On Estimating Edit Distance: Alignment, Dimension Reduction, and Embeddings

Moses Charikar∗
Stanford University
moses@cs.stanford.edu

Ofir Geri†
Stanford University
ofirgeri@cs.stanford.edu

Michael P. Kim‡
Stanford University
mpk@cs.stanford.edu

William Kuszmaul
Stanford University
kuszmaul@cs.stanford.edu

Abstract

Edit distance is a fundamental measure of distance between strings and has been widely studied in computer science. While the problem of estimating edit distance has been studied extensively, the equally important question of actually producing an alignment (i.e., the sequence of edits) has received far less attention. Somewhat surprisingly, we show that any algorithm to estimate edit distance can be used in a black-box fashion to produce an approximate alignment of strings, with modest loss in approximation factor and small loss in run time. Plugging in the result of Andoni, Krauthgamer, and Onak, we obtain an alignment that is a \((\log n)^{O(1/\epsilon^2)}\) approximation in time \(\tilde{O}(n^{1+\epsilon})\).

Closely related to the study of approximation algorithms is the study of metric embeddings for edit distance. We show that min-hash techniques can be useful in designing edit distance embeddings through three results: (1) An embedding from Ulam distance (edit distance over permutations) to Hamming space that matches the best known distortion of \(O(\log n)\) and also implicitly encodes a sequence of edits between the strings; (2) In the case where the edit distance between the input strings is known to have an upper bound \(K\), we show that embeddings of edit distance into Hamming space with distortion \(f(n)\) can be modified in a black-box fashion to give distortion \(O(f(\text{poly}(K)))\) for a class of periodic-free strings; (3) A randomized dimension-reduction map with contraction \(c\) and asymptotically optimal expected distortion \(O(c)\), improving on the previous \(\tilde{O}(c^{1+2/\log\log\log n})\) distortion result of Batu, Ergun, and Sahinalp.

1 Introduction

The edit distance \(\Delta_{ed}(x, y)\) between two strings \(x\) and \(y\) is the minimum number of character insertions, deletions, and substitutions needed to transform \(x\) into \(y\). This is a fundamental distance measure on strings, extensively studied in computer science [2, 5, 10, 12, 21, 25, 28]. Edit distance has applications in areas including computational biology, signal processing, handwriting recognition,

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and image compression [23]. One of its oldest and most important uses is as a tool for comparing differences between genetic sequences [1, 23, 24].

The textbook dynamic-programming algorithm for edit distance runs in time $O(n^2)$ [24, 27, 28], and can be leveraged to recover a sequence of edits, also known as an alignment. The quadratic runtime is prohibitively large for massive datasets (e.g., genomic data), and conditional lower bounds suggest that no strongly subquadratic time algorithm exists [5].

The difficulty of computing edit distance has motivated the development of fast heuristics [1, 11, 16, 23]. On the theoretical side, the tradeoff between run time and approximation factor (or distortion) is an important question (see [18, Section 6], and [20, Section 8.3.2]). Andoni and Onak [4] (building on beautiful work of Ostrovsky and Rabani [25]) gave an algorithm that estimates edit distance within a factor of $2^{\tilde{O}(\sqrt{\log n})}$ in time $n^{1+o(1)}$. The current best known tradeoff was obtained by Andoni, Krauthgamer and Onak [3], who gave an algorithm that estimates edit distance to within factor $(\log n)^{O(1/\varepsilon)}$ with run time $\tilde{O}(n^{1+\varepsilon})$.

Alignment Recovery While these algorithms produce estimates of edit distance, they do not produce an alignment between strings (i.e., a sequence of edits). By decoupling the problem of numerical estimation from the problem of alignment recovery, the authors of [4] and [3] are able to exploit techniques such as metric space embeddings [1] and random sampling in order to obtain better approximations. The algorithm of [3] runs in phases, with the $i$-th phase distinguishing between whether $\Delta_{\text{ed}}(x, y)$ is greater than or significantly smaller than $\frac{n}{2^i}$. At the beginning of each phase, a nuanced random process is used to select a small fraction of the positions in $x$, and then the entire phase is performed while examining only those positions. In total, the full algorithm samples an $\tilde{O}\left(\frac{n}{\Delta_{\text{ed}}(x,y)}\right)$-fraction of the letters in $x$. Note that this is a polynomially small fraction as we are interested in the case where $\Delta_{\text{ed}}(x, y) > n^{1/2}$ (if the edit distance is small, we can run the algorithm of Landau et al. [22] in linear time). Given that the algorithm only views a small portion of the positions in $x$, it is not clear how to recover a global alignment between the two strings.

We show, somewhat surprisingly, that any edit distance estimator can be turned into an approximate aligner in a black box fashion with modest loss in approximation factor and small loss in run time. For example, plugging the result of [3] into our framework, we get an algorithm with distortion $(\log n)^{O(1/\varepsilon^2)}$ and run time $\tilde{O}(n^{1+\varepsilon})$. To the best of our knowledge, the best previous result that gave an approximate alignment was the work of Batu, Ergun, and Sahinalp [8], which has distortion that is polynomial in $n$.

Embeddings of Edit Distance Using Min-Hash Techniques The study of approximation algorithms for edit distance closely relates to the study of embeddings [18, 25]. An embedding from a metric space $M_1$ to a metric space $M_2$ is a map of points in $M_1$ to $M_2$ such that distances are preserved up to some factor $D$, known as the distortion. Loosely speaking, low-distortion embeddings from a complex metric space $M_1$ to a simpler metric space $M_2$ allow algorithm designers to focus on

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1The algorithm of [4] has a recursive structure in which at each level of recursion, every substring $\alpha$ of some length $l$ is assigned a vector $v_\alpha$ such that the $\ell_1$ distance between vectors closely approximates edit distance between substrings. The vectors at each level of recursion are constructed from the vectors in lower levels through a series of procedures culminating in an application of Bourgain’s embedding to a sparse graph metric. As a result, although the distances between vectors in the top level of recursion allow for a numerical estimation of edit distance, it is not immediately clear how one might attempt to extract additional information from the vectors in order to recover an alignment.
the simpler metric space, rather than directly handling the more complex one. Embeddings from edit distance to Hamming space have been widely studied \cite{9,10,12,25} and have played pivotal roles in the development of approximation algorithms \cite{4} and streaming algorithms \cite{9,10}.

The second contribution of this paper is to introduce new algorithms for three problems related to embeddings for edit distance. The algorithms are unified by the use of min-hash techniques to select pivots in strings. We find this technique to be particularly useful for edit distance because hashing the content of strings allows us to split strings in places where their content is aligned, thereby getting around the problem of insertions and deletions misaligning the strings. In several of our results, this allows us to obtain algorithms which are either more intuitive or simpler than their predecessors. The three results are summarized below.

**Efficiently Embedding the Ulam Metric into Hamming Space:** For the special case of the Ulam metric (edit distance on permutations), we present a randomized embedding $\phi$ of permutations of size $n$ to poly($n$)-dimensional Hamming space with distortion $O(\log n)$. Given strings $x$ and $y$, the Hamming differences between $\phi(x)$ and $\phi(y)$ not only approximate the edit distance between $x$ and $y$, but also implicitly encode a sequence of edits from $x$ to $y$. If the output string of our embedding is stored using a sparse vector representation, then the embedding can be computed in linear time, and its output can be stored in linear space. The logarithmic distortion of our embedding matches that of Charikar and Krauthgamer’s embedding into $\ell_1$-space \cite{12}, which did not encode the actual edits and needed quadratic time and number of dimensions. Our embedding also supports efficient updates, and can be modified to reflect an edit in expected time $O(\log n)$ (as opposed to the deterministic linear time required by \cite{12}).

**Embedding Edit Distance in the Low-Distance Regime:** Recently, there has been considerable attention devoted to edit distance in the low-distance regime \cite{9,10}. In this regime, we are interested in finding algorithms that run faster or perform better given the promise that the edit distance between the input strings is small. This regime is of considerable interest from the practical point of view. Landau, Myers and Schmidt \cite{22} gave an exact algorithm for strings with edit distance $K$ that runs in time $O(n + K^2)$. Recently, Chakraborty, Goldenberg and Koucký \cite{10} gave a randomized embedding of edit distance into Hamming space that has distortion linear in the edit distance with probability at least $2/3$.

Given an embedding with distortion $\gamma(n)$ (a function of the input size), could one obtain an embedding whose distortion is a function of $K$, the edit distance, instead of $n$? We answer this question in the affirmative for the class of $(D,R)$-periodic free strings. We say that a string is $(D,R)$-periodic free if none of its substrings of length $D$ are periodic with period of at most $R$. For $D \in \text{poly}(K)$ and $R = O(K^3)$, we show that the embedding of Ostrovsky and Rabani \cite{25} can be used in a black-box fashion to obtain an embedding with distortion $2^O(\sqrt{\log K \log \log K})$ for $(D,R)$-periodic free strings with edit distance of at most $K$. Our result can be seen as building on the min-hash techniques of \cite{12,Section 3.5} (which in turn extends ideas from \cite{6}). The authors of \cite{12} give an embedding for $(t,180tK)$-non-repetitive strings with distortion $O(t \log(tK))$ \cite{12}. The key difference is that our notion of $(D,R)$-periodic free is much less restrictive than the notion of non-repetitive strings studied in \cite{12}.

**Optimal Dimension Reduction for Edit Distance:** The aforementioned work of Batu et al. \cite{8} introduced and studied an interesting notion of dimension reduction for edit distance: An embedding of edit distance on length-$n$ strings to edit distance on length-$n/c$ strings (with larger alphabet size) is called a *dimension-reduction map* with contraction $c$. By first performing dimension reduction, one can then apply inefficient algorithms to the contracted strings at a relatively small
overall cost. This idea was used in \[\mathbb{S}\] to design an approximation algorithm with approximation factor \(O(n^{1/3+\omega(1)})\). We provide a dimension-reduction map with contraction \(c\) and asymptotically optimal expected distortion \(O(c^e)\), improving on the distortion of \(\tilde{O}(c^{1/2}/\log \log \log n)\) obtained by the deterministic map of \(\mathbb{S}\).

2 Preliminaries

Throughout the paper, we will use \(\Sigma\) to denote an alphabet\(\mathbb{S}\) and \(\Sigma^n\) to denote the set of words of length \(n\) over that alphabet. Additionally, we use \(\mathcal{P}_n\) to denote the set of permutations of length \(n\) over \(\Sigma\), or equivalently, the subset of \(\Sigma^n\) containing words whose letters are distinct. When referring to strings and permutations of length at most \(n\), we use \(\Sigma^{\leq n}\) and \(\mathcal{P}_{\leq n}\) respectively. Given a string \(w\) of length \(n\), we denote its letters by \(w_1, w_2, \ldots, w_n\), and we use \(w[i:j]\) to denote the substring \(w_i w_{i+1} \cdots w_j\) (which is empty if \(j < i\)).

An edit operation is either an insertion of a letter into a word, a deletion of a letter from a word, or a substitution of a letter with another letter. Given words \(x\) and \(y\), an alignment from \(x\) to \(y\) is a sequence of edits transforming \(x\) to \(y\). The edit distance \(\Delta_{ed}(x,y)\) is the minimum number of edits needed to transform \(x\) to \(y\). Alternatively, it is the length of an optimal alignment.

The following definition and lemma concerning periodic strings will often be useful.

Definition 2.1. A string \(w\) is periodic with period \(p\) (or \(p\)-periodic) if \(w_i = w_{i+p}\) for all \(i \in \{1, \ldots, |w| - p\}\).

Lemma 2.2. Suppose \(w \in \Sigma^n\) is \(p\)-periodic and \(q\)-periodic, with \(n \geq p + q\). Then \(w\) is \(\gcd(p,q)\)-periodic.

Proof. See Appendix\(\mathbb{C}\).

A family \(\mathcal{H}\) of hash functions from \(S\) to \(T\) is said to be \(n\)-wise independent for every distinct \(x_1, \ldots, x_n \in S\), and every \(y_1, \ldots, y_n \in T\) (not necessarily distinct), \(\Pr[x_1 = y_1, \ldots, x_n = y_n] = \frac{1}{|T|^n}\).

Several of our algorithms assume access to a family of hash function \(\mathcal{H}\) from \(\Theta(\log n)\) bits to \(\Theta(\log n)\) bits such that \(h \in \mathcal{H}\) is \(n\)-wise independent and can be evaluated in constant time. One can simulate this using the randomly generated hash family \(\mathcal{G}\) by Pagh and Pagh \(\cite{20}\) which is with high probability independent on any given set of \(n\) elements \(S\); although it is a slight abuse of notation, we often refer to \(\mathcal{G}\) as being \(n\)-wise independent with high probability. The family \(\mathcal{G}\) requires \(O(n)\) preprocessing time and \(O(n \log n)\) random bits, but allows for constant-time evaluation\(\mathbb{F}\).

Finally, when we have a pairwise independent hash function \(h\), it will often be useful to treat it as though whenever \(x \neq y\), \(\Pr[h(x) = h(y)] = 0\). This is called the no-collisions assumption, and the following lemma shows that for a polynomial-time algorithm we can assume it with high probability.

Lemma 2.3. Let \(S\) be a set of objects with \(|S| \in \text{poly}(n)\) and pick any \(p \in \text{poly}(n)\). Then there exists a sufficiently large \(b \in \Theta(\log n)\) such that the following holds. If we select \(h\) at random from a pairwise independent family of hash functions mapping \(S\) to \(b\) bits, then \(h\) has probability at least

\(^2\)When comparing these distortions, one should note that \(2/\log \log \log n\) goes to zero very slowly; in particular, \(c^{1+2/\log \log \log n} \geq c^{1.66}\) for all \(n \leq 10^{82}\), the number of atoms in the universe.

\(^3\)We assume that characters in \(\Sigma\) can be represented in \(\Theta(\log n)\) bits, where \(n\) is the size of input strings.

\(^4\)Note that we use the model in which \(O(\log n)\) random bits can be generated in constant time.
1−\frac{1}{p(n)} of being injective on S. That is, the no-collisions assumption holds with probability at least 1−\frac{1}{p(n)}.

Proof. See Appendix C.

3 Alignment Recovery Using a Black-Box Approximation Algorithm

In this section, we show how to transform a black-box algorithm A for approximating edit distance into an algorithm B (with different approximation ratio and run time) for finding an approximately optimal alignment between strings. The algorithm B appears here as Algorithm 1. In the description of the algorithm, we rely on the following definition of a partition.

Definition 3.1. A partition of a string u into m parts is a tuple P = (p_0, p_1, p_2, \ldots, p_m) such that p_0 = 0, p_m = \lvert u \rvert, and p_0 \leq p_1 \leq \cdots \leq p_m. For i \in \{1, \ldots, m\}, the i-th part of the partition P is the subword P_i := u[p_{i-1} + 1 : p_i], which is considered to be empty if p_i = p_{i-1}. A partition of a string u into m parts is an equipartition if each of the parts is of size either \lceil \lvert u \rvert \rceil/m or \lfloor \lvert u \rvert \rceil/m.

Algorithm 1 Black-Box Approximate Alignment Algorithm

Input: Strings u, v with \lvert u \rvert + \lvert v \rvert \leq n.
Parameters: m \in \mathbb{N} satisfying m \geq 2 and an approximation algorithm A for edit distance.

1. If \lvert u \rvert \leq 1, then find an optimal alignment in time O(\lvert v \rvert) naively.

2. Let P = (p_0, p_1, \ldots, p_m) be an equipartition of u.

3. Let S consist of the positions in v which can be reached by adding or subtracting a power of (1 + \frac{1}{m}) to some p_i. Formally, define

   S = \left(\{p_0, \ldots, p_m\} \cup \{\lvert v \rvert\} \cup \left\{p_i \pm (1 + \frac{1}{m})^j \left| i, j \geq 0 \right\} \right) \cap \{0, \ldots, \lvert v \rvert\}.

4. Using dynamic programming, find a partition Q = (q_0, \ldots, q_m) of v such that each q_i is in S, and such that the cost \sum_{i=1}^m A(P_i, Q_i) is minimized:

   (a) For l \in S, let f(l, j) be the subproblem of returning a choice of q_0, q_1, \ldots, q_j with q_j = l which minimizes \sum_{i=1}^j A(P_i, Q_i).

   (b) Solve each f(l, j) by examining precomputed answers for each of the subproblems of the form f(l', j - 1) with l' \leq l \in S: if f(l', j - 1) gives a choice of q_0, q_1, \ldots, q_{j-1} with \sum_{i=1}^{j-1} A(P_i, Q_i) = t, then we can set q_j = l to get \sum_{i=1}^j A(P_i, Q_i) = t + A(P_j, v[l'+1 : l]). (Here, A(P_j, v[l'+1 : l]) is computed using A.)

5. Recurse on each pair (P_i, Q_i). Combine the resulting alignments between each P_i and Q_i to obtain an alignment between u and v.
Formally, we assume that the approximation algorithm $\mathcal{A}$ has the following properties:

1. There is some non-decreasing function $\gamma$ such that for all $n > 0$, and for any two strings $u, v$ with $|u| + |v| \leq n$,
   \[ \Delta_{ed}(u, v) \leq \mathcal{A}(u, v) \leq \gamma(n) \cdot \Delta_{ed}(u, v) \]

2. $\mathcal{A}(u, v)$ runs in time at most $T(n)$ for some non-decreasing function $T$ which is super-additive in the sense that $T(j) + T(k) \leq T(j + k)$ for $j, k \geq 0$.

We are now ready to state the main theorem of this section.

**Theorem 3.2.** For all $u, v$ with $|u| + |v| \leq n$ and $m \geq 2$, Algorithm 1 outputs an alignment from $u$ to $v$ such that the number of edits is at most $(3\gamma(n))^{O(\log_m n)} \cdot \Delta_{ed}(u, v)$. Moreover, the algorithm runs in time $\tilde{O}(m^5 \cdot T(n))$.

Before continuing, we provide a brief discussion of Algorithm 1. The algorithm first breaks $u$ into a partition $P$ of $m$ equal parts. It then uses the black-box algorithm $\mathcal{A}$ to search for a partition $Q$ of $v$ such that $\sum_i \Delta_{ed}(P_i, Q_i)$ is near minimal; after finding such a $Q$, the algorithm recurses to find approximate alignments between $P_i$ and $Q_i$ for each $i$. Rather than considering every option for the partition $Q = (q_0, \ldots, q_m)$, the algorithm limits itself to those for which each $q_i$ comes from a relatively small set $S$.

The set $S$ is carefully designed so that although it is small (which enables good run time), any optimal partition $Q^{opt}$ of $v$ can be in some sense well approximated by some partition $Q$ using only $q_i$ values from $S$. The result is that the multiplicative error introduced at each level of recursion will be bounded by $3\gamma(n)$; across the $O(\log_m n)$ level of recursion, the total multiplicative error will then be $3\gamma(n)^{O(\log_m n)}$. The fact that the recursion depth appears in the exponent of the multiplicative error is why we partition $u$ and $v$ into many parts at each level.

Next we discuss several implications of Theorem 3.2. The parameter $m$ allows us to trade off the approximation factor and the run time of the algorithm. When taken to the extreme, this gives two particularly interesting results.

**Corollary 3.3.** Let $0 < \varepsilon < 1$ (not necessarily constant). Then $m$ can be chosen so that Algorithm 1 has approximation ratio $(3\gamma(n))^{O(\frac{1}{\varepsilon})}$ and run time $\tilde{O}(T(n) \cdot n^\varepsilon)$.

**Proof.** Set $m = n^{\varepsilon/5}$. Then the approximation ratio of Algorithm 1 is $(3\gamma(n))^{O(\log_m n)} = (3\gamma(n))^{O(\frac{1}{\varepsilon})}$ and the run time on $u, v$ with $|u| + |v| \leq n$ is $\tilde{O}(m^5 \cdot T(n)) = \tilde{O}(T(n) \cdot n^\varepsilon)$. \qed

**Corollary 3.4.** Let $0 < \varepsilon < 1$ (not necessarily constant). Then $m$ can be chosen so that Algorithm 1 has approximation ratio $n^{O(\varepsilon)}$ and run time $\tilde{O}(T(n)) \cdot (3\gamma(n))^{O(1/\varepsilon)}$.

**Proof.** Set $m = (3\gamma(n))^{\frac{1}{\varepsilon}}$. The approximation ratio is $(3\gamma(n))^{O(\log_m n)} = (3\gamma(n))^{O(\varepsilon \log_3(n))} = n^{O(\varepsilon)}$, and the run time on $u, v$ with $|u| + |v| \leq n$ is $\tilde{O}(m^5 \cdot T(n)) = \tilde{O}(T(n)) \cdot (3\gamma(n))^{O(1/\varepsilon)}$. \qed

Note that one can also apply our results to approximation algorithms $\mathcal{A}$ which are randomized in the sense that they succeed with probability $2/3$, meaning that $\Pr[\Delta_{ed}(x, y) \leq A(x, y) \leq \gamma(n)\Delta_{ed}(x, y)] \geq \frac{2}{3}$. In particular, given such an algorithm $\mathcal{A}$, we can amplify the probability of success to $1 - \frac{1}{\text{poly}(n)}$ by performing $\Theta(\log n)$ independent computations of $\mathcal{A}(x, y)$ and keeping the median return value. Moreover, by the union bound, the amplified $\mathcal{A}$ will succeed with probability $1 - \frac{1}{\text{poly}(n)}$ for every pair $(x, y)$ we run it on, as long as we invoke it only polynomially many times.
times. Hence we can apply our results to such algorithms $A$ without modification, at the cost of an additional logarithmic factor in run time and failure probability $\frac{1}{\text{poly}(n)}$.

