Buckling and Vibration Behavior of Composite Beam due to Axially Varying In-plane Loads

Sangram Anil Kanade¹, Lenin Babu Mailan Chinnapandi¹,²,³ P Jeyaraj³ and Jeyanthi Subramanian¹
¹School of Mechanical Engineering, Vellore Institute of Technology, Chennai, India
²Electric Vehicles-Incubation, Testing and Research Centre, Vellore Institute of Technology, Chennai, India
³Department of Mechanical Engineering, NIT Surathkal, Mangalore
Corresponding author’s e-mail address: lenin.babu@vit.ac.in

Abstract:
In this work buckling and vibration behavior of composite beam is analyzed based on the shear and normal deformable beam theory (SNDBT) by using Ritz method. The effects of aspect ratios, boundary conditions, fiber angles on the buckling and vibration subjected to non-uniform axial loads studied in detail. The obtained numerical results in terms of dimensionless fundamental frequencies and dimensionless first critical buckling loads are compared with the results from previous research for convergence studies.

Keywords: shear and normal deformable beam theory; Ritz method; Dimensionless critical buckling load

1. Introduction:
Composite laminates have been used increasingly in a variety of industrial areas due to their high stiffness and strength-to-weight ratios, long fatigue life, resistance to electrochemical corrosion, and other superior material properties of composites. A true understanding of their structural behaviour is required, such as the deflections, buckling loads and modal characteristics, the through thickness distributions of stresses and strains, the large deflection behaviour and, of extreme importance for obtaining strong, reliable multi-layered structures, the failure characteristics[1]. Therefore, the researchers have developed various theories to present the bending, free vibration and buckling behaviours of LCBs. A review of these theories can be found in [2–4].

In some real applications, the beam structure is subjected to non-uniform in-plane loads. For example, the load exerted on the aircraft wings, load on the stiffened plate in the ship structures, or on the slabs of a multi-storey building by the adjoining structures usually are non-uniform in-plane loads, Panda and Ramachandra [6]. So, the behavior of the structures under the action of non-uniform in-plane compressive loading and shear loading is important in aircraft, civil and ship-building industries. Kang and Leissa [7] developed exact solutions to study buckling stability of rectangular plates under linearly varying in-plane loading. Panda and Ramachandra [6] investigated the effect of non-uniform in-plane loads on buckling stability of rectangular higher order shear deformation plates. Kim et al. [8] studied coupled stability analysis of thin-walled composite beams with closed cross-section subjected to various varying forces. Farajpour et al. [6].

The laminated plate theories are essential to provide accurate analysis of laminated composite plates, and a variety of laminated plate theories have been developed and reported in a large amount of literatures. It should be noted that the higher order beam theories (HBTs) which assume a higher-order variation of the axial displacement through the height of the beam, predict the structural behavior of the LCBs more accurate than classical beam theory (CBT) and the first-order shear deformable beam theories (FSDTs). The CBT neglects the shear deformation effect, underestimates the deflections and overestimates the natural frequencies and critical buckling loads. Moreover, the FSDTs require a shear correction factor which depends on the BC, loading condition and geometrical properties. Eventually, the use of HBTs has been increasing due to their advantages Over CBT and FSDTs.

According to literature and author’s knowledge no researchers have attempted to study the buckling loads and their mode-shapes of composite laminated beam under varying axial load by using unified beam theories. The present study is intended to fill this gap in the literature. Six distributions of axial load are proposed, which are constant load, Linear Load-zero from left side, Linear Load-zero from right side, Parabolic Load-zero from left side, Parabolic Load-zero from right side, and Symmetric Parabolic Load. Unified higher order beam theories are proposed to consider all slenderness ratios of the beam ranging from thin, moderated thick, to thick cases. The differential quadrature method (DQM) is exploited to discretize the space and convert the governing differential equations into a set of algebraic equations that will be solved as an eigenvalue problem[10].
Applied new shape functions using the Ritz method for the thermomechanical buckling analysis of LCBs. It is noteworthy that the elastic buckling may also occur under the non-uniform variable in plane loading conditions. Some reported works can be found for the elastic buckling of isotropic columns with non-uniform loading conditions [11–14]. This work aims to study the buckling behavior of isotropic, laminated composite and sandwich beams based on a SNDBT [15-18] by using Ritz method. In this paper objective is to check effects and influence of BCs, aspect ratios, fiber angles on the buckling response of isotropic, laminated composite beams subjected to axially variable in-plane loads.

