Generalized abelian coset conformal field theories.

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December 1999

Abstract

The reductions of conformal field theories which lead to generalized abelian cosets are studied. Primary fields and correlation functions of arbitrary abelian coset conformal field theory are explicitly expressed in terms of those of the original theory. The coset theory has global abelian symmetry.

PACS 11.25.Hf.

Keywords: conformal field theory, coset models, extended symmetry algebras.

In this paper we study the conformal field theory based on the generalized coset construction [1, 2]

\[ K(m) = L(m) - L_u(m), \quad m \in \mathbb{Z}, \]

where \( L(m) \), \( L_u(m) \) are Virasoro generators and \( L_u(m) \) are quadratic in affine \( \hat{u}(1)^d \) currents. (For a review see [3, 4]).

When \( L(m) \) are given by Sugawara construction for the affine Lie algebra \( \hat{\mathfrak{g}} \) we have the \( g/u(1)^d, 1 \leq d \leq \text{rank} \, g \), coset. The \( su(2)/u(1) \) theory

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is equivalent to the $Z_k$ invariant parafermion theory whose studies were initiated by Fateev and Zamolodchikov [3]. Various aspects of $g/u(1)^d$ coset theories for arbitrary simple $g$ were investigated in Refs. [6-10].

There are many other conformal models which have $\hat{u}(1)^d$ affine subalgebras and can be reduced to coset (1). The class of such models includes $N = 2$ superconformal algebras [11], $N$-extended superconformal algebras [12, 13], non-semi-simply affine-Sugawara constructions [14, 15] and other. Classical algebras with Virasoro generators (1) were studied in Ref. [16]. Some currents of the reduced $N = 2$ and $W_{2,1}^{sl(3)}$ algebras were obtained in Ref. [17]. As well as the original models the reduced ones can be used as building blocks in string theories. Therefore it is important to investigate their properties.

The object of this paper is to construct a class of primary fields of (1) and corresponding correlation functions in terms of the original theory based on $L(m)$. To find the primary fields we use their transformation properties under coset conformal algebra. Solving corresponding equations we express primary fields of (1) in terms of the original primary fields and $\hat{u}(1)^d$ currents. This enables us to express correlation functions in terms of the original ones. All the coset fields under consideration commute with the Heisenberg algebra generated by non-zero modes of $\hat{u}(1)^d$ and form (non-trivial) representations of $u(1)^d$. As a consequence of this construction we obtain a new realization of $Z_k$ invariant parafermion currents.

Let $\Omega$ be the (super)algebra which has the Virasoro and $\hat{u}(1)^d$ subalgebras

$$[L(m), L(n)] = (m - n)L(m + n) + c \left[ \frac{1}{12} (m^3 - m) \right] \delta_{m,-n}, \quad (2)$$

$$[J^i(m), J^j(n)] = km \delta^{ij} \delta_{m,-n},$$

$$[L(m), J^i(n)] = -n J^i(m + n),$$

where $m, n \in Z$ and $1 \leq i, j \leq d$; $c$ and $k$ are the central charges.

Let $G_h(w)$ be the primary field of $\Omega$ which satisfies the equations

$$[L(m), G_h(w)] = w^{m+1} \partial_w G_h(w) + h(m + 1)w^m G_h(w),$$

$$[J^i(m), G_h(w)] = w^m t^i G_h(w).$$

2
Here $h$ is the conformal dimension of $G_h(w)$ and $t^i$ is the (reducible) representation of the generators of $u(1)^d$ for the field $G_h(w)$.

