Higher order Painlevé system of type $D_{2n+2}^{(1)}$

arising from integrable hierarchy

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Abstract

A higher order Painlevé system of type $D_{2n+2}^{(1)}$ was introduced by Y. Sasano. It is an extension of the sixth Painlevé equation ($P_{VI}$) for the affine Weyl group symmetry. It is also expressed as a Hamiltonian system of order $2n$ with a coupled Hamiltonian of $P_{VI}$. In this paper, we discuss a derivation of this system from a Drinfeld-Sokolov hierarchy.

1 Introduction

The Drinfeld-Sokolov hierarchies are extensions of the KdV (or mKdV) hierarchy for the affine Lie algebras [DS]. It is known that they imply several Painlevé equations by similarity reduction [AS, FS, KK1, KIK, KK2]. On the other hand, two types of extensions of the Painlevé equations for the affine Weyl group symmetry have been studied, type $A_n^{(1)}$ [NY1] and type $D_{2n+2}^{(1)}$ [S]. For type $A_n^{(1)}$ among them, the relation to the Drinfeld-Sokolov hierarchies is already clarified. In this paper, we investigate the relation for type $D_{2n+2}^{(1)}$.

Recall that the higher order Painlevé system of type $D_{2n+2}^{(1)}$ given in [S] is a Hamiltonian system of order $2n$ with a coupled Hamiltonian of $P_{VI}$. Let $q_i, p_i$ ($i = 1, \ldots, n$) be dependent variables on $s$ and $\alpha_i$ ($i = 0, \ldots, 2n + 2$) complex parameters satisfying

$$\alpha_0 + \alpha_1 + \sum_{j=2}^{2n} 2\alpha_j + \alpha_{2n+1} + \alpha_{2n+2} = 1.$$
We also set
\[
H_i = q_i(q_i - 1)(q_i - s)p_i^2 - \{(\beta_{i,1} - 1)q_i(q_i - 1) \\
+ \beta_{i,3}(q_i - 1)(q_i - s) + \beta_{i,4}q_i(q_i - s)\}p_i + \alpha_{2i}(\alpha_{2i} + \beta_{i,0})q_i,
\]
for \(i = 1, \ldots, n\), where
\[
\beta_{i,0} = \alpha_1 + \sum_{j=1}^{i-1} \alpha_{2j+1}, \quad \beta_{i,1} = \alpha_0 + \sum_{j=1}^{i-1} 2\alpha_{2j} + \sum_{j=1}^{i-1} \alpha_{2j+1},
\]
\[
\beta_{i,3} = \sum_{j=i}^{n-1} \alpha_{2j+1} + \sum_{j=i+1}^{n} 2\alpha_{2j} + \alpha_{2n+1}, \quad \beta_{i,4} = \sum_{j=i}^{n-1} \alpha_{2j+1} + \alpha_{2n+2}.
\]
We consider a Hamiltonian system
\[
s(s - 1)dq_i = \{H, q_i\}, \quad s(s - 1)dp_i = \{H, p_i\} \quad (i = 1, \ldots, n), \quad (1.1)
\]
with a Hamiltonian
\[
H = \sum_{i=1}^{n} H_i + \sum_{1 \leq i < j \leq n} 2(q_i - s)p_i q_j \{(q_j - 1)p_j + \alpha_{2j}\}, \quad (1.2)
\]
where \(\{\cdot, \cdot\}\) stands for the Poisson bracket defined by
\[
\{p_i, q_j\} = \delta_{i,j}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0 \quad (i, j = 1, \ldots, n).
\]
Note that each \(H_i\) is equivalent to the Hamiltonian of \(P_{\text{VI}}\) (see [IKSY]). In fact, the parameters satisfy the following relations:
\[
\beta_{i,0} + \beta_{i,1} + 2\alpha_{2i} + \beta_{i,3} + \beta_{i,4} = 1 \quad (i = 1, \ldots, n).
\]
The system (1.1) with (1.2) admits affine Weyl group symmetry of type \(D_{2n+2}^{(1)}\). Denoting the dependent variables by
\[
\varphi_0 = \frac{1}{2n+2}, \quad \varphi_1 = q_1 - s, \quad \varphi_{2i+1} = q_{i+1} - q_i \quad (i = 1, \ldots, n - 1),
\]
\[
\varphi_{2j} = -\frac{p_j}{2n+2} \quad (j = 1, \ldots, n), \quad \varphi_{2n+1} = 1 - q_n, \quad \varphi_{2n+2} = -q_n,
\]
we consider birational canonical transformations
\[
r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i, \quad r_i(\varphi_j) = \varphi_j + \frac{\alpha_i}{\varphi_i}\{\varphi_i, \varphi_j\}, \quad (1.3)
\]
Then the system (1.1) with (1.2) is invariant under the action of them. Furthermore, a group of symmetries $\langle r_0,\ldots,r_{2n+2} \rangle$ is isomorphic to the affine Weyl group of type $D^{(1)}_{2n+2}$.

In this paper, we show that the system (1.1) with (1.2) is derived from a Drinfeld-Sokolov hierarchy by similarity reduction. The Drinfeld-Sokolov hierarchies are characterized by graded Heisenberg subalgebras of the affine Lie algebras. For a derivation of (1.1), we choose the affine Lie algebra $\mathfrak{g}(D^{(1)}_{2n+2})$ and its graded Heisenberg subalgebra of type $(1,1,0,1,0,\ldots,1,0,1,1)$. It is suggested by the fact that $PVI$ is derived from the hierarchy associated with the graded Heisenberg subalgebra of $\mathfrak{g}(D^{(1)}_4)$ of type $(1,1,0,1,1)$.

This paper is organized as follows. In Section 2 we recall the affine Lie algebra $\mathfrak{g}(D^{(1)}_{2n+2})$ and its graded Heisenberg subalgebra. In Section 3 we formulate a similarity reduction of a Drinfeld-Sokolov hierarchy of type $D^{(1)}_{2n+2}$. In Section 4 we derive the system (1.1) with (1.2) from the similarity reduction. In Section 5 we discuss a derivation of the group of symmetries (1.3).
2 Affine Lie algebra

In this section, we introduce the affine Lie algebra of type $D^{(1)}_{2n+2}$ and its Heisenberg subalgebra of type $(1,1,0,1,0,\ldots,1,0,1,1)$, following the notation of [Kac].

