1. Introduction

The Gauss hypergeometric differential equation
\[ E(a, b, c) : z(z-1)u'' + \{(a+b+1)z-c\}u' + aub = 0 \]
is regular on \( \mathbb{C} - \{0, 1\} \) for general parameters \( a, b \) and \( c \), and the solution space is spanned by Euler type integrals
\[ \int \gamma x^{a-c}(x-1)^{c-b-1}(x-z)^{-a}dx, \]
that are regarded period integrals for algebraic curves if \( a, b, c \in \mathbb{Q} \). Two independent solutions \( f_0(z), f_1(z) \) define a multi-valued analytic function \( s(z) = f_0(z)/f_1(z) \) (Schwarz map), and monodromy transformations for \( s(z) \) are given by fractional linear transformations.

If parameters satisfy the conditions
\[ |1-c| = \frac{1}{p}, \quad |c-a-b| = \frac{1}{q}, \quad |a-b| = \frac{1}{r}, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1, \]
with \( p, q, r \in \mathbb{N} \cup \{\infty\} \), the monodromy group is isomorphic to a triangle group
\[ \Delta(p, q, r) = \langle M_0, M_1, M_\infty | M_0^q = M_1^p = M_\infty^r = M_0M_1M_\infty = 1 \rangle \]
(the condition \( M_0^q = 1 \) is omitted if \( p = \infty \), and so on). In this case, the upper half plane is mapped to a triangle with vertices \( s(0), s(1) \) and \( s(\infty) \), angles \( \pi/p \), \( \pi/q \) and \( \pi/r \) respectively by \( s \), and so is the lower half plane. Copies of these two triangles give a tessellation of a disk \( \mathbb{D} \) by the monodromy action, and we have an isomorphism \( \mathbb{D}/\Delta(p, q, r) \cong \mathbb{C} - \{0, 1\} = \mathbb{P}^1 \). For example, \( E(1/2, 1/2, 1) \) is known as the Picard-Fuchs equation for the Legendre family of elliptic curves \( y^2 = x(x-1)(x-z) \) and the monodromy group \( \Delta(\infty, \infty, \infty) \) is projectively isomorphic to the congruence subgroup \( \Gamma(2) \) of level 2. Also a triangle group \( \Delta(n, n, n) \) with \( n \geq 4 \) is interesting, since its commutator subgroup \( N_n \) gives a uniformization of the Fermat curve \( F_n \) of degree \( n \). More precisely, the natural projection \( \mathbb{D}/N_n \to \mathbb{D}/\Delta(n, n, n) = \mathbb{P}^1 \) is an Abelian covering branched at 0, 1 and \( \infty \) with the covering group \( \Delta(n, n, n)/N_n \cong (\mathbb{Z}/n\mathbb{Z})^2 \) (see [CIW94]).

In [177], Takeuchi determined all arithmetic triangle groups. According to it, \( \Delta(n, n, n) \) is arithmetic (and hence the Fermat curve \( F_n \) is a Shimura curve) for \( n \in FT = \{4, 5, 6, 7, 8, 9, 12, 15\} \). These groups come from the Picard-Fuchs equation for algebraic curves \( X_t : y^n = x(x-1)(x-t) \) with \( m = n \) (resp. \( m = 2n \)) if \( n \in FT \) is odd (resp. even). Among them, \( n = 5 \) and \( 7 \) are special in the sense that a Jacobian \( J(X_t) \) is simple in general, and Picard-Fuchs equations describe variations of Hodge structure on the whole of \( H^1(X_t, \mathbb{Q}) \), rather than sub Hodge structures. These two families are treated by Shimura as examples of PEL families in [Sm64]. Also de Jong and Noot studied them as counter examples of Coleman's conjecture (which asserts the finiteness of the number of CM Jacobians for a fixed genus \( g \geq 4 \)) for \( g = 4, 6 \) in [JIN91] (see also [R99] and [MO13] for this direction).

For \( n = 5 \), we gave \( s^{-1} \) by theta constants in [KO3] as a byproduct of study of the moduli space of ordered five points on \( \mathbb{P}^2 \). In present paper, we compute the monodromy group, Riemann’s period matrices and the Riemann constant with an explicit symplectic basis for \( n = 7 \). Using them, we express the Schwarz inverse map \( s^{-1} \) by Riemann’s theta constants (Theorem 4.10). As a consequence, we give explicit moduli interpretations of the Klein quartic curve \( K_4 \) and the Fermat septic curve \( F_7 \) as modular...
2. Uniformization of Fermat Curves

2.1. Hypergeometric integral. We compute monodromy groups and invariant Hermitian forms for hypergeometric integrals

\[ u(t) = \int \Omega_\alpha(x), \quad \Omega_\alpha(x) = \{x(x-1)(x-t)\}^{-\alpha}dx \]

according to [Y97, Chap. IV], for \( \alpha = \frac{k}{2k+1} \) and \( \frac{2k-1}{2k+1} \) with \( k \geq 2 \). They satisfy differential equations

\[ E\left(\frac{k}{2k+1}, \frac{k-1}{2k+1}\right) \quad \text{and} \quad E\left(\frac{2k-1}{4k}, \frac{2k-3}{4k}, \frac{2k-1}{2k}\right) \]

with monodromy groups \( \Delta(n, n, n) \), \( n = 2k + 1 \) and \( 2k \) respectively. Let us consider decompositions

\[ P^1(\mathbb{C}) = \mathbb{H}_+ \cup P^1(\mathbb{R}) \cup \mathbb{H}_-, \quad P^1(\mathbb{R}) = I_0 \cup I_1 \cup I_2 \cup I_3, \]

where \( \mathbb{H}_+ \) and \( \mathbb{H}_- \) are the upper and lower half planes respectively, and \( I_k \) are (oriented) real intervals.

(As the initial position of \( t \), we assume that \( 0 < t < 1 \).)

Modifying boundaries \( \partial \mathbb{H}_+ \) and \( \partial \mathbb{H}_- \) to avoid 0, t, 1 and \( \infty \) as follows, we fix a branch of \( \Omega_\alpha(x) \) on a simply connected domain \( \mathbb{H}_- \) and define integrals \( u_k(t) = \int_{I_k} \Omega_\alpha(x) \) by this branch.

By the Cauchy integral theorem, they satisfy

\[ 0 = \int_{\partial \mathbb{H}_-} \Omega_\alpha(x) = u_0(t) + u_1(t) + u_2(t) + u_3(t), \]

\[ 0 = \int_{\partial \mathbb{H}_+} \Omega_\alpha(x) = u_0(t) + cu_1(t) + c^2u_2(t) + c^3u_3(t), \quad c = \exp(2\pi i \alpha) \]

since \( \Omega_\alpha(x) \) is multiplied by \( \exp(2\pi i \alpha) \) if \( x \) travels around 0, t or 1 in clockwise direction. Hence we have

\[ u_2(t) = -\frac{1}{1 + c} \{ u_1(t) + (1 + c + c^2)u_3(t) \}. \]
2.2. Monodromy. Now let $\delta_0$ and $\delta_1$ be paths to make a half turn around 0 and 1 respectively in counter clockwise direction, starting from the initial point of $t$.

Corresponding analytic continuations are represented by connection matrices $h_0$ and $h_1$:

$$
\delta_0 : \begin{bmatrix} u_1(t) \\ u_3(t) \end{bmatrix} \rightarrow \begin{bmatrix} -c^{-1}u_1(t') \\ u_3(t') \end{bmatrix} = h_0 \begin{bmatrix} u_1(t') \\ u_3(t') \end{bmatrix}, \quad h_0 = \begin{bmatrix} -c^{-1} & 0 \\ 0 & 1 \end{bmatrix},
$$

$$
\delta_1 : \begin{bmatrix} u_1(t) \\ u_3(t) \end{bmatrix} \rightarrow \begin{bmatrix} u_1(t') + u_2(t') \\ c^{-1}u_2(t') + u_3(t') \end{bmatrix} = h_1 \begin{bmatrix} u_1(t') \\ u_3(t') \end{bmatrix}, \quad h_1 = \begin{bmatrix} \frac{c}{c^2 + c} & -\frac{c^2 + c + 1}{c^2 + c} \\ \frac{c^2 + c + 1}{c^2 + c} & \frac{c}{c^2 + c} \end{bmatrix},
$$

where $u_1(t'), \ldots, u_4(t')$ are integrals over oriented intervals $I'_1, \ldots, I'_4$ defined for new configurations $-\infty < t' < 0 < 1 < \infty$ and $-\infty < 0 < 1 < t' < \infty$. The monodromy group $\text{Mon}$ is generated by

$$
g_0 = h_0^2 = \begin{bmatrix} c^{-2} & 0 \\ 0 & 1 \end{bmatrix}, \quad g_1 = h_1^2 = \begin{bmatrix} \frac{c^2 + 1}{c^2 + c} & -\frac{1 - c^2}{c^2 + c} \\ \frac{1 - c^2}{c^2 + c} & \frac{c^2 + 1}{c^2 + c} \end{bmatrix}.
$$

