CONSTRUCTIONS OF MAJORIZING MEASURES, BERNOLLI PROCESSES AND COTYPE

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ABSTRACT. We present three methods to construct majorizing measures in various settings. These methods are based on direct constructions of increasing sequences of partitions through a simple exhaustion procedure rather than on the construction of well separated ultrametric subspaces. The first scheme of construction provides a simple unified proof of the Majorizing Measure Theorem for Gaussian processes and of the following fact. If \( A, B \) are balanced convex sets in a vector space, and if \( A \) is sufficiently convex, a control of the covering numbers \( N(A, \varepsilon B) \) for all \( \varepsilon > 0 \) implies the (a priori stronger) existence of a majorizing measure on \( A \) provided with the distance induced by \( B \). This establishes, apparently for the first time, a clear link between geometry and majorizing measures, and generalizes the earlier results on majorizing measures on ellipsoids in Hilbert space, that were obtained by specific methods. Much of the rest of the paper is concerned with the structure of bounded Bernoulli (=Radmacher) processes. The main conjecture on their structure is reformulated in several ways, that are shown to be equivalent, and to be equivalent to the existence of certain majorizing measures. Two schemes of construction of majorizing measures related to this problem are presented. One allows to describe Bernoulli processes when the index set, provided with the supremum norm, is sufficiently small. The other allows to prove a weak form of the main conjecture. This result, while not sufficient to characterize boundedness of Bernoulli processes, allows to prove the remarkable fact that for any continuous operator \( T \) from \( C(K) \) to \( E \), the Rademacher cotype-2 constant of \( T \) is controlled by the maximum of the Gaussian cotype-2 constant of \( T \) and of its \((2,1)\)-summing norm. It is also proved, as a consequence of one of the main inequalities on Bernoulli processes, that in a Banach space \( E \) of dimension \( n \), at most \( n \log n \log \log n \) vectors suffices to compute the Rademacher cotype 2 constant of \( E \) within a universal constant.

1 - Introduction

The notion of majorizing measure has allowed considerable progress in the study of certain stochastic processes, in particular Gaussian processes.

Given a metric space \((T, d)\), and a probability measure \( \mu \) on \( T \), we set

\[
\gamma_{1/2}(T, d, \mu) = \sup_{x \in T} \int_0^\infty \sqrt{\frac{1}{\mu(B(x, \varepsilon))}} d\varepsilon,
\]
where $B(x, \varepsilon)$ is the closed ball for $d$ centered at $x$ of radius $\varepsilon$. One should observe that, since $\log 1 = 0$, the integrand is zero when $\varepsilon$ is larger than the diameter of $T$. We set

$$\gamma_{1/2}(T, d) = \inf_{\mu} \gamma_{1/2}(T, d, \mu),$$

(1.2)

where the infimum is taken over all possible choices of $\mu$. It is implicitly assumed in (1.1) that the ball $B(x, \varepsilon)$ is $\mu$-measurable. It must be pointed out that in (1.1) it is equivalent to assume that $\mu$ is supported by a countable subset of $T$. Actually, despite their name, majorizing measures have little connection with measure theory, and are actually a kind of weight system to measure the size of $T$. The set $T$ will often be a subset of a Hilbert space $H$, $d$ being the distance induced from $H$. In that case we will write $\gamma_{1/2}(T)$ rather than $\gamma_{1/2}(T, \| \cdot \|_2)$.

Consider an integer $M$, and standard independent Gaussian random variables $(g_i)_{i \leq M}$. For a subset $T$ of $\mathbb{R}^M$, we set

$$G(T) = \sup \{ E \sup_{t \in S} \sum_{i \leq M} t_i g_i; \ S \subset T; \ S \text{ finite} \}$$

where $t \in \mathbb{R}^M$ is written as $t = (t_i)_{i \leq M}$. It is proved in [T1] that for a certain constant $K$, we have

$$\frac{1}{K} \gamma_{1/2}(T) \leq G(T) \leq K \gamma_{1/2}(T),$$

(1.3)

when $T$ is provided by the distance induced by $\ell_2^M$. (The right-hand inequality is an earlier result of X. Fernique; the left hand side is known as the majorizing measure theorem). We observe that (1.3) does not depend on $M$. By approximation one could thus work in $\mathbb{R}^N$. But the present setting offers the advantage that one does not have to bother about infinite series.

A simpler proof of (1.3) is given in [T4]. An even simpler proof will be given in Section 2 of the present paper. While (1.3) is now fairly easy to prove, the construction of majorizing measures (i.e. of measures on $T$ that witness the left-hand side inequality of (1.3) in practical situations) is a difficult question. One reason is that, while (1.3) is in principle (as explained in [T2]) a theorem about geometry of the Hilbert space, this geometric aspect is not understood. There are situations (in particular some concrete classes of functions on $[0, 1]^2$ that are studied in [T7]) where one has a rather geometrical knowledge of $T$ and where the precise computation of $\gamma_{1/2}(T)$ is currently intractable. The first main
contribution of the present paper will be the description in a simple but important situation of a precise link between the geometry of $T$ and the value of certain functionals $\gamma_{\alpha, \beta}(T)$ that generalize $\gamma_{1/2}(T)$ and that have been introduced in [T7]. Given a metric space $(T, \delta)$, numbers $\alpha, \beta > 0$, and a probability measure $\mu$ on $T$, we set

$$\gamma_{\alpha, \beta}(T, \delta, \mu) = \sup_{x \in T} \int_{0}^{\infty} \varepsilon^\beta \left( \log \frac{1}{\mu(B(x, \varepsilon))} \right)^{\alpha \beta} \frac{d\varepsilon}{\varepsilon}^{1/\beta},$$

and

$$\gamma_{\alpha, \beta}(T, \delta) = \inf_{\mu} \gamma_{\alpha, \beta}(T, \delta, \mu)$$

the infimum being taken on all probability measures. Where $\beta = 1$, we write $\gamma_{\alpha}$ rather than $\gamma_{\alpha, 1}$. When $T$ is a subset of a Hilbert space, $\delta$ will be the distance induced by the norm, and we will write simply $\gamma_{\alpha, \beta}(T)$. Thus $\gamma_{1/2, 1}(T) = \gamma_{1/2}(T)$. The motivation for the introduction of these functionals is not a desire of empty generality, but the existence of concrete situations where these functionals are easy to manipulate.

It is shown in [T7] how to compute $\gamma_{\alpha, \beta}(E)$ when $E$ is an ellipsoid in Hilbert space, and it is shown that this computation is at the root of deep matching theorems of Ajtai, Komlos, Tusnady and Leighton and Shor on random samples in $[0, 1]^2$. This computation is done in [T7] by an explicit construction. We will show that what actually only matters is the fact that the ellipsoid is 2-convex. Equally irrelevant is the fact that we try to cover the ellipsoid with balls of a Hilbert space. Only the covering numbers are relevant. We recall that for two convex sets $B, U$ in a vector space, $N(B, U)$ denotes the minimum number of translates of $U$ needed to cover $B$.

**Theorem 1.1.** Consider balanced convex sets $B, U$ in a vector space, and denote by $\| \cdot \|_B, \| \cdot \|_U$ their gauges. Assume that

$$\| \cdot \|_B \text{ is } 2 - \text{convex.}$$

Then, for any $\alpha > 0$,

$$\gamma_{\alpha, 2}(B, \| \cdot \|_U) \leq K \sup_{\varepsilon > 0} \varepsilon (\log N(B, \varepsilon U))^{\alpha},$$

where $K$ depends only on the constant implicit in (1.5) and on $\alpha$.

It is also easier in that case to get convinced that the left-hand side of (1.6) should a priori be of bigger order than the right-hand side. In the case of the ellipsoids of [T7], the computation of the right-hand side of (1.7) is a standard exercise (by
volume estimates). We should also mention that there is a converse to Theorem 1.1 when one replaces (1.5) by the condition that $\| \cdot \|_{B}$ is 2-smooth. We will not prove it.

There are good reasons to believe that the scheme of proof common to Theorem 1.1 and to the majorizing measure theorem is conceptually the correct approach. But our attempts to use this approach on the most interesting classes of functions considered in [T7] have failed due to impassable technical (and combinatorial) difficulties. Nonetheless in Section 7 we will demonstrate how to use that scheme to study a non-trivial situation (that could also be handled by the methods of [T7]) with the belief that our approach contains some of the essential ingredients needed for the final solution.

The rest of the paper is devoted to Bernoulli processes. (The link with the previous material being that the crucial ingredients in each of the results we will present are closely related to the basic scheme of Section 2.)

Consider an independent sequence $(\varepsilon_i)_{i \leq M}$ of Bernoulli random variables, i.e. $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = 1/2$. For a subset $T$ of $\mathbb{R}^M$, we set

$$b(T) = \sup \{ E \sup_{t \in S} \sum_{i \leq M} \varepsilon_i t_i; \; S \subset T, \; S \text{ finite} \}. \quad (1.7)$$

(The “Bernoulli process” is the collection of random variables $X_t = \sum_{i \leq M} \varepsilon_i t_i$). The importance of this quantity is that sequences of ones and minus ones are one of the oldest and most fundamental structures of Probability. How to describe $b(T)$ the way (1.1) describes $G(T)$? A first simple observation is that, by a simple comparison result, we have

$$b(T) \leq KG(T). \quad (1.8)$$

There, as well as in the rest of the paper, $K$ denotes a universal constant, not necessarily the same at each occurrence. (When the constant depends only on, say, $\alpha, \beta$, we will write $K(\alpha, \beta)$, etc.).

Another simple observation is that, using the bound $| \sum_{i \leq M} \varepsilon_i t_i | \leq \sum_{i \leq M} |t_i|$, we have

$$b(T) \leq \sup_{t \in T} \| t \|_1 \quad (1.9)$$

where $\| t \|_1 = \sum_{i \leq M} |t_i|$. Thus (1.8) and (1.9) represent two different ways to control $b(T)$. One can interpolate between these two bounds, i.e.

$$b(T) \leq K \inf \{ u > 0; T \subset U + uB_1; \; G(U) \leq u \} \quad (1.10)$$

where $B_1 = \{ t \in \mathbb{R}^M; \sum_{i \leq M} |t_i| \leq 1 \}$.

Thus, it is natural to ask whether the right-hand side of (1.10) is actually of the same order as the left-hand side.
**The Bernoulli Problem.** Is it true that there exists a universal constant $K$ such that, given a subset $T$ of $\mathbb{R}^M$, one can find a subset $U$ of $\mathbb{R}^M$ such that $T \subset U + Kb(T)B_1$, where $\gamma_{1/2}(U) \leq Kb(T)$?

(We have replaced in this statement $G(U)$ by the equivalent quantity $\gamma_{1/2}(U)$).

The difficulty is of course that the decomposition is neither unique nor canonical. The Bernoulli Problem has been the main motivation behind the papers [T4] [T5] that study related questions, some of them being discussed in Section 4, where the Bernoulli Problem will be commented in detail.

The control on $\gamma_{1/2}(U)$ involves the structure of $U$ for the $\ell_2$ norm, and the difficulty of the Bernoulli problem is that we have to separate a structure involving the $\ell_2$ norm with one involving the $\ell_1$ norm. In the special situation where one has a control on the $\ell_\infty$ norm, the $\ell_1$ part disappears, making the problem easier. In Section 4, we will explain why the correct formulation of this phenomenon is as follows.

**Theorem 1.2.** $\gamma_{1/2}(T) \leq K(b(T) + \gamma_1(T, \| \cdot \|_\infty))$.

A basic tool in the proof of Theorem 1.2 is an extension of the construction scheme of Section 2 to a two parameter situation. This construction incidentally allows to recover some of the most technical results of [T5] with a significantly simpler proof. This tool is presented in Section 5.

It turns out, (somewhat unexpectedly) that Theorem 1.2 is the starting point for an apparently new comparison principle between gaussian and Bernoulli averages.

**Theorem 1.3.** Consider vectors $(x_i)_{i \leq M}$ is a Banach $X$ of dimension $n$. Then there is a subset $I$ of $\{1, \cdots, M\}$ such that

$$\text{card} I \leq Kn \log n \log \frac{E \| \sum_{i \leq M} g_i x_i \|}{E \| \sum_{i \leq M} \varepsilon_i x_i \|} \leq Kn \log n \log \log n$$

such that

$$E \| \sum_{i \notin I, i \leq M} g_i x_i \| \leq KE \| \sum_{i \leq M} \varepsilon_i x_i \|.$$

While we have been unable to solve the Bernoulli problem, we did succeed in proving the following weaker result.

**Theorem 1.4.** Consider $p > 1$. Then there is a constant $K(p)$, that depends on $p$ only, such that for each subset $T$ of $\mathbb{R}^M$, we can find a subset $U$ with $T \subset U + K(p)b(T)B_p$, where $\gamma_{1/2}(U) \leq K(p)b(T)$ and

$$B_p = \{ t \in \mathbb{R}^M; \sum |t_i|^p \leq 1 \}.$$
We do not know whether in Theorem 1.4 it is possible to replace $B_p$ by the weak-$\ell_1$, ball

$$B_{1,\infty} = \{t \in \mathbb{R}^N; \sup_{u>0} u \text{ card } \{i; |t_i| \geq u\} \leq 1\}$$

(a question that is apparently easier that the full Bernoulli problem). Apparently, all the available techniques are hopelessly inadequate to approach this question. Thus it seems of interest to state a particularly attractive special case (that was pointed out to me by S. Montgomery-Smith). Consider a finite group $G$, a function $f$ on $G$ such that $\|f\|_\infty \leq 1$, and

$$E \sup_{t \in G} \left| \sum_{x \in G} \varepsilon_x f(tx) \right| \leq 1.$$

There $(\varepsilon_x)_{x \in G}$ denotes an independent Bernoulli sequence, and $tx$ denotes the product in $G$. For $i \geq 0$, consider the distance $d_i$ on $G$ given by

$$d_i^2(s, t) = \sum_{x \in G} \min(2^{-4i}, (f(sx) - f(tx))^2)$$

The question is to decide whether, for some universal constant $K$, we have

$$\sum_{i \geq 0} 2^{-i} \sqrt{\log N(G, d_i, 2^{-i})} \leq K.$$

(The relevance of this inequality is not obvious and will be explained in Section 4).

The importance of Theorem 1.4 is that, while this result does not describe the structure of bounded Bernoulli processes, it is sufficient to describe the Rademacher cotype 2 constant of operators from $C(L)$ (where $L$ is a compact space). Let us recall that the Rademacher cotype 2 constant $C_2^r(V)$ of an operator $V$ from $C(L)$ to a Banach space $X$ is the infimum of the numbers $A$ such, for each $M$, and each sequence $(f_i)_{i \leq M}$ of continuous functions on $L$, we have

$$\left( \sum_{i \leq M} \|V(f_i)\|^2 \right)^{1/2} \leq AE \| \sum_{i \leq M} \varepsilon_i f_i \|_\infty.$$

Similarly, one defines the Gaussian cotype 2 constant $C_2^g(V)$ of $V$ as the infimum of the numbers $A$ such that for each $M$ and each sequence of continuous functions $(f_i)_{i \leq M}$ on $L$ we have

$$\left( \sum_{i \leq M} \|V(f_i)\|^2 \right)^{1/2} \leq AE \| \sum_{i \leq M} g_i f_i \|.$$

Recall also that the $(2,1)$-summing norm $\|V\|_{2,1}$ of $V$ is defined as the infimum of the numbers $A$ such that for each $M$ and each sequence $(f_i)_{i \leq M}$ of $C(L)$,

$$\left( \sum_{i \leq M} \|V(f_i)\|^2 \right)^{1/2} \leq A \| \sum_{i \leq M} |f_i| \|_\infty.$$
Theorem 1.5. For some universal constant $K$, we have

\begin{equation}
K^{-1}(C_2^g(V) + \|V\|_{2,1}) \leq C_2^r(V) \leq K(C_2^g(V) + \|V\|_{2,1}).
\end{equation}

To help the reader to appreciate this result, we should state the following corollary.

Corollary 1.6. Consider an operator $V$ from $\ell^N_\infty$ to a Banach space $X$. Then

\begin{equation}
C_2^r(V) \leq K(LLN)^{1/2}\|V\|_{2,1},
\end{equation}

where $LLN = \max(1, \log(\log N))$.

This result is optimal, and represents an improvement of order $(LLN)^{1/2}$ over the earlier result of [MS-T].

As a last result, we will mention the following corollary of Theorem 1.3.

Corollary 1.7. In a Banach space $X$, of dimension $n$, there exists vectors $(x_i)_{i \leq N}$ where

\begin{equation}
N \leq Kn \log nLLn
\end{equation}

and

\begin{equation}
C_2^r(X)E\left\| \sum_{i \leq N} \varepsilon_i x_i \right\| < K\left(\sum_{i \leq N} \|x_i\|^2\right)^{1/2}.
\end{equation}

In other words, the rademacher cotype 2 constant of $X$ can be “computed with $N$ vectors” within a universal constant.

Having stated the most easily understood results, we now describe in detail the organization of the paper. Section 2 gives the basic partitioning scheme and explains its relationship with the construction of majorizing measures. As a first application we prove the Majorizing Measure Theorem for Gaussian processes. In Section 3, the scheme of Section 2 is used to prove (a generalization of) Theorem 1.1. The study of Bernoulli processes starts in Section 4. There the main problem is reformulated in several ways, that are shown to be equivalent. The structure of Bernoulli processes does not depend (in contrast with the Gaussian case) of one single distance. We recall the method (introduced in [T5]) of measuring the size of a set, provided with a family of distance-like functionals, with respect to the existence of majorizing measures. In the rest of Section 4, we construct two families of functionals such that the resulting measure of size coincide exactly with the measure of size occurring in the Bernoulli problem (resp. a weaker form of the Bernoulli problem). The motivation for these rather technically difficult results is that they demonstrate that the type of decomposition occurring in the Bernoulli problem is not intractable. However, these results are (mostly) not used in the sequel and should be omitted at first reading. In Section 5, we establish the basic
partitioning scheme in the case of a family of distances. This method extends, and significantly simplifies, crucial sections of the paper [T5]. This scheme is then applied to the proof of Theorem 1.2, which in turn is applied to the proof of Theorem 1.3. In Section 6, we prove Theorem 1.4. Unfortunately, it is apparently impossible to use the scheme of Section 5, so we develop a specialized method. We also prove Corollary 1.6. Section 7 depends only on the material of Section 2. Its purpose is to demonstrate how to use the basic partitioning scheme to study a concrete class of functions.

2. A general partitioning construction.

We consider a metric space \((T, d)\). We denote by \(B(x, a)\) the ball centered at \(x\) of radius \(a\). For a subset \(S\) of \(T\), the diameter \(\Delta(S)\) is defined as

\[
\Delta(S) = \sup \{d(x, y); x, y \in S\}.
\]

We assume that \(T\) is of finite diameter. Assume that, for \(k \in \mathbb{Z}\), we are given a map \(\varphi_k : T \to \mathbb{R}^+\). We assume that

\[
\forall x \in T, \forall k \in \mathbb{Z}, \varphi_k(x) \leq \varphi_{k+1}(x),
\]

\[
A = \sup \{\varphi_k(x); x \in T, k \in \mathbb{Z}\} < \infty.
\]

Consider a function \(\theta : \mathbb{N} \to \mathbb{R}^+\), and assume that

\[
\lim_{n \to \infty} \theta(n) = \infty.
\]

Assume that, for certain numbers \(r \geq 4, \beta > 0\), the following holds:

(2.4) Given any point \(x\) of \(T\), any \(k \in \mathbb{Z}\), any \(n \geq 1\), and any points \(y_1, \ldots, y_n\) of \(B(x, r^{-k})\) such that

\[
\forall i, j \leq n, \ i \neq j \Rightarrow d(y_i, y_j) \geq r^{-k-1}
\]

we have

\[
\max_{j \leq n} \varphi_{k+2}(y_j) \geq \varphi_k(x) + r^{-\beta k} \theta(n).
\]

The reader should observe that on the left we have \(\varphi_{k+2}\) rather than \(\varphi_{k+1}\). This is the crucial point of the condition.

