It is shown that for controlled Moran constructions in $\mathbb{R}$, including the (sub) self-similar and more generally, (sub) self-conformal sets, the quasi-Assouad dimension coincides with the upper box dimension. This can be extended to some special classes of self-similar sets in higher dimensions. The connections between quasi-Assouad dimension and tangents are studied. We show that sets with decreasing gaps have quasi-Assouad dimension 0 or 1 and we exhibit an example of a set in the plane whose quasi-Assouad dimension is smaller than that of its projection onto the $x$-axis, showing that quasi-Assouad dimension may increase under Lipschitz mappings.

1. Introduction

Recently, a number of authors have investigated the Assouad and lower-Assouad dimensions of subsets of $\mathbb{R}^d$; see [8] and the many papers referenced there. These dimensions differ from the well-known Hausdorff and box dimensions as they provide information about the extreme behaviour of the local geometry of the set. The quasi-Assouad dimensions give less extreme, but still local, geometric information. As in the case of Assouad dimensions, quasi-Assouad dimensions take into account relative scales locally at any position in the set, but the difference is that the ‘minimal depth’ of the relative scales increases as the size of the considered neighbour decreases, and this has a moderating effect. These dimensions were introduced by Lü and Xi in [19] and shown to be an invariant quantity under quasi-Lipschitz maps, unlike the Assouad dimension. In this note, we develop further properties of the quasi-Assouad dimensions.

To give their definitions we first introduce notation. For a subset $A \subseteq \mathbb{R}^d$, let $N_r(A)$ be the least number of balls of radius $r$ needed to cover $A$ and let $N_r,R(E) = \max_{x \in E} N_r(E \cap B(x, R))$. The upper and lower box dimensions, denoted $\dim_B E$ and $\dim_B E$ respectively, are given by

$$\dim_B E = \limsup_{r \to 0} \left( \frac{\log N_r(E)}{\log r} \right), \quad \dim_B E = \liminf_{r \to 0} \left( \frac{\log N_r(E)}{\log r} \right).$$

If the two values coincide, we refer to it as the box dimension, $\dim_B E$. Given $0 \leq \delta < 1$, define

$$\overline{h}(E)(\delta) = \inf \left\{ \alpha \geq 0 : \exists b, c > 0 \text{ such that } \forall 0 < r \leq R^{1+\delta} \leq R \leq b, N_{r,R}(E) \leq c \left( \frac{R}{r} \right)^{\alpha} \right\}$$

and

$$\underline{h}(E)(\delta) = \sup \left\{ \alpha \geq 0 : \exists b, c > 0 \text{ such that } \forall 0 < r \leq R^{1+\delta} \leq R \leq b, N_{r,R}(E) \geq c \left( \frac{R}{r} \right)^{\alpha} \right\}.$$

These functions are monotone and by taking limits we obtain the quasi-Assouad dimension of $E$, $\dim_{qA} E$, and the quasi-lower Assouad dimension of $E$, $\dim_{qL} E$:

$$\dim_{qA} E = \lim_{\delta \to 0} \overline{h}(E)(\delta), \quad \dim_{qL} E = \lim_{\delta \to 0} \underline{h}(E)(\delta).$$

The Assouad and the lower-Assouad dimensions are given by

$$\dim_A E = \overline{h}(E)(0), \quad \dim_L E = \underline{h}(E)(0).$$

2010 Mathematics Subject Classification. Primary 28A80, 28A78.

Key words and phrases. Assouad dimension, self-similar set, self-conformal set, tangent.

The work of I. García was partially supported by a grant from the Simons Foundation.

The work of K. Hare was supported by NSERC 2016 03719.
For any bounded set $E$ these dimensions are ordered in the following way:

$$\dim_L E \leq \dim_{QL} E \leq \dim_{QL C} E \leq \dim_B E \leq \dim_{qA} E \leq \dim_A E.$$  

(1)

It is known that all these dimensions coincide for self-similar sets satisfying the open set condition, but for more general sets, strict inequalities are possible throughout, c.f. [8] [19] and Example 16.

In order to better understand how the inhomogeneity of a set depends on the scale, Fraser and Yu [11] also recently introduced the following refined parametric variants for $\theta \in (0, 1)$, known as the Assouad spectrum and lower spectrum of $E$ respectively:

$$\dim_{qA} E = \inf \left\{ \alpha : \exists b, c > 0 \text{ such that } \forall 0 < R \leq b, N_{R^{1/\theta}}(E) \leq c \left( \frac{R}{R^{1/\theta}} \right)^\alpha \right\},$$

$$\dim_{QL} E = \sup \left\{ \alpha : \exists b, c > 0 \text{ such that } \forall 0 < R \leq b, N_{R^{1/\theta}}(E) \geq c \left( \frac{R}{R^{1/\theta}} \right)^\alpha \right\}.$$  

In general,  

$$\dim_{qL} E \leq \dim_{QL} E \leq \dim_{QL C} E \leq \dim_B E \leq \dim_{qA} E \leq \dim_A E$$

(2)

for all such $\theta$ (11) and it is unknown if $\dim_{qA} E = \sup_{\theta < 1} \dim_{A}^\theta E$ and $\dim_{qL} E = \inf_{\theta < 1} \dim_{L}^\theta E$.

In section 2 of this note, we consider limit sets of suitably controlled Moran constructions in $\mathbb{R}$, a class which includes the family of self-similar and more generally, self-conformal, sets. In [8], Fraser proved that self-conformal sets have the same lower-Assouad and box dimension. However, he gave an example of a self-similar set $E \subseteq \mathbb{R}$ with $\dim_B E < \dim_A E$. In contrast, we prove that the quasi-Assouad dimension and upper box dimension of these limit sets always agree. This shows, in particular, that the Assouad spectrum and lower spectrum of self-conformal sets in $\mathbb{R}$ coincide with their box dimension for every $\theta \in (0, 1)$, regardless of separation conditions, and answers questions posed by Balka and Peres in [2] and Fraser and Yu (for the case of $\mathbb{R}$) in [11]. The equality of these dimensions also holds for certain self-similar carpets in $\mathbb{R}^2$, but it is unknown if the result is true, in general, even for self-similar sets in higher dimensions.

Generalizations of tangents of a set are a useful concept when considering the Assouad dimension. These are essentially limits, in the Hausdorff metric, of sequences of magnifications of local parts of the set. Tangents often have simpler structure than the original set and their Assouad dimensions are always lower bounds for the Assouad dimension of the original set. From this, we show that the quasi-lower Assouad dimensions of self-affine carpets are the same as their Assouad dimensions.

In section 4 we observe that if $\overline{\dim}_B E = 0$, then the same is true for the quasi-Assouad (but not necessarily, the Assouad) dimension of $E$ and show that the quasi-Assouad dimension of a sequence in $\mathbb{R}$ with decreasing gaps is either 0 or 1. This same dichotomy was shown to hold for Assouad dimensions (although not necessarily with the same value for the Assouad and quasi-Assouad dimensions) in [10]. We also give an example to illustrate that, like the Assouad dimension (but not the Hausdorff or box dimensions), the quasi-Assouad dimension can rise when taking projections.

2. Quasi-Assouad dimension of controlled Moran constructions

In this section we consider controlled Moran constructions defined in [18] [22]. We begin by introducing notation.

Fix an integer $m > 1$ and let $\Omega = \{1, \ldots, m\}^\mathbb{N}$. For $\omega = \omega_1 \omega_2 \ldots \in \Omega$ let $\omega_n = \omega_1 \ldots \omega_n$ for all $n \geq 1$. Let $\omega_0$ be the empty word, which has length 0 by definition. Also, let $\Omega^k = \{1, \ldots, m\}^k$ be the set of words of length $k$ and $\Omega^* = \bigcup_{k \geq 0} \Omega^k$ the set of finite words in the alphabet $\{1, \ldots, m\}$. For finite words $\omega = \omega_1 \ldots \omega_k$ and $\tau = \tau_1 \ldots \tau_l$, we let $\tau \omega := \tau_1 \ldots \tau_l \omega_1 \ldots \omega_k$ denote the concatenation of the words and let $\omega^- := \omega_1 \ldots \omega_{k-1}$ for $\omega \in \Omega^k$, $k \geq 1$. 


