Abstract

Using families of curves to generalize vector fields, the Lie bracket is defined on a metric space, $M$. For $M$ complete, versions of the local and global Frobenius theorems hold, and flows are shown to commute if and only if their bracket is zero.

An example is given showing $L^2(\mathbb{R})$ is controllable by two elementary flows.

Key Words: metric space, Banach space, flow, nonsmooth, foliation, integral surface

MSC: 51F99; 93B29; 53C12

1 Introduction

The main goal of this monograph is to further the point of view that many beautiful geometrical and analytical results valid on differentiable manifolds hold on general metric spaces. Besides the wider relevance gained by generalization, the foundations of the subject are clarified when the limits of applicability are explored. This effort has a long and often disjointed history, only one sliver of which is relevant here. The approach in this paper, which has been used by several others, is to use the well-known characterization of a vector in a tangent space as an equivalence class of curves which are tangent to each other. A curve $c$ on a metric space $(M, d)$ is a continuous map $c : (\alpha, \beta) \rightarrow M$ where $(\alpha, \beta) \subset \mathbb{R}$. Two curves $c_i : (\alpha_i, \beta_i) \rightarrow M$ for $i = 1, 2$ are tangent at $t \in (\alpha_1, \beta_1) \cap (\alpha_2, \beta_2)$ if

$$
\lim_{h \to 0} \frac{d(c_1(t + h), c_2(t + h))}{h} = 0.
$$

In this way we may generalize a vector field (a family of vectors) on a manifold as an arc field (a family of curves) on a metric space--Definition 1 below.

It has been said the three pillars of differential geometry are: ($I$) the Inverse Function Theorem, ($II$) the Existence Theorem for ordinary differential equations (ODEs) and ($III$) Frobenius' Theorem. All of these classical theorems may be written with vector fields on manifolds and so may also be written with arc fields on metric spaces. We expect any result on manifolds which has
a sufficiently geometrically realized proof can be generalized to metric spaces using curves in place of vectors. A metric space version of (I) is contained in [2], e.g.; and versions of (II) have been proven several times independently in e.g., [9], [2], and [4]—see Theorem 2 below. A version of (III) is the main result of this paper, Theorem 27, an involutive distribution on a complete metric space is integrable. Since the result is for complete metric spaces, it generalizes the classical result on Banach manifolds (proven, e.g., in [1]). Theorem 27 further generalizes the classical result by assuming only Lipschitz-type regularity instead of smoothness, which is of interest in, for example, control theory.

As far as I have been able to determine, this particular approach to the proof of Frobenius’ classical theorem has not been vetted in the literature—though it uses basic, well-known ideas. We outline the approach in this paragraph, simplified to vector fields on a manifold. The terminology and assumptions will be clarified in the main body of the paper, and Figures 2 and 3 from Section 5 may aid intuition. The crux of the local Frobenius result in two dimensions is as follows: Given two transverse vector fields \( f, g : M \to TM \) there exists an integral surface (tangent to linear combinations of \( f \) and \( g \)) through any point \( x_0 \in M \) when the Lie bracket satisfies \( \left[ f, g \right] = af + bg \) for some choice of functions \( a, b : M \to \mathbb{R} \) (involutivity of \( f \) and \( g \)). To prove this, define

\[
S := \left\{ F_t G_s(x_0) \in M : |s|, |t| < \delta \right\}
\]

where \( F \) and \( G \) are the local flows of \( f \) and \( g \) guaranteed to exist by (II). Since \( f \) and \( g \) are transverse, we may choose \( \delta > 0 \) small enough for \( S \) to be a well-defined surface. \( S \) will be shown to be the desired integral surface through \( x_0 \). Notice \( S \) is tangent to \( f \) by construction, but it is not immediately clear \( S \) is tangent to \( a'f + b'g \) for arbitrarily chosen \( a', b' \in \mathbb{R} \). Notice, though, that by construction \( S \) is tangent to \( g \) at any point \( x = G_s(x_0) \), and also \( S \) is tangent to \( a''f + b''g \) at \( x \) for functions \( a'' \) and \( b'' \). Therefore establishing

\[
(F_t)^* \left( a'f + b'g \right) = a''f + b''g \quad \text{at} \quad x = G_s(x_0)
\]

for some functions \( a'' \) and \( b'' \), proves \( S \) is tangent to \( a'F + b'G \) at an arbitrary point \( z = F_t G_s(x_0) \in S \), since the push-forward \( (F_t)_* \) and the pull-back \( (F_t)^* \) are inverse to each other and preserve tangency since they are local diffeomorphisms. Next since the Lie bracket equals the Lie derivative,

\[
\lim_{h \to 0} \frac{F_h^* (g) - g}{h} = [f, g] = af + bg
\]

by involutivity so

\[
F_h^* (g) = g + h (af + bg) + o(h) = \tilde{a}f + \tilde{b}g + o(h).
\]

Using the fact that \( F_t^* (f) = f \) for any \( h \), and the linearity of pullback for fixed \( t \), we have for functions \( a_t \) and \( b_t : M \to \mathbb{R} \)

\[
F_{t/n}^* \left( a_t f + b_t g \right) = (a_{t+1} f + b_{t+1} g) + o(1/n)
\]
for some functions \(a_{i+1}\) and \(b_{i+1}\). Then since

\[
F_t^* = \left( F_{t/n}^* F_{t/n}^* \ldots F_{t/n}^* \right)^{(n)}
\]

(where the superscript in round brackets denotes composition \(n\) times) we get (1) as follows:

\[
F_t^* (a_0 f + b_0 g) = \lim_{n \to \infty} \left( F_{t/n}^* \right)^{(n)} (a_0 f + b_0 g)
= \lim_{n \to \infty} a_n f + b_n g + o(1/n) = a_\infty f + b_\infty g + 0
\]

completing the sketch for manifolds.

A pivotal fact on which the metric space version relies is that arc fields which satisfy certain Lipschitz-type conditions generate unique local flows (proven in [4] and reviewed in Section 2). Also a natural linear structure may be associated with a metric space (though it has no *a priori* linear structure) using compositions of flows which faithfully generalizes the linearity of vector fields; this was introduced in [6]. We present this in Section 3 along with the generalization of the Lie bracket for vector fields which uses the well-known asymptotic characterization of the Lie bracket; i.e., successively follow the flows forward and backward for time \(\sqrt{t}\). This investigation further clarifies for us the surprising fact Sussman and others have noted: smoothness is not necessary to define a geometrically meaningful Lie bracket. In Section 4 the pull-back along a flow is shown to behave naturally with linearity and the bracket, which mimics properties of the Lie derivative on manifolds. Many more such algebraic properties are valid than are contained in these sections, but in this monograph we present only the minimum machinery directly relevant to proving Frobenius’ Theorem in Section 5.

Section 6 applies this local Frobenius theorem to study foliations yielding a global theorem on metric spaces. A metric space generalization of the Nagumo-Brezis Invariance Theorem is proven, which is used to show integrable distributions are involutive. We do not discuss the facet of the classical Global Frobenius Theorem which guarantees local coordinates on which there exist coordinate vector fields tangent or perpendicular to an involutive distribution. In light of these results, however, this now seems ripe for exploration.

Section 7 proves a well-known result from Hamiltonian dynamics is also valid for metric spaces: two flows commute if and only if the bracket is 0. This is not exactly a corollary of the metric space Frobenius Theorem, but the proof is a mere simplification of that from Theorem 27.

Finally in Section 8 an almost trivial example applying these ideas has a result which astounded me: Any Lebesgue square-integrable function may be approximated using successive compositions of two elementary flows, starting from the constant zero function. In other words, \(L^2(\mathbb{R})\) is controllable by two flows. You may skip straight to this Example 47 after perusing the following review and the definitions in Section 3. [12] is an accessible text introducing the
2 Review of terminology and basic results

The proofs of all of the results from this section are contained in [4] for forward flows, also called semi-flows. Minimal changes, stated here, give us the corresponding results for (bidirectional) flows.

A metric space \((M, d)\) is a set of points \(M\) with a function \(d : M \times M \to \mathbb{R}\) called the metric which has the following properties:

(i) \(d(x, y) \geq 0\) positivity
(ii) \(d(x, y) = 0\) iff \(x = y\) nondegeneracy
(iii) \(d(x, y) = d(y, x)\) symmetry
(iv) \(d(x, y) \leq d(x, z) + d(z, y)\) triangle inequality

for all \(x, y, z \in M\). The open ball of radius \(r\) about \(x \in M\) is denoted by \(B(x, r) := \{y : d(x, y) < r\}\). We assume throughout this paper that \((M, d)\) is a locally complete metric space, i.e., there exists a complete neighborhood of each point in \(M\). Denote the open ball in \(M\) about \(x_0 \in M\) with radius \(r\) by

\[B(x_0, r) := \{x \in M : d(x, x_0) < r\}.\]

A map \(f : (M, d_M) \to (N, d_N)\) between metric spaces is Lipschitz continuous if there exists \(K_f \geq 0\) such that

\[d_N(f(x_1), f(x_2)) \leq K_f d_M(x_1, x_2)\]

for all \(x_1, x_2 \in M\). A homeomorphism is an invertible Lipschitz map whose inverse is also Lipschitz (i.e., stronger than a homeomorphism, weaker than a diffeomorphism).

The following definition is made in analogy with vector fields on manifolds, where vectors are represented as curves on the manifold.

**Definition 1** An arc field on \(M\) is a continuous map \(X : M \times [-1, 1] \to M\) such that for all \(x \in M\), \(X(x, 0) = x\),

\[\rho(x) := \sup_{s \neq t} \frac{d(X(x, s), X(x, t))}{|s - t|} < \infty,\]

(i.e., \(X(x, \cdot)\) is Lipschitz), and the function \(\rho(x)\) is locally bounded so

\[\rho(x, r) := \sup_{y \in B(x, r)} \{\rho(y)\} < \infty,\]

for \(r > 0\) sufficiently small.

A solution curve to \(X\) is a curve \(\sigma\) tangent to \(X\), i.e., \(\sigma : (\alpha, \beta) \to M\) for some open interval \((\alpha, \beta) \subseteq \mathbb{R}\) has the following property for each \(t \in (\alpha, \beta)\)

\[\lim_{h \to 0} \frac{d(\sigma(t + h), X(\sigma(t), h))}{h} = 0,\]

i.e., \(d(\sigma(t + h), X(\sigma(t), h)) = o(h)\).
\( \rho \) is a bound on the speed of the arcs. \( \alpha \) and \( \beta \) are members of the extended reals \( \mathbb{R} \cup \{ \pm \infty \} \).

The two variables for arc fields and flows which are usually denoted by \( x \) and \( t \) are often thought of as representing space and time. In this paper \( x, y, \) and \( z \) are used for space variables, while \( r, s, t, \) and \( h \) may fill the time variable slot. An arc field \( X \) will often have its variables migrate liberally between parentheses and subscripts

\[
X(x,t) = X_x(t) = X_t(x)
\]

depending on which variable we wish to emphasize in a calculation. We also use this convention for flows \( F \) defined below.

The following conditions guarantee existence and uniqueness of solutions.

**Condition E1:** For each \( x_0 \in M \), there are positive constants \( r, \delta \) and \( \Lambda_X \) such that for all \( x, y \in B(x_0,r) \) and \( t \in (-\delta,\delta) \)

\[
d(X_t(x),X_t(y)) \leq d(x,y)(1 + |t| \Lambda_X).
\]

**Condition E2:**

\[
d(X_{s+t}(x),X_t(X_{s}(x))) = O(st)
\]
as \( st \to 0 \) locally uniformly in \( x \); in other words, for each \( x_0 \in M \), there are positive constants \( r, \delta \) and \( \Omega_X \) such that for all \( x \in B(x_0,r) \) and \( s, t \in (-\delta,\delta) \)

\[
d(X_{s+t}(x),X_t(X_{s}(x))) \leq |st| \Omega_X.
\]

![Figure 1: Conditions E1 and E2](image)

**Theorem 2** Let \( X \) be an arc field satisfying E1 and E2 on a locally complete metric space \( M \). Then given any point \( x \in M \), there exists a unique solution \( \sigma_x : (\alpha_x, \beta_x) \to M \) with \( \sigma_x(0) = x \).
Several remarks are in order. Here, $x$ is called the \textbf{initial condition} for the solution $\sigma_x$ in the above theorem. Uniqueness of solutions means that for any $x \in M$, the curve $\sigma_x : (\alpha_x, \beta_x) \to M$ has maximal domain $(\alpha_x, \beta_x)$ in the sense that for any other solution $\tilde{\sigma}_x : (\tilde{\alpha}_x, \tilde{\beta}_x) \to M$ also having initial condition $x$, we have $(\tilde{\alpha}_x, \tilde{\beta}_x) \subset (\alpha_x, \beta_x)$ and $\tilde{\sigma}_x = \sigma_x|_{(\tilde{\alpha}_x, \tilde{\beta}_x)}$ (i.e., $\sigma_x$ is the \textbf{maximal solution curve}).