We denote by \(k_0\) the largest integer such that the diameter of \(T\) is \(\leq r^{-\beta k_0}\).
Theorem 2.1. Under conditions (2.1) to (2.4), one can find an increasing sequence of finite partitions \((C_k)_{k \geq k_0}\) of \(T\), and for each atom \(C\) of \(C_k\) one can find an index \(\ell_k(C) \geq 1\), such that the following properties hold.

(2.5) Each set \(C\) of \(C_k\) has diameter \(\leq 2 \cdot r^{-k}\).

(2.6) For any \(k \geq k_0 + 1\), and any two sets \(C, D\) of \(C_{k+1}\) that are included in the same element of \(C_k\), we have \(\ell_{k+1}(C) \neq \ell_{k+1}(D)\).

(2.7) If, for \(x \in T\), we denote by \(C_k(x)\) the unique element of \(C_k\) that contains \(x\), we have

\[
\forall x \in T, \quad \sum_{k \geq k_0} r^{-\beta_k} \theta(\ell_{k+1}(C_{k+1}(x))) \leq 4A.
\]

Comment. (2.7) is of course a “smallness condition” on the sequence \((C_k)_{k \geq k_0}\).

Proof. The construction goes by induction over \(k\). For each set \(C \in C_k\), we also construct a distinguished point \(z_k(C) \in C\), such that

\[
\forall y \in C, \quad d(y, z_k(C)) \leq r^{-k}.
\]

Observe that this condition implies (2.5).

At the first stage, we set \(C_{k_0} = \{T\}\), \(\ell_{k_0}(T) = 1\), and we choose \(z_{k_0}(T)\) such that

\[
\varphi_{k_0+2}(z_{k_0}(T)) \leq 2^{-1}A + \inf\{\varphi_{k_0+2}(y); y \in T\}.
\]

We assume now that the partition \(C_k\) has been constructed, as well as the points \(z_k(C)\) for \(C \in C_k\). To construct \(C_{k+1}\), it suffices to show how to partition any given element \(C\) of \(C_k\). This will be done in turn by an inductive argument. The index of any piece will simply be the rank at which it is constructed.

First, we choose \(y_1 \in C\) such that

\[
\varphi_{k+2}(y_1) \leq 2^{k_0-k-1}A + \inf\{\varphi_{k+2}(y); y \in C\}.
\]

We then pick inductively \(y_2, \ldots, y_{\ell}\) such that

\[
y_{\ell} \in C \setminus \bigcup_{i < \ell} B(y_i, r^{-k-1})
\]

and

\[
\varphi_{k+2}(y_{\ell}) \leq 2^{k_0-k-1}A + \inf\{\varphi_{k+2}(y); y \in C \setminus \bigcup_{i < \ell} B(y_i, r^{-k-1})\}.
\]

The construction continues as long as possible. It eventually stops according to (2.2), (2.3), (2.4). (This point will be detailed later.) We set

\[
D_{\ell} = (C \cap B(y_{\ell}, r^{-k-1})) \setminus \bigcup B(y_i, r^{-k-1}).
\]
Let us observe immediately the following crucial fact, that follows from (2.9)

\[(2.10) \quad \forall \, y \in D_\ell, \quad \varphi_{k+2}(y_\ell) \leq 2^{k_0-k-1}A + \varphi_{k+2}(y).\]

The sets \(D_\ell\) form the partition of \(C\) that we look for. We set

\[(2.11) \quad z_{k+1}(D_\ell) = y_\ell\]

\[(2.12) \quad \ell_{k+1}(D_\ell) = \ell.\]

Thus (2.10) can be rewritten as

\[(2.13) \quad \forall \, y \in D_\ell, \quad \varphi_{k+2}(z_{k+1}(D_\ell)) \leq 2^{k_0-k-1}A + \varphi_{k+2}(y).\]

This completes the construction. It is obvious that (2.8) (hence (2.5)) holds, and we proceed to prove (2.7).

We observe that \(d(z_k(C), y_i) \leq r^{-k}\), and that for \(i < j\), we have \(d(y_i, y_j) \geq r^{-k-1}\). Thus, by (2.4), for each \(\ell\) we have

\[(2.14) \quad \max_{i \leq \ell} \varphi_{k+2}(y_i) \geq \varphi_k(z_k(C)) + r^{-\beta k} \theta(\ell).\]

On the other hand, by (2.10), for \(i \leq \ell\), and since \(y_\ell \in D_1\), we have

\[\varphi_{k+2}(y_i) \leq 2^{k_0-k-1}A + \varphi_{k+2}(y_\ell)\]

so that, combining with (2.14) we get

\[(2.15) \quad \varphi_{k+2}(y_\ell) + 2^{k_0-k-1}A \geq \varphi_k(z_k(C)) + r^{-\beta k} \theta(\ell).\]

Consider now any \(x \in \ell\), and apply (2.15) to \(C = C_k(x)\), \(\ell = \ell_{k+1}(C_k(x))\), \(D_\ell = C_{k+1}(x)\), \(y_\ell = z_{k+1}(D_\ell) = z_{k+1}(C_{k+1}(x))\), so that

\[(2.16) \quad d^{-k-1}A + \varphi_{k+2}(z_{k+1}(C_{k+1}(x))) \geq \varphi_k(z_k(C_k(x))) + r^{-\beta k} \theta(\ell_{k+1}(C_{k+1}(x))).\]

Since

\[z_{k+2}(C_{k+2}(x)) \in C_{k+2}(x) \subset C_{k+1}(x) = D_\ell,\]

by (2.13) we have

\[\varphi_{k+2}(z_{k+1}(C_{k+1}(x))) \leq 2^{k_0-k-1}A + \varphi_{k+2}(z_{k+2}(C_{k+2}(x))).\]

Combining with (2.16) we get

\[(2.17) \quad 2^{k_0-k}A + \varphi_{k+2}(z_{k+2}(C_{k+2}(x))) \geq \varphi_k(z_k(C_k(x))) + r^{-\beta k} \theta(\ell_{k+1}(C_{k+1}(x))).\]

If we sum these relations for \(k_0 \leq k \leq m\), we obtain

\[
\sum_{k_0 \leq k \leq m} \, r^{-k\beta} \theta(\ell_{k+1}(C_{k+1}(x))) \leq 2A + \varphi_{m+1}(z_{m+1}(C_{m+1}(x)))) + \varphi_{m+2}(z_{m+2}(C_{m+2}(x))) \leq 4A
\]

by (2.2). This completes the proof. \(\Box\)

The following result clarifies the relationship between the situation of Theorem 2.1 and majorizing measures.
Theorem 2.2. Assume that we have an increasing sequence of finite partitions 
\((C_k)_{k \geq k_0} \) on \(T\), and assume that for each atom \(C\) of \(C_k\), we have an index \(\ell_k(C) \geq 1\) that satisfies (2.6). Assume moreover that for some numbers \(\alpha > 0, M\), we have

\[
\forall x \in T, \sum_{k \geq k_0} r^{-\beta k} \left( \log \ell_{k+1}(C_{k+1}(x)) \right)^{\alpha} \leq M, \tag{2.18}
\]

where \(C_k(x)\) denotes again the unique element of \(C_k\) that contains \(x\).

Then, we can find a probability measure \(\mu\) on \(T\) such that

\[
\forall x \in T, \sum_{k \geq k_0} r^{-\beta k} \left( \log \frac{1}{\mu(C_k(x))} \right)^{\alpha} \leq K(\alpha, \beta, r)(M + r^{-\beta k_0}) \tag{2.19}
\]

where \(K(\alpha, \beta, r)\) depends on \(\alpha, \beta, r\) only.

Proof. We first observe the elementary fact that

\[
\sum_{\ell \geq 1} \frac{1}{(\ell + 1)^2} \leq 1. \tag{2.20}
\]

We set \(w_{k_0}(T) = \frac{1}{2}\), and, for \(k > k_0\) we define inductively a number \(w_k(C)\) for \(C \in C_k\) by

\[
w_k(C) = \frac{1}{2(\ell_k(C) + 1)^2} w_{k-1}(C') \tag{2.21}
\]

where \(C'\) is the element of \(C_{k-1}\) that contains \(C\). Using (2.20) and (2.6) we see inductively that

\[
\sum_{C \in C_k} w_k(C) \leq 2^{k_0-k-1}.
\]

Thus, we can find a probability measure \(\mu\) on \(T\) that gives mass \(\geq w_k(C)\) to an arbitrary point \(z_k(C)\), of \(C\), for \(k \geq k_0\), \(C \in C_k\). Thus we have

\[
\sum_{k > k_0} r^{-\beta k} \left( \log \frac{1}{\mu(C_k(x))} \right)^{\alpha} \leq H =: \sum_{k > k_0} r^{-\beta k} \left( \log \frac{1}{w_k(C_k(x))} \right)^{\alpha}.
\]

Using (2.21), we get

\[
H \leq \sum_{k > k_0} r^{-\beta k} \left[ \log \frac{1}{w_{k-1}(C_{k-1}(x))} + \log(2(\ell(C(x)) + 1)^2) \right]^\alpha. \tag{2.22}
\]

Now, we observe that, if we set \(\delta = (1 + r^\beta)/2\), for all \(x, y > 0\), we have, since \(\delta > 1\)

\[
(x + y)^\alpha \leq \delta x^\alpha + K(\alpha, \beta, r)y^\alpha.
\]
Thus, from (2.22), we get
\[ H \leq \delta \sum_{k > k_0} r^{-\beta k} \left( \log \frac{1}{w_{k-1}(C_{k-1}(x))} \right)^\alpha + K(\alpha, \beta, r) \sum_{k > k_0} r^{-\beta k} \left( \log 2(\ell(C_k(x)) + 1)^2 \right)^\alpha. \]
Since the first summation is at most
\[ \delta r^{-\beta k_0}(\log 2)^\alpha + \delta r^{-\beta} H \]
and \( \delta r^{-\beta} \leq 1 - (1 - r^{-\beta})/2, \) the result follows easily. \( \square \)

**Remark.** When \( \alpha \leq 1, \) using that fact that \((x + y)^\alpha \leq x^\alpha + y^\alpha, \) we see that we can take \( K(\alpha, \beta, r) = K, \) independent of \( \alpha, \beta, r. \)

As a first application, we prove the left-hand side of (1.3). For this, we use Theorem 2.1 with \( \theta(n) = K^{-1}\sqrt{\log n} \) (for a large enough \( K \)) and
\[ \varphi_k(x) = G(T) - G(B(x, r^{-k})). \]

The fact that (2.4) holds for \( r \) sufficiently large is proved in [T4]. The left-hand side of (1.3) then follows from Theorem 2.2. It is of interest to compare the approach of Theorem 1.1 with that of [T4]. The main idea is identical but the argument is, so to say, reversed. One gain in this approach is that we no longer need the analogue of Lemma of [T6]; this is fortunate, since this analogue would not hold for all the values of \( \alpha, \beta \) of interest.

### 3. Majorizing measures on sufficiently convex sets.

In this section \( T = B \) is the unit ball of a normed space \( X. \) We assume that the norm of \( X \) has a modulus of convexity with a \( p^{\text{th}} \) power estimate. More precisely we assume that for some number \( p (p \geq 2) \) and some number \( \gamma > 0, \) we have
\[
\inf \left\{ 1 - \frac{\|x + y\|}{2}; \|x\| = \|y\| = 1; \|x - y\| \leq \varepsilon \right\} \geq \gamma \varepsilon^p,
\]
for all \( 0 < \varepsilon \leq 2. \) The choice of parameters \( (p = \beta = 2) \) made in the statement of Theorem 1.1 is important for applications, but it is instructive (and require no further effort) to perform the proof in a more general setting.

Let us first note the following simple fact

**Lemma 3.12.** If \( \|x\|, \|u\| \leq t \) and \( \|x + y\| \geq 2u, \) then
\[ \gamma \|x - y\|^p \leq t^{p-1}(t - u). \]

**Proof.** In [L-T], p. 60 it is shown that in (3.1) one can replace the condition \( \|x\| = \|y\| = 1 \) by \( \|x\|, \|y\| \leq 1. \) If we use (3.1) for \( x/t, y/t, \) we then obtain
\[ \frac{1}{t} - \frac{\|x + y\|}{2} \geq \gamma \frac{\|x - y\|^p}{t^{p-1}}. \]
from which the result follows.

The proof of Theorem 3.1 consists of two main steps. In the first, we will apply Theorem 2.1, and in the second we will apply Theorem 2.2.

To apply Theorem 2.1, we define the functions \( \varphi_k(x) \) for \( x \in B \) as follows. We set \( \varphi_k(x) = 0 \) if \( x \in 2r^{-k}U \). Otherwise, we set

\[
\varphi_k(x) = \sup\{t > 0; tB \cap (x + 2r^{-k}U) = \emptyset\}.
\]

It is obvious that (2.1) and (2.2) hold, with \( A = 1 \).

For \( n \geq 2 \), we set

\[
\varepsilon(n) = \sup\{\varepsilon > 0; \exists y_1, \ldots, y_n \in B; \forall \ell, \ell', 1 \leq \ell < \ell' \leq n, y_\ell \notin y_{\ell'} + \varepsilon U\}.
\]

We observe that \( \varepsilon(n) \leq 2 \). We observe the following simple relations

\[
(3.3) \quad \varepsilon < \frac{\varepsilon(n)}{2} \Rightarrow N(B, \varepsilon U) \geq n
\]

\[
(3.4) \quad \varepsilon' > \varepsilon(n) \Rightarrow N(B, \varepsilon' U) \leq n.
\]

**Lemma 3.3.** Assume \( r \geq 8 \). Consider \( x \in B \), \( n \geq 2 \), and points \( y_1, \ldots, y_n \in B(x, r^{-k}) \), such that

\[
i < j \leq n \Rightarrow y_j \notin y_i + r^{-k-1}B.
\]

Then we have

\[
\sup_{\ell \leq n} \varphi_{k+2}(y_\ell) \geq \varphi_k(x) + \frac{\gamma r^{-kp}}{(2r\varepsilon(n))^p}.
\]

**Proof.** By definition of \( \varphi_{k+2} \), for \( t > \sup_{i \leq n} \varphi_{k+2}(y_i) \) and any \( i \leq n \), we can find a point

\[
z_i \in tB \cap (y_i + 2r^{-k-2}U).
\]

We note that

\[
z_i \in y_i + 2r^{-k-2}U \subset x + (2r^{-k-2} + r^{-k})U \subset x + 2r^{-k}U.
\]

Thus, by convexity of \( U \), we have

\[
\forall i, j \leq n, \quad \frac{z_i + z_j}{2} \in x + 2r^{-k}U
\]

and thus, by definition of \( \varphi_k \), we have

\[
\|\frac{z_i + z_j}{2}\| \geq \varphi_k(x).
\]
Since $z_i \in tB$ for all $i \leq n$, it follows from Lemma 3.1 that
\[ \gamma \|z_i - z_j\|^p \leq t^{p-1}(t - \varphi_k(x)). \]

It thus follows that the points $z_i$, for $i \leq n$, belong to the ball $z_1 + RB$, where $R = (t - \varphi_k(x))/\gamma$.

Now, since $z_i \in y_i + 2r^{-k-2}U$, since $y_i \not\in r^{-k-1}U + y_j$ for $j \neq i$, and since $r \geq 8$, we have
\[ z_i - z_j \not\in \frac{r^{-k-1}}{2}U. \]

By definition of $\varepsilon(n)$, we thus have
\[ \frac{r^{-k-1}}{2R} \leq \varepsilon(n) \]
which means
\[ t^{p-1}(t - \varphi_k(x)) \geq \frac{\gamma}{(2r)^p \varepsilon(n)^p}. \]

Since $t > \sup_{\ell \leq n} \varphi_{k+2}(y_\ell)$ is arbitrary, and since $A = 1$, this completes the proof. \(\square\)

We now can apply Theorem 2.1 with $\theta(1) = 0$, $\theta(n) = \gamma/(2r\varepsilon(n))^p$ for $n \geq 2$. We observe that we have $k_0 = 0$.

**Corollary 3.4.** We can find an increasing sequence of finite partitions $(C_k)_{k \geq 0}$ of $B$, and indexes $\ell_k(C)$ for $C \in C_k$, that satisfy conditions (2.5) and (2.6) of Theorem 2.1, and such that
\begin{equation}
\forall x \in B, \quad \sum_{k \geq 0} \frac{r^{-kp}}{\varepsilon(\ell_{k+1}(C_{k+1}(x)))^p} \leq \frac{2^{p+2}r^p \gamma}{\alpha \beta'}
\end{equation}
where we make the convention that, when $\ell_{k+1}(C_{k+1}(x)) = 1$, the corresponding term of the series is zero.

The next main step in the proof of Theorem 3.1 is to interpret condition (3.5) when we suitably control the covering numbers $N(B, \varepsilon U)$.

**Proposition 3.5.** Consider the number $\beta' > 0$ such that
\begin{equation}
\frac{1}{\beta} = \frac{1}{\beta'} + \frac{1}{p}
\end{equation}
and assume that
\begin{equation}
M^{\beta'} = \sum_{k \geq 0} r^{-k\beta'}(\log N(B, r^{-k}U))^{\alpha \beta'} < \infty.
\end{equation}

Then the partition of Corollary 3.4 satisfies
\begin{equation}
\sum_{k \geq 0} r^{-k\beta}(\log \ell_{k+1}(C_{k+1}(x)))^{\alpha \beta} \leq K(\alpha, \beta, r) \frac{M^{\beta}}{\gamma^{\beta/p}}.
\end{equation}
Comment. In (3.6) we certainly allow the case \( \beta' = \infty \) \( (\beta = p) \), in which case (3.7) has to be interpreted as

\[
M = \sup_{k \geq 0} r^{-k}(\log N(B, r^{-k}U))^\alpha < \infty.
\]

The necessary modifications to the proof in that case are left to the reader.

Proof. Step 1. We fix \( x \). We observe that in (3.8) the contribution of the terms for which \( \ell_{k+1}(C_{k+1}(x)) = 1 \) is zero. For \( m \geq 0 \), we set

\[
I(m) = \{k \geq 0; 2^m \leq \log_2 \ell_{k+1}(C_{k+1}(x)) < 2^{m+1}\}.
\]

When \( I(m) \) is not empty, we denote by \( i(m) \) its smallest element, and we make the convention that when \( I(m) \) is empty the corresponding term does not appear. We have

\[
\sum_{k \in I(m)} r^{-k\beta}(\log_2 \ell_{k+1}(C_{k+1}(x)))^{\alpha\beta} \leq \sum_{k \in I(m)} r^{-k\beta 2^{(m+1)\alpha\beta}} \leq \sum_{k \geq i(m)} r^{-k\beta 2^{(m+1)\alpha\beta}} \leq K(\alpha, \beta) r^{-i(m)\beta 2^{m\alpha\beta}}.
\]

Thus

\[
\sum_{k \geq 0} r^{-k\beta}(\log \ell_{k+1}(C_{k+1}(x)))^{\alpha\beta} \leq K(\alpha, \beta) \sum_{m \geq 0} r^{-i(m)\beta 2^{m\alpha\beta}}.
\]

Step 2. We set \( u = \beta/\beta' \), \( v = \beta/p \), so that, by (3.6), \( u + v = 1 \). We observe the identity

\[
r^{-i(m)\beta 2^{m\alpha\beta}} = (2^{m\alpha\beta} \varepsilon_m^{\beta'})^u (r^{-pi(m)} \varepsilon_m^{-p})^v
\]

where \( \varepsilon_m = \varepsilon(2^{2^m}) \).

