Tachyonic Preheating*

Lev Kofman

Canadian Institute for Theoretical Astrophysics, University of Toronto, ON M5S 3H8
E-mail: kofman@cita.utoronto.ca

(July 26, 2001)

Abstract

I review the theory of preheating after inflation, focussing on the recently found tachyonic preheating in the theories with spontaneous symmetry breaking. This occurs due to the tachyonic instability of the scalar field near the top of its effective potential. Contrary to the common expectation, tachyonic instability converts most of the energy into that of colliding classical waves very rapidly, within a single oscillation. Efficient tachyon preheating is typical for the hybrid inflationary scenario, including SUSY motivated and brane inflation models.

I. PREHEATING AFTER INFLATION

The best-fit universe is the uniform and homogeneous flat $\Omega_{\text{tot}} = 1$ Friedmann-Lemaître expanding universe with gaussian scale-free metric fluctuations. The model is in remarkable agreement with the observations of the CMB anisotropy, the large scale structure of the universe and the tests of the global geometry.

Inflationary theory suggests that the expansion of the universe was preceded not by the Big Bang singularity, but rather by the exponentially fast expansion of the universe when four dimensional geometry of the universe is close to the nonsingular De Sitter geometry. Almost De Sitter geometry (where the Hubble parameter is slow varying with time, $\dot{H} \ll H^2$) can be realized by gravity of the matter with the vacuum-like equation of state $p \approx -\rho$, where $p$ and $\rho$ are the pressure and energy density. There are many specific realizations of this equation of state, motivated by the different aspects of the elementary particle physics. The vacuum-like equation of state can be provided by the out-of-equilibrium scalar fields or by the other effects (like the quantum gravity mechanisms) which behave like an effective scalar field.

Although scalar fields are not yet discovered experimentally, they are the vital ingredients of the high energy physics theories, and a plethora of scalar fields exists in the supergravity and superstring theories. Fundamental M-theory should encompass both supergravity and

*Invited talk at the Conference PASCOS2001
string theory. At present the low-energy phenomenology is described by the $N = 1 d = 4$ supergravity. Its Lagrangian begins with the scalar fields terms

$$e^{-1} \mathcal{L} = -\frac{1}{2} M^2 P R - \partial_\mu \Phi^i \partial^\mu \Phi_i + e^K \left( D^i W D_i W - 3 \frac{W W^*}{M^2} \right) + ...$$  \hspace{1cm} (1)

where a scalar field $\Phi^i$ is the complex conjugate of $\Phi_i$. Some preferable choices of the Kähler potentials $K$, superpotentials $W$ and Yang–Mills couplings hopefully will be selected at the level of the fundamental theory. Until the fundamental theory of all interactions is well understood, one may try to address the issues of the early universe cosmology in the context of the most general phenomenological $N = 1$ supergravity–Yang–Mills–matter theory. This, in fact, was the case during the last decade.

At first glance it seems hard to gain useful insights or predictions from a loose pool of inflationary models embedded in (1). Fortunately, theory of inflation, independently on its concrete model, provides us with several universal mechanisms which make it immensely successful as the theory of the initial conditions for the observable Friedmann-Lemmetre universe. These mechanisms are related to the properties of the DeSitter solution in General Relativity and to the properties of classical and quantum fields in the background of De Sitter space-time. Remarkably, these properties are not directly related to the microscopic physics behind the inflation. This is not the first time when we can successfully do cosmology at the phenomenological level without the knowledge of the microscopic physics. For example, celebrated Cold Dark Matter theory of the Large Scale Structure of the universe relies on the simple assumption that there is a nonbaryonic invisible dark matter component with the ”dust-like” equation of state $p_{cdm} \approx 0$. This is enough to develop the theory of the structure formation even without understanding the microscopic origin of dark matter.

There are several major predictions of inflation:

- Inflation erases pre-inflationary classical inhomogeneities and entropy.
- The total mass density in the universe $\Omega_{tot} = 1$.
- Vacuum quantum fluctuations of the scalar field(s) generate almost scale free gaussian scalar metric perturbations.
- Vacuum quantum fluctuations of gravitational waves generate almost scale free gaussian tensor metric perturbations.
- All the particles in the universe are created from the decay of inflaton energy $\rho$ in the process of (p)reheating after inflation.

The ratio of amplitudes of scalar and tensor modes, the spectral index of scalar and tensor modes, and the character of (p)reheating after inflation can be model dependent.

In this contribution I will review the theory of preheating after inflation, concentrating on the recently found tachyonic preheating [10,11].

According to the inflationary scenario, the Universe initially expands quasi-exponentially in a vacuum-like (DeSitter) state without entropy or particles. At the stage of inflation, all energy is contained in a classical slowly moving fields $\Phi$ in the inflaton sector. The last term of the 1st line of (1) is the scalar potential $V(\Phi_i)$. The equations of motion based on the first line should describe inflation, which is a challenging problem by itself. The Lagrangian (1) contains also other fields which give subdominant contributions to gravity. The Friedmann equation for the scale factor $a(t)$ and the equation for $\Phi(t)$ determine the evolution of the background fields.
In the chaotic inflation models, soon after the end of inflation, an almost homogeneous inflaton field $\Phi(t)$ coherently oscillates with a very large amplitude of the order of the Planck mass $M_P$ around the minimum of its potential $V(\Phi)$. This scalar field can be considered as a coherent superposition of inflatons with zero momenta. The amplitude of oscillations gradually decreases not only because of the expansion of the universe, but also because energy is transferred to particles created by the oscillating field. At this stage we shall recall the rest of the fundamental Lagrangian which includes all the fields interacting with inflaton. These interactions lead to the creation of many ultra-relativistic particles from the inflaton. Gradually, the inflaton field decays and transfers all of its energy to the created particles. In this scenario all the matter constituting the universe is created from this process of reheating. Typically particle production from coherently oscillating inflatons occurs not in the non-perturbative regime of preheating \([4]\).

