ON THE ENTROPY FOR GROUP ACTIONS ON THE CIRCLE

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Abstract. We show that for a finitely generated group of $C^2$ circle diffeomorphisms, the entropy of the action equals the entropy of the restriction of the action to the non-wandering set.

1. Introduction

Let $(X, \text{dist})$ be a compact metric space and $G$ a group of homeomorphisms of $X$ generated by a finite family of elements $\Gamma = \{g_1, \ldots, g_n\}$. To simplify, we will always assume that $\Gamma$ is symmetric, that is, $g^{-1} \in \Gamma$ for every $g \in \Gamma$. For each $n \in \mathbb{N}$ we denote by $B_\Gamma(n)$ the ball of radius $n$ in $G$ (w.r.t. $\Gamma$), that is, the set of elements $f \in G$ which may be written in the form $f = g_{m_1} \cdots g_{m_k}$ for some $m \leq n$ and $g_{i} \in \Gamma$. For $g \in G$ we let $\|f\| = \|f\|_\Gamma = \min\{n : f \in B_\Gamma(n)\}$

As in the classical case, given $\varepsilon > 0$ and $n \in \mathbb{N}$, two points $x, y$ in $X$ are said to be $(n, \varepsilon)$-separated if there exists $g \in B_\Gamma(n)$ such that $\text{dist}(g(x), g(y)) \geq \varepsilon$. A subset $A \subset X$ is $(n, \varepsilon)$-separated if all $x \neq y$ in $A$ are $(n, \varepsilon)$-separated. We denote by $s(n, \varepsilon)$ the maximal possible cardinality (perhaps infinite) of a $(n, \varepsilon)$-separated set. The topological entropy for the action at the scale $\varepsilon$ is defined by

$$h_\Gamma(G \circ X, \varepsilon) = \limsup_{n \to \infty} \frac{\log(s(n, \varepsilon))}{n},$$

and the topological entropy is defined by

$$h_\Gamma(G \circ X) = \lim_{\varepsilon \to 0} h_\Gamma(G \circ X, \varepsilon).$$

Notice that, although $h_\Gamma(G \circ X, \varepsilon)$ depends on the system of generators, the properties of having zero, positive, or infinite entropy, are independent of this choice.

The definition above was proposed in [5] as an extention of the classical topological entropy of single maps (the definition extends to pseudo-groups of homeomorphisms, and hence is suitable for applications in Foliation Theory). Indeed, for a homeomorphism $f$, the topological entropy of the action of $\mathbb{Z} \cong \langle f \rangle$ equals two times the (classical) topological entropy of $f$. Nevertheless, the functorial properties of this
notion remain unclear. For example, the following fundamental question is open.

**General Question.** Is it true that \( h_{\Gamma}(G \circ X) \) is equal to \( h_{\Gamma}(G \circ \Omega) \)?

Here \( \Omega = \Omega(G \circ X) \) denotes the *non-wandering* set of the action, or in other words
\[
\Omega = \{ x \in X : \text{ for every neighborhood } U \text{ of } x, \text{ we have } f(U) \cap U \neq \emptyset, \text{ for some } f \neq id \text{ in } G. \}
\]
This is a closed invariant set whose complement \( \Omega^c \) corresponds to the *wandering set* of the action.

The notion of topological entropy for group actions is quite appropriate in the case where \( X \) is a one–dimensional manifold. In fact, in this case, the topological entropy is necessarily finite (cf. §2). Moreover, in the case of actions by diffeomorphisms, the dichotomy \( h_{\text{top}} = 0 \) or \( h_{\text{top}} > 0 \) is well understood. Indeed, according to a result originally proved by Ghys, Langevin, and Walczak, for groups of \( C^2 \) diffeomorphisms \( [5] \), and extended by Hurder to groups of \( C^1 \) diffeomorphisms (see for instance \( [9] \)), we have \( h_{\text{top}} > 0 \) if and only if there exists a resilient orbit for the action. This means that there exist a group element \( f \) contracting by one side to a fixed point \( x_0 \), and another element \( g \) which sends \( x_0 \) into its basin of contraction by \( f \).

The results of this work give a positive answer to the General Question above in the context of group actions on one–dimensional manifolds under certain mild assumptions.

