Pure extensions of locally compact abelian groups

Peter Loth

Abstract. In this paper, we study the group \( \text{Pext}(C, A) \) for locally compact abelian (LCA) groups \( A \) and \( C \). Sufficient conditions are established for \( \text{Pext}(C, A) \) to coincide with the first Ulm subgroup of \( \text{Ext}(C, A) \). Some structural information on pure injectives in the category of LCA groups is obtained. Letting \( \mathcal{E} \) denote the class of LCA groups which can be written as the topological direct sum of a compactly generated group and a discrete group, we determine the groups \( G \) in \( \mathcal{E} \) which are pure injective in the category of LCA groups. Finally we describe those groups \( G \) in \( \mathcal{E} \) such that every pure extension of \( G \) by a group in \( \mathcal{E} \) splits and obtain a corresponding dual result.

1. Introduction

In this paper, all considered groups are Hausdorff topological abelian groups and will be written additively. Let \( \mathfrak{L} \) denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. The Pontrjagin dual group of a group \( G \) is denoted by \( \hat{G} \) and the annihilator of \( S \subseteq G \) in \( \hat{G} \) is denoted by \( (\hat{G}, S) \). A morphism is called proper if it is open onto its image, and a short exact sequence

\[
0 \rightarrow A \overset{\phi}{\rightarrow} B \overset{\psi}{\rightarrow} C \rightarrow 0
\]

in \( \mathfrak{L} \) is said to be proper exact if \( \phi \) and \( \psi \) are proper morphisms. In this case, the sequence is called an extension of \( A \) by \( C \) (in \( \mathfrak{L} \)), and \( A \) may be identified with \( \phi(A) \) and \( C \) with \( B/\phi(A) \). Following Fulp and Griffith \([FG1]\), we let \( \text{Ext}(C, A) \) denote the (discrete) group of extensions of \( A \) by \( C \). The elements represented by pure extensions of \( A \) by \( C \) form a subgroup of \( \text{Ext}(C, A) \) which is denoted by \( \text{Pext}(C, A) \). This leads to a functor \( \text{Pext} \) from \( \mathfrak{L} \times \mathfrak{L} \) into the category of discrete abelian groups. The literature shows the importance of the notion of pure extensions (see for instance \([E]\)). The concept of purity in the category of locally compact abelian groups has been studied by several authors (see e.g. \([A], [B], [Fu1], [HH], [Kh], [L1], [L2], \) and \([V]\)). The notion of topological purity is due to Vilenkin \([V]\): a

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subgroup $H$ of a group $G$ is called topologically pure if $\overline{nH} = H \cap \overline{nG}$ for all positive integers $n$. The annihilator of a closed pure subgroup of an LCA group is topologically pure (cf. [L2]) but need not be pure in $\hat{G}$ (see e.g. [A]). As is well known, $\text{Pext}(C, A)$ coincides with $\text{Ext}(C, A) = \bigcap_{n=1}^{\infty} n\text{Ext}(C, A)$, the first Ulm subgroup of $\text{Ext}(C, A)$, provided that $A$ and $C$ are discrete abelian groups (see [F]). In the category $\mathfrak{L}$, a corresponding result need not hold: for groups $A$ and $C$ in $\mathfrak{L}$, $\text{Ext}(C, A)^1$ is a (possibly proper) subgroup of $\text{Pext}(C, A)$, and it coincides with $\text{Pext}(C, A)$ if (a) $A$ and $C$ are compactly generated, or (b) $A$ and $C$ have no small subgroups (see Theorem 2.4). If $G$ is pure injective in $\mathfrak{L}$, then $G$ has the form $R \oplus T \oplus G'$ where $R$ is a vector group, $T$ is a toral group and $G'$ is a densely divisible topological torsion group having no nontrivial pure compact open subgroups. However, the converse need not be true (cf. Theorem 2.7). Let $\mathfrak{C}$ denote the class of LCA groups which can be written as the topological direct sum of a compactly generated group and a discrete group. Then a group in $\mathfrak{C}$ is pure injective in $\mathfrak{L}$ if and only if it is injective in $\mathfrak{L}$ (see Corollary 2.8). Let $G$ be a group in $\mathfrak{C}$. Then every pure extension of $G$ by a group in $\mathfrak{C}$ splits if and only if $G$ has the form $R \oplus T \oplus A \oplus B$ where $R$ is a vector group, $T$ is a toral group, $A$ is a topological direct product of finite cyclic groups and $B$ is a discrete bounded group. Dually, every pure extension of a group in $\mathfrak{C}$ by $G$ splits exactly if $G$ has the form $R \oplus C \oplus D$ where $R$ is a vector group, $C$ is a compact torsion group and $D$ is a discrete direct sum of cyclic groups (see Theorem 2.11).