Let $\Gamma \subset \Omega$ be a compact set satisfying $\sigma(\Gamma) \subset \Gamma$, where $\sigma : \Omega \to \Omega$ is the left shift operator defined by $\sigma(\omega_1 \omega_2 \ldots) = \omega_2 \omega_3 \ldots$. Set $\Gamma_n = \{\omega_n : \omega \in \Gamma\}$ and $\Gamma_* = \bigcap_{n \geq 0} \Gamma_n$. Note that if $\tau \omega \in \Gamma_*$ then $\omega \in \Gamma_*$ by the shift invariance of $\Gamma$.

We denote by $|J|$ the diameter of the set $J$.

**Definition 1.** A family $\mathcal{J} = \{J_\omega : \omega \in \Gamma_*\}$ of compact subsets of $\mathbb{R}^d$ having positive diameter is a Moran construction if the following properties hold:

M1) $J_\omega \subset J_{\omega^i}$ for all $\omega \in \Gamma_* \setminus \{\emptyset\}$,
M2) $|J_{\omega^n}| \to 0$ as $n \to \infty$ for all $\omega \in \Gamma$.

Moreover, $\mathcal{J}$ is a controlled Moran construction if it also satisfies for some $D > 1$

M3) $|J_\omega| \leq D |J_\tau| |J_\sigma|$ for all $\omega, \tau \in \Gamma_*$,
M4) $|J_\omega| \geq D^{-1} |J_\omega|$ for all $\omega \in \Gamma_* \setminus \{\emptyset\}$.

We will require the following property:

M5) $J_\omega \cap J_\omega = \emptyset$ whenever there is $\tau \in \Gamma_*$ such that $J_{\tau \omega} \cap J_{\tau \omega} = \emptyset$.

Given a Moran construction as above, its limit set is the non-empty compact set $E$ defined as

$$E = \bigcap_{n \geq 1} \bigcup_{\omega \in \Gamma_n} J_\omega.$$ 

A simple example of a controlled Moran construction is the classical Cantor set. More generally, other examples of Moran constructions satisfying M3) to M5) are given by special classes of iterated function systems, as follows. Given a non-empty closed set $X \subset \mathbb{R}^d$, an iterated function system (IFS) is a family of contractions $\{f_1, \ldots, f_m\}$ where $f_i : X \to X$. Denote by $f_\omega = f_{\omega_1} \circ \cdots \circ f_{\omega_n}$ for $\omega = \omega_1 \ldots \omega_n \in \Omega^n$, for $n \in \mathbb{N}$. Assume that the maps are bi-Lipschitz contraction with Lipschitz constants $a_i$ and $b_i$, that is, $0 < a_i \leq b_i < 1$ are such that

$$a_i \|x - y\| \leq \|f_i(x) - f_i(y)\| \leq b_i \|x - y\|$$

for all $x, y \in X$. It is known that there is a set $W \subset X$ such that if $\Gamma \subset \Omega$ and $\sigma(\Gamma) \subset \Gamma$, then $\{f_\omega(W) : \omega \in \Gamma\}$ is a Moran construction. Furthermore, if for some constant $C$ the following inequality holds

$$\|f_\omega(x) - f_\omega(y)\| \leq C \|f_\omega(W)\| \|x - y\|$$

for all $x, y \in W$ and $\omega \in \Gamma_*$, then the Moran construction verifies M3), M4); see Lemma 2.1 in [18]. Clearly M5) holds because of the injectivity of the maps. When $\Gamma = \Omega$, the corresponding limit set is the attractor of the IFS, that is the unique non-empty compact set $E$ that satisfies the identity

$$E = \bigcup_{i=1}^m f_i(E).$$

In particular, if the maps of the IFS are contracting similarities, or more generally, conformal maps, the above conditions are satisfied. In these cases, the attractors are called self-similar and self-conformal, respectively. Recall that in the case of $\mathbb{R}$, a conformal IFS is a family of $C^{1+\epsilon}$-diffeomorphisms $\{f_1, \ldots, f_m\}$ such that there is a non-empty open interval $I$ where, for each $1 \leq i \leq m$, $f_i$ is a bi-Lipschitz contraction on $I$.

On the other hand, if $\Gamma \subset \Omega$, then the limit set verifies

$$E \subset \bigcup_{i=1}^m f_i(E).$$

In this case, if the maps of the IFS are similarities, the set $E$ is called sub-self-similar (see [5] for examples of sub-self similar sets), while if the maps are conformal the set is called sub-self-conformal.

It is known ([4 Ex. 2], [8 Thm. 2.10]) that for a self-conformal set $E,$

$$\dim_L E = \dim_H E = \dim_B E,$$

(3)
where \( \dim_H \) denotes the Hausdorff dimension. However, for sub-self-similar and sub-self-conformal sets it is still an open problem if the equality holds (\([8]\)).

For \( 0 < r < \min\{|J_i|: 1 \leq i \leq m\} \) define
\[
\Gamma_r = \{ \omega \in \Gamma_s : |J_\omega| < r \leq |J_{\omega^-}| \}.
\]
For any \( \omega \in \Gamma_r \) we have by M4) that
\[
|J_\omega| < r \leq D|J_\omega|.
\]
Moreover, \( \Gamma_r \) is a finite set that verifies \( E = \bigcup_{\omega \in \Gamma_r} E \cap J_\omega \). In particular, the family of basic sets of approximate length \( r \), defined by
\[
C_r = \{ J_\omega : \omega \in \Gamma_r \},
\]
is a covering of \( E \) with sets of diameter comparable to \( r \).

**Theorem 1.** Let \( J \) be a controlled Moran construction satisfying the additional assumption M5) and let \( E \subset \mathbb{R} \) be the limit set. Then \( \dim_A E = \overline{\dim}_B E \).

**Proof.** Since the inequality \( \overline{\dim}_B X \leq \dim_A X \) holds for any bounded subset \( X \), we only need to show the opposite inequality.

Given \( 0 < \alpha < \dim_A E \), by definition there is \( \delta_0 > 0 \) such that \( \overline{\dim}_B(\delta_0) > \alpha \). Then, there are two positive sequences \( \{r_n\} \) and \( \{R_n\} \) such that \( R_n \uparrow N \), \( r_n = R_n^{1+\delta_0} \) with \( \delta_n \geq \delta_0 \) and
\[
N_r(E \cap B(x_n, R_n)) \geq R_n^{-\delta_n \alpha}
\]
for some \( x_n \in E \).

By fixing \( n \), we can cover \( E \cap B(x_n, R_n) \) with at most three elements in \( C_{2DR_n} \) in the following way. First, pick any \( I_1 \in C_{2DR_n} \) that contains \( x_n \); this set contains at least one of the intervals \([x_n - R_n, x_n]\) or \([x_n, x_n + R_n]\) and we assume that it contains the latter. Then pick \( I_2 \in C_{2DR_n} \), the interval that intersects \( I_1 \) and has the smallest left endpoint. It may occur that \( I_1 = I_2 \). Finally, pick, if possible, \( I_3 \in C_{2DR_n} \), the interval to the left of \( I_1 \), disjoint from it and with largest right endpoint. If \( I_3 \) does not exist, then clearly \( E \cap B(x_n, R_n) \subset I_1 \cup I_2 \). Note that if \( x \in E \cap B(x_n, R_n) \setminus \{ I_1 \cup I_2 \cup I_3 \} \), then \( x \) is to the left of \( I_2 \). If \( x \) is to the right of \( I_3 \), then by considering an interval in \( C_{2DR_n} \) that contains \( x \), we arrive at a contradiction of either the definition of \( I_2 \) or \( I_3 \). On the other hand, if \( x \) is to the left of \( I_3 \) then \( x_n - x > |I_3| \geq 2R_n \), and this is a contradiction. Therefore \( E \cap B(x_n, R_n) \subset I_1 \cup I_2 \cup I_3 \).