**Theorem A.** If \( G \) is a finitely generated subgroup of \( \text{Diff}^2_+(S^1) \), then for every finite system of generators \( \Gamma \) of \( G \), we have
\[
h_{\Gamma}(G \circ S^1) = h_{\Gamma}(G \circ \Omega).
\]

Our proof for Theorem A actually works in the Denjoy class \( C^{1+bv} \), and applies to general codimension-one foliations on compact manifolds. In the class \( C^{1+Lip} \), it is quite possible that we could give an alternative proof using standard techniques from Level Theory \( [2] [6] \).

It is unclear whether Theorem A extends to actions of lower regularity. However, it still holds under certain algebraic hypotheses. In fact, (quite unexpectedly) the regularity hypothesis is used to rule out the existence of elements \( f \in G \) that fix some connected component of the wandering set and which are *distorted*: that is, those elements which satisfy
\[
\lim_{n \to \infty} \frac{\|f^n\|}{n} = 0.
\]
Actually, for the equality between the entropies it suffices to require
that no element in $G$ be sub-exponentially distorted. In other words,
it suffices to require that, for each element $f \in G$ with infinite order,
there exist a non-decreasing function $q : \mathbb{N} \to \mathbb{N}$ (depending on $f$)
with sub-exponential growth satisfying $q(||f^n||) \geq n$, for every $n \in \mathbb{N}$.
This is an algebraic condition which is satisfied by many groups, as for
example nilpotent or free groups. (We refer the reader to [1] for a nice
discussion on distorted elements.) Under this hypothesis, the following
result holds.

**Theorem B.** If $G$ is a finitely generated subgroup of $\text{Homeo}_+(S^1)$ without
sub-exponentially distorted elements, then for every finite system
of generators $\Gamma$ of $G$, we have

$$h_\Gamma(G \circ S^1) = h_\Gamma(G \circ \Omega).$$

The entropy of general group actions and distorted elements seem
to be related in an interesting manner. Indeed, though the topological
entropy of a single homeomorphism $f$ may be equal to zero, if this
map appears as a sub-exponentially distorted element inside an acting
group, then this map may create positive entropy for the group action.

2. Some background

In this work we will consider the normalized length on the circle, and
every homeomorphism will be orientation preserving.

We begin by noticing that if $G$ is a finitely generated group of circle
homeomorphisms and $\Gamma$ is a finite generating system for $G$, then for all
$n \in \mathbb{N}$ and all $\varepsilon > 0$ one has

$$s(n, \varepsilon) \leq \frac{1}{\varepsilon} \#B_\Gamma(n).$$

Indeed, let $A$ be a $(n, \varepsilon)$-separated set of cardinality $s(n, \varepsilon)$. Then for
every two adjacent points $x, y$ in $A$ there exists $f \in B_\Gamma(n)$ such that
$\text{dist}(f(x), f(y)) \geq \varepsilon$. For a fixed $f$, the intervals $[f(x), f(y)]$ which
appear have disjoint interior. Since the total length of the circle is 1,
any given $f$ can be used in this construction at most $1/\varepsilon$ times, which
immediately gives (1).

Notice that, taking the logarithm at both sides of (1), dividing by $n$,
and passing to the limits, this gives

$$h_\Gamma(G \circ S^1) \leq \text{gr}_\Gamma(G),$$
where $\text{gr}_\Gamma(G)$ denotes the growth of $G$ with respect to $\Gamma$, that is,

$$\text{gr}_\Gamma(G) = \lim_{n \to \infty} \frac{\log(#B_\Gamma(n))}{n}.$$ 

Some easy consequences of this fact are the following ones:

- If $G$ has sub-exponential growth, that is, if $\text{gr}_\Gamma(G) = 0$ (in particular, if $G$ is nilpotent, or if $G$ is the Grigorchuk-Maki’s group considered in [8]), then $h_\Gamma(G \circlearrowleft S^1) = 0$ for all finite generating systems $\Gamma$.

