Reduced genus-two Gromov-Witten Invariants
for complex manifolds

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Abstract

In this article, we construct the reduced genus-two Gromov-Witten invariants for certain almost Kähler manifold \((X,\omega,J)\) such that \(J\) is integrable and satisfies some regularity conditions. In particular, the standard projective space \((\mathbb{P}^n,\omega_0,J_0)\) of dimension \(n\leq 7\) satisfies these conditions. This invariant counts the number of simple genus-two \(J\)-holomorphic curves that satisfy appropriate number of constraints.

Key words: Reduced Gromov-Witten invariant, pseudocycle, orbifold, obstruction, gluing.

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Running head: Reduced genus-two Gromov-Witten Invariants

1 Introduction and main results

This article is devoted to a study on the reduced genus-two Gromov-Witten invariants.

In symplectic topology and algebraic geometry, Gromov-Witten invariants are rational numbers that, in certain situations, count pseudo holomorphic curves satisfying prescribed conditions in a given symplectic manifold. These invariants have been used to distinguish symplectic manifolds. They also play a crucial role in enumerative algebraic geometry and were inspired by the closed

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type IIA string theory. An early form of the invariants was used by Gromov in [G] (also see [M1]) to obtain important results on symplectic manifolds. The genus zero Gromov-Witten invariants for semi-positive symplectic manifolds were first studied by Ruan in distinguishing symplectic manifolds in [R1] and [R2]. The first general construction of the Gromov-Witten invariants were constructed by Ruan and Tian in [RT1] and [RT2] for semi-positive symplectic manifolds. They constructed these invariants by using solutions of the inhomogeneous Cauchy-Riemann equation. The invariants with fixed marked points also arose in the context of sigma models and were considered by Witten in [W] in early 90’s. Later, in [KM], Kontsevich and Manin formulated the Gromov-Witten invariants in a more algebraic setting. In 1995/1996, using the new technique of virtual cycle constructions, the Gromov-Witten invariants were constructed for general algebraic manifolds first by Li-Tian [LT1] and then for general symplectic manifolds by Fukaya-Ono [FO], Li-Tian [LT2], Siebert [Si] and Ruan [R3].

Let \((X, \omega)\) be a compact symplectic manifold of dimension \(2n\). The Gromov-Witten invariants are given as homomorphisms

\[
GW^X_{g,k,A} : H^*(X, \mathbb{Q})^\otimes k \to H^*(\mathcal{M}_{g,k}; \mathbb{Q}).
\]

To construct them, let \(J\) be an almost complex structure on \(X\) which is tamed by \(\omega\). In this case \((X, \omega, J)\) is called a almost Kähler manifold. For \(A \in H_2(X, \mathbb{Z})\) and a pair \((g, k)\) of nonnegative integers, denote by \(\mathcal{M}_{g,k}(X, A; J)\) the moduli space of equivalence classes of stable \(J\)-holomorphic maps from nodal genus-\(g\) Riemann surfaces with \(k\) marked points in the homology class \(A\). This determines a rational virtual fundamental class of dimension

\[
\dim \mathcal{M}^{vir}_{g,k}(X, A) \equiv \dim^{vir} \mathcal{M}_{g,k}(X, A) = 2(c_1(TX), A) + 2(n-3)(1-g) + 2k.
\]

Then one can pull back cohomology classes on \(X\) and integrate them against the virtual fundamental class to get the invariants.

Denote by \(\mathcal{M}^0_{g,k}(X, A; J)\) the subspace of \(\mathcal{M}_{g,k}(X, A; J)\) consisting of stable maps \([\mathcal{C}, u]\) whose domain \(\mathcal{C}\) is a smooth Riemann surface of genus \(g\). If \((\mathbb{P}^n, \omega_0, J_0)\) is the complex projective space of complex dimension \(n\) with the standard Kähler structure and \(\ell\) is the homology class of a complex line in \(\mathbb{P}^n\), the space \(\mathcal{M}^0_{g,k}(\mathbb{P}^n, d) \equiv \mathcal{M}^0_{g,k}(\mathbb{P}^n, d\ell; J_0)\) is a smooth orbifold of dimension \(\dim \mathcal{M}^{vir}_{g,k}(\mathbb{P}^n, d\ell)\) at least for \(d \geq 2g - 1\). Moreover, \(\mathcal{M}^0_{0,k}(\mathbb{P}^n, d) \equiv \mathcal{M}^0_{0,k}(\mathbb{P}^n, d\ell, J_0)\) is a compact topological orbifold stratified by smooth orbifolds of even dimensions and \(\mathcal{M}^0_{0,k}(\mathbb{P}^n, d)\) is its main stratum. In particular, \(\mathcal{M}^0_{0,k}(\mathbb{P}^n, d)\) is a dense open subset of \(\mathcal{M}^0_{0,k}(\mathbb{P}^n, d)\).

When \(g \geq 1\), the moduli space \(\mathcal{M}_{g,k}(\mathbb{P}^n, d)\) has many irreducible components of various dimensions. In particular, \(\mathcal{M}^0_{g,k}(\mathbb{P}^n, d)\) is not dense in \(\mathcal{M}_{g,k}(\mathbb{P}^n, d)\). In fact, some components of
$\overline{M}_{g,k}(\mathbb{P}^n, d)$ have dimensions strictly larger than $\dim \overline{M}^{vir}_{g,k}(\mathbb{P}^n, d')$. Thus in general we cannot use the space $\overline{M}_{g,k}(X, A; J)$ directly to define Gromov-Witten invariants.

In [Z3], A. Zinger constructed the reduced genus-one Gromov-Witten invariants for $(X, \omega, J)$ under some regularity conditions which are satisfied for the standard $(\mathbb{P}^n, \omega_0, J_0)$. In fact, he proved that the closure $\overline{M}^0_{1,k}(X, A; J)$ of the subspace $\mathcal{M}^0_{1,k}(X, A; J)$ in $\overline{M}_{1,k}(X, A; J)$ has the structure of a compact topological orbifold stratified by smooth orbifolds of even dimensions and $\overline{M}^0_{1,k}(X, A; J)$ is the main stratum of $\overline{M}_{1,k}(X, A; J)$. Thus one can use the space $\overline{M}^0_{1,k}(X, A; J)$ to define the reduced genus-one Gromov-Witten invariants.

For the higher genus case, the space $\overline{M}_{g,k}(X, A; J)$ has many more irreducible components with dimensions strictly larger than $\dim \overline{M}^{vir}_{g,k}(X, A)$. Thus in order to define the reduced genus-$g$ Gromov-Witten type invariants, one need to construct a suitable subspace of $\overline{M}_{g,k}(X, A; J)$ containing $\mathcal{M}^0_{g,k}(X, A; J)$ which has better properties, e.g., it represents a pseudocycle of the desired dimension. Moreover, the newly defined invariants should have precisely geometric meaning, e.g., it counts the number of simple genus-$g$ pseudo holomorphic curves that pass appropriate number of constraints.

It is well-known from algebraic geometry that $\mathcal{M}^0_{2,k}(\mathbb{P}^n, d)$ is smooth of expected dimension for $d \geq 3$ and its complement in $\overline{M}^0_{2,k}(\mathbb{P}^n, d)$ is certainly of complex codimension at least one. While for general symplectic manifolds, this may not be true. In order to derive a structure analogous to the moduli space $\overline{M}^0_{2,k}(\mathbb{P}^n, d)$, we introduce the following regularity conditions:

If $u : \mathcal{C} \to X$ is a smooth map from a Riemann surface and $A \in H_2(X, \mathbb{Z})$, we write

$$u \leq_\omega A \quad \text{if} \quad u_*[\mathcal{C}] = A \quad \text{or} \quad \langle \omega, u_*[\mathcal{C}] \rangle < \langle \omega, A \rangle.$$

**Definition 1.1.** Suppose $(X, \omega, J)$ is a compact almost Kähler manifold such that $J$ is integrable and $A \in H_2(X, \mathbb{Z})$. Then complex structure $J$ is **A-regular** if the following hold:

(i) For every $J$-holomorphic map $u : \mathbb{P}^1 \to X$ such that $u \leq_\omega A$, we have $H^1(\mathbb{P}^1, u^*TX) = 0$ and $H^1(\mathbb{P}^1, u^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$.

(ii) For every nonconstant $J$-holomorphic map $u : \mathbb{P}^1 \to X$ such that $u \leq_\omega A$, we have

- (ii-a) $H^1(\mathbb{P}^1, u^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$;
- (ii-b) if there exists $z \in \mathbb{P}^1$ such that $du(z) = 0$, then $H^1(\mathbb{P}^1, u^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-3)) = 0$;
- (ii-c) if there exist $z_1, z_2 \in \mathbb{P}^1$ such that $z_1 \neq z_2$ and $u(z_1) = u(z_2)$, then $H^1(\mathbb{P}^1, u^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-3)) = 0$;
- (ii-d) if there exists $z \in \mathbb{P}^1$ such that $du(z) = 0$, then either $H^1(\mathbb{P}^1, u^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-4)) = 0$ or $u$ factor through a branched covering $\overline{u} : S^2 \to X$, i.e., there exists a holomorphic branched covering $\phi : S^2 \to S^2$ such that $u = \overline{u} \circ \phi$ and $\deg(\phi) \geq 2$;
(ii-e) if there exist \(z_1,z_2 \in \mathbb{P}^1\) such that \(z_1 \neq z_2\) and \(u(z_1) = u(z_2)\), then either \(H^1(\mathbb{P}^1, u^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-4)) = 0\) or \(u\) factor through a branched covering \(\tilde{u} : S^2 \to X\) as (ii-d).

(iii) For every nonconstant \(J\)-holomorphic map \(u : T^2 \to X\) such that \(u \leq_\omega A\), we have

(iii-a) \(H^1(T^2, u^*TX) = 0\) and \(H^1(T^2, u^*TX \otimes \mathcal{O}_{T^2}(-1)) = 0\);

(iii-b) either \(H^1(T^2, u^*TX \otimes \mathcal{O}_{T^2}(-2)) = 0\) or \(u\) factor through a branched covering \(\tilde{u} : \Sigma \to X\), i.e., there exists a holomorphic branched covering \(\phi : T^2 \to \Sigma\) such that \(u = \tilde{u} \circ \phi\) and \(\deg(\phi) \geq 2\).

(iv) For every nonconstant \(J\)-holomorphic map \(u : \Sigma \to X\) from a smooth Riemann surface of genus two such that \(u \leq_\omega A\), either \(H^1(\Sigma, u^*TX) = 0\) or \(u\) factor through a branched covering \(\tilde{u} : \Sigma' \to X\), i.e., there exists a holomorphic branched covering \(\phi : \Sigma \to \Sigma'\) such that \(u = \tilde{u} \circ \phi\) and \(\deg(\phi) \geq 2\).

(v) For every pair of nonconstant \(J\)-holomorphic maps \(u_1, u_2 : \mathbb{P}^1 \to X\) such that \(u_1, u_2 \leq_\omega A\), we have:

(v-a) for all \(z_1, z_2 \in \mathbb{P}^1\), satisfying \(u_1(z_1) = u_2(z_2)\) and \(du_2(z_2) = \lambda du_1(z_1)\) for some \(\lambda \in \mathbb{C} \setminus \{0\}\), one of the following holds: \(H^1(\mathbb{P}^1, u_1^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-3)) = 0\), \(H^1(\mathbb{P}^1, u_2^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-3)) = 0\), \(u_1(\mathbb{P}^1) = u_2(\mathbb{P}^1)\).

(v-b) for all \(z_1, z_2, z_1', z_2' \in \mathbb{P}^1\) and \(z_1 \neq z_1', z_2 \neq z_2'\) satisfying \(u_1(z_1) = u_2(z_2)\), \(u_1(z_1') = u_2(z_2')\), one of the following holds: \(H^1(\mathbb{P}^1, u_1^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-3)) = 0\), \(H^1(\mathbb{P}^1, u_2^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-3)) = 0\), \(u_1(\mathbb{P}^1) = u_2(\mathbb{P}^1)\).

It is well-known that the standard complex structure \(J_0\) on \(\mathbb{P}^n\) satisfies the above regularity conditions.

For any almost Kähler manifold \((X, \omega, J)\), let \(N = \inf\{\langle c_1(TX), u_*[\Sigma]\rangle : \partial_Ju = 0, u_*[\Sigma] \neq 0\}\), where \(\Sigma\) is a smooth Riemann surface of genus at most one.

The following are main results in this paper:

**Theorem 1.2.** Suppose \((X, \omega, J)\) is a compact almost Kähler manifold such that \(J\) is integrable, \(A \in H^2(X, \mathbb{Z}) \setminus \{0\}\) and the complex structure \(J\) is \(A\)-regular in the sense of Definition 1.1. Denote the closure of the space \(\mathcal{M}_{2,k}^0(X, A, J)\) in \(\overline{\mathcal{M}}_{2,k}(X, A, J)\) under the stable map topology by \(\overline{\mathcal{M}}_{2,k}^0(X, A, J)\). Then one of the following must hold:

(i) An element \(b \equiv [C, u] \in \overline{\mathcal{M}}_{2,k}^0(X, A, J) \setminus \mathcal{M}_{2,k}^0(X, A, J)\) if and only if \(\{D_{h_i}^{(1)}b, \ldots, D_{h_{i}}^{(k)}b\}_{h_i \in \Omega}\) satisfy a set of linear equations, where \(D_{h_i}^{(l)}b\) denotes the covariant derivative of \(u_{h_i}\) of order \(l\) at the corresponding nodal point of \(C\) with respect to some metric on \(\Sigma_P\) and \(X\), \(\Omega\) is a set consisting of certain irreducible components of \(C\), and \(k\) is at most 3. In particular, \(\overline{\mathcal{M}}_{2,k}^0(X, A, J) \setminus \mathcal{M}_{2,k}^0(X, A, J)\) is a smooth orbifold of dimension at most \(\dim \overline{\mathcal{M}}_{2,k}^{vir}(X, A) - 2\) in a neighborhood of \([C, u]\).
(ii) An element $b \equiv [C, u] \in \mathcal{M}_{2,k}^0(X, A, J) \setminus \mathcal{M}_{2,k}^0(X, A, J)$ must satisfy: there exists $b_1 \equiv [C_1, u_1] \in \mathcal{M}_{g,k}^0(X, A_1, J)$ and $b_2 \equiv [C_2, u_2] \in \mathcal{M}_{g,0}^0(X, A_2, J)$ such that $u(C) = u_1(C_1)$, $g \leq 1$, and $A - A_1 = mA_2 \neq 0$, where $m$ is a positive integer.

**Theorem 1.3.** Under the assumption of Theorem 1.2, suppose $\dim X \equiv 2n < \min\{N + 6, 2N + 6\}$. Then the evaluation map

$$ev: \mathcal{M}_{2,k}^0(X, A, J) \rightarrow X^k$$

represents a pseudocycle of dimension $\dim \mathcal{M}_{2,k}^{vir}(X, A)$, which can be used to define the reduced genus-two Gromov-Witten invariants $GW_{2,k}^0(X; \cdot)$ for $(X, \omega, J)$.

**Theorem 1.4.** Under the assumption of Theorem 1.3, suppose $(\mu_1, \ldots, \mu_k)$ is a $k$-tuple of proper submanifolds of $X$ of total codimension $\dim \mathcal{M}_{2,k}^{vir}(X, A)$ in general position. Then the invariant $GW_{2,k}^0(X; (\mu_1, \ldots, \mu_k))$ counts the signed number of simple (cf. §2.5 of [MS]) genus-two $J$-holomorphic curves with smooth domains that pass $(\mu_1, \ldots, \mu_k)$.

**Remark 1.5.** The proof of Theorem 1.2 is based on understanding the conditions under which a stable map $[C, u]$ lies in $\mathcal{M}_{2,k}^0(X, A, J)$. The precise description of these conditions are contained in §5 and §6. In the genus one case, the conditions were found in [Z3]. We believe that the reduced genus-two Gromov-Witten invariants are closely related to the standard genus-two Gromov-Witten invariants. We are going to study their relation in a separate paper.

In this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively.

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## 2 Structure of the moduli space $\mathcal{M}_{2,k}^0(X, A, J)$

In this section, we study the structure of the moduli space $\mathcal{M}_{2,k}^0(X, A, J)$ and the obstruction bundle on it. In the following of this paper, we assume $(X, A, J)$ satisfies the regularity conditions in Definition 1.1.

An element $[C, u]$ in $\mathcal{M}_{2,k}^0(X, A, J)$ is the equivalence class of a pair consisting of a connected $k$-pointed nodal genus-two Riemann surface $C$ and a $J$-holomorphic map $u : C \rightarrow X$ such that every contracted genus-0 component of $C$ contains at least three special points (i.e. node-branches and marked points) and every contracted genus-1 component contains at least one special point,
cf. Chapter 24 of [MirSym]. In general, one can use the associated graph $T_{\mathcal{C}}$ of $\mathcal{C}$ to describe $[\mathcal{C}, u]$ as in Chapter 2 of [FO].

In order to study the structure of the moduli space, we make the following definition:

**Definition 2.1.** The principle component $\Sigma_P$ of a stable map $[\mathcal{C}, u]$ in $\overline{M}_{2,k}(X, A, J)$ is the union of irreducible components $\{\Sigma_i\}_{1 \leq i \leq l}$ of $\mathcal{C}$ such that $\bigcup_{1 \leq i \leq l} \Sigma_i$ is a connected nodal surface of genus two and $l$ is the least number satisfying this property. In other words, $\Sigma_P$ is the smallest connected nodal surface in $\mathcal{C}$ of genus-two.

**Remark 2.2.** Note that by Definition 2.1, $\mathcal{C}$ is obtained from $\Sigma_P$ by attaching bubble trees, (cf. §3 below). It is easy to see that $\Sigma_P$ belongs to one of the following cases:

(i) A smooth Riemann surface of genus two.
(ii) Two smooth tori and a set of spheres.
(iii) A torus with only one node.
(iv) A smooth torus and a set of spheres, they together contains exactly one circle.
(v) A set of spheres contain exactly two circles.

We illustrate each case in the following propositions separately.

In the following, we denote by $n_{nod}$ the number of nodes in $\mathcal{C}$ and $\mathcal{M}_T(X, A, J)$ the stratum of $\overline{M}_{2,k}(X, A, J)$ of type $T$, where $T$ is the combinatorial type of $[\mathcal{C}, u]$ as in Chapter 2 of [FO].

By (iv) in Definition 1.1, we have the following:

**Proposition 2.3.** The moduli space $\mathcal{M}_{2,k}^{0,\text{simp}}(X, A, J)$ consisting of simple genus-two $J$-holomorphic curves with smooth domains is a smooth orbifold with the desired dimension $\dim \overline{M}_{2,k}^{\text{vir}}(X, A)$. □

In the following, we separate our study into several cases according the behavior of a stable map $[\mathcal{C}, u]$ in $\overline{M}_{2,k}(X, A, J)$ restricted to its principle component $\Sigma_P$.

**Proposition 2.4.** Suppose the principle component $\Sigma_P$ of a stable map $[\mathcal{C}, u]$ in $\overline{M}_{2,k}(X, A, J)$ is described in (i) of Remark 2.2, i.e., $\Sigma_P$ is a smooth Riemann surface of genus two. Then we have the following:

(i) If $u|_{\Sigma_P} \not\equiv \text{const}$ and $H^1(\Sigma, u^*TX) = 0$, then $\mathcal{M}_T(X, A, J)$ is a $\dim \overline{M}_{2,k}^{\text{vir}}(X, A) - 2n_{nod}$ dimensional smooth orbifold.

(ii) If $u|_{\Sigma_P} \not\equiv \text{const}$ and $H^1(\Sigma, u^*TX) \neq 0$, then $u|_{\Sigma_P}$ factor through a branched covering $\tilde{u} : \Sigma' \to X$, i.e., there exists a holomorphic branched covering $\phi : \Sigma_P \to \Sigma'$ such that $u|_{\Sigma_P} = \tilde{u} \circ \phi$ and $\deg(\phi) \geq 2$.

(iii) If $u|_{\Sigma_P} \equiv \text{const}$, then $\mathcal{M}_T(X, A, J)$ is a $\dim \overline{M}_{2,k}^{\text{vir}}(X, A) + 2(2n - n_{nod})$ dimensional smooth orbifold.
Proof. The proposition follows from (i) and (iv) in Definition 1.1 and the implicit function theorem. For the reader’s convenience, here we give the proof of (iii). We prove the simplest case, the general case follows similarly. Suppose there are \( m \) bubbles \( \{ C_i \}_{1 \leq i \leq m} \) attached directly to \( \Sigma_P \) and \( u_i \equiv u|_{C_i} \) is non-constant for \( 1 \leq i \leq m \). Then there is a natural isomorphism

\[
\mathcal{M}_T(X, A, J) \cong \left( \mathcal{M}_{2, k_0 + m} \times \left\{ \prod_{i=1}^{m} \mathcal{M}_{0, k_{i+1}}(X, A_i, J) : ev_{k_i + 1}(u_i) = ev_{k_{j+1}}(u_j), 1 \leq i, j \leq m \right\} \right) / S_m,
\]

where \( \mathcal{M}_{g,l} \) denotes the moduli space of smooth Riemann surfaces of genus \( g \) with \( l \) marked points. \( k_0 \) denotes the number of marked points on \( \Sigma_P \) and \( k_i \) denotes the number of marked points on \( C_i \) for \( 1 \leq i \leq m \). In particular, we have \( \sum_{i=0}^{m} k_i = k \). \( A_i \in H_2(X, \mathbb{Z}) \) and \( \sum_{i=1}^{m} A_i = A \). \( ev_{k_i+1}(u_i) \) is the evaluation map of \( u_i \) at the \( (k_i+1) \)-th marked point. \( S_m \) is the permutation group of order \( m \).

By (i) of Definition 1.1, the evaluation map

\[
ev_{k_1+1} \times \cdots \times ev_{k_m+1} : \prod_{i=1}^{m} \mathcal{M}_{0, k_{i+1}}(X, A_i, J) \to X^m
\]

is transversal to the diagonal \( \Delta \equiv \{(x, \ldots, x) \in X^m\} \). Hence the right hand side of (2.1) is a smooth orbifold by the implicit function theorem. By the index theorem, we have

\[
\dim \mathcal{M}_T(X, A, J) = \dim \mathcal{M}_{2, k_0 + m} + \sum_{i=1}^{m} \dim \mathcal{M}_{0, k_{i+1}}(X, A_i, J) - \text{codim} \Delta
\]

\[
= 2(3 + k_0 + m) + \sum_{i=1}^{m} 2((c_1(TX), A_i) + n - 3 + k_i + 1) - 2n(m - 1)
\]

\[
= 2((c_1(TX), A) + k + 3 + n - m) = \dim \mathcal{M}^{vir}_{2, k}(X, A) + 2(2n - m).
\]
Hence (iii) holds in this case. The general case follows by a similar argument: once there is one more node, the dimension decreases by 2. The proof of the proposition is complete.

**Proposition 2.5.** Suppose the principle component \( \Sigma_P \) of a stable map \([C, u]\) in \( \overline{\mathcal{M}}_{2, k}(X, A, J) \) is described in (ii) of Remark 2.2, i.e., it consists of two smooth tori \( T_1 \) and \( T_2 \) and a set of spheres \( \{ S_i \}_{1 \leq i \leq l} \). Then we have the following:

(i) If \( \deg(u|_{T_1}) \neq 0 \) and \( \deg(u|_{T_2}) \neq 0 \), then \( \mathcal{M}_T(X, A, J) \) is a \( \dim \overline{\mathcal{M}}^{\text{vir}}_{2, k}(X, A) - 2n_{\text{nod}} \) dimensional smooth orbifold.

(ii) If \( \deg(u|_{T_1}) = 0 \) and \( \deg(u|_{T_2}) \neq 0 \) or \( \deg(u|_{T_1}) \neq 0 \) and \( \deg(u|_{T_2}) = 0 \), then \( \mathcal{M}_T(X, A, J) \) is a \( \dim \overline{\mathcal{M}}^{\text{vir}}_{2, k}(X, A) + 2(n - n_{\text{nod}}) \) dimensional smooth orbifold.

(iii) If \( \deg(u|_{T_1}) = 0 \) and \( \deg(u|_{T_2}) = 0 \), then \( \mathcal{M}_T(X, A, J) \) is a \( \dim \overline{\mathcal{M}}^{\text{vir}}_{2, k}(X, A) + 2(2n - n_{\text{nod}}) \) dimensional smooth orbifold.

**Proof.** The proposition follows from (i) and (iii-a) in Definition 1.1 and the implicit function theorem.

**Proposition 2.6.** Suppose the principle component \( \Sigma_P \) of a stable map \([C, u]\) in \( \overline{\mathcal{M}}_{2, k}(X, A, J) \) is described in (iii) of Remark 2.2, i.e., a torus with only one node. Let \((T^2, z_1, z_2)\) be the normalization of \( \Sigma_P \). Then we have the following:

(i) If \( u|_{\Sigma_P} \neq \text{const} \) and \( H^1(T^2, u^*TX \otimes \mathcal{O}_{T^2}(-z_1-z_2)) = 0 \), then \( \mathcal{M}_T(X, A, J) \) is a \( \dim \overline{\mathcal{M}}^{\text{vir}}_{2, k}(X, A) - 2n_{\text{nod}} \) dimensional smooth orbifold.

(ii) If \( u|_{\Sigma_P} \neq \text{const} \) and \( H^1(T^2, u^*TX \otimes \mathcal{O}_{T^2}(-z_1-z_2)) \neq 0 \), then \( u|_{\Sigma_P} \) factor through a branched covering \( \bar{u} : \Sigma' \to X \), i.e., there exists a holomorphic branched covering \( \phi : T^2 \to \Sigma' \) such that \( u|_{\Sigma_P} = \bar{u} \circ \phi \) and \( \deg(\phi) \geq 2 \).

(iii) If \( u|_{\Sigma_P} = \text{const} \), then \( \mathcal{M}_T(X, A, J) \) is a \( \dim \overline{\mathcal{M}}^{\text{vir}}_{2, k}(X, A) + 2(2n - n_{\text{nod}}) \) dimensional smooth orbifold.

**Proof.** The proposition follows from (i) and (iii-b) in Definition 1.1 and the implicit function theorem.

**Proposition 2.7.** Suppose the principle component \( \Sigma_P \) of a stable map \([C, u]\) in \( \overline{\mathcal{M}}_{2, k}(X, A, J) \) is described in (iv) of Remark 2.2, i.e., a smooth torus \( \Sigma \equiv S_0 \) together with a set of spheres \( \{ S_i \}_{1 \leq i \leq l} \) and \( \Omega = \{ S_0, \ldots, S_i \} \subset \{ S_0, \ldots, S_l \} \) form a circle. Let \( A_1 = u_*[S_0] \) and \( A_2 = \sum_{i \in \Omega \backslash S_0} u_*[S_i] \). Then we have the following:

(i) If \( A_1 \neq 0 \) and \( A_2 \neq 0 \), then \( \mathcal{M}_T(X, A, J) \) is a \( \dim \overline{\mathcal{M}}^{\text{vir}}_{2, k}(X, A) - 2n_{\text{nod}} \) dimensional smooth orbifold.

(ii) If \( A_1 = 0 \) and \( A_2 \neq 0 \), then \( \mathcal{M}_T(X, A, J) \) is a \( \dim \overline{\mathcal{M}}^{\text{vir}}_{2, k}(X, A) + 2(n - n_{\text{nod}}) \) dimensional smooth orbifold.
Figure 2.2: Domains in Propositions 2.6 and 2.7

(iii) If \( A_1 \neq 0, A_2 = 0 \) and \( S_0 \notin \Omega \), then \( \mathcal{M}_T(X, A, J) \) is a \( \dim \overline{\mathcal{M}}_{2,k}^{vir}(X, A) + 2(n - n_{nod}) \) dimensional smooth orbifold.

(iv) If \( A_1 \neq 0, A_2 = 0 \) and \( S_0 \in \Omega \), let \( \{z_1, z_2\} = S_0 \cap \Sigma_P \setminus S_0 \), then we have:

(iv-a) If \( H^1(S_0, u^*TX \otimes \mathcal{O}_{S_0}(-z_1 - z_2)) = 0 \), then \( \mathcal{M}_T(X, A, J) \) is a \( \dim \overline{\mathcal{M}}_{2,k}^{vir}(X, A) - 2n_{nod} \) dimensional smooth orbifold.

(iv-b) If \( H^1(S_0, u^*TX \otimes \mathcal{O}_{S_0}(-z_1 - z_2)) \neq 0 \), then \( u|_{S_0} \) factor through a branched covering \( \tilde{u} : \Sigma' \to X \), i.e., there exists a holomorphic branched covering \( \phi : S_0 \to \Sigma' \) such that \( u|_{S_0} = \tilde{u} \circ \phi \) and \( \deg(\phi) \geq 2 \).

(v) If \( A_1 = 0 = A_2 \), then \( \mathcal{M}_T(X, A, J) \) is a \( \dim \overline{\mathcal{M}}_{2,k}^{vir}(X, A) + 2(2n - n_{nod}) \) dimensional smooth orbifold.

Proof. The proposition follows from (i), (ii-a), (ii-c), (iii-a) and (iii-b) in Definition 1.1 and the implicit function theorem. The simplest case that the domain \( C \) is a smooth torus \( \Sigma \equiv S_0 \) and only one sphere \( S_1 \) with \( S_0 \in \Omega \) or \( S_0 \notin \Omega \) are illustrated in Figure 2.3.

Proposition 2.8. Suppose the principle component \( \Sigma_P \) of a stable map \([C, u]\) in \( \overline{\mathcal{M}}_{2,k}(X, A, J) \) is described in (v) of Remark 2.2, i.e., a set of spheres \( \{S_i\}_{1 \leq i \leq l} \) and \( \Omega_1 = \{S_{i_1}, \ldots, S_{i_s}\} \) and \( \Omega_2 = \{S_{j_1}, \ldots, S_{j_t}\} \) form two circles. Then we have the following:

Case 1. If \( \Omega_1 \subset \Omega_2 \), we let \( A_1 = \sum_{i \in \Omega_1} u_*[S_i] \) and \( A_2 = \sum_{i \in \Omega_2 \setminus \Omega_1} u_*[S_i] \). Then we have

(i) If \( A_1 \neq 0 \) and \( A_2 \neq 0 \), then \( \mathcal{M}_T(X, A, J) \) is a \( \dim \overline{\mathcal{M}}_{2,k}^{vir}(X, A) - 2n_{nod} \) dimensional smooth orbifold.

