A SINGULAR LIMIT PROBLEM
FOR THE IBRAZIMOV-SHABAT EQUATION

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Abstract. We consider the Ibragimov-Shabat equation, which contains nonlinear dispersive effects. We prove that as the diffusion parameter tends to zero, the solutions of the dispersive equation converge to discontinuous weak solutions of a scalar conservation law. The proof relies on deriving suitable a priori estimates together with an application of the compensated compactness method in the $L^p$ setting.

1. Introduction

Bäcklund transformations have been useful in the calculation of soliton solutions of certain nonlinear evolution equations of physical significance [7, 18, 23, 24] restricted to one space variable $x$ and a time coordinate $t$. The classical treatment of the surface transformations, which provide the origin of Bäcklund theory, was developed in [9]. Bäcklund transformations are local geometric transformations, which construct from a given surface of constant Gaussian curvature $-1$ a two parameter family of such surfaces. To find such transformations, one needs to solve a system of compatible ordinary differential equations [8].

In [12, 13], the authors used the notion of differential equation for a function $u(t, x)$ that describes a pseudo-spherical surface, and they derived some Bäcklund transformations for nonlinear evolution equations which are the integrability condition $sl(2, R)$-valued linear problems [11, 10, 15, 16, 24].

In [17], the authors had derived some Bäcklund transformations for nonlinear evolution equations of the AKNS class. These transformations explicitly express the new solutions in terms of the known solutions of the nonlinear evolution equations and corresponding wave functions which are solutions of the associated Ablowitz-Kaup-Newell-Segur (AKNS) system [1, 26].

In [14], the authors used Bäcklund transformations derived in [12, 13] in the construction of exact soliton solutions for some nonlinear evolution equations describing pseudospherical surfaces which are beyond the AKNS class. In particular, they analyzed the following equation [2]:

\begin{equation}
\partial_x (\partial_t u + \alpha g(u) \partial_x u + \beta \partial_x u) = \gamma g'(u), \quad \alpha, \beta, \gamma \in \mathbb{R},
\end{equation}

where $g(u)$ is any solution of the linear ordinary differential equation

\begin{equation}
g''(u) + \mu g(u) = \theta, \quad \mu, \theta \in \mathbb{R}.
\end{equation}

(1.1) include the sine-Gordon, sinh-Gordon and Liouville equations, in correspondence of $\alpha = 0$.

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In [22], Rabelo proved that the system of the equations (1.1) and (1.2) describes pseudo-spherical surfaces and possesses a zero-curvature representation with a parameter.

In [3], the authors investigated the well-posedness in classes of discontinuous functions of (1.1), when \( \alpha = -1, \beta = 0, \mu = 0, \theta = 1. \)

Moreover, in [4], the authors investigated the well-posedness in classes of discontinuous functions of (1.1), when \( \alpha = 1, \beta = 0, \mu = -1, \theta = 1, \gamma = -1. \)

One more equation, that describes pseudo-spherical surface, is the following one [25]:

\[
\partial_t u_{\varepsilon, \beta} - \frac{3}{5} \partial_x (u_{\varepsilon, \beta}^5) + \beta \varepsilon \partial^3_{xxx} u = 3 \beta \varepsilon u_{\varepsilon, \beta}^2 \partial^2_{xx} u_{\varepsilon, \beta} - 9 u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2, 
\]

which is the Ibraginov-Shabat equation. Following [5, 6], we consider the following diffusive approximation of (1.3)

\[
\partial_t u_{\varepsilon, \beta} - \frac{3}{5} \partial_x (u_{\varepsilon, \beta}^5) + \beta \varepsilon \partial^3_{xxx} u = 3 \beta \varepsilon u_{\varepsilon, \beta}^2 \partial^2_{xx} u_{\varepsilon, \beta} - 9 \beta u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2 + \varepsilon \partial^2_{xx} u_{\varepsilon, \beta},
\]

where \( \beta \) is the dispersive parameter.

We consider the initial value problem for (1.4), so we augment (1.4) with the initial condition

\[
u(0, x) = u_0(x),
\]

on which we assume that

\[
u_0 \in L^2(\mathbb{R}) \cap L^{10}(\mathbb{R}).
\]

We are interested in the no high frequency limit, i.e., we send \( \beta \to 0 \) in (1.4). In this way, we pass from (1.4) to

\[
\partial_t u - \frac{3}{5} \partial_x (u^5) = 0, \quad t > 0, \ x \in \mathbb{R},
\]

\[
u(0, x) = u_0(x), \quad x \in \mathbb{R},
\]

which is a scalar conservation law.

We study the dispersion-diffusion limit for (1.4). Therefore, we fix two small numbers \( 0 < \varepsilon, \beta < 1 \), and consider the following third order problem

\[
\begin{cases}
\partial_t u_{\varepsilon, \beta} - \frac{3}{5} \partial_x (u_{\varepsilon, \beta}^5) + \beta \varepsilon \partial^3_{xxx} u_{\varepsilon, \beta} = 3 \beta u_{\varepsilon, \beta}^2 \partial^2_{xx} u_{\varepsilon, \beta} - 9 \beta u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2 + \varepsilon \partial^2_{xx} u_{\varepsilon, \beta}, & t > 0, \ x \in \mathbb{R}, \\
u_{\varepsilon, \beta}(0, x) = u_{\varepsilon, \beta, 0}(x), & x \in \mathbb{R},
\end{cases}
\]

where \( u_{\varepsilon, \beta, 0} \) is a \( C^\infty \) approximation of \( u_0 \) such that

\[
u_{\varepsilon, \beta, 0} \to u_0 \quad \text{in} \ L^p_{\text{loc}}(\mathbb{R}), \ 1 \leq p < 10, \ \text{as} \ \varepsilon, \beta \to 0,
\]

\[
\|u_{\varepsilon, \beta, 0}\|^2_{L^p(\mathbb{R})} + \|u_{\varepsilon, \beta, 0}\|^2_{L^{10}(\mathbb{R})} + (\beta + \varepsilon^2) \|\partial_x u_{\varepsilon, \beta, 0}\|^2_{L^2(\mathbb{R})} \leq C_0, \ \varepsilon, \beta > 0,
\]

where \( C_0 \) is a constant independent on \( \varepsilon \) and \( \beta \).

The main result of this paper is the following theorem.

**Theorem 1.1.** Assume that (1.6) and (1.9) hold. If

\[
\beta = O(\varepsilon^2),
\]

then, there exist two sequences \( \{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}} \) with \( \varepsilon_n, \beta_n \to 0 \), and a limit function

\[
u \in L^\infty(0, T; L^2(\mathbb{R}) \cap L^{10}(\mathbb{R})), \quad T > 0,
\]

such that

i) \( \nu \) is a distributional solution of (1.7),

ii) \( u_{\varepsilon_n, \beta_n} \to u \) strongly in \( L^p_{\text{loc}}((0, \infty) \times \mathbb{R}) \), for each \( 1 \leq p < 10 \).
Moreover, if
\[
(1.11) \quad \beta = \mathcal{O}(\varepsilon^{2+\alpha}), \quad \text{for some } \alpha > 0,
\]
then,
iii) \( u \) is the unique entropy solution of (1.7).

The paper is organized in three sections. In Section 2 we prove some a priori estimates, while in Section 3 we prove Theorem 1.1.

2. A priori Estimates

This section is devoted to some a priori estimates on \( u_{\varepsilon,\beta} \). We denote with \( C_0 \) the constants which depend only on the initial data, and with \( C(T) \) the constants which depend also on \( T \).

