On the Classification of Accidental Symmetries of the Two Higgs Doublet Model Potential

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ABSTRACT

Recently, it has been shown\cite{1} that the two-Higgs-doublet-model potential may exhibit a maximum of 13 distinct accidental symmetries. Such a classification is based on a six-dimensional bilinear scalar field formalism realizing the SO(1,5) symmetry group. This note presents the transformation relations for each of the 13 symmetries in the original scalar field space and their one-to-one correspondence to the space of scalar bilinears, thereby providing firm support for the completeness of the classification.

KEYWORDS: Symmetries, extended Higgs sector
There are several theoretical and cosmological reasons that motivate us to go beyond the Standard Model (SM) Higgs sector. In particular, the so-called Two Higgs Doublet Model (2HDM), where the SM is minimally extended with a second Higgs doublet, can provide new sources of CP violation of spontaneous [2] or explicit origin, predict stable scalars as Dark Matter candidates [3], and give rise to electroweak baryogenesis [4] through a strong first order phase transition [5]. Unlike the SM, the 2HDM potential may realize a large number of different symmetries [6], global or discrete, whose breaking may result in pseudo-Goldstone bosons [7], mass hierarchies, flavour-changing neutral currents [8] and CP violation [9, 10, 11]. The systematic analysis of the different possible symmetries and their phenomenology have been the subject of many recent studies [12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

For given choices of its theoretical parameters, the 2HDM potential may exhibit three different classes of accidental symmetries. The first class of symmetries pertains to transformations of two Higgs doublets $\phi_{1,2}$ only, but not their complex conjugates $\phi^*_{1,2}$, and are therefore called Higgs Family (HF) symmetries [12, 21]. Known HF symmetries include the $Z_2$ discrete symmetry [8], the Peccei–Quinn symmetry $U(1)_{PQ}$ [22] and the HF symmetry $SU(2)_{HF}$ [6, 20] acting on the Higgs doublets $\phi_{1,2}$.

The second class of transformations relates the fields $\phi_{1,2}$ to their CP-conjugates $\phi^*_{1,2}$ and are termed CP symmetries [21]. Known examples of this kind are the CP1 symmetry which describes the standard CP transformation $\phi_{1(2)} \to \phi^*_{1(2)}$ [2, 6, 9], the CP2 symmetry where $\phi_{1(2)} \to (-\phi^*_{2(1)})$ [14] and the CP3 symmetry which combines CP1 with an $SO(2)_{HF}$ transformation of the fields $\phi_{1,2}$ [20, 21].

Nevertheless, there is a third class of symmetries which utilize mixed HF and CP transformations that leave the $SU(2)_L$ gauge kinetic terms of $\phi_{1,2}$ canonical [11]. Examples of this kind are the $O(8)$ and $O(4) \otimes O(4)$ symmetries in the real field space [6]. As we will show in this note, these mixed HF/CP transformations play an important role to properly identify all the 13 accidental symmetries [11] that may occur in the 2HDM potential. In particular, based on the bilinear scalar field formalism realizing the $SO(1,5)$ symmetry group, we derive explicit transformation relations for each of the 13 symmetries in the original scalar field space and give their one-to-one correspondence to the space of scalar bilinears, thereby proving the self-consistency and the completeness of the classification conducted in [11].

To start with, let us write down the general bare, local structure of the 2HDM potential $V$ in the usual doublet field space $\phi_{1,2}$:

$$V = -\mu_1^2(\phi^\dagger_1 \phi_1) - \mu_2^2(\phi^\dagger_2 \phi_2) - m_{12}^2(\phi^\dagger_1 \phi_2) - m_{12}^* (\phi^\dagger_2 \phi_1) + \lambda_1(\phi^\dagger_1 \phi_1)^2 + \lambda_2(\phi^\dagger_2 \phi_2)^2$$

$$+ \lambda_3(\phi^\dagger_1 \phi_1)(\phi^\dagger_2 \phi_2) + \lambda_4(\phi^\dagger_1 \phi_2)(\phi^\dagger_2 \phi_1) + \frac{\lambda_5}{2}(\phi^\dagger_1 \phi_2)^2 + \frac{\lambda_6}{2}(\phi^\dagger_2 \phi_1)^2$$

$$+ \lambda_6(\phi^\dagger_1 \phi_1)(\phi^\dagger_1 \phi_2) + \lambda_6^*(\phi^\dagger_2 \phi_1)(\phi^\dagger_2 \phi_2) + \lambda^*_7(\phi^\dagger_2 \phi_2)(\phi^\dagger_1 \phi_1) \quad (1)$$

The potential $V$ contains 4 real mass parameters, $\mu_1^2, \mu_2^2, \Re m_{12}^2$ and $\Im m_{12}^2$, and 10 real quartic couplings, $\lambda_{1,2,3,4}, \Re \lambda_{5,6,7}$ and $\Im \lambda_{5,6,7}$.