Indeed, let us consider a simple toy model of chaotic inflation with the quadratic potential $V(\Phi) = \frac{1}{2} m_\phi \Phi^2$ and $L_{\text{int}} = -\frac{1}{2} g^2 \Phi^2 \chi^2$ describing the interaction between the inflatons $\Phi = \Phi_1$ and other massless Bose particles $\chi = \Phi_2$. We can consider quantum fluctuations of the field $\chi$ interacting with the classical homogeneous background field $\Phi(t)$. The quantum scalar field $\hat{\chi}$ in a flat FRW background has the eigenfunctions $\chi_k(t) e^{-ikx}$ with comoving momentum $k$. The temporal part of the eigenfunction obeys the equation

$$\ddot{\chi}_k + 3\frac{\dot{a}}{a} \dot{\chi}_k + \left( \frac{k^2}{a^2} - \xi R + g^2 \dot{\phi}^2 \right) \chi_k = 0 \quad (2)$$

with vacuum-like initial conditions: $\chi_k \simeq e^{-ikt}$ in the far past. The coupling to the curvature $\xi R$ will not be important in the presence of the interaction (but would lead to gravitational preheating in the absence of the interaction). In this model, the inflaton field $\Phi(t)$ coherently oscillates as $\Phi(t) \approx \tilde{\Phi}(t) \sin (m_\phi t)$, with the amplitude $\tilde{\Phi}(t) = \frac{M_p}{\sqrt{3\pi}} \cdot \frac{1}{m_\phi t}$ decreasing as the universe expands.

Equation (2) describes excitation of the quantum fluctuation $\chi_k$. At first glance the effect of particle creation $n_\chi$ could be treated perturbatively with respect to the small coupling $g^2$. However, the smallness of $g^2$ alone does not necessarily lead to the perturbative approach to describe the excitation of $\chi_k$ modes. To check whether the interaction term $g^2 \dot{\phi}^2$ in eq. (4) is perturbative or not, we have to use a new time variable $z = mt$ and the essential dimensionless coupling parameter $q = \frac{g^2 \phi^2}{m^2}$. Scalar metric fluctuations in this model are compatible with cosmology if the inflaton mass is $m_\phi \simeq 10^{-6} M_p$; therefore, typically $q \simeq 10^{10} g^2 \gg 1$ for not negligibly small $g^2$. In fact, a consistent setting for the problem of $\chi$-particle creation from the $\phi$-inflaton requires $q \gg 1$ even without additional assumptions about $g^2$. It is known that if we have two scalars $\phi$ and $\chi$, then the latest stage of inflation will be driven by the lightest scalar. The square of the effective mass of the $\chi$-field includes a term $g^2 \dot{\phi}^2$. Inflation is driven by the $\phi$-field if its square mass $m^2$ is smaller than $g^2 \phi^2$. This leads to the condition $q \gg 1$.

Solutions of the equation (2) for $q \gg 1$ are unstable for a range of $k$ and $\chi$ particles are created exponentially fast. Indeed, suppose that there is no expansion of the universe. In this case background oscillations are harmonic, $\Phi(t) \sim \sin (m_\phi t)$, and equation (2) is reduced to the Mathieu equation. Solutions of this equation are exponentially unstable within the
set of resonance bands, $\chi_k \sim e^{\mu_k m \phi t}$, where the characteristic exponent $\mu_k$ depends on $(k, q)$. In realistic case of the expanding universe, parameter $q$ is time dependent. For the broad resonance case $q \gg 1$ this parameter jumps over a number of instability bands within a single background oscillation, so the concept of stability/instability bands is inapplicable here. Parametric resonance in this case, described by (2), is a stochastic process [2]. In the regime of stochastic resonance particle are created exponentially fast as $n_k \sim e^{\int dt \mu_k}$. Characteristic exponent here is also stochastic with typical value $\bar{\mu}_k \sim 0.1$ in the resonance momentum range $0 < k < q^{1/4} m$.

Due to the copious particle creation from the background field, very soon, within few dozens of the background oscillations, we have to take into account backreaction of the created particles. Unfortunately, known theoretical approaches to include backreaction effects, like the Hartree approximation, are not enough in this situation. First fully nonlinear lattice simulation of the preheating [3] revealed that the leading backreaction effect is rescattering of $\chi$ particles on the background inlatons, $\chi_k \Phi_0 \rightarrow \chi_k' \Phi_{k-k'}$. Lattice simulations demonstrated that there is a first, resonant stage of preheating, where initial excitation of $\chi$ field occurs exponentially fast as it is expected from the linear theory of equation (2). Then the system enters the nonlinear stage of preheating where classical waves of $\chi$ and $\Phi$ fields are rescattering and gradually relaxing towards equilibrium.

Different in technical details, but qualitatively similar theory was developed for other types of the chaotic inflation potentials, e.g. see [4] for $V(\Phi) = \frac{1}{4} \lambda \Phi^4$ theory.

II. PREHEATING IN HYBRID INFLATION

Another popular class of inflationary models – hybrid inflation – involve multiple scalar fields $\Phi_i$ in the inflaton sector [5]. In particular, hybrid inflation can be realized in supergravity for certain choice of superpotential $W$ [6], and in the theory of brane inflation [7].

Previous studies of preheating in hybrid models were concentrated on particle creation by parametric resonance that may occur when homogeneous background fields oscillate around the minimum of the potential $V$ [8]. Technically the linear fluctuations were considered around the time-dependent homogeneous background fields. Such parametric resonance may or may not be strong depending on the coupling parameters. However, we recently found [10] that there is strong preheating in hybrid inflation, but its character is quite different from preheating based on parametric resonance. It turns out that there is very efficient tachyonic instability that appears in the hybrid inflation models. The backreaction of rapidly generated fluctuations does not allow homogeneous background oscillations to occur because all energy of the oscillating field is transferred to the energy of long-wavelength scalar field fluctuations within a single oscillation! However, this does not preclude the subsequent decay of the Higgs and inflaton inhomogeneities into other particles, and thus reheating without parametric resonance.

Let us first consider the background evolution and the results of the naive perturbative approach to describe the quantum fluctuations around the background solutions. Consider the simple potential for the two-field hybrid inflation is

$$V(\phi, \sigma) = \frac{\lambda}{4} (\sigma^2 - v^2)^2 + \frac{g^2}{2} \phi^2 \sigma^2,$$  

(3)
where we used notations $\Phi_1 = \phi$, $\Phi_2 = \sigma$. Inflation in this model occurs while the homogeneous $\Phi_1$ field slow rolls from large $\phi$ towards the bifurcation point at $\phi = \frac{\sqrt{\lambda}}{g} v$ (due to the slight lift of the potential in $\phi$ direction). Once $\phi(t)$ crosses the bifurcation point, the curvature of the $\sigma$ field, $m_\sigma^2 \equiv \partial^2 V/\partial \sigma^2$, becomes negative. This negative curvature results in exponential growth of $\sigma$ fluctuations. Inflation then ends abruptly in a “waterfall” manner.

