Influence of surface effects on stress state in a body with two circular holes

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1. Introduction

Problems, related to surface elasticity attract attention nowadays. The general methodology of obtaining analytical solutions for problems involving effects of surface elasticity consisting in using the specific boundary conditions corresponding to the equations of surface elasticity is rather clear. Moreover, such an approach is almost prescribed by the nature of equations of surface elasticity. Rather general method using complex variables in 2-D was developed in the work of Grekov and co-authors [1], [2]. Meanwhile, the amount of obtained analytical solution for particular problems is limited, may be due to rather awkward appearance of the boundary conditions. The majority of the solutions obtained are restricted to simple geometries like spheric pore [3], [4], [5], circular hole [6], simple plate [7]. However the used shapes of sphere or circular hole are the least interesting from the point of view of influence of surface effects, because for such forms the effect is minimal. Here we consider a problem of two equal holes in a plane under uniform comprehensive tension. The problem may be of interest for the case of rather close holes.

The parameters of the model are the following: $R$ is the holes radii; $2d$ is the distance between the holes centers; $\lambda, \mu$ are Lame’s elastic constants; $\lambda^s, \mu^s$ are the similar surface constants.

2. Bipolar coordinates; the general solution

To solve the problem of such a geometry it is convenient to use bipolar coordinates [8]. Actually the problem of a plate with two free of stress holes were solved by Ling [9] in 1948 using Jeffery [8] general solution for an area bounded by two non-coaxial circles. Here we extend Ling’s solution [9] (see also [10]) to account for the surface elasticity effects.
Let us following [8] introduce bipolar coordinates $\alpha, \beta$ related to Cartesian coordinates $x, y$ as follows

\[ x + iy = -a \coth \left( \frac{i \alpha + i \beta}{2} \right); \quad \alpha + i \beta = \ln \left( \frac{x + i(y + a)}{x + i(y - a)} \right), \quad (1) \]

the scale factor being

\[ g \equiv \frac{1}{\sqrt{\left( \frac{\partial x}{\partial \alpha} \right)^2 + \left( \frac{\partial y}{\partial \alpha} \right)^2}} = \frac{\cosh \alpha - \cos \beta}{a}. \quad (2) \]

The coordinate frame and the problem geometry is shown on Figure 1. Points $O_1, O_2$ have coordinates $(0, -a), (0, a)$ in Cartesian frame. For any point $P$ the radii from points $O_1, O_2$ to this point having lengths $r_1, r_2$, and angles between $x$ -axis and the radii being $\theta_1, \theta_2$, respectively, the bipolar coordinates are

\[ \alpha = \ln \frac{r_1}{r_2}; \quad \beta = \theta_1 - \theta_2. \quad (3) \]

Lines $\alpha = constant$ are a set of co-axial circles with limiting points $O_1, O_2$. The circles corresponding to the positive values of $\alpha$ lie above $x$ -axis, and those corresponding to negative values below, while $x$ -axis itself, being the common radical axis, corresponds to $\alpha = 0$.

Circles $\beta = constant$ are arcs of circles passing through points $O_1, O_2$ and crossing sets of circles $\alpha = constant$ orthogonally. On the right hand side of $y$-axis $\beta$ is positive, and on the left hand side negative, while $y$-axis itself corresponds to $\beta = 0$ except segment $O_1, O_2$ where $\beta = \pm \pi$.

![Figure 1: The coordinate frame and the problem geometry.](image)

At infinity $\alpha = 0, \beta = 0$; at points $O_1, O_2$ $\alpha = \pm \infty$, while $\beta$ being undetermined.
The constant value $\alpha = \pm \gamma$ correspond to two hole’s contours so that for the hole radius $R$, the distance between the holes centres, $d$, and the value of $\gamma$ the following relations take place [8]

$$R = a / \sinh \gamma; \quad d = a \coth \gamma; \quad d/R = \cosh \gamma.$$  \hspace{1cm} (4)

