Classical Aspects of the Abelian Higgs Model on the Light Front

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Abstract

We investigate canonical structure of the Abelian Higgs model within the framework of DLCQ. Careful boundary analysis of differential equations, such as the Euler-Lagrange equations, leads us to a novel situation where the canonical structure changes in a drastic manner depending on whether the (light-front) spatial Wilson line is periodic or not. In the former case, the gauge-field ZM takes discrete values and we obtain so-called “Zero-Mode Constraints” (ZMCs), whose semiclassical solutions give a nonzero vev to the scalar fields. Contrary, in the latter case, we have no ZMC and the scalar ZMs remain dynamical as well as the gauge-field ZM. In order to give classically the nonzero vev to the scalar field, we work in a background field which minimizes the light-front energy.

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1 Introduction

Light-front (LF) quantization [1] and particularly the method of Discretized Light-Cone Quantization (DLCQ) [2] have a great impact not only upon nonperturbative study in field theories but also upon the M-theory as Matrix model in string theory [3]. A desirable property that the vacuum is simple or trivial enables us to calculate mass spectra of bound states as well as their wave functions using various nonperturbative techniques. Such a simplification, however, is allowed only if we neglect the longitudinal zero momentum modes (simply abbreviated as ZMs) of fields. This subtlety was first discussed seriously by Maskawa and Yamawaki [4] by setting the longitudinal direction $x^−$ finite, which is the basic strategy of DLCQ, with periodic boundary condition. The longitudinal momenta of fields are discretized and the ZMs can be safely treated. They found that the ZMs of scalar fields are not dynamical and in fact subject to constraint relations called the Zero-Mode Constraints (ZMCs). After considerable study, it is now widely believed that nonperturbative treatment of the ZMCs is crucial for describing the spontaneous symmetry breaking (SSB) on the LF [3].

Compared to the scalar ZM, our understanding of the gauge-field ZM is still insufficient though it might play an important role in 3+1 dimensional QCD. While its relevance for the topological structure such as the $\theta$-vacua, has been discussed in simple field theories [3], more complicated situation where the topological effect coexists with SSB has never been investigated. It can be best studied in the 1+1 dimensional Abelian Higgs model:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu \phi|^2 - V(\phi),$$  
$$V(\phi) = \frac{\lambda}{4} \left( \phi^* \phi - v^2 \right)^2,$$  

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative and $v^2 > 0$. This is a very interesting model entangled with both “Higgs mechanism” and “instanton” associated to $\pi_1(S^1) = \mathbb{Z}$. Indeed the effect of instantons dramatically changes a naive perturbative picture: confinement of fractionally charged particles occurs even in broken phase [3]. So it is an intriguing question how the interplay of these two effects can be observed on the LF.

As the first step to attack this problem, we study in this letter classical aspects of the Abelian Higgs model in general dimensions. Within the Hamiltonian formulation, the appearance of the topological effect in 1+1 dimensions cannot be discussed until we quantize the system. Even at the classical level, however, we find a remarkable property
of the model in any dimension which has never been discussed.

We work within a standard DLCQ method, i.e., the LF space $x^- = (x^0 - x^1)/\sqrt{2}$ is made compact $x^- \in [-L, L]$ and we impose periodic boundary conditions for all fields. The reason why we treat the periodic boundary conditions is that our standard knowledge in the LF formalism tells us that ZMs of scalar and gauge field are expected to be closely related to SSB and topological structure, respectively.

The content is as follows. In the next section, we introduce the notion of the spatial Wilson line using the Euler-Lagrange (EL) equation. The canonical structure of the model crucially depends on the periodicity of the Wilson line. The periodic and non-periodic cases will be separately discussed in Secs. 3 and 4, respectively. The last section is devoted to discussions.

