EFFECT OF DIFFUSION IN A SPATIAL SIS EPIDEMIC MODEL WITH SPONTANEOUS INFECTION

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Abstract. This paper is concerned with an SIS epidemic reaction-diffusion model with mass-action incidence incorporating spontaneous infection in a spatially heterogeneous environment. The main goal of this article is to study the influence of spontaneous infection on the endemic equilibrium (EE) of the model. To achieve this, first the existence of EE is investigated. Furthermore, we discuss the asymptotic behavior of endemic equilibrium if the migration rate of the susceptible or infected population is sufficiently small. Compared to the case without spontaneous infection, our theoretical results show that spontaneous infection can enhance persistence of infectious disease.

1. Introduction. In the past decades, various types of epidemic models, using differential equations, have been developed to understand the mechanisms of infectious diseases [4, 7, 12, 14, 15, 19]. Recently, considerable susceptible-infected-susceptible (SIS) epidemic reaction-diffusion models have been developed to study the impact of the migrating movement and spatial heterogeneity on disease transmission, see, for instance, [1, 2, 8, 9, 10, 20, 25, 26, 28, 30, 31, 32].

Allen et al. [2] proposed a frequency-dependent SIS (susceptible-infected-susceptible) reaction-diffusion model for a population living in a spatially heterogeneous environment:

\[
\begin{align*}
\frac{\partial S}{\partial t} - dS \Delta S &= -\frac{\beta(x)SI}{S+I} + \gamma(x)I, \quad x \in \Omega, \ t > 0, \\
\frac{\partial I}{\partial t} - dI \Delta I &= \frac{\beta(x)SI}{S+I} - \gamma(x)I, \quad x \in \Omega, \ t > 0, \\
\partial_n S &= \partial_n I = 0, \quad x \in \partial \Omega, \ t > 0, \\
S(x, 0) &= S_0(x), \ I(x, 0) = I_0(x).
\end{align*}
\]

(1)

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In [2], the authors investigated the existence, uniqueness and stabilization of the disease-free equilibrium of system (1), and proved that the infected population vanishes on the entire habitat as the mobility of the susceptible population trends zero. Moreover, they speculated that the unique EE (endemic equilibrium) should be globally asymptotically stable. The article [26] confirmed this conjecture under some appropriate conditions. Further the results relating to the progressive behavior of the EE can be obtained in the literature [25, 28]. The studies on model in [2] and related problems can be found, for instance, in [1, 8, 9, 18, 20, 27].

Replacing the standard incidence $\frac{\beta(x)SI}{S+I}$ in (1) by the mass-action incidence $\beta(x)SI$, the authors in [10] studied the following SIS reaction-diffusion model

$$
\begin{cases}
\frac{\partial S}{\partial t} - d_S \Delta S = -\beta(x)SI + \gamma(x)I, & x \in \Omega, \ t > 0, \\
\frac{\partial I}{\partial t} - d_I \Delta I = \beta(x)SI - \gamma(x)I, & x \in \Omega, \ t > 0, \\
\partial_n S = \partial_n I = 0, & x \in \partial \Omega, \ t > 0, \\
S(x, 0) = S_0(x), \ I(x, 0) = I_0(x).
\end{cases}
$$

In [10], the authors defined a basic reproduction number $R_0$ for model (2), and studied the existence and non-existence of the EE under some cases. Furthermore, they considered the global attractiveness of the DFE (disease-free equilibrium) and the EE. In [32], the authors testified the asymptotic profile of the EE of (2) for large or small diffusion rate when $R_0 > 1$.

In the most works for SIS in the literature, the study has been carried out on the assumption that infectious disease can be transmitted from infected individual to susceptible population. In reality, however, infectious diseases can be transmitted by voluntary factors along with contact between two classes. In fact, many things can play a role as voluntary factors in epidemiological modeling. For example, the spontaneous factor can be explained by brief contacts with individuals outside of the population [24], or can also describe the spontaneous (automatic) infection of the uninfected, independent of the contact with the infected [17]. As another example, it represents mutual infection between individuals and media [3, 29, 33]. In [16], the spontaneous contraction of emotions has been used as a voluntary factor.

In this paper, we assume that, in addition to directed transmission between two classes, infection may also occur through contact with infectious imports at a rate $\eta(x)$, either by susceptible individual simply leaving the population and making contact with infectious individuals located elsewhere or by infectious visitors briefly entering the population and making contact with susceptible individual [24]. Thus, we consider the following model which includes these spontaneous infection factors in addition to infections caused by infected individual:

An infection can occur at an infection rate relative to the rate at which an infected object leaves the population and comes in contact with an infected object located elsewhere or an infected visitor briefly visits the population.

$$
\begin{cases}
\frac{\partial S}{\partial t} - d_S \Delta S = -\beta(x)SI - \eta(x)S + \gamma(x)I, & x \in \Omega, \ t > 0, \\
\frac{\partial I}{\partial t} - d_I \Delta I = \beta(x)SI + \eta(x)S - \gamma(x)I, & x \in \Omega, \ t > 0, \\
\partial_n S = \partial_n I = 0, & x \in \partial \Omega, \ t > 0, \\
S(x, 0) = S_0(x) \geq 0, \ I(x, 0) = I_0(x) \geq 0,
\end{cases}
$$

where $S(x,t)$ and $I(x,t)$ represent the density of susceptible and infected individuals at location $x$ and time $t$, respectively; positive constants $d_S$ and $d_I$ are the diffusion rates for the susceptible and infected populations, respectively; $\beta(x), \gamma(x),$ and $\eta(x)$
are positive and Hölder continuous functions on $\overline{\Omega}$ that represent the rate of disease transmission, recovery and spontaneous infection at $x$, respectively. The habitat $\Omega \subset \mathbb{R}^m$ ($m \geq 1$) is a bounded domain with smooth boundary $\partial \Omega$, where the homogeneous Neumann boundary condition means that no population flux crosses the boundary $\partial \Omega$. Regarding the initial data $S_0$ and $I_0$, we assume throughout the paper that

$$(A1) \quad \int_\Omega I_0(x)dx > 0 \text{ with } S_0 \geq 0, I_0 \geq 0 \text{ on } \overline{\Omega}, \text{ and } S_0, I_0 \text{ are continuous on } \Omega.$$ 