The additive topological group of real numbers is denoted by $\mathbb{R}$, $\mathbb{Q}$ is the group of rationals, $\mathbb{Z}$ is the group of integers, $\mathbb{T}$ is the quotient $\mathbb{R}/\mathbb{Z}$, $\mathbb{Z}(n)$ is the cyclic group of order $n$ and $\mathbb{Z}(p^\infty)$ denotes the quasicyclic group. By $G_d$ we mean the group $G$ with the discrete topology, $tG$ is the torsion part of $G$ and $bG$ is the subgroup of all compact elements of $G$. Throughout this paper the term “isomorphic” is used for “topologically isomorphic”, “direct summand” for “topological direct summand” and “direct product” for “topological direct product”. We follow the standard notation in [F] and [HR].

2. Pure extensions of LCA groups

We start with a result on pure extensions involving direct sums and direct products.

**Theorem 2.1.** Let $G$ be in $\mathfrak{L}$ and suppose $\{H_i : i \in I\}$ is a collection of groups in $\mathfrak{L}$. If $H_i$ is discrete for all but finitely many $i \in I$, then

$$\text{Pext}(\bigoplus_{i \in I} H_i, G) \cong \prod_{i \in I} \text{Pext}(H_i, G).$$

If $H_i$ is compact for all but finitely many $i \in I$, then

$$\text{Ext}(C, A)^1 = \bigcap_{n=1}^{\infty} n\text{Ext}(C, A),$$

the first Ulm subgroup of $\text{Ext}(C, A)$, provided that $A$ and $C$ are discrete abelian groups (see [F]). In the category $\mathfrak{L}$, a corresponding result need not hold: for groups $A$ and $C$ in $\mathfrak{L}$, $\text{Ext}(C, A)^1$ is a (possibly proper) subgroup of $\text{Pext}(C, A)$, and it coincides with $\text{Pext}(C, A)$ if (a) $A$ and $C$ are compactly generated, or (b) $A$ and $C$ have no small subgroups (see Theorem 2.4). If $G$ is pure injective in $\mathfrak{L}$, then $G$ has the form $R \oplus T \oplus G'$ where $R$ is a vector group, $T$ is a toral group and $G'$ is a densely divisible topological torsion group having no nontrivial pure compact open subgroups. However, the converse need not be true (cf. Theorem 2.7). Let $\mathfrak{C}$ denote the class of LCA groups which can be written as the topological direct sum of a compactly generated group and a discrete group. Then a group in $\mathfrak{C}$ is pure injective in $\mathfrak{L}$ if and only if it is injective in $\mathfrak{L}$ (see Corollary 2.8). Let $G$ be a group in $\mathfrak{C}$. Then every pure extension of $G$ by a group in $\mathfrak{C}$ splits if and only if $G$ has the form $R \oplus T \oplus A \oplus B$ where $R$ is a vector group, $T$ is a toral group, $A$ is a topological direct product of finite cyclic groups and $B$ is a discrete bounded group. Dually, every pure extension of a group in $\mathfrak{C}$ by $G$ splits exactly if $G$ has the form $R \oplus C \oplus D$ where $R$ is a vector group, $C$ is a compact torsion group and $D$ is a discrete direct sum of cyclic groups (see Theorem 2.11).

