ON CONVERGENCE RATES IN APPROXIMATION THEORY FOR OPERATOR SEMIGROUPS

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Abstract. We create a new, functional calculus, approach to approximation of $C_0$-semigroups on Banach spaces. As an application of this approach, we obtain optimal convergence rates in classical approximation formulas for $C_0$-semigroups. In fact, our methods allow one to derive a number of similar formulas and equip them with sharp convergence rates. As a byproduct, we prove a new interpolation principle leading to efficient norm estimates in the Banach algebra of Laplace transforms of bounded measures on the semi-axis.

1. Introduction

Approximation theory is a classical chapter in the theory of $C_0$-semigroups with various applications to PDEs and their numerical analysis. By approximating a $C_0$-semigroup with exponentials of bounded operators or with rational functions of its generator one can often reduce the study of a difficult problem to a simpler one (otherwise intractable). An instance of such an approach is the famous Hille-Yosida generation theorem where Yosida’s approximation arises.

The two core results in approximation theory, the Trotter-Kato theorem and the Chernoff product formula, proved to be very helpful in many areas of analysis, including differential operators, mathematical physics and probability theory. The following particular cases of the Trotter-Kato theorem representing different approaches to semigroup approximation became well-known and pave their way to the most of books on semigroup theory, see e.g. [16, Chapters III.4, III.5], [18, Chapters 1.7, 1.8], [37, Chapters 3.4–3.6], [9, Chapter 5].

Theorem 1.1. Let $(e^{-tA})_{t \geq 0}$ be a bounded $C_0$-semigroup on a Banach space $X$. Then the following statements hold.

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a) [Yosida’s approximation] For every $x \in X$, 
\[ e^{-tA}x = \lim_{n \to \infty} e^{-ntA(n+A)}^{-1}x \]
uniformly in $t$ from compacts in $\mathbb{R}_+$.

b) [Dunford-Segal approximation] For every $x \in X$, 
\[ e^{-tA}x = \lim_{n \to \infty} e^{-nt(1-e^{-A/n})}x \]
uniformly in $t$ from compacts in $\mathbb{R}_+$.

c) [Euler’s approximation] For every $x \in X$, 
\[ e^{-tA}x = \lim_{n \to \infty} (1 + tA/n)^{-n}x \]
uniformly in $t$ from compacts in $\mathbb{R}_+$.

(The name for the approximation formula in b) is not well-established, although some authors use this terminology. We find it natural too since the formula was introduced for the first time by Dunford and Segal in [12].)

While theorems on semigroup approximation are very useful, with a few exceptions, they still have a merely qualitative character and the natural problem of finding optimal rates of approximation remains open. The aim of our paper is to fill this gap.

The approximations introduced in Theorem 1.1 will be of primary importance for us. So, before describing our approach, we give a short account of known rate estimates for these approximations. Among the three, Euler’s formula attracted most of attention and relevant results can be summarized as follows.

**Theorem 1.2.** Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$. Then the following hold.

(i) [6, Theorem 4] There exists $c > 0$ such that for all $n \in \mathbb{N}$ and $t > 0$, 
\[ \|e^{-tA}x - (1 + tA/n)^{-n}x\| \leq c \left( \frac{t}{\sqrt{n}} \right)^2 \|A^2 x\|, \quad x \in \text{dom}(A^2); \]

(ii) [13, Theorem 1.7] There exists $c > 0$ such that for all $n \in \mathbb{N}$ and $t > 0$, 
\[ \|e^{-tA}x - (1 + tA/n)^{-n}x\| \leq c \frac{t}{\sqrt{n}} \|Ax\|, \quad x \in \text{dom}(A); \]

(iii) [31, Corollary 4.4] There exists $c > 0$ such that for all $n \in \mathbb{N}$, $t > 0$ and $0 < \alpha \leq 2$, 
\[ \|e^{-tA}x - (1 + tA/n)^{-n}x\| \leq c \left( \frac{t}{\sqrt{n}} \right)^\alpha \|x\|_{\alpha,2,\infty}, \quad x \in X_{\alpha,2,\infty}, \]

where the Banach space $X_{\alpha,2,\infty}$ (called a Favard space) is defined as 
\[ X_{\alpha,2,\infty} := \left\{ x \in X : \|x\|_{\alpha,2,\infty} := \|x\| + \sup_{t > 0} \frac{\|(e^{-tA} - I)^2 x\|}{t^\alpha} < \infty \right\}. \]
Recall that \( \text{dom}(A^\alpha) \) is embedded continuously in \( X_{\alpha,2,\infty} \), \( \alpha \in (0,2) \), but there are examples (see e.g. [28, p. 340]) showing the the inclusion \( \text{dom}(A^\alpha) \subset X_{\alpha,2,\infty} \) is in general strict. The results similar to Theorem 1.2 were also obtained in [15]. Note however that in [6], [13], [15] and [31] rational approximations more general than Euler’s formula were studied, and Theorem 1.2 is a partial case of more general statements proved there.

Much less is known about Yosida’s and Dunford-Segal approximation formulas for bounded \( C_0 \)-semigroups on Banach spaces. Some partial results for analytic semigroups can be found in [3] and [40] (see also [42]). To the best of our knowledge, there are no papers devoted to convergence rates in Yosida’s and Dunford-Segal formulas on our level of generality.

In this paper we propose a general approach to approximation formulas for \( C_0 \)-semigroups on Banach spaces (as in Theorem 1.1 and similar ones). Our basic observation is that behind each of the approximation formulas in Theorem 1.1 there is a specific Bernstein function and one may look at approximation issues through the lens of asymptotic relations

\[
e^{-nt\varphi(z/n)} \to e^{-tz}, \quad n \to \infty, \quad z > 0,
\]

or

\[
e^{-n\varphi(tz/n)} \to e^{-tz}, \quad n \to \infty, \quad z > 0,
\]

with \( \varphi \) being a Bernstein function. In particular, Yosida’s and Dunford-Segal approximations arise when one puts \( \varphi(z) = z/(z+1) \) and \( \varphi(z) = 1 - e^{-z} \) in (1.1), respectively, and Euler’s approximation corresponds to the choice \( \varphi(z) = \log(1+z) \) in (1.2). By the Hille-Phillips functional calculus machinery and subordination, approximations (1.1) and (1.2) give rise to their respective operator versions

\[
e^{-nt\varphi(A/n)} \to e^{-tA}, \quad n \to \infty,
\]

and

\[
e^{-n\varphi(tA/n)} \to e^{-tA}, \quad n \to \infty,
\]

where \(-A\) is the generator of a bounded \( C_0 \)-semigroup \( (e^{-tA})_{t \geq 0} \) on a Banach space \( X \), and convergence takes place in the strong topology of \( X \). Our approach extends the functional calculus approach to approximation issues started in the foundational papers [6] and [24], and developed further in many subsequent articles. For comparatively recent contributions to this area, see [13], [14], [15], and [35].

Theorem 1.2 suggests that it is natural to study a semigroup approximation formula by restricting it to the domains of powers of generators, and to hope that smoothness of the elements will correspond to a certain rate of convergence in (1.3) and (1.4). The present paper develops this observation in a comprehensive way. Our Bernstein functions setting allows one to unify the approximations formulas for \( C_0 \)-semigroups and to equip them with rates which, moreover, are optimal under natural spectral assumptions. As a result, we quantify classical approximation formulas in Theorem 1.3.
restricted to the domains of fractional powers of $A$, thus extending a number of known results and proving several new ones. In particular, the following is true.

**Theorem 1.3.** Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$, and let $\alpha \in (0, 2]$. Then for all $x \in \text{dom}(A^\alpha)$, $t > 0$, and $n \in \mathbb{N}$,

a) [Yosida’s approximation] \[
\|e^{-tA}x - e^{-ntA(n+A)^{-1}x}\| \leq 16M \left(\frac{t}{n}\right)^{\alpha/2} \|A^\alpha x\|;
\]

b) [Dunford-Segal approximation] \[
\|e^{-tA}x - e^{-nt(1-e^{-A/n})x}\| \leq 8M \left(\frac{t}{n}\right)^{\alpha/2} \|A^\alpha x\|;
\]

c) [Euler’s approximation] \[
\|e^{-tA}x - (1 + tA/n)^{-n}x\| \leq 8M \left(\frac{t^2}{n}\right)^{\alpha/2} \|A^\alpha x\|.
\]

where $M := \sup_{t \geq 0} \|e^{-tA}\|$. 

Theorem 1.3(c) was obtained in [20] with a different proof and a better constant. Note that the domain $(0, 2]$ for $\alpha$ in Theorem 1.3 cannot in general be enlarged, see Remark 5.3 below.

Another advantage of putting approximation theory into a Bernstein functions framework is that such a setting reduces the study of rates to norm estimates in the algebra $A_1^+ (\mathbb{C}_+)$ of Laplace transforms of bounded Radon measures on the semi-axis. In turn, the estimates in $A_1^+ (\mathbb{C}_+)$ can be done in an elegant and transparent way by means of a new interpolation principle in $A_1^+ (\mathbb{C}_+)$, Theorem 4.2 below.

Moreover, invoking a certain matrix construction and yet another interpolation result, we improve convergence rates in Theorem 1.3 if $(e^{-tA})_{t \geq 0}$ is a bounded analytic $C_0$-semigroup. Namely, the following result holds.

**Theorem 1.4.** Let $(e^{-tA})_{t \geq 0}$ be a bounded analytic $C_0$-semigroup on $X$, with generator $-A$. Then there exists $C > 0$ such that for every $\alpha \in [0, 1]$,

a) [Yosida’s approximation] \[
\|e^{-tA}x - e^{-ntA(n+A)^{-1}x}\| \leq C(nt^{1-\alpha})^{-1} \|A^\alpha x\|;
\]

b) [Dunford-Segal approximation] \[
\|e^{-tA}x - e^{-nt(1-e^{-A/n})x}\| \leq C(nt^{1-\alpha})^{-1} \|A^\alpha x\|;
\]

c) [Euler’s approximation] \[
\|e^{-tA}x - (1 + t/nA)^{-n}x\| \leq C(nt^{-\alpha})^{-1} \|A^\alpha x\|,
\]

for all $t > 0$, $n \in \mathbb{N}$ and $x \in \text{dom}(A^\alpha)$. 


In particular, Theorem 1.4(c) extends well-known results from [33] and [11] on convergence rates in Euler’s formula for bounded analytic semigroups restricted to domains of integer powers of their generators, see also [13], [26], [40] and [44]. (One may also consult [5] and [36] where the results from [33] and [11] were reproved in an alternative way.) Theorem 1.4(a) and (b) improves substantially the results from [3] and [41].

2. Notations

For a closed linear operator \( A \) on a complex Banach space \( X \) we denote by \( \text{dom}(A) \), \( \text{ran}(A) \), \( \rho(A) \) and \( \sigma(A) \) the domain, the range, the resolvent set and the spectrum of \( A \), respectively, and let \( \overline{\text{ran}}(A) \) stand for the norm-closure of the range of \( A \). We will write the Lebesgue integral \( \int_{0}^{\infty} \) as \( \int_{0}^{\infty} \). (Such a notation is established in [38] and, as far as we rely on [38] essentially, we decided to use this notation too.) With a slight abuse of notation, we use \( z^{a}f \) to denote the function \( z \to z^{a}f(z) \) as an element of a function space. Finally, we let

\[
\mathbb{C}_{+} = \{ z \in \mathbb{C} : \text{Re} z > 0 \}, \quad \mathbb{R}_{+} = [0, \infty).
\]

3. Completely monotone, Stieltjes and Bernstein functions

In this section we collect notions and facts from function theory needed in the sequel.

Let \( \mathcal{M}(\mathbb{R}_{+}) \) be a Banach algebra of bounded Radon measures on \( \mathbb{R}_{+} \). It will be convenient to work with the image of \( \mathcal{M}(\mathbb{R}_{+}) \) under the Laplace transform. Recall that the Laplace transform of \( \mu \in \mathcal{M}(\mathbb{R}_{+}) \) is given by

\[
(\mathcal{L}\mu)(z) := \int_{0}^{\infty} e^{-sz} \mu(ds), \quad z \in \mathbb{C}_{+}.
\]

If

\[
A_{1}^{1}(\mathbb{C}_{+}) := \{ \mathcal{L}\mu : \mu \in \mathcal{M}(\mathbb{R}_{+}) \}
\]

\[
\|\mathcal{L}\mu\|_{A_{1}^{1}(\mathbb{C}_{+})} := \|\mu\|_{\mathcal{M}(\mathbb{R}_{+})} = |\mu|(\mathbb{R}_{+}),
\]

where \( |\mu|(\mathbb{R}_{+}) \) denotes the total variation of \( \mu \) on \( \mathbb{R}_{+} \), then it is easy to prove that \( (A_{1}^{1}(\mathbb{C}_{+}), \|\cdot\|_{A_{1}^{1}(\mathbb{C}_{+})}) \) is a commutative Banach algebra with pointwise multiplication, and the Laplace transform

\[
\mathcal{L} : \mathcal{M}(\mathbb{R}_{+}) \mapsto A_{1}^{1}(\mathbb{C}_{+})
\]

is an isometric isomorphism.

