Exact Hole-induced Resonating-Valence-Bond Ground State in Certain \( U = \infty \) Hubbard Models

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We prove that the motion of a single hole induces the nearest-neighbor resonating-valence-bond (RVB) ground state in the \( U = \infty \) Hubbard model on a triangular cactus – a tree-like variant of a kagome lattice. The result can be easily generalized to \( t-J \) models with antiferromagnetic interactions \( J \geq 0 \) on the same graphs. This is a weak converse of Nagaoka’s theorem of ferromagnetism on a bipartite lattice.

A resonating-valence-bond (RVB) state is an exotic spin liquid state originally envisioned by Anderson [1]. It was revisited after the discovery of high-\( T_c \) superconductivity [2, 3], which gave rise to the notion that by doping the RVB, holons, the fractionalized excitations carrying charge \( e \) and spin 0, can condense to become a superconductor [4–6]. In this picture, the background antiferromagnetic interaction, \( J \), plays an essential role as a mediator of valence-bond formation and thus of “preformed Cooper pairs.”

Even in the absence of explicit exchange interactions, however, magnetism can still arise upon doping of the Hubbard model at half-filling in the \( U = \infty \) limit (where \( J = 0 \)). The idea is that the motion of a doped hole (or electron) shuffles the background spin ordering, leading to the magnetism [7]. In particular, the celebrated “Nagaoka’s theorem” states that for a bipartite system (e.g. a square lattice), introducing a single hole leads to a fully polarized ferromagnetic ground state due to the constructive interference of the hole motion in a ferromagnetic background [8]. This result was generalized to a wider class of graphs by Tasaki [9] – the only requirement is that the product of hopping matrix elements around any loop in the graph is positive. (See also [10] [11] for a related theme on kinetically induced magnetism.) On a non-bipartite lattice, however, the product of hopping matrix elements around loops with an odd number of bonds is negative, frustrating the kinetic energy of a hole in a ferromangetic background. Indeed, recent numerical studies have concluded that the ground state of the \( U = \infty \) Hubbard model on a triangular lattice in the presence of a single hole has total spin zero (\( S_{\text{tot}} = 0 \)) and has 120° order as in the case of triangular lattice antiferromagnets [12, 13].

In this paper, starting from a simple problem on a single triangle, we study the \( U = \infty \) Hubbard model on a certain class of graphs known as a triangular cactus (also known as a Husimi cactus), on which the kinetic motion of a hole is unfrustrated (frustrated) in an RVB (ferromagnetic) background. The ground state of this model is rigorously proven to be a nearest-neighbor RVB state with a delocalized holon. Such a graph has a property that the product of hopping matrix elements around any cycle (a loop of length \( l \geq 3 \) in which only the first and the last vertices are equal) is negative. We also remark that the system is integrable thanks to the existence of extensive number of conserved quantities – this is an example of Hilbert space fragmentation [16–18].

A hole in a triangle. We start by solving the two-electron problem for the Hubbard model on a triangle with \( U = \infty \) and \( t > 0 \):

\[
H = -t \sum_{i=1}^{3} \sum_{\sigma=\uparrow,\downarrow} \left[ c_{i\sigma}^\dagger c_{i+1,\sigma} + H.c. \right] + [U = \infty],
\]

where the site \( i = 4 \) is identified with \( i = 1 \) (\( c_{4,\sigma} \equiv c_{1,\sigma} \)). In the total \( S = 1 \) (triplet) sector, energy eigenvalues are \( E_n = 2t \cos(\frac{2\pi n}{3}) \), where \( n = 0, 1, 2 \), with three-fold degeneracies due to the spin-rotational symmetry (corresponding to the total \( S^z = \pm 1, 0 \)). In the \( S = 0 \) (singlet) sector, energy eigenvalues are \( E_n = -2t \cos(\frac{2\pi n}{3}) \), where \( n = 0, 1, 2 \). The ground state is the singlet state:

\[
|\text{GS}\rangle = \frac{1}{\sqrt{3}} \left( |\Delta\rangle + |\Delta\rangle + |\Delta\rangle \right),
\]

where a circle on a vertex denotes the location of a hole and the magenta bond denotes the singlet state on two sites. The singlet state is oriented in a counter-clockwise direction on a triangle. In the \( S = 0 \) ground state, the hole’s kinetic energy has its minimum possible value \(-2t\), whereas it is frustrated in a spin-polarized background, with the lowest energy being \(-t\).

