Residual Gauge Fixing in Light-Front QCD

Wei-Min Zhang and Avaroth Harindranath
Department of Physics, The Ohio State University
Columbus, Ohio 43210, USA

abstract

Understanding the nontrivial features of light-front QCD is a central goal in current investigations of nonperturbative light-front field theory. We find that, with the choice of light-front gauge with antisymmetric boundary conditions for the field variables, the residual gauge freedom is fixed and the light-front QCD vacuum is trivial. The nontrivial structure in light-front QCD is determined by non-vanishing asymptotic physical (transverse) gauge fields at longitudinal infinity, which are responsible for nonzero topological winding number.
Recently, the search for nonperturbative solutions of quantum chromodynamics (QCD) in light-front coordinates has become a very active subject in hadronic and nuclear physics \[1\]. In the light-front quantization of QCD, one usually chooses the light-front gauge, \(A^+ = 0\). As is well known in non-abelian gauge theory, the advantage of choosing a physical gauge, such as temporal gauge, axial gauge or light-front gauge, is that no ghost field is introduced. However, there still exists residual gauge freedom in a physical gauge, which is manifested differently in equal-time and light-front quantizations. Fixing the residual gauge freedom is the first step in nonperturbative calculations in the canonical Hamiltonian formalism.

In equal-time quantization, one usually chooses the temporal gauge \((A^0 = 0)\). The residual gauge freedom in \(A^0 = 0\) gauge is determined by Gauss’s law (the generators of residual gauge transformations annihilate physical states). However, the implementation of Gauss’s law is complicated, and in practice, it has been solved only on the lattice \[2\]. Sometimes one chooses the axial gauge \((A^3 = 0)\) in equal-time canonical quantization. In axial gauge, the time-component of gauge potentials can be computed explicitly, and thus there is apparently no residual gauge freedom. However, as noticed first by Schwinger \[3\], there indeed exist residual gauge transformations in \(A^3 = 0\), which are generated by the longitudinal color electric fields at infinity in the \(z\)-direction. Despite this insight, investigations of the residual gauge fixing in axial gauge have remained obscure \[4\]. In light-front quantization, it is convenient to choose the light-front gauge. To the best of our knowledge, the issue of residual gauge fixing in \(A^+ = 0\) gauge has not been explored in light-front coordinates.\[1\]

In the light-front canonical quantization of QCD, the residual gauge freedom in \(A^+ = 0\) is associated with the \(x^-\)-independent gauge transformations which operate on the modes with zero light-front longitudinal momentum, i.e., the \(k^+ = 0\) modes. By choosing antisymmetric boundary conditions for physical field variables at light-front longitudinal infinity, the \(k^+ = 0\) modes are removed and the residual gauge freedom is fixed. However, the antisymmetric boundary conditions imply that the physical fields do not vanish at longitudinal infinity. In this letter, we show that the asymptotic gauge fields at longitudinal infinity give rise to a non-vanishing topological winding number. Thus, the asymptotic gauge fields induced by the residual gauge fixing may be the source of nontrivial properties in light-front QCD \[1\].

1. **Light-front QCD.** We begin by recalling the essential features of the light-front formulation of QCD (for more detailed discussions see refs.\[3,7\]). In light-front coordinates \(x^+ = x^0 \pm x^3\), \(x_{\perp} = (x^1, x^2)\), with the light-front gauge \(A^+_a \equiv A^0_a + A^3_a = 0\), \(^1\)

\(^1\)One may find some discussion about residual gauge freedom in \(A^+ = 0\) in equal-time quantization \[3\]. In this case, the residual gauge fixing could, in principle, be determined by Gauss’s law, as in the case of temporal gauge.
the QCD Hamiltonian can be written as follows:

\[
H = \int dx \cdot d^2 x_{\perp} \left\{ \frac{1}{2} (E - a^2 + B^2) + \psi^\dagger \{ \alpha_{\perp} \cdot \left[ (i \partial_{\perp} + g A_{\perp}) \right] + \beta m \} \psi \right\},
\]

