On stability of radial collapse of cylindrical shell filled with viscous incompressible fluid

D A Fursova\(^1\) and Yu G Gubarev\(^1,2\)

\(^1\)Department for Differential Equations, Novosibirsk State University, Novosibirsk, 630090, Russian Federation
\(^2\)Laboratory for Fluid and Gas Vortex Motions, Lavrentyev Institute for Hydrodynamics, Novosibirsk, 630090, Russian Federation

E-mail: gubarev@hydro.nsc.ru

Abstract. We study nonlinear stability of radial collapse of a cylindrical shell filled with a viscous incompressible fluid homogeneous in density. We have made the following assumptions: 1) there is a vacuum inside the shell, 2) there is the layer of compressed polytropic gas outside the shell, the gas serves as a product of instant detonation and causes nonzero constant pressure on the outer surface of the shell, 3) there is a vacuum beyond the layer of gas. By the direct Lyapunov method, we state the absolute stability of radial collapse of the viscous cylindrical shell relative to finite disturbances of the same type of symmetry. Namely, we construct a Lyapunov function satisfying all conditions of the Lyapunov first theorem (stability theorem) regardless of radial collapse mode. Thus, we confirm the Trishin hypothesis and prove that cumulation of the fluid kinetic energy during the radial collapse of the cylindrical shell near its geometric axis never originates.

1. Introduction
Cumulation phenomena with concentration of the force (or energy, or other physical quantities) in a point, or along a line, or on a plane are still of interest for experts, especially in applications [1-3].

In the paper, we study nonlinear stability of radial collapse of a cylindrical shell filled with a viscous incompressible fluid homogeneous in density relative to finite disturbances of the same type of symmetry [2, 4]. On the one hand, this problem deals with cumulative processes of energy concentration along a line. On the other hand, it is related to the problem of acceleration of bodies by detonation products of explosives. The last problem, in its turn, is one of key problems in high energy density physics, mechanics of impulse processes and explosion physics. By impact, accelerated bodies cause shock waves of required intensity and form inside considered samples. The shock waves help us to study media properties and behavior of the constructions under extremal conditions, they are used in technological processes of welding and punching after detonation, in hardening and parts machining.

We restrict ourselves with cylindrical cumulative linings for explosive dynamic loading [2]. This kind of loading is produced by powerful condensed explosives causing the pressure 20-50 GPa while detonation happens. The detonation products impact the shell and accelerate it. Note that inertial forces determine the process during the deformation of a thin shell. However, we need to take into consideration effects of the material strength for thicker shells. So, we have different types of problem statements [2, 4].
We may transfer the problem to the one-dimensional case under the following conditions: the shell is relatively thin and long in the axial direction, and its deformation under the impact of detonation products of explosives is homogeneous, i.e., there are no significant local deformations.

Since inertial forces play the key role during explosive throwing of thin linings, and wave processes have little influence on the final rate of the linings, the model of incompressible fluid is widely used for the description of the motion of thin linings.

However, experiments showed that the dependencies of the shell radius on time calculated within the model of ideal incompressible fluid are in good agreement with the experimental ones only until a particular moment. One can observe a significant difference between the data obtained from the model and experiments respectively on the final stage of the radial collapse of the shell. This difference goes away if to apply the model of viscous incompressible fluid for the lining material. This model explains several characteristic physical effects stated at first by experiments like the stop of the motion of the shell when its inner surface reaches some nonzero radius, explosive evaporation of the shell because of quick transition of its kinetic energy into heat causing by viscous forces, dynamic loss of stability of the shell form [5, 6].

In this paper, we justify the effect of the stop of inner surface of the viscous cylindrical shell on some nonzero radius. For this, we construct a Lyapunov function [7] satisfying all conditions of the Lyapunov first theorem (stability theorem) [8] regardless of radial collapse mode. Thus, we prove that cumulation of the kinetic energy of a viscous incompressible fluid homogeneous in density during the radial collapse of the cylindrical shell near its geometric axis actually never originates.

2. Equation for radius of inner surface of viscous cylindrical shell and its transformation

Hereinafter, we consider a cylindrical shell with viscous incompressible fluid of constant density \( \rho_1 \) surrounded by a layer of compressed polytropic gas. There is a vacuum inside the shell and beyond the layer.

In the monograph [2], there is a detailed derivation of the equation describing how the process of radial collapse of the cylindrical shell develops on time. Below, we reproduce particular steps of the derivation.