The above covering with three intervals, together with inequality (\([4]\)), imply that there are \( \tau \in \Omega_{2DR_n} \) and points \( \{z_j\}_{j=1}^M \subset E \cap J_\tau \) which are \( 2R_n \) apart; here \( M \geq cR_n^{-\delta_n \alpha} \) and \( c > 0 \) is a constant independent of \( n \). For each \( 1 \leq j \leq M \), we choose \( \omega \in \Omega^* \) such that \( \tau \omega \in \Omega_{r_n} \) and \( z_j \in E \cap J_{\tau \omega} \). It follows from (\([5]\)) that \( J_{\tau \omega} \cap J_{\tau \omega'} = \emptyset \) for \( j \neq j' \), and by (M5) we get that \( \{J_{\tau \omega}, \ldots, J_{\tau \omega_M}\} \) is a disjoint family of intervals. Moreover, by definition of \( \tau \), \( \omega \) and by (M3), we have, for any \( 1 \leq j \leq M \), that
\[
1 \leq |J_{\tau \omega}| \leq D|J_{\tau}| |J_{\omega}| < 2D^2 R_n |J_{\omega}|,
\]
or
\[
\frac{1}{2D^3} R_n^{\delta_n} < |J_{\omega}|.
\]
This implies that we need at least \( M/2 \) intervals of length \( 1/(2D^3) R_n^{\delta_n} \) to cover \( E \cap (J_{\tau \omega_1} \cup \ldots \cup J_{\tau \omega_M}) \). Hence
\[
N_{\frac{1}{2D^3} R_n^{\delta_n}}(E) \geq \frac{M}{2} \geq \frac{c}{2} R_n^{-\delta_n \alpha}.
\]
Since \( \delta_n \geq \delta_0 > 0 \), \( R_n^{\delta_n} = R_n/r_n \to 0 \), thus it is immediate that \( \overline{\dim}_B E \geq \alpha \). This concludes the proof since \( \alpha \) can be taken arbitrarily close to \( \dim_A E \).

From the inequalities in (\([3]\)) we immediately obtain the following consequence.

**Corollary 2.** Let \( E \subset \mathbb{R} \) be the limit set of a Moran construction satisfying M3), M4) and M5). Then \( \dim_A E = \overline{\dim}_B E \) for any \( 0 < \theta < 1 \).
Corollary 3. If \( E \subset \mathbb{R} \) is a self-conformal set, then \( \dim^\theta E = \dim^\theta_A E = \dim_H E \) for any \( 0 < \theta < 1 \), regardless of separation conditions.

Remark 1. If a self-conformal set \( E \subset \mathbb{R}^d \) is the attractor of an IFS that satisfies the weak separation condition, then \( \dim_H E = \dim_A E \) ([15] Rem. 3.7 and 3.9; see also [9] for the self-similar case), thus Corollary 3 is already known in that case. The second equality in the statement was obtained in [11] Cor. 4.2 for self-similar sets in \( \mathbb{R} \), but under the additional hypothesis that the IFS does not have super-exponential concentration of cylinders. Corollary 3 answers Question 4.3 in [11] in the case of \( \mathbb{R} \), regarding the Assouad spectrum of self-similar and self-conformal sets.

Remark 2. The modified Assouad dimension \( \dim_{MA} \) is another concept of dimension inspired by the Assouad dimension. It was defined in [23] and satisfies \( \dim_H X \leq \dim_{MA} X \leq \dim_{qA} X \) for any totally bounded metric space. In particular, by Theorem 1 above, we have \( \dim_H E = \dim_{MA} E \) for any self-similar set \( E \subset \mathbb{R} \), which answers in the affirmative Problem 1.16 in [2].

Moreover, from [3] we have:

**Corollary 4.** Any self-conformal subset of \( \mathbb{R} \) of Hausdorff dimension strictly smaller than one is quasi-uniformly disconnected. Moreover, it is quasi-uniformly disconnected, but not uniformly disconnected if and only if it is the attractor of an IFS that does not verify the weak separation property.

The extension of Theorem 1 to higher dimensions seems to be more complicated. Even in the self-similar case we do not know if it remains true. Below, we give two examples of self-similar sets in \( \mathbb{R}^2 \) where the equality still holds. The first are product sets, for which we make use of the following lemma. The proof is similar to that given in [8] Thm. 2.1 for (lower) Assouad dimension and the details are left for the reader.

**Lemma 5.** Suppose \( X, Y \) are metric spaces and that \( X \times Y \) is a metric space with the product metric 
\[
d((x_1, y_1), (x_2, y_2)) = \max (d_X(x_1, x_2), d_Y(y_1, y_2)).
\]
Then
\[
\dim_{qL} X + \dim_{qA} Y \leq \dim_{qA} X \times Y \leq \dim_{qA} X + \dim_{qA} Y,
\]
\[
\dim_{qL} X + \dim_{qL} Y \leq \dim_{qL} X \times Y \leq \dim_{qL} X + \dim_{qL} Y.
\]

Suppose that \( F, G \subset \mathbb{R} \) are self-similar sets and consider their cartesian product \( S = F \times G \). From the previous Lemma we have
\[
\dim_{qA} S \leq \dim_{qA} F + \dim_{qA} G = \dim_{MA} F + \dim_{MA} G \leq \dim_H S \leq \dim_{qA} S.
\]
Note that \( S \) is a self-similar set when all the maps involved in the IFS of \( F \) and \( G \) have the same contraction ratios, but need not satisfy the weak separation property.

Another family of self-similar sets in \( \mathbb{R}^2 \) whose structure is less rigid than product sets, and where there is still coincidence of the dimensions, is the family of horizontal self-similar carpets, where overlaps are allowed only on the horizontal direction. These are defined as follows. We begin with a self-similar set \( F \) on the line arising from an IFS \( \mathcal{F} = \{ f_1, \ldots, f_m \} \) with \( f_i([0,1]) \subset [0,1] \) and satisfying the weak separation property. For each \( j \), consider the rectangle \( [0,1] \times f_j([0,1]) \) and let \( i_j \) be any natural number. Inside this rectangle choose \( i_j \) squares, \( Q_{i,j}, i = 1, \ldots, i_j \), of side length equal to the diameter of \( f_j([0,1]) \). Define \( S_{(i,j)} \) to be the unique orientation-preserving similarity sending \([0,1]^2 \)
to $Q_{i,j}$ and let $S \subseteq \mathbb{R}^2$ be the self-similar set associated to the IFS $S = \{S_{i,j} : 1 \leq i \leq i_j, 1 \leq j \leq m\}$. The set $S$ is what we mean by a horizontal self-similar carpet; see Figure 1.

For this class, the argument in the proof of Theorem 1 can be extended.

![Figure 1. Basic intervals $f_i([0,1])$ from the first step of $F$ and a possible pattern for an horizontal self-similar carpet.](image)

**Proposition 6.** If $S$ is an horizontal self-similar carpet, then $\dim_q S = \dim_H S$.

**Proof.** Denote by $C_F^r$ and $C_s^r$ the set of basic intervals and basic squares of approximate length $r$ corresponding to $F$ and $S$, respectively. Given $0 < R < 1$ and $x \in S$, we claim that the relative ball $S \cap B(x, R)$ can be covered by at most $T$ basic squares in $C_s^R$, where $T$ is a finite constant depending only on $F$. Using this fact we proceed exactly as in the proof of Theorem 1 to deduce that $\dim_q S = \dim_B S$, and as $S$ is self-similar, $\dim_B S = \dim_H S$.

In order to prove the claim, note that the orthogonal projection of $S$ onto the $y$-axis is (an isometric copy of) $F$, and the orthogonal projection of any basic cube in the construction of $S$ corresponds to a basic interval in the construction of $F$. Hence, if $\pi_y$ denotes that orthogonal projection, one of the equivalent definitions of the weak separation property (see [24, Thm. 1-(4a)]; in this case $a = 1$, $M = [0, 1]$) implies the interval $F \cap \pi_y(B(x, R))$ is intersected by at most $T$ (= $\gamma_{a,M}$ in the notation of [24]) basic intervals in $C_F^R$.

Denoting those intervals by $I = J_1, \ldots, J_t$, with $t \leq T$, we have that any cube in $C_s^{2R}$ that intersects $S \cap B(x, R)$ has the form $I_i \times J_j$ for some $1 \leq j \leq t$ and some interval $I_i$. For each such $j$ we can argue as in the three intervals argument in the proof of Theorem 1 to get at most three basic cubes in $C_s^{2R}$ that cover $(S \cap B(x, R)) \cap \pi_y^{-1}(J_j)$. Therefore, there is a covering of $S \cap B(x, R)$ with at most $3T$ cubes in $C_s^{2R}$. \qed

### 3. Quasi-Assouad Dimension and Tangent Structure

In this section we find bounds on the quasi-Assouad dimension of a set by dimensions of its ‘tangents’. We begin by recalling some definitions and results. Given $X, Y \subset \mathbb{R}^d$ compact subsets, their Hausdorff distance is

\[ \text{dist}_H(X, Y) = \max\{p_H(X, Y), p_H(Y, X)\}, \]

where

\[ p_H(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|x - y\|. \]
**Definition 2.** Let $F$ and $\hat{F}$ be compact subsets of $\mathbb{R}^d$. We say that $\hat{F}$ is a weak tangent of $F$ if there is a compact subset $X \subset \mathbb{R}^d$, that contains both $F$ and $\hat{F}$, and a sequence of bi-Lipschitz maps $T_k : \mathbb{R}^d \to \mathbb{R}^d$, with Lipschitz constants $a_k, b_k$ satisfying

$$a_k \|x - y\| \leq \|T_k(x) - T_k(y)\| \leq b_k \|x - y\|$$

and $\sup b_k/a_k < \infty$, such that $\text{dist}_H(T_k(F) \cap X, \hat{F}) \to 0$ as $k \to \infty$. In the case that the contraction ratios, $b_k$, are unbounded we call $\hat{F}$ a generalized tangent of $F$.