Indeed, in the singlet subspace (\( S^2 = 0 \)), unique basis states can be identified with the location of the holon, i.e. the state \(|\Delta\rangle\) can be identified as the state with a holon (with its creation operator \( h_i^\dagger \)) at the circled site. In the triplet sector (\( S = 1 \)), with a fixed total \( S^z = \pm 1, 0 \), the basis states can similarly be identified by the position of the hole. It is then easy to see that the Hamiltonian of a hole in the singlet sector is given by \( H_{\text{eff}}^{(s)} = -t \sum_{i=1}^{3} \left( h_i^\dagger h_{i+1} + H.c. \right) \), whereas in the triplet sector with a fixed total \( S^z \), \( H_{\text{eff}}^{(t)} = +t \sum_{i=1}^{3} \left( h_i^\dagger h_{i+1} + H.c. \right) = -t \sum_{i=1}^{3} \left( e^{-i\pi} h_i^\dagger h_{i+1} + H.c. \right) \). Effectively, the hole sees a \( \pi \)-flux through the triangle when the background spins form a triplet pair [19].
The main result of the paper (the theorem below) is that the motion of a single hole lifts such degeneracy and induces the RVB ground state.

Before going into technical details, we first define the convenient many-body basis of the problem. For this, we make a direct contact with quantum dimer models [1,24,25], and consider the states of hard-core (nearest-neighbor) dimers on a triangular cactus graph, with a single monomer (that is, all sites but one are touched by a dimer). Once the location of the monomer is specified, it is easy to see that there is a unique dimer covering, which has exactly one dimer fully contained in every triangle (see Fig. 1(c) for the illustration of such a configuration). Now consider the Hamiltonian describing the hopping of a monomer:

$$H_{\text{hop}} = -t \sum_{\triangle} \left( |\triangle\rangle \langle \triangle| + |\triangle| \langle \triangle| \right) + |\triangle\rangle \langle \triangle| + \text{H.c.},$$

where a circle on a vertex denotes the location of monomer. The dimer is colored black to differentiate it from a singlet bond. In any step in which the monomer hops to a nearest-neighbor site, one dimer is moved, but in such a way that it remains interior to the same triangle. Thus, we can label the dimers uniquely by a plaquette index \( f \), and this index is preserved under the specified dynamics.

Now let us consider the corresponding electron problem. Given the location of the hole, \( i \), and the corresponding unique dimer covering, let \( \hat{S}_f \) and \( \hat{S}_f^z \) be the total spin and spin component in the \( z \)-direction, respectively, of the two electrons touched by the dimer contained in the plaquette \( f \). The two spins form either a singlet or triplet state: \( \hat{S}_f = 0, 1, \hat{S}_f^z \) be the total spin and spin component in the \( z \)-direction, respectively, of the two electrons touched by the dimer contained in the plaquette \( f \). The two spins form either a singlet or triplet state: \( \hat{S}_f = 0, 1, \hat{S}_f^z \). Again, we choose to orient valence-bonds in counter-clockwise direction around each triangle, \( f \), whenever \( \hat{S}_f^z = 0 \). (This introduces a sign convention for resonating-valence-bond-type wave-functions [23].) Of these basis states, the state corresponding to the unique valence-bond covering with the holon at site \( i \) will be denoted by

$$|i, \nu \rangle = |i, \{\hat{S}_f = 0\}, \{\hat{S}_f^z = 0\}\rangle.$$  

Then, the following theorem is the main result of this paper.

