Inertia as the “Threshold of Elasticity” of Quantum States

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Abstract

The principle of general relativity (GR) of Einstein is based on the possibility “to create” or “to compensate” locally the gravitation field by a choice of the accelerated frame in spacetime. In quantum field theory do not exist pointwise particles—“material points” (what we need for the locality), but we have rather extended wave packets (quantum particles) with “polarizations” of different kinds that correspond, in principle, to indefinite number of internal degrees of freedom. In this case both the spacetime manifold is very poor and the factorization of all motions in accordance with the classification—“uniform/accelerated” is very rough, for the desirable possibility “to create” or “to compensate” any physical field. Hence, we should deal with a quantum state space. Therefore, it is natural to think that creation and compensation (annihilation) of the quantum field may be related to the choice of the functional frame in the state space and that must be some new principle of classification of quantum motions. This hypothesis has been called “superrelativity”. This leads to a new gauge theory of the geometric type in the projective Hilbert space.

Here I try to connect the inertia property of “elementary” quantum particles with the deformation of quantum modes which prevent these particles from the flying apart and to correct some formulas of my previous articles.

1 Introduction. The Quantum Origin of Inertia

Einstein stated in his attempt to constitute the principle of the general covariance that the reason of the difference in the behavior of the two liquid bodies $S_1$ and $S_2$ lies outside of the system, namely—in the system of distant stars. I think, however, that
here is some confusion since the system of the distant stars in this discussion takes place of the \textit{absolut reference frame as a true discriminator} of the character of motions in spite of Einstein statement that under a correct formulation of physical laws, any frame must be available [1]. It is not enough, however, to conclude that there is some inconsequence in the foundations of \textit{classical} general relativity. Furthermore, I think that even the \textit{notion of “acceleration” should be avoided in a consistent quantum gravity} because, for instance, the physical state of a free falling down droplet of mercury in gravitational field is indistinguishable from the physical state of the similar droplet which is far from masses. If and only if there exists some acceleration force which evokes changing of the physical state of the droplet (temperature, surface tension etc.), one can think about “accelerated motion”, although in particular case the droplet may even rest relative, say, surface of a table. This example shows that instead of indefinite notion of “acceleration”, an immanent characteristic of the spacetime motion should be used–internal (as a matter of fact–quantum) states of system. Hence, we can point out the physical reason of the different behavior of the bodies $S_1$ and $S_2$, if we will take into account a \textit{physical state of these bodies}. But, if the distant stars can not play the role even of a “true” reference frame for the measurement of acceleration, how one can think that they are a source of the masses of “elementary” particles, droplets of mercury etc.? We have to find some different source of the inertia properties of matter.

How we can do it? Now it is well known that the existence and stability of extended macroscopic objects are supported by Goldstone’s modes [3]. I think \textbf{we can connect the inertia property rather with the quantum conditions of excitations and reconstruction of these modes} [4] than with the Mach principle [1, 2, 4]. Then, in some sense, one can say that the source of inertia lies, in fact, “outside” the spatial boundary of body, namely–in the Hilbert state space. This means that the deformation of the self-consistent boundary leads to the generation of the collective mode as a dynamical quantum reaction on external force. Therefore \textbf{an acceleration is only external (macroscopic) “exhibit” of internal quantum reaction} [9, 10, 11]. That is one may treat the inertia as an “elasticity” of the self-consistent boundary at the quantum level. The deformations of this boundary leads to deformations of the quantum states of electron’s shells and these excitations one can interpret as evidence of the inertial properties of a physical system.

Here we try to connect the inertia property of an “elementary particle” and the deformation of the internal quantum collective modes which prevent this system from flying apart under some kinds of repulsive forces. Dirac applied a surface tension force of some non-Maxwellian type as a stabilizer in the extensible model of the electron

\footnote{As far as I know only in the two letters from Schrödinger to Einstein (18.11.50) and from Einstein to Schrödinger (22.12.50) the problem of acceleration in quantum theory has been discussed. Einstein wrote in his letter that “full description can not be built on the notion of acceleration”.
}
K.R.W. Jones pursuing to reach in his beautiful works [13, 14] the objective interpretation of the wave function, have used the Newtonian gravitational self-energy in order to find bound and stable solitary wave solution of the nonlinear gravitational Schrödinger equation of the scalar Bose-Einstein condensate. That is this nonlinearity should play the role of the physical mechanism of the suppressing of the dispersion which leads to spread out wave packets. I think that for the relativistic realization of these ideas, a geometrical approach is unavoidable and that geometrical spirit of Einstein program should be conserved. Instead of nonrelativistic scheme of Jones I propose to use the superrelativity principle which return us to the geometrical origin (in projective Hilbert space) of the unified interactions in accordance with Einstein’s program. Then the curvature $R^{*}_{kj}i$ of the CP(N) is the quantum source of the inertia of the nonlocal elementary particles. Thereby the self-interacting fields of Barut’s type [15] has a geometric origin. They arise as a state-dependent gauge fields “of the third kind” of Doebner-Goldin [16].

