PIERCING FAMILIES OF CONVEX SETS IN THE PLANE THAT AVOID A CERTAIN SUBFAMILY WITH LINES

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Abstract. We define a $C(k)$ to be a family of $k$ sets $F_1, \ldots, F_k$ such that $\text{conv}(F_i \cup F_{i+1}) \cap \text{conv}(F_j \cup F_{j+1}) = \emptyset$ when $\{i, i+1\} \cap \{j, j+1\} = \emptyset$ (indices are taken modulo $k$). We show that if $\mathcal{F}$ is a family of compact, convex sets that does not contain a $C(k)$, then there are $k - 2$ lines that pierce $\mathcal{F}$. Additionally, we give an example of a family of compact, convex sets that contains no $C(k)$ and cannot be pierced by $\lceil \frac{k}{2} \rceil - 1$ lines.

1. Introduction

Let $\mathcal{F}$ be a family of sets in the plane; $\mathcal{F}$ is said to have a line transversal if there is a line that intersects each set in $\mathcal{F}$. If every $r$ sets in $\mathcal{F}$ have a line transversal, then $\mathcal{F}$ is said to have the $T(r)$-property, and $\mathcal{F}$ is said to be $T^n$ if there are $n$ lines whose union intersects each set in $\mathcal{F}$. In this case we say $\mathcal{F}$ is pierced by these lines. In 1969, Eckhoff showed that if $\mathcal{F}$ is a family of compact, convex sets that has the $T(r)$-property where $r \geq 4$, then $\mathcal{F}$ is $T^2$ [1]. A result of Santalo shows that this result is best possible [12], i.e. for all $r$, there exists a family of compact, convex sets with the $T(r)$-property that does not have a line transversal. Eckhoff also showed in 1973 that the $T(3)$-property does not imply $T^2$ [2]. In 1975, Kramer proved that the $T(3)$-property implies $T^3$ [9]. Eckhoff later showed in 1993 that the $T(3)$-property implies $T^4$, and conjectured that the $T(3)$-property in fact implies $T^3$ [3]. This conjecture has recently been verified by McGinnis and Zerbib [11]. In fact, they proved a stronger statement, which we now explain.

Three sets $F_1, F_2, F_3$ in the plane are said to be a tight triple if $\text{conv}(F_1 \cup F_2) \cap \text{conv}(F_2 \cup F_3) \cap \text{conv}(F_3 \cup F_1) \neq \emptyset$. This was first defined by Holmsen [6]. A family of planar sets will be called a family of tight triples if every three sets in the family are a tight triple. If three sets have a line transversal, then they are a tight triple as the convex hull of two of the three sets intersects the third. McGinnis and Zerbib showed that a family of tight triples consisting of compact, convex sets is $T^3$, which implies Eckhoff’s conjecture.

The main purpose of this paper is to prove an extension of the result that families of tight triples are $T^3$. We define a certain type of family of sets, which we call a $C(k)$.

Definition 1.1. For $k \geq 4$, we define a $C(k)$ to be a family of $k$ distinct sets in the plane together with a linear ordering, say $F_1, \ldots, F_k$ where the sets are ordered by their indices, such that $\text{conv}(F_i \cup F_{i+1}) \cap \text{conv}(F_j \cup F_{j+1}) = \emptyset$ when $\{i, i+1\} \cap \{j, j+1\} = \emptyset$ (indices are taken modulo $k$). We show that if $\mathcal{F}$ is a family of compact, convex sets that does not contain a $C(k)$, then there are $k - 2$ lines that pierce $\mathcal{F}$. Additionally, we give an example of a family of compact, convex sets that contains no $C(k)$ and cannot be pierced by $\lceil \frac{k}{2} \rceil - 1$ lines.

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Figure 1. An example of a $C(5)$. The sets of the $C(k)$ are in dark gray, and the ordering on these sets is indicated by the numbers $1, \ldots, 5$. The dark gray and light gray together depict the union $\bigcup_{i=1}^{5} \text{conv}(F_i \cup F_{i+1})$.

