ANNULUS MAXIMAL AVERAGES ON VARIABLE HYPERPLANES

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Abstract. By giving a thin width of $0 < \delta \ll 1$ to both a unit circle and a unit line, we set an annulus and a tube on the Euclidean plane $\mathbb{R}^2$. Consider the maximal means $M_\delta$ over dilations of the annulus, and $N_\delta$ over rotations of the tube. It is known that their operator norms on $L^2(\mathbb{R}^2)$ are $O((\log 1/\delta)^{1/2})$. In this paper, we study the maximal averages $M^A_\delta$ and $N^A_\delta$ over those annuli and tubes now imbedded on the variable hyperplanes $(x, x_3) + \{(y, \langle A(x), y \rangle) : y \in \mathbb{R}^2 \} \subset \mathbb{R}^3$ where $A$ is a $2 \times 2$ matrix. The model hyperplane is the horizontal plane of the Heisenberg group when $A$ is the skew-symmetric matrix denoted by $E$. It turns out that a rank of matrix $EA + (EA)^T$ or $A + A^T$ determines $\|M^A_\delta\|_{op}$ or $\|N^A_\delta\|_{op}$ respectively. In the higher dimension, the corresponding spherical maximal means is bounded in $L^p$ if $A$ has only complex eigenvalues.

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1. Introduction

Let $\delta > 0$. Set an annulus $S_\delta = \{y \in \mathbb{R}^2 : 1 - \delta/2 \leq |y| \leq 1 + \delta/2\}$ with its radius and width $(1, \delta)$, and a tube $T_\delta = \{y \in \mathbb{R}^2 : |y_1| < 1/2, |y_2| < \delta/2\}$ with the dimensions $1 \times \delta$. Let $R_\theta$ be a $2 \times 2$ matrix of the rotation by angle $\theta \in [0, 2\pi]$. For a locally integrable function $f \in L^1_{loc}(\mathbb{R}^2)$,
consider the annulus maximal average of \( f \) over the dilations \( tS_\delta \) of the annulus \( S_\delta \),

\[
M_\delta f(x) = \sup_{t > 0} \frac{1}{|S_\delta|} \int_{y \in S_\delta} |f(x - ty)|dy
\]

and the tube (Nikodym) maximal average of \( f \) over the rotations \( R_\theta T_\delta \) of the tube \( T_\delta \),

\[
N_{\delta} f(x) = \sup_{\theta \in [0,2\pi]} \frac{1}{|T_\delta|} \int_{y \in T_\delta} |f(x - R_\theta y)|dy.
\]

The operator norms of \( N_\delta \) and \( M_\delta \) have the same growth rate in \( \delta \) as

\[
\|N_\delta\|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} = O(|\log(1/\delta)|^{1/2}) \quad \text{and} \quad \|M_\delta\|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} = O(|\log(1/\delta)|^{1/2}).
\]

The former was obtained by Córdoba [5] and the latter by Bourgain [11, 12] and Schlag [22].

1.1. Maximal Average along Variable Hyperplanes in \( \mathbb{R}^3 \). In this paper, we place the above annuli and tubes to be imbedded on the hyperplanes \( \pi_A(x,x_3) \) in \( \mathbb{R}^3 \)

\[
\pi_A(x,x_3) := \{(x,y) : y \in \mathbb{R}^2\} \quad \text{where} \quad A \text{ is a } 2 \times 2 \text{ matrix}.
\]

Let \( (x,x_3) \in \mathbb{R}^2 \times \mathbb{R} \). Define the annulus maximal function of \( f \in L^1_{loc}(\mathbb{R}^3) \) as

\[
M^A_\delta f(x,x_3) = \sup_{t > 0} \frac{1}{|S_\delta|} \int_{y \in S_\delta} |f(x - ty, x_3 - A(x)ty)|dy
\]

and the tube (Nikodym) maximal function of \( f \in L^1_{loc}(\mathbb{R}^3) \) as

\[
N^A_\delta f(x,x_3) = \sup_{\theta \in [0,2\pi]} \frac{1}{|T_\delta|} \int_{y \in T_\delta} |f(x - R_\theta y, x_3 - A(x)R_\theta y)|dy.
\]

Our main purpose is to classify the operator norms of \( M^A_\delta \) on \( L^2(\mathbb{R}^3) \) according to \( 2 \times 2 \) matrices \( A \), which leads us to compare those of \( M^A_\delta \) and \( N^A_\delta \).

Definition 1.1 (Skew–Symmetric Matrix). Let \( M_{d \times d}(\mathbb{R}) \) be the set of \( d \times d \) real matrices. In \( M_{2 \times 2}(\mathbb{R}) \), we select the following three matrices:

\[
E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I_c = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad c \neq 0.
\]

The skew–symmetric matrix is usually denoted by \( J \). However, we use \( E \) rather than \( -J \), and write as \( E(x) \) the rotation of \( x \in \mathbb{R}^2 \) by \( \pi/2 \) counterclockwise.

Definition 1.2 (Symmetric and Skew–Symmetric Ranks). Let \( A \in M_{2 \times 2}(\mathbb{R}) \). Define the symmetric rank and the skew–symmetric rank by

\[
\text{rank}_{\text{sym}}(A) = \text{rank} \ (A + A^T) \quad \text{and} \quad \text{rank}_{\text{skw}}(A) = \text{rank} \ (EA + (EA)^T).
\]

Roughly speaking, \( \text{rank}_{\text{skw}}(A) \) measures the extent to which \( A \) is close to the skew–symmetric matrix \( E \). Indeed,

\[
\text{rank}_{\text{skw}}(E) = 2, \quad \text{rank}_{\text{skw}}(I_c) = 1 \quad \text{and} \quad \text{rank}_{\text{skw}}(I) = 0.
\]
Moreover, we shall see that $\operatorname{rank}_{\text{skw}}(A)$ measures a non-overlapping property of the eigenvalues of $A$ and the moderateness of their eigenspace dimensions in Proposition 2.1.