The following article is organized as follows: Section 2 focusing on problem formulation, which includes kinematics assumptions of displacement field, constitutive equations of composite laminated beams, axial load functions, and equilibrium equations. Section 3 presents the solution strategy and discretization method of the beam using Ritz method. Section 4 is devoted to validation and parametric studies. These results preset effects of beam theories, type of loading, Aspect ratio, on critical buckling loads and corresponding mode-shapes. Main remarks and conclusion points are highlighted and summarized in Section 2.

2. Kinematics, stress and strain relations :

The displacement field of the SNDBT to be used within this study is given with the following equations [15-18]:

\[ U(x, z) = u(x) - z \frac{dw_b(x)}{dx} - \frac{az^2}{2} \frac{dw_b(x)}{dx} = u(x) - zw_b(x) - f(z)w_s(x) \]  \hspace{1cm} Eq. 1a

\[ W(x, z) = w_b(x) + w_s(x) + \left(1 - \frac{az^2}{2}\right)w_s(x) = w_b(x) + w_s(x) + g(z)w_s(z) \]  \hspace{1cm} Eq. 1b

where \( u = \) axial displacement, \( w_b = \) bending components of the transverse displacement, \( w_s = \) shear components of the transverse displacement, \( w_n = \) an unknown which is defined to obtain the normal deformation effect.

When \( g(z) \) is set to zero, the present SNDBT is reduced to a HBT. With the linearity assumption for the deformations, the kinematic relations associated with the displacement field can be given by:

\[ \varepsilon_{xx} = \frac{du}{dx} = u - zw_b - f(z)w_s \]  \hspace{1cm} Eq. 2a

\[ \varepsilon_{zz} = \frac{dw_b}{dx} = g(z) \]  \hspace{1cm} Eq. 2b

\[ \gamma_{xz} = \frac{dw_s}{dx} + \frac{du}{dx} = g(z)(w_s - w_x) \]  \hspace{1cm} Eq. 2c

For each orthotropic lamina, the stress-strain relation is given by:

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{zz} \\
\sigma_{xz}
\end{bmatrix}^b =
\begin{bmatrix}
Q_{11} & Q_{13} & 0 \\
Q_{13} & Q_{33} & 0 \\
0 & 0 & Q_{55}
\end{bmatrix}^b
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{zz} \\
\gamma_{xz}
\end{bmatrix}
\]  \hspace{1cm} Eq. 3

where \((xx, zz, xz)\) and \((xx, zz, xz)\) are the components of the stress and strain tensors, respectively. \(Q_i\)'s are reduced transformed stiffness and can be given by:

\[ Q_{11} = C_{11}\cos^4\theta + 2(C_{12} + 2C_{66})\cos^2\theta\sin^2\theta + C_{66}\sin^4\theta \]

\[ Q_{13} = C_{23}\sin^2\theta + C_{13}\cos^2\theta \]

\[ Q_{33} = C_{33} \]

\[ Q_{55} = C_{55}\sin^2\theta + C_{44}\cos^2\theta \]  \hspace{1cm} Eq. 4

Where,

\[ C_{11} = \frac{E_1}{\Delta}; \quad C_{12} = \frac{E_1\nu_{21}}{\Delta}; \quad C_{22} = \frac{E_2(1-\nu_{23}\nu_{13})}{\Delta}; \quad C_{13} = \frac{E_1\nu_{13}}{\Delta}; \]

\[ C_{23} = \frac{E_2\nu_{21}(1-\nu_{23})}{\Delta}; \quad C_{33} = \frac{E_3(1-\nu_{31}\nu_{13})}{\Delta}; \quad C_{66} = G_{12}; \quad C_{44} = G_{13}; \quad C_{55} = G_{23}; \]
\[ \Delta = 1 - \theta_{12}\theta_{21} - \theta_{13}\theta_{31} \]

E1, E2, G12, G13, G23, 12 and 21 are the six independent material properties.

| Problem | Structure | Material properties |
|---------|-----------|---------------------|
| LCB     | [15,19]   | E1/E2 = Open; E3 = E2; G12 = G13 = 0.5E2; G23 = 0.2 E2; \( \theta_{12} = \theta_{13} = \theta_{23} = 0.25 \) |

3. Laminated composite beam:

A laminated composite beam (LCB) shown in Fig. 1 has the top and bottom layer thickness (\( h_1 \)) and core thickness (\( h_2 \)). The equations given above for laminated composite beam is used in the formulation of LCB. The main difference between laminated composite beam and laminated sandwich beam formulations are the material properties and the thickness of the layers. In the laminated composite beam formulation, all of the layers have the same thickness and material properties, but the core and face layers have different material properties and different thicknesses in the LCB formulation. It is noteworthy that the Zig-Zag theories where the discontinuity of the displacement at each layer interface is captured, can provide more accurate results than the classical higher order shear and normal deformable beam theories, especially for the sandwich or layered composite laminates.

