An application of Prekopa’s theorem on the log-concavity of warranty reserves in reliability engineering

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Abstract. The log-concavity of the expected refund rate for an item with a nonrenewing pro-rata rebate warranty policy, and of the expected total reserve to service the warranty over the item life cycle under the same policy are proved. Upper bounds of warranty reserves are derived. The results are extended to other measures for spares and repairs under other warranty policies. Applications are given to ranking items’ prices and production and inventory control.

1. Introduction

Log-concave densities have played a prominent role in applied probability for both discrete and absolutely continuous univariate distributions, since log-concave functions have a large number of good mathematical properties. A starting point in this article is to emphasize on some properties of log-concave functions in connection with Polya frequency functions (see Karlin (1968)). The aim of this note is to prove the log-concavity of the expected refund rate for an item and of the expected total reserve to service the warranty over the item’s life-cycle with a nonrenewing pro-rata rebate warranty policy. The log-concavity of these functionals is applied for obtaining their upper bounds being useful for optimal pricing under the same warranty policy. Similar results are given for the expected demand for spares per unit time over the warranty period and expected total number of spares over the item’s life-cycle, and measures of repairs under a nonrenewing free replacement warranty policy. Throughout the article, given a measure space \((\Omega, A, \lambda)\) we denote by \(f\) a probability density function on \(A\) with respect to \(\lambda\), and assume \(f\) belongs to the space of functions defined on \(A \subset \mathbb{R}\), whose derivatives of all orders \(n \leq 2\) exist and are continuous in \(A\), whenever required.

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2. Preliminaries

A non-negative function \( f : \mathcal{A} \rightarrow [0, +\infty) \), for \( \mathcal{A} \subset \mathbb{R}^n \) is said to be log-concave (log-convex) in \( x \in \mathcal{A} \), if for all \( x_1, x_2 \in \mathcal{A} \) and \( \alpha \in [0, 1] \) such that \( \alpha x_1 + (1 - \alpha) x_2 \in \mathcal{A} \), we have that
\[
 f(\alpha x_1 + (1 - \alpha) x_2) \geq (\leq) f(x_1)^\alpha f(x_2)^{1-\alpha}.
\]

Taking log(0) = -\infty, then an equivalent condition is that log(f) is concave (convex). For the basic properties of log-concave functions, we may refer to Eaton (1987).

Log-concavity can also be characterized in terms of Polya frequency of order 2 functions (see Karlin (1968) for a comprehensive study). A function \( g : \mathcal{A} \rightarrow \mathbb{R} \) is said to be total positive of order 2, for short \( g(x, y) \) is \( TP_2 \) in \( (x, y) \), if for all if for all \( x_1 < x_2 \in \mathcal{A} \) and \( y_1 < y_2 \in \mathcal{A} \), then
\[
 g(x_1, y_2)g(x_2, y_1) \leq g(x_1, y_1)g(x_2, y_2).
\]

Also, a function \( f : \mathcal{A} \rightarrow [0, +\infty) \), for \( \mathcal{A} \subset \mathbb{R}^2 \) is said to be Polya frequency of order 2, for short \( g(x, y) = TP_2 \) in \( (x, y) \in \mathcal{A}^2 \). From nomenclature by Karlin (1968), a \( PF_2 \) density is distinguished from a \( PF_2 \) frequency function by the additional condition that \( f \) is integrable, in this event \( f \) is usually normalized by a multiplicative constant, so that \( \int_{-\infty}^{+\infty} f(x)\,dx = 1 \). The log-concavity of a measurable function \( f \) is equivalent to \( f \) being a \( PF_2 \) density function vanishing on the negative axis. Several techniques have been developed to construct log-concave functions, as Prekopa theorem in Prekopa (1973), that we recall next.

**Proposition 1.** Let \( \phi(t, x) \) be a convex function in \( \mathbb{R}^{n+1} \) and let \( \psi(x) \) be a convex function in \( \mathbb{R}^n \). Assume that
\[
 \int_{\mathbb{R}^n} e^{\psi(x) - \phi(0, x)}\,dx = 1.
\]

Then the function \( f(t) = \int_{\mathbb{R}^n} e^{-\phi(t, x)}\,dx \) is log-concave.