Before stating the main result, we introduce a few notations. Given two scalar expressions $A, B$, we write $A \lesssim B$ if $A \leq CB$ for a constant $C > 0$ independent of $A, B$. The notation $A \approx B$ indicates $A \lesssim B$ and $B \lesssim A$. In particular, we write $\| \cdot \| \approx \delta^{-c}$ if $\delta^{-c} \lesssim \| \cdot \| \lesssim \delta^{-c}$ for an arbitrary small $\epsilon > 0$.

**Main Theorem 1** (Annulus Maximal Function). For $2 \times 2$ nonzero matrices $A$,

- If $\operatorname{rank}_{\text{skw}}(A) = 2$, then $\|M^A_\delta\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \approx_{\epsilon} 1$.
- If $\operatorname{rank}_{\text{skw}}(A) = 1$, then $\|M^A_\delta\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \approx_{\epsilon} \begin{cases} \delta^{-1/6} & \text{if } \operatorname{rank}(A) = 2 \\ \delta^{-1/4} & \text{if } \operatorname{rank}(A) = 1, \end{cases}$
- If $\operatorname{rank}_{\text{skw}}(A) = 0$, then $\|M^A_\delta\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \approx_{\epsilon} \delta^{-1/2}$.

If $A = 0$, then $M^A_\delta$ is reduced to the Euclidean case $\mathcal{M}^0_\delta$. For the case of the Heisenberg group given by $A = E$, we obtain the $L^p$ boundedness of $\mathcal{M}^A_\delta$ for $p > 2$ in [13].

Let $S^1$ be a unit circle. We next consider the average operator $f \to \mathcal{A}_{S^1(A)}f$ given by

$$\mathcal{A}_{S^1(A)}f(x, x_3, t) = \int_{y \in S^1} f(x - ty, x_3 - \langle A(x), ty \rangle) \, d\sigma(y).$$

**Corollary 1.1.** Suppose that $A \in M_{2 \times 2}(\mathbb{R})$ is invertible. We then have the following $L^2$-Sobolev inequalities:

$$\|\mathcal{A}_{S^1(A)}f\|_{L^2(\mathbb{R}^2 \times [0, 1])} \lesssim \|f\|_{L^{2+\alpha}(\mathbb{R}^{2+1})} \begin{cases} \text{for } \alpha > 0 & \text{if } \operatorname{rank}_{\text{skw}}(A) = 2 \\ \text{for } \alpha > 1/6 & \text{if } \operatorname{rank}_{\text{skw}}(A) = 1 \\ \text{for } \alpha > 1/2 & \text{if } \operatorname{rank}_{\text{skw}}(A) = 0. \end{cases}$$

### 1.2. Comparison with Nikodym Maximal functions

The result of Main Theorem 1 states that $\mathcal{M}^A_\delta$ has the best bound $O(\delta^{-c})$ when $\operatorname{rank}_{\text{skw}}(A) = 2$ whereas $\mathcal{M}^A_\delta$ has the worst bound $\approx \delta^{-1/2}$ when $\operatorname{rank}_{\text{skw}}(A) = 0$ (exactly when $A = cI$). We compare the annulus maximal operator $\mathcal{M}^A_\delta$ with the Nikodym maximal operator $\mathcal{N}^A_\delta$ of [14] from the author’s previous results of [11] [12]. The norm of $\mathcal{N}^A_\delta$ is determined by $\operatorname{rank}_{\text{sym}}(A) = \operatorname{rank}(A + A^T)$ as follows.

**Theorem 1.1** (Nikodym Maximal Functions in [12]). For $2 \times 2$ nonzero matrices $A$,

- If $\operatorname{rank}_{\text{sym}}(A) = 2$, then $\|\mathcal{N}^A_\delta\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \approx_{\epsilon} 1$.
- If $\operatorname{rank}_{\text{sym}}(A) = 1$, then $\|\mathcal{N}^A_\delta\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \approx_{\epsilon} \delta^{-1/6}$.
- If $\operatorname{rank}_{\text{sym}}(A) = 0$, then $\|\mathcal{N}^A_\delta\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \approx_{\epsilon} \delta^{-1/4}$.

If $A = 0$, then $\mathcal{N}^A_\delta$ is reduced to the Euclidean case $\mathcal{N}^0_\delta$ as $\|\mathcal{N}^0_\delta\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} = O((\log 1/\delta)^{1/2})$.

**Remark 1.2.** In [12], the theorem is stated in terms of $D := (a_{12} + a_{21})^2 - 4a_{11}a_{22} = \det(A + A^T)$. 

Theorem 1.1 states that $\mathcal{N}_\delta^A$ has the best bound $O(\delta^{-\epsilon})$ when $\text{rank}_{\text{sym}}(A) = 2$, however $\mathcal{N}_\delta^A$ has the worst bound $\approx \delta - \frac{1}{4}$ when $\text{rank}_{\text{sym}}(A) = 0$ (exactly when $A = cE$). The novelty of this paper is to demonstrate that the skew–symmetric or symmetric rank condition of $A$ plays a strikingly opposite role between annuli and tubes on the variable planes $\pi_A(x, x_3)$ when classifying the operator norms of $\mathcal{M}_\delta^A$ and $\mathcal{N}_\delta^A$.

