A NOTE ON SUBGROUPS IN A DIVISION RING THAT ARE LEFT ALGEBRAIC OVER A DIVISION SUBRING

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Abstract. Let \( D \) be a division ring with center \( F \) and \( K \) a division subring of \( D \). In this paper, we show that a non-central normal subgroup \( N \) of the multiplicative group \( D^* \) is left algebraic over \( K \) if and only if so is \( D \) provided \( F \) is uncountable and contained in \( K \). Also, if \( K \) is a field and the \( n \)-th derived subgroup \( D^{(n)} \) of \( D^* \) is left algebraic of bounded degree \( d \) over \( K \), then \( \dim_F D \leq d^2 \).

1. Introduction

Let \( D \) be a division ring with center \( F \) and \( K \) a division subring of \( D \). Recall that an element \( a \in D \) is left algebraic over \( K \) if there exist \( a_0, a_1, \ldots, a_n \in K \) not all zeros such that \( a_0 + a_1a + \cdots + a_na^n = 0 \), or equivalently, there exists a non-zero polynomial \( f(t) \in K[t] \) whose coefficients are written on the left such that \( f(a) = 0 \). Here, we have to emphasize the convention that for polynomials over a ring \( R \) (commutative or not), a polynomial \( f(t) \in R[t] \) can be written in two ways such as

\[
    f(t) = \sum_{\text{finite}} a_it^i = \sum_{\text{finite}} t^ia_i.
\]

However, if \( a \in S \), where \( S \) is a ring containing \( R \), the substitution functions may give the different values, i.e., we may have

\[
    \sum_{\text{finite}} a_ia^i \neq \sum_{\text{finite}} a^ia_i.
\]

In this paper, we always mean

\[
    f(a) = \sum_{\text{finite}} a_ia^i,
\]

and then \( a \) is called a right root of \( f(t) \). A left root of \( f(t) \) and a right algebraic element are defined similarly. A subset \( S \) is called left algebraic (resp., right algebraic) over \( K \) if every element in \( S \) is left algebraic (resp., right algebraic) over \( K \). If \( K \) is central, that is, \( K \subseteq F \), then the right algebraicity coincides with the left one. However, if \( K \) is not central, then there are division rings \( K \subseteq D \) such that \( D \) is left algebraic but not right algebraic over \( K \). Observe that with division rings \( K \subseteq D \) which were presented by Cohn in [6, Page 548], we can show that the division ring \( D \) is left algebraic but not right algebraic over \( K \). In Section 2, we give a natural and simple example of division rings \( K \subseteq D \) such that \( D^* \) contains a proper normal subgroup \( N \) which is left algebraic but not right algebraic over \( K \).

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In the case when $K$ is a subfield of $D$, it is not known whether $D$ is left algebraic over $K$ provided it is right algebraic over $K$ (see [4, Page 1610]).

The notion of one-sided (right or left) algebraicity has been introduced to study polynomials over division rings. For example, [9] (see Chapter 7) is one of the oldest books mentioning this notion. Some special cases of one-sided algebraicity were studied by several authors. For instance, C. Faith [7] investigated division rings that are radical over their division subrings, P. M. Cohn and A. H. Schofield [6, 13] studied the left and right dimension of a division ring $D$ over its division subring $K$. Recently (e.g., see [11, 13]), division rings whose elements are algebraic (left or right) over some division subring have been received considerable attention. In [11], Mahdavi-Hezavehi conjectured that in a division ring $D$ with center $F$, if a non-central subnormal subgroup $N$ of $D^*$ is algebraic over $F$, then so is $D$. Hazrat [8, Theorem 2.4] shows that this conjecture is true if $N$ contains some term in the descending central series of $D^*$. However, in general, the conjecture remains still without the answer even if $N$ is normal in $D^*$. This inspired us to pose the following more general conjecture.

**Conjecture 1.** Let $D$ be a division ring with center $F$, $K$ a division subring of $D$ containing $F$, and $N$ a subnormal subgroup of $D^*$. If $N$ is non-central, then $N$ is left algebraic (resp., right algebraic) over $K$ if and only if so is $D$.

