The Sign of Fourier Coefficients of Half-Integral Weight Cusp Forms

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From a result of Waldspurger [6], it is known that the normalized Fourier coefficients $a(m)$ of a half-integral weight holomorphic cusp eigenform $f$ are, up to a finite set of factors, one of $\pm \sqrt{L(1/2, f, \chi_m)}$ when $m$ is square-free and $f$ is the integral weight cusp form related to $f$ by the Shimura correspondence [8]. In this paper we address a question posed by Kohnen: which square root is $a(m)$? In particular, if we look at the set of $a(m)$ with $m$ square-free, do these Fourier coefficients change sign infinitely often? By partially analytically continuing a related Dirichlet series, we are able to show that this is so.

1. Introduction

Let $k$ be an odd integer and $f \in S_{k/2}(\Gamma_0(4))$; that is, a cusp form of half-integral weight $k/2$ and level 4 as described by Shimura [8]. We will discuss the automorphic properties of $f$ in more detail in Section 2. Let the Fourier expansion of $f$ at $\infty$ be

$$f(z) = \sum_{m=1}^{\infty} a(m)m^{k/4 - 1/4} e(mz). \quad (1.1)$$

The Shimura correspondence provides a holomorphic modular form $f$ of weight $k - 1$ such that if $f$ is an eigenform of the Hecke operator $T_{1/2}(p^2)$, then $f$ is an eigenform of $T_{k-1}(p)$ with the same eigenvalue. It was proven in [7] that the level of $f$ is 2. For the definitions of the Hecke operators in the half-integral weight case, see [8]. Waldspurger proved in [9] that for any square-free $t$,

$$a(t)^2 = c \cdot L\left(\frac{1}{2}, f, \chi_t\right) \quad (1.2)$$

where

$$\chi_t(n) = \left(\frac{-1}{n}\right)^{k/2 - 1/2} \quad (1.3)$$

is the unique real primitive character modulo $t$. The constant $c$ is dependent only on $f$. Later this result was made explicit by Kohnen and Zagier in the case of $f \in S_{k+1/2}(\Gamma_0(4))$, see [6] for the definition of the space $S_{k+1/2}(\Gamma_0(4))$. For a fundamental discriminant $D$ satisfying

$$(-1)^{k+1/4} D > 0, \quad (1.4)$$
they proved that
\[
\frac{a(|D|)^2}{\langle f, f \rangle} = \frac{(\frac{k}{2} - 1)! \ L(\frac{k}{2}, f, \chi_D)}{\pi^{(k-1)/2} \langle f, f \rangle}.
\] (1.5)
Here \( \langle f, f \rangle \) and \( \langle f, f \rangle \) are the normalized Petersson inner products, and
\[
\chi_D(n) = \left( \frac{D \ n}{n} \right)
\] (1.6)
is the Kronecker symbol.

The relationship (1.2) prompts the questions posed by Kohnen: which square root of \( L(\frac{k}{2}, f, \chi_t) \) is \( a(t) \) proportional to, and how often? In [5], Kohnen in fact proves that for any half-integral weight cusp form \( f \in S_{k+\frac{1}{2}}(N, \chi) \), not necessarily an eigenform, the sequence of Fourier coefficients \( a(tn^2) \) for a fixed \( t \) square-free has infinitely many sign changes. A natural next question one may ask is whether all the Fourier coefficients \( a(t) \) with \( t \) running over square-free integers change sign infinitely often. In the following theorem we prove that this is indeed the case for eigenforms.

**Theorem 1.1.** Given \( f \in S_{k+\frac{1}{2}}(\Gamma_0(4)) \), an eigenform of all Hecke operators \( T_p^k(p^2) \) for \( p \) prime, where \( k \) is an odd integer, with Fourier expansion
\[
f(z) = \sum_{m=1}^{\infty} a(m) m^{k/4 - 1/2} e(mz);
\] (1.7)
the Fourier coefficients \( a(t) \), with \( t \) running over square-free integers, change sign infinitely often.

Inspired by the methods in [1] and [2], we will prove Theorem 1.1 by analytically continuing the Dirichlet series
\[
\sum_{t \geq 1} \frac{a(t)}{t^s}
\] (1.8)
to \( \Re(s) > 3/4 \) by exploiting the analytic continuations of a family of Mellin transforms related to \( f \); we then prove our claim by contradiction.

