Spatial Moduli of Non-Differentiability for Linearized Kuramoto–Sivashinsky SPDEs and Their Gradient

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Abstract: We investigate spatial moduli of non-differentiability for the fourth-order linearized Kuramoto–Sivashinsky (L-KS) SPDEs and their gradient, driven by the space-time white noise in one-to-three dimensional spaces. We use the underlying explicit kernels and symmetry analysis, yielding spatial moduli of non-differentiability for L-KS SPDEs and their gradient. This work builds on the recent works on delicate analysis of regularities of general Gaussian processes and stochastic heat equation driven by space-time white noise. Moreover, it builds on and complements Allouba and Xiao’s earlier works on spatial uniform and local moduli of continuity of L-KS SPDEs and their gradient.

Keywords: L-KS SPDEs; space-time white noise; modulus of non-differentiability; Hölder regularity

1. Introduction

The fourth-order linearized Kuramoto–Sivashinsky (L-KS) SPDEs are used to model the pattern formation phenomena accompanying the appearance of turbulence (see [1–6]). Among other things, the authors of [1,2] investigated classical examples of deterministic and stochastic pattern formation PDEs, and the authors of [1–4] investigated the L-KS class and its connection to many classical and new examples of deterministic and stochastic pattern formation PDEs.

In [7,8], motivated by [1], Allouba introduced and gave the explicit kernel stochastic integral equation formulation for L-KS SPDEs. The fundamental kernel associated with the deterministic version of this class is built on the Brownian-time process in [3,7,8]. In this article, we give exact, dimension-dependent, spatial moduli of non-differentiability for the important class of stochastic equation:

\[
\begin{align*}
\frac{\partial U}{\partial t} &= -\frac{\varepsilon}{8} (\Delta + 2\beta)^2 U + \frac{\partial^{d+1}W}{\partial t \partial x}, \\
U(0,x) &= u_0(x), \\
(t,x) &\in \mathbb{R}_+ \times \mathbb{R}^d,
\end{align*}
\]

(1)

where \(\Delta\) is the \(d\)-dimensional Laplacian operator, \(\mathbb{R}_+ = (0,\infty), (\varepsilon, \beta) \in \mathbb{R}_+ \times \mathbb{R}\) is a pair of parameters, and the noise term \(\partial^{d+1}W/\partial t\partial x\) is the space-time white noise corresponding to the real-valued Brownian sheet \(W\) on \(\mathbb{R}_+ \times \mathbb{R}^d, d = 1, 2, 3\). The initial data \(u_0\) here are assumed Borel measurable, deterministic, and 2-continuously differentiable on \(\mathbb{R}^d\), whose 2-derivative is locally Hölder continuous with some exponent \(0 < \gamma \leq 1\).

Of course, Equation (1) is the formal (and non-rigorous) equation. Its rigorous formulation, which we work with in this article, is given in mild form as kernel stochastic integral equation (SIE). This SIE was first introduced and studied in [1–3,7–10]. We give it below in Section 2, along with some relevant details.

The existence/uniqueness as well as sharp dimension-dependent \(L^p\) and Hölder regularity of the linear and nonlinear noise version of (1) were investigated in [1,2,9,11]. The exact uniform and local moduli of continuity for the L-KS SPDE in the time variable \(t\) and space variable \(x\) were investigated in [4]. In fact, in [4], the exact spatio-temporal, dimension-dependent, uniform, and local moduli of continuity for the fourth order of the
L-KS SPDEs and their gradient were established. It was studied in [11] that the solutions to the fourth order L-KS SPDEs and their gradient, driven by the space-time white noise in one-to-three dimensional spaces, in time, have infinite quadratic variation, and also investigated temporal central limit theorems for the realized power variations of the L-KS SPDEs with space-time white noise in one-to-three dimensional spaces in times.

In a series of articles [1,5–7], Allouba investigated the existence/uniqueness, sharp dimension-dependent $L^p$, and Hölder regularity of the linear and nonlinear noise version of Equation (1). These results naturally lead to the following list of motivating questions:

- Are the solutions to Equation (1) spatial continuously differentiable?
- What are the exact moduli of non-differentiability?
- What are the exact moduli of continuity?

Allouba and Xiao [4] investigated the exact, spatio-temporal, dimension-dependent, uniform, and local moduli of continuity for the fourth order L-KS SPDEs and their gradient. These results gave the answers to spatial continuity and exact moduli of continuity of the solutions to Equation (1), and gave partial answers to above questions. In this article, we are concerned with the exact moduli of non-differentiability of the process $U$ and its gradient in space.

Our paper is organized as follows. In Section 2, the rigorous L-KS SPDE kernel SIE formulation, spatial spectral density, and spatial small ball probability estimates for L-KS SPDEs and their gradient are discussed by using the L-KS SPDE kernel SIE formulation and symmetry analysis.

