Chapter 1

**CHARGE INHOMOGENEITIES IN STRONGLY CORRELATED SYSTEMS**

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**Abstract**

We review the problem of stripe states in strongly correlated systems, and some of the theoretical, numerical and experimental methods used in the last few years to understand these states. We compare these states to more traditional charge-ordered states such as charge density waves (CDW) and phase-separated systems. We focus on the origin of stripe states as an interplay between magnetic and kinetic energy, and argue that the stripe state is generated via a mechanism of kinetic energy release that can be described via strongly correlated models such as the t-J model. We also discuss phenomenological models of stripes, and their relevance for magnetism and for the pinning of stripes by the underlying lattice and by disordered impurities. Recent experimental evidence for the existence of stripe states in different cuprate systems is also reviewed.

**Keywords:** charge inhomogeneities, strongly correlated electronic systems, charge-density-wave states, striped phase, doped Mott insulators, high-$T_c$ superconductors, pattern formation in low-dimensional systems.
1. Introduction

The problem of strong electron correlations in transition-metal oxides and U and Ce intermetallics has been a subject of intense research in the last 20 years. This interest has been driven mostly by puzzling experimental findings in materials such as organic conductors and superconductors, heavy-fermion alloys, and high-temperature superconductors. These systems are characterized by large Coulomb interactions, low dimensionality, strong lattice coupling, and competition between different phases: antiferromagnetism, ferromagnetism, spin density waves (SDW), superconductivity, and charge density waves (CDW). The strong interplay between different order parameters is believed to lead to charge and spin inhomogeneities, and to a myriad of energy and length scales that makes the problem very difficult to treat with the methods and techniques used for the study of Fermi-liquid metals. Although the problem of the quantum critical behavior of metals in the proximity of an isolated zero-temperature phase transition has been subject of much study and heated debate [1], our understanding of the problem of electrons close to multiple phase transitions is still in its infancy. Here we will review both the experimental evidence for the existence of certain inhomogeneous states called "stripe" states and some of the current theoretical approaches used to understand their origin, nature, and importance in the context of magnetism and superconductivity [2].

The formation of static or dynamic spin-charge stripes in strongly correlated electronic systems has been corroborated recently by several experiments, especially in manganites [3, 4], nickelates [5-8], and cuprates [9-34]. The experiments span a large variety of techniques, from scanning tunneling microscopy (STM) [4,9-14], neutron (and x-ray) scattering [5,6,15-25], nuclear magnetic (and quadrupole) resonance (NMR and NQR) [7, 26, 27], muon spin rotation ($\mu$SR) [28, 29], optical and Raman spectroscopy [8, 30, 31], transport [32], angle-resolved photoemission (ARPES) [33], and ion channeling [34].

Charge and spin modulated states, such as CDWs, Wigner crystals, SDWs, antiferromagnetism, and ferrimagnetism are common occurrences in many transition-metal compounds. These systems are characterized by an order parameter (such as the charge and/or spin density) that is modulated with a well-defined wave vector $Q$. Because of the modulations and the coupling to the lattice, these states usually present lattice distortions which are easily observed in diffraction experiments such as neutron scattering. In this regard the stripe states discussed here are very much like CDW/SDW instabilities, except that in CDW systems the ordered state is driven by a Fermi-surface instability (usually generated by nesting and/or Van Hove singularities), and Coulomb effects are secondary because of good screening. The clearest example of such Fermi-surface effects occurs in Cr alloys, where the system undergoes a phase
transition into a CDW/SDW state [35]. The CDW and SDW transitions occur at the same temperature, and the charge order has a period that is 1/2 that of the SDW. The main difference between the phase transition in Cr and the stripe states to be discussed here is that the charge order in stripe systems occurs at higher temperature than the spin order [16]. Thus, on reducing the temperature the onset of charge order occurs first and the spins simply follow. In a weak-coupling analysis of Fermi-surface instabilities, this type of transition is not possible because of reconstruction of the Fermi surface due to the appearance of long-range order. Thus there are different energy scales for the charge and spin order that characterizes the materials discussed here.

We should stress that the CDW and/or SDW instabilities in metallic systems are not trivial, and although we understand the basic mechanisms which drive these instabilities [36] our knowledge of their origin and effects on the electronic degrees of freedom is far from complete. Systems such as transition-metal dichalcogenides [37] have a high temperature CDW transition with a very anomalous metallic phase and show in addition the phenomenon of "stripe formation" [38]. The stripes in these CDW systems are understood, however, because the CDW order is incommensurate with the lattice and therefore phase fluctuations of the CDW order parameter are allowed energetically. Local CDW phase-slips give rise to a filamentary stripe phase which, in fact, has a Fermi-surface origin.

To understand the origin of CDW stripes one may consider the complex order parameter $\Delta$ for a CDW with incommensurate ordering wave vector $Q$. The free energy of the problem may be expressed as [39]

$$F = F_0[\Re(\Delta)] + \int dr \left\{ \frac{1}{2m^*Q^2} \left[ \|Q \cdot (\nabla - iQ)\Delta\|^2 + \kappa \|Q \times \nabla \Delta\|^2 \right] \right\}$$  \hspace{1cm} \text{(1.1)}$$

where $F_0[x]$ is a minimal polynomial of $x$ that respects the symmetry of the lattice and renders the free energy bounded from below. For a triangular lattice, for instance, it can be written as [39]

$$F_0[\Re(\Delta)] = a(r, T)\Re(\Delta)^2 + b(r, T)\Re(\Delta)^3 + c(r, T)\Re(\Delta)^4,$$  \hspace{1cm} \text{(1.2)}$$

where the coefficients of the expansion are smooth functions of temperature. In particular for the quadratic term $a(r, T) = a_0(r)/(T - T_{ICDW})$ where $T_{ICDW}$ is the transition temperature of the incommensurate CDW (ICDW) state. In (1.1) $m^*$ and $\kappa$ are parameters specific to the material under consideration and the derivative terms are written such that the free energy of the CDW is minimal when the ordering wave vector lies in the correct direction and has the correct wave length. In the case of an ICDW the order parameter is obtained by minimizing (1.1) to give

$$\Delta_{ICDW}(r) = \Delta_{0,I}e^{iQ \cdot r},$$  \hspace{1cm} \text{(1.3)}$$
where $\Delta_{0,I} = \sqrt{2} a_0 (T_{ICDW} - T)/(3c_0)$ for $T < T_{ICDW}$ and zero otherwise. The parameter $c(r,T) = c_0 = constant$. Here we have assumed that the parameters in (1.2) may be expanded in a form such as $b(r) = b_0 + b_1 \exp\{iK_i \cdot r\}$ where $K_i$ are the shortest reciprocal-lattice vectors characteristic of the lattice symmetry.

In the commensurate CDW (CCDW) case the wave vector of the order parameter "locks" with the lattice so that its modulation becomes a fraction of the lattice wave vector $K_1$. Then one would replace $Q$ in (1.3) by $K_1/q$, where $q$ is an integer, and $\Delta_0$ by a value $\Delta_{0,C}$ which must be calculated from the free energy and depends in general on various coefficients of $F_0$ in (1.2). The transition temperature $T_{CCDW}$ is usually smaller than $T_{ICDW}$ so that the generical behavior of the system consists of two transitions, first into an incommensurate phase and then into a commensurate phase [39]. In many systems the ICDW-CCDW transition does not occur and the system remains incommensurate down to very low temperatures [37].

In order to study the problem of the commensurate-incommensurate transition, and the topological defects which appear due to incommensurability, one must generalize (1.3) to include phase fluctuations. These may be incorporated by writing the order parameter in the form

$$\Delta(r) = \Delta_0 e^{i\frac{2}{q} K_1 \cdot r + i\theta(r)},$$

(1.4)

where $\theta(r)$ is the angle variable which determines the commensurability of the system: for the ICDW $\theta(r) = (Q - K_1/q) \cdot r$, while for a CCDW $\theta = 0$. Because we are considering the simplest problem of a single CDW wave vector the problem becomes effectively one-dimensional if the variables are redefined in a new, rotated, rescaled, reference frame defined by $s = (x, y) = |Q - K_1/q| r$. In this case it is obvious that $\theta(x) = \theta(x)$, and by direct substitution of (1.1) we find that the dimensionless free energy per unit of length relative to the commensurate case becomes

$$\delta f = \int dx \frac{1}{2} \left\{ |\partial_x \theta(x) - 1|^2 + g [1 - \cos(q\theta)] \right\}$$

(1.5)

where $g$ is the coupling constant of the system and depends on the parameters of (1.1). The free energy in (1.5) describes a sine-Gordon model where the cosine term favors the commensurate state ($\theta(x) = 0$) while the gradient term favors the incommensurate state ($\theta(x) = x$). Thus $\theta$ is the order parameter and the discrete symmetry $\theta \rightarrow \theta + 2\pi/q$ is broken in the ordered phase. There is therefore a critical coupling value $g_c$ that separates these two phases. However, it is easy to see that there are other solutions which minimize the free energy. In fact, variation of (1.5) with respect to $\theta$ yields

$$\frac{d^2 \theta}{dx^2} = g q \sin(q\theta)$$

(1.6)
which has a particular solution

\[ \theta_K(x) = \frac{2}{q} \arctan \left( e^{\sqrt{g} x} / 2 \right), \]  

where the boundary conditions are \( \theta(x = -\infty) = d\theta(x = -\infty)/dx = 0 \). Notice that (1.7) changes smoothly from \( \theta = 0 \) at \( x = -\infty \) to \( \theta = 2\pi/q \) when \( x \gg 1/\sqrt{g} \). In the context of the sine-Gordon model this is called a topological soliton or kink, while in the CDW literature [39] it is called a discommensuration. In general, the solution of (1.6) is given by [40]:

\[ \theta(x) = \frac{2}{q} \arcsin[\eta \text{sn}(\sqrt{g} x; k)] \]  

where \( \eta = \pm 1 \) and \( \text{sn}(u, k) \) denotes the sine-amplitude, which is a Jacobian elliptic function of modulus \( k \). The sine-amplitude is an odd function of its argument \( u \) and has period \( 4K(k) \), where \( K(k) \) is the complete elliptic integral of the first kind. By substituting (1.8) in (1.5) and minimizing with respect to \( k \) one finds that \( k \approx 2\sqrt{g} \). In the limit \( g \to 0 \) one has also \( k \to 0 \) and \( \text{sn}(u, k) = \sin(u) \), whence \( \theta(x) \approx x \) as expected. On the other hand, when \( g \gg 1 \) one finds \( k \approx 1 \) and \( \text{sn}(u, k) \approx \tanh(u) \), in which case (1.7) is obtained. For a generic value of \( g \) the solution has the form of a staircase. The plateaus in the staircase are multiples of \( 2\pi/q \) and correspond to regions where the CDW is in phase with the lattice, while in the transitions between the plateaus the CDW is not locked, leading to discommensurations. For large values of \( g \) the discommensurations are rather narrow and we find stripe states. These states have been observed experimentally in CDW systems [38].

In contrast to these CDW stripes, the stripe systems which we will discuss here have their origin in Mott insulators with very large Coulomb energies, whereas typical CDW/SDW systems are very good metals in their normal phase (Cr is a shiny metal while La\(_2\)CuO\(_4\) is opaque and grayish). It is exactly the “mottness” of these systems which complicates the theoretical understanding of their nature. If we take seriously the analogy between dichalcogenides and cuprates we could think of the stripes as phase-slips of an incommensurate order parameter associated with the Mott phase. The primary question concerns the order parameter of a Mott phase. Antiferromagnetic order usually occurs in a Mott insulator but is essentially a parasitic phase (systems of spinless electrons at 1/2-filling with strong next-nearest neighbor repulsion, and frustrated magnetic systems, can be Mott insulators without showing any type of magnetic order [41]). Unfortunately, the order parameter which characterizes the Mott phase in a finite number \( d \) of spatial dimensions is not known. Within the dynamical mean-field theory \( (d \to \infty) \) the order parameter of the Mott transition has been identified with a zero mode of an effective Anderson model [42], but generalizations for the case of finite dimensions have not been
established. The search for the order parameter which characterizes “mottness” is one of the important problems of modern condensed matter physics.

The aim of this work is to summarize the current literature on the stripe phase in high-$T_c$ superconductors. Although there is a consensus for the existence of stripes in manganites and nickelates, no agreement has yet been achieved concerning the superconducting cuprates. Despite this controversy, the presence of stripes is now firmly established in La$_{2-x}$Sr$_x$CuO$_4$ (LSCO) [16, 17, 30–33]. In addition, recent experiments suggest that they may also be present in YBa$_2$Cu$_3$O$_{7-\delta}$ (YBCO) [19-26,34], as well as in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ (BSCCO) [12, 13]. In this work we discuss some of the different experimental techniques which prove or suggest the presence of stripes in cuprates, and we present some theoretical ideas on the existence and relevance of the stripe state.

