On stability of tangent bundles of Fano manifolds with $b_2 = 1$.  

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Introduction

An important outstanding problem in differential geometry asks which Fano manifolds $X$ with $b_2(X) = 1$ admit a Kähler—Einstein metric. The only known general result (besides some necessary conditions as the reductivity of the Lie algebra of holomorphic vector fields and the Futaki invariant) is the positive answer for rational-homogeneous manifolds, which holds also if $b_2(X) > 1$. A weaker and more algebraic question would ask whether the tangent bundle $T_X$ is stable (with respect to the anticanonical bundle $-K_X$). For the connection between both notions see e.g. [Ti]. If $b_2(X) > 1$, $T_X$ is not necessarily stable due to the geometry of contractions of extremal rays; for 3-folds this has been studied completely by A. Steffens in his thesis [St]. There are several papers considering the 2-dimensional case, i.e. del Pezzo surfaces, see [Ti] for references. This paper contributes to the stability problem in case $b_2(X) = 1$.

Of course the stability of $T_X$ is very much related to cohomology vanishings of type

$$H^p(X, \Omega^q(t)) = 0.$$  

These vanishings are studied in Section 1. We say that a pair $(Y, \mathcal{O}(1))$ consisting of a projective manifold $Y$ and an ample line bundle $\mathcal{O}(1)$ has special cohomology if

(a) $H^p(Y, \Omega^q_Y(t)) = 0$ for $0 < p < \dim Y$, $p + q \neq \dim Y$ and $t \in \mathbb{Z} \setminus \{0\}$

(b) $H^p(Y, \Omega^q_Y) = 0$ for $0 < p < \dim Y, p + q \neq \dim Y, p \neq q$

(c) $H^p(Y, \Omega^p_Y) = C$ for $0 \leq p \leq \dim Y$,

In other words, conditions (b) and (c) say that the odd Betti numbers except the are 0 while the even Betti numbers are 1.

Of course $(\mathbb{P}^n, \mathcal{O}(1))$ has special cohomology. But due to the following “functorial” theorem we have much more examples.

**Theorem 1.** Let $(Y, \mathcal{O}(1))$ have special cohomology.

(1) Let $X \in |\mathcal{O}_Y(d)|$ be a smooth divisor for some $d > 0$. Then $(X, \mathcal{O}(1)|_X)$ has special cohomology.

(2) Let $\pi : X \rightarrow Y$ be the $k$-cyclic cover branched along some $D \in |\mathcal{O}(kd)|$. Then $(X, \pi^*(\mathcal{O}_Y(1)))$ has special cohomology.

This theorem concerns vanishing of higher cohomology groups. The same methods yield a $H^0$-vanishing theorem which can be stated as
Theorem 2. Let $(Y, \mathcal{O}(1))$ have special cohomology, moreover assume the tangent (or cotangent) bundle of $Y$ to be semi-stable.

(1) Every smooth $X \in |\mathcal{O}_Y(d)|$ has stable (co-)tangent bundle, if $\dim X \geq 3$

(2) Let $\pi : X \to Y$ be the $k$-cyclic cover branched along some $D \in |\mathcal{O}(kd)|$, $k \geq 2$. Then $X$ has stable (co-)tangent bundle, if $\dim Y \geq 3$.

Of course Theorem 2 does not solve the stability problem for Fano manifolds completely. However we can prove in Section 2:

Theorem 3. Let $X$ be a Fano $n$-fold of index $r$ with $b_2(X) = 1$.

(1) If $r \geq n - 1$, then $T_X$ is stable.

(2) If $r = n - 2$ and if there are “enough” smooth divisors in $|H|$, $H$ being the ample generator of $\text{Pic}(X) = \mathbb{Z}$, then $T_X$ is stable.

(3) Every Fano 4-fold with $b_2(X) = 1$ has stable tangent bundle

The existence of enough smooth divisors (for the precise statement see Section 2) in (2) is expected to hold on every Fano $n$-fold of index $n - 2$.

It should be mentioned that the methods of Section 2 are to some extend ad hoc methods relying on classification results of Fujita, Mukai and others, so they do not apply immediately to manifolds of lower index.

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0. Preliminaries

For a complex manifold $X$ we denote by $\Omega^q_X$ the sheaf of holomorphic $q$-forms; we will use often $\Omega_X$ instead of $\Omega^1_X$. $T_X$ will always denote the tangent bundle (sheaf) of $X$ and for a smooth subvariety $Z \subset X$, $N_{Z/X}$ will denote the conormal bundle. The restriction of a coherent sheaf $F$ over $X$ to $Z$ will be denoted by $F|_Z$. We will frequently identify divisors on $X$ with line bundles associated to them. For a line bundle $O(1)$ and a coherent sheaf $F$ on $X$ by $F(t)$ we will denote $F \otimes O(1)^{\otimes t}$.

Stability or semi-stability is understood in the sense of Mumford—Takemoto, so a vector bundle (= locally free sheaf) $E$ is (semi-)stable with respect to an ample line bundle $L$ if for all coherent sheaves $F \subset E$, $0 < rk F < rk E$, it holds

$$\mu(F) < \mu(E)$$

(respectively, $\mu(F) \leq \mu(E)$), where $\mu(F) = \frac{c_1(F).L^{n-1}}{rk(F)}$.

If $b_2(X) = 1$ and the ample line bundle is not mentioned explicitly we mean of course “the” ample generator of $PicX$.

A Fano manifold has by definition ample anticanonical bundle $-K_X = det(T_X)$. Its index is the largest positive integer $r$ such that $\frac{1}{r}K_X \in PicX$. The coindex $c$ is given by $c = dimX - r + 1$. By the Kobayashi-Ochiai theorem $c$ is always non-negative, $c = 0$ iff $X = P^n$, and $c = 1$ iff $X = Q^n$, the $n$-dimensional smooth quadric.

In Section 2 we will use informations on various analytic cohomology groups of Grassmannians, in particular for $G(1, 5)$, the Grassmannian of lines in $P^5$.

Lemma 0.1. Let $G = G(1, n)$ be the Grassmann variety of lines in projective $n$-space. For a real number $s$ we denote by $[s]$ its round up, by $\lfloor s \rfloor$ its round down. Let $O(1)$ be the ample generator of $Pic(G)$. Then $H^p(G, \Omega^q_{G}(t)) \neq 0$ if and only if one of the following conditions is satisfied

1. $t = 0$ and $p = q$
2. $p = 0$ and $t \geq \min(q + 1, \lceil \frac{q}{2} \rceil + 2)$
3. $p = dimG = 2(n - 1)$ and $t \leq \max(-2n + q + 1, \lceil \frac{q}{2} \rceil - n - 1)$
4. $0 < p < 2(n - 1)$, $p + 1 \leq t \leq n - p$ and $q = 2p + t - 1$
5. $0 < p < 2(n - 1)$, $-p - 1 \leq t \leq -n + p$ and $q = 2p + t - 2n + 3$.