2 The Physical Quantum Projective Space

I had already discussed the dynamical sense of the projective Hilbert space and the Fubini-Study metric [5, 6, 7, 8, 9, 10, 11]. Of course, any appropriate choice of the local coordinates is acceptable, but II coordinates is as natural as, for example, the Cartesian coordinates in Euclidean space. The using of CP(N) geometry is closely connected with both the new point of view on the role of spacetime structure and physical field concept.

2.1 Spacetime and Field Concept

It has been mentioned early [5, 7, 9, 10] that we should sacrifice the priority of spacetime manifold and to build quantum field theory over state space. From the technical point of view this necessitate a new construction of quantum physical fields over state space. Only after that the spacetime propagation of their “envelopes” should be described.

When we build our theories we usualy begin with some fields on spacetime manifold. Thus, we think that it is the most realistic approach to the achievement of an agreement between theory and experiment. But we should remember that the measurement of the spacetime position of the “event” is only the “label” of the fact of some quantum transition in a quantum state space [5, 8, 12]. Therefore we can assume that from the physical point of view just the quantum dynamics in the state space is interesting for us. Hence, it is more natural to deal with functions on quantum state space than with the ordinary spacetime functions (classical or ordinary quantum fields).
That is, instead of a material point and functions of its coordinates in space-time, one should use functions of the coordinates of a quantum state. These tensor fields take place of dynamical variables which have reasonable analytic properties. Otherwise we obtain singular results because a pointwise (in space-time) dynamical variables contain singular functions. In fact, the all known variants of the regularization procedure are the processes of a “delocalization” of the pointwise dynamical variables. Thus, in our case we search functions of “quantum transition” in the space of the “internal” degrees of freedom, not functions of coordinates of the “event” in spacetime which (coordinates) are, after all, only obtrusive illusion. Dynamical (habitual) spacetime arises only as a manifold of the “centrum of mass” of quantum “droplets”—the soliton-like solutions of some nonlinear wave equations.

I would like to emphasize the fact that what was historically called “coupling constants” are, as a matter of fact, spacetime functions under renormalization procedure. For example, the Newton’s constant $G_N$ should be subjected a renormalization in quantum area because, if the mass of some particle $m$ corresponds to the oscillation process with the frequency $\omega$ by the equation $mc^2 = \hbar \omega$, it should have (in accordance with the Einstein’s formula for a frequency) a larger value—“blue shift of de Broglie wave” in the vicinity of a massive body $M$

$$m_0 = \frac{\hbar \omega_0}{c^2} = \frac{\hbar \omega}{c^2} \left(1 + \frac{G_N M}{c^2 d}\right),$$

(2.1)

where $\frac{G_N M}{d}$ is a gravitational potential of a massive body. This result was interpreted by Einstein as an affirmation of Mach principle [2]. Thus, the Mach mechanism of mass generation must have fantastic, almost infinite resolution $r_0/L_G = 10^{-13} cm/10^{20} cm = 10^{-33}$ for the spatial separate generation of quite different masses of elementary particles at nuclear distance. It is defined by the relation of the nuclear size to the remote galaxy distance. Besides that, on the quantum level it leads to the infinite self-interacting gravitation field which we can not see in experiments. Similar arguments are applicable to the electroweak and to the set of strong coupling “constants”. That is any field construction which based on spacetime dependent functions (fields) and pointwise charges as sources of these fields leads to difficulties and requires renormalization of these charges. Therefore one have to use a new set of primordial elements in the quantum area [3, 4, 10]. The program of the overcoming of these obstacles must be based on the isometry between the coset structure of quantum interactions and the projective quantum state space CP(N).

### 2.2 Coset Structure of Universal Quantum Interactions

As was mentioned early [3, 11] the physical deformations of the pure quantum state have the geometric structure of a coset, i.e. the structure of the CP(N): $G/H = SU(N+1)/S[U(1)_{el} \times U(N)] = CP(N)$. This paves the way to the invariant study
of the spontaneously broken unitary symmetry \[9\]. This statement has a general character and does not depend on particular properties of the pure quantum state. The reason for the change of motion of material point is an existence of a force. The reason for the change of a pure quantum state is an interaction which is evoked by unitary transformations from the coset \(G/H\). The reaction of a material point is an acceleration. The reaction of a pure quantum state is its deformation whose geometry is in fact the geometry of the coset transformations \(SU(N+1)/SU(1) \times U(N)\). That is a universal quantum interactions at the fundamental level are rooting in the Hilbert projective state space \(CP(N)\).