This review is structured as follows: in section II, a survey of the theoretical derivations of the stripe phase as a ground state of models which are appropriate for describing doped Mott insulators is presented. In section III we discuss the experimental observation of stripes, and in section IV the role of kinetic energy. In section V we consider some phenomenological models, which provide a means to go beyond the question of existence of stripes and allows one to predict measurable quantities. Finally, in section VI, we present the most recent experimental results for YBCO and BSCO, as well as the open questions and topics of debates. In section VII we draw our conclusions.

2. Origin of stripes

One of the main problems in condensed matter theory since the discovery of high temperature superconductors in 1986 [43] is related with the possible dilute phases of Mott insulators [44]. These materials have a large charge transfer gap, so at half-filling are insulating two-dimensional (2D) antiferromagnets well described by the isotropic Heisenberg model [45]. These are trademarks of “mottness” [46] and led Anderson [47] to propose that cuprates may be well described by a Hubbard model with large intra-site repulsion $U$. Later studies showed that close to half-filling and infinite $U$ the model maps into the $t$-$J$ model [48]

$$H = -t \sum_{\langle i,j \rangle,\alpha} P c_{i,\alpha}^\dagger c_{j,\alpha} P + J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j,$$  \hspace{1cm} (1.9)

where $t$ is the hopping energy and $J \approx 4t^2/U \ll t, U$ is the exchange interaction between neighboring electron spins, $\mathbf{S}_i = c_{i,\alpha}^\dagger \bar{\sigma}_{\alpha,\beta} c_{i,\beta}$ ($c_{i,\alpha}$ is the electron annihilation operator at the site $i$ with spin projection $\alpha = \uparrow, \downarrow$, and $\sigma^a$ with $a = x, y, z$ is a Pauli matrix). In Eq. (1.9) $P$ is the projection operator onto states with only single site occupancy (double occupancy is forbidden). Eq. (1.9) reduces trivially to the Heisenberg model at half-filling and describes the
direct interplay between the two main driving forces in the system, magnetism (characterized by $J$) and kinetic energy (characterized by $t$).

The existence of a stripe phase in cuprates was first suggested in the context of Hartree-Fock studies of the Hubbard model close to half-filling and at zero temperature ($T = 0$) [49]. This calculation is essentially analogous to the one used to study CDW/SDW transitions in metallic materials. For $U < t$, vertical stripes (parallel to the $x$- or $y$-axis of the crystal) were shown to be lower in energy [49–51], whereas for large $U > t$ diagonal stripes were found to be energetically more favorable [51, 52]. The crossover from vertical to diagonal stripes was calculated numerically to occur at $U/t \sim 3.6$ [51]. Such calculations have recently been generalized to finite temperatures, and the phase diagram was derived as a function of $T$ and doping $n_h$, (Fig. 1.1 [53]). One important feature of these mean-field calculations is that they predict the formation of charge-ordered domain walls at which the staggered magnetization changes phase by $\pi$. The magnetic order parameter is therefore maximal not at $(\pi/a, \pi/a)$ as in an ordinary antiferromagnet, but at an incommensurate vector $(\pi/a \pm \delta, \pi/a)$ where $\delta = \pi/\ell \ll 1$ and $\ell$ denotes the charge stripe spacing. This incommensurability is an important feature of the stripe problem because, as we will demonstrate below, it leads to a reduction in the kinetic energy of the holes. These calculations, however, always predict that the stripe states possess a gap. The simple reason for this effect is that the only way in Hartree-Fock to reduce the energy of the system is by opening a gap at the Fermi surface. Furthermore, Hartree-Fock calculations in strongly interacting systems are not quantitatively reliable because they are unable to take fluctuation effects into account, and therefore should be considered only as providing some qualitative insight into the ground-state properties. They, however, do provide a "high energy" guide (snapshot picture) of the possible phases of the problem, and in fact they have been fundamental for the interpretation of certain experiments such as neutron scattering.

Recent theoretical efforts have focused not only on determining the ground-state properties of the Hubbard, but also of the $t$-$J$ model. Numerical techniques such as density matrix renormalization group (DMRG) [54], quantum Monte Carlo [55], and exact diagonalization [56], have been applied to the 2D $t$-$J$ model with differing degrees of success. The main problem is that a strongly interacting problem like the $t$-$J$ model is subject to strong finite-size and boundary-condition effects which are difficult to control.

**2.1 Numerical Studies**

Early numerical calculations on the $t$-$J$ model have shown that for physical values of $J/t$ and close to half-filling there is a tendency for phase separation [57]. This phase separation can be pictured in the limit of $t = 0$ (classical
Figure 1.1. Phase diagram in the plane of temperature $T$ and hole concentration $x$ ($n_h = 2x$ for LSCO) obtained by Machida and Ichioka [53] from mean-field studies of the Hubbard model. In the figure, N indicates the normal phase and C denotes the commensurate antiferromagnetic phase. In addition, two other incommensurate phases exist, with vertical (VIC) or diagonal (DIC) stripe order. The VIC phase is metallic, but the DIC is insulating at zero temperature.

limit) as a lowering of the system energy by placing all the holes together in order to minimize the number of broken antiferromagnetic bonds. This simple picture leads to separation into two distinct phases: a commensurate, insulating region and an insulating, hole-rich region. It naturally overestimates the importance of the magnetic energy relative to the kinetic energy, and therefore can be correct only when $J \gg t$. For finite values of $t$, the hole wave function delocalizes and this picture breaks down. The main question is for which values of $J/t$ a phase separation may arise. Emery, Kivelson and Lin [57] found that phase separation can occur for infinitesimal values of $J/t$ sufficiently close to half-filling. These results, however, have been questioned in the light of more recent numerical data. There is no doubt that the $t$-$J$ model undergoes phase separation for $J \gg t$ as all numerical calculations indicate. Close to the physical region of $J < t$, the current evidence for phase separation is weak, and so the issue remains controversial.

DMRG calculations in large clusters [54] indicate the presence of stripe correlations in the $t$-$J$ model. These studies, however, have been criticized on the basis of the special role of boundary conditions. Recent work on the Ising $t$-$J$ model indicate that stripe formation does occur in this system, independent of the boundary conditions [58]. It was shown via non-perturbative analytical
calculations that minimization of the hole kinetic energy is the driving force behind stripe formation. This result has been confirmed by a number of numerical calculations in the $t$-$J$ model [59], as well as in the $t$-$J_z$ model [60]. Another important conclusion from these studies is that in the stripe phase the superconducting correlations are extremely weak. In fact DMRG calculations show that the stripe state is a CDW/Luttinger-liquid state with vanishing density of states at the chemical potential, and thus is naturally insulating [54]. The DMRG calculations then support the idea that stripe correlations compete with superconductivity instead of enhancing it. This is consistent with experimental findings in Nd-doped LSCO, where the superconducting transition is reduced when static stripe order sets in [16].

Although the DMRG calculations were originally performed in an $t$-$J$ model, essentially the same physics is found in the $t$-$J_z$ model. The reason for the similarity between the $t$-$J$ (where the spins are dynamical) and the $t$-$J_z$ (where the spins are static) may be understood on the basis of the fluctuation timescales for each component in the problem. In the $t$-$J$ model the spins fluctuate with a rate $\tau_s \approx \hbar/J$, while the timescale for hole motion is $\tau_h \approx \hbar/t$. When $J/t < 1$ (the physical regime of the model) one has $\tau_s > \tau_h$, that is, the holes move “faster” than spins. In this case a Born-Oppenheimer approximation is reasonable, since the spins have slow dynamics, and the two problems become essentially identical [56]. The advantage of working with the $t$-$J_z$ model is that many of its properties are significantly easier to study both numerically and analytically.

The introduction of next-nearest-neighbor hopping $t'$ favors mobile $d$-wave pairs of holes for $t' > 0$, and single-hole excitations (spin-polarons) for $t' < 0$ [61]. At $t' = 0$ the stripe state is very close in energy to the $d$-wave pair state, and thus a small change in the boundary conditions or inclusion of small perturbations in the Hamiltonian can easily favor one many-body state relative the other. The quasi-degeneracy of different many-body states is an important characteristic of strongly correlated systems. Moreover, in real materials other effects may also be responsible for the selection of the ground state, that is, for lifting of the quasi-degeneracy. Indeed, by including lattice anisotropies, which arise in the low-temperature tetragonal (LTT) phase of LSCO co-doped with Nd, the stripe state can be easily selected [62–64]: Hartree-Fock calculations of the Hubbard model have shown that a very small anisotropy (on the order of ten percent) in the hopping parameter $t$ is already sufficient to stabilize the striped phase, independent of the boundary conditions (open or periodic) [63]. Monte Carlo studies of the $t$-$J$ model have also confirmed these results [64].
2.2 Stripes and phase separation

The problem of the formation of inhomogeneous states in a system with phase separation can be easily understood from a classical point of view by studying the Ginzburg-Landau free energy functional. Let \( \psi \) be the order parameter of a system described by a free energy, \( F \), of the form

\[
F[\psi] = \alpha(T,x)|\psi|^2 + \frac{\beta(x)}{2}|\psi|^4 + \frac{\gamma}{3}|\psi|^6, \tag{1.10}
\]

where \( \psi \) may be complex for a superconductor, \( \alpha(T,x) \), \( \beta(x) \), and \( \gamma > 0 \) are functions of the temperature \( T \) and some parameter \( x \) (such as doping or pressure). In the theory of second-order phase transitions, the \( |\psi|^6 \) term is neglected close to the critical line because \( \beta > 0 \) in this region of the parameter space. Here, however, we assume that \( \beta \) may be negative, and therefore this term is required so that the free energy is bounded from below. The critical line in the \( (T,x) \) plane is given by \( T_c(x) \) (as shown in Fig. 1.2) and we assume the existence of a quantum critical point (QCP) at \( x = x_a \) (that is, \( T_c(x_a) = 0 \)).

Close to the critical line we introduce the parameterization:

\[
\alpha(T,x) = \alpha_0 \left[ \frac{T}{T_c(x_a)} - 1 \right],
\]

\[
\beta(x) = \beta_0 \left[ \frac{T_c(x)}{T_c(x_a)} - 1 \right], \tag{1.11}
\]

while \( \gamma \) is approximately independent of \( x \) and \( T \). Notice that with this choice the parameter \( \beta(x) \) is positive for \( x > x_s \), signaling that in this regime the transition is of second order. However, \( \beta(x) \) vanishes at \( x = x_s \) and becomes negative for \( x < x_s \), indicating that the nature of the phase transition changes at small \( x \). In fact the point \( (x_s, T_c(x_a)) \) is a tricritical point.

For \( x > x_s \) the \( |\psi|^6 \) term is irrelevant close to the phase transition, and the transition is of second order depending on whether \( T \) is greater or smaller than \( T_c(x) \). Minimizing the free energy with respect to the order parameter yields

\[
|\psi_0(x,T)|^2 \approx \frac{\alpha_0 \left( 1 - T/T_c(x) \right)}{\beta_0 \left( T_c(x)/T_c(x_a) - 1 \right)} \tag{1.12}
\]

for \( T < T_c(x) \) and \( x > x_s \). Notice, however, that at \( x < x_s \) the parameter \( \beta(x) \) vanishes, and one must include the \( |\psi|^6 \) term. In this case, the free energy has two minima (instead of one) at the critical line indicating that the system has two phases, one with \( \psi = 0 \) (normal) and another with \( \psi = \psi_0 \) (ordered). Minimization of \( F \) with respect to the order parameter provides the condition for the phase transition

\[
\beta^2(x) = \frac{16}{3} \alpha(T^*,x) \gamma, \tag{1.13a}
\]

\[
T^*(x) = T_c(x) + \frac{3\beta_0^2 T_c(x)}{16 \alpha_0 \gamma} \left( \frac{T_c(x)}{T_c(x_a)} - 1 \right)^2. \tag{1.13b}
\]
The solution of these equations gives two critical lines, $T_1(x)$ and $T_2(x)$ on Fig.1.2. These lines terminate at $x_1$ and $x_2$, and for $x_1 < x < x_2$ there is a coexistence region with two phases (normal and ordered).