### 2. Automorphic Properties

Before we proceed, we will review the automorphic properties of half-integral weight cusp forms. Let \( f \) be as above. Given \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(4) \), we have
\[
j(\gamma, z) := \varepsilon_d^{-1} \left( \frac{c}{d} \right) (cz + d)^{\frac{k}{2}} = \theta(\gamma(z))/\theta(z)
\] (2.1)
where \( \left( \frac{c}{d} \right) \) is Shimura’s extension of the Jacobi symbol as in [3]. Setting \( \xi := (\gamma, j(\gamma, z)) \), \( f \) satisfies
\[
f|_{\xi_h} (z) := j(\gamma, z)^{-k} f(z) = \varepsilon_d^h \left( \frac{c}{d} \right) (cz + d)^{-\frac{k}{2}} f \left( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) z \right) = f(z).
\] (2.2)
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Here $\varepsilon_d$ is the sign of the Gaussian sum $\sum_{n=1}^{d} e(n^2/d)$:

$$
\varepsilon_d = \begin{cases} 
1 & \text{if } d \equiv 1 \pmod{4} \\
i & \text{if } d \equiv 3 \pmod{4}.
\end{cases}
$$

Furthermore, we fix the following expressions for $f_{1/2}$ and $f_0$, which are evaluations of $f$ at the respective cusps $1/2$ and 0, as

$$
f_{1/2}(z) := f|_{\gamma = \left( \begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix} \right)}(z) = (-2z + 1)^{-1/2}f \left( \frac{1}{-2} \right) z.
$$

$$
f_0(z) := f|_{\gamma = \left( \begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix} \right)}(z) = (-2iz)^{-1/2}f \left( \frac{0}{4} \right) z.
$$

Also note that $f(r \cdot z) = f(z)$ for all $r \in \mathbb{R}^\ast$.

3. Arguing by Contradiction

Our proof by contradiction proceeds as follows. Take the Dirichlet series

$$
M(s) = \sum_{t \geq 1} \frac{a(t)}{t^s}
$$

as described in [1, 8] and assume that $a(t)$ changes sign finitely many times. Assume for a contradiction that $a(t) \geq 0$ for $t > T$ where $T$ is sufficiently large. Throughout this section, we let $t$ denote a square-free positive integer.

Suppose that $M(s)$ analytically continues to $\Re(s) > 3/4$ with polynomial growth in $\Im(s)$, as this work will demonstrate. Using a well-known inverse Mellin transform, we get

$$
\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} M(s)\Gamma(s)x^s \ ds = \sum_t a(t)e^{-t/x}.
$$

The integral on the left-hand side above is $O(x^{3/4+\varepsilon})$ for any $\varepsilon > 0$, as we can move the line of integration to $\Re(s) = 3/4 + \varepsilon$. On this vertical line, the gamma function decreases exponentially, whereas the analytic continuation of $M(s)$ only has polynomial growth, as will be shown below in Proposition 4.4. Since the integrand has no poles for $\Re(s) > 3/4$, we don’t pick up any residues in moving the line of integration. Thus we arrive at the inequality

$$
\sum_t a(t)e^{-t/x} \ll x^{3/4+\varepsilon}.
$$

The completed Eisenstein series for level 4 is

$$
E^*(z, s) = 2^{2s-1}\zeta^*(2s)E(z, s) = 2^{2s-1}\zeta^*(2s) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \Im(\gamma z)^s,
$$

where $\zeta^*(s)$ is given by

$$
\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).
$$
The completed Eisenstein series is holomorphic except for simple poles at $s = 0, 1$, with residues $-\frac{1}{4}$ and $\frac{1}{4}$ respectively. We have the identity

$$
\int_{\Gamma_0(4),b} |f(z)|^2 E^s(z, s) y^{k/2} \frac{dx dy}{y^2} = \Gamma \left( s + \frac{k}{2} - 1 \right) 2^{1-k} \pi^{-(s+\frac{k}{2}-1)} \zeta^*(2s) L^{(2)}(f, s),
$$

where

$$
L^{(2)}(f, s) = \sum_{m=1}^{\infty} \frac{a(m)^2}{m^s}
$$

which follows after a Rankin-Selberg unfolding. This implies that $L^{(2)}(f, s)$ has a pole at $s = 1$ with a non-zero residue. In fact, due to the integral representation above, the Rankin-Selberg convolution $L$-series extends to a meromorphic function with the only pole at $s = 1$ when $\Re(s) \geq \frac{1}{2}$.

