We study the onset of a classical order parameter after a second-order phase transition in quantum field theory. We consider a quantum scalar field theory in which the system-field (long-wavelength modes), interacts with its environment, represented both by a set of scalar fields and by its own short-wavelength modes. We compute the decoherence times for the system-field modes and compare them with the other time scales of the model. We analyze different couplings between the system and the environment for slow quenches. Within our approximations decoherence is in general a short time event.

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I. INTRODUCTION

The emergence of classical physics from quantum behaviour is important for several physical phenomena in the early Universe. This is beyond the fundamental requirement that only after the Planck time can the metric of the Universe be assumed to be classical. For example,

- The inflationary era is assumed to have been induced by scalar inflaton fields, with simple potentials [1]. Such fields are typically assumed to have classical behaviour, although in principle a full quantum description should be used.
- The origin of large scale structure in the Universe can be traced back to quantum fluctuations that, after crossing the horizon, were frozen and became classical, stochastic, inhomogeneities [2].
- It is generally assumed that several phase transitions have occurred during the expansion of the Universe [3]. As in the case for the inflaton fields, the (scalar) order parameter fields that describe these transitions are described classically. However, the description of early universe phase transitions from first principles is intrinsically quantum mechanical [4].
- As a specific application [5] of the previous point, the very notion of topological defects (e.g. strings and monopoles) that characterize the domain structure after a finite-time transition, and whose presence has consequences for the early universe, is based on this assumption of classical behaviour for the order parameter [6], as it distributes itself between the several degenerate ground states of the ordered system.

In the present paper we are concerned with the third point above, the quantum to classical transition of the order parameters in second order phase transitions. Any approach must take into account both the quantum nature of the order parameter and the non-equilibrium aspects of the process. The problem of the quantum to classical transition in the context of inflationary models was first addressed by Guth and Pi [7]. In that work, the authors used an inverted harmonic oscillator as a toy model to describe the early time evolution of the inflaton, starting from a Gaussian quantum state centered on the maximum of the potential. They subsequently showed that, according to Schrödinger’s equation, the initial wave packet maintains its Gaussian shape (due to the linearity of the model). Since the wave function is Gaussian, the Wigner function is positive for all times. Moreover, it peaks on the classical trajectories in phase space as the wave function spreads. The Wigner function can then be interpreted as a classical probability distribution for coordinates and momenta, showing sharp classical correlations at long times. In other
words, the initial Gaussian state becomes highly squeezed and indistinguishable from a classical stochastic process. In this sense, one recovers a classical evolution of the inflaton rolling down the hill.

A similar approach has been used by many authors to describe the appearance of classical inhomogeneities from quantum fluctuations in the inflationary era [8]. Indeed, the Fourier modes of a massless free field in an expanding universe satisfy the linear equation

$$\phi''_k + \left( k^2 - \frac{a''}{a} \right) \phi_k = 0. \quad (1)$$

For sufficiently long-wavelengths ($k^2 \ll a''/a$), this equation describes an unstable oscillator. If one considers an initial Gaussian wave function, it will remain Gaussian for all times, and it will spread with time. As with the toy model of Guth and Pi, one can show that classical correlations do appear, and that the Wigner function can again be interpreted as a classical probability distribution in phase space. [It is interesting to note that a similar mechanism can be invoked to explain the origin of a classical, cosmological magnetic field from amplification of quantum fluctuations].

However, classical correlations are only one aspect of classical behaviour. It was subsequently recognized that, in order to have a complete classical limit, the role of the environment is crucial, since its interaction with the system distinguishes the field basis as the pointer basis [9]. [We are reminded that, even for the fundamental problem of the space-time metric becoming classical, simple arguments based on minisuperspace models suggest that the classical treatment is only correct because of the interaction of the metric with other quantum degrees of freedom [10].]

While these linear instabilities cited above characterise free fields, the approach fails when interactions are taken into account. Indeed, as shown again in simple quantum mechanical models (e.g. the anharmonic inverted oscillator), an initially Gaussian wave function becomes non-Gaussian when evolved numerically with the Schrödinger equation. The Wigner function now develops negative parts, and its interpretation as a classical probability breaks down [11]. One can always force the Gaussianity of the wave function by using a Gaussian variational wave function as an approximate solution of the Schrödinger equation, but this approximation deviates significantly from the exact solution as the wave function probes the non-linearities of the potential [11,12].

When interactions are taken into account, classical behaviour is recovered only for "open systems", in which the relevant degrees of freedom interact with their environment. When this interaction produces both a diagonalization of the reduced density matrix and a positive Wigner function, the quantum to classical transition is completed [13].

Going from quantum mechanical toy models to quantum field theory is, of course, extremely difficult. For this reason, several authors [14] have considered different approximations in quantum mechanics, and compared them to the exact results. If successful in quantum mechanics, they could be implemented with greater confidence in field theory. As already mentioned, when this procedure is applied in the context of closed systems, the conclusion is that, in general, mean-field approximations do not reproduce the evolution of the system at late time [11,12,14]. Therefore, in principle, there is not reason to believe they will do so in field theory. In spite of this, computational necessity has lead many authors to perform calculations in closed field theories with Hartree, mean field, or 1/N approximations, since they are non-perturbative, well-defined and suitable for numerical calculations [15–18]. In such calculations, classical correlations do appear in some field theory models [16,17]. However, since such decoherence (in a time-averaged sense) takes place at long times after the transition has been achieved initially, when the mean field approximation has broken down, this may be an artifact of the Gaussian-like approximations [11].

In a previous paper [11] we considered similar arguments for quantum mechanics, but for open systems. When an anharmonic inverted oscillator is coupled to a high temperature environment, it becomes classical very quickly, even before the wave function probes the non-linearities of the potential. Being an early time event, the quantum to classical transition can now be studied perturbatively. In general, recoherence effects are not expected [19]. Taking these facts into account, we have extended the approach to field theory models [20]. In field theory, one is usually interested in the long-wavelengths of the order parameter. Moreover, the early universe is replete with fields of all sorts which comprise a rich environment. For this reason, we considered a model in which the order parameter interacts with a large number of environmental fields, including its own short-wavelengths. Assuming weak coupling and high critical temperature, we have shown that decoherence is a short time event, shorter than the spinodal time $t_{sp}$, which is essentially that time by which the order parameter field has sampled the degenerate ground states. As a result, perturbative calculations are justified [20]. Subsequent dynamics can be described by a stochastic Langevin equation, the details of which are only known for early times.

Our approach in Ref. [20] has some connections with well-established classical behaviour of thermal scalar field theory [21] at high temperature. In many articles it has been shown that, at high temperatures, the behaviour of long-wavelength modes is determined by classical statistical field theory. The effective classical theory is obtained after integrating out the hard modes with $k \geq gT$. Although similar, this approach has some important differences from ours: the "classical behaviour" in this soft thermal mode analysis is defined through the coincidence of the quantum and the statistical correlation functions. In particular, thermal equilibrium is assumed to hold at all times.
Finally, the cutoff that divides system and environment depends on the temperature, which is externally fixed. In our approach, the quantum to classical transition is defined by the diagonalization of the reduced density matrix, which is not assumed to be thermal. In phase transitions the separation between long and short-wavelengths is determined by their stability, which depends on the parameters of the potential.

In Ref. [20] we only considered the case of an instantaneous quench, and bi-quadratic coupling between the system and the environment. In Ref. [22] we began to extend those results to the case of a slow quench. In this paper we provide the details of that analysis and extend it to other couplings between system and environment. The consideration of slow quenches is very important since the Kibble-Zurek mechanism predicts the relation between the subsequent domain structure and the quench time [23–26] (by indirectly counting defects).

The paper is organized as follows. In Section 2 we introduce our models. These are theories containing a real system field $\phi$, which undergoes a transition, coupled to other scalar fields $\chi_a$ ($a = 1, 2, ..., N$), which constitute the external part of the environment. Gauge fields bring their own specific difficulties and we shall not discuss them here. We compute the influence functional by integrating out the environmental fields for different couplings. Section 3 is dedicated to reviewing the evaluation of the master equation and the diffusion coefficients which are relevant in order to study decoherence. In Section 4 we evaluate upper bounds on the decoherence time for slow quenches. As we will see, provided quenches are not to slow, decoherence takes place before the field samples the minimum of the potential, i.e. decoherence time is typically shorter than the spinodal time. However, since all relevant timescales depend only logarithmically upon the parameters of the theory, it is necessary to keep track of $O(1)$ prefactors carefully, something that we rather took for granted in Ref. [20] and in Ref. [22]. We will also show that the bi-quadratic coupling is the most relevant for the quantum to classical transition. Section 5 contains the conclusions of our work. Two short appendices fill in some of the detail.