![Figure 1: Geometry of a laminated composite beam](image)

**Variational formulation**

The potential energy of a shear and normal deformable beam can be written as:

\[
U = \frac{1}{2} \int_0^L \left( \sigma_{xx} \varepsilon_{xx} + \sigma_{zz} \varepsilon_{zz} + \sigma_{xy} \varepsilon_{xy} \right) dV \quad \text{Eq. 5}
\]

where \( V \) is the volume of the beam. The potential energy of the external axially variable in-plane load shown in Fig. 2 is given by:
Here, $N_0^x(x) = \text{external axially variable in-plane load. The following polynomial axial load distribution is chosen in the present study:}$

$$N_0^x(x) = N_0 \left[ \alpha_1 \left( x + \frac{L}{2} \right)^2 + \alpha_2 \left( x + \frac{L}{2} \right) + \alpha_3 \right] \quad \text{.................................Eq. 7a}$$

$$N_0^x(x) = N_0 P(x) \quad \text{.........................................................................................Eq. 7b}$$

Positive load is assumed compressive. The distributed external loads obtained based on the different values of $i (i = 1, 2, 3)$ given in Table 1 are used for the numerical computations.

| Load Type               | Load Symbol | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ |
|-------------------------|-------------|------------|------------|------------|
| Constant Load           | $N_0^1$     | 0          | 0          | 1          |
| Linear Load-zero from left side | $N_0^2$     | 0          | 2          | 0          |
| Linear Load-zero from right side | $N_0^3$    | 0          | -2         | 2          |
| Parabolic Load-zero from left side | $N_0^4$     | 3          | 0          | 0          |
| Parabolic Load-zero from right side | $N_0^5$     | 3          | -6         | 3          |
| Symmetric Parabolic Load | $N_0^6$     | -6         | 6          | 0          |

| Table 1: Coefficients of external axial distributed loads (Karamanli and Aydogdu [20]) |

The integral of the each axially variable in-plane load through the length of the beam is equal to integral of the uniformly distributed in-plane load. This may allow making the comparisons in a fair ground. Using Eqs. (1)–(7), the total potential energy can be obtained as follows:

$$\pi = u + V$$

Dimensionless critical buckling load (DCBLs) of isotropic beams subjected to various axially varying in plane loads for different BCs and fibre angles.
The stiffness coefficients of the present SNDBT are defined as:

\[(A, B, B_s, D, D_s, H) = \int_{-h/2}^{h/2} Q_{11} b(1, z, f, z^2, f, x, g^2) dz \] ................................................................. Eq. 9a

\[A_s = \int_{-h/2}^{h/2} Q_{55} g^2 dz \] ................................................................. Eq. 9b

\[(X, Y, Y_s) = \int_{-h/2}^{h/2} Q_{13} b g(1, z, f) dz \] ................................................................. Eq. 9c

\[Z = \int_{-h/2}^{h/2} Q_{33} b g \] ................................................................. Eq. 9d

where \(A = \) the classical extensional
\(B = \) bending-stretching coupling
\(D = \) bending stiffness coefficients
\(D_s = \) shear bending stiffness coefficients
\(Z = \) normal bending stiffness coefficients
\(B_s = \) stretching-shear bending coupling coefficients
\(X = \) the and stretching-normal bending stiffness coefficients
\(Y = \) shear-bending coupling,
\(Y_s = \) bending-normal bending coupling
\(A_s = \) is the shear stiffness coefficient.

4. Ritz procedure
Since the differential equations derived for the buckling analysis of LCBs consist of variable coefficients, the exact solution cannot be obtained. Various numerical techniques can be employed for the solution of this complicated problem like FEM, DQM, meshless methods, Ritz method etc. Among them, the Ritz procedure has been found accurate and simple by the researchers [21-22]. In this study, the Ritz procedure is employed for the solution of the problem. The kinematic boundary conditions for the studied problem are given in Table 2.

| BC | \(x = -L/2\) | \(x = L/2\) |
|-----|----------------|----------------|
| SS  | \(u = 0, w_b = 0, w_r = 0, w_z = 0\) | \(w_b = 0, w_r = 0, w_z = 0\) |
| CS  | \(u = 0, w_b = 0, w_r = 0, w_z = 0\) | \(u = 0, w_b = 0, w_r = 0, w_z = 0\) |
| CC  | \(u = 0, w_b = 0, w_r = 0, w_z = 0\) | \(u = 0, w_b = 0, w_r = 0, w_z = 0\) |
| CF  | \(u = 0, w_b = 0, w_r = 0, w_z = 0\) | \(u = 0, w_b = 0, w_r = 0, w_z = 0\) |

