Algebraic and Topological Structures of Complex Numbers

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Abstract. Complex numbers play a fundamental role in every branch of mathematics. First, we review the definition of the complex numbers, discuss their arithmetic operation properties and give proof of the Lagrange identity. At last, we take the investigation of topological notations of the complex plane.

1. Introduction
Complex numbers play a fundamental role in every branch of mathematics such as the number theory, and the probability theory [1, 2]. Due to this, we can deal with practical problems in science and engineering in a very concise way.

In the first section, we review the computational properties of complex numbers and discuss their representation forms (general form, trigonometric form, polar form). And we give a detailed proof of the Lagrange identity.

In the second section, we mainly summarize the topological notations and properties of the complex plane. The complex plane is a good example to understand abstract definitions and fundamental theorems from the point topology.

2. The Main Body of Work

2.1. Complex Number
Definition 2.1.1: A complex number is a mathematical number extended from real numbers. Its general form is \( x + i \cdot y : x, y \in \mathbb{R}, \text{where } i = \sqrt{-1}. \)

Remark 2.1.2: The introduction of the symbol \( i, \text{which represent } \sqrt{-1}, \) is meaningful because \( i \) can be viewed as a root of the equation

\[
X^2 + 1 = 0
\]  

Remark 2.1.3 We use notation \( \mathbb{C} \) to represent the set of all complex numbers. That is,
\[ \mathbb{C} = \{ x + i \cdot y : x, y \in \mathbb{R} \} \]  \hspace{1cm} (2)

**Definition 2.1.4** For a given complex number \( z = x + i \cdot y \in \mathbb{C} \), we define the real part \( \text{Re}(z) \) of \( z \) as \( \text{Re}(z) = x \); we define its imaginary part \( \text{Im}(z) \) as \( \text{Im}(z) = y \); we define its absolute value \( |z| \) as \( |z| = \sqrt{x^2 + y^2} \).

**Definition 2.1.5** If \( \text{Im}(z) = 0 \), then we say \( z \) is a real number, and if \( \text{Re}(z) = 0 \), we say \( z \) is imaginary.

### 2.2. Operations on \( \mathbb{C} \)

In this section, we admit several operations on \( \mathbb{C} \). We can find that defined operations can be viewed as a natural extension of arithmetic operations on \( \mathbb{R} \).

**Definition 2.2.1** Given two complex numbers, \( z_1 = x_1 + i \cdot y_1, z_2 = x_2 + i \cdot y_2 \), where \( x_1, y_1 \) are real numbers, we define the addition of these two complex numbers as

\[
(X_1 + i \cdot y_1) + (x_2 + i \cdot y_2) = (x_1 + x_2) + i \cdot (y_1 + y_2);
\]  \hspace{1cm} (3)

We define multiplication of two complex numbers as

\[
(x_1 + i \cdot y_1) \cdot (x_2 + i \cdot y_2) = (x_1 \cdot x_2 - y_1 \cdot y_2) + i \cdot (x_1 \cdot y_2 + x_2 \cdot y_1).
\]  \hspace{1cm} (4)

Also, the conjugate of a complex number can be defined as

\[ \overline{x + i \cdot y} = x - i \cdot y \]  \hspace{1cm} (5)

**Definition 2.2.2** We say a set is a field if it admits two operations, addition and multiplication, and every non-zero element has a unique inverse in sense of multiplication.

**Theorem 2.2.3** The set \( \mathbb{C} \) of complex numbers endowed with addition and multiplication is a field.

**Proof:** It suffices to show every non-zero complex number has a unique inverse. Our strategy to obtain the inverse of a non-zero complex number is by multiplication of conjugation of one term on both the denominator and the nominator. Assume \( z = x + i \cdot y, (x, y \in \mathbb{R}, x^2 + y^2 \neq 0) \), we can obtain its inverse as follow:

\[
Z^{-1} = (x + i \cdot y)^{-1} = \frac{1}{x + i \cdot y} = \frac{x - i \cdot y}{(x + i \cdot y)(x - i \cdot y)} = \frac{x - i \cdot y}{x^2 + y^2}
\]  \hspace{1cm} (6)