2.3. Hermitian form and period domain. It is known that there exists a unique monodromy-invariant Hermitian form up to constant (see e.g. [B07] and [Y97]). In fact, we can easily check that $h_0$ and $h_1$ belong to a unitary group

$$
U_H = \{ g \in \text{GL}_2(\mathbb{C}) \mid g^* H g = H \}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 + c + c^{-1} \end{bmatrix},
$$

and hence $\text{Mon} \subset U_H$. The value of $1 + c + c^{-1}$ is negative for $c = \exp(2\pi i\alpha)$ with $\alpha = \frac{k}{2k+1}$ and $\frac{2k-1}{4k}$ ($k \geq 2$), and $H$ is indefinite. Therefore two domains

$$
\mathbb{D}^+_H = \{ u \in \mathbb{C}^2 \mid \pm \bar{u}H u < 0 \}/\mathbb{C}^* \subset \mathbb{P}^1(\mathbb{C}).
$$

are disks, and $U_H$ acts on each domain. Now the image of the Schwarz map

$$
\mathcal{s} : \mathbb{C} - \{ 0, 1 \} \rightarrow \mathbb{P}^1(\mathbb{C}), \quad t \mapsto [u_1(t) : u_3(t)]
$$

is contained in either $\mathbb{D}^+_H$ or $\mathbb{D}^-_H$, which is tessellated by Schwarz triangles. Since we have

$$
\mathcal{s}(0) = \lim_{t \rightarrow 0} [u_1(u) : u_3(t)] = [0 : u_3(0)] \in \mathbb{D}^-_H,
$$

we see that $\mathbb{D}^+_H/\text{Mon} \cong \mathbb{P}^1(\mathbb{C})$ and $\mathbb{D}^-_H/[\text{Mon}, \text{Mon}] \cong \mathcal{F}_n$, where $\mathcal{F}_n$ is the Fermat curve of degree $n$ with $n = 2k + 1$ (resp. $2k$) if $\alpha = \frac{k}{2k+1}$ (resp. $\frac{2k-1}{4k}$).

2.4. Remark. (1) Putting $\zeta_d = \exp(2\pi i/d)$, we have

$$
1 + c + c^{-1} = \begin{cases} 1 + (\zeta_{2k+1})^k + (\zeta_{4k+1})^{k+1} & (n = 2k + 1) \\ 1 + (\zeta_{4k})^{2k-1} + (\zeta_{4k})^{2k+1} & (n = 2k) \end{cases}
$$

(2) In the case of $n = 2k + 1$, we have

$$
g_0 = \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix}, \quad g_1 = \frac{1}{1 + \zeta^k} \begin{bmatrix} \zeta^k + \zeta^{k+1} - \zeta^{2k} & 1 + \zeta \\ \zeta - \zeta^{k+1} & \zeta \end{bmatrix}
$$

where $\zeta = \zeta_{2k+1}$. Since $1/(1 + \zeta^k) = -(\zeta + \zeta^2 + \cdots + \zeta^k)$ and det $g_1 = \zeta$, the monodromy group $\text{Mon}$ is a subgroup of $U_H \cap \text{GL}_2(\mathbb{Z}[\zeta])$.

(3) In the case of $n = 2k$, we have

$$
g_0 = \begin{bmatrix} \zeta^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad g_1 = \frac{1}{1 + \zeta^{2k-1}} \begin{bmatrix} \zeta^{2k+1} + \zeta^{2k-1} - \zeta^{4k-2} & 1 + \zeta^2 \\ \zeta^2 - \zeta^{2k+1} & \zeta \end{bmatrix}
$$

where $\zeta = \zeta_{4k}$. Note that the cyclotomic polynomial $\Phi_{4k}(x)$ satisfies $\Phi_{4k}(1) = 1$ if $4k \neq 2^m$. In this case, $1 - \zeta$ is a unit in $\mathbb{Z}[\zeta]$, and so is $1/(1 + \zeta^{2k-1}) = \zeta/(\zeta - 1)$. Hence $\text{Mon}$ is a subgroup of $U_H \cap \text{GL}_2(\mathbb{Z}[\zeta])$ if $4k \neq 2^m$. 
2.5. **Fermat curve as a Shimura variety.** A triangle group \( \Delta(n, n, n) \) is arithmetic for
\[ n \in FT = \{ 4, 5, 6, 7, 8, 9, 12, 15 \}, \]
and the Fermat curve \( F_n \) is a Shimura curve. Let us see corresponding families of hypergeometric curves
\[ X_t : y^n = x(x - 1)(x - t) \]
for these case. By the Riemann-Hurwitz formula, the genus of \( X_t \) is \( g = m - 1 \) if \( 3 \nmid m \), and \( g = m - 2 \) if \( 3 \mid m \). Let \( \rho \) be the covering automorphism \( (x, y) \to (x, \zeta_m y) \) where \( \zeta_m = \exp(2\pi i / m) \). By this action, we can decompose \( \text{H}^1(X_t, \mathbb{Q}) \) into irreducible representations of \( \rho \), and \( \text{H}^1(X_t, \mathbb{C}) \) into eigenspaces of \( \rho \). Let \( V(\lambda) \) be the \( \lambda \)-eigenspace of \( \rho \). If \( m \) is not prime, the covering \( X_t \to \mathbb{P}^1 \) has intermediate curves \( Y_t \), and the pullback of \( \text{H}^1(Y_t, \mathbb{C}) \) consists of \( \zeta_m^k \) such that \( (m, k) \neq 1 \). Conversely, such \( V(\zeta_m^k) \) descends to a quotient curve. From explicit basis of \( \text{H}^{1,0}(X_t) \), we see that the Prym part
\[ \text{H}_{\text{Prym}}^1(X_t, \mathbb{Q}) = \left[ \oplus_{(k, m) = 1} V(\zeta_m^k) \right] \cap \text{H}^1(X_t, \mathbb{Q}) \]
has a Hodge structure of type
\[ \text{H}_{\text{Prym}}^1(X_t, \mathbb{C}) = V(\lambda_1) \oplus \cdots \oplus V(\lambda_{2d}) \]
where \( 2d = [\mathbb{Q}(\zeta_m) : \mathbb{Q}] \), \( \lambda_1, \ldots, \lambda_{2d} \) are primitive roots of unity \( \zeta_m, \ldots, \zeta_m^{2d-1} \) such that \( \lambda_t = \lambda_{2d+1-i} \) and \( \dim V(\lambda_i) = 2 \) for \( i = 1, \ldots, 2d \). Therefore the Hodge structure on \( \text{H}_{\text{Prym}}^1(X_t, \mathbb{Q}) \) with the action of \( \rho \) is determined by a decomposition \( V(\lambda) = V(\lambda_1) \oplus \cdots \oplus V(\lambda_{2d}) \) (the decomposition of \( V(\lambda_t) \) is automatically determined as the complex conjugate of \( V(\lambda_d) \), and vice versa), that is, determined by periods of \( \Omega_\alpha(x) \in V(\lambda_d)^{1,0} \).

| \( \Delta(n, n, n) \) | \( m \) | \( g \) | \([\mathbb{Q}(\zeta_m) : \mathbb{Q}] \) | \( x^n dx/y^n \) with the following \((a, b)\) give a basis of \( \text{H}^{1,0}(X_t)_{\text{Prym}} \) |
|----------------|------|------|----------------|--------------------------------------------------|
| (4, 4, 4) | 8 | 7 | 4 | \((0, 3), (0, 5), (0, 7), (1, 7)\) |
| (5, 5, 5) | 5 | 4 | 4 | \((0, 2), (0, 3), (0, 4), (1, 4)\) |
| (6, 6, 6) | 12 | 10 | 4 | \((0, 5), (0, 7), (0, 11), (1, 11)\) |
| (7, 7, 7) | 7 | 6 | 6 | \((0, 3), (0, 4), (0, 5), (1, 5), (0, 6), (1, 6)\) |
| (8, 8, 8) | 16 | 15 | 8 | \((0, 7), (0, 9), (0, 11), (1, 11), (0, 13), (1, 13), (0, 15), (1, 15)\) |
| (9, 9, 9) | 9 | 7 | 6 | \((0, 4), (0, 5), (0, 7), (1, 7), (0, 8), (1, 8)\) |
| (12, 12, 12) | 24 | 22 | 8 | \((0, 11), (0, 13), (0, 17), (1, 17), (0, 19), (1, 19), (0, 23), (1, 23)\) |
| (15, 15, 15) | 15 | 13 | 8 | \((0, 7), (0, 8), (0, 11), (1, 11), (0, 13), (1, 13), (0, 14), (1, 14)\) |

2.6. In the cases \( n = 5 \) and \( 7 \), the monodromy group has a nice representaiton. Put
\[ \Gamma = U_H \cap \text{GL}_2(\mathbb{Z}[n]), \]
\[ \Gamma(m) = \{ g \in \Gamma \mid g \equiv 1 \mod m \}. \]
The arithmetic quotient \( \mathbb{D}_H^+ / \Gamma \) is the moduli space of Jacobians of curves
\[ y^n = x^3 + ax + b \]
as a PEL-family (see [Sm64]). Therefore we have the following diagram
\[ \mathbb{D}_H^+ / \text{Mon} \longrightarrow \mathbb{P}^1 = \{ \text{ordered distinct 3 + 1 points (0, 1, t, \infty)} \} \]
\[ \downarrow \]
\[ \mathbb{D}_H^+ / \Gamma \longrightarrow \mathbb{P}^1 / S_3 = \{ \text{unordered distinct 3 points in } \mathbb{C} \} / \sim \]
where horizontal arrow are isomorphisms, and \( \sim \) is the equivalence relation by affine transformations. From this fact, we see that \( \Gamma / \text{Mon} \) is isomorphic to \( S_3 \) up to the center.