We are interested primarily in the equilibrium solutions of (3), i.e., the solution of the steady state problem

$$\begin{cases}
-d_S \Delta S = -\beta(x)SI - \eta(x)S + \gamma(x)I, & x \in \Omega, \\
-d_I \Delta I = \beta(x)SI + \eta(x)S - \gamma(x)I, & x \in \Omega, \\
\partial_n S = \partial_n I = 0, & x \in \partial \Omega.
\end{cases} \quad (4)$$

A componentwisely nonnegative and nontrivial solution $(S, I) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ to (4) is called an endemic equilibrium (EE). By the strong maximum principle and Hopf’s lemma for elliptic equations, it is easy to see that any EE $(S, I)$ satisfies $S(x) > 0$ and $I(x) > 0$ for all $x \in \overline{\Omega}$.

The aim of our present work is to study the influence of spontaneous infection on the qualitative behavior of the solution to problem (3). At first, we discuss the linear stability of the unique positive constant EE when the parameters $\beta$, $\eta$ and $\gamma$ are positive constants. Next, the existence of non-constant EE of system (4) is shown and their asymptotic profile is investigated when the movement rate of the susceptible or infected population is small. The organization of this paper is as follows. In Section 2, we present the main results of this paper. Section 3 is devoted to the proof of Theorem 2.1. Section 4 provides the proofs of Theorems 2.2 and 2.3.

2. Main results. In this section, we introduce the main results of this article. We mainly focus on the existence of EE and the asymptotic behavior of the EE of problem (4) when the diffusion rate $d_S$ or $d_I$ tends to zero.

First, we obtain the existence of EE as described below.

**Theorem 2.1.** System (4) has at least one EE.

In [10], the authors found a basic reproduction number $R_0$ and discussed the existence of the EE when $R_0 > 1$. They proved that if $\beta$ and $\gamma$ are positive constants or $d_S = d_I \equiv d$, the EE is globally attractive when $R_0 > 1$. In this paper, we show that EE also exists and linearly stable.

Next, we describe the results of the asymptotic behavior of EE. When $d_I > 0$ is fixed and $d_S \to 0$, we can show the following.

**Theorem 2.2.** Let $d_S \to 0$. Every positive solution $(S, I)$ of system (4) satisfies

$$(S, I) \to (S^*, I^*) \text{ uniformly on } \overline{\Omega},$$

where $I^*$ is a positive constant and satisfies

$$\int_\Omega \left( \frac{\gamma(x)I^*}{\eta(x) + \beta(x)I^*} + I^* \right) dx = N,$$

and

$$S^* = \frac{\gamma(x)I^*}{\eta(x) + \beta(x)I^*}.$$
In [32], when migration rate of the susceptible population is small enough, the authors only proved that the infectious disease exists if $\beta$ is a positive constant with $N - \int_{\Omega} \frac{\gamma(x)}{\beta(x)} dx > 0$, and the infectious disease goes extinct if $N - \int_{\Omega} \frac{\gamma(x)}{\beta(x)} dx < 0$. In [31], the authors improved the result of [32], namely, they showed that if migration rate of the susceptible population is small enough, the infectious disease always exists provided that $N - \int_{\Omega} \left[ \frac{\gamma(x)}{\beta(x)} \right] dx > \frac{1}{4} \int_{\Omega} \left[ \frac{\|\nabla \beta\|^2}{\beta^3} \right] dx$ or $N |\Omega| > \frac{\gamma(x)}{\beta(x)}$ on $\Omega$. In this paper, we prove that the disease also exists as the movement of the susceptible population tends to zero.

Biologically, Theorem 2.2 shows that even though the mobility of the susceptible population $d_S$ is very small, the disease still exists on the entire habitat. Moreover, the density of the susceptible is positive and heterogeneous, while the density of the infected is positive and homogeneous.

Finally, we analyze the asymptotic behavior of the EE $(S, I)$ as $d_I \to 0$ while $d_S > 0$ is fixed and $\Omega \subset \mathbb{R}^1$. Our result can be stated as follows.

**Theorem 2.3.** Let $d_I \to 0$. Every positive solution $(S, I)$ of problem (4) satisfies (up to a sequence of $d_I$) $S \to S^* \text{ uniformly on } \Omega$, where $S^* \in C(\Omega)$ with $S^* > 0$, and

$$\int_{\Omega} I dx \to C_0$$

where $C_0$ is a positive constant.

In [32], the authors studied the asymptotical profile of the EE as $d_I \to 0$ with $\frac{d_I}{d_S} \to d > 0$, and speculated that the infected population is positive when $d_I \to 0$ with $d_S > 0$ is fixed. In [31], the authors proved this guess in one space dimension with some conditions. In this paper, we also study the asymptotic behavior of the EE as $d_I \to 0$ with $d_S > 0$ be fixed in one space dimension.