In Section 3, we investigate spatial zero-one laws for L-KS SPDEs and their gradient were established. It was studied in [11] that the solutions to Equation (1), and gave partial answers to above questions. In this article, we are concerned with the exact moduli of non-differentiability of the process $U$ and its gradient in space.

Our paper is organized as follows. In Section 2, the rigorous L-KS SPDE kernel SIE formulation, spatial spectral density, and spatial small ball probability estimates for L-KS SPDEs and their gradient are discussed by using the L-KS SPDE kernel SIE formulation and symmetry analysis. In Section 3, we investigate spatial zero-one laws and the exact spatial moduli of non-differentiability for L-KS SPDEs and their gradient by making use of the Gaussian correlation inequality [12] and the theory on limsup random fractals [13]. In Section 4, the results are summarized and discussed.

2. Methodology
2.1. Rigorous Kernel Stochastic Integral Equations Formulations

As shown in [1–3], the L-KS kernel is the fundamental solution to the deterministic version of (8) ($a \equiv 0$ and $\sigma \equiv 0$) below, and is given by

$$K_{LKS}^{BM} = \int_0^{\infty} e^{-is/2} \frac{1}{(2\pi is)^{d/2}} K_{BM} ds + \int_0^{\infty} e^{-is/2} \frac{1}{(2\pi is)^{d/2}} K_{BM} ds$$

(2)

where $t^2 = -1$ and $K_{BM}^{\beta} = e^{-\beta/(2\pi)} \sqrt{2\pi}$. The nonlinear drift diffusion L-KS SPDE is

$$\begin{align*}
\frac{\partial U}{\partial t} &= -\frac{\epsilon}{8} (\Delta + 2\beta)^2 U + a(U) + \sigma(U) \frac{\partial^{d+1} W}{\partial t \partial x}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d; \\
U(0,x) &= u_0(x),
\end{align*}$$

(3)

Then, the rigorous L-KS kernel SIE (mild) formulation is the following SIE:

$$U(t,x) = \int_{\mathbb{R}^d} K_{LKS}^{\beta} u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} K_{LKS}^{\beta} [a(U(s,y)) dsdy + \sigma(U(s,y)) W(ds \times dy)]$$

(4)

(see p. 530 in [5] and Definition 1.1 and Equation (1.11) in [1]). Of course, the mild formulation of (1.1) is then obtained by setting $\sigma \equiv 1$ and $a \equiv 0$ in (4).
Notation 1. Positive and finite constants (independent of $x$) in Section i are numbered as $c_{i,1}, c_{i,2}, \ldots$

2.2. Spatial Spectral Density for L-KS SPDEs and Their Gradient

Our spatial results are crucially dependent on the following spatial spectral density for L-KS SPDEs and their gradient. In this lemma, (a) is Lemma 3.1 in [4], and (b) follows from (3.28) in [4].

Lemma 1. (L-KS SPDE spatial spectral density). Fix $(\epsilon, \beta) \in \mathbb{R}_+ \times \mathbb{R}$ and fix $t \in \mathbb{R}_+$. Assume that $u_0 = 0$ in Equation (1).

(a) Let the spatial dimension $d \in \{1, 2, 3\}$. Then, the L-KS SPDE solution $\{U(t, x), x \in \mathbb{R}^d\}$ is stationary with spectral density

$$f(t, \xi) = \frac{4}{\epsilon (2\pi)^d} \frac{1 - e^{-\frac{\epsilon}{4}(\beta^2 + |\xi|^2)^2}}{(-2\beta + |\xi|^2)^2}, \quad \forall \xi \in \mathbb{R}^d. \tag{5}$$

(b) Let $d = 1$. Then, the gradient of the L-KS SPDE solution $\{\partial_x U(t, x), x \in \mathbb{R}\}$ is stationary with spectral density

$$\tilde{f}(t, \xi) = \frac{4}{\epsilon (2\pi)^d} \frac{\xi^2 (1 - e^{-\frac{\epsilon}{4}(\beta^2 + |\xi|^2)^2})}{(-2\beta + |\xi|^2)^2}, \quad \forall \xi \in \mathbb{R}. \tag{6}$$

2.3. Extremes for L-KS SPDEs and Their Gradient

Our spatial results are also dependent on the following small ball probability estimates for L-KS SPDEs and their gradient. Fix $t \in \mathbb{R}_+$. For $x \in \mathbb{R}^d$, $r \in \mathbb{R}_+$ and compact rectangle $I_{\text{space}} \subset \mathbb{R}^d$, we define

$$I(x, r) = \sup_{y \in I_{\text{space}}: |y| \leq 1} |U(t, x + ry) - U(t, x)|$$

and

$$\mathcal{I}(x, r) = \sup_{y \in I_{\text{space}}: |y| \leq 1} |\partial_x ry U(t, x + ry) - \partial_x U(t, x)|.$$

Lemma 2. Fix $(\epsilon, \beta) \in \mathbb{R}_+ \times \mathbb{R}$ and fix $t \in \mathbb{R}_+$. Assume that $u_0 = 0$ in Equation (1).

