Modification of the Bel-Robinson type energy-momentum

Lau Loi So
Department of Physics, National Central University, Chung-Li 320, Taiwan

Abstract
For describing the non-negative gravitational energy-momentum in terms of a pure Bel-Robinson type energy-momentum in a quasi-local 2-surface, both the Bel-Robinson tensor $B$ and tensor $V$ are suitable. We have found that this Bel-Robinson type energy-momentum can be modified such that it satisfies the Lorentz covariant, future pointing and non-spacelike properties. We find that these particular quasi-local energy-momentum properties can be obtained from (i): $B$ or $V$ plus a tensor $S$ in a small sphere limit, or (ii): directly evaluating the energy-momentum of $B$ or $V$ in a small ellipsoid region. (iii): calculate the total energy using the Landau-Lifshitz pseudotensor in a small ellipsoid, from Jupiter’s tidal force to Io in Schwarzschild spacetime, in an elliptic orbit.

1 Introduction
According to the Living Review article, Szabados said (see 4.2.2 in [1]): “Therefore, in vacuum in the leading $r^5$ order any coordinate and Lorentz-covariant quasi-local energy-momentum expression which is non-spacelike and future pointing must be proportional to the Bel-Robinson ‘momentum’ $B_{\mu\lambda\xi\kappa} t^\lambda t^\xi t^\kappa$.” Note that here $t^\alpha$ is the timelike unit vector and ‘momentum’ means 4-momentum. Previously, we believed that the Bel-Robinson type energy-momentum was the natural choice and indeed the only choice for describing the non-negative gravitational quasi-local energy-momentum expression. However, we have now found that it is not the case.

In the past, we thought there were only two gravitational energy-momentum expressions that have the positive definite energy (i.e., causal) since they give a positive multiple of the Bel-Robinson type energy-momentum in a small sphere limit. They are the Papapetrou pseudotensor [2, 3, 4] and tetrad-teleparallel energy-momentum gauge current expression [5, 6]. We even had concluded that both the Einstein and Landau-Lifshitz pseudotensors cannot guarantee positive definite [2], but now we discovered that the Landau-Lifshitz pseudotensor ensure positivity while Einstein does not. The motivation why we review the argument given by Szabados [1] is that we suspect there may exists a relaxation such that the desired physical requirements can be satisfied, i.e., the four-momentum are Lorentz covariant, future pointing and non-spacelike. We find that the explanation given by Szabados is necessary but not sufficient.

Positive gravitational energy is required for the stability of the spacetime [7] and any quasi-local stress expression which gives the Bel-Robinson type energy-momentum is the desirable candidate. Moreover, evaluate the quasi-local energy-momentum around a closed 2-surface, we can use the Bel-Robinson type energy-momentum to test whether the expression can have a chance to give the positivity at the large scale or not. Since negative quasi-local energy guarantees negative for a large scale, while positive quasi-local energy might have a chance for the large scale. Checking the result for the gravitational energy in a small regions is an economy way because the positivity energy prove is not easy.

Basically, quasi-local methods are not fundamentally different than pseudotensor methods [8, 9]. We will use the pseudotensor to illustrate our modified quasi-local Bel-Robinson type energy-momentum in three cases in section 3. Although pseudotensor

1email address: s0242010@gmail.com
is an coordinates dependent object, it stills a practical way to calculate the work done for an isolated system from an external universe, e.g., tidal heating through transferring the gravitational field from Jupiter to its satellite Io \cite{10}. Tidal heating means when an external tidal field $E_{ij}$ interacts with the evolving quadrupole moment $I_{ij}$ of an isolated body, the tidal work per unit time is \[ \frac{dW}{dt} = -\frac{c^3}{2} E_{ij} \frac{dI_{ij}}{dt}, \] where $I_{ij} \propto a_0^2 E_{ij}$ and $a_0$ is the radius of Io. This work rate formula is the same for the Newtonian energy and general relativistic Landau-Lifshitz pseudotensor \cite{11}. Tidal heating is a real physical observable irreversible process that Jupiter distorts and heats up Io \cite{12}, it should be unambiguous of how one’s choice to localize the energy, Purdue used the Landau-Lifshitz pseudotensor to calculate the tidal heating for Io in 1999 \cite{10}. Two years later, Favata examined different classical pseudotensors (i.e., Einstein, Landau-Lifshitz, Møller and Bergmann conserved quantities) and discovered the tidal heating formula \cite{13}. Moreover, in 2000, Booth and Creighton modified the Brown and York quasi-local energy formalism and obtained the same result for the tidal dissipation formula \cite{9}.