With this in mind, we can apply Corollary 3.3 to the randomized algorithm of Andoni et al. with approximation ratio $(\log n)^{O(1/\varepsilon^2)}$ and run time $O(n^{1+\varepsilon})$, in order to obtain the following concrete result. (Note that $\varepsilon$ may be $o(1)$.)

**Corollary 3.5.** There exists an approximate-alignment algorithm which runs in time $\tilde{O}(n^{1+\varepsilon})$, and has approximation factor $(\log n)^{O(1/\varepsilon^2)}$ with probability $1 - \frac{1}{\text{poly}(n)}$.

The remainder of the section is devoted to proving Theorem 3.2.

### 3.1 Proof of Theorem 3.2

The proof of the theorem will follow from Proposition 3.6, which bounds the run time of Algorithm 1, and Proposition 3.11, which bounds the approximation ratio.

Throughout this section, let $u, v$ and $m$ be the values given to Algorithm 1. Let $P = (p_0, \ldots, p_m)$ be the equipartition of $u$ into $m$ parts, and let $S$ be the set defined by Algorithm 1. We begin by bounding the run time of Algorithm 1.

**Proposition 3.6.** Algorithm 1 runs in time $\tilde{O}(T(|u| + |v|) \cdot m^5)$.

**Proof.** If $|u| \leq 1$, then we can find an optimal alignment in time $O(|v|)$ naively.

Suppose $|u| > 1$. Notice that $|S| \leq O(m^2 \log n)$. In particular, because $(1 + \frac{1}{m})(m+1)\ln n \geq n$,

$$S \subseteq \{p_0, \ldots, p_m\} \cup \{|v|\} \cup \left\{p_i \pm \left(1 + \frac{1}{m}\right)^j \mid i \in [0 : m], j \in [0 : (m+1)\ln n]\right\},$$

which has size at most $O(m^2 \log n)$.

Finding an equipartition of $u$ can be done in linear time, and constructing $S$ takes time $O(|S|) = \tilde{O}(m^2)$. In order to perform the fourth step which selects $Q$, we must compute $f(l, j)$ for each $l \in S$ and $j \in [0 : m]$. This results in $O(m|S|) \leq O(m^3 \log n)$ subproblems. To evaluate $f(l, j)$, we must consider each $l' \in S$ satisfying $l' \leq l$, and then compute the cost of $f(l', j - 1)$ plus $A(P_j, v[l' + 1 : l])$ (which takes time at most $T(|u| + |v|)$ to compute). Therefore, each $f(l, j)$ is computed in time $O(|S| \cdot T(|u| + |v|)) \leq \tilde{O}(T(|u| + |v|) \cdot m^2)$. Because there are $O(m \cdot |S|) = O(m^3)$ subproblems of the form $f(l, j)$, the total run time of the dynamic program is $\tilde{O}(T(|u| + |v|) \cdot m^5)$.

So far we have shown that the first level of recursion takes time $\tilde{O}(T(|u| + |v|)m^5)$. We will now extend this to consider the $i$-th level of recursion for any $i$. The sum of the lengths of the inputs to Algorithm 1 at a particular level of the recursion is at most $|u| + |v|$. Therefore, by the super-additivity of $T(n)$, it follows that the time spent in any given level of recursion is at most $\tilde{O}(T(|u| + |v|)m^5)$. Because each level of recursion reduces the sizes of the parts of $u$ by a factor of $\Omega(m)$, the number of levels is at most $O(\log_m n) \leq O(\log n)$. Therefore, the run time is $\tilde{O}(T(|u| + |v|) \cdot m^5)$.

When discussing the approximation ratio of Algorithm 1, it will be useful to have a notion of edit distance between partitions of strings.

**Definition 3.7.** Given two partitions $C = (c_0, \ldots, c_m)$ and $D = (d_0, \ldots, d_m)$ of strings $a$ and $b$ respectively, we define $\Delta_{ed}(C, D) := \sum_i \Delta_{ed}(C_i, D_i)$.  

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In order to bound the approximation ratio of Algorithm 1, we will introduce, for the sake of analysis, a partition \(Q^{\text{opt}} = (q_0^{\text{opt}}, \ldots, q_m^{\text{opt}})\) of \(u\) satisfying \(\Delta_{\text{ed}}(P, Q^{\text{opt}}) = \Delta_{\text{ed}}(u, v)\). Recall that \(P\) is fixed, which allows us to use it in the definition of \(Q^{\text{opt}}\).

We claim that some partition \(Q^{\text{opt}}\) satisfying \(\Delta_{\text{ed}}(P, Q^{\text{opt}}) = \Delta_{\text{ed}}(u, v)\) must exist. If \(u\) and \(v\) differed by only a single edit, one could start from \(P\) and explicitly define \(Q^{\text{opt}}\) so that \(\Delta_{\text{ed}}(P, Q^{\text{opt}}) = 1\) (by a case analysis of which type of edit was performed). It can then be shown by induction on the number of edits that, in general, we can obtain a partition \(Q^{\text{opt}}\) satisfying \(\Delta_{\text{ed}}(P, Q^{\text{opt}}) = \Delta_{\text{ed}}(u, v)\).

Our strategy for bounding the approximation ratio of Algorithm 1 will be to compare \(\Delta_{\text{ed}}(P, Q)\) for the partition \(Q\) selected by our algorithm to \(\Delta_{\text{ed}}(P, Q^{\text{opt}})\). We do this through three observations.

The first observation upper bounds \(\Delta_{\text{ed}}(P, Q)\). Informally, it shows that the cost in edit distance which Algorithm 1 pays for selecting \(Q\) instead of \(Q^{\text{opt}}\) is at most \(2 \sum_{i=0}^{m} |q_i - q_i^{\text{opt}}|\).

**Lemma 3.8.** Let \(Q = (q_0, \ldots, q_m)\) be a partition of \(v\). Then

\[\Delta_{\text{ed}}(P, Q) \leq \Delta_{\text{ed}}(u, v) + 2 \sum_{i=1}^{m} |q_i - q_i^{\text{opt}}|\]

**Proof.** Observe that

\[\Delta_{\text{ed}}(P, Q) \leq \Delta_{\text{ed}}(P, Q^{\text{opt}}) + \Delta_{\text{ed}}(Q^{\text{opt}}, Q) = \Delta_{\text{ed}}(u, v) + \sum_{i=1}^{m} \Delta_{\text{ed}}(Q_i^{\text{opt}}, Q_i).

Because \(Q\) and \(Q^{\text{opt}}\) are both partitions of \(v\), \(\Delta_{\text{ed}}(Q_i, Q_i^{\text{opt}}) \leq |q_i - q_i^{\text{opt}}| + |q_i^{\text{opt}} - q_i^{\text{opt}}|\). In particular, \(|q_i - q_i^{\text{opt}}|\) insertions to the left side of one of \(Q_i\) or \(Q_i^{\text{opt}}\) (whichever has its start point further to the right) will result in the two substrings having the same start-point; and then \(|q_i^{\text{opt}} - q_i^{\text{opt}}|\) insertions to the right side of one of \(Q_i\) or \(Q_i^{\text{opt}}\) (whichever has its end point further to the left) will result in the two substrings having the same end-point. Thus

\[\Delta_{\text{ed}}(u, v) + \sum_{i=1}^{m} \Delta_{\text{ed}}(Q_i^{\text{opt}}, Q_i) \leq \Delta_{\text{ed}}(u, v) + \sum_{i=1}^{m} |q_i - q_i^{\text{opt}}| + |q_i^{\text{opt}} - q_i^{\text{opt}}| \leq \Delta_{\text{ed}}(u, v) + 2 \sum_{i=1}^{m} |q_i - q_i^{\text{opt}}|,

where we are able to disregard the case of \(i = 0\) because \(q_0 = q_0^{\text{opt}} = 0\). \(\square\)

The next observation establishes a lower bound for \(\Delta_{\text{ed}}(u, v)\).

**Lemma 3.9.** \(\Delta_{\text{ed}}(u, v) \geq \frac{1}{m} \sum_{i=1}^{m} |q_i - q_i^{\text{opt}}|\).

**Proof.** Because \(\Delta_{\text{ed}}(P, Q^{\text{opt}}) = \Delta_{\text{ed}}(u, v)\), we must have that for each \(i \in [m]\),

\[\Delta_{\text{ed}}(u, v) = \Delta_{\text{ed}}(u[1 : p_i], v[1 : q_i^{\text{opt}}]) + \Delta_{\text{ed}}(u[p_i + 1 : |u|], v[q_i^{\text{opt}} + 1 : |v|]).

Notice, however, that the strings \(u[1 : p_i]\) and \(v[1 : q_i^{\text{opt}}]\) differ in length by at least \(|q_i^{\text{opt}} - p_i|\). Therefore, their edit distance must be at least \(|q_i^{\text{opt}} - p_i|\), implying that \(\Delta_{\text{ed}}(u, v) \geq |q_i^{\text{opt}} - p_i|\).

It follows that \(\frac{1}{m} \Delta_{\text{ed}}(u, v) \geq \frac{1}{m} |q_i^{\text{opt}} - p_i|\). Summing over \(i \in [m]\) gives the desired equation. \(\square\)
So far we have shown that the cost in edit distance which Algorithm 1 pays for selecting \( Q \) instead of \( Q^{\mathrm{opt}} \) is at most \( 2 \sum_{i=0}^{m} |q_i - q_i^{\mathrm{opt}}| \) (Lemma 3.8), and that the edit distance from \( u \) to \( v \) is at least \( \frac{1}{m} \sum_{i=1}^{m} |p_i - q_i^{\mathrm{opt}}| \) (Lemma 3.9). Next we compare these two quantities. In particular, we show that if \( Q \) is chosen to mimic \( Q^{\mathrm{opt}} \) as closely as possible, then each of the \( |q_i - q_i^{\mathrm{opt}}| \) will become small relative to each of the \( |p_i - q_i^{\mathrm{opt}}| \).

**Lemma 3.10.** There exists a partition \( Q = (q_0, \ldots, q_m) \) of \( v \) such that each \( q_i \) is in \( S \), and such that for each \( i \in [0 : m] \), \( |q_i - q_i^{\mathrm{opt}}| \leq \frac{1}{m} |p_i - q_i^{\mathrm{opt}}| \).

**Proof.** Consider the partition \( Q \) in which \( q_i \) is chosen to be the largest \( s \in S \) satisfying \( s \leq q_i^{\mathrm{opt}} \). Observe that: (1) because \( 0 \in S \), each \( q_i \) always exists; (2) because \( |v| \in S \), we will have \( q_m = |v| \); (3) and because \( q_0^{\mathrm{opt}} \leq q_1^{\mathrm{opt}} \leq \cdots \leq q_m^{\mathrm{opt}} \), we will have that \( q_0 \leq q_1 \leq \cdots \leq q_m \). Therefore, \( Q \) is a well-defined partition of \( v \).

It remains to prove that \( |q_i - q_i^{\mathrm{opt}}| \leq \frac{1}{m} |p_i - q_i^{\mathrm{opt}}| \). We consider three cases:

**Case 1:** \( q_i^{\mathrm{opt}} = p_i \). If \( q_i^{\mathrm{opt}} = p_i \), then \( q_i^{\mathrm{opt}} \in S \) and thus \( q_i = q_i^{\mathrm{opt}} \) as well, meaning that \( 0 = |q_i - q_i^{\mathrm{opt}}| \leq \frac{1}{m} |p_i - q_i^{\mathrm{opt}}| = 0 \) trivially.

**Case 2:** \( p_i < q_i^{\mathrm{opt}} \). Consider the largest non-negative integer \( j \) such that \( p_i + (1 + \frac{1}{m})^j \leq q_i^{\mathrm{opt}} \). By definition of \( j \),

\[
\left( 1 + \frac{1}{m} \right)^j \leq q_i^{\mathrm{opt}} - p_i \leq \left( 1 + \frac{1}{m} \right)^{j+1}.
\]  

(3.1)

It follows that

\[
q_i^{\mathrm{opt}} - \left( p_i + \left( 1 + \frac{1}{m} \right)^j \right) \leq \left( 1 + \frac{1}{m} \right)^{j+1} - \left( 1 + \frac{1}{m} \right)^j.
\]  

Simplifying, this becomes,

\[
q_i^{\mathrm{opt}} - \left( p_i + \left( 1 + \frac{1}{m} \right)^j \right) \leq \frac{1}{m} \left( 1 + \frac{1}{m} \right)^j.
\]  

(3.2)

Since \( [p_i + (1 + \frac{1}{m})^j] \in S \), the definition of \( q_i \) ensures that \( q_i \) is between \( p_i + (1 + \frac{1}{m})^j \) and \( q_i^{\mathrm{opt}} \) inclusive. Therefore, (3.2) implies

\[
q_i^{\mathrm{opt}} - q_i \leq \frac{1}{m} \left( 1 + \frac{1}{m} \right)^j.
\]  

(3.3)

Combining (3.1) with (3.3), it follows that \( q_i^{\mathrm{opt}} - q_i \leq \frac{1}{m} (q_i^{\mathrm{opt}} - p_i) \), as desired.

**Case 3:** \( p_i > q_i^{\mathrm{opt}} \). This case is similar to Case 2. Consider the smallest positive integer \( j \) such that \( p_i - (1 + \frac{1}{m})^j \leq q_i^{\mathrm{opt}} \). By definition of \( j \),

\[
\left( 1 + \frac{1}{m} \right)^{j-1} \leq p_i - q_i^{\mathrm{opt}} \leq \left( 1 + \frac{1}{m} \right)^j.
\]  

(3.4)
Manipulating this yields $q_i^{\text{opt}} - p_i - \left(1 + \frac{1}{m}\right)^j \leq \left(1 + \frac{1}{m}\right)^j - \left(1 + \frac{1}{m}\right)^{j-1}$, which in turn implies

$$q_i^{\text{opt}} - \left( p_i - \left(1 + \frac{1}{m}\right)^j \right) \leq \frac{1}{m} \left(1 + \frac{1}{m}\right)^j - \left(1 + \frac{1}{m}\right)^{j-1}. \quad (3.5)$$

Simplifying, this becomes,

$$q_i^{\text{opt}} - \left(p_i - \left(1 + \frac{1}{m}\right)^j\right) \leq \frac{1}{m} \left(1 + \frac{1}{m}\right)^j \cdot (3.6)$$

Since $\max(\lceil p_i - (1 + \frac{1}{m})^j \rceil, 0) \in S$, the definition of $q_i$ ensures that $q_i$ is between $p_i - (1 + \frac{1}{m})^j$ and $q_i^{\text{opt}}$ inclusive. Therefore, (3.5) implies

$$q_i^{\text{opt}} - q_i \leq \frac{1}{m} \left(1 + \frac{1}{m}\right)^j. \quad (3.6)$$

Combining (3.4) with (3.6), it follows that

$$q_i^{\text{opt}} - q_i \leq \frac{1}{m} (p_i - q_i^{\text{opt}}), \text{ as desired.}$$

\[\Box\]

We are now equipped to bound the approximation ratio of Algorithm 1, thereby completing the proof of Theorem 3.2. In particular, the preceding lemmas will allow us to bound the approximation ratio at each level of recursion to $O(\gamma(n))$. The approximation ratio will then multiply across the $O(\log m n)$ levels of recursion, giving total approximation ratio $O(\gamma(n))^{O(\log m n)}$.

**Proposition 3.11.** Let $E(u, v)$ be the number of edits returned by Algorithm 1. Then

$$\Delta_{\text{ed}}(u, v) \leq E(u, v) \leq \Delta_{\text{ed}}(u, v) \cdot (3\gamma(n))^{O(\log m n)}.$$

**Proof.** Because Algorithm 1 finds a sequence of edits from $u$ to $v$, clearly $\Delta_{\text{ed}}(u, v) \leq E(u, v)$.

In order to show that $E(u, v) \leq \Delta_{\text{ed}}(u, v) \cdot (3\gamma(n))^{O(\log m n)}$, we use the previous lemmas. By Lemma 3.10 there is some partition $Q = (q_0, \ldots, q_m)$ of $v$ such that each $q_i$ is in $S$, and such that for each $i \in [0 : m]$, $|q_i - q_i^{\text{opt}}| \leq \frac{1}{m} |p_i - q_i^{\text{opt}}|$. By Lemma 3.9 it follows that

$$\sum_{i=1}^{m} |q_i - q_i^{\text{opt}}| \leq \frac{1}{m} \sum_{i=1}^{m} |p_i - q_i^{\text{opt}}| \leq \Delta_{\text{ed}}(u, v).$$

Applying Lemma 3.8 we then get that

$$\Delta_{\text{ed}}(P, Q) \leq \Delta_{\text{ed}}(u, v) + 2 \sum_{i=1}^{m} |q_i - q_i^{\text{opt}}| \leq 3\Delta_{\text{ed}}(u, v).$$

Thus there is some $Q$ which Algorithm 1 is allowed to select such that $\Delta_{\text{ed}}(P, Q) \leq 3\Delta_{\text{ed}}(u, v)$. Since the approximation ratio of $\mathcal{A}$ is $\gamma(n)$, the partition $Q$ which the algorithm actually chooses at the first level of recursion must therefore satisfy

$$\Delta_{\text{ed}}(P, Q) \leq 3\gamma(n) \Delta_{\text{ed}}(u, v). \quad (3.7)$$

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Recall from the proof of Proposition 3.6 that Algorithm 1 has \( O(\log m n) \) levels of recursion. After the \( i \)-th level of recursion, \( u \) has implicitly been split into a large partition \( P^i \), \( v \) has implicitly been split into a large partition \( Q^i \), and the recursive subproblems are searching for edits between pairs of parts of \( P^i \) and \( Q^i \). Using (3.7), we get by induction that 
\[
\Delta_{ed}(P^i, Q^i) \leq (3\gamma(n))^i \Delta_{ed}(u, v).
\]
Since there are \( O(\log m n) \) levels of recursion, it follows that the number of edits returned by the algorithm is at most 
\[
\Delta_{ed}(u, v) \cdot (3\gamma(n))^{O(\log m n)}.
\]

We conclude the section with two remarks.

Remark 3.12. Because \( m \) is typically asymptotically small, we have not made an effort to optimize the run time exponent of \( m \). In particular, the run time of \( \tilde{O}(T(n)m^5) \) can be improved to \( \tilde{O}(T(n)m^3) \) by limiting each \( q_i \) to be in the set 
\[
\{p_i\} \cup \{|v|\} \cup \left\{ p_i \pm \left(1 + \frac{1}{m}\right)^j \right\} \mid i, j \geq 0 \}.
\]
This comes at the expense of making it slightly more difficult to prove a variant of Lemma 3.10, since it is no longer easy to ensure that the selection of \( q_0, q_1, \ldots, q_m \) satisfies \( q_0 \leq q_1 \leq \cdots \leq q_m \).

Remark 3.13. If \( \gamma(n) \) is \( 1 + o(1) \), then the 3 appearing in the approximation ratio \((3\gamma(n))^{O(\log m n)}\) becomes a bottleneck. This can be handled by redefining 
\[
S := \left(\{p_0, \ldots, p_m\} \cup \{|v|\} \cup \left\{ p_i \pm \left(1 + \frac{1}{mc}\right)^j \right\} \mid i, j \geq 0 \}\right) \cap \{0, \ldots, |v|\}
\]
for some constant \( c > 1 \) so that the options for each \( q_i \) are at a finer granularity. This will achieve approximation ratio \(((1 + O(1/m^{c-1})))^\gamma(n))^{O(\log m n)}\) at the expense of increasing \( m \)'s exponent in the run time.

4 Alignment Embeddings for Permutations

In this section we present a randomized embedding from \( \mathcal{P}_n \), the set of permutations of length \( n \), into Hamming space with expected distortion \( O(\log n) \). The embedding has the surprising property that it implicitly encodes alignments between strings. Moreover, if the output of the embedding is stored using run-length encoding\(^5\) then the size of the output and the run time are both \( O(n) \).

For convenience, in this subsection, we make the Simple Uniform Hashing Assumption \(^13\), which allows us to assume a fully independent family \( \mathcal{H} \) of hash functions mapping \( \Theta(\log n) \) bits to \( \Theta(\log n) \) bits with constant time evaluation. This can be simulated using the family of \(^{26}\), which is independent on any given set of size \( n \) with high probability, and which is further discussed in Section 2.

The description of the embedding appears as Algorithm 2. For simplicity, we assume \( 0 \notin \Sigma \), which allows us to use 0 as a null character. The algorithm takes two parameters: \( \varepsilon \) and \( m \). The parameter \( \varepsilon \) controls a trade-off between the distortion and the output dimension. The parameter \( m \)

\(^5\)In run-length encoding, runs of identical characters are stored as a pair whose first entry is the character and the second entry is the length of the run.

