CAUCHY PROBLEM OF STOCHASTIC KINETIC EQUATIONS

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ABSTRACT. In this paper we establish the optimal regularity estimates for the Cauchy problem of stochastic kinetic equations with random coefficients in anisotropic Besov spaces. As applications, we study the nonlinear filtering problem for a degenerate diffusion process, and obtain the existence and regularity of conditional probability densities under few assumptions. Moreover, we also show the well-posedness for a class of super-linear growth stochastic kinetic equations driven by velocity-time white noises, as well as a kinetic version of Parabolic Anderson Model with measure as initial values.

Keywords: Stochastic kinetic equations, Anisotropic Besov spaces, Itô-Wentzell’s formula, filtering problem

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1. INTRODUCTION

Let $\{(W_t^k)_{t \geq 0}, \; k \in \mathbb{N}\}$ be a sequence of independent one-dimensional standard Brownian motions defined on some stochastic basis $(\Omega, \mathcal{F}, \mathcal{P}; (\mathcal{F}_t)_{t \geq 0})$. Let $\mathcal{P}$ be

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the \( (\mathcal{F}_t) \)-predictable \( \sigma \)-algebra over \( \mathbb{R}_+ \times \Omega \), and \( M_{\text{sym}}^d \) the set of all \( d \times d \)-symmetric positive-definite matrices. Given \( P \times B(\mathbb{R}^{2d}) \)-measurable processes

\[
(a, b, \sigma) : \mathbb{R}_+ \times \Omega \times \mathbb{R}^{2d} \to (M_{\text{sym}}^d, \mathbb{R}^d, \mathbb{R}^d \otimes \ell^2),
\]

we introduce the following operators: for a function \( u \) of \( (x, v) \in \mathbb{R}^{2d} \),

\[
\mathcal{L}_v u := \text{tr}(a \cdot \nabla^2_v u) + b \cdot \nabla_v u, \quad \mathcal{M}^k u := \sigma^k \cdot \nabla_v u, \quad k \in \mathbb{N},
\]

where \( \nabla_v \) (resp. \( \nabla_x \)) stands for the gradient operator that acts only on the variable \( v \) (resp. \( x \)). In this paper we consider the following stochastic kinetic equation (SKE for short) of Itô’s type:

\[
du = \left[ \mathcal{L}_v u + v \cdot \nabla_x u + f \right] dt + \left[ \mathcal{M}^k u + g^k \right] dW^k_t, \quad (1.1)
\]
as well as its adjoint form

\[
du = \left[ \mathcal{L}^*_v u - v \cdot \nabla_x u + f \right] dt + \left[ (\mathcal{M}^k)^* u + g^k \right] dW^k_t, \quad (1.2)
\]
subject to the initial condition \( u(0, \omega, x, v) = u_0(\omega, x, v) \) being \( \mathcal{F}_0 \)-measurable, where the asterisk stands for the adjoint operator, the unknown \( u \) is a function of

\[
t, \omega, x, v \quad \text{(resp. } t, \omega, x, v) \quad \text{(resp. } t, \omega, x, v), \quad (x, v) \quad \text{represents the position and velocity, the nonhomogeneous or free terms}
\]

\[
(f, g) : \mathbb{R}_+ \times \Omega \times \mathbb{R}^{2d} \to (\mathbb{R}, \ell^2)
\]
are \( P \times B(\mathbb{R}^{2d}) \)-measurable processes. Here and below we use the usual Einstein convention for summation: An index appearing in a product will be summed automatically. Notice that SPDEs \((1.1) \) and \((1.2) \) are highly degenerate in \( x \)-direction.

When \( \sigma, g \) are zero and \( a, b, f \) do not depend on \( \omega \), PDEs \((1.1) \) and \((1.2) \) are degenerate deterministic equations and called kinetic Fokker-Planck-Kolmogorov equations in the literature since they are just the associated forward and backward Kolmogorov equations of the following SDE

\[
dX_t = V_t dt, \quad dV_t = b(t, X_t, V_t) dt + \sqrt{2} a(t, X_t, V_t) dB_t, \quad (1.3)
\]
where \( B \) is a \( d \)-dimensional standard Brownian motion. Some backgrounds of non-random PDEs \((1.1) \) and \((1.2) \) are referred to [17, 27, 31] and references therein. We mention that the two-sides estimates of the distribution density of SDE \((1.3) \) (also called heat kernel of \( \mathcal{L}_v + v \cdot \nabla_x \)) were studied by Delarue and Menozzi [8]. The Schauder estimates for deterministic kinetic equation \((1.1) \) were established in [19, 20, 25], and the maximal \( L^p \)-regularity estimates in \((x, v) \) were obtained in [3] (see also [5], [12] and [11] for nonlocal versions). Moreover, through studying PDE \((1.1) \) with rough drift \( b \), the strong and weak well-posedness of SDE \((1.3) \) with irregular drift \( b \) was also studied in [4, 30, 35].

On the other hand, SKE \((1.1) \) has a close connection with the filtering problem associated with SDE \((1.3) \). In fact, let us consider the following simple stochastic Langevin equation

\[
dX_t = V_t dt, \quad dV_t = dB_t - dW_t,
\]
where \( B \) and \( W \) are two independent Brownian motions. In this model, \( W \) is regarded as the observable noise, \( B \) is the hidden noise, \( (X, V) \) stands for the position and velocity of particles, whose distribution needs to be predicted. Let \( \pi(t, \omega, x, v) \) be the conditional probability density of \( \mathbb{P}(X_t, V_t) \in \cdot | \mathcal{F}_t^W \), i.e., for any \( \varphi \in L^\infty(\mathbb{R}^{2d}) \),

\[
\int_{\mathbb{R}^{2d}} \pi(t, \omega, x, v) \varphi(x, v) dx dv = \mathbb{E}(\varphi(X_t, V_t) | \mathcal{F}_t^W)(\omega),
\]
where $\mathcal{F}^W_t$ is the $\sigma$-algebra generated by $\{W_s, s \leq t\}$. It is well known that (see [26])
\[(\partial_t + v \cdot \nabla_x)\pi = \Delta_v \pi dt + \nabla_v \pi dW_t.\]  
\[(1.4)\]
This is just a model equation of SKE $(1.1)$. Note that the nonlinear filtering problem of SDE $(1.3)$ is naturally related to SKE $(1.2)$ (see Section 6 below).

The study of SPDEs has a long history since the earlier work of Pardoux [23] about the SPDEs in Hilbert spaces, Krylov and Rozovskii [13] about the SPDEs in the framework of Gelfand triple, and Walsh [29] about the study of stochastic wave equations. Nowadays, the theory of SPDEs has been greatly developed, and there are vast literatures, see, for examples, the monographs [6, 15, 18, 26], etc. Here we only mention parts of the related work. In the case of smooth coefficients, Rozovsky and Lototsky [26] systematically developed a complete linear $L^2$-theory about parabolic SPDEs, especially, addressed the application in nonlinear filtering problems (see also [33] for a non-smooth, degenerate $L^2$-theory). A complete $L^p$-theory about semi-linear SPDEs was established by Krylov [15]. Here by a theory, according to Krylov’s words [15], we mean not only results that, for $f, g^k$ belonging to a space $S$, the solution belongs to some stochastic spaces $S'$, but also that every element of $S'$ can be obtained as a solution for certain $f, g^k$ belonging to the same $S$. In other words, we have a bijection $F : S \rightarrow S'$ between $f, g^k$ and the solutions. In [21], Mikulevicius obtained the Schauder estimates for nondegenerate SPDEs when the leading coefficient $a$ is nonrandom and $\sigma = 0$. The full Schauder theory for nondegenerate second order SPDEs with random coefficients was established recently by Du and Liu [9]. The main difficulty, when we consider the random leading coefficients, is that the direct Duhamel formulation is no longer applicable due to the non-adaptedness of the integrands in the stochastic integrals since we have random kernels in this case. In [9], the authors adopt Campanato’s energy characterization of Hölder spaces to overcome this difficulty.

The aim of this paper is to establish the optimal regularity estimates for SKE $(1.1)$ in anisotropic Besov spaces under some Hölder regularity assumptions on $a, b, \sigma$. Our optimal regularity estimates are not only for nondegenerate velocity component $v$, but also for degenerate position component $x$. As discussed above, when $a$ is nonrandom, by Duhamel’s formulation, and using completely the same argument developed in [11], one can establish a satisfactory Schauder theory. However, for random leading coefficient $a$, the method adopted by Du and Liu [9], if it is not impossible, seems hard to be used for SKE $(1.1)$. We mention that when $d = 1$, Pascucci and Pesce [24] obtained the Schauder estimates for SKE $(1.1)$ by using Itô-Wentzell’s formula to reduce SKE $(1.1)$ into a deterministic PDE with random coefficients. Thus the price to pay is that the coefficients have to be at least $C^3$-differentiable in $x, v$. Moreover, we are also interested in the following nonlinear SKE with super-linear growth coefficient in $\mathbb{R}^2$:
\[du = [\Delta_x u + v \cdot \nabla_x u] dt + |u|^\gamma dB, \quad \gamma \in (0, \frac{1}{2}),\]  
\[(1.5)\]
where $dB$ stands for the velocity-time white noise. In particular, it includes a kinetic version of continuous Parabolic Anderson Model (abbreviated as PAM) when $\gamma = 0$ and $u \geq 0$. We referred to [2] and references therein for some background about PAM. In the nondegenerate case, i.e., $u$ does not depend on $x$, the global well-posedness to the above super-linear growth SKEs with Dirichlet boundary conditions for $\gamma \in (0, \frac{1}{2})$ was first established by Mueller [22]. His proof is based on the large deviation estimates and only for positive solutions. While in [15, Section
8.4], Krylov provides a quite different and more direct proof, and still for positive solutions, where the key observation is that for any bounded stopping time \( \tau \),

\[
\mathbb{E} \int_{\mathbb{R}} u(\tau, v) dv \leq \int_{\mathbb{R}} u(0, v) dv.
\]

Here we aim to show the existence and uniqueness of weak solutions and the optimal regularity of \( u \) in \( t, x, v \) to nonlinear SKE (1.5). As above, the key point is to show

\[
\mathbb{E} \int_{\mathbb{R}} |u(\tau, x, v)| dv \leq \int_{\mathbb{R}} |u(0, x, v)| dv.
\]

See Lemma 7.9 below.

Now we describe the strategies of proving the optimal Besov regularity to SKE (1.1). First of all, we study the model equation

\[
du = [\Delta_v u + v \cdot \nabla_x u + f] dt + g^k dW^k_t.
\]

Using some tools from [11], we establish the optimal regularity estimates of the solution in anisotropic Besov spaces with respect to the time and spatial-velocity variables by Duhamel’s formula. We would like to emphasize that the semigroup \( \mathcal{P}_t \) associated with \( \Delta_v + v \cdot \nabla_x \) behaves unlike the Gaussian heat semigroup. One has to consider the transport semigroup \( \Gamma_t f(x, v) := f(x + tv, v) \). The non-commutativity \( \Gamma_t \mathcal{P}_t \neq \mathcal{P}_t \Gamma_t \) brings us many difficulties. Moreover, although \( \mathcal{P}_t \) is a strongly continuous semigroup in \( L^p \)-space, there seems not exist a good characterization for the domain of \( \Delta_v + v \cdot \nabla_x \) in \( L^p \)-space. Thus, SKE (1.6) does not fall into the abstract framework studied in [6]. We have to carefully handle the anisotropy caused by degenerate term \( v \cdot \nabla_x \). Next, by a generalized Itô-Wentzell’s formula ([15,16]), we transform SKE (1.1) with random but constant \( a, \sigma \) into model equation (1.6), and then obtain the regularity estimates for (1.1), where the key point is that we need to work in Besov spaces with finite integrability exponent. Finally, we shall use the freezing coefficient argument to derive the optimal regularity estimates for variable random coefficients \( a, \sigma \). In order to make the perturbation term can be absorbed by the freezing term, we introduce a new localized anisotropic Besov norm \( \| \cdot \|_{\tilde{B}^{s}_{p,q}} \) in (2.18) below. Unlike \( L^\infty \not\subset L^p \) for \( p < \infty \), one obvious advantage of using this localized norm is that for any \( s > 0 \),

\[
\| \cdot \|_{\tilde{B}^{s}_{p,q}} \leq C \| \cdot \|_{\tilde{B}^{s}_{p',q'}}, \quad p' \geq p.
\]

Thus our initial value can be a constant. Such a norm was also used in [36]. It is noticed that if the coefficients \( a, b, \sigma \) and \( f, g \), initial value \( u_0 \) do not depend on the position variable \( x \), then SKEs (1.1) and (1.2) naturally reduce to the classical SPDEs studied in [15]. To the authors’ knowledge, even in this classical case, our Besov estimates are also new.

This paper is organized as follows: In Section 2, we introduce some anisotropic function spaces, and prepare some useful results for later use. In Section 3, we establish the optimal regularity estimates for random constant coefficients case, namely, \( a \) and \( \sigma^k \) are independent of space and velocity variables. In Sections 4 and 5, we prove our main results (see Theorems 5.2, 5.3, 5.5). In Section 6, we apply our results to the filtering problem of degenerate diffusion processes, and show the existence and regularity of conditional density processes, which satisfies a nonlinear SPDE in classical sense under some assumptions. In Section 7, we also show a well-posedness result for a class of nonlinear SKEs driven by velocity-time white noises, and obtain the optimal Besov regularity, which seems to be new even in the classical nondegenerate case (cf. [15,29]).
Throughout this paper, we use the following conventions: The letter $C = C(\cdots)$ denotes an unimportant constant, whose value may change in different places, and which is increasing with respect to its arguments. We use $A \asymp B$ and $A \lesssim B$ to denote $C^{-1}B \leq A \leq CB$ and $A \leq CB$, respectively, for some unimportant constant $C \geq 1$. As usual, we use $:= \text{or} =: \text{as a way of definition},$ and for any $a, b \in \mathbb{R}$,

$$a \wedge b := \min(a, b), \quad a \vee b := \max(a, b), \quad a^+ := a \vee 0, \quad a^- := -(a \wedge 0).$$

Moreover, for $p \in [1, \infty]$, we use $\ell^p$ to denote the usual space of a sequence of real numbers that is $p$-order summable.

2. Preliminaries

2.1. Vector-valued anisotropic Besov and Hölder spaces. In this subsection we introduce the vector-valued anisotropic Hölder and Besov spaces and their basic properties for later use. Let $m = (m_1, \cdots, m_n) \in \mathbb{N}^n$ with $m_1 + \cdots + m_n = N$ and $\theta = (\theta_1, \cdots, \theta_n) \in [1, \infty)^n$ be fixed. We denote $\theta \cdot m = \theta_1 m_1 + \cdots + \theta_n m_n$. For $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$, we introduce the following distance in $\mathbb{R}^N$

$$|x - y|_\theta := \sum_{i=1}^n |x_i - y_i|^{1/\theta_i}, \quad x_i, y_i \in \mathbb{R}^{m_i}.$$

Let $\mathbb{B}$ be a Banach space. For $p \in [1, \infty]$, let $L^p_{\mathbb{B}}(\mathbb{B}) := L^p(\mathbb{R}^N, dx; \mathbb{B})$ be the usual vector-valued $L^p$-space over $\mathbb{R}^N$. For $h \in \mathbb{R}^N$ and a map $f : \mathbb{R}^N \to \mathbb{B}$, the first order difference operator is defined by

$$\delta_h^{(1)} f(x) := f(x + h) - f(x),$$

and for $M \in \mathbb{N}$, the $M$-order difference operator is defined recursively by

$$\delta_h^{(M)} f(x) := \delta_h^{(1)} \delta_h^{(M-1)} f(x).$$

By induction, it is easy to see that

$$\delta_h^{(M)} f(x) = \sum_{k=0}^M (-1)^{M-k} \binom{M}{k} f(x + kh), \quad h \in \mathbb{R}^N, \quad (2.1)$$

where $\binom{M}{k}$ is the binomial coefficient.

**Definition 2.1** (Vector-valued anisotropic Hölder spaces). For $s \in (0, \infty)$, the $\mathbb{B}$-valued anisotropic Hölder space is defined by norm

$$\|f\|_{C^s_\mathbb{B}(\mathbb{B})} := \|f\|_{L^\infty(\mathbb{B})} + \|f\|_{C^s_{\mathbb{B}}(\mathbb{B})} < \infty,$$

where

$$[f]_{C^s_{\mathbb{B}}(\mathbb{B})} := \sup_h \|\delta_h^{[s]+1} f\|_{L^\infty(\mathbb{B})}/|h|^s.$$

Here $[s]$ denotes the greatest integer less than $s$.

In order to introduce the anisotropic Besov space, we need a symmetric nonnegative $C^\infty$-function $\phi_0^\theta$ on $\mathbb{R}^N$ with

$$\phi_0^\theta(\xi) = 1 \text{ for } \xi \in B_1^0 \text{ and } \phi_0^\theta(\xi) = 0 \text{ for } \xi \notin B_2^0,$$

where for $r > 0$ and $y \in \mathbb{R}^N$,

$$B_r^0(y) := \{x \in \mathbb{R}^N : |x - y|_\theta \leq r\}, \quad B_r^0 := B_r^0(0).$$

For $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$ and $j \in \mathbb{N}$, we define

$$\phi_j^\theta(\xi) := \phi_0^\theta(2^{-\theta j} \xi) - \phi_0^\theta(2^{-\theta(j-1)} \xi),$$

where $\phi_j^\theta$ is the $j$th order difference of $\phi_0^\theta$. 

Furthermore, we introduce the vector-valued anisotropic Besov spaces $B^s_{p,\mathbb{B}}(\mathbb{B})$ and $B^s_{p,\mathbb{B}}(\mathbb{B})$ defined by

$$B^s_{p,\mathbb{B}}(\mathbb{B}) := \left\{ f \in L^p_{\mathbb{B}}(\mathbb{B}) : \|f\|_{B^s_{p,\mathbb{B}}(\mathbb{B})} < \infty \right\},$$

where

$$\|f\|_{B^s_{p,\mathbb{B}}(\mathbb{B})} := \left( \sum_{k=0}^\infty 2^{ks} [f]_{C^k_{p,\mathbb{B}}(\mathbb{B})}^p \right)^{1/p}.$$
where
\[2^{-\theta_j} \xi := (2^{-\theta_j} \xi_1, \ldots, 2^{-\theta_j} \xi_n)\].

By the very definition, one sees that for \(j \in \mathbb{N}\), \(\phi_j^\theta(\xi) = \phi_0^\theta(2^{-\theta(j-1)} \xi) \geq 0\) and
\[
\text{supp} \phi_j^\theta \subset B_{2^{j+1}}, \quad \sum_{j=0}^n \phi_j^\theta(\xi) = \phi_0^\theta(2^{-n} \xi) \to 1, \quad n \to \infty.
\]

For \(f \in \mathbb{L}_2^1(\mathbb{B})\), let \(\hat{f}\) be the Fourier transform of \(f\) defined by
\[
\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} f(x) dx, \quad \xi \in \mathbb{R}^N,
\]
and \(\hat{f}\) the Fourier inverse transform of \(f\) defined by
\[
\hat{f}(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^N} e^{i\xi \cdot x} \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}^N.
\]

Let \(\mathcal{S}\) be the space of all Schwartz functions on \(\mathbb{R}^N\) and \(\mathcal{S}'(\mathbb{B})\) the space of all continuous linear operators from \(\mathcal{S}\) to \(\mathbb{B}\), called vector-valued distribution space. Note that \(\mathcal{S}' = \mathcal{S}'(\mathbb{R})\) is just the tempered distribution space. In a natural way, \(\mathbb{L}_2^1(\mathbb{B}) \subset \mathcal{S}'(\mathbb{B})\), and we can extend the Fourier transform to the element in \(\mathcal{S}'(\mathbb{B})\) by duality. For given \(j \in \mathbb{N}_0\), the block operator \(\mathcal{R}_j^\theta\) is defined on \(\mathcal{S}'(\mathbb{B})\) by (see [1])
\[
\mathcal{R}_j^\theta f(x) := (\phi_j^\theta \hat{f})(x) = \phi_j^\theta \ast f(x) = 2^{\theta \cdot (j-1)} \int_{\mathbb{R}^N} \hat{\phi}_j^\theta(2^{\theta(j-1)}y) f(x-y) dy,
\]
where the convolution is understood in the distributional sense. In particular, by the symmetry of \(\phi_j^\theta\), we have
\[
\langle \mathcal{R}_j^\theta f, g \rangle = \langle f, \mathcal{R}_j^\theta g \rangle, \quad f \in \mathcal{S}', \ g \in \mathcal{S},
\]
where \(\langle \cdot, \cdot \rangle\) stands for the dual pair between \(\mathcal{S}'\) and \(\mathcal{S}\).

**Definition 2.2** (Vector-valued anisotropic Besov spaces). For \(s \in \mathbb{R}\) and \(p \in [1, \infty]\), the \(\mathbb{B}\)-valued anisotropic Besov space is defined by
\[
\mathbf{B}_{p,\theta}^s(\mathbb{B}) := \left\{ f \in \mathcal{S}'(\mathbb{B}) : \|f\|_{\mathbf{B}_{p,\theta}^s(\mathbb{B})} := \sup_{j \geq 0} \left( 2^{sj} \|\mathcal{R}_j^\theta f\|_{\mathbb{L}_p^\theta(\mathbb{B})} \right) < \infty \right\}.
\]

When \(\theta = (1, \ldots, 1)\), we shall simply write
\[
\mathbf{B}_p^s(\mathbb{B}) := \mathbf{B}_{p,1}^s(\mathbb{B}), \quad \mathcal{R}_j := \mathcal{R}_j^1, \quad \phi_j := \phi_j^1.
\]
When \(\mathbb{B} = \mathbb{R}\), we shall simply write \(\mathbf{B}_{p,\theta}^s := \mathbf{B}_{p,\theta}^s(\mathbb{R})\).

**Remark 2.3.** In the literature, there are usually two subscripts \(p, q\) in the definition of Besov spaces, where \(p\) stands for the integrability of spatial variables and \(q\) denotes the \(\ell^q\)-norm of frequency index \(j\). Since we only use the Besov space of \(q = \infty\), in our definition, we take \(q = \infty\) and drop it for simplicity.

**Remark 2.4.** In application below, we usually take \(\mathbb{B} = \mathbb{L}_p^\theta := L^p(\omega, \mathcal{F}, \mathbb{P})\). In this case, by Fubini’s theorem, for any \(s < s'\) and \(p \in [1, \infty)\), one has
\[
\mathbf{B}_{p,\theta}^{s'}(\mathbb{L}_p^\theta) \subset \mathbb{L}_p^\theta(\mathbf{B}_{p,\theta}^s) \subset \mathbf{B}_{p,\theta}^s(\mathbb{L}_p^\theta).
\]

Indeed, by definition and Fubini’s theorem,
\[
\|f\|_{\mathbb{L}_p^\theta(\mathbf{B}_{p,\theta}^s)} = \mathbb{E}\|f(\omega, \cdot)\|_{\mathbb{B}_{p,\theta}^s}^p = \mathbb{E} \left( \sup_{j \geq 0} 2^{sjp} \|\mathcal{R}_j^\theta f(\omega, \cdot)\|_{\mathbb{L}_p^\theta}^p \right)
\leq \sum_{j \geq 0} 2^{sjp} \mathbb{E} \left( \|\mathcal{R}_j^\theta f(\omega, \cdot)\|_{\mathbb{L}_p^\theta}^p \right) = \sum_{j \geq 0} 2^{sjp} \|\mathcal{R}_j^\theta f\|_{\mathbb{L}_p^\theta(\mathbb{L}_p^\theta)}^p.
\]
\[ \leq \sum_{j \geq 0} 2^{(s-s')j} p \| f \|_{B^p_{\theta,0}(L^q_\beta)}^p = 1/(1 - 2^{(s-s')p}) \| f \|_{B^p_{\theta,0}(L^q_\beta)}, \]

and

\[ \| f \|_{B^p_{\theta,0}(L^q_\beta)} = \sup_{j \geq 0} 2^{sjp} \| R^\theta_j f \|_{L^q_\beta} \leq E \left( \sup_{j \geq 0} 2^{sjp} \| R^\theta_j f(\omega, \cdot) \|_{L^q_\beta} \right) = \| f \|_{B^p_{\theta,0}(L^q_\beta)}. \]

We recall the following result whose proof is completely the same as in [1, p.52, Lemma 2.1]. We omit the details.

**Lemma 2.5** (Bernstein’s type inequalities). For any \( 1 \leq p \leq q \leq \infty, k \in \mathbb{N}_0 \) and \( i = 1, \cdots, n \), there is a constant \( C = C(\theta, m, p, q, k, i) > 0 \) such that for all \( j \geq 0 \),

\[ \| \nabla^k x_i f \|_{L^q_\beta(\mathbb{B})} \leq C 2^{j(\theta, k+\theta m)(\frac{1}{q} - \frac{1}{p})} \| R^\theta_j f \|_{L^q_\beta(\mathbb{B})}, \]

where \( \nabla^k x_i \) denotes the \( k \)-order gradient with respect to \( x_i \).

As easy consequences of the above lemma, we have

**Lemma 2.6.** Let \( 1 \leq p \leq q \leq \infty, s \in \mathbb{R} \) and \( k \in \mathbb{N}_0 \), \( i = 1, \cdots, n \).

(i) If \( 0 < \theta_i k + (\theta \cdot m)(\frac{1}{p} - \frac{1}{q}) < s \), then for some \( C = C(\theta, m, p, q, s, k, i) > 0 \),

\[ \| \nabla^k x_i f \|_{L^q_\beta(\mathbb{B})} \leq C \| f \|_{B^p_{\theta,0}(\mathbb{B})}, \]

and for \( s' = s_i + \theta_i k + (\theta \cdot m)(\frac{1}{p} - \frac{1}{q}) \),

\[ \| \nabla^k x_i f \|_{B^q_{\theta,0}(\mathbb{B})} \leq C \| f \|_{B^p_{\theta,0}(\mathbb{B})}. \]

(ii) For any \( s_1 < s < s_2 \), there is a \( C = C(\theta, m, p, s, s_2, s_1) > 0 \) such that

\[ \| f \|_{B^q_{\theta,s}(\mathbb{B})} \leq C \| f \|_{B^q_{\theta,s_1}(\mathbb{B})} \| f \|_{B^q_{\theta,s_2}(\mathbb{B})} \]

**Proof.** (i) Noting that \( f = \sum_j R^\theta_j f \), by Bernstein’s inequality (2.4), we have

\[ \| \nabla^k x_i f \|_{L^q_\beta(\mathbb{B})} \leq \sum_j \| \nabla^k x_i R^\theta_j f \|_{L^q_\beta(\mathbb{B})} \leq \sum_{j \geq 0} 2^{j(\theta, k+\theta m)(\frac{1}{q} - \frac{1}{p})} \| R^\theta_j f \|_{L^q_\beta(\mathbb{B})} \]

\[ \leq \sum_{j \geq 0} 2^{j(\theta, k+\theta m)(\frac{1}{q} - \frac{1}{p}) - js} \| f \|_{B^q_{\theta,0}(\mathbb{B})} \leq \| f \|_{B^q_{\theta,0}(\mathbb{B})}. \]

Thus we get (2.5). Estimate (2.6) is direct by (2.4) and \( R^\theta_j \nabla^k x_i = \nabla^k x_i R^\theta_j \).

(ii) It follows by definition. \( \square \)

The following lemma is elementary.

