Topological Holography

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Abstract
We study a topological field theory in four dimensions on a manifold with boundary. A bulk-boundary interaction is introduced through a novel variational principle rather than explicitly. Through this scheme we find that the boundary values of the bulk fields act as external sources for the boundary theory. Furthermore, the full quantum states of the theory factorize into a single bulk state and an infinite number of boundary states labeled by loops on the spatial boundary. In this sense the theory is purely holographic. We show that this theory is dual to Chern-Simons theory with an external source. We also point out that the holographic hypothesis must be supplemented by additional assumptions in order to take into account bulk topological degrees freedom, since these are apriori invisible to local boundary fields.
There has been much recent interest in the interplay between bulk and boundary dynamics. The two main directions being explored presently are (i) the Maldacena conjecture[1], which postulates a relationship between a bulk string/M-theory and a boundary conformal field theory (also known as the AdS/CFT correspondence), and (ii) the holographic hypothesis [2, 3, 4], which states that all information about a theory in the bulk of a bounded region is available, in some sense, on the boundary of the region. In particular, the AdS/CFT correspondence has been viewed as an example of the holographic hypothesis [3].

The first of these directions is based in part on the observation that the symmetry group of $d+1$-dimensional anti-deSitter space-time $SO(2,d)$ is the same as the conformal group of Minkowski space-time in $d$ dimensions. More specifically, a statement of the conjecture is [6, 7]

$$\exp \left[ - \int_{AdS_{d+1}} \mathcal{L}_{SUGRA}(\phi_i(\phi^B_i)) \right] = \exp \left[ \int_{\partial AdS_{d+1}} \mathcal{O}^i(\phi^B_i) \right]_{CFT}.$$  (1)

The left hand side of this equation is the evaluation of the Euclidean supergravity action on the classical solutions for which the background is the $(d+1)$-dimensional anti-deSitter metric. $\phi_i$ represent the bulk dynamical fields in the solution. The surface integral in the supergravity action, which is a functional of the boundary value $\phi^B_i$ of $\phi_i$, is a crucial input here. The right hand side is the quantum expectation value of the primary field $\mathcal{O}^i$ of some conformal field theory on the boundary of AdS, where the boundary value of the bulk field $\phi^B_i$ acts as an external source. Thus, this conjectured equality provides a way of computing conformal field theory correlation functions from classical supergravity. It therefore provides a classical-quantum duality for a sector of the solution space of supergravity, (– the sector for which the metric is anti-deSitter). A key feature of this prescription is that a classical bulk field provides, via its boundary value, an external source for a boundary quantum theory. This feature appears in the model we discuss below.

The second direction in this bulk-boundary interplay is (at least partly) motivated by arguments concerning black holes: The fact that the entropy of a black hole is proportional to its area suggests the possibility that the theory describing microstates of a black hole is either (i) a surface theory, or, (ii) a bulk theory whose states are “visible” on the bounding surface in such a way that the entropy becomes proportional to the surface area. This is closely connected to and motivated by the Beckenstein bound argument [8, 9].

There are in fact (at least) four possible definitions of what holography may mean:

(i) For a theory defined in a bounded spatial region, all bulk degrees of freedom are

\[ \text{ There is a more general, and fully quantum mechanical statement of this conjecture, where the left hand side includes functional integrals over bulk fields } \phi_i \text{ which have boundary values } \phi^B_i, \text{ and over asymptotically anti-deSitter metrics. The statement of the correspondence given above is effectively the tree level evaluation of the left hand side, and represents all its tests to date! } \]
“visible” on the boundary of the spatial region through a physical process. This may be done via a “screen mapping” [4, 10].

(ii) For an $n$-dimensional theory, $T_1$, containing bulk and boundary degrees of freedom, there is an $(n-1)$-dimensional theory, $T_2$, which captures all the degrees of freedom of $T_1$. Then the possibility is open that $T_2$ is itself defined on a manifold with boundary, having both bulk and surface degrees of freedom. This is obviously different from (i).