The proof of (0.1) can be deduced from [La, Cor. 4.1], see also [We, 2.4] and [Sn].

In section 1 we will discuss some extensions of the following result of Flenner.

Theorem 0.2. [F, Satz 8.11] Let $X$ be a weighted complete intersection of dimension $n$ in a weighted projective space. Let $O(1)$ denote the restriction to $X$ of the universal $O(1)$-sheaf from the weighted projective space. If $X$ is smooth then

(A) $H^q(X, \Omega^q_X) = \mathbb{C}$ for $0 \leq q \leq n$, $2q \neq n$, and

(B) $H^p(X, \Omega^q_X(t)) = 0$ in the following situations:

(a) $0 < p < n$, $p + q \neq n$, $p \neq q$;
(b) $0 < p < n$, $p + q \neq n$, $t \neq 0$;

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(c) \( p + q > n, t > q - p \);
(d) \( p + q < n, t < q - p \).

The knowledge of the above cohomology allows to set the stability of the tangent bundle, so that we have the following:

**Corollary 0.3.** Let \( X \) be as above. Assume moreover that \( n \geq 3 \). Then \( \Omega_X \) (as well as \( T_X \)) is stable.

The argument to prove the corollary is typical and will be used several times throughout the paper: For \( n \geq 3 \) we have \( b_2(X) = 1 \) and thus we may assume that \( det \Omega_X \equiv O(-r) \) and moreover that \( r \leq n - 1 \) because otherwise \( X \) is either a projective space or a quadric. By \( d \) det us denote the self-intersection \( (O(1))^n \). Let \( F \subset \Omega_X \) be a subsheaf of \( \Omega_X \) of rank \( q, 1 \leq q < n = rk \Omega_X \) and \( det F \equiv O(k) \). Then

\[
\mu(\Omega) = -\frac{r \cdot d}{n} \quad \text{and} \quad \mu(F) = \frac{k \cdot d}{q}
\]

hence we are to prove that

\[
-k > q \cdot \frac{r}{n}.
\]

We may assume that \( F \) is reflexive so that \( det F \subset \Omega_X^q \) is an invertible subsheaf. Therefore we have a non-zero section in \( \Omega_X^q (-k) \) and from the previous theorem, point (d), it follows that \( -k \geq q \). Since \( r/n < 1 \) this proves the desired inequality.

1. Vanishing theorems on complete intersections and cyclic coverings.

In the present section we discuss some results on vanishing of cohomology of sheaves of twisted holomorphic differential forms and their wedge products. To make citations easier let us introduce the following

**Definition.** A pair \((Y, O(1))\) consisting of a projective manifold \( Y \) of dimension \( \geq 3 \) and an ample line bundle \( O(1) \) over \( Y \) has special cohomology if the following conditions are satisfied

(a) for \( 0 < p < dim Y, p + q \neq dim Y \) and \( t \in Z \setminus \{0\} \)

\[
H^p(Y, \Omega_Y^q(t)) = 0,
\]

(b) for \( 0 < p < dim Y, p + q \neq dim Y, p \neq q \)

\[
H^p(Y, \Omega_Y^q) = 0
\]

(c) for \( 0 \leq p \leq dim Y, 2p \neq dim Y \)

\[
H^p(Y, \Omega_Y^p) = C.
\]

Note that, due to the well-known Bott formulae, the pair \((P^n, O(1))\) has special cohomology. Flenner in [F, Satz 8.11] proved that smooth weighted complete intersections in weighted projective spaces have special cohomology, too. Our point is to prove that the class of manifolds having special cohomology contains also complete intersections in them and cyclic coverings of them.
Theorem 1.1. Let $Y$ be a projective manifold of dimension $n+1$ and $\mathcal{O}(1)$ an ample line bundle over $Y$. Assume that the pair $(Y, \mathcal{O}(1))$ has special cohomology. Let $X \subset Y$ be a smooth divisor from a linear system $|\mathcal{O}(d)|$. Then the pair $(X, \mathcal{O}_X(1))$ has also special cohomology.

Remark. If $p+q > n$ and $t > 0$, or $p+q < n$ and $t < 0$, the vanishing in the condition (a) of the above definition is immediate from the Kodaira-Nakano vanishing theorem. Also let us note that for $t = 0$ and $p+q < n$ we have an isomorphism

$$H^p(Y, \Omega^q_Y) \to H^p(X, \Omega^q_X)$$

(restricting of $(p,q)$-forms) which follows from Lefschetz hyperplane section theorem. Thus, the pair $(X, \mathcal{O}_X(1))$ as in the theorem satisfies the conditions (b) and (c) from the above definition and only (a) has to be proved outside the range which is settle Kodaira-Nakano.

Also, let us note that the theorem is clearly true for $q = 0$; this follows from the cohomology of the sequence

$$0 \to \mathcal{O}_Y(t-d) \to \mathcal{O}_Y(t) \to \mathcal{O}_X(t) \to 0.$$

Before proving the theorem let us recall two exact sequences of sheaves, the first one on $Y$:

$$0 \to \Omega^q_Y(t) \to \Omega^q_Y(t+d) \to \Omega^q_{Y|X}(t+d) \to 0 \quad (1)$$

and the next one which is a twisted exterior power of cotangent—conormal sequence on $X$:

$$0 \to \Omega^q_X(t) \to \Omega^{q+1}_Y|X(t+d) \to \Omega^{q+1}_X(t+d) \to 0. \quad (2)$$

We will use these sequences to express cupping with $c_1(\mathcal{O}(d))$, compare with [Mt, (2.2)] and also [So, (0.7)]:

Lemma 1.2. The composition of appropriate maps on cohomology of the above sequences:

$$H^{p-1}(Y, \Omega^{q-1}_Y) \to H^{p-1}(X, \Omega^{q-1}_{Y|X}) \to H^{p-1}(X, \Omega^{q-1}_X)$$

$$\to H^{p-1}(X, \Omega^q_{Y|X}(d)) \to H^p(Y, \Omega^q_Y)$$

is cupping with $c_1(\mathcal{O}(d))$ (and thus is an isomorphism for $p + q < n + 1$).

Proof of lemma. The above sequences come by wedging those for $q = 1$ so first we examine the situation for $q = 1$. Let us assume that the divisor $X$ is defined locally on $Y$ on a covering $(U_i)$ by functions $(f_i)$. The morphism $H^0(Y, \mathcal{O}) \cong H^0(X, \mathcal{O}) \to H^0(X, \Omega^1_{Y|X}(d))$ in (2) is defined by differentials $(df_i)$ which glue to a section over $X$. Since the first map in (1) is locally defined by multiplying by $f_i$, the boundary map in the cohomology of (1) associates to the unit section the following 1-cocycle

$$(df_i - df_j)/f_i = (d(log(f_i/f_j))).$$
This, however, is the first Chern class of the divisor $X$, see [H, exercise III.7.4] and [Mt].

