Residual vanishing for blowup solutions to 2D Smoluchowski-Poisson equation

Takashi Suzuki

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Abstract

We study Smoluchowski-Poisson equation in two space dimensions provided with Dirichlet boundary condition for the Poisson part. For this equation several profiles of blowup solution have been noticed. Here we show the residual vanishing.

1 Introduction

We study parabolic-elliptic system proposed in statistical physics to describe the motion of mean field of many self-gravitating Brownian particles [20]. It is composed of the Smoluchowski part

\[ u_t = \Delta u - \nabla \cdot u \nabla v \quad \text{in } \Omega \times (0,T) \]  

with null-flux boundary condition

\[ \frac{\partial u}{\partial \nu} - v \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0,T) \]  

and the Poisson part in the form of

\[ -\Delta v = u, \quad v|_{\partial \Omega} = 0, \]  

where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with smooth boundary \( \partial \Omega \) and \( \nu \) is the outer unit normal vector. Initial condition is given as

\[ u|_{t=0} = u_0(x) \geq 0 \quad \text{in } \Omega, \]  

where \( u_0 = u_0(x) \) is a smooth function.

System (1)-(4) is subject to thermodynamical laws, total mass conservation and free energy decreasing,

\[ \frac{d}{dt} \int_{\Omega} u = \int_{\Omega} \nabla \cdot (\nabla u - u \nabla v) = \int_{\partial \Omega} \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \, ds = 0 \]  

\[ \frac{d}{dt} F(u) = -\int_{\Omega} |\nabla (\log u - v)|^2 \leq 0 \]
where $ds$ denotes the surface element and

$$ F(u) = \int_{\Omega} u(\log u - 1) \, dx - \frac{1}{2} \langle (-\Delta)^{-1} u, u \rangle $$

with $v = (-\Delta)^{-1} u$ standing for (3).

A related model arises in the context of chemotaxis in theoretical biology [6, 9], where the Poisson part is provided with the Neumann boundary condition such as

$$ -\Delta v = u - \frac{1}{|\Omega|} \int_{\Omega} u, \frac{\partial v}{\partial \nu} \bigg|_{\partial \Omega} = 0, \quad \int_{\Omega} v = 0. $$

Concerning (1)-(2), (4), and (8), there is a threshold of $\|u_0\|_1 = \lambda$ for the blowup of the solution. More precisely, if $\lambda < 4\pi$ the solution exists global-in-time [1, 4, 11]. If a local mass greater than $4\pi$ is concentrated on a boundary point, on the contrary, there arises blowup in finite time [10, 17]. Underlying blowup mechanisms were suspected from the study of stationary solutions [2]. This attempt was followed by [5, 15], using radially symmetric and general stationary solutions, respectively. Up to now several properties have been known [16, 14, 12]. See our previous work [24] and the references therein. System (1)-(4), provided with Dirichlet condition for the Poisson part, is taken by [23]. It excludes boundary blowup points. Here we continue the study [24] on interior blowup points.

Fundamental features of system (1)-(4) are the following. First, local-in-time unique existence of the classical solution is standard, given smooth initial value $u_0 = u_0(x) \geq 0$. Henceforth, $T \in (0, +\infty]$ denotes its maximal existence time. If $u_0 \not\equiv 0$, which we always assume, the strong maximum principle and the Hopf lemma guarantee $u(\cdot, t) > 0$ on $\overline{\Omega}$ for $t > 0$. Maximal existence time $T$ of non-stationary solution $u = u(\cdot, t)$, on the other hand, is estimated from below by $\|u_0\|_\infty$. Hence $T < +\infty$ implies

$$ \lim_{t \uparrow T} \|u(\cdot, t)\|_\infty = +\infty $$

and the blowup set

$$ S = \{ x_0 \in \overline{\Omega} \mid \exists x_k \to x_0, \exists t_k \uparrow T \text{ such that } u(x_k, t_k) \to +\infty \} $$

is non-empty. Since boundary blowup points are excluded [23], if $T < +\infty$ in (1)-(4) we have

$$ u(x, t) \, dx \to \sum_{x_0 \in S} m(x_0) \delta_{x_0}(dx) + f(x) \, dx \quad \text{in } M(\overline{\Omega}) = C(\overline{\Omega})' $$

as $t \uparrow T$. Here, the blowup set satisfies $S \subset \Omega$ with $2S < +\infty$, and it holds that $0 \leq f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus S)$.

The blowup mechanism at each inner blowup point, described by [24], is more complicated than the ones suspected before. Let $x_0 \in S$ and $R(t) = (T - t)^{1/2}$. As is shown in [22, 25] it holds that

$$ \lim_{b \uparrow +\infty} \limsup_{t \uparrow T} \|u(\cdot, t)\|_{L^1(B(x_0, bR(t)))} = 0. $$

Henceforth, $C_i, i = 1, 2, \cdots, 15,$ denote positive constants.

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Theorem 1 ([21]). Any \( t_k \uparrow T \) admits a sub-sequence, denoted by the same symbol, and \( m \in \mathbb{N} \cup \{0\} \), provided with the following property. Thus, given \( 0 < \varepsilon < 1 \) and \( R \gg 1 \), there is \( \tilde{s} \geq 1 \) such that, for \( t_k' \uparrow T \) defined by \( R(t_k') = \tilde{s}R(t_k) \) we have \( 0 < r_j \leq C_1R(t_k') \) and \( x_j \in B(x_0, C_1R(t_k')) \), \( 1 \leq j \leq m \).

