The high temperature CP-restoring phase transition at $\theta = \pi$

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The CP-restoring phase transition at $\theta = \pi$ and high temperature is investigated using two related models that aim to describe the low-energy phenomenology of QCD, the NJL model and the linear sigma model coupled to quarks. Despite many similarities between the models, different predictions for the order of the phase transition result. Using the Landau-Ginzburg formalism, the origin of this difference is traced back to a non-analytic vacuum term at zero temperature that is present in the NJL model, but usually not included in the linear sigma model. Due to the absence of explicit CP violation, this term always alters the qualitative aspects of the high temperature phase transition at $\theta = \pi$, just as for $\theta = 0$ in the chiral limit.

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It is well known that there is a possibility of CP violation in the strong interaction due to instanton contributions. These contributions are incorporated in the QCD Lagrangian through the topological $\frac{g_\pi}{32\pi^2}F\tilde{F}$-term, where $\theta$ is the QCD vacuum angle. This term violates CP, unless $\theta = 0 \mod \pi$. The case $\theta = \pi$ is special, because then Dashen’s phenomenon can occur, i.e., spontaneous CP violation at $\theta = \pi$.

From experiments it is known that in nature $\theta$ is very small [2, 3, 4, 5]. The reason for this is unknown and is commonly referred to as the strong CP problem. However, it has been argued that in heavy-ion collisions meta-stable CP-violating states could be created corresponding to states with an effective $\theta \neq \pi$. Studying the behavior of the strong interactions at nonzero $\theta$ is therefore of interest and has been done quite extensively using chiral Lagrangians, see for example Refs [14, 15, 16, 17, 18, 19, 20].

Recently the $\theta$-dependence of two models describing the chiral dynamics of low energy QCD have been studied, the NJL model [21] and the linear sigma model coupled to quarks (LSM$q$) [22]. In both models the effects of instantons are included through an additional interaction, the ‘t Hooft determinant interaction [23, 24]. It was found that both models exhibit Dashen’s phenomenon, which turns out to be temperature dependent. This is to be expected, since at high temperature the effects of instantons, which are needed for the CP violation, are exponentially suppressed [25]. In both models the spontaneous CP violation at $\theta = \pi$ disappears at a critical temperature between 100 and 200 MeV, however, the order of the phase transition differs. In case of the NJL model the transition is of second order, whereas in the LSM$q$ model it is of first order. Clearly this difference is important, because a first order transition allows meta-stable phases, in contrast to a second order transition.

Although the NJL and LSM$q$ model are not the same, they are closely related. Eguchi [26] has shown that when the NJL model is bosonized, a linear sigma model is obtained (see also [27]). However, the effects of quarks are treated differently in both models, which was already discussed in Ref. [28] for $\theta = 0$. In the case of the LSM$q$ model the effects of the quarks are usually only taken into account for nonzero temperatures, whereas in the NJL model their effects are necessarily incorporated also at zero temperature. Ref. [28] found that the order of the chiral symmetry restoring phase transition at $\theta = 0$ was the same in both models, but the critical temperatures differ. While the qualitative aspects of the phase transition are similar at $\theta = 0$, this is not the case for the high temperature CP-restoring phase transition at $\theta = \pi$ as we will discuss in detail. We should mention here that the situation at $\theta = 0$ depends on the amount of explicit chiral symmetry breaking. In Ref. [29] it was observed that when the pion mass is reduced in order to study the chiral limit, neglecting the effects of the quarks at zero temperature can affect the order of the high temperature phase transition at $\theta = 0$ too.

Although there is a CP-restoring phase transition at high chemical potential also, in this paper we will restrict to the temperature dependence of this phase transition at $\theta = \pi$, because there the differences between the two models are most pronounced. The paper is organized as follows. First, the effective potentials of both models are analyzed analytically, which will allow the determination of the order of the phase transitions using standard Landau-Ginzburg type of arguments. A comparison to numerical results obtained earlier corroborates these conclusions. Subsequently, we will discuss the bosonification procedure of Eguchi, which relates the NJL model to a linear sigma model and allows us to further pinpoint the origin of the similarities and differences with the LSM$q$ model.
I. NJL Model

The Nambu-Jona-Lasinio (NJL) model, introduced in Refs. [30, 31], is a model for low energy QCD that contains four-point interactions between the quarks. In this paper the following form of the NJL model is used, in the notation of Ref. [21]

\[ \mathcal{L}_{NJL} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi + \mathcal{L}_{\bar{q}q} + \mathcal{L}_{\text{det}}, \]

where \( m \) is the current quark mass. In contrast to Ref. [21], here the up and down quark masses are taken equal, which matters little for our present purposes. Furthermore,

\[ \mathcal{L}_{\bar{q}q} = G_1 \left[ (\bar{\psi} \tau_3 \psi)^2 + (\bar{\psi} \tau_5 \gamma_5 \psi)^2 \right], \]

is the attractive part of the \( \bar{q}q \) channel of the Fierz transformed color current-current interaction [32] and

\[ \mathcal{L}_{\text{det}} = 8G_2 e^{i\theta} \det (\bar{\psi} R \psi_L) + \text{h.c.}, \]

is the 't Hooft determinant interaction which depends on the QCD vacuum angle \( \theta \) and describes the effects of instantons [23, 24]. In the literature \( G_1 \) and \( G_2 \) are often taken equal, which at \( \theta = 0 \) means that the low energy spectrum consists of \( \sigma \) and \( \pi \) fields only, but here we will allow them to be different. We will restrict to the two flavor case, using \( \tau_a \) with \( a = 0, ..., 3 \) as generators of \( U(2) \). We will not consider nonzero baryon or isospin chemical potential.