One reason to be interested in hybrid inflation is that it can be implemented in supersymmetric theories (\textsuperscript{1}). In particular, for illustration we will consider preheating in the supersymmetric F-term inflation as an example of a hybrid model.

The simplest F-term hybrid inflation model (without undesirable domain walls) is based on a superpotential with three left-chiral superfields $\Phi_i = (\Phi_1, \Phi_2, \Phi_3)$ in the Lagrangian (\textsuperscript{1})

$$W = \frac{\sqrt{\lambda}}{2} \Phi_1 \left(4 \Phi_2 \Phi_3 - v^2\right).$$

In this case, the spontaneous breaking of the local (global) $U(1)$ symmetry between the $\Phi_2$ and $\Phi_3$ fields will lead to gauge (global) string formation.

In global SUSY, using the same notation for superfields and their complex scalar components, this superpotential contributes

$$V_F = \frac{\lambda}{4} |4 \Phi_2 \Phi_3 - v^2|^2 + 4 \lambda |\Phi_1|^2 \left(|\Phi_3|^2 + |\Phi_2|^2 \right).$$

(5)

to the scalar potential. In general, $\Phi_3$ and $\Phi_3$ could be (oppositely) charged under a local $U(1)$ symmetry, in which case we should include a D-term, $V_D$, which we neglect here.

In this model, inflation occurs when chaotic initial conditions lead to $\langle |\Phi_1| \rangle \gg v$. When this happens, the fields $\Phi_2$ and $\Phi_3$ acquire large effective masses and roll to their local minimum at $\langle \Phi_2 \rangle = \langle \Phi_3 \rangle = 0$. In this limit, the potential (\textsuperscript{5}) becomes $V \approx \frac{\lambda v^4}{4}$, which gives rise to a non-vanishing effective cosmological constant. However, this is a false vacuum state; the true vacuum corresponds to $\langle \Phi_2 \Phi_3 \rangle = \frac{v^2}{4}$ and $\langle \Phi_1 \rangle = 0$. The slow-roll potential drives the evolution of the inflaton towards its true VEV. When its magnitude reaches the value $\langle |\Phi_3| \rangle = \frac{v}{2}$, spontaneous symmetry breaking occurs.

For further discussion of symmetry breaking in this model, let us rewrite (\textsuperscript{4}) in terms of polar fields: $\Phi_3 = |\Phi_3| e^{i \theta}$, $\Phi_2 = |\Phi_2| e^{i \bar{\theta}}$. The potential becomes

$$V_F = \frac{\lambda}{4} \left(16 |\Phi_2|^2 |\Phi_3|^2 - 8 v^2 |\Phi_2||\Phi_3| \cos(\theta + \bar{\theta}) + v^4 \right) + 4 \lambda |\Phi_2|^2 \left(|\Phi_2|^2 + |\Phi_3|^2 \right).$$

(6)

At the stage of symmetry breaking, when $\langle |\Phi_2 \Phi_3| \rangle$ begins to move away from zero, the absolute phase $\text{Arg}(\Phi_2 \Phi_3) = \theta + \bar{\theta}$ acquires a mass and is forced to zero. Note, however, that the potential is independent of the relative phase, $\theta - \bar{\theta}$, reflecting the $U(1)$ symmetry. Thus, in a quasi-homogeneous patch, the $U(1)$ symmetry allows us to choose the relative phase of the $\Phi_2$, $\Phi_3$ fields to be zero without any loss of generality. This choice, combined with the vanishing of the absolute phase, is equivalent to choosing the two complex $\Phi_2$, $\Phi_3$ fields to be real. In order to leave canonical kinetic terms, we define $\sigma_\pm \equiv |\Phi_3| \pm |\Phi_2|$. Furthermore, as inflation has left the inflaton homogeneous across all the patches, we may choose it to be real: $\phi \equiv \sqrt{2} |\Phi_1|.$

In terms of these three real fields, the potential now becomes
\[
V_F = \frac{\lambda}{4} \left( \sigma_+^2 - \sigma_-^2 - v^2 \right)^2 + \lambda \phi^2 \left( \sigma_+^2 + \sigma_-^2 \right).
\]

(7)

In the symmetric phase, when \(\sigma_\pm = 0\), the \(\sigma\) fields have an effective mass-squared: \(m_\pm^2(\phi) = \lambda (2\phi^2 \mp v^2)\). We can now see that spontaneous symmetry breaking occurs in this model exactly as in the two field model (3). For \(\phi < \phi_c = \frac{v}{\sqrt{2}}\), the \(\sigma_+\) field has a tachyonic mass that triggers symmetry breaking and the end of inflation. On the other hand, the \(\sigma_-\) field has always a large and positive effective mass-squared, pinning it to zero. Thus, during inflation and at the initial stages of symmetry breaking, this model behaves just like the standard two field hybrid model (3). We have only to apply the constraint \(g^2 = 2\lambda\) and identify the Higgs field with \(\sigma_+\). The equations for the homogeneous background components \(\phi(t)\) and \(\sigma_\pm(t)\) admits simple solution

\[
\phi(t) + \frac{1}{\sqrt{2}} \sigma_+(t) = \phi_c, \quad \sigma_-(t) = 0.
\]

(8)

To study preheating in the F-term inflation, we have to analyse evolution of the vacuum fluctuations. Consider vacuum fluctuations in the inflaton sector \(\Phi_i\) of the theory (6). Usual description in terms of a homogeneous background plus small fluctuations gives us equations for fluctuations around the background solution (5). We define the variances of fields:

\[
\langle |\sigma_\pm - \langle \sigma_\pm \rangle|^2 \rangle_{\text{ren}} = \int \frac{d^3k}{(2\pi)^3} \left[ |\delta \sigma_{k\pm}(t)|^2 - |\delta \sigma_{k\pm}(0)|^2 \right] \equiv \int \frac{dk}{k} \mathcal{P}_\pm(k, t)
\]

(9)

and similar for \(\phi\) field. Here \(\mathcal{P}_{\pm, \phi}(k, t)\) are the spectra of the fluctuations.
FIGURES

FIG. 1. Evolution of the background fields \( \sigma_+(t) \) and \( \phi(t) \) after symmetry breaking and the log of \( \mathcal{P}_+(k) \) for the mode with the momentum \( k = 0.2\sqrt{\lambda}v \).