(ii) If \( A_1 = 0, A_2 \neq 0 \), then \( \mathcal{M}_T(X, A, J) \) is a \( \dim \overline{\mathcal{M}}_{2,k}^{vir}(X, A) + 2(n - n_{nod}) \) dimensional smooth orbifold.
Figure 2.3: Domains for $S_0 \in \Omega$ and $S_0 \notin \Omega$ in Proposition 2.7

(iii) If $A_1 \neq 0$, $A_2 = 0$, then one of the following holds:

(iii-a) $\mathcal{M}_T(X, A, J)$ is a $\dim \mathcal{M}^\text{vir}_{2,k}(X, A) - 2n_{\text{nod}}$ dimensional smooth orbifold .

(iii-b) There exists $i_0 \in \Omega_1$ such that $A_1 = u_*[s_{i_0}]$ and $u|_{S_{i_0}}$ factor through a branched covering $\tilde{u}: S^2 \to X$, i.e., there exists a holomorphic branched covering $\phi: S_{i_0} \to S^2$ such that $u|_{S_{i_0}} = \tilde{u} \circ \phi$ and $\deg(\phi) \geq 2$.

(iii-c) There exist $i_1, i_2 \in \Omega_1$ such that $A_1 = u_*[S_{i_1}] + u_*[S_{i_2}]$ and $u_{i_1}(S_{i_1}) = u_{i_2}(S_{i_2})$.

(iv) If $A_1 = 0 = A_2$, then $\mathcal{M}_T(X, A, J)$ is a $\dim \mathcal{M}^\text{vir}_{2,k}(X, A) + 2(2n - n_{\text{nod}})$ dimensional smooth orbifold.

Case 2. If $\Omega_1 \setminus \Omega_2 \neq \emptyset$ and $\Omega_2 \setminus \Omega_1 \neq \emptyset$ hold, Let $A_1 = \sum_{i \in \Omega_1} u_*[S_i]$ and $A_2 = \sum_{i \in \Omega_2 \setminus \Omega_1} u_*[S_i]$. Then we have the following:

(v) If $A_1 \neq 0$ and $A_2 \neq 0$, then $\mathcal{M}_T(X, A, J)$ is a $\dim \mathcal{M}^\text{vir}_{2,k}(X, A) - 2n_{\text{nod}}$ dimensional smooth orbifold.

(vi) If $A_1 \neq 0$, $A_2 = 0$ and $\sum_{i \in \Omega_2} u_*[S_i] = 0$, then $\mathcal{M}_T(X, A, J)$ is a $\dim \mathcal{M}^\text{vir}_{2,k}(X, A) + 2(n - n_{\text{nod}})$ dimensional smooth orbifold.

(vii) If $A_1 \neq 0$, $A_2 = 0$ and $\sum_{i \in \Omega_2} u_*[S_i] \neq 0$, then one of the following holds:

(vii-a) $\mathcal{M}_T(X, A, J)$ is a $\dim \mathcal{M}^\text{vir}_{2,k}(X, A) - 2n_{\text{nod}}$ dimensional smooth orbifold .

(vii-b) There exists $i_0 \in \Omega_1$ such that $A_1 = u_*[s_{i_0}]$ and $u|_{S_{i_0}}$ factor through a branched covering $\tilde{u}: S^2 \to X$, i.e., there exists a holomorphic branched covering $\phi: S_{i_0} \to S^2$ such that $u|_{S_{i_0}} = \tilde{u} \circ \phi$ and $\deg(\phi) \geq 2$. 

10
(vii-c) There exist $i_1, i_2 \in \Omega_1$ such that $A_1 = u_1[S_{i_1}] + u_2[S_{i_2}]$ and $u_{i_1}(S_{i_1}) = u_{i_2}(S_{i_2})$.

(viii) If $A_1 = 0 = A_2$, then $\mathcal{M}_T(X, A, J)$ is a dimension $\dim \mathcal{M}_{2,k}^{vir}(X, A) - 2n_{nod}$ dimensional smooth orbifold.

**Proof.** The proposition follows from (i), (ii-a), (ii-d) and (v-b) in Definition 1.1 and the implicit function theorem. The simplest case that the domain $C$ are exactly two sphere $S_1$ and $S_2$ are illustrated in Figure 2.5.

Summing up the results above, we have the following:

**Theorem 2.9.** Suppose the regularity conditions in Definition 1.1 hold. Then we have the following:

(i) Suppose $T$ belongs to (i) of Propositions 2.4-2.8 or (iv-a) of Proposition 2.7 or (iii-a), (v), (vii-a) of Proposition 2.8. then $\mathcal{M}_T(X, A, J)$ is a dimension $\dim \mathcal{M}_{2,k}^{vir}(X, A) - 2n_{nod}$ dimensional smooth orbifold.

(ii) Suppose $T$ belongs to (ii) of Propositions 2.4, 2.6 or (iv-b) of Proposition 2.7 or (iii-b), (iii-c), (vii-b), (vii-c) of Proposition 2.8. Denote by $A_P = u_*[\Sigma_P]$. Then we have: there exists $b_1 \equiv [C_1, u_1] \in \mathcal{M}_{g,k}(X, A_1, J)$ and $b_2 \equiv [C_2, u_2] \in \mathcal{M}_{g,0}(X, A_2, J)$ such that $u(\Sigma_P) = u_1(C_1)$, $g \leq 1$, and $A_P - A_1 = mA_2 \neq 0$.

(iii) If $T$ belongs to (ii) of Propositions 2.5 or (ii), (iii) of Proposition 2.7 or (ii), (vi) of Proposition 2.8, then $\mathcal{M}_T(X, A, J)$ is a dimension $\dim \mathcal{M}_{2,k}^{vir}(X, A) - 2(n - n_{nod})$ dimensional smooth orbifold.

(iv) If $T$ belongs to (iii) of Propositions 2.4-2.6 or (v) of Proposition 2.7 or (iv), (viii) of Proposition 2.8, then $\mathcal{M}_T(X, A, J)$ is a dimension $\dim \mathcal{M}_{2,k}^{vir}(X, A) - 2(2n - n_{nod})$ dimensional smooth orbifold.
Figure 2.5: Domains for (i) and (v) with exactly two spheres in Proposition 2.8

Note that for the special case $(\mathbb{P}^n, \omega_0, J_0)$, we have

**Theorem 2.10.** Suppose $d \geq 3$, then each stratum $\mathcal{M}_T(\mathbb{P}^n, d)$ of $\overline{\mathcal{M}}_{2,k}(\mathbb{P}^n, d)$ is a smooth orbifold. More precisely, we have the following:

(i) If $T$ belongs to (i) of Theorem 2.9, then $\dim \mathcal{M}_T(\mathbb{P}^n, d) = \dim \overline{\mathcal{M}}_{2,k}(\mathbb{P}^n, d) - 2n_{\text{nod}}$.

(ii) If $T$ belongs to (ii) of Theorem 2.9, then $\dim \mathcal{M}_T(\mathbb{P}^n, d) = \dim \overline{\mathcal{M}}_{2,k}(\mathbb{P}^n, d) + 2(n - 1 - n_{\text{nod}})$.

(iii) If $T$ belongs to (iii) of Theorem 2.9, then $\dim \mathcal{M}_T(\mathbb{P}^n, d) = \dim \overline{\mathcal{M}}_{2,k}(\mathbb{P}^n, d) + 2(n - n_{\text{nod}})$.

(iv) If $T$ belongs to (iv) of Theorem 2.9, then $\dim \mathcal{M}_T(\mathbb{P}^n, d) = \dim \overline{\mathcal{M}}_{2,k}(\mathbb{P}^n, d) + 2(2n - n_{\text{nod}})$.

**Proof.** It remains to prove (ii). We only prove the case that $\Sigma_P$ is smooth, the proof can be generalized to the case that $\Sigma_P$ is not smooth. It is sufficient to show that the moduli space $\mathcal{M}_{0,2,k}(\mathbb{P}^n, 2)$ is a smooth orbifold of dimension $4n + 8 + 2k = \dim \overline{\mathcal{M}}_{2,k}(\mathbb{P}^n, 2) + 2(n - 1)$.

By Castelnuovo’s bound (cf. P116 of [ACGH]), the image of each $[C, u] \in \mathcal{M}_{0,2,k}(\mathbb{P}^n, 2)$ in $\mathbb{P}^n$ has genus zero. Hence it must factor through a degree-one map $\tilde{u} : S^2 \to \mathbb{P}^n$, i.e., there exists a holomorphic branched covering $\phi : \Sigma \to S^2$ such that

$$u = \tilde{u} \circ \phi, \quad \deg(\phi) = 2. \quad (2.2)$$

Thus there is a natural identification of

$$\mathcal{M}_{0,2,k}(\mathbb{P}^n, 2) \cong \mathcal{M}_{0,0}(\mathbb{P}^n, 1) \times \mathcal{M}_{0,2,k}(\mathbb{P}^1, 2). \quad (2.3)$$

In fact, the first factor describes the position of $\text{im}(u)$ in $\mathbb{P}^n$ and the second factor describes the branched covering from $\Sigma$ to $S^2$.  

12
By the index theorem, we have
\[
\dim \mathcal{M}_{2,k}(\mathbb{P}^n, 2) = \dim \mathcal{M}_{0,0}(\mathbb{P}^n, 1) + \dim \mathcal{M}_{2,k}(\mathbb{P}^1, 2) = 4n - 4 + 12 + 2k = 4n + 8 + 2k.
\]

Note that in the last equality, we have used the fact that
\[
H^1(\Sigma, \mathcal{O}(\phi^*T\mathbb{P}^1)) \cong H^0(\Sigma, \mathcal{O}(\phi^*T\mathbb{P}^1)^* \otimes K_\Sigma)^* \cong H^0(\Sigma, \mathcal{O}(-2))^* = 0.
\]
for any \( \phi \in \mathcal{M}_{2,k}(\mathbb{P}^1, 2) \). Thus the linearization \( D_\phi \) of the \( \overline{\partial} \)-operator for the bundle \( \phi^*T\mathbb{P}^1 \)
\[
D_\phi : \Gamma(\Sigma, \phi^*T\mathbb{P}^1) \to \Gamma(\Sigma, \Lambda^{0,1}\Sigma^* \otimes \phi^*T\mathbb{P}^1)
\]
is surjective. Hence \( \mathcal{M}_{2,k}(\mathbb{P}^1, 2) \) is a smooth orbifold of dimension 12 by the index theorem and the implicit function theorem. Clearly the smooth case has a rather simple proof: every degree 2 map from a smooth genus 2 curve to \( \mathbb{P}^n \) is a double cover of a line; such a double cover is determined by 6 points on the line, and the dimension of \( G(2, n + 1) \) is \( 4(n - 1) \).

Note that the linearized operator \( D_u \) of the \( \overline{\partial} \)-operator at \([C, u] \in \mathcal{M}_{T}(X, A, J)\) is not surjective in general. Hence we need to study the obstructions \( H^1_{\overline{\partial}}(C, u^*TX) \). We have the following:

**Theorem 2.11.** Suppose the regularity conditions in Definition 1.1 hold. Then the obstruction at \([C, u] \in \mathcal{M}_T(X, A, J)\) is one of the following cases:

(i) If \( T \) belongs to (i) of Theorem 2.9, then \( H^1_{\overline{\partial}}(C, u^*TX) = 0 \).

(ii) If \( T \) belongs to (ii) of Theorem 2.9, then \( H^1_{\overline{\partial}}(C, u^*TX) = \text{coker} \ D_u \).

(iii) If \( T \) belongs to (iii) of Theorem 2.9, then \( H^1_{\overline{\partial}}(C, u^*TX) \cong \mathbb{C}^n \).

(iv) If \( T \) belongs to (iv) of Theorem 2.9, then \( H^1_{\overline{\partial}}(C, u^*TX) \cong \mathbb{C}^{2n} \).

**Proof.** It follows directly from the regularity conditions in Definition 1.1. The proof for the special case \((\mathbb{P}^n, \omega_0, J_0)\) is given below. The proof of the general case is the same.

Note that for the special case \((\mathbb{P}^n, \omega_0, J_0)\), we have

**Lemma 2.12.** (cf. Corollary 6.5 of [Z1]) Let \( \Sigma \) be a smooth Riemann surface. If \( u : \Sigma \to \mathbb{P}^n \) is a holomorphic map, then the linearization \( D_u \) of the \( \overline{\partial} \)-operator for the bundle \( u^*T\mathbb{P}^n \)
\[
D_u : \Gamma(\Sigma, u^*T\mathbb{P}^n) \to \Gamma(\Sigma, \Lambda^{0,1}\Sigma^* \otimes u^*T\mathbb{P}^n)
\]
is surjective provided \( d + \chi(\Sigma) > 0 \), where \( d \) is the degree of \( u \).

**Lemma 2.13.** Let \( \Sigma \) be a smooth Riemann surface. If \( u : \Sigma \to \mathbb{P}^n \) is a holomorphic map of degree \( d \), then for any tuple of pairwise distinct points \( \{p_0, \ldots, p_l\} \in \Sigma^{l+1} \), the map
\[
\varphi^{(l)} : \ker D_u \to \bigoplus_{0 \leq m \leq l} T_{u(p_m)}\mathbb{P}^n, \quad \varphi^{(l)}(\xi) = (\xi(p_0), \xi(p_1), \ldots, \xi(p_l))
\]

is surjective provided 
\[ d + \chi(\Sigma) \geq l + 2. \]

**Theorem 2.14.** Suppose \( d \geq 3 \), then the obstruction at \([C, u] \in \mathcal{M}_T(\mathbb{P}^n, d)\) is one of the following cases:

(i) If \( T \) belongs to (i) of Theorem 2.10, then \( H^1_\partial(C, u^*T\mathbb{P}^n) = 0. \)

(ii) If \( T \) belongs to (ii) of Theorem 2.10, then \( H^1_\partial(C, u^*T\mathbb{P}^n) \cong \mathbb{C}^{n-1}. \)

(iii) If \( T \) belongs to (iii) of Theorem 2.10, then \( H^1_\partial(C, u^*T\mathbb{P}^n) \cong \mathbb{C}^n. \)

(iv) If \( T \) belongs to (iv) of Theorem 2.10, then \( H^1_\partial(C, u^*T\mathbb{P}^n) \cong \mathbb{C}^{2n}. \)

**Proof.** Note that any \([C, u] \in \overline{\mathcal{M}}_{2,k}(\mathbb{P}^n, d)\) is obtained from an element in \( \mathcal{M}_T(\mathbb{P}^n, \text{deg}(u|_{\Sigma_P})) \) by attaching bubble trees. While there are no obstructions for attaching bubble trees. Thus the obstructions come from \( u_{\Sigma_P} \equiv u|_{\Sigma_P}. \)

By the proof of Propositions 2.4-2.8, the operator \( D_u \) is surjective when \( T \) belongs to (i) of Theorem 2.9. Hence (i) holds.

We prove (ii) as follows. By Propositions 2.4-2.8, when \( T \) belongs to (ii) of Theorem 2.9, we must have \( \text{deg}(u|_{\Sigma_P}) = 2 \) and we can write

\[ \mathcal{M}_T(\mathbb{P}^n, 2) \cong \mathcal{M}_{0,0}^0(\mathbb{P}^n, 1) \times \mathcal{M}_T(\mathbb{P}^1, 2). \]  

(2.4)

First we consider the case that \( \Sigma_P \) is smooth. In this case, any \([C, u] \in \mathcal{M}_T(\mathbb{P}^n, 2)\) must factor through a degree-one map \( \tilde{u} : S^2 \to \mathbb{P}^n \), i.e., there exists a holomorphic branched covering \( \phi : \Sigma_P \to S^2 \) such that

\[ u = \tilde{u} \circ \phi, \quad \text{deg}(\phi) = 2. \]  

(2.5)

Since \( \text{im}(\tilde{u}) \) is a line in \( \mathbb{P}^n \), it is a complex submanifold of \( \mathbb{P}^n \). We denote it by \( \mathbb{P}^1_{\tilde{u}}. \) Hence we have

\[ u_{\Sigma_P}^*T\mathbb{P}^n = u_{\Sigma_P}^*(T\mathbb{P}^1_{\tilde{u}} \oplus N_{\mathbb{P}^1_{\tilde{u}}}) = u_{\Sigma_P}^*(T\mathbb{P}^1_{\tilde{u}} \oplus \bigoplus_{i=1}^{n-1} H_{\mathbb{P}^1_{\tilde{u}}, i}), \]  

(2.6)

where we denote by \( N_{\mathbb{P}^1_{\tilde{u}}/\mathbb{P}^n} \) the normal bundle of \( \mathbb{P}^1_{\tilde{u}} \) in \( \mathbb{P}^n \) and \( H_{\mathbb{P}^1_{\tilde{u}}, i} \) its decomposition into \( n - 1 \) line bundles. Thus by Dolbeault Theorem, we have

\[ H^1_\partial(\Sigma_P, u_{\Sigma_P}^*T\mathbb{P}^n) \cong H^1(\Sigma_P, \mathcal{O}(u_{\Sigma_P}^*T\mathbb{P}^n)) \]

\[ \cong H^1(\Sigma_P, \mathcal{O}(u_{\Sigma_P}^*(T\mathbb{P}^1_{\tilde{u}}) \oplus \bigoplus_{i=1}^{n-1} H_{\mathbb{P}^1_{\tilde{u}}, i})) \]

\[ \cong H^1(\Sigma_P, \mathcal{O}(u_{\Sigma_P}^*(T\mathbb{P}^1_{\tilde{u}})) \bigoplus \bigoplus_{i=1}^{n-1} H^1(\Sigma_P, \mathcal{O}(u_{\Sigma_P}^*(H_{\mathbb{P}^1_{\tilde{u}}, i})))) \]

\[ \cong H^0(\Sigma_P, \mathcal{O}(-2))^* \bigoplus \bigoplus_{i=1}^{n-1} H^0(\Sigma_P, \mathcal{O})^* \cong \mathbb{C}^{n-1} \]  

(2.7)

by Kodaira-Serre duality. The proof of the other cases are similar. For the reader’s convenience, here we give the proof of case (ii) in Proposition 2.6 and omit the proofs of the others.
Suppose $\Sigma_P$ is a torus $\Sigma$ with only one node and $(\Sigma, x_1, x_2)$ is the normalization of $\Sigma_P$. Let $\xi \in L^p(\Sigma_P, \Lambda^{0,1}T^*\Sigma_P \otimes u^*_\Sigma_P T\mathbb{P}^n)$ and we want to find $\sigma \in L^p(\Sigma, u^*_\Sigma T\mathbb{P}^n)$ such that $D_{u^*_\Sigma} \sigma = \xi$. Since $\xi \in L^p(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes u^*_\Sigma T\mathbb{P}^n)$, we can find $\tilde{\sigma} \in L^p(\Sigma, u^*_\Sigma T\mathbb{P}^n)$ such that $D_{u^*_\Sigma} \tilde{\sigma} = \xi$ by Lemma 2.12. Since $\deg(u^*_\Sigma) = 2$, we may assume $\tilde{\sigma}(x_1) = 0$ by Lemma 2.13. Hence if $\sigma$ exists, we must have $\tilde{\sigma}(x_2) = 0$ also. We consider the short exact sequence of sheaves on $\Sigma$

$$0 \to O(u^*_\Sigma T\mathbb{P}^n \otimes (-x_1 - x_2)) \to O(u^*_\Sigma T\mathbb{P}^n \otimes (-x_1)) \to O((u^*_\Sigma T\mathbb{P}^n \otimes (-x_1))_{x_2}) \to 0$$

(2.8)

where we view $O((u^*_\Sigma T\mathbb{P}^n \otimes (-x_1))_{x_2})$ as a sheaf on $\Sigma$ via extension by 0 (cf. p.38 of [GH]). Taking the corresponding long exact sequence in cohomology, we obtain

$$\cdots \to H^0(\Sigma, O(u^*_\Sigma T\mathbb{P}^n \otimes (-x_1))) \to H^0(\Sigma, O((u^*_\Sigma T\mathbb{P}^n \otimes (-x_1))_{x_2})) \to H^1(\Sigma, O((u^*_\Sigma T\mathbb{P}^n \otimes (-x_1))_{x_2})) \to H^1(\Sigma, O(u^*_\Sigma T\mathbb{P}^n \otimes (-x_1))) \to H^2(\Sigma, O(u^*_\Sigma T\mathbb{P}^n \otimes (-x_1))) \to \cdots$$

(2.9)

Note that $H^1(\Sigma, O(u^*_\Sigma T\mathbb{P}^n \otimes (-x_1))) = 0$ by Lemma 2.13. Thus $\partial$ is surjective. Thus we have

$$\text{coker}D_{u^*_\Sigma} \cong \text{coker} \gamma \cong H^1(\Sigma, O(u^*_\Sigma T\mathbb{P}^n \otimes (-x_1 - x_2)))$$

$$\cong H^1(\Sigma, O(u^*_\Sigma T\mathbb{P}^1_{u^*_\Sigma} \otimes (-x_1 - x_2))) \bigoplus \bigoplus_{i=1}^{n-1} H^1(\Sigma, O(u^*_\Sigma (H^1_{u^*_\Sigma} \otimes (-x_1 - x_2))))$$

$$\cong H^0(\Sigma, O(-2)^*) \bigoplus \bigoplus_{i=1}^{n-1} H^0(\Sigma, O)^* \cong \mathbb{C}^{n-1}$$

(2.10)

The proof of (iii) is obvious and we omit it here.

We prove (iv). Denote by $\Sigma$ the union of components of $\Sigma_P$ which are mapped to constants such that each connected component of $\Sigma$ has genus greater than zero. Then $\Sigma$ contains one or two connected components and each one is mapped to a constant. In the second case, we can write $\Sigma = \Sigma_1 \cup \Sigma_2$. Then we have

$$H^1_{\overline{\mathcal{O}}}(\mathcal{C}, u^* T\mathbb{P}^n) \cong H^1_{\Sigma_1} \otimes T_{ev(\Sigma)} \mathbb{P}^n \cong \mathbb{C}^{2n}$$

(2.11)

provided $\Sigma$ is connected. Here we denote by $H^0_{\Sigma_1}$ the space of harmonic $(0,1)$-forms on $\Sigma$, cf. §22.3 of [MirSym].

$$H^1_{\overline{\mathcal{O}}}(\mathcal{C}, u^* T\mathbb{P}^n) \cong (H^0_{\Sigma_1} \otimes T_{ev(\Sigma_1)} X) \oplus (H^0_{\Sigma_2} \otimes T_{ev(\Sigma_2)} X) \cong \mathbb{C}^{2n}$$

(2.12)

provided $\Sigma$ is disconnected. Now the theorem follows.

\[\n\]

### 3 Gluing construction

Given a stable map $[\mathcal{C}, u] \in \mathcal{M}_{T}(X, A, J) \subset \mathcal{M}_{2,k}(X, A, J)$. Our goal in this section is to construct approximately $J$-holomorphic maps $[\Sigma_v, u_v] \in \mathcal{X}^0_{2,k}(X, A, J)$ by using the gluing technique, where
$X^0_{2,k}(X, A, J)$ denotes the space of equivalence classes of smooth maps from $\Sigma_v$ to $X$ with $k$ marked points in the homology class $A$ and $\Sigma_v$ is a smooth Riemann surface of genus two depending on the gluing parameter $v$. Roughly speaking, $\Sigma_v$ is obtained from $C$ by replacing each attaching node of the corresponding two components by thin necks connecting them. Thus geometrically $\Sigma_v$ is a smooth Riemann surface of genus two, but should be viewed as a Riemann surface close to $C$.

While $u_v$ equals to $u$ away from the thin necks and $u_v$ is close to $u$ in an appropriate sense. Thus $u_v$ is $J$-holomorphic away from the thin necks.

3.1 Gluing in bubble trees

In this section, we describe the gluing construction in bubble trees. We proceed as in [Z2] and [Z3] in this section. Let $q_N, q_S : \mathbb{C} \to S^2 \subset \mathbb{R}^3$ be the stereographic projections mapping the origin of $\mathbb{C}$ to the north and south poles respectively. Explicitly, we have

$$q_N(z) = \left( \frac{2z}{1 + |z|^2}, \frac{1 - |z|^2}{1 + |z|^2} \right), \quad q_S(z) = \left( \frac{2z}{1 + |z|^2}, \frac{-1 + |z|^2}{1 + |z|^2} \right).$$  \hfill (3.1)

We denote the south pole of $S^2$, i.e., the point $(0, 0, -1) \in \mathbb{R}^3$ by $\infty$ and $e_\infty = dq_S|_0 \left( \frac{\partial}{\partial s} \right)$, where we write $z = s + it \in \mathbb{C}$. We identify $\mathbb{C}$ with $S^2 \setminus \{\infty\}$ via the map $q_N$.

**Definition 3.1.** A rooted tree $I$ is a finite partially ordered set satisfying: if $h, h_1, h_2 \in I$ such that $h_1, h_2 < h$, either $h_1 \leq h_2$ or $h_2 \leq h_1$ holds; moreover, $I$ has a unique minimal element $\hat{0}$, i.e., $\hat{0} < h$ for all $h \in \hat{I} \equiv I \setminus \{\hat{0}\}$.

For any $h \in \hat{I}$, denote by $\iota_h \in I$ the largest element of $I$ which is smaller that $h$. We call $\iota : \hat{I} \to I$ the attaching map of $I$.

**Definition 3.2.** Suppose $M$ is a finite set. A $X$-valued bubble tree with $M$-marked points is a tuple

$$b = (M, I, x, (j, y), u), \quad x : \hat{I} \to S^2 \setminus \{\infty\}, \quad j : M \to I,$$

$$y : M \to S^2 \setminus \{\infty\}, \quad u : I \to C^\infty(S^2, X)$$  \hfill (3.2)

such that $u_h(\infty) = u_h(x_h)$ for all $h \in \hat{I}$. The special points on each bubble $\Sigma_h \equiv \{h\} \times S^2$, i.e., $(j_i, y_i) \in \Sigma_h$ and $(j_i, x_i)$ with $t_i = h$ together with the point $(h, \infty)$, are pairwise distinct. In addition, if $u_h[S^2] = 0 \in H_2(X, \mathbb{Z})$, then $\Sigma_h$ should contain at least three special points. $u$ is $J$-holomorphic if its restriction to each component is.

We associate such a tuple with a nodal Riemann surface

$$\Sigma_b = \left( \bigsqcup \{h\} \times S^2 \right)_{h \in I} / \sim$$  \hfill (3.3)
where \((h, \infty) \sim (i_h, x_h)\) for \(h \in \hat{I}\). We call \(x_h\) the attaching node of the bubble \(h\). Clearly we obtain a continuous map \(u_b : \Sigma_b \to X\).

The general structure of bubble trees is described by tuples \(T_B = (M, I, j, A)\), where \(I\) and \(j\) are maps as described in Definitions 3.1 and 3.2, while \(A_h = u_{h*}[\Sigma_h] \in H_2(X,\mathbb{Z})\) for \(h \in I\). We call such tuples bubble types. Denote by \(\mathcal{H}_{T_B}\) the space of holomorphic maps of type \(T_B\) and \(\mathcal{M}_{T_B}\) its equivalence classes.

For each \(h \in I\), let
\[
\chi_{T_B}h = \begin{cases} 
0, & \text{if } A_i = 0 \ \forall i \leq h; \\
1, & \text{if } A_h \neq 0, \text{ and } A_i = 0 \ \forall i < h; \\
2, & \text{otherwise.}
\end{cases} \tag{3.4}
\]

Let \(\beta : \mathbb{R} \to [0, 1]\) be a smooth function such that
\[
\beta(t) = \begin{cases} 
0 & \text{if } t \leq 1, \\
1 & \text{if } t \geq 2
\end{cases} \quad \text{and} \quad \beta'(t) > 0, \quad \text{for } t \in (1, 2) \tag{3.5}
\]
and \(\beta_r(t) = \beta(r^{-1}t)\) for any \(r > 0\).

Given a bubble type \(T_B = (M, I, j, A)\), let \(d(T_B) : I \to \mathbb{R}\) be given by
\[
d_i(T_B) = \langle \omega, A_i \rangle + \{|l \in M : j_l = i|\} + \sum_{i_h = i} d_h(T_B), \quad \forall i \in I. \tag{3.6}
\]

Note that \(d_i(T_B)\) is uniquely determined by Definition 3.1 and (3.6). A bubble tree in (3.2) is called balanced if for all \(i \in \hat{I}\) the following conditions hold:
\[
\begin{align*}
(B1) \int_C |du_i \circ q_N|^2 z + \sum_{i_h = i} d_h(T_B) x_h + \sum_{j_l = i} y_l &= 0; \\
(B2) \int_C |du_i \circ q_N|^2 \beta(|z|) + \sum_{i_h = i} d_h(T_B) \beta(|x_h|) + \sum_{j_l = i} \beta(|y_l|) &= \frac{1}{2}.
\end{align*}
\]
It is called completely balanced if (B1) and (B2) hold for all \(i \in I\).

