JACOB’S LADDERS, REVERSE ITERATIONS AND NEW INFINITE SET OF $L_2$-ORTHOGONAL SYSTEMS GENERATED BY THE RIEMANN $\zeta\left(\frac{1}{2} + it\right)$-FUNCTION

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Abstract. It is proved in this paper that continuum set of $L_2$-orthogonal systems generated by the Riemann zeta-function on the critical line corresponds to every fixed $L_2$-orthogonal system on a fixed segment. This theorem serves as a resource for new set of integrals not accessible by the current methods in the theory of the Riemann zeta-function.

Dedicated to the 100th anniversary of G.H. Hardy’s fundamental theorem: the function $\zeta\left(\frac{1}{2} + it\right)$ has an infinite set of zeros, \[1\].

1. Introduction

1.1. In this paper we obtain new properties of the signal

$$Z(t) = e^{i\vartheta(t)}\zeta\left(\frac{1}{2} + it\right), \tag{1.1}$$

$$\vartheta(t) = -\frac{t}{2}\ln\pi + \text{Im} \ln \Gamma\left(\frac{1}{4} + \frac{t}{2}\right),$$

which is generated by the Riemann zeta-function. In connection with (1.1) we have introduced (see \[5\], (9.1), (9.2)) the formula

$$\tilde{Z}^2(t) = \frac{d\varphi_1(t)}{dt}, \tag{1.2}$$

where

$$\tilde{Z}^2(t) = \frac{Z^2(t)}{2\Phi'[\varphi(t)]} = \frac{|\zeta\left(\frac{1}{2} + it\right)|^2}{\omega(t)}, \tag{1.3}$$

$$\omega(t) = \left\{1 + \mathcal{O}\left(\frac{\ln\ln t}{\ln t}\right)\right\}\ln t.$$ 

The function $\varphi_1(t)$ that we call Jacob’s ladder (see our paper \[2\]) according to the Jacob’s dream in Chumash, Bereishis, 28:12, has the following properties:

(a) $$\varphi_1(t) = \frac{1}{2}\varphi(t),$$

(b) function $\varphi(t)$ is solution of the non-linear integral equation (see \[2\], \[5\])

$$\int_0^{\mu[T]} Z^2(t)e^{-\frac{\lambda(T)}{2}t}dt = \int_0^T Z^2(t)dt,$$

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where each admissible function $\mu(y)$ generates a solution

$$y = \varphi_\mu(T) = \varphi(T); \quad \mu(y) \geq 7y \ln y.$$  

**Remark 1.** The main reason to introduce Jacob’s ladders in [2] lies in the following: the Hardy-Littlewood integral (1918)

$$\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \, dt$$

has – in addition to the Hardy-Littlewood (and other similar) expression possessing unbounded errors at $T \to \infty$ – the following infinite set of almost exact expressions

$$\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \, dt = \varphi_1(T) \ln \varphi_1(T) + (c - \ln 2\pi)\varphi_1(T) + c_0 + O \left( \frac{\ln T}{T} \right), \quad T \to \infty,$$

where $c$ is the Euler’s constant, and $c_0$ is the constant from the Titchmarsh-Kober-Atkinson formula (see [7], p. 141).

**Remark 2.** Simultaneously with (1.4) we have proved that the following transcendental equation

$$\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \, dt = V(T) \ln V(T) + (c - \ln 2\pi)V(T) + c_0$$

has an infinite set of asymptotic solutions

$$V(T) = \varphi_1(T), \quad T \to \infty.$$  

**Remark 3.** The Jacob’s ladder $\varphi_1(T)$ can be interpreted by our formula (see [2], (6.2))

$$T - \varphi_1(T) \sim (1 - c)\pi(T); \quad \pi(T) \sim \frac{T}{\ln T}, \quad T \to \infty,$$

where $\pi(T)$ is the prime-counting function, as an asymptotic complement function to the function