An alternative formulation of the 2HDM potential may be obtained by introducing the
8-dimensional (8D) complex multiplet \[\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ i\sigma^2 \phi_1^* \\ i\sigma^2 \phi_2^* \end{pmatrix}, \tag{2}\]

where \(\sigma^2\) is the second matrix of the Pauli matrices \(\sigma^{1,2,3}\). Observe that each doublet component of \(\Phi\) transforms covariantly under a SU(2)\(_L\) gauge transformation \(U_L\), i.e. \(\Phi \rightarrow U_L \Phi\). Under charge conjugation, the multiplet \(\Phi\) satisfies the Majorana-type property \[\Phi = C \Phi^*, \tag{3}\]

where \(C = \sigma^2 \otimes \sigma^0 \otimes \sigma^2\), with \(C = C^{-1} = C^*\) and \(\sigma^0 = 1\) being the 2 \(\times\) 2 identity matrix.

Given the \(\Phi\)-multiplet, we may go over to the bilinear field space which realizes an SO(1,5) symmetry group, by defining the null 6-vector \[R^A \equiv \Phi^\dagger \Sigma^A \Phi = \begin{pmatrix} \phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 \\ \phi_1^\dagger \phi_2 + \phi_2^\dagger \phi_1 \\ -i [\phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_1] \\ \phi_1^\dagger \phi_1 - \phi_2^\dagger \phi_2 \\ \phi_1^\dagger i\sigma^2 \phi_2 - \phi_2^\dagger i\sigma^2 \phi_1^* \\ -i [\phi_1^\dagger i\sigma^2 \phi_2 + \phi_2^\dagger i\sigma^2 \phi_1^*] \end{pmatrix}, \tag{4}\]

with \(A = 0, 1, 2, 3, 4, 5 = \mu, 4, 5\). The six 8 \(\times\) 8-dimensional matrices \(\Sigma^A\) may be expressed in terms of double tensor products as follows:

\[
\Sigma^{0,1,3} = \frac{1}{2} \sigma^0 \otimes \sigma^{0,1,3} \otimes \sigma^0, \quad \Sigma^2 = \frac{1}{2} \sigma^3 \otimes \sigma^2 \otimes \sigma^0, \quad \Sigma^4 = -\frac{1}{2} \sigma^2 \otimes \sigma^2 \otimes \sigma^0, \quad \Sigma^5 = -\frac{1}{2} \sigma^1 \otimes \sigma^2 \otimes \sigma^0. \tag{5}\]

Note that the above six matrices satisfy the Majorana condition: \(C^{-1} \Sigma^A C = (\Sigma^A)^T\). Further details of the matrices \(\Sigma^A\) are given in \[II\].

Having introduced the field-bilinear 6-vector \(R^A\), the potential \(V\) in \[II\] can now be written down in the quadratic form

\[V = -\frac{1}{2} M_A R^A + \frac{1}{4} L_{AB} R^A R^B, \tag{6}\]
where

\[ M_A = \begin{pmatrix} \mu_1^2 + \mu_2^2, & 2\Re(m_{12}^2), & -2\Im(m_{12}^2), & \mu_1^2 - \mu_2^2, & 0, & 0 \end{pmatrix}, \quad (7) \]

\[ L_{AB} = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 & \Re(\lambda_6 + \lambda_7) & -\Im(\lambda_6 + \lambda_7) & \lambda_1 - \lambda_2 & 0 & 0 \\ \Re(\lambda_6 + \lambda_7) & \lambda_4 + \Re(\lambda_5) & -\Im(\lambda_5) & \Re(\lambda_6 - \lambda_7) & 0 & 0 \\ -\Im(\lambda_6 + \lambda_7) & -\Im(\lambda_5) & \lambda_4 - \Re(\lambda_5) & -\Im(\lambda_6 - \lambda_7) & 0 & 0 \\ \lambda_1 - \lambda_2 & \Re(\lambda_6 - \lambda_7) & -\Im(\lambda_6 - \lambda_7) & \lambda_1 + \lambda_2 - \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (8) \]

Evidently, for a U(1)_Y-invariant 2HDM potential all the elements of M_A and L_{AB} involving the components A, B = 4, 5 vanish, where the non-zero elements M_µ and L_{µν} have originally been calculated in [16, 17, 18]. As we will see, however, this apparent redundancy plays an important role to properly identify all accidental symmetries that may take place in a U(1)_Y-invariant 2HDM potential.