**Lemma 2.1.** For each \( t > 0 \),
\[
(2.1) \quad \|u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|\partial_x u_{\varepsilon,\beta}(s,\cdot)\|_{L^2(\mathbb{R})}^2 \, ds + 36\beta \int_0^t u_{\varepsilon,\beta}^2(\partial_x u_{\varepsilon,\beta})^2 \, ds \leq C_0.
\]

**Proof.** Multiplying (1.8) by \( u_{\varepsilon,\beta} \), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u_{\varepsilon,\beta}^2 \, dx = \int_{\mathbb{R}} u_{\varepsilon,\beta} \partial_t u_{\varepsilon,\beta} \, dx
\]
\[
= 3 \int_{\mathbb{R}} \partial_x u_{\varepsilon,\beta}^3 \, dx - \beta \int_{\mathbb{R}} \partial_x^3 u_{\varepsilon,\beta} \, dx
\]
\[
+ 3\beta \int_{\mathbb{R}} \partial_x^2 u_{\varepsilon,\beta}^2 \, dx - 9\beta \int_{\mathbb{R}} u_{\varepsilon,\beta}^2(\partial_x u_{\varepsilon,\beta})^2 \, dx
\]
\[
+ \varepsilon \int_{\mathbb{R}} \partial_x^2 u_{\varepsilon,\beta} \, dx
\]
\[
= \beta \int_{\mathbb{R}} \partial_x u_{\varepsilon,\beta} \partial_x^2 u_{\varepsilon,\beta} \, dx - 9\beta \int_{\mathbb{R}} u_{\varepsilon,\beta}^2(\partial_x u_{\varepsilon,\beta})^2 \, dx
\]
\[
- 9\beta \int \partial_x^2 u_{\varepsilon,\beta} \, dx - \varepsilon \int (\partial_x u_{\varepsilon,\beta})^2 \, dx,
\]
that is
\[
(2.2) \quad \frac{d}{dt} \|u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\beta}(t,\cdot)|^2 \, dx + 18\beta \int_{\mathbb{R}} u_{\varepsilon,\beta}^2(\partial_x u_{\varepsilon,\beta})^2 \, dx = 0.
\]

An integration on \((0, t)\) and (1.9) give (2.1). \(\square\)

**Lemma 2.2.** For each \( t > 0 \),
\[
\beta \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{5} \|u_{\varepsilon,\beta}(t,\cdot)\|_{L^6(\mathbb{R})}^6
\]
\[
+ 2\varepsilon \int_0^t \|\partial_x^2 u_{\varepsilon,\beta}(s,\cdot)\|_{L^2(\mathbb{R})}^2 \, ds + 6\beta^2 \int_0^t \|\partial_x u_{\varepsilon,\beta}(s,\cdot)\|_{L^4(\mathbb{R})}^4 \, ds
\]
\[
+ 6\varepsilon \int_0^t u_{\varepsilon,\beta}^4(\partial_x u_{\varepsilon,\beta})^2 \, ds + 6\beta^2 \int_0^t u_{\varepsilon,\beta}(\partial_x^2 u_{\varepsilon,\beta})^2 \, ds \, dx
\]
\[
+ \frac{42}{5} \beta \int_0^t u_{\varepsilon,\beta}^6(\partial_x u_{\varepsilon,\beta})^2 \, dx \leq C_0.
\]

In particular, we have
\[
(2.4) \quad \|u_{\varepsilon,\beta}(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C_0\beta^{-\frac{1}{4}}.
\]
Moreover, fixed \( T > 0 \),

\[
\|u_{\epsilon, \beta}\|_{L^p((0,T) \times \mathbb{R})} \leq C_0 T^\frac{1}{4},
\]

\[
\|u_{\epsilon, \beta}\|_{L^4((0,T) \times \mathbb{R})} \leq C_0 T^\frac{1}{4}.
\]

**Proof.** Multiplying \((1.8)\) by \(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\), we have

\[
\left(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\right) \partial_t u_{\epsilon, \beta} - \frac{3}{5} \left(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\right) \partial_x u_{\epsilon, \beta}^5
\]

\[
+ \beta \left(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\right) \partial^3_{xxx} u_{\epsilon, \beta}
\]

\[
= 3\beta \left(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\right) u_{\epsilon, \beta}^2 \partial^2_{xx} u_{\epsilon, \beta}
\]

\[
- 9\beta \left(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\right) u_{\epsilon, \beta} \left(\partial_x u_{\epsilon, \beta}\right)^2
\]

\[
+ \epsilon \left(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\right) \partial^2_{xx} u_{\epsilon, \beta}.
\]

Observe that

\[
\int_{\mathbb{R}} \left(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\right) \partial_t u_{\epsilon, \beta} dx = \frac{d}{dt} \left(\beta \int_{\mathbb{R}} \partial_x u_{\epsilon, \beta}(t, \cdot)^2 dx + \frac{1}{10} \int_{\mathbb{R}} \partial_x u_{\epsilon, \beta}(t, \cdot) dx^6\right),
\]

\[
-\frac{3}{5} \int_{\mathbb{R}} \left(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\right) \partial_x u_{\epsilon, \beta}^5 dx = 3\beta \int_{\mathbb{R}} u_{\epsilon, \beta}^4 \partial_x u_{\epsilon, \beta} \partial^2_{xx} u_{\epsilon, \beta} dx,
\]

\[
\beta \int_{\mathbb{R}} \left(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\right) \partial^3_{xxx} u_{\epsilon, \beta} dx = -3\beta \int_{\mathbb{R}} u_{\epsilon, \beta}^4 \partial_x u_{\epsilon, \beta} \partial^2_{xx} u_{\epsilon, \beta} dx,
\]

\[
3\beta \int_{\mathbb{R}} \left(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\right) u_{\epsilon, \beta}^2 \partial^2_{xx} u_{\epsilon, \beta} dx = -3\beta^2 \int_{\mathbb{R}} u_{\epsilon, \beta}^2 \left(\partial_x u_{\epsilon, \beta}\right)^2 dx
\]

\[
+ \frac{9}{5} \beta \int_{\mathbb{R}} u_{\epsilon, \beta}^7 \partial^2_{xx} u_{\epsilon, \beta} dx,
\]

\[
-9\beta \int_{\mathbb{R}} \left(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\right) u_{\epsilon, \beta} \left(\partial_x u_{\epsilon, \beta}\right)^2 dx = 9\beta^2 \int_{\mathbb{R}} u_{\epsilon, \beta} \left(\partial_x u_{\epsilon, \beta}\right)^2 \partial^2_{xx} u_{\epsilon, \beta} dx
\]

\[
- \frac{27}{5} \beta \int_{\mathbb{R}} u_{\epsilon, \beta}^6 \left(\partial_x u_{\epsilon, \beta}\right)^2 dx,
\]

\[
\epsilon \int_{\mathbb{R}} \left(-\beta \partial^2_{xx} u_{\epsilon, \beta} + \frac{3}{5} u_{\epsilon, \beta}^5\right) \partial^2_{xx} u_{\epsilon, \beta} dx = -\epsilon \beta \int_{\mathbb{R}} \partial^2_{xx} u_{\epsilon, \beta}(t, \cdot)^2 dx
\]

\[
- 3\epsilon \int_{\mathbb{R}} u_{\epsilon, \beta}^4 \left(\partial_x u_{\epsilon, \beta}\right)^2 dx.
\]
Therefore, integrating (2.7) over $\mathbb{R}$,

$$
\frac{d}{dt}\left( \frac{\beta}{2} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{1}{10} \| u_{\varepsilon, \beta}(t, \cdot) \|^6_{L^6(\mathbb{R})} \right)
+ \beta \varepsilon \| \partial^{2}_{xx} u_{\varepsilon, \beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + 3 \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \beta}^4(\partial_x u_{\varepsilon, \beta})^2 \, dx
+ 3 \beta^2 \int_{\mathbb{R}} u_{\varepsilon, \beta}^2(\partial^{2}_{xx} u_{\varepsilon, \beta})^2 \, dx + \frac{27}{5} \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^6(\partial_x u_{\varepsilon, \beta})^2 \, dx
= \frac{9}{5} \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^7(\partial_x u_{\varepsilon, \beta})^2 \, dx + 9 \int_{\mathbb{R}} u_{\varepsilon, \beta}^2(\partial^{2}_{xx} u_{\varepsilon, \beta})^2 \, dx + 27 \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^6(\partial_x u_{\varepsilon, \beta})^2 \, dx + 3 \beta^2 \int_{\mathbb{R}} u_{\varepsilon, \beta}^4(\partial_x u_{\varepsilon, \beta})^2 \, dx.
$$