\(^6\)With high probability, there are no hash collisions.
Algorithm 2 Alignment Embedding for Permutations

Input: A string \( w = w_1 \cdots w_n \in \mathcal{P}_n \).
Parameters: \( \varepsilon \) and \( m \geq \log_{1/2+\varepsilon} \frac{1}{n} + 1 \).

1. At the first level of recursion only:
   (a) Initialize an array \( A \) of size \( 2^m - 1 \) (indexed starting at one) with zeros. The array \( A \) will contain the output embedding.
   (b) Select a hash function \( h \) mapping \( \Sigma \) to \( r \log n \) bits for a sufficiently large constant \( r \).

2. Let \( i \) minimize \( h(w_i) \) out of the \( i \in [n/2 - \varepsilon n : n/2 + \varepsilon n] \). We call \( w_i \) the pivot in \( w \).

3. Set \( A[2^m-1] = w_i \).

4. Recursively embed \( w_1 \cdots w_{i-1} \) into \( A[1 : 2^m - 1] \).

5. Recursively embed \( w_{i+1} \cdots w_n \) into \( A[2^m - 1 + 2^m - 1] \).

dictates the maximum depth of recursion that can be performed within the array \( A \). In particular, \( m \) needs to be chosen such that the algorithm does not run out of space for the embedding in the recursive calls.

Since each recursive step takes as input words of size in the range \([1/2 - \varepsilon)n : (1/2 + \varepsilon)n] \), the input size at the \( i \)-th level of the recursion is at most \( (1/2 + \varepsilon)^i n \). We need to choose \( m \) such that at the \( m \)-th level of recursion, the input size will be at most 1. Therefore, it suffices to pick \( m \) satisfying

\[
m \geq \log_{1/2+\varepsilon} \frac{1}{n} + 1.
\]

We denote the resulting embedding of the input string \( w \) into the output array \( A \) by \( \phi_{\varepsilon,m}(w) \). Moreover, for \( m = \lceil \log_{1/2+\varepsilon} \frac{1}{n} + 1 \rceil \), we define \( \phi_\varepsilon(w) \) to be \( \phi_{\varepsilon,m}(w) \). Note that \( \phi_\varepsilon \) embeds \( w \) into an array \( A \) of size

\[
O \left( 2^{\log_{1/2+\varepsilon} 1/n} \right) = O \left( n^{-1/\log(1/2+\varepsilon)} \right),
\]

which one can verify for \( \varepsilon \leq \frac{1}{4} \) is \( O(n^{1+6\varepsilon}) \).

We call \( \phi_\varepsilon \) an alignment embedding because \( \phi_\varepsilon \) maps a string \( x \) to a copy of \( x \) spread out across an array of zeros. When we compare \( \phi_\varepsilon(x) \) with \( \phi_\varepsilon(y) \) by Hamming differences, \( \phi_\varepsilon \) encodes an alignment between \( x \) and \( y \); it pays for every letter which it fails to match up with another copy of the same letter. In particular, every pairing of a letter with a null corresponds to an insertion or deletion, and every pairing of a letter with a different letter corresponds to a substitution. As a result, \( \text{Ham}(\phi_\varepsilon(x), \phi_\varepsilon(y)) \) will always be at least \( \Delta_{sd}(x,y) \).

The rest of this section is dedicated to prove the following theorem which summarizes the properties of \( \phi_\varepsilon \).

**Theorem 4.1.** For \( \varepsilon \leq \frac{1}{4} \), there exists a randomized embedding \( \phi_\varepsilon \) from \( \mathcal{P}_n \) to \( O(n^{1+6\varepsilon}) \)-dimensional Hamming space with the following properties.

- For \( x, y \in \mathcal{P}_n \), \( \phi_\varepsilon(x) \) and \( \phi_\varepsilon(y) \) encode a sequence of \( \text{Ham}(\phi_\varepsilon(x), \phi_\varepsilon(y)) \) edits from \( x \) to \( y \). In particular, \( \text{Ham}(\phi_\varepsilon(x), \phi_\varepsilon(y)) \geq \Delta_{sd}(x,y) \).
• For \( x, y \in P_n \), \( \mathbb{E}[\text{Ham}(\phi_\varepsilon(x), \phi_\varepsilon(y))] \leq O \left( \frac{1}{\varepsilon} \log n \right) \cdot \Delta_{ed}(x, y) \).

• For \( x \in P_n \), \( \phi_\varepsilon(x) \) is sparse in the sense that it only contains \( n \) non-zero entries. Moreover, if \( \phi_\varepsilon(x) \) is stored with run-length encoding, it can be computed in time \( O(n) \).

The first property in the theorem follows from the discussion above. In order to prove that
\[
\mathbb{E}[\text{Ham}(\phi_\varepsilon(x), \phi_\varepsilon(y))] \leq \Delta_{ed}(x, y)O \left( \frac{1}{\varepsilon} \log n \right),
\]
we will consider a series of at most \( 2\Delta_{ed}(x, y) \) insertions or deletions that are used to transform \( x \) into \( y \). Each substitution operation can be emulated by an insertion and a deletion.

This results in a series of intermediate strings starting from \( x \) and ending in \( y \) that differ by one insertion or deletion. Moreover, note that by ordering deletions before insertions, each of the intermediate strings will still be a permutation. In the following key lemma, we will bound the expected Hamming distance between pairs of strings that differ by one insertion (or equivalently, one deletion). By the triangle inequality, we get the bound on \( \mathbb{E}[\text{Ham}(\phi_\varepsilon(x), \phi_\varepsilon(y))] \).

**Lemma 4.2.** Let \( x \in P_n \) be a permutation, and let \( y \) be a permutation derived from \( x \) by a single insertion. Let \( 0 < \varepsilon \leq \frac{1}{4} \) and let \( m \) be large enough so that \( \phi_{\varepsilon, m} \) is well-defined on \( x \) and \( y \). Then
\[
\mathbb{E}[\text{Ham}(\phi_{\varepsilon, m}(x), \phi_{\varepsilon, m}(y))] \leq O \left( \frac{1}{\varepsilon} \log n \right).
\]

**Proof.** Observe that the set of letters in position-range \( [(1/2-\varepsilon)|x| : (1/2+\varepsilon)|x|] \) in \( x \) differs by at most \( O(1) \) elements from the set of letters in position-range \( [(1/2-\varepsilon)|y| : (1/2+\varepsilon)|y|] \) in \( y \). Thus with probability \( 1 - O(1/(\varepsilon n)) \), there will be a letter \( l \) in the overlap between the two ranges whose hash is smaller than that of any other letter in either of the two ranges.

In other words, the pivot in \( x \) (i.e., the letter in the position range with minimum hash) will differ from the pivot in \( y \) with probability \( O(1/(\varepsilon n)) \).

Therefore,
\[
\mathbb{E}[\text{Ham}(\phi_{\varepsilon, m}(x), \phi_{\varepsilon, m}(y))] = \Pr[\text{pivots differ}] \cdot \mathbb{E}[\text{Ham}(\phi_{\varepsilon, m}(x), \phi_{\varepsilon, m}(y)) \mid \text{pivots differ}]
+ \Pr[\text{pivots same}] \cdot \mathbb{E}[\text{Ham}(\phi_{\varepsilon, m}(x), \phi_{\varepsilon, m}(y)) \mid \text{pivots same}]
\leq O \left( \frac{1}{\varepsilon n} \right) \cdot \mathbb{E}[\text{Ham}(\phi_{\varepsilon, m}(x), \phi_{\varepsilon, m}(y)) \mid \text{pivots differ}]
+ \mathbb{E}[\text{Ham}(\phi_{\varepsilon, m}(x), \phi_{\varepsilon, m}(y)) \mid \text{pivots same}].
\]

In general, \( \text{Ham}(\phi_{\varepsilon, m}(x), \phi_{\varepsilon, m}(y)) \) cannot exceed \( O(n) \). Thus
\[
\mathbb{E}[\text{Ham}(\phi_{\varepsilon, m}(x), \phi_{\varepsilon, m}(y))] \leq O \left( \frac{1}{\varepsilon n} \right) \cdot O(n) + \mathbb{E}[\text{Ham}(\phi_{\varepsilon, m}(x), \phi_{\varepsilon, m}(y)) \mid \text{pivots same}]
\leq O \left( \frac{1}{\varepsilon} \right) + \mathbb{E}[\text{Ham}(\phi_{\varepsilon, m}(x), \phi_{\varepsilon, m}(y)) \mid \text{pivots same}].
\]

If the pivot in \( x \) is the same as in \( y \), then the insertion must take place to either the left or the right of the pivot. Clearly \( \phi_{\varepsilon, m}(x) \) and \( \phi_{\varepsilon, m}(y) \) will agree on the side of the pivot in which the edit does not occur. Inductively applying our argument to the side on which the edit occurs, we

\[\text{Note by Lemma 2.3 the probability of any collisions is negligible.}\]
incur a cost of $O(1/\varepsilon)$ once for each level in the recursion. The maximum depth of the recursion is $O \left( \log_{1/2+\varepsilon} \frac{1}{n} \right) = O(\log n)$. This gives us

$$E[\text{Ham}(\phi_{\varepsilon,m}(x), \phi_{\varepsilon,m}(y))] \leq O \left( \frac{1}{\varepsilon} \right) \cdot \log n,$$

as desired.

It remains only to analyze the run time of computing the embedding. Notice that for $x \in P_n$, $\phi_\varepsilon(x)$ is a string with $n$ non-zero entries. Consequently, $\phi_\varepsilon(x)$ can be stored in space $\Theta(n)$ if runs of zeros are stored using run-length encoding. Moreover, if we store $\phi_\varepsilon(x)$ with run-length encoding, we will show that we can compute $\phi_\varepsilon(x)$ in time $O(n)$. Recall that the Range Minimum Query problem can be solved with linear preprocessing time and constant query time \cite{15}. Consequently, with linear preprocessing time, we can build a data structure which supports constant-time queries that take a contiguous substring of $x$ and return which letter has minimum hash. Using this, each recursive step in the embedding can be performed in constant time. Since each such step writes one letter to the output, there are only $n$ steps in total, and thus the run time is $O(n)$.

Remark 4.3. Simulating a fully independent hash function $h$ can be done using $O(n \log n)$ random bits \cite{26}. It turns out that if we are willing to tolerate a run time of $O(n \log n)$, then this can be reduced to polylog($n$) bits, as follows.

Select hash functions $h_1, \ldots, h_m$ from the 1/2-min-wise independent family $H$ of \cite{19} mapping $\Sigma$ to $r \log n$ bits for a sufficiently large constant $r$. Then modify Algorithm 2 to use $h_i$ at the $i$-th level of recursion. By using a separate hash function at each level of recursion, we allow ourselves to analyze the levels independently. Because each $h_i$ is 1/2-min-wise independent, the analysis then follows similarly as in the proof of Lemma 4.2.

Notice that each $h_i$ uses $O(\log n)$ random bits \cite{19}, resulting in $O(\log^2 n)$ random bits in total. Moreover, a naive implementation of the algorithm yields run time $O(n \log n)$.

5 Embedding Periodic-Free Substrings in the Low Edit Distance Regime

We say that a string is $(D,R)$-periodic if it is length at least $D$ and is periodic with period at most $R$ (for $R \leq D$). A string is $(D,R)$-periodic free if it contains no contiguous $(D,R)$-periodic substrings.

Suppose we have an embedding $\psi$ from edit distance in $\Sigma^n$ into Hamming space with sub-polynomial distortion $\gamma(n)$, meaning there is some value $T \geq 1$ such that for all $x, y \in \Sigma^n$, we have $\Delta_{ed}(x, y) \leq \frac{1}{T} \text{Ham}(\psi(x), \psi(y)) \leq \gamma(n) \cdot \Delta_{ed}(x, y)$. Our goal is to use such an embedding as a black box in order to obtain a new embedding for the so-called low edit distance regime. The new embedding, which would be parameterized by a value $K$, would take any two strings $x, y \in \Sigma^n$ with $\Delta_{ed}(x, y) \leq K$ and map $x$ and $y$ to Hamming space with distortion $\gamma'(K)$, a function of $K$ rather than a function of $n$.

In this section we make progress toward such an embedding with the added constraints that our strings $x$ and $y$ are $(D,R)$-periodic free for $D \in \text{poly}(K)$ of our choice and $R \in O(K^3)$. Our embedding $\alpha$ takes two such strings with $\Delta_{ed}(x, y) \leq K$ and maps them into Hamming space with
Let \( \alpha \) be an embedding. The primary embedding succeeds with some probability, but its distortion is not well bounded in expectation. By applying techniques from \([12]\), we are able to take any high-probability low-distance-regime embedding, and turn it into an embedding with good expected distortion (Theorem 5.10), thereby completing the full construction of our embedding. The following theorem states the main result in this section.

**Theorem 5.2.** Suppose we have an embedding \( \psi \) from edit distance in \( \Sigma^n \) to Hamming space with subpolynomial distortion \( \gamma(n) \) (and with \( \gamma(n) \geq 2 \) for all \( n \)). Let \( K \in \mathbb{N} \) and pick some \( D \in \text{poly}(K) \). Then there exists \( R \in O(K^3) \) and an embedding \( \alpha : \Sigma^n \to \{0,1\}^n \) such that for \((D,R)\)-periodic free \( x,y \in \Sigma^n \) with \( \Delta_{ed}(x,y) \leq K \), we have

\[
\frac{1}{\gamma(O(K^3D))} \Omega(\Delta_{ed}(x,y)) \leq K \Pr[\alpha(x) \neq \alpha(y)] \leq O(\Delta_{ed}(x,y)).
\]

In other words, \( \alpha \) is an embedding from edit distance (at most \( K \)) between \((D,R)\)-periodic free strings to scaled Hamming distance with expected distortion at most \( O(\gamma(K^3D)) \).

The following result follows by plugging in the embedding of Ostrovsky and Rabani \([25]\), which is defined over binary strings, and maps edit distance to \( \ell_1 \)-distance with distortion \( 2O(\sqrt{\log K \log \log K}) \).

**Corollary 5.3.** Let \( K \in \mathbb{N} \) and pick some \( D \in \text{poly}(K) \). Then there exists \( R \in O(K^3) \) and an embedding \( \alpha : \{0,1\}^n \to \{0,1\}^n \) such that for \((D,R)\)-periodic free \( x,y \in \{0,1\}^n \) with \( \Delta_{ed}(x,y) \leq K \), we have

\[
\frac{1}{2O(\sqrt{\log K \log \log K})} \cdot \Omega(\Delta_{ed}(x,y)) \leq K \Pr[\alpha(x) \neq \alpha(y)] \leq O(\Delta_{ed}(x,y)).
\]

In other words, \( \alpha \) is an embedding from edit distance (at most \( K \)) to scaled Hamming distance with expected distortion \( 2O(\sqrt{\log K \log \log K}) \).

**Proof.** Although the embedding of Ostrovsky and Rabani embeds into \( \ell_1 \) rather than into Hamming space, it can be converted using standard techniques to an embedding into Hamming space with
additional blow-up in dimension and constant additional multiplicative distortion. Therefore, applying Theorem \(5.2\) to the embedding of Ostrovsky and Rabani, we get \(\alpha\) satisfying
\[
\frac{1}{2^{O\left(\sqrt{\log (K^2 D) \log \log (K^2 D)}\right)}} \Omega (\Delta_{ed}(x,y)) \leq K \Pr[ \alpha(x) \neq \alpha(y) ] \leq O(\Delta_{ed}(x,y)) .
\]
Since \(D \in \text{poly}(K)\), it follows that
\[
\frac{1}{2^{O\left(\sqrt{\log K \log \log K}\right)}} \Omega (\Delta_{ed}(x,y)) \leq K \Pr[ \alpha(x) \neq \alpha(y) ] \leq O(\Delta_{ed}(x,y)) ,
\]
as desired.

The primary embedding is given by Algorithm \(3\). The algorithm uses as parameters a window-size \(W\), a sub-window size \(W'\), and a sub-sub-window size \(W''\). We will use the term window to refer to \(W\) consecutive letters in a word, the term sub-window to refer to \(W'\) consecutive letters in a window, and the term sub-sub-window to refer to \(W''\) consecutive letters within a sub-window. We will assign values to \(W, W', W''\) at the end of our analysis.

Algorithm \(3\) is designed so that when two strings \(x\) and \(y\) differ by a small number of edits, they are likely to be broken into similar window sequences. In particular, suppose that \(A_t(x)\) and \(A_t(y)\) are aligned with each other, meaning that some optimal sequence of edits from \(x\) to \(y\) maps the first letter of \(A_t(x)\) to the first letter of \(A_t(y)\) without editing the letter itself. Then consider what happens when we construct \(A_{t+1}(x)\) and \(A_{t+1}(y)\). By picking a sub-window uniformly at random, we guarantee that with high probability (as a function of \(W, W'\)) no edits occur directly within that sub-window. However, if \(x\) and \(y\) differ by an insertion or deletion prior to that sub-window, the sub-window within \(x\) may be misaligned with the sub-window within \(y\). Nonetheless, the set of \(W''\)-letter sub-sub-windows of the sub-window of \(x\) will be almost the same as the set of \(W''\)-letter sub-sub-windows of the sub-window of \(y\). By selecting the sub-sub-window with minimum hash as the start position for the next window, we are then able to guarantee that with high probability (as a function of \(W'\)) we pick start positions for \(A_{t+1}(x)\) and \(A_{t+1}(y)\) which are aligned with each other. Note that in order for the mini-hash technique to work, one needs the sub-sub-windows of any given sub-window to all be distinct; for appropriately selected \(W, W', W''\), this will be a consequence of Lemma \(5.1\).

Before formally analyzing Algorithm \(3\), we discuss its use of hash functions. For each \(t \in \{1, \ldots, n^{\frac{1}{2}}\}\), the algorithm requires a \(2^n\)-wise independent hash function \(h_t\) mapping \(\Sigma^{W''}\) to \(\Theta(\log n)\) bits. Note that the \(h_t\) hash functions can be efficiently simulated using the family of \(26\). In particular, one can first use a pairwise independent hash function \(g\) to map elements of \(\Sigma^{W''}\) to \(\Theta(\log n)\) bits (while avoiding collisions with high probability). Then, one can then select a function \(h\) mapping \(\Theta(\log n)\) bits to \(\Theta(\log n)\) bits from the family of \(26\). Finally one can then define \(h_t(u) = h((t,g(u)))\), where \((t,g(u))\) represents the tuple containing \(t\) and \(g(u)\). For strings \(x, y \in \Sigma^n\), and for each \(t\), with high probability \(h_t\) will be independent on the \(W''\)-letter substrings.

---

\(8\)In particular, if \(x \in \mathbb{R}^\ell\) is the output of the embedding of \(25\), then one can transform it as follows. First by shifting each coordinate’s value, one can assume that the values of coordinates are always between 0 and \(O(n^2)\), since if the embedding ever had two outputs differing by more than \(\Omega(n^2)\) in some coordinate, then the distortion of the embedding would be at least \(\Omega(n)\). Then after rescaling up by \(\ell\) and rounding each of the coordinates, one can choose each coordinate to be a positive integer of magnitude at most \(O(n^2\ell)\), while incurring at most constant additional distortion. Each coordinate can then be encoded in \(O(n^2\ell)\) Hamming coordinates by mapping the value \(i\) to the adjacent placed values \(1, 1, \ldots, 1, 0, 0, \ldots, 0\), where the 1 is repeated \(i\) times.
Algorithm 3 The Primary Embedding

Given: (i) an input string \( w \) of length at most \( n \), (ii) parameters \( W'' \ll W' \ll W \), (iii) an embedding \( \psi \) from edit distance into Hamming space, where the input strings are of length at most \( W \), and the output has a fixed length, (iv) \( s_1, \ldots, s_{2n/W} \) randomly selected elements of \( \{1, \ldots, \frac{W}{W'}\} \), and (v) for each \( t \in \{1, \ldots, \frac{W'}{2W'}\} \), a \( 2n \)-wise independent hash function \( h_t \) mapping \( \Sigma_{W''} \) to \( \Theta(\log n) \) bits.

1. Let \( b = b_1 \cdots b_{2W} \) consist of dummy letters not in \( \Sigma \). Define \( w' \) to be the concatenation \( wb \).

2. Construct a window sequence \( A(w) = A_1(w), \ldots, A_{2n/W}(w) \) of windows in \( w' \), by first setting \( A_1(w) \) to consist of the first \( W \) letters \( w'_1 \cdots w'_{W} \) of \( w' \), and then constructing each \( A_{t+1}(w) \) from \( A_t(w) \) as follows:
   
   (a) If \( A_t(w) \) consists entirely of dummy letters, then define \( A_{t+1}(w) = A_t(w) \). Otherwise, continue to the next step.

   (b) Divide the second half of the window \( A_t(w) \) into \( \frac{W}{2W'} \) non-overlapping sub-windows of size \( W' \), and consider the \( s_t \)-th such sub-window.