## 2 Technical background

The Bel-Robinson tensor $B$ and the recently proposed tensor $V$ \cite{14} both fulfil the Lorentz covariant, future pointing and non-spacelike requirements in a small sphere limit. They are defined in empty space as follows:

\[
B_{\alpha\beta\xi\kappa} := R_{\alpha\lambda\xi\sigma} R_{\beta\lambda\kappa} - R_{\alpha\lambda\kappa} R_{\beta\lambda\xi} - \frac{1}{8} g_{\alpha\beta} g_{\xi\kappa} R^2,
\]

\[
V_{\alpha\beta\xi\kappa} := R_{\alpha\lambda\xi\sigma} R_{\beta\kappa} + R_{\alpha\lambda\kappa} R_{\beta\lambda\xi} + R_{\alpha\lambda\xi} R_{\beta\lambda\kappa} + R_{\alpha\lambda\kappa} R_{\beta\lambda\xi} - \frac{1}{8} g_{\alpha\beta} g_{\xi\kappa} R^2,
\]

where $R^2 = R_{\mu\nu\xi\kappa} R^{\mu\nu\xi\kappa}$, Greek letters mean spacetime and the signature we use is $+2$. The associated known energy-momentum density is

\[
B_{\mu\lambda\sigma\tau} t^\lambda t^\sigma t^\tau \equiv V_{\mu\lambda\sigma\tau} t^\lambda t^\sigma t^\tau = (E_{ab} E^{ab} + H_{ab} H^{ab}, 2\epsilon_{cab} E^a_d H^{bd}),
\]

where Latin denotes spatial indices. The electric part $E_{ab}$ and magnetic part $H_{ab}$, are defined in terms of the Weyl curvature \cite{15}: $E_{ab} := C_{ambn} t^m t^n$ and $H_{ab} := * C_{ambn} t^m t^n$, where $t^m$ is the timelike unit vector and $* C_{\mu\nu\xi\kappa}$ indicates its dual for the evaluation. Here we emphasize that both $B$ and $V$ are totally traceless $t_{\mu\nu\sigma\tau} = 0$, which means $t_{\mu000} = t_{\mu0ij} \delta^{ij}$, where $t$ can be replaced by $B$ or $V$. Moreover, the energy component in \cite{3} is non-negative definite for all observers, which is well known, and the linear momentum component is a kind of cross product between $E$ and $H$:

\[
\epsilon_{cab} E^a_d H^{bd} = (\epsilon_{1ab} E^a_d H^{bd}, \epsilon_{2ab} E^a_d H^{bd}, \epsilon_{3ab} E^a_d H^{bd}) = (E_{2a} H^{3a} - E_{3a} H^{2a}, E_{3a} H^{1a} - E_{1a} H^{3a}, E_{1a} H^{2a} - E_{2a} H^{1a}) = (A_x, A_y, A_z),
\]

where $A := (E_{1a} \times H^{1a} + E_{2a} \times H^{2a} + E_{3a} \times H^{3a})$. The cross product can be well-defined if we treat $E_{1a}$ as a 3-dimensional vector, explicitly $E_{1a} = (E_{11}, E_{12}, E_{13})$. Similarly for $E_{2a}$, $E_{3a}$, $H_{1a}$, $H_{2a}$ and $H_{3a}$. Referring to \cite{4}, the momentum magnitude can be interpreted as follows

\[
|\epsilon_{cab} E^a_d H^{bd}| = |E_{1a} \times H^{1a} + E_{2a} \times H^{2a} + E_{3a} \times H^{3a}|
\leq |E_{1a} \times H^{1a}| + |E_{2a} \times H^{2a}| + |E_{3a} \times H^{3a}|
= |E_{1a}| \|H_{1b}\| \sin \theta_1 + |E_{2a}| \|H_{2b}\| \sin \theta_2 + |E_{3a}| \|H_{3b}\| \sin \theta_3,
\]

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= |E_{1a}| \|H_{1b}\| \sin \theta_1 + |E_{2a}| \|H_{2b}\| \sin \theta_2 + |E_{3a}| \|H_{3b}\| \sin \theta_3,
\]
where $\theta_1$ is the angle between $E_{1a}$ and $H_{1a}$; similarly for $\theta_2$ and $\theta_3$.