We assume that the motion of the viscous cylindrical shell satisfies the law of conservation of mass

\[
r^2(t) - R^2(t) = C \equiv \text{const} > 0
\]

(1)

and the continuity equation

\[
r_1(t) \dot{r}_1(t) = R(t) \dot{R}(t) \equiv rv
\]

where \( r_1 \) and \( R \) denote the outer and the inner radius of the shell respectively, \( t \) denotes time,

\[
\dot{r}_1(t) \equiv \frac{dr_1}{dt}, \quad \dot{R}(t) \equiv \frac{dR}{dt}, \quad r \text{ is a radial coordinate of a particle from viscous incompressible fluid homogeneous in density, } \nu \text{ is the radial component of the velocity of the fluid particle.}
\]

Then the author of [2] writes the general form of the law of conservation of mechanical energy for a volume \( V \) of solid medium:

\[
\frac{d}{dt} \int_{V} \rho_1 \nu^2 \, dV + \int_{V} D_{ij} \sigma_{ij} \, dV = \int_{S} \nu_i t_i^{(n)} \, dS
\]

(3)

Here \( \rho_1 \) denotes the density of the medium, \( \nu_i \) are components of the medium velocity field, \( D_{ij} \) and \( \sigma_{ij} \) are components of the strain-rate and the stress tensor respectively, \( S \) denotes the surface area bounding the volume \( V \), \( t_i^{(n)} = \sigma_{ij} n_j \) are components of the stress vector acting on the elementary square \( dS \) with unit outward normal vector \( n \), \( n_j \) are components of the vector \( n \). Lower indexes \( i \) and \( j \) denote the corresponding space coordinates. The relation (3) states the connection between the change rate of total mechanical energy of the medium and the power of surface forces.
Further, the equation (3) is applied to radial collapse of a cylindrical shell with viscous incompressible fluid of constant density $\rho_1$ toward its axis under the action of the pressure $P \equiv \text{const}$ of compressed polytropic gas on the outer surface of the shell, the pressure on the inner surface is equal to zero.

Consider a viscous incompressible Newtonian fluid homogeneous in density as the material of the cylindrical shell. So, the following equation holds for the stress tensor components:

$$\sigma_{ij} = -P \delta_{ij} + 2\mu D_{ij}$$

where $\delta_{ij}$ denotes the Kronecker delta [9], $\mu \equiv \text{const}$ is a coefficient of dynamic viscosity. Moreover, the cylindrical symmetry implies that $D_{rr} = \partial v/\partial r$, $D_{\varphi \varphi} = v/r$ (here $\varphi$ denotes the azimuth coordinate), and all other $D_{ij}$ are zero. By constancy of density $\rho_1$ and the continuity equation (2), we get $D_{ii} \equiv 0$. Thus,

$$D_{ij} \sigma_{ij} = -PD_{ii} + 2\mu D_{ij}D_{ij} = 2\mu \left[ \left( \frac{\partial v}{\partial r} \right)^2 + \left( \frac{v}{r} \right)^2 \right] = 4\mu R^2 \frac{R^2}{r^4}$$  \hspace{1cm} (4)

By the relations (4) and (1), we may rewrite the equation (3) for the unit length of the viscous cylindrical shell in the axial direction in the following intermediate form:

$$\dot{E} + \frac{4\pi \mu C R^2}{\rho_1} = -2\pi RRP$$  \hspace{1cm} (5)

where $E \equiv \pi \rho_1 (R R')^2 \ln(r_i/R)$ denotes the kinetic energy, $\dot{E} \equiv dE/dt$, and $\pi$ is the classical constant equal to the ratio of the length of a circle to its diameter.

The expression (5) is an ordinary differential equation of the second order with unknown function $R(t)$ which denotes the inner radius of the cylindrical shell, see the relation (1). If $P = 0$, then we may integrate the equation (5) once [2]. By this reason, we want to transform the relation (5) to a system of two ordinary differential equations of the first order such that the integral of the expression (5) immediately follows from it under zero pressure on the outer surface of the shell. About the integral of the relation (5), this is an ordinary differential equation of the first order, see the relations (4.4.6) and (4.4.11) in [2]. Since there exist more known methods how to construct a Lyapunov function for a system of ordinary differential equations of the first order than for a single one of high order [10], it is more convenient to deal with such a system. Already constructed for the expression (4.4.11) [2] Lyapunov function (see the relation (7) from [4]) gives us an additional argument to convert the expression (5) to a system of ordinary differential equations of the first order.