Denote by \(PSL(2, \mathbb{C})\) the group of Möbius transformations. Let
\[
PSL(2, \mathbb{C})^{(0)} = \{g \in PSL(2, \mathbb{C}) : g(\infty) = \infty\}, \quad \mathcal{G}_{T_B} = \prod_{h \in I} PSL(2, \mathbb{C})^{(0)} \tag{3.7}
\]

For \(b = (M, I, x, (j, y), u) \in \mathcal{H}_{T_B}\) and \(g \in PSL(2, \mathbb{C})^{(0)}\), define \(gb = (M, I, gx, (j, gy), gu)\) by
\[
(gx)_h = g_{i_h}x_h, \quad (gy)_i = g_{j_l}y_l, \quad (gu)_i = g_iu_i, \tag{3.8}
\]
where for a map \(u : S^2 \to X\) and \(g \in PSL(2, \mathbb{C})\), we define \((gu)(z) = u(g^{-1}z)\).
Let $M_{TB}^{(0)} \subset H_{TB}$ denote the subset of completely balanced bubble trees. Then the group
\[ \prod_{h \in I} S^1 \times \text{Aut}(T_B) \] acts on $M_{TB}^{(0)}$ and all the stabilizers are finite. Then we have
\[ M_{TB} \cong M_{TB}^{(0)} / \left( \prod_{h \in I} S^1 \times \text{Aut}(T_B) \right). \] (3.9)

By Proposition 3.3 in \cite{Z2}, $M_{TB}^{(0)}$ is a smooth oriented manifold and then $M_{TB}$ is a smooth orbifold. One may think of elements in $M_{TB}^{(0)}$ as good representatives of $M_{TB}$. In fact, $M_{TB}^{(0)} = \Psi^{-1}((0, \frac{1}{2})^I)$, where $\Psi_{TB} \equiv (\Psi_{TB,i})_{i \in I} : H_{TB} \to (\mathbb{C} \times \mathbb{R})^I$ is defined by
\[ \Psi_{TB,i}(M,I,x,(j,y),u) = \left( \int_C |du_i \circ q_N|^2 z + \sum_{i_k = i} d_h(T_B)x_h + \sum_{j_l = i} y_l, \int_C |du_i \circ q_N|^2 \beta(|z|) + \sum_{i_k = i} d_h(T_B)\beta(|x_h|) + \sum_{j_l = i} \beta(|y_l|) \right) \] (3.10)
and $\Psi_{TB}$ is smooth and transversal to every point $(0,r_i)_{i \in I}$ such that $|r_i - \frac{1}{2}| \leq \frac{1}{4}$ for all $i \in I$. Let
\[ \tilde{M}_{TB}^{(0)} = \Psi^{-1}_{TB}\left\{(0,r_i)_{i \in I} : r_i \in \left(\frac{1}{4},\frac{3}{4}\right) \text{ if } \chi_{TB}^i = 1, \ r_i = \frac{1}{2} \text{ otherwise} \right\}, \] (3.11)
\[ \tilde{F}_B = \tilde{M}_{TB}^{(0)} \times \mathbb{C}[|i|] \to \tilde{M}_{TB}^{(0)}, \quad \tilde{F}_B^\theta = \tilde{M}_{TB}^{(0)} \times \{ \mathbb{C} \setminus \{0\} \}[\theta]. \] (3.12)

If $\pi_{\mathcal{F}} : \mathcal{F} \to \mathcal{X}$ is a normed vector bundle and $\delta : \mathcal{X} \to \mathbb{R}$ is any function, let $\mathcal{F}_{\delta} = \{ v \in \mathcal{F} : |v| < \delta(\pi_{\mathcal{F}}(v)) \}$ be the $\delta$-disk bundle.

Now we describe the basic gluing construction in bubble trees. For each sufficiently small element $v = (b,v) \in \tilde{F}_B^\theta$, where $(\Sigma_b, u_b)$ is an element of $M_{TB}^{(0)}$, let $q_v : \Sigma_v \to \Sigma_b$ be the basic gluing map constructed in \cite{Z2}. Let
\[ b(v) = (\Sigma_v, u_v), \quad u_v = u_b \circ q_v \] (3.13)
be the approximately $J$-holomorphic map corresponding to $v$. The primary marked point $y_0(v)$ of $\Sigma_v$ is the point $\infty$ of $\Sigma_v \cong S^2$. By the construction of $q_v$, it factors through each of the maps $q_{v,i} : \Sigma_v \to \Sigma_b$ for $i \in I$. Let $g_v$ be the Riemannian metric on $\Sigma_b$ such that its restriction to each component is the standard metric on $S^2$. By §3.3 of \cite{Z2}, We can construct a Riemannian metric $g_v$ on $\Sigma_v$ such that:

(G1) $q_v : (\Sigma_v, g_v) \to (\Sigma_b, g_b)$ is an isometry (and thus holomorphic) outside of the annuli
\[ A_{v,h}^{+} = q_{v,h}^{-1}\left\{ z \in \Sigma_{b,h} : 1 \leq |v_h|^{-\frac{1}{2}}|z| \leq 2 \right\}, \] (3.14)
\[ A_{v,h}^{-} = q_{v,h}^{-1}\left\{ z \in \Sigma_{b,h} : \frac{1}{2} \leq |v_h|^{-\frac{1}{2}}|z| \leq 1 \right\}. \] (3.15)
where \( \phi_x z = z - x \equiv q_N^{-1}(z) - q_N^{-1}(x) \in \mathbb{C} \) for \( x, z \in S^2 \setminus \{ \infty \} \).

(G2) \( q_{v, i} : (A_{v, h}^\pm, g_v) \rightarrow (q_{v, i}(A_{v, h}^\pm), g_v) \) is an isometry.

Moreover, the map \( q_v \) collapses \( \hat{I} \) disjoint circles on \( \Sigma_v \) and is a diffeomorphism away from them. These circles are mapped to the \( \hat{I} \) nodal branches. Alternatively, \( (\Sigma_v, g_v) \) can be viewed as the surface obtained by smoothing the nodes of \( \Sigma_b \). An explicit construction of \( q_{v, i} \) may be described as follows. For a rooted tree \( I \) and a tuple \( v \equiv (v_h)_{h \in \hat{I}} \in \mathbb{C}^I \) such that \( \sum_{h \in \hat{I}} |v_h| \) is sufficiently small, choose any ordering \( \prec \) of \( I \) consistent with its partial ordering. If \( v_h \in \mathbb{C} \) with \( 0 < |v_h| < \delta \), let \( p_{h, (x_h, v_h)} : B_{x_h}(\delta^{\frac{1}{2}}) \equiv \{ \phi_{x_h} z < \delta^{\frac{1}{2}} \} \rightarrow \mathbb{C} \cup \{ \infty \} \) be given by

\[
p_{h, (x_h, v_h)}(z) = (1 - \beta_{|v_h|}(2|\phi_{x_h} z|)) \left( \frac{v_h}{\phi_{x_h} z} \right)
\]

and define \( q_{v, (x_h, v_h)} : \Sigma_T^h \rightarrow \Sigma_T^h \cup \Sigma_h \) by

\[
q_{v, (x_h, v_h)}(z) = \begin{cases} (h, q_{s}(p_{h, (x_h, v_h)}(z))), & \text{if } |v_h|^{-\frac{1}{2}}|\phi_{x_h} z| \leq 1; \\ (t_h, \phi_{x_h}^{-1}(\beta_{|v_h|}(|\phi_{x_h} z|)\phi_{x_h} z)), & \text{if } 1 \leq |v_h|^{-\frac{1}{2}}|\phi_{x_h} z| \leq 2; \\ (t_h, z), & \text{otherwise,} \end{cases}
\]

where \( \Sigma_T^h \) is obtained from \( T_B \) by dropping the bubble \( h \) together with all bubbles descendent from it. Thus \( q_{v, (x_h, v_h)} \) is a diffeomorphism except on the circle \( |v_h|^{-\frac{1}{2}}|\phi_{x_h} z| = 1 \) and the circle is mapped to the point \( (h, \infty) = (t_h, x_h) \). Moreover, \( q_{v, (x_h, v_h)} \) is holomorphic outside the annulus \( \frac{1}{2} \leq |v_h|^{-\frac{1}{2}}|\phi_{x_h} z| \leq 2 \). Taking \( q_{v, 0} = Id \) and \( q_{v, h} = q_{v, (x_h, v_h)} \circ q_{v, i} \) inductively according to the ordering \( \prec \), we obtain \( q_{v, h} \) for all \( h \in I \).

By (G1), \( u_v \) is \( J \)-holomorphic outside \( A_{v, h}^\pm \). For \( p > 2 \), we define norms \( \| \cdot \|_{v, p, 1} \) and \( \| \cdot \|_{v, p} \) on \( \Gamma(v) \equiv L^p(\Sigma_v, u_v^*TX) \) and \( \Gamma^{0, 1}(v) \equiv L^p(\Sigma_v, A^{0, 1}T^*\Sigma \otimes u_v^*TX) \) respectively as in §3.3 of [Z2]. These norms are equivalent to the ones used in [IT2]. Let \( D_v : \Gamma(v) \rightarrow \Gamma^{0, 1}(v) \) be the linearization of the \( \overline{\partial} \)-operator at \( b(v) \). Since the linearization \( D_b \) of the \( \overline{\partial} \)-operator at \( b \) is surjective by (i) of Definition 1.1, if \( v \in \mathcal{F}T_B^0 \) is sufficiently small, \( D_v \) is also surjective. In particular, we have a decomposition

\[
\Gamma(v) = \Gamma_-(v) \oplus \Gamma_+(v) \equiv \{ \xi \circ q_v : \xi \in \Gamma_-(b) \equiv \text{ker } D_b \} \oplus \{ \zeta \in \Gamma(v) : \zeta(\hat{0}, \infty) = 0; \langle \zeta, \xi \rangle_{v, 2} = 0, \forall \xi \in \Gamma_-(v) \text{ s.t. } \xi(\hat{0}, \infty) = 0 \}, \tag{3.18}
\]

where \( (\hat{0}, \infty) \equiv y_0(v) \) is the primary marked point of \( \Sigma_v \). Note that the choice of \( \Gamma_+(v) \) is permissible by (i) of Definition 1.1. Moreover, the operator \( D_v : \Gamma_+(v) \rightarrow \Gamma^{0, 1}(v) \) is an isomorphism and the norms of \( D_v \) and of the inverse of its restriction to \( \Gamma_+(v) \) depend only on \( b \) and not on \( v \). Let

\[
\pi_{v,-} : \Gamma(v) \rightarrow \Gamma_-(v), \quad \pi_{v,+} : \Gamma(v) \rightarrow \Gamma_+(v) \tag{3.19}
\]
be the projection maps corresponding to the decomposition (3.18).

Denote by $\mathcal{X}_{0,M}(X,A,J)$ the space of equivalence classes of smooth maps into $X$ from genus zero Riemann surfaces with marked points indexed by the set $\{0\} \cup M$ in the homology class $A$ and by $\mathcal{X}_{0,M}^0(X,A,J)$ its subset consisting of those maps with smooth domains. If $K \subset \mathcal{M}_{TB}$ denote by $K^{(0)}$ and $\tilde{K}^{(0)}$ the preimages of $K$ under the projections $\mathcal{M}_{TB}^{(0)} \to \mathcal{M}_{TB}$ and $\tilde{\mathcal{M}}_{TB}^{(0)} \to \mathcal{M}_{TB}$ respectively. We have the following Lemma for gluing in bubble trees.

**Lemma 3.3.** (cf. Lemma 3.3 of [Z3]) For every precompact open subset $K$ of $\mathcal{M}_{TB}$, there exist $\delta_K, \epsilon_K, C_K \in \mathbb{R}^+$ and an open neighborhood $U_K$ of $K$ in $\mathcal{X}_{0,M}(X,A,J)$ such that

(i) For all $v = (b,v) \in \mathcal{T}_{TB}^0|_{\bar{K}^{(0)}}$, the equation

$$\bar{\theta}_J \exp_{a,v} \zeta = 0, \quad \zeta \in \Gamma_+(v), \quad ||\zeta||_{v,p,1} < \epsilon_K,$$

has a unique solution $\zeta_v$.

(ii) The map $\bar{\phi} : \mathcal{T}_{TB}^0|_{\bar{K}^{(0)}} \to \mathcal{M}_{0,\{0\} \cup M}^0(X,A,J) \cap U_K, v \mapsto [\exp_{b(v)} \zeta_v]$ is smooth.

(iii) For all $v = (b,v) \in \mathcal{T}_{TB}^0|_{\bar{K}^{(0)}}$, we have $\exp_{v_0}(\bar{\phi}(v)) = \exp_{b}(v)$.

(iv) For all $v = (b,v) \in \mathcal{T}_{TB}^0|_{\bar{K}^{(0)}}$, we have $||\zeta_v||_{v,p,1}, ||\nabla^T \zeta_v||_{v,p,1} \leq C_K||v||^{1/p}$, where $\nabla^T \zeta_v$ denotes the covariant derivative with respect to the connection defined in §3 in [Z3].

### 3.2 Gluing in the principle component

The general structure of genus-two bubble maps is described as a tuple

$$T = ((I_1 \cup M_P, \Sigma_P, A_P), (T_B^{(l)})_{l \in I_1}),$$

where $\Sigma_P$ is a nodal Riemann surface of genus-two as in Remark 2.2, $M_P$ denotes the marked points on $\Sigma_P$, $A_P \in H_2(X,\mathbb{Z})$ and $T_B^{(l)}$'s are bubble trees defined in §3.1 for $l \in I_1$. We denote by $\{x_I\}_{l \in I_1}$ the $I_1$ points on $\Sigma_P$ where the corresponding bubble trees are attached. Let

$$\mathcal{C} = \left(\Sigma_P \bigcup (T_B^{(l)})_{l \in I_1}\right) \sim, \quad (\Sigma_P, x_I) \sim (T_B^{(l)}, (0, \infty)), \quad \forall l \in I_1,$$

where $(0, \infty)$ is the primary marked point on $\Sigma_B^{(l)}$ and $\Sigma_B^{(l)}$ is the nodal Riemann surface of genus zero corresponding to the bubble tree $T_B^{(l)}$. We call $\Sigma_P$ the principle component of $\mathcal{C}$.

Note that we have a natural isomorphism

$$\mathcal{M}_T(X,A,J) \cong \{(b_P, (b_B^{(l)})_{l \in I_1}) \in \mathcal{M}_{TB}(X,A,J) \times \prod_{l \in I_1} \mathcal{M}_{TB} : \exp_{v_0}(b_B^{(l)}) = \exp_{v_1}(b_P), \forall l \in I_1\} \bigg/ \text{Aut}^*(T),$$
where \( T_P = (I_1 \cup M_P, \Sigma_P, A_P) \), \( ev_0(b^{(l)}) \) is the evaluation map at the primary marked point \((\hat{0}, \infty)\) and \( ev_{i_1}(b_P) \) is the evaluation map at the attaching node \( x_l \) of the bubble tree \( T_B^{(l)} \) and \( \text{Aut}^*(T) = \text{Aut}(T)/\{g \in \text{Aut}(T) : g \cdot h = h, \forall h \in I_1\} \). Let

\[
\mathcal{M}_T^{(0)} = \{(b_P, (b^{(l)})_{l \in I_1}) \in \mathfrak{M}_{T_P}(X, A_P, J) \times \prod_{l \in I_1} \mathcal{M}_{T_B^{(l)}}^{(0)} : ev_0(b^{(l)}) = ev_{i_1}(b_P), \forall l \in I_1\}, \tag{3.23}
\]

\[
\widetilde{\mathcal{M}}_T^{(0)} = \{(b_P, (b^{(l)})_{l \in I_1}) \in \mathfrak{M}_{T_P}(X, A_P, J) \times \prod_{l \in I_1} \widetilde{\mathcal{M}}_{T_B^{(l)}}^{(0)} : ev_0(b^{(l)}) = ev_{i_1}(b_P), \forall l \in I_1\}. \tag{3.24}
\]

Then clearly we have \( \mathfrak{M}_T(X, A, J) = \mathcal{M}_T^{(0)}/(\text{Aut}^*(T) \times (S^1)^{|I|}) \), where \( I = \cup_{l \in I_1} I_B^{(l)} \) and \( I_B^{(l)} \) is the partially ordered set associated to the bubble tree \( T_B^{(l)} \) as in Definition 3.1. Let

\[
\mathcal{F}T = \mathcal{F}P \mathcal{T} \oplus \mathcal{F}_0 \mathcal{T} \oplus \mathcal{F}_1 \mathcal{T} \rightarrow \mathcal{M}_T^{(0)},
\]

\[
\mathcal{F}_P \mathcal{T} = \mathcal{F}_P \mathcal{T}_P, \quad \mathcal{F}_0 \mathcal{T} = \bigoplus_{h \in I_1} \mathcal{F}_h \mathcal{T}, \quad \mathcal{F}_1 \mathcal{T} = \mathcal{F}_1 \mathcal{T} = \mathcal{M}_T^{(0)} \times \mathbb{C}^{|I|\setminus I_1}, \tag{3.25}
\]

where \( \mathcal{F}_P : \mathcal{M}_T^{(0)} \rightarrow \mathfrak{M}_{T_P}(X, A_P, J) \) is the projection map, \( L_h \mathcal{T}_0 \) is the universal tangent line bundle at the marked point \( x_h \) for \( h \in I_1 \) and \( \mathcal{F}T_P \) is the bundle of gluing parameters in the principle component. In particular, \( \text{rank} \mathcal{F}T_P = n_{\text{mod}}(\Sigma_P) \), i.e., the number of nodes in \( \Sigma_P \). As before, let \( \mathcal{F}T^0 \) be the subset of \( \mathcal{F}T \) consisting of those elements with all components nonzero.

Now we describe the gluing construction in the principle component \( \Sigma_P \). For each sufficiently small element \( v = (b, v) \in \mathcal{F}T^0 \), we have

\[
v = (b, v) \equiv (b, v_P, v_0, \{v^{(l)}\}_{l \in I_1}) \in \mathcal{F}_P \mathcal{T} \oplus \mathcal{F}_0 \mathcal{T} \oplus \mathcal{F}_1 \mathcal{T} \tag{3.26}
\]

We smooth out all the nodes in \( \Sigma_P \) by the parameter \( v_P \). The bundle of gluing parameters in the principle component \( \mathcal{F}T_P \) over \( \mathfrak{M}_{T_P}(X, A_P, J) \) has the form

\[
\mathcal{F}T_P = \bigoplus_{x \in \text{nod}(\Sigma_P)} T_{x,0} \Sigma_{x,0} \otimes T_{x,1} \Sigma_{x,1}, \tag{3.27}
\]

where we denote by \( \Sigma_{x,0} \) and \( \Sigma_{x,1} \) the two components corresponding to the node \( x \). Here in order to simplify notations, we omit the use of a finite cover of \( \mathfrak{M}_{T_P}(X, A_P, J) \) as in [RT2]. For any nonzero \( v_x \in T_{x,0} \Sigma_{x,0} \otimes T_{x,1} \Sigma_{x,1} \), define the map

\[
\Phi_{x,v_x} : T_{x,0} \Sigma_{x,0} \setminus \{0\} \rightarrow T_{x,1} \Sigma_{x,1} \setminus \{0\}, \quad X \otimes \Phi_{x,v_x} X = v_x. \tag{3.28}
\]

Now let \( \phi_{x,0} : \Sigma_{x,0} \rightarrow T_{x,0} \Sigma_{x,0} \) and \( \phi_{x,1} : \Sigma_{x,1} \rightarrow T_{x,1} \Sigma_{x,1} \) be holomorphic coordinates near \( x \) on the two components respectively. Let \( \tilde{p}_{h,(x,v_x)} : \{\phi_{x,0} z < \delta \} \rightarrow T_{x,1} \Sigma_{x,1} \) be given by

\[
\tilde{p}_{h,(x,v_x)}(z) = (1 - \beta_{v_x}(2|\phi_{x,0} z|)) \Phi_{x,v_x}(\phi_{x,0} z) \tag{3.29}
\]
and define

\[ q_{v,(x,v_x)} : (\Sigma_{x,0} \setminus \{|v_x|^{-\frac{1}{2}}|\phi_{x,0}| \leq \frac{1}{2}\}) \cup (\Sigma_{x,1} \setminus \{|v_x|^{-\frac{1}{2}}|\phi_{x,1}| \leq 2\}) \to \Sigma_{x,0} \cup \Sigma_{x,1} \]

by

\[ q_{v,(x,v_x)}(z) = \begin{cases} 
(0, z), & \text{if } z \in \Sigma_{x,0} \setminus \{|v_x|^{-\frac{1}{2}}|\phi_{x,0}| \leq 2\}; \\
(0, \phi^{-1}_{x,0}(\beta_{v_x}(|\phi_{x,0}|)|\phi_{x,0}z)), & \text{if } 1 \leq |v_x|^{-\frac{1}{2}}|\phi_{x,0}| \leq 2; \\
(1, \phi^{-1}_{x,1}(\beta_{h,(x,v_x)})(z)), & \text{if } \frac{1}{2} \leq |v_x|^{-\frac{1}{2}}|\phi_{x,0}| \leq 1; \\
(1, z), & \text{if } z \in \Sigma_{x,1} \setminus \{|v_x|^{-\frac{1}{2}}|\phi_{x,1}| \leq 2\}. 
\end{cases} \quad (3.30) \]

Note that by (3.28) and (3.29), the map \( q_{v,(x,v_x)} \) is well-defined. We smooth out all the nodes in \( \Sigma_P \) as above and obtain \( q_{b,v_p} \). Then we define

\[ q_{v_p} : \Sigma_{(b,v_p)} \to \Sigma_b \equiv \mathcal{C} \quad (3.31) \]

to be the extension of \( q_{b,v_p} \) by identity to the bubble components. Geometrically, \( \Sigma_{(b,v_p)} \) is obtained from \( \Sigma_b \) by replacing all the nodes in the principle component by thin necks connecting the corresponding two components.

Let \( v = (b,v) \equiv (b,v_p,v_0,\{v^{(l)}\}_{i \in I_1}) \) be given by (3.26). If \( i \in \tilde{I}_B^{(h)} \), we put

\[ \rho_i(v) = \left( b, v_0,h \prod_{\{i' \in \tilde{I}_B^{(h)} : i' \leq i\}} v_{i'}^{(h)} \right) \in T_{x,h} \Sigma_P, \quad (3.32) \]

where the product term is defined to be 1 if \( \{i' \in \tilde{I}_B^{(h)} : i' \leq i\} = \emptyset \). We denote by

\[ \chi(T) = \{i : \chi_{\tilde{T}_B^{(h)}}i = 1, h \in I_1\}. \quad (3.33) \]

4 Study for \( \mathcal{M}_T(X,A,J) \) in (i) of Theorem 2.9

In the remaining of this paper, we prove the main theorem by looking for conditions under which the approximately \( J \)-holomorphic map \( [\Sigma_v,u_v] \in \mathcal{X}^0_{2,k}(X,A,J) \) can be deformed into a \( J \)-holomorphic map, where \( \Sigma_v \) is the smooth Riemann surface of genus two and \( u_v \) is the approximately \( J \)-holomorphic map constructed below. We will separate the proof into several sections according to the classification of stable maps in Theorem 2.9.

In this section we study stable maps in (i) of Theorem 2.9. In these cases we have \( H^1_{\partial}(\mathcal{C}, u^*TX) = 0 \) by Theorem 2.10.

Let \( \Sigma_{(b,v_p)} \) be the Riemann surface constructed via (3.30) in §3.2. By construction, its principle component is a smooth Riemann surface \( \Sigma_{b,v_p} \) of genus-two and \( \Sigma_{(b,v_p)} \) is obtained from \( \Sigma_{b,v_p} \)
by attaching $|I_1|$ bubble trees at the points $\{x_h\}_{h \in I_1}$. For each $h \in I_1$, we identify a small neighborhood $U(x_h)$ of $x_h$ in $\Sigma_{b,p,v^p}$ with a neighborhood $\tilde{U}(x_h)$ of 0 in $T_{x_h} \Sigma_{b,p,v^p}$ biholomorphically and isometrically. In fact, we can choose a Kähler metric $g_{b,p,v^p}$ on $\Sigma_{b,p,v^p}$ to be flat on each $U(x_h)$. We assume that all of these neighborhoods are disjoint from each other and from the $n_{nod}(\Sigma_P)$ thin necks of $\Sigma_{b,v^p}$. if $z \in U(x_h)$, denote by $|z - x_h|$ its norm with respect to the metric $g_{b,p,v^p}$. Then we define the map

$$q_{v_0} : \Sigma(b,v^p, \nu_0) \to \Sigma(b,v^p)$$

via the formula (3.17) by replacing the term $\phi_{x_h} z$ there by $z - x_h \in T_{x_h} \Sigma_{b,p,v^p}$. Then we smooth out all the nodes in the bubble trees as in §3.1 to obtain

$$q_{v_1} : \Sigma_v \equiv \Sigma(b,v) \to \Sigma(b,v_p, \nu_0)$$

At last, we define

$$q_v = q_{v_0} \circ q_{v_0} \circ q_{v_1} : \Sigma_v \to \Sigma_b \equiv C.$$  

By construction, $q_v$ is a homeomorphism outside of $n_{nod}(\Sigma_b)$ circles of $\Sigma_v$ and is biholomorphic outside of $n_{nod}(\Sigma_b)$ thin necks. We take

$$b(v) = (\Sigma_v, j_v, u_v), \quad \text{where} \quad u_v = u_b \circ q_v,$$

(4.4) to be the approximately $J$-holomorphic map corresponding to the basic gluing map $q_v$, where $j_v$ is the complex structure on $\Sigma_v$. We denote by

$$\Gamma(v) \equiv L^p_1(\Sigma_v, u_v^* TX), \quad \Gamma^{0,1}(v) \equiv L^p(\Sigma_v, \Lambda^{0,1}_v T^* \Sigma_v \otimes u_v^* TX),$$

(4.5) the Banach completions of the corresponding spaces of smooth sections with respect to the norms $\| \cdot \|_{v,p,1}$ and $\| \cdot \|_{v,p}$ induced from the basic gluing map $q_v$ as in [22]. Let

$$\Gamma_-(v) = \{ (\xi \circ q_v) : \xi \in \Gamma_-(b) \equiv \ker D_b \} \subset \Gamma(v)$$

(4.6) and $\Gamma_+(v)$ the $(L^2, v)$-orthogonal complement of $\Gamma_-(v)$ in $\Gamma(v)$. Let $\pi_{v, \pm}$ be the $(L^2, v)$-orthogonal projections onto $\Gamma_\pm(v)$ respectively.

The following is the main theorem in this section.

**Theorem 4.1.** Suppose $T = (I_1 \cup M_P, \Sigma_P, A_P), (T_B^{(l)})_{l \in I_1}$ is a bubble type belongs to (i) of Theorem 2.9, then for every precompact open subset $K$ of $\mathfrak{M}_T(X, A, J)$, there exist $\delta_K, \epsilon_K, C_K \in \mathbb{R}^+$ and an open neighborhood $U_K$ of $K$ in $X_{2,k}(X, A, J)$ with the following properties:

(i) For all $v = (b,v) \in \overline{\mathcal{F} T^{(l)}_{\delta_K}} |_{K^{(l)}}$, the equation

$$\mathcal{D}_J \exp_{uv} \zeta = 0, \quad \zeta \in \Gamma_+(v), \quad \| \zeta \|_{v,p,1} < \epsilon_K,$$

(4.7)
has a unique solution $\zeta_v$.

(ii) The map $\tilde{\phi} : \mathcal{F}^T_\delta K_{(0)} \to \mathcal{M}^0_{2,k}(X, A, J) \cap U_K$, $v \mapsto [\exp_b(v) \zeta_v]$ is smooth.

In particular, we have $\mathcal{M}_T(X, A, J) \subset \mathcal{M}^0_{2,k}(X, A, J)$ is a smooth orbifold of dimension at most $\dim \mathcal{M}^{\text{vir}}_{2,k}(X, A) - 2$.

**Proof.** Note that in these cases, the operator $D_b$ is surjective. Hence $D_v$ is also surjective provided $v = (b, v) \in \mathcal{F}^T_\delta K_{(0)}$ is sufficiently small by continuity, where $D_v : \Gamma(v) \to \Gamma^{0,1}(v)$ is the linearization of the $\overline{\partial}_J$-operator at $b(v)$. Moreover, by the choice of the norms $\| \cdot \|_{v,p,1}$ and $\| \cdot \|_{v,p}$, we have the following estimates similar to Theorem 4.1 in [Z3]

$$\|\pi_v - \xi\|_{v,p,1} \leq C_K \|\xi\|_{v,p,1}, \forall \xi \in \Gamma(v); \quad \| D_v \xi\|_{v,p} \leq C_K |v|^\frac{1}{2} \|\xi\|_{v,p,1}, \forall \xi \in \Gamma_-(v); \quad (4.8)$$

$$C_K^{-1} \|\xi\|_{v,p,1} \leq \| D_v \xi\|_{v,p} \leq C_K \|\xi\|_{v,p,1}, \forall \xi \in \Gamma_+(v). \quad (4.9)$$

Thus by a standard argument as in the genus-zero case, cf. [MS] or [Z3], the theorem follows.

## 5 Study for $\mathcal{M}_T(X, A, J)$ in (iv) of Theorem 2.9

In this section we study stable maps in (iv) of Theorem 2.9. In these cases we have $H^1_{\overline{\partial}}(\mathcal{C}, u^*TX) \cong \mathbb{C}^{2n}$ by Theorem 2.10. In [Z1], A. Zinger studied the enumerative problem of genus-two curves with a fixed complex structure in $\mathbb{P}^n$ for $n = 2, 3$. In [Z3], A. Zinger defined the reduced genus-one Gromov-Witten invariants. Our present paper uses similar arguments as these two papers, but our cases are more complicated, we need to study all the boundary components $\mathcal{M}_T(X, A, J)$, while in [Z1], the author only need to consider two cases for $\mathbb{P}^2$ and five cases for $\mathbb{P}^3$.

First we study the case that $\Sigma_P$ is a smooth Riemann surface of genus two in §5.1-5.3. In this case the obstruction bundle has the form $\mathcal{H}_{\Sigma_P}^{0,1} \otimes T_{\text{ev}(\Sigma_P)}X \cong \mathbb{C}^{2n}$, where $\mathcal{H}_{\Sigma_P}^{0,1}$ is the space of harmonic $(0, 1)$-forms on $\Sigma_P$ and $\text{ev}(\Sigma_P)$ is the evaluation map at the principle component. We study the general cases in §5.4-5.5, which are modifications of the methods in §5.1-5.3.

Now let us assume $\Sigma_P$ is smooth. Given $\psi \in \mathcal{H}_{\Sigma_P}^{0,1}$, $b = (b_P, (b(l_1)^{0,1}_{l_1})) \in \tilde{\mathcal{M}}_{T}^{(0)}$, $x \in \Sigma_P$, $m \geq 1$ and a Kähler metric $g_{b, \Sigma_P}$ on $\Sigma_P$ which is flat near $x$. Define $D_{b, x}^{(m)} \psi \in T_{x}^{0,1} \Sigma_P^m$ as follows: If $(s, t)$ are conformal coordinates centered at $x$ such that $s^2 + t^2$ is the square of the $g_{b, \Sigma_P}$-distance to $x$. Let

$$\left\{ D_{b, x}^{(m)} \psi \right\} \left( \frac{\partial}{\partial s} \right) \equiv \left\{ D_{b, x}^{(m)} \psi \right\} \left( \frac{\partial}{\partial s}, \ldots, \frac{\partial}{\partial s} \right) = \frac{\pi}{m!} \left\{ \frac{D_{b, x}^{m-1} \psi}{ds^{m-1}} |_{(s, t) = 0} \right\} \left( \frac{\partial}{\partial s} \right), \quad (5.1)$$

where the covariant derivatives are taken with respect to the metric $g_{b, \Sigma_P}$. Since $\psi \in \mathcal{H}_{\Sigma_P}^{0,1}$, we have $\psi = f(ds - idt)$ for some anti-holomorphic function $f$. Because $g_{b, \Sigma_P}$ is flat near $x$, it follows
that $D_{(m)}^{b,x} \psi \in T_x^{m,1} \Sigma_P^{\otimes m}$. For an orthogonal basis $\{\psi_1, \psi_2\}$ of $\mathcal{H}_P^{0,1}$, let $s_{b,x}^{(m)} \in T_x^{*} \Sigma_P^{m} \otimes \mathcal{H}_P^{0,1}$ be given by

$$s_{b,x}^{(m)}(v) \equiv s_{b,x}^{(m)}(v, \ldots, v) = \sum_{1 \leq j \leq 2} \{D_{b,x}^{(m)} \psi_j\}(v) \psi_j. \quad (5.2)$$

The section $s_{b,x}^{(m)}$ is independent of the choice of a basis for $\mathcal{H}_P^{0,1}$ but is dependent on the choice of the metric $g_{b,\Sigma_P}$ when $m > 1$. However, $s_{b,x}^{(1)}$ depends only on $(\Sigma_P, j_{\Sigma_P})$, we denote it by $s_{\Sigma_P, x}$. By p.246 in [GH], $s_{\Sigma_P, x}$ does not vanish and thus spans a subbundle of $\Sigma_P \times \mathcal{H}_P^{0,1} \rightarrow \Sigma_P$. Denote this subbundle by $\mathcal{H}_P^{+}$ and its orthogonal complement by $\mathcal{H}_P^{-}$. In particular, $\mathcal{H}_P^{-}$ is independent of the choice of the metric $g_{b,\Sigma_P}$ on $\Sigma_P$.

