{\beta} R_{\gamma \mu \alpha} = 0.
\] (165)

Concerning the Ricci identity, let us consider a one-form which may have a discontinuity across $\Sigma$. It can be written as a 1-covariant tensor and as a one-form distribution as
\[
t_\alpha = t_\alpha^\gamma \theta + t_\alpha^\gamma (1 - \theta); \quad l_\alpha = l_\alpha^\gamma \theta + l_\alpha^\gamma (1 - \theta)
\]
To compute the derivatives, we need to take $l_\alpha$ as a distribution. Then, from (138) we first have
\[
\nabla_\alpha l_\alpha = \nabla_\alpha t_\alpha^\gamma \theta + \nabla_\alpha t_\alpha^\gamma (1 - \theta) + [l_\alpha] n_\mu \delta^\Sigma
\]
and applying (138) to the first part not proportional to $\delta^\Sigma$ we derive
\[
\nabla_\alpha \nabla_\beta l_\alpha = \nabla_\alpha \nabla_\beta t_\alpha^\gamma \theta + \nabla_\alpha \nabla_\beta t_\alpha^\gamma (1 - \theta) + [\nabla_\alpha t_\alpha^\gamma] n_\beta \delta^\Sigma + \nabla_\alpha [[l_\alpha]] n_\beta \delta^\Sigma).
\] (166)
Formula (163) gives the last term here:
\[
\nabla_\alpha [[l_\alpha]] n_\beta \delta^\Sigma) = \nabla_\alpha (l_\alpha n_\mu n_\rho \delta^\Sigma) + (n_\mu h^\rho_{\beta \gamma} \nabla_\gamma [l_\alpha] + [l_\alpha] K_{\beta \mu} - K^\Sigma_{\beta \mu} n_\rho [l_\alpha] n_\mu) \delta^\Sigma.
\]
Introducing (159) into (166) and using this last result we arrive at
\[
(\nabla_\alpha \nabla_\mu - \nabla_\mu \nabla_\alpha) l_\alpha = (\nabla_\alpha \nabla_\mu - \nabla_\mu \nabla_\alpha) t_\alpha^\gamma \theta + (\nabla_\alpha \nabla_\mu - \nabla_\mu \nabla_\alpha) t_\alpha^\gamma (1 - \theta)
- t_\alpha^\gamma [l_\alpha] \delta^\Sigma)
\]
and using here the Bianchi identity on both $\pm$ regions and expression (4) we finally get
\[
(\nabla_\alpha \nabla_\mu - \nabla_\mu \nabla_\alpha) l_\alpha = -l_\alpha^\rho R_{\alpha \beta \gamma \mu}.
\] (167)
Of course, this can be extended to tensor fields of any $(p, q)$ type which may have discontinuities across $\Sigma$.

What about the Ricci identity for tensor distributions not associated to tensor fields? The answer now is much more involved, and it must be treated case by case, because taking covariant derivatives presents several problems. As an illustrative example, let us analyze the case of the second covariant derivative of $\delta^\Sigma$. For the first derivative we have from (163):
\[
\nabla_\mu \delta^\Sigma = \nabla_\mu (n_\mu n_\rho \delta^\Sigma) - K^\Sigma_{\rho \mu} n_\rho \delta^\Sigma
\] (168)
so that defining a one-form distribution $\Delta_\mu$ with support on $\Sigma$ as follows:
\[
\langle \Delta_\mu, \gamma^\rho \rangle := -\int_\Sigma n_\mu n_\rho \nabla_\mu \gamma^\rho d\sigma; \quad \Delta_\mu = \nabla_\mu (n_\mu n_\rho \delta^\Sigma)
\] (169)
we can also write
\[
\nabla_\mu \delta^\Sigma = \Delta_\mu - K^\Sigma_{\rho \mu} n_\rho \delta^\Sigma.
\]
Note, however, that $\Delta_\mu$, and therefore $\nabla_\mu \delta^\Sigma$ too, is only well defined when acting on test vector fields whose covariant derivative is locally integrable on $\Sigma$. Thus, the second covariant derivative of $\delta^\Sigma$ is not defined in the general case with a discontinuous connection $\Gamma^\alpha_{\beta \mu}$. To see this, observe that to define $\nabla_\alpha \nabla_\mu \delta^\Sigma$ we need to define $\nabla_\alpha \Delta_\mu$, but this should be according to definition A.5:
\[ \langle \nabla_{\lambda} \Delta_{\mu}, Y^{\lambda \nu} \rangle = -\langle \Delta_{\mu}, \nabla_{\lambda} Y^{\lambda \nu} \rangle \]  
(170)

and this is ill-defined because \( \nabla_{\lambda} Y^{\lambda \mu} \) does not have a locally integrable covariant derivative in the sense of functions: actually, its covariant derivative can only be defined in the sense of distributions.