2.7. **Remark.** For \( n = 5 \), the Hermitian form \( H \) is same with one given in [Sm64]:
\[ H = \begin{bmatrix} 1 & 0 \\ 0 & 1 + \zeta_7^3 + \zeta_7^6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & (1 - \sqrt{5})/2 \end{bmatrix}. \]
For \( n = 7 \), the Hermitian form given in [Sm64] is
\[ S = \begin{bmatrix} 1 & 0 \\ -\frac{\zeta_{24}^6 + \zeta_{24}^9}{\zeta_{24}^6 - \zeta_{24}^9} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-\zeta_7^7 + \zeta_7^6) \end{bmatrix} = t A H A, \]
\[ A = \begin{bmatrix} 1 & 0 \\ 0 & \zeta_7 + \zeta_7^6 \end{bmatrix} \in \text{GL}_2(\mathbb{Z}[\zeta_7]). \]
2.8. Proposition ([YY84] for \( n = 5 \)). Let us denote the image of \( G \subset \text{GL}_2(\mathbb{Z}[\zeta_n]) \) in \( \text{PGL}_2(\mathbb{Z}[\zeta_n]) \) by \( \overline{G} \). For \( n = 5 \) and \( 7 \),
\begin{enumerate}
\item the projective modular group \( \overline{\Gamma} \) is projectively generated by \( h_0 \) and \( h_1 \),
\item we have \( \text{Mon} = \overline{\Gamma}(1 - \zeta_n) \), \( [\text{Mon, Mon}] = \overline{\Gamma}((1 - \zeta_n)^2) \) as automorphisms of \( \mathbb{D}^+_H \).
\end{enumerate}

\textbf{Proof.} We show these facts only for \( n = 7 \), but the case \( n = 5 \) is shown by the same way (also see [YY84] and [K03] for \( n = 5 \)). The quotient group \( \Gamma / \Gamma(1 - \zeta_7) \) is isomorphic to a subgroup of the finite orthogonal group
\[ \text{O}(Q, \mathbb{F}_7) = \{ g \in \text{GL}_2(\mathbb{F}_7) \mid t g Q g = Q \}, \quad \mathbb{F}_7 = \mathbb{Z}[\zeta_7]/(1 - \zeta_7), \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}. \]

The group \( \text{O}(Q, \mathbb{F}_7) \) is isomorphic to \( S_3 \times \{ \pm 1 \} \), since elements of \( \text{O}(Q, \mathbb{F}_7)/\{ \pm 1 \} \) are
\begin{align*}
\text{order 2:} & \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}, \quad \text{order 3:} \quad \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 5 \\ 3 & 3 \end{bmatrix}.
\end{align*}

Since we have
\[ h_0 \equiv \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad h_1 \equiv \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} \mod 1 - \zeta_7, \]
the group \( \Gamma / \Gamma(1 - \zeta_7) \) is generated by \( h_0, h_1 \) and \( \pm 1 \), and isomorphic to \( S_3 \times \{ \pm 1 \} \). Therefore \( \text{Mon} \) coincides with \( \Gamma(1 - \zeta_7) \) since we have \( \text{Mon} \subset \Gamma(1 - \zeta_7) \) and \( \overline{\Gamma} / \text{Mon} = S_3 \). Note that \( \text{Mon} \) is generated by \( h_0^3 \) and \( h_1^3 \), and hence \( \overline{\Gamma} \) is generated by \( h_0 \) and \( h_1 \). A homomorphism
\[ \nu : \Gamma(1 - \zeta_7) \rightarrow \text{M}_2(\mathbb{F}_7), \quad \nu(g) = \frac{1}{1 - \zeta_7}(g - 1) \mod 1 - \zeta_7 \]
has the kernel \( \Gamma((1 - \zeta_7)^2) \), and the image is generated by
\[ \nu(g_0) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \nu(g_1) = \begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix}. \]
Therefore we have \( \Gamma(1 - \zeta_7)/\Gamma((1 - \zeta_7)^2) \cong (\mathbb{Z}/7\mathbb{Z})^2 \). Since we have
\[ [\Gamma(1 - \zeta_7), \Gamma(1 - \zeta_7)] \subset \Gamma((1 - \zeta_7)^2), \quad \text{Mon}/[\text{Mon, Mon}] \cong (\mathbb{Z}/7\mathbb{Z})^2, \]
we conclude that \( [\text{Mon, Mon}] = \Gamma((1 - \zeta_n)^2) \).

\[ \square \]

3. Heptagonal Curves

3.1. From now, we concentrate in the case \( n = 7 \), that is, a 1-dimensional family of algebraic curves
\[ X_7 : y^7 = x(x - 1)(x - t). \]

We denote \( \zeta_7 = \exp(2\pi i/7) \) simply by \( \zeta \). As a Riemann surface, \( X_7 \) is obtained by gluing seven sheets \( \Sigma_1, \cdots, \Sigma_7 \), each of which is a copy of \( \mathbb{P}^1 \) with cuts (see Figure 1) and satisfying \( \rho(\Sigma_i) = \Sigma_{i+1} \) where indices are considered modulo 7. Let \( t_1(x_1, x_2) \) be an oriented real interval from \( x_1 \) to \( x_2 \) on \( \Sigma_i \). We define 1-cycles
\[ \gamma_1 = i_1(0, t) + i_2(t, 0) = (1 - \rho)i_1(0, t), \quad \gamma_2 = i_1(t, 1) + i_2(1, t) = (1 - \rho)i_1(t, 1), \quad \gamma_3 = i_1(1, \infty) + i_2(\infty, 1) = (1 - \rho)i_1(1, \infty). \]

For computation of intersection numbers, we use deformations of \( \gamma_1 \) and \( \gamma_3 \) as in Figure 1.

Let \( \text{Int}_k \) be the intersection matrix \( [\rho^j(\gamma_k) \cdot \rho^i(\gamma_k)]_{0 \leq i,j \leq 5} \).

\[ \text{Int}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad \text{Int}_3 = \begin{bmatrix} 0 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \end{bmatrix} \]

and \( \det \text{Int}_1 = \det \text{Int}_3 = 1 \). Since \( \rho^k(\gamma_1) \cdot \rho^k(\gamma_3) = 0 \), the intersection matrix of twelve 1-cycles \( \gamma_1, \rho(\gamma_1), \cdots, \rho^5(\gamma_1) \) and \( \gamma_3, \rho(\gamma_3), \cdots, \rho^5(\gamma_3) \) is unimodular, and they form a basis of \( H_1(X_7, \mathbb{Z}) \). Hence

\[ \square \]
The associated period matrix is \[ \Pi = \begin{bmatrix} f_{\gamma_1}R & f_{\gamma_2}R & f_{\gamma_3}R \\ f_{\gamma_1}R^3 & f_{\gamma_2}R^5 & f_{\gamma_3}(I + R^2) \\ f_{\gamma_2}R^3 & f_{\gamma_3}(I + R + R^2) & f_{\gamma_3}(I + R - R^4 + R^5) \end{bmatrix}, \]

where \( \gamma = (\omega_1, \ldots, \omega_6) \) and \( R = \text{diag}(\zeta^4, \zeta^3, \zeta^2, \zeta, 1) \). The normalized period matrix \( \tau = \Pi A \Pi_B^{-1} \) belongs to the Siegel upper half space \( \mathbb{H}_6 \), consisting of symmetric matrices of degree 6 whose imaginary part is positive definite. The symplectic group

\[ Sp_{12}(\mathbb{Z}) = \{ \gamma \in GL_{12}(\mathbb{Z}) \mid \gamma J \gamma = J \}, \quad J = \begin{bmatrix} 0 & I_6 \\ -I_6 & 0 \end{bmatrix}, \]

acts on \( \mathbb{H}_6 \) by \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \tau = (at + b)(ct + d)^{-1} \), and \( \mathcal{A}_6 = \mathbb{H}_6/Sp_{12}(\mathbb{Z}) \) is the moduli space of principally polarized abelian varieties (p.p.a.v.) of dimension 6.
3.4. Remark. For a suitable choice of a branch of $\Omega_a(x)$ in the previous section, we have

$$\int_{\gamma_k} \omega_1 = (1 - \zeta^k)u_k(t) \quad (k = 1, 2, 3).$$

Since we use $u_k$ for projective coordinates mainly, hereafter we denote $\int_{\gamma_k} \omega_1$ by $u_k$ for simplicity.

