A $d$-dimensional Brownian motion as a weak limit from a one-dimensional Poisson process

Xavier Bardina$^\dagger$, Carles Rovira $^*$

$^\dagger$ Dept. Matemàtiques, Edifici C, Universitat Autònoma de Barcelona, 08193-Bellaterra

$^*$ Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, 08007-Barcelona

E-mail: Xavier.Bardina@uab.cat, Carles.Rovira@ub.edu

Abstract

We show how from an unique standard Poisson process we can build a family of processes that converges in law to a $d$-dimensional standard Brownian motion for any $d \geq 1$.

Keywords: $d$-dimensional Brownian motion, weak approximations, Poisson process

1 Introduction and main result

Consider the sequences of processes

$$\{z^\theta_\varepsilon(t) = \varepsilon \int_0^{2\pi} \cos(\theta N_s) ds, \quad t \in [0, T]\},$$

$$\{y^\theta_\varepsilon(t) = \varepsilon \int_0^{2\pi} \sin(\theta N_s) ds, \quad t \in [0, T]\},$$

where $\{N_s, s \geq 0\}$ is a standard Poisson process.

When $\theta = 0$, the processes $z^\theta_\varepsilon$ are deterministic and go to infinity when $\varepsilon$ tends to zero. On the other hand, when $\theta = \pi$, the processes $z^\theta_\varepsilon$ can be written as

$$z^\theta_\varepsilon(t) = \varepsilon \int_0^{2\pi} (-1)^{N_s} ds.$$

This case was studied by Stroock in [4], where he proved that the laws of these processes in the space of continuous functions on $[0, T]$ converge weakly towards the law of $\sqrt{2} W_t$, where $\{W_t; t \in [0, T]\}$ is a standard Brownian motion. When $\theta = 0$ or $\theta = \pi$, the processes $y^\theta_\varepsilon$ are constant and equal to zero.

$^*$Corresponding author.
When $\theta \in (0, \pi) \cup (\pi, 2\pi)$ it is proved in [1] that when $\varepsilon$ tends to zero the processes $x_\theta^\varepsilon := (z_\theta^\varepsilon, y_\theta^\varepsilon)$ converge weakly towards two independent standard Brownian motions.

The aim of this paper is to extend this result to a $d$-dimensional case for any $d \geq 1$.

Now, given $\theta_1, \ldots, \theta_n, \theta_{n+1}, \ldots, \theta_{n+m}$ let us consider the process:

$$\{x_{\varepsilon}^{\theta_1, \ldots, \theta_n, \theta_{n+1}, \ldots, \theta_{n+m}}(t) = (z_{\varepsilon}^{\theta_1}, \ldots, z_{\varepsilon}^{\theta_n}, y_{\varepsilon}^{\theta_{n+1}}, \ldots, y_{\varepsilon}^{\theta_{n+m}})(t), \quad t \in [0, T]\}.$$  

For simplicity we will denote by $\theta$ the $n+m$ values $\theta_1, \ldots, \theta_n, \theta_{n+1}, \ldots, \theta_{n+m}$.

We will assume the following hypothesis $(H)$ on $\theta$:

- $\theta_i \in (0, \pi) \cup (\pi, 2\pi), 1 \leq i \leq n+m$,
- $\theta_i + \theta_j \neq 2\pi$ for all $1 \leq i, j \leq n+m$.
- $\theta_i - \theta_j \neq 0$ for all $1 \leq i, j \leq n$ and $n+1 \leq i, j \leq n+m$.

Let us point out the meaning of the last hypothesis: two parameters $\theta_i$ and $\theta_j$ can only be equal if $i \leq n$ and $j \geq n+1$. In other words we can deal with $\varepsilon \int_0^{2\pi} \cos(\theta_i N_s)ds$ and $\varepsilon \int_0^{2\pi} \sin(\theta_i N_s)ds$, but the results will not be possible (obviously) if we have two times $\varepsilon \int_0^{2\pi} \cos(\theta_i N_s)ds$ or $\varepsilon \int_0^{2\pi} \sin(\theta_i N_s)ds$.

Our result states as follows,

**Theorem 1.1** Consider $P_\theta^\varepsilon$ the image law of $x_\varepsilon^\theta$ in the Banach space $C([0, T], \mathbb{R}^{n+m})$ of continuous functions on $[0, T]$. If $\theta$ satisfies hypothesis $(H)$ then $P_\theta^\varepsilon$ converges weakly as $\varepsilon$ tends to zero towards the law on $C([0, T], \mathbb{R}^{n+m})$ of a $n+m$-dimensional standard Brownian motion.

