Novel Aspects in $p$-Brane Theories: Weyl-Invariant Light-Like Branes

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Abstract

We consider a novel class of Weyl-conformally invariant $p$-brane theories which describe intrinsically light-like branes for any odd world-volume dimension, hence the acronym WILL-branes (Weyl-Invariant Light-Like branes). We discuss in some detail the properties of WILL-brane dynamics which significantly differs from ordinary Nambu-Goto brane dynamics. We provide explicit solutions of WILL-membrane (i.e., $p = 2$) equations of motion in arbitrary $D = 4$ spherically symmetric static gravitational backgrounds, as well as in product spaces of interest in Kaluza-Klein context. In the first case we find that the WILL-membrane materializes the event horizon of the corresponding black hole solutions, thus providing an explicit dynamical realization of the membrane paradigm in black hole physics. In the second “Kaluza-Klein” context we find solutions describing WILL-branes wrapped around the internal (compact) dimensions and moving as a whole with the speed of light in the non-compact (space-time) dimensions.

Keywords: Weyl-conformal invariant $p$-brane actions, light-like $p$-branes, non-Riemannian volume forms, variable string/brane tension, Kaluza-Klein, event horizons, membrane paradigm.

1 Introduction

The idea of replacing the standard Riemannian integration measure (Riemannian volume-form) with an alternative non-Riemannian volume-form or, more generally, employing on equal footing both Riemannian and non-Riemannian volume-forms to construct new classes of models involving gravity, called two-measure theories, has been proposed few years ago [1] and since then it is a subject of active research and developments [2] (for related ideas, see [3]).

Two-measure theories address various basic problems in cosmology and particle physics, and provide plausible solutions for a broad array of issues, such as: scale invariance and its dynamical breakdown; spontaneous generation of dimensionful fundamental scales; the cosmological constant problem; the problem of fermionic families; applications in modern brane-world scenarios. For a detailed discussion we refer to the series of papers [1, 2].

Subsequently, the idea of employing an alternative non-Riemannian integration measure was applied systematically to string, $p$-brane and $Dp$-brane models [4] (for a background on string and brane theories, see refs.[5]). The main feature of these new classes of modified string/brane theories is the appearance of the pertinent string/brane tension as an additional dynamical degree of freedom beyond the usual string/brane physical degrees of freedom, instead of being introduced ad hoc as a dimensionful scale. The dynamical string/brane tension acquires the physical meaning of a world-sheet electric field strength (in the string case) or world-volume $p + 1$-form field strength (in the $p$-brane case) and obeys Maxwell (Yang-Mills) equations of motion or their higher-rank antisymmetric tensor gauge
field analogues, respectively. As a result of the latter property the modified-measure string model with dynamical tension yields a simple classical mechanism of “color” charge confinement.

In the next section we proceed to our main task which is the study of a novel class (first proposed in our preceding work [6]) of p-brane theories which are Weyl-conformal invariant for any p and which describe intrinsically light-like branes for any odd (p + 1). Thus, their dynamics significantly differs both from the standard Nambu-Goto (or Dirac-Born-Infeld) branes as well as from their modified versions with dynamical string/brane tensions [4] mentioned above.

2 Weyl-Invariant p-Brane Theories

2.1 Standard Nambu-Goto Branes

Let us first briefly recall the standard Polyakov-type formulation of the bosonic p-brane action:

\[ S = -\frac{T}{2} \int d^{p+1}\sigma \sqrt{-\gamma} \left[ \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - \Lambda (p-1) \right] . \]  

(1)

Here \( \gamma_{ab} \) is the ordinary Riemannian metric on the \( p + 1 \)-dimensional brane world-volume with \( \gamma \equiv \det ||\gamma_{ab}|| \). The world-volume indices \( a, b = 0, 1, \ldots, p \) : \( G_{\mu\nu} \) denotes the Riemannian metric in the embedding space-time with space-time indices \( \mu, \nu = 0, 1, \ldots, D - 1 \). \( T \) is the given ad hoc brane tension; the constant \( \Lambda \) can be absorbed by rescaling \( T \) (see below Eq.(7)). The equations of motion w.r.t. \( \gamma^{ab} \) and \( X^\mu \) read:

\[ T_{ab} \equiv \left( \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \gamma_{ab} \gamma^{cd} \partial_c X^\mu \partial_d X^\nu \right) G_{\mu\nu} + \gamma_{ab} \frac{\Lambda}{2} (p-1) = 0 , \]  

(2)

\[ \partial_a \left( \sqrt{-\gamma} \gamma^{ab} \partial_b X^\mu \right) + \sqrt{-\gamma} \gamma^{ab} \partial_a X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu\lambda} = 0 , \]  

(3)

where:

\[ \Gamma^\mu_{\nu\lambda} = \frac{1}{2} \gamma^{\mu cd} \left( \partial_c G_{\kappa\lambda} + \partial_\kappa G_{\nu\lambda} - \partial_\nu G_{\kappa\lambda} \right) \]  

(4)

is the affine connection for the external metric.