**Lemma 2.7.** For any \( 0 < \beta < \alpha < \infty \), it holds that for all \( \lambda > 0 \),

\[ \sum_{j \geq 0} (\lambda 2^{\alpha j} \land 1) 2^{-\beta j} \leq \left( \frac{2^\beta}{\ln 2} \int_0^\infty (r^\alpha \land 1) r^{-\beta - 1} dr \right) \lambda^{\frac{\beta}{\alpha}}. \]

**Proof.** Note that

\[ \sum_{j \geq 0} (\lambda 2^{\alpha j} \land 1) 2^{-\beta j} \leq 2^\beta \int_0^\infty (\lambda 2^{\alpha s} \land 1) 2^{-\beta s} ds = \frac{2^\beta \lambda^{\frac{\beta}{\alpha}}}{\ln 2} \int_0^\infty (r^\alpha \land 1) r^{-\beta - 1} dr, \]

which gives the desired estimate. \( \square \)

Now we show the following important characterization of anisotropic Besov space \( B^q_{\theta,0}(\mathbb{B}) \) (cf. [28]).
Lemma 2.8. For any $s \in (0, \infty)$ and $p \in [1, \infty]$, there exists a constant $C = C(\theta, m, p, s) \geq 1$ such that for all $f \in \mathcal{B}_{p,\theta}^s(\mathbb{R})$, 
\[
\|f\|_{\mathcal{B}_{p,\theta}^s(\mathbb{R})} \lesssim_C \sup_{h \in \mathbb{R}^N} \left( |h|_\theta^{-s} \|\delta_h^s\|_{L_p^s(\mathbb{R})} \right) \|f\|_{L_p^s(\mathbb{R})},
\]
(2.8) 
In particular, $\mathcal{C}_p^s(\mathbb{R}) = \mathcal{B}_{\infty,\theta}^s(\mathbb{R})$ and 
\[
\|f\|_{\mathcal{B}_{p,\theta}^s(\mathbb{R})} \lesssim_C \|f\|_{\mathcal{B}_{p,\theta}^{s_1}(\mathbb{R})} + \cdots + \|f\|_{\mathcal{B}_{p,\theta}^{s_n}(\mathbb{R})},
\]
(2.9) 
where for $i = 1, \cdots, n$, 
\[
\|f\|_{\mathcal{B}_{p,\theta}^{s_i}(\mathbb{R})} := \sup_{j \geq 0} \left[ 2^{sj} \left( \int_{\mathbb{R}^n} \|\mathcal{R}_j f(x)\|_p^p \, dx \right)^{\frac{1}{p}} \right],
\]
and for $x = (x_1, \cdots, x_n)$, 
\[
\mathcal{R}_j^x f(x) := \mathcal{R}_j f(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots)(x_i).
\]
(2.10) 
Proof. (i) For simplicity, we set $M := [s] + 1$. We first prove that for any $h \in \mathbb{R}^N$, 
\[
\|\delta_h^M f\|_{L_p^s(\mathbb{R})} \lesssim |h|_\theta \|f\|_{\mathcal{B}_{p,\theta}^s(\mathbb{R})}.
\]
By definition, without loss of generality, it suffices to prove that for $h_1 \in \mathbb{R}^{m_1}$, 
\[
\|\delta_{h_1}^M f\|_{L_p^s(\mathbb{R})} \lesssim |h_1|^{s/\theta} \|f\|_{\mathcal{B}_{p,\theta}^s(\mathbb{R})},
\]
(2.11) 
where $h_1 := (h_1, 0, \cdots, 0) \in \mathbb{R}^N$. Since $\mathcal{R}_j^x \delta_h^M f = \delta_h^M / \mathcal{R}_j f$, by (2.1) with Taylor’s expansion and Bernstein’s inequality (2.4), we have 
\[
\|\mathcal{R}_j^x \delta_h^M f\|_{L_p^s(\mathbb{R})} \lesssim |h_1|^M \|\nabla_x \mathcal{R}_j^x f\|_{L_p^s(\mathbb{R})} \lesssim |h_1|^M 2^M (2^M - s)^j \|f\|_{\mathcal{B}_{p,\theta}^s(\mathbb{R})},
\]
and also, 
\[
\|\mathcal{R}_j^x \delta_h^M f\|_{L_p^s(\mathbb{R})} \lesssim \sum_{k=0}^M \binom{M}{k} \|\mathcal{R}_j^x f\|_{L_p^s(\mathbb{R})} \lesssim 2^M 2^{-sj} \|f\|_{\mathcal{B}_{p,\theta}^s(\mathbb{R})},
\]
Hence, 
\[
\|\delta_h^M f\|_{L_p^s(\mathbb{R})} \lesssim \sum_{j \geq 0} \|\mathcal{R}_j^x \delta_h^M f\|_{L_p^s(\mathbb{R})} \lesssim \sum_{j \geq 0} \left( |h_1|^M 2^M (2^M - s)^j \right) \|f\|_{\mathcal{B}_{p,\theta}^s(\mathbb{R})}.
\]
which gives (2.11) by Lemma 2.7. 

(ii) For $j \geq 1$, since $\int_{\mathbb{R}^N} \delta_h^M f(x) \, dh = (2\pi)^d/2 \phi_j^0 (0) = 0$, by (2.1) and the change of variable, we have 
\[
\int_{\mathbb{R}^N} \delta_h^M f(x) \, dh = \sum_{k=0}^M (-1)^{M-k} \binom{M}{k} \int_{\mathbb{R}^N} \phi_j^0 (h) f(x + kh) \, dh
\]
\[
= \sum_{k=1}^M (-1)^{M-k} \binom{M}{k} \int_{\mathbb{R}^N} \phi_j^0 (h) f(x + kh) \, dh,
\]
\[
= \sum_{k=1}^M (-1)^{M-k} \binom{M}{k} \int_{\mathbb{R}^N} \phi_j^0 (k \cdot) f(x + h) \, dh.
\]
In particular, if we define for $j \in \mathbb{N}_0$, 
\[
\phi_j^M (\xi) := (-1)^{M+1} \sum_{k=1}^M (-1)^{M-k} \binom{M}{k} \phi_j^0 (k \xi),
\]
then

\[ (-1)^{M+1} \int_{\mathbb{R}^N} \tilde{\phi}_j^\theta(h) \delta_h^{(M)} f(x) dh = [\phi_j^\theta, \cdot] f(x) =: \mathcal{R}_j^\theta f(x), \]

and for \( j \geq 1, \)

\[
\| \mathcal{R}_j^\theta f \|_{L^p_\mathcal{B}} \leq \int_{\mathbb{R}^N} |\tilde{\phi}_j^\theta(h) | \| \delta_h^{(M)} f \|_{L^p_\mathcal{B}} dh \\
\leq \left( \int_{\mathbb{R}^N} |\tilde{\phi}_j^\theta(h) | \| h \|_{L^p_\mathcal{B}} dh \right) \sup_h \frac{\| \delta_h^{(M)} f \|_{L^p_\mathcal{B}}}{|h|^{s}_\theta} \\
\lesssim 2^{-sj} \sup_h \left( |h|^{-s} \| \delta_h^{(M)} f \|_{L^p_\mathcal{B}} \right). \tag{2.12}
\]

On the other hand, noting that

\[
\phi_j^\theta, M (\xi) = \phi_0^\theta, M (2^{-\theta j} \xi) - \phi_0^\theta, M (2^{-\theta(j-1)} \xi)
\]

and

\[
\phi_0^\theta, M (\xi) = 1 \text{ for } \xi \in B_1^{\theta/M} \text{ and } \phi_0^\theta, M (\xi) = 0 \text{ for } \xi \notin B_2^\theta,
\]

we have

\[
\text{supp } \phi_j^\theta, M \subset B_{2^j+1}^\theta \setminus B_{(2^{j-1})/M}^\theta.
\]

Hence, for any \( i, j \in \mathbb{N}_0 \) with \( |j - i| > \log_2 M + 2 =: \gamma, \)

\[
\mathcal{R}_j^\theta f = 0.
\]

Moreover, noting that for any \( \xi \in \mathbb{R}^N, \)

\[
\sum_{j \geq 0} \phi_j^\theta, M (\xi) = (-1)^{M+1} \sum_{k=1}^{M} \sum_{j \geq 0} (-1)^M \binom{M}{k} \phi_j^\theta (k\xi) \\
= (-1)^{M+1} \sum_{k=1}^{M} (-1)^{M-k} \binom{M}{k} = 1,
\]

we have

\[
\mathcal{R}_i^\theta f = \sum_{j \geq 0} \mathcal{R}_j^\theta \mathcal{R}_i^\theta, M f = \sum_{|j-i| \leq \gamma} \mathcal{R}_i^\theta \mathcal{R}_j^\theta, M f.
\]

Therefore, for \( i \geq \gamma + 1, \) by (2.12),

\[
\| \mathcal{R}_i^\theta f \|_{L^p_\mathcal{B}} \leq \sum_{|i-j| \leq \gamma} \| \mathcal{R}_i^\theta \mathcal{R}_j^\theta, M f \|_{L^p_\mathcal{B}} \\
\lesssim \sum_{|i-j| \leq \gamma} 2^{-sj} \sup_h \left( |h|^{-s} \| \delta_h^{(M)} f \|_{L^p_\mathcal{B}} \right) \\
\lesssim 2^{-si} \sup_h \left( |h|^{-s} \| \delta_h^{(M)} f \|_{L^p_\mathcal{B}} \right).
\]

For \( i < \gamma + 1, \) we always have

\[
\| \mathcal{R}_i^\theta f \|_{L^p_\mathcal{B}} \lesssim \| f \|_{L^p_\mathcal{B}}.
\]

Thus we obtain another side estimate and complete the proof. \( \square \)

For a Banach space \( \mathcal{B}, \) let \( \mathcal{L}_\mathcal{B} \) be the Banach space of all bounded linear operators from \( \mathcal{B} \) to \( \mathcal{B}. \)
Lemma 2.9. For any $s \in (0, 2)$ and $p \in [1, \infty]$, there exists a constant $C = C(\theta, m, p, s) > 0$ such that for all $T \in C_0^s(\mathcal{L}_B)$ and $g \in B^s_{p,\theta}(B)$,
\[
\|Tg\|_{B^s_{p,\theta}(B)} \lesssim C \left\{ \|T\|_{L^\infty(\mathcal{L}_B)} \|g\|_{B^s_{p,\theta}(B)} + \|T\|_{C^s_0(\mathcal{L}_B)} \|g\|_{L^\infty(\mathcal{B})}, \quad s \in (0, 1),
\|T\|_{L^\infty(\mathcal{L}_B)} \|g\|_{B^s_{p,\theta}(B)} + \|T\|_{C^s_0(\mathcal{L}_B)} \|g\|_{B^s_{p,\theta}(B)}, \quad s \in [1, 2), \right. \tag{2.13}
\]
where $(Tg)(x) := T(x)g(x)$ for $x \in \mathbb{R}^N$.

Proof. For $s \in (0, 1)$, by definition and (2.8), we have
\[
\|\delta_h^{(1)}(Tg)\|_{L^\infty(\mathcal{B})} \lesssim \|T\|_{L^\infty(\mathcal{L}_B)} \|\delta_h^{(1)}g\|_{L^\infty(\mathcal{B})} + \|\delta_h^{(1)}T\|_{L^\infty(\mathcal{L}_B)} \|g\|_{L^\infty(\mathcal{B})} \\
\lesssim |\delta_h^{(1)}g| \|T\|_{L^\infty(\mathcal{L}_B)} \|B^{s}_{p,\theta}(B)\| + |\delta_h^{(1)}T\| \|C_0^s(\mathcal{L}_B)\| \|g\|_{L^\infty(\mathcal{B})}.
\]
For $s \in [1, 2)$, noting that
\[
\delta_h^{(2)}(Tg) = T(\cdot + h)^{(2)} - T\delta_h^{(2)}g + \delta_h^{(1)}Tg(\cdot + h) + \delta_h^{(1)}g,
\]
by (2.8), we have
\[
\|\delta_h^{(2)}(Tg)\|_{L^\infty(\mathcal{B})} \lesssim \|T\|_{L^\infty(\mathcal{L}_B)} \|\delta_h^{(2)}g\|_{L^\infty(\mathcal{B})} + \|\delta_h^{(2)}T\|_{L^\infty(\mathcal{L}_B)} \|g\|_{L^\infty(\mathcal{B})} + \|\delta_h^{(1)}T\|_{L^\infty(\mathcal{L}_B)} \|\delta_h^{(1)}g\|_{L^\infty(\mathcal{B})} \\
\lesssim |\delta_h^{(2)}g| \|T\|_{L^\infty(\mathcal{L}_B)} \|B^{s}_{p,\theta}(B)\| + |\delta_h^{(1)}T\| \|C_0^s(\mathcal{L}_B)\| \|g\|_{L^\infty(\mathcal{B})} + |\delta_h^{(1)}g| \|T\|_{L^\infty(\mathcal{L}_B)} \|B^{s}_{p,\theta}(B)\|.
\]
The estimate (2.13) follows by $\|Tg\|_{L^\infty(\mathcal{B})} \lesssim \|T\|_{L^\infty(\mathcal{L}_B)} \|g\|_{L^\infty(\mathcal{B})}$ and (2.8) again. \hfill \Box

Remark 2.10. For any $s \in \mathbb{R}$ and $s' > |s|$, we in fact have (cf. [1])
\[
\|Tg\|_{B^{s'}_{p,\theta}(B)} \lesssim C \|T\|_{C_{p,\theta}^s(\mathcal{L}_B)} \|g\|_{B^s_{p,\theta}(B)}, \tag{2.14}
\]
where $C = C(\theta, m, p, s, s') > 0$.

The following commutator estimates will be used to improve the regularity in $x$.

Lemma 2.11. Let $p \in [1, \infty]$, $\gamma \in (0, 1)$ and $s \in (0, 2)$. For any $i = 1, \ldots, n$, there is a constant $C = C(\theta, m, p, s, \gamma, i) > 0$ such that for all $T \in C_{p,\theta}^s(\mathcal{L}_B)$, $g \in B^s_{p,\theta}(\mathcal{B})$ and $j \geq 0$,
\[
\|([R^x_{j, i}, T]g)\|_{B^s_{p,\theta}(B)} \lesssim C 2^{-\gamma j} \|T\|_{C_{p,\theta}^s(\mathcal{L}_B)} \|g\|_{B^s_{p,\theta}(B)}, \tag{2.15}
\]
where $[R^x_{j, i}, T]g := R^x_{j, i}(Tg) - TR^x_{j, i}(g)$ and $R^x_{j, i}$ is defined by (2.10). Moreover,
\[
\|([R^x_{j, i}, T]g)\|_{L^\infty(\mathcal{B})} \lesssim C 2^{-\gamma j} \|T\|_{C_{p,\theta}^s(\mathcal{L}_B)} \|g\|_{L^\infty(\mathcal{B})}. \tag{2.16}
\]

Proof. Without loss of generality, we assume $i = 1$. For simplicity, we write
\[
x_1 := (x_2, \ldots, x_n), \quad \tau_h^{(x_1)}g(x) := g(x_1 - h, x_1), \quad h \in \mathbb{R}^m,
\]
and
\[
\delta_h^{(x_1)}T(x) := T(x - h, x_1) - T(x, x_1).
\]
By definition we have
\[
[R^x_{j, 1}, T]g(x) = \int_{\mathbb{R}^m} \delta_j(h) \delta_h^{(x_1)}T(x) \tau_h^{(x_1)}g(x)dh.
\]
Note that by (2.13),
\[
\|\delta_h^{(x_1)}T\|_{B^s_{p,\theta}(B)} \lesssim \|\delta_h^{(x_1)}T\|_{C^s_0(\mathcal{L}_B)} \|g\|_{B^s_{p,\theta}(B)} \lesssim |h|^\gamma \|T\|_{C_{0}^s(\mathcal{L}_B)} \|g\|_{B^s_{p,\theta}(B)}.
\]
Hence,
\[
\| [R_{j}^{2}; T] g \|_{B^{s}_{p,q}(B)} \lesssim \left( \int_{R_{m=1}} \delta_{j}(h)[h]\gamma dh \right) [T] C_{j}^{\gamma}(C_{\delta_{j}}(L_{\delta})) \| g \|_{B^{s}_{p,q}(B)} \lesssim 2^{-\gamma j}[T] C_{j}^{\gamma}(C_{\delta_{j}}(L_{\delta})) \| g \|_{B^{s}_{p,q}(B)}.
\]
Thus we get (2.15). Similarly, we can prove (2.16).
\[\Box\]

Let \( \chi : \mathbb{R}^{N} \to [0, 1] \) be a smooth function with \( \chi = 1 \) on \( B_{\theta}^{0} \) and \( \chi = 0 \) on the complement of \( B_{\theta}^{0} \). For \( \delta > 0 \) and \( x_{0} \in \mathbb{R}^{N} \), define
\[
\chi_{x_{0}}^{\delta}(x) := \chi((x - x_{0})/\delta).
\]
We also introduce the following localized Besov space (see [36]).

**Lemma 2.12.** Let \( p \in [1, \infty] \) and \( s \in \mathbb{R} \). For any fixed \( \delta, \delta' > 0 \), there is a constant \( C = C(\theta, m, p, s, \delta, \delta') \geq 1 \) such that
\[
\| f \|_{\tilde{B}^{s}_{p,q}(B)} := \sup_{x_{0}} \| \chi_{x_{0}}^{\delta} f \|_{B^{s}_{p,q}(B)} \approx C \sup_{x_{0}} \| \chi_{x_{0}}^{\delta'} f \|_{B^{s}_{p,q}(B)}.
\]
In particular, the following localized Besov space is independent of the choice of \( \delta \):
\[
\tilde{B}^{s}_{p,q}(B) := \{ f \in S'(B) : \| f \|_{\tilde{B}^{s}_{p,q}(B)} < \infty \}.
\]

**Proof.** Without loss of generality, we assume \( \delta < \delta' \). For fixed \( x_{0} \in \mathbb{R}^{N} \), note the support of \( \chi_{x_{0}}^{\delta'} \) is contained in \( B_{2\delta'}^{0}(x_{0}) \). By a finite covering technique, one can find a number \( M \) independent of \( x_{0} \) and points \( \{x_{i}, i = 1, \ldots, M\} \) such that
\[
B_{2\delta'}^{0}(x_{0}) \subset \cup_{i=1}^{M} B_{\delta}^{0}(x_{i}).
\]
Let \( \varphi_{i}, i = 1, \ldots, M \) be the partition of unity associated with \( \{B_{\delta}^{0}(x_{i}), i = 1, \ldots, M\} \) so that
\[
\supp(\varphi_{i}) \subset B_{\delta}^{0}(x_{i}), \quad \sum_{i=1}^{M} \varphi_{i}(x) = 1 \quad \text{on} \quad \cup_{i=1}^{M} B_{\delta}^{0}(x_{i}).
\]
Since \( \chi_{x_{i}}^{\delta} = 1 \) on \( B_{\delta}^{0}(x_{i}) \), by (2.14) we have
\[
\| \delta_{i}^{\gamma} f \|_{B^{s}_{p,q}(B)} \lesssim \sum_{i=1}^{M} \| \chi_{x_{0}}^{\delta'} \varphi_{i} f \|_{B^{s}_{p,q}(B)} = \sum_{i=1}^{M} \| \chi_{x_{0}}^{\delta'} \varphi_{i} \chi_{x_{i}}^{\delta} f \|_{B^{s}_{p,q}(B)} \lesssim \sum_{i=1}^{M} \| \chi_{x_{0}}^{\delta'} \varphi_{i} \|_{C^{\gamma}_{\delta}} \| \chi_{x_{i}}^{\delta} f \|_{B^{s}_{p,q}(B)} \lesssim \sup_{x_{0}} \| \chi_{x_{0}}^{\delta} f \|_{B^{s}_{p,q}(B)}.
\]
The proof is complete. \[\Box\]

**Remark 2.13.** By (2.6) and the definition, we have
\[
\| \nabla_{x_{i}}^{k} f \|_{B^{s}_{p,q}(B)} \lesssim \| f \|_{B^{s+k\theta}_{p,q}(B)},
\]
and for \( s > 0 \) and \( 1 \leq p \leq p' \leq \infty \),
\[
\| f \|_{B^{s}_{p,q}(B)} \lesssim \| f \|_{B^{s'}_{p,q}(B)}.
\]
2.2. Itô-Wentzell’s formula for distribution-valued processes. In this subsection we recall the generalized Itô-Wentzell’s formula of distribution-valued Itô’s processes in [16]. Let $\mathbf{D}$ be the set of all $\mathcal{S}'$-valued function $u(t,\omega,\cdot) : \mathbb{R}_+ \times \Omega \to \mathcal{S}'$ with that, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$(t,\omega) \mapsto \langle u(t,\omega,\varphi) \rangle$$

is predictable. For $p \geq 1$, we denote by $\mathbf{D}^p$ the subset of $\mathbf{D}$ consisting of all $u$ such that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $T,R \in \mathbb{R}_+$,

$$\int_0^T \sup_{|x| \leq R} |\langle u(t,\omega,\varphi),\varphi(\cdot - x) \rangle|^p dt < \infty \quad (a.s.-\omega). \quad (2.19)$$

In the same way, we define $\mathbf{D}^p(\ell^2)$ by $u = (u_1,\cdots)$ with $u_i \in \mathbf{D}$ and replacing $| \cdot |$ in (2.19) by $\| \cdot \|_{\ell^2}$.

**Definition 2.14** (Generalized Itô’s process). Let $u,f \in \mathbf{D}$, $g \in \mathbf{D}(\ell^2)$ and $\tau$ be a stopping time. One says that the equality

$$du(t,x) = f(t,x)dt + g^k(t,x)dW_t^k, \quad t \leq \tau, \quad (2.20)$$

holds in the distribution sense if $1_{[0,\tau]}f \in \mathbf{D}^1$, $1_{[0,\tau]}g \in \mathbf{D}^2(\ell^2)$ and for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, with probability one, for all $t \in \mathbb{R}_+$,

$$\langle u(t \land \tau),\varphi \rangle = \langle u(0),\varphi \rangle + \int_0^{t \land \tau} \langle f(s),\varphi \rangle ds + \int_0^{t \land \tau} \langle g^k(s),\varphi \rangle dW_s^k. \quad (2.21)$$

Let $X_t$ be an $\mathbb{R}^N$-valued stochastic process given by

$$X_t^i = \int_0^t b_t^i ds + \int_0^t \sigma_t^{ik} dW_s^k, \quad i = 1,\cdots,N,$$

where $b_t, \sigma_t^{ik}$ are predictable $\mathbb{R}^N$-valued processes with

$$\int_0^T (|b_t| + \|\sigma_t\|^2_{\mathbb{R}^N \otimes \ell^2}) dt < \infty, \quad a.s., \quad \forall T > 0.$$

The following generalized Itô-Wentzell’s formula are proven in [16, Theorem 1.1].

**Lemma 2.15** (Itô-Wentzell’s formula). Let $f,u \in \mathbf{D}$, $g \in \mathbf{D}(\ell^2)$ and $\tau$ be a stopping time, $X$ an Itô’s process. Suppose (2.20) holds. Then $w(t,x) = u(t,x+X_t)$ satisfies

$$dw(t,x) = \left[ \frac{1}{2} \sigma_t^{ik} \sigma_t^{jk} \partial_j w(t,x) + b_t \cdot \nabla w(t,x) \right] dt$$

$$+ \left[ f(t,x+X_t) + \partial_t g_t^k(x+X_t) \sigma_t^{ik} \right] dt$$

$$+ \left[ g_t^k(x+X_t) + \sigma_t^{ik} \partial_i w(t,x) \right] dW_t^k, \quad t \leq \tau,$$

in the distribution sense.

3. Model equations

Let $W_t$ be a $d$-dimensional standard Brownian motion. Define

$$(X_t, V_t) := \left( \sqrt{2} \int_0^t W_s ds, \sqrt{2} W_t \right)$$

and the kinetic semigroup

$$P_t f(x,v) := E f(x + tv + X_t, v + V_t) = (\Gamma_{tp_t}) \ast (\Gamma_{t} f)(x,v), \quad (3.1)$$
where \( p_t \) is the density of \((X_t, V_t)\) given by

\[
p_t(x, v) = \left( \frac{2\pi t^4}{3} \right)^{\frac{d}{2}} \exp \left( -\frac{3|x|^2 + |3x - 2tv|^2}{4t^3} \right),
\]

and

\[
\Gamma_t f(x, v) := f(x + tv, v).
\]

It is easy to see that for any \( \lambda > 0 \),

\[
p_{\lambda t}(x, v) = \lambda^{-2d} p_{t}(\lambda^{-3/2} x, \lambda^{-1/2} v)
\]

and for \( f \in \mathcal{S} \), by Itô’s formula,

\[
\partial_t P_t f = [\Delta_v + v \cdot \nabla_x] P_t f,
\]

which also holds for \( f \in \mathcal{S}'(\mathbb{B}) \) in the distributional sense by duality.

In the remainder of this paper, we take the parameters in Subsection 2.1 as

\[
N = 2d, \quad n = 2, \quad m_1 = m_2 = d, \quad \theta = (3, 1).
\]

For notational simplicity, we use \( z = (x, v) \) to denote a generic point in \( \mathbb{R}^{2d} \), and for a Banach space \( \mathbb{B} \), and for \( T > 0 \) and \( p \in [1, \infty] \), we write

\[
L^p_T(\mathbb{B}) := L^p([0, T]; \mathbb{B}), \quad L^p_{x,\omega}(\mathbb{B}) := L^p_T(L^p(\mathbb{B})).
\]

The operator \( \Gamma_t \) in (3.2) will play a crucial role below. Note that it is in general not a bounded operator in \( B^s_{p, \theta} \), but we obviously have

\[
\|\Gamma_t f\|_{L^p_T} = \|f\|_{L^p_T}.
\]

### 3.1. Estimates of kinetic semigroup \( P_t \)

In this subsection we show some basic estimates about \( P_t \) in anisotropic Besov spaces. First of all we have the following estimate about the heat kernel \( p_t \) that is similar to [11, Lemma 5.1].

**Lemma 3.1.** For any \( l \geq 0 \), there is a constant \( C = C(d, l) > 0 \) such that

\[
\|\mathcal{R}^\theta_1 \Gamma_t p_l\|_{L^1} \leq C (4t^l)^{-1}, \quad \forall j \in \mathbb{N}, \quad t > 0.
\]

**Proof.** Note that

\[
\Gamma_t p_l(x, v) = t^{-2d} p_1(t^{-\frac{2}{3}} x + t^{-\frac{1}{2}} v, t^{-\frac{1}{2}} v).
\]

Let \( h := t^{-\frac{2}{3}} 2^{-j} \). Denote the left hand side of (3.4) by \( \mathcal{J} \). By the change of variable, we have

\[
\mathcal{J} = \int_{\mathbb{R}^{2d}} \left| \int_{\mathbb{R}^{2d}} \partial_1^\theta (\bar{x}, \bar{v}) p_1(x - h^3 \bar{x} + v - h\bar{v}, v - h\bar{v}) d\bar{x} d\bar{v} \right| \, dx dv.
\]

We define the following operators: for \( m \in \mathbb{N} \),

\[
\Lambda^m f(x, v) := (\Delta_x + \Delta_v^3)^m f(x, v),
\]

and for \( \hat{f} \in \mathcal{S} \) with \( 0 \notin \text{support of } \hat{f} \),

\[
\hat{\Lambda}^m f(\xi, \eta) := (|\xi|^2 + |\eta|)^{-m} \hat{f}(\xi, \eta).
\]

Let

\[
H(\bar{x}, \bar{v}, x, v) := p_1(x - h^3 \bar{x} + v - h\bar{v}, v - h\bar{v})
\]

Since \( \phi_1^\theta(\xi, \eta) \in B^\theta_{1} \setminus B^\theta_{1} \), for \( m \in \mathbb{N} \), we have

\[
\mathcal{J} = \int_{\mathbb{R}^{2d}} \left| \int_{\mathbb{R}^{2d}} \Lambda^{-m} \phi_1^\theta (\bar{x}, \bar{v}) \Lambda^m H(\cdot, \cdot, x, v)(\bar{x}, \bar{v}) d\bar{x} d\bar{v} \right| \, dx dv
\]

\[
\leq \int_{\mathbb{R}^{2d}} |\Lambda^{-m} \phi_1^\theta (\bar{x}, \bar{v})| d\bar{x} d\bar{v} \int_{\mathbb{R}^{2d}} |\Lambda^m H(\cdot, \cdot, x, v)| (\bar{x}, \bar{v}) d\bar{x} d\bar{v}.
\]
By the chain rule and elementary calculations, we have
\[
\sup_{\bar{x},\bar{v}} \int_{\mathbb{R}^{2d}} |\Lambda^m H(\cdot, \cdot, x, v)| (\bar{x}, \bar{v}) dx dv \lesssim h^6m.
\]
Since $\Lambda^{-m} \tilde{\phi}^0_1$ is a Schwartz function, we thus have
\[
\mathcal{I} \lesssim h^6m \int_{\mathbb{R}^{2d}} |\Lambda^{-m} \tilde{\phi}^0_1(\bar{x}, \bar{v})| d\bar{x} d\bar{v} \lesssim h^6m.
\]
Moreover, it is easy to see that
\[
\mathcal{I} \lesssim \int_{\mathbb{R}^{2d}} |\tilde{\phi}^0_1(\bar{x}, \bar{v})| d\bar{x} d\bar{v} \lesssim 1.
\]
The desired estimate now follows by the above two estimates.