   (c) Each substring of length \( W'' \) inside the sub-window is called a sub-sub-window (sub-sub-windows may overlap). Compute the hash \( h_t(v) \) of each sub-sub-window \( v \).

   (d) The window \( A_{t+1}(w) \) will consist of \( W \) letters starting at the same position as the sub-sub-window with the smallest hash. (Ties can be broken arbitrarily but, it turns out, will not occur with high probability.)

3. Define \( P(w) \) to be a partition of \( w \) where the \( i \)-th part starts at the first letter of \( A_i(w) \) and ends prior to the first letter of \( A_{i+1}(w) \), excluding any dummy letters.

4. Embed each part of \( P(w) \) using \( \psi \), and concatenate the embedding results. The dimension of the concatenation is \( \ell \cdot |P(w)| \) where \( \ell \) denotes the output dimension of \( \psi \). In order to ensure that the outputs of our algorithm are of a fixed dimension, pad the concatenated result with zeros as needed to bring its length to \( \ell \cdot \lceil 2n/W \rceil \). Return the padded string.

of \( x \) and \( y \). By the union bound, the independence of \( h_t \) holds simultaneously for all \( t \) with high probability.

We now present a formal analysis of Algorithm 3. For simplicity, we will ignore floors and ceilings when convenient. We start with two definitions that will aid us in our discussion of the algorithm.

**Definition 5.4.** Let \( x \) and \( y \) be words over \( \Sigma \cup \{b_1, b_2, \ldots\} \). Fix some optimal sequence of edits between \( x \) and \( y \). Then a letter in \( x \) or \( y \) is said to be touched if it is involved in an edit, or if an insertion or deletion occurs immediately to its right in the sequence of edits. A letter \( a \) in \( x \) is said to be siblings with a letter \( b \) in \( y \) if either \( a \) and \( b \) are the first letters of \( x \) and \( y \), or the edits map \( a \) to \( b \) while leaving both untouched.

**Example 5.5.** Consider \( x = abcdefghi \) and \( y = bpeqrsfgbia \). Moreover, consider the following
sequence of edits:

\[
\begin{array}{cccccccc}
  a & b & c & d & e & f & g & h & i \\
  b & p & d & e & q & r & s & f & g & h & i & a
\end{array}
\]

That is, we delete and reinsert the \(a\), we substitute the \(c\) with a \(p\), and we insert the \(q\), \(r\), and \(s\). Between \(x\) and \(y\), the pairs of siblings are \((b, b)\), \((d, d)\), \((f, f)\), \((g, g)\), and \((h, h)\).

In general, if there is a sequence of adjacent untouched letters in \(x\), then the siblings of those letters will form the same adjacent sequence in \(y\). This property of touched letters is the main motivation for the above definition.

**Definition 5.6.** Let \(P\) and \(Q\) be partitions of strings \(x\) and \(y\) into \(j\) parts \(P_1, \ldots, P_j\) and \(Q_1, \ldots, Q_j\) respectively. We call \((P, Q)\) edit-preserving if there exists an optimal sequence of edits from \(x\) to \(y\) such that the first letter of \(P_i\) is siblings with the first letter of \(Q_i\) for each \(i\).

Intuitively, if \((P, Q)\) is edit-preserving, then we can focus on the parts of \(P\) and \(Q\) separately when trying to embed \(x\) and \(y\) into Hamming space. This is formalized by the following lemma.

**Lemma 5.7.** Let \(P\) and \(Q\) be partitions of strings \(x\) and \(y\) into \(j\) parts \(P_1, \ldots, P_j\) and \(Q_1, \ldots, Q_j\) respectively. If \((P, Q)\) is edit-preserving, then

\[
\Delta_{ed}(x, y) = \sum_{i=1}^{j} \Delta_{ed}(P_i, Q_i).
\]

**Proof.** Consider an optimal sequence of edits from \(x\) to \(y\) such that the first letter of \(P_i\) is siblings with the first letter of \(Q_i\) for each \(i\). We say that an edit occurs within \(P_1\) if the first letter of \(P_2\) is to the right of the edit. For \(i > 1\), we say that an edit occurs within \(P_i\) if it takes place between the first letter of \(P_i\) and the first letter of \(P_{i+1}\) (or the end of the word, if \(i = j\)). Notice that because the first letter of each \(P_i\) is siblings with the first letter of each \(Q_i\), the edits occurring within \(P_i\) transform \(P_i\) into \(Q_i\). It follows that

\[
\Delta_{ed}(x, y) = \sum_{i=1}^{j} \Delta_{ed}(P_i, Q_i),
\]

as desired. \(\square\)

For the rest of the section, for any string \(w\), we use \(P(w)\) to denote the partition constructed in Algorithm 3. The following key lemma establishes that under certain conditions, \((P(x), P(y))\) is edit-preserving with high probability.

**Lemma 5.8.** Let \(x, y \in \Sigma^n\) be words such that \(\Delta_{ed}(x, y) \leq K\). Moreover, suppose that for each sub-window of either \(x\) or \(y\), no two of its sub-sub-windows are equal. Fix a minimal sequence of edits between \(x\) and \(y\). Then \((P(x), P(y))\) is edit-preserving with probability at least

\[
1 - 16K \cdot \left( \frac{W'}{W} - \frac{1}{W' - W'' + 1} \right).
\]
Proof. Notice that, as long as the window $A_t(x)$ is not entirely contained within the dummy letters $b$, then the window $A_{t+1}(x)$ is constructed to overlap it by at most $W/2$ letters. It follows that each letter $l$ in $x$ appears in at most two windows in $x$’s window sequence. For each $A_t(x)$ which is not contained within the dummy letters $b$, define $B_t(x)$ to be the sub-window of $A_t(x)$ used to determine $A_{t+1}(x)$’s starting point. Recall that $B_t(x)$ is chosen uniformly at random (independently for each $t$) out of the sub-windows making up the second half of $A_t(x)$. Thus a given letter in the second half of $A_t$ has probability $W/W/2$ of appearing within $B_t(x)$. Since each letter appears in the second half of at most one window, a given letter in $x$ has probability at most $\frac{2W}{W}$ of appearing in any $B_t(x)$. There are at most $2K$ touched letters in $x$. By the union bound, the probability of any touched letter in $x$ appearing in any $B_t(x)$ is at most $\frac{4KW'}{W}$.

Define $B_t(y)$ similarly as to $B_t(x)$. We shall assume for the rest of the proof that none of the letters in any $B_t(x)$ or $B_t(y)$ are touched by the edits. We additionally condition on each one of the $h_t$ functions having no collisions (on the elements we hash), which happens with probability $1 - \frac{1}{\text{poly}(n)}$ (by Lemma 2.3). By the analysis provided so far, these conditions will occur with probability at least

$$1 - \frac{8KW'}{W} - \frac{1}{\text{poly}(n)} \geq 1 - \frac{16KW'}{W}. \quad (5.1)$$

Consider some window $A_t(x)$ (with $t < 2n/W$) and the corresponding window $A_t(y)$. Suppose that the first letters of $A_t(x)$ and $A_t(y)$ are siblings. In other words, either $t = 1$, or the edits between $x$ and $y$ transform the first letter of $A_t(x)$ to become the first letter of $A_t(y)$, without ever touching either letter. We will show that

$$\Pr \left[ \text{First letters of } A_{t+1}(x) \text{ and } A_{t+1}(y) \text{ siblings} \mid \text{First letters of } A_t(x) \text{ and } A_t(y) \text{ siblings} \right] \geq 1 - \frac{2k_t}{W' - W'' + 1}. \quad (5.2)$$

where $k_t$ is defined to be the number of touched letters contained in either $A_t(x)$ or $A_t(y)$. Also recall that these probabilities are conditioned on lack of hash collisions and on the event that touched letters do not appear in any $B_t(x)$ or $B_t(y)$.

The easy case for (5.2) occurs when either of $A_t(x)$ or $A_t(y)$ consists entirely of dummy letters. Because the first letters of $A_t(x)$ and $A_t(y)$ are siblings, it follows that both consist entirely of dummy letters, and $A_t(x) = A_t(y)$. Therefore, with probability 1, the first letter of $A_{t+1}(x)$ will be siblings with the first letter of $A_{t+1}(y)$.

Suppose, on the other hand, that neither $A_t(x)$ nor $A_t(y)$ consists entirely of dummy letters. Then, because the sub-window $B_t(x)$ of $A_t(x)$ consists of untouched letters, the same sub-window must appear in $y$ (we will denote the image of $B_t(x)$ in $y$ by $B_t'(x)$). Since $A_t(x)$ and $A_t(y)$ start with sibling letters, the offset (due to insertions or deletions) of the sub-window $B_t(x)$ in $A_t(x)$ must be within $k_t$ of the offset of $B_t'(x)$ in $A_t(y)$. Since $B_t(x)$ appears at the same offset (determined by $s_t$) in $A_t(y)$ as does $B_t(x)$ in $A_t(x)$, it follows that $B_t(y)$ must overlap $B_t'(x)$ in at least $W' - k_t$ letters. This means that, for each one of $B_t(x)$ and $B_t(y)$, out of the $W' - W'' + 1$ sub-sub-windows, all but at most $k_t$ of them appear in the overlap between $B_t(x)$ and $B_t(y)$. It follows that with probability at least $1 - \frac{2k_t}{W' - W'' + 1}$, there is a unique sub-sub-window $u$ in the overlap of $B_t(y)$ and $B_t'(x)$ whose $h_t$-hash is smaller than that of any other sub-sub-window in $B_t(y)$ or $B_t'(x)$. This, in turn, guarantees that the first elements of $A_{t+1}(x)$ and $A_{t+1}(y)$ are siblings, completing the proof of (5.2).
By repeatedly applying (5.2),
\[
\Pr \left[ \text{First letters of } A_t(x) \text{ and } A_t(y) \text{ siblings } \forall t \right] \geq \prod_{t=1}^{2n/W-1} \left( 1 - \frac{2k_t}{W' - W'' + 1} \right) \\
\geq 1 - \sum_{t=1}^{2n/W-1} \left( \frac{2k_t}{W' - W'' + 1} \right).
\]

Since each of \( x \) and \( y \) contain at most 2\( K \) touched letters. Each touched letter in \( x \) or \( y \) appears in at most two windows of \( A(x) \) or \( A(y) \), respectively. It follows that \( \sum_{t=1}^{2n/W-1} k_t \leq 8K \), and
\[
\Pr \left[ \text{First letters of } A_t(x) \text{ and } A_t(y) \text{ siblings } \forall t \right] \geq 1 - \frac{16K}{W' - W'' + 1}. \tag{5.3}
\]

By definition, if the first letters of \( A_t(x) \) and \( A_t(y) \) are siblings for all \( t \), then \((P(x), P(y))\) will be edit-preserving. Thus \((5.1)\) and \((5.3)\) combine to tell us that \((P(x), P(y))\) are edit-preserving with probability at least
\[
1 - 16K \cdot \left( \frac{W'}{W} - \frac{1}{W' - W'' + 1} \right).
\]

The following theorem shows that with high probability, Algorithm 3 behaves well on \((D, R)\)-periodic free strings of distance at most \( K \) from each other.

**Theorem 5.9.** Suppose we have an embedding \( \psi \) from edit distance in \( \Sigma^n \) to Hamming space with distortion \( \gamma(n) \), meaning there is some value \( T \geq 1 \) such that for all \( x, y \in \Sigma^n \), we have \( \Delta_{ed}(x, y) \leq \frac{1}{T} \text{Ham}(\psi(x), \psi(y)) \leq \gamma(n) \cdot \Delta_{ed}(x, y) \). Let \( K, D, R, C \) be positive variables satisfying \( R \geq 32KC \). Then there exists a randomized embedding \( \phi \) from \( \Sigma^n \) to Hamming space satisfying the following property. If \( x, y \in \Sigma^n \) are \((D, R)\)-periodic free and \( \Delta_{ed}(x, y) \leq K \), then
\[
\Delta_{ed}(x, y) \leq \frac{1}{T} \text{Ham}(\phi(x), \phi(y)) \leq \Delta_{ed}(x, y) \cdot \gamma(64DKC),
\]
with probability at least \( 1 - \frac{1}{C} \).

**Proof.** Note that if \( n < D \), then we can achieve \( \gamma(D) \) distortion using the embedding \( \psi \), and that if \( n \geq D \) but \( D < R \), then there are no \((D, R)\)-periodic free strings \( x, y \in \Sigma^n \), again making the result trivial. Thus we can assume that \( R \leq D \leq n \).

Select \( W'' = D \), \( W' = W'' + 32KC \) and \( W = W'.32KC \). Let \( \phi(x) \) and \( \phi(y) \) be the outputs of Algorithm 3.

Because \( x \) and \( y \) are \((D, R)\)-periodic free, Lemma 5.1 tells us that for any substring \( a \) of either \( x \) or \( y \) satisfying \( |a| \leq D + R \), the \( D \)-letter substrings of \( a \) are distinct. Since \( W' = D + 32KC \leq D + R \) (from the statement of the theorem) and \( W'' = D \), it follows that for each sub-window of \( x \) and \( y \), the sub-sub windows are distinct. Hence we can apply Lemma 5.8 to see that \((P(x), P(y))\) is edit-preserving with probability at least
\[
1 - 16K \cdot \left( \frac{W'}{W} - \frac{1}{W' - W'' + 1} \right) = 1 - 16K \cdot \left( \frac{1}{32KC} + \frac{1}{32KC + 1} \right) \\
> 1 - \frac{1}{C}.
\]

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Supposing that \((P(x), P(y))\) is edit-preserving, Lemma 5.7 tells us that the distortion of \(\phi\) on the edit distance between \(x\) and \(y\) is at most the distortion of the embedding \(\alpha\) on words of length \(W = (D + 32KC) \cdot 32KC\). Since \(32KC \leq R \leq D\), we have that \((D + 32KC) \cdot 32KC \leq 64DKC\). It follows that
\[
\Delta_{ed}(x, y) \leq \frac{1}{T} \text{Ham}(\phi(x), \phi(y)) \leq \Delta_{ed}(x, y) \cdot \gamma(64DKC),
\]
as desired.

So far we have obtained an embedding which behaves well with probability \(1 - \frac{1}{T}\). The next result, which is implicitly present in Section 3.5 of [12] takes any high-probability low-edit-distance-regime embedding from edit distance to Hamming space, and turns it into an embedding with good expected distortion.

**Theorem 5.10.** Let \(S\) be a subset of \(\Sigma^{\leq n}\). Suppose we have a randomized embedding \(\phi: S \rightarrow \Sigma^n\) such that for all \(x, y \in S\) satisfying \(\Delta_{ed}(x, y) \leq K\),
\[
\Delta_{ed}(x, y) \leq \frac{1}{T} \text{Ham}(\phi(x), \phi(y)) \leq F(K) \Delta_{ed}(x, y)
\]
with probability at least \(1 - \frac{1}{K \cdot F(K)}\) for some distortion function \(F(K)\) with \(F(K) \geq 2\), for some \(T \geq 1\), and for all \(K \in \mathbb{N}\).

Then there exists \(\alpha: S \rightarrow \{0, 1\}\) such that for all \(x, y \in S\) satisfying \(\Delta_{ed}(x, y) \leq K\),
\[
\Omega(\Delta_{ed}(x, y)) \leq K \cdot F(K) \Pr[\alpha(x) \neq \alpha(y)] \leq O(\Delta_{ed}(x, y) F(K)).
\]
In other words, \(\alpha\) is an embedding from edit distance (at most \(K\)) over \(S\) to Hamming distance (scaled) with expected distortion at most \(O(F(K))\).

**Proof.** Implicit in [12]. The full proof is deferred to Appendix A.

Combining Theorems 5.9 and 5.10 we can now prove the main result for this section.

**Proof of Theorem 5.2** Because \(\gamma\) is subpolynomial, we can pick \(C \in O(K^2)\) and \(R = 32KC \in O(K^3)\) such that \(K \cdot \gamma(64DKC) \leq C\). Because \(R \geq 32KC\), we can apply Theorem 5.9 to obtain an embedding with distortion \(\gamma(64DKC)\) with probability \(1 - \frac{1}{K}\). Because \(K \cdot \gamma(64DKC) \leq C\), we can then apply Theorem 5.10 to obtain a map \(\alpha: \Sigma^n \rightarrow \{0, 1\}\) as described.

**Remark 5.11.** The embedding described by Theorem 5.2 maps strings to only a single bit. By concatenating together multiple independent iterations of the embedding, one can obtain a map to multiple bits for which the output distances are bounded with high probability in addition to just in expectation. In fact, although stated as a randomized result, the above embedding be iterated over all possible choices of random bits in order to give a deterministic embedding from edit distance to scaled Hamming space for \((D, R)\)-periodic free strings whose distances are upper bounded by \(K\).

**Remark 5.12.** The embedding described by Theorem 5.2 can also be performed on pairs of \((D, R)\)-periodic strings \(x, y\) with \(\Delta_{ed}(x, y) > K\). In this case, the embedding will encode the fact that \(\Delta_{ed}(x, y) \geq \Omega(K)\). In particular, the embedding of Theorem 5.9 will output strings of Hamming distance at least \(T \cdot K\); and one can check that the embedding of Theorem 5.10 will then satisfy
\[
\frac{\Omega(K)}{(O(K^2))} \leq K \Pr[\alpha(x) \neq \alpha(y)]\]

This follows by the same logic as in Case 2 of the proof of Theorem 5.10.
6 Dimension Reduction

A mapping of edit distance on length-\(n\) strings to edit distance on strings of length at most \(n/c\) (with larger alphabet size) is called a dimension-reduction map with contraction \(c\). In this section we present a randomized dimension-reduction map for edit distance whose contraction is \(c\) and whose distortion is \(O(c)\), which is within a constant factor of optimal. This marks an improvement over the previous state of the art [8], which achieved distortion \(\tilde{O}(c^{1+\frac{2}{\log \log \log n}})\). Moreover, our dimension-reduction map can be computed in time \(\tilde{O}(n)\).

We begin with a formal definition of a dimension-reduction map.

**Definition 6.1.** We say that a randomized map \(\phi\) from strings in \(\Sigma^n\) to strings of length at most \(n/c\) over a different alphabet \(\Sigma'\) is a dimension-reduction map with contraction \(c\) and expected distortion at most \(\alpha \cdot \beta\) if for all \(x,y \in \Sigma^n\),

\[
\Delta_{ed}(x,y) \leq \alpha \Delta_{ed}(\phi(x),\phi(y)),
\]

and

\[
E[\Delta_{ed}(\phi(x),\phi(y))] \leq \beta \Delta_{ed}(x,y).
\]

Note, in particular, that the lower bound \((6.1)\) is required to hold regardless of the random bits used to compute \(\phi\). This is similar to the way distortion was defined in [7].

Before continuing, we present a lower bound for the expected distortion of a dimension-reduction map \(\phi\). This is a simple generalization of the analogous lower bound for deterministic dimension reduction provided in [8].

**Lemma 6.2.** Suppose \(\phi\) is a randomized dimension-reduction map with contraction \(c\). Then the expected distortion of \(\phi\) is at least \(c\).

**Proof.** Consider \(x,y \in \Sigma^n\) with \(\Delta_{ed}(x,y) = 1\). In order so that \((6.1)\) holds for some \(\alpha\), it must be that \(\Delta_{ed}(\phi(x),\phi(y)) \neq 0\). Therefore, \(\Delta_{ed}(\phi(x),\phi(y)) \geq 1\), meaning that \(\beta \geq 1\).

Now consider \(x,y \in \Sigma^n\) with \(\Delta_{ed}(x,y) = n\). Since \(\phi(x),\phi(y)\) are of length at most \(n/c\), it follows that \(\Delta_{ed}(\phi(x),\phi(y)) \leq n/c\). Therefore, from \((6.1)\), \(\alpha\) is at least \(c\).

Since \(\alpha \geq c\) and \(\beta \geq 1\), the expected distortion of \(\phi\) is \(\alpha \beta \geq c\).

The main result in this section will be constructing a dimension-reduction map \(\phi\) for which the following theorem holds.

**Theorem 6.3.** Let \(c,n \in \mathbb{N}\). There exists a dimension-reduction map \(\phi\) such that for every two strings \(x,y \in \Sigma^\leq n\),

- The lengths of \(\phi(x)\) and \(\phi(y)\) are at most \(O(|x|/c)\) and \(O(|y|/c)\), respectively.
- \(\Delta_{ed}(x,y) \leq 2c \cdot \Delta_{ed}(\phi(x),\phi(y))\).
- The expected distortion of \(\phi\) is \(O(c)\). In particular,

\[
\Pr[\Delta_{ed}(\phi(x),\phi(y)) > m\Delta_{ed}(x,y)] \leq \left(\frac{1}{2}\right)^{\Omega(m)} + \frac{1}{\text{poly}(n)},
\]

for all \(m \in \mathbb{N}\) and for a polynomial \(\text{poly}(n)\) of our choice.
• Each of $\phi(x)$ and $\phi(y)$ can be computed in time $O(n \log c)$.