Eqs.(2) when \( p \neq 1 \) imply:

\[ \Lambda \gamma_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} , \]  

(5)

which in turn allows to rewrite Eq.(2) as:

\[ T_{ab} \equiv \left( \partial_a X^\mu \partial_b X^\nu - \frac{1}{p+1} \gamma_{ab} \gamma^{cd} \partial_c X^\mu \partial_d X^\nu \right) G_{\mu\nu} = 0 . \]  

(6)

Furthermore, using (5) the Polyakov-type brane action (1) becomes on-shell equivalent to the Nambu-Goto-type brane action:

\[ S = -T \Lambda^{-\frac{1}{p+1}} \int d^{p+1}\sigma \sqrt{-\det ||\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}||} . \]  

(7)

2.2 Weyl-Invariant Branes: Action and Equations of Motion

In ref.[6] we proposed the following novel p-brane actions:

\[ S = - \int d^{p+1}\sigma \Phi(\varphi) \left[ \frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - \sqrt{F_{ab}(A) F_{cd}(A) \gamma^{ac} \gamma^{bd}} \right] \]  

(8)

with \( F_{ab}(A) = \partial_a A_b - \partial_b A_a \), and:

\[ \Phi(\varphi) = \frac{1}{(p+1)!} \varepsilon_{i_1 \ldots i_{p+1}} \varepsilon^{a_1 \ldots a_{p+1}} \partial_{a_1} \varphi^{i_1} \ldots \partial_{a_{p+1}} \varphi^{i_{p+1}} , \quad i, j = 1, \ldots, p + 1 . \]  

(9)

Here \( \gamma_{ab} \) and \( G_{\mu\nu} \) have the same meaning as in (1).

Let us notice the following significant differences of (8) w.r.t. the standard Nambu-Goto p-branes (in the Polyakov-like formulation) (1):
• New non-Riemannian integration measure density (volume-form) $\Phi(\varphi)$ (9) instead of the usual $\sqrt{-\gamma}$, built entirely in terms of auxiliary world-sheet scalar fields $\varphi^i$ independent of the Riemannian metric $\gamma_{ab}$.

• There is no "cosmological-constant" term $((p-1)\sqrt{-\gamma})$ in (8).

• The action (8) is manifestly Weyl-conformal invariant for any $p$; here Weyl-conformal symmetry is given by Weyl rescaling of $\gamma_{ab}$ supplemented with a special diffeomorphism in the target space of auxiliary $\varphi$-fields:

$$\gamma_{ab} \rightarrow \gamma'_{ab} = \rho \gamma_{ab} \quad , \quad \varphi^i \rightarrow \varphi'^i(\varphi) \quad \text{with} \quad \det\left| \frac{\partial \varphi'^i}{\partial \varphi^i} \right| = \rho \, .$$

• There are no ad hoc dimensionfull constants in (8); the variable brane tension $\chi \equiv \frac{\Phi(\varphi)}{\sqrt{-\gamma}}$ is Weyl-conformal gauge dependent: $\chi \rightarrow \rho^{\frac{1}{p}(1-p)}\chi$.

• The action (8) contains an additional world-volume gauge field $A_a$ in a “square-root” Maxwell (Yang-Mills) Lagrangian\(^1\); the latter can be straightforwardly generalized to the non-Abelian case: $\sqrt{-\text{Tr}(F_{ab}(A)F_{cd}(A))} \, \gamma^{ac} \gamma^{bd}$ with $F_{ab}(A) = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$.

• The presence of the world-volume gauge field $A_a$ allows for natural (linear) optional couplings both to external world-volume as well as to space-time "color" charge currents in a Weyl-conformally invariant way (see Eq.(53) below).

• The action (8) describes intrinsically light-like $p$-branes for any odd $(p+1)$ (see Eq.(17) below).

The action (8) yields the following equations of motion w.r.t. auxiliary scalars $\varphi^i$:

$$\frac{1}{2} \gamma^{cd} (\partial_c X \partial_d X) - \sqrt{FF\gamma} = M \left( = \text{const} \right) ,$$

with the short-hand notations:

$$(\partial_a X \partial_b X) \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \quad , \quad \sqrt{FF\gamma} \equiv \sqrt{F_{ab} F_{cd} \gamma^{ac} \gamma^{bd}} .$$

The equations of motion w.r.t. $\gamma^{ab}$ are:

$$\frac{1}{2} (\partial_a X \partial_b X) + \frac{F_{ac} \gamma^{cd} F_{db}}{\sqrt{FF\gamma}} = 0 \, ,$$

which upon taking the trace imply $M = 0$ in Eq.(11).