(1)

where \( E - a^2 = -\frac{1}{2} \partial^2 - A_{\perp} \) and \( B^2 = \partial^1 A^2_1 - \partial^2 A^1_1 + g f^{abc} A^1_a A^2_b \) are the longitudinal components of color electric and magnetic fields, and \( \psi_{\pm} = \frac{1}{2} \gamma^0 \gamma^\pm \psi = \Lambda_{\pm} \psi \) the light-front quark field variables. In fact, the light-front QCD is a two-component theory, where all the physical quantities depend only on the two-component physical (transverse) gauge fields \( A^i_a, i = 1, 2 \) and two-component quark fields \( \psi_{\pm} \). The dependent field variables \( (A^a_-, \psi_-) \) are determined by the following constraint equations,

\[
\psi_- = \frac{1}{i \partial^+} \{ \alpha_{\perp} \cdot \left[ (i \partial_{\perp} + g A_{\perp}) \right] + \beta m \} \psi_+,
\]

(2)

\[
A_a^- = -\frac{2}{\partial^+} E_a^- = \frac{1}{(\partial^+)^2} \left( \partial^+ A^a_+ + g f^{abc} (A^b_+ A^c_+ + 2 \psi^\dagger \psi_{\pm} T^a \psi_{\pm}) \right).
\]

(3)

In the above formulation, one has to define the operator \( \left( \frac{1}{\partial^+} \right) \). A typical definition in light-front field theory is

\[
\left( \frac{1}{\partial^+} \right)^n f(x^+, x^-, x_{\perp}) = \frac{1}{4^n} \int_{-\infty}^{\infty} dx^- dx^- \cdots dx^- \cdots \varepsilon(x^+ - x_{\perp}) \cdots \varepsilon(x_{\perp} - x_{\perp}) f(x^+, x^-, x_{\perp}),
\]

(4)

where \( \varepsilon(x) = 1, 0, -1 \) for \( x > 0, = 0, < 0 \). Eq.(4) requires that all the field variables satisfy antisymmetric boundary conditions. Thus, the basic commutation relations in phase space quantization \([7, 9]\) become:

\[
[A^i_a(x), \partial^+ A^j_b(y)]_{x^+ = y^+} = i \delta_{ab} \delta^i_0 \delta^3(x - y),
\]

(5)

\[
[A^i_a(x), A^j_b(y)]_{x^+ = y^+} = -i \delta_{ab} \delta^{ij} \frac{1}{4} \varepsilon(x^+ - y^+) \delta^2(x_{\perp} - y_{\perp}),
\]

(6)

\[
\{ \psi_+(x), \psi^\dagger_+(y) \}_{x^+ = y^+} = \Lambda_{\perp} \delta^3(x - y),
\]

(7)

and all other commutators between the physical degrees of freedom vanish. A consistent definition for eq.(6) implies that

\[
\lim_{x^+ \to \infty} \varepsilon(x^+ - y^+) \equiv \eta(y^-) = \begin{cases} 0 & y^- \to \pm \infty \\ 1 & \text{otherwise} \end{cases}.
\]

(8)

As a consequence of eq.(4), the choice of antisymmetric boundary conditions remove the \( k^+ = 0 \) modes \([7]\). Meanwhile, the light-front quantization of QCD ensures that the light-front longitudinal momentum of quarks and gluons must be positive semidefinite \( (k^+ \geq 0) \) \([10]\). This property implies that the light-front QCD vacuum,
which has zero total momentum, only contains particles with zero longitudinal momentum [10]. Thus, with the antisymmetric boundary condition, the light-front QCD vacuum is trivial, i.e., it is identical to the light-front bare vacuum. On the other hand, it is known that QCD in equal-time quantization has a nontrivial vacuum associated with topological gauge solutions [11].

The question is: where are the nontrivial features hidden in the above formulation of light-front QCD? Note that the light-front bare vacuum is not identical to the equal-time bare vacuum. The trivial light-front vacuum originates from the choice of antisymmetric boundary conditions at light-front longitudinal infinity, and with such boundary conditions the physical field variables do not vanish on the boundary surfaces. Since antisymmetric boundary conditions for the physical fields imply possible existence of nontrivial topological soliton solutions [12], we suggest that the nontrivial QCD structure must be carried purely by the boundary behavior of gauge fields. The antisymmetric boundary condition for the physical (transverse) gauge fields fixes the residual gauge freedom, and the boundary behavior of gauge fields determines the nontrivial topological properties of QCD in $A^+_a = 0$ gauge.