Considering the inverse Mellin transform

$$
I = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L^{(2)}(f, s) \Gamma(s) x^s ds = \sum_{m=1}^{\infty} a(m)^2 e^{-m/x},
$$

and shifting the line of integration to $\Re(s) = \frac{1}{2}$ past the pole at $s = 1$, we get

$$
I = (\text{Res}_{s=1} L^{(2)}(f, s)) x + \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L^{(2)}(f, s) \Gamma(s) x^s ds,
$$

which implies, as the contribution from the integral above is $O(x^{\frac{1}{2}})$, that

$$
x \ll \sum_{m} a(m)^2 e^{-m/x}.
$$

In the above sum, the square-free integers play a nontrivial role. Indeed, using Lemma 4.1 we will be able to conclude that for any $\varepsilon > 0$ and $\Re(s) = \sigma$

$$
L^{(2)}(f, s) = \sum_{m=1}^{\infty} \frac{a(m)^2}{m^s} = \sum_{t} \sum_{n=1}^{\infty} \frac{a(tn)^2}{(tn)^s} \ll \sum_{t} \sum_{n=1}^{\infty} \frac{a(t)^2}{t^{\sigma n^{2\sigma - 2\varepsilon}}} = \zeta(2\sigma - 2\varepsilon) M^{(2)}(\sigma)
$$

where

$$
M^{(2)}(s) = \sum_{t} \frac{a(t)^2}{t^s}.
$$

Therefore, as we move $\sigma$ towards $1^+$ along the real line, since $L^{(2)}(f, s)$ has a pole at $1$ and $\zeta(2\sigma - 2\varepsilon) \ll 1$ for $\sigma \geq 1$, it follows from (3.11) that $\sum_{t} a(t)^2 / t^\sigma$ tends to infinity. Therefore the function $M^{(2)}(s)$, which is an analytic function on the region $\Re(s) > 1$, has a singularity as $s$ tends to $1$. Put

$$
A(x) = \sum_{T < t \leq x} a(t)^2.
$$
We claim that the singularity of $M^{(2)}(s)$ forces the partial sums, $A(x)$, to not be of slow growth. Indeed, assume that for some $c < 1$,

$$A(x) = O(x^c),$$

(3.14)

from the partial summation formula we get, for $\Re(s) > 1$

$$\sum_{t > T} a(t)^2 x^{-t} = s \int_T^\infty \frac{A(u)}{u^{s+1}} du.$$

(3.15)

Then, since we assumed $A(x) \ll x^c$, the right-hand side above is an analytic function on the right half plane $\Re(s) > c$, but the left-hand side $M^{(2)}(s)$ has a singularity at $s = 1$, giving our contradiction. Thus, for every $c$ with $0 < c < 1$, every constant $\alpha > 0$, and every $x$, there is an $x_0 > x$ such that

$$A(x_0) \geq \alpha x_0^c.$$

(3.16)

Now from (3.3) we have a constant $\beta > 0$ such that,

$$\beta x^{3/4 + \varepsilon} \geq e \left| \sum a(t) e^{-t/x} \right| \geq e \left| \sum_{t > T} a(t) e^{-t/x} \right| - e \left| \sum_{t \leq T} a(t) e^{-t/x} \right|,$$

(3.17)

so using our assumption on the eventual non-negativity of $a(t)$, we have that

$$\beta x^{3/4 + \varepsilon} + O(1) \geq e \sum_{t > T} a(t) e^{-t/x} \geq \sum_{T < t \leq x} a(t),$$

(3.18)

where $\varepsilon > 0$ is arbitrarily small, and $\beta$ depends on $\varepsilon$. Thus increasing $\beta$ to accommodate the constant term, we get

$$\beta x^{3/4 + \varepsilon} \geq \sum_{T < t < x} a(t).$$

(3.19)

Let $\lambda n^{\theta}$ be an upper bound on the individual coefficient $a(n)$ of the half-integral weight modular form $f$; according to [3] one may take $\theta = 3/14$. Now apply (3.16) and get that for some $x_0$ as above, which we may choose to be arbitrarily large,

$$\lambda/\beta x_0^{3/4 + \varepsilon} \geq \lambda \sum_{T < t \leq x_0} a(t) \geq \sum_{T < t \leq x_0} \frac{a(t)^2}{t^\theta} \geq x_0^{-\theta} \sum_{T < t \leq x_0} a(t)^2 \geq \alpha x_0^{-\theta + \varepsilon},$$

(3.20)

again by our non-negativity assumption. This implies,

$$x_0^{c - \varepsilon - 27/28} \leq \frac{\lambda\beta}{\alpha}.$$

(3.21)

By chosing $c$ and $\varepsilon$ appropriately we may make the exponent on the left hand side greater than 0, giving our contradiction. Therefore, the assumption that all Fourier coefficients $a(t)$ change sign finitely many times for square-free $t$ must be false. Thus, in order to show that the Fourier coefficients $a(t)$ change sign infinitely often for square-free $t$, we need only show that $M(s)$ can be analytically continued up to the line $\Re(s) = 3/4$, and grows slowly on vertical strips. The remainder of this paper is devoted to proving this.
4. Analytic Continuation

