Abstract—Wireless network coding (WNC) has been proposed for next generation networks. In this contribution, we investigate WNC schemes with embedded strong secrecy by exploiting structured interference in relay networks with two users and a single relay. In a practical scenario where both users employ finite, uniform signal input distributions we compute the corresponding strong secrecy capacity, and make this explicit when PAM modems are used. We then describe a simple triple binning encoder that can provide strong secrecy close to capacity with respect to an untrustworthy relay in the single antenna and single relay setting. An explicit encoder construction is described when \(M\)-PAM or \(M\)-QAM modulators are employed at the users’ transmitters. Lastly we generalize to a MIMO relay channel where the relay has more antennas than the users, and optimal precoding matrices are studied. Our results establish that the design of WNC transmission schemes with enhanced throughput and guaranteed data confidentiality is feasible in next generation systems.

Index Terms—Wireless network coding, secrecy capacity, strong secrecy, signal space alignment, triple binning encoder.

I. INTRODUCTION

Recently, the ideas of network coding [1] have been extended to the wireless physical medium; notably, in [2], [3], among others, the concept of harnessing interference through structured codes was explored in the framework of wireless network coding (WNC). These technologies can be proven instrumental in enabling the envisaged multi-fold increase in data throughput in fifth generation (5G) networks [4]. The generic WNC system model with two independent sources and one relay is depicted in Fig. 1 and assumes that communication is executed in two cycles. In the first cycle, the nodes A, referred to as Alice, and B, referred to as Bob, transmit simultaneously codewords \(x_A\) and \(x_B\), respectively, to the relay node R, referred to as Ray. In the second cycle, Ray, broadcasts to Alice and Bob a function \(f(x_A + x_B)\) of the total received signal; Alice and Bob then retrieve each other messages by canceling off their corresponding transmissions.

Depending on the transformation \(f(\cdot)\) executed by Ray, one of the following relaying strategies can be employed: amplify and forward, decode and forward, compress and forward [5], or the recently introduced compute and forward [6] approach.

Nevertheless, despite the potential for substantial increase of the transmission rates in wireless networks, a major obstacle in the widespread deployment of WNC and generally of relay networks arises due to security concerns, i.e., the confidentiality of the exchanged data with respect to an untrustworthy relay. A straightforward approach would be employing encryption at upper layers of the communication network or encryption at the physical layer [7]. However, the management of secret keys used by the crypto algorithms depends on the structure of the access network and already fourth generation systems (4G) have a key hierarchy of height five (5) for each individual end-user, while there exist multiple keys in each layer of the hierarchy [8]. Extrapolating from the experience of 4G systems, it is expected that the management of secret keys in 5G would become an even more complicated task [9]. The generation, the management and the distribution of secret keys in decentralized settings, such as device-to-device WNC networks, without an infrastructure that supports key management and authentication will impose new security challenges.

An alternative theoretical framework for the study of data confidentiality in the physical layer of wireless networks, dubbed as physical layer security [10], [11], [12], has recently become a focal point of research in the wireless community. The metric of interest, referred to as "the channel secrecy capacity" is the supremum of transmission rates at which data can be exchanged reliably while satisfying a weak secrecy [13], [14] or a strong secrecy constraint [15]. Both weak and strong secrecy can be achieved asymptotically as the code’s blocklength becomes arbitrarily long, while Shannon’s definition of perfect secrecy in [16] and the related concept of semantic security on the other hand explicitly assume finite blocklengths.

The first study of the secrecy capacity of relay channels with confidential messages has appeared in [17] while further analyses followed [18], [19]; these contributions established that the secrecy capacity of one-way relay channels is zero, unless, the source-destination channel is better than the source-relay channel. In essence, relay topologies of practical interest

![Fig. 1. Physical layer network coding (PNC) with two transmitter and one relay node.](image-url)
in which the link to the relay is better than the direct link were shown to be inherently insecure. Due to this limiting result, subsequent work focused entirely on cooperative relay channels with trustworthy relays, [20], [21], [22], [23] to cite but a few.

However, unlike one-way relay networks, systems employing network coding can on the other hand benefit from the simultaneity of transmissions to an untrustworthy relay to achieve data confidentiality as noted in [24]. In essence, the structured interference observed by the relay can be exploited to achieve perfect secrecy in the wireless transmissions [25]. In [26] and [27] the role of interference in achieving strong secrecy was demonstrated using lattice encoders; in these works the superposition of the interference to the data was viewed as a modulo addition operation, i.e., the superposition was assumed to take place in the code space and not in the signal space.

In the present study, WNC networks in which Ray can observe superpositions in the signal space (real sums of signals transmitted by Alice and Bob as opposed to modulo sums in the code space) is investigated assuming all nodes have multiple input multiple output (MIMO) M-ary quadrature phase shift keying (QAM) transceivers; this realistic scenario is fundamentally more demanding than previously investigated settings [23]. We propose an explicit triple binning coding scheme for the M-QAM modes that achieves strong secrecy [28]. The proposed technique is outlined with a constructive example of coset coding in the single input single output (SISO) setting that allows Ray to obtain estimates of linear combinations of the transmitted QAM symbols but not to retrieve any of the secret bits they carry, thus achieving perfect secrecy, i.e., zero information leakage per QAM symbol.

Next, our results are extended to the MIMO case in which we study optimal precoding matrices which achieve the required signal alignment at the relay, while preserving secrecy. Our study differs from earlier work on interference alignment for secrecy [29], [30] as the problem formulation in the WNC setting with secrecy constraints is fundamentally different. Our work also differs in spirit from, for example, signal space alignment for the relay channel [31] and interference alignment for the MIMO channel [32] in that the required secrecy conditions demand equality of matrices rather than just the subspaces generated by their columns. This stricter notion of equality has the subtle effect of altering the structure and dimension of the solution space of the precoder problem, even though the degrees of freedom is the same as is found in [31].

The paper is organized as follows. In Section II the SISO system model is presented. In Section III we compute the strong secrecy capacity of the SISO system given finite constellations, and provide an explicit formula for the strong secrecy capacity in the case of PAM modems. In Section IV an explicit encoder is constructed that achieves strong secrecy, allowing both users to transmit with strong secrecy at close to capacity when employing PAM or QAM modems. In Section V we generalize to a MIMO channel in which the users and the relay have multiple antennas, and study optimal precoding matrices. Finally in section VI the conclusions of this contribution are drawn and future directions of the work are discussed.

II. SECURE WNC SYSTEM MODEL AND SECRECY RATE

Communication between Alice and Bob with the help of Ray takes place into two cycles as depicted in Fig. 1. In what follows, we use the subscript $A$ to denote quantities and variables (source symbols, codewords, etc.) corresponding to Alice and the subscript $B$ for those belonging to Bob. All channel coefficients and encoding/decoding algorithms are public, i.e., known by Alice, Bob, and Ray. The notation $x^n$ denotes the sequence $[x(1), x(2), \ldots, x(n)]$ while lower case letters denote realizations of respective random variables that are represented with the corresponding upper case letters, e.g., $x$ denotes a realization of the random variable $X$ with probability mass function (pmf) $p_X(x)$.

We assume that Alice’s and Bob’s source symbols (secret messages) are drawn from discrete alphabets. Under an average power constraint, the use of Gaussian encoders has been demonstrated to achieve the secrecy capacity of the interference channel [33]. However when transmission is constrained by a joint amplitude-variance constraint, it has been shown that the capacity is on the contrary achieved by employing codebooks with finite supports; a recent extension of these results in the wiretap channel has shown that this holds true for the secrecy capacity as well [34]. Due to this reason, in the following we exclusively operate under the assumption that all codebooks have finite supports.

