Regulators in the Arithmetic of Function Fields
Quentin Gazda

Abstract
As a natural sequel for the study of $A$-motivic cohomology, initiated in [Gaz22], we develop a notion of regulator for rigid analytically trivial mixed Anderson $A$-motives. In accordance with the conjectural number field picture, we define it as the morphism at the level of extension modules induced by the exactness of the Hodge-Pink realization functor. The purpose of this text is twofold: we first prove a finiteness result for $A$-motivic cohomology and, under a weight assumption, we then show that the source and the target of regulators have the same dimension. It came as a surprise to the author that the image of this regulator might not have full rank, preventing the analogue of a renowned conjecture of Beilinson to hold in our setting.

Contents
1 Introduction 1
2 Hodge-Pink structures and their extensions 6
  2.1 Hodge-Pink structures 6
  2.2 Infinite Frobenii 7
3 Rigid analytically trivial mixed $A$-motives 12
  3.1 Definitions 12
  3.2 The Betti realization functor 14
  3.3 Analytic continuation 15
  3.4 The associated mixed Hodge-Pink structure 15
4 Regulators and finiteness theorems 21
  4.1 Regulators of $A$-motives 21
  4.2 The complex $G$ and the fundamental exact sequence 23
  4.3 Strategy of proofs 24
5 Shtuka models à la Mornev 25
  5.1 $C$-shtuka models 25
  5.2 $C \times C$-shtuka models 26
  5.3 Shtuka models and extensions of mixed Hodge-Pink structures 31
6 Proof of the main theorems 35
  6.1 Cohomological computations 35
  6.2 Proof of Theorems 4.1 and 4.4 39

1 Introduction
Very recently in [ANDTR22], Anglès-Ngo Dac-Tavares-Ribeiro made the tremendous exploit of establishing a class formula for a large family of Anderson $A$-modules. This achievement sits at the top of a long list of work, dating back to Carlitz [Car35] in 1935, and culminating with the breakthroughs of Taelman [Tae12] and V. Lafforgue [Laf09].

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In the classical picture of number fields, class formulas in this level of generality are rather stated in terms of mixed motives in the form of Beilinson’s conjectures. Those are far reaching conjectures on special L-values formulated by Beilinson in the eighties in two celebrated papers [Bei84, Bei86]. The whole picture is rooted over the notion of a Beilinson regulator, conjecturally defined as follows. Let $M$ be a mixed motive over the field of rational numbers $\mathbb{Q}$. Here, "motive" is understood in the wording of Deligne (e.g. [Del89, §1]). Consider the $\mathbb{Q}$-vector space $\text{Ext}^1_{\mathbb{R}}(\mathbb{1}, M)$ of 1-fold extension of the neutral motive $\mathbb{1}$ by $M$ in the category of mixed motives over $\mathbb{Q}$. It is expected that one can define a natural subspace $\text{Ext}^1_{\mathbb{R}}(\mathbb{1}, M)$ consisting of *extensions having everywhere good reduction* (e.g. [Sch91]). The Hodge realization functor $\mathcal{H}+$, from the category of mixed motives and with values in the category of mixed Hodge structures enriched with infinite Frobenii, is expected to be exact and, in this respect, should induce a morphism of the extension spaces:

$$\text{Ext}^1_{\mathbb{R}}(\mathbb{1}, M) \rightarrow \text{Ext}^1_{\mathbb{R}-\text{Hdg}}(\mathbb{1}^+, \mathcal{H}^+(M)). \quad (1.1)$$

Observe that the right-hand side is an $\mathbb{R}$-vector space of finite dimension. The above map is - conjecturally - the *Beilinson’s regulator of $M$*. We denote it $\text{Reg}(M)$.

The following is expected, although far from being proved.

(I) The space $\text{Ext}^1_{\mathbb{R}}(\mathbb{1}, M)$ has finite dimension over $\mathbb{Q}$.

(II) If $M$ is pure of weight $< -2$, then $\text{Reg}(M)$ has dense image.

(III) There is a $\mathbb{Q}$-structure $V(M)$ on the target of $\text{Reg}(M)$, natural in $M$, such that we have $\det(\text{im} \text{ Reg}(M)) = L^*(M, 0) \det V(M)$ as $\mathbb{Q}$-structures on $\det \text{Ext}^1_{\mathbb{R}-\text{Hdg}}(\mathbb{1}^+, \mathcal{H}^+(M))$. Conjectures [I] and [II] are referred to as Beilinson’s conjectures. This text is concerned with the function field analogue of [I] and [II], so we do not further comment on the (conjectural) definition of the special L-value $L^*(M, 0)$ nor on the $\mathbb{Q}$-structure $V(M)$. We rather refer the reader to the survey [Nek94] for a complete account of Beilinson’s conjectures and their history.

The present work grew out as an attempt to bridge these two pictures, and understand how Beilinson’s conjectures could be stated in the language of Anderson A-motives. The study of motivic cohomology for A-motives was initiated in [Gaz22] and this text consists in a natural sequel. Our primary interest is the definition of a *Beilinson’s regulator* in this context, and the study of the counterpart of conjectures [I] and [II]. We hope to make conjecture [III] the subject of a subsequent work.

**Anderson A-motives.** Let $(C, \mathcal{O}_C)$ be a geometrically irreducible smooth projective curve over a finite field $F$ of characteristic $p$, and fix $\infty$ a closed point of $C$. The $\mathbb{F}$-algebra

$$A = H^0(C \setminus \{\infty\}, \mathcal{O}_C)$$

consists of functions of $C$ that are regular away from $\infty$. We denote by $K$ its fraction field. The notion of A-motives dates back to the pioneer work of Anderson [And86], and generalizes prior ideas of Drinfeld [Drd74]. Let us state the definition of an *Anderson A-motives*, leaving details for Section 3. Throughout this text, unlabeled tensor or fiber products are over $F$.

**Definition** [Gaz22]. An *Anderson A-motive* $M$ over $K$ consists of a finite locally-free $A \otimes K$-module $M$ together with an isomorphism of $A \otimes K$-modules

$$\tau_M : (\tau^* M)|_{(\text{Spec } A \otimes K) \setminus V(i)} \xrightarrow{\sim} M|_{(\text{Spec } A \otimes K) \setminus V(i)}.$$

where $\tau : A \otimes K \rightarrow A \otimes K$ is the ring endomorphism acting as the identity on $A$ and as the $|F|$-th power map on $K$, and where $V(i)$ is the effective Cartier divisor on $\text{Spec } A \otimes K$ associated to the locally-free ideal $1 := \ker(A \otimes K \rightarrow K, a \otimes r \mapsto ar)$. 

2
Remark 1.1. In general, $A$-motives do not carry a weight filtration; those that do are called mixed (see [Gaz22 §3]). However, as the main results of this text do not require mixedness - aside from obvious analogy purposes with the number field theory - we voluntarily forget the notion of mixedness. Yet, that of weights of an $A$-motive (see [Gaz22 Def. 3.20]) plays a major role.

A crucial assumption throughout this text however is that of rigid analytic triviality (Subsection 3.2 for details). Given an $A$-motive $\mathcal{M}$ over $K$, we consider the $A$-module

$$M_B := \{ f \in M \otimes_{A \otimes K} C_\infty(A) \mid f = \tau_M(\tau^* f) \}.$$  

It is the Betti realization of $\mathcal{M}$. Here, $C_\infty(A)$ is an affinoid algebra over $C_\infty$, the completion of an algebraic closure of the local field $K_\infty$ of $(C, \mathcal{O}_C)$ at $\infty$ (see [HJ20 §2.3.3], [GM22 §3] or Section 3 below for details). To ensure good properties of the functor $\mathcal{M} \mapsto M_B$ (e.g. exactness) we focus on rigid analytically trivial $A$-motives; i.e. those $\mathcal{M}$ for which $M_B$ generates $M \otimes_{A \otimes K} C_\infty(A)$ over $C_\infty(A)$ (Definition 3.8). The category of rigid analytically trivial $A$-motives over $K$, denoted by $AMot^{rat}_A$ along this text, is an exact $A$-linear category which shall play the role of the classical category of mixed motives over $\mathbb{Q}$ and the functor $\mathcal{M} \mapsto M_B$ that of the singular realization. The reader is invited to consult [BGHP20] for surveys on this analogy.

Let $\mathcal{M}$ be an object of $AMot^{rat}_A$. The $A$-module $M_B$ carries a natural action of the Galois group $G_\infty := \text{Gal}(K_\infty^s/K_\infty)$, $K_\infty^s$ being the separable closure of $K_\infty$ inside $C_\infty$. We have (see Proposition 3.11 and Corollary 3.17):

**Proposition.** The profinite group $G_\infty$ acts continuously on the discrete $A$-module $M_B$.

Further, the functor $\mathcal{M} \mapsto M_B$, having for source the category $AMot^{rat}_A$ and for target the category of continuous $A$-linear representations of $G_\infty$, is exact.

The action of $G_\infty$ on $M_B$ should be interpreted as the counter-part of the complex conjugation acting on the Betti realization of classical mixed motives, although here $G_\infty$ is a way more convoluted that its number field avatar $\text{Gal}(\mathbb{C}/\mathbb{R})$.

In [Gaz22], we established the definition of $\text{Ext}_A^1(\mathbb{1}, \mathcal{M})$, a natural $A$-module consisting of integral extensions of the neutral $A$-motive $\mathbb{1}$ by $\mathcal{M}$ in the category $AMot^{rat}_A$. However, there are at least two reasons why this module is not finitely generated in general, preventing the naive analogue of conjecture (1) to hold in our context:

1. The first reason, mentioned in details in [Gaz22 §5], is related to the fact that taking Hodge filtrations is not an exact operation on the full class of exact sequences. This is solved by the notion of regulated extensions, introduced in Definition 5.7 in loc. cit. The sub-$A$-module $\text{Ext}_A^{1, \text{reg}}(\mathbb{1}, \mathcal{M})$ of regulated extensions, however, might still not be finitely generated.

2. The second reason, more subtle, is due to the infinite nature of the absolute Galois group $G_\infty$. We solve this second point by introducing the notion of analytic reduction at $\infty$ defined as follows.

The exactness of the Betti realization functor induces a morphism at the level of extension modules:

$$r_B : \text{Ext}_A^{1, \text{reg}}(\mathbb{1}, \mathcal{M}) \longrightarrow H^1(G_\infty, M_B) \quad (1.2)$$

where the target denotes the continuous Galois cohomology module. We say that an extension $[E]$ in $\text{Ext}_A^{1, \text{reg}}(\mathbb{1}, \mathcal{M})$ has analytic reduction at $\infty$ if it lies in the kernel of $r_B$; equivalently, if $[E]$ splits as representations of $G_\infty$. We denote by $\text{Ext}_A^{1, \text{reg}}(\mathbb{1}, \mathcal{M})$ the kernel of $r_B$, and by $\text{Cl}(\mathcal{M})$ its cokernel. Our first main theorem is the following (repeated from Theorem 4.1).

**Theorem** [4.1]. The $A$-modules $\text{Ext}_A^{1, \text{reg}}(\mathbb{1}, \mathcal{M})$ and $\text{Cl}(\mathcal{M})$ are finitely generated. If in addition the weights of $\mathcal{M}$ are all negative, then $\text{Cl}(\mathcal{M})$ is finite.
Let us throw some comments on Theorem [4.1]\

— The above theorem should be understood as the analogue of conjecture [1.1] for rigid analytically trivial Anderson A-motives. Indeed, all extensions of classical mixed motives are regulated in the obvious sense, and hence [1.2] corresponds classically to the morphism of $\mathbb{Q}$-vector spaces

$$r_B : \text{Ext}^1_{\mathbb{A}^1}(\mathbb{I}, M) \rightarrow H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), M_B)$$

induced by the exactness of the Betti realization (above, $M$ is a mixed motive over $\mathbb{Q}$, $M_B$ its Betti realization). Yet, in the $\mathbb{Q}$-linear category of mixed motives, the right-hand side is zero, which amounts to say that all extensions in $\text{Ext}^1_{\mathbb{A}^1}(\mathbb{I}, M)$ have analytic reduction at $\infty$.

— As a second remark, let us mention that Theorem [4.1] also shows that $\text{Ext}^1_{\mathbb{A}^1}(\mathbb{I}, M)$ is almost never finitely generated. To wit, its size is approximately the same as $H^1(G_{\infty}, M_B)$. Yet, $G_{\infty}$ is not topologically finitely generated: by class field theory, its wild inertia group is topologically isomorphic to the group of 1-unit in $\mathcal{O}_{\infty}$, itself isomorphic to a countable product of $\mathbb{Z}$.

— Finally, let us mention the work of Mornev [Mor18] Thm. 1.1), supersiding prior results of Taelman in [Tae10], where he obtained a similar version of Theorem [4.1] for Drinfeld modules having everywhere good reduction. We strongly suspect a relation with our work.

This discussion hints that the module $\text{Ext}^1_{\mathbb{A}^1}(\mathbb{I}, M)$ is the right source of a regulator. This speculation is in chords with recent computations obtained jointly with Maurischat [GM22a] for the Carlitz tensor powers. The side of Hodge-Pink structures, that we portray next, seems to confirm this insight.

**Hodge-Pink structures.** In an innovative unpublished monograph [Pin97], Pink defined and studied the general theory of Hodge structures in function fields arithmetic. The right object of study, highlighted in loc. cit. and nowadays called Hodge-Pink structures, consists in couples $\mathcal{H} = (H, q_H)$, where

1. $H$ is a finite dimensional $K_{\infty}$-vector space,

2. $q_H$ is a $K_{\infty}([1])$-lattice in the $K_{\infty}[[t]]$-vector space $M_H := H \otimes_{K_{\infty}, v} K_{\infty}[[t]]$ (see Section 2 for details). Here $K_{\infty}[[t]]$ denotes the completion of $A \otimes K_{\infty}$ for the $t$-adic topology, $K_{\infty}[[t]]$ its fraction field, and $v : K_{\infty} \rightarrow K_{\infty}[[t]]$ the morphism $a \mapsto a \otimes 1$ (which is well-defined by [Gaz22a, Lem. 5.1]).

**Remark 1.2.** In [Pin97], Pink is rather concerned with mixed Hodge-Pink structures which are triples $(H, W_H, q_H)$, where $W = WH$ is a $\mathbb{Q}$-graded finite filtration of $H$ by subspaces, satisfying a certain semi-stability condition.

Classically, Hodge structures arising from motives naturally carry an involuion called the infinite Frobenius which is obtained functorially from the action of the complex conjugation. We suggest the following function field counterpart:

**Definition (2.9).** An infinite Frobenius for $\mathcal{H}$ is a $K_{\infty}$-linear continuous representation $\phi : G_{\infty} \rightarrow \text{End}_{K_{\infty}}(H)$, $H$ carrying the discrete topology, such that for all $\sigma \in G_{\infty}$,

$$\phi(\sigma) \otimes_A \sigma : H_{K_{\infty}[[t]]} \rightarrow H_{K_{\infty}[[t]]}$$

preserves the Hodge-Pink lattice $q_H$. We denote by $\text{HPk}^+$ the category of pairs $(\mathcal{H}, \phi_H)$ where $\mathcal{H}$ is an Hodge-Pink structure and $\phi_H$ is an infinite Frobenius for $\mathcal{H}$. 
Let \( H^+ \) be a mixed Hodge-Pink structure equipped with an infinite Frobenius, and let \( \mathcal{I}^+ \) denote the neutral object in \( \mathcal{H}P^+ \). Contrary to the number field picture, yet similar to what we observed for mixed \( A \)-motives, the space of extensions \( \text{Ext}^1_{\mathcal{H}P^+}(\mathcal{I}^+, H^+) \) is generally not of finite dimension over \( K_{1\infty} \). Reason is almost identical to those for \( A \)-motives: taking Hodge filtrations is not an exact operation, and this space of extensions is intertwined with the Galois cohomology of the profinite group \( G_{1\infty} \) of \( K_{1\infty} \). Using Pink’s notion of Hodge additivity - which inspired that of \textit{regulated} extensions in \textbf{Gaz22} - we considered the subspace

\[
\text{Ext}^1_{\text{reg}}(\mathcal{I}^+, H^+)(1.3)
\]

of Hodge additive extensions. We denote by \( \text{Ext}^1_{\mathcal{H}P^+, \infty}(\mathcal{I}^+, H^+) \), or simply \( \text{Ext}^1_{\text{reg}} \), when the category \( \mathcal{H}P^+ \) is clear from context, the subspace of \( (1.3) \) of extensions whose infinite Frobenius splits.

From Proposition \textbf{3.26} below, there is an exact functor \( \mathcal{H}^+ : \mathcal{A}\text{Mot}_{\infty}^{\text{at}} \rightarrow \mathcal{H}P^+ \) (see Hartl-Juschka \textbf{HJ20} for mixedness analogue of this statement), playing the role of the Hodge realization functor (see Definition \textbf{3.25}). As a corollary of Theorem \textbf{3.28} below, we record:

**Proposition.** The space \( \text{Ext}^1_{\text{reg}}(\mathcal{I}^+, \mathcal{H}^+(M)) \) is finite dimensional over \( K_{1\infty} \).

The exactness of \( \mathcal{H}^+ \) induces an \( A \)-linear morphism

\[
r_{\mathcal{H}^+} : \text{Ext}^1_{\mathcal{A}\text{Mot}_{\infty}^{\text{at}}}(\mathcal{I}, M) \rightarrow \text{Ext}^1_{\mathcal{H}P^+}(\mathcal{I}^+, \mathcal{H}^+(M))
\]

which, almost by design, maps \( \text{Ext}^1_{\mathcal{A}\text{Mot}_{\infty}^{\text{at}}}(\mathcal{I}, M) \) to \( \text{Ext}^1_{\text{reg}}(\mathcal{I}^+, \mathcal{H}^+(M)) \) (Lemma \textbf{3.2}). All our efforts are worth to justify the next definition.

**Definition \textbf{1.3}.** We call the regulator of \( M \), and denote it \( \mathcal{R}_{\mathcal{H}^+}(M) \), the \( A \)-linear morphism

\[
\mathcal{R}_{\mathcal{H}^+}(M) : \text{Ext}^1_{\mathcal{A}\text{Mot}_{\infty}^{\text{at}}}(\mathcal{I}, M) \rightarrow \text{Ext}^1_{\text{reg}}(\mathcal{I}^+, \mathcal{H}^+(M))
\]

induced by \( r_{\mathcal{H}^+} \).

It is the counterpart of Beilinson’s regulator in this setting. Our second main result is the following.

**Theorem \textbf{1.4}.** Assume that the weights of \( M \) are all negative. The rank of \( \text{Ext}^1_{\mathcal{A}\text{Mot}_{\infty}^{\text{at}}}(\mathcal{I}, M) \) over \( A \) equals the dimension of \( \text{Ext}^1_{\text{reg}}(\mathcal{I}^+, \mathcal{H}^+(M)) \) over \( K_{1\infty} \).

In view of Conjectures \textbf{[I]} and the above, it is natural to ask whether the image of \( \mathcal{R}_{\mathcal{H}^+}(M) \) has full rank in its target. It is surprisingly false stated as it is, even in the simplest case of the \( p \)-th Carlitz twist (function field analogue of Tate twists); this will be the subject of a subsequent work. The right formulation of Conjectures \textbf{[II]} and \textbf{[III]} in this context still is an open problem to the author and source of current investigations.

**Methods.** Our proof of the main theorems took strong inspirations in the work of Laforgue \textbf{Laf09} and Mornev \textbf{Mor18}, and hinges on the concept of \textit{shtuka models}. We associate non-canonically to \( M \) - which sits at the level of the affine curve \( \text{Spec} \, A \otimes K \) - a \textit{shtuka model} \( M_0 = (M_0, \mathcal{N}_0, \tau_0) \) over the surface \( (\text{Spec} \, A) \times C \) (Proposition \textbf{5.24}). Attached to it is a complex of \( A \)-modules\textsuperscript{1} sitting in degrees 0 and 1:

\[
G_M = R\Gamma \left( \text{Spec} \, A \times C, M_0 \rightarrow^\tau \mathcal{N}_0 \right),
\]

\textsuperscript{1} \( G_M \) is in fact defined differently in Definition \textbf{4.5} below, but is quasi-isomorphic to the next definition by Proposition \textbf{4.7}.
which only depends on \( M \) up to quasi-isomorphisms. It plays a similar role to that of shtuka cohomology as defined and studied extensively in [Mor18]. We show in Proposition 4.3 that there is a natural long exact sequence of \( A \)-modules:

\[
0 \to \text{Hom}_{\text{AMot}, F}(\mathbb{1}, M) \to M_{\text{reg}, \infty} \to H^1(G \mathbb{M}) \\
\to \text{Ext}^1_{A, \infty}(\mathbb{1}, M) \to \mathbb{H}^1(G \mathbb{M}, M_B) \to H^1(G \mathbb{M}) \to 0
\]

where \( r_B \) is the map defined in (1.2). The first part of Theorem 4.1 is implied by the perfectness of the complex \( G_{\mathbb{M}} \) which itself follows from finiteness of coherent cohomology of the proper \( A \)-scheme (Spec \( A \) \times \( C \) \to Spec \( A \)).

To some extent, the proof of Theorem 1.4 and the second part of Theorem 4.1 is similar but more involved. When the weights of \( M \) are non-positive, we further associate to \( \mathbb{M} \) an \( A \)-shtuka, which itself follows from finiteness of coherent cohomology.

In this section we provide all the necessary material on Hodge-Pink structures and their extensions.\[ \text{Acknowledgment: } \]

I learnt a posteriori that the question of constructing natural finitely generated extension modules out of the category of \( A \)-motives - as \( \text{Ext}^1_{A, \infty}(\mathbb{1}, M) \) is - remained an open problem suggested by Dinesh Thakur. I am much indebted to him for his interest in my work. In early versions of the manuscript, I have benefitted much from multiple exchanges and discussions with the following people to whom I wish to reiterate my gratitude: Gebhard Böckle, Bhargav Bhatt, Christopher Deninger, Urs Hartl, Annette Huber-Klawitter, Maxim Mornev and Federico Pellarin. I am also grateful to Max Planck Institute for Mathematics in Bonn for its hospitality and financial support.

## 2 Hodge-Pink structures and their extensions

In this section we provide all the necessary material on Hodge-Pink structures and their extension modules. Our main source of inspiration is Pink’s monograph [Pin97]. An innovation of ours is the introduction of infinite Frobenii in this context.

### 2.1 Hodge-Pink structures

**Definitions**

Let \((C, \mathcal{O}_C)\) is a geometrically irreducible smooth projective curve over \( F \) and \( \infty \) is a closed point of \( C \). The \( F \)-algebra

\[ A = H^0(C \setminus \{\infty\}, \mathcal{O}_C) \]
consists of functions of $C$ that are regular away from $\infty$.

Let $v$ be a closed point of $C$ and denote by $K_v$ the local function field at $v$. Let $R$ be a Noetherian subring of $K_v$ which contains $A$ and such that $R \otimes_A K$ is a field (which is then identified with a subfield of $K_v$). The ring $R$ will play the role of a \textit{coefficient ring}. In practice, $R$ should be either $A$, $K$ or $K_v$. Let $L$ be a separable extension of $K_v$: $L$ will play the role of a \textit{base field}.

If $k$ is a field which is an $A$-algebra through a morphism $\kappa$, we let $j = j_k$ denote the maximal ideal of $A \otimes k$ generated by the set of differences $\{a \otimes 1 - 1 \otimes a | a \in A\}$ (equivalently, the kernel of $A \otimes k \to k$). We denote by $k[[j]]$ the completion of $A \otimes k$ along the ideal $j_k$, and by $k[[j]]^\prime$ its field of fractions. $k[[j]]$ is a discrete valuation ring with maximal ideal $j$ and residual field $k$. By a $k[[j]]$-lattice $h$ in a finite dimensional $k((j))$-vector space $V$ we shall mean a finitely generated sub-$k[[j]]$-module of $V$ that contains a basis.