In Section 3 (see Theorem 3.3), we give the answer to this conjecture in the case when $N$ is a normal subgroup of $D^*$ and $F$ is uncountable. Further, in Section 4 we study a division ring $D$ whose $n$-th derived subgroup $D^{(n)}$ of $D^*$ is left algebraic of bounded degree over some subfield (recall that the $n$-th derived subgroup $D^{(n)}$ of $D^*$ is defined as follow: $D^{(1)} = D'$ is the commutator subgroup of $D^*$ and $D^{(n)}$ is the commutator subgroup of $D^{(n-1)}$ for $n > 1$). Note that in [4], it was proved that if $D$ is a division ring with center $F$ and there exists a subfield $K$ of $D$ such that $D$ is left algebraic over $K$ of bounded degree $d$, then $\dim_F D \leq d^2$. This result was extended for the commutator subgroup $D'$ instead of $D$ in [11, Theorem 17]. The result we get in Theorem 4.4 generalizes this fact by considering $D^{(n)}$ for an arbitrary $n \geq 1$ instead of $D'$.

## 2. Example

In this section, we give an example of division rings $K \subseteq D$ such that $D^*$ contains a proper normal subgroup $N$ which is left algebraic but not right algebraic over $K$.

Let $F$ be a field with an endomorphism $\sigma$ and $t$ an indeterminate. We denote by $F((t, \sigma)) = \{ \sum_{i=0}^{\infty} a_i t^i \mid a_i \in F, n \in \mathbb{Z} \}$ the ring of Laurent skew series in $t$ over $F$ with respect to $\sigma$ in which the addition is defined as usual, and the multiplication is an extension of the rule $ta = \sigma(a)t$. In general, $F((t, \sigma))$ is not a division ring (for example, if $\sigma$ is not injective, then there exists $a \in F^*$ such that $\sigma(a) = 0$, so $ta = \sigma(a)t = 0$). However, if $\sigma$ is an automorphism, then $F((t, \sigma))$ is a division ring. For $\alpha = \sum_{i=0}^{\infty} a_i t^i$, the lowest power appearing in $\alpha$ is denoted by $\degmin(\alpha)$. That is, $\degmin(\alpha) = \min\{i \mid a_i \neq 0\}$.

Now let $k$ be a field of characteristic 0 and $\{x_0, x_1, \ldots\}$ a countable set of commuting indeterminates. Consider the field of fractions $F = k[x_0, x_1, \ldots]$ of the polynomial ring $k[x_0, x_1, \ldots]$, and the endomorphism $\sigma : F \to F$ defined by $\sigma(x_i) = x_{i+1}$ for $i \in \mathbb{N}$. Then, we have the following easy lemma.
Lemma 2.1. Let $F = k(x_0, x_1, \ldots)$ and $\sigma : F \to F$ be defined above. Then $D = F((t, \sigma))$ is a division ring and
\[
\degmin(\alpha, \beta) = \degmin(\alpha) + \degmin(\beta)
\]
for every $\alpha, \beta \in D$. In particular, $\degmin(\alpha^{-1}) = -\degmin(\alpha)$ for every $\alpha \in D^*$.

Proof. The proof of first conclusion is essentially due to that of [11, Example 1.7 and Proposition 14.2]. The proof of second one is elementary by the fact that $\sigma$ is injective. □

Now, we are ready to give an example we have mentioned in the beginning of this section.

Example 2.2. Let $D = F((t, \sigma))$ be as in Lemma 2.1 and consider the following subset in $D$:
\[
K = F((t^2, \sigma)) = \left\{ \sum_{i=n}^{\infty} a_i t^{2i} \mid n \in \mathbb{Z}, a_i \in F \right\}.
\]
It is easy to see that $K$ is a division subring of $D$. It is obvious that if $\alpha = \sum_{i=n}^{\infty} a_i t^i \in D$, then
\[
\alpha = \sum_{i=2j \geq n} a_i t^i + \sum_{i=2j+1 \geq n} a_i t^i,
\]
that is, $\alpha = \alpha_1 + \alpha_2 t$, where $\alpha_1, \alpha_2 \in K$. Hence, $\{1, t\}$ is a basis of the left vector space $D$ over $K$, which implies that the dimension of the left vector space $D$ over $K$ is 2. Hence, every element of $D$ is left algebraic of degree $\leq 2$ over $K$.

Now, let $N = \{ \alpha \in D^* \mid \degmin(\alpha) = 0 \}$. We claim that $N$ is a proper normal subgroup of $D^*$. Indeed, it is trivial that $N \neq D^*$. For $\alpha, \beta \in N$, the condition $\degmin(\alpha) = \degmin(\beta) = 0$ implies $\degmin(\alpha \beta) = \degmin(\alpha) + \degmin(\beta) = 0$ and $\degmin(\alpha^{-1}) = -\degmin(\alpha) = 0$. Therefore, $\alpha \beta, \alpha^{-1} \in N$, which shows that $N$ is subgroup of $D^*$. Assume that $\alpha \in N$ and $\beta \in D^*$. Then,
\[
\degmin(\beta^{-1} \alpha \beta) = \degmin(\beta^{-1}) + \degmin(\alpha) + \degmin(\beta)
\]
\[= -\degmin(\beta) + \degmin(\beta) = 0.
\]
As a corollary, $\beta^{-1} \alpha \beta \in N$, so $N$ is normal in $D^*$.

