Conformal biderivations of loop $W(a, b)$ Lie conformal algebra

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Abstract We study conformal biderivations of a Lie conformal algebra. First, we give the definition of a conformal biderivation. Next, we determine the conformal biderivations of loop $W(a, b)$ Lie conformal algebra, loop Virasoro Lie conformal algebra, and Virasoro Lie conformal algebra. Especially, all conformal biderivations on Virasoro Lie conformal algebra are inner conformal biderivations.

Keywords Lie conformal algebras, conformal biderivations, Virasoro Lie conformal algebra, loop Virasoro Lie conformal algebra, loop $W(a, b)$ Lie conformal algebra

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1 Introduction

The notion of Lie conformal algebras was introduced by Kac [6] as a formal language describing the singular part of the operator product expansion in conformal field theory. It is useful to research infinite dimensional Lie algebras satisfying the locality property. The structure theory and representation theory of some Lie conformal algebras have been extensively studied in [1,6].

In recent years, biderivations have been aroused many scholars’ great interests. Brešar [2,3] showed that all biderivations on commutative prime rings are inner biderivations and determined the biderivations of semiprime rings. Brešar and Zhao [4] proved all skew-symmetric biderivations on a perfect and centerless Lie algebra are inner biderivations. References [5,8–12] gave biderivations of specific examples of Lie algebras.

The main object investigated in this paper is the loop $W(a, b)$ Lie conformal algebra, denoted by $\text{CLW}(a, b)$, which is a free $\mathbb{C}[\partial]$-module...
CLW\((a, b)\) = \(\bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]L_i \oplus \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]G_i\)

with a \(\mathbb{C}[\partial]\)-basis \(\{L_i, G_i \mid i \in \mathbb{Z}\}\) satisfying the following \(\lambda\)-brackets:

\[
\begin{align*}
[L_i \lambda L_j] & = (\partial + 2\lambda)L_{i+j}, \\
[L_i \lambda G_j] & = (\partial + (1-b)\lambda)G_{i+j}, \\
[G_i \lambda L_j] & = -(b\partial + (b-1)\lambda)G_{i+j}, \\
[G_i \lambda G_j] & = 0.
\end{align*}
\]

The relations between \(W(a, b)\) and \(CLW(a, b)\) can be found in [7].

This paper is organized as follows. First, we give the definitions of conformal biderivations and inner conformal biderivations on a Lie conformal algebra. Next, we determine the conformal biderivations of loop \(W(a, b)\) Lie conformal algebra, loop Virasoro Lie conformal algebra, and Virasoro Lie conformal algebra. Especially, all conformal biderivations on Virasoro Lie conformal algebra are inner conformal biderivations.

Throughout this paper, all vector spaces, linear maps, and tensor products are over the complex field \(\mathbb{C}\). In addition to the standard notations \(\mathbb{Z}\) and \(\mathbb{R}\), we use \(\mathbb{Z}_{\geq 0}\) to denote the set of nonnegative integers.

## 2 Conformal biderivations of a Lie conformal algebra

The following notion was due to [1].

**Definition 1** A Lie conformal algebra is a \(\mathbb{C}[\partial]\)-module \(\mathcal{R}\) endowed with a bilinear map

\[
\mathcal{R} \times \mathcal{R} \to \mathcal{R}[\lambda],
\]

\[
(x, y) \mapsto [x \lambda y],
\]

called the \(\lambda\)-bracket, satisfying the following axioms \((z \in \mathcal{R})\):

(i) conformal sesquilinearity: \([(\partial x) \lambda y] = -\lambda[x \lambda y];

(ii) skew-symmetry: \([x \lambda y] = -[y - \partial - \lambda x];

(iii) Jacobi identity: \([x \lambda [y \mu z]] = [[x \lambda y] \lambda + \mu z] + [y \mu [x \lambda z]].