The symmetry structure of the NJL model is very similar to that of QCD. In the absence of quark masses and the instanton interaction, there is a global \( SU(3)_c \times U(2)_L \times U(2)_R \)-symmetry. The instanton interaction breaks it to \( SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_B \). For nonzero, but equal quark masses this symmetry is reduced to \( SU(3)_c \times SU(2)_V \times U(1)_B \).

We choose the parameters the same way as in Refs. [21, 32]. This means we write

\[ G_1 = (1 - c)G_0, \quad G_2 = cG_0, \]

where the parameter \( c \) controls the instanton interaction, while the value for the quark condensate at \( \theta = 0 \) (which is determined by the combination \( G_1 + G_2 \)) is kept fixed. For our numerical studies we will use the following values for the parameters: \( m = 6 \) MeV, a three-dimensional momentum UV cut-off \( \Lambda = 590 \) MeV/c and \( G_0\Lambda^2 = 2.435 \). These values lead to a pion mass of 140.2 MeV, a pion decay constant of 92.6 MeV and finally, a quark condensate \( \langle \bar{u}u \rangle = \langle \bar{d}d \rangle = (-241.5 \) MeV\(^3 \) [32], all in reasonable agreement with experimental determinations.

A. The effective potential

To calculate the ground state of the theory, the effective potential has to be minimized. In this section the effective potential is calculated in the mean-field approximation. In the following we will only consider the case of unbroken isospin symmetry, such that only nonzero \( \langle \bar{\psi} \psi \rangle \) and/or \( \langle \bar{\psi} \gamma_5 \psi \rangle \) can arise. At \( \theta = 0 \) only \( \langle \bar{\psi} \psi \rangle \) becomes nonzero. A nonzero \( \langle \bar{\psi} \gamma_5 \psi \rangle \) signals that CP invariance is broken, i.e., it serves as an order parameter for the CP-violating phase.

To obtain the effective potential in the mean-field approximation, first the interaction terms are “linearized” in the presence of the \( \langle \bar{\psi} \psi \rangle \) and \( \langle \bar{\psi} \gamma_5 \psi \rangle \) condensates (this is equivalent to the procedure with a Hubbard-Stratonovich transformation used in Ref. [21])

\[ (\bar{\psi} \psi)^2 \simeq 2 \langle \bar{\psi} \psi \rangle \bar{\psi} \psi - \langle \bar{\psi} \psi \rangle^2, \]
\[ (\bar{\psi} \gamma_5 \psi)^2 \simeq 2 \langle \bar{\psi} \gamma_5 \psi \rangle \bar{\psi} \gamma_5 \psi - \langle \bar{\psi} \gamma_5 \psi \rangle^2, \]
\[ (\bar{\psi} \psi)(\bar{\psi} \gamma_5 \psi) \simeq \langle \bar{\psi} \psi \rangle \bar{\psi} \gamma_5 \psi + \langle \bar{\psi} \gamma_5 \psi \rangle \bar{\psi} \gamma_5 \psi - \langle \bar{\psi} \psi \rangle \langle \bar{\psi} \gamma_5 \psi \rangle, \]

leading to

\[ \mathcal{L}^{\text{vac}}_{\text{NJL}} = \bar{\psi} (i \gamma^\mu \partial_\mu - \mathcal{M}) \psi - \frac{(G_1 - G_2 \cos \theta) \alpha_0^2}{4(G_1^2 - G_2^2)} - \frac{(G_1 + G_2 \cos \theta) \beta_0^2}{4(G_1^2 - G_2^2)} - \frac{(G_2 \sin \theta) \alpha_0 \beta_0}{2(G_1^2 - G_2^2)}, \]

where \( \mathcal{M} = (m + \alpha_0) + \beta_0 \gamma_5 \) and

\[ \alpha_0 = -2(G_1 + G_2 \cos \theta) \langle \bar{\psi} \psi \rangle + 2G_2 \sin \theta \langle \bar{\psi} \gamma_5 \psi \rangle, \]
\[ \beta_0 = -2(G_1 - G_2 \cos \theta) \langle \bar{\psi} \gamma_5 \psi \rangle + 2G_2 \sin \theta \langle \bar{\psi} \psi \rangle. \]
This Lagrangian is quadratic in the quark fields, so the integration can be performed. After going to imaginary time the thermal effective potential in the mean-field approximation is obtained\cite{34}

\[
\psi_{\text{N}1L} = \frac{\alpha_0^2 (G_1 - G_2 \cos \theta)}{4(G_1^2 - G_2^2)} + \frac{\beta_0^2 (G_1 + G_2 \cos \theta)}{4(G_1^2 - G_2^2)} + \frac{G_2 \alpha_0 \beta_0 \sin \theta}{2(G_1^2 - G_2^2)} + \psi_q, \tag{8}
\]

with

\[
\psi_q = -T N_c \sum_{p_0=(2n+1)\pi T} \frac{d^3p}{(2\pi)^3} \log \det K, \tag{9}
\]

and where \(K\) is the inverse quark propagator,

\[
K = (i\gamma_0 p_0 + \gamma_\alpha p_\alpha - M). \tag{10}
\]