We check that this description extends for other $p$ and $q$: for a Čech cocycle $\alpha \in \check{Z}^{p-1}((U_i), \Omega^q_Y)$ the composition of first 3 arrows associates a cocycle which on $U_{i_0} \cap \ldots \cap U_{i_{p-1}}$ assumes value $df_{i_0} \wedge \alpha_{i_0 \ldots i_{p-1}}$ (note that $df_{i_0}$ is restricted to the intersection $U_{i_0} \cap \ldots \cap U_{i_{p-1}}$). The boundary map then gives the cocycle

$$\frac{df_{i_1}}{f_{i_1}} \wedge \alpha_{i_1 \ldots i_p} = \sum_{k=1}^{k=p} (-1)^{k+1} \frac{df_{i_k}}{f_{i_k}} \wedge \alpha_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_p}$$

which, since $\alpha$ is a cocycle, is equal to

$$\left(\frac{df_{i_1}}{f_{i_1}} - \frac{df_{i_0}}{f_{i_0}}\right) \wedge \alpha_{i_1 \ldots i_p}$$

and this is just the $\wedge$-product of the first Chern class of $X$ with the cocycle $\alpha$.

**Remark.** Alternatively, one may use the Dolbeault complex to prove the above lemma.

**Proof of Theorem 1.1.** First let us note that we may assume $p + q < n$ because then the other case is set by Serre duality.

Since we know the vanishing of $H^p(X, \Omega_Y^q(t))$ for $t < 0$ and $p + q < n$ we can proceed by induction on $t$. Namely, to prove the vanishing of $H^p(X, \Omega_Y^q(t + d))$, $t + d > 0$, we may assume that either $t = 0$ or $H^{p+1}(X, \Omega_Y^q(t)) = 0$.

Consider Diagram 1 with exact row and columns. The horizontal arrows in the diagram are coming from the cohomology of (2), so are arrows in the second and fifth (counting from the left) columns of the diagram (for appropriate $p$, $q$ and $t$). The other two columns are using arrows from (1).

From our assumptions concerning cohomology of $\Omega_Y^q(t)$ it follows that (remember that $t + d$ is assumed to be positive)

$$H^p(\Omega_Y^q(t + d)) = H^{p+1}(\Omega_Y^q(t + d)) = 0,$$

and thus

$$H^{p+1}(\Omega_Y^q(t)) \simeq H^p(\Omega_{Y|X}^q(t + d)).$$

If $t \neq 0$ then also $H^{p+1}(\Omega_Y^q(t)) = 0$ and by inductive assumption $H^{p+1}(\Omega_Y^{q-1}(t)) = 0$ which yields the desired vanishing of $H^p(\Omega_Y^q(t + d))$

If $t = 0$ then the map

$$H^p(\Omega_Y^{q-1}) \to H^p(\Omega_{Y|X}^q(d)) \simeq H^{p+1}(\Omega_Y^q)$$

is just a part of the cupping (3) described in lemma 1.2 so that it is surjective. On the other hand, because of the hard Lefschetz theorem the cupping map

$$H^{p+1}(\Omega_Y^{q-1}) \simeq H^{p+1}(\Omega_X^{q-1}) \to H^{p+1}(\Omega_{Y|X}^q(d)) \to H^{p+2}(\Omega_Y^q)$$

is injective. This yields the desired vanishing in this case, too.

As a corollary we obtain a special case of a theorem of Flenner [F, 8.11]:
Corollary 1.3. Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of dimension $n$. Then for $0 < p < n$, $p + q \neq n$ and $t \in \mathbb{Z} \setminus \{0\}$ we have

$$H^p(X, \Omega^q_X(t)) = 0.$$  

Theorem 1.4. Let $X$ and $Y$ be as in the previous theorem ($n \geq 3$) If $\Omega_Y$ is semistable then $\Omega_X$ is stable.

Proof. We may assume $\text{det} \Omega_Y \equiv \mathcal{O}(-s)$ and $s \geq d > 0$, because otherwise $X$ is of general type and the stability of $\Omega_X$ is easily set by Kodaira–Nakano vanishing. Thus $Y$ is Fano and one may assume that $\text{Pic}Y \simeq \mathbb{Z}[\mathcal{O}(1)]$. Also, we may assume that $s \leq n + 2$ and $(s, d) \neq (n + 2, 1), (n + 1, 1)$ since these boundary cases are clear and related to the hyperplane sections of the projective space $Y = \mathbb{P}^{n+1}$ and of the smooth quadric $Y = Q^{n+1}$.

Now we will use a result of Maruyama [Mr, 2.6.1] which asserts that semistability is preserved under the operation of taking wedge powers. Therefore, from the semistability of $\Omega_Y$ it follows that

$$H^0(Y, \Omega^q_Y(t)) = 0 \text{ for } t < \frac{q \cdot s}{n + 1}.$$  

Since $\text{Pic}X \simeq \mathbb{Z}[\mathcal{O}(1)]$, to prove the stability of $\Omega_X$ we have to show

$$H^0(X, \Omega^q_X(t)) = 0 \text{ for } t \leq \frac{q \cdot (s - d)}{n}.$$  

Consider Diagram 2 which is a version of the diagram used to prove Theorem 1.1. We argue as before. Namely, because of our assumptions on $n$, $s$ and $d$,

$$\frac{q \cdot s}{n + 1} > \frac{q \cdot (s - d)}{n}$$  

and the semistability of $\Omega_Y$ implies vanishing of $H^0(\Omega^q_Y(t))$, for $t$ in question. So we have only to discuss the situation for $t = 0$ or $t = d$ because only then the groups neighbouring $H^0(\Omega^q_X(t))$ are non-zero. But we make exactly the same argument as before, using cupping, to conclude the appropriate vanishing.

As a corollary we get the following result for complete intersections in projective spaces which completes a partial result obtained by Subramanian in [Sb].

Corollary 1.5. Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of dimension $n$, $n \geq 3$. Then the cotangent bundle of $X$ is stable.

Now we make similar argument for cyclic coverings of manifolds with special cohomology. First we recall some generalities.

Let $\mathcal{L}$ be a line bundle over $Y$ and $D \subset Y$ a smooth divisor from $|\mathcal{L}^\otimes k|$. By $L$ we denote the total space $L := \text{Spec}(\text{Sym}\mathcal{L}^\vee)$ of the bundle $\mathcal{L}$ with the projection $\bar{\pi} : L \to Y$. We have a natural coordinate $\xi$ on fibers of $\bar{\pi}$ coming from the zero section $Y \to Y_0 \subset L$. Note that $\xi$ is a section of $\bar{\pi}^* \mathcal{L}$ vanishing along $Y_0$, in other words $\xi$ descends to $Y$ as
the section of $\tilde{\pi}_*(\tilde{\pi}^*\mathcal{L}) = \mathcal{L} \oplus \mathcal{O} \oplus \mathcal{L}^\vee \ldots$ associated with the $\mathcal{O}$-summand. Let us assume that the divisor $D$ is defined on a covering $(U_i)$ by functions $(f_i)$. Then a cyclic covering associated to $D$ is a variety $X$ defined locally in each of $\tilde{\pi}^{-1}(U_i)$ by the equation $\xi_i^k = f_i$, where $\xi_i$ is the restriction of $\xi$ to $\tilde{\pi}^{-1}(U_i)$. These equations patch up nicely and define $X$ globally with a projection $\pi : X \to Y$. For more information see [BPV], Section I.17.