$G_h(w)$ can be decomposed in the set of one-dimensional irreducible representations of $\hat{u}(1)^d$. Let $G_h^\mu(w) = P_\mu G_h(w)$ be the irreducible component of $G_h(w)$ which satisfies the equation

$$[J^i(m), G_h^\mu(w)] = \mu^i w^m G_h^\mu(w).$$

Here $P_\mu$ is the projector and $\mu = (\mu^i)$ is the weight. As well as $G_h(w)$ the field $G_h^\mu(w)$ is primary with respect to Virasoro algebra (2)

$$[L(m), G_h^\mu(w)] = w^{m+1} \partial_w G_h^\mu(w) + h(m + 1) w^m G_h^\mu(w).$$

(3)

Correlation functions of these fields can be computed using correlation functions of primary fields of $\Omega$

$$< G_{h_1}^{\mu_1}(w_1) \ldots G_{h_n}^{\mu_n}(w_n) > = \prod_{i=1}^n P_{\mu_i} < G_{h_1}(w_1) \ldots G_{h_n}(w_n) >$$

We shall use the following properties of the vacuum state $|0>$

$$< 0| J^i(m < 0) = J^i(m \geq 0)|0> = 0.$$  

(4)

The operator $L^\mu(m)$ is given by

$$L^\mu(m) = \frac{1}{2k} \sum_{i=1}^d : J^i(m - n) J^i(n) :.$$ 

(5)

The normal-ordering symbol $:\cdot:\$ means that negative modes of the currents are on the left. The operators $L^\mu(m)$ satisfy Virasoro algebra (3) with the central charge $c = d$ and

$$[L^\mu(m), J^i(n)] = -n J^i(m + n).$$

We shall use the relation

$$[L^\mu(m), G_h^\mu(w)] = \frac{1}{k} w^{m+1} : \mu \cdot J(w) G_h^\mu(w) : + \frac{\mu^2}{2k} (m + 1) w^m G_h^\mu(w),$$

(5)

where

$$: J^i(w) G_h^\mu(w) : = \sum_{m<0} J^i(m) w^{-m-1} G_h^\mu(w) + G_h^\mu(w) \sum_{m \geq 0} J^i(m) w^{-m-1}.$$
The coset generators $K(m)$ [1] satisfy Virasoro algebra [2] with the central charge $c - d$ and

$$[J^i(m), K(n)] = 0.$$  

(6)

This equation suggests that all the fields of the coset model commute with $J^i(m)$. However, the primary fields $\tilde{G}^\mu_h(w)$ which we present in this paper commute only with the Heisenberg algebra generated by $J^i(m)$, $m \neq 0$,

$$[J^i(m), \tilde{G}^\mu_h(w)] = 0$$  

(7)

and transform under $u(1)^d$ as

$$[J^i(0), \tilde{G}^\mu_h(w)] = \mu^i \tilde{G}^\mu_h(w),$$  

(8)

Eqs. (6) and (8) express the fact that the coset theory has global abelian symmetry.

By the definition the coset primary field $\tilde{G}^\mu_h(w)$ satisfies the equation

$$[K(m), \tilde{G}^\mu_h(w)] = w^{m+1} \partial_w \tilde{G}^\mu_h(w) + \tilde{h}(m+1)w^m \tilde{G}^\mu_h(w),$$  

(9)

where $\tilde{h}$ is the conformal dimension of $\tilde{G}^\mu_h(w)$.

Here we give a solution of this equation

$$\tilde{G}^\mu_h(w) = U^\mu_< (w) G^\mu_h(w) U^\mu(w),$$  

$$\tilde{h} = h - \frac{\mu^2}{2k},$$  

(10)

where

$$U^\mu_< (w) = e^{-\frac{1}{k} \mu \cdot Q_< (w)}, \quad Q^i_< (w) = - \sum_{m<0} \frac{1}{m} J^i(m) w^{-m},$$  

$$U^\mu(w) = U^\mu_0 (w) U^\mu_> (w), \quad U^\mu_0 (w) = w^{-\frac{1}{k} \mu \cdot J(0)},$$  

$$U^\mu_> (w) = e^{-\frac{1}{k} \mu \cdot Q_> (w)}, \quad Q^i_> (w) = - \sum_{m>0} \frac{1}{m} J^i(m) w^{-m}.$$  