3.5. Symplectic representation. Let $M \in Sp_{12}(\mathbb{Z})$ be the symplectic representation of $\rho$ with respect to the above basis:

$$(\rho(A_1), \ldots, \rho(A_6), \rho(B_1), \ldots, \rho(B_6)) = (A_1, \ldots, A_6, B_1, \ldots, B_6)^t M.$$ Explicit form of $M$ is given in Appendix. By definition, we have $M \begin{bmatrix} \Pi_A \\ \Pi_B \end{bmatrix} = \begin{bmatrix} \Pi_A \\ \Pi_B \end{bmatrix} R$. Therefore $\Pi_A \Pi_B^{-1}$ belongs to a domain $\mathbb{H}_0^M = \{ \tau \in \mathbb{H} \mid M \cdot \tau = \tau \}$, which parametrizes p.p.a.v of dimension 6 with an automorphism $M$ (see section 5 in [BG92]). We know that this domain is 1-dimensional, and hence isomorphic to $\mathbb{D}_H^\uparrow$ ([BL92], Chap. 9 and [Sm64]). The centralizer of $M$ in $Sp_{12}(\mathbb{Z})$

$$Sp_{12}^M(\mathbb{Z}) = \{ g \in Sp_{12}(\mathbb{Z}) \mid gM = Mg \}.$$ acts on the domain $\mathbb{H}_0^M$.

3.6. Proposition. There exist a group isomorphisms $\phi : \Gamma \rightarrow Sp_{12}^M(\mathbb{Z})$ and an analytic isomorphism $\Phi : \mathbb{D}_H^\uparrow \rightarrow \mathbb{H}_0^M$ such that $\Phi(gu) = \phi(g)\Phi(u)$, that give the following commutative diagram.

$$\begin{array}{ccc}
\mathbb{D}_H^\uparrow & \longrightarrow & \mathbb{H}_0^M \\
\downarrow & & \downarrow \\
\mathbb{D}_H^\uparrow/\Gamma & \longrightarrow & \mathbb{H}_0^M / Sp_{12}^M(\mathbb{Z})
\end{array}$$

Proof. Now we have

$$\Pi_{A1} = \begin{bmatrix} \int_{A_1} \omega_1, \ldots, \int_{A_6} \omega_1 \end{bmatrix} = \begin{bmatrix} \zeta^4u_1, \zeta^5u_1, \zeta^6u_1, (1 + \zeta)u_3, (\zeta^2 - \zeta^4 + \zeta^6)u_3, (1 + \zeta + \zeta^4)u_3 \end{bmatrix},$$

$$\Pi_{B1} = \begin{bmatrix} \int_{B_1} \omega_1, \ldots, \int_{B_6} \omega_1 \end{bmatrix} = \begin{bmatrix} u_1, (1 + \zeta)u_1, (1 + \zeta + \zeta^2)u_1, \zeta^6u_3, \zeta^5u_3, (1 + \zeta^4 - \zeta^2 - \zeta^6)u_3 \end{bmatrix}.$$ This correspondence $\begin{bmatrix} u_1 \\ u_3 \end{bmatrix} \mapsto \begin{bmatrix} \Pi_{A1} \\ \Pi_{B1} \end{bmatrix}$ define a linear map $\Phi_1 : \mathbb{C}^2 \rightarrow \mathbb{C}^{12}$. Since coefficients of $u_1$ (or $u_3$) in $\Pi_{A1}$ and $\Pi_{B1}$ give a $\mathbb{Z}$-basis of $\mathbb{Z}[\zeta]$, there exists a homomorphism $\phi : GL_2(\mathbb{Z}[\zeta]) \rightarrow GL_{12}(\mathbb{Z})$ such that $\Phi_1(gu) = \phi(g)\Phi_1(u)$. Especially, we have $\phi(\zeta^4I_2) = M$ and the image of $\phi$ is the centralizer of $M$. We can easily check that the condition

$$|u_1|^2 + (1 + \zeta^3 + \zeta^4)|u_3|^2 < 0$$

for $\mathbb{D}_H^\uparrow$ is equivalent to Riemann’s relation ([M83])

$$\text{Im} \left( \sum_{i=1}^6 \int_{B_i} \omega_1 \int_{A_i} \omega_1 \right) > 0,$$

and hence $\phi(\Gamma) = Sp_{12}^M(\mathbb{Z})$. We give the map $\Phi$, which is compatible with $\Phi_1$, explicitly in Appendix.

3.7. Remark. Let us define a homomorphism

$$\lambda : H_4(X_1, \mathbb{Z}) = \langle \gamma_1, \gamma_3 \rangle_{\mathbb{Z}[\rho]} \rightarrow \mathbb{Z}[\zeta]^2, \quad F_3(\rho)\gamma_1 + F_4(\rho)\gamma_3 \rightarrow (F_1(\zeta^4), F_3(\zeta^4)).$$

By explicit computation, we see that the intersection form (which gives the polarization) on $H_4(X_1, \mathbb{Z})$ is given by

$$E(x, y) = \frac{1}{7} \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}((\zeta^3 - \zeta^4)\overline{x}H^{-1}y).$$
4. SCHWARZ INVERSE AND THETA FUNCTION

4.1. Abel-Jacobi map. For the normalized holomorphic 1-forms

\[ \tilde{\xi} = (\xi_1, \ldots, \xi_6) = (\omega_1, \ldots, \omega_6)B^{-1} \]

with respect to \( A_i \) and \( B_i \) in the previous section, period integrals satisfy

\[ \tau = \left[ \int_{A_i} \tilde{\xi} \right]_{1 \leq i \leq 6} \in \mathbb{H}_M^6, \quad \left[ \int_{B_i} \tilde{\xi} \right]_{1 \leq i \leq 6} = I_6. \]

Let \( \text{Div}(X_i) \) be the group of divisors on \( X_i \), and \( J(X_i) \) be the Jacobian variety \( \mathbb{C}^6/\mathbb{Z}^6\tau + \mathbb{Z}^6 \). The Abel-Jacobi map with the base point \( P_\infty \) is

\[ \text{Div}(X_i) \to J(X_i), \quad \sum m_i Q_i \mapsto \sum m_i \int_{P_\infty}^Q \tilde{\xi} \mod \mathbb{Z}^6\tau + \mathbb{Z}^6. \]

We denote this homomorphism by \( \mathfrak{A} \) and a lift of \( \mathfrak{A}(D) \) by \( \mathfrak{A}(D) \) (Hence \( \mathfrak{A} : \text{Div}(X_i) \to \mathbb{C}^6 \) is a multi-valued map). As is well known, \( \mathfrak{A} \) factors through

\[ \text{Div}(X_i) \to \text{Pic}(X_i) = \text{Div}(X_i)/\{\text{principal divisors}\}. \]

Since the base point is fixed by \( \rho \), the map \( \mathfrak{A} \) is \( \rho \)-equivariant. Therefore the image of a \( \rho \)-invariant divisor belongs to the fixed points of \( \rho \), that is, the \((1 - \rho)\)-torsion subgroup

\[ J(X_i)_{1-\rho} = \{ z \in J(X_i) \mid (1 - \rho)z = 0 \}. \]

4.2. Lemma. The \((1 - \rho)\)-torsion subgroup is

\[ J(X_i)_{1-\rho} = \{ \mathfrak{A}(mP_0 + nP_1) \mid m, n \in \mathbb{Z} \} \cong (\mathbb{Z}/7\mathbb{Z})^2 \]

More explicitly, we have

\[ \mathfrak{A}(mP_0 + nP_1) \equiv a_{m,n}\tau + b_{m,n} \mod \mathbb{Z}^6\tau + \mathbb{Z}^6 \]

with

\[ a_{m,n} = \frac{1}{7}(m, 2m, 3m, 2m + 3n, 2m + 3n, 0) \in \mathbb{Z}^6, \]

\[ b_{m,n} = \frac{1}{7}(-m, -m, -m, 3m + n, 5m + 4n, m + 5n) \in \mathbb{Z}^6. \]

**Proof.** It is obvious that \( \text{Ker}(1 - \rho) \cong (\mathbb{Z}[\zeta]/(1 - \zeta))^2 \cong (\mathbb{Z}/7\mathbb{Z})^2 \). Recall that

\[ \gamma_1 = (1 - \rho)i_1(0, 1), \quad \gamma_2 = (1 - \rho)i_1(1, \infty), \quad \gamma_3 = (1 - \rho)i_1(t, 1). \]

Computing intersection numbers, we see that

\[ \gamma_2 = A_1 + A_2 + A_3 + B_4 + B_5 = \rho(\gamma_1) + \rho^3(\gamma_1) + \rho^5(\gamma_1) + \rho^5(\gamma_1) + \rho^3(\gamma_3). \]

