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Left Riemann–Liouville Fractional Sobolev Space on Time Scales and Its Application to a Fractional Boundary Value Problem on Time Scales

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Abstract: First, we show the equivalence of two definitions of the left Riemann–Liouville fractional integral on time scales. Then, we establish and characterize fractional Sobolev space with the help of the notion of left Riemann–Liouville fractional derivative on time scales. At the same time, we define weak left fractional derivatives and demonstrate that they coincide with the left Riemann–Liouville ones on time scales. Next, we prove the equivalence of two kinds of norms in the introduced space and derive its completeness, reflexivity, separability, and some embedding. Finally, as an application, by constructing an appropriate variational setting, using the mountain pass theorem and the genus properties, the existence of weak solutions for a class of Kirchhoff-type fractional p-Laplacian systems on time scales with boundary conditions is studied, and three results of the existence of weak solutions for this problem is obtained.

Keywords: Riemann–Liouville derivatives; time scales; left fractional Sobolev’s spaces; boundary value problems; mountain pass theorem; genus properties

1. Introduction

To unify the discrete analysis and continuous analysis, and allow a simultaneous treatment of differential and difference equations, Stefan Hilger [1] proposed the time scale theory and established its related basic theory [2,3]. To date, the study of dynamic equations on time scales is studied in [4–7].

On the one hand, the classical derivatives are local in nature; i.e., using classical derivatives, we can describe changes in the neighborhood of a point, but using fractional derivatives we can describe changes in an interval. Namely, fractional derivatives are non-local in nature. Fractional derivatives are non-local, so the $\frac{1}{2}$ derivative cannot have a local meaning such as tangent or curvature but would have to take into account the properties of the curve to a large extent (boundary conditions). This property makes these derivatives suitable to simulate more physical phenomena such as earthquake vibrations, polymers, etc.

The geometrical meaning of ordinary derivatives is simple and intuitive: for the smooth function $f$, which is differentiable at $x$, it shows local behavior of $f$ around point $x$. A simple definition can be provided directly from the geometrical meaning: one can expect that the fractional derivative could give a nonlinear (power law) approximation of the local behavior of non-differentiable functions. Fractional order derivatives are related to memory and hereditary properties of various real materials [8–13]. In the past few decades, fractional calculus and fractional differential
equations have attracted widespread attention in the field of differential equations, as well as in applied mathematics and science. In addition to true mathematical interest and curiosity, this trend is also driven by interesting scientific and engineering applications that have produced fractional differential equation models to better describe (time) memory effects and (space) non-local phenomena [14–19]. There are various definitions of the fractional derivative [20–23]. Wang et al. introduced the theory of fractional Sobolev spaces on time scales by conformable fractional derivatives on time scales in [6]. The rise in these applications gives new vitality to the field of fractional calculus and fractional differential equations and calls for further research in this field.

On the other hand, recently, based on the concept of the fractional derivative of Riemann–Liouville on time scales [24], the authors of [7] established the fractional Sobolev space on time scales. However, in a recent work, the authors of [25] pointed out that the definition of fractional integral on time scales proposed in [24] is not the natural one on time scales. Furthermore, they developed a new notion of Riemann–Liouville fractional integral on time scales, which can effectively unify the discrete fractional calculus [26,27] and its continuous counterpart [28].

Motivated by the above discussion, in order to fix this defect in the fractional Sobolev space on time scales established in [7], in this paper, we want to contribute with the development of this new area in terms of theories of fractional differential equations on time scales. More precisely, we first show that the concept of the Riemann–Liouville fractional integral on time scales from [7] coincides with the ones from [29], which is significant as it allows us to prove the semigroup properties of the Riemann–Liouville fractional integral on time scales. Next, the left fractional Sobolev space in the sense of weak Riemann–Liouville derivatives on time scales was constructed and characterized via Riemann–Liouville derivatives on time scales. Then, as an application of our new theory, we study the solvability of a class of Kirchhoff-type fractional \( p \)-Laplacian systems on time scales with boundary conditions by using variational methods and the critical point theory. As far as we know, no one has studied this problem using other methods.

The rest of present paper is organized as follows: In Section 2, we review some symbols, basic notions and basic results of time-scale calculus that will be used later. In Section 3, we study some basic properties of left Riemann–Liouville fractional integral and differential operators on time scales, including the equivalence between fractional integrals and fractional derivatives on time scales defined by the Laplace transform and the inverse Laplace transform. In Section 4, we provide the definition of left fractional Sobolev spaces on time scales and study some of their important properties. In Section 5, as an application of the results of this paper, we study the solvability of Kirchhoff-type fractional \( p \)-Laplacian systems on time scales by using the mountain pass theorem and the genus properties. In Section 6, we give a concise conclusion.

2. Preliminaries

In this section, we briefly collect some basic known notations, definitions, and results that will be used later.

A time scale \( \mathbb{T} \) is an arbitrary nonempty closed subset of the real set \( \mathbb{R} \) with the topology and ordering inherited from \( \mathbb{R} \). Throughout this paper, we denote by \( \mathbb{T} \) a time scale.

**Definition 1** ([2]). For \( t \in \mathbb{T} \), we define the forward jump operator \( \sigma : \mathbb{T} \to \mathbb{T} \) by \( \sigma(t) := \inf \{ s \in \mathbb{T} : s > t \} \), while the backward jump operator \( \rho : \mathbb{T} \to \mathbb{T} \) is defined by \( \rho(t) := \sup \{ s \in \mathbb{T} : s < t \} \).

We will use the following notations: \( f^0_\mathbb{R} = [a, b], f_\mathbb{R} = [a, b], f^0 = f^0_\mathbb{R} \cap \mathbb{T}, f = f_\mathbb{R} \cap \mathbb{T}, f^k = [a, \rho(b)] \cap \mathbb{T} \).
Remark 1 ([2]). (1) In Definition 1, we put \( \inf \emptyset = \sup T \) (i.e., \( \sigma(t) = t \) if \( T \) has a maximum \( t \)) and \( \sup \emptyset = \inf T \) (i.e., \( \rho(t) = t \) if \( T \) has a minimum \( t \)), where \( \emptyset \) denotes the empty set.

(2) If \( \sigma(t) > t \), we say that \( t \) is right-scattered, while if \( \rho(t) < t \), we say that \( t \) is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated.

(3) If \( t < \sup T \) and \( \sigma(t) = t \), we say that \( t \) is right-dense, while if \( t > \inf T \) and \( \rho(t) = t \), we say that \( t \) is left-dense. Points that are right-dense and left-dense at the same time are called dense.

(4) The graininess function \( \mu : T \to [0, \infty) \) is defined by \( \mu(t) := \sigma(t) - t \).

(5) The derivative makes use of the set \( T^k \), which is derived from the time scale \( T \) as follows: If \( T \) has a right-scattered maximum \( M \), then \( T^k := T \setminus \{M\} \); otherwise, \( T^k := T \).

Definition 2 ([2]). Assume that \( f : T \to \mathbb{R} \) is a function and let \( t \in T^k \). Then we define \( f^A(t) \) to be the number (provided it exists) with the property that given any \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( t \) (i.e., \( U = (t - \delta, t + \delta) \cap T \) for some \( \delta > 0 \)) such that

\[
|f(\sigma(t)) - f(s) - f^A(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|
\]

for all \( s \in U \). We call \( f^A(t) \) the delta (or Hilger) derivative of \( f \) at \( t \). Moreover, we say that \( f \) is delta (or Hilger) differentiable (or in short, differentiable) on \( T^k \) provided \( f^A(t) \) exists for all \( t \in T^k \). The function \( f^A : T^k \to \mathbb{R} \) is then called the (delta) derivative of \( f \) on \( T^k \).

Definition 3 ([2]). A function \( f : T \to \mathbb{R} \) is called rd-continuous provided it is continuous at right-dense points in \( T \) and its left-sided limits exist (finite) at left-dense points in \( T \). The set of rd-continuous functions \( f : T \to \mathbb{R} \) will be denoted by \( C_{rd}(T) = C_{rd}(T, \mathbb{R}) \). The set of functions \( f : T \to \mathbb{R} \) that are differentiable and whose derivative is rd-continuous is denoted by \( C^1_{rd}(T) = C^1_{rd}(T, \mathbb{R}) \).

Theorem 1 ([3]). If \( a, b \in T \) and \( f, g \in C_{rd}(T) \), then

\[
\int_a^b f^A(t)g^A(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^A(t)g(t)\Delta t.
\]

Theorem 2 ([3]). If \( f \) is \( \Delta \)-integrable on \( a, b \in T \), then so is \( |f| \), and

\[
\left| \int_a^b f(t)\Delta t \right| \leq \int_a^b |f(t)|\Delta t.
\]

Definition 4 ([24]). Let \( J \) denote a closed bounded interval in \( T \). A function \( F : J \to \mathbb{R} \) is called a delta antiderivative of function \( f : J \to \mathbb{R} \) provided \( F \) is continuous on \( J \), delta-differentiable at \( J \), and \( F^A(t) = f(t) \) for all \( t \in J \). Then, we define the \( \Delta \)-integral of \( f \) from \( a \) to \( b \) by

\[
\int_a^b f(t)\Delta t := F(b) - F(a).
\]

Theorem 3 ([30]). The convolution is commutative and associative, that is, for \( f, g, h \in \mathcal{F} \),

\[
f * g = g * f, \quad (f * g) * h = f * (g * h).
\]

Proposition 1 ([31]). \( f \) is an increasing continuous function on \( J \). If \( F \) is the extension of \( f \) to the real interval \( ]a, b[ \) given by

\[
F(s) := \begin{cases} 
  f(s), & \text{if } s \in T, \\
  f(t), & \text{if } s \in (t, \sigma(t)) \not\in T,
\end{cases}
\]

then

\[
\int_a^b f(t)\Delta t \leq \int_a^b F(t)dt.
\]
Theorem 4 ([32]). \( y(t,s) = h_{n-1}(t,\sigma(s)) \) is the Cauchy function of \( y^\Delta = 0 \), where
\[
h_0(t,s) = 1, \quad h_n(t,s) = \int_s^t h_{n-1}(\tau,s) \Delta \tau, \quad n \in \mathbb{N}.
\]

Theorem 5 ([32]). For all \( n \in \mathbb{N}_0 \), we have
\[
\mathcal{L}_\Delta (h_n(x,0))(z) = \frac{1}{2^{n-1}}, \quad x \in \mathbb{T}_0,
\]
for all \( z \in \mathbb{C} \setminus \{0\} \) such that \( 1 + z\mu(x) \neq 0, x \in \mathbb{T}_0 \), and
\[
\lim_{x \to \infty} (h_n(x,0)e^{cz}(x,0)) = 0.
\]

Definition 5 ([32], shift (delay) of a function). For a given function \( f : [t_0, \infty) \to \mathbb{C} \), the solution of the shifting problem
\[
u^\Delta(t,\sigma(s)) = -\nu^\Delta(t,s), \quad t, s \in \mathbb{T}, \quad t \geq t \geq t_0,
\]
\[
u(t,t_0) = f(t), \quad t \in \mathbb{T}, \quad t \geq t_0,
\]
is denoted by \( \hat{f} \) and is called the shift or delay of \( f \).

Definition 6 ([32], \( \Delta \) power function). Suppose that \( \alpha \in \mathbb{R} \); we define the generalized \( \Delta \)-power function \( h_\alpha(t,t_0) \) on \( \mathbb{T} \) as follows:
\[
h_\alpha(t,t_0) = \mathcal{L}_\Delta^{-1} \left( \frac{1}{2^{\alpha+1}} \right)(t), \quad t \geq t_0,
\]
for all \( z \in \mathbb{C} \setminus \{0\} \) such that \( \mathcal{L}_\Delta^{-1} \) exists, \( t \geq t_0 \). The fractional generalized \( \Delta \)-power function \( h_\alpha(t,s) \) on \( \mathbb{T} \), \( t \geq s \geq t_0 \) is defined as the shift of \( h_\alpha(t,t_0) \), i.e.,
\[
h_\alpha(t,s) = \hat{h_\alpha}(\cdot,t_0)(t,s), \quad t, s \in \mathbb{T}, \quad t \geq s \geq t_0.
\]

Definition 7 ([25,33], fractional integral on time scales). Suppose \( h \) is an integrable function on \( f \). Let \( 0 < \alpha \leq 1 \). Then, the left fractional integral of order \( \alpha \) of \( h \) is defined by
\[
\mathcal{I}_a^\alpha h(t) := \int_a^t \frac{(t - \sigma(s))^{\alpha-1}}{\Gamma(\alpha)} h(s) \Delta s.
\]
The right fractional integral of order \( \alpha \) of \( h \) is defined by
\[
\mathcal{I}_b^\alpha h(t) := \int_t^b \frac{(s - \sigma(t))^{\alpha-1}}{\Gamma(\alpha)} h(s) \Delta s,
\]
where \( \Gamma \) is the gamma function.

Definition 8 ([25,33], Riemann–Liouville fractional derivative on time scales). Let \( t \in \mathbb{T}, \) \( 0 < \alpha \leq 1, \) and \( h : \mathbb{T} \to \mathbb{R} \). The left Riemann–Liouville fractional derivative of order \( \alpha \) of \( h \) is defined by
\[
\mathcal{D}_a^\alpha h(t) := \left( \mathcal{I}_a^{\alpha-\alpha} h(t) \right)^\Delta = \frac{1}{\Gamma(1-\alpha)} \left( \int_a^t (t - \sigma(s))^{-\alpha} h(s) \Delta s \right)^\Delta.
\]
Actually, \( \mathcal{T}_a^b h(t) \) can be rewritten as \( \Delta \circ \mathcal{T}_a^b t_1^{-\alpha} h(t) \). The right Riemann–Liouville fractional derivative of order \( \alpha \) of \( h \) is defined by
\[
\mathcal{T}_a^b \mathcal{D}^\alpha_h(t) := -\left( \mathcal{T}_a^b t_1^{-\alpha} h(t) \right)^\Delta = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (s-\sigma(t))^{-\alpha} h(s)\Delta s.
\]

**Definition 9** ([25,33], Caputo fractional derivative on time scales). Let \( t \in \mathbb{T}, 0 < \alpha \leq 1 \) and \( h : \mathbb{T} \to \mathbb{R} \). The left Caputo fractional derivative of order \( \alpha \) of \( h \) is defined by
\[
\mathcal{T}_a^b \mathcal{D}_C^\alpha_h(t) := \mathcal{T}_a^b t_1^{-\alpha} h^\alpha(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\sigma(s))^{-\alpha} h^\alpha(s)\Delta s.
\]

The right Caputo fractional derivative of order \( \alpha \) of \( h \) is defined by
\[
\mathcal{T}_a^b \mathcal{D}_C^\alpha_h(t) := -\mathcal{T}_a^b t_1^{-\alpha} h^\alpha(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (s-\sigma(t))^{-\alpha} h^\alpha(s)\Delta s.
\]

**Definition 10** ([34]). For \( f : \mathbb{T} \to \mathbb{R} \), the time scale or generalized Laplace transform of \( f \), denoted by \( \mathcal{L}_\mathbb{T}\{f\} \) or \( F(z) \), is given by
\[
\mathcal{L}_\mathbb{T}\{f\}(z) = F(z) := \int_0^\infty f(t)g^\alpha(t)\Delta t,
\]
where \( g(t) = e_{\mathbb{T}c}(t,0) \).

