Large monochromatic components of small diameter

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Abstract
Gyárfás conjectured in 2011 that every r-edge-colored $K_n$ contains a monochromatic component of bounded (“perhaps three”) diameter on at least $n/(r-1)$ vertices. Letzter proved this conjecture with diameter four.

In this note we improve the result in the case of $r = 3$: We show that in every 3-edge-coloring of $K_n$ either there is a monochromatic component of diameter at most three on at least $n/2$ vertices or every color class is spanning and has diameter at most four.

KEYWORDS
diameter, monochromatic component, Ramsey theory

MATHEMATICAL SUBJECT CLASSIFICATION
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1 | MONOCHROMATIC COMPONENTS

An easy exercise in an introductory graph theory course—a remark by Erdős and Rado, see [4]—states that any 2-coloring of the edges of $K_n$ has a monochromatic spanning component. In general, Gyárfás [3] proved that the largest monochromatic component in an r-edge-coloring of $K_n$ has order at least $n/(r-1)$ and equality holds if an affine plane of order $r-1$ exists and $(r-1)^2$ divides $n$. 

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Füredi [2] proved the significantly larger lower bound \( n/(r - 1 - (r - 1)^{-1}) \) in the case that there is no affine plane of order \( r - 1 \). This connection to the existence of affine planes suggests that to determine exactly the maximum size of a monochromatic component is extremely difficult in general.

A double (triple) star is the tree obtained by joining the centers of two (three) stars by a path of length one (two). Clearly, double or triple stars have diameter 3 or 4, respectively. Additional structure on large monochromatic components has been conjectured by Gyárfás [4].

**Conjecture 1** (Gyárfás [4, Problem 4.2]). For \( r \geq 3 \), is there a monochromatic double star on at least \( n/(r - 1) - o(n) \) vertices in every \( r \)-coloring of \( K_n \)?

A weaker version of the problem reads as follows.

**Conjecture 2** (Gyárfás [4, Problem 4.3]). Given positive numbers \( n, r \). Is there a constant \( d \) (perhaps \( d = 3 \)) such that in every \( r \)-coloring of \( K_n \) there is a monochromatic subgraph of diameter at most \( d \) with at least \( n/(r - 1) \) vertices?

The assumption \( r \geq 3 \) in Conjecture 1 is necessary, since a random two-coloring will give a monochromatic double star of size \( \approx 3n/4 \) only. The best result for double stars is due to Gyárfás and Sárközy.

**Theorem 3** (Gyárfás and Sárközy [5]). Every \( r \)-edge-coloring of \( K_n \) contains a monochromatic double star on at least \( \frac{n(r+1)+r-1}{r^2} \) vertices.

The bipartite Ramsey number of the double star has been determined by Mubayi [7]. The result of Theorem 4 is tight if each color class is biregular.

**Theorem 4** (Mubayi [7]). In every \( r \)-edge-coloring of the complete bipartite graph \( K_{k,\ell} \) there is a monochromatic double star of order \( \frac{k+\ell}{r} \).

The weaker Conjecture 2 was later shown to be true by Ruszinkó [8] with \( d = 5 \).

**Theorem 5** (Ruszinkó [8]). In every \( r \)-edge-coloring of \( K_n \) there is a monochromatic subgraph of diameter at most 5 on at least \( n/(r - 1) \) vertices.

This was further improved and was shown to be true for \( d = 4 \) by Letzter [6].

**Theorem 6** (Letzter [6]). In every \( r \)-edge-coloring of \( K_n \) there is a monochromatic triple star on at least \( n/(r - 1) \) vertices.

For the case of \( d = r = 2 \), the following tight bound was proved by Erdős and Fowler [1].

**Theorem 7** (Erdős and Fowler [1]). Every 2-edge-coloring of \( K_n \) contains a monochromatic connected subgraph of diameter at most 2 on at least \( 3n/4 \) vertices.
Moreover, for \( r = 3, 4, 5, 6 \), Ruszinkó, Song, and Szabo [9] constructed colorings where the maximum size of a monochromatic, diameter 2 subgraph is strictly less than \( n/(r - 1) \), suggesting that \( d = 3 \) is best possible for diameter in Conjecture 2.

In this note we further improve (in terms of diameter) Theorem 6 for three colors. Let \( G_\alpha, G_\beta, \) and \( G_\gamma \) be the subgraphs of \( K_n \) induced by the edges that have color \( \alpha, \beta, \) and \( \gamma, \) respectively.

**Theorem 8.** In every 3-edge-coloring of \( K_n \) either there is a monochromatic connected subgraph of diameter at most 3 on at least \( n/2 \) vertices or each of \( G_\alpha, G_\beta, \) and \( G_\gamma \) is spanning and has a diameter at most 4.