Notation. We frequently use the smooth cutoff functions:

1. $\psi$ supported in $\{u : |u| \leq 1\} \subset \mathbb{R}^m$ with $\psi(u) \equiv 1$ in $|u| < 1/2$,

2. $\chi$ supported in $\{u : 1/2 \leq |u| \leq 2\} \subset \mathbb{R}^m$ for $m \in \mathbb{Z}^+$ allowing slight line by line changes of $\chi$ and $\psi$. We denote the phase functions by $\Phi, \Psi$ and the integral kernels by $K, L$, which can be different in cases.

1.3. Rotational Curvature and Heisenberg group. Given $A \in M_{d \times d}(\mathbb{R})$ and $(x, x_{d+1}) \in \mathbb{R}^d \times \mathbb{R}$, set the following hyperplanes

\begin{equation}
\pi_A(x, x_{d+1}) = (x, x_{d+1}) - \{(y, \langle A(x), y \rangle) : y \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}
\end{equation}

as in (1.2). Consider the average of $f$ over a ball embedded in the plan $\pi_A(x, x_{d+1})$ given by

$$A_{\pi_A}(f)(x, x_{d+1}) = \int_{\mathbb{R}^d} f(x - y, x_{d+1} - \langle A(x), y \rangle) \psi(y) dy.$$ 

Whenever $A$ is invertible, the smoothing effect of the average from the variable planes $\pi_A(x, x_{d+1})$ is due to $\det(A) \neq 0$ which is the rotational curvature developed by Phong and Stein in 1980s. They used the concept of the rotational curvature for establishing the $L^p$ theory of the singular Radon transforms and generalized Radon transforms [19]. This was preceded by the model case study of the horizontal plane of the Heisenberg group $\mathbb{H}^n$ by Geller and Stein [9]. These effects of the curvature arising from the $x$-side were culminated in the study of the maximal average on the variable hyper-surfaces conducted by Sogge and Stein [23, 24]. Their theory covers the maximal averages associated with the variable surfaces of co-dimension one. More general average operators including a large class of Fourier integral operators and generalized Radon transforms have been studied in the context of canonical relations in [7, 8, 20, 21].

1.4. Spherical Maximal Average along Variable Hyperplanes. To study the maximal average along the surfaces of co-dimension two, we consider the $(d-1)$-dimensional unit sphere $S^{d-1}$ contained in the $d$-dimensional hyperplane $\pi_A(x, x_{d+1}) \subset \mathbb{R}^{d+1}$ as in (1.6). Let $d\sigma$ be the measure on $S^{d-1}$, and set for $(x, x_{d+1}, t) \in \mathbb{R}^{d+1} \times \mathbb{R}_+$,

\begin{equation}
A_{S^{d-1}(A)}(f)(x, x_{d+1}, t) := \int_{y \in S^{d-1}} f(x - ty, x_{d+1} - \langle A(x), ty \rangle) d\sigma(y)
\end{equation}
Heisenberg group $H$ for $n > 1$, Narayanan and Thangavelu \cite{18} proved (1.10) by using the spectral theory of the Heisenberg two sided fold singularities of \cite{7}, which can cover the step two nilpotent groups. In the same by observing that the phase function of the corresponding Fourier integral operators satisfies the (1.10)

$$\Lambda(\mathbb{R}^n)$$

where $[d\sigma]_t \otimes \delta_{2n+1}$ supported on the horizontal plane $\mathbb{R}^{2n} \times \{0\}$ of the Heisenberg group $\mathbb{H}^n$:

$$\mathbb{H}^n \cong \mathbb{R}^{2n+1}$$

In 1997, Nevo and Thangavelu \cite{14} initiated a study on the maximal average of (1.9) for $A = E$ in (1.8) and obtained the maximal and pointwise ergodic theorems for the radial average on the Heisenberg group $\mathbb{H}^n$ with $n \geq 2$. In 2004, Müller and Seeger \cite{16} proved that, for $n \geq 2$,

$$\|M_{S^{2n-1}(E)}f\|_{L^p(\mathbb{H}^n)} \lesssim \|f\|_{L^p(\mathbb{H}^n)} \text{ if and only if } p > \frac{2n-1}{2n}$$

by observing that the phase function of the corresponding Fourier integral operators satisfies the two sided fold singularities of \cite{7}, which can cover the step two nilpotent groups. In the same year, Narayanan and Thangavelu \cite{18} proved (1.10) by using the spectral theory of the Heisenberg group for $n \geq 2$. Recently, Anderson, Cladek, Pramanik and Seeger in \cite{6} replaced the Heisenberg horizontal plane $\mathbb{R}^{2n} \times \{0\}$ by general hyperplanes $\{(y, \Lambda(y)) : y \in \mathbb{R}^{2n}\}$ in $\mathbb{H}^n$ where $\Lambda$ is a linear functional $\Lambda : \mathbb{R}^{2n} \to \mathbb{R}$, and obtain the $L^p$ estimate for the same $p$ as (1.10). In this paper, we treat the general matrix $A \in M_{d \times d}(\mathbb{R})$ and obtain the $L^p(\mathbb{R}^{d+1})$ boundedness of $M_{S^{d-1}(A)}$ of (1.8) for $d \geq 3$.

**Main Theorem 2.** Let $d \geq 3$. Suppose that $A \in M_{d \times d}(\mathbb{R})$ has only complex eigenvalues. Then

$$\|M_{S^{d-1}(A)}f\|_{L^p(\mathbb{R}^{d+1})} \lesssim \|f\|_{L^p(\mathbb{R}^{d+1})} \text{ if and only if } p > \frac{d}{d-1}.$$  

This is a generalization of the case $A = E$ of the Heisenberg group for $d = 2n \geq 4$ in (1.10).