Long-range interactions are readily introduced by modifying the $|\psi|^4$ term in the free energy to

$$F_C = \int d\mathbf{r} \int d\mathbf{r}' |\psi(\mathbf{r})|^2 \frac{e^2}{|\mathbf{r} - \mathbf{r}'|} |\psi(\mathbf{r}')|^2.$$  (1.14)

In this case of fully phase-separated states the cost in electrostatic energy is too high and phase separation is frustrated to a finite length scale, $\ell_{PS}$, that depends on the coefficient of the $|\psi|^6$ term. The formation of finite droplets with $\psi = \psi_0$ and size $\ell_{PS}$ is therefore more favorable than the separation of the system into two homogeneous phases with $\psi = 0$ and $\psi = \psi_0$. Stripes can also be generated in this model if one adds terms which break the rotational symmetry

$$F_\pm = \sum_\mathbf{q} [\cos(q_x) \pm \cos(q_y)] |\psi(\mathbf{q})|^2,$$  (1.15)

depending on whether the interaction with the lattice may be represented in terms of $L = 0$ (plus sign) or $L = 2$ (minus sign) angular momentum states ($s$- and $d$-wave, respectively). In the $L = 0$ case a checkerboard state is favored, but even a small $d$-wave term generates stripes along the $x$- or $y$-directions.
Disorder can also frustrate the phase separation as one may show by adding a “random mass” term

\[ F_D = \int d\mathbf{r} \ m(\mathbf{r}) |\psi(\mathbf{r})|^2 \]  

(1.16)

to the free energy, where \( m(\mathbf{r}) \) is a gaussian variable with average zero and variance \( u \). Using the replica-technique with \( n \) replicas (\( n \to 0 \) at the end of the calculation) and averaging over disorder gives

\[ F = \sum_{a=1}^{n} F^{(0)}_a - u \sum_{a,b=1}^{n} \int d\mathbf{r} |\psi_a(\mathbf{r})|^2 |\psi_b(\mathbf{r})|^2, \]  

(1.17)

where \( F^{(0)}_a \) is the free energy without disorder and with \( n \) fields \( \psi_n \). Notice that in the replica-symmetric case \( (\psi_n = \psi \text{ for all } n) \) the disorder generates a term of the order \( -u |\psi|^4 \) which decreases the effective value of \( \beta \), therefore reducing the value of \( T_c(x_s) \). In a renormalization-group (RG) sense this term is relevant, and if the disorder is sufficiently strong it will bring the tricritical point to zero temperature (that is, the tricritical point becomes a quantum critical point (QCP) and will completely destroy the first-order phase transition). However, in the ordered phase the system may still possess a coexistence phase with different values of the order parameter. Once again we would have a situation where the system forms droplets of the paramagnetic phase \( (\psi = 0) \) inside the ordered phase. The size of these droplets depends on the strength of the disorder and it is easy to make them adopt a stripe conformation by adding a term of the form (1.15) that breaks the rotational symmetry.

Although the phenomenology of the problem is quite clear, what is not so evident is how to apply this theory to the cuprates. Emery et al. [65] proposed that a model similar to the one discussed here (the Blume-Emery-Griffiths model) may be applied to the cuprates if one defines a pseudo-spin \( S_i \) which takes the values \( S_i = +1 \) and \( S_i = -1 \) on regions corresponding to hole-rich and hole-poor, respectively, whereas \( S_i = 0 \) indicates a local density equal to the average value. In this case \( \psi(\mathbf{r}) \) is the coarse-grained version of \( S_i \) and the above discussion is applicable. Notice, once again, that this model completely disregards the kinetic energy of the problem and can be applied only in the situation where \( t = 0 \). It is therefore not at all surprising that stripes appear. The inclusion of itinerant degrees of freedom is not straightforward. One of the main effects of the presence of itinerant degrees of freedom is the generation of dissipation which can change the dynamical properties (and exponents) of the system. This problem has been the object of recent intensive study in the context of quantum phase transitions when the coupling between the magnetic order parameter and the electrons is weak, and the electrons may be treated as a Fermi liquid [66]. In this case the electronic system serves as a heat bath for
the relaxation of the magnetic order parameter. In Mott insulators, the mere existence of a Fermi surface and a Fermi-liquid state can be questioned and thus it is clear that the weak-coupling formalism cannot be applied to these systems. The charge degrees of freedom cannot be modeled purely as a heat bath because their feedback effect in the magnetic system is very strong.

It is interesting to compare the two mechanisms for stripe formation quoted previously. In one mechanism (represented by the Hartree-Fock calculations), stripes are long period CDWs arising from Fermi-surface nesting in a weakly incommensurate system [49–52]. There are four features which arise from a Fermi-surface instability: 1) the transition is spin driven, i.e., there is a single transition temperature $T_c$ below which the broken-symmetry solution of the Hartree-Fock equations is stable; 2) in the low-$T$ phase there are gaps or pseudo-gaps on the Fermi surface; 3) the spacing between domain walls is equal to $\pi/x$, where $x$ is the hole concentration; 4) the high-$T$ phase should be a Fermi liquid.

In the case of stripes arising from Coulomb-frustrated phase separation, the situation is quite different [67]: 1) the transition is charge driven, i.e., local spin order between the antiphase domain walls can develop only after the holes are expelled from the magnetic regions. Ginzburg-Landau considerations indicate either a first-order transition, in which spin and charge order arise simultaneously, or a sequence of transitions in which first the charge order and then the spin order appears as $T$ is lowered [68]; 2) the stripe spacing is not necessarily a simple function of $x$, and there is no reason to expect the Fermi energy to lie in a gap or pseudo-gap; 3) a high-$T$ Fermi-liquid phase is not a prerequisite.

3. Experimental detection of stripes in high-$T_c$ cuprates: LSCO

Although the first predictions for stripe formation in doped Mott insulators were made 13 years ago, not much attention was paid to these results in connection with high-$T_c$ superconductors until 1995, when experimental data from neutron-scattering measurements in cuprates were interpreted consistently within a stripe picture [16]. Co-doping of cuprates has been extremely important for revealing the modulated charge states. However, the inclusion of co-dopant usually reduces $T_c$, raising doubts about the coexistence of superconductivity and the striped phase [16, 69]. The first experimental detection of stripes in the cuprates was achieved in a Nd co-doped compound $\text{La}_{2-x-y}\text{Nd}_y\text{Sr}_x\text{CuO}_4$. For $y = 0.4$ and $x = 0.12$, Tranquada et al. [16] found that the commensurate magnetic peak at $\mathbf{Q} = (\pi/a, \pi/a)$ shifts by a quantity $\delta = \pi/\ell$, giving rise to four incommensurate peaks. In addition, new Bragg peaks appear at the points $(\pm 2\delta, 0)$ and $(0, \pm 2\delta)$, indicating that the charges form domain walls separated by a distance $\ell$, and that the staggered magneti-
zation undergoes a phase-shift of $\pi$ when crossing them. The position of the peaks indicates that the stripes are oriented along the vertical and horizontal directions, with a density of one hole per two Cu sites (quarter-filled).

The reason why static stripes could be detected in this compound is based on a structural transition induced by the Nd atoms. Indeed, co-doping with many rare-earth species, including Nd and Eu, produces a buckling of the oxygen octahedra around the Cu sites and a corresponding transition from the low-$T$ orthorhombic (LTO) to a low-$T$ tetragonal (LTT) phase. The critical concentration of Nd needed to destroy superconductivity is a function of the charge-carrier density, i.e., the concentration $x$ of Sr atoms. However, the buckling angle of the octahedra is a universal parameter: for tilts above a critical angle $\theta \sim 3.6^\circ$, superconductivity is completely suppressed in these materials [69]. For the values of Nd co-doping $y$ used in the first neutron-scattering experiments, superconductivity was still present, and the authors claimed that in their samples $T_c \sim 5$ K. However, coexistence of static stripe order and superconductivity in LSCO is an issue that remains controversial, although recent experiments in La$_2$CuO$_{4+y}$ have shown the coexistence of these two phases [70] in the same volume of the sample.

Another important factor assisting the detection of charge stripes in LSCO systems by elastic neutron scattering was the selected doping concentration $x \approx 1/8$. The 1/8 anomaly was known since 1988, when electrical resistivity measurements in La$_{2-x}$Ba$_x$CuO$_4$ (LBCO) were first performed [71]. A mysterious reduction of $T_c$ was detected around $x = 1/8$, but the understanding of this phenomenon was possible only recently, in the light of the stripe picture. Indeed, Ba substitution also induces a structural transition, similar to rare-earth co-doping, which probably acts to pin the stripe structure, stabilizing the charge ordering, and hence reducing $T_c$. Recently, Koike et al. have shown that the 1/8 phenomenon is common to all the cuprates [72] and that a similar effect must occur for $x \sim 1/4$ [73].

Though the first neutron scattering experiment was performed in a Nd doped sample with the “magic” hole concentration $x = 1/8$, further measurements on samples with $x = 0.10$ and $x = 0.15$ confirmed the existence of incommensurate peaks in the spin and charge sectors, giving support to the stripe picture (Fig. 1.3 [16]). Moreover, systematic studies of superconducting LSCO samples with a range of doping values $x$ has been performed by inelastic neutron scattering [17]. The detected incommensurability is exactly the same as that obtained in co-doped samples (Fig. 1.4 [16]). Both elastic and inelastic neutron scattering, in addition to NMR [27], NQR, $\mu$SR [28], Hall transport [32], and ARPES [33] measurements indicate that stripes are present in LSCO. A linear dependence of the incommensurability $\delta$ as a function of the doping concentration $x$ has been detected for $x < 1/8$, indicating that the stripes behave as “incompressible” quantum fluids in this regime, that is, for $0.05 < x < 0.12$.
Charge Inhomogeneities in Strongly Correlated Systems

Figure 1.3. Experimental phase diagram obtained by Ichikawa et al. [16] for Nd-doped LSCO. $T_{ch}$ and $T_m$ denote, respectively, the temperatures below which charge and spin ordering could be detected in this system by elastic neutron scattering measurements. The superconducting transition temperature $T_c$ obtained by susceptibility measurements is also shown. In addition, the structural transition lines from the low-temperature orthorhombic (LTO) to the low-temperature tetragonal (LTT) and to the low-temperature-less-orthorhombic (LTLO) phases are displayed.

The hole density in each stripe is fixed (one hole per two Cu sites), and by increasing the amount of charge in the system one consequently increases the number of stripes and reduces their average separation $\ell(x)$. Moreover, Yamada et al. [74] showed that in this regime $T_c$ is also proportional to $\delta$, i.e., $T_c \propto x \propto \delta \propto 1/\ell(x)$. Above $x = 1/8$, however, the behavior of the system changes and $\delta$ nearly saturates, indicating a transition to a more homogeneous phase. Recently, neutron scattering experiments were performed within the spin-glass regime, for $0.02 < x < 0.05$ [18]. The result was surprising: the incommensurate peaks are rotated by $45^\circ$ in reciprocal space, suggesting that the stripes are diagonal and half-filled, with one hole per Cu site, analogous to nickelate stripes [18]. However, this conclusion may be premature. Because
the incommensurate peaks are observed only in the spin, but not in the charge sector, other explanations of the phenomenon are plausible, such as the formation of a spiral phase [75]. We will discuss this topic below in Secs. 4.2 and 5. A summary of available experimental data concerning the incommensurability is presented in Fig. 1.4.

It is interesting to compare the two mechanisms proposed theoretically [49-53,67] for stripe formation in the light of the experimental results [16]. Charge order indeed appears before spin order, favoring the Emery-Kivelson proposal of frustrated phase separation [67], but the stripe separation clearly displays a linear dependence on the inverse of the hole density, as predicted by the Hartree-Fock analysis [49-53]. Concerning the stripe filling, the Hartree-Fock predictions are observed in the spin-glass regime, whereas the Emery-Kivelson proposition holds within the superconducting underdoped regime. However, recent slave-boson studies of the 3-band Hubbard model have shown that if the oxygen-oxygen hopping integral $t_{pp}$ is finite, quarter-filled stripes are more stable than half-filled ones [76]. DMRG studies of the $t$-$J$ model found also that quarter-filled stripes are the lowest-energy configuration [54].

Figure 1.4. Summary of data concerning incommensurability $\delta = \pi/\ell$ as a function of doping concentration $x$. Data were obtained from neutron-scattering measurements by several groups: open and full small circles are from [18] and [17], respectively; dark squares are from [111], the grey one from [17], and the white one from [16]; large circles are from Refs. [17-22].

4. The single-hole problem: the role of kinetic energy

The problem of a single hole in a 2D antiferromagnet has a long history and it is probably one of the best-studied cases of strongly correlated systems. This problem is by nature single particle because it deals with a single particle interacting with a complex magnetic environment. Physically this is realized
only in physical systems with extremely small carrier densities, and many of their properties can be related to the polaron problem [80]. Here we will not review the problem in any way (there are very good reviews on the subject) but we would like to stress the important role that the hole kinetic energy plays in determining the possible phases. Moreover, as we are going to show, the same physics may be responsible for the stripe phases in transition-metal oxides.