- In the general case, if $\#\Gamma = q \geq 1$, then from the relations

$$#B_\Gamma(n) \leq 1 + \sum_{j=1}^{n} 2q(2q-1)^{j-1} = \begin{cases} 1 + \left(\frac{q}{q-1}\right)((2q-1)^n - 1), & q \geq 2, \\ 1 + 2n, & q = 1, \end{cases}$$ 

one concludes that

$$h_\Gamma(G \circlearrowleft S^1) \leq \log(2q - 1).$$

This shows in particular that the entropy of the action of $G$ on $S^1$ is finite. Notice that this may be also deduced from the probabilistic arguments of [3] (see Théorème D therein). However, these arguments only yield the weaker estimate $h_\Gamma(G \circlearrowleft S^1) \leq \log(2q)$ when $\Gamma$ has cardinality $q$.

3. SOME PREPARATION FOR THE PROOFS

The statement of our results are obvious when the non-wandering set of the action equals the whole circle. Hence, we will assume in what follows that $\Omega$ is a proper subset of $S^1$, and we will currently denote by $I$ some of the connected components of the complement of $\Omega$. Let $\text{Est}(I)$ denote the stabilizer of $I$ in $G$.

**Lemma 1.** The stabilizer $\text{Est}(I)$ is either trivial or infinite cyclic.

*Proof.* The (restriction to $I$ of the) nontrivial elements of $\text{Est}(I)|_I$ have no fixed points, for otherwise these points would be non-wandering. Thus $\text{Est}(I)|_I$ acts freely on $I$, and according to Hölder Theorem [4, 7], its action is semiconjugate to an action by translations. We claim that, if $\text{Est}(I)|_I$ is nontrivial, then it is infinite cyclic. Indeed, if not then the corresponding group of translations is dense. This implies that the preimage by the semiconjugacy of any point whose preimage is a single point corresponds to a non-wandering point for the action. Nevertheless, this contradicts the fact that $I$ is contained in $\Omega^c$. 

If \( \text{Est}(I)|_I \) is trivial then \( f|_I \) is trivial for every \( f \in \text{Est}(I) \), and hence \( f \) itself must be the identity. We then conclude that \( \text{Est}(I) \) is trivial.

Analogously, \( \text{Est}(I) \) is cyclic if \( \text{Est}(I)|_I \) is cyclic. In this case, \( \text{Est}(I)|_I \) is generated by the restriction to the interval \( I \) of the generator of \( \text{Est}(I) \). \( \square \)

**Definition 1.** A connected component \( I \) of \( \Omega^c \) will be called of type 1 if \( \text{Est}(I)|_I \) is trivial, and will be called of type 2 if \( \text{Est}(I)|_I \) is infinite cyclic.

Notice that the families of connected components of type 1 and 2 are invariant, that is, for each \( f \in G \) the interval \( f(I) \) is of type 1 (resp. of type 2) if \( I \) is of type 1 (resp. of type 2). Moreover, given two connected components of type 1 of \( \Omega^c \), there exists at most one element in \( G \) sending the former into the latter. Indeed, if \( f(I) = g(I) \) then \( g^{-1}f \) is in the stabilizer of \( I \), and hence \( f = g \) if \( I \) is of type 1.

**Lemma 2.** Let \( x_1, \ldots, x_m \) be points contained in a single type 1 connected component of \( \Omega^c \). If for some \( \varepsilon > 0 \) the points \( x_i, x_j \) are \((\varepsilon, n)\)-separated for every \( i \neq j \), then \( m \leq 1 + \frac{1}{\varepsilon} \).

**Proof.** Let \( I = ]a, b[ \) be the connected component of type 1 of \( \Omega^c \) containing the points \( x_1, \ldots, x_m \). After renumbering the \( x_i \)'s, we may assume that \( a = x_1 < x_2 < \ldots < x_m < b \). For each \( 1 \leq i \leq m - 1 \) one can choose an element \( g_i \in B_T(n) \) such that \( \text{dist}(g_i(x_i), g_i(x_{i+1})) \geq \varepsilon \).

