BALDWIN-OZSVÁTH-SZABÓ COHOMOLOGY IS A LINK INVARIANT

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Abstract. In their recent preprint, Baldwin, Ozsváth and Szabó defined a twisted version (with coefficients in a Novikov ring) of a spectral sequence, previously defined by Ozsváth and Szabó, from Khovanov homology to Heegaard-Floer homology of the branched double cover along a link. In their preprint, they give a combinatorial interpretation of the $E_3$-term of their spectral sequence. The main purpose of the present paper is to prove directly that this $E_3$-term is a link invariant. We also give some concrete examples of computation of the invariant.

1. Introduction

The last decade or so has been a fruitful time for invention of a new generation of knot invariants. This includes Khovanov homology [7, 5], which is a sequence of homology groups whose Euler characteristic is the Jones polynomial, and knot Floer homology of Ozsváth and Szabó [11, 12, 13, 9], which is similarly related to the Alexander polynomial. In [3], Ozsváth and Szabó considered yet another link invariant, namely the Heegaard-Floer homology of the branched double cover of $S^3$ along $L$, and discovered a spectral sequence from Khovanov homology to $\tilde{HF}(\Sigma(L))$. Baldwin [1] proved that every $E^r$-term of this spectral sequence is a link invariant.

More recently still, Baldwin, Ozsváth and Szabó [3] introduced a variant, namely perturbed Heegaard Floer homology with coefficients in a “Novikov ring”. They also constructed a spectral sequence analogous to [3] in this new setting. Curiously, the behavior of this modified construction is in a way quite distinct from [3]. Instead of the $E_2$-term being Khovanov homology, $d_1$ is, in fact, trivial, and the cochain complex $(E_2, d_2)$ has a combinatorial description given in [3]. In fact, basis

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elements of $E_2$ can be identified with Kauffman states for the Alexander polynomial \cite{Khovanov}, the set of which is considerably smaller than the basis of the chain complex calculating Khovanov homology.

The main purpose of this paper is to show that the $E_3$-term of the spectral sequence mentioned in the last paragraph, which we call Baldwin-Ozsváth-Szabó cohomology, is an invariant of oriented links. This was conjectured by John Baldwin. It is proved in \cite{Baldwin} that the next possible differential in this spectral sequence is $d_6$. It is therefore natural to ask if the spectral sequence collapses. This is not known at present. Even if the spectral sequence collapses, the $E_3$ term is a new invariant, since it is graded, while the spectral sequence is, at least a priori, not.

We aim for the present paper to be entirely self-contained. In fact, we use no Floer homology techniques, our methods are entirely algebraic. We define all the concepts we are using in Section 2 below, and state our main result precisely. We also prove from first principles that the Baldwin-Ozsváth-Szabó $d_2$-differential satisfies $dd = 0$, without referring to the spectral sequence. In Section 3 we prove a fundamental lemma which allows us to vary the field of coefficients. This is a key step in proving invariance under the Reidemeister moves, which is proved in Sections 4, 5. Ultimately, the main tool used in those proofs are algebraic identities involving Möbius transformations over fields of characteristic 2.

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# 2. Preliminaries, and statement of the main result

Consider an oriented link $L$ in $S^3$ with generic projection $\mathcal{D}$. Throughout this paper, we will use the following assumption:

(A) Every connected component of $S^2 \setminus \mathcal{D}$ is simply connected.

Following \cite{Baldwin}, we denote by $(C(\mathcal{D}), \Psi)$ the cochain complex which is the $E_2$-term of the spectral sequence \cite{Khovanov} converging to the Heegaard-Floer twisted cohomology $\widehat{HF}(\Sigma(L))$ where $\Sigma(L)$ is the branched double cover of $S^3$ along the link $L$. In particular, with $\mathcal{D}$, there is associated a planar **black graph** $B(\mathcal{D})$, and a dual planar **white graph** $W(\mathcal{D})$. $C(\mathcal{D})$ is the $\Lambda$-module on the basis $K(\mathcal{D})$, which is the set of all Kauffman...
states, which are the spanning trees of $B(\mathcal{D})$. We color the connected components of $S^2 \setminus \mathcal{D}$ (called faces) black and white so that a black and white face never share an edge. The vertices of $B(\mathcal{D})$ consist of faces colored black, and edges go through crossings of $\mathcal{D}$. The white graph is defined dually where the vertices are the faces which are colored white. Note that two vertices of the graph $B(\mathcal{D})$ may have connected by multiple edges and loops are also possible (similarly for the graph $W(\mathcal{D})$). Because of this, technically, $G = B(\mathcal{D}), W(\mathcal{D})$ must be defined as 1-dimensional CW complexes, i.e. there are sets of vertices $V(G)$ and edges $E(G)$ and source and target maps $S, T : E(G) \to V(G)$. However, when it is clear which edge connecting two vertices $x, y \in V(G)$ we have in mind, we will also abuse notation to write $\{x, y\} \in E(G)$.

Note that there is a canonical bijection

$$(1) \quad \tau : E(B(\mathcal{D})) \to E(W(\mathcal{D}))$$

sending a black edge to the white edge passing over the same crossing of $\mathcal{D}$. We may additionally speak of orientations related by $\tau$ when the white edge orientation is obtained by rotating the black edge orientation by 90° degrees counter-clockwise. Note also that for a black spanning tree $T$, there is a unique dual white spanning tree $\tau(T)$ which contains precisely the edges $\tau(e)$ where $e \notin E(T)$. We also note that $B(\mathcal{D})$ and $W(\mathcal{D})$ are planar graphs; by the Assumption (A), together with their faces, these graphs specify “Poincaré-dual” CW-decompositions of $S^2$, which will be denoted by $B(S^2), W(S^2)$, respectively.

To each edge $e$ of $B(\mathcal{D})$ there is now assigned a height $h(e) \in \{0, 1\}$ which depends on the direction $e$ crosses the crossing of $\mathcal{D}$. The convention is arbitrary, but must be fixed. Actually, more precisely, there is another convention which must be fixed, namely positive and negative crossings, and both conventions must be related appropriately. Let us say that a crossing is positive when the upper edge of the crossing is oriented in the direction 90° clockwise from the orientation of the bottom edge. In the other case, we speak of a negative crossing (see Figure 1). Let the number of positive resp. negative crossings of the projection $\mathcal{D}$ be $n_+$ resp. $n_-$. Now the height of a black edge $e$ passing through a crossing is 0 if the upper arc of the crossing is 45° counter-clockwise from the edge $e$ and 1 otherwise (this is independent of orientation; see Figure 2).

We may make different conventions regarding heights of white edges. It is perhaps most natural to set

$$(2) \quad h(\tau(e)) = 1 - h(e).$$
Figure 1. A positive crossing and a negative crossing

\[ e_1 \rightarrow e_2 \]

Figure 2. The height of an edge through a crossing

\[ h(e) = 0 \quad h(e) = 1 \]

(That way, if we swap faces colored white and black, we obtain manifestly isomorphic cochain complexes.) For a spanning tree \( T \) of \( B(D) \), we set

\[ h(T) = \sum_{e \in E(T)} h(e) + \sum_{e \notin E(T)} (1 - h(e)). \]

We now construct a cochain complex whose summand in degree \( d \) is the free \( \Lambda \)-module (where \( \Lambda \) is a field specified below) on all spanning trees of height \( h = 2d + n_- \). In other words,

\[ d = \frac{1}{2}(h - n_-), \]

and we notice that \( \in \frac{1}{2}\mathbb{Z} \). It is not difficult to see, however, that for a given projection \( D \), all degrees which can occur differ by integers, or, in other words, heights of any two spanning trees \( T, T' \) differ by even numbers (this is shown by induction on the number of edges in \( E(T) \setminus E(T') \)). The differential \( \Psi \) additionally depends on weights which are \( \mathbb{Z} \)-linearly independent (except as explicitly specified below) real numbers \( w(e) \) assigned to each oriented black edge \( e \). Reversing orientation of an edge has the effect of reversing the sign of \( w(e) \). We set

\[ w(\tau(e)) = w(e). \]

To define \( \Psi \), we also choose a base point which is an arc of \( D \). Then there is precisely one adjacent black vertex and one adjacent white
vertex which are called the black base point and white base point. Now let \( T \in K(D) \) and \( T' \in K(D) \) where there exist black edges \( e, f \in E(B(D)) \) with \( h(e) = 0, h(f) = 1 \),
\[
E(T') = (E(T) \setminus \{e\}) \cup \{f\}.
\]
(Note that \( h(T') = h(T) + 2 \).) Consider then the unique black circuit \( c \) specified by the edges of \( E(T) \cup E(T') \). We orient the circuit consistently (clockwise or counterclockwise) so that \( f \) is oriented from the connected component \( C \) of \( E(T) \cap E(T') \) not containing the base point to the connected component \( C' \) containing the base point. Then let \( A(T,T') \) be the sum of the weights of the edges of the circuit \( c \), oriented as specified above. We obtain another number \( B(T,T') \) as the sum of the weights of all black edges from a vertex of \( C \) to a vertex of \( C' \). Then define
\[
\Psi(T) = \sum_{T'} \left( \frac{1}{1 + T A(T,T')} + \frac{1}{1 + T B(T,T')} \right) T'.
\]

The coefficient is calculated in the Novikov field \( \Lambda \), by which we mean the set of elements of the form
\[
\sum_{n=0}^{\infty} T^{a_i}
\]
where \( a_1 < a_2 < ... \in \mathbb{R} \). Addition and multiplication is performed term-wise over \( \mathbb{Z}/2 \), by which operations \( \Lambda \) forms a ring, which is readily seen to be a field.

