Abstract

A monic polynomial in $\mathbf{F}_q[t]$ of degree $n$ over a finite field $\mathbf{F}_q$ of odd characteristic can be written as the sum of two irreducible monic elements in $\mathbf{F}_q[t]$ of degrees $n$ and $n-1$ if $s$ is sufficiently large. Such a representation is possible in $\mathbf{F}_q[t]$ if $q$ is larger than an explicitly given bound in terms of $n$.

1 Introduction

Which monic elements $F$ in a polynomial ring $\mathbf{F}_q[t]$ can be written as the sum of two monic irreducibles of unequal degrees not larger than the degree of $F$? In a nutshell, our result says that such a representation is possible in odd characteristic if $q$ is sufficiently large relative to the degree of the polynomial $F$.

More precisely, we prove the following results.

Theorem 1.1. Let $\mathbf{F}_q$ be a finite field of odd characteristic and cardinality $q$ and let $F$ be a monic polynomial in $\mathbf{F}_q[t]$ whose degree is at least 2.

Then for any sufficiently large integer $s$, there exist irreducible monic polynomials $F_1$ and $F_2$ in $\mathbf{F}_{q^s}[t]$ with $\deg(F_1) = \deg(F) - 1$ and $\deg(F_2) = \deg(F)$ such that

$$F = F_1 + F_2.$$  

Theorem 1.2. Let $\mathbf{F}_q$ be a finite field of odd characteristic and cardinality $q$ and let $F$ be a monic polynomial in $\mathbf{F}_q[t]$ whose degree $n$ is at least 2.

Then if $q > 8(n+6)2^{2n^2}$, there exist irreducible monic polynomials $F_1$ and $F_2$ in $\mathbf{F}_q[t]$ with $\deg(F_1) = \deg(F) - 1$ and $\deg(F_2) = \deg(F)$ such that

$$F = F_1 + F_2.$$  

The statements and a brief summary of the proofs of the results of this paper appeared in [1].

A significant part of the proof of theorem 1.1 consists of a slight adaptation of the proof of the main theorem in [2], which we quote below as theorem 2.1. We therefore assume familiarity with the proof of that result.

The proof of theorem 1.2 essentially consists of adding explicit estimates in terms of $n$ to the existence statements in the proof of theorem 1.1.

The source of inspiration for the results above is a letter sent to Leonhard Euler and dated 7th June 1742 [13], in which Christian Goldbach conjectured that every
integer greater than 5 is equal to the sum of three primes. In his answer \[12\], Euler noted that this conjecture is equivalent to the claim that every even integer greater than 2 can be written as the sum of two primes.

As for partial results towards the Goldbach conjecture, Vinogradov (\[20\] or \[8,\] chpt. 26) proved that every sufficiently large odd integer is the sum of three primes. Furthermore, Chen (\[6\] or \[18,\] chpt. 10) showed that every sufficiently large even integer is the sum of a prime and a product of at most two primes.

Goldbach’s question can also be asked in the ring \(\mathbb{F}_q[t]\) rather than in the ring of integers. Preparing for the formulation, we note that the absolute value of \(F \in \mathbb{F}_q[t]\) is defined as the integer \(|F| = q^{\deg(F)}\). Furthermore, the analogon to an even integer is a polynomial in \(\mathbb{F}_q[t]\) which is divisible by an irreducible polynomial of absolute value equal to 2. Even polynomials can therefore exist only in \(\mathbb{F}_2[t]\).

The function field version of the Goldbach conjecture can then be stated as follows:

**Conjecture 1.3** (Conjecture 1.20 in \[10\]). Let \(\mathbb{F}_q\) be a finite field of characteristic \(p\) and cardinality \(q\) and let \(F\) be a monic polynomial in \(\mathbb{F}_q[t]\) of degree \(n\) at least 2 which is even if \(q = 2\) and not of the form \(t^2 + t + a\) in characteristic 2.