Now, since \( I \) is of type 1, the intervals \( ]g_i(x_i), g_i(x_{i+1})[ \) are two by two disjoint. Therefore, the number of these intervals times the minimal length among them is less than or equal to 1. This gives \((m-1)\varepsilon \leq 1\), thus proving the lemma. \( \square \)

The case of connected components \( I \) of type 2 of \( \Omega^c \) is much more complicated than the one of type 1 connected components. The difficulty is related to the fact that, if the generator of the stabilizer of \( I \) is sub-exponentially distorted in \( G \), then this would imply the existence of exponentially many \((n, \varepsilon)\)-separated points inside \( I \), and hence a relevant part of the entropy would be “concentrated” in \( I \). To deal with this problem, for each connected component \( I \) of type 2 of \( \Omega^c \) we denote by \( p_I \) its middle point, and then we define \( \ell_I : G \to \mathbb{N}_0 \) as follows. Let \( h \) be the generator of the stabilizer of \( I \) such that \( h(x) > x \) for all \( x \) in \( I \). For each \( f \in G \) the element \( fhf^{-1} \) is the generator of the stabilizer of \( f(I) \) with the analogous property. We then let \( \ell_I(f) = |r| \), where \( r \)
is the unique integer number such that
\[ fh^r f^{-1}(p_f) \leq f(p_I) < fh^{r+1} f^{-1}(p_f). \]

**Lemma 3.** For all \( f, g \) in \( G \) one has
\[ \ell_I(g \circ f) \leq \ell_{f(I)}(g) + \ell_I(f) + 1. \]

*Proof.* Let \( r \) be the unique integer number such that
\[ (fh f^{-1})^r(p_{f(I)}) \leq f(p_I) < (fh f^{-1})^{r+1}(p_{f(I)}), \]
and let \( s \) be the unique integer number such that
\[ (gfh f^{-1} g^{-1})^s(p_{gf(I)}) \leq g(p_{f(I)}) < (gfh f^{-1} g^{-1})^{s+1}(p_{gf(I)}), \]
so that
\[ \ell_I(f) = |r|, \quad \ell_{f(I)}(g) = |s|. \]

We then have
\[ g^{-1}(gfh f^{-1} g^{-1})^s(p_{gf(I)}) \leq p_{f(I)} < g^{-1}(gfh f^{-1} g^{-1})^{s+1}(p_{gf(I)}), \]
that is
\[ (fh f^{-1})^s g^{-1} p_{f(I)} \leq p_{f(I)} < (fh f^{-1})^{s+1} g^{-1}(p_{gf(I)}). \]
Therefore,
\[ (fh f^{-1})^{r+s} g^{-1}(p_{gf(I)}) \leq f(p_I) < (fh f^{-1})^{r+s+1} g^{-1}(p_{gf(I)}), \]
and hence
\[ (fh f^{-1})^{r+s} g^{-1}(p_{gf(I)}) \leq f(p_I) < (fh f^{-1})^{r+s+2} g^{-1}(p_{gf(I)}). \]
This easily gives
\[ g(fh f^{-1})^{r+s} g^{-1}(p_{gf(I)}) \leq g(f(p_I) < g(fh f^{-1})^{r+s+2} g^{-1}(p_{gf(I)}), \]
and thus
\[ (gfh f^{-1} g^{-1})^{r+s}(p_{gf(I)}) \leq g(f(p_I) < (gfh f^{-1} g^{-1})^{r+s+2}(p_{gf(I)}). \]
This shows that \( \ell_I(gf) \) equals either \(|r+s|\) or \(|r+s+1|\), which concludes the proof. \qed

The following corollary is a direct consequence of the preceding lemma, but may be proved independently.

**Corollary 1.** For every \( f \in G \) one has
\[ |\ell_I(f) - \ell_{f(I)}(f^{-1})| \leq 1. \]
Proof. From (2) one obtains
\[ h^{-r+1}(p_I) < f^{-1}(p_{f(I)}) \leq h^{-r}(p_I) < h^{-r+1}(p_I), \]
and hence \( \ell_{f(I)}(f^{-1}) \) equals either \( |r| \) or \( |r+1| \). Since \( \ell_I(f) = |r| \), the corollary follows.