More precisely, we replace the radial velocity $\dot{R}$ of the inner surface of the cylindrical shell in the relation (5) by the kinetic energy $E$ of the unit length of the shell in the axial direction due to the expression $R = R^{-1} E^{1/2} \left[ \pi \rho_1 \ln(r_i/R) \right]^{-1/2}$:

$$\dot{E} + \frac{4\pi \mu CE}{\rho_1 (r_i R)^2 \ln(r_i/R)} = -2P \left( \frac{\pi E}{\rho_1 \ln(r_i/R)} \right)^{1/2}$$  \hspace{1cm} (6)

We exclude $E$ from the relation (6) by the substitution $u \equiv \sqrt{E}$:

$$u \frac{du}{dR} = -\pi RP - \frac{2\nu Cu}{r_i^2 R} \left( \frac{\pi \rho_1}{\ln(r_i/R)} \right)^{1/2}$$  \hspace{1cm} (7)

Here $u$ is a new required function of an independent variable $R$, $v \equiv \mu / \rho_1$ is a coefficient of the kinematic viscosity.
The expression (7) is an ordinary differential equation of the first order and the particular case of the Abel equation of the second kind [11]. Now, by the change of desired function 
\[ y = u - 2\nu \sqrt{2\pi} \times \sqrt{\rho_l \ln(1 + CR^{-2})}, \]
we rewrite the relation (7) as follows
\[ y + 2\nu \left[ 2\pi \rho_l \ln\left(1 + \frac{C}{R^2}\right)^{1/2} \right] \frac{dy}{dR} = -\pi RP \quad (8) \]
By the definition of the functions \( y, u \) and the kinetic energy \( E \) of the unit length of the viscous cylindrical shell in the axial direction, we return in the expression (8) from the argument \( R \) to the initial independent variable \( t \). Thus, we get the first relation of the required system of two ordinary differential equations of the first order:

\[ y = \left( \frac{2\pi}{\rho_l} \right)^{1/2} \rho \left[ \ln\left(1 + \frac{C}{R^2}\right)^{-1/2} \right] \quad (9) \]
Hereinafter, the dot sign over the function denotes its total derivative with respect to time.

Let us obtain the second relation. For the function \( u \), we have two expressions, \( u = y + 2\nu \sqrt{2\pi} \times \sqrt{\rho_l \ln(1 + CR^{-2})} \) and \( u = \sqrt{E} = RR\sqrt{\pi \rho_l \ln(r_1 R^{-1})} \). Applying the equation (1), we get
\[ y + 2\nu \left[ 2\pi \rho_l \ln\left(1 + \frac{C}{R^2}\right)^{1/2} \right] = RR \left[ \pi \rho_l \ln\left(\frac{r_1}{R}\right)^{1/2} \right] = RR \left[ \frac{\pi \rho_l}{2} \ln\left(1 + \frac{C}{R^2}\right)^{1/2} \right] \]
Finally, we have
\[ \dot{R} = R^{-1} \left\{ y \left( \frac{2}{\pi \rho_l} \right)^{1/2} \left[ \ln\left(1 + \frac{C}{R^2}\right)^{-1/2} \right] + 4\nu \right\} \quad (10) \]
As it was planned, we have replaced the relation (5) by the desired system (9), (10) of ordinary differential equations of the first order. Indeed, if \( P = 0 \), then the expression (9) has the form \( \dot{y} = 0 \), which is integrated as \( y = C_1 \equiv \text{const} \). By the initial data, we calculate the constant \( C_1 \):
\[ C_1 = \sqrt{\pi \rho_l} \left[ R(0) \dot{R}(0) - 4\nu \left[ \ln\left(\frac{r_1(0)}{R(0)}\right) \right]^{1/2} \right] \]
Substituting the value of \( C_1 \) instead of the function \( y \) in the equation (10), we get the relation (4.4.11) from the monograph [2].

3. The Lyapunov function and absolute stability of radial collapse of viscous cylindrical shell
Below, we assume that the system (9), (10) has exact solution \( y = \psi(t), R = \chi(t) \) describing the radial collapse of the cylindrical shell, i.e., \( \chi(t) \geq 0, \dot{\chi}(t) < 0 \), and satisfying the fixed initial data \( \chi(0) = R(0), \psi(0) = y(0) \).