Then there exist irreducible monic polynomials \(F_1\) and \(F_2\) in \(\mathbb{F}_q[t]\) with \(\deg(F_1) < \deg(F)\) and \(\deg(F_2) = \deg(F)\) such that

\[F = F_1 + F_2.\]

Car \[3\] gave an upper bound in terms of \(n\) and \(q\) for the number of polynomials \(F \in \mathbb{F}_q[t]\) of degree at most \(n\) which are not the sum of two irreducibles whose degrees are at most equal to the degree of \(F\). She also proved a function field version of Chen’s theorem \[4\] and in \[5\], she derived an asymptotic formula for the number of triples \((P_1, P_2, P_3)\) of irreducible polynomials in \(\mathbb{F}_q[t]\) of equal degree and with \(P_1 = P_2 + P_3\).

Using sieve methods, Cherly \[4\] established that in \(\mathbb{F}_q[t]\) with \(q > 2\), every polynomial \(F\) of sufficiently high degree can be expressed as the sum of two polynomials of unequal degrees not larger than the degree of \(F\) and each having at most four prime factors.

Effinger and Hayes \[9\] proved that with the exception of polynomials of the form \(t^2 + a\) in characteristic 2, every noneven monic polynomial in \(\mathbb{F}_q[t]\) of degree \(n\) at least 2 is the sum of three irreducible monic polynomials, one of degree \(n\) and the other two of degree strictly smaller than \(n\).

Hayes \[15\] gave an elementary proof that for \(q\) sufficiently large relative to the degree \(n\) of a polynomial \(F \in \mathbb{F}_q[t]\), we can write \(F\) as the sum of two irreducibles both of degree \(n + 1\). Furthermore, he proved an asymptotic formula with \(q \to \infty\) for the number of ways a given monic polynomial \(F \in \mathbb{F}_q[t]\) of degree \(n\) can be written as the sum of two irreducible monic polynomials both of degree \(n + 1\), assuming that \(F\) is squarefree or \(\text{char}(\mathbb{F}_q) \nmid n + 1\).

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**Notation.** We write \(\mathbf{F}\) for an algebraic closure of \(\mathbb{F}_q\) and let \(\mathbf{F}_{q^r}\) denote its unique subfield of cardinality \(q^r\). For \(k\) a field, \(X\) a \(k\)-scheme and \(k'\) a field extension of \(k\), the scheme \(X \times_{\text{Spec}(k)} \text{Spec}(k')\) will be denoted by \(X_{k'}\). We say that a point on a curve is of order \(n\) with some \(n \geq 2\) if the scheme-theoretic intersection of the curve and its tangent at that point is of multiplicity \(n\).
2 Proof of Theorem 1.1

The main tool is a slight variant of the following theorem.

**Theorem 2.1** (Theorem 1.1 in [2]). Let $F_q$ be a finite field of characteristic $p$ and cardinality $q$. Let $f_1, \ldots, f_n \in F_q[t, x]$ be irreducible polynomials whose total degrees $\deg(f_i)$ satisfy $p \nmid \deg(f_i)(\deg(f_i) - 1)$ for all $i$. Assume that the curves $C_i \subseteq P^2_{F_q}$ defined as the Zariski closures of the affine curves

$$f_i(x, t) = 0$$

are smooth. Then, for any sufficiently large $s \in \mathbb{N}$, there exist $a, b \in F_q^s$ such that the polynomials $f_1(at + b, t), \ldots, f_n(at + b, t) \in F_q^s[t]$ are all irreducible.

The idea is to show that for $s$ large enough, there exists an $f_1 \in F_q[x, t]$ of degree $n - 1$ such that a slight variant of theorem 2.1 can be applied to the two polynomials $f_1(x, t)$ and $f_2(x, t) = -f_1(x, t) + F(t)$. We also have to impose the condition $a = 1$ to ensure that the resulting polynomials in $F_q[t]$ are monic.

