CONTACTOMORPHISMS WITH $L^2$ METRIC ON STREAM FUNCTIONS

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Abstract. Here we investigate some geometric properties of the contactomorphism group $D_\theta(M)$ of a compact contact manifold with the $L^2$ metric on the stream functions. Viewing this group as a generalization to the $D(S^1)$, the diffeomorphism group of the circle, we show that its sectional curvature is always non-negative and that the the Riemannian exponential map is not locally $C^1$. Lastly, we show that the quantomorphism group is a totally geodesic submanifold of $D_\theta(M)$ and talk about its Riemannian submersion onto the symplectomorphism group of the Boothby-Wang quotient of $M$.

1. Introduction

Let $M$ be an orientable compact, contact manifold (without boundary) of odd dimension $2n + 1$. Recall that a manifold $M$ is a contact manifold if there exists a 1-form $\theta$ which satisfies the non-degeneracy condition that $\theta \wedge d\theta^n \neq 0$ everywhere\[4\]. We call $\theta$ the contact form. If we let $D(M)$ be the group of diffeomorphisms of $M$, we say that $\eta \in D(M)$ is a contactomorphism if $\eta^*\theta$ is some positive functional multiple of $\theta$. We will denote the contactomorphism group by $D_\theta(M)$. $D_\theta(M)$ can be thought of as an infinite dimensional Riemannian manifold using the framework of Arnold\[1\].

The diffeomorphism group of the circle, $D(S^1)$, has been heavily studied and has interesting applications to fluid mechanics. Depending on the metric, some classical PDE arise as the geodesic equation on $D(S^1)$ such as the right-invariant Burgers’ equation and the Camassa-Holm equation. It was shown that the Riemannian exponential map is not a local $C^1$ map for the $L^2$ metric\[6\]. This is not the case when they considered the $H^1$ metric. Later it was shown that $D(S^1)$ has vanishing geodesic distance for the $H^s$ metric if and only if $s \leq 1/2$\[3, 11\].

The contactomorphism group has been studied before but in many different contexts. Smolentsev\[14, 15\] worked with the quantomorphism group $D_\theta(M)$, which is the group of diffeomorphisms which exactly preserve the contact form, with the bi-invariant $L^2$ metric on stream functions. In \[5, 9\], $D_\theta(M)$ was studied with the $L^2$ metric on velocity fields which in turn becomes the full $H^1$ metric on stream functions. In this paper, we consider $D_\theta(M)$ with the $L^2$ metric on its stream functions. The geometric differences of $D_\theta(M)$ with these two metrics are apparent just as in the case of $D(S^1)$. As $D(S^1)$ coincides with $D_\theta(S^1)$ trivially, we view $D_\theta(M)$ as a natural generalization to $D(S^1)$. In \[13\], Shelukhin considers the $L^\infty$ norm on the contactomorphisms isotopic to the identity and shows how that induces a bi-invariant distance function on the full $D_\theta(M)$.

We summarize the results of this paper as follows. First we show that $D_\theta(M)$ has non-negative sectional curvature. Next we prove that the Riemannian exponential map is not a local $C^1$ map. Lastly, we show that the quantomorphism group $D_q(M)$ is a totally geodesic submanifold of $D_\theta(M)$.

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2. Geometric Background

We will be working primarily on the Lie algebra of $D_\theta(M)$, and we will use the following well-known fact that the Lie algebra $T_eD_\theta(M)$ can be identified with the space of smooth functions $f : M \to \mathbb{R}$.

Proposition 2.1. [9] The Lie algebra $T_eD_\theta(M)$ consists of vector fields $u$ such that $\mathcal{L}_u \theta = \lambda \theta$ for some function $\lambda : M \to \mathbb{R}$. Any such field is uniquely determined by the function $f = \theta(u)$, and we write $u = S_\theta f$. Thus we have that

$$T_eD_\theta(M) = \{S_\theta f : f \in C^\infty(M)\}.$$

Here we call $S_\theta$ the contact operator. The Lie bracket on $T_eD_\theta(M)$ is given by

$$[S_\theta f, S_\theta g] = S_\theta \{f, g\}, \text{ where } \{f, g\} = S_\theta f(g) - gE(f).$$

Here $E$ denotes the Reeb vector field, uniquely specified by the conditions $\theta(E) = 1$, $\iota_E d\theta = 0$. We call $\{\cdot, \cdot\}$ the “contact Poisson bracket”; it is not a true Poisson bracket since it does not satisfy Leibniz’s rule.