**Remark 1.3** (Homogeneous Group). To each $d \times d$ matrix $A$, we can assign a homogeneous group $G = G_{d+1}(A)$ identified with $\mathbb{R}^d \times \mathbb{R}$ endowed with the group multiplication

$$(x, x_{d+1}) \cdot (y, y_{d+1}) = (x + y, x_{d+1} + y_{d+1} + (A(x), y)).$$

Then $G$ is the group with the inverse element of $(x, x_{d+1})$ given by $(x, x_{d+1})^{-1} = (-x, -x_{d+1} + (A(x), x))$.

We can check that $G$ is abelian if and only if $A^T = A$. The Heisenberg group $\mathbb{H}^n$ is the non-abelian
group $G_{2n+1}(E)$ for the skew-symmetric matrix $E$. Let $g_{A}(y, y_{d+1}) = g(y, y_{d+1} + \langle A(y), y \rangle)$. Then the spherical average (1.7) is expressed as the group convolution

$$A_{S^{d-1}}(f)(x, x_{d+1}, t) = f *_{G_{d+1}(A)} [d\sigma_{t}(\cdot) \otimes \delta_{d+1}(\cdot)]A(x, x_{d+1})$$

where $d\sigma_{t}$ is the spherical measure on $\mathbb{R}^d$ and $\delta_{d+1}$ is a Dirac mass in $\mathbb{R}$ as in (1.9).

**Organization.** In Section 2, we classify the matrices of $M_{2 \times 2}$ according to rank $skw(A)$, and discuss why $AE + (AE)^{T}$ appears in Main Theorem 1. In Sections 3 and 4, we reduce matters to the estimates of the oscillatory integral operators given by

$$T^{\lambda} f(x, t) = \lambda^{d/2} \int_{\mathbb{R}^d} e^{2\pi i \lambda \Phi(x, t, \xi)} \chi(t) \psi(x) \xi \hat{f}(\xi) d\xi$$

for $(x, t) \in \mathbb{R}^d \times \mathbb{R}$ with $\Phi(x, t, \xi) = \langle A(x), \xi \rangle + t|x + \xi|$. In Sections 5 through 7, we establish the estimate of $\|T^{\lambda}\|_{op}$ to show the sufficient part of Main Theorems 1 and 2. In Section 8, we obtain the lower bounds of Main Theorem 1. In Section 9, we prove Corollary 9.1. In Section 10, we briefly display why $A + A^{T}$ arises in the estimate of our Nikodym maximal average, which contrasts with $AE + (AE)^{T}$ in the case of the annulus maximal average.

2. Skew–Symmetric Ranks

2.1. Properties of Skew–Symmetric Ranks.

**Lemma 2.1.** Let $A \in M_{d \times d}(\mathbb{R})$. When defining $\text{rank}_{skw}(A)$ in Definition 1.2, we can switch $AE$ and $EA$:

(2.1) \quad \text{rank}(AE + (AE)^{T}) = \text{rank}(EA + (EA)^{T}).$

Let $Q$ be an orthogonal matrix. Then we have the following invariance properties:

(2.2) \quad \text{rank}_{skw}(A) = \text{rank}_{skw}(A^{T}) = \text{rank}_{skw}(Q^{T}AQ) = \text{rank}_{skw}(A^{-1})$

where the last equality holds if $A$ is invertible.

**Proof of Lemma 2.1** By $EE^{T} = E^{T}E = I$ and $E^{T} = -E$, we have

$$EA + (EA)^{T} = E(AE + (AE)^{T})E^{T} \quad \text{and} \quad EA + (EA)^{T} = -A^{T}E + (A^{T}E)^{T}.$$ 

Then the first formula implies (2.1), and the second implies the first part of (2.2). The second of (2.2) follows from the commutativity as $QE = \pm EQ$. Finally, insert $AE$ into $A$ below

(2.3) \quad (A + A^{T})(A^{-1}) = (A^{T})(A^{-1})^{T} + A^{-1}$

to obtain the last equality of (2.2). \qed

We see that $\text{rank}_{skw}(A)$ is determined by the maximal multiplicity of the eigenvalues of $A$ and its eigenspace dimension.
Proposition 2.1. Let $A \in M_{2 \times 2}(\mathbb{R})$ and let $E(\lambda)$ be the eigenspace of an eigenvalue $\lambda$ of $A$. Denote $Q$ by an orthogonal matrices below.

1-1) $\text{rank}_{\text{skw}}(A) = 2$ if and only if $A$ has two different nonzero eigenvalues $\lambda_i$ with $\dim(E(\lambda_i)) = 1$. For this case, $\text{rank}(A)$ can be 1 or 2.

1-2) $\text{rank}_{\text{skw}}(A) = 1$ and $\text{rank}(A) = 2$ if and only if $A$ is a multiple of $Q^T \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} Q$ for $c \neq 0$ if and only if $A$ has only one eigenvalue $\lambda \neq 0$, whose multiplicity is 2 with $\dim(E(\lambda)) = 1$.

1-3) $\text{rank}_{\text{skw}}(A) = 1$ and $\text{rank}(A) = 1$ if and only if $A$ is a multiple of $Q^T \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} Q$ for $c \neq 0$ if and only if $A$ has only one eigenvalue $\lambda = 0$, whose multiplicity is 2 with $\dim(E(\lambda)) = 1$.