Key to the existence of such an $f_1$ is the observation that theorem 2.1 demands genericity conditions of the polynomials $f_i$. Indivisibility by $p$ is assumed to ensure separability of the Gauss maps of the $C_i$, which is a genericity condition as well.

In the proof of theorem 2.1, the intersection of the line $x = at + b$ and a curve $C_i$ is interpreted as the fibre of a projection $C_i \rightarrow P^1$ from a point $M$ in $P^2$. For the purpose of that proof, it turns out that $M$ can be chosen freely in a Zariski open subscheme of $P^2$. However, our condition $a = 1$ is equivalent to choosing $M$ as the point at infinity with $x = 1$ and $t = 1$. This choice of $M$ imposes some further necessary conditions on the two polynomials $f_i$.

Subject only to its own lower bound on $s$, the said variant of theorem 2.1 then produces an element $b_0 \in F_q^s$ such that both $f_1(t + b_0, t)$ and $f_2(t + b_0, t)$ are monic and irreducible in $F_q[t]$; in view of $F = f_1 + (-f_1 + F)$, this proves theorem 1.1.

Let $X$ denote the family of all plane algebraic curves $f_1(x, t) = 0$ over $F$ of degree at most $n - 1$ in $A^2$ and fix an embedding of $X$ by a map

$$\begin{align*}
X &\quad \longrightarrow \quad A^I \times A^2 \\
 f_1(x, t) &\quad \mapsto \quad (c) \times \{f_1(c)(x, t) = 0\},
\end{align*}$$

where $(c)$ is the coefficient vector of $f_1(c)$.

We denote by $\mathfrak{f}$ the family of affine curves in $A^2$ of degree $n - 1$ such that $f_1(t + b, t)$ is monic for any $b$. This means that for every member of $\mathfrak{f}$, the coefficients of the terms of total degree $n - 1$ sum up to one, so the coefficients $(c)$ of such a curve correspond to a point on an affine subspace $H \subset A^I$ of codimension one.

For any element $f_1(c) \in \mathfrak{f}$ with coordinates $(c) \in H_F$, we set

$$f_2(c)(x, t) = -f_1(c)(x, t) + F(t).$$

We then consider the families of curves in $H \times P^2_F$

$$\begin{align*}
C_1(c) &\quad \xrightarrow{\alpha_1} \quad H_F \\
C_2(c) &\quad \xrightarrow{\alpha_2} \quad H_F,
\end{align*}$$

where $C_i(c) = \alpha_i^{-1}(c)$ denotes the Zariski closure of the affine curve $f_i(c)(x, t) = 0$ in $P^2_F$.
We let $\beta_i$ denote the rational map $C_i \to \mathbb{P}^1$ constituted by projection from $M = (1, 1, 0)$ in homogenised coordinates $(x, t, z)$. We need to show that $\beta_1$ and $\beta_2$ are in fact morphisms and this amounts to checking that the point $M$ does not lie on $C_{1(c)}$ or $C_{2(c)}$. We assume that $(c) \in H$ and so monicity of the two polynomials $f_i(t + b, t)$ for any $b$ implies that indeed $M \notin C_{i}$. 

If $k$ is a field, a finite $k$-scheme $X$ is said to have at most one double point if $n(X) \geq r(X) - 1$, where $r(X)$ denotes the rank and $n(X)$ the geometric number of points of $X$ (this paragraph is quoted from [2]).

**Definition 2.2 ([11]).** A finite morphism $f : C \to \mathbb{P}^1_k$ is called generic if $f^{-1}(x)$ has at most one double point for all $x \in \mathbb{P}^1_k$.

With $s$ sufficiently large, we want to choose a point $(c)$ in $H_{F_q^s(\mathbb{F}_q^s)}$ such that the following conditions are satisfied:

1. Both $C_{1(c)}$ and $C_{2(c)}$ are smooth.
2. Both $C_{1(c)}$ and $C_{2(c)}$ are absolutely irreducible.
3. The Gauss maps of both $C_{1(c)}$ and $C_{2(c)}$ are separable.
4. The morphism $\beta_1$ is generic.
5. The morphism $\beta_2$ is generic over an affine subscheme $A^1 \subset \mathbb{P}^1$.
6. No line $x = t + b$ is tangent to both $C_{1(c)}$ and $C_{2(c)}$.
7. The line at infinity is not tangent to $C_1$.

