A holographic description of heavy-flavoured baryonic matter decay involving glueball

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Abstract

We holographically investigate the decay of heavy-flavoured baryonic hadron involving glueball by using the Witten-Sakai-Sugimoto model. Since baryon in this model is recognized as the D4-brane wrapped on $S^4$ and the glueball field is identified as the bulk gravitational fluctuations, the interaction of the bulk graviton and the baryon brane could be naturally interpreted as glueball-baryon interaction through the holography which is nothing but the close-open string interaction in string theory. In order to take account into the heavy flavour, an extra pair of heavy-flavoured branes separated from the other flavour branes with a heavy-light open string is embedded into the bulk. Due to the finite separation of the flavour branes, the heavy-light string creates massive multiplets which could be identified as the heavy-light meson fields in this model. As the baryon brane on the other hand could be equivalently described by the instanton configuration on the flavour brane, we solve the equations of motion for the heavy-light fields with the Belavin-Polyakov-Schwarz-Tyupkin (BPST) instanton solution for the $N_f = 2$ flavoured gauge fields. Then with the solutions, we evaluate the soliton mass by deriving the flavoured onshell action in strongly coupling limit and heavy quark limit. After the collectivization and quantization, the quantum mechanical system for glueball and heavy-flavoured baryon is obtained in which the effective Hamiltonian is time-dependent. Finally we use the standard technique for the time-dependent quantum mechanical system to analyze the decay of heavy-flavoured baryon involving glueball and we find one of the decay process might correspond to the decay of baryonic B-meson involving the glueball candidate $f_0(1710)$. This work is a holographic approach to study the decay of heavy-flavoured hadron in nuclear physics.

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1 Introduction

Quantum Chromodynamics (QCD) as the fundamental theory of nuclear physics predicts the bound state of pure gluons [1, 2, 3] because of its non-Abelian nature. Such bound state is always named as “glueball” which is believed as the only possible composite particle state in the pure Yang-Mills theory. In general glueball states could have various Lorentz structures e.g. a scalar, pseudoscalar or a tensor glueball with either normal or exotic $J^{PC}$ assignments. Although the glueball state has not been confirmed by the experiment, its spectrum has been studied by the simulation of lattice QCD [4, 5, 6]. According to the lattice calculations, it indicates that the lightest glueball state is a scalar with assignment of 0$^{++}$ and its mass is around 1500-2000MeV [4, 7]. These results also suggest that the scalar meson $f_0(1710)$ could be considered as a glueball state. Glueball may be produced by the decay of various hadrons in the heavy-ion collision [8, 9, 10], so the dynamics of glueball is very significant. However lattice QCD involving real-time quantities is extremely complexed and phenomenological models usually include a large number of parameters with some corresponding uncertainties. Thus it is still challenging to study the dynamics of glueball with traditional quantum field theory.

Fortunately there is an alternatively different way to investigate the dynamics of glueball based on the famous AdS/CFT correspondence or the gauge-gravity duality pioneered in [11] where a top-down holographic approach from string theory by Witten [12] and Sakai and Sugimoto [13] (i.e. the WSS model) is employed. Analyzing the AdS/CFT dictionary with the WSS model, the glueball field is identified as the bulk gravitational fluctuations carried by the close strings while the meson states are created by the open strings on the $N_f$ probe flavour branes. Hence this model naturally includes the interaction of glueball and meson through the holography which is nothing but the close-open string interaction in string theory. Along this direction, decay of glueball into mesons has been widely studied with this model e.g. in [14, 15, 16].

Keeping the above information in mind and partly motivated by [8, 9, 10], in this work we would like to holographically explore the glueball-baryon interaction particularly involving the heavy flavour as an extension to the previous study in [19]. In the WSS model, baryon is identified as the D4'-brane wrapped on $S^4$ [13, 20] (namely the baryon vertex) which could be equivalently described by the instanton configuration on the flavoured D8-branes according to the string theory [21, 22]. The configuration of various D-branes is illustrated in Table 1. In order to take account into the heavy flavour, we embed an extra pair of flavoured D8/D8-brane into the bulk geometry which is separated from the other $N_f$ (light-flavoured) D8/D8-branes with an open string (the heavy-light string) stretched between them [23, 24] as illustrated in Figure 1. In this configuration there would be additional multiplets created by the heavy-light (HL) string

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2 Since the WSS model is based on AdS$_7$/CFT$_6$ correspondence, several previous work is also relevant to this model e.g. [17, 18].

3 We will use “D4’-brane” to distinguish the baryon brane from those $N_c$ D4-branes as colour branes throughout the manuscript.
and they would acquire mass due to the finite separation of the flavour branes. Hence we could interpret these multiplets as the HL meson fields and the instanton configuration on the D8-branes with the multiplets would include heavy flavour thus can be identified as heavy-flavoured baryon \cite{25, 26}. So similarly as the case of glueball and meson, there must be glueball-baryon interaction in holography as close string interacting with D4′-brane carrying the heavy-flavour through the HL string, or namely graviton interacting with the heavy-flavoured instantons.

\[
\begin{array}{cccccccc}
 & 0 & 1 & 2 & 3 & (4) & 5(U) & 6 & 7 & 8 & 9 \\
\hline
\text{Coloured } N_c \text{ D4} & - & - & - & - & - & - & - & - & - & - \\
\text{Flavoured } N_f \text{ D8/D8} & - & - & - & - & - & - & - & - & - & - \\
\text{Baryon vertex D4'} & - & - & - & - & - & - & - & - & - & - \\
\end{array}
\]

Table 1: The brane configuration of the WSS model: “-” denotes the world volume directions of the D-branes.

Let us outline the content and the organization of this manuscript here. We consider the baryonic bound states created by the baryon vertex in this model with two flavours i.e. \( N_f = 2 \). Following \cite{22, 24, 25, 26}, baryons are identified as Skyrmions in the WSS model and they can be described by a quantum mechanical system for their collective modes in the moduli space. The effective Hamiltonian could be obtained by evaluating the classical mass of the soliton \( S^{\text{onshell}} = - \int dt M_{\text{soliton}} \). So the main goal of this paper is to evaluate the effective Hamiltonian involving glueball-baryon interaction with heavy flavour. In section 2, we specify the setup with the heavy flavour in this model and solve the classical equations of motion for the HL meson field on the flavour brane. In section 3, we search for the analytical solutions of the bulk gravitational fluctuations then explicitly compute the onshell action with these solutions. All the calculations are done in the limitation of large ’t Hooft coupling constant \( \lambda \) where the holography is exactly valid. The final formulas of the effective Hamiltonian depend on the glueball field and the number of heavy-flavoured quarks so that it is time-dependent. Therefore the method for the time-dependent system in quantum mechanics would be suitable to describe the decay of heavy-flavoured baryons under the classical glueball field. Resultantly we obtain several possible decay processes with the effective Hamiltonian and pick out one of them which might probably correspond to the decay of baryonic B-meson involving the glueball candidate \( f_0 (1710) \) as discussed in \cite{8, 9, 10}.

Since the WSS model has been presented in many lectures, for reader’s convenience we only collect some relevant information about this model in the Appendix A, B, C which can be also reviewed in \cite{13, 22, 25, 26, 27}. Respectively the gravitational polarization used in this paper are collected in Appendix A. Some useful formulas about the D-brane action and the embedding of the probe branes and string in our setup can be found in Appendix B. In Appendix C, it reviews the effective quantum mechanical system for the collective modes of baryon. At the end of this manuscript some messy but essential calculations about our main discussion have been
summarized in the Appendix D.

2 Baryon as instanton with heavy flavour

The baryon spectrum with pure light flavours in this model is reviewed in the Appendix C, so we only outline how to include the heavy flavour in this section. Some necessary information about the embedding of the probe branes and string in our setup could be reviewed in Appendix B.