Biologically, Theorem 2.3 shows that the disease also exists when the movement rate of the infected population is very small. Spontaneous infection increases the persistence of infectious diseases and makes it more hard to control disease.

### 3. Proof of Theorem 2.1.

At first, let

$$\int_{\Omega} (S_0 + I_0) dx = N \quad (5)$$

be the total number of individuals in $\Omega$ at $t = 0$. Adding the equations in (3) and integrating over $\Omega$ give that

$$\frac{\partial}{\partial t} \int_{\Omega} (S + I) dx = \int_{\Omega} \Delta (d_S S + d_I I) = 0, \quad t > 0.$$

Hence the total population size is a positive constant, i.e.,

$$\int_{\Omega} (S + I) dx = N, \quad t \geq 0. \quad (6)$$

Corresponding to (3) and (6), the equilibrium (i.e., steady state) problem satisfies the elliptic system (4).

By the standard theory for semi-linear parabolic systems [5], problem (3) admits a unique solution $(S(x, t), I(x, t))$ for $x \in \Omega$ and $t \in (0, T_{\text{max}})$, where $T_{\text{max}}$ is the maximal existence time for the solutions of the problem. Moreover, the strong
maximum principle for parabolic equations yields that the solution is positive on $\Omega \times (0, T_{\text{max}})$. By using the comparison principle and $S$-equation, we can get that $S(x, t)$ is uniformly bounded in $\Omega \times (0, T_{\text{max}})$. According to uniform bound of $S(x, t)$ and the $L^1$-norm of $I(x, t)$ for $t \in (0, T_{\text{max}})$, using Lemma 3.1 [27] and $I$-equation, we derive that $I(x, t)$ is bounded in $\Omega \times (0, T_{\text{max}})$. By the standard theory for semi-linear parabolic systems again, we have that $T_{\text{max}} = \infty$.

Before we give the proof of Theorem 2.1, we discuss the asymptotical stability of the constant EE when the parameters $\beta, \gamma$ and $\eta$ are positive constants. It is obvious that $(\hat{S}, \hat{I})$ is the unique positive constant EE of (4), where

\[
\hat{S} = \frac{N \beta |\Omega| + \gamma + \eta - \sqrt{(N \beta |\Omega| + \gamma + \eta)^2 - 4 \beta \gamma N |\Omega|^2 \beta}}{2 \beta},
\]

\[
\hat{I} = \frac{N \beta |\Omega| - \gamma - \eta + \sqrt{(N \beta |\Omega| - \gamma - \eta)^2 + 4 \beta \eta N |\Omega|^2 \beta}}{2 \beta}.
\]

We now linearize (3) around $(\hat{S}, \hat{I})$. Let $\xi(x, t) = S(x, t) - \hat{S}$ and $\zeta(x, t) = I(x, t) - \hat{I}$, we can obtain the following linearized system:

\[
\begin{aligned}
\frac{\partial \xi}{\partial t} &= d_S \Delta \xi - \xi (\beta \hat{I} + \eta) - \zeta (\beta \hat{S} - \gamma), \quad x \in \Omega, \ t > 0, \\
\frac{\partial \zeta}{\partial t} &= d_I \Delta \zeta + \xi (\beta \hat{I} + \eta) + \zeta (\beta \hat{S} - \gamma), \quad x \in \Omega, \ t > 0.
\end{aligned}
\]

(7)

Substituting $(\xi(x, t), \zeta(x, t)) = (e^{-\lambda t} \phi(x), e^{-\lambda t} \psi(x))$ into (7). We then derive an eigenvalue problem

\[
\begin{aligned}
d_S \Delta \phi - \phi(\beta \hat{I} + \eta) - \psi(\beta \hat{S} - \gamma) + \lambda \phi &= 0, \quad x \in \Omega, \\
d_I \Delta \psi + \phi(\beta \hat{I} + \eta) + \psi(\beta \hat{S} - \gamma) + \lambda \psi &= 0, \quad x \in \Omega
\end{aligned}
\]

(8)

with the boundary conditions

\begin{align*}
\partial_{\nu} \phi &= \partial_{\nu} \psi = 0, \quad x \in \partial \Omega.
\end{align*}

Observe from (6) and the expression of $(\hat{S}, \hat{I})$, we can get that

\[
\int_{\Omega} (\xi + \zeta) dx = e^{-\lambda t} \int_{\Omega} (\phi + \psi) dx = 0.
\]

Therefore,

\[
\int_{\Omega} (\phi + \psi) dx = 0.
\]

(9)

Using the same method as Theorem 2.1 of [30] and (9) yields that the real part of the principal eigenvalue of (8) is greater than zero, so we can get the following result:

**Proposition 1.** The constant EE $(\hat{S}, \hat{I})$ is linearly stable for (3).

Next, we prove Theorem 2.1.
Proof. From now on, we consider the following equations

\[
\begin{align*}
-d_3 \Delta S &= -\left(\tau \beta(x) + (1 - \tau)\beta_0\right)SI \\
&\quad -\left(\tau \eta(x) + (1 - \tau)\eta_0\right)S + \left(\tau \gamma(x) + (1 - \tau)\gamma_0\right)I, \quad x \in \Omega, \\
-d_4 \Delta I &= \left(\tau \beta(x) + (1 - \tau)\beta_0\right)SI \\
&\quad +\left(\tau \eta(x) + (1 - \tau)\eta_0\right)S - \left(\tau \gamma(x) + (1 - \tau)\gamma_0\right)I, \quad x \in \Omega, \\
\frac{\partial S}{\partial \nu} &= \frac{\partial I}{\partial \nu} = 0, \quad x \in \partial \Omega, \\
\int_\Omega (S + I) dx &= N,
\end{align*}
\]  

(10)

where $\beta_0$, $\gamma_0$, $\eta_0$ are positive constants and $\tau \in [0, 1]$. We remark that equations in (10) become equations in (4) when $\tau = 1$.