(iii) The same as (ii) with the extra condition that $T_2$ is a theory defined strictly on the boundary of the region on which $T_1$ is defined. In this case, $T_2$ has only bulk degrees of freedom, since the boundary of a boundary vanishes.

(iv) All the degrees of freedom of a theory in a bounded region are associated with its boundary. In this case holography is automatic.

The AdS/CFT correspondence appears to fall in category (iii). However, the Beckenstein bound argument, which requires entropy to be proportional to the bounding area, appears to be consistent with all four possibilities.

An interesting possibility which should be taken into account in a definition of holography is the case of theories which have bulk topological degrees of freedom, associated for example with handles. It is then possible that these are not visible on the boundary. An example is a Wilson line observable with end points on the boundary which may or may not wrap around a bulk handle (see Figure 1). Then only (ii) seems viable as a definition of holography, and may exclude the AdS/CFT case (iii).

In this paper we describe a theory which is holographic in the sense of both (ii) and (iv) above. The quantum states factorize into bulk and boundary states, with a unique bulk state. It has the unusual property that all its quantum states are effectively associated with loops lying on the spatial boundary. The dynamics of the loop states is trivial, so their worldlines are cylindrical. Furthermore, the bulk states give rise to external sources for the boundary quantum theory in a natural manner. This latter feature is similar to one...
of the key features in the AdS/CFT correspondence, albeit in a simpler context. Models of this type may be used to probe the limitations of the holographic hypothesis and perhaps the AdS/CFT correspondence. One such limit, as briefly mentioned above, appears to be that bulk topological degrees of freedom are not captured by the values of local fields on boundary.

**The Model:** The theory we consider is a topological field theory on a four-dimensional manifold \( \mathbb{R} \times \Sigma^3 \), defined by an unusual specification of the variational principle. The action is

\[
S[A, B; a] = \int_{\mathbb{R} \times \Sigma^3} \text{Tr} \left\{ B \wedge \mathcal{F}(A) + \frac{\Lambda}{2} B \wedge B \right\} + k \int_{\mathbb{R} \times \Sigma^2} \text{Tr} \left\{ a \wedge da + \frac{2}{3} a \wedge a \wedge a \right\},
\]

where \( B \) is a Lie-algebra valued two form, \( A \) and \( a \) are Lie-algebra valued one forms, and \( \mathcal{F}(A) \) denotes the curvature two-form. \( \Sigma^2 \) is the 2-boundary of the “spatial” surface \( \Sigma^3 \).

This action contains no explicit interaction terms between the bulk and boundary fields. However, the action alone does not determine the equations of motion or the subsequent canonical structure, since it must be supplemented by a variational principle. Particular choices of the variational principle can lead to a situation in which the bulk BF-theory is coupled to the boundary Chern-Simons theory. We will invoke the particular scheme where the field \( a \) must be varied in accord with the variations of the field \( A \) on the boundary.

The variation of this action is

\[
\delta S = \text{Tr} \int_{\mathbb{R} \times \Sigma^3} \left[ \delta B \wedge (\mathcal{F}(A) + \Lambda B) + \delta A \wedge (\mathcal{D}_A B) \right] + \text{Tr} \int_{\mathbb{R} \times \Sigma^2} \left[ k \mathcal{F}(a) \wedge \delta a + B \wedge \delta A \right].
\]

The variational principle is well-defined if we require that

\[
\delta a = \delta A|_{bd},
\]

and the usual requirement that all surface terms in the variation vanish. The constraints on the variations of the fields may be viewed as giving rise to the equations of motion for the boundary theory. Ordinarily, a variational principle is supplemented by conditions such as the vanishing of the variations of certain fields on the boundary. Our prescription is unusual only in that it fixes the variation of certain fields to equal certain other variations on the boundary.