**Theorem:** The ground state of the Hamiltonian Eq. (3) in the presence of a single hole (\( 2N_f \) electrons on \( 2N_f + 1 \) sites).
sites) is unique and is the positive \(a(i) > 0\) superposition of all the possible valence-bond coverings \(|i, \text{VBC}\rangle\). This is the nearest-neighbor “resonating-valence-bond (RVB) state” with a delocalized holon:

\[
|\Psi_0\rangle = \sum_i a(i) |i, \text{VBC}\rangle.
\]

(see Fig. 1 (c) for the illustration of this RVB state.) The Theorem can be easily proven with the following well-known lemma (see e.g., Ref. [27]).

Lemma (diamagnetic inequality): Consider a single particle hopping problem under a magnetic field on a general 2-edge-connected planar graph in the presence of an arbitrary on-site potential term:

\[
T[\phi_f] + V_0 \equiv - \sum_{(i,j)} t_{ij} e^{-i\theta_{ij}} |i\rangle \langle j| + \sum_i \epsilon_i |i\rangle \langle i|,
\]

where we assume \(t_{ij} > 0\) and \(\theta_{ij}\) is an induced Berry phase on an edge \(|i, j\rangle\) due to a flux \(\phi_f\) through a plaquette \(f\) to which \(|i, j\rangle\) belongs. We will simply denote by \(T_0\) the hopping matrix in the absence of a magnetic field: \(T_0 \equiv T[\phi_f = 0]\). Here, a 2-edge-connected graph is a connected graph in which every edge belongs to at least one plaquette. Formally, it is defined to be a connected graph that cannot be disconnected by deleting any single edge. Then, the flux configuration that minimizes the ground state energy of \(T[\phi_f]\) is unique and is the one without any flux: \(\phi_f = 0\) for all \(f\), i.e., when \(T[\phi_f] = T_0\). The physical meaning is that “a magnetic field raises the energy.”

Proof of the lemma: Let \(|\psi\rangle\) be the normalized ground state of \(T[\phi_f] + V_0\) for a given non-trivial flux configuration \(\{\phi_f\}\) with the energy \(E_0\), and \(|\psi\rangle\) be the normalized ground state of \(T_0 + V_0\) with the energy \(E_0\). It is easy to see that \(E_0 \leq E'_0\) by using the triangle inequality:

\[
E'_0 = - \sum_{(i,j)} t_{ij} e^{-i\theta_{ij}} |\psi_i^f\rangle \langle \psi_j^f| + \sum_i \epsilon_i |\psi_i^f|^2 \\
\geq - \sum_{(i,j)} t_{ij} |\psi_i^f| \cdot |\psi_j^f| + \sum_i \epsilon_i |\psi_i^f|^2 \\
= \langle |\psi_i^f| (T_0 + V_0) |\psi_i^f| \rangle \geq E_0.
\]

(10)

Here, \(|\cdot|\) denotes the matrix with every entry replaced by its absolute value: e.g., \((|A|)_{ij} = |A_{ij}|\).

In order to prove the uniqueness, it is enough to show that the first inequality above is a strict inequality. Let us assume otherwise, in which case each term in \(- \langle |\psi_i^f| T[\phi_f] |\psi_i^f| \rangle\) is real and positive:

\[
e^{-i\theta_{ij}} |\psi_i^f| \cdot |\psi_j^f| > 0
\]

(11)

for all \(|i, j\rangle\). Now, let \(\phi_f \neq 0\) for some plaquette \(f\), with its vertices \(i_1, i_2, \ldots, i_n\) \((i_{n+1} \equiv i_1)\). From Eq. 11 we obtain

\[
\prod_{k=1}^n e^{-i\theta_{ik+1}} |\psi_i^f| \cdot |\psi_k^f| > 0,
\]

which is in contradiction to the assumption that \(\phi_f \neq 0\). This completes the proof. □