2 Dilemma in Canonical Structure

Consider the EL equation for the Higgs field:

$$D_+ D_- \phi + D_- D_+ \phi - D_i D_i \phi + \frac{\partial V}{\partial \phi^*} = 0.$$  \hfill (2.1)

The fastest way to derive ZMC in a (non-gauged) scalar theory is just integrate the EL equation over $x^-$. In our case, however, naive integration of (2.1) does not yield constraint relation because of the covariant derivative $D_-$. In order to replace $D_-$ by the ordinary derivative $\partial_-$, let us introduce the (LF) spatial Wilson line

$$W(x^+, x^-, x_\perp) \equiv \exp\left\{i e \int_{-L}^{x^-} dy^- A_-(x^+, y^-, x_\perp)\right\}, \hfill (2.2)$$

where $x_\perp$ denotes the transverse directions. Then we have

$$\partial_- [2W^{-1}D_+ \phi] = W^{-1}\left[D_i D_i \phi + i e \phi \Pi^- - \frac{\partial V}{\partial \phi^*}\right], \hfill (2.3)$$

where we used a formula $D_- f = W \partial_-(W^{-1} f)$ and $\Pi^- = F_{+\perp} = \partial_+ A_- - \partial_- A_+$ is the conjugate momenta of $A_-$. Note that the integration of the left-hand-side does not necessarily vanish because the Wilson line is not periodic in general. Therefore we find that only if the Wilson line is periodic $W(-L) = W(L)$, the space integration of (2.3) generates a constraint analogous to the usual ZMC:

$$\int_{-L}^{L} dx^- W^{-1}\left[D_i D_i \phi + i e \phi \Pi^- - \frac{\partial V}{\partial \phi^*}\right] = 0.$$  \hfill (2.4)
A few comments are in order. First of all, there is no problem in regarding this equation
as a constraint though it includes the time derivative of $A_-$ through $\Pi^-$. This is because
(gauge-fixed) $A_-$ giving periodic Wilson line is restricted to discrete values. More details
are in the next section.

Next, this somewhat heuristic observation suggests that the canonical structure itself
depends on the periodicity of the Wilson line. In fact, the necessity to distinguish these
two cases is required when we try to determine Lagrange multipliers in the usual canonical
procedure. The consistency condition of a Lagrange multiplier $\lambda$ for a (primary) constraint
$\theta = \Pi_{\phi^*} - D_- \phi$ is

$$D_- \lambda = \frac{1}{2} \left( D_i D_i \phi + i e \phi \Pi^- - \frac{\partial V}{\partial \phi^*} \right) \equiv K. \tag{2.5}$$

Ambiguity in the Lagrange multiplier is intimately related to a new constraint. Therefore,
it is essential to obtain a general solution to this differential equation. Indeed, the counter-
part in the usual scalar theory is just a differential equation $2 \partial_- \lambda = \partial_i \partial_i \phi - \partial V / \partial \phi$ and its general solution subject to $\lambda(-L) = \lambda(L)$ allows a “zero mode” to be undetermined, which implies the existence of the ZMC [4]. The general solution to eq.(2.5) without boundary consideration for $\lambda$ is given by

$$\lambda(x) = W(x) \int_{-L}^x dy^- W^{-1}(y) K(y) + C W(x), \tag{2.6}$$

where $C$ is an integral constant. The second term comes from a homogeneous equation
$D_- \lambda = 0$. Imposing further $\lambda(-L) = \lambda(L)$, a natural requirement from the periodicity of the Hamiltonian, we must distinguish two cases depending on the periodicity of $W(x)$. In the case where $W(x)$ is not periodic, i.e., $W(L) \neq W(-L)$, we can completely determine $\lambda$ from the requirement $\lambda(-L) = \lambda(L)$. So there is no residual constraint in this case and therefore both the scalar and gauge ZMs are dynamical. On the other hand, in the case of the periodic Wilson line, i.e., $W(L) = W(-L) = 1$, the integral constant $C$ leaves undetermined and we have the following secondary constraint