By this lemma, we can show the following crucial estimates.

**Lemma 3.2.** (i) For any $p \in [1, \infty]$, $\alpha \in \mathbb{R}$, $l \geq 0$ and $T \geq 1$, there is a constant $C = C(T, d, \alpha, l) > 0$ such that for all $t \in [0, T]$ and $j \in \mathbb{N}_0$,
\[
\|R_j^0 P_t f\|_{L^p_2(\mathcal{B})} \lesssim C 2^{-j\alpha} h_{l,\alpha}(t) \|f\|_{\mathcal{B}^{\alpha,\alpha}_p(\mathcal{B})}, \tag{3.5}
\]
where
\[
h_{l,\alpha}(t) := (1 + t^{-1})(1 + t)^{|\alpha|}.
\]
(ii) Let $p \in [1, \infty]$, $\alpha \in \mathbb{R}$, $r \in [1, \infty)$, $q \in [r, \infty]$ and $\kappa, \kappa' \in (0, 1]$ with $\frac{\kappa}{r} > \frac{\kappa'}{q}$ when $q > r$ and $\kappa' \geq \kappa$ when $q = r$. For any $T \geq 1$, there is a constant $C = C(T, d, \alpha, r, q, \kappa, \kappa') > 0$ such that for all $j \in \mathbb{N}_0$ and $t \in [0, T]$,
\[
\left( \int_0^t (t - s)^{\kappa'-1} \|R_j^0 P_{t-s} f(s)\|_{L^q_p(\mathcal{B})}^q ds \right)^{1/q} \lesssim C 2^{-(\alpha + 2(\frac{q'}{q} - \frac{\kappa}{q}))j} \left( \int_0^t (t - s)^{\kappa-1} \|f(s)\|_{\mathcal{B}^{\alpha,\alpha}_p(\mathcal{B})}^q ds \right)^{1/q}. \tag{3.6}
\]
**Proof.** (i) By definition (3.1), we have (see [11, Lemm 6.7]),
\[
R_j^0 P_t f = \sum_{l \in \Theta^j_0} (R_j^0 \Gamma_t p_l) * (\Gamma_t R_j^0 f), \quad j \in \mathbb{N}_0,
\]
where
\[
\Theta^j_0 := \left\{ \ell \in \mathbb{N}_0 \mid 2^\ell \leq 2^4(1 + t) \right\}
\]
and for $j \geq 1$,
\[
\Theta^j := \left\{ \ell \in \mathbb{N}_0 \mid 2^\ell \leq 2^4(2^j + t2^3), \quad 2^\ell \leq 2^4(2^\ell + t2^{3\ell}) \right\}.
\]
If $j \geq 1$, then by Young’s inequality, we have for any $l \geq 0$ and $\alpha \neq 0$,
\[
\|R_j^0 P_l f\|_{L^p_2(\mathcal{B})} \leq \|R_j^0 \Gamma_t p_l\|_{L^2} \sum_{\ell \in \Theta^j_0} \|\Gamma_t R_j^0 f\|_{L^p_2(\mathcal{B})} \leq \sum_{\ell \in \Theta^j_0} (t4^j)^{-l} \|f\|_{\mathcal{B}^{\alpha,\alpha}_p(\mathcal{B})} \lesssim (t4^j)^{-l} \|f\|_{\mathcal{B}^{\alpha,\alpha}_p(\mathcal{B})}.
\]
where the last step is due to [11, Lemma 6.7]. By the arbitrariness of $l \geq 0$,
\[
\|R_j^0 P_l f\|_{L^p_2(\mathcal{B})} \leq 2^{-j\alpha} (1 + (t4^j)^{-l}) \|f\|_{\mathcal{B}^{\alpha,\alpha}_p(\mathcal{B})}.
\]
If $j = 0$, then we similarly have for $\alpha \neq 0$,
\[
\|R_0^{\alpha} P_t f\|_{L^p_{\alpha}(B)} \lesssim \|R_0^{\alpha} \Gamma_l P_t\|_{L^p_{\alpha}(B)} \sum_{l \in \Theta_0^c} \|\Gamma_l R_0^{\alpha} f\|_{L^p_{\alpha}(B)} \lesssim \sum_{l \in \Theta_0^c} 2^{-\alpha l} \|f\|_{B^\alpha_{p,\theta}(B)} \lesssim (1 + t)^{|\alpha|} \|f\|_{B^\alpha_{p,\theta}(B)}.
\]
Thus we obtain (3.5) for $\alpha \neq 0$. For $\alpha = 0$, it follows by interpolation theorem.

(ii) Denote the left hand side of (3.6) by $\mathcal{J}$. For $q \in (r, \infty)$, let $q' \in [r, \infty)$ with $\frac{1}{q'} + \frac{1}{q} = \frac{1}{r}$. By (3.5) and Hölder’s inequality, we have
\[
\mathcal{J} \lesssim 2^{-j\alpha} \left( \int_0^t (t - s)^{\alpha'-1} h_{\ell,\alpha}'((t-s)^{4^{l}}) \|f(s)\|_{B^\alpha_{p,\theta}(B)}^2 ds \right)^{1/r}.
\]
\[
\lesssim 2^{-j\alpha} \left( \int_0^t h_{\ell,\alpha}'((t-s)^{4^{l}}) (t-s)^{\frac{\alpha'}{4} - \frac{\alpha'}{4} (1 + |\theta|)} ds \right)^{1/q'}
\times \left( \int_0^t \|f(s)\|_{B^\alpha_{p,\theta}(B)}^q (t-s)^{\alpha'-1} ds \right)^{1/q}.
\]
For the first integral denoted by $\mathcal{J}_0$, by the change of variable we have
\[
\mathcal{J}_0 = 4^{-\left(\frac{\alpha}{q} + \frac{\alpha}{q'} - \frac{\alpha'}{4} - \frac{\alpha'}{4} \right)} \left( \int_0^{t4^{l}} h_{\ell,\alpha}'(s) s^{\frac{\alpha}{4} - \frac{\alpha}{4} (1 + |\theta|)} ds \right)^{1/q'}
\lesssim 4^{-\left(\frac{\alpha + 2\alpha'}{4} - \frac{\alpha'}{4}\right)} \left( \int_0^{\infty} (1 + s^{-q'}) (1 + s)^{|\theta| q'} s^{\frac{\alpha}{4} - \frac{\alpha}{4} (1 + |\theta|)} ds \right)^{1/q'}
\]
In particular, for $l$ large enough, the last integral is finite. Thus,
\[
\mathcal{J} \lesssim 2^{-j\alpha} \left( \int_0^t \|f(s)\|_{B^\alpha_{p,\theta}(B)}^q (t-s)^{\alpha'-1} ds \right)^{1/q}.
\]
For $q = r$, it is similar. The proof is complete. 

Next we show the following estimate about the kinetic semigroup $P_t$.

**Lemma 3.3.** For any $\alpha \in \mathbb{R}$, $\kappa \geq 0$ and $T > 0$, there exists a constant $C = C(T, d, \alpha, \kappa) > 0$ such that for all $t \in (0, T]$,
\[
\|P_t f\|_{B^{\alpha + \kappa}_{p,\theta}(B)} \lesssim C \ t^{-\frac{\alpha}{2}} \|f\|_{B^\alpha_{p,\theta}(B)}, \quad \|P_t f\|_{B^{\alpha - \kappa}_{p,\theta}(B)} \lesssim C \ t^{-\frac{\alpha}{2}} \|f\|_{B^\alpha_{p,\theta}(B)}.
\]  

**Proof.** Let $T > 0$. By (3.5), we have for any $j \in \mathbb{N}_0$ and $l \geq |\alpha| + \frac{\alpha}{2}$,
\[
\|R_0^{\alpha} f\|_{L^p_{\alpha}(B)} \lesssim 2^{-\alpha j} \|h_{\ell,\alpha}(t^{4^l})\|_{\text{sup}_{s \geq 0} s^{\frac{\alpha}{2}} h_{\ell,\alpha}(s)}.
\]
where $\text{sup}_{s \geq 0} s^{\frac{\alpha}{2}} h_{\ell,\alpha}(s) < \infty$ for $l \geq |\alpha| + \frac{\alpha}{2}$ and $\kappa \geq 0$. Thus by definition,
\[
\|P_t f\|_{B^{\alpha + \kappa}_{p,\theta}(B)} = \sup_{j \geq 0} 2^{j(\alpha + \kappa)} \|R_j^{\alpha} P_t f\|_{L^p_{\alpha}(B)} \lesssim t^{-\frac{\alpha}{2}} \|f\|_{B^\alpha_{p,\theta}(B)}.
\]
which gives the first estimate in (3.7).

Next we prove the second estimate in (3.7). Let $\chi$ be as in (2.17). Fix $\delta > 0$, $z_0 := (x_0, v_0) \in \mathbb{R}^d \times \mathbb{R}^d$, and for $t > 0$, let
\[
z_{0,t} := (x_0 - tv_0, v_0), \quad \chi_{z_0}^\delta(t, z) := \chi((t - z_{0,t})/\delta)
\]
and define
\[ u(t, z) := P_t f(z), \quad u^\delta_{z_0}(t, z) := \chi^\delta_{z_0}(t, z)u(t, z). \] (3.10)
By definition and (3.3), it is easy to see that
\[ \partial_t u^\delta_{z_0} = \Delta_v u^\delta_{z_0} + v \cdot \nabla_x u^\delta_{z_0} + F^\delta_{z_0}, \]
where
\[ F^\delta_{z_0} := -2\nabla_v u \cdot \nabla_v \chi^\delta_{z_0} - ((v - v_0) \cdot \nabla_x \chi^\delta_{z_0} + \Delta_v \chi^\delta_{z_0})u. \]

By Duhamel’s formula, we have
\[ u^\delta_{z_0}(t) = P_t(u^\delta_{z_0}(0)) + \int_0^t P_{t-s} F^\delta_{z_0}(s)ds =: I_1(t) + I_2(t). \]
For \( I_1(t) \), since \( u^\delta_{z_0}(0, z) = f(z)\chi^\delta_{z_0}(0, z) \), by (3.8), we clearly have
\[ \| R^\theta_j F^\delta_{z_0}(t) \|_{L^r_\alpha(B)} \lesssim 2^{-(\alpha+j)} t^{-\frac{\alpha}{2}} \| u^\delta_{z_0}(0) \|_{B^\alpha_{p,\theta}(B)} \lesssim 2^{-(\alpha+j)} t^{-\frac{\alpha}{2}} \| f \|_{B^\alpha_{p,\theta}(B)}. \]

For \( I_2(t) \), noting that by \( \chi^\delta_{z_0} = 1 \) on the support of \( \chi^\delta_{z_0} \),
\[ F^\delta_{z_0} = -2\nabla_v u^\delta_{z_0} \cdot \nabla_v \chi^\delta_{z_0} - ((v - v_0) \cdot \nabla_x \chi^\delta_{z_0} + \Delta_v \chi^\delta_{z_0})u^\delta_{z_0}, \]
by (3.6) with \( q = r = 1 \) and (2.14), we have
\[ \| R^\theta_j I_2(t) \|_{L^r_\alpha(B)} \lesssim 2^{-(\alpha+j)} \int_0^t (t-s)^{-\frac{\alpha}{2}} \| F^\delta_{z_0}(s) \|_{B^\alpha_{p,\theta}(B)} ds \]
\[ \lesssim 2^{-(\alpha+j)} \int_0^t (t-s)^{-\frac{\alpha}{2}} \| u^\delta_{z_0}(s) \|_{B^{\alpha+1}_{p,\theta}(B)} ds \]
\[ \lesssim 2^{-(\alpha+j)} \int_0^t (t-s)^{-\frac{\alpha}{2}} \| u(s) \|_{B^{\alpha+1}_{p,\theta}(B)} ds. \]

Thus, combining the above calculations, we obtain that for any \( \kappa \in [0, 2) \),
\[ \| u(t) \|_{B^{\alpha+\kappa}_{p,\theta}(B)} = \sup_{z_0} \sup_{j \geq 0} 2^{\alpha+j} \| R^\theta_j u^\delta_{z_0}(t) \|_{L^r_\alpha(B)} \]
\[ \lesssim t^{-\frac{\alpha}{2}} \| f \|_{B^\alpha_{p,\theta}(B)} + \int_0^t (t-s)^{-\frac{\alpha}{2}} \| u(s) \|_{B^{\alpha+1}_{p,\theta}(B)} ds. \] (3.11)

In particular, letting \( \kappa = 1 \), we get
\[ \| u(t) \|_{B^{\alpha+1}_{p,\theta}(B)} \lesssim t^{-\frac{\alpha}{2}} \| f \|_{B^\alpha_{p,\theta}(B)} + \int_0^t (t-s)^{-\frac{\alpha}{2}} \| u(s) \|_{B^{\alpha+1}_{p,\theta}(B)} ds, \]
which implies by Gronwall’s inequality of Volterra’s type (see [32]) that
\[ \| u(t) \|_{B^{\alpha+1}_{p,\theta}(B)} \lesssim t^{-\frac{\alpha}{2}} \| f \|_{B^\alpha_{p,\theta}(B)}. \]

Substituting this into (3.11), we obtain the second estimate in (3.7) for \( \kappa \in (0, 2) \).
For general \( \kappa \geq 2 \), it follows by the semigroup property of \( P_t \). \( \square \)

The following lemma will be used to show the regularity in time variable.

Lemma 3.4. (i) For any \( p \in [1, \infty) \), \( \alpha \in (0, 1) \) and \( R > 0 \), there is a constant \( C = C(d, p, \alpha, R) > 0 \) such that for all \( t > 0 \),
\[ \| \Gamma_t(\chi_0^R f) - \chi_0^R f \|_{L^r_\alpha(B)} \lesssim C t^{\frac{\alpha}{2}} \| \chi_0^R f \|_{B^\alpha_{p,\theta}(B)}, \] (3.12)
and for any \( \gamma \in (0, \frac{\alpha}{2}] \) and \( T > 0 \), there is a \( C = C(T, d, p, \alpha, \gamma, R) > 0 \) such that
\[ \| \Gamma_t(\chi_0^R f) - \chi_0^R f \|_{B^\gamma_{p,\theta}(B)} \lesssim C t^{\frac{\alpha+\gamma}{2}} \| \chi_0^R f \|_{B^\alpha_{p,\theta}(B)}, \quad t \in [0, T]. \] (3.13)
(ii) Let \( p \in [1, \infty) \). For any \( f \in L_p^x(\mathbb{B}) \), it holds that
\[
\lim_{t \to 0} \| P_t f - f \|_{L_p^x(\mathbb{B})} = 0, \quad (3.14)
\]
and for any \( \gamma < \alpha \) and \( f \in B_{p,\rho}^\gamma(\mathbb{B}) \),
\[
\lim_{t \to 0} \| P_t f - f \|_{B_{p,\rho}^\gamma(\mathbb{B})} = 0. \quad (3.15)
\]
(iii) For any \( p \in [1, \infty] \) and \( \alpha \in (0, 1) \), there is a constant \( C = C(d, \alpha, p) > 0 \) such that for all \( t \geq 0 \),
\[
\| P_t f - \Gamma_t f \|_{L_p^x(\mathbb{B})} \lesssim C t^{\frac{\alpha}{2}} \| f \|_{B_{p,\rho}^\gamma(\mathbb{B})}, \quad (3.16)
\]
and for any \( \gamma \in (0, \frac{\alpha}{3}] \), there is a \( C = C(d, \alpha, \gamma, p) > 0 \) such that for all \( t \geq 0 \),
\[
\| P_t f - \Gamma_t f \|_{B_{p,\rho}^\gamma(\mathbb{B})} \lesssim C t^{\frac{\alpha - \gamma}{2}} \| f \|_{B_{p,\rho}^\gamma(\mathbb{B})}. \quad (3.17)
\]
Proof. (i) Write \( R_t := \chi^R \). Let \( R_j^x \) be defined as in (2.10). Note that
\[
\| \Gamma_t R_j^x f_R(x, v) - R_j^x f_R(x, v) \|_{L_p^x(\mathbb{B})} \leq \int_0^1 \| \nabla_x R_j^x f_R(x + s t v, v) \| ds.
\]
Since \( f_R \) has support in \( B_{2R}^\theta \), by Bernstein’s inequality (2.4), we have
\[
\| \Gamma_t R_j^x f_R - R_j^x f_R \|_{L_p^x(\mathbb{B})} \leq 2 R t \| \nabla_x R_j^x f_R \|_{L_p^x(\mathbb{B})} \lesssim t 2^j \| R_j^x f_R \|_{L_p^x(\mathbb{B})},
\]
and also,
\[
\| \Gamma_t R_j^x f_R - R_j^x f_R \|_{L_p^x(\mathbb{B})} \leq 2 \| R_j^x f_R \|_{L_p^x(\mathbb{B})}.
\]
Therefore,
\[
\| \Gamma_t f_R - f_R \|_{L_p^x(\mathbb{B})} \leq \sum_{j \geq 0} \| \Gamma_t R_j^x f_R - R_j^x f_R \|_{L_p^x(\mathbb{B})} \lesssim \sum_{j \geq 0} \left( \left( t 2^j \right) \wedge 1 \right) \| R_j^x f_R \|_{L_p^x(\mathbb{B})}
\]
\[
\lesssim \sum_{j \geq 0} \left( \left( t 2^j \right) \wedge 1 \right) 2^{-j} \| f_R \|_{B_{p,\rho}^{x/3}(\mathbb{B})} \lesssim t^{\frac{\alpha}{2}} \| f_R \|_{B_{p,\rho}^\gamma(\mathbb{B})},
\]
where the last step is due to Lemma 2.7. For (3.13), noting that by (2.8),
\[
\| \delta_h^{(1)}(\Gamma_t f_R) \|_{L_p^x(\mathbb{B})} = \| \delta_h^{(1)}(\Gamma_t h) f_R \|_{L_p^x(\mathbb{B})} \lesssim \| \Gamma_t h \|_{\mathfrak{g}} \| f_R \|_{B_{p,\rho}^\gamma(\mathbb{B})},
\]
where for \( h = (h_1, h_2) \in \mathbb{R}^{2d}, \Gamma_t h = (h_1 + t h_2, h_2) \), we have
\[
\| \delta_h^{(1)}(\Gamma_t f_R - f_R) \|_{L_p^x(\mathbb{B})} \lesssim \| \delta_h^{(1)}(\Gamma_t f) \|_{L_p^x(\mathbb{B})} + \| \delta_h^{(1)} f_R \|_{L_p^x(\mathbb{B})}
\]
\[
\lesssim (|\Gamma_t h |_{\mathfrak{g}} + | h_2 |_{\mathfrak{g}}) \| f_R \|_{B_{p,\rho}^\gamma(\mathbb{B})}.
\]
Moreover, by (3.12) we also have
\[
\| \delta_h^{(1)}(\Gamma_t f_R - f_R) \|_{L_p^x(\mathbb{B})} \leq 2 \| \Gamma_t f_R - f_R \|_{L_p^x(\mathbb{B})} \lesssim t^{\frac{\alpha}{2}} \| f_R \|_{B_{p,\rho}^{x/3}(\mathbb{B})}.
\]
Hence,
\[
\| \delta_h^{(1)}(\Gamma_t f_R - f_R) \|_{L_p^x(\mathbb{B})} \lesssim \left( t^{\frac{\alpha}{2}} \wedge (|\Gamma_t h |_{\mathfrak{g}} + | h_2 |_{\mathfrak{g}}) \right) \| f_R \|_{B_{p,\rho}^\gamma(\mathbb{B})}
\]
\[
\lesssim C t^{\frac{\alpha}{2}} \| h_2 |_{\mathfrak{g}} \| f_R \|_{B_{p,\rho}^\gamma(\mathbb{B})}.
\]
Thus we obtain (3.13) by (2.8).
(ii) Note that by (3.1),
\[
\| P_t f - f \|_{L_p^x(\mathbb{B})} \leq \| \Gamma_t P_t * \Gamma_t f - \Gamma_t f \|_{L_p^x(\mathbb{B})} + \| \Gamma_t f - f \|_{L_p^x(\mathbb{B})}
\]
\[
= \| P_t * f - f \|_{L_p^x(\mathbb{B})} + \| \Gamma_t f - f \|_{L_p^x(\mathbb{B})},
\]
which in turn yields (3.14). Moreover, for \( \gamma < \alpha \), we have
\[
\|P_t f - f\|_{B^\gamma_{p,\alpha}(B)} = \sup_{j \geq 0} 2^j \|R_j^\gamma (P_t f - f)\|_{L^\infty(B)} \leq \sum_{j \geq 0} 2^j \|R_j^\gamma (P_t f - f)\|_{L^\infty(B)}.
\]
Since \( \|R_j^\gamma P_t f\|_{L^\infty(B)} \lesssim 2^{-\alpha j} \|f\|_{B^\alpha_{p,\alpha}(B)} \), by the dominated convergence theorem,
\[
\lim_{t \downarrow 0} \|P_t f - f\|_{B^\gamma_{p,\alpha}(B)} = \sum_{j \geq 0} 2^j \lim_{t \downarrow 0} \|R_j^\gamma (P_t f - f)\|_{L^\infty(B)} \overset{(3.14)}{=} 0.
\]
(iii) Note that by (2.8),
\[
\|\Gamma_t f(\cdot + \bar{x}, \cdot + \bar{v}) - \Gamma_t f\|_{L^\infty(B)} = \|f(\cdot + \bar{x} + t\bar{v}, \cdot + \bar{v}) - f\|_{L^\infty(B)} \lesssim (|\bar{x} + t\bar{v}|^\frac{\gamma}{\alpha} + |\bar{v}|^\alpha) \|f\|_{B^\alpha_{p,\alpha}(B)}.
\]
(3.18)
Thus, by definition we have
\[
\|P_t f - \Gamma_t f\|_{L^\infty(B)} \leq \int_{\mathbb{R}^{2d}} \Gamma_t p_t(\bar{x}, \bar{v}) \|\Gamma_t f(\cdot + \bar{x}, \cdot + \bar{v}) - \Gamma_t f\|_{L^\infty(B)} d\bar{x} d\bar{v}
\]
\[
\leq \int_{\mathbb{R}^{2d}} \Gamma_t p_t(\bar{x}, \bar{v})(|\bar{x} + t\bar{v}|^\frac{\gamma}{\alpha} + |\bar{v}|^\alpha) \|f\|_{B^\alpha_{p,\alpha}(B)} d\bar{x} d\bar{v}
\]
\[
= \int_{\mathbb{R}^{2d}} p_t(\bar{x}, \bar{v})(|\bar{x}|^\frac{\gamma}{\alpha} + |\bar{v}|^\alpha) d\bar{x} d\bar{v} \|f\|_{B^\alpha_{p,\alpha}(B)} \lesssim t^\frac{\gamma}{\alpha} \|f\|_{B^\alpha_{p,\alpha}(B)}.
\]
(For 3.17, set \( h := t^{-\frac{\gamma}{2}} 2^{-j} \). Since \( \|\Gamma_t p_t\|_{L^1} = 1 \), we have for \( j \geq 0 \),
\[
\|R_j^\gamma P_t f - R_j^\gamma \Gamma_t f\|_{L^\infty(B)} = \|(\Gamma_t p_t) * (R_j^\gamma \Gamma_t f) - R_j^\gamma \Gamma_t f\|_{L^\infty(B)}
\]
\[
\leq \int_{\mathbb{R}^{2d}} \Gamma_t p_t(\bar{x}, \bar{v}) \|R_j^\gamma \Gamma_t f(\cdot + \bar{x}, \cdot + \bar{v}) - R_j^\gamma \Gamma_t f\|_{L^\infty(B)} d\bar{x} d\bar{v}
\]
\[
\leq \int_{\mathbb{R}^{2d}} \Gamma_t p_t(\bar{x}, \bar{v}) \left( |\bar{x}| \|\nabla_x R_j^\gamma \Gamma_t f\|_{L^\infty(B)} + |\bar{v}| \|\nabla_v R_j^\gamma \Gamma_t f\|_{L^\infty(B)} \right) d\bar{x} d\bar{v}
\]
\[
\lesssim (t^{1/2} 2^j + t^{1/2} 2^j) \|R_j^\gamma \Gamma_t f\|_{L^\infty(B)} = (h^{-3} + h^{-1}) \|R_j^\gamma \Gamma_t f\|_{L^\infty(B)}.
\]
Moreover, we clearly have
\[
\|(R_j^\gamma \Gamma_t p_t) * (\Gamma_t f) - R_j^\gamma \Gamma_t f\|_{L^\infty(B)} \lesssim 2 \|R_j^\gamma \Gamma_t f\|_{L^\infty(B)}.
\]
Thus
\[
\|R_j^\gamma P_t f - R_j^\gamma \Gamma_t f\|_{L^\infty(B)} \lesssim ((h^{-3} + h^{-1}) \wedge 1) \|R_j^\gamma \Gamma_t f\|_{L^\infty(B)}.
\]
On the other hand, for \( j \geq 1 \), noting that
\[
R_j^\gamma \Gamma_t f(x, v) = \int_{\mathbb{R}^{2d}} \tilde{\phi}_j^\gamma(\bar{x}, \bar{v}) \Gamma_t f(x - \bar{x}, v - \bar{v}) d\bar{x} d\bar{v},
\]
by (3.18), we have
\[
\|R_j^\gamma \Gamma_t f\|_{L^\infty(B)} \leq \int_{\mathbb{R}^{2d}} \tilde{\phi}_j^\gamma(\bar{x}, \bar{v}) \|\Gamma_t f(\cdot - \bar{x}, \cdot - \bar{v}) - \Gamma_t f\|_{L^\infty(B)} d\bar{x} d\bar{v}
\]
\[
\lesssim \int_{\mathbb{R}^{2d}} \tilde{\phi}_j^\gamma(\bar{x}, \bar{v})(|\bar{x} + t\bar{v}|^\frac{\gamma}{\alpha} + |\bar{v}|^\alpha) \|f\|_{B^\alpha_{p,\alpha}(B)} d\bar{x} d\bar{v}
\]
\[
\lesssim (t^{1/2} 2^j + t^{-\alpha j}) \|f\|_{B^\alpha_{p,\alpha}(B)} = t^\frac{\gamma}{\alpha} (h^\frac{\gamma}{\alpha} + h^\alpha) \|f\|_{B^\alpha_{p,\alpha}(B)}.
\]
Therefore, for \( j \geq 1 \) and \( \gamma \in (0, \frac{\alpha}{2}) \),
\[
\|R_j^\gamma P_t f - R_j^\gamma \Gamma_t f\|_{L^\infty(B)} \lesssim t^\frac{\gamma}{\alpha} ((h^{-3} + h^{-1}) \wedge 1) (h^\frac{\gamma}{\alpha} + h^\alpha) \|f\|_{B^\alpha_{p,\alpha}(B)}
\]
\[
\leq 2t^\frac{\gamma}{\alpha} h^\gamma \|f\|_{B^\alpha_{p,\alpha}(B)} = 2t^\frac{\gamma}{\alpha} 2^{-\gamma j} \|f\|_{B^\alpha_{p,\alpha}(B)}.
\]
which together with
\[\|\mathcal{R}_n^\alpha P_t f - \mathcal{R}_n^\alpha \Gamma_t f\|_{L^p_x(B)} \lesssim (\langle t^2 \rangle + 1)\|\mathcal{R}_n^\alpha \Gamma_t f\|_{L^p_x(B)} \lesssim \frac{\alpha - \gamma}{\gamma} \|f\|_{L^p_x(B)}\]
yields (3.17).

