ORIENTATIONS OF INFINITE GRAPHS

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Abstract. Building on recent work by Thomassen, we show that Nash-Williams’ orientation theorem, that every finite $2k$-edge-connected multigraph has a $k$-arc-connected orientation, also holds for all infinite multigraphs.

1. Introduction

A directed multigraph is $k$-arc-connected if from any vertex $v$ to any other vertex $w$ of the graph there exist $k$ arc-disjoint forwards directed paths. Clearly, the underlying undirected graph of a $k$-arc-connected multigraph must be $2k$-edge-connected. The classic orientation theorem of Nash-Williams from 1960 asserts that for finite multigraphs, also the following converse is true.

Theorem 1.1 (Nash-Williams’ orientation theorem [8]). Every finite $2k$-edge-connected multigraph has a $k$-arc-connected orientation.

In the same paper, Nash-Williams claimed that his result also holds for infinite graphs – but the promised proof was never published and the claim was not repeated in [9]. Despite significant effort, it has remained open ever since whether the orientation theorem holds for infinite graphs as well. The purpose of this paper is to establish that it does.

So far, for arbitrary infinite graphs, only the case $k = 1$ was known, proved by Egyed by a Zorn’s lemma argument already in 1941 [4].

To appreciate the difficulty of the general case, note that a priori it is not even clear whether any sufficiently large edge-connectivity implies the existence of a $k$-arc-connected orientation. This is different for finite multigraphs, where a simple argument shows that every $4k$-edge-connected multigraph has a $k$-arc-connected orientation: By the Nash-Williams/Tutte tree packing theorem [3, Corollary 2.4.2], any such graph has $2k$ edge-disjoint spanning trees, so after fixing a common root, we may simply orient half of the trees away from and the other half towards the root. This approach, however, is blocked for infinite graphs: there exist locally finite graphs of arbitrarily large finite (edge-)connectivity that do not even posses three edge-disjoint spanning trees [1].

Motivated by the above considerations, Thomassen has asked in 1985 whether there is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that any $f(k)$-edge-connected multigraph has a $k$-arc-connected orientation [11]. This conjecture has been featured again in [2, Conjecture 8], where also a topological variation

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of the problem was suggested by allowing directed topological arcs in the space consisting of $G$ together with all its ends; this topological version has been recently solved by Jannasch [6].

More than 50 years after Nash-Williams finite orientation theorem and about 30 years after posing his own conjecture, Thomassen achieved a marvellous breakthrough towards the orientation theorem by proving that every finite $8k$-edge-connected multigraph has a $k$-arc-connected orientation [12], giving $f(k) \leq 8k$. In the present note, we show how to improve Thomassen's argument in order to get the best possible bounds, thereby establishing Nash-Williams’ orientation theorem for all infinite graphs in its optimal form.

We remark that our proof employs successively more refined variants of Mader’s lifting theorem from 1978 [7], 1992 [5] and 2016 [10], results that were certainly not available to Nash-Williams in 1960.

2. Immersions of finite graphs of prescribed connectivity

If $G$ is a graph and $H$ is a graph with vertices $x_1, x_2, \ldots, x_n$, then an immersion of $H$ in $G$ is a subgraph consisting of $n$ distinguished vertices $y_1, y_2, \ldots, y_n$ and a collection of pairwise edge-disjoint paths in $G$ such that for each edge $x_i x_j$ in $H$ there is a corresponding path in the collection from $y_i$ to $y_j$. This immersion is said to be on $\{y_1, \ldots, y_n\}$. This subgraph of $G$ is a strong immersion of $H$ if additionally, any such path joining $y_i, y_j$ avoids all other vertices $y_k$.

Thomassen proved in [12, Theorem 4] that for any finite set of vertices $A_0$ in a $4k$-edge-connected locally finite graph $G$, there is an immersion in $G$ of a finite Eulerian $2k$-edge-connected graph on $A_0$. The main result of this section shows that by foregoing the requirement of being Eulerian, we can find an immersion that reflects the original connectivity.