Therefore we have

\[ i_1(0, t) = \frac{1}{7}(6 + 5\rho + 4\rho^2 + 3\rho^3 + 2\rho^4 + \rho^5)\gamma_1 = \frac{1}{7}(5A_1 + 3A_2 + A_3 + 2B_1 + 2B_2 + 2B_3), \]

\[ i_1(1, \infty) = \frac{1}{7}(6 + 5\rho + 4\rho^2 + 3\rho^3 + 2\rho^4 + \rho^5)\gamma_3 = \frac{1}{7}(-3A_4 + 4A_5 + 7A_6 - B_4 + 3B_5 + 2B_6), \]

\[ i_1(t, 1) = \frac{1}{7}(6 + 5\rho + 4\rho^2 + 3\rho^3 + 2\rho^4 + \rho^5)\gamma_2 \]

\[ = \frac{1}{7}(A_1 + 2A_2 + 3A_3 - B_1 - B_2 - B_3) + \frac{1}{7}(A_4 + A_5 - 7A_6 + 5B_4 - B_5 + 4B_6), \]

namely,

\[ \int_0^t \tilde{\xi} \equiv \frac{1}{7}(5, 3, 1, 0, 0, 0)\tau + \frac{1}{7}(2, 2, 2, 0, 0, 0), \]

\[ \int_1^\infty \tilde{\xi} \equiv \frac{1}{7}(0, 0, 0, 4, 4, 0)\tau + \frac{1}{7}(0, 0, 0, 6, 3, 2), \]

\[ \int_1^t \tilde{\xi} \equiv \frac{1}{7}(1, 2, 3, 1, 1, 0)\tau + \frac{1}{7}(6, 6, 5, 6, 4) \] \mod \mathbb{Z}^6 + \tau \mathbb{Z}^6.

As combinations of these integrals, we obtain explicit values of \( \mathfrak{A}(P_0) \) and \( \mathfrak{A}(P_1) \). \( \square \)
4.3. **Theta function and Riemann constant.** Let us consider Riemann’s theta function

\[
\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}^g} \exp[\pi i n^t \zeta + 2 \pi i n^t z], \quad (z, \tau) \in \mathbb{C}^g \times \mathbb{H}_g.
\]

The Abel-Jacobi map \( \overline{\mathfrak{M}} \) induces a birational morphism from \( \text{Sym}^6 X_t \) to \( J(X_t) \), and \( W^5_\emptyset = \overline{\mathfrak{M}}(\text{Sym}^5 X_t) \) is a translation of the theta divisor

\[
\Theta = \{ z \in J(X_t) \mid \vartheta(z) = 0 \}.
\]

More precisely, there exist a constant vector \( \kappa \in \mathbb{C}^g \) such that \( \vartheta(e, \tau) = 0 \) if and only if

\[
e \equiv \kappa - \mathfrak{A}(Q_1 + \cdots + Q_5) \mod \mathbb{Z}^g \tau + \mathbb{Z}^g
\]

for some \( Q_1, \ldots, Q_5 \in X_t \). The constant \( \kappa \) (or its image \( \kappa \) in \( J(X_t) \)) is called the Riemann constant. It is the image of a half canonical class by \( \mathfrak{A} \) ([M53], Chap. II, Appendix to §3), and depends only on a symplectic basis \( A_1, B_1 \) and the base point of \( \mathfrak{A} \). Since \( \text{div}(\omega_5) = 10P_\infty \), the image of the canonical class by \( \mathfrak{M} \) is 0 and \( \kappa \) must be a half period. Hence we have \( \kappa = a\tau + b \) for some meets \( a, b \in \mathbb{Z}^g \). By the same argument as the proof of Lemma 5.4 in [K03], the corresponding theta characteristic \( (a, b) \) is invariant under the action of \( M \) on \( \mathbb{Q}^{12}/\mathbb{Z}^{12} \):

\[
M \cdot (a, b) = (a, b)M^{-1} + \frac{1}{2}(\text{diag}(C^t D), \text{diag}(A^t B)), \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]

By explicit computation, we have

4.4. **Lemma.** The \( M \)-invariant theta characteristics are \( (a_m, a_n, b_m, n, b_0) \) with

\[
a_0 = \frac{1}{2}(1, 0, 0, 0, 1), \quad b_0 = \frac{1}{2}(1, 1, 0, 0, 0).
\]

Especially, we have \( \kappa \equiv a_0 \tau + b_0 \). Since \( \vartheta(-e) = \vartheta(e) \) and \( \kappa \) is a half period, we have

\[
\kappa - W^5_{\emptyset} = \Theta = -\Theta = \kappa + W^5_{\emptyset},
\]

that is \( W^5_{\emptyset} = -W^5_{\emptyset} \).

4.5. **Lemma.** Let us consider \( J(X)_{1-\rho} \cap W^5_{\emptyset} \). By definition, we have \( \overline{\mathfrak{M}}(mP_0 + nP_1) \in W^5_{\emptyset} = -W^5_{\emptyset} \) for \( 0 \leq m, n \leq 6 \) such that \( m + n \leq 5 \) or \( (7 - m) + (7 - n) \leq 5 \). The rest of \( J(X)_{1-\rho} \) are \( \overline{\mathfrak{M}}(mP_0 + nP_1) \) with the following \( (m, n) \):

(1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5),
(4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (6, 1), (6, 2).

Moreover we have the following reduction:

\[
(6P_0 + P_1) = (2P_1 + P_1 + 4P_\infty) + \text{div}(\frac{x}{y}), \quad (3P_0 + 3P_1) = (4P_1 + 2P_\infty) + \text{div}(\frac{x(x - 1)}{y^4}),
\]

\[
(5P_0 + P_1) = (3P_1 + 2P_1 + P_\infty) + \text{div}(\frac{x}{y^4}), \quad (4P_0 + 3P_1) = (P_0 + 4P_1 + 2P_\infty) + \text{div}(\frac{x(x - 1)}{y^4}),
\]

that is, \( \overline{\mathfrak{M}}(6P_0 + P_1), \overline{\mathfrak{M}}(3P_0 + 3P_1), \overline{\mathfrak{M}}(5P_0 + P_1), \overline{\mathfrak{M}}(4P_0 + 3P_1) \in W^5_{\emptyset} \).

By the equality \( W^5_{\emptyset} = -W^5_{\emptyset} \) and symmetry for \( P_0, P_1 \), we see that \( \overline{\mathfrak{M}}(mP_0 + nP_1) \in W^5_{\emptyset} \) if

\((m, n) \neq (2, 4), (2, 5), (3, 5), (4, 2), (5, 2), (5, 3)\).

The converse is also true:
4.6. Lemma. We have \( \overline{\lambda}(mP_0 + nP_1) \not\in W^5_{\mathbb{Q}} \) for
\[(m, n) = (2, 4), (2, 5), (3, 5), (4, 2), (5, 2), (5, 3).\]

Proof. To prove this, note that
\[(5P_0 + 2P_1) = (4P_1 + 2P_1 + P_\infty) + \text{div}(\frac{x}{y^2}), \quad \therefore \overline{\lambda}(5P_0 + 2P_1) = \overline{\lambda}(4P_1 + 2P_1)\]
and
\[\overline{\lambda}(3P_i + 5P_j) = -\overline{\lambda}(4P_1 + 2P_j), \quad i, j \in \{0, 1\}.\]
By symmetry for \( P_0, P_1 \) and \( P_i \), it suffices to prove that \( \overline{\lambda}(4P_0 + 2P_1) \not\in W^5_{\mathbb{Q}} \).

Applying the Riemann–Roch formula for \( 4P_0 + 2P_1 \), we have
\[\ell(4P_0 + 2P_1) = \ell(K - 4P_0 - 2P_1) + 1\]
where \( \ell(D) = \dim H^0(X_t, \mathcal{O}(D)) \) and \( K \) is the canonical class. From the vanishing order of \( \omega_i \):
\[
\begin{array}{cccccccc}
\text{at } P_0 & \omega_5 & \omega_3 & \omega_2 & \omega_1 & \omega_0 & \omega_3 & \omega_2 & \omega_1 \\
0 & 1 & 2 & 3 & 7 & 8 & 3 & 2 & 1
\end{array}
\begin{array}{cccccccc}
\text{at } P_1 & \omega_5 & \omega_3 & \omega_2 & \omega_1 & \omega_0 & \omega_3 & \omega_2 & \omega_1 \\
0 & 0 & 1 & 1 & 1 & 2 & 3 & 1 & 3
\end{array}
\]
we see that there does not exist a holomorphic 1-form \( \omega \) such that \( \text{div}(\omega) - 4P_0 - 2P_1 \) is positive. Therefore we have \( \ell(4P_0 + 2P_1) = 1 \) and \( H^0(X_t, \mathcal{O}(4P_0 + 2P_1)) \) contains only constant functions. This implies \( \overline{\lambda}(4P_0 + 2P_1) \not\in W^5_{\mathbb{Q}} \). \( \square \)