Then, for \( B_k^i = B(x_k^i, b_j) \) and \( G_k = \bigcup_{j=1}^{m} B_k^i \), it holds that

\[
B_k^i \cap B_k^j = \emptyset, \quad i \neq j, \quad k \gg 1
\]

\[
\lim_{k \to \infty} \sup_{b \uparrow +\infty} \left| \| u(\cdot, t_k') \|_{L^1(B_k^i)} - 8\pi \right| < \varepsilon, \quad 1 \leq j \leq m
\]

\[
\lim_{b \uparrow +\infty} \limsup_{t \to T} R(t_k')^2 \| u(\cdot, t_k') \|_{L^\infty(B(0_{bR(t_k')}) \setminus G_k)} \leq C_2 R^{-2}. \tag{12}
\]

In this paper we show that if \( (12) \) is extended continuously then it holds that \( m(x_0) \in 8\pi \mathbb{N} \). Namely, here we assume that any \( \varepsilon > 0 \) admits \( m(t) \in \mathbb{N} \cup \{0\}, x_j(t) \in B(x_0, C_3 R(t)) \), and \( 0 < r_j(t) \leq C_3 R(t) \) for \( 0 < T - t \ll 1 \) and \( 1 \leq j \leq m(t) \in \mathbb{N} \cup \{0\} \), such that, for \( B_j(t) = B(x_j(t), r_j(t)) \) and \( G(t) = \bigcup_{j=1}^{m(t)} B_j(t) \) it holds that

\[
B_i(t) \cap B_j(t) = \emptyset, \quad i \neq j
\]

\[
\lim_{b \uparrow +\infty} \limsup_{t \to T} \max_{1 \leq j \leq m(t)} \left| \| u(\cdot, t) \|_{L^1(B_j(t))} - 8\pi \right| < \varepsilon
\]

\[
\lim_{b \uparrow +\infty} \limsup_{t \to T} R(t)^2 \| u(\cdot, t) \|_{L^\infty(B(0_{bR(t)}) \setminus G(t))} \leq C_4. \tag{13}
\]

Then our result arises as follows.

Theorem 2. If \( (13) \) holds in \( [1, T] \) with \( T < +\infty \), then there is \( \tilde{t}_k \uparrow T \) such that

\[
\lim_{b \uparrow +\infty} \lim_{k \to \infty} \\| u(\cdot, \tilde{t}_k) \|_{L^1(B(0_{bR(\tilde{t}_k)}) \setminus G(\tilde{t}_k))} < \varepsilon. \tag{14}
\]

Hence we obtain \( m(x_0) = 8\pi m, m \in \mathbb{N} \), in \( (10) \), and in particular, \( m(t) = m \) in \( (13) \).

We call \( (13) \) the residual vanishing. Although \( m(x_0) \in 8\pi \mathbb{N} \) follows from \( (11), (13) \), and \( (14) \), we shall show \( m(x_0) \in 8\pi \mathbb{N} \) first, and then \( (14) \). In future we shall discuss the problems to derive \( (13) \) from \( (12) \) and also to refine \( (14) \) to

\[
\lim_{b \uparrow +\infty} \limsup_{t \to T} \\| u(\cdot, t) \|_{L^1(B(0_{bR(t)}) \setminus G(t))} < \varepsilon.
\]

This paper is composed of three sections. Taking preliminaries in the next section, we prove Theorem 2 in the final section.
2 Prelimiaries

Weak solution is a fundamental tool in later arguments. This notion was introduced first for the pre-scaled Smoluchowski-Poisson equation [18]. Let $G = G(x, x')$ be the Green’s function to (3). Then we say that $0 \leq \mu = \mu(dx, t) \in C^1_s([0, T], \mathcal{M}(\Omega))$ is a weak solution to (1)-(3) if there is $0 \leq \nu = \nu(\cdot, t) \in L^\infty(0, T; \mathcal{E}')$ called multiplicate operator satisfying the following properties, where $\mathcal{E}$ is the closure of the linear space $E_0 = \{ \psi + \rho \phi \mid \psi \in C(\Omega \times \Omega), \phi \in \mathcal{X} \}, \mathcal{X} = \{ \phi \in C^2(\Omega) \mid \partial x \phi \partial x = 0 \}$ in $L^\infty(\Omega \times \Omega)$ and $\rho \phi(x, x') = \nabla \phi(x) \cdot \nabla x G(x, x') + \nabla \phi(x') \cdot \nabla x' G(x, x')$:

1. For $\phi \in \mathcal{X}$ the mapping $t \in [0, T] \mapsto \langle \phi, \mu(dx, t) \rangle$ is absolutely continuous and there holds
   \[
   \frac{d}{dt} \langle \phi, \mu(dx, t) \rangle = \langle \Delta \phi, \mu(dx, t) \rangle + \frac{1}{2} \langle \rho \phi, \nu(\cdot, t) \rangle_{\mathcal{E}, \mathcal{E}'} \text{ a.e. } t. \tag{15}
   \]

2. We have
   \[
   \nu(\cdot, t)_{C(\Omega \times \Omega)} = \mu(dx, t) \otimes \mu(dx', t) \text{ a.e. } t. \tag{16}
   \]

Here we confirm that the property $\nu \geq 0$ of $\nu \in \mathcal{E}'$ means
   \[
   |\langle f, \nu \rangle_{\mathcal{E}, \mathcal{E}'}| \leq \langle g, \nu \rangle
   \]
for any $f, g \in \mathcal{E}$ satisfying $|f| \leq g$ a.e. in $\Omega \times \Omega$.