The rest of the section proceeds as follows. In Subsection 6.1, we present a simplified version of Theorem 6.3 for permutations. This motivates our more general construction, and yields a linear-time $O(\log n)$-approximation algorithm for finding an optimal sequence of edits between permutations. Then in Subsection 6.2 we present preliminaries for the more general dimension-reduction map $\phi$. In Subsection 6.3 we define $\phi$ and prove Theorem 6.3. Finally, in Subsection 6.4 we discuss an application of our dimension-reduction map to approximation algorithms for edit distance.

6.1 Dimension Reduction for Permutations

In this section we present a dimension-reduction map for permutations. This motivates the techniques which we will use in general, and has the interesting property that the map can be computed in strictly linear time.

For convenience, in this subsection we assume access to a fully independent family $H$ of hash functions mapping $\Theta(\log n)$ bits to $\Theta(\log n)$ bits with constant time evaluation. This can be simulated using the family of [26], which is independent on any given set of size $n$ with high probability, and which is further discussed in Section 2.

Algorithm 4 Dimension Reduction for Permutations with Edit Distance

Input: a string $w \in P_{\leq n}$, a parameter $c \in \mathbb{N}$ with $c$ even, and a constant $b$.

1. Select $h$ at random from a family of $n$-wise independent hash functions mapping $\Sigma$ to $\{0,1\}^{b \log n}$.

2. Call a letter $w_i$ a marker if $c < i \leq n - c$ and if $h(w_i)$ is smaller than any of $h(w_j)$ for $j \in \{i - c, \ldots, i + c\} \setminus \{i\}$. That is, $w_i$ is a marker if $h(w_i)$ is the unique minimum in the sequence $h(w_{i-c}), \ldots, h(w_{i+c})$.

3. Partition $w$ into blocks in the following way: Each marker indicates the beginning of a block. For each block of size at least $c$, subdivide it into blocks such that all of them are of size exactly $c$, except for the last one which will be of size at most $c$.

4. Return a string over the alphabet $\Sigma^*$ (i.e., whose letters are strings in $\Sigma$) where each block from the preceding step is a letter.

Given $w \in P_{\leq n}$, Algorithm 4 shows how to compute $\phi(w)$. The embedding selects a letter $w_i$ to be a marker if $w_i$'s hash is smaller than that of any of the preceding $c$ letters or following $c$ letters. The algorithm then splits $w$ based on the markers, and then further splits each of the resulting chunks into blocks of size $c$, except that the last block in each chunk may be of size between 1 and $c$.

The following theorem establishes that $\phi$ is a dimension-reduction map.

**Theorem 6.4.** Let $p \in \text{poly}(n)$. Then there exists a sufficiently large constant $b$ such that for every two permutations $x, y \in P_{\leq n}$,
• The lengths of $\phi(x)$ and $\phi(y)$ are at most $O(|x|/c)$ and $O(|y|/c)$, respectively.

• $\Delta_{ed}(x, y) \leq c \cdot \Delta_{ed}(\phi(x), \phi(y))$.

• The distortion of $\phi$ is $O(c)$. In particular,
  \[
  \Pr[\Delta_{ed}(\phi(x), \phi(y)) > m\Delta_{ed}(x, y)] \leq \left(\frac{1}{2}\right)^{\Omega(m)} + \frac{1}{p(n)},
  \]
  for all $m \in \mathbb{N}$.

• Each of $\phi(x)$ and $\phi(y)$ can be computed in time $O(n)$.

We prove each of the parts of the theorem individually. The first part of Theorem 6.4 is established by the following lemma.

**Lemma 6.5.** Let $w \in \mathcal{P}_{\leq n}$. Then for any contiguous $m \cdot c$ letters in $w$, $\phi(w)$ breaks the letters into at most $O(m)$ blocks.

**Proof.** Notice that no window of $c + 1$ letters in $w$ can contain two distinct markers $w_i$ and $w_j$. Indeed, in order for $w_i$ to be a marker, we would need that $h(w_i) < h(w_j)$. But then in order for $w_j$ to be a marker, we would also need that $h(w_j) < h(w_i)$, a contradiction.

It follows that the total number of markers in the $mc$ letters is at most $O(m)$. Recall that once the markers are placed, the string is partitioned into blocks of size $c$, except that the block preceding a marker may be smaller. Thus in the $mc$ letters there are at most $O(m)$ blocks of length $c$ and then at most one additional block for each marker, totaling to $O(m)$ blocks.

The second part of Theorem 6.4 is established by the following lemma.

**Lemma 6.6.** For $x, y \in \mathcal{P}_{\leq n}$,
\[
\Delta_{ed}(x, y) \leq c \cdot \Delta_{ed}(\phi(x), \phi(y)).
\]

**Proof.** Consider a sequence of edits from $\phi(x)$ to $\phi(y)$. Since the blocks in $\phi(x)$ and $\phi(y)$ are all of size at most $c$, each edit to $\phi(x)$ corresponds with a sequence of at most $c$ edits to $x$. Therefore, we get that $\Delta_{ed}(x, y) \leq c \cdot \Delta_{ed}(\phi(x), \phi(y))$. \(\square\)

In order to prove the third part of Theorem 6.4 which bounds the distortion of $\phi$, we must first upper bound the probability of having many consecutive letters without any markers. In the following analysis we will condition on $h$ being injective (i.e., collision free). Later, we will apply the no-collisions assumption (Lemma 2.3) to justify this with high probability.

**Lemma 6.7.** Consider $w \in \mathcal{P}_{\leq n}$, $k \in \{1, \ldots, b\}$, and $m \in \mathbb{N}$. Consider some $3cm$ consecutive letters, and condition on $h$ being injective on those letters. Then the algorithm places a marker due to $h$ in the $3cm$ letters with probability at least $1 - \left(\frac{2}{3}\right)^m$.

**Proof.** Consider a sequence of $3c$ consecutive characters $w_i \cdots w_{i+3c-1}$ in $w$. With probability $\frac{1}{3}$, the letter $w_j$ with minimum hash will be in the middle $c$ out of the $3c$ letters; that is, we will have $i + c \leq j < i + 2c$. Should this occur, then $w_j$ will be a marker, since its hash will be smaller
than any of the $c$ letters to its left or right. Hence with probability at least $\frac{1}{3}$, the sequence of $3c$ consecutive letters will contain a marker.

Now consider $3cm$ consecutive characters of $w$. If we break these characters into $m$ disjoint chunks of $3c$ letters, then the argument above demonstrates that the probability of the no chunk containing a marker is at most $\left(\frac{2}{3}\right)^m$ (by independence of $h$).

In order to bound the distortion of $\phi$, we begin by considering the case where $x$ and $y$ differ by a single insertion.

**Lemma 6.8.** Let $m \in \mathbb{N}$. Let $x \in \mathcal{P}_{\leq n}$ be a permutation and $y \in \mathcal{P}_{\leq n}$ be a permutation that is derived from $x$ by a single insertion. Condition on the fact that $h$ is injective on the letters in $x$. Then, $\Pr[\Delta_{ed}(\phi(x), \phi(y)) > m] \leq \left(\frac{2}{3}\right)^{\Omega(m)}$.

**Proof.** Let $i$ be the index such that $y$ is generated from $x$ by adding a character at index $i$. That is, $x_1 \cdots x_{i-1} = y_1 \cdots y_{i-1}$ and $x_1x_{i+1} \cdots = y_{i+1}y_{i+2} \cdots$.

Whether a letter is a marker is determined entirely by the $c$ letters immediately to its left and right. Thus for any $j < i$, there is a marker at $x_j$ if and only if there is a marker at $y_j$, and for any $j \geq i + c$, there is a marker at $x_j$ if and only if there is a marker at $y_j+1$.

It follows that the blocks containing only characters prior to $x_{i-c-1}$ and $y_{i-c-1}$ are entirely identical in $\phi(x)$ and $\phi(y)$. This is because, in general, any edit which modifies only markers more than one character to the right of a block will not affect that block.

Moreover, it follows that if there is a marker in some position $x_j$ with $j \geq i + c$, then there will also be a marker in $y_{j+1}$, and the blocks starting from $x_jx_{j+1} \cdots$ will be identical to those starting from $y_{j+1}y_{j+2} \cdots$. By Lemma 6.7 with probability at least $1 - \left(\frac{2}{3}\right)^m$, there is a marker in the $3mc$ letters following $x_{i-c-1}$. Consequently with probability at least $1 - \left(\frac{2}{3}\right)^m$, the blocks in $x_jx_{j+1} \cdots$ will be identical to those in $y_{j+1}y_{j+2} \cdots$ for some $j < i + 3mc + c$.

So far we have shown that with probability at least $1 - \left(\frac{2}{3}\right)^m$, the blocks in $x$ and $y$ are the same except for those intersecting $x_{i-c-1}, \ldots, x_{i+3mc+c-1}$ or $y_{i-c-1}, \ldots, y_{i+3mc+c}$. By Lemma 6.5 it follows that the number of edits needed to transform $\phi(x)$ to $\phi(y)$ is at most $O(m)$, completing the proof.

The preceding lemma can be used to bound $\Delta_{ed}(\phi(x), \phi(y))$ with high probability given that $x$ and $y$ differ by a single insertion. In order to extend the lemma to hold for many edits simultaneously, we must first make an observation about sums of dependent geometric random variables.

**Lemma 6.9.** Let $X_1, X_2, \ldots, X_k$ be (not necessarily independent) random variables over $\mathbb{N}$ satisfying $\Pr[X_i > m] \leq \frac{1}{2m}$ for all $i$ and $m \geq 1$. Then for all $\lambda \in \mathbb{N}$,

$$\Pr\left[\sum_{i=1}^{k} X_i \geq \lambda k\right] \leq \frac{1}{2^{\Omega(\lambda)}}.$$

**Proof.** See Appendix C

In fact, we will need a slightly specialized application of the above lemma.
Lemma 6.10. Let \( p \) be some (small) probability. Let \( X_1, X_2, \ldots, X_k \) be (not necessarily independent) random variables over \( \mathbb{N} \) satisfying \( \Pr[X_i > \lambda] \leq \frac{1}{2^{\Omega(\lambda)}} + \frac{1}{p} \) for all \( i \) and \( \lambda \in \mathbb{N} \). Then for all \( \lambda \in \mathbb{N} \),

\[
\Pr\left[ \sum_{i=1}^{k} X_i > k \lambda \right] \leq \frac{1}{2^{\Omega(\lambda)}} + \frac{2k}{p}.
\]

Proof. See Appendix C.

We are now prepared to bound the distortion of our dimension-reduction map \( \phi \).

Lemma 6.11. Select \( p \in \text{poly}(n) \), and pick \( b \) to be a sufficiently large constant. Let \( x \) and \( y \) be permutations in \( \mathcal{P}_{\leq n} \). Then for \( m \in \mathbb{N} \),

\[
\Pr[\Delta_{ed}(\phi(x), \phi(y)) > m \Delta_{ed}(x, y)] \leq \left( \frac{1}{2} \right)^{\Omega(m)} + \frac{1}{p(n)}.
\]

Proof. There is a sequence of \( 2\Delta_{ed}(x, y) + 1 \) permutations \( x^0, x^1, x^2, \ldots, x^{2\Delta_{ed}(x, y)} \) such that \( x^0 = x, x^{2\Delta_{ed}(x, y)} = y \), and each \( x^{i+1} \) is either equal to \( x^i \) or is obtained from \( x^i \) by either a single insertion or a single deletion.\(^{10}\)

By the no-collisions assumption (Lemma 2.3), we can select \( b \) to be a sufficiently large constant such that \( h \) is injective on all of the letters in all of the \( x_i \)'s with probability at least \( 1 - \frac{1}{4np(n)} \).

Therefore, by Lemma 6.8 for any \( x^i, x^{i+1} \), we have

\[
\Pr[\Delta_{ed}(\phi(x^{i+1}), \phi(x^i)) > m] \leq \left( \frac{1}{2} \right)^{\Omega(m)} + \frac{1}{4np(n)},
\]

where the constant in the \( \Omega(m) \) is independent of \( p(n) \).

If we apply Lemma 6.10, then we see that

\[
\Pr\left[ \sum_{i=0}^{2\Delta_{ed}(x, y) - 1} \Delta_{ed}(x^i, x^{i+1}) > 2m \cdot \Delta_{ed}(x, y) \right] \leq \frac{1}{2^{\Omega(m)}} + \frac{4\Delta_{ed}(x, y)}{4n \cdot p(n)} \leq \frac{1}{2^{\Omega(m)}} + \frac{1}{p(n)}.
\]

By the triangle inequality, it follows that

\[
\Pr[\Delta_{ed}(\phi(x), \phi(y)) > m \Delta_{ed}(x, y)] \leq \left( \frac{1}{2} \right)^{\Omega(m)} + \frac{1}{p(n)}.
\]

\( \Box \)

It remains only to bound the run time necessary to compute \( \phi(w) \) for \( w \in \mathcal{P}_{\leq n} \).

Lemma 6.12. Let \( w \in \mathcal{P}_{\leq n} \). Then we can compute \( \phi(w) \) in time \( O(n) \).

\(^{10}\)In order to ensure that each successive string is still a permutation of length at most \( n \), it is useful to perform all of the deletions prior to the insertions.
Proof. This is straightforward except for one subtlety. In order to identify markers, we wish to identify for each window of size $2c+1$ which letter in that window has minimum hash (and whether that minimum is unique). This can be accomplished in linear time (and practically) using the Minimum on a Sliding Window Algorithm [17].

We now present the full proof of Theorem 6.4.

Proof of Theorem 6.4. Lemma 6.12 bounds the run time to $O(n)$. By Lemma 6.5, the lengths of $\phi(x)$ and $\phi(y)$ are at most $O(|x|/c)$ and $O(|y|/c)$. It remains to analyze the distortion of $\phi$. By Lemma 6.6, $\Delta_{ed}(x, y) \leq c \cdot \Delta_{ed}(\phi(x), \phi(y))$. In order to complete the proof, we therefore wish to upper bound $\Delta_{ed}(\phi(x), \phi(y))$ in expectation by $O(\Delta_{ed}(x, y))$. By Lemma 6.11,

$$\Pr[\Delta_{ed}(\phi(x), \phi(y)) > m\Delta_{ed}(x, y)] \leq \left(\frac{1}{2}\right)^{\Omega(m)} + \frac{1}{p(n)},$$

for all $m \in \mathbb{N}$. It follows that

$$\mathbb{E}[\Delta_{ed}(\phi(x), \phi(y))] = \sum_{i=0}^{\infty} \Pr[\Delta_{ed}(\phi(x), \phi(y)) > i]$$

$$\leq \Delta_{ed}(x, y) \left(1 + \sum_{m=1}^{\infty} \Pr[\Delta_{ed}(\phi(x), \phi(y)) > m\Delta_{ed}(x, y)]\right)$$

$$\leq \Delta_{ed}(x, y) \left(1 + \sum_{m=1}^{n} \left(\left(\frac{1}{2}\right)^{\Omega(m)} + \frac{1}{p(n)}\right)\right) [\text{Using } \Delta_{ed}(\phi(x), \phi(y)) \leq n]$$

$$= \Delta_{ed}(x, y) \left(O(1) + \frac{n}{p(n)}\right).$$

For $p(n)$ sufficiently large, this is $O(\Delta_{ed}(x, y))$, as desired.

\[\square\]

6.2 Dimension-Reduction Preliminaries

Before presenting our dimension-reduction map $\phi$ in the general case, we provide preliminary definitions and lemmas.

Definition 6.13. Let $w \in \Sigma^{\leq n}$ be a string. A contiguous substring $a$ in $w$ is a maximally periodic substring (with parameter $c$) if the length of $a$ is at least $8c$, it is periodic with minimum period at most $c$, and if extending $a$ one letter in either the left or the right prohibits $a$ from having a minimum period at most $c$.

Note that in a maximally periodic substring, the repeated substring need not appear an integer number of times.

Lemma 6.14. Suppose $a$ and $b$ are maximally periodic substrings (with parameter $c$) of $w$. If $a \neq b$, then $a$ and $b$ overlap in fewer than $2c$ letters.
Proof. Suppose $a$ and $b$ overlap in at least $2c$ letters. Let $p$ be the minimum period of $a$ and $q$ be the minimum period of $b$. Because $a$ has period $p$, $b$ has period $q$, and the overlap between $a$ and $b$ is of size at least $2c \geq p + q$, Lemma 2.2 tells us that the overlap is $gcd(p, q)$-periodic. Since any $p$ consecutive letters in $a$ completely determine the values of the other letters in $a$, and since the overlap of $a$ and $b$ consists of at least $p$ consecutive letters, it follows that $a$ is $gcd(p, q)$-periodic. Similarly, $b$ is $gcd(p, q)$-periodic. Thus $p = q$.

Since $a$ and $b$ both have minimum period $p$ and have overlap length at least $2c$, it follows that the string union of $a$ and $b$ has minimum period $p$ as well. This contradicts the statement that $a$ and $b$ are maximally periodic substrings of $w$. \hfill \square

Definition 6.15. Let $w \in \Sigma^{\leq n}$ be a string. A contiguous substring $a$ in $w$ is a maximally non-periodic substring (with parameter $c$) if $a$ is a maximal substring of $w$ that does not intersect with any maximally periodic substrings.

Lemma 6.16. Let $w \in \Sigma^{\leq n}$ and let $a$ be a maximally non-periodic substring (with parameter $c$). Then, any $c + 1$ subsequent overlapping blocks (i.e., blocks starting at $c + 1$ consecutive letters) of at least $8c$ characters in $a$ are distinct.

Proof. Assume toward contradiction that there are two identical blocks $x, y$ of length at least $8c$ that start at indices $i, j$ (respectively) such that $i < j$ and $j \leq i + c$. Then since $x_k = y_k$ for all $k \leq |x|$, and since $y_k = x_{k+(j-i)}$ for all $k \leq |x| + i - j$, it follows that $x_k = x_{k+(j-i)}$ for all $k \leq |x| + i - j$. Since $j - i \leq c$, this means that $x$ is a $8c$-letter substring with period at most $c$, contradicting the fact that $a$ is a maximally non-periodic substring of $w$. \hfill \square

6.3 The Embedding in the General Case

The embedding in the general case first has to identify all the maximally periodic and non-periodic substrings. Then, it has different rules as to where to place markers in each of the substrings. For maximally non-periodic substrings, looking at overlapping $8c$-letter substrings will result in something which is locally a permutation, and will allow us to place markers similarly to the previous case (the embedding for permutations). For maximally periodic substrings, the period of the substring will determine the markers. The description of the embedding appears as Algorithm 5.

Our algorithm assumes access to an $n$-wise independent family $H$ of hash functions mapping strings of length $8c$ to the set $\{0, 1\}^{b \log n}$. In Section 6.3.3 we will discuss how to implement the algorithm in time $O(n)$, including how to efficiently construct a family $H$ which behaves as $n$-wise independent with high probability. (Because the family $H$ behaves as $n$-wise independent only with high probability, the final algorithm differs slightly from Algorithm 5, but maintains the original algorithm’s high-probability results.)

In order to prove Theorem 6.3, we divide the analysis into three parts. First, we bound the size of the output strings $\phi(x), \phi(y)$ (Subsection 6.3.1). Second, we bound the distortion in Subsection 6.3.2. We show that $\frac{1}{8} \Delta_{ed}(x, y) \leq \Delta_{ed}(\phi(x), \phi(y))$, and then we show that the probability of $\Delta_{ed}(\phi(x), \phi(y))$ being greater than $\Delta_{ed}(x, y)$ by a factor of $m$ decreases exponentially in $m$; this is the most difficult step in the analysis. Finally, we demonstrate how to construct $\phi(x)$ in time $O(n \log c)$ (Subsection 6.3.3). Combining these, we are able to prove Theorem 6.3 in Subsection 6.3.4.
Algorithm 5 Dimension Reduction for Strings with Edit Distance

Input: a string \( w \in \Sigma^{\leq n} \), a positive even integer \( c \), and a constant parameter \( b \in \mathbb{N} \).

1. Select \( h \) at random from a family \( \mathcal{H} \) of \( n \)-wise independent hash functions mapping \( \Sigma^{8c} \) to \( \{0, 1\}^{b \log n} \).

2. Find all maximally periodic substrings (with parameter \( c \)) in \( w \).

3. For each maximally periodic substring \( a \) that was found in the previous step:
   (a) Let \( p \) be the minimum period of \( a \) and \( s \) be the smallest multiple of \( p \) satisfying \( s \geq c \).
   (b) Let \( a_t \) be the first letter in \( a \) such that \( a_t a_{t+1} \cdots a_{t+p-1} \) is the lexicographically smallest cyclic shift of the \( p \)-letter substring that \( a \) repeats. (Note that it appears at least once in \( a \) since \( |a| \geq 8c \) and \( p \leq c \).)
   (c) Place markers at each of the letters \( a_t, a_t+s, a_t+2s, \ldots \) within the maximally periodic substring \( a \). Additionally, place a marker after the final letter of \( a \).