The components of stress tensor $\sigma_{ij}$ are expressed in terms of one biharmonic function $\Phi$

$$\Delta^2 \Phi = 0; \quad \left[ \frac{\partial^4}{\partial \alpha^4} + 2 \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + \frac{\partial^4}{\partial \beta^4} - 2 \frac{\partial^2}{\partial \alpha^2} + 2 \frac{\partial^2}{\partial \beta^2} + 1 \right] (g\Phi) = 0,$$  \hspace{1cm} (5)

as follows

$$a\sigma_{\alpha\alpha} = \left[ (\cosh \alpha - \cos \beta) \frac{\partial^2}{\partial \alpha^2} - \sinh \alpha \frac{\partial}{\partial \alpha} - \sin \beta \frac{\partial}{\partial \beta} + \cosh \alpha \right] (g\Phi); \hspace{1cm} (6)$$

$$a\sigma_{\alpha\beta} = \left[ (\cosh \alpha - \cos \beta) \frac{\partial^2}{\partial \alpha^2} - \sinh \alpha \frac{\partial}{\partial \alpha} - \sin \beta \frac{\partial}{\partial \beta} + \cos \beta \right] (g\Phi); \hspace{1cm} (7)$$

$$a\sigma_{\beta\beta} = -(\cosh \alpha - \cos \beta) \frac{\partial^2}{\partial \alpha \partial \beta} (g\Phi). \hspace{1cm} (8)$$

The general solution symmetrical with respect of both $\alpha$ and $\beta$ for the case two holes has the form [9]

$$\frac{g\Phi}{ap} = \frac{1}{2} (\cosh \alpha + \cos \beta) + K (\cosh \alpha - \cos \beta) \ln(\cosh \alpha - \cos \beta) + \sum_{n=1}^{\infty} f_n(\alpha) \cos n\beta; \hspace{1cm} (9)$$

$$f_n(\alpha) = A_n \cosh(n + 1)\alpha + B_n \cosh(n - 1)\alpha. \hspace{1cm} (10)$$

Here the first term corresponds to equi-component tension $p$ applied at infinity; the second term is chosen to ensure the necessary character of the additional stresses decay at infinity; the third term serves to satisfy the boundary conditions at the hole contours.

In addition to (9) the following condition should be satisfied [9]

$$\sum_{n=1}^{\infty} (A_n + B_n) = 0, \hspace{1cm} (11)$$

corresponding to decaying additional stresses at infinity.

In order to satisfy the boundary conditions on the contour (due to symmetry only one contour $\alpha = \gamma$ may be considered), the stresses at the contour should be represented as Fourier series, which in case of symmetry in $\alpha, \beta$ are

$$a\sigma_{\alpha\alpha} = c_0 + \sum_{n=1}^{\infty} c_n \cos n\beta; \hspace{1cm} (12)$$

$$a\sigma_{\alpha\beta} = \sum_{n=1}^{\infty} b_n \sin n\beta; \hspace{1cm} (13)$$

$$a\sigma_{\beta\beta} = d_0 + \sum_{n=1}^{\infty} d_n \cos n\beta. \hspace{1cm} (14)$$

Here coefficients $b_k, c_k, d_k$ are expressed in terms of $A_n, B_n, K$. 

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3. Boundary conditions; problem formulation

To solve the problem it is necessary to specify the boundary conditions at the hole contour. For the case in question they correspond to the generalized Young-Laplace equation [11]

$$\sigma \cdot n = -\nabla_s \sigma_s. \tag{15}$$

Here $\sigma$ is the bulk stress tensor; $n$ is the unit normal vector to the boundary; $\sigma_s$ is the surface stress tensor; $\nabla_s$ stays for the surface divergence [11].

For a curved surface with two orthogonal unit base vectors $e_1, e_2$ operator $\nabla_s$ is expressed as follows

$$\nabla_s \sigma_s = -\left[ \frac{\sigma_{11}^s}{R_1} + \frac{\sigma_{22}^s}{R_2} \right] + \frac{e_1}{h_1 h_2} \left[ \frac{\partial (h_2 \sigma_{11}^s)}{\partial \alpha_1} + \frac{\partial (h_1 \sigma_{21}^s)}{\partial \alpha_2} + \frac{\partial h_1}{\partial \alpha_1} \sigma_{12}^s - \frac{\partial h_2}{\partial \alpha_2} \sigma_{22}^s \right] + \frac{e_2}{h_1 h_2} \left[ \frac{\partial (h_1 \sigma_{22}^s)}{\partial \alpha_2} + \frac{\partial (h_2 \sigma_{12}^s)}{\partial \alpha_1} + \frac{\partial h_2}{\partial \alpha_1} \sigma_{21}^s - \frac{\partial h_1}{\partial \alpha_2} \sigma_{11}^s \right]. \tag{16}$$

