Abstract

In Hliněná et al. (2014) the authors, inspired by Karaçal and Kesicioğlu (2011), introduced a pre-order induced by uninorms. This contribution is devoted to a classification of families of uninorms by means of types of pre-orders (and orders) they induce. Philosophically, the paper follows the original idea of Clifford (1954).

Keywords: Pre-order induced by uninorm, Representable uninorm, Uninorm, Uninorm with continuous underlying operations, Locally internal uninorm.

1 Introduction

In this paper we study pre-orders generated by uninorms. The main idea is based on that of Karaçal and Kesicioğlu [19], and follows the original idea of Clifford [4]. The main idea of authors is to show a relationship between families of uninorms and families of pre-orders (partial orders, in some cases) they induce (see [16]). In some sense, the pre-order (see Definition 11) follows the original idea by Clifford [4]. Another relation induced by uninorms, that is always a partial order (see Definition 12), was proposed by Erteğrul et al. [11]. Here, the main intention of authors was to get a partial order. But this relation (partial order) does not extend the relation introduced by Clifford [4].

2 Preliminaries

In this section we review some well-known types of monotone commutative monoidal operations on $[0,1]$ and provide an overview of, from the point of view of this contribution, important steps in introducing orders (and pre-orders) induced by semigroups.

2.1 Known types of monotone commutative monoidal operations on $[0,1]$

In this part we give just very brief review of well-known types of monotone commutative monoidal operations on $[0,1]$. For more details we recommend monographs [2, 20].

Definition 1 (see, e.g., [20]). A triangular norm $T$ (t-norm for short) is a commutative, associative, monotone binary operation on the unit interval $[0,1]$, fulfilling the boundary condition $T(x,1) = x$, for all $x \in [0,1]$.

Definition 2 (see, e.g., [20]). A triangular conorm $S$ (t-conorm for short) is a commutative, associative, monotone binary operation on the unit interval $[0,1]$, fulfilling the boundary condition $S(x,0) = x$, for all $x \in [0,1]$.

Remark 1. If $T$ is a t-norm, then $S(x,y) = 1 - T(1-x,1-y)$ is a t-conorm and vice versa. We obtain a dual pair $(T,S)$ of a t-norm and a t-conorm.

Example 1. Well-known examples of triangular norms and their dual t-conorms are:

- $T_M(x,y) = \min(x,y)$, $S_M(x,y) = \max(x,y)$,
- $T_P(x,y) = x \cdot y$, $S_P(x,y) = x + y - x \cdot y$,
- $T_L(x,y) = \max(x+y-1,0)$, $S_L(x,y) = \min(x+y,1)$.

Casasnovas, Mayor [3] introduced divisible t-norms.

Definition 3 ([3]). Let $L$ be a bounded lattice and $T : L \times L \to L$ be a t-norm. $T$ is said to be divisible if the following conditions are satisfied for all $(x,y) \in L^2$:

$$ (x \leq y) \Rightarrow (\exists z \in L)(T(y,z) = x) \quad (1) $$

Of course, a t-norm $T : [0,1]^2 \to [0,1]$ is divisible if and only if it is continuous.
Definition 4 (see, e.g., [2]). Let \( \ast : [0, 1]^2 \to [0, 1] \) be a binary commutative operation. Then

(i) element \( e \) is said to be idempotent if \( e \ast e = e \),

(ii) element \( e \) is said to be neutral if \( e \ast x = x \) for all \( x \in L \),

(iii) element \( a \) is said to be annihilator if \( a \ast x = a \) for all \( x \in L \).

Definition 5 ([26]). A uninorm \( U \) is a function \( U : [0, 1]^2 \to [0, 1] \) that is increasing, commutative, associative and has a neutral element \( e \in [0, 1] \).