Given an $R$-module $M$, there are (at least) two ways to obtain an $L[[j]]$-module out of $M$: the base-wise way would be $M \otimes_R L[[j]]$ seeing $R$ as a subfield of $L$; the coefficient-wise way would be to consider $L[[j]]$ as an $A$-module via the morphism $\nu : A \to A \otimes L$, $a \mapsto a \otimes 1$. By [Gaz22, Lem. 5.1], $\nu$ extends uniquely to a morphism $\nu : K_v \to L[[j]]$ which coincides modulo $j$ with the inclusion $K_v \to L$, and it thus makes sense to consider the module $M \otimes_{R,\nu} L[[j]]$.

Following Pink’s [Pin97] Def. 3.2, we define:

**Definition 2.1.** An Hodge-Pink structure $\mathcal{H}$ with base field $L$ and coefficient ring $R$ consists of the data of $(H, q)$ where

1. $H$ is a finitely generated $R$-module,
2. $q = q_H$ is an $L[[j]]$-lattice in the $L((j))$-vector space $H_{L((j))} := H \otimes_{R,\nu} L((j))$.

We call $q$ the Hodge-Pink lattice of $\mathcal{H}$. We let $p_H := H \otimes_{R,\nu} L[[j]]$ and name it the \textit{tautological lattice} of $\mathcal{H}$.

We gather Hodge-Pink structures into an $R$-linear category $\mathcal{HPk}_L$ (the leftscript "$R$" should disappear from the notations when clear from the context). A \textit{morphism from $\mathcal{H} = (H, W, q)$ to $\mathcal{H}' = (H', W', q')$ in $\mathcal{HPk}_L$} is a morphism $f : H \to H'$ of $R$-modules such that $f_{L((j))} := f \otimes_{R,\nu} id_{L((j))}$ preserves the Hodge-Pink lattices; i.e. $f_{L((j))}(q) \subseteq q'$. We call $f$ \textit{strict} whenever we have:

$$f_{L((j))}(q) = q' \cap \text{im } f_{L((j))}.$$

**Extensions**

We are interested in computing extension modules in the category $\mathcal{HPk}_L$. However, $\mathcal{HPk}_L$ is not abelian and we have to clarify the notion of exact sequences we would like to consider.

**Definition 2.2.** A sequence $S : 0 \to H' \to H \to H'' \to 0$ in $\mathcal{HPk}_L$ is called \textit{exact} if its underlying sequence of $R$-module is, and \textit{strict} if morphisms of $S$ are.

It is formal to check that the category $\mathcal{HPk}$ together with the notion of strict exact sequences form an exact category. Given $X, Y$ two Hodge-Pink structures, we denote by $\text{Ext}_{\mathcal{HPk}_L}(X, Y)$ the $R$-module of classes of strict exact sequences of the form $0 \to Y \to H \to X \to 0$.

We turn to the description of those extension modules. We write $X = (X, q_X)$ and $Y = (Y, q_Y)$. Given an $L((j))$-linear morphism $\phi : X_{L((j))} \to Y_{L((j))}$, consider the object

$$E_{\phi} = (Y \oplus X, \{(q_y + f(q_x)), q_x) \mid (q_y, q_x) \in q_Y \oplus q_X\}) \in \text{Ob } \mathcal{HPk}_L.$$  \hspace{1cm} (2.1)
Let also $S_f$ be the canonical short sequence $0 \to Y \to E_f \to X \to 0$ with obvious morphisms. A simple computation shows that $S_f$ is strict exact. In particular, the assignment $f \mapsto [S_f]$ defines a map
\[ \varphi : \text{Hom}_{L((i))}(X_{L((i))}, Y_{L((i))}) \to \text{Ext}^1_{\text{HP}^k}(X, Y) \]  
(2.2)
The following proposition suffices to describe the extension module when the underlying module of $X$ is projective:

**Proposition 2.3.** Suppose that $X$ is a projective $R$-module. The morphism $\varphi$ is a surjective $R$-linear morphism whose kernel is
\[ \text{Hom}_R(X, Y) + \text{Hom}_{L((i))}(q_X, q_Y). \]
Here, the subscripts "$q$" refers to the submodule of morphisms compatible with the Hodge-Pink lattice.

**Proof.** Let $f$ and $g$ be elements of $\text{Hom}_{L((i))}(X_{L((i))}, Y_{L((i))})$. A formal computation shows that the Baer sum of $[S_f]$ and $[S_g]$ is $[S_f + g]$, and that the pullback of $[S_f]$ by $a \cdot \text{id} : X \to X$, for $a \in R$, is $[S_f]$. This proves that $\varphi$ is $R$-linear.

To show that $\varphi$ is surjective, fix an extension $[S]$ of $X$ by $Y$. By our assumption that $X$ is projective, the underlying sequence of $R$-modules splits. In other words, $S$ is equivalent to an exact sequence of the form
\[ 0 \to Y \to (Y \oplus X, q) \to X \to 0 \]  
(2.3)
where $q \subset Y_{L((i))} \oplus X_{L((i))}$ is an $L[[i]]$-lattice. Let us define a morphism $f : X_{L((i))} \to Y_{L((i))}$ as follows. As morphisms of $L[[i]]$ are strict, the underlying sequence of lattices is exact:
\[ 0 \to q_Y \to q \to q_X \to 0. \]

For $q_x \in q_X$, choose one of its lift $\tilde{q}_x$ in $q$, unique up to an element in $q_Y$. The assignment $f(q_x) := \tilde{q}_x + q_Y$ defines an $L[[i]]$-linear morphism $f : q_X \to Y_{L((i))}/q_Y$. As $q_X$ is a projective $L[[i]]$-module, $f$ lifts to an $L((i))$-linear morphism $f : X_{L((i))} \to Y_{L((i))}$. It is clear from construction that $S$ is isomorphic to $S_f$, hence $[S] = \varphi(f)$.

It remains to describe the kernel of $\varphi$. Observe that $S_f$ is equivalent to $S_q$ if and only if there exists an $R$-linear map $u : X \to Y$ such that $f = g + u$. Besides, $S_f$ splits if and only if $f$ preserves the Hodge-Pink lattices. This proves that
\[ \ker \varphi = \text{Hom}_{R}^W(X, Y) + \text{Hom}_{L((i))}(q_X, q_Y) \]
as desired.\[ \square \]

Let $\mathbb{1}$ be the Hodge-Pink structure over $L$ whose underlying $R$-module is $R$ itself and whose mixed Hodge-Pink lattice is $q_{\mathbb{1}} = p_{\mathbb{1}} = L[[i]]$. We call $\mathbb{1}$ the neutral Hodge-Pink structure. We end this paragraph with the following corollary:

**Corollary 2.4.** Let $H$ be a Hodge-Pink structure over $L$. We have a natural isomorphism of $R$-modules
\[ \varphi : H_{L((i))} \to \text{Ext}^1_{\text{HP}^k}(\mathbb{1}, H). \]

**Hodge-additive extensions**

Contrary to the number fields setting, the $R$-module $\text{Ext}^1_{\text{HP}^k}(X, Y)$ is almost never finitely generated over $R$, even when $R = K_v$. This is an issue regarding regulators, which classically are morphism of finite dimensional vector spaces. Following Pink, we now discuss the notion of Hodge additivity on extensions which solve this nuisance.
Let \( H \) be an Hodge-Pink structure over \( L \). We first recall how to associate a finite decreasing filtration - the Hodge filtration - on \( H_L \). For \( p \in \mathbb{Z} \), let \( \text{Fil}^p H_L \) denote the image of \( \mathcal{P}_H \cap \mathfrak{P}_{q_H} \) through the composition:

\[
\mathcal{P}_H = H \otimes_R L[[t]] \mod t \rightarrow H \otimes_R L = H_L.
\]

We call \( \text{Fil} H_L = (\text{Fil}^p H_L)_p \) the Hodge filtration of \( H_L \). The Hodge polygon of \( H \) is defined as the polygon of the filtration \( \text{Fil} H_L \).

**Definition 2.5** ([Pin97, §8]). A strict exact sequence \( 0 \rightarrow H' \rightarrow H \rightarrow H'' \rightarrow 0 \) is said to be Hodge additive if the Hodge polygon of \( H' \) coincide with that of \( H'' \oplus H' \).

As the Hodge polygon is invariant under isomorphism, the property of being Hodge additive respects equivalences of extensions. Hence, the following definition makes sense:

**Definition 2.6.** Let \( X, Y \) be objects in \( \text{HPk}_L \). We denote by \( \text{Ext}^1_{\text{HPk}_L}(X,Y) \) the subset of Hodge additive extensions of \( X \) by \( Y \) in \( \text{HPk}_L \).

One easily shows that \( \text{Ext}^1_{\text{HPk}_L}(X,Y) \) is a sub-\( R \)-module. The following result due to Pink shows that \( \text{Ext}^1_{\text{HPk}_L}(X,Y) \) is finitely generated.

**Proposition 2.7** ([Pin97 Prop. 8.7]). Let \( X, Y \) be objects in \( \text{HPk}_L \) where the underlying module of \( X \) is projective. Let \( f : X_L[[t]] \rightarrow Y_L[[t]] \) be an \( [[t]] \)-linear map. Then, the sequence \( S_f \) is Hodge additive if and only if \( f \) preserves the tautological lattices. In particular, \( \text{Ext}^1_{\text{HPk}_L}(X,Y) \) is an sub-\( R \)-module of \( \text{Ext}^1_{\text{HPk}_L}(X,Y) \), and \( \varphi \) induces an isomorphism:

\[
\text{Hom}_{L[[t]]}(p_X,p_Y) \rightarrow \text{Hom}_R(X,Y) + \text{Hom}_{L[[t]]}(q_X,q_Y) \cap \text{Hom}_{L[[t]]}(p_X,p_Y).
\]

In the particular case of \( X \) being \( 1 \), we get:

**Corollary 2.8.** Let \( H \) be a mixed Hodge-Pink structure over \( L \). The morphism \( \varphi \) of Corollary 2.4 induces:

\[
\varphi : \frac{\mathcal{P}_H}{H + (q_H \cap \mathcal{P}_H)} \rightarrow \text{Ext}^1_{\text{HPk}_L}(1,H).
\]

### 2.2 Infinite Frobenii

Before introducing infinite Frobenii for Hodge-Pink structures, let us shortly recall the classical story.

#### The classical picture

According to Nekovar [Nek94 (2.4)] and Deligne [Del89 §1.4 (M7)], an infinite Frobenius \( \phi_\infty \) for a mixed Hodge structure \( (H,W,H,\text{Fil} H) \) (with coefficients \( \mathbb{R} \) and base \( \mathbb{C} \)) is an involution of the \( \mathbb{R} \)-vector space \( H \) compatible with \( WH \), and such that \( \phi_\infty \otimes_R c \) preserves \( \text{Fil} H \). Mixed Hodge Structures arising from the singular cohomology groups of a variety \( X \) over \( \mathbb{R} \) are naturally equipped with an infinite Frobenius, induced by functoriality of the action of the complex conjugation on the complex points \( X(\mathbb{C}) \).

We let \( \text{MHS}_R \) be the category whose objects are pairs \((H,\phi_\infty)\) where \( H \) is a mixed Hodge structures and \( \phi_\infty \) is an infinite Frobenius for \( H \). Morphisms in \( \text{MHS}_R \) are the morphisms of mixed Hodge structures over \( \mathbb{C} \) which commute to infinite Frobenii.

Extension modules in the category \( \text{MHS}_C \) of mixed Hodge structures over \( \mathbb{C} \) are well known. Given an object \( H \) of \( \text{MHS}_C \), the complex of \( \mathbb{R} \)-vector spaces

\[
\mathbb{R}H \otimes F^0W_0H_C \xrightarrow{(\frac{1}{2})_0} W_0H_C
\]
represents the cohomology of $\text{RHom}_{\text{MHS}}(\mathbb{1}, H)$ (e.g. [Bei86] §1, [Car80] Prop. 2, [PS08] Thm. 3.31]). We obtain an $\mathbb{R}$-linear morphism

$$\frac{W_0H_C}{W_0H + F^0W_0H_C} \xrightarrow{\sim} \text{Ext}^1_{\text{MHS}}(\mathbb{1}, H).$$

If now $H^+$ denotes an object in the category $\text{MHS}^+_{\mathbb{R}}$ with infinite Frobenius $\phi_\infty$, the complex $\text{RHom}_{\text{MHS}}^+((1^+, H^+))$ is rather represented by

$$\left[(W_0H)^+ \oplus (F^0W_0H_C)^+ \xrightarrow{\frac{1}{2}} (W_0H_C)^+\right]$$

where the subscript $+$ means the corresponding $\mathbb{R}$-subspace fixed by $\phi_\infty \otimes \epsilon$ (e.g. [Bei86] §1, [Nek94] (2.5)]. We obtain an $\mathbb{R}$-linear morphism

$$\frac{(W_0H_C)^+}{(W_0H)^+ + (F^0W_0H_C)^+} \xrightarrow{\sim} \text{Ext}^1_{\text{MHS}}^+\left(\mathbb{1}, H\right).$$

Infinite Frobenii for Hodge-Pink structures

We keep notations from the previous subsection. Assume from now on that $L$ is a finite separable extensions of $K$, (hence complete), and fix $L^*$ a separable closure of $L$. In this subsection, we enrich Hodge-Pink structures with a compatible continuous action of the profinite Galois group $G_L := \text{Gal}(L^*|L)$. In several cases, we compute extensions.

Let $H = (H, q_H)$ be an Hodge-Pink structure with coefficient ring $R$ and base field $L^*$.

**Definition 2.9.** An infinite Frobenius for $H$ is an $R$-linear continuous representation $\phi : G_L \to \text{End}_R(H)$, $G_L$ carrying the profinite topology and $H$ the discrete topology, such that, for all $\sigma \in G_L$,

$$\phi(\sigma) \otimes_A \sigma : H_{L^*}^{\sigma(\mathbb{1})} \to H_{L'}^{\sigma(\mathbb{1})}$$

preserves the Hodge-Pink lattice $q_H$.

**Remark 2.10.** Above, we denoted $\sigma$ its extension to $L^*\langle (\mathbb{1}) \rangle$, i.e. obtained by the functoriality of the assignment $k \mapsto k\langle (\mathbb{1}) \rangle$, from the category of $A$-fields to rings, applied to $\sigma : L^* \to L^*$.

We let $\mathcal{HPk}^+_L$ (or $\mathcal{HPk}^+_L$) be the category whose objects are pairs $(H, \phi_H)$ where $H$ is a mixed Hodge-Pink structure over $L^*$ and $\phi_H$ is an infinite Frobenius for $H$. Morphisms in $\mathcal{HPk}^+_L$ are the one in $\mathcal{HPk}^+_L$, commuting the infinite Frobenii.

**Definition 2.11.** A sequence $S : 0 \to H' \to H \to H'' \to 0$ in $\mathcal{HPk}^+_L$ is called exact (resp. strict) if the underlying sequence in $\mathcal{HPk}^+_L$ is exact (resp. strict) as in Definition 2.2.

As well, $\mathcal{HPk}^+_L$ with the above class of strict exact sequences is an exact category and we now study its extension modules. While the ingenious analogue of (2.4) holds for Hodge-Pink structures (this was Corollary 2.8), a description similar as (2.5) does not hold in our setting as the action of the complex conjugation is replaced by that of the (infinite) profinite group $G_L$. Therefore, the extension modules are intertwined with the Galois cohomology of $G_L$, preventing an isomorphism as simple as (2.5) to exist. In order to clarify how Galois cohomology interferes with the computation of extension spaces, we introduce next an $R$-linear morphism $\text{d}_{H^+}$.

By definition, we have a forgetful functor from $\mathcal{HPk}^+_L$ to the category of $R$-linear continuous representation of $G_L$, assigning $H^+ = (H, \phi_H)$ to $\phi_H$. Being exact, it induces a natural $R$-linear morphism at the level of extensions:

$$\text{Ext}^1_{\mathcal{HPk}^+_L}(G^+, H^+) \to \text{Ext}^1_{G_L}(\phi_G, \phi_H),$$

the right-hand side denoting extensions of continuous $R$-linear $G_L$-representations.
Definition 2.12. An extension $[E] \in \text{Ext}^1_{\text{HPk}^L_+}(G^+, H^+)\text{-modules}$ is said to have analytic reduction if it splits as a representation of $G_L$; i.e., lies in the kernel of (2.6). We denote the latter by $\text{Ext}^1_{\text{HPk}^L_+\nu}(G^+, H^+)$ or simply by $\text{Ext}^1(G^+, H^+)$. Let $\mathbb{I}$ denote the neutral mixed Hodge-Pink structure over $L^*$, and denote by $\mathbb{I}^+$ the object of $\text{HPk}^L_+$ given by the pair $(\mathbb{I}, \phi_1)$, where $\phi_1 : G_L \to R$, $\sigma \mapsto 1$ is the neutral representation.

Definition 2.13. Given an object $H^+$ of $\text{HPk}^L_+$, we denote by

$$d_{H^+} : \text{Ext}^1_{\text{HPk}^L_+}(\mathbb{I}^+, H^+) \to H^1(G_L, H)$$

the $R$-linear morphism (2.6) with $G^+ = \mathbb{I}^+$, where $H^1$ here denotes the continuous group cohomology.

Recall that $H_{L^*(\mathbb{I})} = H \otimes_{R, \nu} L^*(\mathbb{I})$ is endowed with a continuous action of $\sigma \in G_L$ given by $\phi_H(\sigma) \otimes \sigma$. For $S$ a subset of $H_{L^*(\mathbb{I})}$ we denote by $S^+$ the subset of elements fixed under this action of $G_L$. Let $\varphi$ be the isomorphism of Corollary 2.8. There is an $R$-linear morphism:

$$\varphi^+ : \frac{(H_{L^*(\mathbb{I})})^+}{H^+ + q_H} \to \text{Ext}^1_{\text{HPk}^L_+}(\mathbb{I}^+, H^+),$$

mapping the class of $h \in (H_{L^*(\mathbb{I})})^+$ to the extension $(\varphi(h), (\phi_H, 0))$. By definition, the image of $\varphi^+$ lands in $\text{Ext}^1_{\text{HPk}^L_+}(\mathbb{I}^+, H^+)$. Under some assumption, we can say more.

Proposition 2.14. Suppose that $H^1(G_L, q_H)$ is trivial. Then, the sequence of $R$-modules:

$$0 \to \frac{(H_{L^*(\mathbb{I})})^+}{H^+ + q_H} \xrightarrow{\varphi^+} \text{Ext}^1_{\text{HPk}^L_+}(\mathbb{I}^+, H^+) \xrightarrow{d_{H^+}} H^1(G_L, H) \to 0$$

is exact. In particular, under the same assumption, $\varphi^+$ induces an isomorphism of $R$-modules:

$$\varphi^+ : \frac{(H_{L^*(\mathbb{I})})^+}{H^+ + q_H} \cong \text{Ext}^1_{\text{HPk}^L_+}(\mathbb{I}^+, H^+).$$

Remark 2.15. It will appear that the condition $H^1(G_L, q_H) = (0)$ is always satisfied for mixed Hodge-Pink structures arising from rigid analytically trivial mixed $A$-motives. We refer to Lemma 3.30 below.

Proof. We first show that $d_{H^+}$ is surjective. Let $c : G_L \to H$ be a cocycle. We denote by $[c]$ the $R$-linear $G_L$-representation of $H \oplus R$ given by

$$[c] : G_L \to \text{End}_R(H \oplus R), \quad \sigma \mapsto \left( \begin{array}{cc} \phi_H(\sigma) & c(\sigma) \\ 0 & 1 \end{array} \right).$$

Because $H^1(G_L, q_H) = 0$, there exists $h \in q_H$ such that $c(\sigma) = h - (\phi_H(\sigma) \otimes \sigma)(h)$. Recall that $\varphi(h)$ is represented by the extension of $\mathbb{I}$ by $H$ whose middle term is

$$E_h := \left( H \oplus R, \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} q_H \oplus L^*[1] \right).$$

It is formal to check that $[c]$ defines an infinite Frobenius for $E_h$, so that the extension given by the pair $(E_h, [c])$ defines an element of $\text{Ext}^1_{\text{HPk}^L_+}(\mathbb{I}^+, H^+)$. Its image through $d_{H^+}$ is $c$, as desired.

Before computing the kernel of $d_{H^+}$, we begin with an observation. Let $c$ be a cocycle $G_L \to H$ such that $(E_h, [c])$ defines an extension of $\mathbb{I}^+$ by $H^+$ in $\text{HPk}^L_+$. For $m \in H$, the
Let us compute the kernel of $d_{PH}^k$. Regulators in the Arithmetic of Function Fields Q. Gazda

A rigid analytically trivial mixed

ϕ is exact. In particular, under the same assumption, implies $h_{m} \in \mathbb{H}$.

In this subsection we review the usual setup of

3.1 Definitions

Given two objects $G^+$ and $H^+$ of $HPk^+_L$, denote by

$$\text{Ext}_{HA}^1(1^+, H^+) := \text{Ext}_{HA}^1(1^+, H^+) \cap \text{Ext}_{HA}^1(1^+, H^+)$$

the sub-$R$-module of Hodge-additive extensions having analytic reduction. The same argument that of the proof of Proposition [2.14] applies to show:

**Proposition 2.16.** Suppose that $H^1(G_L, q_H \cong p_H)$ is trivial. Then, the sequence of $R$-modules:

$$0 \longrightarrow \frac{p_H}{H^+} \mathop{\longrightarrow}^{\varphi^+} \frac{\text{Ext}_{HA}^1(1^+, H^+)}{H^1(G_L, H)} \longrightarrow 0$$

is exact. In particular, under the same assumption, $\varphi^+$ induces an isomorphism of $R$-modules:

$$\varphi^+: \frac{p_H}{H^+} \mathop{\longrightarrow}^{\sim} \frac{\text{Ext}_{HA}^1(1^+, H^+)}{H^1(G_L, H)},$$

3 Rigid analytically trivial mixed $A$-motives

3.1 Definitions

In this subsection we review the usual setup of $A$-motives. Recall that $(C, O_C)$ is a geometrically irreducible smooth projective curve over a finite field $F$ with $q$ elements, $\infty$ is a closed point of $C$ and $A$ is the $F$-algebra $O_C(C \setminus \{\infty\})$.

Let $R$ be an $A$-algebra through a $F$-algebra morphism $\kappa: A \rightarrow R$. We denote by $A \otimes R$ the tensor product over $F$, and we let $j = j_k$ be the kernel of the multiplication map $A \otimes R \rightarrow R$, $a \otimes r \mapsto \kappa(a)r$. The following observation appears in [Har19].
Lemma 3.1. The ideal \( j \) is a projective \( A \otimes R \)-module of rank 1. In particular, \( V(j) \) defines a Cartier divisor on \( \text{Spec} \, A \otimes R \).

Proof. Denote by \( j_{id} \) the kernel of the multiplication map \( A \otimes A \to A \). Being projective of rank 1 is a property stable by base-change, and as \( j_{id} = j_{id} \otimes_{A \otimes A} (A \otimes R) \), it suffices to show that \( j_{id} \) is projective of rank 1 over \( A \otimes A \).

We first observe that \( j_{id} \) is finite projective. Indeed, \( A \otimes A \) is a Noetherian domain, so it suffices to show that \( j_{id} \) is flat. We use Bourbaki’s flatness criterion: \( j_{id} \) is torsion free, and \( j_{id}/j_{id}^2 \) is flat as it is isomorphic to \( \Omega_{A/F} \) itself flat over \( A \cong (A \otimes A)/j_{id} \). The criterion applies to show that \( j_{id} \) is flat. To conclude that it has constant rank 1, it suffices to observe that \( \text{Spec}(A \otimes A) \) is connected and that \( j_{id}/j_{id}^2 \) has rank 1 over \( A \).