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The proof of the second assertion is similar. To prove the last statement, let \( E \) be a prime and \( P_{\text{ext}}(\bigoplus G_i) \) be a group defined by \( \text{Ext}(\bigoplus H_i, G) \in I \). Since an extension equivalent to a pure extension is pure, \( G \) are stripped of their topology, the corresponding isomorphism maps the group \( P_{\text{ext}}(\bigoplus G_i) \) onto \( \prod P_{\text{ext}}(H_i, G) \). If the groups \( H_i \) and \( G \) are exact, hence \( \hat{\phi} = \text{Ext}(\bigoplus H_i) \rightarrow \prod \text{Ext}(H_i, G) \) (see [FG1] Theorem 53.7 and p. 231, Exercise 6). Since an extension equivalent to a pure extension is pure, \( \hat{\phi} \) maps \( P_{\text{ext}}(\bigoplus H_i, G) \) onto \( \prod P_{\text{ext}}(H_i, G) \), establishing the first statement. The proof of the second assertion is similar. To prove the last statement, let \( p \) be a prime and \( H = \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n) \), taken discrete. Assume \( \text{Ext}(\hat{Q}, H) = 0 \). By [FG2] Corollary 2.10, the sequences

\[
\text{Ext}(\hat{Q}, H) \rightarrow \text{Ext}(\hat{Q}, H/tH) \rightarrow 0
\]

and

\[
0 = \text{Hom}(\hat{Q}/\mathbb{Z}, H/tH) \rightarrow \text{Ext}(\hat{Z}, H/tH) \rightarrow \text{Ext}(\hat{Q}, H/tH)
\]

are exact, hence [FG1] Proposition 2.17 yields \( H/tH \cong \text{Ext}(\hat{Z}, H/tH) = 0 \) which is impossible. Since \( \hat{Q} \) is torsion-free, it follows that \( \text{Ext}(\hat{Q}, H) = \text{Ext}(\hat{Q}, H) \neq 0 \). On the other hand, we have

\[
\prod_{n=1}^{\infty} \text{Ext}(\hat{Q}, \mathbb{Z}(p^n)) = \prod_{n=1}^{\infty} \text{Ext}(\hat{Q}, \mathbb{Z}(p^n)) = 0
\]

by [FG1] Theorem 2.12 and [F] Theorem 21.1. Note that this example shows that Proposition 6 in [FU1] is incorrect.

**Proposition 2.2.** Suppose \( E_0: 0 \rightarrow A \xrightarrow{\phi} B \rightarrow C \rightarrow 0 \) is a proper exact sequence in \( \mathcal{L} \). Let \( \alpha: A \rightarrow A \) be a proper continuous homomorphism and \( \alpha_* \) the induced endomorphism on \( \text{Ext}(C, A) \) given by \( \alpha_*(E) = \alpha E \). Then \( E_0 \in \text{Im} \alpha_* \) if and only if \( \text{Im} \phi/\text{Im} \phi \alpha \) is a direct summand of \( B/\text{Im} \phi \alpha \).

**Proof.** If \( \alpha: A \rightarrow A \) is a proper morphism in \( \mathcal{L} \), then

\[
0 \rightarrow \text{Im} \alpha \rightarrow A \rightarrow \text{Im} \phi/\text{Im} \phi \alpha \rightarrow 0
\]

are proper exact sequences in \( \mathcal{L} \) (cf. [HR] Theorem 5.27). Now [FG2] Corollary 2.10 and the proof of [F] Theorem 53.1 show that \( E_0 \in \text{Im} \alpha_* \) if and only if the induced proper exact sequence

\[
0 \rightarrow \text{Im} \phi/\text{Im} \phi \alpha \rightarrow B/\text{Im} \phi \alpha \rightarrow C \rightarrow 0
\]

splits.

If \( A \) and \( C \) are groups in \( \mathcal{L} \), then \( \text{Ext}(C, A) \cong \text{Ext}(\hat{A}, \hat{C}) \) (see [FG1] Theorem 2.12). We have, however:
Lemma 2.3. Let $A$ and $C$ be in $\mathcal{L}$. Then:

(i) In general, $\text{Pext}(C, A) \not\cong \text{Pext}(\hat{A}, \hat{C})$.

(ii) Let $\mathfrak{R}$ denote a class of LCA groups satisfying the following property: If $G \in \mathfrak{R}$, then $\hat{G} \in \mathfrak{R}$ and $nG$ is closed in $G$ for all positive integers $n$. Then $\text{Pext}(C, A) \cong \text{Pext}(\hat{A}, \hat{C})$ whenever $A$ and $C$ are in $\mathfrak{R}$.

