GLOBAL SOLUTION FOR A COUPLED PARABOLIC SYSTEM WITH DEGENERATE COEFFICIENTS AND TIME-WEIGHTED SOURCES

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ABSTRACT. In this paper, we obtain the so-called Fujita exponent to the following parabolic system with time-weighted sources and degenerate coefficients

\[ u_t - \text{div}(\omega(x)\nabla u) = t^r v^p \quad \text{and} \quad v_t - \text{div}(\omega(x)\nabla v) = t^s u^p \]

in \( \mathbb{R}^N \times (0, T) \) with initial data belonging to \( [L^\infty(\mathbb{R}^N)]^2 \). Where \( p, q > 0 \) with \( pq > 1 \); \( r, s > -1 \); and either \( \omega(x) = |x|^a \), or \( \omega(x) = |x|^b \) with \( a, b > 0 \).

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1. INTRODUCTION

Many problems that emerge in several branches of science are associated with elliptic and parabolic partial differential equations, which present a diffusion operator of the form \( \text{div}(\omega(x)\nabla \cdot) \). Where \( \text{div} \) is the divergent, \( \nabla \) is the gradient, and the spatial function \( \omega : \mathbb{R}^N \rightarrow [0, \infty) \) is a weight representing the part of thermal diffusion, which can degenerate (see, e.g., \([2, 10, 14, 15, 13, 21, 25, 37, 26, 32, 33, 41, 42, 43]\) and the references therein). Several authors have extensively studied models related to those problems, and the literature is well known. For example, see the works of Kohn and Nirenberg \([28]\); Fabes, Kenig, and Serapioni \([17]\); Gutierrez and Nelson \([22]\); Fujishima, Kawakami, and Sire \([18]\); Dong and Phan \([12]\); Sire, Terracini, and Vita \([30]\).
We are interested in the following degenerate parabolic problem with time-weighted sources.

\[
\begin{align*}
    u_t - \text{div}(\omega(x)\nabla u) &= t^ru^p & \text{in } \mathbb{R}^N \times (0,T), \\
    v_t - \text{div}(\omega(x)\nabla v) &= t^st^q & \text{in } \mathbb{R}^N \times (0,T), \\
    u(0) &= u_0 & \text{in } \mathbb{R}^N, \\
    v(0) &= v_0 & \text{in } \mathbb{R}^N,
\end{align*}
\]

(1.1)

where \((u_0, v_0) \in L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N) \equiv [L^\infty(\mathbb{R}^N)]^2; u_0, v_0 \geq 0; p, q > 0 \text{ with } pq > 1;\)

\(r, s > -1;\) and the weighted function \(\omega : \mathbb{R}^N \to [0, \infty)\) either

\(A\quad \omega(x) = |x|^a \quad \text{with } a \in [0, 1) \text{ if } N = 1, 2; \text{ and } a \in [0, 2/N) \text{ if } N \geq 3, \text{ or}\)

\(B\quad \omega(x) = |x|^b \quad \text{with } b \in [0, 1).\)

Note that the function \(\omega\) with these characteristics belongs to the Muckenhoupt class of functions \(\mathcal{A}_{1+\frac{2}{N}},\) and the operator \(\text{div}(\omega(x)\nabla \cdot)\) is not self-adjoint (Fujishima et al. in [18] p.6) comment on this particularity).

When \(\omega(x) = |x|^a\), it admits a line of singularities; thus, the problem (1.1) is related to the fractional Laplacian through the Caffarelli-Silvestre extension, see [4], [34], and [18]. The fractional Laplacian is associated with nonlocal diffusion and appears in the Levy diffusion process; for example, see [11] [27].

In [18], Fujishima et al. studied the following problem

\[
\begin{align*}
    W_t - \text{div}(\omega(x)\nabla W) &= W^p & \text{in } \mathbb{R}^N \times (0,T), \\
    W(0) &= W_0 & \text{in } \mathbb{R}^N,
\end{align*}
\]

(1.2)

and obtained the following Fujita exponent

\[p^*(\alpha) = 1 + \frac{2 - \alpha}{N},\]

where \(\alpha = a\) in case (A) and \(\alpha = b\) in case (B).

Several authors extensively studied problem (1.2) when \(\omega(x) = 1\). Hirose Fujita pioneered the approach of associating a critical exponent with the global existence of solutions [19]. Specifically, he showed that if \(1 < p < p^*(0)\), then problem (1.2) does not admit any non-negative global solution. For \(p > p^*(0)\), there exist both global and nonglobal solutions, depending on the size of the initial data; see also [29] [34] for more details. In the critical case \(p = p^*(0)\), Hayakawa [23] (when \(N = 1, 2\)), and later Aronson and Weinberger [3] (when \(N \geq 3\)) showed that problem (1.2) has no global solution.

When \(\omega(x) = 1\) and \(r = s = 0\), problem (1.1) was studied by Escobedo and Herrero [16]; they showed that

\[(pq)^* = 1 + \frac{2}{N}(\max\{p, q\} + 1)\]

is the Fujita exponent for the problem (1.1). That means that if \(pq > (pq)^*\) then any non-trivial nonnegative solution blows up in finite time, and when \(pq \leq (pq)^*\), there exist both global and nonglobal solutions.

Later, when \(\omega(x) = 1\) and the time-weighted sources are \((1 + t)^r\) and \((1 + t)^s\) instead of \(t^r\) and \(t^s\), respectively, Cao et al. [7] showed the existence of the following

\footnote{The Muckenhoupt classes \(A_p\), with \(p > 1\), is defined as the class of locally integrable non-negative functions \(w\) that satisfies \(\left(\int_Q wdx\right)\left(\int_Q w^{\frac{1}{p-1}}\right) < K\) for every cube \(Q\) and some constant \(K\).}
Fujita exponent
\[(pq)^* = 1 + \frac{2\max\{(r + 1)q + s + 1, (s + 1)p + r + 1\}}{N}\]
for the problem (1.1). See also [5], [6], [24] and the references therein for other related results.

The main contribution of the current work is to guarantee the existence of the so-called Fujita exponent for the problem (1.1); for this purpose, we adapted the approach used in [18] and [16]. Nevertheless, difficulties inherent to the degenerate coupled system (1.1) appear, and the case \(pq > 1\) with \(0 < p < 1\) (or \(0 < q < 1\)) merit more effort. Also, note that when \(p = q > 1\), \(r = s = 0\), and \(u = v\), (1.1) is reduced to problem (1.2) studied recently in [18].

We want to mention that our approach can be applied to study the critical Fujita exponent of the following coupled systems:

\[
\begin{align*}
\frac{du_i}{dt} - \text{div}(\omega(x)\nabla u_i) &= t^{r_i}u_{i+1}^q \quad i = 1, \ldots, m-1 \quad \text{in } \mathbb{R}^N \times (0, T), \\
\frac{du_m}{dt} - \text{div}(\omega(x)\nabla u_m) &= t^{r_m}u_{m+1}^q \quad \text{in } \mathbb{R}^N \times (0, T),
\end{align*}
\]

and

\[
\begin{align*}
\frac{dv}{dt} - \text{div}(\omega(x)\nabla v) &= t^{r_1}u^p + t^{r_2}v^q \quad \text{in } \mathbb{R}^N \times (0, T), \\
v(t) &= S(t)v_0 + \int_0^t S(t-\sigma)\sigma^p u^q d\sigma,
\end{align*}
\]

When \(\omega(x) = 1\), problem (1.3) was studied in [36], [39] and [6], whereas that (1.4) was studied in [8], [38], and [5].

Solutions to the problem (1.1) are understood in the following sense.

