Incompatibility breaking quantum channels

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Abstract
A typical bipartite quantum protocol, such as EPR-steering, relies on two quantum features, entanglement of states and incompatibility of measurements. Noise can delete both of these quantum features. In this work we study the behavior of incompatibility under noisy quantum channels. The starting point for our investigation is the observation that compatible measurements cannot become incompatible by the action of any channel. We focus our attention to channels which completely destroy the incompatibility of various relevant sets of measurements. We call such channels \textit{incompatibility breaking}, in analogy to the concept of entanglement breaking channels. This notion is relevant especially for the understanding of noise-robustness of the local measurement resources for steering.

Keywords: incompatibility, quantum measurement, quantum channel, quantum observable, steering

(Some figures may appear in colour only in the online journal)

1. Introduction
In a typical quantum information task such as quantum key distribution or teleportation, one party, Alice, prepares a bipartite quantum system in some state, sends one of the systems to
another party, Bob, after which they perform some measurements on their respective component systems. In order for such a setup to yield some advantage over a corresponding classical scenario, it is crucial that it relies on some genuine quantum feature of its constituent parts. Usually this means that the state shared by Alice and Bob should be entangled. In order to make use of an entangled state, Alice and Bob need to perform appropriate quantum measurements, and there can be a qualitative feature that these measurements must have. A notable requirement is incompatibility, i.e., both Alice and Bob need to be able to choose between measurements that cannot be performed jointly with a common device. In particular, incompatibility is essential for one-sided quantum key distribution protocols based on Einstein–Podolsky–Rosen steering [1–3]. For this reason, both entanglement and incompatibility can be viewed as quantum resources whose understanding is of utmost importance in view of emerging technological applications.

Incompatible quantum measurements, understood mathematically as POVMs without a common refinement, have been studied for a long time; see e.g. [4–7] for early studies and [8–10] for some recent contributions. Their importance was further emphasized by the recently observed [2, 3] tight connection to EPR-steering, which is currently under active investigation; see e.g. [11–15].

A delicate step in almost any quantum information protocol is the transmission of quantum systems in a noisy environment. These processes typically induce decoherence on the quantum state, degrading its quality as a resource for some quantum protocol under consideration. Motivated by the steering protocols, we look at situations where the noise is local, and acts only on Alice’s side. Such an effect is described by a quantum channel $\Lambda$ which in the Schrödinger picture maps an initial bipartite state $\varrho$ into the final state $(\Lambda_{a} \otimes \text{id})(\varrho)$. (Throughout this paper we consider quantum channels $\Lambda$ primarily in the Heisenberg picture, and use the notation $\Lambda_{a}$ for the predual map representing the channel in the Schrödinger picture.) A particular instance is the depolarizing channel which turns maximally entangled states into isotropic states [16].

In order to evaluate the performance of a protocol, it is essential to study the effect of noise channels on its resources. An important step in this direction is to characterize those channels which destroy a resource completely; in the case of entanglement, the relevant objects are entanglement breaking channels, i.e., channels $\Lambda$ such that $(\Lambda_{a} \otimes \text{id})(\varrho)$ is a separable state for every bipartite state $\varrho$. The structure of entanglement breaking channels is well known [17], and the concept has also been generalized into various directions [18, 19].

We now wish to change the viewpoint from entanglement breaking channels to something more appropriate for e.g. steering-based protocols for which entanglement is necessary but not sufficient. Owing to the duality relation

$$\text{tr}\left[(\Lambda_{a} \otimes \text{id})(\varrho)A \otimes B\right] = \text{tr}\left[\varrho \Lambda(A) \otimes B\right],$$

we can alternatively view the effect of the noise channel in the Heisenberg picture as a cause of disturbance on Alice’s measurements. In other words, instead of seeing the effect of the noise as decoherence on the nonlocal resource $\varrho$, we interpret it as a distortion on Alice’s local measurement resource, that is, incompatibility. Study of this resource can be done without any regard to the bipartite setting.

In this work we regard incompatibility as a general multi-purpose quantum resource that may be lost due to an action of a noisy quantum channel; the purpose of the paper is to initiate the study of the properties of channels relevant for this phenomenon. The starting point for our investigation is the observation that compatible measurements cannot become incompatible by the action of any channel, while the reverse happens typically. In our investigation
we focus our attention to channels which completely destroy the incompatibility of various relevant sets of measurements. We call such channels *incompatibility breaking*, in analogy to the concept of entanglement breaking channels. More generally, one can also consider channels that partially destroy incompatibility, as quantified e.g. by an *incompatibility monotone* introduced in [10], or the *joint measurability degree* [20]; this will be considered in a forthcoming paper. Furthermore, we restrict ourselves to the finite-dimensional setting, thereby excluding continuous variable Gaussian systems, which we treat in another work [21].

We show below that entanglement breaking channels are all incompatibility breaking in the strongest sense, i.e. they destroy incompatibility of every set of measurements. We then demonstrate examples of channels that destroy the incompatibility of any set of $n \geq 2$ observables but are not entanglement breaking. We also show that there exist channels that are incompatibility breaking in the strongest sense, but nevertheless not entanglement breaking. In this sense, entanglement is more robust to noise than incompatibility.

The paper is organized as follows. In section 2 we first state the definition of incompatibility, and then review its role as a local resource for EPR-steering. We then proceed to introduce the concepts relevant for incompatibility breaking channels in section 3. In section 4 we investigate the example of depolarizing channels. Using the derived results we prove in section 5 that every entanglement breaking channel is incompatibility breaking, but the converse does not hold. Finally, the structural connections between different notions of incompatibility breaking channels are summarized in section 6.

### 2. Incompatibility as a quantum resource

The notion of incompatibility is best approached by first defining its opposite, compatibility. The general intuitive idea behind compatibility is very simple: a collection of quantum measurements is compatible, if they can all be simulated by a single common quantum measurement. From the practical point of view, this means that there exists *in principle* a single measurement device, which can be used in place of any of the original devices by only adjusting the classical post-processing of the outcomes.

For a given set of measurements, the impossibility of such a device renders the set a quantum resource. In order to briefly explain why this is so, recall that quantum features commonly appear in the context of correlation experiments which cannot be described by local hidden variable models [38], and the point is that joint measurements provide exactly such models. In the context of CHSH Bell inequalities, this has been noted e.g. in [8]; more importantly, the connection is explicit in EPR-steering, which is perhaps the most notable application where incompatibility appears as a quantum resource. Hence, we consider this case in detail, especially explaining how noisy steering provides motivation for the study of incompatibility breaking channels.

#### 2.1. Incompatibility of quantum observables

In order to precisely define incompatibility, we first need to introduce the basic concepts.

The state dependence of measurement outcome distributions in a quantum measurement is described by the associated *observable*, mathematically represented as a *positive operator valued measure* (POVM). A POVM $A$ with a finite outcome set $\Omega_A$ is defined as a map $a \mapsto A(a)$ that assigns a bounded operator to each outcome and satisfies
for all $a \in \Omega_A$. Given a state $\rho$ of the system, the probability to get a particular outcome $a \in \Omega_A$ is then $\text{tr}[\rho A(a)]$. It is convenient to denote $A(X) = \sum_{a \in X} A(a)$ for any set $X \subseteq \Omega_A$. The labeling of outcomes is not relevant for the questions that we will investigate. For this reason, we may assume $\Omega_A \subseteq \mathbb{Z}$ whenever this is convenient.