Figure 2. A $C(5)$-free and $C(3)$-free family that is not $C(4)$-free.

$\{i, i+1\} \cap \{j, j+1\} = \emptyset$ (indices are taken modulo $k$). Additionally, we define a $C(3)$ to be a family of three disjoint sets in the plane that is not a tight triple.

Roughly speaking, if $F_1, \ldots, F_k$ is a $C(k)$, then the union $\bigcup_{i=1}^{k} \text{conv}(F_i \cup F_{i+1})$ resembles a closed loop that does not cross itself (see Figure 1). Notice that the sets in a $C(k)$ are pairwise disjoint.

A family $\mathcal{F}$ is said to be $C(k)$-$\text{free}$ if $\mathcal{F}$ does not contain a $C(k)$ as a subfamily. We note that a $C(k)$-free family may not be $C(k-1)$-free, and similarly, a $C(k-1)$-free family need not be $C(k)$-free. (see Figure 2).

Let $L(k)$ be the smallest integer such that any $C(k)$-free family of compact, convex sets can be pierced by $L(k)$ lines. The following is the main result of this paper.
Theorem 1.2. Let \( k \geq 4 \). We have the following:

\[
\left\lceil \frac{k}{2} \right\rceil \leq L(k) \leq k - 2.
\]

For \( k = 4 \), the lower bound of Theorem 1.2 follows from the result of Santaló 12 that there are families with the \( T(4) \)-property that do not have a line transversal. This is due to the fact that a \( C(k) \) cannot have a line transversal, so a family with the \( T(4) \)-property is in particular \( C(4) \)-free. We also note that the upper bound for \( k = 4 \) was essentially proved in the concluding remarks of 11. Indeed, the proof outlined in 11 shows that if \( \mathcal{F} \) is a family of compact, convex sets in the plane that is not \( T^2 \), then there are non-parallel lines such that each quadrant defined by these two lines contains a set from \( \mathcal{F} \). These 4 sets then make up a \( C(4) \).

For \( k = 5 \), we get a tight result.

Corollary 1.3. The following equality holds:

\[
L(5) = 3.
\]

The proof of Theorem 1.2 is split into two sections. Section 2 is dedicated to proving the lower bound of Theorem 1.2 and Section 3 is dedicated to the upper bound. For 2 points \( p \) and \( q \), we denote by \([p, q]\) to be the line segment connecting \( p \) and \( q \). If \( p = q \), then \([p, q]\) consists of a single point.

2. The lower bound

In this section we exhibit a \( C(k) \)-free family \( \mathcal{F} \) that is not \( T^{\left\lceil \frac{k}{2} \right\rceil - 1} \). The inspiration for the construction of such a family comes from 2 where an example of a family of compact, convex sets with the \( T(3) \)-property that is not \( T^2 \) is exhibited by Eckhoff. As mentioned earlier, the result is already established for \( k = 4 \) so we may assume that \( k \geq 5 \).

For \( k \) odd, we will present a family \( \mathcal{F} \) that is both \( C(k) \)-free and \( C(k + 1) \)-free and is not \( T^{\left\lceil \frac{k}{2} \right\rceil - 1} \). This will establish the lower bound of Theorem 1.2. We note that for \( k \) even, an example of a \( C(k) \)-free family that is not \( T^{\left\lceil \frac{k}{2} \right\rceil - 1} \) is given simply by \( k - 1 \) points in general position. However, in this example, the family is \( C(k) \)-free for the seemingly trivial reason that are only \( k - 1 \) sets in the family. For each \( k \geq 5 \), we present a family demonstrating the lower bound of Theorem 1.2 that contains more than \( k \) sets.

Let \( p_1, \ldots, p_{3(k-1)} \) be equidistant points on the unit circle, arranged clockwise. For \( 1 \leq i \leq 3 \), let \( p_i^k \) be a point lying slightly counterclockwise to \( p_i \) and \( p_i^c \) to be a point lying slightly clockwise to \( p_i \) in such a way that the points \( p_1^k, p_1^c, p_2^k, p_2^c, p_3^k, p_4, \ldots, p_{3(k-1)} \) are arranged in clockwise order.

