ON THE FUNCTIONAL EQUATION $Af^2 + Bg^2 = 1$ ON THE FIELD OF COMPLEX $p$-ADIC NUMBERS

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Abstract. For a fixed prime $p$, let $\mathbb{C}_p$ denote the complex $p$-adic numbers. For polynomials $A, B \in \mathbb{C}_p[x]$ we consider decompositions $A(x)f^2(x) + B(x)g^2(x) = 1$ of entire functions $f, g$ on $\mathbb{C}_p$ and try to improve an impossibility result due to A. Boutabaa concerning transcendental $f, g$. We also provide an independent proof of a $p$-adic diophantic statement due to D. N. Clark, which is an important ingredient of Boutabaa’s method.

1. Introduction

For a prime $p$, let $\mathbb{C}_p$ denote the field of complex $p$-adic numbers, and let $\mathbb{Q} \subset \mathbb{C}_p$ be the field of algebraic numbers over $\mathbb{Q}$. Furthermore, we denote by $\mathcal{A}(\mathbb{C}_p)$ the ring of entire functions on $\mathbb{C}_p$. For a field $K$ we denote by $K(x)$ the ring of rational functions over $K$, and $K[x]$ the ring of polynomials.

A. Boutabaa [1] has shown:

**Theorem 1.1.** Let $A, B \in \mathbb{Q}(x)$ be not identically zero. Suppose that $f, g \in \mathcal{A}(\mathbb{C}_p)$ have coefficients in $\mathbb{Q}$ and satisfy

$$A(x)f^2(x) + B(x)g^2(x) = 1. \quad (1.1)$$

Then $f, g \in \mathbb{C}_p[x]$.

The main aim of this paper is to provide a self-contained proof of the following more general result in the case that $A, B$ are polynomials:

**Theorem 1.2.** Let $A, B \in \mathbb{Q}[x]$ be not identically zero. Suppose that $f, g \in \mathcal{A}(\mathbb{C}_p)$ satisfy (1.1) and $g(0) \neq 0$. If $f^{(i)}(0) \in \mathbb{Q}, \, (0 \leq i \leq \deg A + \deg B - 1)$ and $g^{(i)}(0) \in \mathbb{Q}, \, (0 \leq i < \frac{\deg A + \deg B - 1}{2})$, then $f, g \in \mathbb{C}_p[x]$.

When $\deg A$ and $\deg B$ do not have the same parity, we have the following stronger conclusion:

**Remark 1.3.** Let $A, B$ be in $\mathbb{C}_p[x]$ such that $\deg A \neq \deg B \mod 2$. Then equation (1.1) has no solutions in $(\mathcal{A}(\mathbb{C}_p))^2 \setminus \mathbb{C}_p^2$.

We note that the method of this paper allows for a similar assertion as that of Theorem 1.2 in the more abstract setting, when $\mathbb{C}_p$ is replaced by an algebraically closed and topologically complete ultrametric field $K$ of characteristic zero. This purely academic generalization however does not give more insight into the problem but only complicates notation and proofs. On the other hand, the case of $K$ with non-zero characteristic clearly makes no sense here.

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1.1. History. The problem of decomposing complex meromorphic functions $f, g$ in the form $A(x)f^n + B(x)g^m = 1$, where $A, B$ are certain meromorphic coefficients, has been thoroughly studied in the sixties and seventies of the last century. F. Gross [6] shows that $f^n + g^n = 1$ has no meromorphic solutions, when $n > 3$ and no entire solutions, if $n > 2$. Moreover, for $n = 2$ he characterizes all meromorphic solutions $f, g$. Generalizations of these results are due to C.-C. Yang [13] and N. Toda [12], and they are essentially applications of Nevanlinna Theory [8], in particular of the second Nevanlinna Theorem. To see the match with our work, let us specialize C.-C. Yang’s quite general result [13] as follows:

**Theorem 1.4.** For non-constant complex entire functions $f, g$, the functional equation

$$A(x)f^m(x) + B(x)g^n(x) = 1,$$

(1.2)

where $A, B \in \mathbb{C}[x]$ cannot hold, unless $n \leq 2, m \leq 3$.

Note that in the case $n = m = 2, A = B = 1$, a well known pair of entire solutions are the sine and cosine.

In the $p$-adic domain, A. Boutabaa considers decompositions of this kind in [1]. Building on an improved version of the $p$-adic second Nevanlinna Theorem [3] it could be shown that compared with Theorem 1.4, the respective $p$-adic results are stronger:

**Theorem 1.5.** For non-constant $p$-adic entire functions $f, g \in \mathcal{A}(\mathbb{C}_p)$, the functional equation (1.2) cannot hold with $A, B \in \mathbb{C}_p[x]$, unless $n = m = 2$.

