We analyze the modulational instability of nonlinear Bloch waves in topological photonic lattices. In the initial phase of the instability development captured by the linear stability analysis, long wavelength instabilities and bifurcations of the nonlinear Bloch waves are sensitive to topological band inversions. At longer timescales, nonlinear wave mixing induces spreading of energy through the entire band and spontaneous creation of wave polarization singularities determined by the band Chern number. Our analytical and numerical results establish modulational instability as a tool to probe bulk topological invariants and create topologically nontrivial wave fields.

Topological photonic bands can be combined with appreciable mean-field nonlinear interactions in a variety of platforms [13], including exciton-polariton condensates in structured microcavities [4–5], waveguide arrays [6], metasurfaces [7], and ring resonators [8]. These nonlinear topological photonic systems are of growing interest due to not only their ability to host novel effects with no analogue in electronic topological materials, but also potential applications such as lasers, optical isolators, and frequency combs. Previous studies focused on the use of nonlinearities to create localized wavepackets such as edge and bulk solitons [9–16]. As complex nonlinear wave systems are typically sensitive to perturbations, precise control over excitation conditions is required to create these solitons. However, it remains unclear to what extent solitons in topological bands are robust to disorder, potentially limiting their utility.

In this paper we study the nonlinear dynamics of delocalized Bloch waves in topological bands, establishing their sensitivity to topological invariants such as the Chern number. We show that modulational instability of nonlinear Bloch waves can lead to the spontaneous formation of wave fields characterized by non-trivial Chern numbers inherited from the linear Bloch bands. The underlying mechanism is the energy-dependent parametric gain provided by the modulational instability [17–22], which enables selective population of a single Bloch band starting from a simple plane wave initial state. In addition to paving a simple way to sculpture novel structured light fields, the modulational instability also enables measurement of bulk topological invariants of bosonic wave systems. This is generally a difficult task unless the band eigenstates are known a priori, time-consuming Bloch band tomography is performed [24–27], or the bulk-edge correspondence is employed [28–30]. Our approach is based on the generic phenomenon of modulational instability and insensitive to the precise form and type of the nonlinearity (i.e., whether interactions are attractive or repulsive).

We first characterize the short time dynamics of nonlinear Bloch waves using the linear stability analysis. We find that although the Bloch waves at high symmetry points of the Brillouin zone are unstable in the presence of weak nonlinearities, they become stable at a critical nonlinearity strength. This critical strength coincides with the bifurcation of a nonlinear Dirac cone [37], where additional symmetry-breaking nonlinear Bloch waves emerge. We show that their long wavelength instabilities are sensitive to the band topology. Second, we use numerical simulations to study the modulational instability at longer propagation times. For weak nonlinearities the instability remains confined to the initially-excited band. Nonlinear wave mixing processes lead to the excitation of all the band’s linear modes, imprinting the band’s Chern number on the wave field’s polarization [35–36]. Interestingly, the polarization field converges to a quasi-equilibrium state well before the system is able to truly thermalize [31–34]. Thus, the topological properties of the band affect the modulational instability at small and large time and nonlinearity scales.

We consider a two-dimensional photonic lattice governed by the nonlinear Schrödinger equation,

$$i\partial_t |\psi(r,t)\rangle = (\hat{H}_L + \hat{H}_{NL}) |\psi(r,t)\rangle,$$

where $t$ is the evolution variable (time or propagation distance), $|\psi(r,t)\rangle$ is the wave field profile, $\hat{H}_L$ and $\hat{H}_{NL}$ are linear and nonlinear parts of the Hamiltonian, and $r = (x,y)$ indexes the lattice sites. We consider the chiral-$\pi$-flux model illustrated in Fig. 1(a). This is a two band tight binding model for a Chern insulator on a square lattice with two sublattices $a$ and $b$, i.e. $|\psi\rangle = (\psi_a, \psi_b)^T$, described by the Bloch Hamiltonian [38]

$$\hat{H}_L(k) = d(k) \cdot \hat{\sigma}, \quad d_x = \Delta + 2J_2 (\cos k_x - \cos k_y),$$

$$d_x + id_y = J_1 [e^{-i\pi/4} (1 + e^{i(k_y-k_x)}) + e^{i\pi/4} (e^{-i(k_y+k_x)} + e^{i(k_y-k_x)})],$$

where the wavevector $k = (k_x, k_y)$ is restricted to the first Brillouin zone $k_{x,y} \in [-\pi, \pi]$, $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ are
Pauli matrices acting on the sublattice (pseudospin) degree of freedom, $J_{1,2}$ are nearest and next-nearest neighbour hopping strengths, and $\Delta$ is a detuning between the sublattices. We will fix $J_2 = J_1/\sqrt{2}$, which enhances nonlinear effects by maximizing the band flatness [38]. For the nonlinear part of the Hamiltonian $\hat{H}_{NL}$ we consider an on-site nonlinearity of the form

$$\hat{H}_{NL} = \Gamma \text{diag}[f(|\psi_a(r)|^2), f(|\psi_b(r)|^2)],$$  

where $\Gamma$ is the nonlinear interaction strength and $f$ is the nonlinear response function.

The Bloch wave eigenstates of Eq. (2) form two energy bands $E_{\pm}(\mathbf{k})$, i.e. $\hat{H}_L(\mathbf{k})|u_{+}(\mathbf{k})\rangle = E_{+}(\mathbf{k})|u_{+}(\mathbf{k})\rangle$. Their topology is characterized by the quantized Chern number $[41]$,

$$C = \frac{1}{2\pi} \int_{\text{BZ}} \mathcal{F}(\mathbf{k}),$$  

where $\mathcal{F}(\mathbf{k})$ is the Berry curvature. In a two band system $\mathcal{F}(\mathbf{k})$ can be expressed in terms of the polarization field $\mathbf{n}_\pm(\mathbf{k}) = \langle u_\pm(\mathbf{k})|\hat{\sigma}|u_\pm(\mathbf{k})\rangle = \pm \mathbf{d}/|\mathbf{d}|$, i.e.

$$\mathcal{F}(\mathbf{k}) = -\frac{1}{2} \hat{\mathbf{n}} \cdot [(\partial_{k_x} \hat{\mathbf{n}}) \times (\partial_{k_y} \hat{\mathbf{n}})],$$  

with the Chern number counting the number of times $\mathbf{n}_\pm$ covers the unit sphere. We note that the interpretation of $C$ in terms of the wave polarization field can also be generalized to multi-band systems [39].