(A cyclic covering is a special $k$-section of the line bundle $\mathcal{L}$. Let us recall that $\pi : X \to Y$ is a $k$-section of a line bundle $\mathcal{L}$ if $X$ embeds over $Y$ into the total space bundle $\mathcal{L}$.)

**Theorem 1.6.** Let us assume that the pair $(Y, \mathcal{O}(1))$ has special cohomology. By $\mathcal{L}$ let us denote the line bundle $\mathcal{O}$, where $\mathcal{O}$ is a divisor from $[\mathcal{L}^\otimes k]$. If $\mathcal{O}_X(1)$ is the pull-back of $\mathcal{O}(1)$ to $X$ then the pair $(X, \mathcal{O}_X(1))$ has special cohomology.

Proof is very similar to the proof of the Theorem 1.1. First, we want to understand cupping, now in $L$. For this purpose we consider a sequence of normal — conormal sheaves on $L$:

$$0 \to \tilde{\pi}^*\Omega_Y \to \Omega_L \to \Omega_{L/Y}^1 \otimes \tilde{\pi}^*\mathcal{L}^{-1} \to 0$$

(4)

where $\tilde{\pi} : L \to Y$ is the projection. If $(U_i)$ is a covering trivializing both $\mathcal{L}$ and $\Omega_Y$ the sequence splits as follows

$$(dy_i) \mapsto (dy_i, d\xi_i) \mapsto (d\xi_i)$$

where $y_i$ are coordinates in $U_i$.

If we twist the above sequence by $\tilde{\pi}^*(\mathcal{L}^t)$, restrict to $X$ and project to $Y$ the result is

$$0 \to \Omega_Y \otimes (\mathcal{L}^t \oplus \mathcal{L}^{t-1} \oplus \ldots \oplus \mathcal{L}^{t-k+1}) \to \pi_* (\Omega_{L/X}) \otimes \mathcal{L}^t \to \mathcal{L}^{t-1} \oplus \mathcal{L}^{t-2} \oplus \ldots \oplus \mathcal{L}^{t-k} \to 0$$

(5)

**Lemma 1.7.** For $t = 1 \ldots k - 1$ the split maps and the above sequence induce a map

$$H^0(Y, \mathcal{O}) \to H^1(Y, \Omega_Y \otimes (\mathcal{L}^t \oplus \mathcal{L}^{t-1} \oplus \ldots \oplus \mathcal{L}^{t-k+1})) \to H^1(Y, \Omega_Y)$$

which defines the first Chern class of $\mathcal{L}$ (that is, the image of the unit 1 $H^0(Y, \mathcal{O}_Y)$ is a multiple of the first Chern class of $\mathcal{L}$). For $t = k$ the induced map

$$H^0(Y, \mathcal{O}) \to H^1(Y, \Omega_Y \otimes (\mathcal{L}^t \oplus \mathcal{L}^{t-1} \oplus \ldots \oplus \mathcal{L}^{t-k+1}))$$

is trivial.

**Proof of lemma.** Any section of $\tilde{\pi}^*(\mathcal{L}^t) = \mathcal{O}(tY_0)$ over $\tilde{\pi}^{-1}(U)$ can be written as

$$\sum_{s \geq -t} g_{is} \xi_i^s$$

where $g_{is} \in \Gamma(U, \mathcal{L}^{-s})$, see [BPV, I.17.2]. Thus, since $(U_i)$ trivializes $\mathcal{L}$, any section of $\Omega_{L/Y} \otimes \tilde{\pi}^*\mathcal{L}^t$ over $\tilde{\pi}^{-1}(U_i)$ can be written as

$$\sum_{s \geq -t} g_{is} \xi_i^s d\xi_i$$

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where \( g_{is} \in \Gamma(U_i, \mathcal{O}_Y) \).

Over \( X \subset L \) we have the following two relations

\[
\xi_i^k = f_i \quad \text{and} \quad k \xi_i^{k-1} d \xi_i = df_i
\]

so that any section of \( \Omega^*_L/X \otimes \pi^* L \) over \( \pi^{-1}(U_i) \) can be written as

\[
-t+k-2 \sum_{s \geq -t} g_{is} \xi_i^s d \xi_i + g_{is}^0 \xi^{-t} df_i.
\]

The split map

\[
L^s \to \pi_* (\Omega^*_L/X) \otimes L^t \cong L^{t-1} \oplus \ldots \oplus L^{t-k+1}
\]

for \( t-1 \leq s \leq t-k+1 \) associates to the “unit” section of \( L^s \) over \( U_i \) the section

\[
\xi_i^{-s-1} d \xi_i
\]

while for \( s = t - k \) the result is

\[
\xi_i^{k-t-1} d \xi_i = \frac{1}{k} \xi^{-t} df_i.
\]

Consequently, the map

\[
H^0(Y, \mathcal{O}) \to H^1(Y, \Omega_Y \otimes (L^t \oplus L^{t-1} \oplus \ldots \oplus L^{t-k+1}))
\]

for \( t = 1 \ldots k \) can be described as follows

\[
1 \mapsto \left( \frac{d \xi_i}{\xi} - \frac{d \xi_j}{\xi} \right) = \frac{1}{k} \left( \frac{df_i}{f_i} - \frac{df_j}{f_j} \right) = \frac{1}{k} d(\log(\frac{f_i}{f_j})).
\]

For \( t = k \) the above cocycle is trivial as it is the Čech boundary of

\[
\left( \frac{df_i}{f_i} \right) \in \check{C}^0((U_i), \Omega_Y \otimes L^k).
\]

For \( t \leq k - 1 \), however, the resulting cocycle is non-trivial and it defines the first Chern class of \( L \) so that we are done with the lemma.

Now we consider an exterior power of the sequence (4), we restrict it to \( X \), project it to \( Y \) and twist by a multiple of \( \mathcal{O}(1) \). The result is

\[
0 \to \Omega^0_Y(t) \oplus \Omega^0_Y(t-d) \oplus \ldots \oplus \Omega^0_Y(t-(k-1)d) \to \pi_* (\Omega^0_L/X)(t)
\]

\[
\to \Omega^0_Y(t-d) \oplus \Omega^0_Y(t-2d) \oplus \ldots \oplus \Omega^0_Y(t-kd) \to 0.
\]

Arguing as in the proof of Lemma 1.2 we have
Lemma 1.8. For $t = d, 2d, \ldots (k - 1)d$ the map induced from (6)

$$H^{p-1}(\Omega_Y^{q-1}) \to H^p(\Omega_Y^q)$$

is the cupping with the first Chern class of $L$.