We first define three families of sets, which consists only of line segments (see Figure 3).

\[
\mathcal{F}_1 = \{ [p_1^k, p_1^c], [p_4, p_6], [p_7, p_9], \ldots, [p_{3(k-1)}-2, p_{3(k-1)}] \}
\]

\[
\mathcal{F}_2 = \{ [p_2, p_4], [p_5, p_7], [p_8, p_{10}], \ldots, [p_{3(k-1)}-4, p_{3(k-1)}-2], [p_{3(k-1)}-1, p_1^c] \}
\]

\[
\mathcal{F}_3 = \{ [p_3, p_5], [p_6, p_8], [p_{10}, p_{12}], \ldots, [p_{3(k-1)}-3, p_{3(k-1)}-1], [p_{3(k-1)}, p_2^c] \}.
\]

Finally, we take \( \mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \) (see Figure 3). We now show that \( \mathcal{F} \) is \( C(k) \)-free and \( C(k + 1) \)-free, and we show it is not \( T^{\left\lceil \frac{k}{2} \right\rceil - 1} \). For a set \([p, q] \in \mathcal{F}\) we say that \( p \) comes clockwise before \( q \) if the clockwise distance along the unit circle from \( p \)
Figure 3. An example demonstrating the lower bound of Theorem 1.2 for \( k = 5 \). The sets in \( F_1 \) are red, in \( F_2 \) are blue, and in \( F_3 \) are green.

to \( q \) is less than the clockwise distance from \( q \) to \( p \). When \([p, q] \in F\) and \( p \) comes clockwise before \( q \), we denote \( \text{arc}[p, q] \) to be the arc along the unit circle that goes clockwise from \( p \) to \( q \). When \([p, q] \in F\) and \( p \) comes before \( q \), we denote \( \text{arc}[p, q] \) to be the arc along the unit circle that goes clockwise from \( p \) to \( q \). Also, we use \( I([p, q]) \) to denote the set of indices \( i \) such that \( p_i \in \text{arc}[p, q] \), \( p_i' \in \text{arc}[p, q] \), or \( p_i'' \in \text{arc}[p, q] \). For example, \( I(p_5', p_4) = \{2, 3\} \). Note that if \( F_1, F_2 \in F \) intersect, then \( I(F_1) \cap I(F_2) \neq \emptyset \).

Lemma 2.1. The family \( F \) is \( C(k) \)-free and \( C(k + 1) \)-free.

Proof. Let \([p, q] \in F\) where \( p \) comes clockwise before \( q \). There is no set of \( F \) that is disjoint from \([p, q]\) and contains a point on \( \text{arc}[p, q] \). Since for each such set \([p, q]\), \( \text{arc}[p, q] \) contains at least 3 of the \( 3k \) points in

\[ P = \{p_1^k, p_1', p_2^k, p_2', p_3^k, p_3', p_4, \ldots, p_{3(k-1)}\}, \]

a \( C(k) \) of \( F \) has the property that each set \([p, q]\) in the \( C(k) \) contains exactly 3 points in \( \text{arc}[p, q] \), and every point of \( P \) is in \( \text{arc}[p, q] \) for some \([p, q]\) of the \( C(k) \). However, each set \([p', q']\) of \( F \) that contains \( p_5^k \) in \( \text{arc}[p', q'] \) has the property that at least 4 points of \( P \) are contained in \( \text{arc}[p', q'] \), a contradiction. Also, \( F \) is \( C(k + 1) \)-free by the same reasoning. \( \square \)

Lemma 2.2. The family \( F \) is not \( T[\frac{k}{2}] \)-free.

Proof. Notice that for any \( F \in F \), if a line \( L \) pierces \( F \), then \( L \) intersects \( \text{arc}F \). Any point on the unit circle is contained in \( \text{arc}F \) for at most 3 sets \( F \in F \). If a point is contained in \( \text{arc}F \) for 3 such sets \( F \in F \), then this point must be of the form \( p_i \) for some \( 4 \leq i \leq 3(k - 1) \). Since a line intersects the unit circle in at most 2 points, any line intersects at most 6 sets in \( F \).