Table 2: Kinematic boundary conditions used for the numerical computations

| BC | \(P_{1}\) | \(P_{2}\) | \(P_{12}\) | \(P_{22}\) | \(q_{1}\) | \(q_{2}\) | \(q_{12}\) | \(q_{22}\) |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|
| SS  | 1      | 1      | 1      | 1      | 0      | 1      | 1      | 1      |
| CS  | 1      | 2      | 2      | 1      | 0      | 1      | 1      | 1      |
| CC  | 1      | 2      | 2      | 1      | 1      | 2      | 2      | 1      |
| CF  | 1      | 2      | 2      | 1      | 0      | 1      | 0      | 0      |

Table 3: Boundary exponents for various BCs

The following displacement functions \(u(x), w_b (x), w_r (x)\) and \(w_z (x)\) are defined for the present composite beam buckling problem:
where $A_0$, $B_0$, $C_0$ and $D_0$ are undetermined coefficients, $j(x)$, $j(x)$, $j(x)$ and $j(x)$ are the trial functions, $p$ and $q$ ($\xi = u, wb, ws, w2$) are the boundary exponents used to satisfy the BCs given in Table 3. The governing equations for the LCBs under the in-plane loads can be derived by substituting Eqs. (10a)–(10d) into Eq. (8) and then employing the minimum energy principle:

$$\frac{\partial}{\partial \xi} = 0$$

with $q_0$ representing the undetermined coefficients, that leads to,

$$\begin{bmatrix}
    [K_{11}] & [K_{12}] & [K_{13}] & [K_{14}] \\
    [K_{12}]^T & [K_{22}] & [K_{23}] & [K_{24}] \\
    [K_{13}]^T & [K_{23}]^T & [K_{33}] & [K_{34}] \\
    [K_{14}]^T & [K_{24}]^T & [K_{34}]^T & [K_{44}]
\end{bmatrix}
- N_0
\begin{bmatrix}
    [0] & [0] & [0] & [0]
\end{bmatrix}
= \begin{bmatrix}
    [A] \\
    [B] \\
    [C] \\
    [D]
\end{bmatrix}
$$

where is the dimensionless critical buckling load (DCBL). $[K_{ii}]$ and $[M_{ii}]$ are the stiffness and geometry matrices, respectively. The above mentioned matrices are symmetric. The DCBL of the LCBs can be defined by:

$$\lambda = \frac{N_0 d^2}{E \beta h^3}$$

The symmetric matrices are given by:

$$\begin{align*}
K_{11}(i,j) &= A \int_{-L/2}^{L/2} \theta_{ix} \theta_{ix} \, dx \\
K_{12}(i,j) &= -B \int_{-L/2}^{L/2} \theta_{ix} \phi_{ijx} \, dx \\
K_{13}(i,j) &= -B_0 \int_{-L/2}^{L/2} \theta_{ix} \xi_{ijx} \, dx \\
K_{14}(i,j) &= X \int_{-L/2}^{L/2} \theta_{ix} \psi_{ij} \, dx \\
K_{22}(i,j) &= D \int_{-L/2}^{L/2} \phi_{ij} \phi_{ij} \, dx \\
K_{23}(i,j) &= H \int_{-L/2}^{L/2} \psi_{ij} \xi_{ijx} \, dx \\
K_{24}(i,j) &= -Y \int_{-L/2}^{L/2} \psi_{ij} \psi_{ij} \, dx \\
K_{33}(i,j) &= D_0 \int_{-L/2}^{L/2} \xi_{ij} \xi_{ijx} \, dx + A_0 \int_{-L/2}^{L/2} \xi_{ij} \xi_{ijx} \, dx \\
K_{34}(i,j) &= -Y_0 \int_{-L/2}^{L/2} \xi_{ij} \phi_{ijx} \, dx + A_0 \int_{-L/2}^{L/2} \xi_{ij} \phi_{ijx} \, dx \\
K_{44}(i,j) &= Z \int_{-L/2}^{L/2} \psi_{ij} \psi_{ijx} \, dx + A_0 \int_{-L/2}^{L/2} \psi_{ij} \psi_{ijx} \, dx \\
M_{23}(i,j) &= -N_0 \int_{-L/2}^{L/2} P(x) \left( \int_{-L/2}^{L/2} \psi_{ij} \phi_{ijx} \, dx \right) \, dx
\end{align*}$$
$M_{23}(i, j) = -N_0 I_{i-1/2}^{L/2} p(x) \left( I_{i-1/2}^{L/2} P_{i-1/2} x_j dx \right) \text{dx} \quad \text{Eq. 14l}$