The usefulness of tangents is that they can be used to obtain lower bounds for the Assouad dimension of the original set, namely

**Theorem 7.** If $\hat{F}$ is a weak tangent of $F$, then

$$\dim_A \hat{F} \leq \dim_A F. \quad (8)$$

If, in addition, there is some $\theta > 0$ such that for all $r > 0$ and $x \in \hat{F}$ there is some $y \in \hat{F}$ such that $B(y, r\theta) \subseteq B(x, r) \cap X$, then $\dim_L F \leq \dim_L \hat{F}$. \hfill □

The above result is due to Fraser [8 Prop. 7.7]. A version using similarities to define tangents (the usual definition) was previously obtained by Mackay and Tyson, see [24 Prop. 2.9] or [21 Prop. 6.1.5].

There is no loss of generality in assuming that $X = [0, 1]^d$ since the Assouad dimension of any set can be characterized by its tangents in this sense. To be specific

$$\dim_A F = \max\{\dim_H \hat{F} : \hat{F} \text{ is a generalized tangent of } F \text{ with } X = [0, 1]^d\}. \quad (9)$$

We refer the reader to [3], [14], [15] and [17], noting that the terminology there is quite different: microsets and star dimension are used instead of tangents and Assouad dimension. Observe that from [8] and [3] any compact set $F$ has a generalized tangent $\hat{F}$ for which $\dim_H \hat{F} = \dim_A \hat{F} = \dim_A F$.

We call the set $\hat{F}$ a pseudo-tangent of $F$ if there is a sequence of bi-Lipschitz maps $T_k$, as above, with $\sup b_k = \infty$, such that we have the ‘one-sided’ pseudo-distance, $p_H(\hat{F}, T_k(F)) \to 0$ as $k \to \infty$. The convenience of this definition is that it does not require the intersection with the set $X$ or the two-sided comparison of distance, and still the inequality $\dim_A \hat{F} \leq \dim_A F$ holds in this case; see [9], where the definition and proof are made for similarities, but the same proof applies in the bi-Lipschitz setting.

Returning to the quasi-Assouad dimension, we first give an example where [8] fails when Assouad is replaced by quasi-Assouad dimension. The following result is useful for this task; a more general version in $\mathbb{R}^d$ was obtained recently ([132]), but here we include a different proof.

**Proposition 8.** A subset $F \subseteq \mathbb{R}$ has Assouad dimension 1 if and only if $[0, 1]$ is a generalized tangent of $F$.

**Proof.** By [8], the Assouad dimension of any tangent is a lower bound for the Assouad dimension of the set. On the other hand, by [21] Thm. 5.1.8], the assumption that $\dim_A F = 1$ is equivalent to the fact that $F$ is not uniformly disconnected. Thus for each $k$ there are distinct points, $z_0^k, \ldots, z_n^k \in F$, such that $|z_i - z_{i+1}| < |z_0 - z_n|/k$, where $z_0 < z_i < z_n$ and $|z_0 - z_n|$ goes to 0. The last condition ensures that $[0, 1]$ is a generalized tangent (not just a weak tangent).

Let $T_k$ be the affine transformation that maps $[z_0, z_n]$ to $[0, 1]$. For $x_i = T_k(z_i)$ we have

$$|x_i - x_{i+1}| = \frac{|z_i^k - z_{i+1}^k|}{|z_n^k - z_0^k|} < \frac{1}{k},$$

and hence for $0 \leq j < k$ each interval $[j/k, (j+1)/k)$ contains at least one of the $x_i$'s. This shows the sets $T_k(\{z_0^k, \ldots, z_n^k\})$ converge in the Hausdorff metric to $[0, 1]$ and therefore, $\text{dist}_H(T_k(F) \cap [0, 1], [0, 1]) \to 0$ as $k \to \infty$. \hfill □
Corollary 9. If $F \subseteq \mathbb{R}$ has $\dim_{qA} F < 1 = \dim_{a} F$, then $\hat{F} = [0, 1]$ is a generalized tangent with

$$1 = \dim_{H} \hat{F} = \dim_{qA} \hat{F} > \dim_{qA} F.$$

Hence, not only \([8]\), but also \([9]\), fails when Assouad dimension is replaced by quasi-Assouad dimension. An explicit example can be given by considering the Cantor-like sets $F_{\alpha}$ constructed in the following way.

Example 10. Fix $0 < \alpha < 1$ and for each $k \geq 1$ put $D^{k} = \{(i_{1}, \ldots, i_{k}) : 1 \leq i_{j} \leq 2^{j}, 1 \leq j \leq k\}$. We construct the set inductively, beginning with $I_{0} = [0, 1]$ at step 0. Having constructed the step $k - 1$ Cantor intervals, $I_{i_{1} \ldots i_{k-1}}$, for $(i_{1}, \ldots, i_{k-1}) \in D^{k-1}$, to construct the Cantor intervals of step $k$ we take the $2^{k}$ subintervals, $I_{i_{1} \ldots i_{k-1}l}$, $1 \leq l \leq 2^{k}$, uniformly distributed inside $I_{i_{1} \ldots i_{k-1}}$, with equal lengths $2^{-k/\alpha}|I_{i_{1} \ldots i_{k-1}}|$. The Cantor-like set $F_{\alpha}$ is given by

$$F_{\alpha} = \bigcap_{k \geq 1} \bigcup_{I_{i}}.$$

Example 1.17 from [10] is a special case and it follows from the arguments given there that $\dim_{qA} F_{\alpha} = \alpha$, while $\dim_{A} F_{\alpha} = 1$.

Moreover, this example turns out to be quite pathological with respect to tangents.

Proposition 11. Any generalized tangent of the set $F_{\alpha}$ of the example above is either a finite set or an interval.

Proof. For notational ease, we will omit the subscript $\alpha$. Let $\hat{F}$ be a generalized tangent of $F$ and denote by $T_{k}$ the associated bi-Lipschitz maps. By a gap of $\hat{F}$ we mean a bounded complementary (maximal) open interval of the complement of $\hat{F}$. If $\hat{F}$ has no gaps, then it is an interval (which may be a singleton). So, assume that $\hat{F}$ contains at least one gap. Choose one of maximal length, say $J = (a, b)$, where $a, b \in \hat{F}$. Below we show that i) if $x \in \hat{F}$ and $x < a$, then $a - x \geq |J|$, or ii) if $x \in \hat{F}$ and $x > b$, then $x - b \geq |J|$. These statements imply, by the maximality of $J$, that $\hat{F}$ is a finite equidistributed set in some subinterval of $[0, 1]$.

We show only i) since ii) follows by a symmetric argument. So assume that $x < a$ with $x \in \hat{F}$. One consequence of the Hausdorff convergence is that for each sufficiently small $\epsilon > 0$ (much smaller than $|J|$ and $a - x$), and all sufficiently large, the sets $(a - \epsilon, a + \epsilon)$ and $(b - \epsilon, b + \epsilon)$ contain points of $T_{k}(F)$, while $(a + \epsilon, b - \epsilon) \cap T_{k}(F) = \emptyset$. Hence there is a gap $G_{k}$ of $F$ from some step $k$ in the construction of $F$ such that

$$(a + \epsilon, b - \epsilon) \subseteq T_{k}(G_{k}) \subseteq (a - \epsilon, b + \epsilon).$$

Suppose $I_{k}^{j}$ is the closed interval from step $k$ in the construction of $F$ that shares an endpoint with $G_{k}$ and is placed on its left. The set $F$ has the property that $|I_{k}^{j}|/|G_{k}| \rightarrow 0$ and as the maps $T_{k}$ are bi-Lipschitz with constants satisfying $\sup b_{k}/a_{k} < \infty$, this ensures that $T_{k}(I_{k}^{j}) \subseteq (a - 2\epsilon, a + 2\epsilon)$ for large enough $k$. Note that the gap $G_{k}$, adjacent to $I_{k}^{j}$ but on its left, and whose existence is guaranteed because of the existence of $x$ and the choice of $\epsilon$, is a gap from some step $\bar{k} < k$. Note also that the lengths $g_{k-1}$ and $g_{k}$ of any gaps from steps $k - 1$ and $k$ verify $g_{k}/g_{k-1} \rightarrow 0$ as $k \rightarrow \infty$. In particular, for every $k$ sufficiently large we get $b_{k}|G_{k}| \leq a_{k}|G_{k}|^{\alpha}$.