The structure of Polya densities of order 2 that vanish on the negative axis provides their moment inequalities and bounds, and they can be characterized by Laplace transforms that exist in an open strip of the complex plane containing the imaginary axis, by different inversion and representation theorems in the literature (see Karlin (1968) and references therein). Finally, we recall how computing Polya densities of order 2 from their Laplace transforms. Let \( \mathcal{Z} \) denote the class of entire functions
\[
 \mathcal{Z} = \{ \varphi(s) = C s^k e^{\delta s} \prod_{i=1}^\infty (1 + \lambda_i s) | \varphi(0) = 1, \delta \geq 0, k \in \mathbb{Z}_+, \lambda_i \geq 0, \sum_{i=1}^{\infty} \lambda_i < \infty \}.
\]

A necessary and sufficient condition that a density function \( f \) for which \( f(u) = 0 \), for \( u < 0 \) be \( PF_2 \) is that the reciprocal of its Laplace transform \( \phi(s) \) be an entire function in the class \( \mathcal{Z} \) with \( \sum_{i=1}^{\infty} \lambda_i > 0 \). Then consider
\[
 \frac{1}{\phi(s)} = \varphi(s) = Ce^{\delta s - \gamma s^2} s^r \prod_{i=1}^\infty (1 + a_i s) e^{-a_i s}
\]
where \( \gamma, \delta, a_i \) and \( C \) are real, \( \gamma > 0 \), and \( r \) is a nonnegative integer. Assume that \( \phi(s) \) is positive for \( \alpha < s < \beta \) where \( \alpha \) and \( \beta \) denote consecutive zeros of \( \varphi \). By the Laplace inversion formula, for \( \alpha < \sigma < \beta \)
\[
 h(x) = \left( \frac{1}{2\pi i} \right) \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\sigma s} \phi(s)\,ds
\]
and \( h(x) = Ce^{\sigma x} f(x) \), where \( f \) is a \( PF_2 \) density and \( C \) is a positive normalizing constant.
3. Log-concavity of warranty reserves

In this section, we obtain the main results on the log-concavity of the behaviour of some warranty expenses’ measures. Warranty assures the buyer that the manufacturer will either repair or replace items that do not perform satisfactorily or refund a fraction or the whole of the sale price. This is an important issue in terms of pricing the items. Different policies have been defined in the literature, with criteria as the coverage (or remaining warranty time from the time of the item is replaced, leading to renewing or nonrenewing warranties); the dimensions of the warranty (age and usage of the item); the form of refund or reimbursement to the customer on the failure of the item or dissatisfaction with service; the requirements for the manufacturer, and other diverse factors. For a comprehensive study, we may refer to Blischke and Murthy (2000), Chapter 17. The warranties determined from the reimbursement’s point of view, can be classified in: 1. A lump-sum rebate, e.g., 'money-back guarantee', which is usually assigned for a relatively small time interval immediately after the purchase of the item or service. 2. A free repair of the failed item. It is called a free repair warranty (FRW). 3. A repair provided at reduced cost to the buyer. The cost reduction is usually a decreasing function of the time to failure. It is called pro-rata warranty (PRW). 4. A combination of the preceding terms that starts with an FRW up to a specified time and switches to repair at pro-rated cost for the remainder of the warranty period. It is called FRW/PRW. On the other hand, there are two types of warranty from the coverage’s point of view: 1. Non-renewing warranty (NR): A newly sold item is covered by a warranty for some calendar time of duration \( w \), called the warranty period, which usually starts ate the time of the purchase of the item. For this policy, during the warranty period, the warrantor assumes all, which is a non-renewing free repair warranty (NRFRW), or a portion of the expenses which is a non-renewing pro-rata warranty (NRPRW), associated with the failure of the item. 2. Renewing warranty (R): The warrantor repairs any faulty item from the time of the purchase up to time \( w \), at the time of each repair within an existing warranty, the item is warranted anew for a period of length \( w \), and the warranty coverage expires when the lifetime of the item (the original one or its repaired version) exceeds \( w \). It can be RFRW or RPRW.