These simple speculations can help presumably to solve the problem of hierarchy of the fundamental interaction and masses. The fine structure constant \(\alpha\) in the Lorentz-radial Klein-Gordon equation (in the Lommel form) \[8, 10\] takes place of the dimensionless mass (or energy) of a fundamental scalar field. The geodesic excitations of this scalar field which are induced by transformations from the coset should be somehow related to the spectrum of masses of nonlinear particle-like solution of nonlinear wave equation which arises under the geodesic variation of the Fourier components of the solution of the Lommel equation \[8, 9, 10\]. It was be shown \[10, 11\] that excitations which was generated by the geodesic flow in \(CP(N)\) correspond to quite concrete elements of the invariant subspace in the algebra \(AlgSU(N+1)\)

\[
\hat{B} = \begin{pmatrix}
0 & f_1^* & f_2^* & \ldots & f_N^* \\
f_1 & 0 & 0 & \ldots & 0 \\
f_2 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f_N & 0 & 0 & \ldots & 0 
\end{pmatrix}
\tag{2.2}
\]
as functions of the Fourier components of the initial state (solution of the Lommel equation) \(|f^i| = g|\Phi^i|(R^2 - |\Phi^0|^2)^{-1/2}\), where \(\arg f^i = \arg \Phi^i\) (up to the general phase), and angle from the equation \(\cos \Theta = |\Phi^0|/R\). This is a natural type of the “second quantization” which has, however, dynamical character \[11\]. That is, instead of the Dirac’s identification of Fourier components with operators which obey Fermi or Bose commutation rules \[17\] we have a invariant correspondence between them. This “second quantization” can not be related to some ensemble of the identical quantum particles but rather to internal oscillations near the bottom of the valley of the intrinsic quantum potential of the natural connection in \(CP(N)\)

\[
\Gamma_{kl}^i = -2\frac{\delta_k^i \Pi^* + \delta_l^i \Pi^*}{R^2 + \sum_{s=1}^{N} |\Pi^s|^2},
\tag{2.3}
\]

\[11\] The physical unity of theory demands only single true fundamental interaction constant \(\alpha = \frac{e^2}{\hbar c}\); that, from the geometric point of view, is the sectional
The curvature of the $\text{CP}(N)$ $\kappa = \alpha = R^{-2}$. Then all physical coupling constants should arise as characteristics of the deformation of quantum state in different directions of the quantum state space $\text{CP}(N)$ under action of the coset transformations in the form $\frac{f^k}{g^k} (\cos \Theta - 1)$. This requires not only topological equivalence of the coset manifold $SU(N + 1)/S[U(1) \times U(N)]$ and $\text{CP}(N)$. There is isometry between these manifolds [18]. In order to identify the metrics of the coset manifold and $\text{CP}(N)$ it is enough to identify the tangent space $T_0 SU(N + 1)/S[U(1) \times U(N)]$ which contains matrices $F, G$ like (2.2) and the tangent space $T_0 \text{CP}(N) = \mathbb{C}^N$ which contains vectors $(f_1, f_2, \ldots, f_N), (g_1, g_2, \ldots, g_N)$. Then one should define the “Lagrangian of the geodesic velocities of deformation” as a scalar product

$$L_{fg} = (F, G) = \frac{1}{2} \text{Tr} FG^* = \text{Re}(f, g)_{\mathbb{C}^N} = \text{Re} \sum_{i=1}^{N} f^i g^{i*}.$$  \hspace{1cm} (2.4)

In order to establish the isometry globally over whole group we have to introduce the Killing metric by using left-invariant vector fields $A_F(u) = uF, B_G(u) = uG, u \in SU(N + 1), F, G \in \text{AlgSU}(N + 1)$. The Killing scalar product is

$$< A, B > = \text{Re} \text{Tr} AB^* |_0 \hspace{1cm} (2.5)$$

for $A, B \in SU(N + 1)$ is $Ad[U(N + 1)]$-invariant. Then $r^2 = ||g||^2 = < g, g > = \text{Re} \text{Tr} E = N + 1$. This Killing metric should be in the “harmony” with the Fubini-Study metric in $\text{CP}(N)$. That is $SU(N+1)$ and the coset are embedded into the sphere with the radius $r = \sqrt{N + 1} = \sqrt{2S + 1}$, where multilevel $N + 1$ system (the multiplet of excitations, for example) may be expressed in terms of spin $S$. The radius of the sectional curvature of $\text{CP}(N)$ is, in accordance with our assumption, $R = \alpha^{-1/2}$. Thus, the radius $R$ has a physical meaning. Therefore we should normalize correspondence between $R$ and $r$. Thereby, we have here some kind of the superselection rule for the “mass–spin” relationship. The question, of course, is, whether physically correct the identification of $S$ with the spin of particles.