The proof of Theorem 2 is constructive and shows the Euler curves $X^{(n)}_{\frac{t}{n}}(x)$ converge to the solution. Here we are using $f^{(n)}$ to denote the composition of a map $f : M \to M$ with itself $n$ times so

$$X^{(n)}_{\frac{t}{n}}(x) = \underbrace{X_{\frac{t}{n}} \circ X_{\frac{t}{n}} \circ \ldots \circ X_{\frac{t}{n}}}_{n \text{ times}}(x)$$

and we have

$$\lim_{n \to \infty} X^{(n)}_{\frac{t}{n}}(x) = \sigma_x(t).$$

for suitably small $|t|$. Other, slightly different formulations of Euler curves also lead to the same result, $\sigma$, under Conditions E1 and E2, e.g.,

$$\xi_n(t) := X_{t-i2^{-n}}X^{(i)}_{i2^{-n}}(x) \quad \text{for} \quad i \cdot 2^{-n} \leq t \leq (i+1)2^{-n}$$

also has

$$\lim_{n \to \infty} \xi_n(t) = \sigma_x(t)$$

for suitably small $|t|$. Theorem 2 and those that follow are true under more general conditions outlined in [4] and [9]. But throughout this paper and in every application I’ve seen, E1 and E2 are satisfied and are easier to use.

**Example 3** A Banach space $(M, \|\cdot\|)$ is a normed vector space, complete in its norm (e.g., $\mathbb{R}^n$ with Euclidean norm). A Banach space is an example of a metric space with $d(u,v) := \|u - v\|$. A \textbf{vector field} on a Banach space $M$ is a map $f : M \to M$. A \textbf{solution} to a vector field $f$ with \textbf{initial condition} $x$ is a curve $\sigma_x : (\alpha, \beta) \to M$ defined on an open interval $(\alpha, \beta) \subset \mathbb{R}$ containing 0 such that $\sigma_x(0) = x$ and $\sigma_x'(t) = f(\sigma_x(t))$ for all $t \in (\alpha, \beta)$. The classical Picard-Lindelöf Theorem guarantees unique solutions for any locally Lipschitz $f$. With a few tricks, most differential equations can be represented as vector fields on a suitably abstract space.

Every Lipschitz vector field $f : M \to M$ gives rise to an arc field $X(x,t) := x + tf(x)$ and it is easy to check $X$ satisfies E1 and E2 (cf. [4]). Further the solutions to the arc field are exactly the solutions to the vector field. Therefore Theorem 2 is a generalization of the classical Picard-Lindelöf Theorem.

**Remark 4** Of prime import for this monograph, the proof of Theorem 2 actually shows solutions are \textbf{locally uniformly tangent} to $X$:

$$d(X_x(t), \sigma_x(t)) = o(t)$$
locally uniformly for $x \in M$, i.e., for each $x_0 \in M$ there exists a constant $r > 0$ such that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in B(x_0, r)$

$$\frac{d(X_x(t), \sigma_x(t))}{|t|} < \varepsilon$$

whenever $0 < |t| < \delta$.

More than this, the proof also shows solutions are tangent uniformly for all arc fields $X$ which satisfy $E1$ and $E2$ for specified $\Lambda$ and $\Omega$.

We denote local uniform tangency of two arc fields $X$ and $Y$ by $X \sim Y$. It is easy to check $\sim$ is an equivalence relation. E.g., transitivity follows from the triangle inequality:

$$\frac{d(X_t(x), Z_t(x))}{|t|} \leq \frac{d(X_t(x), Y_t(x))}{|t|} + \frac{d(Y_t(x), Z_t(x))}{|t|}.$$

We use the symbol $\sim$ in many contexts in this paper (particularly Section 6), but there is always a local-uniform-tangency property associated with it.

**Corollary 5** Assume the conditions of Theorem 2 and let $s \in (\alpha_x, \beta_x)$ and $y = \sigma_x(s)$. Then $\alpha_y = \alpha_x - s$ and $\beta_y = \beta_x - s$ so

$$(\alpha_y, \beta_y) = (\alpha_{\sigma_x(s)}, \beta_{\sigma_x(s)}) = \{t : \alpha_x - s < t < \beta_x - s\}.$$

Thus $t \in (\alpha_y, \beta_y)$ if and only if $t + s \in (\alpha_x, \beta_x)$, and then we have

$$\sigma_{\sigma_x(s)}(t) = \sigma_x(s + t).$$

Defining $W \subset M \times \mathbb{R}$ by

$$W := \{(x, t) \in M \times \mathbb{R} : t \in (\alpha_x, \beta_x)\}$$

and $F : W \to M$ by $F(x, t) := \sigma_x(t)$ we have:

(i) $M \times \{0\} \subset W$ and $F(x, 0) = x$ for all $x \in M$.

(ii) For each (fixed) $x \in M$, $F(x, \cdot) : (\alpha_x, \beta_x) \to M$ is the maximal solution $\sigma_x$ to $X$.

(iii) $F(t, F(s, x)) = F(t + s, x)$.

$F$ is called the **local flow** generated by the arc field $X$. Compare Condition $E2$ with property $(iii)$ above to see why an arc field might be thought of as a “pre-flow”.

**Theorem 6** Let $\sigma_x : (\alpha_x, \beta_x) \to M$ and $\sigma_y : (\alpha_y, \beta_y) \to M$ be two solutions to an arc field $X$ which satisfies $E1$. Assume $(\alpha_x, \beta_x) \cap (\alpha_y, \beta_y) \supset I$ for some interval $I$, and assume $\Lambda_X$ holds on a set containing

$$\{\sigma_x(t) : t \in I\} \cup \{\sigma_y(t) : t \in I\}.$$
Then
\[ d(\sigma_x(t), \sigma_y(t)) \leq e^{\Lambda x|t|}d(x,y) \text{ for all } t \in I, \]
i.e.,
\[ d(F_t(x), F_t(y)) \leq e^{\Lambda x|t|}d(x,y). \] (3)

**Theorem 7** For \( F \) and \( W \) as above, \( W \) is open in \( M \times \mathbb{R} \) and \( F \) is continuous on \( W \).

For fixed \( t \) it is clear \( F_t \) is a local lipeomorphism, when defined, by Theorem 6. Compare Condition E1 with line (3) to see why E1 may be thought of as a local linearity property for \( X \), needed for the continuity of \( F \).

**Definition 8** An arc field \( X \) on a metric space \( M \) is said to have **linear speed growth** if there is a point \( x \in M \) and positive constants \( c_1 \) and \( c_2 \) such that for all \( r > 0 \)
\[ \rho(x,r) \leq c_1 r + c_2, \] (4)
where \( \rho(x,r) \) is the local bound on speed given in Definition 1.

**Theorem 9** Let \( X \) be an arc field on a complete metric space \( M \), which satisfies E1 and E2 and has linear speed growth. Then \( F \) is a (full) \textbf{flow} with domain \( W = M \times \mathbb{R} \).

**Example 10** Every local flow on a metric space is generated by an arc field. Any local flow \( F \) gives rise to an arc field \( \overline{F} : M \times [-1,1] \to M \) defined by

\[
\overline{F}(x,t) := \begin{cases} 
F(x,t) & \text{if } t \in \left( \frac{\alpha_x}{2}, \frac{\beta_x}{2} \right) \\
F \left( x, \frac{\alpha_x}{2} \right) & \text{if } t \in \left[ -1, \frac{\alpha_x}{2} \right] \\
F \left( x, \frac{\beta_x}{2} \right) & \text{if } t \in \left[ \frac{\beta_x}{2}, 1 \right].
\end{cases}
\]

(The issue here is that \( F \), being a local flow, may have \( \alpha_x \) or \( \beta_x < 1 \).) Clearly the local flow generated by \( \overline{F} \) is \( F \). Since all our concerns with arc fields are local, we will never focus on \( t \notin \left( \frac{\alpha_x}{2}, \frac{\beta_x}{2} \right) \) and henceforth we will not notationally distinguish between \( \overline{F} \) and \( F \) as arc fields.

With this identification of flows being arc fields (but not necessarily vice-versa) we may simplify Remark 8 to: \( X \sim F \) if \( X \) satisfies E1 and E2.

### 3 The bracket and linearity

To simplify notation we drop parentheses for expressions such as \( Y_t \circ X_s(x) = Y_t(X_s(x)) \) and write \( Y_tX_s(x) \) since the composition of arbitrary maps is associative.
Definition 11. The **bracket** of two arc fields $X$ and $Y$ is the map $[X,Y] : M \times [-1,1] \to M$ with

$$ [X,Y](x,t) := \begin{cases} Y_{-\sqrt{t}}Y_{-\sqrt{t}}Y_{\sqrt{t}}(x) & \text{for } t \geq 0 \\ X_{-\sqrt{|t|}}X_{-\sqrt{|t|}}Y_{\sqrt{|t|}}(x) & \text{for } t < 0. \end{cases} \quad (5) $$

There are many different equivalent characterizations of the Lie bracket on a manifold. \[\] uses the obvious choice of the **asymptotic** characterization to generalize the concept to metric spaces. $[X,Y](x,t)$ traces out a small “parallelogram” in $M$ starting at $x$, which hopefully almost returns to $x$. The bracket measures the failure of $X$ and $Y$ to commute as will be made clear in Theorems \[\] and \[\].

Definition 12. We say $X \mathcal{E} Y$ close if

$$ d(Y_sX_t(x),X_sY_t(x)) = O(|st|) $$

locally uniformly in $x$, i.e., if for each $x_0 \in M$ there exist positive constants $C_{XY}, \delta,$ and $r$ such that for all $x \in B(x_0,r)$

$$ d(Y_sX_t(x),X_sY_t(x)) \leq C_{XY}|st| $$

for all $|s|,|t| < \delta$.

Lemma 13. If $X \mathcal{E} Y$ close and satisfy E1 and E2 then

$$ d(Y_tX_tY_t(x),x) = O(t^2) $$

locally uniformly for $x \in M$.

**Proof.**

$$ d(Y_{-s}X_{-t}Y_{-t}X_{t}(x),x) $$

$$ \leq d(Y_{-s}X_{-t}Y_{-t}X_{t}(x),Y_{-s}X_{-t}X_{t}Y_{s}(x)) + d(Y_{-s}X_{-t}X_{t}Y_{s}(x),Y_{-s}Y_{s}(x)) + d(Y_{-s}Y_{s}(x),x) $$

$$ \leq d(Y_{-s}X_{t}(x),X_{t}Y_{s}(x))(1 + |s| \Lambda_Y)(1 + |t| \Lambda_X) + t^2 \Omega_X(1 + |s| \Lambda_Y) + s^2 \Omega_Y $$

$$ \leq C_{XY}|st|(1 + |s| \Lambda_Y)(1 + |t| \Lambda_X) + t^2 \Omega_X(1 + |s| \Lambda_Y) + s^2 \Omega_Y \leq C( |st| + t^2 + s^2 ) $$

where

$$ C := \max \{ C_{XY}(1 + \Lambda_Y)(1 + \Lambda_X), \Omega_X(1 + \Lambda_Y), \Omega_Y \} $$

Letting $s = t$ gives the result. \[\]

Proposition 14. If $X \mathcal{E} Y$ close and satisfy E1 and E2 then $[X,Y]$ is an arc field.

**Proof.** We establish the local bound on speed. The purpose of Lemma \[\] is to give $d([X,Y](x,t),x) = O(t)$ for $t \geq 0$. Similarly, for $t < 0$

$$ d(X_tY_tX_{-t}Y_{-t}(x),x) $$

$$ \leq d(X_tY_tX_{-t}Y_{-t}(x),X_tX_{-t}(x)) + d(X_tX_{-t}(x),x) $$

$$ \leq d(Y_tX_{-t}Y_{-t}(x),X_{-t}(x))(1 + |t| \Lambda_X) + t^2 \Omega_X $$

9
which, using this trick again, gives
\[
\leq d \left( X_{-t} Y_t (x), Y_{-t} X_t (x) \right) (1 + |t| \Lambda_X) (1 + |t| \Lambda_Y)
+ t^2 \Omega_Y (1 + |t| \Lambda_Y) + t^2 \Omega_X = O \left( t^2 \right)
\]
since $X$ & $Y$ close.

Therefore
\[
d \left( [X, Y]_t (x), x \right) = O \left( t \right)
\]
for both positive and negative $t$. Then since $\sqrt{|t|}$ is Lipschitz except at $t = 0$ we see $[X, Y]$ has bounded speed.