Fig. 1(b) shows the spectrum of $\hat{H}_L$ as a function of $\Delta/J_1$, which exhibits topological transitions at $\Delta/J_1 = 2\sqrt{2}$ and $-2\sqrt{2}$, where the gap closes at $\mathbf{k}_0 = (\pi, 0)$ and $(0, \pi)$ respectively. Let us consider the former point. The linear Bloch wave can be continued as a nonlinear Bloch wave [39, 40] $|\phi(r)\rangle = (\sqrt{T}_0, 0)^T e^{i\pi x} + \text{c.c.}$ with energy $E_{NL} = \Delta - 4J_2 + \Gamma f(I_0)$ bifurcating from the lower band when $\Delta < 4J_2$ and from the upper band when $\Delta > 4J_2$ [see purple line in Fig. 1(b)]. Performing the standard linear stability analysis [41], we compute the eigenvalue spectrum $\lambda(\mathbf{k})$ of its perturbation modes; perturbations with $\text{Im}(\lambda) > 0$ are linearly unstable. While the qualitative features of the perturbation spectrum do not depend on the precise form of the nonlinear response function, to be specific we consider pure Kerr nonlinearity $f(I) = I$.

Figs. 1(c,d) plot the growth rate and wavevector of the most unstable perturbation mode as a function of $\Delta$ and $\Gamma$. For weak nonlinearities $\Gamma$ we observe behaviour qualitatively similar to the scalar nonlinear Schrödinger equation: Bloch waves at the band edge exhibit a long wavelength instability under self-focusing nonlinearity, i.e. when $\Gamma m_{\text{eff}} < 0$, where $m_{\text{eff}} = \Delta - 4J_2$ is the wave effective mass at $\mathbf{k}_0$. Interestingly, a second long wavelength instability also occurs for stronger nonlinearities in the vicinity of the stable line $\Gamma I_0/2 = -m_{\text{eff}}$. This critical line occurs when the nonlinearity-induced potential closes the band gap and corresponds to a transition from an exponential instability at weak $\Gamma$ to an oscillatory instability at strong $\Gamma$.

To reveal the generic behavior in the vicinity of the critical line we consider the effective Dirac model obtained as a long wavelength expansion of Eq. (2), i.e. $\mathbf{k} = \mathbf{k}_0 + \mathbf{p}$ with $|\mathbf{p}| \ll 1$ [41],

$$\hat{H}_D = J_1 \sqrt{2}(p_x \hat{\sigma}_y - p_y \hat{\sigma}_x) + (m_{\text{eff}} + J_2[p_x^2 + p_y^2])\hat{\sigma}_z.$$  

The quadratic $J_2[p_x^2 + p_y^2]\hat{\sigma}_z$ term is essential to correctly reproduce the Chern number $C = \frac{1}{2}(1 - \text{sgn}[J_2 m_{\text{eff}}])$ and the main features of the linear perturbation spectrum.

The nonlinear Bloch wave solutions of Eq. (6) can be obtained analytically [41] and are shown in Figs. 1(c,d). The critical line coincides with the formation of a nonlinear Dirac cone at $\mathbf{k}_0$ [37], i.e. a symmetry-breaking bifurcation of the nonlinear Bloch waves. At the bifurcation new modes $|\phi(r)\rangle = (\sqrt{T_0}e^{i\pi x}, \sqrt{T_0}e^{i\pi x}) + \text{c.c.}$ emerge, with the relative phase between the two sublattices $\varphi$ forming a free parameter. Moreover, in the non-trivial phase an additional bifurcation occurs at higher intensities at $|\mathbf{p}| = \sqrt{4 - \Delta/J_2}$, corresponding to $d_c(\mathbf{p}) = 0$. The new branches emerging from this bifurcation merge with the lower band as $\Gamma I_0$ is increased, producing a gapless non-
FIG. 2. (a–c) The transition in the nonlinear Bloch wave spectrum across the critical line in the nontrivial (solid blue; $m_{\text{eff}} = -1/2$) and trivial (dashed red; $m_{\text{eff}} = 1/2$) phases of the effective Dirac model Eq. (6). (d,e) Growth rate (d) and magnitude (e) of the most unstable perturbation wavevector along the critical line. (f) Instability growth rate as a function of the polar angle measured from the symmetry-breaking nonlinear Bloch wave vector in the nontrivial (solid blue) and trivial (dashed red) phases.

linear Bloch wave spectrum, while in the trivial phase the nonlinear Bloch wave spectrum remains gapped [41].

While the nonlinear Dirac cone occurs in both topological phases, the modes’ stability in the vicinity of the bifurcation point is sensitive to the linear band topology. For example, in both the tight binding and continuum models the critical stable line terminates abruptly in the trivial phase at $\Delta = \Delta_c = 4J_2 + \frac{J_1^2}{J_2}$, as shown in Fig. 2(d,e). Beyond this critical detuning the most unstable wavevector is $|p_r| = \sqrt{|\Gamma I_0|J_2 - J_1^2 / J_2^2}$; the length scale of the instability is dictated by the quadratic $J_2(p_x^2 + p_y^2)$ term and vanishes in the usual linear Dirac approximation, which neglects $p_x^2$ terms. As a second example, Fig. 2(f) shows the angular (directional) dependence of the maximal instability growth rate of the symmetry-breaking nonlinear Bloch wave. In the nontrivial phase the instability is strongly anisotropic, with wavevectors in the direction perpendicular to the direction of the pseudospin remaining stable, whereas in the trivial phase instabilities occur for all angles.