To check (9) we have the following computations

$$[K(m), \tilde{G}^\mu_h(w)] = U^\mu_< (w) [K(m), G^\mu_h(w)] U^\mu(w)$$  

$$= w^{m+1} U^\mu_< (w) \left( \partial_w G^\mu_h(w) - \frac{1}{k} : \mu \cdot J(w) G^\mu_h(w) : \right) U^\mu$$  

$$+ \tilde{h}(m+1)w^m \tilde{G}^\mu_h(w)$$  

$$= w^{m+1} \partial_w \tilde{G}^\mu_h(w) + \tilde{h}(m+1)w^m \tilde{G}^\mu_h(w).$$
Here we used eqs. (3),(4) and at the last step the relations
\[ \partial_w U_\mu^\nu(w) = \frac{1}{k} U_\mu^\nu(w) \sum_{m<0} \mu \cdot J(m) w^{-m-1}, \]
\[ \partial_w U^\mu(w) = \frac{1}{k} U^\mu(w) \sum_{m\geq0} \mu \cdot J(m) w^{-m-1}. \]

It is easy to check that \( \tilde{G}^\mu_h(w) \) also satisfies eqs. (7) and (8).

The original primary field \( G^\mu_h(w) \) can be expressed as follows
\[ G^\mu_h(w) = (U_\mu^\nu(w))^{-1} \tilde{G}^\mu_h(w) (U^\mu(w))^{-1}. \]

Computations show that
\[ U_0^\nu(z) G^\mu_h(w) = G^\nu_h(z) U_0^\mu(w) z^{-\frac{\mu \nu}{k}}, \]
\[ G^\mu_h(z) U_\nu^\mu(w) = U_\nu^\mu(w) G^\mu_h(z) \left( 1 - \frac{w}{z} \right)^{-\frac{\mu \nu}{k}}. \] (11)

From this, eqs.(8) and (7) it follows that the \( n \)-point coset correlation function can be written as
\[ < \tilde{G}^\mu_{h_1}(w_1) \ldots \tilde{G}^\mu_{h_n}(w_n) >= G^\mu_{h_1}(w_1) \ldots G^\mu_{h_n}(w_n) > \prod_{i<j} (w_i - w_j)^{\frac{\mu_i \mu_j}{k}}. \]

In the case of \( su(2)/u(1) \) this relation was obtained in Ref. [5].

Using eqs.(8) and (11) one can find the operator product expansion of two coset primary fields
\[ \tilde{G}^\mu_{h_1}(z) \tilde{G}^\mu_{h_2}(w) = (z - w)^{-\frac{\mu_1 \mu_2}{k}} U_\nu^\mu_1(z) U_\nu^\mu_2(w) G^\mu_{h_1}(z) G^\mu_{h_2}(w) U^\mu_1(z) U^\mu_2(w). \] (12)

As an example let us consider the \( su(2)/u(1) \) coset. The \( su(2) \) algebra is generated by the currents \( E^+(w) \), \( E^-(w) \) and \( J(w) \) which satisfy the operator product expansions
\[ E^+(z) E^+(w) = \text{reg.}, \quad E^-(z) E^-(w) = \text{reg.}, \]
\[ E^+(z) E^-(w) = \frac{k}{(z-w)^2} + \frac{\sqrt{2} J(w)}{z-w} + \text{reg.}, \]
\[ J(z) J(w) = \frac{k}{(z-w)^2} + \text{reg.} \] (13)
The $su(2)/u(1)$ Virasoro generator $K(w)$ is given by

$$K(w) = \sum_m K(m)w^{-m-2} = L^{su(2)}(w) - L^{u(1)}(w), \quad (14)$$

where

$$L^{su(2)}(w) = \frac{1}{2(k + 2)} \left( : J(w)J(w) : + : E^+(w)E^-(w) : + : E^-(w)E^+(w) : \right),$$