1-4) $\text{rank}_{\text{skw}}(A) = 0$ if and only if $A$ is a multiple of an identity matrix $I$ if and only if $A$ has only one eigenvalue $\lambda \neq 0$, whose multiplicity is 2 with $\dim(E(\lambda)) = 2$.

In (1-2) and (1-3) above, matrices $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$ can be replaced with $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ respectively.

For the proof of Proposition 2.1 for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, observe that

$$EA + (EA)^T = \begin{pmatrix} -2a_{21} & a_{11} - a_{22} \\ a_{11} - a_{22} & 2a_{12} \end{pmatrix}.$$  

Note also that $-\det(EA + (EA)^T) = \text{trace}(A)^2 - 4 \det(A) = (a_{22} + a_{11})^2 - 4(a_{11}a_{22} - a_{12}a_{21})$ is inside the square root of the following expression of the eigenvalue $\lambda$ of $A$:

$$\lambda = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{22} + a_{11})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}$$

being zeros of $\det(A - \lambda I) = 0$.

Proof of (1-1). We have (1-1) directly from (2.4) and (2.5). □

By (2.4) and (2.5), $\det(EA + (EA)^T) = 0$ if and only if $A$ has eigenvalue $\lambda^*$ of multiplicity 2.

Proof of (1-2). The matrix $A$ having the eigenvalue $\lambda^* \neq 0$ of multiplicity 2 has the Jordan form of $A$ as above with $c \neq 0$ because $\text{rank}(A) = 2$. □

Proof of (1-3). The condition $\text{rank}(A) = 1$ with $\lambda^* = 0$ implies the above expression for some $c \neq 0$. □

Proof of (1-4). Note $EA + (EA)^T = 0$ in (2.4) implies that $a_{12} = a_{21} = 0$ and $a_{11} = a_{22}$, namely $A = cI$. □

Proposition 2.1 states that the smaller $\text{rank}_{\text{skw}}(A)$ is, the bigger the multiplicity of eigenvalues and their eigenspace dimensions are.

Lemma 2.2. In proving Main Theorems 1 and 2, we can replace a matrix $A$ by $Q^TAQ$ for any orthogonal matrix $Q$. 

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Proof. To deal with $\mathcal{M}_A^\delta f(x, x_3) = \sup_{t>0} A_\delta f(x, x_3, t)$, we can set $\tilde{f}(x, x_3) = f(Qx, x_3)$. Then by change of variable $y \to Qy$,

$$A_\delta f(Qx, x_3, t) = \frac{1}{|S_1^\delta|} \int_{S_1^\delta} \tilde{f} \left( x - y, x_3 - \langle Q^T A Q x, y \rangle \right) dy.$$

This yields the desired result. □

2.2. Why $EA + (EA)^T$ arises. Let $T^\lambda$ in (1.12) with $d = 2$. For this case $\lambda^{-1} T^\lambda$ is the standard form of an oscillatory integral operator. The first estimate for $\text{Ma}$ in Theorem 1 will be

$$\| \lambda^{-1} T^\lambda \|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2 \times \mathbb{R})} = O(\lambda^{-1} \lambda')$$

under the assumption of $\text{rank}_{\text{skw}}(A) = 2$. Compute the mixed hessian of the phase function $\Phi$,

$$\Phi''(x_1, x_2, t)(\xi_1, \xi_2) = \begin{pmatrix} \Phi_{x_1 \xi_1} & \Phi_{x_2 \xi_1} \\ \Phi_{x_1 \xi_2} & \Phi_{x_2 \xi_2} \\ \Phi_{t\xi_1} & \Phi_{t\xi_2} \end{pmatrix}$$

where $\Phi(x, t, \xi) = \langle A(x), \xi \rangle + t|x + \xi|$ in (1.12).

The first obstacle is existence of a $2 \times 2$ matrix $A$ of rank $\text{rank}_{\text{skw}}(A) = 2$ satisfying

$$\text{rank}(\Phi''(x_1, x_2, t)(\xi_1, \xi_2)) = 1$$

which will be checked in (6.7). For this case, a direct application of the Hörmairder condition or the corank 2 condition with fold singularities does not seem to give the decay rate $\lambda^{-1}$ in (2.6). Instead, our estimate will be based on measuring the singular region:

$$|\{ \xi : |(EA + (EA)^T)\xi, \xi| \lesssim 1/\lambda \}| \lesssim 1/\lambda$$

if and only if $\text{rank}(EA + (EA)^T) = 2$.

In Sections 6 and 7, the main estimate is to compute the integral kernel $\lambda^{-2}[T^\lambda]^* T^\lambda$ of (1.12) given by

$$K(\xi, \eta) = \int e^{2\pi i \lambda \Psi_{\text{circle}}(x, t, \xi, \eta)} \psi(x)^2 \chi(t)^2 \chi(t|\xi + x|) \chi(t|\eta + x|) dx dt.$$

The phase function is

$$\Psi_{\text{circle}}(x, t, \xi, \eta) = \langle A(\xi - \eta), x \rangle + t(|\xi + x| - |\eta + x|).$$