Remark 2. For any uninorm with neutral element equal to \( e \) we denote

\[
A(e) = [0, e] \times [e, 1] \cup [e, 1] \times [0, e].
\]

1. If \( e \notin \{0, 1\} \) is the neutral element of \( U \), we say that \( U \) is a proper uninorm.

2. Every uninorm \( U \) has a distinguished element \( a \) called annihilator, for which the following holds \( U(a, x) = U(0, 1) = a \). A uninorm \( U \) is said to be conjunctive if \( U(x, 0) = 0 \), and \( U \) is said to be disjunctive if \( U(1, x) = 1 \), for all \( x \in [0, 1] \).

Lemma 1 ([12]). Let \( U \) be a uninorm with the neutral element \( e \). Then, for \( (x, y) \in [0, 1]^2 \) the following holds

(i) \( T(x, y) = \frac{U(ex, ey)}{e} \) is a t-norm,

(ii) \( S(x, y) = \frac{U((1-e)x+(1-e)y+e)-e}{1-e} \) is a t-conorm.

For all \((x, y) \in A(e)\) we have

\[
\min(x, y) \leq U(x, y) \leq \max(x, y).
\]

Definition 6. Let \( U \) be a uninorm. We say that \( U \) is internal if \( U(x, y) \in \{x, y\} \) for all \( (x, y) \in [0, 1]^2 \).

A uninorm \( U \) is locally internal on a set \( G \subseteq [0, 1]^2 \) if \( U(x, y) \in \{x, y\} \) for all \( (x, y) \in G \).

Remark 3. (a) Particularly, a uninorm \( U \) is locally internal on the boundary if \( U(x, 0) \in \{x, 0\} \) and \( U(x, 1) \in \{x, 1\} \) holds for all \( x \in [0, 1] \). Some examples of uninorms which are not locally internal on the boundary can be found, e.g., in [14, 15, 16], see also Fig. 1.

(b) An important family of uninorms is that of internal ones. From results by Drewniak and Drygaś [6] follows that the family of all internal uninorms is identical with that of idempotent uninorms. Some further study of locally internal uninorms can be found, e.g., in [8] and in literature referenced therein.

From results in [6, 21, 25] we have the following.

\[ A(e) = [0, e] \times [e, 1] \cup [e, 1] \times [0, e]. \]

1. If \( e \notin \{0, 1\} \) is the neutral element of \( U \), we say that \( U \) is a proper uninorm.

2. Every uninorm \( U \) has a distinguished element \( a \) called annihilator, for which the following holds \( U(a, x) = U(0, 1) = a \). A uninorm \( U \) is said to be conjunctive if \( U(x, 0) = 0 \), and \( U \) is said to be disjunctive if \( U(1, x) = 1 \), for all \( x \in [0, 1] \).

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(ii) \( S(x, y) = \frac{U((1-e)x+(1-e)y+e)-e}{1-e} \) is a t-conorm.

For all \((x, y) \in A(e)\) we have

\[
\min(x, y) \leq U(x, y) \leq \max(x, y).
\]

Lemma 2. Let \( U \) be a uninorm. \( U \) is idempotent if and only if it is \( U \) internal.

Proposition 1 (E.g., [10]). Let \( f : [-\infty, \infty] \to [0, 1] \) be an increasing bijection. Then

\[
U(x, y) = f^{-1}(f(x) + f(y))
\]

is a uninorm that is continuous everywhere except at points \((0, 1)\) and \((1, 0)\), and is strictly increasing on \([0, 1]^2 \). \( U \) is conjunctive if we adopt the convention \( -\infty + \infty = -\infty \), and \( U \) is disjunctive adopting the convention \( -\infty + \infty = \infty \).

Definition 7 (E.g., [10]). The uninorm \( U \) fulfilling formula \((2)\) for an increasing bijection \( f : [-\infty, \infty] \to [0, 1] \) adopting either of the conventions, \( -\infty + \infty = -\infty \) or \( -\infty + \infty = \infty \), is said to be a representable uninorm.