**Example 2** [6] *Virasoro Lie conformal algebra* \(\text{Vir}\) is a free \(\mathbb{C}[\partial]\)-module \(\text{Vir} = \mathbb{C}[\partial]L\), satisfying the following \(\lambda\)-bracket:

\[
[L \lambda L] = (\partial + 2\lambda)L.
\]

**Example 3** [13] *Loop Virasoro Lie conformal algebra* \(\text{CW}\) is a free \(\mathbb{C}[\partial]\)-module \(\text{CW} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]L_i\) satisfying the following \(\lambda\)-brackets:

\[
[L_i \lambda L_j] = (\partial + 2\lambda)L_{i+j}.
\]

**Example 4** [7] *Loop \(W(a, b)\) Lie conformal algebra* \(\text{CLW}(a, b)\) is a free \(\mathbb{C}[\partial]\)-module \(\text{CLW}(a, b)\) defined as in (1), satisfying the \(\lambda\)-brackets in (2).
Definition 5 [6] Let $U$ and $V$ be two $\mathbb{C}[\partial]$-modules. A *conformal linear map* from $U$ to $V$ is a $\mathbb{C}$-linear map $f: U \to V[\lambda]$, denoted by $f_\lambda: U \to V[\lambda]$, such that

$$f_\lambda(\partial x) = (\partial + \lambda)f_\lambda(x), \quad x \in U.$$ 

Moreover, let $W$ also be a $\mathbb{C}[\partial]$-module. A *conformal bilinear map* from $U\times V$ to $W$ is a $\mathbb{C}$-bilinear map $f: U\times V \to W[\lambda]$, denoted by $f_\lambda: U\times V \to W[\lambda]$, such that

$$f_\lambda(\partial x, y) = -\lambda f_\lambda(x, y), \quad f_\lambda(x, \partial y) = (\partial + \lambda)f_\lambda(x, y), \quad x \in U, y \in V.$$ 

Definition 6 [6] Let $\mathcal{R}$ be a Lie conformal algebra. A conformal linear map $d_\lambda: \mathcal{R} \to \mathcal{R}[\lambda]$ is a conformal derivation, if for any $x, y \in \mathcal{R}$, it holds that

$$d_\lambda([x_\mu y]) = [(d_\lambda(x))_{\lambda+\mu} y] + [x_\mu (d_\lambda(y))].$$ 

Definition 7 Let $\mathcal{R}$ be a Lie conformal algebra. We call a conformal bilinear map $\phi_\lambda: \mathcal{R}\times \mathcal{R} \to \mathcal{R}[\lambda]$ *skew-symmetric*, if it satisfies

$$\phi_\lambda(x, y) = -\phi_{-\partial-\lambda}(y, x), \quad x, y \in \mathcal{R}.$$ 

Definition 8 Let $\mathcal{R}$ be a Lie conformal algebra. We call a conformal bilinear map $\phi_\lambda: \mathcal{R}\times \mathcal{R} \to \mathcal{R}[\lambda]$ a *conformal biderivation of \mathcal{R},* if it satisfies the following equations:

$$\begin{align*}
\phi_\lambda(x, y) &= -\phi_{-\partial-\lambda}(y, x), \\
\phi_\lambda(x, [y_\mu z]) &= \left((\phi_\lambda(x, y))_{\lambda+\mu} z\right) + [y_\mu \phi_\lambda(x, z)], \quad x, y, z \in \mathcal{R}. \tag{3}
\end{align*}$$

Remark 9 If $\phi_\lambda$ is a conformal biderivation of Lie conformal algebra $\mathcal{R}$, then $\phi_\lambda(x, \cdot)$ is a conformal derivation obviously.

Lemma 10 Let $\phi_\lambda$ be a skew-symmetric conformal bilinear map of Lie conformal algebra $\mathcal{R}$. Then (3) is equivalent to

$$\phi_{\lambda+\mu}([x_\mu y], z) = [x_\mu \phi_{\lambda}(y, z)] - [y_\lambda \phi_{\mu}(x, z)], \quad x, y, z \in \mathcal{R}. \tag{4}$$

Proof If (4) holds, changing the left-hand side of the equation by skew-symmetry, we have

$$-\phi_{-\partial-\lambda-\mu}(z, [x_\mu y]) = [x_\mu \phi_\lambda(y, z)] - [y_\lambda \phi_\mu(x, z)], \quad x, y, z \in \mathcal{R}.$$ 