A simple way to involve the heavy flavour in this model is to embed an extra pair of flavoured D8/\overline{D8}-brane separated from the other \(N_f\) (light-flavoured) D8/\overline{D8}-branes with an open string (the heavy-light string) stretched between them as illustrated in Figure 1. The HL string creates additional multiplets according to the string theory [27] since it connects to the separated branes. And these multiplets could be approximated by local vector fields near the worldvolume of the light flavour branes. Note that the multiplets acquire mass due to the finite length, or namely the non-zero vacuum expectation value (VEV) of the HL string. Therefore we could interpret the multiplets created by the HL string as the heavy-flavoured mesons with massive flavoured (heavy-flavoured) quarks. Actually this mechanism to acquire mass is nothing but the “Higgs mechanism” in string theory. So let us replace the gauge fields on the light flavour branes by its matrix-valued form to involve the heavy flavour,

\[ A_a \rightarrow \mathcal{A}_a = \begin{pmatrix} A_a & \Phi_a \\ -\Phi_\dagger_a & 0 \end{pmatrix} \]  

(2.1)

In our notation \(A_a\) is an \(N_f \times N_f\) matrix-valued 1-form while \(\mathcal{A}_a\) is an \((N_f+1) \times (N_f+1)\) matrix-valued 1-form. \(\Phi_a\) is an \(N_f \times 1\) matrix-valued vector which represents HL meson field and the index runs over the light flavour brane. Thus the field strength of \(\mathcal{A}_a\) also becomes matrix-valued as a 2-form,

\[ F_{ab} \rightarrow \mathcal{F}_{ab} = \begin{pmatrix} F_{a[b] - \Phi_\dagger_a \Phi_b] \partial_{[a} \Phi_{b]} + \mathcal{A}_{[a} \Phi_{b]} \\ -\partial_{[a} \Phi_\dagger_{b]} - \Phi_\dagger_{[a} \mathcal{A}_{b]} \end{pmatrix} \]  

(2.2)

where \(F_{ab}\) refers to the field strength of \(A_a\). Imposing (2.1) (2.2) into D8-brane action (C-1) and keep the quadric terms of \(\Phi_a\), it leads to a Yang-Mills (YM) action\(^4\)

\[^4\text{We do not given the explicit formula of the CS term with HL fields since it is independent on the metric thus it is irrelevant to the glueball-baryon interaction.}\]
Figure 1: The various D-brane configurations in the WSS model. **LEFT**: The configuration of the standard WSS model according to Table 1. The bulk geometry is produced by $N_c$ coincident D4-branes which represent “colours” in QCD. The flavours are introduced into the model by embedding $N_f$ pairs of coincident D8/D8-branes at the antipodal position of the bulk geometry. $U$ refers to the holographic direction and $X^4$ is compactified on $S^1$. The D4'-brane as the baryon vertex looks like a point in the $U - X^4$ plane. Mesons are created by the open string on the flavoured D8/D8-branes while baryons are created by the wrapped D4'-branes. **RIGHT**: The WSS model with heavy flavour. An additional pair of flavoured D8/D8-brane (denoted by the red line) as the heavy-flavoured (H) brane separated from the other $N_f$ pairs of light-flavoured (L) D8/D8-branes with a HL string (denoted by the green line) is embedded. The baryon vertex contains heavy flavour in this configuration through the HL string.
\[ S_{DBI}^{YM} = -\frac{1}{4} (2\pi\alpha')^2 T_8 \text{Tr} \int_{D8/D8} d^9 x e^{-\Phi} \sqrt{-g} g^{ab} g^{cd} \mathcal{F}_{ac} \mathcal{F}_{bd} \]

\[
= -\frac{1}{4} (2\pi\alpha')^2 T_8 \int_{D8/D8} d^9 x e^{-\Phi} \sqrt{-g} \left[ g^{ab} g^{cd} \text{Tr} (\mathcal{F}_{ac} \mathcal{F}_{bd} - \alpha_{ac} \mathcal{F}_{bd} - \mathcal{F}_{ac} \alpha_{bd}) - 2 g^{ab} g^{cd} f_{ac} f_{bd} \right],
\]

(2.3)

where

\[ f_{ab} = \partial_a \Phi_b + A_{[a} \Phi_{b]}, \quad f_{ab}^\dagger = -\partial_{[a} \Phi^{\dagger}_{b]} - \Phi^{\dagger}_{[a} A_{b]}, \quad \alpha_{ab} = \Phi_{[a} \Phi^{\dagger}_{b]}, \]

(2.4)

We should notice that from the full formula of the DBI action, it would contain an additional term of the transverse mode \( \Psi \) of D8/D8-branes as shown in Appendix B. And this term could be written as,

\[ S_{\Psi}^{D8} = -\tilde{T}_8 \text{Tr} \int_{D8/D8} d^9 x e^{-\Phi} \sqrt{-\det g} \left\{ \frac{1}{2} D_a \Psi D^a \Psi + \frac{1}{4} [\Psi, \Psi]^2 \right\}, \]

(2.5)

with \( D_a \Psi = \partial_a \Psi + [A_a, \Psi] \) and \( \tilde{T}_8 = (2\pi\alpha')^2 T_8 \). In the case of \( N_f \) pairs of light-flavoured D8/D8 branes separated from one pair of heavy-flavoured D8/D8 brane, we can define the moduli solution of \( \Psi \) with a finite VEV \( v \) by the extrema of the potential contribution or \( [\Psi, [\Psi, \Psi]] = 0 \) \[27, 28\] as,

\[ \Psi = \left( \begin{array}{cc} -\frac{v}{N_f} & 1_{N_f} \\ 0 & v \end{array} \right). \]

(2.6)

So the action (2.5) could be rewritten by plugging the solution (2.6) into (2.5) as,

\[ S_{\Psi} = -\tilde{T}_8 v^2 \left( \frac{N_f + 1}{N_f} \right)^2 \text{Tr} \int d^4 x \int_{-\infty}^{+\infty} dZ e^{-\Phi} \sqrt{-\det g} g^{ab} \Phi_a \Phi_b, \]

(2.7)

which is exactly the mass term of the HL field. Then perform the rescaling (C-3), we could obtain the classical equations of motions for \( \Phi_a \) from (2.3) (2.7) as,

\[ D_M D_M \Phi_N - D_N D_M \Phi_M + 2 \mathcal{F}_{NM} \Phi_M + \mathcal{O} (\lambda^{-1}) = 0. \]

\[ D_M (D_0 \Phi_M - D_M \Phi_0) - \mathcal{F}^{0M} \Phi_M - \frac{1}{64\pi^2 a} \epsilon_{MNPQ} K_{MNPQ} + \mathcal{O} (\lambda^{-1}) = 0, \]

(2.8)

where \( x^M = \{ x^i, Z \}, i = 1, 2, 3 \) and the 4-form \( K_{MNPQ} \) is given as,

\[ K_{MNPQ} = \partial_M A_N \partial_P \Phi_Q + A_M A_N \partial_P \Phi_Q + \partial_M A_N A_P \Phi_Q + \frac{5}{6} A_M A_N A_P \Phi_Q. \]

(2.9)
Since the holographic approach is valid in the strongly coupling limit $\lambda \to \infty$, the contributions from $O\left(\lambda^{-1}\right)$ have been dropped off. Note that the light flavoured gauge field $A_a$ satisfies the equations of motion obtained by varying the action \((C-1)\), so their solution remains to be \((C-2)\) in the large $\lambda$ limit. And we could further define $\Phi_a = \phi_a e^{\pm im_H x^0}$ in the heavy quark limit i.e. $m_H \to \infty$ as in [25, 26] so that $D_0 \Phi_M = (D_0 \pm im_H) \phi_M$ where “$\pm$” corresponds to quark and anti-quark respectively. By keeping these in mind, altogether we find the full solution for \((2.8)\) as,

$$
\phi_0 = -\frac{1}{1024a\pi^2} \left[ \frac{25\rho}{2(x^2 + \rho^2)^{5/2}} + \frac{7}{\rho(x^2 + \rho^2)^{3/2}} \right] \chi,
$$

$$
\phi_M = \frac{\rho}{(x^2 + \rho^2)^{3/2}} \sigma_M \chi,
$$

\((2.10)\)

where $\chi$ is a spinor independent on $x^M$. Then in the double limit i.e. $\lambda \to \infty$ followed by $m_H \to \infty$, the Hamiltonian for the collective modes involving the heavy flavour could be calculated as in \((C-7)\) by following the procedures in Appendix C.