To understand these topological phase-dependent stability properties, we note that in the trivial phase the perturbation modes maintain a similar polarization to the nonlinear Bloch wave, enabling efficient nonlinear wave mixing and promoting instabilities. On the other hand, in the non-trivial phase the perturbation modes’ polarization rotates away from the opposite pole of the Bloch sphere, thereby reducing the strength of the nonlinear wave mixing due to poor spatial overlap between the nonlinear Bloch wave and the perturbation modes. While this difference may seem minor, it can play a critical role close to bifurcation points by lifting the degeneracy between the bands of perturbation modes. Thus, the modulational instability does not just depend on the dispersion of the energy eigenvalues, but is also sensitive to the geometrical properties of the Bloch waves, i.e., their polarization, and the band topology. This is our first key result.

Next, we carry out numerical simulations of Eq. (1) to study the modulational instability beyond the initial linearized dynamics. To characterize the complex multimode dynamics, we compute the following observables: (i) The normalized real space participation number,

$$ P_r = \frac{\mathcal{P}^2}{2N} \left( \sum_r \left| \psi_n(r)^4 + |\psi_\sigma(r)|^4 \right| \right)^{-1} 
$$

where $\mathcal{P} = \sum_r |\psi_n(r)|^2$ is the total power. $P_r$ measures the fraction of strongly excited lattice sites. (ii) The Fourier space participation number $P_k$, which measures similarly the fraction of excited Fourier modes. (iii) The polarization direction $\hat{n}(k) = \langle \psi(k) | \hat{\sigma} | \psi(k) \rangle / \langle \psi(k) | \psi(k) \rangle$, which exhibits singularities sensitive to the band topology. We average these observables over an ensemble of small random initial perturbations to the nonlinear Bloch wave. The average polarization $\langle \hat{n}(k) \rangle$ in general describes a mixed state with $n^2 = \langle \hat{n}(k) \rangle \cdot \langle \hat{n}(k) \rangle < 1$. When $n^2 > 0$ for all $k$, i.e., the “purity gap” $\text{min}_k (n^2)$ remains open, the wave field is characterized by a quantized Chern number $\mathcal{C}$ [14,15].

Fig. 3 illustrates the dynamics of the $k_0 = (\pi, 0)$ nonlinear Bloch wave with initial intensity $I_0 = 1$, when each lattice site is subjected to a random perturbation with amplitude not exceeding $0.01\sqrt{I_0}$. We use saturable nonlinearity of the form $f(I) = 2I/(1 + I)$, which takes into account the inevitable saturation of nonlinear response at high intensities, a system size of $N = 32 \times 32$ unit cells with periodic boundary conditions [42], and average observables over 100 initial perturbations. We consider parameters corresponding to different instability regimes: exponential focusing, exponential defocusing, and oscillatory defocusing. The focusing instability generates a collection of localized solitons, resulting in a decrease in $\langle P_r \rangle$ in Fig. 3(a). On the other hand, the defocusing nonlinearity spreads energy over both sublattices, resulting in a small increase in $\langle P_r \rangle$. In all cases $\langle P_k \rangle$ increases due to other Fourier modes being populated via nonlinear wave mixing. For the exponential instabilities this is
accompanied by the purity gap opening and emergence of a well-defined Chern number corresponding to the band Chern number. Interestingly, the purity gap opens prior to \( \langle F_{r,k} \rangle \) reaching a steady state. Under the oscillatory instability the purity gap remains negligible due to competition between pairs of instability modes with the same growth rates.

To explore the emergence of a purity gap in more detail, we present in Fig. 3(a) its value at \( t = t_f = 40J_1 \) as a function of \( \Delta \), which tunes between the trivial and non-trivial phases [38]. For \( \Delta > 0 \) we observe good correspondence with the results of the linear stability analysis in Fig. 1, the size of the purity gap follows the instability growth rate, and the purity gap vanishes when the Bloch wave is linearly stable or exhibits an oscillatory instability, because the polarization becomes sensitive to the initial random perturbation. Interestingly, the purity gap also closes at \( \Delta \approx -2 \) despite no change in the fastest growing instability mode. This corresponds to a nonlinearity-induced closure of the band gap at the other high symmetry point \( k = (0, \pi) \).

While the trivial and non-trivial phases exhibit similar purity gaps, their differing topology can be observed by measuring the field polarization \( \langle \hat{n}(k) \rangle \) at long times, as illustrated in Fig. 3(b,c). Employing the method of Ref. [38], the Chern number can be measured by summing the charges of the phase singularities of the polarization azimuth \( \theta = \frac{1}{2} \tan^{-1}(n_x/n_z) \) weighted by \( \text{sgn}(n_y) \). In the trivial phase (large \( \Delta \)) the field is predominantly localized to a single sublattice, such that \( n_z \) remains nonzero and there are no phase singularities in \( \theta \); hence \( C = 0 \). In the non-trivial phase \( \langle \hat{n}(k) \rangle \) spans the entire Bloch sphere, corresponding to a pair of opposite charge phase singularities with opposite weights \( \text{sgn}(n_y) = \pm 1 \), and hence \( C = 1 \). Thus, the long time instability dynamics can be used to measure the band Chern number. This is our second important finding.

In conclusion, we have studied how the modulational instability of nonlinear Bloch waves can be used to probe band topology. The linear stability spectrum describing the short time instability dynamics exhibits bifurcations and a re-emergence of stability which are sensitive to topological band inversions. At longer evolution times nonlinear wave mixing can populate an entire band, enabling the spontaneous creation of topologically non-trivial wave fields from simple plane wave initial states. Since the timescales involved are shorter than the wave thermalization time, these effects should be experimentally observable in nonlinear waveguide arrays [9], Bose-Einstein condensates in optical lattices [21, 22], or exciton-polariton condensates [41, 5]. While we focused on the chiral-\( \pi \)-flux model, we have observed similar behaviour in other topological tight binding models. Lattices with a larger band flatness typically exhibit emergence of a purity gap and well-defined Chern number for a wider range of nonlinearity strengths. It will be interesting to generalize our findings to periodically-driven Floquet systems such as the nonlinear waveguide array employed in Ref. [6], where perfectly flat topological bands have been demonstrated.