Thus

$$|J| - 2\epsilon \leq |T_{k}(G_{k})| \leq |T_{k}(G_{k})^{\alpha}|,$$

so any point in $\hat{F}$ to the left of $a$, and not contained in $(a - 2\epsilon, a + 2\epsilon)$, must be at least distance $|J| - 2\epsilon$ from $a$ for some constant $c > 0$. But since $\epsilon$ can be made arbitrarily small, an easy argument shows that $\hat{F} \cap (a - 2\epsilon, a + 2\epsilon) = \{a\}$, and therefore we conclude that i) holds by letting $\epsilon \rightarrow 0$. \(\square\)

Note that the above example also illustrates that there is no way to select a subfamily from the tangents to $F$ so that the identity \([9]\) remains valid for the quasi-Assouad dimension. However, we
can extend Theorem 7 above to the quasi-Assouad dimensions if we restrict to tangents for which the convergence is sufficiently quick.

**Definition 3.** We say that the generalized tangent \( \hat{F} \) to the set \( F \) is a *generalized fast tangent* if the following decay condition is satisfied: there are constants \( C, \epsilon > 0 \) such that

\[
\text{dist}_H(T_k(F) \cap [0,1]^d, \hat{F}) \leq Cb_k^{-\epsilon},
\]

where \( T_k \) and \( b_k \) are as in the definition of a generalized tangent. In this case we say that \( \hat{F} \) is a *tangent of order \( \epsilon \).* We similarly define fast *pseudo-tangents* by the requirement that the pseudo-tangent \( \hat{F} \) verifies \( p_H(\hat{F}, T_k(F)) \leq Cb_k^{-\epsilon} \).

The relation between tangents and quasi-Assouad dimensions is given in the next result, where we have chosen to weaken some hypotheses for clarity of the exposition; see Remark 4 for more general statements.

**Theorem 12.** Suppose \( \hat{F} \subset \mathbb{R}^d \) is a non-empty, generalized fast tangent of \( F \subset \mathbb{R}^d \) given by bi-Lipschitz maps \( T_k \) with Lipschitz constants \( a_k, b_k \) satisfying \( \sup b_k = \infty \) and \( \sup b_k/a_k < \infty \).

(i) For the quasi-Assouad dimension of \( F \) we have the lower bound

\[
\dim_{qA} \hat{F} \leq \dim_{qA} F.
\]

If, in addition, there is some \( C' \) such that \( b_{k+1} \leq C'b_k \), then

\[
\dim_{qA} \hat{F} \leq \dim_{qA} F.
\]

(ii) Assume that \( \hat{F} \) contains an interior point of \([0,1]^d\). Then, for the quasi-lower Assouad dimension we have the upper bound

\[
\dim_{qL} F \leq \dim_{qL} \hat{F}.
\]

If, in addition, there is some \( C' \) such that \( b_{k+1} \leq C'b_k \) and furthermore, there is \( \theta > 0 \) such that for any \( r \in (0,1] \) and \( x \in \hat{F} \) there is \( y \in \hat{F} \) such that \( B(y, r\theta) \subset B(x, r) \cap [0,1]^d \), then

\[
\dim_{qL} F \leq \dim_{qL} \hat{F}.
\]

**Remark 3.** Similar statements can be made for the dimensions \( \dim^{\theta}_{A} F \) and \( \dim^{\theta}_{L} F \) where the allowable \( \theta \) depend on the choice of \( \epsilon \). For example, if \( \hat{F} \) is a generalized fast tangent of order \( \epsilon \) such that \( \dim_{A} \hat{F} = s \), then \( \dim^{\theta}_{A} F \geq s \) for any \( 1/(1+\epsilon) \leq \theta < 1 \). We leave the technical details for the reader.

Notation: When we write \( x_k \approx X_k \) we mean there are positive constants \( a, b \) such that \( aX_k \leq x_k \leq bX_k \) for all \( k \).

**Proof.** (i) Let \( \dim_{B} \hat{F} = s \). We may assume \( s > 0 \), else the result is trivial. Temporarily fix \( \eta > 0 \). Then

\[
N_r(\hat{F}) \geq r^{-(s-\eta)}
\]

for all sufficiently small \( r \).

Since \( \hat{F} \) is a generalized fast tangent, there are constants \( C, \epsilon > 0 \) such that \( \text{dist}_H(T_k(F) \cap [0,1]^d, \hat{F}) \leq Cb_k^{-\epsilon} \). Put \( r_k = Cb_k^{-\epsilon} \), let \( R = \text{diam} \hat{F} \) and pick any \( y_0 \in \hat{F} \). Then, for each \( k \) we can find \( m = m_k \approx \left( r_k^{-(s-\eta)} \right) \) points, \( y_1, \ldots, y_m \in B(y_0, R) \) that are \( 3r_k \)-separated. The assumption on \( \hat{F} \) implies, in particular, that \( p_H(\hat{F}, T_k(F)) \leq r_k \), and this ensures that we can choose \( x_0, x_1, \ldots, x_m \in T_k(F) \) such that \( \|x_i - y_i\| \leq r_k \) for each \( i = 0, \ldots, m \). We have \( x_i \in B(x_0, R + 2r_k) \) and \( \|x_i - x_j\| \geq r_k \) for all \( 1 \leq i \neq j \leq m \).

Taking preimages under \( T_k \), we can find \( z_0, z_1, \ldots, z_m \in F \) such that

\[
z_i \in B\left(z_0, \frac{1}{a_k}(R + 2r_k)\right) \quad \text{and} \quad \|z_i - z_j\| \geq \frac{r_k}{b_k}.
\]

This shows that

\[
N_{\frac{r_k}{b_k}}(F \cap B(z_0, \frac{1}{a_k}(R + 2r_k))) \geq m.
\]
Note that \( m \approx \left( \frac{(R^2 + 2r)/a_k}{r/b_k} \right)^{1-\eta} \) (and hence \( \dim_A F \geq s \)). An easy calculation shows \( C_{y_k} \leq \left( \frac{1}{\eta k} (R + 2r_k) \right)^{1+\sigma/2} \) for large \( k \), and, of course, \((R + 2r_k)/a_k \to 0 \), consequently \( \dim_{qA} F \geq s - \eta \). As \( \eta > 0 \) is arbitrary, \( \dim_{qA} F \geq s \).

Now suppose that \( \dim_{qA} \hat{F} = t \) and there is some \( C' \) such that \( b_{k+1} \leq C'b_{k} \) for all \( k \). Again, temporarily fix \( \eta > 0 \). Then there is some \( 0 < \delta < 1 \) and arbitrarily small \( r, R \) with \( r \leq R^{1+\delta} \) and \( y_0 \in \hat{F} \) such that \( N_r(B(y_0, R) \cap \hat{F}) \geq (R/r)^{1-\eta} \). Choose \( k \) such that \( b_{k+1} \leq R \leq b_k^{-\epsilon} \). As above, we deduce that for suitable \( z_0 \in F \) we have

\[
N_{\epsilon/k} \left( F \cap B(z_0, \frac{1}{d_k} (R + 2r)) \right) \geq \left( \frac{R}{r} \right)^{1-\eta}.
\]

Since \( R \approx b_k^{-\epsilon} \) and \( r \leq R^{1+\delta} \), one can easily verify that \( r/b_k \geq ((R + 2r)/a_k)^{1+\sigma} \) for a choice of \( \sigma > 0 \) (depending on \( \delta \) and \( \epsilon \)). It follows that \( \dim_{qA} F \geq t \).

(ii) Let \( \dim_B \hat{F} = s \). Given any \( \eta > 0 \) we have \( N_{r_k}(\hat{F}) \leq r_k^{-(s+\eta)} \) if \( k \) is sufficiently large, where as before \( r_k = Cb_k^{\epsilon} \) with \( C \) and \( \epsilon \) given by the definition of the generalized fast tangent. The generalized fast tangent hypothesis implies that for each \( x \in T_k(F) \cap [0,1] \) there is some \( \hat{x} \in \hat{F} \) such that \( \|x - \hat{x}\| \leq r_k \) and this ensures that \( N_{3r_k}(T_k(F) \cap [0,1]) \leq r_k^{-(s+\eta)} \).