**Theorem 6** ([34], Inversion formula of the Laplace transform). Suppose that \( F(z) \) is analytic in the region \( \text{Re}_\mu(z) > \text{Re}_\mu(c) \) and \( F(z) \to 0 \) uniformly as \( |z| \to \infty \) in this region. Suppose \( F(z) \) has finitely many regressive poles of finite order \( \{z_1, z_2, \ldots, z_n\} \) and \( \mathcal{F}_{\mathbb{T}}(z) \) is the transform of the function \( \tilde{f}(t) \) on \( \mathbb{R} \) that corresponds to the transform \( F(z) = \mathcal{F}_{\mathbb{T}}(z) \) of \( f(t) \) on \( \mathbb{T} \), if
\[
\int_{c-i\infty}^{c+i\infty} |\mathcal{F}_{\mathbb{T}}(z)||dz| < \infty,
\]
then
\[
f(t) = \sum_{i=1}^n \text{Res}_{z=z_i} e_{\mathbb{T}}(t,0) F(z),
\]
has transform \( F(z) \) for all \( z \) with \( \text{Re}(z) > c \).

**Definition 11** ([29], Riemann–Liouville fractional integral on time scales). Let \( \alpha > 0, \mathbb{T} \) be a time scale, and \( f : \mathbb{T} \to \mathbb{R} \). The left Riemann–Liouville fractional integral of \( f \) of order \( \alpha \) on the time scale \( \mathbb{T} \), denoted by \( \mathcal{I}_{\mathbb{T}}^\alpha f \), is defined by
\[
\mathcal{I}_{\mathbb{T}}^\alpha f(t) = \mathcal{L}_\mathbb{T}^{-1}\left[ \frac{F(z)}{z^\alpha} \right](t).
\]

**Theorem 7** ([25], Cauchy result on time scales). Let \( n \in \{1,2\}, \mathbb{T} \) be a time scale with \( a_1, t_1, \ldots, t_n \in \mathbb{T}, t_i > a, i = 1, \ldots, n \), and \( f \) an integrable function on \( \mathbb{T} \). Then,
\[
\int_{a}^{t_1} \cdots \int_{a}^{t_n} f(t_0)\Delta t_0 \cdots \Delta t_{n-1} = \frac{1}{(n-1)!} \int_{a}^{t_n} (t_n - \sigma(s))^{n-1}\Delta s.
\]
Theorem 8 ([5]). A function $f : J \to \mathbb{R}^N$ is absolutely continuous on $J$ iff $f$ is $\Delta$-differentiable $\Delta$-a.e. on $J$ and
\[ f(t) = f(a) + \int_{[a,t)_\Delta} f^\Delta(s) \Delta s, \quad \forall t \in J. \]

Theorem 9 ([35]). A function $f : \mathbb{T} \to \mathbb{R}$ is absolutely continuous on $\mathbb{T}$ iff the following conditions are satisfied:
(i) $f$ is $\Delta$-differentiable $\Delta$-a.e. on $J$ and $f^\Delta \in L^1(\mathbb{T})$.
(ii) The equality
\[ f(t) = f(a) + \int_{[a,t)_\Delta} f^\Delta(s) \Delta s \]
holds for every $t \in \mathbb{T}$.

Theorem 10 ([36]). A function $q : J_{\mathbb{R}} \to \mathbb{R}^m$ is absolutely continuous iff there exist a constant $c \in \mathbb{R}^m$ and a function $\varphi \in L^1$ such that
\[ q(t) = c + (I^{1}_{a^+}, \varphi)(t), \quad t \in J_{\mathbb{R}}. \]
In this case, we have $q(a) = c$ and $q'(t) = \varphi(t)$, $t \in J_{\mathbb{R}}$ a.e.

Theorem 11 ([5], integral representation). Let $\alpha \in (0,1)$ and $q \in L^1$. Then, $q$ has a left-sided Riemann–Liouville derivative $D^\alpha_{a^+} q$ of order $\alpha$ iff there exist a constant $c \in \mathbb{R}^m$ and a function $\varphi \in L^1$ such that
\[ q(t) = \frac{1}{\Gamma(\alpha)} \frac{c}{(t-a)^{1-\alpha}} + (I^{\alpha}_{a^+}, \varphi)(t), \quad t \in J_{\mathbb{R}} \quad a.e.. \]
In this case, we have $I^{1-\alpha}_{a^+} q(a) = c$ and $(D^\alpha_{a^+}, q)(t) = \varphi(t)$, $t \in J_{\mathbb{R}}$ a.e.

Lemma 1 ([4]). Let $f \in L^1_\Delta(J^0)$. Then, the following
\[ \int_{J^0} (f \cdot q^\Delta)(s) \Delta s = 0, \quad \text{for every } \varphi \in C^1_{0,rd}(J^k) \]
holds iff there exists a constant $c \in \mathbb{R}$ such that
\[ f \equiv c \quad \Delta \text{-a.e. on } J^0. \]

Definition 12 ([4]). Let $p \in \mathbb{R}$ be such that $p \geq 1$ and $u : J \to \mathbb{R}$. Say that $u$ belongs to $W^1,p_\Delta(J)$ iff $u \in L^1_\Delta(J^0)$ and there exists $g : J^k \to \mathbb{R}$ such that $g \in L^p(J^0)$ and
\[ \int_{J^0} (u \cdot q^\Delta)(s) \Delta s = - \int_{J^0} (g \cdot q^\Delta)(s) \Delta s, \quad \forall \varphi \in C^1_{0,rd}(J^k), \]
with
\[ C^1_{0,rd}(J^k) := \left\{ f : J \to \mathbb{R} : f \in C^1_{rd}(J^k), f(a) = f(b) \right\}, \]
where $C^1_{rd}(J^k)$ is the set of all continuous functions on $J$ such that they are $\Delta$-differential on $J^k$ and their $\Delta$-derivatives are rd-continuous on $J^k$. 

Theorem 12 ([4]). Let \( p \in \mathbb{R} \) be such that \( p \geq 1 \). Then, the set \( L^p_{\Lambda}(f^0) \) is a Banach space together with the norm defined for every \( f \in L^p_{\Lambda}(f^0) \) as

\[
\|f\|_{L^p_{\Lambda}} := \begin{cases} \left[ \int_0^1 |f|^p(s) \Delta s \right]^{\frac{1}{p}}, & \text{if } p \in \mathbb{R}, \\ \inf \{C \in \mathbb{R} : |f| \leq C \Delta - \text{a.e. on } f^0\}, & \text{if } p = +\infty. \end{cases}
\]

Moreover, \( L^2_{\Lambda}(f^0) \) is a Hilbert space together with the inner product given for every \( (f,g) \in L^2_{\Lambda}(f^0) \times L^2_{\Lambda}(f^0) \) by

\[
(f,g)_{L^2_{\Lambda}} := \int_{f^0} f(s) \cdot g(s) \Delta s.
\]

Theorem 13 ([28]). Fractional integration operators are bounded in \( L^p(f^0) \); i.e., the following estimate

\[
\|f_{\alpha}\|_{L^p(a,b)} \leq \frac{(b-a)^{\Re \alpha}}{\Re |\Gamma(\alpha)|} \|\varphi\|_{L^p(a,b)}, \quad \Re \alpha > 0
\]

holds.

Proposition 2 ([4]). Suppose \( p \in \mathbb{R} \) and \( p \geq 1 \). Let \( p' \in \mathbb{R} \) be such that \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then, if \( f \in L^p_{\Lambda}(f^0) \) and \( g \in L^{p'}_{\Lambda}(f^0) \), then \( f \cdot g \in L^1_{\Lambda}(f^0) \) and

\[
\|f \cdot g\|_{L^1_{\Lambda}} \leq \|f\|_{L^p_{\Lambda}} \cdot \|g\|_{L^{p'}_{\Lambda}}.
\]

This expression is called Hölder’s inequality and Cauchy–Schwarz’s inequality whenever \( p = 2 \).

Theorem 14 ([3]). (the first mean value theorem). Let \( f \) and \( g \) be bounded and integrable functions on \( f \), and let \( g \) be nonnegative (or nonpositive) on \( f \). Let us set

\[
m = \inf \{f(t) : t \in f^0\} \quad \text{and} \quad M = \sup \{f(t) : t \in f^0\}.
\]

Then, there exists a real number \( \Lambda \) satisfying the inequalities \( m \leq \Lambda \leq M \) such that

\[
\int_{f^0} f(t)g(t) \Delta t = \Lambda \int_{f^0} g(t) \Delta t.
\]

Corollary 1 ([3]). Let \( f \) be an integrable function on \( f \), and let \( m \) and \( M \) be the infimum and supremum, respectively, of \( f \) on \( f^0 \). Then, there exists a number \( \Lambda \) between \( m \) and \( M \) such that \( \int_{f^0} f(t) \Delta t = \Lambda (b - a) \).

Theorem 15 ([3]). Let \( f \) be a function defined on \( f \) and let \( c \in \mathbb{T} \) with \( a < c < b \). If \( f \) is \( \Delta \)-integrable from \( a \) to \( c \) and from \( c \) to \( b \), then \( f \) is \( \Delta \)-integrable from \( a \) to \( b \) and

\[
\int_{f^0} f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t.
\]

Lemma 2 ([3]). Assume that \( a, b \in \mathbb{T} \). Every constant function \( f : \mathbb{T} \rightarrow \mathbb{R} \) is \( \Delta \)-integrable from \( a \) to \( b \) and

\[
\int_{f^0} c \Delta t = c(b - a).
\]
Lemma 3 ([37], A time-scale version of the Arzela–Ascoli theorem). Let $X$ be a subset of $C(J, \mathbb{R})$ satisfying the following conditions:

(i) $X$ is bounded;

(ii) For any given $\epsilon > 0$, there exists $\delta > 0$ such that for all $f \in X$, $|f(t_1) - f(t_2)| < \epsilon$ implies $|t_1 - t_2| < \delta$.

Then, $X$ is relatively compact.

3. Some Fundamental Properties of Left Riemann–Liouville Fractional Operators on Time Scales

Inspired by [38], we can obtain the consistency of Definitions 7 and 11 by using the above theory of the Laplace transform on time scales and the inverse Laplace transform on time scales.

Theorem 16. Let $\alpha > 0$, $T$ be a time scale, $J$ be an interval of $T$, and $f$ be an integrable function on $J$. Then,

$$\left(\int_a^t I_\alpha^b f\right)(t) = I_\alpha^{T} f(t).$$

Proof. Using the Laplace transform on time scale $T$ for (1), in view of Definitions 6, 7, Theorem 5, the proof of Theorem 4.14 in [32], and Definition 10, we have

$$L_T\left\{\left(\int_a^t I_\alpha^b f\right)(t)\right\}(z) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \sigma(s))^{\alpha-1} f(s) \Delta s(z) = L_T\left(h_{\alpha-1}(\cdot,a) * f\right)(t)(z) = L_T\left(h_{\alpha-1}(\cdot,a)\right)(z) L_T(f)(t)(z) = \frac{1}{z^\alpha} L_T\left\{f\right\}(z) = \frac{F(z)}{z^\alpha}(t).$$

Taking the inverse Laplace transform on time scales for (3), with an eye to Definition 11, one arrives at

$$\left(\int_a^t I_\alpha^b f\right)(t) = L_T^{-1}\left[\frac{F(z)}{z^\alpha}\right](t) = I_\alpha^b f(t).$$

The proof is complete.

Combining [24,29] with Theorem 16, we see that Propositions 15–17, Corollary 18, and Theorems 20 and 21 from [24] remain intact under the new Definition 7.

Proposition 3. Let $h$ be $\Delta$-integrable on $J$ and $0 < \alpha \leq 1$. Then $\int_a^t D_\alpha^b h(t) = \Delta \circ \int_a^t I_\alpha^{1-\alpha} h(t)$.

Proof. Let $h : T \rightarrow \mathbb{R}$. In view of (1) and (2), we obtain

$$\int_a^t D_\alpha^b h(t) = \frac{1}{\Gamma(1 - \alpha)} \left(\int_a^t (t - \sigma(s))^{-\alpha} h(s) \Delta s\right)^\alpha = \left(\int_a^t I_\alpha^{1-\alpha} h(t)\right)^\alpha = \Delta \circ \int_a^t I_\alpha^{1-\alpha} h(t).$$

The proof is complete.

Proposition 4. For any function $h$ that is integrable on $J$, the Riemann–Liouville $\Delta$-fractional integral satisfies \(\int_a^t I_\alpha^\beta h(t) = \int_a^t I_\alpha^{1+\beta} = \int_a^t I_\beta^\alpha \circ I_\alpha^\beta h(t)\) for $\alpha > 0$ and $\beta > 0$. 
Proof. Combining with Proposition 3.4 in [29] and Theorem 16, one obtains
\[ T_a^{\alpha} \circ T_a^{\beta} = T_a^{\alpha + \beta}. \]

In a similarly way, one arrives at
\[ T_a^{\beta} \circ T_a^{\alpha} = T_a^{\alpha + \beta}. \]
Consequently, we obtain that
\[ T_a^{\alpha} \circ T_a^{\beta} = T_a^\alpha + \beta = T_a^{\beta} \circ T_a^{\alpha}. \]
The proof is complete. \( \square \)

Proposition 5. For any function \( h \) that is integrable on \( J \) one has
\[ T_a^{\alpha} D_t^{\alpha} \circ T_a^{\alpha} h = h. \]
Proof. Taking account of Propositions 3 and 4, one can get
\[ T_a^{\alpha} D_t^{\alpha} \circ T_a^{\alpha} h(t) = \left( T_a^{\alpha} I_1^{\alpha} (T_a^{\alpha} h(t)) \right)^{\Delta} = \left( T_a h(t) \right)^{\Delta} = h. \]
The proof is complete. \( \square \)

Corollary 2. For \( 0 < \alpha \leq 1 \), we have \( T_a^{\alpha} D_t^{\alpha} \circ T_a^{\alpha} D_t^{-\alpha} = Id \) and \( T_a^{\alpha} I_t^{-\alpha} \circ T_a^{\alpha} I_t^{\alpha} = Id \), where \( Id \) denotes the identity operator.

Proof. In view of Proposition 5, we have
\[ T_a^{\alpha} D_t^{\alpha} \circ T_a^{\alpha} D_t^{-\alpha} = T_a^{\alpha} D_t^{\alpha} \circ T_a^{\alpha} I_t^{\alpha} = Id \] and \[ T_a^{\alpha} I_t^{-\alpha} \circ T_a^{\alpha} I_t^{\alpha} = T_a^{\alpha} D_t^{\alpha} \circ T_a^{\alpha} I_t^{\alpha} = Id. \]
The proof is complete. \( \square \)

Theorem 17. Let \( f \in C(J) \) and \( \alpha > 0 \), then \( f \in T_a^{\alpha} I_t^{\alpha} (J) \) iff
\[ T_a^{\alpha} I_t^{-\alpha} f \in C(J) \] (4)
and
\[ \left. \left( T_a^{\alpha} I_t^{-\alpha} f(t) \right) \right|_{t=a} = 0. \] (5)
where \( T_a^{\alpha} I_t^{\alpha} (J) \) denotes the space of functions that can be represented by the left Riemann–Liouville \( \Delta \)-integral of order \( \alpha \) of a \( C(J) \)–function.