We note the following properties. First, the total mass conservation of this weak solution
   \[
   \mu(\Omega, t) = \mu(\Omega, 0), \quad t \in [0, T]
   \]
is obvious. Next, this weak solution cannot be a measure-valued solution constructed in [3, 8, 13]. In fact, any collision of collapses are not admitted here, and more precisely, we have the following property.

**Lemma 2.1** [18]. If the initial measure $\mu_0(dx) \in \mathcal{M}(\Omega)$ admits $x_0 \in \Omega$ such that
   \[
   \mu_0(\{x_0\}) > 8\pi, \quad \lim_{R \rightarrow 0} \frac{1}{R^2} \langle |x - x_0|^2 \chi_{B(x_0, R)}, \mu_0(dx) \rangle = 0
   \]
then there is no weak solution to (1)-(3) even local-in-time.

It is, however, provided with the following property, derived from the fact that $\mathcal{E}$ is separable.

**Lemma 2.2** [18]. Let $\{\mu_k(dx, t)\} \subset C^1_s([0, T], \mathcal{M}(\Omega))$ be a sequence of weak solutions to (1)-(3). Let the associated multiplicate operator of $\mu_k(dx, t)$ be $\nu_k(\cdot, t) \in L^\infty(0, T; \mathcal{E}')$, and assume
   \[
   \mu_k(\Omega, 0) + \sup_{t \in [0, T]} \|\nu_k(\cdot, t)\|_{\mathcal{E}'} \leq C_0, \quad k = 1, 2, \ldots. \tag{17}
   \]
Then we have a subsequence denoted by the same symbol, \( \mu(dx,t) \in C([0,T], \mathcal{M}(\Omega)) \), and \( \nu(\cdot,t) \in L^\infty(0,T; \mathcal{E}') \) such that

\[
\mu_k(dx,t) \rightharpoonup \mu(dx,t) \quad \text{in} \quad C([0,T], \mathcal{M}(\Omega))
\]

\[
\nu_k(\cdot,t) \rightharpoonup \nu(\cdot,t) \quad \text{in} \quad L^\infty(0,T; \mathcal{E}')
\]

up to a sub-sequence, and this \( \mu(dx,t) \) is a weak solution to (1)-(3) with the multiplicate operator \( \nu(\cdot,t) \) satisfying

\[
\mu(\Omega,0) + \|\nu(\cdot,t)\|_{\mathcal{E}'} \leq C_5.
\]

We agree with the following notations. First, if \( \mu(dx,t) \) has a density as

\[
\mu(dx,t) = u(x,t)dx,
\]

then the multiplicate operator is always taken as

\[
\nu(\cdot,t) = u(x,t)u(x',t)dx'dx',
\]

recalling \( \mathcal{E} \subset L^\infty(\Omega \times \Omega) \). Under this agreement, condition (17) is reduced to

\[
\mu_k(\Omega,0) \leq C_6, \quad k = 1, 2, \ldots
\]

(18)

if each \( \mu_k(dx,t) \) takes density in \( [0,T) \) such as

\[
\mu_k(dx,t) = u_k(x,t)dx,
\]

then the multiplicate operator is always taken as

\[
\nu_k(\cdot,t) = u_k(x,t)u_k(x',t)dx'dx'
\]

in this case, inequality (18) means

\[
\|u_k(\cdot,t)\|_1 = \|u_k(\cdot,0)\|_1 = \mu_k(\Omega,0) \equiv \lambda_k \leq C_6.
\]

Therefore, (17) follows with

\[
\|\nu(\cdot,t)\|_{\mathcal{E}'} = \lambda_k^2 \leq C_6^2.
\]

Using this property, we shall derive a hierarchy of weak solutions in later arguments.

We can define also the regularity of the weak solution. First, given \( \mu = \mu(\cdot,t) \in \mathcal{M}(\Omega) \), we have a unique \( v = v(\cdot,t) \in W^{1,q}(\Omega) \), \( 1 \leq q < 2 \), such that

\[
-\Delta v = \mu, \quad v|_{\partial\Omega} = 0.
\]

Let \( I \subset (0,T) \) be an open interval and \( \omega \subset \Omega \) an open set. If the weak solution \( \mu(dx,t) \) has a density \( u = u(\cdot,t) \in L^p(\omega) \) in \( \omega \subset \Omega \), \( 1 < p < \infty \), for \( t \in I \), the above \( v = v(\cdot,t) \) is in \( W^{2,p}_{\text{loc}}(\omega) \) from the elliptic regularity. By Sobolev’s and Morrey’s imbedding theorems, this regularity implies \( (u\nabla v)(\cdot,t) \in L^1_{\text{loc}}(\omega) \). Then we assign