Proof. (i) The finite torsion part of a group in $\mathcal{L}$ need not be a direct summand (see for instance [Kh]), so there is a finite group $F$ and a torsion-free group $C$ in $\mathcal{L}$ such that $\text{Pext}(C, F) = \text{Ext}(C, F) \neq 0$. On the other hand, $\text{Pext}(\hat{F}, \hat{C}) \cong \text{Pext}(F, \hat{C}) = 0$ by [F] Theorem 30.2.

(ii) Let $A$ and $C$ be in $\mathfrak{R}$ and consider the isomorphism $\text{Ext}(C, A) \cong \text{Ext}(\hat{A}, \hat{C})$ given by $E : 0 \to A \to B \to C \to 0 \mapsto \hat{E} : 0 \to \hat{C} \to \hat{B} \to \hat{A} \to 0$. The annihilator of a closed pure subgroup of $B$ is topologically pure in $\hat{B}$ (cf. [L2] Proposition 2.1) and for all positive integers $n$, $nA$ and $n\hat{C}$ are closed subgroups of $A$ and $\hat{C}$, respectively. Therefore, $E$ is pure if and only if $\hat{E}$ is pure. □

Recall that a topological group is said to have no small subgroups if there is a neighborhood of 0 which contains no nontrivial subgroups. Moskowitz [M] proved that the LCA groups with no small subgroups have the form $\mathbb{R}^n \oplus \mathbb{T}^m \oplus D$ where $n$ and $m$ are nonnegative integers and $D$ is a discrete group, and that their Pontrjagin duals are precisely the compactly generated LCA groups.

Theorem 2.4. For groups $A$ and $C$ in $\mathcal{L}$, we have:

(i) $\text{Pext}(C, A) \supseteq \text{Ext}(C, A)^1$.

(ii) $\text{Pext}(C, A) \neq \text{Ext}(C, A)^1$ in general.

(iii) Suppose (a) $A$ and $C$ are compactly generated, or (b) $A$ and $C$ have no small subgroups. Then $\text{Pext}(C, A) = \text{Ext}(C, A)^1$.

Proof. (i) Let $\alpha : A \to A$ be the multiplication by a positive integer $n$ and let $E : 0 \to A \xrightarrow{\phi} X \to C \to 0 \in n\text{Ext}(C, A)$. Since Ext is an additive functor, there exists an extension $0 \to A \to B \to C \to 0$ such that

\[
\begin{array}{cccccc}
0 & \to & A & \to & B & \to & C & \to & 0 \\
\alpha & \downarrow & & \downarrow & & \parallel & & \\
0 & \to & A & \xrightarrow{\phi} & X & \to & C & \to & 0
\end{array}
\]

is a pushout diagram in $\mathcal{L}$. An easy calculation shows that $nX \cap \phi(A) = n\alpha(A)$, hence $\text{Ext}(C, A)^1$ is a subset of $\text{Pext}(C, A)$.

(ii) Let $\text{Pext}(C, F)$ be as in the proof of Lemma 2.3. Then $\text{Pext}(C, F) \neq 0$ but $\text{Ext}(C, F)^1 = 0$.

(iii) Suppose first that $A$ and $C$ are compactly generated. If $\alpha : A \to A$ is the multiplication by a positive integer $n$, then $\alpha(A) = nA$ is a group in $\mathcal{L}$. Since $A$ is $\sigma$-compact, $\alpha$ is a proper morphism by [HR] Theorem...
5.29. Let $E : 0 \to A \xrightarrow{\phi} B \to C \to 0 \in \text{Ext}(C, A)$. By Proposition 2.2, $E \in \text{Im} \alpha_{\ast} = n\text{Ext}(C, A)$ if and only if $\phi(A)/n\phi(A)$ is a direct summand of $B/n\phi(A)$. Now assume that $E$ is a pure extension. Then $\phi(A)/n\phi(A)$ is pure in the group $B/n\phi(A)$ which is compactly generated (cf. [M] Theorem 2.6). Since the compact group $\phi(A)/n\phi(A)$ is topologically pure, it is a direct summand of $B/n\phi(A)$ (see [L1] Theorem 3.1). Consequently, $E$ is an element of the first Ulm subgroup of $\text{Ext}(C, A)$ and by (i) the assertion follows. To prove the second part of (iii), assume that $A$ and $C$ have no small subgroups. By what we have just shown and Lemma 2.3, we have $\text{Pext}(C, A) \cong \text{Ext}(\hat{A}, \hat{C}) = \text{Ext}(\hat{A}, \hat{C})^1 \cong \text{Ext}(C, A)^1$. □