(a) Let the spatial dimension $d = 3$. Then, there exist positive and finite constants $c_{2,3}$ and $c_{2,2}$ depending only on $\beta$ such that for all $x_0 \in [0, 1]^3$, $r > 0$, and $u \in (0, 1)$,

$$\exp \left( - \frac{c_{2,3} r^3}{u^6} \right) \leq \mathbb{P}(I(x_0, r) \leq u) \leq \exp \left( - \frac{c_{2,2} r^3}{u^6} \right). \tag{7}$$

(b) Let $d = 1$. Then, there exist positive and finite constants $c_{2,3}$ and $c_{2,4}$ depending only on $\beta$ such that for all $x_0 \in [0, 1]$, $r > 0$, and $u \in (0, 1)$,

$$\exp \left( - \frac{c_{2,3} r^3}{u^2} \right) \leq \mathbb{P}(\mathcal{I}(x_0, r) \leq u) \leq \exp \left( - \frac{c_{2,4} r^3}{u^2} \right). \tag{8}$$

Proof. It follows from Lemma 3.2 in [4] that for every fixed $t \in \mathbb{R}_+$, the L-KS SPDE solution $\{U(t, x); x \in \mathbb{R}^3\}$ is spatial strongly locally nondeterministic. That is, for every $T > 0$, there exists a finite constant $c_{2,5} > 0$ (depending on $t$ and $T$) such that for every $n \geq 1$ and for every $x, y_1, \ldots, y_n \in [-T, T]^3$,

$$\text{Var}[U(t, x) | U(t, y_1), \ldots, U(t, y_n)] \geq c_{2,5} \min_{1 \leq j \leq n} \{|x - y_j|\}. \tag{9}$$
where \( y_0 = 0 \). Also, it follows from Lemma 4.4 in [4] that
\[
c_{2,6}|x - y| \leq \mathbb{E}[(U(t, x) - U(t, y))^2] \leq c_{2,7}|x - y|; \forall x, y \in [-T, T]^3.
\] (10)

Thus, by Theorem 3.1 in [14] or Lemma 2.2 in [15], Equations (9) and (10) yield Equation (7).

Similarly, as by (3.29) and (3.30) in [4], Equations (9) and (10) also hold for the gradient of the L-KS SPDE solution \( \partial_x U \) instead of \( U \), one has that Equation (8) holds. This completes the proof.

We also need the following lemma, which is Theorem 1.1 in [12].

**Lemma 3.** Let \( x' = (x'_1, x'_2) \) be an \( \mathbb{R}^n \)-valued Gaussian random vector with mean 0, where \( x_1 = (x_1, ..., x_k)' \), \( x_2 = (x_{k+1}, ..., x_n)' \) and \( 1 \leq k < n \). Then,
\[
\mathbb{P}(\|x\|_\infty \leq u) \leq \kappa \mathbb{P}(\|x_1\|_\infty \leq u) \mathbb{P}(\|x_2\|_\infty \leq u),
\] (11)
where \( \|x\|_\infty \) denotes the maximum norm of a vector \( x \) and
\[
\kappa = \left( \frac{\det(\mathbb{E}[x_1x'_1]) \det(\mathbb{E}[x_2x'_2])}{\det(\mathbb{E}[xx'])} \right)^{1/2}.
\] (12)

We also need the following lemma, which is Lemma 2.4 in [16].

**Lemma 4.** Let \( B = (b_{ij}, 1 \leq i, j \leq 2n) \) be a positive semidefinite symmetric matrix given by
\[
B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},
\]
where \( B_{ij}, 1 \leq i, j \leq 2 \), are \( n \times n \) matrices. Put \( A_i = \sum_{j=n+1}^{2n} |b_{ij}| \) for \( 1 \leq i \leq n \) and \( = \sum_{j=1}^{n} |b_{ij}| \) for \( n + 1 \leq i \leq 2n \). Assume the following conditions are satisfied:
(i) There is a constant \( \lambda \) such that for all \( 1 \leq i \leq 2n \),
\[
A_i < \lambda.
\] (13)
(ii) There exists a finite constant \( K > 0 \) such that for all \( 1 \leq i \leq 2n \),
\[
\frac{\det(B^{(i)})}{\det(B)} \leq K,
\] (14)
where \( B^{(i)} \) is the submatrix of \( B \) obtained by deleting the \( i \)th row and \( i \)th column.

Then
\[
\det(B) \geq e^{-2\lambda K n} \det(B_{11}) \det(B_{22}).
\] (15)

3. Results

3.1. Spatial Zero-One Laws for L-KS SPDEs and Their Gradient

We consider spatial zero-one laws of moduli of non-differentiability for L-KS SPDEs and their gradient.

**Proposition 1.** Fix \( (\epsilon, \beta) \in \mathbb{R}_+ \times \mathbb{R} \) and fix \( t \in \mathbb{R}_+ \). Assume that \( u_0 = 0 \) in Equation (1).