Replacing $z, x, y$ by $x, y, z$, respectively, we obtain

$$-\phi_{-\partial-\lambda-\mu}(x, [y_\mu z]) = [y_\mu \phi_\lambda(z, x)] - [z_\lambda \phi_\mu(y, x)].$$ 

Changing the right-hand side of the above equation by skew-symmetry, we get

$$-\phi_{-\partial-\lambda-\mu}(x, [y_\mu z]) = -[y_\mu \phi_{-\partial-\lambda}(x, z)] + [z_\lambda \phi_{-\partial-\mu}(x, y)].$$
Replacing $\lambda$ and $\mu$ by $-\partial - \lambda' - \mu'$ and $\mu'$, respectively, by conformal sesqui-linearity and skew-symmetry, we obtain

$$
\phi_{\lambda'}(x, [y_{\mu'} z]) = [(\phi_{\lambda'}(x, y))_{\lambda + \mu} z] + [y_{\mu'} \phi_{\lambda'}(x, z)].
$$

The reverse conclusion follows similarly. □

**Definition 11** Denote by $\text{BDer}(\mathcal{R})$ the set of all conformal biderivations of $\mathcal{R}$.

**Lemma 12** If the map $\phi^\lambda_t : \mathcal{R} \times \mathcal{R} \to \mathcal{R}[\lambda]$ is defined by

$$
\phi^\lambda_t(x, y) = t[x\lambda y], \quad \forall \ x, y \in \mathcal{R},
$$

where $t \in \mathbb{C}$, then $\phi^t$ is a conformal biderivation of $\mathcal{R}$. We call this class conformal biderivations inner conformal biderivations.

**Proof** It is straightforward by the definition of conformal biderivations. □

**Lemma 13** Let $\phi_{\lambda}$ be a conformal biderivation of Lie conformal algebra $\mathcal{R}$. Then

$$
[(\phi_{\mu}(x, y))_{\mu+\gamma}[u\lambda v]] = [x\mu y]_{\mu+\gamma} \phi_{\lambda}(u, v), \quad \forall \ x, y, u, v \in \mathcal{R}.
$$

(5)

**Proof** On the one hand, for $x, y, u, v \in \mathcal{R}$, using (4), we have

$$
\phi_{\lambda+\mu}([x\mu u], [y\gamma v]) = [x\mu \phi_{\lambda}(u, [y\gamma v])] - [u\lambda \phi_{\mu}(x, [y\gamma v])] \\
= ([x\mu ([\phi_{\lambda}(u, y)\lambda+\gamma v]) + [x\mu y\phi_{\lambda}(u, v)]) \\
- ([u\lambda (\phi_{\mu}(x, y))_{\mu+\gamma v}] + [u\lambda [y\gamma \phi_{\mu}(x, v)].
$$

On the other hand, using (3), we have

$$
\phi_{\lambda+\mu}([x\mu u], [y\gamma v]) = ((\phi_{\lambda+\mu}([x\mu u], y\gamma v])_{\lambda+\mu+\gamma v} + [y\gamma \phi_{\lambda+\mu}([x\mu u], v)] \\
= ([x\mu \phi_{\lambda}(u, y)]_{\lambda+\mu+\gamma v} - [u\lambda \phi_{\mu}(x, [y\gamma v])] \\
+ ([y\gamma [x\mu \phi_{\lambda}(u, v)] - [y\gamma [u\lambda \phi_{\mu}(x, v)])]
$$

Comparing two sides of the above equations, the right-hand sides of the two equations are equal, that is,

$$
([x\mu ([\phi_{\lambda}(u, y)\lambda+\gamma v]) - [x\mu \phi_{\lambda}(u, y)]_{\lambda+\mu+\gamma v}) \\
+ ([x\mu [y\gamma \phi_{\lambda}(u, v)] - [y\gamma [x\mu \phi_{\lambda}(u, v)])
$$

$$
= ([u\lambda (\phi_{\mu}(x, y))_{\mu+\gamma v}] - [u\lambda \phi_{\mu}(x, y)]_{\lambda+\mu+\gamma v}) \\
+ ([y\gamma [u\lambda \phi_{\mu}(x, v)] - [y\gamma [u\lambda \phi_{\mu}(x, v)])
$$