$$L^{u(1)}(w) = \frac{1}{2k} : J(w)J(w) : .$$

The current $J(w) = \sum_m J(m)w^{-m-1}$ has unit conformal dimension with respect to $L^{su(2)}(w) = \sum_m L^{su(2)}(m)w^{-m-2}$ and $L^{u(1)}(w) = \sum_m L^{u(1)}(m)w^{-m-2}$

$$[L^{su(2)}(m), J(n)] = [L^{u(1)}(m), J(n)] = -nJ^i(m + n).$$

According to eq. (10) the coset currents are given by

$$\tilde{E}^+(w) = e^{-\frac{2k}{\sqrt{k}}} Q_<(w) E^+(w)e^{-\frac{2k}{\sqrt{k}}} Q_>(w)w^{-\frac{2k}{\sqrt{k}}} J(0),$$

$$\tilde{E}^-(w) = e^{\frac{2k}{\sqrt{k}}} Q_<(w) E^-(w)e^{\frac{2k}{\sqrt{k}}} Q_>(w)w^{\frac{2k}{\sqrt{k}}} J(0).$$

These fields are closely related with $Z$ operators for $su(2)$ [18, 19].

Using eqs. (12-14) one can check that the currents

$$\psi_1(w) = (1/\sqrt{k}) \tilde{E}^+(w), \quad \psi_1^+(w) = (1/\sqrt{k}) \tilde{E}^-(w) \quad (15)$$

satisfy the parafermion algebra of Ref. [3]

$$\psi_1(z)\psi_1^+(w) = (z - w)^{-2 + \frac{k}{2}} \left( I + \frac{k + 2}{k} K(w)(z - w)^2 + O((z - w)^3) \right),$$

where $K(w)$ is given by (14).

Eqs. (15) give a new realization of the parafermion currents $\psi_1(w)$ and $\psi_1^+(w)$.

I would like to thank J. Lepowsky for useful correspondence.
References

[1] K.Bardakci and M.B.Halpern, Phys. Rev. D3 (1971) 2493.
   M.B.Halpern, Phys. Rev. D4 (1971) 2398.
[2] P.Goddard, A.Kent and D.Olive, Phys. Lett. B152 (1985) 88.
[3] P.Bouwknegt and K.Schoutens, Phys. Rept. 223 (1993) 183.
[4] M.B.Halpern at al., Phys. Rept. 265 (1996) 1.
[5] V.A.Fateev and A.B.Zamolodchikov, Sov. Phys. JETP 82 (1985) 215.
[6] G.V.Dunne, I.G.Halliday and P.Suranyi, Nucl. Phys. B325 (1989) 526.
[7] A.Gerasimov at al., Nucl. Phys. B328 (1989) 664.
[8] C.Dong and J.Lepowsky, Generalized vertex algebras and relative vertex
   operators, Birkhäuser, Boston, 1993.
[9] S.Hwang and H.Rhedin, Mod. Phys. Lett. A10 (1995) 823.
[10] A.V.Bratchikov, g/u(1)d parafermions from constrained WZNW theo-
    ries, hep-th/9712043, Phys. Lett. B420 (1998) 285.
[11] M.Ademollo et al., Nucl. Phys. B111 (1976) 77.
[12] M.Bershadsky, Phys. Lett. B174 (1986) 255.
[13] V.G.Knizhnik, Theor. Math. Phys. 66 (1986) 68.
[14] D.I.Olive, E.Rabinovici and A.Schwimmer, Phys. Lett. B321 (1994) 361.
[15] J.M.Figueroa-O’Farrill and S.Stanciu, Phys. Lett. B327 (1994) 40.
[16] F.Delduc at al., Phys. Lett. B318 (1993) 457.
[17] R.Blumenhagen at al., Int. J. Mod. Phys. A10 (1995) 2367.
[18] J.Lepowsky and R.Wilson, Proc. Natl. Acad. Sci. USA 78 (1981) 7254.
[19] J.Lepowsky and M.Prims, Standart modules for type one affine Lie
    algebras, in: Number Theory, New York, 1982, Lecture Notes in Math.
    1052, Springer-Verlag, 1984, 194.