According to (3), both $B$ and $V$ have the same Bel-Robinson type energy-momentum in a small sphere region, which exhibits the desired causal relationship:

$$t_{0000} - |t_{000c}| = (E_{ab}E^{ab} + H_{ab}H^{ab}) - |2\epsilon_{cab}E^a_dH^{bd}| \geq 0,$$

and $t$ can be either $B$ or $V$. Here we consider two more possibilities for the comparison with the energy and still obtain the non-negative condition:

$$(E_{ab}E^{ab} + H_{ab}H^{ab}) + k_1(E_{ab}E^{ab} - H_{ab}H^{ab}) \geq 0, \quad \Rightarrow \quad |k_1| \leq 1, \quad (7)$$

$$(E_{ab}E^{ab} + H_{ab}H^{ab}) + k_2E_{ab}H^{ab} \geq 0, \quad \Rightarrow \quad |k_2| \leq 2. \quad (8)$$

The above two extra invariant terms come from

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = 8(E_{ab}E^{ab} - H_{ab}H^{ab}), \quad R_{\alpha\beta\mu\nu} * R^{\alpha\beta\mu\nu} = 16E_{ab}H^{ab}. \quad (9)$$

These two terms are scalar and satisfy the Lorentz covariant property. The first term can be classified as the energy density (i.e., see (20)−(21)) and the second as the momentum density (i.e., look (22)−(23)). Moreover, the momentum density $EH$ can be classified as a dot product between $E$ and $H$:

$$E_{ab}H^{ab} = E_{1a}H^{1a} + E_{2a}H^{2a} + E_{3a}H^{3a} = |E_{1a}| |H_{1b}| \cos \theta_1 + |E_{2a}| |H_{2b}| \cos \theta_2 + |E_{3a}| |H_{3b}| \cos \theta_3, \quad (10)$$

Combining the inequalities from (6) to (8)

$$(E_{ab}E^{ab} + H_{ab}H^{ab}) + k_1(E_{ab}E^{ab} - H_{ab}H^{ab}) + k_2E_{ab}H^{ab} - |2\epsilon_{cab}E^a_dH^{bd}| \geq 0. \quad (11)$$

Based on the argument from Szabados [1], the above non-negative inequality should hold only if $k_1$ and $k_2$ are both zero. However, we can demonstrate that this is not true. Let $|H_{Ia}| = \alpha_I |E_{Ia}|$ and $\alpha_I \geq 0$, where $I = 1, 2, 3$, consider (11) again

$$\begin{align*}
(E_{ab}E^{ab} + H_{ab}H^{ab}) + k_1(E_{ab}E^{ab} - H_{ab}H^{ab}) + k_2E_{ab}H^{ab} - 2|\epsilon_{cab}E^a_dH^{bd}| & \\
\geq & (E^2 + H^2) + k_1(E^2 - H^2) - |k_2||E_{ab}H^{ab}| - 2|\epsilon_{cab}E^a_dH^{bd}| \\
\geq & (1 + k_1)E^2_{1a} + (1 - k_1)H^2_{1a} - |k_2||E_{1a}||H_{1b}| \cos \theta_1 - 2|E_{1a}||H_{1b}| \sin \theta_1 \\
\quad + (1 + k_1)E^2_{2a} + (1 - k_1)H^2_{2a} - |k_2||E_{2a}||H_{2b}| \cos \theta_2 - 2|E_{2a}||H_{2b}| \sin \theta_2 \\
\quad + (1 + k_1)E^2_{3a} + (1 - k_1)H^2_{3a} - |k_2||E_{3a}||H_{3b}| \cos \theta_3 - 2|E_{3a}||H_{3b}| \sin \theta_3 \\
= & \left\{ (1 - \alpha_I)^2 \left[ 1 + \frac{k_1(1 + \alpha_I)}{(1 - \alpha_I)} \right] + 2\alpha_I \left( 1 - \frac{1}{2} |k_2| \cos \theta_I \right) \right\} E^2_{1a} \\
\geq & 0, \quad (12)
\end{align*}$$

provided that

$$k_1 \geq \frac{\alpha_I - 1}{\alpha_I + 1}, \quad |k_2| \leq \frac{2(1 - |\sin \theta_I|)}{|\cos \theta_I|}. \quad (13)$$

Thus (12) is non-negative for some non-vanishing $k_1$ and $k_2$. The component with $k_1$ varies the energy density, while the component with $k_2$ alters the momentum value. One may question the purpose for this kind of modification, but for the present discussion we note that we do not change the energy-momentum relationship indicated in (6) through the introduction of the two terms multiplied by $k_1$ and $k_2$. The detailed physical consequences will be discussed in section 3, i.e., see (16), (19) and (24).