3.2. Simple model equations. We consider the following simple model equation:
\[du = [\nu \Delta v u + v \cdot \nabla_x u + f]dt + g^k dW_t^k, \quad u(0) = u_0, \tag{3.19}\]
where \(\nu > 0\) and \(f \in D^1, \ g \in D^2(\ell^2)\), and \(u_0\) is an \(\mathcal{F}_0\)-measurable \(\mathcal{S}'\)-valued random variable.

**Definition 3.5.** An \(\mathcal{S}'\)-valued predictable process \(u\) is called a distribution solution of the above SKE if for any \(\varphi \in \mathcal{S}\) and \(t \geq 0\),
\[\langle u(t), \varphi \rangle = \langle u_0, \varphi \rangle + \int_0^t \langle u(s), \nu \Delta \varphi - v \cdot \nabla_x \varphi \rangle ds + \int_0^t (f(s), \varphi) ds + \int_0^t (g^k(s), \varphi) dW_s^k, \quad \mathbb{P} - a.s.\]

We have the following regularity estimate for this model equation.

**Theorem 3.6.** For any \(p \geq 2, \ q \in [2, \infty], \ \beta \in \mathbb{R}, \ \kappa \in (0, 1], \ \alpha \leq \beta + 2(1 - \frac{\kappa}{q})\) and \(T > 0\), there is a constant \(C = C(T, d, p, q, \kappa, \alpha, \beta, \nu) > 0\) such that for any distribution solution \(u\) of SKE (3.19) and \(t \in (0, T]\),
\[\|u(t)\|_{B^\beta_{p,q}(L^q_x)} \lesssim C t^{\frac{\alpha - \beta - \frac{1}{q}}{\beta}} \|u_0\|_{B^\beta_{p,q}(L^q_x)} + \int_0^t \left(\|f(s)\|_{B^\beta_{p,q}(L^q_x)} + \|g(s)\|_{B^\beta_{p,q}(L^q_x)}\right)^{\frac{q}{q}} ds + \int_0^t (t - s)^{\kappa - 1} \left(\|f(s)\|_{B^\beta_{p,q}(L^q_x)} + \|g(s)\|_{B^\beta_{p,q}(L^q_x)}\right)^{\frac{q}{q}} ds, \tag{3.20}\]
and also,
\[\|u(t)\|_{B^\beta_{p,q}(L^q_x)} \lesssim C t^{\frac{\alpha - \beta - \frac{1}{q}}{\beta}} \|u_0\|_{B^\beta_{p,q}(L^q_x)} + \int_0^t (t - s)^{\kappa - 1} \left(\|f(s)\|_{B^\beta_{p,q}(L^q_x)} + \|g(s)\|_{B^\beta_{p,q}(L^q_x)}\right)^{\frac{q}{q}} ds. \tag{3.21}\]

**Proof.** Without loss of generality, we assume \(\nu = 1\) in equation (3.19). Let \(P_t\) be defined by (3.1) and \(\tilde{u} := P_t u_0\). Since \(\partial_t \tilde{u} = \Delta \tilde{u} + v \cdot \nabla_x \tilde{u}\), one sees that \(\tilde{u} = u - \tilde{u}\) solves
\[d\tilde{u} = [\Delta \tilde{u} + v \cdot \nabla_x \tilde{u} + f]dt + g^k dW_t^k, \quad \tilde{u}(0) = 0.\]
Thus by (3.7), we may assume \(u_0 \equiv 0\).

(i) By Duhamel’s formula, we have in the distributional sense,
\[u(t) = \int_0^t P_{t-s} f(s) ds + \int_0^t P_{t-s} g^k(s) dW_s^k =: I_1(t) + I_2(t).\]
For \(I_1(t)\), by (3.6) with \((r, \alpha) = (1, \beta)\) and \(B := L^p_y\), we have
\[\|\mathcal{R}_n^\beta I_1(t)\|_{L^p_y} \lesssim \int_0^t \|\mathcal{R}_n^\beta P_{t-s} f(s)\|_{L^p_y} ds \lesssim 2^{-(\beta + 2(1 - \frac{\kappa}{q}))^j} \left(\int_0^t (t - s)^{\kappa - 1} \|f(s)\|_{B^\alpha_{p,q}(L^q_y)}^q \right)^{1/q}.\]
For $I_2(t)$, by BDG’s inequality and Minkowskii’s inequality, we have
\[ \|\mathcal{R}_j^g I_2(t)\|_{L^p_{\omega,x}}^p = \mathbb{E} \left[ \int_0^t \mathcal{R}_j^g P_{t-s} g^k(s) dW_s^k \right]^{p/2} \leq \mathbb{E} \left( \int_0^t \|\mathcal{R}_j^g P_{t-s} g(s)\|_{L^2_{\omega}}^2 ds \right)^{p/2} \leq \left( \int_0^t \|\mathcal{R}_j^g P_{t-s} g(s)\|_{L^2_{\omega}}^r ds \right)^{p/2}. \]
By (3.6) with $(r, \alpha) = (2, \beta + 1)$ and $\mathbb{B} := L^p_{\omega}(\mathbb{F}^2)$, we obtain
\[ \|\mathcal{R}_j^g I_2(t)\|_{L^p_{\omega,x}} \lesssim 2^{-(\beta + 2(1 - \frac{p}{q}))} \left( \int_0^t (t-s)^{\kappa-1} \|g(s)\|_{B^{\beta+1}_{p,q}(L^p_{\omega}(\mathbb{F}^2))}^q ds \right)^{1/q}. \]
Combining the above calculations, we obtain (3.20).

(ii) Let $\chi_{x}^\delta$ be as in (3.9) and define
\[ u_{x_0}^\delta(t, \omega, z) := u(t, \omega, z) \chi_{x_0}^\delta(t, z). \]
It is easy to see that
\[ du_{x_0}^\delta = [\Delta_v u_{x_0}^\delta + v \cdot \nabla x u_{x_0}^\delta + F_{x_0}^\delta] dt + g^k \chi_{x_0}^\delta dW^k, \]
where
\[ F_{x_0}^\delta := f \chi_{x_0}^\delta - 2 \nabla_v u \cdot \nabla x \chi_{x_0}^\delta - ((v - v_0) \cdot \nabla x \chi_{x_0}^\delta + \Delta_v \chi_{x_0}^\delta) u. \]
Note that by (2.14),
\[ \sup_{x_0} \|F_{x_0}^\delta(s)\|_{B_{p,q}^\beta(L^p_{\omega})} \lesssim \|f(s)\|_{B_{p,q}^\beta(L^p_{\omega})} + \|u(s)\|_{B_{p,q}^{\beta+1}(L^p_{\omega})}. \]
By (3.20), we have
\[ \|u(t)\|_{B_{p,q}^{\beta+2(1 - \frac{\alpha}{p})}(L^p_{\omega})} = \sup_{x_0 \in \mathbb{R}^d} \|u_{x_0}^\delta(t)\|_{B_{p,q}^{\beta+2(1 - \frac{\alpha}{p})}(L^p_{\omega})} \lesssim \left( \int_0^t (t-s)^{\kappa-1} \sup_{x_0} \left[ \|F_{x_0}^\delta(s)\|_{B_{p,q}^\beta(L^p_{\omega})} + \|g(s)\|_{B_{p,q}^{\beta+1}(L^p_{\omega})} \right] ds \right)^{1/q} \lesssim \left( \int_0^t (t-s)^{\kappa-1} \left[ \|u(s)\|_{B_{p,q}^{\beta+1}(L^p_{\omega})} + \|f(s)\|_{B_{p,q}^\beta(L^p_{\omega})} + \|g(s)\|_{B_{p,q}^{\beta+1}(L^p_{\omega})} \right] ds \right)^{1/q}. \]
In particular, taking $q = 2$, $\kappa = 1$ and then by Gronwall’s inequality, we get
\[ \|u(t)\|_{B_{p,q}^{\beta+1}(L^p_{\omega})} \lesssim \int_0^t \left[ \|f(s)\|_{B_{p,q}^\beta(L^p_{\omega})} + \|g(s)\|_{B_{p,q}^{\beta+1}(L^p_{\omega})} \right] ds. \]
Substituting this into the above inequality we obtain (3.21).

The following stochastic convolution lemma provides the existence of a continuous version of the solution in time variable, and also will be used to show the boundedness of $\|u(\omega, \cdot)\|_{L^\infty_{\omega,x}}$ in the probability sense.

**Lemma 3.7. (Stochastic convolutions)** Let $p \geq 2$, $T \geq 1$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta + 1 - \frac{2}{p}$. For any $g \in \mathbf{D}(\mathbb{F}^2) \cap L^p_T(\mathbb{B}_{p,q}^\beta(L^p_{\omega}(\mathbb{F}^2)))$, there is a continuous version $\tilde{S}_g$ of $[0, T] \ni t \mapsto S_g(t) := \int_0^t P_{t-s} g^k(s) dW_s^k \in \mathbb{B}_p^\alpha$ so that for some $C = C(p, T, \alpha, \beta) > 0$ and any stopping time $\tau$,
\[ \mathbb{E} \left( \sup_{t \in [0, T \wedge \tau]} \|\tilde{S}_g(t)\|_{B_{p,q}^\beta}^p \right) \lesssim C \int_0^T \|g(s \wedge \tau)\|_{B_{p,q}^\beta}^p (L^p_{\omega}(\mathbb{F}^2)) ds. \]
Proof. We use the factorization method in [7]. Noting that
\[ \int_s^t (t-r)^{-\kappa}(r-s)^{-\kappa} dr = \frac{\pi}{\sin(\kappa \pi)}, \quad 0 \leq s < t < \infty, \quad \kappa \in (0,1), \]
by stochastic Fubini’s theorem (cf. [16]), we have for each \( t \in (0,T) \),
\[ S_g(t) = \int_0^t P_t-s g^k(s) dW^k_s = \frac{\sin(\kappa \pi)}{\pi} \int_0^t (t-s)^{-\kappa-1} P_t-s G(s) ds =: \tilde{S}_g(t), \quad \mathbb{P} - a.s. \]
where
\[ G(t) := \int_0^t (t-s)^{-\kappa} P_t-s g^k(s) dW^k_s. \]
We show the above defined \( \tilde{S}_g \) has the desired property. Since \( \alpha < \beta + 1 - \frac{2}{p} \), one can choose \( \kappa, \varepsilon \in (0,1) \) small enough so that
\[ \alpha = \beta + 1 - \frac{2}{p} - \varepsilon. \tag{3.25} \]
By Minkowski’s inequality and (3.6), we have
\[
\left\| \mathcal{R}_j^0 \tilde{S}_g(t) \right\|_{L^p_{\mathcal{P},\theta}} \lesssim \int_0^t (t-s)^{-\kappa-1} \left\| \mathcal{R}_j^0 P_t-s G(s) \right\|_{L^p_{\mathcal{P},\theta}} ds
\lesssim 2^{-\alpha j} \left( \int_0^t \left\| G(s) \right\|_{B_{p,\theta}^{\alpha-2(\kappa-\frac{1}{p})}} ds \right)^{1/p},
\]
which implies that for all \( t \in [0,T] \),
\[
\left\| \tilde{S}_g(t) \right\|_{B_{p,\theta}^{\beta+1-2\kappa-\frac{2}{p}}} \lesssim \int_0^t \left\| G(s) \right\|_{B_{p,\theta}^{\alpha-2(\kappa-\frac{1}{p})}} ds \lesssim \int_0^T \left\| G(s) \right\|_{B_{p,\theta}^{\alpha-2(\kappa-\frac{1}{p})}} ds.
\]
From this a priori estimate, as in [7, Lemma 1], one sees that \( t \mapsto \tilde{S}_g(t) \) is continuous as long as the last integral is finite a.s.. In particular,
\[
E \left( \sup_{t \in [0,T]} \left\| \tilde{S}_g(t) \right\|_{B_{p,\theta}^{\beta+1-2\kappa-\frac{2}{p}}} \right)^p \lesssim \int_0^T \left\| G(s) \right\|_{B_{p,\theta}^{\alpha-2(\kappa-\frac{1}{p})}} ds \tag{3.26}
\]
On the other hand, by BDG’s inequality and (3.6) again, we have
\[
E \left( \mathcal{R}_j^0 G(t) \right)_{L^p_{\mathcal{P},\theta}}^p \lesssim \left( \int_0^t (t-s)^{-2\kappa} \left\| \mathcal{R}_j^0 P_t-s g(s) \right\|_{L^p_{\mathcal{P},\theta}(\mathbb{R}^d)}^2 ds \right)^{p/2}
\lesssim \left( \int_0^t (t-s)^{-2\kappa} \left\| \mathcal{R}_j^0 P_t-s g(s) \right\|_{L^p_{\mathcal{P},\theta}(\mathbb{R}^d)}^2 ds \right)^{p/2}
\lesssim 2^{-(\beta + 2(1-\frac{2\kappa}{p}))} \int_0^t (t-s)^{-\kappa-1} \left\| g(s) \right\|_{B_{p,\theta}^{\alpha-2(\kappa-\frac{1}{p})}}^p ds.
\]
Hence,
\[
\left\| G(t) \right\|_{B_{p,\theta}^{\beta+1-2\kappa-\frac{2}{p}}} \lesssim \int_0^t (t-s)^{-\kappa-1} \left\| g(s) \right\|_{B_{p,\theta}^{\alpha-2(\kappa-\frac{1}{p})}}^p ds.
\]
Substituting this into (3.26), we obtain
\[
E \left( \sup_{t \in [0,T]} \left\| \tilde{S}_g(t) \right\|_{B_{p,\theta}^{\beta+1-2\kappa-\frac{2}{p}}} \right)^p \lesssim \int_0^T \int_0^t (t-s)^{-\kappa-1} \left\| g(s) \right\|_{B_{p,\theta}^{\alpha-2(\kappa-\frac{1}{p})}}^p ds dt.
\]
= \frac{1}{\kappa} \int_0^T (T - s)^{\kappa} \|g(s)\|_{B^{p}_{\rho,\theta}(L^p(\mathbb{F}))}^p \, ds \\
\leq \frac{T^{\kappa}}{\kappa} \int_0^T \|g(s)\|_{B^{p}_{\rho,\theta}(L^p(\mathbb{F}))}^p \, ds.

Finally, for any stopping time \( \tau \), since \( \tilde{S}_g(t \wedge \tau) = \tilde{S}_g(t \wedge \tau) \) a.s., we have

\[ \mathbb{E}
\left( \sup_{t \in [0, T \wedge \tau]} \|\tilde{S}_g(t)\|_{B^{p}_{\rho,\theta}(L^p)}^p \right) \leq \mathbb{E}
\left( \sup_{t \in [0, T]} \|\tilde{S}_g(t)\|_{B^{p}_{\rho,\theta}(L^p)}^p \right) \leq \int_0^T \|g(s \wedge \tau)\|_{B^{p}_{\rho,\theta}(L^p(\mathbb{F}))}^p \, ds. \]

The proof is complete. \( \square \)

The following corollary is a direct consequence of Theorem 3.6 and Lemma 3.7.

**Corollary 3.8.** Let \( p \geq 2 \), \( T > 0 \) and \( \alpha \leq \beta + 2 - \frac{2}{p} \). Suppose that \( u_0 \in B^{\gamma}_{p,\theta}(L^p) \) and

\[ \mathcal{F}^T_{f,g} := \int_0^T \left[ \|f(s)\|_{B^{p}_{\rho,\theta}(L^p)} + \|g(s)\|_{B^{p+1}_{\rho,\theta}(L^p(\mathbb{F}))} \right] \, ds < \infty. \]

Then for any \( \gamma < \beta + 2 - \frac{2}{p} \), the unique weak solution \( u \) in Theorem 3.6 admits a continuous version \( \tilde{u}(\omega, \cdot) \in C((0, T]; B^\gamma_{p,\theta}) \). Moreover, if \( \gamma < \alpha \), then \( \tilde{u}(\omega, \cdot) \in C((0, T]; B^\gamma_{p,\theta}) \), and in this case, there is a constant \( C = C(p, T, \alpha, \gamma, \beta) > 0 \) such that

\[ \mathbb{E}
\left( \|\tilde{u}\|_{C([0, T]; B^{\gamma}_{p,\theta}(L^p))}^p \right) \leq \|u_0\|_{B^{p}_{p,\theta}(L^p)}^p + \mathcal{F}^T_{f,g}. \]

Although the above corollary asserts the continuity of \( u \) in time variable, it does not say any H"older regularity in the time variable. The following lemma is useful.

**Lemma 3.9.** For any \( p \geq 2 \), \( \beta \in (0, 1) \), \( \gamma \in (0, \frac{\beta}{2}) \) and \( T > 0 \), there is a constant \( C = C(T, d, p, \beta, \gamma) > 0 \) such that for any distribution solution \( u \) of SKE (3.19) and \( 0 < t_1 < t_2 \leq T \),

\[ \|u(t_2) - \Gamma_{t_2-t_1} u(t_1)\|_{B^{\gamma}_{p,\theta}(L^p)} \lesssim_C (t_2 - t_1)^{\frac{\beta - \gamma}{2}} \times \left( \|u(t_1)\|_{B^{\beta}_{p,\theta}(L^p)} + \|f\|_{L^\infty_{t_1,t_2}B^{\beta-2}_{p,\theta}(L^p)} + \|g\|_{L^\infty_{t_1,t_2}B^{\beta-1}_{p,\theta}(L^p(\mathbb{F}))} \right), \tag{3.27} \]

where \( L^\infty_{t_1,t_2}(\mathbb{B}) := L^\infty([t_1, t_2]; \mathbb{B}) \).

**Proof.** For \( 0 < t_1 < t_2 \leq T \), noting that

\[ u(t_2) = \int_{t_1}^{t_2} P_{t_2-s} f(s) \, ds + \int_{t_1}^{t_2} P_{t_2-s} g^k(s) \, dW^k_s + P_{t_2-t_1} u(t_1), \]

we have

\[ u(t_2) - \Gamma_{t_2-t_1} u(t_1) = \int_{t_1}^{t_2} P_{t_2-s} f(s) \, ds + \int_{t_1}^{t_2} P_{t_2-s} g^k(s) \, dW^k_s \]

\[ + (P_{t_2-t_1} - \Gamma_{t_2-t_1}) u(t_1) \]

\[ =: I_1 + I_2 + I_3. \]

For \( I_1 \), by (3.5) with \( l = 3 + (\gamma - \beta)/2 \), we have for all \( j \in \mathbb{N}_0 \),

\[ \|\mathcal{R}_j^p I_1\|_{L^p(L^p)} \lesssim \int_{t_1}^{t_2} \|\mathcal{R}_j^p P_{t_2-s} f(s)\|_{L^p(L^p)} \, ds \]
\[ \leq 2^{j(2-\beta)} \int_{t_1}^{t_2} h_{t,\beta-2}((t_2-s)4^j) \|f(s)\|_{B_{p,\theta}^{\beta-2}(L^p)} ds \]
\[ \leq 2^{-j\beta} \int_{0}^{(t_2-t_1)4^j} h_{t,\beta-2}(s) ds \|f\|_{L_{t_1}^{\infty},t_2(B_{p,\theta}^{\beta-2}(L^p))} \]
\[ \leq 2^{-j\beta} \int_{0}^{(t_2-t_1)4^j} \frac{s}{s-1} ds \|f\|_{L_{t_1}^{\infty},t_2(B_{p,\theta}^{\beta-2}(L^p))} \]
\[ \leq 2^{-j\gamma} (t_2-t_1)^{\frac{\beta-2}{\beta-1}} \|f\|_{L_{t_1}^{\infty},t_2(B_{p,\theta}^{\beta-2}(L^p))}, \]
where we have used that for \( l = 3 + (\gamma - \beta)/2, \)
\[ h_{l,\beta-2}(s) = (1 \wedge s^{-l})(1+s)^{2-\beta} \leq (1 \wedge s^{-l})(1+s)^2 \leq 4s^{\frac{\beta-2}{\beta-1}}. \]

For \( I_2, \) as in (3.22), by (3.5) with \( l = (3+\gamma-\beta)/2, \) we have for all \( j \in \mathbb{N}_0, \)
\[ \|R_j I_2\|_{L^p(L^p)} \leq \left( \int_{t_1}^{t_2} \|R_j P_{t_2-s}g(s)\|^2_{L^p(L^p(t^2))} ds \right)^{1/2} \]
\[ \leq 2^{j(1-\beta)} \left( \int_{t_1}^{t_2} h_{l,\beta-1}((t_2-s)4^j) \|g(s)\|^2_{B_{p,\theta}^{\beta-1}(L^p(t^2))} ds \right)^{1/2} \]
\[ \leq 2^{-j\beta} \left( \int_{0}^{(t_2-t_1)4^j} h_{l,\beta-1}(s) ds \right)^{1/2} \|g\|_{L_{t_1}^{\infty},t_2(B_{p,\theta}^{\beta-1}(L^p(t^2)))} \]
\[ \leq 2^{-j\beta} \left( \int_{0}^{(t_2-t_1)4^j} s^{\beta-\gamma-1} ds \right)^{1/2} \|g\|_{L_{t_1}^{\infty},t_2(B_{p,\theta}^{\beta-1}(L^p(t^2)))} \]
\[ \leq 2^{-j\gamma} (t_2-t_1)^{\frac{\beta-1}{\beta-\gamma}} \|g\|_{L_{t_1}^{\infty},t_2(B_{p,\theta}^{\beta-1}(L^p(t^2)))}. \]

For \( I_3, \) since \( \gamma \in (0, \frac{4}{3}], \) by (3.17) we have
\[ \|I_3\|_{B_{p,\theta}^{\beta}(L^p)} \leq (t_2-t_1)^{\frac{\beta-2}{\beta-1}} \|u(t_1)\|_{B_{p,\theta}^{\beta}(L^p)} . \]

Combining the above estimates, we obtain (3.27).

**Remark 3.10.** In the kinetic case, it is not possible to show the Hölder continuity of \( t \mapsto u(t) \) in \( B_{p,\theta}^{\gamma}(L^p) \) due to the transport term of \( v \cdot \nabla x. \) In other words, one can not show the same estimates for \( \|P_t f - f\|_{B_{p,\theta}^{\gamma}} \) as in (3.16) and (3.17). However, (3.27) together with (3.13) still says some Hölder continuity of \( t \mapsto u(t,x,v) \) in time variable locally in \( x,v \) (see Corollary 7.6 below).

## 4. SKEs with Constant Random Coefficients

In this section we consider the following SKE with constant random coefficients:
\[ du = [\text{tr}(a \cdot \nabla u) + v \cdot \nabla u + f] dt + [\sigma k \cdot \nabla u + g^k] dW^k_t, \quad u(0) = u_0. \] (4.1)

where \( u_0 : \Omega \to S' \) is \( \mathscr{F}_0 \)-measurable, \( f \in D^1, g \in D^2(\ell^2) \) and
\[ (a, \sigma) : \mathbb{R}_+ \times \Omega \to (\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^d \otimes \ell^2) \]
are predictable processes. Suppose that for some \( c_0, c_1 > 0 \) and all \( (t, \omega) \in \mathbb{R}_+ \times \Omega, \)
\[ 2\alpha(t, \omega) - \sigma^k \sigma^k(t, \omega) \geq c_0 \mathbb{I} \] (4.2)
and
\[ \|a(t, \omega)\|_{\mathbb{R}^d \otimes \mathbb{R}^d} + \|\sigma(t, \omega)\|_{\mathbb{R}^d \otimes \ell^2} \leq c_1. \] (4.3)
Definition 4.1. We call a predictable process $u(t, \omega) : \mathbb{R}_+ \times \Omega \to \mathcal{S}'$ a distribution solution of SKE (4.1) if for any $\varphi \in C^\infty_0(\mathbb{R}^{2d})$ and $t \geq 0$,
\[
\langle u(t), \varphi \rangle = \langle u_0, \varphi \rangle + \int_0^t \left[ \langle u, \text{tr}(a \cdot \nabla^2 \varphi) \rangle - \langle u, v \cdot \nabla \varphi \rangle + \langle f, \varphi \rangle \right] ds
\]
\[
+ \int_0^t \langle g^k, \varphi \rangle - \langle u, \sigma^k \cdot \nabla \varphi \rangle \rangle dW^k_s.
\]

We use Krylov’s trick [15] to give a representation for the solution of SKE (4.1) in terms of the solution of the model equation (3.19). Let
\[
\tilde{\sigma} := (2a - \sigma \sigma^* - c_0I)^{1/2} \in \mathbb{R}^d \otimes \mathbb{R}^d
\]
and
\[
\tilde{f} := f - \sigma^k \cdot \nabla g^k.
\]
Let $(B^k)_{k=1, \ldots, d}$ be another Brownian motion that is independent of $W$. Define
\[
\tilde{I}^k_t := \int_0^t \sigma^{ik}(s) dW^k_s, \quad \tilde{I}^k_t := \int_0^t \tilde{\sigma}^{ik}(s) dB^k_s, \quad i = 1, \ldots, d.
\]
We introduce the following auxiliary equation
\[
d\tilde{u} = \left[ \frac{\sqrt{2}}{4} \Delta \tilde{u} + v \cdot \nabla \tilde{u} + \tilde{f} \right] dt + g^k dW^k_t, \quad \tilde{u}(0) = u_0,
\]
where
\[
\tilde{f}(t, x, v) := \tilde{f} \left( t, x + \int_0^t (\tilde{I}_s + \tilde{\tilde{I}}_s) ds, v - \tilde{I}_t - \tilde{\tilde{I}}_t \right)
\]
and
\[
\tilde{g}(t, x, v) := g \left( t, x + \int_0^t (I_s + \tilde{I}_s) ds, v - I_t - \tilde{I}_t \right).
\]
Let
\[
w(t, x, v) = \tilde{u} \left( t, x - \int_0^t (I_s + \tilde{I}_s) ds, v + I_t + \tilde{I}_t \right).
\]
By Itô-Wentzell’s formula (see Lemma 2.15), we have
\[
dw = \left[ \frac{1}{2} \text{tr}(\tilde{a} \cdot \nabla^2 w) + v \cdot \nabla w + \tilde{f} + \sigma^k \cdot \nabla g^k \right] dt
\]
\[
+ \left( g^k + \sigma^k \cdot \nabla w \right) dW^k_t + (\tilde{\sigma}^k \cdot \nabla g^k ) dB^k_t,
\]
where
\[
\tilde{a} := \frac{1}{2}(\tilde{\sigma} \tilde{\sigma}^* + \sigma \sigma^*) + c_0 I/2.
\]
Now we define
\[
u(t, x, v) := \mathbb{E}(w(t, x, v)|\mathcal{F}^W_t),
\]
where $\mathcal{F}^W_t$ is the $\sigma$-algebra generated by $\{W_s, s \leq t\}$ and $\mathcal{F}_0$. By (4.4) and (4.5), one sees that
\[
\tilde{a} = a, \quad \tilde{f} + \sigma^k \cdot \nabla g^k = f.
\]
Note that (cf. [26, p.28 Theorem 1.15])
\[
\mathbb{E} \left( \int_0^t \xi(s) dW^k_s \mid \mathcal{F}^W_t \right) = \int_0^t \mathbb{E} \left( \xi(s) \mid \mathcal{F}^W_s \right) dW^k_s
\]
and
\[
\mathbb{E} \left( \int_0^t \eta(s) dB^k_s \mid \mathcal{F}^W_t \right) = 0.
\]
In particular, one sees that $u$ solves SKE (4.1). Now by Theorem 3.6 and Fubini’s theorem, we have
Theorem 4.2. Let $p \geq 2$, $q \in [2, \infty]$, $\beta \in \mathbb{R}$, $\kappa \in (0, 1]$, $\alpha \leq \beta + 2(1 - \frac{q}{p})$ and $T > 0$. Under (4.2) and (4.3), there is a constant $C = C(T, d, p, q, \kappa, \alpha, \beta, c_0, c_1) > 0$ so that for any distribution solution $u$ of SKE (4.1) and $t \in (0, T]$,

$$
\| u(t) \|_{B_{p, \theta}^{2+2(1 - \frac{q}{p})} (L^\theta_x)} \leq C \left( I \right) + \left( \int_0^t (t - s)^{\kappa - 1} \left[ \| f(s) \|_{L^\theta_x}^q + \| g(s) \|_{L^\theta_x}^q \right] ds \right)^{1/q}.
$$

Proof. Note that by Fubini’s theorem,

$$
\| R^u_0 \|_{L^\theta_x} = \| R^\theta_j w(t) \|_{L^\theta_x} \leq \mathbb{E}(\| R^\theta_j w(t) \|_{L^\theta_x} | \mathcal{F}_t) \|_{L^\theta_x} \leq \| R^\theta_j w(t) \|_{L^\theta_x} = \| R^\theta_j u(t) \|_{L^\theta_x}.
$$

Therefore, by Theorem 3.6,

$$
\| u(t) \|_{B_{p, \theta}^{2+2(1 - \frac{q}{p})} (L^\theta_x)} = \sup_j \left( 2^{(\beta + 2(1 - \frac{q}{p}))} \| R^\theta_j u(t) \|_{L^\theta_x} \right) \leq \left( \bar{u}(t) \right)_{B_{p, \theta}^{2+2(1 - \frac{q}{p})} (L^\theta_x)} \leq C \left( I \right) + \left( \int_0^t (t - s)^{\kappa - 1} \left[ \| f(s) \|_{L^\theta_x}^q + \| g(s) \|_{L^\theta_x}^q \right] ds \right)^{1/q}.
$$

Finally, it is just noticed that by Fubini’s theorem again,

$$
\| \tilde{f}(s) \|_{B_{p, \theta}^{2+2(1 - \frac{q}{p})} (L^\theta_x)} = \| \bar{f}(s) \|_{B_{p, \theta}^{2+2(1 - \frac{q}{p})} (L^\theta_x)} \leq \| f(s) \|_{B_{p, \theta}^{2+2(1 - \frac{q}{p})} (L^\theta_x)} + \| \nabla_v g \|_{B_{p, \theta}^{2+2(1 - \frac{q}{p})} (L^\theta_x)}
$$

and

$$
\| \tilde{g}(s) \|_{B_{p, \theta}^{2+2(1 - \frac{q}{p})} (L^\theta_x)} = \| g(s) \|_{B_{p, \theta}^{2+2(1 - \frac{q}{p})} (L^\theta_x)}.
$$

Thus we obtain (4.6).