**Definition 1.5.** Let \(u\) and \(v\), a.e. finite, measurable functions in \(\mathbb{R}^N \times (0, T)\) for some \(T > 0\). Then we call that \((u, v)\) is a solution of (1.1), if \((u, v) \in L^\infty((0, T); L^\infty(\mathbb{R}^N)) \times L^\infty((0, T); L^\infty(\mathbb{R}^N))\) and satisfies

\[
\begin{align*}
u(t) &= S(t)v_0 + \int_0^t S(t-\sigma)\sigma^p u^q d\sigma,
\end{align*}
\]

for almost \(x \in \mathbb{R}^N\) and \(t > 0\). If \(T = \infty\), then we say that \((u, v)\) is a global-in-time solution. Where \(S(t)\phi(x) := \int_{\mathbb{R}^N} \Gamma(x, y, t)\phi(y)dy\), here \(\Gamma(x, y, t)\) is the fundamental solution of (2.1).

In what follows, we consider the following values

\[
\begin{align*}
\gamma_1 &= \frac{(r + 1) + (s + 1)p}{pq - 1} \\
\gamma_2 &= \frac{(s + 1) + (r + 1)q}{pq - 1} \\
r_{1*} &= \frac{N}{(2 - \alpha)\gamma_1} \\
r_{2*} &= \frac{N}{(2 - \alpha)\gamma_2}.
\end{align*}
\]
Our main result is the following.

**Theorem 1.11.** Let \( r, s > -1, \ p, q > 0, \ p \cdot q > 1, \) and the values (1.7)-(1.10). Suppose \( \alpha = a \) in the case that \( \omega \) satisfies the condition (A), and \( \alpha = b \) in the case that \( \omega \) satisfies the condition (B).

(i) If \( \gamma := \max \{ \gamma_1, \gamma_2 \} \geq \frac{N}{2 - \alpha}, \) then problem (1.1) has no nontrivial global-in-time solutions.

(ii) If \( \gamma := \max \{ \gamma_1, \gamma_2 \} < \frac{N}{2 - \alpha}, \) then there exists nontrivial global-in-time solutions to (1.1). Also, there exists a constant \( \delta > 0 \) such that for any

\[
(u_0, v_0) \in [L^\infty(\mathbb{R}^N) \cap L^{r_1, \infty}(\mathbb{R}^N)] \times [L^\infty(\mathbb{R}^N) \cap L^{r_2, \infty}(\mathbb{R}^N)]
\]

with

\[
\max \{ \| u_0 \|_{r_1, \infty}, \| v_0 \|_{r_2, \infty} \} < \delta,
\]

problem (1.1) has a global-in-time solution \((u, v)\) satisfying:

\[
\sup_{t > 0} (1 + t)^{\frac{N}{r} \left( \frac{1}{r_1} + \frac{1}{r_2} - 1 \right)} \| u(t) \|_{\mu, \infty} < \infty
\]

and

\[
\sup_{t > 0} (1 + t)^{\frac{N}{r} \left( \frac{1}{r_1} + \frac{1}{r_2} - 1 \right)} \| v(t) \|_{\mu, \infty} < \infty
\]

for all \( 0 < \mu \) such that \( \max \{ r_1, r_2 \} < \mu \leq \infty. \)

**Remark 1.12.** Here are some comments on Theorem 1.11.

(i) When \( \alpha = 0, \) Theorem 1.11 coincides with result in [7] Theorem 1.

(ii) When \( \alpha = 0 \) and \( r = s = 0, \) this Theorem agrees with the results that appear in [16].

(iii) This result shows the existence of the critical value of Fujita and is given by

\[
(pq)^* (\alpha) = 1 + \frac{(2 - \alpha) \max \{(s + 1)p + r + 1, (r + 1)q + s + 1\}}{N}
\]

The work is organized in the following way. In section 2, we present the necessary preliminaries. In section 3, we prove the non-global existence. Finally, in section 4, we demonstrate global existence.

2. Preliminaries and Toolbox

In that follows, \( C \) denotes a generic positive constant that may vary in different places, and its change is not essential to the analysis.

The positive part of \( \phi(x) \) is defined by

\[ \phi^+(x) = \max\{ \phi(x), 0 \}. \]

The negative part of \( \phi \) is defined analogously.

Here \( x_1 \) is the first coordinate of \( x = (x_1, ..., x_N) \in \mathbb{R}^N, \) and \( \| \cdot \| \) is the Euclidean norm of \( \mathbb{R}^N. \) The spaces \( L^\infty(\mathbb{R}^N) \) and \( L^\xi(\mathbb{R}^N)(\xi \geq 1) \) are defined as usual, and its norms are denoted by \( \| \cdot \|_{\infty} \) and \( \| \cdot \|_{\xi}, \) respectively.

The Lorentz space \( L^{\xi, \infty}(\mathbb{R}^N), \xi > 1, \) is defined as follows

\[
L^{\xi, \infty} := \left\{ \psi : \mathbb{R}^N \to \mathbb{R}, \| \psi \|_{L^{\xi, \infty}(\mathbb{R}^N)} = \sup_{\rho > 0} \mu \left\{ x \in \mathbb{R}^N : |\psi(x)| > \rho \right\} ^{1/\xi} < \infty \right\},
\]
where \( \mu \) is Lebesgue measure \( \mathbb{R}^N \) (see [20]).

We will denote by \( \Gamma := \Gamma(x,y,t) \) the fundamental solution of the following homogeneous problem

\[
W_t - \text{div}(\omega(x)\nabla W) = 0
\]

in \( \mathbb{R}^N \times (0,T) \), with a pole at point \((y,0)\), with \( \omega \) fulfilling either \((A)\) or \((B)\). Since \( \omega(x) \) belongs to the class \( A_1+2\mathbb{R}^N \) of Muckenhoupt functions (see, e.g. [31]), we have that the fundamental solution \( \Gamma = \Gamma(x,y,t) \) verifies the following properties (for more details, see [22] and [18]):

\[
\begin{align*}
(K_1) & \quad \int_{\mathbb{R}^N} \Gamma(x,y,t)dx = \int_{\mathbb{R}^N} \Gamma(x,y,t)dy = 1 \text{ for } x,y \in \mathbb{R}^N \text{ and } t > 0; \\
(K_2) & \quad \Gamma(x,y,t) = \int_{\mathbb{R}^N} \Gamma(x,\xi, t-s)\Gamma(\xi, y, s)d\xi \text{ for } x,y \in \mathbb{R}^N \text{ and } t > s > 0; \\
(K_3) & \quad \text{Suppose that } c_0 := \sup_Q \left( \frac{1}{|Q|} \int_Q \omega(x)dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{-1}dx \right) < \infty, \text{ where the supremum is taken over all cubes } Q \in \mathbb{R}^N, \text{ and } \\
& \quad h_x(r) = \left( \int_{B_r(x)} \omega(y)^{-\frac{N}{2}}dy \right)^{\frac{1}{N}}.
\end{align*}
\]

Then there exist constants \( C_0, c_0 > 0 \), depending only on \( N \) and \( c_0 \), such that

\[
\begin{align*}
C_0^{-1} \left( \frac{1}{|h_x^*(t)|^N} + \frac{1}{|h_y^*(t)|^N} \right) e^{-c_0 \left( \frac{h_x^*(|x-y|)}{t} \right)^{\frac{1}{N}}} & \leq \Gamma(x,y,t) \\
& \leq C_0^{-1} \left( \frac{1}{|h_x^*(t)|^N} + \frac{1}{|h_y^*(t)|^N} \right) e^{-c_0 \left( \frac{h_x^*(|x-y|)}{t} \right)^{\frac{1}{N}}}
\end{align*}
\]

for \( x,y \in \mathbb{R}^N, t > 0, \) and \( \alpha \in \{a,b\} \). Where \( h_x^{-1} \) denotes the inverse function of \( h_x \).