**Example 15** As in Example 4 let $f, g : B \to B$ be Lipschitz vector fields on a Banach space $B$, and let $X$ and $Y$ be their corresponding arc fields
\[
X (x, t) := x + tf(x)
Y (x, t) := x + tg(x)
\]
It is easy to check $X \& Y$ close:
\[
d (Y_s X_t (x), x) = \| x + tf(x) + sg(x + tf(x)) - [x + sg(x) + tf(x + sg(x))] \|
\leq |t| \|f(x) - f(x + sg(x))\| + |s| \|g(x + tf(x)) - g(x)\|
\leq |t| K_f \|x - (x + sg(x))\| + |s| K_g \|x + tf(x) - x\|
\leq |st| (K_f \|g(x)\| + K_g \|f(x)\|)
\]
so $C_{XY} := (K_f \|g(x)\| + K_g \|f(x)\|)$.

Therefore, even though the vector fields may not be smooth, so their Lie bracket is undefined, their metric space bracket is meaningful and will give us geometric information as we shall see in Theorem 27.

**Definition 16** If $X$ and $Y$ are arc fields on $M$ then define $X + Y$ to be the arc field on $M$ given by
\[
(X + Y)_t (x) := Y_t X_t (x).
\]
For any function $a : M \to \mathbb{R}$ define the arc field $aX$ by
\[
aX (x, t) := X (x, a(x) t).
\]
If $a$ is Lipschitz, then $aX$ is an arc field.

To be fastidiously precise we need to define $aX_t (x)$ for all $t \in [-1, 1]$ so technically we must specify
\[
aX (x, t) := \begin{cases} 
X (x, a(x) t) & -\frac{1}{|a(x)|} \leq t \leq \frac{1}{|a(x)|} \\
X (x, 1) & t > 1/|a(x)| \\
X (x, -1) & t < -1/|a(x)| \\
x & for \ -1 \leq t \leq 1
\end{cases}
\]
when $a(x) \neq 0$
\[
(7)
\]
using the trick from Example 10. Again, we will not burden ourselves with this detail; in all cases our concern with the properties of an arc field $X_x(t)$ is only near $t = 0$.

It is a simple definition check to prove $aX$ is an arc field when $a$ is Lipschitz, since $aX_x(t) = X_x(a(x)t)$ is Lipschitz in $t$ if $X_x(t)$ is; assuming $a(x) \neq 0$,

$$\rho_{aX}(x) := \sup_{s \neq t} \frac{d(X_x(a(x)s), X_x(a(x)t))}{|s - t|} = \sup_{s \neq t} \frac{d(X_x(s), X_x(t))}{\|a(x) - a(x)\|}$$

$$= a(x) \sup_{s \neq t} \frac{d(X_x(s), X_x(t))}{|s - t|} = a(x) \rho_X(x)$$

so

$$\rho_{aX}(x,r) := \sup_{y \in B(x,r)} \{\rho_{aX}(y)\} = \sup_{y \in B(x,r)} \{a(y) \rho(y)\}$$

$$\leq (a(x) + rK_a) \rho_X(x,r) < \infty.$$ 

Now we have the beginnings of a linear structure associated with $M$. For instance, expressions such as $X - Y$ make sense:

$$X - Y := X + (-1)Y$$

where $-1$ is a constant function on $M$. Further, $0$ is an arc field defined as the constant map

$$0(x,t) := x$$

and satisfies $0 + X = X = X + 0$ for any $X$. Notice from the definition, we have $[X,Y] = - [Y,X]$. Another trivial definition check shows this multiplication is associative and commutative:

$$(a \cdot b)X = a(bX) \quad \text{and} \quad (a \cdot b)X = (b \cdot a)X$$

where $\cdot$ denotes multiplication of functions.

**Proposition 17** Assume $X \in Y$ close and satisfy E1 and E2. Then their sum $X + Y$ satisfies E1 and E2.

**Proof.** Checking Condition E1:

$$d((X + Y)_t(x), (X + Y)_t(y))$$

$$= d(Y_tX_x(x), Y_tX_y(y)) \leq d(X_t(x), X_t(y))(1 + |t| \Lambda_Y)$$

$$\leq d(x,y)(1 + |t| \Lambda_X)(1 + |t| \Lambda_Y) \leq d(x,y)(1 + |t|)(\Lambda_X + \Lambda_Y) + t^2 \Lambda_X \Lambda_Y$$

$$\leq d(x,y)(1 + |t| \Lambda_X + \Lambda_Y)$$

where $\Lambda_{X+Y} := \Lambda_X + \Lambda_Y + \Lambda_X \Lambda_Y < \infty$. 

11
Condition E2:
\[
d((X + Y)_{s+t}, (X + Y)_s) = \d(Y_{s+t}X_{s+t}, Y_tX_tX_s) \\
\leq d(Y_{s+t}X_{s+t}, Y_tY_sX_{s+t}) + d(Y_tY_sX_{s+t}, Y_tX_tY_sX_s) \\
\leq |st| \Omega_Y + d(Y_sX_{s+t}, X_sY_s X_s)((1 + |t| \Lambda_Y) \\
\leq |st| \Omega_X + [d(Y_sX_{s+t}, Y_sX_t X_s) + d(Y_sY_t, X_t Y_s)](1 + t \Lambda_X) \\
\] (8)

where \(y := X_s(x)\). Notice
\[
d(Y_sX_{s+t}, Y_sX_t X_s) \leq d(X_{s+t}, X_s X_s)((1 + |s| \Lambda_Y) \\
\leq |st| \Omega_X (1 + |s| \Lambda_Y) = O(|st|)
\]

and the last summand of (8) is also \(O(|st|)\) since \(X \& Y\) close, so E2 is satisfied.

So in this case, the flow \(H\) generated by \(X + Y\) is computable with Euler curves as
\[
H(x, t) = \lim_{n \to \infty} (X + Y)_{t/n}^{(n)}(x) = \lim_{n \to \infty} (Y_{t/n} X_{t/n})^{(n)}(x).
\] (9)

Therefore, this definition of \(X + Y\) using compositions is a direct generalization of the concept of adding vector fields on a differentiable manifold (see Section 4.1A). One of the inspirations for this paper, [3], introduced the sum of semigroups on a metric space in the same spirit as defined here, with commensurable conditions.

When \(X \& Y\) close and satisfy E1 and E2, we also have \((X + Y) \sim (Y + X)\) since
\[
(Y_{t/n}X_{t/n})^{(n)} = Y_{t/n} (X_{t/n}Y_{t/n})^{(n-1)} X_{t/n}
\]
whence both arc fields \(X + Y\) and \(Y + X\) are (locally uniformly) tangent to the flow \(H\) using [2].

**Proposition 18** If \(X\) satisfies E1 and E2 and \(a : M \to \mathbb{R}\) is a Lipschitz function, then \(aX\) satisfies E1 and E2.

**Proof.** E1:
\[
d(aX_t, aX_y(t)) \\
= d(X_t(a(x)t), X_y(a(y)t)) \\
\leq d(X_t(a(x)t), X_x(a(y)t)) + d(X_x(a(y)t), X_y(a(y)t)) \\
\leq |a(x) - a(y)| t \rho + d(x,y) (1 + a(y) |t| \Lambda_X) \\
\leq d(x,y)(K_a |t| \rho + 1 + a(y) |t| \Lambda_X) = d(x,y) (1 + |t| \Lambda_{aX})
\]
where \(\Lambda_{aX} := K_a \rho + a(y) \Lambda_X < \infty\).
E2: For all \( x_0 \in M \) and \( \delta > 0 \) we know \( a \) is bounded by some \( A > 0 \) on \( B(x_0, \delta) \) since \( a \) is Lipschitz.

\[
\begin{align*}
\Omega_a &:= A^2 \Omega_X + \rho^2 K_a A. \quad \blacksquare \\
&
\end{align*}
\]

Combining these results gives

**Theorem 19** If \( a \) and \( b \) are locally Lipschitz functions and \( X \not\equiv Y \) close and satisfy \( E1 \) and \( E2 \), then \( aX + bY \) is an arc field which satisfies \( E1 \) and \( E2 \) and so has a unique local flow.

If in addition \( a \) and \( b \) are globally Lipschitz and \( X \) and \( Y \) have linear speed growth, then \( aX + bY \) generates a unique flow.

**Proof.** We haven’t proven \( aX \) and \( bY \) close, but this is a straightforward definition check, as is the fact that \( aX + bY \) has linear speed growth. \blacksquare

Local flows have the following useful linearity property:

**Proposition 20** If \( F \) is a local flow then interpreting \( F \) as an arc field we can perform the following operations:

1. if \( a \) and \( b \) are constant then \( aF + bF = (a + b) F \)
2. if \( a \) and \( b \) are real functions then \( (aF + bF)_t (x) = (a + b \circ (aF)_t ) F_t (x) \).

**Proof.** This is another obvious definition check:

\[
\begin{align*}
2. \quad & (aF + bF)_t (x) = (bF)_t (aF)_t (x) = F_{b((aF)_t (x))}(F_{a(x)}(x)) = F_{(a(x) + (b\circ(aF)_t))}(x) = (a + b \circ (aF)_t ) F_t (x)
\end{align*}
\]

and 1. follows from 2. \blacksquare

## 4 Contravariance

If \( \phi : M_1 \to M_2 \) is a lipeomorphism (i.e., an invertible Lipschitz map with Lipschitz inverse), then the pull-back of an arc field \( X \) on \( M_2 \) is the arc field \( \phi^*X \) on \( M_1 \) given by

\[
\phi^*X (x, t) := \phi^{-1} (X (\phi (x), t))
\]
or in other notation,

\[
(\phi^*X)_t (x) = \phi^{-1}X_{\phi (x)}
\]
which is a direct analog of the pull-back of a vector field on a manifold using curves to represent vectors. The definition for flows is identical, replacing $X$ with $F$. The pull-back to $M_1$ of a solution $\sigma$ to an arc field on $M_2$ is analogous:

$$(\phi^* \sigma)_x(t) := \phi^{-1}(\sigma_{\phi(x)}(t)).$$

The pull-back of a function $a : M_2 \to \mathbb{R}$ is the function $\phi^* a : M_1 \to \mathbb{R}$ defined as $\phi^* a(x) := a(\phi(x))$.

**Proposition 21** If $\phi : M_1 \to M_2$ is a lipeomorphism and the arc field $X$ on $M_2$ has unique solutions then $\phi^* X$ has unique solutions. The solutions to $\phi^* X$ are the pull-backs of solutions to $X$.

**Proof.** This is obvious: if $F$ is a local flow for $X$ then

$$d (\phi^* X (\phi^* F(x,t), s), \phi^* F(x,t+s))$$

$$= d (\phi^{-1} X [\phi \phi^{-1} F(\phi(x),t), s], \phi^{-1} F(\phi(x),t+s))$$

$$= d (\phi^{-1} X [F(\phi(x),t), s], \phi^{-1} F(\phi(x),t+s))$$

$$\leq K_\phi d (X[F(\phi(x),t), s], F(\phi(x),t+s)) = K_\phi o(s) = o(s)$$

so $\phi^* F$ is a flow (solution) for $\phi^* X$.

Similarly if $\sigma$ is a solution to $\phi^* X$ then $(\phi^{-1})^* \sigma$ is a solution to $X$ so by uniqueness there can be only one such $\sigma$.

The push-forward of any function, curve or flow is defined similarly, e.g.,

$$\phi_* F(x,t) := \phi(F(\phi^{-1}(x), t)).$$

It is easy to check push-forward is covariant (i.e., $(\phi \circ \psi)_* = \phi_* \circ \psi_*$) and pull-back is contravariant (i.e., $(\phi \circ \psi)^* = \psi^* \circ \phi^*$). It is also clear that push-forward and pull-back are inverse operations and Proposition 21 holds mutatis mutandis for push-forward in place of pull-back.

**Proposition 22 (Linearity of Pull-back)** If $X$ and $Y$ are arc fields on $M$ and $\phi : M_1 \to M_2$ is a lipeomorphism, then

(i) $\phi^* (X + Y) = \phi^* (X) + \phi^* (Y)$

(ii) $\phi^* (aX) = (a \circ \phi) \phi^* (X) = \phi^* (a) \phi^* (X)$.