FIG. 2. Means and variances in units of \( \phi_c \). The squared means \( \langle \phi \rangle^2 \) and \( \langle \sigma_+ \rangle^2 \) are ordinary solid lines while the field variances are thick lines. The mean of \( \phi \) starts at \( \phi_c \), oscillates once, and then decays. The mean of \( \sigma_+ \) grows in antiphase to \( \phi \) and freeses at \( \phi_c \).

Numerical solutions of the equations for the background fields and for the time evolution of the linear fluctuations of \( \sigma_+ \) for the mode \( k \) where their spectrum is maximum are plotted as the bold line at the Figure 1. Notice an enormous exponential growth of the fluctuations within a single background oscillation. Indeed, the amplitude \( \mathcal{P}_+(k) \) increases by factor \( 10^{10} \)! There are two factors which contributes to such a strong instability of fluctuations in the model. First, oscillating background fields are crossing the region with significant negative curvature of the effective potential, which results in tachyonic instability. Second, this region turn out to be a turning point for the background oscillations, where the fields spend significant portion of the oscillation. As a result, tachyonic instability is lasting long enough to make the backreaction of the fluctuations to be significant already within single background oscillation. The regime of background oscillations will even not be settled. Therefore practically from the beginning we have to use the lattice simulations to study nonlinear dynamics of the fields.

The results of full nonlinear lattice simulations in the model derived in [10] are plotted in Figure 2. The simulations showed that the homogeneous fields \( \phi \) and \( \sigma_+ \) initially followed the classical trajectory (8) but, within one oscillation of the inflaton field, fluctuations grew too large to speak meaningfully of the fields as homogeneous oscillators. These fluctuations grew in such a way that \( \sigma_+ = \sigma_-^* \) almost exactly throughout the simulation. In other words
Re $\delta \sigma_+$ and Im $\delta \sigma_-$ were excited while Im $\delta \sigma_+$ and Re $\delta \sigma_-$ were not. Because of this we only plot the fields $\phi$ and $\sigma_+$.

In \cite{10,11} we develop a general theory of tachyonic preheating, which occurs due to tachyonic instability in the theories with spontaneous symmetry breaking. Our approach combines analytical estimates with lattice simulations taking into account all backreaction effects. The process of spontaneous symmetry breaking involves transfer of the potential energy into the energy of fluctuations produced due to the tachyonic instability. We show that this process is extremely efficient and requires just a single oscillation of the scalar field falling from the top of the effective potential. In what follows I will illustrate tachyonic preheating using the model (4).

III. SPONTANEOUS SYMMETRY BREAKING AND TACHYONIC PREHEATING

To understand physics of tachyonic preheating, we have to go back to the basics of the spontaneous symmetry breaking \cite{11}. The simplest model of spontaneous symmetry breaking is based on the theory with the effective potential $V_F = \frac{\lambda}{4} v^4 + \frac{\lambda}{4} \sigma_+^4 - \frac{\lambda}{4} \sigma_+^3 + \lambda \sigma_+^2$, dominated by the quadratic term near the top of the potential. The development of tachyonic instability in this model depends on the initial conditions. We will assume that initially the symmetry is completely restored. Initially scalar field fluctuations in this model in the symmetric phase $\phi = 0$ are the same as for a massless field, $\phi_k = \frac{1}{\sqrt{2k}} e^{-ikt + ik \vec{x}}$. Then at $t = 0$ we ‘turn on’ the term $-m^2 \phi^2 / 2$ corresponding to the negative mass squared $-m^2$. The modes with $k = |\vec{k}| < m$ grow exponentially. Initial dispersion of all growing fluctuations with $k < m$ was given by $\langle \delta \phi^2 \rangle = \int_0^m \frac{dk^2}{8 \pi^2} = \frac{m^2}{8 \pi^2}$ and the average initial amplitude of all fluctuations with $k < m$ was given by $\delta \phi = \frac{m}{2 \pi}$. The dispersion of the growing modes at $t > 0$ is growing exponentially. This means that the average amplitude $\delta \phi(k)$ of quantum fluctuations with momenta $\sim k$ initially was $\delta \phi(k) \sim k / 2\pi$, and then it started growing as $e^{\sqrt{m^2-k^2}}$. The tachyonic growth of all fluctuations with $k < m$ continues until $\sqrt{\langle \delta \phi^2 \rangle}$ reaches the value $\sim v/2$, since at $\phi \sim v/\sqrt{3}$ the curvature of the effective potential vanishes and instead of tachyonic growth one has the usual oscillations of all the modes. This happens within a time $\Delta t \sim \frac{1}{2m} \ln \frac{C}{\lambda}$, where $C \sim 10^2$.

The process of symmetry breaking will occur in a somewhat different way in theories where the curvature of the effective potential near its maximum depends on $\phi$. Consider for example $V = -\frac{1}{4} \phi^4$ near the top of the potential. In this case there is an instanton solution corresponding to the tunneling from $\phi = 0$ to nonzero value \cite{12,13}.

Let us now consider the tachyonic instability for the theory (4). The scalar fields potential (3) for the solution (8) in the fields space is reduced to the potential

$$V_F = \frac{\lambda}{4} v^4 - \lambda \sigma_+^3 + \frac{3}{4} \lambda \sigma_+^4$$

(10)

Let us simplify notation $\sigma_+ \to \phi$ and make rescaling $\lambda \to \lambda / 3$. Then the potential (10) once again is reduced to the form

$$V = -\frac{\lambda}{3} \phi^3 + \frac{\lambda}{4} \phi^4 + \frac{\lambda}{12} v^4$$

(11)
Thus the theory (11) is a prototype of the theory (I), which initially looked much more complicated.