Remark 4. Representable uninorms, under the name aggregative operators were studied already by Dombi [5].

Another important class of uninorms is that of continuous ones on \([0, 1]^2 \). These uninorms were characterized by Hu and Li [17], and further studied by Drygaś [7]. From results in [17] we have the following characterization.

Proposition 2. A uninorm \( U \) with neutral element \( e \in [0, 1] \) is continuous on \([0, 1]^2 \) if and only if one of the following conditions is satisfied:

(i) \( U \) is representable,

(ii) there exists \( 0 < a < e \), a continuous t-norm \( T \) a representable uninorm \( U_r \) and an increasing bijection \( \phi : [a, 1] \to [0, 1] \) such that

\[ U(x, y) = U_r(\phi(x), \phi(y)). \]
\[ U(x, y) = \varphi^{-1}(U_r(\varphi(x), \varphi(y))) \] for \( (x, y) \in [a, 1]^2 \),
\[ U(x, y) = aT\left(\frac{y}{a}, \frac{y}{a}\right) \] for \( (x, y) \in [0, a]^2 \),
\[ U(x, y) = \min\{x, y\} \] for \( (x, y) \in [0, a] \cap [a, 1] \cup [a, 1] \cap [0, a] \).
and \( U \) is locally internal on the boundary,

(iii) or there exists \( c < b < 1 \) a continuous \( t \)-conorm \( S \) and a representable unimodular \( U_r \) and an increasing bijection \( \varphi : [0, b] \to [0, 1] \) such that
\[ U(x, y) = \varphi^{-1}(U_r(\varphi(x), \varphi(y))) \] for \( (x, y) \in [0, b]^2 \),
\[ U(x, y) = b + (1 - b)S\left(\frac{y}{b}, \frac{y}{b}\right) \] for \( (x, y) \in [b, 1]^2 \),
\[ U(x, y) = \max\{x, y\} \] for \( (x, y) \in [b, 1] \cap [0, b] \cup [b, 1] \cap [0, b] \),
and \( U \) is locally internal on the boundary.

2.2 An overview of pre-orders induced by a semigroup

The study of orders (pre-orders) induced by a semigroup operation had started by Clifford [4]. Later, Hartwig [13] and independently also Nambooripad [23], defined a partial order on regular semigroups. Their definition is the following.

Definition 8 ([13, 23]). Let \((S, \oplus)\) be a semigroup and \(\mathcal{E}_S\) the set of its idempotent elements. Then
\[ a \leq_b \iff (\exists e, f \in \mathcal{E}_S) (a = b \oplus e = f \oplus b). \]
If the relation \(\leq_b\) is a partial order on \(S\), it is called natural.

Definition 8 was generalized by Mitch [22].

Definition 9 ([22]). Let \((S, \oplus)\) be an arbitrary semigroup. By \(\preceq_S\) we denote the following relation
\[ a \preceq_S b \iff a = b \oplus z_1 = z_2 \oplus b, a \oplus z_1 = a \]
for some \(z_1, z_2 \in \mathcal{E}_S\), where
\[ \mathcal{E}_S^1 = \begin{cases} S & \text{if } S \text{ has a neutral element,} \\ S \cup \{e\} & \text{otherwise, where } e \text{ plays the role of the neutral element,} \end{cases} \]
and \(\mathcal{E}_S^1\) is the set of all idempotents of \(S^1\).

Lemma 3 ([22]). Let \((S, \oplus)\) be an arbitrary semigroup. The relation \(\leq_S\) is reflexive and antisymmetric on \(S\).

Proposition 3 ([22]). Let \((S, \oplus)\) be an arbitrary semigroup. The relation
\[ a \preceq_S b \iff a = x \oplus b = b \oplus y \] (3)
for some \(x, y \in S^1\), is a partial order on \(S\).

From now on, we restrict our attention to commutative semigroups. Lemma 3 and Proposition 3 immediately imply the following.