(2.8)

Since

$$
\frac{9}{5} \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^7(\partial_x u_{\varepsilon, \beta})^2 \, dx = - \frac{63}{5} \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^6(\partial_x u_{\varepsilon, \beta})^2 \, dx,
9 \beta^2 \int_{\mathbb{R}} u_{\varepsilon, \beta}(\partial_x u_{\varepsilon, \beta})^2(\partial^{2}_{xx} u_{\varepsilon, \beta}) \, dx = - 3 \beta^2 \int_{\mathbb{R}} (\partial_x u_{\varepsilon, \beta})^4 \, dx,
$$

it follows from (2.8) that

$$
\frac{d}{dt}\left( \frac{\beta}{2} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{1}{10} \| u_{\varepsilon, \beta}(t, \cdot) \|^6_{L^6(\mathbb{R})} \right)
+ \beta \varepsilon \| \partial^{2}_{xx} u_{\varepsilon, \beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + 3 \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \beta}^4(\partial_x u_{\varepsilon, \beta})^2 \, dx
+ 3 \beta^2 \int_{\mathbb{R}} u_{\varepsilon, \beta}^2(\partial^{2}_{xx} u_{\varepsilon, \beta})^2 \, dx + \frac{27}{5} \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^6(\partial_x u_{\varepsilon, \beta})^2 \, dx
+ 3 \beta^2 \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|^4_{L^4(\mathbb{R})} = 0.
$$

An integration on $(0, t)$ and (1.9) give (2.3).

We prove (2.4). Due to (2.1), (2.3) and the H"older inequality,

$$
u_{\varepsilon, \beta}^2(t, x) = 2 \int_{-\infty}^{x} u_{\varepsilon, \beta}(\partial_x u_{\varepsilon, \beta}) \, dx \leq 2 \int_{\mathbb{R}} |u_{\varepsilon, \beta}(\partial_x u_{\varepsilon, \beta})| \, dx
\leq 2 \| u_{\varepsilon, \beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \leq C_0 \beta^{-\frac{1}{2}},
$$

which gives (2.4).

We prove (2.5). From (2.2), we have

$$\| u_{\varepsilon, \beta}(t, \cdot) \|^6_{L^6(\mathbb{R})} \leq C_0.$$

An integration on $(0, T)$ gives (2.5).

Finally, we prove (2.6). Due to (2.1), (2.3) and the Young inequality,

$$
\int_{\mathbb{R}} u_{\varepsilon, \beta}^4 \, dx \leq \frac{1}{2} \int_{\mathbb{R}} u_{\varepsilon, \beta}^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} u_{\varepsilon, \beta}^6 \, dx \leq C_0.
$$

(2.9)

Therefore, fix $T > 0$, (2.6) follows from (2.9) and an integration on $(0, T)$. □
Lemma 2.3. Let $T > 0$. Assume (1.10) holds true. There exists $C(T) > 0$, independent on $\varepsilon$ and $\beta$, such that

\begin{align*}
&\frac{1}{10} \left\| u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^1_0(\mathbb{R})}^2 + \frac{3\varepsilon^2}{2} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\quad + 45\varepsilon^2 \beta \int_0^t \left\| \partial_x u_{\varepsilon, \beta}(s, \cdot) \right\|_{L^1(\mathbb{R})}^4 ds + 4\varepsilon \int_0^t \int_{\mathbb{R}} \left( u_{\varepsilon, \beta}^8 (\partial_x u_{\varepsilon, \beta})^2 \right) ds dx \\
&\quad + \frac{1}{2} \varepsilon^3 \beta \int_0^t \left\| \partial_{xx} u_{\varepsilon, \beta}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + 15\varepsilon^2 \beta \int_0^t \int_{\mathbb{R}} \left( u_{\varepsilon, \beta}^2 (\partial_x^2 u_{\varepsilon, \beta})^2 \right) dx \\
&\quad + 48\beta \int_0^t u_{\varepsilon, \beta}^{10} (\partial_x u_{\varepsilon, \beta})^2 dx \leq C(T).
\end{align*}

(2.10)

for every $0 < t < T$. Moreover,

\begin{align*}
\beta \left\| \partial_x u_{\varepsilon, \beta} \partial_{xx} u_{\varepsilon, \beta} \right\|_{L^1(0,T) \times \mathbb{R}} &\leq C(T), \\
\beta^2 \int_0^T \left\| \partial_{xx}^2 u_{\varepsilon, \beta}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds &\leq C(T)\varepsilon.
\end{align*}

(2.11) (2.12)

Proof. Let $0 < t < T$. Multiplying (1.8) by $u_{\varepsilon, \beta}^0 - 5\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}$, we have

\begin{align*}
(u_{\varepsilon, \beta}^0 - 5\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}) \partial_t u_{\varepsilon, \beta} - \frac{3}{5} (u_{\varepsilon, \beta}^0 - 5\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta}^5 \\
+ \beta (u_{\varepsilon, \beta}^0 - 5\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}) \partial_{xx}^3 u_{\varepsilon, \beta} \\
= -3\beta (u_{\varepsilon, \beta}^0 - 5\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}) u_{\varepsilon, \beta}^2 \partial_{xx}^2 u_{\varepsilon, \beta} \\
- 9\beta (u_{\varepsilon, \beta}^0 - 5\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}) u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2 \\
+ \varepsilon (u_{\varepsilon, \beta}^0 - 5\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}) \partial_{xx}^2 u_{\varepsilon, \beta}.
\end{align*}

(2.13)