Nevertheless, if the connection is continuous, that is, \( [\Gamma^a_{\mu \nu}] = 0 \), then (170) makes perfect sense because the covariant derivative \( \nabla_{\rho} \nabla_{\lambda} Y^{\lambda \nu} \) is a locally integrable tensor field. Thus, in this case we can write

\[ \langle \nabla_{\lambda} \Delta_{\mu}, Y^{\lambda \nu} \rangle = \int_{S} n_{\rho} n_{\nu} \nabla_{\rho} \nabla_{\lambda} Y^{\lambda \nu} d\sigma \]  
(171)

and we can prove the Ricci identity for distributions such as \( \delta^{S} \). To that end, a straightforward, if somewhat lengthy calculation, using the Ricci identity under the integral and the rest of techniques hitherto explained, leads to the following explicit expression:

\[ \nabla_{\lambda} \nabla_{\nu} \delta^{S} = \nabla_{\nu} n_{\rho} (n_{\lambda} n_{\mu} n_{\nu} n_{\sigma} \delta^{S}) + \nabla_{\rho} \{ (K^{a}_{\mu} - K^{a}_{\nu} n_{\lambda} n_{\mu}) n_{\rho} \delta^{S} \} + \delta^{S} \{ n_{\rho} (\nabla_{\lambda} K^{a}_{\mu} - \nabla_{\nu} K^{a}_{\rho} - n_{\sigma} R^{S}_{\mu \rho \sigma}) + n_{\mu} n_{\lambda} K^{a}_{\nu} K^{b}_{\rho} \} \]

where all the summands are obviously symmetric in \( (\lambda \mu) \) except for those in the last line which, by virtue of the contracted Codazzi relation (155), become simply \( n_{\sigma} n_{\mu} R^{S}_{\nu \rho \sigma} n_{\lambda} n_{\rho} \delta^{S} \), so that finally one arrives at the desired result

\[ \nabla_{\lambda} \nabla_{\nu} \delta^{S} - \nabla_{\nu} \nabla_{\lambda} \delta^{S} = 0. \]

### Appendix E. Problems with the use of Gaussian coordinates based on the matching hypersurface \( \Sigma \)

In the literature on junction conditions [7] or in general when dealing with braneworlds, it is customary to simplify the difficulties of dealing with tensor distributions by using Gaussian coordinates based on the matching hypersurface and a classical Dirac delta ‘function’. This leads to some subtleties very often ignored and, in fact, to unsolvable problems if one is to describe gravitational double layers. In this appendix we clarify this situation and provide a useful translation between the rigorous and the simplified methods. Choose local Gaussian coordinates \( \{ y, u^{a} \} \) based on the matching hypersurface \( \Sigma \) given by

\[ \Sigma : \{ y = 0 \} \]

so that the metric reads locally around \( \Sigma \) as

\[ ds^{2} = dy^{2} + g_{ab}(y, u^{c}) dx^{a}dx^{b}. \]

We can identify the local coordinates of \( \Sigma \) as \( \xi^{a} = u^{a} \), or in other words, the parametric representation of \( \Sigma \) and the tangent vector fields \( \partial_{a} \) are simply

\[ \{ y = 0, u^{a} = \xi^{a} \}, \quad \partial_{a} = \frac{\partial}{\partial u^{a}} \bigg|_{y=0}. \]

The unit normal is in this case

\[ n = dy|_{y=0}. \]
and the first fundamental form (1) becomes simply
\[ h_{ab} = g_{ab}(0, u'). \]

In what follows, \( h \) denotes the determinant of \( h_{ab} \). The two regions matched are represented by \( y > 0 \) and by \( y < 0 \). A trivial calculation proves that the second fundamental forms inherited from both sides are
\[ K^\pm_{ab} = \lim_{y \to 0^\pm} \partial_y g_{ab} \quad \Longrightarrow \quad [K_{ab}] = [\partial_y g_{ab}]|_{y=0}. \]