Note again that \( \Psi \) raises \( h \) by 2. It can be verified directly that the differential \( \Psi \) of (6) satisfies
\[
\Psi \circ \Psi = 0,
\]
which we will prove at the end of this section after some re-statements. Nevertheless, it may be difficult to guess the formula (6) directly. Baldwin, Ozsváth and Szabó \cite{BO} obtained the complex \((C(D), \Psi)\) as the \( E_2 \)-term of a spectral sequence calculating twisted Heegaard-Floer cohomology of the branched double cover \( \Sigma(L) \) of \( S^3 \) along the link \( L \), which implies (7).

It is worth noting that in the definition of the differential \( \Psi \), black and white do not play a symmetrical role: if we interpret \( B(T,T') \) as the sum of weights of white edges on a consistently oriented white circuit \( w \), then the orientation of \( w \) does not depend on the choice of edges \( e, f \), as long as they cross two edges of \( w \) of the required heights. On
the other hand, the orientation of the black circuit $c$ discussed above clearly can depend on the choice of the edges $e, f$ in it.

Nevertheless, it turns out that the definition of $\Psi$ is symmetrical in black and white, when interpreted properly. Let us first observe another property, namely invariance of the choice of base point. Clearly, the definition presented above only depends on the choice of black base point. Now when the black base point moves from the connected component $C$ to the component $C'$, the both of the numbers $A(T, T'), B(T, T')$ get multiplied by $-1$. Thus, the differential remains the same by the formula

\[
\frac{1}{1+k} + \frac{1}{1+\ell} = \frac{1}{1+k^{-1}} + \frac{1}{1+\ell^{-1}},
\]

which is valid in fields of characteristic 2. Let us now turn to the question of swapping black and white. By definition, the differential after the swap will be equal to the original differential when $T, T'$ are such that the white base point is inside the black circuit $c$ if and only if $c$ is oriented clockwise (note that the roles of $e, f$ are the opposite from the roles of the white edges crossing them). By (8), then, again, the differential doesn’t change when the white base point is in the other connected component of $S^2 \setminus c$, and hence is equal to the original differential.

It is worth noting that there is one variant $\Psi'$ of the definition of $\Psi$ which does produce possibly different cohomology, namely if we change the convention so that one of the numbers $A(T, T'), B(T, T')$ remains the same, and the other is multiplied by $-1$. We see that one way of achieving this is by swapping the roles of $e$ and $f$ in determining the orientation of $c$. Therefore, by the universal coefficient theorem, the cohomology of the complex modified in this way is isomorphic to the dual of the $\Psi$-cohomology of the mirror projection $\mathcal{D}'$ to $\mathcal{D}$ of the mirror link $L'$ of $L$. More precisely, counting the number of positive and negative crossings, and keeping in mind that a positive crossing turns into negative and vice versa in the mirror projection, the sign of the cohomological degree gets reversed, i.e.

\[
H^i(C(\mathcal{D}, \Psi') = H^{-i}(C(\mathcal{D}', \Psi)).
\]

It may be tempting to call the cohomology of $(C(\mathcal{D}), \Psi)$ twisted Khovanov cohomology, but this is, in fact, inaccurate, since it is the $E_3$-term (and not $E_2$-term) of the twisted analogue of the spectral sequence \cite{10} from $E_2 = \text{Khovanov cohomology}$ to Heegaard Floer cohomology
of $\Sigma(L)$. Because of this, we use the term Baldwin-Ozsváth-Szabó cohomology.

The field $\Lambda$ and the selection of arbitrary weights with the requirement that they be linearly independent over $\mathbb{Q}$ may seem unnatural. In fact, it can be restated. First observe that in computing the numbers $A(T, T')$, we always sum the weights of edges of a consistently oriented circuit $c$. The circuit determines a cellular 1-cycle, i.e. an element

$$\tau \in Z^\text{cell}_1(B(S^2), \mathbb{Z}).$$

Now since $H^1(S^2, \mathbb{Z}) = 0$, we have $\tau = dx$ where $x \in C^\text{cell}_2(B(S^2), \mathbb{Z})$. The generators of $C^\text{cell}_2(B(S^2))$ are faces $f$, which, by convention, we orient so that the circuit $df$ is oriented counter-clockwise for the bounded faces and clockwise for the unbounded face. Then the sum $\sum f$ of all the faces of $B(S^2)$ is a 2-cycle (representing the fundamental class of $S^2$, and $x$ is determined uniquely up to adding integral multiples of $\sum f$. This means that if for each face $f$ we choose a formal variable $u_f$, with the relation

$$\prod_f u_f = 1,$$

we may assign to $c$ a well defined element

$$\alpha(T, T') = \prod_f (u_f)^{\epsilon(f)}$$

where

$$x = \sum_f \epsilon_f f.$$

Similarly, $B(T, T')$ may be interpreted as the cellular 1-cochain in $C^1_\text{cell}(B(S^2))$ which is of the form $\delta(y)$ where the value of $y$ is 1 on all the vertices of $C$, and 0 on all the vertices of $C'$. The sum $\sum v$ of all vertices of $B(S^2)$ satisfies $\delta(\sum v) = 0$ (it represents the unit element in $H^0(S^2, \mathbb{Z})$, so if we choose, again, a formal variable $z_v$ for every vertex $v$, subject to the relation

$$\prod_v z_v = 1,$$

then we may assign to $T, T'$ a well defined element

$$\beta(T, T') = \prod_v (z_v)^{g(v)}.$$
and define $\Psi$ by
\begin{equation}
\Psi(T) = \sum_{T'} \left( \frac{1}{1 + \alpha(T, T')} + \frac{1}{1 + \beta(T, T')} \right) T'.
\end{equation}
This is clearly a generalization of the definition of $\Psi$ given by (6).

With precise definitions in place, we may now state our main result:

**Theorem 1.** Let $F$ be a field of characteristic 2 with elements $u_f, z_v$ satisfying the relations (9), (10), and such that if we choose one face $f_0$ and one vertex $v_0$, then all the elements $u_f, f \neq f_0$ are algebraically independent and all the elements $z_v, v \neq v_0$ are algebraically independent. Then for each $i$,
\begin{equation}
\text{rank}_F(H^i(C(D), \Psi))
\end{equation}
defined by (11) is independent of the choice of $F$, and of the projection $D$ of an oriented link $L$, subject to the condition (A). If, further, $L$ is a knot, then (12) is independent of orientation. If $L$ is a link which has a projection with more than 1 connected component, then (12) is equal to 0.

The remainder of this paper consists of work toward the proof of Theorem 1. We conclude this section with a

**Proof of the formula** (7) **for $\Psi$ generalized by the definition** (11): Our aim is to compute
\begin{equation}
\Psi\Psi(T)
\end{equation}
for a spanning tree $T$, and prove that its coefficient on any tree $T''$ is equal to 0. The key observation is that it actually suffices to consider the case when $T$ has only two edges of height 0, since otherwise we may contract each component of the complement of the two open edges in $T$ to a point and obtain the same coefficient.

Now up to isomorphism, there is only one tree with two edges, with vertices $x, y, z$ and edges $\{x, z\}, \{y, z\}$. Then there are two non-isomorphic choices of the tree $T''$ (consisting of two edges of height 1): the edges of $T''$ may be either $\{x, z\}, \{y, z\}$, or $\{x, z\}, \{x, y\}$. In the first case, the coefficient of (13) at $T''$ is a sum of two equal terms (each a product of two terms in opposite orders), so the sum is 0 since we are in characteristic 2.

The second case is non-trivial. Assuming, without loss of generality, that the edges $\{x, y\}, \{y, z\}$ and the edge $\{x, z\}$ of height 0 form a face $u$, and if the other bounded face $v$ is bounded by the two $\{x, z\}$ edges,
then (identifying vertices and faces with their corresponding variables),
the formula we need to prove is

\[
\left(\frac{1}{1+u} + \frac{1}{1+x}\right) \left(\frac{1}{1+uv} + \frac{1}{1+y}\right) + \\
\left(\frac{1}{1+v^{-1}} + \frac{1}{1+x}\right) \left(\frac{1}{1+uv} + \frac{1}{1+xy}\right) + \\
\left(\frac{1}{1+u} + \frac{1}{1+xy}\right) \left(\frac{1}{1+v} + \frac{1}{1+y}\right) = 0
\]

(14)

(the left hand side being the coefficient of (13) at \(T''\)). To verify (14), notice that

\[
\frac{1}{1+u} + \frac{1}{1+uv} + \frac{1}{1+v} + \frac{1}{1+uv} + \frac{1}{1+y} = 0, \\
\frac{1}{1+x} + \frac{1}{1+xy} + \frac{1}{1+y} + \frac{1}{1+xy} + \frac{1}{1+v^{-1}+xy} = 1, \\
\frac{1}{1+u} + \frac{1}{1+xy} + \frac{1}{1+u} + \frac{1}{1+xy} + \frac{1}{1+x} + \frac{1}{1+uv} + \frac{1}{1+xy} + \frac{1}{1+uv} + \frac{1}{1+xy} = 1.
\]

\[\square\]

3. THE FUNDAMENTAL LEMMA

Recall that we assume (A). For a finite CW-complex \(X\), note that we have a canonical isomorphism between cellular chains and cellular cochains:

\[
C^{cell}_k(X, \mathbb{R}) \cong C^{k}_{cell}(X, \mathbb{R})
\]

which sends

\[\sum \lambda_i e_i\]

for cells \(e_i\) to the cochain whose value, on a \(k\)-cell \(e\), is

\[\sum \lambda_i\]

We will treat this isomorphism as an identification. It does not, of course, in general send cycles to cocycles, but it is important to note that it is independent of choice of orientation of cells (provided that we choose the same orientation in homology and cohomology).
Lemma 2. Suppose

\[ c := \sum_{e_i \in E(D)} \lambda_i e_i \in Z^1_{cell}(B(S^2), \mathbb{R}) \]

and also

\[ \tau c := \sum_{e_i \in E(D)} \lambda_i \tau(e_i) \in Z^1_{cell}(W(S^2), \mathbb{R}). \]

Then

\[ c = 0 \in C^1_{cell}(B(D, \mathbb{R}). \]

Proof: We have \( H^1_{cell}(B(S^2), \mathbb{R}) = 0 \), so by (16), there exists a function \( u : VB(D) \rightarrow \mathbb{R} \) such that

\[ \delta u = c. \]

Now the condition (17) using (18), translates to the equations

\[ \sum_{y : \{y, x\} \in EB(D)} u(y) - u(x) = 0 \text{ for } x \in VB(D), \]

or

\[ u(x) = \frac{1}{\sharp(S_x)} \sum_{y \in S_x} u(y) \]

where

\[ S_x = \{ e \in EB(D) \mid e \text{ has vertices } x, y \}. \]

(Note that (19) can be interpreted as a discrete analogue of \( u \) being “harmonic”.)