Here $R_1, R_2$ are the radii of principle curvatures; $\alpha_1, \alpha_2$ are two parameters determining the surface, so that $\alpha_1 = \text{const}, \alpha_2 = \text{const}$ give two sets of mutualy orthogonal curves on the surface. For the case under consideration, when the surface is formed by circle (cylinder) $\alpha = \gamma$ in bipolar (bicylindrical) coordinate system (15), we have (e.g. [12])

$$\begin{align*}
\alpha_1 &= \alpha; \quad \alpha_2 = z; \\
R_1 &= R; \quad 1/R_2 = 0; \quad h_1 = 1/g; \quad h_2 = 1.
\end{align*} \tag{17}$$

$$\begin{align*}
\sigma_{\alpha\alpha} &= \frac{1}{R} \sigma_{\beta\beta}^s; \\
\sigma_{\alpha\beta} &= \frac{\cosh \alpha - \cos \beta}{\sinh \alpha} \frac{\partial \sigma_{\beta\beta}^s}{\partial \beta}.
\end{align*} \tag{18}$$

The second equation of (18) with the help of the first equation may be rewritten as

$$\sigma_{\alpha\beta} = \frac{\cosh \alpha - \cos \beta}{\sinh \alpha} \frac{\partial \sigma_{\alpha\alpha}}{\partial \beta}. \tag{19}$$

However the surface stress $\sigma_{\beta\beta}^s$ involved are still unknown. To obtain the boundary conditions appropriate to use let us express the surface stress in terms of kinematics variable according to Shuttleworth and Hooke’s law

$$\sigma_{\beta\beta}^s = C_{\beta\beta\beta\beta}^s \epsilon_{\beta\beta}; \quad \epsilon_{\beta\beta} = \frac{1}{E} \left[ \sigma_{\beta\beta} - \nu \sigma_{\alpha\alpha} \right]. \tag{20}$$

Here $C_{\beta\beta\beta\beta}^s$ is the surface elastic modulus; $\epsilon_{\beta\beta}$ is the circumferential strain; $E$ and $\nu$ are the bulk Young modulus and Poisson’s ratio, respectively (the values related to plane strain or plane stress cases should be chosen according to the problem in question).

Composing equations in (20), we obtain

$$\frac{\sigma_{\beta\beta}^s}{R} = \epsilon \left[ \sigma_{\beta\beta} - \nu \sigma_{\alpha\alpha} \right]; \quad \epsilon = \frac{C_{\beta\beta\beta\beta}^s}{ER}. \tag{21}$$
Accounting for (21) the boundary conditions (18) are written as follows

\[
\sigma_{\alpha\alpha} = \varepsilon \left[ \sigma_{\beta\beta} - \nu \sigma_{\alpha\alpha} \right]; \quad (22)
\]

\[
\sigma_{\alpha\beta} = \varepsilon \frac{\cosh \alpha - \cos \beta}{\sinh \alpha} \frac{\partial}{\partial \beta} \left[ \sigma_{\beta\beta} - \nu \sigma_{\alpha\alpha} \right]. \quad (23)
\]

Here instead of the second equation, equation (19) may be also used.

The solution of the stated problem may be achieved by equating factors at sines and cosines of \( \beta \), but that result in rather awkward manipulation with series. An alternative way consists in developing all functions involved in series on \( \varepsilon \)

\[
\sigma_{ij} = \sum_{m=0}^{N} \varepsilon^m \sigma_{ij}^{(m)}; \quad b = \sum_{m=1}^{N} \varepsilon^m b^{(m)}; \quad c = \sum_{m=0}^{N} \varepsilon^m c^{(m)}; \quad d = \sum_{m=0}^{N} \varepsilon^m d^{(m)}. \quad (24)
\]

Then, equating the terms of equal \( \varepsilon \) we obtain the recurrent systems of boundary conditions

\[
\sigma_{\alpha\alpha}^{(0)} = 0; \quad \sigma_{\alpha\beta}^{(0)} = 0;
\]

\[
\sigma_{\alpha\alpha}^{(1)} = \sigma_{\beta\beta}^{(0)}; \quad \sigma_{\alpha\beta}^{(1)} = \frac{\cosh \alpha - \cos \beta}{\sinh \alpha} \frac{\partial}{\partial \beta} \left[ \sigma_{\beta\beta}^{(0)} - \nu \sigma_{\alpha\alpha}^{(0)} \right], \quad m \geq 1. \quad (25)
\]