The simplest limit of the $t$-$J$ model with a single hole is the limit of $J = 0$, that is, $U \to \infty$. This limit was studied by Nagaoka [81], who showed that a single hole makes the system unstable toward a ferromagnetic phase. The origin of ferromagnetism in this case lies in the minimization of the hole kinetic energy: because double occupancy is precluded, the kinetic energy conserves the spin projection, and there is no energy penalty for the formation of ferromagnetic bonds, the kinetic energy is minimal when all the spins have the same direction. This process is essentially the same as that occurring in double-exchange systems such as manganites, where a ferromagnetic coupling between the electron spin and a magnetic host produces a ferromagnetic state by minimizing the kinetic energy. For finite but small $J$ ($J/t \ll 1$) the same effect occurs, but instead of polarizing the entire plane of spins a single hole produces a ferromagnetic polarization cloud of size $R$: the hole gains a kinetic energy of order $4t - t/(R/a)^2$ by being free to move in the ferromagnetic region but has to pay a magnetic energy cost of order $J(R/a)^{d-1}$ for the generation of a frustrated magnetic surface with ferromagnetic bonds. Minimization of the total energy of the hole indicates that the radius of the ferromagnetic region decreases according to $R/a \sim (t/J)^{1/(d+1)}/(d - 1)$ as $J/t$ increases (notice that for the case $d = 1$ this estimate always produces $R = \infty$ for all values of $t/J$) [82].

As $J/t$ increases the magnetic energy generated by the frustrated magnetic surface becomes too large, and $R$ shrinks to zero. Thus, larger values of $J/t$ lead to a change in the physics. A simple way to reduce the magnetic frustration is to reduce the frustrated surface of the spin configuration. Instead of a frustrated surface of misaligned spins it becomes energetically favorable to create strings of ferromagnetic pairs of spins due to the retracing motion of the holes. It is clear that in this case the energy of the string grows linearly with its size $L$, and therefore that the energy required to generate a string is approximately $JL$. On the other hand, the hole kinetic energy changes from $-t$ to a quantity of order $-t + t/(L/a)^2$, and the problem is essentially equivalent to that of a single particle in a linearly confining potential. The solution of this quantum mechanical problem is straightforward and minimization of the total energy shows that the size of the strings varies according to $L \sim (t/J)^{1/3}$ independent of the dimensionality. Thus, on increasing $J/t$ the single hole case exhibits a crossover from a ferromagnetic polaron to the so-called spin-polaron. The confinement described here is not completely correct because the hole has
been considered as a semi-classical entity whereas in fact it is a fully quantum-mechanical object, which could undergo quantum tunneling over classically forbidden regions. This tunneling gives rise to “Trugman loops” where the hole can move diagonally but with a very small tunnel splitting (that is, very large effective mass) [83].

In any case, the true situation lies between these extremes and ferromagnetic polarization is concomitant with string processes. It is clear that the problem of the doped antiferromagnet centers on the compensation of magnetic frustration by reduction of the kinetic energy. Furthermore, the string mechanism provides a clear way to release kinetic energy, namely the retraceable motion of the hole. By incorporating both the kinetic energy (creation of ferromagnetic bonds) and the magnetic energy (generation of strings) one may understand how holes can move in a system with strong antiferromagnetic correlations. We have, however, discussed only the case of a single hole but for superconductivity it is important to understand the situation when the density of holes increases. The first crossover in these systems occurs when a finite linear density of holes (say $N/L$ is finite but $N/L^2$ zero) is reached. The second crossover occurs when $N/L^2$ becomes finite. Thus, in such strongly correlated systems one has always at least 2 crossovers: from single particle to 1D and from 1D to 2D. In the next section we will discuss the first crossover and show that it is related to the formation of stripes.

4.1 Crossover from single particle to 1D: stripes in the t-J model

Consider an infinite antiphase domain wall oriented along one of the crystal axes directions of the system (Fig.1.5). The cost in energy per hole to create such a state is $J/2(n^{-1} - 1)$, where $n$ is the linear density of holes along the stripe. However, the hole wave function is translationally invariant along the stripe and the kinetic energy gain due to longitudinal hopping is $\frac{-2t \sin(\pi n)}{\pi n}$. For $J/t = 0.4$ one may show that the energy is minimized for $n = 0.32$ with an energy $E_b \approx -1.255t$ which is larger than the energy of a hole in the bulk (spin polaron), given by $E_{sp} = -2\sqrt{3}t \approx -2.37t$ [56]. Here we have not included the transverse motion of the hole perpendicular to the stripe, which further reduces the kinetic energy of the system but gives also a finite width to the hole wave function. Using a retraceable-path approximation (but ignoring hole-hole interactions) one may calculate analytically the Green function for the holes [58].

Holes are confined to an antiphase domain wall by the potential generated by strings of overturned spins (that is, there is a linearly growing potential transverse to the stripe direction). One may show also that in this configuration the hole is actually a holon, i.e., it carries charge but no spin and any motion
of the hole away from the stripe produces a spinon, a particle with a spin of 1/2 but no charge, of energy $J$. In the bulk the hole carries both spin and charge and therefore is a spin-polaron. Spin-charge separation is thus local, but not macroscopic [58]. Because of this effect, Trugman loops [83], which are responsible for hole deconfinement in the absence of antiphase domain walls, are not effective because the motion of holes away from the wall always produces an excitation of finite energy.

One finds that for $J/t = 0.4$ and $n \approx 0.3$ the energy of the stripe state is $E_0 \approx -2.5t$, and therefore lower than the energy of the spin polaron. Furthermore, the width of the hole wave function has a value on the order of 3 - 4 lattice units [58], and therefore extends a considerable distance from the antiphase domain wall, in contrast to the “cartoon” picture where the stripe has a width of only one lattice spacing [16]. Thus, considering a stripe as a completely 1D object is somewhat misleading because each hole may make long incursions into the antiferromagnetic regions. Moreover, it is clear from these analytic calculations that it is the single-hole kinetic energy which is responsible for the stabilization of the stripe state. These results have been confirmed numerically by DMRG and exact-diagonalization studies [59]. In previous works semi-phenomological field theoretical models were proposed to explain the formation of anti- and in-phase domain walls and stripes [84–86]. However, from the studies on the $t$-$J_z$ model it becomes clear that stripe formation is a short distance problem (that is, it involve high energy states) and cannot not be properly addressed with the use of field theories that can only describe the low-energy, long-wavelength physics.

Figure 1.5. Antiphase domain wall with one hole. Thick lines represent broken bonds, while dashed lines mark the position of the topological defect.
We should stress that we are discussing the ground state, that is, the lowest-energy stationary state and therefore the concept of "stripe fluctuations" refers, in this context, to excitations which are separated from the ground state by an energy of order $t(J/t)^{2/3}$ because of confinement in the transverse direction [58]. These results are consistent not only with the DMRG results for the $t$-$J$ model for small $J/t$ [59] but also with those for the $t$-$J_z$ model [60]. As a consequence the stripe is metallic, in contrast to Hartree-Fock results which always produce a gap [87]. As in a Luttinger liquid [88] one expects that hole-hole interactions drive the system toward a CDW phase, which would become insulating in the presence of any amount of disorder [89]. Moreover, interactions cause the density of states vanish at the chemical potential following a non universal power of the interaction parameter [88].

4.2 Stripes, magnetism and kinetic energy release

At half-filling, cuprates are antiferromagnetic Mott insulators with the Cu atom carrying a spin of $1/2$. La$_2$CuO$_4$ is one of the most striking examples of a layered antiferromagnetic Mott insulator. The Neel temperature, instead of taking a value on the order of the planar magnetic exchange $J$ ($\approx 1500$ K), is approximately 300K. This occurs because of the low dimensionality of the system and indeed, an $O(3)$ invariant 2D Heisenberg model may order only at zero temperature due to the breaking of continuous symmetry (Mermin-Wagner theorem) [90]. However, the small inter-planar coupling, $J_{\perp}$ ($\approx 10^{-4}J$), stabilizes antiferromagnetic order at finite temperature. Magnetism in these systems is evident essentially across the entire phase diagram, although long-range order is lost with only 2% doping by Sr. Since the Sr atoms are located out of the CuO$_2$ planes and their effect is to introduce holes, the doped holes are essential for the destruction of long-range order. More importantly, it is the minimization of kinetic energy of the holes which is the dominant mechanism for the suppression of magnetism. In the following, we discuss several different ways to understand the importance of hole motion in these systems.

A first indication for the importance of hole kinetic energy is given by magnetic measurements for $x < 0.02$ which observe the recovery of the magnetization when the system is cooled below the so-called freezing temperature, $T_F(x)$ [28]. The staggered magnetization $M_S(x, T)$ vanishes at the Neel temperature $T = T_N(x)$ and is a smooth function for $T_F(x) < T < T_N(x)$. However for $T < T_F(x)$ the magnetization seems to recover to the full value expected at $x = 0$ and $T = 0$. This effect can be ascribed to localization of the holes after which they affect the magnetization only locally. Exactly where the localization of holes occurs remains unresolved. However, soft X-ray (oxygen K-edge) absorption experiments indicate that the holes are probably in the oxygen sites [91]. NMR experiments appear to indicate that holes would
localize preferentially close to the charged Sr atoms for electrostatic reasons [92]. However, because the system is annealed as temperature is reduced, it is quite possible that the unscreened Coulomb interaction between holes plays an important role in the localization process. If this is the case, localized stripe patterns may form even at low doping, although disorder effects from Sr doping are very strong in this region of the phase diagram and one would expect any stripe pattern to be random in the CuO$_2$ planes [62]. Recent neutron-scattering experiments at low doping find diagonal incommensurate peaks in the magnetic sector [18]. However, as no charge peak has yet been observed in the spin-glass regime, these measurements may be interpreted within the stripe model but the question remains open: the antiferromagnetic peaks could also be interpreted as the formation of a spiral phase [75]. In order to resolve this question unequivocally neutron-scattering experiments in samples heavily doped with spin-zero impurities such as Zn are required. If, as expected for a spiral state [75], the slope of the incommensurability as a function of Sr concentration $x$ changes by a factor $(1 - 2z)$, where $z$ denotes the Zn concentration, the stripe hypothesis would be excluded in the spin-glass regime.

Independent of the pattern of localization, holes may be localized either on the O or the Cu sites. If hole localization occurs at the O sites, one would expect a spin-glass phase to be observed at temperatures below $T_F(x)$ because a localized hole at an O site liberates one spin (configuration $p^5$) which frustrates the antiferromagnetic coupling between neighboring Cu atoms [93]. If, on the other hand, holes are localized on the Cu sites, the magnetization of the system would be reduced by one quantum of spin. Presumably, because of the delocalization of the hole wave function between O and Cu atoms the two effects can occur simultaneously [48]. The key feature of these experiments is that they indicate the importance of the hole motion for the destruction of long-range order.

The rapid suppression of magnetism with hole doping can be contrasted with the slow suppression of antiferromagnetic order when Cu is replaced by a non-magnetic atoms such as Zn or Mg [94]. In this case, long-range order seems to be lost only at 41% doping, that is, at exactly the classical percolation threshold for a 2D Heisenberg system. As has been shown in recent theoretical and numerical studies of the diluted quantum Heisenberg model, magnetic order seems to disappear only close to the classical percolation threshold even in the quantum system [95]. These results have been investigated experimentally and in fact the quantum fluctuations introduced by the dopants apparently are not sufficient to produce a quantum critical point (QCP) below the value of classical percolation [94]. Although the comparison between the problem of hole doping and Zn doping is not at first obvious, from the chemical point of view Zn introduces a static hole in the Cu plane because it has the same valence, but also has one extra proton. The situation on doping by Zn is there-
fore similar to the problem of holes localized at Cu sites. Indeed the recovered magnetization in the hole-doped case, $M(x, 0)$, is very close to the value which one would obtain on replacing Cu by a density $x$ of Zn atoms. Thus, further confirmation is obtained that the hole kinetic energy is the driving force behind the suppression of antiferromagnetic order in these systems.

5. Phenomenological models: stripes and antiferromagnetism

In the previous sections we have argued that the kinetic energy of holes is fundamental for understanding the stripe phenomenon in cuprates. However, dimensional crossovers are very difficult to measure experimentally. Phase transitions, on the other hand are easy to observe, because they produce strong effects in the thermodynamic properties. Because Mott insulators are usually antiferromagnetic one may ask if such dimensional crossovers in the hole motion affect the antiferromagnetic phase? In the previous section we provided evidence that the kinetic energy of the holes is responsible for the destruction of antiferromagnetism in these systems when the hole concentration is of order $0.01-0.02$. How is it possible that such small doping levels can destroy a robust antiferromagnetic phase, with a Neel temperature which is of order 300K?