### 3 Glueball-baryon interaction with heavy flavour

The dynamic of free glueball is reviewed in Appendix A, so in this section we will focus on the interaction of glueball and baryon with heavy flavour characterized by the collective Hamiltonian. As the glueball field is included by the metric fluctuations, the Chern-Simons (CS) term is independent on the metric thus it does not involve the glueball-baryon interaction. Hence let us start with the five dimensional (5d) YM action plus the mass term which are collected in \((2.3) - (2.7)\). The onshell form of \((2.3) - (2.7)\) corresponds to the effective interaction Hamiltonian of glueball and heavy-flavoured baryon through the relation $S_{\text{onshel}} = -\int dt H_{G-B}$, accordingly we first need to solve the eigenvalue equations for function $H_{E,D,T}$ in large $\lambda$ limit.

The eigenvalue equations for $H_{E,D,T}$ are given in \((A-9)\) and \((A-14)\). In the rescaled coordinate $Z \to \lambda^{-1/2}Z$, the equations are written as,

$$
H''_E(Z) + \left( \frac{1}{Z} + \frac{2Z}{\lambda} \right) H'_E(Z) + \left( \frac{16}{3\lambda} + \frac{M_E^2}{M_{KK}^2} \frac{1}{\lambda} \right) H_D (Z) + O(\lambda^{-2}) = 0,
$$

$$
H''_{D,T}(Z) + \left( \frac{1}{Z} + \frac{2Z}{\lambda} \right) H'_{D,T}(Z) + \frac{M_D^2}{M_{KK}^2} \frac{1}{\lambda} H_{D,T} (Z) + O(\lambda^{-2}) = 0,
$$

\((3.1)\)

and they could be easily solved as,
\[ H_E(z) = C_E \left( 1 - \frac{3M_E^2 + 16M_{KK}^2}{12M_{KK}^2 \lambda^2} \right) \lambda^{-1/2} N_c^{-1} M_{KK}^{-1} + \mathcal{O} \left( \lambda^{-3/2} \right), \]

\[ H_{D,T} (z) = C_{D,T} \left( 1 - \frac{M_{D,T}^2}{4M_{KK}^2 \lambda^2} \right) \lambda^{-1/2} N_c^{-1} M_{KK}^{-1} + \mathcal{O} \left( \lambda^{-3/2} \right). \] (3.2)

Next we perform the rescaling as in (C-3), then insert the BPST solution (C-2) for the gauge field \( A \) and (A-9) for the heavy-light meson field \( \Phi \) into the action (2.3), (2.7). Afterwards by plugging the metric (A-6) plus the dilaton (A-7) with the solution (3.2) and various fluctuations which include the exotic scalar, dilatonic scalar and tensor glueball field all given in the Appendix A into the action (2.3), (2.7), the onshell form of action (2.3), (2.7) could be obtained with the dimensionless variable \( x^\mu \to x^\mu / M_{KK}, A_\mu \to A_\mu M_{KK} \) as,

\[
S_{G,E,D,T-B}^{\text{onshell}} = a C_{E,D,T} \int d^4 x dZ \text{Tr} \left[ \lambda^{1/2} L_{1/2}^{E,D,T} + L_0^{E,D,T} + \lambda^{-1/2} L_{-1/2}^{E,D,T} + \lambda^{-1/2} L^E_{\Psi} \right. \\
\left. + \mathcal{O} \left( \lambda^{-1} m_H^0 \right) \right]. \] (3.3)

where \( a = \frac{1}{216 \pi^2} \), “E,D,T” refers respectively to “exotic scalar, dilatonic scalar and tensor glueball”. Although the above calculation is very straightforward, the explicit forms of \( L_1^{E,D,T} \) and \( L_2^{E,D,T} \) are quite lengthy. So we summarize the full formulas of \( L_1^{E,D,T} \) and \( L_2^{E,D,T} \) with some essential instructions in Appendix D and here skip to the final results. Using the relation \( S_{G-B}^{\text{onshell}} = - \int dt H_{G-B} (t, \chi^s) \), the dimensionless interaction Hamiltonians are computed as:

\[
H_{G_E-B} (t, \chi^s) = - C_E \lambda^{-1/2} M_{KK}^{-1} \left( 5m_H^2 \pi^2 a + \frac{15m_H}{32 \rho^2} \right) G_E \chi^\dagger \chi + \mathcal{O} \left( \lambda^{-1} m_H^0 \right) \\
H_{G_D-B} (t, \chi^s) = C_D \lambda^{-1/2} M_{KK}^{-1} \left( \frac{3m_H}{8 \rho^2} - 6m_H^2 \pi^2 a \right) G_D \chi^\dagger \chi + \mathcal{O} \left( \lambda^{-1} m_H^0 \right) \\
H_{G_T-B} (t, \chi^s) = - C_T \lambda^{-1/2} M_{KK}^{-1} \left( \frac{3}{2} m_H^2 \pi^2 a + \frac{m_H}{4 \rho^2} \right) G_T \chi^\dagger \chi + \mathcal{O} \left( \lambda^{-1} m_H^0 \right), \] (3.4)

The constants \( C_{E,D,T} \) are determined by the eigenvalue equations for \( H_{E,D,T} \) and they depends on the mass of the various glueballs. We numerically evaluate \( C_{E,D,T} \) in Table 2 with the corresponding glueball mass. Notice that the operator \( G_{E,D,T} \) satisfies the equations of motion by varying action \( \text{[A-10]} \) \( \text{[A-15]} \), thus its classical solution is \( G_{E,D,T} = \frac{1}{2} (e^{-iM_{E,D,T} t} + c.c) \) and it is time-dependent. On the other hand, the spinor \( \chi \) has to be however quantized by its anti-commutation relation \( \{ \chi_\alpha, \chi_\beta^\dagger \} = \delta_{\alpha \beta} \) in the full quantum field theory so \( \chi^\dagger \chi \) is the number operator of heavy quarks. Therefore in our theory the glueball field could be treated as the

\footnote{The glueball field \( G_{E,D,T} \) in (3.4) is dimensional which is in the unit of \( M_{KK} \) while the other parameters are dimensionless.}
Table 2: The glueball mass spectrum $M_{E,T}^{(n)}$ in the WSS model in the units of $M_{KK}$ is collected from [14] and the numerical values of the associated coefficients presented in (3.4) $C_{E,D,T}$ are evaluated.

The classical field while baryon is quantized in the moduli space and we can identify $\chi^\dagger \chi = N_Q$ as the number of heavy quarks in a baryonic bound state. Moreover the Hamiltonians in (3.4) is definitely suitable to be perturbations to the quantum mechanics (C-7) since they are all proportional to $\lambda^{-1/2}$ in the large $\lambda$ limit. Then the interaction of glueball and heavy-flavoured baryon could be accordingly described by using the method of time-dependent perturbation in the quantum mechanical system. Last but not least, the decay rates/width $\Gamma$ can be evaluated by using the standard technique for the time-dependent perturbation in quantum mechanics, which is given as,

$$
\Gamma_{B \rightarrow G+\chi \over m_H} = \frac{1}{m_H} \left| \int dt \langle i | H_{G-B} (t, x^s) | j \rangle e^{-i(E_i-E_j)t} \right|^2, \\
= \frac{1}{m_H} \langle i | H_{G-B} (x^s) | j \rangle^2 \delta (E_j - E_i - M_{E,D,T}),
$$

(3.5)

where $|i\rangle, |j\rangle, E_{i,j}$ refers to the eigenstate and the associated eigenvalue of (C-7). And the above decay occurs only if several physical quantities e.g. energy, total angular momentum $J$, are also conserved. Note that the interaction Hamiltonians in (3.4) are independent on $Z$, so $\langle i | H_{G-B} (x^s) | j \rangle$ would be vanished unless the states $|i\rangle, |j\rangle$ take the same quantum number of $n_Z$ and $l$. The Hamiltonians in (3.4) can also describe the decay of an anti-baryon if we replace $m_H$ by $-m_H$.