It suffices to deal with

$$\mathcal{E} = \{ |\xi + x| - |\eta + x| \lesssim \lambda'/\lambda \}$$

because $\partial_t \Psi_{\text{circle}}(x, t, \xi, \eta)$ leads to a good estimate on $\mathcal{E}^c$. The gradient of the phase function on $\mathcal{E}$ is

$$\nabla_x \Psi_{\text{circle}}(x, t, \xi, \eta) = A(\xi - \eta) + (\frac{\xi - \eta}{|\xi + x|} + \text{small error}).$$
Let $E$ be the skew-symmetric matrix (rotation by the angle $\pi/2$). To avoid the critical region in applying the integration by parts for (2.8), take the directional derivative along $E \frac{(\xi - \eta)}{||\xi - \eta||}$ in (2.10):

$$\left\langle \nabla_x \Psi_{\text{circle}}(x, t, \xi, \eta), E \frac{(\xi - \eta)}{||\xi - \eta||} \right\rangle = \left\langle A(\xi - \eta), E \frac{(\xi - \eta)}{||\xi - \eta||} \right\rangle = \left\langle -\frac{1}{2} (EA + (EA)^T)(\xi - \eta), \frac{(\xi - \eta)}{||\xi - \eta||} \right\rangle$$

with small errors. The integral of $|K(\xi, \eta)|$ in (2.8) is evaluated by the measuring the region of small oscillation:

$$(2.11) \quad \mathcal{E}_{\text{circle}} = \left\{ (\xi, \eta) : \left| \left\langle (EA + (EA)^T)(\xi - \eta), \frac{(\xi - \eta)}{||\xi - \eta||} \right\rangle \right| \lesssim \lambda' / \lambda \right\}.$$

If $\text{rank}(EA + (EA)^T) = 2$, then use (2.7) and $|\xi - \eta| \lesssim 1$ to have

$$|\mathcal{E}_{\text{circle}}| \lesssim \lambda' / \lambda \quad \text{so that} \quad |\mathcal{E} \cap \mathcal{E}_{\text{circle}}| \lesssim \lambda' / \lambda^2 \quad \text{which is a desired estimate for (2.6).}$$

**Remark 2.1.** To obtain $\|\lambda^{-1} TA\| = O(\lambda^{-1} \lambda^{1/6})$ in (2.6) under $\text{rank}_{\text{ku}}(A) = 1$ and $\text{rank}(A) = 2$, we apply the two sided fold singularities of $\mathcal{E}$ in Section 7, which were used in (16) for a different situation.

### 3. Basic Decompositions

#### 3.1. Symbol Expression

Let $\delta = 2^{-j}$ in (1.3). Replace the sphere measure $d\sigma(y)$ in (1.7) by

$$(3.1) \quad d\sigma_j(y) = (d\sigma * 2^j \psi(2^{j+1} \cdot))(y) \quad \text{for} \quad y \in \mathbb{R}^d \quad \text{where} \quad * \text{ is the Euclidean convolution in} \mathbb{R}^d.$$

The support of $d\sigma_j$ is the annulus $S_{2^{-j}}^d \subset \mathbb{R}^d$ with width $2^{-j}$. For $A \in M_{d \times d}(\mathbb{R})$, set

$$A_j f(x, x_{d+1}, t) = \int_{\mathbb{R}^d} f(x - ty, x_{d+1} - \langle A(x), ty \rangle) d\sigma_j(y)$$

By applying the Fourier inversion formula for $f$ in the Euclidean space $\mathbb{R}^{d+1}$,

$$A_j f(x, x_{d+1}, t) = \int_{\mathbb{R}^{d+1}} e^{2\pi i (x \cdot x_{d+1})} \widehat{d\sigma_j}(t(x + \xi_{d+1} A(x))) \hat{f}(\xi, \xi_{d+1}) d\xi d\xi_{d+1}$$

where $\widehat{d\sigma}_j$ is the Euclidean Fourier transform of $d\sigma_j$ of (3.1) in $\mathbb{R}^d$. Note that

$$\widehat{d\sigma}_j(t(x + \xi_{d+1} A(x))) = \widehat{d\sigma}(t(x + \xi_{d+1} A(x))) \hat{\psi} \left( \frac{t(x + \xi_{d+1} A(x))}{2t} \right)$$

where $\widehat{d\sigma}$ is the Fourier transform of the measure $d\sigma$ on the unit sphere $S^{d-1}$ satisfying

$$\widehat{d\sigma}(\xi) = 2\pi J_{(n-2)/2}(2\pi |\xi|) |\xi|^{-(n-2)/2} = e^{\pm 2\pi i |\xi|} m(\xi).$$

Indeed, by asymptotic expansion of the Bessel function $J_{(n-2)/2}$, we can find a smooth function $m_0$ supported on $|\xi| \leq 2$ and $m_1$ is supported on $|\xi| \geq 1$ splitting

$$(3.2) \quad m(\xi) = m_0(\xi) + m_1(\xi) \quad \text{where} \quad m_1(\xi) = c_d |\xi|^{-(d-1)/2} + O(|\xi|^{-(d+1)/2}).$$
The maximal average along the measure \((3.1)\) corresponding to \(m_0\) is bounded in \(L^p\) for \(p > 1\) due to the non-vanishing rotational curvature \(\det(A)\) in \([19]\). Moreover, because \(|\xi|^{-(d+1)/2}\) behaves better than \(|\xi|^{-(d-1)/2}\) when \(|\xi| \geq 1\), we reduce matters to the first term of \(m_1\) and reset

\[
m(\xi) = |\xi|^{-(d-1)/2}(1 - \psi(\xi)).
\]

With \(m\) in \((3.3)\), we express \(A_jf(x, x_{d+1}, t)\) as the Fourier integral operator

\[
\mathcal{A}_j f(x, x_{d+1}, t) = 2^{-(d-1)/2} \int e^{2\pi i [\langle x, x_{d+1} \rangle \langle \xi, \xi_{d+1} \rangle \pm t \langle \xi + \xi_{d+1}, A(x) \rangle]} m\left(t(\xi + \xi_{d+1}A(x))\right) \hat{\psi}\left(\frac{t(\xi + \xi_{d+1}A(x))}{2^j}\right) \hat{f}(\xi, \xi_{d+1}) d\xi d\xi_{d+1}.
\]