$M_{33}(i, j) = -N_0 I_{i-1/2}^{L/2} p(x) \left( I_{i-1/2}^{L/2} P_{i-1/2} x_j dx \right) \text{dx} \quad \text{Eq. 14m}$

**5. Buckling Modes**

This section is devoted to the buckling mode-shapes analysis and its effect by loading type, fiber angle and boundary conditions. Influences of load functions and boundary conditions on buckling mode-shapes of laminated composite beams are presented through this section. To investigate the characteristics of buckling mode shapes corresponding to different load types, the first buckling mode shapes of a SS, CC, CS composite laminated symmetric beams are strategized. Effects of the 6 loading types ($N_{11}^1, N_{21}^1, N_{12}^1, N_{13}^1, N_{22}^1, N_{33}^1$) on the first buckling mode-shapes on SS, CC, CF, CS and fiber angle ($0, 15, 30, 45, 60, 75, 90$) are presented. It can be concluded that, load type has a significant effect on the buckling mode shapes. Further, it is observed that for all boundary conditions, the first critical buckling is high in CC and minimum in CF. It is important to note that, referring to all previous results, values of critical buckling for different loading types are ordered in the same sequence with highest buckling load.

### $N_{11}^1$ - alpha1 = 0, alpha2 = 0, alpha3 = 1

| Angle: 0/0/0 | Angle: 0/15/0 |
|---------------|---------------|
| BL 0          | 0.25          |
| SS 46.4351    | 40.5255       |
| CC 105.2815   | 92.1837       |
| CF 16.5445    | 14.3355       |
| CS 72.5537    | 63.2658       |
| Angle: 30/0/0 | Angle: 0/45/0 |
| BL 0          | 0.25          |
| SS 46.0907    | 40.2250       |
| CC 104.4993   | 91.4988       |
| CF 16.4217    | 14.2291       |
| CS 72.0149    | 62.7960       |
| Angle: 60/0/0 | Angle: 0/75/0 |
| BL 0          | 0.25          |
| SS 45.7124    | 39.8949       |
| CC 103.6343   | 90.7416       |
| CF 16.2864    | 14.1119       |
| CS 71.4200    | 62.2773       |
| Angle: 90/0/0 | Angle: 0/90/0 |
| BL 0          | 0.25          |
| SS 45.6745    | 39.8618       |
| CC 103.5442   | 90.6627       |
| CF 16.2726    | 14.0999       |
| CS 71.3585    | 62.2238       |

### $N_{22}^2$ - alpha1 = 0, alpha2 = 2, alpha3 = 0

| Angle: 0/0/0 | Angle: 0/15/0 |
|---------------|---------------|
| BL 0          | 0.25          |
| SS 46.4351    | 40.4331       |
| CC 105.2815   | 92.0541       |
| CF 16.5445    | 14.3486       |
| CS 72.5537    | 63.1913       |
| Angle: 30/0/0 | Angle: 0/45/0 |
| BL 0          | 0.25          |
| SS 46.0907    | 40.0893       |
| CC 104.4993   | 91.3702       |
| | Angle: 0/60/0 | Angle: 0/75/0 |
|---|---|---|
| **BL** | 0 | 0.25 | 0.5 | 0.75 | 0.99 |
| **SS** | 45.7124 | 39.7708 | 32.8486 | 23.2682 | 4.6857 |
| **CC** | 103.6343 | 90.5393 | 75.0178 | 53.8095 | 10.9361 |
| **CF** | 16.2864 | 14.1376 | 11.5286 | 8.1567 | 1.6323 |
| **CS** | 71.4200 | 62.1674 | 51.2382 | 36.5420 | 7.3768 |

| **N_{\alpha}^2:** alpha1 = 0, alpha2 = -2, alpha3 =2 |
|---|---|---|---|---|---|
| **BL** | 0 | 0.25 | 0.5 | 0.75 | 0.99 |
| **SS** | 46.4351 | 40.7162 | 33.6911 | 24.1681 | 4.9061 |
| **CC** | 105.2815 | 92.7290 | 77.1790 | 55.7927 | 11.4390 |
| **CF** | 16.5445 | 14.3583 | 11.7495 | 8.3275 | 1.6694 |
| **CS** | 72.5537 | 63.6423 | 52.7137 | 37.8846 | 7.7136 |

| **N_{\alpha}^2:** alpha1 = 3, alpha2 = 0, alpha3 = 0 |
|---|---|---|---|---|---|
| **BL** | 0 | 0.25 | 0.5 | 0.75 | 0.99 |
| **SS** | 46.4351 | 40.3848 | 32.8486 | 23.2682 | 4.6675 |
| **CC** | 103.5442 | 90.4606 | 74.5886 | 53.3278 | 10.7961 |
| **CF** | 16.2726 | 14.1256 | 11.5622 | 8.1968 | 1.6437 |
| **CS** | 71.3585 | 62.1140 | 50.9992 | 36.2816 | 7.3034 |