The class of completely monotone functions can be considered as a generalization of \( A_{1}^{1}(\mathbb{C}_{+}) \) to the setting of Laplace transforms of, in general, unbounded positive measures. Recall that a function \( f \in C^{\infty}(0, \infty) \) is said to be completely monotone if

\[
f(t) \geq 0 \quad \text{and} \quad (-1)^{n} \frac{d^{n}f(t)}{dt^{n}} \geq 0 \quad \text{for all} \ n \in \mathbb{N} \text{ and} \ t > 0.
\]
By the Bernstein theorem [38, Theorem 1.4] any complete monotone function $f$ is the Laplace transform of a unique positive Radon measure $\nu$ on $\mathbb{R}_+$, i.e. for all $z > 0$, 

$$f(z) = \int_0^\infty e^{-zt} \nu(dt).$$

Note that any bounded completely monotone function $f$ belongs to $A_1^+(\mathbb{C}_+)$ by Fatou’s lemma. Moreover, if we denote $f_\tau(\cdot) = f(\tau \cdot), \tau > 0$, then $f_\tau \in A_1^+(\mathbb{C}_+)$ and

$$\|f_\tau\|_{A_1^+(\mathbb{C}_+)} = f(0+), \quad \tau > 0. \tag{3.1}$$

The notion of completely monotone function is closely related to the notion of Bernstein function which will be central for the studies in this paper. A positive function $\varphi \in C^\infty(0, \infty)$ is called Bernstein function if its derivative is completely monotone. By [38, Theorem 3.2], $\varphi$ is Bernstein if and only if there exist $a, b \geq 0$ and a positive Radon measure $\mu$ on $(0, \infty)$ satisfying

$$\int_0^\infty \frac{s}{1+s} \mu(ds) < \infty$$

such that

$$\varphi(z) = a + bz + \int_0^\infty (1 - e^{-zs}) \mu(ds), \quad z > 0. \tag{3.2}$$

The formula (3.2) is called the Levy-Khintchine representation of $\varphi$. The triple $(a, b, \mu)$ is uniquely determined by the corresponding Bernstein function $\varphi$ and is called the Levy-Khintchine triple. Thus we will write occasionally $\varphi \sim (a, b, \mu)$. Every Bernstein function extends analytically to $\mathbb{C}_+$ and continuously to $\overline{\mathbb{C}_+}$. In the following a Bernstein function will be identified with its continuous extension to $\overline{\mathbb{C}_+}$.

The class of Bernstein functions can also be viewed in terms of the algebra $A_1^+(\mathbb{C}_+)$ as the following statement shows, see [19, Lemma 2.5].

**Lemma 3.1.** Every Bernstein function $\varphi$ can be written in the form

$$\varphi(z) = \psi_1(z) + z \psi_2(z), \quad z \geq 0,$$

where $\psi_1, \psi_2 \in A_1^+(\mathbb{C}_+)$. 

From the definition it follows that a Bernstein function $\varphi$ is increasing, and if $\varphi \sim (a, b, \mu)$ then

$$a = \varphi(0) \quad \text{and} \quad b = \lim_{t \to \infty} \frac{\varphi(t)}{t}. \tag{3.3}$$

Since we will often deal with bounded Bernstein functions, we collect several simple characterizations of such functions in the following lemma.

**Lemma 3.2.** Let $\varphi$ be a Bernstein function. Then the following statements are equivalent.

(i) $\varphi$ is bounded on $\mathbb{R}_+$. 

(ii) \( \varphi' \in L^1(0, \infty) \).

(iii) \( \varphi \) has a Levy-Hintchine representation of the form \((a, 0, \mu)\) with \( \mu \) being a bounded positive Radon measure on \((0, \infty)\).

For the proof it suffices to note that the equivalence of (i) and (ii) follows from the positivity of \( \varphi' \) and Newton-Leibnitz formula, while the equivalence of (i) and (iii) is proved in [38, Corollary 3.8, (v)]. (It is a direct consequence of Fatou’s lemma and the Levy-Hintchine representation (3.2).)

Moreover, from (3.2) it follows that if \( \varphi \) is a bounded Bernstein function then \( b = 0 \) and

\[
\varphi(\infty) - \varphi(0) := \lim_{z \to +\infty} \varphi(z) - a = \int_{0+}^{\infty} \mu(ds) < \infty,
\]

hence \( \varphi \in A^1_+(\mathbb{C}_+) \) and

\[
\|\varphi\|_{A^1_+(\mathbb{C}_+)} = a + 2 \int_{0+}^{\infty} \mu(ds) = 2\varphi(\infty) - \varphi(0).
\]

It will be crucial for the sequel that if \( f \) is completely monotone and \( \varphi \) is Bernstein function, then \( f \circ \varphi \) is completely monotone by [38, Theorem 3.6]. If, in addition, \( f \) is bounded, then \( f \circ \varphi \) is bounded and completely monotone, and Fatou’s theorem yields the following estimate for the \( A^1_+(\mathbb{C}_+) \)-norm of \( f \circ \varphi \):

\[
\|f \circ \varphi\|_{A^1_+(\mathbb{C}_+)} = (f \circ \varphi)(0+) \leq \|f\|_{A^1_+(\mathbb{C}_+)}. \tag{3.5}
\]

Note finally that \( f \) and \( \varphi \) are Bernstein functions then \( f \circ \varphi \) is Bernstein as well. For these properties of compositions see [38, Theorem 3.6 and Corollary 3.7].

One of the most important properties of Bernstein functions is that their exponentials one-to-one correspond to convolution semigroups of subprobability measures and the correspondence is given by the Laplace transform. The following statement can be found e.g. in [38, Theorem 5.2]. Recall that a family of Radon measures \( (\mu_t)_{t \geq 0} \) on \( \mathbb{R}_+ \) is called a vaguely continuous convolution semigroup of subprobability measures if \( \mu_t(\mathbb{R}_+) \leq 1 \) for all \( t \geq 0, \mu_{t+s} = \mu_t * \mu_s \) for all \( t, s \geq 0 \) and vague-lim_{t \to 0+} \mu_t = \delta_0 \), where \( \delta_0 \) stands for the Dirac measure at zero.

**Theorem 3.3.** The function \( \varphi \) is Bernstein if and only if there exists a vaguely continuous semigroup \( (\mu_t)_{t \geq 0} \) of subprobability measures on \( \mathbb{R}_+ \) such that

\[
(L\mu_t)(z) = e^{-t\varphi(z)}, \quad z \in \mathbb{C}_+, \tag{3.6}
\]

for all \( t \geq 0 \).

The subclass of Bernstein functions introduced below will be fundamental for our studies.

**Definition 3.4.** Let

\[
\Phi := \{ \varphi \text{ is Bernstein} : \varphi(0) = 0, \varphi'(0+) = 1, \ |\varphi''(0+)\| < \infty \}. \tag{3.7}
\]
3.5. If $\varphi$ has the Levy-Hintchine representation $(0, 0, \mu)$ then the assumptions (3.7) are equivalent to
\begin{equation}
\int_{0^+}^{\infty} s \mu(ds) = 1, \quad \int_{0^+}^{\infty} s^2 \mu(ds) < \infty.
\end{equation}

It will be essential for us that the functions $\varphi_1(z) = z/(z + 1)$, $\varphi_2(z) = 1 - e^{-z}$ and $\varphi_3(z) = \log(1 + z)$ are Bernstein (see [38, Chapter 15]), and moreover $\{\varphi_1, \varphi_2, \varphi_3\} \subset \Phi$.

An important subclass of completely monotone functions is formed by Stieltjes functions. A function $f : (0, \infty) \rightarrow \mathbb{R}_+$ is called a Stieltjes function if it can be written as
\begin{equation}
f(z) = \frac{a}{z} + b + \int_{0^+}^{\infty} \frac{\nu(ds)}{z + s}, \quad z > 0,
\end{equation}
where $a, b \geq 0$ and $\nu$ is a positive Radon measure on $(0, \infty)$ satisfying
\[\int_{0^+}^{\infty} \frac{\nu(ds)}{1 + s} < \infty.\]

In this case, $\mu$ is called a Stieltjes measure and (3.9) is called the Stieltjes representation for $f$, since such a representation is unique. Moreover, every nonzero Stieltjes function $f$ is completely monotone, and the functions $zf$ and $1/f$ are Bernstein, see [38, Chapters 6-7] for more details. It follows from the definition, that the functions $z^{-\alpha}, \alpha \in [0, 1]$, are Stieltjes.

We will also need a generalization of Stieltjes functions called sometimes generalized Stieltjes functions. Recall that a function $f : (0, \infty) \mapsto \mathbb{R}_+$ is generalized Stieltjes of order $\alpha > 0$ if it can be written as
\begin{equation}
f(z) = \frac{a}{z^\alpha} + b + \int_{0^+}^{\infty} \frac{\nu(ds)}{(z + s)^\alpha}, \quad z > 0,
\end{equation}
where $a, b \geq 0$ and $\nu$ is a positive Radon measure on $(0, \infty)$ satisfying
\[\int_{0^+}^{\infty} \frac{\nu(ds)}{(1 + s)^\alpha} < \infty.\]

Observe that if $f$ is generalized Stieltjes (of any positive order), then $f$ admits an (unique) analytic extension to $\mathbb{C} \setminus (-\infty, 0]$ which will be identified with $f$ and denoted by the same symbol. The class of generalized Stieltjes functions of order $\alpha$ will be denoted by $S_\alpha$. In this terminology, Stieltjes functions constitute precisely the class $S_1$ of generalized Stieltjes functions of order 1. For a thorough treatment of generalized Stieltjes functions one may consult [27].

3.1. Functional Calculus. Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$. Recall that the mapping
\[g = L\mu \mapsto g(A) := \int_{\mathbb{R}_+} e^{-tA} \mu(dt)\]
(with a strong Bochner integral in the definition of \( g(A) \)) defines a continuous algebra homomorphism from \( A_1^+(\mathbb{C}_+) \) into the Banach space of bounded linear operators on \( X \) satisfying
\[
\|g(A)\| \leq \left( \sup_{t \geq 0} \left\| e^{-tA} \right\| \right) \|g\|_{A_1^+(\mathbb{C}_+)}, \quad g \in A_1^+(\mathbb{C}_+).
\]
This homomorphism is called the Hille–Phillips (HP) functional calculus for \( A \), see [25, Chapter XV]. It admits an extension to a class of functions larger than \( A_1^+(\mathbb{C}_+) \). This extension is constructed in the following way.

If \( f: \mathbb{C}_+ \to \mathbb{C} \) is holomorphic such that there exists \( e \in A_1^+(\mathbb{C}_+) \) with \( ef \in A_1^+(\mathbb{C}_+) \) and the operator \( e(A) \) is injective, then we set
\[
\text{dom}(f(A)) := \{ x \in X : (ef)(A)x \in \text{ran}(e(A)) \}
\]
$$f(A) := e(A)^{-1}(ef)(A).$$

In this case \( f \) is called regularizable, and \( e \) is called a regularizer for \( f \). Such a definition of \( f(A) \) does not depend on the regularizer \( e \) and \( f(A) \) is a closed operator on \( X \). Moreover, the set of all regularizable functions \( f \) forms an algebra \( A \) (depending on \( A \)). (See e.g. [23, p. 4-5] for its proof.

Theorem 3.6. Let \( f \in A_1^+(\mathbb{C}_+) \) and \(-A\) be the generator of a bounded \( C_0 \)-semigroup on a Banach space \( X \). Then
\[
\{ f(\lambda) : \lambda \in \sigma(A) \} \subset \sigma(f(A)).
\]

Bernstein functions are, in fact, part of the extended HP functional calculus. Indeed, as a consequence of Lemma 3.1, we obtain that every Bernstein function is regularizable by any of the functions \( e_\lambda(z) = (\lambda + z)^{-1}, \Re \lambda > 0 \). Moreover, the next operator-valued analogue of (3.2) holds, see e.g. [19, Corollary 2.6] and [38, Corollary 12.21].

Theorem 3.7. Let \(-A\) be the generator of a bounded \( C_0 \)-semigroup \( (e^{-tA})_{t \geq 0} \) on \( X \), and let \( \varphi \sim (a,b,\mu) \) be a Bernstein function. Then \( \varphi(A) \) is defined in
the extended HP functional calculus. Moreover, $\text{dom}(A) \subseteq \text{dom}(\varphi(A))$ and

\begin{equation}
\varphi(A)x = ax + bAx + \int_{0^+}^{\infty} (1 - e^{-tA})x \mu(dt)
\end{equation}

for each $x \in \text{dom}(A)$, and $\text{dom}(A)$ is a core for $\varphi(A)$.

Similarly, there is the next operator-valued counterpart of the representation (3.10) for Stieltjes functions, see e.g. [21, Theorem 2.5, (ii)].

**Proposition 3.8.** Let $-A$ be the generator of a bounded $C_0$-semigroup on a Banach space $X$. If $f \sim (0,0,\mu)$ is a Stieltjes function and $A$ has dense range, then $f$ belongs to the extended HP calculus and

\begin{equation}
f(A)x = \int_{0^+}^{\infty} (s + A)^{-1}x \mu(ds)
\end{equation}

for every $x \in \text{ran}(A)$. Moreover, $\text{ran}(A)$ is a core for $f(A)$.

It would be instructive to recall that, by the mean ergodic theorem, $A$ is injective if it has dense range. Thus, Proposition 3.8 requires, in particular, the injectivity of $A$.

Finally, we note that if $-A$ generates a bounded $C_0$-semigroup on $X$, then for any Bernstein function $\varphi$ the operator $-\varphi(A)$ generates a bounded $C_0$-semigroup on $X$ as well and the latter semigroup can be represented in terms of $\varphi$ and $(e^{-tA})_{t\geq 0}$ as the formula (3.16) suggests, see e.g. [38] Theorem 12.6] which we quote below.

**Theorem 3.9.** Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t\geq 0}$ on a Banach space $X$, and let $\varphi$ be a Bernstein function with the corresponding vaguely continuous convolution semigroup $(\mu_t)_{t\geq 0}$ of subprobability measures on $\mathbb{R}_+$. Then the formula

\begin{equation}e^{-t\varphi(A)} := \int_0^\infty e^{-sA} \mu_t(ds), \quad t \geq 0,\end{equation}

defines a bounded $C_0$-semigroup $(e^{-t\varphi(A)})_{t\geq 0}$ on $X$ with generator $-\varphi(A)$, where $\varphi(A)$ is given by (3.14) on $\text{dom}(A)$.