Thereby physical effects of a quantum dynamics in $\text{CP}(N)$ are curvature-dependent. The projective symmetry is broken. The unitary symmetry is local and hidden. In this case, what variables $\Psi$ say about? Naturally think that they describe not amplitudes of probability of some ensemble, but they should be “classical” Fourier amplitudes of an internal quantum dynamics of a single particle. This idea corresponds to the Barut’s approach to the nonperturbative “self-field” version of the quantum electrodynamics [15]. These related to the Goldston and Higgs excitations of the geometric type which arise under deformation of quantum states in the self-interacting potential of the local unitary rotations of the functional frame in the Hilbert space [11]. It is some analog of the “string excitations” if somebody like to think in the framework of this model. Then one can say that the geodesic of the Fubini-Study metric plays the role of a closed string [11]. But, of course, this mechanism has a quite different physical meaning. I intend to develop here the
differential-geometric aspects of a gauge theory over CP(N) with the metric tensor in the local coordinates \( \Pi \)

\[
G_{ik} = R^2 \left[ \frac{\left( \sum_{s=1}^{N} |\Pi^i|^2 + R^2 \right) \delta_{ik} - \Pi^i \Pi^k}{\left( \sum_{s=1}^{N} |\Pi^i|^2 + R^2 \right)^2} \right].
\]

(2.6)

3 The Geometric Origin of Elasticity of Quantum States

In our approach, the projective Hilbert space CP(N) of the generalized coherent states is treated as the configuration space of wave packets. That is, we regard the Fourier components of the scalar (real or complex) classical field as a “multiplet” relative to \( SU(N+1) \) symmetry broken down to the isotropy group \( U(1) \times U(N) \) of the “vacuum” vector.

The essentially a new element of our approach is the action of geodesic flow in the configuration space on the relative Fourier components CP(N) [10]. The principle of least action arises in CP(N) as a principle of minimal distance. The integral curves of the geodesic flow (geodesics) are stable and closed (periodic).

The key idea proposed here associated with a model for a quantum particle as an extended field dynamical system. Internal “hidden” coordinates of a “pre-system” is local coordinates in the projective Hilbert space CP(N). Self-evolution of this “pre-system” appears as a motion along closed path in CP(N). This path is a geodesic of the Fubini-Study metric. The positive curvature of the Hilbert projective space is the reason for the stability of geodesics in CP(N). We interpret this internal local coordinates as Fourier components of some wave packet in the reference Minkowski spacetime. Therefore the “pre-system” is “wrapped” in the shell of a surrounding field in ordinary spacetime and one has, hence, a non-local extended dynamical field configuration—“droplet”.

The flow is then given by the unitary matrix \( \hat{T}(\tau, g) = \exp(i\tau \hat{B}) = \)

\[
\begin{bmatrix}
    \cos \Theta & -\frac{f_1}{g} \sin \Theta & \cdots & -\frac{f_N}{g} \sin \Theta \\
    \frac{f_1}{g} \sin \Theta & 1 + \left[ \frac{|f_1|^2}{g^2} \right] (\cos \Theta - 1) & \cdots & \frac{f_1 f_N}{g^2} (\cos \Theta - 1) \\
    \cdots & \cdots & \cdots & \cdots \\
    \frac{f_N}{g} \sin \Theta & \frac{f_1 f_N}{g^2} (\cos \Theta - 1) & \cdots & 1 + \left[ \frac{|f_N|^2}{g^2} \right] (\cos \Theta - 1)
\end{bmatrix},
\]

(3.1)

where \( g = \sqrt{\sum_{k=1}^{N} |f_k|^2}, \Theta = g \tau \). The form of the periodic geodesic “deformation” of the Fourier components of the initial solution

\[
\Phi^a = \frac{a!}{\Gamma(a + 1/2)} \int_0^\infty y^{-1/2} L_a^{-1/2} e^{-y} J_1(\alpha \sqrt{y}) dy.
\]