4.7. Jacobi inversion. We apply Theorem 4 in [SI1] Chap. 4, §11, for rational functions
\[f : X_t \rightarrow \mathbb{P}^1, (x, y) \mapsto x, \quad g : X_t \rightarrow \mathbb{P}^1, (x, y) \mapsto 1 - x\]
on \( X_t \). Then we have
\[
\begin{align*}
f(Q_1) \times \cdots \times f(Q_6) &= \frac{1}{E'} \prod_{k=1}^{7} \frac{\partial(k - A(Q_1 + \cdots + Q_6) + \int_{i_{k}(\infty, 0)} \xi, \tau)}{\partial(k - A(Q_1 + \cdots + Q_6), \tau)}, \\
g(Q_1) \times \cdots \times g(Q_6) &= \frac{1}{E'} \prod_{k=1}^{7} \frac{\partial(k - A(Q_1 + \cdots + Q_6) + \int_{i_{k}(\infty, 1)} \xi, \tau)}{\partial(k - A(Q_1 + \cdots + Q_6), \tau)},
\end{align*}
\]
where constants \( E \) and \( E' \) are independent of \( Q_1, \ldots, Q_6 \), integrals \( \int_{i_{k}(\infty, 0)} \xi, \tau = 0, \)
and \( \mathfrak{A}(Q_1 + \cdots + Q_6) \in \mathbb{C}^6 \) takes the same value in the numerator and the denominator.

Substituting \( 4P_1 + 2P_2 \) and \( 2P_1 + 4P_2 \) for \( Q_1 + \cdots + Q_6 \) in [1], and taking the ratio of resultant equations, we have an expression of \( t^2 \) by theta values:
\[t^2 = f(P_0)^2 f(P_1)^4 / f(P_1)^4 f(P_2)^2 \]
\[= \prod_{k=1}^{7} \frac{\partial(k - A(2P_1 + 4P_2) + \int_{i_{k}(\infty, 0)} \xi, \tau)}{\partial(k - A(2P_1 + 4P_2), \tau)} / \prod_{k=1}^{7} \frac{\partial(k - A(4P_1 + 2P_2) + \int_{i_{k}(\infty, 0)} \xi, \tau)}{\partial(k - A(4P_1 + 2P_2), \tau)},
\]
\[= \prod_{k=1}^{7} \frac{\partial(k + a_{2,4} \tau + b_{2,4} + \int_{i_{k}(\infty, 0)} \xi, \tau)}{\partial(k + a_{2,4} \tau + b_{2,4}, \tau)} / \prod_{k=1}^{7} \frac{\partial(k + a_{5,2} \tau + b_{5,2} + \int_{i_{k}(\infty, 0)} \xi, \tau)}{\partial(k + a_{5,2} \tau + b_{5,2}, \tau)}.
\]
Similarly, substituting \( 4P_1 + 2P_2 \) and \( 2P_1 + 4P_2 \) for \( Q_1, \ldots, Q_6 \) in [2], we have
\[1 - t^2 = g(P_0)^2 g(P_1)^4 / g(P_1)^4 g(P_2)^2 \]
\[= \prod_{k=1}^{7} \frac{\partial(k - A(2P_0 + 4P_1) + \int_{i_{k}(\infty, 1)} \xi, \tau)}{\partial(k - A(2P_0 + 4P_1), \tau)} / \prod_{k=1}^{7} \frac{\partial(k - A(4P_0 + 2P_1) + \int_{i_{k}(\infty, 1)} \xi, \tau)}{\partial(k - A(4P_0 + 2P_1), \tau)},
\]
\[= \prod_{k=1}^{7} \frac{\partial(k + a_{2,4} \tau + b_{2,4} + \int_{i_{k}(\infty, 1)} \xi, \tau)}{\partial(k + a_{2,4} \tau + b_{2,4}, \tau)} / \prod_{k=1}^{7} \frac{\partial(k + a_{5,2} \tau + b_{5,2} + \int_{i_{k}(\infty, 1)} \xi, \tau)}{\partial(k + a_{5,2} \tau + b_{5,2}, \tau)}.
\]
4.8. theta functions with characteristics. The above expressions are simplified by introducing theta functions with characteristics \(a, b \in \mathbb{Q}^2\):
\[
\vartheta_{a,b}(z, \tau) = \exp[\pi i a^t \tau^t a + 2\pi i a^t (z + b)] \vartheta(z + a \tau + b, \tau) = \sum_{n \in \mathbb{Z}^2} \exp[\pi i (n + a)^t \tau^t (n + a) + 2\pi i (n + a)^t (z + b)].
\]

We denote a theta constant \(\vartheta_{a,\mu}(0, \tau)\) by \(\vartheta_{a,\mu}(\tau)\). Let \(\vartheta_{m,n}(z, \tau)\) be \(\vartheta_{a,b}(z, \tau)\) with characteristics \(a = a_{m,n} + a_0, \ b = b_{m,n} + b_0\) in Lemma 4.3. With this notation, theta expressions (3) and (4) are
\[
I^2 = \prod_{k=1}^{7} \frac{\vartheta_{[2,5]}(\tau) \vartheta_{[3,2]}(I_{x_k(\infty,0)} \xi, \tau)}{\vartheta_{[2,4]}(\tau) \vartheta_{[3,2]}(I_{x_k(\infty,0)} \xi, \tau)}, \quad (1 - t)^2 = \prod_{k=1}^{7} \frac{\vartheta_{[2,4]}(\tau) \vartheta_{[2,4]}(I_{x_k(\infty,1)} \xi, \tau)}{\vartheta_{[2,4]}(\tau) \vartheta_{[5,2]}(I_{x_k(\infty,1)} \xi, \tau)}.
\]

Putting
\[
\int_{I_{x_k(\infty,x)}} \tilde{x} = \begin{cases} a_{1,0} \tau + b_{1,0} & (x = 0) \\ a_{0,1} \tau + b_{0,1} & (x = 1) \end{cases} \quad (1 \leq k \leq 6), \quad \int_{I_{\tau(\infty,x)}} \tilde{x} = \begin{cases} -6(a_{1,0} \tau + b_{1,0}) & (x = 0) \\ -6(a_{1,0} \tau + b_{1,0}) & (x = 1) \end{cases}
\]

and using formulas
\[
\vartheta_{a,b}(a \tau + b', \tau) = \exp[-\pi i a^t a' - 2\pi i a^t b] \vartheta_{a+a', b+b'}(0, \tau), \quad a', b' \in \mathbb{Q}^2,
\]

\[
\vartheta_{a+a', b+b'}(z, \tau) = \exp(2\pi \sqrt{-1} a'b') \vartheta_{a,b}(z, \tau), \quad a', b' \in \mathbb{Z}^2,
\]

we see that
\[
\prod_{k=1}^{7} \frac{\vartheta_{[2,5]}(\tau) \vartheta_{[3,2]}(I_{x_k(\infty,0)} \xi, \tau)}{\vartheta_{[2,4]}(\tau) \vartheta_{[3,2]}(I_{x_k(\infty,0)} \xi, \tau)} = \zeta^3 \frac{\vartheta_{[5,2]}(\tau)^7}{\vartheta_{[2,4]}(\tau)^7}, \quad \prod_{k=1}^{7} \frac{\vartheta_{[2,4]}(I_{x_k(\infty,1)} \xi, \tau)}{\vartheta_{[5,2]}(I_{x_k(\infty,1)} \xi, \tau)} = \frac{\vartheta_{[2,5]}(\tau)^7}{\vartheta_{[3,2]}(\tau)^7}.
\]

Since \(\vartheta_{a,b}(-z, \tau) = \vartheta_{a,b}(z, \tau)\), we have
\[
I^2 = \zeta^3 \frac{\vartheta_{[5,2]}(\tau)^{14}}{\vartheta_{[2,4]}(\tau)^{14}}, \quad (1 - t)^2 = \frac{\vartheta_{[2,5]}(\tau)^{14}}{\vartheta_{[2,4]}(\tau)^{14}},
\]

namely, there exist constants \(\varepsilon_1 = \pm 1\) and \(\varepsilon_2 = \pm 1\) such that
\[
(5) \quad t = \zeta^3 \frac{\vartheta_{[5,2]}(\tau)^{14}}{\vartheta_{[2,4]}(\tau)^{14}}, \quad 1 - t = \frac{\vartheta_{[2,5]}(\tau)^{14}}{\vartheta_{[2,4]}(\tau)^{14}}.
\]

4.9. Theta transformation. For \(g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}_{2g}(\mathbb{Z})\), theta constants \(\vartheta_{a,b}(\tau)\) satisfy the transformation formula
\[
\vartheta_{g(a,b)}(g \tau) = \mu(g) \exp[2\pi i \lambda_a(b)] \det(C \tau + D) \vartheta_{a,b}(\tau)
\]

where
\[
\lambda_a(b) = -\frac{1}{2} \left( a^t D B a - 2a^t B C b + b^t C A b \right) + \frac{1}{2} \left( a^t D - i b^t C \right) \text{diag}(A') B
\]

and \(\mu(g)\) is a certain 8-th root of 1 depending only on \(g\). Therefore, as coordinates of \(\mathbb{P}^2(\mathbb{C})\), we have
\[
(6) \quad \vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}(g \cdot \tau) = [\mathbf{e}[\lambda_{2,4}(g)] \vartheta_{[2,4]} : \mathbf{e}[\lambda_{2,5}(g)] \vartheta_{[2,5]} : \mathbf{e}[\lambda_{3,5}(g)] \vartheta_{[3,5]}](\tau)
\]

where \(\mathbf{e}[-] = \exp[2\pi i -] \).