$$(1 - c)\pi(T)$$

in the sense

$$\varphi_1(T) + (1 - c)\pi(T) \sim T, \quad T \to \infty.$$  

1.2. In the paper [3] we have proved that the following continuum set $S(T, 2l)$ of the systems

$$\left\{ |\tilde{Z}(t)|, |\tilde{Z}(t)| \cos \left[ \frac{\pi}{T}(\varphi_1(t) - T) \right], |\tilde{Z}(t)| \sin \left[ \frac{\pi}{T}(\varphi_1(t) - T) \right], \ldots, |\tilde{Z}(t)| \cos \left[ \frac{\pi}{T}n(\varphi_1(t) - T) \right], |\tilde{Z}(t)| \sin \left[ \frac{\pi}{T}n(\varphi_1(t) - T) \right], \ldots \right\},$$

$$t \in \left[ \frac{1}{T}, \frac{1}{T} + 2l \right];$$

$$\varphi_1 \left\{ \left[ \frac{1}{T}, \frac{1}{T} + 2l \right] \right\} = \left[ T, T + 2l \right]$$

is the set of orthogonal system on the segment

$$\left[ \frac{1}{T}, \frac{1}{T} + 2l \right]$$
for all 
\[ T \geq T_0[\varphi_1], \quad 2l \in \left(0, \frac{T}{\ln T}\right). \]

Next, in the paper \[4\] we have constructed corresponding continuum set of orthogonal systems generated by Jacobi’s polynomials.

In this paper we give essential generalization of above mentioned. Namely, to every fixed \( L_2 \)-orthogonal system
\[
\{f_n(t)\}_{n=1}^{\infty}, \quad t \in [0, 2l]
\]
we assign continuum set of \( L_2 \)-orthogonal systems
\[
\{F_n(t; T, k, l)\}_{n=1}^{\infty}, \quad t \in \left[\varphi_1(t), \varphi_2(t)\right], \quad T \to \infty, \quad k = 1, \ldots, k_0,
\]
\[ l = o\left(\frac{T}{\ln T}\right); \]
\[
\varphi_1 \left\{\frac{k}{T}, \frac{k}{T + 2l}\right\} = \left[\frac{k - 1}{T}, \frac{k - 1}{T + 2l}\right],
\]
where \( k_0 \in \mathbb{N} \) is an arbitrary fixed number.

2. RESULT

2.1. Let us remind that (see \[6\])
\[
\varphi^r_1(t): \quad \varphi^0_1(t) = t, \quad \varphi^1_1(t) = \varphi_1(t), \quad \varphi^2_1(t) = \varphi_1(\varphi_1(t)), \ldots
\]
The following Theorem holds true.

**Theorem.** For every fixed \( L_2 \)-orthogonal system
\[
\{f_n(t)\}_{n=1}^{\infty}, \quad t \in [0, 2l], \quad l = o\left(\frac{T}{\ln T}\right), \quad T \to \infty
\]
there is continuum set of \( L_2 \)-orthogonal systems
\[
\{F_n(t; T, k, l)\}_{n=1}^{\infty} = \left\{f_n(\varphi^k_1(t) - T)\prod_{r=0}^{k-1} Z[\varphi^r_1(t)]\right\}_{n=1}^{\infty}, \quad t \in \left[\frac{k}{T}, \frac{k}{T + 2l}\right],
\]
where
\[
\varphi_1 \left\{\frac{k}{T}, \frac{k}{T + 2l}\right\} = \left[\frac{k - 1}{T}, \frac{k - 1}{T + 2l}\right], \quad k = 1, \ldots, k_0,
\]
\[
\left[\frac{k}{T}, \frac{k}{T + 2l}\right] = \left[\frac{k - 1}{T}, \frac{k - 1}{T + 2l}\right], \quad k = 1, \ldots, k_0,
\]
\[
and k_0 \in \mathbb{N} \) is arbitrary number, i.e. the following formula is valid
\[
\int_{\frac{k}{T}}^{\frac{k}{T + 2l}} f_m(\varphi^k_1(t) - T)f_n(\varphi^k_1(t) - T)\prod_{r=0}^{k-1} Z^2[\varphi^r_1(t)]dt =
\]
\[
\left\{ 0, \quad m \neq n, \quad A_n, \quad m = n, \quad A_n = \int_0^{2l} f_n^2(t)dt. \right.
\]
Next, we have the following properties

\[ l = o \left( \frac{T}{\ln T} \right) \Rightarrow \]

\[ \frac{k}{|T, T + 2l|} = \frac{k}{T + 2l - T} = o \left( \frac{T}{\ln T} \right), \]