(3.2)
of the Lommel equation
\[ \frac{d^2 \Phi^*}{d \rho^2} + \frac{3}{\rho} \frac{d \Phi^*}{d \rho} + \alpha^2 \Phi^* = 0, \] (3.3)
is represented by the formula
\[ |\Psi(\tau, g, y)\rangle = \sum_{a,b=0}^{N} \Phi^a [\hat{T}^{-1}(\tau, g)]^b_a |b, y\rangle, \] (3.4)
where in the particular case \( \tau = 0 \) one has
\[ |\Psi(0, g, y)\rangle = \sum_{a,b=0}^{N} \Phi^a [\hat{T}^{-1}(0, g)]^b_a |b, y\rangle = |\Phi\rangle, \] (3.5)
and for \( \tau'g' = \Theta' \) when \( \cos \Theta' = |\Phi|/R \)
\[ |\Psi(\tau', g', y)\rangle = \sum_{m,n=0}^{\infty} \Phi^m [\hat{T}^{-1}(\tau', g')]^b_a |b, y\rangle = \sum_{a,b=0}^{N} R \exp[i\omega(\Phi)] \delta^b_a |b, y\rangle = R \exp[i\omega(\Phi)] |0, y\rangle = |\Psi_0\rangle. \]
Thus direction of the transformation of the vacuum vector into the solution of the Lommel equation is determined by the matrix \( \hat{P} = \hat{G}^{-1}(\Phi) \hat{B}(\Phi) \hat{G}(\Phi) \), that is all algebra \( \text{Alg}SU(N+1) \) subjected to the proper (in the Hilbert space) similarity transformation for the adaptation of the algebra structure to any state vector.

Briefly speaking, one can define a proper state-dependent images of the Cartan’s decomposition of the elements of Lie algebra \( \text{Alg}SU(N+1) \) \( h_\Phi = \hat{G}^{-1}(\Phi) \hat{h}(\Phi) \hat{G}(\Phi) \), and \( b_\Phi = \hat{G}^{-1}(\Phi) \hat{b}(\Phi) \hat{G}(\Phi) \). It is easy to proof that the proper commutation rules are commonly known:
\[ [h_\Phi, h_\Phi] \subset h_\Phi, \quad [b_\Phi, b_\Phi] \subset h_\Phi, \quad [h_\Phi, b_\Phi] \subset b_\Phi. \] (3.6)
But such approach is neither convenient nor logical for our purposes because on every step of the ansatz of squeezing one should to solve the “elimination equations”, and this approach appeals an artificial “vacuum” state
\[ |\Psi_0\rangle = \begin{pmatrix} e^{i\omega(\Phi)} & \sqrt{\sum_{a=0}^{N} |\Psi_a|^2} \\ 0 & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots \\ 0 & & & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \] (3.7)
which is not in general the real vacuum state of a concrete physical problem. The most consistent approach is the using of the local coordinates

\[ \Pi^1 = R \frac{\Psi^1}{\Psi^0}, \quad \Pi^2 = R \frac{\Psi^2}{\Psi^0}, \ldots, \quad \Pi^i = R \frac{\Psi^i}{\Psi^0}, \ldots, \quad \Pi^N = R \frac{\Psi^N}{\Psi^0}, \ldots \] (3.8)

That is, we now refer to the term "local" as a fact of a dependence on the local coordinates in the CP(N). One should find the relationship between the linear representations of SU(N+1) group by an "polarization operator" \( \hat{P} = \mu H^\sigma \lambda^\sigma \in \text{AlgSU}(N + 1) \) which depends on external "multipole magnetic" or "gluon" field \( H^\sigma, 1 \leq \sigma \leq N^2 + 2N \) and does not depend on the state of the quantum system, and the nonlinear representation (realization) of the group symmetry in which the infinitesimal operators of the transformations depend on the state. In the linear representation of the action of \( SU(N + 1) \) we have

\[ |\Psi(\tau)\rangle = \exp(-i\frac{\tau}{\hbar} \hat{P})|\Psi\rangle. \] (3.9)

For a full description of a group dynamics by pure quantum states, we shall use coherent states in CP(N). If \( \hat{P}_\sigma \) is one of the \( N^2 + 2N \) directions in the group manifold one has

\[ D_\sigma(\hat{P}) = \Phi^i_\sigma(\Pi, P) \frac{\delta}{\delta \Pi^i} + \Phi^{i*}_\sigma(\Pi, P) \frac{\delta}{\delta \Pi^{i*}}, \] (3.10)

where

\[ \Phi^i_\sigma(\Pi; P) = R \lim_{\epsilon \to 0} \epsilon^{-1} \left\{ \frac{\exp(i\epsilon P^i_{\sigma})}{\Psi^m} \frac{\Psi^m}{\Psi^k} - \frac{\Psi^i}{\Psi^k} \right\} = \lim_{\epsilon \to 0} \epsilon^{-1} \left\{ \Pi^i(\epsilon P^i_{\sigma}) - \Pi^i \right\} \] (3.11)

are the local (in CP(N)) state-dependent components of the SU(N+1) group generators, which are studied in \[5, 6, 7\].