Also, by estimates (2.11)-(2.12) in [18], we have

\[
(K_4)
\int_{|x| \leq t^{\frac{1}{2-N}}} \Gamma(x,y,t)dx \geq C,
\]

for all \( |y| \leq t^{\frac{1}{2-N}} \), and some constant \( C > 0 \).

\[
(K_5)
\Gamma(x,y,t) \geq Ct^{-\frac{N}{2-N}},
\]

for \( |x|, |y| \leq t^{\frac{1}{2-N}}, t > 0, \) and some constant \( C > 0 \).

Remark 2.2. Notice that a consequence of property \((K_3)\) is the nonnegativity of the fundamental solution \( \Gamma \).

We will use the following results to show the global-in-time existence of the solutions to (1.1).

Proposition 2.3 ([18]).
Proposition 2.4 Let $\phi \in L^n(\mathbb{R}^N)$ and $1 \leq q_1 \leq q_2 \leq \infty$, then
\[
\|S(t)\phi\|_{q_2} \leq c_1 t^{-\frac{N}{q_2}}\left(\frac{1}{q_1} - \frac{1}{q_2}\right)\|\phi\|_{q_1}, \ t > 0.
\]
Where the constant $c_1$ can be taken so that it depends only on $N, \alpha \in \{a, b\}$.

Lemma 2.6 Assume now that $0 < p < 1$. Since $u_0$ is a nonnegative function, the conclusion follows as the anterior case replacing $p$ by $1/p$. \hfill \Box
3. Nonglobal Existence

We need the following Proposition to prove the first part of our main result.

Proposition 3.1. Assume that \( \omega \) satisfies either (A) or (B), and \( u_0, v_0 \in L^\infty(\mathbb{R}^N) \) with \( u_0, v_0 \geq 0 \). Suppose that \((u, v) \in \left[ L^\infty((0, T), L^\infty(\mathbb{R}^N)) \right]^2 \) is a solution of (1.1) with \( 0 < T \leq \infty \), and \( p, q > 0 \) with \( pq > 1 \). Then, there exists a constant \( C^* \geq 0 \) (which depends only on \( p, q, r, \) and \( s \)), such that

\[
\begin{align*}
    \frac{t (r+1)+(s+1)p}{pq-1} \| S(t)u_0 \|_\infty &\leq C^* \quad \text{if } q > 1 \\
    \frac{t (r+1)+(s+1)p}{pq-1} \| S(t)u_0^q \|_\infty &\leq C^* \quad \text{if } 0 < q < 1, \\
    \frac{t (r+1)+(s+1)p}{pq-1} \| S(t)v_0 \|_\infty &\leq C^* \quad \text{if } p > 1 \\
    \frac{t (r+1)+(s+1)p}{pq-1} \| S(t)v_0^p \|_\infty &\leq C^* \quad \text{if } 0 < p < 1.
\end{align*}
\]

for all \( t \in [0, T) \).

Proof. We will first prove the first inequality in (3.2). To do this, we will show the following estimate

\[
u(t) \geq C_k t^{(\beta_k - 1)\gamma_1} [S(t)u_0]^{\beta_k}
\]

for all \( t \in (0, T) \) and \( k \in \mathbb{N} \). Where \( C_0 = 1, \beta = pq, \gamma_1 = \frac{(r+1)+(s+1)p}{pq-1} \) and

\[
C_k = C_k^{\beta_k} \left[ ((\beta_k - 1)q\gamma_1 + s + 1)^-p[ (\beta_k - 1)\gamma_1 \beta + p(s + 1) + (r + 1) ] \right]^{-1},
\]

for \( k \in \mathbb{N} \). Indeed, we argue by induction. From (1.6) and property \((K_3)\), we get that \( u(t) \geq S(t)u_0 \), for \( t > 0 \), thus (3.3) holds for \( k = 0 \). Now, assume that (3.3) holds for \( k \geq 1 \), then from (1.6), \((K_1), (K_2), (K_3)\), and Lemma 2.7, we have

\[
v(t) \geq \int_0^t S(t - \sigma) \sigma^a[v(\sigma)]^q d\sigma
\]

\[
\geq \int_0^t S(t - \sigma) \sigma^a \left[ C_k \sigma^{(\beta_k - 1)\gamma_1} [S(\sigma)u_0]^{\beta_k} \right]^q d\sigma
\]

\[
\geq C_k^q \int_0^t \sigma^{(\beta_k - 1)\gamma_1 q + s} S(t - \sigma)[S(\sigma)u_0]^{q \beta_k} d\sigma
\]

\[
\geq C_k^q [S(t)u_0]^{q \beta_k} \int_0^t \sigma^{(\beta_k - 1)\gamma_1 q + s} d\sigma
\]

\[
= C_{k,1} t^{(\beta_k - 1)\gamma_1 q + s + 1} [S(t)u_0]^{q \beta_k}
\]

for \( t > 0 \), where \( C_{k,1} = C_k^q / ((\beta_k - 1)\gamma_1 q + s + 1) \). Similarly, from (3.5), we obtain

\[
u(t) \geq \int_0^t S(t - \sigma) \sigma^a[v(\sigma)]^p d\sigma
\]

\[
\geq \int_0^t S(t - \sigma) \sigma^a \left[ C_{k,1} \sigma^{(\beta_k - 1)\gamma_1 q + s + 1} [S(\sigma)u_0]^{q \beta_k} \right]^p d\sigma
\]

\[
\geq C_{k,1}^p [S(t)u_0]^{q \beta_k + 1} \int_0^t \sigma^{(\beta_k - 1)\gamma_1 \beta + (s + 1)p + r} d\sigma
\]

\[
= C_{k,2} t^{(\beta_k - 1)\gamma_1 q + (s + 1)p + (r + 1)} [S(t)u_0]^{q \beta_k + 1}
\]
for \( t > 0 \), where \( C_{k,2} = C_{k,1}^{p}/[(\beta^{k} - 1)\gamma_{1}\beta + (s + 1)p + (r + 1)] \). Since that

\[
(\beta^{k} - 1)\gamma_{1}\beta + (s + 1)p + (r + 1) = (\beta^{k+1} - 1)\gamma_{1},
\]

we have

\[
u(t) \geq C_{k,2}t^{(\beta^{k+1} - 1)\gamma_{1}}[S(t)u_{0}]^{\beta^{k+1}},
\]

for \( t > 0 \). Also, denoting \( C_{k+1} = C_{k,2} \) and inserting the value of \( C_{k,1} \), we get (3.4).