\footnote{Under this realistic assumption the amplitude of the transmitted signals is strictly finite, as in all actual communication systems.}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Nested encoder for strong secrecy in WNC systems: Alice encodes the secret messages using an inner encoder $\theta_A$ for reliability and an outer encoder $\varphi_A$ for secrecy. On the other hand, Bob employs the corresponding decoders $\phi_B$ and $\varphi_B$ to obtain an estimate of the secret messages transmitted by Alice.}
\end{figure}
We start by examining the scenario in which all nodes have single antennas while the multi-antenna case will be covered in a later section. Our goal is to identify encoding strategies that guarantee strong secrecy in the WNC setting. To highlight the great potential of using WNC for secrecy, in the present work we treat separately the design of (strong) secrecy encoders from error correction encoding. As we will show, this can be straightforwardly achieved with the nested structure depicted in Fig. 2. Including an inner encoder for reliability and an outer encoder for secrecy. The reason we propose this approach is twofold. First, contrary to error correction, in the noiseless WNC setting it is possible to achieve strong secrecy without introducing any delay as will be demonstrated, i.e., strong secrecy can be achieved on a per symbol basis and does not rely on the existence of noise to increase the equivocation at Ray rather than on the existence of structured interference. Secondly, it is possible to achieve strong secrecy employing deterministic rather than stochastic encoders as is customary for broadcasting systems with secrecy constraints [14].

In the presentation of the proposed nested encoder we employ the following notation: Alice (respectively Bob) employs a rate $\frac{q}{n}$, $q, n \in \mathbb{N}$, $q \leq n$ inner encoder for reliability denoted by $\theta_A$ (respectively $\theta_B$) and corresponding decoder denoted by $\vartheta_B$ (respectively $\vartheta_A$). Furthermore, to ensure strong secrecy Alice (Bob) uses a unit symbol rate outer secrecy encoder denoted by $\varphi_A$ (respectively $\varphi_B$) with a corresponding decoder $\phi_B$ (respectively $\phi_A$). On the other hand Ray employs a WNC wrapping function (e.g., “compress and forward”) denoted by $f$. In subsections II-A to II-C, we explain the above setting in further detail and discuss the necessary secrecy conditions.

A. First transmission cycle

Inner encoder for reliability: In the first cycle Alice maps length-$q$ sequences of secret messages $s_A \in S_A$, selected uniformly at random from the set $S_A$ of secret messages to length-$n$ codewords. To this end, Alice first employs an encoding $\theta_A : S_A^n \rightarrow U^n_A$, $\theta_A(s_A^n) = u^n_A$. Similarly Bob maps length-$q$ sequences of secret messages $s_B \in S_B$ using an encoder $\theta_B : S_B^n \rightarrow U^n_B$, $\theta_B(s_B^n) = u^n_B$.

Outer encoder for strong secrecy: Alice and Bob encode the sequences $u^n_A$ and $u^n_B$ respectively to codewords $x_A^n$ and $x_B^n$ element by element using corresponding encoders $\varphi_A : U^n_A \rightarrow X_A$ and $\varphi_B : U^n_B \rightarrow X_B$ with $\varphi_A(u) = x_A$ and $\varphi_B(u) = x_B$. We define $M_A$ and $M_B$ to be the sizes of the codebooks $X_A$ and $X_B$, denoted by $|X_A| = M_A$ and $|X_B| = M_B$.

Ray’s observation (assuming perfect synchronization at Alice and Bob) can be expressed as

$$y = h_A x_A + h_B x_B + w_R = x_R + w_R,$$

where $h_A$ (respectively $h_B$) is the channel (fading) coefficient in the link from Alice (Bob) to Ray, $x_R = h_A x_A + h_B x_B$, and $w_R$ is the noise at Ray, modeled as a realization of a zero-mean circularly symmetric Gaussian complex random variable with variance $\sigma_R^2$.

B. Second transmission cycle

WNC wrapping: In the second cycle of the communication, Ray performs a wrapping of the received WNC observation $y$ to compressed WNC observations using a mapping $f : Y \rightarrow \mathcal{Z}$, where $f(y) = z$ (e.g., possible options for this mapping include “compress and forward” and “compute and forward”). We assume that $f$ is invertible given either $x_A$ or $x_B$, that is, that Alice can recover $y$ from $z$ given that she knows $x_A$, and similarly for Bob. An obvious choice is to select $\mathcal{Z} = \mathcal{Y}$ and have $f$ be the identity function, i.e., Ray forwards exactly what he receives, but we give a more energy-efficient wrapping function in Section III-C for PAM modulators.

Finally, Ray transmits $z$ to Alice and Bob, who then observe

$$y_A = \tilde{h}_A z + w_A,$$

$$y_B = \tilde{h}_B z + w_B,$$

where $\tilde{h}_A$ (respectively $\tilde{h}_B$) is the channel (fading) coefficient from Ray to Alice (Ray to Bob), and $w_A$ (respectively $w_B$) is the noise at Alice (Bob), modeled as a circularly symmetric complex Gaussian random variable with variance $\sigma_A^2$ (respectively $\sigma_B^2$).

Alice uses a decoder $\phi_A : \mathcal{Y}_A \rightarrow U_B$, to produce estimates $\phi_A(y_A) = \hat{u}_B$ of Bob’s transmitted secret codewords. Respectively, Bob uses a function $\phi_B : \mathcal{Y}_B \rightarrow U_A$, to produce estimates $\phi_B(y_B) = \hat{u}_A$ of Alice’s transmitted secret codewords.

For the purposes of error correction Alice (Bob) employs a decoding function $\vartheta_A : U^n_B \rightarrow S^n_A$ with $\vartheta_A(\hat{u}_B^n) = \hat{s}_A^n$ ($\vartheta_B : U^n_A \rightarrow S^n_B$ with $\vartheta_B(\hat{u}_A^n) = \hat{s}_B^n$). Focusing exclusively on the secrecy of the WNC scheme, we assume that $\lim_{n \rightarrow \infty} P_{e,A} = 0$ ($\lim_{n \rightarrow \infty} P_{e,B} = 0$) where the probability of a decoding error at Alice (respectively Bob) is $P_{e,A} = Pr[\vartheta_A(\hat{u}_B^n) \neq s_A^n]$ (respectively $P_{e,B} = Pr[\vartheta_B(\hat{u}_A^n) \neq s_B^n]$).

C. Strong Secrecy and Secrecy Rate

Perfect secrecy (in the strong sense) can be achieved with respect to Ray if the mutual information between Ray’s observation and the secret source symbols is zero, i.e.,

$$I(Y; S_A) = 0 \text{ (strong secrecy condition for Alice)}$$
$$I(Y; S_B) = 0 \text{ (strong secrecy condition for Bob)}.$$

The input and output random variables in the WNC system model form respective Markov chains $S_A \rightarrow U_A \rightarrow X_A \rightarrow X_R \rightarrow Y^n$ and $S_B \rightarrow U_B \rightarrow X_B \rightarrow X_R \rightarrow Y^n$. As a result, due to the data processing inequality, to satisfy conditions (4) and (5) it suffices to show that

$$I(X_R; S_A) = 0 \text{ (equivalent condition to (4))}$$
$$I(X_R; S_B) = 0 \text{ (equivalent condition to (5))}.$$

For fixed input distributions $X_A$ and $X_B$, the following definition measures the upper limit as to the number of bits Alice and Bob can transmit in secret, that is, the maximum possible strong secrecy rate.

**Definition 1**: Given random variables $X_A$ and $X_B$ with known pmfs and some fixed collection of channels $h = (h_A, h_B, \tilde{h}_A, \tilde{h}_B)$, we define the constrained strong secrecy capacities $C_s(A)$ and $C_s(B)$ by

$$C_s^{(A)} = [I(Y_B; X_A | X_B) - I(Y; X_A)]^+,$$
$$C_s^{(B)} = [I(Y_A; X_B | X_A) - I(Y; X_B)]^+.$$
III. STRONG SECRECY RATES IN THE NOISELESS SCENARIO

Throughout this section we assume that (i) the pmfs of $X_A$ and $X_B$ are uniform, (ii) all channels are fixed and invertible, and (iii) $\sigma^2_A = \sigma^2_B = 0$, i.e., we separate the problem of error correction from the design of the strong secrecy encoder. One could similarly think of the results in the noiseless scenario as modeling what is possible asymptotically in the high signal-to-noise ratio (SNR) region for $\text{SNR} \gg 1$. Under these assumptions we will evaluate the constrained strong secrecy capacities.