Next, we conclude that $(S, I)$ of (10) have positive upper and lower bounds independent of $\tau \in [0, 1]$.

In fact, assuming $S(x_0) = \max \limits_\Omega S(x)$, we apply [22, Proposition 2.2] and use the first equation of (10) to obtain

\[
(\tau \beta(x_0) + (1 - \tau)\beta_0)S(x_0)I(x_0) + (\tau \eta(x_0) + (1 - \tau)\eta_0)S(x_0) \leq (\tau \gamma(x_0) + (1 - \tau)\gamma_0)I(x_0).
\]

That is,

\[
\min \{\min_\Omega \beta(x), \beta_0\} S(x_0)I(x_0) + \min \{\min_\Omega \eta(x), \eta_0\} S(x_0) \leq \max \{\max_\Omega \gamma(x), \gamma_0\} I(x_0).
\]

So,

\[
S(x_0) \leq \frac{\max \{\max_\Omega \gamma(x), \gamma_0\} I(x_0)}{\min \{\min_\Omega \beta(x), \beta_0\} I(x_0) + \min \{\min_\Omega \eta(x), \eta_0\}} \leq \frac{\max \{\max_\Omega \gamma(x), \gamma_0\}}{\min \{\min_\Omega \beta(x), \beta_0\}}. \tag{11}
\]

Integrating the second equation of (10) over $\Omega$ gives

\[
\int_\Omega \left(\tau \beta(x) + (1 - \tau)\beta_0\right)SI + (\tau \eta(x) + (1 - \tau)\eta_0)S \right) dx = \int_\Omega \left(\tau \gamma(x) + (1 - \tau)\gamma_0\right)I dx,
\]

therefore,

\[
\min \{\min_\Omega \eta(x), \eta_0\} (N - \int_\Omega I dx) \leq \max \{\max_\Omega \gamma(x), \gamma_0\} \int_\Omega I dx,
\]

that is,

\[
\int_\Omega I dx \geq \frac{N \min \{\min_\Omega \eta(x), \eta_0\}}{\max \{\max_\Omega \gamma(x), \gamma_0\}}. \tag{12}
\]

Applying Theorem 8.18 from [13] with $q = 1$ to the second equation of (4), we can get

\[
\|I\|_{L^1(\Omega)} \leq C \inf \limits_\Omega I,
\]

so

\[
I(x) \geq C_1, \quad \forall \ x \in \Omega, \tag{13}
\]

where $C_1$ is a positive constant independent of $\tau$. 

Letting $S(y) = \min \prod \gamma(x), \gamma_0 \} I(y)$, we apply [22, Proposition 2.2] to the first equation of (10) and use (13) to deduce
\[ S(x) \geq S(y) \geq \frac{\min \{ \prod \gamma(x), \gamma_0 \} I(y)}{\max \{ \prod \beta(x), \beta_0 \} I(y) + \max \{ \prod \eta(x), \eta_0 \}} \geq C_2, \quad \forall x \in \overline{\Omega}, \tag{14} \]
where $C_2$ is a positive constant independent of $\tau$.

In view of (11), (13) and (6), it is obvious that
\[ \int_{\Omega} (\tau \beta + (1 - \tau) \beta_0) SI dx + \int_{\Omega} (\tau \eta + (1 - \tau) \eta_0) S dx \leq C_3. \]
Therefore, we can use the same method as step 1 of Theorem 3.1 in [30] to obtain
\[ \| I \|_{L^\infty(\Omega)} \leq C_4, \tag{15} \]
where $C_5$ is a positive constant independent of $\tau$.

According to (11), (13), (14) and (15), there exists two positive constants $C^*$ and $C_*$ with independent of $\tau$ and $C_* < C^*$, such that every positive solution $(S, I)$ of problem (10) satisfies
\[ C_* < S(x), \quad I(x) < C^*, \quad \forall x \in \overline{\Omega}. \]

Denote
\[ \Gamma = \{(S, I) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : C_* < S(x), \quad I(x) < C^*, \quad \forall x \in \overline{\Omega}\}. \]
So, (10) has no positive solution $(S, I) \in \partial \Gamma$. We define an operator
\[ B(\tau, (S, I)) = (-\Delta + I)^{-1} \left( \hat{f}(\tau, (S, I)), \hat{f}(\tau, (S, I)) \right) \]
with $\tau \in [0, 1]$, where the inverse operator of $-\Delta + I$ is expressed as $(-\Delta + I)^{-1}$ satisfies Neumann boundary condition over $\partial \Omega$ and
\[
\hat{f}(\tau, (S, I)) = I + d_S^{-1} \left\{ [\tau \beta - (1 - \tau) \beta_0] SI - [\tau \eta + (1 - \tau) \eta_0] S + [\tau \gamma + (1 - \tau) \gamma_0] I \right\},
\]
\[
\hat{f}(\tau, (S, I)) = I + d_I^{-1} \left\{ [\tau \beta + (1 - \tau) \beta_0] SI + [\tau \eta + (1 - \tau) \eta_0] S - [\tau \gamma + (1 - \tau) \gamma_0] I \right\}.
\]
Therefore, the existence of positive solutions to (4) is equivalent to the existence of fixed point of the operator $B(\tau, \cdot)$ in $\Gamma$.