2. Residual gauge transformations. In light-front quantization, the residual gauge transformations may be generated by operators

$$ R_a = -\frac{1}{2} \int_{-\infty}^{\infty} dx^- \left\{ 2\partial^+ \partial^i A^i_a + g(f^{abc} A^i_b \partial^+ A^i_c + 2 \xi T^a \xi) \right\}. $$

The definition of $R_a$ here is different from the residual gauge transformations in axial gauge ($A_3^a = 0$), where the corresponding generators are defined as the operators $E^a_\pm \big|_{x^- = \pm \infty}$ do not generate the correct gauge transformations. The $R_a$ in eq.(9) is different by a factor of 2 from the first term in $E^a_\pm \big|_{x^- = \pm \infty}$ [see eq.(3)]. This difference follows because the $E^i_a = -\frac{1}{2} \partial^+ A^i_a$ are not dynamical variables. We shall first show how the $R_a$ generate the transverse residual gauge transformations in light-front QCD.

The transverse gauge transformation in light-front coordinates can be generally defined by

$$ \psi_+(x) \rightarrow \psi'_+(x) = u(x_\perp) \psi_+(x), $$

$$ A^i(x) \rightarrow A'^i(x) = u(x_\perp) A^i(x) u^{-1}(x_\perp) - i g (\partial^i u(x_\perp)) u^{-1}(x_\perp) $$

where $u(x) = \exp(-i \theta_a(x_\perp) T^a)$ are $SU(3)$ gauge group elements. For the infinitesimal $\theta_a(x_\perp)$, the above transformations lead to

$$ \delta \psi_+(x) \equiv \psi'_+(x) - \psi_+(x) = -i T^a \theta_a(x_\perp) \psi_+(x), $$

4
\[ \delta A_i^a(x) \equiv A_i^a(x) - A_i^a(x) = f^{abc} \theta_b(x_\perp) A_c^a(x) - \frac{1}{g} \partial_i \theta_a(x_\perp). \]  

(13)

The gauge transformations generated by \( R_a \) are defined in quantum theory such that the quark and gluon field operators and states (wave functions) transform as follows \[13\],

\[ \psi_+(x) \longrightarrow \psi'_+(x) \equiv U \psi_+(x) U^{-1}, \]  

(14)

\[ A_i^a(x) \longrightarrow A_i'^a(x) \equiv U A_i^a(x) U^{-1} \]  

(15)

\[ \vert \Phi \rangle \longrightarrow \vert \Phi' \rangle \equiv U \vert \Phi \rangle, \]  

(16)

where

\[ U = \exp \left\lbrace - \frac{i}{g} \int d^2 x_\perp \theta_a(x_\perp) R_a(x_\perp) \right\rbrace \]  

(17)

and the \( \theta_a(x_\perp) \) vanish at transverse spatial infinity. For infinitesimal \( \theta_a(x_\perp) \), eqs.(14–15) are reduced to

\[ \delta \psi_+ = i [G_\theta, \psi_+] \ , \ \delta A_i^a = i [G_\theta, A_i^a], \]  

(18)

where

\[ G_\theta = - \frac{1}{g} \int d^2 x_\perp \theta_a(x_\perp) R_a(x_\perp). \]  

(19)

Using eqs.(5-7)\[2\] we have

\[ [R_a(x_\perp), \psi_+(y_-, y_\perp)]_{x^+ = y^+} = g \delta^2(x_\perp - y_\perp) T^a \psi_+(y_-, y_\perp) \]  

(20)

\[ [R_a(x_\perp), A_i^b(y_-, y_\perp)]_{x^+ = y^+} = -ig f^{abc} \delta^2(x_\perp - y_\perp) A_c^a(y_-, y_\perp) \]  

+ \[ i \delta_{ab} \partial_x^i \delta^2(x_\perp - y_\perp) \]  

(21)

\[ [R_a(x_\perp), R_b(y_\perp)]_{x^+ = y^+} = ig f^{abc} \delta^2(x_\perp - y_\perp) R_c(x_\perp). \]  

(22)

From eqs.(20–22), it is easy to verify that eq.(18) produces the local gauge transformations of eqs.(12–13) for quark and gluon fields. Hence, the transformations (14–17) manifest a gauge symmetry of the theory, and the \( R_a \) are the generators of the transverse residual gauge transformations eqs.(12–13).