By the structure theorem for locally compact abelian groups, any group $G$ in $L$ can be written as $G = V \oplus \widetilde{G}$ where $V$ is a maximal vector subgroup of $G$ and $\widetilde{G}$ contains a compact open subgroup. The groups $V$ and $\widetilde{G}$ are uniquely determined up to isomorphism (see [HR] Theorem 24.30 and [AA] Corollary 1).

**Lemma 2.5.** A group $G$ in $L$ is torsion-free if and only if every compact open subgroup of $\widetilde{G}$ is torsion-free.

**Proof.** Only sufficiency needs to be shown. Suppose every compact open subgroup of $\widetilde{G}$ is torsion-free and assume that $G$ is not torsion-free. Then $\widetilde{G}$ contains a nonzero element $x$ of finite order. If $K$ is any compact open subgroup of $\widetilde{G}$, then $K + \langle x \rangle$ is compact (see [HR] Theorem 4.4) and open in $G$ but not torsion-free, a contradiction. □

Dually, we obtain the following fact which extends [A] (4.33). Recall that a group is said to be densely divisible if it possesses a dense divisible subgroup.

**Lemma 2.6.** A group $G$ in $L$ is densely divisible if and only if $\widetilde{G}/K$ is divisible for every compact open subgroup $K$ of $\widetilde{G}$.

**Proof.** Again, only sufficiency needs to be proved. Assume that $\widetilde{G}/K$ is divisible for every compact open subgroup $K$ of $\widetilde{G}$ and let $C$ be a compact open subgroup of $(\widetilde{G})$. Since $(\widetilde{G}) \cong (G/V)$ where $V$ is a maximal vector subgroup of $G$, there exists a compact open subgroup $X/V$ of $G/V$ such that $C \cong ((G/V), X/V) \cong ((G/V)/(X/V))$ (see [HR] Theorems 23.25, 24.10 and 24.11). By our assumption, $(\widetilde{G}/V)/(X/V)$ is divisible. But then $C$ is torsion-free (cf. [HR] Theorem 24.23), so by Lemma 2.5, $\widetilde{G}$ is torsion-free. Finally, [R] Theorem 5.2 shows that $G$ is densely divisible. □

Let $G$ be in $L$. Then $G$ is called pure injective in $L$ if for every pure extension $0 \to A \xrightarrow{\phi} B \to C \to 0$ in $L$ and continuous homomorphism $f : A \to G$ there is a continuous homomorphism $\overline{f} : B \to G$ such that the
is commutative. Following Robertson \[R\], we call \(G\) a topological torsion group if \((n!)x \to 0\) for every \(x \in G\). Note that a group \(G\) in \(\mathfrak{L}\) is a topological torsion group if and only if both \(G\) and \(\hat{G}\) are totally disconnected (cf. \[R\] Theorem 3.15). Our next result improves \[Fu1\] Proposition 9.

**Theorem 2.7.** Consider the following conditions for a group \(G\) in \(\mathfrak{L}\):

(i) \(G\) is pure injective in \(\mathfrak{L}\).
(ii) \(\text{Pext}(X,G) = 0\) for all groups \(X\) in \(\mathfrak{L}\).
(iii) \(G \cong \mathbb{R}^n \oplus T^m \oplus G'\) where \(n\) is a nonnegative integer, \(m\) is a cardinal and \(G'\) is a densely divisible topological torsion group possessing no nontrivial pure compact open subgroups.

Then we have: \((i) \Leftrightarrow (ii) \Rightarrow (iii)\) and \((iii) \not\Rightarrow (ii)\).