Now, since \( \text{arc}[p_1^k, p_3^k] \) does not contain \( p_i \) for any \( 4 \leq i \leq 3(k - 1) \), there is no line that intersects \([p_1^k, p_3^k]\) and intersects 6 sets in \( F \).
It follows that if a lines pierce $F$, then $6(a-1)+5 \geq 3(k-1)$, so $a > \frac{k+1}{2} = \left\lfloor \frac{k}{2} \right\rfloor - 1$. This completes the proof. 

\[ \square \]

3. The upper bound

In this section, we prove that every $C(k)$-free family $F$ can be pierced by $k-2$ lines, and again, we may assume that $k \geq 5$. Because the sets of $F$ are compact, it is the case that if every finite subfamily of $F$ is $T^n$, then $F$ is $T^n$. This is stated for instance in \[3\]. Therefore, throughout this section we may assume that $F$ is finite, and thus, we may scale the plane so that each set of $F$ is contained in the open unit disk.

First, we will need to introduce a topological tool known as the KKM Theorem \[8\].

Let $\Delta^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$ denote the $(n-1)$-dimensional simplex in $\mathbb{R}^n$, whose vertices are the canonical basis vectors $e_1, \ldots, e_n$. A face $\sigma$ of $\Delta^{n-1}$ is a subset of $\Delta^{n-1}$ of the form $\text{conv} \{(e_i : i \in I)\}$ for some $I \subset [n]$.

**Theorem 3.1.** Let $A_1, \ldots, A_n$ be open sets such that for every face $\sigma$ of $\Delta^{n-1}$ we have $\sigma \subset \bigcup_{i \in \sigma} A_i$. Then we have that $\cap_{i=1}^{n} A_i \neq \emptyset$.

We remark that the original KKM Theorem was stated for when the sets $A_i$ are closed, and the statement where the $A_i$’s are open as stated here appears in e.g. \[10\].

Let $U$ be the unit circle, and let $f : [0, 1] \to U$ be a parameterization of $U$ defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$.

A point $x = (x_1, \ldots, x_{2(k-2)}) \in \Delta^{2(k-2)-1}$ corresponds to $2(k-2)$ points on $U$ given by $f_i(x) = f(\sum_{j=1}^{i} x_j)$ for $0 \leq i \leq 2(k-2)$ (note that $f_0(x) = f_{2(k-2)}(x)$). We define the line segments $\ell_i(x) = [f_i(x), f_i+k-2(x)]$ for $0 \leq i \leq 2(k-2)-1$ where addition is taken modulo $2(k-2)$ (see Figure 4). Note that $\ell_i(x) = \ell_{i+k-2}(x)$.

For $1 \leq i \leq k-3$, we define the region $R_x^i$ to be the open set bounded by the lines $\ell_i(x)$ and $\ell_{i-1}(x)$ and by the arc from $f_{i-1}(x)$ to $f_i(x)$. For $k-2 \leq i \leq 2(k-2)$, we define $R_x^i$ to be the intersection of the region in the open unit disk bounded by $\ell_{i-1}(x)$, $\ell_i(x)$ and the arc from $f_{i-1}(x)$ to $f_i(x)$ with the open halfspaces defined by $\ell_j(x)$ for those $1 \leq j \leq 2(k-2)$ for which $\ell_j(x)$ is a line segment and (not a point) containing the open arc from $f_{i-1}(x)$ to $f_i(x)$ (see Figures 4-7).

**Lemma 3.2.** Assume that each $R_x^i$ is non-empty. Let $Q_1$ be the open quadrant defined by the lines $\ell_{k-3}(x)$ and $\ell_{k-2}(x)$ that contains the open arc along the unit circle from $f_0(x)$ to $f_{k-3}(x)$. There is some $1 \leq j \leq k-3$ such that $R_x^j$ is contained in $Q_1$.

