A REMARK ON A CONJECTURE OF BORWEIN AND CHOI

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Abstract. We prove the remaining case of a conjecture of Borwein and Choi concerning an estimate on the square of the number of solutions to \( n = x^2 + Ny^2 \) for a squarefree integer \( N \).

1. Introduction

We consider the positive definite quadratic form \( Q(x, y) = x^2 + Ny^2 \) for a squarefree integer \( N \). Let \( r_{2,N}(n) \) denote the number of solutions to \( n = Q(x, y) \) (counting signs and order). In this note, we estimate \( \sum_{n \leq x} r_{2,N}(n)^2 \).

A positive squarefree integer \( N \) is called solvable (or more classically “numerus idoneus”) if \( x^2 + Ny^2 \) has one form per genus. Note that this means the class number of the form class group of discriminant \(-4N\) equals the number of genera, \( 2t \), where \( t \) is the number of distinct prime factors of \( N \). Concerning \( r_{2,N}(n) \), Borwein and Choi [1] proved the following:

Theorem 1.1. Let \( N \) be a solvable squarefree integer. Let \( x > 1 \) and \( \epsilon > 0 \). We have
\[
\sum_{n \leq x} r_{2,N}(n)^2 = \frac{3}{N} \left( \prod_{p|2N} \frac{2p}{p+1} \right) x \log x + \alpha(N)x + O(N\frac{1}{4}x^{\frac{3}{4}+\epsilon})
\]
where the product is over all primes dividing \( 2N \) and
\[
\alpha(N) = -1 + 2\gamma + \sum_{p|2N} \frac{\log p}{p+1} + \frac{2L'(1, \chi_{-4N})}{L(1, \chi_{-4N})} - \frac{12}{\pi^2} s'(2)
\]
where \( \gamma \) is the Euler-Mascheroni constant and \( L(1, \chi_{-4N}) \) is the L-function corresponding to the quadratic character mod \(-4N\).

Based on this result, Borwein and Choi posed the following:

Conjecture 1.2. For any squarefree \( N \),
\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p|2N} \frac{2p}{p+1} \right) x \log x
\]

The main result in [10] was the following.

Theorem 1.3. Let \( Q(x, y) = x^2 + Ny^2 \) for a squarefree integer \( N \) with \(-N \not\equiv 1 \mod 4\). Let \( r_{2,N}(n) \) denote the number of solutions to \( n = Q(x, y) \) (counting signs and order). Then
\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p|2N} \frac{2p}{p+1} \right) x \log x.
\]

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In this note, we settle the conjecture in the remaining case, namely

**Theorem 1.4.** For $-N \equiv 1 \mod 4$, we have

$$
\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p|2N} \frac{2p}{p+1} \right) x \log x.
$$

2. Preliminaries

Let $Q(x, y) = ax^2 + bxy + cy^2$ denote a positive definite integral quadratic form with discriminant $D = b^2 - 4ac$ and $gcd(a, b, c) = 1$. Given $Q$, let $\kappa$ be the largest positive integer with $D/\kappa^2$ an integer congruent to 0 or 1 modulo 4. We call $\kappa$ the *conductor* of $Q$ and set $\kappa = D/\kappa^2$. Let $r(Q, n)$ be the number of representations of the integer $n$ by the form $Q$. We now relate $r(Q, n)$ to counting the number of integral ideals of norm $n$ in a given class in a generalized ideal class group.

Given $D = \kappa^2 d$ we consider ideals in $\mathcal{O}_K$ where $K = \mathbb{Q}(\sqrt{d})$. Let $I_\kappa$ be the group of fractional ideals of $\mathcal{O}_K$ which are quotients of ideals coprime to $\kappa$ and $P_\kappa$ be the subgroup of fractional ideals which are quotients of principal ideals $\langle \alpha \rangle \in I_\kappa$ where $\alpha \in \mathbb{Z} + \kappa \mathcal{O}$. Then set $\text{CL}_\kappa(K) = I_\kappa/P_\kappa$. The elements of $\text{CL}_\kappa(K)$ correspond bijectively to proper equivalence classes of positive definite quadratic forms of discriminant $D = \kappa^2 d$. If the proper equivalence class of $Q$ corresponds to the ideal class $\mathfrak{c}$, then by [3], page 219, we have

$$
r(Q, n) = \sum_{r|\kappa} w((\kappa/r)^2 d) J(\mathfrak{c}, n/r^2)
$$

where

$$
w(D) = \begin{cases} 6 & \text{if } D = -3 \\ 4 & \text{if } D = -4 \\ 2 & \text{otherwise.} \end{cases}
$$