With the perturbed Hamiltonian in (3.4), this model includes various decays of heavy-flavoured hadrons involving the glueball. So we are going to examine the possible transitions involving one glueball with the leading low-energy excited baryon states $n_\rho \leq 5$. Since our concern is the situation of two-flavoured meson, we could follow [22] by setting $l = J = 0, N_Q = 1, N_c = 3$ in order to fit the experimental data of the (pseudo) scalar meson states with one heavy flavour. Then let us first take account into the energy conservation $E (n_\rho = n'_\rho + \Delta n_\rho, l = 0, N_B = 1, n_Z) - E (n'_\rho, l = 0, N_B = 1, n_Z) \equiv \Delta E (\Delta n_\rho) = M_{E,D,T}^{(n)}$ if the
transition of hadron decay would happen, where $M_{E,D,T}^{(n)}$ refers to the glueball mass given in Table 2 and $E(n_\rho,l,N_E,n_Z)$ refers to the baryonic spectrum in (C-9). By keeping these in mind, the following relations are picked out,

$$\mathcal{E}(\Delta n_\rho = 3) / M_D^{(n=1)} \simeq 0.986, \quad \mathcal{E}(\Delta n_\rho = 4) / M_E^{(n=2)} \simeq 1.008, \quad (3.6)$$

while $\mathcal{E}(\Delta n_\rho), n_\rho \leq 5$ with $\Delta n_\rho = 0, 1, 2$ does not match to any $M_{E,D,T}^{(n)}$. Hence we could find the following possible decays involving glueball according to (3.6),

I: Baryonic $|J = 0, n_\rho = 3\rangle \to G_D^{(n=1)}, J^{PC} = 0^{++} + \text{Baryonic} |J = 0, n_\rho = 0\rangle$

II: Baryonic $|J = 0, n_\rho = 4\rangle \to G_D^{(n=1)}, J^{PC} = 0^{++} + \text{Baryonic} |J = 0, n_\rho = 1\rangle$

III: Baryonic $|J = 0, n_\rho = 5\rangle \to G_D^{(n=1)}, J^{PC} = 0^{++} + \text{Baryonic} |J = 0, n_\rho = 2\rangle$

IV: Baryonic $|J = 0, n_\rho = 3\rangle \to G_D^{(n=1)}, J^{PC} = 2^{++} + \text{Baryonic} |J = 0, n_\rho = 0\rangle$

V: Baryonic $|J = 0, n_\rho = 4\rangle \to G_D^{(n=1)}, J^{PC} = 2^{++} + \text{Baryonic} |J = 0, n_\rho = 1\rangle$

VI: Baryonic $|J = 0, n_\rho = 5\rangle \to G_D^{(n=1)}, J^{PC} = 2^{++} + \text{Baryonic} |J = 0, n_\rho = 2\rangle$

VII: Baryonic $|J = 0, n_\rho = 4\rangle \to G_E^{(n=2)}, J^{PC} = 0^{++} + \text{Baryonic} |J = 0, n_\rho = 0\rangle$

VIII: Baryonic $|J = 0, n_\rho = 5\rangle \to G_E^{(n=2)}, J^{PC} = 0^{++} + \text{Baryonic} |J = 0, n_\rho = 1\rangle, \quad (3.7)$

where we have denoted the states by their quantum numbers and the associated decay rates $\Gamma$ are numerically evaluated in Table 3 by using the effective Hamiltonian in (3.4). Notice that the mass of the dilatonic and exotic scalar glueball in (3.7) are given as $M_{E}^{(n=2)} / M_{D}^{(n=1)} \simeq 1.30$ which is close to the mass ratio of the glueball candidates $f_0(1710)$ and $f_0(1500)$ as $M_{f_0(1710)} / M_{f_0(1500)} \simeq 1.14$, moreover all of them should be the state of $J^{PC} = 0^{++}$. Accordingly we could identify the dilatonic and exotic scalar glueball in (3.7) to $f_0(1500)$ and $f_0(1710)$ respectively which are the two glueball candidates discussed frequently in many lectures.

|     | I       | II      | III      | IV      |
|-----|---------|---------|----------|---------|
| $\Gamma$ ($10^{-1}$) | 0.0392 | 0.0628 | 0.0785  | 0.1046  |
|     | V       | VI      | VII      | VIII    |
| $\Gamma$ ($10^{-1}$) | 0.1674 | 0.2093 | 0.6316  | 1.0527  |

Table 3: The corresponding decay rates in the units of $m_H$ to the transitions in (3.7) by setting $l = 0, N_Q = 1, N_c = 3, N_f = 2$.

If we furthermore consider the parity of baryonic states as discussed in [22], the above states with odd $n_Z$ in this model would correspond to the meson states with odd parity since the parity transformation is $Z \to -Z$. In this sense, the transition II, V, VII describes the decay of
the heavy-flavoured scalar (non-glueball) meson involving glueball while the pure scalar meson
with even parity is less evident according to the current experimental data. On the other hand,
as the glueball states we discussed in this manuscript all have even parity, it implies that the
parity of the transition I, III, IV, VI may be violated. We also notice that if \( \frac{l}{2} = J \) is identified
as the quantum number of the spin, the decay processes IV, V, VI in (3.7) involving tensor

\[ J^{PC} = 2^{++} \]

may be probably forbidden since the initial and final baryonic states are all pure scalars i.e. the total angular momentum may not be conserved in these transitions\(^6\) and this result would be in agreement with the previous discussion in [19]. Therefore we could
conclude that only the decay process VIII in (3.7) might be realistic. This transition describes
the decay of the baryonic meson consisted of one heavy- and one light- flavoured quark. So
while the identification of the other transitions might be less clear, the transition VIII could
be interpreted as the decay of the baryonic B-meson involving the glueball candidate \( f_0 (1710) \)
as discussed e.g. in [8, 9, 10] since the corresponding quantum numbers of the states could be
identified.

4 Summary

In this paper, with the top-down approach of WSS model, we propose a holographic description
of the decay of heavy-flavoured meson involving glueball. The HL field is introduced into the
WSS model to describe the dynamics of heavy flavour and it is created by the HL string with
a pair of heavy-flavoured D8/\( \overline{D8} \)-brane separated from the other light flavoured D8/\( \overline{D8} \)-brane.
Since baryon in this model could be equivalently represented by the instanton configurations
on the light-flavoured brane and the glueball field is identified as the bulk gravitational waves,
we solve the classical equations of motion for the HL field with instanton solution for the gauge
fields. In the limitation of large \( \lambda \) followed by large \( m_H \), we derive the mass formula of the soliton
as the onshell action of the flavour brane by taking account of the HL field and bulk gravitational
waves. Then following the collectivization and quantization of the soliton in [22, 25, 26], the
effective Hamiltonian for the collective modes of heavy-flavoured baryons is obtained which
includes the interaction with glueball. Afterwards, we examine the possible decay processes and
compute the associated decay rates with the effective Hamiltonian. We find these decay rates
are in agreement with the previous works by using this model as in [14, 15, 16, 19] since they
are proportional to \( \lambda^{-1} \). Then by comparing the quantum numbers of the baryonic states with
some experimental data and employing the identification of baryonic states in [22], we find that
one decay process might be realistic and could be interpreted as the decay of baryonic B-meson
involving the glueball candidate \( f_0 (1710) \) as discussed in [8, 9, 10]. Noteworthily according to

\(^6\)For a tensor glueball, we suggest to consider a tensor field dependent on the coordinates of the moduli space
\( y_i \) in order to obtain the correct decay process. We would like to leave it as a future study and focus on the scalar

\[ \text{glueball in the current work.} \]
lattice QCD $f_0 (1710)$ is an excited state in the glueball candidates which is just consistent with that the glueball state discussed in transition VIII is also an excitation.