In order to calculate the effective potential, it is convenient to multiply \(K\) with \(\gamma_0\), which does not change the determinant, but gives a new matrix \(\tilde{K}\) with \(ip_\alpha\)'s on the diagonal. It follows that \(\det \tilde{K} = \prod_{i=1}^8 (\lambda_i - ip_0)\), where \(\lambda_i\) are the eigenvalues of \(K\) with \(p_0 = 0\). Because of the symmetries of the inverse propagator, half of the eigenvalues are equal to \(E_p = \sqrt{p^2 + M^2}\) and the other half to \(E_p = -\sqrt{p^2 + M^2}\), with \(M^2 = (m + \alpha_0)^2 + \beta_0^2\). After the summation over the Matsubara frequencies, we obtain

\[
\psi_q = -8N_c \int \frac{d^3p}{(2\pi)^3} \left[ \frac{E_p}{2} + T \log \left(1 + e^{-E_p/T}\right) \right]. \tag{11}
\]

At \(T = 0\) this integral can be performed analytically. A conventional non-covariant three-dimensional UV cut-off is used to regularize the integral and yields:

\[
\psi_q^{T=0} = \nu_q \left[ M^3 \log \left( \frac{\Lambda^2}{M^2} + \sqrt{1 + \frac{\Lambda^2}{M^2}} \right) - M \left( M^2 + 2\Lambda^2 \right) \sqrt{\frac{\Lambda^2}{M^2} + 1} \right], \tag{12}
\]

where the degeneracy factor \(\nu_q = 24\).

\[\text{B. The CP-restoring phase transition}\]

In this section the high-\(T\) CP-restoring phase transition at \(\theta = \pi\) is investigated. As was shown in Ref.\cite{21} the phenomenon of spontaneous CP violation is governed by the strength \(c\) of the \('t\) Hooft determinant interaction. It will be assumed that \(c = 0.2\), which following the arguments of Ref.\cite{23} is considered realistic. But in fact, the critical temperature is too very good approximation \(c\)-independent for \(c\) above \(\sim 0.05\), as can be seen from the \((T, c)\) phase diagram given in Ref.\cite{21}.

We will start with a numerical minimization as a function of the temperature, the results of which, together with those for the LSM model, are shown in Fig.\[1\]. One observes that the critical temperature of the NJL model is significantly larger than the one of the linear sigma model, in agreement with the results of Ref.\cite{28} for the chiral phase transition at \(\theta = 0\). Furthermore, the order of the phase transition is clearly different, contrary to the results of Ref.\cite{28} for \(\theta = 0\).

Next we will derive an analytic expression for the effective potential for the NJL model. Two important observations which can be made from the numerical study will be help. First, we note that \(\alpha_0\) is very small and constant as long as \(\beta_0\) is nonzero, which allows us to approximate \(M^2 \approx \beta_0^2\). Furthermore, \(\beta_0\) and hence \(M\) can be considered much smaller than \(\pi T\) and \(\Lambda\), allowing expansions. These observations simplify our study considerably.

The phase transition occurs for \(M\) much smaller than \(\Lambda\), so Eq.\cite{12} can be expanded in \(M^2/\Lambda^2\) at \(T = 0\):

\[
\psi_q^{T=0} = \nu_q \left[ \frac{-M^4 \log M^2}{64\pi^2} + \frac{M^4 \log (4\Lambda^2)}{64\pi^2} - \frac{M^4}{128\pi^2} - \frac{\Lambda^2 M^2}{16\pi^2} \right] \tag{13}\]

For the phase transition, the non-analytic term \(M^4 \log M^2\) turns out to be very relevant. We will see that it is exactly the absence of this term at finite temperatures in the NJL model that causes the differences between the two models.

Usually the temperature-dependent part of the potential has to be evaluated numerically, however when \(M < \pi T\) the integral can be expanded in \(M/\Lambda\). As can be inferred from Fig.\[1\]it is exactly this regime which is relevant for the
phase transition. Note that the temperature-dependent part of the potential is UV finite, which means that for this part the cut-off can be taken to infinity. In Ref. [21] this was not done, leading to a slightly larger critical temperature. Performing the expansion, we obtain [35]

\[ V_T^q = -\nu_q \int \frac{d^3p}{(2\pi)^3} \log \left( 1 + e^{-E_p/T} \right) = \nu_q \left[ \frac{7\pi^2T^4}{720} + \frac{M^2T^2}{48} + \frac{M^4}{32\pi^2} \left( \frac{\gamma_E}{4} + \frac{1}{2} \log \frac{M^2}{T^2} - \log \pi \right) + \cdots \right]. \tag{14} \]

From this expansion one can see that also the temperature dependence contains a logarithmic term, that will precisely cancel the one of Eq. (13) when added together.

Using that \( M^2 \approx \beta_0^2 \), we end up with the effective potential

\[ V_{\mathrm{NJL}}^{\mathrm{vac}}(T) = A_{\mathrm{NJL}}(T) + B_{\mathrm{NJL}}(T)\beta_0^2 + C_{\mathrm{NJL}}(T)\beta_0^4, \tag{15} \]

where

\[ A_{\mathrm{NJL}}(T) = -\frac{(7\pi^2T^4 + 45\Lambda^4)}{720\pi^2} \nu_q, \tag{16} \]

\[ B_{\mathrm{NJL}}(T) = \frac{(\pi^2T^2 - 3\Lambda^2)}{48\pi^2} \nu_q + \frac{1}{4\Lambda_0}, \tag{17} \]

\[ C_{\mathrm{NJL}}(T) = \frac{(\log (4\Lambda^2) - \log T^2) \nu_q + (-1 + \gamma_E - \log \pi)\nu_q}{64\pi^2} + \frac{1}{32\pi^2}. \tag{18} \]

One observes that the logarithm at zero temperature is cancelled by the logarithm in the temperature dependence. As long as \( \beta_0 < \pi T, \Lambda \) the potential contains no logarithms and is fully analytic. We note that this expression is the same as the chiral limit at \( \theta = 0 \), with \( \beta_0 \) replaced by \( \alpha_0 \).