Development of instability in the theory (11) presents us with a new challenge. The curvature of the effective potential at $\phi = 0$ in this theory vanishes, which means that, unlike in the theory $-m^2 \phi^2$, infinitesimally small perturbations in this theory do not grow. On the other hand, in this theory, unlike in the theory $-\lambda \phi^4$, there are no instantons which would describe tunneling from $\phi = 0$. Thus, in the theory $-\lambda v \phi^3$, which occupies an intermediate position between $-m^2 \phi^2$ and $-\lambda \phi^4$, both mechanisms which could lead to the development of instability do not work. Does it mean that the state $\phi = 0$ in this theory is, in fact, stable?

The answer to this question is no, the state $\phi = 0$ in the theory $-\lambda v \phi^3$ is unstable. Indeed, even though $\langle \phi \rangle$ initially is zero, long wavelength fluctuations of the field $\phi$ are present, and they may play the same role as the homogeneous field $\phi$ in triggering the instability.

The scalar field fluctuations with momentum $k < k_0$ have initial amplitude $\langle \delta \phi^2 \rangle \sim \frac{k^2}{8\pi^2}$. Thus the short wavelength fluctuations with momenta $k > k_0$ live on top of the long wavelength field with an average amplitude $\delta \phi_{\text{rms}}(k_0) \sim \sqrt{\langle \delta \phi^2 \rangle} \sim \frac{\lambda v}{\sqrt{2\pi}}$.

The curvature of the effective potential $V'' = |m_{\text{eff}}|^2$ at $\phi = \delta \phi_{\text{rms}}(k_0)$ in the theory (11) is given by $-2\lambda v \delta \phi_{\text{rms}}(k_0) \sim -\lambda v \frac{k_0^2}{\sqrt{2\pi}}$. Consider the fluctuations with momentum $k$ somewhat greater than $k_0$, so that the amplitude of the long wavelength field $\delta \phi$ does not change significantly on a scale $k^{-1}$. Short wavelength fluctuations with $k = Ck_0$ with $C$ somewhat greater than 1 will grow on top of the field $\phi \sim \delta \phi_{\text{rms}}(k_0)$ if $k^2 < |m_{\text{eff}}|^2 \sim \frac{\lambda v k_0}{\sqrt{2\pi}}$.

Taking for definiteness $C \sim \sqrt{2}$, one may argue that fluctuations with $k < \frac{\lambda v}{2\pi}$ may enter a self-sustained regime of tachyonic growth. Small fluctuations rapidly grow large, which justifies semi-classical methods used for the description of this process. The average initial amplitude of the growing tachyonic fluctuations with momenta smaller than $\frac{\lambda v}{2\pi}$ is

$$\delta \phi_{\text{rms}} \sim \frac{\lambda v}{4\pi^2}.$$  \hspace{1cm} (12)

These fluctuations grow until the amplitude of $\delta \phi$ becomes comparable to $2v/3$, and the effective tachyonic mass vanishes. At that moment the field can be represented as a collection of waves with dispersion $\sqrt{\langle \delta \phi^2 \rangle} \sim v$, corresponding to coherent states of scalar particles with occupation numbers $n_k \sim \left(\frac{4\pi^2}{\lambda} \right)^2 \gg 1$. A more accurate investigation shows that the initial value of the field is few times greater than $\sqrt{\langle \delta \phi^2 \rangle} \sim v$, and therefore the occupation numbers will be somewhat smaller,

$$n_k \sim O(10) \lambda^{-2}.$$  \hspace{1cm} (13)

Because of the nonlinear dependence of the tachyonic mass on $\phi$, a detailed description of this process is more involved than in the quadratic theory. Indeed, even though the typical amplitude of the growing fluctuations is given by (12), the speed of the growth of the fluctuations increases considerably if the initial amplitude is somewhat bigger than (12). Thus even though the fluctuations with amplitude a few times greater than (12) are exponentially suppressed, they grow faster and may therefore have greater impact on the process than the fluctuations with amplitude (12).
FIG. 3. Three panels above show growth of quantum fluctuations of the field $\phi$ in the theory $V = -\frac{1}{3} v \phi^3 + \frac{1}{4} \phi^4$ looks like bubble formation. Preheating occurs due to a combined effect of bubble production, tachyonic instability and bubble wall collisions.
Low probability fluctuations with $\delta \phi \gg \delta \phi_{\text{rms}}$ correspond to peaks of the initial Gaussian distribution of the fluctuations of the field $\phi$. Such peaks tend to be spherically symmetric [14]. As a result, the whole process looks not like a uniform growth of all modes, but more like bubble production (even though there are no instantons in this model).

To study this issue in a more detailed way, one may use a stochastic approach to tunneling and bubble formation developed in [15]. The main idea of this approach can be explained as follows. Tunneling can be represented as a result of accumulation of quantum fluctuations with the amplitude greatly exceeding their usual value determined by uncertainty principle. This happens when the long wavelength quantum fluctuations responsible for the tunneling correspond to bosonic excitations with large occupation numbers. In such cases one can treat these fluctuations as classical fields experiencing Brownian motion due to their interaction with the short wavelength quantum fluctuations.

Suppose that the large fluctuations of the scalar field responsible for reheating in the model (11) initially look like spherically symmetric bubbles (which is the case if the probability of such fluctuations is strongly suppressed, see above). Equation of motion for a bubble of a scalar field $\phi(r)$ in Minkowski space is

$$\ddot{\phi} = \phi'' + 2\phi' r^{-1} - V'(\phi).$$

(14)

Here $r$ is a distance from the center of the bubble, $\phi' = \frac{\partial \phi}{\partial r}$. At the moment of its formation, the bubble wall does not move, $\dot{\phi} = 0$, $\ddot{\phi} = 0$ (critical bubble). Then it gradually starts growing, $\ddot{\phi} > 0$, which requires that

$$|\phi'' + 2\phi' r^{-1}| < -V'(\phi).$$

(15)

A bubble of a classical field is formed only if it contains a sufficiently large field $\phi$, and if the bubble itself is sufficiently large. If the size of the bubble is too small, the gradient terms are greater than the term $|V'(\phi)|$, and the field $\phi$ inside the bubble does not grow.

