The pathwise uniqueness of solution to a SPDE driven by $\alpha$-stable noise

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Abstract. In this paper we study the pathwise uniqueness of solution to the following stochastic partial differential equation

$$
\frac{\partial X_t(x)}{\partial t} = \frac{1}{2} \Delta X_t(x) + X_t(x)^\beta \dot{L}_t(x), \quad X_0 \geq 0, \quad t > 0, \quad x \in \mathbb{R},
$$

where $1 < \alpha < 2$, $0 < \beta < 1$ and $\dot{L}$ denotes an $\alpha$-stable white noise on $\mathbb{R}_+ \times \mathbb{R}$ without negative jumps. In the special case of $\alpha \beta = 1$, where solution to the above equation is the density of a super-Brownian motion with $\alpha$-stable branching (see Mytnik (2002)), our result leads to its pathwise uniqueness for $1 < \alpha < 4 - 2\sqrt{2}$. The local Hölder continuity of the solution is also obtained for fixed time $t > 0$ and $\alpha \beta \neq 1$.

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1 Introduction

It was proved by Konno and Shiga (1988) as well as Reimers (1989) that one-dimensional binary branching super-Brownian motion has a jointly continuous density that is a random field \{\(X_t(x) : t > 0, x \in \mathbb{R}\}\} satisfying the following continuous-type stochastic partial differential equation (SPDE):

$$
\frac{\partial X_t(x)}{\partial t} = \frac{1}{2} \Delta X_t(x) + \sqrt{X_t(x)} \dot{W}_t(x), \quad X_0 \geq 0, \quad t > 0, \quad x \in \mathbb{R},
$$

(1.1)

where $\Delta$ denotes the one-dimensional Laplacian operator and \{\(\dot{W}_t(x) : t > 0, x \in \mathbb{R}\}\} denotes the derivative of a space-time Gaussian white noise. The weak uniqueness of solution to (1.1) follows from that of a martingale problem for super-Brownian motion. The pathwise uniqueness

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for SPDE (1) remains open even though it has been studied by many authors; see [14, 15, 21, 6]. The main difficulty comes from the unbounded drift coefficient and the non-Lipschitz diffusion coefficient.

Mytnik (2002) considered the following jump-type SPDE and constructed a weak solution:

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2} \Delta X_t(x) + X_t(x)^\beta \hat{L}_t(x), \quad X_0 \geq 0, \quad t > 0, \quad x \in \mathbb{R}, \quad (1.2)$$

where $\hat{L}$ is a one sided $\alpha$-stable white noise on $\mathbb{R}^+ \times \mathbb{R}$ without negative jumps, $1 < \alpha < 2$ and $0 < \beta < 1$. Put $p := \alpha \beta < 2$. For the case $p = 1$, the solution is the density of a super-Brownian motion with $\alpha$-stable branching. The weak uniqueness of the solution to (1.2) also follows from a martingale problem and the pathwise uniqueness is still unknown. Mytink and Perkins (2003) showed that the density has a continuous version at any fixed time. Recently, Fleischmann et al. (2010) showed that this continuous version is locally Hölder continuous with index $\eta_c := 2/\alpha - 1$, and Fleischmann et al. (2011) showed that it is also Hölder continuous with index $\eta_c := (3/\alpha - 1) \wedge 1$ at any given spatial point. For $p \neq 1$, the weak uniqueness of solution to SPDE (1.2) and the regularities of solution $X_t(\cdot)$ at fixed time $t$ are unknown.

Throughout this paper, we always assume that $1 < \alpha < 2$ and $0 < \beta < 1$. Our goal is to establish the pathwise uniqueness of solution to (1.2) under certain conditions of $\alpha$ and $\beta$. In particular, for $p = 1$ we show that the pathwise uniqueness holds for $1 < \alpha < 4 - 2\sqrt{2}$. To prove the pathwise uniqueness we need the local Hölder continuity of the solution at fixed time $t > 0$ (Fleischmann et al. 2010) only proved this result for the case $p = 1$.

To continue with our introduction we present some notation. Let $\mathcal{B}(\mathbb{R})$ be the set of Borel functions on $\mathbb{R}$. Let $B(\mathbb{R})$ denote the Banach space of bounded Borel functions on $\mathbb{R}$ furnished with the supremum norm $\| \cdot \|$. We use $C(\mathbb{R})$ to denote the subset of $B(\mathbb{R})$ of bounded continuous functions. For any integer $n \geq 1$, $C^n(\mathbb{R})$ be the subset of $C(\mathbb{R})$ of functions with bounded continuous derivatives up to the $n$th order. Let $C^n_c(\mathbb{R})$ be the subset of $C^n(\mathbb{R})$ with compact support. We use the superscript “+” to denote the subsets of positive elements of the function spaces, e.g., $B(\mathbb{R})^+$. For $f, g \in \mathcal{B}(\mathbb{R})$ write $\langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x)dx$ if it exists. Let $M(\mathbb{R})$ be the space of finite Borel measures on $\mathbb{R}$ endowed with the weak convergence topology. For $\mu \in M(\mathbb{R})$ and $f \in B(\mathbb{R})$ we also write $\mu(f) = \int f d\mu$.

Equation (1.2) is a formal SPDE that is understood in the following sense: For any $f \in \mathcal{S}(\mathbb{R})$, the (Schwartz) space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}$,

$$\langle X_t, f \rangle = X_0(f) + \frac{1}{2} \int_0^t \langle X_s, f'' \rangle ds + \int_0^t \int_{\mathbb{R}} X_s(x)^\beta f(x)L(ds, dx), \quad t \geq 0, \quad (1.3)$$

where $L(ds, dx)$ is a one sided $\alpha$-stable white noise on $\mathbb{R}^+ \times \mathbb{R}$ without negative jumps. Let $m(dz) := c_0 z^{1-\alpha} 1_{\{z > 0\}} dz$ where $c_0 := \alpha(\alpha - 1)/\Gamma(2 - \alpha)$ and $\Gamma$ denotes the Gamma function. By the proof of Theorem 1.1(a) of Mytink and Perkins (2003), there is a Poisson random measure $N(ds, dz, dx)$ on $(0, \infty)^2 \times \mathbb{R}$ with intensity $dsm(dz)dx$ such that

$$L(ds, dx) = \int_0^\infty z\tilde{N}(ds, dz, dx), \quad (1.4)$$

where $\tilde{N}(ds, dz, du)$ is the compensated measure for $N(ds, dz, dx)$. Thus, if $\{X_t : t \geq 0\}$ is a weak solution of (1.3), then for each $f \in \mathcal{S}(\mathbb{R})$ we have

$$\langle X_t, f \rangle = X_0(f) + \frac{1}{2} \int_0^t \langle X_s, f'' \rangle ds + \int_0^t \int_{\mathbb{R}} X_s(x)^\beta f(x)z\tilde{N}(ds, dz, dx), \quad t \geq 0. \quad (1.5)$$
Definition 1.1 We say that SPDE (1.3) has a weak solution \((X, L)\) with initial value \(X_0 \in M(\mathbb{R})\) if there is a pair \((X, L)\) defined on the same filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\), which satisfies the usual conditions, so that

(i) \(L\) is an \(\alpha\)-stable white noise on \(\mathbb{R}_+ \times \mathbb{R}\) without negative jumps.

(ii) The two-parameter nonnegative process \(X = \{X_t(x) : t > 0, x \in \mathbb{R}\}\) is progressively measurable on \(\mathbb{R}_+ \times \mathbb{R} \times \Omega\), and \(\{X_t(x)dx : t > 0\}\) is a \(M(\mathbb{R})\)-valued càdlàg process.

(iii) For each \(f \in \mathcal{F}(\mathbb{R})\), \((X, L)\) satisfies (1.3).

(iv) If \(p = \alpha\beta \neq 1\), then

\[
\mathbb{P}\left\{ \int_0^t ds \int \mathbb{R} X_s(x)^p dx < \infty \text{ for all } t > 0 \right\} = 1.
\]

Our main result, Theorem 1.1, is that the pathwise uniqueness holds for solution to equation (1.3) under certain conditions of \(\alpha\) and \(\beta\). To prove the pathwise uniqueness we need the local Hölder continuity of the solution at fixed time \(t > 0\). For \(p = 1\), the proof for the local Hölder continuity of \(X_t(x)\) is based on the following equation (Fleischmann et al. (2010)):

\[
\langle X_t, f \rangle = X_0(f) + \frac{1}{2} \int_0^t \langle X_s, f'' \rangle ds + \int_0^t \int_0^\infty \int_0^\infty f(x)z M(ds, dz, dx),
\]

where \(M(ds, dz, dx)\) denotes a random measure on \((0, \infty)^2 \times \mathbb{R}\) with compensator \(\hat{M}(ds, dz, dx) = dsm(dz)X_s(x)dx\). Equation (1.6) was established for super-Brownian motion. But for \(p \neq 1\) the solution to (1.3) is not a density of super-Brownian motion and we cannot obtain the equivalent of equation (1.6). So, inspired by Dawson and Li (2006, 2012), we transform (1.3) into the following SPDE (see Proposition 2.1):

\[
\langle X_t, f \rangle = X_0(f) + \frac{1}{2} \int_0^t \langle X_s, f'' \rangle ds + \int_0^t \int_0^\infty \int_0^\infty X_{s-}(u)^p z f(u)\tilde{N}_0(ds, dz, du, dv), \quad t > 0,
\]

where \(f \in \mathcal{F}(\mathbb{R})\) and \(\tilde{N}_0(ds, dz, du, dv)\) is a compensated Poisson measure on \((0, \infty)^2 \times \mathbb{R} \times (0, \infty)\) with intensity \(dsm(dz)du dv\). By modifying the proof of Theorem 1.2(a) and using (1.7), we can obtain the local Hölder continuity (in the spatial variable) with index \(\eta_c := 2/\alpha - 1\) for the continuous version of the solution to (1.3). Notice that \(\eta_c \uparrow 1\) as \(\alpha \downarrow 1\), which is quite different from that of continuous-type SPDE whose local Hölder index is typically smaller than \(\frac{1}{2}\). This fact is key to proving the pathwise uniqueness.

We outline our approach here. By an infinite-dimensional version of the Yamada-Watanabe argument for ordinary stochastic differential equations (see Mytnik et al. (2006)), the problem of showing the pathwise uniqueness is reduced to showing the analogue of the local time term is zero (see proof of Theorem 1.1 and Lemma 1.6). That is to show that

\[
I_{4,1,n,k}^{m,n,k}(t) := \mathbb{E}\left\{ \int_0^{t \wedge \tau_k} ds \int_{-K}^K \Psi_s(x)dx \int_0^1 z^2 m(dz) \int_{-1}^1 \tilde{V}_s(x - y/m)^2 \Phi(y)^2 dy \times \int_0^1 \psi_n((\tilde{U}_s, \Phi_m^p) + zhm\tilde{V}_s(x - y/m)\Phi(y))(1 - h)dh \right\}
\]

goes to zero as \(m, n \to \infty\). Here \(\tilde{U}_s\) is the difference of continuous versions of the two weak solutions to (1.3), \(\tilde{V}_s\) denotes the difference of continuous versions of these two solutions with power \(\beta\), \(\psi_n\) and \(\Psi\) are two positive test functions, \(\Phi_m\) is a nonnegative function and \(\tau_k\) is a stopping time.
Inspired by an argument of Mytnik and Perkins (2011), for fixed $s, m$ and $x$, denote by $x_{s,m}(x) \in [-1,1]$ a value satisfying
\[
|\tilde{V}_s(x - \frac{x_{s,m}(x)}{m})| = \inf_{y \in [-1,1]} |\tilde{V}_s(x - \frac{y}{m})|.
\]
The key to the proof of Lemma 4.6 is to split the local time term $I_{4,1,k}(t)$ into two terms, where one term is bounded from above by
\[
I_{4,1,k}(t) := 4m(na_n)^{-1}E\left\{ \int_0^{t \wedge \tau_k} ds \int_{-K}^K \Psi_s(x)dx \int_0^1 z^2 m(dz) \times \int_{-1}^1 |\tilde{V}_s(x - \frac{y}{m}) - \tilde{V}_s(x - \frac{x_{s,m}(x)}{m})|^2 \Phi(y)^2 dy \right\},
\]
and the other term is bounded from above by
\[
I_{4,1,2}(t) := 4m(na_n)^{-1}E\left\{ \int_0^t ds \int_{-K}^K \Psi_s(x)H_{m,n}(x)dx \right\}
\]
with $H_{m,n}(x)$ defined by
\[
H_{m,n}(x) := \sup_{s \in M} \int_{-1}^1 |\tilde{V}_s(x - \frac{x_{s,m}(x)}{m})\Phi(y)|^2 dy \int_{0}^1 z^2 m(dz) \times \int_{-1}^1 1_{\{|\tilde{V}_s,\Phi_n| + zm\tilde{V}_s(x - \frac{x_{s,m}(x)}{m}) \Phi(y) < a_{n-1}\}}(1 - h)dh,
\]
where $M$ is a countable dense subset of $[0,T]$.

Observe that for fixed $s$, the continuous version $\tilde{X}_s$ of the weak solution to (1.3) satisfies
\[
\sup_{|x| \leq K, |y| \leq 1} |\tilde{X}_s(x - \frac{y}{m}) - \tilde{X}_s(x - \frac{v}{m})|^{2\beta} ds \leq (2/m)^{2\eta}\beta \sup_{|x| \leq K, |y| \leq 1, y \neq v} \frac{|\tilde{X}_s(x - \frac{y}{m}) - \tilde{X}_s(x - \frac{v}{m})|^{2\beta}}{|y/m - v/m|^{2\eta}}.
\]
So, it is natural to apply the Hölder continuity of $x \mapsto \tilde{X}_s(x)$ to find a collection of suitable stopping times $(\tau_k)$ so that $\lim_{k \to \infty} \tau_k = \infty$ almost surely, and the first term $I_{4,1,1,k}(t)$ can be bounded by $4m(na_n)^{-1}(2/m)^{2\eta}\beta$ which tends to zero as $m$ and $n$ jointly go to infinity in a certain way (if $\eta > \frac{1}{2}$). It is hard to show that the supremum or integral with respect to $s \in [0,T]$ on the right side of (1.8) is finite. To this end the time $\tau_k$ is chosen so that a Riemann type “integral” of the right hand side of (1.8) over $s \in [0,\tau_k]$ is finite. Concerning the second term $I_{4,1,2}(t)$, if $\tilde{V}_s(x - \frac{x_{s,m}(x)}{m}) \neq 0$, then the function $[-1,1] \ni y \mapsto \tilde{V}_s(x - \frac{x_{s,m}(x)}{m})$ is bounded away from zero. Now we can show that the integrand of $I_{4,1,2}(t)$ is bounded from above by a deterministic function which also converges to zero as $m, n \to \infty$ under certain conditions of $\alpha$ and $\beta$.

The paper is organized as follows. In Section 2 we first present some properties of the weak solution to equation (1.3). The local Hölder continuity for the continuous version of the solution is established in Section 3. In Section 4 we prove the main result of pathwise uniqueness of solutions to (1.3). In Section 5, the proofs of Proposition 2.3 and Lemma 2.4 are presented.

**Notation:** Throughout this paper, we adopt the conventions
\[
\int_x^y = \int_{(x,y]} \quad \text{and} \quad \int_x^\infty = \int_{(x,\infty]}
\]
for any $y \geq x \geq 0$. Let $C$ denote a positive constant whose value might change from line to line. We write $C_\varepsilon$ or $C_\varepsilon'$ if the constant depends on another value $\varepsilon \geq 0$. We sometimes write $\mathbb{R}_+$ for $[0, \infty)$. Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}$ and $(P_t)_{t \geq 0}$ denote the transition semigroup of a one-dimensional Brownian motion. For $t > 0$ and $x \in \mathbb{R}$ write $p_t(x) := (2\pi t)^{-\frac{1}{2}} \exp\{-x^2/(2t)\}$. We always use $N_0(ds,dz,du, dv)$ to denote the Poisson random measure corresponding to the compensated Poisson measure $\tilde{N}_0(ds,dz,du, dv)$.

## 2. Weak solution

In this section we establish some properties of the weak solution for (\ref{1.3}), which will be used in next two sections.