The standard elliptic regularity theory confirms that $B$ is a compact operator from $[0, 1] \times \Gamma$ to $C(\overline{\Omega}) \times C(\overline{\Omega})$. Moreover, we have
\[ (S, I) \neq B(\tau, (S, I)), \quad \forall \tau \in [0, 1] \text{ and } (S, I) \in \partial \Gamma. \]
Therefore, the topological degree $\operatorname{deg}(\Gamma - B(\tau, \cdot), \Gamma)$ is well-defined, which does not rely on $\tau \in [0, 1]$.

Denote
\[ \tilde{S} = \frac{N \beta_0}{\| I \|} + \gamma_0 + \eta_0 - \sqrt{(\frac{N \beta_0}{\| I \|} + \gamma_0 + \eta_0)^2 - \frac{4 \beta_0 \gamma_0 N}{\| I \|}} \quad \text{and} \]
\[ \tilde{I} = \frac{N \beta_0}{\| I \|} - \gamma_0 - \eta_0 + \sqrt{(\frac{N \beta_0}{\| I \|} - \gamma_0 - \eta_0)^2 + \frac{4 \beta_0 \gamma_0 N}{\| I \|}}. \]
According to Proposition 1, we know that $(\tilde{S}, \tilde{I})$ is linearly stable for the unique positive constant steady state of (4) with $(\beta_0, \gamma_0, \eta_0)$ instead of $(\beta, \gamma, \eta)$. Now, by
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the well-known Leray-Schauder degree index formula [23, Theorem 2.8.1], we can obtain
\[ \text{deg} \left( I - B(0, \cdot), \tau \right) = \text{index} \left( I - B(0, \cdot), (\tilde{S}, \tilde{I}) \right) = 1. \]
Hence, in view of the homotopy invariance of the Leray-Schauder degree, we get
\[ \text{deg} \left( I - B(1, \cdot), \tau \right) = \text{deg} \left( I - B(0, \cdot), \tau \right) = 1, \]
which hints that \( B(1, \cdot) \) exists at least one fixed point in \( \Gamma \). Therefore, there exists at least one positive solution for problem (4).

4. Asymptotic profile of EE.

4.1. Proof of Theorem 2.2.

Proof. To study the impact of the diffusion coefficient \( d_S \) on the solution \( (S, I) \), we first derive the upper and lower bounds of \( S \) and \( I \), which are independent of \( d_S \).

Integrating the second equation of (4) by parts over \( \Omega \) gives
\[ \int_{\Omega} \left( \beta(x)SI + \eta(x)S \right) dx = \int_{\Omega} \gamma(x)Idx, \]
and so
\[ \min \frac{\eta(x)}{\bar{\eta}} \int_{\Omega} Sdx \leq \max \frac{\gamma(x)}{\bar{\gamma}} \int_{\Omega} Idx. \]
Substituting (6) into the above inequality, we can get
\[ \min \frac{\eta(x)}{\bar{\eta}} N - \min \frac{\eta(x)}{\bar{\eta}} \int_{\Omega} Idx \leq \max \frac{\gamma(x)}{\bar{\gamma}} \int_{\Omega} Idx, \]
that is,
\[ \int_{\Omega} Idx \geq \frac{\min \frac{\eta(x)}{\bar{\eta}} N}{\max \frac{\gamma(x)}{\bar{\gamma}} + \min \frac{\eta(x)}{\bar{\eta}}}. \]
Recalling that
\[ -d_I \Delta I = \beta(x)SI + \eta(x)S - \gamma(x)I \geq -\gamma(x)I \]
and using Theorem 8.18 from [13] with \( q = 1 \) yield
\[ \|I\|_{L^1(\Omega)} \leq C \inf \frac{\gamma(x)}{\bar{\gamma}}. \]
Therefore, we have
\[ I(x) \geq C_1, \quad (16) \]
where \( C_1 \) is a positive constant independent of \( d_S \) and depends on \( d_I \).

Now taking \( \min \frac{S(x)}{\bar{S}} = S(x_0) \), we apply [22, Proposition 2.2] to the first equation of (4) to deduce
\[ \gamma(x_0)I(x_0) \leq \beta(x_0)S(x_0)I(x_0) + \eta(x_0)S(x_0), \]
which in turn yields
\[ \min \frac{S(x)}{\bar{S}} = S(x_0) \geq \frac{\gamma(x_0)I(x_0)}{\beta(x_0)I(x_0) + \eta(x_0)}, \quad (17) \]
and it follows from (16) that
\[ \min \frac{S(x)}{\bar{S}} \geq C_2 := \frac{\gamma(x_0)C_1}{\beta(x_0)C_1 + \eta(x_0)}, \quad (18) \]
where the positive constant \( C_2 \) is independent of \( d_S \).
Letting \( \max \frac{S(x)}{p} = S(x^*) \), we use [22, Proposition 2.2] again to the first equation of (4) to obtain

\[
\gamma(x^*) I(x^*) \geq \beta(x^*) S(x^*) I(x^*) + \eta(x^*) S(x^*),
\]

that is,

\[
S(x) \leq S(x^*) \leq \frac{\gamma(x^*) I(x^*)}{\beta(x^*) I(x^*) + \eta(x^*)} \leq \max \frac{\gamma(x)}{\beta(x)},
\]

(19)

We now rewrite the equation of \( I \) and consider the problem

\[
\begin{cases}
-d_I \Delta I = \left( \beta(x) S + \frac{\eta(x) S}{I} - \gamma(x) \right) I, & x \in \Omega, \\
\partial_I I = 0, & x \in \partial\Omega.
\end{cases}
\]

(20)

For fix \( d_I > 0 \), in view of (16) and (19), we have that

\[
\left\| \frac{\beta(x) S + \frac{\eta(x) S}{I} - \gamma(x)}{d_I} \right\|_{L^\infty(\Omega)} \leq C_3.
\]

Using the Harnack inequality [21] for the equation in (20) yields that

\[
\max_{\Omega} I(x) \leq C_3 \min_{\Omega} I(x),
\]

(21)

where \( C_3 \) is a positive constant depending on \( d_I \) and \( \Omega \).

