Graphical Nonlinear System Analysis

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Abstract—We use the recently introduced concept of a scaled relative graph (SRG) to develop a graphical analysis of input–output properties of feedback systems. The SRG of a nonlinear operator generalizes the Nyquist diagram of an LTI system. In the spirit of classical control theory, important robustness indicators of nonlinear feedback systems are measured as distances between SRGs.

Index Terms—Graphical analysis, incremental input/output stability, nonlinear systems, Nyquist criterion.

I. INTRODUCTION

The graphical analysis of a feedback system via the Nyquist diagram of its return ratio is a foundation of classical control theory. It underlies the analysis concept of stability margins and the design concept of loop shaping, which themselves provide the grounds for the gap metric [1], [2], and $H_{\infty}$ control [3].

The Nyquist diagram also has a fundamental place in the theory of nonlinear systems of the Lur’e form (that is, systems composed of an LTI forward path in feedback with a static nonlinearity). The circle and Popov criteria allow the stability of a Lur’e system to be proved by verifying a geometric condition on the Nyquist diagram of the LTI component [4]. The geometric condition is determined by the properties of the static nonlinearity. Notably, only the Nyquist diagram of the LTI component is defined, owing to a lack of a suitable definition of phase for nonlinear systems. At best, the frequency response of a nonlinear system may be computed approximately. The describing function [5], [6], [7] gives rise to a family of Nyquist curves for a nonlinearity, parameterized by the amplitude of the input. Other efforts to generalize frequency response to nonlinear systems include the work of [8] on Bode diagrams for convergent systems, and the recently introduced notions of nonlinear phase and singular angle by Chen et al. [9], [10].

In this article, we show that the scaled relative graph (SRG) of [11] generalizes the Nyquist diagram of an LTI transfer function, and may be plotted for nonlinear input/output operators. The SRG has been introduced in the theory of optimization to visualize incremental properties of nonlinear operators, that is, properties that are measured between pairs of input/output trajectories, such as Lipschitz continuity and maximal monotonicity. Such properties may be verified by checking geometric conditions on the SRG of an operator. Algebraic manipulations to the operator correspond to geometric manipulations to the SRG. The SRG gives rise to simple, intuitive, and rigorous proofs of the convergence of many algorithms in convex optimization. Furthermore, the tool is particularly suited to proving tightness of convergence bounds, and has been used to prove novel tightness results [11], [12].

The objective of this article is to provide a bridge between SRG analysis and the incremental input-output analysis of feedback systems [4]. Our main result (Theorem 2) establishes a generalization of the Nyquist theorem for stable nonlinear operators. Based on the homotopy argument central to IQC analysis [13], this result enables SRG analysis to address well-posedness issues—in contrast to [11] for example, where well-posedness of operators is assumed. In the context of nonlinear feedback system analysis, this result enables an elegant and classical definition of stability margins for nonlinear feedback systems. We illustrate the generality of the approach by recovering several classical results.

The second part of the article aims at providing concrete illustrations of the theory. As a first step, we show that the SRG of an LTI system is derived from its Nyquist curve, and we also provide an analytical derivation of the SRG of scalar-valued static nonlinearities. Preliminary results were presented in the conference paper [14]. We then illustrate the application of SRG analysis to three representative examples from the literature. Our first example (see Section VII) is a feedback loop involving delays and saturations, a classical benchmark of IQC system analysis [13]. Our SRG analysis provides an analytical bound on the feedback gain which guarantees stability, and closely matches previous numerical bounds obtained by IQC analysis. We stress, however, that the SRG analysis characterizes the incremental gain whereas previous bounds were nonincremental. We furthermore give an analytical bound on the incremental $L_2$ gain of the closed loop. This example illustrates the strength and potential of SRG analysis for verifying incremental properties, which is of considerable importance, even though it seems to have received little attention in IQC analysis [13], [15], [16]. Our second example illustrates SRG analysis in so-called cyclic feedback systems [17], [18], [19], [20]. Here also SRG analysis
suggests a strong potential: on top of providing an elegant graphical interpretation of existing results, we illustrate that SRG analysis provides analytical bounds on the incremental gain of the feedback system and stability margins against dynamical uncertainties. Our final example (see Section IX) combines cascades and delays in the analysis of a congestion control model previously studied in [21]. Here again, SRG analysis provides novel bounds on the incremental gain, and generalizes the equilibrium-independent passivity analysis proposed in [21].

The rest of this article is organized as follows. We begin by introducing some background material in Section II, before defining the SRG in Section III, and summarizing its main properties. Section IV presents our main theoretical results on the SRG analysis of feedback systems. Section V connects the SRG to the Nyquist diagram of an LTI transfer function, and Section VI describes the SRGs of important classes of static nonlinearities. Three detailed examples are then developed in Sections VII–IX. Finally, Section X concludes this article.

II. BACKGROUND AND PRELIMINARIES

A. Signal Spaces, Operators, and Relations

Let \( L \) denote a Hilbert space, equipped with an inner product, \( \langle \cdot, \cdot \rangle : L \times L \to \mathbb{C} \), and the induced norm \( \|x\| := \sqrt{\langle x|x \rangle} \).

We will pay particular attention to Lebesgue spaces of square-integrable functions. Given a time axis, which for brevity we will always consider to be \( \mathbb{R}_{\geq 0} \), and a field \( \mathbb{F} \subseteq \{\mathbb{R}, \mathbb{C}\} \), we define the space \( L^2_0(\mathbb{F}) \) by bounding the space of all signals \( u : \mathbb{R}_{\geq 0} \to \mathbb{F}^n \) such that

\[
\|u\| := \left( \int_0^\infty \|\bar{u}(t)\|dt \right)^{\frac{1}{2}} < \infty
\]

where \( \bar{u}(t) \) denotes the conjugate transpose of \( u(t) \). The inner product of \( u, y \in L^2_0(\mathbb{F}) \) is defined by

\[
\langle u, y \rangle := \int_0^\infty \bar{u}(t)y(t)dt.
\]

The Fourier transform of \( u \in L^2_0(\mathbb{F}) \) is defined as

\[
\hat{u}(j\omega) := \int_0^\infty e^{-j\omega t}u(t)dt.
\]

We omit the dimension and field when they are immaterial or clear from context.

For some \( T \in \mathbb{R}_{\geq 0} \), define the truncation operator \( P_T \) by

\[
(P_Tu)(t) := \begin{cases} u(t) & t \leq T \\ 0 & t > T \end{cases}
\]

where \( t \in \mathbb{R}_{\geq 0} \) and \( u \) is an arbitrary signal. Define the extension of \( L^2_0(\mathbb{F}) \) [22, 23, p. 22], [4, p. 172] to be the space

\[
L^2_{0,e}(\mathbb{F}) := \{u : \mathbb{R}_{\geq 0} \to \mathbb{F}^n \mid \|P_Tu\| < \infty \text{ for all } T \in \mathbb{R}_{\geq 0}\}.
\]

An operator, or system, on a space \( \mathcal{X} \), is a possibly multivalued map \( R : \mathcal{X} \to \mathcal{X} \). The identity operator, which maps \( u \in \mathcal{X} \) to itself, is denoted by \( I \). The graph, or relation, of an operator, is the set \( \{(u, y) \mid u \in \text{dom } R, y \in R(u)\} \subseteq \mathcal{X} \times \mathcal{X} \). We use the notions of an operator and its relation interchangeably, and denote them in the same way. The relation of an operator may be thought of as an input/output partition of a behavior [24, Def. 3.3.1].

The usual operations on functions can be extended to relations

\[
S^{-1} = \{(y, u) \mid y \in S(u)\}
\]

\[
S + R = \{(x, y + z) \mid (x, y) \in S, (x, z) \in R\}
\]

\[
SR = \{(x, z) \mid \exists y \text{ s.t. } (x, y) \in R, (y, z) \in S\}.
\]

Note that \( S^{-1} \) always exists, but is not an inverse in the usual sense. In particular, it is in general not the case that \( S^{-1}S = I \). If, however, \( S \) is an invertible function, the relational inverse and functional inverse coincide, so the notation \( S^{-1} \) can be used without ambiguity.

An operator \( R \) on \( L_2 \) or \( L_{2,e} \) is said to be causal if \( P_T(R(u)) = P_T(R(P_Tu)) \) for all \( u \).

B. Incremental Input/Output Analysis

The input/output approach to nonlinear systems analysis originated in the dissertation of Zames [25] and early work by Sandberg [26]. Noting that the amplification of a nonlinear system was, in general, dependent on the input, Zames introduced the notion of the incremental gain of a system, which characterizes the worst case amplification a system is capable of. Incremental properties feature heavily in Desoer and Vidyasagar’s classic text [4]. The general pattern is that requiring a property to be verified for every possible input, rather than just a single distinguished input (i.e., \( u = 0 \), for example), leads to much stronger results, often comparable to the results that may be proved for linear systems. This is perhaps unsurprising, as any property of a linear system is automatically incremental. The study of properties relative to the zero input, however, has dominated nonlinear input/output theory since these early days.