4. **Interpolation estimates in $A_+^1(\mathbb{C}_+)$**

We start with with a lemma providing estimates for the $A_+^1(\mathbb{C}_+)$-norms of a pair of functions used in the proof of our interpolation principle for $A_+^1(\mathbb{C}_+)$, Theorem [4.2] below.

**Lemma 4.1.** Let

\begin{align}
u_{\beta,\tau}(z) &:= \left(\frac{z}{z + \tau}\right)^\beta, \quad v_{\beta,\tau}(z) := \frac{1}{(z + \tau)^\beta}, \quad z \in \mathbb{C}_+,
\end{align}
for $\beta > 0$ and $\tau > 0$. Then $u_{\beta, \tau}, v_{\beta, \tau} \in A^1_+(C_+)$ for all $\beta, \tau > 0$ and
\begin{equation} 2^{\beta+1} \geq \|u_{\beta, \tau}\|_{A^1_+(C_+)} \geq 1, \tag{4.2} \end{equation}
\begin{equation} \|v_{\beta, \tau}\|_{A^1_+(C_+)} = \frac{1}{\tau^\beta}. \tag{4.3} \end{equation}

Proof. Let $\tau > 0$ be fixed. Note that $u_{1, \tau}$ is a Bernstein function. If $\beta \in (0, 1)$, then the function $z \mapsto z^\beta$ is Bernstein, and $u_{\beta, \tau} = (u_{1, \tau})^\beta$ is a Bernstein function too as a composition of Bernstein functions. Moreover, it is bounded since $u_{1, \tau}$ is bounded. Thus $u_{\beta, \tau} \in A^1_+(C_+)$ and by (3.4),
\begin{equation} \|u_{\beta, \tau}\|_{A^1_+(C_+)} \leq \|u_{1, \tau}\|_{A^1_+(C_+)} \|u_{\alpha, \tau}\|_{A^1_+(C_+)} \leq 2^{n+1} \leq 2^{\beta+1}. \tag{4.4} \end{equation}

Moreover,
\begin{equation} \|u_{\beta, \tau}\|_{A^1_+(C_+)} \geq \sup_{z \in C_+} |u_{\beta, \tau}(z)| \geq \lim_{z \to \infty} |u_{\beta, \tau}(z)| = 1. \end{equation}

This gives the second estimate in (4.2).

To prove the assertion about $v_{\beta, \tau}$, note that
\begin{equation} v_{\beta, \tau}(z) = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-zs} e^{-\tau s} s^{\beta-1} ds, \quad z \in C_+, \tag{4.5} \end{equation}
where $\Gamma$ is the gamma-function. Thus $v_{\beta, \tau}$ is a bounded completely monotone function for all $\beta > 0$ and $\tau > 0$. Hence
\begin{equation} \|v_{\beta, \tau}\|_{A^1_+(C_+)} = v_{\beta, \tau}(0+) = \frac{1}{\tau^\beta}. \tag{4.6} \end{equation}

Define now
\begin{equation} c_\beta := \|u_{\beta, \tau}\|_{A^1_+(C_+)} = \|u_{\beta, 1}\|_{A^1_+(C_+)}, \quad \beta > 0. \tag{4.7} \end{equation}
Note that the bounds for $c_\beta$ in (4.2) are far from being optimal and better estimates requiring however much more involved arguments are given in Appendix. For instance, since
\begin{equation} u_{2,1}(z) = \left(1 - \frac{1}{z+1}\right)^2 = 1 + \int_0^\infty e^{-zs}e^{-s}(s-2) ds, \end{equation}
it follows that
\begin{equation} c_2 = \|u_{2,1}\|_{A^1_+(C_+)} = 1 + \int_0^\infty e^{-s} |s-2| ds = 2(1 + e^{-2}) \leq \frac{5}{2}. \tag{4.8} \end{equation}
On the other hand, (4.2) is completely sufficient for our purposes. The inequality (4.7) will be used in the next section.
Our arguments depend essentially on sharp estimates of \( \|Ff\|_{A^1_+(\mathbb{C}_+)} \) for \( F \in A^1_+(\mathbb{C}_+) \) and \( f \in S_\beta \). The next result of independent interest yields bounds for \( \|Ff\|_{A^1_+(\mathbb{C}_+)} \) in terms of \( \|F\|_{A^1_+(\mathbb{C}_+)} \) and \( \|z^{-\beta}F\|_{A^1_+(\mathbb{C}_+)} \) thus allowing us to simplify technical details and to make our presentation more transparent.

**Theorem 4.2.** Let \( F \in A^1_+(\mathbb{C}_+) \) be such that \( z^{-\beta}F \in A^1_+(\mathbb{C}_+) \) for some \( \beta > 0 \), and denote

\[
\|F\|_{A^1_+(\mathbb{C}_+)} = a \quad \text{and} \quad \|z^{-\beta}F\|_{A^1_+(\mathbb{C}_+)} = b.
\]

Then for any \( f \in S_\beta \),

\[
\|Ff\|_{A^1_+(\mathbb{C}_+)} \leq 2^\beta a f \left( \left( a/b \right)^{1/\beta} c^{-1/\beta}_\beta \right). \tag{4.9}
\]

**Proof.** Fix \( \beta > 0 \) and consider first \( f_0 \in S_\beta \) of the form

\[
f_0(z) = \int_{0+}^{\infty} \frac{\nu(d\tau)}{(z + \tau)^{\beta}}, \quad z \in \mathbb{C}_+,
\]

where \( \mu \) is a Radon measure on \((0, \infty)\) satisfying

\[
\int_{0+}^{\infty} \frac{\nu(d\tau)}{(1 + \tau)^{\beta}} < \infty.
\]

For \( \tau > 0 \) let \( u_{\beta,\tau} \) and \( v_{\beta,\tau} \) be the functions defined in (4.1). Then

\[
F(z)f_0(z) = \int_{0+}^{\infty} v_{\beta,\tau}(z) F(z) \nu(d\tau). \tag{4.11}
\]

We use that \( u_{\beta,\tau}, v_{\beta,\tau} \in A^1_+(\mathbb{C}_+) \) by Lemma 4.1. First, note that

\[
\|v_{\beta,\tau}F\|_{A^1_+(\mathbb{C}_+)} \leq \|v_{\beta,\tau}\|_{A^1_+(\mathbb{C}_+)} \|F\|_{A^1_+(\mathbb{C}_+)} \leq \frac{a}{\tau^{\beta}}.
\]

On the other hand,

\[
\|v_{\beta,\tau}F\|_{A^1_+(\mathbb{C}_+)} \leq \|u_{\beta,\tau}\|_{A^1_+(\mathbb{C}_+)} \|z^{-\beta}F\|_{A^1_+(\mathbb{C}_+)} \leq c_{\beta}b.
\]

Then using the elementary inequality

\[
\min \left( x^{-\beta}, y^{-\beta} \right) \leq \frac{2^\beta}{(x + y)^{\beta}}, \quad x, y > 0,
\]

we obtain

\[
\|v_{\beta,\tau}F\|_{A^1_+(\mathbb{C}_+)} \leq a \min \left( \tau^{-\beta}, c_{\beta}b/a \right) \leq \frac{2^\beta a}{((c_{\beta}b/a)^{-1/\beta} + \tau)^{\beta}}. \tag{4.13}
\]

Observe further that, in view of (4.5), \( \tau \mapsto Fv_{\beta,\tau} \) is a continuous, \( A^1_+(\mathbb{C}_+) \)-valued function on \((0, \infty)\). Moreover, by (4.13), the function \( \tau \mapsto \|Fv_{\beta,\tau}\|_{A^1_+(\mathbb{C}_+)} \)
is $\nu$-integrable on $(0, \infty)$. Thus the $A^1_+(C_+)$-valued Bochner integral
\[ \int_{0+}^\infty Fv_{\beta, \tau} \nu(d\tau) \]
is well-defined. Since point evaluations $f \mapsto f(z)(z \in C_+)$ are bounded linear functionals on $A^1_+(C_+)$, and separate elements of $A^1_+(C_+)$, (4.11) implies that the integral coincides with $Ff_0$. Hence by (4.11), (4.13), and a standard inequality for Bochner integrals we obtain
\[ (4.14) \]
\[ \|Ff_0\|_{A^1_+(C_+)} \leq \int_{0+}^\infty \|Fv_{\beta, \tau}\|_{A^1_+(C_+)} \nu(d\tau) \leq 2^\beta a f_0 \left( (a/b)^{1/\beta} c^{-1/\beta}_\beta \right). \]

Let now $f \in S_\beta$. Then $f(z) = c_0 z^{-\beta} + c_1 + f_0$, where $c_0, c_1 \geq 0$ and $f_0$ is given by (4.10). Using (4.14) and and taking into account that $c_\beta \geq 1$, we have
\[ \|Ff\|_{A^1_+(C_+)} \leq c_0 b + c_1 a + 2^\beta a f_0 \left( (a/b)^{1/\beta} c^{-1/\beta}_\beta \right) = 2^\beta a \left( c_0 \frac{b}{2^\beta a} + \frac{c_1}{2^\beta} + f_0((a/b)^{1/\beta} c^{-1/\beta}_\beta) \right) \leq 2^\beta a f \left( (a/b)^{1/\beta} c^{-1/\beta}_\beta \right). \]

\[ \square \]

In our studies of approximation formulas, we will be using the following straightforward corollary of Theorem 4.2 rather than Theorem 4.2 itself. The corollary clarifies the interpolation nature of Theorem 4.2.

**Corollary 4.3.** Let $F \in A^1_+(C_+)$ be such that $z^{-\beta} F \in A^1_+(C_+)$ for some $\beta > 0$, and denote
\[ (4.15) \]
\[ \|F\|_{A^1_+(C_+)} = a \quad \text{and} \quad \|z^{-\beta} F\|_{A^1_+(C_+)} = b. \]
Then for every $\alpha \in [0, \beta]$ one has $z^{-\alpha} F \in A^1_+(C_+)$ and
\[ (4.16) \]
\[ \|z^{-\alpha} F\|_{A^1_+(C_+)} \leq 2^\beta c^{\alpha/\beta}_\beta a^{1-\alpha/\beta} b^{\alpha/\beta}. \]

For the proof of Corollary 4.3 it suffices to note that $z^{-\alpha} \in S_\beta$ if $\alpha \in [0, \beta]$ (see [27], Theorem 3).

For a function $\varphi \in \Phi$ define
\[ (4.17) \]
\[ \Delta_\varphi^z(t) := e^{-te(z)} - e^{-tz}, \quad z \in C_+, \quad t > 0. \]

We proceed with deriving several consequences of Corollary 4.3 which will be instrumental in Sections 5 and 6 dealing with convergence rates in approximation formulas for $C_{0}$-semigroups. The convergence rates will follow from estimates of the $A^1_+(C_+)$-norm of $z^{-\alpha} \Delta_\varphi^z$, $\alpha \in [0, 2]$. To get bounds for $\|z^{-\alpha} \Delta_\varphi^z\|_{A^1_+(C_+)}$ using Corollary 4.3 we first compute the $A^1_+(C_+)$-norm of $z^{-2} \Delta_\varphi^z$. 


Proposition 4.4. Let $\varphi \in \Phi$. For every $t > 0$, the function $z^{-2} \Delta_t^\varphi$ belongs to $A^1_+(\mathbb{C}_+)$ and

$$
\|z^{-2} \Delta_t^\varphi\|_{A^1_+(\mathbb{C}_+)} = \frac{t}{2} |\varphi''(0+)|, \quad t > 0.
$$

Proof. Fix $t > 0$. By Theorem 3.3 there exists a convolution semigroup of subprobability Radon measures $\nu_t$ on $[0, \infty)$ such that

$$
ee^{-t\varphi(z)} = \int_0^\infty e^{-zs} \nu_t(ds).
$$

Since $e^{-t\varphi}$ is completely monotone and $\varphi(0) = 0$, (3.1) implies that

$$
\|e^{-t\varphi}\|_{A^1_+(\mathbb{C}_+)} = e^{-t\varphi(0)} = \int_0^\infty \nu_t(ds) = 1.
$$

Let us further show that

$$
z^{-2} \Delta_t^\varphi(z) = \int_0^\infty e^{-zs} G_t(s) ds,
$$

where

$$
G_t(s) = \chi(t-s) \int_0^s (s-\tau) \nu_t(d\tau) + \chi(s-t) \int_s^\infty (\tau-s) \nu_t(d\tau),
$$

and $\chi(\cdot)$ stands for the characteristic function of $\mathbb{R}_+$. Taking into account (4.18) and that

$$
\frac{1}{z^2} = \int_0^\infty e^{-z\tau} \tau d\tau, \quad z > 0,
$$

we have

$$
z^{-2} \Delta_t^\varphi(z) = \int_0^\infty e^{-z\tau} d\tau \int_0^\infty e^{-zs} \nu_t(ds) - \int_t^\infty e^{-zs} (s-t) ds
$$

$$
= \int_0^\infty e^{-zs} \int_0^s (s-\tau) \nu_t(d\tau) ds - \int_t^\infty e^{-zs} (s-t) ds
$$

$$
= \int_0^\infty e^{-zs} H_t(s) ds,
$$

where

$$
H_t(s) = \int_0^s (s-\tau) \nu_t(d\tau) - \chi(s-t)(s-t).
$$

It is clear that if $t \geq s$ then $H_t(s) = G_t(s)$. We prove that $H_t(s) = G_t(s)$ for $t < s$ as well. If $s > t$, then using (4.19) and

$$
\int_0^\infty \tau \nu_t(d\tau) = -(e^{-t\varphi})'(0+) = t,
$$

we have

$$
\int_t^\infty \nu_t(ds) = \int_0^\infty \nu_t(ds) - \int_0^t \nu_t(ds) = 1 - \int_0^t \nu_t(ds) = 0.
$$

This implies that $H_t(s) = G_t(s)$ for $t < s$. If $t = s$, then $H_t(s) = G_t(s)$. Therefore, for any $t > 0$, we have

$$
z^{-2} \Delta_t^\varphi(z) = \int_0^\infty e^{-zs} H_t(s) ds.
$$

It follows that

$$
z^{-2} \Delta_t^\varphi(z) = \int_0^\infty e^{-zs} \nu_t(ds).
$$

This completes the proof.
we infer that
\[ H_t(s) = s \left( \int_0^s \nu(t) \, dt - 1 \right) + t - \int_0^s \tau \nu(t) \, dt \]
\[ = -s \int_s^\infty \nu(t) \, dt + \int_s^\infty \tau \nu(t) \, dt \]
\[ = \int_s^\infty (\tau - s) \nu(t) \, dt, \]
hence \( H_t(s) = G_t(s) \). Thus (4.20) holds. Since \( G_t(s) > 0, s > 0 \), the function \( z^{-2} \Delta_t^\varphi \) is completely monotone. Hence, applying Lopital’s rule twice, we get
\[
\| z^{-2} \Delta_t^\varphi \|_{A^1_+(\mathbb{C}_+)} = \lim_{{z \to 0^+}} \frac{e^{-t\varphi(z)} - e^{-t\varphi(0+)} - z^2}{z^2} = \frac{t}{2} |\varphi''(0^+)|.
\]
\[ \square \]

Now we are in position to use Corollary 4.3 for estimates of \( \| z^{-\alpha} \Delta_t^\varphi \|_{A^1_+(\mathbb{C}_+)} \) when \( \alpha \in [0, 2] \).