We also need a Riemannian structure on $(M, \theta)$, and we will require that the Riemannian metric be associated to the contact form. It will also be convenient to assume that $E$ is a Killing field (i.e., its flow consists of isometries).

Definition. If $(M, \theta)$ is a contact manifold and $E$ is the Reeb field, a Riemannian metric $(\cdot, \cdot)_g$ is associated if it satisfies the following conditions:

1. $\theta(u) = (u, E)_g$ for all $u \in TM$, and
2. there exists a $(1, 1)$-tensor field $\phi$ such that $\phi^2(u) = -u + \theta(u)E$ and $d\theta(u, v) = (u, \phi v)_g$ for all $u$ and $v$.

If in addition $E$ is a Killing field, we say that that $(M, \theta, g)$ is $K$-contact.

Now if we have a $K$-contact manifold $(M, \theta, g)$, we define a right-invariant metric $(\cdot, \cdot)$ on $D_\theta(M)$ by

$$\langle S_\theta f, S_\theta g \rangle = \int_M fgd\mu$$

Lemma 2.2. With $X = S_\theta f$ and $Y = S_\theta g$ we have that

$$\text{ad}_X^* Y = S_\theta [S_\theta f(g) + g(n+2)E(f)]$$

Proof. Let $X = S_\theta f$, $Y = S_\theta g$, and $Z = S_\theta h$ so we have

$$\langle \text{ad}_X^* Y, Z \rangle = \langle \text{ad}_S_\theta f S_\theta g, S_\theta h \rangle = \langle S_\theta g, \text{ad}_S_\theta f S_\theta h \rangle = -\int_M gS_\theta f(h)d\mu + \int_M gE(f)hd\mu =$$

$$\int_M hS_\theta f(g) + hg\text{div}S_\theta f + hgE(f)d\mu = \int_M (S_\theta f(g) + g(n+1)E(f) + gE(f))hd\mu.$$

Thus we have that

$$\text{ad}_X^* Y = S_\theta [S_\theta f(g) + g(n+2)E(f)]$$

□
On any Lie group with a right-invariant Riemannian metric, the geodesic equation\[1\] can be written in terms of the flow equation
\[
\frac{d\eta}{dt} = u \circ \eta
\]
and the Euler-Arnold equation
\[
\frac{du}{dt} + \text{ad}^*_u u = 0.
\]
In this case, the Euler-Arnold equation becomes
\[
\frac{df}{dt} + f(n + 3)E(f) = 0.
\]

**Example 2.3.** For \( M = S^1 \) with the coordinate being \( \alpha \) and the standard 1-form being \( d\alpha \) we get that the Reeb field is \( E = \frac{d}{d\alpha} \) and the contact operator is \( S_\theta f = fE \). Thus the geodesic equation on the circle becomes
\[
\frac{df}{dt} + 3ff_{\alpha} = 0.
\]
This is the right-invariant Burgers’ equation which is studied in \[6\]. It is usual Euler-Arnold equation on \( \mathcal{D}(S^1) \), the diffeomorphism group of the circle.

In the above example of the circle, the diffeomorphism group, which is studied in \[6\] \[11\], coincides with the contactomorphism group. As we will see later in this paper, the contactomorphism group shares many properties with the diffeomorphism group of the circle with the \( L^2 \) right-invariant metric. Thus, we view \( D_\theta(M) \) as a generalization of \( \mathcal{D}(S^1) \).