Now (19) implies that \( u \) is constant on connected components \( C \) of \( B(D) \) (actually, by our assumption, \( B(D) \) is connected). To see this, consider

\[ m_C = \min_{x \in VC} u(x). \]

By induction, we see that \( u(y) = m_C \) for all \( y \in VC \). This implies that \( c = \delta u = 0. \)

Lemma 3. Suppose \( c_1, ..., c_n \) are consistently oriented circuits in \( B(D) \) and \( q_1, ..., q_m \) are consistently oriented circuits in \( W(D) \) and

\[ \sum_{i=1}^{n} \lambda_i w(c_i) = \sum_{j=1}^{m} \mu_j w(q_j) \text{ for } \lambda_i, \mu_j \in \mathbb{Z}. \]
Then

\[
\sum_{i=1}^{n} \lambda_i w(c_i) = \sum_{j=1}^{m} \mu_j (q_j) = 0.
\]

Proof: We have

\[
c := \sum_{j=1}^{m} \sum \{ \mu_j e \mid \tau(e) \text{ is an edge of } q_j \} \in Z^1_{cell}(B(S^2), \mathbb{R}),
\]

\[
q := \sum_{i=1}^{n} \sum \{ \lambda_i \tau(e) \mid e \text{ is an edge of } c_i \} \in Z^1_{cell}(W(S^2), \mathbb{R}),
\]

and by \(\mathbb{Z}\)-linear independence of weights,

\[
q = \tau c.
\]

Apply Lemma 2.

Corollary 4. The quantity (12) does not depend on the choice of the field \(F\) subject to the conditions of Theorem 1.

Proof: For two different choices of fields \(F, F'\), consider the two complexes \(F(K(D)), F'(K(D))\) with generators \(K(D)\). Performing cancellation (cf. [1, 2]) simultaneously on the same edge of both complexes, Lemma 3 implies by induction on the number of steps of cancellation that an edge disappears in one of the complexes if and only if it disappears it disappears in the others. This implies that the cohomologies are isomorphic.

4. Reidemeister 1 and 2

Next, we shall prove that Baldwin-Ozsváth-Szabó cohomology is invariant under the three Reidemeister moves (see Figure 3). Note first that if a generic projection \(D'\) is obtained from a generic projection \(D\) by performing a Reidemeister 1 move creating a new crossing, then we either added a new vertex \(v\) and an edge \(e\) originating in \(v\) to the black graph, or a loop \(f\) without adding a new vertex. The complex \((C(D'), \Psi)\) is therefore isomorphic to \((C(D), \Psi)\) up to shift of degrees. To compute the shift of degrees, note that height of corresponding states increases by 1 if and only if \(e\) has height 1 or \(f\) has height 0; otherwise, heights of corresponding states stay the same as in \(C(D)\).
However, by our conventions, the first case arises if and only if the new crossing was negative. Thus, by the formula (1), the degree of corresponding states remains unchanged in either case.

Let us now turn to the Reidemeister 2 move. Let $D'$ be the projection after a Reidemeister 2 move. By the isomorphism of black and white complexes, we may assume that the number of black vertices increases by 2. More precisely, there exists a vertex

$$u \in VB(D)$$

such that

$$VB(D') = (VB(D) \setminus \{u\}) \amalg \{u_1, u_2, v\}.$$  

Additionally, if $S$ is the set of all edges in $EB(D)$ adjacent to $x$, there exists a decomposition

$$S = S_1 \amalg S_2$$

such that for every edge in $S_i$ with vertices $u, w$, there is an edge in $EB(D')$ with vertices $u_i, w, i = 1, 2$. Additionally, for every edge $e \in EB(D)$ neither vertex of which is $u, e \in EB(D')$, and we also have edges

$$\{u_i, v\} \in EB(D')$$

where $\{u_i, v\}$ has height $i - 1$. Finally, $EB(D')$ contains no other edges other than specified above. Note that we have a bijection

$$\phi : EB(D) \rightarrow EB(D') \setminus \{\{u_1, v\}, \{u_2, v\}\}$$

which sends $\{z, t\}$ to itself for $z, t \neq u$ and $\{z, u\}$ to the appropriate $\{z, u_i\}$. Furthermore, $\phi$ preserves height. The main purpose of this section is to prove the following

**Proposition 5.** The chain complexes $(C(D), \Psi), (C(D'), \Psi)$ have isomorphic cohomology groups.

There are two complementary approaches to studying the complex $C(D')$ (two spectral sequences corresponding to two different decreasing filtrations). We will discuss both, which we feel is necessary for telling the whole story. The reason is that the first approach applies
uniformly to the Reidemeister 2 and Reidemeister 3 moves. In the case of the Reidemeister 2 move, the first approach is strictly speaking not necessary, since the second approach gives a stronger result and points more clearly to the solution of the problem. In the case of the Reidemeister 3 move, however, the second approach is not available directly; we will use an analogue of the first approach to reduce the problem to a situation where the second approach to the Reidemeister 2 move applies.

The first approach: Denote by $F^p C(D')$ the free $\Lambda$-module on all Kauffman states of $D'$ (=spanning trees $T$ of $B(D')$) for which

$$\sum_{e \in EB(D') : \phi(e) \not\in ET} h(e) + \sum_{e \in EB(D') : \phi(e) \in ET} (1 - h(e)) \geq p. \tag{22}$$

(Note that the left hand side is the formula for $h(T)$ modified by excluding the terms for the edges $\{u_i, v\}$.) Then

$$\Psi F^p C(D') \subseteq F^p C(D')$$

(as the differential never decreases the height contribution of any single edge of the black graph).

Let us consider the spectral sequence associated with the filtration $F^p$. To identify it, we need some additional notation. Let

$$K_i = \{T \in K(D') \mid \{u_i, v\} \in ET, \{u_{2-i}, v\} \notin ET\},$$

$$L = \{T \in K(D') \mid \{u_i, v\} \in ET, i = 1, 2\}.$$

We have a bijection $\kappa : K_1 \cong K_2$ with

$$E(\kappa(T)) = (E(T) \setminus \{\{u_1, v\}\}) \cup \{\{u_2, v\}\}.$$  \(\kappa\)

Then clearly, for $T \in K_1$,

$$d_0(T) = c_T \kappa(T), \ c_T \neq 0 \in \Lambda,$$

while for $T \in L$,

$$d_0(T) = 0.$$

Now note that we have a canonical bijection $\iota : K(D) \rightarrow L$,

$$E(\iota(T)) = ET \cup \{\{u_1, v\}, \{u_2, v\}\}.$$  \(\iota\)

Noting carefully that $\iota$ raises height by 1, but the number of negative crossings of $D'$ is also greater by 1 than the number of negative crossings of $D$, we see that we have an isomorphism of graded modules

$$E_1 \cong C(D). \tag{23}$$
However, we need to understand the bigrading. To this end, simply note that the filtration degree of $\iota(T)$ is twice its degree minus 1 plus the number of negative crossings of $D'$, in other words,

$$E_{1}^{p,q} \neq 0 \text{ implies } p = 2(p + q) + n_-(D).$$

Thus, the $E_1$-term lies on a line of slope $-1/2$, and it follows that the only possible differential is $d_2$, and the spectral sequence collapses to $E_3$.

Clearly, we can compute $d_2$, but it is important to note that despite the suggestive formula (23), $d_2$ is not simply an obvious modification of $\Psi_D$ by changing the field $F$: this is because of the fact that a $d_2$ in a spectral sequence associated to a decreasing filtration of a cochain complex is not computed simply by implying the differential $d$ to a cocycle $c$ of the complex of filtration degree $p$, even when $d_1$ is trivial: one must add a counter-term to eliminate the summand of $d(x)$ in filtration degree $p + 1$. In the present case, the differential $d_2$ is more cleanly computed by the second approach, which, in fact, gives a stronger result.

**The second approach:** Introduce another decreasing filtration $G^p$ on $C(D')$ defined by

$$\sum_{e \in (EB(D') \setminus \phi(EB(D))) \cap ET} h(e) + \sum_{e \in (EB(D') \setminus \phi(EB(D))) \setminus ET} (1 - h(e)) \geq p.$$

Roughly speaking, then, in the $G$-filtration, we are counting height contributions of the edges $\{u_i, v\}$, i.e. exactly the edges not counted in the $F$-filtration. We see that

$$G^0(C(D')) = C(D'), \quad G^3(C(D')) = 0,$$

Thus, the associated graded cochain complex is non-trivial only in filtration degrees $0, 1, 2$.