(a) Let the spatial dimension \( d \in \{1, 2, 3\} \). Then, for any compact rectangle \( I_{\text{space}} \subset \mathbb{R}^d \), there exists a constant \( 0 \leq c_{3,1} \leq \infty \) such that
\[
\liminf_{r \to 0^+} \rho(r) \inf_{x \in I_{\text{space}}} I(x, r) = c_{3,1} \ a.s.
\] (16)
where
\[ \rho(r) = (r \ln r)^{-1/3} - 1/2. \] (17)

(b) Let \( d = 1 \). Then, for any compact rectangle \( I_{space} \subset \mathbb{R} \), there exists a constant \( 0 \leq c_{3,2} \leq \infty \) such that
\[ \lim_{r \to 0^+} \inf_{x \in I_{space}} \frac{\gamma(r)}{\rho(r)} \geq c_{3,2} \quad \text{a.s.} \] (18)

where
\[ \gamma(r) = (r \ln r)^{-1/2}. \] (19)

Remark 1. Equation (16) establishes zero-one law of moduli of non-differentiability of the sample function \( x \mapsto U(t,x) \) over the compact rectangle \( I_{space} \). Equation (18) establishes zero-one law of moduli of non-differentiability of the sample function \( x \mapsto \partial_x U(t, x) \) over the compact rectangle \( I_{space} \).

Remark 2. From the proof below, Equation (16) holds for any \( \rho(r) \) whenever \( \rho(r) \ln r \to 0 \), and Equation (18) holds for any \( \gamma(r) \) whenever \( \gamma(r) \ln r \to 0 \). Here, \( \rho(r) \) and \( \gamma(r) \) in Equations (17) and (19) come from Theorem 1 below for convenience.

Proof of Proposition 1. As the proof of Equation (18) is similar to Equation (16), we only prove Equation (16). Let \( A_1 := O(0, 1) \subset \mathbb{R}^d \) and for \( n \geq 2 \), \( A_n := O(0, n) \setminus O(0, n - 1) \subset \mathbb{R}^d \) such that \( A_1, A_2, \ldots \), are mutually disjoint, where the following notation is used: \( O(h, \delta) = \{ z \in \mathbb{R}^d : \| z - h \| \leq \delta \} \). For \( n \geq 1 \) and \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \), let
\[ X_n(t, x) := \int_{A_n} \int_0^1 I_{K_LK_R^d} W(dr \times dz), \]
then \( X_n = \{ X_n(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \}, n = 1, 2, \ldots \), are independent Gaussian fields. By Equation (4), we express
\[ U(t, x) = \sum_{n=1}^{\infty} X_n(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d. \]

Equip \( I_{space} = [0, 1]^d \) with the canonical metric
\[ d_X(x, y) = (E[X_n(t, x) - X_n(t, y)]^2)^{1/2}, \quad x, y \in I_{space}, \] (20)
and denote by \( N(d_X, I_{space}, \delta) \) the smallest number of \( d_X \)-balls of radius \( \delta > 0 \) needed to cover \( I_{space} \). It follows from Lemma 1 that
\[ d_X(x, y) = \left(2 \int_{A_n} (1 - \cos(\langle x - y, \tau \rangle))f(t, \tau)d\tau \right)^{1/2} \]
\[ \leq |x - y| \left( \int_{A_n} |\tau|^2 f(t, \tau)d\tau \right)^{1/2} \]
\[ =: |x - y|K_n, \quad x, y \in \mathbb{R}^d, \] (21)
where for obtaining the last inequality, we bound \( 1 - \cos(\langle x, y \rangle) \) by \( |x|^2 |y|^2 / 2 \) for \( x, y \in \mathbb{R}^d \). It follows from Theorem 4.1 in Meerschaert et al. [17] that
\[ \sup_{x, y \in I_{space}, |y| \leq r} \frac{|X_n(t, x + y) - X_n(t, x)|}{\lambda(r)} \leq c_{3,3} \quad \text{a.s.}, \] (22)
where \( \lambda(r) = r \sqrt{|\ln r|} \). Put

\[
\zeta_M(t, x) = \sum_{n=1}^{M} X_n(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d.
\]

Then, by Equation (22), one has

\[
\lim_{r \to 0^+} \sup_{x, y \in I_{\text{space}}: |y| \leq r} \rho(r) |\zeta_M(t, x + y) - \zeta_M(t, x)| = 0 \text{ a.s.} \tag{23}
\]

Therefore, the random variable

\[
\liminf_{r \to 0^+} \rho(r) \inf_{x \in I_{\text{space}}} I(x, r)
\]

is measurable with respect to the tail field of \( \{X_n\}_{n=1}^{\infty} \) and thus is constant almost surely. This implies Equation (16). \( \square \)

### 3.2. Spatial Moduli of Non-Differentiability for L-KS SPDEs and Their Gradient

We establish the exact spatial moduli of non-differentiability for L-KS SPDE solution \( U(t, x) \) and the gradient process \( \partial_x U(t, x) \).