The phase transition occurs when \( B_{\mathrm{NJL}}(T) \) changes sign. As the potential is symmetric and quartic in the order parameter, we conclude (following Landau-Ginzburg arguments) that the phase transition is of second order, which the numerical analysis corroborates. The critical temperature is equal to

\[ T_{c}^{\mathrm{NJL}} = \sqrt{\frac{3\nu_q G_0 \Lambda^2 - 12\pi^2}{G_0\pi^2 \nu_q}} = 185 \text{ MeV}. \tag{19} \]

As long as \( T < \frac{2\Lambda}{\pi} \exp(-1 + \gamma_E) = 246 \text{ MeV}, \) \( C_{\mathrm{NJL}} \) is positive, such that higher order terms in \( \beta_0 \) are not needed in the analysis.

FIG. 1: The temperature dependence of the condensates in the NJL and linear sigma model.
II. LSMq MODEL

The linear sigma model coupled to quarks, like the NJL model, is an effective low-energy model for QCD \[28, 29, 36, 37\], similar in form to the Gell-Mann-Lévy model \[38\]. It is a hybrid model that includes both meson and constituent quark degrees of freedom, the latter only at nonzero temperature however. As was the case in the NJL model, the effects of instantons are included via the ’t Hooft determinant interaction. In this paper the analysis of Ref. \[22\] is followed.

We will start with the $T = 0$ case, when only mesons are considered. The Lagrangian, which contains all Lorentz invariant terms allowed by symmetry and renormalizability has the following form, using a slightly different notation than Ref. \[22\]

$$\mathcal{L}_{LS} = \frac{1}{2} \text{Tr}(\partial \phi^{\dagger} \partial \phi) + \frac{\mu^2}{2} \text{Tr}(\phi^{\dagger} \phi) - \frac{\lambda_1}{4} \left[ \text{Tr}(\phi^{\dagger} \phi) \right]^2 - \frac{\lambda_2}{4} \left[ \text{Tr}[\phi^{\dagger} \phi] \right]^2 + \frac{\kappa}{2} e^{i\theta} \text{det}(\phi) + e^{-i\theta} \text{det}(\phi^\dagger)$$

where $\phi$ is chiral field, defined as

$$\phi = \frac{1}{\sqrt{2}} (\sigma + i\eta) + \frac{1}{\sqrt{2}} (a_0 + i\pi) \cdot \tau.$$  \hspace{1cm} (21)

The Lagrangian incorporates both spontaneous and explicit breaking of chiral symmetry, the latter through the term proportional to $H$. To study this symmetry breaking, we can concentrate on the potential corresponding to Eq. (20), expressed in the meson fields

$$\mathcal{V}_{T=0}^{T=0} = -\frac{\mu^2}{2} (\sigma^2 + \pi^2 + \eta^2 + a_0^2) - \frac{\kappa}{2} \cos \theta (\sigma^2 + \pi^2 - \eta^2 - a_0^2) + \kappa \sin \theta (\sigma \eta - \pi \cdot a_0) - H \sigma + \frac{1}{4} (\lambda_1 + \frac{\lambda_2}{2}) (\sigma^2 + \eta^2 + \pi^2 + a_0^2)^2 + \frac{2\lambda_2}{4} (\sigma a_0 + \eta \pi + \pi \times a_0)^2.$$  \hspace{1cm} (22)

The spontaneous symmetry breaking manifests itself through nonzero $\sigma$ and $\eta$ condensates and are obtained by minimizing the potential. We allow for these condensates by shifting the fields

$$\sigma \rightarrow \sigma_0 + s, \quad \eta \rightarrow \eta_0 + h,$$  \hspace{1cm} (23)

where $\sigma_0$ and $\eta_0$ are the values that minimize the potential and $s$ and $h$ are the fluctuations. These $\sigma_0$ and $\eta_0$ are proportional to the condensates $\alpha_0$ and $\beta_0$ of the NJL model, respectively.

The potential can now be split in two parts, a vacuum part and one that depends on the fluctuations, i.e.,

$$\mathcal{V}_{LS}^{T=0} = \mathcal{V}_{LS}^{\text{vac}, T=0} + \mathcal{V}_{LS}^{\text{fluc}}.$$  \hspace{1cm} (24)

First we concentrate on the vacuum part, which is given by the following expression:

$$\mathcal{V}_{LS}^{\text{vac}, T=0} = \frac{\lambda}{4} (\sigma_0^2 - v_\sigma^2)^2 - H \sigma_0 + \frac{\lambda}{4} (\eta_0^2 - u_\eta^2)^2 + \kappa \sin \theta \sigma_0 \eta_0 + \frac{\lambda}{2} \sigma_0^2 \eta_0^2 - \frac{\lambda}{4} (v_\eta^4 + u_\eta^4).$$

where we have defined the combination of couplings $\lambda \equiv \lambda_1 + \lambda_2/2$, and follow the notation of Ref. \[22\]:

$$v_\sigma^2 \equiv \frac{\mu^2 + \kappa \cos \theta}{\lambda}; \quad u_\eta^2 \equiv v_\eta^2 - \frac{2\kappa}{\lambda} \cos \theta.$$  \hspace{1cm} (26)