**Theorem 2.1.** Let $k \geq 2$ be a natural number, let $G$ be a $2k$-edge-connected locally finite graph, and let $A_0$ be a finite set of vertices in $G$. Then $G$ contains a strong immersion of a finite $2k$-edge-connected multigraph on $A_0$.

2.1. Mader’s lifting theorem and the lifting graph. Our proof of Theorem 2.1 needs two main ingredients. The first is Mader’s lifting theorem.

Lifting two distinct edges $vx, vy$ incident with a common vertex $v$ in a graph $G$ means deleting them and adding a new edge $xy$ to $G$ (possibly parallel to existing edges between $x$ and $y$).

Suppose $G = (V + v, E)$ is a finite multigraph such that any two vertices in $V$ are joined by $k$ pairwise edge-disjoint paths in $G$. A pair of edges $vx, vy$ is called admissible for edge-connectivity $k$, or simply admissible if the connectivity constant $k$ is understood from context, if after lifting $vx, vy$ we obtain a graph $G'$ in which still any two vertices in $V$ are joined by $k$ pairwise edge-disjoint paths in $G'$. We use Mader’s Lifting theorem in the following version of Frank [5].
Theorem 2.2 (Mader, Frank). Suppose that $G = (V + v, E)$ is a finite connected multigraph such that any two vertices in $V$ are joined by $k$ pairwise edge-disjoint paths in $G$. If $v$ is not incident with a bridge and $d(v) \neq 3$, there are $\left\lfloor \frac{d(v)}{2} \right\rfloor$ pairwise disjoint admissible pairs of edges incident to $v$.

The lifting graph $L(G, v, k)$ is the graph whose vertices are the edges incident with $v$, and two vertices $e_i, e_j$ are adjacent if $(e_i, e_j)$ is an admissible pair for edge-connectivity $k$. From this perspective, Theorem 2.2 implies that under the above assumptions on $G$, if $d(v)$ is even, then $L(G, v, k)$ has a perfect matching. We will use the following result due to Ok, Richter and Thomassen describing the structure of the lifting graph [10].

Theorem 2.3 (Ok, Richer and Thomassen). Let $k \geq 2$ be even, and $G = (V + v, E)$ be a finite connected multigraph such that any two vertices in $V$ are joined by $k$ pairwise edge-disjoint paths in $G$. If $v$ is not incident with a bridge and $d(v)$ is even, then $L(G, v, k)$ is a connected, complete multipartite graph (in particular, it has a disconnected complement).

2.2. Boundary-linked decompositions. Our second ingredient for Theorem 2.1 is the following decomposition result due to Thomassen, developed for his proof that $8k$-edge connectivity yields a $k$-arc-connected orientation.

Recall that an end of a graph $G$ is an equivalence class of rays, where two rays of $G$ are equivalent if there are infinitely many vertex-disjoint paths between them in $G$. The boundary of a set of vertices $A$ is the collection of edges in $G$ with one endvertex in $A$ and the other outside of $A$. Now let $G = (V, E)$ be a locally finite connected graph. A set of vertices $A \subset V$ is called boundary-linked if the induced subgraph $G[A]$ together with its boundary has a collection of pairwise edge-disjoint equivalent rays $R_1, R_2, \ldots$ such that each edge in the boundary is the first edge of one of the rays $R_i$.

Theorem 2.4 (Thomassen [12, Theorem 1]). Let $G$ be a locally finite connected graph, and let $A_0$ be a finite set of vertices in $G$. Then $V(G) \setminus A_0$ can be partitioned into finitely many sets each of which is either a singleton or a boundary-linked vertex set with finite boundary in $G$.

2.3. Proof of the immersion theorem. We are now ready to prove our main immersion Theorem 2.1, extending the proof of [12, Theorem 4].

Proof of Theorem 2.1. In order to construct the strong immersion for our given finite set of vertices $A_0$, we apply Theorem 2.4. Let $G'$ be the finite graph obtained by contracting each of the boundary-linked sets into a single vertex (keeping all parallel edges that arise). Then $G'$ is $2k$-edge-connected; hence, every vertex has degree strictly bigger than 3, and $G'$ has no bridge. We shall modify $G'$ into an immersion using liftings of suitable edges.