A. Strong Secrecy Capacity in the Noiseless Scenario

**Lemma 1:** For uniform $X_A$ and $X_B$ with support sizes $M_A$ and $M_B$ respectively, the constrained strong secrecy capacities in the noiseless scenario are given by

$$C_s^{(A)} = \left[ \log_2(M_A) - I(Y; X_A) \right]^+,$$  

$$C_s^{(B)} = \left[ \log_2(M_B) - I(Y; X_B) \right]^+.$$  

**Proof:** We can compute easily that

$$I(Y_B; X_A | X_B) = I(Z; X_A | X_B) = I(X_R; X_A | X_B) = \log_2(M_A)$$

The first equality follows from the equation $Y_B = h_B Z$ and the invertibility of the channel $h_B$. The second follows from the invertibility of the WNC wrapping function $f$ when either $x_A$ or $x_B$ are known. The third follows from the fact that given $x_B$ and $x_A$, $x_R$ is uniquely determined via $x_A = h_A^{-1}(x_R - h_B x_B)$. The result for $C_s^{(A)}$ follows immediately from substitution into (8), and the result for $C_s^{(B)}$ follows by switching the roles of Alice and Bob.

For fixed channels $h_A$ and $h_B$, the set of all possible observations at Ray in the noiseless scenario is

$$\mathcal{Y} = X_R = \{h_A x_A + h_B x_B | x_A \in \mathcal{X}_A, x_B \in \mathcal{X}_B\}$$

which comes with an addition function

$$\psi : h_A \mathcal{X}_A \times h_B \mathcal{X}_B \rightarrow \mathcal{Y}$$

$$\psi(h_A x_A, h_B x_B) = h_A x_A + h_B x_B.$$  

Let us use the notation

$$\psi^{-1}(y) = \{(h_A x_A, h_B x_B) | y = h_A x_A + h_B x_B \}$$

for the set of all pairs of points which add to a given observation $y$ at Ray.

**Lemma 2:** In the noiseless scenario the random variable

$$Y = X_R = h_A X_A + h_B X_B$$

describing Ray’s observation has pmf

$$p_Y(y) = \frac{|\psi^{-1}(y)|}{M_A M_B}$$

and entropy

$$H(Y) = \log_2(M_A M_B) - \frac{1}{M_A M_B} \sum_{y \in \mathcal{Y}} |\psi^{-1}(y)| \log_2 |\psi^{-1}(y)|.$$  

**Proof:** See appendix A.

The following proposition says that the number of bits Alice and Bob fail to communicate in secret is determined by $H(Y)$.

**Proposition 1:** In the noiseless scenario the strong secrecy capacities $C_s^{(A)}$ and $C_s^{(B)}$ are equal and furthermore satisfy

$$C_s^{(A)} = \log_2(M_A M_B) - H(Y) = \frac{1}{M_A M_B} \sum_{y \in \mathcal{Y}} |\psi^{-1}(y)| \log_2 |\psi^{-1}(y)|$$

for any collection $h$ of channels.

**Proof:** See appendix B.

In light of the above proposition, in the noiseless scenario we can simply define the strong secrecy capacity to be

$$C_s = C_s^{(A)} = C_s^{(B)}.$$  

In the following subsection we shall see that $C_s$ is asymptotically achievable using standard pulse amplitude modulation (PAM) if the channel gains are equal. This latter assumption requires simple appropriate precoding, e.g., implemented as channel inversion.

B. Constrained Strong Secrecy Capacity for PAM Modems

Let us now study a familiar scenario in which $\mathcal{X}_A$ and $\mathcal{X}_B$ are, respectively, $M_A$- and $M_B$-PAM constellations. Throughout this section we assume that $M_A \leq M_B$. The main theorem of this subsection states the conditions for the bound $C_s \leq \log_2(M_A)$ to be asymptotically achievable with simple PAM modems.

**Theorem 1:** In the noiseless WNC setting with $h_A = h_B$ (or using suitable channel precoding), if Alice and Bob employ $M_A$-PAM and $M_B$-PAM modulators with $M_A \leq M_B$, the constrained strong secrecy capacity is given by

$$C_s = \log_2(M_A) \left(1 - \frac{M_A + 1}{M_B}\right) + \frac{2}{M_A M_B} \sum_{a=1}^{M_A} a \log_2(a).$$

In particular, we have the asymptotic behavior

$$\lim_{M_B \to \infty} C_s = \log_2(M_A),$$

as the size of the larger constellation increases.

**Proof:** See appendix C.

The first summand in the above expression for $C_s$ corresponds to, roughly, the number of secret bits Alice and Bob can pass through Ray when he observes a signal in the “flat” part of his pmf, see Fig. 3. Analogously, the second summand in the above corresponds to the number of secret bits Alice and Bob can transmit when Ray observes a signal in the “linear” part of his pmf.
Fig. 3. The pmf of Ray’s observation \( y = x_A + x_B \) in the noiseless scenario with channel gains \( h_A = h_B = 1 \). Here Alice employs a 4-PAM modulator and Bob a 16-PAM modulator.

Fig. 4. Bob’s secrecy rate \( R_s^{(B)} \) using the simple encoding scheme of Section III-C. The solid lines represent the values \( R_s^{(B)} \), and the dashed lines represent the corresponding values of the constrained strong secrecy capacity \( C_s^{(B)} \) calculated using Theorem 1.

C. Explicit Encoder Construction - a Simple Example

In this subsection we outline a simple, explicit scheme for allowing Bob to transmit \( \log_2(M_A) \) secret bits, in an asymptotic sense as \( M_B \to \infty \). The scheme is based on Bob transmitting \( \log_2(M_A) \) secret bits when he can guarantee that Ray will observe a \( y \) with \( |\psi^{-1}(y)| = M_A \) and thus maximal entropy, regardless of the symbol transmitted by Alice. When Bob cannot guarantee that Ray will observe a \( y \) with maximal entropy, he transmits no secret bits. For fixed \( M_A \) and increasing \( M_B \), the “flat” part of Ray’s pmf (where maximal entropy is experienced) gets arbitrarily large while the “linear” part (where Ray experiences suboptimal entropy) has constant size, and thus Bob can transmit \( \log_2(M_A) \) bits arbitrarily often as \( M_B \to \infty \). We make all of this precise by describing \( \varphi_B \) explicitly as follows.

Let us fix \( M_A \) and \( M_B \) such that \( M_A \leq M_B \). We label Bob’s PAM symbols with bits by viewing his \( M_B \)-PAM constellation as a union of translates of Alice’s \( M_A \)-PAM constellation. We assign the first \( \log_2(M_A) \) bits of each of Bob’s signals according to the corresponding Gray labeling on the \( M_A \)-PAM. The last \( \log_2(M_B/M_A) \) bits of Bob’s PAM symbols are then used to declare to which of the \( M_B/M_A \) translates a given symbol belongs.

Let us define the following subset \( \mathcal{X}_B^{(M_A)} \):

\[
\mathcal{X}_B^{(M_A)} = \{ x_B \mid |\psi^{-1}(x_A + x_B)| = M_A \forall x_A \in \mathcal{X}_A \}
\]

Thus \( \mathcal{X}_B^{(M_A)} \) is the set of all signals Bob can transmit which guarantee that Ray experiences \( \log_2(M_A) \) bits of entropy, and thus Bob transmits secret bits only when his signal is in this set. A simple calculation reveals that

\[
|\mathcal{X}_B^{(M_A)}| = M_B - 2M_A + 2.
\]

For Bob to transmit \( M_A \) secret symbols uniformly, we must “trim” the edges of \( \mathcal{X}_B^{(M_A)} \) so that its size is divisible by \( M_A \). If \( M_A \geq 4 \) we thus remove the outer two points and define

\[
\mathcal{X}_B^{(s)} = \{ -M_B + 2M_A + 1, \ldots, M_B - 2M_A - 1 \},
\]

which has size \( M_B - 2M_A \). When \( M_A = 2 \) we simply choose \( \mathcal{X}_B^{(s)} = \mathcal{X}_B^{(M_A)} \). Bob now defines the encoder \( \varphi_B \) of his secret bits by declaring that the first \( \log_2(M_A) \) bits of all symbols in \( \mathcal{X}_B^{(s)} \) are secret. Because Ray experiences maximal entropy at all points which encode secret bits, we see that \( I(Y; S_B) = 0 \) and thus perfect secrecy is preserved.