8
|    | Angle: 0/60/0 | Angle: 0/75/0 |
|----|--------------|--------------|
| BL | 0            | 0.25         |
|    | 0.25         | 0.5          |
|    | 0.75         | 0.75         |
|    | 0.99         | 0.99         |
| SS | 45.7124      | 39.7708      |
| CC | 103.6343     | 90.5393      |
| CF | 16.2864      | 14.1376      |
| CS | 71.4200      | 62.1674      |

| Angle: 0/90/0 | Angle: 0/15/0 |
|--------------|--------------|
| BL | 0            | 0.25         |
|    | 0.25         | 0.5          |
|    | 0.75         | 0.75         |
|    | 0.99         | 0.99         |
| SS | 45.6745      | 39.7273      |
| CC | 103.5442     | 90.4606      |
| CF | 16.2726      | 14.1256      |
| CS | 71.3585      | 62.1140      |

| N2, alpha1 = 3, alpha2 = -6, alpha3 = 3 |
|-----------------------------------------|
| Angle: 0/0/0 | Angle: 0/45/0 |
|--------------|--------------|
| BL | 0            | 0.25         |
|    | 0.25         | 0.5          |
|    | 0.75         | 0.75         |
|    | 0.99         | 0.99         |
| SS | 46.44        | 45.05        |
| CC | 105.28       | 104.35       |
| CF | 16.54        | 14.38        |
| CS | 72.55        | 71.56        |

| Angle: 0/30/0 | Angle: 0/75/0 |
|--------------|--------------|
| BL | 0            | 0.25         |
|    | 0.25         | 0.5          |
|    | 0.75         | 0.75         |
|    | 0.99         | 0.99         |
| SS | 46.09        | 44.72        |
| CC | 104.50       | 103.57       |
| CF | 16.42        | 14.27        |
| CS | 72.01        | 71.03        |

| Angle: 0/60/0 | Angle: 0/15/0 |
|--------------|--------------|
| BL | 0            | 0.25         |
|    | 0.25         | 0.5          |
|    | 0.75         | 0.75         |
|    | 0.99         | 0.99         |
| SS | 45.7124      | 45.05        |
| CC | 103.6343     | 102.716        |
| CF | 16.2864      | 14.152       |
| CS | 71.4200      | 70.443       |

| N2, alpha1 = -6, alpha2 = 6, alpha3 = 0 |
|-----------------------------------------|
| Angle: 0/0/0 | Angle: 0/15/0 |
|--------------|--------------|
| BL | 0            | 0.25         |
|    | 0.25         | 0.5          |
|    | 0.75         | 0.75         |
|    | 0.99         | 0.99         |
| SS | 46.4351      | 40.4135      |
| CC | 105.2815     | 92.0173      |
| CF | 16.5445      | 14.3539      |
| CS | 72.5537      | 63.1723      |
### Table 5: Buckling Modes

| Angle: 0/30/0          | Angle: 0/45/0          | Angle: 0/60/0          | Angle: 0/75/0          | Angle: 0/90/0          |
|------------------------|------------------------|------------------------|------------------------|------------------------|
| BL                     | BL                     | BL                     | BL                     | BL                     |
| 0                      | 0.25                   | 0.5                    | 0.75                   | 0.99                   |
| SS                     | 46.0907                | 40.1138                | 32.9301                | 23.4235                | 4.7141                 |
| CC                     | 104.4993               | 91.3336                | 75.3447                | 53.9021                | 10.9201                |
| CF                     | 16.4217                | 14.2474                | 11.6550                | 8.2578                 | 1.6549                 |
| CS                     | 72.0149                | 62.7031                | 51.4990                | 36.6521                | 7.3812                 |
|                        |                        |                        |                        |                        |                        |
| SS                     | 45.8505                | 39.9047                | 32.7585                | 23.3014                | 4.6896                 |
| CC                     | 103.9514               | 90.8547                | 74.9497                | 53.6196                | 10.8629                |
| CF                     | 16.3359                | 14.1729                | 11.5941                | 8.2147                 | 1.6463                 |
| CS                     | 71.6378                | 62.3748                | 51.2294                | 36.4603                | 7.3425                 |
|                        |                        |                        |                        |                        |                        |
| SS                     | 45.8505                | 39.9047                | 32.7585                | 23.3014                | 4.6896                 |
| CC                     | 103.9514               | 90.8547                | 74.9497                | 53.6196                | 10.8629                |
| CF                     | 16.3359                | 14.1729                | 11.5941                | 8.2147                 | 1.6463                 |
| CS                     | 71.6378                | 62.3748                | 51.2294                | 36.4603                | 7.3425                 |
|                        |                        |                        |                        |                        |                        |
| SS                     | 45.8505                | 39.9047                | 32.7585                | 23.3014                | 4.6896                 |
| CC                     | 103.9514               | 90.8547                | 74.9497                | 53.6196                | 10.8629                |
| CF                     | 16.3359                | 14.1729                | 11.5941                | 8.2147                 | 1.6463                 |
| CS                     | 71.6378                | 62.3748                | 51.2294                | 36.4603                | 7.3425                 |