As an improvement of [19], this work provides an alternative way to investigate the interaction of glueball and heavy-flavoured baryons in strongly coupling system through the holographic approach of the underlying string theory. Although this approach is quite principal and contains few parameters, it is actually valid in the large $N_c$ limit. So phenomenological theories or models are always needed as a comparison with holography.

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**Appendix**

**A. The bulk supergravity and glueball dynamics in the WSS model**

The WSS model is based on the AdS$_7$/CFT$_6$ correspondence of $N_c$ M5-branes in string theory which can be reduced to $N_c$ D4-branes compactified on $S^1$ in 10d bulk. So taking the large $N_c$ limit, the bulk dynamic is described by the 10d type IIA supergravity action which is given as,

$$S_{\text{IIA}} = \frac{1}{2 k_{10}^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left( \mathcal{R} + 4 \nabla_M \Phi \nabla^M \Phi - \frac{1}{2} |F_4|^2 \right),$$

where $\Phi$ denotes the dilaton field, $2k_{10}^2 = 16G_{10}/g_s^2 = (2\pi)^7 l_s^8$. $\mathcal{R}, G_{10}$ is 10d scalar curvature and Newton constant respectively. $F_4 = dC_3$ is the field strength of the Romand-Romand (R-R) 3-form $C_3$. The geometrical solution for the bulk metric is given as,

$$ds^2 = \left( \frac{U}{R} \right)^{3/2} \left[ \eta_{\mu\nu} dX^\mu dX^\nu + f(U) (dX^4)^2 \right] + \left( \frac{R}{U} \right)^{3/2} \left[ \frac{dU^2}{f(U)} + U^2 d\Omega_4^2 \right],$$

$$f(U) = 1 - \frac{U_{KK}^3}{U^3}, \quad e^{-\Phi} = \left( \frac{U}{R} \right)^{3/4}, \quad F_4 = \frac{2\pi N_c}{V_4} \epsilon_4, \quad R^3 = \pi g_s N_c l_s^3,$$

with a periodic condition for $X^4$,

$$X^4 \sim X^4 + 2\pi \delta X^4, \quad \delta X^4 = \frac{1}{M_{KK}}. \quad (A-3)$$

And the $r, z, Z$ coordinate used in the paper is defined as,

$$U^3 = U_{KK}^3 + U_{KK} z^2, \quad Z = \frac{z}{U_{KK}}, \quad 1 + Z^2 = \frac{r_6^6}{r_{KK}^6}, \quad U_{KK} = \frac{r_{KK}^2}{4 R}. \quad (A-4)$$
Note that $\epsilon_4$ represents a unit volume element on $S^4$, $g_s, l_s$ denotes the string coupling constant and the length of string. The indices $\mu, \nu$ in (A-2) run from 0 to 3. Additionally we could define the QCD variables in terms of,

$$\lambda = g_{YM}^2 N_c, \quad g_{YM}^2 = 2\pi g_s l_s M_{KK},$$  \[(A-5)\]

where $g_{YM}, \lambda$ respectively denotes the Yang-Mills and the 't Hooft coupling constant.

In this model the glueball fields are identified as the gravitational fluctuations to the bulk solution (A-2), thus we could rewrite the metric as $G_{MN} \rightarrow G^{(0)}_{MN} + \delta G_{MN}$ in order to involve the glueball field. The 10d metric reduced from 11d supergravity with gravitational fluctuations is,

$$g_{\mu\nu} = \frac{r^3}{L^3} \left[ 1 + \frac{L^2}{2r^2} \delta G_{11,11} \right] \eta_{\mu\nu} + \frac{L^2}{r^2} \delta G_{\mu\nu},$$
$$g_{44} = \frac{r^3 f}{L^3} \left[ 1 + \frac{L^2}{2r^2} \delta G_{11,11} + \frac{L^2}{r^2 f} \delta G_{44} \right],$$
$$g_{rr} = \frac{L}{r f} \left[ 1 + \frac{L^2}{2r^2} \delta G_{11,11} + \frac{r^2 f}{L^2} \delta G_{rr} \right],$$
$$g_{r\mu} = \frac{r}{L} \delta G_{r\mu}, \quad g_{\Omega\Omega} = \frac{r}{L} \left( \frac{L}{r} \right)^2 \left[ 1 + \frac{L^2}{2r^2} \delta G_{11,11} \right],$$ \[(A-6)\]

with the dilaton,

$$e^{4\Phi} = \frac{r^2}{L^2} \left( 1 + \frac{L^2}{r^2} \delta G_{11,11} \right).$$  \[(A-7)\]

Since different formulas of $\delta G_{MN}$ corresponds to various glueball, in this paper we consider the following forms of $\delta G_{MN}$:

**The exotic scalar glueball**

The exotic scalar glueball corresponds to the exotic polarizations of the bulk graviton whose quantum number is $J^{CP} = 0^{++}$. The 11d components of $\delta G_{MN}$ are given as ,
\[ \delta G_{44} = -\frac{r^2}{L^2} f(r) H_E(r) G_E(x), \]
\[ \delta G_{\mu\nu} = \frac{r^2}{L^2} H_E(r) \left[ \frac{1}{4} \eta_{\mu\nu} - \left( \frac{1}{4} + \frac{3r_{KK}^6}{5r^6 - 2r_{KK}^6} \right) \frac{\partial_\mu \partial_\nu}{M_E^2} \right] G_E(x), \]
\[ \delta G_{11,11} = \frac{r^2}{4L^2} H_E(r) G_E(x), \]
\[ \delta G_{rr} = -\frac{L^2}{r^2} \frac{1}{f(r)} \frac{3r_{KK}^6}{5r^6 - 2r_{KK}^6} H_E(r) G_E(x), \]
\[ \delta G_{r\mu} = \frac{90 r r_{KK}^6}{M_E^2 L^2 (5r^6 - 2r_{KK}^6)^2} H_E(r) \partial_\mu G_E(x), \]  
(A-8)

with the eigenvalue equation for function \( H_E(r) \) as,
\[ \frac{1}{r^3} \frac{d}{dr} \left[ r \left( r^6 - r_{KK}^6 \right) \frac{d}{dr} H_E(r) \right] + \left[ \frac{432 r^2 r_{KK}^{12}}{(5r^6 - 2r_{KK}^6)^2} + L^4 M_E^2 \right] H_E(r) = 0. \]  
(A-9)

In 10d bulk the above components in (A-8) satisfy the asymptotics \( \delta G_{44} = -4\delta G_{11} = -4\delta G_{22} = -4\delta G_{33} = -4\delta G_{11,11} \) for \( r \to \infty \). Plugging the solution (A-2) and the fluctuations (A-8) with the eigenvalue equation (A-9) into the action (A-1), it leads to the kinetic term of the exotic scalar glueball,
\[ S_{G_E(x)} = -\frac{1}{2} \int d^4 x \left[ (\partial_\mu G_E)^2 + M_E^2 G_E^2 \right], \]  
(A-10)

where the pre-factor in (A-10) has been normalized to \(-1/2\) by choosing the boundary value of \( H_E(r) \).