Next, we certify convergence of \( I = I(x, d_S) \) as \( d_S \to 0 \). Observe that \( I \) satisfies

\[
\begin{cases}
-d_I \Delta I = \left( \beta(x) S + \frac{\eta(x) S}{I} - \gamma(x) \right) I, & x \in \Omega, \\
\partial_I I = 0, & x \in \partial\Omega.
\end{cases}
\]

(22)

The standard \( L^p \)-estimate for elliptic equations enables one to assert that

\[
\|I\|_{W^{2,p}(\Omega)} \leq C, \quad \forall p \geq 1.
\]

Taking \( p \) to be large enough, we can obtain

\[
\|I\|_{C^1(\Omega)} \leq C.
\]

Therefore, there exists a sequence \( \{d_{S,n}\} \) of \( \{d_S\} \), denoted by \( d_{S,n} = d_n \), with \( d_n \to 0 \) as \( n \to \infty \), and a corresponding positive solution sequence \( (S_n, I_n) \) of (4) with \( d_S = d_n \), such that

\[
I_n \to I^* \text{ uniformly on } \Omega, \quad \text{as } n \to \infty,
\]

(23)

where \( I^* \in C(\Omega) \) and \( I^* > 0 \) on \( \Omega \) due to (16).

Finally, we consider the convergence of \( S_n \) as \( d_S \to 0 \), where \( S_n \) satisfies

\[
\begin{cases}
-d_n \Delta S_n = -\beta(x) S_n I_n - \eta(x) S_n + \gamma(x) I_n, & x \in \Omega, \\
\partial_n S_n = 0, & x \in \partial\Omega.
\end{cases}
\]

(24)

In view of (23), we know that for any small \( \epsilon > 0 \), it holds

\[
0 < I^* - \epsilon \leq I_n(x) \leq I^* + \epsilon,
\]

(25)

on \( \Omega \) for all large \( n \). Hence, for all sufficiently large \( n \), we get

\[
\begin{align*}
&-\beta(x) S_n[I^*(x) + \epsilon] - \eta(x) S_n + \gamma(x) [I^*(x) - \epsilon] \\
\leq & \quad -\beta(x) S_n I_n - \eta(x) S_n + \gamma(x) I_n \\
\leq & \quad -\beta(x) S_n[I^*(x) - \epsilon] - \eta(x) S_n + \gamma(x) [I^*(x) + \epsilon].
\end{align*}
\]

(26)
For the above fixed large \( n \), we consider the following auxiliary problem
\[
\begin{aligned}
- d_n \Delta u_n &= - \beta(x) u_n (I^*(x) + \epsilon) - \eta(x) u_n + \gamma(x)(I^*(x) - \epsilon), & x \in \Omega, \\
\partial_n u_n &= 0, & x \in \partial \Omega.
\end{aligned}
\] (27)

It is clear that (27) has a unique positive solution, denoted by \( \pi_n \). Using similar arguments to those in the proof of [11, Lemma 2.4], we know that
\[
\lim_{n \to \infty} \pi_n (x) = \frac{\gamma(x)(I^* - \epsilon)}{\eta(x) + \beta(x)(I^* + \epsilon)} \quad \text{uniformly on } \overline{\Omega}, \quad n \to \infty.
\] (28)

It is obvious that \( S_n \) is an upper solution of (27), and so we obtain
\[
\lim_{n \to \infty} S_n (x) \geq \frac{\gamma(x)(I^* - \epsilon)}{\eta(x) + \beta(x)(I^* + \epsilon)} \quad \text{uniformly on } \overline{\Omega}.
\] (29)

Similar to the discussion above, we get
\[
\limsup_{n \to \infty} S_n (x) \leq \frac{\gamma(x)(I^* + \epsilon)}{\eta(x) + \beta(x)(I^* - \epsilon)} \quad \text{uniformly on } \overline{\Omega}.
\] (30)

Combining (29) and (30) with the arbitrariness of small \( \epsilon > 0 \), we obtain that
\[
\lim_{n \to \infty} S_n (x) = S^*(x) := \frac{\gamma(x)I^*}{\eta(x) + \beta(x)I^*} \quad \text{uniformly on } \overline{\Omega}.
\] (31)

In response to (23) and (31), we have
\[
\beta(x) S_n I_n + \eta(x) S_n - \gamma(x) I_n \to \beta(x) S^* I^* + \eta(x) S^* - \gamma(x) I^* = 0
\quad \text{uniformly on } \overline{\Omega} \text{ as } n \to \infty.
\]

Hence, it is easily seen that \( I^* \) satisfies
\[
\begin{aligned}
- d_I \Delta I^* &= 0, & x \in \Omega, \\
\partial_n I^* &= 0, & x \in \partial \Omega.
\end{aligned}
\]

Therefore, \( I^* \) must be a positive constant. Recalling (6), it yields that
\[
\int_{\Omega} (S^* + I^*) dx = N.
\]

Inserting \( S^* \) into the above equality gives
\[
\int_{\Omega} \left( \frac{\gamma(x)I^*}{\eta(x) + \beta(x)I^*} + I^* \right) dx = N.
\]