4. For each maximally non-periodic substring \( x = x_1 \cdots x_d \):
   (a) Define a word \( y \) of length \( d - 8c + 1 \) such that \( y_i = x_i x_{i+1} \cdots x_{i+8c-1} \). Notice that the letters of \( y \) are substrings of \( x \).
   (b) For each \( i = \frac{c}{2} + 1, \ldots, d - 8c + 1 - \frac{c}{2} \), place a marker at \( x_i \) if, \( h(y_i) < h(y_j) \) for all \( j \neq i \) satisfying \( i - \frac{c}{2} \leq j \leq i + \frac{c}{2} \). That is, place a marker at \( x_i \) if \( h(y_i) \) is the unique minimum out of \( h(y_{i - \frac{c}{2}}), \ldots, h(y_{i + \frac{c}{2}}) \).

5. Partition \( w \) into blocks in the following way: Each marker indicates the beginning of a block. For each block of size at least \( c \) in a maximally non-periodic substring, subdivide it into blocks of size exactly \( c \), except for the last one which will be of size at most \( c \).

6. Return a string over the alphabet \( \Sigma^* \) (i.e., whose letters are strings in \( \Sigma \)) where each block from the preceding step is a letter.

6.3.1 Bounding the Length of the Output Strings

Lemma 6.17. Let \( x \in \Sigma^{\leq n} \). When computing \( \phi(x) \), at most 5 markers are placed in any given \( c/2 \) consecutive letters of \( x \).

Proof. Consider some \( c/2 \) consecutive letters \( a = a_1 \cdots a_{c/2} \) of \( x \). The only way for two markers placed by a maximally periodic substring to be within the same \( c/2 \) positions is for one of those markers to be after the end of the periodic substring. Thus any maximally periodic substring intersecting \( a \) contributes at most two markers. By Lemma [6.14] the letters in \( a \) can appear in at most two maximally periodic substrings, which therefore contribute at most four markers.

On the other hand, we claim that at most one marker can be placed in a given \( c/2 \) letters by maximally non-periodic substrings. Indeed, if markers \( w_i \) and \( w_j \) are placed in the same \( c/2 \) letters, then we get that for the corresponding \( y_i, y_j \), we have \( h(y_i) < h(y_j) \) and \( h(y_j) < h(y_i) \), a contradiction.
Corollary 6.18. Let $w \in \Sigma^{\leq n}$. When computing $\phi(w)$, the total number of blocks that intersect any $mc$-letter substring is at most $O(m)$. Hence $|\phi(w)| \leq O(n/c)$.

Proof. By the preceding lemma, every chunk of $c/2$ letters contains at most 5 markers, bounding the number of markers to $10m \in O(m)$.

After markers are placed, blocks between markers in non-periodic substrings may be further partitioned. This further partitioning can be seen as adding a new sub-marker every $c$ letters until the next marker. Note that this process never places two sub-markers in the same $c/2$ letters, bounding the number of submarkers in our $mc$-letter substring to $2m$.

Therefore, the total number of blocks that intersect the $mc$-letter substring is $O(m)$.

6.3.2 Bounding the Distortion

Observation 6.19. Each block is of size at most $c$ within maximally non-periodic substrings, and is size at most $2c - 1$ within maximally periodic substrings. Blocks that start in a maximally non-periodic substring and end in a maximally periodic substring are also of size at most $2c - 1$.

Lemma 6.20. For any $x, y$, $\Delta_{ed}(x, y) \leq 2c\Delta_{ed}(\phi(x), \phi(y))$.

Proof. The proof follows exactly as that of Lemma 6.13 except with $c$ replaced with $2c$.

Lemma 6.21. Consider $W \in \mathbb{N}$. Let $x$ be a string of length $m \cdot 2W$ such that any contiguous substring of length at most $2W$ is a permutation. Consider an $|x|$-wise independent hash function $h$, and condition on $h$ having no collisions on the letters of $x$. For $W < i \leq |x| - W$, we say that $x_i$ is a minimal letter if $h(x_i) = \min_{i-W \leq j \leq i+W} h(x_j)$. Then,

$$\Pr[x \text{ does not contain a minimal letter}] \leq \left(\frac{1}{2}\right)^{m-2}.$$ 

Proof. Break $x$ into subwords $x^1, x^2, \ldots, x^{2m}$ of size $W$. Let $h_{\min}(x^i)$ denote the minimum hash of the letters in $x^i$. Because each window of $2W$ letters is a permutation in $x$, $x^i$ and $x^{i+1}$ are supported by disjoint sets of letters for all $i$. Therefore $h_{\min}(x^i) \neq h_{\min}(x^{i+1})$ for all $i$.

We begin by analyzing the structure of $x$ under the condition that $x$ contains no minimal letters. Suppose there is some $s$ such that $h_{\min}(x^{s-1}) > h_{\min}(x^s)$, and pick a minimal such $s$. Then in order so that $x^s$ not contain a minimal letter, it must be that either $s = 2m$ or that $h_{\min}(x^s) > h_{\min}(x^{s+1})$. Continuing like this, we see that

$$h_{\min}(x^{s-1}) > h_{\min}(x^s) > h_{\min}(x^{s+1}) > \cdots > h_{\min}(x^{2m}). \quad (6.3)$$

Moreover, since $s$ is the minimal $s$ satisfying $h_{\min}(x^{s-1}) > h_{\min}(x^s)$, we must have

$$h_{\min}(x^1) < h_{\min}(x^2) < h_{\min}(x^3) < \cdots < h_{\min}(x^{s-1}). \quad (6.4)$$

Notice that if there does not exist an $s$ for which $h_{\min}(x^{s-1}) > h_{\min}(x^s)$, then (6.4) holds for $s = 2m + 1$. Therefore, as long as $x$ contains no minimal letters, then (6.3) and (6.4) hold for some $s \in \{2, \ldots, 2m + 1\}$.

To complete the proof, we will show that, given an arbitrary $x$, the probability of (6.3) and (6.4) holding for some $s$ is at most $1/2^{m-2}$. If we consider the cases of $s \leq m + 1$ and $s > m + 1$ separately, then it suffices to prove in each case that the probability of (6.3) and (6.4) holding is
at most $1/2^{m-1}$. Since the problem is symmetrically defined for these two cases, we will consider only $s \leq m + 1$. In particular, we will show that for a given $x$,

$$
\Pr \left[ h_{\min}(x^{m+1}) > h_{\min}(x^{m+2}) > h_{\min}(x^{m+3}) > \cdots > h_{\min}(x^{2m}) \right] \leq \left( \frac{1}{2} \right)^{m-1} \tag{6.5}
$$

Define $a_i$ to be the number of letters appearing in $x^{m+i}$ but not in any of $x^{m+1}, \ldots, x^{m+i-1}$. Define $b_i = \sum_{j=1}^{i} a_i$ to be the number of distinct letters appearing in $x^{m+1}, x^{m+2}, \ldots, x^{m+i}$. Suppose that

$$
h_{\min}(x^{m+1}) > h_{\min}(x^{m+2}) > h_{\min}(x^{m+3}) > \cdots > h_{\min}(x^{m+i-1}) \tag{6.6}
$$

Then in order for $h_{\min}(x^{m+i})$ to be smaller than $h_{\min}(x^{m+i-1})$, some letter $l$ in $x^{m+i}$ not appearing in any earlier $x^{m+j}$ must have the smallest hash out of the letters in $x^{m+1}, x^{m+2}, \ldots, x^{m+i}$. For a given $l \in x^{m+i}$ not appearing in any earlier $x^{m+j}$, the probability of $l$’s hash being smallest is $1/b_i$. By the union bound, it follows that the probability of $h_{\min}(x^{m+i}) < h_{\min}(x^{m+i-1})$ given (6.6) is at most $a_i/b_i$.

Hence the probability of (6.5) occurring for a given $x$ is no greater than

$$
\prod_{i=2}^{m} \frac{a_i}{b_i} = \prod_{i=2}^{m} \frac{a_i}{\sum_{j=1}^{i} a_i}.
$$

Because each window of $2W$ letters in $x$ is a permutation, $a_1$ and $a_2$ are both $W$. This gives us

$$
\prod_{i=2}^{m} \frac{a_i}{2W + \sum_{j=3}^{i} a_i} \leq \prod_{i=2}^{m} \frac{a_i}{2W} \leq \left( \frac{1}{2} \right)^{m-1},
$$

as desired. \hfill \square

The above claim allows us to bound the probability of our algorithm failing to place any markers in a long substring of consecutive characters.

**Lemma 6.22.** Consider $w \in \Sigma^{\leq n}$. Consider a $(m + 10) \cdot c$ letter substring $u$ of $w$, and condition on $h$ being injective on the $8c$-letter contiguous substrings of $u$. (Here, $h$ is as defined in Algorithm 3.) Then the algorithm places a marker in $u$ with probability at least $1 - \left( \frac{1}{7} \right)^{m-2}$.

**Proof.** Consider a substring $u$ of $(m + 10) \cdot c$ consecutive letters. If $u$ intersects a maximally periodic substring by at least $2c$ letters, or contains the letter after a maximally periodic substring, then $u$ must contain a marker from that maximally periodic substring. Thus in order for $u$ to be at risk of not containing a marker, all but the final $2c - 1$ letters of $u$ must be elements of the same maximally non-periodic substring.

Thus we need only consider the case where $u$ contains at least $(m + 8) \cdot c$ consecutive letters all in the same maximally non-periodic substring. Let $v$ be the string of $y$-values corresponding to the first $mc$ of those $(m + 8)c$ consecutive letters. Because the letters are in a maximally non-periodic substring, each consecutive $c$ $y$-values in $v$ are distinct (by Lemma 6.16). By the assumption that $h$ is injective on the $8c$-character substrings of $u$, we have that $h_k$ is injective on the letters of $v$. By Lemma 6.21, with probability at least $1 - \left( \frac{1}{7} \right)^{m-2}$, there is a minimal $y$ value in $v$, which leads to a marker being placed by the algorithm at the corresponding letter in $u$. \hfill \square
Lemma 6.23. Let \( m \in \mathbb{N} \). Let \( x \) be a string and \( y \) be a string that is derived from \( x \) by a single insertion before letter \( x_i \). Condition on the fact that \( h \) is injective on the \( 8c \)-letter substrings of \( x \) and \( y \).

Then, \( \Pr[\Delta_{ed}(\phi(x), \phi(y)) > m] \leq \left( \frac{1}{2} \right)^{\Omega(m)} \).

Proof. Let \( i \) be the index such that \( y \) is generated from \( x \) by adding a character at index \( i \). That is, \( x_1 \cdots x_{i-1} = y_1 \cdots y_{i-1} \) and \( x_{i+1} \cdots = y_{i+1} y_{i+2} \cdots \).

We claim that for any \( j < i - 16c - c/2 \), there will be a marker at \( x_j \) if and only if there is a marker at \( y_j \). In particular, if \( x_j \) belongs to a maximally periodic substring if and only if \( y_j \) belongs to the same maximally periodic substring, except that the periodic substring containing \( y_j \) may be truncated or extended by the insertion of \( y_i \). (Note that the insertion occurs more than \( 8c \) characters after position \( j \), and thus cannot get rid of the maximally periodic substring entirely.) Moreover, for the same reason, \( x_j \) appears immediately after a maximally periodic substring if and only if \( y_j \) appears immediately after the same maximally periodic substring. Therefore, for \( j < i - 16c - c/2 \), \( x_j \) will be a marker due a maximally periodic substring if and only if \( y_j \) is a marker due to a maximally periodic substring. On the other hand, if both \( x_j \) and \( y_j \) belong to maximally non-periodic substrings, then the non-periodic substring containing \( x_j \) will extend to position \( x_{j+8c+c/2} \) if and only if the non-periodic substring containing \( y_j \) extends to position \( y_{j+8c+c/2} \). (Here, we are using that the insertion occurs more than \( 8c \) characters after position \( j + 8c + c/2 \) and thus cannot affect for letters in positions \( \leq j + 8c + c/2 \) whether they are contained in a maximally non-periodic substring). Therefore, for \( j < i - 16c - c/2 \), given that \( x_j \) and \( y_j \) are contained in maximally non-repetitive substrings, one will be a marker if and only if the other is as well.

Now, we claim that the blocks that contain only characters before \( x_{i-16c-c/2-1} \) or \( y_{i-16c-c/2-1} \) are identical and appear in both \( \phi(x) \) and \( \phi(y) \). This is because, in general, any edit which modifies only markers more than one character to the right of a block will not affect that block.

In order to examine which blocks after the insertion are affected by the insertion, we consider two cases.

Case 1: \( x_i \) does not belong to a maximally periodic substring that extends until at least \( x_{i+8c-1} \). In turn, this means that \( y_{i+1} \) does not belong to a maximally periodic substring that extends until at least \( y_{i+8c} \), since otherwise, that periodic substring would still be present in \( x \) beginning at \( x_i \). Therefore, the maximally periodic substrings in \( x \) which intersect \( x_{i+8c-1} x_{i+8c} \cdots \) are identical to those in \( y \) which intersect \( y_{i+8c} y_{i+8c+1} \cdots \). It follows that any for \( j \geq i + 8c \), \( x_j \) will be a marker due a maximally periodic substring if and only if \( y_{j+1} \) is a marker due to a maximally periodic substring. Moreover, it follows that for \( j \geq i + 8c \), \( x_j \) will be a member of a maximally non-periodic substring if and only if \( y_{j+1} \) is a member of a maximally non-periodic substring. Therefore, for \( j \geq i + 8c + c/2 \), \( x_j \) will be a marker in a maximally non-periodic substring if and only if \( y_{j+1} \) is a marker in a maximally non-periodic substring.

So far we have established that for \( j \geq i + 8c + c/2 \), \( x_j \) is a marker if and only if \( y_{j+1} \) is a marker. Recalling by assumption that \( h \) is injective on the \( 8c \)-letter substrings of \( x \) and \( y \), by Lemma 6.22 with probability at least \( 1 - \left( \frac{1}{2} \right)^{m-2} \), there will be a marker in the \( (m+10) \cdot c \) characters after \( x_{i+8c+c/2} \) (if such characters exist). Assume that this marker is located at \( x_j \). Then a marker will also be placed at \( y_{j+1} \), and since all of the markers to the right of \( x_j \) are identical to those to the right of \( y_{j+1} \), all the blocks that contain the characters starting from \( x_j \) or \( y_{j+1} \) will be identical in \( \phi(x) \) and \( \phi(y) \). We get that with probability at least \( 1 - \left( \frac{1}{2} \right)^{m-2} \), only the blocks that contain the characters from \( x_{i-16c-c/2-1} \cdots x_{i+(m+18+1/2)c} \) or \( y_{i-16c-c/2-1} \cdots y_{i+(m+18+1/2)c+1} \) can differ
between $\phi(x)$ and $\phi(y)$. By Corollary 6.18 there are at most $O(m)$ blocks that need to be edited in order to transform $\phi(x)$ to $\phi(y)$.

**Case 2:** $x_i$ belongs to a maximally periodic substring that extends until at least $x_i+8c-1$. By Lemma 6.14 there is only a single maximally periodic substring $u$ containing both $x_i$ and $x_i+8c-1$. Because $u$ contains at least $8c$ letters to the right of the insertion, those $8c$ characters will still be part of a maximally periodic substring in $y$, and the endpoint of the maximally periodic substring will be unchanged. Consequently, if we denote by $y_{i^*}$ the last character of $u$, then $y_{i^*+1}$ will be the last character in $y$ which is contained in a maximally periodic substring also containing $y_{i+1}$. As a result, the maximally periodic substrings and the maximally non-periodic substrings in $x$ which intersect the letters to the right of $x_{i^*}$ are identical to those in $y$ which intersect the letters to the right of $y_{i^*+1}$. Thus the markers to the right of $x_{i^*}$ will be identical to those to the right of $y_{i^*+1}$.

Moreover, there will be a marker at each of $x_{i^*+1}$ and $y_{i^*+2}$ because they each marks the end of a maximally periodic substring. Therefore, the blocks beginning at $x_{i^*+1}$ will be identical to those beginning at $y_{i^*+2}$.

Now we consider letters between $x_i$ and $x_{i^*}$ and the letters between $y_i$ and $y_{i^*}$ (inclusive). Note that $x_i \ldots x_{i^*} = y_{i+1} \ldots y_{i^*+1}$, and both are suffixes of maximally periodic substrings with the same periods as each other. Recall that $u$ is the maximally periodic substring of which $x_i \ldots x_{i^*}$ is a suffix; and let $v$ denote the maximally periodic substring of which $y_{i+1} \ldots y_{i^*+1}$ is a suffix. By Lemma 6.14 the only markers in the substrings $x_{i+2c} \ldots x_{i^*−2c}$ and $y_{i+2c+1} \ldots y_{i^*−2c+1}$ are due to $u$ and $v$, respectively (and not some other periodic substring overlapping $u$ or $v$). Denote by $j_1$ the index of the first marker placed in $x_{i+2c} \ldots x_{i^*−2c}$, and by $j_2^*$ the index of the first marker placed in $y_{i+2c+1} \ldots y_{i^*−2c+1}$. Additionally, denote by $j_2$ the index of the last marker placed before or at $x_{i^*−2c}$ and by $j_2^*$ the index of the last marker placed before or at $y_{i^*−2c+1}$.

Each of the blocks in $x$ between $x_{j_1}$ and $x_{j_2}$ and each of the blocks in $y$ between $y_{j_1^*}$ and $y_{j_2^*}$ are identical.\(^{11}\) With this in mind, we propose the following sequence of edits from $\phi(x)$ to $\phi(y)$:

1. The blocks that contain only letters before $x_{i−16c−c/2−1}$ or $y_{i−16c−c/2−1}$ are identical, and therefore incur no edits.

2. There are $O(c)$ characters in the subwords $x_{i−16c−c/2−1} \ldots x_{j_1−1}$ and $y_{i−16c−c/2−1} \ldots y_{j_1−1}$. By Corollary 6.18 the blocks containing characters from the subword $x_{i−16c−c/2−1} \ldots x_{j_1−1}$ can be transformed into the blocks containing characters from the subword $y_{i−16c−c/2−1} \ldots y_{j_1−1}$ in $O(1)$ operations.

3. The blocks that correspond to $x_{j_1} \ldots x_{j_2−1}$ and $y_{j_1^*} \ldots y_{j_2^*−1}$ are identical, except that one of these substrings may include $O(1)$ additional blocks (since the blocks in the maximally periodic substring $u$ may appear at different offsets than the blocks in $v$). Thus we can map the blocks in these two substrings to each other in $O(1)$ operations.

4. There are $O(c)$ characters in the subwords $x_{j_2} \ldots x_{i^*}$ and $y_{j_2} \ldots y_{i^*+1}$. By Corollary 6.18 the blocks containing characters from the subword $x_{j_2} \ldots x_{i^*}$ can therefore be transformed into the blocks containing characters from the subword $y_{j_2} \ldots y_{i^*+1}$ in $O(1)$ operations.

5. The blocks beginning at $x_{i^*+1}$ are identical to those beginning at $y_{i^*+2}$, requiring no additional edits.

\(^{11}\)In particular, if $p$ is the period of the maximally periodic substrings containing the blocks, then the length of each block is the smallest multiple $s$ of $p$ satisfying $s \geq c$; and the contents of each block is oriented to begin on the lexicographically smallest cyclic shift of the $p$-letter substring being repeated.
The total number of edits needed to transform \( \phi(x) \) to \( \phi(y) \) in this case is \( O(1) \).

In conclusion, in either case, the total number of edits is \( O(m) \) (with probability at least \( 1 - \left( \frac{1}{2} \right)^{m-2} \)), as desired. \( \square \)

We are now prepared to bound the distortion of our dimension-reduction map \( \phi \).

**Lemma 6.24.** Let \( x \) and \( y \) be strings. Then for \( m \in \mathbb{N} \),

\[
\Pr[\Delta_{\text{ed}}(\phi(x), \phi(y)) > m \Delta_{\text{ed}}(x, y)] \leq \left( \frac{1}{2} \right)^{\Omega(m)} + \frac{1}{\text{poly}(n)},
\]

where \( \text{poly}(n) \) is an arbitrarily large polynomial in \( n \) of our choice controlled by the constant \( b \).

**Proof.** The proof follows exactly as that of Lemma 6.11 for permutations. The only difference is that here Lemma 6.23 should be used in place of 6.8. \( \square \)

### 6.3.3 Bounding the Run Time by \( O(n \log c) \)

In this section, we discuss how to implement our dimension reduction algorithm in time \( O(n \log c) \).

We first discuss how to identify maximally periodic substrings. The following lemma allows us to quickly detect whether a 4c-letter string is periodic with a small period.

**Lemma 6.25.** Let \( a = a_1 \cdots a_{4c} \) be a string of length 4c. We can detect whether \( a \) is periodic with a period \( \leq c \) in time \( O(|a|) \). Additionally, if \( a \) is periodic, we return its (minimum) period \( p \).

**Proof.** Consider the suffix array \( A \) of \( a \). The \( i \)-th entry of \( A \) is the \( i \)-th lexicographically smallest suffix of \( a \). It is well known that \( A \) can be constructed in time \( O(|a|) \) [14].