Using the Jacobi identity of Lie conformal algebra, we obtain

$$
[(\phi_{\lambda}(u, y))_{\lambda+\gamma} [x\mu v]] + [x\mu y]_{\mu+\gamma} \phi_{\lambda}(u, v) \\
= ([\phi_{\mu}(x, y))_{\mu+\gamma} [u\lambda v]] + [u\lambda y]_{\lambda+\gamma} \phi_{\lambda}(x, v),
$$

(5)
and it is equivalent to

\[
([\phi_\mu(x, y)]_{\mu+\gamma}[u_\lambda v]) - [[x_\mu y]_{\mu+\gamma}\phi_\lambda(u, v)] = ([\phi_\lambda(u, y)]_{\lambda+\gamma}[x_\mu v]) - [[u_\lambda y]_{\lambda+\gamma}\phi_\mu(x, v)].
\]  

(6)

Now, let

\[
\Phi_{\lambda,\mu,\gamma}(x, y; u, v) = ([\phi_\mu(x, y)]_{\mu+\gamma}[u_\lambda v]) - [[x_\mu y]_{\mu+\gamma}\phi_\lambda(u, v)].
\]

Then it follows from (6) that

\[
\Phi_{\lambda,\mu,\gamma}(x, y; u, v) = \Phi_{\mu,\lambda,\gamma}(u, y; x, v).
\]

Obviously, by the skew-symmetry and conformal sesquilinearity, we get

\[
\Phi_{\lambda,\mu,\gamma}(x, y; u, v) = \Phi_{\mu,\lambda,\gamma}(y, x; u, v).
\]

For one thing, we have

\[
\Phi_{\lambda,\mu,\gamma}(x, y; u, v) = -\Phi_{\lambda,\gamma,\mu}(y, x; u, v) = -\Phi_{\gamma,\lambda,\mu}(u, x; y, v) = \Phi_{\gamma,\mu,\lambda}(x, u; y, v).
\]

For another thing, we also have

\[
\Phi_{\lambda,\mu,\gamma}(x, y; u, v) = \Phi_{\mu,\lambda,\gamma}(y, x; u, v) = -\Phi_{\mu,\gamma,\lambda}(y, u; x, v) = -\Phi_{\gamma,\mu,\lambda}(x, u; y, v).
\]

Then

\[
\Phi_{\lambda,\mu,\gamma}(x, y; u, v) = 0,
\]

which implies (5).

\[\square\]

**Remark 14** Let \(\phi_\lambda\) be a conformal biderivation of Lie conformal algebra \(\mathcal{R}\). If \([x_\lambda y] = 0\), then

\[\phi_\lambda(x, y) \in Z([\mathcal{R}_X\mathcal{R}]), \quad x, y \in \mathcal{R},\]

where

\[Z([\mathcal{R}_X\mathcal{R}]) = \{u \in \mathcal{R} \mid [u_\lambda [v_\lambda w]] = 0, \forall v, w \in \mathcal{R}\}.
\]

3 Conformal biderivations of Lie conformal algebras \(CW, Vir,\) and \(CLW(a, b)\)

**Theorem 15** Every conformal biderivation \(\phi_\lambda\) on the loop Virasoro Lie conformal algebra \(CW\) has the forms

\[
\phi_\lambda(L_i, L_j) = (\partial + 2\lambda) \sum_{k \in \mathbb{Z}} a_{k-i-j} L_k
\]

(7)
for some complex numbers set \( \{a_k, k \in \mathbb{Z}\} \), where \( \sum_{k \in \mathbb{Z}} \) is a finite sum.

**Proof** Suppose that
\[
\phi_\lambda(L_i, L_j) = \sum_{k \in \mathbb{Z}} f_{i,j}^k(\partial, \lambda) L_k
\]
is a conformal biderivation of the loop Virasoro Lie conformal algebra CW. Then for any \( x, y, u, v \in \mathcal{R} \), (5) holds.