Beginning with the common formulation, a finite collection $\mathcal{M} = \{ A_1, \ldots, A_n \}$ of observables is said to be compatible (or jointly measurable) if there exists a joint observable $G$, which is an observable defined on the product outcome space $\Omega_{A_1} \times \cdots \times \Omega_{A_n}$ and satisfies the marginal conditions

$$G(X_1 \times \Omega_{A_2} \times \cdots \times \Omega_{A_n}) = A_1(X_1)$$

$$\vdots$$

$$G(\Omega_{A_1} \times \cdots \times \Omega_{A_{n-1}} \times X_n) = A_n(X_n)$$

for all $X_1 \subseteq \Omega_{A_1}, \ldots, X_n \subseteq \Omega_{A_n}$. Here we have used the notation $G(Y) = \sum_{g \in Y} G(g)$, so e.g. the condition that the first equation is valid for all $X_1 \subseteq \Omega_{A_1}$ is equivalent to the requirement that

$$\sum_{a_1, a_2, \ldots, a_n} G(a_1, a_2, \ldots, a_n) = A_1(a_1)$$

holds for all $a_1 \in \Omega_{A_1}$.

This formulation of compatibility often appears in the literature; see e.g. [23]. However, an equivalent formulation in terms of general postprocessing is more convenient for our purposes, and it also allows to formulate compatibility of an infinite number of observables. To properly formulate joint measurements for infinite number of observables, we first recall the general definition of a POVM. A POVM $G$ with infinite outcome set $\Omega$ must be understood literally as an operator-valued measure, i.e., a map that associates a positive operator $G(X)$ to each Borel subset $X \subseteq \Omega$ (an event), has $\sum_{g \in X} G(g)$ for any disjoint collection of sets ($\sigma$-additivity), and satisfies the normalization $G(\Omega) = 1$.

In order to motivate the idea of postprocessing, suppose that we perform a measurement of an observable $G$ and obtain an outcome $g \in \Omega$. We can make as many copies of this outcome as we want since this is just classical information. After copying, we can process each copy of $g$ in an independent way. In particular, we can assign to it a new outcome $a$ with a conditional probability $f(a, g)$ of $a$ given $g$. These are normalized as $\sum_a f(a, g) = 1$, and we can think of the new outcomes arising from a usual finite-outcome observable $A$, whose elements are thus defined by

$$A(a) = \int_{\Omega_0} f(a, g) G(dg), \quad \text{for all } a \in \Omega_A. \quad (2)$$

In general, an arbitrary collection $\mathcal{M}$ of finite-outcome observables is said to be compatible (or jointly measurable) if there exists an observable $G$ such that every $A \in \mathcal{M}$ can be obtained from $G$ by a postprocessing of the form (2). If a collection $\mathcal{M}$ is not compatible, it is said to be incompatible. It can be shown [24] that in the case of a finite collection $\mathcal{M}$, this formulation of compatibility is equivalent to the usual one given above. Moreover, it is well-known that projective measurements are compatible if and only if they commute with each other.
2.2. Example: loss of incompatibility due to noise

We now consider a simple example of a compatible collection of infinite number of observables, which also provides a preliminary demonstration of how noise destroys incompatibility.

On a spin-1/2 quantum system, the measurement of the spin component in the direction $\vec{n} \in \mathbb{R}$ is described by the two-outcome observable

$$S^\dag(\pm) = \frac{1}{2}(1 \pm \vec{n} \cdot \vec{\sigma}),$$

which is sharp (in the sense of being projective). The collection

$$\mathcal{M} = \{ S^\dag \vec{n} \in \mathbb{R}^3, \| \vec{n} \| = 1 \}$$

is non-commutative, and since these observables are all projective, they are therefore also incompatible. From the physical point of view, $\mathcal{M}$ constitutes a quantum resource for e.g. standard CHSH Bell experiments.

By adding completely depolarizing noise to these observables, we obtain the corresponding noisy versions, considered e.g. in [25]:

$$S^\dag(\pm) = tS^\dag(\pm) + \frac{1-t}{2} = \frac{1}{2}(1 \pm t\vec{n} \cdot \vec{\sigma}).$$

This noise addition can be written as an action of a quantum channel, hence fits into the framework introduced in the next section. In this noisy case, noncommutativity is no longer relevant for incompatibility, as we shortly see.

The spin direction observable $\mathcal{D}$ with outcomes on the surface of the unit sphere $S^2$ in $\mathbb{R}^3$ is defined as

$$\mathcal{D}(X) = \frac{1}{4\pi} \int_X (1 + \vec{k} \cdot \vec{\sigma})d\vec{k}.$$

For any direction $\vec{n}$, we define the postprocessing function $f_{\vec{n}}$ by

$$f_{\vec{n}}(\pm, \vec{k}) = \begin{cases} 1, & \text{if } \pm \vec{k} \cdot \vec{n} > 0, \\ 0, & \text{otherwise}. \end{cases}$$

A direct calculation then shows that

$$\int_{f_{\vec{n}}} (\pm, \vec{k})d\mathcal{D}(\vec{k}) = \frac{1}{4\pi} \int_{f_{\vec{n}}} (\pm, \vec{k})(1 + \vec{k} \cdot \vec{\sigma})d\vec{k} = S^\dag_{1/2}(\pm).$$

In conclusion, the noisy collection

$$\mathcal{M}_{\text{noisy}} = \{ S^\dag_{t/2}: \vec{n} \in \mathbb{R}^3, \| \vec{n} \| = 1 \}$$

is compatible. This collection can no longer be used in CHSH Bell experiments as the joint observable provides a hidden variable model which ensures that all Bell inequalities hold [8]. Hence, the noise addition with $t = 1/2$ has destroyed the quantum resource. It can be shown that $1/2$ is actually the critical value; the corresponding set is still incompatible for all $t > 1/2$ [3].

2.3. Motivation: incompatibility and noisy EPR-steering

While incompatibility of quantum measurements is an interesting topic by itself, its use in quantum information protocols provides motivation to the idea of incompatibility as a
quantum resource. While CHSH Bell experiments mentioned in the above example constitute one physical example of such a resource, EPR-steering provides a much more general application, because it can be used with arbitrary measurements (not just binary ones). The purpose of this subsection is to demonstrate how noisy EPR-steering naturally motivates the study of incompatibility breaking channels.

The setting for EPR-steering [22] consists of two local parties, Alice and Bob, sharing a bipartite state \( \rho \), and having access to their respective collections \( A \) and \( B \) of local measurements. Incompatibility of the set \( A \) is known to be a clearly defined resource for steering, in the sense that the latter does not work without it [2, 3]. Obviously also the state needs to be suitably chosen, so we effectively have two different quantum objects, \( A \) and \( \rho \), 'containing' the resources. Accordingly, it makes sense to try to describe them separately; this approach is further motivated by the fact that resource theories for states and measurements can be developed independently of each other [10, 26, 27].

This separation of resources is especially compelling in noisy scenarios. In fact, one can think of two types of noise: (1) imprecision in the preparation of the bipartite state \( \rho \), and (2) subsequent dynamical noise leading to decoherence. The essential point is then the following: dynamical noise (2) is local in the steering setup, acting independently in the spatially separated laboratories of Alice and Bob. Consequently, it is conceptually better to associate dynamical decoherence with the local resource \( A \) (Heisenberg picture) rather than the nonlocal resource \( \rho \) (Schrödinger picture), and leave \( \rho \) with the preparation noise.

Hence, the topic of this paper, i.e. the effects of noise channels on measurement incompatibility, is naturally motivated by the resource-theoretic approach to EPR-steering where the local measurement resource \( A \) is considered separately from the state \( \rho \). This becomes especially clear in the situation where the original state resource \( \rho \) is perfect, i.e. maximally entangled. In fact, we will demonstrate below that noise needed to destroy EPR-steering is exactly one described by an incompatibility breaking channel to be defined in the next section.

As a basic reference for steering we use the seminal paper [22] by Wiseman et al. As mentioned in the introduction, the connection between steering and incompatibility was observed in [2, 3]; our presentation here is slightly different from the formal point of view due to our definition of incompatibility in terms of post-processing functions. The advantage of this approach is the direct connection to local hidden state models (see below), which are central to steering, and will also be referred to in the subsequent sections of this paper.