Proof. Suppose \( f \in T_a^{\alpha} I_t^{\alpha} (J) \), \( f(t) = T_a^{\alpha} I_t^{\alpha} g(t) \) for some \( g \in C(J) \), and
\[ T_a^{\alpha} I_t^{-\alpha} (f(t)) = T_a^{\alpha} I_t^{-\alpha} (T_a^{\alpha} I_t^{\alpha} g(t)). \]
In view of Proposition 4, one gets
\[ T_a^{\alpha} I_t^{-\alpha} (f(t)) = T_a^{\alpha} I_t^{\alpha} g(t) = \int_a^t g(s) \Delta s. \]
As a result, \( T_a^{\alpha} I_t^{-\alpha} f \in C(J) \) and
\[ \left( T_a^{\alpha} I_t^{-\alpha} f(t) \right) \bigg|_{t=a} = \int_a^a g(s) \Delta s = 0. \]
Inversely, suppose that \( f \in C(J) \) satisfies (4) and (5). Then, by applying Taylor’s formula to function \( \frac{T}{a} T_{\frac{1}{a}} f \), we obtain
\[
\frac{T}{a} T_{\frac{1}{a}} f(t) = \int_a^t \frac{\Delta}{\Delta s} \frac{T}{a} T_{\frac{1}{a}} f(s) \Delta s, \quad \forall t \in J.
\]
Let \( \varphi(t) = \frac{\Delta}{\Delta a} \frac{T}{a} T_{\frac{1}{a}} f(t) \). Note that \( \varphi \in C(J) \) by (4). Now by Proposition 4, one sees that
\[
\frac{T}{a} T_{\frac{1}{a}} (f(t)) = \frac{T}{a} T_{\frac{1}{a}} \varphi(t) = \frac{T}{a} T_{\frac{1}{a}} \left[ \frac{T}{a} T_{\frac{1}{a}} \varphi(t) \right]
\]
and hence
\[
\frac{T}{a} T_{\frac{1}{a}} (f(t)) - \frac{T}{a} T_{\frac{1}{a}} \left[ \frac{T}{a} T_{\frac{1}{a}} \varphi(t) \right] = 0.
\]
Therefore, we have
\[
\frac{T}{a} T_{\frac{1}{a}} [f(t) - \frac{T}{a} T_{\frac{1}{a}} \varphi(t)] = 0.
\]
From the uniqueness of the solution to Abel’s integral Equation ([39]), this implies that \( f - \frac{T}{a} T_{\frac{1}{a}} \varphi = 0 \). Hence, \( f = \frac{T}{a} T_{\frac{1}{a}} \varphi \) and \( f \in \frac{T}{a} T_{\frac{1}{a}} (f) \). The proof is complete.

**Theorem 18.** Let \( \alpha > 0 \) and \( f \in C(J) \) satisfy the condition in Theorem 17. Then,
\[
\left( \frac{T}{a} \frac{T}{a} \circ \frac{T}{a} \frac{D}{a} \right)(f) = f.
\]

**Proof.** Combining Theorem 17 with Proposition 5, we can see that
\[
\frac{T}{a} \frac{T}{a} \circ \frac{T}{a} \frac{D}{a} f(t) = \frac{T}{a} \frac{T}{a} \circ \frac{T}{a} \frac{D}{a} \left( \frac{T}{a} \frac{T}{a} \varphi(t) \right) = \frac{T}{a} \frac{T}{a} \varphi(t) = f(t).
\]
The proof is complete.

**Theorem 19.** Let \( \alpha > 0, p, q \geq 1, \) and \( \frac{1}{p} + \frac{1}{q} = 1 + \alpha \), where \( p \neq 1 \) and \( q \neq 1 \) in the case when \( \frac{1}{p} + \frac{1}{q} = 1 + \alpha \). Moreover, let
\[
\frac{T}{a} \frac{T}{a} (L^p) := \left\{ f : f = \frac{T}{a} \frac{T}{a} g, \; g \in L^p(J) \right\}
\]
and
\[
\frac{T}{a} \frac{T}{a} (L^p) := \left\{ f : f = \frac{T}{a} \frac{T}{a} g, \; g \in L^p(J) \right\},
\]
then the following integration by part formulas hold.
(a) If \( \varphi \in L^p(J) \) and \( \psi \in L^q(J) \), then
\[
\int_{\rho} \varphi(t) \left( \frac{T}{a} \frac{T}{a} \psi \right)(t) \Delta t = \int_{\rho} \psi(t) \left( \frac{T}{a} \frac{T}{a} \varphi \right)(t) \Delta t.
\]
(b) If \( g \in \frac{T}{a} \frac{T}{a} (L^p) \) and \( f \in \frac{T}{a} \frac{T}{a} (L^q) \), then
\[
\int_{\rho} g(t) \left( \frac{T}{a} \frac{T}{a} f \right)(t) \Delta t = \int_{\rho} f(t) \left( \frac{T}{a} \frac{T}{a} g \right)(t) \Delta t.
\]
(c) For Caputo fractional derivatives, if \( g \in \mathbb{T}^a I^b_t L^p \) and \( f \in \mathbb{T}^a I^b_t L^q \), then

\[
\int_0^b g(t) \left( t \mathcal{C} D^\alpha_t f \right) (t) \Delta t = \left[ t \mathcal{C}^{1-a} g(t) \cdot f(t) \right]_{t=a}^b + \int_a^b f(\sigma(t)) \left( t \mathcal{D}^\alpha_t g \right) (t) \Delta t.
\]

and

\[
\int_0^b g(t) \left( t \mathcal{C} D^\alpha_b f \right) (t) \Delta t = \left[ t \mathcal{C}^{1-a} g(t) \cdot f(t) \right]_{t=a}^b + \int_a^b f(\sigma(t)) \left( t \mathcal{D}^\alpha_b g \right) (t) \Delta t.
\]

**Proof.**

(a) It follows from Definition 7 and Fubini’s theorem on time scales that

\[
\int_a^b \phi(t) \left( t \mathcal{C}^\alpha I^\beta_t \psi \right) (t) \Delta t = \int_a^b \phi(t) \left( \frac{1}{\Gamma(\alpha)} \int_a^t (t - \sigma(s))^{\alpha-1} \psi(s) \Delta s \right) \Delta t = \int_a^b \psi(s) \int_s^b \frac{(t - \sigma(s))^{\alpha-1}}{\Gamma(\alpha)} \phi(t) \Delta t \Delta s
\]

\[
= \int_a^b \psi(t) \int_t^b \frac{(s - \sigma(t))^{\alpha-1}}{\Gamma(\alpha)} \phi(s) \Delta s \Delta t
\]

\[
= \int_a^b \psi(t) \left( t \mathcal{C} I^\beta_t \phi \right) (t) \Delta t.
\]

(b) It follows from Definition 8 and Fubini’s theorem on time scales that

\[
\int_a^b g(t) \left( t \mathcal{D}^\alpha f \right) (t) \Delta t = \int_a^b g(t) \left( \frac{1}{\Gamma(1-\alpha)} \left( \int_a^t (t - \sigma(s))^{-\alpha} f(s) \Delta s \right)^, \Delta s \right) \Delta t
\]

\[
= \int_a^b f(s) \left( \frac{1}{\Gamma(1-\alpha)} \left( \int_s^b (t - \sigma(s))^{-\alpha} g(t) \Delta t \right)^, \Delta s \right) \Delta t
\]

\[
= \int_a^b f(t) \left( \frac{1}{\Gamma(1-\alpha)} \left( \int_t^b (s - \sigma(t))^{-\alpha} g(s) \Delta s \right)^, \Delta t \right) \Delta t
\]

\[
= \int_a^b g(t) \left( t \mathcal{D}^\alpha I^\beta_b f \right) (t) \Delta t.
\]
(c) It follows from Definition 9, Fubini’s theorem on time scales and Theorem 1 that

\[
\int_{0}^{T} g(t) \left( \frac{T}{a} \right)^\alpha \Delta t = \int_{0}^{T} g(t) \left( \frac{1}{\Gamma(1-\alpha)} \right) \int_{a}^{b} (t - \sigma(s))^{-\alpha} f^\Delta(s) \Delta s \Delta t
\]

\[
= \int_{0}^{T} f^\Delta(s) \left( \frac{1}{\Gamma(1-\alpha)} \right) \int_{a}^{b} (t - \sigma(s))^{-\alpha} g(t) \Delta t \Delta s
\]

\[
= \int_{0}^{T} f^\Delta(t) \left( \frac{1}{\Gamma(1-\alpha)} \right) \int_{a}^{b} (s - \sigma(t))^{-\alpha} g(s) \Delta s \Delta t
\]

\[
= \left[ \frac{T}{T} \right]^{\alpha} \left( \begin{array}{c}
  f(t) \\
  \end{array} \right) \bigg|_{t=a}^{b} = \int_{0}^{T} f(\sigma(t)) \left( \frac{1}{\Gamma(1-\alpha)} \right) \int_{a}^{b} (s - \sigma(t))^{-\alpha} g(s) \Delta s \Delta t
\]

Then, we have the following result.

\[
D^\alpha \left( \begin{array}{c}
  f(t) \\
  \end{array} \right) = \left[ \frac{T}{T} \right]^{\alpha} \left( \begin{array}{c}
  f(t) \\
  \end{array} \right) + \int_{0}^{T} f(\sigma(t)) \left( \frac{1}{\Gamma(1-\alpha)} \right) \int_{a}^{b} (s - \sigma(t))^{-\alpha} g(s) \Delta s \Delta t.
\]

The second relation is obtained in a similar way. The proof is complete.

\(\Box\)

4. Fractional Sobolev Spaces on Time Scales and Their Properties

In this section, we present and prove some lemmas, propositions, and theorems, which are of utmost significance for our main results.

In the following, let \(0 < a < b\). Inspired by Theorems 8–11, we give the following definition.

**Definition 13.** Let \(0 < \alpha \leq 1\). By \(AC^{\alpha,1}_{\Delta,a^+}(J, \mathbb{R}^N)\) we denote the set of all functions \(f : J \to \mathbb{R}^N\) that have the representation

\[
f(t) = \frac{1}{\Gamma(\alpha)} \frac{c}{(t-a)^{1-\alpha}} + \frac{T}{\alpha} \Delta(t) \phi(t), \quad t \in J \quad \Delta - a.e.
\]

with \(c \in \mathbb{R}^N\) and \(\phi \in L^1_{\Delta}\).

Then, we have the following result.

**Theorem 20.** Let \(0 < \alpha \leq 1\) and \(f \in L^1_{\Delta}\). Then, function \(f\) has the left Riemann–Liouville derivative \( \frac{T}{a} D^\alpha f \) of order \(\alpha\) on the interval \(J\) iff \(f \in AC^{\alpha,1}_{\Delta,a^+}(J, \mathbb{R}^N)\); that is, \(f\) has the representation (6). In such a case,

\[
\left( \frac{T}{a} \right)^\alpha f(a) = c, \quad \left( \frac{T}{a} D^\alpha f \right)(t) = \phi(t), \quad t \in J \quad \Delta - a.e.
\]

**Proof.** Let us assume that \(f \in L^1_{\Delta}\) has a left-sided Riemann–Liouville derivative \( \frac{T}{a} D^\alpha f \). This means that \( \frac{T}{a} D^\alpha f \) is (identified to) an absolutely continuous function. From the integral representation of Theorems 8 and 10, there exists a constant \(c \in \mathbb{R}^N\) and a function \(\phi \in L^1_{\Delta}\) such that

\[
\left( \frac{T}{a} \right)^\alpha f(t) = c + \left( \frac{T}{a} \right)^\alpha \Delta(t) \phi(t), \quad t \in J,
\]

with \(\left( \frac{T}{a} \right)^\alpha f(a) = c\) and \(\left( \frac{T}{a} \right)^\alpha \Delta(t) \phi(t) = \frac{T}{a} D^\alpha f = \phi(t), t \in J \quad \Delta - a.e.\).
By Proposition 4 and applying \( T_{a} L_{t}^{\alpha} \) to (7), we obtain
\[
(T_{a} L_{t}^{\alpha} f)(t) = (T_{a} L_{t}^{\alpha} c)(t) + (T_{a} L_{t}^{\alpha} \varphi)(t), \quad t \in J \quad \text{\( \Delta \)-a.e..}
\] (8)

The result follows from the \( \Delta \)-differentiability of (8).

Conversely, let us assume that (6) holds true. From Proposition 4 and applying \( T_{a} L_{t}^{\alpha} \) to (6), we obtain
\[
(T_{a} L_{t}^{\alpha-a} f)(t) = c + (T_{a} L_{t}^{\alpha} \varphi)(t), \quad t \in J \quad \text{\( \Delta \)-a.e..}
\]
and then \( T_{a} L_{t}^{\alpha-a} f \) has an absolutely continuous representation. Further, \( f \) has a left-sided Riemann–Liouville derivative \( T_{a} D_{t}^{\alpha} f \). This completes the proof. \( \square \)

**Remark 2.** (i) By \( AC_{\Delta, a^{*}}^{k, p} (1 \leq p < \infty) \) we denote the set of all functions \( f : J \to \mathbb{R}^{N} \) possessing representation (6) with \( c \in \mathbb{R}^{N} \) and \( \varphi \in L_{p}^{\Delta} \).

(ii) It is easy to see that Theorem 20 implies that for any \( 1 \leq p < \infty \), \( f \) has the left Riemann–Liouville derivative \( T_{a} D_{t}^{\alpha} f \in L_{p}^{\Delta} \) iff \( f \in AC_{\Delta, a^{*}}^{k, p} \); that is, \( f \) has the representation (6) with \( \varphi \in L_{p}^{\Delta} \).

**Definition 14.** Let \( 0 < \alpha \leq 1 \) and let \( 1 \leq p < \infty \). By left Sobolev space of order \( \alpha \) we will mean the set \( W_{\Delta, a^{+}}^{k, p} = W_{\Delta, a^{+}}^{k, p} (J, \mathbb{R}^{N}) \) given by
\[
W_{\Delta, a^{+}}^{k, p} := \left\{ u \in L_{\Delta}^{p}, \exists \varphi \in C_{c, \text{rad}}^{\infty}, \text{such that } \int_{J} u(t) \cdot T_{a} D_{t}^{\alpha} \varphi(t) d\Delta t = \int_{J} g(t) \cdot \varphi(t) d\Delta t \right\}.
\]

**Remark 3.** A function \( g \) given in Definition 16 will be called the weak left fractional derivative of order \( 0 < \alpha \leq 1 \) of \( u \); let us denote it by \( T_{a} u_{a^{+}}^{\alpha} \). The uniqueness of this weak derivative follows from [4].

We have the following characterization of \( W_{\Delta, a^{+}}^{k, p} \).

**Theorem 21.** If \( 0 < \alpha \leq 1 \) and \( 1 \leq p < \infty \), then \( W_{\Delta, a^{+}}^{k, p} = AC_{\Delta, a^{+}}^{k, p} \cap L_{p}^{\Delta} \).

**Proof.** On the one hand, if \( u \in AC_{\Delta, a^{+}}^{k, p} \cap L_{p}^{\Delta} \) then from Theorem 20 it follows that \( u \) has derivative \( T_{a} D_{t}^{\alpha} u \in L_{p}^{\Delta} \). Thus, Theorem 19 implies that
\[
\int_{J} u(t) \cdot T_{a} D_{t}^{\alpha} \varphi(t) d\Delta t = \int_{J} (T_{a} D_{t}^{\alpha} u)(t) \varphi(t) d\Delta t
\]
for any \( \varphi \in C_{c, \text{rad}}^{\infty} \). So, \( u \in W_{\Delta, a^{+}}^{k, p} \) with \( T_{a} u_{a^{+}}^{\alpha} = g = T_{a} D_{t}^{\alpha} u \in L_{p}^{\Delta} \).