Also $J(\mathfrak{c}, n)$ is the number of integral ideals of norm $n$ in the class $\mathfrak{c}$, where $\mathfrak{c}$ is the image of $\mathfrak{c}$ under the natural homomorphism $\text{CL}_\kappa(K) \to \text{CL}_\kappa/K(K)$. For the form $Q(x, y) = x^2 + Ny^2$ where $-N \equiv 1 \mod 4$, the conductor $\kappa = 2$ and so we have

$$
r_{2,N}(n) = w(-4N)J(\mathfrak{c}, n) + w(-N)J(\mathfrak{c}_2, n/4) = 2J(\mathfrak{c}, n) + w(-N)J(\mathfrak{c}_2, n/4)
$$

where $\mathfrak{c}_2$ is the image under $\text{CL}_2(K) \to \text{CL}_1(K)$, that is, $\mathfrak{c}_2$ is a class in the ideal class group of $K = \mathbb{Q}(\sqrt{-N})$.

We now discuss a classical result of Rankin [11] and Selberg [12] which estimates the size of Fourier coefficients of a modular form. Specifically, if $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i nz}$ is a nonzero cusp form of weight $k$ on $\Gamma_0(N)$, then

$$
\sum_{n \leq x} |a(n)|^2 = \alpha(f, f)x^k + O(x^{k-\delta})
$$

where $\alpha > 0$ is an absolute constant and $\langle f, f \rangle$ is the Petersson scalar product. In particular, if $f$ is a cusp form of weight 1, then $\sum_{n \leq x} |a(n)|^2 = O(x)$. One can adapt their result to say the following. Given two cusp forms of weight $k$ on a suitable congruence subgroup of $\Gamma = SL_2(\mathbb{Z})$, say $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i nz}$ and $g(z) = \sum_{n=1}^{\infty} b(n)e^{2\pi i nz}$, then
\[
\sum_{n \leq x} a(n)b(n)n^{1-k} = Ax + O(x^{\frac{3}{5}})
\]

where \(A\) is a constant. In particular, if \(f\) and \(g\) are cusp forms of weight 1, then
\[
\sum_{n \leq x} a(n)b(n) = O(x).
\]

We conclude this section with a relationship between genus characters of generalized ideal class groups and the poles of the Rankin-Selberg convolution of L-functions. Recall that a group homomorphism \(\chi : I_2 \to S^1\) is an ideal class character if it is trivial on \(P_2\), i.e.
\[
\chi((a)) = 1
\]
for \(a \equiv 1\) mod \(2\). Thus an ideal class character is a character on the generalized class group \(I_2 \backslash P_2\). Recall also that a genus character (see Chapter 12, section 5 in [5]) is an ideal class character of order at most two.

Let us also recall the notion of the Rankin-Selberg convolution of two L-functions. For squarefree \(N\), consider two ideal class characters \(\chi_1, \chi_2\) for \(CL_2(K)\), the generalized ideal class group of \(K = \mathbb{Q}(\sqrt{-N})\) and their associated Hecke L-series
\[
L_2(s, \chi_1) = \sum_{(a,2)=1} \frac{\chi_1(a)}{N(a)^s}
\]
\[
L_2(s, \chi_2) = \sum_{(a,2)=1} \frac{\chi_2(a)}{N(a)^s}
\]
which converge absolutely in some right half-plane. We form the convolution L-series by multiplying the coefficients,
\[
L_2(s, \chi_1 \otimes \chi_2) = \sum_{(a,2)=1} \frac{\chi_1(a)\chi_2(a)}{N(a)^s}
\]
The following result describes a relationship between genus characters \(\chi\) and the orders of poles of \(L_2(s, \chi \otimes \chi)\). The proof is similar to that of Proposition 2.4 in [10].

**Proposition 2.1.** Let \(\chi\) be an ideal class character for \(CL_2(K)\), \(-N \equiv 1\) mod 4, and \(L_2(s, \chi)\) the associated Hecke L-series. Then \(\chi\) is a genus character if and only if \(L_2(s, \chi \otimes \chi)\) has a double pole at \(s = 1\).