**Proof.** By Theorem 4, we may assume that each of \( G_\alpha, G_\beta, \) and \( G_\gamma \) is both spanning and connected because if one is not, then the union of the other two color classes is a complete bipartite graph on \( n \) vertices.

Suppose, towards a contradiction and without loss of generality, that the distance between \( w_1 \) and \( w_2 \) is at least 5 in \( G_\alpha \) and \( w_1w_2 \in E(G_\beta) \). The set of the vertices \( U \) of the double star centered by \( w_1 \) and \( w_2 \) in \( G_\beta \) must contain less than \( n/2 \) vertices, otherwise the theorem is proven.

Note that there are no \( \beta \)-colored edges from \( \{w_1, w_2\} \) to \( V \setminus U \) by definition. Split the remaining vertices of \( V \setminus U \) into three parts:

\[
X = \{v \in V \setminus U : vw_1 \in E(G_\gamma), vw_2 \in E(G_\alpha)\},
Y = \{v \in V \setminus U : vw_1 \in E(G_\gamma), vw_2 \in E(G_\gamma)\},
Z = \{v \in V \setminus U : vw_1 \in E(G_\alpha), vw_2 \in E(G_\gamma)\}.
\]

Note that there are no vertices \( v \), such that \( vw_1 \in E(G_\alpha) \) and \( vw_2 \in E(G_\alpha) \), or else the distance between \( w_1 \) and \( w_2 \) in \( G_1 \) would be 2 < 5. Clearly, neither \( X \) nor \( Z \) is empty, or else there is a star in \( G_\gamma \) (centered at either \( w_1 \) or \( w_2 \)) of order greater than \( n/2 \). Furthermore, no edge between \( X \) and \( Z \) is colored \( \alpha \), or else we have a path of length 3 in \( G_\alpha \) between \( w_1 \) and \( w_2 \).

In addition, there is a length 2 path in color \( \gamma \) between each pair of vertices in \( X \) (through \( w_1 \)), between each vertex in \( X \) with each vertex in \( Y \) (through \( w_1 \)), between each pair of vertices in \( Y \) (through either \( w_1 \) or \( w_2 \)), between each vertex in \( Y \) and each vertex in \( Z \) (through either \( w_1 \) or \( w_2 \)), and between each pair of vertices in \( Z \) (through \( w_2 \)).

Since \( X \cup Y \cup Z \cup \{w_1, w_2\} \) contains more than \( n/2 \) vertices, there must exist some vertices, \( v_X \in X, v_Z \in Z \), such that their distance in color \( \gamma \) within the vertex set \( X \cup Y \cup Z \cup \{w_1, w_2\} \) is at least 4, otherwise we have found a vertex set of diameter 3 in color \( \gamma \) of size larger than \( n/2 \). For this to be the case, neither \( v_X \) nor \( v_Z \) can have an edge colored \( \gamma \) connecting it to any vertex in \( Y \), otherwise there would be a path of length 3 connecting \( v_X \) and \( v_Z \) in \( G_\gamma \). Furthermore, since there is no edge of color \( \alpha \) between \( X \) and \( Z \), \( v_X \) must have only edges colored \( \beta \) between itself and all vertices of \( Z \), and \( v_Z \) must have only edges colored \( \beta \) between itself and all vertices of \( X \), otherwise, again \( v_X \) and \( v_Z \) would have distance at most 3 in \( G_\gamma \).

Now, we have a double star in color \( \beta \), anchored at \( v_X \) and \( v_Z \), containing all of \( X \cup Z \). If \( Y \) is empty, the theorem is proven. Therefore, there must be some \( v_Y \in Y \) such that
neither \( v_Yv_X \) nor \( v_Yv_Z \) has color \( \beta \), otherwise there is a double star in color \( \beta \), anchored at \( v_X \) and \( v_Z \), containing \( X \cup Y \cup Z \), which is a double star on at least \( n/2 \) vertices.

So, the edges \( v_Yv_X \) and \( v_Yv_Z \) have neither color \( \beta \) nor color \( \gamma \). Therefore, both such edges have color \( \alpha \). This produces a path in \( G_\alpha \), namely, \( w_1v_Zv_Yv_Xw_2 \), of length 4, which is a contradiction to the assumption that \( w_1 \) and \( w_2 \) have distance at least 5 in \( G_\alpha \). \( \square \)

2 | CONCLUSION

Though Theorem 8 does not prove Conjecture 2 for \( d = 3 \) in the case of three colors, it gives support to this very natural and surprisingly difficult question.

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