**Corollary 4.5.** Let \( \varphi \in \Phi \). Then for every \( \alpha \in [0, 2] \),
\[
(4.21) \quad \| z^{-\alpha} \Delta_t^\varphi \|_{A^1_+(\mathbb{C}_+)} \leq 8 \left( t |\varphi''(0+)\right)^{\alpha/2}.
\]

**Proof.** First note that
\[
\| \Delta_t^\varphi \|_{A^1_+(\mathbb{C}_+)} \leq \| e^{-t\varphi(\cdot)} \|_{A^1_+(\mathbb{C}_+)} + \| e^{-t\cdot} \|_{A^1_+(\mathbb{C}_+)} = 2, \quad t > 0.
\]
Moreover, Proposition 4.4 yields
\[
\| z^{-2} \Delta_t^\varphi \|_{A^1_+(\mathbb{C}_+)} = \frac{t}{2} |\varphi''(0^+)|.
\]
Then, using Corollary 4.3 (with \( \beta = 2 \)) and the bound for \( c_2 \) in (4.7), we obtain
\[
\| z^{-\alpha} \Delta_t^\varphi \|_{A^1_+(\mathbb{C}_+)} \leq 8 \left( t |\varphi''(0+)\right)^{\alpha/2}.
\]
\[ \square \]

For all \( n \in \mathbb{N}, t > 0, \) and \( z \in \mathbb{C}_+ \) define the scaled versions of (4.17):
\[
(4.22) \quad \Delta_{t,n}^\varphi(z) := e^{-nt\varphi(z/n)} - e^{-tz},
\]
and
\[
(4.23) \quad E_{t,n}^\varphi(z) := e^{-n\varphi(zt/n)} - e^{-tz},
\]
and note that
\[
(4.24) \quad \Delta_{t,n}^\varphi(z) = \Delta_{n,t}^\varphi(z/n), \quad E_{t,n}^\varphi(z) = E_{n,t}^\varphi(tz/n).
\]
The next statement is a scaled version of the preceding corollary.
Corollary 4.6. Let \( \varphi \in \Phi \). Then for all \( \alpha \in [0, 2], n \in \mathbb{N} \) and \( t > 0 \),

\[
\| z^{-\alpha} \Delta_{t,n}^{\varphi} \|_{A_{1}^{+}(\mathbb{C}+)} \leq 8 \left( |\varphi''(0+)| \frac{t}{n} \right)^{\alpha/2},
\]

and

\[
\| z^{-\alpha} E_{t,n}^{\varphi} \|_{A_{1}^{+}(\mathbb{C}+)} \leq 8 \left( |\varphi''(0+)| \frac{t^{2}}{n} \right)^{\alpha/2}.
\]

Proof. Fix \( n \in \mathbb{N} \) and \( t > 0 \). By (4.24),

\[
z^{-\alpha} \Delta_{t,n}^{\varphi}(z) = n^{-\alpha}(z/n)^{-\alpha} \Delta_{n}(z/n),
\]

hence

\[
\| z^{-\alpha} \Delta_{t,n}^{\varphi} \|_{A_{1}^{+}(\mathbb{C}+)} = n^{-\alpha} \| z^{-\alpha} \cdot \Delta_{n} \|_{A_{1}^{+}(\mathbb{C}+)} \leq 8n^{-\alpha}(nt|\varphi''(0+)|)^{\alpha/2} = 8 \left( \frac{t}{n} |\varphi''(0+)| \right)^{\alpha/2},
\]

and (4.25) is proved.

To prove (4.26) observe that

\[
z^{-\alpha} E_{t,n}^{\varphi}(z) = (t/n)^{\alpha} (tz/n)^{-\alpha} \Delta_{n}(tz/n).
\]

Thus

\[
\| z^{-\alpha} E_{t,n}^{\varphi} \|_{A_{1}^{+}(\mathbb{C}+)} = (t/n)^{\alpha} \| z^{-\alpha} \Delta_{n} \|_{A_{1}^{+}(\mathbb{C}+)} \leq 8 (t/n)^{\alpha} (n|\varphi''(0+)|)^{\alpha/2} = 8 \left( |\varphi''(0+)| \frac{t^{2}}{n} \right)^{\alpha/2},
\]

and (4.26) follows. \( \square \)

Let us now apply Corollary 4.6 to the approximations of the exponential function considered in Theorem 1.1.

Example 4.7. a) Yosida’s approximation. Let \( \varphi_{1} \) be a Bernstein function defined by

\[
\varphi_{1}(z) := \frac{z}{z+1}, \quad z > 0.
\]

Note that \( \varphi_{1} \in \Phi \) since

\[
\varphi_{1}(0) = 0, \quad \varphi_{1}'(0+) = 1, \quad \varphi_{1}''(0+) = -2.
\]

Then (4.25) and (4.26) imply that

\[
\| z^{-\alpha} \Delta_{t,n}^{\varphi_{1}} \|_{A_{1}^{+}(\mathbb{C}+)} \leq 16 \left( \frac{t}{n} \right)^{\alpha/2},
\]

(4.27)
and

\[(4.28) \quad \| z^{-\alpha} E_{t,n}^{\varphi_1} \|_{A^1_+ (C_+)} \leq 16 \left( \frac{t^2}{n} \right)^{\alpha/2}, \]

for all \( t > 0 \) and \( n \in \mathbb{N} \).

b) Dunford-Segal approximation. Define a Bernstein function \( \varphi_2 \) as

\[ \varphi_2(z) := 1 - e^{-z}, \quad z > 0. \]

Since \( \varphi_2(0) = 0, \quad \varphi_2'(0+) = 1, \quad \varphi_2''(0+) = -1, \)
we have \( \varphi_2 \in \Phi \). Now (4.25) and (4.26) yield

\[ \| z^{-\alpha} \Delta_{t,n}^{\varphi_2} \|_{A^1_+ (C_+)} \leq 8 \left( \frac{t}{n} \right)^{\alpha/2}, \]

and

\[ \| z^{-\alpha} E_{t,n}^{\varphi_2} \|_{A^1_+ (C_+)} \leq 8 \left( \frac{t^2}{n} \right)^{\alpha/2}, \]

for all \( t > 0 \) and \( n \in \mathbb{N} \).

c) Euler’s approximation. Consider the Bernstein function

\[ \varphi_3(z) := \log(1 + z), \quad z > 0. \]

Since \( \varphi_3(0) = 0, \quad \varphi_3'(0+) = 1, \quad \varphi_3''(0+) = -1, \)
we have \( \varphi_3 \in \Phi \). By (4.25) and (4.26),

\[ \| z^{-\alpha} \Delta_{t,n}^{\varphi_3} \|_{A^1_+ (C_+)} \leq 8 \left( \frac{t}{n} \right)^{\alpha/2}, \]

and

\[ \| z^{-\alpha} E_{t,n}^{\varphi_3} \|_{A^1_+ (C_+)} \leq 8 \left( \frac{t^2}{n} \right)^{\alpha/2}, \]

for all \( t > 0 \) and \( n \in \mathbb{N} \).

5. Applications: rates of approximation of bounded \( C_0 \)-semigroups

Using Corollary 4.6 and the HP functional calculus set up in Section 3.1 we are now able to get convergence rates in abstract and concrete approximation schemes for \( C_0 \)-semigroups on Banach spaces. Without loss of generality, we can consider only bounded \( C_0 \)-semigroups. (The general case can be reduced to this case by rescaling.)

If \(-A\) is the generator of a bounded \( C_0 \)-semigroup \( (e^{-tA})_{t \geq 0} \) on a Banach space \( X \), then for \( t > 0 \) and \( n \in \mathbb{N} \) let

\[ \Delta_{t,n}^{\varphi} (A) := e^{-nt\varphi(A/n)} - e^{-tA}, \]
and

\[ E_{t,n}^{\varphi}(A) := e^{-n\varphi(A/n)} - e^{-tA}, \]

where the right-hand sides are well-defined by Theorem 3.9.

We need to define the fractional powers of \(-A\) in the framework of the extended HP calculus. Note that

\[ (5.1) \quad z^{\alpha}v_{\alpha,1}(z) = u_{\alpha,1}(z) \quad \text{and} \quad z^{-\alpha}v_{\alpha,1}(z) = v_{\alpha,1}(z), \quad z \in \mathbb{C}_+, \quad \alpha > 0, \]

and \(v_{\alpha,1}, u_{\alpha,1} \in A_{1}^{+}(\mathbb{C}_{+})\) by Lemma 4.1. Thus, by the first equality in (5.1), the function \(z \to z^{\alpha}, \alpha > 0\), belongs to the extended HP calculus for \(A\). If \(A\) is injective then \(A^{\alpha}\) is injective as well (see [23, Proposition 3.1.1, d]), and by the second equality in (5.1), the function \(z \to z^{-\alpha}, \alpha > 0\), belongs to the extended HP calculus for \(A\) too. It is instructive to observe that the fractional powers defined in this way coincide with the fractional powers defined by means of the extended holomorphic functional calculus developed e.g. in [23, Section 3]. We omit the details and refer to [4], where, in particular, compatibility of various calculi is discussed. (See also [23, Proposition 1.2.7 and Section 3.3].)

Now we are in position to recast estimates of scalar functions from the previous section into norm estimates of functions of generators.

**Theorem 5.1.** Let \(-A\) be the generator of a bounded \(C_{0}\)-semigroup \((e^{-tA})_{t\geq 0}\) on a Banach space \(X\), and let \(\varphi \in \Phi\). If \(\alpha \in (0, 2]\), then for all \(x \in \text{dom}(A^{\alpha}), t > 0, \text{and } n \in \mathbb{N}\),

\[ (5.2) \quad \|\Delta_{t,n}^{\varphi}(A)x\| \leq 8M \left(\frac{t|\varphi''(0+)|}{n}\right)^{\alpha/2} \|A^{\alpha}x\|, \]

and

\[ (5.3) \quad \|E_{t,n}^{\varphi}(A)x\| \leq 8M \left(\frac{t^2|\varphi''(0+)|}{n}\right)^{\alpha/2} \|A^{\alpha}x\|, \]

where \(M := \sup_{t \geq 0} \|e^{-tA}\|\).

**Proof.** Let \(n \in \mathbb{N}, t > 0, \alpha \in (0, 2]\) and \(x \in \text{dom}(A^{\alpha})\) be fixed. We use the HP calculus to estimate the norm of \((A + \delta)^{-\alpha}\Delta_{t,n}(A + \delta), \delta > 0\). Since \(\text{dom}(A^{\alpha}) = \text{dom}(A + \delta)^{\alpha}\) for every \(\delta > 0\), (3.11) and (4.25) imply that

\[ (5.4) \quad \|\Delta_{t,n}^{\varphi}(A + \delta)x\| \leq 8M \left(\frac{t|\varphi''(0+)|}{n}\right)^{\alpha/2} \|(A + \delta)^{\alpha}x\|. \]

By [23, Proposition 3.1.9],

\[ (5.5) \quad \lim_{\delta \to 0+} (A + \delta)^{\alpha}x = A^{\alpha}x. \]

Furthermore, from Theorem 3.3 it follows that

\[ (5.6) \quad \Delta_{t,n}^{\varphi}(\cdot + \delta) - \Delta_{t,n}^{\varphi}(\cdot) = \int_{0}^{\infty} e^{-zs}(e^{-\delta s} - 1) d\nu_{t,n}(s), \]
where \( \nu_{n,t} \) are bounded Radon measures on \( \mathbb{R}_+ \). Hence, by the Lebesgue bounded convergence theorem,
\[
A_+^1(\mathbb{C}_+) = \lim_{\delta \to 0^+} \Delta_{t,n}^{\varphi}(\cdot + \delta) = \Delta_{t,n}^{\varphi},
\]
and by (3.11),
\[
\lim_{\delta \to 0^+} \Delta_{t,n}^{\varphi}(A + \delta)x = \Delta_{t,n}^{\varphi}(A)x.
\]
Letting now \( \delta \to 0^+ \) in (5.4) we get
\[
\|\Delta_{t,n}^{\varphi}(A)x\| \leq 8M \left( \frac{t}{n} |\varphi(0^+)| \right)^{\alpha/2} \|A^\alpha x\|
\]
and (5.2) is proved.