Let \( \tau \) be the ring endomorphism of \( A \otimes R \) acting as the identity on \( A \) and as raising to the \( q \)th power on \( R \) (i.e. \( \tau(a \otimes r) = a \otimes r^q \)).

Definition 3.2. An \( A \)-motive of rank \( r \) over \( R \) is a pair \( \underline{M} = (M, \tau_M) \) where \( M \) is a locally-free module over \( A \otimes R \) together with a \( \tau \)-linear isomorphism outside the zero locus \( V(j) \) of \( j \):

\[
\tau_M : (\tau^*M)|_{(\text{Spec} \, A \otimes R) \setminus V(j)} \xrightarrow{\sim} M|_{(\text{Spec} \, A \otimes R) \setminus V(j)}.
\]

We call \( \underline{M} \) effective whenever \( \tau_M \) comes from the pullback by \( \text{Spec} \, A \otimes R \to (\text{Spec} \, A \otimes R) \setminus V(j) \) of a morphism \( \tau^*M \to M \).

A morphism \( (M, \tau_M) \to (N, \tau_N) \) of \( A \)-motives over \( R \) is an \( A \otimes R \)-linear morphism \( f : M \to N \) such that \( f \circ \tau_M = \tau^*f \circ \tau_N \). We let \( \text{AMot}_R \) denote the \( A \)-linear category of \( A \)-motives over \( R \).

Remark 3.3. Along this text, we shall denote by \( M[j^{-1}] \) the module \( M|_{(\text{Spec} \, A \otimes R) \setminus V(j)} \).

This agrees with the convention taken in [Gaz22], where \( M[j^{-1}] \) is rather described as the submodule of \( M \otimes_{A \otimes R} \text{Quot}(A \otimes R) \) - where \( \text{Quot}(A \otimes R) \) denotes the localization of \( A \otimes R \) at the non-zero divisor consisting in elements \( x \) for which there exists a positive integer \( n \geq 0 \) such that \( j^n x \in M \).

Most of the results in this text are concern with \( A \)-motives over \( K \) (with \( \kappa : A \to K \) being the inclusion). For those, one has a notion of weights that we recall briefly (the reader will find all details in [Gaz22, §3]). There is an (exact) functor

\[
\text{AMot}_K \to \text{Isoc}_K, \quad \underline{M} \mapsto I_{\infty}(\underline{M})
\]

where the targeted category is that of \( \infty \)-isocrystals over \( K \). The category \( \text{Isoc}_K \) admits a slope function in the sense of [And09], hence any object admits a unique slope filtration (the Harder-Narasimhan filtration) [Gaz22, Thm. 3.9]. It is said that \( \underline{M} \) has weights \( \nu_1 < \nu_2 < \ldots < \nu_s \) if \( I_{\infty}(\underline{M}) \) has slopes \( -\nu_1 > -\nu_2 > \ldots > -\nu_s \).

3.2 The Betti realization functor

We let \( K \) be the fraction field of \( A \) (equivalently, the function field of \( C \)). Here, we introduce the Betti realization of an \( A \)-motive (Definition 3.7) and discuss rigid analytically triviality (Definition 3.8). One chief aim is to define the full subcategory \( \text{AMot}_K^{\text{rat}} \) of \( \text{AMot}_K \) consisting of rigid analytically trivial \( A \)-motives over \( K \), which shall be the source of the Hodge-Pink realization functor to be defined in Subsection 3.4. Historically, the notion of rigid analytic triviality dates back to Anderson [And88 §2], and most of this subsection owes to his work. A novelty of our account is the consideration of a natural continuous action of \( G_\infty \) - the absolute Galois group at \( \infty \) - on the Betti realization \( A \)-modules. The existence of canonical infinite Frobenii attached to the associated Hodge-Pink structures will follow from this construction.
Tate algebras

Let \( L \) be a field over \( \mathbb{F} \) complete with respect to an non-archimedean norm \(|\cdot|\), and let \( \mathcal{O}_L \) be its valuation ring with maximal ideal \( m_L \).

**Definition 3.4.** We denote by \( O_L(A) \) the \( O_L \)-algebra given by the completion of \( A \otimes O_L \) with respect to the \( A \otimes m_L \)-adic topology. We denote by \( L(A) \) the \( L \)-algebra \( L \otimes O_L \), \( O_L(A) \). We again denote by \( \tau \) the continuous extension of \( A \otimes O_L \to A \otimes O_L \), \( a \otimes c \mapsto a \otimes c^\tau \) to \( O_L(A) \) and \( L(A) \).

**Remark 3.5.** The notation \( L(A) \) is here to stress that it generalizes the classical Tate algebra over \( L \); if \( A \cong \mathbb{F}[t] \), then \( L(A) \cong L(t) \). For general rings \( A \), one can show that \( L(A) \) is an affinoid algebra (see [GM22b, Prop. 3.2]).

The following preliminary lemma will be used next, in the definition of the Betti realization functor.

**Lemma 3.6.** Let \( \kappa : A \to L \) be an \( \mathbb{F} \)-algebra morphism with discrete image. We have \( j_\ast L(A) = L(A) \).

**Proof.** Because \( \kappa(A) \) is discrete in \( L \), it contains an element \( \alpha \) of norm \( |\alpha| > 1 \). Let \( a \in A \) be such that \( \alpha = \kappa(a) \). Then, \( \kappa(a)^{-1} \in m_L \) and the series
\[
- \sum_{n \geq 0} a^n \otimes \kappa(a)^{-1}
\]
converges in \( O_L(A) \) to the inverse of \( (a \otimes 1 - 1 \otimes \kappa(a)) \). \( \square \)

The Betti realization of an \( A \)-motive

Let \( K_\infty \) be the completion of \( K \) with respect to the \( \infty \)-adic topology, and denote by \( O_\infty \) its ring of integers with maximal ideal \( m_\infty \). We fix \( K_\infty^* \) a separable closure of \( K_\infty \), and denote \( C_\infty \) its completion (which is now algebraically closed and complete, by Krasner’s Lemma). The canonical norm on \( K_\infty \) extends uniquely to a norm \(|\cdot|\) on \( C_\infty \). The action of \( G_\infty = \text{Gal}(K_\infty^*/K_\infty) \) extends by continuity to \( C_\infty \).

Let \( M = (M, \tau_M) \) be an \( A \)-motive over \( K \). By Lemma 3.6 the ideal \( j \subset A \otimes K \) becomes invertible in \( C_\infty(A) \), and thus \( \tau_M \) induces an isomorphism of modules over \( C_\infty(A) \):
\[
\tau_M : \tau^*(M \otimes_{A \otimes K} C_\infty(A)) \xrightarrow{\sim} M \otimes_{A \otimes K} C_\infty(A)
\]
which commutes with the action of \( G_\infty \) on \( M \otimes_{A \otimes K} C_\infty(A) \), inherited from the right-hand side of the tensor. We still denote by \( \tau_M \) the isomorphism.

**Definition 3.7.** The **Betti realization of** \( M \) is the \( A \)-module
\[
M_B := \{ \omega \in M \otimes_{A \otimes K} C_\infty(A) \mid \omega = \tau_M(\tau^*\omega) \}
\]
endowed with the compatible action of \( G_\infty \) it inherits as a submodule of \( M \otimes_{A \otimes K} C_\infty(A) \). We let \( M_B^+ \) denote the sub-\( A \)-module of \( M_B \) of elements fixed by \( G_\infty \).

The next definition is borrowed from [And86, §2.3].

**Definition 3.8.** The \( A \)-motive \( M \) is called **rigid analytically trivial** if the map
\[
M_B \otimes_A C_\infty(A) \to M \otimes_{A \otimes K} C_\infty(A), \quad \omega \otimes c \mapsto \omega \cdot c
\]
given by the multiplication is bijective.

**Remark 3.9.** Not every \( A \)-motive is rigid analytically trivial. An example of \( A \)-motive which is not rigid analytically trivial is given in [And86, 2.2], or [Tae09, Ex. 3.2.10].
The following proposition rephrases [BH87 Cor. 4.3]:

**Proposition 3.10.** Let $M$ be an $A$-motive over $F$ of rank $r$. Then $M_B$ is a finite projective $A$-module of rank $r'$ satisfying $r' \leq r$ with equality if and only if $M$ rigid analytically trivial.

When $M$ is rigid analytically trivial, in Definition 3.3 the field $C_\infty$ can be replaced by a much smaller field. This is the subject of the next proposition.

**Proposition 3.11.** Let $M$ be a rigid analytically trivial $A$-motive over $K$. There exists a (complete) finite separable field extension $L$ of $K_\infty$ in $C_\infty$ such that $M_B$ is contained in $M \otimes_{A \otimes K} L(A)$. In particular, the action of $G_\infty$, equipped with the profinite topology, on $M_B$, equipped with the discrete topology, is continuous.

**Proof.** For effective $t$-motives this is proven in [And86 Thm 4], so we explain how one may reduces to this case. Let $t$ be a non constant element of $A$. The inclusion $F[t] \subset A$ makes $A$ into a finite flat $F[t]$-module, and therefore $M$ defines an $F[t]$-motive of rank deg($t$) · rank $M$ over $K$. Using the identification $F[t] \otimes K = K[t]$, we rather write $t$ for $t \otimes 1$ and $\theta$ for $1 \otimes t$. Let $n > 0$ be an integer so that $(t - \theta)^n \tau_M(\tau^s M) \subset M$. Let $N_B$ be the $F[t]$-motive over $K$ whose underlying module is $N = K[t]$ and where $\tau_N$ is the multiplication by $(t - \theta)^n$ (this is the dual of Carlitz $n$th twist). If $\sqrt[n]{-\theta}$ denotes a $q - 1$-root of $-\theta$ in $C_\infty$, we have

$$N_B = (\sqrt[n]{-\theta})^{-n} \prod_{i=0}^{\infty} \left(1 - \frac{t}{\sqrt[n]{-\theta}}\right)^n \cdot F[t] \subset K_\infty(\sqrt[n]{-\theta}) \langle t \rangle.$$

The $F[t]$-motive $N$ has been chosen so that $M \otimes N$ is effective (see Definition 3.2). We are thus in the range of application [And86 Thm 4] which states that there exists a finite extension $H$ of $K_\infty$ in $C_\infty$ such that

$$M_B \otimes_{F[t]} N_B = (M \otimes N)_B \subset (M \otimes_{K[t]} N) \otimes_{K[t]} H(t) = M \otimes_{K[t]} H(t).$$

It follows that there exists a finite extension $L'$ of $K_\infty$ such that $M_B \subset M \otimes_{K[t]} L'(t)$ (e.g. one can take $L' := H(\sqrt[n]{-\theta})$).

We now show that one can choose $L'$ separable over $K_\infty$. Note that $M \otimes_{K[t]} K_\infty(t)$ is free of finite rank over $K_\infty(t)$. Therefore, $(M \otimes_{K[t]} K_\infty(t))/(t^n)$ is a finite dimensional $K_\infty$-vector space for all positive integers $n$. By Lang’s isogeny Theorem (e.g. [Kat73 Prop. 1.1]), the multiplication map

$$\{m \in (M \otimes_{K[t]} K_\infty(t))/(t^n) \mid m = \tau_M(\tau^s m)\} \otimes K_\infty \longrightarrow (M \otimes_{K[t]} K_\infty(t))/(t^n)$$

is an isomorphism. In particular, the inclusion

$$\{m \in (M \otimes_{K[t]} K_\infty(t))/(t^n) \mid m = \tau_M(\tau^s m)\} \subseteq \{m \in (M \otimes_{K[t]} C_\infty(t))/(t^n) \mid m = \tau_M(\tau^s m)\}$$

is an equality. This shows that $M_B$ is both a submodule of $M \otimes_{K[t]} K_\infty(t)$ and $M \otimes_{K[t]} L'(t)$. Because $M$ is free over $K[t]$, it follows that $M_B \subset M \otimes_{K[t]} L(t)$ where $L = L' \cap K_\infty$ is a finite separable extension of $K_\infty$. As $(A \otimes K) \otimes_{K[t]} L(t)$ is isomorphic to $L(A)$, we deduce that $M_B \subset M \otimes_{A \otimes K} L(A)$.

As a consequence of Proposition 3.11 by the faithful flatness of the inclusion $L(A) \rightarrow C_\infty(A)$ ([Bou07 AC I\S 3.5 Prop. 9]), we have:

**Proposition 3.12.** Let $M$ be a rigid analytically trivial $A$-motive over $K$. Let $L$ be as in Proposition 3.11. The multiplication map

$$M_B \otimes_A L(A) \rightarrow M \otimes_{A \otimes K} L(A)$$

is an isomorphism of $L(A)$-modules.
The first part of the next result is inspired by [BH07, Prop. 6.1]. We have adapted its proof to allow the smaller field $K^s_\infty$ instead of $C_\infty$. This is needed in order to compute the $A$-module $H^1(G_\infty, M_B)$ of continuous Galois cohomology.

**Theorem 3.13.** Let $M$ be a rigid analytically trivial $A$-motive over $K$. There is an exact sequence of $A[G_\infty]$-modules:

$$0 \rightarrow M_B \rightarrow M \otimes_{A \otimes K} C_\infty(A)^{ir \tau_M} \rightarrow M \otimes_{A \otimes K} C_\infty(A) \rightarrow 0.$$  \hspace{1cm} (3.2)

Furthermore, it induces a long exact sequence of $A$-modules

$$0 \rightarrow M^+_B \rightarrow M \otimes_{A \otimes K} K_\infty(A) \rightarrow M \otimes_{A \otimes K} K_\infty(A) \rightarrow H^1(G_\infty, M_B) \rightarrow 0.$$  \hspace{1cm} (3.3)

**Remark 3.14.** The fact that $\sum_{n \geq 0} c_n t^n \in K^s_\infty(t)$ implies $\sum_{n \geq 0} c_n t^n \in K^s_\infty$. This follows from [BH07, Prop. 6.1]. We shall use the same argument as in loc. cit. to show that the sequence

$$0 \rightarrow M_B \rightarrow M \otimes_{K[t]} K^s_\infty(t)^{ir \tau_M} \rightarrow M \otimes_{K[t]} K^s_\infty(t) \rightarrow 0, \hspace{1cm} (3.4)$$

where the first inclusion is well-defined by Proposition 3.11 is exact. It suffices to show the surjectivity of $\text{id} \rightarrow M \otimes_{K[t]} K^s_\infty(t)$. Let $f \in M \otimes_{K[t]} K^s_\infty(t)$. Since $M$ is rigid analytically trivial, without loss of generality we can assume that $f = c \cdot \omega$ for $c = \sum_{n \geq 0} c_n t^n \in K^s_\infty(t)$ and $\omega \in M_B$. For every $n \geq 0$, let $b_n \in K^s_\infty(t)$ be a solution of $x - x^n = c_n$. The condition $|c_n| \rightarrow 0$ as $n$ grows implies $|b_n| \rightarrow 0$. Hence, the element

$$g := \left( \sum_{n=0}^{\infty} b_n t^n \right) \cdot \omega$$

belongs to $M \otimes_{K[t]} K^s_\infty(t)$ and satisfies $\text{id} \rightarrow M \otimes_{K[t]} K^s_\infty(t)$. Surjectivity follows.

We turn to the second part of the statement. By Proposition 3.11 $G_\infty$ acts continuously on $\sum_{n \geq 0} c_n t^n \in K^s_\infty(t)$, and taking invariants yields a long exact sequence of $A$-modules:

$$0 \rightarrow M^+_B \rightarrow M \otimes_{A \otimes K} K_\infty(A)^{ir \tau_M} \rightarrow M \otimes_{A \otimes K} K_\infty(A) \rightarrow H^1(G_\infty, M_B) \rightarrow \cdots$$

We shall prove that the $K[t]$-module

$$H^1(G_\infty, M \otimes_{K[t]} K^s_\infty(t))$$

is isomorphic to $M \otimes_{K[t]} H^1(G_\infty, K^s_\infty(t))$, hence it suffices to show that $H^1(G_\infty, K^s_\infty(t))$ vanishes. By continuity, it is enough to show the vanishing of $H^1(G_\infty, L(t))$ for any subfield $L \subset K^s_\infty$ that is a finite Galois extension of $K_\infty$. For such $L$, we denote by $H$ the finite Galois group $\text{Gal}(L/K_\infty)$. By the additive version of Hilbert’s 90 Theorem [Ser68] $x \in \sum_{\sigma \in H} \alpha^\sigma$ is nonzero. Thus, $f$ can be written as

$$f = \left( \sum_{\sigma \in H} \alpha^\sigma f^\sigma \right) - \left( \sum_{\sigma \in H} \alpha^\sigma c(\sigma) \right) \in K^s_\infty(t) + L(t).$$

It follows that $c$ is trivial, and that $H^1(G_\infty, L(t)) = 0$. This concludes the proof. 

\[\square\]
We are now ready to introduce the category of rigid analytically trivial $A$-motives over $K$, as mentioned in the introduction.

**Definition 3.15.** We let $A\text{Mot}^\text{rat}_K$ be the full subcategory of $A\text{Mot}_K$ whose objects are rigid analytically trivial.

Recall that a sequence of objects in $A\text{Mot}_K$ is called exact if the underlying sequence of modules is (see [Gaz22, Def. 2.5]), and that the category $A\text{Mot}_K$ endowed with the class of exact sequences forms an exact category (Gaz22 Prop. 2.6). The next proposition, which ensures that extension modules in the category $A\text{Mot}^\text{rat}_K$ are well-defined, is borrowed from [HJ20, Lem. 2.3.25].

**Proposition 3.16.** Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in $A\text{Mot}_K$. Then $M$ is rigid analytically trivial if and only if $M'$ and $M''$ are. In particular, the category $A\text{Mot}^\text{rat}_K$ is exact.

We finally record that the Betti realization functor having $A\text{Mot}^\text{rat}_K$ as its source is exact (this is not true on the full category $A\text{Mot}_K$).

**Corollary 3.17.** The functor $M \mapsto M_B$ from $A\text{Mot}^\text{rat}_K$ to the category $\text{Rep}_A(G_\infty)$ of continuous $A$-linear representations of $G_\infty$ is exact.

**Proof.** This follows from Theorem [3.13] together with the Snake Lemma. 

**Extensions with analytic reduction**

In view of Corollary 3.17, the following definition makes sense:

**Definition 3.18.** Let $S : 0 \to M' \to M \to M'' \to 0$ be an exact sequence in $A\text{Mot}^\text{rat}_K$. We say that $S$ has analytic reduction at $\infty$ if the sequence $S_B$ splits in $\text{Rep}_A(G_\infty)$.

**Remark 3.19.** The above notion is crucial to obtain finitely generated extension modules out of the category $A\text{Mot}^\text{rat}_K$ (see Section 6). It is the counterpart of Definition 2.13 for Hodge-Pink structures.

Given two rigid analytically trivial $A$-motives $N, M$, the exactness of the Betti realization functor induces a morphism of $A$-modules:

$$r_B(N, M) : \text{Ext}^1_{A\text{Mot}^\text{rat}_K}(N, M) \to \text{Ext}^1_{\text{Rep}_A(G_\infty)}(N_B, M_B)$$

which is the counterpart of (2.6) in the context of Hodge-Pink structures. By definition, the kernel of $r_B(N, M)$ consists of extensions having analytic reduction at $\infty$. We set:

$$\text{Ext}^1_\infty(N, M) := \ker r_B(N, M).$$

**3.3 Analytic continuation**

To associate an Hodge-Pink structure to a rigid analytically trivial $A$-motive $M$, it is crucial to understand the behaviour of elements in $M_B$ — which can be seen as functions over the affinoid subdomain $\text{Spm} \ C_\infty(A)$ with values in $M \otimes_K C_\infty$ — near the point $j$. However, the latter does not belong to the maximal spectrum of $C_\infty(A)$, as one deduces from Lemma 3.6. Hence it is necessary to extend elements of $M_B$ to a larger domain. In this subsection, we show that elements of $M_B$ can be meromorphically continued to the whole rigid analytification of the affine curve $\text{Spec} \ A \otimes C_\infty$, with their only poles supported at $j$ and its iterates $\tau_j, \tau^2_j, \ldots$. In the $\deg(\infty) = 1$-case, this is treated in [HJ20 §2.3.4], for special functions of Anderson $A$-modules this is the subject of [GM22b].
We recall some material from [GM22b, §3]. Let \( L \) be any complete subfield of \( \mathbb{C}_\infty \) that contains \( K_\infty \) and let \( | \cdot | \) be the norm on \( L \) it inherits as a subfield of \( \mathbb{C}_\infty \). We define Gauss norms on \( A \otimes L \): let \( c \in L^{\times} \) and let \( \rho := |c| > 0 \). For \( f \in A \otimes L \), we set:

\[
\| f \|_\rho := \inf \left( \max \{ |\ell_i|^{\deg a_i} \} \right)
\]

where the infimum is taken over all representations of \( f \) as finite sums \( \sum_i a_i \otimes \ell_i \) in \( A \otimes L \). The application \( \| \cdot \|_\rho \) is indeed a Gauss norm [GM22b Prop. 3.2], and we denote by \( L(A)_\rho \) the affinoid algebra over \( L \) obtained by completing \( A \otimes L \) with respect to it. Observe that \( L(A) \) as in Definition 3.4 coincides with \( L(A)_1 \) (see [GM21 Prop. 2.2]). If \( \rho < \rho' \), there is a canonical inclusion \( L(A)_{\rho'} \hookrightarrow L(A)_{\rho} \) and we set:

\[
L\langle A \rangle := \lim_{\rho \to \rho'} L(A)_{\rho}
\]

where \( \rho \) runs over \( |L^{\times}| \). If \( X = \mathcal{X}_L \) denote the rigid analytic variety \( \text{Spec} A \otimes L \)ris over \( L \), then \( X \) is isomorphic to the glueing \( \lim_{\rho \to \rho} \mathcal{X}_\rho \) where \( \mathcal{X}_\rho := \text{Sp} L(A)_{\rho} \). In particular, \( L\langle A \rangle \) is the ring of global sections of \( \mathcal{O}_X \).

As \( \| \tau(f) \|_\rho = \| f \|_{\rho/\rho}^q \) for \( f \in A \otimes L \), \( \tau \) induces a morphism \( L(A)_{\rho} \to L(A)_{\rho'/\rho} \) which are compatible: we still denote by \( \tau \) the resulting morphism \( L\langle A \rangle \to L\langle A \rangle \) and \n
\[
\tau : \mathcal{X}_L \longrightarrow \mathcal{X}_L.
\]

For \( m \geq 1 \), we denote by \( J_m \) the divisor corresponding to \( j + j_1 + \ldots + j_m \) on \( X \) and by \( J^\text{reg} \) the rigid analytic divisor \( J^\text{reg} = j + j_1 + \ldots \). For \( n \geq 0 \), we introduce

\[
\mathcal{O}_X(n \cdot J) := \text{colim}_m \mathcal{O}_X(n \cdot J_m)
\]

as the sheaf of meromorphic functions on \( X \) with a pole of order at most \( n \) at the support of \( J \).