For $h \in \cup_{l \in I_1} I_B^{(l)} \equiv I$ and $m \in \mathbb{N}$, define

$$D_{(m)}^{h} b = \frac{2}{(m - 1)!} \frac{D^{m-1} d}{ds^m} (u_h \circ q_S) \bigg|_{(s,t)=0}, \quad (5.3)$$

where the covariant derivatives are taken with respect to the standard metric $s + it \in \mathbb{C}$ and a metric $g_{X,b}$ on $X$ which is flat near $u(\Sigma_P)$, e.g. we may assume $X$ is isomorphic to $\mathbb{C}^n$ near $u(\Sigma_P)$.

Let $\delta_T \in C^\infty(\tilde{\mathcal{M}}_T^{(0)}, \mathbb{R}^+)$ satisfying $4\delta_T(b)\|du_i\|_{b,C^{0}} < r_X$ for any $b \in \tilde{\mathcal{M}}_T^{(0)}$, where $r_X$ is the injectivity radius of $X$ with respect to the metric $g_{X,b}$. We use Kähler metrics $g_{b,\Sigma_P}$ on $\Sigma_P$ which are flat near $x_h$ for $h \in I_1$.

For $h \in I$ and $\epsilon > 0$, denote by

$$\tilde{A}_{b,h}^{-} (\epsilon) = \{ (h,z) \in \Sigma_{b,h} \equiv \{ h \} \times S^2 : |z| > e^{-h^2/2} \}, \quad (5.4)$$

$$\tilde{A}_{b,h}^{+} (\epsilon) = \{ (t_h,z) \in \Sigma_{b,t_h} : |z - x_h| < 2e^{h^2/2} \}, \quad (5.5)$$

$$A_{v,h}^{-} (\epsilon) = q^{-1}_v (\tilde{A}_{b,h}^{-} (\epsilon)) \subset \Sigma_v, \quad (5.6)$$

where $\epsilon_h = \Sigma_P$ for $h \in I_1$.

Now for $v = (b,v) \in \tilde{\mathcal{T}}^{0}_\delta T$ sufficiently small and $V \psi \in T_{u(\Sigma_P)}X \otimes \mathcal{H}_P^{0,1}$, define $R_0 V \psi \in \mathcal{E}_v^{0,1}(u_v)$ as follows: If $z \in \Sigma_v$ is such that $q_v(z) \in \Sigma_{b,h}$ for some $h \in \chi(T)$ as defined in (3.33) and $|q^{-1}_S(q_v(z))| \leq 2\delta_T(b)$, we define $\Pi_{v}(z) \in T_{u(\Sigma_P)}X$ by $\exp_{u(\Sigma_P)} \Pi_{v}(z) = u_v(z)$. Given $z \in \Sigma_v$, let $h$ be such that $q_v(z) \in \Sigma_{b,h}$. If $w \in T_{z} \Sigma_v$, put

$$R_0 V \psi|_w w = \begin{cases} 0, & \text{if } \chi_T h_z = 2; \\ \beta(\delta_T(b)|q_v(z)|(|\psi|_w) \Pi_{v}(z))V, & \text{if } \chi_T h_z = 1; \\ (\psi|_w)V, & \text{if } \chi_T h_z = 0, \end{cases} \quad (5.7)$$

where $\chi_T$ is the natural extension of $\chi_{T_B}^{(l)}$ to $T$ and $\Pi_{v}(z)$ is the parallel transport along the geodesic $t \mapsto \exp_{u(\Sigma_P)} t \Pi_{v}(z)$ with respect to the Levi-Civita connection of the metric $g_{X,b}$.
By the same proof of Lemma 4.3 of [Z1], we have the following expansion:

**Lemma 5.1.** Suppose $T$ is a bubble type given by (iv) of Theorem 2.9 with $\Sigma_P$ being a smooth Riemann surface of genus two. Then there exists $\delta \in C^{\infty}(\mathfrak{M}_T(X, A, J), \mathbb{R}^+) \text{ such that for all } v = (b, v) \in \mathcal{F}_{\delta}T_0, V \in T_u(\Sigma_P)X$ and $\eta \in \mathcal{H}^{0,1}_{\Sigma_P}$ we have

$$\langle \langle \overline{\partial}_j u_v, R_v V \eta \rangle \rangle_{\nu, 2} = - \sum_{m \geq 1, h \in \chi(T)} \langle D_h^{(m)} h, V \rangle \left( \left\{ D_b^{(m)} h, \xi_h(v) \right\} \left( (d\phi_h, \tau(h) \vert \xi_h(v))^{-1} \rho_h(v) \right) \right)$$

where $T(h)$ is determined by $h \in T^T(h), \xi_h(v) = q_{v, t_h}^{-1}(t_h, x_h) \in \Sigma_v$ and $\phi_{b, h}$ is a holomorphic identification of neighborhoods of $x_h$ in $\Sigma_{b, t_h}$ and $T_{x_h} \Sigma_{b, t_h}$, and $\rho_h(v)$ is given by (3.32).

Next we estimate the formal adjoint $D_v^*$ of the linearization $D_v$ of the $\overline{\partial}_J$-operator at $u_v$ with respect to the above $(L^2, v)$-inner product.

**Lemma 5.2.** (cf. Lemma 2.2 of [Z1]) Suppose $T$ is a bubble type given by (iv) of Theorem 2.9 with $\Sigma_P$ being a smooth Riemann surface of genus two. Then there exists $\delta \in C^{\infty}(\mathfrak{M}_T(X, A, J), \mathbb{R}^+)$ such that for all $v = (b, v) \in \mathcal{F}_{\delta}T_0$ and $V \eta \in T_u(\Sigma_P)X \otimes \mathcal{H}^{0,1}_{\Sigma_P}$, we have $D_v^* R_v V \eta$ vanishes outside of the annuli

$$\tilde{A}_{v, h} = q_v^{-1}(\{(h, z) \in \Sigma_{b, h} : \delta_T(b) \leq |q^{-1}_S(q_v(z))| \leq 2\delta_T(b)\})$$

with $h \in \chi(T)$. Moreover, there exists $C \in C^{\infty}(\mathfrak{M}_T(X, A, J), \mathbb{R}^+)$ such that

$$\|D_v^* R_v V \eta\|_{\nu, C^0} \leq C(b) \left( \sum_{h \in \chi(T)} |\rho_h(v)| \right) \|V\| \|\eta\|_2. \quad (5.8)$$

**Proof.** Since we will use the expression of $D_v^* R_v V \eta$ below, we give the proof of the lemma here. Let $(s, t)$ be the conformal coordinates on $\tilde{A}_{v, h}$ given by $q_v(s, t) = s + it \in \mathbb{C}$. Write $g_v = \theta^{-2}(s, t)(ds^2 + dt^2)$, then we have $\theta = \frac{1}{2}(1 + s^2 + t^2)$ by Riemannian geometry. Let $\xi(s, t)$ be given by

$$\xi(s, t) = \{R_v V \eta\}_{(s, t)} \partial_s = \beta \left( \delta_T(b) \sqrt{s^2 + t^2} \right) \left( \eta(s, t) \partial_s \right) \big|_{\tilde{A}_v(s, t)} V.$$

Then by Remark C.1.4 of [MS] we have

$$D_v^* R_v V \eta \vert_z = \theta^2 \left( -\frac{D}{ds} \xi + J \frac{D}{dt} \xi \right),$$

where $\frac{D}{ds}$ and $\frac{D}{dt}$ denote covariant derivations with respect to the metric $g_{X, b}$ on $X$. Since this metric is flat on the support of $\xi$ by assumption and $\eta \in \mathcal{H}^{0,1}_{\Sigma_P}$, we have

$$D_v^* R_v V \eta \vert_z = \frac{(1 + s^2 + t^2)^2}{4} \left( \beta \delta_T(b) \sqrt{s^2 + t^2} \right) \left( \eta(s, t) \partial_s \right) \big|_{\tilde{A}_v(s, t)} V. \quad (5.9)$$
Note that the right hand side of (5.9) vanishes unless \( \delta_T(b) \leq \sqrt{s^2 + t^2} \leq 2\delta_T(b) \) by (3.5). Hence we have

\[
|D_v^s R_v \eta|_{v,z} \leq C(b_v)|\eta|_{(s,t)}|\partial_s| |V| \leq C(b)|\rho_\eta(v)||\eta||_2|V|.
\] (5.10)

Hence (5.8) holds.

Next we describe our choice for a tangent-space model as §2.3 of [Z1]. Let \( \Gamma_{\pm}(v) \) and \( \pi_{v,\pm} \) be given by the formula (4.9). Now we fix an \( h^* \in \chi(T) \) and let

\[
\bar{\Gamma}_-(v) = D_v^s R_v(H^+_{\Sigma_P}(\overline{x}_{h^*}(v)) \otimes T_v(\Sigma_P)X),
\] (5.11)

where \( H^+_{\Sigma_P} \) is defined below (5.2) and \( \overline{x}_{h^*}(v) = q_{\overline{v},\zeta}(h_{h^*}, x_{h^*}) \in \Sigma_v \). Denote by \( \Gamma_+(v) \) the \( (L^2, v) \)-orthogonal complement of \( \Gamma_-(v) \) in \( \Gamma(v) \) and \( \pi_{v,\pm} \) the \( (L^2, v) \)-orthogonal projections onto \( \Gamma_\pm(v) \). Let \( \tilde{\Gamma}_+(v) \) be the image of \( \Gamma_+(v) \) under \( \pi_{v,\pm} \) and \( \tilde{\Gamma}_-(v) \) be its \( (L^2, v) \)-orthogonal complement. Denote by \( \tilde{\pi}_{v,\pm} \) the \( (L^2, v) \)-orthogonal projections onto \( \tilde{\Gamma}_\pm(v) \). Then we have the following:

**Lemma 5.3.** Suppose \( T \) is a bubble type given by (iv) of Theorem 2.9 with \( \Sigma_P \) being a smooth Riemann surface of genus two. Then for every precompact open subset \( K \) of \( \mathfrak{M}_T(X,A,J) \), there exist \( \delta_K, \epsilon_K, C_K \in \mathbb{R}^+ \) and an open neighborhood \( U_K \) of \( K \) in \( \mathfrak{X}_{2,k}(X,A,J) \) with the following properties:

(i) For all \( v = (b, v) \in \mathfrak{F}T_{\delta_K}|_{\tilde{K}(v)} \), we have

\[
\| \pi_{v,\pm} \xi \|_{v,p,1} \leq C_K \| \xi \|_{v,p,1}, \quad \forall \xi \in \Gamma(v)
\] (5.12)

\[
C_K^{-1} \| \xi \|_{v,p,1} \leq \| D_v \xi \|_{v,p} \leq C_K \| \xi \|_{v,p,1}, \quad \forall \xi \in \tilde{\Gamma}_+(v).
\] (5.13)

(ii) For all \( \tilde{b} \in \mathfrak{X}_{2,k}(X,A,J) \cap U_K \), there exist \( v = (b, v) \in \mathfrak{F}T_{\delta_K}|_{\tilde{K}(v)} \) and \( \zeta \in \tilde{\Gamma}_+(v) \) such that

\[
\| \zeta \|_{v,p,1} < \epsilon_K \text{ and } [\exp b(v) \zeta_v] = [\tilde{b}].
\]

**Proof.** The proof of (i) follows from §2.3 in [Z1]. In fact, \( \tilde{\Gamma}_-(v) \) is a tangent-space model in the sense of Definition 3.11 in [Z2]. Then (5.12) and (5.13) follow from Lemmas 3.5, 3.12 and 3.16 in [Z2]. The argument of §4 in [Z2] can be modified to show the existence of \( (v, \zeta) \) satisfying (ii) and this pair is unique up to the action of the automorphism group \( \text{Aut}^*(T) \times (S^1)^{|\Gamma|} \), cf. the proof of Lemma 4.4 in [Z3].

For any \( v = (b, v) \in \mathfrak{F}T_{\delta_K} \) and \( h \in \chi(T) \), let

\[
\alpha_{T,h}^{(k)}(v) = (D_h^{(k)} b)s_{b,z_T(h)}(v), \quad \alpha_T^{(k)}(v) = \sum_{h \in \chi(T)} \alpha_{T,h}^{(k)}(v).
\] (5.14)

We denote \( \alpha_{T,h}^{(1)}(v) \) and \( \alpha_T^{(1)}(v) \) by \( \alpha_{T,h}(v) \) and \( \alpha_T(v) \) respectively.
We want to analyze the conditions under which a stable map can be deformed to a $J$-holomorphic map whose domain is smooth. We have the following first-order estimate:

**Lemma 5.4.** Suppose $T$ is a bubble type given by (iv) of Theorem 2.9 with $\Sigma_P$ being a smooth Riemann surface of genus two. Then for every precompact open subset $K$ of $\mathfrak{M}_T(X, A, J)$, there exist $\delta_K, C_K \in \mathbb{R}^+$ and an open neighborhood $U_K$ of $K$ in $\mathfrak{X}_{2,k}(X, A, J)$ satisfying: For every $v = (b, v) \in \mathcal{F}\mathcal{T}^0_{\delta_K}|_{\tilde{K}(0)}$ and $V \eta \in T_{u(\Sigma_P)}X \otimes \mathcal{H}_{\Sigma_P}^{0,1}$, we have

$$\left| \langle (\overline{\partial}_J u_v, R_v V \eta) \rangle_{v, 2} + \langle (\alpha_T(v), V \eta) \rangle_{v, 2} \right| \leq C_K |v| \cdot |\rho(v)| \cdot \|V \eta\|.$$

**Proof.** As in Lemma 4.5 of [Z1], we have

$$\|s_{b, \tilde{z}_h(v)}((d\phi_{b, T(h)}|_{\tilde{z}_h(v)})^{-1})\rho_h(v)) - s_{b, x_T(h)}(\rho_h(v))\|_2 \leq C_K |\phi_{b, T(h)}(\tilde{z}_h(v)|_b|\rho_h(v)|$$

$$\leq C_K |v| \cdot |\rho_h(v)|,$$

$$\sum_{m \geq 2} |D_h^{(m)} b| |\rho_h(v)|^m \leq C_K |\rho_h(v)|^2,$$

for all $h \in \chi(T)$ and $v = (b, v) \in \mathcal{F}\mathcal{T}^0_{\delta_K}|_{\tilde{K}(0)}$ with $\delta_K$ being sufficiently small. Thus the lemma follows from Lemma 5.1.

Let $\delta_K$ be given by Lemma 5.4. For each $v = (b, v) \in \mathcal{F}\mathcal{T}^0_{\delta_K}|_{\tilde{K}(0)}$, we define the homomorphism

$$\pi^{0,1}_{v, -} : \Gamma^{0,1}(v) \rightarrow \Gamma_{-}^{0,1}(b_P),$$

$$\pi^{0,1}_{v, -} \xi = - \sum_{1 \leq i \leq n, 1 \leq j \leq 2} \langle \xi, R_v e_i \psi_j \rangle e_i \psi_j \in \Gamma_{-}^{0,1}(b_P),$$(5.15)

where $\{\psi_1, \psi_2\}$ is an orthonormal basis for $\mathcal{H}_{\Sigma_P}^{0,1}$ as in (5.2) and $\{e_j\}_{1 \leq j \leq n}$ is an orthonormal basis for $T_{u(\Sigma_P)}X$. Denote the kernel of $\pi^{0,1}_{v, -}$ by $\Gamma^{0,1}_{+}(v)$. Then we have the following:

**Lemma 5.5.** Suppose $T$ is a bubble type given by (iv) of Theorem 2.9 with $\Sigma_P$ being a smooth Riemann surface of genus two. Then an element $b \equiv [C, u] \in \mathfrak{M}_T(X, A, J) \cap \mathfrak{M}_2^{0}(X, A, J)$ must satisfy $\alpha_T(v) = 0$ for some $v = (b, v) \in \mathcal{F}\mathcal{T}^0_{\delta_K}|_{\tilde{K}(0)}$, where $\delta_K$ is given by Lemma 5.4.

**Proof.** By (ii) of Lemma 5.3,

$$U_T = \{[\exp_{u_v} \zeta] : v = (b, v) \in \mathcal{F}\mathcal{T}_{\delta_K}|_{\tilde{K}(0)} \cap \tilde{K}(0), \|\zeta\|_{v, 1} < \delta_K\}$$

is an open neighborhood of $K$ in $\mathfrak{X}_{2,k}(X, A, J)$. Now suppose $\overline{\partial}_J \exp_{u_v} \zeta = 0$ for a pair $(v, \zeta)$. Write

$$\overline{\partial}_J \exp_{u_v} \zeta = \overline{\partial}_J u_v + D_v \zeta + N_v \zeta,$$(5.16)

where $N_v$ is a quadratic form satisfying (cf. Theorem 3a of [E])

$$N_v \zeta = N_v \zeta' \|_{v, 1} \leq C_K (\|\zeta\|_{v, 1} + \|\zeta'\|_{v, 1}) \|\zeta - \zeta'\|_{v, 1},$$ (5.17)
Now we estimate $\|\overline{\partial} J u_v\|_{v,p}$. By the construction of $q_v$, we have $\overline{\partial} J u_v = 0$ outside the annuli $A_{v,h}^- (\{v_h\})$ for $h \in \chi(T)$ and $A_{v,h}^\pm (\{v_h\})$ for $\{h \in I : \chi_T h = 2\}$. By the construction of $q_v$, we have $\|dq_v\|_{C^0} < C(b)$ for some $C \in C^\infty(\mathcal{M}_T(X, A, J), \mathbb{R}^+)$. Thus we have

$$\|\overline{\partial} J u_v\|_{v,p} \leq C_K \sum_{h \in I} \|du_h\|_{A_{v,h}^\pm (\{v_h\})}^{2(2-p)} \leq C_K |v_h|^\frac{1}{p}.\quad (5.19)$$

Moreover, we have

$$\|du_h\|_{A_{v,h}^\pm (\{v_h\})} \leq C_K \left(\int_{A_{v,h}^\pm (\{v_h\})} |du_h|^p d\mu\right)^{\frac{1}{p}} + C_K \left(\int_{A_{v,h}^\pm (\{v_h\})} |z|^{2(2-p)} |du_h|^2 d\mu\right)^{\frac{1}{2}} \leq C_K |v_h|^\frac{1}{p}.\quad (5.19)$$

Now (5.16)-(5.19) and (5.13) yield

$$\|\zeta\|_{v,p,1} \leq C_K |v|^\frac{1}{p}.\quad (5.20)$$

Note that

$$\overline{\partial} J \exp u_v \zeta = 0 \iff \begin{cases} \pi_{v,-}^{0,1} (\overline{\partial} J u_v + D_v \zeta + N_v \zeta) = 0 \in \Gamma_0^{0,1} (b_P), \\ \overline{\partial} J u_v + D_v \zeta + N_v \zeta = 0 \in \Gamma_0^{0,1} (v). \end{cases}\quad (5.21)$$

Since the structure on $X$ near $u(\Sigma_P)$ is isomorphic to $\mathbb{C}^n$ by assumption, we have $N_v \zeta = 0$ on the support of $R_v V \eta$. Hence we have

$$\langle \langle N_v \zeta, R_v V \eta \rangle \rangle_{v,2} = 0.\quad (5.22)$$

Thus by Lemmas 5.2 and 5.4 we have

$$\pi_{v,-}^{0,1} (v, \zeta) \equiv \pi_{v,-}^{0,1} (\overline{\partial} J u_v + D_v \zeta + N_v \zeta) = \alpha_T (v) + \epsilon(v, \zeta),\quad (5.23)$$

where

$$\|\epsilon(v, \zeta)\| \leq C_K (|v| + \|\zeta\|_{v,p,1}) |\rho(v)| \leq C_K |v|^{\frac{1}{p}} |\rho(v)|.\quad (5.24)$$

Note that we may choose a basis $\{\psi_1, \psi_2\}$ of $\mathcal{H}_{\Sigma_P}^{0,1}$ such that $\psi_i(x_{\chi(h)}) \neq 0$ for $h \in \chi(T)$ and $i = 1, 2$. Thus in order to satisfies (5.21), we must have $\alpha_T (v) = 0$ provided $|v|$ is sufficiently small. This proves the lemma. 

Now we separate our study into several subsections according to the number of bubble trees.
5.1 There is only one bubble tree

In this subsection, we consider the case $I_1 = 1$. Note that in this case $\text{rank}_{\overline{M}}(v) = 2n$ which is less than the dimension of $\text{coker}D_h = 4n$. Thus we need further expansions according the position of the attaching node of the bubble tree. More precisely, we have the following cases:

**Theorem 5.1.1.** Suppose $T$ is a bubble type given by (iv) of Theorem 2.9 with $\Sigma_P$ being smooth and $|\chi(T)| = 1$. Denote by $h$ the single bubble in $\chi(T)$. Assume the attaching node $x_T(h)$ of the bubble tree is not one of the six branch points of the canonical map $\Sigma_P \to \mathbb{P}^1 : x \mapsto s_{\Sigma_P, x}$. Then an element $b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \overline{\mathcal{M}}^0_{2,k}(X, A, J)$ if and only if $D_h(1)b = D_h(2)b = 0$. In particular, $\mathcal{M}_T(X, A, J) \cap \overline{\mathcal{M}}^0_{2,k}(X, A, J)$ is a smooth orbifold of dimension at most $\dim \overline{\mathcal{M}}^\text{vir}_{2,k}(X, A) - 2$.

**Proof.** By Lemma 5.5, an element $b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \overline{\mathcal{M}}^0_{2,k}(X, A, J)$ must satisfy $\alpha_T(v) = 0$ for some $v = (b, v) \in \overline{\mathcal{F}}_{\delta_k}|\overline{K}(0)$, thus we have $D_h(1)b = 0$. Denote by $\mathcal{S} = \{b \in \mathcal{M}_T(X, A, J) : D_h(1)b = 0\}$. Then by (ii-a) of Definition 1.1, $\mathcal{S}$ is a smooth suborbifold of $\mathcal{M}_T(X, A, J)$ of codimension $2n$. Let $\mathcal{NS}$ denote the normal bundle of $\mathcal{S}$ in $\mathcal{M}_T(X, A, J)$ and identify a small neighborhood of its zero section with a tubular neighborhood of $\mathcal{S}$ in $\mathcal{M}_T(X, A, J)$.

Suppose $(b, N) \in \mathcal{NS}$ and $v = ((b, N), v) \in \overline{\mathcal{F}}_{\delta_k}|\overline{K}(0)$. We consider the second-order expansion of $\langle \langle \overline{\mathcal{J}}_u((b, N); v), R_v\overline{V}\eta \rangle \rangle_{v, 2}$. Note that we have

$$
\|s_b(2)_{h(h)}((d\phi_b, \tau(h)|\overline{\theta}_h(v))^{-1} \rho_h(v)) - s_b(2)_{h(h)}(\rho_h(v))\|_2 \leq C_K |\phi_b, \tau(h)(\overline{\theta}_h(v))|_b |\rho_h(v)|^2
$$

$$
\leq C_K |v| \cdot |\rho_h(v)|^2,
$$

$$
\sum_{m \geq 3} |D_h(m)(b, N)| |\rho_h(v)|^m \leq C_K |\rho_h(v)|^3,
$$

$$
|D_h(2)(b, N) - D_h(2)(b, 0)| \leq C_K |N|,
$$

for $v, N$ sufficiently small by continuity.

Now by Lemma 5.1 we have

$$
\left| \langle \langle \overline{\mathcal{J}}_u((b, N); v), R_v\overline{V}\eta \rangle \rangle_{v, 2} + \langle \langle D_h(1)(b, N)\overline{\theta}_h(v)\overline{\theta}_h(v)^{-1} \rho_h(v) \rangle \rangle_{v, 2} + \alpha_T((b, 0); v, \overline{V}\eta) \rangle \rangle_{v, 2} \right| \leq C_K |\rho(v)|^2(|v| + |N|) \|

$$

(5.25)

Let $\{\psi_j\}$ be an orthogonal basis for $\mathcal{H}_{\Sigma_P}^{0,1}$ such that $\psi_1 \in \mathcal{H}_{\Sigma_P}^{+}(\overline{\theta}_h(v))$, $\psi_2 \in \mathcal{H}_{\Sigma_P}^{-}(\overline{\theta}_h(v))$ and $\{V_i\}$ an orthogonal basis for $T_{ev_{\Sigma_P}(b, N)}X$. Note that since $\zeta \in \overline{\Gamma}^+(v)$, we have

$$
\langle \langle \zeta, D_v^*R_v\overline{V}\psi_1 \rangle \rangle_{v, 2} = 0
$$

(5.26)

by the construction of $\overline{\Gamma}^+(v)$ before Lemma 5.3. Here we use notations in Lemma 5.5. Since $\psi_2 \in \mathcal{H}_{\Sigma_P}^{-}(\overline{\theta}_h(v))$, we have $\psi_2(\overline{\theta}_h(v)) = 0$. Hence by (5.10) we have

$$
|D_v^*R_v\overline{V}\psi_2|_{v, z} \leq C_K |\psi_2|_{(s, t)} |\partial_v| |V_i| \leq C_K |\rho(v)|^2.
$$

(5.27)
By (5.25)-(5.27), (5.20) and (5.22), we have

\[
\pi_{v,-}^{0,1}(v, \zeta) \equiv \pi_{v,-}^{0,1}(\partial f u_v + D u \zeta + N_v \zeta)
\]

\[
= D_h^{(1)}(b, N) s_{b, \overline{\varepsilon}_h(v)}^{(1)} (\partial f_{b, T(h)}|\varepsilon_h(v))^{-1} \rho_h(v) + \alpha^{(2)}_T (b, 0; v) + \epsilon(v, \zeta),
\]

\[
= D_h^{(1)}(b, N) s_{b, \overline{\varepsilon}_h(v)}^{(1)} (\partial f_{b, T(h)}|\varepsilon_h(v))^{-1} \rho_h(v)) + D_h^{(2)} b \left( D^{(2)}_{b, x_T(h)} \psi_1(\rho_h(v)) \psi_1 + \{D^{(2)}_{b, x_T(h)} \psi_2(\rho_h(v)) \psi_2\} + \epsilon(v, \zeta), \right. \]

where

\[
\|\epsilon(v, \zeta)\| \leq C_K |\rho(v)|^2 (|v| + |N| + \|\zeta\|_{v, p, 1}) \leq C_K |\rho(v)|^2 (|N| + \|v\|_P^2). \tag{5.29}
\]

Since the attaching node \(x_T(h)\) of the bubble tree is not one of the six branch points of the canonical map \(\Sigma_F \to \mathbb{P}^1: x \mapsto s_{\Sigma_F, x}\), we have \(D^{(2)}_{b, x_T(h)} \psi_2 \neq 0\). Hence in order to satisfy (5.21), we must have \(D^{(2)}_h b = 0\) provided \(|v|\) and \(|N|\) are sufficiently small.

Conversely, suppose \(D^{(1)}_h b = D^{(2)}_h b = 0\). We want to construct \(\phi(v) \in \mathcal{M}^0_{x,k}(X, A, J)\) converging to \(b\). We have the following: In a small neighborhood of \(b\), for any

\[
v = ((\tilde{b}, N); v) \in \mathcal{F} \mathcal{T}^0_{\delta K} |_{\Sigma_F(0)}, \quad \|\zeta\|_{v, p, 1} \leq 2C_K |v|_P^2
\]

the equation

\[
\pi_{v,-}^{0,1}((\tilde{b}, N); v, \zeta) \equiv D^{(1)}_h(\tilde{b}, N) s_{b, \overline{\varepsilon}_h(v)}^{(1)} (\partial f_{b, T(h)}|\varepsilon_h(v))^{-1} \rho_h(v) + \alpha^{(2)}_T (\tilde{b}, 0; v) + \epsilon(v, \zeta) = 0 \tag{5.30}
\]

has a unique small solution \((b^*, N^*) \in \mathcal{M}^{(0)}_{\delta K}\).

In fact, let \(\pi_{\varepsilon_h(v)}^\pm\) be the projections to \(\mathcal{H}^\pm_{\Sigma_F} (\varepsilon_h(v))\) respectively and consider the equation

\[
\pi_{\varepsilon_h(v)}^\pm \circ \pi_{v,-}^{0,1}((\tilde{b}, N); v, \zeta) \equiv D^{(2)}_h b \left( D^{(1)}_h \psi_1(\rho_h(v)) \psi_1 + \{D^{(1)}_h \psi_2(\rho_h(v)) \psi_2\} + \pi_{\varepsilon_h(v)}^- \circ \epsilon(v, \zeta) = 0. \right. \tag{5.31}
\]

Note that \(D^{(1)}_h b = d\bar{u}_h|_\infty = 0\), thus by (ii-b) of Definition 1.1, and \(D^{(2)}_b x_T(h) \psi_2 \neq 0\), the map

\[
\Phi^\theta : \mathcal{F} \mathcal{T}^0_{\delta K} \to T_{u(\Sigma_F)} X \otimes \mathcal{H}^0_{\Sigma_F} (\varepsilon_h(v)), \quad v \mapsto (D^{(2)}_h b \left( D^{(2)}_h \psi_1(\rho_h(v)) \psi_1 + \{D^{(2)}_h \psi_2(\rho_h(v)) \psi_2\} \right)
\]

is transversal to the zero section. This together with (5.29) yields a unique solution \((b^*, N)\) of (5.31) for each \(N\) provided \(|v|\) and \(|N|\) are sufficiently small.