Recall that $\mathfrak{g} = \mathfrak{g}(D^{(1)}_{2n+2})$ is a Lie algebra generated by the Chevalley generators $e_i, f_i, \alpha_i^\vee (i = 0, \ldots, 2n + 2)$ and the scaling element $d$ with the fundamental relations

$$(\text{ad} e_i)^{1-a_{ij}}(e_j) = 0, \quad (\text{ad} f_i)^{1-a_{ij}}(f_j) = 0 \quad (i \neq j),$$

$$[\alpha_i^\vee, \alpha_j^\vee] = 0, \quad [\alpha_i^\vee, e_j] = a_{ij}e_j, \quad [\alpha_i^\vee, f_j] = -a_{ij}f_j, \quad [e_i, f_j] = \delta_{i,j}\alpha_i^\vee;$$

$$[d, \alpha_i^\vee] = 0, \quad [d, e_i] = \delta_{i,0}e_0, \quad [d, f_i] = -\delta_{i,0}f_0,$$

for $i, j = 0, \ldots, 2n + 2$. The generalized Cartan matrix $A = (a_{ij})_{i,j=0}^{2n+2}$ for $\mathfrak{g}$ is defined by

$$a_{ii} = 2 \quad (i = 0, \ldots, 2n + 2),$$
$$a_{02} = a_{i,i+1} = a_{2n2n+2} = -1 \quad (i = 1, \ldots, 2n),$$
$$a_{ij} = 0 \quad (\text{otherwise}).$$

We denote the Cartan subalgebra of $\mathfrak{g}$ by

$$\mathfrak{h} = \bigoplus_{j=0}^{2n+2} \mathbb{C}\alpha_j^\vee \oplus \mathbb{C}d.$$

The canonical central element of $\mathfrak{g}$ is given by

$$K = \alpha_0^\vee + \alpha_1^\vee + \sum_{i=2}^{2n} 2\alpha_i^\vee + \alpha_{2n+1}^\vee + \alpha_{2n+2}^\vee.$$

The normalized invariant form $(\cdot | \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is determined by the conditions

$$(\alpha_i^\vee | \alpha_j^\vee) = a_{ij}, \quad (e_i | f_j) = \delta_{i,j}, \quad (\alpha_i^\vee | e_j) = (\alpha_i^\vee | f_j) = 0,$$
$$\quad (d | d) = 0, \quad (d | \alpha_i^\vee) = \delta_{0,j}, \quad (d | e_j) = (d | f_j) = 0,$$

for $i, j = 0, \ldots, 2n + 2$.

Consider a gradation $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$ of type $(1,1,0,1,0,\ldots,1,0,1,1)$ by setting

$$\deg \mathfrak{h} = \deg e_i = \deg f_i = 0 \quad (i \in \mathcal{I}),$$
$$\deg e_j = 1, \quad \deg f_j = -1 \quad (j \in \mathcal{J}),$$

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where $I = \{2, 4, \ldots, 2n\}$ and $J = \{0, 1, 3, 5, \ldots, 2n + 1, 2n + 2\}$. With an element $\vartheta \in \mathfrak{h}$ such that

$$(\vartheta | \alpha_i^y) = 0, \quad (\vartheta | \alpha_j^y) = 1 \quad (i \in I; j \in J),$$

this gradation is defined by

$$\mathfrak{g}_k = \{ x \in \mathfrak{g} \mid [\vartheta, x] = kx \} \quad (k \in \mathbb{Z}).$$

We denote by

$$\mathfrak{g}_{<0} = \bigoplus_{k<0} \mathfrak{g}_k, \quad \mathfrak{g}_{\geq 0} = \bigoplus_{k \geq 0} \mathfrak{g}_k.$$

Such gradation implies the Heisenberg subalgebra of $\mathfrak{g}$

$$\mathfrak{s} = \{ x \in \mathfrak{g} \mid [x, \Lambda_1] = [x, \Lambda_2] = \mathbb{C}K \},$$

with elements of $\mathfrak{g}_1$

$$\Lambda_1 = e_0 + e_{1,2} + \sum_{j \in J'} (e_j + e_{j-1,j,j+1}) + e_{2n+1} + e_{2n,2n+2},$$

$$\Lambda_2 = e_1 + e_{0,2} + \sum_{j \in J'} (e_{j-1,j} + e_{j,j+1}) + e_{2n+2} + e_{2n,2n+1},$$

where $J' = \{3, 5, \ldots, 2n - 1\}$ and

$$e_{i_1,i_2,\ldots,i_{n-1},i_n} = ade_{i_1}ade_{i_2} \ldots ade_{i_{n-1}}e_{i_n}.$$  

Note that $\mathfrak{s}$ admits the gradation of type $(1, 1, 0, 1, 0, \ldots, 1, 0, 1, 1)$, namely

$$\mathfrak{s} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{s}_k, \quad \mathfrak{s}_k \subset \mathfrak{g}_k.$$

We also remark that the positive part of $\mathfrak{s}$ has a graded bases $\{\Lambda_k\}_{k=1}^{\infty}$ satisfying

$$[\Lambda_k, \Lambda_l] = 0, \quad [\vartheta, \Lambda_k] = n_k \Lambda_k \quad (k, l = 1, 2, \ldots),$$

where $n_k$ stands for the degree of element $\Lambda_k$ defined by

$$n_k = \begin{cases} 
  k & (k : \text{odd}) \\
  k - 1 & (k : \text{even})
\end{cases}.$$

The explicit formulas of $\Lambda_k$ ($k \geq 3$) are given in Appendix A.

In the last, we introduce the Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{n}_+$ and $\mathfrak{n}_-$ be the subalgebras of $\mathfrak{g}$ generated by $e_i$ and $f_i$ ($i = 0, \ldots, 2n + 2$) respectively.
Then the Borel subalgebra \( b_+ \) of \( g \) is defined by \( b_+ = \mathfrak{h} \oplus \mathfrak{n}_+ \). Note that we have the triangular decomposition
\[
g = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{n}_- \oplus \mathfrak{b}_+.
\]
We also remark that
\[
\mathfrak{n}_- = g_{<0} \bigoplus \bigoplus_{i \in \mathcal{I}} \mathbb{C} f_i, \quad g_{\geq 0} = \bigoplus_{i \in \mathcal{I}} \mathbb{C} f_i \oplus \mathfrak{b}_+.
\]

3 Drinfeld-Sokolov hierarchy

In this section, we formulate a Drinfeld-Sokolov hierarchy of type \( D_{2n+2}^{(1)} \) and its similarity reduction associated with the Heisenberg subalgebra \( \mathfrak{s} \).

In the following, we use the notation of infinite dimensional groups
\[
G_{<0} = \exp(\hat{\mathfrak{g}}_{<0}), \quad G_{\geq 0} = \exp(\hat{\mathfrak{g}}_{\geq 0}),
\]
where \( \hat{\mathfrak{g}}_{<0} \) and \( \hat{\mathfrak{g}}_{\geq 0} \) are completions of \( \mathfrak{g}_{<0} \) and \( \mathfrak{g}_{\geq 0} \) respectively.