Since

\begin{align*}
\int_{\mathbb{R}} (u_{\varepsilon, \beta}^0 - 3\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}) \partial_t u_{\varepsilon, \beta} dx &= \frac{d}{dt} \left( \frac{1}{10} \left\| u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^1_0(\mathbb{R})}^2 + \frac{5\varepsilon^2}{2} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right), \\
-\frac{3}{5} \int_{\mathbb{R}} (u_{\varepsilon, \beta}^0 - 5\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta}^5 dx &= 9\varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon, \beta}^4 \partial_x u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} dx, \\
\beta \int_{\mathbb{R}} (u_{\varepsilon, \beta}^0 - 5\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}) \partial_{xx}^3 u_{\varepsilon, \beta} dx &= -9\beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^8 \partial_x u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} dx \\
&= 36\beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^7 \partial_x u_{\varepsilon, \beta} dx, \\
3\beta \int_{\mathbb{R}} (u_{\varepsilon, \beta}^0 - 5\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}) u_{\varepsilon, \beta}^2 \partial_{xx}^2 u_{\varepsilon, \beta} dx &= 3\beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^1 \partial_{xx}^2 u_{\varepsilon, \beta} dx \\
&- 15\beta \varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon, \beta}^2 \partial_{xx}^2 u_{\varepsilon, \beta}^2 dx \\
-9\beta \int_{\mathbb{R}} (u_{\varepsilon, \beta}^0 - 5\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}) u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2 dx &= -9\beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^10 (\partial_x u_{\varepsilon, \beta})^2 dx \\
&+ 45\varepsilon^2 \beta \int_{\mathbb{R}} u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2 \partial_{xx}^2 u_{\varepsilon, \beta} dx \\
\varepsilon \int_{\mathbb{R}} (u_{\varepsilon, \beta}^0 - 5\varepsilon^2 \partial_{xx} u_{\varepsilon, \beta}) \partial_{xx}^2 u_{\varepsilon, \beta} dx &= -9\varepsilon \int_{\mathbb{R}} u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2 dx - 5\varepsilon^3 \left\| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\end{align*}
integrating (2.13) over $\mathbb{R}$,
\[
\frac{d}{dt} \left( \frac{1}{10} \| u_{\varepsilon, \beta}(t, \cdot) \|_{L^1_0(\mathbb{R})}^{10} + \frac{5 \varepsilon^2}{2} \| \partial_t u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) + 9 \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \beta}^8 (\partial_x^2 u_{\varepsilon, \beta})^2 \, dx + 5 \varepsilon^3 \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
+ 15 \beta \varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon, \beta}^2 (\partial_x^2 u_{\varepsilon, \beta})^2 \, dx + 15 \beta \varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon, \beta}^10 (\partial_t u_{\varepsilon, \beta})^2 \, dx \\
= 45 \varepsilon^2 \beta \int_{\mathbb{R}} u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2 \partial_x^2 u_{\varepsilon, \beta} \, dx + 3 \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^11 \partial_x^2 u_{\varepsilon, \beta} \, dx \\
- 9 \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \beta}^4 \partial_x^2 u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \, dx + 36 \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^7 (\partial_x u_{\varepsilon, \beta})^2 \, dx.
\]

(2.14)

Since
\[
45 \varepsilon^2 \beta \int_{\mathbb{R}} u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2 \partial_x^2 u_{\varepsilon, \beta} \, dx = -45 \varepsilon^2 \beta \int_{\mathbb{R}} (\partial_x u_{\varepsilon, \beta})^4 \, dx,
\]
\[
3 \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^11 \partial_x^2 u_{\varepsilon, \beta} \, dx = -33 \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^10 (\partial_x u_{\varepsilon, \beta})^2 \, dx,
\]
from (2.14), we have
\[
\frac{d}{dt} \left( \frac{1}{10} \| u_{\varepsilon, \beta}(t, \cdot) \|_{L^1_0(\mathbb{R})}^{10} + \frac{3 \varepsilon^2}{2} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) \\
+ 45 \varepsilon^2 \beta \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^1(\mathbb{R})}^4 + 9 \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \beta}^8 (\partial_x u_{\varepsilon, \beta})^2 \, dx \\
+ 5 \varepsilon^3 \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 15 \beta \varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon, \beta}^2 (\partial_x^2 u_{\varepsilon, \beta})^2 \, dx \\
+ 48 \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^10 (\partial_t u_{\varepsilon, \beta})^2 \, dx \\
= -9 \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \beta}^4 \partial_x^2 u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \, dx + 36 \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^7 (\partial_x u_{\varepsilon, \beta})^2 \, dx.
\]

(2.15)

Due to the Young inequality,
\[
9 \varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon, \beta}^4 \partial_x u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta} \, dx \leq 9 \int_{\mathbb{R}} \left| \varepsilon^2 u_{\varepsilon, \beta}^4 \partial_x u_{\varepsilon, \beta} \right| \left| \varepsilon^3 \partial_x^2 u_{\varepsilon, \beta} \right| \, dx \\
\leq \frac{9 \varepsilon}{2} \int_{\mathbb{R}} u_{\varepsilon, \beta}^8 (\partial_x u_{\varepsilon, \beta})^2 \, dx + \frac{9 \varepsilon^3}{2} \| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2.
\]

(2.16)

It follows from (2.15) and (2.16) that
\[
\frac{d}{dt} \left( \frac{1}{10} \| u_{\varepsilon, \beta}(t, \cdot) \|_{L^1_0(\mathbb{R})}^{10} + \frac{3 \varepsilon^2}{2} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) \\
+ 45 \varepsilon^2 \beta \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^1(\mathbb{R})}^4 + \frac{9 \varepsilon}{2} \int_{\mathbb{R}} u_{\varepsilon, \beta}^8 (\partial_x u_{\varepsilon, \beta})^2 \, dx \\
+ \frac{1}{2} \varepsilon^3 \| \partial_x^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 15 \beta \varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon, \beta}^2 (\partial_x^2 u_{\varepsilon, \beta})^2 \, dx \\
+ 48 \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^10 (\partial_t u_{\varepsilon, \beta})^2 \, dx = 36 \beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^7 (\partial_x u_{\varepsilon, \beta})^2 \, dx.
\]

(2.17)
Since $0 < \varepsilon < 1$, thanks to (1.10), (2.4) and the Young inequality,
\[
36\beta \int_{\mathbb{R}} u_{\varepsilon, \beta}^7 (\partial_x u_{\varepsilon, \beta})^2 \, dx \leq 36\beta \int_{\mathbb{R}} |u_{\varepsilon, \beta}|^7 (\partial_x u_{\varepsilon, \beta})^2 \, dx \\
\leq 36\beta \|u_{\varepsilon, \beta}(t, \cdot)\|_{L^{\infty}(\mathbb{R})} \int_{\mathbb{R}} u_{\varepsilon, \beta}^6 (\partial_x u_{\varepsilon, \beta})^2 \, dx \\
\leq C_{0}\beta \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \beta}^6 (\partial_x u_{\varepsilon, \beta})^2 \, dx \\
\leq \varepsilon^2 C_0 \int_{\mathbb{R}} u_{\varepsilon, \beta}^4 (\partial_x u_{\varepsilon, \beta})^2 \, dx + \frac{\varepsilon}{2} \int_{\mathbb{R}} u_{\varepsilon, \beta}^8 (\partial_x u_{\varepsilon, \beta})^2 \, dx \\
\leq \varepsilon C_0 \int_{\mathbb{R}} u_{\varepsilon, \beta}^4 (\partial_x u_{\varepsilon, \beta})^2 \, dx + \frac{\varepsilon}{2} \int_{\mathbb{R}} u_{\varepsilon, \beta}^8 (\partial_x u_{\varepsilon, \beta})^2 \, dx.
\]  
(2.18)

Therefore, from (2.17) and (2.18),
\[
\frac{d}{dt} \left( \frac{1}{10} \|u_{\varepsilon, \beta}(t, \cdot)\|^2_{L^1(\mathbb{R})} + \frac{3\varepsilon^2}{2} \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|^2_{L^2(\mathbb{R})} \right) \\
+ 45\varepsilon^2 \beta \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|^4_{L^1(\mathbb{R})} + 4\varepsilon \int_{\mathbb{R}} u_{\varepsilon, \beta}^8 (\partial_x u_{\varepsilon, \beta})^2 \, dx \\
+ \frac{1}{2} \varepsilon^3 \|\partial_{xx}^2 u_{\varepsilon, \beta}(t)\|^2_{L^2(\mathbb{R})} + 15\beta \varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon, \beta}^2 (\partial_{xx}^2 u_{\varepsilon, \beta})^2 \, dx \\
+ 48\varepsilon \int_{\mathbb{R}} u_{\varepsilon, \beta}^{10} (\partial_x u_{\varepsilon, \beta})^2 \, dx \leq \varepsilon C_0 \int_{\mathbb{R}} u_{\varepsilon, \beta}^4 (\partial_x u_{\varepsilon, \beta})^2 \, dx.
\]

An integration on $(0, t)$ and (2.3) give (2.10).