At small $r$ the shape of the bubble can be approximated by $\phi = \phi(0) - \alpha r^2/2$. In this approximation, the bubble has a typical size $r_0 \sim \sqrt{\frac{2\alpha \phi(0)}{\alpha}}$, and $\phi' r^{-1} = \phi'' = -\alpha$. Therefore at the moment of the bubble formation, when $\ddot{\phi} = 0$, one has

$$\phi'' = V'(\phi(0))/3.$$  

(16)

Replacing $\phi''$ by $k_0^2 \phi(0)$ one finds that the bubble can be considered a result of overlapping of quantum fluctuations with typical momenta $k < k_0 \sim r_0^{-1}$, where

$$k_0^2 = C^2 \frac{V'(\phi(0))}{3\phi(0)}.$$  

(17)

Here $C = O(1)$ is some numerical factor reflecting uncertainty in our estimate of $k_0$.

Let us estimate the probability of an event when vacuum fluctuations occasionally build up a configuration of the field satisfying this condition. In order to do it one should remember that the dispersion of quantum fluctuations of the field $\phi$ with $k < k_0$ is given by $\langle \delta \phi^2 \rangle \sim \frac{k_0^2}{8\pi^2}$. This gives

$$\langle \phi^2 \rangle_{k < k_0} \sim \frac{k_0^2}{8\pi^2} = C^2 \frac{V'(\phi(0))}{24\pi^2 \phi(0)}.$$  

(18)
This is an estimate of the dispersion of perturbations which may sum up to produce a bubble of the field $\phi$ that satisfies the condition (15). Of course, this estimate is rather crude. But let us nevertheless use eq. (18) to evaluate the probability that these fluctuations build up a bubble of a radius $r > k_0^{-1}$ containing the field $\phi$ at its center. Assuming, in the first approximation, that the probability distribution is gaussian, one finds:

$$P(\phi) \sim \exp \left( -\frac{\phi^2}{2\langle \phi^2 \rangle_{k<k_0}} \right) = \exp \left( -\frac{12\pi^2 \phi^3}{C^2 V'(\phi)} \right).$$

The general formula (19), being applied to the theory $-\lambda \phi^4/4$ to within a factor of $C \approx 2$ coincides with the Euclidean action for the instanton in this theory. Taking into account the very rough method we used to estimate $k_0$ and calculate the dispersion of the perturbations responsible for tunneling, the coincidence is rather impressive. As it was shown in [16,15], in application to the tunneling during inflation in the potentials with $V'' \ll H^2$ this approach gives exactly the same answer as the Euclidean approach. Most importantly, this methods allows to investigate tunneling and development of instability in the theories where the instanton solutions do not exist [15]. In particular, for the tunneling in the theory $-\lambda v \phi^3/3$ one finds

$$P(\phi) \sim (\lambda v \phi)^2 \exp \left( -\frac{12\pi^2 \phi}{C^2 \lambda v} \right).$$

We included here the subexponential factor $O(k_0^4) \sim (\lambda v \phi)^2$, which is necessary to describe the probability of tunneling per unit time per unit volume.

This means that the tunneling is not suppressed for $\phi \sim \frac{C^2 \lambda v}{12\pi^2}$. This result is in agreement with our previous estimate (12). Now let us take into account that the total time of the development of instability is a sum of the time of tunneling plus the time necessary for rolling of the field down. One can show that the time of rolling down is inversely proportional to $m(\phi) \sim \sqrt{\lambda v \phi}$, i.e. it decreases at large $\phi$. Also, the subexponential factor $(\lambda v \phi)^2$ grows at large $\phi$, which makes tunneling to large $\phi$ faster. Consequently, as we already discussed above, the main contribution to the development of instability is given by the fluctuations with $\phi > \sim \frac{C^2 \lambda v}{12\pi^2}$. Exponential suppression of the probability of such fluctuations leads to their approximate spherical symmetry.

The results of our lattice simulations for this model [11] are shown in three panel of Fig. 3. In this model bubbles form quickly enough that we were able to start with quantum fluctuations centered at $< \phi >= 0$ and allow the bubbles to form automatically. The bubbles (high peaks of the field distribution) grow, change shape, and interact with each other, rapidly dissipating the vacuum energy $V(0)$.

IV. DEVELOPMENT OF EQUILIBRIUM AFTER PREHEATING

The character of preheating may vary from model to model, e.g. parametric excitation in chaotic inflation [1] and tachyonic preheating in hybrid inflation [10], but its distinct feature remains the same: rapid amplification of one or more bosonic fields to exponentially large occupation numbers. This amplification is eventually shut down by backreaction of the produced fluctuations. The end result of the process is a turbulent medium of coupled,
inhomogeneous, classical waves far from equilibrium. Despite the development of our understanding of preheating after inflation, the transition from this stage to a hot Friedmann universe in thermal equilibrium has remained relatively poorly understood. The details of this thermalization stage depend on the constituents of the fundamental Lagrangian (1) and their couplings, so at first glance it would seem that a description of this process would have to be strongly model-dependent. Recently we performed a fully nonlinear study of the development of equilibrium after preheating [17]. We have performed lattice simulations of the evolution of interacting scalar fields during and after preheating for a variety of inflationary models. We have found, however, that many features of this stage seem to hold generically across a wide spectrum of models. Indeed, at the end of preheating and beginning of the turbulent stage \( t_* \), the fields are out of equilibrium. We have examined many models and found that at \( t_* \) there is not much trace of the linear stage of preheating and conditions at \( t_* \) are not qualitatively sensitive to the details of inflation. We therefore expect that this second, highly nonlinear, turbulent stage of preheating may exhibit some universal, model-independent features. Although a realistic model would include one or more Higgs-Yang-Mills sectors, we treat the simpler case of interacting scalars.

We have numerically investigated the processes of preheating and thermalization in a variety of models and determined a set of rules that seem to hold generically. These rules can be formulated as follows (in this section we use notations \( \phi = \Phi_1 \) for the inflaton field and \( \chi, \sigma \) for other scalars \( \Phi_i \))

1. In many, if not all viable models of inflation there exists a mechanism for exponentially amplifying fluctuations of at least one field \( \chi \). These mechanisms tend to excite long-wavelength excitations, giving rise to a highly infrared spectrum.