**Remark 1.2** It will be also possible to consider the case $\theta_i = \pi$ for some $i \in \{1, \ldots, n\}$. In this case, we need to deal with $\frac{1}{\sqrt{2}}z_\varepsilon^{\theta_i}$ instead of $z_\varepsilon^{\theta_i}$. Nevertheless, the proof follows the same computations.

The structure of the paper is the following. In Section 2 we recall the basic results of [1] that implies our theorem when $m = n = 1$ and $\theta_1 = \theta_2$. In Section 3 we give the proof of our main theorem.

Along the paper $K$ denote positive constants, not depending on $\varepsilon$, which may change from one expression to another one.

**2 The two-dimensional case**

In [1] it is proved an approximation in law of the complex Brownian motion by processes constructed from a unique standard Poisson process.
Theorem 2.1 [\textit{Theorem 1.1}] Define for any $\varepsilon > 0$

$$\{v_\varepsilon^\theta(t) = \varepsilon \int_0^t e^{i\theta N_s} ds, \quad t \in [0,T]\}$$

where $\{N_s, s \geq 0\}$ is a standard Poisson process. Consider $P_\varepsilon^\theta$ the image law of $v_\varepsilon^\theta$ in the Banach space $C([0,T],C)$ of continuous functions on $[0,T]$. Then, if $\theta \in (0,\pi) \cup (\pi,2\pi)$, $P_\varepsilon^\theta$ converges weakly as $\varepsilon$ tends to zero towards the law on $C([0,T],C)$ of a complex Brownian motion.

In fact, it corresponds to our two-dimensional case with $n = m = 1$ and $\theta_1 = \theta_2 = \theta$ for $\theta \in (0,\pi) \cup (\pi,2\pi)$. Set $P_\varepsilon^\theta$ the image law of $x_\varepsilon^\theta := (z_\varepsilon^\theta, y_\varepsilon^\theta)$ in the space $C([0,T],\mathbb{R}^2)$.

The proof of the weak convergence is obtained checking that the family $P_\varepsilon^\theta$ is tight and that the law of all possible weak limits of $P_\varepsilon^\theta$ is the law of two independent standard Brownian motions.

The tightness is proved using the Billingsley criterium (see Theorem 12.3 of [\textit{2}]). Since the processes are null on the origin it suffices to prove the following lemma (see Lemma 2.1 in [\textit{1}]).

Lemma 2.2 There exists a constant $K$ such that for any $s < t$

$$\sup_\varepsilon \left( E(\varepsilon \int_0^s \cos(\theta N_x) dx)^4 + E(\varepsilon \int_0^s \sin(\theta N_x) dx)^4 \right) \leq K(t-s)^2.$$ 

In order to identify the limit law, it is considered $\{P_{\varepsilon_n}^\theta\}_n$ a subsequence of $\{P_\varepsilon^\theta\}_\varepsilon$ (that is also denoted by $\{P_\varepsilon^\theta\}_\varepsilon$) weakly convergent to some probability $P^\theta$. Then, it is checked that the two components of the canonical process $X = (Z,Y) = \{X_t(x) = x(t) = (z(t),y(t))\}$ under the probability $P^\theta$ are two independent Brownian motions.

Using Paul Lévy’s theorem (see Theorem 3.1 below) it suffices to prove that under $P^\theta$, $Z$ and $Y$ are both martingales with respect to the natural filtration, $\{F_t\}$, with quadratic variations $<Z,Z> = t$, $<Y,Y> = t$ and covariation $<Z,Y> = 0$.

To check the martingale property with respect to the natural filtration $\{F_t\}$, it is proved (see subsection 3.1 in [\textit{1}]) that for any $s_1 \leq s_2 \leq \cdots \leq s_k \leq s < t$ and for any bounded continuous function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$E_{P^\theta} [\varphi(X_{s_1},...,X_{s_k}) (Z_t - Z_s)] = 0,$$

$$E_{P^\theta} [\varphi(X_{s_1},...,X_{s_k}) (Y_t - Y_s)] = 0.$$

The computation of the quadratic variations and covariation is done in the following proposition (see Proposition 3.1 in [\textit{1}]).