The operational starting point for the discussion is that of a correlation experiment consisting of two parties, Alice and Bob, choosing observables from sets \( M_A \) and \( M_B \), respectively; see figure 1. All observables \( A \in M_A \) and \( B \in M_B \) are assumed to have finite

![Figure 1. A correlation experiment consists of two parties, Alice and Bob, who choose observables from sets \( M_A \) and \( M_B \), respectively. The experiment contains both non-local resources (state \( \rho \)) and local resources (sets \( M_A \) and \( M_B \)). In the one-way steering scenario (from Alice to Bob), the relevant local resource is Alice’s set \( M_A \), which needs to be incompatible.](image-url)
outcome sets \( \Omega_A \) and \( \Omega_B \). The experiment is described by the conditional probabilities
\[
P(a, b | A, B) = \text{tr}[\rho A(a) \otimes B(b)], \quad A \in \mathcal{M}_A, \ a \in \Omega_A, \ B \in \mathcal{M}_B, \ b \in \Omega_B,
\]
associated with the different choices of measurements and the resulting outcomes. The setting is said to have a \textit{local hidden variable model}, if there exists a probability space \( (\Omega, p) \), and ‘response’ functions \( f_A(a, g) \) and \( h_B(b, g) \) such that
\[
\sum_a f_A(a, g) = \sum_b h_B(b, g) = 1 \quad \text{for all } g,
\]
and
\[
P(a, b | A, B) = \int_{\Omega} f_A(a, g) h_B(b, g) p(\text{d}g).
\]
If such a model does not exist, it is customary to say that the correlations are \textit{nonclassical}.

We then define for each \( A(a) \) the corresponding conditional states
\[
\sigma_{aA} = \text{tr}_A[\rho (A(a) \otimes I)]
\]
on Bob’s side. This family satisfies the \textit{no-signaling conditions}
\[
\sigma_B = \sum_a \sigma_{aA} = \sum_a \sigma_{aA'}, \quad \text{for all } A, A',
\]
and the normalization \( \text{tr}[\sigma_B] = 1 \). We assume that Bob’s measurements \( \mathcal{M}_B \) are informationally complete, so that he can construct the conditional states \( \sigma_{aA} \). Bob’s measurements \( \mathcal{M}_B \) are not otherwise important in this setting.

We can now define one-sided EPR-steering [22] from Alice to Bob as follows: the setting is said to be \textit{non-steerable} if it has a local hidden variable model, where Bob’s response functions are of particular form, namely \( h_B(b, g) = \text{tr}[\varrho \otimes B(b)] \) for some family \( \{\varrho_b\}_{b \in \Omega} \) of local states \( \varrho_b \). Such a model is often called \textit{local hidden state model} [22].

Clearly, non-steerability is equivalent to the existence of a decomposition
\[
\sigma_{aA} = \int_{\Omega} f_A(a, g) \varrho_b \ p(\text{d}g),
\]
meaning that the states \( \sigma_{aA} \) can be classically simulated using the hidden states \( \varrho_b \); we refer to [22] for a more detailed discussion on this interpretation. When the setting is not non-steerable, we say that it is \textit{steerable} (from Alice to Bob or by Alice), or that Alice can steer Bob’s state.

The above described idea of separating the resources now concretely means writing
\[
\sigma_{aA} = S(A(a)),
\]
where \( S \) is the map from Alice’s observable algebra to the set of subnormalized states on Bob’s side, defined by
\[
S(A) = \text{tr}_A[\rho (A \otimes I)].
\]
This map is linear, maps every positive operator to a positive operator, and satisfies the normalization condition \( S(I) = \varrho_B \). Suppose that
\begin{enumerate}
\item \( S \) is invertible,
\item \( S^{-1} \) maps positive operators to positive operators.
\end{enumerate}

This is true, for instance, when \( \rho \) is a pure state with full Schmidt rank, in which case there exists an invertible matrix \( R \) such that \( S(A) = R^T A^T R \), when the bases of the two Hilbert spaces are identified in a suitable way. (This follows, e.g., from lemma 2 in [28], by direct computation. Here \( A^T \) denotes the transpose of \( A \).)

The connection between steering and incompatibility is now the following: assuming steerability, we can define a unique POVM \( \mathcal{G} \) on Alice’s side by setting
\[
S(G(X)) := \int_X \varphi_g \rho(dg);
\]
then
\[
A(a) = \int_{\Omega} f_A(a, g)G(dg),
\]
for all \( A \in \mathcal{M}_A \). But this means precisely that the set \( \mathcal{M}_A \) is compatible according to the definition we gave in section 2.1. Conversely, if \( \mathcal{M}_A \) is compatible, we can express their joint POVM \( G \) as an integral \( G(X) = \int_X D(g)p(dg) \), and define the hidden states \( \rho_k := S(D(g)) \), showing that the setting is steerable.

Now a typical way of ending up with an \( S \) without properties (i) or (ii) is to have local noise, as discussed in the introduction: suppose we begin with an \( S \) coming from a state with pure state with full Schmidt rank, and apply a noise channel \( \Lambda \) on Alice's side. If we would consider the noise acting in the Schrödinger picture on the state resource, we adjust the map \( S \):
\[
S_{\text{noisy}}(A) = S \circ \Lambda.
\]
In particular, if the channel is entanglement breaking, then \( S_{\text{noisy}} \) represents a useless resource. However, since the noise is local, it is more appropriate to consider it as acting on the local resource and hence in the Heisenberg picture; this is described by adjusting the collection \( \mathcal{M}_A \):
\[
\mathcal{M}_{A,\text{noisy}} = \left\{ \Lambda \circ A | A \in \mathcal{M}_A \right\}
\]
From this we immediately get the following ‘noisy version’ of the connection between incompatibility and steering:

**Proposition 1.** Suppose Alice and Bob share a bipartite pure state \( \varphi_0 \) of full Schmidt rank.

(a) (Ideal scenario). Alice can steer Bob with a set of measurements \( \mathcal{M}_A \), if and only if \( \mathcal{M}_A \) is incompatible.

(b) (Noisy scenario). Suppose that Alice has an incompatible set \( \mathcal{M}_A \) of measurements, but subjected to a local noise channel \( \Lambda \) on her system. Then Alice can steer Bob if and only if \( \Lambda \) does not map \( \mathcal{M}_A \) into a compatible set of observables.

In conclusion, we have identified two separate resources for EPR-steering, the nonlocal state resource appearing in the form of the map \( S \), and local measurement resource \( \mathcal{M}_A \) consisting of incompatible measurements. This separation of resources fits naturally with scenarios involving local decoherence, because it allows one to describe the effects of noise solely in terms of the measurement resource \( \mathcal{M}_A \). In particular, the above proposition shows that noise channels destroying steerability for an ideal state resource (maximally entangled), are exactly the ones which break the incompatibility of \( \mathcal{M}_A \). This provides a strong motivation for the detailed investigation of such channels, which we begin in the next section. Study of the properties of a non-ideal state resource (the map \( S \)) is out of the scope of the paper.

3. Incompatibility breaking channels

3.1. \( n \)-incompatibility breaking channels

Quantum evolution is generally described by a quantum channel, which is a unital completely positive linear map \( \Lambda \) on the observable algebra \( \mathcal{L}(\mathcal{H}) \) (the set of bounded operators on the
system Hilbert space $\mathcal{H}$). Quantum channels describe operations that can be performed on the system. In practice, this can take place passively via environmental interaction, actively via controlled interactions, or in a combination of both. We denote by $\mathcal{C}$ the set of all quantum channels on $\mathcal{L}(\mathcal{H})$ and use the notation $\Lambda_1 \circ \Lambda_2$ for the functional composition of two channels $\Lambda_1, \Lambda_2 \in \mathcal{C}$, i.e.,

$$(\Lambda_1 \circ \Lambda_2)(T) = \Lambda_1(\Lambda_2(T)).$$

This is called concatenation of $\Lambda_2$ and $\Lambda_1$.

A quantum channel $\Lambda$ maps any observable $A$ into another observable $\Lambda(A)$ by way of composition:

$$\Lambda(A) := \Lambda \circ A, \quad (\Lambda \circ A)(x) := \Lambda(A(x)).$$

We remark that already the positivity and unitality of $\Lambda$ are sufficient for observables to be mapped to observables. The meaning of the following simple observation is that such maps cannot create incompatibility [10]. (Implications of this result to steering have been noticed earlier; see [29] and the references therein.)