(2.5)

\[ \frac{k-1}{|[T, T + 2l]|} = \frac{k-1}{T - T + 2l} = (1 - c)\pi(T); \quad \pi(T) \sim \frac{T}{\ln T}, \]

(2.6)

\[ [T, T + 2l] \prec [\frac{1}{T}, \frac{1}{T + 2l}] \prec \cdots \prec [\frac{k}{T}, \frac{k}{T + 2l}] \prec \cdots, \]

where \( \pi(T) \) stands for the prime-counting function.

2.2.

Remark 4. We obtain from (2.2) by (1.3) that

\[ F_n(t; T, k, l) = f_n(\varphi^1(t) - T) \prod_{r=0}^{k-1} \left| \frac{\zeta \left( \frac{1}{2} + i \varphi^1_r(t) \right)}{\sqrt{\omega[\varphi^1_r(t)]}} \right|, \]

i. e. our formula (2.8) shows direct connection between the Riemann function

\[ \zeta \left( \frac{1}{2} + it \right) \]

and an arbitrary \( L_2 \)-orthogonal system

\[ \{f_n(t)\}_{n=1}^{\infty}, \quad t \in [0, 2l]. \]

Remark 5. Asymptotic behavior of the disconnected set (see (2.6), (2.7))

\[ \Delta(T, k, l) = \bigcup_{r=0}^{k} [\frac{r}{T}, \frac{r}{T} + 2l] \]

is as follows: if \( T \to \infty \), then the components of the set recedes unboundedly each from other and all together are receding to infinity. Hence, if \( T \to \infty \) the set behaves as one dimensional Friedmann-Hubble expanding universe.

2.3. Since (see (2.4))

\[ t \in \left[ \frac{1}{T}, \frac{1}{T + 2l} \right] \Rightarrow \]

\[ \varphi_1(t) \in [\varphi_1(T), \varphi_1(T + 2l)] = \left[ \frac{1}{T}, \frac{1}{T + 2l} \right] \Rightarrow \]

\[ \varphi_2(t) \in [\varphi_1(T), \varphi_1(T + 2l)] = \left[ \frac{k-1}{T}, \frac{k-1}{T + 2l} \right] \Rightarrow \]

\[ \vdots \]

we point-out the following...
Property 1. If

\[ t \in \left[ T, T + 2l \right], \ k = 1, \ldots, k_0 \]

then

\[ (2.10) \quad \phi_1^r(t) \in \left[ kT, k\dot{T} + 2l \right], \ r = 0, 1, \ldots, k \]

holds true for the arguments of the functions (see (2.2), (2.8))

\[ f_n(\phi_1^r(t) - T), |\tilde{Z}[\phi_1^r(t)]|, \omega[\phi_1^r(t)], \left| \zeta \left( \frac{1}{2} + i\phi_1^r(t) \right) \right|. \]

3. Examples

3.1. For the classical Fourier orthogonal system

\[ \left\{ 1, \cos \frac{\pi t}{l}, \sin \frac{\pi t}{l}, \ldots, \cos \frac{\pi nt}{l}, \sin \frac{\pi nt}{l} \right\} \]