Actually, we are repeating the same comparison with Szabados [1]. However, we have found a different result; one that is strictly forbidden according to the conclusion
of Szabados’s article. A natural question if (12) is correct, is what are the allowed ranges for \( k_1 \) and \( k_2 \)? More precisely, looking at (11) again, we consider what ranges for constants \( k_1 \) and \( k_2 \) may be selected such that the Lorentz covariant and future directed non-spacelike qualities can be kept. For this purpose we use the 5 Petrov types \([16]\) Riemann curvature for the verification. After some simple algebra, we find a different results from Szabados \([1]\):

\[
|k_1| \leq 1, \quad |k_2| \leq 2(1 - |k_1|). \tag{14}
\]

This indicates that, in terms of a quasi-local energy-momentum expression, \( B \) and \( V \) are not the only candidates that satisfy the Lorentz covariant and future directed non-spacelike requirements in a small sphere limit. There exists some relaxation freedom for the modification, the detail will be discussed in three cases in section 3. Here we list out the accompanied tensor \( S \) with \( B \) or \( V \) as follows:

\[
S_{\alpha\beta\xi\kappa} = R_{\alpha\xi\lambda\sigma} R_{\beta\kappa}^{\lambda\sigma} + R_{\alpha\kappa\lambda\sigma} R_{\beta\xi}^{\lambda\sigma} + \frac{1}{4} g_{\alpha\beta} g_{\xi\kappa} R^2. \tag{15}
\]

### 3 Quasi-local energy-momentum

We now examine the positive definite gravitational quasi-local energy-momentum, which satisfies the Lorentz covariant and future directed non-spacelike conditions.

Case (i): Consider a simple physical situation such that within a small sphere limit we define: \( t + sS \), where \( t \) can be replaced by \( B \) or \( V \), and \( s \) is a constant. For constant time \( t_0 = 0 \), the energy-momentum in vacuum with radius \( r \)

\[
2\kappa \mathcal{P}_\mu = \int_{t_0} (t^0_{\mu\xi\kappa} + sS^0_{\mu\xi\kappa}) x^\xi x^\kappa dV = \frac{4\pi}{15} r^5 (t^0_{\mu ij} + sS^0_{\mu ij}) \delta^{ij}, \tag{16}
\]

where \( \kappa = 8\pi G/c^4 \), \( G \) is the Newtonian constant and \( c \) the speed of light. According to \([1]\), the only possibility is \( s = 0 \) in order to produce the Lorentz covariant, future pointing and non-spacelike properties. However, we can show that there are some \( s \neq 0 \) such that these properties are preserved. As the 4-momentum of \( S_{0\mu ij} \delta^{ij} = -10(E_{ab}^2 - H_{ab}^2, 0, 0, 0) \), we only vary the energy and without affecting the momentum. After the substitution, the energy for (16) is

\[
- \mathcal{P}_0 = \mathcal{E} = \frac{2\pi}{15\kappa} r^5 \left[ (E_{ab} E^{ab} + H_{ab} H^{ab}) - 10s(E_{ab} E^{ab} - H_{ab} H^{ab}) \right], \tag{17}
\]