For this, consider a vertex $v \in G' - A_0$. If $v$ is a singleton in the decomposition, then we use Theorem 2.2 to lift all but at most one edge incident with $v$. By that point, $v$ has degree at
most 1. So after deleting \( v \), the resulting graph will continue to be \( 2k \)-edge-connected. Proceed in the same way with the remaining vertices \( v \in G' - A_0 \) that are singletons in the decomposition.

Next, we consider a vertex \( v \) in \( G' \) which in the decomposition corresponds to a boundary-linked set \( A \). Let us first argue that we may assume that \( v \) has even degree. Indeed, otherwise, use Theorem 2.2 to lift all but one edge \( f \) incident with \( v \), and then delete \( v \). As before, the resulting graph will be \( 2k \)-edge-connected. Now reverse the liftings – in the resulting graph \( G' - f \), the vertex \( v \) will have even degree, and any two vertices different from \( v \) are joined by \( 2k \) pairwise edge-disjoint paths in \( G' \).

So now that we may assume that \( v \) has even degree, we may proceed as in the proof of [12, Theorem 4] and carefully lift the edges incident with \( v \) by focusing on lifting graph \( L(G', v, 2k) \). The vertices of this graph \( L(G', v, 2k) \) are the edges \( e_1, e_2, ..., e_{2q} \) incident with \( v \). We now define another graph \( M \) on this vertex set. We consider edge-disjoint rays \( R_1, R_2, ..., R_{2q} \) in \( G[A] \) starting with the edges \( e_1, e_2, ..., e_{2q} \) in the boundary of \( A \), and add an edge between two vertices \( e_i, e_j \) in \( M \) if \( G[A] \) has a collection of infinitely many pairwise disjoint paths joining \( R_i, R_j \) having only their endpoints in common with \( R_1 \cup R_2 \cup ... \cup R_{2q} \). Since these rays belong to the same end of \( G \), it follows that \( M \) is connected. Since \( d(v) \) and our edge-connectivity \( 2k \) are both even, we know by Theorem 2.3 that \( L(G', v, 2k) \) has a disconnected complement, and so the graphs \( L(G', v, 2k) \) and \( M \) have a common edge joining \( e_i, e_j \), say. Let \( P' \) be a path in \( G[A] \) joining \( R_i, R_j \) with only its endvertices in common with \( R_1 \cup R_2 \cup ... \cup R_{2q} \). Let \( P_{i,j} \) be a path in \( R_i \cup R_j \cup P' \) starting and terminating with \( e_i, e_j \). Now delete the edges of \( P_{i,j} \) from \( G[A] \), lift \( e_i, e_j \) in \( G' \) and define a new graph \( M \) (where we disregard \( R_i, R_j \)) and a new lifting graph (without the edges \( e_i, e_j \)). The new \( M \) and the new lifting graph have a common edge, and we repeat the above argument to find a new path in \( G[A] \) and lift the corresponding edges in the new \( G' \).

After we have done this for every vertex \( v \) in \( G' \) not in \( A_0 \), we obtain a \( 2k \)-edge-connected graph \( H \) with vertex set \( A_0 \). By construction, when we reverse the liftings, each edge in this graph corresponds to a path in \( G \) internally disjoint from \( A_0 \), and hence we obtain the desired strong immersion of \( H \) on \( A_0 \) in \( G \). \( \square \)

3. Extending orientations of Eulerian subgraphs

For a multigraph \( G = (V, E) \), a set of edges \( F \subset E \) and a non-trivial bipartition \((A, B)\) of \( V \), we write \( F(A, B) \) for the set of edges in \( F \) with one endvertex in \( A \) and the other in \( B \).