**Proposition 2.** Let $\Lambda$ be a unital and positive linear map on $\mathcal{L}(\mathcal{H})$. If the observables $A_1, \ldots, A_n$ are compatible, then also the transformed observables $\Lambda(A_1), \ldots, \Lambda(A_n)$ are compatible.

**Proof.** Assuming $\{A_1, \ldots, A_n\}$ is compatible, there exists a joint observable $G$ for $A_1, \ldots, A_n$. From the definition it immediately follows that $\Lambda(G)$ is a joint observable for $\Lambda(A_1), \ldots, \Lambda(A_n)$. □

A unitary channel $\sigma_U(T) := UTU^*$ is reversible in the sense that $\sigma_U$ has an inverse map which is a channel. It follows that observables $A_1, \ldots, A_n$ are compatible if and only if $\sigma_U(A_1), \ldots, \sigma_U(A_n)$ are compatible. In other words, unitary channels preserve incompatibility. As an extreme example of the opposite kind, a channel $T \mapsto \text{tr} [\eta T]1$ maps every operator to a multiple of the identity operator. Hence, the image of any collection of observables is compatible. A typical channel is something between these two extremes, and we thus expect that it maps some subsets of observables into compatible ones, but not all.

**Definition 1.** Let $\Lambda$ be a channel and $\mathcal{A}$ an incompatible subset of observables. If the image $\Lambda(\mathcal{A})$ is compatible, we say that $\Lambda$ breaks the incompatibility of $\mathcal{A}$.

As an example, let us consider quantum channels $\Gamma_{t,\Theta,\eta} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ of the form

$$\Gamma_{t,\Theta,\eta}(T) = t\Theta(T) + (1 - t)\text{tr} [\eta T]1,$$

where $\eta$ is a fixed state, $\Theta$ is a fixed channel, and $0 \leq t \leq 1$. The channel $\Gamma_{t,\Theta,\eta}$ is thus a mixture of $\Theta$ and the channel $T \mapsto \text{tr} [\eta T]1$, which in the Schrödinger picture maps each state to the fixed state $\eta$. It is natural to regard $\Gamma_{t,\Theta,\eta}$ as a noisy version of $\Theta$, and since the channel $T \mapsto \text{tr} [\eta T]1$ breaks the incompatibility of any collection of observables, we expect that the same is true for any $\Gamma_{t,\Theta,\eta}$ with $t \leq t_0$ for some critical value $t_0$. In the following we give a bound that is independent of $\Theta$. 

**Proposition 3.** Let \( n \geq 2 \) be an integer. For all \( 0 \leq t \leq \frac{1}{n} \), the quantum channel \( \Gamma_{t,\Theta,n} \) breaks the incompatibility of an arbitrary set \( \{A_1, \ldots, A_n\} \) of \( n \) observables.

**Proof.** For a collection of \( n \) observables \( A_1, \ldots, A_n \), we define a map \( G \) by

\[
G(a_1, \ldots, a_n) = t \Theta(a_1) \cdot \prod_{j=1}^{n} \text{tr} \left[ \eta A_j(a_j) \right] \\
+ \cdots + t \Theta(a_n) \cdot \prod_{j=1}^{n} \text{tr} \left[ \eta A_j(a_j) \right] \\
+ (1 - tn) \cdot \prod_{j} \text{tr} \left[ \eta A_j(a_j) \right]
\]

Then \( G(a_1, \ldots, a_n) \geq 0 \) and

\[
G(X_1 \times \Omega_{k_1} \times \cdots \times \Omega_{k_n}) = \Gamma_{t,\Theta,n}(A(X_1))
\]

and similarly for the other marginals. Hence, \( G \) is a joint observable for the observables \( \Gamma_{t,\Theta,n}(A_1), \ldots, \Gamma_{t,\Theta,n}(A_n) \).

Motivated by this example we now introduce the following concept.

**Definition 2.** Let \( \Lambda \) be a quantum channel on \( \mathcal{L}(\mathcal{H}) \). If \( \Lambda \) breaks the incompatibility of every collection of \( n \) observables, then \( \Lambda \) is called \( n \)-incompatibility breaking. We let \( n-\text{IBC} \) denote the set of all \( n \)-incompatibility breaking channels.

The following basic properties of the sets \( n-\text{IBC} \) are immediate.

**Proposition 4.** Each \( n-\text{IBC} \) is a convex subset of \( C \). With respect to the channel concatenation relation, \( n-\text{IBC} \) is an ideal of the semigroup \( C \), that is, if \( \Lambda \in C \) and \( \Lambda' \in n-\text{IBC} \), then \( \Lambda \circ \Lambda' \in n-\text{IBC} \). The following inclusions hold:

\[
n-\text{IBC} \subseteq \cdots \subseteq 3-\text{IBC} \subseteq 2-\text{IBC}.
\]

**Proof.** In order to show convexity, we take \( \Lambda, \Lambda' \in n-\text{IBC} \) and \( 0 \leq t \leq 1 \), and fix \( n \) observables \( A_1, \ldots, A_n \). Then the sets \( \{\Lambda(A_1), \ldots, \Lambda(A_n)\} \) and \( \{\Lambda'(A_1), \ldots, \Lambda'(A_n)\} \) have joint observables \( G \) and \( G' \), respectively. The mixture \( t G + (1-t) G' \) is then a joint observable for the observables \( (t \Lambda + (1-t) \Lambda')(A_1), \ldots, (t \Lambda + (1-t) \Lambda')(A_n) \). Hence \( t \Lambda + (1-t) \Lambda' \in n-\text{IBC} \), i.e. \( n-\text{IBC} \) is convex. The fact that \( n-\text{IBC} \) is an ideal follows immediately from the definition, together with proposition 2. The inclusions are obvious.

We will show in section 4.2 that every subset \( n-\text{IBC} \) is strictly containing some higher subset \( m-\text{IBC} \), at least if the dimension of the Hilbert space is large enough.

**3.2. Incompatibility breaking channels and complete incompatibility**

We now proceed to introduce the key concept of the paper in its general form.

**Definition 3.** Let \( \Lambda \) be a quantum channel on \( \mathcal{L}(\mathcal{H}) \). If \( \Lambda \) breaks the incompatibility of the set of all observables, then \( \Lambda \) is called incompatibility breaking. We denote by \( \text{IBC} \) the set of all such channels.
Since the above definition requires that A maps all observables into a compatible set, the
task of determining if a given channel is incompatibility breaking appears tedious. In view of
this, it would be desirable to have some smaller ‘test sets’ of observables. This motivates the
next definition.

**Definition 4.** A set $\mathcal{A}$ of observables is said to have complete incompatibility if any channel
that breaks its incompatibility is necessarily in IBC.

We remark that any set having complete incompatibility is an optimal resource for noisy
EPR-steering, assuming that the form of the noise is unknown. In order to demonstrate basic
examples of sets having complete incompatibility, we make the following observation.

**Proposition 5.** If $\Lambda$ breaks the incompatibility of a set of observables $\mathcal{A}$, then it also breaks
the incompatibility of the convex hull of the set of all postprocessings of elements of $\mathcal{A}$ (into
other finite-outcome POVMs).

**Proof.** The claim concerning postprocessing follows immediately from the transitivity of the
postprocessing relation, i.e. if $\mathcal{M}$ is compatible with $A(a) = \int f_A(a, g)G(dg)$ for each
$A \in \mathcal{M}$, and we postprocess each $A$ further into $B(b) = \sum h(b, a)A(a)$ using an arbitrary
postprocessing function $h$, the resulting $B(b) = \int f_B(b, g)G(dg)$ is a postprocessing of $G$ with
$f_B(b, g) = \sum h(b, a)f_A(a, g)$. Furthermore, given two of such observables with the same
outcome set, the convex combination $B = \lambda B_1 + (1 - \lambda)B_2$ is also a postprocessing of $G$
with function $f(b, g) = \lambda f_{B_1}(b, g) + (1 - \lambda)f_{B_2}(b, g)$. Hence, the total set of observables
obtained from $G$ in this way is compatible.