3. Residual gauge invariance and gauge fixing. However, the above derivation is not consistent with the choice of antisymmetric boundary conditions. From eq.(21),

\footnote{If the commutation involves \( \partial^+ A_i^a \), one must use eq.(5) rather than (6). Otherwise, the ordering of differentiation and integration may cause a problem.}
we see that the residual gauge transformations generated by $R_a$ break the antisymmetric boundary condition for $A^i_a$ at longitudinal infinity, and therefore are not allowed. By using the antisymmetric boundary condition, the first term in eq.(9) can be integrated out explicitly and the $R_a$ are reduced to

$$ R'_a = \pm 4 \partial^i A^i_a |_{x^- = \pm \infty} - \frac{g}{2} \int_{-\infty}^{\infty} dx^- \left( f^{abc} A^i_b \partial^+ A^i_c + 2 \psi^+ \gamma^a \psi^+ \right). $$  

We can explicitly show that for the $R'_a$, 

$$ [R'_a (x^-), \psi^+(y^-, y^-)]_{x^+ = y^+} = g \delta^2 (x_- - y_-) T^a \psi^+(y^-, y^-), $$  

$$ [R'_a (x^-), A^i_b (y^-, y^-)]_{x^+ = y^+} = -ig f^{abc} \delta^2 (x_- - y_-) A^i_c (y^-, y^-) $$

$$ + i \eta (y^-) \delta_{ab} \partial^+ \delta^2 (x_- - y_-), $$  

$$ [R'_a (x^-), R'_b (y^-)]_{x^+ = y^+} = ig f^{abc} \delta^2 (x_- - y_-) R'_c (x^-). $$

The commutator of eq.(25) is different from eq.(21) by a factor of $\eta (y^-)$ in the second term. From the definition of $\eta (y^-)$ [see eq.(8)], it follows that the residual gauge transformations generated by $R'_a$ preserve the antisymmetric boundary condition. But, unfortunately, we find that

$$ [R'_a , H] \neq 0. $$

There are two possible interpretations for eq.(27): either eq.(27) indicates that the residual gauge invariance is broken due to the antisymmetric boundary condition of $A^i_a$ at longitudinal infinity, or it implies that the $R'_a$ are not proper generators of the residual gauge transformations and that the antisymmetric boundary condition of $A^i_a$ at longitudinal infinity fixes completely the residual gauge freedom.

In perturbative light-front QCD, the breaking of residual gauge invariance is associated with the non-cancellation of light-front infrared divergences in gauge invariant sectors. The antisymmetric boundary conditions lead to a principal value prescription, which regularizes light-front infrared singularities. At tree level, one can check that the principal value prescription derived from eq.(3) removes all light-front infrared singularities. In loop calculations, with the principal value prescription, it is known that there still exist spurious poles, which lead to a mixing of light-front ultraviolet and infrared divergences. In these cases, it has been demonstrated that the most severe divergences (mixing of logarithmic ultraviolet and infrared divergences) are cancelled in the higher-order corrections to the scale evolution of the hadronic structure functions. Therefore, the antisymmetric boundary condition does not obviously break the residual gauge invariance.

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3 The light-front spurious poles can also be removed by use of the Mandelstam-Leibbrandt (ML) prescription [14]. Unfortunately, the ML prescription cannot be applied directly to light-front quantization in the Hamiltonian formalism.
In fact, the second term in the commutator of eq.(25) indicates that the residual
gauge transformations generated by $R_a$ are $x^-$ dependent, which is not allowed, as we
have pointed out in the beginning. Thus, Eq.(27) strongly suggests that one does not
have additional gauge freedom to choose other $A^i_a$ such that the resulting Hamiltonian
remains invariant. In other words, the antisymmetric boundary condition of $A^i_a$ at
longitudinal infinity does fix the residual gauge freedom in $A^+_a = 0$ gauge. We now
consider the nonperturbative consequences of the asymptotic gauge fields induced by
the residual gauge fixing.