Further we obtain the following equations of motion w.r.t. world-volume gauge field $A_a$ and w.r.t. brane embedding coordinates $X^\mu$, respectively:

$$\partial_b \left( \frac{F_{cd} \gamma^{ac} \gamma^{bd}}{\sqrt{FF\gamma}} \Phi(\varphi) \right) = 0 \, ,$$

$$\partial_a \left( \Phi(\varphi) \gamma^{ab} \partial_b X^\mu + \Phi(\varphi) \gamma^{ab} \partial_a X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu\lambda} \right) = 0 \, ,$$

where $\Gamma^\mu_{\nu\lambda}$ is the same as in (4).

### 2.3 Light-Like Branes

Now, let us consider the $\gamma^{ab}$-equations of motion (13). Since $F_{ab}$ is an anti-symmetric $(p+1) \times (p+1)$ matrix, it is therefore not invertible in any odd $(p+1)$, i.e. $F_{ab}$ has at least one zero-eigenvalue vector $V^a$ ($F_{ab} V^b = 0$). Thus, for any odd $(p+1)$ the induced metric:

$$g_{ab} \equiv (\partial_a X \partial_b X) \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

\(^1\) "Square-root" Maxwell (Yang-Mills) action in $D = 4$ was originally introduced in [7] and later formulated in dual variables and generalized to "square-root" actions of higher-rank antisymmetric tensor gauge fields in $D \geq 4$ in refs.[8]; see also ref.[9].
on the world-volume of the Weyl-invariant brane (8) is *singular as opposed* to the ordinary Nambu-Goto brane where the induced metric is proportional to the intrinsic Riemannian world-volume metric (cf. Eq.(5)). In other words:

\[
(\partial_a X \partial_b X) V^b = 0 \quad \text{and} \quad (\partial_a X \partial_b X = 0, \quad (\partial_X X) \partial_b X = 0,
\]

where \(\partial_X \equiv V^\alpha \partial_{a}\) and \(\partial_{\perp}\) are derivates along the tangent vectors in the complement of the tangent vector field \(V^\alpha\).

The constraints (17) imply the following important conclusion: every point on the (fixed-time) world-surface of the Weyl-invariant \(p\)-brane (8) (for odd \((p+1)\)) moves in orthogonal direction w.r.t. itself with the speed of light in a time-evolution along the zero-eigenvalue vector-field \(V^\alpha\) of the world-volume electromagnetic field-strength \(F_{ab}\). Therefore, we will call (8) (for odd \((p+1)\)) by the acronym \textit{WILL-brane} (Weyl-Invariant Light-Like-brane) model.

### 2.4 Dual Formulation of WILL-Branes

The \(A_a\)-equations of motion (14) can be solved in terms of \((p-2)\)-form gauge potentials \(A_{a_1...a_{p-2}}\) dual w.r.t. \(A_a\). The respective field-strengths are related as follows:

\[
F_{ab}(A) = -\frac{1}{\chi} \frac{\sqrt{-\gamma}}{2(p-1)} \gamma^{ab} \cdots \gamma^{c_{p-1}d_{p-1}} F_{c_1...d_{p-1}}(\Lambda) \chi^{cd} (\partial_c X \partial_d X),
\]

where:

\[
F_{a_1...a_{p-1}}(\Lambda) = (p-1)\partial_{[a_1} A_{a_2...a_{p-1}]}
\]

is the \((p-1)\)-form dual field-strength, and \(\chi \equiv \frac{\sqrt{\gamma}}{\sqrt{-\gamma}}\) is the variable brane tension, which we find to be explicitly expressed in terms of the dual field-strength:

\[
\chi^2 \equiv \chi^2(\gamma, \Lambda) = \frac{2}{(p-1)^2} \gamma^{a_1 b_1} \cdots \gamma^{a_{p-1} b_{p-1}} F_{a_1...a_{p-1}}(\Lambda) F_{b_1...b_{p-1}}(\Lambda).
\]

Now, the Bianchi identities for \(A_a\) turn into dynamical equations of motion for the dual \((p-2)\)-form gauge potentials \(A_{a_1...a_{p-2}}\):

\[
\partial_a \left( \frac{\sqrt{-\gamma}}{\chi(\gamma, \Lambda)} \gamma^{ab} \cdots \gamma^{a_{p-1} b_{p-1}} F_{a_1...a_{p-2}}(\Lambda) \chi^{cd} (\partial_c X \partial_d X) \right) = 0
\]

All equations of motion (13),(15) and (21) can be equivalently derived from the following *dual WILL-brane* action:

\[
S_{\text{dual}} = -\frac{1}{2} \int d^{p+1} \sigma \chi(\gamma, \Lambda) \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}
\]

with \(\chi(\gamma, \Lambda)\) given in (20) above.

