VANISHING OF THE CYCLIC COHOMOLOGY OF INFINITE VON NEUMANN ALGEBRAS

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Abstract. We prove that if $A$ is an infinite von Neumann algebra (i. e., the identity can be decomposed as a sum of a sequence of pairwise disjoint projections, all equivalent to the identity) then the cyclic cohomology of $A$ vanishes. We show that the method of the proof applies to certain algebras of infinite matrices.

1. Introduction

Cyclic cohomology is a theory introduced by Alain Connes in [1] with the intent to do the analogue of de Rham cohomology in the context of noncommutative operator algebras (also independently discovered by B. Tsygan [3] and J.-L. Loday and D. Quillen [4] in the context of homology of Lie algebras).

In the paper [1] Part II, pp 310-360], A. Connes develops the theory from scratch, and the result we prove here was an effort to make precise the idea sketched in the end of page 319 and the beginning of the page 320, in which he indicates the proof of this fact for the algebra of infinite matrices introduced in [2]. This we do explicitly in Corollary 2 below. More precisely, we show by an elementary argument that all the cyclic cohomology groups of an infinite von Neumann algebra vanish. We recall that an infinite von Neumann algebra is one which have two (and therefore a pairwise disjoint sequence) disjoint projections whose sum is the identity.

This implies in particular that if $H$ is an infinite dimensional Hilbert space, then the cyclic cohomology of the algebra $L(H)$ of bounded operators in $H$ has trivial cyclic cohomology (as observed by M. Wodzicki in [7], in the context of homology). On the other hand, since $H^0_\lambda(A)$ is the set of tracial functionals in $A$, if the algebra $A$ has a finite trace, then the cohomology does not vanish.

Since a von Neumann algebra $M$ can be decomposed uniquely as a direct sum of subalgebras $M_I$, $M_{II_1}$, $M_{II_\infty}$ and $M_{III}$, of types $I$, $II_1$, $II_\infty$, and $III$.
II_1, II_\infty, and III, respectively (see [5, chapter 2]), and M_{II_\infty}, M_{III}
and the infinite part of M_I are infinite, then the cyclic cohomology
of M is determined by the summands M_{II_i} (for which there are finite
tracial states) and the finite dimensional parts of M_I (which are Morita
equivalent to \mathbb{C}) and so they have nontrivial cyclic cohomology.

In the following section we prove the main result and derive some
corollaries. We assume that the reader is familiar with [1, Part II, pp
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2. THE MAIN RESULT

The main ingredient we use is the following result by A. Connes.

**FACT 1.** Let \( A \) be a unital \( C \)-algebra. Suppose that there is a homo-
morphism \( \rho : A \to A \) and a matrix \( V \in M_2(A) \), such that
\[
V \begin{bmatrix} a & 0 \\ 0 & \rho(a) \end{bmatrix} V^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \rho(a) \end{bmatrix}.
\]
Then the cyclic cohomology of \( A \) vanishes. (See [1, Proposition 5.2,
page 319].)

Now, some basic facts about von Neumann algebras.

**FACT 2.** If \( A \) is an infinite von Neumann algebra, then there are par-
tial isometries \( w_n, v_n \in A, n \in \mathbb{N} \), satisfying \( p_n = v_n w_n \) is a projection,
\( w_n v_n = 1 \), \( p_n p_m = 0 \) if \( n \neq m \), and \( 1 = \sum_n p_n \). (See [5, Proposition
2.2.4, page 84].)

We list some properties of the elements \( v_n, w_n \in A \):

1. \( w_n p_n = w_n \),
2. \( p_n v_n = v_n \),
3. \( w_n v_m = 0 \), if \( n \neq m \),
4. we can assume that \( \|v_n\|, \|w_n\| \leq 1 \).

**Proposition 1.** If \( A \) is an infinite von Neumann algebra, then the
cyclic cohomology of \( A \) vanishes.

**Proof.** Let \( \rho : A \to A, \rho(a) = \sum_{i \geq 0} v_{2i+1} a w_{2i+1} \)

Let \( \theta_1, \theta_2, \theta_3 \in A \) be the elements
\[
\theta_1 = \sum_{i \geq 1} p_i = 1 - p_0,
\]
\[
\theta_2 = p_0 + \sum_{i \geq 1} (v_{2i} w_{2i-1} + v_{2i-1} w_{2i})
\]
\[ \theta_3 = \sum_{i \geq 0} (v_{2i}w_{2i+1} + v_{2i+1}w_{2i}) \]

All these sums converge in the strong operator topology and \( \| \rho(a) \| \leq \| a \| \).