Let us begin by studying this situation in complete generality, and gradually add information specific to $C(D')$. Therefore, let us first consider a cochain complex $\tilde{Q}$ with a decreasing filtration $G^i$ where $G^0 \tilde{Q} = \tilde{Q}, \ G^3 \tilde{Q} = 0$. Let us denote the associated graded pieces in filtration degrees $0, 1, 2$ by $U_0, \tilde{Q}, U_2$, respectively. We shall assume we are working in the category of $F$-modules where $F$ is a field of characteristic 2. In particular, since $F$ is a field, we may choose (arbitrarily) splittings of the maps $G^i(\tilde{Q}) \to G^i(\tilde{Q})/G^{i+1}(\tilde{Q})$. After this choice, we see that the most general form $\tilde{Q}$ can take is expressed in the following
diagram:

\[
\begin{array}{ccc}
U_0 & \overset{i}{\rightarrow} & Q \\
\eta & \downarrow & \eta \\
U_2.
\end{array}
\]

Here \(U_0, Q, U_2\) are considered cochain complexes by the differential on the associated graded pieces of \(\tilde{Q}\), and the total differential is the sum of that differential and all applicable arrows of (27) (roughly, but note that not exactly, a totalization of a double complex). The necessary and sufficient condition for this to work is that \(i, j\) be chain maps, and \(\eta\) be a chain homotopy between \(ji\) and 0. Actually, our convention (which we hope to justify later) is to denote the differentials on \(U_0\) and \(Q\) by \(d\), and the differential on \(U_2\) by \(d'\), so the homotopy condition reads

\[
d'\eta + \eta d = ji.
\]

In the most general situation thus described, little can be said beyond the spectral sequence associated with the filtration.

In the case \(\tilde{Q} = C(D')\), we choose the splitting so that \(U_0, Q, U_2\) are generated by \(K_1, L, K_2\) respectively. Since we know from Corollary 4 that the choice of the field \(F\) is quite flexible, consider a choice of the field \(F\) for the definition of \(C(D)\), and adjoin to it two variables \(x, y\) corresponding to the edges \((u_1, v), (v, u_2)\) (and also the corresponding white edges). We will allow the possibility of an algebraic relation between \(x\) and \(y\) (although we will see quickly that certain algebraic relations, such as \(xy^{-1} = 1\) are forbidden), but we will assume that each of the variables \(x, y\) is algebraically independent from \(F\). Denote the resulting field by \(\tilde{F}\), which we will use for the definition of \(C(D')\): our convention is that the element of \(\tilde{F}\) associated to a black circuit \(c\) in \(B(D')\) is equal to the element associated with the corresponding circuit in \(B(D)\) (obtained by contracting the new edges), times any of the variables \(x, y\) (or their inverses) corresponding to any of the new edges \(c\) may contain with the appropriate orientations. The element of \(\tilde{F}\) associated with a white circuit \(w\) in \(B(D')\) is by our convention the product of all the vertex variables of \(B(D)\) inside (resp. outside) the circuit when oriented counterclockwise (resp. clockwise), times the
product of any of the variables $x, y$ or their inverses corresponding to the white edges $w$ may contain.

With these conventions, if $T$ is a spanning tree in $K_1$, adding $\{u_2, v\}$ to $T$ specifies a black circuit. If we denote the element associated with the corresponding black circuit in $B(\mathcal{D})$ by $b$, then

$$\eta(T) = \left(\frac{1}{1 + bxy} + \frac{1}{1 + xy^{-1}}\right) \kappa(T).$$

The coefficient is non-zero, which translates to the first special condition on $\tilde{Q}$:

$$\eta \text{ is an isomorphism of } \tilde{F}\text{-modules.}$$

With this special condition, we see immediately from (28) that

$$d' = \eta d \eta^{-1} + j i \eta^{-1}$$

(keep in mind that we are in characteristic 2). However, we can say even more:

**Lemma 6.** Under the assumption (30),

$$d + i \eta^{-1}j$$

is a differential on $Q$, and $\tilde{Q}$ is quasiisomorphic to $(Q, d + i \eta^{-1}j)[1]$ (the square bracket denotes dimensional shift by the specified number).

**Proof:** We have

$$(d + i \eta^{-1}j)(d + i \eta^{-1}j) = dd + din^{-1}j + i \eta^{-1}jd + i \eta^{-1}j i \eta^{-1}j = d i \eta^{-1}j + i \eta^{-1}jd + i \eta^{-1}d \eta^{-1}j + i \eta^{-1}d'j = 0.$$ 

A chain map

$$\tau : (Q, d + i \eta^{-1}j)[1] \to \tilde{Q}$$

is defined by

$$\tau(x) = x + \eta^{-1}j(x).$$

To see that $\tau$ is a chain map, compute

$$\tau(dx + i \eta^{-1}jx) = dx + \eta^{-1}jdx + \eta^{-1}j i \eta^{-1}j(x) = dx + \eta^{-1}d \eta^{-1}jx + \eta^{-1}j i \eta^{-1}jx = dx + d \eta^{-1}jx,$$

while

$$d\tau(x) = dx + i \eta^{-1}jx + d \eta^{-1}j(x) + i \eta^{-1}jx + \eta \eta^{-1}jx + jx = dx + d \eta^{-1} jx.$$
Clearly, additionally, the map $\tau$ is injective. We claim that its cokernel is isomorphic to $U$, which is the totalization of the double complex

$$
\begin{array}{ccc}
U'_2 & \xrightarrow{\lambda} & U_2 \\
\downarrow & & \downarrow \\
\end{array}
$$

where $\lambda$ is an arbitrary chain isomorphism, which is clearly acyclic. To this end, we construct a chain map

$$
\mu : \tilde{Q} \rightarrow U,
$$

given as the identity on $U_2$, by

$$
\mu(x) = \lambda^{-1}\eta(x)
$$

for $x \in U_0$, and by

$$
\mu(x) = \lambda^{-1} jx
$$

for $x \in Q$. To verify that $\mu$ commutes with the differential on $x \in U_0$, we have:

$$
d\mu(x) = d\lambda^{-1}\eta(x) = \eta(x) + \lambda^{-1}d'\eta(x) = \eta(x) + \lambda^{-1}\eta dx + \lambda^{-1}jix = \mu(dx + ix + \eta x).
$$

To verify $\mu$ commutes with the differential on $x \in Q$, we have

$$
d\mu(x) = \lambda^{-1}d'jx + jx = \lambda^{-1}jdx + jx = \mu(dx + jx).
$$

Now obviously the sequence of cochain complexes

$$
0 \rightarrow (Q, d + i\eta^{-1}j)[1] \xrightarrow{\tau} \tilde{Q} \xrightarrow{\mu} U \rightarrow 0
$$

is exact, which implies our statement by the long exact sequence in cohomology. □

It is not difficult to see by definition that $Q$ with the differential (32) is isomorphic to the $E_2$-term of the spectral sequence constructed in The First Approach.

Despite its aesthetic appeal, however, Lemma 6 still does not solve our problem: even though we have a chain complex isomorphic to $Q$ as $\tilde{F}$-modules, there is no a priori reason to suspect any connection between the differentials $d$ and (32): it is not even reasonable to call (32) a deformation of $d$, since in general the two summands of (32) do not commute.
Here is where we need to bring in even more concrete information from the situation at hand. As a warm-up, let us consider the differential $d'$ on $U_2$ (see (31)) instead of the differential (32) on $Q$. Notice that both differentials are of similar form, a sum of two terms, one of which is related to a differential we know ($d$ on $U_0$ in case of (31) and $d$ on $Q$ in case of (32)), and the other is expressed as a composition of the maps $i, j$.

In the case of (31), however, we do have another way of understanding the differential $d'$: Recalling that $U_2$ is isomorphic to $U_0$ as a $\tilde{F}$-module by the bijection $\kappa$, and recalling our conventions regarding $\tilde{F}$, one can see that both the differentials $d$ on $U_0$ and $d'$ on $U_2$ are in fact the Baldwin-Ozsváth-Szabó differential with different choice of variables for a suitable link projection. Consider first the summands of $\Psi$ which relate only trees in $K_i$ with a fixed $i$. The difference of coefficients is only in white cycles which cross the edges $\{u_i, v\}$; for a white circuit $w$ in $K_1$ crossing the edge $\{u_2, v\}$ and associated element $\alpha$ in $C(D')$, the element associated with the corresponding white circuit in $K_2$ will be $\alpha$ multiplied by $xy^{-1}$ or $yx^{-1}$, depending on the orientation. Both of these are, in fact, forms of $\Psi$ with different choices of variables in $\tilde{F}$ for the projection $E$ obtained by performing a skein move instead on the two arcs on $D$ involved in the R2 move we are studying - see Figure 4. (More explicitly, $B(E)$ is obtained from $B(D')$ by deleting the vertex $v$ and the edges $\{u_i, v\}$.)