**Collaroy 2.2.4 (properties of complex conjugation)**

Given any two-complex number, \( z \) and \( \omega \), we have the following identities:

\[ \overline{z + \omega} = \overline{z} + \overline{\omega}; \overline{z} \overline{\omega} = z \overline{\omega} \]  \hspace{1cm} (7)

**Proof:** Assume \( z = x_1 + i \cdot y_1, \omega = x_2 + i \cdot y_2 \) where \( x_1, y_1 \) are reals. Then

\[
\overline{z + \omega} = (x_1 - i \cdot y_1) + (x_2 - i \cdot y_2) = (x_1 + x_2) - (y_1 + y_2)i
\]

\[ \Rightarrow \overline{z + \omega} = z + \omega \]  \hspace{1cm} (8)

And,

\[
\overline{z \overline{\omega}} = (x_1 - i \cdot y_1)(x_2 - i \cdot y_2) = (x_1 x_2 - y_1 y_2) - i \cdot (x_1 y_2 + y_1 x_2)
\]

\[ \Rightarrow \overline{z \overline{\omega}} = \overline{z \omega} \]  \hspace{1cm} (9)
Other identities

\[ Re(z) = \frac{1}{2} (z - \bar{z}) \]
\[ Im(z) = \frac{1}{2i} (z - \bar{z}) \]
\[ |Z| = \sqrt{(z\bar{z})} \] (10)

Also, \( z \in \mathbb{C} \) is real if \( \bar{z} = z \), and \( z \) is imaginary if \( \bar{z} = -z \).

Theorem 2.2.5 (Lagrange identity)

Let \( z_1, ..., z_n, w_1, ..., w_n \) be any 2n complex numbers, then,

\[ \left| \sum_{j=1}^{n} z_j w_j \right|^2 = \left( \sum_{j=1}^{n} |z_j|^2 \right) \left( \sum_{j=1}^{n} |w_j|^2 \right) - \sum_{1 \leq j < k \leq n} |z_j \bar{w}_k - z_k \bar{w}_j|^2 \] (11)

Proof:

The identity obviously holds when \( n = 1 \).
Assume it holds when \( n = k \), then when \( n = k + 1 \),

\[ \text{LHS} = \left| \sum_{j=1}^{k} z_j w_j + z_{k+1} w_{k+1} \right|^2 \]
\[ = \left| \sum_{j=1}^{k} z_j w_j \right|^2 + |z_{k+1} w_{k+1}|^2 + \sum_{j=1}^{k} z_j w_j \cdot z_{k+1} w_{k+1} + \bar{z}_{k+1} \bar{w}_{k+1} \cdot \bar{z}_j \bar{w}_j \] (12)

\[ \text{RHS} = \left( \sum_{j=1}^{k} |z_j|^2 + |z_{k+1}|^2 \right) \left( \sum_{j=1}^{k} |w_j|^2 + |w_{k+1}|^2 \right) - \sum_{1 \leq j < k \leq n} |z_j \bar{w}_k - z_k \bar{w}_j|^2 \] (13)

So, we need to show that

\[ \sum_{j=1}^{k} z_j w_j \cdot z_{k+1} w_{k+1} + \bar{z}_{k+1} \bar{w}_{k+1} \cdot \sum_{j=1}^{k} z_j w_j = \]
\[ \left( \sum_{j=1}^{k} |z_j|^2 \right) \cdot |w_{k+1}|^2 + \left( \sum_{j=1}^{k} |w_j|^2 \right) \cdot |z_{k+1}|^2 - \sum_{j=1}^{k} |z_j \bar{w}_{k+1} - z_{k+1} \bar{w}_j|^2 \] . (14)

We can find that
So, the equality holds.

2.3. Visualization of The Complex Plane \( \mathbb{C} \)

The complex plane \( \mathbb{C} \) can be shown as a coordinate plane \( \mathbb{R}^2 \) where \( z = x + iy \) with the coordinate pair \( (x, y) \) due to the existence of a bijection between \( \mathbb{R}^2 \) and \( \mathbb{C} \) [3]. When we view complex numbers as points on the coordinate plane \( \mathbb{R}^2 \), the x-axis is called the real axis, the y-axis is called the imaginary axis, and the coordinate plane is called the complex plane [4, 5].