We want to clarify if the exact solution \( y = \psi(t), R = \chi(t) \) of the system (9), (10) is stable with respect to finite cylindrically-symmetric perturbations \( y'(t), R'(t) \).

Changing the required functions \( y = \psi + y', R = \chi + R' \) in the system (9), (10), we get the new system of ordinary differential equations for the finite perturbations \( y' \) and \( R' \):
\[ \dot{y}' = -P \left( \frac{2\pi}{\rho_l} \right)^{1/2} \left[ \ln\left(1 + \frac{C}{(\chi + R')^2}\right) \right]^{-1/2} \quad (11) \]
The relations (11) form a nonlinear non-autonomous system of two ordinary differential equations of the first order with unknown non-autonomy [12]. For such systems, the methods of constructing Lyapunov functions have not yet been developed. However, in this case, we may apply the Barbashin method [10] for constructing Lyapunov functions to study stability for the zero solution to one subclass of nonlinear autonomous systems of two ordinary differential equations of the first order.

Namely, by the substitution $x \equiv y, \ y_1 \equiv -P\sqrt{2\pi\rho_1\ln(1+CR^2)}$, we rewrite the system (9), (10) in the form

$$\dot{x} = y_1, \ \dot{y}_1 = -\frac{\rho_1y_1^3}{2\pi^2CP}\exp\left(\frac{2\pi P^2}{\rho_1y_1^2}\right) x + \frac{2\nu \rho_1 y_1^3}{\pi CP^2}\exp\left(\frac{2\pi P^2}{\rho_1y_1^2}\right) (12)$$

It is easy to check that the relations (12) satisfy the conditions for the particular class of nonlinear autonomous systems of two ordinary differential equations of the first order studied by Barbashin in his monograph [10].

Due to [10], we may consider the following function as the candidate in a role of a Lyapunov function,

$$w = \frac{x^2}{2} + \frac{\pi}{2}CP\exp\left(-\frac{2\pi P^2}{\rho_1y_1^2}\right)$$

or, in terms of the functions $y$ and $R$,

$$w = \frac{y^2}{2} + \frac{\pi}{2}PR^2 (13)$$

Differentiating the function $w$ from the expression (13) with respect to time along the solution of the system (9), (10), we get the relation

$$\dot{w} = 4\pi \nu P \equiv \text{const} > 0$$

Let us define the Lyapunov function as follows

$$w_1 \equiv \frac{1}{2}(y^2 - y_1^2) + \frac{\pi}{2}P(R^2 - x^2) = \frac{y'}{2}(y' + 2y) + \frac{\pi}{2}PR'(R' + 2x) (14)$$

Calculating the derivative of $w_1$ from the expression (14) with respect to $t$ along the solution of the system of two ordinary differential equations (11) of the first order, we have $\dot{w}_1 = 0$. So, $w_1 \equiv \text{const}$, and the function $w_1$ is the integral of motion of the system (11). Moreover, we may clarify the definition of $w_1$,

$$w_1 \equiv k^2 > 0 \ (y' \neq 0, R' \neq 0), \ \ w_1 = 0 \ (y' = 0, R' = 0) (15)$$

where $k$ is a constant.

The function $w_1$ is positively defined and satisfies all conditions of the Lyapunov first theorem (stability theorem) [7, 8, 10] regardless of the form of the exact solution $y = \psi(t), \ R = \chi(t)$ of the system (9), (10) of ordinary differential equations of the first order.

Thus, with the help of the Lyapunov function $w_1$ from the relations (14), (15), we have proved that the radial collapse $y = \psi(t), \ R = \chi(t)$ of the cylindrical shell of viscous incompressible Newtonian fluid of constant density $\rho_1$ under the outer pressure $P \equiv \text{const}$ of compressed polytropic gas is absolutely stable relative to finite radial disturbances $y'(t), \ R'(t)$ (11). It implies that an unlimited growth of radial velocity $\dot{R}$ of the inner surface of the viscous cylindrical shell with time during a
cylindrically-symmetric collapse of the shell toward its geometric axis never originates. So, we get no cumulation for the kinetic energy of the fluid fulfilled the cylindrical shell. It confirms completely the Trishin hypothesis formulated by him in the monograph [2].

Note that the system (9), (10) when \( P = 0 \) implies the equation (4.4.11) [2]. However, the Lyapunov function from [4], see the relation (7), does not follow from \( w_1 \). It is so since we have the outer pressure \( P \) in the differential operator of the equation (9).