Lemma 1.8 implies that some nonzero cohomology groups of $\Omega_Y^q$ do not con to the cohomology of $\Omega_{L|X}$. Thus, as a corollary we get the follo

Lemma 1.9. In the situation of Theorem 1.6 for $p + q < \text{dim} Y$, $p > 0$

(i) $H^p(X, \Omega_{L|X}^q(t)) = 0$ if $t \neq 0, kd$.

(ii) the induced map $H^p(Y, \pi_*(\Omega_{L|X}^q)(kd)) \to H^p(Y, \Omega_Y^{q-1})$ is injective (and bijective for $p + q < \text{dim} Y - 1$.)

(iii) the map $H^p(Y, \Omega_Y^q) \to H^p(Y, \pi_*(\Omega_{L|X}^q))$ is bijective.

To conclude Theorem 1.6 we use the following sequence which is a version of (2) for $X \subset L$

$$0 \to \Omega_{X}^{q-1}(t) \to \Omega_{L|X}^q(t + kd) \to \Omega_{X}^q(t + kd) \to 0. \quad (7)$$

We proceed by induction with respect to $t$ : we may assume that the vanishing is $\Omega_{X}^{q-1}(t)$ (for $t$ negative this is Kodaira—Nakano vanishing) and we want to conclude the vanishing for

For $t \neq -kd, 0$ and $p + q < \text{dim} X, p > 0$ we have by (1.9)

$$H^{p+1}(X, \Omega_{X}^{q-1}(t)) = H^p(X, \Omega_{L|X}^q(t + kd)) = 0.$$

For $t = -kd$ we have from 1.9(iii) that $H^p(X, \Omega_{X}^q) = H^p(Y; \Omega_Y^q)$ For $t = 0$ the induced map from the cohomology sequence of (7)

$$H^p(X, \Omega_{X}^{q-1}) \to H^p(X, \Omega_{L|X}^q \otimes \mathcal{O}(X))$$

is a part of cupping (see the proof of (1.1)), so is injective and thus, because 1.9(ii), it is an isomorphism, therefore $H^p(X, \Omega_{X}^q(kd)) = 0$.

Now we can prove an extension of Theorem 1.4.

Theorem 1.10. Let $\pi : X \to Y$ be a cyclic covering which satisfies the assumptions from Theorem 1.6. If $\Omega_Y$ is semistable and $k \geq 2$ then $\Omega_X$ is stable.

Proof. Again, as in the proof of (1.4), we may assume $K_Y = \mathcal{O}(-s)$ for some positive $s$. Also, we have $\text{dim} Y + 1 \geq s \geq (k - 1)d > 0$ and $(s, k, d) \neq (\text{dim} Y + 1, 2, 1)$ because then we have a double covering of a projective space by a quadric. Then

$$K_X = \mathcal{O}_X(-s + (k - 1)d)$$

and thus we have to prove

$$H^0(X, \Omega_X^q(t)) = 0 \text{ for } t \leq \frac{q \cdot (s - (k - 1)d)}{\text{dim} X}.$$
However, using the semistability of $Y$ (and again Maruyama’s result [Mr, 2.6.1]), and because of (6) we get the vanishing of $H^0(X, \Omega^q_{|X}(t))$ for

$$t < \min \left( \frac{qs}{\dim X}, \frac{(q-1)s}{\dim X} + d \right).$$

On the other hand, in the cohomology of the sequence (7) (twisted by $O(-kd)$) either $H^1(X, \Omega^{q-1}_X(t-kd))$ vanishes or the map $H^1(X, \Omega^q_X \otimes O(X)) \to H^1(X, \Omega^{q+1}_{|X}(t+k))$ is injective (as a part of cupping), so that, since

$$t \leq \frac{q \cdot (s - (k-1)d)}{\dim X} < \min \left( \frac{qs}{\dim X}, \frac{(q-1)s}{\dim X} + d \right)$$

we have the desired vanishing.

**Remark.** Note that the “obvious” estimate of $H^0(X, \Omega^q_X(t))$ in terms of $H^0(\Omega^{q+1}_{|X}(t+k))$ which comes from (7) does not give the vanishing in the range we are interested in.

2. **Stability of the tangent bundle of Fano manifolds with $b_2 = 1$.**

Let $X$ always denote a Fano manifold with $b_2(X) = 1$ and index $r$. In this section we investigate the problem whether the tangent bundle $T_X$ is stable, at least for large index. Let $H$ be the ample generator of $\text{Pic}(X) = \mathbb{Z}$. For a coherent sheaf $F$ we write for short

$$\mu(F) = \frac{c_1(F)}{rk(F)}$$

instead of

$$\frac{c_1(F) \cdot H^{n-1}}{rk(F)}.$$

If the index $r \geq n$, then either $X \simeq \mathbb{P}^n$ or $X \simeq \mathbb{Q}^n$, the $n$-dimensional quadric, and hence $T_X$ is well-known to be stable; as a rational homogeneous manifold it even carries a Kähler-Einstein metric. In order to treat the case of lower index it is suitable to have a certain condition (ES). This condition is defined as follows:

(ES) for $k = 1, \ldots, r - 1$ there exist smooth members $H_1, \ldots, H_k \in |H|$ such that $H_1 \cap \ldots \cap H_k$ is a smooth $(n-k)$-fold (hence a Fano manifold of index $r-k$).

The condition (ES) is important for induction purposes and holds for 3-folds (Shokurov), for 4-folds of index 2 (Wilson, [Wi]) and generally for index $n-1$ (= coindex 2) (Fujita).

**Lemma 2.1.** Assume (ES) for the Fano manifold $X$ and let $r \leq n - 1$. Let $F \subset T_X$ be a reflexive subsheaf of $T_X$. Then $\mu(F) < 1$.

**Proof.** Assume $\mu(F) \geq 1$. Let $m = rkF$, $1 \leq m \leq n - 1$. Choose $H_1 \in |H|$ smooth and let $G_1 = (F|_{H_1})^{**}$. Then we have a morphism

$$\varphi_1 : G_1 \to T_{X|H_1}.$$
which is generically injective, hence injective (a priori \( \varphi_1 \) exists only outside codimension 2, then extend). Via the epimorphism

\[ T_{X|H_1} \longrightarrow H_{|H_1}, \]

\( \varphi_1 \) induces a map \( \alpha_1 : G_1 \longrightarrow H_{|H_1} \). Now either \( \alpha_1 = 0 \), then we can realize \( G_1 \) as a subsheaf of \( T_{H_1} \), or \( \alpha_1 \neq 0 \), and then \( c_1(\text{Im } \alpha_1) = \lambda H \mid H_1 \) for some \( \lambda \leq 1 \), hence \( \text{Ker } \alpha_1 \)

is a subsheaf of \( T_{H_1} \) with \( c_1(\text{Ker } \alpha_1) \geq c_1(F_{|H_1}) - 1 \) (note that \( \text{Pic}(X) = \mathbb{Z} \) by Lefschetz, and consider Chern classes as numbers). In this second case we have \( \mu(\text{Ker } \alpha_1) \geq 1 \).