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FIG. 3. Long time instability dynamics in the different instability regimes: focusing exponential (blue), defocusing exponential (red), and oscillatory instability (brown). (a) Real space participation number. (b) Fourier space participation number. (c) Purity gap.

FIG. 4. (a) Purity gap time \( t = t_f = 40/J_1 \) as a function of \( \Delta \). (b,c) Field polarization textures at \( t_f \) in the (b) trivial (\( \Delta = -3/J_1 \)) and (c) non-trivial (\( \Delta = 0 \)) phases. The Chern number is obtained by summing the charges of the polarization azimuth vortices (indicated by arrows) weighted by \( \text{sgn}(n_y) \) at the vortex core (indicated by \( \pm 1 \)) [35].
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**Supplementary Material for**

**“Probing band topology using modulational instability”**

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**LINEAR STABILITY ANALYSIS**

Here we consider the linear stability of nonlinear Bloch waves in a generic tight binding lattice described by the nonlinear evolution equation

\[
i\partial_t |\psi(r,t)\rangle = (\hat{H}_L + \hat{H}_{NL}) |\psi(r,t)\rangle.
\]

We assume that the nonlinear part of the Hamiltonian \(\hat{H}_{NL}\) is a diagonal matrix with real elements dependent only on the local on-site intensity, i.e. \(\hat{H}_{NL} = \Gamma\text{diag} [f(|\psi_a|^2),f(|\psi_b|^2),...]\), where \(f\) describes the intensity-dependent nonlinear frequency shift and \(a,b,...\) indexes the sublattice degree of freedom. The linear part of the Hamiltonian \(\hat{H}_L\) can be expanded in real space as

\[
\hat{H}_L |\psi(r)\rangle = \sum_\delta \hat{C}(\delta) |\psi(r + \delta)\rangle,
\]

where summation \(\delta\) is over neighbouring unit cells. Transforming to Fourier space, \(|\psi(r)\rangle = \sum_k |\psi(k)\rangle e^{ik\cdot r}\), we obtain the Bloch Hamiltonian

\[
\hat{H}(k) |\psi(k)\rangle = \left( \sum_\delta \hat{C}(\delta)e^{ik\cdot \delta} \right) |\psi(k)\rangle.
\]

Note that under the Fourier transform \(\hat{H}_L^* |\psi(r)\rangle \rightarrow \hat{H}_L^* (-k) |\psi(k)\rangle\).

To perform the linear stability analysis we consider small perturbations about some nonlinear steady state \(|\phi(r)\rangle\) with energy \(E\), i.e. \(|\psi(r,t)\rangle = (|\phi(r)\rangle + |p(r,t)\rangle) e^{-iEt}\). First, by Taylor expansion of the diagonal nonlinear term and neglecting terms quadratic in the perturbation, we obtain a linearised evolution equation for the perturbation, perturbation,

\[
(i\partial_t + E)|p(r)\rangle = \hat{H}_L |p(r)\rangle + \Gamma \sum_{j=a,b,...} \left( [f(|\phi_j|^2) + f'(|\phi_j|^2)\phi_j^2]p_j(r) + f'(|\phi_j|^2)\phi_j^2 p_j^*(r) \right) |j\rangle,
\]

The solution to this set of coupled first order linear differential equations can be expanded in terms of exponential functions as \(|p(r,t)\rangle = |u(r)\rangle e^{-i\lambda t} + |v(r)\rangle e^{i\lambda t}\). We collect terms with the same time dependence to obtain the eigenvalue problem

\[
\lambda |u(r)\rangle = (\hat{H}_L - E) |u(r)\rangle + \Gamma \sum_{j=a,b,...} \left( [f(|\phi_j|^2) + f'(|\phi_j|^2)\phi_j^2]u_j(r) + f'(|\phi_j|^2)\phi_j^2 v_j(r) \right) |j\rangle,
\]

\[
\lambda |v(r)\rangle = -(\hat{H}_L^* - E) |v(r)\rangle - \Gamma \sum_{j=a,b,...} \left( [f(|\phi_j|^2) + f'(|\phi_j|^2)\phi_j^2]v_j(r) + f'(|\phi_j|^2)\phi_j^2 u_j(r) \right) |j\rangle.
\]
Now we assume that steady state is a nonlinear Bloch wave such that $|\phi(\mathbf{r})\rangle = |\phi\rangle e^{i\mathbf{k}_0 \cdot \mathbf{r}}$. Fourier transforming the above equations, there is coupling between perturbation fields $u(\mathbf{k})$ and $|v(\mathbf{k} - 2\mathbf{k}_0)\rangle$. We obtain the coupled equations

$$
\lambda |u(\mathbf{k} + \mathbf{k}_0)\rangle = (\hat{H}(\mathbf{k}_0 + \mathbf{k}) - E) |u\rangle + \Gamma \sum_{j=a,b,...} [(f(|\phi_j|^2) + f'(|\phi_j|^2)|\phi_j|^2)u_j + f'(|\phi_j|^2)|\phi_j|^2v_j] |j\rangle, $$
(S7)

$$
\lambda |v(\mathbf{k} - \mathbf{k}_0)\rangle = -(\hat{H}^*(\mathbf{k}_0 - \mathbf{k}) - E) |v\rangle - \Gamma \sum_{j=a,b,...} [(f(|\phi_j|^2) + f'(|\phi_j|^2)|\phi_j|^2)u_j + f'(|\phi_j|^2)|\phi_j|^2v_j] |j\rangle. $$
(S8)

This eigenvalue problem has a built-in particle hole symmetry: eigenvalues $\lambda$ must occur in complex conjugate pairs. Real $\lambda$ correspond to stable perturbation modes, purely imaginary $\lambda$ result in exponential instabilities, and complex $\lambda$ correspond to oscillatory instabilities.