We show complex number \( 3 + 2i \) on the complex plane as in Figure 1,

![Figure 1 3+2i on the complex plane.](image)

Definition 2.3.1 For \( z \in \mathbb{C} \), \( z = |z|e^{i\theta} \) is called the polar form of \( z \). The real number \( \theta \) is unique up to a multiple of \( 2\pi \), which is called the argument of \( z \), and denoted as \( \text{arg } z \).

Using trigonometric identities, we can check that

\[
e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}
\]

Thus, for \( z \), for \( \mathbb{C} \) we can get

\[
Z \cdot w = |z| \cdot e^{i\text{arg } z} |w| \cdot e^{i\text{arg } w} = |zw| \cdot e^{i(\text{arg } z + \text{arg } w)}
\]

So, multiplication of a complex number \( z \) with a complex number \( w \) consists of scaling \( z \) by \( |w| \) and rotating counterclockwise by \( \text{arg } w \).

Specifically, \( z \rightarrow iz \) corresponds to a counterclockwise rotation of \( \text{arg } i \), since \( \text{arg } i = \frac{\pi}{2} \).

Finally, complex conjugation

\[
\overline{x + iy} = x - iy
\]

corresponds to reflection over the real axis.
2.3.1. Remark: one might object to the existence of $i$ and therefore $\mathbb{C}$.

2.4. Inequalities
Since the absolute value corresponds to distance in the complex place, one obtains the triangle inequality:

$$|z + w| \leq |z| + |w|$$  \hspace{1cm} (19)

This further implies the reverse triangle inequality:

$$||z| - |w|| \leq |z - w|$$  \hspace{1cm} (20)

Indeed, from the triangle inequalities we have

$$|z| = |z - w + w| \leq |z - w| + |w|$$  \hspace{1cm} (21)

So that

$$|z| - |w| \leq |z - w|$$  \hspace{1cm} (22)

Also, on the other hand

$$|w| = |w - z + z| \leq |w - z| + |z|$$  \hspace{1cm} (23)

So that

$$-||z| - |w|| \leq |w| - |z| \leq |w - z| = |z - w|$$  \hspace{1cm} (24)

Hence

$$||Z| - |w|| \leq |z| - |w|$$  \hspace{1cm} (25)

We can also obtain

$$|\text{Re}(z)| \leq |z| \quad \text{and} \quad |\text{Im}(z)| \leq |z|$$  \hspace{1cm} (26)

2.5. Topology of Complex Plane
We establish some notations and terminology for subsets $\Omega \subset \mathbb{C}$

Definition 2.5.1 For $z_0 \in \mathbb{C}$ and $r > 0$, the open disc of radius $r$ centered at $z_0$ is

$$D_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$$  \hspace{1cm} (27)

The closed disc with radius $r$ centered at $z_0$ is

$$\overline{D}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq r\}$$  \hspace{1cm} (28)

The circle of radius $r$ centered at $z_0$ is

$$C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$$  \hspace{1cm} (29)
In this case, we could presume the notation for the radius $r$ or center $z_0$ for any of the above when they are unimportant to the discussion. And finally, we reserve the following notation for the unit disk is represented:

$$\mathbb{D} = D(0) = \{ z \in \mathbb{C} : |z| < 1 \} \quad (30)$$

Now we turn to the topological properties of $\mathbb{C}$. Since the complex plane can be identified with $\mathbb{R}^2$ and has the same notions of distance and convergence, these properties are simply a translation from $\mathbb{R}^2$ to $\mathbb{C}$ [6].

Definition 2.5.2 We say $\Omega \subset \mathbb{C}$ is open if for all $z \in \mathbb{C}$ there exists $r > 0$ such that $D_r(z) \cap \Omega$

So that we could say $\Omega$ is closed if its complement

$$\Omega^c = \mathbb{C} \setminus \Omega \quad (31)$$

is open.

The importance from these examples is to see whether the contour is opened or closed is not a dichotomy, as a set could be open, closed, both, and neither [7]. The one could usually circle whether or not a set is open directly from the definition. The following offers a convenient to circle whether or not a set is closed [8].