We now define the input/output properties of systems considered in this article. We begin with a definition of incremental stability.

**Definition 1:** Let \( R : L_2 \to L_2 \). The incremental \( L_2 \) gain of \( R \) is

\[
\mu := \sup_{u_1,u_2 \in \text{dom } R} \frac{\|y_1 - y_2\|}{\|u_1 - u_2\|}
\]

where \( y_1 \in R(u_1), y_2 \in R(u_2) \). If \( \mu < \infty \), \( R \) is said to have \textit{finite incremental \( L_2 \) gain}, or be \textit{incrementally \( L_2 \) stable}.

The second class of properties relate to passivity.

**Definition 2:** Let \( R : L_2 \to L_2 \). Then:

1) \( R \) is said to be \textit{incrementally positive} if

\[
\langle u_1 - u_2, y_1 - y_2 \rangle \geq 0
\]

for all \( u_1, u_2 \in \text{dom } R \) and \( y_1 \in R(u_1), y_2 \in R(u_2) \).

2) \( R \) is said to be \textit{\( \lambda \)-input-strict incrementally positive} if

\[
\langle u_1 - u_2, y_1 - y_2 \rangle \geq \lambda \|u_1 - u_2\|^2
\]

for all \( T \geq 0 \), all \( u_1, u_2 \in \text{dom } R \) and \( y_1 \in R(u_1), y_2 \in R(u_2) \).

3) \( R \) is said to be \textit{\( \gamma \)-output-strict incrementally positive} if

\[
\langle u_1 - u_2, y_1 - y_2 \rangle \geq \gamma \|y_1 - y_2\|^2
\]

for all \( u_1, u_2 \in \text{dom } R \) and \( y_1 \in R(u_1), y_2 \in R(u_2) \).
For causal operators on $L_2$, incremental positivity coincides with the stronger property of incremental passivity (the proof follows the same lines as [4, Lemma 2, p. 200]). In the language of optimization, incremental positivity is called monotonicity. Monotone operator theory originated in the study of networks of nonlinear resistors. The prototypical monotone operator was Duffin's quasi-linear resistor [27], a resistor with increasing, but not necessarily linear, $i - v$ characteristic. The modern notion of a monotone operator was introduced by [28] and [29]. Following the influential paper of [30], monotone operators have become a cornerstone of optimization theory. Monotone operator methods have seen a surge of interest in the last decade, due to their applicability to large-scale and nonsmooth problems [31], [32], [33], [34]. SRGs have been developed to prove convergence of these optimization methods.

Monotone operator theory is closely related to the classical input/output theory of nonlinear systems. All of the properties studied in the theory of monotone operators correspond to a property in input/output system theory, the difference being that the former are defined for an arbitrary Hilbert space, while the latter are defined on $L_2$ or $L_2,c$. Table I shows these equivalences.

### III. Scaled Relative Graphs

We define SRGs in the same way as [11], with the minor modification of allowing complex valued inner products.

Let $\mathcal{L}$ be a Hilbert space. The angle between $u, y \in \mathcal{L}$ is defined as

$$\angle(u, y) := \arccos \frac{\Re(u^*y)}{|u^||y|} \in [0, \pi].$$

Let $R : \mathcal{L} \to \mathcal{L}$ be an operator. Given $u_1, u_2 \in \mathcal{U} \subseteq \mathcal{L}$, $u_1 \neq u_2$, define the set of complex numbers $z_R(u_1, u_2)$ by

$$z_R(u_1, u_2) := \left\{ \frac{|y_1 - y_2|}{|u_1 - u_2|}, \pm j\angle(u_1 - u_2, y_1 - y_2) : y_1 \in R(u_1), y_2 \in R(u_2) \right\}.$$

If $u_1 = u_2$ and there are corresponding outputs $y_1 \neq y_2$, then $z_R(u_1, u_2)$ is defined to be $\{\infty\}$. If $R$ is single valued at $u_1$, $z_R(u_1, u_1)$ is the empty set.

The scaled relative graph (SRG) of $R$ over $\mathcal{U} \subseteq \mathcal{L}$ is then given by

$$\text{SRG}_\mathcal{U}(R) := \bigcup_{u_1, u_2 \in \mathcal{U}} z_R(u_1, u_2).$$

If $\mathcal{U} = \mathcal{L}$, we write $\text{SRG}(R) := \text{SRG}_\mathcal{L}(R)$.

If $R$ is linear and dom $R$ is a linear subspace of $\mathcal{L}$, $Ru_1 - Ru_2 = R(u_1 - u_2) = Rv$ for some $v \in \text{dom} R$, and we can define

$$z_R(v) := \frac{|Rv|}{|v|} e^{\pm j\angle(v, Ru)}$$

and

$$\text{SRG}_{\text{dom}R}(R) := \{z_R(v) : v \in \text{dom} R, v \neq 0\}.$$

In the special case that $R$ is linear and time invariant with transfer function $R(s)$, and $v(t) = e^{j\omega t}$, $\lim_{T \to \infty} \|R(P_T v)\|/\|P_T v\| = |R(j\omega)|$ and $\lim_{T \to \infty} \angle(P_T v, R(P_T v)) = |\arg R(j\omega)|$ (where $\arg R(j\omega)$ is measured between $\pi$ and $\pi$). Thus, the gain and phase of the SRG generalize the classical notions of the gain and phase of an LTI transfer function.

### A. System Properties From SRGs

If $\mathcal{A}$ is a class of operators, we define the SRG of $\mathcal{A}$ by

$$\text{SRG}(\mathcal{A}) := \bigcup_{R \in \mathcal{A}} \text{SRG}(R).$$

Note that a class of operators can be a single operator.

A class $\mathcal{A}$, or its SRG, is called SRG-full if

$$R \in \mathcal{A} \iff \text{SRG}(R) \subseteq \text{SRG}(\mathcal{A}).$$

By construction, the implication $R \in \mathcal{A} \Rightarrow \text{SRG}(R) \subseteq \text{SRG}(\mathcal{A})$ is true. The value of SRG-fullness is in the reverse implication: $\text{SRG}(R) \subseteq \text{SRG}(\mathcal{A}) \Rightarrow R \in \mathcal{A}$. This allows class membership to be tested graphically. If $\mathcal{A}$ is the class of systems with a particular system property, SRG-fullness of $\mathcal{A}$ allows this property to be verified for a particular operator $R$ by plotting its SRG. If $\text{SRG}(R) \subseteq \text{SRG}(\mathcal{A})$, $R$ has the property.

The following proposition gives the SRGs of the classical system properties introduced in Section II-B.

**Proposition 1:** The SRGs of incrementally positive systems (top left), input-strict incrementally positive systems (top right), output-strict incrementally positive systems (bottom...
right), and incrementally $L_2$ bounded systems (bottom left) are shown below. 

![Diagram](image)

$$\langle u_1 - u_2 | y_1 - y_2 \rangle \geq 0 \quad \langle u_1 - u_2 | y_1 - y_2 \rangle \geq \lambda \| u_1 - u_2 \|^2$$

$$\| y_1 - y_2 \| \leq \mu \| u_1 - u_2 \| \quad \langle u_1 - u_2 | y_1 - y_2 \rangle \geq \gamma \| y_1 - y_2 \|^2$$

All of these classes are SRG-full.

**Proof:** These SRGs are proved in [11], and all of the shapes follow from quick calculations. SRG-fullness follows from [11, Th. 3.5].

SRG-fullness of the classes of Proposition 1 means, for example, that if the SRG of a system lies in the right-half plane, the system is incrementally positive, and if the SRG of a system is bounded, the system has finite incremental $L_2$ gain. These are reminiscent of the facts that an LTI system is passive if its Nyquist diagram lies in the right-half plane, and has finite $H_\infty$ norm if its Nyquist diagram is bounded. We will show in Section V that Proposition 1 is indeed a generalization of these classical facts.

The properties of finite incremental $L_2$ gain and incremental positivity are particular examples of incremental integral quadratic constraints (IQCs) [13]. A striking corollary of [11, Th. 3.5] is that any SRG defined by a static incremental IQC is SRG-full.

**Proposition 2:** Let $u_i(t)$ denote the input to an arbitrary operator on $L_2$, and $y_i(t)$ denote a corresponding output. Let $\Delta u = u_1 - u_2$ and $\Delta y = y_1 - y_2$, and $\hat{\chi}(\omega)$ denote the Fourier transform of signal $x(t)$. Then the classes of operators which obey either of the constraints

$$\int_{-\infty}^{\infty} \left( \begin{array}{c} \Delta \hat{u}(\omega) \\ \Delta \hat{y}(\omega) \end{array} \right) \begin{array}{cc} a & b \\ c & d \end{array} \left( \begin{array}{c} \Delta \hat{u}(\omega) \\ \Delta \hat{y}(\omega) \end{array} \right) d\omega \geq 0 \quad (1)$$

$$\int_{0}^{\infty} \left( \begin{array}{c} \Delta u(t) \\ \Delta y(t) \end{array} \right) \begin{array}{cc} a & b \\ c & d \end{array} \left( \begin{array}{c} \Delta u(t) \\ \Delta y(t) \end{array} \right) dt \geq 0 \quad (2)$$

where $a, b, c, d \in \mathbb{R}$, are SRG-full.