In order to establish relationships between "internal" parameters in CP(N) and a propagation of the scalar field near the light cone in the "reference spacetime", we should "lift" a geodesic deformation of the initial Fourier components into the fiber bundle. Namely, if we assume that in accordance with the "superequivalence principle" \[9\] an infinitesimal geodesic "shift" of field dynamical variables could be compensated by an infinitesimal transformations of the basis in Hilbert space, then one can get some effective self-interaction potential as an addition term in original Klein-Gordon equation in the Lommel form \[3, 3\]. We will label hereafter vectors of the Hilbert space by Dirac’s notations |... > and tangent vectors to CP(N) (field dynamical variables) by arrows over letters, \( \xi \in T_{\Pi CP(N)} \), for example. Then one has a definition of the rate of a state vector changing

\[ |v(y)\rangle = -(i/\hbar) \hat{P} |\Psi(y)\rangle. \] (3.12)
The “descent” of the vector field $|v(y)| >$ onto the base manifold $\text{CP}(N)$ is a mapping by the two formulas: $f : \mathcal{H} \to \text{CP}(N)$, i.e. (3.8)

$$f : (\Psi^0, ..., \Psi^i, ..., \Psi^N) \to (R \frac{\Psi^1}{\Psi^0}, ..., R \frac{\Psi^N}{\Psi^0}, ...) = (\Pi^1, ..., \Pi^N),$$

and

$$\tilde{\xi} = f_* (\Psi^0, ..., \Psi^N)|v(y) > = \left. \frac{d}{d\tau} (R \frac{\Psi^1}{\Psi^0}, ..., R \frac{\Psi^N}{\Psi^0}) \right|_0 = -\left(\frac{i}{\hbar}\right)[RP_0^1 - P_0^0 \Pi^1 + (P_k^1 - (1/R) P_k^0 \Pi^1)\Pi^k, ..., RP_0^N - P_0^0 \Pi^N + (P_k^N - (1/R) P_k^0 \Pi^N)\Pi^k].$$

That is the operator $\hat{P}$ determines a field dynamical variable $\tilde{\xi}$ (3.14). On the geodesic which spans both the vacuum vector (3.7) and the solution of the Lommel equation there is a “natural” vector field $\tilde{\xi}(\Pi(\tau))$ which, of course, is, in general, non-parallel along the geodesic. One can look on the integral curve of this vector field as on some “excited string” in $\text{CP}(N)$ under a perturbation of the “geodesic string”. We propose the following guide of realization of “superequivalence” principle:

1. To use the covariant derivative of the vector field $\tilde{\xi}(\Pi(\tau))$ relative the Fubini-Study metric in order to keep the tangent vector field. It is well known that at a point $\Pi + \Delta \Pi$ in $\text{CP}(N)$ the “shifted” field $\tilde{\xi} + \delta \tilde{\xi} = \tilde{\xi} + \frac{\delta \xi}{\delta \tau} \delta \tau$ contains the derivative $\frac{\delta \xi}{\delta \tau}$, which is not, in the general case, a tangent vector to $\text{CP}(N)$, but the covariant derivative $\frac{\Delta \xi}{\delta \tau} = \frac{\delta \xi}{\delta \tau} + \Gamma^i_{km} \xi^k \frac{\delta \Pi^m}{\delta \tau}$ is a tangent vector to $\text{CP}(N)$.

2. To make a small shift along the covariant derivative $\frac{\Delta \xi}{\delta \tau}$.

3. It may be proved that for the shape of the ellipsoid of polarization at the point $\Pi + \delta \Pi$ along the direction of the covariant derivative $\frac{\Delta \xi}{\delta \tau}$ on the geodesic from $\{0\}$ to $\Pi$ there is a point $\Pi + \Delta \Pi$ where ellipsoid of polarization has the same shape. That is instead of the shift along integral curve of the vector field $\tilde{\xi}(\Pi(\tau))$ one can use a shift along the geodesic which was mentioned above. Thereby one can avoid difficulties with zeroth modes which arise under the general variation involving tansformation from the isotropy group.

4. Now we should “lift” the new tangent vector $\xi^i + \Delta \xi^i$ into the original Hilbert space $\mathcal{H}$, that is, one needs to realize two inverse mappings: $f^{-1} : \text{CP}(N) \to \mathcal{H}$ at point $\Pi^i + \Delta \Pi^i$ by the formula

$$\Psi^0 = \frac{R^2}{\sqrt{\sum_{s=1}^{N} |\Pi^s + \Delta \Pi^s|^2 + R^2}} \cdots, \quad \Psi^i = (\Pi^i + \Delta \Pi^i) \frac{R}{\sqrt{\sum_{s=1}^{N} |\Pi^s + \Delta \Pi^s|^2 + R^2}},$$

or in the first approximation

$$f^{-1} : (\Pi^1 + \Delta \Pi^1, ..., \Pi^N + \Delta \Pi^N) \to [\Psi^0 + \frac{\partial \Psi^0}{\partial \Pi^i} \Delta \Pi^i, ..., \Psi^N + \frac{\partial \Psi^N}{\partial \Pi^i} \Delta \Pi^i].$$