**Example 3.20.** Let \( C = \mathbb{P}^1_F \) and let \( \infty \) be the point \( [0 : 1] \). We identify \( A \) with \( F[t] \) and the tensor product \( A \otimes K \) with \( K[t] \). Let \( \theta \in K \) denote the image of \( t \) so that \( \tau^j \) corresponds to the principal ideal \( (t - \theta^j) \) of \( K[t] \). We have

\[
L\langle A \rangle = L\langle t \rangle = \left\{ \sum_{n=0}^{\infty} a_n t^n : a_n \in A; \lim_{n \to \infty} a_n \to 0 \right\},
\]

\[
L\langle A \rangle = L\langle t \rangle = \left\{ \sum_{n=0}^{\infty} a_n t^n : a_n \in A; \forall \rho > 1 : \lim_{n \to \infty} a_n \rho^n \to 0 \right\}.
\]

The ring \( L(A) \) corresponds to series converging in the closed unit disc, whereas \( L\langle A \rangle \) consists of entire series. The morphism \( \tau \) acts on both rings by mapping

\[
f = \sum_{n=0}^{\infty} a_n t^n \mapsto f^{(1)} = \sum_{n=0}^{\infty} a_n^2 t^n.
\]

Now, \( \mathcal{O}_X(n \cdot J)(X) \) is identified with the subring of \( \text{Quot} L\langle t \rangle \) consisting of elements \( f \) such that

\[
\prod_{j=0}^{\infty} \left( 1 - \frac{t}{\theta^j} \right) \cdot f \in L\langle t \rangle.
\]

By Lemma 3.6, \( J \) is not supported on \( \text{Sp} L(A) \) and we have a morphism

\[
\mathcal{O}_X(n \cdot J)(X_L) \longrightarrow \mathcal{O}_X(n \cdot J)(\text{Sp} L(A)) = L(A)
\]

given by sheaf restriction. We are now in position to prove the main result of this subsection (compare with [HJ20 Prop. 2.3.30]).
Theorem 3.21. Let $M$ be a rigid analytically trivial $A$-motive over $K$. Let $L$ be a finite separable extension of $K_{\infty}$ such that $M_B$ is contained in $M \otimes_{A \otimes K} L(A)$ (whose existence is guaranteed by Proposition 3.11). Then, there exists a large enough integer $n$ for which canonical morphism
\[(M \otimes_{A \otimes K} \mathcal{O}_X(n \cdot J)(\mathcal{X}_L))^\tau_{M=1} \to (M \otimes_{A \otimes K} L(A))^\tau_{M=1} = M_B\]
is an isomorphism.

Before engaging in the proof, we set some definitions. Let $t \in A$ be a non-constant element and denote by $\theta$ the element $1 \otimes t$ in $A \otimes K$. As $\mathbb{F}[t] \to A$ is finite flat, the multiplication maps
\[A \otimes_{\mathbb{F}[t]} L(t) \to L(A), \quad A \otimes_{\mathbb{F}[t]} L(t) \to L(A)\]
are isomorphisms. For $i \geq 0$, the converging product (we abbreviated $t \otimes 1$ in $t$)
\[\Pi_t := \prod_{j=0}^{\infty} \left(1 - \frac{t}{\theta^q} \right)\]
defines elements in $L(A)$ whose only zeros in $X$ are supported at $\bigcup_{j \geq 0} V(t - \theta^q)$.

Proof of Theorem 3.21. Let $n \geq 0$ be such that $j^n \tau_M(\tau^* M) \subset M$ and $t \in A$ as above. Let us first show that $\Pi_t^n \cdot M_B \subset M \otimes_{A \otimes K} L(A)$. We use again notations of the proof of Proposition 3.11 $\nu^q - \theta$ is a $q - 1$-root of the image of $-t$ in $L$ and, introducing
\[\omega_t := \nu^q - \theta \prod_{i=0}^{\infty} \left(1 - \frac{t}{\theta^q} \right)^{-1} = \nu^q - \theta \cdot \Pi_t^{-1} \in \text{Quot} L(A)\],

we have $N_B = \omega_t^{-n} \cdot \mathbb{F}[t]$ where $N$ is the $\mathbb{F}[t]$-motive over $K$ whose underlying module is $N = K[t]$ and where $\tau_N$ is the multiplication by $(t - \theta)^n$. The $\mathbb{F}[t]$-motive $N$ has been chosen so that $M \otimes N$ is effective and from [BH07] Prop. 3.4 we deduce that $(M \otimes N)_B \subset M \otimes_{K[t]} L(A)$. Hence $\omega_t^{-n} \cdot M_B \subset M \otimes_{A \otimes K} L(A)$ as desired.

Now, from [GM22] Prop. 3.8, we have
\[\mathcal{O}_X(n \cdot J)(\mathcal{X}_L) = \{ f \in \text{Quot} L(A) \mid \forall t \in A \setminus \mathbb{F} : \Pi_t^n \cdot f \in L(A) \}\].

As $M$ is finite projective, we conclude that $M_B \subset M \otimes_{A \otimes K} \mathcal{O}_X(n \cdot J)(\mathcal{X}_L)$. The theorem follows.

Corollary 3.22. Under the assumptions of Theorem 3.21 the multiplication map
\[M_B \otimes_A \mathcal{O}_X(n \cdot J)(\mathcal{X}_L) \to M \otimes_{A \otimes K} \mathcal{O}_X(n \cdot J)(\mathcal{X}_L), \quad \omega \circ c \mapsto \omega \cdot c\]
is bijective.

Proof. This follows from the fully faithful flatness of $\mathcal{O}_X(n \cdot J)(\mathcal{X}_L) \to L(A)$.

3.4 The associated mixed Hodge-Pink structure

Let $M$ be a rigid analytically trivial $A$-motive over $K$ (Definition 3.7). Let $M_B$ be the Betti realization of $M$ (Definition 3.7). By Corollary 3.22 there exists a finite separable extension $L$ of $K_{\infty}$ such that the multiplication map
\[M_B \otimes_A \mathcal{O}_X(n \cdot J)(\mathcal{X}_L) \to M \otimes_{A \otimes K} \mathcal{O}_X(n \cdot J)(\mathcal{X}_L), \quad \omega \circ c \mapsto \omega \cdot c \quad (3.8)\]
is bijective. Localizing at $j$, we obtain an isomorphism of $K_\infty(j)$-modules:
\[M_B \otimes_{A \otimes K} K_\infty(j) \simto M \otimes_{A \otimes K} K_\infty(j), \quad (3.9)\]
where $\nu$ denote the morphism $A \to K_\infty(j)$, $a \mapsto a \otimes 1$, introduced earlier in Section 2 in the context of Hodge-Pink structures.
Definition 3.23. We denote by $\gamma_M$ the isomorphism (3.9).

A trivial yet important remark is the following:

Lemma 3.24. The morphism $\gamma_M$ is $G_\infty$-equivariant, where $\sigma \in G_\infty$ acts on the right-hand side of (3.9) via $\sigma \otimes \sigma$ and on the left via $\text{id}_M \otimes \sigma$.

Let $R$ be a Noetherian subring of $K_\infty$ containing $A$. In the next definition, attributed to Pink, we attach an Hodge-Pink structure to $M$ (see also [HJ20, Def. 2.3.32]).

Definition 3.25. We let $\mathcal{HR}(M)$ be the Hodge-Pink structure (with base field $K_\infty^s$, coefficients ring $R$)

- whose underlying $R$-module is $M_R \otimes_A R$,
- whose Hodge-Pink lattice is $\mathfrak{p}_M = \gamma_M^{-1}(M \otimes_{A_{\otimes K}} K_\infty^s[i])$.

The tautological lattice of $\mathcal{HR}(M)$ is $\mathfrak{p}_M = M_R \otimes_{A_{\otimes K}} K_\infty^s[i]$. The action of $G_\infty$ on $M_R$ is continuous (Proposition 3.11) and defines an infinite Frobenius $\phi_M$ for $\mathcal{HR}(M)$. We denote by $\mathcal{HR}^+(M)$ the pair $(\mathcal{HR}(M), \phi_M)$.

Proposition 3.26. The assignment $M \mapsto \mathcal{HR}(M)$ (resp. $\mathcal{HR}^+(M)$) defines an exact functor $\mathcal{HR}^+_R : \text{AMot}^{\text{rat}}_R \to \text{HPk}_{K_\infty^s}$ (resp. $\text{HPk}_{K_\infty^s}^+$) of exact categories.

Proof. Let $S : 0 \to M' \to M \to M'' \to 0$ be an exact sequence in $\text{AMot}^{\text{rat}}_R$. By Corollary 3.17 the sequence $S_B$ is exact hence $\mathcal{HR}(S)$ and $\mathcal{HR}^+(S)$ are exact. To show that they are strict, observe that the sequence

$$0 \to \mathfrak{p}_{M'} \to \mathfrak{p}_M \to \mathfrak{p}_{M''} \to 0$$

is isomorphic to $0 \to M' \otimes_{A_{\otimes K}} K_\infty^s[i] \to M \otimes_{A_{\otimes K}} K_\infty^s[i] \to M'' \otimes_{A_{\otimes K}} K_\infty^s[i] \to 0$ through $\gamma_S$, and the latter is exact by flatness of $A \otimes K \to K_\infty^s$. \hfill $\square$

We recall the notion of regulated extensions in $\text{AMot}^{\text{rat}}_R$, as introduced in [Gaz22, Def. 5.7]:

Definition 3.27. Let $S : 0 \to M' \to M \to M'' \to 0$ be an exact sequence in $\text{AMot}^{\text{rat}}_R$. We say that $S$ is regulated if $\mathcal{HA}(S)$ is Hodge additive (Definition 2.5).

We conclude this section by giving a description of the extension modules of Hodge-Pink structures arising from $A$-motives. It consists mainly in the reformulation of Propositions 2.14 and 2.16 in the case of $H = \mathcal{HR}^+(M)$ for a rigid analytically trivial $A$-motive $M$ over $K$.

Theorem 3.28. Let $M$ be a rigid analytically trivial mixed $A$-motive over $K$ whose weights are all non-positive. Let $H^+ = \mathcal{HR}^+(M)$. In the notations of Proposition 2.14, we have an exact sequence

$$0 \to M \otimes_{A_{\otimes K}} K_\infty^s[i] \to \text{Ext}^1_{\text{HPk}_{K_\infty^s}}(\mathcal{HR}^+(M), H^+) \to \text{Ext}^{1, \text{ha}}_{\text{HPk}_{K_\infty^s}}(\mathcal{HR}^+(M), H^+) \to 0$$

where $(M_R) := M_R \otimes_A R$. The Hodge additive version of this exact sequence holds:

$$0 \to (M + \tau_M(M)) \otimes_{A_{\otimes K}} K_\infty^s[i] \to \text{Ext}^1_{\text{HPk}_{K_\infty^s}}(\mathcal{HR}^+(M), H^+) \to \text{Ext}^{1, \text{ha}}_{\text{HPk}_{K_\infty^s}}(\mathcal{HR}^+(M), H^+) \to 0.$$

The theorem follows from the next two lemmas, the first of which precise the form of $\mathfrak{p}_M$ seen as a submodule of $M \otimes_{A_{\otimes K}} K_\infty^s[i]$.

Lemma 3.29. We have $\gamma_M(\mathfrak{p}_M) = \tau_M(M) \otimes_{A_{\otimes K}} K_\infty^s[i]$. 

Proof. Let $L$ be a finite separable extension of $K_\infty$ as in Theorem 3.21 and let $\tau J_m$ denote the divisor $\tau_1 + \tau_2 + \ldots + \tau_m$ on $X_L$ and $\tau J := \tau J_\infty = \tau_1 + \tau_2 + \ldots$. For $n \geq 0$, we also denote by $\mathcal{O}_X(n \cdot \tau J)$ the sheaf on $X_L$ given by $\colim_m \mathcal{O}_X(n \cdot \tau J_m)$; namely the sheaf of meromorphic functions on $X_L$ that are holomorphic away from the support of $\tau J$. The morphism $\tau$ on $X$ induces a map
\[ \tau : \mathcal{O}_X(n \cdot J)(X_L) \to \mathcal{O}_X(n \cdot \tau J)(X_L). \]
Taking the pullback of (3.8) by the above morphism produces an isomorphism:
\[ M_B \otimes_A \mathcal{O}_X(n \cdot \tau J)(X_L) \cong (\tau^* M) \otimes_{A \otimes K} \mathcal{O}_X(n \cdot \tau J)(X_L) \]
which, localized at $j$, produces:
\[ \delta_M : M_B \otimes_A K_* \mathcal{O}_X[j] \cong (\tau^* M) \otimes_{A \otimes K} K_* \mathcal{O}_X[j]. \]
The latter inserts in a commutative diagram
\[ \begin{array}{ccc}
M_B \otimes_A K_* \mathcal{O}_X(j) & \xrightarrow{\delta_M} & (\tau^* M) \otimes_{A \otimes K} K_* \mathcal{O}_X(j) \\
\gamma_M & \downarrow & \gamma_M \\
M \otimes_{A \otimes K} K_* \mathcal{O}_X(j) & \xrightarrow{\tau_M \otimes \text{id}_{K_* \mathcal{O}_X(j)}} & M \otimes_{A \otimes K} K_* \mathcal{O}_X(j) 
\end{array} \]
Note that this already appears in [HJ20 Prop.2.3.30] under different notations. The equality $\gamma_M(p_M) = \tau_M(\tau^* M) \otimes_{A \otimes K} \mathcal{O}_X[j]$ follows from the commutativity of the above diagram together with the fact that both $\gamma_M$ and $\delta_M$ are isomorphisms.

To apply Proposition 2.16 and conclude the proof of Theorem 3.28, we need a vanishing result of Galois cohomology, supplied by the next lemma.

Lemma 3.30. Let $l$ be a $K_* \mathcal{O}_X[j]$-lattice in $M_B \otimes_A K_* \mathcal{O}_X(j)$. Then, $l$ is $G_\infty$-equivariant and $H^1(G_\infty, l) = (0)$.

Proof. The $K_* \mathcal{O}_X[j]$-lattice $l$ is isomorphic to a $K_* \mathcal{O}_X[j]$-lattice in $M \otimes_{A \otimes K} K_* \mathcal{O}_X(j)$ via $\gamma_M$. By the elementary divisor Theorem in the discrete valuation ring $K_* \mathcal{O}_X[j]$, there exists a $G_\infty$-equivariant $K_* \mathcal{O}_X(j)$-linear automorphism $\psi$ of the $K_* \mathcal{O}_X(j)$-vector space $M \otimes_{A \otimes K} K_* \mathcal{O}_X(j)$ such that
\[ \gamma_M(l) = \psi(M \otimes_{A \otimes K} K_* \mathcal{O}_X[j]). \]
This implies that $l$ is $G_\infty$-equivariant and further that $l$ is isomorphic to $M \otimes_{A \otimes K} K_* \mathcal{O}_X[j]$ as a $K_* \mathcal{O}_X[j] \otimes_{G_\infty}$-module. By the additive Hilbert’s 90 Theorem we have $H^1(G_\infty, K_* \mathcal{O}_X[j]) = 0$ and it follows that $H^1(G_\infty, l) = 0$.

4 Regulators and finiteness theorems

4.1 Regulators of $A$-motives

Let $M = (M, \tau_M)$ be an $A$-motive over $K$. Recall from [Gaz22 §4] that there exists a unique $A \otimes A$-submodule $M_A \subset M$ with the following properties:
\begin{enumerate}
  \item $M_A$ is finitely generated and generates $M$ over $A \otimes K$,
  \item $M_A$ is stable through $\tau_M$; namely, $\tau_M(\tau^* M_A) \subset M_A^{-1}$,
  \item $M_A$ is maximal; i.e. not strictly included in another submodule satisfying 1 and 2
\end{enumerate}
The module $M_A$ is called the *maximal integral model* of $M$ over $A$. One can show that $M_A$ is a projective $A \otimes A$-module [Gaz22] Prop. 4.43 although this will not be needed below. We also give a name to the following submodule of $M[1^{-1}]$:

$$N_A := (M + \tau_M(\pi^* M)) \cap M_A[1^{-1}].$$

(4.1)

In *loc. cit.*, we introduced the sub-$A$-module of *integral* and *regulated* extensions of $\mathbb{I}$ by $M$ in $\text{AMot}_K$, denoted $\text{Ext}_A^{1, \text{reg}}(\mathbb{I}, M)$, and proved that the map

$$\iota : \frac{N_A}{(id - \tau_M)(M)} \longrightarrow \text{Ext}_A^{1, \text{reg}}(\mathbb{I}, M),$$

(4.2)

assigning to the class of $m \in N_A \subset M[1^{-1}]$ the extension whose middle object has underlying module $M \oplus (A \otimes K)$ and $\tau$-morphism $(\tau_M m)$ (with obvious arrows), is a natural isomorphism of $A$-modules [Gaz22] Thm. D+Cor. 5.10]. The author also formulated a conjecture [Gaz22, Con. 5.13] relating the above to the submodule of regulated extensions having everywhere good reduction.

Some computations suggested that $\text{Ext}_A^{1, \text{reg}}(\mathbb{I}, M)$ is generally not finitely generated (see below), contrary to what is expected in the number fields setting. Assuming $M$ rigid analytically trivial, this defect is measured by the morphism $r_B$ introduced in [35]: that there is a map:

$$r_B(M) : \text{Ext}_A^{1, \text{reg}}(\mathbb{I}, M) \longrightarrow H^1(G_\infty, M_B)$$

(4.3)

assigning to an extension of rigid analytically trivial $A$-motives the class of the continuous cocycle associated to the induced extension of $A$-linear representations of $G_\infty$. We introduce the following notations:

$$\text{Ext}_A^{1, \text{reg}}(\mathbb{I}, M) := \ker r_B(M), \quad \text{Cl}(M) := \coker r_B(M).$$

We are ready to state our first main result.

**Theorem 4.1.** Suppose that $M$ is rigid analytically trivial. Both $\text{Ext}_A^{1, \text{reg}}(\mathbb{I}, M)$ and $\text{Cl}(M)$ are finitely generated $A$-modules. If the weights of $M$ are all negative, then $\text{Cl}(M)$ is finite.

The proof of Theorem 4.1 (and Theorem 4.4 below) is postponed to the last section. Let us comment the above statement.

---

- Classically, the $\mathbb{Q}$-vector space $\text{Ext}_A^1(\mathbb{I}, M)$, consisting in extensions having everywhere good reduction of the unit motive by a mixed motive $M$ over $\mathbb{Q}$, is expected to be finite dimensional (e.g. [Sch91] §III). First observe that, in our analogy, Theorem 4.1 is the function field counterpart of this expectation: classically, the analogue of $r_B(M)$, given in [13], would rather have targeted the finite 2-group $H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), M_B)$, $M_B$ denoting the Betti realization of $M$. To that respect, the finite generation of the kernel of $r_B(M)$ is the counterpart of the statement that $\text{Ext}_A^1(\mathbb{I}, M)$ has finite dimension.

- A second observation, already announced and corroborating the analogy made in the above paragraph, is that the $A$-module $\text{Ext}_A^{1, \text{reg}}(\mathbb{I}, M)$ is typically not finitely generated. To wit, Theorem 4.1 roughly tells that a set of generators has - up to a finite set - the same cardinality as one for $H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), M_B)$. Yet, $G_\infty$ is not topologically finitely generated: by class field theory, its wild inertia group is topologically isomorphic to the group of 1-unit in $\mathcal{O}_\infty$, itself isomorphic to a countable product of $\mathbb{Z}_p$.

- The modules $\text{Ext}_A^{1, \text{reg}}(\mathbb{I}, M)$ and $\text{Cl}(M)$ seem to contain function field arithmetic informations, as showed in the case of Carlitz twists in [GM22a].

Let $M$ be rigid analytically trivial. We move to the definition of the regulator of $M$. Let $H^+$ denote the mixed Hodge-Pink structure $H^+_K(\mathbb{I}, M)$ attached to $M$ with coefficient ring
Let \( M = K_{\infty} \). In view of Proposition 3.26, the Hodge-Pink realization functor \( \mathcal{H}^+ \) is exact and hence induces an \( A \)-linear morphism of the corresponding extension groups:

\[
r_{\mathcal{H}^+}(M) : \text{Ext}^1_{A_{\text{Mot}}_{\mathcal{K}}}(1, M) \longrightarrow \text{Ext}^1_{\text{HPk}_{K_{\infty}}}(\mathbb{I}^+, H^+).
\]

From Definition 3.27, \( r_{\mathcal{H}^+}(M) \) maps the class of regulated exact sequences in \( A_{\text{Mot}}_{\mathcal{K}}^{\text{rat}} \) to that of Hodge additive strict exact sequences in \( \text{HPk}_{K_{\infty}}^+ \). Therefore, there is square:

\[
\begin{array}{ccc}
\text{Ext}^1_{A_{\text{reg}}}(1, M) & \xrightarrow{r_B(M)} & H^1(G_{\infty}, M_B) \\
\uparrow r_{\mathcal{H}^+}(M) & & \downarrow c \\
\text{Ext}^1_{\text{HPk}_{K_{\infty}}}(\mathbb{I}^+, H^+) & \xrightarrow{d_B^+} & H^1(G_{\infty}, (M_B)_{K_{\infty}})
\end{array}
\]

The left vertical arrow \( d_B^+ \) was introduced in Definition 2.13. The next lemma stems from the definitions:

**Lemma 4.2.** The square (4.5) commutes in the category of \( A \)-modules.

As such, \( r_{\mathcal{H}^+}(M) \) induces a morphism from the corresponding kernels. Accordingly, we fix the following namings:

**Definition 4.3.** We call the regulator of \( M \), and denote it by \( \mathcal{R}_{\mathcal{H}^+}(M) \), the restriction of \( r_{\mathcal{H}^+}(M) \):

\[
\mathcal{R}_{\mathcal{H}^+}(M) : \text{Ext}^1_{A_{\text{reg}}}(1, M) \longrightarrow \text{Ext}^1_{\text{HPk}_{K_{\infty}}}(\mathbb{I}^+, H^+)
\]

which is well-defined according to Lemma 4.2.

Our second main result is the following:

**Theorem 4.4.** Let \( M \) be a rigid analytically trivial \( A \)-motive over \( K \) whose weights are all negative. Then, the rank of \( \text{Ext}^1_{A_{\text{reg}}}(1, M) \) as an \( A \)-module equals the dimension of the \( K_{\infty} \)-vector space \( \text{Ext}^1_{\text{HPk}_{K_{\infty}}}(\mathbb{I}^+, H^+) \).

Regarding the above, it is natural to ask whether the image of \( \mathcal{R}_{\mathcal{H}^+}(M) \) forms a lattice of full rank in the \( K_{\infty} \)-vector space \( \text{Ext}^1_{\text{HPk}_{K_{\infty}}}(\mathbb{I}^+, H^+) \). This would be an analogue of Beilinson’s conjecture [Beilinson 8.4.1]. Surprisingly, this fails to hold in many situations: this fails in the simple case where \( M \) is the Carlitz \( n \)th twist, \( n \geq 1 \) being a multiple of the characteristic \( p \) (and will be studied in a subsequent work). Fortunately enough, such a situation does not occur for number fields.

### 4.2 The complex \( G \) and the fundamental exact sequence

Let \( M \) be a rigid analytically trivial \( A \)-motive over \( K \). In this subsection we introduce a chain complex \( G_M \) which will play a fundamental role in the proof of Theorems 4.1 and 4.4. It deserves its own definition:

**Definition 4.5.** Let \( G_M \) denote the complex of \( A \)-modules sitting in degrees 0 and 1:

\[
G_M = \left[ \frac{M \otimes_{A \otimes_{K} K_{\infty}} K_{\infty}(A)}{M_{\mathcal{A}}} \right]_{0}^{1}
\]

where the arrow is induced by \( m \mapsto m - \tau_M(m) \) on \( M \otimes_{A \otimes K} K_{\infty}(A) \).