Thus, by Hölder’s inequality, we have

\[
S \leq S_1^u S_2^v,
\]

where

\[
S = \sum_{m \geq 0} r^{-i(m)\beta 2^{m\alpha\beta}}; \quad S_1 = \sum_{m \geq 0} 2^{m\alpha\beta} \varepsilon_m^{\beta'}; \quad S_2 = \sum_{m \geq 0} r^{-pi(m)} \varepsilon_m^{-p}.
\]

Step 3. Since

\[
\ell_{i(m)+1}(C_{i(m)+1}(x)) \geq 2^{2^m}
\]

and since \( \varepsilon(n) \geq \varepsilon(n') \) for \( n \leq n' \), we have

\[
\varepsilon(\ell_{i(m)+1}(C_{i(m)+1}(x)))^{-p} \geq \varepsilon^{-p}.
\]
Since, from the definition, we have \( i(m) \neq i(m') \) for \( m \neq m' \), it follows from (3.5) that

\[
S_2 \leq \frac{2^{p+2p^p}}{\gamma}.
\]

**Step 4.** Control of \( S_1 \).

For \( i \geq 0 \), consider the set

\[
J(i) = \{ m \geq 0; 2r^{-i-1} < \varepsilon_m \leq 2r^{-i} \}.
\]

When \( J(i) \) is non-empty, we denote by \( m(i) \) its largest element, and when \( J(i) = \emptyset \), we make the convention that the corresponding term does not appear.

We have

\[
\sum_{m \in J(i)} 2^{m \alpha \beta'} \varepsilon_m^{\beta'} \leq \sum_{\ell \leq m(i)} 2^{\ell \alpha \beta'} 2^{\beta'} r^{-\beta' i}
\]

\[
\leq K(\alpha, \beta) 2^{m(i) \alpha \beta'} r^{-\beta' i}.
\]

Since \( \varepsilon_m > 2r^{-i-1} \), by (3.2) we have

\[
N(B, r^{-i-1}U) \geq 2^{2m(i)}
\]

since \( \varepsilon_m = \varepsilon(2^{2m}) \). Thus, using (3.7) we get

\[
\sum_{i \geq 0} 2^{m(i) \alpha \beta'} r^{-\beta' i} \leq \sum_{i \geq 0} r^{-\beta' i} (\log_2 N(B, r^{-i-1}U))^{\alpha \beta'}
\]

\[
\leq r^{\beta'} \sum_{i \geq 0} r^{-\beta' i} (\log_2 N(B, r^{-i-1}U))^{\alpha \beta'}
\]

\[
\leq r^{\beta'} K(\alpha, \beta) M^{\beta'}.
\]

Combining with (3.12), we get

\[
S_1 \leq r^{\beta'} K(\alpha, \beta) M^{\beta'}.
\]

The result then follows from (3.9) to (3.13). \( \square \)

Theorem 3.1 is now a consequence of Theorem 2.2 and Proposition 3.5.

**4. Functionals on classes of functions and the Bernoulli Problem.**

For \( \tau \geq 1 \), let us consider the measure \( \mu_\tau \) of density \( a_\tau e^{-|t|^\tau} \) with respect to Lebesgue measure (where \( a_\tau \) is a normalizing constant) and let us consider a family \( (h_i)_{i \leq N} \) of independent random variables distributed like \( \mu_\tau \). For a subset \( T \) of \( \mathbb{R}^N \), we can consider the “canonical process” \( (X_t)_{t \in T} \), where

\[
X_t = \sum_{i \leq N} t_i h_i
\]
and the quantity
\[ F_\tau(T) = E \sup_{t \in T} X_t =: \sup_{S \subset T, S \text{ finite}} E \sup_{t \in S} \sum_{i \leq N} t_i h_i. \]

It is a remarkable fact that the quantity \( F_\tau(T) \) can be characterized in terms of the geometry of \( T \). This is a generalization of the majorizing measure theorem. In the following we write \( A \sim B \) to mean \( A \leq K(\tau)B, B \leq K(\tau)A \). We denote by \( \tau' \) the conjugate exponent of \( \tau \).

**Theorem 4.1.** [T6]

a) If \( 1 \leq \tau \leq 2 \), we have
\[ F_\tau(T) \sim \gamma_{1/2}(T, \| \cdot \|_2) + \gamma_{1/\tau}(T, \| \cdot \|_{\tau'}). \]

b) If \( \tau \geq 2 \), then
\[ F_\tau(T) \sim \min \{ A; T \subset U + V; \gamma_{1/2}(T, \| \cdot \|_2) \leq A, \gamma_{1/\tau}(T, \| \cdot \|_{\tau'}) \leq A \}. \]

The case \( \tau = 2 \) is the majorizing measure theorem for Gaussian processes. Part b is strikingly similar with the Bernoulli Conjecture. Actually this conjecture can be thought of to be like the “limiting case” \( \tau \to \infty \) of Theorem 4.1. (It should be noted that the constants implicit in the symbol \( \sim \), as given by the arguments of [T6], go to infinity with \( \tau \).)

Thus the majorizing measure theorem appears as one given element of a continuous family of theorems. One may then wonder whether the Bernoulli conjecture is properly formulated, and why Gaussian processes should play a prominent part in that conjecture. As it turns out, there is no reason to distinguish Gaussian process (other than their intrinsic importance). Actually, one could also choose to distinguish the canonical processes for \( \tau = 1 \). This is also natural, since, by comparison, we have \( F_\tau(T) \leq K(\tau)F_{\tau'}(T) \) for \( \tau \geq \tau' \), so that control of \( F_1(T) \) is the strongest of this family of conditions. As it turns out, these formulations are equivalent.

**Theorem 4.2.** The following are equivalent:

a) For any subset \( T \) of \( \mathbb{R}^N \), one can find \( U \subset \mathbb{R}^N \) such that \( T \subset U + Kb(T)B_1 \), and \( \gamma_{1/2}(U) \leq Kb(T) \).

b) For any subset \( T \) of \( \mathbb{R}^N \), one can find \( U \subset \mathbb{R}^N \) such that \( T \subset U + Kb(T)B_1 \), and
\[ \gamma_{1/2}(U) \leq Kb(T); \quad \gamma_1(U, \| \cdot \|_{\infty}) \leq Kb(T). \]

c) For any subset \( T \) of \( \mathbb{R}^N \), one can find \( U \subset \mathbb{R}^N \) such that \( T \subset U + Kb(T)B_1 \) and \( \gamma_1(U, \| \cdot \|_{\infty}) \leq Kb(T) \).

**Proof.** a) is the original formulation. It is obvious that b) \( \Rightarrow \) c) \( \Rightarrow \) a).
We prove that \( c \Rightarrow a \). Consider \( U \) as given by \( c \) and consider the set
\[
V = U \cap (T + Kb(T)B_1).
\]
Thus \( b(V) \leq Kb(T) \), and, since \( V \subset U \), we have \( \gamma_1(V, \| \cdot \|_\infty) \leq Kb(T) \). Thus, by Theorem 1.2 (that we will prove in Section 5) we have \( \gamma_{1/2}(V) \leq Kb(T) \). Now, since \( T \subset U + Kb(T)B_1 \), it is easy to check that \( T \subset V + Kb(T)B_1 \), which proves \( a \).

Thus, it remains to prove that \( a \Rightarrow b \). For this it suffices to show that for any subset \( U \) of \( \mathbb{R}^N \), we can write \( U \subset W + aB_1 \), where \( a = K\gamma_{1/2}(U) \) and
\[
\gamma_{1/2}(W) \leq Ka, \quad \gamma_1(W, \| \cdot \|_\infty) \leq Ka.
\]

The idea to prove this goes back to [T1], and has subsequently been used numerous times by this author. We will not reproduce the argument, since a stronger fact will be proved in Proposition 4.3 below. □

One of the difficulties of studying the Bernoulli problem is that the corresponding measure
\[
\inf\{u; T \subset U + uB_1; \gamma_{1/2}(U) \leq u\}
\]
of the size of a subset \( T \) of \( \mathbb{R}^N \) is very cumbersome to manipulate. The corresponding difficulty was solved in [T6] in the case of Theorem 4.1 and of the canonical processes. It is not obvious at all how to adopt these ideas to the case of Bernoulli processes. We will nonetheless show that there exists indeed a seemingly more manageable functional that is exactly equivalent to the quantity (4.1). This is certainly an encouraging fact.

First we must recall a convenient notion (introduced in [T5]) that allows one to study the size of a set relatively to a family of functionals. Consider a set \( T \), and assume that we are given a number \( r \geq 4 \) and that for \( j \in \mathbb{Z} \) we have a function \( \varphi_j \) on \( T \times T \), \( \varphi_j \geq 0 \), that satisfies \( \varphi_j(s, t) = \varphi_j(t, s) \). Typically \( \varphi_j \) will be the square of a distance, so that we can assume
\[
\varphi_j(s, t) \leq 4(\varphi_j(s, u) + \varphi_j(u, t))
\]
for all \( s, t, u \) of \( T \). For a subset \( U \) of \( T \), we define
\[
D_j(U) = \sup\{\varphi_j(s, t); s, t \in U\}.
\]

Consider \( i \in \mathbb{Z} \) and an increasing sequence of finite partitions \((C_j)_{j \geq i}\) of \( T \). For \( x \in T, j \geq i \), we denote as usual by \( C_j(x) \) the unique element of \( C_j \) that contains \( x \). For a probability measure \( \mu \) on \( T \) we consider the quantity
\[
\sup_{x \in T} \sum_{j \geq i} r^{-j} \left( D_j(C_j(x)) + \log \frac{1}{\mu(C_j(x))} \right)
\]
and we define the functional $\theta_i(T)$ as the infimum of the previous quantity over all possible choices of $\mu$ and the sequence $(C_j)_{j \geq i}$. (The reader should observe that, in contrast with the definition of [T5], we do not require that $C_i = \{T\}$. This is however only a minor technical point.) The idea of these functionals, as explained in [T5], Section 3 is that they are related to the usual notion of majorizing measures through a change of variable. In particular we will use the following fact, that is proved in [T5] (and can also be deduced from Theorem 5.1 below).

\[(4.3) \text{ When one uses the functionals } \varphi_j(s, t) = r^{2j} \|s - t\|_2^2 \text{ then } \theta_i(T) \leq Kr(\gamma_{1/2}(T) + r^{-i}). \]

It will be notationwise more convenient to work now in the space of measurable functions on a measure space $(\Omega, \Sigma, \lambda)$. We do not assume that $\lambda$ is a probability; indeed the most important case is $\Omega = \{1, \ldots, N\}$, $\lambda$ being the counting measure. The more general formulation has also some intrinsic interest. We denote by $B_1$ (resp. $B_2$, $B_\infty$) the unit ball of $L_1(\lambda)$ (resp. $L_2(\lambda), L_\infty(\lambda)$).

On $L_2(\lambda)$, we consider the functions

\[(4.4) \varphi_j(f, g) = \int_{\Omega} \min(1, r^{2j}(f - g)^2) d\lambda. \]

An immediate observation is that

\[(4.5) \varphi_j(f, g) \leq r^{2j} \|f - g\|_2^2. \]

**Proposition 4.3.** Consider a subset $T$ of $L_2(\lambda)$ and $i \in \mathbb{Z}$. Assume that $T \subset r^{-i}B_\infty/4$. Then we have

\[T \subset U + K\theta_i(T)B_1 \]

where

\[(4.6)_{1/2}(U) \leq Kr(\theta_i(T) + r^{-i} + r^{-i+1}D_{i-1}(T)); \quad \gamma_{1/2}(U, \| \cdot \|_\infty) \leq K(\theta_i(T) + r^{-i}). \]

**Comments.** 1) In the case where $\lambda$ the counting measure, we have $B_1 \subset B_2 \subset B_\infty$. For this choice of $\lambda$, the condition $T \subset r^{-i}B_\infty/4$ is not very restrictive in practice. Indeed, the Proposition will be used for values of $i$ such that $r^{-i}$ is of order $\theta_i(T)$. For these values, and since $\text{diam}_2(U) \leq K\gamma_{1/2}(U)$, (4.6) implies in any case that $T \subset Kr^{-i}B$. 
2) Clearly the value of $\theta_i(T)$ can only decrease when the functionals $\varphi_j$ decrease. Thus, by (4.3), we see that $\theta_i(T) \leq K(\gamma_{1/2}(T) + r^{-i})$ so that Proposition 4.3 can be applied to complete the proof of $a \Rightarrow b$ in Theorem 4.2.

Proof. **Step 1.** We start with a simple observation. By Markov’s inequality, if $0 \leq a \leq r^{-j}$, we have

$$
\lambda(\{|f| \geq a\}) \leq \frac{1}{a^2} \int \min(f^2, r^{-2j}) d\lambda = \frac{r^{-2j}}{a^2} \int \min(r^{2j} f^2, 1) d\lambda.
$$

(4.7)

**Step 2.** Consider an increasing sequence $(C_j)_{j \geq i}$ of finite partitions of $T$ and a probability measure $\mu$ on $T$ such that

$$
\forall x \in T, \sum_{j \geq i} r^{-j} \left(D_j(C_j(x)) + \log \frac{1}{\mu(C_j(x))}\right) < 2\theta_i(T).
$$

(4.8)

For $C \in C_j$, $j \geq i$, we chose one element $y(C) \in C$. We select one element $y(T) \in T$. For $x \in T$, $j \geq i$, we set $\pi_j(x) = y(C_j(x))$. We set $\pi_{i-1}(x) = y(T)$.

For each $\omega \in \Omega$, we define

$$
\ell(x, \omega) = \inf\{j \geq i - 1; |\pi_j(x)(\omega) - \pi_{j+1}(x)(\omega)| > r^{-j}\}.
$$

(4.9)

(When the set on the right is empty, we set $\ell(x, \omega) = \infty$). We set $\pi_\infty(x) = x$, and we set

$$
u(x)(\omega) = x(\omega) - u(x)(\omega).
$$

(4.11)

**Step 3.** We fix $x$ and we show that $\|v(x)\|_1 \leq K\theta_i(T)$. We define

$$
m(x, \omega) = \inf\left\{j \geq i - 1; |x(\omega) - \pi_{j+1}(x)(\omega)| > \frac{r^{-j-1}}{2}\right\}
$$

(4.12)

(when the set on the right is empty, we set $m(x, \omega) = \infty$).

Since we assume $T \subset r^{-i}B_\infty/4$, we have

$$
|x(\omega) - \pi_i(x)(\omega)| \leq \frac{r^{-i}}{2}
$$

and this shows that $m(x, \omega) \geq i$. The definition of $m(x, \omega)$ thereby implies that

$$
|x(\omega) - \pi_{m(x, \omega)}(\omega)| \leq \frac{r^{-m(x, \omega)}}{2}.
$$

For $j < m(x, \omega)$, we have

$$
|\pi_j(x)(\omega) - \pi_{j+1}(x)(\omega)| \leq |x(\omega) - \pi_j(x)(\omega)| + |x(\omega) - \pi_{j+1}(x)(\omega)|
\leq r^{-j} + \frac{r^{-j-1}}{2} \leq r^{-j}.
$$
so that \( j < \ell(x, \omega) \). This shows that \( m(x, \omega) \leq \ell(x, \omega) \).

We show now that

\[
|v(x)(\omega)| \leq 2r^{-m(x, \omega)}. \tag{4.13}
\]

Indeed

\[
|v(x)(\omega)| = |x(\omega) - \pi_\ell(x, \omega)(x)(\omega)| \\
\leq |x(\omega) - \pi_m(x, \omega)(x)(\omega)| + \sum_{m(x, \omega) \leq \ell < \ell(x, \omega)} |\pi_\ell(x)(\omega) - \pi_{\ell+1}(x)(\omega)| \\
\leq \frac{1}{2} r^{-m(x, \omega)} + \sum_{\ell \geq m(x, \omega)} r^{-\ell} \leq 2r^{-m(x, \omega)}.
\]

For \( j \geq i - 1 \), we set \( A_j = \{ \omega; m(x, \omega) = j \} \). By definition of \( m(x, \omega) \), we have

\[
\omega \in A_j \Rightarrow |x(\omega) - \pi_{j+1}(x)(\omega)| > \frac{r^{-j-1}}{2}.
\]

Since \( x, \pi_{j+1}(x) \) both belong to \( C_{j+1}(x) \), we use (4.7) with \( a = r^{-j-1}/2 \) to see that

\[
\lambda(A_j) \leq 4D_{j+1}(C_{j+1}(x)).
\]

Thus

\[
\int r^{-m(x, \omega)} d\lambda(\omega) = \sum_{j \geq i-1} r^{-j} \lambda(A_j) \\
\leq 4r \sum_{j \geq i-1} r^{-j-1} D_{j+1}(C_{j+1}(x)) \leq 8r \theta_i(T).
\]

This proves that \( \|v(x)\|_1 \leq K r \theta_i(T) \) and finishes this step.

**Step 4.** We set \( U = \{ u(x); x \in T \} \), and we proceed to show that

\[
\gamma_1(U, \| \cdot \|_\infty) \leq K(\theta_i(T) + r^{-i}). \tag{4.14}
\]

The main observation is that

\[
x, y \in C \in C_j \Rightarrow \|u(x) - u(y)\|_\infty \leq 4r^{-j}. \tag{4.15}
\]

Indeed, we have \( \pi_\ell(x) = \pi_\ell(y) \) for \( \ell \leq j \). Thus, the definition of \( \ell(x, \omega) \) shows that if either \( \ell(x, \omega) < j \) or \( \ell(y, \omega) < j \) we have \( \ell(x, \omega) = \ell(y, \omega) \), so that \( u(x)(\omega) = u(y)(\omega) \). If \( \ell(x, \omega) \geq j, \ell(y, \omega) \geq j \), we write

\[
|u(x)(\omega) - u(y)(\omega)| \leq \sum_{j \leq \ell < \ell(x, \omega)} |\pi_\ell(x)(\omega) - \pi_{\ell+1}(x)(\omega)| \\
+ \sum_{j \leq \ell < \ell(y, \omega)} |\pi_\ell(y)(\omega) - \pi_{\ell+1}(y)(\omega)| \\
\leq 2 \sum r^{-j} \leq 4r^{-j}.
\]
This proves (4.15). The result now follows simply by considering e.g. the probability that gives mass $2^{-j+i-1}\mu(C)$ to $u(y(C))$, for all $C \in C_j$, $j \geq i$, and by a routine computation.

**Step 5.** We show that

$$\gamma_{1/2}(U) \leq K(\theta_i(T) + r^{-i} + r^{-i+1}D_{i-1}(T)^{1/2}).$$

First, we observe that we can actually assume the following

If $C \supset D$, $C \in C_j$, $D \in C_{j+1}$, and if $y(C) \in D$, then $y(D) = y(C)$.

This implies in particular that

(4.16) \hspace{1cm} \pi_{\ell+1}(\pi_{\ell}(x)) = \pi_{\ell}(x).

for each $x, \ell$.