In two band tight binding models the Bloch Hamiltonian can be parameterized using the Pauli matrices as $\hat{H}(k) = \mathbf{d}(k) \cdot \hat{\sigma}$, where $\mathbf{d}(k)$ is a real 3 component vector. We obtain the explicit matrix form of the above linear stability equations,

$$
\lambda \begin{pmatrix}
|u\rangle \\
|v\rangle
\end{pmatrix} = \begin{pmatrix}
\mathbf{d}(\mathbf{k}_0 + \mathbf{k}) \cdot \hat{\sigma} & -E + \Gamma \sum_j (f_j + f_j^*|\phi_j|^2)|j\rangle \langle j| \\
-E - \Gamma \sum_j f_j^2|\phi_j|^2|j\rangle \langle j| & -E + \Gamma \sum_j (f_j + f_j^*|\phi_j|^2)|j\rangle \langle j|
\end{pmatrix} \begin{pmatrix}
|u\rangle \\
|v\rangle
\end{pmatrix},
$$
(S9)

where $f_j = f(|\phi_j|^2)$ and $f_j^* = f'(|\phi_j|^2)$.

For the case of a nonlinear Bloch wave with intensity $I_0$ localized to the $a$ sublattice analyzed in the main text, we have $|\phi\rangle = (\sqrt{I_0}, 0, d_{x,y}(\mathbf{k}_0) = 0, d_z(\mathbf{k}_0) = \Delta - 4J_2, \text{and } E = d_z(\mathbf{k}_0) + \Gamma f(I_0)$. The $\mathbf{k} = 0$ eigenvalue problem takes the simple form

$$
\lambda \begin{pmatrix}
\begin{pmatrix} u_a \\ u_b \\ v_a \\ v_b \end{pmatrix} = \begin{pmatrix}
\Gamma f'I_0 & 0 & 0 & 0 \\
0 & -2d_z(\mathbf{k}_0) - \Gamma f & 0 & 0 \\
-\Gamma f'I_0 & 0 & -\Gamma f'I_0 & 0 \\
0 & 0 & 0 & 2d_z(\mathbf{k}_0) + \Gamma f(I_0)
\end{pmatrix} \begin{pmatrix} u_a \\ u_b \\ v_a \\ v_b \end{pmatrix}.
\end{pmatrix}
(S10)

yielding $\lambda = 0, 0, \pm [2d_z(\mathbf{k}_0) + \Gamma f(I_0)]$ and a fourfold degeneracy when $\Gamma f(I_0)/2 = -d_z(\mathbf{k}_0)$, i.e. when the nonlinear energy shift on the $a$ sublattice is sufficient to close the band gap. Complex instability eigenvalues emerge beyond this threshold intensity. Note that this transition is independent of $f'(I_0)$, i.e. the precise form of the nonlinear response function.

### NONLINEAR DIRAC MODEL

In this section, we examine linear stability of the high-symmetry nonlinear Bloch wave with the in-plane wave vector $\mathbf{k}_0 = [\pi, 0]$. In the vicinity of $\mathbf{k}_0$, $\mathbf{k} = \mathbf{k}_0 + \mathbf{p}$, the series expansion in the Bloch Hamiltonian $\hat{H}_L(\mathbf{k})$ at $|\mathbf{p}| \ll 1$ leads to the Dirac-like Hamiltonian

$$
\hat{H}_D = -J_1 \sqrt{2} (-p_x \hat{\sigma}_y + p_y \hat{\sigma}_x) + (\Delta - 4J_2 + J_2 [p_x^2 + p_y^2]) \hat{\sigma}_z.
(S11)
$$

The corresponding evolution equations including the local Kerr nonlinearity $f(I) = I$ can be formulated in the real space in terms of spatial derivatives by substituting $p_{x,y} = -i\partial_{x,y}$:

$$
i \partial_t \psi = \left( \begin{array}{c} \Delta - 4J_2 - J_2 [\partial_x^2 + \partial_y^2] + \Gamma |\psi|^2 | \\
J_1 \sqrt{2} (i\partial_y - \partial_x) & -\Delta + 4J_2 + J_2 [\partial_x^2 + \partial_y^2] + \Gamma |\psi|^2 \end{array} \right) \psi.
(S12)
$$

**Nonlinear dispersion of bulk modes**

We search for the solution of (S12) in the form of weakly nonlinear Bloch waves:

$$
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{-iEt + ip_x x + ip_y y}.
(S13)$$
Plugging the spinor $|S20\rangle$ into Eq. (S12) results in the system of equations for the amplitudes $A$ and $B$
\[
\begin{aligned}
(E + \Delta - 4J_2 + J_2 \left[ p_x^2 + p_y^2 \right] + \Gamma |A|^2)A - J_1 \sqrt{2} \left( p_y + ip_x \right) B &= 0, \\
\left( E + \Delta - 4J_2 + J_2 \left[ p_x^2 + p_y^2 \right] - \Gamma |B|^2 \right)B + J_1 \sqrt{2} \left( p_y - ip_x \right) A &= 0,
\end{aligned}
\] (S14)
where the wave vector $p = (p_x, p_y) = p(\cos \theta, \sin \theta)$ can be defined in the polar coordinate system, $p_y + ip_x = ipe^{-i\theta}$.