4. The proof in the smooth case

To rule out the possibility of “concentration” of the entropy on a type 2 connected component \( I \) of \( \Omega^c \), in the \( C^2 \) case we will use classical control of distortion arguments in order to construct, starting from the function \( \ell_I \), a kind of quasi-morphism from \( G \) into \( \mathbb{N}_0 \). Slightly more generally, let \( \mathcal{F} \) be any finite family of connected components of type 2 of \( \Omega^c \). We denote by \( \mathcal{F}^G \) the family formed by all the intervals contained in the orbits of the intervals in \( \mathcal{F} \). For each \( f \in G \) we then define
\[
\ell_\mathcal{F}(f) = \sup_{I \in \mathcal{F}^G} \ell_I(f).
\]
A priori, the value of \( \ell_\mathcal{F} \) could be infinite. We claim however that, for groups of \( C^2 \) diffeomorphisms, its value is necessarily finite for every element \( f \).

**Proposition 1.** For every finite family \( \mathcal{F} \) of type 2 connected components of \( \Omega^c \), the value of \( \ell_\mathcal{F}(f) \) is finite for each \( f \in G \).

To show this proposition, we will need to estimate the function \( \ell_I(f) \) in terms of the distortion of \( f \) on the interval \( I \).

**Lemma 4.** For each fixed type 2 connected component \( I \) of \( \Omega^c \) and every \( g \in G \), the value of \( \ell_I(g) \) is bounded from above by a number \( L(V) \) depending on \( V = \text{var}(|\log(g'|_I)|) \), the total variation of the logarithm of the derivative of the restriction of \( g \) to \( I \).

**Proof.** Denote \( |a,b| = I \) and \( |\tilde{a},\tilde{b}| = g(I) \). If \( h \) is a generator for the stabilizer of \( I \), then for every \( f \in G \) the value of \( \ell_I(f) \) corresponds (up to some constant \( \pm 1 \)) to the number of fundamental domains for the dynamics of \( fhf^{-1} \) on \( f(I) \) between the points \( p_{f(I)} \) and \( f(p_I) \), which in its turn corresponds to the number of fundamental domains for the dynamics of \( h \) on \( I \) between \( f^{-1}(p_{f(I)}) \) and \( p_I \). Therefore, we need to show that there exists \( c < d \) in \( |a,b| \) depending on \( V \) and such that \( g^{-1}(p_{g(I)}) \) belongs to \( [c,d] \). We will show that this happens for the values
\[
c = a + \frac{|I|}{2e^V} \quad \text{and} \quad d = b - \frac{|I|}{2e^V}.
\]
We will just check that the first choice works, leaving the second one to the reader. By the Mean Value Theorem, there exists $x \in g(I)$ and $y \in [\bar{a}, p_g(I)]$ such that

$$(g^{-1})'(x) = \frac{|I|}{|g(I)|}$$

and

$$(g^{-1})'(y) = \frac{|g^{-1}([\bar{a}, p_f(I)])|}{|[\bar{a}, p_g(I)]|} = \frac{g^{-1}(p_g(I)) - a}{|g(I)|/2}.$$ 

By the definition of the constant $V$, we have $(g^{-1})'(x)/(g^{-1})'(y) \leq e^V$. This gives

$$e^V \geq \frac{|I|/|g(I)|}{2(g^{-1}(p_g(I)) - a)/|g(I)|} = \frac{|I|}{2(g^{-1}(p_g(I)) - a)},$$

thus proving that $g^{-1}(p_g(I)) \geq a + \frac{|I|}{2e^V}$, as we wanted to show. □

Proof of Proposition 1. Let $J = ]\bar{a}, \bar{b}[$ be an interval in the orbit by $G$ of $I = ]a, b[$. If $g = g_i \cdots g_1, g_i \in \Gamma$, is an element of minimal length sending $I$ into $J$, then the intervals $I, g_i(I), g_i g_i(I), \ldots, g_i \cdots g_i(I)$ have two by two disjoint interiors. Therefore,

$$\text{var}(\log((g')|I)) \leq \sum_{j=0}^{n-1} \text{var}(\log(g_{i+j} \cdots g_1(I))) \leq \sum_{h \in \Gamma} \text{var}(\log(h')) =: W.$$ 

Moreover, denoting $V = \text{var}(\log(f')),$

$$\text{var}(\log((fg)'|I)) \leq \text{var}(\log(g'|I)) + \text{var}(\log(f')) = W + V.$$ 