Taking
\[
x = u = L_i, \quad y = v = L_j,
\]
in (5), we get
\[
[\phi_\mu(L_i, L_j)]_{\mu+\gamma}[L_i \lambda L_j] = [[L_i \mu L_j]_{\mu+\gamma} \phi_\lambda(L_i, L_j)],
\]
that is,
\[
(\partial + \mu + \gamma + 2\lambda) \sum_{k \in \mathbb{Z}} f_{i,j}^k(-\mu - \gamma, \mu) L_{i+j+k} = (\mu - \gamma) \sum_{k \in \mathbb{Z}} f_{i,j}^k(\partial + \mu + \gamma, \lambda) L_{i+j+k}.
\]

Obviously, we have
\[
(\partial + \mu + \gamma + 2\lambda) f_{i,j}^k(-\mu - \gamma, \mu) = (\mu - \gamma) f_{i,j}^k(\partial + \mu + \gamma, \lambda), \quad \forall i, j, k \in \mathbb{Z}.
\]

Fixing \( i, j, k \in \mathbb{Z} \), since the highest power of \( \partial \) on the left-hand side of (9) is probably 1, we have
\[
f_{i,j}^k(\partial, \lambda) = a_0(\lambda) + a_1(\lambda) \partial,
\]
and substituting it into (9), we get
\[
(\partial + \mu + \gamma + 2\lambda)(a_0(\mu) + a_1(\mu)(-\mu - \gamma)) = (\mu - \gamma)(a_0(\lambda) + a_1(\lambda)(\partial + \mu + \gamma)).
\]

Considering the coefficients of \( \partial \), we have
\[
a_0(\mu) + a_1(\mu)(-\mu - \gamma) = a_1(\lambda)(\mu - \gamma).
\]

By (10), we can assume \( a_1(\lambda) = a_1 \in \mathbb{C} \). Then
\[
a_0(\lambda) = 2\lambda a_1, \quad f_{i,j}^k(\partial, \lambda) = a_1(\partial + 2\lambda).
\]

So
\[
\phi_\lambda(L_i, L_j) = (\partial + 2\lambda) \sum_{k \in \mathbb{Z}} f_{i,j}^k L_k, \quad f_{i,j}^k \in \mathbb{C}.
\]

Taking
\[
x = L_i, \quad y = L_j, \quad u = L_k, \quad v = L_l,
\]
in (5), we can get
\[
f_{i,j}^m = f_{k,l}^{m+k+l-i-j},
\]
and especially,
\[ f^m_{0,0} = f^{m+k+l}_{k,l}, \quad \forall m \in \mathbb{Z}. \]

Set \( f^k_{0,0} := a_k \), i.e.,
\[ \phi_\mu(L_0, L_0) = (\partial + 2\lambda) \sum_{k \in \mathbb{Z}} a_k L_k. \]

Then (7) holds.

Conversely, it is easy to prove that any skew-symmetric conformal bilinear map satisfying (7) is a conformal biderivation. \( \square \)

**Corollary 16** Every conformal biderivation on the Virasoro Lie conformal algebra is an inner conformal biderivation.

**Proof** By the proof of Theorem 15, we can get the conclusion immediately. \( \square \)

**Theorem 17** Let \( R \) be the Lie conformal algebra \( \text{CLW}(a,b) \). Then a skew-symmetric conformal bilinear map \( \phi_\lambda \) of \( R \) is a conformal biderivation if and only if \( \phi_\lambda \) satisfies the conditions

\[
\begin{align*}
\phi_\lambda(L_i, L_j) &= (\partial + 2\lambda) \left( \sum_{k \in \mathbb{Z}} a_{k-i-j} L_k + \delta_{b+1,0} \sum_{k \in \mathbb{Z}} b_{k-i-j} G_k \right), \\
\phi_\lambda(G_i, G_j) &= 0, \\
\phi_\lambda(L_i, G_j) &= (\partial + (1 - b)\lambda) \sum_{k \in \mathbb{Z}} a_{k-i-j} G_k, \\
\phi_\lambda(G_i, L_j) &= (\partial + (1 - b)\lambda) \sum_{k \in \mathbb{Z}} a_{k-i-j} L_k,
\end{align*}
\]

for complex numbers sets \( \{a_k, k \in \mathbb{Z}\} \) and \( \{b_k, k \in \mathbb{Z}\} \), where \( \sum_{k \in \mathbb{Z}} \) is a finite sum.