Proposition 2.3 Consider $\{P_{\varepsilon_n}^\theta\}$ the laws on $C([0,T],\mathbb{R}^2)$ of the processes $x_\varepsilon^\theta$ and assume that $P_{\varepsilon_n}^\theta$ is a subsequence weakly convergent to $P^\theta$. Let $X = (Z,Y)$ be the canonical process and let $\{F_t\}$ be its natural filtration. Then, under $P^\theta$, if $\theta \in (0,\pi) \cup (\pi,2\pi)$ it is hold that the quadratic variations $<Z,Z> = t$, $<Y,Y> = t$, and the covariation $<Z,Y> = 0$. 

3
3 Proof of the main result

In this section we will give the proof of Theorem 1.1. We will follow the same method than in [1]. So, it suffices to check the tightness of the family $P_\varepsilon^d$ and to identify the law of all possible weak limits of $P_\varepsilon^d$.

The tightness is proved also using the Billingsley criterium, and it is an obvious consequence of Lemma 2.2.

The identification of the limit law will be done using Paul Lévy’s theorem.

**Theorem 3.1** Let $X = \{X_t = (X^{(1)}_t, \ldots, X^{(d)}_t), \mathcal{F}_t, 0 \leq t < \infty\}$ be a continuous, adapted process in $\mathbb{R}^d$ such that, for any component $1 \leq k \leq d$ the process

$$M^{(k)}_t := X^{(k)}_t - X^{(k)}_0, \quad 0 \leq t < \infty$$

is a continuous local martingale relative to $\{\mathcal{F}_t\}$ and the cross-variations are given by $< M^{(k)}, M^{(j)} >_{t} = \delta_{k,j}t$, for $1 \leq k,j \leq d$. Then $\{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ is a $d$-dimensional Brownian motion.

Let us consider $\{P_\varepsilon^d\}_\varepsilon$ a subsequence of $\{P_\varepsilon^d\}_\varepsilon$ (that we also will denote by $\{P^d\}_\varepsilon$) weakly convergent to some probability $P^0$. Consider the canonical process $X = (Z^1, \ldots, Z^n, Y^{n+1}, \ldots, Y^{n+m})$ under the probability $P^0$. It suffices to check that under $P^0$, $Z^i, 1 \leq i \leq n$, and $Y^j, n+1 \leq j \leq n+m$, are martingales with respect to the natural filtration, $\{\mathcal{F}_t\}$, with quadratic variations $< Z^i, Z^i >_{t=1} = 1 \leq i \leq n$, $< Y^j, Y^j >_{t=1} = n + 1 \leq j \leq n + m$, and covariance $< Z^i, Z^j >_{t=1} = n, n+1 \leq j \neq h \leq n+m$.

In order to check the martingale property with respect to the natural filtration $\{\mathcal{F}_t\}$ it suffices to prove that for any $s_1 \leq s_2 \leq \cdots \leq s_k \leq s < t$ and for any bounded continuous function $\varphi : \mathbb{R}^{(n+m)k} \rightarrow \mathbb{R}$,

$$E_{P^0}[\varphi(X_{s_1}, \ldots, X_{s_k})(Z^i_s - Z^i_t)] = 0, \quad \text{for} \quad 1 \leq i \leq n,$$

$$E_{P^0}[\varphi(X_{s_1}, \ldots, X_{s_k})(Y^j_s - Y^j_t)] = 0, \quad \text{for} \quad n+1 \leq j \leq n+m.$$

These computations has been done in subsection 3.1 in [1].

The proof of the quadratic variations can be done following exactly the proof of Proposition 3.1 in [1]. So, it remains only to compute all the covariations.

First we have to prove that $< Z^i, Z^j >_{t=1} = 0$, for $1 \leq i \neq j \leq n$. It suffices to prove that for any $s_1 \leq s_2 \leq \cdots \leq s_k \leq s < t$ and for any bounded continuous function $\varphi : \mathbb{R}^{(n+m)k} \rightarrow \mathbb{R}$,

$$E[\varphi(x^{(i)}_s(s_1), \ldots, x^{(i)}_s(s_k)) (z^{(i)}_s(t) - z^{(i)}_t(s)) (z^{(j)}_s(t) - z^{(j)}_t(s))]$$

converges to zero when $\varepsilon$ tends to zero. Notice that this last expression can be written as