and the associated momentum is \( \mathcal{P}_c = \frac{2\pi}{15\kappa} r^5 (2\epsilon_{cab} E^{ad} H^{bd}) \). Since the values of \( E_{ab} \) and \( H_{ab} \) can be arbitrary at a given point, the sign of the energy component of \( S \) is uncertain and obviously \( S \) affects the desired Bel-Robinson type energy-momentum inequality: \( \mathcal{E} \geq |\vec{\mathcal{P}}| \). Previously, our preference was achieving a multiple of pure Bel-Robinson type energy-momentum in a small sphere \([14]\), and we thought the result in (16) required \( s = 0 \). However, we have now shown that this is not true: we have found that certain linear combinations of \( t \) and \( S \) are legitimate. Comparing (12) and (17), we observe that \( |k_1| = 10|s| \leq 1 \) and \( k_2 = 0 \) produce results that satisfy the non-negative energy, Lorentz covariant and future directed non-spacelike requirements. Here we give a remark: previously we thought both Einstein \( t^E_{\alpha\beta} \) and Landau-Lifshitz \( t^{LL}_{\alpha\beta} \) pseudotensors cannot give the positive (i.e., causal) definite quasi-local energy in Riemann normal coordinates \([2]\):

\[
t^E_{\alpha\beta} = \frac{2}{9} \left( B_{\alpha\beta\xi\kappa} - \frac{1}{4} S_{\alpha\beta\xi\kappa} \right) x^\xi x^\kappa, \quad t^{LL}_{\alpha\beta} = \frac{7}{18} \left( B_{\alpha\beta\xi\kappa} + \frac{1}{14} S_{\alpha\beta\xi\kappa} \right) x^\xi x^\kappa. \tag{18}
\]
This implies that the Landau-Lifshitz pseudotensor (i.e., corresponding $|s| = \frac{1}{14} < \frac{1}{10}$) is a suitable candidate for the Lorentz covariant and future directed non-spacelike requirements, while Einstein pseudotensor does not (i.e., associated $|s| = \frac{1}{7} > \frac{1}{10}$).

Case (ii): Evaluate the energy-momentum in a small ellipsoid, replacing $t$ by $B$ or $V$. Consider a simple dimension $(a, b, c) = (\sqrt{1 + \Delta}, 1, 1)r_0$ for non-zero $|\Delta| << 1$ and $r_0$ finite. For constant time $t_0 = 0$, the corresponding 4-momentum are

$$2K\mathcal{P}_\mu = \int_{t_0} t^0_{\mu ij}x^ix^j dV = \frac{4\pi}{15}(t^0_{\mu ij}\delta^{ij} + \Delta t^0_{\mu 11})r_0^5\sqrt{1 + \Delta}. \tag{19}$$

Here we list out the energy component for $B$ and $V$

$$B_{0011} = E_{ab}E^{ab} + H_{ab}H^{ab} - 2E_{1a}E^{1a} - 2H_{1a}H^{1a}, \tag{20}$$
$$V_{0011} = 3E_{ab}E^{ab} - H_{ab}H^{ab} - 8E_{1a}E^{1a} + 4H_{1a}H^{1a}, \tag{21}$$

and the associated momenta are

$$B_{0c11} = 2\epsilon_{cab}(E^{ad}H_d - 2E^{a1}H_1), \tag{22}$$
$$V_{0c11} = 2\epsilon_{1ab}(E^{ad}H_d - 2E^{a1}H_1, 2E^{a}H_2 - 4E^{a2}H_1, 2E^{a}H_3 - 4E^{a3}H_1). \tag{23}$$

Looking at (19), $\Delta t^0_{\mu 11}$ varies the energy and momentum of $t^0_{\mu ij}\delta^{ij}$ simultaneously, i.e., making it analogous with (12): $k_1 \neq 0 \neq k_2$. Using the 5 Petrov types Riemann curvature to compare the energy and momentum in (19), we find that if $t$ is replaced by $B$ the Lorentz covariant and future directed non-spacelike properties require $\Delta \in (-1, 1]$. Similarly, if we replace $t$ by $V$, it is also true provided $\Delta \in [-\frac{1}{3}, \frac{1}{3}]$. However, as far as the quasi-local small 2-surface is concerned, practically, we only need the non-zero $\Delta$ to be sufficiently small. Therefore, the result in (19), a linear combination for $t^0_{\mu ij}\delta^{ij}$ with an extra $t^0_{\mu 11}$, is a physically reasonable candidate for describing the quasi-local energy-momentum.