We now consider GL(8, C) scalar-field transformations acting on the Φ multiplet that leave the SU(2)_L gauge-kinetic term of the Higgs doublets, \( \frac{1}{2} (D_µΦ) \dagger (D^µΦ) \), invariant, where \( D_µ = \sigma^0 \otimes \sigma^0 \otimes (\sigma^0 \partial_µ + ig_w W_µ^I σ^I / 2) \) is the covariant derivative in the Φ-space. This restriction reduces the GL(8, C) transformations to unitary rotations U ∈ U(4) in the Φ-space, subject to the Majorana constraint [1]: U* = C⁻¹ U C. The latter condition implies that the generators K^a of the Majorana-constrained U(4) group, denoted hereafter as U_M(4), should satisfy the important relation:

\[ C^{-1} K^a C = -K^{a*} = -K^a^T. \quad (9) \]

As it can be easily checked, the identity matrix Σ^0 given in [5] does not obey the above condition, so Σ^0 cannot be one of the generators of U_M(4). Likewise, none of the other 5 matrices, Σ^1,2,3,4,5, satisfies (9), as one obtains the opposite sign (see remark after (5)). However, a careful analysis yields the following 10 generators K^a (with a = 0, 2, . . . , 9) of SU_M(4):

\[ K^0 = \frac{1}{2} σ^3 \otimes σ^0 \otimes σ^0, \quad K^1 = \frac{1}{2} σ^3 \otimes σ^1 \otimes σ^0, \quad K^2 = \frac{1}{2} σ^0 \otimes σ^2 \otimes σ^0, \]

\[ K^3 = \frac{1}{2} σ^3 \otimes σ^3 \otimes σ^0, \quad K^4 = \frac{1}{2} σ^1 \otimes σ^0 \otimes σ^0, \quad K^5 = \frac{1}{2} σ^1 \otimes σ^3 \otimes σ^0, \]

\[ K^6 = \frac{1}{2} σ^0 \otimes σ^0 \otimes σ^0, \quad K^7 = \frac{1}{2} σ^2 \otimes σ^3 \otimes σ^0, \quad K^8 = \frac{1}{2} σ^1 \otimes σ^1 \otimes σ^0, \]

\[ K^9 = \frac{1}{2} σ^2 \otimes σ^1 \otimes σ^0. \quad (10) \]

Note that the generator K^0 is related to U(1)_Y hypercharge rotations. Moreover, the 5 matrices Σ^I (with I = 1, 2, 3, 4, 5) in (5) and the 10 matrices K^a in (10) represent the 15 generators of the complete SU(4) group. In particular, because of the different even and odd transformation properties of Σ^I and K^a under C conjugation, the Lie commutators have the following structure:

\[ [K^a, Σ^I] = 2i f^{aIJ} Σ^J, \quad (11) \]
where $f^{aIJ}$ is a subset of the structure constants of the SU(4) group. Note that the Lie commutators involving $\Sigma^I$ or $K^a$ only close within themselves.

Since unitary transformations leave the zero component $R^0$ of the 6-vector $R^A$ invariant, we only consider the action of $SU_M(4)$ on its ‘spatial’ components $R^I$. Specifically, an infinitesimal $SU_M(4)$ transformation of $\Phi$ changes $R^I$ by an amount

$$\delta R^I = i \theta^a \Phi^I [\Sigma^I, K^a] \Phi = 2 \theta^a f^{aIJ} R^J,$$

where $\theta^a$ are the group parameters of $SU_M(4)$. Observe that we used (11) to arrive at the last equality in (12). The corresponding 10 generators $T^a$ in the 5-dimensional bilinear space $R^I$ may be calculated by

$$(T^a)^{IJ} = -i f^{aIJ} = \text{Tr} ([\Sigma^I, K^a] \Sigma^J).$$

In detail, we obtain

$$T^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T^3 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T^6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \end{pmatrix}, \quad T^7 = \begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix},$$

$$T^9 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ (14)