$$E \left( \varphi (x^{(i)}_s(s_1), \ldots, x^{(i)}_s(s_k)) \left( \varepsilon \int_{\frac{s_1}{\varepsilon}}^{\frac{s}{\varepsilon}} \cos(\theta_1 N_x) dx \right) \left( \varepsilon \int_{\frac{s_1}{\varepsilon}}^{\frac{s}{\varepsilon}} \cos(\theta_1 N_x) dx \right) \right). \quad (1)$$
Similarly, to prove that $< Y^j, Y^h >_t = 0$, for $n + 1 \leq j \neq h \leq n + m$, it is enough to prove that for any $s_1 \leq s_2 \leq \cdots \leq s_k \leq s < t$ and for any bounded continuous function $\varphi : \mathbb{R}^{(n+m)k} \rightarrow \mathbb{R}$,

$$E \left( \varphi \left( x^\theta_0(s_1), \ldots, x^\theta_0(s_k) \right) \left( \varepsilon \int_{\frac{s}{2}}^{\frac{s}{2}} \sin(\theta_1 N_x)dx \right) \left( \varepsilon \int_{\frac{s}{2}}^{\frac{s}{2}} \sin(\theta_h N_x)dx \right) \right)$$

converges to zero when $\varepsilon$ tends to zero.

Finally, to prove that $< Z^j, Y^j >_t = 0$, for $1 \leq i \leq n < n + 1 \leq j \leq n + m$, it is enough to prove that for any $s_1 \leq s_2 \leq \cdots \leq s_k \leq s < t$ and for any bounded continuous function $\varphi : \mathbb{R}^{(n+m)k} \rightarrow \mathbb{R}$,

$$E \left( \varphi \left( x^\theta_0(s_1), \ldots, x^\theta_0(s_k) \right) \left( \varepsilon \int_{\frac{s}{2}}^{\frac{s}{2}} \cos(\theta_i N_x)dx \right) \left( \varepsilon \int_{\frac{s}{2}}^{\frac{s}{2}} \sin(\theta_j N_x)dx \right) \right)$$

converges to zero when $\varepsilon$ tends to zero.

Let us finish the proof of the theorem checking the convergence to zero of (1), (2) and (3) when $\varepsilon$ tends to zero. For simplicity we will use only $\theta_1$ and $\theta_2$.

**Study of (3).** Notice that (2) is equal to

$$\varepsilon^2 \int_{\frac{s}{2}}^{\frac{s}{2}} \int_{\frac{s}{2}}^{\frac{s}{2}} E \left( \varphi \left( x^\theta_0(s_1), \ldots, x^\theta_0(s_k) \right) \sin(\theta_1 N_x_1) \sin(\theta_2 N_x_2) \right) dx_1 dx_2$$

$$+ \varepsilon^2 \int_{\frac{s}{2}}^{\frac{s}{2}} \int_{\frac{s}{2}}^{\frac{s}{2}} E \left( \varphi \left( x^\theta_0(s_1), \ldots, x^\theta_0(s_k) \right) \sin(\theta_1 N_x_1) \sin(\theta_2 N_x_2) \right) dx_2 dx_1$$

$$:= I_1 + I_2.$$ 

Using that $\sin(a) \sin(b) = \frac{\cos(a-b) - \cos(a+b)}{2}$ we obtain that

$$I_1 = \frac{1}{2} \varepsilon^2 \int_{\frac{s}{2}}^{\frac{s}{2}} \int_{\frac{s}{2}}^{\frac{s}{2}} E \left( \varphi \left( x^\theta_0(s_1), \ldots, x^\theta_0(s_k) \right) \cos(\theta_1 N_x_1 - \theta_2 N_x_2) \right) dx_1 dx_2$$

$$- \frac{1}{2} \varepsilon^2 \int_{\frac{s}{2}}^{\frac{s}{2}} \int_{\frac{s}{2}}^{\frac{s}{2}} E \left( \varphi \left( x^\theta_0(s_1), \ldots, x^\theta_0(s_k) \right) \cos(\theta_1 N_x_1 + \theta_2 N_x_2) \right) dx_1 dx_2$$

$$:= I_{1,1} - I_{1,2}.$$ 

We start with the term $I_{1,1}$. Notice that

$$I_{1,1}$$

$$= \frac{1}{2} \text{Re} \left[ \varepsilon^2 \int_{\frac{s}{2}}^{\frac{s}{2}} \int_{\frac{s}{2}}^{\frac{s}{2}} E \left( \varphi \left( x^\theta_0(s_1), \ldots, x^\theta_0(s_k) \right) e^{i(\theta_1 N_x_1 - \theta_2 N_x_2)} \right) dx_1 dx_2 \right]$$