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{skein_move.png}
\caption{The skein move}
\end{figure}

Therefore, we know that

\begin{equation}
(34) \quad \text{the cohomology groups of } (U_0, d), (U_2, d') \text{ are isomorphic by Corollary 4.}
\end{equation}

Based on this, one could hope to apply an analogous principle to the differential (32) if we can somehow swap the roles of $Q$ and $U$. This, in fact, can be done by considering an R2 move on the projection $E$, and relating all the new variables appropriately. Let $E'$ be the projection obtained from $E$ by an R2 move on the arcs related to the arcs of $D$ on which we performed the original R2 move by a skein move. Then $B(E')$ is obtained from $B(E)$ by adding two new edges $(u_1, u_2)$. Denote these edges by $e, f$. Let, additionally, $h(e) = 0, h(f) = 1$, and extend the field $\tilde{F}$ further into a field $F'$ by attaching two new variables $z, t$ associated
with the edges $e$, $f$, respectively, each algebraically independent of $\tilde{F}$. Our conventions regarding calculating the elements $\alpha(T, T')$, $\beta(T, T')$ for $C(\mathcal{E}')$ are the same as in the case of $C(\mathcal{D}')$: specifically, a black circuit $c$ in $B(\mathcal{E}')$ which does not contain any of the edges $e$, $f$ is assigned the same element as in $C(\mathcal{E})$; if $c$ contains one or both of the edges $e$, $f$, and if the corresponding circuit in $C(\mathcal{D})$ (obtained by contracting the edges $e$, $f$) is assigned an element $b$, then $c$ is assigned the element $b$ multiplied by some of the elements $z$, $z^{-1}$, $t$, $t^{-1}$, depending on which of the edges $e$, $f$ $c$ contains, and orientation. Regarding white circuits $w$, again, take the product of all the vertex variables of $B(\mathcal{D})$ inside (resp. outside) of $w$ depending on whether $w$ is oriented counter-clockwise or clockwise, times, possibly, some of the elements $z$, $z^{-1}$, $t$, $t^{-1}$, depending on which of the edges $e$, $f$ the circuit $w$ crosses, and orientation.

Denote by $L'_1$, resp. $L'_2$ resp. $K'$ the sets of spanning trees of $B(\mathcal{E})$ which contain $e$ resp. $f$ resp. neither $e$ nor $f$. Now, analogously as above, filtering $C(\mathcal{E}')$ by the total height contribution of the edges $e$, $f$ only, and performing the same analysis as we did for $C(\mathcal{D}')$, we see that $C(\mathcal{E}')$ is isomorphic to a cochain complex of the form

\[
\begin{array}{c}
Q_0 \\
\xi \downarrow \\
Q_2
\end{array}
\]

\[
\begin{array}{c}
\downarrow j' \\
U \downarrow i' \\
\downarrow j'
\end{array}
\]

where $Q_0$ resp. $Q_2$ resp. $U$ are generated by $L'_1$ resp. $L'_2$ resp. $K'$. Once again, $\xi$ is an isomorphism of $F'$-modules. Specifically if $T$ is a spanning tree in $L'_1$ which, by deleting the edge $e$, creates a white circuit with associated element $v$, and if

\[
\kappa' : L'_1 \rightarrow L'_2
\]

is the canonical bijection (obtained by replacing the edge $e$ with the edge $f$), then

\[
(36) \quad \xi(T) = \left( \frac{1}{1 + ztv} + \frac{1}{1 + zt^{-1}} \right) \kappa'(T).
\]

If we denote the differentials on $Q_0$, $U$ by $d$ and the differential on $Q_2$ by $d'''$ (justified, again, by the idea that the last of the three differentials must be distinguished while the others can be understood from the context), (31) translates to

\[
d''' = \xi d \xi^{-1} + i' j' \xi^{-1},
\]
or, equivalently,

\[(37) \quad \xi d\xi^{-1} = d'' + i' j' \xi^{-1}. \]

Our strategy is to set up relations between the variables \(x, y, z, t\) so that the right hand side of (37) is equal to (32), and

\[(38) \quad d'' \text{ and } d_Q \text{ are related by a change of elements assigned to circuits in } F'. \]

If we are successful, then, analogously to (34), \(d''\) and \(d_Q\) in (37) are related by change of elements in \(F''\) assigned to circuits, and hence give isomorphic cohomology by Corollary 4. Hence, (32), which is related by conjugation by the \(F''\)-module isomorphism \(\xi\) to \(d\), has isomorphic cohomology to \(d''\), which, by (37), has the same cohomology as \(d_Q\). This is what we are trying to prove.

To obtain this scenario, we impose, in addition to (38), the equations

\[(39) \quad i = i', \]
\[(40) \quad \eta^{-1} j = j' \xi^{-1}. \]

Let first interpret (38). Consider two spanning trees \(T, T'\) in \(L\) where \(T'\) is obtained from \(T\) by omitting an edge of height 0 and adding an edge of height 1. Then the coefficient at \(d_Q(T)\) at \(T'\) is

\[
\frac{1}{1 + bxy} + \frac{1}{1 + w}
\]

where \(w\) is the appropriate white circuit, while the coefficient of \(d''\) between the corresponding trees in \(L_2\) is

\[
\frac{1}{1 + bt} + \frac{1}{1 + w}
\]

so we see that (38) is satisfied no matter what additional relations on \(x, y, z, t\) we impose.

Next, we impose the equality (39). When performing \(i\) on a spanning tree \(T\), we delete an edge, thus creating a white circuit. Denote the corresponding element of \(F''\) by \(v\). Then the corresponding coefficient in \(i\) is

\[
\frac{1}{1 + yvw} + \frac{1}{1 + bxy}
\]

and the corresponding coefficient in \(i'\) is

\[
\frac{1}{1 + ztvw} + \frac{1}{1 + bt}
\]

Thus, (39) will hold if we impose

\[(41) \quad zt = y, \ xy = t. \]
It is interesting that then also equality arises in (38), so no change of assigned elements is needed there.

Next, however, we must consider the equation (40), which translates to

\[
\left( \frac{1}{1 + bxy} + \frac{1}{1 + xy^{-1}} \right)^{-1} \left( \frac{1}{1 + bxy} + \frac{1}{1 + vx} \right) = \left( \frac{1}{1 + ztv} + \frac{1}{1 + bx} \right) \left( \frac{1}{1 + ztv} + \frac{1}{1 + zt^{-1}} \right)^{-1}.
\]

This is non-trivial. We first prove

**Lemma 7.** In a field of characteristic 2, we have the identity

\[
\left( \frac{1}{1 + a} + \frac{1}{1 + k} \right)^{-1} \left( \frac{1}{1 + a} + \frac{1}{1 + \ell} \right) = \left( \frac{1}{1 + \ell^{-1}k} + \frac{1}{1 + k} \right)^{-1} \left( \frac{1}{1 + \ell^{-1}k} + \frac{1}{1 + a^{-1}k} \right).
\]

**Proof:** Bringing the terms on the left hand side of (43) to common multiplier, we get

\[
\frac{(1 + k)(a + \ell)}{(1 + \ell)(a + k)}.
\]

Doing the same on the right hand side gives

\[
\frac{(1 + k)(\ell^{-1}k + a^{-1}k)}{(1 + a^{-1}k)(\ell^{-1}k + k)}.
\]

Now (44) is gotten from (45) by dividing both numerator and denominator by \(a^{-1}\ell^{-1}k\).

To apply (43) to (42), (using also (8)), we set

\[
k = x^{-1}y \quad \ell = v^{-1}x^{-1} \quad ztv = \ell^{-1}k \quad k = zt^{-1} \quad a = b^{-1}x^{-1}y^{-1} \quad a^{-1}k = bz.
\]

These equations are overdetermined but consistent, and have a solution

\[
x = t^2, \quad y = t^{-1}, \quad z = t^{-2}.
\]

Note that this solution also satisfies (41), and clearly is legal from the point of view of applying Corollary 4, so we are done.

To be completely precise, we have solved the “non-trivial” case of the equations (38), (39), (40): There is another “trivial” case when the
black cycle does not go through the vertex $v$ (resp. any of the edges $e$, $f$). In this case, the corresponding components of the differentials $d_Q$, $d$, $d''$, (32) coincide (in particular, the corresponding component of the $i\eta^{-1}j$ summand is 0 and the corresponding component of $d$ commutes with $\xi$, so the equations remain true in that case as well. This completes the proof of Proposition 5.

5. Reidemeister 3

Ironically, the methods of the last section do not apply directly to the Reidemeister 3 move because the projections before and after an R3 move play symmetrical roles: there is no obvious candidate inside the Baldwin-Ozsváth-Szabó complex of one projection for a part which would be isomorphic to some modification of the complex of the other. To get around this, we use the following idea suggested to us by John Baldwin: let us study braids on three strands labelled, from left to right, 1, 2, 3. Let $a$ resp. $b$ be the braid crossing strand 1 over strand 2 (resp. strand 2 over strand 3). Then the famous braid relation can be written in the form

\[(47) \quad aba^{-1} = b^{-1}ab,\]

which means that we have an unbraid

\[(48) \quad b^{-1}a^{-1}baba^{-1}\]

(see Figure 5)
Now consider a generic projection $D$ with three arcs 1, 2, 3 such that 1, 2 bound a component of $S^2 \setminus D$ labelled black, and 2, 3 bound a component of $S^2 \setminus D$ labelled white. Then by a BR move we shall mean an operation where we replace the arcs 1, 2, 3 by the unbraid (48).

Now suppose we can prove the following

**Proposition 8.** Suppose a generic projection $D'$ of an oriented link $L$ (satisfying (A)) is obtained from a generic projection $D$ of $L$ using the BR move. Then $H^i(C(D')) \cong H^i(C(D))$.

We claim this proves invariance under the Reidemeister 3 move: To see this, simply note that change from $b^{-1}ab$ to $aba^{-1}$ is an R3 move (see Figure 6). If we want to make this move inside a projection $D$, first change $b^{-1}ab$ to

$$b^{-1}abb^{-1}a^{-1}baba^{-1}$$

using the BR move, and then change (49) to $aba^{-1}$ using a sequence of R2 moves (which we can do by Proposition 5) - see Figure 7, thus implying invariance of Baldwin-Ozsváth-Szabó cohomology under the R3 move.
Thus, we will focus on proving Proposition 8. The method of proof is, in fact, more or less analogous to the proof of Proposition 5, but unfortunately, the situation is more complicated. Let $D'$ be a projection obtained from a projection $D$ by the BR move. By isomorphism of black and white, we may assume that $B(D)$ has a vertex $u$ in the component of $S^2 \setminus D$ shared by arcs 1 and 2. Thus, there is a white component shared by arcs 2 and 3, and there is another face colored black adjacent to 3, which corresponds to a vertex $w$ of $B(D)$.