Suppose \( a \) is periodic with minimum period \( p \leq c \). Then

\[ a_1 \cdots a_{4c-p} = a_{p+1} \cdots a_{4c}. \]

Moreover, we claim that, out of all the proper suffixes of \( a \), \( a_{p+1} \cdots a_{4c} \) has the longest common prefix with \( a \). Indeed suppose there is some \( 1 \leq q < p \) such that \( a_1 \cdots a_{4c-p} = a_{q+1} \cdots a_{4c-p+q} \). Then the first \( 4c-p \) letters of \( a \) are both \( p \)-periodic and \( q \)-periodic; by Lemma 2.2, since \( 4c-p \geq p+q \), it follows that the first \( 4c-p \) letters of \( a \) are \( \gcd(p, q) \)-periodic. Since all of \( a \) is \( p \)-periodic, it follows that all of \( a \) is \( \gcd(p, q) \)-periodic, contradicting that \( p \) is the minimum period of \( a \). Thus no such \( q \) exists, meaning that out of all the proper suffixes of \( a \), \( a_{p+1} \cdots a_{4c} \) has the longest common prefix with \( a \). It follows that \( a_{p+1} \cdots a_{4c} \) will appear adjacently to \( a_1 a_2 \cdots a_{4c} \) in the suffix array \( A \) (and will, in fact, appear right before \( a_1 a_2 \cdots a_{4c} \)).

Thus, in the event that \( a \) is periodic with minimum period \( p \leq c \), we can determine \( p \) by examining the element in \( A \) preceding \( a_1 a_2 \cdots a_{4c} \). Therefore, the following algorithm can be used to determine whether \( a \) has minimum period \( \leq c \): Define \( p+1 \) to be the index in \( a \) of the first letter of the element of \( A \) preceding \( a_1 a_2 \cdots a_{4c} \); check whether \( p \leq c \) and whether \( a \) is \( p \)-periodic; if \( a \) is, return \( p \), and otherwise return that \( a \) is not periodic with period \( \leq c \). \( \square \)

Armed with the preceding lemma, we can identify the maximally periodic substrings of \( w \) in time \( O(n) \).

**Proposition 6.26.** The maximally periodic substrings of \( w \) can be identified in time \( O(n) \).
Proof. We identify the maximally periodic substrings as follows. Break w into disjoint chunks $u_1, \ldots, u_m$ of size $4c$ (ignoring up to $4c-1$ letters at the end). Because maximally periodic substrings are length $\geq 8c$, each maximally periodic substring must entirely contain some chunk $u_i$. Therefore, we can detect the maximally periodic substrings as follows. For $i$ from 1 to $m$, determine whether $u_i$ is periodic with period $\leq c$, and if it is let $p$ be its period; we can do this in time $O(|u_i|) = O(c)$ per $u_i$ by Lemma 6.25. If $u_i$ is part of a larger maximally periodic substring, then that substring must also have period $p$. Thus we can detect whether $u_i$ is part of a maximally periodic string by extending $u_i$ to the left and right as far as we can go while still maintaining period $p$ (This is easy to do in time proportional to the length of the resulting extended $u_i$). If the result is length $\geq 8c$, then we have identified a maximally periodic substring. If we identify a maximally periodic substring, then we move on to the first $u_j$ with $j > i$ such that $u_j$ is not contained in the maximally periodic substring; otherwise, we move on to $u_{i+1}$ and repeat the process.

For each of the $\lfloor n/(4c) \rfloor u_i$’s we spend $O(c)$ work finding the period $p$ of $u_i$ (if there is one). We then either spend time $O(c)$ extending $u_i$ to a periodic string of length $< 8c$, or we extend $u_i$ to a maximally periodic substring $x$ in time $O(|x|)$. The total time spent on $u_i$’s which don’t yield maximally periodic substrings is $O(c \cdot c) = O(n)$. Since we spend $O(|x|)$ time successfully finding each maximally periodic substring $x$, the total time spent extending $u_i$’s to maximally periodic substrings is $O(n)$. Therefore, the total time spent to detect the maximally periodic substrings is $O(n)$.

For each maximally periodic substring $x$ with period $p$, we need to determine the offset $i \in [p]$ which minimizes $x_i x_{i+1} \cdots x_{i+p-1}$ lexicographically. This can be performed in time $O(c)$ by building the suffix array $A$ for the substring $x_1 x_2 \cdots x_{2p-1}$, and then comparing the positions within the suffix array of $x_i x_{i+1} \cdots x_{2p-1}$ for each $i \in [1, p]$. Thus we can complete the third step of the algorithm for all of the maximally periodic substrings in time $O(n)$.

So far we have explained how to determine the markers in maximally periodic strings in time $O(n)$. In order to show that the entire algorithm can be performed in time $O(n \log c)$, we must show how to efficiently determine the markers within maximally non-periodic strings. The main difficulty in this is computing the hashes of each of the $8c$-letter substrings. In particular, once these are computed, the markers can be determined using the Minimum on a Sliding Window Algorithm as in the proof of Lemma 6.12.

Therefore, we devote the remainder of the section to constructing a family $H$ of hash functions from $\Sigma^{8c}$ to $\{0, 1\}^{b \log n}$ such that with high probability $H$ behaves as $n$-wise independent (that is, for any set $A$ of size $n$, with high probability the family $H$ is independent on $A$), and such that $h \in H$ can be evaluated on every $8c$-letter substring of $w$ in total time $O(n \log c)$.

Recall from [28] that in time $O(n)$ one can construct a family $S$ of hash functions from $\{0, 1\}^{\Theta(\log n)}$ to $\{0, 1\}^{b \log n}$ such that with high probability, $S$ behaves as $n$-wise independent, and such that $s \in S$ can be evaluated in constant time. It therefore suffices to construct a family $G$ of hash functions mapping $\Sigma^{8c}$ to $\{0, 1\}^{\Theta(\log n)}$ so that with high probability, $g \in G$ is injective on the $8c$-letter substrings of $w$. Given such a family $G$, we can then define $h = g \circ s$ in order to obtain a family $H$ which behaves as $n$-wise independent from $\Sigma^{8c}$ to $\{0, 1\}^{b \log n}$ with high probability.

For convenience, we will assume that $8c$ is a power of two, though the argument can easily be

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12Note that this offset is unique. In particular, if $x_i \cdots x_{i+p-1} = x_j \cdots x_{j+p-1}$ for $i < j \leq p$ then it follows that $x_i \cdots x_{j+p-1}$ is $(j-i)$-periodic. By Lemma 2.2 we get that $x_i \cdots x_{j+p-1}$ is gcd($p, j-i$)-periodic. Thus $x$ is gcd($p, j-i$)-periodic, a contradiction.
adapted to the more general setting.\footnote{This is done by first hashing the windows of size \(2^{\lceil \log 8c \rceil}\), and then computing a hash of each window of size \(8c\) by hashing together the hash for the overlapping windows of size \(2^{\lceil \log 8c \rceil}\) which constitute the larger window of size \(8c\).} Our construction relies on the idea of a Merkle tree. An element \(g\) of \(G\) is determined by \(\log 8c + 1\) pairwise independent hash functions \(f_0, f_1, f_2, \ldots, f_{\log 8c}\), with each \(f_i\) mapping \(2r \log n\) bits to \(r \log n\) bits (where \(r\) is a constant of our choice). Given these \(f_i\) functions, we recursively define a function \(g^*\) such that for a string \(a = a_1 \cdots a_{2^k}\) (with \(2^k \leq 8c\)) we have that \(g^*(a) = f_k(g^*(a_1 \cdots a_{2^{k-1}}), g^*(a_{2^{k-1}} + 1 \cdots a_{2^k}))\) when \(k \geq 1\) and by \(g^*(a) = f_0(a_1)\) when \(k = 0\). Finally, we define our function \(g\) by \(g(a) = g^*(a)\) for strings of length \(8c\). The next lemma establishes that \(g \in G\) is injective on the \(8c\)-letter substrings of \(w\) with high probability, as desired.

**Lemma 6.27.** Suppose \(r\) is sufficiently large (remember each \(f_i\) maps to \(r \log n\) bits), and let \(g\) be selected at random from the family \(G\) described above. Then \(g\) is injective on the \(8c\)-letter substrings of \(w\) with probability \(1 - \frac{1}{\text{poly}(n)}\).

**Proof.** Let \(u, v \in \Sigma^{8c}\) be distinct. By Lemma 2.3 if we select \(r\) large enough, then with probability at least \(1 - \frac{1}{\text{poly}(n)}\) for a polynomial of our choice, there will be no collisions in any of the applications of hash function \(f_0\) when computing \(g(u)\) and \(g(v)\). The same holds for each of the other \(f_i\)'s. In particular, if there are no collisions when we apply \(f_{i-1}\), then \(f_i\) will be fed distinct inputs and with high probability will also have no collisions (this is why we use a different hash function at each level). By the union bound, it follows that with probability \(1 - \frac{1}{\text{poly}(n)}\), \(g^*(u) \neq g^*(v)\).

Applying the union bound again to all pairs \(u, v\) of \(8c\)-letter substrings of \(w\), it follows that with probability \(1 - \frac{1}{\text{poly}(n)}\), \(g\) is injective on them. \(\square\)

By defining \(G\) in this way, we are able to quickly compute all of the \(8c\)-letter window hashes using dynamic programming.

**Theorem 6.28.** Let \(g \in G\) be selected at random. Suppose \(a_1 \cdots a_m\) is a string over an alphabet \(\Sigma\). Then we can compute \(g(a_i \cdots a_{i+8c-1})\) for all \(i\) in time \(O(m \log c)\).

**Proof.** This is a simple application of dynamic programming. In particular, we perform \(O(\log c)\) iterations. In the \(i\)-th iteration we compute the \(g^*\) hashes of all the substrings of length \(2i-1\) (allowing overlap). Using the results of the previous iteration, each iteration takes time \(O(m)\). Thus the total run time is \(O(m \log c)\). \(\square\)

### 6.3.4 Proof of Theorem 6.3

We are now prepared to prove that \(\phi\) is a dimension-reduction map satisfying the properties claimed in Theorem 6.3.

**Proof.** Consider two strings \(x, y \in \Sigma^n\). By Corollary 6.18 \(\phi(x)\) and \(\phi(y)\) are lengths \(O(|x|/c)\) and \(O(|y|/c)\). By Lemma 6.20 \(\Delta_{\text{ed}}(x, y) \leq 2c \cdot \Delta_{\text{ed}}(\phi(x), \phi(y))\). By Lemma 6.24 for \(m \in \mathbb{N}\),

\[
\Pr[\Delta_{\text{ed}}(\phi(x), \phi(y)) > m \Delta_{\text{ed}}(x, y)] \leq \left(\frac{1}{2}\right)^{\Omega(m)} \frac{1}{\text{poly}(n)},
\]

for all \(m \in \mathbb{N}\) and for a polynomial \(\text{poly}(n)\) of our choice. Exactly as in the proof of Theorem 6.4, as long as we pick \(\text{poly}(n)\) sufficiently large, it follows that \(\mathbb{E}[\Delta_{\text{ed}}(\phi(x), \phi(y))] \leq O(\Delta_{\text{ed}}(x, y))\), and
subsequently the expected distortion of $\phi$ is $O(c)$. Finally, the discussion in Section 6.3.3 establishes that $\phi(x)$ and $\phi(y)$ can each be computed in time $O(n\log c)$, as desired.

6.4 An Approximation Algorithm

Prior to the approximation algorithms of [4] and [3], the state of the art for approximating edit distance was the algorithm of [8], which obtained distortion $\tilde{O}\left(\min\left(n^{1/3} + \frac{2}{3\log \log \log n}, \Delta_{ed}(x,y)^{1/2} + \frac{1}{\log \log \log n}\right)\right)$. Moreover, prior to our results in Section 3, the algorithm of [8] remained the best approximation algorithm for the approximate alignment problem.

The approximation algorithm of [8] is built on top of the dimension-reduction map presented in the same paper. Consequently, the improvements in dimension-reduction distortion presented in this section can be used to improve the approximation algorithm. In particular, the algorithm can be updated to achieve distortion $O\left(\min\left(n^{1/3}, \Delta_{ed}(x,y)^{1/2}\right)\right)$.

In order so that our dimension-reduction map $\phi$ can be used by the algorithm of [8], one must prove a subtle property of $\phi$. Because the property may be useful in future work, we state it here. The proof can be found in Appendix B.

**Lemma 6.29.** Consider $x, y \in \Sigma^{\leq n}$. Pick $m \in \mathbb{N}$. Then with probability $1 - \left(\frac{1}{2}\right)^{\Omega(m)} - \frac{1}{\text{poly}(n)}$ (for a polynomial of our choice), there is a sequence of at most $m\Delta_{ed}(x,y)$ edits from $\phi(x)$ (with parameter $c$) to $\phi(y)$ with the following property. If a block $u$ in $\phi(x)$ is not modified by the edits and is mapped to a block $v$ in $\phi(y)$, then the start position of block $u$ in $x$ is within $O(\Delta_{ed}(x,y) + c)$ of the start position of block $v$ in $y$.

A Transforming High Probability into Expected Distortion for Embeddings into Hamming Space

In this section, we prove Theorem 5.10, which is implicitly present in Section 3.5 of [12]. It takes any high-probability low-edit-distance-regime embedding from edit-distance to Hamming space, and turns it into an embedding with good expected distortion.

We will need the following routine lemma.

**Lemma A.1.** Let $j \in \mathbb{N}$ and $k \in \mathbb{R}$ with $k \geq 2$. Then

$$\frac{j}{4k} \leq 1 - (1 - 1/k)^j \leq \frac{j}{k}.$$

**Proof.** Because $(1 - 1/k)^k$ is an increasing function in $k$ we must have $(1 - 1/k)^k \geq (1 - 1/2)^2 = 1/4$. Therefore, for $i \leq k + 1$, we have that $\frac{1}{i}(1 - 1/k)^{i-1} \geq \frac{1}{4k}$, and thus $(1 - 1/k) \cdot (1 - 1/k)^{i-1} \leq (1 - 1/k)^{i-1} - \frac{1}{4k}$. Applying this repeatedly, it follows that $(1 - 1/k)^j \leq 1 - \frac{j}{4k}$. On the other
hand, when we compute \((1 - 1/k)^j\), each multiplication by \((1 - 1/k)\) subtracts at most \(1/k\) from the previous term, giving that
\[
\frac{j}{4k} \leq 1 - (1 - 1/k)^j \leq \frac{j}{k}.
\]

We now prove Theorem \([5.10]\). It is restated below.

**Theorem A.2** (Restatement of Theorem \([5.10]\)). Let \(S\) be a subset of \(\Sigma^{\leq n}\). Suppose we have a randomized embedding \(\phi : S \to \Sigma^m\) such that for all \(x, y \in S\) satisfying \(\Delta_{ed}(x, y) \leq K\),
\[
\Delta_{ed}(x, y) \leq \frac{1}{T} \text{Ham}(\phi(x), \phi(y)) \leq F(K)\Delta_{ed}(x, y)
\]
with probability at least \(1 - \frac{1}{K \cdot F(K)}\) for some distortion function \(F(K)\) with \(F(K) \geq 2\), for some \(T \geq 1\), and for all \(K \in \mathbb{N}\).

Then there exists \(\alpha : S \to \{0, 1\}\) such that for all \(x, y \in S\) satisfying \(\Delta_{ed}(x, y) \leq K\),
\[
\Omega(\Delta_{ed}(x, y)) \leq K \cdot F(K) \Pr[\alpha(x) \neq \alpha(y)] \leq O(\Delta_{ed}(x, y)F(K)).
\]

In other words, \(\alpha\) is an embedding from edit distance (at most \(K\)) over \(S\) to Hamming distance (scaled) with expected distortion at most \(O(F(K))\).

**Proof of Theorem \([5.10]\).** For \(x \in S\), we define \(\alpha(x)\) as follows. First define \(w = \phi(x)\). Then, use (pairwise independent) hash functions \(h_1, \ldots, h_m\) from \(\Sigma\) to \(\{0, 1\}\) to map \(w\) to a binary string \(w' := h_1(w_1) \cdots h_m(w_m)\). Next, select a binary string \(r \in \{0, 1\}^m\) such that each position in \(r\) takes value one with probability \(\frac{1}{T \cdot K \cdot F(K)}\). Finally, define
\[
\alpha(x) := \langle r, w' \rangle \equiv 2 \left(\sum_i r_i w'_i\right) \mod 2.
\]

Now consider some \(y \in \Sigma^n\) which is distinct from \(x\) such that \(\Delta_{ed}(x, y) \leq K\). Then, following the same progression as for \(x\), we define \(u := \phi(y)\), \(u'\) to be \(u\) with its \(i\)-th letter hashed to \(\{0, 1\}\) by \(h_i\), and \(\alpha(y) := \langle r, u' \rangle \equiv 2\).

We wish to analyze \(\Pr[\alpha(x) \neq \alpha(y)]\). Let \(j\) be the number of positions in which \(w\) and \(u\) differ. There are two cases.

- **Case 1** is that \(\frac{j}{T} \not\in \Delta_{ed}(x, y), F(K) \Delta_{ed}(x, y)\). Because \(w = \phi(x)\) and \(u = \phi(y)\), this occurs with probability at most \(\frac{1}{K \cdot F(K)}\). Therefore, this case affects \(\Pr[\alpha(x) \neq \alpha(y)]\) by an additive factor of at most \(\frac{1}{K \cdot F(K)}\).

- **Case 2** is that \(\Delta_{ed}(x, y) \leq \frac{j}{T} \leq F(K) \Delta_{ed}(x, y)\).

If all of the ones in \(r\) correspond with positions in which \(w\) and \(u\) agree, then we are guaranteed that \(\alpha(x) = \alpha(y)\). In order for this to occur, each of the \(j\) positions in which \(w\) and \(u\) disagree must be assigned a zero in \(r\); this occurs with probability \(\left(1 - \frac{1}{T \cdot K \cdot F(K)}\right)^j\).

On the other hand, with probability \(1 - \left(1 - \frac{1}{T \cdot K \cdot F(K)}\right)^j\), \(r\) takes value one in at least one position in which \(x\) and \(y\) disagree. When this happens, we have \(\Pr[\alpha(x) = \alpha(y)] = \frac{1}{2}\).
Therefore, if we restrict ourselves to Case 2, then

\[ \Pr[\alpha(x) \neq \alpha(y)] = \left(1 - \left(1 - \frac{1}{T \cdot K \cdot F(K)}\right)^j\right) \frac{1}{2}. \]

By Lemma A.1 this implies

\[ \frac{1}{8} \cdot \frac{j}{T \cdot K \cdot F(K)} \leq \Pr[\alpha(x) \neq \alpha(y)] \leq \frac{1}{2} \cdot \frac{j}{T \cdot K \cdot F(K)}. \]

Because \( j \) is between \( T \cdot \Delta_{ed}(x, y) \) and \( T \cdot F(K) \cdot \Delta_{ed}(x, y) \) (inclusive),

\[ \frac{1}{8} \cdot \frac{\Delta_{ed}(x, y)}{K \cdot F(K)} \leq \Pr[\alpha(x) \neq \alpha(y)] \leq \frac{1}{2} \cdot \frac{\Delta_{ed}(x, y)}{K}. \quad (A.1) \]

Since Case (2) occurs with probability at least \( 1 - \frac{1}{K \cdot F(K)} \geq \frac{1}{2} \), we can lower bound \( \Pr[\alpha(x) \neq \alpha(y)] \) by \( \frac{1}{2} \Pr[\alpha(x) \neq \alpha(y) \mid \text{Case 2}] \). This gives us

\[ \frac{1}{16} \cdot \frac{\Delta_{ed}(x, y)}{K \cdot F(K)} \leq \Pr[\alpha(x) \neq \alpha(y)]. \]

Since Case (1) occurs with probability at most \( \frac{1}{K \cdot F(K)} \), we can apply (A.1) to bound \( \Pr[\alpha(x) \neq \alpha(y)] \) above by

\[ \Pr[\alpha(x) \neq \alpha(y)] \leq \frac{1}{K \cdot F(K)} + \frac{1}{2} \cdot \frac{\Delta_{ed}(x, y)}{K}. \]

Multiplying our lower and upper bounds by \( 16K \cdot F(K) \), we get

\[ \Delta_{ed}(x, y) \leq 16K \cdot F(K) \Pr[\alpha(x) \neq \alpha(y)] \leq 16 + 8\Delta_{ed}(x, y)F(K), \]

completing the proof.

\[ \square \]

### B Proof of Lemma 6.29

In this section, we present a proof of Lemma 6.29.

Notice that for \( x, y \in \Sigma^n \), there exists a sequence \( e_1, e_2, \ldots, e_s \) of edits from \( x \) to \( y \) such that \( s \leq 2\Delta_{ed}(x, y) \); such that each edit is an insertion or a deletion; and such that for each \( e_i \) none of the edits \( e_1, e_2, \ldots, e_s \) take place to the right of \( e_i \). Define \( x^0 = x \) and \( x^i \) to be \( x \) after the first \( i \) edits. In particular, \( x^s = y \). A letter in \( x^i \) is \emph{untouched} if it was not involved in any of the edits \( e_1, \ldots, e_i \). For each untouched letter \( l \) in \( x^i \), define \( p(l) \) to be the position in \( x \) where \( l \) originates (prior to the edits).