These are exactly the 10 generators of the orthogonal SO(5) group. Consequently, the relation (13) represents one of the central results of this note, as it gives an one-to-one correspondence between the generators of $SU_M(4)$ and those of SO(5). Hence, we get the isomorphism: $SO(5) \cong SU_M(4)/Z_2$, between the $\Phi$- and the $R^I$-space. This result offers firm proof of the equivalence relation, between $SU_M(4)$ and SO(5), presented in [1].

It is now obvious that the maximal reparameterization group acting on the $\Phi$-space in
Table 1: Parameter relations for the 13 accidental symmetries [1] related to the U(1)_Y-invariant 2HDM potential in the diagonally reduced basis, where \( \text{Im} \lambda_5 = 0 \) and \( \lambda_6 = \lambda_7 \). A dash signifies the absence of a constraint.

| No | Symmetry | \( \mu_1^2 \) | \( \mu_2^2 \) | \( m_{12}^2 \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \lambda_4 \) | Re\( \lambda_5 \) | \( \lambda_6 = \lambda_7 \) |
|----|----------|---------|---------|-----------|--------|--------|--------|--------|-----------|--------|
| 1 | \( Z_2 \times O(2) \) | - | - | Real | - | - | - | - | Real | - |
| 2 | \( (Z_2)^2 \times SO(2) \) | - | - | 0 | - | - | - | - | 0 | - |
| 3 | \( (Z_2)^3 \times O(2) \) | - | \( \mu_1^2 \) | 0 | - | \( \lambda_4 \) | - | - | 0 | - |
| 4 | \( O(2) \times O(2) \) | - | - | 0 | - | - | - | - | 0 | - |
| 5 | \( Z_2 \times (O(2))^2 \) | - | \( \mu_1^2 \) | 0 | - | \( \lambda_1 \) | - | - | 2\( \lambda_1 - \lambda_{34} \) | 0 |
| 6 | \( O(3) \times O(2) \) | - | \( \mu_1^2 \) | 0 | - | \( \lambda_1 \) | - | 2\( \lambda_1 - \lambda_3 \) | 0 | 0 |
| 7 | SO(3) | - | - | Real | - | - | - | - | \( \lambda_4 \) | Real |
| 8 | \( Z_2 \times O(3) \) | - | \( \mu_1^2 \) | Real | - | \( \lambda_1 \) | - | - | \( \lambda_4 \) | Real |
| 9 | \( (Z_2)^2 \times SO(3) \) | - | \( \mu_1^2 \) | 0 | - | \( \lambda_1 \) | - | - | \( \pm \lambda_4 \) | 0 |
| 10 | \( O(2) \times O(3) \) | - | \( \mu_1^2 \) | 0 | - | \( \lambda_1 \) | 2\( \lambda_1 \) | - | 0 | 0 |
| 11 | SO(4) | - | - | 0 | - | - | - | 0 | 0 | 0 |
| 12 | \( Z_2 \times O(4) \) | - | \( \mu_1^2 \) | 0 | - | \( \lambda_1 \) | - | 0 | 0 | 0 |
| 13 | SO(5) | - | \( \mu_1^2 \) | 0 | - | \( \lambda_1 \) | 2\( \lambda_1 \) | 0 | 0 | 0 |

The 2HDM potential, which leaves the SU(2)_L gauge kinetic term of \( \Phi \) canonical, is

\[
G^\Phi_{2\text{HDM}} = (SU_M(4)/Z_2) \otimes SU(2)_L. \tag{15}
\]

The group \( G^\Phi_{2\text{HDM}} \) includes the U(1)_Y hypercharge group through the generator \( K^0 \) of SU_M(4), as well as 9 other generators related to HF/CP transformations. On the other hand, the SU(2)_L group generators may be represented as \( \sigma^0 \otimes \sigma^0 \otimes (\sigma^{1,2,3}/2) \), which manifestly commute with all generators of SU_M(4). Finally, the quotient factor \( Z_2 \) appearing in (15) is needed to avoid double covering the group \( G^\Phi_{2\text{HDM}} \) in the \( \Phi \)-space.