**Proof:** Equation (1) gives

$$a \| \Delta \hat{u} \|^2 + (b + c) \langle \Delta \hat{u} | \Delta \hat{y} \rangle + d \| \Delta \hat{y} \|^2 \geq 0.$$  

By Parseval’s theorem, this is equivalent to

$$a \| \Delta u \|^2 + (b + c) \langle \Delta u | \Delta y \rangle + d \| \Delta y \|^2 \geq 0,$$

which is also implied by (2). The result then follows from [11, Th. 3.5].

A class of operators defined defined by a geometric region is SRG-full.

**Proposition 3:** Let $C \subseteq \mathbb{C}$. The class of operators $A$ defined by

$$A := \{ R \text{ an operator} | \text{SRG}(R) \subseteq C \}$$

is SRG-full.

**Proof:** The definition of $A$ can be written as

$$R \in A \iff \text{SRG}(R) \subseteq C,$$

which is the definition of SRG-fullness.

This fact is particularly useful for system analysis, as it allows the SRG of an operator to be over-approximated by a geometric region if, for example, the precise SRG is unknown, or the SRG does not obey the properties necessary to apply a theorem. Over-approximating an SRG simply amounts to making the analysis more conservative.

**B. Interconnections**

Under mild conditions on the SRG, system interconnections correspond to geometric manipulations of their SRGs. These interconnection results are proved by [11, Ths. 4.1–4.5]. We recall the statements of these theorems in the following five propositions.

**Proposition 4:** If $A$ and $B$ are SRG-full, then $A \cap B$ is SRG-full, and

$$\text{SRG}(A \cap B) = \text{SRG}(A) \cap \text{SRG}(B).$$

**Proposition 5:** Let $\alpha \in \mathbb{R}, \alpha \neq 0$. If $A$ is a class of operators

$$\text{SRG}(\alpha A) = \alpha \text{SRG}(A),$$

$$\text{SRG}(I + A) = 1 + \text{SRG}(A).$$

Furthermore, if $A$ is SRG-full, then $\alpha A$, $\alpha A$, and $I + A$ are SRG-full.

We define inversion in the complex plane by the Möbius transformation $re^{j\omega} \mapsto (1/r)e^{j\omega}$. This is “inversion in the unit circle” points outside the unit circle map to the inside, and vice versa. The points 0 and $\infty$ are exchanged under inversion.

**Proposition 6:** If $A$ is a class of operators, then

$$\text{SRG}(A^{-1}) = (\text{SRG}(A))^{-1}.$$  

Furthermore, if $A$ is SRG-full, then $A^{-1}$ is SRG-full.

Define the line segment between $z_1, z_2 \in \mathbb{C}$ as $[z_1, z_2] := \{ (1-\alpha)z_1 + \alpha z_2 | \alpha \in [0, 1] \}$. A class of operators $A$ is said to satisfy the chord property if $z \in \text{SRG}(A) \setminus \{\infty\}$ implies $[z, \bar{z}] \subseteq \text{SRG}(A)$.

**Proposition 7:** Let $A$ and $B$ be classes of operators, such that

$$\infty \notin \text{SRG}(A) \text{ and } \infty \notin \text{SRG}(B).$$

Then

1) if $A$ and $B$ are SRG-full, then $A + B \supseteq \text{SRG}(A) + \text{SRG}(B)$;
2) if either $A$ or $B$ satisfies the chord property, then

$$\text{SRG}(A + B) \subseteq \text{SRG}(A) + \text{SRG}(B).$$

Under additional assumptions, $\infty$ can be allowed—see the discussion following [11, Th. 4.4].

Define the right-hand arc, $\text{Arc}^+(z, \bar{z})$, between $z$ and $\bar{z}$ to be the arc between $z$ and $\bar{z}$ with center on the origin and real part
greater than or equal to \( \text{Re}(z) \). The left-hand arc, \( \text{Arc}^{-}(z, \bar{z}) \), is defined the same way, but with real part less than or equal to \( \text{Re}(z) \). Formally

\[
\text{Arc}^{+}(z, \bar{z}) := \{ r e^{j(1-2\theta)} \phi \mid z = r e^{j\phi}, \\
\phi \in (-\pi, \pi], \theta \in [0,1], r \geq 0 \}
\]

\[
\text{Arc}^{-}(z, \bar{z}) := - \text{Arc}^{+}(-z, -\bar{z}).
\]

A class of operators \( \mathcal{A} \) is said to satisfy the right hand (respectively, left hand) arc property if, for all \( z \in \text{SRG}(\mathcal{A}) \), \( \text{Arc}^{+}(z, \bar{z}) \in \text{SRG}(\mathcal{A}) \) (resp. \( \text{Arc}^{-}(z, \bar{z}) \in \text{SRG}(\mathcal{A}) \)).

Proposition 8: Let \( \mathcal{A} \) and \( \mathcal{B} \) be classes of operators, such that

1. if \( \mathcal{A} \) and \( \mathcal{B} \) are SRG-full, then \( \text{SRG} (\mathcal{A} \mathcal{B}) \supseteq \text{SRG}(\mathcal{A}) \text{SRG}(\mathcal{B}) \);
2. if either \( \mathcal{A} \) or \( \mathcal{B} \) satisfies an arc property, then \( \text{SRG}(\mathcal{A} \mathcal{B}) \subseteq \text{SRG}(\mathcal{A}) \text{SRG}(\mathcal{B}) \).

Under additional assumptions, \( \infty \) and \( \emptyset \) can be allowed—see the discussion following [11, Th. 4.5].

C. Scaled Graphs About Particular Solutions

Scaled relative graphs capture the behavior of an operator with respect to any possible operating point. However, the behavior of one or several specific inputs (for example, stable equilibria) may be of particular interest. The methods of this article apply equally to the analysis of properties with respect to particular inputs, via the scaled graph (SG). For notational convenience, we only define the SG over the full space, but the SG can be restricted to a subset of the input space in the same way as the SRG.

Definition 3: Let \( R : \mathcal{L} \rightarrow \mathcal{L} \). The scaled graph of \( R \) over \( \mathcal{L} \) with respect to the input \( u^* \) is

\[
\text{SG}_{u^*}(R) := \bigcup_{u \in \mathcal{L}} \{ z \mid p(u, u^*) \}.
\]

Note that the SG of an LTI operator with respect to any input is equal to its SRG.

The graphical algebra of SRGs applies to SGs with very little modification—the only requirement is that interconnected SGs are defined with respect to compatible inputs. In the remainder of this section, we highlight the difference between incremental and input-specific properties, using the example of positivity.

Definition 4: An operator \( H : \mathcal{L}_2 \rightarrow \mathcal{L}_2 \) is said to be positive about \( u^* \in \mathcal{L}_2 \) if, for all \( u \in \mathcal{L}_2 \), \( y \in H(u) \) and \( y^* \in H(u^*) \), \( \langle u - u^*, y - y^* \rangle > 0 \).

From this definition, it follows immediately that an operator is positive about \( u^* \) if and only if its SG about \( u^* \) belongs to the closed right-half plane. However, this does not mean its SG about any other input necessarily lies in the right-half plane—Fig. 1 gives such an example.

Taking the union of SGs over multiple trajectories allows properties that lie between trajectory-dependent and incremental to be verified. For example, [35] define equilibrium-independent passivity to be passivity with respect to every constant input to the system (under assumptions on the system that ensure that there is a constant output for every constant input). This can be verified by checking that the union of SGs over constant inputs lies in the right-half plane.

IV. FEEDBACK ANALYSIS WITH SCALED RELATIVE GRAPHS

In this section, we demonstrate the use of scaled relative graphs for the stability analysis of feedback interconnections. We begin by using the SRG to generalize the Nyquist criterion to a stable nonlinear operator in unity gain negative feedback, and introduce a nonlinear stability margin. We then formulate a general stability theorem by inflating the point –1 to the negative of the SRG of an operator in the feedback path, and show that this theorem encompasses both the incremental small gain and incremental passivity theorems.

This stability theorem relies on viewing a feedback interconnection as the inverse of a parallel interconnection. The conditions of the theorem ensure that the parallel interconnection has a strictly positive lower bound on its incremental gain; it then follows that its inverse has an upper bound on its incremental gain. In order to show that a feedback interconnection leads to a well-defined operator, we use a homotopy argument similar to [13]. We place a gain \( \tau \in [0,1] \) in the feedback loop, and assume stability for \( \tau = 0 \) (no feedback). We then use SRGs to show stability for every \( \tau \in (0,1] \), which implies that there is no loss of stability as the feedback is introduced. This allows us to use SRG analysis to prove not only stability but also well-posedness.