We fix $x$ and we estimate $\|u(x) - u(\pi_{\ell}(x))\|_2$. We set $G_{\ell} = \{\omega; \ell(x, \omega) = \ell\}$. Since

$$\omega \in G_{\ell} \Rightarrow |\pi_{\ell}(x)(\omega) - \pi_{\ell+1}(x)(\omega)| > r^{-\ell},$$

it follows from (4.12), taking $a = r^{-\ell}$ that (since $\pi_{\ell}(x), \pi_{\ell+1}(x) \in C_{\ell}(x)$)

(4.17) \hspace{1cm} \lambda(G_{\ell}) \leq D_{\ell}(C_{\ell}(x)).

We have seen in the proof of (4.15) that $u(x)(\omega) - u(\pi_j(x))(\omega) = 0$ unless $\ell(x, \omega) \geq j$. We have, when $\ell(x, \omega) \geq j$

(4.18) \hspace{1cm} |u(x)(\omega) - u(\pi_j(x))(\omega)| \leq \sum_{j \leq \ell < \ell(x, \omega)} |\pi_{\ell}(x)(\omega) - \pi_{\ell+1}(x)(\omega)|

$$\leq \sum_{\ell \geq j} z_{\ell}(\omega)$$

where we define

$$z_{\ell}(\omega) = |\pi_{\ell}(x)(\omega) - \pi_{\ell+1}(x)(\omega)|$$

if the right hand side is $\leq r^{-\ell}$, and $z_{\ell}(\omega) = 0$ otherwise. In particular

$$\|z_{\ell}\|_2^2 \leq \int \min(r^{-\ell}, |\pi_{\ell}(x)(\omega) - \pi_{\ell+1}(x)(\omega)|)^2 d\lambda(\omega) \leq r^{-2\ell}D_{\ell}(C_{\ell}(x))$$

since $C_{\ell}(x)$ contains both $\pi_{\ell}(x)$ and $\pi_{\ell+1}(x)$.

Thus, by (4.17) and the triangle inequality we have

(4.19) \hspace{1cm} \|u(x) - u(\pi_j(x))\|_2 \leq \sum r^{-\ell}D_{\ell}(C_{\ell}(x))^{1/2}.\]
Let us observe that this inequality holds in particular for $j = i - 1$ so that

$$
\|u(x) - u(\pi_{i-1}(x))\|_2 = \|u(x) - u(y(T))\|_2 \leq r^{-i+1}D_{i-1}(T)^{1/2} + Kr^{-i/2}\theta_i(T)^{1/2}
$$

since, for $\ell \geq i$, we have

$$
D_\ell(C_\ell(x))^{1/2} \leq r^{\ell/2}\theta_i(T)^{1/2}.
$$

Thus using the inequality $\sqrt{ab} \leq a + b$ we have shown the following:

(4.20) The diameter of $U$ for $L^2$ is at most $2r^{-i+1}D_{i-1}(T)^{1/2} + K(\theta_i(T) + r^{-i})$.

Consider now the measure $\nu$ that gives mass $2^{-j+i-1}\mu(C)$ to each point $u(y(C))$, $C \in C_j$, $j \geq i$, and set

$$
w(x, j) = \log \frac{1}{2^{-j+i-1}\mu(C_j(x))}.
$$

Since $u(\pi_j(x))$ has mass at least $2^{-j+i-1}\mu(C_j(x))$, setting $r(x) = \|u(x) - u(\pi_i(x))\|_2$, we have, using the inequality $\sqrt{ab} \leq a + b$, as well as (4.19),

$$
I(x) = \int_0^{r(x)} \left( \log \frac{1}{B_2(u(x), t)} \right)^{1/2} dt \leq \sum_{j \geq i} \|u(x) - u(\pi_j(x))\|_2 \sqrt{w(x, j + 1)}
$$

$$
\leq 4 \sum_{j \geq i} \left( \sum_{\ell \geq j} r^{-\ell}(D_\ell(C_\ell(x)))^{1/2} \right) \sqrt{w(x, j + 1)}
$$

$$
\leq 4 \sum_{j \geq i} \sum_{\ell \geq j} r^{-(\ell-j)/2} \sqrt{r^{-\ell}D_\ell(C_\ell(x))} r^{-j} w(x, j + 1)
$$

$$
\leq 8 \sum_{j \geq i} \sum_{\ell \geq j} r^{-(\ell-j)/2}(r^{-\ell}D_\ell(C_\ell(x)) + r^{-j}w(x, j + 1))
$$

$$
\leq K\left( \sum_{\ell \geq i} r^{-\ell}D_\ell(C_\ell(x)) + \sum_{j \geq i} r^{-j}w(x, j + 1) \right)
$$

$$
\leq K(r\theta_i(T) + r^{-i}).
$$

The conclusion then follows from (4.20). \qed

The following is a kind of converse to Proposition 4.3 and proves that the functional considered in this Proposition is indeed a sharp way to study the decompositions $T \subset U + uB_1$. We now on assume that $\lambda$ is the counting measure on $\{1, \ldots, N\}$.

**Proposition 4.4.** Consider $i \in \mathbb{Z}$ and $U \subset L^2(\lambda)$ with $\gamma_{1/2}(U) \leq r^{-i}$. Then

$$
\theta_i(U + r^{-i}B_1) \leq K(r)r^{-i}.
$$
Proof. **Step 1.** From the discussion prior to Proposition 4.3 follows that $\theta_i(U) \leq Kr^{-i}$. Thus we can find an increasing sequence $(A_j)_{j \geq i}$ of finite partitions of $U$, and a probability measure $\nu$ on $U$ such that

$$\forall x \in U, \sum_{j \geq i} r^{-j} \left( D_j(A_j(x)) + \log \frac{1}{\nu(A_j(x))} \right) \leq Kr^{-i}. $$

**Step 2.** To each $y \in B_1$, we associate a sequence of integers $p_\ell(y) = (p_\ell(y))_{\ell \geq 1}$ that satisfies the following properties.

$$(4.21) \quad \lambda(\{r^{-\ell} \leq |y| \leq r^{-\ell+1}\}) \leq r^{\frac{3}{2}p_\ell(y)}$$

$$(4.22) \quad \sum_{\ell \geq 1} r^{\frac{3}{2}p_\ell(y) - \ell} \leq K(r)$$

$$(4.23) \quad \forall \ell \geq 1, \quad |p_{\ell+1}(y) - p_\ell(y)| \leq 1. $$

To do this we denote by $q_\ell(y)$ the smallest integer for which the left-hand side of (4.21) is $\leq r^{\frac{3}{2}q_\ell(y)}$, and we set

$$p_\ell(y) = \max_{m \geq 1} (q_m(y) - |m - \ell|).$$

Thus (4.23) holds. To prove (4.22), we simply observe that

$$r^{\frac{3}{2}p_\ell(y) - \ell} \leq \sum_{m \geq 1} r^{\frac{3}{2}q_m(y) - \frac{3}{2}|m - \ell| - \ell} \leq \sum_{m \geq 1} r^{-|m - \ell|/2} r^{\frac{3}{2}q_m(y) - m}$$

and we invert the summation signs.

**Step 3.** For $j \geq i$, we consider the family $S_j$ of sequences $\overline{q}$ of integers, $\overline{q} = (q_1, \ldots, q_{j-i+1})$ that satisfy

$$\sum_{1 \leq m \leq j-i+1} r^{\frac{3}{2}q_m - m} \leq K(r)$$

where $K(r)$ is the constant of (4.22).

For each point $t \in U + r^{-i}B_1$, we choose once for all a decomposition $t = x(t) + r^{-i}y(t)$, where $x(t) \in U$, $y(t) \in B_1$. (The choice is made arbitrarily among all possible decompositions.) To $A \in A_j$, $\overline{q} \in S_j$, we associate the set $C(A, \overline{q})$ that consists of all the points for which $x(t) \in A$ and $y = y(t)$ satisfies

$$(4.24) \quad p_1(y) = q_1, \ldots, p_{j-i+1}(y) = q_{j-i+1}. $$

These sets $C(A, \overline{q})$ form a finite partition of $U + r^{-i}B_1$, that we denote by $C_j$. The sequence $(C_j)$ is increasing.
Step 4. We show that

\[(4.25) \quad D_j(C(A, \overline{q})) \leq K[D_j(A) + r^{3q_{j-i+1}}].\]

Since \(\varphi_j^{1/2}\) is a distance, it suffices to show that

\[\varphi_j(r^{-i}y, 0) \leq 2r^{3q_{j-i+1}}\]

whenever (4.24) holds.

To simplify notations, we write \(p_m\) rather than \(p_m(y)\). We have

\[(4.26) \quad \varphi_j(r^{-i}y, 0) = \int \min(r^{2j-2i}y^2, 1) d\lambda \]
\[\leq \sum_{\ell \geq 1} r^{-2\ell+2} \lambda(\{r^{2j-2i}y^2 \geq r^{-2\ell}\}) \cdot \]

Now,

\[\lambda(\{r^{2j-2i}y^2 \geq r^{-2\ell}\}) = \lambda(\{y \geq r^{-\ell+i-j}\}) \leq r^{3q_{\ell+j-i}}.\]

Thus, by (4.26)

\[\varphi_j(r^{-j}y, 0) \leq r^2 \sum_{\ell \geq 1} r^{-2\ell+4} p_{\ell+j-i} \]
\[= r^{2j-2i+2} \sum_{\ell \geq 1} r^3 p_{\ell+j-i} - 2(\ell+j-i).\]

It follows from (4.23) that this sum is at most twice its first term, so that

\[\varphi_j(r^{-j}y, 0) \leq 2r^{3p_{j-i+1}} = 2r^{3q_{j-i+1}}\]

since \(p_{j-i+1} = p_{j-i+1}(y) = q_{j-i+1}\).

Step 5. The definition of \((C_j)\) shows that if \(x = x(t), y = y(t)\), then, for \(j \geq i\), we have \(C_j(t) = C(A_j(x), \overline{q}^j(y))\), where

\[\overline{q}^j(y) = (p_1(y), \ldots, p_{j-i+1}(y)).\]

Thus, by (4.25)

\[r^{-j}D_j(C_j(t)) \leq K[r^{-j}D_j(A_j(x)) + r^{3p_{j-i+1}}(y-j)].\]

Combining with (4.22) yields

\[\sum r^{-j}D_j(C_j(t)) \leq K(r(\theta_i(U) + r^{-i}).\]
Step 6. We observe that $\text{card } S_j \leq (K(r)(j - i + 1))^{j-i}$. There is a probability $\mu$ on $A + r^{-i}B_1$ that gives mass at least

$$\nu(C)2^{-j+i-1}(\text{card } S_j)^{-1}$$

to each set $C(A, \mathbf{q})$, $\mathbf{q} \in S_j$, $A \in \mathcal{A}_j$. The fact that

$$\sum_{j \geq i} r^{-j} \log \frac{1}{\mu(C_{j}(t))} \leq K(r)(\theta_{i}(U) + r^{-i})$$

then follows by a routine computation. \hfill \Box

The measure of the size of a set of functions by the quantity $\theta_{i}(T)$ associated to functionals $\varphi_{j}$ given by (4.9) seems to be the correct way to capture decompositions $T \subset U + uB_{1}$, in Propositions 4.3 and 4.4. However, in order to prove Theorem 1.3 it will be easier to use a cruder tool, that is exactly adapted to the study of weaker decompositions $T \subset U + uB_{p, \infty}$, where

$$B_{p, \infty} = \{f \in L^0(\lambda); \sup_{t \geq 0} t^p \lambda(\{|f| \geq t\}) \leq 1\}.$$

We consider $1 \leq p < 2$, and we define $\gamma$ by $\gamma(2-p) = 1$. We set

$$d_i(f, g) = \left(\int \min((f - g)^2, r^{-4\gamma_j})d\lambda\right)^{1/2}.$$  

Proposition 4.5. Consider a subset $T$ of $L^2(\lambda)$. Assume that $T \subset \frac{1}{4}B_{\infty}$. Assume that there is an increasing sequence of finite partitions $(C_{j})_{j \geq 0}$ of $T$ and a probability measure $\mu$ on $T$ such that the following holds for a certain number $S \geq 1$.

(4.27) Each set $C \in C_{j}$ is of diameter $\leq r^{-j}$ for $d_{j}$.

(4.28) $\forall x \in T$, \hspace{1em} $\sum_{j \geq 0} r^{-j} \left(\log \frac{1}{\mu(C_{j}(x))}\right)^{1/2} \leq S$.

Then we can find a set $U$ with $\gamma_{1/2}(U) \leq K(p)S$ such that

$$T \subset U + K(p, r)B_{p, \infty}.$$

Proof. Step 1. For $C \in C_{j}$, $j \geq 0$, we pick a point $y(C) \in C$. For $x \in T$, we set $\pi_{j}(x) = y(C_{j}(x))$. We define, for $x \in T$, $\omega \in \Omega$

$$\ell(x, \omega) = \inf\{j \geq 0; |\pi_{j}(x)(\omega) - \pi_{j+1}(x)(\omega)| \geq r^{-2\gamma_j}\}$$

and $\ell(x, \omega) = \infty$ when the set on the left is empty. We set $\pi_{\infty}(x) = x$, and we define

$$u(x)(\omega) = \pi_{\ell(x, \omega)}(x)(\omega)$$

$$v(x) = \pi_{\infty}(x).$$
Step 2. We set \( U = \{ u(x); x \in T \} \), and we proceed to prove that \( \gamma_{1/2}(U) \leq K(p)S \). The basic fact is that, if \( x,y \in C \subset C_k \), we have
\[
\| u(x) - u(y) \|_2 \leq K(p)r^{-k}.
\]
Indeed, as in the proof of Proposition 4.3, we have \( u(x)(\omega) = u(y)(\omega) \) unless \( \ell(x,\omega) \geq k; \ell(y,\omega) \geq k \). Thus since \( \pi_k(x) = \pi_k(y) \), it suffices to show that
\[
(4.29) \quad \| (u(x) - \pi_k(x))1_{\{\ell(x,\omega) \geq k\}} \|_2 \leq 2r^{-k}.
\]
Now
\[
(4.30) u(x)(\omega) - \pi_k(x)(\omega)1_{\{\ell(x,\omega) \geq k\}} \leq \sum_{\ell \geq k} |\pi_\ell(x)(\omega) - \pi_{\ell+1}(x)(\omega)|1_{\{\ell(x,\omega) > \ell\}}.
\]
Since \( \pi_\ell(x), \pi_{\ell+1}(x) \) belong to \( C_\ell(x) \), and since \( |\pi_\ell(x)(\omega) - \pi_{\ell+1}(x)(\omega)| \leq r^{-2\gamma \ell} \) when \( \ell < \ell(x,\omega) \), by (4.27) and the definition of \( d_i \) we have
\[
\| \pi_\ell(x)(\omega) - \pi_{\ell+1}(x)(\omega)1_{\{\ell(x,\omega) > \ell\}} \|_2 \leq r^{-\ell}
\]
so that (4.29) follows by the triangle inequality.

That \( \gamma_{1/2}(U) \leq K(p)S \) follows by the usual computation, putting mass \( 2^{-j-1} \mu(C) \) at each point \( u(y(C)) \).

Step 3. We prove that for each \( x \), we have \( \| v(x) \|_{p,\infty} \leq K(p)S \). Consider
\[
m(x,\omega) = \sup \left\{ j \geq 0; |x(\omega) - \pi_j(x)(\omega)| \leq \frac{r^{-2\gamma j}}{2} \right\}.
\]
(Observe that the set on the right is not empty since \( T \subset \frac{1}{4}B_\infty(\lambda) \).) As in the proof of Proposition 2.3, we see that \( m(x,\omega) \leq \ell(x,\omega) \). Thus
\[
(4.31) |v(x)(\omega)| \leq |u(x)(\omega) - \pi_m(x,\omega)(x)| + \sum_{m(x,\omega) \leq \ell < \ell(x,\omega)} |\pi_\ell(x)(\omega) - \pi_{\ell+1}(x)(\omega)|
\]
\[
\leq K(p)r^{-2\gamma m(x,\omega)}.
\]
If we set \( H_k = \{ m(x,\omega) = k \} \), we see that, by definition of \( m(x,\omega) \), we have
\[
\omega \in H_k \Rightarrow |x(\omega) - \pi_{k+1}(x)(\omega)| \geq \frac{r^{-2\gamma (k+1)}}{2}
\]
so that, using (4.7),
\[
\frac{r^{-4\gamma(k+1)}}{4} \lambda(H_k) \leq \int \min((x - \pi_{k+1}(x))^2, r^{-4\gamma(k+1)})d\lambda \leq r^{-2(k+1)}
\]
by the argument of (4.12) and since both \( x \) and \( \pi_{k+1}(x) \) belong to \( C_{k+1}(x) \). Thus
\[
(4.32) \quad \lambda(H_k) \leq 4 r^{4\gamma k - 2k} = 4r^{2(2\gamma - 1)(k+1)}.
\]
Since, by the choice of \( \gamma = (2-p)^{-1} \), we have \( 2\gamma - 1 = p\gamma \), the result follows from (4.31) (4.32), since
\[
\lambda(\{ m(x,\omega) \leq k \}) \leq \sum_{\ell \leq k} \lambda(H_\ell) \leq K(p,r)r^{2(2\gamma - 1)k}. \quad \square
\]

The following shows that the method of Proposition 4.4 is indeed the correct approach to study the decompositions \( T \subset U + \mu B \).
Proposition 4.4. Consider a subset $T$ of $L^0(\lambda)$, and assume that $T \subset U + B_{p,\infty}$, where $\gamma_{1/2}(U) \leq 1$. Then one can find an increasing family of partitions $(C_i)_{i \geq 0}$ of $C$ and a probability measure $\mu$ on $T$ such that, for each $C \in C_i$, the diameter of $C$ for $d_i$ is $\leq K(p)r^{-i}$, and that

$$\forall x \in T, \sum_{i \geq 0} r^{-i} \left( \log \frac{1}{\mu(C_i(x))} \right)^{1/2} \leq K.$$

Proof. Using (4.3), it suffices to show that if a set $C$ has a diameter for $L^2(\lambda)$ that is $\leq 2r^{-i}$, then $C + B_{p,\infty}$ has a diameter for $d_i$ that is $\leq K(p)r^{-i}$. Since $d_i$ is a distance, it suffices to observe that, for $f \in B_{p,\infty}$

$$d_i(f,0)^2 = \int_0^{r^{-2\gamma_i}} \lambda(\{f \geq t\}) dt \leq \int_0^{r^{-2\gamma_i}} t^{-p} 2t dt = \frac{2}{2-p} r^{-2\gamma_i(2-p)} = \frac{2}{2-p} r^{-2i}.$$

5. Construction of majorizing measures in the two-parameters situation.

We consider a set $T$, and a family $\varphi_j$ of functions from $T \times T$ to $\mathbb{R}^+ (j \in \mathbb{Z})$. We assume $\varphi_j(s,t) = \varphi_j(t,s)$, and

$$\forall s,t,u \in T, \quad \varphi_j(s,t) \leq 2(\varphi_j(s,u) + \varphi_j(u,t)).$$

(5.1)

For a subset $S$ of $T$, we set $D_j(S) = \sup \{ \varphi_j(s,t); s,t \in S \}$. Given $t \in T$, $a > 0$, we set

$$B_j(t,a) = \{ s \in T; \varphi_j(t,s) \leq a \}.$$

Thus, by (5.1) we have

$$D_j(B_j(t,a)) \leq 4a.$$

(5.2)

We consider a number $r \geq 2$, and for simplicity we assume that $r$ is a power of 2 ($r = 2^\tau, \tau \in \mathbb{N}$). We make the crucial assumption that, for some $\delta > 0$,

$$\forall s,t \in T, \forall j \in \mathbb{Z}, \quad \varphi_{j+1}(s,t) \geq r^{1+\delta} \varphi_j(s,t).$$

(5.3)

The functions $\varphi_i$ given by (4.10) do not satisfy this condition.