A question concerning our approach so far is why we do not, perhaps more simply, consider the above action with \( a = A \) at the outset. The reason is that doing this gives a different theory: functional differentiability requires that the fields \( A \) and \( B \) satisfy the condition \( \mathcal{F}(A) = B \) on the boundary. In our case on the other hand, the bulk field \( B \) provides a source for the *independent* boundary curvature \( \mathcal{F}(a) \).
The phase space variables are identified by writing

\[ S[A, A_0, B, B_0; a, a_0] = \int dt \left[ \int_{\Sigma^3} \text{Tr} \left\{ B \wedge \dot{A} + B_0 \wedge (F(A) + \Lambda B) + A_0 \wedge D_A B \right\} ight. \\
+ k \int_{\Sigma^2} \text{Tr} \left\{ a \wedge \dot{a} + a_0 \wedge F(a) \right\} + \int_{\Sigma^2} \text{Tr} A_0 \wedge B \right] \]  

(5)

In this expression we have written \( A = A_0 + A, B = B_0 + B \) and \( a = a_0 + a \), where the fields carrying a subscript 0 contain the ‘time’ components, and the other fields contain the spatial components of the respective forms; \( \dot{d} \) represents the time component of the exterior derivative; \( F(\cdot) \) is the spatial part of the curvature two form; and \( D_A \) is the spatial part of the covariant derivative. The boundary contribution has two terms: the first is the Chern-Simons action in Hamiltonian form, and the second is from an integration by parts in the bulk part of the action. This latter term provides a bulk source for the boundary curvature.

The canonical structure of the theory is obvious from (5): \( A^a_i \) and \( E^{ai} \equiv \epsilon^{abc} B^i_{bc} \) are the canonically conjugate variables in the bulk, \( a^i_1 \) and \( a^i_2 \) are canonically conjugate on the boundary; \( a, b, \cdots \) are spatial indices 1, 2, 3 in the bulk, and 1, 2 on the boundary; \( i, j, \cdots \) are Lie algebra indices). The time component fields \( (a_0, A_0 \text{ and } B_0) \) appear as Lagrange multipliers, and varying these fields gives the phase space constraints. Since \( \Sigma^3 \) has a boundary, there is an additional boundary constraint arising from functional differentiability of the action (recall that \( a_0 \) must be varied with \( A_0 \)). The relevant variations are

\[ \delta_{A_0} S = \int dt \left\{ \int_{\Sigma^3} \text{Tr} \left\{ \delta A_0 \wedge D_A B \right\} + \int_{\Sigma^2} \text{Tr} \left\{ \delta A_0 \wedge (B + k F(a)) \right\} \right\} \]  

(6)

\[ \delta_{B_0} S = \int dt \int_{\Sigma^3} \text{Tr} \left\{ \delta B_0 \wedge (F(A) + \Lambda B) \right\} \]  

(7)

From the above it is clear that the Hamiltonian is a linear combination of constraints. The bulk constraints are

\[ G^i \equiv (D_a E^{ai}) \big|_{\Sigma^3} = 0, \quad J^{ai} \equiv (\epsilon^{abc} F(A)^i_{bc} + \Lambda E^{ai}) \big|_{\Sigma^3} = 0 \]  

(8)

In addition to these there is a surface constraint

\[ H^i \equiv \left( E^{3i} + k \epsilon^{ab} F(a)^i_{ab} \right) \big|_{\Sigma^2} = 0, \]  

(9)

due to the presence of the surface integral in (5).

It is clear that both the bulk and boundary constraints are first class and provide the complete prescription for classical Hamiltonian evolution. The bulk phase space variable \( E^{3i} \) provides a source for the boundary curvature, and the boundary fields \( a \) evolve via the first class constraint (8). This evolution is a gauge transformation on \( a \). As a result the bulk evolution of \( (A, E) \) is consistent with the boundary evolution of \( a \). To see this more explicitly, start from an initial classical configuration where \( E^{3i} \) fixes \( F(a) \). We must now
ensure that the evolved $E^{3i}$ and $F(a)$ also satisfy the constraint \( \mathcal{E} \). That this is indeed the case is ensured by our variational principle, and is easiest to see directly in the covariant picture.