A. Nyquist Stability Criterion for Stable Nonlinear Operators

The Nyquist criterion characterizes the stability of a transfer function \( L \) in unity gain negative feedback (see Fig. 2) in terms of the distance between the Nyquist diagram of \( L \) and the point -1. This distance is called the stability margin, and is the inverse of the \( H_\infty \) norm of the sensitivity transfer function [36, p. 50]. In this section, we show that the Nyquist criterion and stability margin can be generalized to stable nonlinear operators.
by replacing the Nyquist diagram with an SRG. For such stable nonlinear operators, the closed loop system is stable if the SRG of the loop operator leaves $-1$ on the left.

**Theorem 1:** Let $L : L_2 \rightarrow L_2$ be an operator with finite incremental $L_2$ gain. The closed loop operator shown in Fig. 2 maps $L_2$ to $L_2$ and has finite incremental $L_2$ gain from $u$ to $y$ if

$$0 \notin 1 + \tau \text{SRG}(L) \quad \text{for all } \tau \in (0, 1].$$

The closed loop gain from $u$ to $e$ in Fig. 2 is bounded above by $1/s_m$, where $s_m$ is the shortest distance between $\text{SRG}(L)$ and $-1$.

**Proof:** We place a gain of $\tau$ in the feedback path, and show that the mapping from $u$ to $y$ is continuous if (3) holds. The operator from $u$ to $y$ is given by $(L^{-1} + \tau I)^{-1}$. Let the distance between $\text{SRG}(L^{-1})$ and $-\tau$ be $r_\tau > 0$. Then $\text{SRG}(L^{-1} + \tau I)$ is at least a distance of $r_\tau$ from the origin, so its inverse is at most $1/r_\tau$ from the origin, giving a bound of $1/r_\tau$ on the incremental gain from $u$ to $y$, as illustrated below. Condition (3) guarantees that $r_\tau > 0$ for all $\tau \in (0, 1]$.

Let $\epsilon > 0$ be smaller than $r_\tau$. Then there exists a $\delta$ (positive or negative) such that, if $\tau$ is changed to $\tau + \delta$, the distance $r_\tau$ decreases by $\epsilon$. Furthermore, as $\epsilon \rightarrow 0$, $\delta \rightarrow 0$. This is a statement of the fact that the distance between a set and a point varies continuously with the position of the point. The closed loop incremental gain bound then increases to $1/(r_\tau - \epsilon)$. This is provided $\epsilon < r_\tau$ (in which case $\delta$ small enough that $-(\tau + \delta)$ does not intersect $\text{SRG}(L^{-1})$) and approaches $r_\tau$ as $\delta \rightarrow 0$. This shows continuity from $\tau$ to the closed loop incremental gain from $u$ to $y$, and shows that finite incremental gain is preserved provided $\text{SRG}(L^{-1})$ never intersects $1/\tau$. In particular, all inputs in $L_2$ continue to map to outputs in $L_2$. We conclude finite incremental gain from $u$ to $y$ by setting $\tau = 1$.

To prove the second part of the theorem, note that the relation from $u$ to $e$ is given by

$$e = (I + L)^{-1}u.$$

If $\text{SRG}(I + L)$ is bounded away from 0 by a distance $s_m$, then $(I + L)^{-1}$ has an $L_2$ gain bound of $1/s_m$.

**B. General Feedback Stability Theorem**

The Nyquist stability criterion presented in the previous section can be generalized to allow a second nonlinear operator in the feedback path, by inflating the point $-1$ into the SRG of the feedback operator. The following theorem encompasses the classical small gain and passivity theorems as special cases.

Let $\mathcal{H}$ be a class of operators. By $\tilde{\mathcal{H}}$, we will denote a class of operators such that $\mathcal{H} \subseteq \tilde{\mathcal{H}}$ and $\text{SRG}(\mathcal{H})$ satisfies the chord property.

**Theorem 2:** Consider the feedback interconnection shown in Fig. 3 between any pair of operators $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$, where $\mathcal{H}_1$ is a class of operators on $L_2$ with finite incremental gain, and $\mathcal{H}_2$ is a class of operators on $L_2$. If, for all $\tau \in (0, 1]$ $\text{SRG}(H_1)^{-1} \cap -\tau \text{SRG}(H_2) = \emptyset$, then the feedback interconnection maps $L_2$ to $L_2$ and has an incremental $L_2$ gain bound from $u$ to $y$ of $1/r_m$, where $r_m$ is the shortest distance between $\text{SRG}(H_1^{-1})$ and $-\tau \text{SRG}(H_2)$.

The choice of which SRG to over-approximate is arbitrary. In the theorem, we have chosen $\text{SRG}(H_2)$, but it could just as well be $\text{SRG}(H_1^{-1})$.

**Proof of Theorem 2:** For a gain of $\tau$ in the feedback path, the class of operators from $u$ to $y$ is given by $(H_1^{-1} + \tau H_2)^{-1}$.

Suppose there exists a positive number $r_\tau$ such that $|z - w| \geq r_\tau$ for all $z \in \text{SRG}(H_1^{-1})$, $w \in \text{SRG}(-\tau H_2)$.

Since $\text{SRG}(H_\infty^{-1} + \tau H_2) \subseteq \text{SRG}(H_\infty^{-1}) + \tau \text{SRG}(H_2)$, where $H_2 \in \mathcal{H}_2$, it follows that $\text{SRG}(H_\infty^{-1} + \tau H_2)$ is bounded away from zero by a distance of $r_\tau$ for all $H_2 \in \mathcal{H}_2$. In particular, this holds for every operator $H_2 \in \mathcal{H}_2$.

Applying the inverse transformation gives an incremental $L_2$ gain bound of $1/r_\tau$.

Ensuring this holds for all $\tau \in (0, 1]$ means the finite incremental gain of $\mathcal{H}_1$ is never lost, so the feedback interconnection remains defined on $L_2$. $r_m$ corresponds to $r_1$.

One case where the criteria of Theorem 2 are automatically satisfied is the classical small gain setting.

**Corollary 1:** Consider the feedback interconnection shown in Fig. 3 between any pair of operators $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the classes of operators on $L_2$ with finite
incremental $L_2$ gain bounds of $\gamma$ and $\lambda$, respectively. If $\gamma \lambda < 1$, then the feedback interconnection maps $L_2$ to $L_2$ and has an incremental $L_2$ gain bound from $u$ to $y$ of $\gamma/(1 - \gamma \lambda)$.

Proof: The result follows directly from Theorem 2. The conditions of the theorem are shown to be satisfied by the geometry below.

![Diagram showing SRG of $H_1^{-1}$ and $H_2$](image)

The second case where the conditions of Theorem 2 are automatically satisfied is in the feedback interconnection of incrementally positive systems. The classical incremental passivity theorem [22] is proved in the following corollary.

Corollary 2: Consider the feedback interconnection shown in Fig. 3 between any pair of operators $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$, where $\mathcal{H}_1$ is the class of $\lambda$-input-strict incrementally positive operators which have an incremental $L_2$ gain bound of $\mu$, and $\mathcal{H}_2$ is the class of incrementally positive operators. Assume $\lambda > 0$. Then the feedback interconnection maps $L_2$ to $L_2$ and has an incremental $L_2$ gain bound from $u$ to $y$ of $\mu^2/\lambda$.

Proof: The SRGs of $H_1$ and $H_2$ are contained in the SRGs shown below. Note that these both satisfy the chord property.

![Diagram showing SRG of $H_1^{-1}$ and $H_2$](image)

The SRG of the inverse of the class of $\lambda$-input-strict incrementally positive operators is the circle with center $1/(2\lambda)$ and radius $1/(2\lambda)$ (Proposition 1). This circle is parameterized as $\{(1/\lambda) \cos(\theta) \exp(j\theta), \quad |0 \leq \theta \leq 2\pi\}$. The semicircle with center at the origin, positive real part, and radius $\mu$, which is the SRG of the class of incrementally positive operators with an incremental $L_2$ gain bound of $\mu$, is parameterized as $\{\mu \exp(j\phi), \quad |\pi/2 \leq \phi \leq \pi/2\}$. The result then follows from Theorem 2 and the geometry below.

![Diagram showing SRG of $H_1^{-1}$ and $H_2$](image)

Corollary 2 characterizes the incremental gain of the closed loop. We can also characterize the incremental positivity of the closed loop, with another form of the classical passivity theorem. The following theorem generalizes [14, Prop. 8].

Theorem 3: Consider the feedback interconnection shown in Fig. 3 between any pair of operators $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$, where $\mathcal{H}_1$ is the class of operators which are $\gamma$-output-strict incrementally positive, and $\mathcal{H}_2$ is the class of operators which are $\lambda$-input-strict incrementally positive. If

$$\lambda + \gamma \geq 0$$

then the operator from $u$ to $y$ is $(\gamma + \lambda)$-output-strict incrementally positive.

Proof: Assume, without loss of generality, that $\lambda < 0$. We first prove the case where $\lambda + \gamma > 0$. This follows from the geometry shown below.

![Diagram showing SRG of $H_1^{-1}$ and $H_2$](image)

The case where $\lambda + \gamma = 0$ then follows by taking the limit $\lambda \to -\gamma$, and allowing the radius of the circle in the final panel above to tend to $\infty$.