**Proof.** Trivial definition check.

Since the pull-back and linearity are established for arc fields, we can now explore another characterization of the bracket. In the context of $M$ being a smooth manifold, let $F$ and $G$ be local flows generated by smooth vector fields $f$ and $g$. There it is well known the following “dynamic” characterization of the Lie bracket is equivalent to the asymptotic characterization

$$[f, g] = \frac{d}{dt} (F_t)^* g \bigg|_{t=0}. \quad (10)$$
Using
\[ \frac{d}{dt} (F_t)^* g \bigg|_{t=0} = \lim_{t \to 0} \frac{(F_t)^* g - g}{t} = [f, g] \]
for inspiration, we return to the context of metric spaces where, with \( F \) and \( G \) viewed as arc fields, their bracket \([F, G]\) is defined, and then
\[
F_t^* G_t (x) = (t [F, G] + G)_t (x) \quad \text{for } t \geq 0 \quad \text{(11)}
\]
\[
F_s^* G_s (x) = (-s [-F, -G] - G)_{-s} (x) \quad \text{for } s < 0 \quad \text{(12)}
\]
which hold because
\[
(t [F, G] + G)_t (x) = G_t [F, G]_{|t} (x)
\]
\[
= G_t G_{-t} F_{-t} G_t F_t (x) = F_{-t} G_t F_t (x) = F_t^* G_t (x)
\]
and
\[
(-s [-F, -G] - G)_{-s} (x)
\]
\[
= G_s [-F, -G]_{|s} (x) = G_s (-G)_{-|s|} (-F)_{-|s|} (-G)_{|s|} (-F)_{|s|} (x)
\]
\[
= G_s G_{|s|} F_{|s|} G_{-|s|} F_{-|s|} (x) = F_{-s} G_s F_s (x) = F_s^* G_s (x).
\]
These facts will be used in the heart of the proof of our main result, Theorem 27, as will the following

**Proposition 23** \((F_s)^* X \sim X\).

**Proof.** Using the properties of flows \( F_t = F_{-s+t+s} = F_{-s} F_t F_s \) and \( F_t^{-1} = F_t \) we get
\[
d \left( \left( (F_s)^* X \right)_t (x) , X_t (x) \right) \\
\leq d \left( F_{-s} X_t F_s (x) , F_{-s} F_t F_s (x) \right) + d \left( F_t (x) , X_t (x) \right) \\
\leq e^{s\Lambda_X} d \left( X_t (y) , F_t (y) \right) + o (t) = o (t)
\]
where \( y := F_s (x) \) and the exponential comes from Theorem 6. \( \blacksquare \)

## 5 Local Frobenius Theorem

**Definition 24** Two arc fields \( X \) and \( Y \) are (locally uniformly) **transverse** if for each \( x_0 \in M \) there exists \( \delta > 0 \) such that
\[
d \left( X_s (x) , Y_t (x) \right) \geq \delta \left( |s| + |t| \right)
\]
for \( |t| < \delta \) for all \( x \in B (x_0, \delta) \).
Example 25 On the plane $\mathbb{R}^2$ with Euclidean norm $\|\cdot\|$ any two linearly independent vectors $u, v \in \mathbb{R}^2$ give us the transverse arc fields

$$X_t(x) := x + tu \quad \text{and} \quad Y_t(x) := x + tv.$$  

To check this, it is easiest to define a new norm on $\mathbb{R}^2$ by

$$\|x\|_{uv} := |x_1| + |x_2|$$

where $x = x_1 u + x_2 v$ and $x_1, x_2 \in \mathbb{R}$. Since all norms on $\mathbb{R}^2$ are metrically equivalent there must exist a constant $C > 0$ such that $\|x\|_{uv} \leq C \|x\|$ for all $x \in \mathbb{R}^2$. Then taking $\delta := \frac{1}{C}$

$$d(X_s(x), Y_t(x)) = \|su - tv\| \geq \delta \|su - tv\|_{uv} = \delta (|s| + |t|).$$

A localization argument shows any pair of continuous vector fields $f$ and $g$ on a differentiable manifold give transverse arc fields if $f$ and $g$ are non-colinear at each point.

A (2-dimensional) surface is a 2-dimensional topological manifold, i.e., locally homeomorphic to $\mathbb{R}^2$.

For any subset $N \subset M$ and element $x \in M$ the distance from $x$ to $N$ is defined as

$$d(x, N) := \inf \{d(x, y) : y \in N\}.$$  

This function $d$ is not a metric, obviously, but it does satisfy the triangle inequality:

$$d(x, N) \leq d(x, y) + d(y, N)$$

for all $x, y \in M$.

Definition 26 A surface $S \subset M$ is an integral surface of two arc fields $X$ and $Y$ if for any Lipschitz functions $a, b : M \to \mathbb{R}$ then $S$ is locally uniformly tangent to $aX + bY$ for $x \in S$, i.e.,

$$d((aX + bY)_t(x), S) = o(t)$$

locally uniformly in $x$. Locally uniform tangency is denoted $S \sim aX + bY$.

Theorem 27 Assume $X \& Y$ close, are transverse, and satisfy $E1$ and $E2$ on a locally complete metric space $M$. Let $F$ and $G$ be the local flows of $X$ and $Y$. If $[F, G] \sim aX + bY$ (locally uniform tangency) for some Lipschitz functions $a, b : M \to \mathbb{R}$, then for each $x_0 \in M$ there exists an integral surface $S$ through $x_0$.

Proof. It may be beneficial to review the outline of this proof from the third paragraph of the introduction. The metric space constructs of the previous sections will now be inserted into the manifold outline. A rigorous verification of the analytic estimates requires some tedious, but straightforward, calculations detailed here.
Figure 2: integral surface $S$

Define

$$S := \{ F_t G_s (x_0) : |s|, |t| < \delta \}$$

where $\delta > 0$ is chosen small enough for $S$ to be a well-defined surface (Figure 2). I.e., $F_{t_1} G_{s_1} (x_0) = F_{t_2} G_{s_2} (x_0)$ implies $t_1 = t_2$ and $s_1 = s_2$ so

$$\phi : (-\delta, \delta) \times (-\delta, \delta) \subset \mathbb{R}^2 \rightarrow S \subset M$$

defined by $\phi (s, t) := F_t G_s (x_0)$ is a homeomorphism. Finding such a $\delta$ is possible since $X$ and $Y$ are transverse. To see this, assume the contrary. Then there are different choices of $s_i$ and $t_i$ which give $F_{t_1} G_{s_1} (x_0) = F_{t_2} G_{s_2} (x_0)$ which implies $G_{s_1} (x_0) = F_{t_3} G_{s_2} (x_0)$ and letting $y := G_{s_2} (x_0)$ we must also then have

$$F_t (y) = G_s (y). \quad (13)$$

If this contrary assumption were true, then for all $\varepsilon > 0$ there would exist $s$ and $t$ with $|s|, |t| < \varepsilon$ such that (13) holds. Since $X$ and $Y$ are transverse, this cannot be so.

We will show $S$ is the desired integral surface through $x_0$. Assume $\delta$ is also chosen small enough so throughout $S$ the functions $|a|$ and $|b|$ are bounded, while the constants $\Lambda$, $\Omega$, and $\rho$ for $X$ and $Y$ hold uniformly, and that the closure of $B (x, 2\delta (\rho + 1))$ is complete. This is possible because $F$ and $G$ have locally bounded speeds, since $X$ and $Y$ do.

Notice $S \sim X$ by construction, but it is not immediately clear $S \sim a'X + b'Y$ for arbitrarily chosen $a', b' \in \mathbb{R}$. Notice we can use

$$a'X + b'Y \sim a'F + b'G \sim b'G + a'F \sim b'Y + a'X$$

and so we will show $S \sim a'F + b'G$. We need to show this is true for an arbitrary point $z \in S$, so assume $z := F_t G_s (x_0)$ for some $s$ and $t \in \mathbb{R}$. Notice by the
construction of $S$ we have $S \sim a''F + b''G$ at $x := G_s(x_0)$ for an arbitrary choice of Lipschitz functions $a''$ and $b''$ since $a''F + b''G \sim b''G + a''F$ and

$$(b''G + a''F)_h(x) = F_{a''(G_{b''(x)}h)(x)}G_{b''(x)}h(x) = F_{a''(G_{b''(x)}h)(x)}G_{b''(x)}h(x) = F_{a''(G_{b''(x)}h)(x)}(G_{b''(x)}h)_hG_{b''(x)}h(x) \in S$$

when $h$ is small.

$(x_0, x, z, s$ and $t$ are now fixed for the remainder of the proof; however, we only explicitly check the case $t > 0$, indicating the changes where needed to check the $t < 0$ case.)

If we prove

$$(F_t)_*(a'F + b'G) \sim S \quad \text{at} \quad x = G_s(x_0) \quad (14)$$

this will prove $S \sim a'F + b'G$ at $z$, since the push-forward $(F_t)_*$ and the pull-back $(F_t)^*$ are inverse and local lipschmorphisms and so preserve tangency. See Figure 3.

![Figure 3: pull-back to $G_s(x_0)$](image)

Restating (11):

$$F_t^*G_t(x) = (t [F,G] + G)_t(x)$$

so

$$F_{t/n}^*G_{t/n}(x) = (\frac{t}{n} [F,G] + G)_{t/n}(x) \quad (15)$$

for our previously fixed small $t \geq 0$ and arbitrary positive integer $n \in \mathbb{N}$. (For $t < 0$ use (12) instead.) For any arc fields $Z$ and $\overline{Z}$ clearly

$$d\left(Z_t(x), \overline{Z}_t(x)\right) = o(t) \quad \text{implies}$$

$$d\left((tZ)_t(x), \overline{(Z)}_t(x)\right) = d\left((Z)_{t^2}(x), (\overline{Z})_{t^2}(x)\right) = o(t^2) \quad (16)$$
and so

\[ [F,G] \sim aF + bG \quad \text{implies} \quad d \left( \left( \frac{1}{n} [F,G] \right)_{t/n}(x), \left( \frac{1}{n} (aF + bG) \right)_{t/n}(x) \right) = o \left( \frac{1}{n} \right) \quad (17) \]

since \( t \) is fixed.

We use these facts to establish (14), first checking

\[ d \left( (F_t^* (a'F + b'G))_{t/n}(x), S \right) = o \left( \frac{1}{n} \right) \]

as \( n \to \infty \). At the end of the proof we will replace \( t/n \) by arbitrary \( r \to 0 \).

Using the linearity of pull-back (Proposition 22) we get

\[
\begin{align*}
&d \left( (F_t^* (a'F + b'G))_{t/n}(x), S \right) \\
= &d \left( \left( (a' \circ F_t) F_t^* (F) + (b' \circ F_t) F_t^*(G) \right)_{t/n}(x), S \right) \\
= &d \left( \left( a_0 F + b_0 F_{t/n}^*(G) \right)_{t/n}(x), S \right)
\end{align*}
\]

where \( a_0 := a' \circ F_t \) and \( b_0 := b' \circ F_t \). Using (15) means this last estimate is

\[
\begin{align*}
&d \left( \left( a_0 F + b_0 F_{t/n}^*(G) \right)_{t/n}(x), S \right) \\
\leq &d \left( \left( \left( \frac{1}{n} [F,G] \right)_{t/n}(x) \right), \left( a_0 F + b_0 F_{t/n}^*(G) \right)_{t/n}(x) \right) \\
&+ d \left( \left( \left( \frac{1}{n} (aF + bG) + G) \right)_{t/n}(x), S \right)
\end{align*}
\]

\[ (18) \]

We estimate the first term as

\[
\begin{align*}
&d \left( \left( a_0 F + b_0 F_{t/n}^*(G) \right)_{t/n}(x), \left( a_0 F + b_0 F_{t/n}^*(G) \right)_{t/n}(x) \right) \\
= &d \left( \left( b_0 F_{t/n}^*(n-1)_{t/n} \left( \frac{1}{n} [F,G] + G) \right)_{t/n}(y), \left( b_0 F_{t/n}^*(n-1)_{t/n} \left( \frac{1}{n} (aF + bG) + G) \right)_{t/n}(y) \right)
\end{align*}
\]

where \( y := a_0 F_{t/n}(x) \)

\[
\begin{align*}
&d \left( \left( F_{(n-1)t/n}^* \left( \frac{1}{n} [F,G] + G) \right)_{t/n}(y), \left( F_{(n-1)t/n}^* \left( \frac{1}{n} (aF + bG) + G) \right)_{t/n}(y) \right) \\
= &d \left( \left( F_{(n-1)t/n}^* \left( \frac{1}{n} [F,G] + G) \right)_{t/n}(y), \left( F_{(n-1)t/n}^* \left( \frac{1}{n} (aF + bG) + G) \right)_{t/n}(y) \right)
\end{align*}
\]

\[ (19) \]
where \( z := F_{(n-1)t/n}(y) \). Then by Theorem 20, \( r \) is

\[
\leq d \left( \left( \frac{1}{r} \right) \left[ F, G \right] \right)_{b_0(y)t/n}(z), \left( \frac{1}{n} \right) (a F + b G)_{b_0(y)t/n}(z) \right) \right) e^{\lambda x (n-1)t/n} \]

\[
d \left( G_{b_0(y)t/n} \left( \frac{1}{r} \right) \left[ F, G \right] \right)_{b_0(y)t/n}(z), G_{b_0(y)t/n} \left( \frac{1}{n} \right) (a F + b G)_{b_0(y)t/n}(z) \right) \right) e^{\lambda x (n-1)t/n} \]

\[
\leq d \left( \left( \frac{1}{r} \right) \left[ F, G \right] \right)_{b_0(y)t/n}(z), \left( \frac{1}{n} \right) (a F + b G)_{b_0(y)t/n}(z) \right) \right) e^{\lambda x (n-1)t/n} e^{\lambda y b_0(y)t/n} \]

\[
\leq r \left( b_0 (y) \left( \frac{1}{r} \right)^2 \right) e^{\lambda x (n-1)t/n + \lambda y b_0(y)t/n} =: o_1 \left( \frac{1}{n^2} \right) \quad (20)
\]

where we define

\[
r(s) := d \left( \left[ F, G \right] \right)_{b_0(y)t/n}(z), (a F + b G)_{b_0(y)t/n}(z) \right) \right) .
\]

By the main assumption of the theorem, \( r(s) = o(s) \) so notice we have \( o_1 \left( \frac{1}{n^2} \right) = o \left( \frac{1}{n^2} \right) \) but we need to keep a careful record of this estimate as we will be summing \( n \) terms like it: the subscript distinguishes \( o_1 \) as a specific function.