**Proposition 2.1** (i) If $(X,L)$ is a weak solution of (\ref{1.3}), then there is, on an enlarged probability space, a compensated Poisson measure $\tilde{N}_0(ds,dz,du, dv)$ on $(0, \infty)^2 \times \mathbb{R} \times (0, \infty)$ with intensity $dsm(dz)du$ so that (\ref{1.5}) holds. Define a predictable $(0, \infty)$-valued process $\theta(s,z,u,v)$ by $\theta(s,z,u,v) = (\theta_1(s,z), \theta_2(s,u,v))$ with

$$
\theta_1(s,z,u) := \frac{z}{X_{s-}(u)} 1_{\{X_{s-}(u) > 0\}} + z 1_{\{X_{s-}(u) = 0\}}
$$

and

$$
\theta_2(s,u,v) := \tilde{\theta}(s,u,v) 1_{\{X_{s-}(u) > 0\}} + \tilde{\theta}(u,v) 1_{\{X_{s-}(u) = 0\}},
$$

where

$$
\tilde{\theta}(s,u,v) := \begin{cases}
  u, & v \leq X_{s-}(u)^p, \\
  \infty, & v > X_{s-}(u)^p
\end{cases}
$$

and

$$
\tilde{\theta}(u,v) := \begin{cases}
  u, & v \in (0,1), \\
  \infty, & v \in (0,1]^c
\end{cases}
$$

and we use the convention that $0 \cdot \infty = 0$. Then for all $B \in \mathcal{B}(0, \infty)$ and $a \leq b \in \mathbb{R},$

$$
1_{B \times (a,b)}(\theta(s,z,u,v)) = 1_{\{X_{s-}(u) > 0\}} 1_B\left(\frac{z}{X_{s-}(u)^p}\right) 1_{(a,b)}(u) 1_{v \leq X_{s-}(u)^p} + 1_{\{X_{s-}(u) = 0\}} 1_B(z) 1_{(a,b)}(u) 1_{(0,1)}(v).
$$

Moreover,

$$
\int_0^\infty \int_0^\infty \int_0^\infty 1_{B \times (a,b)}(\theta(s,z,u,v))m(dz)dudv = \int_0^\infty m(dz) \int_a^b du \int_0^{X_{s-}(u)^p} 1_{\{X_{s-}(u) > 0\}} 1_B\left(\frac{z}{X_{s-}(u)^p}\right) dv + \int_0^\infty \int_a^b 1_{\{X_{s-}(u) = 0\}} 1_B(z) m(dz) du = \int_0^\infty \int \int_0^\infty 1_{B \times (a,b)}(z,u)m(dz)du.
$$
Then by [7, p.93], on an extension of the probability space, there exists a Poisson random measure $N_0(ds, dz, du, dv)$ on $(0, \infty)^2 \times \mathbb{R} \times (0, \infty)$ with intensity $dsm(z)du dv$ so that

$$N(\{(0, t] \times B \times (a, b]\}) = \int_0^t \int \int \int \int_0^\infty 1_{B \times (a, b]}(\theta(s, z, u, v))N_0(ds, dz, du, dv).$$

Let $\tilde{N}_0(ds, dz, du, dv) = N_0(ds, dz, du, dv) - ds(z)du dv$. Then by (1.4) it is easy to see that for each $f \in \mathcal{S}(\mathbb{R})$,

$$\int_0^t \int \int X_s^\beta(u)f(u)L(ds, du) = \int_0^t \int \int \int_0^\infty X_s^\beta(u) f(u) z f(u) N_0(ds, dz, du, dv).$$

(ii) The proof is essentially the same as that of [10, Theorem 9.32]. Suppose that $\{X_t : t > 0, x \in \mathbb{R}\}$ satisfies (1.7). Define the random measure $N(ds, dz, du, dv)$ on $(0, \infty)^3$ by

$$N((0, t] \times B \times (a, b]) := \int_0^t \int \int \int_a^b X_s^\beta(u) 1_{\{X_s^\beta(u) > 0\}} 1_B \left( \frac{z}{X_s^\beta(u)} \right) N_0(ds, dz, du, dv)$$

$$+ \int_0^t \int \int \int_a^b 1_{\{X_s^\beta(u) = 0\}} 1_B(z) N_0(ds, dz, du, dv).$$

It is easy to see that $N(ds, dz, du)$ has a predictable compensator

$$\tilde{N}(\{(0, t] \times B \times (a, b]\}) = \int_0^t \int \int \int_a^b X_s^\beta(u) 1_{\{X_s^\beta(u) > 0\}} 1_B \left( \frac{z}{X_s^\beta(u)} \right) ds(z)du dv$$

$$+ \int_0^t \int \int \int_a^b 1_{\{X_s^\beta(u) = 0\}} 1_B(z) ds(z)du dv$$

$$= \int_0^t \int \int \int_a^b 1_B(z)ds(z)du dv.$$

Then $N(ds, dz, du)$ is a Poisson random measure with intensity $ds(z)du$ (see [3, Theorems II.1.8 and II.4.8]). Define the $a$-stable white noise $L$ by

$$L_t(a, b] = \int_0^t \int \int_a^b z \tilde{N}(ds, dz, du).$$

We then have

$$\int_0^t \int X_s^\beta(u)f(u)L(ds, du) = \int_0^t \int \int \int a^b X_s^\beta(u) f(u) z \tilde{N}(ds, dz, du)$$

$$= \int_0^t \int \int \int \int_0^\infty X_s^\beta(u) z f(u) N_0(ds, dz, du, dv)$$

for each $f \in \mathcal{S}(\mathbb{R})$. $(X, L)$ is thus a weak solution to (1.3). \hfill \Box

We need the following assumption on the weak solution:

**Assumption 2.2** Suppose that $(X, L)$ is a weak solution to (1.3). For $p = \alpha \beta > 1$, there exists a constant $q > \frac{3p}{3p - \alpha}$ so that

$$\mathbb{P}\left\{ \int_0^t ds \int X_s(x)^q dx < \infty \text{ for all } t > 0 \right\} = 1.$$
In the rest of this section, we always assume that $(X, L)$ is a weak solution to (1.3) satisfying Assumption 2.2 with deterministic initial value $X_0 \in M(\mathbb{R})$. Then it follows from Proposition 2.1 $\{X_t(x) : t > 0, x \in \mathbb{R}\}$ satisfies (1.7).

**Proposition 2.3** For any $t > 0$ and $f \in B(\mathbb{R})$ we have

$$\langle X_t, f \rangle = X_0(P_t f) + \int_0^t \int_0^x \int_0^\infty \int_0^\infty zP_{t-s}f(u)\tilde{N}_0(ds, dz, du, dv) \ P\text{-a.s.} \tag{2.1}$$

Moreover,

$$X_t(x) = \int_\mathbb{R} p_t(x-z)X_0(dz) + \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty zp_{t-s}(x-u)\tilde{N}_0(ds, dz, du, dv),$$

$$P\text{-a.s., } \lambda\text{-a.e. } x. \tag{2.2}$$

The proof is given in Appendix.

**Lemma 2.4** Let $0 < \bar{p} < \alpha$ and $T > 0$ be fixed. Then for each $0 < t \leq T$ there is a set $K_t \subset \mathbb{R}$ of Lebesgue measure zero so that

$$E\{X_t(x)^p\} \leq C_T t^{-\frac{\bar{p}}{2}}, \quad x \in \mathbb{R}\setminus K_t. \tag{2.3}$$

The proof is also given in Appendix.

**Lemma 2.5** Suppose that $T > 0$, $\delta \in (1, \alpha)$, $\delta_1 \in (\alpha, 2)$ and $0 < r < \min\{1, \frac{3-\delta}{\delta}, \frac{3-\delta_1}{\delta_1}\}$. Then for each $0 < t \leq T$ and the set $K_t \subset \mathbb{R}$ from Lemma 2.4 we have

$$E\{|X_t(x_1) - X_t(x_2)|^\delta\} \leq C_T t^{-\frac{(r+1)\delta}{2}}|x_1 - x_2|^{r\delta}, \quad x_1, x_2 \in \mathbb{R}\setminus K_t. \tag{2.4}$$

**Proof.** For $t > 0$ and $x \in \mathbb{R}$ let

$$Z_1(t, x) := \int_0^t \int_0^1 \int_\mathbb{R} \int_0^\infty zp_{t-s}(x-u)\tilde{N}_0(ds, dz, du, dv)$$

and

$$Z_2(t, x) := \int_0^t \int_0^\infty \int_\mathbb{R} \int_0^\infty zp_{t-s}(x-u)\tilde{N}_0(ds, dz, du, dv).$$

By (2.4e) of [15], for all $t > 0$, $\theta \in [0, 1]$ and $u \in \mathbb{R}$ we have

$$|p_t(x_1 - u) - p_t(x_2 - u)| \leq C|x_1 - x_2|^\theta t^{-\theta/2}|p_t(x_1 - u) + p_t(x_2 - u)|. \tag{2.5}$$

Then by (1.6) of [19] and Lemma 2.4

$$E\{|Z_1(t, x_1) - Z_1(t, x_2)|^\delta_1\} \leq C \int_0^1 z^{\delta_1}m(dz) \int_0^t ds \int_\mathbb{R} E\{X_s(u)^p\}|p_{t-s}(x_1 - u) - p_{t-s}(x_2 - u)|^{\delta_1} du$$

$$\leq C_T |x_1 - x_2|^{r\delta_1} \int_0^t s^{-p/2}(t-s)^{-\frac{r\delta_1 + \delta_1 - 1}{2}} ds \int_\mathbb{R} [p_t(x_1 - u) + p_t(x_2 - u)] du.$$
\[ \leq C_T|x_1 - x_2|^\delta \int_0^t s^{-p/2}(t-s)^{-\frac{\alpha_1 + \beta_1 - 1}{2}} \, ds. \]  

(2.6)

It follows from Hölder inequality that

\[ E\{|Z_1(t, x_1) - Z_1(t, x_2)|^\delta\} \leq \left\{ E\{|Z_1(t, x_1) - Z_1(t, x_2)|^{\delta_1}\}\right\}^{\frac{\delta}{\delta_1}}. \]

(2.7)

Similar to (2.6) we have

\[ E\{|Z_2(t, x_1) - Z_2(t, x_2)|^\delta\} \leq C_T|x_1 - x_2|^\delta \int_0^t s^{-p/2}(t-s)^{-\frac{\alpha_1 + \beta_1 - 1}{2}} \, ds. \]

(2.8)

Combining (2.6)–(2.8) one has

\[ E\{|Z_1(t, x_1) - Z_1(t, x_2)|^\delta + |Z_2(t, x_1) - Z_2(t, x_2)|^\delta\} \leq C_T t^{-\frac{\alpha_1 + \beta_1 - 1}{2}}|x_1 - x_2|^\delta. \]

(2.9)

By (2.5) we have

\[ \left| \int \rho_t(x_1 - y)X_0(dy) - \int \rho_t(x_2 - y)X_0(dy) \right| \]
\[ \leq \int |\rho_t(x_1 - y) - \rho_t(x_2 - y)|X_0(dy) \leq C|x_1 - x_2|^r t^{-\frac{r + 1}{2}}X_0(1), \]

which together with (2.2) and (2.9) implies (2.4). \qed

Lemma 2.6 For each \( t > 0 \) and \( t_n > 0 \) satisfying \( t_n \to t \) as \( n \to \infty \), there is a set \( K_t \subset \mathbb{R} \) of Lebesgue measure zero so that

\[ \lim_{n \to \infty} E\{|X_{t_n}(x) - X_t(x)|\} = 0, \quad x \in \mathbb{R} \setminus K_t. \]

Proof. For \( t_0, t > 0 \), by (2.2),

\[ E\{|X_{t_0+t}(x) - X_{t_0}(x)|\} \]
\[ \leq \int \left| \int_{t_0}^{t_0+t} \rho_t(x-y) - \rho_{t_0}(x-y) \right| X_0(dy) \]
\[ + \left| \int_{t_0}^{t_0+t} \int_0^{\infty} \int_0^{X_{t_0-}(u)^p} \rho_{t_0+t-s}(x-u) \tilde{N}_0(ds, dz, du, dv) \right| \]
\[ + \left| \int_{t_0}^{t_0+t} \int_0^{\infty} \int_0^{X_{t_0-}(u)^p} \rho_{t_0+s}(x-u) \tilde{N}_0(ds, dz, du, dv) \right| \]
\[ =: I_1(t_0, t) + |I_2(t_0, t)| + |I_3(t_0, t)|. \]

By dominated convergence, \( I_1(t_0, t) \) tends to zero as \( t \to 0 \).

Let

\[ I_{2,1}(t_0, t) := \int_{t_0}^{t_0+t} \int_0^1 \int_0^1 \int_0^{X_{t_0-}(u)^p} \rho_{t_0+t-s}(x-u) \tilde{N}_0(ds, dz, du, dv) \]

and

\[ I_{3,1}(t_0, t) := \int_{t_0}^{t_0+t} \int_0^1 \int_0^1 \int_0^{X_{t_0-}(u)^p} \rho_{t_0+s}(x-u) \tilde{N}_0(ds, dz, du, dv). \]
Let $I_{2,2}(t_0, t) := I_2(t_0, t) - I_{2,1}(t_0, t)$ and $I_{3,1}(t_0, t) := I_3(t_0, t) - I_{3,1}(t_0, t)$. Then by Lemma 2.4 and dominated convergence, both

$$
\mathbb{E}\{I_{2,1}(t_0, t)^2\} = \int_0^1 \int_{t_0}^{t_0+t} \mathbb{E}\{X_s(u)^p\} p_{t_0+t-s}(x-u)^2 du \\
\quad \leq C_T \int_{t_0}^{t_0+t} s^{-\frac{p}{2}}(t_0 + t - s)^{-\frac{1}{2}} ds
$$

and

$$
\mathbb{E}\{I_{3,1}(t_0, t)^2\} = \int_0^1 \int_{t_0}^{t_0+t} \mathbb{E}\{X_s(u)^p\} [p_{t_0+t-s}(x-u) - p_{t_0-s}(x-u)]^2 du \\
\quad \leq C_T \int_{t_0}^{t_0+t} s^{-\frac{p}{2}} ds \int_{t_0}^{t_0+t} [p_{t_0+t-s}(x-u) - p_{t_0-s}(x-u)]^2 du
$$

tend to zero as $t \to 0$.

Similarly, both

$$
\mathbb{E}\{|I_{2,2}(t_0, t)|\} \leq \mathbb{E}\left\{ \int_{t_0}^{t_0+t} \int_0^\infty \int_0^x z p_{t_0+t-s}(x-u) N_0(ds, dz, du, dv) \right\} \\
\quad + \mathbb{E}\left\{ \int_0^\infty \int_{t_0}^{t_0+t} z m(ds) \int_{t_0+t}^0 ds \int_0^x X_s(u)^p p_{t_0+t-s}(x-u) du \right\} \\
\quad = 2 \int_0^\infty \int_{t_0}^{t_0+t} z m(ds) \int_{t_0+t}^0 ds \int_0^x \mathbb{E}\{X_s(u)^p\} p_{t_0+t-s}(x-u) du
$$

and

$$
\mathbb{E}\{|I_{3,2}(t_0, t)|\} \leq 2 \int_0^\infty \int_{t_0}^{t_0+t} z m(ds) \int_{t_0+t}^0 ds \int_0^x \mathbb{E}\{X_s(u)^p\} [p_{t_0+t-s}(x-u) - p_{t_0-s}(x-u)] du
$$

tend to zero as $t \to 0$. The proof is completed. \hfill \Box

3 Hölder continuities

In this section we establish the local Hölder continuity of the weak solution to (1.3) at fixed time $t > 0$. Throughout this section we always assume that $(X, L)$ is a weak solution to (1.3) satisfying Assumption 2.2 with deterministic initial value $X_0 \in M(\mathbb{R})$. For $n \geq 1$ and $0 \leq k \leq 2^n$, put $n_k := k/2^n$.

**Theorem 3.1** For fixed $t > 0$, with probability one $X_t$ has a continuous version $\tilde{X}_t$. Moreover, for each $\eta < \eta_c = \frac{2}{\alpha} - 1$, the continuous version $\tilde{X}_t$ is locally Hölder continuous of order $\eta$, i.e. for any compact set $K \subset \mathbb{R}$,

$$
\sup_{x, z \in K, x \neq z} \frac{|\tilde{X}_t(x) - \tilde{X}_t(z)|}{|x - z|^{\eta}} < \infty \text{ \ P-a.s.}
$$

Moreover, for each $T > 0$ and subsequence $\{n' : n' \geq 1\}$ of $\{n : n \geq 1\}$, we have

$$
\liminf_{n' \to \infty} \frac{1}{2^{n'}} \sum_{k=1}^{2^{n'}} \sup_{x, z \in K, x \neq z} \frac{|\tilde{X}_{n'T}(x) - \tilde{X}_{n'T}(z)|}{|x - z|^{\eta}} < \infty \text{ \ P-a.s.}
$$
For any \( x \in \mathbb{R} \) and \( s > 0 \), let
\[
Z_s(x) := \int_0^s \int_0^\infty \int_0^\infty \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} 1_{\{v \leq x - (u)^p\}} z p_{s-1}(x - u) \tilde{N}_0(ds, dz, du, dv).
\]

**Lemma 3.2** The results of Theorem 3.1 also hold with \( \eta_c \) replaced by \( \eta_c' = \eta_c 1_{(\alpha \geq \frac{3}{2})} + \frac{2-\alpha}{\alpha} 1_{(\alpha < \frac{3}{2})} \).