II. THE INFLUENCE ACTION FOR AN EXTERNAL ENVIRONMENT

For the infinite degree of freedom quantum field $\phi$ undergoing a continuous transition, the field ordering after the transition begins is due to the growth in amplitude of its unstable long-wavelength modes. For these modes the environment consists of the short-wavelength modes of the field, together with all the other fields with which $\phi$ inevitably interacts [27–29] in the absence of selection rules. The inclusion of explicit environment fields is both a reflection of the fact that a scalar field in isolation is physically unrealistic, as well as providing us with a systematic approximation scheme.

The $\phi$-field describes the scalar order parameter, whose $Z_2$ symmetry is broken by a double-well potential. Specifically, we take the simplest classical action with scalar and environmental fields $\chi_a$

$$S[\phi, \chi] = S_{\text{syst}}[\phi] + S_{\text{env}}[\chi] + S_{\text{int}}[\chi_a, \phi],$$

where (with $\mu^2, m^2 > 0$)

$$S_{\text{syst}}[\phi] = \int d^4x \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \mu^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right\},$$

$$S_{\text{env}}[\chi_a] = \sum_{a=1}^N \int d^4x \left\{ \frac{1}{2} \partial_\mu \chi_a \partial^\mu \chi_a - \frac{1}{2} m_a^2 \chi_a^2 \right\}.$$  

The most important interactions will turn out to be of the biquadratic form

$$S_{\text{int}}[\chi_a, \phi] = S_{\text{qu}}[\phi, \chi] = -\sum_{a=1}^N \frac{g_a}{8} \int d^4x \phi^2(x) \chi_a^2(x).$$

Even if there were no external $\chi$ fields with a quadratic interaction of kind (5), the interaction between short and long-wavelength modes of the $\phi$-field can be recast, in part, in this form (see later), showing that such a term is obligatory. The generalization to a complex field $\phi$ is straightforward, and has been considered elsewhere [22].

Later, we shall consider additional interactions to the biquadratic interaction of a bilinear form

$$S_{\text{bilin}}[\phi, \chi] = -\sum_{a=1}^N \frac{g_{\text{bilin}}}{4} \int d^4x \phi^2(x) \chi_a(x),$$

Finally, the cutoff that divides system and environment depends on the temperature, which is externally fixed. In our approach, the quantum to classical transition is defined by the diagonalization of the reduced density matrix, which is not assumed to be thermal. In phase transitions the separation between long and short-wavelengths is determined by their stability, which depends on the parameters of the potential.

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$$S_{\text{syst}}[\phi] = \int d^4x \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \mu^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right\},$$

$$S_{\text{env}}[\chi_a] = \sum_{a=1}^N \int d^4x \left\{ \frac{1}{2} \partial_\mu \chi_a \partial^\mu \chi_a - \frac{1}{2} m_a^2 \chi_a^2 \right\}.$$  

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in which the environment couples linearly.

We could also have included fermionic Yukawa interactions but these provide a smaller contribution to the diffusion constant because of Fermi statistics (which gives a diffusion coefficient relatively $O(\mu^2/T^2)$). However, because the biquadratic self-interaction is the overwhelming term in the diffusion coefficient, in general we are not interested in effects that are overshadowed by it, and we shall not consider Yukawa interactions here.

Our exception to this rule of ignoring small contributions is in the the inclusion of linear couplings of the form

$$S_{\text{lin}}[\phi, \chi_a] = -\sum_{a=1}^{N} \frac{g_a \mu^2}{4} \int d^4x \phi(x) \chi_a(x), \quad (7)$$

in which the order parameter field couples linearly to a linear environment. Much if not most of the work on decoherence has been for linear coupling to the environment of this type (e.g. see [30,31]). While it is sensible in quantum mechanics, in the context of quantum field theory such linear couplings signal an inappropriate field diagonalisation, although they are exactly solvable in some circumstances [31].

Although the system field $\phi$ can never avoid the decohering environment of its own short-wavelength modes, to demonstrate the effect of an environment we first consider the case in which the environment is taken to be composed only of the fields $\chi_a$. We are helped in this by the fact that environments have a cumulative effect on the onset of classical behaviour. That is, the inclusion of a further component of the environment reduces the time $t_D$ it takes for the system to behave classically (for the definition of $t_D$ see later). Thus it makes sense to include the environment one part after another, since we can derive an upper bound on $t_D$ at each step. The short-wavelength modes of the $\phi$ field will be considered last.

To keep our calculations tractable, we need a significant part of the environment to have a strong impact upon the system-field, but not vice-versa, from which we can bound $t_D$. The simplest way to implement this is to take a large number $N \gg 1$ of scalar $\chi_a$ fields with comparable masses $m_a \simeq \mu$ weakly coupled to the $\phi$, with $\lambda, g_a, g'_a, g''_a \ll 1$. Thus, at any step, there are $N$ weakly coupled environmental fields influencing the system field, but only one weakly self-coupled system field to back-react upon the explicit environment.

We first consider the case in which the fields $\chi_a$ interact through the biquadratic interaction (5) alone. For one-loop consistency at second order in our calculation of the diffusion coefficient (that enforces classicality) it is sufficient, at order of magnitude level, to take identical $g_\alpha = g/\sqrt{N}$. Further, at the same order of magnitude level, we take $g \simeq \lambda$. This is very different from the more usual large-$N \mathcal{O}(N+1)$-invariant theory with one $\phi$-field and $N$ $\chi_a$ fields, dominated by the $O(1/N)$ ($\chi^2$) interactions, that has been the standard way to proceed for a closed system. With our choice there are no direct $\chi^4$ interactions, and the indirect ones, mediated by $\phi$ loops, are depressed by a factor $g/\sqrt{N}$. In this way the effect of the external environment qualitatively matches the effect of the internal environment provided by the short-wavelength modes of the $\phi$-field, but in a more calculable way.

For small $g$ the model has a continuous transition at a temperature $T_c$. The environmental fields $\chi_a$ reduce $T_c$ and, in order that $T_c^2/\mu^2 = 24/(\lambda + \sum g_a) \gg 1$, we must take $\lambda + \sum g_a \ll 1$, whereby $1 \gg 1/\sqrt{N} \gg g$. Further, with this choice the dominant hard loop contribution of the $\phi$-field to the $\chi_a$ thermal masses is

$$\delta m^2_T = O(g T_c^2/\sqrt{N}) = O(\mu^2/N) \ll \mu^2.$$

Similarly, the two-loop (setting sun) diagram which is the first to contribute to the discontinuity of the $\chi$-field propagator is of magnitude

$$g^2 T_c^2 / N = O(\mu^2 N^{3/2}) \ll \delta m^2_T,$$

in turn. That is, the effect of the thermal bath on the propagation of the environmental $\chi$-fields is ignorable. This was our intention: to construct an environment that reacted on the system field, but was not reacted upon by it to any significant extent. In particular, the infinite $N$ limit does not exist. Dependence on $N$ is implicit through $T_c$, as well as through the couplings, for initial temperatures $T_0 = O(T_c)$. With $\eta = \sqrt{6\mu^2/\lambda}$ determining the position of the minima of the potential and the final value of the order parameter, this choice of coupling and environments gives the hierarchy of scales necessary for establishing a reliable approximation scheme, as has been shown in [20].

We shall assume that the initial states of the system and environment are both thermal, at a high temperature $T_0 > T_c$. We then imagine a change in the global environment (e.g. expansion in the early universe) that can be characterised by a change in temperature from $T_0$ to $T_1 < T_c$. That is, we do not attribute the transition to the effects of the environment-fields.

Given our thermal initial conditions it is not the case that the full density matrix has $\phi$ and $\chi$ fields uncorrelated initially, since it is the interactions between them that leads to the restoration of symmetry at high temperatures. Rather, on incorporating the hard thermal loop 'tadpole' diagrams of the $\chi$ (and $\phi$) fields in the $\phi$ mass term leads to the effective action for $\phi$ quasiparticles,
\[ S_{\text{eff}}^{\text{syst}}[\phi] = \int d^4 x \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_\phi^2(T_0) \phi^2 - \frac{\lambda}{4!} \phi^4 \right\} \]

where \( m_\phi^2(T_0) \propto (1 - T_0/T_c) \) for \( T \approx T_c \). As a result, we can take an initial factorised density matrix at temperature \( T_0 \) of the form \( \rho(T_0) = \hat{\rho}_0[T_0] \times \hat{\rho}_\chi[T_0] \), where \( \hat{\rho}_0[T_0] \) is determined by the quadratic part of \( S_{\text{eff}}^{\text{syst}}[\phi] \) and \( \hat{\rho}_\chi[T_0] \) by \( S_{\text{env}}[\chi_a] \). That is, the many \( \chi_a \) fields have a large effect on \( \phi \), but the \( \phi \)-field has negligible effect on the \( \chi_a \).

Provided the change in temperature is not too slow the exponential instabilities of the \( \phi \)-field grow so fast that the field has populated the degenerate vacua well before the temperature has dropped significantly below \( T_c \). Since the temperature \( T_c \) has no particular significance for the environment fields, for these early times we can keep the temperature of the environment fixed at \( T_\chi \approx T_c \) (our calculations are only at the level of orders of magnitude). Meanwhile, for simplicity the \( \chi_a \) masses are fixed at the common value \( m \approx \mu \).