We now proceed to obtain an analytic continuation of the Dirichlet series (1.8) to the region $\Re(s) > 3/4$. First, note that

$$\sum_{t \geq 1} \frac{a(t)}{t^s} = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} \sum_{r \mid m} \mu(r) = \sum_{r=1}^{\infty} \mu(r) D_r(s), \quad (4.1)$$

where

$$D_r(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}, \quad (4.2)$$

Lemma 4.2 shows that the series $D_r(s)$ converges for $\Re(s) > 1$, and in fact $D_r(1 + \varepsilon + it) \ll \varepsilon 1/r^2$. With this fact, we easily see that our Dirichlet series over the square-free integers converges on the half-plane $\Re(s) > 1$. We now further examine the series $D_r(s)$:

$$D_r(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} \left( \frac{1}{r^2} \sum_{u \mod r^2} e\left(\frac{mu}{r^2}\right) \right)$$

$$= \frac{1}{r^2} \sum_{u \mod r^2} \sum_{m=1}^{\infty} \frac{a(m)e\left(\frac{mu}{r^2}\right)}{m^s}. \quad (4.3)$$

The innermost Dirichlet series can be expressed in terms of an additive ely twisted Mellin integral of $f$. For rational $q$, denote

$$\Lambda(f, q, s) = \int_0^{\infty} f(iy + q) y^{s+\left(\frac{k}{4} - \frac{1}{2}\right)} \frac{dy}{y}. \quad (4.4)$$

This integral converges for every $s \in \mathbb{C}$ because $q \in \mathbb{Q}$ is a cusp and since $f$ is a cusp form, $f(iy + q)$ decreases exponentially as $y \to \infty$ and as $y \to 0$. Thus $\Lambda(f, q, s)$ is an entire function of $s$. Let $f_s$ denote $f, f_{\frac{k}{4}},$ or $f_0$. Now expanding $f_s$ in the integral as its Fourier series, with respective coefficients $a_s(n)$, we get:

$$\int_0^{\infty} f_s(iy + q)y^{s+\left(\frac{k}{4} - \frac{1}{2}\right)} \frac{dy}{y} = \int_0^{\infty} \sum_{m=1}^{\infty} a_s(m)m^{\left(\frac{k}{4} - \frac{1}{2}\right)} e(m(iy + q))y^{s+\left(\frac{k}{4} - \frac{1}{2}\right)} \frac{dy}{y}$$

$$= \sum_{m=1}^{\infty} a_s(m)m^{\left(\frac{k}{4} - \frac{1}{2}\right)} e(mq) \int_0^{\infty} e^{-2\pi my} y^{s+\left(\frac{k}{4} - \frac{1}{2}\right)} \frac{dy}{y}$$

$$= \frac{\Gamma(s + \left(\frac{k}{4} - \frac{1}{2}\right))}{(2\pi)^{s+\left(\frac{k}{4} - \frac{1}{2}\right)}} \sum_{m=1}^{\infty} \frac{a_s(m)e(mq)}{m^s}. \quad (4.5)$$

For ease of notation call, from now on, $s' := s + \left(\frac{k}{4} - \frac{1}{2}\right)$. Using the integral representation of our Dirichlet series, which is that of $L(f, s)$ twisted by an additive character, we obtain
\( D_r(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} \) 
\[
= (2\pi)^s \frac{1}{\Gamma(s')} \frac{1}{r^2} \sum_{u \mod r^2} A \left( \frac{f}{r^2}, \frac{u}{r^2}, s \right) 
\]
\[
= (2\pi)^s \frac{1}{\Gamma(s')} \frac{1}{r^2} \sum_{d \mid r^2} \sum_{(u,d)=1} \sum_{u \mod d} A \left( \frac{f}{d}, \frac{u}{d}, s \right), \tag{4.6}
\]

where the fraction \( u/d \) is in lowest terms, and by abuse of notation, we continue to call the numerator \( u \). Equation (4.6) allows us to express \( D_r(s) \) as a finite sum of entire functions, hence \( D_r(s) \) itself is an entire function. Therefore it makes sense to talk about the growth properties of \( D_r(s) \) in \( r \) for any fixed \( s \) on the complex plane. Also note that we only need to estimate \( D_r(s) \) for square-free \( r \), due to the existence of \( \mu(r) \) in (4.1).

**Lemma 4.1.** The Fourier coefficients of a half-integral weight cusp eigenform \( f \in S_{k/2}(\Gamma_0(4)) \), with \( k \geq 5 \) as above satisfy the following bound

\[
a(tn^2) \ll |a(t)|n^\varepsilon \tag{4.7}
\]

where \( t, n \in \mathbb{N} \) and \( t \) is square-free, with the implied constant dependent only on \( f \) and \( \varepsilon > 0 \).