Proof of the Theorem: Since \(\{\tilde{S}_f, H\} = \{\tilde{S}_f^\dagger, H\} = 0\), let us consider the Hamiltonian Eq. 5 in a given \(\{\tilde{S}_f\} \) and \(\{\tilde{S}_f^\dagger\}\) sector, \(H|\{\tilde{S}_f\},\{\tilde{S}_f^\dagger\}\rangle\). As shown in the single triangle problem above, the hole sees effective \(\pi\)-fluxes (no-fluxes) on triangles, \(f\), at which \(\tilde{S}_f\) is a triplet (singlet). Hence, \(H|\{\tilde{S}_f\},\{\tilde{S}_f^\dagger\}\rangle\) is the Hamiltonian of a single hole hopping problem in the presence of \(\pi\)-fluxes through the triangle plaquettes, \(f\), with \(\tilde{S}_f = 1\). According to the lemma (diamagnetic inequality), the energy minimizing flux configuration is unique and is the one without any flux, and hence, \(\tilde{S}_f = 0\) and \(\tilde{S}_f = 0\) for all \(f\). Also,

\[
H|\{\tilde{S}_f=0\},\{\tilde{S}_f^\dagger=0\}\rangle = - \sum_{(i,j)} t_{ij} |i\rangle \langle j| + \sum_i \tilde{V}_i |i\rangle \langle i|,
\]

(13)

where \(\tilde{V}_i \equiv V(|n_i = 0, n_j \neq 1\rangle\) is the effective on-site potential felt by the hole at site \(i\). Since the off-diagonal elements of \(H|\{\tilde{S}_f=0\},\{\tilde{S}_f^\dagger=0\}\rangle\) are all negative, the Perron-Frobenius theorem ensures that the ground state, \(|\Psi_0\rangle\), of \(H|\{\tilde{S}_f=0\},\{\tilde{S}_f^\dagger=0\}\rangle\) (and hence of \(H\)) is the superposition of all the basis states (Eq. 7) with positive coefficients, Eq. 8 □

\(t-J\) model. The nearest-neighbor RVB state of the form Eq. 8 with \(a(i) > 0\) is still a ground state in the presence of nearest-neighbor antiferromagnetic Heisenberg interactions, \(J > 0\) of the following form:

\[
H_f = \sum_f J_f \sum_{l=1}^3 S_l^f \cdot S_{l+1}^f \\
= \sum_f \frac{J_f}{2} \left[ S_f (S_f + 1) - \frac{3}{4} n_f \right].
\]

(14)

Here, \(S_l^f\) is the spin operator on site \(l\) of a triangle \(f\) with \(S_3^f = S_f^\dagger\), \(\tilde{S}_f = \sum_{l=1}^3 S_l^f\), and \(n_f\) is the total number operator on a triangle \(f\). Antiferromagnetic interactions \(J\) are uniform for bonds of the same triangle \(f\), while they can differ on different triangles.

Proof of the Theorem in the presence of \(J > 0\): Observe that each \(|i, \text{VBC}\rangle\) describing a valence-bond covering with the holon at site \(i\) is an eigenstate of \(H_f\) with the lowest possible energy eigenvalue (for a fixed \(i\)):

\[
H_f |i, \text{VBC}\rangle = \left( -\frac{3}{4} \sum_f J_f \right) |i, \text{VBC}\rangle.
\]

(15)

This means that the ground state of the total Hamiltonian including \(H_f\) is still in the \(\{\tilde{S}_f = 0\}\) sector. Moreover, since \(H_f|\{\tilde{S}_f=0\},\{\tilde{S}_f^\dagger=0\}\rangle\) is diagonal in the basis \(|i, \text{VBC}\rangle\), it follows from the Perron-Frobenius theorem
that the ground state is still of the form Eq. [8] with \( a(i) \) modified but remaining positive. □