$$\int_{-L}^L dx^- W^{-1}(x) K(x) = 0. \tag{2.7}$$

This is the same as eq.(2.4). The canonical structure of the model changes drastically
depending on the periodicity of the Wilson line. As is evident from these arguments,
this somewhat strange situation is directly related to a problem whether we can define
the inverse of the differential operator $D_-$ and, in other words, whether $D_-$ has zero
eigen-value or not.
The final comment is on physics. From our conventional knowledge in the LF formalism, we naively expect that ZMC for scalars may be necessary for describing the Higgs mechanism whereas the (dynamical) gauge-field ZM will be responsible for nontrivial vacuum structure in 1+1 dimensions. Nevertheless, as we saw, ZMC does not necessarily exist and it rather emerges in a very special case with measure zero in the phase space. When the ZMC exists, the gauge-field ZM is restricted to discrete values. On the other hand, the more general case allows gauge-field ZM to be fully dynamical but there is no ZMC and now we cannot follow the conventional discussion to describe the symmetry breaking. So we are in a very dilemmatic situation.

In the following sections, we discuss these two cases separately and restrict our consideration to the 1+1 dimensional case for brevity. We can do that without loss of generality.

3 Periodic Wilson line and the Zero-Mode Constraint

Let us discuss an appropriate decomposition of the scalar field. The existence of ZMC does not necessarily imply that the naive decomposition of scalars into zero and non-zero modes is a good one. To see this, we first perform the gauge fixing. Since the Light-Cone axial gauge in compact space inevitably misses the ZM of $A_-$, i.e., \( \hat A_- = (1/2L) \int_L^L dx^- A_- (x) \) remains unfixed, zero-mode separation of $A_-$ has a physics meaning. After gauge fixing, requiring periodicity to the Wilson line restricts \( \hat A_- \) to be discrete value

\[
\hat A_- = \frac{\pi n}{eL}, \quad n \in \mathbb{Z}.
\]  

(3.1)

Since the nonZM $\tilde A_-$ is fixed to be zero in the LC gauge and $\partial_+ \tilde A_- = 0$, eq.(2.4) does not include time derivative and thus can be understood as a constraint relation. Now the periodic Wilson line becomes $W(x) = e^{i\pi n(x^- + L)/L}$. This means that the ambiguous mode of the Lagrange multiplier is not a zero mode but a mode with frequency $\pi n/L$ (see eq.(2.6)), and also that a scalar field with frequency $\pi n/L$ should be taken as a constrained variable. It will be convenient to redefine the scalar field so that the zero mode of a new variable becomes constrained:

\[
\phi_n(x) \equiv e^{-i\frac{\pi n}{L} x^-} \phi(x).
\]  

(3.2)

More strictly, you can impose $W(L) = 1$ as a constraint in this case. Then, from the consistency condition of the constraint, the secondary constraint $\Pi^- = \partial_+ \hat A_- = 0$ is derived.
New field also satisfies the periodic boundary condition and this change of variable is essentially equivalent to the large gauge transformation. The ZMC (2.4) is rewritten in terms of $\phi_n$ as
\[
\int_{-L}^{L} dx^- \left( ie\phi_n \tilde{\Pi}^- - \frac{\partial V}{\partial \phi_n^*} \right) = 0. \tag{3.3}
\]
Explicit form of $\tilde{\Pi}^-$ is given by solving the Gauss law $\partial_- \Pi^- = ie[(D_- \phi^*)^* \phi - \phi^* D_- \phi]$
\[
\tilde{\Pi}^- = \int_{-L}^{L} dy^- \tilde{\theta}(x^- - y^-)ie(\partial_- \phi_n^* \phi_n - \phi_n^* \partial_- \phi_n), \tag{3.4}
\]
where $\tilde{\theta}(x^- - y^-)$ is the periodic step function $\partial_x \tilde{\theta}(x^- - y^-) = \delta(x^- - y^-) - 1/2L$. Equation (3.3) can be understood as a constraint relation for the zero mode of the new variable and the decomposition $\phi_n = \hat{\phi}_n + \tilde{\phi}_n$ becomes useful. That is why we called eq.(3.3) the ZMC from the beginning. Now all the ZMs ($\hat{\phi}_n$, $\hat{\phi}_n^*$ and $\hat{A}_-$) can be treated as non-dynamical. Following the standard procedure, we obtain the Dirac brackets between physical variables:
\[
\{ \tilde{\phi}_n(x), \tilde{\phi}_n^*(y) \}_{DB} = -\frac{1}{4} \epsilon(x^- - y^-), \tag{3.5}
\]
\[
\{ \tilde{\phi}_n(x), \tilde{\phi}_n(y) \}_{DB} = \{ \tilde{\phi}_n^*(x), \tilde{\phi}_n^*(y) \}_{DB} = 0. \tag{3.6}
\]
We can easily go to quantum theory by replacing these by commutators. Let us evaluate eq.(3.3) in a semiclassical treatment ($\hbar$-expansion). In the lowest order, any ZM should not have operator part. Since operator contribution will come from the nonzero modes, we here assume that the lowest (classical) ZM does not depend on the nonzero mode $\tilde{\phi}_n$. So in this simple approximation, eq.(3.3) becomes just
\[
\left( \hat{\phi}_n^* \hat{\phi}_n - v^2 \right) \hat{\phi}_n = 0, \tag{3.7}
\]
and the solution is
\[
\hat{\phi}_n = 0, \quad v e^{i\alpha}, \tag{3.8}
\]
where $\hat{\phi}_n$ denotes the classical part of $\hat{\phi}_n$ and $\alpha$ is an arbitrary constant. Rewritten in terms of the original variable, the solution $\hat{\phi}_n = ve^{i\alpha}$ corresponds to nonzero mode. Nevertheless, it can be made into a zero mode by the large gauge transformation and thus gives nonzero vacuum expectation value. This is equivalent to evaluating the ZMC with $n = 0$ from the beginning. In addition, the Hamiltonian has no $n$-dependence after we insert the solution $\hat{\phi}_n$. 