**Proof.** The Shimura correspondence \( 3 \) gives us that for \( t \in \mathbb{N} \) square-free we have that

\[
\sum_{n=1}^{\infty} \frac{a(tn^2)}{n^{s-\frac{k}{2}+1}} = a(t) \left( \sum_{m_1=1}^{\infty} \chi_t(m_1)\mu(m_1) \right) \left( \sum_{m_2=1}^{\infty} \frac{A(m_2)}{m_2^s} \right), \tag{4.8}
\]

where \( \chi_t(m_1) = \left( \frac{-1}{m_1} \right)^{\frac{k}{2}-\frac{1}{2}} \left( \frac{4}{m_1} \right) \) and the \( A(m_2) \) are the Fourier coefficients of the weight \( k-1 \) cusp form \( f \in S_{k-1}(\Gamma_0(2)) \) such that

\[
f(z) = \sum_{m_2=1}^{\infty} A(m_2)e^{2\pi im_2z} \tag{4.9}
\]

where \( f \) is associated to \( f \) by the Shimura correspondence. Expanding the right-
hand side of (4.8) we get
\[
\sum_{n=1}^{\infty} \frac{a(tn^2)}{n^{s-\frac{3}{2}+\frac{\varepsilon}{2}}} = a(t) \sum_{n=1}^{\infty} \sum_{m \mid n} \frac{\chi_i(m)\mu(m)}{m^{s-\frac{1}{2}+\frac{\varepsilon}{2}}} A(m) m^2
\]
\[
= a(t) \sum_{n=1}^{\infty} \sum_{m \mid n} \frac{\chi_i\left(\frac{n}{m}\right)\mu\left(\frac{n}{m}\right)}{n^{\frac{1}{2}}} A(m) m^{\frac{1}{2}-\frac{\varepsilon}{2}}.
\]
Comparing coefficients term-by-term we see that for each \( n \),
\[
a(tn^2) = a(t) \sum_{m \mid n} \frac{\chi_i\left(\frac{n}{m}\right)\mu\left(\frac{n}{m}\right)}{n^{\frac{1}{2}}} A(m) m^{\frac{1}{2}-\frac{\varepsilon}{2}}.
\]
(4.11)
Since the Ramanujan-Petersson conjecture is known for integral weight cusp forms, we have \( A(m) \ll m^{(\frac{1}{2}+\varepsilon)\frac{3}{2}+\varepsilon} \) with the implied constant dependent on \( \varepsilon \) (and thus \( f \) and \( \varepsilon \)). Using this bound and taking absolute values of (4.11) we get
\[
a(tn^2) \ll |a(t)| n^{-\frac{1}{2}} \sum_{m \mid n} m^{\frac{3}{2}+\varepsilon} \ll |a(t)| n^{-\frac{1}{2}} \sum_{m \mid n} m^{\frac{3}{2}+\varepsilon} \sigma_{\frac{1}{2}}(n).
\]
(4.12)
Since \( \sigma_{\frac{1}{2}}(n) \leq d(n)\sqrt{n} \), where \( d(n) \) is the divisor function, we have \( \sigma_{\frac{1}{2}}(n) \ll n^{1/2+\varepsilon} \) with the implied constant dependent on \( \varepsilon \). Putting this into (4.12) gives the desired result.

Lemma 4.2. Letting \( r \in \mathbb{N} \) and \( \tau \in \mathbb{R} \),
\[
D_r(1+\varepsilon+i\tau) = \sum_{n=1}^{\infty} \frac{a(m)}{m^{1+\varepsilon+i\tau}} \ll \frac{1}{r^2}
\]
where the implied constant depends only on \( f \) and \( \varepsilon \).

Proof. In the sum, we write \( m = nr^2 \), and let \( n_0 \) be the square-free part of \( n \). Then by Lemma 4.1 \( a(nr^2) \ll |a(n_0)| \left(\frac{n_0}{n_0}\right)^\varepsilon \), and therefore for \( s = \sigma + i\tau \) with \( \sigma \geq 1 \),
\[
\sum_{n=1}^{\infty} \frac{a(nr^2)}{n^{s+2\varepsilon}} \ll r^{2\varepsilon} \sum_{n=1}^{\infty} \frac{|a(n_0)| |n/n_0|^{\varepsilon}}{n^{\sigma+2\varepsilon}} \leq r^{2\varepsilon} \sum_{n=1}^{\infty} \frac{|a(n_0)|}{n^{\sigma+\varepsilon}},
\]
(4.14)
where the implied constant only depends on \( f \) and \( \varepsilon \). Now,
\[
\sum_{n=1}^{\infty} \frac{|a(n_0)|}{n^{\sigma+\varepsilon}} = \sum_{d=1}^{\infty} \sum_{n \text{ square-free}} \frac{|a(n)|}{(nd^2)^{\sigma+\varepsilon}}
\]
\[
= \zeta(2\sigma+2\varepsilon) \sum_{n \text{ square-free}} \frac{|a(n)|}{n^{\sigma+\varepsilon}} \ll \zeta(2\sigma+2\varepsilon) \sum_{n=1}^{\infty} \frac{|a(n)|}{n^{\sigma+\varepsilon}} = O(1)
\]
(4.15)
where the implied constant again depends on \( f \) and \( \epsilon \). Therefore, using both (4.14) and (4.15) we have