This part of the potential determines the phase structure and has to be compared with the NJL expression \[13\]. The main difference is that this potential is fully analytic and does not contain any logarithmic terms.
The part of the potential that depends on the fluctuations is used to determine the parameters $\mu^2$, $\kappa$, $H$, $\lambda_1$ and $\lambda_2$ in Eq. (20). They are obtained by fitting the masses contained in $V_{\text{fluc}}^{\text{LS}}$ and the pion decay constant at $\theta = 0$ such that the model reproduces the low-energy phenomenology of QCD. At $\bar{\theta} = 0$ $V_{\text{fluc}}^{\text{LS}}$ has the following form

$$V_{\text{fluc}}^{\text{LS}} = \frac{1}{2} \left[ m_\pi^2 \pi^2 + m_\sigma^2 \sigma^2 + m_\eta^2 \eta^2 + m_{a_0}^2 a_0^2 \right]$$

$$+ \left( \lambda_1 + \frac{1}{2} \lambda_2 \right) \sigma_0 s \left( s^2 + \pi^2 + \eta^2 \right) + \left( \lambda_1 + \frac{3}{2} \lambda_2 \right) \sigma_0 s a_0^2 + \lambda_2 \sigma_0 \eta \pi \cdot a_0$$

$$+ \left( \frac{1}{4} \lambda_1 + \frac{1}{8} \lambda_2 \right) \left( s^2 + \pi^2 + \eta^2 + a_0^2 \right)^2 + \frac{1}{2} \lambda_2 \left[ \left( s a_0 + \eta \pi \right)^2 + \left( \pi \times a_0 \right)^2 \right]. \quad (27)$$

The masses depend on the parameters of the model as follows:

$$m_\pi^2 = -\mu^2 - \kappa + \frac{1}{2} (2\lambda_1 + \lambda_2) \sigma_0^2,$$

$$m_\sigma^2 = -\mu^2 - \kappa + \frac{3}{2} (2\lambda_1 + \lambda_2) \sigma_0^2,$$

$$m_{a_0}^2 = -\mu^2 + \kappa + (\lambda_1 + \frac{3}{2} \lambda_2) \sigma_0^2,$$

$$m_\eta^2 = -\mu^2 + \kappa + (\lambda_1 + \frac{1}{2} \lambda_2) \sigma_0^2. \quad (28)$$

The mass values used are: $m_\pi = 138\text{MeV}$, $m_\sigma = 600\text{MeV}$, $m_{a_0} = 980\text{MeV}$, and $m_\eta = 574\text{MeV}$.

At nonzero $\theta$, $\eta_5$ becomes nonzero, which alters the mass relations. Furthermore, cross terms like $\sigma \eta$ become nonzero, signalling that the mass eigenstates are no longer CP eigenstates, as discussed for the NJL model in Ref. [21]. As a consequence, the $\sigma$-field mixes with the $\eta$-field and the $\pi$-field mixes with the $a_0$-field. We will not give these expressions explicitly here.

A. Nonzero temperature

In the LSMq model the quarks start to contribute at nonzero temperatures. In fact, it is assumed that all the temperature dependence comes from the quarks. In Ref. [28] it is argued that this approach is more justified for studying high $T$ phenomena than considering only thermal fluctuations of the meson fields, because at high $T$ constituent quarks become light and mesonic excitations heavy. For the study of the chiral phase transition at $\theta = 0$ this approach yields results that are qualitatively similar to those of the NJL model.

The part of the LSMq Lagrangian that depends on the quark fields is:

$$\mathcal{L}_q = \bar{\psi} \left[ i \partial - g (\sigma + i \gamma_5 \eta) + \mathbf{a_0} \cdot \mathbf{\tau} + i \gamma_5 \mathbf{\pi} \cdot \mathbf{\tau} \right] \psi. \quad (29)$$

The quark thermal fluctuations are incorporated in the effective potential for the mesonic sector, by means of integrating out the quarks to one loop [22]. The resulting quark contribution to the potential is given by

$$V_q^T = -\nu q \int \frac{d^3p}{(2\pi)^3} T \log \left( 1 + e^{-E_p/T} \right). \quad (30)$$

This expression is equal to the temperature dependent part of the NJL model Eq. [13], with again $E_p = \sqrt{\mathbf{p}^2 + M^2}$ and the constituent quark mass $M$ depends on the vacuum expectation values of the meson fields in the following way: $M = g \sqrt{\langle \sigma_0^2 + \eta_0^2 \rangle}$, where $g$ is the Yukawa coupling between the quarks and the mesons. A reasonable value for the constituent quark mass at $\theta = 0$ fixes this coupling constant. In Ref. [22] (and here) $g = 3.3$ is used, which leads to a cross-over for the chiral phase transition as a function of temperature at $\theta = 0$ and to a constituent quark mass of approximately 1/3 of the nucleon mass.