**Proof.** Let \( r, \delta \) and \( \delta_1 \) satisfy the conditions in Lemma 2.5 and \( r \delta > 1 \). By (2.9) and Corollary 1.2(ii) of [20], for each \( 0 < \varepsilon < r - 1/\delta \) and \( T > 0 \),
\[
\sup_{n \geq 1} \mathbb{E} \left\{ \frac{1}{2^n} \sum_{k=1}^{2^n} \sup_{x,y \in K, x \neq y} \frac{|\tilde{Z}_{n_k} t(x) - \tilde{Z}_{n_k} t(y)|}{|x-y|^{\varepsilon}} \right\} < \infty,
\]
where \( \tilde{Z}_t(x) \) denotes a continuous version of \( Z_t(x) \) for each \( t > 0 \). This implies that for each subsequence \( \{n': n' \geq 1\} \) of \( \{n : n \geq 1\} \),
\[
\lim \inf_{n' \to \infty} \frac{1}{2^{n'}} \sum_{k=1}^{2^{n'}} \sup_{x,z \in K, x \neq z} \frac{|\tilde{Z}_{n'_k} t(x) - \tilde{Z}_{n'_k} t(z)|}{|x-z|^{\eta}} < \infty
\]
almost surely by the Markov inequality. By (2.5) we have
\[
\int_\mathbb{R} |p_t(x-u) - p_t(z-u)|X_0(du) \leq C|x-z|^t t^{-\frac{\varepsilon+1}{2}} X_0(1), \quad t > 0.
\]
Then
\[
\lim \sup_{n' \to \infty} \frac{1}{2^{n'}} \sup_{x,z \in K, x \neq z} \int_\mathbb{R} |p_{n'_k} t(x-u) - p_{n'_k} t(z-u)|X_0(du) dt < \infty.
\]
Therefore, Theorem 3.1 holds with \( \eta_c \) replaced by \( \varepsilon \). Now via the same argument as in Proposition 2.9 of [3] one can finish the proof.

**Lemma 3.3** For fixed \( t > 0 \), with probability one \( X_t \) has a continuous version \( \tilde{X}_t \). Moreover, for any compact subset \( K \) of \( \mathbb{R} \) and \( \delta \in (1, \alpha) \), we have
\[
\sup_{t \in (0,T]} t^\delta \mathbb{E} \left\{ \sup_{x \in K} \tilde{X}_t(x)^\delta \right\} < \infty. \tag{3.1}
\]

**Proof.** The first assertion follows immediately from Corollary 1.2(i) of [20] and Lemma 2.5. It follows from Lemma 2.4 that
\[
\sup_{t \in (0,T]} t^\delta \mathbb{E} \left\{ |\tilde{X}_t(0)|^\delta \right\} < \infty. \tag{3.2}
\]
Then one can complete the proof of (3.1) by (3.2), Lemma 2.5 and Corollary 1.2(iii) of [20].

By Proposition 2.1, \( \{X_t(x) : t > 0, x \in \mathbb{R}\} \) satisfies (1.7) with \( X_0 \in M(\mathbb{R}) \). Similar to [3, Lemma 2.12] we can prove the following lemma.
Lemma 3.4 Fix \( \delta \in [1, 3) \), \( r \in [0, 1) \) with \( r < \frac{3-\delta}{\delta} \) and a nonempty compact \( K \subset \mathbb{R} \). Define

\[
V_n := \int_0^1 \frac{1}{2^n} \sum_{k=1}^{2^n} 1_{\{nk_{k-1},nk_k\}}(s) \sum_{i=k}^{2^n} (n_i - s)^{-\frac{\delta+\delta-1}{2}} V_{n,i}(s) ds
\]

with

\[
V_{n,i}(s) := \sup_{x_1,x_2 \in K} \int_{\mathbb{R}} X_s(u)^p |p_{n_i-s}(x_1-u) + p_{n_i-s}(x_2-u)| du.
\]

Then we have

\[
\sup_{n \geq 1} P\{V_n \geq C_\varepsilon\} \leq \varepsilon, \quad \varepsilon > 0 \tag{3.3}
\]

and

\[
\int_0^1 ds \int_{\mathbb{R}} X_s(u)^p \frac{1}{2^n} \sum_{k=1}^{2^n} 1_{\{nk_{k-1},nk_k\}}(s) \sum_{i=k}^{2^n} |p_{n_i-s}(x_1-u) - p_{n_i-s}(x_2-u)|^\delta du \leq CV_n |x_1-x_2| r^\delta, \quad x_1, x_2 \in K. \tag{3.4}
\]

Proof. We assume that \( K \subset [0, 1] \) for simplicity. Observe that

\[
V_{n,i}(s) \leq \sup_{x_1,x_2 \in [0,1]} \int_{|y| \geq 2} X_s(y)^p \left[p_{n_i-s}(x_1-y) + p_{n_i-s}(x_2-y)\right] dy + 2 \sup_{|y| \leq 2} \bar{X}_s(y)^p
\]

\[
\leq 2 \int_{\mathbb{R}} X_s(y)^p \left[p_{n_i-s}(y+1) + p_{n_i-s}(y-1)\right] dy + 2 \sup_{|y| \leq 2} \bar{X}_s(y)^p
\]

almost surely. Then by Lemmas 2.4 and 3.3

\[
E\{V_{n,i}(s)\} \leq Cs^{-\frac{p}{2}}.
\]

It is elementary to check that

\[
\sup_{n \geq 1} E\{V_n\} \leq \sup_{n \geq 1} C \int_0^1 \frac{1}{2^n} \sum_{k=1}^{2^n} 1_{\{nk_{k-1},nk_k\}}(s) \sum_{i=k}^{2^n} (n_i - s)^{-\frac{\delta+\delta-1}{2}} s^{-\frac{p}{2}} ds
\]

\[
\leq \sup_{n \geq 1} C \frac{2^n}{2^n} \sum_{k=1}^{2^n} \sum_{i=k}^{2^n} \int_{nk_{k-1}}^{nk_k} (n_i - s)^{-\frac{\delta+\delta-1}{2}} s^{-\frac{p}{2}} ds
\]

\[
= \sup_{n \geq 1} C \frac{2^n}{2^n} \sum_{i=1}^{2^n} \int_0^{n_i} (n_i - s)^{-\frac{\delta+\delta-1}{2}} s^{-\frac{p}{2}} ds \leq \infty.
\]

Then (3.3) follows from the Markov inequality. Using (2.5) one gets

\[
\int_0^1 ds \int_{\mathbb{R}} X_s(u)^p \sum_{k=1}^{2^n} 1_{\{nk_{k-1},nk_k\}}(s) \frac{1}{2^n} \sum_{i=k}^{2^n} |p_{n_i-s}(x_1-u) - p_{n_i-s}(x_2-u)|^\delta du
\]

\[
\leq C|x_1-x_2|^\delta \frac{1}{2^n} \sum_{k=1}^{2^n} 1_{\{nk_{k-1},nk_k\}}(s) ds \int_{\mathbb{R}} X_s(u)^p \sum_{i=k}^{2^n} (n_i - s)^{-\frac{\delta+\delta-1}{2}} |p_{n_i-s}(x_1-u) + p_{n_i-s}(x_2-u)| du,
\]

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which implies (3.24).

For \( t \geq 0 \) and \( \psi \in B(\mathbb{R}) \) define the discontinuous martingales
\[
t \mapsto M_t^1(\psi) := \int_0^t \int_0^\infty \int_0^\infty X_{s-}^{(u)^p} \psi(u) 1_{\{|u| \leq |K|+1\}} \tilde{N}_0(ds, dz, du, dv)
\]
and
\[
t \mapsto M_t^2(\psi) := \int_0^t \int_0^\infty \int_0^\infty X_{s-}^{(u)^p} \psi(u) \frac{1}{|u|} 1_{\{|u| > |K|+1\}} \tilde{N}_0(ds, dz, du, dv).
\]

For \( i = 1, 2 \) let \( \Delta M^i_t(y) \) denote the jumps of \( M^i(ds, dy) \). Write \(|K| := \sup_{x \in K} |x|\). Similar to Lemma 2.14 one can show the following result.

**Lemma 3.5** Let \( \varepsilon > 0 \) and \( \gamma \in (0, \alpha^{-1}) \). Then for each \( \varepsilon > 0 \) and \( n \geq 1 \),
\[
\mathbb{P}\left( \bigcup_{k=1}^{2^n} \left\{ \Delta M^1_k(y) > 2^{\lambda n} C_\varepsilon (n_k - s)^{\lambda} \text{ for some } s \in [n_{k-1}, n_k) \text{ and } |y| \leq |K| + 1 \right\} \right) \leq \varepsilon
\]
and
\[
\mathbb{P}\left( \bigcup_{k=1}^{2^n} \left\{ \Delta M^2_k(y) > 2^{\lambda n} C_\varepsilon (n_k - s)^{\lambda} \text{ for some } s \in [n_{k-1}, n_k) \text{ and } |y| > |K| + 1 \right\} \right) \leq \varepsilon.
\]
where \( \lambda := \alpha^{-1} - \gamma \).

**Proof.** Since the proofs are similar, we only present the first one. Let \( c > 0 \). For \( n \geq 1 \) and \( 1 \leq k \leq 2^n \) put
\[
Y^1_{n,k} = N_0\left( (s, z, u, v) : s \in [n_{k-1}, n_k), z \geq 2^{\lambda n} c(n_k - s)^{\lambda}, |u| \leq |K| + 1, v \leq Y_{s-}^{(u)^p}\right).
\]
Then by the Markov inequality for all \( n \geq 1 \) and \( 1 \leq k \leq 2^n \),
\[
\mathbb{P}\left\{ \frac{\Delta M^1_k(y)}{2^{\lambda n}(n_k - s)^{\lambda}} > c \text{ for some } s \in [n_{k-1}, n_k) \text{ and } |y| \leq |K| + 1 \right\} = \mathbb{P}\{Y^1_{n,k} \geq 1\} \leq \mathbb{E}\{Y^1_{n,k}\}.
\]
By Lemma 2.4,
\[
\mathbb{E}\{Y^1_{n,k}\} = \int_{n_{k-1}}^{n_k} ds \int_0^\infty m(dz) \int_{-(|K|+1)}^{(|K|+1)} du \int_0^\infty X_{s-}^{(u)^p} 1_{\{|z| > 2^{\lambda n}(n_k - s)^{\lambda}\}} dv
\]
\[
= Cc^{-\alpha} \int_{n_{k-1}}^{n_k} ds \int_{-(|K|+1)}^{(|K|+1)} \mathbb{E}\{X_{s}^{(u)^p}\} 2^{-\alpha \lambda n} (n_k - s)^{-\alpha} du
\]
\[
\leq Cc^{-\alpha} 2^{-\alpha \lambda n} \int_{n_{k-1}}^{n_k} (n_k - s)^{-\alpha} s^{-\frac{\alpha}{2}} ds \leq Cc^{-\alpha} n_k^{-\frac{\alpha}{2}} 2^{-n},
\]
which implies
\[
\sum_{k=1}^{2^n} \mathbb{E}\{Y^1_{n,k}\} \leq C c^{-\alpha}.
\]
Proof of Theorem 3.1.

The proof is similar to that of \[3, \text{Lemma 2.15}\].

where \( T \) modification of that of \[3, \text{Theorem 1.2(a)} \] which proceeds as follows. Let \( \psi \) be the space of measurable functions \( \psi \) on \( \mathbb{R}_+ \times \mathbb{R} \) so that

\[
\int_0^t s^{-\frac{\alpha}{2}} ds \int \left| \psi(s,x) \right| + |\psi(s,x)|^2 dx < \infty, \quad t > 0.
\]

Similar to \[3, \text{Lemma 2.15}\], we can prove the next result.

**Lemma 3.6** Given \( \psi \in \mathcal{L} \) with \( \psi \geq 0 \), there exists a spectrally positive \( \alpha \)-stable process \( \{L_t : t \geq 0\} \) so that

\[
Z_t(\psi) := \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty 1_{\{v \leq X_{s-}(u)p\}} z\psi(s,u)\tilde{N}_0(ds,dz,du,dv) = L_{T(t)}, \quad t \geq 0.
\]

where \( T(t) := \int_0^t ds \int \mathbb{R} X_s(u)p\psi(s,u)\alpha du \).

The proof is similar to that of \[3, \text{Lemma 2.15}\].

**Proof of Theorem 3.1.** By Lemma 3.2 we only consider the case \( \alpha < 3/2 \). The proof is a modification of that of \[3, \text{Theorem 1.2(a)} \] which proceeds as follows. Let \( X_0 \in \mathcal{M}(\mathbb{R}) \) be fixed. We assume that \( T = 1 \) in this proof. Recall that \( \lambda = \frac{1}{\alpha} - \gamma \) and

\[
Z_s(x) = \int_0^s \int_0^\infty \int_0^\infty \int_0^\infty 1_{\{v \leq X_{s-}(u)p\}} z p_{s-s_1}(x-u)\tilde{N}_0(ds, dz, du, dv).
\]

Also recall that \( n_k = \frac{k}{2^n} \) for \( n \geq 1 \) and \( 0 \leq k \leq 2^n \). Then

\[
\frac{1}{2^n} \sum_{k=1}^{2^n} [Z_{n_k}(x_1) - Z_{n_k}(x_2)]
\]

\[
= \frac{1}{2^n} \sum_{k=1}^{2^n} \int_0^\infty \int_0^\infty \int_0^\infty z[p_{n_k-s}(x_1-u) - p_{n_k-s}(x_2-u)] \tilde{N}_0(ds, dz, du, dv)
\]

\[
= \int_0^1 \int_0^\infty \int_0^\infty \int_0^\infty 1_{\{v \leq X_{s-}(u)p\}} \frac{1}{2^n} \sum_{k=1}^{2^n} 1_{\{0,n_k\}}(s) z[p_{n_k-s}(x_1-u) - p_{n_k-s}(x_2-u)] \tilde{N}_0(ds, dz, du, dv)
\]

\[
= \int_0^1 \int_0^\infty \int_0^\infty \int_0^\infty 1_{\{v \leq X_{s-}(u)p\}} \frac{1}{2^n} \sum_{k=1}^{2^n} 1_{\{n_k-1,n_k\}}(s) \sum_{i=k}^{2^n} z[p_{s_1-s}(x_1-u) - p_{s_1-s}(x_2-u)] \tilde{N}_0(ds, dz, du, dv)
\]

\[
= \int_0^1 \int_0^\infty \int_0^\infty \int_0^\infty 1_{\{v \leq X_{s-}(u)p\}} \frac{1}{2^n} \sum_{k=1}^{2^n} 1_{\{n_k-1,n_k\}}(s) \sum_{i=k}^{2^n} z[p_{s_1-s}(x_1-u) - p_{s_1-s}(x_2-u)] \tilde{N}_0(ds, dz, du, dv)
\]

\[
- \int_0^1 \int_0^\infty \int_0^\infty \int_0^\infty 1_{\{v \leq X_{s-}(u)p\}} \frac{1}{2^n} \sum_{k=1}^{2^n} 1_{\{n_k-1,n_k\}}(s) \sum_{i=k}^{2^n} z[p_{s_1-s}(x_2-u) - p_{s_1-s}(x_1-u)] \tilde{N}_0(ds, dz, du, dv),
\]

where \( \psi_n^+(s,u) \) and \( \psi_n^-(s,u) \) are, respectively, the positive part and the negative part of

\[
\frac{1}{2^n} \sum_{k=1}^{2^n} 1_{\{n_k-1,n_k\}}(s) \sum_{i=k}^{2^n} [p_{n_i-s}(x_1-u) - p_{n_i-s}(x_2-u)].
\]
One can see that both $\psi_n^+(s,u)$ and $\psi_n^-(s,u)$ satisfy the assumptions of Lemma 3.6, and there is a stable process $L^1$ and $L^2$ so that

$$\frac{1}{2^n} \sum_{k=1}^{2^n} |Z_{n_k}(x_1) - Z_{n_k}(x_2)| = L^1_{T^+_n} - L^2_{T^-_n}, \quad (3.5)$$

where $T^\pm_n := \int_0^1 ds \int_\mathbb{R} X_s(u)^p[\psi_n^\pm(s,u)]^\alpha du$. Let $\varepsilon \in (0,1)$ be fixed. By Lemma 3.4

$$\sup_{n \geq 1} \mathbb{P}(V_n > C_\varepsilon) \leq \varepsilon. \quad (3.6)$$

Set

$$A_n^\varepsilon := \bigcap_{k=1}^{2^n} \left\{ \frac{\Delta M_1^1(y)}{2^{\lambda n}(n_k - s)^\lambda} \leq C_\varepsilon \text{ for all } s \in [n_{k-1}, n_k) \text{ and } |y| \leq |K| + 1 \right\} \cap \left\{ \frac{\Delta M_2^2(y)}{2^{\lambda n}(n_k - s)^\lambda} \leq C_\varepsilon \text{ for all } s \in [n_{k-1}, n_k) \text{ and } |y| > |K| + 1 \right\} \cap \left\{ V_n \leq C_\varepsilon \right\}.$$  