### 3. The Curvature

In \[10\] it was shown that the contactomorphism group is a regular smooth Lie group. The curvature of a Lie group \( G \) with right-invariant metric in the section determined by a pair of vectors \( X, Y \) in the Lie algebra \( \mathfrak{g} \) is given by the following formula\[2\].

\[
C(X, Y) = \langle d, d \rangle + 2\langle a, b \rangle - 3\langle a, a \rangle - 4\langle B_X, B_Y \rangle
\]

where
\[
2d = B(X, Y) + B(Y, X), \quad 2b = B(X, Y) - B(Y, X),
\]
\[
2a = \text{ad}_X Y, \quad 2B_X = B(X, X), \quad 2B_Y = B(Y, Y),
\]
where \( B \) is the bilinear operator on \( \mathfrak{g} \) given by the relation \( \langle B(X, Y), Z \rangle = \langle X, \text{ad}_Y Z \rangle \), i.e., \( B(X, Y) = \text{ad}_Y^* X \). Note that in terms of the usual Lie bracket of vector fields, we have \( \text{ad}_X Y = -[X, Y] \). The sectional curvature is then given by the normalization \( K(X, Y) = C(X, Y)/|X \wedge Y|^2 \). But here we only care about the sign so we will work with \( C \) only.

Next we will show that the sectional curvature will always be non-negative.

**Theorem 3.1.** The sectional curvature is nonnegative.
Proof. With $X = S_\theta f$ and $Y = S_\theta g$, we use the above formula and (1) to compute
\[
C(X, Y) = \frac{1}{4} \int_M \left[ (n + 3)(fE(g) + gE(f)) \right]^2
\]
\[
- 2([\{f, g\}][S_\theta g(f) - S_\theta f(g) + (n + 2)(fE(g) - gE(f))]
\]
\[
- 3([f, g]^2] - 4[(n + 3)^2fE(f)gE(g)]d\mu
\]
\[
= \frac{1}{4} \int_M \left[ (n + 3)(fE(g) + gE(f)) \right]^2
\]
\[
- 2(n + 3)[f, g](fE(g) - gE(f)) + [f, g]^2
\]
\[
- 4[(n + 3)^2fE(f)gE(g)]d\mu
\]

because $S_\theta g(f) - S_\theta f(g) + fE(g) - gE(f) = 2\{f, g\}$ by antisymmetry of the contact Poisson bracket. Now since
\[
(fE(g) + gE(f))^2 - 4fE(f)gE(g) = (fE(g) - gE(f))^2,
\]
we have that the non-normalized sectional curvature is given by
\[
C(X, Y) = \frac{1}{4} \int_M \left[ [f, g] - (n + 3)(fE(g) - gE(f)) \right]^2d\mu.
\]

\[\square\]

Here we can immediately see how the geometry of $D_\theta(M)$ changes when we consider the $L^2$ metric on stream functions rather than the $H^1$ metric where it was shown in [5] that the curvature can take on any sign.

4. Geodesics

From (4) we have that the flow equation and Euler-Arnold equation are given by
\[
\frac{\partial \varphi}{\partial t} = u \circ \varphi
\]
\[
\frac{\partial f}{\partial t} + 3fE(f) = 0
\]
where $u = S_\theta f$.

Let $(x, z) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ be Darboux coordinates for our contact manifold and thus the contact form is given by
\[
\alpha = dz - \sum y_i dx_i
\]
and the Reeb field is given by
\[
E = \frac{\partial}{\partial z}.
\]

Now given an initial condition, $f_0$, we are able to solve this first order PDE implicitly in these coordinates to get
\[
f(t, x, z) = f_0(x, z).
\]

Note that this solution does not describe trajectories.

In [13], it was shown that given the metric (2), the energy functional is in fact degenerate. Thus, just as in the case of the diffeomorphism group of the circle with the $L^2$ metric[11], $D_\theta(M)$ has vanishing geodesic distance.
5. The Exponential Map

Let \( \varphi(t; v) \) be the geodesic starting at the identity and in the direction of \( v \). Recall that the exponential map \( \exp_p \) on a Riemannian manifold \( M \) at a point \( p \in M \) is defined by the geodesic flow at time 1. Explicitly, it is defined as \( \exp_p(v) = \varphi(1, v) \). Next we will show that as with the case of the diffeomorphism group of the circle [6], \( D_\theta(M) \) with the \( L^2 \) metric on stream functions has an exponential map which is not locally \( C^1 \).

**Theorem 5.1.** The Riemannian exponential map of the \( L^2 \) right invariant metric on stream functions of \( D_\theta(M) \) is not a \( C^1 \) map from a neighborhood of zero in \( T_e D_\theta(M) \) to \( D_\theta(M) \).