Our interests in the transitions in the early universe (domain structure, defect formation and, indirectly, structure formation) and the subsequent field evolution [6] are naturally couched in terms of fields, rather than particles. Since we need to be able to distinguish between different classical system-field configurations evolving after the transition, we will only be interested in the field-configuration basis for this reduced density matrix (in analogy with the usual quantum Brownian motion model, e.g. [32]). The resulting reduced density matrix \( \rho_f[\phi^+, \phi^-, t] = \langle \phi^+ | \hat{\rho}_f(t) | \phi^- \rangle \) describes the evolution of the system under the influence of the environment, and is defined by

\[
\rho_f[\phi^+, \phi^-, t] = \int \prod_{a=1}^N \mathcal{D}_{\chi_a} \rho[\phi^+, \chi_a, \phi^-, \chi_a, t],
\]

where \( \rho[\phi^+, \chi_a^+, \phi^-, \chi_a^-, t] = \langle \phi^+ | \chi_a^+ | \phi^- | \chi_a^- \rangle \) is the full density matrix. For the reasons given above, the environment will have had the effect of making the system effectively classical once \( \rho_f(t) \) is, approximately, diagonal. Quantum interference can then be ignored and we obtain a classical probability distribution from the diagonal part of \( \rho_f(t) \), or equivalently, by means of the reduced Wigner functional, which is positive definite after \( t_D \) [11]. For weak coupling there will be no 'recoherence' at later times in which the sense of classical probability will be lost [19].

The temporal evolution of \( \rho_f \) is given by

\[
\rho_f[\phi^+_t, \phi^-_t, t] = \int d\phi^+_t \int d\phi^-_t J_t[\phi^+_t, \phi^-_t, t] \rho_f[\phi^+_t, \phi^-_t, t_0] \rho_f[\phi^+_t, \phi^-_t, t_0],
\]

where \( J_t \) is the reduced evolution operator

\[
J_t[\phi^+_t, \phi^-_t, t] = \int \mathcal{D} \phi^+ \mathcal{D} \phi^- e^{i\{S[\phi^+] - S[\phi^-]\}} F[\phi^+, \phi^-].
\]

The Feynman-Vernon [33] influence functional (IF) \( F[\phi^+, \phi^-] \) in (9) can be written in terms of influence action \( \delta A[\phi^+, \phi^-] \) and the coarse grained effective action (CGEA) \( A[\phi^+, \phi^-] \) by (see [28] for formal definitions)

\[
F[\phi^+, \phi^-] = \exp i\delta A[\phi^+, \phi^-],
\]

\[
A[\phi^+, \phi^-] = S[\phi^+] - S[\phi^-] + \delta A[\phi^+, \phi^-].
\]

Beginning from the initial distribution, peaked around \( \phi = 0 \), we follow the evolution of the system under the influence of the environment fields. We will calculate the influence action to lowest non-trivial order (two vertices) for large \( N \). Similarly to the biquadratic interaction, we assume weak coupling \( \lambda \approx g' \approx g'' \ll 1 \), where we have defined \( g', g'' \) by the order of magnitude relations \( g'_a = g' / \sqrt{N} \) and \( g''_a = g'' / \sqrt{N} \) respectively.

As we are considering weak coupling with the environment fields, we may expand the influence functional \( F[\phi^+, \phi^-] \) up to second non-trivial order in coupling strengths. The general form of the influence action is then [28,34]

\[
\delta A[\phi^+, \phi^-] = \{ \langle S_{\text{int}}[\phi^+_a, \chi_a^+] \rangle_0 - \langle S_{\text{int}}[\phi^-, \chi_a^-] \rangle_0 \} + \frac{i}{2} \{ \langle S_{\text{int}}[\phi^+_a, \chi_a^+] \rangle_0 - \langle S_{\text{int}}[\phi^+_a, \chi_a^+] \rangle_0^2 \} - \frac{i}{2} \{ \langle S_{\text{int}}[\phi^+_a, \chi_a^+] S_{\text{int}}[\phi^-_a, \chi_a^-] \rangle_0 - \langle S_{\text{int}}[\phi^+_a, \chi_a^+] \rangle_0 \langle S_{\text{int}}[\phi^-_a, \chi_a^-] \rangle_0 \} \]

\[
+ \frac{i}{2} \{ \langle S_{\text{int}}[\phi^-_a, \chi_a^-] \rangle_0 - \langle S_{\text{int}}[\phi^-_a, \chi_a^-] \rangle_0^2 \}. \]

\]

\]
For weak couplings it is relatively easy to compute the upper bound on the decoherence time due to the interactions $\mathcal{S}_{\text{int}}[\phi, \chi]$ of (5), (6) and (7). We shall find that, in our particular model, it is in general shorter than the spinodal time $t_{\text{sp}}$, defined as the time for which

$$\langle \phi^2 \rangle_{t_{\text{sp}}} \sim \eta^2 = 6\mu^2/\lambda.$$  \hspace{1cm} (13)

[More exactly, $t_{\text{sp}}$ should be defined as the time that the r.m.s. of the field takes to reach the spinodal point, the point of inflection in the potential, but the difference is only logarithmically small, so we stay with (13).]

It is for this reason that, with the qualifications below, we can use perturbation theory. In consequence, by the time that the field is ordered it can be taken to be classical. This has implications [22] for the formation of the defects that are a necessary byproduct of transitions.

III. MASTER EQUATIONS AND THE DIFFUSION COEFFICIENTS

In this Section we will obtain the evolution equation for the reduced density matrix (master equation), paying particular attention to the diffusion term, which is responsible for decoherence. We will closely follow the quantum Brownian motion (QBM) example [30,35], translated into quantum field theory [20,28].

This Section contains a generalization of the result already given in [20], adapted to the slow quenches that are physically relevant, in which $m(T)$ varies linearly in time. The formal deduction of the master equation is not altered by the duration of the quench. For the details we will follow Ref. [28]. The first step in the evaluation of the master equation is the calculation of the density matrix propagator $J_i$ from Eq. (9). In order to solve the functional integration which defines the reduced propagator, we perform a saddle point approximation

$$J_i[\phi^+_1, \phi^-_1, t|\phi^+_0, \phi^-_0, t_0] \approx \exp iA[\phi^+_1, \phi^-_1],$$  \hspace{1cm} (14)

where $\phi^+_1$ is the solution of the equation of motion $\frac{\delta H_{\text{cl}}}{\delta \phi} = 0$ with boundary conditions $\phi^+_0(t_0) = \phi^+_1$ and $\phi^-_0(t) = \phi^-_1$. Even then, it is very difficult to solve this equation analytically. We are helped by the observation that the ordering of the field is due to the growth of the long-wavelength unstable modes. Unstable long-wavelength modes start growing exponentially as soon as the quench is performed, whereas short-wavelength modes will oscillate. As a result, the field correlation function rapidly develops a peak (Bragg peak) at wavenumber $k = \vec{k} \ll \mu$. Specifically [36], initially as $k^2 = \mathcal{O}(\mu/\sqrt{T_\eta})$, where $T_\eta^{-1}$ is the quench rate. Assuming that a classical description can be justified post-hoc, a domain structure forms quickly with a characteristic domain size $O(\vec{k}^{-1})$, determined from the position of this peak. [As an example, see the numerical results of [25], where this classical behaviour has been assumed through the use of stochastic equations (see later)]. With this in mind, we adopt an approximation in which the system-field contains only one Fourier mode with $\vec{k} = \vec{k}_0 = O(\vec{k}^{-1})$, characteristic of the domain size.

As “trial” classical solutions, we take

$$\phi_{\text{cl}}(\vec{x}, s) = f(s, t) \cos(k_0 x) \cos(k_0 y) \cos(k_0 z),$$  \hspace{1cm} (15)

where $f(s, t)$ satisfies $f(0, t) = \phi_1$ and $f(t, t) = \phi_1$. Eq. (15) is an exemplary configuration that mimics domain formation. As intended, it represents domains of finite size $k_0^{-1}$. Although such a regular domain structure is idealised, numerical results (as in [25]) suggest that it is a reasonable first step. In practice, after this detailed motivation, we shall find that the decoherence time $t_D$ is insensitive to wavelength for $k_0$ small. For the purpose of calculation, it is sufficient to set $k_0 = 0$, even though the physical situation requires a finite $k_0$. An exact matching of $k_0$ to $k$ is unnecessary. [We note that, in [20] and [22], we adopted a simpler form $\phi_{\text{cl}}(\vec{x}, s) = f(s, t) \cos(k_0 \vec{x})$. Although showing a domain structure in one dimension is less physical, our conclusions are unchanged although details differ.]