Consider the element $x_0 + t \in N$. To finish example, we will show that $x_0 + t$ is not right algebraic over $K$. Suppose that there exist $h_0(t^2), h_1(t^2), \ldots, h_n(t^2) \in K$ such that $h_n(t^2) \neq 0$ and
\[
h_0(t^2) + (x_0 + t) h_1(t^2) + \cdots + (x_0 + t)^n h_n(t^2) = 0.
\]
We seek a contradiction. Indeed, observe that, after expanding, $(x_0 + t)^i$ is written as a linear sum of term $x_0^m x_1^m \cdots x_{i-1}^m t^m$ over $\mathbb{Z}$, so one sees that the term $x_0^{n-1} t$ appears in $(x_0 + t)^n$ but does not in $(x_0 + t)^i$ with $i < n$. Moreover, all powers of $t$ appearing in $h_0(t^2), \ldots, h_{n-1}(t^2)$ is even, so $x_0^{n-1} t h_n(t^2) = 0$, equivalently, $h_n(t^2) = 0$, a contradiction. Thus, $x_0 + t$ is not right algebraic over $K$.

3. LEFT ALGEBRAIC NORMAL SUBGROUPS IN A DIVISION RING

For a division ring $D$ and its division subring $K$, $KD$ and $D_K$ denote the left and right vector space over $K$ respectively. In this section, we give the affirmative answer to Conjecture 1 in the case when $F$ is uncountable and $N$ is a non-central normal subgroup of $D^*$.

We need some lemmas.
Let $D$ be a division ring with center $F$ and $N$ a normal subgroup of $D^*$. If $N$ is non-central, then $C_D(N) = F$.

**Lemma 3.2.** Let $D$ be a division ring with center $F, K$ a division subring of $D$ containing $F$ and $a$ an element of $D$. Assume that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct elements in $F$ such that all elements $a - \alpha_i$ are non-zeros. Then, either $a$ is left (resp. right) algebraic over $K$ or the set $\{(a - \alpha_i)^{-1} \mid i = 1, 2, \ldots, n\}$ is left (resp. right) linearly independent over $K$.

**Proof.** It is enough to prove the lemma for the left case since the right case is similar. Assume that $a$ is not left algebraic over $K$ and

$$\beta_1(a - \alpha_1)^{-1} + \beta_2(a - \alpha_2)^{-1} + \cdots + \beta_n(a - \alpha_n)^{-1} = 0$$

for some $\beta_i \in K$. Consider the polynomials

$$f(t) = (t - \alpha_1)(t - \alpha_2)\cdots(t - \alpha_n) \in F[t] \subseteq K[t]$$

and $f_i(t) = f(t)/(t - \alpha_i)$ for $1 \leq i \leq n$. Multiplying both sides of (1) on the right by $f(a)$, we get $\beta_1 f_1(a) + \beta_2 f_2(a) + \cdots + \beta_n f_n(a) = 0$. This shows that $a$ is a right root of the polynomial $g(t) = \beta_1 f_1(t) + \beta_2 f_2(t) + \cdots + \beta_n f_n(t)$, so $g(t) \equiv 0$ because $a$ is not left algebraic over $K$. Then, for every $1 \leq i \leq n$, we have $0 = g(\alpha_i) = \beta_i f_i(\alpha_i)$. Therefore, $\beta_i = 0$ for all $i$. Hence, the set $\{(a - \alpha_i)^{-1} \mid i = 1, 2, \ldots, n\}$ is left linearly independent over $K$. \hfill \Box

Note that the special case of Lemma 3.2 when $K = F$ was considered in [12, Proposition 5.2.21].

**Theorem 3.3.** Let $D$ be a division ring with uncountable center $F$, $K$ a division subring of $D$ containing $F$ and $N$ a normal subgroup of $D^*$. If $N$ is non-central, then $N$ is left algebraic (resp., right algebraic) over $K$ if and only if so is $D$.