**Graphs:**

- **Graph 1:** Frequency vs. Buckling Load for angle 0/30/0
- **Graph 2:** Frequency vs. Buckling Load for angle 0/75/0

**Legend:**
- SS
- CC
- CF
- CS

**Equation:**
- \( \alpha_1 = 3, \alpha_2 = 0, \alpha_3 = 0 \)

**Note:**
- The data represents the buckling modes under different angles and frequencies for a given buckling load.
6. Numerical results

This section is concerned with DCBL and effects of in-plane distribution functions, aspect ratio, fiber orientation, boundary conditions, on SNDBT on LCBs structure.

**Figure 3:** Effects of the fiber angles on the variation of the CBLs of the symmetric \([0^\circ/\theta/0^\circ]\) LCBs for various BCs \(L/h = 100\)

**Figure 4:** The effects of the aspect ratio on the variation of the DCBLs of the symmetric \([0^\circ/90^\circ/0^\circ]\) LCSBs

---

11
The effect of shear deformation function on the critical buckling load of composite beam with different boundary conditions and fiber angle will be investigated firstly. After that, shear normal deformation beam theory (SNDBT), will be exploited in investigation of other effects. The buckling analysis of the symmetric \([0°/θ/0°]\) LCBs under the variable in-plane loads are performed in this section. It is assumed that all the laminate of the composite beams has the same material properties and thickness. the DCBLs of the symmetric \([0°/θ/0°]\) LCBs are given for various axially variable in plane loads, BCs, fiber angles \((θ)\) and aspect ratios. It can be seen that the minimum DCBL is always obtained for the LCBs subjected to the thickness. The DCBLs of the symmetric \([0°/θ/0°]\) LCBs are given for various axially variable in plane loads, BCs, fiber angles \((θ)\) and aspect ratios. It can be seen that the minimum DCBL is always obtained for the LCBs subjected to the thickness. It is assumed that all the laminate of the composite beams has the same material properties and thickness. the DCBLs of the symmetric \([0°/θ/0°]\) LCBs are given for various axially variable in plane loads, BCs, fiber angles \((θ)\) and aspect ratios. It can be seen that the minimum DCBL is always obtained for the LCBs subjected to the thickness. Moreover, the effect of shear deformation function on the critical buckling load of composite beam with different boundary conditions is investigated. The buckling analysis of composite laminated beams are investigated under distributed in-plane axial load described by six functions. Unified higher order shear deformation theories are proposed to include the shear effects, extension bending, and rigidity of the beam structure. Ritz method is exploited to solve the govern equilibrium equations and derive the critical buckling loads and their mode-shapes. The most finding are summarized as:

- The distributed axial in-plane load tends to distort the mode-shapes and deviate them from symmetry.
- CC is the most boundary condition significant for distortion of its mode rather than other boundaries under distributed axial load.
- Aspect ratio and fiber orientations have significant effects on the DCBLs.
- It is seen that \(N_2\) and \(N_3\) types load distributions give higher critical buckling loads.
- The type of axially variable in-plane load is significantly affecting the DCBL of the beam structures. The fiber angle value \((θ)\) is important for the DCBL. Moreover, its effect on the DCBL is more pronounced especially for CS and CC symmetric LCBs.
- The buckling mode shapes of LCBs are significantly affected by the type of axially variable in-plane load for all BCs.
- The DCBL of the LCSBs depend on not only the aspect ratio but also the BCs and fiber orientation angles. The effect of the thickness stretching on the buckling mode shapes may vary depending on the type of the axially variable in-plane load.

7. Conclusion:

In this article, the static critical buckling loads and mod-shapes of composite laminated beams are investigated under distributed in-plane axial load described by six functions. Unified higher order shear deformation theories are proposed to include the shear effects, extension bending, and rigidity of the beam structure. Ritz method is exploited to solve the govern equilibrium equations and derive the critical buckling loads and their mode-shapes.