Bob’s secrecy rate given the above encoding scheme is

\[
R_s^{(B)} = \log_2(M_A) \left( \frac{1}{M_B} \right)^{\frac{1}{2} - \frac{2}{M_B}} \frac{1}{M_B} \left[ \log_2(M_A) \left( 1 - \frac{1}{M_B} \right) \right] + \frac{1}{M_B} \left( \log_2(M_A) \right) + \frac{1}{M_B} \left( \log_2(M_A) \right) - \frac{1}{M_B}
\]

In particular, \( R_s^{(B)} \to \log_2(M_A) \) as \( M_B \to \infty \). On the other hand, Alice cannot transmit at any secrecy rate since she can never be certain that Ray’s observation will be on the “flat” region as all of her symbols can generate elements on the “linear” regions, i.e., \( R_s^{(A)} = 0 \).

In Fig. 3 we plot the above function \( R_s^{(B)} \) from (27) for various \( M_A \leq M_B \). The dashed lines represent the secrecy capacity \( C_s^{(B)} \) as computed by Theorem 1 and the solid lines are the corresponding values of \( R_s^{(B)} \). The scheme is designed to only take advantage of the “flat” part of Ray’s pmf, and thus leaves large gaps to capacity when \( M_B \) is not sufficiently larger than \( M_A \), resulting in a large proportion of Ray’s observations landing on the “linear” part of his pmf. In particular, notice that given this simple scheme we need \( M_B \geq 4M_A \) to achieve any non-trivial secrecy.

In the next section we alleviate some of the drawbacks of this simple scheme and provide a more general construction which allows Bob to transmit at a secrecy rate much closer to capacity, while simultaneously giving Alice a non-trivial secrecy rate.

IV. EXPLICIT ENCODER CONSTRUCTION WITH PAM AND QAM MODEMS IN THE NOISELESS SCENARIO

In the noiseless scenario we can simply set \( u_A = s_A \) and \( u_B = s_B \), i.e., remove the reliability encoders \( \vartheta_A \) and
Furthermore, without loss of generality we assume that $M_A \leq M_B$. The central idea behind the proposed approach in designing the secrecy encoders \( \varphi_A \) and \( \varphi_B \) stems from the following observation. The superposition of two PAM signals is equivalent to the convolution of two uniform pmfs; the resulting pmf contains a “flat” region in which Ray’s observations are equiprobable and two “linear” regions in which combinations of symbols occur with increasing/decreasing probabilities.

Ray experiences the most confusion when \( |\psi^{-1}(y)| \) is maximal, namely equal to \( M_A \). As previously discussed, this occurs exactly on the support of the “flat” part of the pmf of Ray’s observation. Given an observation \( y \) such that \( |\psi^{-1}(y)| = M_A \), Ray’s equivocation is \( \log_2(M_A) \) bits and Alice and Bob design their encoders \( \varphi_A \) and \( \varphi_B \) to exploit this fact. On the other hand, for observations \( y \) that live in the support of the “linear” regions of Ray’s pmf, Alice and Bob can transmit a varying number of secret bits depending on the number of preimages \( |\psi^{-1}(y)| \) of \( y \). In the following we discuss an approach to exploit this effect.

### A. Encoder Construction with Feedback

More elaborate encoding schemes can be built by introducing feedback from Alice to Bob and vice versa, allowing the transmission of secret bits also for observations \( y \) that lie in the support of the “linear” regions of Ray’s pmf. To achieve strong secrecy in the sense of \( \theta \) and \( \theta^* \) the construction of the respective encoders can rely on the fact that out of a total of \( \log_2(M_A + M_B) \) bits exchanged by Alice and Bob in each transmission, \( \log_2(M_A) \) bits can be used for the transmission of data (secret/public) bits and \( \log_2(M_B) \) bits can be used to generate suitable prefixes for feedback.

Alice and Bob can transmit a varying number of secret bits depending on the number of preimages \( |\psi^{-1}(y)| \) of \( y \). When the number of preimages \( |\psi^{-1}(y)| \) is even then \( y \) could have been generated by \( \log_2(\psi^{-1}(y)) \) equiprobable pairs \( \{x_A, x_B\} \) and as a result it is possible to encode Bob’s (Alice’s) secret symbols so that for every bit position \( b_i \), in a partition of size \( \log_2(\psi^{-1}(y)) \) out of \( \log_2(M_A) \) bits we have that

\[
\Pr(b_i = 0) = 0.5, i = 1, \ldots, \log_2(\psi^{-1}(y)).
\]

A straightforward labeling to meet condition (28) is summarized in the following three-steps:

1. List in increasing order Bob’s PAM amplitudes.
2. Group them into \( \frac{M_A}{2} \) pairs of increasing amplitudes (e.g. for a 16-PAM group them into the following eight pairs \((-15, -13), (-11, -9), (9, 11), (13, 15))\).
3. Map the PAM amplitudes to specific symbols so that the first \( \log_2(M_A) \) bits are conjugated in the two symbols in each pair.

On the other hand, if the number of preimages of \( y \) is odd then condition (28) cannot be met unless we introduce delay by distributing the codewords into more than one symbols (this scenario is not examined in the present contribution).

Using this simple approach, we can transmit a varying number of secret bits in all but the odd multiplicity preimages of \( y \). As a result, the maximum achievable secrecy rate can be expressed as:

\[
R_s^{(f)} = \log_2(M_A) \left(1 - \frac{M_A - 1}{M_B}\right) + \frac{4}{M_AM_B} \sum_{a=2}^{M_A-1} a \log_2 a.
\]

Table I

| BPSK | -1 | 1 |
|------|----|---|
| \( x_A \) | 1 | 0 |

| 8-PAM | -7 | -5 | -3 | -1 | 1 | 3 | 5 | 7 |
|-------|----|----|----|----|----|----|----|----|
| \( x_B \) | 111 | 001 | 100 | 010 | 110 | 000 | 101 | 011 |

We note that in the case \( M_A = 2, M_B > M_A \), then \( R_s^{(f)} = C_s \) as the linear regions contain only two elements, both with null equivocation. In the following subsection we discuss an explicit encoder design to achieve \( R_s^{(f)} \) as well as intermediate rates.

### B. Implementation of Feedback with Triple Binning Encoding

Based on the previous remarks, the outputs \( x_A \) and \( x_B \) of the encoders \( \varphi_A, \varphi_B \) are proposed to be jointly partitioned into three separate bins, denoted by \( K_A, K_C, K_B \) (so that each partition might include bits from both or either \( x_A \) and \( x_B \)). The role of each bin is explained below:

- **Bin \( K_A \) of secret bits:** Alice (Bob) can transmit up to \( K_s = |K_A| \) secret bits in bin \( K_A \) whenever the number of preimages of \( x_A + x_B \) is even; otherwise \( K_s \) public (non-secret) bits are transmitted. Based on Theorem 1, \( K_s \leq \log_2(M_A) \) with maximum equivocation is achieved when \( K_A = \log_2(M_A) \).

- **Bins \( K_A \) and \( K_B \) of feedback bits:** In order to allow for feedback to be sent from Alice (Bob) to Bob (Alice) a second bin \( K_C \) containing Alice’s feedback and a third bin \( K_B \) carrying Bob’s feedback are required with sizes \( K_A = |K_A| \) and \( K_B = |K_B| \) respectively; these can be used either to allow Alice (Bob) to alert Bob (Alice) whether secret bits should be transmitted or not, e.g., in the next transmission cycle, or to simply properly index the secret/common bits in the current transmission cycle. Apparently,

\[
K_s + K_A + K_B \leq \log_2(M_A) + \log_2(M_B).
\]

For illustration purposes, we demonstrate the proposed approach with a specific example in which at most \( C_s = R_s^{(f)} = 0.875 \) bits/sec/Hz can be transmitted with strong secrecy assuming that Alice employs a BPSK modulator and Bob an 8-PAM modulator. A possible mapping of the BPSK and the 8-PAM signals to the respective codewords \( x_A \) and \( x_B \) employing the three-step approach outlined previously is...
shown in Table I. Here, Bob does not use standard Gray
mapping of PAM amplitudes to symbols but rather a mapping
in which all but two neighboring codewords have Hamming
distance 2; the goal behind this approach is to increase the
equivocation at Ray (instead of minimizing the bit error rate).