**Observables:** The Hamiltonian of the bulk theory is a linear combination of first class constraints. Therefore the gauge invariant observables are phase space functionals that have weakly vanishing Poisson brackets with the constraints. For $\Lambda \neq 0$ we have not been able to find any observables. While we do not have a proof of this, the fact that there is a unique solution of the bulk Dirac quantization constraint (described below) suggests there are no bulk observables. On the other hand, for $\Lambda = 0$ there are two types of bulk observables. One type is parametrized by loops (and are traces of holonomies), while the other is parametrized by both loops and surfaces. In this paper we will be interested in the case of non-vanishing $\Lambda$.

Contrasted with the bulk case, there are an infinite set of boundary observables for non-zero values of $\Lambda$. Since the boundary constraint generates Yang-Mills gauge transformations on $a$, the boundary observables are traces of the holonomy of $a$ for *all* loops lying on the spatial boundary $\Sigma^2$. Denoting these observables by

$$T_\gamma[a] \equiv \text{Tr P} \exp \int_\gamma a$$

for loops $\gamma$, their Poisson algebra is

$$\{T_\alpha[a], T_\beta[a]\} = \Delta(\alpha, \beta) (T_{\alpha\beta}[a] - T_{\alpha\beta^{-1}}[a]),$$

where

$$\Delta(\alpha, \beta) = \int ds \int dt \epsilon_{ab} \dot{\alpha}^a(s) \dot{\beta}^b(t) \delta^2(\alpha(s) - \beta(t))$$

measures a weighted intersection number of the loops $\alpha$ and $\beta$, and $\beta^{-1}$ denotes traversal of the loop in the opposite sense ($\dot{\alpha}^a(s)$ is the tangent vector to the loop at the parameter value $s$).

Consequently, the 4-dimensional theory we have outlined has an infinite number of boundary observables parameterized by loops lying in the 2-boundary $\Sigma^2$ of $\Sigma^3$. The observables form a closed infinite dimensional Poisson algebra.

On the constraint surface, the bulk contains no local degrees of freedom: for gauge group $SU(N)$ there are $3(N^2 - 1)$ configuration variables $A_{\alpha}^i$ and $4(N^2 - 1)$ first class bulk

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3 A proof may be devised along the following lines: Write down all the basic local and non-local Gauss law invariant variables. The local ones are combinations of the electric and magnetic fields with internal indices contracted; there are four such variables. The non-local ones are the traces of electric field insertions between holonomy segments such as $\text{Tr}[E^a(x_0)U_\gamma(x_0, x_1)E^b(x_1)U_\gamma(x_2, x_3)E^c(x_3)\cdots]$; these are a countably infinite set. (In the limit of the loop $\gamma$ shrinking to a point, these become functions of the local variables). Consider the general Gauss law invariant function to be an arbitrary function of these variables, and calculate its Poisson bracket with $J^a$, and see if the result can be made to vanish.
constraints. This means that there can be at most a finite number of bulk topological degrees of freedom. However, since there appear to be no bulk observables to evaluate on the constraint surface, it is likely that there are no bulk degrees of freedom in this theory. This means that the large gauge freedom may be used to set all bulk fields to zero. Nevertheless there may still be an infinite number of surface observables on the reduced phase space. To see this take $E^{3i}$ to be zero everywhere in the bulk but keep it arbitrary and non-zero on the boundary. This gives the curvature of a via the surface constraint (9). Conversely, given any boundary field $a$, the boundary field $E^{3i}$ is determined. This field may then be arbitrarily extended into the interior. A particularly simple case is where $E^{3i}$ vanishes on the boundary. Then the reduced phase space of our four-dimensional theory is the (finite dimensional) moduli space of flat connections on the 2-boundary $\Sigma^2$, which may be a surface of arbitrary genus.