The most finding are summarized as:

- The distributed axial in-plane load tends to distort the mode-shapes and deviate them from symmetry.
- CC is the most boundary condition significant for distortion of its mode rather than other boundaries under distributed axial load.
- Aspect ratio and fiber orientations have significant effects on the DCBLs.
- It is seen that \(N_2\) and \(N_3\) types load distributions give higher critical buckling loads.
- The type of axially variable in-plane load is significantly affecting the DCBL of the beam structures. The fiber angle value \((θ)\) is important for the DCBL. Moreover, its effect on the DCBL is more pronounced especially for CS and CC symmetric LCBs.
- The buckling mode shapes of LCBs are significantly affected by the type of axially variable in-plane load for all BCs.
- The DCBL of the LCSBs depend on not only the aspect ratio but also the BCs and fiber orientation angles. The effect of the thickness stretching on the buckling mode shapes may vary depending on the type of the axially variable in-plane load.

References

1. Y.X. Zhang a,*, C.H. Yang Recent developments in finite element analysis for laminated composite plates Composite Structures 88 (2009) 147–157.
2. Ghugal YM, Shimpi RP. A review of refined shear deformation theories for isotropic and anisotropic laminated beams. J Reinfr Plast Compos 2001;20(3):255–72.
3. Aguiar R, Moleiro F, Soares CM. Assessment of mixed and displacement-based models for static analysis of composite beams of different cross-sections. Compos Struct 2012;94(2):601–16.
4. Zhen W, Wanji C. An assessment of several displacement-based theories for the vibration and stability analysis of laminated composite and sandwich beams. Compos Struct 2008;84(4):337–49.
5. Mathew TC, Singh G, Rao GV. Thermal buckling of cross-ply composite laminates. Comput Struct 1992;42(2):281–7.
6. S.K. Panda, L.S. Ramachandra, Buckling of rectangular plates with various boundary conditions loaded by non-uniform in-plane loads, Int. J. Mech. Sci. 52 (6) (2010) 819–828.
7. J.H. Kang, A.W. Leissa, Exact solutions for the buckling of rectangular plates having linearly varying in-plane loading on two opposite simply supported edges, Int. J. Solids Struct. 42 (14) (2005) 4220–4238.
8. N.I. Kim, D.K. Shin, Y.S. Park. Coupled stability analysis of thin-walled composite beams with closed cross-section, Thin-Walled Struct. 48 (8) (2010) 581–596.
9. A. Farajpour, A.R. Shahidi, M. Mohammadi, M. Mahzoon, Buckling of orthotropic micro/nanoscale plates under linearly varying in-plane load via nonlocal continuum mechanics, Compos. Struct. 94 (5) (2012) 1605–1615.
10. M.A. Eltaher a,b,*, S.A. Mohamed c, A. Melaibari a Thin–Walled Structures 147 (2020) 106488 Static stability of a unified composite beams under varying axial loads
11. Eisenberger M. Buckling loads for variable cross-section members with variable axial forces. Int J Solids Struct 1991;27(2):135–44.
12. Lee K. Buckling of fibers under distributed axial load. Fibers Polym 2008;9(2):200–2.
13. Wang CM, Wang CY, Reddy JN. Exact solutions for buckling of structural members. Boca Raton, Florida: CRC Press; 2004.
14. Li QS. Exact solutions for the generalized Euler’s problem. J Appl Mech 2009;76(4):041015.
15. Vo TP, Thai HT, Nguyen TK, Lane D, Karamanli A. Flexural analysis of laminated composite and sandwich beams using a four-uniform shear and normal deformation theory. Compos Struct 2017;176:388–97.
16. Vo TP, Thai HT, Aydogdu M. Free vibration of axially loaded composite beams using a four-unknown shear and normal deformation theory. Compos Struct 2017;178:406–14.
17. Karamanli A, Vo T. Size dependent bending analysis of two directional functionally graded microbeams via a quasi-3D theory and finite element method. Composites Part B 2018;144:171–83.
18. Trinh LC, Vo T, Thai HT, Nguyen TK. Size-dependent vibration of bi-directional functionally graded microbeams with arbitrary boundary conditions. Composites Part B 2018;134:225–45.
19. Pagano N. Exact solutions for composite laminates in cylindrical bending. J Compos Mater 1969;3(3):398–411.
20. A. Karamanli, M. Aydogdu, Buckling of laminated composite and sandwich beams due to axially varying in-plane loads, Compos. Struct. 210 (2019) 391–408.
21. Aydogdu M. Buckling analysis of cross-ply laminated beams with general boundary conditions by Ritz method. Compos Sci Technol 2006;66(10):1248–55.
22. Aydogdu M. Thermal buckling analysis of cross-ply laminated composite beams with general boundary conditions. Compos Sci Technol 2007;67(6):1096–104.