We then completes the proof. \( \square \)

4.2. Proof of Theorem 2.3.

Proof. According to preliminary estimate for positive solution of (4), we have
\[
S(x) \leq C^*
\] (32)

for some positive constant \( C^* \) independent of \( d_I > 0 \). In view of (6), it is easily seen that
\[
\int_{\Omega} I dx \leq N.
\] (33)

Multiplying the first equation of (4) by \( \frac{1}{S} \) and integrating the resulting equality over \( \Omega \), we get
\[
- \int_{\Omega} \beta(x) I dx - \int_{\Omega} \eta(x) dx + \int_{\Omega} \gamma(x) \frac{I}{S} dx = - \int_{\Omega} d_S |\nabla S^2| dx \leq 0,
\]
which shows that
\[
\int_{\Omega} \frac{\gamma(x)I}{S} \, dx \leq \int_{\Omega} \beta(x)I \, dx + \int_{\Omega} \eta(x) \, dx \leq \max_{\Omega} \beta(x)N + \max_{\Omega} \eta(x)|\Omega|.
\] (34)

Notice that \( S \) solves
\[
\begin{cases}
-\Delta S + \left[ \frac{\beta(x)I + \eta(x) - \gamma(x)I}{S} \right] S = 0, & x \in \Omega, \\
\partial_{\nu} S = 0, & x \in \partial \Omega.
\end{cases}
\] (35)

For fixed \( d_S > 0 \), it follows from (34) that
\[
\left\| \frac{\beta(x)I + \eta(x) - \gamma(x)I}{S} \right\|_{L^1(\Omega)} \leq C.
\]

Applying the Harnack inequality with \( q = 1 > \frac{1}{2} \), we have
\[
\max_{\Omega} S(x) \leq C \min_{\Omega} S(x),
\] (36)

where \( C \) is a positive constant determined by \( d_S, q \) and \( \Omega \).

Now, integrating the first equation (4) over \( \Omega \), we obtain
\[
\int_{\Omega} \gamma(x)I \, dx = \int_{\Omega} \beta(x)SI \, dx + \int_{\Omega} \eta(x)S \, dx,
\]

that is,
\[
\min_{\Omega} \beta(x) \int_{\Omega} SI \, dx \leq \max_{\Omega} \gamma(x) \int_{\Omega} I \, dx \leq \max_{\Omega} \gamma(x)N.
\]

Hence,
\[
\int_{\Omega} SI \, dx \leq \frac{\max_{\Omega} \gamma(x)N}{\min_{\Omega} \beta(x)}.
\] (37)

In view of (33) and (37), we have
\[
\left\| \frac{\gamma(x)I - \beta(x)SI - \eta(x)S}{d_S} \right\|_{L^1(\Omega)} \leq C_1,
\]

where \( C_1 \) is a positive constant independent of \( d_I \). Applying for the [6, Theorem 8], we obtain that for any given \( q \in [1, \frac{n}{n-1}) \),
\[
\| S \|_{W^{1,q}(\Omega)} \leq C_2.
\]

Next, for any \( q > n = 1 \), the standard Sobolev embedding theorem tells us that \( W^{1,q}(\Omega) \) is embedded into \( C^\alpha(\Omega), 0 < \alpha \leq 1 - \frac{1}{q} \). On the other hand, \( C^\alpha(\Omega) \) is compactly embedded into \( C(\Omega) \). Thus, there exist a sequence of \( d_I \) such that corresponding EE sequence of (4) satisfying
\[
S \to S_* \text{ uniformly on } \Omega \tag{38}
\]
as \( d_I \to 0 \), where \( S_* \in C(\Omega) \) and \( S_* \geq 0 \). According to (36),
\[
\text{either } S_* \equiv 0 \text{ or } S_* > 0 \text{ uniformly on } \Omega. \tag{39}
\]

Suppose \( S_* \equiv 0 \), that is,
\[
S \to 0 \text{ uniformly on } \Omega. \tag{40}
\]

Then for arbitrarily small \( \epsilon \) with \( 0 < \epsilon < \min_{\Omega} \frac{\gamma(x)}{\beta(x)} \), we have
\[
0 \leq S_n \leq \epsilon \text{ for all large } n.
\]
So, for all sufficiently large $n$, we obtain that
\[-\gamma(x)I_n \leq \beta(x)S_n I_n + \eta(x)S_n - \gamma(x)I_n \leq \beta(x)\epsilon I_n + \eta(x)\epsilon - \gamma(x)I_n.
\]
Using the same argument as in Theorem 4.1, we get
\[I_n \to 0 \text{ uniformly on } \Omega,
\]
which is a contradiction with (5). Therefore, $S_* > 0$ on $\Omega$.

In light of (33), we can verify that there exist a sequence of $d_I$, such that
\[\int_\Omega I dx \to C_0 \quad (41)
\]
as $d_I \to 0$, where $C_0$ is a positive constant. We argue indirectly and suppose that $C_0 = 0$. Integrating the first equation of (4) by parts over $\Omega$, we have
\[\int_\Omega \gamma(x)I dx = \int_\Omega \beta(x)S I dx + \int_\Omega \eta(x)S dx.
\]
In view of (38) and $C_0 = 0$, we obtain
\[\lim_{d_I \to 0} \int_\Omega \eta(x)S(x)dx = 0,
\]
which is a contradiction with $S_* > 0$ on $\Omega$. Therefore, (41) holds.

\[\square\]

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REFERENCES

[1] L. J. S. Allen, B. M. Bolker, Y. Lou and A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic disease patch model, SIAM J. Appl. Math., 76 (2007), 1283–1309.