Note, that the pair of $p$-adic functions $\sin(x), \cos(x)$ is not entire, see also Example 3.2. This shortcoming motivates A. Boutabaa’s “impossibility result” Theorem 1.5 when $n = m = 2$. Since the $p$-adic Nevanlinna Theory is not applicable in this case, Boutabaa employs a Diophantic approximation result for algebraic numbers due to D.N.Clark [4].

1.2. Program of paper. We start by recalling some elementary facts on entire functions and diophantic approximations of algebraic numbers in section 2. We then state a new proof of Clark’s result (Proposition 2.5, see also Remark 2.6). In the final section 3 we present the proof of Theorem 1.2.

2. Preliminaries

**Notation 2.1.**
- $|.|_p$ denotes the $p$-adic absolute value on the field of complex $p$-adic numbers $\mathbb{C}_p$, and the respective additive valuation is given by $\text{ord}_p(x) := -\log_p(|x|_p)$, where $\log_p(\cdot)$ is the logarithm with base $p$.
- The “closed” disk with center $a \in \mathbb{C}_p$ and radius $r > 0$ equals $d(a, r) := \{x \in \mathbb{C}_p : |x - a| \leq r\}$.
- We denote by $\mathbb{Z}_p$ the ring of $p$-adic integers, which is the topological completion of $\mathbb{N}$ with respect to the metric induced by the $p$-adic absolute value $|.|_p$.
- For $\xi \in \mathbb{C}_p$ we define the hypergeometric coefficient $(\xi)_k$ inductively as $(\xi)_1 := \xi, \ (\xi)_k := (\xi)_{k-1}(\xi - k + 1), \ \text{when} \ k > 1$. 


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- For a natural number \(n\), \(\sigma(n)\) is the sum of the digits in its expansion as a \(p\)-adic integer and the factorial satisfies
  \[
  \text{ord}_p(n!) = \frac{n - \sigma_p(n)}{p - 1},
  \]
  (2.1)
  see [10], p. 241.

- For a set \(D\), \(|D|\) denotes its cardinality. Furthermore, for a real number \(x\), \([x]\) denotes its integral part. Accordingly, \(\langle x \rangle := x - [x]\) is the fractional part of \(x\).

- Finally, we denote by \(A(C_p)\) the algebra of entire functions on \(C_p\), that is the set of formal power series \(f(x) = \sum_{i \geq 0} c_i x^i\) with coefficients \(c_i\) in \(C_p\) for any \(i \geq 0\) such that either \(\{i \in N : c_i \neq 0\}\) is finite (that is \(f \in C_p[x]\)), or such that \(\forall \lambda \in R : \lim_{n \to \infty} \text{ord}_p(c_n) - \lambda n = \infty\).

- \(\overline{Q}\) is the algebraic closure of \(Q \subset C_p\).

It is well known, that for any \(r > 0\), \(A(C_p)\) can be endowed by a multiplicative ultrametric norm \(|| \cdot ||(r)\) defined as
\[
||f||(r) := \max_{i \geq 0} |c_i|_p r^i = \max_{x \in d(0, r)} |f(x)|_p.
\]
Also, as in classical complex analysis, we have “Liouville’s Theorem”: If for \(f \in A(C_p)\), \(||f||(r)\) is bounded as \(r \to \infty\), then \(f \in C_p\). Indeed, this is an immediate consequence of the following useful result on entire functions [7]:

**Lemma 2.2.** For any \(f \in A(C_p)\) there exists \(\alpha_0 \in C_p\) and \(r \in N\) such that \(f\) can be decomposed as
\[
\alpha_0 x^r \prod_{f(\alpha) = 0} \left(1 - \frac{x}{\alpha}\right), \alpha_0 \in C_p, r \in N.
\]

By means of this result we can prove the assertion of Remark 1.3.

**Proof.** Let \(a_s, b_t\) be the leading coefficients of \(A, B\) respectively and let \(d(0, R)\) be a disk containing all zeros of \(A\) and \(B\). It is well known, that when \(|x| > R\), we have \(|\text{det}(x)| = |a_s||x|^s, |B(x)| = |b_t||x|^t\). Consider now \(\Gamma := \{x \mid r_1 < |x| < r_2\}\) with \(r_2 > r_1 > R\) such that \(f, g\) have no zero inside \(\Gamma\). Let \(k\) be the number of zeros of \(f\) and let \(l\) be the number of zeros of \(g\) in \(d(0, r_1)\). So the number of zeros of \(f^2\) (resp. \(g^2\)) is \(2k\) (resp. \(2l\)). Due to Lemma 2.2 \(|f(x)|\) is of the form \(|x^s|\) inside \(\Gamma\) and \(|g(x)|\) is of the form \(|x^t|\) inside \(\Gamma\). Therefore \(|A(x)f^2(x)| = |B(x)g^2(x)|\) is of the form \(|a_x x^{s+2k}l, |B(x)g^2(x)|\) is of the form \(|b_x x^{t+2l}|\). Since \(Af^2 + Bg^2 = 1\), the two functions \(|a_x x^{s+2k}|l, |b_x x^{t+2l}|\) must be equal in \([r_1, r_2]\). This contradicts that \(s, t\) have different parity. \(\square\)