Introducing the time variables \( t_k \) \( (k = 1, 2, \ldots) \), we consider a system of partial differential equations
\[
\partial t_k - B_k = W(\partial t_k - \Lambda_k)W^{-1} \quad (k = 1, 2, \ldots), \tag{3.1}
\]
for a \( G_{<0} \)-valued function \( W \), where \( B_k \) stands for the \( \mathfrak{g}_{\geq 0} \)-component of \( W\Lambda_k W^{-1} \). The Zakharov-Shabat equations
\[
[\partial t_k - B_k, \partial t_l - B_l] = 0 \quad (k, l = 1, 2, \ldots), \tag{3.2}
\]
follows from the system (3.1). We call the system (3.2) the Drinfeld-Sokolov hierarchy of type \( D_{2n+2}^{(1)} \).

Under the system (3.1), we consider the operator
\[
\mathcal{M} = W \exp \left( \sum_{k=1,2,\ldots} t_k \Lambda_k \right) \vartheta \exp \left( - \sum_{k=1,2,\ldots} t_k \Lambda_k \right) W^{-1}.
\]
Then the operator \( \mathcal{M} \) satisfies
\[
[\partial t_k - B_k, \mathcal{M}] = 0 \quad (k = 1, 2, \ldots). \tag{3.3}
\]
Also \( \mathcal{M} \) is expressed as
\[
\mathcal{M} = W \vartheta W^{-1} - \sum_{k=1,2,\ldots} n_k t_k W \Lambda_k W^{-1}.
\]
Now we require that the similarity condition $M \in \mathfrak{g}_{\geq 0}$ is satisfied. Note that it is equivalent to
\[
\vartheta + \sum_{k=1,2,...} n_k t_k B^c_k = W \vartheta W^{-1},
\]
where $B^c_k$ stands for the $\mathfrak{g}_{<0}$-component of $W \Lambda_k W^{-1}$. Then we have
\[
M = \vartheta - \sum_{k=1,2,...} n_k t_k B_k.
\]
We also assume that $t_k = 0$ for $k \geq 3$. Then the systems (3.2) and (3.3) are equivalent to
\[
\begin{align*}
[\partial_{t_1} - B_1, \partial_{t_2} - B_2] &= 0, \\
[\partial_{t_k} - B_k, \vartheta - t_1 B_1 - t_2 B_2] &= 0 \quad (k = 1, 2).
\end{align*}
\] (3.4)
We regard the system (3.4) as a similarity reduction of the Drinfeld-Sokolov hierarchy of type $D_{2n+2}^{(1)}$.

The $\mathfrak{g}_{\geq 0}$-valued functions $B_k$ ($k = 1, 2$) are expressed in the form
\[
B_k = U_k + \Lambda_k, \quad U_k = \sum_{i=0}^{2n+2} u_{k,i} \alpha^i + \sum_{i \in I} x_{k,i} e_i + \sum_{i \in I} y_{k,i} f_i.
\]
In terms of the operators $U_k \in \mathfrak{g}_0$, this similarity reduction can be expressed as
\[
\begin{align*}
\partial_{t_1}(U_2) - \partial_{t_2}(U_1) + [U_2, U_1] &= 0, \\
[\Lambda_1, U_2] - [\Lambda_2, U_1] &= 0, \\
t_1 \partial_{t_1}(U_k) + t_2 \partial_{t_2}(U_k) + U_k &= 0 \quad (k = 1, 2).
\end{align*}
\]
In the following, we use the notation of a $\mathfrak{g}_{\geq 0}$-valued 1-form $\mathcal{B} = B_1 dt_1 + B_2 dt_2$ with respect to the coordinates $t = (t_1, t_2)$. Then the similarity reduction (3.4) is expressed as
\[
d_t M = [\mathcal{B}, M], \quad d_t \mathcal{B} = \mathcal{B} \wedge \mathcal{B},
\] (3.5)
where $d_t$ stands for an exterior differentiation with respect to $t$. Denoting by
\[
\mathcal{M}_1 = -t_1 \Lambda_1 - t_2 \Lambda_2, \quad \mathcal{B}_1 = \Lambda_1 dt_1 + \Lambda_2 dt_2,
\]
we can express the operators $\mathcal{M}$ and $\mathcal{B}$ in the form
\[
\begin{align*}
\mathcal{M} &= \vartheta + \sum_{i \in I} \xi_i e_i + \sum_{i \in I} \psi_i f_i + \mathcal{M}_1, \\
\mathcal{B} &= u + \sum_{i \in I} x_i e_i + \sum_{i \in I} y_i f_i + \mathcal{B}_1.
\end{align*}
\]
where
\[ \theta = \vartheta + \sum_{i=0}^{2n+2} \theta_i \alpha_i^\vee, \quad u = \sum_{i=0}^{2n+2} u_i \alpha_i^\vee. \]

The system \( (3.5) \) is expressed in terms of these variables as follows:
\[
\begin{align*}
 d_t \theta_i &= x_i \psi_i - y_i \xi_i, \quad d_t \theta_j = 0, \\
 d_t \xi_i &= (u | \alpha_i^\vee) \xi_i - x_i (\theta | \alpha_i^\vee), \\
 d_t \psi_i &= - (u | \alpha_i^\vee) \psi_i + y_i (\theta | \alpha_i^\vee), \\
 d_t u_i &= x_i \wedge y_i + y_i \wedge x_i, \quad d_t u_j = 0, \\
 d_t x_i &= (u | \alpha_i^\vee) \wedge x_i, \quad d_t y_i = - (u | \alpha_i^\vee) \wedge y_i,
\end{align*}
\]
and for \( i \in I \) and \( j \in J \).

4 Coupled Painlevé VI system

In this section, we show that the system \( (1.1) \) with \( (1.2) \) is derived from the similarity reduction \( (3.5) \).