We have a commutative diagram of \( A \)-modules with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & M_{\mathcal{A}} & \longrightarrow & M \otimes_{A \otimes K} K_{\infty}(A) & \longrightarrow & G_M^0 \\
\downarrow \text{id} - \tau_M & & \downarrow \text{id} - \tau_M & & \downarrow d^1 \\
0 & \longrightarrow & N_{\mathcal{A}} & \longrightarrow & M \otimes_{A \otimes K} K_{\infty}(A) & \longrightarrow & G_M^1 \\
\end{array}
\]

The next proposition introduces what we believe to be fundamental for the sequel:
Proposition 4.6. The Snake Lemma applied to the diagram (4.6) results in a long-exact sequence of $A$-modules:

\[
0 \to \text{Hom}_{A^\text{Mot}_K}(1, M) \to M_B^+ \to H^0(G_M) \to \text{Ext}^1_A(1, M) \to r_B(M) \mapsto H^1(G_\infty, M_B) \to H^1(G_\infty, M_B^+) \to 0.
\]

Proof. The complex $[M, \text{id} - \tau M, N_A]$ has $\text{Hom}_{A^\text{Mot}_K}(1, M)$ and $\text{Ext}^1_A(1, M)$ for 0th and 1st-cohomology modules (see [Gaz22 Prop. 4.23] and (4.6)). The kernel and the cokernel of the middle vertical arrow is computed by Theorem 3.13. Hence, everything is clear but, perhaps, that the map $r_B(M)$ is the true one that appears at the level of the cokernels. To prove the latter, first observe that Theorem 3.13 gives an isomorphism of $A$-modules:

\[
\frac{M \otimes_{A^\otimes K} K_\infty(A)}{(\text{id} - \tau M)(M \otimes_{A^\otimes K} K_\infty(A))} \cong H^1(G_\infty, M_B).
\]

The above map assigns to the class of $f \in M \otimes_{A^\otimes K} K_\infty(A)$ the class of the cocycle $cf : \sigma \mapsto \sigma\xi_f - \xi_f$, where $\xi_f \in M \otimes_{A^\otimes K} K_\infty(A)$ is any solution $\xi$ of the equation $\xi - \tau M(\tau^*\xi) = f$. The class of $cf$ then does not depend on the choice of $\xi_f$, whose existence is provided by Theorem 3.13.

We turn to an explicit description of the map $r_B(M)$. Choose $m \in N_A$ and let $[E] \in \text{Ext}^1_A(1, M)$ be the extension $(m)$. The set underlying the Betti realization of $[E]$ consists of pairs $(\xi, a)$, $\xi \in M \otimes_{A^\otimes K} \mathbb{C}_\infty(A)$ and $a \in K_\infty(A)$, solution of the system

\[
\begin{pmatrix}
\tau M & m \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\tau^*\xi \\
\tau^*a
\end{pmatrix}
= \begin{pmatrix}
\xi \\
a
\end{pmatrix}.
\]

It follows that $a \in A$ and $\xi - \tau M(\tau^*\xi) = am$. A splitting of $0 \to M_B \to E_B \to A \to 0$ in the category of $A$-modules corresponds to the choice of a solution $\xi_m \in M \otimes_{A^\otimes K} \mathbb{C}_\infty(A)$ of the equation $\xi - \tau M(\tau^*\xi) = m$. To the choice of $\xi_m$, corresponds the decomposition

\[
M_B \oplus 1_B \cong E_B, \quad (\omega, a) \mapsto (\omega + a\xi_m, a).
\]

An element $\sigma \in G_\infty$ acts on the right-hand side by

\[
(\omega + a\xi_m, a) \mapsto (\sigma(\omega + a\xi_m), a) = (\sigma\omega + a(\sigma\xi_m - \xi_m) + a\xi_m, a)
\]

where $\sigma\xi_m - \xi_m \in M_B$. Hence, $\sigma$ acts as the matrix $\left(\begin{smallmatrix} \sigma & -\xi_m \\ 0 & 1 \end{smallmatrix}\right)$. We deduce from this computation that the square

\[
\begin{array}{ccc}
N_A & \xrightarrow{i} & M \otimes_{A^\otimes K} K_\infty(A) \\
\downarrow (\text{id} - \tau M)(M_A) & & \downarrow (\text{id} - \tau M)(M \otimes_{A^\otimes K} K_\infty(A)) \\
\text{Ext}^1_A(1, M) & \xrightarrow{r_B(M)} & H^1(G_\infty, M_B)
\end{array}
\]

where the top arrow stems from the inclusion $N \subset M \otimes_{A^\otimes K} K_\infty(A)$, commutes in the category of $A$-modules. This ends the proof of the Proposition.

4.3 Strategy of proofs

In the following section, we develop the theory of shtuka models for $M$. Roughly speaking, they are the data of $(M, N, \tau M)$ where $M \to N$ are two coherent sheaves on $\text{Spec } A \times C$, and where $\tau M : \tau^*M \to N$ is a morphism which restricts to $\tau_M : \tau^*M \to N_A$ on the affine part $\text{Spec } A \times \text{Spec } C$ together with some vanishing condition along $\infty$ (Definition 5.1). Shtuka models for $M$ always exist (Proposition 5.2), though they are far from being unique. If the weights of $M$ are negative, our construction extends to sheaves on $C \times C$. 

\[\text{Page 24}\]
One of the main steps towards the proof of the theorems is that there is a quasi-isomorphism
\[ G_M \overset{\sim}{\longrightarrow} \text{R}^\cdot \left( \text{Spec} \ A \times C, \mathcal{M}^{-\tau} \mathcal{N} \right) \quad \text{(Proposition 5.7).} \] (Definition 5.3, Theorem 5.6)

That \( C \) is proper over \( \mathbb{F} \) leads to the perfectness of \( G_M \). The fundamental exact sequence of Proposition 4.4 then implies that both \( \text{Ext}^{1, \text{reg}}_{A, \mathbb{F}}(1, M) \) and \( \text{Cl}(M) \) are finitely generated \( A \)-modules, thus completing the proof of the first part of Theorem 4.1.

Theorem 4.3 and the second part of Theorem 4.1 follow from subtler observations on \( C \times C \)-shtuka models along \( \text{Spf} \ \mathcal{O}_\mathbb{F} \times \text{Spf} \ \mathcal{O} \). Denoting by \( \mathcal{M} \) and \( \mathcal{N} \) the corresponding formal sheaves and by \( \mathcal{M}_\mathbb{F} \) and \( \mathcal{N}_\mathbb{F} \) the \( \mathcal{O}_\mathbb{F} \otimes \mathcal{O} \)-modules of their global sections, the **coup de théâtre** of Subsection 5.3 is an exact sequence
\[ 0 \longrightarrow \mathcal{M}_\mathbb{F} \otimes_A K \longrightarrow \frac{\mathcal{N}_\mathbb{F}}{(1 - \tau_M)(\mathcal{M}_\mathbb{F})} \otimes \mathcal{O} K \longrightarrow \text{Ext}^{1, \text{ha}}_{A, \mathbb{F}}(1^+, \mathcal{M}_\mathbb{F})(\mathcal{M}_\mathbb{F}) \longrightarrow 0 \] (see Corollary 5.19). The material used to construct this sequence resembles much to the techniques employed by V. Lafforgue in [Laf09, §4]. Although we do use the theory of function fields Fontaine rings as developed in [GL11] or [Har09], the results involved might be reminiscent of such a theory at the place \( \infty \).

Now, using classical derived algebro-geometric methods, we show that there is a quasi-isomorphism of complexes of \( K_\mathbb{F} \)-vector spaces:
\[ \text{R}^\cdot \left( \text{Spf} \ \mathcal{O}_\mathbb{F} \times C, \mathcal{M}^{-\tau} \mathcal{N} \right) \cong \frac{\mathcal{N}_\mathbb{F}}{(1 - \tau_M)(\mathcal{M}_\mathbb{F})}_{[0]} \] (4.9)

The last ingredient in our proof is Grothendieck’s comparison Theorem between Zariski and formal coherent cohomology, which amounts to a quasi-isomorphism among the left-hand side of (4.9) tensored along \( \mathcal{O} \rightarrow K \) and the right-hand side of (4.8) tensored along \( A \rightarrow K \).

This reads both that \( G_M \otimes_A K \) sits in degree 0 - hence that \( \text{Cl}(M) \) is torsion - and that the dimension of \( \text{Ext}^{1, \text{ha}}_{A, \mathbb{F}}(1^+, \mathcal{M}_\mathbb{F})(\mathcal{M}_\mathbb{F}) \) is the same as that of \( \text{Ext}^{1, \text{reg}}_{A, \mathbb{F}}(1, M) \otimes_A K \); hence achieving the proof of the main Theorems.

## 5 Shtuka models à la Mornev

Let \( M \) be an \( A \)-motive over \( K \) (we may not need rigid analytically triviality). In this section, we associate non-canonically to \( M \) a **shtuka model** on \( (\text{Spec} \ A) \times C \) and, whenever \( M \) only has non-positive weights, on \( C \times C \). This powerful technique was, to the knowledge of the author, first introduce in [Mor18 §12] in the context of Drinfeld modules with everywhere good reduction. An incarnation of this construction seems already to appear in [Fan15] Def. 1.13 in the setting of Anderson \( t \)-modules. Although our motivations owe much to Mornev’s work, our definition of \( C \times C \)-shtuka models differs. The one presented below in Definition 5.4 has the nice feature to carry an existence result by simply assuming that the weights of \( M \) are non-positive (Theorem 5.6).

Let us introduce some notations. We still denote by \( \tau : C \times C \rightarrow C \times C \) the morphism of \( \mathbb{F} \)-schemes which acts as the identity on the left-hand factor \( C \) and as the absolute \( q \)-Frobenius on the right-hand one. Because \( C \) is separated over \( F \), the diagonal morphism \( C \rightarrow C \times C \) is a closed immersion and its image defines a closed subscheme \( \Delta \) of \( C \times C \) of codimension 1. It defines an effective divisor \( \Delta \) on \( C \times C \). The evaluation of \( \mathcal{O}(\Delta) \) at the
affine open subscheme Spec(A ⊗ A) of C × C recovers the ideal j of A ⊗ A.

We also borrow notations from [Gaz22]. For R a Noetherian F-algebra, A_∞(R) stand
for the ring
\[ A_∞(R) = \varprojlim_n (O_∞ ∩ R)/(m^n_∞ ⊗ R). \]
This ring was considered to define isocrystals and mixedness in loc. cit. Let also B_∞(R) be
the ring K_∞ ⊗ A_∞(R). Geometrically, the formal spectrum Spec A_∞(R) corresponds to
the completion of the Noetherian scheme C × Spec R along the closed subscheme \{∞\} × R,
that is:
\[ \text{Spf} A_∞(R) = \text{Spf} O_∞ × \text{Spec} R. \]
Dually, when the completion is done at the level of the base ring instead of at the level of
coefficients, we obtain the algebra introduced in the context of Betti realizations, namely
\[ O_∞(A), \text{defined in Section 3 as} \]
\[ O_∞(A) = \varprojlim_n (A ⊗ O_∞)/(A ⊗ m^n_∞). \]
Similarly, Spf O_∞(A) is the completion of Spec(A ⊗ O_∞) along Spec A × {∞}.

The closed subscheme C × {∞} defines an effective divisor on C × C which we denote
∞C. Similarly, we let ∞A be the effective divisor (Spec A) × {∞} of (Spec A) × C.

### 5.1 C-shtuka models

**Definition 5.1.** A C-shtuka model for \( M \) is the datum \((N, M, \tau_M)\) of

(a) A coherent sheaf \( N \) on \( \text{Spec}(A) \times C \) such that \( N(\text{Spec } A ⊗ A) = N_A \),

(b) A coherent subsheaf \( M \) of \( N \) such that \( M(\text{Spec } A ⊗ A) = M_A \) and for which the

cokernel of the inclusion \( \iota : M → N \) is supported at \( \Delta \),

(c) A morphism \( \tau_M : τ^*M → N(-∞A) \) which coincides with \( \tau_M : τ^*M_A → N_A \) on the

affine open subscheme Spec \( A ⊗ A \).

**Proposition 5.2.** A C-shtuka model for \( M \) exists.

*Proof.* Let \( B \) be a sub-F-algebra of \( K \) such that \( (\text{Spec } A) ∪ (\text{Spec } B) \) forms an affine open
covering of \( C \) in the Zariski topology. Let \( D \) be the sub-F-algebra of \( K \) containing both \( A \) and
\( B \) and such that Spec \( D = (\text{Spec } A) \cap (\text{Spec } B) \). For \( S ∈ \{A, B, D\} \), we let \( j_S \) be the
ideal of \( A ⊗ S \) given by either \( j_A := 1, j_D := j(A ⊗ D) \) and \( j_B := j_D ∩ (A ⊗ B) \). Note that
\( \mathcal{O}(\Delta)(\text{Spec } A ⊗ S) = j_S \).

Let \( M_D \) be the \( A ⊗ D \)-module \( M_A ⊗_A D \), and let \( M_B^j \) be an \( A ⊗ B \)-lattice in \( M_D \)
(for instance, if \( m_1, ..., m_s \) are generators of \( M_D \), consider \( M_B^j \) to be the \( A ⊗ B \)-submodule
spanned by \( m_1, ..., m_s \)).

Since \( τ_M(τ^*M_A) ⊂ M_A[j_A^{-1}] \), we have \( τ_M(τ^*M_D) ⊂ M_D[j_D^{-1}] \). However, it might not be
true that \( τ_M(τ^*M_B^j) ⊂ M_B^j[j_B^{-1}] \). Yet, there exists \( d ∈ B \) invertible in \( D \) such that
\[ τ_M(τ^*M_B^j) ⊂ d^{-1}M_B^j[j_B^{-1}]. \]

Let \( r ∈ B \) invertible in \( D \) which vanishes \( 2 \) at \( \infty \) and let \( M_B := (rd)M_B^j \). We now have
\[ τ_M(τ^*M_B) ⊂ rM_B[j_B^{-1}]. \]

---

2Let \( x \) be a closed point on \( C \) distinct from \( ∞ \). Then \( B := H^0(C \setminus \{x\}, O_C) \) works. In the latter case,
we have \( D := H^0(C \setminus \{∞, x\}, O_C) \).

3Such an \( r \) always exists: the divisor \( D := \deg(x) \cdot ∞ − \deg(∞) \cdot x \) has degree zero so that \( nD \) is principal
for \( n \) large enough \((C^0(K)) \) is finite [Hos22, Lem. 5.6]). Chosing \( r \) such that \( (r) = nD \), then \( r ∈ B \) and \( r \) is
invertible in \( D \).
Since $r$ is invertible in $D$, the multiplication maps furnish \textit{gluing} isomorphisms

$$M_A \otimes_A D \xrightarrow{\sim} M_D \xleftarrow{\sim} M_B \otimes_B D.$$  \hfill (5.1)

For $S \in \{A, B, D\}$, we set $N_S := (M + \tau_M(\tau^*M)) \cap M_S[1]$. $N_S$ is an $A \otimes S$-module of finite type which contains $M_S$. By flatness of $D$ over $A$ (resp. $B$), the multiplication maps also are isomorphisms:

$$N_A \otimes_A D \xrightarrow{\sim} N_D \xleftarrow{\sim} N_B \otimes_B D.$$  \hfill (5.2)

Let $\mathcal{M}$ (resp. $\mathcal{N}$) be the coherent sheaf on $\text{Spec} \times C$ resulting from the glueing (5.1) (resp. (5.2)) by Zariski descent. Since $M_A \subset N_A$ and $M_B \subset N_B$, $\mathcal{M}$ is a subsheaf of $\mathcal{N}$. We further have $M_A[j^{-1}] = N_A[j^{-1}]$ and $M_B[j^{-1}] = N_B[j^{-1}]$ which implies that the cokernel of $\mathcal{M} \subset \mathcal{N}$ is supported at $\Delta$.

Because $\tau_{\mathcal{M}}(\tau^*\mathcal{M}) \subset N_S$ for all $S \in \{A, B, D\}$, one obtains by glueing a morphism of $\mathcal{O}_{(\text{Spec} A) \times C}$-modules $\tau_{\mathcal{M}} : \tau^*\mathcal{M} \to \mathcal{N}$. Since $\tau_{\mathcal{M}}(\tau^*M_B) \subset rN_B$ and $r$ vanishes at $\infty$, we also have $\tau_{\mathcal{M}}(\tau^*\mathcal{M}) \subset \mathcal{N}(-\infty A)$.

Let $\mathcal{M} = (N, \mathcal{M}, \tau_{\mathcal{M}})$ be a $C$-shtuka model for $\mathcal{M}$. The assumption that the image of the $N$ lands in $\mathcal{N}(-\infty A)$ is crucial for the incoming cohomological considerations. In much of what follows, this is materialized by the next lemma.

\textbf{Lemma 5.3.} Let $i : \text{Spec} \mathcal{O}_\infty(A) \to \text{Spec} A \otimes \mathcal{O}_\infty \leftarrow (\text{Spec} A) \times C$ be the canonical composition of $A$-schemes. The inclusion of sheaves $i^*\mathcal{M} \subset i^*\mathcal{N}$ is an equality and the induced morphism

$$\iota - \tau_{\mathcal{M}} : i^*\mathcal{M}(\text{Spec} \mathcal{O}_\infty(A)) \to i^*\mathcal{N}(\text{Spec} \mathcal{O}_\infty(A))$$

is an isomorphism of $\mathcal{O}_\infty(\mathcal{A})$-modules.

\textbf{Proof.} By Lemma 3.6, we have $i^*\mathcal{O}_\infty(A) = \mathcal{O}_\infty(A)$. In particular, $i^*\Delta$ is the empty divisor of $\text{Spec} \mathcal{O}_\infty(A)$. The equality between $i^*\mathcal{M}$ and $i^*\mathcal{N}$ follows.

Let $\pi_\infty$ be a uniformizer of $\mathcal{O}_\infty$. We denote by $\Xi$ the $\mathcal{O}_\infty(\mathcal{A})$-module $i^*\mathcal{M}(\text{Spec} \mathcal{O}_\infty(A))$. Because $\tau_{\mathcal{M}}(\tau^*\mathcal{M}) \subset \mathcal{N}(-\infty A)$, we have $\tau_{\mathcal{M}}(\tau^*\Xi) \subset \pi_\infty \Xi$. In particular, for all $\xi \in \Xi$, the series

$$\psi := \sum_{n=0}^{\infty} \tau_{\mathcal{M}}^{\pi_n}(\tau^*\xi)$$

converges in $\Xi$. The assignment $\xi \mapsto \psi$ defines an inverse of $id - \tau_{\mathcal{M}}$ on $\Xi$. \hfill \Box

\subsection{5.2 \textit{C} \times \textit{C}-shtuka models}

We want to extend the construction of Proposition 5.2 from $(\text{Spec} A) \times C$ to $\textit{C} \times \textit{C}$.

\textbf{Definition 5.4.} A $\textit{C} \times \textit{C}$-shtuka model $\mathcal{M}$ for $\mathcal{M}$ is the datum $(\mathcal{N}, \mathcal{M}, \tau_{\mathcal{M}})$ of

(a) a coherent sheaf $\mathcal{N}$ on $\textit{C} \times \textit{C}$ such that $\mathcal{N}(\text{Spec} \times \mathcal{A}) = N_A$,

(b) a coherent subsheaf $\mathcal{M}$ of $\mathcal{N}$ such that $\mathcal{M}(\text{Spec} \times \mathcal{A}) = M_A$ and such that the cokernel of the inclusion $\iota : \mathcal{M} \to \mathcal{N}$ is supported at $\Delta$,

(c) a morphism of sheaves $\tau_{\mathcal{M}} : \tau^*\mathcal{M} \to \mathcal{N}(-\infty C)$ which coincides with $\tau_{\mathcal{M}} : \tau^*M_A \to N_A$ on $\text{Spec} \times \mathcal{A}$.

\textbf{Remark 5.5.} Clearly, the restriction of a $\textit{C} \times \textit{C}$-shtuka model for $\mathcal{M}$ on $(\text{Spec} A) \times C$ is a $\textit{C}$-shtuka model for $\mathcal{M}$.

The main result of this subsection is the following:

\textbf{Theorem 5.6.} If the weights of $\mathcal{M}$ are non-positive, a $\textit{C} \times \textit{C}$-shtuka model for $\mathcal{M}$ exists.

Before initiating the proof of Theorem 5.6, which will take us the remaining of this subsection, we shall supply some ingredients on function fields isocrystals with negative weights.
Lemma 5.7. Let $M$ be an $A$-motive over $K$ whose weights are all non-positive. Then $M \otimes_{A \otimes K} \mathcal{B}_\infty(K)$ contains an $A_{\infty}(K)$-lattice stable by $\tau_M$.

Proof. The proof for the $A = F[t]$-case is essentially [Har11, Lem. 1.5.9]. The general $A$-case follows the same lines, using Dieudonné Theory as in [Gaz22, §3.1].

If the weights of $M$ are further negative, the same strategy shows the following:

Lemma 5.8. Let $M$ be an $A$-motive over $K$ whose weights are all negative. There exist an $A_{\infty}(K)$-lattice $T$ in $M \otimes_{A \otimes K} \mathcal{B}_\infty(K)$ and two positive integers $d$ and $h$ such that $\tau_M^h(\tau^h T) \subset m_d^T$.

Proof of Theorem 5.6. We use the notations and definitions of the proof of Proposition 5.2. That is, $B$ is a sub-$F$-algebra of $K$ such that $(\text{Spec } A) \cup (\text{Spec } B)$ forms an open affine cover of $C$, $D$ is the sub-$F$-algebra of $K$ containing $A$ and $B$, and $\text{Spec } D = (\text{Spec } A) \cap (\text{Spec } B)$.

Let $\mathcal{M}_0 = (N_0, M_0, \tau_0)$ be a $C$-shtuka model for $M$. We recover the notations of the proof of Proposition 5.2 by setting:

$$M_A := \mathcal{M}_0(\text{Spec } A \otimes A), \quad M_B := \mathcal{M}_0(\text{Spec } A \otimes B), \quad M_D := \mathcal{M}_0(\text{Spec } A \otimes D),$$

and similarly for $N$ and $N_0$. Because the weights of $M$ are non-positive, there exists by Lemma 5.8 an $A_{\infty}(K)$-lattice $T$ in $M \otimes_{A \otimes K} \mathcal{B}_\infty(K)$ stable by $\tau_M$. We introduce six sub-modules of $T$, namely:

(i) two sub-$A_{\infty}(A)$-modules of $T$:

$$T_A := T \cap (M_A \otimes_{A \otimes A} \mathcal{B}_\infty(A)), \quad U_A := T \cap (N_A \otimes_{A \otimes A} \mathcal{B}_\infty(A)).$$

(ii) two sub-$A_{\infty}(B)$-modules of $T$:

$$T_B := T \cap (M_B \otimes_{A \otimes B} \mathcal{B}_\infty(B)), \quad U_B := T \cap (N_B \otimes_{A \otimes B} \mathcal{B}_\infty(B)).$$

(iii) and two sub-$A_{\infty}(D)$-modules of $T$:

$$T_D := T \cap (M_D \otimes_{A \otimes D} \mathcal{B}_\infty(D)), \quad U_D := T \cap (N_D \otimes_{A \otimes D} \mathcal{B}_\infty(D)).$$

The first two $A_{\infty}(A)$-modules are in fact equal. Indeed, as $j\mathcal{B}_\infty(A) = \mathcal{B}_\infty(A)$, and since the inclusion $A \otimes A \to \mathcal{B}_\infty(A)$ is flat, we have

$$N_A \otimes_{A \otimes A} \mathcal{B}_\infty(A) = [(M + \tau_M(\tau^* M)) \cap M_A[1]] \otimes_{A \otimes A} \mathcal{B}_\infty(A)$$

$$= [(M + \tau_M(\tau^* M)) \otimes_{A \otimes A} \mathcal{B}_\infty(A)] \cap [M_A[1]] \otimes_{A \otimes A} \mathcal{B}_\infty(A)$$

$$= [M \otimes_{A \otimes K} \mathcal{B}_\infty(K)] \cap [M_A \otimes_{A \otimes A} \mathcal{B}_\infty(A)]$$

$$= M_A \otimes_{A \otimes A} \mathcal{B}_\infty(A).$$

Our aim is to glue together $M_A, M_B, T_A$ and $T_B$ (resp. $N_A, N_B, U_A$ and $U_B$) to obtain $\mathcal{M}$ (resp. $\mathcal{N}$) along the covering $\text{Spec } A \otimes A, \text{Spec } A \otimes B, \text{Spec } A_{\infty}(A)$ and $\text{Spec } A_{\infty}(B)$ of $C \times C$.