An observable $A$ is called rank-1 if each nonzero operator $A(x)$ has rank 1. Using
proposition 5 we get the following result, exhibiting one basic set having complete
incompatibility.

**Proposition 6.** The set of all rank-1 observables with at most $d^2$ outcomes has complete
incompatibility, where $d$ is the dimension of the Hilbert space.

**Proof.** We can represent each finite-outcome observable as a convex combination of
extremal POVMs. All extremals have at most $d^2$ outcomes, and every extremal is a relabelling
of some rank one extremal [30]. Since relabelling is a special case of postprocessing,
proposition 5 completes the proof.

With this result, we are able to prove a connection between incompatibility breaking
channels, and the previously defined concept of $n$-incompatibility breaking channels.

**Proposition 7.** $IBC = \bigcap_{n \geq 2} n-IBC$. 
Proof. The inclusion $\text{IBC} \subseteq \bigcap_{n \geq 2} \mathbf{n} - \text{IBC}$ is trivial. For the converse, let $\Lambda$ denote the set of all observables with at most $d^2$ outcomes. By proposition 6, $\Lambda$ has complete incompatibility, so in order to establish $\bigcap_{n \geq 2} \mathbf{n} - \text{IBC} \subseteq \text{IBC}$ it suffices to prove that $\Lambda(\mathcal{A})$ is compatible for $\Lambda \in \bigcap_{n \geq 2} \mathbf{n} - \text{IBC}$. By adding trivial outcomes if necessary, we can assume that each $\mathcal{A} \in \Lambda$ has the outcome set $\Omega = \{1, 2, \ldots, d^2\}$. By Tychonov’s theorem (a standard result in general topology, see e.g. [31]), the cartesian product set $\Omega_0 = \Omega^4$ is compact in the product topology. For each $\mathcal{A} \in \Lambda$, let $\pi_\mathcal{A}: \Omega_0 \to \Omega$ be the canonical projection. If $\Lambda \in \bigcap_{n \geq 2} \mathbf{n} - \text{IBC}$, then for any finite collection $\mathcal{F} \subseteq \mathcal{A}$, the image $\Lambda(\mathcal{F})$ is compatible, and there exists a joint observable $\mathcal{G}$ on the finite product space $\Omega^\mathcal{F}$. Since $\Omega_0 = \Omega^\mathcal{F} \times \Omega^\mathcal{A} \setminus \mathcal{F}$, we can trivially extend $\mathcal{G}$ to an observable $\mathcal{G}_{\mathcal{F}}$ on $\Omega_0$ by defining $\mathcal{G}_{\mathcal{F}}(X \times Y) = \mathcal{G}(X)\mu(Y)$ for all $X \subseteq \Omega^\mathcal{F}$ and $Y$ in the Borel $\sigma$-algebra $\mathcal{B}(\Omega^\mathcal{F})$ of subsets of the (possibly infinite) set $\Omega^\mathcal{F}$, where $\mu$ is any probability measure. Then $\Lambda = \mathcal{G}_{\mathcal{F}} \circ \pi_\mathcal{A}^{-1}$ for all $\mathcal{A} \in \mathcal{F}$.

Now for any $\mathcal{A} \in \Lambda$, let $\mathcal{M}_\mathcal{A}$ denote the set of all observables $\mathcal{G}$ on $\Omega_0$ such that $\Lambda(\mathcal{A}) = \mathcal{G} \circ \pi_\mathcal{A}^{-1}$. Then the above results imply that the collection $\{\mathcal{M}_\mathcal{A}\}_{\mathcal{A} \in \Lambda}$ has the outcome set $\bigcap_{n \geq 2} \mathbf{n} - \text{IBC}$ and this subset is closed because the limit of a pointwise convergent net of bilinear maps is clearly bilinear. Hence $\mathcal{M}_\mathcal{A}$ is compact in this topology. By Tychonov’s theorem, the cartesian product of these topological spaces is also compact. Thus for any $\mathcal{A} \in \Lambda$, let $\mathcal{M}_\mathcal{A}$ be the closed set of all observables $\mathcal{G}$ on $\Omega_0$ such that $\Lambda(\mathcal{A}) = \mathcal{G} \circ \pi_\mathcal{A}^{-1}$. Then the above results imply that the collection $\{\mathcal{M}_\mathcal{A}\}_{\mathcal{A} \in \Lambda}$ is compact in this topology. The subset $\mathcal{M}_\mathcal{A}$ is trivial. For the converse, let $\mathcal{M}_\mathcal{A}$ be a net in $\mathcal{M}_\mathcal{A}$ converging to a map $\mathcal{G}: \Omega \to \mathcal{C}$ is a function, then $\mathcal{M}_\mathcal{A}$ is a net of elements of $\mathcal{M}_\mathcal{A}$ converging to a map $\mathcal{M}: \Omega \to \mathcal{C}$ is a function, then $\mathcal{M} \circ \pi_\mathcal{A}^{-1} = \mathcal{A}$. Hence $\mathcal{M}_\mathcal{A}$ is compact.

The above result thus gives another way of checking whether or not a channel is incompatibility breaking or not, namely, by checking the compatibility of the images $\Lambda(\mathcal{A}_1), \ldots, \Lambda(\mathcal{A}_n)$ of finite sets of observables.

5 A regular Borel POVM on $\Omega_0$ is a positive operator valued measure $\mathcal{G}: \mathcal{B}(\Omega_0) \to \mathcal{L}(\mathcal{H})$ such that the total variation of each complex measure $X \mapsto \text{tr}[\mathcal{G}(X)]$ is a regular Borel measure for each (trace class) operator $T$.

6 In fact, for each pair $(T, f) \in \mathcal{L}(\mathcal{H}) \times C(\Omega_0)$, define the compact sets $\mathcal{S}_{T, f} = \{ \lambda \in \mathbb{C}, \langle \lambda \rangle \leq \| T \| / \| f \| \}$ (where $\| T \|$ is the trace norm). By Tychonov’s theorem, the cartesian product of these topological spaces is also compact. The product is the set of functions $B: \mathcal{L}(\mathcal{H}) \times C(\Omega_0) \to \mathcal{C}$ with $\| B(T, f) \| \leq \| T \| / \| f \|$, equipped with the topology of pointwise convergence. By the standard duality, the subset of bilinear functions $B$ correspond exactly to the elements of $\mathcal{L}(C(\Omega_0), \mathcal{L}(\mathcal{H}))$, and this subset is closed because the limit of a pointwise convergent net of bilinear maps is clearly bilinear. Hence $\mathcal{M}_\mathcal{A}$ is compact.
4. Breaking incompatibility with white noise

As a special instance of the previously defined class of noisy channels $\Gamma_{\theta, n}$, we have the class of\textit{ depolarizing channels} $\Gamma_{dep}^t \equiv \Gamma_{\theta, n, 1/d}$, $0 \leq t < 1$. The action of these types of channels is very simple:

$$\Gamma_{dep}^t(A) = tA + (1 - t) \frac{1}{d} \text{tr}[A]I,$$

In the Schrödinger picture the parameter $1 - t$ represents the amount of white noise (understood as the completely depolarizing channel) mixed into the state $\rho$. With $t = 1$ the channel is identity, hence clearly does not break the incompatibility of any set. The values of $1 - t$ for which $\Gamma_{dep}^t$ becomes $n$-incompatibility breaking and incompatibility breaking, respectively, represent the robustness of the incompatibility of the corresponding sets of observables against white noise. Similar ideas on noise-robustness of incompatibility have been recently considered in [10, 33] for pairs of observables.

We note that the noisy spin measurements in section 2.2 are obtained precisely by applying $\Gamma_{dep}^t$ to the ideal ones. In particular, this means that $\Gamma_{dep}^1$ breaks the incompatibility of the set of all projective spin-1/2 measurements. We now proceed to study more general sets of observables.