The \(p\)-adic version of Liouville’s Theorem (and improvements of it, e.g. the \(p\)-adic Thue-Siegel-Roth Theorem, [9]) is well known:

**Lemma 2.3.** For any \(\alpha \in d(0, 1) \cap \overline{Q} \setminus N\) there exists a constant \(C\) such that for all \(n \in N\),
\[
\text{ord}_p(\alpha - n) \leq C + k \log_p(n),
\]
where \(k\) is the degree of \(\alpha\) over \(Q\).

We prepare Clark’s statement with the following elementary observation:
Proposition 2.4. For any $s \in \mathbb{N}$ we have

$$\lim_{n \to \infty} \frac{1}{N} \sum_{0 \leq j \leq N, \ord_p(j) \leq s} \ord_p(j) = \frac{1-p^{-s}}{p-1} - \frac{s}{p^{s+1}}.$$ 

Proof. For $i \in \mathbb{N}$ we introduce the sets $L_i^N := \{ k \leq N : p^i \mid k, p^{i+1} \nmid k \}$. Then the power of this set is $|L_i^N| = \left[ \frac{N}{p^i} \right] - \left[ \frac{N}{p^{i+1}} \right]$. For any pair $i \neq j$, $L_i^N$ and $L_j^N$ are disjoint, and $\bigcup_{i=0}^s L_i^N = \{ j \mid j \leq N \}$. Also note that for $k \in L_i^N$, we have $\ord_p(k) = i$. Therefore, due to $\sum_{j \leq N} \ord_p(j) = \sum_{i=0}^s |L_i^N|(i)$ we have

$$\lim_{n \to \infty} \frac{1}{N} \sum_{0 \leq j \leq N} \ord_p(j) = \lim_{n \to \infty} \frac{1}{N} \sum_{i=0}^s \left( \frac{N}{p^i} - \frac{N}{p^{i+1}} \right)(i) = \sum_{i=0}^s \left( \frac{1}{p^i} - \frac{1}{p^{i+1}} \right)(i) = \frac{1-p^{-s}}{p-1} - \frac{s}{p^{s+1}}.$$ 

\[\square\]

The following Lemma is essentially due to D. N. Clark ([4]). However taking into account a comment by M. Setoyanagi ([11]) on Clark's conclusions we find it advisable to include an independent proof of this result. Compared with the original formulation in [4], also numbers not in $d(0,1)$ are involved, and the notion 'non-Liouville number' is avoided.

Lemma 2.5. If $\alpha \in \mathbb{C}_p$ is algebraic over $\mathbb{Q}$ or $\alpha \notin \mathbb{Z}_p$, then for sufficiently large $m$

$$\lim_{N \to \infty} \frac{\sum_{i=m}^N \ord_p(\alpha - i)}{N} = w(\alpha) \tag{2.2}$$

where $w(\alpha)$, the so-called 'weight' of $\alpha$, is given by

$$w(\alpha) = \begin{cases} 
\frac{1}{p-1} - \langle r(\alpha) \rangle p^{-\langle r(\alpha) \rangle - 1} & \alpha \in d(0,1) \setminus \mathbb{Z}_p \\
\frac{1}{p-1} & \alpha \in \mathbb{Z}_p \\
\ord_p(\alpha) & \alpha \notin d(0,1)
\end{cases}$$

with $r(\alpha) := \sup_{i \geq 0} \ord_p(\alpha - i)$.

Proof. We distinguish 4 cases:

Case 1: $\alpha \in \mathbb{N}$. With $m = \alpha + 1$ we have $\ord_p(\prod_{j=m}^N (\alpha - j)) = \frac{N-m+1-\sigma(N-m+1)}{p-1}$. Since $\sigma(j) = O(\log(j))$ when $j \to \infty$, we obtain

$$\frac{1}{N} \sum_{j=m}^N \ord_p(\alpha - j) \to \frac{1}{p-1}$$

as $N \to \infty$.