In order to classify all possible HF/CP accidental symmetries of the 2HDM potential, it is more convenient to go over to the 5-dimensional bilinear space \( R^I \), where the maximal reparameterization group is \( G^R_{2\text{HDM}} = SO(5) \), which leaves \( R^0 \) invariant. Given that SO(5) is the maximal symmetry group in the \( R^I \)-space, Ref. [1] classifies all possible symmetries derived from SO(5), including all its proper, improper and semi-simple subgroups. Such an analysis led to a maximum of 13 accidental symmetries for the 2HDM potential, which are presented in Table 1. The same table shows the parameter restrictions for each of the 13 symmetries in a specific bilinear basis [15], where \( L_{11} \) is made diagonal by an SO(3) \( \subset SO(5) \) rotation [24]. In this diagonally reduced basis, one has the restrictions:

\[
\text{Im} \lambda_5 = 0, \quad \lambda_6 = \lambda_7, \tag{16}
\]

thus reducing to 7 the number of independent quartic couplings for the 2HDM potential. From Table 1 we observe that all 13 symmetries include \( SO(2) \cong U(1)_Y \) as a subgroup. Note that the parameter relations pertinent to the 13 symmetries are chosen, so as to manifestly lead to CP-invariant scalar potentials.
It is worth commenting that only two discrete factors, $(Z_2)^2$ and $(Z_2)^4$, are allowed, as being the only admissible subgroups of SO(5), where $Z_2$ is the reflection group of one of the components $R^4$. More explicitly, the standard CP (or CP1) discrete symmetry may be represented as $\Delta_{\text{CP}1} = C = \sigma^2 \otimes \sigma^0 \otimes \sigma^2$ in the $\Phi$-space, and the usual discrete ‘$Z_2$’ (CP2) symmetry as $\Delta_{Z_2} = \sigma^0 \otimes \sigma^3 \otimes \sigma^0$ ($\Delta_{\text{CP}2} = \sigma^2 \otimes \sigma^2 \otimes \sigma^0$). In the $R^4$-space, the transformation matrices (or the generating group elements) associated with the CP1, ‘$Z_2$’ and CP2 discrete symmetries are respectively given by

$$D_{\text{CP}1} = \text{diag}(1, -1, 1, -1), \quad D_{Z_2} = \text{diag}(1, -1, 1, -1),$$
$$D_{\text{CP}2} = \text{diag}(-1, -1, 1, -1).$$

(17)

As a consequence, both the traditional ‘$Z_2$’ symmetry and CP2 are actually isomorphic to the $(Z_2)^4$ symmetry.

It is straightforward to identify the generators pertinent to the continuous HF/CP symmetries of the 2HDM potential in the diagonally reduced basis (16). Specifically, the 2HDM potential possesses a continuous symmetry, if \[ \left[ T^a, L \right] = 0, \quad T^a M = 0, \] where $L$ and $M$ denote the $5 \times 5$ matrix $L_{ij}$ and the 5-dimensional vector $M_i$ in the reduced basis, respectively. Given the one-to-one correspondence between $T^a$ and $K^a$ generators, it is not difficult to determine the transformation relations associated with a given continuous HF/CP symmetry in the $\Phi$-space through:

$$\Phi \rightarrow \Phi' = e^{i\theta^a K^a} \Phi,$$

(19)

where $\theta^a \in [0, 2\pi]$ are the group parameters of the $\text{SU}_M(4)/Z_2$ group.

It is interesting to determine the SO(5) generators related to a particular accidental symmetry that remain (un)broken after electroweak symmetry breaking. In this way, we can find the number of pseudo-Goldstone bosons predicted, according to the Goldstone theorem. In the 5-dimensional bilinear $R^4$-space, a neutral vacuum solution in its standard basis implies that $\phi_1^T i\sigma^2 \phi_2 = 0$, i.e. $R^4 = R^5 = 0$, or equivalently $R^\mu R^\mu = 0$. Alternatively, a standard basis for writing down a neutral vacuum solution $R^4_0$ may be defined through the relation: $T^a_{ij} R^4_0 = 0$. Consequently, an SO(5) generator $T^a$ remains unbroken after electroweak symmetry breaking, if it satisfies the condition:

$$T^a_{ij} R^4_0 = 0.$$

(20)

By definition, the hypercharge generator $T^0$ will always be unbroken when acting on a neutral vacuum solution $R^4_0$. This should not be too surprising, as $T^0$ is equivalent to the electromagnetic generator, given by $Q_{\text{em}} = \sigma^0 \otimes \sigma^0 \otimes (\sigma^3/2) + K^0$ in the $\Phi$-space, once we notice that the weak isospin generator $\sigma^0 \otimes \sigma^0 \otimes (\sigma^3/2)$ has no effect on the $\text{SU}(2)_L$ gauge-invariant 5-vector $R^4$.