By explicit form of \(\sigma_0 = \phi(h_0)\) and \(\sigma_1 = \phi(h_1)\) in Appendix, we see that
\[
\lambda_{2,4}(\sigma_0) = 53/56, \quad \lambda_{2,5}(\sigma_0) = 53/56, \quad \lambda_{3,5}(\sigma_0) = 7/8,
\]

\[
\lambda_{2,4}(\sigma_1) = 25/56, \quad \lambda_{2,5}(\sigma_1) = 19/392, \quad \lambda_{3,5}(\sigma_1) = 79/392,
\]

and
\[
\vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}(\sigma_0 \cdot \tau) = [-\vartheta_{[2,5]} : \vartheta_{[3,5]}(\sigma_0)], \quad \vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}(\sigma_1 \cdot \tau) = [\vartheta_{[2,4]} : \vartheta_{[3,5]}(\sigma_1)].
\]

Applying these for (6), we obtain
\[
(7) \quad \vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}(\tau) = [-\vartheta_{[2,5]} : \vartheta_{[3,5]}(\tau)], \quad \vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}(\sigma_0 \cdot \tau) = [\vartheta_{[2,4]} : \vartheta_{[3,5]}(\sigma_0)].
\]
4.10. **Theorem.** (1) The inverse of the Schwarz map

\[ s : \mathbb{C} \setminus \{0, 1\} \rightarrow \mathbb{D}^+_H, \quad t \mapsto u = [u_1(t) : u_3(t)] \]

is given by \( \Gamma(1 - \zeta) \)-invariant function \( t(u) = \varepsilon_1 \zeta^2 \vartheta_{[2, 5]}(\tau)^7 \vartheta_{[3, 5]}(\tau)^7 \), where \( \Phi : \mathbb{D}^+_H \rightarrow \mathbb{H}^M_0 \) is the modular embedding given in Appendix. In other word, \( \Phi(u) \in \mathbb{H}^M_0 \) is the period matrix of an algebraic curve \( y^7 = x(x - 1)(x - t(u)) \).

(2) The analytic map

\[ Th : \mathbb{D}^+_H \rightarrow \mathbb{P}^2(\mathbb{C}), \quad u \mapsto [e^{5/49} \vartheta_{[2, 4]} \vartheta_{[2, 5]} : \vartheta_{[2, 5]} \vartheta_{[3, 5]} : -\vartheta_{[2, 4]} \vartheta_{[3, 5]}](\Phi(u)) \]

induces an isomorphism \( \mathbb{D}^+_H / \Gamma(1 - \zeta)^2 \) and the Fermat septic curve

\[ \mathcal{F}_7 : X^7 + Y^7 + Z^7 = 0, \quad [X : Y : Z] \in \mathbb{P}^2(\mathbb{C}). \]

**Proof.** From (4), we have

\[ 1 = \varepsilon_1 \zeta^2 \vartheta_{[2, 5]}(\tau)^7 \vartheta_{[3, 5]}(\tau)^7 + \varepsilon_2 \vartheta_{[2, 4]}(\tau)^7. \]

Since this equation must be invariant under actions of \( \sigma_0 = \phi(h_0) \) and \( \sigma_1 = \phi(h_1) \) in (7) (otherwise, the image of \( Th \) is not irreducible), we see that \( \varepsilon_1 = \varepsilon_2 = 1 \) and

\[ t = \zeta^2 \vartheta_{[2, 5]}(\Phi(u))^7 \vartheta_{[3, 5]}(\Phi(u))^7. \]

Let us recall that \( \Gamma(1 - \zeta) \) is projectively generated by \( h_0^2 \) and \( h_1^2 \), and \( \Gamma((1 - \zeta)^2) \) is projectively isomorphic to the commutator subgroup of \( \Gamma(1 - \zeta) \). From (4), we see that

\[ \begin{aligned}
[\vartheta_{[2, 4]} : \vartheta_{[2, 5]} : \vartheta_{[3, 5]}][\sigma_0^2 \cdot \tau] &= [\zeta \vartheta_{[2, 4]} : \zeta \vartheta_{[2, 5]} : \vartheta_{[3, 5]}](\tau), \\
[\vartheta_{[2, 4]} : \vartheta_{[2, 5]} : \vartheta_{[3, 5]}][\sigma_1^2 \cdot \tau] &= [\vartheta_{[2, 4]} : \vartheta_{[2, 5]} : \zeta \vartheta_{[3, 5]}](\tau).
\end{aligned} \]

Therefore the commutator subgroup of \( \Gamma(1 - \zeta) \) acts trivially on

\[ [\vartheta_{[2, 4]}(\Phi(u)) : \vartheta_{[2, 5]}(\Phi(u)) : \vartheta_{[3, 5]}(\Phi(u))] \in \mathbb{P}^2, \]

and the map \( Th \) gives a \((\mathbb{Z}/2\mathbb{Z})^2\)-equivariant isomorphism of \( \mathbb{D}^+_H / \Gamma((1 - \zeta)^2) \) and the Fermat septic curve. \( \square \)

4.11. **Klein quartic.** It is known that the Klein quartic curve

\[ \mathcal{K}_4 : X^3 Y + Y^3 Z + Z^3 X = 0, \quad [X : Y : Z] \in \mathbb{P}^2(\mathbb{C}). \]

is the quotient of \( \mathcal{F}_7 \) by an automorphism

\[ \alpha : \mathcal{F}_7 \rightarrow \mathcal{F}_7, \quad [X : Y : Z] \mapsto [\zeta X : \zeta^3 Y : Z] \]

which is induced by \( g_0 g_1^3 \in \Gamma(1 - \zeta) \) via the map \( Th \). The quotient map is given by

\[ \mathcal{F}_7 \rightarrow \mathcal{K}_4, \quad [X : Y : Z] \mapsto [XY^3 : YZ^3 : ZX^3]. \]

The Klein quartic \( \mathcal{K}_4 \) is isomorphic to the elliptic modular curve \( \mathcal{X}(7) \) of level 7, and also to a Shimura curve parametrizing a family of QM Abelian 6-folds (see [E99]). The following Corollary gives a new moduli interpretation of \( \mathcal{K}_4 \).

4.12. **Corollary.** The Klein quartic curve \( \mathcal{K}_4 \) is isomorphic to \( \mathbb{D}^+_H / \Gamma_{Klein} \) where

\[ \Gamma_{Klein} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1 - \zeta) \mid a \equiv 1 \mod (1 - \zeta)^2 \right\}. \]

**Proof.** Let us recall the homomorphism

\[ \nu : \Gamma(1 - \zeta) \rightarrow M_2(\mathbb{F}_7), \quad \nu(g) = \frac{1}{1 - \zeta}(g - 1) \mod 1 - \zeta \]

in the proof of Proposition 2.8. The kernel of \( \nu \) is \( \Gamma((1 - \zeta)^2) \) and the image is generated by

\[ \nu(g_0) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \nu(g_1) = \begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix}. \]
Since we have $\nu(g_0 g_4^2) = \left[ -a + 5b \quad b \quad 5b \right]$, the group $\Gamma_{Klein}$ is generated by $\Gamma((1 - \zeta)^2)$ and $g_0 g_4^2$. Namely we have $\mathbb{D}_H^+ / \Gamma_{Klein} = \mathcal{F}_7/\langle \alpha \rangle$.

### 4.13. PEL-family.
Let $(A, E, \rho, \lambda)$ be a 4-tuple.

1. $A$ is a 6-dimensional complex Abelian variety $V/A$, where $V$ is isomorphic to the tangent space $T_0A$ and $A$ is isomorphic to $H_1(A, \mathbb{Z})$.
2. $E : \Lambda \times \Lambda \to \mathbb{Z}$ is a principal polarization.
3. $\rho$ is an automorphism of order 7 preserving $E$, and the induced action on $T_0A$ has eigenvalues $\zeta, \zeta^2, \zeta^3, \zeta^4$.
4. $\lambda : \Lambda \to \mathbb{Z}[\zeta]^2$ is an isomorphism such that $\lambda(\rho(x)) = \zeta^4 \lambda(x), \quad E(x, y) = \frac{1}{7} \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}((\zeta^3 - \zeta^4)^7 \lambda(x) H^{-1} \lambda(y))$

(See Remark 3.7). Note that $\lambda$ induces an isomorphism of the torsion subgroup $A_{tor}$ and $(\mathbb{Q}(\zeta)/\mathbb{Z}[\zeta])^2$.