We show that (2.12) holds. Thanks to (1.10), (2.1), (2.10) and Hölder inequality,
\[
\beta \int_{0}^{T} \int_{\mathbb{R}} |\partial_x u_{\varepsilon, \beta}|^2 \, dx \, ds \leq \beta \int_{0}^{T} \int_{\mathbb{R}} |\partial_x u_{\varepsilon, \beta}|^2 \, dx \, ds \\
\leq \beta \int_{0}^{T} \int_{\mathbb{R}} |\partial_x u_{\varepsilon, \beta}|^2 \, dx \, ds \\
\leq C(T) \beta \varepsilon^2 \leq C(T),
\]

that is (2.11).

Finally, we prove (2.12). Due to (1.10) and (2.10), we have
\[
\beta^2 \int_{0}^{T} \|\partial_{xx}^2 u_{\varepsilon, \beta}(s, \cdot)\|^2_{L^2(\mathbb{R})} \, ds \leq C_0 \varepsilon^4 \int_{0}^{T} \|\partial_{xx}^2 u_{\varepsilon, \beta}(s, \cdot)\|^2_{L^2(\mathbb{R})} \, ds \leq C(T) \varepsilon,
\]

which gives (2.12). □

3. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. The following technical lemma is needed [20].

**Lemma 3.1.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$. Suppose that the sequence $\{L_n\}_{n \in \mathbb{N}}$ of distributions is bounded in $W^{-1, \infty}(\Omega)$. Suppose also that
\[
L_n = L_{1,n} + L_{2,n},
\]
where \( \{L_{1,n}\}_{n \in \mathbb{N}} \) lies in a compact subset of \( H^{-1}_\text{loc}(\Omega) \) and \( \{L_{2,n}\}_{n \in \mathbb{N}} \) lies in a bounded subset of \( M_{\text{loc}}(\Omega) \). Then \( \{L_n\}_{n \in \mathbb{N}} \) lies in a compact subset of \( H^{-1}_\text{loc}(\Omega) \).

Moreover, we consider the following definition.

**Definition 3.1.** A pair of functions \( (\eta, q) \) is called an entropy–entropy flux pair if \( \eta : \mathbb{R} \to \mathbb{R} \) is a \( C^2 \) function and \( q : \mathbb{R} \to \mathbb{R} \) is defined by

\[
q(u) = - \int_0^u 3\xi^2 \eta'(\xi) d\xi.
\]

An entropy-entropy flux pair \( (\eta, q) \) is called convex/compactly supported if, in addition, \( \eta \) is convex/compactly supported.

We begin by proving the following result.

**Lemma 3.2.** Assume that \( (1.6), (1.9), \) and \( (1.10) \) hold. Then for any compactly supported entropy–entropy flux pair \( (\eta, q) \), there exist two sequences \( \{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}} \), with \( \varepsilon_n, \beta_n \to 0 \), and a limit function

\[
u \in L^\infty(0,T;L^2(\mathbb{R}) \cap L^{10}(\mathbb{R})),
\]

such that

\[
u_{\varepsilon_n,\beta_n} \to \nu \quad \text{in} \quad L^p_{\text{loc}}((0,\infty) \times \mathbb{R}), \quad \text{for each} \quad 1 \leq p < 10
\]

and \( \nu \) is a distributional solution of \( (1.4) \).

**Proof.** Let \( \mathbb{R}^+ = (0, \infty) \), and let us consider a compactly supported entropy–entropy flux pair \( (\eta, q) \). Multiplying \( (1.8) \) by \( \eta'(u_{\varepsilon, \beta}) \), we have

\[
\partial_t \eta(u_{\varepsilon, \beta}) + \partial_x q(u_{\varepsilon, \beta}) = \varepsilon \eta'(u_{\varepsilon, \beta}) \partial_x^2 u_{\varepsilon, \beta} - \beta \eta'(u_{\varepsilon, \beta}) \partial_x^3 u_{\varepsilon, \beta} + 3 \beta \eta'(u_{\varepsilon, \beta}) u_{\varepsilon, \beta}^2 \partial_x^2 u_{\varepsilon, \beta} - 9 \beta \eta'(u_{\varepsilon, \beta}) u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2
\]

\[
= I_1, \varepsilon, \beta + I_2, \varepsilon, \beta + I_3, \varepsilon, \beta + I_4, \varepsilon, \beta + I_5, \varepsilon, \beta + I_6, \varepsilon, \beta + I_7, \varepsilon, \beta,
\]

where

\[
I_{1, \varepsilon, \beta} = \partial_x (\varepsilon \eta'(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta}),
I_{2, \varepsilon, \beta} = - \varepsilon \eta''(u_{\varepsilon, \beta}) (\partial_x u_{\varepsilon, \beta})^2,
I_{3, \varepsilon, \beta} = - \partial_x (\beta \eta'(u_{\varepsilon, \beta}) \partial_x^2 u_{\varepsilon, \beta}),
I_{4, \varepsilon, \beta} = \beta \eta''(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta},
I_{5, \varepsilon, \beta} = \partial_x (3 \beta \eta'(u_{\varepsilon, \beta}) u_{\varepsilon, \beta}^2 \partial_x u_{\varepsilon, \beta}),
I_{6, \varepsilon, \beta} = - 3 \beta \eta''(u_{\varepsilon, \beta}) u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2,
I_{7, \varepsilon, \beta} = - 15 \beta \eta''(u_{\varepsilon, \beta}) u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2.
\]

We have

\[
I_{1, \varepsilon, \beta} \to 0 \quad \text{in} \quad H^{-1}((0,T) \times \mathbb{R}), \quad T > 0, \quad \text{as} \quad \varepsilon \to 0.
\]

Thanks to Lemma 2.1,

\[
\left\| \varepsilon \eta'(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \right\|_{L^2((0,T) \times \mathbb{R})}^2 \leq \left\| \eta' \right\|_{L^\infty(\mathbb{R})}^2 \varepsilon^2 \int_0^T \left\| \partial_x u_{\varepsilon, \beta} (s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds
\]