**Theorem 1.** (Spatial moduli of non-differentiability) Fix \((\epsilon, \beta) \in \mathbb{R}_+ \times \mathbb{R} \) and fix \( t \in \mathbb{R}_+ \). Assume that \( u_0 = 0 \) in Equation (1).

(a) Let the spatial dimension \( d = 3 \). Then, for any compact rectangle \( I_{\text{space}} \subset \mathbb{R}^3 \),

\[
\lim_{r \to 0^+} \rho(r) \inf_{x \in I_{\text{space}}} I(x, r) = c_{3,4} \text{ a.s.} \tag{25}
\]

Consequently, the sample paths of \( U(t, x) \) are almost surely nowhere differentiable in all directions of \( x \).

(b) Let \( d = 1 \). Then, for any compact rectangle \( I_{\text{space}} \subset \mathbb{R} \),

\[
\lim_{r \to 0^+} \rho(r) \inf_{x \in I_{\text{space}}} I(x, r) = c_{3,5} \text{ a.s.} \tag{26}
\]

Consequently, the sample paths of \( \partial_x U(t, x) \) are almost surely nowhere differentiable in \( x \).

**Proof of Theorem 1.** As the proof of Equation (26) is similar to Equation (25), we only prove Equation (25). To show Equation (25), we first claim the following two inequalities:

\[
\lim_{r \to 0^+} \rho(r) \inf_{x \in I_{\text{space}}} I(x, r) \geq c_{3,6} \text{ a.s.} \tag{27}
\]

and

\[
\lim_{r \to 0^+} \rho(r) \inf_{x \in I_{\text{space}}} I(x, r) \leq c_{3,7} \text{ a.s.} \tag{28}
\]

where \( c_{3,6} < c_{3,7} \) and \( c_{3,7} > (c_{2,1}/2)^{1/6} \). By Equations (27) and (28) and the zero-one law Equation (16), one has that Equation (25) holds and thereby we complete the proof.

It remains to show Equations (27) and (28). We first show Equation (27). Without loss of generality, we assume \( I_{\text{space}} = [0, 1]^3 \).

For \( n \in \mathbb{Z}_+ \), we define \( r_n = \theta^{-n} \) and \( a_n = \theta^{bn} \), where \( \theta > 1 \) is an arbitrary constant and will be specified latter on. For \( i = (i_1, i_2, i_3) \in \mathbb{Z}_+^3 \) and \( n \geq 1 \), we put

\[
\Omega_n = \{ r < (0, 1): r_{n+1} < r \leq r_n \},
\]

\[
\Omega_{n,1} = \{ x = (x_1, x_2, x_3)' \in I: i a_n^{-1} < x \leq (i + 1)a_n^{-1} \}.
\]
where 1 is a vector with elements 1. Observe that for all \( r \in (0, 1) \), there exists a set \( \Omega_n \) such that \( r \in \Omega_n \), and for all \( x \in I \), there exists a set \( \Omega_{n,i} \) such that \( x \in \Omega_{n,i} \). Let \( x_{i,n} := i \theta_n^{-1} \) be a point in \( \Omega_{n,i}, 1 \in [0, a_n^2] \cap \mathbb{B}_3^+ \). Then, by Equation (7), one has

\[
\mathbb{P}\left( \rho(r_n) \min_{i \in [0, a_n^2] \cap \mathbb{B}_3^+} I(x_{i,n}, r_n) \leq c_{3,8}\right) \leq \sum_{i \in [0, a_n^2] \cap \mathbb{B}_3^+} \mathbb{P}(I(x_{i,n}, r_n) \leq c_{3,8} \rho(r_n)^{-1}) \leq 3 \theta^{-3,8} n.
\]

Therefore, by Borel–Cantelli lemma, one has

\[
\liminf_{n \to \infty} \rho(r_n) \min_{i \in [0, a_n^2] \cap \mathbb{B}_3^+} I(x_{i,n}, r_n) = c_{3,8} \text{ a.s.} \tag{29}
\]

It follows from Theorem 4.1 in Meerschaert et al. [17] that

\[
\limsup_{n \to \infty} \rho(r_n) \sup_{x \in [0,2]^3} I(x, r_n) = 0 \text{ a.s.} \tag{30}
\]

As the function \( x \mapsto \rho(x) \) is decreasing for \( x \in (0, 1) \), one has

\[
\liminf_{r \to 1+} \rho(r) \inf_{r \to 1+} I(x, r) \geq \liminf_{n \to \infty} \left( \rho(r_n) \inf_{x \in [0,1]^3} I(x, r) \right) \geq \liminf_{n \to \infty} \left( \rho(r_n) \inf_{x \in [0,1]^3} \frac{\rho(r_n+1)}{\rho(r_n)} I(x, r_n) \right) \geq \liminf_{n \to \infty} \left( \rho(r_n) \inf_{x \in [0,1]^3} \frac{\rho(r_n+1)}{\rho(r_n)} I(x, r_n) \right) \geq \liminf_{n \to \infty} \left( \rho(r_n) I(x_{i,n}, r_n) - 2 \limsup_{n \to \infty} \sup_{x \in [0,2]^3} \rho(r_n) I(x, r_n) \right). \tag{31}
\]

It follows from Equations (29)–(31) that Equation (27) holds.