By Lemmas 3 and 4 and Corollary 1

$$\ell_J(f) \leq \ell_J(g^{-1}) + \ell_I(fg) + 1 \leq \ell_I(g) + \ell_I(fg) + 2 \leq L(W) + L(W + V) + 2.$$ 

This shows the proposition when $F$ consists of a single interval. The case of general finite $F$ follows easily. □

For a given $\varepsilon > 0$ we define $\ell_\varepsilon = \ell_{F_\varepsilon}$, where $F_\varepsilon = \{I_1, \ldots, I_k\}$ is the family of the connected components of $\Omega^\varepsilon$ having length greater than or equal to $\varepsilon$, with $k = k(\varepsilon)$. Notice that, by Lemma 3 for every $f, g$ in $\Gamma$ one has

$$\ell_\varepsilon(gf) \leq \ell_\varepsilon(g) + \ell_\varepsilon(f) + 1$$

(3)
Lemma 5. There exists constants $A(\varepsilon) > 0$ and $B(\varepsilon)$ satisfying the following property: If $x_1, \ldots, x_m$ are points contained in a single connected component of type 2 of $\Omega^c$ and $x_i, x_j$ are $(\varepsilon, n)$-separated for every $i \neq j$, then $m \leq A(\varepsilon)n + B(\varepsilon)$.

Proof. Denote $c_\varepsilon = \max\{\ell_\varepsilon(g) : g \in \Gamma\}$ (according to Proposition 1, the value of $c_\varepsilon$ is finite). Let $I$ be the type 2 connected component of $\Omega^c$ containing $x_1, \ldots, x_m$. We may assume that $x_1 < x_2 < \ldots < x_m$. For each $1 \leq i \leq k$ let $h_i$ be the generator of $Est(I_i)$. Notice that $\ell_\varepsilon(h_i^r) \geq |r|$ for all $r \in \mathbb{Z}$.

If $f$ is an element in $B_\varepsilon(n)$ sending $I$ into some $I_i$, then the number of points which are $\varepsilon$-separated by $f$ is less than or equal to $1/\varepsilon + 1$. We claim that the number of elements in $B_\varepsilon(n)$ sending $I$ into $I_i$ is bounded above by $4nc_\varepsilon + 4n - 1$. Indeed, if $g$ also sends $I$ onto $I_i$ then $gf^{-1} \in Est(I_i)$, hence $gf^{-1} = h_i^r$ some $r$. Therefore, using (3) one obtains $|r| \leq \ell_\varepsilon(h_i^r) \leq 2nc_\varepsilon + 2n - 1$.

Since the previous arguments apply to each type 2 interval $I_i$, we have

$$m \leq k\left(\frac{1}{\varepsilon} + 1\right)(4nc_\varepsilon + 4n - 1).$$

Therefore, letting

$$A(\varepsilon) = \left(4k + \frac{4k}{\varepsilon}\right)(1 + c_\varepsilon) \quad \text{and} \quad B(\varepsilon) = -(k + \frac{k}{\varepsilon}),$$

this concludes the proof. \qed

To conclude the proof of Theorem A, the following notation will be useful.

Notation 1. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, we will denote by $s(n, \varepsilon)$ the largest cardinality of a $(n, \varepsilon)$-separated subset of $S^1$. Likewise, $s_\Omega(n, \varepsilon)$ will denote the largest cardinality of a $(n, \varepsilon)$-separated set contained in the non-wandering set.

Proof of Theorem A. Fix $0 < \varepsilon < 1/2L$, where $L$ is a common Lipschitz constant for the elements in $\Gamma$. We will show that, for some function $p_\varepsilon$ growing linearly on $n$ (and whose coefficients depend on $\varepsilon$), one has

$$s(n, \varepsilon) \leq p_\varepsilon(n)s_\Omega(n, \varepsilon) + p_\varepsilon(n).$$

Actually, any function $p_\varepsilon$ with sub-exponential growth and verifying such an inequality suffices. Indeed, taking the logarithm in both sides, dividing by $n$, and passing to the limit, this implies that

$$h_\Gamma(G \odot S^1, \varepsilon) = h_\Gamma(G \odot \Omega, \varepsilon).$$
Letting $\varepsilon$ go to zero, this gives
\[ h_\Gamma(G \cap S^1) \leq h_\Gamma(G \cap \Omega). \]

Since the opposite inequality is obvious, this shows the desired equality between the entropies.