Now we to show that there exists \( \kappa_{0} > 0 \) such that \( C_{k} \geq \kappa_{0}^{\beta_{k}} \) for all \( k \geq 2 \). Let \( \theta_{k} = -\beta^{-k}\ln(C_{k}) \), it is sufficient to prove that the sequence \( \{\theta_{k}\}_{k \in \mathbb{N}} \) is limited from above. From (3.4), we have

\[
\theta_{i} - \theta_{i-1} = -\beta^{-i}\ln\left(\frac{C_{i}^{\gamma - 1}}{C_{i-1}^{\gamma - 1}}\right)
= -\beta^{-i}\ln\left(\frac{[(\beta^{i-1} - 1)\gamma_{1} + s + 1]^{p}[(\beta^{i} - 1)\gamma_{1}\beta + p(s + 1) + (r + 1)]}{(\beta^{i} - 1)\gamma_{1}\beta + p(s + 1) + (r + 1)}\right)
\leq -\beta^{-i}\ln\left(\begin{cases} [\gamma_{1}(\beta^{i} - 1)]^{p+1} & \text{if } p > 1, \\ q[\gamma_{1}(\beta^{i} - 1)]^{2} & \text{if } 0 < p \leq 1 \end{cases}\right)
\leq C\beta^{-i}(i + 1),
\]

this implies \( \theta_{k} - \theta_{1} = \sum_{i=1}^{k}(\theta_{i} - \theta_{i-1}) \leq C\sum_{i=1}^{k}\beta^{-i}(i + 1) < \infty \). Thus, from this and (3.3) we have

\[
u(t)^{1/\beta} \geq \kappa_{0}t^{\gamma_{1}(1-1/\beta^{k})}S(t)u_{0},
\]

for all \( t \in (0, T) \). Since \( \beta > 1 \), letting \( k \to \infty \), we get the first inequality in (3.2).

We argue similarly to the above case to prove the second inequality of (3.2). We apply Lemma 2.7 iteratively, starting with \( v(t) \geq t^{\gamma_{1}}S(t)u_{0}^{\gamma} \), until getting the following inequality

\[
u(t) \geq D_{k}t^{(\beta^{k} - 1)\gamma_{1}}[S(t)u_{0}^{p}]^{\beta^{k-1}}
\]

for all \( t \in (0, T) \) and \( k \in \mathbb{N} \). Where \( \beta = pq, \gamma_{1} = \frac{(r+1)(s+1)}{pq-1}, \) and \( D_{k} \geq \eta_{1}^{\beta_{k}} (\eta_{1} > 0) \). So, from (3.6), we get

\[
u(t)^{q/\beta} \geq \eta_{1}t^{\gamma_{1}(1-1/\beta^{k})}S(t)u_{0}^{q},
\]

for all \( t \in (0, T) \) and some positive constant \( \eta_{1} \). Then, letting \( k \) tend to infinity, we obtain the second inequality of (3.2).

By symmetry, the other inequalities follow immediately. \( \square \)

The following Corollary is a direct consequence of the above Proposition

**Corollary 3.7.** Assume that \( \omega \) satisfies either (A) or (B), and \( u_{0}, v_{0} \in L^{\infty}(\mathbb{R}^{N}) \) with \( u_{0}, v_{0} \geq 0 \). If \( (u, v) \in \left[L^{\infty}((0, \infty), L^{\infty}(\mathbb{R}^{N}))\right]^{2} \) is a global-in-time solution of (1.1). Then, there exists a constant \( C^{**} > 0 \) (which depends only on \( p, q, r, \) and \( s \)),
such that
\[
\begin{align*}
& t^{\frac{(r+1)+(x+1)p}{p-1}} \|S(t)u(t)\|_\infty \leq C^{**} \quad \text{if } q > 1, \\
& t^q \|S(t)u(t)^q\|_\infty \leq C^{**} \quad \text{if } 0 < q < 1, \\
& t \left(\frac{(r+1)+(x+1)q}{p-1}\right) \|S(t)v(t)\|_\infty \leq C^{**} \quad \text{if } p > 1, \\
& t^p \|S(t)v(t)^p\|_\infty \leq C^{**} \quad \text{if } 0 < p < 1.
\end{align*}
\]
for all \( t \in (0, \infty) \).

**Proof.** Since \((u,v) \in [L^\infty((0,\infty), L^\infty(\mathbb{R}^N))]^2\) is a global-in-time solution of (1.1), then \((u(t+\sigma), v(t+\sigma))\) for \( t > 0 \) and for \( \sigma > 0 \) is solution for problem (1.1) with initial condition \((u(\sigma), v(\sigma))\). Thus, the estimate (3.2) with \((u(\sigma), v(\sigma))\) instead of \((u_0, v_0)\) is hold. Therefore the result follows by taking \( \sigma = t \) in this estimate. \( \square \)

**Proof of Nonglobal Existence (Theorem 1.11 - (i)).** Without loss of generality, we can assume that \( \gamma_1 = \gamma \). Thus, we have two cases:

**Case I :** \( q > 1 \). We argue by contradiction. Suppose that there exists \((u,v)\), a non-trivial global-in-time solution of (1.1) with initial condition \((u_0, v_0)\), thus, \( u_0 \) or \( v_0 \) is a non-trivial function. Suppose that \( u_0 \neq 0 \), thus by Lemma 2.6 and arguing as the proof of Proposition 3.1, we have
\[
\begin{align*}
& u(t) \geq [S(t)u_0] > 0 \quad \text{and} \quad v(t) \geq (s + 1)^{-1}[S(t)u_0]t^{s+1} > 0,
\end{align*}
\]
for \( t > 0 \).

Let \( w(t) := u(t + \tau) \) and \( z(t) := v(t + \tau) \) for \( t \geq 0 \) and some \( \tau \geq 1 \). Note that, (3.8) implies that \( w(0) \neq 0, z(0) \neq 0 \). Since \((w, z)\) satisfies (1.0) with initial condition \((w(0), z(0)) = (u(\tau), v(\tau))\), then by Proposition 5.1 we have
\[
\begin{align*}
& t^{\frac{(r+1)+(x+1)p}{p-1}} \|S(t)w(0)\|_\infty \leq C^*,
\end{align*}
\]
for all \( t \geq 0 \).

We can find a non-trivial function \( 0 \leq U_1 \in L^\infty(\mathbb{R}^N) \) such that \( \text{supp } U_1 \subset B(t_0^{-\frac{1}{\alpha}}) \) (the ball of center 0 and radius \( t_0^{-\frac{1}{\alpha}} \)) for some \( t_0 \geq 1 \), and \( 0 \leq U_1 \leq w(0) \). By Lemma 2.6, we have
\[
\begin{align*}
& S(t)U_1(x) \geq CMt^{-\frac{N}{\gamma_1}}, \quad M := \int_{B(t_0^{-\frac{1}{\alpha}})} U_1(y)dy, \quad \text{for } t \geq t_0, |x| \leq t_0^{-\frac{1}{\alpha}}, \text{ and } C > 0.
\end{align*}
\]
Let us first assume that \( \gamma_1 > \frac{N}{\alpha} \). From (3.10) and \((K_3)\), we have
\[
\begin{align*}
& t^{\gamma_1} \|S(t)w(0)\|_\infty \geq t^{\gamma_1} \|S(t)U_1\|_\infty \geq CT^{\gamma_1-\frac{N}{\gamma_1}},
\end{align*}
\]
for all \( t \geq t_0 \). But this contradicts (3.9).