5. SKEs with Variable Random Coefficients

In this section we consider SKEs (1.1) and (1.2) with variable coefficients, and assume the following super-parabolic conditions:

(H0) There are $c_0, c_1 > 0$ such that for all $(t, \omega, x, v)$,

$$
2a(t, \omega, x, v) - \sigma \sigma^*(t, \omega, x, v) \geq c_0
$$

and

$$
\| a(t, \omega, x, v) \|_{\mathbb{R}^d} + \| \sigma(t, \omega, x, v) \|_{\mathbb{R}^d} \leq c_1.
$$

5.1. Optimal Besov Regularity Estimates. In this subsection we consider SKE (1.1) and assume (H0) and (H1) and

(H1) We suppose that for some $\beta \in (0, 1)$ and $c_2 > 0$,

$$
\left\| a \right\|_{L^\infty_T (C^\beta_p)} + \left\| b \right\|_{L^\infty_T (C^\beta_p)} + \left\| \sigma \right\|_{L^\infty_T (C^{1+\beta}_p)} \leq c_2, \forall T > 0,
$$

where $L^\infty_T (B) := L^\infty([0, T] \times \Omega, \mathcal{P}, dt \times \mathcal{F}; B)$.

We first introduce the following notion about the solution of SKE (1.1).

Definition 5.1. Let $T > 0$, $\beta \in (0, 1)$ and $p \in [2, \infty)$. A predictable process $u \in L^\infty_T (B_{p, \theta}^{2+2(1 - \frac{q}{p})} (L^\theta_x))$ is called a weak solution of SKE (1.1) with initial value $u_0 \in \tilde{B}_{p, \theta}^{2+2(1 - \frac{q}{p})} (L^\theta_x)$ if for any $\varphi \in C_c^\infty (\mathbb{R}^d)$,

$$
\langle u(t), \varphi \rangle = \langle u_0, \varphi \rangle + \int_0^t [\langle tr(a \cdot \nabla^2 u), \varphi \rangle - \langle u, v \cdot \nabla_x v \rangle + \langle b \cdot \nabla u, \varphi \rangle] ds.
$$
\[ u \in \mathbf{B}_{p,q}^{2+\beta}(L^p_\nu), \quad f \in L^\infty(L^p_\nu(\mathbb{L}^2)), \quad g \in L^\infty(\mathbf{B}_{p,q}^{1+\beta}(L^p_\nu(\mathbb{L}^2))). \]

Under \((H_0)\) and \((H_1')\), there is a unique weak solution \( u \) to SKE (1.1) so that
\[
\| u \|_{L^\infty(\mathbf{B}_{p,q}^{2+\beta}(L^p_\nu))} \lesssim \| u_0 \|_{\mathbf{B}_{p,q}^{2+\beta}(L^p_\nu)} + \| f \|_{L^\infty(\mathbf{B}_{p,q}^{2+\beta}(L^p_\nu))} + \| g \|_{L^\infty(\mathbf{B}_{p,q}^{1+\beta}(L^p_\nu(\mathbb{L}^2)))}, \tag{5.1}
\]
where the constant \( C = C(d,p,\beta,\kappa,T,c_0,c_1,c_2) > 0 \).

Proof. Let \( \tilde{u} := P_t u_0 \) solve \( \partial_t \tilde{u} = \Delta_v \tilde{u} + v \cdot \nabla_x \tilde{u} \). By (3.7) we have
\[
\| \tilde{u} \|_{L^\infty(\mathbf{B}_{p,q}^{2+\beta}(L^p_\nu))} \lesssim \| u_0 \|_{\mathbf{B}_{p,q}^{2+\beta}(L^p_\nu)}.
\]
If we consider \( \tilde{u} = u - \bar{u} \), then \( \tilde{u} \) solves SKE (1.1) with
\[
\tilde{f} = f + \mathcal{L}_v \tilde{u} - \Delta_v \tilde{u}, \quad \tilde{g} = g + \sigma \cdot \nabla_v \tilde{u},
\]
and \( \tilde{f}, \tilde{g} \) satisfy the same assumptions as \( f, g \). Thus, without loss of generality we may assume \( u_0 \equiv 0 \). We divide the proof into three steps. In the first two steps, we prove the a priori estimate (5.1) by freezing coefficient argument. In the last step, we use the continuity method to show the existence of weak solutions.

(i) Let \( \chi_\delta^{z_0} \) be as in (3.9) and define
\[
a_\delta^{z_0}(t, \omega) := a(t, \omega, z_0,t), \quad \sigma_\delta^{z_0}(t, \omega) := \sigma(t, \omega, z_0,t)
\]
and
\[
u_\delta^{z_0}(t, \omega, z) := u(t, \omega, z) \chi_\delta^{z_0}(t, z).
\]
Let \( u \) be any weak solution of SKE (1.1). By Definition 5.1, it is easy to see that \( u_\delta^{z_0} \) is a weak solution of the following freezing SKE:
\[
du_\delta^{z_0} = \left[ \text{tr}(a_{\delta^{z_0}} \cdot \nabla_v u_\delta^{z_0}) + v \cdot \nabla_x u_\delta^{z_0} + \tilde{f} \right] dt + \left[ \sigma_{\delta^{z_0}} \cdot \nabla_v u_\delta^{z_0} + \tilde{g} \right] dW^k_t
\]
with \( u_\delta^{z_0}(0) = 0 \), where
\[
\tilde{f} := \text{tr}(a - a_{\delta^{z_0}} \cdot \nabla_v u_{\delta^{z_0}}) - 2 \text{tr}(a \cdot (\nabla_v u \otimes \nabla_v \chi_\delta^{z_0})) + \chi_\delta^{z_0} b \cdot \nabla_v u
\]
\[- \left[ (a \cdot \nabla_v \chi_\delta^{z_0}) + (v - v_0) \cdot \nabla_x \chi_\delta^{z_0} \right] u + f \chi_\delta^{z_0},
\]
and
\[
\tilde{g} := (\sigma - \sigma_{\delta^{z_0}}) \cdot \nabla_v u_{\delta^{z_0}} - \sigma \cdot \nabla_v \chi_\delta^{z_0} u + g \chi_\delta^{z_0}.
\]
For simplicity of notations, if we let
\[
\bar{a} := (a - a_{\delta^{z_0}}) \chi_\delta^{z_0}, \quad \bar{b} := 2a \cdot \nabla_v \chi_\delta^{z_0} - \chi_\delta^{z_0} b, \quad \bar{\sigma} := (\sigma - \sigma_{\delta^{z_0}}) \chi_\delta^{z_0},
\]
and
\[
\bar{c} := - \left( \text{tr}(a \cdot \nabla_v \chi_\delta^{z_0}) + (v - v_0) \cdot \nabla_x \chi_\delta^{z_0} \right), \quad \bar{h} := \sigma \cdot \nabla_v \chi_\delta^{z_0},
\]
then due to \( \chi_\delta^{z_0} = 1 \) on the support of \( \chi_\delta^{z_0} \), we can write
\[
\tilde{f} = \text{tr}(\bar{a} \cdot \nabla_v u_{\delta^{z_0}}) - \bar{b} \cdot \nabla_v u_{\delta^{z_0}} + \bar{c} \cdot u_{\delta^{z_0}} + \bar{f} \chi_\delta^{z_0},
\]
and
\[
\tilde{g} = \bar{\sigma} \cdot \nabla_v u_{\delta^{z_0}} - \bar{h} u_{\delta^{z_0}} + g \chi_\delta^{z_0}.
\]
(ii) Let $T > 0$, $p \in [2, \infty)$ and $\beta \in (0, 1)$. For any $q \in [2, \infty]$, by Theorem 4.2 with $\kappa = 1$, there is a constant $C > 0$ such that for all $\delta \in (0, 1)$, $z_0 \in \mathbb{R}^d$ and $t \in (0, T]$,

$$
\|u_\delta(t)\|_{B^\delta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} \lesssim C \left( \int_0^t \left[ \|f(s)\|_{B^\theta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))}^q + \|g(s)\|_{B^\theta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))}^q \right] ds \right)^{1/q}.
$$

(5.4)

Now let us estimate $\|\tilde{f}(s)\|_{B^\delta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))}$ and $\|\tilde{g}(s)\|_{B^\delta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))}$. Below we drop the variable $s$. For each $(t, z) \in \mathbb{R}_+ \times \mathbb{R}^d$, we introduce a bounded linear operator:

$$
T_{t, z}^\delta : L^p_{\omega} \to L^p_{\omega}, \quad T^\delta_{t, z} f(\omega) := \tilde{a}(t, z, \omega) f(\omega).
$$

Noting that by the assumptions,

$$
\|T^\delta_{t, z} f\|_{L^p_{\omega}} \leq c_2(2\delta)^{\beta} \|f\|_{L^p_{\omega}}, \quad \|T^\delta_{t, z} - T^\delta_{t, z'} f\|_{L^p_{\omega}} \leq c_2 |z - z'|^{\beta} \|f\|_{L^p_{\omega}},
$$

by (2.13), (2.7) and Young’s inequality, we have

$$
\|\text{tr}(\bar{a} \cdot \nabla^2 u_\delta)\|_{B^\delta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} \lesssim \|\nabla^2 u_\delta\|_{B^\delta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} \lesssim \|\nabla^2 u_\delta\|_{L^q_{\omega}(\mathbb{R}^d)} \lesssim 1.
$$

Similarly, since $\|\tilde{g}\|_{L^q_{\omega}(\mathbb{R}^d)} \lesssim 1$, by (2.13) again, we have

$$
\|\bar{g} \cdot \nabla u_{2\delta z_0}\|_{B^\delta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} \lesssim \|u_{2\delta z_0}\|_{L^q_{\omega}(\mathbb{R}^d)} \lesssim 1.
$$

Therefore, for any $\delta \in (0, 1)$ and some $C_0, C_1 > 0$ independent of $\delta$ and $z_0$,

$$
\|\bar{f}\|_{B^\delta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} \lesssim C_0 \delta^{\beta} \|u_\delta\|_{B^\delta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} + C_0 \|u_{2\delta z_0}\|_{B^\delta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))}
$$

$$
\lesssim C_0 \|u_\delta\|_{B^\delta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} + C_0 \|u_{2\delta z_0}\|_{B^\delta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} + C_0 \|u_{2\delta z_0}\|_{B^\delta_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))}.
$$

(5.5)

Next we estimate $\|\tilde{g}\|_{B^{1+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))}$. As above, by (2.13), we have

$$
\|\bar{\sigma} \cdot \nabla u_{\delta z_0}\|_{B^{1+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} \lesssim \|\bar{\sigma}\|_{L^q_{\omega}(\mathbb{R}^d)} \|\nabla u_{\delta z_0}\|_{B^{1+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} \lesssim \|u_{\delta z_0}\|_{B^{1+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} + \|u_{\delta z_0}\|_{B^{1+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))},
$$

and so,

$$
\|\tilde{g}\|_{B^{1+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} \lesssim C_0 \delta \|u\|_{B^{1+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} + C_0 g_{1+\beta}(L^q_{\omega}(\mathbb{R}^d)) \lesssim C_0 \|u\|_{B^{1+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} + C_0 g_{1+\beta}(L^q_{\omega}(\mathbb{R}^d)).
$$

(5.6)

By (5.4), (5.5), (5.6) and Lemma 2.12, we obtain that for any $q \in [2, \infty]$, $\delta \in (0, 1)$ and $t \in [0, T]$,

$$
\|u(t)\|_{B^{\delta+2(1-\frac{1}{q})}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} = \sup_{z_0 \in \mathbb{R}^d} \|u_\delta(t)\|_{B^{\delta+2(1-\frac{1}{q})}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} \lesssim \left( \int_0^t \left[ C_0 \delta^{\beta} \|u(s)\|_{B^{\delta+2(1-\frac{1}{q})}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d))} + C_0 g_{1+\beta}(L^q_{\omega}(\mathbb{R}^d)) \right] ds \right)^{1/q} + K_T,
$$

(5.7)

where

$$
K_T := \|f\|_{L^p_{\omega}(B^{\delta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d)))} + \|g\|_{L^p_{\omega}(B^{1+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d)))}.
$$

In particular, taking $q = \infty$ and $\delta$ being small enough, we obtain for all $t \in [0, T]$,

$$
\|u\|_{L^\infty_{\omega}(B^{2+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d)))} \lesssim C \|u\|_{L^\infty_{\omega}(B^{2+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d)))} + K_T.
$$

(5.8)

Substituting this into (5.7) and taking $q = \frac{2}{\gamma}$, we obtain

$$
\|u\|_{L^\infty_{\omega}(B^{2+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d)))} \lesssim \left( \int_0^t \|u\|_{L^\infty_{\omega}(B^{2+\beta}_{p,\theta}(L^q_{\omega}(\mathbb{R}^d)))} ds \right)^{1/q} + K_T,
$$

(5.9)
which implies by Gronwall’s inequality that
\[
\|u\|_{L^\infty_T(\tilde{B}^2_{p,\theta}(L^p_w))} \lesssim K_T.
\]
Substituting this into (5.8), we obtain (5.1).

(iii) We use the continuity method to show the existence of weak solutions. Let
\[
a_\lambda := (\lambda a + (1 - \lambda)\mathbb{I}), \quad \sigma^k_\lambda := \lambda \sigma^k, \quad \lambda \in [0, 1].
\]
Note that \(a_\lambda, \sigma_\lambda\) satisfy (H_0) and (H_2') uniformly in \(\lambda \in [0, 1]\). For instance,
\[
2a_\lambda - \sigma^k_\lambda = 2(1-\lambda)\mathbb{I} + \lambda(2a - \lambda \sigma^\ast) \geq (2(1 - \lambda) + \lambda c_0)\mathbb{I} \geq (2 \wedge c_0)\mathbb{I}.
\]
Given \((f, g) \in L^\infty_T(\tilde{B}^2_{p,\theta}(L^p_w)) \times L^\infty_T(\tilde{B}^{1+\beta}_{p,\theta}(L^p_w))\), we consider the following SKE:
\[
\begin{align*}
du &= \{\text{tr}(a_\lambda \cdot \nabla^2 u) + v \cdot \nabla u + \lambda b \cdot \nabla u + f \} \, dt + [\sigma^k_\lambda \cdot \nabla u + g^k] \, dW^k_t
\end{align*}
\]
with \(u(0) = u_0\). Suppose that for some \(\lambda_0 \in [0, 1]\), the above SKE admits a unique weak solution \(u \in L^\infty_T(\tilde{B}^{2+\beta}_{p,\theta}(L^p_w))\) so that (5.1) holds. We want to show that there is a \(\delta\) independent of \(\lambda\) such that for any \(\lambda \in [\lambda_0, \lambda_0 + \delta]\), the above SKE still has a unique weak solution \(u \in L^\infty_T(\tilde{B}^{2+\beta}_{p,\theta}(L^p_w))\). Indeed, given \(w \in L^\infty_T(\tilde{B}^{2+\beta}_{p,\theta}(L^p_w))\), consider SKE
\[
\begin{align*}
du &= \{\text{tr}(a_{\lambda_0} \cdot \nabla^2 u) + v \cdot \nabla u + \lambda_0 b \cdot \nabla u + \lambda_0 f \} \, dt + [\sigma^k_{\lambda_0} \cdot \nabla u + (\sigma^k_{\lambda_0} - \sigma^k_0) \cdot \nabla u + g^k] \, dW^k_t
\end{align*}
\]
with \(u(0) = u_0\). Note that by (2.13),
\[
\begin{align*}
&\|\text{tr}(a_{\lambda_0} \cdot \nabla^2 u)\|_{L^\infty_T(\tilde{B}^2_{p,\theta}(L^p_w))} \lesssim |\lambda_0 - \lambda_0| \cdot \|w\|_{L^\infty_T(\tilde{B}^{2+\beta}_{p,\theta}(L^p_w))}, \\
&\|\nabla u\|_{L^\infty_T(\tilde{B}^2_{p,\theta}(L^p_w))} \lesssim |\lambda_0 - \lambda_0| \cdot \|w\|_{L^\infty_T(\tilde{B}^{1+\beta}_{p,\theta}(L^p_w))}, \\
&\|\nabla u\|_{L^\infty_T(\tilde{B}^2_{p,\theta}(L^p_w))} \lesssim |\lambda_0 - \lambda_0| \cdot \|w\|_{L^\infty_T(\tilde{B}^{2+\beta}_{p,\theta}(L^p_w))}.
\end{align*}
\]
By the assumption, SKE (5.10) admits a unique solution denoted by \(Q_\lambda w\). In other words, we obtain a mapping
\[\begin{align*}
L^\infty_T(\tilde{B}^{2+\beta}_{p,\theta}(L^p_w)) \ni w \rightarrow Q_\lambda w \in L^\infty_T(\tilde{B}^{2+\beta}_{p,\theta}(L^p_w)).
\end{align*}\]
Moreover, by the linearity of SKE (5.10) and (5.1), (5.11)-(5.13), there is a constant \(C_0 > 0\) independent of \(\lambda_0\) and \(\lambda\) such that
\[\begin{align*}
&\|Q_\lambda (w - \bar{w})\|_{L^\infty_T(\tilde{B}^{2+\beta}_{p,\theta}(L^p_w))} \leq C_0 |\lambda - \lambda_0| \|w - \bar{w}\|_{L^\infty_T(\tilde{B}^{2+\beta}_{p,\theta}(L^p_w))}.
\end{align*}\]
In particular, if we let \(\delta = 1/(2C_0)\), we obtain a contraction map \(Q_\lambda\). By the fixed point theorem, for any \(\lambda \in [\lambda_0, \lambda_0 + \delta]\), there is a unique \(u \in L^\infty_T(\tilde{B}^{2+\beta}_{p,\theta}(L^p_w))\) so that
\[\begin{align*}
Q_\lambda u = u.
\end{align*}\]
That is, SKE (5.9) is uniquely solvable in \(L^\infty_T(\tilde{B}^{2+\beta}_{p,\theta}(L^p_w))\). Since for \(\lambda = 0\), SKE (5.9) is uniquely solvable (see Theorem 3.6), by iteration, (5.9) is also uniquely solvable for \(\lambda = 1\). The proof is complete. \(\square\)
5.2. Improvement of the regularity in $x$. Note that by (2.9),
\[ B_{p,q}^{2+\beta}(L^p_t) = B_{p,q}^{2+\beta}(\mathbb{R}^d) \cap B_{p,v}^{2+\beta}(L^v_t). \]
For $\beta \in (0, 1)$, the regularity of weak solutions in variable $x$ in Theorem 5.2 does not exceed 1. Naturally, we may ask under what weakest conditions on the coefficients, weak solutions $u$ of SKE (1.1) become strong solutions, i.e., $u$ is $C^1$-differentiable in variable $x$. For this aim, we make the following assumptions:

\((H^{[\gamma]}_{[\beta]})\) We suppose that for some $\gamma, \beta \in (0, 1)$,
\[ \|a\|_{L^\infty_T}^{\gamma,\beta}(C_1^0) + \|b\|_{L^\infty_T}^{\gamma,\beta}(C_2^0) + \|\sigma\|_{L^\infty_T}^{\gamma,\beta}(C_3^0) \leq C_3. \]
We also introduce the following mixed space: for $\beta, \gamma > 0$ and $p \in [1, \infty]$,
\[ \|f\|_{B^\gamma,\beta}_{p,r} := \sup_j 2^{j\gamma}\|R^\gamma_j f\|_{B^\beta}_{p,r}(\mathbb{R}^d) \leq \|2^{j\gamma}\|\|R^\gamma_j R^\beta_j f\|_{L^r(\mathbb{R}^d)} < \infty, \]
where $R^\gamma_j$ is defined in (2.10), and
\[ \|f\|_{B^\gamma,\beta}_{p,r}(\mathbb{R}^d) := \sup_z \|\chi^\gamma_z f\|_{B^\beta}_{p,r}(\mathbb{R}^d) < \infty, \]
where $\chi^\gamma_z$ is defined in (2.17). We have

**Theorem 5.3.** Let $T > 0$, $p \in [2, \infty)$, $\beta, \gamma \in (0, 1)$ and
\[ u_0 \in B_{p,\infty}^{2+\beta}(L^2_\mathbb{R}^d), f \in L^p_T(B_{p,\infty}^{\gamma,\beta}(L^2_\mathbb{R}^d)), g \in L^\infty_T(B_{p,\infty}^{\gamma,1+\beta}(L^2_\mathbb{R}^d)). \]
Under $(H_0)$ and $(H^{[\gamma]}_{[\beta]})$, the unique weak solution $u$ in Theorem 5.2 also satisfies
\[ \|u\|_{L^p_T(B_{p,\infty}^{\gamma,\beta}(L^2_\mathbb{R}^d))} \leq C \|u_0\|_{B_{p,\infty}^{2+\beta}(L^2_\mathbb{R}^d)} + \|f\|_{L^p_T(B_{p,\infty}^{\gamma,\beta}(L^2_\mathbb{R}^d))} + \|g\|_{L^\infty_T(B_{p,\infty}^{\gamma,1+\beta}(L^2_\mathbb{R}^d))}, \]
where $C = C(d, p, \beta, \gamma, T, c_0, c_1, c_3) > 0$.

**Proof.** Following the proof of Theorem 5.2, noting that
\[ dR^\beta_j u^\delta_{z_0} = \left[ \text{tr}(a_{z_0} \cdot \nabla^2_v R^\beta_j u^\delta_{z_0}) + v \cdot \nabla_x R^\beta_j u^\delta_{z_0} + R^\beta_j \tilde{f} \right] dt \]
\[ + (R^\beta_j g^k \cdot a^k_{z_0} \cdot \nabla_v R^\beta_j u^\delta_{z_0})dW^k_t, \]
for any $q \in [2, \infty]$, by Theorem 4.2 with $\alpha = \beta + 2(1 - \frac{1}{q})$ and $\kappa = 1$, we have
\[ \|R^\beta_j u^\delta_{z_0}(t)\|_{B_{p,q}^{\beta+2(1-\frac{1}{q})}(L^q_\mathbb{R}^d)} \leq C \|R^\beta_j u^\delta_{z_0}(0, \cdot)\|_{B_{p,q}^{\beta+2(1-\frac{1}{q})}(L^q_\mathbb{R}^d)} \]
\[ + \left( \int_0^t \left[ \|R^\beta_j \tilde{f}(s)\|_{B_{p,q}^{\beta}}(L^q_\mathbb{R}^d) + \|\tilde{g}(s)\|_{B_{p,q}^{\beta}}(L^q_\mathbb{R}^d) \right] ds \right)^{1/q}. \]
We estimate $\|R^\beta_j \tilde{f}(s)\|_{B_{p,q}^{\beta}}(L^q_\mathbb{R}^d)$ and $\|R^\beta_j \tilde{g}(s)\|_{B_{p,q}^{\beta+1}}(L^q_\mathbb{R}^d))$. Note that
\[ R^\beta_j \text{tr}(\tilde{a} \cdot \nabla^2_v u^\delta_{z_0}) = \text{tr}(\tilde{a} \cdot R^\beta_j \nabla^2_v u^\delta_{z_0}) + \text{tr}(\nabla^2_v u^\delta_{z_0}) \]
and
\[ R^\beta_j (\tilde{a} \cdot \nabla_v u^\delta_{z_0}) = \tilde{a} \cdot R^\beta_j u^\delta_{z_0} + [R^\beta_j, \tilde{a}] \cdot \nabla_v u^\delta_{z_0}. \]
By Lemma 2.11, $(H^{[\gamma]}_{[\beta]})$ and Young’s inequality, it is easy to see that
\[ \|R^\beta_j \text{tr}(\tilde{a} \cdot \nabla^2_v u^\delta_{z_0})\|_{B_{p,q}^{\beta}}(L^q_\mathbb{R}^d) + \|R^\beta_j (\tilde{b} \cdot \nabla_v u^{2\delta}_{z_0})\|_{B_{p,q}^{\beta}}(L^q_\mathbb{R}^d) + \|R^\beta_j (\tilde{c} \cdot u^{2\delta}_{z_0})\|_{B_{p,q}^{\beta}}(L^q_\mathbb{R}^d) \]
\[ \leq C \delta^\beta \|R^\beta_j u^\delta_{z_0}\|_{B_{p,q}^{\beta}}(L^q_\mathbb{R}^d) + \|R^\beta_j u^\delta_{z_0}\|_{L^q_\mathbb{R}^d} + 2^{-\gamma_j} \|u^{2\delta}_{z_0}\|_{B_{p,q}^{\beta}}(L^q_\mathbb{R}^d), \]
and
\[ \|R^\beta_j (\tilde{a} \cdot \nabla_v u^\delta_{z_0})\|_{B_{p,q}^{\beta+1}}(L^q_\mathbb{R}^d) + \|R^\beta_j (\tilde{h} u^{2\delta}_{z_0})\|_{B_{p,q}^{\beta+1}}(L^q_\mathbb{R}^d) \]
\[ \leq C \delta^\beta \|R^\beta_j u^\delta_{z_0}\|_{B_{p,q}^{\beta+1}}(L^q_\mathbb{R}^d) + \|R^\beta_j u^\delta_{z_0}\|_{L^q_\mathbb{R}^d} + 2^{-\gamma_j} \|u^{2\delta}_{z_0}\|_{B_{p,q}^{\beta+1}}(L^q_\mathbb{R}^d). \]
Recall the definitions of $\tilde{f}$ and $\tilde{g}$ in (5.2) and (5.3). Substituting these two estimates into (5.15), we obtain

$$
\|R^*_j u^\delta_{x_0}(t)\|_{\beta + 2\gamma + 1 - \frac{1}{q}}(\mathbb{L}^q_p) \lesssim C \|R^*_j (u_0 \chi^\delta_{x_0}(0, \cdot))\|_{\beta + 2\gamma + 1 - \frac{1}{q}}(\mathbb{L}^q_p) + \left( \int_0^t \left[ \delta^\beta \|R^*_j u^\delta_{x_0}\|_{\mathcal{B}^{\beta + 2\gamma,p}(\mathbb{L}^p_q)} + \|R^*_j u^\delta_{x_0}(s)\|_{\mathbb{L}^q_p} \right] ds \right)^{1/q} + 2^{-\gamma j} \left( \int_0^t \|u^\delta_{x_0}(s)\|_{\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q)} ds \right)^{1/q} + \left( \int_0^t \|R^*_j \delta^\delta_{x_0}(s)\|_{\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q)}^q \right)ds^{1/q},
$$

which further implies by definition and (5.1) that for any $q \in [2, \infty]$,

$$
\|u(t)\|_{\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q)} \lesssim C \|u_0\|_{\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q)} + \left( \int_0^t \|u(s)\|_{\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q)}^q \right)ds^{1/q} + K_T,
$$

where

$$K_T := \|f\|_{L^\infty_p(\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q))} + \|g\|_{L^\infty_p(\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q))}.
$$

In particular, letting $q = \infty$ and $\delta$ be small enough, we get

$$
\|u\|_{L^\infty_p(\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q))} \lesssim C \|u_0\|_{\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q)} + \|u\|_{L^\infty_p(\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q))} + K_T.
$$

Substituting this into (5.16) and taking $q = 2$, we have

$$
\|u\|_{L^p_p(\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q))} \lesssim C \|u_0\|_{\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q)} + \left( \int_0^t \|u(s)\|_{\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q)}^2 \right)^{1/2} + K_T.
$$

By Gronwall’s inequality, we get

$$
\|u\|_{L^p_p(\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q))} \lesssim \|u_0\|_{\mathcal{B}^{\gamma + 1,p}(\mathbb{L}^p_q)} + K_T.
$$

Substituting this into (5.17), we obtain the desired estimate. \( \square \)

### 5.3. Solvability of adjoint equation (1.2)

In this subsection we study the adjoint equation (1.2), and assume

$$(H_{\phi}^\beta)^T$$

We suppose that for some $\beta \in (0, 1)$ and $c_2 > 0$,

$$
\|a\|_{L^\infty_{\omega,\omega}(\mathbb{C}_p^\beta)} + \|b\|_{L^\infty_{\omega,\omega}} + \|\sigma\|_{L^\infty_{\omega,\omega}(\mathbb{C}_p^\beta)} \leq c_2, \quad T > 0.
$$

We also introduce the following notion about the solution of SKE (1.2).