The definition of a stability margin for nonlinear operators leads us naturally to pose an “$H_\infty$ design problem,” in the same vein as [3], to do with the maximization of the stability margin over a set of admissible controllers. A generalization of the $H_\infty$ design question to nonlinear operators is as follows: given a plant $G$ (modeled by an operator on $L_2$) in feedback with an uncertain block $\Delta$ known to be bounded by a particular SRG, design a controller $C$ to maximize the distance between SRG $(CG)^{-1}$ and $-\text{SRG}(\Delta)$.

V. SCALED RELATIVE GRAPH OF AN LTI TRANSFER FUNCTION

In this section, we show that the SRG of a stable LTI transfer function is the convex hull of its Nyquist diagram, under the Beltrami–Klein mapping. We first presented this result in [14], and it was noted by [37] that this is a special case of a more general phenomenon involving the numerical range of a linear operator. This allows computational methods for the numerical
range to be applied directly to the computation of the boundary of an SRG.

We begin by introducing some preliminaries from hyperbolic geometry in Section V-A, before giving the main result in Section V-B.

A. Hyperbolic Geometry

We recall some necessary details from hyperbolic geometry.

The notation is consistent with [38].

**Definition 5:** Let \( z_1, z_2 \in \mathbb{C}_{\geq 0} \) := \{ z \in \mathbb{C} | \Re(z) \geq 0 \}, the upper half complex plane. We define the following sets.

1) \( \text{Circ}(z_1, z_2) \) is the circle through \( z_1 \) and \( z_2 \) with center on the real axis. If \( \Re(z_1) = \Re(z_2) \), this is defined as the infinite line passing through \( z_1 \) and \( z_2 \).

2) \( \text{Arc}_{\min}(z_1, z_2) \) is the arc of \( \text{Circ}(z_1, z_2) \) in \( \mathbb{C}_{\geq 0} \). If \( \Re(z_1) = \Re(z_2) \), then \( \text{Arc}_{\min}(z_1, z_2) \) is \([z_1, z_2]\).

3) Given \( z_1, \ldots, z_m \in \mathbb{C}_{\geq 0} \), the \textit{arc-edge polygon} is defined by: \( \text{Poly}(z_1) := \{z_1\} \) and \( \text{Poly}(z_1, \ldots, z_m) \) is the smallest simply connected set containing \( S \), where

\[
S = \bigcup_{i,j=1 \ldots m} \text{Arc}_{\min}(z_i, z_j).
\]

Note that, as \( \text{Poly}(z_1, \ldots, z_{m-1}) \subseteq \text{Poly}(z_1, \ldots, z_m) \subseteq \mathbb{C}_{\geq 0} \), the set \( \text{Poly}(Z) \), where \( Z \) is a countably infinite sequence of points in \( \mathbb{C}_{\geq 0} \), is well defined as the limit \( \lim_{m \to \infty} \text{Poly}(Z_m) \), where \( Z_m \) is the length \( m \) truncation of \( Z \) (see [39, p. 111]).

Definition 5 forms the basis of the Poincaré half-plane model of hyperbolic geometry. Under the Beltrami–Klein mapping, \( f \circ g \), where

\[
f(z) = \frac{2z}{1 + |z|^2}
g(z) = \frac{z - j}{z + j}
\]

\( \mathbb{C}_{\geq 0} \) is mapped onto the unit disk, and \( \text{Arc}_{\min}(z_1, z_2) \) is mapped to a straight line segment. We make the following definitions of convexity and the convex hull in the Poincaré half-plane model.

**Definition 6:** A set \( S \subseteq \mathbb{C}_{\geq 0} \) is called hyperbolic-convex or \( h \)-convex if

\[
z_1, z_2 \in S \Rightarrow \text{Arc}_{\min}(z_1, z_2) \in S.
\]

Given a set of points \( P \in \mathbb{C}_{\geq 0} \), the \textit{h-convex hull} of \( P \) is the smallest \( h \)-convex set containing \( P \).

Note that \( h \)-convexity is equivalent to Euclidean convexity under the Beltrami–Klein mapping. \( \text{Arc}_{\min}(z_1, z_2) \) is the minimal geodesic between \( z_1 \) and \( z_2 \) under the Poincaré metric, so \( h \)-convexity may be thought of as geodesic convexity with respect to this metric. We recall the following useful lemma of [38].

**Lemma 1:** (Lemma 2.1 [38]): Given a sequence of points \( Z \in \mathbb{C}_{\geq 0} \), Poly \( (Z) \) is \( h \)-convex.

In our terminology, given a sequence of points \( Z \in \mathbb{C}_{\geq 0} \), Poly \( (Z) \) is the \( h \)-convex hull of \( Z \).

B. SRGs of LTI Transfer Functions

Let \( g : L_2 \to L_2 \) be linear and time invariant, and denote its transfer function by \( G(s) \). \( g \) maps a complex sinusoid \( u(t) = a e^{jωt} \) to the complex sinusoid \( y(t) = a |G(jω)| e^{j(∠G(jω) + jωt)} \).

These signals do not belong to \( L_2 \), but are treated as limits of sequences in \( L_2 \). Precisely, we define the points on the SRG corresponding to sinusoidal signals by taking the gain and phase to be

\[
\lim_{T \to \infty} \frac{\|P_T y\|}{\|P_T u\|},
\]

\[
\lim_{T \to \infty} \angle(P_T u, P_T y).
\]

Both these limits exist when \( u \) and \( y \) are sinusoidal. The Nyquist diagram \( \text{Nyquist}(G) \) of an operator \( g : L_2(\mathbb{C}) \to L_2(\mathbb{C}) \) is the locus of points \( \{G(jω) | ω \in \mathbb{R}\} \).

**Theorem 4:** Let \( g : L_2(\mathbb{C}) \to L_2(\mathbb{C}) \) be linear and time invariant, with transfer function \( G(s) \). Then \( \text{SRG}(g) \cap \mathbb{C}_{\geq 0} \) is the h-convex hull of Nyquist \( (G) \cap \mathbb{C}_{\geq 0} \).

The proof of Theorem 4 is closely related to the proof of [38, Th. 3.1], and may be found in Appendix. A consequence of Theorem 4 is that the SRG of an LTI operator is bounded by its Nyquist diagram. For example, the SRG of the transfer function \( 1/(s^3 + 5 s^2 + 2 s + 1) \) is illustrated in Fig. 4. Further examples are given in [14].

Given Theorem 4, we recover two familiar properties of the Nyquist diagram as special cases of Proposition 1, namely, that passivity is equivalent to the Nyquist diagram lying in the right-half plane, and the \( H_\infty \) gain is the maximum magnitude of the Nyquist diagram.

VI. SCALLED RELATIVE GRAPHS OF STATIC NONLINEARITIES

LTI systems map complex sinusoids to complex sinusoids, and the behavior of an LTI system on \( L_2 \) can be fully characterized by its behavior on complex sinusoids.

Similarly, static nonlinearities map square waves to square waves. Here, we show that the behavior of single input, single output static nonlinearities on \( L_2 \), insofar as it is captured by the scaled relative graph, is fully characterized by their behavior on a 2-D subspace of \( L_2 \) spanned by two Haar wavelets (truncations of a square wave to a single period). In particular, we show that the SRGs of the saturation and ReLU are identical, and closely
related to the SRG of a first-order lag. The use of square waves allows us to test the effect of different input amplitudes on the output, which is analogous to the use of sinusoids to test the effect of different input frequencies on the output of an LTI system.

**Proposition 9:** Suppose \( S : L_2 \rightarrow L_2 \) is the operator given by a SISO static nonlinearity \( s : \mathbb{R} \rightarrow \mathbb{R} \), such that for all \( u_1, u_2 \in \mathbb{R}, y_1 \in s(u_1), y_2 \in s(u_2) \)

\[
\mu(u_1 - u_2)^2 \leq (y_1 - y_2)(u_1 - u_2) \leq \lambda(u_1 - u_2)^2. \tag{4}
\]

Then the SRG of \( S \) is contained within the disk centered at \((\lambda + \mu)/2\) with radius \((\lambda - \mu)/2\).

For a static nonlinearity obeying Condition (4), we say that it is incrementally in the sector \([\mu, \lambda]\).