\[
D_r(s + \epsilon) = \sum_{m=1}^{\infty} \frac{a(m)}{m^{s+\epsilon}} = \sum_{n=1}^{\infty} \frac{a(nr^2)}{(nr^2)^{s+\epsilon}} = \frac{1}{r^{2s+2\epsilon}} \sum_{n=1}^{\infty} a(nr^2) \ll \frac{1}{r^{2s}}.
\]

(4.16)

Letting \( s = 1 + \epsilon \), we have

\[
D_r(1 + \epsilon + i\tau) \ll \frac{1}{r^\epsilon},
\]

(4.17)

completing our proof.

In order to show that

\[
M(3/4 + \epsilon) = \sum u(r)D_r(3/4 + \epsilon)
\]

(4.18)

converges, we will bound \( D_r(-\epsilon + it) \) by a polynomial in \( t \) and apply a Phragmén-Lindelöf convexity argument.

The twisted Mellin integrals \( \Lambda(f, u/d, s) \) have functional equations. Depending on the class of equivalent cusps that \( u/d \) belongs to, we get slightly different functional equations. They are as follows:

**Lemma 4.3.** If \( 4 \mid d \),

\[
\Lambda\left( f, \frac{u}{d}, s \right) = d^{1-2s}(-i)^{\frac{s}{2}}\varepsilon_v \left( \frac{d}{v} \right)^{1-s} \Lambda\left( f, \frac{v}{d}, 1-s \right),
\]

where \( v \) is chosen so that \( uv \equiv -1 \pmod{d} \).

If \( 2 \mid d \), the functional equation has the type,

\[
\Lambda\left( f, \frac{u}{d}, s \right) = d^{1-2s}(-i)^{\frac{s}{2}}\varepsilon_v \left( \frac{d}{v} \right)^{1-s} \Lambda\left( f, \frac{v}{d}, 1-s \right),
\]

where once again \( v \) is chosen to satisfy \( uv \equiv -1 \pmod{d} \).

Finally, if \( 2 \nmid d \), then

\[
\Lambda\left( f, \frac{u}{d}, s \right) = (2d)^{1-2s}\varepsilon_d^{-k} \left( \frac{v}{d} \right)^{1-s} \Lambda\left( f_0, \frac{v}{d}, 1-s \right),
\]

where \( v \) be such that \( 4uv \equiv -1 \pmod{d} \).

**Proof.** First note that

\[
\int_0^\infty \left( iy + \frac{u}{d} \right) y^{s+(\frac{k}{2}-\frac{1}{2})} \frac{dy}{y} = \int_0^\infty \left( \left( \frac{1}{0} \frac{u/d}{1} \right) iy \right) y^{s+(\frac{k}{2}-\frac{1}{2})} \frac{dy}{y}.
\]

(4.22)
We observe that $u/d$ is equivalent to to the cusps $\infty, 1/2, 0$ depending on the conditions $4|d, 2||d$ or $2 \not| d$ respectively. We consider the following matrix decompositions in each case. If $4|d$,

$$
\begin{pmatrix}
1 & u/d \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
4/d & 0 \\
0 & d
\end{pmatrix}
= 
\begin{pmatrix}
-u & e \\
-d & v
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & -1 \\
4 & 0
\end{pmatrix},
\tag{4.23}
$$

and if $2||d$,

$$
\begin{pmatrix}
1 & u/d \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
4/d & 0 \\
0 & d
\end{pmatrix}
= 
\begin{pmatrix}
2e - u & e \\
2v - d & v
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & -1 \\
4 & 0
\end{pmatrix},
\tag{4.24}
$$

where $v$ and $e$ are chosen to satisfy $uv - de = -1$. Finally for $d$ odd, with $v$ and $e$ chosen to satisfy $4uv - de = -1$, we consider the following matrix decomposition:

$$
\begin{pmatrix}
1 & u/d \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1/d & 0 \\
0 & d
\end{pmatrix}
= 
\begin{pmatrix}
e & u \\
4v & d
\end{pmatrix}
\begin{pmatrix}
0 & 1/4 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}1 & v/d \\
0 & 1
\end{pmatrix}
\begin{pmatrix}0 & -1 \\
4 & 0
\end{pmatrix}.
\tag{4.25}
$$

Note that in all of the matrix decompositions, the leftmost matrix is in $\Gamma_0(4)$.