$$= \frac{1}{2} \text{Re} \left[ \varepsilon^2 \int_{\frac{s}{2}}^{\frac{s}{2}} \int_{\frac{s}{2}}^{\frac{s}{2}} E \left( \varphi \left( x^\theta_0(s_1), \ldots, x^\theta_0(s_k) \right) \right. \right.$$

$$\left. \times e^{i(\theta_1 - \theta_2)N_x_1} e^{-i(\theta_1 - \theta_2)(N_x_1 - N_x_2)} e^{-i\theta_2(N_x_2 - N_x_1)} \right) dx_1 dx_2 \right].$$
Using the independence of the increments of the Poisson process and that \( E(e^{i\theta N_s}) = e^{-\theta(1-e^{\pi})} \) we get

\[
I_{1,1} = \frac{1}{2} \text{Re} \left[ \epsilon^2 \int_{\frac{2\pi}{N}}^{i\theta} \int_{\frac{2\pi}{N}}^{i\theta} e \left( e^{i(\theta_1-\theta_2)(N_{s_1}-N_{s_2})} \right) E \left( e^{i(\theta_1-\theta_2)(N_{x_1}-N_{x_2})} \right) dx_1 dx_2 \right] 
\]

\[
\leq K\epsilon^2 \int_{\frac{2\pi}{N}}^{i\theta} \int_{\frac{2\pi}{N}}^{i\theta} e^{-\left(1-e^{i(\theta_1-\theta_2)}\right)} e^{-\left(1-e^{-i\theta_2}\right)} dx_1 dx_2 
\]

\[
= K\epsilon^2 \int_{\frac{2\pi}{N}}^{i\theta} \int_{\frac{2\pi}{N}}^{i\theta} e^{-\left(1-e^{i(\theta_1-\theta_2)}\right)} e^{-\left(1-e^{-i\theta_2}\right)} dx_1 dx_2 
\]

\[
\leq K\epsilon^2 \frac{1}{1-\cos(\theta_1)1-\cos(\theta_1-\theta_2)} 
\]

that clearly converges to zero when \( \epsilon \) tends to zero because \( \theta_2 \neq 0 \) and \( \theta_1 - \theta_2 \neq 0 \).

Using the decomposition

\[
e^{i(\theta_1 N_{x_1} + \theta_2 N_{x_2})} = e^{i\theta_2(N_{x_2}-N_{x_1})} e^{i(\theta_1+\theta_2)(N_{x_1}-N_{x_2})} e^{i(\theta_1+\theta_2)N_{x_2}}
\]

and following the same computations we obtain that

\[
I_{1,2} \leq K\epsilon^2 \frac{1}{1-\cos(\theta_1)1-\cos(\theta_1-\theta_2)} 
\]

This last expression also goes to zero because \( \theta_2 \in (0, \pi) \cup (\pi, 2\pi) \) and \( \theta_1 + \theta_2 \neq 2\pi \).

On the other hand, the expression \( I_2 \) is equal to the expression \( I_1 \) interchanging the roles of \( \theta_1 \) and \( \theta_2 \). So, we obtain that,

\[
I_2 \leq K\epsilon^2 \frac{1}{1-\cos(\theta_1) \left( \frac{1}{1-\cos(\theta_2-\theta_1)} + \frac{1}{1-\cos(\theta_1+\theta_2)} \right)} 
\]

This last expression also goes to zero because \( \theta_1 \in (0, \pi) \cup (\pi, 2\pi) \), \( \theta_2 - \theta_1 \neq 0 \) and \( \theta_1 + \theta_2 \neq 2\pi \).
**Study of (1).** By the same computations and using that \( \cos(a) \cos(b) = \frac{\cos(a+b)+\cos(a-b)}{2} \) we have also that,

\[
E \left( \varphi \left( x^\theta_1(s_1), \ldots, x^\theta_2(s_k) \right) \left( \varepsilon \int_{2\pi}^{2\pi} \cos(\theta_1 N_x) \, dx \right) \left( \varepsilon \int_{2\pi}^{2\pi} \cos(\theta_2 N_x) \, dx \right) \right)
= \varepsilon^2 \int_{2\pi}^{2\pi} \int_{2\pi}^{2\pi} E \left( \varphi \left( x^\theta_1(s_1), \ldots, x^\theta_2(s_k) \right) \cos(\theta_1 N_{x_1}) \cos(\theta_2 N_{x_2}) \right) \, dx_1 \, dx_2
+ \varepsilon^2 \int_{2\pi}^{2\pi} \int_{2\pi}^{2\pi} E \left( \varphi \left( x^\theta_1(s_1), \ldots, x^\theta_2(s_k) \right) \sin(\theta_1 N_{x_1}) \cos(\theta_2 N_{x_2}) \right) \, dx_2 \, dx_1
= I_{1,1} + I_{1,2} + I_{2,1} + I_{2,2},
\]

where \( I_{1,1} \) and \( I_{1,2} \) has been defined in the study of (2) and \( I_{2,1} \) and \( I_{2,2} \) will be the analogous decomposition of \( I_2 \). So, we have the convergence to zero when \( \varepsilon \) tends to zero.