An isomorphism $f : (A, E, \rho, \lambda) \to (A', E', \rho', \lambda')$ is defined as an isomorphism of Abelian varieties $f : A \to A'$ such that $f^*E' = E$, $f \circ \rho = \rho' \circ f$ and $\lambda = \lambda' \circ f$. Then we see that

### 4.14. Corollary.
We have isomorphisms

$$
\mathbb{D}_H^+ / \Gamma(m) \cong \left\{ \text{Set of (A, E, \rho, \lambda) modulo isomorphisms \ f such that } \lambda^{-1} \equiv (\rho' \circ f)^{-1} \mod (m^{-1} \mathbb{Z}[\zeta]/\mathbb{Z}[\zeta])^2 \right\},
$$

$$
\mathbb{D}_H^+ / \Gamma_{Klein} \cong \left\{ \text{Set of (A, E, \rho, \lambda) modulo isomorphisms \ f such that } \lambda^{-1}(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = (\rho' \circ f)^{-1}(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \right\}.
$$

5. **K3 surface**

5.1. In this final section, we construct K3 surfaces with a non-symplectic automorphism of order 7 attached to $X_t$, according to Garbagnati and Penegini (GP). For generalities on K3 surfaces and elliptic surfaces, see [SS10] and references in it. Let us consider two curves

$$
X_t : y_1^2 = x_1(x_1 - 1)(x_1 - t), \quad X_\infty : y_2^2 = x_2^2 - 1
$$

and an affine algebraic surface

$$
S_t : y^2 = x(x - z)(x - tz) + z^10.
$$

$X_t$ is a hyperelliptic curve of genus 3. The surface $S_t$ is birational to the quotient of $X_t \times X_\infty$ by an automorphism

$$
\rho \times \rho : X_t \times X_\infty \to X_t \times X_\infty, \quad (x_1, y_1) \times (x_2, y_2) \mapsto (x_1, \zeta y_1) \times (x_2, \zeta y_2),
$$

and the rational quotient map $X_t \times X_\infty \to S_t$ is given by

$$
z = y_1/y_2, \quad y = x^5 x_2, \quad x = x \sqrt{z}.
$$

The minimal smooth compact model of $S_t$ (denoted by the same symbol $S_t$) is a K3 surface with an elliptic fibration

$$
\pi : S_t \to \mathbb{P}^1, \quad (x, y, z) \mapsto z.
$$

To see this, let us consider a minimal Weierstrass form

$$
S'_t : y^2 = x^3 + G_2(z)x + G_3(z)
$$

and the discriminant

$$
\Delta(z) = 4G_2(z)^3 + 27G_3(z)^2 = z^6(27z^{14} - 2(2t - 1)(t + 1)(t - 2)z^7 - t^2(t - 1)^2).
$$

From this, we see that $S_t$ is a K3 surface, and it has a singular fiber of type $I_0^*$ at $z = 0$, of type $IV$ at $z = \infty$ and fourteen fibers of type $I_1$ on $\mathbb{P}^1 \setminus \{0, \infty\}$. Note that

$$
\frac{dx_1}{y_1^2} \otimes \frac{y_2 dy_2}{x_2} \in H^0(X_t, \Omega^1) \otimes H^0(X_\infty, \Omega^1)
$$
is the unique \((\rho \times \rho)\) - invariant element up to constants, and descents to a holomorphic 2-form on \(S_t\) (see [GP, Section 3]). Therefore the period map for a family of K3 surface \(S_t\) is given by the Schwarz map \(s\). Note also that an automorphism \(\rho \times \text{id}\) of \(X_t \times X_\infty\) descents to \(S_t\):

\[
\rho \times \text{id} : S_t \rightarrow S_t, \quad (x, y, z) \mapsto (\zeta x, \zeta^5 y, \zeta z).
\]

Since \(S_t/\langle \rho \times \text{id} \rangle\) is birational to a rational surface \(X_t/\langle \rho \rangle \times X_\infty/\langle \rho \rangle\), the automorphism \(\rho \times \text{id}\) is non-simplyctic. Hence the transcendental lattice \(T_{S_t}\) is a free \(\mathbb{Z}[\rho \times \text{id}]\)-module ([NT79]). Since our family has positive dimensional moduli, we have rank \(T_{S_t} \geq 12\) and rank \(NS(S_t) \leq 10\) for a general \(t \in \mathbb{C} - \{0, 1\}\), where \(NS(S_t)\) is the Néron-Severi lattice.

5.2. Let us compute the Néron-Severi lattice and the Mordell-Weil group \(MW(S_t)\). Let \(o\) be the zero section of \(\pi : S_t \rightarrow \mathbb{P}^1\). We have three sections

\[
s_a : \mathbb{P}^1 \rightarrow S_t, \quad z \mapsto (x, y, z) = (az^5, z), \quad a = 0, 1, t
\]

such that \(s_0 + s_1 + s_t = o\) in \(MW(S_t)\). Let \(2\ell_0 + \ell_1 + \ell_2 + \ell_3 + \ell_4\) be the irreducible decomposition of \(\pi^{-1}(0)\), and \(\ell_1' + \ell_2' + \ell_3'\) be that of \(\pi^{-1}(\infty)\). For a suitable choice of indeces, intersection numbers of these curves are given by the following graph; self intersection number of each curve is \(-2\), two curves are connected by an edge if they intersect and intersection numbers are 1 except \(s_a \cdot s_b = 2\).

Let \(N \subset NS(S_t)\) be the lattice generated by \(o, s_0, s_1, s_t, \ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_1'\). The rank of \(N\) is 10 and the discriminant is \(-49\). Hence the Picard number of \(S_t\) is generically 10 and the rank of \(MW(S_t)\) is 2 by the Shioda-Tate formula ([SS10, Corollary 6.13]). Since the fixed locus \(S_{t,\text{id}}\) is contained in \(\pi^{-1}(0) \cup \pi^{-1}(\infty)\) and no elliptic curve contained in \(S_{t,\text{id}}\), we see that \(NS(S_t) = U(7) \oplus \mathbb{E}_8\) by the classification theorem of Artebani, Sarti and Taki ([AST11, § 6]). Therefore we have \(NS(S_t) = N\). Let \(L\) be the lattice generated by the zero section and vertical divisors. It is known that \(MW(S_t) \cong NS(S_t)/L\) ([SS10, Theorem 6.3]). Now it is obvious that \(MW(S_t) = \mathbb{Z}_{s_0} \oplus \mathbb{Z}s_1 \cong \mathbb{Z}^2\).

**APPENDIX A.**

A.1. Symplectic representation.
A.2. Period matrix. Let \( z \) be \( \exp[2\pi i/7] \) and put
\[
a = 1 + z + z^2 + z^4 = \frac{1 + \sqrt{7}}{2}, \quad b_1 = z - 2z^2 - 2z^4, \quad b_2 = -(2z^3 + 1 - z^6 + 2z^5).
\]
The modular embedding \( \Phi : \mathcal{D}_H^+ \to \mathbb{H}_0^M \) in Proposition 3.6 is given by
\[
\Phi(u) = \frac{1}{\Delta} \begin{bmatrix}
A_{11} & O \\
O & D_{11}
\end{bmatrix} u_1^2 + \begin{bmatrix}
O & B_{12} \\
B_{12} & O
\end{bmatrix} u_1 u_2 + \begin{bmatrix}
A_{22} & O \\
O & D_{22}
\end{bmatrix} u_2^2
\]
where
\[
\Delta = (z^2 + z + 1)(2z^2 - z + 2)u_1^2 + 3(z + 1)u_2^2
\]
and
\[
A_{11} = (z^2 + z + 1)(2z^2 - z + 2) \begin{bmatrix}
a & 0 & -1 \\
0 & a - 1 & -a \\
-1 & -a & 1
\end{bmatrix}, \quad D_{11} = (z^2 + 1) \begin{bmatrix}
2z^6 + z^5 - z^3 - 1 & 2z^6 - z^3 & -z^3 \\
2z^6 - z^3 & z^3 - z^3 & z^6 - a \\
-z^3 & z^6 - a & a
\end{bmatrix}
\]
\[
B_{12} = (z^3 - z^5) \begin{bmatrix}
-b_1 & -b_2 & -1 \\
z^5b_1 & z^5b_2 & z^5 \\
(1 + z^3)b_1 & (1 + z^3)b_2 & (1 + z^5)
\end{bmatrix}
\]
\[
A_{22} = -3 \begin{bmatrix}
z(z^5 - z^4 - 1) & z - 1 & z^2 + 1 \\
z^2 - z^5 + 1 & z^5 + 1 & -z^5(z + 1)
\end{bmatrix}, \quad D_{22} = (z + 1) \begin{bmatrix}
3a - 2 & a - 1 & a \\
a - 1 & 2a - 1 & -2 \\
-2 & a + 1 & 2
\end{bmatrix}
\]

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