This covering is not Zariski, so we will use the Beauville-Laszlo Theorem [BL05] to carry out the gluing process. By functoriality, the morphism $\tau_M$ will result as the glueing of

$$\begin{array}{cccc}
\tau^* M_A & \tau^* M_B & \tau^* T_A & \tau^* T_B \\
N_A & N_B & U_A & U_B
\end{array}$$

along the corresponding covering. Note that the first two arrows glue together as they arise from $\mathcal{M}_0$. 

28
Step 1: the modules $T_A$, $T_B$, $U_A$ and $U_B$ are finitely generated. We prove finite generation for $T_A$ (the argument for $T_B$, $U_A$ and $U_B$ being similar).

Let $L$ be a finite free $A \otimes A$-module containing $M_A$ and let $n = (n_1, \ldots, n_s)$ be a basis of $L$. For any element $m$ in $L \otimes A \otimes A \otimes B_\infty(K)$, we denote by $v_\infty(m)$ the minimum of the $\infty$-valuations of the coefficients of $m$ in $n$. Let also $\Lambda \subset L \otimes A \otimes A \otimes B_\infty(K)$ be the finite free $A_\infty(A)$-module generated by $n$. Clearly, $v_\infty(\lambda) \geq 0$ for any $\lambda \in \Lambda$. As $T \subset \Lambda \otimes A \otimes K \otimes B_\infty(K)$ is finitely generated over $A_\infty(K)$, there exists a positive integer $v_T$ such that $v_\infty(t) \geq -v_T$ for all $t \in T$.

Let $x \in T_A \setminus \{0\}$. Because $A \otimes A \to B_\infty(A)$ is flat, we have the inclusions

$$T_A \subset M_A \otimes A \otimes B_\infty(A) \subset \Lambda \otimes A \otimes B_\infty(A) = \bigcup_{n=0}^{\infty} \pi_\infty^{-n} \Lambda,$$

and there exists a non-negative integer $n$ such that $x = \pi_\infty^{-n} m$ for some $m \in \Lambda \setminus \pi_\infty \Lambda$. Comparing valuations yields

$$n = v_\infty(m) - v_\infty(x) \leq v_\infty(m) + v_T.$$

The number $v_\infty(m)$ cannot be positive, otherwise we would have $m \in \pi_\infty (\Lambda \otimes A \otimes A_\infty(K))$, which contradicts our assumption $m \notin \pi_\infty \Lambda$. Thus, $n \leq v_T$ and it follows that

$$x \in \bigcup_{n=0}^{v_T} \pi_\infty^{-n} \Lambda.$$

Consequently, $T_A \subset \bigcup_{n=0}^{v_T} \pi_\infty^{-n} \Lambda$ and, because $A_\infty(A)$ is Noetherian, $T_A$ is finitely generated.

Step 2: $T_A \otimes A D$ and $T_B \otimes B D$ (resp. $U_A \otimes A D$ and $U_B \otimes B D$) are dense in $T_D$ (resp. $U_D$) for the $m_\infty$-adic topology. We only prove the density of $T_A \otimes A D$ in $T_D$ since the argument for the others follows along the same lines.

Let $t \in T_D = T \cap (M_D \otimes A \otimes B_\infty(D))$. Let $(m_1, \ldots, m_s)$ be generators of $M_A$ as an $A \otimes A$-module. $t$ can be written as a sum $\sum_{i=1}^r m_i \otimes b_i$ with coefficients $b_i \in B_\infty(D)$. For $i \in \{1, \ldots, r\}$, let $(b_{i,n})_{n \in \mathbb{Z}}$ be a sequence in $B_\infty(A) \otimes A D$, such that $b_{i,n} = 0$ for $n \ll 0$, satisfying $b_i - b_{i,n} \in m_\infty^n A_\infty(D)$ for all $n \in \mathbb{Z}$. In particular, $(b_{i,n})_{n \in \mathbb{Z}}$ converges to $b_i$ when $n$ tends to infinity. For $n \in \mathbb{Z}$, we set:

$$t_n := \sum_{i=1}^r m_i \otimes b_{i,n} \in (M_A \otimes A \otimes B_\infty(A)) \otimes A D.$$

Then, $t - t_n$ belongs to $m_\infty^n \Xi$ where $\Xi$ is the $A_\infty(D)$-module generated by $(m_1, \ldots, m_s)$. For $n$ large enough, $m_\infty^n \Xi \subset T$, hence $t - t_n \in T$ and $t_n \in T$. We deduce that $t_n \in T_A \otimes A D$ for large value of $n$ and that $(t_n)_{n \in \mathbb{Z}}$ converges to $t$ when $n$ goes to infinity. We conclude that $T_A \otimes A D$ is dense in $T_D$.

Steps 1&2 $\implies$ compatibility. Because $T_A$ and $T_B$ are finitely generated over $A_\infty(A)$ and $A_\infty(B)$ respectively, $T_A \otimes A_\infty(A)$ is isomorphic to $T_B \otimes A_\infty(B) \otimes A_\infty(D)$ with the completion of $T_B \otimes B D$ (by [Bou07], (AC)§3 Thm. 3.4.3). Therefore, the multiplication maps are isomorphisms:

$$T_A \otimes A_\infty(A) \otimes A_\infty(D) \cong T_D \otimes B \otimes A_\infty(B) \otimes A_\infty(D),$$

$$U_A \otimes A_\infty(A) \otimes A_\infty(D) \cong U_D \otimes U_B \otimes A_\infty(B) \otimes A_\infty(D).$$
Step 3: the glueing. We consider the morphisms of formal schemes over $\text{Spf } \mathcal{O}_\infty$

$$\text{Spf } \mathcal{A}_\infty(A) = \text{Spf } \mathcal{O}_\infty \hat{\otimes} A \xrightarrow{i} \text{Spf } \mathcal{O}_\infty \hat{\otimes} C \xrightarrow{j} \text{Spf } \mathcal{O}_\infty \hat{\otimes} B = \text{Spf } \mathcal{A}_\infty(B).$$

By the Beauville-Laszlo Theorem [BL95], there exists a unique pair of coherent sheaves $(\mathcal{M}, \mathcal{N})$ of $\mathcal{O}_{C \times C}$-modules such that

$$\mathcal{M}(\text{Spec } A \otimes A) = M_A \quad \mathcal{N}(\text{Spec } A \otimes A) = N_A$$
$$\mathcal{M}(\text{Spec } A \otimes B) = M_B \quad \mathcal{N}(\text{Spec } A \otimes B) = N_B$$
$$(i^* \mathcal{M})(\text{Spf } \mathcal{O}_\infty \hat{\otimes} A) = T_A \quad (i^* \mathcal{N})(\text{Spf } \mathcal{O}_\infty \hat{\otimes} A) = U_A$$
$$(j^* \mathcal{M})(\text{Spf } \mathcal{O}_\infty \hat{\otimes} B) = T_B \quad (j^* \mathcal{N})(\text{Spf } \mathcal{O}_\infty \hat{\otimes} B) = U_B$$

Since, for each row of the above table, the left-hand side is canonically a submodule of the right-hand side, we have $\mathcal{M} \subset \mathcal{N}$. Because these inclusions become equalities away from $\Delta$, we deduce that the cokernel of the inclusion $\mathcal{M} \subset \mathcal{N}$ is supported at $\Delta$. The glueing of (5.3) results in a morphism $\tau_M : \tau^* \mathcal{M} \to \mathcal{N}$. Finally, we recall that there exists $r \in B$ invertible in $D$ and vanishing at infinity such that $\tau_M(\tau^* M_B) \subset r N_B$ and thus $\tau_M(\tau^* T_B) \subset r U_B$. Hence, the image of $\tau_M$ lands in $\mathcal{N}(-\infty C)$. \hfill $\Box$

Remark 5.9. It is noteworthy that the converse of Theorem 5.8 do hold: if $\mathcal{M}$ admits a $C \times C$-shtuka model, then all the weights of $\mathcal{M}$ are non-positive. Indeed, one easily shows that the existence of such a shtuka model implies the existence of a stable $\mathcal{A}_\infty(K)$-lattice in $M \otimes_{A \otimes K} \mathcal{B}_\infty(K)$. Such an event happens only if the slopes of the $\infty$-isocrystal attached to $\mathcal{M}$ has non-negative slopes, i.e. $\mathcal{M}$ only has non-positive weights.

We fix $\mathcal{M}$ a $C \times C$-shtuka model of $\mathcal{A}_\infty$. Let $i : \text{Spec } \mathcal{A}_\infty(A) \to C \times C$. We denote by:

$$L_A := (i^* \mathcal{M})(\text{Spf } \mathcal{A}_\infty(A)) = (i^* \mathcal{N})(\text{Spf } \mathcal{A}_\infty(A))$$
$$L := L_A \otimes_{\mathcal{A}_\infty(A)} \mathcal{A}_\infty(K)$$

(for the first equality, we used that $\Delta$ is not supported at $\text{Spf } \mathcal{A}_\infty(A)$). $\tau_M$ induces an $\mathcal{O}_\infty$-linear endomorphism of $L$ (resp. $L_A$). The next lemma records the additional pleasant feature of shtuka models when the weights of $\mathcal{M}$ are all negative (this will be used later in Subsection 6.2).

Lemma 5.10. Assume that all the weights of $\mathcal{M}$ are negative. Then, the morphism $\text{id} - \tau_M$ induces an $\mathcal{O}_\infty$-linear automorphism of $L$ and $L_A$.

Proof. The statement for $L_A$ implies the one for $L$. Because the weights of $\mathcal{M}$ are negative, there is, by Lemma 5.8, an $\mathcal{A}_\infty(K)$-lattice $T$ in $M \otimes_{A \otimes K} \mathcal{B}_\infty(K)$ and two positive integers $h$ and $d$ such that $\tau_M^h(\tau^* T) = m_d T$.

To show that $\text{id} - \tau_M$ is injective on $L_A$, let $x$ be an element of $\ker(\text{id} - \tau_M[L_A])$. We may assume without loss that $x \in T$. For all positive integer $n$, $x = \tau_M^n(x) \in m_d T$.

Because $d > 0$, $x = 0$.

We turn to surjectivity. Let $T'$ be the $\mathcal{A}_\infty(K)$-lattice generated by the elements of $T$, $\tau_M(\tau^* T)$, ..., and of $\tau_M^{h-1}(\tau(\tau^* T))$. Then $T'$ is stable by $\tau_M$. Let $y \in L_A$ and let $k \geq 0$ be such that $\tau_M^k y \in T$. For all $n \geq 0$, we have $\tau_M^{nh} y \in m_d^{-k} T'$ and, in particular, for all $u \in \{0, 1, ..., h - 1\}$,

$$\tau_M^{nh+u}(\tau(\tau^* T)) y \in m_d^{-k} T'.$$

Therefore, the series

$$\sum_{n=0}^{\infty} \tau_M^n(\tau^* T) y = \sum_{n=0}^{h-1} \left( \sum_{u=0}^{h-1} \tau_M^{nh+u}(\tau(\tau^* T)) y \right)$$

calculates in $L_A$ to $f$ satisfying $f - \tau_M(\tau^* f) = y$. \hfill $\Box$
5.3 Shtuka models and extensions of mixed Hodge-Pink structures

Let $\mathcal{M}$ be a mixed and rigid analytically trivial $A$-motive over $K$ whose weights are all non-positive. Let $(N, \mathcal{M}, \pi_M)$ be a $C \times C$-shtuka model for $\mathcal{M}$, whose existence is ensured by Theorem 1.6. Let $\iota : \mathcal{M} \to N$ be the inclusion of sheaves. We consider the inclusion of ringed spaces

$$\text{Spf}\, \mathcal{A}_\infty(\mathcal{O}_\infty) = \text{Spf}\, \mathcal{O}_\infty \otimes_{\mathcal{O}_\infty} C \to C \times C$$

and denote respectively $\hat{N}$ and $\hat{M}$ the pullback of $N$ and $M$ through $\hat{t}$. Finally, denote by $\hat{N}_\infty$ and $\hat{M}_\infty$ the finitely generated $\mathcal{A}_\infty(\mathcal{O}_\infty)$-modules:

$$\hat{N}_\infty := \hat{N}(\mathcal{O}_\infty)\otimes_{\mathcal{O}_\infty} K, \quad \hat{M}_\infty := \hat{M}(\mathcal{O}_\infty)\otimes_{\mathcal{O}_\infty} K.$$

The aim of this subsection is to prove that there is an exact sequence of $K_\infty$-vector spaces (Corollary 6.11):

$$0 \to \hat{M}_\infty \otimes_A K_\infty \to \hat{N}_\infty \otimes_{\mathcal{O}_\infty} \mathcal{M}_\infty \otimes_{\mathcal{O}_\infty} K_\infty \to \text{Ext}^{1,\text{ha}}_{\hat{M}_\infty}(\mathcal{M}_\infty) \to 0.$$

This above sequence appeared to the author as the most miraculous part of the proof of Theorem 1.4. A surprising feature is that this property does not depend on the choice of the shtuka model. The reader will have no trouble to notice how much this subsection relies on ideas from V. Lafforgue in [La09 §4].

We start by a proposition.

**Proposition 5.11.** There is an isomorphism of $K_\infty$-vector spaces

$$\hat{N}_\infty \otimes_{\mathcal{O}_\infty} (\iota - \tau_M)(\mathcal{M}_\infty) \isom (\hat{N}_\infty/\hat{M}_\infty) \otimes_{\mathcal{O}_\infty} K_\infty.$$

We split the proof of Proposition 5.11 into several lemmas.

**Lemma 5.12.** There exists an injective $A_\infty(\mathcal{O}_\infty)$-linear morphism $\iota' : \hat{N}_\infty \to \hat{M}_\infty$ and a positive integer $e$ such that $\iota' t$ and $\iota t'$ coincide with the multiplication by $(\pi_\infty \otimes 1 - 1 \otimes \pi_\infty)^e$ on $\hat{M}_\infty$ and $\hat{N}_\infty$ respectively.

**Proof.** Let $\delta := O(\Delta)(\mathcal{O}_\infty)$ as an ideal of $A_\infty(\mathcal{O}_\infty)$. The cokernel of the inclusion $\iota : \hat{M}_\infty \to \hat{N}_\infty$ is $\delta$-torsion. It is also finitely generated, and since $\pi_\infty \otimes 1 - 1 \otimes \pi_\infty \in \delta$, there exists $e \geq 0$ such that $(\pi_\infty \otimes 1 - 1 \otimes \pi_\infty)^e v \in \hat{M}_\infty$ for all $v \in \hat{N}_\infty$. We let $\iota' : \hat{N}_\infty \to \hat{M}_\infty$ be the multiplication by $(\pi_\infty \otimes 1 - 1 \otimes \pi_\infty)^e$ and the lemma follows.

**Lemma 5.13.** Let $t$ be a positive integer. Then, $\iota - \tau_M$ and $\iota$ respectively induce isomorphisms of $K_\infty$-vector spaces:

$$\left(\frac{\hat{M}_\infty}{(1 \otimes \pi_\infty)^t \hat{M}_\infty}\right) \otimes_{\mathcal{O}_\infty} K_\infty \isom \left(\frac{\hat{N}_\infty}{(1 \otimes \pi_\infty)^t \hat{N}_\infty}\right) \otimes_{\mathcal{O}_\infty} K_\infty,$$

$$\left(\frac{\hat{M}_\infty}{(1 \otimes \pi_\infty)^t \hat{M}_\infty}\right) \otimes_{\mathcal{O}_\infty} K_\infty \isom \left(\frac{\hat{N}_\infty}{(1 \otimes \pi_\infty)^t \hat{N}_\infty}\right) \otimes_{\mathcal{O}_\infty} K_\infty.$$

**Proof.** Let $\iota'$ and $e \geq 0$ be as in Lemma 5.12. The multiplication by

$$\left(\sum_{k=0}^{e} \pi_\infty^{-(k+1)} \otimes \pi_\infty^k\right)^e = \left(\frac{1 - \pi_\infty^t}{\pi_\infty \otimes 1 - 1 \otimes \pi_\infty}\right)^e \isom (\pi_\infty \otimes 1 - 1 \otimes \pi_\infty)^{-e}$$

on $(\hat{M}_\infty/(1 \otimes \pi_\infty)^t \hat{M}_\infty) \otimes_{\mathcal{O}_\infty} K_\infty$ defines an inverse of $\iota' t$. The same argument shows that $\iota t'$ is an automorphism of $(\hat{N}_\infty/(1 \otimes \pi_\infty)^t \hat{N}_\infty) \otimes_{\mathcal{O}_\infty} K_\infty.$
On the other-hand, we have \((t'\tau_M)^k(t'\tau_M) = (1\oplus\pi_\infty)(t'\tau_M)_k \subset (1\oplus\pi_\infty)(t'\tau_M)_k\) for \(k\) large enough. Hence, \((t'\tau_M)\) is nilpotent on \(\hat{M}_\infty/((1\oplus\pi_\infty)^2M_\infty,\) and so is \((t')^{-1}(t'\tau_M)\). In particular, 
\[
(t'-(t\tau_M)) = (t')(\text{id}-(t')^{-1}(t'\tau_M))
\]
is an isomorphism. It follows that \(t-(t\tau_M)\) is injective and \((t')^{-1}\) surjective. Since \(t'\) is invertible, \(t'\) is injective. We deduce that \(t-(t\tau_M)\) and \((t')^{-1}\) and thus \(t\) are isomorphisms.

**Lemma 5.14.** Let \(t\) be a non-negative integer. Then, the canonical maps

\[
\begin{align*}
(1\oplus\pi_\infty)^2\hat{N}_\infty \otimes_{\mathcal{O}_\infty} K_\infty &\rightarrow (1\oplus\pi_\infty)^2\hat{M}_\infty \otimes_{\mathcal{O}_\infty} K_\infty, \\
(t-(t\tau_M))((1\oplus\pi_\infty)^2\hat{M}_\infty) \otimes_{\mathcal{O}_\infty} K_\infty &\rightarrow (1\oplus\pi_\infty)^2\hat{N}_\infty \otimes_{\mathcal{O}_\infty} K_\infty
\end{align*}
\]

are isomorphisms of \(K_\infty\)-vector spaces.

**Proof.** In the category of \(O_\infty\)-vector spaces, we have a diagram exact on lines and commutative on squares:

\[
\begin{array}{cccc}
0 & \rightarrow & (1\oplus\pi_\infty)^t\hat{M}_\infty & \rightarrow & \hat{M}_\infty & \rightarrow & \hat{M}_\infty/(1\oplus\pi_\infty)^t\hat{M}_\infty & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (1\oplus\pi_\infty)^t\hat{N}_\infty & \rightarrow & \hat{N}_\infty & \rightarrow & \hat{N}_\infty/(1\oplus\pi_\infty)^t\hat{N}_\infty & \rightarrow & 0
\end{array}
\]

By Lemma 5.12, the third arrow is the tensor product with \(K_\infty\) over \(O_\infty\). The second isomorphism comes from Lemma 5.12. We have \(t-(t\tau_M)\).

**Lemma 5.15.** For \(t\) large, we have \((t-(t\tau_M))((1\oplus\pi_\infty)^t\hat{M}_\infty) = (1\oplus\pi_\infty)^t\hat{M}_\infty\).

**Proof.** Let \(t'\) and \(e\) be as in Lemma 5.12. We chose \(t\) such that \((q-1)t > e\). For \(s \geq t\), let \(\hat{M}_s := (1\oplus\pi_\infty)^t\hat{M}_\infty\). \((\hat{M}_s)_{s \geq t}\) forms a decreasing family of \(A_\infty(O_\infty)\)-modules for the inclusion. It suffices to show that

\[
(t'-(t\tau_M))(\hat{M}_t) = (t')(\hat{M}_t).
\]

By our assumption on \(t\), we have \((t')^{-1}\tau_M(M_s) \subset \hat{M}_{s+1}\) for all \(s \geq t\). Hence, the endomorphism \(\text{id}-(t')^{-1}\tau_M\) of \(\hat{M}_t\) becomes an automorphism over the completion of \(\hat{M}_t\) with respect to the \((1\oplus\pi_\infty)\)-adic topology (equivalently, the topology which makes \((\hat{M}_s)_{s \geq t}\) a neighbourhood of 0 for all \(s \geq t\)). To conclude, it suffices to show that \(\hat{M}_t\) is already complete for this topology. Because \(\hat{M}_t\) is Noetherian, we have

\[
(\hat{M}_t)_s \cong \hat{M}_t \otimes_{A_\infty(O_\infty)} A_\infty(O_\infty)_{s\oplus(1\oplus\pi_\infty)},
\]

and it suffices to show that \(A_\infty(O_\infty)\) is complete for the \((1\oplus\pi_\infty)\)-adic topology. We have the identifications

\[
A_\infty(O_\infty) = (F_\infty \otimes O_\infty)[\pi_\infty \otimes 1] = (F_\infty \otimes F_\infty)[1 \oplus \pi_\infty, \pi_\infty \otimes 1]
\]

which allows us to conclude that \(A_\infty(O_\infty)\) is complete for the \((1\oplus\pi_\infty)\)-adic topology. 

**Proof of Proposition 5.11.** The desired isomorphism results of the composition

\[
\begin{array}{cccc}
\hat{N}_\infty & \otimes_{\mathcal{O}_\infty} K_\infty & \rightarrow & (1\oplus\pi_\infty)^2\hat{N}_\infty \otimes_{\mathcal{O}_\infty} K_\infty \\
\uparrow & & & \uparrow \\
\hat{N}_\infty/t(\hat{M}_\infty) \otimes_{\mathcal{O}_\infty} K_\infty & \rightarrow & (1\oplus\pi_\infty)^2\hat{N}_\infty \otimes_{\mathcal{O}_\infty} K_\infty
\end{array}
\]

32
For $v \in \hat{\mathcal{N}}_{\infty} \otimes_{\mathcal{O}_\infty} K_{\infty}$, the dashed morphism maps
\[ v + (t - \tau_M)(\hat{\mathcal{M}}_{\infty} \otimes_{\mathcal{O}_\infty} K_{\infty}) \mapsto v' + t(\hat{\mathcal{M}}_{\infty} \otimes_{\mathcal{O}_\infty} K_{\infty}), \]
where $v'$ is any element of $\hat{\mathcal{N}}_{\infty} \otimes_{\mathcal{O}_\infty} K_{\infty}$ satisfying
\[ v' - v \in t(\hat{\mathcal{M}}_{\infty} \otimes_{\mathcal{O}_\infty} K_{\infty}) + (t - \tau_M)(\hat{\mathcal{M}}_{\infty} \otimes_{\mathcal{O}_\infty} K_{\infty}). \]

Recall that the morphism $\nu : A \to K_{\infty}[i]$, $a \mapsto a \otimes 1$, extends to $K_{\infty}$ (e.g. [Gaz22, Lem. 5.1]). We record:

**Lemma 5.16.** The kernel $\nu$ of $\nu \otimes \text{id} : K_{\infty} \otimes K_{\infty} \to K_{\infty}[i]$ corresponds to the ideal generated by the set $\{ f \otimes 1 - 1 \otimes f \mid f \in \mathbb{F}_\infty \}$.