Now consider the equation

\[
\pi_{\varepsilon_h(v)}^\pm \circ \pi_{v,-}^{0,1}((\tilde{b}, N); v, \zeta) \equiv D^{(1)}_h(\tilde{b}, N) s_{\varepsilon_h(v)}^{(1)} (\partial f_{b^*, T(h)}|\varepsilon_h(v))^{-1} \rho_h(v)) + D^{(2)}_h b \left( D^{(2)}_h \psi_1(\rho_h(v)) \psi_1 + \pi_{\varepsilon_h(v)}^\pm \circ \epsilon(v, \zeta) = 0 \right. \tag{5.32}
\]
By (ii-a) of Definition 1.1, the map
\[ \Phi^+ : \mathcal{FT}^0_{\delta K} \to T_{u(\Sigma_P)} X \otimes \mathcal{H}_{\Sigma_P}^2 (\tilde{x}_h(v)), \]
\[ N \mapsto D_h^{(1)} (b^*, N) s_{b^*}^{(1)} (v) \]
is transversal to the zero section. Note that
\[ \left\| D_h^{(2)} b^* (D_h^{(2)} \psi_1) (\rho_h (v)) \psi_1 \right\| \leq C_K |v|^2. \]
This together with (5.29) yields a unique solution \((b^*, N^*)\) of (5.32) provided \(|v|\) is sufficiently small. Thus \((b^*, N^*)\) is the unique solution of (5.30) as desired. Denote by \(\mu(v, \zeta) = (b^*, N^*), v)\).

Now we define the map
\[ \Psi_v : \{ \zeta \in \tilde{\Gamma}_+ (\mu(v, \zeta)) : \zeta \leq 2C_K |v|^{\frac{1}{p}} \} \to \Gamma_+^{0,1} (\mu(v, \zeta)) \]
\[ \Psi_v (\zeta) = \overline{\partial_J} u_{\mu(v, \zeta)} + D_{\mu(v, \zeta)} \zeta + N_{\mu(v, \zeta)} \zeta \]
Since the derivative
\[ D \Psi_v (0) : \tilde{\Gamma}_+ (\mu(v, \zeta)) \to \Gamma_+^{0,1} (\mu(v, \zeta)) \]
is an isomorphism and \(\|\Psi_v (0)\|_{\mu(v, 0), p, 1} \leq 2C_K |v|^{\frac{1}{p}}\). The equation \(\Psi_v (\zeta) = 0\) has a unique small solution \(\zeta_v\) by the contraction principle. We define the map
\[ \phi : \{ (b, v) \in \mathcal{FT}^0_{\delta K} |_{\tilde{K}(0)} \} : D_h^{(1)} b = 0 = D_h^{(2)} b \} \to \mathcal{M}_{2,k}^0 (X, A, J), \]
\[ \phi (v) = [\exp_{\mu(v, \zeta_v)} \zeta_v]. \]
Then \(\phi (v)\) converges to \(b\) in the stable map topology. Hence \(b \in \mathcal{M}_T (X, A, J) \cap \mathcal{M}_{2,k}^0 (X, A, J)\). The proof of the theorem is complete.

**Theorem 5.1.2.** Suppose \(T\) is a bubble type given by (iv) of Theorem 2.9 with \(\Sigma_P\) being smooth and \(|\chi(T)| = 1\). Denote by \(h\) the single bubble in \(\chi(T)\). Assume the attaching node \(x_T(h)\) of the bubble tree is one of the six branch points \(\{z_m\}_{1 \leq m \leq 6}\) of the canonical map. Then we have the following:

(i) If \(H^1 (\mathbb{P}^1, u_h^* TX \otimes \mathcal{O}_{\mathbb{P}^1} (-4\infty)) = 0\), then an element \(b \equiv [C, u] \in \mathcal{M}_T (X, A, J) \cap \mathcal{M}_{2,k}^0 (X, A, J)\) if and only if \(\alpha_T (v) = 0\) for some \(v = (b, v) \in \mathcal{FT}^0_{\delta K} |_{\tilde{K}(0)}\) and \(\tilde{\alpha}_T (w, v) = 0\), where \(\tilde{\alpha}_T (w, v)\) is a linear combination of \(D_h^{(2)} b\) and \(D_h^{(3)} b\) with coefficients depending on \(w \in T_{z_m} \Sigma_P, v\) and the position of the nodes in bubble trees. In particular, \(\mathcal{M}_T (X, A, J) \cap \mathcal{M}_{2,k}^0 (X, A, J)\) is a smooth orbifold of dimension at most \(\dim \mathcal{M}_{2,k}^{vir} (X, A) - 2\).

(ii) If \(H^1 (\mathbb{P}^1, u_h^* TX \otimes \mathcal{O}_{\mathbb{P}^1} (-4\infty)) \neq 0\), then an element \(b \equiv [C, u] \in \mathcal{M}_T (X, A, J) \cap \mathcal{M}_{2,k}^0 (X, A, J)\) must satisfy: \(u_h\) factor through a branched covering \(\tilde{u} : S^2 \to X\), i.e., there exists a holomorphic branched covering \(\phi : S^2 \to S^2\) such that \(u_h = \tilde{u} \circ \phi\) and \(\deg (\phi) \geq 2\).
Proof. By Lemma 5.5, an element $b \equiv [C, u] \in \mathfrak{M}_T(X, A, J) \cap \mathfrak{M}_2^0(X, A, J)$ must satisfy $\alpha_T(v) = 0$ for some $v = (b, v) \in \mathcal{F} \mathcal{T}^0_{\delta t_k} |_{\tilde{K}(0)}$, thus we have $D^{(1)}_h b = 0$. Denote by $\mathcal{S}_m = \{b \in \mathfrak{M}_T(X, A, J): D^{(1)}_h b = 0, x_T(h) = z_m\}$, where $\{z_m\}_{1 \leq m \leq 6}$ are the six branch points of $s_{\Sigma P}$. Note that $z_m$ depends on the complex structure $j_{\Sigma P}$ on $\Sigma P$, in order to simplify notations, we use $z_m$ to denote them. Clearly, each $\mathcal{S}_m$ is a smooth suborbifold of $\mathfrak{M}_T(X, A, J)$ of codimension $2n + 2$. Let $\mathcal{N} \mathcal{S}_m$ denote the normal bundle of $\mathcal{S}_m$ in $\mathcal{S}$, where $\mathcal{S}$ is given by Theorem 5.1.1 and identify a small neighborhood of its zero section with a tubular neighborhood of $\mathcal{S}_m$ in $\mathcal{S}$.

Suppose $(b, w) \in \mathcal{N} \mathcal{S}_m, (b, w, N) \in \mathcal{N} \mathcal{S}$ and $v = ((b, w, N); v) \in \mathcal{F} \mathcal{T}^0_{\delta t_k} |_{\tilde{K}(0)}$. We consider the third-order expansion of $\langle [J_\xi u((b, w, N); v), R_v \eta \rangle \rangle v, 2$. Note that we have

$$
\left| s^{(3)}_{b, \tilde{x}_h(v)}((d\phi_b, T(h)|_{\tilde{x}_h(v)})^{-1} \rho_h(v)) - s^{(3)}_{b, z_m}(\rho_h(v)) \right|_2 \leq C_K |\phi_b, T(h)|_{\tilde{x}_h(v)}||\rho_h(v)||^3
\leq C_K |v : \rho_h(v)|^3,
$$

$$
\sum_{m \geq 4} \left| D^{(m)}_h (b, w, N) |\rho_h(v)|^m \leq C_K |\rho_h(v)|^4,
$$

$$
| D^{(3)}_h (b, w, N) - D^{(3)}_h (b, 0, 0) | \leq C_K |N| + |w|,
$$

for $v, w, N$ sufficiently small by continuity.

Now by Lemma 5.1 we have

$$
\left| \langle [J_\xi u((b, w, N); v), R_v \eta \rangle \rangle v, 2 + \langle D^{(1)}_h (b, w, N)s^{(1)}_{b, \tilde{x}_h(v)}((d\phi_b, T(h)|_{\tilde{x}_h(v)})^{-1} \rho_h(v)) \right. \\
+ \left. D^{(2)}_h (b, w, N)s^{(2)}_{b, \tilde{x}_h(v)}((d\phi_b, T(h)|_{\tilde{x}_h(v)})^{-1} \rho_h(v)) + \alpha_T^{(3)}((b, 0, 0); v), V \eta \rangle \rangle v, 2 \right|
\leq C_K |\rho_h(v)|^3(|v| + |w| + |N|)||V \eta||. \tag{5.33}
$$

Note that we have $\tilde{x}_h(v) = \tilde{x}_h((b, w, N); v) = \tilde{x}_h((b, w, 0); v) \equiv \tilde{x}_h(w, v) \in \Sigma P$. Identify a small neighborhood of $z_m$ in $\Sigma P$ with a small neighborhood of 0 in $T_{\tilde{x}_m} \Sigma P$, then we have $|\tilde{x}_h(w, v)| \leq C_K (|v| + |w|)$. Let $s^{(2,-)} \in \Gamma(\Sigma P, T^* \Sigma P \otimes \mathcal{H}_{\Sigma P})$ be the projection of the section $s^{(2,-)}_{b, w}$ onto the subbundle $\mathcal{H}_{\Sigma P}$. Then we have $s^{(2,-)}$ is independent of the metric on $\Sigma P$ and has transversal zeros at the six points $\{z_m\}_{1 \leq m \leq 6}$ (cf. P.402 of [Z4]).

Let $\{\psi_j\}$ be an orthonormal basis for $\mathcal{H}_{\Sigma P}^{0,1}$ such that $\psi_1 \in \mathcal{H}_{\Sigma P}^+(\tilde{x}_h(w, v)), \psi_2 \in \mathcal{H}_{\Sigma P}^-(\tilde{x}_h(w, v))$ and $\{V_i\}$ an orthonormal basis for $T_{exp(b, w, N)}X$. Note that since $\zeta \in \hat{\Gamma}^+(v)$, we have

$$
\langle [\zeta, D^c_v R_v V \psi_1] \rangle v, 2 = 0 \tag{5.34}
$$

Since $\psi_2 \in \mathcal{H}_{\Sigma P}^-(\tilde{x}_h(w, v), v)$, we have $\psi_2(\tilde{x}_h(w, v)) = 0$. Note that

$$
\psi_2(\tilde{x}_h(w, v)) = \left( D^{(2)}_{b, \tilde{x}_h(w, v)} \psi_2 \right) \psi_2, \quad s^{(2,-)}(z_m) = 0.
$$
Thus we have \( \| \nabla \psi^2 \| \leq C_K |\tilde{x}_h(w,v)| \) for \( |v| \) and \( |w| \) small enough. Hence by (5.10) we have

\[
|D_v^* R_v \psi^2|_{v,z} \leq C_K |\psi^2|_{(s,t)} \| \theta_s \|_{V_t} \leq C_K |\rho(v)|^2 (|\rho(v)| + |\tilde{x}_h(w,v)|).
\]

(5.35)

Note that since \( s^{(2,-)}(z_m) = 0 \) and \( |\pi_{\tilde{x}_h(w,v)}^- - \pi_{z_m}| \leq C_K |\rho_h(v)|^2 \), we have

\[
\left| s^{(2,-)}_{b, \tilde{x}_h(w,v)} \left( (d\phi_b, T(h)|\tilde{x}_h(v))^{-1} \rho_h(v) \right) - s^{(3,-)}_{b, z_m} \right| \leq C_K |\tilde{x}_h(w,v)|^2 |\rho_h(v)|^2.
\]

(5.36)

By (5.35), (5.20) and (5.22), we have

\[
\pi_{v,-}^{0,1}(v, \zeta) \equiv \pi_{v,-}^{0,1}(\partial_f u_v + D_v \zeta + N_v \zeta)
\]

\[
= D_h^{(1)}(b, w, N)s^{(1)}_{b, \tilde{x}_h(v)}((d\phi_b, T(h)|\tilde{x}_h(v))^{-1} \rho_h(v)) + D_h^{(2)}(b, w, N)s^{(2,+)_{b, \tilde{x}_h(v)}}((d\phi_b, T(h)|\tilde{x}_h(v))^{-1} \rho_h(v))
\]

\[
+ D_h^{(2)}(b, 0, 0)s^{(3,-)}_{b, z_m}(\tilde{x}_h(w, v), \rho_h(v), \rho_h(v)) + D_h^{(3)}(b, 0, 0)s^{(3,+)_{b, z_m}}(\rho_h(v))
\]

\[
+ D_h^{(3)}(b, 0, 0)s^{(3,-)}_{b, z_m}(\rho_h(v)) + \epsilon(v, \zeta),
\]

(5.37)

where

\[
\| \epsilon(v, \zeta) \| \leq C_K |\rho(v)|^3(|v| + |w| + |N|) + C_K |\rho(v)|^2 (|\rho(v)| + |\tilde{x}_h(w, v)|) \cdot \| \zeta \|_{v, p, 1}
\]

\[
+ C_K |\tilde{x}_h(w, v)| |\rho(v)|^2 (|\tilde{x}_h(w, v)| + |w| + |N|)
\]

(5.38)

Since \( s^{(2,-)} \) has transversal zeros at \( z_m \), we have \( s^{(3,-)}_{b, z_m} \neq 0 \). Hence in order to satisfy (5.21), we must have

\[
\tilde{\alpha}_T(w, v) \equiv D_h^{(2)}(b, 0, 0)s^{(3,-)}_{b, z_m}(\tilde{x}_h(w, v), \rho_h(v), \rho_h(v)) + D_h^{(3)}(b, 0, 0)s^{(3,-)}_{b, z_m}(\rho_h(v)) = 0
\]

(5.39)

provided \( |v|, |w| \) and \( |N| \) are sufficiently small.

Conversely, suppose \( \alpha_T(v) = 0 \) and \( \tilde{\alpha}_T(w, v) = 0 \). We want to construct \( \phi(w, v) \in \mathcal{M}^0_{\tilde{\alpha}_T}(X, A, J) \) converging to \( b \). Note that by (ii-d) of Definition 1.1, (ii) of Theorem 5.1.2 holds. Hence we only need to consider the case \( H^1(\mathbb{P}^1, w^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-4\infty)) = 0 \). We have the following: In a small neighborhood of \( b \), for any

\[
v = ((\tilde{b}, w, N); v) \in \mathcal{F}_T \delta_{K(0)} \delta_K, \quad |v| \leq \delta_K, \quad \| \zeta \|_{v, p, 1} \leq 2C_K |v|^{1/2}
\]

the equation (5.37) with \( b \) replaced by \( \tilde{b} \) has a unique small solution \( (b^*, w, N^*) \in \tilde{\mathcal{M}}_{\tilde{T}}^{(0)} \).

In fact, let \( \pi_{\tilde{x}_h(w,v)}^\pm \) be the projections to \( \mathcal{H}_{\tilde{T}}^\pm(\tilde{x}_h(w,v)) \) respectively and consider the equation

\[
\pi_{\tilde{x}_h(w,v)}^- \circ \pi_{v,-}^{0,1}((\tilde{b}, w, N); v), \zeta) \equiv D_h^{(2)}(\tilde{b}, 0, 0)s^{(3,-)}_{b, z_m}(\tilde{x}_h(w, v), \rho_h(v), \rho_h(v))
\]

\[
+ D_h^{(3)}(\tilde{b}, 0, 0)s^{(3,-)}_{b, z_m}(\rho_h(v)) + \pi_{\tilde{x}_h(w,v)}^- \circ \epsilon(v, \zeta) = 0
\]

(5.40)
By the assumption $H^1(\mathbb{P}^1, u^*_n TX \otimes \mathcal{O}_{\mathbb{P}^1}(-4\infty)) = 0$, the map

$$\Phi^- : \tilde{\mathcal{T}}_\delta \to T_{u(\Sigma_P)} X \otimes \mathcal{H}_{\Sigma_P}^-(\bar{x}_h(w, v)),$$

$$v \mapsto D_h^{(2)}(\tilde{b}, 0, 0)s^{(3, -)}_{b, zm}(\bar{x}_h(w, v), \rho_h(v)) + D_h^{(3)}(\tilde{b}, 0, 0)s^{(3, -)}_{b, zm}(\rho_h(v))$$

is transversal to the zero section. This together with (5.38) yields a unique solution $(b^*, w, N)$ of (5.30) for each $N$ small enough.

Now consider the equation

$$\pi^+_{\bar{x}_h(w, v)} \circ \pi^{0,1}_{\bar{x}_h(w, v)} \cdot ((b^*, w, N); v), \zeta) \equiv D_h^{(1)}(b^*, w, N)s^{(1)}_{b, h(h(w, v))}((d\phi_{b^*}, \tau(h)|\bar{x}_h(w, v))^{-1}\rho_h(v))$$

$$+ D_h^{(2)}(b^*, w, N)s^{(2, +)}_{b, h(h(w, v))}((d\phi_{b^*}, \tau(h)|\bar{x}_h(w, v))^{-1}\rho_h(v))$$

$$+ D_h^{(3)}(b^*, 0, 0)s^{(3, +)}_{b, zm}(\rho_h(v)) + \pi^+_{\bar{x}_h(w, v)} \circ \epsilon(v, \zeta) = 0.$$  (5.41)

By (ii-a) of Definition 1.1, the map

$$\Phi^+ : \tilde{\mathcal{T}}_\delta \to T_{u(\Sigma_P)} X \otimes \mathcal{H}_{\Sigma_P}^+(\bar{x}_h(v)),$$

$$N \mapsto D_h^{(1)}(b^*, w, N)s^{(1)}_{b, h(h(w, v))}((d\phi_{b^*}, \tau(h)|\bar{x}_h(w, v))^{-1}\rho_h(v))$$

is transversal to the zero section. Note that

$$\left\| D_h^{(2)}(b^*, w, N)s^{(2, +)}_{b, h(h(w, v))}((d\phi_{b^*}, \tau(h)|\bar{x}_h(w, v))^{-1}\rho_h(v)) + D_h^{(3)}(b^*, 0, 0)s^{(3, +)}_{b, zm}(\rho_h(v)) \right\| \leq C_K|\rho(v)|^2. $$

This together with (5.38) yields a unique solution $(b^*, w, N^*)$ of (5.41) provided $\delta_K$ is sufficiently small. Thus $(b^*, w, N^*)$ is the unique solution of (5.37) as desired. Let $\mu(v, \zeta) = ((b^*, w, N^*), v)$. Then the same argument in Theorem 5.1.1 yields the $J$-holomorphic map $\phi(w, v) = [\exp_{\mu(v, \zeta)} \zeta_v] \in \mathcal{M}_{2,k}^0(\mathbb{P}^n, d)$. The proof of the theorem is complete.

Now we consider the general bubble tree case. Denote by $\chi(T) = \{h_1, h_2, \ldots, h_p\}$.

**Theorem 5.1.3.** Suppose $T$ is a bubble type given by (iv) of Theorem 2.9 with $\Sigma_P$ being smooth together with $|I_1| = 1$ and $|\chi(T)| \geq 2$. Assume the attaching node $\bar{x}$ of the bubble tree is not one of the six branch points of the canonical map. Then an element $b \equiv [C, u] \in \mathcal{M}_{T}(X, A, J) \cap \mathcal{M}_{2,k}^1(X, A, J)$ if and only if $\alpha_T(v) = 0$ for some $v = (b, v) \in \tilde{\mathcal{T}}_\delta \to T_{\Sigma_P} |_{K(0)}$ and $\tilde{\alpha}_T(v) = 0$, where $\tilde{\alpha}_T(v)$ is a linear combination of $\{D_{h_1}^{(1)}b, D_{h_2}^{(2)}b\}_{h_i \in \chi(T)}$ with coefficients depending on $v$ and the position of the nodes in bubble trees. Moreover, $\alpha_T(v)$ and $\tilde{\alpha}_T(v)$ are linear independent. In particular, $\mathcal{M}_{T}(X, A, J) \cap \mathcal{M}_{2,k}^1(X, A, J)$ is a smooth orbifold of dimension at most $\dim \mathcal{M}_{2,k}^{\text{vir}}(X, A) - 2$.

**Proof.** We only give the proof of the simplest case that all of $\{h_1, h_2, \ldots, h_p\}$ are attached to the bubble $\tilde{h} \in I_1$. The general case follows by a similar argument and we will sketch its proof in the end.
By Lemma 5.5, an element \( b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^0(X, A, J) \) must satisfy \( \alpha_T(v) = 0 \) for some \( v = (b, v) \in F_T^0 |_{\tilde{K}(0)} \), where
\[
\alpha_T(v) = \sum_{h_i \in \chi(T)} (D_{h_i}^{(1)} b) s_{\Sigma_P, \tilde{z}} (\rho_{h_i}(v)).
\] (5.42)

Denote by \( \mathcal{S} = \{ b \in \mathcal{M}_T(X, A, J) : \alpha_T(v) = 0 \} \). Then by (ii-a) of Definition 1.1, \( \mathcal{S} \) is smooth suborbifold of \( \mathcal{M}_T(X, A, J) \). Let \( \mathcal{NS} \) denote the normal bundle of \( \mathcal{S} \) in \( \mathcal{M}_T(X, A, J) \).

Suppose \( (b, N) \in \mathcal{NS} \) and \( v = ((b, N); v) \in F_T^0 |_{\tilde{K}(0)} \). We consider the second-order expansion of \( \langle \tilde{\partial}_J u((b, N); v), R_u V \eta \rangle \rangle_{v, 2} \). As in Theorem 5.1.1, we have
\[
\begin{align*}
\left| \langle \tilde{\partial}_J u((b, N); v), R_u V \eta \rangle \rangle_{v, 2} + \langle \sum_{h_i \in \chi(T)} D_{h_i}^{(1)}(b, N)s_{h_i, \tilde{z}}^{(1)}((d\phi_{b, T(h_i)}|_{\tilde{z}}^{-1}\rho_{h_i}(v)) \right. \\
+ \alpha_T^{(2)}((b, 0); v), V \eta \rangle \rangle_{v, 2} \leq C_K |\rho(v)|^2 (|v| + |N|) ||V \eta||.
\end{align*}
\] (5.43)

Let \( \tilde{h}_i((b, N); v) = \tilde{h}_i((b, 0); v) = \tilde{h}_i(v) \in \Sigma_P \) for \( h_i \in \chi(T) \). Identify a small neighborhood of \( \tilde{h} \) in \( \Sigma_P \) with a small neighborhood of 0 in \( T_{\tilde{h}} \Sigma_P \) and let \( x_i^1(v) = \tilde{h}_i(v) - \tilde{h}_i(v) = v_i^h(x_{h_i} - x_{h_i}) \) and \( x_i^2(v) = (x_i^2(v), \ldots, x_i^p(v)) \).

Let \( \{ \psi_j \} \) be an orthonormal basis for \( \mathcal{H}_{\Sigma_P}^{0,1} \) such that \( \psi_1 \in \mathcal{H}_{\Sigma_P}^{1,0}((\tilde{h}_i(v)), \psi_2 \in \mathcal{H}_{\Sigma_P}^{1,0}((\tilde{h}_i(v)) \) and \( \{ V_i \} \) an orthonormal basis for \( T_{\psi P(b, N)}X \). Note that since \( \zeta \in \tilde{\partial}^+ \), we have
\[
\langle \zeta, D^*_v R_u V \psi_1 \rangle \rangle_{v, 2} = 0
\] (5.44)

Since \( \psi_2 \in \mathcal{H}_{\Sigma_P}^{1,0}((\tilde{h}_i(v)) \), we have \( \psi_2((\tilde{h}_i(v)) = 0 \). Hence by (5.10) we have
\[
|D^*_v R_u V \psi_2|_{v, 2} \leq C_K |\psi_2|_{s, v} |\partial_s| |V_i| \leq C_K |\rho(v)|(|\rho(v)| + |x_i^s(v)|). \] (5.45)

Note that we have
\[
\begin{align*}
&\left| s_{h_i, \tilde{z}}^{(1)} ((d\phi_{b, T(h_i)}|_{\tilde{z}}^{-1}\rho_{h_i}(v)) - s_{h_i, \tilde{z}}^{(1)} (\rho_{h_i}(v)) - s_{h_i, \tilde{z}}^{(2)} (x_i^1(v), \rho_{h_i}(v))) \right| \\
&\leq C_K (|x_i^1(v)| |\rho(v)|(|x_i^1(v)| + |v|). \] (5.46)

By (5.43) - (5.46), we have
\[
\pi_{v, -}^{0,1} (v, \zeta) \equiv \pi_{v, -}^{0,1} (\tilde{\partial}_J u_v + D_v \zeta + N_v \zeta) = D_{h_i}^{(1)}(b, N)s_{h_i, \tilde{z}}^{(1)}((d\phi_{b, T(h_i)}|_{\tilde{z}}^{-1}\rho_{h_i}(v)) \\
+ \sum_{h_i \in \chi(T)} (D_{h_i}^{(1)}(b, N)s_{h_i, \tilde{z}}^{(1)} (\rho_{h_i}(v)) + D_{h_i}^{(1)}(b, 0)s_{h_i, \tilde{z}}^{(2)} (x_i^1(v), \rho_{h_i}(v))) \\
+ \sum_{h_i \in \chi(T)} (D_{h_i}^{(2)}(b, 0)s_{h_i, \tilde{z}}^{(2)} (\rho_{h_i}(v)) + D_{h_i}^{(2)}(b, 0)s_{h_i, \tilde{z}}^{(2)} (\rho_{h_i}(v))) + \varepsilon(v, \zeta), \] (5.47)
where

\[ \| e(v, \zeta) \| \leq C_K |\rho(v)|^2 |v| + |N| + C_K |\rho(v)| (|\rho(v)| + |x^*(v)|) : \| \zeta \|_{v,p,1} \]

\[ + C_K |x^*(v)||\rho(v)|(|x^*(v)| + |v| + |N|). \]

(5.48)

Note that \( \rho_{h_i}(v) = v_h v_{h_i} \) and \( x^*_i(v) = v_h (x_{h_i} - x_{h_1}) \neq 0 \) for \( h_i \in \chi(T) \setminus h_1 \). Hence

\[ |s_{b,\tilde{x}}(\rho_{h_i}(v))| = o(|s_{b,\tilde{x}}(x^*_i(v), \rho_{h_i}(v))|), \quad \forall h_i \in \chi(T) \setminus h_1, \]

(5.49)

where we denote by \( o(x) \) the higher order term of \( x \) as \( x \to 0 \). By changing the order of \( \{h_1, \ldots, h_p\} \), we may assume \( \rho_{h_1}(v) = \min_{1 \leq i \leq p} \rho_{h_i}(v) \). Thus we have

\[ |s_{b,\tilde{x}}^{(2,-)}(\rho_{h_1}(v))| = \sum_{h_1 \in \chi(T) \setminus h_1} o(|s_{b,\tilde{x}}^{(2,-)}(x^*_i(v), \rho_{h_i}(v))|). \]

(5.50)

Since \( \tilde{x} \) is not one of the six branch points of the canonical map, we have \( s_{b,\tilde{x}}^{(2,-)} \neq 0 \). Thus in order to satisfy (5.21), we must have

\[ \alpha_T(v) = 0 = \tilde{\alpha}_T(v) \] We want to construct \( \phi(v) \in \mathfrak{M}_{2,k}^0(X, A, J) \) converging to \( b \). We have the following: In a small neighborhood of \( b \), for any

\[ v = ((\tilde{b}, N); v) \in \mathcal{F}T^0_{\delta_R \delta_{\tilde{K}(0)}}, \quad \| \zeta \|_{v,p,1} \leq 2C_K |v|^{\frac{1}{p}} \]

the equation (5.47) with \( b \) replaced by \( \tilde{b} \) has a unique small solution \((b^*, N^*) \in \tilde{\mathcal{M}}^0_T\).

In fact, let \( \pi_{\tilde{x}_h_1(v)}^\pm \) be the projections to \( \mathcal{H}_{\Sigma_{x_1}}(\tilde{x}_{h_1}(v)) \) respectively and consider the equation

\[ \pi_{\tilde{x}_h_1(v)}^- \circ \pi_{\tilde{x}_{h_1}(v)}^{0,1} (((\tilde{b}, N); v), \zeta) \equiv \tilde{\alpha}_T(v) + \pi_{\tilde{x}_h_1(v)}^- \circ e(v, \zeta) = 0. \]

(5.52)

Note that by (ii-a) of Definition 1.1 and \( s_{b,\tilde{x}}^{(2,-)} = D_{b,\tilde{x}}^{(2)} \psi_2 \neq 0 \), the map \( \tilde{\alpha}_T \) is transversal to the zero section. This together with (5.48) yields a unique solution \((b^*, N) \) of (5.52) for each \( N \) provided \( |v| \) and \( |N| \) are sufficiently small.

Now consider the equation

\[
\begin{align*}
\pi_{\tilde{x}_h_1(v)}^+ \circ \pi_{\tilde{x}_{h_1}(v)}^{0,1} (((b^*, N); v), \zeta) & \equiv D^{(1)}_{h_1}(b^*, N)s^{(1)}_{b^*,\tilde{x}_h_1(v)}((d\phi_{b^*,T(h_1)}|\tilde{x}_{h_1}(v))^{-1} \rho_{h_1}(v)) \\
& + \sum_{h_i \in \chi(T) \setminus h_1} (D^{(1)}_{h_i}(b^*, N)s^{(1)}_{b^*,\tilde{x}_h_1(v)}(\rho_{h_i}(v)) + D^{(1)}_{h_i}(b^*, 0)s^{(2,-)}_{b^*,\tilde{x}}(x^*_i(v), \rho_{h_i}(v))) \\
& + \sum_{h_i \in \chi(T)} D^{(2)}_{h_i}(b^*, 0)s^{(2,+)}_{b^*,\tilde{x}}(\rho_{h_i}(v)) + \pi_{\tilde{x}_{h_1}(v)}^+ \circ e(v, \zeta) = 0
\end{align*}
\]
By (ii-a) of Definition 1.1, the map
\[
\Phi^+ : \mathcal{FT}_{\delta_K}^0 \to T_u(\Sigma_P)X \otimes \mathcal{H}_{\delta_P}^{\pm}(\bar{x}_h(v)),
\]
\[
N \to D_{h_1}(b^*, N)\delta_{b^*, \bar{x}_h(v)}((d\phi_{b^*}, T(h_1)|\bar{x}_h(v))^{-1}\rho_{h_1}(v)) + \sum_{h_1 \in \chi(T)\setminus h_1} D_{h_1}(b^*, N)s_{b^*, \bar{x}_h(v)}(\rho_{h_1}(v))
\]
is transversal to the zero section. This together with (5.48) yields a unique solution \((b^*, N^*)\) of (5.53) provided \(|v|\) is sufficiently small. Thus \((b^*, N^*)\) is the unique solution of (5.47) as desired. Denote by \(\mu(v, \zeta) = ((b^*, N^*), v)\). Then the same argument in Theorem 5.1.1 yields the holomorphic map \(\phi(v) = [\exp_{\mu(v, \zeta)} \zeta v] \in M_{2,k}(X, A, J)\).