By Lemmas 3.4 and 3.5

$$\sup_{n \geq 1} \mathbb{P}(A_n^{\varepsilon,c}) \leq 3\varepsilon,$$

where $A_n^{\varepsilon,c}$ denotes the complement of $A_n^{\varepsilon}$. Define $Z_t^{n,\varepsilon}(x) = Z_t(x)1_{A_n^{\varepsilon}}$. From (3.5), for $r > 0$ we have

$$\mathbb{P}\left\{ \frac{1}{2^n} \sum_{k=1}^{2^n} |Z_{n_k}^{n,\varepsilon}(x_1) - Z_{n_k}^{n,\varepsilon}(x_2)| \geq 2r|x_1 - x_2|\right\}$$

$$\leq \mathbb{P}\left\{ L^1_{T^+_n} > r|x_1 - x_2|^\eta, A_n^\varepsilon \right\} + \mathbb{P}\left\{ L^2_{T^-_n} > r|x_1 - x_2|^\eta, A_n^\varepsilon \right\}. \quad (3.7)$$

Note that on $A_n^{\varepsilon}$ the jumps of $M_1^1(x)$ do not exceed

$$C_\varepsilon 2^{\lambda n}(n_k - s)^\lambda, \quad s \in [n_{k-1}, n_k).$$

Then the jumps of

$$(0,1) \ni t \rightarrow \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty 1_{\{ 0 < X_s(u)^p \} } \tilde{\psi}_n^+(s,u) 1_{\{|u| \leq |K| + 1\}} \tilde{N}_0(ds,dz,du,dv)$$

are bounded by

$$I_n := C_\varepsilon 2^{\lambda n} \sup_{1 \leq k \leq 2^n} \sup_{(s,y) \in [n_{k-1}, n_k) \times \mathbb{R}} (n_k - s)^\lambda \psi_n^+(s,y)$$

$$\leq C_\varepsilon 2^{\lambda n} \sup_{1 \leq k \leq 2^n} \sup_{(s,y) \in [n_{k-1}, n_k) \times \mathbb{R}} \frac{1}{2^n} \sum_{i=k}^{2^n} (n_k - s)^\lambda |p_{n_i-s}(x_1 - y) - p_{n_i-s}(x_2 - y)|. \quad (3.8)$$

Applying (3.8) with $\theta = \eta_c - 2\gamma$ gives

$$\sup_{y \in \mathbb{R}} \frac{1}{2^n} \sum_{i=k}^{2^n} (n_k - s)^\lambda |p_{n_i-s}(x_1 - y) - p_{n_i-s}(x_2 - y)|$$
Applying Lemma 3.4 with $\delta$ are bounded by

$$C|\eta - 2\gamma| \frac{1}{2^n} \sum_{i=k}^{2^n} (n_i - s)^{-\eta/2+\gamma} (n_k - s)^{\lambda} \sup_{y \in \mathbb{R}} p_{n_i-s}(y)$$

$$\leq C|\eta - 2\gamma| \frac{1}{2^n} \sum_{i=k}^{2^n} (n_k - s)^{\lambda}$$

$$\leq C|\eta - 2\gamma| \frac{1}{2^n} \sum_{i=k}^{2^n} \left( \frac{1}{i - k + 1} \right)^{\lambda} \leq C|x_1 - x_2|^{\eta - 2\gamma}$$

for $s \in [n_{k-1}, n_k)$. This implies

$$I_n \leq C_\epsilon |x_1 - x_2|^{\eta - 2\gamma}. \quad (3.9)$$

Similarly, one can see that the jumps of

$$l \mapsto \int_0^l \int_0^\infty \int_\mathbb{R} \int_0^\infty 1_{v \leq X_{s-}(u)^p} 1_{|u| > |K| + 1} z\psi^n_\alpha(s, u) \tilde{N}_0(ds, dz, du, dv)$$

are bounded by

$$C_\epsilon |x_1 - x_2|^{\eta - 2\gamma}.$$ Combining with (3.9) we conclude that the jumps of

$$l \mapsto \int_0^l \int_0^\infty \int_\mathbb{R} \int_0^\infty 1_{v \leq X_{s-}(u)^p} z\psi^n_\alpha(s, u) \tilde{N}_0(ds, dz, du, dv)$$

on $A_\epsilon^\delta$ are bounded by

$$C_\epsilon |x_1 - x_2|^{\eta - 2\gamma}.$$ By an abuse of notation we write $L_{T_n^\pm}$ for $L_{T_n^1}$ and $L_{T_n^2}$. Then

$$\mathbb{P}\left\{ L_{T_n^\pm} \geq r |x_1 - x_2|^\eta, A_\epsilon^\delta \right\}$$

$$= \mathbb{P}\left\{ L_{T_n^\pm} \geq r |x_1 - x_2|^\eta, \sup_{u \leq T_n^\pm} \Delta L_u \leq C_\epsilon |x_1 - x_2|^{\eta - 2\gamma}, A_\epsilon^\delta \right\}$$

$$\leq \mathbb{P}\left\{ \sup_{v \leq T_n^\pm} L_v 1_{\left\{ \sup_{u < v} \Delta L_u \leq C_\epsilon |x_1 - x_2|^{\eta - 2\gamma} \right\}} \geq r |x_1 - x_2|^\eta, A_\epsilon^\delta \right\}. \quad (3.10)$$

Observe that

$$T_n^\pm \leq \int_0^1 ds \int_\mathbb{R} X_s(u)^p \sum_{k=1}^{2^n} 1_{(n_{k-1}, n_k)}(s) \left| \frac{1}{2^n} \sum_{i=k}^{2^n} |p_{n_i-s}(x_1 - u) - p_{n_i-s}(x_2 - u)|^\alpha \right| du$$

$$\leq \int_0^1 ds \int_\mathbb{R} X_s(u)^p \sum_{k=1}^{2^n} 1_{(n_{k-1}, n_k)}(s) \left| \frac{1}{2^n} \sum_{i=k}^{2^n} |p_{n_i-s}(x_1 - u) - p_{n_i-s}(x_2 - u)|^\alpha \right| du$$

Applying Lemma 3.4 with $\delta = \alpha$ and $r = 1$ one gets

$$T_n^\pm \leq C_\epsilon |x_1 - x_2|^\alpha \quad \text{on} \{ V_n \leq C_\epsilon \}. $$
Combining this with (3.10) we have
\[
P\left\{ L_{T_n^+}^r \geq r|x_1 - x_2|^\eta, A_n^e \right\} 
\leq P\left\{ \sup_{u \leq C\varepsilon|x_1 - x_2|^\alpha} L_v 1_{\{\sup_{u < v} \Delta L_u \leq C\varepsilon|x_1 - x_2|^{\alpha - 2\gamma}\}} \geq r|x_1 - x_2|^\eta \right\}.
\]
Using (3.14) of [3], and [3 Lemma 2.3] with \(\kappa = \alpha\), \(t = C\varepsilon|x_1 - x_2|^\alpha\), \(x = r|x_1 - x_2|^\eta\), and \(y = C\varepsilon|x_1 - x_2|^{\alpha - 2\gamma}\), one obtains
\[
P\left\{ L_{T_n^+}^r \geq r|x_1 - x_2|^\eta, A_n^e \right\} \leq \left(C\varepsilon r^{-1}|x_1 - x_2|^{2\alpha - 2}\right) C_n^c r|x_1 - x_2|^\eta + 2\gamma.
\] (3.11)
Taking \(\gamma := \frac{\eta - \eta}{4}\), we have\[
P\left\{ L_{T_n^+}^r \geq r|x_1 - x_2|^\eta, A_n^e \right\} \leq \left(C\varepsilon r^{-1}|x_1 - x_2|\right) C_n^c r|x_1 - x_2|^{(\eta - \eta_\kappa)/2}.
\]
which together with (3.7) gives\[
P\left\{ \frac{1}{2^n} \sum_{k=1}^{2^n} |Z_{n_k}^{\varepsilon}(x_1) - Z_{n_k}^{\varepsilon}(x_2)| \geq 2r|x_1 - x_2|^\eta \right\} \leq 2(C\varepsilon r^{-1}|x_1 - x_2|) C_n^c r|x_1 - x_2|^{(\eta - \eta_\kappa)/2}.
\]
By [3 Lemma III.5.1], it is easy to see that\[
P\left\{ \frac{1}{2^n} \sum_{k=1}^{2^n} \sup_{x,z \in K, |x-z| \leq \delta} |\tilde{Z}_{n_k}^{\varepsilon}(x) - \tilde{Z}_{n_k}^{\varepsilon}(z)| \geq rG\left( \left\lfloor \log_2 \frac{|K|}{2\delta} \right\rfloor \right) \right\} \leq Q\left( \left\lfloor \log_2 \frac{|K|}{2\delta} \right\rfloor, r \right) \quad (3.12)
\]
for all \(n \geq 1\) and \(\delta > 0\), where \(\tilde{Z}_{n_k}^{\varepsilon}(x)\) is a continuous version of \(Z_{n_k}^{\varepsilon}(x)\),
\[
G(m) := \sum_{l=m}^{\infty} 2(2^{-l}|K|)^\eta = \frac{2|K|^\eta}{1 - 2^{-\eta} 2^{-\eta m}}
\]
and
\[
Q(m, r) := \sum_{l=m}^{\infty} 2^l 2(C\varepsilon r^{-1}2^{-l}|K|) C_n^c r(2^{-l}|K|)(\eta - \eta_\kappa)/2.
\]
It is easy to check that\[
Q(r) := \sum_{m=0}^{\infty} Q(m, r) < \infty \quad \text{and} \quad Q(r) \to 0 \text{ as } r \to \infty. \quad (3.13)
\]
Observe that for each \(m, n \geq 1\) and \(1 \leq k \leq 2^n\)
\[
\sup_{x,z \in K, \frac{1}{2^{m+1}} < |x-z| \leq \frac{\delta}{2^{m+1}}} \frac{|\tilde{Z}_{n_k}^{\varepsilon}(x) - \tilde{Z}_{n_k}^{\varepsilon}(z)|}{|x-z|^\eta} \leq \sup_{x,z \in K, |x-z| \leq \frac{\delta}{2^{m+1}}} |\tilde{Z}_{n_k}^{\varepsilon}(x) - \tilde{Z}_{n_k}^{\varepsilon}(z)| \left( \frac{\delta}{2^{m+1}} \right)^{-\eta}.
\]
This implies
\[
\left\{ \frac{1}{2^n} \sum_{k=1}^{2^n} \sup_{x,z \in K, \frac{1}{2^{m+1}} < |x-z| \leq \frac{\delta}{2^{m+1}}} \frac{|\tilde{Z}_{n_k}^{\varepsilon}(x) - \tilde{Z}_{n_k}^{\varepsilon}(z)|}{|x-z|^\eta} \geq r \right\}
\]
\[ \{ 1 \leq \sum_{k=1}^{2^n} \sup_{x-z \leq \delta} |\tilde{Z}_{n_k}(x) - \tilde{Z}_{n_k}(z)| \geq r \left( \frac{\delta}{2^{m+1}} \right)^{\eta} \} \]

\[ \subset \{ 1 \leq \sum_{k=1}^{2^n} \sup_{x-z \leq \delta} |\tilde{Z}_{n_k}(x) - \tilde{Z}_{n_k}(z)| \geq rc_1 G \left( \left\lfloor \log_2 \frac{|K|}{2^{-m} \delta} \right\rfloor \right) \}, \]

where \( c_1 := \inf_{m \geq 0} G \left( \left\lfloor \log_2 \frac{|K|}{2^{-m} \delta} \right\rfloor \right) > 0 \). It follows from (3.12) that for each \( m \geq 0 \),

\[ P \left\{ \frac{1}{2^n} \sum_{k=1}^{2^n} \sup_{x-z \leq \delta} \left| \tilde{Z}_{n_k}(x) - \tilde{Z}_{n_k}(z) \right| \geq r \right\} \]

\[ \leq P \left\{ \frac{1}{2^n} \sum_{k=1}^{2^n} \left| \tilde{Z}_{n_k}(x) - \tilde{Z}_{n_k}(z) \right| \geq rc_1 G \left( \left\lfloor \log_2 \frac{|K|}{2^{-m} \delta} \right\rfloor \right) \right\} \]

\[ \leq Q \left( \left\lfloor \log_2 \frac{|K|}{2^{-m} \delta} \right\rfloor, rc_1 \right), \]

which implies

\[ P \left\{ \frac{1}{2^n} \sum_{k=1}^{2^n} \sup_{x-z \leq \delta} \left| \tilde{Z}_{n_k}(x) - \tilde{Z}_{n_k}(z) \right| \geq r \right\} \leq Q(rc_1). \]

Then by Fatou’s lemma and (3.6) for each subsequence \( \{n'\} \) of \( \{n\} \),

\[ P \left\{ \liminf_{n' \to \infty} \frac{1}{2^n'} \sum_{k=1}^{2^n'} \sup_{x-z \leq \delta} \left| Z_{n_k}(x) - Z_{n_k}(z) \right| \geq r \right\} \]

\[ \leq \liminf_{n' \to \infty} \left\{ P \left( \frac{1}{2^n'} \sum_{k=1}^{2^n'} \sup_{x-z \leq \delta} \left| Z_{n_k}(x) - Z_{n_k}(z) \right| \geq r \right) \right\} \]

\[ \leq \liminf_{n' \to \infty} \left\{ \left( P \left( \frac{1}{2^n'} \sum_{k=1}^{2^n'} \sup_{x-z \leq \delta} \left| \tilde{Z}_{n_k}(x) - \tilde{Z}_{n_k}(z) \right| \geq r \right) + P(A_{n'}^{\varepsilon,c}) \right) \right\} \]

\[ \leq Q(rc_1) + 3\varepsilon. \]

First letting \( r \to \infty \) and then letting \( \varepsilon \to 0 \) we immediately have

\[ \lim_{n' \to \infty} \frac{1}{2^n'} \sum_{k=1}^{2^n'} \sup_{x-z \leq \delta} \left| Z_{n_k}(x) - Z_{n_k}(z) \right| < \infty \]

almost surely by (3.13). It follows from (2.5) that

\[ \int_{\mathbb{R}} |p_t(x-u) - p_t(z-u)|X_0(du) \leq C|x-z|^{\eta}e^{-\frac{a+1}{2}}X_0(1), \quad t > 0. \]

Then

\[ \limsup_{n' \to \infty} \frac{1}{2^n'} \sum_{k=1}^{2^n'} \sup_{x-z \neq z} \frac{\int_{\mathbb{R}} |p_{n_k}(x-u) - p_{n_k}(z-u)|X_0(du)}{|x-z|^{\eta}} < \infty. \]

This completes the proof. \( \square \)
4 Pathwise uniqueness

In this section we prove the pathwise uniqueness for (4.3). Recall that $\eta_c = \frac{2}{\alpha} - 1$.

**Theorem 4.1** Suppose that there exists a $\delta > 0$ satisfying

$$2\beta\eta_c > 1 + 1/\delta \text{ and } \frac{\alpha\beta}{\alpha - 1} > \delta + 1. \quad (4.1)$$

If $(X, L)$ and $(Y, L)$, with $X_0 = Y_0 \in M(\mathbb{R})$, are two weak solutions to equation (4.3) defined on the same filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying Assumption 2.2, then with probability one, for each $t > 0$ we have

$$X_t(x) = Y_t(x), \quad \lambda\text{-a.e. } x. \quad (4.2)$$

**Remark 4.2** Assumption (4.1) is equivalent to $p\alpha^2 - 2p(p + 3)\alpha + 4p(p + 1) > 0$ and $\alpha > 1$. If $p = 1$, it is further equivalent to $1 < \alpha < 4 - 2\sqrt{2}$.