**Proof.** Let $d_{\infty} := [\mathbb{F}_\infty : \mathbb{F}]$. For $i \in \mathbb{Z}/d_{\infty}\mathbb{Z}$, we consider the ideal of $K_{\infty} \otimes K_{\infty}$ given by
\[ \mathfrak{d}^{(i)} = \{(f \otimes 1 - 1 \otimes f^i) \mid f \in \mathbb{F}_\infty \}. \]
It is the kernel of the map $K_{\infty} \otimes K_{\infty} \to K_{\infty}$, $a \otimes b \mapsto ab^i$, hence is a maximal ideal. For $f \in \mathbb{F}_\infty$, the polynomial $\prod_{i \in \mathbb{Z}/d_{\infty}\mathbb{Z}} (x - f^i)$ belongs to $\mathbb{F}[x]$, and thus the product of the $\mathfrak{d}^{(i)}$ is zero. By the chinese remainders Theorem, we have
\[ K_{\infty} \otimes K_{\infty} = K_{\infty} \otimes K_{\infty}/\mathfrak{d}^{(0)}\mathfrak{d}^{(1)} \cdots \mathfrak{d}^{(d_{\infty} - 1)} = \prod_{i \in \mathbb{Z}/d_{\infty}\mathbb{Z}} K_{\infty} \otimes K_{\infty}/\mathfrak{d}^{(i)} \]
which is a product of $d_{\infty}$ fields. Because $\mathfrak{v}$ is a prime ideal of $K_{\infty} \otimes K_{\infty}$, we have $\mathfrak{v} = \mathfrak{d}^{(i)}$ for some $i$. If $f \in \mathbb{F}_\infty$, then $f \otimes 1 - 1 \otimes f$ belongs to $\mathfrak{v}$ by definition. We deduce that $i = 0$. \( \square \)

We are almost in position to prove the main result of this section.

**Theorem 5.17.** Let $(\mathcal{M}, \mathcal{N}, \tau_M)$ be a $C \times C$-shtuka model for $\hat{\mathcal{M}}$. Then, there is an isomorphism of $K_{\infty}$-vector spaces
\[ \frac{\hat{\mathcal{N}}(\text{Spf} \mathcal{O}_\infty \otimes \mathcal{O}_\infty)}{(t - \tau_M)\mathcal{M}(\text{Spf} \mathcal{O}_\infty \otimes \mathcal{O}_\infty) \otimes_{\mathcal{O}_\infty} K_{\infty}} \cong (M + \tau_M(\tau^* M)) \otimes_{A \otimes K} K_{\infty}[i] \]
where the $K_{\infty}$-vector space structure on the right-hand side is given through $\nu$.

We begin with two preliminary lemmas concerning the ring $B_{\infty}(\mathcal{O}_\infty)$.

**Lemma 5.18.** Let $d \subset \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ be the ideal generated by elements of the form $a \otimes 1 - 1 \otimes a$ for $a \in \mathcal{O}_\infty$. The canonical morphism
\[ \overline{\mathcal{O}_\infty} \otimes \mathcal{O}_\infty \overline{d}^m \to \overline{A}_m(\mathcal{O}_\infty) \]
is an isomorphism for all $m \geq 1$.

**Proof.** The sequence of $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$-modules $0 \to d \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty \to 0$ is exact, and taking completion with respect to the $\pi_\infty$-adic topology on $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$ and the $\pi_\infty$-adic topology on $\mathcal{O}_\infty$ reads:
\[ 0 \to d \mathcal{A}_{\infty}(\mathcal{O}_\infty) \to \mathcal{A}_{\infty}(\mathcal{O}_\infty) \to K_{\infty} \to 0 \]
(e.g. [0315]) and the case $m = 1$ follows. Before treating the general $m$-case, observe that $\overline{d}/\overline{d}^2$ is the $\mathcal{O}_{\infty}$-module $\Omega_{\mathcal{O}_\infty/F}^1$ of Kähler differentials. In particular, $\overline{d}/\overline{d}^2$ is a free $\mathcal{O}_{\infty}$-module of rank 1. We deduce that for any $r \in d \setminus d^2$, the multiplication by $r$ induces an isomorphism of one-ranked $\mathcal{O}_{\infty}$-modules:
\[ (\mathcal{O}_\infty \otimes \mathcal{O}_\infty)/d \cong d/\overline{d}^2. \]
It follows that $\mathfrak{d} = \mathfrak{d}^2 + r \cdot \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ and hence $\mathfrak{d}^{m-1} = \mathfrak{d}^m + r \mathfrak{d}^{m-1}$ for all $m \geq 1$. From Nakayama’s Lemma, $\mathfrak{d}^{m-1} \neq \mathfrak{d}^m$ and we deduce from the sequence of isomorphisms

$$\mathfrak{d}/\mathfrak{d}^2 \xrightarrow{\times r} \mathfrak{d}^2/\mathfrak{d}^3 \xrightarrow{\times r} \cdots \xrightarrow{\times r} \mathfrak{d}^{m-1}/\mathfrak{d}^m$$

that $\mathfrak{d}^{m-1}/\mathfrak{d}^m$ is free of rank 1 over $\mathcal{O}_\infty$. It follows that there is an exact sequence

$$0 \longrightarrow \mathfrak{d}^m \longrightarrow \mathfrak{d}^{m-1} \longrightarrow \mathcal{O}_\infty \longrightarrow 0.$$

Similarly, taking completions yields:

$$0 \longrightarrow \mathfrak{d}^m A_\infty(\mathcal{O}_\infty) \longrightarrow \mathfrak{d}^{m-1} A_\infty(\mathcal{O}_\infty) \longrightarrow \mathcal{O}_\infty \longrightarrow 0.$$  

Hence, for all $m \geq 1$, the canonical map

$$\mathfrak{d}^{m-1}/\mathfrak{d}^m \xrightarrow{\sim} \mathfrak{d}^{m-1} A_\infty(\mathcal{O}_\infty)/\mathfrak{d}^m A_\infty(\mathcal{O}_\infty).$$  

is an isomorphism.

Back to the proof of the lemma, where we so far only proved the case $m = 1$. The general $m$-case follows by induction using the Snake Lemma on the diagram

$$\begin{array}{c}
\mathfrak{d}^{m-1}/\mathfrak{d}^m \\
\mathcal{O}_\infty \otimes \mathcal{O}_\infty/\mathfrak{d}^m \\
\mathfrak{d}^{m-1} A_\infty(\mathcal{O}_\infty)/\mathfrak{d}^m A_\infty(\mathcal{O}_\infty) \\
\mathcal{O}_\infty \otimes \mathcal{O}_\infty/\mathfrak{d}^{m-1}
\end{array} \xrightarrow{\sim} \begin{array}{c}
\mathcal{O}_\infty \otimes \mathcal{O}_\infty/\mathfrak{d}^m \\
\mathfrak{d}^{m-1} A_\infty(\mathcal{O}_\infty)/\mathfrak{d}^m A_\infty(\mathcal{O}_\infty) \\
\mathcal{O}_\infty \otimes \mathcal{O}_\infty/\mathfrak{d}^{m-1}
\end{array} \bigg\downarrow \text{induction hypothesis}
$$

where our induction hypothesis implies that the middle vertical map is an isomorphism. \( \square \)

**Proof of Theorem 5.17** Let $\text{Spec } A \cup \text{Spec } B \rightarrow C$ be an affine Zariski cover of $C$. Let $N_B := \mathcal{N}(\text{Spec } B \otimes B)$ and $M_B := \mathcal{M}(\text{Spec } B \otimes B)$. From the sheaf property, we have an isomorphism of $K_\infty \otimes K_\infty$-modules:

$$\frac{N_A}{M_A} \otimes_{A \otimes A} (K_\infty \otimes K_\infty) \xrightarrow{\sim} \frac{N_B}{M_B} \otimes_{B \otimes B} (K_\infty \otimes K_\infty).$$  

(5.10)

By definition, there exists a positive integer $m$ for which the right-hand side of (5.10) is annihilated by $\mathfrak{d}^m$, where $\mathfrak{d} := \text{ker}(\mathcal{O}_\infty \otimes \mathcal{O}_\infty \rightarrow \mathcal{O}_\infty)$. Hence,

$$\frac{N_B}{M_B} \otimes_{B \otimes B} (K_\infty \otimes K_\infty) \cong \frac{N_B}{M_B} \otimes_{B \otimes B} \left( \frac{K_\infty \otimes K_\infty}{\mathfrak{d}^m} \right) \cong \frac{N_B}{M_B} \otimes_{B \otimes B} \left( \frac{K_\infty \otimes \mathcal{O}_\infty}{\mathfrak{d}^m} \right).$$  

(5.11)

The last isomorphism above comes from the fact the any element of the form $(1 \otimes a) \in K_\infty \otimes \mathcal{O}_\infty$ is coprime to $\mathfrak{d}^m$, thus invertible in $K_\infty \otimes \mathcal{O}_\infty/\mathfrak{d}^m$. Applying Lemma 5.18, one gets

$$\frac{N_B}{M_B} \otimes_{B \otimes B} (K_\infty \otimes K_\infty) \cong \frac{N_B}{M_B} \otimes_{B \otimes B} \left( \frac{K_\infty}{\mathfrak{d}^m} \right) \cong \frac{N_B}{M_B} \otimes_{B \otimes B} \frac{B_\infty(\mathcal{O}_\infty)}{\mathfrak{d}^m} \cong \frac{N_B}{M_B} \otimes_{B \otimes B} \mathcal{B}_\infty(\mathcal{O}_\infty) \cong \frac{N_\infty}{M_\infty} \otimes_{\mathcal{O}_\infty} K_\infty.$$  

(5.12)

On the other-hand, the left-hand side of (5.10) is annihilated by both $i^m$ and $\mathfrak{v}^m$ for a large enough integer $m$, where $\mathfrak{v}$ is the ideal of $K_\infty \otimes K_\infty$ generated by $\{f \otimes (1-f) \mid f \in \mathcal{F}_\infty\}$:
by Lemma 5.16 $v = \ker(K_\infty \otimes K_\infty \rightarrow K_\infty[1])$. Taking $m$ to be a large enough power of $q$, we can even assume $v = v^m$, so that

$$\frac{N_A}{M_A} \otimes_{A \otimes A} (K_\infty \otimes K_\infty) \cong \frac{N_A}{M_A} \otimes_{A \otimes A} \left( \frac{K_\infty \otimes K_\infty}{(v + y^m)K_\infty \otimes K_\infty} \right) \cong \frac{N_A}{M_A} \otimes_{A \otimes A} K_\infty[1] \cong \frac{N_A \otimes_{A \otimes A} K_\infty[1]}{M_A \otimes_{A \otimes A} K_\infty[1]}.$$

(5.13)

The composition of (5.13), (5.10) and (5.12) gives the desired isomorphism:

$$\frac{\hat{N}_\infty}{M_\infty} \otimes_{O_\infty} K_\infty \sim \frac{(M + \tau_M(\tau^*M)) \otimes_{A \otimes K} K_\infty[1]}{M \otimes_{A \otimes K} K_\infty[1]}.$$

Pre-composition with the isomorphism of Proposition 5.11 gives the desired isomorphism.

As announced, we have:

**Corollary 5.19.** There is an exact sequence of $K_\infty$-vector spaces:

$$0 \rightarrow \frac{M^\perp_+ \otimes_A K_\infty}{(\tau - \tau_M)(\hat{M}_\infty \otimes_{O_\infty} K_\infty)} \rightarrow \hat{N}_\infty \otimes_{O_\infty} K_\infty \rightarrow \Ext^{1,ha}_{\infty} (\hat{H}^+_\infty(\hat{M}), M_\infty) \rightarrow 0.$$

**Proof.** This is a consequence of the combination of Theorem 5.17 and Theorem 3.28.

6 Proof of the main theorems

6.1 Cohomological computations

In this subsection, we recall general facts on coherent cohomology of schemes. This will subsequently be applied to sthuka models in the next subsection to achieve the proof of Theorems 4.1 and 4.4. We refer to [Wei94] for the definitions of homological algebra (cones, distinguished triangles, derived categories, etc.)

To fix the setting, we consider the following commutative square in the category of schemes over $C$:

$$\begin{array}{ccc}
\Spec K_\infty \times C & \rightarrow & \Spec O_\infty \times C \\
\downarrow^i & & \downarrow^q \\
\Spec K \times C & \rightarrow & C \times C \\
\end{array}$$

(6.1)

The next result is classical:

**Proposition 6.1.** Let $\mathcal{F}$ be a sheaf of $O_{C \times C}$-modules. In the derived category of $K_\infty$-modules, there is a quasi-isomorphism

$$R\Gamma(\Spec A \times C, \mathcal{F}) \otimes_A K_\infty \cong R\Gamma(\Spec O_\infty \times C, q^* \mathcal{F}) \otimes_{O_\infty} K_\infty$$

which is functorial in $\mathcal{F}$.

Let now $S$ be a scheme and let $T$ be a separated scheme over $S$. Let $U$, $V$ and $W$ be affine schemes over $S$ which insert in a commutative diagram of $S$-schemes

$$\begin{array}{ccc}
U & \rightarrow & T \\
\uparrow^i & & \uparrow^j \\
W & \rightarrow & V \\
\end{array}$$

35
such that \( \{U \to T, V \to T\} \) forms a covering of \( T \).

For \( \mathcal{F} \) a sheaf of \( \mathcal{O}_T \)-modules, we denote by \( S(\mathcal{F}) \) the sequence of \( \mathcal{O}_T \)-modules:

\[
0 \to \mathcal{F} \to i_* i^* \mathcal{F} \oplus j_* j^* \mathcal{F} \to k_* k^* \mathcal{F} \to 0
\]

where the morphisms are given by the adjunction unit (note that the data of \( S(\mathcal{F}) \) is functorial in \( \mathcal{F} \)). The next lemma is of fundamental importance for our cohomological computations:

**Lemma 6.2.** Assume that \( S(\mathcal{O}_T) \) is exact. Then, for any finite locally free sheaf \( \mathcal{F} \) of \( \mathcal{O}_T \)-modules, \( S(\mathcal{F}) \) is exact. In particular, the natural map

\[
R^i(T, \mathcal{F}) \to [\mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(W)], \tag{6.2}
\]

where the right-hand side is a complex concentrated in degrees 0 and 1, is a quasi-isomorphism.

**Proof.** We show that \( S(\mathcal{F}) \) is an exact sequence (the second assertion follows, since applying \( R^i(T, -) \) to \( S(\mathcal{F}) \) yields the distinguished triangle computing (6.2)). To prove exactness of \( S(\mathcal{F}) \), first note that \( i, j \) and \( k \) are affine morphisms because \( T \) is separated (01SG). Thus, the pushforward functors appearing in \( S(\mathcal{F}) \) are naturally isomorphic to their right-derived functor (06G9R). Thereby, \( S(\mathcal{F}) \) is naturally isomorphic in \( D_{qc}(T) \), the derived category of quasi-coherent sheaves over \( T \), to the triangle

\[
\mathcal{F} \to R_i i^* \mathcal{F} \oplus R_j j^* \mathcal{F} \to R_k k^* \mathcal{F} \to [1] \tag{6.3}
\]

and it is sufficient to show that the latter is distinguished. Yet, because \( \mathcal{F} \) is finite locally-free, the projection formula (01E8) implies that (6.3) is naturally isomorphic to

\[
\mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{O}_T \to \mathcal{F} \otimes_{\mathcal{O}_T} (R_i i_\ast \mathcal{O}_U \oplus R_j j_\ast \mathcal{O}_V) \to \mathcal{F} \otimes_{\mathcal{O}_T} R_k k_\ast \mathcal{O}_W \to [1]
\]

Because \( \mathcal{F} \) is locally-free, the functor \( \mathcal{F} \otimes_{\mathcal{O}_T} - \) is exact on \( D_{qc}(T) \) and it suffices to show the distinguishness of

\[
\mathcal{O}_T \to R_i i_\ast \mathcal{O}_U \oplus R_j j_\ast \mathcal{O}_V \to R_k k_\ast \mathcal{O}_W \to [1].
\]

But because \( \mathcal{O}_U = i^* \mathcal{O}_T, \mathcal{O}_V = j^* \mathcal{O}_T \) and \( \mathcal{O}_W = k^* \mathcal{O}_T \), this follows from our assumption that \( S(\mathcal{O}_T) \) is exact. We conclude that (6.3) is distinguished. 

Assuming that \( T \) is a smooth variety\(^4\) over a field allows us to relax the "locally free" assumption in Lemma 6.2 to "coherent".

**Proposition 6.3.** Let \( k \) be a field and assume that \( S = \text{Spec} \, k \). Assume further that \( T \) is a smooth variety over \( k \), and that \( i, j \) and \( k \) are flat. Let \( \mathcal{F} \) be a coherent sheaf on \( X \). Then, \( S(\mathcal{F}) \) is exact. In particular, the natural map

\[
R^i(T, \mathcal{F}) \to [\mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(W)]
\]

is a quasi-isomorphism.

**Proof.** Choose a resolution of \( \mathcal{F} \) by finite locally free sheaves \( 0 \to \mathcal{F}_n \to \cdots \to \mathcal{F}_0 \to \mathcal{F} \to 0 \). Because \( i \) (resp. \( j \), \( k \)) is flat, \( i^* \) (resp. \( j^*, k^* \)) is an exact functor on quasi-coherent sheaves. Because it is affine, \( i_* \) (resp. \( j_*, k_* \)) is an exact functor on quasi-coherent sheaves. Thereby, for all \( s \in \{0, \ldots, n\} \), the sequence \( S(\mathcal{F}_s) \) is exact by Lemma 6.2. Using the \( n \times n \)-Lemma in the abelian category of quasi-coherent sheaves of \( \mathcal{O}_T \)-modules, we deduce that \( S(\mathcal{F}) \) is exact. 

The main result of this subsection is:

\(^4\)By variety over \( k \), we mean that \( T \) is integral and that \( T \to \text{Spec} \, k \) is separated and of finite type.
Theorem 6.4. Assume the setting of Proposition 6.3. Let $F'$ be a coherent sheaf of $O_T$-module and let $f : F \to F'$ be a morphism of sheaves of abelian groups. Then, the rows and the lines of the following diagram

\[
\begin{align*}
R\Gamma(T, F) &\longrightarrow F(U) \oplus F(V) \longrightarrow F(W) \longrightarrow [1] \\
R\Gamma(T, F') &\longrightarrow F'(U) \oplus F'(V) \longrightarrow F'(W) \longrightarrow [1] \\
\text{cone}(f_T) &\longrightarrow \text{cone}(f_U) \oplus \text{cone}(f_V) \longrightarrow \text{cone}(f_W) \longrightarrow [1]
\end{align*}
\]

form distinguished triangles in the derived category of abelian groups, where $f_Y := R\Gamma(Y, f)$ (for $Y \in \{T, U, V, W\}$).

Proof. We lift the first two lines in the category of chain complexes: by Lemma 6.2, the diagram

\[
\begin{align*}
0 &\longrightarrow F \longrightarrow i_*i^*F \oplus j_*j^*F \longrightarrow k_*k^*F \longrightarrow 0 \\
0 &\longrightarrow F' \longrightarrow i_*i^*F' \oplus j_*j^*F' \longrightarrow k_*k^*F' \longrightarrow 0
\end{align*}
\]

is exact on lines and commutative on squares in the category of quasi-coherent sheaves of $O_T$-modules. From (013T) we can find injective resolutions $F \to I^*_1$, $i_*i^*F \oplus j_*j^*F \to I^*_2$ and $k_*k^*F \to I^*_3$ (respectively $F' \to J^*_1$, $i_*i^*F' \oplus j_*j^*F' \to J^*_2$ and $k_*k^*F' \to J^*_3$) such that

\[
\begin{align*}
0 &\longrightarrow I^*_1 \longrightarrow I^*_2 \longrightarrow I^*_3 \longrightarrow 0 \\
0 &\longrightarrow J^*_1 \longrightarrow J^*_2 \longrightarrow J^*_3 \longrightarrow 0
\end{align*}
\]

is an injective resolution of the whole diagram (6.5). Completing the vertical maps into distinguished triangles gives:

\[
\begin{align*}
0 &\longrightarrow \text{cone}(i^*_1) \longrightarrow \text{cone}(i^*_2) \longrightarrow \text{cone}(i^*_3) \longrightarrow 0 \\
0 &\longrightarrow \text{cone}(j^*_1) \longrightarrow \text{cone}(j^*_2) \longrightarrow \text{cone}(j^*_3) \longrightarrow 0
\end{align*}
\]

where the rows are distinguished triangles. The third line is a direct sum of exact sequences and therefore is exact. The horizontal exact sequences transform to distinguished triangles in the derived category of abelian module. This concludes.

Under Noetherianity assumptions, Theorem 6.4 can be extended to the case of formal schemes. Our main reference is [FK18, §I]. From now on, we assume that $T, U, V$ and $W$ are Noetherian schemes over $S$. Let $T'' \subseteq T$, $U'' \subseteq U$, $V'' \subseteq V$ and $W'' \subseteq W$ be closed subschemes such that $i^{-1}(T'') = U''$, $j^{-1}(T'') = V''$, and $p^{-1}(U'') = W'' = q^{-1}(V'')$. It follows
that \( k^{-1}(T') = W' \). Let \( \hat{T}, \hat{U}, \hat{V} \) and \( \hat{W} \) be the formal completions along the corresponding closed subschemes [FK18 §I.1.4]. We obtain a commutative diagram of formal schemes

\[
\begin{array}{ccc}
\hat{U} & \xrightarrow{i} & \hat{T} \\
\downarrow & & \downarrow \gamma \\
\hat{W} & \xrightarrow{j} & \hat{V}
\end{array}
\]

Given an adically quasi-coherent sheaf \( \mathcal{F} \) of \( O_T \)-modules [FK18 §I,Def.3.1.3], we consider the sequence

\[
\hat{S}(\mathcal{F}) : 0 \to \mathcal{F} \to \hat{i}_i^* \mathcal{F} \oplus \hat{j}_j^* \mathcal{F} \to \hat{k}_k^* \mathcal{F} \to 0.
\]

**Lemma 6.5.** Let \( \mathcal{F} \) be a quasi-coherent sheaf on \( T \). Then \( \hat{S}(\mathcal{F}) \cong \hat{S}(\mathcal{F}) \), where \( \mathcal{G} \mapsto \hat{\mathcal{G}} \) denotes the formal completion functor along \( T' \). In particular, if \( S(\mathcal{F}) \) is exact, then \( \hat{S}(\mathcal{F}) \) is exact.

**Proof.** This almost follows from the flat-base change Theorem [02KH]. Indeed, the diagram

\[
\begin{array}{ccc}
\hat{U} & \xrightarrow{f_U} & U \\
\downarrow & & \downarrow \iota \\
\hat{T} & \xrightarrow{f_T} & T
\end{array}
\]

where \( f_U \) and \( f_T \) are the canonical maps, is Cartesian. Because \( \iota \) is affine, \( \iota \) is quasi-compact and quasi-separated [01S7]. On the other-hand, \( f_T \) is flat and the flat-base change Theorem applies. It states that for any quasi-coherent sheaf \( \mathcal{G} \) of \( O_U \)-modules, the natural map

\[
f_T^* R\hat{i}_i^* \mathcal{G} \longrightarrow R\hat{i}_i^* (f_U^* \mathcal{G})
\]

is a quasi-isomorphism in the derived category of \( O_U \)-modules. Because \( \iota \) is affine, the functors \( R\hat{i}_i^* \) and \( \hat{i}_i^* \) are isomorphic on the category of coherent sheaves [0G9R]. Similarly, but in the setting of formal geometry, \( \iota \) is also affine [FK18 §I,Def.4.1.1], and the formal analogue of the previous argument [FK18 §I, Thm.7.1.1] reads that the functors \( \hat{R}\hat{i}_i^* \) and \( \hat{i}_i^* \) are isomorphic on the category of adically quasi-coherent sheaves. Therefore, in the derived category of \( O_U \)-modules, we have an isomorphism

\[
f_T^* \hat{i}_i^* \mathcal{G} \cong \hat{i}_i^* f_U^* \mathcal{G}.
\]

Applied to \( \mathcal{G} = i^* \mathcal{F} \) for a quasi-coherent \( \mathcal{F} \) on \( T \), we obtain \( f_T^* \hat{i}_i^* \mathcal{F} \cong \hat{i}_i^* f_T^* \mathcal{F} \) functorially in \( \mathcal{F} \). In other words,

\[
\hat{i}_i^* \mathcal{F} \cong \hat{i}_i^* \mathcal{F}.
\]

The very same argument for \( j \) and \( k \) in place of \( i \) yields respectively \( \hat{j}_j^* \mathcal{F} \cong \hat{j}_j^* \mathcal{F} \) and \( \hat{k}_k^* \mathcal{F} \cong \hat{k}_k^* \mathcal{F} \). It follows that \( \hat{S}(\mathcal{F}) \cong \hat{S}(\mathcal{F}) \). Since the formal completion functor is exact, \( \hat{S}(\mathcal{F}) \) is exact if \( S(\mathcal{F}) \) is.