For the general case, let \(\hat{h} = \min_{h < h_i, h_i \in \chi(T)} \{ h : |\{ l \in I : u_l = h \} | \geq 2 \} \). Then a similar argument as above obtains the required conditions \(\alpha_T(v) = 0 = \hat{\alpha}_T(v)\). The proof of the theorem is complete.

**Theorem 5.1.4.** Suppose \(T\) is a bubble type given by (iv) of Theorem 2.9 with \(\Sigma_P\) being smooth together with \(|I_1| = 1\) and \(|\chi(T)| \geq 2\). Assume the attaching node \(\hat{x}\) of the bubble tree is one of the six branch points of the canonical map. Then one of the following must hold:

(i) An element \(b \equiv [C, u] \in M_T(X, A, J) \cap \overline{M}_{2,k}(X, A, J)\) if and only if \(\alpha_{\hat{T}}(v) = 0\) for some \(v = (b, v) \in \mathcal{FT}_{\delta_K}^{0}|_{T_{\hat{x}}(v)}\) and \(\alpha_T(w, v) = 0\), where \(\alpha_T(w, v)\) is a linear combination of \(\{D_{h_1}(b, D_{h_1}(b^*, D_{h_1}(b^*, v))\}_{h_1 \in \chi(T)}\) with coefficients depending on \(w \in T_{\hat{x}}(\Sigma_P), v\) and the position of the nodes in bubble trees. In particular, \(M_T(X, A, J) \cap \overline{M}_{2,k}(X, A, J)\) is a smooth orbifold of dimension at most \(\dim \overline{M}_{2,k}(X, A) - 2\).

(ii) An element \(b \equiv [C, u] \in M_T(X, A, J) \cap \overline{M}_{2,k}(X, A, J)\) must satisfy: there exists \(h \in \chi(T)\) such that \(u_h\) factor through a branched covering \(\hat{u} : S^2 \to X, i.e., there exists a holomorphic branched covering \(\phi : S^2 \to S^2\) such that \(u_h = \hat{u} \circ \phi and \deg(\phi) \geq 2\). In particular, \(M_T(X, A, J) \cap \overline{M}_{2,k}(X, A, J)\) is contained in a smooth orbifold of dimension at most \(\dim \overline{M}_{2,k}(X, A) - 2\).

**Proof.** We only give the proof of the simplest case that all of \(\{h_1, h_2, \ldots, h_p\} \equiv \chi(T)\) are attached to the bubble \(\hat{h} \in I_1\). The general cases follow similarly.

As in Theorem 5.1.3, we have (5.42). Denote by \(S_m = \{ b \in S : x^*_b = z_m \}\), where \(S\) is given by Theorem 5.1.3, and \(NS_m\) the normal bundle of \(S_m\) in \(S\).

Suppose \((b, w) \in NS_m\), \((b, w, N) \in NS\) and \(v = ((b, w, N); v) \in \mathcal{FT}_{\delta_K}^{0}|_{T_{\hat{x}}(v)}\). We consider the third-order expansion of \(\langle (\hat{\partial} ju_{((b, w, N); v), R_v V\eta}) \rangle_{v, 2}\). As in Theorem 5.1.2, we have
\[
\left| \langle (\hat{\partial} ju_{((b, w, N); v), R_v V\eta}) \rangle_{v, 2} + \langle \sum_{h_1 \in \chi(T)} (D_{h_1}(b, v) \delta_{b, \bar{x}_h(v)}((d\phi_{b}, T(h_1)|\bar{x}_h(v))^{-1}\rho_{h_1}(v)) + D_{h_1}(b, w, N)s_{b, \bar{x}_h(v)}(\rho_{h_1}(v)) \right.
\]
\[
+ D_{h_1}(b, w, N)s_{b, \bar{x}_h(v)}((d\phi_{b}, T(h_1)|\bar{x}_h(v))^{-1}\rho_{h_1}(v)) + \alpha_{\hat{T}}(b, 0, 0; v, V\eta) \rangle_{v, 2} \right| \leq C_K |\rho(v) |^3 (|v| + |w| + |N|)|V\eta|. \tag{5.54}
\]
Let $\tilde{x}_h_i(v) = \tilde{x}_h_i((b, w, N); v) = \tilde{x}_h_i((b, w, 0); v) \equiv \tilde{x}_h_i(w, v) \in \Sigma_P$ for $h_i \in \chi(T)$. Identify a small neighborhood of $z_m$ in $\Sigma_P$ with a small neighborhood of 0 in $T_{z_m} \Sigma_P$ and let $x^*_i(w, v) = \tilde{x}_h_i(w, v) - \tilde{x}_h_i(w, v) = \tilde{v}_i (x_h_i - x_h_i)$ and $x^*(w, v) = (x^*_1(w, v), \ldots, x^*_p(w, v))$.

Let $\{\psi\}$ be an orthogonal basis for $H_{e,\Sigma_P}^1(\tilde{x}_h_i(w, v))$, $\psi_2 \in H_{\Sigma_P}^1(\tilde{x}_h_i(w, v))$ and $\{V_i\}$ an orthogonal basis for $T_{eu,\Sigma_P(b, w, N)}X$. Note that since $\zeta \in \tilde{\Gamma}^+(v)$, we have

$$ \langle \langle \zeta, D^*_v R_v V_i \psi_1 \rangle \rangle_{v, 2} = 0 $$

(5.55)

Since $\psi_2 \in H_{\Sigma_P}^1(\tilde{x}_h_i(w, v))$, we have $\psi_2(\tilde{x}_h_i(w, v)) = 0$. Note that $s^{(2, -)}(z_m) = 0$, thus by (5.10) we have (cf. Lemma 4.28 in [ZL])

$$ |D^*_v R_v V_i| \psi_2|_{v, z} \leq C_K |\psi_2|_{(s, t)} |\partial_s||V_i| \leq C_K (|\rho_h(v)|^2 (|\rho_h(v)| + |\tilde{x}_h_i(w, v)|)). $$

(5.56)

Let $\pi^\pm_{x_h_i}(w, v)$ be the projections to $H_{\Sigma^\pm}_{x_h_i}(\tilde{x}_h_i(w, v))$ respectively. Then by Taylor expansion with respect to $x^*_i(w, v) = \tilde{x}_h_i(w, v) - \tilde{x}_h_i(w, v)$, we have

$$ \left| \begin{array}{c}
(1) s_{b, \tilde{x}_h_i(w, v)}((d\phi_b, \tau(h_i))|\tilde{x}_h_i(v))^{-1} \rho_h(v) - s_{b, \tilde{x}_h_i(w, v)}(\rho_h(v)) - s_{b, \tilde{x}_h_i(w, v)}(x^*_i(w, v), \rho_h(v)) \\
- s_{b, \tilde{x}_h_i(w, v)}(x^*_i(w, v), x^*_i(w, v), \rho_h(v)) \\
\end{array} \right| \leq C_K |x^*_i(w, v)|^2 |\rho_h(v)|. $$

(5.57)

Since $\pi^\pm_{x_h_i}(w, v) s_{b, \tilde{x}_h_i(w, v)} = 0$, $s^{(2, -)}_{b, z_m} = 0$ and $\tilde{x}_h_i(w, v) = w + v \cdot x_h_i$, we have

$$ \left| \begin{array}{c}
(2) s_{b, \tilde{x}_h_i(w, v)}(x^*_i(w, v), \rho_h(v)) - s_{b, z_m}(\tilde{x}_h_i(w, v), x^*_i(w, v), \rho_h(v)) \\
\end{array} \right| \leq C_K |\tilde{x}_h_i(w, v)|^2 |\rho_h(v)| |x^*_i(w, v)|. $$

(5.58)

Summing up (5.57)-(5.59), we obtain

$$ \left| \begin{array}{c}
\pi^\pm_{x_h_i}(w, v) s_{b, \tilde{x}_h_i(w, v)}((d\phi_b, \tau(h_i))|\tilde{x}_h_i(v))^{-1} \rho_h(v) - s_{b, z_m}(\tilde{x}_h_i(w, v), x^*_i(w, v), \rho_h(v)) \\
\end{array} \right| \leq C_K (|x^*_i(w, v)|^3 |\rho_h(v)| + |\tilde{x}_h_i(w, v)||\rho_h(v)||x^*_i(w, v)|| |x^*_i(w, v)| + |\tilde{x}_h_i(w, v)||\rho_h(v)|). $$

(5.60)

Similarly, we have

$$ \left| \begin{array}{c}
\pi^\pm_{x_h_i}(w, v) s_{b, \tilde{x}_h_i(w, v)}(d\phi_b, \tau(h_i)|\tilde{x}_h_i(v))^{-1} \rho_h(v) - s_{b, z_m}(\tilde{x}_h_i(w, v), \rho_h(v), \rho_h(v)) \\
\end{array} \right| \leq C_K (|x^*_i(w, v)|^2 |\rho_h(v)|^2 + |\tilde{x}_h_i(w, v)|^2 |\rho_h(v)|^2 + |x^*_i(w, v)||\tilde{x}_h_i(w, v)||\rho_h(v)|^2). $$

(5.61)
By \((5.54)-(5.61)\), we have

\[
\pi_{v,-}^{0,1}(v, \zeta) \equiv \pi_{v,-}^{0,1}(D_vu_v + D_v\zeta + N_v\zeta) = D^{(1)}_{h_1}(b, w, N)s^{(1)}_{b, \tilde{z}_{h_1}(v)}((d\phi_{b, T(h_1)}|\tilde{z}_{h_1}(v))^{-1}\rho_{h_1}(v)) \\
+ \sum_{h_i \in \chi(T) \backslash h_1} (D^{(1)}_{h_i}(b, w, N)\pi_{x_{h_i}(w,v)}^{+} s^{(1)}_{b, \tilde{z}_{h_i}(w,v)}((d\phi_{b, T(h_i)}|\tilde{z}_{h_i}(v))^{-1}\rho_{h_i}(v))) \\
+ D^{(1)}_{h_1}(b, 0, 0)s^{(3,-)}_{b, z_m}(\tilde{h}_i(w, v), x^*_i(w, v), \rho_{h_i}(v))) \\
+ \sum_{h_i \in \chi(T)} (D^{(2)}_{h_i}(b, w, N)\pi_{x_{h_i}(w,v)}^{+} s^{(2)}_{b, \tilde{z}_{h_i}(w,v)}((d\phi_{b, T(h_i)}|\tilde{z}_{h_i}(v))^{-1}\rho_{h_i}(v))) \\
+ D^{(2)}_{h_1}(b, 0, 0)s^{(3,-)}_{b, z_m}(\tilde{h}_i(w, v), \rho_{h_i}(v), \rho_{h_i}(v))) + a^T((b, 0, 0); v) + \epsilon(v, \zeta),
\]

where

\[
\|\epsilon(v, \zeta)\| \leq C_K(|\rho(v)|^3 + |x^*(w, v)||\rho(v)||\tilde{x}(w, v)| + |\rho(v)|^2|\tilde{x}(w, v)|)(|v| + |w| + |N|) \\
+ C_K\left(|\rho_{h_i}(v)|^2(|\rho_{h_i}(v)| + |\tilde{x}_{h_i}(w, v)|) + \sum_{2 \leq i \leq p} |\rho_{h_i}(v)||v^*_h| + |\tilde{x}_{h_i}(w, v)||v^*_h|\right)\cdot \|\zeta\|_{v, p, 1} \\
+ C_K(|x^*_i(w, v)|^3|\rho_{h_i}(v)| + |\tilde{x}_{h_i}(w, v)||\rho_{h_i}(v)||x^*_i(w, v)||x^*_i(w, v)| + |\tilde{x}_{h_i}(w, v)|) \\
+ C_K(|x^*_i(w, v)|^2|\rho_{h_i}(v)|^2 + |\tilde{x}_{h_i}(w, v)|^2|\rho_{h_i}(v)|^2 + |x^*_i(w, v)||\tilde{x}_{h_i}(w, v)||\rho_{h_i}(v)|^2).
\]

where \(|\tilde{x}(w, v)| = \sum_{1 \leq i \leq p} |\tilde{x}_{h_i}(w, v)|\).

By replacing the order of \(\{h_1, \ldots, h_p\}\) if necessary, we may assume that \(|\tilde{x}_{h_i}(w, v)| \leq |\tilde{x}_{h_j}(w, v)|\) for \(2 \leq i \leq p\). Note that \(|\tilde{x}_{h_i}(w, v) - \tilde{x}_{h_j}(w, v)| = v^*_h(x_{h_i} - x_{h_j})\) and \(x_{h_i} \neq x_{h_j}\) for \(i \neq j\). Thus we have \(|v^*_h| \leq C_K|\tilde{x}_{h_i}(w, v)|\) for \(2 \leq i \leq p\). Hence we have

\[
|s^{(3,-)}_{b, z_m}(\tilde{x}_{h_i}(w, v), \rho_{h_i}(v), \rho_{h_i}(v))| = a(s^{(3,-)}_{b, z_m}(\tilde{x}_{h_i}(w, v), x^*_i(w, v), \rho_{h_i}(v))) \\
|s^{(3,-)}_{b, z_m}(\rho_{h_i}(v), \rho_{h_i}(v), \rho_{h_i}(v))| = a(s^{(3,-)}_{b, z_m}(\tilde{x}_{h_i}(w, v), x^*_i(w, v), \rho_{h_i}(v)))
\]

for \(2 \leq i \leq p\).

We have the following two cases:

**Case 1.** We have \(D^{(1)}_{h_1}b \neq 0\).

In this case we can represent \(v_{h_1}\) as linear combination of \(v_{h_i}\) for \(h_i \in \chi(T) \backslash h_1\). In particular, we have \(|v_{h_1}| \leq C_K \sum_{h_i \in \chi(T) \backslash h_1} |v_{h_i}|\. Hence we have

\[
|s^{(3,-)}_{b, z_m}(\tilde{x}_{h_1}(w, v), \rho_{h_1}(v), \rho_{h_1}(v))| = \sum_{h_i \in \chi(T) \backslash h_1} o(s^{(3,-)}_{b, z_m}(\tilde{x}_{h_i}(w, v), x^*_i(w, v), \rho_{h_i}(v))) \\
|s^{(3,-)}_{b, z_m}(\rho_{h_1}(v), \rho_{h_1}(v), \rho_{h_1}(v))| = \sum_{h_i \in \chi(T) \backslash h_1} o(s^{(3,-)}_{b, z_m}(\tilde{x}_{h_i}(w, v), x^*_i(w, v), \rho_{h_i}(v)))
\]
Thus the dimension of \( \alpha \) provided by (5.62) and (5.63).

In this subsection, we consider the case 5.2 There are exactly two bubble trees

Note that \( s_{b, z_m} \neq 0 \), thus in order to satisfy (5.21), we must have

\[
\overline{\alpha}(w, v) \equiv \sum_{2 \leq i \leq \rho} D_{h_i}^{(1)}(b, 0, 0)s_{b, z_m}^{(3,-)}(\overline{x}_{h_i}(w, v), x_i^*(w, v), \rho_{h_i}(v)) = 0
\]

by (5.62) and (5.63).

**Case 2.** We have \( D_{h_1}^{(1)}b = 0 \).

In this case we have

\[
\overline{\alpha}(w, v) \equiv \sum_{2 \leq i \leq \rho} D_{h_i}^{(1)}(b, 0, 0)s_{b, z_m}^{(3,-)}(\overline{x}_{h_i}(w, v), x_i^*(w, v), \rho_{h_i}(v)) \\
+ D_{h_1}^{(2)}(b, 0, 0)s_{b, z_m}^{(3,-)}(\overline{x}_{h_1}(w, v), \rho_{h_1}(v)) + D_{h_1}^{(3)}(b, 0, 0)s_{b, z_m}^{(3,-)}(\rho_{h_1}(v)) = 0
\]

provided \(|v|, |w| \) and \(|N| \) are sufficiently small.

Note that \( \alpha_T(v) \) and \( \overline{\alpha}(w, v) \) are linear independent in Case 1. While in Case 2, we have

\[
\alpha_T(v) = \sum_{h_i \in \chi(T) \setminus h_1} (D_{h_i}^{(1)}b)s_{\Sigma_P, \tilde{x}}(\rho_{h_i}(v)) = 0.
\]

Thus the dimension of \( \alpha_T^{-1}(0) \) is at most \( \dim \mathcal{M}_{vir}^{2,k}(X, A) - 2 \).

The proof of the converse is similar to the previous theorems and we omit it here. We only remark that in Case 1, (i) holds; while in Case 2, (i) holds if \( H^1(\mathbb{P}^1, u_{h_1}^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-4\infty)) = 0 \) and (ii) holds if \( H^1(\mathbb{P}^1, u_{h_1}^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-4\infty)) \neq 0 \). These follows by (ii-d) in Definition 1.1.

#### 5.2 There are exactly two bubble trees

In this subsection, we consider the case \(|I_1| = 2 \). Note that in this case the rank of \( \alpha_T(v) \) may not equal to the dimension of \( \text{coker}D_b \). Thus we need further expansions.

**Theorem 5.2.1.** Suppose \( T \) is a bubble type given by (iv) of Theorem 2.9 with \( \Sigma_P \) being smooth and \(|I_1| = 2 \), i.e., there are exactly two bubble trees. Assume \( x_1 \) and \( x_2 \) are not conjugate via the map \( s_{\Sigma_P} \), where \( x_1 \) and \( x_2 \) are attaching nodes of the two bubble trees. Then an element \( b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^{\theta}(X, A, J) \) if and only if \( \alpha_T(v) = 0 \) for some \( v = (b, v) \in \mathcal{F}^\theta T_{\delta_K}\). In particular, \( \mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^{\theta}(X, A, J) \) is a smooth orbifold of dimension at most \( \dim \mathcal{M}_{vir}^{2,k}(X, A) - 2 \).

**Proof.** By Lemma 5.5, an element \( b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^{\theta}(X, A, J) \) must satisfy \( \alpha_T(v) = 0 \) for some \( v = (b, v) \in \mathcal{F}^\theta T_{\delta_K} \). Since \( x_1 \) and \( x_2 \) are not conjugate via the map \( s_{\Sigma_P} \), we have \( \text{rank}(\alpha_T) = 4n = \dim(\text{coker}D_b) \). Hence

\[
\alpha_T : \mathcal{F}^\theta T_{\delta_K} \to T_{u(\Sigma_P)}X \otimes \mathcal{H}_{\Sigma_P}^{0,1}, \quad v \mapsto \alpha_T(v),
\]
is transversal to the zero section in a small neighborhood of $b$ by (ii-a) in Definition 1.1. Hence by (5.23) and (5.24), in a small neighborhood of $b$, for any

$$v = (b, v) \in \overline{F_{\delta K}}^0|_{\tilde{K}(0)}, \quad \|\zeta\|_{v,p,1} \leq 2C_K|v|^{\frac{1}{p}},$$

the equation

$$\pi_{v,-}^0((\tilde{b}, v), \zeta) = \alpha_T(v) + \epsilon(v, \zeta) = 0$$

has a unique small solution $b^* \in \tilde{M}_{T}^{(0)}$. Denote by $\mu(v, \zeta) = (b^*, v)$. Then the theorem follows by the same argument as in Theorem 5.1.1.

Next we consider the case $x_1$ and $x_2$ are conjugate via the map $s_{\Sigma_P}$.

**Theorem 5.2.2.** Suppose $T$ is a bubble type given by (iv) of Theorem 2.9 with $\Sigma_P$ being smooth and $|I_1| = |\chi(T)| = \{|(h_1, h_2)| = 2$. Assume the attaching nodes $x_{T(h_1)}$ and $x_{T(h_2)}$ differ by the nontrivial holomorphic automorphism of $\Sigma_P$ (cf. p. 254 of [GH]). Then one of the following must hold:

(i) An element $b \equiv [C, u] \in M_{T}(X, A, J) \cap \overline{M}_{2,k}^{0}(X, A, J)$ if and only if $\alpha_T(v) = 0$ for some $v = (b, v) \in \overline{F_{\delta K}}^0|_{\tilde{K}(0)}$ and $\alpha_T(w, v) = 0$, where $\alpha_T(w, v)$ is a linear combination of $\{D_{h_1}^{(1)}b, D_{h_1}^{(2)}b\}_{i=1,2}$ with coefficients depending on $w \in T x_{T(h_1)} \Sigma_P, v$ and the position of the nodes in bubble trees. In particular, $M_T(X, A, J) \cap \overline{M}_{2, k}^{0}(X, A, J)$ is a smooth orbifold of dimension at most $\dim \overline{M}_{2,k}^{\text{vir}}(X, A) - 2$.

(ii) An element $b \equiv [C, u] \in M_{T}(X, A, J) \cap \overline{M}_{2, k}^{0}(X, A, J)$ must satisfy $u_1(\Sigma_{h_1}) = u_2(\Sigma_{h_2})$.

**Proof.** By Lemma 5.5, an element $b \equiv [C, u] \in M_{T}(X, A, J) \cap \overline{M}_{2, k}^{0}(X, A, J)$ must satisfy $\alpha_T(v) = 0$ for some $v = (b, v) \in \overline{F_{\delta K}}^0|_{\tilde{K}(0)}$. Note that

$$\alpha_T(v) = (D_{h_1}^{(1)}b)s_{\Sigma_P, x_{T(h_1)}}(\rho_{h_1}(v)) + (D_{h_2}^{(1)}b)s_{\Sigma_P, x_{T(h_2)}}(\rho_{h_2}(v)), \quad (5.66)$$

and $s_{\Sigma_P, x_{T(h_1)}} = s_{\Sigma_P, x_{T(h_2)}}$. Denote by $-x$ the image of $x \in \Sigma_P$ under the nontrivial holomorphic automorphism $\sigma$ of $\Sigma_P$ and identify $T_x \Sigma_P$ with $T_{-x} \Sigma_P$ via $\sigma$. Let $\Sigma_P^* = \Sigma_P \setminus \{z_m\}_{1 \leq m \leq 6}$, where $\{z_m\}_{1 \leq m \leq 6}$ are the fixed points of $\sigma$. Denote by $S = \alpha_T^{-1}(0)$. Then by (ii-a) in Definition 1.1, $S$ is a complex suborbifold of $M_{T}(X, A, J)$. Moreover, we have $S = S_0 \times S_1$, where $S_0 = \{(x_{T(h_1)}, -x_{T(h_1)} : x_{T(h_1)} \in \Sigma_P\}$ and $S_1 = \{D_{h_2}^{(1)}b = \lambda D_{h_1}^{(1)}b, \lambda \in \mathbb{C}\}$. Let $NS$ denote the normal bundle of $S$ in $M_{T}(X, A, J)$ and identify a small neighborhood of its zero section with a tubular neighborhood of $S$ in $M_{T}(X, A, J)$. Then we have

$$NS = NS_0 \oplus NS_1, \quad NS_0 = \pi_{\Sigma_P, x_{T(h_2)}}^* T \Sigma_P, \quad NS_1 = E_1,$$
where \( \pi_{\Sigma_P, \Sigma_{T(h_i)}} : S_0 \subset \Sigma_P \times \Sigma_P \rightarrow \Sigma_P \) is the projection to the \( i \)-th factor and \( E_1 \) is the orthogonal complement of \( D^{(1)} b \) in \( T_{\theta} \mid_{\Sigma_P} X \).

Suppose \((b, w, N) \in NS\) and \( v = ((b, w, N); v) \in \hat{\mathcal{F}}\mathcal{T}_{\delta_k}^{\theta} \mid_{K^0}\). We consider the second-order expansion of \( \langle \langle \partial_j u_{((b, w, N); v)} , R_v V \eta \rangle \rangle \rangle_{v, 2} \). Note that we have

\[
\left\| s^{(2)}_{b, \tilde{x}_h(v)} \left( (d\phi_b, \tau(h))|_{\tilde{x}_h(v)} \right)^{-1} \rho_h(v) \right\|_{2} \leq C_K \left| \phi_b, \tau(h) \right|_{b} \left| \rho_h(v) \right|_{2} \leq C_K |v| \cdot |\rho_h(v)|^2,
\]

\[
\sum_{m \geq 3} |D^{(m)}_{b} (b, w, N) | \left| \rho_h(v) \right|^m \leq C_K |\rho_h(v)|^3,
\]

\[
\left| D^{(2)}_{h} (b, w, N) - D^{(2)}_{h} (b, 0, 0) \right| \leq C_K (|N| + |w|),
\]

for \( h \in \chi(T) \) and \( v, w, N \) sufficiently small by continuity. Thus by Lemma 5.1 we have

\[
\left| \langle \langle \partial_j u_{((b, w, N); v)} , R_v V \eta \rangle \rangle \rangle_{v, 2} + \langle \langle D^{(1)}_{h} (b, w, N) s^{(1)}_{b, \tilde{x}_h(v)} \left( (d\phi_b, \tau(h))|_{\tilde{x}_h(v)} \right)^{-1} \rho_h(v) \rangle \rangle_{v, 2} \right| \leq C_K \left| \rho(v) \right|^2 (|v| + |w| + |N|) ||V\eta||.
\]

Let \( \tilde{x}_h((b, w, N); v) = \tilde{x}_h(b, w, 0); v) \equiv \tilde{x}_h((b, w, v) \in \Sigma_P \) for \( i = 1, 2 \). Identify a small neighborhood of \( x_{T(h_i)} \) in \( \Sigma_P \) with a small neighborhood of \( 0 \) in \( T_{x_{T(h_i)}} \Sigma_P \) and let \( x^*(w, v) = \tilde{x}_h((b, w, v) \in \Sigma_P \), then we have \( |x^*(w, v)| \leq C_K (|v| + |w|) \).

Let \( \{ \psi_j \} \) be an orthonormal basis for \( \mathcal{H}^0_{\Sigma_P} \) such that \( \psi_1 \in \mathcal{H}^+_{\Sigma_P} (\tilde{x}_h(w, v)) \), \( \psi_2 \in \mathcal{H}^-_{\Sigma_P} (\tilde{x}_h(w, v)) \) and \( \{ V_i \} \) an orthonormal basis for \( T_{evp(b, w, N)} X \). Note that since \( \zeta \in \bar{\Gamma}^+ \), we have

\[
\langle \langle \zeta, D^{*}_{v} R_v V \psi_1 \rangle \rangle_{v, 2} = 0
\]

(5.68)

Since \( \psi_2 \in \mathcal{H}^-_{\Sigma_P} (\tilde{x}_h(w, v)) \), we have \( \psi_2 (\tilde{x}_h(w, v)) = 0 \). Note that \( \psi_2 (-\tilde{x}_h(w, v)) = 0 \) since \( \psi_2 \) is invariant under \( \sigma \). Hence by (5.10) we have

\[
|D^{*}_{v} R_v V \psi_2|_{v, 2} \leq C_K |\psi_2|_{(s, t)} |\partial_b||V_i| \leq C_K \left| \rho(v) \right| (|\rho(v)| + |x^*(w, v)|).
\]

(5.69)

Note that we have

\[
\left| s^{(1)}_{b, \tilde{x}_h} ((d\phi_b, \tau(h))|_{\tilde{x}_h(v)} \left( (d\phi_b, \tau(h))|_{\tilde{x}_h(v)} \right)^{-1} \rho_h(v) \right| \leq C_K (|x^*(w, v)| ||\rho(v)|| (|x^*(w, v)| + |v|).
\]

(5.70)

By (5.67)-(5.70), we have

\[
\pi_{v, -}^{0, 1} (v, \zeta) \equiv \pi_{v, -}^{0, 1} (\partial_j u_v + D_v \zeta + N_v \zeta)
\]

43
to the zero section. This together with (5.72) yields a unique solution (5.71) with

\[
\pi_1(x^*, v) = e_1
\]

We have

\[
D_{h_2}^{(1)}(b, 0, 0)s_{b, x_{T(h_1)}}^{(2, -)}(\rho_{h_2}(v)) + D_{h_2}^{(2)}(b, 0, 0)s_{b, x_{T(h_2)}}^{(2, -)}(\rho_{h_2}(v)) + \epsilon(v, \zeta)
\]

where

\[
\|\epsilon(v, \zeta)\| \leq C_K|\rho(v)|^2(|v| + |w| + |N|) + C_K|\rho(v)|(|\rho(v)| + |x^*(w, v)|) \cdot |\zeta|_{v, \rho, 1} + C_K(|x^*(w, v)||\rho(v)||(\rho(v)) + |x^*(w, v)|) \cdot |\zeta|_{v, \rho, 1}.
\]

Since \(x_{T(h_1)} \in \Sigma_\rho\), we have \(s_{b, x_{T(h_1)}}^{(2, -)} \neq 0\). Hence in order to satisfy (5.21), we must have

\[
\tilde{\alpha}_T(w, v) \equiv D_{h_2}^{(1)}(b, 0, 0)s_{b, x_{T(h_1)}}^{(2, -)}(x^*(w, v), \rho_{h_2}(v)) + D_{h_2}^{(2)}(b, 0, 0)s_{b, x_{T(h_1)}}^{(2, -)}(\rho_{h_1}(v))\]

\[
+ D_{h_2}^{(2)}(b, 0, 0)s_{b, x_{T(h_2)}}^{(2, -)}(\rho_{h_2}(v)) = 0
\]

provided \(|v|, |w| \text{ and } |N|\) are sufficiently small.