(3.1)

\[ t \in [0, 2l], \ l = o \left( \frac{T}{\ln T} \right) \]

we have as corresponding (see (2.2), (2.8)) continuous set of orthogonal systems the following

\[ \left\{ \prod_{r=0}^{k-1} \left| \frac{\zeta \left( \frac{1}{2} + i\phi_1^r(t) \right)}{\sqrt{\omega[\phi_1^r(t)]}} \right|, \ldots, \right. \]

\[ \left( \prod_{r=0}^{k-1} \left| \frac{\zeta \left( \frac{1}{2} + i\phi_1^r(t) \right)}{\sqrt{\omega[\phi_1^r(t)]}} \right| \right) \cos \left( \frac{\pi}{l} n(\phi_1^r(t) - T) \right), \]

\[ \left( \prod_{r=0}^{k-1} \left| \frac{\zeta \left( \frac{1}{2} + i\phi_1^r(t) \right)}{\sqrt{\omega[\phi_1^r(t)]}} \right| \right) \sin \left( \frac{\pi}{l} n(\phi_1^r(t) - T) \right), \ldots \}

(3.2)

\[ t \in \left[ T, T + 2l \right], \ k = 1, \ldots, k_0, \ T \to \infty, \]

and, for example,

\[ k_0 = S = 10^{10^{34}}, \ S^S, \ldots \]

where \( S \) is the Skewes constant.

3.2. For the system of Jacobi’s functions

\[ \sqrt{(1 - t)^\alpha (1 + t)^\beta} P_n^{(\alpha, \beta)}(t), \ t \in [-1, 1], \ n = 0, 1, 2, \ldots ; \ \alpha, \beta > -1 \]

(3.3)

generated by the Jacobi’s polynomials \( P_n^{(\alpha, \beta)} \) we have that

\[ \int_{-1}^{1} (1 - t)^\alpha (1 + t)^\beta P_n^{(\alpha, \beta)}(t) P_m^{(\alpha, \beta)}(t) dt = 0, \ m \neq n, \]

(3.4)

\[ \int_{-1}^{1} (1 - t)^\alpha (1 + t)^\beta \left[ P_n^{(\alpha, \beta)}(t) \right]^2 dt = \]

\[ = \frac{2^{\alpha + \beta + 1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{2n + \alpha + \beta + 1} = A_n(\alpha, \beta). \]

Next, the substitution

\[ x = t - T - 1 \]
in (3.3) yields (see (3.4)) the formulae

\[
\int_T^{T+2} (2 + T - t)^\alpha (t - T)^\beta P_m^{(\alpha,\beta)}(t - T - 1)P_n^{(\alpha,\beta)}(t - T - 1)dt = 0, \ m \neq n, \ldots
\]

Consequently, the following continuum set (for each fixed pair \(\alpha, \beta > -1\)) of orthogonal systems

\[
\left\{ P_n^{(\alpha,\beta)}(t - T - 1)\sqrt{(T + 2 - \varphi^k_1(t))^{\alpha} (\varphi^k_1(t) - T)^\beta} \times \prod_{r=0}^{k-1} \left| \frac{\zeta \left( \frac{1}{2} + i \varphi^r_1(t) \right)}{\sqrt{|\varphi^r_1(t)|}} \right| \right\}_{n=0}^\infty,
\]

\(t \in [T, T + 2], \ T \to \infty, \ k = 1, \ldots, k_0.\)

corresponds to the Jacobi’s orthogonal system (3.3) (see (2.2), (2.8)).

3.3. For the system of Bessel’s functions

(3.5)

\[
\left\{ \sqrt{\pi} J_n \left( \frac{\mu_{m_1}}{2t} \right) \right\}_{m=1}^\infty, \ t \in [0, 2l]
\]

generated by Bessel’s function \(J_n(t)\) we have that

\[
\int_0^{2l} tJ_n \left( \frac{\mu_{m_1}}{2t} \right) J_n \left( \frac{\mu_{m_2}}{2t} \right) dt = 0, \ m_1 \neq m_2,
\]

\[
\int_0^{2l} t \left[ J_n \left( \frac{\mu_{m_1}}{2t} \right) \right]^2 dt = 2l^2 \left[ J_n^2 \left( \mu_{m_1} \right) \right]^2,
\]

where

\(\{\mu_{m_1}\}_{m=1}^\infty\)

is the sequence of the roots of equation

\(J_n(\mu) = 0.\)