We introduce below a gauge transformation
\[
\mathcal{M}^+ = \exp(\text{ad}(\Gamma)) \mathcal{M}, \quad d_t - B^+ = \exp(\text{ad}(\Gamma)) (d_t - B),
\]
with \( \Gamma \in g_0 \) such that \( \mathcal{M}^+ \) and \( B^+ \) should take values in \( b_+ \). Then the system \( (3.5) \) is transformed into
\[
\begin{align*}
 d_t \mathcal{M}^+ &= [B^+, \mathcal{M}^+], \\
 d_t B^+ &= B^+ \wedge B^+.
\end{align*}
\]
It is equivalent to the system \( (1.1) \) with \( (1.2) \) under a certain transformation of variables. We recall that the operator \( \mathcal{M} \) is expressed as
\[
\mathcal{M} = \theta + \sum_{i \in I} \xi_i e_i + \sum_{i \in I} \psi_i f_i + \mathcal{M}_1,
\]
where
\[
\mathcal{M}_1 = -t_1 e_0 - t_2 e_1 - \sum_{j \in J'} t_1 e_j - t_1 e_{2n+1} - t_2 e_{2n+2} - t_2 e_{0,2} - t_1 e_{1,2}
- \sum_{j \in J'} t_2 (e_{j-1,j} + e_{j,j+1}) - t_2 e_{2n,2n+1} - t_1 e_{2n,2n+2} - \sum_{j \in J'} t_1 e_{j-1,j,j+1}.
\]
We first consider a gauge transformation
\[ \mathcal{M}' = \exp(\text{ad}(\Gamma_1))\mathcal{M}, \quad d_t - B' = \exp(\text{ad}(\Gamma_1))(d_t - B), \]
with \( \Gamma_1 = \sum_{i \in I} \gamma_i e_i \) defined by
\[ \gamma_2 = \frac{t_2}{t_1}, \quad \gamma_{2i+2} = \frac{t_1 + t_2 \gamma_i}{t_2 + t_1 \gamma_i} \quad (i = 1, \ldots, n-1). \]
Then we obtain
\[ \mathcal{M}'_1 = \exp(\text{ad}(\Gamma_1))(\mathcal{M}_1) \]
\[ = -t_1 e_0 - t_2 e_1 - \sum_{j \in J'} t_1 e_j - t_2 e_{2n+1} - t_1 e_{2n+2} \]
\[ - (t_1 - t_2 \gamma_2)e_{1,2} - \sum_{j \in J'} \{(t_2 + t_1 \gamma_{j-1})e_{j-1,j} + (t_2 - t_1 \gamma_{j+1})e_{j,j+1}\} \]
\[ - (t_1 + t_2 \gamma_{2n})e_{2n,2n+1} - (t_2 + t_1 \gamma_{2n})e_{2n,2n+2}. \]

We next consider a gauge transformation
\[ \mathcal{M}^* = \exp(\text{ad}(\Gamma_2))\mathcal{M}', \quad d_t - B^* = \exp(\text{ad}(\Gamma_2))(d_t - B'), \]
with \( \Gamma_2 \in \mathfrak{h} \) such that
\[ \mathcal{M}'_1^* = \exp(\text{ad}(\Gamma_2))(\mathcal{M}'_1) \]
\[ = e_0 + b_1 e_1 + \sum_{j \in J'} b_j e_j + b_{2n+1} e_{2n+1} + b_{2n+2} e_{2n+2} \]
\[ + e_{1,2} + \sum_{j \in J'} (e_{j-1,j} + e_{j,j+1}) + e_{2n,2n+1} + e_{2n,2n+2}. \]
Note that the coefficients \( b_j \) are algebraic functions in \( t_1 \) and \( t_2 \). Then We have
\[ d_t \mathcal{M}^* = [\mathcal{B}^*, \mathcal{M}^*], \quad d_t \mathcal{B}^* = \mathcal{B}^* \wedge \mathcal{B}^*. \tag{4.1} \]
With the notation
\[ \mathcal{B}'_1 = \exp(\text{ad}(\Gamma_2))\exp(\text{ad}(\Gamma_1))(\mathcal{B}_1), \]
the operators \( \mathcal{M}^* \) and \( \mathcal{B}^* \) are expressed in the form
\[ \mathcal{M}^* = \theta^* + \sum_{i \in I} \xi_i^* e_i + \sum_{i \in I} \psi_i^* f_i + \mathcal{M}_1^*, \]
\[ \mathcal{B}^* = u^* + \sum_{i \in I} x_i^* e_i + \sum_{i \in I} y_i^* f_i + \mathcal{B}_1^*. \]
where
\[ \theta^* = \theta + \sum_{i=0}^{2n+2} \theta_i^* \alpha_i^* \quad \text{and} \quad u^* = \sum_{i=0}^{2n+2} u_i^* \alpha_i^*. \]

We finally consider a gauge transformation
\[ M^+ = \exp(\text{ad}(\Gamma_3)) M^*, \quad d_t - B^+ = \exp(\text{ad}(\Gamma_3))(d_t - B^*), \]
with \( \Gamma_3 = \sum_{i \in I} \eta_i f_i \) such that \( M^+, B^+ \in b_+ \), namely
\[ \xi^*_i \eta^*_i - (\theta^*|\alpha_i^*) \eta_i - \psi^*_i = 0 \quad (i \in I), \]
and
\[ d_t \eta_i = x_i^* \eta^*_i - (u^*|\alpha_i^*) \eta_i - y_i^* \quad (i \in I). \]

Here we have

**Lemma 4.1.** Under the system (4.1), the equation (4.3) follows from the equation (4.2).

**Proof.** The first equation of the system (4.1) can be expressed as

\[
\begin{align*}
 d_t \theta^*_i &= x_i^* \psi^*_i - y_i^* \xi^*_i, \\
 d_t \xi^*_i &= (u^*|\alpha_i^*) \xi^*_i - x_i^* (\theta^*|\alpha_i^*), \\
 d_t \psi^*_i &= -(u^*|\alpha_i^*) \psi^*_i + y_i^* (\theta^*|\alpha_i^*),
\end{align*}
\]

for \( i \in I \) and \( j \in J \). By using (4.4) and \( (d_t \theta^*|\alpha_i^*) = 2d_t \theta_i^* \), we obtain

\[
 d_t \{ \xi^*_i \eta^*_i - (\theta^*|\alpha_i^*) \eta_i - \psi^*_i \} = \{2 \xi^*_i \eta_i - (\theta^*|\alpha_i^*) \}\{d_t \eta_i - x_i^* \eta^*_i + (u^*|\alpha_i^*) \eta_i + y_i^* \} \quad (i \in I).
\]

It follows that the equation (4.2) implies (4.3) or
\[ \eta_i = \frac{(\theta^*|\alpha_i^*)}{2 \xi^*_i} \quad (i \in I). \]

Hence it is enough to verify that the equation (4.3) follows from (4.5). Together with (4.4), the equation (4.5) implies

\[
 d_t \eta_i = \frac{(d_t \theta^*|\alpha_i^*) \xi^*_i - (\theta^*|\alpha_i^*) d_t \xi^*_i}{2(\xi^*_i)^2} \]
\[ = x_i^* \eta^*_i - (u^*|\alpha_i^*) \eta_i - y_i^* + \frac{x_i^* \{4 \xi^*_i \psi^*_i + (\theta^*|\alpha_i^*) \}^2}{4(\xi^*_i)^2} \quad (i \in I). \]