### 3 The WILL-Membrane

The \textit{WILL-membrane} dual action (particular case of (22) for \(p=2\)) reads:

\[
S_{\text{dual}} = -\frac{1}{2} \int d^3 \sigma \chi(\gamma, u) \sqrt{-\gamma} \gamma^{ab} (\partial_a X \partial_b X),
\]

\[
\chi(\gamma, u) \equiv \sqrt{-2} \gamma^{cd} \partial_c u \partial_d u,
\]

where \(u\) is the dual “gauge” potential w.r.t. \(A_a\):

\[
F_{ab}(A) = -\frac{1}{2\chi(\gamma, u)} \sqrt{-\gamma} \varepsilon_{abc} \gamma^{cd} \partial_d u \gamma^{ef} (\partial_e X \partial_f X)\]

\(S_{\text{dual}}\) is manifestly Weyl-invariant (under \(\gamma_{ab} \rightarrow \rho \gamma_{ab}\)).
The equations of motion w.r.t. $\gamma^{ab}$, $u$ (or $A_a$), and $X^\mu$ read accordingly:

$$
(\partial_a X \partial_b X) + \frac{1}{2} \gamma^{cd} (\partial_c X \partial_d X) \left( \frac{\partial_a u \partial_b u}{\gamma^{ef} \partial_e u \partial_f u} - \gamma_{ab} \right) = 0 ,
$$

(26)

$$
\partial_a \left( \sqrt{-\gamma} \gamma^{ab} \partial_b u \right) \gamma^{cd} (\partial_c X \partial_d X) = 0 ,
$$

(27)

$$
\partial_a \left( \chi(\gamma, u) \sqrt{-\gamma} \gamma^{ab} \partial_b X^\mu \right) + \chi(\gamma, u) \sqrt{-\gamma} \gamma^{ab} \partial_a X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu\lambda} = 0 .
$$

(28)

The first equation above shows that the induced metric $g_{ab} \equiv (\partial_a X \partial_b X)$ has zero-mode eigenvector $V^a = \gamma^{ab} \partial_b u$.

The invariance under world-volume reparametrizations allows to introduce the following standard (synchronous) gauge-fixing conditions:

$$
\gamma^{0i} = 0 \ (i = 1, 2) , \quad \gamma^{00} = -1 .
$$

(29)

In spite of the high non-linearity of Eq.(27) for the dual “gauge potential” $u$, we can easily find solutions by using the following ansatz:

$$
u(\tau, \sigma^1, \sigma^2) = \frac{T_0}{\sqrt{2}} \tau ,
$$

(30)

where $T_0$ is an arbitrary integration constant with the dimension of membrane tension. In particular:

$$
\chi \equiv \sqrt{-2} \gamma^{ab} \partial_a u \partial_b u = T_0
$$

(31)

The ansatz (30) means that we take $\tau \equiv \sigma^0$ to be evolution parameter along the zero-eigenvalue vector-field of the induced metric on the brane ($V^a = \gamma^{ab} \partial_b u = \text{const}(1, 0, 0)$).

With the gauge choice for $\gamma_{ab}$ (29) the equations of motion w.r.t. $\gamma^{ab}$ (26) (which are in fact constraints) become (recall $(\partial_a X \partial_b X) \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$):

$$
(\partial_0 X \partial_0 X) = 0 , \quad (\partial_0 X \partial_i X) = 0 ,
$$

(32)

$$
(\partial_i X \partial_j X) - \frac{1}{2} \gamma^{ij} \gamma^{kl} (\partial_k X \partial_l X) = 0 ,
$$

(33)

Note that Eqs.(33) look exactly like the classical (Virasoro) constraints for an Euclidean string theory with world-sheet parameters $(\sigma^1, \sigma^2)$.

The gauge choice for (29) together with the ansatz (30), as well as taking into account (32), bring the the equations of motion w.r.t. $u$ to the form:

$$
\partial_0 \left( \sqrt{\gamma^{(2)}} \gamma^{kl} (\partial_k X \partial_l X) \right) = 0 ,
$$

(34)

where $\gamma^{(2)} = \det ||\gamma_{ij}|| \ (i, j, k, l = 1, 2)$. Eq.(34) is the only remnant from the original $A_a$-equations of motion (14).