We assume that to each subset $S$ of $T$ is associated a number $F(S) \geq 0$. If $S \subset S'$, we assume $F(S) \leq F(S')$. We consider an increasing family $(A_j)_{j \geq i}$ of finite partitions of $T$. We assume the following condition, where $\alpha, \beta$ are $> 0$ (this condition is a substitute for (2.4)).
(5.4) Consider \( j \geq i \), and consider \( p \geq \tau - 1 \). Consider a subset \( C \) of \( T \), and assume that for a certain \( D \in A_j \), we have \( C \subset D \). Assume that \( D_{j-1}(C) \leq 2^{p-\tau+3} \). Set \( N = 2^p \), and consider points \( t_1, \ldots, t_N \) of \( C \), such that

\[
\ell, \ell' \leq N, \ell \neq \ell' \Rightarrow \varphi_j(t_\ell, t_{\ell'}) \geq 2^p.
\]

Consider for each \( \ell \leq N \), a subset \( A_\ell \) of \( C \cap B_j(t_\ell, \alpha 2^p) \). Then

\[
F\left( \bigcup_{\ell \leq N} A_\ell \right) \geq \beta r^{-j} 2^p + \min_{\ell \leq N} F(A_\ell).
\]

**Theorem 5.1.** Suppose, with the notations above that \( \alpha r^\delta \geq 4 \), and consider a probability \( \nu \) on \( T \), and an increasing sequence \( (A_j)_{j \geq i} \) of finite partitions of \( T \). Then there exists a probability \( \mu \) on \( T \) and an increasing sequence of finite partitions \( (C_j)_{j \geq i} \) on \( T \) such that

\[
\forall \ x \in T, \ \sum_{j \geq i} r^{-j} \left( D_j(C_j(x)) + \log \frac{1}{\mu(C_j(x))} \right)
\leq K \left( \frac{1}{\beta} F(T) + r^{-i}(1 + D_{i-1}(T)) + \sup_{y \in T} \sum_{j \geq i} r^{-j} \log \frac{1}{\nu(A_j(y))} \right).
\]

We first present the basic construction. (This construction will then be iterated to prove Theorem 5.1.)

**Proposition 5.2.** Consider \( j \geq i \), a subset \( C \) of \( T \). Assume that \( C \) is contained in a set belonging to \( A_j \). Assume that \( D_{j-1}(C) \leq 2^{n^2} \), and that we are given a number \( a(C) \) that satisfies the following two properties

\[
F(C) - \beta r^{-j} 2^n \leq a(C) \leq F(C)
\]

\[
\forall \ t \in C, \ F(C \cap B_j(t, \alpha 2^{n-1})) \leq a(C) + \varepsilon_j
\]

where \( \varepsilon_j = \beta r^{-i} 2^{-j+i} \).

Set \( n' = n + \tau - 1 \). Then, for \( s \geq n' \), \( \ell \leq N_s = 2^{2^s} \), we can find sets \( V(s, \ell) \) and numbers \( a(V(s, \ell)) \) that satisfy the following conditions

\[
F(V(s, \ell)) - \beta r^{-j-1} 2^s \leq a(V(s, \ell)) \leq F(V(s, \ell))
\]

\[
\forall \ t \in V(s, \ell), \ F(V(s, \ell) \cap B_{j+1}(t, \alpha 2^{s-1})) \leq a(V(s, \ell)) + \varepsilon_{j+1}
\]

\[
F(V(s, \ell)) + a(V(s, \ell)) + \frac{\beta}{4} r^{-j-1} 2^s \leq F(C) + a(C) + \frac{\beta}{8} r^{-j} 2^n + \varepsilon_j.
\]
**Comments.** 1) Conditions (5.10), (5.11) express that \( a(V(s, \ell)) \) is to \( V(s, \ell) \) what \( a(C) \) is to \( C \).

2) The reader observes the different coefficients of \( \beta \) in (5.12) so that summation of such relations does yield information.

**Proof. Step 1. Construction.** Starting with \( s = n' \), we construct points \( t(s, \ell) \) of \( C, \ell \leq N_s \), that satisfy the following conditions.

(5.13) Denote by \( H(s, \ell) \) the union of the sets \( B_{j+1}(t(s', \ell'), 2^s) \) for either \( s' < s \) or \( s = s', \ell' < \ell \). Then

\[
\text{otherwise } t(s, \ell) \notin H(s, \ell).
\]

(5.14) \( C \cap B_{j+1}(t(s, \ell), \alpha 2^s)) \supseteq \sup \{ F(C \cap B_{j+1}(t, \alpha 2^s)); t \in C \setminus H(s, \ell) \} - \varepsilon_{j+2} \).

The construction is immediate. It continues as long as possible. We set

\[
W(s, \ell) = C \cap B_{j+1}(t(s, \ell), 2^s)
\]

\[
V(s, \ell) = W(s, \ell) \cup W(s, \ell')
\]

where the union is over all the choices of \( s' < s \) or \( s' = s, \ell' < \ell \). It is obvious that the sets \( V(s, \ell) \) form a partition of \( C \). Also, by (5.2) we have \( D_{j+1}(V(s, \ell)) \leq 2^{s+2} \).

**Step 2.** If \( s = n' = n + \tau - 1 \), we set \( a(V(s, \ell)) = F(V(s, \ell)) \).

If \( s \geq n' + 1 = n + \tau \), we set

\[
a(V(s, \ell)) = \min(F(V(s, \ell)), F(C) - \beta r^{-j-1} 2^{s-1})
\]

(so that (5.10) holds) and we prove (5.11). It suffices to consider the case \( s \geq n' + 1 \), and to prove that

\[
\forall \forall t \in V(s, t), F(V(s, \ell) \cap B_{j+1}(t, \alpha 2^{s-1})) \leq F(C) - \beta r^{-j-1} 2^{s-1} + \varepsilon_{j+1}.
\]

First, we observe that, by construction of \( t(s - 1, N_{s-1}) \), (condition (5.14)) we have

(5.15) \( C \cap H(s - 1, N_{s-1}), F(C \cap B_{j+1}(t, \alpha 2^{s-1})) \subseteq F(C \cap B_{j+1}(t(s - 1, N_{s-1}), \alpha 2^{s-1})) + \varepsilon_{j+2} \).

By (5.13), we have, for \( \ell, \ell' \leq N_{s-1}, \)

\[
\ell \neq \ell' \Rightarrow \varphi_{j+1}(t(s - 1, \ell), t(s - 1, \ell')) \geq 2^{s-1}.
\]

We set

\[
A_{s-1} = C \cap B_{j+1}(t(s - 1, \ell), \alpha 2^{s-1})
\]
We see that we can use (5.4) (with $j+1$ rather than $j$, $s-1$ rather than $p$). Indeed, $D_j(C) \leq 2^{n+2} \leq 2^{s-\tau+2}$ since $s \geq n+\tau$.

Thus, we have

\begin{equation}
F(C) \geq \beta r^{-j-1}2^{s-1} + \min_{\ell \leq N_{s-1}} F(A_\ell).
\end{equation}

On the other hand, using (5.14) again, we have

\[\ell' < \ell \Rightarrow F(C \cap B_{j+1}(t(s-1,\ell), \alpha 2^{s-1})) \leq F(C \cap B_{j+1}(t(s-1,\ell'), \alpha 2^{s-1})) + \epsilon_{j+2}.\]

Thus

\begin{equation}
\epsilon_{j+2} + \min_{\ell \leq N_{s-1}} F(A_\ell) \geq F(C \cap B_{j+1}(t(s-1,N_{s-1}), \alpha 2^{s-1}))
\end{equation}

and combining (5.16) to (5.18) we get the result.

**Step 3.** We show that if $s \leq n+\tau+1$ we have

\begin{equation}
F(V(s,\ell)) \leq a(C) + \epsilon_j.
\end{equation}

Indeed we have

\[V(s,\ell) \subset C \cap B_{j+1}(t(s,\ell), 2^{s}).\]

Now, by (5.3), we have

\[B_{j+1}(t(s,\ell), 2^{s}) \subset B_j(t(s,\ell), r^{-1-\delta}2^{s}).\]

Since $\alpha r^\delta \geq 4$ and $r = 2^\tau$ we have $r^{-1-\delta}2^{s} \leq \alpha 2^{s-\tau-2} \leq \alpha 2^{n-1}$, so the result follows from (5.7).

**Step 4.** We prove (5.12). For this, we must distinguish cases.

**Case 1.** $s = n' = n + \tau - 1$. We note that by (5.10) we have $a(V(s,\ell)) \leq F(V(s,\ell)) \leq F(C)$. By (5.19), we have

\[F(V(s,\ell)) + a(V(s,\ell)) \leq F(C) + a(C) + \epsilon_j.\]

Since $r^{-j-1}2^{s} = \frac{1}{2}r^{-j}2^{n}$, (5.12) follows.

**Case 2.** $n + \tau \leq s \leq n + \tau + 1$. By definition of $a(V(s,\ell))$, we have

\begin{equation}
a(V(s,\ell)) \leq F(C) - \beta r^{-j-1}2^{s-1}.
\end{equation}

Combining with (5.19) we get

\[F(V(s,\ell)) + a(V(s,\ell)) \leq F(C) + a(C) - \beta r^{-j-1}2^{s-1} + \epsilon_j\]

from which (5.12) follows.
Case 3. $s \geq n + \tau + 2$. By (5.6), we have

$$a(C) \geq F(C) - \beta r^{-j}2^n.$$ 

Since $F(V(s, \ell)) \leq F(C) \leq a(C) + \beta r^{-j}2^n$, we then have

$$a(V(s, \ell)) + F(V(s, \ell)) \leq a(C) + F(C) + \beta r^{-j}2^n - \beta r^{-j-1}2^{s-1}.$$ 

Since $s \geq n + \tau + 2$, we have $r^{-j-1}2^{s-1} \geq 2r^{-j}2^n$, so that

$$\beta r^{-j}2^n - \beta r^{-j-1}2^{s-1} \leq -\frac{\beta}{4} r^{-j-1}s$$

from which (5.12) follows. This completes the proof of Proposition 5.2. \hfill \Box

Proof of Theorem 5.1. The construction of the family $C_j$ goes by induction over $j$. Together with each element $D$ of $C_j$, we will also construct an index $n(D)$ and a positive number $c(D)$, in such a way that the following conditions hold:

1. $D_j(D) \leq 2^{n(D)+2}$
2. $F(D) - \beta r^{-j}2^{n(D)} \leq c(D) \leq F(D)$
3. $\forall t \in D, F(D \cap B_j(t, \alpha 2^{n(t)-1})) \leq c(D) + \varepsilon_j$.

We start the construction with $C_{i-1} = \{T\}$, $c(T) = F(T)$ and for $n(T)$ the smallest integer such that $D_{i-1}(T) \leq 2^{n(T)+2}$.

Assume now that we have constructed $C_j$. We show how to partition a given element $D$ of $C_j$. First, we break $D$ into the pieces $D \cap A$, $A \in A_j$. We fix $A$, and we show how to partition $C = D \cap A$. We set

$$a(C) = \min(c(D), F(C)).$$

Thus, setting $n = n(D)$, we have by (5.22)

1. $F(C) - \beta r^{-j}2^n \leq a(C) \leq F(C)$
2. $\forall t \in C, F(C \cap B_j(t, \alpha 2^{n-1})) \leq a(D) + \varepsilon_j$.

We are then in a position to apply Proposition 5.2. The partition $(V(s, \ell))$, $s \geq n + \tau - 1$, $\ell \leq N_s$ that we obtained is the partition we want. We set

$$n(V(s, \ell)) = s; \quad c(V(s, \ell)) = a(V(s, \ell)).$$

This completes the construction of $C_{j+1}$. We observe that, by construction

$$(\text{5.26}) \quad \forall D \in C_j, \quad \forall A \in A_j, \quad \text{card} \{C \in C_j : C \subseteq D \cap A, n(C) = n\} \leq N_s = 2^{2^n}$$
By (5.9) to (5.11), (5.21) and (5.22) will hold for any element $D'$ of $C_{j+1}$ (so that the construction can continue). We rewrite (5.12) as

$$F(D') + c(D') + \frac{\beta}{4} r^{-j-1} 2^n(D') \leq F(C) + a(C) + \frac{\beta}{8} r^{-j} 2^n(D) + \varepsilon_j$$

$$\leq F(D) + c(D) + \frac{\beta}{8} r^{-j} 2^n(D) + \varepsilon_j.$$  

Thus, for any $x \in T$, we have

$$F(C_{j+1}(x)) + c(C_{j+1}(x)) + \frac{\beta}{4} r^{-j-1} 2^n(C_{j+1}(x)) \leq F(C_j(x)) + c(C_j(x)) + \frac{\beta}{8} r^{-j} 2^n(C_j(x)) + \varepsilon_j.$$  

We sum these inequalities for $j \geq i - 1$. We get

$$\frac{\beta}{8} \sum_{j \geq i} r^{-j} 2^n(C_j(x)) \leq 2F(T) + \frac{\beta}{8} r^{-i-1} 2^n(T) + 4\beta r^{-i}$$  

so that

$$\sum_{j \geq i} r^{-j} 2^n(C_j(x)) \leq \frac{16}{\beta} F(T) + Kr^{-i-1}(1 + D_{i-1}(T)).$$

We now construct the measure $\mu$. First, by induction over $j$ we construct weights $w(D)$ for $D \in C_j$, such that

$$\sum_{D \in C_j} w(D) \leq 2^{i-j-2}. \tag{5.29}$$

To do this, if $D' \in C_{j+1}$ is contained in $C \cap A (D \in C_j, A \in A_j)$, we set

$$w(D') = \frac{1}{4} \nu(A) w(D) 2^{-2^n(D')+1}. \tag{5.30}$$

It follows from (5.26) that

$$\sum_{D' \in D \cap A} 2^{-2^n(D')+1} \leq \sum_{n \geq 0} 2^n 2^{-2^n+1} = \sum_{n \geq 0} 2^{-2^n} \leq \sum_{n \geq 0} 2^{-n} = 2$$

so that (5.29) at rank $j+1$ follows from (5.30) and the fact that $\mu$ is a probability.

It follows from (5.29) that there exists a probability $\mu$ on $T$ that gives mass $\geq w(D)$ to each $D \in D_j$. From (5.30) we have

$$\log \frac{1}{\mu(C_{j+1}(x))} \leq \log \frac{1}{w(C_{j+1}(x))} \leq \log 4 + 2^n(C_{j+1}(x)) + \log \frac{1}{\nu(A(x))} + \log \frac{1}{w(C_j(x))}.$$  

Using (5.28), (5.22), the conclusion follows easily. \hfill \square

Before we prove Theorem 1.2, we should mention that Theorem 5.1 provides a simpler proof of Theorem 5.1 of [T5]. To see this, we take $A_j = \{T\}$ for all $j$, and we define for $F(S)$ the “size” of the largest tree (in the precise sense of Theorem 5.1 of [T4]) that is contained in $S$. That (5.1) holds is obvious from the definition of the “size” of a tree.

An essential ingredient to the proof of Theorem 1.2 is the following, that is a weakening of Corollary 2.7 of [T5].
Proposition 5.3. Consider \( t_1, \ldots, t_N \in \mathbb{R}^M \), and \( a, b > 0 \). Assume

\begin{align*}
\forall \ell, \ell' \leq N, \quad & \| t_{\ell} - t_{\ell'} \|_{\infty} \leq a \quad (5.31) \\
\forall \ell \neq \ell', \quad & \| t_{\ell} - t_{\ell'} \|_2 \geq b. \quad (5.32)
\end{align*}

Consider \( \sigma > 0 \), and sets \( A_\ell \subset B_2(t_\ell, \sigma) \) for \( \ell \leq N \). Then

\[
b(\bigcup_{\ell \leq N} A_\ell) \geq \frac{1}{K} \min \left( b \sqrt{\log N}, \frac{b^2}{a} \right) + \min_{\ell \leq N} b(A_\ell) - K \sigma \sqrt{\log N}.
\]

Corollary 5.4. If \( \sigma \leq b/K_1 \), \( a \leq 2b/\sqrt{\log N} \), we have

\[
b(\bigcup_{\ell \leq N} A_\ell) \geq \frac{1}{K} b \sqrt{\log N} + \min_{\ell \leq N} b(A_\ell).
\]

Proof of Theorem 1.2.

Step 1. We choose \( \alpha = 1/K_1^2 \), where \( K_1 \) occurs in Corollary 5.4, and we chose for \( r \) the smallest power of 2 such that \( r^\alpha \geq 4 \). We define \( i \) as the smallest for which \( r^i \) is larger than the diameter of \( T \) for \( \| \cdot \|_{\infty} \). First, we find an increasing sequence \( (A_j)_{j \geq i} \) of finite partitions of \( T \), and a probability measure \( \nu \) on \( T \) such that

\[
\forall x \in T, \quad \sum_{j \geq i} r^{-j} \log \frac{1}{\nu(A_j(x))} \leq K \gamma_1(T, \| \cdot \|_{\infty})
\]

and that the diameter for \( \| \cdot \|_{\infty} \) of each \( A \in A_j \) is at most \( 2r^{-j} \). We set \( \varphi_j(s, t) = 2^{2j} \| s - t \|_2 \), so that (5.3) holds with \( \delta = 1 \).

We now prove that (5.4) holds for a certain \( \beta > 0 \), when \( F(S) = b(S) \). Since it is assumed in (5.4) that \( C \) is contained in a set of \( A_j \), we see that (5.31) holds for \( a = 2r^{-j} \). The definition of \( \varphi_j \) shows that (5.32) holds for \( b = r^{-j} 2^{p/2} \). Since \( A_\ell \) is contained in \( B_j(T_\ell, \alpha 2^p) \), we see that the number \( \sigma \) of Proposition 5.3 can be taken equal to \( \sqrt{\alpha r^{-j} 2^{p/2}} \), so that, by the choice of \( \alpha \), we have \( \sigma \leq b/K_1 \). Since \( \sqrt{\log N} = 2^{p/2} \sqrt{\log 2} \), we have \( a \leq 2b/\sqrt{\log N} \), so that the result follows from (5.33).

Thus, we can use Theorem 5.1. The right-hand side of (5.5) is at most

\[
K(b(T) + r^{-i} + r^i \Delta^2 + \gamma_1(T, \| \cdot \|_{\infty})
\]

where \( \Delta \) is the \( \ell_2 \) diameter of \( T \). Now, \( r^{-i} \leq K \gamma_1(T, \| \cdot \|_{\infty}) \), and since the \( \ell_2 \) diameter is less than the \( \ell_{\infty} \) diameter, this is at most

\[
K(b(T) + \gamma_1(T, \| \cdot \|_{\infty})).
\]

Proof of Theorem 1.2.