To show (4), fix a $(n, \varepsilon)$-separated set $S$ containing $s(n, \varepsilon)$ points. Let $n_\Omega$ (resp. $n_{\Omega^c}$) be the number of points in $S$ which are in $\Omega$ (resp. in $\Omega^c$). Obviously, $s(n, \varepsilon) = n_\Omega + n_{\Omega^c}$. Let $t = t_S$ be the number of connected components of $\Omega^c$ containing points in $S$, and let $l = \lfloor \frac{t}{2} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part function. We will show that there exists a $(n, \varepsilon)$-separated set $T$ contained in $\Omega$ having cardinality $l$. This will obviously give $s_\Omega(n, \varepsilon) \geq l$. Using the inequalities $t \leq 2l + 1$ and $n_\Omega \leq s_\Omega(n, \varepsilon)$, and by Lemmas 2 and 3 this will imply that
\[ s(n, \varepsilon) = n_\Omega + n_{\Omega^c} \leq n_\Omega + tk\left(1 + \frac{1}{\varepsilon}\right)(4nc_\varepsilon + 4n - 1) \leq s_\Omega(n, \varepsilon) + (2s_\Omega(n, \varepsilon) + 1)k\left(1 + \frac{1}{\varepsilon}\right)(4nc_\varepsilon + 4n - 1), \]

thus showing (4).

To show the existence of the set $T$ with the properties above, we proceed in a constructive way. Let us number the connected components of $\Omega^c$ containing points in $S$ in a cyclic way by $I_1, \ldots, I_t$. Now for each $1 \leq i \leq l$ choose a point $t_i \in \Omega$ between $I_{2i-1}$ and $I_{2i}$, and let $T = \{t_1, \ldots, t_l\}$. We need to check that, for $i \neq j$, the points $t_i$ and $t_j$ are $(n, \varepsilon)$-separated. Now by construction, for each $i \neq j$ there exist at least two different points $x, y$ in $S$ contained in the interval of smallest length in $S^1$ joining $t_i$ and $t_j$. Since $S$ is a $(n, \varepsilon)$-separated set, there exist $m \leq n$ and $g_{i_1}, \ldots, g_{i_m}$ in $\Gamma$ so that $\text{dist}(h(x), h(y)) \geq \varepsilon$, where $h = g_{i_m} \cdots g_{i_2}g_{i_1}$. Unfortunately, because of the topology of the circle, this does not imply that $\text{dist}(h(t_i), h(t_j)) \geq \varepsilon$. However, the proof will be finished if we show that
\[ \text{dist}(g_{i_r} \cdots g_{i_1}(t_i), g_{i_r} \cdots g_{i_1}(t_j)) \geq \varepsilon \text{ for some } 0 \leq r \leq m. \]

This claim is obvious if $\text{dist}(t_i, t_j) \geq \varepsilon$. If this is not the case then, by the definition of the constants $\varepsilon$ and $L$, the length of the interval $[g_{i_1}(t_i), g_{i_1}(t_j)]$ is smaller than $1/2$, and hence it coincides with the distance between its endpoints. If this distance is at least $\varepsilon$, then we are done. If not, the same argument shows that the length of the interval $[g_{i_2}g_{i_1}(t_i), g_{i_2}g_{i_1}(t_j)]$ is smaller than $1/2$ and coincides with the distance between its endpoints. If this length is at least $\varepsilon$, then we are done. If not, we continue the procedure... Clearly, there must be some integer
$r \leq m$ such that the length of the interval $[g_{i_r-1} \cdots g_{i_1}(t_i), g_{i_r-1} \cdots g_{i_1}(t_j)]$ is smaller than $\varepsilon$, but the one of $[g_{i_r-1} \cdots g_{i_1}(t_i), g_{i_r-1} \cdots g_{i_1}(t_j)]$ is greater than or equal to $\varepsilon$. As before, the length of the later interval will be forced to be smaller than $1/2$, and hence it will coincide with the distance between its endpoints. This shows (5) and concludes the proof of Theorem A.