We write $f(s, t)$ as

$$f(s, t) = \phi_1 u_1(s, t) + \phi_1 u_2(s, t),$$  \hspace{1cm} (16)

where, during the quench, $u_1(s, t)$ are solutions of the mode equation with boundary conditions $u_1(0, t) = 1, u_1(t, t) = 0$ and $u_2(0, t) = 0, u_2(t, t) = 1$. In order to obtain the master equation we must compute the final time derivative of the propagator $J_i$. After that, all the dependence on the initial field configurations $\phi_{\text{cl}}^\pm$ (coming from the classical solutions $\phi_{\text{cl}}^\pm$) must be eliminated. The free propagator, defined as

$$J_0[\phi^+_1, \phi^-_1, t|\phi^+_0, \phi^-_0, 0] = \int_{\phi^+_1}^{\phi^+_0} \int_{\phi^-_1}^{\phi^-_0} D\phi^+ D\phi^- \exp\{i[S_0(\phi^+) - S_0(\phi^-)]\};$$  \hspace{1cm} (17)
satisfies the general identities \([28,35]\) (which is valid for instantaneous and for slow quenches)
\[
\phi_{cl}^+(s)J_0 = \left[ \phi_{cl}^+ [u_2(s,t) - \frac{u_2(t,t)}{u_1(t,t)} u_1(s,t)] + i \frac{u_1(s,t)}{u_1(t,t)} \partial_{\phi_{cl}^+} \right] J_0. \tag{18}
\]

These identities allow us to remove the initial field configurations \(\phi_{cl}^\pm\), by expressing them in terms of the final fields \(\phi_{cl}^\pm\) and the derivatives \(\partial_{\phi_{cl}^\pm}\), and obtain the master equation.

The full equation is very complicated and, as for quantum Brownian motion, it depends on the system-environment coupling.

As we are solely interested in decoherence, it is sufficient to calculate the correction to the usual unitary evolution coming from the noise kernels (imaginary part of the influence action). The result reads
\[
i \dot{\rho}_l = \langle \phi_{cl}^+ | [H, \rho_l] | \phi_{cl}^- \rangle - iV \sum_j \Gamma_j D_j (\omega_0,t) \rho_l + ... \tag{19}
\]
where \(j\) labels the different couplings. \(D_j\) are the diffusion coefficients. The ellipsis denotes other terms coming from the time derivative that do not contribute to the diffusive effects. \(V\) is understood as the minimal volume inside which there are no coherent superpositions of macroscopically distinguishable states for the field. The \(\Gamma_j\), which depend on the sums and differences \(\phi_{cl}^\pm \pm \phi_{cl}^-\) of the field amplitudes, have to be calculated case by case.

The effect of the diffusion coefficient on the decoherence process can be seen considering the following approximate solution to the master equation
\[
\rho_l [\phi^+, \phi^-; t] \approx \rho_{cl}^\mu [\phi^+, \phi^-; t] \exp \left[ -V \sum_j \Gamma_j \int_0^t ds \ D_j (k_0, s) \right], \tag{20}
\]
where \(\rho_{cl}^\mu\) is the solution of the unitary part of the master equation (i.e. without environment). The system will decohere when the non-diagonal elements of the reduced density matrix are much smaller than the diagonal ones.

This decoherence time \(t_D\) sets the scale after which we have a classical system-field configuration, and depends strongly on the properties of the environment. It is constrained by
\[
1 \approx V \sum_j \Gamma_j \int_0^{t_D} ds \ D_j (k_0, s) \gtrsim V \Gamma_1 \int_0^{t_D} ds \ D_1 (k_0, s), \tag{21}
\]
for any particular \(j = l\), and corresponds to the time after which we are able to distinguish between two different field amplitudes, inside a given volume \(V\). Our conservative choice is that this volume factor is \(O(\mu^{-3})\) since \(\mu^{-1}\) (the Compton wavelength) is the smallest scale at which we need to look.

In the next Section we will compute the corresponding diffusion coefficients, from which we will estimate the decoherence time bounds.

\section*{IV. DECOHERENCE TIMES FOR SLOW QUENCHES}

Our earliest results from Ref. [20] were for instantaneous quenches, for which the quench time is effectively \(\mu^{-1} \ll t_{sp}\). A priori, it was not obvious that this slow quench preserves these results about the decoherence time, that \(t_D \lesssim t_{sp}\). This is assumed in the Kibble scenario \([5]\) for domain formation as determined by the appearance of defects, in which classical defects are assumed to have appeared as soon as the transition is effected. This was first addressed by us in a rather schematic way in \([22]\), to which what follows provides a more sophisticated analysis and extensions to other couplings.

To tackle this problem, we assume that the quench begins at \(t = 0\) and ends at time \(t = 2\tau_q\), with \(\tau_q \gg \tau_r \sim \mu^{-1}\). At the qualitative level at which we are working it is sufficient to take \(m_\phi^2(T_0) = \mu^2\) exactly. Most simply, we consider a quench linear in time, with temperature \(T(t)\), for which the mass function is of the following form \([37]\)
\[
m_\phi^2(t) = m_\phi^2(T(t)) = \begin{cases} 
\mu^2 & \text{for } t \leq 0 \\
\mu^2 - \frac{\mu^2}{\tau_q} \bigg \{ & \text{for } 0 < t \leq 2\tau_q \\
-\mu^2 & \text{for } t \geq 2\tau_q 
\end{cases}
\]
Note that our \(\tau_q\) is the inverse quench rate \(T_c^{-1} dT/dt\bigg|_{T=T_c}\), and so differs from that of \([37]\) by a factor of 2.
The field behaves as a free field in an inverted parabolic potential for an interval of approximately $t_{sp}$ [36], where

$$\langle \phi^2 \rangle_{t_{sp}} \sim \eta^2 .$$

We have rederived $t_{sp}$, using the exact results of [37] (see Appendix). The end result is that $t_{sp}$ is the solution to

$$\frac{\eta^2}{C} \approx \frac{T_c}{\mu^2} \exp \left[ \frac{4}{3} \left( \frac{\Delta_0(t_{sp})}{t} \right)^{3/2} \right],$$

where $\Delta_0(t) = t - \tau_q$, $C = (64 \sqrt{2} \pi^{3/2})^{-1}$, and $\bar{t} = (\tau_q/\mu^2)^{1/3}$ (see details in the Appendix A).

In order to find the master equation (or more strictly the diffusion coefficient) we follow the same procedure as in Section III, assuming a dominant wavenumber $k_0$. The classical solution is given in Eq. (16), where $u_i$ satisfy the mode equation

$$\left[ \frac{d^2}{ds^2} + k_0^2 + \mu^2 - \frac{\mu^2 s}{\tau_q} \right] u_i(s, t) = 0.$$

Since we are neglecting the self-interaction term, our conclusions are only believable for $t \lesssim t_{sp}$.

The solution of Eq.(24) is given by

$$u_1(s, t) = -\frac{Ai(\Delta_k(s)/t)Bi(\Delta_k(s)/t) + Ai(\Delta_k(t)/t)Bi(\Delta_k(t)/t)}{Ai(-\omega_0^2 t^2)Bi(\Delta_k(t)/t) - Ai(\Delta_k(t)/t)Bi(-\omega_0^2 t^2)},$$

$$u_2(s, t) = -\frac{Ai(\Delta_k(s)/t)Bi(-\omega_0^2 t^2) + Ai(-\omega_0^2 t^2)Bi(\Delta_k(s)/t)}{Ai(-\omega_0^2 t^2)Bi(\Delta_k(t)/t) - Ai(\Delta_k(t)/t)Bi(-\omega_0^2 t^2)},$$

where $Ai[s]$ and $Bi[s]$ are the Airy functions, with $\Delta_k(s) = s - \omega_0^2 t^2$, $\Delta_k(t) = t - \omega_0^2 t^2$, and $\omega_0^2 = \mu^2 + k_0^2$. We note that $\Delta_0(t) = t - \tau_q$. In the causal analysis of Kibble [5] $\bar{t} \left( \mu^{-1} \ll \bar{t} \ll \tau_q \right)$ is the time at which the adiabatic field correlation length collapses at the speed of light, the earliest time in which domains could have formed. Our analysis suggests that this earliest time is not $t$, but $t_{sp}$.