The proof of (5.3) is completely analogous to the proof of (5.2) being based on (3.11) and (4.26), and is therefore omitted. \( \square \)

**Remark 5.2.** If \( \alpha = 2 \) then (5.2) and (5.3) hold with better constants. Indeed, arguing as in the proof of Theorem 5.1 and using Theorem 4.2, we obtain
\[
\|\Delta_{t,n}^{\varphi}(A)x\| \leq M \left( \frac{t}{n} |\varphi'(0^+)| \right) \|A^2 x\|,
\]
and
\[
\|E_{t,n}^{\varphi}(A)x\| \leq M \left( \frac{t}{n} |\varphi'(0^+)| \right) \|A^2 x\|
\]
for all \( n \in \mathbb{N}, t > 0 \) and \( x \in \text{dom}(A^2) \).

**Remark 5.3.** Observe that Theorem 5.1 does not in general hold if \( \alpha > 2 \). To see this, we first note that if \( \varphi \in \Phi \) and \( t > 0 \) then
\[
\lim_{z \to 0^+} \frac{\Delta_{t,n}^{\varphi}(z)}{z^2} = \lim_{z \to 0^+} \frac{e^{-nt\varphi(z/n)} - e^{-tz}}{z^2} = \frac{t}{2n} |\varphi''(0^+)|.
\]
Hence if \( \varphi''(0^+) \neq 0 \) (that is if \( \varphi(z) \neq z \)), and if \( -A \) is a generator of a bounded \( C_0 \)-semigroup on a Banach space \( X \) such that \( \text{ran}(A) = X \) and \( \sigma(A) \) has accumulation point at zero, then \( A^{-\alpha} \Delta_{t,n}^{\varphi}(A) \) is not bounded for all \( t > 0 \) and \( n \in \mathbb{N} \). Indeed, if \( A^{-\alpha} \Delta_{t,n}^{\varphi}(A) \) is bounded for some \( t > 0 \) and \( n \in \mathbb{N} \), then, using the product rule (5.12), for any \( \tau > 0 \) we have
\[
\|v_{0,\tau}(A)\Delta_{t,n}^{\varphi}(A)\| = \|A^{-\alpha} \Delta_{t,n}^{\varphi}(A)u_{0,\tau}(A)\| \leq Mc_0 \|A^{-\alpha} \Delta_{t,n}^{\varphi}(A)\|.
\]
On the other hand, choosing \( \{\lambda_k\}_{k=1}^{\infty} \subset \sigma(A) \) such that \( \lambda_k \to 0 \) as \( k \to \infty \), and \( \tau_k := 2^{-1} |\lambda_k|, k \in \mathbb{N} \), and employing Theorem 5.6 we get
\[
\|v_{0,\tau_k}(A)\Delta_{t,n}^{\varphi}(A)\| \geq \sup_{\lambda \in \sigma(A)} |(\lambda + \tau_k)^{-\alpha} \Delta_{t,n}^{\varphi}(\lambda)| \\
\geq |(1 + 2^{-1} e^{-i \arg \lambda_k})^{-\alpha} |\lambda_k^{-\alpha} \Delta_{t,n}^{\varphi}(\lambda_k)| \to \infty, k \to \infty,
\]
a contradiction.

Similarly, \( A^{-\alpha} E_{t,n}^{\varphi}(A) \) is an unbounded operator for \( \alpha > 2 \) if \( \varphi''(0^+) \neq 0 \) \( (\varphi(z) \neq z) \) and \( \sigma(A) \) has accumulation point at zero.
The following quantification of Theorem 1.1 is a direct consequence of Theorem 5.1 and Example 4.7. It is one of the main results of the paper.

**Corollary 5.4.** Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$, and let $\alpha \in (0, 2]$. Then for all $x \in \text{dom}(A^\alpha)$, $t > 0$, and $n \in \mathbb{N}$,

1. **[Yosida’s approximation]**
   \[ \|e^{-tA}x - e^{-ntA(n+A)^{-1}}x\| \leq 16M \left( \frac{t}{n} \right)^{\alpha/2} \|A^\alpha x\|; \]

2. **[Dunford-Segal’s approximation]**
   \[ \|e^{-tA}x - e^{-nt(1-e^{-A/n})}x\| \leq 8M \left( \frac{t}{n} \right)^{\alpha/2} \|A^\alpha x\|; \]

3. **[Euler’s approximation]**
   \[ \|e^{-tA}x - (1 + tA/n)^{-n}x\| \leq 8M \left( \frac{t^2}{n} \right)^{\alpha/2} \|A^\alpha x\|. \]

where $M := \sup_{t \geq 0} \|e^{-tA}\|$.

**Remark 5.5.** We may use Theorem 5.1 and Example 4.7 to get new approximation formulas. In particular, applying (4.28) instead of (4.27) we obtain the following variation upon Yosida’s approximation. Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$, and let $\alpha \in (0, 2]$ on $X$, and let $\alpha \in (0, 2]$. Then for all $x \in \text{dom}(A^\alpha)$, $t > 0$, and $n \in \mathbb{N}$,

\[ (5.9) \quad \|e^{-tA}x - e^{-ntA(n+A)^{-1}}x\| \leq 8M \left( \frac{t^2}{n} \right)^{\alpha/2} \|A^\alpha x\|. \]

The estimate (5.9) is clearly better than Corollary 5.4.a) for small $t$, while Corollary 5.4.a) gives approximation better than (5.9) for big $t$. We leave the formulation of other possible approximation relations to the reader.

We will prove in Section 7 below that the convergence rate estimates provided by Theorem 5.1 (and Corollary 5.4) are sharp under certain spectral conditions on $-A$.

### 6. Rates of Approximation of Analytic Semigroups

In this section, we improve Theorem 5.1 (and Corollary 5.4) if $(e^{-tA})_{t \geq 0}$ is a bounded analytic $C_0$-semigroup on a Banach space $X$. Moreover, we get convergence rates even if the semigroup approximation takes place on the whole of $X$.

Recall that a $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$ is said to be analytic of angle $\theta$ if it has an analytic extension to $S_{\theta} = \{ \lambda \in \mathbb{C} : |\arg \lambda| < \theta \}$ for some $\theta \in (0, \frac{\pi}{2}]$ which is bounded on $S_{\theta'} \cap \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \}$ for all $\theta' \in (0, \theta)$. Moreover, a $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ is called bounded analytic of angle $\theta$
if it has a bounded analytic extension to $S_{\theta'}$ for each $\theta' \in (0, \theta)$. We call a
semigroup $(e^{-tA})_{t \geq 0}$ bounded analytic if it is bounded analytic of angle $\theta$
for some $\theta \in (0, \frac{\pi}{2}]$. For basic properties of analytic $C_0$-semigroups see e.g. [2, Chapter 3.7] and [16, Chapter II.4].

It is well-known that bounded analytic semigroups can be characterized
in terms of their asymptotics on the real axis. Namely, $-A$ is the generator
of a bounded analytic semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$ if and only
if $\sup_{t \geq 0} \|e^{-tA}\|$ and $\sup_{t>0} \|tAe^{-tA}\|$ are finite, see e.g. [16, Theorem 4.6].

In this case, let for the rest of this paper

(6.1) \[ M_0 := \sup_{t \geq 0} \|e^{-tA}\|, \quad M_1 := \sup_{t>0} \|tAe^{-tA}\|, \]

and

(6.2) \[ M := \max (2M_0, M_1). \]

Consider now the Banach space $X' = X \oplus X$ with the norm $\|\cdot\|_{X'}$ given by

$\|(x_1, x_2)\|_{X'} := \max (\|x_1\|, \|x_2\|), \quad (x_1, x_2) \in X'.$

If $A$ is a closed densely defined operator on $X$, then define the operator $A$ on $X'$ by

(6.3) \[ \text{dom}(A) := \text{dom}(A) \oplus \text{dom}(A), \quad A := \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}. \]

By [10] p. 367–368 (see also [9, Chapter 5]), $-A$ is the generator of an analytic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X'$ if and only if $-A$ generates an analytic semigroup on $X$. The semigroup $(e^{-tA})_{t \geq 0}$ is given by

(6.4) \[ e^{-tA} = \begin{pmatrix} e^{-tA} & -tAe^{-tA} \\ 0 & e^{-tA} \end{pmatrix}, \quad t > 0. \]

Combining the two results cited above, we formulate a statement which will be one of the basic tools in this section.

**Theorem 6.1.** The operator $-A$ is the generator of a bounded analytic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$ if and only if $-A$ defined by means of (6.3) is the generator of a bounded analytic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X'$ given by (6.4). Moreover, if $(e^{-tA})_{t \geq 0}$ satisfies (6.1), then

(6.5) \[ \sup_{t \geq 0} \|e^{-tA}\| \leq M_0 + M_1, \]

(6.6) \[ \sup_{t>0} \|tAe^{-tA}\| \leq 2M_1 + 4M_1^2. \]

If $\varphi$ is a Bernstein function, then by Theorem 3.9, $\varphi(A)$ is the generator of a bounded $C_0$-semigroup $(e^{-t\varphi(A)})_{t \geq 0}$ on $X'$. Moreover, subject to a mild restriction, $(e^{-t\varphi(A)})_{t \geq 0}$ has a matrix structure similar to (6.4) as the next proposition shows.
Proposition 6.2. Let $-A$ be the generator of a bounded analytic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$ and let $\varphi$ be a Bernstein function such that
\begin{equation}
\varphi'(0+) < \infty.
\end{equation}
Then $(e^{-t\varphi(A)})_{t \geq 0}$ is a bounded $C_0$-semigroup on $X$ and
\begin{equation}
(e^{-t\varphi(A)})_{t \geq 0} = \int_0^\infty e^{-sA} \nu_t(ds),
\end{equation}
where the integral converges strongly. Hence, by Theorem 6.1,
\begin{equation}
\varphi'(0+) < \infty.
\end{equation}
Then $(e^{-t\varphi(A)})_{t \geq 0}$ is a bounded $C_0$-semigroup on $X$ and
\begin{equation}
(e^{-t\varphi(A)})_{t \geq 0} = \int_0^\infty e^{-sA} \nu_t(ds).
\end{equation}
By (6.7), we have $\varphi' \in A_1^1(C_+).$ Moreover $e^{-t\varphi} \in A_1^1(C_+)$ as well. Hence by the product rule for the Hille-Phillips functional calculus we have
\begin{equation}
-t\varphi'(A)e^{-t\varphi(A)} = \varphi'(A) e^{-t\varphi(A)} = -\int_0^\infty sA e^{-sA} \nu_t(ds),
\end{equation}
and
\begin{equation}
\int_0^\infty sA e^{-sA} \nu_t(ds) = -A \int_0^\infty s e^{-sA} \nu_t(ds) = -tA \varphi'(A) e^{-t\varphi(A)}.
\end{equation}
In view of (6.9), this yields (6.8). \hfill \square

Remark 6.3. One can prove that if $\varphi$ is a Bernstein function then $(e^{-t\varphi(A)})_{t \geq 0}$ is a bounded analytic $C_0$-semigroup if $(e^{-tA})_{t \geq 0}$ is so. This fact however will not be needed in the sequel. For its proof as well as other related statements see [22].

The following result provides convergence rates in approximation formulas for bounded analytic semigroups on the domains of their generators.

Theorem 6.4. Let $-A$ be the generator of a bounded analytic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$, and let $\varphi \in \Phi$. Then for all $t > 0$ and $x \in \text{dom}(A),$
\begin{equation}
\|\Delta_t^\varphi(A)x\| \leq (2M_0 + M_1) |\varphi''(0+)| \|Ax\|.
\end{equation}
Proof. Without loss of generality we may assume that $A$ is injective. (Otherwise we can proceed as in the proof of Theorem 5.1.)
Let $t > 0$ be fixed. We apply (5.7) (with $n = 1$) to the semigroup $(e^{-tA})_{t \geq 0}$ taking into account the matrix representations (6.4) and (6.8) for $(e^{-tA})_{t \geq 0}$.
\[ tAe^{-tA}x - tA\varphi'(A)e^{-t\varphi(A)}x \leq \|e^{-t\varphi(A)}y - e^{-tA}y\|_X \]
\[ \leq 2^{-1}t(M_0 + M_1)|\varphi''(0+)|\|A^2y\|_X \]
\[ \leq t(M_0 + M_1)|\varphi''(0+)|\|A^2x\|. \]

Hence for every \( x \in \text{dom}(A) \),
\begin{equation}
(6.11) \quad \|e^{-tA}x - \varphi'(A)e^{-t\varphi(A)}x\| \leq (M_0 + M_1)|\varphi''(0+)|\|Ax\|.
\end{equation}

Since \( \varphi \in \Phi \),
\[ \varphi'(z) = c + \int_{0^+}^\infty e^{-zs}s\mu(ds), \quad z > 0, \]
for some \( c > 0 \) and a bounded positive Radon measure \( s \cdot \mu \) on \((0, \infty)\) such that
\[ \int_{0^+}^{\infty}s\mu(ds) = 1, \quad \int_{0^+}^{\infty}s^2\mu(ds) < \infty. \]

Therefore, by the HP calculus,
\[ (1 - \varphi'(A))x = (\varphi'(0+) - \varphi'(A))x = \int_{0^+}^{\infty}(1 - e^{-sA})x s\mu(ds), \]
and then, in view of Theorem 3.9
\[ e^{-t\varphi(A)}(1 - \varphi'(A))x = \int_{0^+}^{\infty} \int_{0^+}^{\infty}(e^{-\tau A} - e^{-(s+\tau)A})x s\mu(ds)\nu_t(d\tau) \]
for a subprobability Radon measure \( \nu_t \). Thus, if \( x \in \text{dom}(A) \), then
\begin{equation}
(6.12) \quad \|e^{-t\varphi(A)}(\varphi'(A) - 1)x\| \leq \int_{0^+}^{\infty} \int_{0^+}^{\infty}(e^{-\tau A} - e^{-(s+\tau)A})x s\mu(ds)\nu_t(d\tau) \]
\[ \leq M_0\|Ax\| \int_{0^+}^{\infty}s\mu(ds)\nu_t(d\tau) \]
\[ = M_0\|Ax\||\varphi''(0+)|. \]

Combining (6.11) and (6.12), we obtain
\[ \|e^{-tA}x - e^{-t\varphi(A)}x\| \leq (2M_0 + M_1)|\varphi''(0+)|\|Ax\|, \]
for \( x \in \text{dom}(A) \), which is (6.10).
\[ \square \]

Our aim now is to prove an estimate of the form
\begin{equation}
(6.13) \quad \|e^{-tA} - e^{-t\varphi(A)}\| \leq \frac{C'}{t}, \quad t > 0.
\end{equation}
This will be done by writing, as in the proof of Theorem 6.4
\[
\|e^{-tA} - e^{-t\varphi(A)}\| \leq \|e^{-tA} - \varphi'(A)e^{-t\varphi(A)}\| + \|(1 - \varphi'(A))e^{-t\varphi(A)}\|
\]
and estimating each term in the right-hand side of (6.14) separately. The first term can be handled easily by our techniques.