4.1. Robustness of incompatibility of finite collections of observables

For depolarizing channels we have the following improvement of proposition 3. It describes the amount of white noise needed to turn all finite collections of observables compatible.

**Proposition 8.** The channel $\Gamma_{dep}^t$ is $n$-incompatibility breaking for all

$$0 \leq t \leq \frac{n + d}{n(d + 1)}$$

where $d = \dim \mathcal{H}$.

**Proof.** Following the approximate cloning scheme of [34], we can proceed in the same way as in [35] to find a sufficient condition for the compatibility of noisy versions of any $n$ observables. Let $\{A_1, \ldots, A_n\}$ be a set of observables. We define $G$ as

$$G(x_1, x_2, \ldots, x_n) = \frac{d}{d + n - 1} \text{tr}[S_n A_1(x_1) \otimes A_2(x_2) \otimes \cdots \otimes A_n(x_n) S_n]$$

for all $x_j \in \Omega_j$, where $S_n$ is the projection from $\mathcal{H}_d^{\otimes n}$ to its symmetric subspace and $\text{tr}[\cdot]$ is partial trace over all but one part of the system. A direct calculation shows that $G$ is a joint observable for the observables $\Gamma_{dep}^t(A_1), \ldots, \Gamma_{dep}^t(A_n)$ with $t = \frac{n + d}{n(d + 1)}$.

4.2. 2–IBC $\neq$ 3–IBC

As a consequence of proposition 8 we can demonstrate that there are channels that are 2-incompatibility breaking but not 3-incompatibility breaking. Let us consider three qubit
observables

\[
X(\pm 1) = \frac{1}{2}(1 \pm \sigma_x),
\]

\[
Y(\pm 1) = \frac{1}{2}(1 \pm \sigma_y),
\]

\[
Z(\pm 1) = \frac{1}{2}(1 \pm \sigma_z),
\]

The image of $X$ under the action of $\Gamma_t^{\text{dep}}$ is

\[
\Gamma_t^{\text{dep}}(X)(\pm 1) = \frac{1}{2}(1 \pm t\sigma_x),
\]

and similarly for $Y$ and $Z$. If we choose $t$ such that $1/\sqrt{3} < t$, then the observables $\Gamma_t^{\text{dep}}(X)$, $\Gamma_t^{\text{dep}}(Y)$ and $\Gamma_t^{\text{dep}}(Z)$ are incompatible [37] and hence $\Gamma_t^{\text{dep}}$ is not 3-incompatibility breaking. But if $t$ also satisfies $t \leq 2/3$, then by proposition 8 ($d = 2$, $n = 3$) the channel $\Gamma_t^{\text{dep}}$ is 2-incompatibility breaking. In conclusion, $3 - \text{IBC} \subseteq \text{IBC}$.

In the following we want to extend the previous observation and to show that every set $n - \text{IBC}$ is strictly containing some higher set $m - \text{IBC}$ at least in some Hilbert space with high enough dimension. For this purpose, we recall that for every integer $n \geq 3$, it is possible to construct a set of $n$ incompatible observables such that every subset of $n - 1$ observables is compatible [36]. We say that this kind of set of observables is a Specker set of order $n$ (see [37] for an explanation of Specker’s parable of the overprotective seer).

The explicit construction presented in [36] uses a Clifford algebra $\text{CL}(m)$ of $m$ generators for a Specker set of order $m$, and the observables are very similar to those in (15)–(17). The dimension of the Hilbert space is then the same as the chosen representation of $\text{CL}(m)$. If $m$ is even, then $\text{CL}(m)$ has a single irreducible representation of degree $2^{m/2}$, and if $m$ is odd, then $\text{CL}(m)$ has two irreducible representations, each of degree $2^{(m-1)/2}$. Furthermore, for odd $m$ we can make use of the explicit form of one of the representations that is known.

**Proposition 9.** For every integer $n \geq 3$ and any odd integer $m \geq n^2$, the sets $n - \text{IBC}$ and $m - \text{IBC}$ are different in the Hilbert space of dimension $2^{\frac{m+1}{2}}$.

**Proof.** Fix an integer $p \geq (n^2 - 1)/2$ and a matrix representation of Clifford algebra in the Hilbert space of dimension $d = 2^p$. The representation consists of $m = 2p + 1$ selfadjoint matrices $\delta_j$ that define $m$ different binary observables $A_j$ as

\[
A_j(\pm) := \frac{1}{2}(1 \pm \delta_j).
\]

We have

\[
\Gamma_t^{\text{dep}}(A_j)(\pm) = \frac{1}{2}(1 \pm t\delta_j),
\]

hence from [36] we know that the observables $\Gamma_t^{\text{dep}}(A_1), \ldots, \Gamma_t^{\text{dep}}(A_m)$ are incompatible if and only if $t > \frac{1}{\sqrt{m}}$. That is, the channel $\Gamma_t^{\text{dep}}$ is not $m$-incompatibility breaking for any $t > \frac{1}{\sqrt{m}}$. On the other hand, if $t$ satisfies the inequality in (14), then $\Gamma_t^{\text{dep}}$ is $n$-incompatibility breaking. There exists a $t$ that simultaneously satisfies these two inequalities if
\[ \frac{1}{\sqrt{2p+1}} < \frac{n + 2^p}{n(2^p + 1)}. \]  

which is equivalent to

\[ n < \frac{2^p \sqrt{2p + 1}}{2^p + 1 - \sqrt{2p + 1}}. \]  

This inequality is satisfied since \( n \leq \sqrt{2p + 1}. \)  

### 4.3. Robustness of incompatibility of the set of all measurements

We now proceed to derive a bound for the noise parameter \( t \) for which the depolarizing channel \( \Gamma_{2p}^{t} \) becomes incompatibility breaking. Our goal is to show that this can be done using the hidden state models appearing in the context of steering. The reason why this works is the one-to-one correspondence between steerability and incompatibility, described in detail in section 2.3. In the fundamental level, the correspondence is between hidden state models and joint observables.

We begin with a very brief account of the relevant hidden state models; for more discussion we refer the reader to the comprehensive review [41]. The history begins with Werner’s seminal result [38], adapted to the steering context by Wiseman et al [11]. The idea in the latter was to obtain hidden state models for the isotropic states to derive steerability conditions for the set of projective measurements. A generalisation by Almeida et al [39] (based on [40]) involves rank-1 POVMs; it has been adapted to steering in [29]. In these models, the probability space for the hidden variable is always the unitary group \( U(d) \), with the normalized Haar measure \( U_d \), and the hidden states are simply given by

\[ U \mapsto U \left| \varphi_0 \right> \left< \varphi_0 \right| U^*. \]

where \( \varphi_0 \) is some fixed fiducial state the choice of which does not play a role. The following point is crucial: according to the discussion in section 2.3, in any scenario where EPR-steering via full Schmidt rank pure state is prevented by this particular hidden state model, the \( U(d) \)-covariant POVM

\[ G(Z) = d \int_Z U \left| \varphi_0 \right> \left< \varphi_0 \right| U^* dU \]

defines the joint observable for Alice’s measurements. Using this fact, in combination with propositions 1 and 6, we can therefore translate the above cited hidden state results into statements of incompatibility breaking with the depolarizing channel. In this translation, the response functions associated to Alice’s measurements in the hidden state model become exactly the postprocessing functions connecting these measurements to the above POVM \( G \).

Since the translation is not entirely straightforward (we need e.g. Propositions 5 and 6), and because we wish to stress that all our results on incompatibility breaking channels can be understood independently on the steering connection, we provide direct proofs. However, we omit lengthy computations which can be directly taken from the hidden state context, more specifically [41], where the interested reader can find more details.