For each one of these conditions, we proceed to show that the points $(c)$ which satisfy it make up a nonempty open subscheme of $H_F$. The first three conditions correspond to the assumptions of theorem [2,1] while the other four turn out to be necessary for the proof of theorem [2,1] to go through with the additional condition $a = 1$.

Condition 1 of smoothness: Since a projective curve is complete, the projection to $H_F$ of the singular loci of the two families $C_{i(c)}$ in $H_F \times \mathbb{P}^1_F$ is Zariski closed. For $p \nmid n - 1$, we set $f_1 = 2x^{n-1} - tx^{n-1} + d$ and then a short calculation shows that the corresponding $C_{1(c)}$ is smooth for any nonzero $d$ and that $C_{2(c)}$ is singular for at most $n-1$ different values of $d \in \mathbb{F}$. Likewise, for $p \mid n - 1$ we take $f_1 = 2x^{n-1} + x - t^{n-1} + t^{n-2}$ for which both the resulting $C_{1(c)}$ and $C_{2(c)}$ are smooth. Therefore the points $(c) \in H$ whose fibres $C_{i(c)}$ are smooth curves form a nonempty open subscheme $A_1 \subset H$.

Condition 2 of absolute irreducibility: Absolute irreducibility of the polynomial with coordinates $(c)$ remains unchanged by multiplication with a nonzero scalar, so we can projectivise the vector $(c)$. A standard argument from elimination theory [11, cor. 14.3] then shows that a polynomial over an algebraically closed field of fixed degree and fixed number of variables factorises nontrivially if and only if certain polynomials in its coefficients vanish. Such an argument shows that $f_{i(c)}$ is irreducible over $\mathbb{F}$ if and only if $(c)$ belongs to a Zariski open subscheme $A_2$ of $A^1$ and therefore of $H$.

In order to show that $A_2$ is nonempty, we have to find an $f_{1(c)}$ with $(c) \in H$ such that both $f_{1(c)}$ and $f_{2(c)} = -f_{1(c)} + F(t)$ are absolutely irreducible. We consider $f_1 = x^{n-1} + bt + b_0$ with $b_1$ nonzero, $b_1 \neq a_1$ and $b_0 = a_0$ where $a_0$ is the constant and $a_1$ the linear coefficient of $F(t)$. Then both $f_1$ and $f_2$ are irreducible by Eisenstein’s criterion, the respective primes being $b_1 t + b_0$ and $t$. 

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Condition 3 of separability of the Gauss maps: If \( p \nmid n(n-1)(n-2) \), then separability of the Gauss map for \( C_{1(c)} \) with \( (c) \) any element in \( A_1 \cap A_2 \) is proved in the last paragraph of the proof of proposition 3.1 in [2].

For the case \( p \mid n(n-1)(n-2) \), we note that a smooth plane curve has separable degree 1 over the dual curve via the Gauss map ([17, Cor. 4.5]) and a curve with separable Gauss map has only finitely many points of order at least 3 ([13] IV Ex. 2.3). Conversely, if such a curve has only finitely many points of order greater than 2, the Gauss map is birational and therefore separable. Therefore our assumption \( p > 2 \) simplifies the condition for an inseparable Gauss map to every tangent intersecting the curve in a point of multiplicity at least 3.

This condition is satisfied if there exists a solution to the equation

\[
f_i(at + b, t) = (ut - v)^3 h_i(t),
\]

where we projectivise the coefficients of \( f_i \) and \( h_i \). Intersecting the solutions over all \((a, b) \in A_F^2\) yields the curves \( C_i \) whose every tangent intersects it in a point of multiplicity at least 3.