[2] L. J. S. Allen, B. M. Bolker, Y. Lou and A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model, Discrete Contin. Dyn. Syst., 21 (2008), 1–20.

[3] F. Altarelli, A. Braunstein, L. Dall’Asta, J. R. Wakeling and R. Zecchina, Containing epidemic outbreaks by message-passing techniques, Physical Review X, 4 (2014), 021024.

[4] R. M. Anderson and R. M. May, Population biology of infectious diseases: Part I, Discrete Contin. Dyn. Syst. Ser. A, 280 (1979), 361–367.

[5] H. Amman, Invariant sets and existence theorems for semilinear parabolic and elliptic systems, J. Math. Anal. Appl., 65 (1978), 432–467.

[6] H. Brezis and W. A. Strauss, Semi-linear second-order elliptic equations in $L^1$, J. Math. Soc. Japan, 25 (1973), 565–590.

[7] R. S. Cantrell and C. Cosner, Spatial Ecology via Reaction-diffusion Equations, Ser. Math. Comput. Biol., 2003.

[8] R. H. Cui, K. Y. Lam and Y. Lou, Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments, J. Differential Equations, 263 (2017), 2343–2373.

[9] R. H. Cui and Y. Lou, A spatial SIS model in advective heterogeneous environments, J. Differential Equations, 261 (2016), 3305–3343.

[10] K. Deng and Y. X. Wu, Dynamics of a susceptible-infected-susceptible epidemic reaction-diffusion model, Proc. Roy. Soc. Edinburgh Sect. A, 146 (2016), 929–946.

[11] Y. H. Du, R. Peng and M. X. Wang, Effect of a protection zone in the diffusive Leslie predator-prey model, J. Differential Equations, 246 (2009), 3932–3956.

[12] Z. J. Du and R. Peng, A priori $L^\infty$ estimates for solutions of a class of reaction-diffusion systems, J. Math. Biol., 72 (2016), 1429–1439.
[13] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 2001.
[14] H. W. Hethcote, *The mathematics of infectious diseases*, SIAM Rev., 42 (2000), 599–653.
[15] H. W. Hethcote, Epidemiology models with variable population size, *Mathematical Understanding of Infectious Disease Dynamics*, 16 (2009), 63–89.
[16] A. Hill, D. G. Rand, M. A. Nowak and N. A. Christakis, Emotions as infectious diseases in a large social network: The SI Sa model, *Proceedings of the Royal Society B*, 277 (2010), 3827–3835.
[17] A. Hill, D. G. Rand, M. A. Nowak and N. A. Christakis, Infectious disease modeling of social contagion in networks, *Plos Comput. Biol.*, 6 (2010), e1000968, 15 pp.
[18] W. Z. Huang, M. A. Han and K. Y. Liu, Dynamics of an SIS reaction-diffusion epidemic model for disease transmission, *Math. Biosci. Eng.*, 7 (2010), 51–66.
[19] M. J. Keeling and P. Rohani, *Modeling Infectious Diseases in Humans and Animals*, Princeton University Press, 2008.
[20] H. C. Li, R. Peng and F. B. Wang, Varying total population enhances disease persistence: Qualitative analysis on a diffusive SIS epidemic model, *J. Differential Equations*, 262 (2017), 885–913.
[21] C. S. Lin, W. M. Ni and I. Takagi, Large amplitude stationary solutions to a chemotaxis system, *J. Differential Equations*, 72 (1988), 1–27.
[22] Y. Lou and W. M. Ni, Diffusion, self-diffusion and cross-diffusion, *J. Differential Equations*, 131 (1996), 79–131.
[23] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Providence, RI: American Mathematical Society, Providence, RI, 2001.
[24] S. O’Regan and J. Drake, Theory of early warning signals of disease emergence and leading indicators of elimination, *Theoretical Ecology*, 6 (2013), 333–357.
[25] R. Peng, Asymptotic profiles of the positive steady state for an SIS epidemic reaction-diffusion model I, *J. Differential Equations*, 247 (2009), 1096–1119.
[26] R. Peng and S. Q. Liu, Global stability of the steady states of an SIS epidemic reaction-diffusion model, *Nonlinear Anal.*, 71 (2009), 239–247.
[27] R. Peng and X. Q. Zhao, A reaction-diffusion SIS epidemic model in a time-periodic environment, *Nonlinearity*, 25 (2012), 1451–1471.
[28] R. Peng and F. Q. Yi, Asymptotic profile of the positive steady state for an SIS epidemic reaction-diffusion model: effects of epidemic risk and population movement, *Phys. D*, 259 (2013), 8–25.
[29] H. J. Shi, Z. S. Duan and G. R. Chen, An SIS model with infective medium on complex networks, *Physica A*, 387 (2008), 2133–2144.
[30] Y. C. Tong and C. X. Lei, An SIS Epidemic Reaction-Diffusion Model with Spontaneous Infection in A Spatially Heterogeneous Environment, *Nonlinear Anal. Real World Appl.*, 41 (2018), 443–460.
[31] X. W. Wen, J. P. Ji and B. Li, Asymptotic profiles of the endemic equilibrium to a diffusive SIS epidemic model with mass action infection mechanism, *J. Math. Anal. Appl.*, 458 (2018), 715–729.
[32] Y. X. Wu and X. F. Zou, Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism, *J. Differential Equations*, 261 (2016), 4424–4447.
[33] M. Yang, G. R. Chen and X. C. Fu, A modified SIS model with an infective medium on complex networks and its global stability, *Physica A*, 390 (2011), 2408–2413.

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