**Definition 5.4.** Let $T > 0$, $\beta \in (0, 1)$ and $p \in [2, \infty)$. A predictable process $u \in L^\infty_p(\mathcal{B}^{\beta}(\mathbb{L}^p_q))$ is called a weak solution of SKE (1.2) if for any $\varphi \in C^\infty_c(\mathbb{R}^d)$,

$$
\langle u(t), \varphi \rangle = \langle u_0, \varphi \rangle + \int_0^t \left[ \langle u, \mathcal{L}_v \varphi + v \cdot \nabla \varphi \rangle + \langle f, \varphi \rangle \right] ds + \int_0^t \langle u, \mathcal{M}_{g^k}^k \varphi \rangle + \langle g^k, \varphi \rangle \rangle dW^k.
$$

We have the following result.
Theorem 5.5. Let $T > 0$, $p \in [2, \infty)$, $\beta \in (0, 1)$ and $\nu_0 \in \tilde{B}^{\beta}_{p, \theta}(L_2^p)$, $f \in L^2_T(\tilde{B}^{\beta - 2}_{p, \theta}(L_2^p))$, $g \in L^\infty_T(\tilde{B}^{\beta - 1}_{p, \theta}(L_2^p))$.

Under (H$_0$) and (H$_1$), there is a unique weak solution $u$ to SKE (1.2) so that

$$
\|u\|_{L^\infty_T(\tilde{B}^{\beta}_{p, \theta}(L_2^p))} \lesssim C \|\nu_0\|_{\tilde{B}^{\beta}_{p, \theta}(L_2^p)} + \|f\|_{L^2_T(\tilde{B}^{\beta - 2}_{p, \theta}(L_2^p))} + \|g\|_{L^\infty_T(\tilde{B}^{\beta - 1}_{p, \theta}(L_2^p))},
$$
where the constant $C = C(d, p, q, \beta, T, \alpha_0, c_1, c_2) > 0$.

Proof. We only prove the a priori estimate (5.18). We follow the proof of Theorem 5.2 and use the same notations therein. It is the same reason as in the proof of Theorem 5.2 that we may assume $\nu_0 \equiv 0$. By Definition 5.4, one sees that $u_{20}$ is a weak solution of the following freezing SKE:

$$
du_{20} = \left[\text{tr}(a_{20} \cdot \nabla^2 u_{20}^\delta) - v \cdot \nabla u_{20}^\delta + \tilde{f} \right]dt + \left[a_{20} \cdot \nabla u_{20}^\delta + g^\delta \right]dW_t^k,
$$
where

$$
\tilde{f} := \partial_{v_{ij}v_{ij}}(\alpha_{ij} - a_{ij})u_{20}^\delta - 2\partial_{v_{i}v_{i}}(\alpha_{ij}u_{20}^\delta)\partial_{v_{j}}v_{20}^\delta + \text{div}_v(bu_{20}^\delta) - \left[\text{tr}(a \cdot \nabla^2 \nabla \delta) + (v - v_0) \cdot \nabla \nabla \delta + b \cdot \nabla \nabla \theta \right]u + f \nabla \theta,
$$
and

$$
g := \text{div}_v((\sigma - \nabla \nabla \delta)u_{20}^\delta - \sigma \cdot \nabla \nabla \theta u + g \nabla \theta.
$$

For simplicity of notations, if we set

$$
\tilde{a} := (a - a_{20})\nabla^\delta, \quad \tilde{\sigma} := (\sigma - \nabla \nabla \theta)\nabla^\delta, \quad \tilde{h} := \sigma \cdot \nabla \nabla \theta,
$$
and

$$\tilde{c} := -(\text{tr}(a \cdot \nabla^2 \nabla \theta) + (v - v_0) \cdot \nabla \nabla \theta + b \cdot \nabla \nabla \theta),
$$
then due to $\nabla^\delta = 1$ on the support of $\nabla \theta$, we can write

$$
\tilde{f} = \partial_{v_{ij}v_{ij}}(\tilde{a}_{ij}u_{20}^\delta) - 2\partial_{v_{i}v_{i}}(\alpha_{ij}u_{20}^\delta)\partial_{v_{j}}v_{20}^\delta + \text{div}_v(bu_{20}^\delta) + \tilde{c} u_{20}^\delta + f \nabla \theta,
$$
and

$$
g = \text{div}_v(\tilde{\sigma} u_{20}^\delta) - \tilde{h} u_{20}^\delta + g \nabla \theta.
$$

For any $p \in [2, \infty)$ and $q \in (2, \infty)$, by Theorem 4.2, there is a constant $C = C(T, d, p, q, \beta, \alpha_0, c_1) > 0$ such that for all $\delta \in (0, 1)$ and $\nabla \theta \in \mathbb{R}^{2d}$,

$$
\|u_{20}^\delta\|_{L^\infty_T(\tilde{B}^{\beta}_{p, \theta}(L_2^p))} \lesssim C \left\|\tilde{f}\right\|_{L^1_T(\tilde{B}^{\beta - 2}_{p, \theta}(L_2^p))} + \|\tilde{g}\|_{L^2_T(\tilde{B}^{\beta - 1}_{p, \theta}(L_2^p))},
$$
(5.19)

We estimate each term of the right hand side. By Bernstein’s inequality and (2.13),

$$
\|\partial_{v_{ij}v_{ij}}(\alpha_{ij}u_{20}^\delta)\|_{\tilde{B}^{\beta - 2}_{p, \theta}(L_2^p)} \lesssim \|\alpha u_{20}^\delta\|_{\tilde{B}^{\beta}_{p, \theta}(L_2^p)} \lesssim \delta^\beta \|u_{20}^\delta\|_{\tilde{B}^{\beta}_{p, \theta}(L_2^p)} + C\delta \|u_{20}^\delta\|_{L^\infty_T(L_2^\infty)},
$$
and by (2.14),

$$
\|\partial_{v_{i}v_{i}}(\alpha_{ij}u_{20}^\delta)\|_{\tilde{B}^{\beta - 2}_{p, \theta}(L_2^p)} \lesssim \|\alpha_{ij}u_{20}^\delta\|_{\tilde{B}^{\beta - 2}_{p, \theta}(L_2^p)} \lesssim \|\alpha_{ij}u_{20}^\delta\|_{L^p_T(L_2^p)} \lesssim \|a_{ij}u_{20}^\delta\|_{L^p_T(L_2^p)} \lesssim \|u_{20}^\delta\|_{L^p_T(L_2^p)},
$$
and

$$
\|\text{div}_v(bu_{20}^\delta)\|_{\tilde{B}^{\beta - 2}_{p, \theta}(L_2^p)} \lesssim \|bu_{20}^\delta\|_{\tilde{B}^{\beta - 1}_{p, \theta}(L_2^p)} \lesssim \|b\|_{L^\infty_T(L_2^\infty)} \|u_{20}^\delta\|_{L^\infty_T(L_2^\infty)}.
$$

Since $\tilde{c}$ is bounded, we also have

$$
\|\tilde{c} u_{20}^\delta\|_{\tilde{B}^{\beta - 2}_{p, \theta}(L_2^p)} \lesssim \|\tilde{c} u_{20}^\delta\|_{L^2_T(L_2^p)} \lesssim \|u_{20}^\delta\|_{L^2_T(L_2^p)}.
$$

Therefore,

$$
\|\tilde{f}\|_{\tilde{B}^{\beta - 2}_{p, \theta}(L_2^p)} \lesssim \delta^\beta \|u\|_{\tilde{B}^{\beta}_{p, \theta}(L_2^p)} + C\delta \|u\|_{L^\infty_T(L_2^\infty)} + \|f\|_{\tilde{B}^{\beta - 2}_{p, \theta}(L_2^p)}.
$$
Similarly, one can show that
\[ \| \hat{g} \|_{B_{p,q}^{-\alpha}(L^p, L^q)} \lesssim \delta^\beta \| u \|_{B_{p,q}^\beta(L^p)} + C \delta \| u \|_{L^p_t(L^q)} + \| g \|_{B_{p,q}^{-\alpha}(L^p, L^q)}. \]
Substituting these two estimates into (5.19), and taking supremum in \( z_0 \in \mathbb{R}^{2d} \), we obtain that for any \( q \in [2, \infty] \),
\[ \| u \|_{L^p_t(B_{p,q}^{-\alpha}(L^p, L^q))} \lesssim C \delta^\beta \| u \|_{L^p_t(B_{p,q}^\beta(L^p))} + C \delta \| u \|_{L^p_t(L^q)} + f \|_{L^q_t(L^p)} + \| g \|_{L^p_t(B_{p,q}^{-\alpha}(L^p, L^q))}. \]
The remaining proof is completely the same as in Theorem 5.2.

6. Applications in nonlinear filtering problems

Fix \( d, d_1 \in \mathbb{N} \). Let \( \hat{(B_t)}_{t \geq 0} \) and \( (B_t)_{t \geq 0} \) be two independent \( d \) and \( d_1 \)-dimensional Brownian motions on some stochastic basis \((\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0})\). Let
\[ (\hat{b}, \hat{\sigma}, \sigma, \hat{b}) : \mathbb{R}^+ \times \mathbb{R}^{2d} \times \mathbb{R}^{d_1} \to (\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^{d} \otimes \mathbb{R}^{d_1}, \mathbb{R}^{d_1}) \]
and \( \hat{\sigma} : \mathbb{R}^+ \times \mathbb{R}^{d_1} \to \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_1} \) be bounded Borel measurable functions. Consider the following SDE of Itô’s type:
\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} X_t \\ V_t \\ Y_t \end{pmatrix} &= \begin{pmatrix} V_t \\ \hat{b}(t, U_t) \\ \hat{b}(t, U_t) \end{pmatrix} dt + \begin{pmatrix} 0, & 0, & 0 \\ 0, & \hat{\sigma}(t, U_t), & \sigma(t, U_t) \\ 0, & 0, & \hat{\sigma}(t, Y_t) \end{pmatrix} d\begin{pmatrix} 0 \\ \hat{B}_t \\ B_t \end{pmatrix},
\end{align*}
\]
where
\[ U_t := (X_t, V_t, Y_t) =: (Z_t, Y_t) \in \mathbb{R}^{2d} \times \mathbb{R}^{d_1}, \]
\( Z_t \) stands for the unobservable signal and \( Y_t \) denotes the observable signal, \( U_0 \) is an \( \mathcal{F}_0 \)-measurable random variable. Let \( \mathcal{F}_t^Y \) be the \( \mathbb{P} \)-completed \( \sigma \)-algebra generated by \( \{Y_s, s \leq t\} \), which represents the observation information. In application, we want to predict \( Z_t \) in terms of \( \mathcal{F}_t^Y \). More precisely, we want to calculate the conditional distribution of \( Z_t \) under \( \mathcal{F}_t^Y \):
\[ \Pi_t(\omega, A) := \mathbb{P}(Z_t \in A | \mathcal{F}_t^Y). \]
This is usually called the filtering problem (cf. [26]).

Throughout this section we suppose that
\[
\begin{align*}
(A_1) & \quad \Pi_0(\omega, dz) = \pi_0(\omega, z) dz \quad \text{with} \quad \pi_0 \in \cap_{\alpha \geq 2} B_{p,q}^\beta(L^p_z) \quad \text{for some} \ \beta \in (0, 1). \\
(A_2) & \quad \hat{b}, \hat{\sigma}, b, \sigma \text{ are bounded and Lipschitz in } (x, v, y) \text{ and uniformly in } t. \\
(A_3) & \quad \hat{\sigma} \text{ and } \sigma \text{ are non-singular, and for some } K > 0 \text{ and all } t, x, v, y,
\|\hat{\sigma}^{-1}(t, x, v, y)\|_{M_{4\alpha}^{4\alpha}} + \|\sigma^{-1}(t, y)\|_{M_{4\alpha}^{4\alpha}} \leq K. 
\end{align*}
\]
Under the above assumptions, it is well-known that SDE (6.1) admits a unique strong solution. Let
\[ \hat{h}(t, z, y) := (\hat{\sigma}^{-1} \hat{b})(t, z, y), \quad \hat{b}(t, z, y) := (\hat{b} - \sigma \hat{h})(t, z, y) \]
and
\[ W_t = B_t + \int_0^t \hat{h}(s, U_s) ds. \]
Then we can write SDE (6.1) as
\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} X_t \\ V_t \\ Y_t \end{pmatrix} &= \begin{pmatrix} V_t \\ \hat{b}(t, U_t) \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0, & 0, & 0 \\ 0, & \hat{\sigma}(t, U_t), & \sigma(t, U_t) \\ 0, & 0, & \hat{\sigma}(t, Y_t) \end{pmatrix} d\begin{pmatrix} 0 \\ \hat{B}_t \\ B_t \end{pmatrix},
\end{align*}
\]
Define
\[
\rho_t := \exp \left\{ \int_0^t \tilde{h}(s,U_s)dB_s + \frac{1}{2} \int_0^t \|\tilde{h}(s,U_s)\|^2ds \right\}. \tag{6.2}
\]

Fix $T > 0$. Since $\tilde{h}$ is bounded, by Itô’s formula, it is easy to see that
\[
\mathbb{E}(\rho_T^{-1}|\mathcal{F}_t) = \rho_t^{-1}, \quad t \in [0,T]. \tag{6.3}
\]

Thus, by Girsanov’s theorem, under the new probability measure
\[
\mathbb{P}_T(d\omega) := \rho_T^{-1}(\omega)\mathbb{P}(d\omega), \tag{6.4}
\]

$(W_t)_{t \in [0,T]}$ is still a $d_1$-dimensional Wiener process and independent of $\tilde{B}$. Moreover, since $\tilde{\sigma}$ is invertible, it is easy to see that (see [26, Lemma 6.2])
\[
\mathcal{F}^Y_t = \mathcal{F}^W_t \vee \mathcal{F}^Y_0. \tag{6.5}
\]

This is the crucial point for deriving the density equation of $\Pi_t(\omega, \cdot)$. In other words, by Girsanov’s technique, the observation information is transformed into Brownian filtration information.

Now we let
\[
\Sigma(t,\omega,z) := \sigma(t,z,Y_t(\omega)), \quad \Sigma(t,\omega,z) := \sigma(t,z,Y_t(\omega)),
\]

\[
b(t,\omega,z) := \tilde{b}(t,z,Y_t(\omega)), \quad h(t,\omega,z) := (\tilde{\sigma}^{-1} \tilde{b})(t,z,Y_t(\omega)),
\]

and
\[
\mathcal{L}^{\nu} \varphi := a \cdot \nabla^2 \varphi + b \cdot \nabla \varphi, \quad \mathcal{M}^\nu \psi := \Sigma \cdot \nabla \psi + h \varphi, \tag{6.7}
\]

where $a := \frac{1}{2}(\Sigma \Sigma^* + \Sigma^* \Sigma)$. Consider the following linear SKE:
\[
du = [\mathcal{L}^{\nu} u - v \cdot \nabla u]dt + \mathcal{M}^\nu u dW_t, \quad u(0) = \pi_0, \tag{6.8}
\]

where $\mathcal{L}^{\nu}$ and $\mathcal{M}^\nu$ stand for the adjoint operators of $\mathcal{L}^{\nu}$ and $\mathcal{M}^\nu$, respectively. Under $(A_1)$-$(A_3)$, by Theorem 5.5, there exists a unique generalized solution
\[
u \in \cap_{p \geq 2} \mathbb{L}^p_T(\mathbb{B}^{\beta}_{p;\theta}(\mathbb{L}^p_D)), \quad T > 0, \tag{6.9}
\]

to the above SKE. We first show the following result.

**Lemma 6.1.** For each $t \geq 0$, $R > 0$ and $\beta' < \beta$, it holds that for $\mathbb{P}$-almost all $\omega$,
\[
\lambda_0^R(\cdot)u(t,\omega, \cdot) \in \cap_{p \geq 2} \mathbb{B}^{\beta'}_{p;\theta}, \tag{6.10}
\]

where $\lambda_0^R$ is the cutoff function in (2.17), and for any bounded measurable $\varphi$,
\[
\langle u(t,\omega), \varphi \rangle = \mathbb{E}^\mathbb{P}_T (\varphi(Z_t)\rho_T|\mathcal{F}^Y_t)(\omega), \tag{6.11}
\]

where $\rho_T$ is defined by (6.2).

**Proof.** First of all, for each $R,t > 0$, by (6.9) and (2.3), we have
\[
\lambda_0^R u(t) \in \cap_{p \geq 2} \mathbb{B}^{\beta}_{p;\theta}(\mathbb{L}_D^p) \subset \cap_{p \geq 2} \mathbb{L}^p_D(B^{\beta'}_{p;\theta}),
\]

which in turn gives (6.10). Let
\[
(b_\varepsilon, \tilde{\Sigma}_\varepsilon, \Sigma_\varepsilon, h_\varepsilon) = (b, \tilde{\Sigma}, \Sigma, h) * \varrho_\varepsilon,
\]

where $(\varrho_\varepsilon)_{\varepsilon \in (0,1)}$ is a family of mollifiers in $\mathbb{R}^d$. Let $(\mathcal{L}_{\varepsilon}^\nu, \mathcal{M}_{\varepsilon}^\nu)$ be defined as in (6.7) in terms of the above mollifying coefficients. Let $u_\varepsilon$ be the unique solution of the following SKE:
\[
du_\varepsilon = [(\mathcal{L}_{\varepsilon}^\nu)^* u_\varepsilon - v \cdot \nabla u_\varepsilon]dt + (\mathcal{M}_{\varepsilon}^\nu)^* u_\varepsilon dW_t, \quad u_\varepsilon(0) = \pi_0.
\]
By (5.18), there is a constant $C > 0$ independent of $\varepsilon \in (0, 1)$ such that
\[
\|u_\varepsilon\|_{L^2_p(B_{p,\theta}^0(\mathbb{L}^\varepsilon))} \lesssim C \|\pi_0\|_{B_{p,\theta}^0(\mathbb{L}^\varepsilon)}.
\] (6.12)

Let
\[
U_\varepsilon := u_\varepsilon - u, \quad F_\varepsilon := ((\mathcal{L}^\varepsilon - \mathcal{L}_0)^* u_\varepsilon, G_\varepsilon := (\mathcal{M}^\varepsilon - \mathcal{M}_0)^* u_\varepsilon.
\]

Then one sees that
\[
dU_\varepsilon = [(\mathcal{L}^\varepsilon)^* U_\varepsilon - v \cdot \nabla x U_\varepsilon + F_\varepsilon]dt + [(\mathcal{M}^\varepsilon)^* U_\varepsilon + G_\varepsilon]dW_t, \quad U_\varepsilon(0) = 0.
\]

By (5.18), we have
\[
\|U_\varepsilon\|_{L^2_p(B_{p,\theta}^0(\mathbb{L}^\varepsilon))} \lesssim \|F_\varepsilon\|_{L^2_p(B_{p,\theta}^0(\mathbb{L}^\varepsilon))} + \|G_\varepsilon\|_{L^2_p(B_{p,\theta}^0(\mathbb{L}^\varepsilon))} + \|\mathcal{M}^\varepsilon - \mathcal{M}_0\|_{L^2_p(B_{p,\theta}^0(\mathbb{L}^\varepsilon))}.
\] (6.13)

Here and below, the implicit constant in $\lesssim$ is independent of $\varepsilon \in (0, 1)$. Noting that
\[
F_\varepsilon = \partial^2_{v_i v_j}((a^{ij} - a^{ij})u_\varepsilon) + \partial_{v_i}((b^i - b^i)u_\varepsilon),
\]
by Bernstein's inequality (2.4), (2.13) and (6.12), we have
\[
\|F_\varepsilon\|_{L^2_p(B_{p,\theta}^0(\mathbb{L}^\varepsilon))} \lesssim \|\epsilon_\varepsilon - \epsilon_\|_{L^2_p(C_{p,\theta}^0(\mathbb{L}^\varepsilon))} + \|b_\| - \|b_\|_{L^2_p(\mathbb{L}^\varepsilon)}
\lesssim \|\epsilon_\varepsilon - \epsilon_\|_{L^2_p(C_{p,\theta}^0(\mathbb{L}^\varepsilon))} + \|b_\| - \|b_\|_{L^2_p(\mathbb{L}^\varepsilon)}
\lesssim (\epsilon_1 - \| \epsilon_\|_{L^2_p(\mathbb{L}^\varepsilon))} \lesssim \epsilon_1 - \| \epsilon_\|_{L^2_p(\mathbb{L}^\varepsilon))}.
\]

where in the third inequality we have used that for $|z - z'| \leq 1,
\|(a_\| - a_\|)(t, \omega, z) - \|(a_\| - a_\|)(t, \omega, z')\| \lesssim \|a_\| - a_\|\| \|a_\|_{L^p(\mathbb{L}^\varepsilon)}|z - z'| \lesssim \epsilon_1 - \| z - z' \|_{L^\varepsilon}.
\]

Similarly we have
\[
\|G_\varepsilon\|_{L^2_p(B_{p,\theta}^0(\mathbb{L}^\varepsilon))} \lesssim \|\Sigma_\varepsilon - \Sigma_\|_{L^2_p(\mathbb{L}^\varepsilon)} + \|\gamma_\| - \|\gamma_\|_{L^2_p(\mathbb{L}^\varepsilon)}
\lesssim \|\Sigma_\varepsilon - \Sigma_\|_{L^2_p(\mathbb{L}^\varepsilon)} + \|\gamma_\| - \|\gamma_\|_{L^2_p(\mathbb{L}^\varepsilon)}
\lesssim \epsilon_1 - \| \gamma_\|_{L^2_p(\mathbb{L}^\varepsilon))}.
\]

Thus, by (6.13),
\[
\lim_{\varepsilon \to 0} \|U_\varepsilon\|_{L^2_p(B_{p,\theta}^0(\mathbb{L}^\varepsilon))} = 0.
\] (6.14)

Let $Z^\varepsilon := (X^\varepsilon, V^\varepsilon)$ solve the following SDE
\[
dX^\varepsilon_t = V^\varepsilon_t dt, \quad dV^\varepsilon_t = b_z(t, Z^\varepsilon_t) dt + \Sigma_\varepsilon(t, Z^\varepsilon_t) d\tilde{B}_t + \Sigma_\varepsilon(t, Z^\varepsilon_t) dB_t, \quad Z^\varepsilon_0 = Z_0.
\]

Since $b_z, \Sigma, \Sigma$ are bounded and Lipschitz continuous in $z$ uniformly in $(t, \omega, \varepsilon)$, it is by now standard to show that
\[
\lim_{\varepsilon \to 0} \mathbb{E}|Z^\varepsilon_t - Z_t|^2 = 0.
\] (6.15)

Now by [26, p.201, Theorem 5.6], we have for any $\varphi \in C^\infty_b(\mathbb{R}^d)$,
\[
\langle u_\varepsilon(t, \omega), \varphi \rangle = \mathbb{E}^\varepsilon_t \langle \varphi(Z^\varepsilon_t)\rho^\varepsilon_t, \mathcal{F}^\varepsilon_t \rangle (\omega), \quad \mathbb{P} - a.s.
\] (6.16)

where
\[
\rho^\varepsilon_t := \exp \left\{ \int_0^t \gamma_\varepsilon(s, Z^\varepsilon_s) dB_s + \frac{1}{2} \int_0^t \|\gamma_\varepsilon(s, Z^\varepsilon_s)\|^2 ds \right\}.
\]

By (6.14) and (6.15) and taking limits for (6.16), we get
\[
\langle u(t, \omega), \varphi \rangle = \mathbb{E}^T_t \langle \varphi(Z_t)\rho_t, \mathcal{F}^T_t \rangle (\omega), \quad \mathbb{P} - a.s.
\] (6.17)

On the other hand, for any $A \in \mathcal{F}^T_t$, by (6.4) and (6.3) we have
\[
\mathbb{E}^T_t (1_A \varphi(Z_t)\rho_t) = \mathbb{E} (1_A \varphi(Z_t)\rho_t \rho^{-1}_T) = \mathbb{E} (1_A \varphi(Z_t)\mathbb{E}(\rho_t \rho^{-1}_T | \mathcal{F}_t))
\]
\[
= \mathbb{E} (1_A \varphi(Z_t)) = \mathbb{E} (1_A \varphi(Z_t)\rho_T \rho^{-1}_T) = \mathbb{E}^T_t (1_A \varphi(Z_t)\rho_T),
\]

which means
\[
\mathbb{E}^T_t (\varphi(Z_t)\rho_t | \mathcal{F}^T_t) = \mathbb{E}^T_t (\varphi(Z_t)\rho_T | \mathcal{F}^T_t).
\]
Substituting this into (6.17), we obtain (6.11).

We now state the main result of this section.

**Theorem 6.2.** Under (A1)-(A3), for each $t > 0$, the conditional distribution $\Pi_t(\omega, \cdot)$ has a continuous density $\pi_t(\omega, \cdot)$ for $\mathbb{P}$-almost all $\omega$, and

$$
\pi_t(\omega, \cdot) = \frac{u(t, \omega, \cdot)}{\|u(t, \omega, \cdot)\|_1},
$$

(6.18)

where $u$ is given in (6.9). Moreover, $\pi_t$ solves the following nonlinear SKE:

$$
d\pi_t(\phi) = \pi_t(\mathcal{L}_t\phi + \nu \cdot \nabla_x \phi) dt + (\pi_t(\mathcal{M}_t\phi) - \Pi_t(h_t)(\pi_t(\phi))) dW_t,
$$

(6.19)

for any $\phi \in C_c^\infty(\mathbb{R}^d)$, where $W_t := W_t + \int_0^t \Pi_t(h_s) ds$.