4. Zero approximation; classical elasticity

In case of zero right hand sides in (15), corresponding to the classical elasticity, the solution [9] (see also [10]) is the following

\[
A_{n}^{(0)} = 2K^{(0)} \frac{e^{-n\gamma} \sinh (n\gamma) + ne^{-\gamma} \sinh (2n\gamma)}{n(n+1)(n \sinh 2\gamma + \sinh (2n\gamma))}, \quad n \geq 1;
\]

\[
B_{n}^{(0)} = -2K^{(0)} \frac{(e^{-n\gamma} \sinh (n\gamma) + ne^{-\gamma} \sinh (2n\gamma))}{n(n-1)(n \sinh 2\gamma + \sinh (2n\gamma))}, \quad n \geq 2;
\]

\[
E_{1}^{(0)} = -1 + \frac{K^{(0)}}{2} \tanh \gamma \cosh 2\gamma;
\]

\[
K^{(0)} = \left( \frac{1}{2} + \tanh \gamma \sinh^2 \gamma - 4 \sum_{n=2}^{+\infty} \frac{e^{-n\gamma} \sinh (n\gamma) + n \sinh (n \sinh \gamma + \cosh \gamma)}{n(n^2 - 1)(n \sinh 2\gamma + \sinh (2n\gamma))} \right)^{-1}.
\]

Here the coefficients are marked with the upper index 0 to indicate that they correspond to the case of absence of surface elasticity. They also serves as the first term in series of \( \varepsilon \) corresponding to the solution of the stated problem of surface elasticity.

5. Recurrent solution

In section 3 by means of formulae (25) the problem in question was reduced to the set of successive problems for each term of the series on small parameter \( \varepsilon \), in which each term is to be obtained in a recurrent way. Although such a solution is appeared to be awkward, but a good point consists in the similarity of the structure of all terms, which allows using rather simple algorithms.
Indeed, knowing \( m \)-th solution (i.e knowing \( A_n^{(m-1)}, B_n^{(m-1)}, K^{(m-1)} \)), for stress components similar to (12) with the help of (6), (9) we obtain

\[
a\sigma^{(m-1)}_{\alpha\alpha} = c_0^{(m-1)} + \sum_{n=1}^{\infty} c_n^{(m-1)} \cos n\beta; \quad a\sigma^{(m-1)}_{\alpha\beta} = d_0^{(m-1)} + \sum_{n=1}^{\infty} d_n^{(m-1)} \cos n\beta, \quad (27)
\]

where \( c_n^{(m-1)}, d_n^{(m-1)} \) are

\[
c_0^{(m-1)} = B_1^{(m-1)} + A_1^{(m-1)} \cosh 2\gamma - \frac{1}{2} K^{(m-1)} \cosh 2\gamma;
\]

\[
c_n^{(m-1)} = \delta_{n,1} K^{(m-1)} \cosh \gamma - \frac{1}{2} \delta_{n,2} K^{(m-1)} - A_n^{(m-1)} \left[ (n^2 - 1) \cosh \gamma \cosh(n + 1)\gamma + (n + 1) \sinh \gamma \sinh(n + 1)\gamma \right] - B_n^{(m-1)} \left[ (n^2 - 1) \cosh \gamma \cosh(n - 1)\gamma + (n - 1) \sinh \gamma \sinh(n - 1)\gamma \right] + \frac{1}{2} (n + 2)(n + 1) \left[ A_{n+1}^{(m-1)} \cosh(n + 2)\gamma + B_{n+1}^{(m-1)} \cosh n\gamma \right] + \frac{1}{2} (n - 2)(n - 1) \left[ A_{n-1}^{(m-1)} \cosh n\gamma + B_{n-1}^{(m-1)} \cosh(n - 2)\gamma \right], \quad n \geq 1.
\]

\[
d_0^{(m-1)} = B_1^{(m-1)} - A_1^{(m-1)} \cosh 2\gamma + \frac{1}{2} K^{(m-1)} \cosh 2\gamma;
\]

\[
d_n^{(m-1)} = -\delta_{n,1} K^{(m-1)} \cosh \gamma + \frac{1}{2} \delta_{n,2} K^{(m-1)} + A_{n+1}^{(m-1)} \left[ (n + 2)(n + 1) \cosh(n + 2)\gamma \right] - \frac{1}{2} A_{n-1}^{(m-1)} \left[ (n - 1)(n + 2) \cosh n\gamma \right] - \frac{1}{2} B_{n+1}^{(m-1)} \left[(n - 2)(n + 1) \cosh n\gamma\right] - \frac{1}{2} B_{n-1}^{(m-1)} \left[(n - 1)(n - 2) \cosh(n - 2)\gamma\right], \quad n \geq 1.
\]

Here \( \delta_{n,m} \) is Kronecker’s delta.