Accordingly, the $X^\mu$-equations of motion now read:

$$
\Box^{(3)} X^\mu + \left( -\partial_0 X^\nu \partial_0 X^\lambda + \gamma^{kl} \partial_k X^\nu \partial_l X^\lambda \right) \Gamma^\mu_{\nu\lambda} = 0 ,
$$

(35)

where:

$$
\Box^{(3)} \equiv - \frac{1}{\sqrt{\gamma^{(2)}}} \partial_0 \left( \sqrt{\gamma^{(2)}} \partial_0 \right) + \frac{1}{\sqrt{\gamma^{(2)}}} \partial_i \left( \sqrt{\gamma^{(2)}} \gamma^{ij} \partial_j \right) .
$$

(36)

We recall that everywhere in Eqs.(32)–(36) the space-like part of the internal membrane metric $\gamma_{ij}$ is of the form (42).
4 WILL-Membrane Solutions in Non-Trivial Gravitational Backgrounds

4.1 Example: WILL-Membrane in Spherically-Symmetric Static Backgrounds

Let us consider a general spherically-symmetric static gravitational background in \( D = 4 \) embedding space-time:

\[
(ds)^2 = -A(r)(dt)^2 + B(r)(dr)^2 + r^2[(d\theta)^2 + \sin^2(\theta) (d\phi)^2] .
\]

(37)

Specifically we have:

\[
A(r) = B^{-1}(r) = 1 - \frac{2GM}{r}
\]

(38)

for Schwarzschild black hole,

\[
A(r) = B^{-1}(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}
\]

(39)

for Reissner-Nordström black hole,

\[
A(r) = B^{-1}(r) = 1 - \kappa r^2
\]

(40)

for (anti-) de Sitter space, etc..

To find solutions of the equations of motion (and constraints) (32)–(36) we will use the following ansatz:

\[
X_0 \equiv t = \tau , \quad X_1 \equiv r = r(\tau, \sigma^1, \sigma^2) , \quad X^2 \equiv \theta = \sigma^1 , \quad X^3 \equiv \phi = \sigma^2 ;
\]

(41)

\[
\|\gamma_{ij}\| = a(\tau) \begin{pmatrix}
1 & 0 \\
0 & \sin^2(\sigma^1)
\end{pmatrix}
\]

(42)

In other words, we assume that the underlying WILL-membrane has spherical topology of its fixed-time world-surface.

From Eqs.(32) taking into account (37) we obtain:

\[
\frac{\partial}{\partial \tau} r = \pm A(r) , \quad \frac{\partial}{\partial \sigma^i} r = 0 .
\]

(43)

From Eq.(34) we get \( \frac{\partial}{\partial \tau} r = 0 \) which upon combining with (43) gives:

\[
r = r_0 \equiv \text{const} , \quad \text{where} \quad A(r_0) = 0 .
\]

(44)

The \( X^0 \)-equation of motion (Eq.(35) for \( \mu = 0 \)) implies for the intrinsic WILL-membrane metric:

\[
\|\gamma_{ij}\| = c_0 e^\mp \tau / r_0 \begin{pmatrix}
1 & 0 \\
0 & \sin^2(\sigma^1)
\end{pmatrix}
\]

(45)

where \( c_0 \) is an arbitrary integration constant.

From (44) we conclude that the WILL-membrane with spherical topology (and with exponentially blowing-up/deflating radius w.r.t. internal metric) “sits” on (materializes) the event horizon of the pertinent black hole in \( D = 4 \) embedding space-time.

4.2 Example: WILL-membrane in Product-Space Backgrounds

Here we consider WILL-membrane moving in a general product-space \( D = (d + 2) \)-dimensional gravitational background \( \mathcal{M}^d \times \Sigma^2 \) with coordinates \( (x^\mu, y^m) \) \( (\mu = 0, 1, \ldots, d-1, m = 1, 2) \) and Riemannian metric \( (ds)^2 = f(y)g_{\mu\nu}(x)dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n \).

We assume that the WILL-brane wraps around the “internal” space \( \Sigma^2 \) and use the following ansatz (recall \( \tau \equiv \sigma^0 \)):

\[
X^\mu = X^\mu(\tau) , \quad Y^m = \sigma^m , \quad \gamma_{mn} = a(\tau) g_{mn}(\sigma^1, \sigma^2)
\]

(46)

Then the equations of motion and constraints (32)–(36) reduce to:

\[
\partial_\tau X^\mu \partial_\tau X^\nu g_{\mu\nu}(X) = 0 , \quad \frac{1}{a(\tau)} \partial_\tau \left( a(\tau) \partial_\tau X^\mu \right) + \partial_\tau X^\nu \partial_\tau X^\lambda \Gamma^\mu_{\nu\lambda} = 0
\]

(47)
where \(a(\tau)\) is the conformal factor of the space-like part of the internal membrane metric (last Eq.(46)).