Case 2: $\alpha \in \mathbb{Z}_p \setminus \mathbb{N}$. We first establish lower bounds for the sum (2.2). For every natural number $\beta > N$ one has

$$\left| \frac{\beta(\beta-1)\ldots(\beta-N)}{(N+1)!} \right|_p \leq 1.$$ 

Since $\mathbb{N}$ is dense in $\mathbb{Z}_p$ and due to the ultrametric rule "the strongest one wins", for any natural number $N$ we can find some $\alpha_N \in \mathbb{N}$ such that $\alpha_N > N$ and
Due to Lemma 2.3 there exists $|\alpha_N - l|_p = |\alpha - l|_p$, for all $0 \leq i \leq N$. Therefore,

$$\left| \frac{\alpha(\alpha-1) \ldots (\alpha-N)}{(N+1)!} \right|_p \leq 1$$

and due to formula (2.1) we infer

$$\lim \inf_{N \to \infty} \frac{1}{N} \sum_{j=m}^{N} \text{ord}_p(\alpha - j) \geq \frac{1}{p-1}.$$  

Upper bounds for this sum can be achieved as follows. By induction it holds that

$$\frac{(-1)^{n+1}}{\alpha(\alpha-1) \ldots (\alpha-n)} = \sum_{i+j=n} (-1)^{i+1} i!j!(\alpha - j).$$  

(2.3)

Multiplying (2.3) by $n!$ and evaluating with respect to the norm $| \cdot |_p$ we conclude from $|\binom{n}{i}|_p = \frac{n!}{\prod_{i=0}^{\max(i-n)} |\alpha - j|_p}$ that

$$\left| \frac{n!}{(\alpha)_n} \right|_p \leq \max_{0 \leq j \leq n} \frac{1}{|\alpha - j|_p}.$$  

(2.4)

Due to Lemma 2.3 there exists $C' \in \mathbb{R}^+$ such that for any natural number $j$, $|\alpha - j|_p \geq \frac{C'}{p^j}$ for $k = [Q(\alpha) : \mathbb{Q}]$. After inserting this inequality into (2.4) we take the logarithm to the base $p$ of the $n$-th root of (2.4). Monotonicity of the root yields and formula (2.1) yield

$$\frac{\text{ord}_p((\alpha)_{n+1})}{n} \leq \frac{1}{p-1} + \frac{k \log_p n - \sigma_p(n)}{n} - \frac{\log_p C'}{n}$$  

(2.5)

and taking limits we have

$$\lim_{N \to \infty} \frac{\sum_{i=0}^{N} \text{ord}_p(\alpha - i)}{N} \leq \frac{1}{p-1}.$$

**Case 3:** $\alpha \in d(0,1) \setminus \mathbb{Z}_p$. This is the most tricky part. First, we observe that $r(\alpha) = \sup_{\beta \geq 0} \text{ord}_p(\beta - j) = \max_{i \in \mathbb{Z}_p} \text{ord}_p(\alpha - i')$ exists in $\mathbb{R}$, since $\mathbb{Z}_p$ is compactly contained in $\mathbb{C}_p$. Moreover, the maximum is taken on by a natural number, that is, there exists $m \in \mathbb{N}$ such that $r(\alpha) = \text{ord}_p(\alpha - m)$. By setting $\beta := \frac{\alpha - m}{p^{r(\alpha)}}$, we obtain

$$r(\beta) = \sup_{\beta \geq 0} \text{ord}_p(\beta - j) = \sup_{\beta \geq 0} \text{ord}_p(\frac{\alpha - m}{p^{r(\alpha)}} - j) =$$

$$= \sup_{\beta \geq 0} \text{ord}_p(\alpha - m - j p^{r(\alpha)}) - [r(\alpha)] =$$

$$= \text{ord}_p(\alpha - m) - [r(\alpha)] = r(\alpha) - [r(\alpha)] = \langle r(\alpha) \rangle.$$  

For a fixed natural number $j > m$ we have

$$\text{ord}_p(\alpha - j) = \text{ord}_p(\beta + \frac{m - j}{p^{r(\alpha)}}) + [r(\alpha)] = \langle r(\alpha) \rangle +$$

$$+ \begin{cases} \langle r(\alpha) \rangle, & \text{ord}_p(\frac{m - j}{p^{r(\alpha)}}) \geq \langle r(\alpha) \rangle \\ \text{ord}_p(\frac{m - j}{p^{r(\alpha)}}), & \text{ord}_p(\frac{m - j}{p^{r(\alpha)}}) < \langle r(\alpha) \rangle \end{cases}.$$
The second case in the right hand side of the last formula follows from the ultrametric rule "the strongest one wins". Concerning the first one, we note that \( r(\beta) = \text{ord}_p(\beta) = \langle r(\alpha) \rangle \), hence by the non-archimedean triangle’s inequality, \( \text{ord}_p(\beta + \frac{m-j}{p^{r(\alpha)}}) \geq \langle r(\alpha) \rangle \). The reversed inequality holds due to the maximality of \( r(\beta) \). Rewriting the latter formula we receive

\[
\text{ord}_p(\alpha - j) = \begin{cases} r(\alpha), & \text{ord}_p(j - m) \geq \lfloor r(\alpha) \rfloor + 1 \\ \text{ord}_p(j - m), & \text{ord}_p(j - m) \leq \lfloor r(\alpha) \rfloor \end{cases}.
\]