Conversely, suppose \(\alpha_T(v) = 0\) and \(\tilde{\alpha}_T(w, v) = 0\). We want to construct \(\phi(w, v) \in \mathcal{M}_2^0(X, A, J)\) converging to \(b\). We want to show: In a small neighborhood of \(b\), for any

\[
v = ((\tilde{b}, w, N); v) \in \tilde{\mathcal{F}}_0^{\rho}|K(w)|, \quad |w| \leq \delta_K, \quad \|\zeta\|_{v, \rho, 1} \leq 2C_K|v|^{\frac{1}{\beta}}
\]

the equation (5.71) with \(b\) replaced by \(\tilde{b}\) has a unique small solution \((b^*, w, N^*) \in \tilde{\mathcal{M}}_0\). Let \(\pi_{\pm_{\tilde{x}_{h_1}}(w, v)}\) be the projections to \(\mathcal{H}_{\Sigma_{\rho}}(\tilde{x}_{h_1}(w, v))\) respectively and consider the following equation:

\[
\pi_{\tilde{x}_{h_1}(w, v)} \circ \pi_{\rho_{h_1}(v)} \circ ((\tilde{b}, w, N); v), \zeta \equiv D_{h_2}^{(1)}(\tilde{b}, 0, 0)s_{b, x_{T(h_1)}}^{(2, -)}(x^*(w, v), \rho_{h_2}(v))
\]

\[
+ D_{h_2}^{(2)}(\tilde{b}, 0, 0)s_{b, x_{T(h_1)}}^{(2, -)}(\rho_{h_1}(v)) + D_{h_2}^{(2)}(\tilde{b}, 0, 0)s_{b, x_{T(h_2)}}^{(2, -)}(\rho_{h_2}(v)) + \pi_{\tilde{x}_{h_1}(w, v)} \circ \epsilon(v, \zeta) = 0.
\]

We have the following two cases:

**Case 1.** We have \(D_{h_1}^{(1)}b \neq 0\).

In this case we have \(|\rho_{h_1}(v)| \leq C_K|\rho_{h_2}(v)|\). Note that by \(\alpha_T(v) = 0\) and (v-a) in Definition 1.1, if \(u_{h_1}(\Sigma_{h_1}) \neq u_{h_2}(\Sigma_{h_2})\), we may assume \(H^1(\mathbb{R}^1, u^2_{h_2}TX \otimes \mathcal{O}_{\mathbb{P}_1}(-3z_2)) \neq 0\), thus \(\tilde{\alpha}_T(w, v)\) is transversal to the zero section. By (5.72), the terms in \(\pi_{\tilde{x}_{h_1}(w, v)} \circ \epsilon(v, \zeta)\) contains \(\rho_{h_1}(v)\) can be dominated by \(|\rho_{h_2}(v)|^2\). Thus we obtain a unique solution \((b^*, w, N)\) of (5.74) for each \(N\) when \(\delta_K\) is sufficiently small.

**Case 2.** We have \(D_{h_1}^{(1)}b = 0\).

In this case, we have \(D_{h_2}^{(1)}b = 0\) by (6.66). Hence by (ii-b) in Definition 1.1, \(\tilde{\alpha}_T(w, v)\) is transversal to the zero section. This together with (5.72) yields a unique solution \((b^*, w, N)\) of (5.74) for each \(N\) when \(\delta_K\) is sufficiently small.
Then the same argument in Theorem 5.1.1 yields the holomorphic map for some \( v \) and is transversal to the zero section. Note that the general case follows similarly.

Now consider the equation

\[
\begin{align*}
\pi^+_{\tilde{x}_{b_1}(w,v)} \circ \pi_{v,-}^0((b^*, w, N); v), \zeta) & \equiv D^{(1)}_{h_1}(b^*, w, N)s^{(1)}_{b^*, \tilde{x}_{b_1}(w,v)}((d\phi_{b^*, T(h_1)}|_{\tilde{x}_{b_1}(v)})^{-1}\rho_{h_1}(v)) \\
+ D^{(1)}_{h_2}(b^*, w, N)s^{(1)}_{b^*, \tilde{x}_{b_1}(w,v)}(\rho_{h_2}(v)) + D^{(2)}_{h_2}(b^*, 0, 0)s^{(2,+)}_{b^*, x_{T(h_1)}}(x^*(w, v), \rho_{h_2}(v)) \\
+ D^{(2)}_{h_1}(b^*, 0, 0)s^{(2,+)}_{b^*, x_{T(h_2)}}(\rho_{h_1}(v)) + D^{(2)}_{h_2}(b^*, 0, 0)s^{(2,+)}_{b^*, x_{T(h_2)}}(\rho_{h_2}(v)) \\
+ \pi^+_{\tilde{x}_{h_1}(w,v)} \circ \epsilon(v, \zeta) = 0.
\end{align*}
\]

By (ii-a) in Definition 1.1, the map

\[
\Phi^+ : \mathcal{F}_\delta^0 \to T_u(\Sigma_P)X \otimes \mathcal{H}^1_{\Sigma_P}(\tilde{x}_{h}(v)),
\]

\[
N \mapsto D^{(1)}_{h_1}(b^*, w, N)s^{(1)}_{b^*, \tilde{x}_{b_1}(w,v)}((d\phi_{b^*, T(h_1)}|_{\tilde{x}_{b_1}(v)})^{-1}\rho_{h_1}(v)) + D^{(1)}_{h_2}(b^*, w, N)s^{(1)}_{b^*, \tilde{x}_{b_1}(w,v)}(\rho_{h_2}(v))
\]
is transversal to the zero section. Note that

\[
\left\| D^{(1)}_{h_2}(b^*, 0, 0)s^{(2,+)}_{b^*, x_{T(h_1)}}(x^*(w, v), \rho_{h_2}(v)) + D^{(2)}_{h_1}(b^*, 0, 0)s^{(2,+)}_{b^*, x_{T(h_1)}}(\rho_{h_1}(v)) \\
+ D^{(2)}_{h_2}(b^*, 0, 0)s^{(2,+)}_{b^*, x_{T(h_2)}}(\rho_{h_2}(v)) \right\| \leq C_K|\rho(v)|(|\rho(v)| + |x^*(w, v)|).
\]

This together with (5.72) yields a unique solution \((b^*, w, N^*)\) of (5.75) provided \(|v|\) is sufficiently small. Thus \((b^*, w, N^*)\) is the unique solution of (5.71) as desired. Let \(\mu(v, \zeta) = (b^*, w, N^*), v\).

Then the same argument in Theorem 5.1.1 yields the holomorphic map \(\phi(w, v) = [\exp_{\mu(v, \zeta)} \zeta_v] \in \mathcal{M}^0_{2,k}(X, A, J)\). The proof of the theorem is complete.

Now we consider the general case.

**Theorem 5.2.3.** Suppose \( T \) is a bubble type given by (iv) of Theorem 2.9 with \( \Sigma_P \) being smooth together with \(|I_1| = 2\) and \(|\chi(T)| \geq 3\). Assume the attaching nodes \( x_{h_1} \) and \( x_{h_2} \) of the two bubble trees differ by the nontrivial holomorphic automorphism of \( \Sigma_P \). Then an element \( b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \mathcal{M}^0_{2,k}(X, A, J) \) if and only if \( \alpha_T(v) = 0 \) for some \( v = (b, v) \in \mathcal{F}_\delta^0|_{\delta(k)} \) and \( \alpha_T(w, v) = 0 \), where \( \alpha_T(w, v) \) is a linear combination of \( \{D^{(1)}_{h_i}(b, D^{(2)}_{h_i}b)\}_{h_i \in \chi(T)} \) with coefficients depending on \( w \in T_{x_{h_1}} \Sigma_P, v \) and the position of the nodes in bubble trees. In particular, \( \mathcal{M}_T(X, A, J) \cap \mathcal{M}^0_{2,k}(X, A, J) \) is a smooth orbifold of dimension at most \( \dim \mathcal{M}^\text{vir}_{2,k}(X, A) - 2 \).

**Proof.** Denote by \( \chi(T) = \{h_1, h_2, \ldots, h_p\} \). We only give the proof of the simplest case that all of \( \{h_1, h_2, \ldots, h_q\} \) are attached to the bubble \( h_a \in I_1 \) and \( \{h_{q+1}, \ldots, h_p\} \) are attached to \( h_\beta \in I_1 \). The general case follows similarly.

By Lemma 5.5, an element \( b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \mathcal{M}^0_{2,k}(X, A, J) \) must satisfy \( \alpha_T(v) = 0 \) for some \( v = (b, v) \in \mathcal{F}_\delta^0|_{\delta(k)} \), where

\[
\alpha_T(v) = \sum_{h_i \in \chi(T)} (D^{(1)}_{h_i}(b)s_{\Sigma_P, \tilde{x}}(\rho_{h_i}(v)).
\]

(5.76)
Here we identify \( x_{h_1} \) and \( x_{h_2} \) via the nontrivial holomorphic automorphism \( \sigma \) of \( \Sigma_P \) and denote them simply by \( \tilde{x} \). Denote by \( S = \{ b \in \mathcal{M}_T(X, A, J) : \alpha_T(v) = 0 \} \). Then by (2-a) in Definition 1.1, \( S \) is complex suborbifold of \( \mathcal{M}_T(X, A, J) \). Let \( \mathcal{N}S \) denote the normal bundle of \( S \) in \( \mathcal{M}_T(X, A, J) \). As in Theorem 5.2.2, we have \( S = S_0 \times S_1 \), where

\[
S_0 = \{ (x_{h_1}, -x_{h_1}) : x_{h_1} \in \Sigma_p \}, \quad S_1 = \left\{ \sum_{h_i \in \chi(T)} (D_{h_i}^{(1)}b)\gamma_{h_{\Sigma p}, x_{h_1}}(\rho_{h_i}(v)) = 0 \right\}.
\]

Then we have

\[
\mathcal{N}S = \mathcal{N}S_0 \oplus \mathcal{N}S_1, \quad \text{where} \quad \mathcal{N}S_0 = \pi_{\Sigma p, x_{T(h)}p}^* T_{\Sigma p}, \quad \mathcal{N}S_1 = E_1,
\]

where \( \pi_{\Sigma p, x_{T(h)}p} \) is the projection to the \( i \)-th factor and \( E_1 \) is the orthogonal complement of \( \text{span}_{h_i \in \chi(T)} \{ D_{h_i}^{(1)}b \} \) in \( T_{u(\Sigma p)}X \).

Suppose \( (b, w, N) \in \mathcal{N}S \) and \( v = ((b, w, N); v) \in \mathcal{F}_T^{\mathcal{N}(\kappa)} \). Let \( \tilde{x}_{h_i}(b, w, N; v) = \tilde{x}_{h_i}(b, w, 0); v \equiv \tilde{x}_{h_i}(w, v) \in \Sigma_p \) for \( h_i \in \chi(T) \). Identify a small neighborhood of \( x_{T(h)} \) in \( \Sigma_P \) with a small neighborhood of \( 0 \) in \( T_{x_{T(h)}} \Sigma_P \) and let \( x_i^*(w, v) = \tilde{x}_{h_i}(w, v) - \tilde{x}_{h_i}(w, v) \), then we have \( |x_i^*(w, v)| \leq C_K(|v| + |w|) \).

We consider the second-order expansion of \( \langle \overline{\partial} J u_{(b,w,N);v}, R_v V \eta \rangle \). Similar to Theorem 5.1.3 and Theorem 5.2.2, we have

\[
\pi_{v,-}^0((\overline{\partial} J u + D_v \zeta + N_v \zeta) = D_{h_{1}}^{(1)}(b, w, N)\gamma_{h_{1}(w,v)}((d\phi_{h_{1}}(v))^{-1} \rho_{h_{1}}(v))
\]

\[
+ \sum_{2 \leq i \leq p} (D_{h_{1}}^{(1)}b, w, N)\gamma_{h_{1}(w,v)}((\rho_{h_{1}}(v))) + D_{h_{1}}^{(1)}(b, 0, 0)\gamma_{h_{1}(w,v)}((\rho_{h_{1}}(v))))
\]

\[
+ \sum_{1 \leq i \leq p} (D_{h_{1}}^{(2)}b, 0, 0)\gamma_{h_{1}(w,v)}((\rho_{h_{1}}(v))) + D_{h_{1}}^{(2)}(b, 0, 0)\gamma_{h_{1}(w,v)}((\rho_{h_{1}}(v))) + \epsilon(v, \zeta),
\]

where

\[
\|\epsilon(v, \zeta)\| \leq C_K|\rho(v)|^2(|v| + |w| + |N|) + C_K|\rho(v)|||\rho(v)|| + |x_i^*(w, v)| \cdot \|\zeta\|_{v,p,1}
\]

\[
+ C_K(|x_i^*(w, v)||\rho(v)|||\rho(v)|| + |x_i^*(w, v)| + |v| + |w| + |N|).
\]

Note that \( \rho_{h_{1}}(v) = v_{\alpha}x_{h_{1}} \) and \( \tilde{x}_{h_{1}}(w, v) = v_{\alpha}x_{h_{1}} \) for \( 1 \leq i \leq q \), \( \rho_{h_{i}}(v) = v_{\beta}v_{h_{i}} \) and \( \tilde{x}_{h_{i}}(w, v) = w + v_{\beta}\tilde{x}_{h_{i}} \) for \( q + 1 \leq i \leq p \). Hence we have \( x_i^*(w, v) = v_{\alpha}(x_{h_{i}} - \tilde{x}_{h_{i}}) \neq 0 \) for \( 2 \leq i \leq q \) and \( x_i^*(w, v) = w + v_{\beta}x_{h_{i}} - v_{\alpha}x_{h_{1}} \) for \( q + 1 \leq i \leq p \). Clearly we may assume \( |x_i^*(w, v)| \geq \delta_K|v_{\beta}| \) for \( q + 1 \leq i \leq p - 1 \). Hence we have

\[
|\gamma_{h_{1}}(w, v)| = o(\|\gamma_{h_{1}}(w, v), \rho_{h_{1}}(v))\|), \quad 2 \leq i \leq p - 1
\]

(5.79)
By changing the order of \(\{h_1, \ldots, h_q\}\), we may assume \(\rho_{h_1}(v) = \min_{1 \leq i \leq q} \rho_{h_i}(v)\). Thus we have

\[
|s_{b, \tilde{x}}^{(2,-)}(\rho_{h_1}(v))| = \sum_{2 \leq i \leq q} o\left(|s_{b, \tilde{x}}^{(2,-)}(x_i^s(w, v), \rho_{h_i}(v))|\right). \tag{5.80}
\]

We have the following two cases:

**Case 1.** We have \(D_{h_p}^{(1)} b \neq 0\).

In this case we have \(|\rho_{h_p}(v)| \leq C_K \sum_{1 \leq i \leq p-1} |\rho_{h_i}(v)|\) by \(\alpha_T(v) = 0\). Hence by (5.79) and (5.80), we have

\[
|s_{b, \tilde{x}}^{(2,-)}(\rho_{h_p}(v))| = \sum_{2 \leq i \leq p-1} o\left(|s_{b, \tilde{x}}^{(2,-)}(x_i^s(w, v), \rho_{h_i}(v))|\right). \tag{5.81}
\]

Since \(\tilde{x} \in \Sigma_P\), we have \(s_{b, \tilde{x}}^{(2,-)} \neq 0\). Hence in order to satisfies (5.21), we must have

\[
\tilde{\alpha}_T(w, v) \equiv \sum_{2 \leq i \leq p} D_{h_i}^{(1)}(b, 0, 0)s_{b, \tilde{x}}^{(2,-)}(x_i^s(w, v), \rho_{h_i}(v)) = 0 \tag{5.82}
\]

provided \(|v|, |w|\) and \(|N|\) are sufficiently small.

**Case 2.** We have \(D_{h_p}^{(1)} b = 0\).

In this case, in order to satisfies (5.21), we must have

\[
\tilde{\alpha}_T(w, v) \equiv \sum_{2 \leq i \leq p} D_{h_i}^{(1)}(b, 0, 0)s_{b, \tilde{x}}^{(2,-)}(x_i^s(w, v), \rho_{h_i}(v)) + D_{h_p}^{(2)}(b, 0, 0)s_{b, \tilde{x}}^{(2,-)}(\rho_{h_p}(v)) = 0 \tag{5.83}
\]

provided \(|v|, |w|\) and \(|N|\) are sufficiently small.

The proof of the converse is similar to the previous theorems since the map \(\tilde{\alpha}_T\) and \(\alpha_T\) are transversal to the zero sections in both cases by (ii-a) and (ii-b) in Definition 1.1.

**5.3 There are at least three bubble trees**

In this subsection, we consider the case \(|I_1| \geq 3\). Note that in this case the rank of \(\alpha_T(v)\) is 4n. Hence we have the following:

**Theorem 5.3.1.** Suppose \(T\) is a bubble type given by (iv) of Theorem 2.9 with \(\Sigma_P\) being smooth and \(|I_1| \geq 3\), i.e., there are at least three bubble trees. Then an element \(b \equiv [\mathcal{C}, u] \in \mathcal{M}_T(X, A, J) \cap \overline{\mathcal{M}}_{2,k}^0(X, A, J)\) if and only if \(\alpha_T(v) = 0\) for some \(v = (b, v) \in \overline{\mathcal{F}}_T^\emptyset|_{\overline{K}(v)}\). In particular, \(\mathcal{M}_T(X, A, J) \cap \overline{\mathcal{M}}_{2,k}^0(X, A, J)\) is a smooth orbifold of dimension at most \(\dim \overline{\mathcal{M}}_{2,k}^0(X, A) - 2\).

**Proof.** By Lemma 5.5, an element \(b \equiv [\mathcal{C}, u] \in \mathcal{M}_T(X, A, J) \cap \overline{\mathcal{M}}_{2,k}^0(X, A, J)\) must satisfy \(\alpha_T(v) = 0\) for some \(v = (b, v) \in \overline{\mathcal{F}}_T^\emptyset|_{\overline{K}(v)}\). Note that in this case we have \(\text{rank}(\alpha_T) = 4n = \dim(\text{coker}D_b)\) by (ii-a) in Definition 1.1. Thus the theorem follows by a similar argument as in Theorem 5.2.1. 



47
5.4 The principle component $\Sigma_P$ is not smooth and $u_\ast[\Sigma_P] = 0$

In this subsection, we consider the case that the principle component $\Sigma_P$ is not smooth and $u_\ast[\Sigma_P] = 0$, i.e., $u|_{\Sigma_P} = \text{const}$.

First we use §3.2 to smooth out all the nodes in $\Sigma_P$, then we use the methods in §5.1-5.3 to study the conditions under which an element $b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^0(X, A, J)$. Denote by $(\Sigma_P, v_P)$ the smooth Riemann surface obtained from $\Sigma_P$ by smoothing nodes. By the gluing construction, we have an isomorphism $R_{v_P} : H^0_{\Sigma_P} \to H^0_{(\Sigma_P, v_P)}$ which depends continuously on the parameter $v_P$. In fact, we can choose basis of the Hodge bundle of holomorphic 1-forms over the Deligne-Mumford space $\overline{\mathcal{M}}_{2,k}$ depending continuously on the parameter $v_P$, where $k_P = |I_1 \cup M_P|$. Then this case follows by a similar argument as in the previous sections by replacing the term $R_v$ by $R_{(v_0, v_1)} \circ R_{v_P}$, where $v_1 \equiv \{v^{(i)}\}_{i \in I_1}$.

The following is the main theorem in this section.

**Theorem 5.4.1.** Suppose $T$ is a bubble type given by (iv) of Theorem 2.9 with $u_\ast[\Sigma_P] = 0$. Then one of the following must hold:

(i) An element $b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^0(X, A, J)$ if and only if $\{D_h^{(1)} b, D_h^{(2)} b, D_h^{(3)} b\}_{h \in \chi(T)}$ satisfy a set of linear equations of rank $4n$ whose coefficients depending on $v = (b, v) \in \mathcal{F}_T^{\delta_k} | \mathcal{R}(0)$ and the position of nodes on $C$. In particular, $\mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^0(X, A, J)$ is a smooth orbifold of dimension at most $\dim \mathcal{M}_{2,k}^{vir}(X, A) - 2$.

(ii) An element $b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^0(X, A, J)$ must satisfy: there exists $h \in \chi(T)$ such that $u_h$ factor through a branched covering $\tilde{u} : S^2 \to X$, i.e., there exists a holomorphic branched covering $\phi : S^2 \to S^2$ such that $u_h = \tilde{u} \circ \phi$ and $\deg(\phi) \geq 2$.

(iii) An element $b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^0(X, A, J)$ must satisfy: there exist $h_1, h_2 \in \chi(T)$ such that $u_{h_1}(\Sigma_{h_1}) = u_{h_2}(\Sigma_{h_2})$.

**Proof.** We only give the proof of two cases: $\Sigma_P$ is a torus with only one node and $\Sigma_P$ is obtained from two tori attached at one node. The other cases follow by a similar argument, so we omit their proofs here.

**Case 1.** The principle component $\Sigma_P$ is a torus with only one node.

We will construct an isomorphism $R_{v_P} : H^0_{\Sigma_P} \to H^0_{(\Sigma_P, v_P)}$ depending continuously on the parameter $v_P$. Suppose $(T, x_1, x_2)$ is the normalization of $\Sigma_P$. Then we have

$$H^1(\Sigma_P, O) \cong H^1(T, O(-x_1 - x_2)) \cong H^0(T, O(x_1 + x_2) \otimes K_T)^* \cong \mathbb{C}^2$$

which consists of meromorphic one-forms $\omega$ on $T$ that are holomorphic on $T \setminus \{x_1, x_2\}$ and have at most simple poles at $x_1$ and $x_2$ together with $\text{Res}_{x_1} \omega + \text{Res}_{x_2} \omega = 0$ (cf. §22.3 of [MirSym]).
Note that by the Residue Theorem (cf. P.222 of [GH]), the condition $\text{Res}_{\omega_1} + \text{Res}_{\omega_2} = 0$ is automatically satisfied.

Now we study nonzero sections of $H^1(\Sigma_P, \mathcal{O})$. Note that if $\text{Res}_{\omega_1} = 0 = \text{Res}_{\omega_2}$, then the one-form $\omega_1$ is holomorphic on $T$, i.e., $\omega_1 \in H^0(T, K_T)^* \cong H^0(T, \mathcal{O})^* \cong \mathbb{C}$. Thus $\omega_1$ is nowhere vanishing on $T$. If $\text{Res}_{\omega_1} = -\text{Res}_{\omega_2} \neq 0$, then the one-form $\omega_2$ has exactly two simple poles on $T$. We claim that $\omega_2$ has at most two zeros on $T$. In fact, if $\omega_2$ has $l$ zeros, then we have $\omega_2 \in H^0(T, \mathcal{O}(2-l) \otimes K_T)^* = 0$ provided $l > 2$. Now we fix an $h_* \in \chi(T)$ and let $\{\psi_1, \psi_2\}$ be an orthogonal basis of $\mathcal{H}^{0,1}_{\Sigma_P}$ such that $\psi_1(x_{h_*}(v)) \neq 0$ and $\psi_2(x_{h_*}(v)) = 0$. We extend $\{\psi_1, \psi_2\}$ to be a basis $\{\psi_{v,1}, \psi_{v,2}\}$ of $\mathcal{H}^{0,1}_{(\Sigma_P, v_p)}$ such that $\psi_{v,1}(x_{h_*}(v)) \neq 0$ and $\psi_{v,2}(x_{h_*}(v)) = 0$. Note that $\{\psi_{v,j}\}$ is continuous with respect to the parameter $v_p$ outside small neighborhoods $U_i$ of $x_i$ for any $m \geq 0$ and the attaching nodes of bubble trees lies outside $U_i$. Hence Theorem 5.4.1 holds in this case by a similar argument as in §5.1-5.3 by considering expansions of the approximately $J$-holomorphic curves.

Case 2. The principle component $\Sigma_P$ is two tori attached at one node.

We will construct an isomorphism $R_{v_p} : \mathcal{H}^{0,1}_{\Sigma_P} \rightarrow \mathcal{H}^{0,1}_{(\Sigma_P, v_p)}$ depending continuously on the parameter $v_p$. Suppose $(T_1, x_1), (T_2, x_2)$ is the normalization of $\Sigma_P$. Then we have

$$H^1(\Sigma_P, \mathcal{O}) \cong H^1(T_1, \mathcal{O}(x_1)) \oplus H^1(T_2, \mathcal{O}(x_2))$$

$$\cong H^0(T_1, \mathcal{O}(x_1) \otimes K_{T_1})^* \oplus H^0(T_2, \mathcal{O}(x_2) \otimes K_{T_2})^* \cong \mathbb{C}^2$$

which consists of meromorphic one-forms $\omega$ on $T_1 \cup T_2$ that are holomorphic on $T_1 \setminus \{x_1\}$ and $T_2 \setminus \{x_2\}$ and have at most simple poles at $x_1$ and $x_2$ together with $\text{Res}_{x_1} = 0 = \text{Res}_{x_2}$. Note that by the Residue Theorem applied to each $T_i$, we have $\text{Res}_{x_i} = 0 = \text{Res}_{x_2}$. Hence $\omega$ is holomorphic on each $T_i$. Since $H^0(T_i, K_{T_i})^* \cong H^0(T_i, \mathcal{O})^* \cong \mathbb{C}$, we have $\omega|_{T_i}$ is either nowhere vanishing or identically zero. We choose a basis $\{\psi_1, \psi_2\}$ of $\mathcal{H}^{0,1}_{\Sigma_P}$ such that $\psi_1|_{T_1} \neq 0$, $\psi_1|_{T_2} = 0$ and $\psi_2|_{T_1} = 0, \psi_2|_{T_2} \neq 0$. Then extend $\{\psi_1, \psi_2\}$ to be a basis $\{\psi_{v,1}, \psi_{v,2}\}$ of $\mathcal{H}^{0,1}_{(\Sigma_P, v_p)}$. Since $\psi_2|_{T_1} = 0$, we need to study the behavior of $\psi_{v,2}|_{T_1}$ carefully in order to derive the conditions for $b \equiv [c, u] \in \mathcal{M}_{T}(X, A, J) \cap \mathcal{M}_{T_2}^0(X, A, J)$. Write the local coordinate at the node as $\{(z, w) \in \mathbb{C}^2, zw = v_p\}$. Assume in a small neighborhood of the node $T_1$ and $T_2$ are given by the coordinate planes $\{z = 0\}$ and $\{w = 0\}$ respectively. Suppose $\psi_2|_{T_2} = dz$ and $\psi_2|_{T_1} = 0$. We construct an approximately holomorphic one-form $\hat{\psi}_{v,2}$ on $(\Sigma_P, v_p)$ as follows: First note that the one-form $\psi_2 = dz$ has the form $\psi_2 = \frac{d}{dw} dw$ in the $w$-coordinate. Then we choose a meromorphic one-form $\omega$ on $T_1$ with only one pole at $x_1$ of order two such that its principle part is $\frac{-d}{dw} dw$. By the Residue Theorem applied to $(T_1, \omega)$, we have the expansion of $\omega$ near $x_1$ as $\omega = v_p(-\frac{1}{w^2} + a_0 + a_1 w + \cdots) dw$, where
In this subsection, we consider the case that the principle component $\Sigma$ connected component of $\Sigma$ has genus greater than zero. Since $deg(\omega_0) = 0$. The simplest example in this case is given by the second figure in Figure 2.1.

Then $\tilde{\psi}_{v_P,2}$ is a globally defined one form on $(\Sigma_P, v_P)$ and it is holomorphic outside the annulus $A = \{ |v_P|^{\frac{1}{2}} \leq |z| \leq 2|v_P|^{\frac{1}{2}} \} = \{ \frac{1}{2}|v_P|^{\frac{1}{2}} \leq |w| \leq |v_P|^{\frac{1}{2}} \}$. Inside the annulus $A$, we have

$$\tilde{\psi}_{v_P,2} = v_P \beta_{v_P}(\frac{|v_P|}{|w|})(a_0 + a_1 w + \ldots) dw.$$  

(5.85)

Since $\omega \in H^0(T_1, O(2) \otimes K_{T_1})^*$, it has at most two zeros on $T_1$. In fact, if $\omega$ has $l$ zeros, then we have $\omega \in H^0(T_1, O(2-l) \otimes K_{T_1})^* = 0$ provided $l > 2$. Note that we have $\| \partial_\omega \tilde{\psi}_{v_P,2} \|_{(\Sigma_P, v_P), L^p} \leq C|v_P|^{1+\frac{1}{2p}}$ for some constant $C$ independent of $v_P$ and $p > 2$, where near the node $x$ we use the cylinder-like metric. Since $\partial_\omega$ is a first-order elliptic operator, it follows from the standard elliptic estimate that we can find a one-form $\epsilon$ on $\Sigma_P$ such that $\| \epsilon \|_{(\Sigma_P, v_P), W^{1,p}} \leq C_K|v_P|^{1+\frac{1}{2p}}$ and $\psi_{v_P,2} \equiv \tilde{\psi}_{v_P,2} + \epsilon$ is holomorphic. Thus as $v_P$ converges to 0, we have $\frac{1}{v_P} \psi_{v_P,2}$ will $C^\infty$-converge to $\frac{1}{v_P} \tilde{\psi}_{v_P,2}$ outside small neighborhood $U$ of the node $x_1$ in $T_1$. In fact, this holds since both $\frac{1}{v_P} \psi_{v_P,2}$ and $\frac{1}{v_P} \tilde{\psi}_{v_P,2}$ are holomorphic outside $U$ and two holomorphic functions are $C^0$-close implies they are $C^\infty$-close. We construct $\psi_{v_P,1}$ similarly.

Now we fix an $h_* \in \chi(T)$ and let $\{\psi_{v_P,1}, \psi_{v_P,2}\}$ be a basis of $\mathcal{H}^0_{\Sigma_P, v_P}$ constructed as above such that $\psi_{v_P,1}(\tilde{x}_{h_*}(v)) \neq 0$ and $\psi_{v_P,2}(\tilde{x}_{h_*}(v)) = 0$. Note that in the proof of the theorems in §5.1-5.3, we only use $\frac{\psi_{v_P,2}^{(m+1)}(y)}{\psi_{v_P,2}^{(m)}(y)} \leq C_K$ where $m$ is the order of $y$ as a zero of $\psi_{v_P,2}$ and $l \in \mathbb{N}$. By the above construction, $\frac{\psi_{v_P,2}^{(m+1)}(y)}{\psi_{v_P,2}^{(m)}(y)}$ will $C^0$-converge to $\frac{\psi_{v_P,2}^{(m+1)}(y)}{\psi_{v_P,2}^{(m)}(y)}$ outside small neighborhood $U$ of the node $x_1$ in $T_1$. While $\frac{\psi_{v_P,2}^{(m+1)}(y)}{\psi_{v_P,2}^{(m)}(y)}$ is independent of $v_P$. Hence Theorem 5.4.1 holds in this case by a similar argument as in §5.1-5.3 by considering expansions of the approximately $J$-holomorphic curves.

At last we sketch the proof of the general case. Firstly we glue certain rational components in $\Sigma_P$ to obtain $\Sigma_P'$, which belongs to one of the above cases. Then we use the above to show the theorem holds.

### 5.5 The principle component $\Sigma_P$ is not smooth and $u_\ast[\Sigma_P] \neq 0$

In this subsection, we consider the case that the principle component $\Sigma_P$ is not smooth and $u_\ast[\Sigma_P] \neq 0$. The simplest example in this case is given by the second figure in Figure 2.1.