**Lemma 6.5.** Let $-A$ be the generator of a bounded analytic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$ and let $\varphi \in \Phi$. Then
\[
\|e^{-tA} - \varphi'(A)e^{-t\varphi(A)}\| \leq \frac{2(M_0 + 2M_1) + 4M_2^2}{t}|\varphi''(0+)|\|A\|, \tag{6.15}
\]

Proof. As in the proof of Theorem 6.4 we can assume that $A$ is injective. If $y = (0,x) \in \text{dom}(A)$, $x \in \text{dom}(A)$, then applying (6.10) to the bounded analytic semigroup $(e^{-tA})_{t \geq 0}$ we get
\[
\left\|tAe^{-tA}x - tA\varphi'(A)e^{-t\varphi(A)}x\right\| \leq \left\|e^{-tA}y - e^{-t\varphi(A)}y\right\|_X \leq (2(M_0 + 2M_1) + 4M_2^2)|\varphi''(0+)||Ax|,
\]
so
\[
\left\|e^{-tA}x - \varphi'(A)e^{-t\varphi(A)}x\right\| \leq \frac{2(M_0 + 2M_1) + 4M_2^2}{t}|\varphi''(0+)||x|.
\]
Since $\text{dom}(A)$ is dense in $X$, the latter inequality holds for every $x \in X$, and (6.15) follows. \qed

To estimate the second term in the right-hand side of (6.14) we need to prove several auxiliary inequalities.

**Lemma 6.6.** Let $p \not\equiv\text{const}$ be a Bernstein function and let $q \in L^\infty_{\text{loc}}(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$. Then
\[
\int_0^\infty e^{-tp(s)}|q(s)|\,ds \leq Ct^{-1}, \quad t > 0,
\]
where
\[
C = \inf_{a > 0} \left[ \frac{\|q\|_{L^\infty[0,a]}}{p'(a)} + \frac{\|q\|_{L^1[a,\infty)}}{p(a)} \right].
\]
Proof. Since $p$ is Bernstein, $p$ is increasing on $[0, \infty)$, while $p'$ is decreasing on $(0, \infty)$. Hence, for all $a > \epsilon > 0$,
\[
p(s) = \int_\epsilon^a p'(t)\,dt + p(\epsilon) \geq p'(a)(s - \epsilon), \quad s \in [0,a],
\]
and, letting $\epsilon \to 0^+$, we get
\[
p(s) \geq p'(a)s, \quad s \in [0,a].
\]
Therefore, taking into account that $p'$ is strictly positive on $(0,\infty)$ by assumption,
\[
\int_0^\infty e^{-tp(s)}|q(s)|\,ds \leq \int_0^a e^{-tp(s)}|q(s)|\,ds + \int_a^\infty e^{-tp(s)}|q(s)|\,ds
\]
\[
\leq \|q\|_{L^\infty[0,a]} \int_0^a e^{-tp'(a)s}\,ds + e^{-tp(a)} \int_a^\infty |q(s)|\,ds
\]
\[
\leq \frac{\|q\|_{L^\infty[0,a]}}{p'(a)} + \frac{\|q\|_{L^1[a,\infty)}}{p(a)} t^{-1}, \quad a > 0,
\]
and the statement follows.

\[\square\]

**Lemma 6.7.** Let $-A$ be the generator of a bounded analytic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$. Then
\[
\|(1 - e^{-sA})e^{-\tau A}\| \leq \frac{2Ms}{\tau + s},
\]
where $M$ is given by (6.2).

**Proof.** By assumption for all $s, \tau > 0$,
\[
\|(1 - e^{-sA})e^{-\tau A}\| \leq 2M_0,
\]
and
\[
\|(1 - e^{-sA})e^{-\tau A}\| \leq \int_\tau^{s+\tau} \|Ae^{-\tau A}\|\,dr \leq M_1 \int_\tau^{s+\tau} \frac{dr}{\tau} \leq M_1 \frac{s}{\tau}.
\]
Then, using (4.12) with $\beta = 1$, we obtain
\[
\|(1 - e^{-sA})e^{-\tau A}\| \leq \min (2M_0, M_1 s\tau^{-1}) \leq \frac{2Ms}{\tau + s}.
\]

\[\square\]

Using Lemma 6.7 and Proposition 6.6, we prove now a statement which can be considered as a generalization of the second inequality in (6.6).

**Theorem 6.8.** Let $\psi$ be a bounded Bernstein function satisfying
\[
\psi(0) = 0, \quad \psi'(0+) < \infty,
\]
and let $\varphi \not\equiv$ const be a Bernstein function. Let $-A$ be the generator of a bounded analytic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$. Then
\[
\sup_{t > 0} \|t\psi(A)e^{-t\varphi(A)}\| \leq 2M \left[ \frac{\psi'(0+)}{\varphi'(1)} + \frac{\psi(1)}{\varphi(1)} \right].
\]

**Proof.** Let us start with establishing appropriate integral representations for $\psi(A)$ and $e^{-t\varphi(A)}$, $t \geq 0$.

Since $\psi$ is bounded on $\mathbb{R}_+$ and $\psi(0) = 0$, the Levy-Hintchine representation of $\psi$ is of the form
\[
\psi(z) = \int_{0+}^{\infty} (1 - e^{-zs}) \gamma(ds), \quad z > 0,
\]
where $\gamma$ is a bounded positive Radon measure on $(0, \infty)$. Then by the HP calculus,

$$
\psi(A) = \int_{0+}^{\infty} (1 - e^{-sA}) \gamma(ds), \quad z > 0.
$$

Moreover, by Theorem 3.3

$$
e^{-t\varphi(z)} = \int_0^\infty e^{-\tau \nu_t(d\tau)}, \quad z > 0,
$$

where $(\nu_t)_{t \geq 0}$ is a convolution semigroup of subprobability Radon measures on $\mathbb{R}_+$. From Theorem 3.9 it follows that $-\varphi(A)$ is the generator of a bounded $C_0$-semigroup $(e^{-t\varphi(A)})_{t \geq 0}$ given by

$$
e^{-t\varphi(A)} = \int_0^\infty e^{-\tau A} \nu_t(d\tau), \quad z > 0.
$$

Fix $t > 0$. Then (6.19), (6.21) and Fubini’s theorem imply that

$$
\psi(A)e^{-t\varphi(A)} = \int_{0+}^{\infty} (1 - e^{-sA}) \gamma(ds) \int_0^\infty e^{-\tau A} \nu_t(d\tau)
= \int_{0+}^{\infty} \int_0^\infty (1 - e^{-sA}) e^{-\tau A} \nu_t(d\tau) \gamma(ds).
$$

Hence, using Lemma 6.7 we get

$$
\|\psi(A)e^{-t\varphi(A)}\| \leq 2M \int_0^\infty s \int_0^\infty \frac{\nu_t(d\tau)}{s + \tau} \gamma(ds).
$$

On the other hand, by (6.20), we have

$$
\int_0^\infty \frac{\nu_t(d\tau)}{s + \tau} = \int_0^\infty e^{-zs} e^{-t\varphi(z)} dz, \quad t > 0, \quad s > 0,
$$

so, in view of (6.19),

$$
\int_{0+}^{\infty} s \int_0^\infty \frac{\nu_t(d\tau)}{s + \tau} \gamma(ds) = \int_{0+}^{\infty} s \int_0^\infty e^{-zs} e^{-t\varphi(z)} dz \gamma(ds)
= \int_0^\infty e^{-t\varphi(z)} \int_{0+}^{\infty} se^{-zs} \gamma(ds) dz
= \int_0^\infty e^{-t\varphi(z)} \psi'(z) dz.
$$

From here by (6.22) it follows that

$$
\|\psi(A)e^{-t\varphi(A)}\| \leq 2M \int_0^\infty e^{-t\varphi(z)} \psi'(z) dz,
$$

where, by Lemma 3.2, $\psi' \in L^\infty_{loc}(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$. Now (6.23) and Lemma 6.6 with $p = \varphi$ and $q = \psi'$ yield

$$
t\|\psi(A)e^{-t\varphi(A)}\| \leq 2M \inf_{a > 0} \left\{ \frac{\|\psi'\|_{L^\infty[0,a]} \varphi'(a) + \|\psi'\|_{L^1[0,\infty)}}{\varphi(a)} \right\}, \quad t > 0.
$$
It remains to note that
\[
\inf_{a > 0} \left\{ \frac{\| \psi' \|_{L^\infty[0,a]} + \| \psi' \|_{L^1[a,\infty)}}{\varphi'(a)} \right\} = \inf_{a > 0} \left\{ \frac{\psi'(0^+) + \psi(a)}{\varphi'(a)} \right\} \\
\leq \frac{\psi'(0^+) + \psi(1)}{\varphi'(1)}.
\]
□

In particular, Theorem 6.8 provides a desired estimate of the second term in the right-hand of (6.14).

**Corollary 6.9.** Let \( \varphi \) be a Bernstein function such that
\[
\varphi'(0^+) = 1, \quad |\varphi''(0^+)| < \infty.
\]
Let \(-A\) be the generator of a bounded analytic \( C_0 \)-semigroup \((e^{-tA})_{t \geq 0}\) on \( X \). Then
\[
\| (1 - \varphi'(A)) e^{-t\varphi(A)} \| \leq \frac{2M}{t} \left[ \frac{|\varphi''(0^+)|}{\varphi'(1)} + \frac{\varphi'(1)}{\varphi'(1)} \right],
\]
for all \( t > 0 \).

**Proof.** Note that by (6.24), the function \( \psi(z) = 1 - \varphi'(z) \), \( z > 0 \), is Bernstein and satisfies (6.17). Moreover, \( \psi \) is bounded since \( \varphi' \) is bounded by (6.24). Hence to get (6.9) it suffices to apply Theorem 6.8 to a Bernstein function \( \varphi \) and a bounded Bernstein function \( \psi \). □

As an immediate consequence of (6.14), Lemma 6.5 and Corollary 6.9 we obtain (6.13) which is one of the main results of the paper.

**Theorem 6.10.** Let \(-A\) be the generator of a bounded analytic \( C_0 \)-semigroup \((e^{-tA})_{t \geq 0}\) on \( X \) and let \( \varphi \in \Phi \). Then
\[
\| e^{-tA}x - e^{-t\varphi(A)}x \| \leq \frac{C}{t}, \quad x \in X,
\]
for all \( t > 0 \), where
\[
C = 2(M_0 + 2M_1 + 2M_1^2)|\varphi''(0^+)| + 2M \left[ \frac{|\varphi''(0^+)|}{\varphi'(1)} + \frac{\varphi'(1)}{\varphi'(1)} \right].
\]

**Remark 6.11.** It is interesting to note that Theorem 6.10 provides an estimate for the difference of semigroups \((e^{-tA})_{t \geq 0}\) and \((e^{-t\varphi(A)})_{t \geq 0}\) at infinity. If \( \varphi \) is bounded, then \( \varphi(A) \) is a bounded operator, and \((e^{-tA})_{t \geq 0}\) is asymptotically close to the semigroup \((e^{-t\varphi(A)})_{t \geq 0}\) with bounded generator.

We continue with “interpolating” between (6.10) and (6.25). To this aim we prove an operator analogue of Theorem 4.2 which is of independent interest.

**Theorem 6.12.** Let \(-A\) be the generator of a bounded \( C_0 \)-semigroup \((e^{-tA})_{t \geq 0}\) on \( X \), with dense range. Let \( B \) and \( A^{-1}B \) be bounded linear operators on \( X \), and denote
\[
\| B \| = a \quad \text{and} \quad \| A^{-1}B \| = b.
\]
Then for any Stieltjes function $f$, the operator $f(A)B$ is bounded and
\begin{equation}
\|f(A)B\| \leq 2(1 + M_0)a f(a/b),
\end{equation}
where $M_0 := \sup_{t \geq 0} \|e^{-tA}\|$.