It remains to verify (2.2). Since \( r(\alpha - m) = r(\alpha) \) we may assume without loss of generality \( m = 0 \). With this choice we obtain

\[
\sum_{j=1}^{N} \text{ord}_p(\alpha - j) = r(\alpha) \lfloor j \leq N \mid \text{ord}_p(j) \geq \lfloor r(\alpha) \rfloor + 1 \rfloor + \sum_{1 \leq j \leq N, \text{ord}_p(j) \leq \lfloor r(\alpha) \rfloor} \text{ord}_p(j).
\]

Note that \( \lfloor j \leq N \mid \text{ord}_p(j) \geq \lfloor r(\alpha) \rfloor + 1 \rfloor \) = \( \lfloor j \leq N \mid p^{\lfloor r(\alpha) \rfloor + 1} \mid j \rfloor \) = \( \left\lfloor \frac{N}{p^{\lfloor r(\alpha) \rfloor + 1}} \right\rfloor \).

Furthermore, due to Lemma 2.5

\[
\frac{1}{N} \sum_{1 \leq j \leq N, \text{ord}_p(j) \leq \lfloor r(\alpha) \rfloor} \text{ord}_p(j) \to \frac{1 - p^{-\lfloor r(\alpha) \rfloor}}{p - 1} - \frac{\lfloor r(\alpha) \rfloor}{p^{\lfloor r(\alpha) \rfloor + 1}}
\]

when \( N \to \infty \). Therefore, we conclude that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j \leq N} \text{ord}_p(\alpha - j) = \frac{1 - p^{-\langle r(\alpha) \rangle}}{p - 1} - \langle r(\alpha) \rangle p^{-\langle r(\alpha) \rangle - 1},
\]

which finishes the proof in Case 3.

**Case 4:** \( \alpha \notin d(0, 1) \). This is the trivial case, since due to the ultrametric rule "the strongest one wins" and \( \text{ord}_p(i) \geq 0 \) for all \( i \geq 0 \), one has \( \text{ord}_p(\alpha - i) = \text{ord}_p(\alpha) \) for all \( i \geq 0 \).

The following remark is due:

**Remark 2.6.** The proof of "Case 3" seems to be more transparent than the one given in (31). Also, in "Case 2" no geometric argument using 'Newton Polygons' is involved, as has been in (31). Instead, we decompose the left side term of (2.3) into partial fractions.

An immediate consequence of Lemma 2.5 is the following

**Corollary 2.7.** For any \( P \in \mathbb{Q}[x] \setminus \{0\} \) there exists an integer \( m \geq 0 \) such that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=m}^{N} \text{ord}_p(P(i)) < \infty.
\]

**Proof.** Decompose \( P \) into a product of linear terms \( x - \alpha_j \), where \( \alpha_j \in \mathbb{Q} \ (j = 1, \ldots, \deg P) \) times a constant. Due to the additivity of the valuation we may apply Lemma 2.5 to each of the factors. \( \square \)
3. Proof of the main theorem

**Lemma 3.1.** Let \( f, g \in \mathcal{A}(\mathbb{C}_p) \) and \( A, B \in \mathbb{C}_p[x] \setminus \{0\} \) such that (1.1) holds. Then \( f \) and \( g \) satisfy the system of differential equations

\[
A'f + 2Af = hg, \quad B'g + 2Bg' = -hf
\]

for a certain polynomial \( h \) over \( \mathbb{C}_p \) with \( \deg h \leq \deg A + \deg B \). Furthermore, if \( \deg A > -\infty \) (that is \( A \neq 0 \)) we have for some \( \gamma > 0 \), \( \|A'f + 2Af\|\|r\| \leq \gamma r^{\deg A - 1}\|f\|\|r\| \) when \( r \) sufficiently large.