Next, we show Equation (28). For convenience, we sometimes write a typical parameter ("space point") \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) also as \( (x_i) \), or \( c \), if \( x_1 = x_2 = x_3 = c \). For every \( n \geq 1 \), we define \( r_n = 2^{-n}, \theta_n = r_n, b_n = \ln r_n, \Lambda_n = b^{-1} \), \( \Lambda_n = \lfloor b^{-1} r_n^2 \rfloor \), \( S_n = \{1, ..., \Lambda_n\} \), and \( T_n = \{i = (i_1, i_2, i_3) \in \mathbb{Z}_3^+ : i_k \in \{1, ..., \lfloor b^{-1} \rfloor\}, k = 1, 2, 3\} \), where \( |a| \) denotes the integer part of \( a \in \mathbb{R}_+ \) satisfying \( |a| \geq a < |a| + 1 \). For every \( i \in S_n \), we define

\[
x_{i,n} = ib \theta_n,
\]

and \( Z_{i,n} \) to be 1 or 0 according as the random variable

\[
\max_{(k_j) \in T_n} \rho(r_n) |U(t, x_{i,n} + b(k_j) \theta_n) - U(t, x_{i,n})| \leq c_{3,7}
\]

is or not. Put \( H(y) = \mathbb{E}[(U(t, x + y) - U(t, x))^2] \) and \( U_{i,n}(x) = U(t, x_{i,n} + x) - U(t, x_{i,n}) \). Then, for \( i,j \in S_n \) and \( x, y \in [0,1]^3 \), one has

\[
\mathbb{E}[U_{i,n}(x)U_{j,n}(y)] = \frac{1}{2} \left(-H(x_{i,n} - x_{j,n}) + (y - x) + H((x_{i,n} - x_{j,n}) - x) + H((x_{i,n} - x_{j,n}) + y) - H(x_{i,n} - x_{j,n})\right). \tag{32}
\]
Put $\nabla H = (\partial H/\partial t_1, \partial H/\partial t_2, \partial H/\partial t_3)$, $\omega_{lm} = (\partial^2 H/\partial t_l \partial t_m)(x_{j,\alpha} - x_{i,\alpha}) + \eta_3(y - \eta_1 x + \eta_2 x) (1 \leq l, m \leq 3)$ and the matrix $\Omega = (\omega_{lm})_{3 \times 3}$, where $\eta_1, \eta_2, \eta_3 \in [0, 1]$. It follows from Taylor expansion that

$$
\begin{align*}
\mathbb{E}[\mathcal{U}_n(x)\mathcal{U}_n(y)] &= (\nabla H)(x_{j,\alpha} - x_{i,\alpha}) + (y - \eta_1 x) - \nabla H(x_{j,\alpha} - x_{i,\alpha} - \eta_2 x) x' \\
&= (y - \eta_1 x + \eta_2 x) \Omega x'.
\end{align*}
$$

For convenience, we arrange all points in $\mathcal{T}_n$ according to the following rule: for two points $(k_i') = (k_1, k_2, k_3), (m_j') = (m_1, m_2, m_3) \in \mathcal{T}_n$, we denote $(k_i') < (m_j')$ if there exists $1 \leq j \leq 3$ such that $k_1 = m_1, ..., k_{j-1} = m_{j-1}, k_j < m_j$ with convention $k_0 = m_0 = 0$. Consider Gaussian random vectors $x_1 := (\mathcal{U}_n(b(k_j')), (k_i') \in \mathcal{T}_n', x_2 := (\mathcal{U}_n(b(m_j)), (m_j') \in \mathcal{T}_n'$ and $x = (x_1, x_2)$. Then,

$$
\Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2^T & \Sigma_1 \end{pmatrix},
$$

where $\Sigma$ is the covariance matrix of $x, \Sigma_1 = \mathbb{E}[x_1 x_1']$ and $\Sigma_2 = \mathbb{E}[x_2 x_2']$. Put $g(r_n) = c_{3,2} r_n^{-1}$. It follows from Lemma 3 that

$$
\mathbb{P}\left( \max_{(k_i') \in \mathcal{T}_n} |\mathcal{U}_n(b(k_j'))| \leq g(r_n), \max_{(m_j') \in \mathcal{T}_n} |\mathcal{U}_n(b(m_j'))| \leq g(r_n) \right) \leq \kappa \mathbb{P}\left( \max_{(k_i') \in \mathcal{T}_n} |\mathcal{U}_n(b(k_j'))| \leq g(r_n) \right) \mathbb{P}\left( \max_{(m_j') \in \mathcal{T}_n} |\mathcal{U}_n(b(m_j'))| \leq g(r_n) \right),
$$

where

$$
\kappa = \left( \frac{\det(\Sigma_1)\det(\Sigma_2)}{\det(\Sigma)} \right)^{1/2}.
$$

We first verify that the positive semidefinite $(2b^{-3}) \times (2b^{-3})$ matrix $\Sigma$ satisfies Conditions (i)-(ii) of Lemma 4.