Then we have

$$V(B(D')) = (V(B(D)) \setminus \{u\}) \amalg \{u_1, u_2, v_1, v_2\}.$$  

To describe $E(B(D'))$, we note that, once again, if we denote by $S$ the set of edges adjacent to $u$ in $B(D)$, then

$$S = S_1 \amalg S_2$$

such that to each edge $\{u, q\} \in S_i$ there corresponds, in $B(D')$, an edge $\phi(\{u, q\}) := \{u_i, q\}$. In addition, every edge $\{q, q'\}$ of $B(D)$ where $q, q' \neq u$ is also present in $B(D')$ (including the case when one or both of $q, q'$ are equal to $w$), and the following additional special edges are also in $B(D')$ (we choose orientations to make assignment of elements easier later):

$$e_1 = (u_1, v_1), \quad e_2 = (v_1, v_2), \quad e_3 = (v_1, w),$$
$$e_4 = (w, v_2), \quad e_5 = (v_2, u_2), \quad e_6 = (w, u_2).$$

There are no additional edges in $B(D)$ except the ones just specified. The heights of $e_1, ..., e_6$ are, in this order, 1, 0, 1, 1, 1, 0. (All this is determined by the braid $[18]$ - see Figure 8.)

Now if we attempted to use the Second Approach directly, i.e. filter $C(D')$ by the height contributions from the edges $e_i$, $i = 1, ..., 6$, the associated graded complex will be non-trivial in 5 different degrees (the associated graded piece in filtration degree 0 turns out to be trivial). This situation seems too complicated to analyze directly by a diagram analogous to (27).

This is where the First Approach becomes relevant: Let $F^p C(D')$ be the filtration by height contributions of all the edges except $e_1, ..., e_6$. More precisely, then, again, $F^p C(D')$ is spanned by all spanning trees $T$ of $B(D')$ such that

$$\sum_{e \in EB(D); \phi(e) \in ET} h(e) + \sum_{e \in EB(D); \phi(e) \notin ET} (1 - h(e)) \geq p.$$  

...
Call the sum of the height contributions of the edges $e_i$ the $e$-degree:

$$e(T) := \sum_{i : e_i \in EB(D)} h(e_i) + \sum_{i : e_i \notin EB(D)} (1 - h(e_i)).$$

For a subset $I \subset \{1, \ldots, 6\}$, denote by $K_I$ the set of all spanning trees $T$ of $B(D')$ such that $e_i \in ET$ if and only if $i \in I$.

Then the following table specifies the $e$-degrees of the sets $K_I$:

| $e$-degree | $I$                  |
|-----------|----------------------|
| 1         | $\{126\}, \{236\}, \{246\}, \{256\}$ |
| 2         | $\{12\}, \{23\}, \{24\}, \{25\}, \{1246\}, \{1256\}, \{1236\}$ |
| 3         | $\{125\}, \{123\}, \{124\}, \{235\}, \{245\}, \{156\}, \{356\}, \{146\}, \{346\}$ |
| 4         | $\{1235\}, \{1245\}, \{1346\}, \{1356\}, \{14\}, \{15\}, \{34\}, \{35\}$ |
| 5         | $\{134\}, \{135\}, \{145\}, \{345\}$ |
| 6         | $\{1346\}$ |

Then $d_0$ in the spectral sequence associated with the filtration $F$ is given by all contributions of the differential which raise $e$-degree by 2. This can actually be determined by cancellation, since sets $K_I, K_J$ are bijective when the equivalence relations $\sim_I, \sim_J$ on $\{u_1, u_2, w\}$ coincide where $\sim_I$ is the equivalence relation of being in the same connected component of the forest with edges $\{e_i \mid i \in I\}$.

Cancellation is additionally simplified by the fact that $d_0$ preserves the number of edges. To simplify notation, let us write $I$ instead of $K_I$. In even $e$-degrees, we then easily see that

$$\{12\}, \{23\}, \{24\}, \{25\}$$
of $e$-degree 2 cancel the sets
\{14\}, \{15\}, \{34\}, \{35\}
of $e$-degree 4. Additionally,
\{1246\}, \{1256\}, \{1236\}
in $e$-degree 2 cancel three of the four sets
\{1235\}, \{1245\}, \{1346\}, \{1356\}
of $e$-degree 4, and the remaining set of $e$-degree 4 cancels the set
\{1346\}
of $e$-degree 6. Thus, $d_0$ entirely cancels all trees of even $e$-degree. In odd $e$-degree, similarly, we can use the sets
\{126\}, \{236\}, \{246\}, \{256\}
of $e$-degree 1 to cancel the sets
\{156\}, \{356\}, \{146\}, \{346\}.
Additionally, one can perform the following cancellations between sets of $e$-degree 3 and sets of $e$-degree 5:
\begin{align*}
\{123\} & \rightarrow \{135\} \\
\{124\} & \rightarrow \{134\} \\
\{235\} & \rightarrow \{345\} \\
\{245\} & \rightarrow \{145\}.
\end{align*}
We see that the only set left is
\begin{equation}
\{125\}
\end{equation}
in $e$-degree 3, which is the result we wanted, since adding the edges $e_1, e_2, e_5$ gives a bijection
\[\kappa : K(D) \rightarrow K_{\{125\}}.\]
Additionally, since the $E_1$-term is entirely in $e$-degree 3, we have
\[2(p + q) = p + 3,\]
which means that the $E_1$-term is on a line of slope $-1/2$, the only possible differential is $d_2$, and the spectral sequence collapses to $E_3$. Thus, we have reduced our statement to showing that “a modification of the differential on $C(D)$” has isomorphic cohomology. Additionally, the modification can be computed using the method of (33).

To make this precise, consider the field $\bar{F}$ obtained from $F$ by attaching new variables $A, B, C, D, E, F$ corresponding to the edges $e_1, ..., e_6$ (in that orientation), and the same variables for the corresponding
white edges. On forming \( \alpha(T, T') \), \( \beta(T, T') \) in \( C(D') \), we adapt the same convention as in the case of the R2 move, i.e. for a black or white circuit \( c \) occurring in a graph (a tree plus or minus one edge) containing the edges \( e_i, i \in I \), and not the edges \( e_j, j \in \{1, ..., 6\} \setminus I \), take the appropriate element of \( F \) assigned to the circuit in the graph obtained by contracting each equivalence class of \( \sim_I \) to a point, and omitting the edges \( e_j, j \in \{1, ..., 6\} \setminus I \), and then multiply by those of the variables \( A, B, C, D, E, F \) which occur in \( c \) or which \( c \) crosses, or their inverses, according to orientation.

Now the general recipe for calculating \( d_2 \) is to take a \( d_0 \)-cocycle \( t \), and add to the component of \( dt \) in the same \( e \)-degree the following term: apply to \( t \) the component \( \nu \) of the differential which raises \( e \)-degree by 1. Then \( \nu(t) \in Imd_0 \), say, \( \nu(t) = d_0(s) \). Then we have

\[
(52) \quad d_2(t) = \nu(s).
\]

Starting with a \( T \in K_{\{1,2,5\}} \), applying \( \nu \) gives three summands

\[
(53) \quad \left( \frac{1}{1 + rCDF} + \frac{1}{1 + bACE} \right) T_{15},
\]

\[
(54) \quad \left( \frac{1}{1 + bAB} + \frac{1}{1 + qB^{-1}DF} \right) T_{1235},
\]

\[
(55) \quad \left( \frac{1}{1 + bACD^{-1}} + \frac{1}{1 + qB^{-1}DF} \right) T_{1245}.
\]

Here \( T_I \) indicates a tree in \( K_I \). \( b \in F \) is the element associated to the black circuit arising from \( T \) in the graph obtained by identifying all the edges \( e_i \) to a single point, \( q \in F \) is an element assigned according to the above convention to a white circuit corresponding to the connected component containing the vertex \( w \) and \( r \in F \) is the element assigned to a white cycle containing \( v_2 \) (according to the right hand rule).

In general, we do not know that \( T \) is a \( d_0 \)-cocycle, but this is true if the coefficients in (53) and (54) are equal, i.e. when

\[
(56) \quad BD = C.
\]

Let \( T_{1256} \) be obtained from \( T_{1235} \) by replacing \( e_3 \) with \( e_6 \) (or equivalently from \( T_{1245} \) by replacing \( e_4 \) with \( e_6 \)). Then

\[
(57) \quad d_0T_{1256} = \left( \frac{1}{1 + B^{-1}CEF^{-1}} + \frac{1}{1 + qB^{-1}DF} \right) T_{1235} + \left( \frac{1}{1 + DEF^{-1}} + \frac{1}{1 + qB^{-1}DF} \right) T_{1245}.
\]
The two coefficients in (57) are equal by (56).

Now there exists a tree $T_{125}$ and an element $c \in F$ associated to a black circuit under the above convention such that

$$
\nu(T_{1245}) = \left( \frac{1}{1 + bACEF^{-1}} + \frac{1}{1 + qB^{-1}DF} \right) T_{125} + \text{other terms}.
$$

Next, let $T_{12}$ resp. $T_{25}$ be obtained from $T_{15}$ by replacing $e_5$ with $e_2$ resp. $e_1$ with $e_2$. Then

$$
d_0(T_{12}) = \left( \frac{1}{1 + bACE} + \frac{1}{1 + CDE^{-1}} \right) T_{15} + \text{other terms},
$$

$$
d_0(T_{25}) = \left( \frac{1}{1 + bACE} + \frac{1}{1 + BCA^{-1}} \right) T_{15} + \text{other terms}.
$$

The other terms do occur and will make it necessary to add counter-terms of $e$-degree 2 in $K_J$ for where the cardinality of $J$ is 2. However, it is easy to see that $\{12\}$, $\{25\}$ are the only terms which, after applying $\nu$, can produce a non-zero multiple of a tree in $K_{\{125\}}$.