**The dilatonic and tensor glueball**

The fluctuations of the metric,
\[ \delta G_{11,11} = -3 \frac{r^2}{L^2} H_D(r) G_D(x), \]
\[ \delta G_{\mu\nu} = \frac{r^2}{L^2} H_D(r) \left[ \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{M_D^2} \right] G_D(x), \]  
(A-12)

corresponds to another mode of the scalar glueball \( 0^{++} \). We employ “dilatonic” for the upon mode since \( \delta G_{11,11} \) reduces to the 10d dilaton.

Besides the tensor glueball corresponds to the metric fluctuations with a transverse traceless polarization whose quantum number is \( J^{CP} = 2^{++} \). We can choose the following components of the graviton polarizations as tensor glueball field,
\[ \delta G_{\mu\nu} = -\frac{r^2}{L^2} H_T(r) T_{\mu\nu}(x), \]  
(A-13)

where \( T_{\mu\nu} \equiv T_{\mu\nu} G_T(x) \). \( T_{\mu\nu} \) is a constant symmetric tensor satisfying the normalization and traceless condition \( T_{\mu\nu} T^{\mu\nu} = 1, \eta^{\mu\nu} T_{\mu\nu} = 0 \). The functions \( H_{D,T}(r) \) satisfies the eigenvalue equation,

\[
\frac{1}{r^3} \frac{d}{dr} \left[ r \left( r^6 - r^6_{KK} \right) \right] + L^4 M_{D,T}^2 H_{D,T}(r) = 0. 
\]  
(A-14)

We can also obtain the kinetic action of the dilatonic scalar and tensor glueball as,

\[
S_{G_D}(x) = -\frac{1}{2} \int d^d x \left[ (\partial_a G_D)^2 + M_{G_D}^2 G_D^2 \right],
\]

\[
S_{T}(x) = -\frac{1}{4} \int d^d x \left[ T_{\mu\nu} \left( \partial^2 - M_{T}^2 \right) T^{\mu\nu} \right],
\]  
(A-15)

once the solution \( (A-2) \) and fluctuations \( (A-12) (A-13) \) with eigenvalue equation \( (A-14) \) are imposed to the action \( (A-1) \) and the boundary value of \( H_{D,T} \) has to been determined by the normalization conditions in \( (A-15) \).

B. The full Dp-brane action and the embedding of the probe branes

The complete DBI action

We give the complete formula of the Dp-brane here and it could also be reviewed in many textbooks of string theory, Let us consider \( D \) dimensional spacetime parametrized by \( \{X^\mu\}, \mu = 0, 1...D - 1 \) with a stack of Dp-branes. In this subsection, the indices \( a, b = 0, 1...p \) and \( i, j, k = p + 1...D - 1 \) denote respectively the directions parallel and vertical to the Dp-branes. The complete bosonic action of a Dp-branes is,

\[
S_{D_p-\text{branes}} = S_{\text{DBI}} + S_{\text{CS}}, \]  
(B-1)

where \[ 27 \]

\[
S_{\text{DBI}} = -T_p \text{STr} \int d^{p+1} \xi e^{-\Phi} \sqrt{-\det \left( \left[ \left( Q^{-1} - \delta^i{}_{ij} \right) E_{ji} + 2\pi\alpha' F_{ij} \right] \right)},
\]

\[
S_{\text{CS}} = \mu_p \sum_{n=0,1} \int_{D_p-\text{branes}} C_{p-2n+1} \wedge \frac{(B + 2\pi\alpha' F)^n}{n!},
\]

\[
Q^i{}_{j} = \delta^i{}_{j} + 2\pi\alpha' \left[ \varphi^i, \varphi^k \right] E_{kj}, \ E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}. \]  
(B-2)

We have denoted the metric of the \( D \) dimensional spacetime and the 2-form field as \( g_{\mu\nu}, B_{\mu\nu} \) respectively. \( F \) is the gauge field strength defined on the D-brane and “STr” refers to the
“symmetric trace”. We use $\varphi^i$ ’s to represent the transverse modes of the D$p$-branes which are given by the T-duality relation $2\pi\alpha'^{\prime}\varphi^i = X^i$. So the DBI action in (B-2) could be expanded as,

$$S_{\text{BDI}} = -T_p\text{Tr} \int d^{p+1}\xi e^{-\Phi} \sqrt{-g} \left[ 1 + \frac{1}{4} \left( 2\pi\alpha' \right)^2 F_{ab} F^{ab} + \frac{1}{2} D_a \varphi^i D^a \varphi^i + \frac{1}{4} [\varphi^i, \varphi^j]^2 \right] + \text{high orders}. $$

(B-3)

The 2-form field $B$ has been gauged away. The gauge field $A_a$ and scalar field $\varphi^i$ ’s are all in the adjoint representation of $U(N)$. Note that there is only one transverse coordinate for the D8/$\overline{\text{D8}}$-brane which has been defined as $\Psi \equiv \varphi^9$ in the main text.

Comments about the the probe branes and strings

Here let us briefly outline the embedding of the probe D8/$\overline{\text{D8}}$-brane and the HL string. Using the bulk metric (A-2), the induced metric on the probe D8/$\overline{\text{D8}}$-branes is obtained as,

$$ds_{\text{D8}/\overline{\text{D8}}}^2 = \left( \frac{U}{R} \right)^{3/2} \left[ f(U) + \left( \frac{R}{U} \right)^3 \frac{U'^2}{f(U)} \right] (dX^4)^2 + \left( \frac{U}{R} \right)^{3/2} \eta_{\mu\nu} dX^\mu dX^\nu + \left( \frac{R}{U} \right)^{3/2} U^2 d\Omega_4^2, $$

(B-4)

where $U' = \frac{dU}{dX^4}$. Then insert the metric (B-4) into the DBI action of D8/$\overline{\text{D8}}$-branes, it yields the formula,

$$S_{\text{D8}/\overline{\text{D8}}} \propto \int d^4x dU U^4 \left[ f(U) + \left( \frac{R}{U} \right)^3 \frac{U'^2}{f(U)} \right]^{1/2} . $$

(B-5)

Hence we can obtain the equation of motion for the function $U(X^4)$ as,

$$\frac{d}{dX^4} \left( \frac{U^4 f(U)}{[f(U) + \left( \frac{R}{U} \right)^3 \frac{U'^2}{f(U)}]^{1/2}} \right) = 0. $$

(B-6)

Using the boundary condition in [13], as $U(X^4 = 0) = U_0$ and $U'(X^4 = 0) = 0$, the generic solution for (B-6) is computed as,

$$X^4(U) = E(U_0) \int_{U_0}^U dU \frac{(U)}{f(U) [U^8 f(U) - E^2(U_0)]^{1/2}}, $$

(B-7)

where $E(U_0) = U_0^4 f^{1/2}(U_0)$ and we have used $U_0$ to denotes the connected position of the D8/$\overline{\text{D8}}$-branes. Afterwards let us further introduce the coordinates $(r, \Theta)$ and $(y, z)$ which satisfy,
\begin{align}
  y &= r \cos \Theta, \quad z = r \sin \Theta, \\
  U^3 &= U_{KK}^3 + U_{KW} r^2, \quad \Theta = \frac{2\pi}{\beta} X^4 = \frac{3 U_{KK}^{1/2}}{2 R^{3/2}}.
\end{align} \tag{B-8}

In the standard WSS model, the probe D8/\overline{D8}-branes are embedded at \( \Theta = \pm \frac{1}{2} \pi \) respectively i.e. the position of \( y = 0 \), which exactly corresponds to the antipodal D8/\overline{D8}-branes (blue) in Figure 1. In this case, the solution for the embedding function is \( X^4(U) = \frac{1}{4} \beta \) and \( U_0 = U_{KK} \). In addition, the \( (B-7) \) also allows the non-antipodal solution if we choose \( \Theta = \pm \Theta_H \neq \pm \frac{1}{2} \pi, U_0 = U_H \neq U_{KK} \) which corresponds to the non-antipodal D8/\overline{D8}-branes (red) in Figure 1. On the other hand, while each endpoints of the HL string could move along the flavoured branes, in our setup it is stretched between the heavy- (non-antipodal) and light-flavoured (antipodal) D8/\overline{D8}-branes. So it connects the positions respectively on the heavy- and light-flavoured D8/\overline{D8}-branes which are most close to each other and in the \( U - X^4 \) plane, they are the positions of \( (U_{KK}, 0) \) on the light-flavoured branes and \( (U_H, 0) \) on the heavy-flavoured branes. And this is the configuration of the HL string with minimal length i.e. the VEV.