A possible way to understand the effect of the hole motion is to consider the formation of infinitely long stripes. The first obvious effect is a breaking of the spatial rotational symmetry. In the antiferromagnetic phase the spin rotational symmetry is also broken, indicating that both symmetries must be broken in the ground state of a striped antiferromagnetic phase. As a consequence the Goldstone modes associated with the broken symmetries must carry information about them. For an ordinary (non-striped) antiferromagnetic phase these are spin-wave modes characterized by an energy dispersion $E(k) = c_s k$, where $c_s$ is the spin-wave velocity, and a spin-stiffness $\rho_s$ associated with the twist of the order parameter [45]. In a striped antiferromagnetic phase the Goldstone modes remain spin waves, but because of the broken rotational symmetry their dispersion is different if the mode propagates along the direction of broken symmetry or perpendicular to it, i.e., the energy dispersion is not circularly symmetric and $E(k_||, k_\perp) = \sqrt{c_{s||}^2 k_{||}^2 + c_{s\perp}^2 k_{\perp}^2}$, where $||$ and $\perp$ refer respectively to the directions parallel and perpendicular to the stripes. At wave lengths longer than the stripe separation $\ell$ and energies lower than the first spin-wave gap due to the folding of the Brillouin zone, this kind of dispersion is guaranteed by the nature of the broken symmetries.

In an ordinary antiferromagnet the spin-wave velocity is simply related to the lattice spacing $a$ and the exchange constant $J$ by $c_s = SJa^d$. Thus, a striped antiferromagnet may be modeled simply by assuming that the only ef-
The effect of the stripes is to introduce anisotropy in the exchange constants. Let us consider the case of a spatially anisotropic Heisenberg model with exchange constants $J_x$ and $J_y$ in the $x$- and $y$-directions, respectively, in which case the spin-wave velocities in each direction are given by

$$c_y^2 = 2S^2 a^2 J_y(J_x + J_y),$$
$$c_x^2 = 2S^2 a^2 J_x(J_x + J_y).$$

(1.18)

Microscopically, one may regard the stripes as causing local modification of the exchange across an antiphase domain wall from $J$ to a value $J'(\leq J)$. This alteration of $J$ leads to a macroscopic change in the values of the exchange constants in the same way that the introduction of impurities in a solid leads to an average change in the unit-cell volume.

To relate $J$ and $J'$ to $J_x$ and $J_y$ is not a trivial task. One possibility would be to solve the linear spin-wave theory for the striped antiferromagnet and calculate the derivative of the spin-wave energy at the ordering vector $Q$. For stripes with a separation of $N_s$ lattice sites, this procedure requires the solution of $N_s$ coupled differential equations. Besides being computationally intensive, the solution would not provide significant insight into the origin of $J'$ and would just exchange one phenomenological parameter by another.

The simplest theory describing a striped antiferromagnet is the spatially anisotropic non-linear $\sigma$ model

$$S_{eff} = \frac{1}{2} \int_0^{\beta h} d\tau \int dx \int dy \left\{ S^2 \left[ J_y (\partial_y \hat{n})^2 + J_x (\partial_x \hat{n})^2 \right] + \frac{\hbar^2}{2a^2(J_x + J_y)} (\partial_\tau \hat{n})^2 \right\},$$

(1.19)

where $\hat{n}$ is a unit vector field. The symbols have been chosen to suggest the continuum limit of an underlying effective integer-spin Heisenberg Hamiltonian on a square lattice [97]. The underlying anisotropy parameter is the ratio of the two exchange constants, or of the two velocities,

$$\alpha = J_x / J_y.$$  

(1.20)

The value of $\alpha$ characterizes the theory, but its exact dependence on microscopic parameters is not easy to derive.

We proceed by making use of well-established techniques to analyze the behavior of the field theory described by the action (1.19) to predict the physical properties of the system of interest. It is useful to reexpress (1.19) symmetrically by a dimensionless rescaling of variables $x' = (\alpha)^{-1/4} x \Lambda$, $y' = (\alpha)^{1/4} y \Lambda (\Lambda \sim 1/a$ is a momentum cut-off), $\tau' = \sqrt{2(J_x + J_y)} \sqrt{J_x J_y} S a \tau / \hbar$. 


The effective action (1.19) becomes

\[ S_{\text{eff}} = \frac{\hbar}{(2g_0)} \int_0^{\Lambda \beta c_0} d\tau' \int dx' \int dy' (\partial_{\mu} \hat{n})^2, \quad (1.21) \]

where \( \mu \) denotes \( x', y', \) and \( \tau' \),

\[ g_0(\alpha) = \frac{\hbar c_0 \Lambda}{\rho_s} = \frac{2(1 + \alpha)}{\sqrt{\alpha}} (a\Lambda)/S \quad (1.22) \]

is the bare coupling constant, \( c_0 = [2(J_x + J_y)\sqrt{J_x J_y}]^{1/2} (aS)/\hbar \) the spin wave velocity and

\[ \rho_s^0 = \sqrt{J_x J_y} S^2 \quad (1.23) \]

the classical spin stiffness of the rescaled model. The original anisotropy is now contained in the limits of integration.

Notice that while \( \alpha \) depends on the ratio \( J_x/J_y \), the spin stiffness depends on the product \( J_x J_y \). Thus, given a microscopic model where \( J_x \) and \( J_y \) are expressed in terms of microscopic quantities the field theory is well defined. Unfortunately no calculations yet exist for the microscopic form of these quantities, and certain assumptions are required concerning their behavior. If the spatial rotational symmetry is broken at the macroscopic level, that is, one has infinitely long stripes in the \( y \) direction, a simple choice would be \( J_y = J \) and \( J_x = \alpha J \), whence

\[ \rho_s^0 = \sqrt{\alpha} \rho_I \quad (1.24) \]

where

\[ \rho_I = JS^2 \quad (1.25) \]

is the spin stiffness of the isotropic system. This choice is valid only when the system is composed of a mono-domain of stripes [98]. If the system is broken into domains, in which case the rotational symmetry is broken microscopically the choice (1.24) may not be the most appropriate. At sufficiently long wave lengths the system is essentially isotropic and therefore the spin stiffness is the same in all directions, [96]

\[ \rho_s^0 = \rho_I \quad (1.26) \]

which is obtained by choosing \( J_x = \sqrt{\alpha} J \) and \( J_y = J/\sqrt{\alpha} \). We note that these choices are essentially arbitrary and are based on qualitative expectations concerning the nature of the correlations at very long wave lengths. One may show that the choice (1.26) is appropriate very close to the antiferromagnetic phase \( (x < 0.02) \) where the breaking of the system into domains is quite probably because of disorder effects [96]. However, at larger doping \( (x = 1/8) \)
the choice (1.24) is more appropriate because long-range stripe order is observed [98]. With these two parameterizations one may analyse the problem and calculate physical quantities for comparison with experiments. One general consequence of the anisotropy introduced by the presence of stripes is a growth of quantum fluctuations because of the reduction of effective dimensionality. In fact, by using large $N$ methods and RG calculations, one may demonstrate that the effective spin stiffness is given by

$$
\rho_s(\alpha) = \rho_0^0(\alpha) \left[ 1 - \frac{g_0(1)}{g_c(\alpha)} \right],
$$

(1.27)

where

$$
g_c(\alpha) = \sqrt{8\pi^2 \sqrt{\alpha/(1 + \alpha)}} \left\{ \ln \left( \sqrt{\alpha} + \sqrt{1 + \alpha} \right) + \sqrt{\alpha} \ln \left( 1 + \sqrt{1 + \alpha} \right) / \sqrt{\alpha} \right\}^{-1}
$$

(1.28)

is a critical coupling constant. Notice that (1.27) is reduced from its classical value $\rho_0^0(\alpha)$ for fixed anisotropy $\alpha$, and that it vanishes at some critical value $\alpha_c$, whence $g_c(\alpha_c) = g_0$. Hence, as a function of the anisotropy, the model exhibits quantum critical point where the system undergoes a quantum phase transition from an ordered Neel state to a paramagnetic phase. The loss of antiferromagnetic order at $x = x_0$ can thus be considered as a consequence of the enhancement of quantum fluctuations due to the presence of stripes.

As explained previously, these considerations are valid when the holes move along the stripes and modify the exchange constant across the stripe. However, when localization occurs at low temperatures (as observed in the recovery of magnetization in NQR experiments) the stripes essentially cease to exist and the system undergoes a phase transition into a spin-glass phase. The simplest way to understand this phase is to consider hole localization at the oxygen sites with consequent liberation of one spin $1/2$ [93]. This spin frustrates the antiferromagnetic order, because the superexchange interaction of the O spin with the neighboring Cu spins is antiferromagnetic. This problem may be treated by considering the O spin as a classical localized dipole moment [75].

Finally, the theoretical results concerning the existence of stripes and the appropriate model for describing them can be summarized as follows: the proposal that the doped $t$-$J$ model undergoes a phase separation is supported by variational arguments [77], diagonalization on small clusters [77], and Green-function quantum Monte Carlo calculations [78]. On the other hand, several quantum Monte Carlo calculations [55], series expansions [79], exact diagonalization [56], and DMRG calculations [54] yield results contradicting these claims and supporting the stripe picture. In order to gain more insight into the problem, we begin by considering the simplest possible case, namely, the single-hole problem.
6. Phenomenological models: Transverse fluctuations and pinning of stripes

While much theoretical effort has been concentrated on determining whether stripes are the ground state of models such as the Hubbard and the $t$-$J$ models, which are supposed to describe high-$T_c$ superconductors, a parallel research direction has also developed which consists of phenomenological studies of the striped phase. In this case, one assumes the existence of stripes and discusses further aspects such as their effect on the antiferromagnetic state (see previous section). Motivated by issues such as the static or fluctuating nature of stripes and the mechanism of stripe pinning, a phenomenological theory for the pinning of stripes has been developed. Zaanen et al. have related transverse stripe fluctuations to the restricted solid on solid model (RSOS) which describes the growth of surfaces [99]. In a simplified form of the model, the transverse kink excitations of stripes are mapped to a quantum spin-1 chain model [99], whose Hamiltonian is

$$H = \sum_n \left[-t \left(S^x_n S^x_{n+1} + S^y_n S^y_{n+1}\right) - D S^z_n S^z_{n+1} + J (S^z_n)^2\right].$$ (1.29)

Here, $t$ has the role of a hopping parameter for transverse kinks, $J$ controls the density of kinks and the $D$ term represents a nearest-neighbor interaction of kinks. The spins take the values $S^z_n = 0, \pm 1$, where $+1$ and $-1$ are associated respectively with stripe kinks and anti-kinks and $S^z_n = 0$ describes unperturbed (flat) segments. The full phase diagram for this problem was determined numerically by den Nijs and Rommelse [100] after earlier calculations by Schulz [101], who treated the spin-1 problem as two coupled spin-1/2 chains. We have recently reanalysed the calculations of Schulz [102] and derived the correct phase diagram from this formalism, which agrees with the one obtained in Ref. [100] (see Fig. 1.6). Six different phases can be identified, depending on the values of the $D$ and $J$ parameters. If $J$ is positive, the last term of Eq. (1.29) determines that $S^z_n = 0$ and the stripe is straight (flat phase). If $J$ is negative, both values $S^z_n = \pm 1$ are equally favorable with respect to the $J$ term and the $D$ term determines the value of $S^z_n$. If $D$ is positive, nearest-neighbor segments prefer to be similar and therefore $S^z_n$ and $S^{z+1}_n$ will have the same sign. This gives rise to the ferromagnetic phase (diagonal stripes), with a sequence of kinks or anti-kinks. On the other hand, if $D$ is negative, $S^z_n$ and $S^{z+1}_n$ prefer to have opposite signs, and the stripe will be bond-centered and flat, with a “zig-zag” shape (a kink follows an anti-kink and vice versa). In addition, both the flat and the bond-centered flat phases, which are gapped, can undergo a Kosterlitz-Thouless transition to gapless rough or bond-centered rough phases, respectively. The sixth phase, which was not identified by Schulz, corresponds to a gapped, disordered, flat phase (DOF), (Fig. 1.6). In contrast to the flat phase, this phase has a finite density of kinks and anti-kinks, which are posi-
tionally disordered, but have an antiferromagnetic order in the sense that a kink $S_z = 1$ is on average followed by an anti-kink $S_z = -1$ (rather than another kink), with any number of $S_z = 0$ states in between. The DOF phase is the valence-bond phase which is responsible for the Haldane gap.