On the other hand, if \( u \in W_{\Delta, a^{+}}^{k, p} \), then \( u \in L_{p}^{\Delta} \) and there exists a function \( g \in L_{p}^{\Delta} \) such that
\[
\int_{J} u(t) \cdot T_{a} D_{t}^{\alpha} \varphi(t) d\Delta t = \int_{J} g(t) \varphi(t) d\Delta t
\]
for any \( \varphi \in C_{c, \text{rad}}^{\infty} \). To show that \( u \in AC_{\Delta, a^{+}}^{k, p} \cap L_{p}^{\Delta} \), it suffices to check (Theorem 20 and definition of \( AC_{\Delta, a^{+}}^{k, p} \)) that \( u \) possesses the left Riemann–Liouville derivative of order \( \alpha \), which belongs to \( L_{p}^{\Delta} \); that is, \( T_{a} D_{t}^{\alpha-a} u \) is absolutely continuous on \( J \) and its delta derivative of \( \alpha \) order (existing \( \Delta \)-a.e. on \( J \)) belongs to \( L_{p}^{\Delta} \).
In fact, let \( \varphi \in C_{c,rd}^\infty \) then \( \varphi \in \mathcal{D}_{b}^a(C_{rd}) \) and \( \mathcal{D}_{b}^a \varphi = -\left( \frac{T}{T} I_{b}^{1-a} \right)^{\Delta} \). From Theorem 19, it follows that

\[
\int_{\mathcal{J}} u(t) \mathcal{D}_{b}^a \varphi(t) d\Delta t = \int_{\mathcal{J}} u(t) (-\frac{T}{T} I_{b}^{1-a} \varphi)^{\Delta}(t) d\Delta t
\]

\[
= \int_{\mathcal{J}} \left( \mathcal{D}_{b}^{1-a} I_{b}^{1-a} u \right)(t) (-\frac{T}{T} I_{b}^{1-a} \varphi)^{\Delta}(t) d\Delta t
\]

\[
= \int_{\mathcal{J}} \left( \mathcal{D}_{b}^{1-a} I_{b}^{1-a} u \right)(t) (-\varphi)^{\Delta}(t) d\Delta t
\]

\[
= - \int_{\mathcal{J}} \left( \mathcal{D}_{b}^{1-a} I_{b}^{1-a} u \right)(t) \varphi^{\Delta}(t) d\Delta t.
\]  

In view of (9) and (10), we obtain

\[
\int_{\mathcal{J}} \left( \mathcal{D}_{b}^{1-a} I_{b}^{1-a} u \right)(t) \varphi^{\Delta}(t) d\Delta t = - \int_{\mathcal{J}} g(t) \varphi(t) d\Delta t
\]

for any \( \varphi \in C_{c,rd}^\infty \). So, \( \mathcal{D}_{b}^{1-a} I_{b}^{1-a} u \in W_{\Delta a}^{L_1} \). Consequently, \( \mathcal{D}_{b}^{1-a} I_{b}^{1-a} u \) is absolutely continuous and its delta derivative is equal \( \Delta - a.e. \) on \( [a,b]_{\mathbb{T}} \) to \( g \in L_\Delta^p \). The proof is complete. \( \square \)

From the proof of Theorem 21 and the uniqueness of the weak fractional derivative, the following theorem follows.

**Theorem 22.** If \( 0 < \alpha \leq 1 \) and \( 1 \leq p < \infty \), then the weak left fractional derivative \( ^{a\alpha} D_{\mathbb{T}} u_{\alpha} \) of a function \( u \in W_{\Delta a}^{L_1} \) coincides with its left Riemann–Liouville fractional derivative \( ^{a\alpha} D_{\mathbb{T}} u_{\alpha} \) of order \( \alpha \) on \( J \).

**Remark 4.** (1) If \( 0 < \alpha \leq 1 \) and \( (1-a)p < 1 \), then \( AC_{\Delta a}^{\alpha,p} \subset L_\Delta^p \) and, consequently,

\[
W_{\Delta a}^{L_1} = AC_{\Delta a}^{\alpha,p} \cap L_\Delta^p = AC_{\Delta a}^{\alpha,p}.
\]

(2) If \( 0 < \alpha \leq 1 \) and \( (1-a)p \geq 1 \), then \( W_{\Delta a}^{L_1} = AC_{\Delta a}^{\alpha,p} \cap L_\Delta^p \) is the set of all functions belong to \( AC_{\Delta a}^{\alpha,p} \) that satisfy the condition \( (\mathcal{D}_{b}^{1-a} I_{b}^{1-a} f)(a) = 0 \).

By using the definition of \( W_{\Delta a}^{L_1} \) with \( 0 < \alpha \leq 1 \) and Theorem 22, one can easily prove the following result.

**Theorem 23.** Let \( 0 < \alpha \leq 1, 1 \leq p < \infty \) and \( u \in L_\Delta^p \). Then \( u \in W_{\Delta a}^{L_1} \) iff there exists a function \( g \in L_\Delta^p \) such that

\[
\int_{\mathcal{J}} u(t) \mathcal{D}_{b}^{a} \varphi(t) d\Delta t = \int_{\mathcal{J}} g(t) \varphi(t) d\Delta t, \quad \varphi \in C_{c,rd}^\infty.
\]

In such a case, there exists the left Riemann–Liouville derivative \( \mathcal{D}_{b}^{a} u \) of \( u \) and \( g = \mathcal{D}_{b}^{a} u \).

**Remark 5.** Function \( g \) will be called the weak left fractional derivative of \( u \in W_{\Delta a}^{L_1} \) of order \( \alpha \). Its uniqueness follows from [4]. From the above theorem it follows that it coincides with an appropriate Riemann–Liouville derivative.

Let us fix \( 0 < \alpha \leq 1 \) and consider in the space \( W_{\Delta a}^{L_1} \), a norm \( \| \cdot \|_{W_{\Delta a}^{L_1}} \) given by

\[
\| u \|_{W_{\Delta a}^{L_1}} = \| u \|_{L_\Delta^p} + \| \mathcal{D}_{b}^{a} u \|_{L_\Delta^p}, \quad u \in W_{\Delta a}^{L_1}.
\]
That is to say, the fractional integration operator is bounded in $L^p_\Delta$ (Theorem 12).

Lemma 4. Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$. For any $f \in L^p_\Delta(J, \mathbb{R}^N)$, we have

$$
\|D^\alpha T^\alpha f\|_{L^p_\Delta([a,t]_\tau)} \leq \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p_\Delta([a,t]_\tau)}, \quad \text{for } \xi \in [a,t]_\tau, \ t \in J. \tag{11}
$$

(Here $\| \cdot \|_{L^p_\Delta}$ denotes the delta norm in $L^p_\Delta$ (Theorem 12)).

Proof. Inspired by Theorem 13 and the proof of Lemma 3.1 of [40], we can prove (11).

In fact, if $1 < p < \infty$ and $g \in L^q_\Delta(J, \mathbb{R}^N)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In consideration of Theorems 2 and 3, Fubini’s theorem on time scales, and Propositions 1 and 2, one arrives at

$$
\begin{align*}
\|D^\alpha T^\alpha f\|_{L^p_\Delta([a,t]_\tau)} & = \int_a^t \|D^\alpha T^\alpha f\|_{L^p_\Delta([a,t]_\tau)} d\xi \\
& = \frac{1}{\Gamma(\alpha)} \int_a^t \int_a^\xi (\xi - \sigma(\tau))^{\alpha-1} f(\tau) |d\tau| |d\xi| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_a^t \int_a^\xi (\xi - \sigma(\tau))^{\alpha-1} |f(\tau)| |d\tau| |d\xi| \\
& = \frac{1}{\Gamma(\alpha)} \int_a^t |f(\tau)| |d\tau| \int_a^\tau (\xi - \sigma(\tau))^{\alpha-1} |d\xi| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_a^t |f(\tau)| |d\tau| \int_a^\tau (\xi - \sigma(\tau))^{\alpha-1} |d\xi| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_a^t |f(\tau)| |d\tau| \int_a^\tau (\xi - \sigma(\tau))^{\alpha-1} |d\xi| \\
& \leq \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p_\Delta([a,t]_\tau)}, \quad \text{for } t \in J. \tag{12}
\end{align*}
$$

Now, suppose that $1 < p < \infty$ and $g \in L^q_\Delta(J, \mathbb{R}^N)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In consideration of Theorems 2 and 3, Fubini’s theorem on time scales, and Propositions 1 and 2, one arrives at

$$
\begin{align*}
\int_a^t g(\xi) \int_a^\xi (\xi - \sigma(\tau))^{\alpha-1} f(\tau) |d\tau| |d\xi| & = \int_a^t g(\xi) \int_a^\xi (\xi - \sigma(\tau))^{\alpha-1} f(\tau) |d\tau| |d\xi| \\
& \leq \int_a^t |g(\xi)| \int_a^\xi (\xi - \sigma(\tau))^{\alpha-1} |f(\xi - \sigma(\tau))| |d\tau| |d\xi| \\
& \leq \int_a^t (\xi - \sigma(\tau))^{\alpha-1} |d\tau| \int_a^\tau |g(\xi)| \int_a^\xi f(\xi - \sigma(\tau)) |d\tau| |d\xi| \\
& \leq \int_a^t (\xi - \sigma(\tau))^{\alpha-1} |d\tau| \int_a^\tau |g(\xi)| \int_a^\xi f(\xi - \sigma(\tau)) |d\tau| |d\xi| \\
& \leq \int_a^t (\xi - \sigma(\tau))^{\alpha-1} |d\tau| \int_a^\tau |g(\xi)| \int_a^\xi f(\xi - \sigma(\tau)) |d\tau| |d\xi| \\
& \leq \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p_\Delta([a,t]_\tau)} \|g\|_{L^q_\Delta([a,t]_\tau)}, \quad \text{for } t \in J. \tag{13}
\end{align*}
$$

For any fixed $t \in J$, consider the functional $H_{g,f} : L^p_\Delta(J, \mathbb{R}^N) \to \mathbb{R}$
According to (13), it is obvious that \( H_{\zeta,f}(g) \in \left( L^q_H(J, \mathbb{R}^N) \right)^* \), where \( \left( L^q_H(J, \mathbb{R}^N) \right)^* \) denotes the dual space of \( L^q_H(J, \mathbb{R}^N) \). Therefore, by (13) and (14) and the Riesz representation theorem, there exists \( h \in L^p_H(J, \mathbb{R}^N) \) such that

\[
\int_a^t h(\xi)g(\xi)\Delta \xi = \int_a^t \left[ \int_a^\xi (\xi - \sigma(\tau))^{a-1}f(\tau)\Delta \tau \right] g(\xi)\Delta \xi
\]

and

\[
\|h\|_{L^p_H([a,t])} = \|H_{\zeta,f}\|_{L^p_H([a,t])} \leq \frac{(t-a)^a}{\Gamma(a+1)} \|f\|_{L^p_H([a,t])}
\]

for all \( g \in L^q_H(J, \mathbb{R}^N) \). Hence, we have by (15) and Definition 7

\[
\frac{1}{\Gamma(a)} h(\xi) = \frac{1}{\Gamma(a)} \int_a^\xi (\xi - \sigma(\tau))^{a-1}f(\tau)\Delta \tau = \frac{1}{\Gamma(a)} \int_a^\xi f(\xi)\Delta \xi, \quad \text{for } \xi \in [a,t],
\]

which means

\[
\|I^q_H f\|_{L^p_H([a,t])} = \frac{1}{\Gamma(a)} \|h\|_{L^p_H([a,t])} \leq \frac{(t-a)^a}{\Gamma(a+1)} \|f\|_{L^p_H([a,t])}
\]

according to (16). Combining this with (12) and (17), we obtain inequality (11). The proof is complete. \( \square \)

**Theorem 24.** If \( 0 < \alpha \leq 1 \), then the norm \( \| \cdot \|_{W^{\alpha,p}} \) is equivalent to the norm \( \| \cdot \|_{a,W^{\alpha,p}} \) given by

\[
\|u\|^p_{a,W^{\alpha,p}} = \|I^{\alpha-\sigma}_a u(a)\|^p + \|\partial_t^\alpha u\|^p_{L^p}, \quad u \in W^{\alpha,p}_{a}\cdot
\]

**Proof.** (1) Assume that \( (1 - \alpha)p < 1 \). On the one hand, in view of Remarks 2 and 4, for \( u \in W^{\alpha,p}_{a}\cdot \), we can write it as

\[
u(t) = \frac{1}{\Gamma(a)} \left( \frac{c}{(t-a)^{1-\alpha}} + \frac{T^q_H}{\Gamma(a)} \phi(t) \right)
\]

with \( c \in \mathbb{R}^N \) and \( \phi \in L^p_H \). Since \( (t-a)^{(a-1)p} \) is an increasing monotone function, by using Proposition 1, we can write that \( \int_0^t (t-a)^{(a-1)p} \Delta t \leq \int_0^{\infty} (t-a)^{(a-1)p} dt \). Furthermore, taking into account Lemma 4, we have

\[
\|u\|^p_{L^p_H} = \int_0^t \left[ \frac{1}{\Gamma(a)} \left( \frac{c}{(t-a)^{1-\alpha}} + \frac{T^q_H}{\Gamma(a)} \phi(t) \right) \right]^p \Delta t \leq 2^{p-1} \left( \frac{|c|^p}{\Gamma^p(a)} \int_0^t (t-a)^{(a-1)p} \Delta t + \|I^\alpha u\|^p_{L^p_H} \right)
\]

\[
\leq 2^{p-1} \left( \frac{|c|^p}{\Gamma^p(a)} \int_0^t (t-a)^{(a-1)p} dt + \|I^\alpha u\|^p_{L^p_H} \right)
\]

\[
\leq 2^{p-1} \left( \frac{|c|^p}{\Gamma^p(a)} (a-1)^p + 1(b-a)^{(a-1)p+1} + K^\alpha ||\phi||^p_{L^p_H} \right),
\]

and

\[
\|u\|^p_{L^p_H} = \int_0^t \left[ \frac{1}{\Gamma(a)} \left( \frac{c}{(t-a)^{1-\alpha}} + \frac{T^q_H}{\Gamma(a)} \phi(t) \right) \right]^p \Delta t \leq 2^{p-1} \left( \frac{|c|^p}{\Gamma^p(a)} \int_0^t (t-a)^{(a-1)p} \Delta t + \|I^\alpha u\|^p_{L^p_H} \right)
\]

\[
\leq 2^{p-1} \left( \frac{|c|^p}{\Gamma^p(a)} \int_0^t (t-a)^{(a-1)p} dt + \|I^\alpha u\|^p_{L^p_H} \right)
\]

\[
\leq 2^{p-1} \left( \frac{|c|^p}{\Gamma^p(a)} (a-1)^p + 1(b-a)^{(a-1)p+1} + K^\alpha ||\phi||^p_{L^p_H} \right),
\]

...
where \( K = \frac{(b-a)^p}{\Gamma(a+1)} \). Noting that \( c = \frac{3}{\alpha} \int_a^1 t^{1-\alpha} u(t) dt \), \( \varphi = \frac{3}{\alpha} \frac{D_t^p u}{\Gamma(p+1)} \), one can obtain

\[
\|u\|^p_{L^\alpha} \leq L_{\alpha,0} \left( |c|^p + \|\varphi\|^p_{L^\alpha} \right) \\
\leq L_{\alpha,0} \left( \frac{3}{\alpha} \int_a^1 t^{1-\alpha} u(t) dt \right)^p + \|\frac{3}{\alpha} \frac{D_t^p u}{\Gamma(p+1)}\|^p_{L^\alpha} \\
= L_{\alpha,0} \|u\|^p_{W_{\alpha,a}^p},
\]

where

\[
L_{\alpha,0} = 2^{p-1} \left( \frac{(b-a)^{1-(1-a)p}}{\Gamma(p+1)(1-(1-a)p)} + K^p \right).
\]

Consequently,

\[
\|u\|^p_{W_{\alpha,a}^p}_{W_{\alpha,a}^p} = \|u\|^p_{L^\alpha} + \|\frac{3}{\alpha} \frac{D_t^p u}{\Gamma(p+1)}\|^p_{L^\alpha} \\
\leq L_{\alpha,1} \|u\|^p_{W_{\alpha,a}^p},
\]

where \( L_{\alpha,1} = L_{\alpha,0} + 1 \).