In Table 2, we exhibit the SO(5) ($\text{SU}_M(4)$) symmetry generators $T^a$ ($K^a$) [cf. (14), (10)] and the discrete group elements [cf. (17)] generating the 13 accidental symmetries of the $\text{U}(1)_Y$-invariant 2HDM potential. We also display the maximally broken SO(5) generators compatible
with a neutral vacuum for each symmetry, along with the maximal number of pseudo-Goldstone bosons that result from the Goldstone theorem. The pseudo-Goldstone bosons associated with the maximal breaking of each symmetry have also been identified in the last column of Table 2, using the explicit analytic results presented in [25] for the minimization conditions and the scalar mass matrices. Thus, we find that as well as CP1 \( \equiv Z_2 \times O(2) \), the symmetries SO(3) and \( Z_2 \times O(3) \) can maximally break spontaneously via a CP non-invariant vacuum. Unlike in the CP1 case, spontaneous breakdown of these two new symmetries may lead to two pseudo-Goldstone bosons, i.e. the two charged Higgs bosons \( h^\pm \). For the symmetry \( (Z_2)^2 \times SO(3) \), the maximal breaking pattern leading to the two charged pseudo-Goldstone bosons \( h^\pm \) is obtained, when the restriction \( \lambda_4 = -\text{Re} \lambda_5 > 0 \) is taken from Table 1.

On the other hand, it is worth reiterating that the symmetry SO(5) relates to the larger O(8) group [6] in the real field space, once the latter gets further restricted such that the SU(2)\(_L\) gauge canonical form of the \( \Phi \) kinetic term is maintained. In the 5-dimensional bilinear R\(^4\)-space, SO(5) can break down to SO(3), giving rise to four pseudo-Goldstone bosons: one of the two CP-even Higgs bosons denoted as \( h \), the CP-odd scalar \( a \) and the two charged Higgs bosons \( h^\pm \). This is consistent with breaking pattern of O(8) \( \rightarrow \) O(7) in the \( \Phi \)-space, leading to seven Goldstone bosons, which include the three would-be Goldstone bosons associated with the longitudinal polarizations of the \( W^\pm \) and \( Z \) bosons. However, one gets a different result within the U(1)\(_Y\)-restricted SO(3) bilinear formalism of [16, 17, 18, 21, 24]. The higher HF/CP symmetry SO(5) appears as SO(3)\(_{\text{HF}}\) in the U(1)\(_Y\)-restricted bilinear formalism, and according to Table 2 (symmetry no. 6), it may break down to SO(2), giving rise to only two pseudo-Goldstone bosons.

Table 2: Symmetry generators [cf. (10), (14)] and discrete group elements [cf. (17)] for the 13 accidental symmetries of the U(1)\(_Y\)-invariant 2HDM potential. For each symmetry, the maximally broken SO(5) generators compatible with a neutral vacuum are displayed, along with the pseudo-Goldstone bosons (given in parentheses) that result from the Goldstone theorem.