Suppose we are given a \( 2k \)-edge-connected graph \( G \). In order to define a \( k \)-arc-connected orientation, we need to orient the edges of any cut \( E(A, B) \) in \( G \) such that at least \( k \) edges go from \( A \) to \( B \), and at least \( k \) edges go from \( B \) to \( A \). Then clearly, if we orient some Eulerian subgraph \( H \) of \( G \) consistently (that is, along some closed Euler trail of \( H \)), then from any cut \( E(A, B) \) in \( G \), there will be an equal number of oriented edges in both directions (and possibly
further edges which are yet unoriented). Thus, on our way towards a $k$-arc-connected orientation for $G$, we have made no obvious mistake yet.

And indeed, as our last ingredient, we note that Nash-Williams’ orientation theorem also holds in the following, slightly stronger form, improving the bounds from [12, Theorem 6]:

**Theorem 3.1.** Let $G$ be a finite $2k$-edge-connected multigraph and $H \subseteq G$ an Eulerian subgraph. Then any consistent orientation $\vec{H}$ of $H$ can be extended to a $k$-arc-connected orientation of $G$.

**Proof.** An odd vertex pairing of a finite multigraph $G = (V, E)$ is a partition $P$ of the vertices of odd-degree in $G$ into sets of size two. Interpreting $P$ as edges, we obtain an Eulerian multigraph $G' = (V, E')$ where $E' = E \cup P$. Then $H \subseteq G \subseteq G'$. Nash-Williams showed in [8, Theorem 2] that every $2k$-edge connected multigraph $G = (V, E)$ has an odd-vertex pairing $P$ such that

$$(*) \quad |E(A, B)| - |P(A, B)| \geq 2k$$

for all non-trivial partitions $(A, B)$ of $V$.

We claim that with such an odd-vertex pairing, any consistent orientation $\vec{G}'$ of the Eulerian graph $G'$ that extends $\vec{H}$ restricts to a $k$-arc connected orientation $\vec{G}$ of $G$ as desired.

To this end, consider any non-trivial bipartition $(A, B)$ of $V$. By Menger’s theorem, we need to show that $\vec{E}'(A, B)$, the edges in $\vec{G}'$ from $A$ to $B$, and $\vec{E}(B, A)$, the edges in $\vec{G}$ from $B$ to $A$, both have cardinality at least $k$. For this, recall that any Eulerian orientation is balanced, i.e.

$$|\vec{E}'(A, B)| = \frac{|E'(A, B)|}{2},$$

in other words, that

$$|\vec{E}(A, B)| + |\vec{P}(A, B)| = \frac{|E(A, B)| + |P(A, B)|}{2}.$$

However, since $|\vec{P}(A, B)| \leq |P(A, B)|$, it follows readily that

$$|\vec{E}(A, B)| \geq \frac{|E(A, B)| + |P(A, B)|}{2} - |P(A, B)| = \frac{|E(A, B)| - |P(A, B)|}{2} \geq k. \quad (\star)$$

4. Nash-Williams’ orientation theorem for infinite graphs

We are now ready to extend Nash-Williams’ orientation theorem to all infinite multigraphs. As mentioned in the introduction, our method of proof adapts Thomassen’s [12, Theorem 7].

**Theorem 4.1.** Every $2k$-edge-connected multigraph has a $k$-arc-connected orientation.

**Proof.** By Theorem 1.1, only the infinite case is open. Next, Thomassen has shown that every infinite $2k$-edge-connected graph has a decomposition into locally finite, $2k$-edge-connected subgraphs [12, §7 & §8]; hence, it suffices to prove the assertion for locally finite multigraphs. Further, by Egvedt’s result [4], we may assume that $k \geq 2$ (so we may apply Theorem 2.1 to $G$ when needed).
Enumerate $V = \{v_0, v_1, \ldots\}$. Beginning with $A_0 = \{v_0\}$ and any directed cycle $\vec{W}_0 \subseteq G$ containing $v_0$, we will construct a sequence of finite subgraphs $W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots$ of $G$ with compatible orientations $\vec{W}_0 \subseteq \vec{W}_1 \subseteq \vec{W}_2 \subseteq \cdots$ and sets of vertices $A_0 \subseteq A_1 \subseteq A_2 \cdots$ such that for all $n \geq 0$:

(i) $\{v_0, \ldots, v_n\} \subseteq A_n \subseteq V(W_n)$.
(ii) Every vertex in $V(W_n) \setminus A_n$ has in-degree equalling out-degree in $\vec{W}_n$.
(iii) For every two distinct vertices $x, y$ in $A_n$, there are $k$ arc-disjoint directed paths in $\vec{W}_n$ from $x$ to $y$ and from $y$ to $x$.