It suffices to replace the amplitude \(\hat{\psi}\left(\frac{t(\xi + \xi_{d+1}A(x))}{2^j}\right)\) above with \(\chi\left(\frac{t(\xi + \xi_{d+1}A(x))}{2^j}\right)\) because the desired norms corresponding to the pieces \(\chi\left(\frac{t(\xi + \xi_{d+1}A(x))}{2^j}\right)\) can be summable over \(2^j < 2^j'\) with the desired bound. Thus we redefine \(A_j\) by

\[
\mathcal{A}_j f(x, x_{d+1}, t) = 2^{-(d-1)/2} \int e^{2\pi i [\langle x, x_{d+1} \rangle \langle \xi, \xi_{d+1} \rangle \pm t \langle \xi + \xi_{d+1}, A(x) \rangle]} \chi\left(\frac{t(\xi + \xi_{d+1}A(x))}{2^j}\right) \hat{f}(\xi, \xi_{d+1}) d\xi d\xi_{d+1}.
\]

In the phase, we take \(t|\xi + \xi_{d+1}A(x)|\) and allow \(t \in \mathbb{R}\). For each fixed \(A\), we set

\[
\mathcal{M}_j f(x, x_{d+1}) = \sup_{t \in \mathbb{R}} \mathcal{A}_j f(x, x_{d+1}, t)
\]

which is a reformulation of \(\mathcal{M}_j^A f(x, x_3)\) in \((1.3)\) with \(\delta = 2^{-j}\).

### 3.2. Frequency Decomposition.

A symbol \(\sigma \in C^\infty(\mathbb{R}^{d+1} \times \mathbb{R} \times \mathbb{R}^{d+1})\) corresponds to the Fourier integral operator \(f \rightarrow \mathcal{T}_\sigma f\) defined by

\[
\mathcal{T}_\sigma f(x, x_{d+1}, t) = \int e^{2\pi i [\langle x, x_{d+1} \rangle \langle \xi, \xi_{d+1} \rangle \pm t \langle \xi + \xi_{d+1}, A(x) \rangle]} \sigma(x, x_{d+1}, t, \xi, \xi_{d+1}) \hat{f}(\xi, \xi_{d+1}) d\xi d\xi_{d+1}
\]

In view of \((3.4)\), we consider the case \(\sigma = m_j\) where \(m_j(x, x_{d+1}, t, \xi, \xi_{d+1}) := \chi\left(\frac{t(\xi + \xi_{d+1}A(x))}{2^j}\right)\).

We decompose \(t \in \mathbb{R}\) in a dyadic manner so that

\[
m_j = \sum_{k \in \mathbb{Z}} m_{j,k}(x, x_{d+1}, t, \xi, \xi_{d+1}) = \chi(2^k t) \chi\left(\frac{t(\xi + \xi_{d+1}A(x))}{2^j}\right).
\]

Next, we split \(\xi_{d+1}\) according to an extremely high or moderate frequency with

\[
a_{j,k}(x, x_{d+1}, t, \xi, \xi_{d+1}) = \chi(2^k t) \chi\left(\frac{t(\xi + \xi_{d+1}A(x))}{2^j}\right) \psi\left(\frac{|\xi_{d+1}|}{2(1 + \epsilon_0)2^{2k}}\right)
\]

and

\[
b_{j,k}(x, x_{d+1}, t, \xi, \xi_{d+1}) = \chi(2^k t) \chi\left(\frac{t(\xi + \xi_{d+1}A(x))}{2^j}\right) \left(1 - \psi\left(\frac{|\xi_{d+1}|}{2(1 + \epsilon_0)2^{2k}}\right)\right).
\]
so that $m_{j,k} = a_{j,k} + b_{j,k}$. Define

$$\mathcal{M}_m^p f(x, x_{d+1}) = \sup_{k \in \mathbb{Z}} \sup_{t \in \mathbb{R}} \left| 2^{-j(d-1)/2} \mathcal{F}_{m_{j,k}} f(x, x_{d+1}, t) \right|$$

(3.10)

$$\mathcal{M}_a^p f(x, x_{d+1}) = \sup_{k \in \mathbb{Z}} \sup_{t \in \mathbb{R}} \left| 2^{-j(d-1)/2} \mathcal{F}_{a_{j,k}} f(x, x_{d+1}, t) \right|$$

$$\mathcal{M}_b^p f(x, x_{d+1}) = \sup_{k \in \mathbb{Z}} \sup_{t \in \mathbb{R}} \left| 2^{-j(d-1)/2} \mathcal{F}_{b_{j,k}} f(x, x_{d+1}, t) \right|.$$  (3.11)

We observe that $\mathcal{M}_j f(x, x_{d+1})$ in (3.11) is expressed as

$$\sup_{t \in \mathbb{R}} A_j f(x, x_{d+1}, t) = \mathcal{M}_j^m f(x, x_{d+1}) \leq \mathcal{M}_j^a f(x, x_{d+1}) + \mathcal{M}_j^b f(x, x_{d+1}).$$

3.3. Majorizing $L_1^\infty(\mathbb{R}_+) \text{ by } L_1^p(\mathbb{R}_+)$. By the Sobolev imbedding $\|\sup_t g\|_{L_p(dt)} \leq \|g\|_{L_1^p(dt)}$,

$$\left( \int |\mathcal{M}_j^m f(x, x_{d+1})|^p dx dx_{d+1} \right)^{1/p} \leq 2^{-(d-1)/2} \left( \sum_k \left\| \partial_t^{1/p} \mathcal{F}_{m_{j,k}} f \right\|_{L_p(\mathbb{R}^{d+1} \times \mathbb{R})}^p \right)^{1/p}.$$  (3.12)