Lemma 4. Let \((S, \oplus)\) be a commutative semigroup. By \(\preceq_S\) we denote the set
\[ a \preceq_S = \{z \in S : z \preceq_S a\}, \]
where \(a \in S\). Then for all \(a, b \in S\) it holds that \(a \preceq_S b\) if and only if \(a \preceq_S \subseteq b \preceq_S\).

Directly by Definition 9 we get

Proposition 4. Let \((S, \oplus)\) be a commutative semigroup. Then the set \(a \preceq_S\) is an ideal in \((S, \oplus)\).

Lemma 5. Let \((S, \oplus)\) be a commutative semigroup. Let \(I_S\) be an ideal of \((S, \oplus)\). Then \((I_S, \oplus|_{I_S})\) is a sub-semigroup of \((S, \oplus)\), where \(\oplus|_{I_S} = \oplus \upharpoonright I_S^2\).

Karaçal and Kesicioğlu [19] defined a partial order on bounded lattices \(L\) by means of \(t\)-norms.

Definition 10 ([19]). Let \(L\) be a bounded lattice and \(T : L \times L \to L\) a \(t\)-norm. We write \(x \leq_T y\) for arbitrary \(x, y \in L\) if there exists \(z \in L\) such that \(x = T(y, z)\).

Proposition 5 ([19]). Let \(L\) be a bounded lattice and \(T : L \times L \to L\) a \(t\)-norm. Then the relation \(\leq_T\) is a partial order on \(L\).

Remark 4. For arbitrary \(t\)-norm \(T\), the partial order \(\leq_T\) from Definition 10 extends the partial order \(\preceq_T\) from Definition 9 in the following sense: let \(L\) be arbitrary bounded lattice and \(T\) a commutative semigroup-operation on \(L\) with a neutral element such that \((L, \leq_T)\) is a partially ordered set. Then
\[ a \preceq_T b \iff a \preceq_T b \]
for all \(a, b \in L\).

Remark 6. Concerning a correspondence between properties of binary aggregation function \(A : L^2 \to L\) and relation \(\preceq_A\) (changing a \(t\)-norm \(T\) for \(A\) in Definition 10), the following can be said:

- if \(A\) has a neutral element, or \(A\) is idempotent, then \(\preceq_A\) is reflexive,
- if \(A\) is associative, then \(\preceq_A\) is transitive,
- the anti-symmetry of \(\preceq_A\) fails if there exist elements \(x \neq z\) and \(y_1, y_2\) such that \(z = A(x, y_1)\) and \(x = A(z, y_2)\). Hence, if one of the following
\[ x \preceq_A z \iff x \preceq_L z, \]
\[ x \preceq_A z \iff z \preceq_L x \]
holds then \(\preceq_A\) is anti-symmetric.

Hliněná et al. [16] introduced the following relation \(\preceq_U\).
Definition 11 ([16]). Let $U : [0, 1]^2 \to [0, 1]$ be an arbitrary uninorm. By $\preceq_U$ we denote the following relation

$$x \preceq_U y \text{ if there exists } \ell \in [0, 1] \text{ such that } U(y, \ell) = x.$$ 

Immediately by Definition 11 we get the next lemma.

**Lemma 6.** Let $U$ be an arbitrary uninorm. Then $\preceq_U$ is transitive and reflexive. If $a$ and $e$ are the annihilator and the neutral elements of $U$, respectively, then

$$a \preceq_U x \preceq_U e$$

holds for all $x \in [0, 1]$.

**Remark 7.** In Definition 11 we have used the same notation $\preceq_U$ for the pre-order defined from a uninorm $U$, as in Definition 10 for the corresponding partial order $\preceq_T$ defined from a t-norm $T$. These two relations really coincide if $U = T$, i.e., the notation should not cause any problems.

The pre-order $\preceq_U$ extends the partial order $\preceq_U$ from Definition 9 in the following sense.