**Proof** Let \( \phi_\lambda \) be a conformal biderivation of Lie conformal algebra \( R \). By Remark 14, \( \phi_\lambda(G_i, G_j) = 0 \), obviously. Suppose

\[
\begin{align*}
\phi_\lambda(L_i, L_j) &= \sum_{k \in \mathbb{Z}} f^k_{i,j}(\partial, \lambda) L_k + \sum_{k \in \mathbb{Z}} g^k_{i,j}(\partial, \lambda) G_k, \\
\phi_\lambda(L_i, G_j) &= \sum_{k \in \mathbb{Z}} s^k_{i,j}(\partial, \lambda) L_k + \sum_{k \in \mathbb{Z}} d^k_{i,j}(\partial, \lambda) G_k.
\end{align*}
\]

First, taking \( x = u = L_i \), \( y = L_j \), \( v = G_j \), in (5), we get

\[
[(\phi_\mu(L_i, L_j))_{\mu+\gamma}[L_i L_j]] = [[L_i L_j]_{\mu+\gamma} \phi_\lambda(L_i, G_j)]
\]

if and only if

\[
\left[ \sum_{k \in \mathbb{Z}} f^k_{i,j}(\partial, \mu) L_k + \sum_{k \in \mathbb{Z}} g^k_{i,j}(\partial, \mu) G_k \right]_{\mu+\gamma} (\partial + (1 - b)\lambda) G_{i+j}
\]

\[
= \left[ (\partial + 2\mu) L_{i+j} \right]_{\mu+\gamma} \left( \sum_{k \in \mathbb{Z}} s^k_{i,j}(\partial, \lambda) L_k + \sum_{k \in \mathbb{Z}} d^k_{i,j}(\partial, \lambda) G_k \right),
\]
if and only if
\[
(\partial + \mu + \gamma + (1 - b)\lambda)(\partial + (1 - b)(\mu + \gamma)) \sum_{k \in \mathbb{Z}} f_{i,j}^k (-\mu - \gamma, \mu) G_{i+j+k} \\
= (\mu - \gamma)(\partial + 2\mu + 2\gamma) \sum_{k \in \mathbb{Z}} s_{i,j}^k (\partial + \mu + \gamma, \lambda) L_{i+j+k} \\
+ (\mu - \gamma)(\partial + (1 - b)(\mu + \gamma)) \sum_{k \in \mathbb{Z}} d_{i,j}^k (\partial + \mu + \gamma, \lambda) G_{i+j+k}.
\]

Obviously, we have \( s_{i,j}^k (\partial, \lambda) = 0 \) and
\[
(\partial + \mu + \gamma + (1 - b)\lambda) f_{i,j}^k (-\mu - \gamma, \mu) = (\mu - \gamma) a_{i,j}^k (\partial + \mu + \gamma, \lambda), \quad \forall i, j, k \in \mathbb{Z}.
\]

Since the highest power of \( \partial \) on the left-hand side of the above equation is probably 1, we have
\[
d_{i,j}^k (\partial, \lambda) = a_0(\lambda) + a_1(\lambda) \partial,
\]
and substituting it into the above equation, we get
\[
(\partial + \mu + \gamma + (1 - b)\lambda) f_{i,j}^k (-\mu - \gamma, \mu) = (\mu - \gamma)(a_0(\lambda) + a_1(\lambda)(\partial + \mu + \gamma)).
\]

Considering the coefficients of \( \gamma \) again, we get
\[
f_{i,j}^k (\partial, \lambda) = b_0(\lambda) + b_1(\lambda) \partial,
\]
and substituting it into the above equation, we get
\[
(\partial + \mu + \gamma + (1 - b)\lambda)(b_0(\mu) + b_1(\mu)(-\mu - \gamma)) \\
= (\mu - \gamma)(a_0(\lambda) + a_1(\lambda)(\partial + \mu + \gamma)).
\]