**Proof.** Let $T > 0$. Note that by (6.2),

$$
\rho_t = \exp \left\{ \int_0^t \tilde{h}(s, U_s) dW_s - \frac{1}{2} \int_0^t \|\tilde{h}(s, U_s)\|^2 ds \right\}
$$

satisfies

$$
\rho_t = 1 + \int_0^t \tilde{h}(s, U_s) \rho_s dW_s.
$$

Taking the conditional expectation with respect to $\mathcal{F}_t^Y$, by [26, p.28, Theorem 1.15], we have

$$
E_{\tilde{\mathbb{P}}^T}(\rho_t | \mathcal{F}_t^Y) = 1 + E_{\tilde{\mathbb{P}}^T} \left( \int_0^t \tilde{h}(s, U_s) \rho_s dW_s | \mathcal{F}_t^Y \right)
$$

$$
= 1 + \int_0^t E_{\tilde{\mathbb{P}}^T} \left( \tilde{h}(s, U_s) \rho_s | \mathcal{F}_s^Y \right) dW_s
$$

$$
= 1 + \int_0^t \left( \tilde{h}(s, U_s) \rho_s | \mathcal{F}_s^Y \right) E_{\tilde{\mathbb{P}}^T} \left( \rho_s | \mathcal{F}_s^Y \right) dW_s,
$$

which together with $\Pi_t(h_t) = E(h_t(Z_t) | \mathcal{F}_t^Y) = E(\tilde{h}(t, U_t) | \mathcal{F}_t^Y)$ yields that

$$
E_{\tilde{\mathbb{P}}^T}(\rho_t | \mathcal{F}_t^Y) = \exp \left\{ \int_0^t \Pi_s(h_s) dW_s - \frac{1}{2} \int_0^t \|\Pi_s(h_s)\|^2 ds \right\}.
$$

(6.20)

Now, by Bayes’ formula about the conditional expectations (cf. [26, p.218, Lemma 6.1]) and (6.11), we have

$$
\Pi_t(\phi) = E(\phi(Z_t) | \mathcal{F}_t^Y) = \frac{E_{\tilde{\mathbb{P}}^T}(\phi(Z_t) \rho_T | \mathcal{F}_T^Y)}{E_{\tilde{\mathbb{P}}^T}(\rho_T | \mathcal{F}_T^Y)} = \left< \frac{u(t), \phi}{\langle u(t), 1 \rangle} \right> = \left< \frac{u(t)}{\langle u(t), 1 \rangle}, \phi \right>.
$$

By the arbitrariness of $\phi$, we get (6.18). The continuity of $\pi_t(\omega, \cdot)$ follows by (6.10) and the Sobolev embedding.

Moreover, by (6.16) and (6.20),

$$
\langle u(t), 1 \rangle = \exp \left\{ \int_0^t \Pi_s(h_s) dW_s - \frac{1}{2} \int_0^t \|\Pi_s(h_s)\|^2 ds \right\}
$$

and

$$
\text{d}\langle u(t), \phi \rangle = \langle u(t), \mathcal{L}_t \phi \rangle dt + \langle u(t), \mathcal{M}_t \phi \rangle dW_t.
$$

Thus, by Itô’s formula,

$$
d\pi_t(\phi) = \left[ \pi_t(\mathcal{L}_t \phi) + \pi_t(\mathcal{M}_t \phi) - \pi_t(\phi) \right] \Pi_t(h_t) dt
$$

$$
+ \pi_t(\mathcal{M}_t \phi) - \pi_t(\phi) \Pi_t(h_t) dW_t
$$

$$
= \pi_t(\mathcal{L}_t \phi) dt + \left( \pi_t(\mathcal{M}_t \phi) - \pi_t(\phi) \Pi_t(h_t) \right) (dW_t + \Pi_t(h_t) dt).
$$

From these, we obtain (6.21) and complete the proof.
Remark 6.3. Under further assumptions on \( \tilde{b}, \tilde{\sigma}, \sigma, b, \tilde{\sigma} \) and \( \pi_0 \), by Theorem 5.3, one can show the \( C^1 \) and \( C^2 \)-smoothness of \( \pi \) with respect to \( x \) variable and \( v \) variable so that \( \pi \) solves the following SKE in the point-wise sense of \( x, v \),

\[
\mathrm{d}\pi_t = (\mathcal{L}_\nu^* \pi_t + v \cdot \nabla_x \pi) \mathrm{d}t + (\mathcal{M}_\nu^* \pi_t - \Pi_t(h_t) \pi_t) \mathrm{d}W_t. \tag{6.21}
\]

For instance, we assume that for some \( \gamma, \beta \in (0, 1) \) with \( 3\gamma + \beta > 1 \),

(\( A_1' \)) For almost all \( \omega \), \( \Pi_0(\omega, dz) = \pi_0(\omega, z) \mathrm{d}z \) with \( \pi_0 \in \cap_{p \geq 2} \mathbb{B}^{\gamma,2+\beta}_p(\mathbb{L}^p) \).

(\( A_2' \)) \( \tilde{b}, \tilde{\sigma}, \sigma, b, \tilde{\sigma} \) are bounded and Lipschitz continuous in spatial variables \( (x, v, y) \) and uniformly in time variable \( t \). Moreover, we also suppose that

\[
\sigma, \tilde{\sigma}, \nabla^2_x \sigma, \nabla^2_v \tilde{\sigma}, \mathrm{div}_v \tilde{b}, \nabla_v b, \nabla_v \tilde{\sigma} \in \mathbb{L}^\infty_{T,y}(C^\gamma_x(C^\beta_v))
\]

and

\[
\mathrm{div}_x \sigma \in \mathbb{L}^\infty_{T,y}(C^{\gamma+1}_x(C^\beta_v)).
\]

Thus, under \( A_1' \), \( A_2' \) and \( A_3' \), by Theorem 5.3, there exists a unique solution to SKE (6.8) with regularity

\[
u \in \cap_{p \geq 2} \mathbb{L}^\infty_{T}((\mathbb{B}^{\gamma,2+\beta}_p(\mathbb{L}^p))), \quad T > 0. \tag{6.22}
\]

Due to \( 3\gamma + \beta > 1 \), by (6.18), (6.22), (2.3) and Sobolev’s embedding, we have

\[
\pi \in \cap_{p \geq 2} \mathbb{L}^\infty_{T}(\mathbb{B}^{\gamma,2+\beta}_{p,x,v}(\mathbb{L}^p)) \subset \cap_{p \geq 2} \mathbb{L}^\infty_{T}(C^1_x \cap C^2_v), \quad T > 0,
\]

where \( C^1_x \cap C^2_v \) stands for the space of \( C^1 \) and \( C^2 \)-smooth functions in \( x \) and \( v \).

7. SKEs Driven by Velocity-Time White Noises

Let \( \{B(t,v), (t,v) \in \mathbb{R}_+ \times \mathbb{R}\} \) be a Brownian sheet of time and velocity variables, whose distribution derivative in \( t, v \) is usually considered as velocity-time white noise. Let

\[
(f, g)(t, \omega, x, v, u) : \mathbb{R}_+ \times \Omega \times \mathbb{R}^2 \times \mathbb{R} \rightarrow (\mathbb{R}, \mathbb{R})
\]

be \( \mathcal{P} \times \mathcal{B}(\mathbb{R}^3) \)-measurable functions. We consider the following nonlinear SKE driven by \( B \),

\[
du = [\Delta_u u + v \cdot \nabla_x u + f(u)] \mathrm{d}t + g(u) \mathrm{d}B(t,v), \quad u(0) = u_0, \tag{7.1}
\]

where the stochastic integral is understood in the sense of Walsh [29]. Here we have suppressed the variables \( (t, \omega, x, v) \) of \( f, g \). We introduce the following notion about the solution to the above SKE.

Definition 7.1. Let \( p \geq 2 \) and \( T > 0 \). We call a predictable process \( u \in \mathbb{L}^p_{T}((\mathbb{L}^p_{x,v})) \) being a weak solution of SKE (7.1) if for any \( \varphi \in C^\infty(\mathbb{R}^2) \) and \( t \in [0, T] \),

\[
\langle u(t), \varphi \rangle = \langle u_0, \varphi \rangle + \int_0^t \int_\mathbb{R} \left[ u(s, \nabla \varphi - v \cdot \nabla_x \varphi) + f(u(s), \varphi) \right] \mathrm{d}s \mathrm{d}B(s,v) \tag{7.2}
\]

a.s.

In order to use our previous results to study SKE (7.1), we shall use the following representation of \( B(t,v) \) (cf. [15, Section 8.2]),

\[
B(t,v) = \sum_{k=1}^\infty \left( \int_0^v \eta_k(r) \mathrm{d}r \right) W_t^k, \tag{7.3}
\]
where \( \{\eta_k, k \in \mathbb{N}\} \subset C_0^\infty(\mathbb{R}) \) is an orthonormal basis of \( L_2^2(\mathbb{R}) \). In particular, for any measurable adapted processes \( \xi(s, v) \) with \( \int_0^T E\|\xi(s,)\|^2_{L_2^2} ds < \infty \),

\[
\int_0^T \int_\mathbb{R} \xi(s, v) dB(s, v) = \sum_{k=1}^\infty \int_0^T \left( \int_\mathbb{R} \xi(s, v) \eta_k(v) dv \right) dW_s^k. \tag{7.4}
\]

For \( N = 1, \ldots, \infty \) and a function \( h \), we introduce the following notations:

\[
G_N(h) := (h\eta_1, \ldots, h\eta_N, 0, \ldots), \quad G(h) := G_\infty(h).
\]

By this notation and (7.4), we can write SKE (7.1) as

\[
du = [\Delta_v u + v \cdot \nabla x u + f(u)] dt + G^k(g(u))dW_t^k, \quad u(0) = u_0. \tag{7.5}
\]

We have the following important lemma.

**Lemma 7.2.** For any \( p \geq 2 \) and \( r \in [1, p] \), there is a constant \( C = C(p, r) > 0 \) such that for all \( N = 1, \ldots, \infty \) and \( h \in L_p^p(\mathbb{L}_2^2) \),

\[
\|G_N(h)\|_{B_p^0(\mathbb{L}_2^p(\mathbb{L}_2^2))} \leq C \|h\|_{L_p^p(\mathbb{L}_2^2)}, \tag{7.6}
\]

where \( \beta := 4(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2} \). Moreover, for any \( \beta' < \beta \), we also have

\[
\lim_{N \to \infty} \|G_N(h) - G(h)\|_{B_{p,\beta'}^0(\mathbb{L}_2^p(\mathbb{L}_2^2))} = 0. \tag{7.7}
\]

**Proof.** Let \( h \in L_p^p(\mathbb{L}_2^2) \). Recall (2.2). By Parseval’s identity, we have for any \( j \in \mathbb{N}_0 \),

\[
\|R_j^0 G_N(h)(x, v)\|_{L_2^2}^2 = \sum_{k=1}^N \left| \int_{\mathbb{R}^2} \bar{\phi}_j(x - \bar{x}, v - \bar{v}) h(\bar{x}, \bar{v}) \eta_k(\bar{v}) d\bar{x} d\bar{v} \right|^2
\]

\[
\leq \sum_{k=1}^\infty \left| \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \bar{\phi}_j(x - \bar{x}, v - \bar{v}) h(\bar{x}, \bar{v}) d\bar{v} \right) \eta_k(\bar{v}) d\bar{v} \right|^2
\]

\[
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \bar{\phi}_j(x - \bar{x}, v - \bar{v}) h(\bar{x}, \bar{v}) d\bar{v} \right)^2 d\bar{v}
\]

\[
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \bar{\phi}_j(x, \bar{v}) h(x - \bar{x}, v - \bar{v}) d\bar{v} \right)^2 d\bar{v}.
\]

Let \( q, s \in [1, \infty] \) be defined by

\[
\frac{1}{q} + \frac{1}{s} = \frac{1}{2} + \frac{1}{p}, \quad \frac{1}{s} + \frac{1}{p} = 1 + \frac{1}{p}.
\]

Since \( p \geq 2 \), by Minkowski’s inequality and Young’s inequality, we have

\[
\|R_j^0 G_N(h)\|_{L_2^p(\mathbb{L}_2^p(\mathbb{L}_2^2))} \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \bar{\phi}_j(x, \bar{v}) h(\cdot - \bar{x}, v - \bar{v}) d\bar{x} \right)^2 d\bar{v} \frac{1}{p/2} \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \bar{\phi}_j(\cdot, \bar{v}) h(\cdot, v - \bar{v}) d\bar{v} \right)^2 d\bar{v} \frac{1}{p/2} \]

\[
\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \bar{\phi}_j(\cdot, \bar{v}) \|h(\cdot, v - \bar{v})\|^2_{L_2^2(\mathbb{L}_2^2)} d\bar{v} \right) d\bar{v} \frac{1}{p/2} \]

\[
\leq \|\bar{\phi}_j\|_{L_2^p(\mathbb{L}_2^2)}^p \|h\|_{L_2^p(\mathbb{L}_2^2)}^p \leq 2^{p(\frac{1}{p} - \frac{1}{2})} \|h\|_{L_2^p}^p = 2^{-p\beta_j} \|h\|_{L_2^p},
\]

where the last inequality is due to the scaling property of \( \bar{\phi}_j \) and \( \bar{\phi}_j \in \mathcal{S} \). Thus, by Fubini’s theorem,

\[
\|G_N(h)\|_{B_p^0(\mathbb{L}_2^p(\mathbb{L}_2^2))} = \sup_{j \geq 0} 2^{\beta_j} \|R_j^0 G_N(h)\|_{L_2^p(\mathbb{L}_2^p(\mathbb{L}_2^2))}.
\]
Moreover, for $\beta' < \beta$, by the dominated convergence theorem we have
\[
\lim_{N \to \infty} \|G_N(h) - G(h)\|_{L_p^p(L^p_2)} = \lim_{N \to \infty} \sup_{j \geq 0} 2^{\beta' j} \|R_j^p(G_N(h) - G(h))\|_{L_p^p(L^p_2)} \\
\leq \sum_{j \geq 0} 2^{\beta' j} \|R_j^p(G_N(h) - G(h))\|_{L_p^p(L^p_2)}
\]
which converges to zero since for each $x, v$,
\[
\lim_{N \to \infty} \|R_j^p(G_N(h) - G(h))(x, v)\|_{L^p_2} = 0.
\]
The proof is complete. 

**Remark 7.3.** Fix $\delta > 0$ and $v_0 \in \mathbb{R}$. By (7.3) and BDG’s inequality, it is similar to (7.8) that for any $p \geq 2$ and $j \in \mathbb{N}_0$,
\[
\|R_j(x_\delta \hat{B}(t, \cdot))\|_{L^p_1(L^p_2)} \leq \left( \int_{\mathbb{R}} \left[ \sum_k |R_j(x_\delta \eta_k(v))W_t^k|^p \right] dv \right)^{1/p} \\
\leq t^{\frac{1}{p}} \left( \int_{\mathbb{R}} \|R_j(x_\delta \eta(v))|^p \right) dv^{1/p} \\
\leq t^{\frac{1}{2}} 2^\delta \|x_\delta \|_{L_p^p} = t^{\frac{1}{2}} 2^\delta \|x_\delta \|_{L_p^p},
\]
which implies that
\[
\hat{B}(t, \cdot) \in \cap_{p \geq 2} B^{-1/2}_p(L^p_2).
\]

### 7.1. SKEs with Lipschitz coefficients

In this subsection we make the following Lipschitz assumptions about $f$ and $g$.

\((H^{f,g}_s)\) For some $s \in (8, \infty)$, there is a $\xi \in \cap_{T > 0} \mathbb{L}^\infty_T(L^s_2)$ such that for all $(t, \omega, z) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^2$ and $u, u' \in \mathbb{R}$,
\[
|(f, g)(t, \omega, z, u) - (f, g)(t, \omega, z, u')| \leq \xi(t, \omega, z)|u - u'|.
\]
Since the time variable is not important in the estimates below, we shall drop the time variables in $f$ and $g$.

**Lemma 7.4.** Under \((H^{f,g}_s)\), for any $p \geq 2$, there is a constant $C_0 > 0$ only depending on $p, s$ and $\sup_{t, \omega} \|\xi(t, \omega, \cdot)\|_{L^p_1}$ such that for all $u, u' \in L^p_2(L^p_2)$,
\[
\|f(u) - f(u')\|_{B^{\beta}_{p, \theta}(L^p_2)} + \|G(g(u) - g(u'))\|_{B^{\beta}_{p, \theta}(L^p_2)} \lesssim C_0 \|u - u'\|_{L^p_1(L^p_2)},
\]
where $\beta = -\frac{1}{2} - \frac{4}{s}$, and for all $u, u' \in L^p_2(L^p_2)$,
\[
\|f(u) - f(u')\|_{B^{\beta}_{p, \theta}(L^p_2)} + \|G(g(u) - g(u'))\|_{B^{\beta}_{p, \theta}(L^p_2)} \lesssim C_0 \|u - u'\|_{L^p_1(L^p_2)}.
\]

**Proof.** We only prove the first one. Let $r \in [1, p]$ be defined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{2}$. Fix $\delta > 0$ and $z_0 \in \mathbb{R}^2$. Let $\chi_\delta(z_0)$ be as in (2.17). For $\beta = -\frac{1}{2} - \frac{4}{s}$, we have
\[
\|\chi_\delta(z_0)(f(u) - f(u'))\|_{B^{\beta}_{p, \theta}(L^p_2)} \lesssim \|\chi_\delta(z_0)(f(u) - f(u'))\|_{L^p_1(L^p_2)} \\
\lesssim \|\chi_\delta(z_0)(f(u) - f(u'))\|_{L^p_1(B^{-1/2}_{p, \theta})} \lesssim \|\chi_\delta(z_0)(f(u) - f(u'))\|_{L^p_1(L^p_2)} \\
\lesssim \|\chi_\delta(z_0)(u - u')\|_{L^p_1(L^p_2)} \lesssim \|\chi_\delta(z_0)(u - u')\|_{L^p_1(L^p_2)} \\
\lesssim \|\chi_\delta(z_0)(u - u')\|_{L^p_1(L^p_2)} \lesssim C_0 \|u - u'\|_{L^p_1(L^p_2)}.
\]
where the last step is due to Hölder’s inequality, and also,
\[
\|\chi_{\mathcal{F}_0}^\delta \mathcal{G}(g(u) - g(u'))\|_{B^p_{\alpha,\beta}(L^p_t(L^q_x))} \lesssim \|\chi_{\mathcal{F}_0}^\delta (g(u) - g(u'))\|_{L^\infty(L^p_x)}.
\]
Combining the above two estimates and by the definition of \(\| \cdot \|_{B^p_{\alpha,\beta}(L^p_t(L^q_x))} \), we obtain the first estimate.

Now we can state and prove our main result of this section.

**Theorem 7.5.** Let \( p \geq 2, s \in (8, \infty), \beta = -\frac{1}{2} - \frac{4}{s} \) and \( \alpha \in (-1, 1 + \beta) \). Under \((H_1, g)\), for any \( \mathcal{F}_0 \)-measurable \( u_0 \in B^0_{p,q}(L^p_x) \) and \( T > 0 \), there is a unique weak solution \( u \in L^\infty_t(L^p_x) \) to SKE (7.1) with regularity estimate:
\[
\|u(t)\|_{B^{\alpha+\beta}_{p,q}(L^p_x)} \lesssim C_1 \frac{t^{-\alpha-\beta}}{T} \|u_0\|_{B^0_{p,q}(L^p_x)} + \tilde{T}_p(f, g), \quad t \in (0, T],
\]
where \( C_1 = C_1(C_0, T, p, \alpha, s, \kappa) > 0 \) and
\[
\tilde{T}_p(f, g) := \|f(\cdot, 0)\|_{L^\infty_t(B^{\alpha}_{p,q}(L^p_x))} + \|G(g(\cdot, 0))\|_{L^\infty_t(B^{\beta}_{p,q}(L^p_x))}.
\]
Moreover, if \( u_0 \in B^0_{p,q}(L^p_x) \), then we also have
\[
\|u(t)\|_{B^{\alpha+\beta}_{p,q}(L^p_x)} \lesssim C_1 \frac{t^{-\alpha-\beta}}{T} \|u_0\|_{B^0_{p,q}(L^p_x)} + T_p(f, g), \quad t \in (0, T],
\]
where
\[
T_p(f, g) := \|f(\cdot, 0)\|_{L^\infty_t(B^{\alpha}_{p,q}(L^p_x))} + \|G(g(\cdot, 0))\|_{L^\infty_t(B^{\beta}_{p,q}(L^p_x))}.
\]
In this case, for \( N \in \mathbb{N} \), let \( u_N \) be the unique weak solution of the following SKE driven by finitely many Brownian motions:
\[
du_N = \left[\Delta v u_N + v \cdot \nabla u_N + f(u_N)\right]dt + G_N(g(u_N))dW^k_t, \quad u_N(0) = u_0.
\]
If \( p > \frac{2}{1+\beta} \), then for any \( \gamma \in (0, 1 + \beta - \frac{4}{s}) \),
\[
\lim_{N \to \infty} \left\| u - u_N \right\|_{L^p_t(C([0,T];B^0_{p,q})} = 0.
\]

**Proof.** We divide the proof into four steps.

(i) We use Picard’s iteration to show the existence of a weak solution. Let
\[
u(t) := P_t u_0, \quad t > 0.
\]
For \( n \geq 2 \), let \( u_n \) solve the following linear SKE:
\[
du_n = \left[\Delta_n u_n + v \cdot \nabla u_n + f(u_n-1)\right]dt + G_N(g(u_n-1))dW^k_t
\]
with \( u_n(0) = u_0 \). Since \( s \in (8, \infty) \) and \( \alpha \in (-1, 1 + \beta) \), one can choose \( \kappa \in (0, \frac{1}{2} - \frac{4}{s}) \) such that
\[
0 < \delta := \frac{1}{2} - \frac{4}{s} - \kappa < \alpha + 1.
\]
Note that by Lemma 7.4,
\[
\|f(u)\|_{B^0_{p,q}(L^p_x)} + \|G(g(u))\|_{B^0_{p,q}(L^p_x)} \lesssim \|u\|_{L^\infty_t(L^p_x)} + \tilde{T}_p(f, g),
\]
where \( \tilde{T}_p(f, g) \) is defined by (7.11). Thus by Theorem 3.6 with therein \( \beta = -\frac{3}{2} - \frac{4}{s} \) and \( q = 2 \), there is a constant \( C > 0 \) independent of \( n \) such that for all \( t \in (0, T] \),
\[
\|u_n(t)\|_{B^0_{p,q}(L^p_x)} \lesssim C \left( t^{-\delta} \|u_0\|_{B^0_{p,q}(L^p_x)}^2 + \int_0^t (t-s)^{\kappa-1} \|u_{n-1}(s)\|_{L^p_x}^2 ds + \tilde{T}_p(f, g) \right).
\]
If we let
\[ h_N(t) := \sup_{n=1, \ldots, N} \| u_n(t) \|_{B^\delta_{p,q}(L^\infty)}^2, \]
then
\[ h_N(t) \lesssim C t^{\alpha - \delta} u_0^2 \| B^\delta_{p,q}(L^\infty) \| + \int_0^t (t-s)^{\kappa-1} h_N(s) \, ds + \tilde{T}_p(f, g). \]

Since \( \alpha - \delta > -1 \), by Gronwall’s inequality of Volterra’s type (see [32]), we get
\[ \| u_n(t) \|_{B^\delta_{p,q}(L^\infty)}^2 \lesssim h_N(t) \lesssim C t^{\alpha - \delta} u_0^2 \| B^\delta_{p,q}(L^\infty) \| + \tilde{T}_p(f, g). \]

(ii) Let \( U_{n,m}(t) := u_n(t) - u_m(t) \). By Theorem 3.6 with therein \( \beta = -\frac{3}{2} - \frac{4}{q} \) and \( q = 2 \), and by Lemma 7.4, there is a constant \( C > 0 \) such that for all \( t \in (0, T] \) and \( n, m \geq 2 \),
\[ \| U_{n,m}(t) \|_{B^\delta_{p,q}(L^\infty)}^2 \lesssim C \int_0^t \int_0^s (t-s)^{\kappa-1} \| U_{n-1,m-1}(s) \|_{B^\delta_{p,q}(L^\infty)}^2 \, ds. \]

Let
\[ \psi(t) := \limsup_{n,m \to \infty} \sup_{s \in (0, t]} \left( s^{\delta - \alpha} \| U_{n,m}(s) \|_{B^\delta_{p,q}(L^\infty)}^2 \right). \]

By (7.16) and (7.17), it is easy to derive that
\[ \psi(T) = 0. \]

Thus, there are \( u(t) \in B^\delta_{p,q}(L^\infty) \) with
\[ \| u(t) \|_{B^\delta_{p,q}(L^\infty)} \lesssim C \int_0^t \| u_0 \|_{B^\delta_{p,q}(L^\infty)} + \tilde{T}_p(f, g) \]
so that
\[ \limsup_{n \to \infty} \sup_{s \in (0, t]} \left( s^{\delta - \alpha} \| u_n(s) - u(s) \|_{B^\delta_{p,q}(L^\infty)}^2 \right) = 0. \]

Since for any \( \varphi \in C_c^\infty(\mathbb{R}^2) \),
\[ \langle u(t), \varphi \rangle = \langle u_0, \varphi \rangle + \int_0^t \left( (u_n, \Delta u \varphi - v \cdot \nabla x \varphi) \right) \, ds \]
\[ + \int_0^t (f(u_{n-1}), \varphi) \, ds + \int_0^t (g(u_{n-1}), \varphi) \, dW^k_s, \]
by the dominated convergence theorem and taking limits \( n \to \infty \) for both sides, one sees that the above \( u \) is a weak solution of (7.1). The uniqueness of \( u \in L^\infty_T(\mathbb{R}^2 \mathbb{R}_t^\infty) \) follows from the same calculations as above.

(iii) Fix \( t \in (0, T] \) and \( \varepsilon \in (0, t) \). Note that by (7.18),
\[ \sup_{s \in [\varepsilon, t]} \| u(s) \|_{B^\delta_{p,q}(L^\infty)} \lesssim C \varepsilon^{\frac{q-2}{4}} \| u_0 \|_{B^\delta_{p,q}(L^\infty)} + \tilde{T}_p(f, g). \]

Starting from the time \( \varepsilon \), by Theorem 3.6 with therein \( \beta = -\frac{3}{2} - \frac{4}{q} \) and \( q = \infty \), we can repeat the proof of (7.16) with \( (\delta, \frac{1}{2} - \frac{4}{q}) \) in place of \( (\alpha, \delta) \) there, and obtain
\[ \| u(t) \|_{B^\delta_{p,q}(L^\infty)} \lesssim C (t-\varepsilon)^{\frac{3}{2} - \frac{1}{q} + \frac{3}{2} \varepsilon} \| u(\varepsilon) \|_{B^\delta_{p,q}(L^\infty)} + \sup_{s \in [\varepsilon, t]} \| u(s) \|_{B^\delta_{p,q}(L^\infty)} + \tilde{T}_p(f, g) \]
\[ \lesssim C (t-\varepsilon)^{\frac{3}{2} - \frac{1}{q} + \frac{3}{2} \varepsilon} \varepsilon^{\frac{q-2}{4}} \| u_0 \|_{B^\delta_{p,q}(L^\infty)} + \tilde{T}_p(f, g). \]

In particular, taking \( \varepsilon = \frac{t}{2} \), we obtain (7.10).

(iv) As for (7.12), it is completely the same. We show the limit (7.14). Let \( U_N := u - u_N \). By definition, we have
\[ dU_N = [\Delta u N + v \cdot \nabla x U_N + f(u) - f(u_N)] \, dt \]
which implies (7.14) by Gronwall’s inequality and (7.7).

By Corollary 3.8 and Lemma 7.4, we have

\[ E \left( \sup_{s \in [0, t]} \|U_N(s)\|_{B_{p, \alpha}^\beta(L^p)}^p \right) \lesssim \int_0^t \|f(u) - f(u_N)\|_{B_{p, \alpha}^{\beta, -\varepsilon}}^p \|G_N(g(u))\|_{B_{p, \alpha}^{\beta, -\varepsilon}}^p ds \]

which implies (7.14) by Gronwall’s inequality and (7.7).