Denoting the total wave intensity $|A|^2 + |B|^2 = I_0$, we first find the solutions for the lower and upper bands at the zero wave vector $p = 0$:
\[
\begin{aligned}
A^{(0)} &= 0, \quad |B^{(0)}|^2 = I_0, \quad E^{(0)}_2 = -\Delta + 4J_2 + \Gamma |B^{(0)}|^2 = -\Delta + 4J_2 + \Gamma I_0, \\
B^{(0)} &= 0, \quad |A^{(0)}|^2 = I_0, \quad E^{(0)}_1 = \Delta - 4J_2 + \Gamma |A^{(0)}|^2 = \Delta - 4J_2 + \Gamma I_0.
\end{aligned}
\] (S15)

At the intensities above the critical value $I_0 \geq 2(\Delta - 4J_2)^2$, we get the additional doubly degenerate solution
\[
|A^{(0)}|^2 = \frac{I_0}{2} - \frac{\Delta - 4J_2}{\Gamma}, \quad |B^{(0)}|^2 = \frac{I_0}{2} + \frac{\Delta - 4J_2}{\Gamma}, \quad E^{(0)}_3 = \frac{\Gamma I_0}{2}
\] (S17)
with the eigenvectors:
\[
\begin{pmatrix}
A^{(0)} \\
B^{(0)}
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi} \sqrt{I_0 - \frac{2(\Delta - 4J_2)}{\Gamma}} \\
\pm \sqrt{I_0 + \frac{2(\Delta - 4J_2)}{\Gamma}}
\end{pmatrix},
\] (S18)
where $\varphi$ is an arbitrary phase depending on which direction we approach the degeneracy point $p = 0$. Specifically, this phase uncertainty is lifted if we consider the limit transition to the point $p = 0$ along different directions $p_0(p_0 \to 0) = (p_0x, p_0y) = p_0(\cos \theta_0, \sin \theta_0)$. According to (S14), the phase shift in spinor (S18) is given by $\varphi = \pi/2 - \arctan(p_{0y}/p_{0x}) = \pi/2 - \theta_0$.

To find the dispersion in the neighborhood of the point $p_x = p_y = 0$, we employ the perturbation theory. Treating $p_x$ and $p_y$ as small perturbations, we expand all quantities to the first order $E = E^{(0)} + E^{(1)} + \ldots; A = A^{(0)} + A^{(1)} + \ldots; B = B^{(0)} + B^{(1)} + \ldots.$ We obtain a cross-like solution describing the nonlinear Dirac cone:
\[
E = E^{(0)} + E^{(1)} = \frac{\Gamma I_0}{2} \pm \frac{\sqrt{2} J_1 \sqrt{p_x^2 + p_y^2}}{\sqrt{1 - \frac{4(\Delta - 4J_2)^2}{\Gamma^2}}}.
\] (S19)

Thus, at $\Gamma^2 I_0^2 > 4 (\Delta - 4J_2)^2$ one of the dispersion curves develops a loop.

The intensities of two components nearby the cross point are corrected as follows
\[
|A|^2 = \frac{I_0}{2} - \frac{\Delta - 4J_2}{\Gamma} \left( 1 \pm \frac{2\sqrt{2} J_1 \sqrt{p_x^2 + p_y^2}}{\sqrt{I_0 \Gamma^2 - 4(\Delta - 4J_2)^2}} \right), \quad |B|^2 = \frac{I_0}{2} + \frac{\Delta - 4J_2}{\Gamma} \left( 1 \pm \frac{-2\sqrt{2} J_1 \sqrt{p_x^2 + p_y^2}}{\sqrt{I_0 \Gamma^2 - 4(\Delta - 4J_2)^2}} \right).
\] (S20)

Substituting amplitudes (S20) and energy (S19) into the system (S14), we may introduce the local effective Hamiltonian as:
\[
\begin{pmatrix}
\pm 2\sqrt{2} J_1 \frac{(\Delta - 4J_2)\sqrt{p_x^2 + p_y^2}}{\sqrt{I_0 \Gamma^2 - 4(\Delta - 4J_2)^2}} \\
-\sqrt{2} J_1 (p_y - ip_x)
\end{pmatrix} \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} = E^{(1)} \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} = \pm \frac{\sqrt{2} J_1 \sqrt{p_x^2 + p_y^2}}{\sqrt{1 - \frac{4(\Delta - 4J_2)^2}{\Gamma^2}}} \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}.
\] (S21)

Next, we derive the exact implicit expression for the nonlinear dispersion $E(p_x, p_y)$. To simplify our derivations, we set $p_x = 0$ and rewrite the system in the form:
\[
\begin{pmatrix}
E_n - M_n \\
J_1 \sqrt{2} p_y
\end{pmatrix} \begin{pmatrix}
A \\
B
\end{pmatrix} = 0,
\] (S22)
\[denoting \ E_n = E - \Gamma I_0/2, \ M_n = \Delta - 4J_2 + \frac{\Gamma I_0}{2} (\Delta - 4J_2 + 2J_2 p_y^2) + J_2 p_y^2. \] The nonlinear dispersion is then given by\n\[
E^2 = 2J_1^2 p_y^2 + M_n^2 (p_y^2).
\] (S23)
The eigenvectors’ intensities on the two sublattices satisfy

\[ |A|^2 = \frac{I_0}{2} + \frac{\Delta - 4J_2}{2(-\Gamma I_0 + E)} + \frac{J_2p_y^2 I_0}{2(-\Gamma I_0 + E)}, \]  \hspace{1cm} (S24)

\[ |B|^2 = \frac{I_0}{2} - \frac{\Delta - 4J_2}{2(-\Gamma I_0 + E)} - \frac{J_2p_y^2 I_0}{2(-\Gamma I_0 + E)}. \]  \hspace{1cm} (S25)

The implicit relation (S23) can be posed as

\[ ((E - \Gamma I_0/2)^2 - 2J_1^2 p_y^2) (E - \Gamma I_0)^2 = (\Delta - 4J_2 + J_2p_y^2)^2 \left( E - \frac{\Gamma I_0}{2} \right)^2. \]  \hspace{1cm} (S26)