Step 1. We choose \( \alpha = 1/K_1^2 \), where \( K_1 \) occurs in Corollary 5.4, and we chose for \( r \) the smallest power of 2 such that \( r^\alpha \geq 4 \). We define \( i \) as the smallest for which \( r^{-i} \) is larger than the diameter of \( T \) for \( \| \cdot \|_{\infty} \). First, we find an increasing sequence \( (A_j)_{j \geq i} \) of finite partitions of \( T \), and a probability measure \( \nu \) on \( T \) such that

\[
\forall x \in T, \quad \sum_{j \geq i} r^{-j} \log \frac{1}{\nu(A_j(x))} \leq K \gamma_1(T, \| \cdot \|_{\infty})
\]

and that the diameter for \( \| \cdot \|_{\infty} \) of each \( A \in A_j \) is at most \( 2r^{-j} \). We set \( \varphi_j(s, t) = 2^{2j} \| s - t \|_2 \), so that (5.3) holds with \( \delta = 1 \).

We now prove that (5.4) holds for a certain \( \beta > 0 \), when \( F(S) = b(S) \). Since it is assumed in (5.4) that \( C \) is contained in a set of \( A_j \), we see that (5.31) holds for \( a = 2r^{-j} \). The definition of \( \varphi_j \) shows that (5.32) holds for \( b = r^{-j} 2^{p/2} \). Since \( A_\ell \) is contained in \( B_j(T_\ell, \alpha 2^p) \), we see that the number \( \sigma \) of Proposition 5.3 can be taken equal to \( \sqrt{\alpha r^{-j} 2^{p/2}} \), so that, by the choice of \( \alpha \), we have \( \sigma \leq b/K_1 \). Since \( \sqrt{\log N} = 2^{p/2} \sqrt{\log 2} \), we have \( a \leq 2b/\sqrt{\log N} \), so that the result follows from (5.33).

Thus, we can use Theorem 5.1. The right-hand side of (5.5) is at most

\[
K(b(T) + r^{-i} + r^i \Delta^2 + \gamma_1(T, \| \cdot \|_{\infty})
\]

where \( \Delta \) is the \( \ell_2 \) diameter of \( T \). Now, \( r^{-i} \leq K \gamma_1(T, \| \cdot \|_{\infty}) \), and since the \( \ell_2 \) diameter is less than the \( \ell_{\infty} \) diameter, this is at most

\[
K(b(T) + \gamma_1(T, \| \cdot \|_{\infty})).
\]

We now turn to the proof of Theorem 1.3. Theorem 1.3 is a consequence of the following.
Proposition 5.5. Consider vectors \((x_i)_{i \leq M}\) is a Banach space \(X\) of dimension \(n\). Then there is a subset \(I\) of \(\{1, \cdots, M\}\) with \(\text{card } I \leq Kn \log n\) such that either
\[
E\| \sum_{i \in J, i \leq M} g_i x_i \| \leq \frac{1}{2} E\| \sum_{i \leq M} g_i x_i \|
\]
or
\[
E\| \sum_{i \not\in I, i \leq M} g_i x_i \| \leq K E\| \sum_{i \leq M} \varepsilon_i x_i \|.
\]

Indeed, to obtain Theorem 1.3, we simply iterate use of Proposition 5.5. It is known that
\[
E\| \sum_{i \leq M} g_i x_i \| \leq K \sqrt{\log(n + 1)} E\| \sum_{i \leq M} \varepsilon_i x_i \|.
\]

We start the proof of Proposition 5.5. We set \(X^* \backslash 1 = \{x^* \in X^*; \|x^*\| \leq 1\}\). Consider a subset \(I\) of \(\{1, \cdots, M\}\) that will be chosen later, and set \(J = \{i \leq M; i \not\in I\}\). We set, for \(x^* \in X^*\),
\[
\|x^*\|_\infty = \sup_{i \in J} |x^*(x_i)|; \quad B_\infty = \{x^* \in X^*; \|x^*\|_\infty \leq 1\}.
\]

We can then reformulate Theorem 1.2 as
\[
E\| \sum_{i \in J} g_i x_i \| \leq K (E\| \sum_{i \in J} \varepsilon_i x_i \| + \gamma_1(X^* \backslash 1, \| \cdot \|_\infty)).
\]

For \(x^* \in X^*\), we set
\[
\|x^*\|_2^2 = \sum_{i \leq M} x^*(x_i)^2
\]
(This is the \(L^2\) norm associated to the gaussian random vector \(\sum_{i \leq M} g_i x_i\)). We set \(B_2 = \{x^* \in X^*; \|x^*\|_2 \leq 1\}\). The key to the proof is the following interpolation formulae

Lemma 5.6. We have
\[
\gamma_1(X^* \backslash 1, \| \cdot \|_\infty) \leq K \gamma_{1/2}(X^* \backslash 1, \| \cdot \|_2) \sup_{\varepsilon > 0} \varepsilon \sqrt{\log N(B_2, \varepsilon B_\infty)}
\]

We will prove this later in order not to break the flow of the argument. We plug (5.38) into (5.36) remembering that \(\gamma_{1/2}(X^* \backslash 1, \| \cdot \|_2) \leq KE\| \sum_{i \leq M} g_i x_i \|\) by (1.3). Thus we get
\[
E\| \sum_{i \in J} g_i x_i \| \leq KE\| \sum_{i \in J} \varepsilon_i x_i \| + K \alpha E\| \sum_{i \leq M} g_i x_i \|
\]
where
\[
\alpha = \sup_{\varepsilon > 0} \varepsilon \sqrt{\log N(B_2, \varepsilon B_\infty)}
\]
Thus, if we can arrange that $4K\alpha \leq 1$, whenever $E\| \sum_{i \leq M} g_i x_i \| \leq 2E\| \sum_{i \in J} g_i x_i \|$, (5.39) becomes

$$E\| \sum_{i \in J} g_i x_i \| \leq KE\| \sum_{i \in J} \varepsilon_i x_i \| + \frac{1}{2} E\| \sum_{i \in J} g_i x_i \|$$

and this implies (5.35).

Before we study $\alpha$, we need some preliminaries. The formulae (5.37) defines a semi-norm $\| \cdot \|_2$ on $X^\ast$. By duality, this semi-norm defines a norm $\| \cdot \|_2$ on the linear span $H$ of the vectors $(x_i)_{i \leq M}$. The unit ball of that norm is the set of vectors $\sum_{i \leq M} \alpha_i x_i$ with $\sum_{i \leq M} \alpha_i^2 \leq 1$. If we denote by $\nu$ the law of $\sum_{i \leq M} g_i x_i$, $\nu$ is a gaussian measure, $H$ is its reproducing kernel and $\| \cdot \|_2$ is the associated norm. One way to reformulate (5.37) is to say that $\nu$ is the canonical gaussian measure on $H$, i.e.

$$\langle x^\ast, y^\ast \rangle = \int_H x^\ast(x)y^\ast(x)d\nu(x)$$

For a subset $A$ of $H$, we can measure its size $\ell(A)$ with respect to the canonical gaussian measure $\nu$ by

$$\ell(A) = \int_H \sup_{x \in A} \langle x, y \rangle d\nu(y)$$

(5.40)

Since $\nu$ is the law of $\sum_{i \leq M} x_i g_i$ we have

$$\ell(A) = E \sup_{x \in A} \langle x, \sum_{i \leq M} x_i g_i \rangle$$

(5.41)

although this formulae will not be used in the present proof. An important fact for the rest of this argument is that, denoting by conv$A$ the balanced convex hull of a set $A$, for a sequence $(z_k)$ in $H$ we have

$$\ell(\text{conv}(z_k)) \leq K \sup_k \| z_k \|_2 \sqrt{\log(k+1)}$$

(5.42)

This results from a trivial computation; see e.g. the introduction of [T2]).

It follows from (5.37) that

$$\sum_{i \leq M} \| x_i \|_2^2 \leq n$$

so that we can assume without loss of generality that $\| x_i \| \leq \sqrt{n/i}$. Consider a number $L$ to be adjusted later, and set $J = \{ i \leq M; i \geq Ln \log n \}$, $C = \text{conv}\{ x_i; i \in J \}$. Then, by (5.42), we get

$$\ell(C) \leq K \sup \sqrt{\frac{n \log(k+1)}{k + n \log n}} \leq K \sqrt{\frac{\log L}{L}}$$

(5.43)
as is easily seen by distinguishing the cases \( k \leq nL \log n \) and \( k \geq nL \log n \). What we need to remember from (5.43) is that \( \ell(C) \) can be made arbitrarily small taking \( L \) large enough.

To bound \( \alpha \), we now simply apply the reverse Sudakov minoration as in [L-T], Chapter 4, (3.15), to obtain

\[
\alpha \leq K \ell(B_\infty^c)
\]

where \( B_\infty^c \) is the polar of \( B_\infty \), that, by the bipolar theorem, is exactly \( C \). The proof is complete. \( \square \)

**Proof of Lemma 5.6.** Consider an increasing sequence \( (A_n)_{n \geq n_0} \) of partitions of \( T = X_1^* \), where each element \( A \) of \( A_n \) is of diameter (for \( \| \cdot \|_2 \)) at most \( 2^{-n} \), and a probability measure \( \mu \) on \( A \) such that

\[
\sup_{t \in T} \sum_{n \geq n_0} 2^{-n} \sqrt{\log \frac{1}{\mu(A_n(t))}} \leq K \gamma_{1/2}(T, \| \cdot \|_2).
\]

(5.44)

There \( A_n(t) \) denotes as usual the unique element of \( A_n \) that contains \( t \). Set

\[
\alpha = \sup_{\varepsilon > 0} \varepsilon \sqrt{\log N(B_2, \varepsilon B_\infty^c)}.
\]

Consider the smallest integer \( p_0 \) with \( 2^{p_0} \alpha \geq 1 \). Given any \( n \geq n_0 \), any \( A \in A_n \) and any \( p \geq p_0 \) we can find a subset \( F(A, p) \) such that

\[
\text{card} (F(A, p)) \leq \exp(2^{2p} \alpha^2)
\]

such that each point of \( A \) is within distance \( 2^{-n-p} \) of a point of \( F(A, p) \). For each choice of \( A, p \), we put a mass

\[
\mu(A) 2^{-p+n_0-n+2e^{-\alpha^2 2^p}}
\]

at each point of \( F(A, p) \), for a total mass \( \leq 1 \).

We now prove that the resulting measure witnesses (5.38). Let us fix \( t \) in \( T \), and, for \( n \geq n_0 \), consider

\[
\varepsilon_n = \frac{\alpha 2^{-n}}{\sqrt{\log \frac{3}{\mu(A_n(t))}}}
\]

When \( 2^{-\ell} \leq \varepsilon_n \), the ball \( B_\infty(t, 2^{-\ell}) \) of center \( t \), of radius \( 2^{-\ell} \) for \( \| \cdot \|_\infty \) satisfies by construction

\[
\nu(B_\infty(t, 2^{-\ell})) \geq \mu(A_n(t)) 2^{-n-p_0+n_0-2e^{-\alpha^2 2^p}}
\]

where \( p = \ell - n \) (observe that \( 2^p \alpha \geq 1 \)). Thus

\[
\log \frac{1}{\nu(B_\infty(t, 2^{-\ell}))} \leq \log \frac{1}{\mu(A_n(t))} + \ell - p_0 - n_0 + 2 + \alpha^2 2^{(\ell-n)}
\]
and thus

\[ S_n =: \sum_{\varepsilon_{n-1} < 2^{-\ell} \leq \varepsilon_n} 2^{-\ell} \log \left[ \frac{1}{\nu(B_\infty(t, 2^{-\ell}))} \right] \leq K \varepsilon_n \log \frac{1}{\mu(A_n(t))} + \frac{\alpha^2}{\varepsilon_{n+1}} 2^{-2n} + a_n \]

where \( a_n = \sum_{\varepsilon_{n+1} \leq 2^{-\ell} \leq \varepsilon_n} 2^{-\ell} (\ell - p_0 - n_0 + 2) \), so that

\[ S_n \leq K \alpha \left[ 2^{-n} \sqrt{\log \frac{2}{\mu(A_n(t))}} + 2^{-n-1} \sqrt{\log \frac{2}{\mu(A_{n+1}(t))}} \right] + a_n. \]

Thus

\[ \sum_{n \geq n_0} S_n \leq K \alpha \gamma_{1/2}(T, \| \cdot \|_2) + \sum_{n \geq n_0} a_n. \]

We now observe that \( \varepsilon_{n_0} = \alpha 2^{-n_0} \), and that the diameter of \( T \) for \( \| \cdot \|_\infty \) is at most \( \varepsilon_{n_0} \) (since \( N(B_2, \alpha B_\infty) = 1 \)). Thus

\[ \int_0^\infty \log \frac{1}{\mu(B_\infty(t, \varepsilon))} d\varepsilon \leq K \sum_{n \geq n_0} S_n \]

Also, \( \sum_{n \geq n_0} a_n \leq K \alpha 2^{-n_0} \leq K \alpha \gamma_{1/2}(T, \| \cdot \|_2) \). This completes the proof. \( \square \)

The reader might have noticed that the argument of Proposition 5.5 shows that

\[ E \| \sum_{i \leq M} \varepsilon_i g_i \| \leq K E \| \sum_{i \leq M} \varepsilon_i x_i \| \]

whenever \( \ell(C) \leq 1/K \), where \( C = \{ x_i; i \geq 1 \} \). (The quantity \( \ell(C) \) is defined in the course of the proof of Proposition 5.5). However more is true.

**Theorem 5.7.** For vectors \( (x_i)_{i \leq M} \) in a Banach space, we have

\[ E \| \sum_{i \leq M} g_i x_i \| \leq KE \| \sum_{i \leq M} \varepsilon_i x_i \| (1 + \ell(C))^3. \] (5.45)

A positive solution to the Bernoulli problem would imply that (5.45) holds with a factor \((1 + \ell(C))\) rather than \((1 + \ell(C))^3\). It seems, however, that the difficulties one faces in proving this (even after one has obtained the apparently optimal Lemma 5.8 below) are of the same nature as some of the difficulties one faces when studying the Bernoulli problem. On the other hand, we know how to do better than (5.45), and in particular how to prove

\[ E \| \sum_{i \leq M} g_i x_i \| \leq KE \| \sum_{i \leq M} \varepsilon_i x_i \| (1 + \ell(C)) \log(2 + \ell(C)) \]

(5.46)

The techniques to obtain this improved estimates are however not related to the other material of the present paper, but rather are variations on the “tree extraction” techniques of [T5]. Since, moreover, there is not much conceptual gain in proving the imperfect inequality (5.46) rather than the (slightly more imperfect) inequality (5.45), we will prove (5.45) only.

**Proof of Theorem 5.7.** We set \( \alpha = \ell(C) \). A key estimate is as follows
Lemma 5.8. Consider \( y^*, y_1^*, \cdots, y_N^* \) in \( X^* \). Assume

\[
\forall i, j \leq N, \|y_i^* - y_j^*\|_2 \geq 1
\]

(5.47)

\[
E \sup_{j \leq N} y_j^* \left( \sum_{i \leq M} \varepsilon_i x_i \right) \geq \sqrt{\log N} \frac{\sqrt{\log N}}{K(1 + \alpha)}
\]

(5.48)

Proof. Consider the map \( W \) from \( X^* \) to \( \mathbb{R}^M \) that sends \( y^* \) to \( (y_i^*(x_i))_{i \leq M} \). Set \( T = W(\{y_j^*; j \leq N\}) \), so that the left-hand side of (5.48) is simply \( b(T) \). Denoting as usual by \( B_2 \) and \( B_1 \) the \( \ell_2 \) and the \( \ell_1 \) unit balls of \( \mathbb{R}^M \), for numbers \( \theta, L > 0 \), consider \( D = \theta B_2 + L B_1 \). Thus if \( t \in D \), we can write

\[
t_i = u_i + v_i; \sum_{i \leq M} u_i^2 \leq \theta^2, \sum_{i \leq M} |v_i| \leq L
\]

(5.49)

However, even if \( t \in W(X^*) \) there is no reason why \( (u_i)_{i \leq M} \) or \( (v_i)_{i \leq M} \) should be of the same type. This is why we moved to \( \mathbb{R}^M \) rather than working in \( X^* \).

The key tool is the version of Sudakov minoration for Bernoulli processes proved in [T5] that asserts that

\[
L \geq K b(T) \Rightarrow \theta \sqrt{\log N(T,D)} \leq K b(T)
\]

so that

\[
b(T) \geq \frac{1}{K} \min(L, \theta \sqrt{\log N(T,D)})
\]

(5.50)

Our task is now to find a lower bound for \( N(T,D) \). Consider \( t \in T \), and \( S = T \cap (t + D) \). Set \( R = \text{card} \ S \). By (5.47) and (the usual) Sudakov minoration, we have

\[
\frac{1}{K} \sqrt{\log R} \leq G(S).
\]

(5.51)

To bound \( \log R \), we now find an upper bound for the right-hand side of (5.51). We follow the notation established during the proof of Proposition 5.5. A basic observation is that for any \( y^* \) in \( X^* \) and any \( x \) in \( H \) we have

\[
y^*(x) = \sum_{i \leq M} y_i^*(x_i) \langle x, x_i \rangle.
\]

(5.52)

This is a consequence of (5.37) and of the fact that the map \( U \) from \( H \) to \( X^* \) given by \( u^*(x) = \langle u^*, U(x) \rangle \) defines an isometry from \( (H, \| \cdot \|) \) into \( (X^*, \| \cdot \|) \).
Consider now \( s \in S \), so that \( s - t \in D \). Also, \( s - t \in W(X^*) \). Thus there is \( y^* \) in \( X^* \) such that \( i \leq M \) we have \( s_i - t_i = y^*(x_i) \), and thus, by (5.52)

\[
\sum_{i \leq M} (s_i - t_i)g_i = y^*\left( \sum_{k \leq M} g_k x_k \right) = \sum_{i \leq M} y^*(x_i) \langle x_i, \sum_{k \leq M} g_k x_k \rangle \\
= \sum_{i \leq M} (s_i - t_i) \langle x_i, Z \rangle
\]

where for simplicity we set \( Z = \sum_{k \leq M} g_k x_k \). We now appeal to (5.49), since \( s - t \in D \), to write

\[
s_i - t_i = u_i + v_i; \sum_{i \leq M} u_i^2 \leq \theta^2, \sum_{i \leq M} |v_i| \leq L.
\]

and thus, by (5.53) we get

\[
\sum_{i \leq M} (s_i - t_i)g_i = \sum_{i \leq M} u_i \langle x_i, Z \rangle + \sum_{i \leq M} v_i \langle x_i, Z \rangle
\]

and thus

\[
|\sum_{i \leq M} (s_i - t_i)g_i| \leq |\sum_{i \leq M} u_i x_i| + L \sup_{i \leq M} |\langle x_i, Z \rangle|
\]

The r.v.

\[
Y_s = \langle \sum_{i \leq M} u_i x_i, Z \rangle = U\left( \sum_{i \leq M} u_i x_i \right)(Z)
\]

is gaussian, and by definition on the norm \( \| \cdot \|_2 \) on \( X^* \),

\[
EY_s^2 = \|U(\sum_{i \leq M} u_i x_i)\|_2^2 = \| \sum_{i \leq M} u_i x_i \|_2^2 \leq \sum_{i \leq M} u_i^2 \leq \theta^2
\]

Now, we have, by (5.55)

\[
G(S) = G(S - t) = E \sup_{s \in S} \sum_{i \leq M} (s_i - t_i)g_i \\
\leq E \sup_{s \in S} |\sum_{i \leq M} (s_i - t_i)g_i| \\
\leq E \sup_{s \in S} |Y_s| + LE \sup_{i \leq M} |\langle x_i, Z \rangle|
\]