**Corollary 7.6.** Let \( s \in (8, \infty) \), \( \beta = -\frac{1}{2} - \frac{1}{2}, \gamma \in (0, \frac{1}{1 + \beta} \alpha \in (-1, 1] \) and \( u_0 \in \cup_{\alpha > -1} B_{p, \alpha}^\alpha \), \( \tilde{\tau}_p(f, g) < \infty \), \( \forall p \geq 2 \), (7.19)

where \( \tilde{\tau}_p(f, g) \) is defined by (7.11). Under (H12), for any \( R, T > 0, p \geq 2 \) and \( \varepsilon \in (0, T) \), there is a constant \( C_2 > 0 \) depending on \( C_1, \alpha, \gamma, \varepsilon, R, \tilde{\tau}_p(f, g) \) and \( \|u_0\|_{B_{p, \alpha}^\alpha} \) such that for the solution \( u \) in Theorem 7.5 and for all \( \varepsilon \leq t_1 < t_2 \leq T \),

\[ \|u(t_2) - u(t_1)\|_{B_{p, \alpha}^\gamma(L^p)} \lesssim C_2 (t_2 - t_1)^{1 + \beta - \gamma}, \]

where \( \chi_0^R \) is the cutoff function in (2.17). In particular, for any \( \delta \in (0, \frac{1}{2} - \frac{1}{2} \alpha) \), there is a finite random variable \( C = C(\varepsilon, T, R, \omega) \) such that for all \( \varepsilon \leq t_1 < t_2 \leq T \) and \( x_1, x_2, v_1, v_2 \in B_R \),

\[ |u(t_1, x_1, v_1) - u(t_2, x_2, v_2)| \lesssim C (|t_1 - t_2|^\delta + |x_1 - x_2|^\delta + |v_1 - v_2|^\delta), \text{ a.s.} \]

**Proof.** For fixed \( R > 0 \), by definition it is easy to see that

\[ d(u\chi_0^R) = \left[ \Delta_v(u\chi_0^R) + v \cdot \nabla_x(u\chi_0^R) + F_R \right] dt + G^k(g(u)\chi_0^R)dW^k, \]

where \( F_R := F(u)\chi_0^R - 2\text{div}_v(u\nabla_x\chi_0^R) + (\Delta_v\chi_0^R - v \cdot \nabla_x\chi_0^R). \)

Thus, for \( \gamma \in (0, \frac{1}{1 + \beta} \alpha \)\), by (3.27) we have

\[ \|u(t_2)\chi_0^R - \Gamma_{t_2 \to t_1} (u(t_1)\chi_0^R)\|_{B_{p, \alpha}^\gamma(L^p)} \lesssim C (t_2 - t_1)^{1 + \beta - \gamma} \]

\[ \times \left( \|u(t_1)\|_{B_{p, \alpha}^{\gamma + \log(L^p)}} + \|F_R\|_{L^1_t L^2_x(B_{p, \alpha}^{\gamma - \log(L^p)})} + \|G^k(g(u)\chi_0^R)\|_{L^1_t L^2_x(B_{p, \alpha}^{\gamma - \log(L^p)})} \right). \]

By Bernstein’s inequality (2.6), (7.6) and (7.10), we clearly have

\[ \|F_R\|_{L^1_t L^2_x(B_{p, \alpha}^{\gamma - \log(L^p)})} + \|G^k(g(u)\chi_0^R)\|_{L^1_t L^2_x(B_{p, \alpha}^{\gamma - \log(L^p)})} \]

\[ \lesssim \|u\chi_0^R\|_{L^1_t L^2_x(B_{p, \alpha}^{\gamma - \log(L^p)})} + 1 \lesssim t_1^{\frac{1 + \beta - \gamma}{\gamma}}. \]
Thus,
\[ \|u(t_2)\chi^R_0 - \Gamma_{t_2-t_1}(u(t_1)\chi^R_0)\|_{\mathcal{B}_{p,q}(L^p)} \lesssim C (t_2 - t_1)^{\frac{1+\alpha}{2}}. \]
Moreover, by (3.13) we also have
\[ \|\Gamma_{t_2-t_1}(u(t_1)\chi^R_0) - u(t_1)\chi^R_0\|_{\mathcal{B}_{p,q}(L^p)} \lesssim (t_2 - t_1)^{\frac{1+\alpha}{2}}. \]
Combining the above two estimates, we obtain (7.20). Finally, the Hölder continuity of \( u(t,\omega,x,v) \) in \( t,x,v \) follows by (7.20), (7.10) and the Kolmogorov continuity theorem.

**Example 7.7.** Here we provide an example for (7.19). Suppose that
\[ u_0(x,v) = \rho(x,v)\mu(\mathrm{d}v), \]
where \( \mu \) is a \( \sigma \)-finite measure over \( \mathbb{R} \) with \( \sup_{\omega_0} \mu\{v : |v - v_0| \leq 1\} < \infty \) and \( \rho(x,v) \) is a bounded measurable function on \( \mathbb{R}^2 \). Then
\[ u_0 \in \cap_{p \geq 1} \hat{\mathcal{B}}^{1/p - 1}_{\rho,\theta} \subset \cap_{p \geq 2} \sqcup_{\alpha > -1} \hat{\mathcal{B}}^{\alpha}_{\rho,\theta}. \tag{7.21} \]
Indeed, for \( z_0 = (x_0,v_0) \in \mathbb{R}^2 \), by definition, we have
\[ \mathcal{R}^\theta_j(\chi^\delta_{z_0} u_0)(x,v) = \int_{\mathbb{R}^2} \tilde{\phi}^\theta_j(x - \bar{x}, v - \bar{v})(\chi^\delta_{z_0}(\rho))(\bar{x},\bar{v})d\bar{x}\mu(\mathrm{d}\bar{v}) \]
\[ = \int_{\mathbb{R}^2} \tilde{\phi}^\theta_j(\bar{x}, \bar{v} - v)(\chi^\delta_{z_0}(\rho))(x - \bar{x}, \bar{v})d\bar{x}\mu(\mathrm{d}\bar{v}). \]
Thus by Minkowski’s inequality,
\[ \|\mathcal{R}^\theta_j(\chi^\delta_{z_0} u_0)(\cdot,v)\|_{L^2} \leq \int_{\mathbb{R}^2} |\tilde{\phi}^\theta_j(\bar{x}, \bar{v} - v)|d\bar{x}\|\chi^\delta_{z_0}(\rho)\|_{L^2}(\bar{v})\|_{L^2}\mu(\mathrm{d}\bar{v}) \]
and
\[ \|\mathcal{R}^\theta_j(\chi^\delta_{z_0} u_0)\|_{L^p(L^2)} \leq \int_{\mathbb{R}^2} |\tilde{\phi}^\theta_j(\bar{x}, \cdot)|d\bar{x}\int_{\mathbb{R}^2} \|\chi^\delta_{z_0}(\rho)(\cdot,v)\|_{L^2}\mu(\mathrm{d}\bar{v}) \]
\[ \lesssim 2^{j(1 - \frac{1}{p})} \mu\{\bar{v} : |\bar{v} - v_0| \leq 2\delta\} \lesssim 2^{j(1 - \frac{1}{p})}, \]
which in turn gives (7.21).

### 7.2. SKEs with super-linear growth coefficients
In this subsection we consider the following super-linear growth SKE:
\[ du = [\Delta_v u + v \cdot \nabla_x u] dt + |u|^{\gamma + 1} dB(t,v), \quad u(0) = u_0, \]
or equivalently,
\[ du = [\Delta_v u + v \cdot \nabla_x u] dt + \mathbb{G}^k(|u|^{\gamma + 1})dW_t^k, \quad u(0) = u_0. \tag{7.22} \]

We want to show the following well-posedness result.

**Theorem 7.8.** Let \( \gamma \in (0,\frac{1}{\theta}) \) and \( p > \frac{12}{1 - 2\gamma} \). For any \( u_0 \in L^1_z \mathcal{B}_{\rho,\theta}^{\frac{1}{2} - 4\gamma} \), there exists a unique weak solution \( u \in \mathcal{C}_b([0,T] \times \mathbb{R}^2) \) a.s. to SKE (7.22).

Before giving a proof, we first consider the following linear SKE
\[ du = [\Delta_v u + v \cdot \nabla_x u + f] dt + \mathbb{G}^k(\xi u)dW_t^k, \quad u(0) = u_0, \tag{7.23} \]
where \( \xi : \mathbb{R}_+ \times \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is a bounded predictable function. The following a priori estimate is crucial for proving Theorem 7.8.
**Lemma 7.9.** Suppose that \( \xi \) is a bounded predictable process. Let \( p \geq 2 \), \( u_0 \in L^p \) and \( u \in L^p(\mathbb{R}_+; L^2_p) \) be a weak solution of SKE (7.23). If \( u^+ \in L^1 \), then for any bounded stopping time \( \tau \),

\[
\mathbb{E}[\|u^+(\tau)\|_{L^1}^2] \leq \|u_0^+\|_{L^1}^2 + \mathbb{E}\int_0^\tau \|f^+(s)\|_{L^1}^2 ds. \tag{7.24}
\]

**Proof.** By (7.14) and Fatou’s lemma, it suffices to show (7.24) for the unique solution of the following SKE driven by finitely many Brownian motions:

\[
du = \left[ \Delta_x u + v \cdot \nabla_x u + f \right] dt + \mathbb{G}^k_N(\xi u) dW^k_t, \quad u(0) = u_0,
\]

where \( N < \infty \). Let \( \rho_e(x, v) := \varepsilon^{-4} \rho(\varepsilon^{-1} x, \varepsilon^{-3} v) \), where \( \rho \) is smooth density function with support in the unit ball. Define \( u_\varepsilon := u \ast \rho_\varepsilon \) and \( f_\varepsilon := f \ast \rho_\varepsilon \). Then

\[
du_\varepsilon(t) = (\Delta_x u_\varepsilon + v \cdot \nabla_x u_\varepsilon + [\rho_\varepsilon \ast v \cdot \nabla_x] u + f_\varepsilon) dt + \mathbb{G}^k_N(\xi u) \ast \rho_\varepsilon dW^k_t,
\]

where

\[
[\rho_\varepsilon \ast v \cdot \nabla_x] u := \rho_\varepsilon \ast (v \cdot \nabla_x u) - v \cdot \nabla_x (u \ast \rho_\varepsilon).
\]

Let \( \psi : \mathbb{R} \to [0, \infty) \) be a smooth convex smooth function with bounded derivative of all orders greater than 1. By Itô’s formula, we have

\[
\psi(u_\varepsilon(t)) = \psi(u_\varepsilon(0)) + \int_0^t \dot{\psi}(u_\varepsilon) \, [\Delta_x u_\varepsilon + v \cdot \nabla_x u_\varepsilon + [\rho_\varepsilon \ast v \cdot \nabla_x] u + f_\varepsilon] \, ds + \frac{1}{2} \iint_0^t \dot{\psi}(u_\varepsilon) \, [\mathbb{G}^k_N(\xi u) \ast \rho_\varepsilon]_2^2 \, ds + \iint_0^t \dot{\psi}(u_\varepsilon) \, [\mathbb{G}^k_N(\xi u) \ast \rho_\varepsilon] \, dW^k_t.
\]

Let \( \chi \) be a nonnegative smooth function with compact support. Multiplying both sides by \( \chi \) and then integrating on \( \mathbb{R}^2 \) and noting that

\[
\int \dot{\psi}(u_\varepsilon) (\Delta_x u_\varepsilon + v \cdot \nabla_x u_\varepsilon) \chi = -\int \psi''(u_\varepsilon) (\nabla_x u_\varepsilon)^2 \chi + \int (\Delta_x \chi - v \cdot \nabla_x \chi) \psi(u_\varepsilon),
\]

we obtain that for any bounded stopping time \( \tau \),

\[
\mathbb{E} \int \psi(u_\varepsilon(\tau)) \chi \leq \mathbb{E} \int \psi(u_\varepsilon(0)) \chi + \mathbb{E} \int_0^\tau (\Delta_x \chi - v \cdot \nabla_x \chi) \psi(u_\varepsilon)
\]

\[
+ \frac{1}{2} \mathbb{E} \int_0^\tau \psi''(u_\varepsilon) \, [\mathbb{G}^k_N(\xi u) \ast \rho_\varepsilon]_2^2 \chi.
\]

Here and below we drop the integral variables \( dx \) for simplicity. Letting \( \varepsilon \to 0 \) and by Fatou’s lemma and the dominated convergence theorem, we get

\[
\mathbb{E} \int \psi(u(\tau)) \chi = \mathbb{E} \int \liminf_{\varepsilon \to 0} \psi(u_\varepsilon(\tau)) \chi \leq \liminf_{\varepsilon \to 0} \mathbb{E} \int \psi(u_\varepsilon(\tau)) \chi
\]

\[
\leq \mathbb{E} \int \psi(u(0)) \chi + \mathbb{E} \int_0^\tau (\Delta_x \chi - v \cdot \nabla_x \chi) \psi(u)
\]

\[
+ \frac{1}{2} \mathbb{E} \int_0^\tau \psi''(u) \, [\mathbb{G}^k_N(\xi u)]_2^2 \chi. \tag{7.26}
\]

Now we take

\[
\psi(r) = \psi_\delta(r) := (r + \sqrt{r^2 + \delta})/2, \quad \delta > 0.
\]

Clearly,

\[
\lim_{\delta \to 0} \psi_\delta(r) = r^+, \quad \psi_\delta'(r) \geq 0, \quad r^2 \psi_\delta''(r) \leq \sqrt{\delta}, \quad |\psi_\delta'(r)| \leq 1.
\]
Thus by \((7.26)\),
\[
\mathbb{E} \int \psi_{s}(u(\tau)) \chi \leq \int \psi_{s}(u(0)) \chi + \mathbb{E} \int_{0}^{T} \int (\Delta \psi - v \cdot \nabla \psi) \psi_{s}(u) + \mathbb{E} \int_{0}^{T} \int f^{+} \psi + \frac{\sqrt{\delta}}{2} \mathbb{E} \int_{0}^{T} \int \|G_{N}(\xi)\|_{L^{2}}^{2} \chi.
\]
Letting \(\delta \downarrow 0\), we get for any bounded stopping time \(\tau\),
\[
\mathbb{E} \int u^{+}(\tau) \chi \leq \int u^{+}(0) \chi + \mathbb{E} \int_{0}^{T} \int u^{+}(\Delta \chi - v \cdot \nabla \chi) + \mathbb{E} \int_{0}^{T} \int f^{+} \chi. \tag{7.27}
\]
Now, let \(\chi : \mathbb{R}^{2} \to [0, 1]\) be a nonnegative smooth function with
\[
\chi(x, v) = \begin{cases} 1, & |x|^{1/3} + |v| \leq 1, \\ 0, & |x|^{1/3} + |v| \geq 2. \end{cases}
\]
For \(R \geq 1\), let
\[
\chi_{R}(x, v) := \chi(R^{-3}x, R^{-1}v).
\]
By the chain rule, it is easy to see that
\[
(\Delta \chi_{R} - v \cdot \nabla \chi_{R})(x, v) = R^{-2}(\Delta \chi - v \cdot \nabla \chi)(R^{-3}x, R^{-1}v) \leq \|\Delta \chi - v \cdot \nabla \chi\|_{\infty} R^{-2} \chi_{2R}(x, v). \tag{7.28}
\]
Thus, for any \(n \in \mathbb{N}_{0}\), by \((7.27)\) with \(\chi_{2^n R}\) in place of \(\chi\), we get for any \(t \in [0, T]\),
\[
\mathbb{E} \int u^{+}(t) \chi_{2^n R} \leq \int u^{+}_{0} + C_{0}(2^n R)^{-2} \mathbb{E} \int_{0}^{t} \int u^{+} \chi_{2^{n+1} R} + \mathbb{E} \int_{0}^{T} \int f^{+},
\]
where \(C_{0} := \|\Delta \chi - v \cdot \nabla \chi\|_{\infty}\). Let \(H_{n}(t) := \mathbb{E} \int u^{+}(t) \chi_{2^n R}\). The above inequality means that
\[
H_{n}(t) \leq \int u^{+}_{0} + \mathbb{E} \int_{0}^{T} \int f^{+} + C_{0}(2^n R)^{-2} \int_{0}^{t} \int H_{n+1}(s) ds.
\]
By iteration, we get for any \(N \in \mathbb{N}\),
\[
H_{0}(t) \leq \left( \int u^{+}_{0} + \mathbb{E} \int_{0}^{T} \int f^{+} ds \right) \left( 1 + \sum_{n=1}^{N-1} \frac{(C_{0} R^{-2} t)^{n}}{2^{n(n-1)} n!} \right) + \frac{(C_{0} R^{-2})^{N}}{2^{N(N-1)} N!} \int_{0}^{T} \int f^{+} ds \cdots ds_{1} H_{N}(s_{N}) ds_{N} \cdots ds_{1}.
\]
Since by Hölder’s inequality and the assumption, for \(q = \frac{p}{p-1} \in (1, 2]\),
\[
H_{N}(t) \leq \mathbb{E} \|u(t)\|_{p} \|\chi_{2^n R}\|_{q} \leq C_{1}(2^{N} R)^{\frac{q}{p}},
\]
where \(C_{1}\) does not depend on \(N, R, t\), we have
\[
\frac{(C_{0} R^{-2})^{N}}{2^{N(N-1)} N!} \int_{0}^{T} \int f^{+} ds \cdots ds_{1} H_{N}(s_{N}) ds_{N} \cdots ds_{1} \leq \frac{C_{1}(2^{N} R)^{\frac{q}{p}}}{2^{N(N-1)} N!} (C_{0} R^{-2} t)^{N} \to 0
\]
as \(N \to \infty\). Hence,
\[
\mathbb{E} \int u^{+}(t) \chi_{R} = H_{0}(t) \leq \left( \int u^{+}_{0} + \mathbb{E} \int_{0}^{T} \int f^{+} ds \right) e^{C_{0} R^{-2} t}.
\]
Letting \(R \to \infty\), we obtain that for any \(t \in [0, T]\),
\[
\mathbb{E} \int u^{+}(t) \leq \int u^{+}_{0} + \mathbb{E} \int_{0}^{T} \int f^{+}.
\]
Finally, by this estimate and (7.27), (7.28) again,
\[
E \int u^+(\tau) \chi_R \leq \int u^+(0) \chi_R + \frac{\|\Delta_v \chi - v \cdot \nabla_x \chi\|_2}{R^2} \int_0^\tau \int u^+ + E \int_0^\tau \int f^+ \chi_R,
\]
which implies the desired result by taking limits \( R \to \infty \).

**Remark 7.10.** Suppose \( f^- \equiv 0 \). If \( u(0) \geq 0 \), then \( u(t) \geq 0 \) a.s. by (7.24). This in turn implies the nonnegativity of the solution to SKE (7.22) provided \( u_0 \geq 0 \).

**Remark 7.11.** Estimate (7.24) is also used to derive the local uniqueness of SKE (7.1) under \((H_{f,g}^\infty)\). More precisely, let \( u_1 \) and \( u_2 \) be two solutions of SKE (7.1) before stopping time \( \tau \leq T \) with the same initial values, i.e., for any \( \varphi \in C_c^\infty(\mathbb{R}^2) \),
\[
\langle u_i(t), \varphi \rangle = \langle u_0, \varphi \rangle + \int_0^t \left[ (u_i, \Delta_v \varphi - v \cdot \nabla_x \varphi) + (f(u_i), \varphi) \right] ds
\]
\[+ \int_0^t \langle G^k(g(u_i)), \varphi \rangle dW_s^k, \quad t \leq \tau, \quad a.s., \quad i = 1, 2.
\]
Then it holds that
\[
u_1(t) = u_2(t), \quad \forall t \leq \tau, \quad a.s.
\]
Indeed, let \( U := u_1 - u_2 \). Then \( U \) solves the following SKE: for \( t \leq \tau \),
\[
dU = [\Delta_v U + v \cdot \nabla_x U + \xi_0 U] dt + G^k(\xi_1 U) dW_t^k, \quad U(0) = 0,
\]
where \( \xi_0 := (f(u_1) - f(u_2))/(u_1 - u_2) \) and \( \xi_1 := (g(u_1) - g(u_2))/(u_1 - u_2) \). Here we use the convention \( \frac{0}{0} := 0 \). Under \((H_{f,g}^\infty)\), \( \xi_0 \) and \( \xi_1 \) are bounded predictable. Using (7.24), we have
\[
E\|U(t \wedge \tau)\|_{L^1_\xi} \leq E \int_0^{t \wedge \tau} \|\xi_0 U(s)\|_{L^1_{\xi\xi}} ds \leq \|\xi_0\|_\infty E \int_0^t \|U(s \wedge \tau)\|_{L^1_{\xi\xi}} ds.
\]
By Gronwall’s inequality, \( U(t \wedge \tau) = 0 \) for any \( t \in [0, T] \). We would like to point out that the local uniqueness seems not to be an obvious consequence of the global uniqueness since for \( u(t) := u(t \wedge \tau) \), we only have
\[
\partial_t u_t = 1_{t \leq \tau} [\Delta_v u_t + v \cdot \nabla_x u_t + f(u_t)] dt + G^k(g(u_t)) dW_{t \wedge \tau}^k,
\]
which has a random degenerate coefficient in front of the Laplacian operator.

**Remark 7.12.** By (7.24) and [14, Theorem III 6.8] or [34, Lemma 2.7], when \( f = 0 \), we have
\[
E \left( \sup_{t \in [0,T]} \|u(t)\|_{L^2_{\xi\xi}}^{1/2} \right) \leq 2\|u_0\|_{L^2_{\xi\xi}}^{1/2}.
\]
(7.29)

Now we are in a position to give

**Proof of Theorem 7.8.** For \( m \in \mathbb{N} \), we define
\[
g_m(u) := (|u| \wedge m)^\gamma + 1.
\]
Clearly, \( g_m \) is global Lipschitz. By Theorem 7.5, the following SKE has a unique weak solution \( u_m \in L^\infty_T(\mathcal{B}_{\rho & \gamma}^{-\gamma}(L^p)) \)
\[
du = [\Delta_v u + v \cdot \nabla_x u] dt + G^k(g_m(u)) dW_t^k, \quad u(0) = u_0.
\]
(7.30)
From the definition, one sees that \( u_m \) is also a weak solution of the following linearized equation:
\[
dw = [\Delta_v w + v \cdot \nabla_x w] dt + G^k(\xi_m w) dW_t^k, \quad w(0) = u_0,
\]

where

\[ \xi_m(t, x, v) := g_m(u_m(t, x, v))/u_m(t, x, v), \ \ \emptyset := 0. \]

Since \(|\xi_m| \leq m^\gamma\), by (7.29), we have

\[ E \left( \sup_{t \in [0, T]} \|u_m(t)\|_{L^2}^{1/2} \right) \leq 2\|u_0\|_{L^2}^{1/2}. \]

In the following we shall use this a priori estimate to derive that for any \( T > 0 \),

\[ \lim_{R \to \infty} \sup_m P \left( \omega : \sup_{t \in [0, T]} \|u_m(t, \omega, \cdot)\|_{C^\gamma(\mathbb{R}^2)} \geq R \right) = 0. \]

(7.31)

For \( S > 0 \), define a stopping time

\[ \tau_m^S := \inf \left\{ t \geq 0 : \|u_m(t, \cdot)\|_{L^2} \geq S \right\}. \]

Noting that

\[ \xi_m(t, \omega, z) \leq |u_m(t, \omega, z)|^\gamma, \]

we have for \( s = 1/\gamma \),

\[ \|\xi_m(t \wedge \tau_m^S, \omega, \cdot)\|_{L^2} \leq S^\gamma. \]

On the other hand, by Duhamel’s formula, we also have

\[ u_m(t) = P_t u_0 + \int_0^t P_{t-s} G^k(\xi_m u_m) dW^k_s. \]

Let \( \frac{1}{2} = \frac{1}{p} + \frac{1}{q} = \frac{1}{\gamma} + \gamma \) and \( \beta = 4(\frac{1}{p} - \frac{1}{\gamma}) - \frac{1}{2} = -\frac{1}{2} - 4\gamma \). For any \( \delta \in (0, \beta + \frac{2}{p}) \), by (3.7), (3.24), (7.6) and Hölder’s inequality, we have

\[ E \left( \sup_{s \in [0, t \wedge \tau_m^S]} \|u_m(s)\|_{B^\delta_{p, \theta}}^p \right) \leq \|u_0\|_{B^\beta_{p, \theta}}^{p+1-\frac{2}{p}} + \int_0^t \|G((\xi_m u_m)(\cdot \wedge \tau_m^S))(s)\|_{B^\delta_{p, \theta}(L^r_\theta)}^p ds \]

\[ \leq 1 + \int_0^t \|\xi_m u_m(s \wedge \tau_m^S)\|_{L^2}^p ds \]

\[ \leq 1 + S^{\gamma p} \int_0^t \|u_m(s \wedge \tau_m^S)\|_{L^2}^p ds, \]

which implies by Gronwall’s inequality that

\[ \sup_m E \left( \sup_{s \in [0, t \wedge \tau_m^S]} \|u_m(s)\|_{B^\delta_{p, \theta}}^p \right) \leq C_1 = C_1(S). \]

Since \( p > \frac{12}{1 + \beta - 2/\gamma} \), one can choose \( \delta \in (\frac{4}{p} - 4\gamma - \frac{2}{\gamma}) = (\frac{4}{p}, \frac{1}{2} - 4\gamma - \frac{2}{\gamma}) \) so that by the above estimate and (2.5) with \( q = \infty \),

\[ \sup_m E \left( \sup_{t \in [0, T \wedge \tau_m^S]} \|u_m(t)\|_{C^\gamma(\mathbb{R}^2)}^p \right) \leq C_2 = C_2(S). \]

By Chebyshev’s inequality, we have

\[ P \left( \omega : \sup_{t \in [0, T]} \|u_m(t, \omega, \cdot)\|_{C^\gamma(\mathbb{R}^2)} \geq R \right) \leq P(\tau_m^S \leq T) \]

\[ + P \left( \omega : \sup_{t \in [0, T \wedge \tau_m^S]} \|u_m(t, \omega, \cdot)\|_{C^\gamma(\mathbb{R}^2)} \geq R \right) \]
\[
\begin{align*}
\leq \frac{1}{S^{1/2}}E\left( \sup_{t \in [0,T]} \|u_m(t, \cdot)\|_{L^1}^{1/2} \right) + \frac{1}{R}E\left( \sup_{t \in [0,T \wedge \tau_m^n]} \|u_m(t)\|_{C_b(R^2)}^p \right) \\
\leq \frac{2\|u_0\|_{L^1}^{1/2}}{S^{1/2}} + \frac{C_2}{R},
\end{align*}
\]

which yields (7.31) by firstly letting \( R \to \infty \) and then \( S \to \infty \).

Finally, by (7.31), we can construct a unique solution of (7.22) as in [15]. Indeed, for given \( m, n \in \mathbb{N} \), we define

\[ \tau_m^n := \inf \left\{ t > 0 : \|u_m(t)\|_{C_b(R^2)} \geq n \right\}. \]

Since for \( m > n \), \( u_m \) and \( u_n \) satisfy the same equation (7.30) with coefficient \( g_n \) before \( \tau_m^n \). By the local uniqueness (see Remark 7.11), we have \( u_m|_{[0,\tau_m^n]} = u_n|_{[0,\tau_m^n]} \).

Hence,

\[ \tau_m \geq \tau_m^n \geq \tau_n, \quad \text{a.s.,} \]

and we can define without ambiguity

\[ u(t, x, v) := u_m(t, x, v), \quad t \leq \tau_m. \]

Clearly, \( u \) is a unique weak solution of (7.22) before \( \tau_m \). By (7.31), one sees that

\[ P(\tau_m \leq T) \leq \sup_n P\left( \sup_{t \in [0,T]} \|u_n(t)\|_{C_b(R^2)} \geq m \right) \]

\[ \leq \sup_n \left( \sup_{t \in [0,T]} \|u_n(t)\|_{C_b(R^2)} \geq m \right) \to 0 \]

as \( m \to \infty \). The proof is complete. \( \square \)

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