Note, this dispersion relation supports the existence of 2 more loops in addition to the loop at the point \( p_y = 0 \), described above. This bifurcation occurs at the p\( B \) y:

\[ p_y^B = \pm \sqrt{\frac{4J_2 - \Delta}{J_2}}, \]  \hspace{1cm} (S27)

in the non-trivial phase only, \( |\Delta| < 4J_2 \). The energies at the point \( p_y^B \) are:

\[ E^{(0)B}_3 = \Gamma I_0, \]  \hspace{1cm} (S28)

\[ E^{(0)B}_{2,1} = \Gamma I_0/2 \pm \sqrt{\frac{2J_1^2(4J_2 - \Delta)}{J_2}}. \]  \hspace{1cm} (S29)

Specifically, the energy \( E^{(0)B}_3 \) corresponds to two additional cross points, which appear only in the nontrivial case with the eigenvectors

\[ \left( \begin{array}{c} A^{(0)B} \\ B^{(0)B} \end{array} \right) = \left( \begin{array}{c} e^{i\varphi} \sqrt{\frac{I_0}{2} + \sqrt{\frac{I_0^2}{4} - 2J_1^2(p_y^B)^2}/\Gamma^2} \\ \pm \sqrt{\frac{I_0}{2} - \sqrt{\frac{I_0^2}{4} - 2J_1^2(p_y^B)^2}/\Gamma^2} \end{array} \right). \]  \hspace{1cm} (S30)

The additional crosses appear at the intensities higher \( \Gamma I_0/2 = \pm \sqrt{\frac{2J_1^2(4J_2 - \Delta)}{J_2}} \) (the sign is chosen depending on the sign of the nonlinearity \( \Gamma \)), which is defined by the degeneracy of the cross point and one of the bands at \( p_y = p_y^B \).

As we can see in Fig. 2 of the main text, the next dispersion plane bifurcation (b) \( \rightarrow \) (c) happens when \( E^{(0)}_1 = E^{(0)B}_2 \) or \( E^{(0)}_2 = E^{(0)B}_1 \) for different signs of \( \Gamma \) (according to Eqs. (S15), (S28)), that corresponds to

\[ \Gamma = \frac{2}{I_0} \left( \mp (\Delta - 4J_2) \pm \sqrt{\frac{2J_1^2(4J_2 - \Delta)}{J_2}} \right). \]  \hspace{1cm} (S31)

**Modulation instability**

To examine linear stability of the nonlinear Bloch modes, we introduce small complex-valued perturbations to the amplitudes: \( A = A_0 + \delta a, B = B_0 + \delta b \) and look for the solution in the form:

\[ \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = \left( \begin{array}{c} A_0 + \delta a \\ B_0 + \delta b \end{array} \right) e^{-iEt + ip_x x + ip_y y}. \]  \hspace{1cm} (S32)

The equations for deviations \( \delta a, \delta b \) can be recast as

\[ i\frac{\partial}{\partial t} \left( \begin{array}{c} \delta a \\ \delta b \\ \delta a^* \\ \delta b^* \end{array} \right) = \hat{L} \left( \begin{array}{c} \delta a \\ \delta b \\ \delta a^* \\ \delta b^* \end{array} \right), \]  \hspace{1cm} (S33)
where operator $\hat{L}$ is the $4 \times 4$ matrix

$$
\hat{L} = \begin{pmatrix}
\hat{H}_D(\partial_x, \partial_y) + H_D(p_x, p_y) - E \hat{I} + 2\Gamma & \left( |A_0|^2 \ 0 \right) \\
0 & -\hat{H}_D(\partial_x, \partial_y) - H_D(p_x, p_y) + E \hat{I} - 2\Gamma \left( |A_0|^2 \ 0 \right)
\end{pmatrix}
$$

where

$$
H_D(p_x, p_y) = \begin{pmatrix} J_2 (p_x^2 + p_y^2) & -J_1 \sqrt{2} (p_y + ip_x) \\
-J_1 \sqrt{2} (p_y - ip_x) & -J_2 (p_x^2 + p_y^2)
\end{pmatrix}
$$

To study modulational instability, we take $|\delta a; \delta a^*; \delta b; \delta b^*| = [C_1; C_2; C_3; C_4] e^{i\lambda t + i\kappa_x x + i\kappa_y y} = C e^{i\lambda t + i\kappa_x x + i\kappa_y y}$ and set $\kappa_x = 0$ to simplify further considerations. Eq. (S33) leads to the system of equations for amplitudes $C$: $\left( \hat{L} - \lambda \hat{I} \right) C = 0$, which indicates instability.

For the cross point at $p_x = p_y = 0$, existing at the intensities $\Gamma I_0 > \pm 2(\Delta - 4J_2)$ at the energy $E(0) = \frac{\Gamma I_0}{2\kappa}$, with the amplitudes $|A_0|^2 = \frac{I_0}{2} - \frac{\Delta - 4J_2}{\Gamma}$, $|B_0|^2 = \frac{I_0}{2} + \frac{\Delta - 4J_2}{\Gamma}$, we find the energy detuning $\lambda$ along the straight lines $I_0 \Gamma + C = -2(\Delta - 4J_2)$ in the parameter plane $(\Gamma, \Delta)$:

$$
\lambda = \pm \sqrt{\frac{1}{2} \left[ e^{-2i\varphi} \kappa_0^2 \right] \left[ C^2 (-1 + e^{2i\varphi})^2 J_1^2 + 2C (-1 + e^{2i\varphi})^2 \Gamma I_0 J_1^2 + 2\Gamma^2 I_0^2 J_1^2 \kappa_0^2 + C J_2 \kappa_0^2 + \Gamma I_0 J_2 \kappa_0^2 + 2J_1^2 \kappa_0^2 + J_2^2 \kappa_0^4 \right]}
$$

We analyse Eq. (S37) for $C = 0$ at the line $I_0 \Gamma = -2(\Delta - 4J_2)$, which is the negatively inclined existence boundary of the cross solution:

$$
\lambda_{1,2} = \pm \sqrt{J_2^2 \kappa_0^4 + 2\kappa_0^2 J_1^2}
$$

$$
\lambda_{3,4} = \pm \sqrt{-4(\Delta - 4J_2) J_2 \kappa_0^2 + J_2^2 \kappa_0^4 + 2J_1^2 \kappa_0^4}
$$

The imaginary part $\text{Im}(\lambda_{1,2})$ is zero for all values of the wave number $\kappa_0$, therefore, $\lambda_{1,2}$ do not show any instability.