On the other hand, we will prove that there exists a constant \( M_{\alpha,1} \) such that

\[
\|u\|^p_{W_{\alpha,a}^p} \leq M_{\alpha,1} \|u\|^p_{W_{\alpha,a}^p}, \quad u \in W_{\alpha,a}^p.
\]

Indeed, let \( u \in W_{\alpha,a}^p \) and consider coordinate functions \((\frac{3}{\alpha} \int_a^t t^{1-\alpha} u)\) of \( \frac{3}{\alpha} t^{1-\alpha} u \) with \( i \in \{1, \ldots, N\} \). Lemma 4, Theorem 14 and Corollary 1 imply that there exist constants

\[
\Lambda_i \in \left[ \inf_{t \in [a,b]} (\frac{3}{\alpha} \int_a^t t^{1-\alpha} u)^i(t), \sup_{t \in [a,b]} (\frac{3}{\alpha} \int_a^t t^{1-\alpha} u)^i(t) \right], \quad (i = 1, 2, \ldots, N)
\]

such that

\[
\Lambda_i = \frac{1}{b-a} \int_a^b (\frac{3}{\alpha} \int_a^s t^{1-\alpha} u)^i(s) ds.
\]

Hence, for a fixed \( t_0 \in ]a,b[ \), if \((\frac{3}{\alpha} \int_a^t t^{1-\alpha} u)^i(t_0) \neq 0 \) for all \( i = 1, 2, \ldots, N \), then we can take constants \( \theta_i \) such that

\[
\theta_i(\frac{3}{\alpha} \int_a^t t^{1-\alpha} u)^i(t_0) = \Lambda_i = \frac{1}{b-a} \int_a^b (\frac{3}{\alpha} \int_a^s t^{1-\alpha} u)^i(s) ds.
\]

Therefore, we have

\[
(\frac{3}{\alpha} \int_a^t t^{1-\alpha} u)^i(t) = \frac{\theta_i}{b-a} \int_a^b (\frac{3}{\alpha} \int_a^s t^{1-\alpha} u)^i(s) ds.
\]

From the absolute continuity (Theorem 9) of \((\frac{3}{\alpha} \int_a^t t^{1-\alpha} u)^i \) it follows that

\[
(\frac{3}{\alpha} \int_a^t t^{1-\alpha} u)^i(t) = (\frac{3}{\alpha} \int_a^t t^{1-\alpha} u)^i(t_0) + \int_{[t_0,t]} \left[ (\frac{3}{\alpha} \int_a^s t^{1-\alpha} u)^i(s) \right] ds
\]

for any \( t \in I \). Consequently, combining with Proposition 3 and Lemma 4, we see that
\[(\frac{T}{t})^1-u)(t) = \left|\left(\frac{T}{t}^1-u\right)(t_0) + \int_{[\theta, t)} \left(\frac{T}{s}^1-u\right)(s) \Delta s\right| \leq \frac{\theta}{b-a} ||(\frac{T}{t}^1-u)||_1^L + \int_{[\theta, t)} ||(\frac{T}{s}^1 D_s^\alpha u)||_1^L \Delta s \]
\[\leq \frac{\theta}{b-a} ||(\frac{T}{t}^1-u)||_1^L + \frac{2}{\alpha} ||D_t^\alpha D_t^\alpha u||_1^L \]
\[\leq \frac{\theta}{b-a} \frac{(b-a)^{1-a}}{\Gamma(2-a)} \left[ ||u||_1^L + ||D_t^\alpha D_t^\alpha u||_1^L \right]
\]
for \( t \in J \). In particular,
\[|\left(\frac{T}{t}^1-u\right)(a)| \leq \frac{\theta}{b-a} \frac{(b-a)^{1-a}}{\Gamma(2-a)} ||u||_1^L + ||D_t^\alpha D_t^\alpha u||_1^L.
\]
So,
\[|\left(\frac{T}{t}^1-u\right)(a)| \leq N \left( \frac{\theta}{\Gamma(2-a)} + 1 \right) \left( ||u||_1^L + ||D_t^\alpha D_t^\alpha u||_1^L \right)
\[\leq N M_{a,0}(b-a)^{p-1} \left( ||u||_1^L + ||D_t^\alpha D_t^\alpha u||_1^L \right),
\]
where \( |\theta| = \max_{i \in \{1, 2, \ldots, N\}} |\theta_i| \) and \( M_{a,0} = |\theta(b-a)^{-a}| \frac{1}{\Gamma(2-a)} + 1 \). Thus,
\[|\left(\frac{T}{t}^1-u\right)(a)|^p \leq N^p M_{a,0}^p (b-a)^{p-1} 2^{p-1} \left( ||u||_1^L^p + ||D_t^\alpha D_t^\alpha u||_1^L^p \right),
\]
and, consequently,
\[\|u\|_{W_{\Delta-a}^{\alpha,p}} = \|\left(\frac{T}{t}^1-u\right)(a)|^p + \|D_t^\alpha D_t^\alpha u||_1^L^p
\[\leq \left( N^p M_{a,0}^p (b-a)^{p-1} 2^{p-1} + 1 \right) \left( ||u||_1^L^p + ||D_t^\alpha D_t^\alpha u||_1^L^p \right)
\[= M_{a,1} \|u\|_{W_{\Delta-a}^{\alpha,p}},
\]
where \( M_{a,1} = N^p M_{a,0}^p (b-a)^{p-1} 2^{p-1} + 1 \).

If \( \left(\frac{T}{t}^1-u\right)(t_0) = 0 \) for \( i \) belongs to some subset of \( \{1, 2, \ldots, N\} \), from the above argument process one can easily see that there exists a constant \( M_{a,1} \) such that (18) holds.

(2) When \( (1-a)^p \geq 1 \), then (Remark 4) \( W_{\Delta-a}^{\alpha,p} = A_{\Delta-a}^{\alpha,p} \cap L_1^p \) is the set of all functions belonging to \( A_{\Delta-a}^{\alpha,p} \) that satisfy the condition \( (\frac{T}{t}^1-u)(a) = 0 \). Hence, in the same way as in the case of \( (1-a)^p < 1 \) (putting \( c = 0 \)), we obtain the inequality
\[\|u\|_{W_{\Delta-a}^{\alpha,p}} \leq L_{a,1} \|u\|_{W_{\Delta-a}^{\alpha,p}}, \quad \text{with some } L_{a,1} > 0.
\]
The inequality
\[\|u\|_{W_{\Delta-a}^{\alpha,p}} \leq M_{a,1} \|u\|_{W_{\Delta-a}^{\alpha,p}}, \quad \text{with some } M_{a,1} > 0
\]
is obvious (it is sufficient to put \( M_{a,1} = 1 \) and use the fact that \( (\frac{T}{t}^1-u)(a) = 0 \).

The proof is complete. \( \square \)

Now, we are in a position to prove some basic properties of the space \( W_{\Delta-a}^{\alpha,p} \).
Theorem 25. The space $W_{\Delta,\alpha}^{\alpha, p}$ is complete with respect to each of the norms $\| \cdot \|_{W_{\Delta,\alpha}^{\alpha, p}}$ and $\| \cdot \|_{\delta, W_{\Delta,\alpha}^{\alpha, p}}$ for any $0 < \alpha \leq 1$, $1 \leq p < \infty$.

Proof. In view of Theorem 24, we only need to show that $W_{\Delta,\alpha}^{\alpha, p}$ with the norm $\| \cdot \|_{\delta, W_{\Delta,\alpha}^{\alpha, p}}$ is complete. Let $\{u_k\} \subset W_{\Delta,\alpha}^{\alpha, p}$ be a Cauchy sequence with respect to this norm. The sequences $\{T_d I^{-\alpha} u_k(a)\}$ and $\{D_t^p u_k\}$ are Cauchy sequences in $\mathbb{R}^N$ and $L_{\Delta}^p$, respectively.

Let $c \in \mathbb{R}^N$ and $\varphi \in L_{\Delta}^p$ be the limits of the above two sequences in $\mathbb{R}^N$ and $L_{\Delta}^p$, respectively. Then the function

$$u(t) = \frac{c}{\Gamma(\alpha)}(t - a)^{\alpha - 1} + \frac{c}{\Gamma(\alpha)} \varphi(t), \quad t \in \Delta - a.e.$$ 

belongs to $W_{\Delta,\alpha}^{\alpha, p}$ and it is the limit of $\{u_k\}$ in $W_{\Delta,\alpha}^{\alpha, p}$ with respect to $\| \cdot \|_{\delta, W_{\Delta,\alpha}^{\alpha, p}}$. The proof is complete. \(\Box\)

The proof method of the following two theorems is inspired by the method used in the proof of Proposition 8.1 (b), (c) in [41].

Theorem 26. The space $W_{\Delta,\alpha}^{\alpha, p}$ is reflexive with respect to the norm $\| \cdot \|_{W_{\Delta,\alpha}^{\alpha, p}}$ for any $0 < \alpha \leq 1$ and $1 < p < \infty$.

Proof. Let us consider $W_{\Delta,\alpha}^{\alpha, p}$ with the norm $\| \cdot \|_{W_{\Delta,\alpha}^{\alpha, p}}$ and define a mapping

$$\lambda : W_{\Delta,\alpha}^{\alpha, p} \ni u \mapsto \left( u, \frac{T_d}{\alpha} D_t^p u \right) \in L_{\Delta}^p \times L_{\Delta}^p.$$ 

It is obvious that

$$\| u \|_{W_{\Delta,\alpha}^{\alpha, p}} = \| \lambda u \|_{L_{\Delta}^p \times L_{\Delta}^p},$$

where

$$\| \lambda u \|_{L_{\Delta}^p \times L_{\Delta}^p} = \left( \sum_{i=1}^2 (\| \lambda u_i \|_{L_{\Delta}^p})^p \right)^{\frac{1}{p}}, \quad \lambda u = \left( u, \frac{T_d}{\alpha} D_t^p u \right) \in L_{\Delta}^p \times L_{\Delta}^p,$$

which means that the operator $\lambda : u \mapsto \left( u, \frac{T_d}{\alpha} D_t^p u \right)$ is an isometric isomorphic mapping and the space $W_{\Delta,\alpha}^{\alpha, p}$ is isometric isomorphic to the space $\Omega = \left\{ \left( u, \frac{T_d}{\alpha} D_t^p u \right) : \forall u \in W_{\Delta,\alpha}^{\alpha, p} \right\}$, which is a closed subset of $L_{\Delta}^p \times L_{\Delta}^p$ as $W_{\Delta,\alpha}^{\alpha, p}$ is closed.

Since $L_{\Delta}^p$ is reflexive, the Cartesian product space $L_{\Delta}^p \times L_{\Delta}^p$ is also a reflexive space with respect to the norm $\| v \|_{L_{\Delta}^p \times L_{\Delta}^p} = \left( \sum_{i=1}^2 (\| v_i \|_{L_{\Delta}^p})^p \right)^{\frac{1}{p}},$ where $v = (v_1, v_2) \in L_{\Delta}^p \times L_{\Delta}^p$.

Thus, $W_{\Delta,\alpha}^{\alpha, p}$ is reflexive with respect to the norm $\| \cdot \|_{W_{\Delta,\alpha}^{\alpha, p}}$. The proof is complete. \(\Box\)

Theorem 27. The space $W_{\Delta,\alpha}^{\alpha, p}$ is separable with respect to the norm $\| \cdot \|_{W_{\Delta,\alpha}^{\alpha, p}}$ for any $0 < \alpha \leq 1$ and $1 \leq p < \infty$.

Proof. Let us consider $W_{\Delta,\alpha}^{\alpha, p}$ with the norm $\| \cdot \|_{W_{\Delta,\alpha}^{\alpha, p}}$ and the mapping $\lambda$ defined in the proof of Theorem 26. Obviously, $\lambda(W_{\Delta,\alpha}^{\alpha, p})$ is separable as a subset of separable space $L_{\Delta}^p \times L_{\Delta}^p$. Since $\lambda$ is the isometry, $W_{\Delta,\alpha}^{\alpha, p}$ is also separable with respect to the norm $\| \cdot \|_{W_{\Delta,\alpha}^{\alpha, p}}$. The proof is complete. \(\Box\)
Proposition 6. Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in W_{\alpha,a}^{\alpha,p}$, if $1 - \alpha \geq \frac{1}{p}$ or $\alpha > \frac{1}{p}$, then

$$
\|u\|_{L^p_{\alpha}} \leq \frac{b^a}{\Gamma(\alpha + 1)} \|T^a D^a_t u\|_{L^p_{\alpha}};
$$

(19)

if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
\|u\|_{\infty} \leq \frac{b^{a - \frac{1}{q}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}} \|T^a D^a_t u\|_{L^p_{\alpha}}.
$$

(20)

Proof. In view of Remark 4 and Theorem 18, in order to prove inequalities (19) and (20), we only need to prove that

$$
\|T^a D^a_t u\|_{L^p_{\alpha}} \leq \frac{b^a}{\Gamma(\alpha + 1)} \|T^a D^a_t u\|_{L^p_{\alpha}}
$$

(21)

for $1 - \alpha \geq \frac{1}{p}$ or $\alpha > \frac{1}{p}$, and that

$$
\|T^a D^a_t u\|_{\infty} \leq \frac{b^{a - \frac{1}{q}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}} \|T^a D^a_t u\|_{L^p_{\alpha}}
$$

(22)

for $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Note that $T^a D^a_t u \in L^p_{\alpha}(I, \mathbb{R}^N)$, and the inequality (21) follows directly from Lemma 4. We are now in a position to prove (22). For $\alpha > \frac{1}{p}$, choose $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For all $u \in W_{\alpha,a}^{\alpha,p}$, since $(t - s)^{-(\alpha - 1)q}$ is an increasing monotone function, by using Proposition 1, we find that

$$
\int_a^t (t - \sigma(s))^{-(\alpha - 1)q} \Delta s \leq \int_a^t (t - s)^{-(\alpha - 1)q} ds.
$$

Taking into account of Proposition 2, we have

$$
\left| \int_a^t (t - \sigma(s))^{-(\alpha - 1)q} D^a_t u(s) \Delta s \right|
\leq \frac{1}{\Gamma(\alpha)} \left( \int_a^t (t - \sigma(s))^{-(\alpha - 1)q} \Delta s \right)^\frac{1}{q} \|T^a D^a_t u\|_{L^p_{\alpha}}
\leq \frac{1}{\Gamma(\alpha)} \left( \int_a^t (t - s)^{-(\alpha - 1)q} ds \right)^\frac{1}{q} \|T^a D^a_t u\|_{L^p_{\alpha}}
\leq \frac{b^{a - \frac{1}{q}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}} \|T^a D^a_t u\|_{L^p_{\alpha}}
= \frac{b^{a - \frac{1}{q}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}} \|T^a D^a_t u\|_{L^p_{\alpha}}.
$$

The proof is complete. \(\square\)

Remark 6. (i) According to (19), we can consider $W_{\alpha,a}^{\alpha,p}$ with respect to the norm

$$
\|u\|_{W_{\alpha,a}^{\alpha,p}} = \|T^a D^a_t u\|_{L^p_{\alpha}} = \left( \int_{\Delta t} \|T^a D^a_t u(t)\|_{L^p_{\alpha}}^p \Delta t \right)^\frac{1}{p}
$$

(23)

in the following analysis.

(ii) It follows from (19) and (20) that $W_{\alpha,a}^{\alpha,p}$ is continuously immersed into $C(I, \mathbb{R}^N)$ with the natural norm $\| \cdot \|_{\infty}$. 


Proposition 7. Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence \( \{u_k\} \subset W^{\alpha,p}_{\Delta^+} \) converges weakly to $u$ in $W^{\alpha,p}_{\Delta^+}$. Then, $u_k \to u$ in $C(J,\mathbb{R}^N)$, i.e., \( \|u - u_k\|_\infty = 0 \) as $k \to \infty$.

Proof. If $\alpha > \frac{1}{p}$, then by (20) and (30), the injection of $W^{\alpha,p}_{\Delta^+}$ into $C(J,\mathbb{R}^N)$, with its natural norm $\| \cdot \|_\infty$, is continuous, i.e., $u_k \to u$ in $W^{\alpha,p}_{\Delta^+}$, then $u_k \to u$ in $C(J,\mathbb{R}^N)$.