Consequently, the following continuum set (for each fixed \(n\)) of orthogonal systems

\[
\left\{ J_n \left( \frac{\mu_{m_1}}{2t} \right) (\varphi^k_1(t) - T) \sqrt{\varphi^k_1(t) - T} \prod_{r=0}^{k-1} \frac{\zeta \left( \frac{1}{2} + i \varphi^r_1(t) \right)}{\sqrt{|\varphi^r_1(t)|}} \right\}_{m=1}^\infty,
\]

\(t \in [T, T + 2l], \ T \to \infty, \ k = 1, \ldots, k_0.\)

corresponds to the Bessel orthogonal system (3.5) (see (2.2), (2.8)).

4. Formula (2.4) as a resource of new integrals containing multiples of \(|\zeta|^2\)

We consider the formula (see (2.4), (2.8))

\[
\int_k^{k+2l} f_n^2(\varphi^k_1(t) - T) \prod_{r=0}^{k-1} \frac{\zeta \left( \frac{1}{2} + i \varphi^r_1(t) \right)}{\sqrt{|\varphi^r_1(t)|}} dt = A_n,
\]

\(A_n = \int_0^{2l} f_n^2(t) dt, \ n = 1, 2, \ldots\)
4.1. Let (see (2.9))
\[ t \in \Delta^0(T, k, l) = \bigcup_{r=0}^{k} \left( \frac{T}{r}, \frac{T}{r} + 2l \right), \quad k = 1, \ldots, k_0. \]

Of course
\[ \Delta^0(T, k, l) \subset [T, T + 2l], \]
and (see (2.5) – (2.7))
\[ \left| [T, T + 2l] \right| = k \sum_{r=0}^{k} \left| \left[ \frac{T}{r}, \frac{T}{r} + 2l \right] \right| + k \sum_{r=1}^{k} \left| \left[ \frac{T}{r-1}, \frac{T}{r} \right] \right| = \]
\[ = (k + 1) o \left( \frac{T}{\ln T} \right) + k O \left( \frac{T}{\ln T} \right) = \]
\[ = O \left( \frac{T}{\ln T} \right), \quad k = 1, \ldots, k_0. \]

Thus, we have the following: if
\[ t \in [T, T + 2l], \]
then (see (4.2))
\[ \ln t = \ln(t - T + T) = \ln T + \ln \left( 1 + \frac{t - T}{T} \right) = \ln T + O \left( \frac{1}{\ln T} \right), \]
i. e.
\[ \ln t \sim \ln T, \quad \forall t \in (T, T + 2l), \quad k = 1, \ldots, k_0. \]

4.2. It is sufficient to use, for example, the formula (4.1) in the case (see (3.1))
\[ f(t) = 1, \quad \Rightarrow \quad A_1 = 2l, \]
i. e.
\[ \int_{T}^{T + 2l} \prod_{r=0}^{k} \left| \zeta \left( \frac{1}{2} + i \varphi_1(t) \right) \right| \omega \left| \varphi_1(t) \right| dt = 2l. \]

Next, we obtain from (4.4) (see (1.3) – (4.3)) by the mean-value theorem that
\[ \int_{T}^{T + 2l} \prod_{r=0}^{k} \left| \zeta \left( \frac{1}{2} + i \varphi_1(t) \right) \right|^2 dt \sim 2l \ln^k T, \quad T \to \infty. \]

Consequently, we obtain from (4.5) in the case
\[ 2l = \frac{\Omega}{\ln^k T} = o \left( \frac{T}{\ln T} \right), \quad \Omega > 0 \]
the following
Corollary.

\[ \int_{\frac{T}{k} + \Omega \ln^{k-1} T}^{k} \prod_{r=0}^{k-1} \left| \zeta \left( \frac{1}{2} + i \phi_1(t) \right) \right|^2 \, dt \sim \Omega, \ T \to \infty, \]

where

\[ 0 < \Omega = o(T \ln^{k-1} T), \ k = 1, \ldots, k_0. \]