Integrability. When \( J = 0 \) (i.e., \( H_J = 0 \)), the entire excited state spectra of Eq. [3] can be obtained by exploiting the extensive set of quantum numbers \( \{ \tilde{S}_f \} \) and \( \{ \tilde{S}_f^z \} \) (\( f = 1, 2, \ldots, N_f \)). The spin excitations are \( \tilde{S}_f \) and \( \tilde{S}_f^z \), which amounts to changing the sign of hopping terms \( t_{ij} \rightarrow -t_{ij} \), the ground state manifold consists of the states with \( \tilde{S}_f^z = 1 \) on those triangles and is 3\( N_f \)-fold degenerate; among them is the familiar fully-polarized Nagaoka ferromagnet. If the \( \pi \)-fluxes are present only in some \( \Delta_s \) (\( \Delta_t \)) the set of directed bonds of triangles at which \( \tilde{S}_f \) forms a singlet (triplet). The charge spectrum can be obtained by diagonalizing the single hole problem in the presence of \( \pi \)-fluxes on \( \Delta_t \) [23]:

\[
H|\{\tilde{S}_f\},\{\tilde{S}_f^z\}\rangle = -\sum_{\langle i,j\rangle \in \Delta_s} t_{ij} |i\rangle \langle j| - \sum_{\langle i,j\rangle \in \Delta_t} t_{ij} e^{-i\pi} |i\rangle \langle j| + \sum_i \tilde{V}_i |i\rangle \langle i| , \tag{16}
\]

In the presence of \( H_J \), \( \tilde{S}_f \) and \( \tilde{S}_f^z \) are no longer good quantum numbers, and the system is no longer integrable.

Relevance of the sign of hopping matrix elements. In the presence of the uniform \( \pi \)-flux on each triangle, which amounts to changing the sign of hopping terms \( t_{ij} \rightarrow -t_{ij} \), the ground state manifold consists of the states with \( N_f \) uncorrelated spin triplets, each of which is localized on the triangle \( f \):

\[
\big|\{\tilde{S}_f\}\big\rangle \equiv \sum_i b(i) |i,\{\tilde{S}_f = 1\},\{\tilde{S}_f^z\}\rangle , \tag{17}
\]

where \( b(i) > 0 \) and \( \{\tilde{S}_f^z\} = \pm 1, 0 \). The ground states are \( 3^{N_f} \)-fold degenerate; among them is the familiar fully-polarized Nagaoka ferromagnet. If the \( \pi \)-fluxes are present only in some \( N_f(< N_f) \) number of triangles, the ground state manifold consists of the states with localized triplets \( \tilde{S}_f = 1 \) on those \( N_f \) triangles and is \( 3^{N_f} \)-fold degenerate.

Spin-\( \frac{1}{2} \) bosons. All of the above conclusions remain true for spin-\( \frac{1}{2} \) hard core bosons if the sign of the hopping term is reversed. This is a weak converse to the results of Ref. [29, 30] which show that the ground state of spin-\( \frac{1}{2} \) bosons is a fully-polarized ferromagnet when the hopping matrix elements are all negative.