For $i, j \in \mathcal{S}_\nu, (k_i), (m_j) \in \mathcal{T}_n$, and $1 \leq u, v \leq 3$, we define $\lambda_u = b\theta((j-i) + \eta_3(m_u - \eta_1 k_u + \eta_2 k_u)), \bar{\lambda}_uv = \partial^2 H(\lambda_j)/\partial \lambda_u \partial \lambda_v, f_u = b\theta(m_u - \eta_1 k_u + \eta_2 k_u)$, and $g_u = b\theta k_u$. Noting $|\bar{\omega}_{uv}| \leq c_{3,3} |\lambda_u|^{-1/2} |\lambda_v|^{-1/2}$ if $u \neq v$, and $|\bar{\omega}_{uu}| \leq c_{3,10} |\lambda_u|^{-1}$ if $u = v$, one has that for all $1 \leq u, v \leq 3$ and $i, j \in \mathcal{S}_n$ with $|j-i| \geq \Lambda$,

$$
|f_u\bar{\omega}_{uv}g_v| \leq c_{3,11} b^{11} r_n^2.
$$

By Equations (33) (with $x = b(k_j'), y = b(m_j)$ and $h = (h_j)$) and (35), one has that for $(k_i'), (m_j') \in \mathcal{T}_n$ and $i, j \in \mathcal{S}_n$ with $|j-i| \geq \Lambda$,

$$
\omega_{(k_i'),(m_j')} = |\mathbb{E}[\mathcal{U}_n(b(k_j'))\mathcal{U}_n(b(m_j))]| \\
= b^2 |(m_j - \eta_1 k_j + \eta_2 k_j) \Omega (k_j')'| \\
\leq \sum_{u=1}^{3} \sum_{v=1}^{3} |f_u\bar{\omega}_{uv}g_v| \\
\leq c_{3,12} b^{11} r_n^2.
$$

Thus, by Equation (36),

$$
\sum_{(k_i') \in \mathcal{T}_n} \sum_{(m_j') \in \mathcal{T}_n} \omega_{(k_i'),(m_j')} \leq c_{3,12} b^{5} r_n^2.
$$

This verifies Condition (i) of Lemma 4 with $\lambda = c_{3,12} b^{5} r_n^2$. 
Note that
\[
\text{Var}\left( U_{u}(b(k_{j}\theta))| U_{u}(b(m_{j}\theta)), (m_{j}) \neq (k_{j}), (m_{j}) \in \mathcal{T}_{u}, u \in \{i,j\} \right) = \frac{\text{det}(\Sigma)}{\text{det}(\Sigma^{(i)})},
\]
where \( U_{u}(b(k_{j}\theta)) \) is the \( l \)-th point in \( U \) according to the above mentioned rule. Then, by Equation (9), one has
\[
\frac{\text{det}(\Sigma)}{\text{det}(\Sigma^{(i)})} \geq \text{Var}\left( U(t,b(k_{j}\theta))| U(t,b(m_{j}\theta)), (m_{j}) \neq (k_{j}), (m_{j}) \in \mathcal{T}_{n} \right) \geq c_{3,13} b_{r_{n}}^{2}. \tag{39}
\]
This verifies Condition (ii) of Lemma 4 with \( K = (c_{3,13} b_{r_{n}}^{2})^{-1} \).