The boundary observable algebra given here is reminiscent of, but fundamentally different from, the construction that gives the Kac-Moody boundary observable algebra associated with 3-dimensional gravity. However, the Brown-Henneaux\cite{BrownHenneaux} construction of observables for the latter theory is intrinsically dependent on the fall-off conditions of the bulk fields. In particular, almost all the Brown-Henneaux observables vanish identically on solutions of 3d gravity such as the BTZ black hole \cite{BTZ}. In our construction this is manifestly not the case – the fields in the bulk may be obtained from any connection $a$ (on $\Sigma^2$), whose curvature gives the boundary value of $E^{3i}$ via the boundary constraint (9). Conversely, given bulk fields, the boundary value of $E^{3i}$ fixes the curvature of the boundary connection $a$. As such, all the observables are non-zero on generic solutions, unlike the case of 3-dimensional gravity.

Quantization: We will carry out a quantization in the Hamiltonian formulation described above. There are two ways to approach this: (i) convert the classical constraints into operator equations in a suitable representation and attempt to solve them for the quantum states, or (ii) find a representation of the algebra of classical gauge invariant observables.

In a model such as the one described here, it is possible to carry out a “hybrid” quantization using both of these approaches simultaneously. This is because the bulk and boundary states have a natural separation. Specifically, the bulk constraint can be imposed as a Dirac quantization condition (since we do not have any bulk observables to find a representation of), while the boundary sector can be quantized by finding a representation of the algebra of the $T_{\alpha}(a)$ observables. We will follow this procedure, and define the bulk quantum constraint as acting by the identity on boundary states.

Consider first the bulk constraints and use the connection representation; $A_i$ are treated as configuration variables, and their conjugate momenta $E^i$ are treated as functional derivative operators

$$E^i \rightarrow -i \frac{\delta}{\delta A_i}$$

(13)
We assume that the quantum states may be written as the product

$$\Psi = \psi_{\Sigma^3}(A) \otimes \psi_{\Sigma^2}(\alpha),$$

where $\alpha$ denotes the parameterization of boundary states (to be discussed below). The bulk constraint $J^{ai}$ gives the condition

$$\left\{ \left( \varepsilon^{abc} F_i^{bc} - i\Lambda \frac{\delta}{\delta A_i^a} \right) \Big| \psi_{\Sigma^3} \right\} \otimes \left\{ I_{|\Sigma^2} \psi_{\Sigma^2}(\alpha) \right\} = 0$$

(14)

where $I$ is the boundary identity operator. The unique solution of this constraint is,

$$\psi_{\Sigma^3}(A) \equiv \exp \left\{ -\frac{i}{\Lambda} \int_{\Sigma^3} \text{Tr} \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\} - \frac{i}{2\Lambda} \int_{\Sigma^2} \text{Tr} \left( A_1 A_2 \right) \right\}$$

(15)

This state also satisfies the bulk Gauss constraint. The surface term in the exponential is necessary to guarantee that the functional derivative gives $\varepsilon^{abc} F_{bi}^{bc}$, and does not spoil the bulk Gauss law invariance. The solution in the case where $\Sigma^3$ is compact without boundary is the bulk part of this functional, and has been discussed in [14].

This Chern-Simons state is not directly related to the Chern-Simons part of the original action (5): the state is still a solution of the bulk constraints if $\Sigma^3$ has no boundary. Furthermore, although (15) is a solution of the bulk constraint, the functional does not transform trivially under the generators of the constraint (i.e. the Poisson bracket of the constraint with the functional does not vanish). This is unlike the Wilson loop functional, which is simultaneously a Gauss law invariant classical observable, as well as a quantum state satisfying the quantum Gauss constraint.