   The mechanism of parametric resonance in single-field models of inflation has been studied for a number of years. This effect is quite robust. Adding additional fields (e.g. \( \sigma \) fields) or self-couplings (e.g. \( \chi^4 \)) has little or no effect on the resonant period. Moreover, in many hybrid models a similar effect occurs due to tachyonic instability. The qualitative features of the fields arising from these processes seem to be largely independent of the details of inflation or the mechanisms used to produce the fields.

2. Exciting one field \( \chi \) is sufficient to rapidly drag all other light fields with which \( \chi \) interacts into a similarly excited state.

   We have seen this effect when multiple fields are coupled directly to \( \chi \) and when chains of fields are coupled indirectly to \( \chi \). All it takes is one field being excited to rapidly amplify an entire sector of interacting fields. These second generation amplified fields will inherit the basic features of the \( \chi \) field, i.e. they will have spectra with more energy in the infrared than would be expected for a thermal distribution.

3. The excited fields will be grouped into subsets with identical characteristics (spectra, occupation numbers, effective temperatures) depending on the coupling strengths.

   We have seen this effect in a variety of models. For example in the models (21) which we are going to consider the \( \chi \) and \( \sigma \) fields formed such a group. In general, fields that are interacting in a group such as this will thermalize much more quickly than other fields, presumably because they have more potential to interact and scatter particles into high momentum states.
4. Once the fields are amplified, they will approach thermal equilibrium by scattering energy into higher momentum modes.

This process of thermalization involves a slow redistribution of the particle occupation number as low momentum particles are scattered and combined into higher momentum modes. The result of this scattering is to decrease the tilt of the infrared portion of the spectrum and increase the ultraviolet cutoff of the spectrum. Within each field group the evolution proceeds identically for all fields, but different groups can thermalize at very different rates.

Here we will illustrate these results with a simple chaotic inflation model with a quartic inflaton potential. The inflaton $\phi$ has a four-legs coupling to another scalar field $\chi$, which in turn can couple to two other scalars $\sigma_1$ and $\sigma_2$. The potential for this model is

$$V = \frac{1}{4} \lambda \phi^4 + \frac{1}{2} g^2 \phi^2 \chi^2 + \frac{1}{2} h_1^2 \chi^2 \sigma_1^2 + \frac{1}{2} h_2^2 \chi^2 \sigma_2^2$$

(21)

Preheating in this theory in the absence of the $\sigma_i$ fields is well studied. For nonsmall $\frac{g^2}{\lambda}$ the field $\chi$ will experience parametric amplification, rapidly rising to exponentially large occupation numbers. In the absence of the $\chi$ field (or for sufficiently small $g$) $\phi$ will be resonantly amplified through its own self-interaction, but this self-amplification is much less efficient than the two-field interaction. The results shown here are for $\lambda = 9 \times 10^{-14}$ (for CMB anisotropy normalization) and $g^2 = 200 \lambda$. When we add a third field we use $h_1^2 = 100 g^2$ and when we add a fourth field we use $h_2^2 = 200 g^2$.

One of the most interesting variables to calculate is the (comoving) number density of particles of the fields $n(t)$ and their occupation number $n_k$. The evolution of the total number density of all particles $n_{\text{tot}}$ is an indication of the physical processes taking place. In the weak interaction limit the scattering of classical waves via the interaction $\frac{1}{2} g^2 \phi^2 \chi^2$ can be treated using a perturbation expansion with respect to $g^2$. The leading four-legs diagrams for this interaction corresponds to a two-particle collision ($\phi \chi \rightarrow \phi \chi$), which conserves $n_{\text{tot}}$. The regime where such interactions dominate corresponds to “weak turbulence” in the terminology of the theory of wave turbulence. If we see $n_{\text{tot}}$ conserved it will be an indication that these two-particle collisions constitute the dominant interaction. Conversely, violation of $n_{\text{tot}}(t) = \text{const}$ will indicate the presence of strong turbulence, i.e. the importance of many-particle collisions. Such higher order interactions may be significant despite the smallness of the coupling parameter $g^2$ (and others) because of the large occupation numbers $n_k$. Later, when these occupation numbers are reduced by rescattering, the two-particle collision should become dominant and $n_{\text{tot}}$ should be conserved. For a bosonic field in thermal equilibrium with a temperature $T$ and a chemical potential $\mu$ the spectrum of occupation numbers in the limit of classical waves is given by

$$n_k \approx \frac{T}{\omega_k - \mu}.$$  

(22)
FIG. 4. Time evolution of number density of particles in the model (21). The curves represent \( n_\phi, n_\chi, n_\sigma_1, n_\sigma_2 \) from top to bottom. Unit of (conformal) time is \( a \cdot 10^{-36} \) sec.

Figure 4 shows an exponential increase of \( n(t) \) during preheating, followed by a gradual decrease that asymptotically slows down. This exponential increase is a consequence of explosive particle production due to parametric resonance. After preheating the fields enter a turbulent regime. In our simulations we see \( n(t) \) decreasing during this stage. This decrease is a consequence of the many-particle interactions beyond the four legs rescattering.

An important point is that the interaction of \( \chi \) and \( \sigma_i \) does not affect the preheating of \( \chi_i \), but does drag \( \sigma_i \) exponentially quickly into an excited state. The fields \( \sigma_i \) are exponentially amplified not by parametric resonance, but by their stimulated interactions with the amplified \( \chi \) field. Unlike amplification by preheating, this direct decay nearly conserves particle number, with the result that \( n_\chi \) decreases as \( \sigma_i \) grow.

Interacting waves of scalar fields constitute a dynamical system. Dynamical chaos is one of the features of wave turbulence. In [17] we address the question how and when the onset of chaos takes place after preheating. To investigate the onset of chaos we have to follow the time evolution of two initially nearby points in the phase space. Consider the theory with the potential (21) with two fields \( \phi \) and \( \chi \) only (which we collectively denote as \( f \)). Consider two configurations of a scalar field \( f \) and \( f' \) that are identical except for a small difference of the fields at a set of points \( x_A \). Chaos can be defined as the tendency of such nearby configurations in phase space to diverge exponentially over time. This divergence is parametrized by the Lyapunov exponent for the system, defined as \( \lambda \equiv \frac{1}{t} \log \frac{\Delta(t)}{\Delta_0} \) where \( \Delta \) is a distance between two configurations and \( \Delta_0 \) is the initial distance at time 0. Numerical results shows very fast onset of chaos around the moment \( t_\ast \) where the strong turbulence begins.