As depicted in Table III whenever one of the edge points is
observed by Ray (for example, +8 is received) then \(\hat{x}_A, \hat{x}_B\)
can be uniquely decoded (in this case, as \((+1,+7)\)). In all
other cases two equiprobable alternatives for \((\hat{x}_A, \hat{x}_B)\) exist.
Therefore, in these cases, 1 bit can be transmitted in secret,
 i.e., \(\hat{x}_A\) or one of the bits of \(\hat{x}_B\) are completely ambiguous
and can be used for the transmission of secret messages. The
corresponding symbol pairs are depicted in Table III where
also a “wrapping” approach for Ray’s WNC encoding scheme
\(z = f(y)\) is presented. In the proposed WNC encoding Ray’s
observations are simply re-mapped to an 8-PAM before being
transmitted back to Alice and Bob. For the specific example
at hand, to implement the triple binning scheme in Bob’s side,
the bin of secret bits \(K_a\) contains the most significant of Bob’s
bits (rightmost) while the remaining two bits are partitioned
between \(K_A\) and \(K_B\). Below we discuss two options for this
partitioning.

1) Option 1: \(K_A = 1, K_B = 2\): A first option is for
Alice to use her current transmission to notify Bob of the
symbol she intends to transmit in the next cycle, e.g., by using
differential encoding (transmitting a 0 when the Hamming
distance between the current and the next symbol is 0 and
1 otherwise); as a result, after two initialization cycles, Bob
can know Alice’s next transmission. In the first initialization
cycle Alice transmits an “anchor” symbol and in the second
a differentially encoded symbol that determines the next symbol;
 e.g., in order to transmit the sequence 101101 Alice may
transmit 101101 or 011101, depending on whether the anchor
symbol is set to 1 or 0, respectively. Furthermore, Bob can
use the last two bits of his 8-PAM symbol to index his own
transmission, so that he sends a public bit when his index is
11 and Alice’s current bit is 1 and a secret bit in all other cases,
i.e., Bob can achieve a secrecy rate \(R_s(A) = C_s = \frac{1}{10} = 0.875
\) bits/sec/Hz. In this scheme there is no feedback to Alice
who cannot transmit at any guaranteed strong secrecy, i.e.,
\(R_s(A) = 0\). With the outlined scheme, the point \((0,0.875)\) in
the \(R_s(A) - R_s(B)\) plane is achieved.

2) Option 2: \(K_A = 2, K_B = 1\): As a second option, like
in the previous example we assume that Alice employs an
approach that allows her to notify Bob of her next transmission
(e.g., using differential encoding as above). On the other hand,
the third of Bob’s bits may be used as an index for his current
transmission and the second for feedback to Alice, i.e., overall
we have \(K_A = 2\) and \(K_B = 1\). In this scenario, Bob should
transmit a common (non-secret) bit whenever his index is 1
and Alice’s symbol is a 1 and, a secret bit otherwise, leading
to a secrecy rate \(R_s(A) = \frac{1}{7} = 0.143\) bits/sec/Hz. In parallel,
Alice can be notified to transmit a common bit in the next
transmission interval when the second of Bob’s bits is 1 or a
secret bit when it is 0, leading to a secrecy rate of \(R_s(A) = \frac{1}{7} = 0.143\) bits/sec/Hz. Using this scheme, the point \((0.5,0.75)\) in
the \(R_s(A) - R_s(B)\) plane is achieved.

It follows straightforwardly that using timesharing tech-
niques we can let \(K_A\) and \(K_B\) take any values between \([0,3]\)
as long as they satisfy \(K_A + K_B \leq 3\). The secrecy rates that
are achievable when employing an 8-PAM modulator and a
BPSK modulator can thus be expressed as:

\[
\begin{align*}
R_s(A) & \leq 1 - 2^{-K_A}, \\
R_s(B) & \leq 1 - 2^{-K_B},
\end{align*}
\]

with \(0 \leq K_A + K_B \leq 3\). (32)

This result is generalized in the following theorem for arbitrary
PAM modulators and subsequently extended to the case of
QAM.

**Theorem 2:** Assume that Alice and Bob use \(M_A\) and \(M_B\)
PAM modems with \(M_A \leq M_B\). Then the strong secrecy rates
in the closure of the convex hull of all \((R_s(A), R_s(B))\) satisfying

\[
\begin{align*}
R_s(A) &= \begin{cases} 
\log_2 M_A (1 - 2^{-K_A}), & K_A \geq \log_2 \frac{M_A}{M_A - 1} \\
R_A, & \text{otherwise}
\end{cases} \\
R_s(B) &= \begin{cases} 
\log_2 M_A (1 - 2^{-K_B}), & K_B \geq \log_2 \frac{M_B}{M_B - 1} \\
R_B, & \text{otherwise}
\end{cases}
\end{align*}
\]

\[(33)\]

\[(34)\]

\[
\begin{align*}
R_A &= \log_2 M_A \left( 1 - \frac{M_A - 1}{M_B} \right) + 4 \sum_{a=i_A}^{M_A-1} \sum_{a \equiv 2 \text{ even}} a \log_2 a, \\
R_B &= \log_2 M_A \left( 1 - \frac{M_A - 1}{M_B} \right) + 4 \sum_{a=i_B}^{M_B-1} \sum_{a \equiv 2 \text{ even}} a \log_2 a,
\end{align*}
\]

\[(35)\]

\[(36)\]

\[
\begin{align*}
i_A &= 1 + 2^{\log_2 M_B - K_A}, \\
i_B &= 1 + 2^{\log_2 M_B - K_B},
\end{align*}
\]

\[(37)\]

\[(38)\]

\(K_A, K_B \in \{0, 1, \ldots, \log_2 M_B\}, K_A + K_B = \log_2 M_B\), (39)

are achievable using the triple binning strategy.

**Proof:** Refer to Appendix D

Finally, viewing a square \(M\)-QAM as two orthogonal \(\sqrt{M}\)
PAMs straightforwardly results in the doubling of the strong
secrecy rates when 2-dimensional modulators are employed,
e.g., when Alice employs a QPSK and Bob an \(M_B\)-QAM modulator \((M_B \geq 4)\) then asymptotically 2 secret bits can be
exchanged in each transmission slot as \(M_B\) increases.

**Corollary 1:** Assume that Alice and Bob use square \(M_A\)
and \(M_B\)-QAM constellations with \(M_A \leq M_B\) and denote
\(N_A = \sqrt{M_A}\) and \(N_B = \sqrt{M_B}\). Then the strong secrecy rates
in the closure of the convex hull of all \((R_s(A), R_s(B))\) satisfying

\[
\begin{align*}
R_s(A) &= \begin{cases} 
2 \log_2 N_A (1 - 2^{-K_A}), & K_A \geq \log_2 \frac{N_A}{N_A - 1} \\
2 R_a, & \text{otherwise}
\end{cases} \\
R_s(B) &= \begin{cases} 
2 \log_2 N_A (1 - 2^{-K_B}), & K_B \geq \log_2 \frac{N_B}{N_B - 1} \\
2 R_b, & \text{otherwise}
\end{cases}
\end{align*}
\]

\[(40)\]

\[(41)\]

\[
\begin{align*}
R_a &= \log_2 N_A \left( 1 - \frac{N_A - 1}{N_B} \right) + 4 \sum_{a=i_a}^{N_A-1} \sum_{a \equiv 2 \text{ even}} a \log_2 a, \\
R_b &= \log_2 N_B \left( 1 - \frac{N_B - 1}{N_A} \right) + 4 \sum_{a=i_b}^{N_B-1} \sum_{a \equiv 2 \text{ even}} a \log_2 a
\end{align*}
\]

\[(42)\]
TABLE II
RAY’S OBSERVATION, DETECTION AND GENERATION OF THE WNC CODEWORD

| Observation | -8  | -6  | -4  | -2  | 0   | +2  | +4  | +6  | +8  |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $[\hat{x}_A, \hat{x}_B]$ | (1, 111) | (1, 100) | (1, 110) | (1, 101) | (1, 011) | --- | --- | --- | --- |
| $z = f(y)$ | 111 | 111 | 100 | 111 | 100 | 100 | 101 | 011 | 111 |

In Fig. 5 the achievable strong secrecy rates using the triple-binning approach and PAM modems are depicted for $M_A = 2$ and $M_B = 2, 4, 8, 16, 32, 64$. Furthermore, the achievable secrecy rates when Alice employs a 16-QAM and Bob an $M_B$-QAM modem are shown in Fig. 6.