Figure 1.6. Sketch of the phase diagram for the spin-1 chain. The stripe configurations represented by circles, and the corresponding values of $S^z$ are shown below. There are six different phases: 1) a gapped flat phase, corresponding to straight stripes ($S_z = 0$); 2) a gapless rough phase (spins equal to zero and $\pm 1$ distributed randomly); 3) a gapped bond-centered (BC) flat phase, which has a long-ranged zig-zag pattern (periodic alternation of $S_z = 1$ and $S_z = -1$); 4) a gapless BC rough phase with a zig-zag pattern (antiferromagnetic correlations with disordered $S_z = \pm 1$ but no $S_z = 0$ states); 5) a diagonal stripe phase, corresponding to a ferromagnetic state in the spin language; 6) a gapped disordered flat phase (DOF), where a kink ($S_z = +1$) is followed on average by an anti-kink ($S_z = -1$), but with some $S_z = 0$ states in between.

In the limit of negligible nearest-neighbor interaction ($D \sim 0$), the above model can be related to the $t$-$J$ model [103] by considering the transverse dynamics of a single vertical stripe in a frozen Neel background. The transverse dynamical properties of holes are described by the $t$-$J_z$ model, and they move under the condition that the horizontal separation between two neighboring holes cannot be larger than one lattice constant. Notice that this condition does not restrict the motion of the stripe: it may still perform excursions very far from the initial straight-line configuration, but the line cannot be broken. In
this case the $t$-$J$ Hamiltonian describing hole dynamics may be mapped onto 
a spin-1 chain Hamiltonian analogous to Eq. (1.29), but with $D = 0$ [103]. 
A duality transformation of the initial quantum string Hamiltonian maps the 
problem onto one describing a 1D array of Josephson junctions [104], which 
is known to exhibit an insulator-superconductor transition at $(t/J)_c = 2/\pi^2$ 
[105]. This Kosterlitz-Thouless transition represents the unbinding of vortex-
antivortex pairs in the equivalent $XY$ model, which translates to a roughening 
transition for the stripe problem. In this way, the transition between the flat 
and rough phases along the vertical line $(D = 0)$ of the phase diagram could 
be determined precisely by analytical means [104].

These models are related to the sine-Gordon model, and by studying the 
spectrum of the quantum string in a Hilbert-space sector of zero topological 
charge, the meaning of the transition in the “string” language may be clarified. 
At $(t/J)_c$ the (insulating) pinned phase, which has an energy spectrum with a 
finite gap, turns into a (metallic) depinned phase where the spectrum becomes 
gapless [104]. This procedure allows the connection of two important and 
different classes of problems, namely the transverse dynamics of stripes in 
doped antiferromagnets and a system with the well-known properties of the 
sine-Gordon model.

In all the models discussed hitherto, the pinning potential arises from the 
discrete nature of the lattice. However, the introduction of holes into the MO$_2$ 
planes ($M = \text{Cu or Ni}$) is not the only consequence of doping a Mott insulator 
such as La$_2$MO$_4$. Doping an antiferromagnetic insulator also introduces disor-
der into this material due to the presence of counterions, which act as attractive 
centers for holes. Doping with divalent atoms, such as Sr$^{2+}$ produces quenched 
disorder, because the ionized dopants are located randomly between the CuO$_2$ 
planes. In contrast, doping with excess oxygen generates annealed disorder. 
Indeed, oxygen atoms have a low activation energy and remain mobile down 
to temperatures of 200-300 K.

In order to account for the random pinning potential provided by the Sr 
atoms in nickelates and cuprates, one may add a disorder potential to the pre-
vious phenomenological Hamiltonian (1.29) with $D = 0$. This allows a deter-
mination of the influence of both disorder and lattice effects on the striped 
phase of cuprates and nickelates. We consider the problem of a single stripe 
along the vertical direction confined in a box of size $2\ell$, where $\ell$ denotes the 
stripe spacing. The system is described by the phenomenological Hamiltonian

$$
\hat{H} = \sum_n \left[ -2t \cos \left( \frac{\hat{p}_n}{\hbar} \right) + J (\hat{u}_{n+1} - \hat{u}_n)^2 + V_n(\hat{u}_n) \right],
$$  

(1.30)

with $t$ the hopping parameter, $\hat{u}_n$ the displacement of the n-th hole from the 
equilibrium (vertical) configuration, $\hat{p}_n$ its conjugate transversal momentum, 
$J$ the stripe stiffness, and $V_n(\hat{u}_n)$ an uncorrelated disorder potential satisfying
\[ \langle V_n(u)V_{n'}(u') \rangle_d = d \delta(u - u') \delta_{n,n'}, \] where \( \langle \ldots \rangle_d \) denotes the Gaussian average over the disorder ensemble and \( d \) is the inverse of the impurity scattering time.

Eq. (1.30) may be straightforwardly related to Eq. (1.29) by noting that

\[ S^z_n = u_n + 1 - u_n \] and that the hopping terms \( S^x_n \) and \( S^y_n \) are connected to translation operators, which can be written in the momentum representation

\[ p_n = e^{\pm ip_n/\hbar} \] [104]. We are considering the lattice parameter \( a = 1 \).

A dimensional estimate provides the dominant features of the phase diagram. At large values of the hopping constant \( t \gg J \), the first term may be expanded as

\[ -2t \cos(\hat{p}_n/\hbar) \sim \text{const.} + t(\hat{p}_n/\hbar)^2. \] In the case of no impurity potential, \( V_n(\hat{u}_n) = 0 \), hole dynamics is governed by the competition between the kinetic term \( t(k_n)^2 \), which favors freely mobile holes, and the elastic one, \( J(\hat{u}_{n+1} - \hat{u}_n)^2 \), which acts to keep them together. When the confinement is determined by the lattice pinning potential, the average hole displacement \( \hat{u}_{n+1} - \hat{u}_n \) is of order 1 (the lattice constant is unity) so the wave vector \( k_n \sim 1 \).

A transition from the flat phase, with the stripe pinned by the underlying lattice, to a free phase is then expected at \( t/J \sim 1 \).

We now consider the opposite limit of strong pinning by impurities. In this case, the potential provided by the lattice is irrelevant and the typical hole displacement is on the order of the separation between stripes, \( 1/k_n \sim \hat{u}_{n+1} - \hat{u}_n \sim \ell \). By comparing the kinetic \( t(1/\ell)^2 \) and the elastic \( J(\ell)^2 \) terms, we observe that a transition should occur at \( t/J \sim (\ell)^4 \).

Indeed, by deriving the differential renormalization group (RG) equations to lowest nonvanishing order in the lattice and disorder parameters, one obtains a set of flow equations [89], which indicate that the transition from the flat (lattice-pinned) to the free phase occurs at \( (t/J)_c = 4/\pi^2 \), and the transition from the disorder-pinned to the free phase occurs at \( (t/J)_c = (36/\pi^2)\ell^4 \). The pinning phase diagram of the striped phase is shown in Fig. 1.7, in which \( \delta = 1/2\ell \).

By comparing these results with recent measurements on nickelates and cuprates one concludes that nickelates occupy the lower left corner of the phase diagram, i.e., they have static stripes which are pinned by the lattice and by the impurities. By contrast, cuprates are characterized by freely fluctuating stripes and so appear in the upper right corner. An appropriate treatment of the striped phase in cuprates must therefore include stripe-stripe interactions and the model becomes similar to that for a 2D fluctuating membrane [89, 106]. In this phenomenological framework, both nickelate and cuprate materials can be understood in a unified way, the difference between them being simply the parameter \( t/J \) which measures the strength of quantum fluctuations.

Although the number of holes is intrinsically connected with the number of pinning centers, recent experimental developments show that it is possible to control these two parameters independently. Co-doping the superconducting cuprate material LSCO with Nd or Zn increases the disorder without modify-
Figure 1.7. Zero-temperature pinning phase diagram of the stripe phase in the presence of lattice and impurity pinning. Three phases can be identified: a quantum membrane phase with freely fluctuating Gaussian stripes, a flat phase with the stripes pinned by the lattice, and a disorder-pinned phase [89].

One may then investigate the stripe pinning produced by Zn and Nd co-doping [109]. The two dopants play fundamentally different roles in the pinning process. Nd, as with other rare-earth co-doping, induces a structural transition which produces a correlated pinning potential trapping the stripes in a flat phase. The situation is analogous to the pinning of vortices by columnar defects or screw dislocations [110]. In this case transverse fluctuations are strongly suppressed, long-range order is achieved and thus the incommensurate peaks observed by neutron diffraction become sharper after the introduction of the co-dopant, as observed experimentally [16]. On the other hand, in-plane Zn- and Ni-doping provide randomly distributed point-like pinning centers, similar to oxygen vacancies in the vortex-creep problem. Within the model in which a stripe is regarded as a quantum elastic string, the effect of randomness is to “disorder” the string, promoting line-meandering, destroying the 1D behavior and broadening the incommensurate peaks. A perturbative treatment of the RG equations discussed previously show at the next higher order that this kind of pinning is relevant only in under-doped systems, in agreement with experiments [111].

The number of charge carriers [16, 72, 107]. On the other hand, growing the superconducting film on a ferroelectric substrate and using an electrostatic field as the control parameter allows the number of charge carriers in the plane to be altered for a fixed Sr concentration x [108]. This class of experiments constitutes an important step towards the control of a normal metal-superconducting transition.

ing the number of charge carriers [16, 72, 107]. On the other hand, growing the superconducting film on a ferroelectric substrate and using an electrostatic field as the control parameter allows the number of charge carriers in the plane to be altered for a fixed Sr concentration x [108]. This class of experiments constitutes an important step towards the control of a normal metal-superconducting transition.
Finally, it is essential to go beyond the studies of transverse stripe excitations to consider the coupling between longitudinal and transverse modes. Longitudinal modes may be described by a Luttinger-liquid Hamiltonian, and the coupling between longitudinal and transverse fluctuations was investigated using bosonization [102]. One finds that a longitudinal CDW instability can arise if the stripe is quarter-filled and the underlying lattice potential has a zig-zag symmetry. This result has shed additional light on the connection between the formation of a LTT phase and the subsequent appearance of charge order in high-$T_c$ cuprates (Fig. 1.3).

Experimentally, the suppression of superconductivity in LSCO co-doped with Nd (Fig. 1.3), and also the upturn of the resistivity in the normal phase, are correlated with a structural transition from the LTO to the LTT phase. Indeed, at $x = 1/8$ the charge ordering temperature $T_{co}$ reaches its highest value and the superconducting temperature $T_c$ shows a local minimum. Neutron diffraction experiments indicate that in the underdoped phase of LSCO the stripes are quarter-filled [16]. The formation of the LTT phase favors a zig-zag symmetry of the transverse stripe degrees of freedom. Thus, below $T_{LTT}$ the CDW instability discussed above becomes relevant and stabilizes a bond centered string with zig-zag symmetry. If the stripe spacing is exactly commensurate, as at $x = 1/8$, a long-range-ordered CDW (Wigner crystal) can form, leading to the suppression of $T_c$. On the other hand, for incommensurate doping values, solitonic modes are present in the stripe which prevent long-range charge order. Longitudinal charge order has hitherto not been observed. Nonetheless, the upturn of the in-plane resistivity below $T_{LTT}$ suggests proximity to an insulating phase inside the LTT phase of underdoped cuprates, which is likely to be the bond-centered zig-zag stripe [102].

7. Experimental detection of stripes in YBCO and BSCCO

An important question at this point is whether charge stripes are peculiar to the lanthanates or a generic feature of all the cuprates. The answer is not yet completely settled, and further experiments are required to achieve an unambiguous conclusion. Measurements on YBCO and BSCCO compounds begin to provide a comprehensive picture.

Inelastic neutron scattering experiments recently performed by Mook et al. on YBCO$_{6.6}$ [20], which corresponds to a hole doping of $x = 0.10$, detected a dynamical incommensurability in the magnetic sector of $\delta = 0.105 \pm 0.01$, which is exactly the value measured by Yamada et al. for the corresponding charge concentration in LSCO [17]. Later, magnetic incommensuration was also observed at $\delta = 0.0625$ in YBCO$_{6.35}$, which corresponds to a doping $x = 1/16$ [21]. In addition, measurements of phonon broadening at a wave
vector consistent with the stripe picture (twice the magnetic wave vector) confirmed the previous results for YBCO$_{6.6}$ and YBCO$_{6.35}$ [21]. Eventually, a static charge order peak has been observed in a 21g crystal of YBCO$_{6.35}$, at a wave vector which is exactly double the dynamical magnetic incommensuration, $2\delta = 0.127$, as expected within the stripe picture [22]. Although the charge peaks are small, 6 orders of magnitude below the strongest crystal Bragg peak, their existence is undeniable. However, it should be emphasized that no charge order has been observed in YBCO$_{6.5}$ and YBCO$_{6.6}$, so the situation is not yet resolved.