Now consider \( \gamma_1 = \frac{N}{\alpha} \). Computing similarly as in (3.9), we have
\[
\begin{align*}
& z(t) \geq Ct^{s+1}[S(t)w(0)]^q, \quad \text{for } t > 0 \text{ and some constant } C > 0.
\end{align*}
\]
On the other hand, from (3.10), we have
\[
\begin{align*}
& [S(t)w(0)](x) \geq Ct^{-\frac{N}{\gamma_1}} = Ct^{-\gamma_1}, \quad \text{for } t \geq t_0, \text{ and } |x| \leq t_0^{-\frac{1}{\alpha}}.
\end{align*}
\]
Note that, \( t + 1 - \sigma \leq t \) and \( \sigma \leq t + 1 - \sigma \) for \( 1 \leq \sigma \leq t/2 \). Thus, from (1.6), \((K_1), (K_4), (K_5), (3.11)\), and \((3.12)\), we get

\[
\int_{|x| \leq (t+1)^{1/\alpha}} w(x, t+1)dx \\
\geq \int_{|x| \leq t} w(x, t+1)dx \\
\geq \int_{|x| \leq t} \int_{|y| \leq (t+1-\sigma)^{1/\alpha}} \sigma^r \Gamma(x, y, t+1-\sigma) z(y, \sigma)^p dyd\sigma dx \\
\geq \int_{t}^{\frac{t}{2}} \int_{|y| \leq (t+1-\sigma)^{1/\alpha}} \sigma^r \left( \int_{|x| \leq (t+1-\sigma)^{1/\alpha}} \Gamma(x, y, t+1-\sigma) dx \right) z^p dyd\sigma \\
\geq C \int_{t}^{\frac{t}{2}} \int_{|y| \leq (t+1-\sigma)^{1/\alpha}} \sigma^r (\sigma^{s+1} |S(\sigma) w(0)|^q)^p dyd\sigma \\
\geq C \int_{t}^{\frac{t}{2}} \int_{|y| \leq (t+1-\sigma)^{1/\alpha}} \sigma^{r+(s+1)p} |S(\sigma) w(0)|^{\beta-1} |S(\sigma) w(0)| dyd\sigma \\
\geq C \int_{t}^{\frac{t}{2}} \sigma^{r+(s+1)p} \cdot \sigma^{-(\beta-1)\gamma_1} \left( \int_{|y| \leq \sigma^{1/\alpha}} |S(\sigma) w(0)| dy \right) d\sigma \\
\geq C \int_{t}^{\frac{t}{2}} d\sigma, \text{ for all } t > 0 \text{ sufficiently large } (t > 2t_0 \geq 2).
\]

By \((3.13)\), we see that for every \( R > 0 \) it is possible to find \( t_2 > 1 \) such that the function \( U_2 \) defined by \( U_2(x) := w(x, t_2) \in L^\infty(\mathbb{R}^N) \) satisfies

\[
\int_{|x| \leq t_2} U_2(x)dx \geq R.
\]

Now consider \((w_1(t), z_1(t)) = (w(t+t_2), z(t+t_2))\). Note that \((w_1(t), z_1(t))\) is a global-in-time solution of problem (1.6) with initial condition \((w_1(0), z_1(0)) = (U_2(x), z(t_2))\). Therefore, from Proposition 3.1 we have

\[
t^{\gamma_1} \| S(t) U_2 \|_\infty \leq C^*, \text{ for all } t \geq 0.
\]

On the other hand, from \((3.14)\) and Lemma 2.6 we have

\[
S(t) U_2(x) \geq C(\alpha, N)^{-1} R t^{-\frac{N}{\alpha}},
\]

for \(|x| \leq t^{1/\alpha}\) and \( t > t_2 \). This implies that

\[
t^{\gamma_1} \| S(t) U_2 \|_\infty = t^{\frac{\gamma_1}{\alpha}} \| S(t) U_2 \|_\infty \geq C(\alpha, N)^{-1} R,
\]

for all \( t > t_0 \). This contradicts inequality \((3.15)\) due to arbitrariness of \( R > 0 \).

**Case II :** \( q < 1 \). We argue by contradiction. Suppose that there exists a global-in-time solution \((u, v)\) of problem (1.11) with initial condition \((u_0, v_0) \in L^\infty(\mathbb{R}^N))^2\), \( u_0, v_0 \geq 0 \); without loss of generality, we can assume that \( u_0 \neq 0 \) and \( v_0 \neq 0 \) (see \((3.3)\)).

Suppose that \( \gamma_1 > \frac{N}{2\alpha} \). We can find a non-trivial function \( 0 \leq U_3 \in L^\infty(\mathbb{R}^N) \) such that \( \text{supp} \ U_3 \subset B(t_0^{1/\alpha}) \) for some \( t_0 \geq 1 \) and \( 0 \leq U_3 \leq u_0 \). Thus, arguing
similarly as in (3.10), and since $\Gamma \geq 0$ (by $(K_3)$), we have

$$u(x, t) \geq |S(t)u_0|(x) \geq Ct^{-\frac{N}{2-\alpha}}\mathcal{X}_{\frac{1}{t^{\frac{1}{\alpha}}}}(x),$$

for $t \geq t_0$ and some constant $C > 0$, where $\mathcal{X}_{\frac{1}{t^{\frac{1}{\alpha}}}}$ is the characteristic function on the ball of center 0 and radius $\frac{1}{t^{\frac{1}{\alpha}}}$. It follows from here that

$$(3.16) \quad t^{q\gamma_1}||S(t)u(t)^q||_\infty \geq Ct^{q(\gamma_1 - \frac{N}{2-\alpha})}S(t)\mathcal{X}_{\frac{1}{t^{\frac{1}{\alpha}}}}(x).$$

for $t \geq t_0$. Besides, by $(K_5)$, we have

$$(3.17) \quad S(t)\mathcal{X}_{\frac{1}{t^{\frac{1}{\alpha}}}}(x) \geq \int_{|y| < t^{\frac{1}{\alpha}}} \Gamma(x, y, t)dy \geq Ct^{-\frac{N}{2-\alpha}} \cdot t^{-\frac{N}{2-\alpha}},$$

for all $x \leq t^{\frac{1}{\alpha}}$ and $t > 0$. Thus, (3.10) contradicts the second inequality in Corollary 3.7.

Now assume $\gamma_1 = \frac{N}{2-\alpha}$. Proceeding similarly as in estimates (3.11) and (3.16), we get that

$$v(t) \geq Ct^{s+1}S(t)u^q(t),$$

for all $t > 0$, and

$$t^{s+1}S(t)u^q(x, t) > Ct^{s+1} \cdot t^{-q\gamma_1} \quad \text{(see (3.17))},$$

for $x \leq t^{\frac{1}{\alpha}}$ and $t > t_0$, respectively. From here and proceeding as in the obtention of (3.10), we have

$$\int_{|x| \leq t^{\frac{1}{\alpha}}} u(x, t + 1)dx$$

$$\geq C \int_{1}^{t^{1/2}} \int_{|y| \leq (t^{1/2} - 1)} \sigma^{r \cdot \sigma^{s+1} |S(\sigma)u(y, \sigma)^q|}dy d\sigma$$

$$\geq C \int_{t_0}^{t^{1/2}} \int_{|y| \leq (t^{1/2} - \frac{t_0}{t^{1/2}})} \sigma^{(s+1)p}[\sigma^{-\gamma_1}]^{\beta-1} [\sigma^{-\gamma_1}]dy d\sigma$$

$$\geq C \int_{t_0}^{t^{1/2}} \sigma^{(s+1)p} \cdot \sigma^{-(\beta-1)\gamma_1} \left( \int_{|y| \leq \sigma} \sigma^{-\gamma_1}dy \right) d\sigma$$

$$\geq C \int_{t_0}^{t^{1/2}} d\sigma, \quad \text{for all } t > 0 \text{ sufficiently large. } (t > 2t_0 \geq 2).$$

Thus, from Corollary 3.7 and arguing similarly as in Case II, we obtain a contradiction.

4. Global Existence

4.1. Local Existence. We first establish the local existence of solutions when $p > 1$ and $q > 1$. Later, we show the local existence for the general case using an approximations method (see the proof of Corollary 4.10).