V. GENERALIZATION TO THE MIMO RELAY CHANNEL

In this section we generalize the setup of the previous sections to a MIMO scenario in which Alice and Bob each have $N$ antennas and Ray has $M \geq N$ antennas. In this case we denote the $M \times N$ channel matrices from Alice to Ray and Bob to Ray by $H_A$ and $H_B$, respectively, and the $N \times M$ channel matrices from Ray to Alice and Ray to Bob, respectively, by $\hat{H}_A$ and $\hat{H}_B$. We assume all matrices are selected from a continuous distribution and are therefore full-rank with probability 1. When Alice and Bob employ PAM modems we take real matrix coefficients, and complex coefficients when they employ QAM modems.

It is well-known and not hard to show (see [31], [35], [36]) that the degrees of freedom $d$ of interference-free transmit dimensions available to both Alice and Bob is bounded by $d \leq (2N - M)^+$, and that every $d$ satisfying this inequality admits an interference-free transmission scheme. Let us recall briefly how such schemes are constructed. Let $H_A$ and $H_B$ be the column spans of $H_A$ and $H_B$, respectively, so that $\dim H_A \cap H_B = (2N - M)^+$. If $d \leq (2N - M)^+$ then Alice and Bob decide on a $d$-dimensional subspace $S$ of this intersection, and then choose $N \times d$ precoding matrices $G_A, G_B$ such that $\text{span } H_A G_A = \text{span } H_B G_B = S$. In a similar manner, $d \leq (2N - M)^+$ guarantees that Ray can transmit his WNC codewords back to Alice and Bob without them losing any signal dimensions.

From now on we suppose that $d \leq (2N - M)^+$. Alice and Bob then form length $d$ data vectors $x'_A, x'_B$ of information symbols and set $x_A = G_A x'_A, x_B = G_B x'_B$ where $G_A, G_B$ are their $N \times d$ precoding matrices. Ray then observes

$$y = H_A G_A x'_A + H_B G_B x'_B + w_R$$

where $w_R$ is a length $M$ vector of additive noise, with variance $\sigma^2_R$ per real dimension. To guarantee that Ray only observes sums of information symbols, Alice and Bob then must construct $G_A$ and $G_B$ to satisfy

$$H_A G_A = H_B G_B$$

(48)

Assuming the above is satisfied, Alice and Bob can then obtain all secrecy rates $dR_s^{(A)}$ and $dR_s^{(B)}$ where $R_s^{(A)}$ and $R_s^{(B)}$ are given by Theorem 2 in the case of PAM modems or Corollary 1 in the case of QAM modems.

In Fig. 5, the achievable strong secrecy rates using the triple-binning approach and PAM modems are depicted for $M_A = 2$ and $M_B = 2, 4, 8, 16, 32, 64$. Furthermore, the achievable secrecy rates when Alice employs a 16-QAM and Bob an $M_B$-QAM modem are shown in Fig. 6.
We further impose power constraints on Alice and Bob by first defining their average per-antenna power by

\[ P_A = \mathbb{E}[|x_{A,i}|^2], \quad P_B = \mathbb{E}[|x_{B,i}|^2] \] (49)

To maintain this constraint after precoding we impose the conditions

\[ |G_A|_F^2 \leq N, \quad |G_B|_F^2 \leq N \] (50)

Here we recall that \( |A|_F^2 \) denotes the Frobenius norm of a matrix \( A \), defined by \( \sum \sigma_i^2 \), where \( \sigma_i \) are the singular values of \( A \). Note that assuming \([48] \) and \([49] \), and after appropriate scaling (in expectation) of the channel matrices \( H_A \) and \( H_B \), we have that the average power of the received signal at Ray is \( P_R = P_A + P_B \).

A. Optimizing Precoders at Alice and Bob

Successful transmission between Alice and Bob requires Ray to accurately detect the sum \( x_A' + x_B' \). That is, \( G_A \) and \( G_B \) should be designed to maximize the mutual information at Ray subject to the secrecy constraint. We see that the task at hand is the following optimization problem:

\[
\text{maximize}_{G_A, G_B} \quad I(Y; X_A' + X_B') \\
\text{subject to} \quad \max \{ \|G_A\|_F^2, \|G_B\|_F^2 \} \leq N \\
H_A G_B = H_B G_B
\] (51)

where \( Y = H_A G_A X_A' + H_B G_B X_B' + W_R \) as before.

1) Zero-forcing precoders: The naive zero-forcing scheme which satisfies the power and secrecy constraints of (51) is the following. Let \( \begin{bmatrix} E_A \\ -E_B \end{bmatrix} \) be a \( 2N \times d \) matrix whose columns form a basis of the right nullspace of the \( M \times 2N \) block matrix \( [H_A \ H_B] \). Now set

\[
G_A = \sqrt{N} E_A / \gamma, \quad G_B = \sqrt{N} E_B / \gamma
\] (52)

and it follows immediately that \( H_A G_A = H_B G_B \) as desired. When \( N = M = d \), we can take \( G_A = \sqrt{N} H_A^{-1} / \gamma \) and \( G_B = \sqrt{N} H_B^{-1} / \gamma \), where \( \gamma = \max \{ \|H_A^{-1}\|_F, \|H_B^{-1}\|_F \} \).

Every pair \( G_A, G_B \) of precoding matrices satisfying the secrecy and power constraint can be constructed via the above process, by studying the nullspace of \( [H_A \ H_B] \). However, for certain parameters of \( M \) and \( N \), one may desire to further optimize some zero-forcing scheme.

2) Using the gap approximation: As an attempt at an improvement on the above naive scheme, let us suppose that the power and secrecy constraints of (51) are satisfied and use the gap approximation to approximate \( I(Y; X_A' + X_B') \) by the channel capacity. By a well-known formula \([57] \), the channel capacity (for fixed \( H_A, H_B \)) is then

\[
C_{(H_A, H_B)} = \log_2 \det \left( I_M + \frac{P_A + P_B}{\sigma_R^2} H_A G_A' G_A H_A^\dagger \right)
\] (53)

The corresponding approximation of the optimization problem at hand is then to maximize \( C_{(H_A, H_B)} \) over all pairs \( (G_A, G_B) \) subject to the same constraints of (51).

To perform this optimization numerically, one can perform steepest descent on the coordinates of \( G_A \) and \( G_B \) via the following process. Let \( G = \begin{bmatrix} G_A \\ G_B \end{bmatrix} \), so that the task is to optimize over all possible \( G \) satisfying the power and secrecy constraints. The constraints of (51) restrict the set of all possible \( G \) to a bounded region of \( N = \text{null} [H_A - H_B] \). One performs steepest descent on the coordinates of \( G \) as normal, but after every iteration replaces \( G \) with the projection \( \text{Proj}_G \) and scales the top and bottom block of \( G \) to satisfy the power constraint of (51). We omit further details.

In Fig. 7 we plot the approximation (53) of the mutual information \( I(Y; X_A' + X_B') \), which measures Ray’s ability to decode the sum \( x_A' + x_B' \) of Alice and Bob’s vectors of information symbols. Here the “zero-forcing” precoders were found explicitly by solving \( H_A G_A = H_B G_B \) and scaling \( G_A \) and \( G_B \) to satisfy the power constraint. The “gap approximation” precoders were then numerically optimized according to (53), and results were averaged over \( 10^5 \) pairs \( (H_A, H_B) \) of channel matrices. The entries of \( H_A \) and \( H_B \) were drawn from real i.i.d. Gaussian distributions normalized to variance \( 1/M \).