Proof. Let a Stieltjes $f_0$ be of the form
\begin{equation}
f_0(z) = \int_{0^+}^{\infty} \frac{\nu(d\tau)}{z + \tau},
\end{equation}
where $\nu$ is a Radon measure on $(0, \infty)$ satisfying
\[
\int_{0^+}^{\infty} \frac{\nu(d\tau)}{1 + \tau} < \infty.
\]
First note that by Proposition 3.8,
\[
f_0(A)x = \int_{0^+}^{\infty} (\tau + A)^{-1}x \nu(d\tau)
\]
for every $x \in \text{ran}(A) \subset \text{dom}(f(A))$. Since $A^{-1}B$ is bounded we have $\text{ran}(B) \subset \text{dom}(A^{-1}) = \text{ran}(A)$ and
\begin{equation}
f_0(A)Bx = \int_{0^+}^{\infty} (\tau + A)^{-1}Bx \nu(d\tau), \quad x \in X.
\end{equation}
Now we estimate the operator kernel $(\tau + A)^{-1}B$ in the integral transform above. Observe that
\[
\| (\tau + A)^{-1}B \| \leq \frac{M_0a}{\tau}, \quad \tau > 0.
\]
On the other hand,
\[
\| (\tau + A)^{-1}B \| \leq \| A(\tau + A)^{-1} \| \| A^{-1}B \| \leq (1 + M_0)b.
\]
Then, using (4.12) with $\beta = 1$, we obtain
\begin{equation}
\| (\tau + A)^{-1}B \| \leq \min \left( M_0a\tau^{-1}, (1 + M_0)b \right) \leq \frac{2(1 + M_0)a}{a/b + \tau}.
\end{equation}
Now (6.28) and (6.29) imply
\begin{equation}
\| f_0(A)B \| \leq \int_{0^+}^{\infty} \| (\tau + A)^{-1}B \| \nu(d\tau) \leq 2(1 + M_0)a f_0(a/b).
\end{equation}
Finally, if $f$ is Stieltjes then $f(z) = c_0z^{-1} + c_1 + f_0(z)$, where $c_0, c_1 \geq 0$ and $f_0$ is as in (6.27). Hence, by (6.30),
\[
\| f(A)B \| \leq c_0b + c_1a + 2(1 + M_0)a f_0(a/b)
\]
\[
= 2a(1 + M_0) \left( \frac{c_0}{2(1 + M_0)} b/a + \frac{c_1}{2(1 + M_0)} + f_0(a/b) \right)
\]
\[
\leq 2a(1 + M_0)f(a/b).
\]
Remark 6.13. Recall that a closed densely defined linear operator $A$ on $X$ is called sectorial if $(-\infty, 0) \subset \rho(A)$ and
$$K := \sup_{\lambda > 0} \lambda \| (\lambda + A)^{-1} \| < \infty.$$ As, by [21, Theorem 2.5, (ii)], Proposition 3.8 holds for sectorial operators $A$ with dense range, the inequality (6.26) is true for such operators too if $M_0$ is replaced by $K$. The proof remains the same.

Since the functions $z^{-\alpha}, \alpha \in [0, 1]$, are Stieltjes, the next corollary is a direct consequence of Theorem 6.12 and is similar to Corollary 4.3.

Corollary 6.14. Under the assumptions of Theorem 6.12, if $\alpha \in [0, 1]$, then
$$\| A^{-\alpha}B \| \leq 2(1 + M_0) a^{1-\alpha} b^\alpha.$$ With a special choice of $B$, Corollary 6.14 becomes the next approximation estimate.

Theorem 6.15. Let $-A$ be the generator of a bounded analytic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$, with dense range. If $\varphi \in \Phi$ and $\alpha \in [0, 1]$ then there exists $C > 0$ such that
$$\| A^{-\alpha} \Delta^\varphi_t(A) \| \leq \frac{C}{t^{1-\alpha}}$$ for all $t > 0$.

Proof. For a fixed $t > 0$ set $B = \Delta^\varphi_t(A)$. By Theorems 6.4 and 6.10 there exist $C_1 \geq C_0 > 0$ depending on $M_0, M_1$ and $\varphi''(0+)$ such that (6.32) holds for $\alpha = 0$ and $\alpha = 1$ with $C_0$ and $C_1$ respectively. By applying Corollary 6.14 with
$$a = \frac{C_0}{t}, \quad b = C_1, \quad t > 0,$$
and with $B$ as above, we obtain
$$\| A^{-\alpha} \Delta^\varphi_t(A) \| \leq 2(1 + M_0) C_0^{1-\alpha} C_1^\alpha t^{\alpha-1}, \quad \alpha \in [0, 1].$$ Then (6.32) follows with
$$C = 2(1 + M_0) \max_{\alpha \in [0, 1]} C_0^{1-\alpha} C_1^\alpha = 2(1 + M_0) C_1.$$

The following scaled version of Theorem 6.15 yields (optimal) convergence rates for approximations of bounded analytic semigroups. It complements and extends Theorem 5.1 in the case when a $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ is bounded analytic.

Theorem 6.16. Let $\varphi \in \Phi$, and let $\Delta^\varphi_{t,n}$ and $E^\varphi_{t,n}$ be defined by (4.22) and (4.23), respectively. Let $-A$ be the generator of a bounded analytic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$. Then there exists $C > 0$ such that for every
\[ \alpha \in [0, 1], \]

\begin{align}
\| \Delta^\varphi_{t,n}(A)x \| & \leq \frac{C}{nt^{1-\alpha}} \| A^\alpha x \|, \\
\| E^\varphi_{t,n}(A)x \| & \leq \frac{Ct^\alpha}{n} \| A^\alpha x \|,
\end{align}

for all \( x \in \text{dom}(A^\alpha), n \in \mathbb{N} \) and \( t > 0 \).

**Proof.** Let \( \delta > 0, \alpha \in [0, 1], n \in \mathbb{N} \), and \( x \in \text{dom}(A^\alpha) \) be fixed. Then by (4.24) and Theorem 6.15 there exists \( C > 0 \) such that

\[ \| \Delta^\varphi_{t,n}(A+\delta)x \| = \| \Delta^\varphi_{nt}(((A+\delta)/n)x) \| \leq \frac{C}{nt^{1-\alpha}} \| (A+\delta)^\alpha x \|. \]

Arguing as in the proof of Theorem 5.1 and letting \( \delta \to 0^+ \) we get

\[ \| \Delta^\varphi_{t,n}(A)x \| \leq \frac{C}{nt^{1-\alpha}} \| A^\alpha x \|. \]

Similarly, since

\[ \| E^\varphi_{t,n}(A+\delta)x \| = \| \Delta^\varphi_{nt}(t(A+\delta)/n)x) \| \leq \frac{C}{nt^{1-\alpha}} \| (A+\delta)^\alpha x \|, \]

by (4.24) and Theorem 6.15 we obtain as above

\[ \| E^\varphi_{t,n}(A)x \| \leq \frac{Ct^\alpha}{n} \| A^\alpha x \|. \]

\[ \Box \]

We now specify Theorem 6.16 for the three classical semigroup approximations considered in Section 5. The next result is an improvement of Corollary 5.4 for bounded analytic semigroups.

**Corollary 6.17.** Let \(-A\) be the generator of a bounded analytic \( C_0 \)-semigroup \((e^{-tA})_{t \geq 0}\) on \( X \). Then there exists \( C > 0 \) such that for every \( \alpha \in [0, 1] \),

a) \[ \text{[Yosida approximation]} \]

\[ \| e^{-tA}x - e^{-ntA(n+1)}^{-1}x \| \leq C(nt^{1-\alpha})^{-1} \| A^\alpha x \|; \]

b) \[ \text{[Dunford-Segal approximation]} \]

\[ \| e^{-tA}x - e^{-nt(1-e^{-A/n})}x \| \leq C(nt^{1-\alpha})^{-1} \| A^\alpha x \|; \]

c) \[ \text{[Euler approximation]} \]

\[ \| e^{-tA}x - (1 + t/nA)^{-n}x \| \leq C(nt^{-\alpha})^{-1} \| A^\alpha x \|. \]

for all \( t > 0, n \in \mathbb{N} \) and \( x \in \text{dom}(A^\alpha) \).
7. Optimality of rate estimates

In this section, we show that Theorems 5.1 and 6.15 as well as their respective Corollaries 5.4 and 6.17 are sharp under natural spectral conditions on the generator.

We start with two auxiliary technical lemmas.

**Lemma 7.1.** If $\varphi \in \Phi, \varphi(z) \not\equiv z$, then there exist $c > 0$ and $T > 0$ such that
\begin{equation}
|\Delta_t^\varphi(\pm i/\sqrt{t})| \geq c,
\end{equation}
for all $t \geq T$.

**Proof.** By assumption, $\varphi$ is of the form
\begin{equation}
\varphi(z) = b z + \int_{0^+}^\infty (1 - e^{-z s}) \mu(ds),
\end{equation}
where a positive Radon measure $\mu$ satisfies $\int_{0^+}^\infty s \mu(ds) < \infty$. Since $\mu \not\equiv 0$, there exists $r > 0$ such that $\mu((0,r)) > 0$ and then $d := \int_{0^+}^r s^2 \mu(ds) > 0$. Note that for every $\tau \in (0, \pi T]$,
\begin{equation}
\operatorname{Re}(\varphi(i\tau)) = 2 \int_{0^+}^\infty \sin^2(\tau s/2) \mu(ds) \geq \frac{2\tau^2}{\pi^2} \int_{0^+}^{\pi/\tau} s^2 \mu(ds) \geq \frac{2\tau^2 d}{\pi^2}.
\end{equation}
Hence for any $\tau \in (0, \pi T]$, $|\Delta_t^\varphi(-i\tau)| = |\Delta_t^\varphi(i\tau)| \geq 1 - e^{-\tau \operatorname{Re}(\varphi(i\tau))} \geq 1 - e^{-2d\tau^2/\pi^2}$.

Setting $\tau = 1/\sqrt{t}$, we thus obtain
\begin{equation}
|\Delta_t^\varphi(-i/\sqrt{t})| = |\Delta_t^\varphi(i/\sqrt{t})| \geq 1 - e^{-2d t^2/\pi^2}, \quad t \geq \frac{r^2}{\pi^2}.
\end{equation}
It remains to put $c := 1 - e^{-2d t^2/\pi^2} > 0$ and $T = \frac{r^2}{\pi^2}$. \hfill \Box

**Lemma 7.2.** If $\varphi \in \Phi, \varphi(z) \not\equiv z$, then there exist $\delta > 0$ and $c > 0$ such that
\begin{equation}
e^{-\varphi(\tau)/\tau} - e^{-1} \geq c \tau, \quad \tau \in (0, \delta].
\end{equation}

**Proof.** By applying Lopital’s rule twice, we obtain
\begin{equation}
\lim_{\tau \to 0^+} \frac{e^{-\varphi(\tau)/\tau} - e^{-1}}{\tau} = \lim_{\tau \to 0^+} \frac{d}{d\tau} e^{-\varphi(\tau)/\tau} = e^{-1} \lim_{\tau \to 0^+} \frac{\varphi(\tau) - \tau \varphi'(\tau)}{\tau^2} = \frac{1}{2e} |\varphi''(0^+)| > 0.
\end{equation}
This clearly implies the claim. \hfill \Box

We also need a simple result on functional calculi which is convenient to separate as the next lemma.

**Lemma 7.3.** Let $-A$ be the generator of a bounded $C_0$-semigroup on $X$ such that $\operatorname{ran}(A) = X$. If $\alpha \geq 0$ and $f(z) = z^{-\alpha} \Delta_t(z), z \in \mathbb{C}_+$, then $f$ belongs to the extended Hille-Philips functional calculus for $A$. 

Hence, by Theorem 3.6, 

**Proof.** Let $e(z) := \frac{e^z}{(z+1)^\alpha}$, then $e \in A^1_+(\mathbb{C}_+)$ by Lemma 3.1. Moreover $ef \in A^1_+(\mathbb{C}_+)$, since $(z+1)^{-\alpha} \in A^1_+(\mathbb{C}_+)$ by Lemma 4.1 again. Thus, $f$ belongs to the extended Hille-Philips functional calculus if $e(A)$ is injective. To prove the injectivity of $e(A)$ it suffices to note that, since $A$ is injective, $A^\alpha$ is injective as well by [23, Proposition 3.1.1, d)].

Now we are ready to prove the main result of this section.

**Theorem 7.4.** Let $-A$ be the generator of a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$ such that $\text{ran}(A) = X$, and let $\varphi \in \Phi, \varphi(z) \neq z$.

(i) If $\{|s| : s \in \mathbb{R}, is \in \sigma(A)\} = \mathbb{R}_+$, then there exist $c > 0$ and $T > 0$ independent of $A$ such that for every $\alpha \in (0, 2]$ and all $t \geq T$,

$$
\|A^{-\alpha} \Delta^\varphi_t(A)\| \geq ct^{\alpha/2}.
$$

(ii) If $\mathbb{R}_+ \subset \sigma(A)$, then there exist $c > 0$ and $T > 0$ independent of $A$ such that for every $\alpha \in (0, 2]$ and all $t \geq T$,

$$
\|A^{-\alpha} \Delta^\varphi_t(A)\| \geq ct^{\alpha-1}.
$$

**Proof.** Let $\alpha \in (0, 2]$ be fixed. Recall that by Corollary 4.3, $z^{-\alpha} \Delta^\varphi_t \in A^1_+(\mathbb{C}_+)$ for all $t > 0$. By the product rule (3.12),

$$
A^{-\alpha}(e^{-t\varphi(A)} - e^{-tA}) = (z^{-\alpha} \Delta_t)(A).
$$

Hence, by Theorem 3.6

$$
\|A^{-\alpha}(e^{-t\varphi(A)} - e^{-tA})\| \geq \sup_{\lambda \in \sigma(A)} \|\lambda^{\alpha}(e^{-t\varphi(\lambda)} - e^{-t\lambda})\|,
$$

for every $t > 0$. Let $c$ and $T$ be given by Lemma 7.1 and let $t$ be such that $t \geq T$. By our assumption either $i/\sqrt{t} \in \sigma(A)$ or $-i/\sqrt{t} \in \sigma(A)$. If $i/\sqrt{t} \in \sigma(A)$ then from (7.5) it follows that

$$
\|A^{-\alpha}(e^{-t\varphi(A)} - e^{-tA})\| \geq t^{\alpha/2}\|\Delta^\varphi_t(i/\sqrt{t})\| \geq ct^{\alpha/2}.
$$

Then, by Lemma 7.1 (7.6) implies (7.3).