(3.15)
and then
\[ f_{\pi + \delta \Pi}^1(\xi + \Delta \xi) = [v^0 + \Delta v^0, v^1 + \Delta v^1, ..., v^N + \Delta v^N] \]
\[ = \left[ \frac{\partial \Psi^0}{\partial \Pi^i}(\xi^i + \Delta \xi^i), \frac{\partial \Psi^1}{\partial \Pi^i}(\xi^i + \Delta \xi^i), ..., \frac{\partial \Psi^N}{\partial \Pi^i}(\xi^i + \Delta \xi^i) \right]. \quad (3.17) \]

It is may be shown that under the parallel transport of the $\vec{\xi}$ along a smooth curve, one has
\[ \Delta \xi^i = \xi^i(\tau) - \xi^i(0) = - \int_0^\tau \Gamma_{kl}^i \xi^l d\Pi^k ds, \quad (3.18) \]
and, therefore, in the first approximation
\[ |\delta v(y) > = - \Gamma_{kl}^i(0) \xi^l(0) \Delta \Pi^k \frac{\partial \Psi^a}{\partial \Pi^i} |a, y >, \quad (3.19) \]
where
\[ \Delta \Pi^k = \Pi^k(\tau) - \Pi^k(0) = \int_0^\tau d\Pi^k ds. \quad (3.20) \]

This evolution effectively defines the map of the local field dynamical variables $\xi^i$ in CP(N) to the dynamically shifted states $|\Psi + \Delta \Psi >$ in original Hilbert space just along a geodesic. The mapping of this evolution back to the full shifted set $\{\Psi^m\}$ like (3.15), by which we can identify the spacetime properties of the solution of a new nonlinear field equation, constitutes a continuous renormalization of the vacuum component $\Psi^0$ which maintains the identity of the physical ground (vacuum) state. In order to fulfill it one should use the local coordinates $\{\Pi^i\}$. Then one has
\[ |\delta v(y) > = - \Gamma_{kl}^i(0) \xi^l(0) \Delta \Pi^m \frac{\partial \Psi^a}{\partial \Pi^i} |a, y > = A_m(y) \Delta \Pi^m = \frac{\partial U}{\partial \Pi^m} \Delta \Pi^m \frac{\partial \Psi^a}{\partial \Pi^i} |a, y >, \]
\[ \Delta \Pi^m = \int_0^\tau d\Pi^m ds. \quad (3.21) \]

Here was introduced the matrix $S_a^b : u^a = S_a^b \Psi^b$ which obeys the equation
\[ \frac{\partial S_a^b}{\partial \Pi^m} \Psi^b + \frac{\partial \Psi^b}{\partial \Pi^m} S_a^b = \Gamma_{km}^i(0) \xi^l(0) \frac{\partial \Psi^a}{\partial \Pi^i}, \quad (3.22) \]
in order to write locally linear equation for the shifted rate
\[ v^a + \delta v^a = - \frac{i}{\hbar} (P_a^b + S_a^b) \Psi^b = - \frac{i}{\hbar} (P_a^b \Psi^b + u^a). \quad (3.23) \]

The connection between $\Phi^i_\sigma(\Pi, P)$ and $\xi^i$ is simply $\frac{d\Phi^i_\sigma}{d\tau} = \xi^i = \Phi^i_\sigma(\Pi, P) \omega^\sigma = \Phi^i_\sigma(\Pi, P) \frac{d\xi^i}{d\tau}$. 11
5. The “direct” comparison of the old and new rates of the changing of state vectors is possible only in the original Hilbert space by the compensation of the geodesic shift with the help of rotations of the functional frame \( \{ |a, y > \} \). Therefore one has

\[
|v'(y) >= |v(y) + dv(y) >= -\frac{i}{\hbar}[\hat{P} + d\hat{P}]|\Psi(y) >
\]

then

\[
-|\delta v(y) > = |dv(y) > = |v(y) + dv(y) > - |v(y) > = -\frac{i}{\hbar}d\hat{P}|\Psi(y) > .
\]

That is it is shown in our original Hilbert space \( \mathcal{H} \) the term \( |dv(y) > \) arises as an additional rate of a change of the state vector \( |\Psi > \)

\[
|dv(y) > = -\frac{i}{\hbar}d\hat{P}|\Psi(y) > = \frac{\delta U}{\delta \Pi^i} \Delta \Pi^i = -\Gamma^i_{kl}(0)\xi^l(0)\Delta \Pi^k \frac{\partial \Psi^a}{\partial \Pi^i} |a, y > .
\]

Then \( dU = \frac{\delta U}{\delta \Pi^i} d\Pi^i + \frac{\delta U}{\delta \Pi^a} d\Pi^a \) where

\[
A_m(y) = \frac{\delta U}{\delta \Pi^m} = -\hbar \Gamma^i_{km} \xi^k \frac{\partial \Psi^a}{\partial \Pi^i} |a, y > .
\]

may be treated as an “instantaneous” self-interacting potential of the scalar configuration associated with the infinitesimal gauge transformation of the local frame with the coefficients \( (2.3) \).