Definition 2.5.3 A sequence $\{Z_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ converges to $w \in \mathbb{C}$ if $\lim_{n \to \infty} |z_n - w| = 0$; it can also be written in the following way:

$$\lim_{n \to \infty} z_n = w \quad (32)$$

and we call $\omega$ the limit of the sequence.

In details, $\{Z_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ converges to $w$ if

$$\forall \epsilon > 0 \exists m \in \mathbb{N} \quad (33)$$

So that,

$$\forall n \geq m \ |z_n - w| < \epsilon \quad (34)$$

Since

$$|\text{Re}(z)|, \quad |\text{Im}(z)| \leq |z| = (\text{Re}(z)^2 + \text{Im}(z)^2)^{\frac{1}{2}} \quad (35)$$

One has

$$\lim_{n \to \infty} z_n = w \iff \lim_{n \to \infty} \text{Re}(z_n) = w \quad \text{and} \quad \lim_{n \to \infty} \text{Im}(z_n) = w \quad (36)$$

Thus, this notion of convergence agrees with the visual one on $\mathbb{R}^2$

Definition 2.5.4 $\{Z_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ is a Cauchy sequence if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ So that $\forall_{n,m \in \mathbb{N}}$ has

$$|z_n - z_m| < \epsilon \quad (37)$$

In the other words, the terms in a Cauchy sequence are eventually as close as we would want.

Theorem 2.5.5 The complex number field $\mathbb{C}$ is complete.

Proof:

Let $\{z_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ be a Cauchy sequence. Using $|\text{Re}(z)|, |\text{Im}(z)| \leq |z|$, we can easily see that $\{\text{Re}(z_n)\}_{n \in \mathbb{N}}, \{\text{Im}(z_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}$ are Cauchy sequences. From the real analysis course, we know that $\mathbb{R}$ is complete, so the real and imaginary part can converge to some $x, y \in \mathbb{R}$, respectively.
Define \( w = x + i \cdot y \). Then,

\[
\lim_{n \to \infty} |z_n - w| = \lim_{n \to \infty} ((\text{Re}(z_n) - x)^2 + (\text{Im}(z_n) - y)^2)^{\frac{1}{2}} = 0.
\]  

Thus \((z_n)_{n \in \mathbb{N}}\) can be converted to \( w \).

Proposition 2.5.6 A set \( \Omega \subset \mathbb{C} \) is closed if and only for any convergent sequence \((z_n)_{n \in \Omega}\) one has \( \lim_{n \to \infty} z_n \in \Omega \).

Proof:

Assume \( \Omega \) is closed and let \((z_n)_{n \in \mathbb{N}} \in \Omega \) be a convergent sequence. Denote \( w = \lim_{n \to \infty} z_n \).

If \( w \notin \Omega \), then since \( \Omega^c \) is open there exists \( r > 0 \) so that \( Dr(w) \subset \Omega^c \). Let \( N \in \mathbb{N} \) be such that for all \( N \in \mathbb{N} \), \( |z_n - w| < r \). But then \( z_n \in Dr(w) \cap \Omega^c \), and \( z_n \in \Omega \). Thus, we will have \( w \in \Omega \).

We will show \( \Omega^c \) is open. Let \( w \in \Omega^c \) and assume, towards a contradiction, that for all \( r > 0 \), \( Dr(w) \subset \Omega \). Therefore for all \( N \in \mathbb{N} \) there exists \( z_n \in Dr(w) \cap \Omega \). But then \((z_n)_{n \in \mathbb{N}} \cap \Omega \) converges to \( w \in \Omega^c \), contradicting our hypothesis.

For any \( \Omega \subset \mathbb{C} \) there are always important open and closed sets associated with it.

Definition 2.5.7 Let \( \Omega \subset \mathbb{C} \).

We say \( z \in \mathbb{C} \) is an interior point of \( \Omega \) if there exists \( r > 0 \) such that \( Dr(z) \subset \Omega \). The interior of \( \Omega \) is the set of all of its interior points and is denoted \( \Omega^0 \).