In these coordinates, the \( \Sigma \)-step function (131) can be easily identified with the standard Heaviside step function \( \delta(y) \). Thus, its covariant derivative is easily computed:
\[ \nabla \delta(y) = \delta(y)dy \]
where \( \delta(y) \) is the Dirac delta ‘function’. This can be immediately put in correspondence with (133) in such a way that, in this coordinate system
\[ \delta^{\Sigma} \leftrightarrow \delta(y). \]

Now, if we multiply \( \delta(y) \) for any function then
\[ F \delta(y) = F|_{y=0} \delta(y) \leftrightarrow F \delta^{\Sigma} = F|_{\Sigma} \delta^{\Sigma}. \]
Observe, however, that a first subtlety arises: when we apply \( \delta(y) \) to any test function \( Y(x^v) \), we do not simply get \( \delta(y) Y \), but we also need to integrate on \( \Sigma \), that is:
\[ \langle \delta(y), Y \rangle = \int_{y=0} Y(0, u') \sqrt{-h} \, du^1 \ldots du^n. \]
This corresponds to (132), after the identifications \( \sigma = \sqrt{-h} \, du^1 \ldots du^n \).

The discontinuity of the connection (5) together with (7) can be expressed by giving the non-zero jumps of the Christoffel symbols:
\[ [\Gamma^\gamma_{ab}] = -[K_{ab}], \quad [\Gamma^\nu_{\mu\nu}] = [K^\alpha_{\beta}], \]
and similarly (9), (11) and (15) read (only the non-zero components shown)
\[ H_{uayb} = -[K_{ab}], \quad H_{uy} = -[K^\nu_{\nu}], \quad H_{ab} = -[K_{ab}], \quad \mathcal{G}_{ab} = -[K_{ab}] + [K^\nu_{\nu}] h_{ab} \]
so that, for instance, the Einstein tensor tangent components acquire a term proportional to \( \delta(y) \) given by \( \mathcal{G}_{ab} \delta(y) \).

And now is when the real problems start: if one is to compute covariant derivatives of the curvature tensors, or the Einstein tensor, as distributions, one needs to compute terms such as, say, \( \nabla_u (\mathcal{G}_{ab} \delta(y)) \). How does one do that? Even simpler, how does one compute \( \nabla_u \delta(y) \)? One might naively write
\[ \nabla \delta(y) \propto \delta'(y)dy \quad \text{wrong!} \]
where \( \delta'(y) \) is ‘the derivative’ of the Dirac delta. This is clearly ill-defined, because one does not know how such a \( \delta'(y) \) should act on test functions (as minus the integral on \( \Sigma \) of the \( y \)-derivative of the test function?). Worse, even if one could find a proper definition of such a \( \delta'(y) \), still the formula would miss the second essential term appearing in (168) which is proportional to \( \delta^{\Sigma} \) and depends on the extrinsic properties of the matching hypersurface via the trace of its second fundamental form.

In order to show how to proceed if one insists on using Gaussian coordinates, the computation of \( \nabla \delta(y) \) must go as follows (here \( g \) stands for the determinant of \( g_{\alpha\beta} \)):
\begin{align*}
\langle \nabla \delta (y), \tilde{Y} \rangle &= - \langle \delta (y), \nabla_{\mu} Y^\mu \rangle = - \int_{y=0} \nabla_{\mu} Y^\mu d\sigma = - \int_{y=0} \frac{1}{\sqrt{-h}} \partial_{\mu} (\sqrt{-h} Y^\mu) d\sigma \\
&= - \int_{y=0} \frac{1}{\sqrt{-h}} \partial_{\mu} (\sqrt{-h} Y^\mu) d\sigma = - \int_{y=0} \left( \partial_{\mu} Y^\nu + \frac{1}{\sqrt{-h}} \partial_{\mu} (\sqrt{-h} Y^\nu) + Y^\nu \frac{1}{\sqrt{-h}} \partial_{\mu} \sqrt{-h} \right) d\sigma \\
&= - \int_{y=0} (\partial_{\mu} Y^\nu + Y^\nu K^{\nu\rho}_{ab}) d\sigma.
\end{align*}

In the last step, we have used Gauss’ theorem. This formula corresponds to (168). Observe the fact that the extrinsic curvature $K_{ab}$ is not necessarily equal as computed from either side of $y = 0$ and therefore it is not univocally defined. Hence, a definite prescription of what its value on $\Sigma$ is, i.e. $K_{ab}^\Sigma$, must be provided.

The above subtleties and difficulties when using Gaussian coordinates are probably the reason why double layers were not found in quadratic $F(R)$ or other quadratic theories until they were derived in [19–21].

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