**Proof.** Differentiating the identity \( A'f^2 + Bg^2 = 1 \) yields \( f(A'f + 2Af') = -g(B'g + 2Bg') \). Since \( f \) and \( g \) have no common zeros, there exists \( h \in \mathcal{A}(\mathbb{C}_p) \) due to Lemma 2.2 such that \( A'f + 2Af = hg, B'g + 2Bg' = -hf \). Moreover, due to the identity \( A'f^2 + Bg^2 = 1 \) we have for \( r \) sufficiently large, \( \|A'f\|\|r\| = \|Bg\|\|r\| \), i.e.

\[
\frac{\|h\|\|r\|}{\|g\|\|r\|\|f\|\|r\|} \leq \gamma r^{\deg A + \deg B} - 1
\]

so we see that \( h \) is a polynomial of degree \( \leq \frac{\deg A + \deg B}{2} - 1 \). \( \square \)

**Example 3.2.** The equation \( f^2 + g^2 = 1 \) has no non-constant entire solutions: For, if there exists \( f, g \) s.t. \( f^2 + g^2 = (f + ig)(f - ig) = 1 \), then, by taking norms on both sides yields \( \|f + ig\|\|r\| = 1 \). Now, if \( f + ig \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p \), then for \( r \to \infty \) we have \( \|f + ig\|\|r\| \to \infty \) which implies \( \|f - ig\|\|r\| \to 0 \), and therefore \( f = ig \) which yields the contradiction \( f^2 + g^2 = 0 \).

**Example 3.3.** For \( a \in \mathbb{C}_p \), the equation \( f^2 + (x - a)g^2 = 1 \) has no entire solutions but \( f = \pm 1, g = 0 \): Assume \( g \neq 0 \). Due to Lemma 3.1 the mentioned polynomial \( h \) is identically zero, therefore \( f' = 0 \), but \( (x - a)g^2 \) cannot be a constant.

**Lemma 3.4.** Let \( f, g \in \mathcal{A}(\mathbb{C}_p) \) satisfy \( A'f + 2Af = hg, B'g + 2Bg' = -hf \) for \( A, B, h \in \mathbb{C}_p[x] \), \( h \neq 0 \), then \( f \) satisfies a linear differential equation of second order:

\[
(4ABh)f'' + (6A'Bh + 2AB'h - 4ABh')f' + (A'Bh + 2A''Bh + 2BA'h + h^3)f = 0
\]

(3.2)

**Proof.** Due to Lemma 3.1, the equation \( f^2 + (x - a)g^2 = 1 \) holds for \( f \) and \( g \) which implies \( g = \frac{1}{h'}(A'f + 2Af') \). Differentiation yields \( g' = -\frac{h''}{h^2}(A'f + 2Af') + \frac{1}{h}(A''f + 3A'f' + 2Af'') \). Inserting \( g, g' \) into (3.1) yields (3.2). \( \square \)

**Definition 3.5.** Consider a differential equation of the following form:

\[
Q^{(2)}(x)y'' + Q^{(1)}(x)y' + Q^{(0)}(x)y = 0, \quad Q^{(i)} \in \mathbb{C}_p[x] \quad (i = 0, 1, 2) \quad (E)
\]

with \( Q^{(2)} \) not identically zero. We define the characteristic number of (E) as

\[
N(E) := \max_i=0,1,2 \deg Q^{(i)} - i,
\]

and the characteristic polynomial of (E) as

\[
P_E(\xi) := \sum_{\deg Q^{(j)} - j = N(E)} q^{(j)}_{\deg Q^{(j)}}(\xi) j
\]

where \( q^{(j)}_{\deg Q^{(j)}} \) \( (j = 0, 1, 2) \) are the leading coefficients of the polynomials \( Q^{(j)} \) \( (j = 0, 1, 2) \).

The following remark highlights the meaning of Definition 3.5.
Remark 3.6. Assume that $f(x) = \sum_{i \geq 0} c_i x^i \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[x]$ solves (E). Then there exists a corresponding recurrence relation for the coefficients of $f$:

$$P_i(n)c_{n+i} + \cdots + P_1(n)c_{n+1} + P_E(n)c_n = 0$$

with certain polynomials $P_s \in \mathbb{C}_p[x]$, where $1 \leq s \leq t$, $P_i \neq 0$ and where $P_E$ is the characteristic polynomial of (E).

Notation 3.7. We write the polynomials $A, B$ and $h$ (due to Lemma 3.1) in the following way:

$$A(x) = \sum_{i=0}^\eta a_i x^i, \quad B(x) = \sum_{j=0}^\chi b_j x^j, \quad h(x) = \sum_{k=0}^\mu h_k x^k; \quad a_q, b_\chi, h_\mu \neq 0,$$

i.e., $A, B$ are polynomials of degree $\eta$ resp. $\chi$. Note that $h$ might be identically zero.