We say \( z \) is a limit point of \( \Omega \) if there exists a sequence \((z_n)_{n \in \mathbb{N}} \subset \Omega \setminus \{ z \} \) with \( \lim_{n \to \infty} z_n = z \). The closure of \( \Omega \) is the union of \( \Omega \) and all of its limit points and is denoted \( \overline{\Omega} \).

The boundary of \( \Omega \) is \( \partial \Omega = \overline{\Omega} \setminus \Omega^0 \).

It follows that \( \Omega \subset \mathbb{C} \) is open if \( \Omega = \Omega^0 \), and \( \Omega \) is closed if \( \Omega = \overline{\Omega} \). In particular, \( Dr(z) = Dr(z)^0 \) and \( \overline{Dr(z)} = \overline{Dr(z)} \) for all \( z \in \mathbb{C} \) and \( r > 0 \). We also note that

\[
\overline{Dr(z)} = Dr(z) \hspace{1cm} \overline{Dr(z)^0} = Dr(z) \hspace{1cm} \partial Dr(z) = \partial \overline{Dr(z)} = c_r(z) \]

(39)

Denote \( \mathbb{Q} + i\mathbb{Q} \) be the set \( \{ x + iy : x, y \in \mathbb{Q} \} \).

One has \((\mathbb{Q} + i\mathbb{Q})^0 = \emptyset \), \( \overline{\mathbb{Q} + i\mathbb{Q}} = \partial (\mathbb{Q} + i\mathbb{Q}) = \mathbb{C} \).

Definition 2.5.9 We say \( \Omega \subset \mathbb{C} \) is bounded if there exists \( R \) such that \( |z| \leq R \) for all \( z \in \Omega \); If \( \Omega \) is bounded, its diameter is the quantity

\[
\text{Diam}(\Omega) = \sup_{z,w \in \Omega} |z - w| \]

(40)

Definition 2.5.8 We say \( \Omega \) is compact if it is closed and bounded.

This might be different from the definition of compactness you are used to (which involves “open covers”), but in the context of the complex plane (i.e., \( \mathbb{R}^2 \)) they are equivalent.

Definition 2.5.9 For \( \Omega \subset \mathbb{C} \), an open cover is a collection \( \{ U_i \subset \Omega : i \in I \} \) of open sets satisfying \( \Omega \subset \bigcup_{i \in I} U_i \).

A subcover is a subcollection, \( \{ U_i : i \in J \} \) for \( J \subset I \), that still covers \( \Omega \).

Theorem 2.5.10 If every open cover \( \{ U_i \subset \mathbb{C} : i \in I \} \) of \( \Omega \) has a finite subcover \( U_{i_1}, U_{i_2}, U_{i_3}, \ldots, U_{i_n} \), then \( \Omega \) is compact.

Proof:

Choose any point \( p \in \Omega \). Considering \( B_m(r)(m = 1, 2, 3, \ldots) \), we have \( \{B_m(r), m = 1, 2, \ldots \} \) is a cover of \( \Omega \). Since every open cover has a subcover, it is obvious that \( \Omega \) is bounded.
In order to show that $\Omega$ is closed, it suffices to show the complement of $\Omega$ is open. That is for any point in $\Omega^c$, there is an open neighborhood of this point that is contained in $\Omega^c$ [9, 10]. Choose any point $p'$, which belongs to the complement of $\Omega$. Considering open sets $B_{\frac{1}{m}}(p)$, we have $\bigcup_m B_{\frac{1}{m}}(p) = \mathbb{C} - p'$. Thus $\{B_{\frac{1}{m}}(p)\}$ is an open cover of $\Omega$. Then there exists a finite open subcover $\{B_{\frac{1}{m}}(p')\; m = 1,2,\ldots, M\}$ for $\Omega$. Notice that the interior of the set $\cap_{i=1}^{M} B_{\frac{1}{m}}(p')$ is just an open neighborhood of $p'$ and is contained in $\Omega^c$. Thus the set $\Omega$ is closed.

3. Conclusion
We discuss both the computational and topological properties of complex numbers, also present the definition of the complex field and prove some basic properties, with which we prove the Lagrange Identity. Furthermore, we demonstrate the geometric meaning of complex numbers and discuss the topological properties of the complex plane. In the future, we will report more on equivalent characterizations of compactness of metric space.

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