B. The phase transition

With all parameters fixed, we can study the CP-restoring phase transition at $\theta = \pi$ in the LSMq model. This was studied in detail, along with other values for $\theta$, in Ref. [22]. There also the effect of a magnetic field was discussed, which we will not take into account.
We are now going to follow the same procedure as for the NJL model to study the details of the phase transition. Again, we start the discussion with numerical results of the minimization of the effective potential, this time the results of Ref. [23]. They are shown in Fig. [1]. From this figure, two simplifying assumptions can be inferred. First, as was the case for the NJL model, in the neighborhood of the phase transition $M < \pi T$, allowing Eq. (30) to be expanded in $M/T$ as in Eq. [14]. Second, $\sigma_0$ is much smaller than $\eta_0$, which means that we can neglect the $\sigma_0$-dependence. This assumption leads to a small error near $\eta_0 \approx 0$, but as we checked explicitly this is not important since the structure of the extrema of the potential is not altered.

Summing the contributions at zero and nonzero temperature gives the following form for the effective potential

$$V_{\text{LS}}^\text{vac} (T) = A_{\text{LS}} (T) + B_{\text{LS}} (T) \eta_0^2 + C_{\text{LS}} (T) \eta_0^4 + D_{\text{LS}} \eta_0^4 \log \eta_0^2,$$

(31)

where

$$A_{\text{LS}} (T) = -\frac{7}{720} \pi^2 T^4 \nu_q,$$

(32)

$$B_{\text{LS}} (T) = \frac{1}{48} (g^2 T^2 \nu_q - 24 (\mu^2 + \kappa)),$$

(33)

$$C_{\text{LS}} (T) = \frac{1}{32} \left( \nu_q \left( \log \left( \frac{g}{\pi} \right) + \gamma_E - \frac{4}{3} \right) g^2 \right) + 8 \lambda,$$

(34)

$$D_{\text{LS}} = \frac{g^4 \nu_q}{64 \pi^2}.$$

(35)

The form of this potential is clearly different from the one of the NJL model Eq. [15], the difference being the uncanceled logarithmic term. This term proportional to $D_{\text{LS}}$ will always cause the phase transition to be of first order. As observed for the NJL model, also in this case the potential is exactly the same as the chiral limit at $\theta = 0$, with $\eta_0$ replaced by $\sigma_0$. Beyond the chiral limit the explicit symmetry breaking term $H \sigma_0$ (which has no analogue at $\theta = \pi$) will change the first order transition into a cross-over, unless the Yukawa coupling $g$ is increased sufficiently [29, 37].

We conclude that the absence of explicit CP violation through a linear term in $\eta_0$ at $\theta = \pi$ lies at the heart of the difference between the observations made here and those in Ref. [28].

Like in the NJL model, it is the sign flip of $B_{\text{LS}}$ that modifies the structure of the minima. But instead of a phase transition, now a meta-stable state develops at $\eta_0 = 0$. When $B_{\text{LS}} (T)$ becomes larger than $2 D_{\text{LS}} \exp \left( -\frac{4}{3} \frac{C_{\text{LS}} (T)}{D_{\text{LS}}} \right)$ the original minimum disappears. Between the two spinodals the minimum jumps, signalling a first order transition.

When the parameters of Ref. [23] are used, we obtain the following values for the spinodals: 118 MeV and 129 MeV. To find the exact point of the phase transition, the potential has to be minimized numerically, giving a critical temperature of 126.4 MeV. As already noted, this is significantly lower than $T_{\text{cNJL}}$, but the specific values depend on the parameter choices made. As should be clear from the previous discussion, choosing different parameters would not affect the conclusion about the different orders of the phase transition, at least as long as $M < \pi T, \Lambda$ and $\kappa > -\mu^2$ (equivalently, $m^2_\pi > 3 \xi^2$ at $T = 0$).

III. RELATION BETWEEN THE NJL AND LSM$_q$ MODEL

As mentioned, the LSM$_q$ model is a hybrid model for mesons, which are coupled to quarks at nonzero temperature, and the NJL model is a quark model, where the bosonic states of quark-antiquark fields are interpreted as mesons. Eguchi [20] has shown how to derive from the Lagrangian of the NJL model a Lagrangian for the mesonic excitations for $G_2 \neq 0$. This bosonification procedure is reviewed in Ref. [21]. Here the corresponding meson Lagrangian will be derived for $G_2 \neq 0$, which was also studied in Ref. [22] in the chiral limit.

The situation will be reviewed for $\theta = 0$, when only the $\bar{\psi} \psi$ receives a vacuum expectation. We start with the Lagrangian given in Eq. (1). The generating functional is given by the standard expression

$$Z[\bar{\xi}, \xi] = \frac{1}{N} \int D\bar{\psi} D\bar{\psi} \exp \left( i \int d^4 x \left[ L_{\text{NJL}} (\bar{\psi}, \psi) + \bar{\psi} \xi + \bar{\xi} \xi \right] \right),$$

(36)

where $\bar{\xi}$ and $\xi$ are the antifermion and fermion sources and $N$ is a normalization factor which will be suppressed from now on. Next we introduce auxiliary fields $\sigma$, $\eta$, $\pi$ and $a_0$ and a new Lagrangian $L'$ such that the effective potential can be written as

$$Z[\bar{\xi}, \xi] = \int D\bar{\psi} D\bar{\psi} D\sigma D\eta D\pi D\bar{a}_0 \exp \left( i \int d^4 x \left[ L'_{\text{NJL}} (\bar{\psi}, \psi) + \bar{\psi} \xi + \bar{\xi} \xi \right] \right),$$