On the other hand, we obtain
\[ 4 \xi^*_i \psi^*_i + (\theta^*|\alpha_i^*) = 0 \quad (i \in I), \]
by substituting (4.5) into (4.2). Combining (4.6) and (4.7), we obtain the equation (4.3).
Thanks to Lemma 4.1, the gauge parameters $\eta_i$ ($i \in \mathcal{I}$) are determined by the equation (4.2). Hence we obtain the system on $\mathfrak{b}_+$

$$d_t \mathcal{M}^+ = [\mathcal{B}^+, \mathcal{M}^+], \quad d_t \mathcal{B}^+ = \mathcal{B}^+ \wedge \mathcal{B}^+, \quad (4.8)$$

with dependent variables

$$\lambda_i = \eta_i - \sum_{j=1}^{i-1} b_{2j+1}, \quad \mu_i = \varphi_i^* \quad (i \in \mathcal{I}).$$

The operator $\mathcal{M}^+$ is expressed in the form

$$\mathcal{M}^+ = \kappa + \sum_{i \in \mathcal{I}} \mu_i e_i + e_0 + (c_1 - \lambda_2) e_1 + \sum_{j \in \mathcal{J}'} (\lambda_{j-1} - \lambda_{j+1}) e_j$$

$$+ (\lambda_{2n} - c_{2n+1}) e_{2n+1} + (\lambda_{2n} - c_{2n+2}) e_{2n+2}$$

$$+ e_{1,2} + \sum_{j \in \mathcal{J}'} (e_{j-1,j} + e_{j,j+1}) + e_{2n,2n+1} + e_{2n,2n+2},$$

where $\kappa \in \mathfrak{h}$ and

$$c_1 = b_1, \quad c_i = - \sum_{j=1}^{n-1} b_{2j+1} - b_i \quad (i = 2n + 1, 2n + 2).$$

Note that $d_t \kappa = 0$. We also remark that $c_1$, $c_{2n+1}$ and $c_{2n+2}$ are algebraic functions in $t_1$ and $t_2$.

Let

$$s_1 = \frac{c_{2n+2} - c_1}{2n + 2}, \quad s_2 = \frac{c_{2n+2} - c_{2n+1}}{2n + 2}. $$

We now regard the system (4.8) as a system of ordinary differential equations

$$\left[ s(s - 1) \frac{d}{ds} - B, \mathcal{M}^+ \right] = 0, \quad (4.9)$$

with respect to the independent variable $s = s_1$ by setting $s_2 = 1$. The explicit formula of the $\mathfrak{b}_+$-valued operator $B$ is given below. We also set

$$q_i = \frac{c_{2n+2} - \lambda_{2i}}{2n + 2}, \quad p_i = -\mu_{2i}, \quad \alpha_j = \frac{(\kappa|\alpha_j^\vee)}{2n + 2}.$$

for $i = 1, \ldots, n$ and $j = 0, \ldots, 2n + 2$. Then we obtain

**Theorem 4.2.** The system (4.9) is equivalent to the system (1.1) with (1.2).
The operator $\mathcal{M}^+$ is described as

$$\mathcal{M}^+ = \kappa + \sum_{i=0}^{2n+2} (2n + 2) \varphi_i e_i + \sum_{i=1}^{2n} e_{i,i+1} + e_{2n,2n+2}.$$  

We recall that

$$\varphi_0 = \frac{1}{2n + 2}, \quad \varphi_1 = q_1 - s, \quad \varphi_{2i+1} = q_{i+1} - q_i \quad (i = 1, \ldots, n - 1),$$

$$\varphi_{2j} = -\frac{p_j}{2n + 2} \quad (j = 1, \ldots, n), \quad \varphi_{2n+1} = 1 - q_n, \quad \varphi_{2n+2} = -q_n.$$  

The operator $B$ is described as

$$B = u + \sum_{i=0}^{2n+2} x_i e_i + y_1 e_{0,2} + \sum_{i=2}^{2n} y_i e_{i,i+1} + y_{2n+1} e_{2n,2n+2} + \sum_{j\in J'} y_1 e_{j-1,j,j+1},$$

where

$$x_0 = -\frac{q_1 - s}{2n + 2}, \quad x_1 = 1, \quad x_{2i+1} = s(s-1) - (q_i - s)(q_{i+1} - s),$$

$$x_{2n+1} = -(s-1)q_n, \quad x_{2n+2} = -s(q_n - 1),$$

$$y_1 = -\frac{1}{(2n + 2)^2}, \quad y_{2i} = -\frac{q_{i+1} - s}{2n + 2}, \quad y_{2i+1} = -\frac{q_i - s}{2n + 2},$$

$$y_{2n} = \frac{s-1}{2n+2}, \quad y_{2n+1} = \frac{s}{2n+2},$$

for $i = 1, \ldots, n - 1$ and

$$(2n + 2)x_{2i} = \sum_{j=1}^{i-1} 2\{(q_j - s)p_j + \alpha_{2j}\} + (q_i - s)p_i + \alpha_{2i} + \alpha_0 + \sum_{j=1}^{i-1} \alpha_{2j+1}.$$
for \( i = 1, \ldots, n \). Here \( u = \sum_{i=0}^{2n+2} u_i \alpha_i^\vee \) satisfies

\[
(u|\alpha_i^\vee) = -\alpha_0(q_1 - s),
\]

\[
(u|\alpha_i^\vee) = -\alpha_0(q_1 + s - 1) - \sum_{j=1}^{n} 2q_j \{ (q_j - 1)p_j + \alpha_{2j} \}
- (2\alpha_2 + \beta_{1,3})(s - 1) - \beta_{1,4}s,
\]

\[
(u|\alpha_{2i+1}^\vee) = - \left\{ \sum_{j=1}^{i} 2(q_j - s)p_j + \beta_{i,1} + 2\alpha_{2i} \right\} (q_i + q_{i+1} - 1) - \beta_{i+1,4}s
- \sum_{j=i+1}^{n} 2q_j \{ (q_j - 1)p_j + \alpha_{2j} \} - (2\alpha_{2i+2} + \beta_{i+1,3})(s - 1),
\]

\[
(u|\alpha_{2n+1}^\vee) = - \left\{ \sum_{j=1}^{n} 2(q_j - s)p_j + \beta_{n,1} + 2\alpha_{2n} \right\} q_n - \alpha_{2n+1} s,
\]

\[
(u|\alpha_{2n+2}^\vee) = - \left\{ \sum_{j=1}^{n} 2(q_j - s)p_j + \beta_{n,1} + 2\alpha_{2n} \right\} (q_n - 1) - \alpha_{2n+2}(s - 1),
\]