Discussion. The exact solvability of the present model relies on its “tree-like” structure, i.e. due to the absence of loops other than triangles. Exact generalization of this result to a 2D or higher dimensional lattice is likely to be obstructed by the existence of longer-ranged valence-bonds generated by the hopping of a holon around an additional loop adjacent to a certain triangle. Moreover, the existence of additional even-length loops produces a tendency towards a ferromagnetism, as exemplified by the Nagaoka’s theorem on a bipartite lattice, and frustrates a tendency to a singlet formation, making analytic solution highly unlikely. However, if the number of non-triangular loops is suppressed in comparison to the number of (corner-sharing) triangles, it is likely that a version of a short-ranged RVB state is stabilized: a kagome lattice or a suitably decorated version of it may be such an example. Such an idea is in line with the attempts to reproduce quantum dimer models as a limiting case by suitably decorating each edge of 2D lattices with Majumdar-Ghosh chain [31, 32]. We hope that the present exact result will prove to be a fruitful starting point for a numerical search for a doping-induced RVB state (as opposed to doping an RVB state induced by frustrated antiferromagnetic interactions). In particular, a numerical study of the \( U = \infty \) Hubbard model on a kagome lattice is currently lacking, although such studies have been carried out for the square and triangular lattice [15, 35]. Whether doping dilute holes in the \( U = \infty \) Hubbard model on the kagome lattice leads to superconductivity [34, 37], a holon Fermi liquid, a holon Wigner crystal [38], or some other state is an interesting open question.

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[1] P. W. Anderson, Resonating valence bonds: A new kind of insulator?, Materials Research Bulletin 8, 153 (1973).
[2] P. W. Anderson, The resonating valence bond state in La\(_2\)CuO\(_4\) and superconductivity, Science 235, 1196 (1987).
[3] G. Baskaran, Z. Zou, and P. W. Anderson, The resonating valence bond state and high-\( T_c \) superconductivity—a mean field theory, Solid State Communications 63, 973 (1987).
[4] D. S. Rokhsar and S. A. Kivelson, Superconductivity and the quantum hard-core dimer gas, Physical Review Letters 61, 2376 (1988).
[5] S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Topology of the resonating-valence-bond state: Solitons and high-\( T_c \) superconductivity, Physical Review B 35, 8865 (1987).
[6] S. Kivelson and D. Rokhsar, Bogoliubov quasiparticles, spinons, and spin-charge decoupling in superconductors, Physical Review B 41, 11693 (1990).
[7] D. Thouless, Exchange in solid $^3$He and the Heisenberg Hamiltonian, Proceedings of the Physical Society (1958-1967) 86, 893 (1965).

[8] Y. Nagaoka, Ferromagnetism in a narrow, almost half-filled s band, Physical Review 147, 392 (1966).

[9] H. Tasaki, Extension of Nagaoka's theorem on the large-
U Hubbard model, Physical Review B 40, 9192 (1989).

[10] K.-S. Kim, C. Murthy, A. Pandey, and S. A. Kivelson, Interstitial-induced ferromagnetism in a two-dimensional Wigner crystal, arXiv preprint arXiv:2206.07191 (2022).

[11] R. Moessner and S. L. Sondhi, Slow holes in the triangular ising antiferromagnet, Physical Review B 62, 14122 (2000).

[12] J. O. Haerter and B. S. Shastry, Kinetic antiferromagnetism in the triangular lattice, Physical Review Letters 95, 087202 (2005).

[13] C. N. Sposetti, B. Bravo, A. E. Trumper, C. J. Gazza, and L. O. Manuel, Classical antiferromagnetism in kinetically frustrated electronic models, Physical Review Letters 112, 187204 (2014).

[14] F. T. Lisandrimi, B. Bravo, A. E. Trumper, L. O. Manuel, and C. J. Gazza, Evolution of Nagaoka phase with kinetic energy frustrating hopping, Physical Review B 95, 195103 (2017).

[15] Z. Zhu, D. Sheng, and A. Vishwanath, Doped Mott insulators in the triangular-lattice Hubbard model, Physical Review B 105, 205110 (2022).

[16] Z.-C. Yang, F. Liu, A. V. Gorshkov, and T. Iadecola, Hilbert-space fragmentation from strict confinement, Physical Review Letters 124, 207602 (2020).

[17] P. Sala, T. Rakovszky, R. Verresen, M. Knap, and F. Pollmann, Ergodicity breaking arising from Hilbert space fragmentation in dipole-conserving Hamiltonians, Physical Review X 10, 011047 (2020).