**Proposition 6.** Let $U$ be an arbitrary uninorm. Then

$$x \preceq_U y \Rightarrow x \preceq_U y$$

for all $(x, y) \in [0, 1]^2$.

A different type of partial order induced by uninorms has been defined by Ertuğrul et al. [11].

**Definition 12 ([11]).** Let $U$ be a uninorm and $e \in [0, 1]$ its neutral element. For $(x, y) \in [0, 1]^2$ denote $x \preceq_U y$ if one of the following properties is satisfied:

1. there exists $\ell \in [0, e]$ such that $x = U(y, \ell)$ and $(x, y) \in [0, e]^2$,
2. there exists $\ell \in [e, 1]$ such that $y = U(x, \ell)$ and $(x, y) \in [e, 1]^2$,
3. $0 \leq x \leq e \leq y \leq 1$.

**Proposition 7 ([11]).** For an arbitrary uninorm $U$, the relation $\preceq_U$ from Definition 12 is a partial order.

**Example 2.** Consider the following uninorm $U$

$$U(x, y) = \begin{cases} \min(x, y) & \text{if } (x, y) \in [0, 1]^2, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

Then $\preceq_U$ coincides with the usual order of $[0, 1]$, while $x \preceq_U y$ if one of the following possibilities is satisfied

- $y = 0.5$.
- $y \geq x$ for $x > 0.5$,
- $x \leq y$ for $(x, y) \in [0, 0.5]^2$.

**Remark 8.** Let $U$ be a uninorm. To compare the relation $\preceq_U$ from Definition 11 with $\preceq_U$ from Definition 12, the following should be remarked.

(i) The relation $\preceq_U$, given in Definition 11 is a pre-order, but not necessarily a partial order. Unlike this, the relation $\preceq_U$ defined by Definition 12, is always a partial order.

(ii) As illustrated by Example 2, the partial order $\preceq_U$ does not necessarily extends the partial order $\preceq_U$ on the semigroup $([0, 1], U)$, i.e.,

$$x \preceq_U y \nRightarrow x \preceq_U y.$$

As shown by Proposition 6, the pre-order $\preceq_U$ always extends the partial order $\preceq_U$ on $([0, 1], U)$, see formula (4).

Further in the text, we will consider only the pre-order $\preceq_U$ to distinguish several families of uninorms.

**Definition 13.** Let $U$ be an arbitrary uninorm.

(i) For $(x, y) \in [0, 1]^2$ we denote $x \sim_U y$ if $x \preceq_U y$ and $y \preceq_U x$.

(ii) For $(x, y) \in [0, 1]^2$ we denote $x \parallel_U y$ if neither $x \preceq_U y$ nor $y \preceq_U x$ holds, and $x \parallel_U y$ if $x \preceq_U y$ or $y \preceq_U x$.

(iii) For arbitrary $x \in [0, 1]$ we denote $x \sim_U = \{ z \in [0, 1]; z \sim_U x \}$.  

### 3 Some distinguished families of uninorms and properties of the corresponding pre-orders

We are going to study a relationship between some distinguished families $\mathcal{U}$ of uninorms on the one hand and properties of the corresponding pre-orders $\preceq_U$ for $U \in \mathcal{U}$ on the other hand.

A direct consequence to Lemma 6 is the following.

**Corollary 1.** Let $U$ be a uninorm. The following holds for all $x \in [0, 1]$:

(i) $0 \preceq_U x$ if and only if $U$ is conjunctive,

(ii) $1 \preceq_U x$ if and only if $U$ is disjunctive.

#### 3.1 Locally internal uninorms

In this part we distinguish three types of locally internal uninorms:

- on the boundary,
- on $A(e)$,
- on $[0, e]^2 \cup [e, 1]^2$. 