\[
\leq \left\| \eta' \right\|_{L^\infty(\mathbb{R})}^2 \frac{\varepsilon C_0}{2} \to 0.
\]
We claim that 
\[ \{I_{2, \varepsilon, \beta}\}_{\varepsilon, \beta > 0} \text{ is bounded in } L^1((0, T) \times \mathbb{R}), \ T > 0. \]
Again by Lemma 2.1,
\[
\| \varepsilon \eta''(u_{\varepsilon, \beta})(\partial_x u_{\varepsilon, \beta}) \|^2_{L^1((0, T) \times \mathbb{R})} \leq \| \eta'' \|_{L^\infty(\mathbb{R})} \varepsilon \int_0^T \| \partial_x u_{\varepsilon, \beta}(s, \cdot) \|^2_{L^2(\mathbb{R})} \, ds
\leq \| \eta'' \|_{L^\infty(\mathbb{R})} \frac{C_0}{2}.
\]
We have that
\[ I_{3, \varepsilon, \beta} \to 0 \text{ in } H^{-1}((0, T) \times \mathbb{R}), \ T > 0, \text{ as } \varepsilon \to 0. \]
Thanks to Lemma 2.3,
\[
\| \beta^2 \eta'(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \|^2_{L^2((0, T) \times \mathbb{R})} \leq \| \eta' \|_{L^\infty(\mathbb{R})}^2 \beta^2 \int_0^T \| \partial_x u_{\varepsilon, \beta}(s, \cdot) \|^2_{L^2(\mathbb{R})} \, ds
\leq \| \eta' \|_{L^\infty(\mathbb{R})}^2 C(T) \varepsilon \to 0.
\]
Let us show that 
\[ \{I_{4, \varepsilon, \beta}\}_{\varepsilon, \beta > 0} \text{ is bounded in } L^1((0, T) \times \mathbb{R}), \ T > 0. \]
Again by Lemma 2.3,
\[
\| \beta \eta''(u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \|^2_{L^1((0, T) \times \mathbb{R})} \leq \| \eta'' \|_{L^\infty(\mathbb{R})} \beta \int_0^T \| \partial_x u_{\varepsilon, \beta}(s, \cdot) \|^2_{L^1(\mathbb{R})} \, ds
\leq \| \eta'' \|_{L^\infty(\mathbb{R})} \beta \int_0^T \| \partial_x u_{\varepsilon, \beta}(s, \cdot) \|^2_{L^1(\mathbb{R})} \, ds.
\]
We claim that
\[ I_{5, \varepsilon, \beta} \to 0 \text{ in } H^{-1}((0, T) \times \mathbb{R}), \ T > 0, \text{ as } \varepsilon \to 0. \]
Due to (1.10), (2.4), (2.5) and the Hölder inequality,
\[
\| 3 \beta \eta''(u_{\varepsilon, \beta}) u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \|^2_{L^2((0, T) \times \mathbb{R})} \leq 9 \| \eta'' \|_{L^\infty(\mathbb{R})} \beta^2 \int_0^T \int_{\mathbb{R}} u_{\varepsilon, \beta}(\partial_x u_{\varepsilon, \beta})^2 \, ds \, dx
\leq 9 \| \eta'' \|_{L^\infty(\mathbb{R})} \beta^2 \| u_{\varepsilon, \beta}(t, \cdot) \|^2_{L^\infty(\mathbb{R})} \int_0^T \int_{\mathbb{R}} u_{\varepsilon, \beta}(\partial_x u_{\varepsilon, \beta})^2 \, ds \, dx
\leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \beta^2 \| u_{\varepsilon, \beta}(t, \cdot) \|^2_{L^\infty(\mathbb{R})} \int_0^T \int_{\mathbb{R}} u_{\varepsilon, \beta}(\partial_x u_{\varepsilon, \beta})^2 \, ds \, dx
\leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \beta^2 \beta \| u_{\varepsilon, \beta} \|^3_{L^3((0, T) \times \mathbb{R})} \| \partial_x u_{\varepsilon, \beta} \|^2_{L^4((0, T) \times \mathbb{R})}
\leq C_0 T^\frac{1}{2} \| \eta'' \|_{L^\infty(\mathbb{R})} \frac{\beta^2 \beta \| u_{\varepsilon, \beta} \|^3_{L^3((0, T) \times \mathbb{R})}}{\varepsilon} \to 0.
\]
We have that 
\[ \{I_{6, \varepsilon, \beta}\}_{\varepsilon, \beta > 0} \text{ is bounded in } L^1((0, T) \times \mathbb{R}), \ T > 0. \]
Thanks to (2.1),
\[
\| 3 \beta \eta''(u_{\varepsilon, \beta}) u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \|^2_{L^1((0, T) \times \mathbb{R})} \leq 3 \| \eta'' \|_{L^\infty(\mathbb{R})} \beta \int_0^T \int_{\mathbb{R}} u_{\varepsilon, \beta}(\partial_x u_{\varepsilon, \beta})^2 \, dt \, dx
\]
Let us show that

Due to (1.10), (2.1) and (2.4),

be a test function with compact support. We have to prove that

\[ \int_\mathbb{R} (u_{\epsilon, \beta})_x \phi dx = 0. \]

We claim that

Due to (1.10), (2.1) and (2.4),

\[ \int_\mathbb{R} (u_{\epsilon, \beta})_x \phi dx = 0. \]

Therefore, Lemma 3.1 and the \( L^p \) compensated compactness [21] give (3.1).

We conclude by proving that \( u \) is a distributional solution of (1.4). Let \( \phi \in C^\infty(\mathbb{R}^2) \) be a test function with compact support. We have to prove that

(3.3) \[ \int_0^\infty R \left( u_{\epsilon, \beta} + 3u_{\epsilon, \beta}^5 \partial_x \phi \right) dt dx + \int_\mathbb{R} u_0(x) \phi(0, x) dx = 0. \]

We have that

\[ \int_0^\infty R \left( u_{\epsilon, \beta} \partial_t \phi - \frac{3u_{\epsilon, \beta}^5}{5} \partial_x \phi \right) dt dx + \int_\mathbb{R} u_0 \phi(0, x) dx \]

\[ + \epsilon \int_0^\infty R u_{\epsilon, \beta} \partial_x^2 \phi dt dx + \epsilon \int_0^\infty R u_0 \partial_x^2 \phi(0, x) dx \]

\[ + \beta \int_0^\infty R u_{\epsilon, \beta} \partial_x \phi dt dx + \beta \int_0^\infty R u_0 \partial_x \phi(0, x) dx \]

\[ = -3 \beta \int_0^\infty R u_{\epsilon, \beta} \partial_x \phi dt dx + 9 \beta \int_0^\infty R u_{\epsilon, \beta} \partial_x^2 \phi dt dx \]

\[ = 15 \beta \int_0^\infty R u_{\epsilon, \beta} \partial_x \phi dt dx. \]

Let us show that

(3.4) \[ 15 \beta \int_0^\infty R u_{\epsilon, \beta} \partial_x \phi dt dx \to 0. \]

Due to (1.10), (2.1) and (2.4),

\[ 15 \beta \int_0^\infty R u_{\epsilon, \beta} \partial_x \phi dt dx \]

\[ \leq 15 \beta \int_0^\infty R |u_{\epsilon, \beta}| \partial_x \phi dt dx \]
Due to (1.11), (2.1), (2.10), and the Hölder inequality, $I$ is bounded in $I$; that is (3.4).

Following [19], we prove the following result.

Therefore, (3.3) follows from (1.9), (3.1), (3.4) and (3.5).

Thanks to (2.5) and the Hölder inequality,

$$
\beta_n \int_0^\infty \int_\mathbb{R} u_{\varepsilon, \beta_n}^3 \partial_{xx}^2 \phi dt dx \to 0.
$$

that is (3.5).

Therefore, (3.3) follows from (1.9), (3.1), (3.4) and (3.5).

Following [19], we prove the following result.

**Lemma 3.3.** Assume that (1.6), (1.9), and (1.11) hold. Then,

$$
u_{\varepsilon, \beta_n} \to u \text{ in } L^p_{\text{loc}}((0, \infty) \times \mathbb{R}), \quad \text{for each } 1 \leq p < 10,
$$

where $u$ is the unique entropy solution of (1.7).