**Proof.** Let’s assume for a contradiction that \( \exp \) is a \( C^1 \) map.

Consider the curve given by \( t \mapsto tu_0 \) with \( t > 0 \) and \( u_0 \in T_e D_\theta(M) \). For \( t \) small enough we have that \( \exp(tu_0) = \varphi(1; tu_0) = \varphi(t; u_0) \) we compute

\[
\frac{d}{dt} \exp(tu_0) \bigg|_{t=0} = \frac{d}{dt} \varphi(t; u_0) \bigg|_{t=0} = u_0
\]

so we have that \( D \exp(0) \) is the identity.

Now we would like to show that \( \exp \) is not invertible in a neighborhood of \( u_0 \in T_e D_\theta(M) \) so we consider the Jacobi fields. Let \( \eta(t) \) be a smooth geodesic with \( \eta(0) = e \) and \( \dot{\eta}(0) = u_0 \) so that every Jacobi field satisfies

\[
\frac{\partial}{\partial t} (\operatorname{Ad}_t \operatorname{Ad}_\eta \frac{\partial \eta}{\partial t}) + \operatorname{ad}^* \frac{\partial \eta}{\partial t} u_0 = 0
\]

with \( J(t) = dL_\eta v \). This equation is obtained by left translating the Jacobi equation [8, 12]. Now since \( E \) is a steady state solution to the Euler-Arnold equation, we have that its flow is a steady state solution to the Euler-Arnold equation. Let \( \eta(t) \) be a smooth geodesic with initial condition \( w_0 \) and locally, in Darboux coordinates \((x, z)\) we have that the above equation becomes

\[
\frac{\partial w}{\partial t} + c(n + 2) E \left( \frac{\partial w}{\partial x} \right) = 0.
\]

We set \( w = \frac{\partial w}{\partial t} \) with initial condition \( w_0 \) and locally, in Darboux coordinates \((x, z)\) we have that the above equation becomes

\[
\frac{\partial w}{\partial t} + c(n + 2) \frac{\partial w}{\partial z} = 0
\]

thus solving for \( g \) we get

\[
g(x, z, t) = \frac{1}{(c(n+2))^t} \int_z^{z-c(n+2)t} w_0(x, s) ds.
\]

So letting \( c_m = \frac{1}{m} \), we have that \( w_0 = \sin \left( \frac{2\pi m z}{n+2} \right) \) gets annihilated at the points \( S_\theta c_m = c_mE \). Thus we have that the \( D \exp(c_mE) \) fails to be invertible at points near zero. That is because \( c_m \) is a sequence going to zero so in any topology, \( c_mE \) also approaches zero. This violates the Inverse Function Theorem which gives us our desired contradiction.

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6. The Quantomorphism Group

In this section we will be considering the group of quantomorphisms. That is, the contactomorphisms which exactly preserve the contact form, not just the structure. This can be written as

\[
D_q(M) = \{ \eta \in D_\theta(M) : \eta^* \theta = \theta \}.
\]
Proof. In order to show that a submanifold is totally geodesic, it is equivalent to show that the second fundamental form vanishes identically. To do so, it suffices to show that \( \nabla u = \text{ad}^*_u u \). Thus we would like to show that

\[
\langle u, \text{ad}^*_u v \rangle = 0
\]

whenever \( u \in T_q \mathcal{D}_q(M) \) and \( v \in T_q \mathcal{D}_q(M) \) with \( v \) orthogonal to \( T_q \mathcal{D}_q(M) \).

So let \( u = S_\theta f \in T_e \mathcal{D}_q(M) \) and \( v = S_\theta g \in T_e \mathcal{D}_q(M) \) orthogonal to \( T_e \mathcal{D}_q(M) \).

\[
\langle \nabla u, v \rangle = \langle \text{ad}^*_u u, v \rangle = \langle u, \text{ad}^*_u v \rangle
\]

\[
= -\int_M (S_\theta f, S_\theta \langle f, g \rangle) g d\mu = -\int_M f \{f, g\} g d\mu
\]

\[
= -\int_M f S_\theta f(g) g d\mu = \int_M g(E(f) + f \text{div} S_\theta f) d\mu = 0
\]

\( \square \)

From Smolentsev\[14,15]\, we can see that the quantomorphism group admits a Riemannian submersion onto the symplectomorphism group of the Boothby-Wang quotient. Let \((M, \theta, g)\) be a \(K\)-contact manifold with Reeb field \(E\). If \(\theta\) is a regular contact form, then \(\mathcal{D}_q(M)\) is a closed and totally geodesic submanifold of \(\mathcal{D}_\theta(M)\).