Another important feature measured recently in YBCO is the 1D nature of the stripes [23]. In a 4g crystal of detwinned YBCO$_{6.6}$, one could observe not four, but only two incommensurate magnetic peaks (the second set of perpendicular peaks have nearly vanished, because the sample was almost completely detwinned). The results suggest stripes aligned along the $b-$axis, in agreement with far-infrared spectroscopic measurements by Basov et al. [31], which indicate that the superfluid density is larger along the $b-$direction. This behavior cannot be attributed to the chains, because in underdoped materials the missing atoms in the chains would inhibit the chain contribution to superconductivity.

As a conclusion, one could state that neutron scattering experiments in YBCO at several different doping concentration [24] confirm the stripe picture sketched for LSCO and reveal the universality of the previous results.

Concerning BSCCO, the majority of experimental results are obtained from scanning tunneling microscopy (STM) and spectroscopy (STS) [7-12]. The advantage of STM lies in its ability to measure simultaneously, with atomic resolution, both the surface topography and the local density of states (LDOS) of a material. The topographic image can be realized due to the exponential dependence of the tunneling current $I$ on the separation of tip and sample. In addition, the differential conductance $G = dI/dV$, where $V$ is the sample bias voltage, is proportional to the LDOS of the sample at the tip location.

Low-$T$ STS in BSCCO samples ($T_c = 87K$) revealed the existence of a large number of randomly distributed regions, with characteristic length scales of order 30 A, which have anomalous LDOS features. These features were initially referred to as quasiparticle scattering resonances (QPSRs) and were thought to be due to quasiparticle scattering from atomic-scale defects or impurities [9], but later measurements performed in Zn-doped samples indicated that these inhomogeneities do not originate from impurities. The locations of Zn impurities were identified from the zero-bias conductance map and the LDOS map taken simultaneously at the same location showed no correlation between the intensity of the integrated LDOS and the location of the Zn impurities [10].

Spatial variations of the tunneling spectrum and of the superconducting gap can be observed in pure as well as in impurity-doped BSCCO samples, and
seem to be intrinsic to the electronic structure. The gap ranges from 25 meV to 65 meV and has a gaussian distribution [10]. The average gap is very similar to that reported previously from tunneling measurements. The spectra obtained at points with larger integrated LDOS exhibit higher differential conductance, smaller gap values and sharper coherence peaks, which are the characteristic features of spectra taken in samples with high oxygen doping concentration. This observation suggests the interpretation that the inhomogeneities may arise from differences in local oxygen concentrations [10].

Recent experiments in underdoped BSCCO indicate that these high-$T_c$ materials are granular superconductors, with microscopic superconducting grains separated by non-superconducting regions. By doping the material with Ni impurities, it was observed that the position of Ni atoms coincide with regions of small gap ($\Delta < 50$ meV). In underdoped BSCO these small-gap regions, which have large $G(\Delta)$ are separated by percolative regions with large gap $\Delta$ and low $G(\Delta)$ [11].

Qualitatively new information is provided by STS measurements in magnetic field up to 7.5 T. The quasiparticle states generated by vortices in overdoped BSCCO show a “checkerboard” pattern with a periodicity of four unity cells [12], in agreement with the charge periodicity expected within the stripe picture [16]. Indeed, the magnetic spatial periodicity previously detected by neutron scattering in overdoped LSCO in the presence [112] or absence [17] of a magnetic field is eight unit cells, exactly twice the charge spacing. Shortly after these measurements, the $4a$ periodicity was observed by Kapitulnik et al. in nearly optimally doped BSCCO without magnetic field [13]. Transformation of the real-space data to reciprocal space showed a periodicity of $4a$ in the randomly distributed regions with anomalous LDOS, which was manifest in four distinct peaks in reciprocal space [13]. Two peaks corresponding to a periodicity of $8a$ along the diagonal due to the periodically missing line of Bi atoms in the BiO plane were also seen clearly, confirming the sample quality. However, later studies by Davis et al. have cast doubt on the stripe interpretation: by analyzing the energy dependence of the wave vectors associated with the modulation, they have argued that the checkerboard LDOS modulation in BSCCO is an effect of quasiparticle interference, and not a signature of stripes [14]. Thus there is currently no agreement concerning the interpretation of the STM data.

In summary, one may state that the presence of stripes is now firmly established in LSCO (static in Nd-doped LSCO, dynamical in pure LSCO). Recent neutron scattering experiments indicate that they are also present in YBCO. However, the magnetic YBCO stripes seem to be dynamical, whereas the charge ones are static. Concerning BSCCO, only STM measurements, which are just sensitive to charge and have the disadvantage of being susceptible to surface defects and pinning, were performed hitherto. They suggest that a
static charge stripe is present in this compound, although many further measurements are required for a definitive understanding.

8. Conclusions

Doped antiferromagnetic insulators have recently attracted a great deal of attention because many of the materials in this class exhibit novel and interesting behavior. The cuprates, for example, become metallic at low doping concentration, and even superconducting at relatively high temperatures, whereas other systems, such as nickelates, show metallic behavior only at very high doping and are never superconducting. Manganites, on the other hand, can display the phenomenon of giant magneto resistance. Despite the different electric and magnetic properties exhibited by these compounds, a common perovskite structure connects them. Moreover, spontaneous symmetry breaking and stripe formation seems also to be a shared feature.

Stripe-like ground states were first predicted from Hartree-Fock calculations of the Hubbard model [49-53]. Later studies of the $t-J$ model also confirmed the initial results [54]. However, this theoretical work attracted significant attention only when Tranquada et al. detected the existence of static spin and charge order in Nd-doped LSCO [16]. The peaks measured by elastic neutron scattering were incommensurate with pure antiferromagnetic order, and suggested that the system had undergone a phase-separation into 1D regions rich in holes (stripes), which were acting as domain walls in the staggered magnetization. The fact that the magnetization changes phase by $\pi$ when crossing a domain wall has the consequence that the magnetic periodicity is twice the charge-stripe spacing. The associated incommensurability is therefore one half of the charge incommensurability, and the detection of both magnetic and charge incommensuration gave undeniable support to the stripe theory [16].

Later inelastic neutron scattering experiments in pure LSCO showed that there is a spin gap in these materials, but that incommensurate peaks can still be measured at rather low energies. The actual value of the spin gap depends on the doping. The incommensurability measured in these compounds is exactly the same as that obtained in Nd-doped LSCO, which is understandable considering replacement of La$^{3+}$ with Nd$^{3+}$ does not add charge carriers to the system, but induces a structural transition that helps to pin the stripes. The presence of stripes in LSCO has been confirmed by several different experimental techniques. Inelastic neutron-scattering experiments [17-23] indicate that incommensurate peaks are also detectable in YBCO, with a linear dependence of the incommensurability as a function of doping concentration similar to that observed in lanthanates. These results seem to indicate that the striped phase could be a generic feature of cuprates, instead of a peculiarity of LSCO. The requirement of very large samples imposed by the neutron scattering renders
experimental progress in the field very slow. Concerning BSCCO, at present only STM data is available and while these may support the existence of stripes [12, 13], the question remains open. It is important to note here that STM is a surface probe, whereas neutron scattering measures bulk properties. Thus, surface impurities could play a dominant role in STM experiments, but a secondary one in neutron scattering. The years to come will hopefully show the truth behind all the controversies.

From the theoretical side there has been steady progress in understanding stripe phases. Different approaches have been applied to the problem with differing degrees of success. Mean-field theories of different kinds [113–115], gauge theories [116], and quantum liquid crystal phenomenology [117] have been employed to describe the stripe state. In this review we focused primarily on studies based on the $t$-$J$ model, where DMRG calculations have shown the presence of stripe phases [58, 60]. These studies have the advantage that rather few parameters determine the physics of stripe formation. Although this kind of approach describes very well the nature of the stripes, it seems to indicate that the stripe state is essentially insulating and extra degrees of freedom, such as phonons [118], may be required to explain the experimental data in these systems [119, 120].

Phenomenological models have also contributed to a significant evolution in understanding. Appropriate models for describing stripe fluctuations have been developed, and analogies with other known systems established [99, 103, 104]. The effect of stripe pinning by impurities and by the underlying lattice, as well as the differing roles of rare-earth and planar impurities, has been clarified [89, 102, 104, 106, 109].

An important task remaining for the coming years is to show, both experimentally and theoretically whether and how stripes are connected to superconductivity. Systematic investigation of the different ways to suppress superconductivity may yield answers to this complex question.

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References

[1] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).

[2] For a recent review with a different perspective see, E. W. Carlson, V. J. Emery, S. A. Kivelson, and D. Orgad, in *The Physics of Conventional and Unconventional Superconductors* ed. by K. H. Bennemann and J. B. Ketterson (Springer-Verlag, 2002); cond-mat/0206217.

[3] S. Mori *et al.*, Nature 392, 473 (1998); Phys. Rev. Lett. 81, 3972 (1998).

[4] C. Renner, G. Aeppli, B.-G. Kim, Y.-A. Soh, and S.-W. Cheong, Nature 416, 518 (2002).

[5] J. M. Tranquada *et al.*, Phys. Rev. Lett. 73, 1003 (1994); S-H. Lee and S-W. Cheong, Phys. Rev. Lett. 79, 2514 (1997); H. Yoshizawa *et al.*, Phys. Rev. B 61, R854 (2000).

[6] C-H. Du *et al.*, Phys. Rev. Lett. 84, 3911 (2000).

[7] Y. Yoshinari *et al.*, Phys. Rev. Lett. 82, 3536 (1999).

[8] T. Katsufuji *et al.*, Phys. Rev. B 54, 14230 (1996); G. Blumberg *et al.*, Phys. Rev. Lett. 80, 564 (1998); K. Yamamoto *et al.*, Phys. Rev. Lett. 80, 1493 (1998).

[9] E. W. Hudson, S. H. Pan, A. K. Gupta, K. -W. Ng, and J. C. Davis, Science 285, 88 (1999).

[10] S. H. Pan *et al.*, Nature 413, 282 (2001).

[11] K. M. Lang, V. Madhavan, J. E. Hoffman, E. W. Hudson, H. Eisaki, S. Uchida, and J. C. Davis, Nature 415, 412 (2002).

[12] J. E. Hoffman, E. W. Hudson, K. M. Lang, V. Madhavan, H. Eisaki, S. Uchida, and J. C. Davis, Science 295, 466 (2002).

[13] C. Howald, H. Eisaki, N. Kaneko, and A. Kapitulnik, cond-mat/0201546; C. Howald, H. Eisaki, N. Kaneko, M. Greven, and A. Kapitulnik, Phys. Rev. B 67, 014533 (2003).

[14] J. E. Hoffman, K. McElroy, D. -H. Lee, K. M. Lang, H. Eisaki, S. Uchida, and J. C. Davis, Science 297, 1148 (2002).

[15] S-W. Cheong *et al.*, Phys. Rev. Lett. 67, 1791 (1991); B. O. Wells *et al.*, Science 277, 1070 (1997); G. Aeppli *et al.*, Science 278, 1432 (1998).
[16] J. M. Tranquada et al., Nature 375, 561 (1995); ibid. Phys. Rev. Lett. 78, 338 (1997); ibid. Phys. Rev. B 54, 7489 (1996); N. Ichikawa et al., Phys. Rev. Lett. 85, 1738 (2000).

[17] K. Yamada et al., Phys. Rev. B 57, 6165 (1998); Y. S. Lee et al., Phys. Rev. B 60, 3643 (1999).

[18] M. Matsuda et al., Phys. Rev. B 61, 4326 (2000); ibid. 62, 9148 (2000); S. Wakimoto et al., Phys. Rev. B 61, 3699 (2000).

[19] P. Dai et al., Phys. Rev. Lett. 80, 1738 (1998); ibid. Science 284, 1344 (1999).

[20] H. A. Mook et al., Nature 395, 580 (1998).

[21] H. A. Mook et al., Nature 401, 145 (1999).

[22] H. A. Mook et al., Phys. Rev. Lett. 88, 097004 (2002).

[23] H. A. Mook et al., Nature 404, 729 (2000).

[24] P. Dai et al., Phys. Rev. B 63, 054525 (2001).

[25] M. Arai et al., cond-mat/9912233; ibid. Phys. Rev. Lett. 83, 608 (1999).

[26] B. Grøvin, Y. Berthier, and C. Collin, Phys. Rev. Lett. 85, 1310 (2000).

[27] B. J. Suh et al., Phys. Rev. B 61, R9265 (2000); M.-H. Julien et al., Phys. Rev. Lett. 84, 3422 (2000).

[28] F. Borsa et al., Phys. Rev. B 52, 7334 (1995); B. Nachumi et al., Phys. Rev. B 58, 8760 (1998); C. Niedermayer et al., Phys. Rev. Lett. 80, 3843 (1998).

[29] M. Akoshima et al., Phys. Rev. B 62, 6761 (2000); J. Low Temp. Phys. 117, 1163 (1999).

[30] A. Lucarelli et al., Phys. Rev. Lett. 90, 037002 (2003).