**Proof.** Let $Q_2, Q_3, Q_4$ be the remaining quadrants defined by $\ell_{k-3}(x)$ and $\ell_{k-2}(x)$ so that $Q_1, Q_2, Q_3, Q_4$ occur in counterclockwise order (see Figure 8). Since the regions $R_x^i$ are non-empty, the $Q_i$’s are non-empty. Assume that $R_x^{k-3}$ is not contained in $Q_1$, then $\ell_{k-4}(x)$ intersects $Q_4$. Similarly, if we assume that $R_x^{k}$ is not contained in $Q_1$, then $\ell_1(x)$ intersects $Q_2$. If $k = 5$, then we are done since $k-4 = 1$ and $\ell_1(x)$ cannot intersect both $Q_2$ and $Q_4$. Otherwise, $k \geq 6$, and we take $j$ to be the smallest index such that $\ell_j(x)$ intersects $Q_4$. Such an index exists since we assume that $\ell_{k-4}(x)$ intersects $Q_4$. Also, $j > 1$, since we assume that $\ell_1(x)$ intersects $Q_2$ (and hence does not intersect $Q_4$). Therefore $j-1 \geq 1$ and $\ell_{j-1}(x)$ intersects $Q_2$, or the intersection of $\ell_{k-2}(x)$ and $\ell_{k-3}(x)$. This implies that $\ell_{j-1}(x)$ and $\ell_j(x)$ intersect in $Q_1$ (see Figure 8). Therefore, $R_x^j$ is contained in $Q_1$. \[ \square \]
Lemma 3.3. If a connected set $F$ contained in the unit disc does not intersect any $\ell_j(x)$, then $F$ is contained in $R_x^i$ for some $i$.

Proof. Let $\tilde{R}_x^i$ be the region bounded by the arc from $f_{i-1}(x)$ to $f_i(x)$ and by the lines $\ell_{i-1}(x)$ and $\ell_i(x)$. Note that $\tilde{R}_x^i = R_x^i$ for $1 \leq i \leq k-3$. Also, we have that $F$ is contained in $\tilde{R}_x^i$ for some $i$. If $1 \leq i \leq k-3$, then we are done since $\tilde{R}_x^i = R_x^i$. So assume that $i \geq k-2$ and $F$ is not contained in $R_x^i$. Since $F$ does not intersect any $\ell_j(x)$, there is some $j$ such that $F$ is contained in the open halfspace defined by $\ell_j(x)$ that does not contain the arc from $f_{i-1}(x)$ to $f_i(x)$.

If $i = 2(k-2)$, then choose the largest index $j \in \{1, \ldots, k-4\}$ such that the open halfspace defined by $\ell_j(x)$ not containing the arc from $f_{2(k-2)-1}(x)$ to $f_0(x)$ contains $F$. Then $F$ is contained in $R_x^{j+1}$. This is because $R_x^{j+1}$ is the region in the open unit disk obtained by taking intersection of the open halfspace defined by $\ell_j(x)$ that does not contain the arc from $f_{2(k-2)-1}(x)$ to $f_0(x)$ (which contains $F$) with the open halfspace defined by $\ell_{j+1}(x)$ that contains the arc from $f_{2(k-2)-1}(x)$ to $f_0(x)$ (which contains $F$ by the maximality of $j$).

If $i = k-2$, then choose the smallest index $j \in \{1, \ldots, k-4\}$ such that the halfspace defined by $\ell_j(x)$ not containing the arc from $f_{k-3}(x)$ to $f_{k-2}(x)$ contains $F$. Then $F$ is contained in $R_x^i$ (by similar reasoning as above).

Hence, we may assume that $i \neq k-2, 2(k-2)$. Let $i' = i - (k-2)$. If there is some $u \in \{0, \ldots, i'-2\}$ such that the halfspace defined by $\ell_u(x)$ not containing the arc from $f_{i-1}(x)$ to $f_i(x)$ contains $F$, then let $j$ be the largest such index. Then $F$ is contained in $R_x^{j+1}$. Otherwise, choose the smallest index $j \in \{k-3, \ldots, i'+1\}$ such that the halfspace defined by $\ell_j(x)$ not containing the arc from $f_{i-1}(x)$ to $f_i(x)$ contains $F$. Then $F$ is contained in $R_x^i$.