\[
\frac{d}{dt} \langle \varphi, \mu(dx,t) \rangle = \langle \Delta \varphi(dx,t) \rangle + \langle \nabla \varphi \cdot \nabla v, \mu(dx,t) \rangle, \quad \text{a.e.} \ t \in I
\]
for any $\varphi \in C_0^2(\omega)$. In such a case we say that $\mu(dx, t)$ is regular in $\omega \times I$. In Lemma \[22\], if $\mu_k(dx, t)$ is regular with the density $u_k(x, t)$ in $\omega \times (0, T)$ satisfying
\[
\sup_{t \in [0, T]} \|u_k(\cdot, t)\|_{L^p(\omega)} \leq C_7
\]
for $p > 1$, then the generated $\mu(dx, t)$ is also regular in $\omega \times (0, T)$. Conversely, we have the following properties by the $\varepsilon$ regularity \[19\] \[24\]. First, if the weak solution $\mu(dx, t) \in C_\ast([0, T], \mathcal{M}(\overline{\Omega}))$ is generated by a sequence of classical solutions $\{u_k(x, t)\}$ then its singular part $\mu_s(dx, t)$ is composed of a finite sum of delta functions. Furthermore, if $\mu(dx, t_0), 0 < t_0 < T$, is regular in the sense of measure in an open set $\hat{\omega} \subset \Omega$, then it is regular in the above sense. More precisely, $\mu(dx, t)$ takes a smooth density function $f = f(x, t)$ in $\omega \times (t_0 - \delta, t_0 + \delta)$, where $\omega$ is an open set satisfying $\overline{\omega} \subset \hat{\omega}$ and $0 < \delta \ll 1$.

The weak solution $a(dx, t) \in C_\ast(-\infty, +\infty; \mathcal{M}(\mathbb{R}^2))$ to
\[
a_t = \Delta a - \nabla \cdot \nabla \Gamma * a \quad \text{in } \mathbb{R}^2 \times (-\infty, +\infty)
\]
is defined similarly, where $\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$ is the fundamental solution to $-\Delta$, $\mathcal{M}(\mathbb{R}^2) = C_0(\mathbb{R}^2)'$ with
\[
C_0(\mathbb{R}^2) = \{f \in C(\mathbb{R}^2 \bigcup \{\infty\}) | f(\infty) = 0\},
\]
and $\mathbb{R}^2 \bigcup \{\infty\}$ denotes one-point compactification of $\mathbb{R}^2$. Then the following property is shown, see \[24\].

**Proposition 2.1** (Liouville property). Let $0 \leq a = a(dx, t) \in C_\ast((-\infty, +\infty), \mathcal{M}(\mathbb{R}^2))$ be a weak solution to \[19\] with uniformly bounded multiplicate operator. Then we have either $a(\mathbb{R}^2, t) = 8\pi$ or $a(\mathbb{R}^2, t) = 0$, exclusively in $t \in \mathbb{R}$.

We can also define the weak solution $\zeta(dy, s) \in C_\ast(-\infty, +\infty; \mathcal{M}(\mathbb{R}^2))$ to
\[
\zeta_s = \Delta \zeta - \nabla \cdot \zeta \nabla (\Gamma * \zeta + \frac{|y|^2}{4}) \quad \text{in } \mathbb{R}^2 \times (-\infty, +\infty),
\]
which arises as the weak scaling limit of $u = u(x, t)$. Thus, given $x_0 \in \mathcal{S} \subset \Omega$, we take the backward self-similar transformation
\[
z(y, s) = (T - t_0)u(x, t), \quad y = (x - x_0) / (T - t)^{1/2}, \quad s = -\log(T - t).
\]
Let $t_k \uparrow +\infty$ and put
\[
s_k = -\log(T - t_k) \uparrow +\infty.
\]
Then, passing to a sub-sequence denoted by the same symbol, we have
\[
z(y, s + s_k) dy \rightarrow \zeta(dy, s) \quad \text{in } C_\ast(-\infty, +\infty; \mathcal{M}(\mathbb{R}^2)),
\]
where $\zeta(dy, s)$ is weak solution to \[20\] provided with a uniformly bounded multiplicate operator. In \[21\], the important property called *parabolic envelope* arises as
\[
\zeta(\mathbb{R}^2, s) = m(x_0) > 0, \quad \langle |y|^2, \zeta(dy, s) \rangle \leq C_8
\]
valid to \( s \in (-\infty, +\infty) \) with \( m(x_0) > 0 \) defined by \([10]\), see \([22, 23]\).

Here we take the scaling back of \( \zeta(dy, s) \), defined by the transformation
\[
A(dy', s') = e^s \zeta(dy, s), \quad y' = e^{-s/2} y, \quad s' = -e^{-s}.
\]

It has an extension as \( 0 \leq A = A(dy, s) \in C_s((-\infty, 0], \mathcal{M}(\mathbb{R}^2)) \) with \( A(dy, 0) = m(x_0) \delta_0(dy) \). It becomes also a weak solution to
\[
A_s = \Delta A - \nabla \cdot A \nabla \Gamma * A \quad \text{in} \quad \mathbb{R}^2 \times (-\infty, 0)
\]
satisfying
\[
A(\mathbb{R}^2, s) = m(x_0), \quad -\infty < s < 0
\]
with a uniformly bounded multiplicate operator.

Now, given \( \tilde{s}_\ell \uparrow +\infty \), we take
\[
A_\ell(dy) = A(dy, -\tilde{s}_\ell)/m(x_0)
\]
to apply concentration compactness principle \([7]\) (see also p. 39 of \([21]\)). Then we obtain the following lemma, which implies Theorem 1.