Proposition 8. Let $U$ be a uninorm. It is locally internal on the boundary if and only if for every element $x \in [0, 1]$
\[ 0 \parallel_U x \quad \text{and} \quad 1 \parallel_U x. \]

Proposition 9. Let $U$ be a uninorm with neutral element $e$. It is locally internal on $A(e)$ if and only if $\preceq_U$ is a partial order and for every element $x \in [0, e]$ and $y \in [e, 1]$ we have
\[ x \parallel_U y. \]

Remark 9. For an arbitrary uninorm $U$ and for a pair $(x, y) \in [0, 1]^2$, we have
\[ U(x, y) = x \implies x \preceq_U y, \]
\[ U(x, y) = y \implies y \preceq_U x. \]

Results in [19] imply the following.

Proposition 11. Let $U$ be a proper uninorm with a neutral element $e$. Then $U$ has continuous underlying $t$-norm and $t$-conorm if and only if the following hold
\[ x \leq y \implies x \preceq_U y \quad \text{for} \quad (x, y) \in [0, e]^2, \]
\[ y \leq x \implies x \preceq_U y \quad \text{for} \quad (x, y) \in [e, 1]^2. \]

Propositions 9 and 11 have the following corollary.

Corollary 2. Let $U$ be a proper uninorm. Then $U$ is locally internal on $A(e)$ and with continuous underlying $t$-norm and $t$-conorm if and only if $\preceq_U$ is a linear order.

Applying Proposition 2 to the pre-order $\preceq_U$ we get the following characterization of representable uninorms.

Proposition 12. A uninorm $U$ is representable if and only if for all $(x, y) \in [0, 1]^2$ we have $x \sim_U y$.

Proposition 2 implies the following characterization of uninorms continuous on $[0, 1]^2$.

Proposition 13. Let $U$ be a proper uninorm with neutral element $e$, which is not representable. Then it is continuous on $[0, 1]^2$ if and only if one of the following is valid.

(i) There exists $0 < a < e$ such that
\[ 1. \quad x \sim_U y \quad \text{for all} \quad (x, y) \in [a, 1]^2, \]
\[ 2. \quad x \preceq_U y \iff x \leq y \quad \text{for all} \quad (x, y) \in [0, a]^2. \]

(ii) There exists $e < b < 1$ such that
\[ 1. \quad x \sim_U y \quad \text{for all} \quad (x, y) \in [0, b]^2, \]
\[ 2. \quad x \preceq_U y \iff x \geq y \quad \text{for all} \quad (x, y) \in [b, 1]^2. \]

3.3 Some other classes of uninorms on $[0, 1]$

First, we provide some results on uninorms with an area of constantness in $[0, e]^2$ or $[e, 1]^2$.

Proposition 14 ([14]). Let $U$ be a proper uninorm having $e$ as neutral element. Let $y > e$ be an idempotent element of $U$. If there exists $x < e$ such that $U(x, y) = \tilde{x} \in [x, e]$ then
\[ U(z, y) = \tilde{x} \quad \text{and} \quad U(z, x) = U(\tilde{x}, x) \quad \text{for all} \quad z \in [x, \tilde{x}]. \]

Dually to Proposition 14 we get

Proposition 15. Let $U$ be a proper uninorm having $e$ as neutral element. Let $y < e$ be an idempotent element of $U$. If there exists $x > e$ such that $U(x, y) = \tilde{x} \in [e, x]$ then
\[ U(z, y) = \tilde{x} \quad \text{and} \quad U(z, x) = U(\tilde{x}, x) \quad \text{for all} \quad z \in [\tilde{x}, x]. \]

Propositions 14 and 15 have the following corollary.
Corollary 3. Let $U$ be a proper uninorm having $e$ as neutral element.

(i) Assume $x < e$ is an idempotent element of $U$. Then either $x \|_{U_1} y$ for all $y \in [e, 1]$ or there exists an interval $[a, b] \subset [e, 1]$ such that $x \|_{U_2} z$ for all $z \in [a, b]$.