Observe that if \( A \) acts as an algebra of operators in the Hilbert space \( H \), then \( \theta_2 \) maps \( p_0 H \) onto itself and maps \( p_{2i}H \) and \( p_{2i+1}H \) (\( i > 0 \)) onto each other. Also, \( \theta_3 \) maps \( p_{2i}H \) and \( p_{2i+1}H \) (\( i \geq 0 \)) onto each other.

We have the following relations: \( \theta_1^2 = 1 - p_0 \), \( \theta_2^2 = \theta_3^2 = 1 \), \( w_0 \theta_1 = \theta_1 v_0 = 0 \), and

\[
\begin{align*}
\theta_1 \rho(a) \theta_1 &= \rho(a) \\
\theta_2 (v_0 aw_0 + \rho(a)) \theta_2 &= \sum_{i \geq 0} v_{2i} a w_{2i} \\
\theta_3 ( \sum_{i \geq 0} v_{2i} a w_{2i} ) \theta_3 &= \rho(a)
\end{align*}
\]

We define the matrices \( X, Y, Z \in M_2(A) \) as follows.

\[
X = \begin{pmatrix} 0 & w_0 \\ v_0 & \theta_1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & \theta_3 \end{pmatrix}.
\]

Then \( X, Y \) and \( Z \) are invertible and \( X^{-1} = X, Y^{-1} = Y, Z^{-1} = Z \), and if \( V = ZYX \), then \( V^{-1} = XYZ \), and

\[
V \begin{bmatrix} a & 0 \\ 0 & \rho(a) \end{bmatrix} V^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \rho(a) \end{bmatrix}.
\]

By Fact 1, this implies the vanishing of the cyclic cohomology of \( A \). \( \square \)

As corollaries to the proof, we have:

**Corollary 1.** The quotient algebra of an infinite von Neumann algebra by a norm closed ideal has vanishing cyclic cohomology. In particular, the Calkin algebra \( Q = B(H)/K(H) \) has zero cyclic cohomology, for \( H \) an infinite dimensional Hilbert space.

**Proof.** If \( u, v \in A \), \( uv = 1 \) and \( vu = p \), a projection, and \( I \) a proper norm closed ideal, then \( p \notin I \). Therefore the computations done in the proof of the proposition goes through the quotient. \( \square \)

Let \( C \) be the algebra introduced in \[2\], of infinite matrices \( (a_{i,j})_{i,j \in \mathbb{N}} \) such that

- the set \( \{a_{i,j} : i, j \in \mathbb{N}\} \) is finite,
- the number of nonzero entries in each line and each column is bounded.
Let $D$ be the algebra of infinite matrices $(a_{i,j})$ such that the number of nonzero entries in each line and each column is finite.

**Corollary 2.** The cyclic cohomology groups of $C$ and of $D$ vanish.

**Proof.** Repeat the proof of the proposition but now with the sums defined componentwise (therefore finite in each component). We only need to define the elements $v_n$ and $w_n$. Let $I_k \subset \mathbb{N}$, $k \in \mathbb{N}$, be pairwise disjoint infinite sets, such that $\mathbb{N} = \bigcup_{k \geq 0} I_k$. Enumerate each $I_k$ as $I_k = \{i_{k,n} : n \geq 0\}$ (for instance in increasing order). We define the matrices $v_n = (a_{i,j}^n)_{i,j \in \mathbb{N}}$ (and $w_n$ as the transpose of $v_n$) as

$$a_{i,j}^n = \begin{cases} 1, & \text{if } i \in I_n \text{ and } i = i_{n,j}; \\ 0, & \text{otherwise}. \end{cases}$$

Then the entry $i,j$ of $w_n v_n$ is $\sum_{k \geq 0} a_{i,k}^n a_{k,j}^n$, which is 1 if $i = j$ and 0 otherwise, that is, $w_n v_n = 1$. On the other hand, the entry $i,j$ of $p_n = v_n w_n$ is $\sum_{k \geq 0} a_{i,k}^n a_{j,k}^n$, which is 1 if $i = j \in I_n$ (for a unique $k \geq 0$) and 0 otherwise. Clearly $1 = \sum_n p_n$, and all these matrices belong to the algebras $C$ and $D$. \qed

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