This completes the proof. □

With the goal of using the KKM Theorem, we define $A_i$ to be the set of points $x \in \Delta^{2(k-2)-1}$ for which $R_x^i$ contains a set in $\mathcal{F}$. Because $R_x^i$ is open and each set in $\mathcal{F}$ is closed, it follows that $A_i$ is open. Let us assume for contradiction that there is no point $x \in \Delta^{2(k-2)-1}$ for which the lines $\ell_j(x)$, $0 \leq j \leq k-3$ pierce $\mathcal{F}$. Then by Lemma 3.3 for each $x \in \Delta^{2(k-2)-1}$, there is some region $R_x^i$ that contains a set in $\mathcal{F}$. It follows that $\Delta^{2(k-2)-1} \subseteq \bigcup_{i=1}^{2(k-2)-1} A_i$. Also, it is clear that for $x = (x_1, \ldots, x_{2(k-2)})$, if $x_i = 0$, then the region $R_x^i$ is empty and hence $x \notin A_i$. It follows from this fact that the sets $A_i$ satisfy the conditions of the KKM Theorem.

Therefore, by the KKM Theorem, there exists a point $x \in \bigcap_{i=1}^{2(k-2)-1} A_i$. Notice in particular that each $R_x^i$ is non-empty.

Let $1 \leq i \leq k-3$ be the index such that $R_x^i$ is contained in $Q_1$, guaranteed by Lemma 3.2 where $Q_1$ is defined as in Lemma 3.2.

Let $F_1$ be the set in $\mathcal{F}$ contained in $R_x^i$, and let $F_j$ be the set of $\mathcal{F}$ contained in $R_x^{k-4+j}$ for $2 \leq j \leq k$. Note that the corresponding regions $R_x^i$ and $R_x^j$ are disjoint, so $F_1, \ldots, F_k$ are pairwise distinct.

Now, conv$(F_1 \cup F_2)$ is separated from $F_3, \ldots, F_k$ by the line $\ell_0(x)$, so conv$(F_1 \cup F_2)$ is disjoint from conv$(F_u \cup F_{u+1})$ for all $3 \leq u \leq k-1$. For $j \in \{2, \ldots, k-1\}$, conv$(F_j \cup F_{j+1})$ is separated from $F_{j+2}, \ldots, F_k$ by $\ell_{k-4+j+1}(x)$, so conv$(F_j \cup F_{j+1})$ is disjoint from conv$(F_u \cup F_{u+1})$ for all $j+2 \leq u \leq k-1$. Finally, conv$(F_k \cup F_1)$ is separated from $F_2, \ldots, F_{k-1}$ by $\ell_{k-3}(x)$, so conv$(F_k \cup F_1)$ is disjoint from conv$(F_u \cup F_{u+1})$ for all $2 \leq u \leq k-2$. It follows that the sets $F_1, \ldots, F_k$ form a $C(k)$, a contradiction.
This completes the proof of Theorem 1.2.

4. Concluding Conjecture

We present a conjecture for the correct value of \( L(k) \), which states that the lower bound of Theorem 1.2 is correct.

**Conjecture 4.1.** We have that \( L(k) = \lceil \frac{k}{2} \rceil \).

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Figure 5. The regions $R_2^2$ (on the left) and $R_3^3$ (on the right).

Figure 6. The regions $R_4^4$ (on the left) and $R_5^5$ (on the right).

Figure 7. The region $R_6^6$. 
Figure 8. The quadrants $Q_1$, $Q_2$, $Q_3$, and $Q_4$.

Figure 9. If $\ell_j(x)$ intersects $Q_4$ and $\ell_{j-1}(x)$ intersects $Q_2$ (or the intersection of $\ell_0(x)$ and $\ell_{k-3}(x)$), then the intersection of $\ell_j(x)$ and $\ell_{j-1}(x)$ is in $Q_1$. 