**Proof:** Define an operator \( \tilde{S} \) by \( u \mapsto \tilde{y} := s(u) - \mu u \). Let \( \Delta u(t) = u_1(t) - u_2(t) \) and \( \Delta \tilde{y}(t) = \tilde{y}_1(t) - \tilde{y}_2(t) \). We drop the \( t \) dependence in the remainder of this proof. By assumption on \( s \), for all \( \Delta u \) and corresponding incremental output \( \Delta \tilde{y} \), we have

\[
0 \leq \Delta u(\Delta \tilde{y} - \mu \Delta u) \leq (\lambda - \mu) \Delta u^2 \tag{5}
\]

\[
0 \leq \Delta u \Delta \tilde{y} \leq (\lambda - \mu) \Delta u^2. \tag{6}
\]

It then follows that \( \Delta u \Delta \tilde{y} \geq 0 \) and \( \Delta u \Delta \tilde{y} - (\lambda - \mu) \Delta u^2 \leq 0 \), from which the following series of equivalent statements follow:

\[
\Delta u \Delta \tilde{y}(\Delta u \Delta \tilde{y} - (\lambda - \mu) \Delta u^2) \leq 0
\]

\[
\Delta u^2(\Delta \tilde{y}^2 - (\lambda - \mu) \Delta u \Delta \tilde{y}) \leq 0
\]

\[
\Delta \tilde{y}^2 \leq (\lambda - \mu) \Delta u \Delta \tilde{y}
\]

\[
\Delta \tilde{u} \Delta \tilde{y} \geq \frac{1}{\lambda - \mu} \Delta \tilde{y}^2.
\]

This shows that \( \tilde{S} \) is output-strict incrementally positive with constant \( 1/(\lambda - \mu) \), so its SRG is the disk with center \((\lambda - \mu)/2\) and radius \((\lambda - \mu)/2\). The result then follows by noting that \( S \) is the parallel interconnection of \( \tilde{S} \) with \( \mu I \), so its SRG is the SRG of \( \tilde{S} \) shifted to the right by \( \mu \).

The same bounding region can be obtained for the SG with respect to an input \( u^* \), by restricting the second input in the proof of Proposition 9 to be \( u^* \). This is stated formally below.

**Proposition 10:** Suppose \( S : L_2 \rightarrow L_2 \) is the operator given by an SISO static nonlinearity \( s : \mathbb{R} \rightarrow \mathbb{R} \), such that, for all \( u_1 \in \mathbb{R}, y_1 \in s(u_1), y^* \in s(u^*) \)

\[
\mu(u_1 - u^*)^2 \leq (y_1 - y^*)(u_1 - u^*) \leq \lambda(u_1 - u^*)^2. \tag{7}
\]

Then the SG of \( S \) with respect to \( u^* \) is contained within the disk centered at \((\lambda + \mu)/2\) with radius \((\lambda - \mu)/2\).

The disks obtained in the previous two propositions are closely related to the disks of the classical incremental circle criterion [40], indeed, taking the negative and inverting transforms one to the other.

We now show that, for a large class of systems, the disk bound on the SRG cannot be improved. If the characteristic curve of \( s \) contains a “maximal elbow,” that is, a point where the slope switches from maximum to minimum, then small signals centred around the elbow can be used to generate the perimeter of the bound of Proposition 9. Furthermore, if the region of minimum slope extends to infinity, then large signals can be used to generate the interior of the bound of Proposition 9. This is formalized in the following two propositions. We treat only an elbow from slope 1 to slope 0, as a loop transformation can be used to convert any other elbow to this form.

**Proposition 11:** Suppose \( S : L_2 \rightarrow L_2 \) is a memoryless nonlinearity defined by a map \( s : \mathbb{R} \rightarrow \mathbb{R} \) which satisfies (4) with \( \mu = 0 \) and \( \lambda = 1 \). Furthermore, suppose there are real numbers \( u^* \) and \( \delta > 0 \), such that

\[
s(u^* + \epsilon) - s(u^*) = 0 \quad \text{for all } \epsilon \in [0, \delta] \tag{8}
\]

\[
s(u^*) - s(u^* - \epsilon) = \epsilon \quad \text{for all } \epsilon \in [0, \delta]. \tag{9}
\]

Then the SRG of \( S \) contains the circle centred at 1/2 with radius 1/2.

**Proof:** We consider two input signals, supported on \([0, 1]\)

\[
u_1(t) = u^*, \quad u_2(t) = \begin{cases} u^* + \epsilon & 0 \leq t < \tau \\ u^* - \epsilon & \tau \leq t \leq 1 \end{cases}
\]

where \( \tau \in [0, 1] \). The corresponding output signals are given by

\[
y_1(t) = s(u^*), \quad y_2(t) = \begin{cases} s(u^* + \epsilon) & 0 \leq t < \tau \\ s(u^*) & \tau \leq t \leq 1 \end{cases}
\]

giving the incremental signals

\[
\Delta y(t) = \begin{cases} -\epsilon & 0 \leq t < \tau \\ \epsilon & \tau \leq t \leq 1 \end{cases}
\]

\( \Delta y \) can be written as \( k(t) \Delta u(t) \), where

\[
k(t) = \begin{cases} 0 & 0 \leq t < \tau \\ 1 & \tau \leq t \leq 1 \end{cases}
\]

Calculating gain then gives

\[
\|\Delta y\| = \left( \int_0^1 k^2(t) \Delta u^2(t) dt \right)^{\frac{1}{2}}
\]

\[
= \left( \int_0^1 \Delta u^2(t) dt \right)^{\frac{1}{2}} = \gamma \|\Delta u\|
\]

for some \( \gamma \) which varies between 0 and 1 as \( \tau \) varies between 1 and 0. It follows that:

\[
\frac{\|\Delta y\|}{\|\Delta u\|} = \gamma.
\]

Calculating the phase gives

\[
\frac{\langle \Delta u \|\Delta y\rangle}{\|\Delta u\| \|\Delta y\|} = \frac{\int_0^1 k(t) \Delta u^2(t) dt}{\gamma \|\Delta u\|^2}
\]

\[
= \frac{\int_0^1 \Delta u^2(t) dt}{\gamma \|\Delta u\|^2} = \cos(\gamma).
\]

Since \( \gamma \in [0, 1] \), we can define \( \theta \) by \( \cos(\theta) = \gamma \). We then have the locus of points on the SRG given by

\[
\cos(\theta) \exp(\pm j \theta), \quad 0 \leq \theta \leq \pi/2
\]

which is the circle with center \( 1/2 \) and radius \( 1/2 \).

**Proposition 12:** Suppose \( S : L_2 \rightarrow L_2 \) is a memoryless nonlinearity defined by a map \( s : \mathbb{R} \rightarrow \mathbb{R} \) which satisfies (4) with
\( \mu = 0 \) and \( \lambda = 1 \), and which satisfies \( s(0) = 0 \). Furthermore, suppose there is a real number \( u^* \) such that
\[
\begin{align*}
  s(u^* + M) - s(u^*) &= 0 \quad \text{for all } M \geq 0 \\
  s(u^*) &> 0.
\end{align*}
\]
Then the SRG of \( S \) is the disk centered at \( s(u^*)/2 \) with radius \( s(u^*)/2 \).

**Proof:** We consider two input signals, supported on \([0, 1] \)
\[
u_1(t) = M, \quad u_2(t) = \begin{cases} M + u^* & 0 \leq t < \tau \\ 0 & \tau \leq t \leq 1 \end{cases}
\]
where \( \tau \in [0, 1] \), and \( M \geq u^* \). Performing the same calculations as in the proof of Proposition 11, and defining \( \beta(M) := s(u^*)/M \), we have
\[
\begin{align*}
  \frac{\|\Delta y\|}{\|\Delta u\|} &= \beta(M) \gamma \\
  \arccos\left( \frac{\Delta u \cdot \Delta y}{\|\Delta u\| \|\Delta y\|} \right) &= \arccos(\gamma).
\end{align*}
\]
Since \( \gamma \in [0, 1] \), we can define \( \theta \) by \( \cos(\theta) = \gamma \). We then have the locus of points on the SRG given by
\[
\beta(M) \cos(\theta) \exp(\pm j\theta), \quad 0 \leq \theta \leq \pi/2.
\]
This is the circle with center \( \beta(M)/2 \) and radius \( \beta(M)/2 \). Varying \( M \) between \( u^* \) and \( \infty \) varies \( \beta(M) \) between \( s(u^*)/u^* \) and 0, so we fill the disk with center \( s(u^*)/2 \) and radius \( s(u^*)/2 \).

Proposition 12 allows us to give an exact characterization of the SRGs of a range of static nonlinearities, including the ideal diode, the ReLU, and the limiting cases of the relay and saturation. The proof of Proposition 11 uses probing signals which have an arbitrarily small magnitude variation about a “worst case” input value. This shows the local or worst case nature of the SRG—the boundary of the SRG is generated by these probing signals.

**Remark 1:** We conclude this section by remarking that the characterization of output-strict incrementally passive static nonlinearities allows the SRGs of a large class of dynamic output-strict incrementally passive nonlinear systems to be characterized. Output-strict incremental passivity is preserved under negative feedback with an incrementally passive system, as shown in Theorem 3. This means that any scalar system of the form
\[
\dot{y} = f(u - y)
\]
where \( f \) is incrementally in a sector with positive constants, is output-strict incrementally passive.