**Remark 2.2.** By Proposition 2.1, if \(\chi\) is a non-genus character, then \(L_2(s, \chi \otimes \chi)\) has at most a simple pole at \(s = 1\).

3. **Proof of Theorem 1.4**

Proof. As the proof is similar to that of Theorem 1.3 in [10], we sketch the relevant details. If \(-N \equiv 1\) mod 4, then the discriminant of \(K = \mathbb{Q}(\sqrt{-N})\) is \(-N\). We also assume that \(t\) is the number of distinct prime factors of \(N\) and so the discriminant \(-N\) also has \(t\) distinct prime factors. For \(K = \mathbb{Q}(\sqrt{-N})\), consider the zeta function
\[
\zeta_K(s, 2) = \sum_{(a,2)=1} \frac{1}{N(a)^s}
\]
where the sum is over those ideals \(a\) of \(O_K\) prime to 2. We now split up \(\zeta_K(s, 2)\), according to the classes \(c_i\) of the generalized ideal class group \(CL_2(K)\), into the partial zeta functions (see page 161 of [7]).
\[
\zeta_c(s) = \sum_{a \in c_i} \frac{1}{N(a)^s}
\]
so that \(\zeta_K(s, 2) = \sum_{i=0}^{h_2-1} \zeta_{c_i}(s)\) where \(h_2\) is the order of \(CL_2(K)\).

Let \(c\) be the ideal class in \(CL_2(K)\) which corresponds to the proper equivalence class of \(Q(x, y) = x^2 + Ny^2\). Now let \(\chi\) be an ideal class character of \(CL_2(K)\) and consider the Hecke L-series for \(\chi\), namely

\[
L_2(s, \chi) = \sum_{(a, 2)=1} \frac{\chi(a)}{N(a)^s}.
\]

We may now rewrite the Hecke L-series as

\[
L_2(s, \chi) = \sum_{i=0}^{h_2-1} \chi(c_i) \zeta_{c_i}(s).
\]

And so summing over all ideal class characters of \(CL_2(K)\), we have

\[
\sum_{\chi} \chi(c)L_2(s, \chi) = \sum_{i=0}^{h_2-1} \zeta_{c_i}(s) \left( \sum_{\chi} \chi(c_i) \right).
\]

The inner sum is nonzero precisely when \(c = c_i\). Thus we have

\[
\zeta_c(s) = \frac{1}{h_2} \sum_{\chi} \chi(c)L_2(s, \chi)
\]
and so

\[
\zeta_c(s) = \frac{1}{h_2} \left( L_2(s, \chi_0) + \chi_1(c)L_2(s, \chi_1) + \cdots + \chi_{h_2-1}(c)L_2(s, \chi_{h_2-1}) \right).
\]

As \(\chi_0\) is the trivial character, \(L_2(s, \chi_0) = \zeta_K(s, 2)\). Comparing \(n^{th}\) coefficients, we have

\[
J(c, n) = \frac{1}{h_2} (a_n + b_1(n) + \cdots + b_{h_2-1}(n)).
\]

where \(a_n\) is the number of integral ideals of \(O_K\) prime to 2 and of norm \(n\) and the \(b_i\)'s are coefficients of weight 1 cusp forms (see [2]). Recall we also have

\[
r_{2, N}(n) = 2J(c, n) + w(-N)J(c_2, n/4)
\]
and so

\[
r_{2, N}(n) = \frac{2}{h_2} \left( a_n + b_1(n) + \cdots + b_{h_2-1}(n) \right) + w(-N)J(c_2, n/4).
\]