Also, by hypothesis, there are \( \hat{y} \in \hat{F} \) and \( \theta > 0 \) such that \( B(\hat{y}, 2\theta) \subset [0,1]^d \) and thus, for \( k \) sufficiently large there is a point \( y_k \in T_k(F) \) so that \( B(y_k, \theta) \subset [0,1]^d \). It follows that \( N_{3r_k}(T_k(F) \cap B(y_k, \theta)) \leq r_k^{-(s+\eta)} \). Defining \( r = 3r_k a_k^{-1}, R = \theta b_k^{-1} \) and \( z_k = T_k^{-1}(y_k) \in F \), we get

\[
N_r(F \cap B(z_k, R)) \leq \left( \frac{R}{r} \right)^{1+\eta}
\]

(and hence \( \dim_{L} F \leq s \)). It is easily seen that \( r \leq R^{1+\epsilon/2} \), therefore, \( \dim_{qL} F \leq s \).

Finally, suppose that \( \dim_{qL} \hat{F} = t \), so given \( \eta > 0 \) there is some \( 0 < \delta < 1 \) and arbitrarily small \( r, R \) with \( r \leq R^{1+\delta} \) and \( \hat{y} \in \hat{F} \) such that

\[
N_r(B(\hat{y}, R) \cap \hat{F}) \leq (R/r)^{1+\eta}.
\]

The geometric condition involving \( \theta \) allows us to assume that \( B(\hat{y}, R) \subset [0,1]^d \). Now choose \( k \) such that \( Cb_k^{\epsilon} \leq r \leq Cb_k^{\epsilon} \), where \( C, \epsilon \) are as before. Then, there is \( y \in T_k(F) \cap [0,1]^d \) such that \( \|y - \hat{y}\| \leq Cb_k^{\epsilon} \) and moreover, \( B(y, \frac{1}{2} R) \subset [0,1]^d \) (for \( k \) sufficiently large). Defining \( r_k = 3Cb_k^{\epsilon} \) and \( R_k = R/2 \), it follows that for \( k \) sufficiently large,

\[
N_r \left( T_k(F) \cap B(y, R_k) \right) \leq N_r(\hat{F} \cap B(\hat{y}, R)) \leq (R/r)^{1+\eta} \approx \left( \frac{R_k}{r_k} \right)^{1+\eta},
\]

where in the last equivalence we used the fact that \( b_{k+1} \leq C'b_k \) for some constant \( C' \). For an appropriate \( z_k \in F \) and a constant \( C'' \), we get

\[
N_{\epsilon/k} \left( F \cap B(z_k, R_k) \right) \leq C'' \left( \frac{R_k/b_k}{r_k/a_k} \right)^{1+\eta}.
\]

It is easily seen that \( r_k/a_k \leq (R_k/b_k)^{1+\delta/2} \) for \( k \) sufficiently large, and therefore \( \dim_{qL} F \leq \dim_{qA} \hat{F} \).

\[ \square \]

**Corollary 13.** Suppose that \( \hat{F} \subset \mathbb{R}^d \) is a generalized fast tangent of \( F \) such that \( \dim_B \hat{F} = s \). Then \( \dim_{qL} F \leq s \leq \dim_{qA} F \) whenever \( \hat{F} \) contains an interior point of \([0,1]^d\).

**Remark 4.** The statements from Theorem 12 can be improved.

a) Part (i) only needs the one-sided hypothesis, \( p(F, T_k(F)) \leq Cb_k^{\epsilon} \to 0 \), i.e., \( \hat{F} \) is a fast pseudo-tangent of \( F \). This is immediate from the proof.
b) A quick inspection of the proof of (ii) shows that we have \( \dim_L F \leq \frac{1}{\dim_B \hat{F}} \) even if the convergence to the generalized tangent is not fast.

**Remark.** Consider the following simple example. Suppose that \( F \subset [0,1] \) and that \( (a,b) \subset [0,1] \) with \( a, b \in F \), but \( (a, b) \cap F = \emptyset \). By considering \( T_k x = 2^k (x - a) \), it is easily seen that \( \hat{F} = \{0\} \) is a fast tangent of \( F \), so, unless \( \dim_{qL} F = 0 \), the conclusion in (ii) is false in this case. This example illustrates that for the quasi-lower Assouad dimension, an additional hypothesis that ensures the tangent ‘carries’ information about the interior of \( F \) is necessary. In the statement of (ii), we have chosen to put this hypothesis directly on \( \hat{F} \). Alternatively, we could have put an additional hypothesis on the approximations of the tangent, for example requiring that \( p_H (F \cap B(z_k, Cu_k^{-1}), \hat{F}) \leq Cb_k^{-\beta} \), where \( z_k \in F \). The proof is a slight modification of the one given here.

A self-similar set \( F \subset \mathbb{R} \) that is not a point and is the attractor of an IFS that does not satisfy the weak separation property has \([0,1]\) as a generalized tangent (\([9]\)). However, combining the results from Sections 2 and 3, we see that if \( \dim_H F < 1 \), then \([0,1]\) must be a ‘slow tangent’ to the self-similar set. This is the case, for instance, with the example given in Section 3.1 of \([8]\).

We finish this section with our results to the calculation of the quasi-Assouad dimensions of a class of planar self-affine sets.

**Example 14.** In \([8]\) Sec. 2.3, Fraser determines the (lower) Assouad dimensions of self-affine carpets that are the attractor of an IFS in the extended Lalley-Gatzouras and Barański classes. These fractals, denoted \( E \), are generated by an IFS of the form \( S_j(x, y) = (c_i x, d_i y) + (a_i, b_i) \), for some \( c_i, d_i \in (0,1) \), \( 1 \leq i \leq m \), where \( c_i \neq d_i \) for at least one \( i \). (See \([8]\) for their complete definitions.) Let \( \pi_i \) denote the projection onto the \( x \)-axis and \( \pi_2 \) the projection onto the \( y \)-axis. Let \( \text{Slice}_{1,i}(E) \) (resp., \( \text{Slice}_{2,i}(E) \)) be the vertical (horizontal) slice of \( E \) through the fixed point of \( S_i \). Fraser proves that if the self-affine carpet \( E \) is of mixed type, i.e., there are \( i \neq i' \) such that \( c_i > d_i \) and \( c_{i'} < d_{i'} \), then

\[
\dim_A E = \max_i \max_{k=1,2} \left( \dim_B \pi_k(E) + \dim_B \text{Slice}_{k,i}(E) \right),
\]

\[
\dim_L E = \min_i \min_{k=1,2} \left( \dim_B \pi_k(E) + \dim_B \text{Slice}_{k,i}(E) \right).
\]

Our corollary above can be used to show that \( \dim_{qA} E = \dim_A E \) for such carpets, and similarly for the quasi-lower Assouad dimension.

To see this, we give the following sketch of the proof (see \([8]\) Sec 7.2 for more details on the definitions). We assume \( \max_i \max_{k=1,2} \left( \dim_B \pi_k(E) + \dim_B \text{Slice}_{k,i}(E) \right) = \dim_B \pi_1(E) + \dim_B \text{Slice}_{1,i}(E) \) for some \( 1 \leq i \leq m \), that now we fix. Consider the approximate square \( Q_k(i, j) \), centred at the point \( \bigcap_{l \geq 1} S^k_i \cap S^k_j([0,1]^2) \), with height \( d_j^k \) and width \( c_j^{l(k)} c_i \), where \( l(k) \) is an integer chosen so that

\[
c_j^l c_i^{l(k)} + 1 \leq d_j^k \leq c_j^l c_i^{l(k)}.
\]

Take the maps \( T_k \) that stretch by \( d_j^{-k} \) in height and by \( \left( c_j^l c_i^{l(k)} \right)^{-1} \) in width, and map the corner of \( Q_k \) to the origin. One can check these maps satisfy the required Lipschitz properties with \( b_k = d_j^{-k} \) and \( a_k = \left( c_j^l c_i^{l(k)} \right)^{-1} \). Take \( F_i = \pi_1(E) \times \pi_2(\text{Slice}_{1,i}(E)) \). This is a product of two self-similar sets satisfying the open set condition and hence

\[
\dim_B F_i = \dim_B \pi_1(E) + \dim_B \text{Slice}_{1,i}(E).
\]

From the structure of the carpet, and since \( l(k) > k \log_c(d_j/c_i) \), it can be seen that

\[
\text{dist}_H(T_k(Q_k), F_i) \leq \max_n d_n^{(k)} \leq \beta \theta^k
\]

for a suitable \( \beta > 0 \). Appealing to the corollary gives the result.
4. Further properties

4.1. Dimensions of sequences with decreasing gaps. In [16] it was shown that sequences in \( \mathbb{R} \) with decreasing gaps have Assouad dimension 0 or 1. It is easy to see that the same statement is true for the quasi-Assouad dimension.