For physical states, the energy density must be finite. From eq.(1), we see that
this requires at least that the field strengths $E^a_-$ vanish at light-front spatial infinity.
As a result, we have the following condition [determined by eq.(3)],
\[
\partial^i A^i_a|_{x^- = \pm \infty} = \mp \frac{g}{4} \int_{-\infty}^{\infty} dx^- (f^{abc} A^b_c \partial^+ A^i_c + 2 \psi^+_T T^a \psi_+).
\] (28)
To accommodate eq.(28), the commutation relations of eqs.(5–7) have to be modified.
Thus, eqs.(24-26) are no longer true and $R_a$ are no longer the generators of the
residual gauge transformation. Furthermore, eq.(28) shows that the $A^i_a$ must satisfy
antisymmetric boundary condition at longitudinal infinity. Thus, in the space of
physical states, the antisymmetric boundary conditions completely fix the residual
gauge freedom, and the requirement of finite energy density provides an additional
condition to determine explicitly the non-vanishing asymptotic physical gauge fields
at longitudinal infinity.

It may be worth mentioning here that Chodos used a condition similar to eq.(28)
to try to fix the residual gauge freedom in axial gauge\[4\]. However, in axial gauge,
the first term in $R_a$ is proportional to the dynamical variables $E^a_+$, and so cannot be
integrated out. Therefore the operator identity is very difficult to solve and the final
formulation is too complicated to be practically useful, as pointed out by Chodos
himself. In the light-front gauge, $E^i_a = -\frac{1}{2} \partial^- A^i_a$, which are not dynamical variables
and their vanishing at longitudinal infinity is reduced to a constraint for $A^i_a|_{x^- = \pm \infty}$
in physical states.

4. Nontrivial topological property. Finally, we shall show an important conse-
quence of the residual gauge fixing by using antisymmetric boundary conditions in
light-front QCD. It has been pointed out that if one chooses the $A^+_a = 0$ gauge with
symmetric boundary conditions, the nontrivial structure associated with a topological
winding number cannot be addressed \[10\]. However, with the symmetric boundary
condition, it is not clear how to fix the residual gauge freedom, which is associated
with the nontrivial structure in light-front QCD. With the antisymmetric boundary
condition, the residual gauge freedom is fixed, and the topological winding number is
determined by the non-vanishing $A^i_a|_{x^- = \infty} = -A^i_a|_{x^- = \infty}$.
Explicitly, we consider the axial current equation (for zero quark mass)

\[ \partial_\mu j_5^\mu = N_f \frac{g^2}{8\pi^2} \Tr(F_{\mu\nu}\tilde{F}^{\mu\nu}), \quad (29) \]

where the axial current is \( j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \), and the dual field strength is \( \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho} \).

The winding number in LFQCD is defined as the net charge between \( x^+ = -\infty \) and \( x^+ = \infty \),

\[ \Delta Q_5 = N_f \frac{g^2}{8\pi^2} \int d^4x \Tr(F_{\mu\nu}\tilde{F}^{\mu\nu}). \quad (30) \]

The integration on the r.h.s. of the above equation is defined in Minkowski space \( (M) \) and can be replaced by a surface integral. It can be found \[7\] that

\[ \Delta Q_5 = -N_f \frac{g^2}{\pi^2} \int dx^+ d^2x_\perp \Tr (A^- [A^1, A^2]) \bigg|_{x^- = -\infty}^{x^- = \infty}, \quad (31) \]

where \( A^-_{\alpha}|_{x=\pm\infty} \) is determined by eq.(3) and satisfies the antisymmetric boundary condition. Eq.(31) shows that a non-vanishing \( \Delta Q_5 \) is generated from the asymptotic fields of \( A^i_\alpha, A^-_{\alpha} \) and their antisymmetric boundary conditions at longitudinal infinity.

For physical states, it is particularly interesting to see from eq.(28) that the asymptotic physical gauge fields are generated by the color charge densities integrated over \( x^- \). Thus, the topological winding number in \( A^+_{\alpha} = 0 \) can be explicitly explored from eq.(28).

To summarize, as a consequence of residual gauge fixing by the antisymmetric boundary condition, the light-front QCD vacuum is trivial. Nontrivial QCD features for physical states are switched to the field operators and are manifested in the asymptotic behavior of physical gauge fields at longitudinal infinity. The trivial vacuum with nontrivial field variables in the light-front QCD may provide a practical framework for describing hadrons. A detailed discussion will be published separately \[7\].

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