Substituting (20) into (18) gives

\[
d \left( F^* \left( a' F + b' G \right) \right)_{t/n}(x), S \right) \]

\[
= d \left( \left( a_0 F + b_0 F^{(n)} \right)_{t/n}(x), S \right) \]

\[
\leq d \left( \left( a_0 F + b_0 F^{(n-1)} \right) \left( \frac{1}{n} \right) (a F + b G) \right)_{t/n}(x), S \right) + o_1 \left( \frac{1}{n^2} \right)
\]

\[
= d \left( \left( a_0 F + b_0 F^{(n-1)} \right) \left( \frac{1}{n} \right) (a F + b G) \right)_{t/n}(x), S \right) + o_1 \left( \frac{1}{n^2} \right)
\]

\[
= d \left( \left( [a_0 + \left( b_0 \left( \frac{1}{n} \right) (a \circ F_{(n-1)t/n}) \right) \circ (a_0 F_{t/n}) \right] \right) \left( \frac{1}{n} \right) (a F + b G) \right)_{t/n}(x), S \right) + o_1 \left( \frac{1}{n^2} \right)
\]

\[
= d \left( \left( a_0 F + b_0 F^{(n-1)} \right)_{t/n}(x), S \right) + o_1 \left( \frac{1}{n^2} \right)
\]

where

\[
a_1 := a_0 + \left( b_0 \left( \frac{1}{n} \right) (a \circ F_{(n-1)t/n}) \right) \circ (a_0 F_{t/n}) \quad \text{and}
\]

\[
b_1 := b_0 \cdot \left( \frac{1}{n} \right) (b \circ F_{(n-1)t/n} + 1) .
\]

This painful calculation from the third line to the fourth line employs the linearity of pull-back (Proposition 22); while the fifth line is due to the linearity of \( F \) (Proposition 20).

After toiling through these many complicated estimates we can relax a bit, since the rest of the proof follows more mechanically by iterating the result of
Therefore, in the region of interest the lines (21) and (22):

\[
\begin{align*}
d \left( \left( a_0F + b_0F^{*}(n) \right)_{t/n} (x), S \right) \\
\leq d \left( \left( a_1F + b_1F^{*}(n-1) \right)_{t/n} (x), S \right) + o_1 \left( \frac{1}{n^2} \right) \\
\leq d \left( \left( a_2F + b_2F^{*}(n-2) \right)_{t/n} (x), S \right) + o_1 \left( \frac{1}{n^2} \right) + o_2 \left( \frac{1}{n^2} \right) \\
\leq ... \leq d \left( (a_nF + b_nG)_{t/n} (x), S \right) + \sum_{i=1}^{n} o_i \left( \frac{1}{n^2} \right) 
\end{align*}
\]

(23)

where

\[
\begin{align*}
a_2 &:= a_1 + \left( b_1 \frac{1}{n} \left( a \circ F_{(n-2)t/n} \right) \right) \circ (a_1F_{t/n}) \\
b_2 &:= b_1 \cdot \left( \frac{1}{n} \left( b \circ F_{(n-2)t/n} \right) + 1 \right) \quad \text{and in general} \\
a_i &:= a_{i-1} + \left( b_{i-1} \frac{1}{n} \left( a \circ F_{(n-i)t/n} \right) \right) \circ (a_{i-1}F_{t/n}) \\
b_i &:= b_{i-1} \cdot \left( \frac{1}{n} \left( b \circ F_{(n-i)t/n} \right) + 1 \right)
\end{align*}
\]

In the region of interest the \(|a|\) and \(|a_0|\) are bounded by some \(A \in \mathbb{R}\) and \(|b|\) and \(|b_0|\) are bounded by some \(B \in \mathbb{R}\) so

\[
\begin{align*}
|b_1| &= |b_0 \cdot \left( \frac{1}{n} \left( b \circ F_{(n-1)t/n} \right) + 1 \right)| \leq B \left( \frac{1}{n}B + 1 \right) \\
|b_2| &= |b_1 \cdot \left( \frac{1}{n} \left( b \circ F_{(n-1)t/n} \right) + 1 \right)| \leq B \left( \frac{1}{n}B + 1 \right)^2 \\
|b_i| &\leq B \left( \frac{1}{n}B + 1 \right)^i \quad \text{and}
\end{align*}
\]

\[
\begin{align*}
|a_1| &= |a_0 + b_0 \frac{1}{n} \left( a \circ F_{(n-1)t/n} \right)| \leq A + B \frac{1}{n} A \\
|a_2| &= |a_1 + b_1 \frac{1}{n} \left( a \circ F_{(n-2)t/n} \right)| \leq \left( A + B \frac{1}{n} A \right) + B \left( \frac{1}{n}B + 1 \right) \frac{1}{n} A \\
|a_3| &= |a_2 + b_2 \frac{1}{n} \left( a \circ F_{(n-3)t/n} \right)| \\
&\leq A + B \frac{1}{n} A + B \left( \frac{1}{n}B + 1 \right) \frac{1}{n} A + B \left( \frac{1}{n}B + 1 \right)^2 \frac{1}{n} A \\
|a_i| &\leq A + \frac{1}{n}AB \sum_{k=0}^{i-1} \left( \frac{1}{n}B + 1 \right)^k = A + \frac{1}{n}AB \frac{\left( \frac{1}{n}B + 1 \right)^i - 1}{\frac{1}{n}B} \\
&= A \left( \frac{1}{n}B + 1 \right)^i.
\end{align*}
\]

Therefore

\[
\begin{align*}
|b_n| &\leq B \left( \frac{1}{n}B + 1 \right)^n \leq Be^{tB} \quad \text{and} \\
|a_n| &\leq A \left( \frac{1}{n}B + 1 \right)^n \leq Ae^{tB}.
\end{align*}
\]

Penultimately, we need to estimate the \(o_1 \left( \frac{1}{n^2} \right)\). Remember from line (20)

\[
o_1 \left( \frac{1}{n^2} \right) := r \left( b_0 (y) \left( \frac{1}{n} \right)^2 \right) e^{A x (n-1)t/n + \lambda y b_0(y)t/n}
\]

21
where \( r(s) = o(s) \), so
\[
o_2\left(\frac{1}{nt}\right) = r\left(b_1(y)\left(\frac{1}{n}\right)^2\right)e^{\Lambda X(n-2)t/n+\Lambda Y b_1(y)t/n} \\
\leq B\left(\frac{1}{n}B + 1\right)o\left(\left(\frac{1}{n}\right)^2\right)e^{\Lambda X(n-2)t/n+\Lambda Y B\left(\frac{1}{n}B1\right)t/n} \\
o_1\left(\frac{1}{nt}\right) = r\left(b_{i-1}(y)\left(\frac{1}{n}\right)^2\right)e^{\Lambda X(n-i)t/n+\Lambda Y b_{i-1}(y)t/n}.
\]

Consequently
\[
\sum_{i=1}^{n} o_i\left(\frac{1}{nt}\right) \leq \sum_{i=1}^{n} r\left(b_{i-1}(y)\left(\frac{1}{n}\right)^2\right)e^{\Lambda X(n-i)t/n+\Lambda Y B\left(\frac{1}{n}B1\right)^{i-1}t/n} \\
\leq o\left(\left(\frac{1}{n}\right)^2\right) B e^{t/n\sum_{i=1}^{n} e^{\Lambda X(n-i)t/n+\Lambda Y B\left(\frac{1}{n}B1\right)^{i-1}t/n}}
\]

since \( r\left(b_{i-1}(y)\left(\frac{1}{n}\right)^2\right) = o\left(\left(\frac{1}{n}\right)^2\right) B e^{t/n} \) for all \( i \). Therefore
\[
\sum_{i=1}^{n} o_i\left(\frac{1}{nt}\right) \leq o\left(\left(\frac{1}{n}\right)^2\right) B e^{t/n} e^{\Lambda X t+\Lambda Y B t/n} = o\left(\frac{1}{n}\right)
\]
as \( n \to \infty \). Putting this into \( 23 \) gives
\[
d\left((F^*_t(a^t F + b^t G))_{t/n}(x), S\right) \leq d\left((a_n F + b_n G)_{t/n}(x), S\right) + o\left(\frac{1}{n}\right) = o\left(\frac{1}{n}\right)
\]
because of the uniform bound on \(|a_n|\) and \(|b_n|\). To see this notice
\[
d\left((a_n F + b_n G)_{t/n}(x), S\right) = o\left(\frac{1}{n}\right)
\]
uniformly for bounded \( a_* \) and \( b_* \) since \( a_* F + b_* G \sim b_* G + a_* F \) and as before \((b_n G + a_n F)_{t/n}(x) \in S\) using the uniform \( \Lambda \) and \( \Omega \) derived in the proofs of Propositions \( 17 \) and \( 18 \) (cf. Remark \( 4 \)).

Finally we need to check
\[
d\left((F^*_t(a^t F + b^t G))_r(x), S\right) = o(r)
\]
when \( r \) is not necessarily \( t/n \). We may assume \( 0 < t < 1 \) and \( 0 < r < t \) so that \( t = nr + \varepsilon \) for some \( 0 \leq \varepsilon < r \) and integer \( n \) with \( \frac{t}{r} - 1 < n \leq \frac{t}{r} \). Therefore the above calculations give
\[
d\left((F^*_t(a^t F + b^t G))_r(x), S\right) = d\left((F^*_t(a^t F + b^t G))_r(x), S\right) \leq o(r) = o(r) .
\]

The \( n \)-dimensional corollary of this 2-dimensional version is given in the next section.
Remark 28 In the assumptions of Theorem 27 \([F, G]\) can be replaced with \([X, Y]\) when they are tangent. Since the brackets use \(\sqrt{t}\) we have \([F, G] \sim [X, Y]\) when \(X\) and \(Y\) are 2nd-order tangent to their flows, i.e.,

\[
\begin{align*}
    d \left( X_t(x), F_t(x) \right) &= O\left(t^2\right) \\
    d \left( Y_t(x), G_t(x) \right) &= O\left(t^2\right)
\end{align*}
\]

locally uniformly. We denote 2nd-order local uniform tangency by \(X \approx F\). This holds, for example, when \(X\) comes from a twice continuously differentiable vector field by Taylor’s theorem. But in formulating our theorem for the nonsmooth case, the two brackets are not interchangeable. Beware: 2nd-order tangency is “big oh” of \(t^2\), not “little oh”.

We might have chosen to define the bracket \([X, Y]\) using the flows instead of the arc fields to simplify the statements of Theorem 27 and those below. However it is often easier to calculate the bracket and to check closure using arc fields instead of the flows.

In light of this remark, Theorem 27 gives

Corollary 29 Assume \(X \& Y\) close, are transverse, and satisfy E1 and E2 on a locally complete metric space \(M\). Further assume \(X\) and \(Y\) are 2nd-order tangent to their local flows \(F\) and \(G\). If \([X, Y] \sim aX + bY\) for some Lipschitz functions \(a, b : M \to \mathbb{R}\), then for each \(x_0 \in M\) there exists an integral surface \(S\) through \(x_0\).

6 Global Frobenius Theorem

The goal of this section is to recast Theorem 27 in the language of distributions and foliations, and so we begin with several definitions. \(M\) is, as ever, a locally complete metric space.

Definition 30 A distribution \(\Delta\) on \(M\) is a set of curves in \(M\).

Example 31 A single arc field \(X\) gives a distribution by forgetting \(X\) is continuous in its space variable \(x\), and defining \(\Delta = \{X(x, \cdot) : x \in M\}\). Any union of arc fields similarly gives a distribution.

Given two arc fields \(X\) and \(Y\), their linear span is a distribution:

\[
\Delta (X, Y) = \{(aX + bY)(x, \cdot) : a, b \in \mathbb{R}, x \in M\}.
\]

The direct sum of an arbitrary collection of arc fields similarly gives a distribution, defined with finite summands. All of the following definitions can, of course, be made with arbitrary indexing sets; but we will only use finite sets of generators in the theorems of this paper.