Eqs.(47) are of the same form as the equations of motion for a massless point-particle with a world-line “einbein” \(e = a^{-1}\) moving in \(M^d\). In other words, the simple solution above describes a membrane living in the extra “internal” dimensions and moving as a whole with the speed of light in “ordinary” space-time.

Notice that although the WILL-brane is wrapping the extra dimensions in a topologically non-trivial way (cf. second Eq.(46)), its modes remain massless from the projected \(d\)-dimensional space-time point of view. This is a highly non-trivial result since we have here particles (membrane modes), which acquire in this way non-zero quantum numbers, while at the same time remaining massless. In contrast, one should recall that in ordinary Kaluza-Klein theory (for a review, see [11]), non-trivial dependence on the extra dimensions is possible for point particles or even standard strings and branes only at a very high energy cost (either by momentum modes or winding modes), which implies a very high mass from the projected \(D = 4\) space-time point of view.

4.3 Example: WILL-Membrane in a PP-Wave Background

As a final non-trivial example let us consider WILL-membrane dynamics in external plane-polarized gravitational wave (pp-wave) background:

\[
(ds)^2 = -dx^+ dx^- - F(x^+, x^I)(dx^+)^2 + dx^I dx^I, \quad (48)
\]

and employ in (32)–(36) the following natural ansatz for \(X^\mu\) (here \(a^0 = \tau; \ I = 1, \ldots, D - 2\)):

\[
X^- = \tau, \quad X^+ = X^+(\tau, \sigma^1, \sigma^2), \quad X^I = X^I(\sigma^1, \sigma^2). \quad (49)
\]

The non-zero affine connection symbols for the pp-wave metric (48) are: \(\Gamma_+ = \partial_+ F, \Gamma_\pm = \partial_I F, \Gamma_{++} = \frac{1}{2} \partial^F F\).

It is straightforward to show that the solution does not depend on the form of the pp-wave front \(F(x^+, x^I)\) and reads:

\[
X^+ = X_0^+ = \text{const}, \quad \gamma_{ij} = \tau - \text{independent}; \quad (50)
\]

\[
\partial_i X^I \partial_j X^J - \frac{1}{2} \gamma_{ij} \gamma^{kl} \partial_k X^I \partial_l X^J = 0, \quad \partial_i \left(\sqrt{\gamma^{(2)}} \gamma^{ij} \partial_j X^I\right) = 0 \quad (51)
\]

where the latter equations describe a string embedded in the transverse \((D - 2)\)-dimensional flat Euclidean space.

5 WILL-Membrane as a Source for Gravity and Electromagnetism

In this section we shall consider the Einstein-Maxwell system coupled to an electrically charged WILL-membrane, i.e., we shall take into account the back-reaction of the WILL-membrane serving as a material and electrically charged source for gravity and electromagnetism. The relevant action reads:

\[
S = \int d^4 x \sqrt{-G} \left[ \frac{R}{16\pi G_N} - \frac{1}{4} F_{\mu\nu}(A) F_{\kappa\lambda}(A) G^{\mu\kappa} G^{\nu\lambda} \right] + S_{\text{WILL-brane}}, \quad (52)
\]

where \(F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu\), and:

\[
S_{\text{WILL-brane}} = - \int d^3 \sigma \Phi(\varphi) \left[ \frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \sqrt{F_{ab} F_{cd} \gamma^{ac} \gamma^{bd}} \right] - q \int d^3 \sigma \varepsilon^{abc} A_\mu \partial_\mu X^\nu F_{bc}. \quad (53)
\]

Note the appearance of a natural Weyl-conformal invariant coupling of the WILL-brane to the external space-time electromagnetic field \(A_\mu\) – the last Chern-Simmons-like term in (53). The latter is a special case of a class of Chern-Simmons-like couplings of extended objects to external electromagnetic fields proposed in ref.[10].