Remark 6. Let us notice explicitly that nor the first two formulae (see (4.6), \( k = 1, 2; \ \Omega = 1 \))

\[ \int_{\frac{T}{k} + \Omega \ln^{k-1} T}^{k} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \, dt \sim 1, \]

(4.7)

\[ \int_{\frac{T}{k} + \Omega \ln^{k-1} T}^{k} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \left| \zeta \left( \frac{1}{2} + i \phi_1(t) \right) \right|^2 \, dt \sim 1, \ T \to \infty \]

are not accessible by the current methods in the theory of the Riemann zeta-function.

Remark 7. The first formula in (4.7) gives us the answer to the question about a form of segments for which the following 

\[ [a(T), b(T)] \to \]

\[ \int_{a(T)}^{b(T)} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \, dt \sim 1, \ T \to \infty \]

holds true. Namely, corresponding segments are as follows

\[ [a(T), b(T)] = \left[ T, T + \frac{1}{\ln T} \right] \leftrightarrow [T, T + \frac{1}{\ln T}]. \]

5. First lemmas

5.1. The sequence

\[ \{T\}_{k=1}^{k} \]

is defined by the formula (comp. (2.3))

\[ \phi_1(T) = T_0^{k-1}, \ k = 1, \ldots, k_0, \ T = T \]

for every \( T \geq T_0[\phi_1] \), where \( k_0 \in \mathbb{N} \) is an arbitrary fixed number. Since the function \( \phi_1(t), \ t \to \infty \)

increases to \( \infty \), then we have from (5.2) that

\[ \left\{ T \to \infty \right\} \Leftrightarrow \left\{ T \to \infty \right\}, \]

i. e.

\[ \left\{ T \to \infty \right\} \Leftrightarrow \left\{ T \to \infty \right\}. \]
Next, we have (see (1.5), (5.2), (5.3)) that
\[ \frac{k}{T} - \frac{k-1}{T} \sim (1 - c) \frac{k}{\ln T} \Rightarrow \frac{1 - \frac{k-1}{k}}{\ln T} \sim \frac{1 - c}{\ln T}, \]
i. e.
\[ \frac{k-1}{T} \sim \frac{k}{T}, \]
and, consequently,
\[ \frac{k}{T} = \{1 + o(1)\}T, \quad T \to \infty, \quad k = 1, \ldots, k_0. \]

Since
\[ \frac{k}{\ln k} \sim \frac{T}{\ln T}, \]
(see (5.3)) then we have (see (5.4)) for the sequence (5.1) that
\[ \frac{k}{T} - \frac{k-1}{T} \sim (1 - c) \frac{T}{\ln T}, \quad T \to \infty, \quad k = 1, \ldots, k_0. \]

Consequently, we have
\[ T < \frac{1}{T} < \cdots < \frac{k_0}{T} \]
and, of course, (see (2.1))
\[ T + H < \frac{1}{T + H} < \cdots < \frac{k_0}{T + H}, \quad 0 < H = o\left(\frac{T}{\ln T}\right). \]

5.2. The following lemma holds true.

**Lemma 1.**
\[ H = o\left(\frac{T}{\ln T}\right) \Rightarrow \]
\[ ||\frac{k}{T}, \frac{k}{T + H}|| = \frac{k}{T + H} - \frac{k}{T} = o\left(\frac{T}{\ln T}\right), \quad T \to \infty, \quad k = 1, \ldots, k_0, \]
i. e. (2.3) holds true.