(37)
with
\[
L'_{\text{NJL}} = \bar{\psi} [i \not{\partial} - m - g (\sigma + i \gamma_5 \eta + a_0 \cdot \tau + i \gamma_5 \pi \cdot \tau)] \psi - \frac{1}{2} \delta \mu_1^2 (\sigma^2 + \pi^2) - \frac{1}{2} \delta \mu_2^2 (\eta^2 + a_0^2),
\]
\[\text{(38)}\]
and
\[
\delta \mu_1^2 = \frac{g^2}{2 (G_1 + G_2)}, \quad \delta \mu_2^2 = \frac{g^2}{2 (G_1 - G_2)}.
\]
\[\text{(39)}\]
Here \(g\) is again the Yukawa coupling between the quarks and mesons, which in the case of the NJL model can be evaluated. It is equal to
\[
g^{-2} = -4N_c i \int \frac{d^4 p}{(2 \pi)^4} \frac{1}{(p^2 - M^2)^2},
\]
\[\text{(40)}\]
which requires some regularization.

Integrating out the quarks gives the following generating functional
\[
\mathcal{Z}[\xi, \xi] = \int D\sigma D\eta D\pi Da_0 \exp \left( i S_{\text{NJL}} + i \int d^4 x \xi \left[ \frac{1}{i \not{\partial} - m - g (\sigma + i \gamma_5 \eta + a_0 \cdot \tau + i \gamma_5 \pi \cdot \tau)} \right] \right)
\]
\[\text{(41)}\]
where the action \(S_{\text{NJL}}\) is equal to
\[
S_{\text{NJL}} = \int d^4 x \left[ - \frac{1}{2} \delta \mu_1^2 (\sigma^2 + \pi^2) - \frac{1}{2} \delta \mu_2^2 (\eta^2 + a_0^2) \right] - i \text{Tr} \log [i \not{\partial} - m - g (\sigma + i \gamma_5 \eta + a_0 \cdot \tau + i \gamma_5 \pi \cdot \tau)].
\]
\[\text{(42)}\]
Assuming that only the \(\sigma\)-field receives a vacuum expectation value \(\sigma_0\), i.e., \(\sigma = \sigma_0 + s\), the action can be split into a vacuum part and a part that depends on the fluctuations, which are the mesons \(s, \eta, \pi, a_0\):
\[
S_{\text{NJL}} = S_{\text{NJL}}^{\text{vac}} + S_{\text{NJL}}^{\text{fluc}},
\]
\[\text{(43)}\]
with
\[
S_{\text{NJL}}^{\text{vac}} = \int d^4 x \left[ - \frac{1}{2} \delta \mu_1^2 \sigma_0^2 \right] - i \text{Tr} \log [i \not{\partial} - M],
\]
\[
S_{\text{NJL}}^{\text{fluc}} = \int d^4 x \left[ - \frac{1}{2} \delta \mu_1^2 (s^2 + 2 \sigma_0 s + \pi^2) - \frac{1}{2} \delta \mu_2^2 (\eta^2 + a_0^2) \right] - i \text{Tr} \log \left[ 1 - \frac{1}{i \not{\partial} - M} g (s + i \gamma_5 \eta + a_0 \cdot \tau + i \gamma_5 \pi \cdot \tau) \right].
\]
\[\text{(44)}\]
and the constituent quark mass \(M = m + g \sigma_0\). In order to obtain a local action for the meson fields, the nonlocal fermionic determinant in \(S_{\text{NJL}}^{\text{fluc}}\) is rewritten using a derivative expansion:
\[
- i \text{Tr} \log \left[ 1 - \frac{1}{i \not{\partial} - M} g (s + i \gamma_5 \eta + a_0 \cdot \tau + i \gamma_5 \pi \cdot \tau) \right] = \sum_{n=1}^{\infty} U^{(n)},
\]
\[\text{(45)}\]
where
\[
U^{(n)} = \frac{1}{n} \text{Tr} \left( \frac{1}{i \not{\partial} - M} g (s + i \gamma_5 \eta + a_0 \cdot \tau + i \gamma_5 \pi \cdot \tau) \right)^n.
\]
\[\text{(46)}\]
From power counting we note that \(U^{(n)}\) with \(n \geq 5\) are convergent and the rest is divergent. Evaluating and retaining only the divergent parts of the \(U^{(n)}\) with \(n = 1, 2, 3, 4\) we end up with the following Lagrangian, which integrated over all space yields \(S_{\text{NJL}}^{\text{fluc}}\):
\[
L_{\text{NJL}}^{\text{fluc}} = \frac{1}{2} \left[ (\partial_\mu s)^2 + (\partial_\mu \eta)^2 + (\partial_\mu \pi)^2 \right] - \frac{1}{2} \left[ m^2_\pi \pi^2 + m^2_\eta \eta^2 + m^2_{a_0} a_0^2 \right] - g_3 s (s^2 + \pi^2 + \eta^2 + 3a_0^2) - 2g_3 \eta \pi \cdot a_0 - \frac{1}{2} g_4 (s^2 + \pi^2 + \eta^2 + a_0^2)^2
\]
\[
- 2g_4 \left[ (sa_0 + \eta \pi)^2 + (\pi \times a_0)^2 \right].
\]
\[\text{(47)}\]
The masses and coupling constants have the following values:

\[
\begin{align*}
m_{\pi}^2 &= \frac{1}{2G_0I_0} - \frac{2I_2}{I_0} = \frac{m}{M} \frac{1}{2G_0I_0}, \\
m_{\sigma}^2 &= m_{\pi}^2 + 4M^2, \\
m_{\eta}^2 &= \frac{1}{2(1-2c)G_0I_0} - \frac{2I_2}{I_0}, \\
m_{a_0}^2 &= m_{\eta}^2 + 4M^2, \\
g_3 &= \frac{2M}{I_0^{1/2}} = 2M g_4^{1/2}, \\
g_4 &= \frac{1}{I_0}, \\
g &= \frac{1}{I_0^{1/2}},
\end{align*}
\]