Given a sequence \( E \) of edits to \( \phi(x^i) \), a block in \( \phi(x^i) \) is \emph{untouched} if \( E \) does not edit it. Moreover, for an untouched block \( a \), we use \( p_E(a) \) to denote the position which the first letter of \( a \) is mapped to by the edits in \( E \).

A sequence of edits \( E \) consisting only of insertions and deletions from \( \phi(x^i) \) to \( \phi(x) \) is called \emph{nearly position preserving} if for each untouched block \( a \) in \( \phi(x^i) \) beginning with an untouched letter
l, we have that \( |p(l) - p_E(a)| \leq 2c \). In other words, the edits in \( E \) map the block \( a \) to a position in \( x \) which is within \( 2c \) of the position in \( x \) to which the edits \( e_1, \ldots, e_t \) map \( l \).

Moreover, for simplicity, we require from a nearly position preserving sequence of edits \( E \) that every block in \( \phi(x^t) \) beginning with a touched letter \( t \) must be touched by \( E \) (though this can be achieved simply by deleting and re-inserting the block). This is simply so that every untouched block \( a \) is guaranteed to have a first letter \( l \) for which \( p(l) \) is defined. Our constructions will satisfy this property trivially.

Finally, define \( d(x^t) \) to be the minimum size of a nearly position preserving sequence of edits from \( \phi(x^t) \) to \( \phi(x) \).

**Lemma B.1.** Let \( m \in \mathbb{N} \) and consider some pair \( x^t \) and \( x^{t+1} \). Condition on the fact that for each \((m + 10) \cdot c \) consecutive letters in \( x^t \), there is some \( h_k \) which is injective on the \( 8c \)-letter substrings.

Then, \( \Pr[d(x^{t+1}) - d(x^t) > m] \leq (\frac{1}{2})^{\Omega(m)} \).

**Proof.** We will consider only the case where \( x^{t+1} \) can be derived from \( x^t \) by an insertion. The case where \( x^{t+1} \) is obtained from \( x^t \) by a deletion follows by the exact same reasoning (except with some indices adjusted by two to account for the fact that they take place after a deletion rather than an insertion).

We will closely follow the proof of Lemma 6.23 with a few small modifications. Let \( E \) be an minimum nearly position preserving sequence of insertions and deletions transforming \( x^t \) to \( x \). In particular, \( |E| = d(x^t) \). Our goal is to show that with probability \( 1 - \frac{1}{2^{\Omega(m)}} \) we can construct a nearly position preserving sequence of \( |E| + O(m) \) edits from \( x^{t+1} \) to \( x \).

Let \( i \) be the index such that \( x^{t+1} \) is generated from \( x^t \) by adding a character at index \( i \). That is, \( x_1 \cdots x_{t-1} = x_1^{t+1} \cdots x_{i-1}^{t+1} \) and \( x_i^{t+1}x_{i+1}^{t+1} \cdots = x_i^{t+1}x_{i+2}^{t+1} \cdots \).

Call a block in \( \phi(x^t) \) (resp. \( \phi(x^{t+1}) \)) a beginning block if it begins before or at \( x_i^t \) (resp. \( x_i^{t+1} \)) and an ending block if it begins after \( x_i^t \) (resp. \( x_i^{t+1} \)). Moreover, we call a block in \( \phi(x) \) a beginning block if it begins before or at \( x_p(x_i^t) \), and an ending block if it begins after \( x_p(x_i^t) \).

Because \( E \) is nearly position preserving, any untouched ending block \( u \) must be mapped by \( E_1 \) to begin in a position at least \( p(x_i^t) - 2c \). Therefore, there is a subset \( E' \) of the edits of \( E \) that maps the beginning blocks in \( \phi(x^t) \) to a prefix of \( x \) containing at least \( x_1 \cdots x_p(x_i^t) - 2c - 1 \). Moreover, because \( E \) is nearly position preserving, any untouched beginning block \( u \) in \( \phi(x^t) \) must be mapped by \( E \) to begin in a position no larger than \( p(x_i^t) + 2c \). Therefore, there is a subset \( E'' \) of \( E' \) mapping the beginning blocks in \( \phi(x^t) \) to a prefix of \( x \) containing at least \( x_1 \cdots x_p(x_i^t) - 2c \) but containing no blocks in \( x \) which begin after \( x_p(x_i^t) + 2c \).

From the logic in the proof of Lemma 6.23 the blocks that contain only characters before \( x_i^{t+1} \) in \( \phi(x^t) \) are identical to those which appear before \( x_i^{t+1} \) in \( \phi(x^{t+1}) \). By Corollary 6.18 it follows that \( O(1) \) deletions and insertions can be used to transform the beginning blocks of \( \phi(x^{t+1}) \) to the beginning blocks of \( \phi(x^t) \). Then using the edits from \( E'' \), we can transform these blocks into a prefix of \( x \) containing at least \( x_1 \cdots x_p(x_i^t) - 2c \) but containing no blocks in \( x \) which begin after \( x_p(x_i^t) + 2c \). Finally, (because of Corollary 6.18 with \( O(1) \) additional edits, we can obtain the beginning blocks of \( x \).

So far we have shown how to perform a series of \( |E| + O(1) \) insertions and deletions to transform the beginning blocks of \( x^{t+1} \) to the beginning blocks of \( x^t \) in a way such that any untouched blocks have their first letter \( l \) mapped to a position within \( 2c \) of \( p_{l+1}(l) \).

---

In particular, since \( E' \) does not map any unedited blocks to begin in a position greater than \( x_p(x_i^t) - 2c \), the only way it can result in such blocks is insert them. \( E'' \) can therefore be obtained from \( E' \) by ignoring such insertions.
Next we will focus on the ending blocks of $x^{t+1}$. We remark that we will construct a sequence of edits between these blocks and the ending blocks of $x$ without any further use of $E$. Just as in the proof of Lemma 6.23 there are two cases to consider.

Case 1: $x_i^t$ does not belong to a maximally periodic substring that extends until at least $x_{i+8c-1}$. Following the same logic as in Lemma 6.23 with probability at least $(\frac{1}{2})^{m-2}$, there will be some $j$ between $i$ and $i + (m + 18 + 1/2)c$ such that the blocks in $\phi(x^{t+1})$ containing letters only from $x_{j+1}^{t+1}x_{j+2}^{t+1} \cdots$ are identical to the blocks of $\phi(x^t)$ containing letters only from $x_j^t x_{j+1}^t \cdots$. Moreover, because each of the edits from $x_0$ to $x^{t+1}$ take place before or at $x_i^{t+1}$, the same logic can be used to conclude that the blocks in $\phi(x^{t+1})$ containing letters only from $x_{j+1}^{t+1}x_{j+2}^{t+1} \cdots$ are identical to the blocks of $\phi(x^t)$ containing letters only from $x_p(x_{j+1}^{t+1})x_p(x_{j+1}^{t+1})+1 \cdots$.

Notice that $p(x_{j+1}^{t+1}) = p(x_i^t) + (j - i)$ because all of the edits from $x_0$ to $x^{t+1}$ occur prior to $x_i^{t+1}$, which is the same letter as $x_i^t$. It follows that $p_{t+1}(x_{j+1}^{t+1}) \leq p_t(x_i^t) + O(mc)$. Therefore, by Lemma 6.23 only $O(m)$ edits are needed to $\phi(x^{t+1})$ to remove the ending blocks which contain letters prior to $x_{j+1}^{t+1}$. Let $\phi(x^{t+1})$ and replace them with the ending blocks in $x$ which contain letters prior to $x_p(x_{j+1}^{t+1})$.

Since the remaining ending blocks are the same between $x^{t+1}$ and $x$, this transforms the ending blocks of $\phi(x^{t+1})$ into those of $x$. Combining this with the edits which transform the beginning blocks of $x^{t+1}$ to those $x$, we have a nearly position preserving series of at most $|E| + O(m)$ edits transforming $x^{t+1}$ to $x$.

Case 2: $x_i^t$ belongs to a maximally periodic substring that extends until at least $x_{i+8c-1}$. Denote by $x_i^t$, the last character of $x^t$ to be in a maximally periodic substring which contains $x_i^t$. Then by the logic from the proof of Lemma 6.23 the blocks beginning at $x_{i+1}^{t+1}$ will be identical to those beginning at $x_{i+2}^{t+1}$. Moreover, because all of the edits from $x_0$ to $x^t$ occur prior to $x_i^t$, the same logic can be used to conclude that the blocks beginning at $x_p(x_{i+1}^{t+1})$ are identical to those beginning at $x_{i+1}^{t+1}$.

So far we have shown that the beginning blocks of $\phi(x^{t+1})$ can be transformed to the beginning blocks of $\phi(x)$ through a nearly position preserving sequence of at most $|E| + O(1)$ edits, and that the ending blocks of $\phi(x^{t+1})$ beginning with $x_{i+1}^{t+1}$ are identical to those of $\phi(x)$ beginning with $x_p(x_{i+1}^{t+1})$. Therefore, it suffices to show that the ending blocks of $\phi(x^{t+1})$ preceding $x_{i+2}^{t+1}$ can be transformed to those in $\phi(x)$ preceding $x_p(x_{i+2}^{t+1})$ through a nearly position preserving sequence of $O(1)$ edits. Denote by $j_1$ the index of the first marker placed in $x_{p(x_{i+1}^{t+1})+2c} \cdots x_{p(x_{i+2c-1}^{t+1})}$ and by $j'_1$ the index of the first marker placed in $x_{i+2c-1}^{t+1} \cdots x_{p(x_{i+2c-1}^{t+1})}$. Additionally, denote by $j_2$ the index of the last marker placed before or at $x_{p(x_{i+2c-1}^{t+1})}$ and by $j'_2$ the index of the last marker placed before or at $x_{i+2c-1}^{t+1}$. By the same logic as from Lemma 6.23 each of the blocks in $\phi(x)$ between $x_{j_1}$ and $x_{j_2}$ and each of the blocks in $\phi(x^{t+1})$ between $x_{j_1}^{t+1}$ and $x_{j_2}^{t+1}$ are identical. Moreover, notice because all of the edits $e_1, \ldots, e_t$ take place before $x_{i+1}^{t+1}$, we have that $p(x_{i+1+r}^{t+1}) = p(x_{i+1}^{t+1}) + r$ for all $r \geq 0$. It follows that for all but the first and last $O(c)$ letters of $x_{j_1}^{t+1} \cdots x_{j_2}^{t+1}$, each letter $l$ has the property that $x_{p(l)}$ appears in $x_{j_1} \cdots x_{j_2}$. Hence, by Corollary 6.13 we can perform $O(1)$ edits to transform the blocks between $x_{j_1}^{t+1}$ and $x_{j_2}^{t+1}$ to the blocks between $x_{j_1}$ and $x_{j_2}$ in a way so that any untouched block beginning with some letter $l$ is mapped to the block in $x$ containing $x_{p(l)}$; it follows these edits are nearly position preserving.

By Corollary 6.13 all but $O(1)$ of the ending blocks in $x^{t+1}$ appear either between $x_{j_1}^{t+1}$ and $x_{j_2}^{t+1}$
or after \(x_{i+1}^{i+2}\), and all but \(O(1)\) of the ending blocks in \(x\) appear either between \(x_j\) and \(x_k\) or after \(x_{p(x_{i+1}^{i+2})}\). Therefore, with \(O(1)\) additional edits, we can eliminate such blocks in \(x_{i+1}^{i+2}\) and replace them with the appropriate blocks in \(x\). This completes the nearly position preserving progression of at most \(|E| + O(1)\) edits from \(x_{i+1}^{i+2}\) to \(x\).

\[\square\]

**Lemma B.2.** Suppose \(c \leq n^{0.4}\). Let \(x\) and \(y\) be strings. Then for \(m \in \mathbb{N}\),

\[
\Pr[d(x^s) > m\Delta_{ed}(x,y)] \leq \left(\frac{1}{2}\right)^{\Omega(m)} + \frac{1}{\text{poly}(n)},
\]

where \(\text{poly}(n)\) is an arbitrarily large polynomial in \(n\) of our choice controlled by the constant \(b\).

**Proof.** It suffices to show that

\[
\Pr\left[\sum_{i=0}^{s-1} (d(x^{i+1}) - d(x^i)) \leq m\Delta_{ed}(x,y)\right] \leq \left(\frac{1}{2}\right)^{\Omega(m)} + \frac{1}{\text{poly}(n)}.
\]

Lemma [B.1] tells us that for each \(i \leq s\), if we condition on the fact that each \((m+9)\cdot c\) consecutive letters in \(x^i\) has some \(h_k\) which is injective on its \(c\)-letter substrings, then \(\Pr[d(x^{i+1}) - d(x^i) > m] \leq \left(\frac{1}{2}\right)^{\Omega(m)}\). Notice that, in comparison, Lemma [6.23] tells us that for under the same conditions, \(\Pr[\Delta_{ed}(x^i, x^{i+1}) > m] \leq \left(\frac{1}{2}\right)^{\Omega(m)}\). Thus we can use the same proof as was used for Lemma [6.24] except with Lemma [B.1] replacing Lemma [6.23]. In particular, whereas the proof previously could be used to obtain the bound

\[
\Pr\left[\sum_{i=0}^{s-1} \Delta_{ed}(x^i, x^{i+1}) \geq m \cdot \Delta_{ed}(x,y)\right] \leq \frac{1}{2^{\Omega(m)}} + \frac{1}{\text{poly}(n)},
\]

the same proof can now be used to provide the bound

\[
\Pr\left[\sum_{i=0}^{s-1} (d(x^{i+1}) - d(x^i)) \geq m\Delta_{ed}(x,y)\right] \leq \left(\frac{1}{2}\right)^{\Omega(m)} + \frac{1}{\text{poly}(n)},
\]

as desired.

\[\square\]

We are now in a position to prove Lemma [6.29].

**Lemma B.3** (Restatement of Lemma [6.29]). Consider \(x, y \in \Sigma^n\). Pick \(m \in \mathbb{N}\). Then with probability \(1 - \left(\frac{1}{2}\right)^{\Omega(m)} - \frac{1}{\text{poly}(n)}\) (for a polynomial of our choice), there is a sequence of at most \(m\Delta_{ed}(x,y)\) edits from \(\phi(x)\) (with parameter \(c\)) to \(\phi(y)\) with the following property. If a block \(u\) in \(\phi(x)\) is not modified by the edits and is mapped to a block \(v\) in \(\phi(y)\), then the start position of block \(u\) in \(x\) is within \(O(\Delta_{ed}(x,y) + c)\) of the start position of block \(v\) in \(y\).

**Proof of Lemma [6.29]**. Lemma [B.2] establishes that with probability \(1 - \left(\frac{1}{2}\right)^{\Omega(m)} - \frac{1}{\text{poly}(n)}\) there is a nearly position preserving sequence of edits from \(\phi(y)\) to \(\phi(x)\). Such a sequence of edits must map each untouched block \(u\) in \(\phi(y)\) starting with some letter \(y_i\) to a position in \(x\) within \(O(c)\) of \(p(y_i)\). Since \(p(y_i)\) can differ from \(i\) by at most \(\Delta_{ed}(x,y)\), it follows that if a block \(u\) in \(\phi(x)\) is not modified by the edits and is mapped to a block \(v\) in \(\phi(y)\), then the start position of block \(u\) in \(x\) is within \(O(\Delta_{ed}(x,y) + c)\) of the start position of block \(v\) in \(y\).
C  Omitted Proofs

Proof of Lemma 2.2. Without loss of generality, assume \( p \leq q \). For convenience, denote the letters of \( w \) by \( w_0, \ldots, w_{n-1} \). Because \( w \) is \( q \)-periodic and is of length at least \( p+q \), we have that \( w_i = w_{i+q} \) for all \( i \in \{0, \ldots, p-1\} \). It follows from the fact that \( w \) is \( p \)-periodic that \( w_i = w_{i+q} = w_{(i+q) \mod p} \) for all \( i \in \{0, \ldots, p-1\} \). By the extended Euclidean algorithm, there exist integers \( a \) and \( b \) such that \( \gcd(p, q) = ap + bq \), and thus \( \gcd(p, q) \equiv bq \pmod{p} \). Repeatedly using the fact that \( w_i \equiv w_{i+bq} \pmod{p} \) for all \( i \in \{0, \ldots, p-1\} \), it follows that \( w_i = w_{(i+\gcd(p, q)) \mod p} \), and thus that the first \( p \) letters of \( w \) are \( \gcd(p, q) \)-periodic. Since \( w \) is \( p \)-periodic, and the first \( p \) letters of \( w \) are \( \gcd(p, q) \)-periodic, we get that all of \( w \) is \( \gcd(p, q) \)-periodic as well. \( \square \)

Proof of Lemma 2.3. Select \( b \geq \log(|S|^2 \cdot p) \in \Theta(\log n) \). Then for distinct \( x, y \in S \), \( \Pr[h(x) = h(y)] \leq \frac{1}{|S|^2 \cdot p(n)} \). Applying the union bound over all pairs \( x, y \in S \), it follows that there is a collision with probability at most \( \frac{1}{p(n)} \). \( \square \)

Proof of Lemma 6.9. Let \( t = 1.5^{1/k} \). Then,

\[
\Pr \left[ \sum_{i=1}^{k} X_i \geq \lambda k \right] = \Pr \left[ t^{\sum_{i=1}^{k} X_i} \geq t^{\lambda k} \right] \leq \frac{\mathbb{E} \left[ t^{\sum_{i=1}^{k} X_i} \right]}{t^{\lambda k}},
\]

where the second inequality follows from Markov's inequality. Since \( t^{1/k} = 1.5 \), it follows that

\[
\Pr \left[ \sum_{i=1}^{k} X_i \geq \lambda k \right] \leq \frac{\mathbb{E} \left[ t^{\sum_{i=1}^{k} X_i} \right]}{2^{O(k)}}.
\]

To complete the proof, it suffices to show that \( \mathbb{E} \left[ t^{\sum_{i=1}^{k} X_i} \right] \leq O(1) \). By Hölder’s inequality,

\[
\mathbb{E} \left[ t^{X_1} \cdot t^{X_2} \cdots t^{X_k} \right] \leq \mathbb{E} \left[ t^{X_1 k} \right]^{1/k} \cdot \mathbb{E} \left[ t^{X_2 k} \right]^{1/k} \cdots \mathbb{E} \left[ t^{X_k k} \right]^{1/k}.
\]

Without loss of generality \( \mathbb{E} \left[ t^{X_i k} \right] \) is maximized by \( i = 1 \). Thus the above expression is at most

\[
\mathbb{E} \left[ t^{X_1 k} \right] = \mathbb{E} \left[ 1.5^{X_1} \right] \leq \sum_{i=1}^{\infty} 1.5^{i} \cdot \Pr[X_1 \geq i] \leq 1.5 + \sum_{i=2}^{\infty} 1.5^{i} \cdot \frac{1}{2^{i-1}} = O(1),
\]

completing the proof. \( \square \)
Proof of Lemma 6.10. Because $\Pr[X_i > \lambda] \leq \frac{1}{2^{\log p}} + \frac{1}{p}$, there exists a positive constant $g$ such that for each $i$ if we define $Y_i = \lceil X_i / g \rceil$ then we get for $\lambda \in \mathbb{N}$ that

$$\Pr[Y_i > \lambda] \leq \frac{1}{2} \left( \frac{1}{2} \right)^\lambda + \frac{1}{p}. \quad (C.1)$$

Thus for $\lambda \leq \log p - 1$, we have that $\Pr[Y_i > \lambda] \leq \left( \frac{1}{2} \right)^\lambda$. Define $Z_i = \min(Y_i, \lfloor \log p \rfloor - 1)$. Then $Z_i$ satisfies that $\Pr[Z_i > \lambda] \leq \left( \frac{1}{4} \right)^\lambda$ for all $\lambda \in \mathbb{N}$. Therefore, by Lemma 6.9 we get that

$$\Pr \left[ \sum_{i=1}^k Z_i \geq \lambda k \right] \leq \left( \frac{1}{2} \right)^{\Omega(\lambda)}. \quad (C.2)$$

For a given $i$,

$$\Pr[Z_i \neq Y_i] = \Pr[Y_i > \lceil \log p \rceil - 1] \leq \frac{2k}{p},$$

where the last inequality comes from (C.1). Thus by the union bound, the probability that $Z_i \neq Y_i$ for any $i$, is at most $\frac{2k}{p}$. Combining this with Equation (C.2), we get that

$$\Pr \left[ \sum_{i=1}^k Y_i > \lambda k \right] \leq \left( \frac{1}{2} \right)^{\Omega(\lambda)} + \frac{2k}{p}.$$

If we recall that $Y_i$ and $X_i$ differ by a constant factor, it follows that

$$\Pr \left[ \sum_{i=1}^k X_i > \lambda k \right] \leq \left( \frac{1}{2} \right)^{\Omega(\lambda)} + \frac{2k}{p}.$$

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