| No | Symmetry | Generators | Discrete Group Elements | Maximally Broken SO(5) Generators | Number of Pseudo-Goldstone Bosons |
|----|----------|------------|-------------------------|-----------------------------------|----------------------------------|
| 1  | \( Z_2 \times O(2) \) | \( T^0 \) | \( D_{CP1} \) | – | 0 |
| 2  | \( (Z_2)^2 \times SO(2) \) | \( T^0 \) | \( D_{Z_2} \) | – | 0 |
| 3  | \( (Z_2)^3 \times O(2) \) | \( T^0 \) | \( D_{CP2} \) | – | 0 |
| 4  | \( O(2) \times O(2) \) | \( T^3, T^9 \) | – | \( T^9 \) | 1 (a) |
| 5  | \( Z_2 \times [O(2)]^2 \) | \( T^2, T^6 \) | \( D_{CP1} \) | \( T^2 \) | 1 (h) |
| 6  | \( O(3) \times O(2) \) | \( T^4, T^9 \) | – | \( T^4, T^9 \) | 2 (h, a) |
| 7  | SO(3) | \( T^0, T^3 \) | \( D_{Z_2} \cdot D_{CP2} \) | \( T^3 \) | 2 (h) |
| 8  | \( (Z_2)^2 \times SO(3) \) | \( T^0, T^3 \) | \( D_{CP1} \cdot D_{CP2} \) | \( T^3 \) | 2 (h) |
| 9  | \( O(2) \times O(3) \) | \( T^2, T^0, T^8, T^9 \) | – | \( T^3 \) | 1 (a) |
| 10 | SO(4) | \( T^0, T^3, T^5, T^7 \) | – | \( T^3, T^5, T^7 \) | 3 (a, \( h^\pm \)) |
| 11 | \( Z_2 \times O(4) \) | \( T^0, T^3, T^5, T^7 \) | \( D_{Z_2} \cdot D_{CP2} \) | \( T^3, T^5, T^7 \) | 3 (a, \( h^\pm \)) |
| 12 | SO(5) | \( T^0, T^3, T^5, T^7 \) | – | \( T^3, T^5, T^7 \) | 4 (h, a, \( h^\pm \)) |
Another illustrative example is the symmetry SO(4), which is equivalent to O(4) ⊗ O(4) in the scalar-field space, where one of the O(4) factors describes gauge-group transformations. As can be seen from Table 2, the symmetry SO(4) may break to SO(3), giving rise to three pseudo-Goldstone bosons: the CP-odd scalar $a$ and the two charged Higgs bosons $h^\pm$. Again, this breaking scenario cannot be clearly distinguished from a scenario based on CP3 ≡ $Z_2 \times [O(2)]^2$, which leads to an erroneous breaking pattern predicting only one pseudo-Goldstone boson, within the U(1)$_Y$-constrained SO(3) bilinear formalism.

It is interesting to remark that the Majorana-constrained unitary group SU$_M(4)$ in (15) contains the custodial symmetry group SU(2)$_C$ (for recent studies, see [27, 23]). In the $\Phi$-basis, there are three independent realizations of SU(2)$_C$ induced by the generators: (i) $K^{0,4,6}$; (ii) $K^{0,5,7}$; (iii) $K^{0,8,9}$. As stated in Table 2, the HF/CP accidental symmetries 7–13 contain at least one of the three generator sets (i), (ii) and (iii), and are therefore custodial symmetric. As a consequence of the custodial symmetry, the $W^\pm$ and $Z$ bosons are degenerate in mass and Veltman’s $\rho$-parameter retains its tree-level value $\rho = 1$, to all orders in perturbation theory. As happens in the SM, however, the U(1)$_Y$ hypercharge and Yukawa interactions violate explicitly the custodial symmetry in the 2HDM.

In summary, we have presented the symmetry generators $K^a$ in (10) that describe the 13 accidental symmetries [1] of the U(1)$_Y$-invariant 2HDM potential (1) in the original scalar field space $\Phi$, by means of (19). We have derived an exact symmetry relation in (13), which gives the one-to-one correspondence between the SU$_M(4)$ generators $K^a$ in the $\Phi$-space and the SO(5) generators $T^a$ in the R$^1$-space. In Table 2, we have explicitly presented all symmetry generators associated with the 13 accidental symmetries, along with possible maximal breaking scenarios. Most importantly, we have explicitly demonstrated how the bilinear formalism based on the SO(5) ⊂ SO(1,5) symmetry group respects the Goldstone theorem, predicting the correct number of pseudo-Goldstone bosons after electroweak symmetry breaking. In conclusion, the results presented in this note provide firm support for the completeness of the classification conducted in [1] for the 13 accidental symmetries of the 2HDM potential.

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References

[1] R. Battye, G. Brawn and A. Pilaftsis, *Vacuum Topology of the Two Higgs Doublet Model*, JHEP 1108 (2011) 020.

[2] T. D. Lee, *A Theory of Spontaneous T Violation*, Phys. Rev. D8 (1973) 1226.

[3] V. Silveira and A. Zee, *Scalar Phantoms*, Phys. Lett. B161 (1985) 136.