(48)

where \(I_0\) and \(I_2\) are two divergent integrals, equal to

\[
\begin{align*}
I_0 &= -4N_c i \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - M^2)^2}, \\
I_2 &= 4N_c i \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - M^2}.
\end{align*}
\]

(49)

If these integrals are regularized using the three-dimensional UV cut-off, the resulting masses are equal to the ones obtained using the random phase approximation used in Ref. [21] (where the dependence on the external momentum of the generalized \(I_0\) defined in, for example, Ref. [27] has been neglected). The Lagrangian (47) without the \(a_0\) and \(\eta\)-fields was also given in Ref. [40]. In the chiral limit the results agree with those of Ref. [39].

Eq. (47) is equal to the fluctuation part of the linear sigma model Lagrangian (20) using the following parameters

\[
\begin{align*}
\lambda_1 &= 0, \\
\lambda_2 &= 4/I_0, \\
\mu^2 &= 2M^2 - \frac{c}{2(1-2c)G_0I_0}, \\
\kappa &= \frac{c}{2(1-2c)G_0I_0}, \\
H &= \frac{m}{2G_0I_0^{1/2}}.
\end{align*}
\]

(50)

Although the bosonification of the NJL model yields \(\lambda_1 = 0\), this is of no consequence for the order of the phase transition, as the effective potential at zero temperature is a quartic polynomial irrespective of whether \(\lambda_1 = 0\). It does however, affect the masses of the mesons. If \(\lambda_1 = 0\), the following relation holds: \(m_{\sigma}^2 - m_{\pi}^2 = m_{a_0}^2 - m_{\eta}^2 = 4M^2\), a property of the NJL model already noted in Ref. [39]. Clearly, the bosonized NJL model does not yield the most general linear sigma model. However, it gives additional contributions to the vacuum that usually are not taken into account in the linear sigma model coupled to quarks [22, 28]. In Ref. [29] it is noted that upon inclusion of fluctuations using an RG flow equation, the transition becomes second order. This boils down to including quark loop effects at zero temperature too and is consistent with our findings.

To conclude, the mesonic part of this bosonized NJL Lagrangian is equal to the mesonic part of the LSMq model. So the mesons are treated in same way in the two models, but the vacuum contributions are treated differently. Since neither model is directly derived from QCD, it is not straightforward to draw a conclusion about the order of the phase transition expected in QCD. If the NJL model is viewed as a model for the microscopic theory underlying the low energy mesonic theory, it would not seem justified to neglect the logarithmic term at zero temperature.

It is straightforward to bosonize the NJL model for \(\theta \neq 0\) when also \(\langle \psi \gamma_5 \psi \rangle\) can become nonzero, leading to cross terms that mix the \(\sigma\)-field with the \(\eta\)-field and \(a_0\)-field with the \(\pi\)-field, but we do not give the expressions here as they do not lead to any additional insights.
IV. CONCLUSIONS

In this paper the high-$T$ CP-restoring phase transition at $\theta = \pi$ was discussed for two different models which aim to describe the low-energy QCD phenomenology, the NJL model and the linear sigma model coupled to quarks. Although the models are related, the philosophy of how the mesons are treated is quite different in both models. In the NJL model they are bosonic states of quark-antiquarks, whereas in the LSM model they are the fundamental degrees of freedom, interacting with quarks at nonzero temperature. Using the bosonification procedure of Eguchi, one can show that a bosonized NJL model gives a linear sigma model, in which mesons are treated in the same way as in the LSM model. However, the vacuum contributions arising from the quark degrees of freedom are different. The LSM model was motivated for high temperatures, when constituent quarks are light and mesons are heavy. Therefore, it is assumed that quarks only play a role at nonzero temperature and do not affect the vacuum contributions at zero temperature. On the other hand, in the NJL model contributions by the quarks are necessarily taken into account also at zero temperature. The temperature dependent contributions to the effective potential are equal in both models, coming exclusively from the quarks. In the end, the effective potentials of the models only differ in their zero temperature contributions. Nevertheless, this directly affects the nature of the phase transition at high temperature at $\theta = \pi$.

The temperature dependence of the ground state of both models was investigated using a Landau-Ginzburg analysis. The difference between the models is that the potential as a function of the order parameter of the LSM model contains a non-analytic logarithmic term, whereas the potential of the NJL model is a quartic polynomial near the phase transition. It is this logarithm that makes the difference, it affects the order of the phase transition. This logarithm comes from the contribution of the quarks at zero temperatures, but neglecting these contributions will affect the high temperature results qualitatively at $\theta = \pi$. A similar effect occurs for the chiral symmetry restoration phase transition at $\theta = 0$ close to the chiral limit, i.e. for sufficiently small explicit symmetry breaking. The absence of explicit CP violation is therefore an important aspect of the physics at $\theta = \pi$.

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