**Proposition 15.** (i) If \( \overline{\dim}_B E = 0 \), then \( \dim_{qA} E = 0 \).

(ii) If \( E = \{a_j\}_j \subseteq [0,1] \), where \( \{a_j - a_{j+1}\}_j \) is a decreasing sequence, then \( \dim_{qA} E = 1 \) if \( \overline{\dim}_B E > 0 \) and otherwise \( \dim_{qA} E = 0 \).

**Proof.** (i) Fix \( \varepsilon, \delta > 0 \). The assumption that \( \overline{\dim}_B E = 0 \) ensures that for all sufficiently small \( r, N_r(E) \leq r^{-\delta \varepsilon} \). Thus for any \( R \geq r^{1-\delta} \) and any \( x \in E \), we have

\[
N_r(B(x, R) \cap E) \leq N_r(E) \leq r^{-\delta \varepsilon} \leq \left(\frac{R}{r}\right)^\varepsilon,
\]

from whence the conclusion is immediate.

(ii) In [11] Thm. 6.2 it is shown that if \( E \) is a sequence with decreasing gaps, then for all \( \theta \in (0,1) \), \( \dim_{qA} E = \min \left( \overline{\dim}_B E, 1 \right) \). Thus if \( \overline{\dim}_B E > 0 \), then \( \dim_{qA} E \geq \sup_{\theta < 1} \dim_{qA} E = 1 \). \( \square \)

**Remark 6.** Although it is also true that \( \dim A E = 0 \) or 1 for sequences with decreasing gaps, the criterion is different. Indeed, as noted in [12] Ex. 6.3, \( E = \{e^{-\sqrt{n}}\}_n \) is a set with decreasing gaps having \( \overline{\dim}_B E = 0 = \text{dim}_{qA} E \), but \( \dim A E = 1 \).

4.2. Different values for different dimensions. Provided the upper and lower box dimensions are to be distinct, then given any six numbers in \([0,1]\), appropriately ordered, there is a compact set \( E \subseteq [0,1] \) which have those numbers as the Assouad-type and box dimensions. We will construct Cantor sets to illustrate this, making use of the following formulas for the box and quasi-Assouad dimensions of a Cantor set \( E \) with ratios of dissection \( r_k \) at step \( k \); see [16] and [19].

\[
\overline{\dim}_B E = \lim sup_n \frac{n \log 2 \cdot |\log r_1 \cdots r_n|}{|\log r_{k+1} \cdots r_{k+n}|}, \quad \dim_{qA} E = \lim sup_k \frac{n \log 2}{|\log r_{k+1} \cdots r_{k+n}|}
\]

If \( \inf r_k > 0 \), then

\[
\dim_{qA} E = \lim_{\delta \to 0} \lim sup_n \frac{n \log 2}{|\log r_{k+1} \cdots r_{k+n}|}
\]

where \( S_{n,\delta} = \{k : r_{k+1} \cdots r_{k+n} \leq (r_1 \cdots r_k)^\delta\} \). For the lower box and (quasi)-lower Assouad dimensions replace sup by inf and lim sup respectively.

**Example 16.** Assume \( 1 \leq a \leq \alpha \leq u < v \leq \beta \leq b < \infty \) are given. There is a Cantor set \( E \subseteq [0,1] \) with \( \dim A E = 1/\alpha \), \( \dim_{qA} E = 1/\alpha \), \( \overline{\dim}_B E = 1/u \), \( \underline{\dim}_B E = 1/v \), \( \dim_{qL} E = 1/\beta \) and \( \dim_{qL} E = 1/b \).

The example we will construct is a generalization of [19] Ex. 1.18 and so we will only sketch the ideas. To begin, we choose a sequence of integers \( s_j \) tending to infinity very rapidly. For convenience, put \( t_{2j} = s_{2j} \left(\frac{u}{v}\right) \) and \( t_{2j+1} = s_{2j+1} \left(\frac{u}{v}\right) \). (If \( u = \alpha \) put \( t_{2j} = s_{2j} \) and similarly if \( v = \beta \).) We will define the ratios of dissection at the various steps as follows:

| Ratio | At steps |
|-------|----------|
| \( 2^{-u} \) | \( t_{2j-1} + j, \ldots, s_{2j} \) |
| \( 2^{-\alpha} \) | \( s_{2j} + 1, \ldots, t_{2j} \) |
| \( 2^{-a} \) | \( t_{2j} + 1, \ldots, t_{2j} + j \) |
| \( 2^{-u} \) | \( t_{2j} + j + 1, \ldots, s_{2j+1} \) |
| \( 2^{-b} \) | \( s_{2j+1} + 1, \ldots, t_{2j+1} + j \) |
| \( 2^{-b} \) | \( t_{2j+1} + 1, \ldots, t_{2j+1} + j \) |

Provided \( s_j \) tends to infinity sufficiently quickly, the ratios at steps \( \{t_{2j} + 1, \ldots, t_{2j} + j\} \) and \( \{t_{2j+1} + 1, \ldots, t_{2j+1} + j\} \) will not influence the long run averages that determine the box and quasi-(lower)
Assouad dimensions. But these ratios will determine the (lower) Assouad dimensions. The construction ensures that the quasi-Assouad dimension is determined by choosing \( R \) to be the length of the Cantor intervals at step \( s_j \) and \( r \) to be the length of Cantor intervals at step \( t_{2j} \), while the quasi-lower dimensions arise with \( R \) the length at step \( s_{2j+1} \) and \( r \) the length at step \( t_{2j+1} \). The choice of \( t_{2j} \) and \( t_{2j+1} \) are made to ensure that the geometric means of the ratios stay within the range \([2^{-r}, 2^{-u}]\) (in the limit) so that the box dimensions are determined along the subsequences of lengths of Cantor intervals at steps \( s_j \). The details are left to the reader.

4.3. Dimensions of orthogonal projections. For our last example, we will show, that as in the case of Assouad dimension (see \( \text{[5]} \) and also \( \text{[10]} \)), the quasi-Assouad dimension may increase under orthogonal projections. As before, we will let \( \pi_x \) (resp., \( \pi_y \)) denote the projection onto the \( x \) (resp. \( y \)) axis.

**Proposition 17.** There is a subset \( E \subseteq \mathbb{R}^2 \) such that

\[
\dim_{qA} \pi_x(E) = 1 > 1/2 = \dim_{qA} E.
\]

**Proof.** We will construct an example to show this. For each \( j \) and \( i = 1, \ldots, 2^j \), let \( x_{ij} = 2^{-j} + (i - 1)2^{-2j} \). The points \( x_{ij} \) belong to \([2^{-j}, 2^{-j+1}]\) and are spaced \( 2^{-2j} \) apart. Let \( y_{ij}, i = 1, \ldots, 2^j \), be the endpoints, ordered from left to right, of the gaps created at step \( j \) in the standard construction of the Cantor set with ratio of dissection \( 1/4 \); these gaps have length \( 2^{-2j+1} \). Put \( E_j = \{ (x_{ij}, y_{ij}) : i = 1, \ldots, 2^j \} \) and \( E = \bigcup_j E_j \). See Figure 2 below. Of course, \( \pi_x(E) = \bigcup_j \{ x_{ij} : i = 1, \ldots, 2^j \} \). By checking \( N_r(B(x_0, R) \cap \pi_x(E)) \) for \( x_0 = 2^{-j}, R = 2^{-j} \) and \( r = 2^{-2j} \), it is easy to see that \( \dim_{qA} \pi_x(E) = 1 \).