**Study of (3).** Again the same computations yield that,

\[
E \left( \varphi \left( x^\theta_1(s_1), \ldots, x^\theta_2(s_k) \right) \left( \varepsilon \int_{2\pi}^{2\pi} \sin(\theta_1 N_x) \, dx \right) \left( \varepsilon \int_{2\pi}^{2\pi} \cos(\theta_2 N_x) \, dx \right) \right)
= \varepsilon^2 \int_{2\pi}^{2\pi} \int_{2\pi}^{2\pi} E \left( \varphi \left( x^\theta_1(s_1), \ldots, x^\theta_2(s_k) \right) \sin(\theta_1 N_{x_1}) \cos(\theta_2 N_{x_2}) \right) \, dx_1 \, dx_2
+ \varepsilon^2 \int_{2\pi}^{2\pi} \int_{2\pi}^{2\pi} E \left( \varphi \left( x^\theta_1(s_1), \ldots, x^\theta_2(s_k) \right) \sin(\theta_1 N_{x_1}) \sin(\theta_2 N_{x_2}) \right) \, dx_2 \, dx_1
= J_1 + J_2.
\]

Using that \( \sin(a) \cos(b) = \frac{\sin(a+b)+\sin(a-b)}{2} \) we obtain that

\[
J_1 = \frac{1}{2} \varepsilon^2 \int_{2\pi}^{2\pi} \int_{2\pi}^{2\pi} E \left( \varphi \left( x^\theta_1(s_1), \ldots, x^\theta_2(s_k) \right) \sin(\theta_1 N_{x_1} - \theta_2 N_{x_2}) \right) \, dx_1 \, dx_2
+ \frac{1}{2} \varepsilon^2 \int_{2\pi}^{2\pi} \int_{2\pi}^{2\pi} E \left( \varphi \left( x^\theta_1(s_1), \ldots, x^\theta_2(s_k) \right) \sin(\theta_1 N_{x_1} + \theta_2 N_{x_2}) \right) \, dx_1 \, dx_2
=: J_{1,1} + J_{1,2}.
\]

Observe that

\[
J_{1,1} = \frac{1}{2} \text{Im} \left[ \varepsilon^2 \int_{2\pi}^{2\pi} \int_{2\pi}^{2\pi} E \left( \varphi \left( x^\theta_1(s_1), \ldots, x^\theta_2(s_k) \right) e^{i(\theta_1 N_{x_1} - \theta_2 N_{x_2})} \right) \, dx_1 \, dx_2 \right],
\]

that is, \( J_{1,1} \) is the imaginary part of the same expression that \( I_{1,1} \) is the real part. So, the same computations for \( I_{1,1} \) show the convergence to zero of \( J_{1,1} \).
when $\varepsilon$ tends to 0. The same parallelism can be done between the terms $J_{1,2}$, $J_{2,1}$, $I_{1,2}$, $I_{2,1}$, $I_{2,2}$, respectively.

The proof of Theorem 1.1 is now complete.

Acknowledgements

Xavier Bardina and Carles Rovira are partially supported by MEC-FEDER grants MTM2006-06427 and MTM2006-01351, respectively.

References

[1] Bardina, X. The complex Brownian motion as a weak limit of processes constructed from a Poisson process. In: Stochastic Analysis and Related Topics VII. Proceedings of the 7th Silivri Workshop, Kusadasi 1998, pp. 149-158. Progress in Probability, Birkhuser. (2001)

[2] Billingsley, P. Convergence of Probability Measures. John Wiley and Sons. (1968).

[3] Lévy, P. Processus Stochastiques et Mouvement Brownien. Gauthier-Villars. (1948).

[4] Stroock, D. Topics in Stochastic Differential Equations (Tata Institute of Fundamental Research, Bombay.) Springer Verlag. (1982).