\section*{C. The collective modes of baryon and its quantization}

As the D4'-brane is identified as baryon in the WSS model, it is equivalent to the instanton configuration on the D8-branes according to the string theory. So the dynamic of the D8/\overline{D8}-brane is given by the Dirac-Born-Infield (DBI) action plus the Chern-Simons (CS) action \( (B-2) \) while the baryonic D4'-brane is identified as the instanton configuration of the gauge field strength on the D8/\overline{D8}-brane. Altogether the action of the flavours with baryons can be simplified as a 5d Yang-Mills (YM) plus CS action by integrating over the \( S^4 \) which is given as,

\begin{align}
  S &= S_{YM} + S_{CS}, \\
  S_{YM} &= -\kappa \text{Tr} \int d^4 x dz e^{-\Phi} \sqrt{-g} g^{ab} g^{cd} F_{ac} F_{bd}, \\
  S_{CS} &= \frac{N_c}{24\pi} \text{Tr} \int d^4 x dz \left( A F^2 - \frac{1}{2} A^2 F^2 - \frac{1}{10} A^5 \right), \tag{C-1}
\end{align}

where the indices \( \alpha, \beta \) run over \( X^\mu \) and \( z \). Particularly in the situation of two flavours i.e. \( N_f = 2 \), the classical instanton configuration could be adopted as the Belavin-Polyakov-Schwarz-Tyupkin (BPST) solution which is given as,

\begin{align}
  A_M &= -\frac{\bar{\sigma}_{MN} x^N}{x^2 + \rho^2}, \quad M, N = 1, 2, 3, z, \\
  A_0 &= -\frac{i}{8\pi^2 a b^{5/2} x^2} \left[ 1 - \frac{\rho^4}{(x^2 + \rho^2)^2} \right], \tag{C-2}
\end{align}
where $A$ is $U(2)$ and $A_0$ is $U(1)$ gauge field. The gauge field strength is defined as $F = dA + [A, A]^2$. And $x^2 = (x^M - X^M)^2$, $X^M$'s are constants. Since the instanton size $\rho$ is of order $\lambda^{-1/2}$, it would be convenient to employ the rescaling,

$$
(x^0, x^M) \rightarrow \left( x^0, \lambda^{-1/2} x^M \right), \quad (A_0, A_M) \rightarrow \left( A_0, \lambda^{1/2} A_M \right),
$$

in order to obtain the explicit dependence of $\lambda$ in the actions in (C-1). Inserting (C-2) into the rescaled gauge field $A$, the mass $M$ of the classical soliton could be evaluated by

$$
S_{\text{onshell}} = - \int \dot{t} M.
$$

Afterwards the baryon states could be identified as Skyrmions so that the characteristics of baryon are reflected by their collective modes. Therefore we could quantize the classical soliton in the moduli space to obtain the baryon spectrum.

In the large $\lambda$ limit, the topology of the moduli space for $N_f = 2$ case is given as $\mathbb{R}^4 \times \mathbb{R}^4 / \mathbb{Z}_2$ since the contribution of $O(\lambda^{-1})$ could be neglected. The the collective coordinates $\{X^M\}$ parameterize the first $\mathbb{R}^4$ while the size $\rho$ and the $SU(2)$ orientation of the instanton parameterize $\mathbb{R}^4 / \mathbb{Z}_2$. Let us denote the $SU(2)$ orientation as $a_I = \frac{y_I}{\rho}$, $I = 1, 2, 3, 4$ with the normalization $\sum_{I=1}^{4} a_I^2 = 1$ so that the size of the instanton satisfies $\rho = \sqrt{y_1^2 + ... y_4^2}$. The quantization procedures of the Lagrangian for the collective coordinates follows those in Ref. Specifically we need to assume that the moduli of the solution is time-dependent. Thus the gauge transformation also becomes time-dependent as,

$$
A_M \rightarrow V \left( A_M^{cl} - i \partial_M \right) V^{-1},
$$

$$
F_{MN} \rightarrow V F_{MN}^{cl} V^{-1}, \quad F_{0M} \rightarrow V \left( \dot{X}^s \partial_0 A_M^{cl} - D_M^{cl} \Phi \right) V^{-1},
$$

The Lagrangian of the collective coordinates in such a moduli space takes the form as,

$$
L = \frac{mX}{2} G_{rs} \dot{X}^s \dot{X}^r - U(X^s) + O(\lambda^{-1}),
$$

where $X^s = \{X^M, a_I\}$. The the kinetic term in (C-5) corresponds to the line element of the moduli space while the potential corresponds to the onshell action of the soliton adopting the time-dependent gauge transformation,

$$
S_{\text{onshell}}^{D8/D8} \simeq S_{\text{onshell}}^{YM + CS} = - \int \dot{t} U(X^s).
$$

Using the solution (C-2), the above integral is easy to calculate in the case of pure light flavours while it becomes quite difficult if the heavy flavour is involved. Without loss of generality, let us consider the large $\lambda$ limit followed by heavy mass limit of the heavy flavour. Hence the dimensionless quantized Hamiltonian corresponding to (C-5) for the collective modes is calculated as,

\[\text{In our notation, } A \text{ is anti-Hermitian which means } A^\dagger = - A.\]
\[ H = M_0 + H_y + H_Z + O \left( \lambda^{-1} m_0^0 \right), \]
\[ H_y = -\frac{1}{2m_y} \sum_{i=1}^{4} \frac{\partial^2}{\partial y_i^2} + \frac{1}{2} m_y \omega_y^2 \rho^2 + \frac{Q}{\rho^2}, \]
\[ H_Z = -\frac{1}{2m_Z} \frac{\partial^2}{\partial Z^2} + \frac{1}{2} m_Z \omega_Z^2 Z^2, \quad \text{(C-7)} \]

where,
\[ M_0 = 8\pi^2 \kappa, \quad \omega_Z^2 = \frac{2}{3}, \quad \omega_{\rho}^2 = \frac{1}{6}, \quad \kappa = \frac{\lambda N_c}{216\pi^2}, \]
\[ Q = Q_L + Q_H, \quad Q_L = \frac{N_c}{40\pi^2 a}, \quad Q_H = \frac{N_Q}{8\pi^2 a} \left( \frac{N_Q}{3N_c} - \frac{3}{4} \right). \quad \text{(C-8)} \]

The value of \( Q \) corresponds to the situation of a baryonic bound state consisting of \( N_Q \) heavy flavoured quarks. The eigenfunctions and mass spectrum of (C-7) can be evaluated by solving its Schrodinger equation, respectively they are obtained as\(^8\)

\[ \psi(y_I) = R(\rho)T^{(l)}(a_I), \quad R(\rho) = e^{\frac{-m_y \omega_{\rho} \rho^2}{2}} \rho^{\tilde{l}} \text{Hypergeometric}_1 F_1 \left( -n_\rho, \tilde{l} + 2; m_y \omega_{\rho} \rho^2 \right) \]
\[ E(l, n_\rho, n_z) = \omega_{\rho} \left( \tilde{l} + 2n_\rho + 2 \right) = \sqrt{\frac{(l+1)^2}{6} + \frac{640}{3} a^2 \pi^4 Q^2 + \frac{2 (n_\rho + n_z) + 2}{\sqrt{6}}}. \quad \text{(C-9)} \]