The Einstein-Maxwell equations of motion are of the standard form:

\[
R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = 8\pi G_N \left( T^{(EM)}_{\mu\nu} + T^{(brane)}_{\mu\nu} \right), \quad (54)
\]
\[\partial_\nu \left( \sqrt{-G} G^{\mu \nu} G^{\rho \lambda} F_{\rho \lambda} \right) + j^\mu = 0, \]  
where:
\[T_{\mu \nu}^{(EM)} \equiv F_{\mu \rho} F_{\nu \lambda} G^{\rho \lambda} - G_{\mu \nu} \frac{1}{4} F_{\rho \sigma} F^{\rho \sigma} G^{\mu \nu}, \]
\[T_{\mu \nu}^{(brane)} \equiv -G_{\mu \nu} G^{\rho \lambda} \int d^3 \sigma \frac{\delta^{(4)}(x - X(\sigma))}{\sqrt{-G}} \Phi(\varphi) \gamma^{ab} \partial_\alpha X^\alpha \partial_b X^\lambda, \]
\[j^\mu \equiv q \int d^3 \sigma \delta^{(4)}(x - X(\sigma)) \varepsilon^{abc} F_{be} \partial_a X^\mu. \]

For the WILL-membrane subsystem we can use instead of the action (53) its dual one (similar to the simpler case Eq.(8) versus Eq.(23)):
\[S_{\text{Will-brane}}^{\text{dual}} = -\frac{1}{2} \int d^3 \sigma \chi(\gamma, u, A) \sqrt{-\gamma} \gamma^{ab} \left( \partial_\alpha X^\alpha \partial_b X \right), \]
where the variable brane tension \( \chi \equiv \frac{\Phi(\varphi)}{\sqrt{\gamma}} \) is given by:
\[\chi(\gamma, u, A) \equiv \sqrt{-2 \gamma^{cd} (\partial_\alpha u - q A_b) (\partial_\alpha u - q A_d)} \quad A_d \equiv A_\alpha \partial_\alpha X^\mu. \]
Here \( u \) is the dual “gauge” potential w.r.t. \( A_\alpha \) and the corresponding field-strength and dual field-strength are related as (cf. Eq.(25)) :
\[F_{ab}(A) = -\frac{1}{2} \frac{1}{\chi(\gamma, u, A)} \sqrt{-\gamma} \varepsilon^{abc} \gamma^{cd} \left( \partial_\alpha u - q A_b \right) \gamma^{ef} \left( \partial_\alpha X^e \partial_\alpha X^f \right). \]

The corresponding equations of motion w.r.t. \( \gamma^{ab}, u \) (or \( A_\alpha \)), and \( X^\mu \) read accordingly:
\[(\partial_\alpha X^\beta \partial_\alpha X) + \frac{1}{2} \gamma^{cd} (\partial_\alpha X \partial_\alpha X) \left( \frac{(\partial_\alpha u - q A_b) (\partial_\alpha u - q A_d)}{\gamma^{ef} (\partial_\alpha u - q A_e) (\partial_\alpha u - q A_f)} - \gamma^{cd} \right) = 0; \]
\[\partial_\alpha \left( \sqrt{-\gamma} \gamma^{ab} (\partial_\alpha u - q A_b) \right) \gamma^{cd} (\partial_\alpha X \partial_\alpha X) = 0; \]
\[\partial_\alpha \left( \chi(\gamma, u, A) \sqrt{-\gamma} \gamma^{ab} \partial_\alpha X^\mu \right) + \chi(\gamma, u, A) \sqrt{-\gamma} \gamma^{ab} \partial_\alpha X^\nu \partial_\beta X^\lambda \Gamma^{\mu}_{\nu \lambda} - q \varepsilon^{abc} F_{be} \partial_a X^\nu \left( \partial_\alpha A_\nu - \partial_\alpha A_\lambda \right) G^{\lambda \mu} = 0. \]

Following steps similar to the ones in the previous section we obtain the following self-consistent spherically symmetric stationary solution for the full coupled Einstein-Maxwell-WILL-membrane system (52). For the Einstein subsystem we have a solution:
\[(ds)^2 = -A(r)(dt)^2 + A^{-1}(dr)^2 + r^2[(d\theta)^2 + \sin^2(\theta)(d\phi)^2], \]
consisting of two different black holes with a common event horizon:

- Schwarzschild black hole inside the horizon:
  \[A(r) \equiv A_-(r) = 1 - \frac{2GM_1}{r}, \quad \text{for} \quad r < r_0 \equiv r_{\text{horizon}} = 2GM_1. \]

- Reissner-Norström black hole outside the horizon:
  \[A(r) \equiv A_+(r) = 1 - \frac{2GM_2}{r} + \frac{GQ^2}{r^2}, \quad \text{for} \quad r > r_0 \equiv r_{\text{horizon}}, \]
where \( Q^2 = 8\pi q^2 G_{\text{horizon}} \equiv 128\pi q^2 G M_1^4; \)

For the Maxwell subsystem we have \( A_1 = \ldots = A_{D-1} = 0 \) everywhere and:
• Coulomb field outside horizon:

\[ A_0 = \sqrt{2} q r_{\text{horizon}}^2 r, \quad \text{for } r \geq r_0 \equiv r_{\text{horizon}}. \quad (68) \]