**Lemma 2.3** (concentration compactness). *Passing to a sub-sequence we have \( m \in \mathbb{N} \cup \{0\} \) such that any \( \varepsilon > 0 \) admits \( y^\ell_j \in \mathbb{R}^2 \) and \( b_j > 0 \), \( 1 \leq j \leq m \), satisfying
\[
\lim_{\ell \to \infty} |y^\ell_i - y^\ell_j| = +\infty, \quad \forall i \neq j \quad (27)
\]
\[
\limsup_{\ell \to \infty} |A_\ell(B^\ell_j)| < \varepsilon, \quad \forall j \quad (28)
\]
\[
|y^\ell_j| \leq C_9(1 + \max_j b_j)\tilde{s}_\ell^{1/2}, \quad \forall \ell \gg 1, \quad \forall j \quad (29)
\]
for \( B^\ell_j = B(y^\ell_j, b_j) \). Furthermore, there arises one of the following alternatives.

1. \( m(x_0) > 8\pi m + \varepsilon \) and \( A_\ell, \ell \gg 1 \), is regular in \( \mathbb{R}^2 \setminus \bigcup_{j=1}^m B(y^\ell_j, b_j) \). It holds that
\[
\liminf_{\ell \to \infty} A_\ell \left( \mathbb{R}^2 \setminus \bigcup_{\ell=1}^m B^\ell_j \right) \geq m(x_0) - 8\pi m - \varepsilon \quad (30)
\]
\[
\lim_{\ell \to \infty} \|A_\ell\|_{L^\infty(\mathbb{R}^2 \setminus \bigcup_{j=1}^m B^\ell_j)} = 0.
\]

2. \( m(x_0) = 8\pi m \) and
\[
\limsup_{\ell \to \infty} A_\ell \left( \mathbb{R}^2 \setminus \bigcup_{j=1}^m B(y^\ell_j, b_j) \right) < \varepsilon.
\]
3 Proof of Theorem 2

From the assumption (13), \(\zeta(dy, s)\) generated in the previous section satisfies an additional condition. Namely, each \(0 < \varepsilon < 1\) admits \(s_1 \gg 1\) provided with the following properties. First, for \(s \geq s_1\) there are \(m(\tilde{s}) \in \mathbb{N} \cup \{0\}\), \(y_j(\tilde{s}) \in \mathbb{R}^2\), and \(b_j(\tilde{s}) > 0\, 1 \leq j \leq m(\tilde{s})\), such that |\(y_j(\tilde{s})| \leq C_{10}\tilde{s}^{1/2}\) and \(b_j(\tilde{s}) \leq C_{10}\). Next, \(\zeta(dy, - \log \tilde{s})\) is regular in \(\bigcup_{\tilde{s} \geq s_1}(\mathbb{R}^2 \setminus E_{\tilde{s}}) \times \{ - \log \tilde{s} \}\) for

\[
E_{\tilde{s}} = \bigcup_{j=1}^{m(\tilde{s})} B_j(\tilde{s}), \quad \tilde{B}_j(\tilde{s}) = B(\tilde{s}^{-1/2}y_j(\tilde{s}), \tilde{s}^{-1/2}b_j(\tilde{s})).
\]

Finally, it holds that \(B_i(\tilde{s}) \cap B_j(\tilde{s}) = \emptyset, i \neq j\), and

\[
\sup_{\tilde{s} \geq s_1} \|\zeta(dy, - \log \tilde{s})\|_{L^\infty(\mathbb{R}^2 \setminus E_{\tilde{s}})} \leq C_{11}
\]

\[
\sup_{\tilde{s} \geq s_1, 1 \leq j \leq m(\tilde{s})} |\zeta(B_j(\tilde{s}), - \log \tilde{s}) - 8\pi| < \varepsilon.
\]

If \(m(x_0) \notin 8\pi\mathbb{N}\), we have always the first alternative in Lemma 2.3 which implies the existence of \(\delta > 0\) such that

\[
\inf_{\tilde{s} \geq s_1} \zeta(\mathbb{R}^2 \setminus E_{\tilde{s}}, - \log \tilde{s}) \geq \delta.
\]

If (35) is not the case there is \(s_2 \geq s_1\) such that

\[
\zeta(\mathbb{R}^2 \setminus E_{s_2}, - \log s_2) < \varepsilon.
\]

Then it holds that (14) with some \(\tilde{t}_k \uparrow T\). Hence Theorem 2 is reduced to the following lemma. For the proof we use (20), particularly, the term \(|y|^2/4\), which attract the density of \(\zeta(dy, s)\) to \(|y| = \infty\).

**Lemma 3.1.** The weak scaling limit \(\zeta(dy, s) \in C_\ast(-\infty, +\infty; \mathcal{M}(\mathbb{R}^2))\), generated by (21), does not satisfy (34) and (35), simultaneously.

**Proof.** Similarly to Lemma 2.2 concerning (1)-(3), given \(\tilde{s}_\ell \uparrow +\infty\), we have a sub-sequence denoted by the same symbol such that

\[
\zeta(dy, s - \tilde{s}_\ell) \rightarrow \tilde{\zeta}(dy, s) \quad \text{in } C_\ast(-\infty, +\infty; \mathcal{M}(\mathbb{R}^2)).
\]

This \(\tilde{\zeta}(dy, s)\) is a weak solution to (20). Furthermore, since \(\{\zeta(dy, s - \tilde{s}_\ell)\}\) is tight by (22), it holds that

\[
\tilde{\zeta}(\mathbb{R}^2, s) = m(x_0), \quad \langle |y|^2, \tilde{\zeta}(dy, s) \rangle \leq C_8.
\]

We have also

\[
\tilde{s}^{-1/2}|y_j(\tilde{s})| \leq C_{10}, \quad \lim_{\tilde{s} \uparrow +\infty} \tilde{s}^{-1/2}b_j(\tilde{s}) = 0
\]

in (33).