Since $u_k \to u$ in $W^{\alpha,p}_{\Delta^+}$, it follows that $u_k \to u$ in $C(J,\mathbb{R}^N)$. In fact, for any $h \in (C(J,\mathbb{R}^N))^*$, if $u_k \to u$ in $W^{\alpha,p}_{\Delta^+}$, then $u_k \to u$ in $C(J,\mathbb{R}^N)$, and thus $h(u_k) \to h(u)$. Therefore, $h \in \left(W^{\alpha,p}_{\Delta^+}\right)^*$, which means that $(C(J,\mathbb{R}^N))^* \subset \left(W^{\alpha,p}_{\Delta^+}\right)^*$. Hence, if $u_k \to u$ in $W^{\alpha,p}_{\Delta^+}$, then for any $h \in (C(J,\mathbb{R}^N))^*$, we have $h \in \left(W^{\alpha,p}_{\Delta^+}\right)^*$, and thus $h(u_k) \to h(u)$, i.e., $u_k \to u$ in $C(J,\mathbb{R}^N)$.

By the Banach–Steinhaus theorem, \( \{u_k\} \) is bounded in $W^{\alpha,p}_{\Delta^+}$ and, hence, in $C(J,\mathbb{R}^N)$. Now, we prove that the sequence \( \{u_k\} \) is equicontinuous. Let $\frac{1}{p} + \frac{1}{q} = 1$ and $t_1, t_2 \in J$ with $t_1 \leq t_2$, for all $f \in L^p(J,\mathbb{R}^N)$, by using Proposition 2, Proposition 1, and Theorem 15, and noting $\alpha > \frac{1}{p}$, we have

\[
\frac{1}{\Gamma(a(1 + (a-1)q))} \bigg( t_1^{(a-1)q+1} - t_2^{(a-1)q+1} \bigg) + \frac{2\|f\|_{L^p}}{\Gamma(a(1 + (a-1)q))} \bigg( t_2 - t_1 \bigg)^{a-1} \frac{1}{q} \leq \frac{1}{\Gamma(a(1 + (a-1)q))} \bigg( t_1^{(a-1)q+1} - t_2^{(a-1)q+1} \bigg) + \frac{2\|f\|_{L^p}}{\Gamma(a(1 + (a-1)q))} \bigg( t_2 - t_1 \bigg)^{a-1} \frac{1}{q}.
\]
Therefore, the sequence \( \{u_k\} \) is equicontinuous since, for \( t_1, t_2 \in J, t_1 \leq t_2 \), by applying (24) and (30), we have

\[
|u_k(t_1) - u_k(t_2)| = \left| \frac{T}{\alpha} \int_{t_1}^{t} \alpha D_{\alpha}^\tau u_k(t_1) - \frac{T}{\alpha} \int_{t_2}^{t} \alpha D_{\alpha}^\tau u_k(t_2) \right|
\]

\[
\leq \frac{2(t_2 - t_1)^{1-p}}{\Gamma(\alpha)(1 + (\alpha - 1)p)^{1}} \left\| \alpha D_{\alpha}^\tau u_k \right\|_{L^p} \\
= \frac{2(t_2 - t_1)^{1-p}}{\Gamma(\alpha)(1 + (\alpha - 1)p)^{1}} \left\| \alpha D_{\alpha}^\tau u \right\|_{L^p} \\
\leq \frac{2(t_2 - t_1)^{1-p}}{\Gamma(\alpha)((\alpha - 1)p + 1)^{1}} \left\| u_k \right\|_{W^{p,\alpha+}_\Delta} \\
\leq C(t_2 - t_1)^{1-p},
\]

where \( \frac{1}{p} + \frac{1}{\alpha} = 1 \) and \( C \in \mathbb{R}^+ \) is a constant. By the Ascoli–Arzela theorem on time scales (Lemma 3), \( \{u_k\} \) is relatively compact in \( C(J, \mathbb{R}^N) \). By the uniqueness of the weak limit in \( C(J, \mathbb{R}^N) \), every uniformly convergent subsequence of \( \{u_k\} \) converges uniformly on \( J \) to \( u \). The proof is complete. \( \square \)

**Remark 7.** It follows from Proposition 7 that \( W^{p,\alpha+}_\Delta \) is compactly immersed into \( C(J, \mathbb{R}^N) \) with the natural norm \( \left\| \cdot \right\|_{\infty} \).

**Theorem 28.** Let \( 1 < p < \infty, \frac{1}{p} < \alpha \leq 1, \frac{1}{p} + \frac{1}{\alpha} = 1, L : J \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}, (t, x, y) \mapsto L(t, x, y) \) satisfies

(i) for each \( (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \), \( L(t, x, y) \) is \( \Delta \)-measurable in \( t \);

(ii) for \( \Delta \), almost every \( t \in J \), \( L(t, x, y) \) is continuously differentiable in \( (x, y) \).

If there exist \( m_1 \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+) \), \( m_2 \in L_1^\alpha(J, \mathbb{R}^+) \) and \( m_3 \in L_1^q(J, \mathbb{R}^+) \), \( 1 < q < \infty \), such that, for \( \Delta \, a.e. \ t \in \mathbb{R} \) and every \( x, y \in \mathbb{R}^N \times \mathbb{R}^N \), one has

\[
|L(t, x, y)| \leq m_1(|x|)(m_2(t) + |y|^p), \\
|D_x L(t, x, y)| \leq m_1(|x|)(m_2(t) + |y|^p), \\
|D_y L(t, x, y)| \leq m_1(|x|)(m_3(t) + |y|^{p-1}).
\]

Then the functional \( \chi \) defined by

\[
\chi(u) = \int_J L(t, u(t), \frac{T}{\alpha} D_{\alpha}^\tau u(t)) \Delta t
\]

is continuously differentiable on \( W^{p,\alpha+}_\Delta \), and \( \forall u, v \in W^{p,\alpha+}_\Delta, \) one has

\[
\langle \chi'(u), v \rangle = \int_J \left[ \left( D_x L(t, u(t), \frac{T}{\alpha} D_{\alpha}^\tau u(t), v(t) \right) + \left( D_y L(t, u(t), \frac{T}{\alpha} D_{\alpha}^\tau u(t), \frac{T}{\alpha} D_{\alpha}^\tau v(t) \right) \right] \Delta t.
\]

**Proof.** It suffices to prove that \( \chi \) has, at every point \( u \), a directional derivative \( \chi'(u) \in (W^{p,\alpha+}_\Delta)^* \) given by (25) and that the mapping

\[
\chi' : W^{p,\alpha+}_\Delta \not\ni u \mapsto \chi'(u) \in (W^{p,\alpha+}_\Delta)^*
\]
is continuous. The rest of proof is similar to the proof of Theorem 1.4 in [42]. We will omit it here. The proof is complete. □

5. An Application

As an application of the concepts we introduced and the results obtained in Section 3, in this section we will use critical point theory to study the solvability of a class of boundary value problems on time scales. More precisely, our goal is to study the following Kirchhoff-type fractional \( p \)-Laplacian system on time scales with boundary condition (KFBVP\(_T\) for short):

\[
\begin{cases}
\left( \beta + \int_{t_0}^T |aD^\alpha_T u(t)|^p \, dt \right) \int_0^T D^\alpha_T \phi_p(aD^\alpha_T u(t)) = \lambda(t) \nabla G(t, u(t)), & \Lambda - \text{a.e. } t \in J, \\
\phi_p(u(0)) = 0,
\end{cases}
\]

where \( \beta, q > 0 \) and \( p > 1 \) are constants, \( \lambda \in L_+^\infty(J, \mathbb{R}^+) \) with \( \text{ess sup}_{t \in J} \lambda(t) := \lambda^0 > \lambda_0 := \text{ess inf}_{t \in J} \lambda(t) > 0 \), \( \frac{\lambda}{a} D^\rho_T \) and \( \frac{\lambda}{a} D^\rho_T \) are the right and the left Riemann–Liouville fractional derivative operators of order \( a \) defined on \( T \), respectively, and \( \phi_p : \mathbb{R} \rightarrow \mathbb{R} \) is the \( p \)-Laplacian ([43]) defined by

\[\phi_p(y) = \begin{cases} |y|^{p-2}y, & \text{if } y \neq 0, \\ 0, & \text{if } y = 0. \end{cases}\]

Furthermore, \( \nabla G \in C(J \times \mathbb{R}, \mathbb{R}) \) denotes the gradient of \( G(t, x) \) in \( x \). When \( T = \mathbb{R} \), \( \text{FBVP}_{T}(26) \) reduces to the following Kirchhoff-type fractional \( p \)-Laplacian system

\[
\begin{cases}
\left( \beta + \int_{t_0}^T |aD^\alpha_T u(t)|^p \, dt \right) \int_0^T D^\alpha_T \phi_p(aD^\alpha_T u(t)) = \lambda(t) \nabla G(t, u(t)), & \text{a.e. } t \in J, \\
\phi_p(u(0)) = 0,
\end{cases}
\]

When \( T = \mathbb{R} \) and \( \lambda(t) = \lambda \in (0, +\infty) \), \( \text{FBVP}_{T}(26) \) reduces to the following Kirchhoff-type fractional \( p \)-Laplacian system

\[
\begin{cases}
\left( \beta + \int_{t_0}^T |aD^\alpha_T u(t)|^p \, dt \right) \int_0^T D^\alpha_T \phi_p(aD^\alpha_T u(t)) = \lambda(t) \nabla G(t, u(t)), & \text{a.e. } t \in J, \\
\phi_p(u(0)) = 0,
\end{cases}
\]

When \( T = \mathbb{R} \), \( \lambda(t) = 1 \), our results further reduce to the following problem

\[
\begin{cases}
\left( \beta + \int_{t_0}^T |aD^\alpha_T u(t)|^p \, dt \right) \int_0^T D^\alpha_T \phi_p(aD^\alpha_T u(t)) = \nabla G(t, u(t)), & \text{a.e. } t \in J, \\
\phi_p(u(0)) = 0,
\end{cases}
\]

which has been studied by [44]. So, in short, our results are improved and generalized [44].

**Definition 15 ([42]).** Let \( E \) be a real Banach space and \( \varphi \in C^1(E, \mathbb{R}) \). If any sequence \( \{u_k\} \subset E \) for which \( \varphi(u_k) \) is bounded and \( \varphi'(u_k) \rightarrow 0 \) as \( k \rightarrow \infty \) possesses a convergent subsequence in \( E \), then we say that \( \varphi \) satisfies the (PS) condition.

**Lemma 5 ([45]).** Let \( E \) be a real Banach space and \( \varphi \in C^1(E, \mathbb{R}) \) satisfying the (PS) condition. Assume that \( \varphi(0) = 0 \) and the following conditions:

(A1) there are constants \( \rho, \sigma > 0 \) such that \( \varphi|_{\partial B_\rho(0)} \geq \sigma; \)

(A2) there exists an \( e \in E \setminus \overline{B_\rho(0)} \) such that \( \varphi(e) \leq 0. \)
Then, \( \varphi \) possesses a critical value \( c \geq \sigma \). Furthermore, \( c \) can be characterized as

\[
c = \inf_{v \in \Gamma} \max_{s \in [0,1]} \varphi(v(s)),
\]

where

\[
\Gamma = \{ v \in C([0,1], E) | v(0) = 0, v(1) = e \}.
\]

**Lemma 6 ([42])**. Let \( E \) be a real Banach space and \( \varphi \in C^1(E, \mathbb{R}) \) satisfying the (PS) condition. If \( \varphi \) is bounded from below, then \( c = \inf_E \varphi \) is a critical value of \( \varphi \).

For the sake of the infinitely many critical points of \( \varphi \), one introduces the genus properties as follows. First, we let

\[
\Xi = \{ A \subset E - \{0\} | A \text{ is closed in } E \text{ and symmetric with respect to } 0 \},
\]

\[
K_c = \{ u \in E | \varphi(u) = c, \varphi'(u) = 0 \},
\]

\[
\varphi^c = \{ u \in E | \varphi(u) \leq c \}.
\]

**Definition 16 ([45])**. For \( A \in \Xi \), we say that the genus of \( A \) is \( n \) denoted by \( \gamma(A) = n \) if there is an odd map \( P \in C(A, \mathbb{R}^N \setminus \{0\}) \) and \( n \) is the smallest integer with this property.

**Lemma 7 ([45])**. Let \( \varphi \) be an even \( C^1 \) functional on \( E \) and satisfy the (PS) condition. For any \( n \in \mathbb{N} \), set

\[
\Xi_n = \{ A \in \Xi | \gamma(A) \geq n \},
\]

\[
c_n = \inf_{A \in \Xi_n} \sup_{u \in A} \varphi(u).
\]

(i) If \( \Xi_n \neq 0 \) and \( c_n \in \mathbb{R} \), then \( c_n \) is a critical value of \( \varphi \).

(ii) If there exists \( l \in \mathbb{N} \) such that \( c_n = c_{n+1} = \cdots = c_{n+l} = c \in \mathbb{R} \) and \( c \neq \varphi(0) \), then \( \gamma(K_c) \geq l + 1 \).

**Remark 8 ([45])**. In view of Remark 7.3 in [45], one sees that if \( K_c \in \Xi \) and \( \gamma(K_c) \) contains infinitely many distinct points. In other words, \( \varphi \) has infinitely many distinct critical points in \( E \).

There have been many results using critical point theory to study boundary value problems of fractional differential equations [46–52] and dynamic equations on time scales [53–57], but results using critical point theory to study boundary value problems of fractional dynamic equations on time scales are still rare [6]. This section will explain that critical point theory is an effective way to deal with the existence of solutions of (26) on time scales.

In this section, we let \( N = 1 \). For the purpose of the presence and proof of our main results, let us first define the functional \( \varphi : W_{\Delta,a^+}^{\alpha,p} \rightarrow \mathbb{R} \) by

\[
\varphi(u) = \frac{1}{q^p} \left( \beta + q \int_{\rho_0} |T_{\Delta}^\alpha u(t)|^p \Delta t \right)^p - \int_{\rho_0} \lambda(t)G(t, u(t)) \Delta t - \frac{p^p}{q^p} \rho \Delta t.
\]

(27)

It is easy to note from (19), condition (G1) (will be stated in Theorem 29) and \( g \in C(J \times \mathbb{R}, \mathbb{R}) \), that the functional \( \varphi \) is well defined on \( W_{\Delta,a^+}^{\alpha,p} \), and \( \varphi \in C(W_{\Delta,a^+}^{\alpha,p}, \mathbb{R}) \). Moreover, for \( \forall u, v \in W_{\Delta,a^+}^{\alpha,p} \), one obtains

\[
\langle \varphi'(u), v \rangle = (\beta + q \|u\|^p)^{p-1} \int_{\rho_0} \phi_{\Delta}^{\alpha e} |T_{\Delta}^\alpha u(t)|^p T_{\Delta}^\alpha v(t) \Delta t - \int_{\rho_0} \lambda(t) \nabla G(t, u(t))v(t) \Delta t,
\]

(28)
which yields
\[
(q'(u), u) = (\beta + e\|u\|^p)^{p-1}\|u\|^p - \int_{\mathcal{P}} \lambda(t) \nabla G(t, u(t)) u(t) \Delta t.
\]
(29)

Now, it is time for us to present and prove our main results as follows:

**Theorem 29.** Let \( \alpha \in \left( \frac{1}{p}, 1 \right) \), and suppose that \( G \) satisfies the following conditions:

\( G(t, x) \) is \( \Delta^- \) measurable and continuously differentiable in \( x \) for \( t \in J \) and there exist \( a \in C(\mathbb{R}^+, \mathbb{R}^+) \), \( b \in L^1_{\Delta^-}(J, \mathbb{R}^+) \) such that
\[
|G(t, x)| \leq a(|x|)b(t), \quad |\nabla G(t, x)| \leq a(|x|)b(t)
\]
for all \( x \in \mathbb{R} \) and \( t \in J \).