4. Conclusion
In the present paper, we have studied the problem of nonlinear stability for radial collapse of a cylindrical shell filled with a viscous incompressible Newtonian fluid homogeneous in density relative to finite radial disturbances. We have made the following specific assumptions: 1) there is a vacuum inside the shell, 2) there is the layer of compressed polytropic gas outside the cylindrical shell, the gas serves as a product of instant detonation and causes nonzero constant pressure on the outer surface of the shell, 3) there is a vacuum beyond the layer of gas. By the direct Lyapunov method, we state the absolute stability for cylindrically-symmetric collapse of the viscous cylindrical shell relative to finite disturbances of the same type of symmetry. More precisely, we have constructed the Lyapunov function satisfying all conditions of the Lyapunov first theorem (stability theorem) regardless of radial collapse mode. So, according to the Trishin supposition [2], cumulation of the kinetic energy of viscous incompressible Newtonian fluid homogeneous in density during the cylindrically-symmetric collapse of the considered shell near its geometric axis never originates.

The fact that cumulation of kinetic energy of viscous fluid during the radial collapse of the cylindrical hollow shell never originate looks surprisingly and paradoxically from a physical point of view. Indeed, viscous forces always stop the motion of the shell upon its inner surface reaches one or the other nonzero radius regardless of the quantities of the initial kinetic energy \( E(0) \) and the outer constant pressure \( P \). Truly, Trishin explained in [2] the stop of the inner surface of the cylindrical shell not only by viscosity. By Trishin, viscous forces cause quick heating of a part of the shell substance, so this part of the shell substance evaporates in explosive way, then gas from the particles of the evaporated substance fills up the inner cavity of the cylindrical shell, produces inside the shell a counter-pressure and, as a result, causes dynamic loss of stability for both surfaces of the cylindrical shell. However, what is even more surprising and paradoxical, the relation (5) involves a coefficient of dynamical viscosity but not temperature. So, we have neither heating nor evaporation! Due to the equation (5), which is the mathematical model of the process of the radial collapse, the stop of the motion of the inner surface of the shell is caused only by viscous forces.

After all, we may conclude that cylindrically-symmetric collapse of the cylindrical shell of a viscous incompressible Newtonian fluid homogeneous in density deserves particular attention of experts, requires painstaking study in the future and has bright prospects for wide applications in science, technology and industry.

Acknowledgments
The authors express their sincere appreciation to M Godin-Boitard, who passed a scientific internship under the direction of Yu G Gubarev in June-September 2016 as a student at the National French Civil Aviation University (Toulouse, France), for his participation and assistance in the work.

References
[1] Zababakhin E I and Zababakhin I E 1988 Infinite Cumulation Phenomena (Moscow: Nauka) [in Russian]
[2] Trishin Yu A 2005 Physics of Cumulative Processes (Novosibirsk: Izd. Inst. Gidrodin. im. M A Lavrentieva, SO RAN) [in Russian]
[3] Yakushev V V, Utkin A V, Zhukov A N, Shakhrai D V and Kim V V 2016 High Temp. 54 197–205
[4] Gubarev Yu G and Sokolov N A 2012 *J. Eng. Phys. Thermophys.* **85** 313–6
[5] Matyushkin N I and Trishin Yu A 1977 *Pis’ma Zh. Tekh. Fiz.* **3** 455–8 [in Russian]
[6] Matyushkin N I and Trishin Yu A 1978 *J. Appl. Mech. Tech. Phy.* **19** 362–71
[7] Lyapunov A M 1992 *The General Problem of the Stability of Motion* (London: Taylor & Francis)
[8] Demidovich B P 1967 *Lectures on the Mathematical Theory of Stability* (Moscow: Nauka) [in Russian]
[9] Pobedrya B E 1986 *Lectures on Tensor Analysis* (Moscow: Izd. Mosk. Gos. Univ.) [in Russian]
[10] Barbashin E A 1970 *Introduction to the Theory of Stability* (Groningen: Wolters-Noordhoff Publishing)
[11] Polyanin A D and Zaitsev V F 2002 *Handbook of Exact Solutions for Ordinary Differential Equations* (Boca Raton, Fla.: Chapman & Hall/CRC)
[12] Egorov A I 2007 *Ordinary Differential Equations with Applications* (Moscow: FIZMATLIT) [in Russian]