Recall that we let $s' = s + (\frac{k}{4} - \frac{1}{2})$. For $4|d$ we use (4.23),

$$
\Lambda \left( f, \frac{u}{d}, s \right) = \int_0^\infty f (iy + \frac{u}{d}) y^{s+\left(\frac{k}{4} - \frac{1}{2}\right)} dy
$$

$$
= \int_0^\infty \int f \left( \begin{pmatrix}1 & u/d \\
0 & 1\end{pmatrix} \begin{pmatrix}4/d & 0 \\
0 & d\end{pmatrix} \frac{id^2y}{4} \right) y^{s'} dy
$$

$$
= 4^{s'} d^{-2s'} \int_0^\infty \int f \left( \begin{pmatrix}-u & e \\
-d & v\end{pmatrix} \begin{pmatrix}1 & v/d \\
0 & 1\end{pmatrix} \frac{0 - 1}{4} \right) y^{s'} dy
$$

$$
= d^{-2s'} \int_0^\infty \int f \left( \begin{pmatrix}-u & e \\
-d & v\end{pmatrix} \left( iy + \frac{v}{d} \right) -4 \right) y^{s'} dy
$$

$$
= d^{-2s'} \int_0^\infty \int f |\xi|_k \left( iy + \frac{v}{d} \right) y^{-s} \frac{dy}{y}
$$

$$
= d^{-2s'} \int_0^\infty \int f |\xi|_k \left( iy + \frac{v}{d} \right) y^{1-s} \frac{dy}{y},
\tag{4.26}
$$

where $\xi = (\gamma, j(\gamma, z))$ for $\gamma = \left( \begin{pmatrix} -u & e \\
-d & v \end{pmatrix} \right)$ and $f |\xi|_k$ denotes the slash operator on half-integral weight forms as described in (2.22). Thus, for $4|d$,

$$
\Lambda \left( f, \frac{u}{d}, s \right) = d^{1-2s} (-i)^{\frac{k}{2}} \frac{d}{v} \int_0^\infty \int f \left( \frac{v}{d} \right) y^{1-s} \frac{dy}{y},
\tag{4.27}
$$

If $2||d$, once again let $v$ and $e$ satisfy $uv - de = -1$. By (4.24) and similar reasoning as above, and using (2.22) and (2.24), we deduce:
The Sign of Fourier Coefficients of Cusp Forms

\[ \Lambda \left( f, \frac{u}{d}, s \right) = \int_0^\infty f \left( iy + \frac{u}{d} \right) y^{s+\left( \frac{d}{d} - \frac{1}{2} \right)} \frac{dy}{y} \]

\[ = d^{-2s'} \int_0^\infty f \left( \frac{2e - u e}{2v - d v} \right) \left( \begin{array}{c} 1 \ 0 \\ -2 \ 1 \end{array} \right) \left( \begin{array}{c} iy \\ 0 \end{array} \right) y^{-s'} \frac{dy}{y} \]

\[ = d^{-2s'} \int_0^\infty f \left( \xi_k \left[ \left( \begin{array}{c} 1 \ 0 \\ -2 \ 1 \end{array} \right) \right] \right) \left( iy + \frac{v}{d} \right) \varepsilon_{d}^{-k} \left( \frac{2v - d v}{v} \right)^k \left( -diy \right)^{\frac{k}{2}} y^{-s'} \frac{dy}{y} \]

\[ = d^{1-2s} (-i)^{\frac{k}{2}} \frac{v}{d} \int_0^\infty f \left( iy + \frac{v}{d} \right) y^{1-s+\left( \frac{k}{2} - \frac{1}{2} \right)} \frac{dy}{y} \] \hfill (4.28)

where, this time, \( \gamma = \left( \frac{2e - u e}{2v - d v} \right) \), and \( \xi = (\gamma, j(\gamma, z)) \). Thus for \( 2 \mid d \),

\[ \Lambda \left( f, \frac{u}{d}, s \right) = d^{1-2s} \left( -i \right)^{\frac{k}{2}} \frac{v}{d} \int_0^\infty f \left( iy + \frac{v}{d} \right) y^{1-s+\left( \frac{k}{2} - \frac{1}{2} \right)} \frac{dy}{y} \]

\hfill (4.29)