Throughout this section we always assume that the assumptions of Theorem 4.1 hold. The proof of Theorem 4.1 adopts the arguments from [14, 15]. By considering a conditional probability, we may assume that the initial states $X_0$ and $Y_0$ are deterministic. For $n \geq 1$ define

$$a_n := \exp\{-n(n + 1)/2\}.$$

Then $a_{n+1} = a_n^{2/m}$. Let $\psi_n \in C^\infty_c(\mathbb{R})$ satisfy $\text{supp}(\psi_n) \subset (a_n, a_{n-1})$, $\int_{a_{n-1}}^{a_n} \psi_n(x)dx = 1$, and $0 \leq \psi_n(x) \leq 2/(nx)$ for all $x > 0$ and $n \geq 1$. For $x \in \mathbb{R}$ and $n \geq 1$ let

$$\phi_n(x) := \int_0^{|x|} dy \int_0^y \psi_n(z)dz.$$

Then $\|\phi_n'\| \leq 1$, $\phi_n(x) \to |x|$, and $\phi_n'(x) \to x/|x|$ for $x \neq 0$ as $n \to \infty$. For $n \geq 1$ and $y, z \in \mathbb{R}$ put

$$D_n(y, z) := \phi_n(y + z) - \phi_n(y) - z\phi_n'(y) \quad \text{and} \quad H_n(y, z) := \phi_n(y + z) - \phi_n(y).$$

Let $\Phi \in C^\infty_c(\mathbb{R})$ satisfy $0 \leq \Phi \leq 1$, $\text{supp}(\Phi) \subset (-1, 1)$ and $\int \Phi(x)dx = 1$. Let $\Phi^m(x, y) = \Phi^m(x, y) := m\Phi(m(x - y))$ for $x, y \in \mathbb{R}$. For $t \geq 0$ let $U_t := X_t - Y_t$ and $V_t := X_t^\beta - Y_t^\beta$. By the argument in Introduction, both $\{X_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$ satisfy assumption (4.3). Using (4.3) and Itô’s formula we have

$$\phi_n([U_t, \Phi^m_x]) = \frac{1}{2} \int_0^t \phi_n([U_s, \Phi^m_x])d\langle U_s, \Delta \Phi^m_x \rangle ds + \int_0^t \int_0^c \int_{\mathbb{R}} H_n([U_{s-}, \Phi^m_x], zV_{s-} \Phi^m_x(y))N(ds, dz, dx) + \int_0^t ds \int_{\mathbb{R}} m(dx) \int_{\mathbb{R}} D_n([U_s, \Phi^m_x], zV_s \Phi^m_x(y))dy. \quad (4.3)$$

For $t > 0$ let $\hat{X}_t$ and $\hat{Y}_t$ denote the continuous versions of $X_t$ and $Y_t$, respectively. Let $\hat{U}_t(y) := \hat{X}_t(y) - \hat{Y}_t(y)$ and $\hat{V}_t(y) := X_t^\beta - Y_t^\beta$. 

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Suppose that $T > 0$ and that $\Psi$ is a nonnegative and compactly supported infinitely differentiable function on $[0, T] \times \mathbb{R}$ satisfying

$$\Psi_s(x) = 0 \text{ for all } (s, x) \in [0, T] \times [-K, K]^c.$$ 

By (4.3) and a stochastic Fubini’s theorem, it is easy to see that

$$\langle \phi_n((U_t, \Phi^m)), \Psi_t \rangle$$

$$= \sum_{i=1}^{n} \langle \phi_n((U_{t_i}, \Phi^m)), \Psi_{t_i-1} \rangle - \langle \phi_n((U_{t_{i-1}}, \Phi^m)), \Psi_{t_{i-1}} \rangle$$

$$= \sum_{i=1}^{n} \langle \phi_n((U_{t_i}, \Phi^m)), \Psi_{t_i-1} \rangle - \langle \phi_n((U_{t_{i-1}}, \Phi^m)), \Psi_{t_{i-1}} \rangle + \sum_{i=1}^{n} \langle \phi_n((U_t, \Phi^m)), \Psi_t - \Psi_{t_{i-1}} \rangle$$

$$= \frac{1}{2} \int_0^t \sum_{i=1}^{n} I_i(s) \langle \phi_n'((U_s, \Phi^m)) \cdot (U_s, \Delta \Phi^m), \Psi_{t_i-1} \rangle ds + \int_0^t \sum_{i=1}^{n} I_i(s) \langle \phi_n((U_{t_i}, \Phi^m)), \Psi_s \rangle ds$$

$$+ \int_0^t \int_{\mathbb{R}} \sum_{i=1}^{n} I_i(s) \langle H_n((U_{s-}, \Phi^m), zV_{s-}(y) \Phi^m(y)), \Psi_{t_{i-1}} \rangle N(ds, dz, dx)$$

$$+ \int_0^t ds \int_{\mathbb{R}} m(dz) \int_{\mathbb{R}} \sum_{i=1}^{n} I_i(s) \langle D_n(U_s, \Phi^m), zV_s(y) \Phi^m(y) \rangle, \Psi_{t_{i-1}} \rangle dy,$$

where $0 = t_0 < t_1 < \cdots < t_n = t$ and $I_i(s) := 1_{(t_{i-1}, t_i]}(s)$. Letting $\max_{1 \leq i \leq n}(t_i - t_{i-1})$ converge to zero we have

$$\langle \phi_n((U_t, \Phi^m)), \Psi_t \rangle = \frac{1}{2} \int_0^t \langle \phi_n'((U_s, \Phi^m)) \cdot (U_s, \Delta \Phi^m), \Psi_s \rangle ds + \int_0^t \langle \phi_n((U_{t_i}, \Phi^m)), \Psi_s \rangle ds$$

$$+ \int_0^t \int_{\mathbb{R}} \langle H_n((U_{s-}, \Phi^m), zV_{s-}(y) \Phi^m(y)), \Psi_s \rangle N(ds, dz, dy)$$

$$+ \int_0^t ds \int_{\mathbb{R}} m(dz) \int_{\mathbb{R}} \langle D_n(U_s, \Phi^m), zV_s(y) \Phi^m(y) \rangle, \Psi_s \rangle dy$$

$$=: I_{1}^{m,n}(t) + I_{2}^{m,n}(t) + I_{3}^{m,n}(t) + I_{4}^{m,n}(t). \quad (4.4)$$

For $k \geq 1$ define a stopping time $\gamma_k$ by

$$\gamma_k := \inf \left\{ t \in (0, T) : \langle X_t, 1 \rangle + \langle Y_t, 1 \rangle > k \right\}$$

with the convention $\inf \emptyset = \infty$. Then $\lim_{k \to \infty} \gamma_k = \infty$ almost surely. Let $\{l' : l' \geq 1\}$ be the subsequence of $\{l : l \geq 1\}$ that will be determined later in Lemma 4.8. For any nonnegative function $f$ define

$$\int_{(0,t]} f(s) ds := \lim_{l' \to \infty} \int_0^{t} \sum_{i=1}^{l'} 1_{(t_{i-1}, T, t_i']}(s) f(t_i') ds$$

and

$$\int_{(0,t]} f(s) ds := \lim_{l' \to t} \int_0^{t'} f(s) ds.$$ 

For fixed $K > 0$ and $\eta \in (0, \eta_\text{c})$ define a stopping time $\sigma_k$ by

$$\sigma_k := \inf \left\{ t \in (0, T] : \int_{(0,t]} \sup_{x \neq z, x, z \in [-K+1, K]} \frac{|X_s(x) - X_s(z)| \lor |Y_s(x) - Y_s(z)|}{|x - z|^\eta} ds > k \right\}.$$
By Theorem 3.4 one sees that \( \lim_{k \to \infty} \sigma_k = \infty \) almost surely.

In the rest of this section we always write

\[
\tau_k := \min\{\gamma_k, \sigma_k\}.
\]

Before proving Theorem 4.1, we state three important lemmas. Similar to [14, Lemma 2.2(b)] we have the following result.

**Lemma 4.3** For any stopping time \( \tau \) and \( t > 0 \), we have

\[
\limsup_{m,n \to \infty} \mathbb{E}\{I_{1}^{m,n}(t \wedge \tau)\} \leq \frac{1}{2} \mathbb{E}\left\{ \int_{0}^{t \wedge \tau} ds \int_{\mathbb{R}} |U_s(x)|\Delta \Psi_s(x)dx \right\}
\]

and

\[
\lim_{m,n \to \infty} \mathbb{E}\{I_{2}^{m,n}(t \wedge \tau)\} = \mathbb{E}\left\{ \int_{0}^{t \wedge \tau} ds \int_{\mathbb{R}} |U_s(x)|\tilde{\Psi}_s(x)dx \right\}.
\]

**Lemma 4.4** For any stopping time \( \tau \) and any \( t > 0 \), \( m, n \geq 1 \), we have

\[
\mathbb{E}\{I_{3}^{m,n}(t \wedge \tau)\} = 0. \tag{4.5}
\]

**Lemma 4.5** If \( m = a_{n-1}^{-\delta} \), then for each \( t > 0 \) and \( k \geq 1 \),

\[
\lim_{m,n \to \infty} \mathbb{E}\{I_{4}^{m,n}(t \wedge \tau_k)\} = 0.
\]

We first present the main proof.

**Proof of Theorem 4.1** By the continuity of \( x \mapsto \tilde{U}_t(x) \), for each \( x \in \mathbb{R} \) and \( t > 0 \),

\[
\lim_{m \to \infty} \langle \tilde{U}_t, \Phi^m_x \rangle = \lim_{m \to \infty} \int_{\mathbb{R}} \tilde{U}_t(x - \frac{y}{m})\Phi(y)dy = \tilde{U}_t(x). \tag{4.6}
\]

Note that \( \|\phi'_n\| \leq 1 \). Then for all \( x_m \to x \) as \( m \to \infty \), we have that

\[
|\phi_n(x_m) - x| \leq |\phi_n(x_m) - \phi_n(x)| + |\phi_n(x) - x| \leq |x_m - x| + |\phi_n(x) - x|,
\]

which converges to zero as \( m, n \to \infty \). Now by (4.6) and Fatou’s lemma

\[
\mathbb{E}\{\langle U_t, \Psi_t\rangle_{1_{\{t \leq \tau_k\}}}\} = \mathbb{E}\left\{ \langle \tilde{U}_t, \Psi_t\rangle_{1_{\{t \leq \tau_k\}}} \right\}
\]

\[
= \mathbb{E}\left\{ \lim_{m,n \to \infty} \phi_n(\langle \tilde{U}_t, \Phi^m_x \rangle, \Psi_t)_{1_{\{t \leq \tau_k\}}} \right\} \leq \liminf_{m,n \to \infty} \mathbb{E}\left\{ \phi_n(\langle \tilde{U}_t, \Phi^m_x \rangle, \Psi_t)_{1_{\{t \leq \tau_k\}}} \right\}
\]

\[
= \liminf_{m,n \to \infty} \mathbb{E}\left\{ \langle \phi_n(\langle U_t \wedge \tau_k, \Phi^m_x \rangle, \Psi_t)_{1_{\{t \leq \tau_k\}}} \right\} \leq \liminf_{m,n \to \infty} \mathbb{E}\left\{ \phi_n(\langle U_t \wedge \tau_k, \Phi^m_x \rangle, \Psi_t)_{1_{\{t \leq \tau_k\}}} \right\}.
\]

Together with (4.4) and Lemmas 4.3 4.5 we have

\[
\mathbb{E}\{\langle U_t, \Psi_t\rangle_{1_{\{t \leq \tau_k\}}}\} \leq \mathbb{E}\left\{ \int_{0}^{t \wedge \tau_k} ds \int_{\mathbb{R}} |U_s(x)|\left(\frac{1}{2} \Delta \Psi_s(x) + \tilde{\Psi}_s(x)\right)dx \right\}.
\]

Letting \( k \to \infty \) in the above inequality we have

\[
\mathbb{E}\{\langle U_t, \Psi_t\rangle\} \leq \int_{0}^{t} ds \int_{\mathbb{R}} \mathbb{E}\{U_s(x)\}(\frac{1}{2} \Delta \Psi_s(x) + \tilde{\Psi}_s(x))dx.
\]
This gives (34) of [14]. By the same argument in the proof of Theorem 1.6 of [14], one has

$$E \left\{ \int_{\mathbb{R}} |U_t(x)| \, dx \right\} = 0,$$

which implies

$$P\{X_t(x) = Y_t(x) \text{ for } \lambda\text{-a.e. } x\} = 1$$

for every $t > 0$. It follows that $\langle X_t, f \rangle = \langle Y_t, f \rangle$ almost surely for every $t > 0$ and $f \in \mathcal{S}$. By the right-continuities of $t \mapsto \langle X_t, f \rangle$ and $t \mapsto \langle Y_t, f \rangle$ we have $P\{\langle X_t, f \rangle = \langle Y_t, f \rangle \text{ for all } t > 0\} = 1$ for every $f \in \mathcal{S}$. Considering a suitable sequence $\{f_1, f_2, \cdots \} \subset \mathcal{S}$ we can conclude (4.2).

We now present the proofs of Lemmas 4.3, 4.5.

**Proof of Lemma 4.3.** By the same argument as Lemma 2.2(b) of [14],

$$\lim_{m,n \to \infty} \sup \mathbb{E}\{I_{1}^{m,n}(t \wedge \tau)\} \leq \lim_{m,n \to \infty} \frac{1}{2} \mathbb{E}\left\{ \int_{0}^{t \wedge \tau} \int_{\mathbb{R}} \phi_n'((\bar{U}_s, \Phi_{x}^m))(\bar{U}_s, \Phi_{x}^m) \Delta \Psi_s(x) \, dx \right\}.$$  \hspace{1cm} (4.7)

By the continuity of $x \mapsto \bar{U}_s(x)$, it is easy to see that

$$\lim_{m,n \to \infty} \phi_n'((\bar{U}_s, \Phi_{x}^m))(\bar{U}_s, \Phi_{x}^m) = \lim_{m,n \to \infty} \phi_n'((\bar{U}_s, \Phi_{x}^m)) \int_{-1}^{1} \bar{U}_s(x - \frac{y}{m}) \Phi(y) \, dy = |\bar{U}_s(x)|.$$

Observe that

$$0 \leq \phi_n'((\bar{U}_s, \Phi_{x}^m))(\bar{U}_s, \Phi_{x}^m) = \phi_n'((\bar{U}_s, \Phi_{x}^m)) \int_{-1}^{1} \bar{U}_s(x - \frac{y}{m}) \Phi(y) \, dy \leq \sup_{|y| \leq K+1} |\bar{X}_s(y) + \bar{Y}_s(y)|.$$

Then by (4.7), (3.1) and dominated convergence

$$\lim_{m,n \to \infty} \sup \mathbb{E}\{I_{1}^{m,n}(t \wedge \tau)\} \leq \lim_{m,n \to \infty} \frac{1}{2} \int_{0}^{t} \int_{\mathbb{R}} \mathbb{E}\left\{ \phi_n'((\bar{U}_s, \Phi_{x}^m))(\bar{U}_s, \Phi_{x}^m) \Delta \Psi_s(x) 1_{\{s \leq \tau\}} \right\} \, dx$$

$$\leq \frac{1}{2} \int_{0}^{t} \int_{\mathbb{R}} \mathbb{E}\left\{ \lim_{m,n \to \infty} \phi_n'((\bar{U}_s, \Phi_{x}^m))(\bar{U}_s, \Phi_{x}^m) \Delta \Psi_s(x) 1_{\{s \leq \tau\}} \right\} \, dx$$

$$= \frac{1}{2} \int_{0}^{t} \int_{\mathbb{R}} \mathbb{E}\left\{ 1_{\bar{U}_s(x) | \Delta \Psi_s(x) 1_{\{s \leq \tau\}}} \right\} \, dx$$

$$= \frac{1}{2} \mathbb{E}\left\{ \int_{0}^{t} \int_{\mathbb{R}} \bar{U}_s(x) | \Delta \Psi_s(x) \right\} \, dx.$$

Note that

$$\left| \mathbb{E}\{I_{2}^{m,n}(t \wedge \tau)\} - \mathbb{E}\left\{ \int_{0}^{t \wedge \tau} \int_{\mathbb{R}} |U_s(x)| \Phi_s(x) \, dx \right\} \right|$$

$$\leq \int_{0}^{t} \int_{\mathbb{R}} \mathbb{E}\left\{ |\phi_n((U_s, \Phi_{x}^m)) - |U_s(x)||\Phi_s(x)| \right\} \, dx$$

$$\leq \int_{0}^{t} \int_{\mathbb{R}} \mathbb{E}\left\{ |\phi_n((U_s, \Phi_{x}^m)) - \phi_n(U_s(x))| + |\phi_n(U_s(x)) - |U_s(x)||\Phi_s(x)| \right\} \, dx$$

$$\leq \int_{0}^{t} \int_{\mathbb{R}} |\Phi_s(x)| \, dx \int_{\mathbb{R}} \mathbb{E}\left\{ |U_s(x - \frac{y}{m}) - U_s(x)| \right\} \, dy.$$
Now by (2.4) and dominated convergence one finishes the proof. \(\square\)

**Proof of Lemma 4.2.** For \(t \geq 0\) and \(m,n \geq 1\) let

\[
I_{3,1}^{m,n}(t) := \int_0^t \int_0^1 \int_{\mathbb{R}} \langle H_n((U_{s-}, \Phi^m), zV_{s-}(y)\Phi^m(y)), \Psi_s \rangle \tilde{N}(ds, dz, dy)
\]

and \(I_{3,2}^{m,n}(t) := I_{3,1}^{m,n}(t) - I_{3,1}^{m,n}(t)\). By the Burkholder-Davis-Gundy inequality, for \(\bar{\alpha} \in (\alpha, \frac{\beta}{\beta} + 2)\) and \(T > 0\),

\[
\mathbb{E} \left\{ \sup_{t \in [0,T]} |I_{3,1}^{m,n}(t \wedge \tau)|^{\bar{\alpha}} \right\} \\
\leq C \mathbb{E} \left\{ \left[ \int_0^T \int_0^1 \int_{\mathbb{R}} |\langle H_n((U_{s-}, \Phi^m), zV_{s-}(y)\Phi^m(y)), \Psi_s \rangle|^2 \tilde{N}(ds, dz, dy) \right]^{\frac{\bar{\alpha}}{2}} \right\} \\
\leq C \mathbb{E} \left\{ \int_0^T \int_0^1 \int_{\mathbb{R}} |\langle H_n((U_{s-}, \Phi^m), zV_{s-}(y)\Phi^m(y)), \Psi_s \rangle|^{\bar{\alpha}} \tilde{N}(ds, dz, dy) \right\}.
\]