**Proof.** If \(G\) is pure injective in \(\mathfrak{L}\), then any pure extension \(0 \to G \to B \to X \to 0\) in \(\mathfrak{L}\) splits because there is a commutative diagram

\[
\begin{array}{ccc}
0 & \to & G \\
\downarrow & & \downarrow \\
B & \to & X \\
\end{array}
\]

hence (i) implies (ii). Conversely, assume (ii). If \(0 \to A \to B \to X \to 0\) is a pure extension in \(\mathfrak{L}\) and \(f: A \to G\) is a continuous homomorphism, then there is a pushout diagram

\[
\begin{array}{ccc}
0 & \to & A \\
\downarrow & & \downarrow \\
B & \to & X \\
\end{array}
\]

The bottom row is an extension in \(\mathfrak{L}\) (cf. \[FG1\]) which is pure. By our assumption, it splits and (i) follows.

To show (ii) \(\Rightarrow\) (iii), let us assume first that \(\text{Pext}(X,G) = 0\) for all groups \(X \in \mathfrak{C}\). Then the proof of \[L1\] Theorem 4.3 shows that \(G\) is isomorphic to \(\mathbb{R}^n \oplus T^m \oplus G'\) where \(n\) is a nonnegative integer, \(m\) is a cardinal and \(G'\) is totally disconnected. Notice that \(G'/bG'\) is discrete (cf. \[HR\] (9.26)(a)) and torsion-free. Since the sequence

\[
0 = \text{Hom}((\mathbb{Q}/\mathbb{Z})^\wedge, G'/bG') \to \text{Ext}(\mathbb{Z}, G'/bG') \to \text{Ext}(\hat{\mathbb{Q}}, G'/bG') = 0
\]

is exact, \(G'/bG'\) is isomorphic to \(\text{Ext}(\mathbb{Z}, G'/bG') = 0\) and therefore \(G' = bG'\). It follows that the dual group of \(G'\) is totally disconnected (cf. \[HR\] Theorem 24.17), thus \(G'\) is a topological torsion group. Suppose that \(\text{Pext}(X,G) = 0\) for all \(X \in \mathfrak{L}\) and let \(K\) be a compact open subgroup of \(G'\). Then \(G'/K\) is a divisible group (see \[Fu2\] Theorem 7 or the proof of \[L1\] Theorem 4.1), so by Lemma 2.6 \(G'\) is densely divisible. Now assume
that $G'$ has a pure compact open subgroup $A$. Since $A$ is algebraically compact, it is a direct summand of $G'$. But then $A$ is divisible, hence connected (see [HR] Theorem 24.25) and therefore $A = 0$. Consequently, (ii) implies (iii).

Finally, (iii) $\not\Rightarrow$ (ii) because for instance, there is a nonsplitting extension of $\mathbb{Z}(p^\infty)$ by a compact group (cf. [A] Example 6.4).

Those groups in $\mathcal{C}$ which are pure injective in $\mathcal{L}$ are completely determined:

**Corollary 2.8.** A group $G$ in $\mathcal{C}$ is pure injective in $\mathcal{L}$ if and only if $G \cong \mathbb{R}^n \oplus T^m$ where $n$ is a nonnegative integer and $m$ is a cardinal.

**Proof.** The assertion follows immediately from [M] Theorem 3.2 and the above theorem.

The following lemma will be needed.

**Lemma 2.9.** Every finite subset of a reduced torsion group $A$ can be embedded in a finite pure subgroup of $A$.

**Proof.** By [F] Theorem 8.4, it suffices to assume that $A$ is a reduced $p$-group. But then the assertion follows from [K] p. 23, Lemma 9 and an easy induction.

A pure extension $0 \to A \to B \to C \to 0$ with discrete torsion group $A$ and compact group $C$ need not split, as [A] Example 6.4 illustrates. Our next result shows that no such example can occur if $A$ is reduced.

**Proposition 2.10.** Suppose $A$ is a discrete reduced torsion group. Then $\text{Pext}(X, A) = 0$ for all compactly generated groups $X$ in $\mathcal{L}$.