**VII. EXAMPLE 1: FEEDBACK WITH SATURATION AND DELAY**

In this section, we use SRGs to analyze feedback systems with delays, dynamic components and static nonlinearities. We will derive incremental stability bounds which depend both on the delay time and the dynamic time constant, similar to the state-of-the-art nonincremental bounds obtained using the roll-off IQC [21]. These bounds are obtained by approximating the SRG of the delay and the dynamics, treated as a single component. In the simple example of this section, where the dynamic component is LTI, this approach reduces to the incremental circle criterion. However, the approach allows for arbitrary dynamic components, as shown in Section IX. One of the advantages of our approach is the derivation of stability margins and incremental \( L_2 \) gain bounds for the closed loop.

We begin with the system of Fig. 5, showing a time delay and an LTI transfer function in feedback with a \( 1/\beta \)-output-strict incrementally passive component \( \Delta \).

We take \( P(s) = s^2/(s^3 + 2 s^2 + 2 s + 1) \), also considered in [13, §3]. The Nyquist diagram of \( P(s) \) cascaded with the delay, and a bounding approximation of the SRG, are shown on the left-hand side of Fig. 6. As the delay is increased, the Nyquist diagram, and hence the SRG, extend further into the left-half plane.

Applying Theorem 2 with \( H_2 = e^{-sT} P(s) \) and \( H_1 = \Delta \), we obtain the right hand side of Fig. 6. Stability is verified if the delay SRG always has real part greater than \( 1/\beta \), which ensures that \( r_m > 0 \). Solving numerically for \( \min_{\Delta} \text{Re}(P(j\omega)e^{j\omega T}) \) gives a stability bound on \( \beta \), as a function of \( T \), shown in Fig. 7, which also shows the nonincremental stability bound obtained by [13] using IQC analysis, for the particular case where \( \Delta \) is a saturation. For short delay times, the nonincremental bound is shown to tend to infinity, using the Zames-Falb-O’Shea multiplier. The incremental bound obtained using SRG analysis has a non-smooth point where the leftmost segment of the Nyquist diagram switches, and is bounded for all delay times.

The SRG analysis gives a bound which guarantees finite incremental \( L_2 \) gain, a stronger property than the \( L_2 \) gain from IQC analysis. Finite incremental \( L_2 \) gain in particular implies input-output Lipschitz continuity. To the best of the authors’
knowledge (and as also noted in [16], [15]) is the only application of incremental IQCs to stability analysis of feedback systems, with only a very weak form of stability guaranteed. As noted by [41], stability results using Zames-Falb-O’Shea and Popov multipliers do not guarantee continuity, as these multipliers do not preserve the incremental passivity of static nonlinear elements. The situation for proving finite incremental \( L_2 \) gain with these multipliers is similar; the loss of incremental passivity of the static nonlinearity means the incremental passivity theorem cannot be applied, so another method of proving stability is needed. One such method would be to apply Theorem 2 to the transformed loop, and indeed there are multipliers which destroy incremental passivity but which still verify an incremental gain bound. For this particular example, the transfer function \( (s + 1)/(s - 1) \) could be used as a multiplier, although it gives a more conservative bound than Fig. 7. Global [16], universal [42] and equilibrium-independent [35] \( L_2 \) gain are weaker than incremental \( L_2 \) gain but stronger than \( L_2 \), and afford differing levels of tractability.

In addition to proving incremental \( L_2 \) stability, we can give an incremental \( L_2 \) gain bound. For a fixed \( \beta, 1/r_m \) is an incremental \( L_2 \) gain bound from \( u \) to \( y \), which depends on the time delay \( T \). For \( \beta = 1 \), this bound is plotted in Fig. 7.

The motivation behind the traditional structure of the Lur’e system is to put all of the “troublesome” elements in the nonlinear component, and all of the dynamics in the LTI component. The availability of explicit SRGs for elements which are usually troublesome, such as saturations and delays, means that this structure is not necessarily ideal for SRG analysis, and the feedback system may be better modeled in a different way. This is illustrated in the following two examples.

VIII. EXAMPLE 2: CYCLIC FEEDBACK SYSTEMS

We now turn to the analysis of cascades. Such systems form the basis of cyclic feedback systems, which are often found in biological models [17], among many other application domains (see, for example, the discussion of [18]). In Theorem 5, we give the SRG of a cascade of output-strict incrementally positive systems, which represents a novel system constraint which cannot be represented as an incremental IQC. A gain margin condition applied to a cascade in unity gain negative feedback gives rise to the incremental secant condition [19]. The cascade SRG, thus, generalizes the incremental secant condition to arbitrary feedback interconnections and disturbances. The result we give here is tight in the sense that stronger conditions than output strict incremental positivity of the plants are required for any stronger bound.

**Theorem 5:** Consider the cascade of \( n \) output-strict incrementally positive systems, with parameters \( 1/\gamma_i, i = 1, \ldots, n \), shown in Fig. 8. The SRG of the cascade is contained within the region with perimeter

\[
|z(\phi)| = \gamma_1 \gamma_2 \ldots \gamma_n \left( \frac{\cos \frac{\phi}{n}}{n} \right)^n e^{-j\phi}, \quad -\pi \leq \phi < \pi. \tag{12}
\]

**Proof:** The SRG of the \( i \)th system is the disk with center \( \gamma_i/2 \) and radius \( \gamma_i/2 \). The perimeter of this disk has the parameterization

\[
z_i(\theta) = \gamma_i \cos(\theta) e^{-j\theta}, \quad -\pi/2 \leq \theta < \pi/2. \tag{13}
\]

As this disk satisfies the right hand arc property, the SRG of the full cascade is the product of \( n \) disks. We claim that the perimeter of this SRG has the parameterization given by (12).

For instance, take any \( z_1, z_2, \ldots, z_n \). Using (13) and Proposition 8 gives the point

\[
w = \gamma_1 \gamma_2 \ldots \gamma_n (\theta_1) \ldots (\theta_n) e^{-j(\theta_1 + \ldots + \theta_n)} \tag{14}
\]

for \(-\pi < \theta_1, \theta_2, \ldots, \theta_n < \pi\). Letting \( \theta_1 = \theta_2 = \ldots = \theta = \theta \), and setting \( \phi = n\theta \) gives the parameterization (12) (noting that \(-\pi < \phi < \pi \) as (12) is \( 2\pi \)-periodic). This shows that all the points \( z(\phi) \) lie within the SRG. To show that they are indeed on the perimeter of the SRG, we take any point \( w \) and show that its magnitude is smaller than the point \( z(\phi) \) with the same argument. This follows from (14) if we can show that

\[
\cos(\theta_1) \cos(\theta_2) \ldots \cos(\theta_n) \leq \cos(\theta_1 + \theta_2 + \ldots + \theta_n).
\]

This is proved in [19]: \( f(\phi) = -\ln \cos(\phi) \) is convex on \((-\pi/2, \pi/2)\). Applying Jensen’s inequality gives \( f(\sum \theta_i) \leq \sum f(\theta_i) \), and the required inequality follows by taking the exponential. Note that the inequality still holds in the limit as one angle \( \theta_i \rightarrow \pm \pi/2 \).

The SRG given by Theorem 5 is illustrated in Fig. 9. For \( n = 2 \), this SRG is a special case of [12, Th. 2]. For \( n > 2 \), this SRG is a novel result. The intercept with the negative real axis is at the point \( z(\pi) = -\gamma_1 \gamma_2 \ldots \gamma_n (\cos \frac{\pi}{n})^n \). A direct application of Theorem 1 to a cascade in unity gain negative feedback thus gives the following incremental secant condition.

**Corollary 3:** Suppose the system of Fig. 8 is placed in unity gain negative feedback, where the \( n \) interconnected systems are each output-strict incrementally positive with parameters \( \gamma_i, i = 1, \ldots, n \). The feedback interconnection has a finite incremental \( L_2 \) gain if

\[
\gamma_1 \gamma_2 \ldots \gamma_n < \left( \frac{\sec \frac{\pi}{n}}{n} \right)^n.
\]
To see that the cascade SRG expresses a more general constraint than possible with an incremental IQC, we can take the $n = 2$ case of Equation (12), and eliminate the parameter $\phi$. This gives the following equality constraint on the boundary of the SRG:

$$\langle u_1 - u_2 \rangle y_1 - y_2 \rangle + \|u_1 - u_2\| y_1 - y_2 \rangle - 2 \|y_1 - y_2\|^2.$$  

The middle term cannot be expressed as an incremental IQC.

The cascade SRG allows several other useful values to be computed. An incremental $L_2$ gain bound can be found by minimizing the distance between $-1$ and $1/(\gamma_1 \ldots \gamma_n \cos(\theta_1) \ldots \cos(\theta_n) e^{-\gamma_1 \ldots \gamma_n \pi / n})$. This distance is shown for $n = 4$ in Fig. 10. Furthermore, we can calculate the shortage of input-strict incremental positivity of the cascade by finding the distance the SRG extends into the left-half plane. For example, for a cascade of two systems, the shortage of input-strict incremental positivity is $\gamma_1 \gamma_2 / 8$ [43, p. 118]. Stan et al. [44] show that if the coupling strength in a network of oscillators modeled as cascade feedback systems is large enough compared to the shortage of each oscillator, the network will synchronize.