The first result, adopted from [11], concerns the noise-robustness of the total set of all projective measurements \( \mathcal{P} \).
Proposition 10. The depolarizing channel $\Gamma_{DP}$ breaks the incompatibility of $\mathcal{P}$ if

$$t \leq t_p \equiv \frac{1}{d - 1} \left( -1 + \sum_{k=1}^{d} \frac{1}{k} \right).$$

(21)

Proof. By proposition 5 we can restrict to nondegenerate observables, i.e., observables given by $A(i) = |\psi_i\rangle\langle\psi_i|$ where $\{|\psi_i\rangle\}$ is an orthonormal basis of $\mathcal{H}$. For any such $A$, define the following postprocessing function (appearing also in equation (6) of [39] and in equation (4.17) of [11]):

$$f^k(i, U) = \begin{cases} 1 & \text{if } \langle U\varphi_0|A(i)U\varphi_0\rangle = \max_j \langle U\varphi_0|A(j)U\varphi_0\rangle \\ 0 & \text{otherwise.} \end{cases}$$

To see what kind of observable is obtained by postprocessing $\mathcal{G}$ with this function, we need to calculate the integral

$$\int \int f^k(i, U) d\mathcal{G}(U) = d \int \int f^k(i, U) U|\varphi_0\rangle\langle\varphi_0|U^* dU.$$

(22)

First, we express the vector $U\varphi_0$ as

$$U\varphi_0 = \sum_{k=1}^{d} (x_k + iy_k)\psi_k = \sum_{k=1}^{d} \sqrt{u_k} e^{i\theta_k}\psi_k,$$

(23)

where $u_k = x_k^2 + y_k^2$, so the integral over the unitary group is replaced with the integral over the unit sphere of $\mathbb{C}^d$, that is,

$$dU = N_d \delta \left( 1 - \sum_{l=1}^{d} u_l \right) \prod_{k=1}^{d} \ du_k \ d\theta_k$$

(24)

where $\delta$ is the delta function and $N_d$ is the normalization factor. Since now $\langle U\varphi_0|A(j)U\varphi_0\rangle = u_j$, the postprocessing function in these new parameters reads

$$f^k(i, u) = \prod_{j=1}^{d} \Theta(u_i - u_j)$$

(25)

where $\Theta$ is the Heaviside step function. We can now calculate the matrix elements of the operator $\int f^k(i, U) d\mathcal{G}(U)$ with respect to the basis $|\psi_i\rangle$ to obtain

$$\int f^k(i, U) d\langle \psi_m| \mathcal{G}(U)|\psi_i\rangle = d \int f^k(i, U) \langle \psi_m|U\varphi_0\rangle\langle U\varphi_0|\psi_i\rangle dU$$

$$= d \int \prod_{j=1}^{d} \Theta(u_i - u_j) \sqrt{u_m u_n} e^{i(\theta_m - \theta_n)} N_d \delta \left( 1 - \sum_{l=1}^{d} u_l \right) \prod_{k=1}^{d} \ du_k \ d\theta_k$$

$$= \delta_{mn} (2\pi)^d N_d \int \prod_{j=1}^{d} \Theta(u_i - u_j) u_m \delta \left( 1 - \sum_{l=1}^{d} u_l \right) \prod_{k=1}^{d} \ du_k.$$

The normalization factor $N_d$ can be obtained via straightforward calculation:

$$1 = \int d\langle \psi_i| \mathcal{G}(U)|\psi_i\rangle = (2\pi)^d N_d \int u_i \delta \left( 1 - \sum_{l=1}^{d} u_l \right) \prod_{k=1}^{d} \ du_k$$

$$= \frac{(2\pi)^d}{(d - 1)!} N_d.$$
From the above expressions we notice that the operator \( \int f_A(i, U) dG(U) \) is diagonal with respect to the basis \( \{ \psi_i \} \), and all of the matrix elements \( \int f_A(i, U) d\langle \psi_m | G(U) | \psi_n \rangle \) with \( m \neq i \) are equal. It is therefore sufficient to calculate the \( i \)th diagonal element which leads to the integral
\[
d! \int \prod_{j=1}^d \Theta(u_j - u_j) u_i \delta \left( 1 - \sum_{j=1}^d w_j \right) \prod_{k=1}^d dw_k
\]
which has been calculated explicitly in equations (A.15)-(A.19) of [41]. Putting all the pieces together, we obtain
\[
\int f_A(i, U) d\langle \psi_m | G(U) | \psi_n \rangle = t_\gamma \delta_{mn} \delta_{mi} + \left( 1 - t_\gamma \right) \frac{1}{d} \delta_{mn}
\]
which gives us the desired operator equality
\[
\int f_A(i, U) dG(U) = t_\gamma A(i) + \left( 1 - t_\gamma \right) \frac{1}{d} 1 = \Gamma_{\gamma \gamma}^{\text{dep}}(A(i)).
\]

We now proceed to consider non-projective observables. For this purpose we denote by \( \mathcal{R}_1 \) the set of all rank-1 observables with finite number of outcomes. The following result is adapted from [39]; we also mention that it has already been noted in the context of incompatibility in [29, 42].

**Proposition 11.** The depolarizing channel \( \Gamma_{\gamma \gamma}^{\text{dep}} \) breaks the incompatibility of \( \mathcal{R}_1 \) if
\[
t \leq t_0 \equiv \frac{(3d - 1)(d - 1)^{d-1}}{(d + 1)d^d}.
\]

**Proof.** Following the method of Barrett [40], for an observable \( A \in \mathcal{R}_1 \) we define the postprocessing functions
\[
f_A(i, U) = \Theta \left( \langle U\varphi_0 | A(i) U\varphi_0 \rangle - \text{tr} \left[ A(i) \right] / d \right) \langle U\varphi_0 | A(i) U\varphi_0 \rangle + \frac{\text{tr} \left[ A(i) \right]}{d} \sum_j \langle U\varphi_0 | A(i) U\varphi_0 \rangle \times \left( 1 - \Theta \left( \langle U\varphi_0 | A(j) U\varphi_0 \rangle - \text{tr} \left[ A(i) \right] / d \right) \right)
\]
where \( \Theta \) is again the Heaviside function. Let \( n \) denote the number of outcomes of \( A \). Since \( A \) is a rank-1 POVM, there exist unit vectors \( \phi_1, \ldots, \phi_n \in \mathcal{H} \) and numbers \( \alpha_1, \ldots, \alpha_n \in (0, 1] \) such that \( A(i) = \alpha_i | \phi_i \rangle \langle \phi_i | \). This implies that
\[
\Theta \left( \langle U\varphi_0 | A(j) U\varphi_0 \rangle - \text{tr} \left[ A(i) \right] / d \right) = \Theta \left( \left| \langle U\varphi_0 | \phi_j \rangle \right|^2 - 1 / d \right)
\]
and we can proceed as in the proof of proposition 10 by fixing an orthonormal basis of \( \mathcal{H} \), expressing \( U\varphi_0 \) in this basis, and switching the integration over the unitary group to integration over the unit sphere of \( \mathbb{C}^d \). This leads to the integrals appearing in p.15 in [41] (computed in the appendix of that paper), and results in
\[
\int f_A(i, U) dG(U) = t_0 A(i) + \left( 1 - t_0 \right) \frac{\text{tr} \left[ A(i) \right]}{d} 1 = \Gamma_{\gamma \gamma}^{\text{dep}}(A(i)).
\]
According to proposition 6, the set $\mathcal{R}_1$ has complete incompatibility, and hence is large enough to determine whether a given channel is incompatibility breaking. We thus conclude the following.

**Theorem 1.** The depolarizing channel $\Gamma_{\text{dep}}$ is incompatibility breaking if (29) holds.

We note that for large $n$ and small $d$, the bound in (29) can be smaller than the one given in (14); see figure 2 for comparison. This demonstrates the fact that at the latter is far from being tight. Hence, when investigating if a channel $\Gamma_{\text{dep}}$ is $n$-incompatibility breaking one should check both bounds.