Denote \(\Delta_x := \{c \in \Delta : c(0) = x\}\).
Definition 32 \( X \) is (locally uniformly) **transverse** to \( \Delta \) if for all \( x_0 \in M \) there exists a \( \delta > 0 \) such that for all \( x \in B(x_0, \delta) \) we have
\[
d(X_t(x), c(s)) \geq \delta (|s| + |t|)
\]
for all \( c \in \Delta_x \) and all \( |s|, |t| < \delta \). The arc fields \( \frac{1}{i} X, \frac{2}{i} X, \ldots, \frac{n}{i} X \) are transverse to each other if for each \( i \in \{1, \ldots, n\} \) we have \( X \) transverse to
\[
\Delta \left( \frac{1}{i} X, \frac{2}{i} X, \ldots, \frac{i-1}{i} X, \frac{i+1}{i} X, \ldots, \frac{n}{i} X \right).
\]
For \( y \in M \) define
\[
d(y, \Delta_x) := \inf \{ d(y, c(t)) : c \in \Delta_x \text{ and } t \in \text{dom}(c) \}.
\]
If \( \Delta_x = \emptyset \) then, as usual, the distance is \( \infty \) by definition. So if \( X \) is transverse to \( \Delta \) then if for all \( x_0 \in M \) there exists a \( \delta > 0 \) such that for all \( x \in B(x_0, \delta) \) we have
\[
d(X_t(x), \Delta) \geq \delta |t|
\]
for all \( |t| < \delta \).

**Definition 33** \( X \) is **tangent** to \( \Delta \) if for each \( x \in M \)
\[
d(X_t(x), \Delta_x) = o(t).
\]
If this distance is \( o(t) \) locally uniformly in \( x \in M \) then \( X \) is **locally uniformly tangent** to \( \Delta \), denoted \( X \sim \Delta \).

Two distributions \( \Delta \) and \( \tilde{\Delta} \) are tangent if for each \( c \in \Delta \) there exists \( \tilde{c} \in \tilde{\Delta} \) such that \( \tilde{c} \) is tangent to \( c \) (at \( t = 0 \)), and vice-versa, for each \( \tilde{c} \in \tilde{\Delta} \) there exists \( c \in \Delta \) such that \( c \) is tangent to \( \tilde{c} \). Local uniform tangency is defined in the obvious way, and denoted \( \Delta \sim \tilde{\Delta} \). Again, \( \sim \) is an equivalence relation.

**Definition 34** A distribution \( \Delta \) is \( n \)-**dimensional** if there exists a set of \( n \) transverse arc fields \( \left\{ \frac{1}{i} X, \frac{2}{i} X, \ldots, \frac{n}{i} X \right\} \) which all mutually close and satisfy E1 and E2 such that \( \Delta \sim \Delta \left( \frac{1}{i} X, \frac{2}{i} X, \ldots, \frac{n}{i} X \right) \).

Given \( X \), if there exist Lipschitz functions \( a_k : M \to \mathbb{R} \) such that \( X \sim \sum_{k=1}^{n} a_k \hat{X} \) then clearly \( X \sim \Delta \left( \frac{1}{i} X, \frac{2}{i} X, \ldots, \frac{n}{i} X \right) \).

**Definition 35** An \( n \)-**dimensional distribution** \( \Delta \sim \Delta \left( \frac{1}{i} X, \frac{2}{i} X, \ldots, \frac{n}{i} X \right) \) is **involutive** if for each choice of \( i, j \in \{1, \ldots, n\} \) we have
\[
\left[ \frac{i}{i} X, \frac{j}{j} X \right] \sim \Delta.
\]
Definition 36 An surface $S$ in $M$ is an $n$-dimensional topological manifold $S \subset M$. A surface is locally uniformly tangent to an arc field $X$, denoted $X \sim S$, if $d(X_{t}(x),S) = o(t)$ locally uniformly in $x$.

A surface is said to be an integral surface for an $n$-dimensional distribution $\Delta \sim \Delta(X,\ldots,X)$ if $\sum_{k=1}^{n} a_k X \sim S$ for any choice of Lipschitz functions $a_k : M \rightarrow \mathbb{R}$.

A distribution is said to be integrable if there exists an integral surface through every point in $M$.

Theorem 27 has the following corollary:

Theorem 37 An $n$-dimensional involutive distribution is integrable.

Proof. $n = 1$ is Theorem 2. $n = 2$ is Theorem 27. Now proceed by induction. We do enough of the case $n = 3$ to suggest the path; and much of this is copied from the proof of Theorem 27.

Choose $x_0 \in M$. Let $X, Y, Z$ be the transverse arc fields guaranteed in the definition of a 3-dimensional distribution. If we find an integral surface $S$ for $\Delta(X,Y,Z)$ through $x_0$ then obviously $S$ is an integral surface for $\Delta$. Let $F, G, H$ be the local flows of $X, Y, Z$ and define

$$S := \{F_{t}G_{s}H_{r}(x_0) : |r|, |s|, |t| < \delta\}$$

with $\delta > 0$ chosen small enough as in the proof of Theorem 27 so that $S$ is a three dimensional manifold. Again we may assume $\delta$ is also chosen small enough so that throughout $S$ the functions $|a_k|$ are bounded by $\Lambda$, the constants $\Lambda, \Omega, \rho$ for $X, Y$ and $Z$ hold uniformly, and the closure of $B(x, 3\delta (\rho + 1))$ is complete. Notice

$$\bar{S} := \{G_{s}H_{r}(x_0) : |r|, |s| < \delta\}$$

is an integral surface through $x_0$ for $\Delta(Y,Z)$ by the proof of Theorem 27. Notice $S \sim X$ by construction, but it is not immediately clear $S \sim a^\prime X + b^\prime Y + c^\prime Z$ for arbitrarily chosen $a^\prime, b^\prime, c^\prime \in \mathbb{R}$. Again we really only need to show $S \sim a^\prime F + b^\prime G + c^\prime H$ for an arbitrary point $z := F_{t}G_{s}H_{r}(x_0) \in S$, and again it is sufficient to prove

$$(F_t)^*(a^\prime F + b^\prime G + c^\prime H) \sim S \quad \text{at} \quad y = G_{s}H_{r}(x_0)$$

by the construction of $S$. Continue as above adapting the same tricks from the proof of Theorem 27 to the extra dimension.

Similar to the definition for a surface, an arbitrary set $S \subset M$ is defined to be locally uniformly tangent to $X$ if

$$d(X_{t}(y),S) = o(t)$$

locally uniformly for $y \in S$, denoted $S \sim X$. 

25
Lemma 38  Let \( \sigma_x : (\alpha, \beta) \to U \subset M \) be a solution to \( X \) which meets Condition E1 with uniform constant \( \Lambda \) on a neighborhood \( U \). Assume \( S \subset U \) is a closed set with \( S \sim X \). Then

\[
d(\sigma_x(t), S) \leq e^{\Lambda t} d(x, S) \quad \text{for all} \quad t \in (\alpha, \beta).
\]

**Proof.** (Adapted from the proof of Theorem 6)

We check only \( t > 0 \). Define

\[
g(t) := e^{-\Lambda t} d(\sigma_x(t), S).
\]

For \( h \geq 0 \), we have

\[
g(t + h) - g(t)
= e^{-\Lambda(t+h)} d(\sigma_x(t+h), S) - e^{-\Lambda t} d(\sigma_x(t), S)
\leq e^{-\Lambda(t+h)} [d(\sigma_x(t+h), X_h(\sigma_x(t))) + d(X_h(\sigma_x(t)), X_h(y)) + d(X_h(y), S)]
\]

\[
- e^{-\Lambda t} d(\sigma_x(t), S)
\]

for any \( y \in S \), which in turn is

\[
\leq e^{-\Lambda(t+h)} [d(X_h(\sigma_x(t)), X_h(y)) + o(h)] - e^{-\Lambda t} d(\sigma_x(t), S)
\leq e^{-\Lambda t} e^{-\Lambda h} d(\sigma_x(t), y)(1 + \Lambda h) - e^{-\Lambda t} d(\sigma_x(t), S) + o(h)
\]

\[
= [e^{-\Lambda h} (1 + \Lambda h) d(\sigma_x(t), y) - d(\sigma_x(t), S)] e^{-\Lambda t} + o(h).
\]

Therefore

\[
g(t + h) - g(t) \leq \left[e^{-\Lambda h} (1 + \Lambda h) - 1\right] e^{-\Lambda t} d(\sigma_x(t), S) + o(h)
\]

since \( y \) was arbitrary in \( S \). Thus

\[
g(t + h) - g(t)
\leq o(h) e^{-\Lambda t} d(\sigma_x(t), S) + o(h) = o(h) (g(t) + 1).
\]

Hence, the upper forward derivative of \( g(t) \) is nonpositive; i.e.,

\[
D^+ g(t) := \lim_{h \to 0^+} \left( \frac{g(t + h) - g(t)}{h} \right) \leq 0.
\]

Consequently, \( g(t) \leq g(0) \) or

\[
d(\sigma_x(t), S) \leq e^{\Lambda t} d(\sigma_x(0), S) = e^{\Lambda t} d(x, S).
\]

Choosing \( x \in S \) in Lemma 38 gives the following metric space generalization of the Nagumo-Brezis Invariance Theorem (Example 3 shows how this generalizes the Banach space setting). We state and prove only the bidirectional case; the case for forward flows is easily adapted \textit{mutatis mutandis}. Cf. 7 for an exposition on general invariance theorems.
Theorem 39  Let $X$ satisfy $E1$ and $E2$ and assume a closed set $S \subset M$ has $S \sim X$. Then for any $x \in S$ we have $F_t(x) \in S$ for all $t \in (\alpha_x, \beta_x)$. I.e., $S$ is an invariant set under the flow $F$.

Theorem 40  The integral surfaces guaranteed by Theorem 37 are unique in the following sense: if $S_1$ and $S_2$ are integral surfaces through $x \in M$, then $S_1 \cap S_2$ is an integral surface.

Proof. The case $n = 1$ is true by the uniqueness of integral curves.

For higher dimensions $n$, Theorem 39 guarantees $S_1$ and $S_2$ contain local integral curves for $\sum_{k=1}^{n} a_k \hat{X}$ for all choices of $a_k \in \mathbb{R}$ with initial condition $x$.

Since the $\hat{X}$ are transverse, there is a small neighborhood of $x$ on which all the choices of the parameters $a_k$ give local non-intersecting curves in $M$ which fill up $n$ dimensions. ■

Therefore, by continuation we have a unique maximal connected integral surface through each point.

Definition 41  A foliation partitions $M$ into a set of subsets $\Phi := \{L_i\}_{i \in I}$ for some indexing set $I$, where the subsets $L_i \subset M$ (called leaves) are disjoint, connected topological manifolds each having the same dimension.

A foliation $\Phi$ is tangent to a distribution $\Delta$ if the leaves are integral surfaces. When a foliation exists which is tangent to a distribution $\Delta$ we say $\Delta$ foliates $M$.

Theorem 42  An $n$-dimensional involutive distribution has a unique tangent foliation.

Proof. Theorems 37 and 40 guarantee the existence of the leaves, i.e., the unique maximal integral surfaces. ■

The converse of this result is easy to prove in the classical context on a Banach space. I do not believe it is true here. Instead we have the following partial converse. Cf. Remark 28.

Proposition 43  Let $\Delta \sim \Delta \left( \frac{1}{k} \hat{X}, \frac{2}{k} \hat{X}, ..., \frac{n}{k} \hat{X} \right)$ be an $n$-dimensional distribution with $\hat{X} \approx \hat{F}$ where $\hat{F}$ is the local flow for $\hat{X}$. If $\Delta$ foliates $M$ then $\Delta$ is involutive.

Proof. Remark 28 gives $\left[ \frac{i}{k} \hat{X}, \frac{j}{k} \hat{X} \right] \sim \hat{F} \Delta \hat{F}$ and Theorem 39 gives $\left[ \hat{F}, \hat{F} \right]^t(x) \in L_i$ if $x \in L_i$ so $\left[ \hat{F}, \hat{F} \right] \sim \Delta$. ■

Collecting all these results we have the following version of the Global Frobenius Theorem.
Theorem 44 Let \( \Delta \sim \Delta \left( \frac{1}{X}, \frac{2}{X}, ..., \frac{n}{X} \right) \) be an \( n \)-dimensional distribution on a locally complete metric space \( M \), with \( X \approx F \) where \( F \) is the local flow for \( X \). The following are equivalent:

1. \( \Delta \) is involutive
2. \( \Delta \) is integrable
3. \( \Delta \) foliates \( M \).