Projecting these solutions to the space of coordinates of \( f_i \) and then restricting to the affine subspace \( H \), we see that the Gauss map is separable for \((c)\) in a Zariski open subscheme \( A_3 \) of \( A_1 \cap A_2 \).

In order to show that \( A_3 \) is nonempty, we let \( F(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_0 \) and set

\[
f_i(x, t) = 2x^{n-1} - x^2t^{n-3} + a_0 + a_1t + (a_2 - 1)t^2.
\]

The case \( n = 3 \) being obvious, for \( n > 3 \) the curve \( C_3 \) and the line at infinity intersect in a point of order 2 as do the curve \( C_2 \) and the line \( x = t \).

Conditions 4 – 7 on the morphisms \( \beta_1 \) and \( \beta_2 \) and tangency conditions: We denote the dual projective plane by \( \mathbb{P}^{2*} \). Let the scheme \( X \) consist of the tangents which intersect \( C_1(c) \cup C_2(c) \) in more than one double point and which we call special tangents, parametrised by the points \((c) \in A_3 \). We then have the projections

\[
\begin{align*}
X & \subset A_3 \times \mathbb{P}^{2*} \xrightarrow{\delta} \mathbb{P}^{2*} \\
& \downarrow \gamma \\
A_3 &
\end{align*}
\]

We now define \( X_1 \subset \gamma(X) \) as follows: \((c) \in X_1 \) if there exists a special tangent whose affine equation is \( x = t + b \). Note that the line at infinity intersects the curve \( C_2 \) in a point of multiplicity \( n \).

In the first paragraph of the proof of proposition 3.1 in [2], it is pointed out that the proposition shows a stronger result which implies the proposition, namely that for all \((c) \in A_3 \), there are only finitely many points in \( \delta((c) \times \mathbb{P}^{2*} \cap X) \). We can therefore choose an \( r \in F \) such that in the new coordinates \( x', t \) with \( x' = rx \), the coefficients \((c)\) are transformed into \((c')\) for whose associated curves \( C_{i(c')} \) there is no special tangent with affine equation \( x = t + b \). It follows that \( X_1 \neq \gamma(X) \).

We have \( H \not\subset X_1 \) since the condition \((c) \in H \) forces one coefficient each of the two polynomials \( f_i(c) \) to be equal to one and this condition cannot be sufficient for the existence of multiple roots of a polynomial.

The standard argument from elimination theory shows that the line at infinity is not tangent to \( C_{1(c)} \) for \((c)\) in an open subscheme of \( H \) and the choice \( f_1 = 2x^{n-1} - xt - t^{n-1} \) with \( j = 1 \) for \( p \mid n - 1 \) and \( j = 0 \) otherwise shows that this open subscheme is indeed nonempty.

We have shown that there exists a nonempty open subscheme \( A_4 \subset A_3 \) such that all \((c) \in A_4 \) satisfy the following conditions: the associated morphism \( \beta_1 \) is
generic, the associated morphism \( \beta_2 \) is generic over an affine subspace \( \mathbb{A}^1 \subset \mathbb{P}^1 \), no line \( x = t + b \) is tangent to both \( C_{1(c)} \) and the line at infinity is not tangent to \( C_{1(c)} \).

We now set \( V = H \cap A_4 \). Let \( s_1 \in \mathbb{N} \) be large enough such that \( V \in A_4 \) contains a point over \( \mathbb{F}_{q^*} \) if \( s \geq s_1 \). By definition of \( A_4 \), every point \((c)\) in \( V \) satisfies the conditions 1 – 7 listed after definition [2.2].

For some \( s \geq s_1 \), we choose \((c) \in V_{\mathbb{F}_{q^*}} \) and consider the two polynomials \( f_1 := f_1(c) \) and \( f_2 := f_2(c) \). We now want to apply an argument analogous to the proof of theorem [2.1] to the two polynomials \( f_1 \) and \( f_2 \) to derive the existence of an element \( b \in \mathbb{F}_{q^*} \) such that \( f_1(t + b, t) \) and \( f_2(t + b, t) \) are both irreducible. In what follows, we deal with the changes necessary to adapt the proof of theorem [2.1] to this purpose.