The question of whether the bound (29) is tight is obviously interesting; we do not know the answer. The basic reason why this converse question is difficult lies in the constructive nature of the proof; if we can find a joint measurement for some value of $t$ (or, equivalently, a hidden state model), then we know that incompatibility has been broken for that value. However, it is obviously much more difficult to prove that there exists no such joint measurement for a given value of $t$.

Actually, even the following more fundamental question appears to be open: is there a $t$ such that $\Gamma_{\text{dep}}$ breaks the incompatibility of $\mathcal{P}$, but is not incompatibility breaking? Clarifying this question would be extremely interesting in view of the following fundamental conjecture: The set of all projective measurements does not have complete incompatibility (in the sense of definition 4).

5. Connection to entanglement breaking channels

In this section we make an important connection to the existing literature on decoherence-inducing channels: we prove that the entanglement breaking channels lie at the 'bottom' of our $n$--IBC hierarchy, forming a proper subset of each of these classes (see figure 3). We then proceed to show that the same is true for IBC. This last observation has been made independently in [42].

We recall that a quantum channel $\Lambda$ is called entanglement breaking if the bipartite state $(\Lambda \otimes \text{Id})(\varrho)$ is separable for all initial states $\varrho$. In particular, this means that the local classical model always exists for any collection of measurements performed after the application of the channel. We denote by EBC the set of all entanglement breaking channels. The structure of these channels is well known: in a finite dimensional Hilbert space every entanglement breaking channel $\Lambda$ can be written in the form

$$\Lambda(T) = \sum_x \text{tr} \left[ \varrho_x T \right] F(x),$$

where $F$ is an observable with a finite number of outcomes and each $\varrho_x$ is a state [17].

It is easy to see that every entanglement breaking channel is $n$-incompatibility breaking for all $n$, i.e., $\text{EBC} \subseteq n$--IBC. Namely, let $\Lambda$ be as in (32) and suppose that $A_1, \ldots, A_n$ are incompatible observables. We define an observable $G$ by

$$G(a_1, \ldots, a_n) = \sum_x \text{tr} \left[ \varrho_x A_1(a_1) \cdots \varrho_x A_n(a_n) \right] F(x).$$

Then

$$G(X_1 \times \Omega_{A_1} \times \cdots \times \Omega_{A_n}) = \sum_x \text{tr} \left[ \varrho_x (X_1) \right] F(x)$$

(34)
for $X_i \subseteq \Omega_i$, and similarly for the other marginals. Hence, $G$ is a joint observable for the observables $\Lambda(A_1), \ldots, \Lambda(A_n)$. This together with proposition 7 implies that every entanglement breaking channel is incompatibility breaking. One can also see this directly by noticing that an observable $A$ is mapped into

$$\Lambda(A(a)) = \sum_x \text{tr} \left[ \rho_x A(a) \right] F(x),$$

that is, each $\Lambda(A)$ is postprocessing of the observable $F$; this means that all observables $\Lambda(A)$ are compatible.
Our next observation is that there are also other $n$-incompatibility breaking channels than just the entanglement breaking channels. To see this, we fix an orthonormal basis $\{ \phi_j \}_{j=1}^d$ of $\mathcal{H}$ and denote

$$\psi_0 = \frac{1}{\sqrt{d}} \sum_{j=1}^d \phi_j \otimes \phi_j.$$  

The pure state $|\psi_0\rangle \langle \psi_0|$ is maximally entangled, and the isotropic state

$$t |\psi_0\rangle \langle \psi_0| + (1-t) \frac{1}{d^2}$$

is entangled if and only if $t > \frac{1}{1+d}$ [16]. We recall that a channel $\Gamma$ is entanglement breaking if and only if the state $(\Gamma_\otimes \mathrm{id})(|\psi_0\rangle \langle \psi_0|)$ is separable [17]. It follows that a depolarizing channel $\Gamma^{\text{dep}}$ is entanglement breaking if and only if $t \leq \frac{1}{1+d}$. A comparison with proposition 8 shows that for each $2 \leq n \leq \dim \mathcal{H}$, we have $\text{EBC} \subset \text{n-IBC}$.

The remaining question is whether there exist other incompatibility breaking channels than just the entanglement breaking ones.

**Theorem 2.** Every entanglement breaking channel is incompatibility breaking, but the converse does not hold.

**Proof.** We consider again the depolarizing channel $\Gamma^{\text{dep}}$. The crucial point is that this is known to be entanglement breaking if and only if $t \leq 1/(d+1)$. The upper bound $1/(d+1)$ is smaller than the upper bound $t_0$ in (29), hence choosing any $t$ between $1/(d+1) < t \leq t_0$ gives a channel $\Gamma^{\text{dep}}$ that is incompatibility breaking but not entanglement breaking.

Therefore we only need to show that such a $t$ can be chosen. We first write $t_0$ as

$$t_0 = \frac{1}{d+1} \left( 3 - \frac{1}{d} \right) \left( 1 - \frac{1}{d} \right)^{d-1}.$$  

We can then see that $(1 - 1/d)^{d-1}$ is a monotonically decreasing function of $d$ with its limit being $1/e$; hence we find

$$t_0 \geq \frac{1}{d+1} \left( \frac{3}{d} - \frac{1}{de} \right).$$  

The term in brackets is strictly larger than one for $d \geq 4$. For $d = 2, 3$ it is a simple exercise to check that $t_0 > 1/(d+1)$.

From the above proof we also see that with increasing dimension $d$ the gap between $t_0$ and $1/(d+1)$ closes, as we can bound $t_0$ easily also from above by $3/(d+1)$. This might suggest that distinguishing $\text{EBC}$ from $\text{IBC}$ becomes more difficult in large-dimensional systems; however it may also be simply due to the bound in theorem 1 not being tight. Hence, this adds further motivation to more detailed investigation of this bound.

6. Conclusion and outlook

As demonstrated by the EPR-steering application, incompatibility of measurements constitutes a crucial resource for quantum information protocols. In view of practical
applications, it is therefore important to establish how it responds to noise. In this paper we have initiated the qualitative study of the evolution of incompatibility under noisy channels, by investigating the structure of the channels that completely destroy the incompatibility of relevant sets of measurements.

In particular, we have defined the set $n-\text{IBC}$ of channels that break the incompatibility of each collection of $n$ measurements. These sets are included within each other forming a chain $n-\text{IBC} \subseteq \cdots \subseteq 3-\text{IBC} \subseteq 2-\text{IBC}$ (see figure 3).

Furthermore, we have defined the set $\text{IBC}$ of channels breaking the incompatibility of the total set of all observables. Understanding the structure of this set is essential for the study of incompatibility on noisy systems, and we have demonstrated how this structure can be probed using the hidden state models appearing in the context of steering. We stress that the latter is not merely a technical tool, but comes with the following conceptually interesting observation: having a bipartite quantum system between Alice and Bob in an initially maximally entangled state subjected to local noise, the resulting loss of quantum nonlocality (in the specific sense of emergence of a hidden state model) can be equivalently ascribed to the loss of a local quantum feature (incompatibility of Alice’s measurements used in the model). We have also demonstrated another aspect of this connection: every entanglement breaking channel is incompatibility breaking, and shown by example that the converse does not hold.

In realistic quantum systems, the basic source of noise comes from their dynamical interaction with environment. While we have not explicitly discussed dynamical aspects in this paper, the above results can naturally be applied in this context. In particular, given any dynamical map, i.e. a family $t \mapsto \Lambda_t$ of channels indexed by time, the basic question is at which times $\Lambda_t$ becomes incompatibility breaking. This question is crucial in view of any practical implementations of, say, quantum key distribution protocols based on EPR-steering (see section 1). In the case where $t \mapsto \Lambda_t$ is a semigroup, we know from proposition 2 that after entering $\text{IBC}$ at some time $t_0$, the dynamical map $\Lambda_t$ will also stay there for all $t \geq t_0$. Hence, the results of propositions 10 and 11, although involving the very special noise addition scheme, are actually expected to illustrate a rather general feature of ‘incompatibility degradation’ by noise. This naturally suggests interesting further research on incompatibility breaking dynamical maps for open quantum systems.

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