For the last inequality we used the fact that

\[
|\sum_{i=1}^n x_i^2|^{\bar{\alpha}} \leq \sum_{i=1}^n |x_i|^\alpha
\]

for all \(x_i \in \mathbb{R}\) and \(n \geq 1\). Since \(|H_n(y,z)| \leq |z|\) for all \(y, z \in \mathbb{R}\) and \(\Psi\) is bounded, by Hölder inequality and Lemma 2.4,

\[
\mathbb{E} \left\{ \sup_{t \in [0,T]} |I_{3,1}^{m,n}(t \wedge \tau)|^{\bar{\alpha}} \right\} \\
\leq C \mathbb{E} \left\{ \int_0^T \int_0^1 \int_{\mathbb{R}} |z| \int_{\mathbb{R}} V_{s-}(y)\Phi^m_x(y)\Psi_s(x) dx |^{\bar{\alpha}} \tilde{N}(ds, dz, dy) \right\} \\
\leq C \mathbb{E} \left\{ \int_0^T \int_0^1 \int_{\mathbb{R}} z^{\bar{\alpha}} |V_{s-}(y)\Phi^m_x(y)\Psi_s(x)|^{\bar{\alpha}} dx N(ds, dz, dy) \right\} \\
\leq C \int_0^T ds \int_0^1 z^{\bar{\alpha}} m(dz) \int_{\mathbb{R}} \Psi_s(x)^{\bar{\alpha}} dx \int_{\mathbb{R}} \mathbb{E} \{ |V_s(y)|^{\bar{\alpha}} \} \Phi^m_x(y)^{\bar{\alpha}} dy \\
\leq C \int_0^T ds \int_{\mathbb{R}} \Psi_s(x)^{\bar{\alpha}} dx \int_{\mathbb{R}} \mathbb{E} \{ X_s(y)^{\bar{\alpha}\beta} + Y_s(y)^{\bar{\alpha}\beta} \} \Phi^m_x(x)^{\bar{\alpha}} dy < \infty.
\]

Similarly,

\[
\mathbb{E} \left\{ \sup_{t \in [0,T]} |I_{3,2}^{m,n}(t \wedge \tau)| \right\} \\
\leq \mathbb{E} \left\{ \int_0^T \int_0^1 \int_{\mathbb{R}} |\langle H_n((U_{s-}, \Phi^m), zV_{s-}(y)\Phi^m(y)), \Psi_s \rangle| N(ds, dz, dy) \right\} \\
+ \mathbb{E} \left\{ \int_0^T ds \int_1^\infty m(dz) \int_{\mathbb{R}} |\langle H_n((U_{s-}, \Phi^m), zV_{s-}(y)\Phi^m(y)), \Psi_s \rangle| dy \right\} \\
= 2 \int_0^T ds \int_1^\infty m(dz) \int_{\mathbb{R}} \mathbb{E} \{ |H_n((U_{s-}, \Phi^m), zV_{s-}(y)\Phi^m(y)), \Psi_s \rangle| \} dy \\
\leq 2 \int_0^T ds \int_1^\infty zm(dz) \int_{\mathbb{R}} \Psi_s(x) dx \int_{\mathbb{R}} \mathbb{E} \{ |V_s(y)| \} \Phi^m_x(y) dy < \infty.
\]
It follows that for $T > 0$

$$\mathbb{E} \left\{ \sup_{t \in [0, T]} |I_{3}^{m,n}(t \wedge \tau)| \right\} < \infty.$$  

Then by [16, p.38], $t \mapsto I_{3}^{m,n}(t \wedge \tau)$ is a martingale, which implies (4.5).

To prove Lemma 4.5, we only need to show the following two lemmata.

**Lemma 4.6** For $m, n, k \geq 1$ and $t \geq 0$ let

$$I_{4,1}^{m,n,k}(t) := \mathbb{E} \left\{ \int_{0}^{t} ds \int_{1}^{1} m(dz) \int_{\mathbb{R}} \langle D_{n}(\langle U_{s}, \Phi_{m}^{m} \rangle, zV_s(y)\Phi_{m}(y)), \Psi_s \rangle dy \right\}.$$  

Then

$$I_{4,1}^{m,n,k}(t) \leq Ck^{2\beta} (na_n)^{-1} m^{1-2\gamma \beta} + Cn^{-1} [a_{n-1}^{-\beta} + a_{n-1}^{\alpha + 1 - \alpha - \frac{2}{\alpha}} m^{\alpha - 1}].$$

**Lemma 4.7** For $m, n \geq 1$ and $t \geq 0$ let

$$I_{4,2}^{m,n}(t) := \mathbb{E} \left\{ \int_{0}^{t} ds \int_{1}^{1} m(dz) \int_{\mathbb{R}} \langle D_{n}(\langle U_{s}, \Phi_{m}^{m} \rangle, zV_s(y)\Phi_{m}(y)), \Psi_s \rangle dy \right\}.$$  

Then $I_{4,2}^{m,n}(t) \to 0$ as $m, n \to \infty$.

*Proof of Lemma 4.5.* We take $1 + 1/\delta < \eta < \eta$. Since $m = a_{n-1}^{-\delta}$, $2\beta \eta > 1 + 1/\delta$ and $\alpha + 1 > \alpha - \frac{2}{\alpha}$, $I_{4,1}^{m,n,k}(t)$ converges to zero as $n \to \infty$ by Lemma 4.6. The desired result follows from Lemma 4.7.

We first present the proof for Lemma 4.7.

*Proof of Lemma 4.7.* Observe that $D_{n}(y, z) \leq 2|z|$ for all $n \geq 1$ and $y, z \in \mathbb{R}$, and

$$\frac{1}{m} D_{n}(\langle \tilde{U}_{s}, \Phi_{m, x}^{m} \rangle, z\tilde{V}_{s}(x - y/m)\Phi(y)) \leq 2z \sup_{|y| \leq K + 1} |\tilde{X}_{s}(y) + \tilde{Y}_{s}(y)|^{\beta}. \quad (4.8)$$

By (4.1), for each $0 < s < t$,

$$\mathbb{E} \left\{ \sup_{|y| \leq K + 1} |\tilde{X}_{s}(y) + \tilde{Y}_{s}(y)|^{\beta} \right\} \leq C_{t}s^{-\frac{\beta}{2}}.$$  

This inequality together with (4.8) and dominated convergence leads to

$$\lim_{m,n \to \infty} I_{4,2}^{m,n}(t)$$

$$= \lim_{m,n \to \infty} \int_{0}^{t} ds \int_{\mathbb{R}} \Psi_{s}(x)dx \int_{1}^{1} m(dz) \int_{\mathbb{R}} \mathbb{E} \left\{ \frac{1}{m} D_{n}(\langle \tilde{U}_{s}, \Phi_{m, x}^{m} \rangle, z\tilde{V}_{s}(x - y/m)\Phi(y)) \right\} dy$$

$$= \int_{0}^{t} ds \int_{\mathbb{R}} \Psi_{s}(x)dx \int_{1}^{1} m(dz) \int_{\mathbb{R}} \mathbb{E} \left\{ \lim_{m,n \to \infty} \frac{1}{m} D_{n}(\langle \tilde{U}_{s}, \Phi_{m, x}^{m} \rangle, z\tilde{V}_{s}(x - y/m)\Phi(y)) \right\} dy.$$  

Then we can finish the proof if we can show

$$\lim_{m,n \to \infty} \frac{1}{m} D_{n}(\langle \tilde{U}_{s}, \Phi_{m, x}^{m} \rangle, z\tilde{V}_{s}(x - y/m)\Phi(y)) = 0.$$  

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Now we prove the above equation. If $\tilde{U}_s(x) = 0$, then by the continuity of $x \mapsto \tilde{U}_s(x)$,
\[
\frac{1}{m} D_n(\langle \tilde{U}_s, \Phi_x^m \rangle, zm\tilde{V}_s(x - \frac{y}{m})\Phi(y)) \leq 2z|\tilde{V}_s(x - \frac{y}{m})\Phi(y)| \leq 2z|\tilde{U}_s(x - \frac{y}{m})|^{\beta}
\]
tends to zero as $m, n \to \infty$. If $\tilde{U}_s(x) > 0$, then there is a strictly positive constant $c_2$ so that
$\tilde{U}_s(x - \frac{y}{m}) \geq c_2$ for all $u \in [-1, 1]$ and for $m$ large enough. Thus,
\[
\langle \tilde{U}_s, \Phi_x^m \rangle = \int_{-1}^{1} \tilde{U}_s(x - \frac{u}{m})\Phi(u)du \geq c_2,
\]
which implies $\langle \tilde{U}_s, \Phi_x^m \rangle + zm\tilde{V}_s(x - \frac{y}{m})\Phi(y) \geq c_2 > 0$ for all $z, h \geq 0$ and for $m$ large enough. Since $\text{supp}(\psi_n) \subset (a_n, a_{n-1})$, by (3.3) of (3.3), we have
\[
\frac{1}{m} D_n(\langle \tilde{U}_s, \Phi_x^m \rangle, zm\tilde{V}_s(x - \frac{y}{m})\Phi(y)) \leq 2m[z\tilde{V}_s(x - \frac{y}{m})\Phi(y)]^2 \int_{0}^{1} \psi_n(\langle \tilde{U}_s, \Phi_x^m \rangle + zm\tilde{V}_s(x - \frac{y}{m})\Phi(y))(1 - h)dh = 0
\]
for $m, n$ large enough. The proof for the case $\tilde{U}_s(x) < 0$ is similar. \hfill \Box

For $m, n, k \geq 1$ and $t > 0$ define
\[
J_{m,n,k}(t) := 1_{\{t \leq \gamma_k\}} \int_{\mathbb{R}} \psi_t(x)dx \int_{0}^{1} z^2m(dz) \int_{\mathbb{R}} V_t(x - \frac{y}{m})^2\Phi(y)^2dy \times \int_{0}^{1} \psi_n(\langle U_t, \Phi_x^m \rangle + zm\tilde{V}_t(x - \frac{y}{m})\Phi(y))(1 - h)dh.
\]
Before proving Lemma 4.8 we need to show two more lemmas.

**Lemma 4.8** There is a subsequence $\{l' : l' \geq 1\}$ of $\{l : l \geq 1\}$ so that for each $m, n, k \geq 1$
\[
\lim_{l' \to \infty} \int_{0}^{t} \sum_{i=1}^{l'} 1_{[u_{i-1}, u_{i}]}J_{m,n,k}(u_{i}, T) - J_{m,n,k}(u_{i-1}, T)ds = 0, \quad t \in (0, T) \tag{4.9}
\]
almost surely.

**Proof.** Observe that $|\psi_n(x)| \leq 2(na_n)^{-1}1_{\{x \leq a_{n-1}\}}$, and on $\{t < \gamma_k\}$
\[
zmV_t(x - \frac{y}{m})\Phi(y) \leq |\langle U_t, \Phi_x^m \rangle + zmV_t(x - \frac{y}{m})\Phi(y)| + |\langle U_t, \Phi_x^m \rangle| \leq |\langle U_t, \Phi_x^m \rangle + zmV_t(x - \frac{y}{m})\Phi(y)| + m(\langle X_t, 1 \rangle + \langle Y_t, 1 \rangle) \leq |\langle U_t, \Phi_x^m \rangle + zmV_t(x - \frac{y}{m})\Phi(y)| + mk.
\]
Then on $\{t < \gamma_k\}$,
\[
zmV_t(x - \frac{y}{m})\Phi(y)\psi_n(\langle U_t, \Phi_x^m \rangle + zmV_t(x - \frac{y}{m})\Phi(y)) \leq 2(na_n)^{-1}(a_{n-1} + mk) =: C(m, n, k).
\]
This implies that for each $r \in (0, 1)$ satisfying $\alpha < 2 - r < \alpha/\beta$,
\[
M_{m,n,k}(x, y, z, h, t) := V_t(x - \frac{y}{m})^2\Phi(y)^2\psi_n(\langle U_t, \Phi_x^m \rangle + zmV_t(x - \frac{y}{m})\Phi(y))1_{\{t \leq \gamma_k\}}
\]
Lemma 4.9

Let \( l' := l'(m, n, k) : l' \geq 1 \) be the subsequence of \( \{ l : l \geq 1 \} \) in Lemma 4.8. Then

\[
\lim_{l' \to \infty} \int_{0}^{T} \sum_{i=1}^{2^l} 1_{\{l_{i-1}T, l_{i}T\}}(s) J_{m, n, k}(l_{i}T) 1_{\{v, T < s \}} ds = \int_{0}^{t} J_{m, n, k}(s) 1_{\{s < \sigma_k\}} ds, \quad t \in (0, T]
\]

almost surely.
Proof. It follows from (4.11) that
\[
E\left\{ \int_0^T J_{m,n,k}(s) ds \right\} < \infty,
\]
which implies
\[
\int_0^T J_{m,n,k}(s) ds < \infty
\]
almost surely. Then by Lemma 4.8 and dominated convergence one obtains
\[
\lim_{t' \to \infty} \int_0^t 1_{[t',T]}(s) J_{m,n,k}(t') ds = \int_0^T J_{m,n,k}(s) ds < \infty
\]
which completes the proof.

Proof of Lemma 4.7. In the following let \( t > 0 \) and \( m, n, k \geq 1 \) be fixed. By (3.3) of [11] and Lemma 4.9 we have
\[
I_{4,1}^{m,n,k}(t) = m E\left\{ \int_0^t \int_{-K}^K \int_{-1}^1 z^2 m(dz) \int_{-1}^1 V_s(x - \frac{y}{m})^2 \Phi(y)^2 dy \right. \\
\left. \times \int_0^1 \psi_n(|(U_s, \Phi_x^m) + zhmV_s(x - \frac{y}{m})\Phi(y)|)(1-h)dh \right\}
\]
\[
= m E\left\{ \int_0^t J_{m,n,k}(s) 1_{s < \sigma_k} ds \right\} = m E\left\{ \int_{(0,t]} J_{m,n,k}(s) 1_{s < \sigma_k} ds \right\}
\]
\[
= m E\left\{ \int_{(0,t]} \tilde{J}_{m,n,k}(s) 1_{s < \sigma_k} ds \right\},
\]
which we now refer to as (4.12).

For fixed \( s \) and \( x \) let \( x_{s,m}(x) \) be a value satisfying
\[
|\tilde{V}_s(x - \frac{x_{s,m}(x)}{m})| = \inf_{y \in [-1,1]} |\tilde{V}_s(x - \frac{y}{m})|.
\]
Recall that \( \psi_n(x) \leq 2(na_n)^{-1} 1_{|x| < a_{n-1}} \). It follows from (4.12) that
\[
I_{4,1}^{m,n,k}(t) \leq 4m(na_n)^{-1} E\left\{ \int_{(0,t]} \int_{-K}^K \int_{-1}^1 z^2 m(dz) \int_{-1}^1 \tilde{V}_s(x - \frac{y}{m})^2 \Phi(y)^2 dy \right. \\
\left. \times \int_0^1 \psi_n(|(\tilde{V}_s, \Phi_x^m) + zhm\tilde{V}_s(x - \frac{y}{m})\Phi(y)|)(1-h)dh \right\}
\]
\[
\leq 4m(na_n)^{-1} E\left\{ \int_{(0,t]} \int_{-K}^K \int_{-1}^1 z^2 m(dz) \int_{-1}^1 \tilde{V}_s(x - \frac{y}{m})^2 \Phi(y)^2 dy \right. \\
\left. \times \int_0^1 \psi_n(|(\tilde{V}_s, \Phi_x^m) + zhm\tilde{V}_s(x - \frac{y}{m})\Phi(y)|)(1-h)dh \right\}.
\]
\[
\times \int_{-1}^{1} |\tilde{V}_s(x - \frac{y}{m}) - \tilde{V}_s(x - \frac{x_{s,m}(x)}{m})|^2 \Phi(y)^2 dy \\
+ 4m(na_n)^{-1} \mathbb{E}\left\{ \int_0^t ds \int_{-K}^{K} \Psi_s(x) H_{m,n}(x) dx \right\} \leq 4m(na_n)^{-1} c_3 \int_0^1 z^2 m(dz) \int_{-K}^{K} dx \\
\times \int_{-1}^{1} \mathbb{E}\left\{ \int_{0,t,\sigma_k} |\tilde{V}_s(x - \frac{y}{m}) - \tilde{V}_s(x - \frac{x_{s,m}(x)}{m})|^2 dy \right\} \Phi(y)^2 dy \\
= I_{4,1,1}^{m,n,k}(t) + I_{4,1,2}^{m,n}(t),
\]

where

\[
H_{m,n}(x) := \sup_{s \in \{n_k T \leq s \leq 2^m n_k \}} \int_{0}^{1} |\tilde{V}_s(x - \frac{x_{s,m}(x)}{m})|^2 dy \int_{0}^{1} z^2 m(dz) \\
\times \int_{0}^{1} 1_{\{|\tilde{U}_s, \Phi + z h \tilde{m} \tilde{V}_s(x - \frac{x}{m})\Phi(y)| < \alpha_n \}} (1 - h) dh.
\]

We can finish the proof in two steps.