Once the construction is complete, we claim that properties (i) and (iii) imply that any orientation $\vec{G}$ of $G$ extending $\vec{W} := \bigcup_{i \in \mathbb{N}} \vec{W}_i$ is $k$-arc-connected. Indeed, for every two distinct vertices $x, y$ in $G$, by (i) there is an $i \in \mathbb{N}$ with $x, y \in A_i$, and so by (iii) there are $k$ arc-disjoint directed paths in $\vec{W}_i$ from $x$ to $y$ and from $y$ to $x$. Since $\vec{W}_i \subseteq \vec{W}$ as oriented subgraphs, these directed paths are directed also in $\vec{W}$, and hence in $\vec{G}$, as desired.

Thus, it remains to describe the inductive construction, and this is where property (ii) is needed. So suppose inductively that we have already constructed $A_n$ and $\vec{W}_n$ according to (i)–(iii). Define $A_{n+1} := V(W_n) \cup \{v_{n+1}\}$ and apply Theorem 2.1 to obtain a strong immersion $W_{n+1}$ in $G$ of a finite $2k$-edge-connected graph $H$, both on the vertex set $A_{n+1}$. Since each of the paths in $W_{n+1}$ that corresponds to an edge of $H$ is either an edge of $W_n$ or is internally disjoint from $W_n$, we may assume that $W_n \subseteq W_{n+1}$ and $W_n \subseteq H$.

After identifying all vertices of $A_n$ in $H$ to obtain a multigraph $\vec{H}$, property (ii) implies that all the edges of $W_n \subseteq \vec{H}$ oriented as in $\vec{W}_n$ form a consistently oriented Eulerian subgraph of $\vec{H}$. Hence we can apply Theorem 3.1 to extend the orientation of this subgraph to all of $\vec{H}$, making this graph $k$-arc-connected. We claim that with this orientation, also $\vec{H}$ is $k$-arc-connected: Indeed, let $E(A, B)$ be any edge-cut in $H$. If $A_n$ lies completely on one side $A$ or $B$, then the cut restricts to a cut in $\vec{H}$, and since $\vec{H}$ is $k$-arc-connected, there exist at least $k$ edges oriented from $A$ to $B$, and also from $B$ to $A$. And if $A_n$ meets both $A$ and $B$, then the cut restricts to a cut of $\vec{W}_n$ separating two vertices from $A_n$, and so by (iii) there again exist at least $k$ edges oriented from $A$ to $B$, and also from $B$ to $A$ in $\vec{W}_n$, and hence in $\vec{H}$. Together, it follows from Menger’s theorem that $\vec{H}$ is indeed $k$-arc-connected.

Finally, we now lift this orientation of $\vec{H}$ to an orientation $\vec{W}_{n+1}$ of the strong immersion $W_{n+1}$ so that $\vec{W}_{n+1}$ satisfies (i)–(iii). Indeed, for each oriented edge in $\vec{H}$, we simply orient the corresponding path in the immersion $W_{n+1}$ accordingly. Then $\vec{W}_n \subseteq \vec{W}_{n+1}$ as directed graphs, and (i) holds by construction. Property (ii) holds since the edges incident with a vertex $v$ in $V(W_{n+1}) \setminus A_{n+1}$ belong to a collection of edge-disjoint, forwards oriented paths containing $v$ in their interior. Finally, property (iii) follows at once from the fact that $\vec{W}_{n+1}$ is an immersion of the $k$-arc-connected graph $\vec{H}$ on the vertex set $A_{n+1}$. \hfill \Box
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