Observing the coefficients of \( \gamma^2 \), we have
\[
b_1(\mu) = a_1(\lambda) = a_1 \in \mathbb{C},
\]
and substituting it into the above equation, we get
\[
b_0(\mu) = 2a_1\mu, \quad a_0(\mu) = a_1(1 - b)\mu.
\]

So we can set
\[
\phi_\lambda(L_i, L_j) = (\partial + 2\lambda) \sum_{k \in \mathbb{Z}} d_{i,j}^k L_k + \sum_{k \in \mathbb{Z}} g_{i,j}^k (\partial, \lambda) G_k,
\]
\[
\phi_\lambda(G_i, G_j) = 0, \quad \phi_\lambda(L_i, G_j) = (\partial + (1 - b)\lambda) \sum_{k \in \mathbb{Z}} d_{i,j}^k G_k,
\]
where \( d_{i,j}^k \in \mathbb{C} \).
Second, taking (8) in (5), we get
\[
[(\phi_\mu(L_i, L_j))_{\mu+\gamma}[L_i\lambda L_j]] = [[L_{i\mu} L_{j\mu+\gamma}]\phi_\lambda(L_i, L_j)]
\]
if and only if
\[
\left[\left((\partial + 2\mu) \sum_{k \in \mathbb{Z}} d^{k}_{i,j} L_k + \sum_{k \in \mathbb{Z}} g^{k}_{i,j}(\partial, \mu)G_k\right)_{\mu+\gamma} (\partial + 2\lambda)L_{i+j}\right]
\]
\[
= \left[\left((\partial + 2\mu)L_{i+j})_{\mu+\gamma}\left((\partial + 2\lambda) \sum_{k \in \mathbb{Z}} d^{k}_{i,j} L_k + \sum_{k \in \mathbb{Z}} g^{k}_{i,j}(\partial, \lambda)G_k\right)\right],
\]
if and only if
\[
(\mu - \gamma)(\partial + \mu + \gamma + 2\lambda)(\partial + 2\mu + 2\gamma) \sum_{k \in \mathbb{Z}} d^{k}_{i,j} L_{i+j+k}
- (\partial + \mu + \gamma + 2\lambda)(b\partial + (b - 1)(\mu + \gamma)) \sum_{k \in \mathbb{Z}} g^{k}_{i,j}(-\mu - \gamma, \mu)G_{i+j+k}
= (\mu - \gamma)(\partial + \mu + \gamma + 2\lambda)(\partial + 2\mu + 2\gamma) \sum_{k \in \mathbb{Z}} d^{k}_{i,j} L_{i+j+k}
+ (\mu - \gamma)(\partial + (1-b)(\mu + \gamma)) \sum_{k \in \mathbb{Z}} g^{k}_{i,j}(\partial + \mu + \gamma, \lambda)G_{i+j+k},
\]
if and only if
\[
- (\partial + \mu + \gamma + 2\lambda)(b\partial + (b - 1)(\mu + \gamma)) \sum_{k \in \mathbb{Z}} g^{k}_{i,j}(-\mu - \gamma, \mu)G_{i+j+k},
= (\mu - \gamma)(\partial + (1-b)(\mu + \gamma)) \sum_{k \in \mathbb{Z}} g^{k}_{i,j}(\partial + \mu + \gamma, \lambda)G_{i+j+k},
\]
and it is easy to see that if and only if
\[
- (\partial + \mu + \gamma + 2\lambda)(b\partial + (b - 1)(\mu + \gamma))g^{k}_{i,j}(-\mu - \gamma, \mu)
= (\mu - \gamma)(\partial + (1-b)(\mu + \gamma))g^{k}_{i,j}(\partial + \mu + \gamma, \lambda), \quad \forall i, j, k \in \mathbb{Z}.
\]
Since the highest power of \(\partial\) on the left-hand side of the above equation is probably 2, we have
\[
g^{k}_{i,j}(\partial, \lambda) = c_0(\lambda) + c_1(\lambda)\partial,
\]
and substituting it into the above equation, we get
\[
- (\partial + \mu + \gamma + 2\lambda)(b\partial + (b - 1)(\mu + \gamma))(c_0(\mu) + c_1(\mu)(-\mu - \gamma))
= (\mu - \gamma)(\partial + (1-b)(\mu + \gamma))(c_0(\lambda) + c_1(\lambda)(\partial + \mu + \gamma)). \quad (13)
\]
Considering the coefficients of \(\gamma^3\) in (13), we have
\[
(b - 1)(c_1(\mu) - c_1(\lambda)) = 0. \quad (14)
\]
(i) For $b = 1$, substituting it in (13), we get
\[-(\partial + \mu + \gamma + 2\lambda)(c_0(\mu) + c_1(\mu)(-\mu - \gamma))
= (\mu - \gamma)(c_0(\lambda) + c_1(\lambda)(\partial + \mu + \gamma)).\]
Observing the coefficients of $\partial$ in the above equation, we have $c_0(\lambda) = c_1(\lambda) = 0$.