Lemma 4.1 (Comparison Principle). Assume either (A) or (B) is in force, and $(u_{0,i}, v_{0,i}) \in [L^\infty(\mathbb{R}^N)]^2 (i = 1, 2)$. Let $f, g : [0, \infty) \to [0, \infty)$ locally Lipschitz continuous functions, $r, s > -1$, and $(u_i, v_i) \in L^\infty((0, T), L^\infty(\mathbb{R}^N)) (i = 1, 2)$ such
\begin{equation}
\begin{aligned}
u_i(x,t) &= \int_{\mathbb{R}^N} \Gamma(t, x, y)u_{0,i}(y)dy + \int_0^t \int_{\mathbb{R}^N} \Gamma(t - \sigma, x, y)\sigma^r f(v_i(y, \sigma))d\sigma dy, \\
u_i(x,t) &= \int_{\mathbb{R}^N} \Gamma(t, x, y)v_{0,i}(y)dy + \int_0^t \int_{\mathbb{R}^N} \Gamma(t - \sigma, x, y)\sigma^s g(u_i(y, \sigma))d\sigma dy,
\end{aligned}
\end{equation}

for almost \( x \in \mathbb{R}^N \) and \( t > 0 \). If \( u_{0,1} \leq u_{0,2} \) and \( v_{0,1} \leq v_{0,2} \), then \( u_1(t) \leq u_2(t) \) and \( v_1(t) \leq v_2(t) \) for all \( t \in (0, T) \).

Proof. Note that it is sufficient to show that \([u_i - v_i]^+ = 0 \) \((i = 1, 2)\). Let \( M_0 = \max\{\|u_i(t)\|_\infty, \|v_i(t)\|_\infty : t \in [0, T], i = 1, 2\} \). Since that \( u_{0,1} \leq u_{0,2} \) and \( v_{0,1} \leq v_{0,2} \), from (K2) we have

\begin{equation}
\begin{aligned}
u_1(t) - u_2(t) &\leq \int_0^t S(t - \sigma)\sigma^r [f(v_1(\sigma)) - f(v_2(\sigma))]d\sigma, \\
u_1(t) - v_2(t) &\leq \int_0^t S(t - \sigma)\sigma^s [g(u_1(\sigma)) - g(u_2(\sigma))]d\sigma.
\end{aligned}
\end{equation}

Thus, since that \( f \) and \( g \) are nondecreasing and Lipschitz continuous on \([0, M_0]\), it follows from (G1) that

\begin{equation}
\begin{aligned}
\left\| [u_1(t) - u_2(t)]^+ \right\|_\infty &\leq C \int_0^t \sigma^r \left\| [v_1(\sigma) - v_2(\sigma)]^+ \right\|_\infty d\sigma, \\
\left\| [v_1(t) - v_2(t)]^+ \right\|_\infty &\leq C \int_0^t \sigma^s \left\| [u_1(\sigma) - u_2(\sigma)]^+ \right\|_\infty d\sigma,
\end{aligned}
\end{equation}

The Lemma is now a direct consequence of Gronwall’s inequality (for example, see [40]).

\hspace{1cm} \Box

\textbf{Theorem 4.2.} Suppose that \( p, q > 1 \) and assume that \( \omega \) satisfies either (A) or (B), and \((u_0, v_0) \in [L^\infty(\mathbb{R}^N)]^2, u_0, v_0 \geq 0\). Then there exists \( T > 0 \) and a constant \( C_0 > 0 \) such that problem (4.3) possesses a unique solution \((u, v)\) on \((0, T)\) satisfying

\begin{equation}
\sup_{0 < t < T} (\|u(t)\|_\infty + \|v(t)\|_\infty) \leq C_0 (\|u_0\|_\infty + \|v_0\|_\infty).
\end{equation}

\textit{Proof.} For any \((u_0, v_0) \in [L^\infty(\mathbb{R}^N)]^2, u_0, v_0 \geq 0\). We define the sequences \( \{u_n\}_{n \geq 1} \) and \( \{v_n\}_{n \geq 1} \) by

\begin{equation}
\begin{aligned}
u_1(x,t) &= \int_{\mathbb{R}^N} \Gamma(x, y, t)u_0(y)dy, \\
u_1(x,t) &= \int_{\mathbb{R}^N} \Gamma(x, y, t)v_0(y)dy
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
u_{n+1}(x,t) &= u_1(x, t) + \int_0^t \sigma^r \int_{\mathbb{R}^N} \Gamma(x, y, t - \sigma)v_n(y, s)^p dy d\sigma, n = 1, 2, \cdots, \\
u_{n+1}(x,t) &= v_1(x, t) + \int_0^t \sigma^s \int_{\mathbb{R}^N} \Gamma(x, y, t - \sigma)u_n(y, s)^q dy d\sigma, n = 1, 2, \cdots,
\end{aligned}
\end{equation}

for almost all \( x \in \mathbb{R}^N \) and all \( t > 0 \). The sequences are non-negative and non-decreasing, that is,

\begin{equation}
0 \leq u_n(x, t) \leq u_{n+1}(x, t) \text{ and } 0 \leq v_n(x, t) \leq v_{n+1}(x, t)
\end{equation}
for almost all $x \in \mathbb{R}^N$, $t > 0$, and all $n \in \mathbb{N}$. This is clear since $\Gamma$, $u_0$, and $v_0$ are non-negative functions ($\Gamma$ is nonnegative by $(K_3)$ property). Thus, we write the limit of the functions.

\begin{equation}
(4.6) \quad u_\infty(x,t) = \lim_{n \to \infty} u_n(x,t), \quad v_\infty = \lim_{n \to \infty} v_n(x,t).
\end{equation}

Furthermore, we see that $u_\infty(x,t), v_\infty(x,t) \in [0, \infty]$. Now, we show that the sequences $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$ are bounded in a small interval of time, that is,

\begin{equation}
(4.7) \quad \sup_{0 < t < T} (\|u_n(t)\|_\infty + \|v_n(t)\|_\infty) \leq 2c_1(\|u_0\|_\infty + \|v_0\|_\infty)
\end{equation}

for all $n \in \mathbb{N}$ and some $T > 0$ small enough. Let $T > 0$, and we argue by induction. For $n = 1$, the inequality (4.7) is hold, and this is due to $(G_1)$. Let us suppose (4.7) holds, for some $k \in \mathbb{N}$, that is,

\begin{equation}
\sup_{0 < t < T} (\|u_k(t)\|_\infty + \|v_k(t)\|_\infty) \leq 2c_1(\|u_0\|_\infty + \|v_0\|_\infty).
\end{equation}

Then, by $(G1)$, we have

\begin{equation}
\begin{split}
\|u_{k+1}(t)\|_\infty &\leq \|u_1(t)\|_\infty + \int_0^t \sigma^\ast \|S(t-\sigma)v_k(\sigma)p\|_\infty d\sigma \\
&\leq c_1\|u_0\|_\infty + c_1 \int_0^t \sigma^\ast \|v_k(\sigma)\|_\infty \, d\sigma \\
&\leq c_1\|u_0\|_\infty + c_1(2c_1(\|u_0\|_\infty + \|v_0\|_\infty))^p \int_0^t \sigma^\ast \, d\sigma,
\end{split}
\end{equation}

for all $t \in (0,T)$. Similarly, we have

\begin{equation}
\begin{split}
\|v_{k+1}(t)\|_\infty &\leq \|v_1(t)\|_\infty + \int_0^t \sigma^\ast \|S(t-\sigma)u_k(\sigma)p\|_\infty d\sigma \\
&\leq c_1\|v_0\|_\infty + c_1 \int_0^t \sigma^\ast \|u_k(\sigma)\|_\infty \, d\sigma \\
&\leq c_1\|v_0\|_\infty + c_1(2c_1(\|u_0\|_\infty + \|v_0\|_\infty))^q \int_0^t \sigma^\ast \, d\sigma,
\end{split}
\end{equation}

for all $t \in (0,T)$. Thus, the inequality (4.7) follows by adding (4.8) and (4.9) and then choosing $T$ as small enough.