By making use of Lemma 4 with \( n = b^{-3}, \lambda = c_{3,13} b^{3} r_{n}^{2} \) and \( K = (c_{3,13} b_{r_{n}}^{2})^{-1} \), one has
\[
\text{det}(\Sigma) \geq e^{-2\Lambda K n}(\text{det}(\Sigma_{1}))^{2}. \tag{40}
\]
This, together with Equation (40), yields that
\[
K \leq e^{c_{3,7} n}. \tag{41}
\]
Notice that \( \Lambda K n \rightarrow 0 \). This, together with Equations (41) and (34), yields that \( \forall \tau > 0 \), whenever \( |i-j| \geq \Lambda \),
\[
\mathbb{P}(Z_{i:n} = 1, Z_{j:n} = 1) \leq (1 + \tau)(\mathbb{P}(Z_{1:n} = 1))^{2}. \tag{42}
\]
For every \( n \geq 1 \), we define \( S_{n} := \sum_{i \in S_{n}} Z_{i:n} \) and \( p_{n} := \mathbb{E}[Z_{1:n}] \). Then, by Equation (7), one has uniformly over \( i \in S_{n} \),
\[
p_{n} = \mathbb{P}(Z_{i:n} = 1) = \mathbb{P}(Z_{1:n} = 1) \geq \mathbb{P}\left( \rho(r_{n}) I(x_{i:n}, \theta) \leq c_{3,7} \right) \geq \exp((c_{2,1} / c_{3,7}) \ln r_{n}). \tag{43}
\]
It follows from Equation (42) that for all \( \tau > 0 \), whenever \( i, j \in S_{n} \) satisfy \( |j-i| \geq \Lambda \), then \( \text{cov}(Z_{i:n}, Z_{j:n}) \leq \tau \mathbb{E}[Z_{i:n}] \mathbb{E}[Z_{j:n}] \). Thus, by Equation (43),
\[
\text{Var}(S_{n}) = \sum_{i,j \in S_{n}} \text{cov}(Z_{i:n}, Z_{j:n}) \leq \tau \lambda_{n}^{2} p_{n}^{2} + \sum_{i,j \in S_{n}: |i-j| \leq \Lambda} \text{cov}(Z_{i:n}, Z_{j:n}).
\]
Thus, noting \( \text{cov}(Z_{i:n}, Z_{j:n}) \leq \mathbb{E}[Z_{i:n}] = p_{n} \), one has
\[
\text{Var}(S_{n}) \leq \tau \lambda_{n}^{2} p_{n}^{2} + \lambda_{n} p_{n} \Lambda.
\]
It follows from the Chebyshev’s inequality and \( \mathbb{E}[S_{n}] = \lambda_{n} p_{n} \) that
\[
\mathbb{P}(S_{n} = 0) \leq \frac{\text{Var}(S_{n})}{\mathbb{E}[S_{n}]^{2}} \leq \tau + \frac{\Lambda}{\lambda_{n} p_{n}}. \tag{44}
\]
As \( c_{3,7} > (c_{2,1}/2)^{1/6} \), one has \( c_{2,1} / c_{3,7}^{6} < 2 \). Thus, by Equation (43), \( \Lambda / (\lambda_{n} p_{n}) \leq r_{n}^{2-c_{2,1}/c_{3,7}^{6}} \ln \ln r_{n}^{13} \rightarrow 0 \). It follows from Equation (44) and the arbitrariness of \( \tau \) that \( \mathbb{P}(S_{n} = 0) \rightarrow 0 \) as \( n \rightarrow \infty \). Finally
\[
\mathbb{P}(S_{n} > 0 \text{ i.o.}) > \limsup_{n \rightarrow \infty} \mathbb{P}(S_{n} > 0) = 1.
\]
This yields

$$\lim \inf_{n \to \infty} \inf_{x \in [0,1]^3} \sup_{(k_j) \in T_n} \rho(r_n) |U(t, x + b(k_j \theta)) - U(t, x)| \leq c_3 \rho$$ \quad \text{a.s.} \quad (45)$$

Thus,

$$\lim \inf_{n \to \infty} \inf_{x \in [0,1]^3} \max_{(k_j) \in T_n} \rho(r_n) |U(t, x + b(k_j \theta)) - U(t, x)| \leq c_3 \rho$$ \quad \text{a.s.} \quad (46)$$

Note that

$$\lim \inf_{n \to \infty} \inf_{x \in [0,1]^3} \rho(r_n) I(x, r_n) \leq \lim \inf_{n \to \infty} \inf_{x \in [0,1]^3} \sup_{(k_j) \in T_n} \rho(r_n) |U(t, x + b(k_j \theta)) - U(t, x)| \quad (47)$$

It follows from Theorem 4.1 in Meerschaert et al. [17] that

$$\lim \sup_{n \to \infty} \sup_{x \in [0,1]^3} \rho(r_n) |U(t, x + y) - U(t, x)| = 0 \quad \text{a.s.} \quad (48)$$

Therefore, by Equations (46)–(48), Equation (28) holds. The proof of Theorem 1 is completed. \Box

4. Conclusions

In this article, we have shown that the solutions to the fourth-order L-KS SPDEs and their gradient, driven by space-time white noise, are almost surely nowhere differentiable in all directions of space variable $x$. We have been concerned with the small fluctuation behavior, with delicate analysis of regularities, and established the exact spatial moduli of non-differentiability, for the above class of SPDEs and their gradient. They complement Allouba’s earlier works on the spatio-temporal Hölder regularity of L-KS SPDEs and their gradient. Together with the Khinchin-type law of the iterated logarithm and the uniform modulus of continuity, they provide complete information on the regularity properties of L-KS SPDEs and their gradient in space.

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Abbreviations

The following abbreviations are used in this manuscript:

- **SPDE**: Stochastic partial differential equation
- **L-KS**: Linearized Kuramoto–Sivashinsky
- **SIE**: Stochastic integral equation

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