• No electric field inside horizon:

\[ A_0 = \sqrt{2} q r_{\text{horizon}} = \text{const}, \quad \text{for } r \leq r_0 \equiv r_{\text{horizon}}. \quad (69) \]

For the WILL-membrane subsystem the corresponding solution reads:

\[ X^0 \equiv t = \tau, \quad \theta = \sigma^1, \quad \phi = \sigma^2, \quad r(\tau, \sigma^1, \sigma^2) = r_{\text{horizon}} = \text{const}, \quad (70) \]

where \( A_{\pm}(r_{\text{horizon}}) = 0 \), i.e., the WILL-membrane “sits” on (materializes) the common event horizon of the pertinent black holes. Furthermore, the presence of the WILL-membrane entails an important matching condition for the metric components along its surface:

\[ \frac{\partial}{\partial r} A_+ \bigg|_{r=r_{\text{horizon}}} - \frac{\partial}{\partial r} A_- \bigg|_{r=r_{\text{horizon}}} = -16\pi G \chi, \quad (71) \]

which yields the following relations between the parameters of the black holes and the WILL-membrane (\( q \) being its surface charge density):

\[ M_2 = M_1 + 32\pi q^2 G^3 M_1^3 \quad (72) \]

and for the brane tension \( \chi \):

\[ \chi \equiv T_0 - 2q^2 r_{\text{horizon}} = q^2 GM_1, \quad \text{i.e. } T_0 = 5q^2 GM_1. \quad (73) \]

Let us stress that the present WILL-brane models provide a systematic description of light-like branes from first principles starting with concise Weyl-conformal invariant actions (8), (52)–(53). As a consequence, these actions also yield additional information impossible to obtain within the phenomenological approach to light-like thin shell dynamics [12] (i.e., where the membranes are introduced \textit{ad hoc}), such as the requirement that the light-like brane must sit on the (common) event horizon(s) of the pertinent black hole(s).

6 Conclusions and Outlook

In the present work we have discussed a novel class of Weyl-invariant \( p \)-brane theories whose dynamics significantly differs from ordinary Nambu-Goto \( p \)-brane dynamics. The principal ingredients of our construction are:

• Alternative non-Riemannian integration measure (volume-form) (9) on the \( p \)-brane world-volume independent of the intrinsic Riemannian metric;

• Acceptable dynamics in the novel class of brane models (Eqs.(8),(53)) naturally requires the introduction of additional world-volume gauge fields.

• By employing square-root Yang-Mills actions for the pertinent world-volume gauge fields one achieves manifest \textit{Weyl-conformal symmetry} in the new class of \( p \)-brane theories \textit{for any} \( p \).

• The brane tension is \textit{not} a constant dimensionful scale given \textit{ad hoc}, but rather it appears as a \textit{composite} world-volume scalar field (Eqs.(20),(24),(60)) transforming non-trivially under Weyl-conformal transformations.

• The novel class of Weyl-invariant \( p \)-brane theories describes intrinsically \textit{light-like} \( p \)-branes for \textit{any} even \( p \) (WILL-\textit{branes}).

\[ ^{2}\text{The matching condition (71) corresponds to the statically soldering conditions in the phenomenological theory of light-like thin shell dynamics in general relativity [12].} \]
• When put in a gravitational black hole background, the WILL-membrane \((p = 2)\) sits on (“materializes”) the event horizon.

• When moving in background product-spaces (“Kaluza-Klein” context) the WILL-membrane describes massless modes, even though the membrane is wrapping the extra dimensions and therefore acquiring non-trivial Kaluza-Klein charges.

• The coupled Einstein-Maxwell-WILL-membrane system \((52)\) possesses self-consistent solution where the WILL-membrane serves as a material and electrically charged source for gravity and electromagnetism, and it “sits” on (materializes) the common event horizon for a Schwarzschild (in the interior) and Reissner-Nordström (in the exterior) black holes. Thus our model \((52)\) provides an explicit dynamical realization of the so called “membrane paradigm” in the physics of black holes \([13]\).

• The WILL-branes could be good representations for the string-like objects introduced by ’t Hooft in ref.[14] to describe gravitational interactions associated with black hole formation and evaporation, since as shown above the WILL-branes locate themselves automatically in the horizons and, therefore, they could represent degrees of freedom associated particularly with horizons.

The novel class of Weyl-conformal invariant \(p\)-branes discussed above suggests various physically interesting directions for further study: quantization (Weyl-conformal anomaly and critical dimensions); supersymmetric generalization; possible relevance for the open string dynamics (similar to the role played by Dirichlet- \((Dp)\)-branes); WILL-brane dynamics in more complicated gravitational black hole backgrounds (e.g., Kerr-Newman).

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