Thus

\[
\sum_{n \leq x} r_{2, N}(n)^2 = \frac{4}{h_2^2} \left( \sum_{n \leq x} a_n^2 + \sum_{n \leq x} b_i(n)^2 + 2 \sum_{n \leq x} a_n b_i(n) + \sum_{i \neq j} b_i(n) b_j(n) \right) + \frac{4}{h_2^2} \sum_{n \leq x} \left( a_n + b_1(n) + \cdots + b_{h_2-1}(n) \right) w(-N)J(c_2, n/4) + \sum_{n \leq x} w(-N)^2 J(c_2, n/4)^2.
\]
Assume $-N \equiv 1 \mod 8$. Applying the main theorem in [6] to the Dirichlet series
$$\sum_{n=1}^{\infty} \frac{a_n^2}{n^s},$$
we obtain
$$\sum_{n \leq x} a_n^2 \sim A x \log x$$
where $A = \frac{1}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p \mid N} \frac{p}{p+1}$. As $-N$ has $t$ distinct prime factors, we have $2^t$ genus characters for $\text{CL}(K)$ where $K = \mathbb{Q}(\sqrt{-N})$. By [7] (see Theorem 1, page 127), we have $2^t$ genus characters for $\text{CL}_2(K)$. We now must estimate
$$\sum_{n \leq x} b_i(n)^2.$$ Let us now assume that the first $2^t - 1$ terms arise from L-functions associated to genus characters. By Proposition 2.1 and an application of Perron’s formula, we obtain
$$\sum_{n \leq x} b_i(n)^2 \sim A x \log x.$$ As this estimate holds for each $i$ such that $1 \leq i \leq 2^t - 1$, the term $Ax \log x$ appears $2^t$ times in the estimate of $\sum_{n \leq x} r_{2,N}(n)^2$. By Remark 2.2 and the Rankin-Selberg estimate, the remaining terms are all $O(x)$. Thus
$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{4}{h_2^2} \left( 2^t \frac{1}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p \mid N} \frac{p}{p+1} \right) x \log x.$$ By [6], we have $L(1, \chi_{-N}) = \frac{h_2}{\sqrt{N}}$ where $h$ is the class number of $K$ and $h_2 = h$. Thus
$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) x \log x.$$ For $-N \equiv 5 \mod 8$, we have $h_2 = 3h$ and again by [6],
$$\sum_{n \leq x} a_n^2 \sim \left( \frac{9}{2\pi^2} L(1, \chi_{-N})^2 \prod_{p \mid N} \frac{p}{p+1} \right) x \log x.$$ Thus
$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) x \log x.$$

**Remark 3.1.** We would like to mention another approach which confirms Theorems 1.3 and 1.4. Let $Q \in \mathbb{Z}^{2 \times 2}$ be a non-singular symmetric matrix with even diagonal entries and $q(x) = \frac{1}{2}Q|x| = \frac{1}{2}x^T Qx$, $x \in \mathbb{Z}^2$, the associated quadratic form in two variables. Let $r(Q, n)$ denote the number of representations of $n$ by the quadratic form $Q$. Now consider the theta function
$$\theta_Q(z) = \sum_{x \in \mathbb{Z}^2} e^{\pi iz Q|x|}.$$ The Dirichlet series associated with the automorphic form $\theta_Q$ is
\[
(4\pi)^{-1/2} \zeta_Q\left(\frac{1}{2} + s\right)
\]

where

\[
\zeta_Q(s) = \sum_{n=1}^{\infty} \frac{r(Q, n)}{n^s} = \sum_{\mathbf{x} \in \mathbb{Z}^2 \setminus \{0\}} q(x)^{-s}
\]

for \(\Re(s) > 1\). A careful and involved application of the Rankin-Selberg method to the above Dirichlet series (see Theorems 2.1 and 5.1 in [8] and Theorem 5.2 in [9]) combined with a Tauberian argument yields the following (see Theorem 6.1 in [8])

\[
\sum_{n \leq x} r(Q, n)^2 \sim A_Q x \log x
\]

where

\[
A_Q = 12 \frac{A(q)}{q} \prod_{p|q} \left(1 + \frac{1}{p}\right)^{-1}.
\]

Here \(q = \det Q\) and \(A(q)\) denotes the multiplicative function defined by

\[
A(p^e) = 2 + \left(1 - \frac{1}{p}\right)(e - 1)
\]

where \(p\) is an odd prime, \(e \geq 1\), and

\[
A(2^e) = \begin{cases} 
1 & \text{if } e \leq 1, \\
2 & \text{if } e = 2, \\
e - 1 & \text{if } e \geq 3.
\end{cases}
\]

Let us now turn to our situation. Consider \(q(x) = x^2 +Ny^2 = \frac{1}{2}x^TQx\) where \(Q = \begin{pmatrix} 2 & 0 \\
0 & 2N \end{pmatrix}\), \(N\) squarefree. Thus \(q = 4N\). Suppose \(N\) has \(t\) distinct prime factors. Then \(A(4N) = 2^{t+1}\) and so

\[
A_Q = \frac{3}{N} 2^{t+1} \prod_{p|2N} \left(1 + \frac{1}{p}\right)^{-1} = \frac{3}{N} \prod_{p|2N} \frac{2p}{p+1}.
\]

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