In the notation of [2], used in the first paragraph after the proof of proposition 3.1, that proposition is used to show the existence of an open subset of \( \mathbb{P}^2 \) from which a point \( M \) can be chosen such that projection from \( M \) of the curves \( C_i \) is a generic morphism to \( \mathbb{P}^1 \).

We have constructed two curves \( C_i \) which are irreducible, smooth, and whose Gauss maps are separable. This latter property was necessary to show that we can choose the point at infinity with coordinates \( x = 1, t = 1 \) as the point \( M \). This choice is equivalent to fixing \( a = 1 \) and necessary to ensure the monicity of the resulting irreducible polynomials \( f_i \) in \( \mathbb{F}_{q^*} \langle t \rangle \) which are to be constructed. The morphisms \( \varphi_1 \) and \( \varphi_2 \) in [2] can then be identified with our morphisms \( \beta_1 \) and \( \beta_2 \). Both \( \beta_i \) are generic over at least \( \mathbb{A}^1 \) and therefore separable. Note that the assumption \( p \neq 2 \) is necessary in the paragraph after the proof of proposition 3.1 in [2].

The proof of proposition 2.2 in [2] goes through for \( C_1 \) without change since \( \beta_1 \) is generic over \( \mathbb{P}^1 \), but for \( C_2 \) it needs the following adaptation. The form of the polynomial \( f_2 \) makes it clear that the line at infinity intersects \( C_2 \) in a point of multiplicity \( n \), while we have shown that since \( C_2 \) satisfies condition 5, the morphism \( \beta_2 \) is generic over \( \mathbb{A}^1 \). Therefore every nontrivial inertia group associated to \( \beta_2 \) is either a group of order 2 or, in one case, a cyclic group of order \( n \). This follows from an argument analogous to the one in the last paragraph of the proof of proposition 2.2 in [2]. Since the symmetric group of \( n \) elements is generated by one element of order \( n \) and a transposition, the rest of the proof of proposition 2.2 in [2] goes through also for the ramification type of \( C_2 \).

The proof of lemma 3.3 in [2] hinges on the fact that there are no tangents to both curves \( C_1 \) and \( C_2 \) which contain the point \( M \); we have shown this directly when we proved that conditions 6 and 7 are satisfied.

From proposition 3.4 in [2] onwards, the proof of theorem 1.1 proceeds as the proof of theorem 2.1. The application of theorem 3.5 in [2] imposes a second condition of the form \( s \geq s_2 \), so we set \( s_0 = \max(s_1, s_2) \).

We get a value \( b_0 \in \mathbb{F}_{q^*} \) with \( s \geq s_0 \) such that

\[
F = f_1(t + b_0, t) + f_2(t + b_0, t),
\]

where both \( f_1(t + b_0, t) \) and \( f_2(t + b_0, t) \) are monic and irreducible elements of \( \mathbb{F}_{q^*} \langle t \rangle \) with \( \deg(f_1) = n - 1 \) and \( \deg(f_2) = n \) and the proof is complete.
3 Proof of theorem 1.2

We unravel the proof of theorem 1.1 and make explicit all lower bounds on $q^*$ in terms of the degree $n$ of the polynomial $F(t)$, using only the simplest nontrivial estimates. We shall show that the resulting lower bound on $q^*$ is satisfied by $q$ provided that $q$ is larger than the lower bound given in the statement of theorem 1.2.