for \( i = 1, \ldots, n - 1 \) and

\[
(u|\alpha_{2i}^\vee) = \left\{ \sum_{j=1}^{i-1} 2(q_j - s)p_j + (q_i - s)p_i + \beta_{i,1} + 2\alpha_{2i} \right\} (2q_i - 1)
+ q_i \{ (q_i - 1)p_i + \alpha_{2i} \} + \sum_{j=i+1}^{n} 2q_j \{ (q_j - 1)p_j + \alpha_{2j} \}
+ (2\alpha_{2i} + \beta_{i+1,3})(s - 1) + \beta_{i+1,4}s,
\]

for \( i = 1, \ldots, n \), where

\[
\beta_{i,0} = \alpha_1 + \sum_{j=1}^{i-1} \alpha_{2j+1}, \quad \beta_{i,1} = \alpha_0 + \sum_{j=1}^{i-1} 2\alpha_{2j} + \sum_{j=1}^{i-1} \alpha_{2j+1},
\]

\[
\beta_{i,3} = \sum_{j=i}^{n-1} \alpha_{2j+1} + \sum_{j=i}^{n} 2\alpha_{2j} + \alpha_{2n+1}, \quad \beta_{i,4} = \sum_{j=i}^{n-1} \alpha_{2j+1} + \alpha_{2n+2}.
\]

**Remark 4.3.** The system (1.1) with (1.2) is derived from a Lax pair associated with the loop algebra \( \mathfrak{so}(4n + 4)[z, z^{-1}] \); see Appendix B.
5 Affine Weyl group symmetry

In this section, we discuss a derivation of the group of symmetries (1.3) following the manner in [NY2].

Recall that the affine Weyl group of type $D^{(1)}_{2n+2}$ is generated by the transformations $r_i$ ($i = 0, \ldots, 2n + 2$) with the fundamental relations

\begin{align*}
  r_i^2 &= 1 \quad (i = 0, \ldots, 2n + 2), \\
  (r_i r_j)^{2 - a_{ij}} &= 0 \quad (i, j = 0, \ldots, 2n + 2; i \neq j).
\end{align*}

acting on the simple roots as

\[ r_i(\alpha_j) = \alpha_j - a_{ij} \alpha_i \quad (i, j = 0, \ldots, 2n + 2), \]

where

\begin{align*}
  a_{ii} &= 2 \quad (i = 0, \ldots, 2n + 2), \\
  a_{02} &= a_{ii+1} = a_{2n2n+2} = -1 \quad (i = 1, \ldots, 2n), \\
  a_{ij} &= 0 \quad (\text{otherwise}).
\end{align*}

Let $X(0) \in G_{<0}G_{\geq 0}$. We consider a $G_{<0}G_{\geq 0}$-valued function

\[ X = X(t_1, t_2, \ldots) = \exp \left( \sum_{k=1,2,\ldots} t_k \Lambda_k \right) X(0). \]

Then we have a system of partial differential equations

\[ X \partial_k X^{-1} = \partial_k - \Lambda_k \quad (k = 1, 2, \ldots), \]

defined through the adjoint action of $G_{<0}G_{\geq 0}$ on $\hat{\mathfrak{g}}_{<0} \oplus \mathfrak{g}_{\geq 0}$. Via a decomposition

\[ X = W^{-1}Z, \quad W \in G_{<0}, \quad Z \in G_{\geq 0}, \]

we obtain the system (3.1).

In the previous section, we have considered the gauge transformation

\[ \mathcal{M}^+ = \exp(\text{ad}(\Gamma))\mathcal{M}, \quad d_t - \mathcal{B}^+ = \exp(\text{ad}(\Gamma))(d_t - \mathcal{B}), \quad \Gamma \in \mathfrak{g}_0, \]

for the derivation of the system (1.1). Note that it arises from

\[ X = (W^+)^{-1}Z^+, \quad W^+ = \exp(\Gamma)W, \quad Z^+ = \exp(\Gamma)Z. \]

Consider transformations

\[ r_i(X) = X \exp(-e_i) \exp(f_i) \exp(-e_i) \quad (i = 0, \ldots, 2n + 2). \]
Under the similarity condition \( \mathcal{M}^+ \in \mathfrak{b}_+ \), their action on \( W^+ \) is given by

\[
    r_i(W^+) = G_i W^+ \quad (i = 0, \ldots, 2n + 2),
\]

where

\[
    G_i = \exp \left( \frac{\alpha_i}{\varphi_i} f_i \right), \quad \alpha_i = \frac{(\alpha_i | \mathcal{M}^+)}{2n + 2}, \quad \varphi_i = \frac{(f_i | \mathcal{M}^+)}{2n + 2}.
\]

It follows that

\[
    r_i(M^+) = G_i M^+ G_i^{-1}, \quad d_t - r_i(B^+) = G_i(d_t - B^+) G_i^{-1},
\]

for \( i = 0, \ldots, 2n + 2 \). Then each \( r_i(M^+) \) and \( r_i(B^+) \) are \( \mathfrak{b}_+ \)-valued and satisfy the system (4.8). Note that the complex parameters \( \alpha_i \ (i = 0, \ldots, 2n + 2) \) can be regarded as the simple roots for \( g(D^{(1)}_{2n+2}) \).

We define a Poisson structure for the operator \( \mathcal{M}^+ \) by

\[
    \{ \varphi_i, \varphi_j \} = \frac{(f_j, f_i | \mathcal{M}^+)}{2n + 2} \quad (i, j = 0, \ldots, 2n + 2).
\]

It is equivalent to

\[
    \{ p_i, q_j \} = \delta_{i,j}, \quad \{ p_i, p_j \} = \{ q_i, q_j \} = 0 \quad (i, j = 1, \ldots, n).
\]

Hence \( p_i, q_i \ (i = 1, \ldots, n) \) give a canonical coordinate system associated with the Poisson structure for \( \mathcal{M}^+ \). Then the action of the transformations \( r_i \ (i = 0, \ldots, 2n + 2) \) on the coefficients of \( \mathcal{M}^+ \) is equivalent to (1.3).