**Proof.** Let us consider a compactly supported entropy–entropy flux pair $(\eta, \varphi)$. Multiplying (1.8) by $\eta'(u_{\varepsilon, \beta})$, we obtain that

$$
\partial_t \eta'(u_{\varepsilon, \beta}) + \partial_x q(u_{\varepsilon, \beta}) = \varepsilon \eta'(u_{\varepsilon, \beta}) \partial_{xx}^2 u_{\varepsilon, \beta} + 3\beta \eta'(u_{\varepsilon, \beta}) u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} - 9\beta \eta'(u_{\varepsilon, \beta}) u_{\varepsilon, \beta} (\partial_x u_{\varepsilon, \beta})^2
$$

$$
= I_1, \varepsilon, \beta + I_2, \varepsilon, \beta + I_3, \varepsilon, \beta + I_4, \varepsilon, \beta + I_5, \varepsilon, \beta + I_6, \varepsilon, \beta + I_7, \varepsilon, \beta,
$$

where $I_1, \varepsilon, \beta, I_2, \varepsilon, \beta, I_3, \varepsilon, \beta, I_4, \varepsilon, \beta, I_5, \varepsilon, \beta, I_6, \varepsilon, \beta, I_7, \varepsilon, \beta$ are defined in (3.2).

Arguing as [19] Lemma 3.3, we obtain that $I_1, \varepsilon, \beta \to 0$ in $H^{-1}((0, T) \times \mathbb{R})$, $\{I_2, \varepsilon, \beta\}_{\varepsilon > 0}$ is bounded in $L^1((0, T) \times \mathbb{R})$, $I_3, \varepsilon, \beta \to 0$ in $H^{-1}((0, T) \times \mathbb{R})$, $I_5, \varepsilon, \beta \to 0$ in $H^{-1}((0, T) \times \mathbb{R})$ and $I_7, \varepsilon, \beta \to 0$ in $L^1((0, T) \times \mathbb{R})$.

Let us show that $I_4, \varepsilon, \beta \to 0$ in $L^1((0, T) \times \mathbb{R})$, $T > 0$ as $\varepsilon \to 0$.

Due to (1.11), (2.1), (2.10), and the Hölder inequality,

$$
\|\beta \eta''(u_{\varepsilon, \beta}) \partial_{xx} u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta}\|_{L^1((0, T) \times \mathbb{R})}
$$

$$
\leq \|\eta''\|_{L^\infty(\mathbb{R})} \beta \int_0^T \|\partial_x u_{\varepsilon, \beta}(s, \cdot) \partial_{xx}^2 u_{\varepsilon, \beta}(s, \cdot)\|_{L^1(\mathbb{R})} ds
$$
Thanks to (1.11), (2.6), (2.10) and the Hölder inequality, \( \eta, q \) a compactly supported entropy–entropy flux pair (negative function. We have to prove that

\[
\int_0^T \int_\mathbb{R} \left| \partial_x u_{\varepsilon, \beta}(s, \cdot) \partial_{xx} u_{\varepsilon, \beta}(s, \cdot) \right| ds dx
\]

\leq C_0 \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \varepsilon \varepsilon^2 \int_0^T \left\| \partial_x u_{\varepsilon, \beta} \right\|_{L^2((0,T) \times \mathbb{R})} ds dx
\]

\leq C_0 \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \varepsilon \varepsilon^2 \left\| \partial_x u_{\varepsilon, \beta} \right\|_{L^2((0,T) \times \mathbb{R})} \varepsilon^2 \left\| \partial_{xx} u_{\varepsilon, \beta} \right\|_{L^2((0,T) \times \mathbb{R})}
\]

\leq C(T) \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \varepsilon^\alpha \to 0.

We have that

\[ I_{6, \varepsilon, \beta} \to 0 \quad \text{is in } L^1((0,T) \times \mathbb{R}), \quad T > 0 \text{ as } \varepsilon \to 0. \]

Thanks to (1.11), (2.6), (2.10) and the Hölder inequality,

\[
\left\| 3 \beta \eta''(u_{\varepsilon, \beta}) u_{\varepsilon, \beta}^2 (\partial_x u_{\varepsilon, \beta})^2 \right\|_{L^1((0,T) \times \mathbb{R})}
\]

\leq 3 \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \beta \int_0^T \int_\mathbb{R} u_{\varepsilon, \beta}^2 (\partial_x u_{\varepsilon, \beta})^2 ds dx
\]

\leq 3 \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \beta \| u_{\varepsilon, \beta} \|_{L^2((0,T) \times \mathbb{R})} \left\| \partial_x u_{\varepsilon, \beta} \right\|_{L^2((0,T) \times \mathbb{R})}^2
\]

\leq C_0 T^{\frac{1}{2}} \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \frac{\beta \varepsilon}{\varepsilon} \left\| \partial_x u_{\varepsilon, \beta} \right\|_{L^2((0,T) \times \mathbb{R})}^2
\]

\leq C(T) \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \varepsilon^{\frac{1}{2}} \to 0.

Therefore, Lemma 3.1 gives (3.6).

We conclude by proving that \( u \) is the unique entropy solution of (1.7). Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\), and \( \phi \in C_0^{\infty}((0, \infty) \times \mathbb{R}) \) a non-negative function. We have to prove that

\[
\int_0^\infty \int_\mathbb{R} (\partial_t \eta(u) + \partial_x q(u)) \phi dt dx \leq 0.
\]

We have that

\[
\int_0^\infty \int_\mathbb{R} (\partial_x \eta(u_{\varepsilon, \beta, \beta}) + \partial_x q(u_{\varepsilon, \beta, \beta})) \phi dt dx
\]

\[ = \varepsilon_n \int_0^\infty \int_\mathbb{R} \partial_x (\eta'(u_{\varepsilon, \beta, \beta}) \partial_x u_{\varepsilon, \beta, \beta}) \phi dt dx - \varepsilon_n \int_0^\infty \eta''(u_{\varepsilon, \beta, \beta}) (\partial_x u_{\varepsilon, \beta, \beta})^2 \phi dt dx
\]

\[ - \beta_n \int_0^\infty \int_\mathbb{R} \partial_x (\eta'(u_{\varepsilon, \beta, \beta}) \partial_{xx} u_{\varepsilon, \beta, \beta}) \phi dt dx
\]

\[ + \beta_n \int_0^\infty \int_\mathbb{R} \eta''(u_{\varepsilon, \beta, \beta}) \partial_x u_{\varepsilon, \beta, \beta} \partial_{xx} u_{\varepsilon, \beta, \beta} \phi dt dx
\]

\[ + 3 \beta_n \int_0^\infty \int_\mathbb{R} \partial_x (\eta'(u_{\varepsilon, \beta, \beta}) u_{\varepsilon, \beta, \beta}^2 \partial_x u_{\varepsilon, \beta, \beta}) \phi dt dx
\]

\[ - 3 \beta_n \int_0^\infty \int_\mathbb{R} \eta''(u_{\varepsilon, \beta, \beta}) u_{\varepsilon, \beta, \beta}^2 \partial_x u_{\varepsilon, \beta, \beta} \phi dt dx
\]

\[ - 15 \beta_n \int_0^\infty \int_\mathbb{R} \eta'(u_{\varepsilon, \beta, \beta}) u_{\varepsilon, \beta, \beta} \partial_x u_{\varepsilon, \beta, \beta} \phi dt dx
\]

\[ \leq -\varepsilon_n \int_0^\infty \eta'(u_{\varepsilon, \beta, \beta}) \partial_x u_{\varepsilon, \beta, \beta} \phi dt dx + \beta_n \int_0^\infty \eta'(u_{\varepsilon, \beta, \beta}) \partial_{xx} u_{\varepsilon, \beta, \beta} \phi dt dx
\]
Due to (1.11), (2.1), (2.4) and the H"older inequality,