Thanks to Lemma 6.5, the proof of Theorem 6.4 blithely applies to the formal situation:

\[\text{e.g. the formal completion of a quasi-coherent sheaf with respect to a closed subscheme of finite presentation is adically quasi-coherent by [FK18 §I, Prop.3.1.5]}\]
Theorem 6.6. Assume the setting of Theorem 6.2. Then, each rows and each lines of the following diagram

\[
\begin{array}{cccccc}
R\Gamma(T, \tilde{\mathcal{F}}) & \longrightarrow & \tilde{F}(U) \oplus \tilde{F}(V) & \longrightarrow & \tilde{F}(W) & \longrightarrow & [1] \\
\downarrow f_T & & \downarrow f_\mathcal{F} & & \downarrow f_W & & \\
R\Gamma(T, \tilde{\mathcal{F}}') & \longrightarrow & \tilde{F}'(U) \oplus \tilde{F}'(V) & \longrightarrow & \tilde{F}'(W) & \longrightarrow & [1] \\
\downarrow \text{cone}(f_T) & & \downarrow \text{cone}(f_\mathcal{F}) & & \downarrow \text{cone}(f_W) & & \downarrow [1] \\
\end{array}
\]

form distinguished triangles in the derived category of abelian groups, where \( \tilde{f}_\mathcal{F} := R\Gamma(Y, \bar{f}) \) (for \( Y \in \{T, U, V, W\} \)).

6.2 Proof of Theorems 4.1 and 4.4

Let \( \mathcal{M} \) be a rigid analytically trivial \( A \)-motive over \( K \). We now assemble the ingredients collected in the last subsectors to end the proof of Theorems 4.1 and 4.3. As promised, the complex \( G_\mathcal{M} \) admits an interpretation in terms of the Zariski cohomology of \( C \)-shtuka models of \( \mathcal{M} \) (Definition 5.1):

Proposition 6.7. Let \( (\mathcal{N}, \mathcal{M}, \tau_M) \) be a \( C \)-shtuka model for \( \mathcal{M} \). Let \( \iota \) denotes the inclusion of \( \mathcal{M} \) in \( \mathcal{N} \). There is a quasi-isomorphism of \( A \)-module complexes

\[
G_\mathcal{M} \sim \text{cone} \left( R\Gamma(\text{Spec} A \times C, \mathcal{M}) \overset{\iota}{\longrightarrow} R\Gamma(\text{Spec} A \times C, \mathcal{N}) \right).
\]

The first part of Theorem 4.1 follows from the above:

Corollary 6.8. The \( A \)-modules \( \text{Ext}^{1}_{A,\infty}(1, \mathcal{M}) \) and \( \text{Cl}(\mathcal{M}) \) are finitely generated.

Proof. As \( \text{Spec} A \times C \) is proper over \( \text{Spec} A \), both \( R\Gamma(\text{Spec} A \times C, \mathcal{M}) \) and \( R\Gamma(\text{Spec} A \times C, \mathcal{N}) \) are perfect complexes of \( A \)-modules. By Proposition 6.7 so is \( G_\mathcal{M} \). We conclude by Proposition 4.6. \( \square \)

Proof of Proposition 6.7. The main ingredients are the cohomological preliminaries of Section 6.1. We consider the particular setting of \( S = \text{Spec} F \) and of the commutative diagram of \( S \)-schemes

\[
\begin{array}{cccccc}
\text{Spec} O_\infty(A) & \overset{i}{\longrightarrow} & \text{Spec} A \times C \\
\downarrow p & & \downarrow j \\
\text{Spec} K_\infty(A) & \overset{k}{\longrightarrow} & \text{Spec} A \otimes A
\end{array}
\]

Because \( A \) is geometrically irreducible over \( F \), \( \text{Spec} A \times C \) is a smooth variety over \( F \). To use the results of Section 6.1, one requires the next two lemmas.

Lemma 6.9. The morphisms \( i, j, k \) are flat.

Proof. We consider the affine open cover \( (\text{Spec} A \otimes A) \cup (\text{Spec} A \otimes B) \) of \( (\text{Spec} A) \times C \). We first show that \( i \) is flat. We have \( i^{-1}(\text{Spec} A \otimes A) = \text{Spec} K_\infty(A) \) and \( i^{-1}(\text{Spec} A \otimes B) = \text{Spec} O_\infty(A) \). The morphism \( A \otimes B \rightarrow O_\infty(A) \) is flat (because it is the completion of the Noetherian ring \( A \otimes B \) at the ideal \( m_\infty \subset B \)) and thus, so is \( A \otimes A \rightarrow K_\infty(A) \). By (01U5), \( i \) is flat.

We have \( j^{-1}(\text{Spec} A \otimes B) = \text{Spec} A \otimes D \), where \( D \subset K \) is the sub-\( F \)-algebra such that \( \text{Spec} D = \text{Spec} A \cap \text{Spec} B \). The inclusion \( B \rightarrow D \) is a localization, and hence \( A \otimes B \rightarrow A \otimes D \) is flat. Therefore, \( j \) is flat.
Because $K_{\infty}(A) \cong K_{\infty} \otimes_{\mathcal{O}_{\infty}} \mathcal{O}_{\infty}(A)$, $p$ is flat. Since compositions of flat morphisms are flat, $k = i \circ p$ is flat.

**Lemma 6.10.** For $T = (\text{Spec } A) \times C$, the sequence $0 \to \mathcal{O}_T \to i_*i^*\mathcal{O}_T \oplus j_*j^*\mathcal{O}_T \to k_*k^*\mathcal{O}_T \to 0$ is exact.

**Proof.** We need to show that the complex $Z := [\mathcal{O}_{\infty}(A) \otimes (A \otimes A) \to K_{\infty}(A)]$, where the morphism is the difference of the canonical inclusions, represents the sheaf cohomology in the Zariski topology of $\mathcal{O}_{\text{Spec } A \times C}$, the latter being quasi-isomorphic to

$$R\Gamma(\text{Spec } A \times C, \mathcal{O}_{\text{Spec } A \times C}) = [(A \otimes B) \oplus (A \otimes A) \to A \otimes D].$$

Let $(t_i)_{i \geq 0}$ be a (countable) basis of $A$ over $\mathbb{F}$. Any element $f$ in $K_{\infty}(A)$ can be represented uniquely by a converging series

$$f = \sum_{i=0}^{\infty} t_i \otimes f_i, \quad t_i \in K_{\infty}, \quad t_i \to 0 \ (i \to \infty).$$

Elements of $\mathcal{O}_{\infty}(A)$ are the ones for which $f_i \in \mathcal{O}_{\infty}$ ($\forall i \geq 0$) and elements of $A \otimes A$ are the ones for which $f_i \in A$ ($\forall i \geq 0$) and $f_i = 0$ for $i$ large enough. Therefore, it is clear that $\mathcal{O}_{\infty}(A) \cap (A \otimes A)$ is $A \otimes (\mathcal{O}_{\infty} \cap A)$. Yet, $\mathcal{O}_{\infty} \cap A$ is the constant field of $C$, showing that $H^0(Z) = H^0(\text{Spec } A \times C, \mathcal{O}_{\text{Spec } A \times C}).$

Because $K_{\infty} = \mathcal{O}_{\infty} + A \otimes A$, the canonical map

$$\frac{A \otimes D}{A \otimes B + A \otimes A} \to \frac{K_{\infty}(A)}{\mathcal{O}_{\infty}(A) + A \otimes A}$$

is surjective. Because $(A \otimes D) \cap \mathcal{O}_{\infty}(A) \subset A \otimes B + A \otimes A$, it is also injective. It follows that $H^1(Z) = H^1(\text{Spec } A \times C, \mathcal{O}_{\text{Spec } A \times C}).$

Now, let $\overline{M} = (\mathcal{N}, M, \tau_M)$ be a $C$-shutka model for $M$. We have

$$(j^*M)(\text{Spec } A \otimes A) = M(\text{Spec } A \otimes A) = M_A,$$

$$(j^*\mathcal{N})(\text{Spec } A \otimes A) = \mathcal{N}(\text{Spec } A \otimes A) = N_A,$$

$$(k^*M)(\text{Spec } K_{\infty}(A)) = (k^*\mathcal{N})(\text{Spec } K_{\infty}(A)) = M \otimes_{A \otimes \mathbb{K}} K_{\infty}(A).$$

**Theorem [6.4] **yields a morphism of distinguished triangles

$$\begin{array}{cccc}
R\Gamma(\text{Spec } A \times C, M) & \to & (i^*M)(\text{Spec } \mathcal{O}_{\infty}(A)) & \to & \frac{M \otimes_{A \otimes \mathbb{K}} K_{\infty}(A)}{M_A} & \to & [1] \\
\downarrow & & \downarrow \text{id} - \tau_M & & \downarrow \text{id} - \tau_M & & \\
R\Gamma(\text{Spec } A \times C, \mathcal{N}) & \to & (i^*\mathcal{N})(\text{Spec } \mathcal{O}_{\infty}(A)) & \to & \frac{M \otimes_{A \otimes \mathbb{K}} K_{\infty}(A)}{N_A} & \to & [1] \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{cone}(\iota - \tau_M) & \to & 0 & \to & G_M & \to & [1]
\end{array}$$

where the cone of the middle upper vertical morphism is zero by Lemma [5.3]. The third row is a distinguished triangle, and the proposition follows.

**Theorems [11] (second part) and [6.4] **will follow from the study of the cohomology of a $C \times C$-shutka model of $\overline{M}$ at $\text{Spf } \mathcal{O}_{\infty} \times C$. The latter corresponds to the completion of the Noetherian scheme $C \times C$ at the closed subscheme $\{\infty\} \times C$. The argument given here is a refinement of the one given in the proof of Proposition 6.7 where we use $C \times C$-shutka models instead of $C$-shutka models. To ensure the existence of a $C \times C$-shutka model, we now assume that all the weights of $\overline{M}$ are negative (Theorem [5.3]).
We apply the results of Section 6.1 under a different setting. We consider the commutative square of schemes over $\text{Spec} \, \mathcal{O}_\infty$:

\[
\begin{array}{ccc}
\text{Spec} \, \mathcal{O}_\infty \otimes A & \xrightarrow{i} & (\text{Spec} \, \mathcal{O}_\infty) \times C \\
\uparrow k & & \uparrow j \\
\text{Spec} \, \mathcal{O}_\infty \otimes K_\infty & \xrightarrow{\iota} & \text{Spec} \, \mathcal{O}_\infty \otimes \mathcal{O}_\infty
\end{array}
\]

Similarly to Lemma 6.9, one shows that $i$, $j$, and $k$ are flat morphisms. For the sake of compatibility of notations with the previous subsection, we let $T = \text{Spec} \, \mathcal{O}_\infty \times C$, $U = \text{Spec} (\mathcal{O}_\infty \otimes A)$, $V = \text{Spec} (\mathcal{O}_\infty \otimes \mathcal{O}_\infty)$ and $W = \text{Spec} (\mathcal{O}_\infty \otimes K_\infty)$. Consider the respective closed subschemes $T' = \{ \infty \} \times C$, $U' = \{ \infty \} \times \text{Spec} \, A$, $V = \{ \infty \} \times \text{Spec} \, \mathcal{O}_\infty$ and $W = \{ \infty \} \times \text{Spec} \, K_\infty$. The compatibility of notations with the previous subsection, we let $\text{Spec} \, \mathcal{O}_\infty \times \text{Spec} \, \mathcal{O}_\infty$, and $W = \text{Spec} \, \mathcal{O}_\infty \times \text{Spec} \, K_\infty$. We obtain the commutative square of formal schemes over $\text{Spec} \, \mathcal{O}_\infty$:

\[
\begin{array}{ccc}
\text{Spf} \, \mathcal{O}_\infty \circ A & \xrightarrow{i} & (\text{Spf} \, \mathcal{O}_\infty) \times C \\
\uparrow k & & \uparrow j \\
\text{Spf} \, \mathcal{O}_\infty \circ K_\infty & \xrightarrow{\iota} & \text{Spf} \, \mathcal{O}_\infty \circ \mathcal{O}_\infty
\end{array}
\]

We let $q : \text{Spec} \, \mathcal{O}_\infty \times C \to C \times C$ be the inclusion of schemes. To the morphism of sheaves $\tau_M : \tau^* (q^* \mathcal{M}) \to (q^* \mathcal{N})$ on $(\text{Spec} \, \mathcal{O}_\infty) \times C$, one associates functorially the morphism of the formal coherent sheaves $\tau_M : \tau^* \mathcal{M} \to \mathcal{N}$ on the formal spectrum $(\text{Spf} \, \mathcal{O}_\infty) \times C$. Because both $q^* \mathcal{M}$ and $q^* \mathcal{N}$ are coherent sheaves, their formal completion corresponds to their pullback along the completion morphism of locally ringed spaces:

\[(\text{Spf} \, \mathcal{O}_\infty) \times C \to (\text{Spec} \, \mathcal{O}_\infty) \times C.
\]

Recall that $\mathcal{N}_\infty$ and $\mathcal{M}_\infty$ were the respective $\mathcal{A}_\infty(\mathcal{O}_\infty)$-modules $\mathcal{N}(\text{Spf} \, \mathcal{O}_\infty \circ \mathcal{O}_\infty)$ and $\mathcal{M}(\text{Spf} \, \mathcal{O}_\infty \circ \mathcal{O}_\infty)$. Let also $L$ and $L_A$ be given respectively by

\[
L := \mathcal{M}(\text{Spf} \, \mathcal{O}_\infty \circ K) = \mathcal{N}(\text{Spf} \, \mathcal{O}_\infty \circ \mathcal{O}_\infty),
\]

\[
L_A := \mathcal{M}(\text{Spf} \, \mathcal{O}_\infty \circ A) = \mathcal{N}(\text{Spf} \, \mathcal{O}_\infty \circ A).
\]

Note that $L$ defines an $\mathcal{A}_\infty(K)$-lattice stable by $\tau_M$ in $L \subset M \otimes_{\mathcal{A}_\infty(K)} B_{\infty}(K)$.

By Theorem 6.6, we have a morphism of distinguished triangles:

\[
R\Gamma(\text{Spf} \, \mathcal{O}_\infty \circ C, \mathcal{M}) \to \mathcal{M}_\infty \to \frac{L \otimes \mathcal{A}_\infty(K)}{L_A} \xrightarrow{\text{id} - \tau_M} [1]
\]

\[
R\Gamma(\text{Spf} \, \mathcal{O}_\infty \circ C, \mathcal{N}) \to \mathcal{N}_\infty \to \frac{L \otimes \mathcal{A}_\infty(K)}{L_A} \xrightarrow{\text{id} - \tau_M} [1]
\]

(6.7)

The third vertical arrow is an isomorphism by Lemma 5.10. The next lemma shows that the middle one is injective once tensored along $\mathcal{O}_\infty \rightarrow K_\infty$:

**Lemma 6.11.** The morphism $i - \tau_M : \mathcal{M}_\infty \otimes_{\mathcal{O}_\infty} K_\infty \rightarrow \mathcal{N}_\infty \otimes_{\mathcal{O}_\infty} K_\infty$ is injective.

**Proof.** For $t > 0$, $a \geq 0$ and $x \in \mathcal{M}_\infty$, we have

\[
(i - \tau_M)((1 \otimes \pi_\infty)^{t+a} x) \equiv (1 \otimes \pi_\infty)^{t+a} i(x) \pmod{(1 \otimes \pi_\infty)^{t+a+1} \mathcal{N}_\infty}.
\]

In particular, the first vertical arrow in diagram (6.7) is injective. The lemma then follows from Lemma 5.11 together with the snake Lemma. \qed
Theorem 5.17 together with Lemma 6.11 implies the existence of a quasi-isomorphism
\[ \text{cone}([\hat{N}_\infty] \to M_{\infty} \to \hat{N}_\infty]) \otimes_{O_{\infty}} K_\infty \cong \frac{(M + \tau M(\tau^*M)) \otimes_{A \otimes K} K_{\infty}[1]}{M \otimes_{A \otimes K} K_{\infty}[1]} \]
We then deduce from Theorem 6.6 that
\[ \text{cone} \left( R\Gamma(Spfr O_{\infty} \times C, \hat{M}) \to R\Gamma(Spfr O_{\infty} \times C, \hat{N}) \right) \otimes_{O_{\infty}} K_\infty \cong \frac{(M + \tau M(\tau^*M)) \otimes_{A \otimes K} K_{\infty}[1]}{M \otimes_{A \otimes K} K_{\infty}[1]} \]
Because \((\text{Spec} A) \times C \to \text{Spec} A\) is proper, Grothendieck’s comparison Theorem \([\text{Gro63}, \text{Thm. 4.1.5}]\) provides natural quasi-isomorphisms
\[ R\Gamma(\text{Spec} O_{\infty} \times C, F) \cong R\Gamma(\text{Spec} O_{\infty} \times C, \hat{F}) \]
for \(F\) being either \(q^*M\) or \(q^*N\). This allows us to rewrite (6.8) as
\[ \text{cone} \left( R\Gamma(Spfr O_{\infty} \times C, q^*M) \to R\Gamma(Spfr O_{\infty} \times C, q^*N) \right) \otimes_{O_{\infty}} K_\infty \cong \frac{(M + \tau M(\tau^*M)) \otimes_{A \otimes K} K_{\infty}[1]}{M \otimes_{A \otimes K} K_{\infty}[1]} \]
and we use Proposition 6.1 to obtain
\[ \text{cone} \left( R\Gamma(Spfr A \times C, M) \to R\Gamma(Spfr A \times C, N) \right) \otimes_{A} K_\infty \cong \frac{(M + \tau M(\tau^*M)) \otimes_{A \otimes K} K_{\infty}[1]}{M \otimes_{A \otimes K} K_{\infty}[1]} \]
From Proposition 6.7 we deduce
\[ G_M \otimes_A K_\infty \cong \frac{(M + \tau M(\tau^*M)) \otimes_{A \otimes K} K_{\infty}[1]}{M \otimes_{A \otimes K} K_{\infty}[1]} \]
In particular, \(G_M \otimes_A K_{\infty}\) sits in degree 0. Therefore, we obtain the second part of Theorem 4.1

**Proposition 6.12.** The \(A\)-module \(H^1(G_M) \cong Cl(M)\) is torsion, and thus finite.

It remains to prove Theorem 4.4. We first introduce a definition (see the next page).
Definition 6.13. We denote by $\rho(M)$ the isomorphism of $K_\infty$-vector spaces

$$\{\xi \in M \otimes_{A \otimes K} K_\infty(A) \mid \xi - \tau_M(\tau^*\xi) \in N_A\} \otimes_A K_\infty \sim \frac{(M + \tau_M(\tau^*M)) \otimes_{A \otimes K} K_\infty[\mathfrak{I}]}{M \otimes_{A \otimes K} K_\infty[\mathfrak{I}]}$$

obtained by the vertical composition of the quasi-isomorphisms of complexes of $K_\infty$-vector spaces:

$$\{\xi \in M \otimes_{A \otimes K} K_\infty(A) \mid \xi - \tau_M(\tau^*\xi) \in N_A\} \otimes_A K_\infty$$

\[\xrightarrow{	ext{Cl}(M) \otimes A K_\infty = 0} \]

$$G_M \otimes_A K_\infty$$

\[\xrightarrow{i \text{ Proposition 6.7}} \]

$$\text{cone} \left[ R\Gamma(\text{Spec } A \times C, M) \xrightarrow{i = -\tau_M} R\Gamma(\text{Spec } A \times C, N) \right] \otimes_A K_\infty$$

\[\xrightarrow{i \text{ Proposition 6.1}} \]

$$\text{cone} \left[ R\Gamma(\text{Spec } O_\infty \times C, M) \xrightarrow{i = -\tau_M} R\Gamma(\text{Spec } O_\infty \times C, q^*N) \right] \otimes_{O_\infty} K_\infty$$

\[\xrightarrow{0.1} \text{ and Lemma 6.10} \]

$$\text{cone} \left[ \mathcal{N}_\infty \xrightarrow{id - \tau_M} \mathcal{N}'_\infty \right] \otimes_{O_\infty} K_\infty$$

\[\xrightarrow{\text{Lemma 6.11}} \]

$$\frac{(i - \tau_M)(M)}{(M + \tau_M(\tau^*M)) \otimes_{A \otimes K} K_\infty[\mathfrak{I}]} \otimes_{O_\infty} K_\infty$$

\[\xrightarrow{\text{Theorem 5.12}} \]

$$\frac{(M + \tau_M(\tau^*M)) \otimes_{A \otimes K} K_\infty[\mathfrak{I}]}{M \otimes_{A \otimes K} K_\infty[\mathfrak{I}]}$$

Proof of Theorem 4.4. We have an exact sequence of $K_\infty$-vector spaces:

$$0 \rightarrow (M_B)_{K_\infty}^+ \rightarrow \frac{\{\xi \in M \otimes_{A \otimes K} K_\infty(A) \mid \xi - \tau_M(\tau^*\xi) \in N_A\} \otimes_A K_\infty}{\xrightarrow{\text{Ext}^{1,\text{reg}}_{A \otimes K}(I, M) \otimes_A K_\infty}} 0.$$

On the other-hand, by Theorem 3.28 we have an exact sequence of $K_\infty$-vector spaces:

$$0 \rightarrow (M_B)_{K_\infty}^+ \rightarrow \frac{(M + \tau_M(\tau^*M)) \otimes_{A \otimes K} K_\infty[\mathfrak{I}]}{\xrightarrow{\text{Ext}^{1,\text{ha}}_{K_\infty}(I^+, \mathcal{M})}} 0.$$

Theorem 4.4 then follows from the fact that $\rho(M)$ is an isomorphism.

From the proof of Theorem 4.4 we see that $\rho(M)$ induces an isomorphism of $K_\infty$-vector spaces:

$$\tilde{\rho}(M) : \text{Ext}^{1,\text{reg}}_{A \otimes K}(I, M) \otimes_A K_\infty \xrightarrow{\sim} \text{Ext}^{1,\text{ha}}_{K_\infty}(I^+, \mathcal{M}).$$

We call $\tilde{\rho}(M)$ the Mock regulator of $M$. The terminology mock is here to emphasize the fact that $\tilde{\rho}(M)$ does not coincide with $\mathcal{R}_{\text{reg}}(M)$ in general. However, we believe that it still have a role to play:
Conjecture 6.14. The map $\rho(M)$ - hence $\bar{\rho}(M)$ - is natural in $M$.

Observe that Conjecture 6.14 is not obvious, as the construction of $\rho(M)$ depends a priori on the choice of a shtuka model for $M$, an operation that is highly non canonical. It would have the following consequence: let $(\mathrm{AMot}_{K_{\infty}}^{\operatorname{rat}})_{<0}$ denotes the full subcategory of $\mathrm{AMot}_{K_{\infty}}^{\operatorname{rat}}$ consisting of objects having all their weights negative. Then, the two functors

$$M \mapsto \operatorname{Ext}^1_{A, \infty}(1, M) \otimes_A K_{\infty}, \quad M \mapsto \operatorname{Ext}^1_{\infty}(1, \mathcal{M}_{K_{\infty}}^+(M)),$$

from $(\operatorname{AMot}_{K_{\infty}}^{\operatorname{rat}})_{<0}$ to the category of $K_{\infty}$-vector spaces, are isomorphic.

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Regulators in the Arithmetic of Function Fields

Q. Gazda

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