7 Commutativity of Flows

Theorem 45 Assume \( X \) and \( Y \) satisfy E1 and E2 on a locally complete metric space \( M \). Let \( F \) and \( G \) be the local flows of \( X \) and \( Y \). Then \( [F, G] \sim 0 \) if and only if \( F \) and \( G \) commute, i.e.,

\[
F_t G_s (x) = G_s F_t (x), \quad \text{i.e.,} \quad F_t^* (G) = G.
\]

Proof. The assumption \([F, G] \sim aX + bY\) with \( a = b = 0\) allows us to copy the approach in the proof of Theorem 27. Let \( \delta > 0 \) be chosen small enough so

1. the functions \(|a|\) and \(|b|\) are bounded
2. the constants \( \Lambda, \Omega, \) and \( \rho \) for \( X \) and \( Y \) hold uniformly
3. \([F, G] \sim 0\) uniformly

all on \( S := B(x, 2\delta (\rho + 1)) \) and that \( S \) is also complete. We check \( t > 0 \). Since \( F_t^* (G) \) and \( G \) are both local flows, we only need to show they are tangent to each other and then they must be equal by uniqueness of solutions.

Imagine being in the context of differentiable manifolds. There, for vector fields \( f \) and \( g \) with local flows \( F \) and \( G \), we would have

\[
\lim_{h \to 0} \frac{F_h^* (g) - g}{h} = \mathcal{L}_f g = [f, g] = 0
\]

so \( F_h^* (g) = g + o(h) \) and thus we expect

\[
F_t^* (g) = g + o(h).
\]

We might use this idea as before with the linearity of pull-back (Proposition 22) to get

\[
F_t^* (g) = \lim_{n \to \infty} F_{t/n}^{*\circ (n)} (g) = \lim_{n \to \infty} g + no (1/n) = g
\]

as desired.

Now in our context of metric spaces with \( t > 0 \), line (11) again gives

\[
F_{t/n}^* (G)_{t/n} (x) = \left( \frac{1}{n} [F, G] + G \right)_{t/n} (x).
\]

For \( t < 0 \) one would use (12). Also we again have

\[
[F, G] \sim 0 \quad \text{implies} \quad d \left( \left( \frac{1}{n} [F, G] \right)_{t/n} (x), x \right) = o \left( \frac{1}{n^2} \right).
\]
Using these tricks (and Theorem 6 in the fourth line following) gives

\[ d \left( (F_t^n(G))_{t/n} (x), G_{t/n} (x) \right) = d \left( (F_t^{n-1}(G))_{t/n} (x), G_{t/n} (x) \right) \]

\[ = d \left( F_t^{n-1} \left( \frac{t}{n} |F,G| + G \right), G_{t/n} (x) \right) \]

\[ \leq d \left( F_t^{n-1} \left( G_{t/n} + \frac{t}{n} |F,G|_{t/n} (x) \right), F_t^{n-1} G_{t/n} (x) \right) + d \left( F_t^{n-1} G_{t/n} (x), G_{t/n} (x) \right) \]

\[ \leq d \left( G_{t/n} + \frac{t}{n} |F,G|_{t/n} (y), G_{t/n} (y) \right) e^{\Lambda X \frac{t(n-1)}{n}} + d \left( F_t^{n-1} G_{t/n} (x), G_{t/n} (x) \right) \]

where \( y := F_{(n-1)t/n} (x) \)

\[ \leq d \left( \frac{t}{n} |F,G|_{t/n} (y), y \right) e^{\Lambda Y t/n} e^{\Lambda X \frac{t(n-1)}{n}} + d \left( F_t^{n-1} G_{t/n} (x), G_{t/n} (x) \right) \]

and so

\[ d \left( (F_t^n(G))_{t/n} (x), G_{t/n} (x) \right) \]

\[ \leq d \left( F_t^{n-1} G_{t/n} (x), G_{t/n} (x) \right) + e^{\Lambda Y t/n + \Lambda X \frac{t(n-1)}{n}} o_1 \left( \frac{1}{n^2} \right) \]

where \( o_1 \left( \frac{1}{n^2} \right) := d \left( \frac{t}{n} |F,G|_{t/n} (y), y \right) \) and \( y := F_{(n-1)t/n} (x) \). Since \( d \left( \frac{t}{n} |F,G|_{t/n} (y), y \right) = o \left( \frac{1}{n^2} \right) \) uniformly for \( y \in B (x, 2 \delta (p + 1)) \) we have

\[ d \left( (F_t^n(G))_{t/n} (x), G_{t/n} (x) \right) \]

\[ \leq e^{\Lambda Y t/n} \sum_{i=1}^{n} o_1 \left( \frac{1}{n^2} \right) e^{\Lambda X \frac{t(n-1)}{n}} = o \left( \frac{1}{n^2} \right) e^{\Lambda Y t/n} \sum_{i=1}^{n} e^{\Lambda X \frac{t(n-1)}{n}} \]

\[ = o \left( \frac{1}{n^2} \right) e^{\Lambda Y t/n} e^{\Lambda X t} \sum_{i=1}^{n} \left( e^{-\frac{t}{n}} \right)^i = o \left( \frac{1}{n^2} \right) e^{\Lambda Y t/n + \Lambda X t} \frac{1 - \left( e^{-\frac{t}{n}} \right)^{n+1}}{1 - \left( e^{-\frac{t}{n}} \right)}. \]
So
\[ d \left( (F^*_t (G))_{t/n} (x), G_{t/n} (x) \right) = o \left( \frac{1}{n} \right) \]
and \( F^*_t (G) \sim G \) by the same argument at the last paragraph of the proof of Theorem 27.

The converse is trivial. ■

Using Example 3, this theorem applies to the non-locally compact setting with nonsmooth vector fields. [10], another paper which inspires this monograph, obtains similar results with a very different approach.

8 Examples

**Example 46** Let \( M \) be a Banach space. First let \( X \) and \( Y \) be translations in the directions of \( u \) and \( v \in M \)

\[
X_t (x) := x + tu \quad Y_t (x) := x + tv
\]

then \( F = X \) and \( G = Y \) for \( |t| \leq 1 \). Obviously \( [F,G] = 0 \) and the flows commute.

Next consider the dilations \( X \) and \( Y \) about \( u \) and \( v \in M \)

\[
X_t (x) := (1 + t) (x - u) + u \quad Y_t (x) := (1 + t) (x - v) + v.
\]

The flows are computable with little effort using Euler curves, e.g.,

\[
F_t (x) = \lim_{n \to \infty} X_t^{(n)} (x) = e^t x - (e^t - 1) u.
\]

Then for \( t \geq 0 \)

\[
[F,G]_{t^2} (x) = G_{-t} F_{-t} G_t F_t (x) = e^{-t} \left[ e^{-t} \left( e^t x - (e^t - 1) u \right) - (e^{-t} - 1) v \right] - (e^{-t} - 1) v
\]

\[
= x - u + e^{-t} u - e^{-t} v + e^{-2t} u + e^{-2t} v - e^{-t} u - e^{-t} v + v
\]

\[
= x + (v - u) (e^{-t} - 1)^2
\]

so \( [F,G] \sim Z \) where \( Z_t (x) := x + t (v - u) \) since, for instance with \( t > 0 \)

\[
d \left( [F,G]_t (x), Z_t (x) \right) = |v - u| \left( e^{-\sqrt{t}} - 1 \right)^2 - t = |t| |v - u| \left( \frac{e^{-\sqrt{t}} - 1}{\sqrt{t}} \right)^2 - 1 = o (t).
\]

Hence the distribution \( \Delta (X,Y) \) is not involutive. However, this shows three dilations \( X,Y,Z \) about linearly independent \( u,v,w \) generate all translations using
their brackets. Using the same tricks we’ve just employed, it is easy to check the bracket of a dilation and a translation is tangent to a translation, e.g., if 
\[ F_t(x) := x + tu \]
and
\[ G_t(x) := e^{tx} \]
then
\[ [F,G] \sim F \]
since for \( t > 0 \)
\[ [F,G]_t(x) = G_{-t}F_{-t}G_tF_t(x) = e^{-t} [e^t [x + tu] - tu] = x + tu (1 - e^{-t}) \]
and so
\[ d([F,G]_t(x),F_t(x)) = |tu| \left| \frac{1-e^{-\sqrt{t}}}{\sqrt{t}} - 1 \right| = o(t) . \]
To summarize:
\[
\begin{align*}
\Delta (\text{translations}) & \quad \text{involutive} \\
\Delta (\text{dilations}) & \quad \text{not involutive} \\
\Delta (\text{dilations,translations}) & \quad \text{involutive}.
\end{align*}
\]
(24)
The previous example holds with minor modifications on the metric space \((H(\mathbb{R}^n), d_H)\) where \( H(\mathbb{R}^n) \) is the set of non-void compact subsets of \( \mathbb{R}^n \) and \( d_H \) is the Hausdorff metric. Theorem 42 gives foliations.

**Example 47 (two parameter decomposition of \( L^2 \))** Now let \( M \) be real Hilbert space \( L^2(\mathbb{R}) \). Since \( M \) is Banach the results of the previous example hold. Denote translation by the function \( h \in L^2(\mathbb{R}) \) by
\[ X_t(f) := f + th. \]
Now however, there is another obvious candidate for an elementary flow: translation with respect to the variable \( x \), i.e.,
\[ Y_t(f)(x) := f(x + t). \]
Unlike dilation and translation, the dynamic engendered by \( Y \) seemingly has nothing to do with the vector space structure of \( L^2(\mathbb{R}) \). In fact, despite appearances, \( Y \) is a nonsmooth flow: notice for example with the characteristic function \( \chi \) as initial condition,
\[ \left. \frac{d}{dt} Y_t(\chi_{[0,1]}) \right|_{t=0} \notin L^2(\mathbb{R}). \]
Interpreted as a flow on a metric space, however, this is no obstacle. We refer to \( X \) as **vector space translation** and \( Y \) as **function translation**. Notice \( X \) and \( Y \) are their own flows (for \(|t| \leq 1\)). It is straightforward to check \( X \not\parallel Y \) close when, for example, \( h \in C^1(\mathbb{R}) \) with derivative \( h' \in L^2(\mathbb{R}) \):
\[
\begin{align*}
d(Y_sX_t(f),X_tY_s(f)) &= \sqrt{\int (f(x + s) + th(x + s) - [f(x + s) + th(x)])^2 dx} \\
&= |st| \sqrt{\int \left( \frac{h(x + s) - h(x)}{s} \right)^2 dx} \\
&= O(|st|)
\end{align*}
\]
uniformly. Since they obviously satisfy E1 and E2, Theorem 1 promises a unique flow for their sum. This was introduced by Colombo and Corli in [6, section 5.2] with other interesting function space examples, which they also characterize with partial differential equations.

Let us now compute the bracket. We check \( t > 0 \) explicitly, skipping the case \( t \leq 0 \) though this is just as easy.

\[
[X,Y]_t^2 (f) (x) = Y_t X_t Y_t X_t (f) (x) = Y_t X_t (f (x + t) + th (x + t))
\]

\[
= f (x) + th (x) - th (x - t) = f (x) + t^2 \frac{h (x) - h (x - t)}{t}.
\]

Defining a new arc field \( Z_t (f) := f + th \) we therefore have

\[
d ([X,Y]_t^2 (f), Z_t (f)) = |t| \sqrt{\int \frac{(h (x) - h (x - t)}{t} - h' (x)}^2 dx = o (t)
\]

when \( h \in C^1 (\mathbb{R}) \) with \( h' \in L^2 (\mathbb{R}) \). Thus \([X,Y] \sim Z\).

This has remarkable consequences. Using the idea of Chow’s Theorem from control theory (also called the Chow-Rashevsky Theorem or Hermes’ Theorem), if the \((n + 1)\)-st derivative \( h^{[n+1]} \) is not contained in \( \text{span} \{ h^i : 0 \leq i \leq n \} \) then iterating the process of bracketing \( X \) and \( Y \) generates a large space reachable via repeated compositions of \( X \) and \( Y \). Denoting

\[
Z_t (f) := f + th^{[n]} \tag{25}
\]

successive brackets of \( X \) and \( Y \) are

\[
[X,Y] \sim Z =: Z \quad Z =: Z \quad \text{for } n \text{ times}
\]

\[
[X,Y] := [[X,Y], Y] \sim Z
\]

\[
[X,Y] := [[[X,Y], Y], ..., Y], Y] \sim Z \tag{26}
\]

For notational purposes we set \([X,Y] := X \). In the particular case \( h (x) := e^{-x^2} \) all of \( L^2 (\mathbb{R}) \) is reachable by \( X \) and \( Y \).