\( G_2 \) There are two constants \( \mu > p^2 \), \( M > 0 \) such that
\[
0 < \mu G(t, x) \leq x \nabla G(t, x), \quad \forall t \in J \text{ and } |x| \geq M.
\]

(\( G_3 \)) \( \nabla G(t, x) = o(|x|^{p-1}) \) as \( |x| \to 0 \) uniformly for \( t \in J \).

Then, \( KFBVP_T (26) \) has at least one nontrivial weak solution.

**Proof.** It is clear that \( \varphi(0) = 0 \), \( \varphi \in C^1(W_{\Delta_a}^{\alpha, p}, \mathbb{R}) \), where \( W_{\Delta_a}^{\alpha, p} \) is a real Banach space from Theorem 25. Therefore, we are now in a position to prove, using Mountain pass theorem (Lemma 5), that

step 1. \( \varphi \) satisfies the \( (PS) \) condition in \( W_{\Delta_a}^{\alpha, p} \). The argument is as follows: Let \( \{u_k\} \subset W_{\Delta_a}^{\alpha, p} \) be a sequence such that
\[
|\varphi(u_k)| \leq K, \quad \varphi'(u_k) \to 0 \quad \text{as} \quad k \to \infty,
\]
for some \( K > 0 \) is a constant. We first prove that \( \{u_k\} \) is bounded in \( W_{\Delta_a}^{\alpha, p} \). From the continuity of \( \mu G(t, x) - xg(t, x) \), we obtain that there is a constant \( c > 0 \) such that
\[
G(t, x) \leq \frac{1}{\mu} x \nabla G(t, x) + c, \quad \forall t \in J, \quad |x| \leq M
\]
Combining with \((G_2)\), we obtain that
\[
G(t, x) \leq \frac{1}{\mu} x \nabla G(t, x) + c, \quad \forall (t, x) \in J \times \mathbb{R}.
\]
(32)

Hence, taking account of (27), (29)–(32), and Lemma 2, we have
\[
K \geq \varphi(u_k)
= \frac{1}{e^{p^2}} \left( \beta + e \int_{\mathcal{P}} \|Du_k(t)\|^{p} \Delta t \right)^{p} - \int_{\mathcal{P}} \lambda(t) G(t, u_k(t)) \Delta t - \frac{\beta^p}{e^{p^2}}
= \frac{1}{e^{p^2}} \left( \beta + e \|u_k\|^{p} \right)^{p} - \int_{\mathcal{P}} \lambda(t) G(t, u_k(t)) \Delta t - \frac{\beta^p}{e^{p^2}}
\geq \frac{1}{e^{p^2}} \left( \beta + e \|u_k\|^{p} \right)^{p} - \int_{\mathcal{P}} \left[ \frac{1}{\mu} u_k(t) \nabla G(t, u_k(t)) + c \right] \Delta t - \frac{\beta^p}{e^{p^2}}
= \frac{1}{e^{p^2}} \left( \beta + e \|u_k\|^{p} \right)^{p} + \frac{1}{\mu} (\varphi'(u_k), u_k) - \frac{1}{\mu} (\beta + e \|u_k\|^{p} \varphi' - c(b - a) - \frac{\beta^p}{e^{p^2}}
\geq (\beta + e \|u_k\|^{p} \varphi' - c(b - a) - \frac{\beta^p}{e^{p^2}}
- cb - \frac{\beta^p}{e^{p^2}}.
\]
(33)
which together with $\varphi'(u_k) \to 0$ as $k \to \infty$ yields

$$K \geq (\beta + \varepsilon \|u_k\|^p)^{p-1} \left[ \left( \frac{1}{p^2} - \frac{1}{\mu} \right) \|u_k\|^p + \frac{\beta}{p^2} \right] - \|u_k\| - cb - \frac{\beta p}{p^2}. \quad (34)$$

Then, combining with $\mu > p^2$ and proof by contradiction, we know that \{u_k\} is bounded in $W_{h,\Delta}^{\alpha,p}$.

Because $W_{h,\Delta}^{\alpha,p}$ is a reflexive Banach space (Theorems 25 and 26), going if necessary to a subsequence, we can assume $u_k \rightharpoonup u$ in $W_{h,\Delta}^{\alpha,p}$. As a result, in view of $\varphi'(u_k) \to 0$ as $k \to \infty$ and the definition of weak convergence, one sees

$$\langle \varphi'(u_k) - \varphi'(u), u_k - u \rangle = \langle \varphi'(u_k), u_k - u \rangle - \langle \varphi'(u), u_k - u \rangle$$

$$\leq \| \varphi'(u_k) \|_{W_{h,\Delta}^{\alpha,p}} \| u_k - u \| - \langle \varphi'(u), u_k - u \rangle$$

$$\leq \| \varphi'(u_k) \|_{W_{h,\Delta}^{\alpha,p}} (\| u_k \| + \| u \|) - \langle \varphi'(u), u_k - u \rangle$$

$$\to 0, \quad \text{as} \quad k \to \infty. \quad (35)$$

Furthermore, it follows from (20), (30), and Remark 7 that \{u_k\} is bounded in $C(J,\mathbb{R})$ and $\| u_k - u \|_{\infty} \to 0$, as $k \to \infty$. Therefore, there is a constant $c_1 > 0$ such that

$$|\nabla G(t, u_k(t)) - \nabla G(t, u(t))| \leq c_1, \quad \forall t \in J, \quad (36)$$

which yields

$$\left| \int_{\rho} (\nabla G(t, u_k(t)) - \nabla G(t, u(t))) (u_k(t) - u(t)) \Delta t \right|$$

$$\leq c_1 b \| u_k - u \|_{\infty}$$

$$\to 0, \quad \text{as} \quad k \to \infty. \quad (37)$$

Furthermore, it follows from the boundedness of \{u_k\} in $W_{h,\Delta}^{\alpha,p}$ that

$$\left[ (\beta + \varepsilon \|u_k\|^p)^{p-1} - (\beta + \varepsilon \|u\|^p)^{p-1} \right] \int_{\rho} \varphi(p_{\alpha} (\frac{T}{\alpha} D_{\alpha}^p u(t))) (\frac{T}{\alpha} D_{\alpha}^p u_k(t) - \frac{T}{\alpha} D_{\alpha}^p u(t)) \Delta t$$

$$= \left[ (\beta + \varepsilon \|u\|^p)^{p-1} - (\beta + \varepsilon \|u\|^p)^{p-1} \right] \left( \frac{1}{p} \int_{\rho} \| \frac{T}{\alpha} D_{\alpha}^p u(t) \|_{p} \Delta t, u_k - u \right)$$

$$\to 0, \quad \text{as} \quad k \to \infty. \quad (38)$$

In consideration of (28), one obtains

$$\langle \varphi'(u_k) - \varphi'(u), u_k - u \rangle + \int_{\rho} \lambda(t) (\nabla G(t, u_k(t)) - \nabla G(t, u(t))) (u_k(t) - u(t)) \Delta t$$

$$= (\beta + \varepsilon \|u_k\|^p)^{p-1} \int_{\rho} \varphi(p_{\alpha} (\frac{T}{\alpha} D_{\alpha}^p u_k(t))) (\frac{T}{\alpha} D_{\alpha}^p u_k(t) - \frac{T}{\alpha} D_{\alpha}^p u(t)) \Delta t$$

$$- (\beta + \varepsilon \|u\|^p)^{p-1} \int_{\rho} \varphi(p_{\alpha} (\frac{T}{\alpha} D_{\alpha}^p u(t))) (\frac{T}{\alpha} D_{\alpha}^p u_k(t) - \frac{T}{\alpha} D_{\alpha}^p u(t)) \Delta t$$

$$= (\beta + \varepsilon \|u_k\|^p)^{p-1} \int_{\rho} (\varphi(p_{\alpha} (\frac{T}{\alpha} D_{\alpha}^p u_k(t))) - \varphi(p_{\alpha} (\frac{T}{\alpha} D_{\alpha}^p u(t)))) (\frac{T}{\alpha} D_{\alpha}^p u_k(t) - \frac{T}{\alpha} D_{\alpha}^p u(t)) \Delta t$$

$$+ [ (\beta + \varepsilon \|u_k\|^p)^{p-1} - (\beta + \varepsilon \|u\|^p)^{p-1} ]$$

$$\times \int_{\rho} (\varphi(p_{\alpha} (\frac{T}{\alpha} D_{\alpha}^p u(t)))) (\frac{T}{\alpha} D_{\alpha}^p u_k(t) - (\frac{T}{\alpha} D_{\alpha}^p u(t)) \Delta t, \quad (39)$$
which together with (35)–(39) yields
\[
\int_0^1 (\phi_p(\frac{7}{a} D_1^t u_k(t)) - \phi_p(\frac{7}{a} D_1^t u(t))) (\frac{3}{a} D_1^t u_k(t) - \frac{3}{a} D_1^t u(t))\Delta t \to 0, \quad \text{as } k \to \infty. \tag{40}
\]

Considering (2.10) in [58], we can find two positive constants \(c_2, c_3\) such that
\[
\int_0^1 (\phi_p(\frac{7}{a} D_1^t u_k(t)) - \phi_p(\frac{7}{a} D_1^t u(t))) (\frac{3}{a} D_1^t u_k(t) - \frac{3}{a} D_1^t u(t))\Delta t \\
\ge \begin{cases} 
  c_2 \int_0^1 \|\frac{7}{a} D_1^t u_k(t) - \frac{3}{a} D_1^t u(t)\|^p \Delta t, & p \ge 2, \\
  c_3 \int_0^1 \left(\frac{\|\frac{7}{a} D_1^t u_k(t) - \frac{3}{a} D_1^t u(t)\|}{\|\frac{7}{a} D_1^t u_k(t) - \frac{3}{a} D_1^t u(t)\|^p}\right)^{\frac{p-2}{2}} \Delta t, & 1 < p < 2.
\end{cases} \tag{41}
\]

When \(1 < p < 2\), with an eye to Proposition 2 and \((\|x\| + |y|)^p \le 2^{p-1}(\|x\|^p + |y|^p) \quad (\forall x, y \in \mathbb{R}, \ p > 0)\), one obtains
\[
\int_0^1 \left(\frac{7}{a} D_1^t u_k(t) - \frac{3}{a} D_1^t u(t)\right)^p \Delta t \\
= \int_0^1 \left(\frac{7}{a} D_1^t u_k(t) - \frac{3}{a} D_1^t u(t)^2 \right)^p \left(\|\frac{7}{a} D_1^t u_k(t) + \|\frac{3}{a} D_1^t u(t)\|^p\right)^{\frac{p-2}{2}} \Delta t \\
\le \left\{ \int_0^1 \left(\frac{7}{a} D_1^t u_k(t) - \frac{3}{a} D_1^t u(t)^2 \right)^p \left(\|\frac{7}{a} D_1^t u_k(t) + \|\frac{3}{a} D_1^t u(t)\|^p\right)^{\frac{p-2}{2}} \Delta t \right\}^{\frac{2}{p}} \\
\times \left\{ \int_0^1 \|\frac{7}{a} D_1^t u_k(t)^2 - \|\frac{3}{a} D_1^t u(t)^2\|^p\right\}^{\frac{2}{p-2}} \Delta t \\
= \left[ \int_0^1 \left(\frac{7}{a} D_1^t u_k(t) - \frac{3}{a} D_1^t u(t)^2 \right)^2 \right]^{\frac{2}{p}} \\
\times \left[ \int_0^1 \left(\|\frac{7}{a} D_1^t u_k(t) + \|\frac{3}{a} D_1^t u(t)\|^p\right)^2 \right]^{\frac{2}{2-p}} \\
\le \left[ \int_0^1 \left(\frac{7}{a} D_1^t u_k(t) - \frac{3}{a} D_1^t u(t)^2 \right)^2 \right]^{\frac{2}{p}} \\
\times \left[ \int_0^1 2^{p-1}(\|\frac{7}{a} D_1^t u_k(t) + \|\frac{3}{a} D_1^t u(t)\|^p) \Delta t \right]^{\frac{2}{2-p}} \\
= \left[ \int_0^1 \left(\frac{7}{a} D_1^t u_k(t) - \frac{3}{a} D_1^t u(t)^2 \right)^2 \right]^{\frac{2}{p}} \\
\times \left[ \int_0^1 2^{p-1}(\|\frac{7}{a} D_1^t u_k(t) + \|\frac{3}{a} D_1^t u(t)\|^p) \Delta t \right]^{\frac{2}{2-p}}. \tag{42}
\]

Therefore, we have
\[
\int_0^1 \left(\frac{7}{a} D_1^t u_k(t) - \frac{3}{a} D_1^t u(t)^2 \right)^2 \left(\|\frac{7}{a} D_1^t u_k(t) + \|\frac{3}{a} D_1^t u(t)\|^p\right)^{\frac{p-2}{2}} \Delta t \\
\ge \frac{1}{\left[ 2^{p-1}(\|\frac{7}{a} D_1^t u_k(t) + \|\frac{3}{a} D_1^t u(t)\|^p) \right]^{\frac{2}{p}}} \left(\|u_k - u\|^p\right)^{\frac{2}{2-p}} \|u_k - u\|^p \tag{43}
\]
\[
= \frac{2^{(p-1)(2-p)}}{\|u_k\|^p + \|u\|^p} \frac{2^{p-2}}{\|u_k - u\|^2}. 
\]
which together with (41) implies
\[
\int_0^\phi (\phi_\phi D_t^\alpha u_k(t) - \phi_\phi D_t^\alpha u(t))(\phi_\phi D_t^\alpha u_k(t) - \phi_\phi D_t^\alpha u(t))\Delta t \\
\geq c_3 \int_0^\phi \|D_t^\alpha u_k(t) - D_t^\alpha u(t)\|^2 \Delta t \\
\geq c_32 \frac{p-1}{p} \left( \|u_k\|_p + \|u\|_p^{p-2}\|u_k - u\|^2 \right), \quad 1 < p < 2. \tag{44}
\]
When \( p > 2 \), taking (41) into account, one obtains
\[
\int_0^\phi (\phi_\phi D_t^\alpha u_k(t) - \phi_\phi D_t^\alpha u(t))(\phi_\phi D_t^\alpha u_k(t) - \phi_\phi D_t^\alpha u(t))\Delta t \\
\geq c_2 \|u_k - u\|^p, \quad p > 2. \tag{45}
\]
As a consequence, combining with (40), (44) and (45), one sees
\[
\|u_k - u\| \to 0, \quad \text{as } k \to \infty. \tag{46}
\]
Therefore, \( \phi \) satisfies the (PS) condition in \( W^{\alpha,p}_{\Delta,a} \).

step 2. \( \phi \) satisfies the \((A_1)\) condition in Lemma 5, which can be explained by the following:

Taking (G3) into account, we can find two positive constants \( \epsilon' \in (0,1) \) and \( \delta \) such that
\[
G(t,x) \leq \frac{(1 - \epsilon')\beta^{p-1}}{\lambda^0 p^{\rho_0^p \Gamma(\alpha+1)}} |x|^p, \quad \forall \ t \in J, \text{ with } |x| \leq \delta. \tag{47}
\]
Let \( \rho = \frac{\delta}{\delta', \beta^{\frac{1}{p}}} \) and \( \delta = \frac{\epsilon' \beta^{p-1} \rho_0^p}{p} \). Hence, taking (30) into consideration, one has
\[
\|u\| \leq \frac{\beta^{p-1}}{\Gamma(\alpha)((\alpha - 1)q + 1)^\frac{p}{p}} \|u\|, \quad \forall \ u \in W^{\alpha,p}_{\Delta,a}, \text{ with } \|u\| = \rho, \tag{48}
\]
which together with (19), (30), (27), and (47) implies
\[
\phi(u) = \frac{1}{\rho^p} \left( \beta + \frac{\rho_0^p}{\Gamma(\alpha+1)} \int_0^\phi \|\phi_\phi D_t^\alpha u(t)\|^p \Delta t \right) - \frac{\beta^p}{\rho^p} \int_0^\phi \lambda(t) G(t,u(t)) \Delta t = \frac{\beta^p}{\rho^p} \\
\geq \frac{\beta^{p-1}}{p} \|u\|^p - \lambda^0 (1 - \epsilon') \beta^{p-1} \frac{\rho_0^p}{\Gamma(\alpha+1)} \int_0^\phi |u(t)|^p \Delta t \\
\geq \frac{\beta^{p-1}}{p} \|u\|^p - \lambda^0 (1 - \epsilon') \beta^{p-1} \frac{\rho_0^p}{\Gamma(\alpha+1)} \frac{\beta^p}{\rho^p} \frac{\|u\|_p^p}{\Gamma(\alpha+1)} \|\phi_\phi D_t^\alpha u\|_p^p \\
= \sigma, \quad \forall \ u \in W^{\alpha,p}_{\Delta,a}, \text{ with } \|u\| = \rho, \tag{49}
\]
which means that the \((A_1)\) condition in Lemma 5 is satisfied.