SRG analysis allows the incremental secant condition to be generalized beyond negative feedback interconnections. For example, if an uncertain gain $k_\Delta$ is placed in feedback with the cascade, as shown in Fig. 11, we can give a bound on $k_\Delta$ for which incremental stability is guaranteed. The inverse SRG of the cascade (see Fig. 10) is shifted to the left by $k_\Delta$; if it does not intersect $-1$, the closed loop has finite incremental gain. This allows us to conclude stability if

$$\left(\gamma_1 \gamma_2 \ldots \gamma_n\right)^{-1} > k_\Delta > 1 - \left(\gamma_1 \gamma_2 \ldots \gamma_n \left(\cos \frac{\pi}{n}\right)^n \right)^{-1}.$$  

Remark 2: Remark 1 showed how the bounding SRG for a static nonlinearity that is incrementally in a sector could be used to determine bounding SRGs for dynamic nonlinearities. The cascade SRG derived in this section allows us to extend this idea to dynamic nonlinearities described by differential equations of the form

$$\dot{y} = f(g(u) - y).$$  

(15)

Suppose that $f$ is incrementally in the sector $[\mu_1, \gamma_1]$ (in the sense of Proposition 9), and $g$ is incrementally in the sector $[\mu_2, \gamma_2]$. For simplicity, assume $\mu_1 = \mu_2 = 0$. The system of (15) can be represented as $f$ in negative feedback with an integrator, with the nonlinearity $g$ at the input. It follows from Remark 1 and the Theorem 5 that this system has a bounding SRG given by the $n = 2$ case of Fig. 9.

IX. EXAMPLE 3: COMBINING CASCADES AND DELAYS

In this final example, we combine a delay with a cascade of two output-strict incrementally positive systems, and revisit the Internet congestion control example of [21]. In that paper, equilibrium-independent IQCs are verified numerically in order to compute a bound on the variables $\delta$, $\beta$, and $N_u$ on the left of Fig. 12, which guarantees (nonincremental) input/output stability of the system. Here, we derive a bound which guarantees finite incremental gain of the system.

In order to combine the delay and first-order lag, we rearrange the forward path as shown in the right of Fig. 12. This is the negative feedback interconnection of $H_1 = e^{-sT} / (s + \beta)$ and $H_2 = -e^{sT} \phi(\cdot)$. Bounding SRGs for $H_1^{-1}$ and $-H_2$, illustrated in the top of Fig. 13, are derived using Theorem 4, Proposition 1 and Proposition 8. Combining these gives a delay-dependent
for nonlinear operators. There are many questions for future work; here, we will list only a couple.

The SRG composition rules rely on a worst-case assumption: that the same signals correspond to the worst-case points in the SRGs of both systems. When dealing with interconnections of individual systems (rather than classes of systems), this can be conservative. For example, applying Theorem 2 to the negative feedback interconnection of $1/(s + 1)$ and $e^{-sT}$ does not give a guarantee of stability; we know from the Nyquist criterion, however, that unity-gain negative feedback around $e^{-sT}/(s + 1)$ does give a stable system. A second case where the analysis appears to be conservative is in the study of multiple input, multiple output systems. Understanding when SRG analysis is tight, and when it is conservative, is a topic of ongoing research. We expect that 20 years of IQC analysis will contribute to further developing SRG analysis.

A second question is concerned with computation of SRGs. Efficient algorithms for computing or approximating the SRGs of nonlinear operators defined by state space models, or directly from input/output data, are an interesting topic for future research.

A third question is concerned with the extension of the Nyquist theorem to the general case of unstable open loop plants.

The SRG characterization of a system can be tightened by taking the intersection of several bounding SRGs, similar to taking the intersection of several IQCs. SRG analysis also allows a characterization to be loosened by taking the union of SRG: for example, a system might either be passive, or have small gain (or both). This is explored in [45].

We hope that the graphical analysis presented in this article will further narrow the gap between linear and nonlinear control theory.

**APPENDIX**

**Proof of Theorem 4**

The proof has three components. We begin by showing that, for an LTI transfer function $G(s)$, the Nyquist diagram at the frequencies $n2\pi/T$ is a subset of the SRG of $G(s)$. We then show, for operators on the space $L_{2,T}$ of $T$-periodic, finite energy signals, the SRG is in the convex hull of the points generated by applying the operator to the basis of $L_{2,T}$ given by $\{e^{jtn2\pi/T}\}_{n \in \mathbb{Z}}$, which are exactly the points on the Nyquist diagram. The result then follows by taking the limit as $T \to \infty$, analogous to the classical derivation of the Fourier transform from the Fourier series.

We begin by observing that the point on the Nyquist diagram of $G$ corresponding to frequency $\omega \in \mathbb{R}$ is precisely $z_G(e^{j\omega T})$. Set $u = ae^{j\omega t}$, then $y = G(u) = aae^{j\omega t + j\psi}$, where $\alpha = |G(j\omega)|$ and $\psi = zG(j\omega)$. A direct calculation gives

$$\langle u|y \rangle = \int_0^T u(t)g(t)dt = \int T\alpha a^2 e^{j\phi}$$

$$\|u\| = \sqrt{T}a,$n

$$\|y\| = \sqrt{T}a$$

The SRG composition rules rely on a worst-case assumption: that the same signals correspond to the worst-case points in the SRGs of both systems.
where $\langle \cdot, \cdot \rangle$ is the inner product on $L_{2,T}$. It follows immediately that:

$$z_G(u) = ae^{i\omega t}$$

that is, the point on the Nyquist diagram of $G$ corresponding to frequency $\omega$.

The next part of the proof closely follows [38, Th. 3.1]. In the interests of brevity, we point out only the main arguments and modifications required to that proof.

Let $G$ be an LTI operator on $L_2$. The restriction of $G$ to $L_{2,T}$ is then an operator on $L_{2,T}$. Let $B$ be the set of functions in $t$ given by $B = \{e^{i\pi n/2T}, n \in \mathbb{Z} \}$. We show that

$$z_G(\text{span}(B) \setminus \{0\}) = \text{Poly}(z_G(B)).$$

(16)

We begin by noting that $B$ is an orthonormal basis for $L_{2,T}$, and in particular, for all $u, v \in B$, $u \neq v$, $\langle v, u \rangle = \langle v, Gu \rangle = \langle Gv, u \rangle = \langle Gv, Gu \rangle = 0$. Therefore, the result of Part 2 of the proof of [38, Th. 3.1] holds: for all such $u, v$, we have

$$z_G(\text{span}(u, v) \setminus \{0\}) = \text{Arcmin}(z_G(u), z_G(v)).$$

The only modification required to the proof is that the inner product here is complex valued, and the real part must be taken. Parts 3 and 4 of the proof of [38, Th. 3.1] show that $z_G(\text{span}(B) \setminus \{0\}) \subseteq \text{Poly}(z_G(B))$ and $z_G(\text{span}(B) \setminus \{0\}) \supseteq \text{Poly}(z_G(B))$ respectively, with the proof requiring only the additional fact that $\text{Poly}(S)$ (in the proof of [38, Th. 3.1]) is defined for a countably infinite set, as described in Section V-A. This concludes the second part of the proof: $z_G(\text{span}(B) \setminus \{0\}) = \text{Poly}(z_G(B))$.

Finally, we extend to aperiodic signals by letting the period $T \to \infty$ and the fundamental frequency $2\pi/T \to 0$. In the interests of brevity, we give the proof here assuming that the Fourier transform of the input $u(t)$ is Riemann integrable. The result can be extended to arbitrary functions on $L_2$ using the same machinery for defining the Fourier transform on $L_2$, see, for instance, [46, Ch. 9]. We first note that $z_G(ae^{i\omega t})$ may be computed using the inner product and norm on $L_2$, rather than $L_{2,T}$, as a limit, and the result will be unchanged. Let $u(t)$ be an input signal on $L_2$, and $y(t)$ the corresponding output. The Fourier inversion theorem gives

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(j\omega)\hat{u}(\omega)e^{j\omega t}d\omega.$$  

(17)

Let

$$\Delta \omega = \frac{\sqrt{2\pi}}{\sum_{n=-\infty}^{\infty} G(jn\Delta\omega)\hat{n}(n\Delta\omega)e^{jn\Delta\omega}}$$

be a Riemann sum approximation of the right hand side of (17), with uniform spacing $\Delta \omega$. By (16), we know this sum belongs to $\text{Poly}\left(\left\{G(j\omega)e^{i\Delta \omega t}\right\}_{n \in \mathbb{Z}}\right) \subseteq \text{Poly}\left(\left\{G(jw)e^{i\omega t}\right\}_{\omega \in \mathbb{R}}\right)$. Letting $\Delta \omega \to 0$, we have that the right hand side of (17) belongs to $\text{Poly}\left(\left\{G(j\omega)e^{i\omega t}\right\}_{\omega \in \mathbb{R}}\right)$, noting that the restriction of the Nyquist diagram to $C_{3\geq 0}$ is compact in $C$. Note that this is precisely the h-convex hull of the Nyquist diagram of $G$. □
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