[18] S. Moudgalya, B. A. Bernevig, and N. Regnault, Quantum many-body scars and Hilbert space fragmentation: A review of exact results, arXiv preprint arXiv:2109.00548 (2021).

[19] This is true even when $t$ varies among different bonds, and when on-site chemical potential disorder and spin-independent interaction terms of the form Eq. [4] are present.

[20] P. Chandra and B. Doucot, Spin liquids on the Husimi cactus, Journal of Physics A: Mathematical and General 27, 1541 (1994).

[21] Z. Hao and O. Tchernyshyov, Fermionic spin excitations in two-and three-dimensional antiferromagnets, Physical review letters 103, 187203 (2009).

[22] Z. Hao and O. Tchernyshyov, Structure factor of low-energy spin excitations in a $S = \frac{1}{2}$ kagome antiferromagnet, Physical Review B 81, 214445 (2010).

[23] R. Moessner and S. L. Sondhi, Resonating valence bond phase in the triangular lattice quantum dimer model, Physical Review Letters 86, 1881 (2001).

[24] G. Misguich, D. Serban, and V. Pasquier, Quantum dimer model on the kagome lattice: Solvable dimer-liquid and Ising gauge theory, Physical Review Letters 89, 137202 (2002).

[25] R. Verresen and A. Vishwanath, Unifying Kitaev magnets, kagome dimer models and ruby Rydberg spin liquids, arXiv preprint arXiv:2205.15302 (2022).

[26] S. Liang, B. Doucot, and P. Anderson, Some new variational resonating-valence-bond-type wave functions for the spin-1/2 antiferromagnetic heisenberg model on a square lattice, Physical review letters 61, 365 (1988).

[27] E. H. Lieb and M. Loss, Fluxes, Laplacians, and Kasteleyn’s theorem, Duke Mathematical Journal 71, 337 (1993).

[28] Similar reasoning is used in obtaining anyon states in the Kitaev model on the honeycomb lattice [39].

[29] E. Eisenberg and E. H. Lieb, Polarization of interacting bosons with spin, Physical Review Letters 89, 220403 (2002).

[30] K. Yang and Y.-Q. Li, Rigorous proof of pseudospin ferromagnetism in two-component bosonic systems with component-independent interactions, International Journal of Modern Physics B 17, 1027 (2003).

[31] K. S. Raman, R. Moessner, and S. L. Sondhi, SU(2)-invariant spin-$\frac{1}{2}$ Hamiltonians with resonating and other valence bond phases, Physical Review B 72, 064413 (2005).

[32] R. Moessner, K. Raman, and S. L. Sondhi, From exotic phases to microscopic Hamiltonians, in AIP Conference Proceedings, Vol. 816 (American Institute of Physics, 2006) pp. 30–40.

[33] L. Liu, H. Yao, E. Berg, S. R. White, and S. A. Kivelson, Phases of the infinite U Hubbard model on square lattices, Physical Review Letters 108, 126406 (2012).

[34] T. Senthil, S. Sachdev, and M. Vojta, Fractionalized Fermi Liquids, Phys. Rev. Lett. 90, 216403 (2003).

[35] S. Sachdev and D. Chowdhury, The novel metallic states of the cuprates: Topological Fermi liquids and strange metals, Progress of Theoretical and Experimental Physics 2016 (2016).

[36] T. Senthil and M. P. Fisher, $Z_2$ gauge theory of electron fractionalization in strongly correlated systems, Physical Review B 62, 7850 (2000).

[37] T. Senthil and M. P. Fisher, Fractionalization, topological order, and cuprate superconductivity, Physical Review B 63, 134521 (2001).

[38] H.-C. Jiang, T. Devereaux, and S. Kivelson, Holon Wigner crystal in a lightly doped kagome quantum spin liquid, Physical Review Letters 119, 067002 (2017).

[39] A. Kitaev, Anyons in an exactly solved model and beyond, Annals of Physics 321, 2 (2006).