Similarly to the remark after Lemma 2.2, the singular part of \(\tilde{\zeta}(dy, s), s \in \mathbb{R}\), denoted by \(\tilde{\zeta}_s(dy, s)\), is composed of a finite sum of delta functions. By applying Lemma 2.1 to the scaling back
\( \hat{A}(dy, s) \) of \( \zeta(dy, s) \), defined as in [29], the coefficient of each delta function of \( \hat{\zeta}(dy, s) \) must be less than or equal to \( 8\pi \). These properties guarantee that the singular support of \( \hat{\zeta}(dy, s) \), denoted by \( S_s \), is composed of a finite number of collisionless accumulating points of \( \{ s^{-1/2} y_j(\tilde{s}) \mid 1 \leq j \leq m(\tilde{s}) \} \) defined for \( -\log \tilde{s} = s + \tilde{s} \) as \( \ell \to \infty \). We may assume also that \( S_s, s \in Q \), is composed of their converging points by a diagonal argument.

Therefore, we have \( m(s) \in N \cup \{0\} \) and \( y^j_\infty(s) \in \mathbb{R}^2, 1 \leq j \leq m(\tilde{s}) \), such that

\[
S_s = \{ y^j_\infty(s) \mid 1 \leq j \leq m(s) \}, \quad \hat{\zeta}(\{ y^j_\infty(s) \}, s) = 8\pi
\]

for \( s \in \mathbb{R} \). Furthermore, the density function \( g = g(y, s) \) of the absolutely continuous part of \( \hat{\zeta}(dy, s) \) is provided with the properties

\[
0 \leq g = g(y, s) \leq C_{11}, \quad \int_{\mathbb{R}^2} |y|^2 g(y, s) \, dy \leq C_8, \quad \|g(\cdot, s)\|_1 \leq m(x_0) \quad \|g(\cdot, s)\|_1 \geq \delta, \ s \in Q. \tag{38}
\]

It holds also that \( S_s \subset B_R \) for any \( R > C_{10} \).

Since \( \hat{\zeta}(dy, s) \in C_s(\mathbb{R}^2, M(\mathbb{R}^2)) \) the set \( G = \bigcup_s (\mathbb{R}^2 \setminus S_s) \times \{ s \} \) is open in \( \mathbb{R}^2 \times (-\infty, +\infty) \).

Furthermore, the above \( g = g(y, s) \) is smooth in \( G \) and it holds that

\[
g_s = \Delta g - \nabla \cdot g \nabla w \quad \text{in } G
\]

\[
w(y, s) = \frac{|y|^2}{4} + 4 \sum_{j=1}^{m(s)} \log \frac{1}{|y - y^j_\infty(s)|} + \Gamma \ast g. \tag{39}
\]

Let

\[
v(y, s) = \frac{|y|^2}{4} + (\Gamma \ast g)(y, s). \tag{40}
\]

Henceforth, we put

\[
\bar{f}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \, d\theta \tag{41}
\]

for \( f = f(y), y \in \mathbb{R}^2 \), where \( y = re^{i\theta} \) is the polar coordinate.

We take \( B_r = B(0, r), r > R, \) and \( s_0 \in \mathbb{R} \), to set

\[
B^\varepsilon_r = B_r \setminus \bigcup_{j=1}^{m(s_0)} B(y^j_\infty(s_0), \varepsilon).
\]

By Lemma [27] again, any collision of the points in \( S_s \) does not occur as \( s \) varies. Therefore, making \( 0 < \varepsilon \ll 1 \), we obtain \( \sharp \left( B(y^j_\infty(s_0), \varepsilon) \cap S_s \right) \leq 1 \) and

\[
y^j_\infty(s) \in B(y^j_\infty(s_0), \varepsilon) \Rightarrow (y - y^j_\infty(s)) \cdot \nu_y \geq 0, \ y \in \partial B(y^j_\infty(s_0), \varepsilon)
\]

for \( 1 \leq j \leq m(s_0) \) and \( |s - s_0| \ll 1 \), where \( \nu_y \) denotes the outer unit normal vector.

Then, (39) implies

\[
\frac{d}{ds} \int_{B^\varepsilon_r} g(y, s) \, dy \leq \int_{\partial B^\varepsilon_r} g_r(y, s) \, d\sigma_y - \int_{\partial B^\varepsilon_r} (g \ast \bar{f})(y, s) \, d\sigma_y \tag{42}
\]
for \( |s - s_0| \ll 1 \), where \( d\sigma = d\sigma_y \) denotes the line element. We integrate (42) in \( t \), to convert it to an inequality on the difference quotient with the mesh \( h > 0 \). Now, making \( \varepsilon \downarrow 0 \) and then \( h \downarrow 0 \), we obtain

\[
\frac{d^+}{ds} \int_{B_r} g \, dy \leq \int_{\partial B_r} g_r \, d\sigma - \int_{\partial B_r} (g v_r) \, d\sigma = \int_{\partial B_r} g_r - \frac{r}{2} g - g(\Gamma \ast g)_r \, d\sigma,
\]

where

\[
\frac{d^+}{ds} A(s) = \limsup_{h \downarrow 0} \frac{1}{h} (A(s + h) - A(s)).
\]

Then, inequality (43) is valid to any \( r > R \) and \(-\infty < s < +\infty\).