The area of the stability can be determined from $\lambda_{3,4}$: it is a purely real quantity for $\Gamma > -\frac{J_1^2}{I_0 J_2}$ or equivalently $2J_1^2 \geq 4(\Delta - 4J_2) J_2$. In the nontrivial case, since $J_2 (\Delta - 4J_2) < 0$, we conclude that $2J_1^2 \geq 4(\Delta - 4J_2) J_2$ for any $J_2, \Delta$. Therefore, the cross point is stable. But in the trivial case, the area of parameters $J_2, J_1$ exists, for which $\text{Im}(\lambda_{3,4}) > 0$, and the cross point becomes unstable. The boundary value of the detuning in the trivial phase is

$$
\Delta_c = 4J_2 + \frac{J_1^2}{2J_2}
$$

Note, for the given intensity, $-\Gamma I_0/2 = \Delta - 4J_2$, the upper branch and the point of the cross are degenerate. Hence, the line of stability $I_0 \Gamma = -2(\Delta - 4J_2)$ appears in Fig. 1 in the main text for the upper branch at $\Delta < \Delta_c$. For $\Delta > \Delta_c$, we analytically obtain the maximum growth rate $\max_{\kappa_y} \text{Im}[\lambda_{3,4}]$ achieved at the wavenumber $\kappa_{y}^{\text{max}}$:

$$
\max_{\kappa_y} \text{Im}[\lambda_{3,4}] = \frac{|J_1^2 + \Gamma I_0 J_2|}{|J_2|};
$$

$$
\kappa_{y}^{\text{max}} = \pm \sqrt{\frac{|\Gamma I_0 J_2 + J_1^2|}{J_2^2}}.
$$

Equation (S37) at the other boundary of the existence of the cross solution ($C = -2\Gamma I_0$) takes the form:

$$
\lambda_{1,2,3,4} = \pm \sqrt{\pm \kappa_0^2 \Gamma I_0 J_2 - \Gamma I_0 J_2 \kappa_0^2 + 2J_1^2 \kappa_0^2 + J_2^2 \kappa_0^4},
$$
from which we obtain the area of stability $\Gamma < \frac{J_2^2}{I_0 J_2}$.

Let us consider Eq. (S37) for the case $\varphi = \pi n, n \in \mathbb{Z}$:

$$\lambda_{1,2} = \pm \sqrt{J_2^2 \kappa_y^4 + CJ_2 \kappa_y^2 + 2J_1^2 \kappa_y^2},$$  \hspace{1cm} (S44)

$$\lambda_{3,4} = \pm \sqrt{J_2^2 \kappa_y^4 + CJ_2 \kappa_y^2 + 2J_2 \Gamma I_0 \kappa_y^2 + 2J_1^2 \kappa_y^2}.$$  \hspace{1cm} (S45)

The boundaries of the cross stability are located on lines with $C = -\frac{2J_1^2 - 2\Gamma I_0}{J_2}$ and $C = -\frac{2J_2^2}{J_2}$. These are the straight lines $I_0 \Gamma = \pm 2(\Delta - 4 J_2 - \frac{J_2^2}{J_2^2})$.

The color maps of the maximum increment value max$_\lambda[\text{Im}(\lambda)]$ in the parameter space for the cross solution $E = \Gamma I_0 / 2$ are plotted in Fig. S1 by using Eq. (S37). On our notations, linear stability of perturbation in $y$ direction depends on the spinor phase angle $\varphi$. Making a generalization about this feature, we note that stability is conditional on the mutual orientation $\Delta \theta = \theta_p - \theta_0$ of the symmetry-broken solution with $p_0$ and perturbation with $p_p$. In the nontrivial phase, it remains stable at $\Delta \theta = 0$ ($\varphi = 0, \theta_p = \pi/2, \theta_0 = \pi/2$) and exhibits the maximum transverse modulational instability at $\Delta \theta = \pi/2$ ($\varphi = \pi/2, \theta_p = \pi/2, \theta_0 = 0$).

FIG. S1. The maximum increment value max$_\lambda[\text{Im}(\lambda)]$ color-coded in the plane of parameters $\Delta / J_1, \Gamma I_0 / J_1$ for the cross point $E = \Gamma I_0 / 2$ at $p = 0$. Parameters are $J_2 = J_1 / \sqrt{2}$, (a) $\varphi = 0$, (b) $\varphi = \pi/2$. Red dashed lines $\Gamma = \pm 2 \frac{(4 J_2 - \Delta)}{I_0}$ highlight the boundaries of the existence of the cross solution. On these lines, the cross point is stable in the nontrivial domain, $|\Delta| < 2\sqrt{2}$, whose upper boundary is marked with the green dashed line. In the trivial domain, the cross point is unstable at detunings larger $\Delta = \frac{J_2^2}{2 J_2} + 4 J_2$ marked with a solid green line. The boundaries of the cross point stability for $\varphi = 0$ are black straight lines $I_0 \Gamma = \pm 2(\Delta - 4 J_2 - \frac{J_2^2}{J_2^2})$. At $\varphi = 0$, the intersection point of the black lines with the boundary of the trivial phase at $\Gamma = \pm 2 \frac{J_1^2}{(J_2 I_0)}$ (straight blue lines) defines the intensity, for which, by changing $\Delta$, we can distinguish the trivial phase from the non-trivial one observing a transition from stability to instability.