Finally, by (4.5), (4.6), and (4.7), we have that the limits functions $u_\infty$ and $v_\infty$ satisfies (4.6) and

\begin{equation}
\sup_{0 < t < T} (\|u_\infty(t)\|_\infty + \|v_\infty(t)\|_\infty) \leq 2c_1(\|u_0\|_\infty + \|v_0\|_\infty).
\end{equation}

Also, by the comparison principle (see Lemma 4.1), $(u_\infty, v_\infty)$ is the unique solution to the problem (1.1). Therefore, Theorem 4.2 is hold. \qed

Now we proof the local existence of solutions of problem (1.1), in the general case, that is, when $p, q > 0$ and $p \cdot q > 1$.

**Corollary 4.10 (Local Existence).** Suppose that $p, q > 0$ with $p \cdot q > 1$, and assume that $\omega$ satisfies either $(A)$ or $(B)$, and $(u_0, v_0) \in [L^\infty(\mathbb{R}^N)]^2$, $u_0, v_0 \geq 0$. Then there
exists \( T > 0 \) and a constant \( C_0 > 0 \) such that problem (1.1) possesses a solution \((u, v)\) on \([0, T]\) satisfying

\[
\sup_{0 < t < T} (\|u(t)\|_\infty + \|v(t)\|_\infty) \leq C_0 (\|u_0\|_\infty + \|v_0\|_\infty).
\]

**Proof.** We use a known approximation method; for example, see [1]. Without loss of the generality, we can assume that \( 1 < q \) and \( 0 < p < 1 \). For each \( n \in \mathbb{N} \), consider a nondecreasing global Lipschitz function \( f_n \) such that

\[
f_n(s) = \begin{cases} 0 & \text{if } s = 0, \\ s^n & \text{if } s > \frac{1}{n}, \end{cases}
\]

and \( |f_n(s_1) - f_n(s_2)| \leq c_n |s_1 - s_2| \) for all \( s_1, s_2 \in [0, \infty) \). Let now see the following approximate problem of (1.1).

\[
\begin{cases}
    u_t - \text{div}(\omega(x) \nabla u) = t^p f_n(v) & \text{in } \mathbb{R}^N \times (0, T), \\
v_t - \text{div}(\omega(x) \nabla v) = t^q u^n & \text{in } \mathbb{R}^N \times (0, T), \\
    u(0) = u_0 & \text{in } \mathbb{R}^N, \\
v(0) = v_0 + \frac{1}{n} & \text{in } \mathbb{R}^N.
\end{cases}
\]

Since that \( f_n \) is globally Lipschitz with \( f_n(0) = 0 \), we can argue similarly as in the proof of Theorem 4.2 for to obtain a unique nonnegative bounded solution \((u_n(t), v_n(t))\) of (4.11), which besides satisfies (4.3). Note that, from \((K_1)\) and \((K_2)\) properties, we obtain

\[
v_k(t) \geq S(t)(v_0 + 1/k) = \int_{\mathbb{R}^N} \Gamma(x, y, t)v_0(y)dy + 1/k \geq 1/k, \text{ for all } k.
\]

Also, by construction, we have that if \( n > m \) then \( f_n(t) = f_m(t) \) for \( t > 1/2m \). Then, from this and (4.12), we have

\[
\begin{align*}
    u_m(t) &= S(t)u_0 + \int_0^t \int_{\mathbb{R}^N} \Gamma(x, y, t - \sigma)f_m(v_m(y, \sigma))dyd\sigma, \\
    v_m(t) &= S(t)(v_0 + 1/m) + \int_0^t \int_{\mathbb{R}^N} \Gamma(x, y, t - \sigma)(u_m(y, \sigma))^\frac{1}{q}dyd\sigma.
\end{align*}
\]

Thus, we have that \((u_m, v_m)\) is also solution of (4.11) with initial condition \((u_0, v_0 + 1/m)\). Therefore, from Lemma 4.1 we have \( u_m \geq u_n \) and \( v_m \geq v_n \) for \( n > m \). That is, the sequences \( \{u_n\}_{n \in \mathbb{N}} \) and \( \{v_n\}_{n \in \mathbb{N}} \) are decreasing and bounded below. Thus the result follows by letting \( n \) go to \( \infty \).

\[\square\]

4.2. Global existence: Proof of Theorem 1.11(ii).

Without loss of the generality, we can suppose that \( 0 < p < 1 \) and \( p \cdot q > 1 \). Consider \( \max\{\|u_0\|_{r_1, \infty}, \|v_0\|_{r_2, \infty}\} < \delta \), where \( \delta > 0 \) will be chosen later small enough.

From (1.7)-(1.10), we obtain the following estimates:

\[
pr_2 = \gamma_1 + (r + 1), \quad qr_2 = \gamma_2 + (s + 1), \quad pr_1 > r_2, \quad qr_1 > r_2.
\]

Also since that \( \gamma < \frac{N}{2-\alpha} \), we have \( r_1, r_2 > 1 \).
Now, similarly to the proof of Theorem 4.2 we define the following sequence

\[ \{(u^n, v^n)\}_{n \geq 0} \]

defined by \( u^0(t) = S(t)u_0, \) \( v^0(t) = S(t)v_0 \) and

\[
\begin{align*}
    u^n(t) &= S(t)u_0 + \int_0^t S(t - \sigma)\sigma^r[v^{n-1}(\sigma)]^p d\sigma, \\
v^n(t) &= S(t)v_0 + \int_0^t S(t - \sigma)\sigma^s[u^{n-1}(\sigma)]^q d\sigma,
\end{align*}
\]  

(4.14)

for all \( t > 0. \) Note that the sequences \( \{u^n\}_{n \geq 0} \) and \( \{v^n\}_{n \geq 0} \) are non-decreasing.

By induction, we prove that.

\[
\begin{align*}
    \|u^n(t)\|_{r_1, \infty} &\leq 2c_{**}\delta, \\
    \|u^n(t)\|_{\infty} &\leq 2c_{*}\delta t^{-1/(2-\alpha)\gamma}, \\
    \|u^n(t)\|_{r_2, \infty} &\leq 2c_{*}\delta,
\end{align*}
\]  

(4.15)

From \( G_2, \) we have

\[
\begin{align*}
    \|u^0(t)\|_{r_1, \infty} &\leq c_{**}\|u_0\|_{r_1, \infty}, \\
    \|u^0(t)\|_{\mu, \infty} &\leq c_{**}t^{-\frac{N}{r_2} \left( \frac{1}{r_2} - \frac{1}{\mu} \right)} \|u_0\|_{r_1, \infty}, \\
    \|v^0(t)\|_{r_2, \infty} &\leq c_{*}\|v_0\|_{r_2, \infty}, \\
    \|v^0(t)\|_{\mu, \infty} &\leq c_{*}t^{-\frac{N}{r_2} \left( \frac{1}{r_2} - \frac{1}{\mu} \right)} \|v_0\|_{r_2, \infty},
\end{align*}
\]  

(4.16)

for all \( t > 0, \) \( \mu \in [r_{**}, \infty], \) and some constant \( c_{**} > 0. \) This implies that (4.16) is held for \( n = 0. \)

Now we assume that (4.15) holds for some \( n \in \mathbb{N}. \) By symmetry we only prove that (4.15) holds for \( u^{n+1}. \) From (2.5) and (4.15), we have

\[
\begin{align*}
    \|v^n(t)\|_{\mu, \infty} &\leq \|v^n(t)\|_{r_2, \infty} \|v^n(t)\|_{\infty} \left( \frac{1}{r_2} - \frac{1}{\mu} \right) \\
    &\leq 2c_{**}\delta t^{-\frac{N}{r_2} \left( \frac{1}{r_2} - \frac{1}{\mu} \right)}
\end{align*}
\]  