[4] V. A. Kuzmin, V. A. Rubakov and M. E. Shaposhnikov, *On the Anomalous Electroweak Baryon Number Nonconservation in the Early Universe*, Phys. Lett. B155 (1985) 36.

[5] A. G. Cohen, D. B. Kaplan and A. E. Nelson, *Progress in electroweak baryogenesis*, Ann. Rev. Nucl. Part. Sci. 43 (1993) 27-70.

[6] N. G. Deshpande and E. Ma, *Pattern of Symmetry Breaking with Two Higgs Doublets*, Phys. Rev. D18 (1978) 2574.

[7] S. Weinberg, *Approximate symmetries and pseudo-Goldstone bosons*, Phys. Rev. Lett. 29 (1972) 1698.

[8] S. L. Glashow and S. Weinberg, *Natural Conservation Laws for Neutral Currents*, Phys. Rev. D15 (1977) 1958.

[9] G. C. Branco, *Spontaneous CP Nonconservation and Natural Flavor Conservation: a Minimal Model*, Phys. Rev. D22 (1980) 2901.

[10] L. Lavoura and J. P. Silva, *Fundamental CP violating quantities in a SU(2) x U(1) model with many Higgs doublets*, Phys. Rev. D50 (1994) 4619.

[11] F. J. Botella and J. P. Silva, *Jarlskog - like invariants for theories with scalars and fermions*, Phys. Rev. D51 (1995) 3870.

[12] I. F. Ginzburg and M. Krawczyk, *Symmetries of two Higgs doublet model and CP violation*, Phys. Rev. D72 (2005) 115013.

[13] G. C. Branco, M. N. Rebelo and J. I. Silva-Marcos, *CP-odd invariants in models with several Higgs doublets*, Phys. Lett. B614 (2005) 187.

[14] S. Davidson and H. E. Haber, *Basis-independent methods for the two-Higgs-doublet model*, Phys. Rev. D72 (2005) 035004.

[15] J. F. Gunion and H. E. Haber, *Conditions for CP-violation in the general two-Higgs-doublet model*, Phys. Rev. D72 (2005) 095002.

[16] C. C. Nishi, *CP violation conditions in N-Higgs-doublet potentials*, Phys. Rev. D74 (2006) 036003.

[17] M. Maniatis, A. von Manteuffel, O. Nachtmann and F. Nagel, *Stability and symmetry breaking in the general two-Higgs-doublet model*, Eur. Phys. J. C48 (2006) 805.
[18] I. P. Ivanov, *Minkowski space structure of the Higgs potential in 2HDM*, Phys. Rev. **D75** (2007) 035001.

[19] C. C. Nishi, *The structure of potentials with N Higgs doublets*, Phys. Rev. **D76** (2007) 055013.

[20] I. P. Ivanov, *Minkowski space structure of the Higgs potential in 2HDM: II. Minima, symmetries, and topology*, Phys. Rev. **D77** (2008) 015017.

[21] P. M. Ferreira, H. E. Haber and J. P. Silva, *Generalized CP symmetries and special regions of parameter space in the two-Higgs-doublet model*, Phys. Rev. **D79** (2009) 116004.

[22] R. D. Peccei and H. R. Quinn, *CP Conservation in the Presence of Instantons*, Phys. Rev. Lett. **38** (1977) 1440.

[23] C. C. Nishi, *Custodial SO(4) symmetry and CP violation in N-Higgs-doublet potentials*, Phys. Rev. **D83** (2011) 095005.

[24] M. Maniatis and O. Nachtmann, *Symmetries and renormalisation in two-Higgs-doublet models*, arXiv:1106.1436.

[25] A. Pilaftsis and C. E. M. Wagner, *Higgs bosons in the minimal supersymmetric standard model with explicit CP violation*, Nucl. Phys. **B553** (1999) 3.

[26] P. Sikivie, L. Susskind, M. B. Voloshin and V. I. Zakharov, *Isospin Breaking in Technicolor Models*, Nucl. Phys. **B173** (1980) 189.

[27] B. Grzadkowski, M. Maniatis and J. Wudka, *The bilinear formalism and the custodial symmetry in the two-Higgs-doublet model*, JHEP **1111** (2011) 030.

[28] M. J. G. Veltman, *Limit On Mass Differences In The Weinberg Model*, Nucl. Phys. B **123** (1977) 89; *Radiative Corrections To Vector Boson Masses*, Phys. Lett. B **91** (1980) 95.