Notice that \( T^{(l)}(a_I) \) satisfies \( \nabla_{S^3}^2 T^{(l)} = -l(l+2)T^{(l)} \) which is the function of the spherical part because \( H_y \) can be written with the radial coordinate \( \rho \) as,
\[ H_y = -\frac{1}{2m_y} \left[ \frac{1}{\rho^3} \partial_\rho (\rho^3 \partial_\rho) + \frac{1}{\rho^2} \left( \nabla_{S^3}^2 - 2m_y Q \right) \right] + \frac{1}{2} m_y \omega_{\rho}^2 \rho^2. \quad \text{(C-10)} \]

**D. Explicit formulas of \( \mathcal{L}_{1/2, 0, -1/2}^{E, D, T} \) and \( \mathcal{L}_{\Psi}^{E, D, T} \)**

Here we collect the explicit formulas of \( \mathcal{L}_{1/2, 0, -1/2}^{E, D, T} \) and \( \mathcal{L}_{\Psi}^{E, D, T} \). For the exotic scalar glueball,
For the dilatonic scalar glueball,
\[ \mathcal{L}_{1/2}^D = -\frac{\partial^2 \partial^2 G_D}{M_D^2 M_{KK}^2} \mathcal{F}_{ik} \mathcal{F}_{j}^k + \frac{3 G_D}{4 M_{KK}} \mathcal{F}_{ij} \mathcal{F}_{ij}^k + \frac{\partial^2 G_D}{4 M_D^2 M_{KK}^2} \mathcal{F}_{ij} \mathcal{F}_{ij}^k \]

\[ - \frac{\partial^2 \partial^2 G_D}{M_D^2 M_{KK}^2} \mathcal{F}_{ij} \mathcal{F}_{j}^k + \frac{1}{2} G_D M_{KK}^{-1} \mathcal{F}_{i} \mathcal{F}_{Zi}^k + \frac{\partial^2 G_D}{2 M_D^2 M_{KK}^2} \mathcal{F}_{Zi} \mathcal{F}_{Zi}^k, \]

\[ \mathcal{L}_{-1/2}^D = \frac{\partial \partial \partial G_D}{4 M_{KK}^3} Z^2 \mathcal{F}_{ik} \mathcal{F}_{j}^k + \frac{\partial \partial G_D}{4 M_{KK}^3} Z^2 \mathcal{F}_{ij} \mathcal{F}_{ij}^k - \frac{\partial^2 G_D}{16 M_{KK}^3} Z^2 \mathcal{F}_{ij} \mathcal{F}_{ij}^k \]

\[ - \frac{3 G_D M_D^2}{16 M_{KK}^3} Z^2 \mathcal{F}_{ij} \mathcal{F}_{ij}^k - \frac{G_D}{12 M_{KK}^3} Z^2 \mathcal{F}_{ij} \mathcal{F}_{ij}^k + \frac{\partial \partial G_D}{8 M_{KK}^3} Z^2 \mathcal{F}_{ij} \mathcal{F}_{ij}^k \]

\[ + \frac{G_D M_D^2}{8 M_{KK}^3} Z^2 \mathcal{F}_{i} \mathcal{F}_{j}^k + \frac{1}{2} G_D M_{KK}^{-1} Z^2 \mathcal{F}_{i} \mathcal{F}_{Zi}^k + \frac{\partial^2 G_D}{2 M_D^2 M_{KK}^2} \mathcal{F}_{Zi} \mathcal{F}_{Zi}^k \]

\[ + \frac{\partial \partial G_D}{M_D^2 M_{KK}^2} \Phi_{0z} \Phi_{0z} - \frac{3 G_D}{2 M_{KK}^2} \Phi_{0z} \Phi_{0z} - \frac{\partial^2 G_D}{M_D^2 M_{KK}^2} \Phi_{0z} \Phi_{0z}, \]

\[ \mathcal{L}_\psi^D = v^2 \frac{(N_f + 1)^2}{N_f^2} \left[ - \frac{\partial \partial G_D}{M_D^2 M_{KK}^2} \Phi_{0z} \Phi_{0z} + \frac{2 G_D}{M_D^2 M_{KK}^2} \Phi_{0z} \Phi_{0z} + \frac{\partial \partial G_D}{M_D^2 M_{KK}^2} \Phi_{0z} \Phi_{0z} \right]. \]  

For the tensor glueball

\[ \mathcal{L}_{1/2}^T = -\frac{T_{ij}^k}{M_{KK}^3} \eta^{ij} F_{ik} F_{j} - T_{ij} \frac{F_{i} F_{j}^k}{M_{KK}^3}, \]

\[ \mathcal{L}_{0}^T = -2 T^{0k}_{i} \frac{\eta^{ij} F_{ik} F_{j}}{M_{KK}^3} - 2 T^{0i}_{j} \frac{F_{i} F_{j}^k}{M_{KK}^3}, \]

\[ \mathcal{L}_{-1/2}^T = \frac{T_{ij}^k}{M_{KK}^3} \eta^{ij} F_{ik} F_{j} + \frac{M_{KK}^2 T_{ij}^k}{M_{KK}^3} \frac{Z^2 \mathcal{F}_{i} \mathcal{F}_{j}^k}{M_{KK}^3} \mathcal{F}_{i} \mathcal{F}_{j}^k - M_{KK} T_{ij} \frac{Z^2 \mathcal{F}_{i} \mathcal{F}_{j}^k}{M_{KK}^3} \mathcal{F}_{i} \mathcal{F}_{j}^k \]

\[ + \frac{M_{KK}^2}{4 M_{KK}^3} T_{ij} \frac{Z^2 \mathcal{F}_{i} \mathcal{F}_{j}^k}{M_{KK}^3} \mathcal{F}_{i} \mathcal{F}_{j}^k - \frac{T_{ij}^k}{M_{KK}^3} \frac{Z^2 \mathcal{F}_{i} \mathcal{F}_{j}^k}{M_{KK}^3} \mathcal{F}_{i} \mathcal{F}_{j}^k \]

\[ - \frac{T_{ij}^k}{M_{KK}^3} \frac{Z^2 \mathcal{F}_{i} \mathcal{F}_{j}^k}{M_{KK}^3} \mathcal{F}_{i} \mathcal{F}_{j}^k, \]

\[ \mathcal{L}_\psi^T = -\frac{(N_f + 1)^2}{3 N_f^2 M_{KK}^3} v^2 \frac{T_{ij}^k}{M_{KK}^3} \Phi_{0z} \Phi_{0z}. \]  

We assume the glueball field is onshell so that \( G_{E,D,T} \) could be chosen as \( G_{E,D,T} = \frac{1}{2} (e^{-i E_{E,D,T} t} + c.c) \) in the rest frame of the glueball, hence we have \( \partial_t G_{E,D,T} = 0, \partial_\mu \partial^\mu G_{E,D,T} = M_{E,D,T}^2 G_{E,D,T} \) which could greatly simplify (D-1) (D-2) (D-3). Since the \( \mathcal{L}_{E,D,T}^E \) refers to the mass term of
the HL field, the mass of the heavy quarks $m_H$ must be related to the separation of the flavour branes i.e. the VEV of $\Psi$. In the heavy quark limit, the explicit relation is given as [24, 25, 26],

$$m_H = \frac{1}{\pi l_s^2} \lim_{z_H \to \infty} \int_0^{z_H} dz \sqrt{-g_{00}g_{zz}},$$

$$\simeq \frac{1}{\pi l_s^2} U^{1/3} z_H^{2/3} + O(z_H^0).$$

$$\equiv \frac{1}{\sqrt{6}} \frac{N_f + 1}{N_f} v,$$

(D-4)

where $z_H$ refers to the position $U = U_H$. Then we further collect the terms of $O(m_H^2)$ and $O(m_H)$ then integral out the part of $z$, it finally leads to the formulas in (3.4).

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