From \( \|g(\cdot, s)\|_1 + \|g(\cdot, s)\|_\infty \leq C_{12} \), it follows that

\[
\sup_{s} \|\langle \nabla \Gamma \ast g \rangle(\cdot, s)\|_\infty < +\infty.
\]

Then, (43) implies

\[
\frac{d^+}{ds} \int_{B_r} g \leq \int_{\partial B_r} g_r - \frac{r}{2} g + C_{13} g \, d\sigma
\]

\[
= \frac{d}{dr} (r \int_0^{2\pi} g \, d\theta) - \int_0^{2\pi} (1 + \frac{r^2}{2}) g \, d\theta + C_{13} r \int_0^{2\pi} g \, d\theta,
\]

which means

\[
\frac{d^+}{ds} \int_0^r r\overline{g} \, dr \leq \frac{d}{dr} (r \overline{g}) - (\frac{r^2}{2} - C_{13} r + 1) \overline{g}, \quad r > R, \ -\infty < s < +\infty,
\]

recalling (41). Here we have

\[
B(r, s) \equiv \int_0^r r\overline{g} \, dr \geq \delta - \frac{C_{14}}{r^2 + 1}, \quad r > R, \ s \in Q
\]

by (33). The strong maximum principle, on the other hand, implies also \( B(r, s) > 0 \) for any \( (r, s) \).

Inequality (45) means

\[
\partial^+_r B \leq B_r - a(r) B_r, \quad a(r) = \frac{r}{2} - C_{13} + \frac{1}{r}.
\]

Using \( A(r) = \frac{r^2}{4} - C_{13} r + \log r \), we have \( a = A' \) and hence

\[
\partial^+_r (e^{-A} B) \leq (e^{-A} B_r)_r, \quad r > R, \ -\infty < s < +\infty
\]

by \( B_r - a(r) B_r = e^A (e^{-A} B_r)_r \). Now we take \( r_1 > R \) such that

\[
a(r) \geq \frac{r}{4} + 1, \quad \delta - \frac{C_{14}}{r^2 + 1} \geq \delta/2, \quad r \geq r_1.
\]
Let \(0 \leq \varphi = \varphi(r)\) be a \(C^1\) function on \([r_1, \infty)\), piecewise \(C^2\), satisfying \(\varphi(r_1) = 0\), \(\varphi(r) > 0\) for \(r > r_1\), and
\[
\int_{r_1}^{\infty} e^{-A \varphi} \, dr < +\infty, \quad \lim_{r \to +\infty} e^{-A (\varphi + \varphi_r)} = 0. \tag{47}
\]
Then we put \(\varphi(r) = 0\) for \(r \in [0, r_1]\). Since \(B \geq 0\), \(B_r \geq 0\), \(B(\infty, t) \leq C_{15}\), each \(s \in \mathbb{R}\) admits \(r_j \uparrow +\infty\) such that \(B_r(r_j, s) \to 0\), which guarantees
\[
\int_{r_1}^{\infty} e^{-A \varphi} \, dr < +\infty, \quad \lim_{r \to +\infty} e^{-A (\varphi + \varphi_r)} = 0. \tag{47}
\]
Then we put \(\varphi(r) = 0\) for \(r \in [0, r_1]\). Since \(B \geq 0\), \(B_r \geq 0\), \(B(\infty, t) \leq C_{15}\), each \(s \in \mathbb{R}\) admits \(r_j \uparrow +\infty\) such that \(B_r(r_j, s) \to 0\), which guarantees
\[
\int_{r_1}^{\infty} e^{-A \varphi} \, dr < +\infty, \quad \lim_{r \to +\infty} e^{-A (\varphi + \varphi_r)} = 0. \tag{47}
\]
We impose, furthermore, that the existence of \(\mu > 0\) such that \(\varphi_{rr} - a \varphi_r \leq -\mu \varphi, \; r \geq r_1\) \(\tag{49}\)
To assure all the above requirements to \(\varphi(r)\), we take \(0 < c_1 \ll 1\), for example, and put
\[
\varphi(r) = \begin{cases} 
c_1 r + c_2, & r \geq r_2 
\sin \beta (r - r_1), & r_1 \leq r < r_2 \equiv r_1 + \frac{\pi}{4}
\end{cases}
\]
where
\[
r_2 = r_1 + \frac{\pi}{4 \sqrt{2} c_1}, \quad c_2 = \frac{1}{\sqrt{2}} (1 - \frac{\pi}{4}) r_1 c_1, \quad \beta = \sqrt{2} c_1.
\]
Then we see \(0 \leq \varphi = \varphi(r) \in C^1([r_1, \infty), \varphi(r_1) = 0, \) and \(\tag{47}\). Making \(c_1 \downarrow 0\), on the other hand, we obtain \(r_2 \uparrow +\infty\). Therefore, \(\tag{49}\) arises for \(0 < \mu \ll 1\) by
\[
\varphi_{rr} = \begin{cases} 
0, & r \geq r_2 
-\beta^2 \varphi, & r_1 \leq r < r_2
\end{cases}
\]
and
\[
a(r) = \frac{r}{2} - C_{13} + \frac{1}{r}, \quad \varphi_r = c_1 > 0, \quad r \geq r_2.
\]
Since \(\tag{48}\) is obtained we have
\[
\frac{d^+}{ds} \int_0^\infty e^{-A B} \varphi \, dr \leq -\mu \int_0^\infty e^{-A B} \varphi \, dr, \quad s \in \mathbb{R}.
\]
This means
\[
\frac{d^+}{ds} \left\{ e^\frac{\mu s}{2} \int_0^\infty e^{-A B} \varphi \, dr \right\} < 0, \quad s \in \mathbb{R}
\]
and hence
\[
\lim_{s \to +\infty} \int_0^\infty e^{-A B(\cdot, s)} \varphi \, dr = 0.
\]
We have, on the other hand,
\[
\int_0^\infty e^{-A B(\cdot, s)} \varphi \, dr \geq \frac{\delta}{2} \int_0^\infty e^{-A \varphi} \, dr > 0, \quad s \in \mathbb{Q}
\]
by \(\tag{46}\), a contradiction. \(\square\)
References