5
4 Non-Periodic Wilson line

In this case, there is no ZMC and thus all the ZMs should be taken as dynamical variables. This means that we cannot utilize the method of solving ZMC to describe SSB. The situation here is rather similar to the usual equal-time calculation: we do not have to have recourse to such a special method restricted to the LF formalism. Since this model shows SSB in the tree (classical) level, we should be able to describe it in a classical treatment. This is possible if we proceed analogously to the background field method. Let us first look for a configuration (in the LC axial gauge) which minimizes the LF energy

\[ P^- = \int_{-L}^{L} dx^- \left[ \frac{1}{2} (\Pi^-)^2 + V(\phi) \right] \geq 0. \] (4.1)

Note that we further need to impose the Gauss law, unlike the equal-time calculation. The LF energy becomes zero if and only if field configuration is given by

\[ \circ A^- = \frac{\pi N}{eL}, \quad N \in \mathbb{Z}, \] (4.2)
\[ \circ \phi = v e^{i(\frac{\pi N}{L} x^- + \alpha)}, \] (4.3)

where \( \alpha \) is an arbitrary constant and can be set to be zero. This configuration is exactly the same as the classical solution of eq.(2.4) in the periodic Wilson line case. Therefore we can give nonzero vev to the scalar field even without ZMC if we expand the fields around this classical energy-minimized configuration. Note also that this configuration gives \( P^+ = 0 \) and therefore is equivalent to the configuration giving zero equal-time energy.

We consider the canonical structure in the energy-minimizing background field. Let us introduce \( V_\pm \) and \( \varphi_N \) by

\[ A_+ = V_- , \] (4.4)
\[ A_- = \frac{\pi N}{eL} + V_-, \] (4.5)
\[ \varphi_N = v + \varphi_N . \] (4.6)

We again introduced \( \varphi_N = e^{-i\frac{\pi N}{L} x^-} \phi \) for convenience. Then we decompose fields into zero and nonzero modes \( V_\pm = \hat{V}_\pm + \tilde{V}_\pm \) and \( \varphi_N = \hat{\varphi}_N + \tilde{\varphi}_N \). To avoid the singularity \( \hat{V}_- = 0 \) which gives the periodic Wilson line, it should be \( 0 < \hat{V}_- < \pi/eL \). This restriction with a given \( N \) means that we work in a fixed sector with respect to the large gauge transformation.
After lengthy but straightforward calculations, the Dirac brackets between dynamical fields are obtained (in the LC axial gauge: $\hat{V}_+ = 0$, $\hat{V}_- = 0$ and $\hat{\Pi}^- + \partial_- \hat{V}_+ = 0$) as follows:

$$\{\hat{V}_-, \hat{\Pi}^-\}_{\text{DB}} = \frac{1}{2L},$$
$$\{\hat{\varphi}_N, \hat{\varphi}_N^*\}_{\text{DB}} = \frac{1}{2L} \frac{1}{2ie} \hat{V}_-,$$
$$\{\hat{\Pi}^-, \hat{\varphi}_N\}_{\text{DB}} = \frac{1}{4L} \hat{\varphi}_N (v + \hat{\varphi}_N),$$
$$\{\hat{\Pi}^-, \hat{\varphi}_N(x)\}_{\text{DB}} = \frac{1}{2L} \int_{-L}^{L} dy^- i e \hat{\varphi}_N(y) \frac{1}{2} G^*(y, x),$$
$$\{\hat{\varphi}_N(x), \hat{\varphi}_N^*(y)\}_{\text{DB}} = -\frac{1}{2} G(x, y).$$

where $\hat{\Pi}^- = \partial_+ \hat{V}_-$ is a conjugate momenta of $\hat{V}_-$ and $G(x, y)$ is Green’s function defined through $(\partial^- - ie \hat{V}_-) G(x, y) = \delta(x^- - y^-) - 1/2L$. From this, it is evident that the number of ZM degrees of freedom is two ($\hat{V}_-$ and $\hat{\varphi}_N$). In quantum theory, these ZMs should be responsible for a nontrivial vacuum structure such as the $\theta$-vacua [10].

5 Discussions

In this letter we introduced a notion of the periodicity of the Wilson line and found that the canonical structure of the ZM sector changes drastically depending on its periodicity. In almost all the phase space except countable points, both the scalar and gauge-field ZM are dynamical. At the exceptional points, the gauge-field ZM takes discrete values and the scalar ZM becomes constrained. We should mention here that this kind of canonical structure has been already pointed out in the 3+1 dimensional $SU(2)$ Yang-Mills theory [1]. At the points where $\hat{A}_-^a = \delta^{a3n\pi}/gL$, there exist two constraint relations:

$$\int_{-L}^{L} dx^- e^{\frac{i \pi n}{4a} x^-} \left(G^1 + iG^2\right) = 0,$$

where $G^a = (D_i F^-_i)^a$ and $(D_i F^+_{+i} + D_j F^j_{ji})^a \ (a = 1, 2)$. These are also originated from the zero eigenvalue problem of the covariant derivative $D_-$. Therefore the same kind of constraint will exist in any gauge theory. In Ref. [1], quantum theory in the case with constraints (5.1) was not developed due to its complexity and also the physics consequences of these constraints are still not clear. In our model, however, the ZMC has a significant
meaning that we could give a nonzero vev to the scalar by solving it semiclassically. We expect that the constraints (5.1) also will give some nontrivial effects on the theory.

One of the most important questions is whether the physics in these two cases, which are canonically distinct from each other, is continuously connected. Its complete solution is not yet given at present. However, as for the symmetry breaking in this model, it should be discussed even in the classical theory. So we managed to construct a classically breaking theory for both cases. In the general case (nonperiodic Wilson line), we could express the symmetry breaking by expanding fields around energy-minimizing configuration. This procedure may correspond to evaluating the effective potential \textit{classically}. With dynamical zero modes in this case, we will be able to evaluate the vacuum energy as in the equal-time calculation. It should be checked after quantization whether our background field is consistent or not.

One more necessary ingredient, the $\theta$-vacua, should also be discussed after quantization. This is because in the Hamiltonian formulation, topological effects such as instantons will be observed as the “quantum tunneling” between multiple vacua. The multiplicity of the vacuum is generated by the large gauge transformation which is a displacement symmetry for the gauge-field $ZM$. We expect that thorough treatment of this symmetry with appropriate continuum limit $L \to \infty$ will lead to a separation of the 1+1 dimensional model from the higher dimensional models, where there should be no topological effects.

All these will be discussed in the future work [10].

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