To see this we apply the theory of orthogonal functions with the Hermite\(^1\) polynomials

\[
H_n (x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = (-1)^n e^{x^2} h^{[n]} (x)
\]

\(^1\)We may of course use other orthogonal families with a different choice of \( h \), particularly when the domain of interest is other than \( \mathbb{R} \); e.g., scaled Chebyshev polynomials for \([0, 2\pi)\), etc. We expect many choices of \( h \) give controllable systems whether the brackets generate orthogonal sets or not.
which have dense span in $L^2(\mathbb{R})$ when multiplied by $e^{-x^2/2}$. Those familiar with orthogonal expansions can predict the rest; we review some of the details.

$$\left\{ \frac{1}{\sqrt{n!2^n \sqrt{\pi}}} H_n(x) e^{-x^2/2} : n \in \mathbb{N} \right\}$$

is a basis of $L^2(\mathbb{R})$ and is orthonormal since

$$\int_{\mathbb{R}} H_m(x) H_n(x) e^{-x^2} \, dx = n!2^n \sqrt{\pi} \delta_{mn}. \quad (27)$$

The Hermite polynomials also satisfy some useful relations

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x) \quad \text{and} \quad H_n'(x) = 2n H_{n-1}(x). \quad (28)$$

Given any $g \in L^2(\mathbb{R})$ it is possible to write

$$g(x) = \sum_{n=0}^{\infty} a_n \frac{1}{\sqrt{n!2^n \sqrt{\pi}}} H_n(x) e^{-x^2/2} \quad (29)$$

(equality in the $L^2$ sense) where

$$a_n := \frac{1}{\sqrt{n!2^n \sqrt{\pi}}} \int_{\mathbb{R}} g(x) H_n(x) e^{-x^2} \, dx \in \mathbb{R}.$$ 

The necessity of this formula for $a_n$ can easily be checked by multiplying both sides of (29) by $H_n(x) e^{-x^2/2}$, integrating and applying (27). However, we want

$$g = \sum_{n=0}^{\infty} c_n h^{[n]}$$

so apply the above process to $g(x) e^{x^2/2}$ instead. Then

$$g(x) e^{x^2/2} = \sum_{n=0}^{\infty} b_n \frac{1}{\sqrt{n!2^n \sqrt{\pi}}} H_n(x) e^{-x^2/2} \quad \text{so}$$

$$g = \sum_{n=0}^{\infty} c_n h^{[n]}$$

where

$$b_n := \frac{1}{\sqrt{n!2^n \sqrt{\pi}}} \int_{\mathbb{R}} g(x) e^{x^2/2} H_n(x) e^{-x^2/2} \, dx \quad \text{so that}$$

$$c_n := \frac{(-1)^n}{n!2^n \sqrt{\pi}} \int_{\mathbb{R}} g(x) h^{[n]}(x) e^{x^2} \, dx.$$ 

\*\*The function $g(x) e^{x^2/2}$ is no longer necessarily $L^2$, of course, but here we lapse into the habit of ignoring convergence issues as they are important for the theoretical proof that all of $L^2(\mathbb{R})$ is reachable with $X$ and $Y$, but not central to this demonstration. This theoretical lapse is easily remedied by multiplying by the characteristic function $\chi_{[-m,m]}$ to guarantee all of the following integrals converge, then letting $m \to \infty$ at the end.\*\*
Therefore when \( N \) is large, \( g \) is approximated by
\[
\sum_{n=0}^{N} c_n h^{[n]} = F_1 (0)
\]
where \( F \) is the flow of the arc field
\[
\tilde{X} := \sum_{n=0}^{N} c_n [X^n Y]
\]
which we follow for unit time starting with initial condition \( 0 \in L^2 (\mathbb{R}) \). \( F \) can, of course, be approximated by Euler curves
\[
F_1 (0) = \lim_{n \to \infty} \tilde{X}^{(n)}_{1/\sqrt{n}} (0)
\]
and since \( \tilde{X} \) is merely a (complicated) composition of \( X \) and \( Y \), this gives us a simple algorithm for approximating any function \( g \) with only two simple flows.

Let us compute a basic example to illustrate this surprising fact. Choosing at random \( g (x) := \chi_{[0,1]} (x) \), the characteristic function of the unit interval, we have
\[
c_n := \frac{(-1)^n}{n! 2^{n} \sqrt{2 \pi}} \int_0^1 H_n (x) \, dx = \frac{(-1)^n}{2(n+1)! 2^{n} \sqrt{2 \pi}} [H_{n+1} (1) - H_{n+1} (0)] \quad \text{so, e.g.,}
\]
\[
c_0 = \frac{1}{\sqrt{2 \pi}}, \quad c_1 = \frac{-1}{2 \sqrt{2 \pi}}, \quad c_2 = \frac{1}{12 \sqrt{2 \pi}}, \quad c_3 = \frac{1}{12 \sqrt{2 \pi}}, \quad c_4 = \frac{1}{480 \sqrt{2 \pi}}, \quad \text{etc.}
\]
by (28). Then stopping for the purposes of illustration at \( N = 3 \) our function \( g \) is approximated by
\[
\sum_{n=0}^{3} c_n h^{[n]}.
\]

Notice the flow of \( \tilde{Z} \) from (25) is locally the same as \( \tilde{Z} \) since it is just vector space translation, so we will use the same symbol. All vector space translations commute under (arc field) addition, and the arc field
\[
\tilde{Z}_t (f) := \left( c_0 Z + c_1 Z + c_2 Z + c_3 Z \right)_t (f)
\]
is locally equal to its flow. Obviously
\[
\tilde{Z}_1 (0) = \sum_{n=0}^{3} c_n h^{[n]}
\]
and \( \tilde{Z} \sim \tilde{X} \) where
\[
\tilde{X}_t (f) := (c_0 X + c_1 [X,Y] + c_2 [[X,Y],Y] + c_3 [[[X,Y],Y],Y])_t (f)
\]
\[
= \left( \sum_{n=0}^{3} c_n [X^n Y] \right)_t (f).
\]
Remember the arc field bracket and the arc field sum are defined as nothing more than compositions of arc fields, e.g.,

\[(c_0 X + c_1 [X,Y] + c_2 [[X,Y],Y])_t = [[X,Y],Y]_{c_2 t} [X,Y]_{c_1 t} X_{c_0 t}\]

and, e.g., when \( t > 0 \)

\[c_2 [[X,Y],Y]_t = Y_{-\sqrt{c_2 t}} (X_{-\sqrt{c_2 t}} Y_{-\sqrt{c_2 t}} X_{\sqrt{c_2 t}} Y_{\sqrt{c_2 t}}) Y_{\sqrt{c_2 t}} (Y_{-\sqrt{c_2 t}} X_{-\sqrt{c_2 t}} Y_{\sqrt{c_2 t}} X_{\sqrt{c_2 t}}).\]

Therefore this approximation of \( g \) is achieved by computing the Euler curves for \( \tilde{X} \) which is a complicated process (with a simple formula) of composing the elementary operations of function translation \( (Y) \) and vector space translation by the Gaussian \( (X) \).

Continuing the example, for choices of \( h \) other than the Gaussian it may be the case that \( h^{[n+1]} \in \text{span}\{h^i : 0 \leq i \leq n\} \). Then the space reachable by \( X \) and \( Y \) is precisely limited. E.g., when \( h \) is a trigonometric function from the orthogonal Fourier decomposition of \( L^2 \) the parameter space is two-dimensional, or when \( h \) is an \( n \)-th order polynomial in the context of \( M = L^2 [a,b] \) then the parameter space is \((n+1)\)-dimensional.

Restating these results in different terminology: Controlling amplitude and phase the 2-parameter system is holonomically constrained. Controlling phase and superposition perturbation \( (Y \text{ and } X) \) generates a larger space of signals; how much \( Y \) and \( X \) deviate from holonomy depends on the choice of perturbation function \( h \). Consequently, a result for signal analysis is: controlling two parameters is enough to generate any signal.

We collect some of the results of the previous example. Denote the reachable set of \( X \) and \( Y \) by

\[R(X,Y) := \{Y_s X_t Y_{s_n-1} X_{t_{n-1}} ... Y_{s_1} X_{t_1} (0) \in L^2(\mathbb{R}) : s_i, t_i \in \mathbb{R}, n \in \mathbb{N}\}\]

where \( 0 \in L^2(\mathbb{R}) \) is the constant function. \( R(X,Y) \) is the set of all finite compositions of \( X \) and \( Y \).

**Theorem 48** Let \( h \in L^2(\mathbb{R}) \) be the Gaussian \( h(x) := e^{-x^2} \) and define

\[X_t(f) := f + th \quad \text{and} \quad Y_t(f)(x) := f(x+t) .\]

Then \( R(X,Y) \) is dense in \( L^2(\mathbb{R}) \).

**Algorithm 49** Let \( g \in L^2(\mathbb{R}) \) be such that \( \int_{\mathbb{R}} [g(x)e^{x^2/2}]^2 \, dx < \infty \). Then

\[g = \lim_{n \to \infty} \tilde{X}^{(n)}_{1/n}(0)\]
where

\[ \tilde{X} := \sum_{n=0}^{\infty} c_n [X^n Y] \quad \text{with} \quad c_n := \frac{(-1)^n}{n! \sqrt{2\pi}} \int_{\mathbb{R}} g(x) h^n(x) e^{x^2} \, dx \]

and \[ [X^n Y] := \left[ \left[ \left[ \left[ X, Y \right], Y \right], \ldots, Y \right], Y \right] \text{ } \quad n \text{ times} \]

and

\[ [X, Y] (f, t) := \begin{cases} Y - \sqrt{t} X - \sqrt{t} Y & \text{for } t \geq 0 \\ X + \sqrt{t} Y - \sqrt{t} Y & \text{for } t < 0 \end{cases} \]

for any \( f \in L^2 (\mathbb{R}) \).

**Example 50** Let us continue Example 47 with \( M = L^2 (\mathbb{R}) \) and

\[ X_t (f) := f + th \quad \text{and} \quad Y_t (f) (x) := f(x + t) \]

which are vector space translation and function translation. Define the arc fields

\[ V_t (f) := e^t f \quad \text{and} \quad W_t (f) (x) := f(e^t x) \]

which may be thought of as vector space dilation (about the point \( 0 \in M \)) and function dilation (about the point \( 0 \in \mathbb{R} \)). Again, \( V \) and \( W \) are coincident with their own flows. Using the same approach as in Example 47, it is easy to check the brackets satisfy

\[
\begin{align*}
[X, Y] (t) &= f + th' + o(t) \\
[X, W]_t (f) (x) &= f(x) + t x h'(x) + o(t) \\
[Y, W]_t (f) (x) &= f(x) - t x + o(t)
\end{align*}
\]

assuming for the \([X, Y]\) and \([X, W]\) calculations that \( h \in C^1 (\mathbb{R}) \) and \( h' \in L^2 (\mathbb{R}) \).

Consequently

\[
\begin{align*}
\Delta (X, Y) &\text{ may be highly non-involutive depending on } h, \\
\Delta (X, V) &\text{ is involutive, but } X \text{ and } V \text{ do not commute,} \\
\Delta (X, W) &\text{ may be highly non-involutive depending on } h, \\
\Delta (Y, V) &\text{ is involutive; } Y \text{ and } V \text{ commute,} \\
\Delta (Y, W) &\text{ is involutive, but } Y \text{ and } W \text{ do not commute,} \\
\Delta (V, W) &\text{ is involutive; } V \text{ and } W \text{ commute.}
\end{align*}
\]

When \( h \) is chosen correctly, \( X \) and \( W \) control many function spaces, similarly to \( X \) and \( Y \). The four involutive distributions foliate \( L^2 (\mathbb{R}) \).

**References**

[1] Ralph Abraham, Jerrold Marsden and Tudor Ratiu, “Manifolds, Tensor Analysis, and Applications”, 2nd Ed., Springer-Verlag, 1988.
[2] J.-P. Aubin, “Mutational and Morphological Analysis,” Birkhauser, Boston, 1999.

[3] Craig Calcaterra, “Arc Fields,” University of Hawaii doctoral dissertation, 1999.

[4] Craig Calcaterra and David Bleecker, Generating Flows on a Metric Space, *Journal of Mathematical Analysis and Applications*, 248, 645-677 (2000)

[5] Craig Calcaterra, Axel Boldt, Michael Green, David Bleecker, Metric Coordinate Systems, preprint at arXiv.org: math.DS/0206253

[6] Rinaldo M. Colombo and Andrea Corli, A Semilinear Structure on Semigroups in a Metric Space, *Semigroup Forum*, 68 (2004), pp. 419-444.

[7] D. Motreanu and N. H. Pavel, “Tangency, Flow Invariance for Differential Equations, and Optimization Problems,” Marcel Decker, 1999.

[8] A. I. Panasyuk, Quasidifferential Equations in a Complete Metric Space under Conditions of the Caratheodory Type. I, *Differential Equations* 31 (1995), pp. 901-910.

[9] A. I. Panasyuk, Quasidifferential Equations in a Complete Metric Space under Caratheodory-type Conditions. II, *Differential Equations* 31, no. 8 (1995), pp. 1308-1317.

[10] Franco Rampazzo and Hector J. Sussman, Commutators of Flow Maps of Nonsmooth Vector Fields, (2006), Journal of Differential Equations (to appear).

[11] Slobodan Simić, Lipschitz Distributions and Anosov Flows. Proceedings of the AMS, 124, no. 6 (1996), pp. 1869-1877.

[12] Eduardo D. Sontag, “Mathematical Control Theory,” 2nd Ed., Springer-Verlag, 1998.