Let \( F_1 = \text{Ker } \alpha_1 \) in both cases and repeat the construction for \( F_1 \subset T_{H_1} \). After \( r - 1 \) steps we end up with a Fano manifold \( Y = H_1 \cap \ldots \cap H_{r-1} \) of index 1, \( \dim Y = n - r + 1 \), and a torsion free sheaf \( F' \subset T_Y \). Observe that at each step we have \( \mu(F_i) \geq 1 \) and \( c_1(F_i) \leq \dim H_i - 1 \), hence it follows immediately that always \( \text{rk } F_i \leq \dim H_i - 1 \). Let \( k \) be the number of steps with \( \alpha_i \neq 0 \). Then \( c_1(F') \geq c_1(F) - k \) with \( \text{rk}(F') = \text{rk}(F) - k \), or \( F' = 0 \).

First assume \( F' \neq 0 \). By Reid (2.2 below), \( T_Y \) is stable, hence we obtain

\[ \frac{c_1(F) - k}{m - k} \leq \frac{1}{n - r + 1} < 1, \]

hence \( c_1(F) < m \), so that \( \mu(F) < 1 \).

If \( F' = 0 \), then in particular \( k = m \), thus \( r - 1 > m \). Then we stop at the step where \( \text{rk}(F_i) = 1 \). By a theorem of Wahl, [Wa], (or by [MS, Thm. 8]) we have \( c_1(F_i) \leq 0 \). Let \( k' \) be the number of steps between the first and the \( i \)-th where the rank drops, so \( k' = m - 1 \). Then \( c_1(F_i) \geq c_1(F) - k' \), hence \( c_1(F) \leq m - 1 \), proving our claim.

For the proof of (2.1) we used

**Proposition 2.2.** (Reid [Re]) Let \( X \) be a Fano manifold of dimension \( n \) with \( b_2(X) = 1 \). Let \( G \subset T_X \) be a proper reflexive subsheaf. Then \( c_1(G) < c_1(X) \) i.e. \( c_1(X) - c_1(G) \) is ample. In particular, \( T_X \) is stable if the index of \( X \) is 1.

**Proof.** Indeed, otherwise we would have a subsheaf \( G \) of rank \( p < n \) with \( c_1(G) \geq c_1(X) \), hence we obtain a non-zero map \( \text{det } G \rightarrow \Lambda^p T_X \). Consequently

\[ 0 \neq H^0(X, \Lambda^p T_X \otimes \text{det } G^*) = H^0(X, \Omega_X^{n-p} \otimes \text{det } G^* \otimes \text{det } T_X), \]

which contradicts Kodaira-Nakano vanishing if \( c_1(G) > c_1(X) \). In case of equality \( c_1(G) = c_1(X) \), use also Hodge duality to obtain a contradiction with vanishing of \( H^{n-p}(X, \mathcal{O}_X) \).

**Theorem 2.3.** Every del Pezzo manifold \( X \) (i.e. \( X \) is a Fano \( n \)-fold of index \( n - 1 \)) with \( b_2 = 1 \) has stable tangent bundle.

**Proof.** Since \( \mu(T_X) = 1 - 1/n \), \( n = \dim X \), this follows from (2.1). Note that (ES) is automatic in this case by Fujita’s result.

In particular every Fano 3-fold with \( b_2 = 1 \) has stable tangent bundle. Since the only Fano \( n \)-folds with index > \( n - 1 \) are projective space and quadric which have stable tangent bundle we now turn to Fano \( n \)-folds with index \( r = n - 2 \) (i.e. coinvariant 3).
Proposition 2.4. Let \( r = n - 2 \) and \( n \geq 4 \). Assume (ES) and the existence of a smooth member \( H_1 \in |H| \) such that \( T_{H_1} \) is stable. Then \( T_X \) is stable or there exists a reflexive subsheaf \( F \subset T_X \) with \( rkF = m \), \( n = 2m \) and \( \mu(F) = \mu(T_X) \). Hence \( T_X \) is semi-stable and stable for \( n \) odd.

Proof. Let \( F \subset T_X \) be a proper reflexive subsheaf. Assume \( \mu(F) \geq \mu(T_X) \), so \( c_1(F)/m \geq (n - 2)/n \). Since \( \mu(F) < 1 \) by (2.1), we get \( c_1(F) < m \), and consequently from both inequalities: \( c_1(F) = m - 2 \) and \( 2m \geq n \).

Now we have a closer look to the proof of lemma (2.1) and use the same notations. If \( \alpha_1 : G_1 \to H_{|H_1} \) is 0, then \( G_1 \subset T_{H_1} \), hence by our assumption we obtain

\[
\mu(F) = \mu(G_1) < \mu(T_{H_1}) < \mu(T_X).
\]

So we may assume \( \alpha_1 \neq 0 \).

Let \( F_1 = \ker \alpha_1 \). Then \( \mu(F_1) \geq \frac{m - 2}{m - 1} \). On the other hand by assumption we have

\[
\mu(F_1) < \frac{n - 3}{n - 1} = \mu(T_{H_1}).
\]

Hence \( 2m < n + 1 \), and we conclude \( 2m = n \), and \( \mu(F) = \mu(T_X) \).

A direct consequence of 2.4 is

Proposition 2.5. Let \( X \) be a Fano \( n \)-fold of coindex 3

1. If \( n = 4 \) and \( H^0(X, \Omega^2_X(1)) = 0 \), then \( T_X \) is stable.
2. Let \( n = 6 \) and assume that (ES) holds. If \( H^0(X, \Omega^3_X(2)) = 0 \) and there exists smooth \( H_1 \in |H| \) such that \( T_{H_1} \) is stable then \( T_X \) is stable.

For the proof just observe that (ES) holds for 4-folds of index 2 by Wilson [Wi].
for \(3 \leq m \leq \frac{n}{2} + 1\), then \(T_X\) is stable.

**Proof.** We may assume that for \(m < n\) the bundle \(T_{X_m}\) is stable and \(T_X\) is not. Then we can perform the construction from the proof of 2.1. In the last part of the proof we considered \(F_i\) which was the “last non-zero sheaf” produced by inductive “slicing”:

\[ c_1(F_i) \geq c_1(F) - rk(F) + 1. \]

Since by the main theorem of [Wa] (see also [MS]) \(c_1(F_i) \leq 0\), it follows from 2.4 that \(c_1(F_i) = 0\) and thus \(F_i\) produces a non-zero section in \(T_{X_m}\) for some \(m \leq n - rk(F) + 1\).

As a direct consequence of 2.6 we obtain

**Proposition 2.7.** Assume \(n = 4\) and \(7 \leq g \leq 10\). Then \(T_X\) is stable.

**Proof.** By [Pr] we have \(H^0(T_Y) = 0\) for every Fano 3-fold \(Y\) with \(7 \leq g \leq 10\). Another proof is easily obtained by applying lemma 2.9.a below.

**Corollary 2.8.** Every Fano \(n\)-fold of coindex 3 satisfying (ES) with \(n \leq 5\) and \(g \geq 7\) has stable tangent bundle.