step 3. \( \phi \) satisfies the \((A_2)\) condition in Lemma 5. Here are some reasons why:
For $s \in \mathbb{R}$, $|x| \geq M$ and $t \in J$, let
\[ F(s) = G(t, sx), \quad H(s) = F'(s) - \frac{\mu}{s} F(s). \] (50)

In view of $(G_2)$, when $s \geq \frac{M}{|x|}$, one obtains
\[ H(s) = \frac{\nabla G(t, sx) sx - \mu G(t, sx)}{s} \geq 0. \]

In addition, taking the expressions of $F(\cdot)$ and $H(\cdot)$ in (50) into account, we can easily obtain the result that $F(s)$ satisfies
\[ F'(s) = H(s) + \frac{\mu}{s} F(s). \]

Therefore, when $s \geq \frac{M}{|x|}$, we have
\[ G(t, sx) = s^p \left[ G(t, x) + \int_1^s \tau^{-\mu} H(\tau) d\tau \right]. \]

So, for $|x| \geq M$ and $t \in J$, together with $(G_1)$, one obtains
\[ \left( \frac{M}{|x|} \right)^\mu G(t, x) \leq G(t, x M/|x|) \leq \max_{|x| \leq M} a(|x|) b(t), \]

which implies that
\[ G(t, x) \leq \frac{|x|^\mu}{M^p} \max_{|x| \leq M} a(|x|) b(t). \]

So, one gets
\[ G(t, x) \geq \frac{|x|^\mu}{M^p} \min_{|x| \leq M} a(|x|) b(t). \] (51)

Therefore, for any $u \in W^{\mu,p}_{\Delta a^+} \setminus \{0\}$, $\xi \in \mathbb{R}^+$, it follows from (27), (30), (51), and $\mu > p^2$ that
\[ \varphi(\xi u) = \frac{1}{\epsilon p^2} \left( \beta + \epsilon \int_0^p \|\xi u\|^p \right)^p - \int_0^p \lambda(t) G(t, \xi u(t)) \Delta t - \frac{\beta p}{\epsilon p^2} \]
\[ \leq \frac{1}{\epsilon p^2} \left( \beta + \epsilon \|\xi u\|^p \right)^p - \int_0^p \lambda(t) G(t, \xi u(t)) \Delta t - \frac{\beta p}{\epsilon p^2} \]
\[ \leq \frac{1}{\epsilon p^2} \left( \beta + \epsilon \|\xi u\|^p \right)^p - \frac{\lambda_0 \min_{|x| \leq M} a(|x|)}{M^\mu} \|\xi u\|_{L^\infty} \int_0^1 b(t) \Delta t - \frac{\beta p}{\epsilon p^2} \]
\[ \to -\infty, \quad \text{as } \xi \to \infty. \] (52)

Therefore, taking $\xi_0$ large enough and letting $\epsilon = \xi_0 u$, we have $\varphi(\epsilon) \leq 0$. As a consequence, $\varphi$ also satisfies the $(A_2)$ condition in Lemma 5.

As a result, we get a critical point $u^*$ of $\varphi$ satisfying $\varphi(u^*) \geq \sigma > 0$, and so $u^*$ is a nontrivial solution of KFBVP$_T$ (26). All in all, Theorem is proved by Step 1–Step 3. □
**Theorem 30.** Let \( a \in \left( \frac{1}{p}, 1 \right] \), and suppose that \( G \) satisfies \((G_1)\) and the following conditions:

\((G_4)\) There are a constant \( 1 < r_1 < p^2 \) and a function \( d \in L^1_\Lambda (J, \mathbb{R}^+) \) such that

\[
|\nabla G(t,x)| \leq r_1 d(t)|x|^r, \quad \forall (t,x) \in J \times \mathbb{R}.
\]

\((G_5)\) There is an open interval \( I \subset J \) and three constants \( \eta, \delta > 0, 1 < r_2 < p^2 \) such that

\[
G(t,x) \geq \eta|x|^r, \quad \forall (t,x) \in \mathbb{I}_T \times |\delta - \delta|.
\]

Then, \( KF_{b,\mathcal{T}} \) (26) has at least one nontrivial weak solution.

**Proof.** It is obvious that \( \varphi(0) = 0, \varphi \in C^1(W_{\Lambda,\alpha}^{\Lambda,p}, \mathbb{R}) \), where \( W_{\Lambda,\alpha}^{\Lambda,p} \) is a real Banach space from Theorem 25. Next, we will finish our proof with the help of Lemma 6.

1. \( \varphi \) is bounded from below in \( W_{\Lambda,\alpha}^{\Lambda,p} \), which can be explained by the following:

Taking account of \((G_4), (20)\) and \((30)\), we get

\[
\varphi(u) = \frac{1}{q_p} \left( \beta + e \int_0^1 \| D_\alpha^t u(t) \|^p dt \right)^p - \int_0^1 \lambda(t) G(t,u(t)) \Delta t - \frac{\beta_p}{q_p^2} \\
= \frac{1}{q_p} (\beta + e \| u \|^p)^p - \int_0^1 \lambda(t) G(t,u(t)) \Delta t - \frac{\beta_p}{q_p^2} \\
\geq \frac{1}{q_p} (\beta + e \| u \|^p)^p - \lambda^0 \int_0^1 d(t)|u(t)|^r \Delta t - \frac{\beta_p}{q_p^2} \\
\geq \frac{1}{q_p} (\beta + e \| u \|^p)^p - \lambda^0 \| d \|_{L^1_\Lambda} \| u \|_{\mathcal{C}}^r - \frac{\beta_p}{q_p^2} \\
\geq \frac{1}{q_p} (\beta + e \| u \|^p)^p - \frac{\lambda^0 \| d \|_{L^1_\Lambda} \Gamma^1(\alpha - \frac{1}{r} + 1)}{\Gamma^1(\alpha)(\alpha - 1)q + 2 \alpha} \| u \|_{\mathcal{C}}^r - \frac{\beta_p}{q_p^2}.
\]

(53)

Since \( 1 < r_1 < p^2 \), \( (53) \) yields \( \varphi(u) \to \infty \) as \( \| u \| \to \infty \). Consequently, \( \varphi \) is bounded from below in \( W_{\Lambda,\alpha}^{\Lambda,p} \).

2. \( \varphi \) satisfies the \((PS)\) condition in \( W_{\Lambda,\alpha}^{\Lambda,p} \). The argument is as follows:

Let \( \{ u_k \} \subset W_{\Lambda,\alpha}^{\Lambda,p} \) be a sequence such that \((31)\) holds. So, together with the proof by contradiction and \((53)\), we can easily see that \( \{ u_k \} \subset W_{\Lambda,\alpha}^{\Lambda,p} \) is bounded in \( W_{\Lambda,\alpha}^{\Lambda,p} \). The remainder of proof is similar to the proof of Step 1 in Proof of Theorem 29. We omit the details.

Consequently, combining with Lemma 6, (1) and (2) in Proof of Theorem 30, one gets \( c = \inf_{W_{\Lambda,\alpha}^{\Lambda,p}} \varphi(u) \), which is a critical value of \( \varphi \). In other words, there is a critical point \( u^* \in W_{\Lambda,\alpha}^{\Lambda,p} \) such that \( \varphi(u^*) = c \).

3. \( u^* \neq 0 \), for the following reasons:

Let \( u_0 \in (W_{\Lambda,\alpha}^{\Lambda,p} \cap W_{\Lambda,\alpha}^{\Lambda,p}) \backslash \{ 0 \} \) [5], and \( \| u_0 \|_{\mathcal{C}} = 1 \), it follows from \((27), (30), (G_5)\) and \((19)\) that

\[
\varphi(\varepsilon u_0) = \frac{1}{q_p} \left( \beta + e \int_0^1 \| D_\alpha^t (\varepsilon u_0)(t) \|^p dt \right)^p - \int_0^1 \lambda(t) G(t,\varepsilon u_0(t)) \Delta t - \frac{\beta_p}{q_p^2} \\
\leq \frac{1}{q_p} (\beta + e \| \varepsilon u_0 \|^p)^p - \int_0^1 \lambda(t) G(t,\varepsilon u_0(t)) \Delta t - \frac{\beta_p}{q_p^2} \\
\leq \frac{1}{q_p} (\beta + e \| \varepsilon u_0 \|^p)^p - \lambda_0 \varepsilon^s \int_0^1 |u_0(t)|^r \Delta t - \frac{\beta_p}{q_p^2}, \quad 0 < s \leq \delta.
\]

(54)

Because of \( 1 < r_2 < p^2 \), \( (54) \) implies \( \varphi(\varepsilon u_0) < 0 \) for \( s > 0 \) that is small enough. Therefore, \( u^* \neq 0 \).
All in all, \( u^* \in W^{\alpha,p}_G \) is a nontrivial critical point of \( \varphi \), and consequently, \( u^* \) is a nontrivial solution of the KFBVP \( (26) \). Hence, we complete the proof of Theorem 30. \( \square \)

**Theorem 31.** Let \( \alpha \in \left( \frac{1}{p}, 1 \right] \), and suppose that \( G \) satisfies (G_1), (G_4), (G_5) and the following conditions:

(\( G_6 \)) There are a constant \( 1 < r_1 < p^2 \) and a function \( d \in L^1_J(J, \mathbb{R}^+) \) such that

\[
\nabla G(t,x) = \nabla G(t,-x), \quad \forall (t,x) \in J \times \mathbb{R}.
\]

Then, the KFBVP \( (26) \) possesses infinitely many nontrivial weak solutions.

**Proof.** Lemma 7 is a powerful tool for us to clarify our conclusion.

Considering (1) and (2) in Proof of Theorem 30, we see that \( \varphi \in C^1(W^{\alpha,p}_G, \mathbb{R}) \) is bounded from below and satisfies the \((PS)\) condition. Furthermore, it follows from (27) and (G_6) that \( \varphi \) is even and \( \varphi(0) = 0 \).

Fixing \( n \in \mathbb{N} \), we take \( n \) disjoint open intervals \( I_i \) such that \( \bigcup_{i=1}^n I_i \subset J \).

Let \( u_i \in (W^{\alpha,p}_G(I_j, \mathbb{R}) \cap W^{\alpha,p}_G) \setminus \{0\} \) and \( \|u_i\| = 1 \), and

\[
W_n = \text{span}\{u_1, u_2, \ldots, u_n\},
\]

\[
D_n = \{u \in W_n ||u|| = 1\}.
\]

For \( u \in W_n \), there are \( \kappa_i \in \mathbb{R} \) such that

\[
u(t) = \sum_{i=1}^n \kappa_i u_i(t), \quad \forall t \in J.\]

Consequently, one obtains

\[
\|u\|^p = \int_0^1 \left( \sum_{i=1}^n \kappa_i u_i(t) \right)^p \Delta t
\]

\[
= \sum_{i=1}^n |\kappa_i|^p \int_0^1 (\sum_{i=1}^n \kappa_i u_i(t))^p \Delta t
\]

\[
= \sum_{i=1}^n |\kappa_i|^p \|u_i\|^p
\]

\[
= \sum_{i=1}^n |\kappa_i|^p, \quad \forall u \in W_n.
\]

In consideration of (20), (30), (27), and (G_5), for \( 0 < t \leq \frac{\delta}{\beta + \frac{\epsilon}{p^2}} \max_{\kappa_i} |\kappa_i| \) and \( u \in D_n \), we obtain

\[
\varphi(u) = \frac{1}{ep^2} \left( \beta + \epsilon \int_0^1 (\sum_{i=1}^n \kappa_i u_i(t))^p \Delta t \right) - \int_0^1 \lambda(t) G(t,u(t)) \Delta t - \frac{\beta^p}{ep^2}
\]

\[
= \frac{1}{ep^2} (\beta + \epsilon \|u\|^p)^p - \sum_{i=1}^n \int_{I_i} \lambda(t) G(t, \kappa_i u_i(t)) \Delta t - \frac{\beta^p}{ep^2}
\]

\[
\leq \frac{1}{ep^2} (\beta + \epsilon \|u_0\|^p)^p - \lambda_0 \epsilon p^2 \sum_{i=1}^n \kappa_i^2 \int_{I_i} |u_i(t)|^2 \Delta t - \frac{\beta^p}{ep^2}.
\]
Since $1 < r_2 < p^2$, together with (58), there are two positive constants $\epsilon, \sigma$ such that
\[
\varphi(\sigma u) < -\epsilon, \quad \forall u \in D_n. \tag{59}
\]
Set
\[
D_n^\sigma = \{ \sigma u | u \in D_n \}, \\
\Pi = \left\{ (\kappa_1, \kappa_2, \cdots, \kappa_n) \in \mathbb{R}^n \left| \sum_{i=1}^n |\kappa_i|^p < \sigma^p \right. \right\}. \tag{60}
\]
Hence, in view of (59), one has
\[
\varphi(u) < -\epsilon, \quad \forall u \in D_n^\sigma. \tag{61}
\]
Together with the fact of $\varphi$ is even and $\varphi(0) = 0$, we obtain
\[
D_n^\sigma \subset \varphi^{-\epsilon} \subset \mathcal{E}. \tag{62}
\]
By (57), we see that the mapping $(\kappa_1, \kappa_2, \cdots, \kappa_n) \to \sum_{i=1}^n \kappa_i u_i(t)$ from $\partial \Pi$ to $D_n^\sigma$ is odd and homeomorphic. As a result, combining with Propositions 7.5 and 7.7 in [45], one gets
\[
\gamma(\varphi^{-\epsilon}) \geq \gamma(D_n^\sigma) = n. \tag{63}
\]
Hence, $\varphi^{-\epsilon} \in \mathcal{E}_n$ and so $\mathcal{E}_n \neq 0$. Let
\[
c_n = \inf_{A \in \mathcal{E}_n} \sup_{u \in A} \varphi(u). \tag{64}
\]
It follows from the fact that $\varphi$ is bounded from below that $-\infty < c_n \leq -\epsilon < 0$. In other words, for any $n \in \mathbb{N}$, $c_n$ is a real negative number.

Consequently, considering Lemma 7, we see that $\varphi$ admits infinitely many nontrivial critical points, and so, KFBVP\(_\mathcal{E}\) (26) possesses infinitely many nontrivial weak solutions. The proof of Theorem 31 is complete. \(\Box\)

6. Conclusions

In present paper, a class of fractional Sobolev spaces on time scales is established with the help of the weak Riemann–Liouville fractional derivative on time scales, and some basic properties of them are obtained. As an application, we study a class of Kirchhoff-type fractional $p$-Laplace boundary value problems on time scales. The existence and multiplicity of nontrivial weak solutions are obtained by using the Mountain path theorem and genus properties. The methods of this paper can also be used to study the solvability of other boundary value problems on time scales. Nowadays, the notions of fractional derivative on time scales in different senses are constantly being put forward. Therefore, our future direction is to study the theory and application of fractional Sobolev spaces on time scales introduced by fractional derivatives in other senses on time scales such as the Caputo, Hadamard, and so on.

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