[1] P. Biler, *Local and global solvability of some systems modelling chemotaxis*, Adv. Math. Sci. Appl. 8 (1998) 715-743.

[2] S. Childress and J.K. Percus, *Nonlinear aspects of chemotaxis*, Math. Biosci. 56 (1981) 217-237.

[3] J. Dolbeault and C. Schmeiser, *The two-dimensional Keller-Segel model after blowup*, Discrete and Contin. Dyn. Syst. Ser. A 25 (2009) 109-121.

[4] H. Gajewski and K. Zacharias, *Global behaviour of a reaction-diffusion system modelling chemotaxis*, Math. Nachr. 195 (1998) 77-114.

[5] M.A. Herrero and J.J.L. Velázquez, *Singularity patterns in a chemotaxis model*, Math. Ann. 306 (1996) 583-623.

[6] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. 329 (1992) 819-824.

[7] P.L. Lions, *The concentration-compactness principle in the calculus of variation. The locally compact case, Part I*, Ann. Inst. H. Poincaré, Analyse nonlinéaire 1 (1984) 109-145.

[8] S. Luckhaus, Y. Sugiyama, and J.J.L. Velázquez, *Measure valued solutions of the 2D Keller-Segel system*, Arch. Rational Mech. Anal. 206 (2012) 31-80.

[9] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci. Appl. 5 (1995) 581-601.

[10] T. Nagai, *Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains*, J. Inequal. Appl. 6 (2001) 37-55.

[11] T. Nagai, T. Senba, and K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac. 40 (1997) 411-433.

[12] Y. Naito and T. Suzuki, *Self-similarity in chemotaxis systems*, Colloquium Mathematics 111 (2008) 11-34.

[13] Y. Seki, Y. Sugiyama, and J.J.L. Velázquez, *Multiple peak aggregations for the Keller-Segel system*, Nonlinearity 26 (2013) 319-352.

[14] T. Senba, *Type II blowup of solutions to a simplified Keller-Segel system in two dimensions*, Nonlinear Analysis 66 (2007) 1817-1839.

[15] T. Senba and T. Suzuki, *Some structures of the solution set for a stationary system of chemotaxis*, Adv. Math. Sci. Appl. 10 (2000) 191-224.

[16] T. Senba and T. Suzuki, *Chemotactic collapse in a parabolic-elliptic system of mathematical biology*, Adv. Differential Equations 6 (2001) 21-50.
[17] T. Senba and T. Suzuki, *Parabolic system of chemotaxis: blowup in a finite and the infinite time*, Meth. Appl. Anal. 8 (2001) 349-368.

[18] T. Senba and T. Suzuki, *Weak solutions to a parabolic-elliptic system of chemotaxis*, J. Funct. Anal. 191 (2002) 17-51.

[19] T. Senba and T. Suzuki, *Time global solutions to a parabolic-elliptic system modelling chemotaxis*, Asymptotic Analysis 32 (2002) 63-89.

[20] C. Sire and P.-H. Chavanis, *Thermodynamics and collapse of self-gravitating Brownian particles in $D$ dimensions*, Phys. Rev. E 66 (2002) 046133.

[21] M. Struwe, *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamilton Systems*, third edition, Springer Verlag, Berlin, 2000.

[22] T. Suzuki, *Free Energy and Self-Interacting Particles*, Birkhäuser, Boston, 2005.

[23] T. Suzuki, *Exclusion of boundary blowup for 2D chemotaxis system provided with the Dirichlet boundary condition for the Poisson part*, J. Math. Pure Appl. 100 (2013) 347-367.

[24] T. Suzuki, *Almost collapse mass quantization in 2D Smoluchowski-Poisson equation*, arXive1311.5679.

[25] T. Suzuki and T. Senba, *Applied Analysis - Mathematical Methods in Natural Science*, 2nd edition, Imperial College Press, 2011.

Takashi Suzuki  
Division of Mathematical Science  
Department of Systems Innovation  
Graduate School of Engineering Science  
Osaka University  
Machikaneyamacho 1-3  
Toyonakashi, 560-8531, Japan  
suzuki@sigmath.es.osaka-u.ac.jp