Then the boundary conditions (the last two equations of (25)) may be written as

\[
a\sigma^{(m)}_{\alpha\alpha} = \sum_{n=0}^{\infty} s_n^{(m)} \cos n\beta; \quad a\sigma^{(m)}_{\alpha\beta} = \sum_{n=1}^{\infty} t_n^{(m)} \sin n\beta, \quad (30)
\]

where

\[
s_n^{(m)} = d_n^{(m-1)} - \nu c_n^{(m-1)};
\]

\[
t_n^{(m)} = \frac{1}{2 \sinh \gamma} \left[ (n - 1)d_{n-1}^{(m-1)} + (n + 1)d_{n+1}^{(m-1)} - 2nd_n^{(m-1)} \cosh \gamma \right] - \frac{\nu}{2 \sinh \gamma} \left[ (n - 1)c_{n-1}^{(m-1)} + (n + 1)c_{n+1}^{(m-1)} - 2nc_n^{(m-1)} \cosh \gamma \right]
\]

appended with condition similar to (11)

\[
\sum_{n=1}^{\infty} \left( A_n^{(m)} + B_n^{(m)} \right) = 0. \quad (32)
\]
These equations form the system to get $f^{(m)}_n, f^{(m)}_N, K^{(m)}$ in terms of $A^{(m-1)}_i, B^{(m-1)}_i, K^{(m-1)}$ as follows [9], [10].

$$a_\sigma^{(m)} = \sum_{n=0}^{\infty} s^{(m)}_n \cos n\beta = -\frac{K^{(m)}}{2} (\cosh 2\gamma - 2 \cosh \gamma \cos \beta + \cos 2\beta) +$$

$$+ f^{(m)}_1(\gamma) + \frac{1}{2} \sum_{n=1}^{\infty} [(n-1)(n-2) f^{(m)}_{n-1}(\gamma) + (n+1)(n+2) f^{(m)}_{n+1}(\gamma) -$$

$$- 2(n^2 - 1) \cosh \gamma f^{(m)}_n(\gamma) - 2 \sinh \gamma f^{(m)}_n(\gamma)] \cos n\beta;$$

$$a_\sigma^{(m)} = \sum_{n=1}^{\infty} t^{(m)}_n \sin n\beta = -K^{(m)} \sinh \gamma \sin \beta -$$

$$- \frac{1}{2} \sum_{n=1}^{\infty} [(n-1) f^{(m)}_{n-1}(\gamma) - 2n \cosh \gamma f^{(m)}_n(\gamma) + (n+1) f^{(m)}_{n+1}(\gamma)] \sin n\beta. \quad (33)$$

Equating coefficients at sines and cosines of similar arguments one founds (hereafter the upper indexes $(m)$ are dropped)

$$2s_0 = 2f_1(\gamma) - K \cosh 2\gamma;$$

$$2n s_n = \psi_{n-1} + \psi_{n+1} - 2\psi_n \cosh \gamma - 2\psi'_n \sinh \gamma + 2K (\delta_{1,n} \cosh \gamma - \delta_{2,n});$$

$$2t_n = \psi'_{n-1} + \psi'_{n+1} - 2\psi'_n \cosh \gamma + 2K \delta_{1,n} \sinh \gamma, \quad (34)$$

where

$$\psi_n = (n-1)n(n+1)f_n(\gamma); \quad \psi'_n = n f'_n(\gamma). \quad (35)$$

For $n > 1$ the values of $\psi_n$, $\psi'_n$ then are found directly

$$\psi'_n = p^{(1)}_n K + p^{(2)}_n; \quad \psi_n = p^{(3)}_n K + p^{(4)}_n, \quad n > 1; \quad (36)$$

$$p^{(1)}_n = 2 e^{-\gamma} \sinh n\gamma - 2 \sinh(n-1)\gamma;$$

$$p^{(2)}_n = 2 \sinh n\gamma \sinh^{-1} \gamma \sum_{k=1}^{\infty} t_k e^{-k\gamma} - 2 \sinh^{-1} \gamma \sum_{k=1}^{\infty} t_k \sinh(n-k)\gamma;$$