(ii) Assume $x > e$ is an idempotent element of $U$. Then either $x \|_{U_1} y$ for all $y \in [0, e)$ or there exists an interval $[a, b] \subset [0, e)$ such that $x \|_{U_2} z$ for all $z \in [a, b]$.

Kalina and Král [18] introduced uninorms which are strictly increasing on $[0, 1]^2$, but not continuous. The construction method was further studied in [1, 27]. Since we are not able to distinguish among continuous t-norms $T$ (t-conorms $S$) by means of the relation $\preceq T (\succeq S)$, we are not able to characterize unambiguously uninorms which are strictly increasing on $[0, 1]^2$. We present the main idea of the construction method, paving, in case the basic ‘brick’ is the product t-norm $T_\pi$:

(a) we split the interval $[0, 1]$ into infinitely countably many disjoint right-closed subintervals $\{I_j; j \in \mathcal{J}\}$, where $\mathcal{J}$ is an index set and $(\mathcal{J}, \oplus, j_0)$ is a commutative increasing monoid and $j_0$ is its neutral element,

(b) $\vartheta_j : I_j \to [0, 1]$ is an increasing bijection.

The resulting uninorm is defined by:

$$U_p(x, y) = \vartheta^{\leftarrow 1}_\vartheta(T_\pi(\vartheta_i(x), \vartheta_j(y))) \quad \text{for } x \in J_i, y \in J_j,$$

$$0 \quad \text{if } \min\{x, y\} = 0,$$

$$1 \quad \text{otherwise.} \quad (6)$$

Concerning the properties of $\succeq_{U_p}$, there are two possibilities depending whether $(\mathcal{J}, \oplus, j_0)$ is a group or not.

Proposition 16. Let $U_p$ be a uninorm defined by (6), $(\mathcal{J}, \oplus, j_0)$ be a commutative group and $\{I_j; j \in \mathcal{J}\}$ be a system of disjoint right-closed intervals whose union is $[0, 1]$. Then:

(i) for every $j \in \mathcal{J}$ and all $(x, y) \in I_j^2$ we have

$$x \succeq_{U_p} y \iff x \leq y,$$

(ii) for all $i, j \in \mathcal{J}$, $i \neq j$, all $x \in I_i$ and $y \in I_j$ we have

$$x \sim_{U_p} y \iff \vartheta_j(y) = \vartheta_i(x),$$

$$x \preceq_{U_p} y \iff \vartheta_i(x) \leq \vartheta_j(y).$$

Proposition 17. Let $U_p$ be a uninorm defined by (6), $(\mathcal{J}, \oplus, j_0)$ be a commutative monoid without inverse elements, with the neutral element $j_0$ and $\{I_j; j \in \mathcal{J}\}$ be a system of disjoint right-closed intervals whose union is $[0, 1]$. Then:

(i) for every $j \in \mathcal{J}$ and all $(x, y) \in I_j^2$ we have

$$x \preceq_{U_p} y \iff x \leq y,$$

(ii) for all $i, j \in \mathcal{J}$, $i \neq j$, all $x \in I_i$ and $y \in I_j$ we have

$$x \sim_{U_p} y \iff \vartheta_j(y) = \vartheta_i(x),$$

$$x \preceq_{U_p} y \iff \vartheta_i(x) \leq \vartheta_j(y).$$

We could formulate dual theorems to Propositions 16 and 17 for the case when the basic ‘brick’ is the probabilistic sum t-conorm.

4 Conclusion

The results presented in this paper are aimed to characterize classes of uninorms by means of a pre-order the induce. We have succeeded in getting full characterization for uninorms which are continuous on $[0, 1]^2$, as well as for uninorms with continuous underlying t-norm and t-conorm, and for those which are locally internal on the boundary and on $A(e)$. Further, it is possible to distinguish whether a uninorm is conjunctive or disjunctive. In some other cases we have partial results.

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