More precisely, from the number of points in the affine space over $F_q$ parametrizing the polynomials $f_1$ and $f_2$ we remove an upper bound for the number of $F_q$--rational points in $H \setminus A_j$ for $j = 1, 2, 3, 4$. The lower bound on $q$ will then imply that the remaining set of points, and therefore the set of usable polynomials $f_1$ and $f_2$, is nonempty. Furthermore, we show that the lower bound on $q$ is also large enough for the Chebotarev Density Theorem in the form of theorem 3.5 in [2] to be applicable to the resulting $f_1$ and $f_2$.

The dimension of the affine space parametrizing the polynomials in two variables of degree $d$ is $(d + 1)(d + 2)/2$. For the degree $d = n - 1$ of $f_1$, we use $A^I$ with $I = n(n+1)/2 - 1$ as parameter space since we only consider normalised polynomials $f_1$ which correspond to points in the affine subspace $H$.

We need the following elementary fact, which we enunciate as a lemma.

**Lemma 3.1.** Let $V$ be a Zariski-closed subset of $A^n$ defined by $m$ equations whose nonzero degrees are all bounded from above by $D$.

Then $V$ is contained in a hypersurface $S \subset A^n$ of degree at most $D^m$.

**Sketch of proof.** The standard result from which the bound $D^m$ follows is contained in [14] I Thm 7.7.

From theorem 1.1 it follows that $H \setminus A_j \neq H$ for $j = 1, 2, 3, 4$. We then use lemma 3.1 to establish that the embedding of $H \setminus A_j$ is contained in a hypersurface of degree bounded by, say, $d$. An upper bound on $|H \setminus A_j|_{F_q}$ is then provided by an upper bound on the number of $F_q$-rational points on one hypersurface of degree $d$. Such a bound is established in the following lemma.

**Lemma 3.2** (IV §3 Lemma 3A in [19]). Let $f(x_1, \ldots, x_n)$ be a nonzero polynomial over $F_q$ of total degree $d$. Then $f$ has at most $dq^{n-1}$ zeros in $F_q^n$.

Condition 1 of smoothness: We have to find an upper bound for the number of points $(c) \in A^I_{F_q}$, such that there exists a singularity on one of the $C_{i(c)}$. The singular locus of a curve of degree $d$ is defined by three equations of degree $d - 1$. By lemma 3.1 the degree of a hypersurface containing the singular locus is bounded by $(d - 1)^3$, and so is the degree of its projection to $A^I$. The points $(c) \in A^I_{F_q}$ which we have to exclude are therefore contained in the projection to $A^I$ of one hypersurface of degree $(n - 2)^3$ associated to $f_1$ and another hypersurface of degree $(n - 1)^3$ associated to $f_2$. By lemma 3.2 we have

$$|H \setminus A_1|_{F_q} \leq (n - 2)^3q^{I-1} + (n - 1)^3q^{I-1}.$$  

Condition 2 of absolute irreducibility: The number of factorizations of a polynomial in two variables of total degree $d$ is bounded by the number of ways we have to split off a factor of total degree between 1 and $(d + 1)/2$. For each degree, such a splitting-off corresponds to a solution of $(d + 1)(d + 2)/2$ equations of degree 2 in the coefficients of the polynomial and of its factors. Lemma 3.1 then gives bounds for the degrees of the reducible loci of $A^I$ and we have

$$|H \setminus A_2|_{F_q} \leq \left[\frac{n}{2}\right] 2^{n(n+1)/2}q^{I-1} + \left[\frac{n+1}{2}\right] 2^{(n+1)(n+2)/2}q^{I-1}.$$  

Condition 3 of separability of the Gauss maps: As detailed in the proof of theorem 1.1 the intersection of an inseparable curve with any tangent has multiplicity
greater than 2. This means that the curve violates the part concerning points of order at least 3 in conditions 4–7 below, which therefore eliminate also inseparable curves.