Using (5.41) we see that

\[
E \sup_{i \leq M} |\langle x_i, Z \rangle| = \ell(C) = \alpha.
\]

Thus, by a standard estimate

\[
G(S) \leq K \theta \sqrt{\log P} + L \alpha.
\]
and plugging back into (5.51) gives

$$(5.56) \quad \frac{1}{K} \sqrt{\log R} \leq K \theta \sqrt{\log R} + L \alpha.$$ 

We now fix $\theta = \frac{1}{2K^2}$, $L = \frac{\sqrt{\log N}}{2K\alpha}$, so that (5.56) implies

$$\sqrt{\log R} \leq \sqrt{\frac{\log N}{2}},$$

and hence $R \leq \sqrt{N}$. This shows that for this choices of $\theta, L, \ell + D$ contains at most $\sqrt{N}$ points of $T$; thus $N(T, D) \geq \sqrt{N}$, and plugging in (5.50) this proves the result. $\square$

 Mimicking the proof of Corollary 2.7 of [T5] (and relying upon (5.7)) we obtain the following, where, for a subset $A$ of $X^*$, we set

$$b(A) = E \sup_{x^* \in A} x^*(\sum_{i \leq M} \varepsilon_i x_i).$$

**Proposition 5.8.** Consider $y_1^*, \ldots, y_N^*$ in $X^*$. Assume

$$\forall \ell \neq \ell', \|y_\ell^* - y_{\ell'}^*\|_2 \geq b.$$ 

Consider $\sigma > 0$ and for $\ell \leq N$ consider $A_\ell \subset B_2(y^*, \sigma) = \{z^* \in X^*; \|y^* - z^*\|_2 \leq \sigma\}$. Then

$$b(\bigcup_{\ell \leq N} A_\ell) \geq \frac{b}{K(1 + \alpha)} \sqrt{\log N} + \min_{\ell \leq N} b(A_\ell) - K\sigma \sqrt{\log N}.$$ 

**Corollary 5.9.** If $\sigma \leq b/K(1 + \alpha)$ we have

$$b(\bigcup_{\ell \leq N} A_\ell) \geq \min_{\ell \leq N} b(A_\ell) + \frac{b}{K(1 + \alpha)} \sqrt{\log N}.$$ 

We now appeal to Theorem 2.1, with $p = 1$. We set $T = X_1^*$, provided with the distance induced by the norm $\| \cdot \|_2$, and we set

$$\varphi_k(x^*) = b(X_1^*) - b(X_1^* \cap B_2(x, r^{-k})).$$

It follows from Corollary 5.9 that (2.4) holds, provided $r = K(1 + \alpha)$ and

$$\theta(n) = \frac{1}{Kr(1 + \alpha)} \sqrt{\log n}.$$ 

Now, we appeal to Theorem 2.2 and the remark that follows its proof. We observe that an extra factor $r$ occurs when comparing the left-hand side of (2.19) with an integral such as the right-hand side of (1.1). This finishes the proof of Theorem 5.6. $\square$
6. Rademacher Cotype.

To proof of Theorem 1.4 relies on a different version of the construction of Section 1. We consider a set $T$, such that on $T$ we have a sequence $(d_j)_{j \geq 0}$ of distances. We assume that this sequence is decreasing, i.e. $d_{j+1}(s, t) \leq d_j(s, t)$ for $s, t \in T$. We denote by $B_j(x, a)$ the ball for $d_j$. We assume that for each $j \geq 0$, each subset $S$ of $T$, we are given a quantity $F_j(S)$ that is increasing in $S$. We assume that the sequence $F_j$ of functionals is decreasing, i.e. $F_{j+1}(S) \leq F_j(S)$ for $S \subset T$, $j \geq 0$.

We assume that for certain $\gamma > 0$, $r \geq 4$, $K_2 > 0$, the following condition (that is a substitute for (2.4)) holds.

\[(6.1) \text{ Consider } j \geq 0, k \geq 0, t \in T, N \geq 2, \text{ and points } (t_\ell)_{\ell \leq N} \text{ in } B_j(t, r^{-k}).\]

Assume that
\[\ell \neq \ell' \text{ implies } d_j(t_\ell, t_{\ell'}) \geq r^{-k-1}.\]

Consider sets $T_\ell \subset B_j(t_\ell, r^{-k-2})$. Then, whenever

\[(6.1.a) \quad r^{-2j\gamma} \leq \frac{r^{-k}}{\sqrt{\log N}}\]

we have

\[(6.1.b) \quad F_j(\bigcup_{\ell \leq N} T_\ell) \geq \frac{1}{K_2} r^{-k} \sqrt{\log N} + \min_{\ell \leq N} F_j(T_\ell).\]

What this means is that each distance $d_j$ satisfies (2.4) provided one considers only values of $N$ that are not too large, i.e. $\sqrt{\log N} \leq r^{2j\gamma-k}$.

**Theorem 6.2.** There exists a number $H$, depending only on $\gamma, K_2$, such that whenever the diameter of $T$ for $d_0$ is at most 1, (6.1) holds and $F_0(T) \leq 1/H$, we can find an increasing sequence $(C_k)_{k \geq 0}$ of finite partitions of $T$, such that the diameter of any $C \in C_k$ for $d_k$ is at most $2r^{-k}$, and a probability measure $\mu$ on $T$ such that

\[\forall x \in T, \quad \sum_{k \geq 0} r^{-k} \left( \log \frac{1}{\mu(C_k(x))} \right)^{1/2} \leq H.\]

**Proof.** The construction of the partitions goes by induction over $k$.

We assume that, for each $C \in C_k$,

\[(6.2) \quad i(C) \leq k.\]

Together with each $C \in C_k$, we will construct an index $i(C)$ such that

\[(6.3) \quad \text{there exists } t \in T \text{ with } C \subset B_{i(C)}(t, r^{-k}).\]
We will also construct an index $\ell(C) \geq 1$ and a number $a(C)$. The basic property of $\ell$ is that if $C, C' \in \mathcal{C}_k$ are contained in the same element of $\mathcal{C}_{k-1}$, $C \neq C'$, then $\ell(C) \neq \ell(C')$. The properties of $a(C)$ are that

\begin{equation}
\forall t \in C, \quad F_i(C \cap B_i(t, r^{-k-1})) \leq a(C) + r^{-k}
\end{equation}

where $i = i(C)$, and

\begin{equation}
\left( \frac{L}{2} \right)^{k-i} r^{-i} \leq F_i(C) - a(C).
\end{equation}

There $L$ is a parameter that will be adjusted later.

To start the construction, we set $C_0 = \{T\}$, $i(T) = 0$, $\ell(T) = 1$, $a(T) = 0$. Then (6.4) holds since we may assume $H \geq 1$.

Suppose now that $C_k$ has been constructed. Consider a set $C \in \mathcal{C}_k$, and set $i = i(C)$. We show how to break $C$ into pieces of $\mathcal{C}_{k+1}$. For that purpose, we perform into $C$ the construction of Theorem 2.1, for the distance $d_i$. Thus, we choose by induction on $\ell$ points $y_\ell$ such that if we set $G_0 = C$, and, for $\ell \geq 1$

\[ G_\ell = C \setminus \bigcup_{m < \ell} B_i(y_m, r^{-k-1}) \]

then $y_\ell \in G_\ell$ and

\begin{equation}
F_i(B_i(y_\ell, r^{-k-2}) \cap C) \geq \sup \{ F_i(B_i(y, r^{-k-2}) \cap C); y \in G_\ell \} - \varepsilon_k.
\end{equation}

where $\varepsilon_k > 0$ will be determined later.

The construction continues as long as possible. We consider the partition of $C$ into the sets

\begin{equation}
V_\ell = G_\ell \cap B_i(y_\ell, r^{-k-1}).
\end{equation}

We set $\ell(V_\ell) = \ell$. Thus

\begin{equation}
\ell(V) \neq \ell(W).
\end{equation}

For any two sets $V, W$ of $\mathcal{C}_{k+1}$ that are contained in $C$, $V \neq W$, we have $\ell(V) \neq \ell(W)$.

We set

\[ T_\ell = B_i(y_\ell, r^{-k-2}) \cap C. \]

We observe from (6.6) that

\begin{equation}
\forall y \in G_\ell, \quad F_i(B_i(y, r^{-k-2}) \cap C) \leq F_i(T_\ell) + \varepsilon_k.
\end{equation}

In particular

\begin{equation}
F_i(T_m) \leq \min F_i(T_\ell) + \varepsilon_k.
\end{equation}
Assume now that $m$ satisfies

$$r^{-2i\gamma} \leq \frac{r^{-k}}{\sqrt{\log m}}. \quad (6.11)$$

Then, by (6.3), (6.10) and (6.1.b), we get

$$F_i(T_m) \leq F_i(C) - \frac{1}{K_2} r^{-k} \sqrt{\log m} + \varepsilon_k. \quad (6.12)$$

Suppose now that the construction of the sets $V_\ell$ has stopped at $\ell = p$ (so that $C = \bigcup_{\ell \leq p} V_\ell$). We show that

$$r^{-2i\gamma} \leq \frac{r^{-k}}{\sqrt{\log p}}.$$

Indeed, otherwise the largest $m < p$ for which (6.11) holds satisfies

$$r^{-2i\gamma} \geq \frac{r^{-k}}{2\sqrt{\log m}},$$

so that

$$r^{-k} \sqrt{\log m} \geq \frac{1}{2} r^{2i\gamma - 2k}. $$

Plugging into (6.12), we get, provided $\varepsilon_k \leq F_0(T)$,

$$\frac{1}{2K_2} r^{2i\gamma - 2k} \leq F_i(C) + \varepsilon_k \leq F_0(T) + \varepsilon_k \leq 2F_0(T). \quad (6.13)$$

From (6.5), and assuming, as we may, that $H \geq 1$, we get

$$\left(\frac{L}{2}\right)^{k-i} r^{-i} \leq F_i(C) - a(C) \leq F_0(T) \leq \frac{1}{H} \leq 1. \quad (6.14)$$

We realize now that if we have selected $L = 2r^{\gamma - i}$, from (6.14) we have $r^{k-i} \leq 1$, so that $r^{\gamma-i-k} \geq 1$. Substituting in (6.13) yields $F_0(T) \geq 1/4K_2$, but this is impossible if we assume, as we may, that $H > 4K_2$.

Thus, we have shown that (6.11), and hence (6.12) holds for all $m \leq p$. We set

$$d(V_m) = \sup \{F_i(V_m \cap B_i(y, r^{-k-2})) : y \in V_m\}. \quad (6.15)$$

Combining (6.9) (used for $\ell = m$) and (6.12) yields

$$d(V_m) \leq F_i(C) - \frac{1}{K_2} r^{-k} \sqrt{\log m} + 2\varepsilon_k. \quad (6.16)$$

Case a. We have

$$F_i(V_m) - d(V_m) \geq \left(\frac{L}{S}\right)^{k+1-i} r^{-i}. \quad (6.17)$$
We set $i(V_m) = i$, $a(V_m) = d(V_m)$. Thus, by definition of $d(V_m)$, (6.4) holds for $V_m$ rather than $C$, $k + 1$ rather than $k$. Since $V_m \subset B_i(y_m, r^{-k-1})$, by (6.4) we have $F_i(V_m) \leq a(C) + r^{-k}$, so that combining with (6.16),
\begin{equation}
F_i(V_m) + a(V_m) + \frac{1}{K_2} r^{-k} \sqrt{\log \ell(V_m)} \leq F_i(C) + a(C) + 2\varepsilon_k + r^{-k}
\end{equation}
where $i = i(C)$.

**Case b.** (6.17) fails. From (6.5) we have
\begin{equation}
F_i(V_m) - d(V_m) \leq \frac{L}{2} (F_i(C) - a(C)).
\end{equation}
We set
\[ i(V_m) = k + 1, \quad a(V_m) = F_{k+1}(V_m) - r^{-k-1} \]
so that (6.3), (6.4), (6.5) will hold at level $k + 1$ for $V_m$. We have, combining (6.19) with (6.16) that
\[ F_i(V_m) - \frac{L}{2} (F_i(C) - a(C)) \leq d(V_m) \leq F_i(C) - \frac{r^{-k}}{K_2} \sqrt{\log m} + 2\varepsilon_k \]
so that
\[ F_i(V_m) \leq \left( 1 + \frac{L}{2} \right) F_i(C) - \frac{L}{2} a(C) - \frac{r^{-k}}{K_2} \sqrt{\log m} + 2\varepsilon_k. \]
Since, by (6.4), $F_i(V_m) \leq a(C) + r^{-k}$ adding $(1 + L) F_i(V_m)$ to the right hand side of this inequality and $(1 + L) (a(C) + r^{-k})$ to the left hand side we have
\[ 2F_i(V_m) \leq F_i(C) + a(C) - \frac{2r^{-k}}{K_2(2 + L)} \sqrt{\log m} + \frac{4\varepsilon_k}{2 + L} + r^{-k} \]
which implies, since $F_{k+1} \leq F_i$,\begin{equation}
F_{k+1}(V_m) + a(V_m) + \frac{2r^{-k}}{K_2(2 + L)} \sqrt{\log \ell(V_m)} \leq F_i(C) + a(C) + r^{-k-1} + \frac{4\varepsilon_k}{2 + L} + r^{-k}.
\end{equation}
The construction is now complete.

It follows from (6.20), (6.18) that for any $x \in T$ we have
\[ F_{i_k+1}(C_{k+1}(x)) + a(C_{k+1}(x)) + \frac{2r^{-k}}{K_2(2 + L)} \sqrt{\log \ell(C_{k+1}(x))} \leq F_{i_k}(C_k(x)) + a(C_k(x)) + 2r^{-k} + 2\varepsilon_k \]
where, for simplicity, we set $i_k = i(C_k(x))$.

By summation of the relations (6.21) over $k \geq 0$, we get (provided $\sum_{k \geq 0} \varepsilon_k \leq 1$)
\[ \sum_{k \geq 1} r^{-k} \sqrt{\log \ell(C_k(x))} \leq K(r, \gamma). \]
The proof is then completed repeating the argument of Theorem 2.2. \qed

We now start the proof of Theorem 1.4. We first observe the following consequence of Corollary 5.4.
Corollary 6.2. There exists a number $r_0$ and a constant $K$, such that, if $r \geq r_0$, whenever we consider elements $t_1, \ldots, t_N$ of $\mathbb{R}^M$, such that
\begin{align}
\forall \ell, \ell' \leq N, \quad \|t_\ell - t_{\ell'}\|_\infty &\leq \frac{2r^{-k}}{\sqrt{\log N}} \tag{6.22} \\
\forall \ell, \ell' \leq N, \quad \|t_\ell - t_{\ell'}\|_2 &\geq r^{-k-1} \tag{6.23}
\end{align}
and whenever we consider sets $T_\ell \subset B_2(t_\ell, r^{-k-2})$, we have
\[ b(\bigcup_{\ell \leq N} T_\ell) \geq \frac{1}{K} r^{-k-1} \sqrt{\log N} + \min_{\ell \leq N} b(T_\ell). \]

We now fix $\gamma > 1$, and we fix $r \geq r_0$ such that $m = r^{2\gamma}$ is an integer. For $k \geq 0$, we consider the map $U_k$ from $[0, 1]$ to $[0, m^{-k}]^m$, defined as follows. We have $U_k(x) = (f^k_\ell(x))_{\ell \leq m^k}$, such that
\begin{align*}
    f^k_\ell &= m^{-k} \quad \text{if } \ell m^{-k} \leq x \\
    f^k_\ell(x) &= x - (\ell - 1)m^{-k} \text{ if } (\ell - 1)m^{-k} < x \leq \ell m^{-k} \\
    f^k_\ell(x) &= 0 \text{ if } x \leq (\ell - 1)m^{-k}.
\end{align*}
We consider the map $V_k$ from $X = [0, 1]^M$ to $[0, m^{-k}]^M \times m^k$ that is obtained by applying $U_k$ to each coordinate. On $X$, we consider the distance $d_j$ given by
\[ d_j(x, y) = \|V_j(x) - V_j(y)\|_2. \]
It should be obvious that the sequence $d_j$ decreases.

For a subset $S$ of $X$, we set
\[ F_j(S) = b(U_j(S)). \]
We now prove the crucial fact that the sequence $F_j$ decreases. We have to show that $b(U_{j+1}(S)) \leq b(U_j(S))$. It should be obvious that $U_{j+1}(S)$ is deduced from $U_j(S)$ the way $U_1(S)$ is deduced from $S$. Thus it suffices to show that $b(U_1(S)) \leq b(S)$. Consider two independent Bernoulli sequences $(\varepsilon_i)_{i \leq M}$, $(\varepsilon_{i\ell})_{i \leq M, \ell \leq m}$ that are independent of each other. Then, writing $t = (t_i)_{i \leq M}$,
\begin{align}
    b(U_1(S)) &= E \sup_{t \in S} \sum_{i \leq m} \varepsilon_i f^1_\ell(t_i) \nonumber \\
    &= E \sup_{t \in S} \sum_{i \leq m} \varepsilon_i \varepsilon_{i\ell} f^1_\ell(t_i). \tag{6.24}
\end{align}
The definition of $f^1_\ell$ shows that
\[ \sum |f^1_\ell(x) - f^1_\ell(y)| \leq |x - y|. \]
Thus, for all choices of \( \varepsilon_{i\ell} \), we have

\[
| \sum_{\ell \leq m} \varepsilon_{i\ell} f^1_{i\ell}(x) - \sum_{\ell \leq m} \varepsilon_{i\ell} f^1_{i\ell}(x) | \leq | x - y |.
\]

In other words, conditionally on the choice of \((\varepsilon_{ij})\) the map \( h_i : x \rightarrow \sum_{\ell \leq m} \varepsilon_{i\ell} f^1_{i\ell}(x) \) is a contraction, and \( h_i(0) = 0 \). Using part a) of Theorem 2.1 of [T5] conditionally on \( \varepsilon_{i\ell} \), we see that

\[
b(U_1(S)) \leq E \sup_{t \in S} \sum_{i \leq M} \varepsilon_i t_i = b(S).
\]

Consider now a subset \( T' \) of \( \mathbb{R}^M \). It is simple to see (using Kinchine’s inequality) that the \( \ell_2 \)-diameter of \( T' \) is \( \leq K b(T') \). Consider \( T = (b(T')L)^{-1} T' \), where \( L \) is a parameter to be adjusted later. If \( L \) is large enough, the \( \ell^2 \) diameter of \( T \) is \( \leq 1 \), and there is then no loss of generality to assume \( T \subset X \).

It follows from Corollary 6.2 (applied to \( V_j(T) \)) that condition (6.1) holds. Indeed we have

\[
\| V_j(x) - V_j(y) \|_\infty \leq m^{1-j} = r^{-2\gamma j}.
\]

Since, by choosing \( L \) appropriately, we can ensure that \( b(T) = F_0(T) \leq 1/H \), (where \( H \) occurs in Theorem 6.2) we see that the conclusion of this Theorem holds. We observe that,

\[
\delta_k(x,y) = \left( \sum_{i \leq M} |x_i - y_i|^2 \wedge r^{-4\gamma k} \right)^{1/2} \leq 2d_k(x,y),
\]

since obviously one can find \( \ell \leq m^k \) for which

\[
| f^k_{\ell}(x_i) - f^k_{\ell}(y_i) | \geq \frac{1}{2} |x_i - y_i| \wedge m^{-k}.
\]

We can then appeal to Proposition 4.4 with \( p \) such that \( \gamma(2-p) = 1 \) to see that \( T' \subset U + K(\gamma)B_p \) where \( \gamma_{1/2}(U) \leq K(\gamma) \). This completes the proof of Theorem 1.4.