(4.17)

for all \( t > 0 \) and \( r_{**} \leq \mu < \infty. \) Also, from (4.13) and (4.17), we have

\[
\begin{align*}
    \|v^n(t)^\eta\|_{\eta, \infty} &= \|v^n(t)^\eta\|_{\eta, \infty} \\
    &\leq (2c_{**}\delta t^{-\frac{N}{r_2} \left( \frac{1}{r_2} - \frac{1}{\eta} \right)})^p \\
    &= C\delta t^{-\frac{N}{r_2} \left( \frac{1}{r_2} - \frac{1}{\eta} \right)} \gamma + (r+1)
\end{align*}
\]  

(4.18)

for any \( \eta > 1 \) with \( r_{**} \leq \eta \). Similarly, from (4.13) and (4.15), we obtain

\[
\begin{align*}
    \|v^n(t)^p\|_{\infty} &= \|v^n(t)^p\|_{\infty} \\
    &\leq (2c_{**}\delta t^{-\frac{N}{r_2} \left( \frac{1}{r_2} - \frac{1}{\eta} \right)})^p \\
    &= (2c_{**}\delta)^pt^{-\gamma}\gamma + (r+1)
\end{align*}
\]  

(4.19)

for all \( t > 0. \)
Thus, by \((G_1), (G_2), (4.13), (4.18)\) (with \(\eta = r_{1*}\)), and \((4.19)\), we have
\[
\left\| \int_{t/2}^{t} S(t - \sigma) \sigma^r v^p(\sigma) \, d\sigma \right\|_{\infty} \leq \int_{t/2}^{t} \sigma^r \|S(t - \sigma)v^p(\sigma)\|_{\infty} d\sigma \\
(4.20)
\leq C \int_{t/2}^{t} \sigma^r \|v^p(\sigma)\|_{\infty} d\sigma \\
\leq C\delta^p t^{-\frac{N}{(2-n)r_{1*}}}
\]
and
\[
\left\| \int_{t/2}^{t} S(t - \sigma) \sigma^p v^p(\sigma) \, d\sigma \right\|_{r_{1*}, \infty} \leq \int_{t/2}^{t} \sigma^r \|S(t - \sigma)v^p(\sigma)\|_{r_{1*}, \infty} d\sigma \\
(4.21)
\leq C \int_{t/2}^{t} \sigma^r \|v^p(\sigma)\|_{p, r_{1*}, \infty} d\sigma \\
\leq C\delta^p
\]
for all \(t > 0\).

On the other hand, since that \(t - \sigma \geq t/2\) for all \(\sigma \in [0, t/2]\), by \((G_2)\) and \((4.18)\) (with \(\eta = \eta_1\), which will be chosen later), we obtain
\[
\left\| \int_{0}^{t/2} S(t - \sigma) \sigma^r v^p(\sigma) \, d\sigma \right\|_{\infty} \\
\leq \int_{0}^{t/2} \|S(t - \sigma)v^p(\sigma)\|_{\infty} d\sigma \\
(4.22)
\leq C \int_{0}^{t/2} (t - \sigma)^{-\frac{N}{(2-n)\eta_1}} \sigma^r \|v^p(\sigma)\|_{\eta_1, \infty} d\sigma \\
\leq C\delta^p t^{-\frac{N}{(2-n)\eta_1}} \int_{0}^{t/2} \sigma^r \sigma^{\frac{N}{(2-n)\eta_1}} \sigma^{-\frac{N}{(2-n)\eta_1}} \, d\sigma \\
\leq C\delta^p t^{-\frac{N}{(2-n)\eta_1}}
\]
for some \(1 < \eta_1 < r_{1*}\) close enough to \(r_{1*}\) so that \(r_{2*} < \eta_1 p\) (it is possible since that \(p \, r_{1*} > r_{2*} > 1\)).

Analogously (using the above \(\eta_1\) again), we obtain
\[
\left\| \int_{0}^{t/2} S(t - \sigma) \sigma^r v^p(\sigma) \, d\sigma \right\|_{r_{1*}, \infty} \\
\leq \int_{0}^{t/2} \|S(t - \sigma)v^p(\sigma)\|_{r_{1*}, \infty} d\sigma \\
(4.23)
\leq C \int_{0}^{t/2} (t - \sigma)^{-\frac{N}{(2-n)\eta_1}} \sigma^r \|v^p(\sigma)\|_{\eta_1, \infty} d\sigma \\
\leq C t^{-\frac{N}{(2-n)\eta_1}} \int_{0}^{t/2} \sigma^r \|v^p(\sigma)\|_{\eta_1, \infty} d\sigma \\
\leq C\delta^p
\]
for all \(t > 0\).
Then, from (4.14), (4.16), (4.20), (4.21), (4.22), (4.23), and taking a $\delta > 0$ sufficiently small, we obtain
\[
\left\| u^{n+1}(t) \right\|_{1.1} \leq c_n \delta + C_\delta \leq 2c_n \delta \quad \| u^{n+1}(t) \|_{r_1, \infty} \leq c_n \delta + C_\delta \leq 2c_n \delta
\]
for all $t > 0$, where $C > 0$ is a constant independent of $n$, $\delta$, and $t$. Thus, we obtain that (4.15) holds for all $u^n$ with $n \in \mathbb{N}$. Arguing similarly, we have that also (4.16) holds for everyone $v^n$ with $n \in \mathbb{N}$.

Now, from (4.15) and arguing similarly as in the proof of Theorem (4.2), we obtain that there exists a global-in-time solution (this solution is unique in the particular case when $p > 1$ and $q > 1$)
\[
(u(t), v(t)) = \left( \lim_{n \to \infty} u^n(t), \lim_{n \to \infty} v^n(t) \right)
\]
of (1.1) such that
\[
\| u(t) \| \leq 2c_n \delta \delta t^{-\frac{N}{2-\alpha}} \quad \| u(t) \|_{r_1, \infty} \leq 2c_n \delta
\]
\[
\| v(t) \| \leq 2c_n \delta \delta t^{-\frac{N}{2-\alpha}} \quad \| v(t) \|_{r_2, \infty} \leq 2c_n \delta.
\]
Using this together with the upper bounded estimate in Corollary 4.10 we obtain a constant $C_0 > 0$ such that
\[
\| u(t) \| \leq C_0 (t+1)^{-\frac{N}{2-\alpha}} \quad \text{and} \quad \| v(t) \| \leq C_0 (t+1)^{-\frac{N}{2-\alpha}}
\]
for $t > 0$. Also, from this and by (2.25), we have
\[
\| u(t) \|_{\mu, \infty} \leq \| u(t) \|_{r_1, \infty} \| u(t) \|_{1, \infty}^{\frac{\mu}{\alpha}} \leq C_1 (t+1)^{-\frac{N}{2-\alpha} \left( \frac{\mu}{\alpha} - \frac{1}{p} \right)},
\]
\[
\| v(t) \|_{\mu, \infty} \leq \| v(t) \|_{r_2, \infty} \| v(t) \|_{1, \infty}^{\frac{\mu}{\alpha}} \leq C_1 (t+1)^{-\frac{N}{2-\alpha} \left( \frac{\mu}{\alpha} - \frac{1}{p} \right)},
\]
for all $\mu$ such that $\max\{r_1, r_2\} < \mu \leq \infty$, $t > 0$ and some constant $C_1 > 0$, thus the proof is concluded.

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