**Proof.** We show that the theorem is true for the left case since the proof for the right case is similar. Thus, assume that $N$ is a non-central normal subgroup of $D^*$ which is left algebraic over $K$. For any $a \in D$, we have to prove that $a$ is left algebraic over $K$. If $a \in C_D(N)$, then by Lemma 3.1 $a \in F \subseteq K$, and there is nothing to prove. Now, assume that $a \notin C_D(N)$. Take $b \in N$ such that $d = ba - ab \neq 0$. For every $\alpha \in F$, one has

$$d = ba - ab = b(a + \alpha) - (a + \alpha)b$$

$$= b(a + \alpha)(1 + (a + \alpha)^{-1}b^{-1}(a + \alpha)b) = b(a + \alpha)(1 + c),$$

where $c = (a + \alpha)^{-1}b^{-1}(a + \alpha)b \in N$. Since $c$ is left algebraic over $K$ and $c + 1 \neq 0$, the element $(c + 1)^{-1}$ is left algebraic over $K$. Consequently, $d^{-1}b(a + \alpha) = (c + 1)^{-1}$ is left algebraic over $K$. Hence, there exist $\beta_1, \beta_2, \cdots, \beta_n \in K$ such that

$$\beta_n(d^{-1}b(a + \alpha))^n + \beta_{n-1}(d^{-1}b(a + \alpha))^{n-1} + \cdots + \beta_1(d^{-1}b(a + \alpha)) + 1 = 0.$$}

We can write this equality as follows:

$$1 = (-\beta_n(d^{-1}b(a + \alpha))^{n-1} - \cdots - \beta_1(d^{-1}b))(a + \alpha).$$

Therefore, $(a + \alpha)^{-1} = -\beta_n(d^{-1}b(a + \alpha))^{n-1} - \cdots - \beta_1d^{-1}b$ is in the left vector $K$-subspace $W$, which is generated by the subgroup $\langle a, b, d \rangle$ of $D^*$ generated by $a, b, d$. Since $\langle a, b, d \rangle$ is a finitely generated subgroup, the cardinality of the basis of $W$ over $K$ is countable. Observe that $F$ is uncountable, so is the set $\{(a + \alpha)^{-1} \mid \alpha \in F\}$.
As a corollary, the set \{(a + a)^{-1} | a \in F\} is left linearly dependent over K. In view of Lemma 3.2, a is left algebraic over K.

The following corollary gives the affirmative answer to [11, Problem 13] in the case of uncountable center F of D and N is normal in D*.

**Corollary 3.4.** Let D be a division ring with uncountable F. Assume that N is a non-central normal subgroup of D*. If N is algebraic over F, then so is D.

4. Left algebraic n-th derived subgroup of bounded degree

Let D be a division ring and K a subfield of D. In this section, we prove that if for some integer n ≥ 1, the n-th derived subgroup D(n) is left algebraic of bounded degree d over K, then dimF D ≤ d^2.

The proof of the following lemma is elementary, so we omit it.

**Lemma 4.1.** Let D be a division ring, K a division subring and x an element in D. The following conditions are equivalent.

1. The element x is left algebraic (resp., right algebraic) over K of degree d.
2. d is the largest integer such that \{xa^n | n = 1, 2, \ldots, d - 1\} is a left (resp., right) linearly independent set in KD (resp., DK).
3. d is the largest integer such that the sum \sum_{i=0}^{d-1} Kx^i (resp., \sum_{i=0}^{d-1} x^iK) is a direct sum in KD (resp., DK).

**Lemma 4.2.** Let D be a division ring with infinite center, N a non-central normal subgroup of D* and K a subfield of D. If N is left (or right) algebraic over K of bounded degree, then D is centrally finite, that is, D is a finite dimensional vector space over its center.

**Proof.** Since N is non-central, there exists a \in N \setminus F. For every x \in D*, one has axa^{-1}x^{-1} = a( axa^{-1}x^{-1}) \in N, so axa^{-1}x^{-1} is left (resp., right) algebraic of bounded degree over K. By [1, Theorem 11], D is centrally finite.

**Corollary 4.3.** Let D be a division ring with infinite center. For any positive integer n, if the n-th derived subgroup D(n) is left (or right) algebraic of bounded degree over some subfield of D, then D is centrally finite.

**Proof.** If D(n) is central, then D* is solvable, so D is a field by [14, 14.4.4, Page 440]. Hence, we can assume that D(n) is non-central. Since D(n) is normal in D*, by Lemma 4.2, D is centrally finite.

The following theorem extends [4, Theorem 1.3] and [11, Theorem 17].

**Theorem 4.4.** Let D be a division ring with infinite center F, K a subfield of D and n be a positive integer. If the n-th subgroup D(n) is left (or right) algebraic of bounded degree d over K, then dimF D ≤ d^2.