### A. Quadratic coupling

We start by considering the biquadratic coupling first, for which the IF is given by

$$\text{Re} \delta A = \frac{g^2}{8} \int d^4x \int d^4y \Delta_2(x)K_q(x-y)\Sigma_2(y),$$

$$\text{Im} \delta A = -\frac{g^2}{16} \int d^4x \int d^4y \Delta_2(x)N_q(x,y)\Delta_2(y),$$

where $K_q(x-y) = \text{Im} G_{++}(x, y) \theta(y^0 - x^0)$ is the dissipation kernel and $N_q(x-y) = \text{Re} G_{++}(x, y)$ is the noise (diffusion) kernel. $G_{++}$ is the relevant closed-time-path correlator of the $\chi$-field at temperature $T_0$. We have defined

$$\Delta_2 = \frac{1}{2}(\phi^{+2} - \phi^{-2}) ; \quad \Sigma_2 = \frac{1}{2}(\phi^{+2} + \phi^{-2}).$$

We can formally find the master equation in the same way that we did in Section III. It is given by Eq.(19), with a diffusion coefficient of the form (details on how to get this coefficient can be found in Appendix B)

$$D_{qu}(k_0, t) = \int_0^t ds \ u(s, t) \ F(k_0, s, t),$$

with

$$u(s, t) = \left[ u_2(s, t) - \frac{\dot{u}_2(t, t)}{\dot{u}_1(t, t)} u_1(s, t) \right]^2,$$
and

\[ F(k_0, s, t) = \frac{1}{64} \left\{ \text{Re}G_{++}^2(0; t - s) + \frac{3}{2}\text{Re}G_{++}^2(2k_0; t - s) + \frac{3}{4}\text{Re}G_{++}^2(2\sqrt{2}k_0; t - s) + \frac{1}{8}\text{Re}G_{++}^2(2\sqrt{3}k_0; t - s) \right\}. \]

It is only in \( u(s, t) \) that the slow quench is apparent. \( G_{++}^2(k, t - s) \) is the Fourier transform of the square of the Feynman propagator (\( \chi \) propagator).

In the high temperature limit \( (T \gg \mu) \), the explicit expression for the kernels can be shown, with a little labour, to be

\[ \text{Re}G_{++}^2(k; t - s) = \frac{T^2}{64\pi^2 k} \int_0^\infty dp \frac{p}{p^2 + \mu^2} \int_0^\infty \frac{du}{u} \cos[(\sqrt{p^2 + \mu^2} + u)(t - s)], \quad (26) \]

and

\[ \text{Re}G_{++}^2(0; t - s) = \frac{T^2}{64\pi^2} \int_0^\infty dp \frac{p^2}{(p^2 + \mu^2)^2} \cos[2\sqrt{p^2 + \mu^2}(t - s)], \quad (27) \]

where \( \mu \) is the thermal \( \chi \)-field mass at temperature \( T \sim T_c \). In our scheme, this is approximately the cold \( \chi \) mass.

It is because the \( \chi \)-field propagator is unaffected by the \( \phi \)-field interactions that the detail of the expressions (26)-(27) are possible. Fortunately, it is inessential. We see that, for times \( \mu t > 1 \), the behaviour of \( D_{qu}(k_0, t) \) is dominated by the exponential growth of \( u(s, t) \), and the integral in Eq.(25) by the interval \( s \approx 0 \). Indeed, it is easy to prove that \( u(0, t) \approx (\frac{\omega_0(t)}{u(0,t)})^2 \gg 1 \), while \( u(t, t) = 1 \). Therefore, although the expressions in Eqs.(25), (26), and (27) are complicated, we can approximate the whole diffusion coefficient by

\[ D_{qu}(k_0, t) \approx F(k_0, 0, t) u(0, t) \int_0^\infty ds \frac{u(s, t)}{u(0,t)} \quad (28) \]

where we used the fact that \( F(k_0, s, t) \) is bounded at \( s = 0 \).

We will assume large \( \Delta_k(t) \) (and \( \Delta_k(s) \)), which means \( \Delta_k(t), \Delta_k(s) \gg \bar{t} \). This condition is satisfied provided \( s \) is larger and not to close to \( \omega_0^2\tau_0/\mu^2 \), and allow us to use the asymptotic expansions of the Airy functions and their derivatives for the evaluation of \( u_k(s, t) \). This will be justified posthoc. In particular we obtain

\[ u(0, t) \approx \frac{1}{4\omega_0 t} \sqrt{\frac{\Delta_k(t)}{t}} \exp \left\{ \frac{1}{3} \left( \frac{\Delta_k(t)}{t} \right)^{\frac{3}{2}} \right\} \]

\[ u(t, t) \approx \frac{1}{2\omega_0^2 t^3} \sqrt{\frac{\Delta_k(t)}{t}} \exp \left\{ \frac{4}{3} \left( \frac{\Delta_k(t)}{t} \right)^{\frac{3}{2}} \right\}. \quad (29) \]

Therefore, it is straightforward to check that in these cases the integral in Eq.(28) is given by

\[ \omega_0 \int_0^{t_0^{-1}} ds \left[ 1 + s \left( \frac{\dot{u}(0, t)}{u(0,t)} + ... \right) \right] \approx \omega_0 \int_0^{t_0^{-1}} ds \left[ 1 + s \frac{2}{\omega_0 t^2} + ... \right], \]

and it is \( O(1) \), due to the fact that \( \mu \bar{t} \gg 1 \).

We can estimate the decoherence time for the quadratic coupling as

\[ 1 \geq VT \int_0^{t_0} dt \ D_{qu}(k_0, t), \quad (30) \]

where \( V \) is the decoherence volume, as before.

In terms of the dimensionless fields \( \bar{\phi} = (\phi^+ + \phi^-)/2\mu \), and \( \delta = (\phi^+ - \phi^-)/2\mu \) it follows that \( \Gamma_{qu} = g^2\mu^4\bar{\phi}^2\delta^2 \). In order to quantify the decoherence time we have to fix the values of \( \delta \), and \( \bar{\phi} \). Inside the volume \( V \) we do not discriminate between field amplitudes which differ by \( O(\mu) \), and therefore, for the sake of argument, we take \( \delta = 1 \). For \( \bar{\phi} \) we set \( \bar{\phi}^2 = \alpha/\lambda \), where \( \lambda \leq \alpha \leq 1 \) is to be determined self-consistently. It is necessary to be as precise as this in the first instance, since \( t_{qu} \), and \( t_D \) only differ by logarithms. It is important that the prefactors in the arguments of these logarithms are determined carefully, given that terms nominally \( O(1) \) can be small. Once we have established that the difference is large enough, this artificial precision can be dropped. Posthoc it is sufficient to take \( \delta \sim O(1) \) and \( \bar{\phi}^2 \sim O(\alpha/\lambda) \), as we did in [20] and [22].
Then, for quadratic interactions alone the decoherence time reads
\[
\exp\left\{\frac{4}{3} \left(\frac{\Delta_k(t_D)}{t}\right)^\frac{3}{2}\right\} \approx 4.10^3 \frac{\omega_0^3}{\Lambda T_c^2 \alpha}.
\]  
(31)

Using \(\langle \phi^2 \rangle\) of (A5) we obtain \(\alpha^2 \approx 7(\mu/T_c)(\mu \tau_q)^{-2/3}\). Thus the decoherence time satisfies
\[
\exp\left\{\frac{4}{3} \left(\frac{\Delta_k(t_D)}{t}\right)^\frac{3}{2}\right\} \approx 2.10^3 \frac{\omega_0^3 \eta^2}{\mu^2 T_c^2 \mu^2},
\]  
(32)

for long-wavelength modes.

If we compare it with the spinodal time of (23), we get,
\[
(\mu \Delta_0(t_{sp}))^\frac{3}{2} - (\mu \Delta_k(t_D))^\frac{3}{2} \approx \frac{3}{4} \sqrt{\mu \tau_q} \left\{2 \ln \frac{\mu}{\omega_0} + \ln \left[\frac{1}{2} \left(\frac{T_c}{\mu}\right)^\frac{1}{2} (\mu \tau_q)^\frac{1}{2}\right]\right\} > 0.
\]  
(33)

We see that the numerical prefactor in the argument of the logarithm is, indeed, \(O(1)\), and our concern for precision was unnecessary, in retrospect. Our approximation scheme depends, as for the instantaneous quench, on the peaking of the power in the field fluctuations at long-wavelength \(k_0 \ll \mu\) by time \(t_{sp}\), and it is sufficient to take \(k_0 \approx 0\). In this case (33) becomes
\[
(\mu \Delta_0(t_{sp}))^\frac{3}{2} - (\mu \Delta_0(t_D))^\frac{3}{2} \approx \frac{3}{4} \sqrt{\mu \tau_q} \ln \left[\frac{1}{2} \left(\frac{T_c}{\mu}\right)^\frac{1}{4} (\mu \tau_q)^\frac{1}{4}\right] > 0,
\]  
(34)

from which \(t_D < t_{sp}\) follows, as in the instantaneous case. The inclusion of further interactions, including the self-interaction with short-wavelength \((k > \mu)\) modes can only reduce \(t_D\) further.

We should point out that, whereas peaking is inevitable for the instantaneous quench for weak enough coupling, it is not the case for very slow quenches. In such cases our approximations break down and a different analysis is required. The details are rather messy, but a sufficient condition for our approximation to be valid is that \(\mu \tau_q \lesssim \eta/\mu\) [36]. Tighter, but less transparent bounds can be given [36].

When these bounds are satisfied the minimum wavelength for which the modes decohere by time \(t_{sp}\) can be shown [22] to be shorter than that which characterises domain size at that time. Although we can talk loosely, but sensibly, about a classical domain structure at time \(t_{sp}\) we cannot yet talk about classical defects on their boundaries, as the naive picture might suggest. Defects (in this case, walls) are described by shorter wavelength modes \((k \lesssim \mu)\). Nonetheless, the classical domain structure is sufficient to determine their density [22].