(ii) For $b \neq 1$, by (14), we have $c_1(\lambda) = c_1 \in \mathbb{C}$, and substituting it in (13), we get
\[-(\partial + \mu + \gamma + 2\lambda)(b\partial + (b - 1)(\mu + \gamma))(c_0(\mu) + c_1(-\mu - \gamma))
= (\mu - \gamma)(\partial + (1 - b)(\mu + \gamma))(c_0(\lambda) + c_1(\partial + \mu + \gamma)).\]
Let us just think about the monomials only containing $\mu$, that is,
\[(b - 1)(c_0(\mu) - c_1\mu) = (b - 1)c_1\mu.\]
So we obtain $c_0(\mu) = 2c_1\mu$, and substituting it in (13) again, we get
\[-c_1(b\partial + (b - 1)(\mu + \gamma)) = c_1(\partial + (1 - b)(\mu + \gamma)),\]
that is, $c_1(b + 1) = 0$. Especially, if $b \neq -1$, then $c_1 = 0$.

From what has been discussed above,
(a) if $b \neq -1$, then we have
\[
\phi(\lambda)(L_i, L_j) = (\partial + 2\lambda)\sum_{k \in \mathbb{Z}} q^k_{i,j} L_k;
\]
\[
\phi(\lambda)(G_i, G_j) = 0, \quad \phi(\lambda)(L_i, G_j) = (\partial + (1 - b)\lambda)\sum_{k \in \mathbb{Z}} q^k_{i,j} G_k;
\]
(b) if $b = -1$, then we have
\[
\phi(\lambda)(L_i, L_j) = (\partial + 2\lambda)\left(\sum_{k \in \mathbb{Z}} q^k_{i,j} L_k + \sum_{k \in \mathbb{Z}} g^k_{i,j} G_k\right),
\]
\[
\phi(\lambda)(G_i, G_j) = 0, \quad \phi(\lambda)(L_i, G_j) = (\partial + (1 - b)\lambda)\sum_{k \in \mathbb{Z}} q^k_{i,j} G_k.
\]
Finally, taking (11) in (5), we can get
\[
d_{i,j}^m = d_{k,l}^{m+k+l-i-j}, \quad g_{i,j}^m = g_{k,l}^{m+k+l-i-j}, \quad \forall m \in \mathbb{Z}.
\]
Especially,
\[
d_{0,0}^m = d_{m,n}^{m+k+l}, \quad g_{0,0}^m = g_{k,l}^{m+k+l}.
\]
Set
\[
d_{0,0}^k := a_k, \quad g_{0,0}^k := b_k.
\]
i.e.,
\[
\phi_{\lambda}(L_0, L_0) = (\partial + 2\lambda) \left( \sum_{k \in \mathbb{Z}} a_k L_k + \delta_{b+1,0} \sum_{k \in \mathbb{Z}} b_k G_k \right).
\]

Then (12) holds.

Conversely, it is easy to prove that any skew-symmetric conformal bilinear map satisfying (12) is a conformal biderivation. □

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