**Theorem 6.1.** If \((M, \theta, g)\) is a \(K\)-contact manifold with Reeb field \(E\). If \(\theta\) is a regular contact form, then \(\mathcal{D}_q(M)\) is a closed and totally geodesic submanifold of \(\mathcal{D}_\theta(M)\).

Now \(T_e \mathcal{D}_\omega(N)\) consists of the vector fields \(V\) such that \(L_V \omega = 0\). We call a vector field \(V\) Hamiltonian if we can associate a function \(H\) such that \(\omega(\cdot, V) = dH(\cdot)\). In order for this definition to be unambiguous, we require that the Hamiltonians have mean zero.

For \(V \in T_e \mathcal{D}_q(M)\), we have that \([V, E] = 0\) and thus \(T_e \mathcal{D}_q(M) \to T_e \mathcal{D}_\omega(N)\) is a projection. We can see that elements of \(T_e \mathcal{D}_q(M)\) are of the form \(V = hE + X\). These vector fields project onto \(T_e \mathcal{D}_\omega(N)\) by \(d\pi \circ X = Y \circ \pi\) with \(Y \in T_e \mathcal{D}_\omega(N)\). Here we have that \(L_V \theta\) implies that \(E(h) = 0\) so that \(h\) is constant in the Reeb direction. Now combined with the fact that we require our stream functions and Hamiltonians to have mean zero, we can see that the map

\[
d\pi : \ker(d\pi)^\perp \to TD_\omega(N)
\]

is an isometry by scaling the one of the volume forms by a constant. Thus the projection of \(\mathcal{D}_q(M)\) onto \(\mathcal{D}_\omega(N)\) is a Riemannian submersion.


References

[1] Arnold, V.I., *On the differential geometry of infinite-dimensional Lie groups and its application to the hydrodynamics of perfect fluids*, in Vladimir I. Arnold: collected works vol. 2, Springer, New York, NY, 2014.

[2] Arnold, V.I. and Khesin, B., *Topological methods in hydrodynamics*, Springer, New York, NY, 1998.

[3] Bauer, M., Bruveris, M., Harms, P., Michor, P.W., *Geodesic distance for right invariant Sobolev metrics of fractional order on the diffeomorphism group*, Ann. Glob. Anal. Geom. 44 (2013).

[4] Blair, D., *Riemannian geometry of contact and symplectic manifolds*, Springer, New York, NY, 2010.

[5] Chhay, B. and Preston, S. C., *Geometry of the contactomorphism group*, Diff. Geo. Appl. 40 (2015).

[6] Contantin, A. Kolev, B., *On the geometric approach to the motion of inertial mechanical systems*, J. Phys. A, 35 (2002).

[7] Ebin, D. G. and Marsden, J., *Diffeomorphism groups and the motion of an incompressible fluid*, Ann. Math. 92 (1970).

[8] Ebin, D. G., Misiolek, G., and Preston, S.C., *Singularities of the exponential map on the volume-preserving diffeomorphism group*, Geom. Funct. Anal. 16 (2006).

[9] Ebin, D. G. and Preston, S.C., *Riemannian geometry of the contactomorphism group*, Arnold Math. J. 1 (2014).

[10] Kriegl, A. and Michor, P. W., *The convenient setting of global analysis*, American Mathematical Society, Providence, RI, 1997.

[11] Michor, P. W. and Mumford, D., *Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms*, Doc. Math. 10 (2005).

[12] Misiolek, G. and Preston, S.C., *Fredholm properties of Riemannian exponential maps on diffeomorphism groups*, Invent. Math. 179 (2009).

[13] Shelukhin, E. *The Hofer norm of a contactomorphism*, arXiv:1411.1457 [math.SG] (2015).

[14] Smolentsev, N.K. *A binevariant metric on the group of symplectic diffeomorphisms and equations* \((\partial/\partial t)\Delta F = \{\Delta F, F\}\), Siberian Math. J. 27 (1986).

[15] Smolentsev, N.K. *Diffeomorphism groups of compact manifolds*, J. Math. Sci. 146 (2007).

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