The most notable feature of Fig. 7 is that further optimization improves the precoding schemes for \((M, N) = (3, 3)\) and \((4, 3)\) quite a bit, but offers no improvement for the other two cases. We explain this phenomenon in the next subsection. Secondly we note that, for example, the performance of the gap approximation precoder for \((M, N) = (4, 3)\) is better than that of the zero-forcing precoder for \((M, N) = (3, 3)\), even though the second scheme offers an additional degree of freedom. At large SNR the scheme with higher degrees of freedom will ultimately have higher capacity, but the plot shows that at the practical, finite SNR regimes of interest, precoding schemes may have more impact on capacity than the degrees of freedom.

B. The dimension of the space of precoders

In this subsection we compute the dimension of the space of all \( G_A \) and \( G_B \) satisfying the secrecy constraint (48) and the
power constraint \( M \geq \). The dimension of this space measures, in essence, the number of independent parameters available when choosing the precoding matrices, and determines the difficulty of optimizing the precoders numerically. For simplicity we restrict to real precoding matrices and real channels, so that the power constraints are polynomials in the matrix entries. To be mathematically precise, ‘dimension’ here means ‘dimension as a real manifold’, but we omit the mathematical technicalities in favor of exposition.

First, note that if \( G_A \) and \( G_B \) are any precoders such that \( \|G_A\|_F^2 < N \) and \( \|G_B\|_F^2 < N \), then we can always improve the performance of the system by multiplying both precoders by a constant so that either \( \|G_A\|_F^2 = N \) or \( \|G_B\|_F^2 = N \). So from now on we assume that one of the inequalities in (50) is actually an equality. On the other hand, the probability that both inequalities are actually equalities, e.g. that both precoders can be chosen to maximize both Alice and Bob’s transmit power, is zero.

**Proposition 2:** Fix \( d \leq N \leq M \) and let \( H_A, H_B \in \mathbb{R}^{M \times N} \) be generic full-rank matrices. Consider the matrix equation \( H_A G_A = H_B G_B \) for some variable matrices \( G_A, G_B \in \mathbb{R}^{N \times d} \) such that \( \|G_A\|_F^2 \leq N \) or \( \|G_B\|_F^2 \leq N \), and that exactly one of these inequalities is an equality. Let \( P \) be the space of all such \( G_A, G_B \) satisfying these conditions. Then

\[
\dim P = (2N - M)d - 1 \quad (54)
\]

In particular, when \( d = 2N - M = 1 \) we have \( \dim P = 0 \) and thus the solution \( G_A, G_B \) is essentially unique.

**Proof:** The dimension of the space of all pairs \( G_A, G_B \in \mathbb{R}^{N \times d} \) is \( 2Nd \). The equation \( H_A G_A - H_B G_B = 0 \) defines \( Md \) linear equations in the entries of \( G_A, G_B \), all of which are independent by the assumptions that \( H_A \) and \( H_B \) are generic and full-rank. Furthermore, suppose without loss of generality that \( \|G_A\|_F^2 = N \). This single additional quadratic equation further reduces the dimension of the total space by one. Putting this all together gives us

\[
\dim P = 2Nd - Md - 1 = (2N - M)d - 1 \quad (55)
\]

as claimed.

When \( d = (2N - M)^+ \) is chosen maximal, as was the case for the simulations depicted in Fig. 11, we have \( \dim P = (2N - M)^2 d - 1 \) dimensions to optimize over when employing precoders designed to maximize (53). When \( (M, N) = (3, 2) \) or \( (5, 3) \) the dimension of \( P \) is zero and thus any solution is essentially unique, meaning that additional optimization yields no benefit in terms of capacity. Preliminary results suggest that the larger \( \dim P \) is, the more benefit numerical optimization yields, but we will refrain from attempting to make this precise.

**C. Remarks on the K-user-pair MIMO relay channel**

In this subsection we very briefly mention a straightforward consequence of our results to the \( K \)-user-pair MIMO relay channel. In the \( K \)-user-pair MIMO relay channel, we have user pairs \( (A_1, B_1), \ldots, (A_K, B_K) \) all attempting communication through a relay. Each user has \( N \) antennas and the relay has \( M \geq N \) antennas. We refer to the precise definition of the channel model in [31], [35], [36] wherein the degrees of freedom of such a scenario has been extensively studied.

**Corollary 2:** For a \( K \)-user-pair MIMO relay channel with parameters satisfying \( Kd = M \) and \( d = 2N - M > 0 \) all users \( A_i \) and \( B_i \) can transmit at secrecy rates

\[
R_s^{(A_i)} = dR_s^{(A)}, \quad R_s^{(B_i)} = dR_s^{(B)} \quad (56)
\]

using PAM modems, with \( R_s^{(A)} \) and \( R_s^{(B)} \) as in Theorem 2 and similarly for QAM modems by Corollary 1.

**Proof:** For the given parameters, \( d \) degrees of freedom are achievable by the main result and signal alignment strategies of [35]. For a given signal alignment strategy, then, each user simply employs the encoding method of Section IV on each of the \( d \) independent signal dimensions.

While this is of course a straightforward corollary to the construction of the signal alignment strategies in [31], [35], it is worth remarking as incorporating non-zero secrecy rates into signal alignment strategies for multi-way MIMO relay channels seems absent from the literature. Future work will concern possible applications to the \( K \)-user MIMO Y-channel [38], in which a single group of users, say \( A_1, \ldots, A_K \) are all attempting simultaneous communication through Ray. Security, then, would be predicated on a generalization of our encoding scheme to the case of an observation of a sum of \( K \) independent PAM constellations.

**VI. CONCLUSIONS AND FUTURE WORK**

We have studied the potential for strong secrecy in a single-relay network with two users. Given finite, uniform input distributions, we have calculated the corresponding strong secrecy capacity, and provided an explicit formula for the strong secrecy capacity when these input distributions are determined by traditional PAM or QAM signaling. A scheme that achieves strong secrecy was presented using standard \( M \)-PAM and \( M \)-QAM modulators, using a novel triple binning approach in the larger of the two constellations. The symbols are generated as the concatenation of i) a bin that carries information bits (secret or common), ii) a bin that carries index bits intended for Bob and iii) a bin that carries index bits for Alice. We have shown that on a per symbol basis it is possible to asymptotically transmit as many secret bits as \( \log_2 \) of the size of the shortest of the PAM/QAM constellations. Our results were generalized to a MIMO setting where the users and the relay have multiple antennas, and optimal precoding matrices were studied. Finally, the potential for lattice encoders, alternative power allocation schemes, and applications to larger relay networks will be examined in the future.

**APPENDIX**

**A. Proof of Lemma 2**

**Proof:** The pmf of \( Y \) is given by the convolution of the pmfs of \( h_A X_A \) and \( h_B X_B \). Explicitly, then, the convolution
of these two is given by
\[ p_Y(y) = \sum_{x_A \in X_A} p_{Y,X_A}(h_A x_A) p_{X_B}(y - h_A x_A) \]
\[ = \frac{1}{M_A} \sum_{x_A \in X_A} p_{X_B}(y - h_A x_A) \]
\[ = \frac{1}{M_A} \sum_{x_A \in X_A} \begin{cases} \frac{1}{M_B}, & y - h_A x_A \in h_B X_B \\ 0, & \text{otherwise} \end{cases} \]
\[ = \frac{1}{M_A M_B} \{ (x_A | y - h_A x_A \in h_B X_B) \} \]
\[ = \frac{1}{M_A M_B} |\psi^{-1}(y)|. \quad (57) \]

To compute the entropy of \( Y \), observe that
\[ H(Y) = -\sum_{y \in Y} \frac{|\psi^{-1}(y)|}{M_A M_B} \log_2 \frac{|\psi^{-1}(y)|}{M_A M_B} \]
\[ = -\frac{1}{M_A M_B} \sum_{y \in Y} |\psi^{-1}(y)| \log_2 |\psi^{-1}(y)| \]
\[ + \log_2(M_A M_B) \frac{1}{M_A M_B} \sum_{y \in Y} |\psi^{-1}(y)|. \quad (58) \]