**B. Bilinear and linear couplings**

Another possibility that needs to be considered is that of a bilinear coupling of the \(\phi\)-field to the environment. This interaction preserves the \(\phi \rightarrow -\phi\) symmetry of \(S_{\text{sys}}[\phi]\) in which, for simplicity we continue to take bilinear couplings equal, as \(g_{\phi}' = g'/\sqrt{N}\). We treat this interactions as an additional set of interactions to the biquadratic interactions, whereby \(T_c\) is qualitatively unchanged.

The influence action is still obtained from Eq.(12), as
\[
\text{Re} \delta A_{\text{bilinear}} = \frac{g'^2 \mu^2}{8} \int d^4x \int d^4y \Delta_2(x) K_b(x - y) \Sigma_2(y),
\]  
(35)

and
\[
\text{Im} \delta A_{\text{bilinear}} = -\frac{g'^2 \mu^2}{16} \int d^4x \int d^4y \Delta_2(x) N_b(x - y) \Delta_2(y),
\]  
(36)

where
\[
K_b = \text{Re} G_{++}(x, y) \theta(y^0 - x^0), \quad N_b = \text{Im} G_{++}(x, y).
\]  
(37)

The temporal diffusion coefficient is now given by
\[
D_{\text{bilin}}(k_0, t) = \int_0^t ds \ u(s, t) F_{\text{bilin}}(k_0, s, t),
\]

where
\[
F_{\text{bilin}}(k_0, s, t) = \frac{1}{64} \left\{ \text{Im} G_{++}(0; t - s) + \frac{3}{2} \text{Im} G_{++}(2k_0; t - s) + \frac{3}{4} \text{Im} G_{++}(2\sqrt{2}k_0; t - s) + \frac{1}{2} \text{Im} G_{++}(2\sqrt{3}k_0; t - s) \right\}.
\]

Following the same arguments as in the previous section, we can evaluate the diffusion coefficient by
\[
D_{\text{bilin}}(k_0, t) \approx F_{\text{bilin}}(k_0, 0, t) \frac{u(0, t)}{\omega_0},
\]

where \(u(0, t)\) is given in Eq.(29). Thus,
\[
D_{\text{bilin}}(k_0, t) \approx 10^{-2} \frac{T_c}{\mu^2 \omega_0} u(0, t).
\]

For the unstable long-wavelengths (\(k_0^2 < \mu^2/2\) approximately) we find, for times \(\mu t \gtrsim 1\), that \(D_{\text{bilin}}(k_0, t)\) again shows the exponential growth
\[
D_{\text{bilin}}(k_0, t) \approx 3 \cdot 10^{-3} \frac{T_c}{\mu^2 \omega_0^2} \sqrt{\frac{\Delta_k(t)}{t}} \exp \left\{ \frac{4}{3} \left( \frac{\Delta_k(t)}{t} \right)^{3/2} \right\}.
\]

It follows that \(\Gamma_{\text{bilin}} = g^2 \mu^4 \phi^2 \delta^2\). As a result, the contribution to the decoherence time from bilinear interaction (for long-wavelength modes) can be evaluated as
\[
\exp \left\{ \frac{4}{3} \left( \frac{\Delta_k(t_D)}{t} \right)^{3/2} \right\} \approx 6 \cdot 10^2 \frac{\omega_0^2}{\lambda T_c \mu \alpha}.
\]

The value of \(\alpha\) is again determined from the condition that, at time \(t_D\), \(\langle \phi^2 \rangle \sim \alpha \eta^2\). That is, \(\alpha^2 = 0.1(\mu \tau_q)^{-2/3}\). Thus, decoherence time is given by
\[
\exp \left\{ \frac{4}{3} \left( \frac{\Delta_k(t_D)}{t} \right)^{3/2} \right\} \approx 3 \cdot 10^{3.692} \frac{\omega_0^2}{\mu^2 T_c} (\mu \tau_q)^{3/4}.
\]

From (23), \(t_D\) and \(t_{sp}\) are related by
\[
(\mu \Delta_0(t_{sp}))^{\frac{3}{4}} - (\mu \Delta_0(t_D))^{\frac{3}{4}} \approx \frac{3}{4} \sqrt{m \tau_q} \ln \left[ 3(\mu \tau_q)^{3/4} \right],
\]

which is positive for a sufficiently slow quench.

As we have already observed, early studies of decoherence were confined largely to quantum mechanical systems, for which the environment was typically a collection of harmonic oscillators, to which the system coupled linearly. Such systems have the virtue of exact solvability (or closed equations) and have been very instructive. However, in the context of quantum field theory linear terms are usually a signal of an inappropriate choice of field basis. Further, as we are considering models with spontaneous symmetry breaking, linear couplings with external fields are not a natural choice since they break the vacuum degeneracy. We include isolated linear couplings for completeness. Again, choosing couplings equal, as \(g_0 = g'/\sqrt{N}\), and defining \(\Delta_1 = (\phi^+ - \phi^-)/2\) and \(\Sigma_1 = (\phi^+ + \phi^-)/2\), we are able to write the real and imaginary parts of the influence functional as
\[
\text{Re} A_{\text{lin}} = \frac{g'^2 \mu^4}{8} \int d^4 x \int d^4 y \Delta_1(x) K_b(x - y) \Sigma_1(y),
\]

and

\[11\]
\[ \text{Im} \delta A_{\text{lin}} = -\frac{g^{*2} \mu^4}{16} \int d^4x \int d^4y \Delta_1(x) N_b(x-y) \Delta_1(y). \]  

(47)

For times \( \mu t \gtrsim 1 \) the diffusion contribution to the master equation is

\[ D_{\text{lin}}(k_0, t) \approx \frac{T_c}{8\omega_0^3} \sqrt{u(0,t)}. \]  

(48)

Note that the exponent is only \textit{half} that of the quadratic and bilinear interactions. In this case, it can be shown that the decoherence time comes from

\[ 1 \approx \frac{V \Gamma_{\text{lin}} T_c}{16\omega_0^3} \sqrt{\frac{t}{\omega_0}} \left[ -\Gamma_1 \gamma^5 + \Gamma_2 \frac{5}{6} - \frac{2}{3} \frac{\Delta_k(t_D)}{t} \right], \]  

(49)

where \( \Gamma[a, z] \) is the incomplete Gamma function and, in this case \( \Gamma_{\text{lin}} = \frac{1}{2} \frac{g^{*2} \mu^2 \delta^2}{6} \). As \( t_D \gg t \), we can approximate Eq.(49) and obtain,

\[ \exp \left\{ \frac{2}{3} \left( \frac{\Delta_k(t_D)}{t} \right)^{\frac{2}{3}} \right\} \approx \frac{16 \omega_0^3 \eta^4}{9 \mu^4} \frac{1}{T_c} \frac{\omega_0^{\frac{2}{3}} \mu^{\frac{1}{3}}}{\tau_q}. \]  

(50)

The decoherence time associated to the linear interaction term, can be written as

\[ \Delta_k(t_D)^{\frac{3}{2}} \approx \frac{3}{2} t^{\frac{3}{2}} \left\{ 3 \ln \frac{\omega_0}{\mu} + \ln \frac{16 \eta^4}{9 \mu^4} \frac{1}{T_c} \frac{\omega_0^{\frac{2}{3}} \mu^{\frac{1}{3}}}{\tau_q} \right\}. \]  

(51)

\( t_D \) and \( t_{sp} \) are related by

\[ (\mu \Delta_0(t_{sp}))^{\frac{3}{4}} = (\mu \Delta_0(t_D))^{\frac{3}{4}} \approx \frac{3}{4} \sqrt{\frac{\eta}{\tau_q}} \ln \left[ 10^4 \frac{\mu^2}{\eta^2} (\mu \tau_q)^{\frac{3}{2}} \right]. \]  

(52)

In this case, due to the bound on the quench time (\( \tau_q \leq \eta/\mu^2 \)), we find that it looks as if \( t_D > t_{sp} \). However this result is not believable as it stands, since the diffusion coefficient has been computed assuming \( t \lesssim t_{sp} \). Whatever, the rapid decoherence of the biquadratic and other couplings is not present. This shows how adopting linear coupling to an environment in mimicry of quantum mechanics can be misleading.