*Proof.* First of all, it follows from (5.6) that
\[ \frac{k}{T} - T \sim (1 - c)k \frac{T}{\ln T}, \]
i. e.
\[ \frac{k}{T} - T = \{1 + o(1)\}(1 - c)k \frac{T}{\ln T} \]
and, simultaneously (see (5.8))
\[ \frac{k}{T + H} - (T + H) = \{1 + o_2(1)\}(1 - c)k \frac{T}{\ln T}. \]
Then we have (see (5.9) – (5.11)) that
\[ 0 < \frac{k}{T + H - \bar{T}} = H + [o_2(1) - o_1(1)](1 - c)k \frac{T}{\ln T} = \]
\[ = H + [o_4(1) - o_3(1)] \frac{T}{\ln T} = \]
\[ = o \left( \frac{T}{\ln T} \right) + o(1) \frac{T}{\ln T} = \]
\[ = o \left( \frac{T}{\ln T} \right), \quad T \to \infty. \]

\[ \square \]

5.3. Next, the following lemma holds true

**Lemma 2.**
\[ H = o \left( \frac{T}{\ln T} \right) \Rightarrow \]
\[ k \frac{T}{T} - k^{-1} \frac{1}{T} + H \sim (1 - c) \frac{T}{\ln T}, \quad T \to \infty, \quad k = 1, \ldots, k_0, \]
\[ \text{i. e. (2.6) holds true.} \]

**Proof.** We have from (5.6) by (5.8), (5.9) that
\[ k \frac{T}{T} - k^{-1} \frac{1}{T} + H \sim (1 - c) \frac{T}{\ln T}, \quad T \to \infty, \quad k = 1, \ldots, k_0, \]
\[ \sim (1 - c) \frac{T}{\ln T} + o \left( \frac{T}{\ln T} \right) \sim \]
\[ \sim (1 - c) \frac{T}{\ln T}, \quad T \to \infty, \quad k = 1, \ldots, k_0. \]

\[ \square \]

**Remark 8.** We have (see (5.12)) that
\[ (5.13) \quad [T, T + H] \prec \left[ T, \frac{1}{T} + H \right] \prec \cdots \prec [T, \frac{1}{k_0} + H], \]
\[ \text{i. e. (2.7) holds true.} \]

6. **Reverse iterations**

6.1. First of all, we have (see (2.3), (5.2)) that
\[ (6.1) \quad \varphi_1(T) = T \Rightarrow \cdots \Rightarrow \varphi_1^k(T) = T, \quad k = 1, \ldots, k_0. \]
Since
\[ (6.2) \quad \varphi_1(T) = T \Rightarrow \frac{1}{T} = \varphi_1^{-1}(T) \]
then we may use the inverse function
\[ \varphi_1^{-1}(T) \]
to generate reverse iterations. We have (see (6.2)) that

\[ \varphi_1(T) = \frac{1}{T} \Rightarrow \frac{2}{T} = \varphi_1^{-1}(T) = \varphi_1^{-1}(\varphi_1^{-1}(T)) = \varphi_1^{-2}(T), \]

(6.3)

\[ k \frac{T}{T} = \varphi_1^{-k}(T), \quad k = 1, \ldots, k_0. \]

Of course, we have (see (6.1), (6.3)) that

\[ \varphi_k(t) \in [\varphi_1^{-k}(T), \varphi_1^{-k}(T + H)] = [\varphi_1^{-k}(T), \varphi_1^{-k}(T + H)] \]

(6.4)

(6.5)

Remark 9. Of course, the following holds true (see (5.13), (6.4))

\[ [T, T + H] \prec [\varphi_1^{-k}(T), \varphi_1^{-k}(T + H)] \prec \ldots \]

\[ \prec [\varphi_1^{-k}(T), \varphi_1^{-k}(T + H)], \quad k = 1, \ldots, k_0. \]

Let us remind for comparison that (see [6], (2.5), Remark 7)

\[ [T, T + H] \succ [\varphi_1(T), \varphi_1(T + H)] \succ \cdots \succ \]

\[ \succ [\varphi_1^{-k}(T), \varphi_1(T + H)], \quad k = 1, \ldots, k_0, \]

where the direct iteration \( \varphi_1^{-k}(T) \) is generated by the function \( \varphi_1(T) \).