**Remark 5.1 (S).** Let

\[
    \sigma_1(i) = (0, 1)(2n + 1, 2n + 2)i, \quad \sigma_2(i) = 2n + 2 - i \quad (i = 0, \ldots, 2n + 2),
\]

where \( (0, 1) \) and \( (2n + 1, 2n + 2) \) stand for the adjacent transpositions. Then the system (1.1) with (1.2) is invariant under the action of transformations \( \pi_1 \) and \( \pi_2 \) defined by

\[
    \pi_1(\alpha_i) = \alpha_{\sigma_1(i)}, \quad \pi_1(q_i) = \frac{s(q_i - 1)}{q_i - s}, \quad \pi_1(p_i) = \frac{(q_i - s)(p_i(q_i - s) + \alpha_{2i})}{s(1 - s)},
\]

and

\[
    \pi_2(\alpha_i) = \alpha_{\sigma_2(i)}, \quad \pi_2(q_i) = \frac{q_i}{q_i}, \quad \pi_2(p_i) = -\frac{q_i(p_i + \alpha_{2i})}{s},
\]

for \( i = 0, \ldots, 2n + 2 \). These transformations generate a group of Dynkin diagram automorphisms of type \( D^{(1)}_{2n+2} \). In fact, they satisfy the fundamental relations

\[
    \pi_i^2 = 1 \quad (i = 1, 2),
\]

\[
    (\pi_1 \pi_2)^3 = 1,
\]

\[
    \pi_i r_j = r_{\sigma_i(j)\pi_i} \quad (i = 1, 2; \ j = 0, \ldots, 2n + 2).
\]

15
A Heisenberg subalgebra

We first introduce the simple Lie algebra $\mathfrak{so}(4n + 4)$ and its loop algebra. Denoting matrix units by $E_{i,j} = (\delta_{i,k}\delta_{j,l})_{k,l=1}^{4n+4}$, we set

$$J = \sum_{i=1}^{4n+4} E_{i,4n+5-i}.$$  

Then the algebra $\mathfrak{so}(4n + 4)$ is defined by

$$\mathfrak{so}(4n + 4) = \{ X \in \text{Mat}(4n + 4; \mathbb{C}) \mid JX + tXJ = 0 \}.$$  

Also let $E_j, F_j, H_j$ ($j = 0, \ldots, 2n + 2$) be the Chevalley generators for the loop algebra $\mathfrak{so}(4n + 4)[z, z^{-1}]$ defined by

$$E_0 = zX_{4n+3,1}, \quad E_i = X_{i,i+1}, \quad E_{2n+2} = X_{2n+1,2n+3},$$

$$F_0 = \frac{1}{z}X_{1,4n+3}, \quad F_i = X_{i+1,i}, \quad F_{2n+2} = X_{2n+3,2n+1},$$

$$H_0 = -X_{1,1} - X_{2,2}, \quad H_i = X_{i,i} - X_{i+1,i+1}, \quad H_{2n+2} = X_{2n+1,2n+1} + X_{2n+2,2n+2},$$

for $i = 1, \ldots, 2n + 1$, where $X_{i,j} = E_{i,j} - E_{4n+5-j,4n+5-i}$. Note that

$$H_0 + H_1 + \sum_{i=2}^{2n} 2H_i + H_{2n+1} + H_{2n+2} = 0.$$  

Under a specialization $K = 0$, we can identify this loop algebra with the affine Lie algebra $\mathfrak{g}(D^{(1)}_{2n+2})$. Note that the scaling element $d$ corresponds to the differential operator $z\partial_z$. We also remark that

$$[X, Y] = XY - YX, \quad (X|Y) = \frac{1}{2}\text{tr} XY.$$  

In a similar manner as [DF], we formulate the Heisenberg subalgebra of type $(1, 1, 0, 1, 0, \ldots, 1, 0, 1, 1)$ in a framework of $\mathfrak{so}(4n + 4)[z, z^{-1}]$. Let $\Lambda_{1,i}$ ($i = 1, 2$) be matrices defined by

$$\Lambda_{1,1} = E_0 + [E_1, E_2] + \sum_{j \in J'} (E_j + [E_{j-1}, [E_j, E_{j+1}]] + E_{2n+1} + [E_{2n}, E_{2n+2}],$$

$$\Lambda_{1,2} = E_1 + [E_0, E_2] + \sum_{j \in J'} ([E_{j-1}, E_j] + [E_j, E_{j+1}]) + E_{2n+2} + [E_{2n}, E_{2n+1}].$$  

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Note that $[\Lambda_{1,1}, \Lambda_{1,2}] = 0$. We also set

$$\Lambda_{(2n+2)k+l,i} = z^k(\Lambda_{1,i})^l \quad (i = 1, 2; k \in \mathbb{Z}; l = 1, 3, \ldots, 2n + 1).$$

Then we have a maximal nilpotent subalgebra $\bigoplus_{k \in \mathbb{Z}} (\mathfrak{C} \Lambda_{2k-1,1} \oplus \mathfrak{C} \Lambda_{2k-1,2})$ of $\mathfrak{so}(4n + 4)[z, z^{-1}]$. It can be identified with the Heisenberg subalgebra $\mathfrak{s}$ given in Section 2 under the specialization $K = 0$.

**Remark A.1.** The isomorphism classes of the Heisenberg subalgebras are in one-to-one correspondence with the conjugacy classes of the finite Weyl group $[KP]$. In the notation of $[C]$, the Heisenberg subalgebra $\mathfrak{s}$ introduced above corresponds to the regular primitive conjugacy class $D_{2n+2}(a_n)$ of the Weyl group of type $D_{2n+2}$.

### B Lax pair

It is known that $P_{VI}$ is derived from the Lax pair associated with the loop algebra $\mathfrak{so}(8)[z, z^{-1}]$ [NY3]. In this section, we propose a Lax pair for the system (1.1) with (1.2) in a framework of $\mathfrak{so}(4n + 4)[z, z^{-1}]$.

In the previous section, we have derived the system (4.9). It can be identified with the system on $\mathfrak{so}(4n + 4)[z, z^{-1}]$

$$s(s - 1) \frac{d}{ds} - B, z \frac{dz}{dz} + M = 0, \quad (B.1)$$

where

$$M = \sum_{i=0}^{2n+2} \varepsilon_i H_i + \sum_{i=0}^{2n+2} \varphi_i E_i + \sum_{i=1}^{2n} \frac{[E_i, E_{i+1}]}{2n + 2} + \frac{[E_{2n}, E_{2n+2}]}{2n + 2} ,$$

$$B = \sum_{i=0}^{2n+2} u_i H_i + \sum_{i=0}^{2n+2} x_i E_i + y_1 [E_0, E_2] + \sum_{i=2}^{2n} y_i [E_i, E_{i+1}]$$

$$+ y_{2n+1} [E_{2n}, E_{2n+2}] + \sum_{j \in J'} y_1 [E_{j-1}, [E_j, E_{j+1}]],$$

under the specialization $K = 0$. Here $\varepsilon_i$ ($i = 0, \ldots, 2n + 2$) are complex parameters such as

$$\alpha_0 = 1 + 2\varepsilon_0 - \varepsilon_2, \quad \alpha_1 = 2\varepsilon_1 - \varepsilon_2, \quad \alpha_2 = -\varepsilon_0 - \varepsilon_1 + 2\varepsilon_2 - \varepsilon_3,$$