The case $\lambda = -i/\sqrt{t} \in \sigma(A)$, $t \geq T$, is considered similarly.

To prove (ii) we argue as in the proof of (i) but use Lemma 7.2 instead of Lemma 7.1. Let $c, \delta > 0$ be given by Lemma 7.2 and let $t > 0$ be such that $t \geq T = 1/\delta$. By assumption $\lambda = 1/t \in \sigma(A)$, hence from (7.3), (7.2) and the product rule (3.12) it follows that

$$
\|A^{-\alpha} \Delta^\varphi_t(A)\| \geq t^\alpha|\varphi(1/t) - e^{-1}| \geq ct^{\alpha-1},
$$

and (ii) is proved.

The following direct corollary of Theorem 7.4 shows that the convergence rates for $\Delta^\varphi_{\alpha}n(A)$ and $E^\varphi_{\alpha}n(A)$ obtained in the previous sections, e.g. the ones in Theorems 5.1, 5.4 6.16 and 6.17 are optimal if the spectrum of $A$ is large enough.
Corollary 7.5. Let \(-A\) be the generator of a bounded \(C_0\)-semigroup \((e^{-tA})_{t \geq 0}\) on a Banach space \(X\) such that \(\text{ran}(A) = X\), and let \(\varphi \in \Phi, \varphi(z) \neq z\).

(i) If \(|s| : s \in \mathbb{R}, is \in \sigma(A)\} = \mathbb{R}_+\), then there exist \(c > 0\) and \(T > 0\) such that for every \(\alpha \in (0,2)\) and all \(t \geq T\),

\[
\|A^{-\alpha} \Delta_{t,n}(A)\| \geq c \left(\frac{t}{n}\right)^{\alpha/2},
\]

\[
\|A^{-\alpha} E_{t,n}(A)\| \geq c \left(\frac{t^2}{n}\right)^{\alpha/2}.
\]

(ii) If \(\mathbb{R}_+ \subset \sigma(A)\), then there exist \(T > 0\) and \(c > 0\) such that for every \(\alpha \in [0,2]\) and all \(t \geq T\),

\[
\|A^{-\alpha} \Delta_{t,n}(A)\| \geq cn^{-1}t^{\alpha-1},
\]

\[
\|A^{-\alpha} E_{t,n}(A)\| \geq cn^{-1}t^{\alpha}.
\]

8. Appendix: Estimates of \(c_\beta\)

In this appendix we give fine estimates for the constants \(c_\beta\) introduced in \((4.6)\).

Fix \(\beta > 0\). Recall that

\[c_\beta = \|u_{\beta,\tau}\|_{A^1_+} = \|u_{\beta,1}\|_{A^1_+}, \quad \beta, \tau > 0,\]

where \(u_{\beta,\tau} = \left(\frac{s}{z+\tau}\right)^\beta\).

We start with the expansion

\[
1 - (1 - t)^\beta = -\sum_{n=1}^\infty (-\beta)_n \frac{t^n}{n!} e^s, \quad |t| < 1,
\]

where

\[(-\beta)_n := \prod_{k=0}^{n-1} (k - \beta), \quad n \geq 1, \quad (-\beta)_0 := 1.\]

Then, since

\[
\frac{1}{(1+z)^n} = \frac{1}{(n-1)!} \int_0^\infty e^{-zs} e^{-s} s^{n-1} ds, \quad z \in \mathbb{C}_+, \quad n \in \mathbb{N},
\]

it follows from \((8.1)\) that

\[
1 - \left(\frac{z}{z+1}\right)^\beta = \sum_{n=1}^\infty (-\beta)_n \frac{1}{n!} (z+1)^n
\]

\[
= -\sum_{n=1}^\infty (-\beta)_n \frac{1}{n!(n-1)!} \int_0^\infty e^{-zs} e^{-s} s^{n-1} ds
\]

\[
= \int_0^\infty e^{-zs} q_\beta(s) ds,
\]
where
\[ q_\beta(s) = \beta e^{-s} \sum_{n=0}^{\infty} \frac{(1-\beta)^n s^n}{(n+1)! n!} = \beta e^{-s} \, _1F_1(1-\beta; 2; s), \]
and \(_1F_1\) is a confluent hypergeometric function. (See [1, Chapter 4] for a background on confluent hypergeometric functions.) Hence
\[ c_\beta = 1 + \int_0^\infty |q_\beta(s)| \, ds = 1 + \beta \int_0^\infty e^{-s} \, _1F_1(1-\beta; 2; s) \, ds. \]
Let now \( \beta = m, m \in \mathbb{N} \). Then
\[ q_m(s) = me^{-s} \, _1F_1(1-m; 2; s) = e^{-s} L_{m-1}^{(1)}(s), \]
where \( L_k^{(1)}(\cdot) \) are the (generalized) Laguerre polynomials, and
\[ (8.2) \quad c_m = 1 + \int_0^\infty e^{-s}|L_{m-1}^{(1)}(s)| \, ds, \quad m \in \mathbb{N}. \]
Thus estimates of \( c_\beta \), at least for integer \( \beta \), reduce to estimates of means of the Laguerre polynomials. One of such estimates is provided by the following result.

**Theorem 8.1.** For every \( m \in \mathbb{N} \),
\[ (8.3) \quad \int_0^\infty e^{-s}|L_{m-1}^{(1)}(s)| \, ds \leq 2\sqrt{m}. \]

**Proof.** Recalling that
\[ \int_0^\infty e^{-s} |L_{m-1}^{(1)}(s)|^2 \, ds = m \]
by [39, Section 5.1], we have
\[ (8.4) \quad \int_0^\infty e^{-s}|L_{m-1}^{(1)}(s)| \, ds \leq \int_0^1 e^{-s}|L_{m-1}^{(1)}(s)|(1-s) \, ds + \int_0^\infty e^{-s}|L_{m-1}^{(1)}(s)| \, ds \]
\[ \leq \left( \int_0^1 e^{-s}(1-s)^2 \, ds \right)^{1/2} \left( \int_0^\infty e^{-s}|L_{m-1}^{(1)}(s)|^2 \, ds \right)^{1/2} \]
\[ + \left( \int_0^\infty se^{-s} \, ds \right)^{1/2} \left( \int_0^\infty e^{-s}|L_{m-1}^{(1)}(s)|^2 \, ds \right)^{1/2} \]
\[ \leq \sqrt{1-2e^{-1}} \left( \int_0^1 e^{-s}|L_{m-1}^{(1)}(s)|^2 \, ds \right)^{1/2} + \sqrt{m}. \]

By Watson’s identity [43, p. 21], if \( s > 0 \) then
\[
\frac{\pi}{2m} e^{-s}|L_{m-1}^{(1)}(s)|^2 = \int_0^\pi L_{m-1}^{(1)}(2s(1+\cos \theta)) e^{-s(1+\cos \theta)} \frac{\sin(s \sin \theta)}{s \sin \theta} \sin^2 \theta \, d\theta
\]
\[ = \int_0^\pi L_{m-1}^{(1)}(4s \cos^2 \frac{\theta}{2}) \, e^{-2s \cos^2 \frac{\theta}{2}} \frac{\sin(s \sin \theta)}{s} \sin \theta \, d\theta, \]
hence
\[
\frac{\pi}{2m} \int_0^1 |L^{(1)}_{m-1}(s)|^2 e^{-s} ds = \int_0^\pi \sin \theta \ G_{m-1}(\theta) \ d\theta,
\]
where
\[
G_{m-1}(\theta) := \int_0^1 L^{(1)}_{m-1} \left( \frac{4s \cos \frac{\theta}{2}}{2} \right) e^{-2s \cos^2 \frac{\theta}{2}} \frac{\sin(s \sin \theta)}{s} \ ds, \quad \theta \in [0, \pi].
\]
Using the properties
\[
L^{(1)}_{m-1}(s) = -\frac{d}{ds} L^{(0)}_{m}(s), \quad L^{(0)}_{m}(0) = 1,
\]
and integrating by parts we obtain
\[
4 \cos^2 \frac{\theta}{2} G_{m-1}(\theta) = -\int_0^1 e^{-2s \cos^2 \frac{\theta}{2}} \frac{\sin(s \sin \theta)}{s} \ d \left( \frac{L^{(0)}_{m}(4s \cos \frac{\theta}{2})}{2} \right)
\]
\[
= -e^{-2 \cos^2 \frac{\theta}{2}} \sin(s \sin \theta) L^{(0)}_{m} \left( \frac{4 \cos \frac{\theta}{2}}{2} \right) + \sin \theta
\]
\[
- 2 \cos^2 \frac{\theta}{2} \int_0^1 e^{-2s \cos^2 \frac{\theta}{2}} \frac{\sin(s \sin \theta)}{s} L^{(0)}_{m}(4s \cos \frac{\theta}{2}) \ ds
\]
\[
+ \int_0^1 e^{-2s \cos^2 \frac{\theta}{2}} \left[ \sin \theta \frac{\cos(s \sin \theta)}{s} - \frac{\sin(s \sin \theta)}{s^2} \right] L^{(0)}_{m}(4s \cos \frac{\theta}{2}) \ ds.
\]
Since by [17, 10.18(14)],
\[|L^{(0)}_{m}(t)| \leq e^{t/2}, \quad t \geq 0,\]
and
\[-\left( \frac{\sin \tau}{\tau} \right)' = \frac{\sin \tau}{\tau^2} - \frac{\cos \tau}{\tau} > 0, \quad \tau \in (0, \pi/2),\]
we have
\[
4 \cos^2 \frac{\theta}{2} G_{m-1}(\theta) \leq \sin(s \sin \theta) + \sin \theta + 2 \cos^2 \frac{\theta}{2} \int_0^1 \frac{\sin(s \sin \theta)}{s} \ ds
\]
\[
+ \sin^2 \theta \int_0^1 \left[ \frac{\sin(s \sin \theta)}{s^2 \sin^2 \frac{\theta}{2}} - \frac{\cos(s \sin \theta)}{s \sin \theta} \right] \ ds
\]
\[
\leq \sin(s \sin \theta) + \sin \theta + 2 \sin \theta \cos^2 \frac{\theta}{2} + \sin \theta \int_0^{\sin \theta} \frac{\sin \tau}{\tau^2} - \frac{\cos \tau}{\tau} \ d\tau
\]
\[
= \sin(s \sin \theta) + \sin \theta + 2 \sin \theta \cos^2 \frac{\theta}{2} + \sin \theta \left[ 1 - \frac{\sin(s \sin \theta)}{s \sin \theta} \right]
\]
\[
= 2 \sin \theta \left( 1 + \cos^2 \frac{\theta}{2} \right), \quad \theta \in (0, \pi),
\]

hence
\[
|G_{m-1}(\theta)| \leq \frac{\sin \frac{\theta}{2} (1 + \cos^2 \frac{\theta}{2})}{\cos \frac{\theta}{2}}.
\]
Then
\[
\frac{\pi}{2m} \int_0^1 |L_{m-1}^{(1)}(s)|^2 e^{-s} ds \leq \int_0^\pi \sin \theta |G_{m-1}(\theta)| d\theta
\]
\[
\leq \int_0^\pi \sin \theta \frac{\sin \frac{\theta}{2}(1 + \cos^2 \frac{\theta}{2})}{\cos \frac{\theta}{2}} d\theta
\]
\[
= 2 \int_0^\pi \sin^2 \frac{\theta}{2}(1 + \cos^2 \frac{\theta}{2}) d\theta
\]
\[
= 4 \int_0^{\pi/2} (1 + \cos^2 \theta) \sin^2 \theta d\theta
\]
\[
= 2 (B(1/2, 3/2) + B(3/2, 3/2))
\]
\[
= \frac{5\pi}{4},
\]
where $B$ is the beta function, and
\[
\int_0^1 |L_{m-1}^{(1)}(s)|^2 e^{-s} ds \leq \frac{5}{2} m.
\]
Therefore, by (8.4) we obtain that
\[
\int_0^\infty e^{-s}|L_{m-1}^{(1)}(s)| ds \leq \sqrt{m} + \sqrt{\left(1 - \frac{2}{e}\right) \frac{5}{2}} \sqrt{m} \leq 2\sqrt{m}.
\]

\section*{Corollary 8.1.} For all $\beta \geq 1$ and $m \in \mathbb{N},$
\begin{equation}
(8.5) \quad c_m \leq 1 + 2\sqrt{m}, \quad c_\beta \leq 2(1 + 2\sqrt{\beta}).
\end{equation}

\section*{Proof.} The bound for $c_m, m \in \mathbb{N},$ is a direct consequence of Theorem 8.1.

To estimate $c_\beta$ for $\beta \geq 1,$ we argue as in the proof of Lemma 4.1. If $\beta = m + \alpha, m \in \mathbb{N}, \alpha \in (0, 1),$ then by (4.4),
\[
c_\beta \leq c_m c_\alpha \leq 2(1 + \sqrt{m}) \leq 2(1 + \sqrt{\beta}).
\]

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