We denote by $\Sigma$ the union of components of $\Sigma_P$ which are mapped to constants such that each connected component of $\Sigma$ has genus greater than zero. Since $deg(u|_{\Sigma_P}) \neq 0$, $\Sigma$ consists of exactly
two connected components $\Sigma_1$ and $\Sigma_2$, each one is mapped to a constant, e.g. the two tori in Figure 2.1. Note that we have

$$H^1_C(\mathcal{C}, u^*TX) \cong (T_{ev(\Sigma_1)}X \otimes \mathcal{H}^{0,1}_{\Sigma_1}) \oplus (T_{ev(\Sigma_2)}X \otimes \mathcal{H}^{0,1}_{\Sigma_2}) \cong \mathbb{C}^{2n}.$$ 

Let $\tilde{\Sigma}_1 \equiv \Sigma_P \setminus \Sigma_2 \supset \Sigma_1$ and $\tilde{\Sigma}_2 \equiv \Sigma_P \setminus \Sigma_1 \supset \Sigma_2$, e.g. the left torus with the central sphere or the right torus with the central sphere respectively in Figure 2.1. Denote by $\Sigma_1$ respectively as illustrated in Figure 2.1.. Then both $\Sigma^{(1)}$ and $\Sigma^{(2)}$ are nodal Riemann surfaces of genus one. Denote by $\Sigma^{(1)}_P$ and $\Sigma^{(2)}_P$ the principle components of $\Sigma^{(1)}$ and $\Sigma^{(2)}$ respectively, e.g. the two tori in Figure 2.1.

We only consider the simplest case that both $\Sigma^{(1)}_P$ and $\Sigma^{(2)}_P$ are smooth tori, i.e., as illustrated in the second figure in Figure 2.1, the general case follows similarly as explained in §5.4. For $v = (b, v) \in \mathcal{F}T^0_{\delta_T}$ sufficiently small and $(V_1 \otimes \psi_1, V_2 \otimes \psi_2) \in (T_{ev(\Sigma_1)}X \otimes \mathcal{H}^{0,1}_{\Sigma_1} \oplus (T_{ev(\Sigma_2)}X \otimes \mathcal{H}^{0,1}_{\Sigma_2}))$, we define $R_v(V_1 \otimes \psi_1, V_2 \otimes \psi_2) \in \Gamma^{0,1}(u_v)$ as the following: if $z \in \Sigma_v$ is such that $q_v(z) \in \Sigma_{b,h}$ for some $h \in \chi^{(1)}$ as defined in (5.83), and $|q_{\delta_T}^{-1}(q_v(z))| \leq 2\delta_T(b)$, we define $\overline{\psi}_v(z) \in T_{u_v}(\Sigma_1)X$ by $\exp_{u(\Sigma_1)}\overline{\psi}_v(z) = q_v(z)$. Given $z \in \Sigma_v$, let $h_z$ be such that $q_v(z) \in \Sigma_{b,h_z}$. If $w \in T_x\Sigma_v$, put

$$R_vV_1\psi_1|w = \begin{cases} 
0, & \text{if } \chi^{\Sigma(v)}h_z = 2; \\
(\beta(\delta_T(b)|q_vz|)(\psi_1|w)\Pi^{u_v}_{\Sigma_v}V_1, & \text{if } \chi^{\Sigma(v)}h_z = 1; \\
(\psi_1|w)V_1, & \text{if } \chi^{\Sigma(v)}h_z = 0, 
\end{cases}
$$

where $\chi^{\Sigma(v)}$ is defined similar to (5.7) with respect to the bubble type $\Sigma^{(1)}$ and $\Pi^{u_v}_{\Sigma_v}(z)$ is the parallel transport along the geodesic $t \mapsto \exp_{u(\Sigma_1)}\overline{\psi}_v(z)$ with respect to the Levi-Civita connection of the metric $g_{X,\{u(\Sigma_1),u(\Sigma_2)\}}$, which are flat near $(u(\Sigma_1)$ and $u(\Sigma_2)$. We define $R_vV_2\psi_2$ similarly. Note that since $u_\ast[\Sigma_P] \neq 0$, there must be a rational component $h$ between $\Sigma_1$ and $\Sigma_2$ such that $u_\ast[\Sigma_h] \neq 0$, thus the map $R_v(V_1 \otimes \psi_1, V_2 \otimes \psi_2)$ is well-defined. Let $\Gamma_\pm(v)$ be given by the formula in §4.

Comparing with the previous sections, we have the following:

**Lemma 5.5.1.** Suppose $T$ is a bubble type given by (iv) of Theorem 2.9 and $u_\ast[\Sigma_P] \neq 0$. Then for every precompact open subset $K$ of $\mathfrak{M}_T(X, A, J)$, there exist $\delta_K, C_K \in \mathbb{R}^+$ and an open neighborhood $U_K$ of $K$ in $\mathfrak{X}_{2,k}(X, A, J)$ with the following properties:

(i) For every $[\tilde{b}] \in \mathfrak{X}_{2,k}^0(X, A, J) \cap U_K$, there exist $v = (b, v) \in \mathcal{F}T^0_{\delta_K}|_{K(\tilde{b})}$, and $\zeta \in \Gamma_\pm(v)$ such that $\|\zeta\|_{v,1} < \delta_K$ and $[\exp_{u_v}\zeta] = [\tilde{b}]$. 

51
(ii) For every \( v = (b, v) \in \widetilde{FT}_{\delta}^{0} |_{\tilde{\mathcal{K}}^{(0)}} \), we have
\[
C^{-1}_{K} \| \xi \|_{v,p,1} \leq \| D_{v} \xi \|_{v,p} \leq C_{K} \| \xi \|_{v,p,1}, \quad \forall \xi \in \Gamma_{+}(v),
\] (5.87)

(iii) For every \( v = (b, v) \in \widetilde{FT}_{\delta}^{0} |_{\tilde{\mathcal{K}}^{(0)}} \) and \( \eta \equiv (V_{1} \otimes \psi_{1}, V_{2} \otimes \psi_{2}) \in H^{1}_{0}(\mathcal{C}, u^{*}TX) \), we have
\[
\| D_{v}^{*} R_{v} \eta \|_{v,C^{0}} \leq C(b) \left( \sum_{h \in \chi(\Sigma^{(1)})} | \rho_{h}^{(1)}(v) | + \sum_{h \in \chi(\Sigma^{(2)})} | \rho_{h}^{(2)}(v) | \right) \| \eta \|_{2}.
\] (5.88)

(iv) For every \( v = (b, v) \in \widetilde{FT}_{\delta}^{0} |_{\tilde{\mathcal{K}}^{(0)}} \) and \( \eta \equiv (V_{1} \otimes \psi_{1}, V_{2} \otimes \psi_{2}) \in H^{1}_{0}(\mathcal{C}, u^{*}TX) \), we have
\[
\left| \langle \partial_{f} u, R_{v} \eta \rangle \right|_{v,2} + \sum_{h \in \chi(\Sigma^{(1)})} \langle D_{h}^{(1)} b, V_{1} \psi_{1}(\rho_{h}^{(1)}(v)) \rangle + \sum_{h \in \chi(\Sigma^{(2)})} \langle D_{h}^{(2)} b, V_{2} \psi_{2}(\rho_{h}^{(2)}(v)) \rangle \leq C_{K} |v| \cdot |\rho(v)| \cdot \| \eta \|,
\] (5.89)

where \( D_{h}^{(1)} b \) is given by (3.32) and \( \rho_{h}^{(i)}(v) \) is given by (3.32) with respect to the bubble type \( \Sigma^{(i)} \).

**Proof.** (i) and (ii) hold by a similar argument in [Z2].

We prove (iii). Note that by §3.2, the gluing construction (3.30) in the principle component \( \Sigma_{\pi} \) coincide with the gluing construction in \( \Sigma^{(1)} \) or \( \Sigma^{(2)} \) as bubble types of genus-one by replacing the term \( q_{S}(p_{h,(x,v_{h})}(z)) \) in (3.17) by \( \phi_{z_{h}}^{-1}(p_{h,(x,v_{h})})(z) \), i.e., we have \( \phi_{z_{h}}^{-1}(\xi) = q_{S}(\tilde{\xi}) \). Thus the proof of Lemma 5.2 remains valid, which yields (iii).

We prove (iv). By the construction of \( q_{c} \) and \( R_{v} \), we have \( \langle \partial_{f} u, R_{v} \eta \rangle = 0 \) outside the annuli \( A_{v}^{-1}(v_{h}) \) for \( h \in \chi(\Sigma^{(1)}) \bigcup \chi(\Sigma^{(2)}) \). Hence we have
\[
\langle \partial_{f} u, R_{v} \eta \rangle_{v,2} = \sum_{h \in \chi(\Sigma^{(1)}) \bigcup \chi(\Sigma^{(2)})} \int_{A_{v}^{-1}(v_{h})} \langle \partial_{f} u, R_{v} \eta \rangle.
\] (5.90)

Now we consider \( \Sigma^{(1)} \) and \( \Sigma^{(2)} \) separately. Then the proof of Lemma 4.3 in [Z1] can be used to obtain an expansion
\[
\langle \partial_{f} u, R_{v} \eta \rangle_{v,2} = - \sum_{m \geq 1, h \in \chi(\Sigma^{(1)})} \langle D_{h}^{(m)} b, V_{1} \rangle \left( \langle D_{h}^{(m)} b, \tilde{\xi}_{h}^{(1)}(v) \rangle \psi_{1}(\rho_{h}^{(1)}(v)) \right) - \sum_{m \geq 1, h \in \chi(\Sigma^{(2)})} \langle D_{h}^{(m)} b, V_{2} \rangle \left( \langle D_{h}^{(m)} b, \tilde{\xi}_{h}^{(2)}(v) \rangle \psi_{2}(\rho_{h}^{(2)}(v)) \right)
\]
where \( T^{(i)}(h), \tilde{\xi}_{h}^{(i)}(v), \rho_{h}^{(i)}(v) \) are the maps as in Lemma 5.1 with respect to the bubble type \( \Sigma^{(i)} \) for \( i = 1 \) or 2. Hence (iv) holds by the same argument as in Lemma 5.4.

For any \( v = (b, v) \in \widetilde{FT}_{\delta}^{0} \), let
\[
\alpha_{T, 1}(v) = \sum_{h \in \chi(\Sigma^{(1)})} \langle D_{h}^{(1)} b \rangle \psi_{1}(\rho_{h}^{(1)}(v)), \quad \alpha_{T, 2}(v) = \sum_{h \in \chi(\Sigma^{(2)})} \langle D_{h}^{(1)} b \rangle \psi_{2}(\rho_{h}^{(2)}(v))
\] (5.91)
The following is the main result in this subsection.

**Theorem 5.5.2.** Suppose $T$ is a bubble type given by (iv) of Theorem 2.9 and $u_*[\Sigma_P] \neq 0$. Then one of the following must hold:

(i) An element $b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^0(X, A, J)$ if and only if $\alpha_{T,1}(v) = 0$ and $\alpha_{T,2}(v) = 0$ for some $v = (b, v) \in \tilde{\mathcal{F}}^\emptyset \delta_K|_{K(0)}$. In particular, $\mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^0(X, A, J)$ is a smooth orbifold of dimension at most $\dim \mathcal{M}_{2,k}^0(X, A) - 2$.

(ii) An element $b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^0(X, A, J)$ must satisfy: there exists $\Sigma_h \subset \Sigma_P$ such that $u_h$ factor through a branched covering $\tilde{u} : S^2 \to X$, i.e., there exists a holomorphic branched covering $\phi : S^2 \to S^2$ such that $u_h = \tilde{u} \circ \phi$ and $\deg(\phi) \geq 2$.

**Proof.** Let $\delta_K$ be given by Lemma 5.5.1. For each $v = (b, v) \in \tilde{\mathcal{F}}^\emptyset \delta_K|_{K(0)}$, we define the homomorphism

$$
\pi_{v,-}^{0,1} : \Gamma_{-}^{0,1}(v) \to \Gamma_{-}^{0,1}(b_P), \quad \pi_{v,-}^{0,1} \xi = - \sum_{1 \leq i \leq \rho(v), 1 \leq j \leq 2} \langle \xi, R_v e_i^j \psi_j \rangle e_i^j \psi_j \in \Gamma_{-}^{0,1}(b_P),
$$

where $\psi_1$ and $\psi_2$ are orthonormal basis for $\mathcal{H}_{\Sigma_1}^{0,1}$ and $\mathcal{H}_{\Sigma_2}^{0,1}$ and $\{e_i^j\}_{1 \leq i \leq \rho(v), 1 \leq j \leq 2}$ are orthonormal basis for $T_{u(\Sigma_j)}X$. Denote the kernel of $\pi_{v,-}^{0,1}$ by $\Gamma_{-}^{0,1}(v)$. Then by Lemmas 5.5.1 and the same argument as in Lemma 5.5, we have

$$
\pi_{v,-}^{0,1}(v, \zeta) = \pi_{v,-}^{0,1}(\partial_J u_v + D_v \zeta + N_v \zeta) = \alpha_{T,1}(v) + \alpha_{T,2}(v) + \epsilon(v, \zeta),
$$

and

$$
\|\epsilon(v, \zeta)\| \leq C_K(\|v\| + \|\zeta\|_{v,p,1})|\rho(v)| \leq C_K|v|^{\frac{1}{2}}|\rho(v)|,
$$

where we use notations in Lemma 5.5. Hence in order to satisfies (5.21), we must have $\alpha_{T,1}(v) = 0$ and $\alpha_{T,2}(v) = 0$ provided $|v|$ is sufficiently small.

The converse is similar to the previous by using (ii-a), (ii-b) and (ii-d) in Definition 1.1. \[\square\]

6 Study for $\mathcal{M}_T(X, A, J)$ in (iii) of Theorem 2.9

In this case we have $\coker D_b = H_{b}^{\emptyset}(C, u^*TX) = \mathbb{C}^n$ by Theorem 2.10. The simplest examples in this case are illustrated in Figure 2.3.

We denote by $\Sigma_1$ the union of components of $\Sigma_P$ which are mapped to constants such that $\Sigma_1$ is connected and has genus one, e.g. the tori on the left hand side of the two figures in Figure 2.3. Then we have

$$
H_{b}^{\emptyset}(C, u^*TX) \cong T_{ev(\Sigma_1)}X \otimes \mathcal{H}_{\Sigma_1}^{0,1}.
$$
Denote by \( \Sigma^{(1)} \equiv \Sigma_1 \cup \{ T^{(h)}_b \}_{x_h \in \Sigma_1} \). Then \( \Sigma^{(1)} \) is a nodal Riemann surface of genus one. Let \( \bar{\chi}(T) = \{ h : [\Sigma_h \subset \Sigma_P \setminus \Sigma_1, nod(h) \cap (\Sigma_1 \cap \Sigma_P \setminus \Sigma_1) \neq \emptyset \} \), where \( nod(h) \) denotes the nodes on \( h \).

For \( h \in \bar{\chi}(T) \), \( x \in \Lambda(h) \equiv nod(h) \cap (\Sigma_1 \cap \Sigma_P \setminus \Sigma_1) \) and \( m \in \mathbb{N} \), define

\[
D^m_x b = \frac{2}{(m-1)!} \left. \frac{D^{m-1} d}{ds} (u_h \circ \phi^{-1}_{x,1}(T)) \right|_{z=(s,t)=0},
\]

where \( \phi_{x,1} \) is given by (3.30) and the covariant derivatives are taken with respect to the standard metric \( s + it \in \mathbb{C} \) and a metric \( g_{x,u(\Sigma_1)} \) on \( X \) which is flat near \( u(\Sigma_1) \). Here \( \Sigma_{x,1} = h \) and \( \Sigma_{x,0} \) is the component in \( \Sigma_1 \) containing \( x \), we identify \( T_{x,0} \Sigma_{x,0} \) and \( T_{x,1} \Sigma_{x,1} \) with \( \mathbb{C} \) via \( \phi_{x,0} \) and \( \phi_{x,1} \) respectively. For example, \( h \) is the right sphere in Figure 2.3 with \( |A(h)| = 2 \) or \( 1 \) respectively in Figure 2.3.

We study the case that the principle component \( \Sigma^{(1)}_P \) of \( \Sigma^{(1)} \) is a smooth torus, the general cases follow similarly as explained in §5.4. For \( v = (b, v) \in \widetilde{\mathcal{F}^0} \) sufficiently small and \( V \psi \in T_{u(\Sigma^{(1)}_P)} X \otimes \mathcal{H}^{0,1}_{\Sigma^{(1)}_P} \), define \( R_v V \psi \in \Gamma^0_1(u_v) \) as follows. If \( z \in \Sigma_v \) is such that \( q_v(z) \in \Sigma_{b,h} \) for some \( h \in \chi(\Sigma^{(1)}) \) and \( |q_v^{-1}(q_v(z))| \leq 2\delta_T(b) \) or \( h \in \bar{\chi}(T), x \in \Lambda(h) 
and \ |\phi_{x,1}(q_v(z))| \leq 2\delta_T(b) \), we define \( \overline{\pi}_v(z) \in T_{u(\Sigma_1)} X \) by \( \exp_{u(\Sigma_1)} \overline{\pi}_v(z) = u_v(z) \). Given \( z \in \Sigma_v \), let \( h_z \) be such that \( q_v(z) \in \Sigma_{b,h_z} \). If \( w \in T_z \Sigma_v \), put

\[
R_v V \psi|_w = \begin{cases} 
0, & \text{if } \chi_{\Sigma^{(1)}} h_z = 0; \\
\beta(\delta_T(b)|q_v(z)|)(\psi|_w) \Pi_{\pi_v(z)} V, & \text{if } \chi_{\Sigma^{(1)}} h_z \neq 0, \\
(\psi|_w) V, & \text{if } \chi_{\Sigma^{(1)}} h_z = 0, 
\end{cases}
\]

where \( \chi_{\Sigma^{(1)}} \) is defined similar to (5.7) with respect to the bubble type \( \Sigma^{(1)} \) and \( \Pi_{\pi_v(z)} \) is the parallel transport along the geodesic \( t \mapsto \exp_{u(\Sigma_1)}(\pi_v(z)) \) with respect to the Levi-Civita connection of the metric \( g_{X,\psi}(\Sigma_1) \) given by Lemma 3.4. Let \( \Gamma^+(v) \) be given by the formula in §4.

Comparing with §5, we have the following lemma.

**Lemma 6.1.** Suppose \( T \) is a bubble type given by (iii) of Theorem 2.9. Then for every precompact open subset \( K \) of \( \mathfrak{M}_T(X,A,J) \), there exist \( \delta_K, C_K \in \mathbb{R}^+ \) and an open neighborhood \( U_K \) of \( K \) in \( \mathcal{X}_{2,k}(X,A,J) \) with the following property:

(i) For every \( \{ \tilde{b} \} \in \mathcal{X}^0_{2,k}(X,A,J) \cap U_K \), there exist \( v = (b, v) \in \widetilde{\mathcal{F}^0} \) and \( \zeta \in \Gamma^+(v) \) such that \( \|\zeta\|_{v,p} < \delta_K \) and \( [\exp_{u_{\zeta}} \zeta] = [\tilde{b}] \).

(ii) For every \( v \in (b, v) \in \widetilde{\mathcal{F}^0} \), we have

\[
C_K \|\zeta\|_{v,p} \leq \|D_v \zeta\|_{v,p} \leq C_K \|\zeta\|_{v,p}, \quad \forall \zeta \in \Gamma^+(v),
\]

(iii) For every \( v = (b, v) \in \widetilde{\mathcal{F}^0} \) and \( V \otimes \psi \in T_{u(\Sigma_1)} X \otimes \mathcal{H}^{0,1}_{\Sigma^{(1)}_P} \), we have

\[
\|D_v^* R_v (V \otimes \psi)\|_{v,c^0}
\]
\[ \leq C(b) \left( \sum_{h \in \chi(\Sigma^{(1)})} |\rho^{(1)}_h(v)| + \sum_{h \in \tilde{\chi}(T), x \in \Lambda(h)} |\rho^{(x)}_h(v)| \right) ||V \otimes \psi||_2. \] (6.4)

(iv) For every \( v = (b, v) \in \tilde{F}T^0_{\delta_K} |_{K(0)} \) and \( V \otimes \psi \in T_{ev(\Sigma_1)}X \otimes H^{0,1}_{\Sigma^{(1)}} \), we have
\[
\left| \langle (\overline{\partial_J} u_v, R_v(V \otimes \psi)) \rangle_{v,2} + \sum_{h \in \chi(\Sigma^{(1)})} \langle D^{(1)}_b, X \rangle \psi(\rho^{(1)}_h(v)) + \sum_{h \in \tilde{\chi}(T), x \in \Lambda(h)} \langle D^{(1)}_x b, X \rangle \psi(\rho^{(x)}_h(v)) \right| \leq C_K |v| \cdot |\rho(v)| \cdot ||V \otimes \psi||, \] (6.5)

where \( \rho^{(1)}_h(v) \) is given by (3.32) with respect to the bubble type \( \Sigma^{(1)} \) for \( h \in \chi(\Sigma^{(1)}) \) and \( \rho^{(x)}_h(v) = \rho_{ih}(v)v \) for \( h \in \tilde{\chi}(T), x \in \Lambda(h) \). Here \( \iota_h \) denotes the component in \( \Sigma_1 \) which contains the node \( x \) and \( \rho_{ih}(v) \) is given by (3.32) with respect to the bubble type \( \Sigma^{(1)} \).

**Proof.** (i) and (ii) hold by a similar argument in [Z2].

As in Lemma 5.5.1, (iii) follows from the proof of Lemma 5.2.

We prove (iv). By the construction of \( q_v \) and \( R_v \), we have \( \langle \overline{\partial_J} u_v, R_v(V \otimes \psi) \rangle = 0 \) outside the annuli \( A_{\nu,h}(\{|v_h|\}) \) for \( h \in \chi(\Sigma^{(1)}) \cup \tilde{\chi}(T) \) where \( A_{\nu,h}(\{|v_h|\}) = \bigcup_{x \in \Lambda(h)} q_v^{-1}(\{|\phi_{x,1}(q_v(z))| \leq 2|v_x|^{\frac{1}{2}}\}) \) for \( h \in \tilde{\chi}(T) \). Hence we have
\[
\langle (\overline{\partial_J} u_v, R_v(V \otimes \psi)) \rangle_{v,2} = \sum_{h \in \chi(\Sigma^{(1)}) \cup \tilde{\chi}(T)} \int_{A_{\nu,h}(\{|v_h|\})} \langle \overline{\partial_J} u_v, R_v(V \otimes \psi) \rangle. \] (6.6)

Then the proof of Lemma 4.3 in [Z1] can be used to obtain an expansion
\[
\langle (\overline{\partial_J} u_v, R_v(V \otimes \psi)) \rangle_{v,2} = - \sum_{m \geq 1, h \in \chi(\Sigma^{(1)})} \langle D^{(m)}_b, X \rangle \left\{ D^{(m)}_{b, \tilde{z}^{(1)}_h(v)} \psi \left( (d\phi_{b, T^{(1)}(h)} \tilde{z}^{(1)}_h(v) - 1) \rho^{(1)}_h(v) \right) \right\} - \sum_{m \geq 1, h \in \tilde{\chi}(T), x \in \Lambda(h)} \langle D^{(m)}_x b, X \rangle \left\{ D^{(m)}_{x, \tilde{z}^{(x)}_h(v)} \psi \left( (d\phi_{b, T^{(x)}(h)} \tilde{z}^{(x)}_h(v) - 1) \rho^{(x)}_h(v) \right) \right\}
\]
where \( T^{(1)}(h), \tilde{z}^{(1)}_h(v), \rho^{(1)}_h(v) \) are the maps as in Lemma 5.1 with respect to the bubble type \( \Sigma^{(1)} \) and \( T^{(x)}(h), \tilde{z}^{(x)}_h(v), \rho^{(x)}_h(v) \) are defined similarly with respect to the node \( x \). Hence (iv) holds by the same argument as in Lemma 5.4.

For any \( v = (b, v) \in \tilde{F}T^0 \), let
\[
\alpha_T(v) = \sum_{h \in \chi(\Sigma^{(1)})} \langle D^{(1)}_b \rangle \psi(\rho^{(1)}_h(v)) + \sum_{h \in \tilde{\chi}(T), x \in \Lambda(h)} \langle D^{(1)}_x b \rangle \psi(\rho^{(x)}_h(v)). \] (6.7)

The following is the main result in this section.

**Theorem 6.2.** Suppose \( T \) is a bubble type given by (iii) of Theorem 2.9. Then one of the following must hold:
(i) An element \( b \equiv [C, u] \in \mathcal{M}_T(X, A) \cap \mathcal{M}_{2,k}^0(X, A, J) \) if and only if \( \alpha_T(v) = 0 \) for some \( v = (b, v) \in \mathcal{F}^\theta_\delta \). In particular, \( \mathcal{M}_T(X, A) \cap \mathcal{M}_{2,k}^0(X, A, J) \) is a smooth orbifold of dimension at most \( \dim \mathcal{M}_{2,k}^0(X, A) - 2 \).

(ii) An element \( b \equiv [C, u] \in \mathcal{M}_T(X, A, J) \cap \mathcal{M}_{2,k}^0(X, A, J) \) must satisfy: there exists \( \Sigma_h \subset \Sigma_P \) such that \( u^* \Sigma_P = u^* \Sigma_h \) and \( u_h \) factor through a branched covering \( \tilde{u} : \Sigma' \to X \), i.e., there exists a holomorphic branched covering \( \phi : \Sigma_h \to \Sigma' \) such that \( u_h = \tilde{u} \circ \phi \) and \( \deg(\phi) \geq 2 \).

**Proof.** Let \( \delta_K \) be given by Lemma 6.1. For each \( v = (b, v) \in \mathcal{F}_\delta^\theta \), we define the homomorphism

\[
\pi_v^0 : \Gamma^0_0(b_P) \to \Gamma^0_0(b_P), \quad \pi_v^0 \xi = - \sum_{1 \leq i \leq n} \langle \xi, R_v e_i \psi \rangle e_i \psi \in \Gamma^0_0(b_P),
\]

where \( \psi \) is an orthonormal basis for \( H^0_0(\Sigma^1 P) \) and \( \{e_i\}_{1 \leq i \leq n} \) is an orthonormal basis for \( T_u(\Sigma) X \). Denote the kernel of \( \pi_v^0 \) by \( \Gamma^0_+^0(b_P) \). Then by Lemmas 6.1 and the same argument as in Lemma 5.5, we have

\[
\pi_v^0(v, \zeta) = \pi_v^0((\overline{\partial} f u_v + D_v \zeta + N_v \zeta) = \alpha_T(v) + \epsilon(v, \zeta),
\]

and

\[\|\epsilon(v, \zeta)\| \leq C_K(\|v\| + \|\zeta\|_{v, p, 1})|\rho(v)| \leq C_K|v|^{\frac{1}{2}}|\rho(v)|,\]

where we use notations in Lemma 5.5. Hence in order to satisfy (5.21), we must have \( \alpha_T(v) = 0 \) provided \( |v| \) is sufficiently small.

The converse is similar to the previous and we only sketch it here. We have the following two cases:

**Case 1.** If \( |\{x \in A(h)\}_{h \in \overline{\chi}(T)}| = \{x_1, x_2\} = 2 \), e.g. the first figure in Figure 2.3.

If \( |\overline{\chi}(T)| = |\{h\}| = 1 \), then by (ii-e) in Definition 1.1, we have (i) or (ii). In other cases, we have (i) by (ii-a), (ii-b) in Definition 1.1.

**Case 2.** If \( |\{x \in A(h)\}_{h \in \overline{\chi}(T)}| = |\{x\}| = 1 \), e.g. the second figure in Figure 2.3.

In this case, by (ii-e) and (iii-b) in Definition 1.1, we have (i) or (ii).

### 7 Proof of the main theorems

In this section we give the proofs of the main theorems.

**Proof of Theorem 1.2.** The theorem follows by §2, 4, 5 and 6.

\[\square\]
Proof of Theorem 1.3. If $b \equiv [C, u]$ belongs to (i) of Theorem 1.2, the boundary component $\mathcal{M}_T(X, A, J) \cap \mathcal{M}^0_{2,k}(X, A, J)$ is a smooth orbifold of dimension at most $\dim \mathcal{M}^{vir}_{2,k}(X, A) - 2$.

If $b \equiv [C, u]$ belongs to (ii) of Theorem 1.2, then we have
\[ \text{ev}(\mathcal{M}_T(X, A, J) \cap \mathcal{M}^0_{2,k}(X, A, J)) \subset \text{ev}(\mathcal{M}_{T'}(X, A', J)) \]
where $T'$ is the bubble type of genus $g$ corresponding to $C'$. By (i) and (iii-a) in Definition 1.1, we have
\[ \dim \mathcal{M}^0_{T'}(X, A', J) = 2\langle c_1(TX), A' \rangle + 2(n - 3)(1 - g) + 2k - 2n_{nod(C')} - 2(n - 3)(1 - g) + 2k - 2n_{nod(C')} \]
(7.1)

On the other hand, by (1.1) we have
\[ \dim \mathcal{M}^{vir}_{2,k}(X, A) = 2\langle c_1(TX), A \rangle - 2(n - 3) + 2k. \]
(7.2)

By (ii) of Theorem 1.2, we have $0 \neq A - A' = mA_2$, where $[C_2, u_2] \in \mathcal{M}^0_{g,0}(X, A_2, J)$. Thus we have
\[ \dim \mathcal{M}_T(X, A', J) \leq \dim \mathcal{M}^{vir}_{2,k}(X, A) - 2 \]
(7.3)
provided $\dim X \equiv 2n < \min\{N + 6, 2N + 6\}$. Hence $\mathcal{M}_T(X, A, J) \cap \mathcal{M}^0_{2,k}(X, A, J)$ serves as a boundary component of $\mathcal{M}^0_{2,k}(X, A, J)$ in the sense of pseudocycle, cf. Definition 6.5.1 of [MS].

Proof of Theorem 1.4. Suppose $u : \Sigma \to X$ factors through an $m$-fold cover $\Sigma' \to X$, where $m \geq 2$. Denote the space of equivalence classes of such maps by $\mathcal{M}_{\Sigma', m}$. Then we have
\[ \text{ev}(\mathcal{M}_{\Sigma', m}) \subset \text{ev}\left(\mathcal{M}^0_{g,k}(X, A/m, J)\right) \]
(7.4)
where $g$ is the genus of $\Sigma'$. Note that we have
\[ \dim \mathcal{M}^0_{g,k}(X, A/m, J) \leq \dim \mathcal{M}^{vir}_{2,k}(X, A) - 2 \]
(7.5)
provided $\dim X \equiv 2n < \min\{N + 6, 2N + 6\}$ since $g \leq 1$. Hence $\text{ev}(\mathcal{M}_{\Sigma', m})$ will not intersect $\mu_1 \times \cdots \times \mu_k$ in general position.

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