$$\alpha_i = -\varepsilon_{i-1} + 2\varepsilon_i - \varepsilon_{i+1} \quad (i = 3, \ldots, 2n - 1),$$

$$\alpha_{2n} = -\varepsilon_{2n-1} + 2\varepsilon_{2n} - \varepsilon_{2n+1} - \varepsilon_{2n+2},$$

$$\alpha_{2n+1} = -\varepsilon_{2n} + 2\varepsilon_{2n+1}, \quad \alpha_{2n+2} = -\varepsilon_{2n} + 2\varepsilon_{2n+2}.$$
Consider a system of linear differential equations

\[ s(s - 1) \frac{dw}{ds} = Bw, \quad z \frac{dw}{dz} + Mw = 0, \]  
(B.2)

for a vector of unknown functions \( w = (w_1, \ldots, w_{4n+4}) \). Then the system (B.1) can be regarded as the compatibility condition of (B.2). In this framework, the group of symmetries (1.3) arise from gauge transformations

\[ r_i(w) = \left( 1 + \frac{\alpha_i}{\varphi_i} F_i \right) w \quad (i = 0, \ldots, 2n + 2). \]

Note that the Lax pair (B.2) of the case \( n = 1 \) is equivalent to one of [NY3].

The Lax pair (B.2) arises from the Drinfeld-Sokolov hierarchy as follows. Under the system (3.1), we consider a \( G_{<0}G_{\geq 0} \)-function \( \Psi = \Psi(t_1, t_2, \ldots) \) defined by

\[ \Psi = W \exp \left( \sum_{k=1,2,\ldots} t_k \Lambda_k \right). \]

Then we obtain

\[ \Psi \partial_{t_k} \Psi^{-1} = \partial_{t_k} - B_k \quad (k = 1, 2, \ldots), \quad \vartheta \Psi \Psi^{-1} = \mathcal{M}. \]  
(B.3)

Note that the system (3.2) can be regarded as the compatibility condition of (B.3). In the following, we use a conventional form of (B.3)

\[ \partial_{t_k} (\Psi) = B_k \Psi \quad (k = 1, 2, \ldots), \quad \vartheta(\Psi) = (\vartheta - \mathcal{M})\Psi. \]

It is equivalent to

\[ \partial_{t_k} (\Psi) = B_k \Psi \quad (k = 1, 2), \quad \vartheta(\Psi) = (t_1 B_1 + t_2 B_2)\Psi, \]  
(B.4)

under the specialization \( \mathcal{M} \in \mathfrak{g}_{\geq 0} \) and \( t_k = 0 \) (\( k \geq 3 \)). Via a gauge transformation \( \Psi'^+ = \exp(\Gamma)\Psi \), the system (B.4) is transformed into

\[ \partial_{s_k} (\Psi'^+) = B_k^+ \Psi'^+ \quad (k = 1, 2), \quad \vartheta(\Psi'^+) = (\vartheta - \mathcal{M}^+)\Psi'^+, \]  
(B.5)

where \( B_k^+ \) (\( k = 1, 2 \)) are defined by \( B^+ = B_1^+ ds_1 + B_2^+ ds_2 \) and

\[ s_1 = \frac{c_{2n+2} - c_1}{2n + 2}, \quad s_2 = \frac{c_{2n+2} - c_{2n+1}}{2n + 2}. \]

The system (B.5) can be identified with (B.2) under the specialization \( s_2 = 1 \) and \( K = 0 \).
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References

[AS] M. J. Ablowitz and H. Segur, Exact linearization of a Painlevé transcendent, Phys. Rev. Lett. 38 (1977), 1103-1106.

[C] R. Carter, Conjugacy classes in the Weyl group, Compositio Math. 25 (1972), 1-59.

[DF] F. Delduc and L. Fehér, Regular conjugacy classes in the Weyl group and integral hierarchies, J. Phys. A: Math. Gen. 28 (1995), 5843-5882.

[DS] V. G. Drinfel’d and V. V. Sokolov, Lie algebras and equations of Korteweg-de Vries type, J. Sov. Math. 30 (1985), 1975-2036.

[FS] K. Fuji and T. Suzuki, The sixth Painlevé equation arising from $D_4^{(1)}$ hierarchy, J. Phys. A: Math. Gen., 39 (2006) 12073-12082.

[IKSY] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, From Gauss to Painlevé — A Modern Theory of Special Functions, Aspects of Mathematics E16 (Vieweg, 1991).

[Kac] V. G. Kac, Infinite dimensional Lie algebras, Cambridge University Press (1990).

[KIK] T. Kikuchi, T. Ikeda and S. Kakei, Similarity reduction of the modified Yajima-Oikawa equation, J. Phys. A: Math. Gen., 36 (2003) 11465-11480.

[KK1] S. Kakei and T. Kikuchi, Affine Lie group approach to a derivative nonlinear Schrödinger equation and its similarity reduction, Int. Math. Res. Not. 78 (2004), 4181-4209.

[KK2] S. Kakei and T. Kikuchi, The sixth Painlevé equation as similarity reduction of $\widehat{gl}_3$ hierarchy, preprint (nlin-si/0508021).

[KP] V. G. Kac and D. Peterson, 112 constructions of the basic representation of the roop group of $E_8$, in Symposium on Anomalies, Geometry ans Topology, ed. W. A. Baedeen and A. R. White, (World Scientific, 1985) 276-298.
[NY1] M. Noumi and Y. Yamada, Higher order Painlevé equations of type $A_l^{(1)}$, Funkcial. Ekvac. 41 (1998), 483-503.

[NY2] M. Noumi and Y. Yamada, Birational Weyl group action arising from a nilpotent Poisson algebra, in Physics and Combinatorics 1999, Proceedings of the Nagoya 1999 International Workshop, ed. A.N.Kirillov, A.Tsuchiya and H.Umemura, (World Scientific, 2001) 287-319.

[NY3] M. Noumi and Y. Yamada, A new Lax pair for the sixth Painlevé equation associated with $\hat{\mathfrak{so}}(8)$, in Microlocal Analysis and Complex Fourier Analysis, ed. T.Kawai and K.Fujita, (World Scientific, 2002) 238-252.

[O] K. Okamoto, Studies on the Painlevé equations, I, Ann. Math. Pura Appl. 146 (1987), 337–381, II, Jap. J. Math. 13 (1987), 47–76, III, Math. Ann. 275 (1986), 221–256, IV, Funkcial. Ekvac. 30 (1987), 305–332.

[S] Y. Sasano, Higher order Painlevé equations of type $D_l^{(1)}$, RIMS Koukyuroku 1473 (2006) 143-163.