We now determine the characteristic polynomial of equation (3.2):

**Lemma 3.8.** Let $A, B \in \overline{\mathbb{Q}}[x]$, $f, g \in \mathcal{A}(\mathbb{C}_p)$ such that (1.1) holds. Suppose the polynomial $h$ of Lemma (3.5) is not identically zero and that $g(0) \neq 0$, $f^{(i)}(0) \in \overline{\mathbb{Q}}$ for $0 \leq i \leq \deg A + \deg B - 1$, and $q^{(i)}(0) \in \mathbb{Q}$ for $0 \leq i < \deg A + \deg B - 1$. Then the characteristic polynomial $P_E$ of the corresponding differential equation (3.2) is an element of $\overline{\mathbb{Q}}[x]$.

**Proof.** We make use of the notation introduced in Definition 3.5 for the polynomial coefficients of (3.2): $Q^{(2)} = 4ABh$, $Q^{(1)} = 6A'B'h - 4AB'h'$, $Q^{(0)} = A'B'h + 2A''h - 2BA'h' + h^3$. Recall that $a_\eta, b_\chi, h_\mu$ denote the coefficients of the leading powers of the resp. polynomials. Note that

$$N(E) := \max_{i=0,1,2} \deg Q(i) - i = \eta + \chi + \mu - 2$$

since $\deg Q^{(1)} \leq \eta + \chi + \mu - 1$ and $\deg Q^{(0)} \leq \eta + \chi + \mu - 2$. First, let us calculate the coefficients $q_j$ of the term $x^{N(E)+j}$ in $Q^{(j)}$ ($j=0,1,2$) depending on the degree of $h$ ($q_j = 0$ might vanish for $j=1$ or $j=0$!)

\begin{itemize}
    \item[(1)] $\mu = \deg h = \deg A + \deg B - 1 = \frac{\eta + \chi}{2} - 1$: Then $q_2 = 4a_\eta b_\chi h_\mu$, $q_1 = a_\eta b_\chi h_\mu (6\eta + 2\chi - 4\mu)$ and finally $q_0 = a_\eta b_\chi h_\mu (2\eta (\eta - 1) - 2\eta \mu + \frac{h_\mu^2}{a_\eta b_\chi})$.
    \item[(2)] $\mu = \deg h < \deg A + \deg B - 1 = \frac{\eta + \chi}{2} - 1$: Then, due to the calculations in the case above $q_2 = 4a_\eta b_\chi h_\mu$, $q_1 = a_\eta b_\chi h_\mu (6\eta + 2\chi - 4\mu)$ and $q_0 = a_\eta b_\chi h_\mu (2\eta (\eta - 1) - 2\eta \mu)$.
\end{itemize}

We derive the characteristic polynomial $P_E(\xi)$:

\begin{itemize}
    \item[(1)] $\mu = \deg h = \deg A + \deg B - 1 = \frac{\eta + \chi}{2} - 1$: $P_E(\xi) = a_\eta b_\chi h_\mu [4\xi (\xi - 1) + 4(\eta + 1)\xi + (\eta^2 + \frac{h_\mu^2}{a_\eta b_\chi})]$.
    \item[(2)] $\mu = \deg h < \deg A + \deg B - 1 = \frac{\eta + \chi}{2} - 1$: $P_E(\xi) = a_\eta b_\chi h_\mu [4\xi (\xi - 1) + (6\eta + 2\chi - 4\mu)\xi + (\eta^2 + 2\eta (\eta - 1) - 2\eta \mu)]$.
\end{itemize}

Due to our assumptions it suffices to show that the leading coefficient $h_\mu$ of $h$ is algebraic over $\mathbb{Q}$. Indeed by differentiating $h = \frac{A'\xi + 2A\xi'}{g} \mu$-times and using Lemma 3.1 we infer

$$\mu^\mu h_\mu = (h(x))^{(\mu)} = \left( \frac{(A(x)f^2(x))^{(\mu)}}{fg} \right).$$

1 This in particular implies $\deg A + \deg B \geq 2$.
∀ \ P for sufficiently large may choose \( \lambda > L \) any real constant \( c \) and some polynomials \( P \).

\[ \begin{align*}
\text{Case 2:} & \quad \gamma \text{ with an appropriate constant } C \\
\text{Case 1:} & \quad \text{such that the following recurrence relation is satisfied for any } n \geq 0 \\
\text{Lemma 3.1, suggesting a number of simplifications which lead to the final form of this paper.}
\end{align*} \]

\[ \begin{align*}
\text{Acknowledgement:} & \quad \text{For reading this manuscript carefully I am indebted to Professor Alain Escassut (Clermont-Ferrand, France). Special thanks to the referee for suggesting a number of simplifications which lead to the final form of this paper.}
\end{align*} \]