More specifically, there exists an additional element $p \in F$ assigned to a white tree $T'_{125}$ such that

$$
\nu(T_{12}) = \left( \frac{1}{1 + bACE} + \frac{1}{1 + rpEF} \right) T'_{125} + \text{other terms},
$$

$$
\nu(T_{25}) = \left( \frac{1}{1 + bACE} + \frac{1}{1 + rpAB^{-1}DF} \right) T'_{125} + \text{other terms}.
$$

Thus, if we impose the additional relation

$$
AD = BE,
$$

the contributions of the trees $T_{12}$ and $T_{25}$ to $d_2$ will be equal, and it suffices to consider one of them.
Thus, to summarize, assuming (56), (63), \(d_2 T\) is obtained by adding to the component of \(d T\) in \(K_{125}\) the term

\[
\sum \left( \frac{1}{1 + bCE} + \frac{1}{1 + rpEF} \right) \left( \frac{1}{1 + bACE} + \frac{1}{1 + CDE^{-1}} \right)^{-1}.
\]

\[
\left( \frac{1}{1 + rCDF} + \frac{1}{1 + bACE} \right) T'_{125} + \sum \left( \frac{1}{1 + bACEF^{-1} + \frac{1}{qB^{-1}DF}} \right) \left( \frac{1}{1 + B^{-1}CEF^{-1} + \frac{1}{qB^{-1}DF}} \right)^{-1}.
\]

\[
\left( \frac{1}{1 + bAB} + \frac{1}{1 + qB^{-1}DF} \right) T_{125}.
\]

Despite the complexity of the expression (in part due to change of variables), we notice that each of the summands of (64) is of the same form as the summand \(i\eta^{-1}j\) in Lemma 6 of the Second Approach to the R2 move.

Therefore, adding the first or second term individually is equivalent to conjugating by an appropriate element \(\xi_1, \xi_2\), assuming we have relations among weights which correspond to (46), i.e.

\[
x = y^{-2}, z = t^{-2}.
\]

These relations translate to

\[
C = E^{-2}, F = D^{-2},
\]

which is clearly consistent with (56), (63).

Now the coefficients of \(\xi_1, \xi_2\) depend only on \(b, r\) and the elements \(A, B, C, D, E, F\), and hence \(\xi_1, \xi_2\) commute. We see that after imposing (56), (63), (66), we have

\[
d_2 = \xi_1\xi_2\kappa^{-1} d_{C(D)}\kappa^{-1}\xi_1^{-1}.
\]

The algebraic relations (56), (63) and (66) can be jointly imposed in \(F'\) without altering the rank of cohomology of \(C(D')\), which concludes our proof of Proposition 8.

**Remark:** It is interesting to note that one can actually have \(T_{125} = T'_{125}\) in (64) when \(q\) resp. \(b\) of the coefficient of \(T_{125}\) is equal to \(p\) resp. \(c\) of the coefficient of \(T'_{125}\) (in that case, the elements labelled \(b\) in the coefficient of \(T_{125}\), \(T'_{125}\) of course won’t be equal).

**Proof of Theorem 1:** We have shown that, subject to the condition (A), the numbers (12) are invariant under the three Reidemeister
moves. However, there are still some minor details left to finish proving the Theorem: When $L$ is a knot, we are claiming that (12) is independent of orientation. This is simply because the number $n_-$ does not depend on orientation in this case.

When, on the other hand, $L$ is a link which has a projection violating the condition (A), we claim that (12) is 0 (and therefore also a link invariant even in this case). To this end, it suffices to prove that (12) is 0 for a projection $\mathcal{D}$ which will become disconnected by a single reversed R2 move.

In such a case, however, there always exist two vertices $v_1, v_2$ of $B(\mathcal{D})$ connected by two edges $e, f$ of heights 0, 1 respectively, where $v_1$ and $v_2$ are in different connected components of $B(\mathcal{D}) \setminus \{e, f\}$. In this case, however, $C(\mathcal{D})$ has the form of the chain complex $\mathcal{U}$ of Lemma 6 where $U'_2$ resp. $U_2$ is generated by spanning trees containing the edge $e$ (resp. $f$). Thus, $C(\mathcal{D})$ is acyclic. \hfill \Box

6. A FEW COMPUTATIONS

The purpose of this Section is to give a few examples of computations of Baldwin-Ozsváth-Szabó (briefly BOS) cohomology (which we will denote by $H_{BOS}$ here), to give a basic idea of its behavior. BOS appears to be a sparse invariant, close in flavor to twisted $\hat{HF}$. There is, (see [3]), a single-graded spectral sequence $E_r$ (i.e. graded like the Bockstein spectral sequence) whose $E_3$-term is $H_{BOS}$, converging to twisted $\hat{HF}$. This spectral sequence is sparse in the sense that the only possible non-zero differentials are of the form $d_{4k+2}$ ([3]). One may ask if this spectral sequence always collapses to $E_3$. This is unknown at present, but even if this is the case, BOS cohomology contains additional information due to the grading, which is intrinsically different from gradings on (twisted) $\hat{HF}$, which are given by spin$^c$-structures.

The simplest computation of BOS cohomology is given by the following

**Proposition 9.** For an alternating link $L$, we have

$$\text{rank}(H^i_{BOS}(L)) = \begin{cases} \text{Det}(L) & \text{when } i = \sigma(L)/2 \\ 0 & \text{else.} \end{cases}$$

**Proof:** Choose an alternating projection $\mathcal{D}(L)$ and a checkerboard coloring so that all black edges have height 1. Then clearly $C(\mathcal{D}(L))$ is
concentrated in a single dimension $i$, and the number of spanning trees is equal to $\text{Det}(L)$ by Kirkhoff’s theorem. To calculate $i$, let $b$ be the number of black vertices. Then

$$i = \frac{(b - 1 - n_{-})}{2}.$$ 

This number is equal to $\sigma(L)/2$ by [15] (see also [4] for more results on that subject).

In [10], Ozsváth and Szabó define a class $\mathcal{A}$ of quasi-alternating links which is the smallest class containing the unknot such that if a link $L$ has resolutions $L_0$ and $L_1$ at a particular crossing in some projection such that $L_0, L_1 \in \mathcal{A}$, and

\begin{equation}
\text{Det}(L_0) + \text{Det}(L_1) = \text{Det}(L),
\end{equation}

then $L \in \mathcal{A}$. For our purposes, it is useful to extend this notion further. Let a class $\tilde{\mathcal{A}}$ of weakly quasi-alternating links be defined the same way as $\mathcal{A}$, except that we replace, in the above definition, “the unknot” by “the unknot and all split links”. It is proved in [10] that all non-split alternating links are quasi-alternating. It follows immediately that all alternating links are weakly quasi-alternating.

We do not know if the statement of Proposition 9 extends to quasi-alternating links. However, we do have a weaker statement. Let us start with the following

Lemma 10. If, for a link $L$, all the values of $i$ for which $H^i_{\text{BOS}}(L) \neq 0$ differ by integral multiples of 2, then

$$\sum_i \text{rank}(H^i_{\text{BOS}}(L)) = \text{rank}(\widehat{HF}_{\text{tw}}(L)) = \text{Det}(L).$$

(Here $\widehat{HF}_{\text{tw}}$ denotes the twisted Heegaard-Floer homology with coefficients in the Novikov ring, as considered in [3].)

**Proof:** Because of sparsity of the Baldwin-Ozsváth-Szabó spectral sequence (the only differentials being $d_{4k+2}$), the spectral sequence collapses under the assumption. Further, in general, the Euler characteristic of $\mathcal{C}(D(L))$ is equal to the Euler characteristic of the based Khovanov complex at $q = -1$, which is $\text{Det}(L)$. Under the given assumption, the Euler characteristic is equal to the total rank. □
Proposition 11. For every weakly quasi-alternating link $L$, all the $i$'s for which $H^i_{BOS}(L) \neq 0$ differ by even integers, and

$$\sum_i \text{rank}(H^i_{BOS}(L)) = \text{Det}(L).$$

Proof: By Lemma [10], the second statement follows from the first. The first statement is proved by induction; for links $L$, $L_0$ and $L_1$ as above, by the induction hypothesis, all non-trivial $H^i_{BOS}$-dimensions of $L_j$ differ by even numbers for $j = 0, 1$, so if it is not true for $L$, then, by the skein long exact sequence, the BOS-cohomologies must contribute to $H^i_{BOS}(L)$ in degrees differing by an odd integer, but then a minus rather than a plus sign occurs in (67). \hfill \Box

In view of these observations, it is natural to ask if there exist a knot whose BOS-cohomology is not concentrated in a single degree. Such knots do indeed exist.

Proposition 12. The torus knot $T(3, 7)$ has non-trivial BOS cohomology in at least two degrees whose difference is not an even integer.