C. Comparison between different couplings

In our present model the environment fields \( \chi_a \) are not the only decohering agents. The environment is also constituted by the short-wavelength modes of the self-interacting field \( \phi \). Therefore, we split the field as \( \phi = \phi_+ + \phi_\chi \), and define the system (\( \phi_+ \)) by those modes with wavelengths longer than the critical value \( \Lambda^{-1} \), while the bath or environment-field (\( \phi_\chi \)) contains wavelengths shorter than \( \Lambda^{-1} \). In order to consider only the unstable modes inside our system, we will set this critical scale \( \Lambda \) of the order of \( \mu \). In practice, where the separation is made exactly is immaterial [36] by time \( t_D \), when the power of the \( \phi \)-field fluctuations is peaked at \( k_0 \ll \mu \). The effect is to give a separation of system from environment through the decomposition of \( S[\phi, \chi] \) of (2) introducing an interaction terms of the form \( \phi_+ \phi_\chi^3, \phi_+^2 \phi_\chi^2, \) and \( \phi_+^3 \phi_\chi^1 \). The relevant interaction term is

\[ S_{\text{couple}}[\phi_+, \chi] = S_{\text{int}}[\phi_+, \chi] - \frac{\lambda}{4} \int d^4x (\phi_+(x) \phi_\chi(x))^2. \]  

(53)

All terms omitted in the expansion [28,29] do not contribute to the one-loop calculations for the long-wavelength modes that we shall now consider. \(^1\)

\(^1\)Strictly speaking, for \( \mu/3 < k_0 < \mu \) one should include an additional term proportional to \( \phi_\chi^2 \phi_\chi \) in the interaction Lagrangian. See Ref. [28] for details.
The net consequence of the separation of the long-wavelength system modes $\phi_<$ of $\phi$ from the short-wavelength modes $\phi_>$ of the environment through the interaction action $S_{\text{couple}}$ of (53) gives an additional one-loop contribution $D_{\phi}(k_0, t)$ to the diffusion function with the same $u(s, t)$ and the same form as in (25). However, $G_{\pm\pm}$ is now constructed from the short-wavelength modes $\phi_>$ of the $\phi$-field as it evolves from the top of the potential hill. Without the additional powers of $N^{-1}$ to order contributions the one-loop calculation is unreliable. In fact, we would not expect the inclusion of the $\phi$-field to give a qualitative change at one-loop level. In the first instance, the approximation (28) remains valid, and the 1-loop term is driven by the exponential growth of $u(0, t)$. Secondly, the effect is that the short-wavelength modes have been kept at the initial temperature $T_0$, on the grounds that passing through the transition quickly has no effect on them. That is, with $g \simeq \lambda$ and no $1/N$ factor, the short-wavelength $\phi$ modes give a contribution comparable, qualitatively, to all the explicit environmental fields put together. At an order of magnitude level there is no change, since the effect is to replace $g^2$ by $g^2 + O(\lambda^2) = O(g^2)$.

However, since the contribution of $D_{\phi}(k_0, t)$ to the overall diffusion function is positive we can derive an upper bound on the decoherence time $t_D$ from the reliable diffusion functions $D_{\text{qu}}(k_0, t)$, $D_{\text{bilin}}(k_0, t)$ (and $D_{\text{lin}}(k_0, t)$). Let us suppose we have a theory where the three considered couplings with the environment are present. Eq.(21) will be satisfied because one of its terms will grow faster than the others, rather than because many terms will each give a small fraction of unity. Specifically, we have (up to numerical factors $O(1)$)

$$
\Gamma_{\text{qu}} \int_0^{t_D} ds \ D_{\text{qu}}(k_0, s) : \Gamma_{\text{bilin}} \int_0^{t_D} ds \ D_{\text{bilin}}(k_0, s) : \Gamma_{\text{lin}} \int_0^{t_D} ds \ D_{\text{lin}}(k_0, s) 
\sim g^2 g' \left( \frac{\mu}{T_c} \right) : g'' \left( \frac{\mu}{T_c} \right) \exp \left\{ \frac{2}{3} \left( \frac{\Delta_0(t_D)}{t} \right)^2 \right\},
$$

(54)

Since $T_c \gg \mu$ and $\delta \approx O(1)$ then, for $\mu t_D \gg 1$, we have

$$
\Gamma_{\text{qu}} \int_0^{t_D} ds \ D_{\text{qu}}(k_0, s) \gg \Gamma_{\text{bilin}} \int_0^{t_D} ds \ D_{\text{bilin}}(k_0, s) \gg \Gamma_{\text{lin}} \int_0^{t_D} ds \ D_{\text{lin}}(k_0, s),
$$

(55)

if $g, g'$ and $g''$ are comparable. Thus, it is sufficient to evaluate the constraint on the time $t_D$ from this biquadratic interaction alone.

V. FINAL REMARKS

We first summarize the results contained in this paper, in which we have shown how the environment leads to the decoherence of the order parameter after a transition. After the integration over the scalar environment-fields $\chi_a$ in Section II, we have obtained the coarse-grained effective action (CGEA) for the system (field $\phi$). From the imaginary part of the CGEA we obtained the diffusion coefficient of the master equation, at 1-loop and in the high temperature environment limit. Terms omitted are relatively $O(N^{-1/2})$ for $N$ weakly coupled environmental fields. Subsequently, we evaluated the decoherence time for the long-wavelength modes of the system-field (for each of the different couplings with the environment considered) for slow quenches. This decoherence time depends on the coupling between system and bath, the self-coupling of the system (through the environment temperature $T$), and the mass. In our model, we have shown that the decoherence time is in general smaller than the spinodal time.

We stress that the inequality $t_D < t_{\text{sp}}$ is insensitive to the strength of the couplings, for weak coupling. The reason is twofold. Firstly, there is the effect that $\Gamma \propto T_0^2$, and $T_0^2 \propto \lambda^{-1}$ is non-perturbatively large for a phase transition. Secondly, because of the non-linear coupling to the environment, obligatory for quantum field theory, $\Gamma \propto \tilde{\phi}^2$. The completion of the transition finds $\tilde{\phi}^2 \approx \eta^2 \propto \lambda^{-1}$ also non-perturbatively large. This suggests that $\Gamma$, and hence $t_D$, can be independent of $\lambda$. In fact, the situation is a little more complicated, but the inequality holds.

This arises because the diffusion coefficients, which trigger classical behaviour when they become large enough, are controlled by the exponential growth of the unstable modes. It is this same exponential growth that determines $t_{\text{sp}}$. This result provides a post-hoc justification of our assertion [36,38] that the spinodal time sets the scale for the onset of classical behaviour (in open systems).

Our emphasis has been on the many weak environments because of the control that this gives us on establishing a robust upper bound on $t_D$. However, we noted earlier that their total contribution was qualitatively that of the short-wavelength modes of the $\phi$ field alone. Environmental fields are an important feature of the early universe, but even had we not included them we would have expected a similar result from the one-loop couplings of short-wavelength to long-wavelength $\phi$ modes. Although, in this case, we have no way to control the higher loop terms, it is quite probable that our prescription of the environments is unnecessarily detailed, and early decoherence is a general feature.
Ideally, we could extend our ideas to gauge theories. This is beyond the scope of this paper, but we would like to emphasize an important point. As diffusion is additive (each extra term in the interaction action gives additional diffusion coefficients to the master equation), the inclusion of further couplings and fields can only reduce $t_D$. Further, the coupling of short to long-wavelength modes of the global theory is omnipresent. However, theories with derivative couplings (scalar QED, for example) tend to produce small additional diffusive effects at low $k$ (scaled by $k$). It is for this reason that, if scalar fields are present, there is no need include gauge field interactions in order to estimate an upper bound to the decoherence time.

The results obtained in this paper also justify in part the use of phenomenological stochastic equations to describe the dynamical evolution of the system field, as we will now discuss. As it is well known [28,29], for $\phi^2\chi^n$ interactions one can regard the imaginary part of $\delta A$ as coming from a noise source $\xi_\alpha(x)$, with a Gaussian functional probability distribution. Taking the biquadratic coupling to the external environment $\chi$-field first leads to a noise, termed $\xi_2$, say, with distribution

$$P[\xi_2(x)] = N_{\xi_2} \exp \left\{ -\frac{1}{2} \int d^4 x \int d^4 y \, \xi_2(x) \left[ g^2 N_q \right]^{-1} \xi_2(y) \right\},$$

where $N_{\xi_2}$ is a normalization factor. This enables us to write the imaginary part of the influence action as a functional integral over the Gaussian field $\xi_2(x)$

$$\int D\xi_2(x) P[\xi_2] \exp \left\{ -i \left\{ \Delta_2(x) \xi_2(x) \right\} \right\} = \exp \left\{ -i \int d^4 x \int d^4 y \left[ \Delta_2(x) \, g^2 N_q(x,y) \, \Delta_2(y) \right] \right\},$$

where

$$A[\phi^+, \phi^-] = -\frac{1}{i} \ln \int D\xi_2 P[\xi_2] \exp \left\{ i S_{\text{eff}}[\phi^+, \phi^-, \xi_2] \right\},$$

where

$$S_{\text{eff}}[\phi^+, \phi^-, \xi_2] = \text{Re} A[\phi^+, \phi^-] - \int d^4 x \left[ \Delta_2(x) \xi_2(x) \right].$$

Therefore, taking the functional variation

$$\frac{\delta S_{\text{eff}}[\phi^+, \phi^-, \xi_2]}{\delta \phi^+} \bigg|_{\phi^+ = \phi^-} = 0,$$

we are able to obtain the "semiclassical-Langevin" equation for the system-field [28,29]

$$\Box \phi(x) - \mu^2 \phi + \frac{\Lambda}{6} \delta^3(x) \phi^3(x) + g^2 \phi(x) \int d^4 y \, K_q(x,y) \phi^2(y) = \phi(x) \xi_2(x),$$

where $\mu$ and $\Lambda$ are constants "renormalized" because of the coupling with the environment. Since (61) is, from (60), the stationary phase approximation, it is only valid once the system has become classical. It can be used to establish domain formation only because $t_D < t_{sp}$.