\[
3 \beta_n \int_{0}^{\infty} \eta''(u_{\varepsilon_n, \beta_n}) \partial_x u_{\varepsilon_n, \beta_n}^2 \partial_x u_{\varepsilon_n, \beta_n} \phi_{\varepsilon_n, \beta_n} \phi_{\varepsilon_n, \beta_n} \, dt \, dx
\]

\[
-3 \beta_n \int_{0}^{\infty} \eta'(u_{\varepsilon_n, \beta_n}) u_{\varepsilon_n, \beta_n}^2 \partial_x u_{\varepsilon_n, \beta_n} \partial_x \phi_{\varepsilon_n, \beta_n} \, dt \, dx
\]

\[
-3 \beta_n \int_{0}^{\infty} \eta''(u_{\varepsilon_n, \beta_n}) u_{\varepsilon_n, \beta_n}^2 (\partial_x u_{\varepsilon_n, \beta_n})^2 \phi_{\varepsilon_n, \beta_n} \, dt \, dx
\]

\[
-15 \beta_n \int_{0}^{\infty} \eta'(u_{\varepsilon_n, \beta_n}) u_{\varepsilon_n, \beta_n} (\partial_x u_{\varepsilon_n, \beta_n})^2 \phi_{\varepsilon_n, \beta_n} \, dt \, dx
\]

\[
\leq \varepsilon_n \int_{0}^{\infty} |\eta'(u_{\varepsilon_n, \beta_n})| (\partial_x u_{\varepsilon_n, \beta_n})^2 |\phi_{\varepsilon_n, \beta_n}| \, dt \, dx
\]

\[
+ \beta_n \int_{0}^{\infty} |\eta'(u_{\varepsilon_n, \beta_n})| (\partial_x u_{\varepsilon_n, \beta_n})^2 |\phi_{\varepsilon_n, \beta_n}| \, dt \, dx
\]

\[
+ \beta_n \int_{0}^{\infty} |\eta'(u_{\varepsilon_n, \beta_n})| (\partial_x u_{\varepsilon_n, \beta_n})^2 |\phi_{\varepsilon_n, \beta_n}| \, dt \, dx
\]

\[
+ \beta_n \int_{0}^{\infty} |\eta'(u_{\varepsilon_n, \beta_n})| (\partial_x u_{\varepsilon_n, \beta_n})^2 |\phi_{\varepsilon_n, \beta_n}| \, dt \, dx
\]

\[
+ \beta_n \int_{0}^{\infty} |\eta'(u_{\varepsilon_n, \beta_n})| (\partial_x u_{\varepsilon_n, \beta_n})^2 |\phi_{\varepsilon_n, \beta_n}| \, dt \, dx
\]

\[
+ \beta_n \int_{0}^{\infty} |\eta'(u_{\varepsilon_n, \beta_n})| (\partial_x u_{\varepsilon_n, \beta_n})^2 |\phi_{\varepsilon_n, \beta_n}| \, dt \, dx
\]

\[
\leq \varepsilon_n \int_{0}^{\infty} |\eta'(u_{\varepsilon_n, \beta_n})| (\partial_x u_{\varepsilon_n, \beta_n})^2 |\phi_{\varepsilon_n, \beta_n}| \, dt \, dx
\]

We show that

\[(3.8) \quad 3 \beta_n \int_{0}^{\infty} u_{\varepsilon_n, \beta_n}^2 (\partial_x u_{\varepsilon_n, \beta_n})^2 |\phi_{\varepsilon_n, \beta_n}| \, dt \, dx \to 0.\]

Due to \([1.11], [2.1], [2.3]\) and the H"older inequality,

\[
3 \beta_n \int_{0}^{\infty} u_{\varepsilon_n, \beta_n}^2 (\partial_x u_{\varepsilon_n, \beta_n})^2 |\phi_{\varepsilon_n, \beta_n}| \, dt \, dx
\]
Lemma 3.2 gives that is (3.10).

\[
\beta \leq C \eta \to 0,
\]

that is (3.9).

We claim that

\[
3\beta_n \| \eta'' \|_{L^\infty(\mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}+\times\mathbb{R})} \| u_{\varepsilon_n, \beta_n} (\partial_x u_{\varepsilon_n, \beta_n})^2 \|_{L^1((0,T) \times \mathbb{R})} \to 0.
\]

Due to (1.11), (2.1) and (2.4),

\[
3\beta_n \| \eta'' \|_{L^\infty(\mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}+\times\mathbb{R})} \int_0^T \int_{\mathbb{R}} u_{\varepsilon_n, \beta_n}^2 (\partial_x u_{\varepsilon_n, \beta_n})^2 \, ds \, dx \\
\leq 3 \| \eta'' \|_{L^\infty(\mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}+\times\mathbb{R})} \beta_n \| u_{\varepsilon_n, \beta_n} (t, \cdot) \|_{L^\infty(\mathbb{R})}^2 \\
\cdot \int_0^T \int_{\mathbb{R}} (\partial_x u_{\varepsilon_n, \beta_n})^2 \, ds \, dx \\
\leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}+\times\mathbb{R})} \beta_n \int_0^T \int_{\mathbb{R}} (\partial_x u_{\varepsilon_n, \beta_n})^2 \, ds \, dx \\
\leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}+\times\mathbb{R})} \frac{3\eta_n}{\varepsilon_n} \int_0^T \int_{\mathbb{R}} (\partial_x u_{\varepsilon_n, \beta_n})^2 \, ds \, dx \\
\leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}+\times\mathbb{R})} \varepsilon_n^{-\frac{3}{4}} \to 0,
\]

that is (3.9).

We have

\[
15\beta_n \| \eta' \|_{L^\infty(\mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}+\times\mathbb{R})} \| u_{\varepsilon_n, \beta_n} (\partial_x u_{\varepsilon_n, \beta_n})^2 \|_{L^1((0,T) \times \mathbb{R})} \to 0.
\]

Again by (1.11), (2.1) and (2.4),

\[
15\beta_n \| \eta' \|_{L^\infty(\mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}+\times\mathbb{R})} \int_0^T \int_{\mathbb{R}} u_{\varepsilon_n, \beta_n} (\partial_x u_{\varepsilon_n, \beta_n})^2 \, ds \, dx \\
\leq 15 \| \eta' \|_{L^\infty(\mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}+\times\mathbb{R})} \beta_n \| u_{\varepsilon_n, \beta_n} (t, \cdot) \|_{L^\infty(\mathbb{R})}^2 \\
\cdot \int_0^T \int_{\mathbb{R}} (\partial_x u_{\varepsilon_n, \beta_n})^2 \, ds \, dx \\
\leq C_0 \| \eta' \|_{L^\infty(\mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}+\times\mathbb{R})} \beta_n \frac{3\eta_n}{\varepsilon_n} \int_0^T \int_{\mathbb{R}} (\partial_x u_{\varepsilon_n, \beta_n})^2 \, ds \, dx \\
\leq C_0 \| \eta' \|_{L^\infty(\mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}+\times\mathbb{R})} \varepsilon_n^{-\frac{2+3\eta_n}{4}} \int_0^T \int_{\mathbb{R}} (\partial_x u_{\varepsilon_n, \beta_n})^2 \, ds \, dx \\
\leq C_0 \| \eta' \|_{L^\infty(\mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}+\times\mathbb{R})} \varepsilon_n^{-\frac{2+3\eta_n}{4}} \to 0,
\]

that is (3.10).

(3.7) follows from (1.11), (3.6), (3.8), (3.9), (3.10), Lemmas 2.1 and 2.3.

Proof of Theorem 1.1. Lemma 3.2 gives i) and ii), while iii) follows from Lemma 3.3.
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