Case (iii): Demonstrate the total energy-momentum on a gravitating system by an external universe, i.e., transferring the gravitational field energy from Jupiter to Io. Referring to second equation of (18), evaluate the energy-momentum for Landau-Lifshitz pseudotensor in a small ellipsoid. It is natural to consider a 2-surface ellipsoid instead of a 2-surface sphere because Jupiter deformed Io from being a perfect sphere through the tidal force. In reality, it is slightly deformed and it suits the quasi-local small 2-surface limit. The detail is follows. Again let $(a, b, c) = (\sqrt{1 + \Delta}, 1, 1)a_0$, constant time $t_0 = 0$ and the 4-momentum are

$$2K\mathcal{P}^{LL}_\mu = \frac{14\pi}{135}\left[(B^0_{\mu ij} + sS^0_{\mu ij})\delta^{ij} + \Delta(B^0_{\mu 11} + sS^0_{\mu 11})\right]a_0^5\sqrt{1 + \Delta}, \tag{24}$$

where $s = \frac{1}{14}$, energy from $S_{0011} = -2(E_{ab}^2 + 2E_{1a}^2 - H_{ab}^2 - 2H_{1a}^2)$ and momentum from $S_{0c11} = 4(0, E_{1a}H_3^a + E_{3a}H_1^a, -E_{1a}H_2^a - E_{2a}H_1^a)$. Looking at (24) for the 4-momentum, we observed that the interval for $\Delta \in [-\frac{1}{3}, \frac{1}{3}]$ satisfies the requirements for the Lorentz covariant and future directed non-spacelike. Recall $\frac{GM}{c^2} = 3.4 \times 10^{-9}$ which is small compare to unity (i.e., weak gravity limit), where $M = 1.90 \times 10^{27}$kg denotes the mass of Jupiter, $r = 4.2 \times 10^{6}$km means the separation between Jupiter and Io. The physical dimension for Io is $(x, y, z) = (3660.0, 3637.4, 3630.6)$ in kilometer. Using our notation: $a = \sqrt{1 + \Delta}a_0$, $b \simeq c \simeq a_0$, where $a_0 = 1817$km and $\Delta = 0.0144$. Indeed this ellipsoid is a little bit deformed from a perfect sphere. In our case, the volume element of Io is the quasi-local 2-surface for evaluating the energy-momentum values. Note that the density of Io is $M_{Io} = 9.83 \times 10^{22}$kg. Let’s use the Schwarzschild metric in spherical coordinates (see §31.2 in [17]) for a simple test. Certainly, there is no momentum since we are dealing with a static spacetime. The non-vanishing
Riemann curvatures are $R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = -\frac{2GM}{c^2 r^3}$ and $R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = -R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = \frac{GM}{c^2 r^3}$. Substitute into (24) and thence the total energy-momentum complex (see (29) in [4] and (45) in [10]) is

$$T_{00}^{LL} = T_{00}^{LL} + (2\kappa)^{-1} T_{00}^{LL} = M_{Io} + \frac{14\pi G^2 M^2}{45\kappa c^4 r_0^6} \left[ (1 - 10s) + \frac{\Delta}{3} (1 - 10s) \right] a_0^5 \sqrt{1 + \Delta} = 1.11M_{Io}. \quad (25)$$

Note that the extra amount of energy received from Jupiter is small but significant.

### 4 Conclusion

To describe the positive quasi-local energy-momentum expression, the Bel-Robinson tensor $B$ and tensor $V$ are suitable because both of them give the Bel-Robinson type energy-momentum in a small sphere region. In the past, it has seemed that only this Bel-Robinson type energy-momentum can manage this specific task: Lorentz covariant, future pointing and non-spacelike. That particular restriction cannot allow even a small amount of energy to be subtracted from this Bel-Robinson type energy-momentum. After some careful comparison and using the 5 Petrov type Riemann curvature for the verification, we have discovered that the Bel-Robinson type energy-momentum implies Lorentz covariant and future directed non-spacelike properties; but the converse is not true. We find that there exists a certain relaxation freedom such that one can (i): add an extra tensor $S$ with $B$ or $V$ in a quasi-local small sphere limit, or (ii): directly evaluate $B$ or $V$ in a small ellipsoid region, (iii): Using the Landau-Lifshitz pseudotensor to calculate the total energy, refer to the Schwarzschild metric, in a small ellipsoid region.

Previously, we thought there are only two classical energy-momentum expressions, Papapetrou pseudotensor and tetrad-teleparallel energy-momentum gauge current expression, that contribute the desired Lorentz covariant and future directed non-spacelike requirements. Now, we have to add one more: Landau-Lifshitz pseudotensor in Riemann normal coordinates.

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