**Proof.** Suppose $E : 0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} X \to 0$ represents an element of $\text{Pext}(X, A)$ where $A$ is a discrete reduced torsion group and $X$ is a compactly generated group in $\mathcal{L}$. By [FG2] Theorem 2.1, there is a compactly generated subgroup $C$ of $B$ such that $\psi(C) = X$. If we set $A' = \phi(A)$, then $A' \cap C$ is discrete, compactly generated and torsion, hence finite, so by Lemma 2.9 $A'$ has a finite pure subgroup $F$ containing $A' \cap C$. Now set $C' = C + F$. Then $F$ is a pure subgroup of $C'$ because it is pure in $B$. But then $F$ is topologically pure in $C'$ since $C'$ is compactly generated. By [L1] Theorem 3.1, there is a closed subgroup $Y$ of $C'$ such that $C' = F \oplus Y$. We have $B = A' + C = A' + C' = A' + Y$ and $A' \cap Y = C' \cap A' \cap Y = (F + C) \cap A' \cap Y = [F + (C \cap A')] \cap Y = F \cap Y = 0$, thus $B$ is an algebraic direct sum of $A'$ and $Y$. Since $Y$ is compactly generated, it is $\sigma$-compact, so by [FG1] Corollary 3.2 we obtain $B = A' \oplus Y$. Consequently, the extension $E$ splits.

**Theorem 2.11.** Let $G$ be a group in $\mathcal{C}$. Then we have:
(i) \( \text{Pext}(X, G) = 0 \) for all \( X \in \mathcal{C} \) if and only if \( G \cong \mathbb{R}^n \oplus \mathbb{T}^m \oplus A \oplus B \) where \( n \) is a nonnegative integer, \( m \) is a cardinal, \( A \) is a direct product of finite cyclic groups and \( B \) is a discrete bounded group.

(ii) \( \text{Pext}(G, X) = 0 \) for all \( X \in \mathcal{C} \) if and only if \( G \cong \mathbb{R}^n \oplus C \oplus D \) where \( n \) is a nonnegative integer, \( C \) is a compact torsion group and \( D \) is a discrete direct sum of cyclic groups.

Proof. Suppose \( G \in \mathcal{C} \) and \( \text{Pext}(X, G) = 0 \) for all \( X \in \mathcal{C} \). By the proof of part (ii) \( \Rightarrow \) (iii) of Theorem 2.7, \( G \) is isomorphic to \( \mathbb{R}^n \oplus \mathbb{T}^m \oplus A \oplus B \) where \( A \) is a compact totally disconnected group and \( B \) is a discrete torsion group. By Lemma 2.3, we have \( \text{Pext}(\hat{A}, X) \cong \text{Pext}(\hat{A}, \hat{X}) = 0 \) for all discrete groups \( X \), hence \( \hat{A} \) is a direct sum of cyclic groups (see [F] Theorem 30.2) and it follows that \( A \) is a direct product of finite cyclic groups. Again, we make use of [A] Example 6.4 and conclude that \( B \) is reduced. But then \( B \) is bounded since it is torsion and cotorsion. Conversely, suppose \( G \) has the form \( \mathbb{R}^n \oplus \mathbb{T}^m \oplus A \oplus B \) as in the theorem and let \( X = \mathbb{R}^m \oplus Y \oplus Z \) where \( Y \) is a compact group and \( Z \) is a discrete group. Then \( \text{Pext}(X, A) \cong \text{Pext}(\hat{A}, \hat{X}) \cong \text{Pext}(\hat{A}, (\hat{X})_d) = 0 \). By Theorem 2.1, Proposition 2.10 and [F] Theorem 27.5 we have

\[
\text{Pext}(X, B) \cong \text{Pext}(\mathbb{R}^m, B) \oplus \text{Pext}(Y, B) \oplus \text{Pext}(Z, B) = 0
\]

and conclude that

\[
\text{Pext}(X, G) \cong \text{Pext}(X, \mathbb{R}^n \oplus \mathbb{T}^m) \oplus \text{Pext}(X, A) \oplus \text{Pext}(X, B) = 0.
\]

Finally, the second assertion follows from Lemma 2.3 and duality. \( \square \)

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Department of Mathematics, Sacred Heart University, 5151 Park Avenue, Fairfield, Connecticut 06825, USA

E-mail address: lothp@sacredheart.edu