Since \( \sum_{y \in Y} |\psi^{-1}(y)| = M_A M_B \) the proof of the lemma is complete. \[ \square \]

B. Proof of Proposition 1

Proof: By Lemma 1 it suffices to compute the mutual information \( I(Y; X_A) \). We first fix a single \( x_A \in X_A \) and compute the marginal mutual information \( I(Y; x_A) \). An easy computation reveals that the joint distribution \((Y, X_A)\) has pmf
\[ p_{Y,X_A}(y,x_A) = \frac{|\psi^{-1}(y) \cap \{ x_A \} \times X_B |}{M_A M_B} \]
\[ = \begin{cases} \frac{1}{M_A M_B}, & \text{if } \psi^{-1}(y) \cap \{ x_A \} \times X_B = \emptyset \\ 0, & \text{otherwise} \end{cases} \quad (59) \]

Computing the mutual information \( I(Y; x_A) \) now gives
\[ I(Y; x_A) = \sum_{y \in Y} p_{Y,X_A}(y,x_A) \log_2 \left( \frac{p_{Y,X_A}(y,x_A)}{p_Y(y) p_{X_A}(x_A)} \right) \]
\[ = \sum_{y \in X_A + X_B} \frac{1}{M_A M_B} \log_2 \frac{M_A}{|\psi^{-1}(y)|} \quad (60) \]

Summing up over all \( x_A \), we arrive at
\[ I(Y; X_A) = \sum_{x_A \in X_A} I(Y; x_A) \]
\[ = \frac{1}{M_A M_B} \sum_{x_A \in X_A} \sum_{y \in Y} \log_2 \frac{M_A}{|\psi^{-1}(y)|} \]
\[ = \log_2(M_A) \]
\[ - \frac{1}{M_A M_B} \sum_{x_A \in X_A} \sum_{y \in Y} \log_2 |\psi^{-1}(y)| \]
\[ = \log_2(M_A) \]
\[ - \frac{1}{M_A M_B} \sum_{y \in Y} |\psi^{-1}(y)| \log_2 |\psi^{-1}(y)|. \quad (61) \]

where the last equality follows by grouping like summands together. The proposition follows. \[ \square \]

C. Proof of Theorem 1

Proof: Without loss of generality we assume that \( M_B \geq M_A \). Furthermore, multiplying the observation at the relay by \( h_A \) does not affect any rate calculations, and hence we may assume \( h_A = h_B = 1 \). The set \( \mathcal{Y} \) of possible observations at Ray now takes the form
\[ \mathcal{Y} = \mathcal{X}_A + \mathcal{X}_B = \{ x_A + x_B | x_A \in \mathcal{X}_A, x_B \in \mathcal{X}_B \}. \quad (62) \]

Considering the convolution of the uniform pmfs on Alice and Bob’s underlying PAM constellations gives the following formula:
\[ |\psi^{-1}(y)| = \begin{cases} 0, & y \text{ odd or } |y| \geq M_B + M_A, \\ \frac{M_B + M_A - |y|}{2}, & M_B - M_A \leq |y| \leq M_B + M_A - 2, \\ M_A, & |y| \leq M_B - M_A - 2. \end{cases} \quad (63) \]

Now let \( S \) be
\[ S = \sum_{y \in \mathcal{Y}} |\psi^{-1}(y)| \log_2 |\psi^{-1}(y)|. \quad (64) \]

We can use the above formula to write \( S \) as
\[ S = \sum_{y \in \mathcal{Y}} |\psi^{-1}(y)| \log_2 |\psi^{-1}(y)| = F + L \quad (65) \]
where
\[ L = 2 \sum_{y = M_B + M_A - 2}^M y \log_2 \left( \frac{M_B + M_A - |y|}{2} \right) \]
and
\[ F = \sum_{y = -M_B + M_A + 2}^{M_B - M_A - 2} M_A \log_2(M_A) \quad (66) \]

Making the change of variables \( a = M_A - \frac{y - (M_B - M_A)}{2} \) transforms the sum \( L \) into
\[ L = 2 \sum_{a = 1}^{M_A} a \log_2(a). \quad (68) \]

To treat \( F \), notice that that the number of even integers in the interval \([-M_B + M_A + 2, M_B - M_A - 2]\) is exactly \( M_B - M_A - 1 \), and hence
\[ F = (M_B - M_A - 1) M_A \log_2(M_A) \quad (69) \]

We can conclude the proof by recalling that \( C_s = \frac{1}{M_A M_B} S = \frac{1}{M_A M_B} (F + L) \), which is easily shown to be equal to the quantity in the theorem given the above calculations. \[ \square \]
The number of elements in the linear part of Ray’s pmf is 2\( \log_2 M_B \) indexing bits (excluding more combinations apparently requires less than \( \log_2 M_B \) bits). As a result, the size of the bins \( K_A \) and \( K_B \) must satisfy \( K_A + K_B \leq \log_2 M_B \). Using \( K_B \) bits for indexing Bob’s transmissions, we can enumerate \( 2^{K_B} \) partitions of \( 2^{\log_2 M_B + \log_2 M_A - K_B} \) combinations each. Of these \( 2^{K_B} \) partitions we need to exclude exactly one (that contains all non-viable combinations). So we are left with \( 2^{K_B} - 1 \) partitions of \( 2^{\log_2 M_B + \log_2 M_A - K_B} \) combinations out of a total of \( 2^{\log_2 M_B + \log_2 M_A} \) combinations.

The number of elements in the linear part of Ray’s pmf is \( 2^{\log_2 M_A + \log_2 (M_A - 1)} \). Depending on the number of elements in the partition to be excluded we distinguish two cases:

**Case 1:** \( 0 \leq K_B \leq \log_2 M_A - 1 \). In this case we exclude all elements in the linear parts of Ray’s pmf and we are left with elements only on the flat region. For each element on the flat region the equivocation is \( \log_2 M_A \) bits. As a result, Bob’s rate in this case is given by

\[
R_s^{(B)} = \log_2 M_A \left( \frac{(2K_B - 1)2^{\log_2 M_B + \log_2 M_A - K_B}}{2^{\log_2 M_B + \log_2 M_A}} \right) = \log_2 M_A (1 - 2^{K_B}). \tag{70}
\]

**Case 2:** \( \log_2 M_A - 1 < K_B \leq \log_2 M_B \). In this case all elements of the flat region are retained for the transmission of secret bits, contributing to the secrecy rate with \( \log_2 M_A (M_B + M_A + 1) \) bits/sec/Hz. On the other hand part of the linear regions is excluded. The optimal exclusion of elements from the linear regions first trims the odd multiplicity elements of the pmf and then trims the parts of the pmf with the less mass. Based on this reasoning the linear parts contribute to the secrecy rate with

\[
R_B = \frac{2}{M_A M_B} \sum_{a=i_A}^{M_A-1} \left( 2 \left( \frac{a}{2} \right) \log_2 \frac{a}{2} \right)
= \frac{4}{M_A M_B} \sum_{a = i_A}^{M_A-1} a \log_2 a,
\tag{71}
\]

\[
i_A = 1 + 2^{\log_2 M_B - K_B}. \tag{72}
\]

As a result, Bob’s rate is given as

\[
R_s^{(B)} = \begin{cases} 
\log_2 M_A (1 - 2^{-K_B}), & K_B \geq \log_2 M_B - M_A - 1, \\
R_B, & \text{otherwise},
\end{cases}
\tag{73}
\]

where

\[
R_B = \log_2 M_A \left( \frac{M_B - M_A + 1}{M_A M_B} \right) + R_B. \tag{74}
\]

Similarly, using an analogous analysis and employing a bin of size \( K_A \) to index Alice’s transmissions we can evaluate the respective expression for Alice’s rate by substituting \( K_B \) by \( K_A \). Finally, we can let \( K_A, K_B \) take any values in \( \{0, 1, \ldots, \log_2 M_B \} \) so that \( K_A + K_B \leq \log_2 M_B \).

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