For \( d \) odd, we choose \( v, e \) such that \( 4uv - de = -1 \). So by (4.25), (2.2) and (2.5) we have

\[ \Lambda \left( f, \frac{u}{d}, s \right) = \int_0^\infty f \left( iy + \frac{u}{d} \right) y^{s+\left( \frac{d}{d} - \frac{1}{2} \right)} \frac{dy}{y} \]

\[ = (2d)^{-2s'} \int_0^\infty f \left( \frac{e}{4v d} \right) \left( \begin{array}{c} 0 \ -1 \\ 4 \ 0 \end{array} \right) \left( iy + \frac{v}{d} \right) y^{-s'} \frac{dy}{y} \]

\[ = (2d)^{-2s'} \int_0^\infty f \left( \xi_k \left[ \left( \begin{array}{c} 0 \ -1 \\ 4 \ 0 \end{array} \right) \right] \right) \left( iy + \frac{v}{d} \right) \varepsilon_{d}^{-k} \left( \frac{v}{d} \right)^k \left( -diy \right)^{\frac{k}{2}} y^{-s'} \frac{dy}{y} \]

\[ = (2d)^{1-2s} \varepsilon_{d}^{-k} \left( \frac{v}{d} \right) \int_0^\infty f \left( iy + \frac{v}{d} \right) y^{1-s+\left( \frac{k}{2} - \frac{1}{2} \right)} \frac{dy}{y} \] \hfill (4.30)

where \( \xi = (\gamma, j(\gamma, z)) \) and \( \gamma = \left( \frac{e}{4v d} \right) \). Thus for \( 2 \nmid d \),

\[ \Lambda \left( f, \frac{u}{d}, s \right) = (2d)^{1-2s} \varepsilon_{d}^{-k} \left( \frac{v}{d} \right) \Lambda \left( f, \frac{v}{d}, 1-s \right) \] \hfill (4.31)

which completes our proof.

We now apply our functional equations to the double sum

\[ \sum_{d \mid r^2} \sum_{(u,d)=1 \atop \text{u mod } d} \Lambda \left( f, \frac{u}{d}, s \right) \] \hfill (4.32)

from (4.6) in order to get an asymptotic bound for \( D_r(s) \) at \( \Re(s) < 0 \) in terms of \( r \). We first split the sum into appropriate parts.
\[
\sum_{d \mid r^2} \sum_{(u,d)=1 \atop u \mod d} \Lambda(f, \frac{u}{d}, s) = \sum_{d \mid r^2} \sum_{(u,d)=1 \atop d \not| r^2} \Lambda(f, \frac{u}{d}, s) + \sum_{d \mid r^2} \sum_{(u,d)=1 \atop 2 \mid d \not| u \mod d} \Lambda(f, \frac{u}{d}, s) + \sum_{d \mid r^2} \sum_{(u,d)=1 \atop d \mid u \mod d} \Lambda(f, \frac{u}{d}, s). \quad (4.33)
\]

In this expression, \(d\) can be assumed cube-free, since \(r\) can be taken to be square-free and \(d\) ranges over \(d \mid r^2\).

Now we estimate this sum for \(s\) slightly to the left of the line \(\Re(s) = 0\). From (4.5) we have
\[
\Lambda(f^*, \frac{v}{d}, 1 - (\varepsilon + i\tau)) = O(\varepsilon, f((1 + |\tau|)^{k/4} + \varepsilon e^{-\pi |\tau|}), \quad (4.34)
\]
where the implied constant is uniform over all \(v\) and \(d\), but is dependent on \(\varepsilon\) and \(f\). Using this along with Lemma 4.3 and (4.33), we get that, for \(\varepsilon > 0\),
\[
\sum_{d \mid r^2} \sum_{(u,d)=1 \atop u \mod d} \Lambda(f, \frac{u}{d}, -\varepsilon + i\tau) \ll \varepsilon, f(1 + |\tau|)^{1+2\varepsilon} + 5\varepsilon. \quad (4.35)
\]

Thus using this estimate in (4.6),
\[
D_r(3/4 + \varepsilon + i\tau) \ll \varepsilon, f(1 + |\tau|)^{1+4\varepsilon}. \quad (4.37)
\]
Using this along with Lemma 4.2, a Phragmén-Lindelöf argument tells us that
\[
D_r(3/4 + \varepsilon + i\tau) \ll \varepsilon, f, \tau 1/\tau^{1+4\varepsilon}
\]
which, when put to use in (4.1), provides
\[
M(s) = \sum_{\text{square-free}} \frac{a(t)}{t^s} = \sum_{r=1}^{\infty} \mu(r) D_r(s) \ll \varepsilon, f, \tau \sum_{r=1}^{\infty} \frac{1}{r^{1+4\varepsilon}} < \infty, \quad (4.38)
\]
where \(s = \sigma + i\tau\) with \(\sigma > 3/4 + \varepsilon\). Therefore, we have proven the following:

**Proposition 4.4.** The series
\[
M(s) = \sum_t \frac{a(t)}{t^s} \quad (4.39)
\]
converges in the half plane \(\Re(s) > \frac{3}{4}\) and also has only polynomial growth in \(\Im(s)\) in the vertical strips in that region.

This was the desired pole-free region to prove Theorem 1.1.
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