Recall again that under a NRPRW policy, the manufacturer is required to refund a fraction of the sale price on failure of an item in the warranty period, from the time of the initial purchase. To carry this out, the manufacturer must set aside a fraction of the sale price, that constitutes a warranty reserve. Formally, let \( w \) denote the length of the warranty period, \( c \) denote the unit sale price, \( l \) denote the item’s life-cycle, and \( \chi(t) \) denote the sales rate (sales per unit time), for \( 0 \leq t \leq l \). Let \( T \) be the failure time of the item, with probability density function \( h \) and distribution function \( H \). Furthermore, we assume that \( \chi(t) \) is a measurable log-concave function for \( 0 \leq t \leq l \). Consider that the item is sold with a NRPRW policy with linear refund function, such that the expected refund rate (the amount refunded per unit time) at \( t \) is given by

\[
\nu(t) = c \int_{b}^{t} \chi(x) \frac{t-x}{w} h(t-x) dx
\]  

(3.1)

where \( b = \max\{0, t-w\} \), for \( 0 \leq t \leq w+l \). We assume that \( \int_{0}^{w+l} \chi(x) = 1 \), then \( \chi \) can be considered the probability density function of a random variable related to the sales intensity.

The log-concavity of the expected refund rate \( \nu \) in (3.1) with respect to \( (0, w+l) \) for an item can be stated when \( \chi \) is a measurable log-concave function. Formally, consider the failure time \( T \) of an item with log-concave probability density function \( h \). Assume that a NRPRW policy is applied for the item with warranty time \( w \), life-cycle \( l \) and sales rate \( \chi(t) \), for any \( 0 \leq t \leq l \). If
\( \chi \) is a measurable log-concave function, then the expected refund rate \( \upsilon \) in (3.1) is log-concave in \((0, w+l)\). Therefore, the previous result means that the log-concavity of this density function implies the log-concavity of the expected refund rate.

Given an item with lifetime \( T \) with a NRPRW policy as above with expected refund rate \( \upsilon \) given in (3.1) for any \( t \in (0, w+l) \), then the expected total reserve needed to service the warranty over the item’s life-cycle is given by
\[
ETR = ETR(w + l) = \int_0^{w+l} \upsilon(t) dt. \tag{3.2}
\]

From the earlier result, and from Prekopa’s theorem, and under the same assumptions, we obtain that the expected total reserve to service the warranty over the item’s life-cycle is log-concave in \( w + l \). Furthermore, some upper bounds for the expected refund rate and the expected total reserve to service the warranty over the item’s life-cycle can be stated.

Given a nonrepairable item that is sold under a non-renewing free replacement warranty (NRFRW) policy, the manufacturer is required to replace all items that fail within the warranty period \( w \). The demand for spares in the interval \([t, t + \delta t]\) is due to failure of items sold in the period \([b, t]\) where \( b = \max\{0, t-w\} \), and is useful for production and inventory control. With the same notation for the parameters of the earlier warranty expenses’s measures, again let \( \chi(t) \) denote the sales rate (sales per unit time), for \( 0 \leq t \leq l \); let \( T \) denote the failure time of the item, with probability density function \( h \) and distribution function \( H \), and assume that \( \chi(t) \) is a measurable log-concave function for \( 0 \leq t \leq l \). The expected demand rate for spares at time \( t \) is given by
\[
\rho(t) = \int_b^t \chi(x)m(t-x)dx \tag{3.3}
\]
where \( m \) is the renewal density function associated with a renewal process with inter-arrival times that have distribution function \( H \) that is defined by
\[
m(t) = h(t) + \int_0^t m(t-x)h(x)dx. \tag{3.4}
\]
The expected total number of spares to service the warranty over the item’s life-cycle is given by
\[
ETS = l + \int_0^{w+l} \rho(t) dt. \tag{3.5}
\]

We provide similar conclusions, i.e, log-concavity, as in the main result in this Section, under NRFRW policy, for the demand for spares, the expected total number of spares required to service the warranty over the item’s life-cycle, the demand for repairs, and the total expected demand for repair over the warranty period.

The performance of warranty policies is an important issue in terms of pricing the items for quality improvement. From our results on log-concavity that are described above, we derive some consequences on ranking items’ prices, and production and inventory control.

4. References
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