5. The proof in the case of non existence of sub-exponentially distorted elements

Recall that the topological entropy is invariant under topological conjugacy. Therefore, due to [3, Théorème D], in order to prove Theorem B we may assume that $G$ is a group of bi-Lipschitz homeomorphisms. Let $L$ be a common Lipschitz constant for the elements in $\Gamma$. Fix again $0 < \varepsilon < 1/2L$, and let $I_1, \ldots, I_k$ be the connected components of $\Omega^c$ having length greater than or equal to $\varepsilon$. Let $h_i$ be a generator for the stabilizer of $I_i$ (with $h_i = Id$ in case where $I_i$ is of type 1). Consider the minimal non decreasing function $q_\varepsilon$ such that, for each of the nontrivial $h_i$'s, one has $q_\varepsilon(\|h_i\|) \geq r$ for all positive $r$. We will show that (4) holds for the function $p_\varepsilon(n) = 2k(1 + \frac{1}{\varepsilon})(2q_\varepsilon(2n) + 1) + 1$.

Notice that, by assumption, this function $p_\varepsilon$ grows at most sub-exponentially on $n$. Hence, as in the case of Theorem A, inequality (4) allows to finish the proof of the equality between the entropies.

The main difficulty for showing (4) in this case is that Lemma 5 is no longer available. However, the following still holds.

**Lemma 6.** If $x_1, \ldots, x_m$ are points contained in a single type 2 connected component $I$ of $\Omega^c$ having length at least $\varepsilon$, and $x_i, x_j$ are ($\varepsilon, n$)-separated for every $i \neq j$, then $m \leq k(\frac{1}{\varepsilon} + 1)(2q_\varepsilon(2n) + 1)$.

**Proof.** Let $I$ be the type 2 connected component of $\Omega^c$ containing $x_1, \ldots, x_m$. We may assume that $x_1 < x_2 < \ldots < x_m$. If $f$ is an element in $B_1(n)$ sending $I$ into some $I_i$, then the number of points which are $\varepsilon$-separated by $f$ is less than or equal to $1/\varepsilon + 1$. We claim that the number of elements in $B_1(n)$ sending $I$ into $I_i$ is bounded above by $q_\varepsilon(r)$. Indeed, if $g$ also sends $I$ onto $I_i$ then $gf^{-1} \in Est(I_i)$, hence $gf^{-1} = h_i^r$ some $r$. Therefore,

$$2n \geq \|gf^{-1}\| = \|h_i^r\|,$$

and hence

$$q_\varepsilon(2n) \geq q_\varepsilon(\|h_i^r\|) \geq |r|.$$
Since the previous arguments apply to each type 2 interval $I_i$, this gives
\[ m \leq k\left(\frac{1}{\varepsilon} + 1\right)(2q_k(2n) + 1), \]
thus proving the lemma.

To show (4) in the present case, we proceed as in the proof of Theorem A. We fix a $(n, \varepsilon)$-separated set $S$ containing $s(n, \varepsilon)$ points. We let $n_{\Omega}$ (resp. $n_{\Omega^c}$) be the number of points in $S$ which are in $\Omega$ (resp. in $\Omega^c$), so that $s(n, \varepsilon) = n_{\Omega} + n_{\Omega^c}$. Let $t = t_S$ be the number of connected components of $\Omega^c$ containing points in $S$, and let $l = \lfloor \frac{1}{\varepsilon} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part function. As before, one can show that there exists a $(n, \varepsilon)$-separated set $T$ contained in $\Omega$ having cardinality $l$. This will obviously give $s_{\Omega}(n, \varepsilon) \geq l$. Inequalities $t \leq 2l + 1$ and $n_{\Omega} \leq s_{\Omega}(n, \varepsilon)$ still holds. Using Lemmas [2] and [6] one now obtains
\[ s(n, \varepsilon) = n_{\Omega} + n_{\Omega^c} \]
\[ \leq n_{\Omega} + tk\left(1 + \frac{1}{\varepsilon}\right)(2q_k(2n) + 1) \]
\[ \leq s_{\Omega}(n, \varepsilon) + (2s_{\Omega}(n, \varepsilon) + 1)k\left(1 + \frac{1}{\varepsilon}\right)(2q_k(2n) + 1). \]
This concludes the proof of Theorem B.

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