Proof: The branched double cover $\Sigma(T(3, 7))$ of $T(3, 7)$ is the Brieskorn homology sphere with multiplicities $2, 3, 7$. Since this is a homology 3-sphere, twistings are homologically trivial and hence $\hat{HF}_{tw}(\Sigma(T(3, 7)))$ and $\hat{HF}(\Sigma(T(3, 7)), \mathbb{Z}/2)$ have the same rank. The latter group is computed in [13], p. 209: one has

$$HF^+(\Sigma(T(3, 7))) = \mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U] \oplus \mathbb{Z},$$

and hence

$$\text{rank}(\hat{HF}(\Sigma(T(3, 7)), \mathbb{Z}/2) = 3$$

by the universal coefficient theorem. Hence, $H^0_{BOS}(\Sigma(T(3, 7)))$ cannot be concentrated in degrees which differ by even integers by Lemma [10]. \hfill \Box

We conclude this paper by computing explicitly the BOS cohomology of alternating knots with the smallest number of crossings, i.e. $8_{19}$, $8_{20}$ and $8_{21}$. Our main tool is the skein long exact sequence: Fix a link projection $\mathcal{D} = \mathcal{D}(L)$ of a link $L$, and choose a crossing $x$. Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be the 0-resolution and 1-resolution of $\mathcal{D}$ at $x$ respectively, and let $L_0$ and $L_1$ be the corresponding links.
Lemma 13. We have a long exact sequence

\begin{equation}
\cdots \to H^{i+n-}(D_1)/2-1/2(L_1) \to H^{i+n-}(D)/2(L) \to \\
H^{i+n-}(D_0)/2(L_0) \to H^{i+n-}(D_1)/2+1/2(L_1) \to \cdots
\end{equation}

Proof: The long exact sequence (68) is the spectral sequence concentrated in filtration degrees 0 and 1 associated with the decreasing filtration on \(C(D)\) where \(F^*C(D)\) is the set of linear combinations of all those spanning trees \(T\) where the edge \(e\) (whether it belongs to \(T\) or not) contributes \(\geq \epsilon\) to the height. \(\square\)

Comment: The reason for the \(n_-/2\) summands in the degrees in (68) is that the grading of the spectral (=exact) sequence is specified by height alone, and cannot include the number of negative crossings which we subtracted in the grading of BOS cohomology, and thus must add back on in the long exact sequence. (To see this, note that the \(L_0\) and \(L_1\) resolutions cannot both preserve the orientation of the link, and hence the numbers of negative crossings can change unpredictably.) Note also that as a result of this, the terms of the long exact sequence (68) are not (oriented or unoriented) link invariants.

The knot 8\textsubscript{21} (see Figure 9). This is the easiest of the examples. Let \(x = c_4\). The 1-resolution is a knot with \(\leq 7\) crossings, hence alternating. There are two spanning trees of height 1 and 15 spanning trees of height 3 in the black graph (independently of how we choose the coloring), so the BOS cohomology of \(D_1\) has rank 13 concentrated in height 3. However, 1 must be added to the height by Lemma 13, so this contributes to \(H_{BOS}^*(D)\) in height 4. On the other hand, the \(L_0\) is the (non-split) alternating link with two crossings. The \(H_{BOS}^*\) of the standard projection \(D_0\) of \(L_0\) is therefore rank 2 in height 1. However, to get from the projection \(D_0\) to \(D'_0\), we must perform some Reidemeister moves. Reidemeister 3 moves can be ignored, since they do not change the number of negative crossings. Undoing a Reidemeister 2 move always eliminates 1 negative crossing (and thus 1 must be added to the height to compensate). There are two types of Reidemeister 1 moves, which we call positive and negative. A positive R1 move does not change the number of negative crossings (and thus may be ignored), while undoing a negative R1 move eliminates 1 negative crossing (and hence 1 must be added to the height to compensate).

To get from \(D_0\) to \(D'_0\), not counting R3 moves, we undo one R2 move, two negative R1 moves and one positive R1 move. Thus, the height must be increased by 3 to adjust, and therefore \(H_{BOS}^*(D_0)\) also
The knot $8_{21}$ in Rolfsen’s table.

The knot $8_{20}$ (see Figure 10). Let $x = b_8$. The 0-resolution $L_0$ is an unknot, but to get the standard projection from $D_0$, we must (ignoring R3 moves) undo two R2 moves, 3 negative R1 moves (and one positive R1 move). Therefore, we must add 5 to the height, so $D_0$ contributes rank 1 in height 5 to the BOS-cohomology of $L = 8_{20}$.

The 1-resolution $M = L_1$ is a link, and we observe immediately that its 0-resolution $M_0$ at $b_3$ is a split link (and therefore has BOS-cohomology 0). The 1-resolution $M_1$ is an alternating link which has an alternating projection with BOS-cohomology of rank 8 in height 3. To get this projection from $D(M)_1$, we must do one and undo one R2 move and do one positive R1 move (ignoring R3 moves). Therefore, there is no change in height. However, since this is the 1-resolution of a 1-resolution, this adds 2 to the height of the contribution to $H^{*}_{BOS}(8_{20})$, so the contribution is of rank 8 in height 3.

We note that $8_{20}$ is therefore weakly quasi-alternating. It has 5 negative crossings, so $H^{*}_{BOS}$ has rank 9 concentrated in degree 0, which is equal to $\sigma(L)/2$.

The knot $8_{19}$ (see Figure 11). This knot (which is also isomorphic to the torus knot $T(3, 4)$ - but that is not helpful here) turns out to be the hardest. It is not quasi-alternating, since it is not Khovanov homology-thin (see [8]). Because of this, we use a slightly different method, using the skein exact sequence (68) only once, and then using
a direct calculation. Let \( x = a_1 \). One observes that the 0-resolution \( L_0 \) is an unknot which contributes to \( H^*_{B\Omega S}(D(L)) \) in dimension 4. It turns out that the generator of \( H_{B\Omega S}(D_0(L)) \) is represented by an element \( a \) of the module \( A \) spanned by all the Kauffman states of \( D_0 \) of height 4 which are 1-resolutions of \( a_7 \) (one has \( \text{rank}(A) = 4 \)).

Denote, on the other hand, by \( B, C_0, C_1 \), respectively, the submodules of \( C(D_1) \) the submodules of \( C(D_1) \) spanned by trees of height 3 resp. the trees of height 5 which are 0-resolutions of \( a_7 \) resp. the trees of height 5 which are 1-resolutions of \( a_7 \). (Keep in mind that 1 must be added to the height to calculate the height of their contribution to \( H_{B\Omega S}(D(L)) \).) One has

\[
\text{rank}(B) = 8 = \text{rank}(C_1), \quad \text{rank}(C_0) = 4.
\]

The BOS differential between \( A, B, C_0, C_1 \) has the form

\[
\begin{array}{ccc}
B & A \\
\Psi_0 & \downarrow \Psi_0 & \downarrow \Psi_1 \\
C_0 & \rightarrow & C_1.
\end{array}
\]

(69)

It turns out that \( \Psi_0 \) is in fact an isomorphism of rank 8 vector spaces, which implies that \( D_1 \) contributes to \( H_{B\Omega S}(D(L)) \) a vector space of dimension 4 in height 6, represented (isomorphically) by \( C_0 \).

However, dimensionally, there is the possibility of a connecting map of the form

\[
\delta : H_{B\Omega S}(D_0) \rightarrow H_{B\Omega S}(D_1)
\]

in the long exact sequence (68). In the present setting, this map is, in fact, calculated directly as

\[
(70) \quad \Psi_0 \Psi_0^{-1} \Psi_1(a).
\]

The computation can be done in the rational function field \( \mathbb{F}_2(a, b, c, x, y, z, t, u) \) where the variables correspond to all the faces of the projection in Figure 11 except the unbounded face and the face bounded by \( a_3, a_4, a_7 \).
By Corollary 4 these 8 variables can be further reduced to 5 variables. In either case, however, this computation was beyond the range of Mathematica on the computers readily available to the authors.

In the present situation, we only need to decide whether the map (70) is non-zero, which can be addressed directly using the following trick. Denote

\[ R = S^{-1}\mathbb{F}_2[a, b, c, x, y, z, t, u] \]

where \( S \) is the set of all the denominators of the coefficients of \( \Psi_0, \Psi_{01}, \Psi_1 \), and let, further,

\[ T = \det(\Psi_{01})^{-1}R. \]

Let

\[ \phi : \mathbb{F}[a, b, c, x, y, z, t, u] \to \mathbb{F}_2(x) \]

be the homomorphism of rings which sends all variables \( a, b, c, x, y, z, t, u \) to \( x \). By direct computation, one verifies

\[ \phi(S) \subseteq \mathbb{F}_2(x)^\times, \]

so \( \phi \) extends to a homomorphism of rings

\[ \overline{\phi} : R \to \mathbb{F}_2(x). \]

It further turns out that

\[ \overline{\phi}(\det \Psi_{01}) \in \mathbb{F}_2(x)^\times, \]

so \( \overline{\phi} \) extends to a homomorphism of rings

\[ \overline{\phi} : T \to \mathbb{F}_2(x). \]

Using Mathematica (which apparently has much more efficient algorithms on one-variable polynomials), we were able to compute

\[ (71) \quad \overline{\phi}(\Psi_0)\overline{\phi}(\Psi_{01}^{-1})\overline{\phi}(\Psi_1) \neq 0. \]

**Lemma 14.** (71) implies \( \Psi_0\Psi_{01}^{-1}\Psi_1(a) \neq 0. \)

**Proof:** Suppose

\[ (72) \quad \Psi_0\Psi_{01}^{-1}\Psi_1(a) = 0. \]

Then all the matrices \( \Psi_0, \Psi_{01}^{-1}, \Psi_1 \) have coefficients which lie in \( T \subseteq \mathbb{F}_2(a, b, c, x, y, z, t, u) \). Hence, (72) must be true in matrices over \( T \), and hence in matrices over \( \mathbb{F}_2(x) \) upon applying the homomorphism \( \overline{\phi} \), contradicting (71). \( \square \)
By the Lemma, then, we see that $H_{BOS}(8_{19})$ has rank 3 and is concentrated in height 6. There are no negative crossings and hence the BOS dimension is $6/2 = 3 = \sigma(8_{19})/2$.

We conclude that the knots $8_{19}$, $8_{20}$, $8_{21}$ all satisfy the conclusion of Proposition 9, even though they are not alternating.

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