Conditions 4–7 on the morphisms $\beta_1$ and $\beta_2$ and tangency conditions: Condition 4 on the morphism $\beta_1$ can be violated by the projective closure of a line with the affine equation $x = t + b, b \in F$ intersecting $C_{1(c)}$ in a point of multiplicity at least 3 or in two points of multiplicity at least 2 each. For the first case, we homogenise the equation

$$f_i(t + b, t) = (ut - v)^3 g_i(t)$$

with respect to the variables $t$ and $b$ and projectivise the coefficients of $f_i$ and $g_i$. Then a point of order at least 3 exists on the solutions of $\deg(f_i) + 1$ equations each of degree at most $\max(\deg(f_i) + 1, 4)$. By lemma 3.1 we remove a hypersurface of degree $(\deg(f_i) + 5)^{\deg(f_i) + 1}$ from $A^1$. The case of two double points is treated analogously.

For the treatment of conditions 5 and 6, observe that we homogenise only with respect to the variable $t$ rather than both $t$ and $b$ since here $b$ is in $A^1$.

Condition 7 stipulating that the line at infinity not be tangent to $C_{1(c)}$ is equivalent to the polynomial consisting of the terms of degree $n - 1$ of $f_1$ not having a double root. A calculation using a resultant shows that we have to remove one hypersurface of degree $2n - 3$ from $H$.

The whole estimate is therefore

$$|H \setminus A_4|_{\mathbb{F}_q} \leq (n + 4)^n q^{f - 1} + (n + 5)^n + (n + 5)^{n+1} q^{f-1} + (n + 6)^{n+1} q^{f-1}$$

$$+ (n + 4)^{2n+1} q^{f-1} + (2n - 3) q^{f-1},$$

where the terms correspond, in that order, to the conditions derived from points of order at least 3 on $f_1, f_2$: two points of order at least 2 on $f_1, f_2$: tangents to both $f_1$ and $f_2$ and from the condition on the line at infinity. Since $(n+5)^2-(n+4)^2 = 2n+9$, we can get rid of the last term and replace this bound by the weaker and simpler bound

$$|H \setminus A_4|_{\mathbb{F}_q} \leq 4(n + 6)^{n+1} q^{f-1} + (n + 4)^{2n+1} q^{f-1}.$$ 

The condition on $q$ which has to be satisfied is

$$q^f - \sum_{j=1}^{4} |H \setminus A_j| > 0,$$

where we set $|H \setminus A_3|_{\mathbb{F}_q} = 0$ for counting purposes. By the above calculations, this is implied by

$$q > (n - 2)^3 + (n - 1)^3 + \frac{1}{2} n 2^{n(n+1)/2} + \frac{1}{2} (n + 1) 2^{(n+1)(n+2)/2}$$

$$+ 4(n + 6)^{n+1} + (n + 4)^{2n+1}.$$ 

This is easily seen to follow from our assumption

$$q > 8(n + 6)^{2n^2} > 8(n + 6)^{(n+1)(n+2)/2+1}.$$ 

We now have to show that the $f_1, f_2 \in \mathbb{F}_q[x, t]$ satisfy the conditions necessary to apply theorem 3.5 in [2]. As detailed in that paper, we construct the function field $L = K_1 \times K_2$. Since $\deg(f_1) = n - 1$ and $\deg(f_2) = n$, we have $N = (n - 1)!n!$. The genus of an irreducible and smooth plane algebraic curve of
degree $d$ is $(d-1)(d-2)/2$, so we have $g_1 = (n-2)(n-3)/2$, $g_2 = (n-1)(n-2)/2$, and by formula (4) in [2] and a short calculation, it follows that
\[
g = 1 + N(n^2 - 2n).
\]
For the application of theorem 3.5 in [2], we then need to satisfy the condition
\[
q - (N + 2g)q^{1/2} - Nq^{1/4} - 2(g + N) > 0.
\]
To simplify the calculation, we take instead
\[
q - 2(N + g)q^{1/2} - 2(N + g) > 0
\]
and this turns out to be implied by
\[
q > 4(N + g)^2 + 4(N + g).
\]
Another short calculation shows that the condition $q > 8(N + g)^2$ follows from $q > 8(n + 6)^{2n^2}$ and the proof is complete.

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