(The latter result might seem surprising at first, and in apparent violation of the intuition derived from the Gauss law. However, it is also illustrated in a simple quantum mechanical example. Consider the constraint equation,

$$\left( \hat{x} + \alpha \hat{p} \right) \psi = 0$$

where $\alpha$ is some dimensionful constant. The solution to this constraint in the $x$ representation is

$$\psi \propto \exp \left\{ -\frac{i}{2\alpha} x^2 \right\}$$

This function is clearly not invariant under the transformation $x \rightarrow x + \alpha$ generated by the constraint. However, the canonical transformation $\tilde{p} = x + \alpha p, \tilde{x} = x/\alpha$ reduces the constraint to $\tilde{p} \tilde{\psi} = 0$ whose solution is $\tilde{\psi}(\tilde{x}) = \text{constant}$, which does commute with constraint.)

Turning now to the boundary dynamics, we choose to quantize this sector by finding a representation of the algebra of the boundary observables $T_\alpha[a]$. This is easiest to do in

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4The authors would like to thank W. Unruh for pointing out this example.
the loop representation, and we follow here the prescription used in the approach to non-perturbative quantum gravity [15, 16]. The holonomy observables $\hat{T}_\alpha$ are defined to act on loop states $|\beta>$ by

$$\hat{T}_\alpha|\beta>: = i\hbar \Delta(\alpha, \beta) \left(|\alpha \circ \beta > - |\alpha \circ \beta^{-1} >\right).$$

From this definition it follows that

$$[\hat{T}_\alpha, \hat{T}_\beta] = \Delta(\alpha, \beta) \left(\hat{T}_{\alpha \circ \beta} - \hat{T}_{\alpha \circ \beta^{-1}}\right).$$

Thus, in this approach the boundary states are the loop kets $|\alpha>$, and the full quantum state of the theory is the product

$$|A, \alpha> = \psi[A]|\alpha>.$$  

The loop states $|\alpha>$ are not all independent: the states are traces of holonomies in the connection representation and are subject to the Mandelstam identities induced by the trace relations on $SU(N)$ matrices. Furthermore, because there is a unique bulk state, the labeling of quantum states is effectively only by loops. An inner product on this space of states may be defined as $<\alpha | \beta > = \delta_{\alpha \beta}$. This completes the description of the quantum theory.

**Discussion:** The model we have described has a number of unusual features which are useful to compare with Chern-Simons theory on a manifold with boundary, and with 2+1 gravity in particular. These latter theories have the property that they are topological in the bulk and, with particular fall-off conditions on the fields, induce a Kac-Moody algebra of observables (which are all constants of motion) on the boundary. These theories thus have non-trivial bulk and boundary observables, (if the bulk has non-trivial topology). The boundary observables may be viewed as the observables of a two-dimensional boundary conformal field theory. Apparently for this reason, these theories have been viewed as an example of the AdS/CFT correspondence [17], and therefore an example of holography. However this does not correspond to holography for any of the possible definitions given above. What would be required for a correspondence with one of these definitions is the specification of a 2-dimensional theory that has both the bulk and boundary observables of 3-dimensional gravity. (See [18] in this regard.)

For the case of our 4-dimensional model, the 3-dimensional theory that has the same observables algebra is Chern-Simons theory coupled to an external source $J^a$ (which plays the role of $E^3$), i.e. the action is the Chern-Simons one with the additional term $\frac{1}{\Lambda} \int_{\Sigma_{1+2}} A_a J^a$. Consequently, in addition to viewing our model as an example of type (iv) holography, we can also view it as type (iii) holography.

In summary, we have discussed some aspects of holography in a 4-dimensional model in which all degrees of freedom are associated with loops on a 2-dimensional boundary. We have:
(i) pointed out that topological bulk observables are missed by any boundary theory that is directly induced by the bulk fields, (ii) suggested that this shortcoming may be side-stepped by broadening sufficiently the definition of holography, and (iii) given a 3-dimensional theory that has the same observable algebra as our 4-dimensional model. This provides a concrete example of duality: theories in different spacetime dimensions having the same classical and quantum observable algebra.

This work was supported by the Natural Science and Engineering Research Council of Canada, and by a University of British Columbia Graduate Fellowship. We thank G. W. Semenoff and W. G. Unruh for discussions.

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