**Proof.** Apply (2.4).

We now deal with all the cases \(6 \leq g \leq 10\) separately. First notice that Fano \(n\)-folds of coindex 3 and \(g = 10\) have dimension at most 5, so this case is already settled. As to \(g = 6\) we have:

**Proposition 2.9.** If \(g = 6\), then \(T_X\) is stable.

**Proof.** In this case, the manifold \(X\) we are to deal with (because of (2.4)) can be described as follows. Either

a) \(X\) is a complete intersection of type (2,1) in \(G = G(1,4)\) or

b) \(X\) is a 2 : 1 covering of \(G\) or of a 4-dimensional linear section of \(G\).

In case b) the covering is easily seen to be cyclic. Using the exact sequences (6) and (7) from Sect.1 and Kodaira-Nakano vanishing we see immediately that it is sufficient to have

\[ H^0(Y, \Omega_Y^3(2)) = 0 \quad \text{or, respectively,} \quad H^0(Y, \Omega_Y^2(1)) = 0 \]

for \(Y = G = G(1,4)\) or, respectively, for \(Y\) a 4-dimensional linear section of \(G\). But this follows from the stability of \(TY\) since \(Y\) is of coindex 2.

In case a) we have \(\dim X = 4\) and by (2.5) it is sufficient to prove

\[ H^0(X, \Omega_X^3(1)) = 0. \]

We have \(X\) contained in a hypersurface \(Y\) of degree 2 in \(G\), so \(X\) is a linear section of \(Y\). Applying lemma 2.9.a below first to \(Y \subset G\) and then to \(X \subset Y\) we obtain

\[ H^0(Y, \Omega_Y^3(1)) = H^0(X, \Omega_X^3(1)) = 0. \]

The vanishings needed for applying 2.9.a are \(H^1(G, \Omega_G^2) = 0 = H^1(Y, \Omega_Y^2)\), which clearly hold.
Lemma 2.9.a. Let $Y$ be a projective manifold with $\text{Pic}(Y) = \mathbb{Z}$. Denote by $\mathcal{O}(1)$ the ample generator. Let $X$ be a smooth member of the linear system $|\mathcal{O}(d)|$ for some $d > 0$. Fix a number $q < \dim Y - 1$ such that $H^1(\Omega^q_Y) = 0$. Then the restriction map

$$H^0(Y, \Omega^q_Y(c)) \to H^0(X, \Omega^q_X(c))$$

is surjective for all $c \leq d$.

**Proof.** We use the following exact sequences (1) and (2) from sect.1.

$$0 \to \Omega^q_Y(c - d) \to \Omega^q_Y(c) \to \Omega^q_{Y|X}(c) \to 0$$

and

$$0 \to \Omega^{q-1}_X(c - d) \to \Omega^q_Y(c) \to \Omega^q_X(c) \to 0.$$ 

First let $c < d$. Then it is sufficient to have

$$H^1(X, \Omega^{q-1}_X(c - d)) = 0 \quad (a_1)$$

$$H^1(Y, \Omega^q_Y(c - d)) = 0 \quad (a_2)$$

Both claims follow from Kodaira-Nakano vanishing, since $q < \dim X$.

If $c = d$, then $(a_2)$ is satisfied by assumption, however $(a_1)$ has to be replaced by the weaker statement that the natural map

$$H^1(X, \Omega^{q-1}_X) \to H^1(X, \Omega^q_Y|X(d))$$

is injective, which follows from (1.2) and Lefschetz.

**Corollary 2.10.** Every Fano 4-fold with $b_2 = 1$ has stable tangent bundle. Every Fano 5-fold with $b_2(X) = 1$ (satisfying (ES)) has stable tangent bundle except possibly for those of index 2.

We now turn to $g = 8$. Then either $X$ is the 8-fold $G(1, 5)$ or its linear section. $G(1, 5)$ having stable tangent bundle as a rational homogeneous manifold (or by (0.1)) it is sufficient by (2.4) to consider 6-dimensional sections.

**Proposition 2.11.** Let $X$ be a 6-dimensional linear section in $G = G(1, 5)$. Then $T_X$ is stable.

**Proof.** By (2.5) it is sufficient to show $H^0(\Omega^3_X(2)) = 0$. Let $Y$ be a linear section in $G$ and $X$ a linear section of $Y$. From the exact sequence

$$0 \to \Omega^3_X(2) \to \Omega^4_{Y|X}(3) \to \Omega^4_X(3) \to 0 \quad (S)$$

we deduce that it is sufficient to prove

$$H^0(X, \Omega^4_{Y|X}(3)) = 0 \quad (1)$$
This in turn follows from
\begin{align}
H^0(Y, \Omega^4_Y(3)) &= 0 \quad (2) \\
H^1(Y, \Omega^4_Y(2)) &= 0. \quad (3)
\end{align}

We first prove (2). Since \( \Omega^4_Y(3) \) is a subbundle of \( \Omega^5_{G|Y}(4) \), it is sufficient to prove
\[ H^0(Y, \Omega^5_{G|Y}(4)) = 0. \]

This in turn is guaranteed by
\[ H^0(G, \Omega^4_G(4)) = H^1(G, \Omega^4_G(3)) = 0. \]

In order to prove (3) we first observe that tensoring (S) by \( \mathcal{O}(-1) \) and substituting \( X \) by \( Y \), \( Y \) by \( G \), and taking cohomology, it suffices to show
\begin{align}
H^1(Y, \Omega^4_{G|Y}(2)) &= 0 \quad (4) \\
H^2(Y, \Omega^3_Y(1)) &= 0 \quad (5)
\end{align}

Now (4) is immediate since from (0.1) it follows
\[ H^1(G, \Omega^4_G(2)) = H^2(G, \Omega^4_G(1)) = 0. \]

To prove (5) we use an analogous sequence to (S) which yields an exact sequence
\[ H^2(Y, \Omega^2_Y) \xrightarrow{\beta} H^2(Y, \Omega^3_{G|Y}(1)) \xrightarrow{\alpha} H^2(Y, \Omega^3_Y(1)) \rightarrow H^3(Y, \Omega^2_Y). \]

Observe \( H^3(\Omega^2_Y) = 0 \) since \( b_5(Y) = 0 \). Moreover \( h^{2,2}(Y) = h^{2,2}(G) = 2 \) and since
\[ H^2(G, \Omega^3_G(1)) = H^3(G, \Omega^3_G(1)) = 0, \]
we have
\[ H^2(G, \Omega^3_{G|Y}(1)) = H^3(G, \Omega^3_G) = \mathbb{C}^2. \]

The composite map
\[ H^2(G, \Omega^2_G) \rightarrow H^2(Y, \Omega^2_Y) \xrightarrow{\beta} H^2(Y, \Omega^3_{G|Y}(1)) \rightarrow H^3(G, \Omega^3_G) \]
is cupping by the fundamental class of \( Y \subset G \) (Lemma 1.2), hence an isomorphism \( (h^{2,2} = h^{3,3}) \). This proves surjectivity of \( \beta \), hence \( \alpha = 0 \), hence (5).