[31] D. N. Basov et al., Phys. Rev. Lett. 74, 598 (1995).

[32] T. Noda et al., Science 286, 265 (1999). A. Gupta et al., Phys. Rev. B 61, 11752 (2000).

[33] X. J. Zhou et al., Science 286, 268 (1999).

[34] R. P. Sharma et al., Phys. Rev. Lett. 77, 4624 (1996); Nature 404, 736 (2000).

[35] E. Fawcett, H. L. Alberts, V. Yu. Galkin, D. R. Noakes, and J. V. Yakhmi, Rev. Mod. Phys. 66, 25 (1994).

[36] G. Gruner, Density Waves in Solids (Perseus Publishing, Cambridge, MA, 1994).

[37] R. L. Withers and J. A. Wilson, J. Phys. C 19, 4809 (1986).

[38] R. M. Fleming, D. E. Moncton, D. B. McWhan, and F. J. DiSalvo, Phys. Rev. Lett. 45, 576 (1980).

[39] W. L. McMillan, Phys. Rev. B 12, 1187 (1975).

[40] A. S. Davidov, Solitons in Molecular Systems (D. Reidel, Dordrecht, 185).
REFERENCES

[41] G. Kotliar, Eur. J. Phys. B 11, 27 (1999).
[42] A. Georges, G. Kotliar, W. Krauth and M. Rozenberg, Rev. Mod. Phys. 68, 13 (1996).
[43] J. G. Bednorz and K. A. Muller, Z. Phys. B 64, 189 (1986).
[44] N. F. Mott, *Metal-Insulator Transitions* (Taylor & Francis, London, 1974).
[45] S. Chakravarty, B. I. Halperin, and D. Nelson, Phys. Rev. Lett. 60, 1057 (1988); Phys. Rev. B 39, 2344 (1989).
[46] T. D. Stanescu and P. Phillips, cond-mat/0104478.
[47] P. W. Anderson, Science 235, 1196 (1987).
[48] F. C. Zhang and T. M. Rice, Phys. Rev. B 37, 3759 (1988).
[49] J. Zaanen and O. Gunnarsson, Phys. Rev. B 40, 7391 (1989).
[50] H. J. Schulz, Europhys. Lett. 4, 609 (1987); Phys. Rev. Lett. 64, 1445 (1990).
[51] M. Inui and P. B. Littlewood, Phys. Rev. B 44, 4415 (1991).
[52] D. Poilblanc and T. M. Rice, Phys. Rev. B 39, 9749 (1989).
[53] K. Machida, Physica C 158, 192 (1989); K. Machida and M. Ichioka, J. Phys. Soc. Jpn. 68, 2168 (1999).
[54] S. R. White and D. J. Scalapino, Phys. Rev. Lett. 80, 1272 (1998); *ibid.* 81, 3227 (1998); Phys. Rev. B 60, R753 (1999); Phys. Rev. B 61, 6320 (2000).
[55] A. Moreo et al., Phys. Rev. B 43, 11442 (1991); H. Fehske et al., Phys. Rev. B 44, 8473 (1991); R. Valenti and C. Gros, Phys. Rev. Lett. 68, 2402 (1992); D. Poilblanc, Phys. Rev. B 52, 9201 (1995); M. Kohno, Phys. Rev. B 55, 1435 (1997).
[56] E. Dagotto, Rev. Mod. Phys. 66, 763 (1994).
[57] V. J. Emery, S. A. Kivelson, and H. Q. Lin, Phys. Rev. Lett. 64, 475 (1990).
[58] A. L. Chernyshev, A. H. Castro Neto, and A. R. Bishop, Phys. Rev. Lett. 84, 4922 (2000).
[59] G. B. Martins, C. Gazza, J. C. Xavier, A. Feiguin, and E. Dagotto, Phys. Rev. Lett. 84, 5844 (2000); G. B. Martins, C. Gazza, and E. Dagotto, Phys. Rev. B 62, 13926 (2000); G. B. Martins, J. C. Xavier, C. Gazza, M. Vojta, and E. Dagotto, *ibid.* 63, 014414 (2001).
[60] A. L. Chernyshev, S. R. White, and A. H. Castro Neto, Phys. Rev. B 65, 214527 (2002).
[61] S. R. White and D. J. Scalapino, Phys. Rev. B 60, R753 (1999); B. Normand and A. P. Kampf, Phys. Rev. B 65, 020509 (2002).
[62] B. P. Stojkovic, Z. G. Yu, A. L. Chernyshev, A. R. Bishop, A. H. Castro Neto, and N. Grønbech-Jensen, Phys. Rev. B 62, 4353 (2000); B. P. Stojkovic, Z. G. Yu, A. R. Bishop, A. H. Castro Neto, and N. Grønbech-Jensen, Phys. Rev. Lett. 82, 4679 (1999).
[63] B. Normand and A. Kampf, Phys. Rev. B 64, 024521 (2001).
[64] F. Becca, L. Capriotti, and S. Sorella, Phys. Rev. Lett. 87, 167005 (2001).
[65] M. Blume, V. J. Emery, and R. B. Griffiths, Phys. Rev. A 4, 1071 (1971); U. L"ow, V. J. Emery, K. Fabricius, and S. A. Kivelson, Phys. Rev. Lett. 72, 1918 (1994).

[66] J. A. Hertz, Phys. Rev. B 14, 1165 (1976).

[67] V. J. Emery et al., Nature 374, 434 (1995); ibid. cond-mat/0001077. V. J. Emery and S. A. Kivelson, Physica C 209, 594 (1993); ibid. 235-240, 189 (1994); ibid. Phys. Rev. Lett. 74, 3253 (1995); L"ow et al., Phys. Rev. Lett. 72, 1918 (1994).

[68] L. P. Pryadko et al., Phys. Rev. B 60, 7541 (1999).

[69] B. Buchner et al., Phys. Rev. Lett. 73, 1841 (1994).

[70] Y. S. Lee, R. J. Birgeneau, M. A. Kastner, Y. Endoh, S. Wakimoto, K. Yamada, R. W. Erwin, S.-H. Lee, G. Shirane, Phys. Rev. B 60, 3643 (1999); B. Khaykovich et al., Phys. Rev. B 67, 054501 (2003); A. T. Savici et al., cond-mat/0202037.

[71] Moodenbaugh et al., Phys. Rev. B 38, 4596 (1988).

[72] Y. Koike et al., Sol. St. Commun. 82, 889 (1992); J. Phys. Soc. Jpn. 67, 3653 (1998); J. Low. Temp. Phys. 105, 317 (1996); ibid., 117, 1163 (1999); T. Adachi et al., ibid. 117, 1151 (1999); M. Akoshima et al., Phys. Rev. B 57, 7491 (1998).

[73] N. Kakinuma, Y. Ono, and Y. Koike, Phys. Rev. B 59, 1491 (1999).

[74] K. Yamada et al., J. Superconductivity 10, 343 (1997).

[75] N. Hasselmann, A. H. Castro Neto, and C. Morais Smith, Europhys. Lett. 56, 870 (2001).

[76] J. Lorenzana and G. Seibold, Phys. Rev. Lett. 89, 136401 (2002).

[77] V. J. Emery et al., Phys. Rev. Lett. 64, 475 (1990).

[78] C. S. Hellberg and E. Manousakis, Phys. Rev. Lett. 78, 4609 (1997).

[79] W. O. Puttika et al., Phys. Rev. Lett. 68, 538 (1992).

[80] A. L. Chernyshev and R. F. Wood, cond-mat/0208541 (unpublished).

[81] Y. Nagaoka, Phys. Rev. 147, 392 (1966).

[82] S. R. White, I. Affleck, Phys. Rev. B 64, 024411 (2001).

[83] S. Trugman, Phys. Rev. B 37, 1597 (1988).

[84] A. H. Castro Neto, Z. Phys. B-Cond. Matter, 103, 185 (1997).

[85] Oron Zachar, Phys. Rev. B 65, 174411 (2002)

[86] Leonid P. Pryadko, Steven A. Kivelson, V. J. Emery, Yaroslav B. Bazaliy, Eugene A. Demler, Phys. Rev. B 60, 7541 (1999).

[87] I. Martin, G. Ortiz, A. V. Balatsky, and A. R. Bishop, Europhys. Lett. 56, 849 (2001); Int. J. Mod. Phys. 14, 3567 (2000).

[88] J. Voit, Rep. Prog. Phys. 58, 977 (1995).
REFERENCES

[89] N. Hasselmann, A. H. Castro Neto, C. Morais Smith, and Yu. A. Dimashko, Phys. Rev. Lett. 82, 2135 (1999).

[90] P. C. Hohenberg, Phys. Rev. 158, 383 (1967); N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966).

[91] N. Nucker et al., Phys. Rev. B 37, 5158 (1988); J.-H. Guo et al., Phys. Rev. B 49, 1376 (1994).

[92] P. C. Hammel, B. W. Statt, R. L. Martin, F. C. Chou, D. C. Johnston, and S-W. Cheong, Phys. Rev. B 57, R712 (1998).

[93] A. Aharony, R. J. Birgeneau, A. Coniglio, M. A. Kastner, and H. E. Stanley, Phys. Rev. Lett. 60, 1330 (1988).

[94] O. P. Vajk, P. K. Mang, M. Greven, P. M. Gehring, and J. W. Lynn, Science 295, 1691 (2002).

[95] A. L. Chernyshev, Y. C. Chen, and A. H. Castro Neto, Phys. Rev. Lett. 87, 067209 (2001); Phys. Rev. B 65, 214527 (2002).

[96] D. Hone and A. H. Castro Neto, J. of Superconductivity, 10, 349 (1997); A. H. Castro Neto and D. Hone, Phys. Rev. Lett. 76, 2165 (1996).

[97] F. D. M. Haldane, Phys. Lett. 93 A, 464 (1983).

[98] C. N. A. van Duin and J. Zaanen, Phys. Rev. Lett. 78, 3019 (1997).

[99] H. Eskes, Y. Osman, R. Grimberg, W. van Saarloo, and J. Zaanen, Phys. Rev. B 58, 6963 (1998); H. Eskes, R. Grimberg, W. van Saarloos, and J. Zaanen, Phys. Rev. B 54, R724 (1996).

[100] M. den Nijs and K. Rommelse, Phys. Rev. B 40, 4709 (1989).

[101] H. J. Schulz, Phys. Rev. B 34, 6372 (1986).

[102] N. Hasselmann, A. H. Castro Neto, and C. Morais Smith, Phys. Rev. B 65, 220511(R) (2002).

[103] C. Morais Smith, Yu. Dimashko, N. Hasselmann, and A. O. Caldeira, Phys. Rev. B 58, 453 (1998).

[104] Yu. A. Dimashko, C. Morais Smith, N. Hasselmann, and A. O. Caldeira, Phys. Rev. B 60, 88 (1999).

[105] R. M. Bradley and S. Doniach, Phys. Rev. B 30, 1138 (1984).

[106] S. Bogner and S. Scheidl, Phys. Rev. B 64, 054517 (2001).

[107] G. Xiao, M. Z. Cieplak, J. Q. Xiao, and C. L. Chien, Phys. Rev. B 42, 8752 (1990).

[108] C. H. Ahn et al., Science 284, 1152 (1999).

[109] C. Morais Smith, N. Hasselmann, and A. H. Castro Neto, AIP Conf. Proceedings 554, 209 (2001).
42

[110] G. Blatter et al., Rev. Mod. Phys. 66, 1125 (1994).

[111] H. Kimura et al., Phys. Rev. B 59, 6517 (1999); K. Hirota et al., Physica B 241-243, 817 (1998).

[112] B. Lake et al., Science 291, 1759 (2001); Nature 415, 299 (2002).

[113] C. Di Castro, L. Benfatto, S. Caprara, C. Castellani, and M. Grilli, Physica C 341, 1715 (2000).

[114] R. S. Markiewicz, J. Phys. Chem. Solids 58, 1179 (1997).

[115] J. Ashkenazi, Int. J. Mod. Phys. B 13, 3422 (1999).

[116] H. V. Kruis, Z. Nussinov, and J. Zaanen, cond-mat/0110055.

[117] V. J. Emery, E. Fradkin, S. A. Kivelson, and T. C. Lubensky, Phys. Rev. Lett. 85, 2160 (2000); D. G. Barci and E. Fradkin, Phys. Rev. B 65, 45320 (2002).

[118] A. H. Castro Neto, Phys. Rev. B 64, 104509 (2001).

[119] R. J. McQueeney, Y. Petrov, T. Egami, M. Yethiraj, G. Shirane, and Y. Endoh, Phys. Rev. Lett. 82, 628 (1999).

[120] Z.-X. Shen, A. Lanzara, S. Ishihara, N. Nagaosa, cond-mat/0108381 and references therein.