**Proof.** We prove the theorem for the left case since the right case is similar. Without loss of generality, we assume that K is a maximal subfield of D. According to Corollary 4.3, dimF D = m^2 < \infty is finite. We must show that m ≤ d. By [2, Theorem 7], there exists x \in D(n) such that L = F(x) is a maximal subfield of D. It is well known that dimF L = m (or see [10, Proposition 15.7 and Theorem 15.8]). One has that D is a left D \otimes_F L-module in which the operator is defined by (\alpha \otimes x^i) \beta = \alpha \beta x^i for every \alpha, \beta \in D and i \in N. Observe that D \otimes_F L is simple,
so $D$ is faithful. On the other side, $D$ may be considered as a left $K$-space. Now, consider $T \in \text{End}_K D$ which is defined by $T(\alpha) = \alpha x$ for every $\alpha \in D$. We claim that the set $\{T^i \mid i = 0, 1, \ldots, m - 1\}$ is left linearly independent over $K$. Indeed, assume that $\sum_{i=0}^{m-1} c_i T^i = 0$ for some $c_0, c_1, \ldots, c_{m-1} \in K$. Then, for every $\alpha \in D$,$$
ul = \left( \sum_{i=0}^{m-1} c_i T^i \right) (\alpha) = \sum_{i=0}^{m-1} c_i \alpha x^i = \left( \sum_{i=0}^{m-1} c_i \otimes x^i \right) \alpha.$$Observe that $D$ is faithful, so $\sum_{i=0}^{m-1} c_i \otimes x^i = 0$, which implies that$$c_0 = c_1 = \cdots = c_{m-1} = 0.$$The claim is proved. The next claim is that there exists $y \in D$ such that $K y + K T(y) + \cdots + K T^{m-1}(y)$ is a direct sum. Indeed, let $t$ be an indeterminate and $K[t]$ be the polynomial ring in $t$ over $K$. Then, we can consider $D$ as a left $K[t]$-module with operator defined by the rule $f(t) \alpha = f(T)(\alpha)$ for every $\alpha \in D$ and $f(t) \in K[t]$. Since $\dim_K D < \infty$, there exists a non-zero element $g(t) \in K[t]$ such that $g(T) = 0$. Hence, for every $\alpha \in D$, one has $g(t) \alpha = g(T)(\alpha) = 0$, so it follows that $D$ is torsion as a left $K[t]$-module. Moreover, it is obvious that $D$ is finitely generated as left $K$-space, so is $D$ as a left $K[t]$-module. Therefore, $D$ is torsion finitely generated as a left module over a PID. Hence, there exist $f_1(t), f_2(t), \cdots, f_\ell(t) \in K[t]$ such that$$\langle f_1(t) \rangle \supseteq \langle f_2(t) \rangle \supseteq \cdots \supseteq \langle f_\ell(t) \rangle$$and an isomorphism$$\phi : K[t]/\langle f_1(t) \rangle \oplus K[t]/\langle f_2(t) \rangle \oplus \cdots \oplus K[t]/\langle f_\ell(t) \rangle \to D,$$where $\langle f(t) \rangle$ denotes the ideal of $K[t]$ generated by some element $f(t) \in K[t]$. Put $y = \phi(1 + \langle f_\ell(t) \rangle) \in D$. We will show that $y$ is the element we need to find. Indeed, assume that $f(t) = c_0 + c_1 t + \cdots + c_{m-1} t^{m-1} \in K[t]$ such that $f(T)(y) = 0$. Then, $f(t)y = 0$, equivalently, $f(t) \in \text{ann}_{K[t]} y$. Observe that $\langle y \rangle \cong R/\langle f_\ell(t) \rangle$ and by direct calculation, one has$$\text{ann}_{K[t]}(D) = \bigcap_{i=1}^{\ell} \text{ann}_{K[t]} K[t]/\langle f_i(t) \rangle = \bigcap_{i=1}^{\ell} \langle f_i(t) \rangle = \text{ann}_{K[t]} y.$$Hence, $f(t) \in \text{ann}_{K[t]} y = \text{ann}_{K[t]} D$. As a corollary, $f(T)(\alpha) = 0$ for every $\alpha \in D$ which contradicts to the fact that $\{T^i \mid i = 0, 1, \ldots, m - 1\}$ is left linearly independent over $K$. Therefore, the claim is proved. Put $u = yxy^{-1}$. Then,$$K + Ku + \cdots + Ku^{m-1} = (K y + K y x + \cdots + K y x^{m-1}) y^{-1}$$is a direct sum. By Lemma 4.1 it follows that $u$ is left algebraic of degree $m$ over $K$. On the other hand, since $x$ is in $D^{(n)}$ and left algebraic of bounded degree $d$ over $K$, so is $u \in D^{(n)}$. Thus, again by Lemma 4.1 $m \leq d$. The proof is now complete.

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