**Step 1.** We first estimate \( I_{4,1,1}^{m,n,k}(t) \). Since \( \tilde{X}_s \) and \( \tilde{Y}_s \) are the continuous versions of \( X_s \) and \( Y_s \) for fixed \( s > 0 \),

\[
\sigma_k = \tilde{\sigma}_k := \inf \left\{ t \in (0, T) : \int_{0,t,\tilde{\sigma}_k} \sup_{|x| \leq K, |y| \leq 1} \frac{|\tilde{X}_s(x) - \tilde{X}_s(z)| \vee |\tilde{Y}_s(x) - \tilde{Y}_s(z)|}{|x - z|^\beta} ds > k \right\}
\]

almost surely. Observe that

\[
\int_{0,t,\tilde{\sigma}_k} \sup_{|x| \leq K, |y| \leq 1} \frac{|\tilde{X}_s(x) - \tilde{X}_s(z)| \vee |\tilde{Y}_s(x) - \tilde{Y}_s(z)|}{|x - z|^\beta} ds \leq 2^{\eta_\beta} m^{-2\eta_\beta} k^{2\beta}.
\]

One can also obtain the same estimation for \( \tilde{Y} \). Then by (4.14) we have

\[
\int_{0,t,\tilde{\sigma}_k} \sup_{|x| \leq K, |y| \leq 1} |\tilde{V}_s(x - \frac{y}{m}) - \tilde{V}_s(x - \frac{v}{m})|^2 ds \leq 2^{\eta_\beta} m^{-2\eta_\beta} \kappa^{2\beta}.
\]

almost surely. We then have

\[
I_{4,1,1}^{m,n,k}(t) \leq 4m(na_n)^{-1} c_3 \int_0^1 z^2 m(dz) \int_{-K}^{K} dx \\
\times \int_{-1}^{1} \mathbb{E}\left\{ \int_{0,t,\sigma_k} |\tilde{V}_s(x - \frac{y}{m}) - \tilde{V}_s(x - \frac{x_{s,m}(x)}{m})|^2 dy \right\} \Phi(y)^2 dy \\
\leq 4m(na_n)^{-1} c_3 \int_0^1 z^2 m(dz) \int_{-K}^{K} dx
\]
\[
\times \int_{-1}^{1} \mathbb{E} \left\{ -\int_{(0,t \wedge \sigma_k)} \sup_{|x| \leq K, |y| \leq v} |\tilde{V}_s(x - \frac{y}{m}) - \tilde{V}_s(x - \frac{v}{m})|^2 ds \right\} \Phi(y)^2 dy \\
\leq 2^{2n^{\beta} + 3} \kappa^{2\beta} c_3 K(na_n)^{-1} m^{1 - 2n^{\beta}} \int_0^1 z^2 m(dz),
\]

where \( c_3 := \sup_{(x,y) \in [0,T] \times [-K,K]} \Psi_s(x) \).

**Step 2.** We then estimate \( I_{4,1,2}(t) \). Observe that

\[
A_{m,n}(x, y, z, s, h) := \{ |\tilde{U}_s, \Phi_{m,x}^y + zm \tilde{V}_s(x - \frac{y}{m})\Phi(y)| \leq a_{n-1} \}
\]

\[
= \{ \left| \int_{-1}^{1} U_s(x - \frac{u}{m})\Phi(u)du + zm \tilde{V}_s(x - \frac{y}{m})\Phi(y) \right| \leq a_{n-1} \}
\]

\[
\subset \{ \left| \tilde{V}_s(x - \frac{x_{s,m}(x)}{m})\Phi(y) \right| \leq \sup_{s,x \in [0,1]} |\tilde{V}_s(x - \frac{x_{s,m}(x)}{m})\Phi(y)| \leq a_{n-1} \}. \quad (4.16)
\]

In fact, if

\[
|\tilde{V}_s(x - \frac{x_{s,m}(x)}{m})\Phi(y)| \leq zm \tilde{V}_s(x - \frac{x_{s,m}(x)}{m})\Phi(y) \geq a_{n-1},
\]

then \( \tilde{V}_s(x - \frac{x_{s,m}(x)}{m}) \neq 0 \). This implies that \( \tilde{V}_s(x - \frac{u}{m}) \neq 0 \) for all \( u \in [-1, 1] \). Then by the continuity of \( u \mapsto \tilde{V}_s(u) \) and the mean value theorem, \( \tilde{V}_s(x - \frac{u}{m}) > 0 \) for all \( u \in [-1, 1] \), or \( \tilde{V}_s(x - \frac{u}{m}) < 0 \) for all \( u \in [-1, 1] \). Since \( |\tilde{V}_s(x - \frac{u}{m})| \leq |\tilde{U}_s(x - \frac{u}{m})|^{\beta} \) for all \( u \in [-1, 1] \),

\[
\left| \int_{-1}^{1} U_s(x - \frac{u}{m})\Phi(u)du + zm \tilde{V}_s(x - \frac{y}{m})\Phi(y) \right|
= \int_{-1}^{1} \left| \int_{-1}^{1} \Phi(u)du \right| + zm \tilde{V}_s(x - \frac{y}{m})\Phi(y)
\geq \int_{-1}^{1} \left| \tilde{V}_s(x - \frac{u}{m})\Phi(u)du \right| + zm \tilde{V}_s(x - \frac{y}{m})\Phi(y)
\geq |\tilde{V}_s(x - \frac{x_{s,m}(x)}{m})\Phi(y)| \geq a_{n-1},
\]

which implies (4.16). It follows from (4.16) that

\[
A_{m,n}(x, y, z, s, h) \subset \{ zm \tilde{V}_s(x - \frac{x_{s,m}(x)}{m})\Phi(y)| < a_{n-1} \} \cap \{ |\tilde{V}_s(x - \frac{x_{s,m}(x)}{m})| < a_{n-1}^{\beta} \}
\]

\[
= : B_{m,n}(x, y, z, s, h) \cap B_{m,n}(x, s), \quad (4.17)
\]

Let \( s, x, y \) be fixed. If \( \tilde{V}_s(x - \frac{x_{s,m}(x)}{m})\Phi(y) \neq 0 \), let \( x_0 := a_{n-1}(m|\tilde{V}_s(x - \frac{x_{s,m}(x)}{m})\Phi(y)|)^{-1} \).

Then it is easy to see that

\[
\int_0^1 z^2 m(dz) \int_0^1 1_{B_{m,n}(x, y, z, s, h)}(1 - h)dh
= \int_0^1 dh \int_0^1 1_{\{z \leq x_0 \}}(1 - h)z^2 m(dz)
\leq \frac{c_0}{2 - \alpha} x_0 + \frac{c_0}{(2 - \alpha)(\alpha - 1)} x_0^{2 - \alpha}. \quad (4.18)
\]

Then by (4.17) and (4.18),

\[
|\tilde{V}_s(x - \frac{x_{s,m}(x)}{m})\Phi(y)|^2 \int_0^1 z^2 m(dz) \int_0^1 1_{A_{m,n}(x, y, z, s, h)}(1 - h)dh
\]
Proof.
We consider a partition \( \Delta \) where \( \tau \) is a stopping time \( (\tau, P, \mathcal{F}, \lambda) := (\tau, P, \mathcal{F}, \lambda) \). By Proposition 2.1 and Lemma 5.1, for any \( \Delta \) and \( \lambda \), we have
\[
\langle X_{t \wedge \tau_k}, P_{t-(t \wedge \tau_k)} f \lambda \rangle = X_0(P_t f \lambda) + \int_0^t \int_0^\infty \int_\mathbb{R} X_{s-(u)}^p z P_{t-s} f \lambda(u) 1_{\{s \leq \tau_k\}} \tilde{N}_0(ds, dz, du, dv),
\]
(5.1)
where \( \tau_k \) is a stopping time defined by
\[
\tau_k := \inf \{ t : |F(p, t)| > k \},
\]
(5.2)
where \( F(p, t) := \int_0^t ds \int_\mathbb{R} X_s(x)^q dx \) for \( p > 1 \) and \( F(p, t) := \langle X_t, 1 \rangle \) for \( p \leq 1 \).

Proof. We consider a partition \( \Delta_n := \{ 0 = t_0 < t_1 < \cdots < t_n = t \} \) of \([0, t]\). Let \( \Delta_n := \max_{1 \leq i \leq n} |t_i - t_{i-1}| \). It is clear that \( \frac{dP_t f \lambda(x)}{dt} = \frac{1}{2} P_t f '' \lambda(x) \) for \( t \geq 0 \). For \( k \geq 1 \) and \( t \in [0, T] \) let \( Z_k(t) = X_{t \wedge \tau_k} \). By Proposition 2.1,
\[
\langle Z_k(t), f \lambda \rangle = X_0(f \lambda) + \int_0^t \int_0^\infty \int_\mathbb{R} X_{s-(u)}^p z f \lambda(u) 1_{\{s \leq \tau_k\}} \tilde{N}_0(ds, dz, du, dv).
\]
(5.4)
It follows that
\[
\langle Z_k(t), P_{t-(t \wedge \tau_k)} f \lambda \rangle
\]

5 Appendix

Before proving Proposition 2.3 we state two lemmas. For \( \lambda > 0 \) let
\[
F_\lambda := \{ fh : f \in C(\mathbb{R}) \} \text{ and } h_\lambda(x) := \exp\{-x^2/(2\lambda)\}, \ x \in \mathbb{R}.
\]

Lemma 5.1 For any \( t \in [0, T] \), \( k, \lambda \geq 1 \) and \( f \lambda := h_\lambda f \) with \( f \in C(\mathbb{R}) \) we have \( P \)-a.s.
\[
\langle X_{t \wedge \tau_k}, P_{t-(t \wedge \tau_k)} f \lambda \rangle = X_0(P_t f \lambda) + \int_0^t \int_0^\infty \int_\mathbb{R} X_{s-(u)}^p z P_{t-s} f \lambda(u) 1_{\{s \leq \tau_k\}} \tilde{N}_0(ds, dz, du, dv),
\]
(5.1)
\[
X_0(P_t f_\lambda) + \sum_{i=1}^n \langle Z_k(t_i), P_{t-(t_i \wedge \tau_k)} f_\lambda - P_{t-(t_i - 1 \wedge \tau_k)} f_\lambda \rangle \\
+ \sum_{i=1}^n \left[ \langle Z_k(t_i), P_{t-(t_i - 1 \wedge \tau_k)} f_\lambda \rangle - \langle Z_k(t_i - 1), P_{t-(t_i - 1 \wedge \tau_k)} f_\lambda \rangle \right]
\]

\[
= X_0(P_t f_\lambda) + \frac{1}{2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \langle Z_k(t), P_s f_\lambda \rangle ds + \frac{1}{2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \langle X_s, P_{t-(t_i - 1 \wedge \tau_k)} f_\lambda \rangle 1_{\{s \leq \tau_k\}} ds \\
+ \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_0^\infty \int_\mathbb{R} \int_0^\infty z P_{t-(t_i - 1 \wedge \tau_k)} f_\lambda(u) 1_{\{s \leq \tau_k\}} \tilde{N}_0(ds, dz, du, dv)
\]

\[
= X_0(P_t f_\lambda) + \frac{1}{2} \sum_{i=1}^n \int_0^t \sum_{i=1}^n I_i(s) \left( \langle X_s, P_{t-(t_i - 1 \wedge \tau_k)} f_\lambda \rangle - \langle Z_k(t_i), P_{t-s} f_\lambda \rangle \right) 1_{\{s \leq \tau_k\}} ds \\
+ \int_0^t \int_0^\infty \int_\mathbb{R} \int_\mathbb{R} z \sum_{i=1}^n I_i(s) P_{t-(t_i - 1 \wedge \tau_k)} f_\lambda(u) 1_{\{s \leq \tau_k\}} \tilde{N}_0(ds, dz, du, dv), \quad (5.3)
\]

where \(I_i(s) := 1_{\{t_{i-1}, t_i\}}(s)\). By dominated convergence we have

\[
\lim_{|\Delta_n| \to 0} \int_0^t \sum_{i=1}^n I_i(s) \langle X_s, P_{t-(t_i - 1 \wedge \tau_k)} f_\lambda \rangle ds = 0.
\]

Observe that \(q > p\) for the case \(p > 1\). By Hölder inequality, for all measurable function \(H(s, u)\) on \([0, t] \times \mathbb{R}\) satifying

\[
\int_0^t ds \int_\mathbb{R} |H(s, u)|^{\frac{q}{p}} du < \infty,
\]

we have

\[
\mathbb{E} \left\{ \int_0^{t \wedge \tau_k} ds \int_\mathbb{R} X_s(u)^p H(s, u) du \right\} \leq \mathbb{E} \left\{ \int_0^{t \wedge \tau_k} ds \int_\mathbb{R} X_s(u)^q du \right\}^{\frac{q}{p}} \left\{ \int_0^t ds \int_\mathbb{R} |H(s, u)|^{\frac{q}{p}} du \right\}^{1 - \frac{q}{p}} \leq C k^{\frac{p}{7}}. \quad (5.5)
\]

Observe that for \(t, s \in [0, T]\)

\[
f(t, s, u, n, k) := \sum_{i=1}^n I_i(s) \left| P_{t-(t_i - 1 \wedge \tau_k)} f_\lambda(u) - P_{t-s} f_\lambda(u) \right|
\]

\[
\leq (2\pi \lambda)^\frac{1}{2} \|f\| \sum_{i=1}^n I_i(s) \left| p_{t+\lambda-(t_i - 1 \wedge \tau_k)}(u) + p_{t-s}(u) \right| \leq 2\|f\| h_{\lambda+T}(u)
\]
tends to zero as \(\Delta_n \to 0\). Then by dominated convergence and \(5.3\), it is easy to see that as \(\Delta_n \to 0\),

\[
\mathbb{E} \left\{ \int_0^t \int_0^\infty \int_\mathbb{R} \int_\mathbb{R} z \sum_{i=1}^n I_i(s) P_{t-(t_i - 1 \wedge \tau_k)} f_\lambda(u) \right\}
\]
One can check that Lemma 5.1 holds with $\tau$ replaced by (2.1). In the following we give the proof of (2.2). By Fubini's theorem, for all $f \in E$ and $|t|=\lambda > 1$, Lemma 5.2 holds for all $\epsilon > 0$.

By monotone convergence, we have

$$\lim_{k \to \infty} \mathbb{E}\left\{ |t|^{\lambda} \mathbb{P}_{t-\tau_k} |h| \right\} = \mathbb{E}\left\{ \lim_{k \to \infty} |t|^{\lambda} \mathbb{P}_{t-\tau_k} |h| \right\} \leq \lim_{k \to \infty} \mathbb{E}\left\{ |t|^{\lambda} \mathbb{P}_{t-\tau_k} |h| \right\} = \mathbb{E}\left\{ |t|^{\lambda} \mathbb{P}_{t-\tau_k} |h| \right\}.$$

By Fatou's lemma, we have

$$\mathbb{E}\left\{ \int_0^t \int_0^t \int_0^t \int_0^t X_{s-}^p f(t,s,u,n,k) 1_{\{s \leq \tau_k\}} \right\} \to 0 \quad (5.7)$$

and

$$\mathbb{E}\left\{ \left( \int_0^t \int_0^t \int_0^t \int_0^t X_{s-}^p f(t,s,u,n,k) 1_{\{s \leq \tau_k\}} \right)^2 \right\} \to 0 \quad (5.7)$$

Now it is obvious that (5.1) follows from (5.3), (5.4), and (5.6), (5.7).

**Lemma 5.2** For all $t \geq 0$ and $f \in B(\mathbb{R})^+$ we have $\mathbb{E}\{\langle X_t, f \rangle\} \leq \mathbb{X}_0(P_t f)$.

**Proof.** By Lemma 5.1 for all $k, \lambda \geq 1$ we have

$$\mathbb{E}\{\langle X_{t \wedge \tau_k}, P_{t-\tau_k} h \lambda \rangle\} = \mathbb{X}_0(P_t h \lambda)$$

By Fatou's lemma, we have

$$\mathbb{E}\{\langle X_t, h \lambda \rangle\} = \mathbb{E}\left\{ \lim_{k \to \infty} \langle X_{t \wedge \tau_k}, P_{t-\tau_k} h \lambda \rangle \right\} \leq \lim_{k \to \infty} \mathbb{E}\left\{ \langle X_{t \wedge \tau_k}, P_{t-\tau_k} h \lambda \rangle \right\} = \mathbb{X}_0(P_t h \lambda).$$

By monotone convergence,

$$\mathbb{E}\{\langle X_t, 1 \rangle\} = \lim_{\lambda \to \infty} \mathbb{E}\{\langle X_t, h \lambda \rangle\} \leq \lim_{\lambda \to \infty} \mathbb{X}_0(P_t h \lambda) = \mathbb{X}_0(1).$$

Then we can finish the proof easily by dominated convergence.