Each part of the environment that we include leads to a further 'dissipative' term on the left hand side of (61) with a countervailing noise term on the right hand side. Once we include the interactions between $\phi_<$ and $\phi_>$ at one-loop level we get a similar equation to (61) from the term $S_{\text{couple}}$ alone (see Eq.(53)). However, although the $\phi_<\phi_3^<$ and $\phi_3^<\phi_>$ terms were ignorable in the bounding of $t_D$, in the Langevin equations they give further terms, with quadratic $\phi^2\chi_3$ noise and linear noise $\xi_1$, respectively.

The inclusion of bilinear interactions leads to the inclusion of further terms in (61) of the same form. For the linear interaction with the environment (to the exclusion of self-interaction) we do recover the additive noise that has been the basis for stochastic equations in relativistic field theory that confirm the scaling behaviour of Kibble’s and Zurek’s analysis [25].

For times later than $t_{sp}$, neither perturbation theory nor more general non-Gaussian methods are valid. It is difficult to imagine an ab initio derivation of the dissipative and noise terms from the full quantum field theory. In this sense, a reasonable alternative is to analyze phenomenological stochastic equations numerically and check the robustness of the predictions against different choices of the dissipative kernels and of the type of noise. We stress that this is only possible because $t_D < t_{sp}$. This will be considered further elsewhere [39].

Finally, we see that the role initially attributed by Kibble [5,23] (and subsequently by others e.g. [40]) to the Ginzburg regime is just not present.
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APPENDIX A: THE SPINODAL TIME

In this Section, we show the estimation of the spinodal time $t_{sp}$ defined from $\langle \phi^2 \rangle_t = t_{sp} \sim \eta^2$.

The equation of motion for the mode $U_k(t)$, with wavenumber $k$ is, in the quench period,

$$\left[ \frac{d^2}{ds^2} + k^2 + \mu^2 - \frac{\mu^2 s}{\tau_q} \right] U_k(t) = 0,$$

subject to the boundary condition $U_k(t) = e^{-i\omega t}$ for $t \leq 0$, where $\omega^2 = \mu^2 + k^2$.

Instead of the simple exponentials of the instantaneous quench, $U_k(t)$ has solution

$$U_k(t) = a_k Ai\left(\frac{\Delta_k(t)}{t}\right) + b_k Bi\left(\frac{\Delta_k(t)}{t}\right),$$

with $Ai[s], Bi[s]$ the Airy functions; $\Delta_k(t) = t - \omega^2 l^3$ and $l = (\tau_q/\mu^2)^{1/3}$. Note that $\Delta_0(t) = t - \tau_Q$, the time since the onset of the transition. In the causal analysis of Kibble [5] $\tau_Q$ is the time at which the adiabatic field correlation length collapses at the speed of light, the earliest time in which domains could have formed. Our analysis suggests that this earliest time is not $\bar{t}$, but $t_{sp}$.

It is straightforward to establish a relationship between $\bar{t}$ and $t_{sp} > \bar{t}$. The constants of integration in (A2) are

$$a_k = \pi [Bi(-\omega^2 \bar{t}^2) + i \omega \bar{t} Ai(-\omega^2 \bar{t}^2)],$$

$$b_k = -\pi [Ai(-\omega^2 \bar{t}^2) + i \omega \bar{t} Ai(-\omega^2 \bar{t}^2)].$$

It follows that, when $\Delta_k(t)/\bar{t}$ is large, then

$$|U_k(t)|^2 \approx \omega \bar{t} \left(\frac{\bar{t}}{\Delta_k(t)}\right)^{1/2} \exp \left[\frac{4}{3} \left(\frac{\Delta_k(t)}{\bar{t}}\right)^{3/2}\right] \approx \mu \bar{t} \left(\frac{\bar{t}}{\Delta_0(t)}\right)^{1/2} \exp \left[\frac{4}{3} \left(\frac{\Delta_0(t)}{\bar{t}}\right)^{3/2}\right] e^{-k^2/\bar{k}^2},$$

where $\bar{k}^2 = \bar{t}^{-3/2}(\Delta_0(t))^{-1/2}/2$.

For large initial temperature $T_0 = O(T_c)$, we find the power spectrum for field fluctuations peaked around $\bar{k}$, and

$$\langle \phi^2 \rangle_t \approx \frac{T_0}{2\pi^2 \mu^2} \int k^2 dk |U_k(t)|^2 \approx \frac{CT_0}{\mu^2} \left(\frac{\Delta_0(t)}{\bar{t}}\right)^{-5/4} \exp \left[\frac{4}{3} \left(\frac{\Delta_0(t)}{\bar{t}}\right)^{3/2}\right].$$

We have intentionally included the prefactor $C$ to show that terms, nominally $O(1)$, can in fact be large or small (in this case $C = (64\sqrt{2}\pi^{3/2})^{-1} = O(10^{-3})$). Note that, although the unstable modes have a limited range of $k$-values, increasing in time, this is effectively no restriction when $\Delta_0(t)/\bar{t}$ is significantly larger than unity.

Finally, we obtain

$$\frac{\eta^2}{C'} \approx \frac{T_0}{\mu^2} \exp \left[\frac{4}{3} \left(\frac{\Delta_0(t_{sp})}{t_{sp}}\right)^{3/2}\right],$$

where $C' = C[\ln(\mu^2 \eta^2/CT_c)^{-5/6}]$. Since the effect on $t_{sp}$ only arises at the level of ‘$\ln \ln$’ terms, $C' \approx C$ is a good estimation in all that follows. Since this choice underestimates $t_{sp}$, it only strengthens our results that $t_{sp} > T_D$. 

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APPENDIX B: THE DIFFUSION COEFFICIENT

In this Appendix we show how to obtain the diffusion coefficient Eq.(25) (for the quadratic coupling) from the reduced propagator. Following the same techniques used for the quantum Brownian motion to obtain the master equation we must compute the time derivative of the propagator $J_r$ (Eq.(14)), and eliminate the dependence on the initial field configurations $\phi^{i}\pm$ that enters through the classical solutions $\phi_{cl}^{i}$. This can be easily done using the propagator identities of Eq.(18). Thus, using Eq.(16) and identities (18), we can write the initial field configuration $\tilde{\phi}_i^\pm$, in terms of the final configurations ($\phi_f^{\pm}$), final field derivatives ($\partial_{\phi_f^{\pm}}$), and the functions of time $u_i(s,t)$,

$$ (\phi_i^+ - \phi_i^-) \ u_1(s,t) \ J_0 = -\frac{\dot{u}_2(t,t)}{u_1(t,t)} \ u_1(s,t) \ (\phi_i^+ - \phi_i^-) \ J_0 + ... , \quad (B1) $$

neglecting terms proportional to derivatives respect to the final field configuration which do not contribute to normal diffusion.

The temporal derivative is given by

$$ i\hbar \partial_t J_r[\phi_i^+, \phi_i^-, t|\phi_i^+, \phi_i^-; 0] = \left\{ h_{ren}[\phi_i^+] - h_{ren}[\phi_i^-] - ig^2 (\phi_i^{2+} - \phi_i^{2-}) V \int_0^t ds \, \Delta_2^r(s) \, F(k_0; s, t) \right. $$

$$ + g^2 (\phi_i^{2+} + \phi_i^{2-}) V \int_0^t ds \, \Delta_2^r(s) \, \tilde{K}_q(k_0; s, t) + ... \right\} J_r[\phi_i^+, \phi_i^-, t|\phi_i^+, \phi_i^-; 0], \quad (B2) $$

where $\tilde{K}_q(k_0; s, t)$ is the Fourier transform of the dissipation kernel. The ellipsis denotes other terms coming from the time derivative which do not contribute to diffusion.

Diffusive effects are associated with terms proportional to $(\phi_i^{2+} - \phi_i^{2-})^2$ in the master equation. Using Eq.(B1) we remove initial conditions from Eq.(B2), and looking only those terms proportional to $\Delta_2^r$, we get the master equation Eq.(19). The diffusion coefficient $D(k_0, t)$ comes from the noise contribution to the influence functional and it is the time dependent coefficient that multiplies $\Delta_2^r$. Thus, for the quadratic coupling example, we find

$$ D_{qu}(k_0, t) = \int_0^t ds \, u(s, t) \, F(k_0; s, t), \quad (B3) $$

where

$$ u(s, t) = \left[ u_2(s, t) - \frac{\dot{u}_2(t,t)}{u_1(t,t)} u_1(s, t) \right]^2. $$

For different couplings between system and environment one should follow the same procedure shown in this Appendix.

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