We now turn to the proof of Theorem 1.5. Since, for a sequence \( f_i \in C(L) \), we have

\[
E \| \sum \varepsilon_i f_i \|_\infty \leq \| \sum |f_i| \|_\infty,
\]

we have \( \| V \|_{2,1} \leq C^*_2(V) \). Since (by comparison of Bernoulli and Gaussian averages) we have

\[
E \| \sum \varepsilon_i f_i \| \leq KE \| \sum g_i f_i \|
\]

we have \( C^*_2(V) \leq K C^*_2(V) \). Thus, the significant part of Theorem 1.4 is the right-hand inequality. It is routine to reduce to the case where \( C(L) = \ell^N_\infty \) (see e.g. [T2]). Consider a sequence \((f_j)_{j \leq M}\) of \( \ell^N_\infty \). Consider the set

\[
T' = \{ t_1, \ldots, t_N \} \subset \mathbb{R}^M.
\]
given by $t_\ell = (t_\ell(j))_{j \leq M}$, where $t_\ell(j) = f_j(\ell)$.

Consider $T = T' \cup \{0\}$. It should be clear that

$$b = b(T) = E \max(0, \sup_{\ell \leq N} \sum_{j \leq M} \varepsilon_j f_j(\ell)) \leq E \| \sum_{j \leq M} \varepsilon_j f_j \|_\infty.$$  

By Theorem 1.4, we can write $T \subset U + KkB_{3/2}$, where $\gamma_{1/2}(U) \leq Kb$. Since $0 \in T$, we can write $0 = u + v$, where $u \in U$, $v \in KkB_{3/2}$. Thus $u = -v \in KkB_{3/2}$. If we replace $U$ by $U - u$, $KkB_{3/2}$ by $KkB_{3/2} - v \subset 2KkB_{3/2}$, we see that we can assume that $0 \in U$.

For $\ell \leq N$, we can write $t_\ell = u_\ell + v_\ell$, where $u_\ell \in U$, $v_\ell \in KkB_{3/2}$. We consider the elements $f_1^1, f_2^2$ of $\ell^N_\infty$, where, for $\ell \leq N$,

$$f_1^1(\ell) = u_\ell(j); \quad f_2^2(\ell) = v_\ell(j).$$

Thus $f_j = f_1^1 + f_2^2$. Since $v_\ell \in KkB_{3/2}$ for each $\ell \leq N$, we have

(6.25) \[ \| (\sum_{j \leq M} |f_j^2|^{3/2})^{2/3} \|_\infty \leq Kb. \]

The key point is a theorem of Maurey (see [P] for a simple proof), according to which for $1 \leq p < 2$, (and in particular $p = 3/2$) we have $\|V\|_p \leq K(p)\|V\|_1$. Thus (6.25) implies

(6.26) \[ \left( \sum_{j \leq M} \| f_j^2 \|^2 \right)^{1/2} \leq Kb\|V\|_2. \]

On the other hand, since $u_\ell \in U$ for each $\ell \leq N$

$$E \| \sum_{j \leq M} g_j f_j^1 \|_\infty = E \sup_{\ell \leq N} \| \sum_{j \leq M} g_j f_j^1(t_\ell) \|$$

$$= E \sup_{\ell \leq N} \| \sum_{j \leq M} g_j u_\ell(j) \|$$

$$\leq K\gamma_{1/2}(U) \leq Kb.$$  

where the first inequality uses the easy well known fact that

$$E \sup_{u \in U} \| \sum_{j \leq M} g_j u(j) \| \leq 2E \sup_{u \in U} \sum_{j \leq M} g_j u(j)$$

whenever $0 \in U$.

Thus, we have

$$\left( \sum_{j \leq M} \| f_j^1 \|^2 \right)^{1/2} \leq K\|f \|_2.$$  

The result follows by combining with (6.26) and using the triangle inequality.
7. An application to a class of functions.

In this section, we prove the following, where \( \lambda \) denotes Lebesgue measure.

**Theorem 7.1.** Consider the class \( F_0 \) of functions on \([0, 1]\) that satisfy \( \int f \, d\lambda = 0, \int |f'| \, d\lambda \leq 1 \). Then \( \gamma_{1,2}(F) < \infty \).

This theorem could be proved using the methods of [T7]. These methods have however intrinsic limitations, and are unable to yield optimal results for the classes of functions on \([0, 1]^2\) considered in [T7]. This is apparently not the case of the approach based on Theorem 1.3 that we will present. While we could not solve any of the questions left open in [T7], this is apparently due to technical problems rather than to an incorrect approach. This is our main motivation for presenting the material of this section.

We start the proof of Theorem 7.1. We fix once and for all two numbers \( \delta, \theta \) such that

\[
1 < \delta < \frac{3}{2} \quad \text{and} \quad \theta > 0, \ (1 + \theta)\delta < 2.
\]

We consider the function \( \xi \) on \( \mathbb{R} \) such that \( \xi(0) = 0, \xi(x) = \xi(-x) \) and

\[
x \geq 0 \Rightarrow \xi'(x) = 1 - \frac{1}{2(1 + x)^\theta}.
\]

Thus, \( |\xi'(x)| \leq 1 \) and \( \xi(x) \leq |x| \).

We consider the functional

\[
\Xi(f) = \int_0^1 \xi(f') \, d\lambda.
\]

It is well defined on the set of functions for which \( f' \) exists a.e. and is integrable (since \( \xi(x) \leq |x| \)).

We will apply Theorem 2.1 (together with the remark following its proof) with \( r = 8 \) and with the functionals (defined on the class \( F \) of functions that satisfy \( \int |f'| \, d\lambda \leq 1 \))

\[
\varphi_k = \inf \{ \Xi(g); \|f - g\|_2 \leq 2r^{-k} \}.
\]

Consider \( f_0 \in F \), and \((f_i)_{i \leq N}\) in \( F \) such that

\[
\forall i, 1 \leq i \leq N, \|f_0 - f_i\|_2 \leq r^{-k};
\]

\[
\forall 1 \leq i, j \leq n, i \neq j, \|f_i - f_j\|_2 \geq r^{-k-1}.
\]

The key point is to prove that

\[
\sup \varphi_{k+2}(f_i) \geq \varphi_k(f_0) + \frac{r^{-2k}}{K} - (\log N)^2 - K r^{-2k/3}.
\]
For each $i \leq N$, consider $\mathcal{F}_i$ such that $\|\mathcal{F}_i - f_i\|_2 \leq 2 \cdot r^{-k-2}$ and

$$\Xi(\mathcal{F}_i) \leq \varphi_{k+2}(f_i) + \varepsilon$$

where $\varepsilon$ will be determined later.

Before going into details, we give the overall idea. The method to prove (7.4) is to show that when $\log N \geq Kr^{2k/3}$, either of the following occurs.

**Case a.** For some $i \leq N$, we can find a function $g_i \in \mathcal{F}$ such that $\|f_i - g_i\|_2 \leq \frac{1}{2} r^{-k}; \quad \Xi(g_i) \leq \Xi(\mathcal{F}_i) - \frac{r^{-2k}}{K} (\log N)^2$.

**Case b.** For all $i \leq N$, we can find a function $h_i \in \mathcal{F}$ such that $\|f_i - h_i\|_2 \leq r^{-k-2}$ and such that $h_i$ “does not depend on too many parameters”.

The functions $h_i$ satisfy $\|f_i - h_i\|_2 \leq 3r^{-k-2}$, so that, since $\|f_i - f_j\|_2 \geq r^{-k-1}$, we have $\|h_i - h_j\|_2 \geq 2r^{-k-2}$. But, since the functions $h_i$ depend on few parameters, it is impossible to have $N$ of them. Thus case a must occur. Now, since $\|f_0 - g_i\|_2 \leq 2r^{-k-2} + \frac{3}{2} r^{-k} \leq 2r^{-k}$ we have by definition of $\varphi_k(f_0)$ and (7.5) that

$$\varphi_k(f_0) \leq \Xi(g_i) \leq \Xi(\mathcal{F}_i) - \frac{r^{-2k}}{K} (\log N)^2$$

$$\leq \varphi_{k+2}(f_i) + \varepsilon - \frac{r^{-2k}}{K} (\log N)^2$$

$$\leq \varphi_{k+2}(f_i) - \frac{r^{-2k}}{2K} (\log N)^2$$

with the choice $\varepsilon = \frac{r^{-2k}}{2} (\log N)^2$, and this proves (7.4).

The technical part of the construction is contained in the following lemma, the proof of which will be delayed in order not to break the flow of the argument.

**Lemma 7.2.** Consider an interval $I \subset [0, 1]$, $f \in \mathcal{F}$, and the function $g$ on $I$ that is obtained by linear interpolation of the values of $f$ at the endpoints of $I$. Then

(7.6) \[ \int_I (f - g)^2 d\lambda \leq 4|I|^3 m_I(f')^2 \]

and

(7.7) \[ \int_I (f - g)^2 d\lambda \leq K|I|^2 (1 + m_I(f'))^{1+\theta} \int_I (\xi(f') - \xi(g')) d\lambda \]

where $m_I(f') = |I|^{-1} \int_I |f'| d\lambda$.

Consider $f = \mathcal{F}_i$, where $i \leq N$ is fixed. We start an approximation procedure that will either lead to the construction of the function $h_i$ of case b, or to the proof...
that (7.5) occurs. Consider the parameter $L$ to be adjusted later, and the largest integer $\ell_0$ such that $2^{\ell_0} \leq L^{-1} \log N$. For $\ell \geq 0$, we denote by $D_\ell$ the dyadic partition of $[0, 1]$ by intervals of length $2^{-\ell}$.

We construct families $(I_\ell)_{\ell \geq \ell_0}$ of dyadic intervals of $[0, 1]$ as follows. First, we consider the family $I_{\ell_0}$ of those intervals $I \in D_{\ell_0}$ that satisfy

$$m_I(f') = |I|^{-1} \int_I |f'| d\lambda \leq 1.$$ 

Having constructed $I_{\ell_0}, \ldots, I_{\ell-1}$, we define $I_\ell$ as the family of those intervals $I$ of $D_\ell$ that are not contained in any interval of $I_{\ell_0}, \ldots, I_{\ell-1}$, and that satisfy

$$m_I(f') \leq 2^{(\ell-\ell_0)\delta}. \quad (7.8)$$

We observe that if $I \in I_\ell$ ($\ell > \ell_0$), the unique interval $I' \in D_{\ell-1}$ that contains $I$ must satisfy $m_{I'}(f') > 2^{(\ell-\ell_0-1)\delta}$ (otherwise $I' \in I_{\ell-1}$.) Thus

$$\int_{I'} |f'| d\lambda \geq 2^{-\ell+1+\delta(\ell-\ell_0-1)}.$$ 

Since $\int_0^1 |f'| d\lambda \leq 1$, there can be at most $2^{\ell-1-\delta(\ell-\ell_0-1)}$ such intervals. Thus the cardinality $M_\ell$ of $I_\ell$ satisfies

$$M_\ell \leq 2^{\ell-\delta(\ell-\ell_0-1)} = 2^{\ell_0 - (\delta-1)(\ell-\ell_0) + \delta}. \quad (7.9)$$

Let us observe that the total number $W$ of ways the sets $I_\ell$ can be chosen satisfies, by crude estimates

$$W \leq \prod_{\ell \geq \ell_0} \left( \frac{2\ell}{M_\ell} \right) \leq \prod_{\ell \geq \ell_0} \left( \frac{e2\ell}{M_\ell} \right)^{M_\ell}$$

$$= \exp \left( \sum_{\ell \geq \ell_0} M_\ell \log \left( \frac{e2\ell}{M_\ell} \right) \right)$$

$$\leq \exp(K2^{\ell_0}) \leq \exp \left( \frac{K}{L} \log N \right), \quad (7.10)$$

using (7.9) and the fact that the function $x \log \left( \frac{e2\ell}{x} \right)$ increases for $x \leq 2\ell$.

A second observation is that, if $I$ is an interval of $I_\ell$, and if $g$ denotes the function on $I$ that linearly interpolates the values of $f$ at the endpoints of $I$, we have, by (7.6), (7.8)

$$d_I =: \int (f - g)^2 d\lambda \leq 4 \cdot 2^{-3\ell} 2^{2(\ell-\ell_0)\delta} \leq 4 \cdot 2^{-3\ell_0} \quad (7.11)$$
since $\delta \leq 3/2$, and also, by (7.7), (7.8)

$$
\begin{align*}
(7.12) \quad d_I &= \int_I (f-g)^2 d\lambda \\
&\leq K 2^{-2\ell_0} \int_I (\xi(f') - \xi(g')) d\lambda \\
&\leq K 2^{-2\ell_0} \int_I (\xi(f') - \xi(g')) d\lambda
\end{align*}
$$

since $\delta(1+\theta) \leq 2$.

**Case 1.** Assume that the sum of the quantities $d_I$ over all possible intervals of $\mathcal{I} = \bigcup_{\ell \geq \ell_0} \mathcal{I}_\ell$ is $> r^{-2(k+2)}$. Then, assuming $\log N \geq 4 L r^{-2k/3}$, we have

$$
4r^{-3\ell_0} \leq 2^5 L^3 (\log N)^{-3} \leq \frac{1}{2} r^{-2k},
$$

so that by (7.11) we can find a subset $\mathcal{I}'$ of $\mathcal{I}$ such that

$$
(7.13) \quad r^{-2(k+2)} < \sum_{I \in \mathcal{I}'} d_I \leq r^{-2(k+2)} + 4r^{-3\ell_0} \leq r^{-2k}.
$$

Consider the function $g$ that coincides with $f$ at the end points of all the intervals of $\mathcal{I}$, as well as in all the intervals $I$ for $I \notin \mathcal{I}'$, and is linear in all the intervals of $\mathcal{I}'$. By (7.13) and summation of the relations (7.12), we get

$$
\|f - g\|_2 \leq r^{-k}
$$

(7.14)

$$
(7.15) \quad r^{-2(k+2)} \leq K 2^{-2\ell_0} (\Xi(f) - \Xi(g)).
$$

From the choice of $\ell_0$, this relation implies

$$
\Xi(f) - \Xi(g) \geq \frac{1}{4KL^2 r^2} r^{-2k} (\log N)^2.
$$

Combined with (7.14) this shows that if case 1 occurs for any $f = \overline{f}_i$, $i \leq N$ then case a above occurs and the proof is finished.

**Case 2.** Assume that case 1 does not occur, and consider the function $h_i$ that coincides with $f = \overline{f}_i$ at the endpoints of the intervals of $\mathcal{I}$ and linearly interpolates $f$ between any two consecutive endpoints. Then $\|f - h_i\|_2 \leq r^{-2(k+2)}$, i.e.

$$
\|\overline{f}_i - h_i\|_2 \leq r^{-(k+2)}.
$$

We prove that it is impossible that case 2 occurs for all $i \leq N$. First we observe that by (7.10), we can choose $L$ large enough that $W \leq \sqrt{N}$. Thus, we can find a collection $H$ of at least $\sqrt{N}$ indices $i$, such that for each $i \in H$, each $\ell$, the family $\mathcal{I}_\ell = \mathcal{I}_{\ell,i}$ constructed from $f_i$ does not depend on $i$. From (7.9), the total number of intervals in $\bigcup_{\ell \geq \ell_0} \mathcal{I}_\ell$ is at most $K 2^{\ell_0} \leq \frac{K}{L} \log N$. Thus the functions $(h_i)_{i \in H}$ all belong to a certain subspace of $C([0,1])$ of dimension $\leq \frac{K}{L} \log N$. Since...
as already observed, the functions $h_i$ satisfy $\|h_i - h_j\|_2 \geq r^{-k-2}$, $\|h_i - h_j\|_2 \leq 2r^{-k}$ ($i \neq j, i, j \in H$) this is impossible for $L$ large enough by standard volume estimates. This completes the proof of Theorem 7.1.

**Proof of Lemma 7.2.** We denote by $g'$ the constant equal to the derivative of $g$ on $I$.

Since

$$\forall x \in I, \quad |f(x) - g(x)| \leq \int_I |f' - g'| d\lambda$$

we have

$$\int_I |f - g|^2 d\lambda \leq |I|(\int_I |f' - g'| d\lambda)^2. \quad (7.16)$$

Since

$$\int_I |f' - g'| d\lambda \leq |I|(|m_I(f') + |g'|) \leq 2|I|m_I(f')$$

this first yields (7.6). To prove (7.7), it suffices to consider the case

$$\int_I (\xi(f') - \xi(g')) d\lambda \leq |I|(1 + m_I(f'))^{-1+\theta}. \quad (7.17)$$

Consider the function

$$\eta(x) = \xi(x) - \xi(g') - (x - g')\xi'(g').$$

By convexity of $\xi$, we have $\eta(x) \geq 0$, and, since $\int_I f'd\lambda = \int_I g'd\lambda$, we have

$$\int_I \eta(f') d\lambda = \int_I (\xi(f') - \xi(g')) d\lambda. \quad (7.18)$$

The main idea is now that, for numbers $\alpha, \beta$, we have

$$\int_I |f' - g'| d\lambda \leq \alpha|I| + \beta\int_I \eta(f') d\lambda, \quad (7.19)$$

provided

$$\forall y, \quad |y - g'| \leq \alpha + \beta\eta(y). \quad (7.20)$$

We observe that, by convexity of $\eta$, given $y_0 > 0$, we have

$$\eta(y) \geq (y - y_0)\eta'(y_0) + \eta(y_0) \geq (y - y_0)\eta'(y_0),$$

so that, for $y \geq g'$, and since $\eta'(y_0) \geq 0$,

$$|y - g'| = y - g' = y - y_0 + y_0 - g' \leq (y_0 - g') + \frac{1}{y_0} \eta(y).$$
A similar consideration when \( y \leq g' \) shows that (7.19) will hold for

\[
\alpha = \max(y_0 - g', g' - y_1); \quad \beta = \max\left(\frac{1}{\eta'(y_0)}, \frac{1}{|\eta'(y_1)|}\right)
\]

when \( y_1 < g' < y_0 \).

We now take \( y_0 = g' + a, \ y_1 = g' - a \), where

\[
a = \left(\frac{(1 + m_I(f'))^{1+\theta}}{|I|} \int_I \eta(f')d\lambda\right)^{1/2}.
\]

We note that \( a \leq 1 \) by (7.17), (7.18). Assuming for definiteness \( g' > 0 \), we see that we get, by definition of \( \xi \),

\[
\beta = \frac{1}{\eta'(y_0)} = \frac{1}{\xi'(g' + a) - \xi'(g')} = 2 \left(\frac{1}{(1 + g')^{\theta}} - \frac{1}{(1 + g' + a)^{\theta}}\right)^{-1}
\]

\[
\leq K\frac{(1 + g' + a)^{\theta+1}}{a}
\]

\[
\leq K\frac{(1 + m_I(f'))^{1+\theta}}{a}.
\]

Substituting in (7.19) yields, by definition of \( a \),

\[
\int_I |f' - g'|d\lambda \leq a|I| + \frac{K}{a} (1 + m_I(f'))^{1+\theta} \int_I \eta(f')d\lambda
\]

\[
\leq K(|I|(1 + m_I(f'))^{1+\theta} \int_I \eta(f')d\lambda)^{1/2}
\]

and in view of (7.16) this completes the proof. \( \Box \)
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