Remark 10. The first two reverse iterations of the segments

\[ [T, T + \frac{1}{\ln T}], \quad [T, T + \frac{1}{\ln^2 T}] \]

are included in integral formulae (4.7).
7. **Main lemma**

7.1. The following Lemma holds true.

**Lemma 3.** If

\[
H = o\left(\frac{T}{\ln T}\right), \quad T \to \infty
\]

then for every Lebesgue-integrable function

\[f(t), \quad t \in [T, T + H]\]

we have that

\[
\int_{T}^{T+H} f(t) dt = \int_{T}^{1} f[\varphi^k_1(t)] \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi^r_1(t)] dt,
\]

\[T \to \infty, \quad k = 1, \ldots, k_0; \quad \varphi^0_1(t) = t,
\]

where \(k_0 \in \mathbb{N}\) is arbitrary fixed number.

**Proof.** In our paper [5], (9.2), (9.5) we have proved the following lemma: if (comp. (2.3))

\[\varphi_1\left[\frac{1}{1}, \frac{1}{T}, T + H\right] = [T, T + H]\]

then for every Lebesgue-integrable function

\[f(t), \quad t \in [T, T + H]\]

we have (comp. (1.2), (1.3))

\[
\int_{T}^{T+H} f(t) dt = \int_{T}^{1} f[\varphi_1(t)] \tilde{Z}^2(t) dt,
\]

\[T \geq T_0[\varphi_1], \quad \varphi_1 \in \left(0, \frac{T}{\ln T}\right).
\]

Another form of (7.3) is expressed by the formula (see (1.3))

\[
\int_{T}^{T+H} f(t) dt = \int_{T}^{1} f[\varphi_1(t)] \tilde{Z}^2(t) dt.
\]

Now, the repeated application of the formula (7.3) (see (1.3)) gives the following: if

\[H = o\left(\frac{T}{\ln T}\right)\]
then
\[
\int_{T}^{T+H} f(t)dt = \int_{T}^{T+H} f[\varphi_1(t)]\tilde{Z}^2(t)dt =
\]
\[
= \int_{T}^{T+H} f[\varphi_1^2(t)]\tilde{Z}^2[\varphi_1(t)]\tilde{Z}^2(t)dt = \cdots =
\]
\[
= \int_{T}^{T+H} f[\varphi_k^1(t)]\prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_r^1(t)]dt,
\]
that is exactly (7.2). \(\Box\)

7.2.

Remark 11. The formula (7.2) can be expressed as follows (see (6.3))
\[
\int_{T}^{T+H} f(t)dt = \int_{T}^{T+H} f[\varphi_k^1(t)]\prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_r^1(t)]dt, \quad T \to \infty.
\]

8. Proof of Theorem

Let
\[
\{f_n(t)\}_{n=1}^{\infty}, \quad t \in [0, 2l], \quad l = o\left(\frac{T}{\ln T}\right)
\]
be arbitrary fixed \(L_2\)-orthogonal system, i. e.
\[
(8.1) \quad \int_{0}^{2l} f_m(t)f_n(t)dt = \begin{cases} 0, & m \neq n, \\ A_n, & m = n, \end{cases} \quad A_n = \int_{0}^{2l} f_n^2(t)dt.
\]
Then we have for corresponding system (2.2) by (7.2), (8.1) that
\[
f(t) \to f_m(\varphi_k^1(t) - T)f_n(\varphi_1^k(t) - T),
\]
\[
\int_{T}^{T+2l} f_m(\varphi_k^1(t) - T)f_n(\varphi_1^k(t) - T)\prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_r^1(t)]dt =
\]
\[
= \int_{T}^{T+2l} f_m(t-T)f_n(t-T)dt =
\]
\[
= \int_{0}^{2l} f_m(t)f_n(t)dt = \begin{cases} 0, & m \neq n, \\ A_n, & m = n, \end{cases}
\]
i. e. (8.2) holds true. Finally, the properties (2.5) – (2.7) follows from (5.9), (5.12),
(5.13).

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