The spacetime dependence of the distribution of this potential is highly anisotropy relative to directions in the projective Hilbert space. The frequencies of oscillating modes should be extracted from the formula

\[
A_{ij}(y) = \frac{\delta^2 U}{\delta \Pi^i \delta \Pi^j} = -\hbar \{ \Gamma^s_{ik} \xi^k \frac{\partial \Psi^a}{\partial \Pi^i} \delta \Pi^s \frac{\partial \Psi^a}{\partial \Pi^j} + \xi^k \left( \frac{\delta^2 \Psi^a}{\partial \Pi^i \partial \Pi^j} \right) + R_{kj}^{s} \xi^k \frac{\partial \Psi^a}{\partial \Pi^s} \} |a, y > ,
\]

where \( R_{kj}^{s} \) is the Riemannian tensor in \( \text{CP}(N) \). That is I think the geometric origin of the “elasticity” of the quantum state which defines the spectrum of quantum mass is evoked by the counteraction of the curvature of \( \text{CP}(N) \) and the “centrifugal” components of the Christoffel symbol. The analysis of this mechanism will be continued.

Strictly speaking, the problem of hierarchy of mass is not absolutely clear. But we have a some hint that mass (energy) is the threshold of the parametric instability of the “homogeneous” modes of oscillations which support the stability of a system and decay of these modes leads to the collective reconstruction to self-consistent boundary. From the phenomenological point of view this reconstruction is the spacetime motion of extended object. For
the relativistic nonlinear scalar field configuration this parametric instability may be studied in the framework of nonlinear Klein-Gordon equation

\[
\frac{d^2(\Psi + \Delta \Psi)^*}{d\rho^2} + \frac{3}{\rho} \frac{d(\Psi + \Delta \Psi)^*}{d\rho} + Y(\Delta \Psi, \rho) + \alpha^2 (\Psi^* + \Delta \Psi^* + \Psi^* \frac{\partial \Delta \Psi}{\partial \Psi}) = 0,
\]

(3.29)

Then, if our assumption is correct, the behavior of the threshold of the parametric instability on the plane of square of mass–quantum action \((m^2, \frac{S}{\hbar})\) is similar to the behavior of Regge trajectories.

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References

[1] A.Einstein, Ann.Phys., 49 769 (1916).

[2] A.Einstein, Vierteljahrschr.geichtl., Med. 44, Ser.3, 37 (1912).

[3] H.Umezawa and H.Matsumoto, M. Tachiki, *Thermo Field Dynamics and Condensed States*, (North-Holland Publishing Company, Amsterdam-New York-Oxford, 1982).

[4] C.W.Misner, K.S.Thorne, J.A. Wheeler, *Gravitation*, (W.H.Freeman and Company, San Francisco, 1973).

[5] P.Leifer, Superrelativity as a unification of quantum theory and relativity, Preprint quant-ph/9610030.

[6] P.Leifer, Superrelativity as a unification of quantum theory and relativity (II), Preprint gr-qc/9612002.

[7] P.Leifer, Quantum theory Requires Gravity and Superrelativity, Preprint gr-qc/9610043.

[8] P.Leifer, Why we can not see the curvature of the quantum state space? Preprint gr-qc/9701006.

[9] P.Leifer, Nonliner modification of quantum mechanics. Preprint hep-th/9702160.

[10] P.Leifer, Found.Phys. 27, (2) 261 (1997).
[11] P. Leifer, The Nonlinear Quantum Gauge Theory– Superrelativity, Preprint [gr-qc/9704054].

[12] P. Dirac, Proc. Roy. Soc. A268 57 (1962).

[13] K. R. W. Jones, Aust. J. Phys. 48, 1055 (1995).

[14] K. R. W. Jones, Mod. Phys. Lett. A 10, (8) 657 (1995).

[15] A. O. Barut, Foundations of Self-Field Quantum Electrodynamics, in New Frontiers in Quantum Electrodynamics and Quantum Optics, Edited by A. O. Barut, Plenum Press: New York (1990).

[16] H.-D. Doebner and G. A. Goldin, Phys. Rev. A 54, (5) 3764 (1996).

[17] P. Dirac, Proc. Roy. Soc. A, 114, 243 (1927).

[18] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol. II, (Interscience Publishers, New York-London-Sydney, 1969).