$$p^{(3)}_n = 2 e^{-\gamma} (n \cosh n\gamma - \coth \gamma \sinh n\gamma) + \sinh^{-1} \gamma [(n+1) \sinh(n-2)\gamma - (n-1) \sinh n\gamma];$$

$$p^{(4)}_n = 2 (n \cosh n\gamma - \coth \gamma \sinh n\gamma) \sinh^{-1} \gamma \sum_{k=1}^{\infty} t_k e^{-k\gamma} +$$

$$+ 2 \sinh^{-1} \gamma \sum_{k=1}^{n-1} [(n-k)t_k \cosh(n-k)\gamma - (ks_k + t_k \coth \gamma) \sinh(n-k)\gamma]. \quad (37)$$

Then using (35) and (10) we find

$$A_1 = \frac{\psi'_1}{2 \sinh 2\gamma}; \quad B_1 = f_1(\gamma) - \frac{\psi'_1}{2} \coth 2\gamma;$$

$$A_n = q^{(1)}_n \psi'_n + q^{(2)}_n \psi_n = \left[q^{(1)}_n p^{(1)}_n + q^{(2)}_n p^{(2)}_n \right] K + q^{(1)}_n p^{(2)}_n + q^{(2)}_n p^{(4)}_n; \quad (38)$$

$$B_n = q^{(3)}_n \psi'_n + q^{(4)}_n \psi_n = \left[q^{(3)}_n p^{(1)}_n + q^{(4)}_n p^{(2)}_n \right] K + q^{(3)}_n p^{(2)}_n + q^{(4)}_n p^{(4)}_n, \quad n > 1.$$
Here

\[ q_n^{(1)} = \frac{\cosh ((n-1)\gamma)}{n(n \sinh 2\gamma + \sinh (2n\gamma))}; \quad q_n^{(2)} = \frac{\sinh ((n-1)\gamma)}{n(n+1)(n \sinh 2\gamma + \sinh (2n\gamma))}; \]

\[ q_n^{(3)} = -\frac{\cosh ((n+1)\gamma)}{n(n \sinh 2\gamma + \sinh (2n\gamma))}; \quad q_n^{(4)} = \frac{\sinh ((n+1)\gamma)}{n(n-1)(n \sinh 2\gamma + \sinh (2n\gamma))}. \]  

Now using condition (32) we may find coefficient \( K \)

\[
K = \left( \frac{\tanh \gamma \sum_{k=1}^{\infty} (t_k e^{-k\gamma} - s_0) - \sum_{k=2}^{\infty} \left[ p_k^{(2)} (q_k^{(1)} + q_k^{(3)}) + p_k^{(4)} (q_k^{(2)} + q_k^{(4)}) \right]}{\frac{1}{2} (\cosh 2\gamma - 2e^{-\gamma \sinh \gamma \tanh \gamma}) + \sum_{k=2}^{\infty} \left[ p_k^{(1)} (q_k^{(1)} + q_k^{(3)}) + p_k^{(3)} (q_k^{(2)} + q_k^{(4)}) \right]} \right). \]

Therefore, formulae of the current section give solution for the problem in question. Note, that unless specifying \( s_n, t_n \) the formulae of the current section gives the solution for an arbitrary tractions applied at the contours. Thus the solution may be considered as a generalization of solution of [9].

6. Results

Below results for a plane with circular holes under uniaxial and allround tension are presented. It is assumed that the elastic properties of the base material determined by the parameters for an isotropic aluminum: \( E = 70.3 \) GPa, \( \nu = 0.345 \), and the corresponding surface properties — elastic constants taken from [1]:

\[ \lambda_s = 6.8511 \text{ H/m}, \quad \mu_s = -0.376 \text{ H/m}. \]

Respectively, the modulus of surface elasticity was

\[ C_{\beta\beta\beta \beta}^s = \lambda_s + 2\mu_s = 6.0991 \text{ H/m}. \]

The radius of the holes was considered \( R = 2 \). The surface stress was calculated for different \( \lambda = \frac{d}{R} \) (\( \lambda \) is a parametr of the location two holes).

![Figure 2: The location of the two holes.](image)
The following are graphs of surface stress at close and far distance between two holes in the case of all-round tension.

Then, there are presented the graphs of the stress component with and without surface stress at different points on the circular holes with uniaxial (longitudinal) tension.
Taking the results for one hole from [1], we compared two graphs of surface stress (the case of one hole and the case of two holes, but far away from each other).

![Figure 9: The comparison of surface stresses.](image)

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