The highlights of our study for early universe phenomenology are the following. The mechanism of preheating after inflation is rather robust and works for many different systems of interacting scalars. There is a stage of turbulent classical waves where the initial conditions for preheating are erased. Initially, before all the fields have settled into equilibrium with a uniform temperature, the reheating temperature may be different in different subgroups of fields. The nature of these groupings is determined by the coupling strengths.
V. CONCLUSION

We considered the dynamics of spontaneous symmetry breaking, which occurs when a scalar field falls down from the top of its effective potential. We have found that the main part of this process typically completes within a single oscillation of the distribution of the scalar field. This is a very unexpected conclusion that may have important cosmological implications.

One of the most efficient mechanisms for the creation of matter after inflation in theories with convex effective potentials ($V''(\phi) > 0$) is the mechanism of parametric amplification of vacuum fluctuations in the process of homogeneous oscillations of the inflaton field, which was called preheating [1]. It has also been noted that in the case where potentials become concave ($V''(\phi) < 0$), preheating may become more efficient [19]. Now we see that this effect is very generic. In many theories with concave potentials the energy of an unstable vacuum state is transferred to the energy of inhomogeneous classical waves of scalar fields within a single oscillation of the field distribution. We emphasize here that we are talking about the oscillations of the field distribution rather than about the oscillations of a homogeneous field $\phi$ because quite often the homogeneous component $\langle \phi \rangle$ of the field $\phi$ remains zero during the process of spontaneous symmetry breaking.

One of the important consequences of our results is the observation [10] that in many models of hybrid inflation [5] the first stage of reheating occurs not due to homogeneous oscillations of the scalar field but due to tachyonic preheating [10]. In particular, this significantly alters the theory of fermionic preheating [18] for the hybrid inflation.

The process of preheating and symmetry breaking may take an especially unusual form in the theory of brane inflation [7] based on the hybrid inflation scenario and the mechanism of tachyon condensation on the brane antibrane system [20].

The situation in models of the type used in the new inflation scenario is somewhat more complicated. In these models the potential is also concave. However, the expansion of the universe stretches inhomogeneities of the field rolling down from the top of the effective potential and makes it homogeneous on an exponentially large scale. Therefore to evaluate a possible significance of tachyonic instability in this regime one must compare the amplitude of the homogeneous component of the field with the amplitude of the quantum fluctuations. The result appears to be very sensitive to the scale of spontaneous symmetry breaking in such models. A preliminary investigation of this issue indicates that in small-field models where the scale of spontaneous symmetry breaking is much smaller than $M_p$, the leading mechanism of preheating typically is tachyonic. If correct, this would be a very interesting conclusion indicating that in large-field models the leading mechanism of preheating typically is related to parametric resonance, whereas in small-field models the main mechanism of preheating is typically tachyonic, at least at the first stages of the process.

Finally we should mention that an interesting application of our methods can be found in the recently proposed ekpyrotic and pyrotechnic scenario [21,22]. Even though we are very skeptical with respect to the ekpyrotic/pyrotechnic scenario for many reasons explained in [22], it is still interesting that the methods developed in the theory of tachyonic preheating provide us with a very simple theory of the generation of density perturbations in these models [24].

I am grateful to Andrei Linde, Gary Felder, Juan J. García-Bellido Patrick Greene and
Igor Tkachev for the collaboration on the tachyonic preheating. I thank NSERC, CIAR and the NATO Linkage Grant 975389 for support.
REFERENCES

[1] L. A. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. Lett. 73, 3195 (1994)
[2] L. A. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. D 56, 3258 (1997).
[3] S. Y. Khlebnikov and I. I. Tkachev, Phys. Rev. Lett. 77, 219 (1996); Phys. Rev. Lett. 79, 1607 (1997)
[4] Greene, P., Kofman, L., Linde, A., & Starobinsky, A. 1997, Phys. Rev.D56, 6175.
[5] A. Linde, Phys. Lett. B259, 38 (1991); Phys. Rev.D49, 748 (1994).
[6] E. J. Copeland, A. R. Liddle, D. H. Lyth, E. D. Stewart and D. Wands, Phys. Rev. D 49, 6410 (1994)
[7] G. Dvali, Q. Shafi and S. Solganik, hep-th/0105203; C. P. Burgess, M. Majumdar, D. Nolte, F. Quevedo, G. Rajesh and R. J. Zhang, hep-th/0105204.
[8] J. García-Bellido and A. Linde, Phys. Rev. D 57, 6065 (1998)
[9] M. Bastero-Gil, S. F. King and J. Sanderson, Phys. Rev. D 60, 103517 (1999)
[10] G. Felder, J. García-Bellido, P. B. Greene, L. Kofman, A. Linde and I. Tkachev, Phys. Rev. Lett. 87, 011601 (2001).
[11] G. Felder, L. Kofman and A. Linde, hep-th/0106179.
[12] S. Fubini, Nuovo Cim. A 34, 521 (1976).
[13] A. D. Linde, Nucl. Phys. B 216, 421 (1983).
[14] J. Bardeen, J.R. Bond, N. Kaiser and A. Szalay, Astrophys. J. 304, 15 (1986).
[15] A. Linde, Nucl. Phys. B 372, 421 (1992)
[16] A.A. Starobinsky, in: Current Topics in Field Theory, Quantum Gravity and Strings, Lecture Notes in Physics, eds. H.J. de Vega and N. Sanchez (Springer, Heidelberg 1986) 206, p. 107.
[17] G. Felder and L. Kofman, Phys. Rev. D 63, 103503 (2001)
[18] J. Baacka, K. Heitmann & C. Pätzold, Phys. Rev. D58, 125013 (1998); P. Greene and L. Kofman, Phys. Lett. B448, 6 (1999).
[19] B. R. Greene, T. Prokopec and T. G. Roos, Phys. Rev. D56, 6484 (1997); G. Felder, L. Kofman and A. Linde, Phys. Rev. D59, 123523 (1999).
[20] A. Sen, JHEP 9808, 010 (1998); JHEP 9808, 012 (1998) hep-th/9805171.
[21] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, hep-th/0103233.
[22] R. Kallosh, L. Kofman and A. Linde, hep-th/0104073.