If now \( g = 9 \), \( X \) is a rational homogeneous 6-fold, thus \( T_X \) is stable. The last case is then \( g = 7 \). Here \( X \) is a 10-dimensional homogeneous manifold, a so-called spinor variety \( S_{10} \) (see [Mu]), hence \( T_X \) is stable, or \( X \) is a linear section of \( S_{10} \). The stability of \( T_X \) of a 6-dimensional section (hence also of a 7-dimensional section by (2.4)) follows from
Lemma 2.12. Let $X_k$ be a smooth $k$-dimensional linear section of the 10-dimensional spinor variety. If $k \geq 5$ then

$$H^0(X_k, \Omega^3_{X_k}(2)) = H^1(X_k, \Omega^3_{X_k}(1)) = H^1(X_k, \Omega^2_{X_k}(1)) = 0.$$ 

Proof of the lemma is by descending induction: for $k = 10$ this follows from Snow’s computations. Therefore we may assume all vanishing to hold for $k + 1$. Also, we will use the knowledge of the following Betti numbers of $X_{10}$ and thus, by Lefschetz, of $X_{k+1}$: $b_2 = b_4 = 1$, $b_3 = b_5 = 0$.

To prove the last vanishing of the lemma we use an exact sequence of sheaves (see the sequence (2) in the proof of theorem 1.1)

$$0 \to \Omega_{X_k} \to \Omega^2_{X_{k+1}|X_{k}}(1) \to \Omega^2_{X_k}(1) \to 0$$

and the resulting exact sequence of cohomology

$$H^1(X_k, \Omega_{X_k}) \to H^1(X_k, \Omega^2_{X_{k+1}|X_{k}}(1)) \to H^1(X_k, \Omega^2_{X_k}(1)) \to H^2(X_k, \Omega_{X_k}).$$

The last term in the above sequence is 0 since $b_3(X_k) = 0$. The first arrow in the sequence is surjective because it is a part of cupping

$$H^1(X_{k+1}, \Omega_{X_{k+1}}) \to H^1(X_k, \Omega_{X_k}) \to H^1(X_k, \Omega^2_{X_{k+1}|X_{k}}(1)) \to H^2(X_{k+1}, \Omega^2_{X_{k+1}})$$

(see lemma 1.2), the cupping is isomorphic since $b_2(X_{k+1}) = b_4(X_{k+1}) = 1$, moreover the restriction $H^1(X_{k+1}, \Omega_{X_{k+1}}) \to H^1(X_k, \Omega_{X_k})$ is an isomorphism by Lefschetz and the kernel of $H^1(X_k, \Omega^2_{X_{k+1}|X_{k}}(1)) \to H^2(X_{k+1}, \Omega^2_{X_{k+1}})$ is $H^1(X_{k+1}, \Omega^2_{X_{k+1}}(1))$ (see diagram 1) so it vanishes by our inductive assumption. This proves vanishing of $H^1(X_k, \Omega^2_{X_k}(1))$.

To prove vanishing of $H^1(X_k, \Omega^3_{X_k}(1))$ a similar exact sequence of cohomology is applied:

$$H^1(X_k, \Omega^3_{X_{k+1}|X_{k}}(1)) \to H^1(X_k, \Omega^3_{X_k}(1)) \to H^2(X_k, \Omega^2_{X_k}) \to H^2(X_k, \Omega^3_{X_{k+1}|X_{k}}(1)).$$

The last arrow in the above sequence is injective as a part of cupping (see 1.2) while the first term above can be bounded by the cohomology of the following exact sequence (see the sequence (1) from the proof of 1.1):

$$0 \to \Omega^3_{X_{k+1}} \to \Omega^3_{X_{k+1}}(1) \to \Omega^3_{X_{k+1}|X_{k}}(1) \to 0.$$

Indeed, by the inductive assumption $H^1(X_{k+1}, \Omega^3_{X_{k+1}}(1)) = 0$ and because

$$b_5(X_{k+1}) = H^2(X_{k+1}, \Omega^3_{X_{k+1}}) = 0$$

it follows that $H^1(X_{k+1}, \Omega^3_{X_{k+1}|X_{k}}(1)) = 0$. Therefore $H^1(X_k, \Omega^3_{X_k}(1)) = 0.$
Now the vanishing of $H^0(X_k, \Omega^3_{X_k}(2))$ is easy: we have the following two exact sequences of cohomology (cf. Diagram 2):

$$H^0(X_{k+1}, \Omega^3_{X_{k+1}}(2)) \to H^0(X_{k+1}, \Omega^3_{X_{k+1}|X_k}(2)) \to H^1(X_{k+1}, \Omega^3_{X_{k+1}}(1))$$

$$H^0(X_k, \Omega^3_{X_{k+1}|X_k}(2)) \to H^0(X_k, \Omega^3_{X_k}(2)) \to H^1(X_k, \Omega^2_{X_k}(1))$$

and the vanishing follows.

It remains to prove the stability of the tangent bundle of an 8-dimensional linear section $X$. From (2.4) we see that for this purpose it is sufficient to prove

**Lemma 2.13** Let $X$ be an 8-dimensional linear section of the 10-dimensional spinor manifold $S$. Then $H^0(\Omega^1_X(3)) = 0$.

**Proof.** Choose a smooth linear section $Y \subset S$ so For both inclusions we will use several times the exact sequences (1) and (2) from sect.1, with $d = 1$. We will quote these sequences just by (1) and (2). By (2) with $q = 4, t = 3$ it is sufficient to show

$$H^0(\Omega^5_Y(4)|X) = 0$$

Using (1) this comes down to show

$$H^0(\Omega^5_Y(4)) = 0$$

$$H^1(\Omega^5_Y(3)) = 0$$

Using again (2), now for $Y \subset X$, (b) comes down to

$$H^0(\Omega^6_S(5)|Y) = 0.$$ 

This is justified by the vanishings (using (1))

$$H^0(\Omega^6_S(5)) = H^1(\Omega^6_S(4)) = 0$$

which both hold by [Sn2,3]. In order to prove (c) we use (2) with $q = 5, t = 3$ and we have to show

$$H^1(\Omega^6_S(4)|Y) = 0$$

$$H^0(\Omega^6_Y(4)) = 0$$

Now (d) follows by (1) from the vanishings

$$H^1(\Omega^6_S(4)) = H^2(\Omega^6_S(3)) = 0$$

and (e) is verified by

$$H^0(\Omega^7_S(5)|Y)) = 0$$

which in turn is guaranteed by

$$H^0(\Omega^7_S(5)) = H^1(\Omega^7_S(4)) = 0.$$ 

All the needed vanishings on $S$ follow again from [Sn2,3].

In summary we obtain

**Theorem 2.14.** Let $X$ be a $n$-dimensional Fano manifold of coindex 3, with $b_2 = 1$. Assume (ES). Then $T_X$ is stable.
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