Now we are ready to present proof of Proposition 2.3.

**Proof of Proposition 2.3** Define stopping time $s_k$ by

$$s_k := \inf \left\{ t \in (0,T) : \int_0^t \int_0^t X_s^p dx \right\}.$$
almost surely. Combining this with Lemma 5.1,

By (1.6) of (19) and (5.5), for 

Choose \( \bar{\varepsilon} \) satisfying \( q > \frac{3p}{3-\alpha} \), by (5.5),

Choose \( \bar{\alpha} \) satisfying \( q > \frac{3p}{3-\alpha} \), by (5.5),

Then (5.10) tends to zero as \( \varepsilon \to 0 \). It follows from (5.8) that

almost surely. Combining this with Lemma 5.1

\[
\int_{\mathbb{R}} X_{t \wedge \tau_k}(x) P_{t \to t \wedge \tau_k} f(x) \, dx 
\]
By the Burkholder-Davis-Gundy inequality, it is obvious that

\[
\int_{\mathbb{R}} p_t(x-z)X_0(dz) + \int_0^{t \wedge \tau_k} \int_0^\infty \int_0^\infty X_{u-}(u)^p zp_{t-s}(x-u)\tilde{N}_0(ds,dz,du) f(x)dx
\]

almost surely. Letting \( k \to \infty \) one completes the proof. \( \square \)

**Proof of Lemma 2.4.** If (2.3) holds for \( \bar{p} = p \lor 1 \), then it is easy to prove the general case by Hölder inequality and (2.2). In the following we establish (2.3) for \( \bar{p} = p \lor 1 \).

(i) We first consider the case \( p \leq 1 \). The result follows from (2.2) immediately if

\[
[0, t_0] \ni t \mapsto \int_0^t \int_0^\infty \int_0^\infty X_{u-}(u)^p zp_{t-s}(x-u)\tilde{N}_0(ds,dz,du)
\]

is a martingale for \( t_0 > 0 \). By [10, p.38] we only need to show that

\[
E\left\{ \sup_{t \in [0, t_0]} \left| \int_0^t \int_0^\infty \int_0^\infty X_{u-}(u)^p zp_{t-s}(x-u)\tilde{N}_0(ds,dz,du) \right| \right\} < \infty.
\]

It is obvious that

\[
E\left\{ \sup_{t \in [0, t_0]} \left| \int_0^t \int_0^\infty \int_0^\infty X_{u-}(u)^p zp_{t-s}(x-u)\tilde{N}_0(ds,dz,du) \right| \right\}
\]

\[
\leq E\left\{ \sup_{t \in [0, t_0]} \left| \int_0^t \int_0^1 \int_0^\infty X_{u-}(u)^p zp_{t-s}(x-u)\tilde{N}_0(ds,dz,du) \right| \right\} + E\left\{ \sup_{t \in [0, t_0]} \left| \int_0^t \int_1^{\infty} \int_0^\infty X_{u-}(u)^p zp_{t-s}(x-u)\tilde{N}_0(ds,dz,du) \right| \right\}
\]

\[
=: E\{I_1(t_0)\} + E\{I_2(t_0)\}.
\]

By the Burkholder-Davis-Gundy inequality,

\[
E\{I_1(t_0)^2\} \leq C \int_0^1 z^2 m(dz) \int_0^{t_0} E\left\{ \int_\mathbb{R} X_{u}(u)^p zp_{t-s}(x-u)^2 du \right\} ds
\]

\[
\leq C \int_0^{t_0} (t_0-s)^{-\frac{p}{2}} \left[ \int_\mathbb{R} X_{u}(u)^p zp_{t-s}(x-u) du \right] ds \tag{5.12}
\]

and

\[
E\{I_2(t_0)\} \leq 2 E\left\{ \int_0^{t_0} ds \int_1^{\infty} z m(dz) \int_\mathbb{R} \int_0^\infty X_{u-}(u)^p zp_{t-s}(x-u) du \right\}
\]

\[
\leq C \int_0^{t_0} E\left\{ \int_\mathbb{R} X_{u}(u)^p zp_{t-s}(x-u) du \right\} ds \tag{5.13}
\]

By Jensen inequality and Lemma [5.2]

\[
E\left\{ \int_\mathbb{R} X_{u}(u)^p zp_{t-s}(x-u) du \right\} ds \leq E\left\{ \left| \int_\mathbb{R} X_{u}(u) zp_{t-s}(x-u) du \right|^p \right\} ds
\]

\[
\leq [2\pi(t_0-s)]^{-\frac{p}{2}} E[|X_s(1)|]^p \leq X_0(1)^{p}[2\pi(t_0-s)]^{-\frac{p}{2}}.
\]

Combining with (5.12) and (5.13) we obtain (2.3) for the case \( p \leq 1 \).

(ii) Now we consider the case \( 1 < p < \alpha \) for \( p \leq 1 \).

\[
E\left\{ \int_\mathbb{R} X_{u}(u)^p zp_{t-s}(x-u) du \right\} ds \leq \left[ \int_\mathbb{R} X_{u}(u) zp_{t-s}(x-u) du \right]^{\frac{p}{2}} ds
\]

\[
\leq [2\pi(t_0-s)]^{-\frac{p}{2}} E[\langle X_s, 1 \rangle]^p \leq X_0(1)^{p}[2\pi(t_0-s)]^{-\frac{p}{2}}.
\]

Combining with (5.12) and (5.13) we obtain (2.3) for the case \( p \leq 1 \).
Step 1. Recalling (5.2), one can see that

\[ I_k(t, x) := \left| \int_0^t \int_0^{\tau_k} \int_0^\infty \int_0^\infty X_{s-}(u)^p z_{t-s}(x-u) \tilde{N}_0(ds, dz, du, dv) \right| \]

\[ \leq \left| \int_0^t \int_0^{1} \int_0^\infty X_{s-}(u)^p z_{t-s}(x-u)1_{\{s \leq \tau_k\}} \tilde{N}_0(ds, dz, du, dv) \right| \]

\[ + \left| \int_0^t \int_0^{\infty} \int_0^\infty \int_0^\infty X_{s-}(u)^p z_{t-s}(x-u)1_{\{s \leq \tau_k\}} \tilde{N}_0(ds, dz, du, dv) \right| \]

\[ := I_{k,1}(t, x) + I_{k,2}(t, x). \]

By (1.6) of [19], for \( \alpha < \hat{p} < 2 \)

\[ \mathbb{E}\left\{ I_{k,1}(t, x)^{\hat{p}} \right\} \leq C \int_0^1 z^{\hat{p}} m(dz) \mathbb{E}\left\{ \int_0^t ds \int_\mathbb{R} X_s(u)^p p_{t-s}(x-u)^\hat{p} 1_{\{s \leq \tau_k\}} du \right\} \]

\[ \leq C \mathbb{E}\left\{ \int_0^t (t-s)^{-\frac{\hat{p}-1}{2}} ds \int_\mathbb{R} X_s(u)^p p_{t-s}(x-u)^\hat{p} 1_{\{s \leq \tau_k\}} du \right\} \]

and

\[ \mathbb{E}\left\{ I_{k,2}(t, x)^{\hat{p}} \right\} \leq C \int_1^\infty z^{\hat{p}} m(dz) \mathbb{E}\left\{ \int_0^t ds \int_\mathbb{R} X_s(u)^p p_{t-s}(x-u)^\hat{p} 1_{\{s \leq \tau_k\}} du \right\} \]

\[ \leq C \mathbb{E}\left\{ \int_0^t (t-s)^{-\frac{\hat{p}-1}{2}} ds \int_\mathbb{R} X_s(u)^p p_{t-s}(x-u)^\hat{p} 1_{\{s \leq \tau_k\}} du \right\} \]

Then one obtains that

\[ \mathbb{E}\left\{ I_k(t, x)^{\hat{p}} \right\} \leq 2 \mathbb{E}\left\{ I_{k,1}(t, x)^{\hat{p}} \right\} + 2 \mathbb{E}\left\{ I_{k,2}(t, x)^{\hat{p}} \right\} \leq C \left\{ \mathbb{E}\left[ I_{k,1}(t, x)^{\hat{p}} \right] + \mathbb{E}\left[ I_{k,2}(t, x)^{\hat{p}} \right] + 1 \right\} \]

\[ \leq C \mathbb{E}\left\{ \int_0^t \left[ (t-s)^{-\frac{\hat{p}-1}{2}} + (t-s)^{-\frac{\hat{p}-1}{2}} \right] ds \int_\mathbb{R} X_s(u)^p p_{t-s}(x-u)^\hat{p} 1_{\{s \leq \tau_k\}} du \right\} + C. \]

Combining this with (2.2), we have

\[ \mathbb{E}\left\{ \int_0^T dt \int_\mathbb{R} X_t(y)^p p_{T-t}(x-y)1_{\{t \leq \tau_k\}} dy \right\} \]

\[ \leq 2 \int_0^T dt \int_\mathbb{R} X_0(p_t(y-\cdot)^p)p_{T-t}(x-y)dy + 2 \int_0^T dt \int_\mathbb{R} \mathbb{E}\left\{ I_k(t, y)^{\hat{p}} \right\} p_{T-t}(x-y)dy \]

\[ \leq C \int_0^T dt \int_0^t \left[ (t-s)^{-\frac{\hat{p}-1}{2}} + (t-s)^{-\frac{\hat{p}-1}{2}} \right] ds \left\{ \int_\mathbb{R} X_s(u)^p p_{T-s}(x-u)^\hat{p} 1_{\{s \leq \tau_k\}} du \right\} \]

\[ + CX_0(1)^p T^{\frac{2p-\hat{p}}{2}} + CT + C \]

\[ \leq C \mathbb{E}\left\{ \int_0^T ds \int_\mathbb{R} X_s(u)^p p_{T-s}(x-u)1_{\{s \leq \tau_k\}} du \int_s^T \left[ (t-s)^{-\frac{\hat{p}-1}{2}} + (t-s)^{-\frac{\hat{p}-1}{2}} \right] dt \right\} \]

\[ + CX_0(1)^p T^{\frac{2p-\hat{p}}{2}} + CT + C \]

\[ \leq C(T^{\frac{3p}{2}} + T^{\frac{3p}{2}}) \mathbb{E}\left\{ \int_0^T ds \int_\mathbb{R} X_s(u)^p p_{T-s}(x-u)1_{\{s \leq \tau_k\}} du \right\} \]

\[ + CX_0(1)^p T^{\frac{2p-\hat{p}}{2}} + CT + C. \]

By (5.11),

\[ \mathbb{E}\left\{ \int_0^T dt \int_\mathbb{R} X_t(y)^p p_{T-t}(x-y)1_{\{t \leq \tau_k\}} dy \right\} < \infty. \]
Taking $T_0 > 0$ with $C' := C(T^{\frac{3-p}{2}} + T^{\frac{1-p}{2}}) < 1$, for all $T \in [0,T_0]$ and $k \geq 1$ we have

$$E\left\{ \int_0^T dt \int_\mathbb{R} X_t(y)^p p_{T-t}(x-y)1_{\{t \leq \tau_k\}} dy \right\} \leq (1-C')^{-1} \left[ CX_0(1)T^{\frac{2-p}{2}} + CT + C \right].$$

Then by Fatou’s lemma

$$\sup_{T \in [0,T_0]} E\left\{ \int_0^T dt \int_\mathbb{R} X_t(x)^p p_{T-t}(x-y) dx \right\} < \infty. \quad (5.14)$$

**Step 2.** In this step we prove that (2.3) holds with $T$ replaced by the $T_0$ in (5.14). For all $0 < r < 2$,

$$\int_0^T (T-t)^{-\frac{\delta}{2}} dt \int_\mathbb{R} X_0(p_t(y-\cdot))p_{T-t}(x-y) dy \leq \left[ (2\pi)^{-1} X_0(1) \right]^p \int_0^T (T-t)^{-\frac{\delta}{2}} t^{\frac{p-1}{2}} dt. \quad (5.15)$$

For all $0 < r < 2$ and $0 < \delta < 3$,

$$\int_0^T (T-t)^{-\frac{\delta}{2}} dt \int_\mathbb{R} p_{T-t}(x-y) dy \int_0^t ds \int_\mathbb{R} X_s(u)^p p_{T-s}(x-u) du \leq C \int_0^T (T-t)^{-\frac{\delta}{2}} dt \int_0^t (t-s)^{-\frac{\delta+1}{2}} ds \int_\mathbb{R} X_s(u)^p p_{T-s}(x-u) du \leq C \int_0^T (T-t)^{-\frac{\delta}{2}} dt \int_\mathbb{R} X_s(u)^p p_{T-s}(x-u) du. \quad (5.16)$$

Combining with (2.2), (5.14), (5.15) it is easy to see that for all $0 < r < 2$

$$\sup_{T \in [0,T_0]} E\left\{ \int_0^T (T-t)^{-\frac{\delta}{2}} dt \int_\mathbb{R} X_t(y)^p p_{T-t}(x-y) dy \right\} < \infty. \quad (5.17)$$

By (1.6) of [19] again,

$$E\left\{ \left| \int_1^t \int_1^\infty \int_\mathbb{R} X_s(u)^p z p_{t-s}(x-u) N_0(ds,dz,du, dv) \right|^p \right\} \leq C \int_1^\infty z^p m(dz) E\left\{ \int_0^t ds \int_\mathbb{R} X_s(u)^p p_{t-s}(x-u)^p du \right\}$$

and

$$E\left\{ \left| \int_0^t \int_0^1 \int_\mathbb{R} X_s(u)^p z p_{t-s}(x-u) N_0(ds,dz,du, dv) \right|^2 \right\} = \int_0^1 z^2 m(dz) E\left\{ \int_0^t ds \int_\mathbb{R} X_s(u)^p p_{t-s}(x-u)^2 du \right\},$$

which implies

$$E\left\{ \left| \int_0^t \int_0^\infty \int_\mathbb{R} X_s(u)^p z p_{t-s}(x-u) N_0(ds,dz,du, dv) \right|^p \right\} \leq C E\left\{ \int_0^t ds \int_\mathbb{R} X_s(u)^p [p_{t-s}(x-u)^p + p_{t-s}(x-u)^2] du \right\} + C.$$
By (2.2) again, we have
\[
E \{ X_t(x)^p \} \leq C t^{-\frac{p}{2}} + C \mathbb{E} \left\{ \int_0^t \int_0^\infty \int_0^\infty \int_\mathbb{R} X_{s-(u)}^p \ z p_{t-s} (x-u) \tilde{N}_0 (ds, dz, du, dv) \right\}^{\frac{p}{2}}
\]
\[
\leq C t^{-\frac{p}{2}} + C \int_0^t \left[ (t-s)^{-\frac{p}{2}} + (t-s)^{-\frac{1}{2}} \right] ds \int_\mathbb{R} p_{t-s}(x-y) X_s(y)^p dy.
\] (5.18)

Then by (5.17) one sees that (2.3) holds with \( T \) replaced by \( T_0 \).

**Step 3.** Similar to Step 1, for \( T \in [0, T_0] \),
\[
E \left\{ \int_0^{2T} \int_\mathbb{R} X_t(y)^p p_{2T-t}(x-y) 1_{\{t \leq \tau_k\}} dy \right\}
\]
\[
\leq C \int_0^T \left[ (t-s)^{-\frac{p}{2}} + (t-s)^{-\frac{1}{2}} \right] ds \int_\mathbb{R} X_s(u)^p p_{2T-s}(x-u) 1_{\{s \leq \tau_k\}} du
\]
\[
+ C X_0(1)^p T^{2-\frac{p}{2}} + C T_0
\]
\[
\leq C (T^{2-\frac{p}{2}} + T^{2-\frac{p}{2}}) \int_0^{2T} ds \int_\mathbb{R} X_s(u)^p p_{2T-s}(x-u) 1_{\{s \leq \tau_k\}} du
\]
\[
+ C X_0(1)^p T^{2-\frac{p}{2}} + C T_0,
\]
which implies
\[
\sup_{T \in [0, T_0]} E \left\{ \int_0^{2T} dt \int_\mathbb{R} X_t(y)^p p_{2T-t}(x-y) dy \right\} < \infty.
\]
It then follows from Step 2 that
\[
\sup_{T \in [0, 2T_0]} E \left\{ \int_0^T dt \int_\mathbb{R} X_t(y)^p p_{T-t}(x-y) dy \right\} < \infty.
\]

Similar to (5.17) we have
\[
\sup_{T \in [0, 2T_0]} E \left\{ \int_0^T (T-t)^{-\frac{p}{2}} dt \int_\mathbb{R} p_{T-t}(x-y) X_t(y)^p dy \right\} < \infty.
\] (5.19)

This together this with (5.18) shows that (2.3) holds with \( T \) replaced by \( 2T_0 \). This completes the proof. \( \square \)

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