Fixed-Orientation Equilateral Triangle Matching
of Point Sets

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Abstract. Given a point set $P$ and a class $C$ of geometric objects, $G_C(P)$ is a geometric graph with vertex set $P$ such that any two vertices $p$ and $q$ are adjacent if and only if there is some $C \in C$ containing both $p$ and $q$ but no other points from $P$. We study $G_\bigtriangleup(P)$ graphs where $\bigtriangleup$ is the class of downward equilateral triangles (i.e., equilateral triangles with one of their sides parallel to the $x$-axis and the corner opposite to this side below that side). For point sets in general position, these graphs have been shown to be equivalent to half-$\Theta_6$ graphs and TD-Delaunay graphs.

The main result in our paper is that for point sets $P$ in general position, $G_\bigtriangleup(P)$ always contains a matching of size at least $\lceil \frac{n-2}{3} \rceil$ and this bound cannot be improved above $\lceil \frac{n-1}{3} \rceil$.

We also give some structural properties of $G_\bigtriangleup(P)$ graphs, where $\bigtriangleup$ is the class which contains both upward and downward equilateral triangles. We show that for point sets in general position, the block cut point graph of $G_\bigtriangleup(P)$ is simply a path. Through the equivalence of $G_\bigtriangleup(P)$ graphs with $\Theta_6$ graphs, we also deduce that any $\Theta_6$ graph can have at most $5n - 11$ edges, for point sets in general position.

Keywords: Geometric graphs, Delaunay graphs, Matchings.

1 Introduction

In this work, we study the structural properties of some special geometric graphs defined on a set $P$ of $n$ points on the plane. An equilateral triangle with one side parallel to the $x$-axis and the corner opposite to this side below (resp. above) that side as in $\bigtriangleup$ (resp. $\bigtriangledown$) will be called a down (resp. up)-triangle. A point set $P$ is said to be in general position, if the line passing through any two points from $P$ does not make angles $0^\circ$, $60^\circ$ or $120^\circ$ with the horizontal. In this paper, we consider only point sets that are in general position and our results assume this pre-condition.

Given a point set $P$, $G_\bigtriangleup(P)$ (resp. $G_\bigtriangledown(P)$) is defined as the graph whose vertex set is $P$ and that has an edge between any two vertices $p$ and $q$ if and only if there is a down-(resp. up-)triangle containing both points $p$ and $q$ but no
other points from \( P \). (See Fig. 1.) We also define another graph \( G_{\bigtriangleup}(P) \) as the graph whose vertex set is \( P \) and that has an edge between any two vertices \( p \) and \( q \) if and only if there is a down-triangle or an up-triangle containing both points \( p \) and \( q \) but no other points from \( P \). In Section 2 we will see that, for any point set \( P \) in general position, its \( G_{\bigtriangleup}(P) \) graph is the same as the well known Triangle Distance Delaunay (TD-Delaunay) graph of \( P \) and the half-\( \Theta_6 \) graph of \( P \) on so-called negative cones. Moreover, \( G_{\bigtriangleup}(P) \) is the same as the \( \Theta_6 \) graph of \( P \). [16].

Given a point set \( P \) and a class \( C \) of geometric objects, the maximum \( C \)-matching problem is to compute a subclass \( C' \) of \( C \) of maximum cardinality such that no point from \( P \) belongs to more than one element of \( C' \) and for each \( C \in C' \), there are exactly two points from \( P \) which lie inside \( C \). Dillencourt [9] proved that every point set admits a perfect circle-matching. Ábrego et al. [1] studied the isothetic square matching problem. Bereg et al. concentrated on matching points using axis-aligned squares and rectangles [3].

A matching in a graph \( G \) is a subset \( M \) of the edge set of \( G \) such that no two edges in \( M \) share a common end-point. A matching of maximum cardinality is called a maximum matching in \( G \). If all vertices of \( G \) appear as end-points of some edge in the matching, then it is called a perfect matching. It is not difficult to see that for a class \( C \) of geometric objects, computing the maximum \( C \)-matching of a point set \( P \) is equivalent to computing the maximum matching in the graph \( G_C(P) \) [1].

The maximum \( \triangle \)-matching problem, which is the same as the maximum matching problem on \( G_{\bigtriangleup}(P) \), was previously studied by Panahi et al. [13]. It was claimed that, for any point set \( P \) of \( n \) points in general position, any maximum matching of \( G_{\bigtriangleup}(P) \) (and \( G_{\bigtriangledown}(P) \)) will match at least \( \lceil \frac{2n}{3} \rceil \) vertices. But we found that their proof of Lemma 7, which is very crucial for their result, has gaps. By a completely different approach, we show that for any point set \( P \) in general position, \( G_{\bigtriangledown}(P) \) (and by symmetric arguments, \( G_{\bigtriangleup}(P) \)) will have a maximum matching of size at least \( \lceil \frac{n-2}{3} \rceil \); i.e., at least \( 2\lceil \frac{n-2}{3} \rceil \) vertices are matched. We also give examples where our bound is tight, in all cases except when \( |P| \) is one less than a multiple of three.

We also prove some structural and geometric properties of the graphs \( G_{\bigtriangledown}(P) \) (and by symmetric arguments, \( G_{\bigtriangleup}(P) \)) and \( G_{\bigstar}(P) \). It will follow that for point sets in general position, \( \Theta_6 \) graphs can have at most \( 5n - 11 \) edges and their block cut point graph is a simple path.

2 Preliminaries

Our notations are similar to those in [4], with minor modifications. A cone is the region in the plane between two rays that emanate from the same point, its apex. Consider the rays obtained by a counter-clockwise rotation of the positive \( x \)-axis by angles of \( \frac{i\pi}{3} \) with \( i = 1, \ldots, 6 \) around a point \( p \). Each pair of successive rays, \( \frac{(i-1)\pi}{3} \) and \( \frac{i\pi}{3} \), defines a cone, denoted by \( A_i(p) \), whose apex is \( p \). For \( i \in \{1, \ldots, 6\} \), when \( i \) is odd, we denote \( A_i(p) \) using \( C_{i+1}(p) \) and the cone