The right hand side involves derivatives up to order \( \mu + 1 \) of \( f \) and of up to order \( \mu \) of \( g \). Due to our assumptions the coefficients of \( f, g \) up to order \( \mu + 1 \) resp. \( \mu \) are algebraic over \( \mathbb{Q} \), and since the coefficients of \( A, B \) have the same property, we are done. \( \square \)

We shall also employ the 'entire version' of \([2] \) Proposition):\

\textbf{Lemma 3.9.} \quad \text{Consider the linear differential equation}

\[ C(x)y'(x) + D(x)y(x) = 0, \quad C(x), D(x) \in K[x] \]

with \( C \) not identically zero. Let \( y(x) \in A(\mathbb{C}_p) \). Then \( y(x) \in \mathbb{C}_p[x] \).

\textbf{Proof of Theorem 1.2} Let \( f, g \in A(\mathbb{C}_p) \setminus \mathbb{C}_p[x] \) such that (1.1) holds. By Lemma 3.1 \( f, g \) satisfy equation (3.1) with a certain polynomial \( h \in \mathbb{C}_p[x] \). We have the two cases:

\textbf{Case 1:} \( h \) identically zero: Due to Lemma 3.9, the solutions of system (3.1) are in \( \mathbb{C}_p[x] \), which is impossible.

\textbf{Case 2:} \( h \) not identically zero: Due to Lemma 3.4, \( f \) satisfies a linear differential equation of the form (3.2). Moreover, due to Lemma 3.8 the characteristic polynomial \( P_E \) of (3.2) lies in \( \mathbb{Q}[x] \). Following the notation of Remark 3.6 we consider a recurrence relation for the coefficients \( c_n \) of \( f(x) = \sum_{n \geq 0} c_n x^n \), \( n \) sufficiently large

\[ P_t(n)c_{n+t} + \cdots + P_1(n)c_{n+1} + P_E(n)c_n = 0, \quad t > 0 \]

(3.4)

We consider the non-trivial case, where \( P_t \) is not identically zero. Clearly \( P_E \) is not identically zero, because \( h \) is not. Since we can multiply equation (3.4) with an appropriate constant \( \gamma \), we may assume without loss of generality that \( \forall i \in \{1, \ldots, t\} \forall n \in \mathbb{N} : \text{ord}_p(P_i(n)) \geq 0 \) where we have set \( P_0 := \gamma P_E \). We have \( P_E \in \mathbb{Q}[x] \) due to Lemma 3.8. Moreover, by induction it follows for any \( k \in \mathbb{N} \) there exist certain polynomials \( P_{i,k}, i \in \{1, \ldots, t\} \) with \( \text{ord}_p(P_{i,k}(n)) \geq 0 \) \( \forall n \geq 1 \) such that the following recurrence relation is satisfied for any \( n \):

\[ P_{1,k}(n)c_{n+t+k} + \cdots + P_{1,k}(n)c_{n+k+1} + P_0(n)P_0(n+1)\cdots P_0(n+k)c_n = 0 \]

(3.5)

Evaluation with respect to \( \text{ord}_p \) yields

\[ \text{ord}_p(P_0(n)P_0(n+1)\cdots P_0(n+k)) + \text{ord}_p(c_n) \geq \min\{\text{ord}_p(c_{n+t+k}), \ldots, \text{ord}_p(c_{n+k+1})\} \]

(3.6)

Moreover, due to Corollary 2.7 \( \exists L \in \mathbb{R} : \lim_{n \to \infty} \text{ord}_p(P_0(n)P_0(n+1)\cdots P_0(n+k)) = L \).

Consider now the growth of \( c_n \), the coefficients of \( f \): the transcendence of \( f \) implies that \( \forall \lambda \in \mathbb{R} : \lim_{n \to \infty} \text{ord}_p(c_n) - n \lambda = \infty \). In other words, for any \( \lambda \in \mathbb{R} \) and for any real constant \( c(\lambda) \) we have \( \text{ord}_p(c_n) \geq c(\lambda) + n \lambda \) for sufficiently large \( n \). We may choose \( \lambda > L \) and some \( c(\lambda) \). Applying this to (3.6) divided by \( n + k \) yields for sufficiently large \( n \):

\[ \frac{\text{ord}_p(s^k P_0(n)P_0(n+1)\cdots P_0(n+k))}{n + k} + \frac{\text{ord}_p(c_n)}{n + k} \geq \frac{c(\lambda) + n \lambda}{n + k} \]

(3.7)

Taking the limits on both sides (\( k \to \infty \)) we derive the contradiction \( L \geq \lambda \).
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