To determine the quasi-Assouad dimension of \( E \), it is convenient to take as the definition of a ‘ball’, \( B(x_0, R) \) in \( \mathbb{R}^2 \), the square with centre \( x_0 \) and sides of length \( R \). Fix such a ball with \( x_0 = (x_{ij}, y_{ij}) \in E \) and assume \( 2^{-2(s+1)} < R \leq 2^{-2s} \) for some \( s \in \mathbb{N} \). The size of \( R \) ensures that the interval \( \pi_y(B(x_0, R)) \) can intersect only one Cantor interval of step \( s \). Choose any \( r < R \), say \( 2^{-2t} < r \leq 2^{-2(t-1)} \).

First, note that

\[
(B(x_0, R) \cap E) \cap ([0, 2^{-2t}) \times [0, 1]) =: \Omega
\]

is contained in \([0, 2^{-2t}) \times (\text{union of Cantor intervals of step } t \text{ contained in } \pi_y(B(x_0, R)) \). There are at most \( 2^{t-s} \) such Cantor intervals, each of length \( 2^{-2t} \). Hence \( N_r(\Omega) \leq 2^{t-s} \).
Next, for each \( m \in \{t+1, \ldots, 2t\} \) consider the elements of
\[
(B(x_0, R) \cap E) \cap (\{2^{-m}, 2^{-(m-1)}\} \times [0, 1]) := \Omega_m.
\]
The \( y \)-coordinates of these points are the endpoints of the gaps at step \( m \) lying within the one Cantor interval of step \( s \) that \( \pi_y(B(x_0, R)) \) intersects. There are \( 2^{m-t} \) of these contained within each Cantor sub-interval of step \( t \). As the distance between consecutive \( x \)-coordinates is \( 2^{-2m} \), the total horizontal distance between the points whose \( y \)-coordinates lie in a (fixed) Cantor interval of step \( t \) is \( 2^{-2m}(2^{m-t}) = 2^{-(m+t)} \leq 2^{-2t} \), while the total vertical distance is the width of the Cantor subinterval, \( 2^{-2t} \). Consequently, such points lie within a square of side length \( 2^{-2t} \) and hence we can cover \( \Omega_m \) with \( 2^{t-s} \) squares of side length \( 2^{-2t} \) for each such \( m \).

Finally, observe that the cardinality of the remainder of \( B(x_0, R) \cap E \), which is contained in \([2^{-t}, 1] \times [0, 1] \), is dominated by \( 2 \) times the number of gaps of step \( \leq t \) within a Cantor interval of step \( s \), and this is bounded above by \( 2^{t-s+1} \). Combining together all these observations, we see that
\[
N_s(B(x_0, R) \cap E) \leq (t+1)2^{-s} + 2^{t-s+1}.
\]
It follows that for each \( \delta > 0 \), \( h_E(\delta) \leq 1/2 \) and thus \( \dim_{qA} E \leq 1/2 \). It is not difficult to see that these estimates are essentially sharp and thus we actually have \( \dim_{qA} E = 1/2 \).

\[\square\]

Remark 7. The classical Marstrand projection theorem, and its more recent variants, (c.f. \cite{6}) states that the orthogonal projections of planar sets have the same dimension at almost every angle, where here dimension can be Hausdorff, upper/lower box or packing. This is not the case for the Assouad dimension. Indeed, it is shown in \cite{10} that for any \( s \) with \( \log_3 3 < s < 1 \), there exists a self-similar set \( F \subset \mathbb{R}^2 \) and two non-empty intervals \( I, J \) such that \( \dim_{qA} \pi_{\theta} F = s \) for all \( \theta \in I \), while \( \dim_{qA} \pi_{\theta} F = 1 \) for almost all \( \theta \in J \). Here \( \pi_{\theta} \) denotes the projection onto the line passing through the origin with angle \( \theta \).

It is unknown what the situation is for the quasi-Assouad dimension. The set \( F \) from \cite{10} is not helpful in resolving this problem since its orthogonal projections are again self-similar sets, and hence the Marstrand projection theorem and Theorem 1 imply that the quasi-Assouad dimensions of the projections of \( F \) are constant almost everywhere.

5. ACKNOWLEDGEMENTS

We thank the referee for his/her helpful suggestions. We also thank Jonathan Fraser for his comments and, in particular, for pointing out that our Theorem 1 answers Problem 1.16 in \cite{2}.

References

[1] J. Angelevska and S. Troscheit, \textit{A dichotomy of self-conformal subsets of \( R \) with overlaps}, arXiv:1602.05821, 2016.
[2] R. Balka and Y. Peres, \textit{Uniform dimension results for fractional Brownian motion}, J. Fractal Geometry, to appear, Arxiv 1509.02979, 2015.
[3] C. Chen, M. Wu and W. Wu, \textit{Accessible values of Assouad and the lower dimensions of subsets}, Arxiv 1602.02180, 2016.
[4] K. Falconer, \textit{Dimensions and measures of quasi self-similar sets}, Proc. Amer. Math. Soc., 106(2):543–554, 1989.
[5] K. Falconer, \textit{Sub-self-similar sets}, Trans. Amer. Math. Soc., 347(8):3121–3129, 1995.
[6] K. Falconer, J. Fraser and X. Jin, \textit{Sixty years of fractal projections} in Fractal geometry and stochastics V, 3-25, Progr. Prob. 70, Birkhauser/Springer, Cham, 2015.
[7] A. Farkas and J. Fraser, \textit{On the equality of Hausdorff measure and Hausdorff content}, J. Fractal Geom., 2(4):403–429, 2015.
[8] J. Fraser, \textit{Assouad type dimensions and homogeneity of fractals}, Trans. Amer. Math. Soc., 336:6687–6783, 2014.
[9] J. Fraser, A. Henderson, E. Olson and J. Robinson, \textit{On the Assouad dimension of self-similar sets with overlaps}, Adv. Math., 273:188-214, 2015.
[10] J. Fraser and T. Orponen, \textit{The Assouad dimensions of projections of planar sets}, Proc. London Math. Soc., 114(2):374–398, 2017.
[11] J. Fraser and H. Yu, \textit{Assouad type spectra for some fractal families}, arXiv:1611.08857, 2016.
[12] J. Fraser and H. Yu, \textit{New dimension spectra: finer information on scaling and homogeneity}, arXiv:1610.02334, 2016.
[13] J. Fraser and H. Yu, \textit{Arithmetic patches, weak tangents, and dimension}, Arxiv: 1611.06960, 2016.
[14] H. Furstenberg, Ergodic fractal measures and dimension conservation, Ergodic Theory Dynam. Systems, 28(02):405-422, 2008.
[15] H. Furstenberg, Ergodic theory and fractal geometry, CBMS Reg. Conf. Ser. Math., AMS 120, 2014.
[16] I. García, K. Hare and F. Mendivil, Assouad dimensions of complementary sets, to appear Proc. Roy. Soc. Edinburgh Sect. A, Arxiv 1604.01234, 2016.
[17] A. Käenmäki, T. Ojala and E. Rossi, Rigidity of quasisymmetric mappings on self-affine carpets, Int. Math. Res. Notices, to appear, Arxiv 1607.02244, 2016.
[18] A. Käenmäki and E. Rossi, Weak separation condition, Assouad dimension, and Furstenberg homogeneity, Ann. Acad. Sci. Fenn. Math., 41(1):465–490, 2016.
[19] F. Lü and L. Xi, Quasi-Assouad dimension of fractals, J. Fractal Geom., 3(2):187-215, 2016.
[20] J. Mackay, Assouad dimension of self-affine carpets, Conform. Geom. Dyn., 15(12):177-187, 2011.
[21] J. Mackay and J. Tyson, Conformal dimension: theory and applications, Univ. Lecture Ser. 54, 2010.
[22] T. Rajala and M. Vilppolainen, Weakly controlled Moran constructions and iterated functions systems in metric spaces, Illinois J. Math., 55(3):1015–1051, 2011.
[23] Q. Wang and L. Xi, Quasi-Lipschitz equivalence of quasi Ahlfors-David regular sets, Sci. China Math., 54(12):2573-2582, 2011.
[24] M. Zerner, Weak separation properties for self-similar sets, Proc. Amer. Math. Soc., 124(11):3529-3539, 1996.

Dept. of Pure Mathematics, University of Waterloo, Waterloo, Ont., Canada N2L 3G1
Current address: Depto. de Matemática, Facultad de Ciencias Exactas y Naturales, Instituto de Investigaciones Físicas de Mar del Plata (CONICET), Universidad Nacional de Mar del Plata, Argentina
E-mail address: nacholma@gmail.com

Dept. of Pure Mathematics, University of Waterloo, Waterloo, Ont., Canada N2L 3G1
E-mail address: kehare@uwaterloo.ca