The derivative $\partial/\partial t$ in (3.6) produces an additional factor

$$|\xi + \xi_{d+1} A(x)| \approx 2^{j+k} \text{ on the support of } (3.8).$$

Hence we observe that

$$|\partial_t^{1/p} \mathcal{F}_{m_{j,k}} f(x, x_{d+1}, t)| \lesssim 2^{(j+k)/p} |\mathcal{F}_{m_{j,k}} f(x, x_{d+1}, t)|$$

where $\bar{m}_{j,k}$ is the similar symbol as $m_{j,k}$ in (3.8). Thus, we denote it by $m_{j,k}$ without confusion. Then in (3.13),

$$(\int |\mathcal{M}_j^m f(x, x_{d+1})|^p dx dx_{d+1})^{1/p} \leq 2^{-(d-1)/2} 2^{(j+k)/p} \left( \sum_k \left\| \mathcal{F}_{m_{j,k}} f \right\|_{L_p(\mathbb{R}^{d+1} \times \mathbb{R})}^p \right)^{1/p} = 2^{j(1/2 - (d-1)/2)} \left\| \left( \sum_k 2^{k/p} \mathcal{F}_{m_{j,k}} f \right)^p \right\|_{L_p(\mathbb{R}^{d+1} \times \mathbb{R})}.$$  (3.13)

Here we can also replace $m$ with $a$ and $b$ of (3.10) in (3.13).

3.4. Littlewood Paley Decompositions.

**Definition 3.1.** Note that $\psi(\xi, \xi_{d+1})$ is supported in $|\xi, \xi_{d+1}| \leq 1$ and $\psi(\xi, \xi_{d+1}) \equiv 1$ on $|\xi, \xi_{d+1}| \leq 1/2$. Using the non-isotropic dilation

$$\chi_j(\xi, \xi_{d+1}) = \psi \left( \frac{\xi}{2^j+1}, \frac{\xi_{d+1}}{2^{j+1}} \right) - \psi \left( \frac{\xi}{2^j}, \frac{\xi_{d+1}}{2^j} \right),$$

we define the Littlewood-Paley projection $\mathcal{P}_j f$ by

$$\mathcal{P}_j f(y, y_{d+1}) = \int e^{2\pi i \langle (y, y+1), \xi_{d+1} \rangle} \chi_j(\xi, \xi_{d+1}) \hat{f}(\eta, \eta_{d+1}) d\eta d\eta_{d+1}. $$
By applying $\sum_{\ell \in \mathbb{Z}} P_{j+k+\ell} = Id$ and the triangle inequality, we have in (3.13),

$$
\|M^m f\|_{L^p(\mathbb{R}^{d+1})} \lesssim 2^{(1/p - (d-1)/2)} \sum_{\ell \in \mathbb{Z}} \left( \sum_k |2^{k/p} T_{m,k} P_{j+k+\ell} f|^p \right)^{1/p} \quad .
$$

Let $g = (g_k)_{k \in \mathbb{Z}}$. We define the $\ell^p$ norm of $g = (g_k)$ with a weight $2^{k/p}$ for each $k^{th}$ component by

$$
\|g\|_{\ell^p} = \left( \sum_k |2^{k/p} g_k|^p \right)^{1/p} .
$$

For each $(j, \ell) \in \mathbb{Z}_+ \times \mathbb{Z}$, let us define a vector valued function

$$
G^m_{j,\ell} f = (T_{m,j} P_{j+k+\ell} f)_{k \in \mathbb{Z}} \quad \text{where } m,j,k \text{ is defined in (3.8).}
$$

Then we can write the $\ell^{th}$ piece of the summation in (3.15) as $\|G^m_{j,\ell} f\|_{L^p(\mathbb{R}^{d+1})}$ so that

$$
\|M^m_j f\|_{L^p(\mathbb{R}^{d+1})} \lesssim 2^{j(1/p - (d-1)/2)} \sum_{\ell \in \mathbb{Z}} \|G^m_{j,\ell} f\|_{L^p(\mathbb{R}^{d+1})} .
$$

where

$$
\|G^m_{j,\ell} f\|_{L^p(\mathbb{R}^{d+1})} = \left( \sum_k |2^{k/p} T_{m,k} P_{j+k+\ell} f|^p \right)^{1/p} .
$$

Replace $(m,j,k)$ in (3.16) by $(a_{j,k}), (b_{j,k})$ of (3.8), (3.9) to define $G^a_{j,\ell}, G^b_{j,\ell}$ satisfying in (3.17)-(3.18).

3.5. $L^2$ Estimate for $G_{j,\ell}$.

**Proposition 3.1.** Let $A \in M_{d \times d}(\mathbb{R})$. Suppose $|\ell| \geq Cj$. Then there exists $c > 0$ such that

$$
\|G^m_{j,\ell} f\|_{L^2(\mathbb{R}^{d+1})} \lesssim 2^{-c|\ell|} \|f\|_{L^2(\mathbb{R}^{d+1})} \quad \text{for all } f \in L^2(\mathbb{R}^{d+1}) .
$$

Suppose that $|\ell| < Cj$ and that $A$ has only complex eigenvalues. Then for a sufficiently small $\epsilon > 0$,

$$
\|G^m_{j,\ell} f\|_{L^2(\mathbb{R}^{d+1})} \lesssim 2^{\epsilon j} \|f\|_{L^2(\mathbb{R}^{d+1})} \quad \text{for all } f \in L^2(\mathbb{R}^{d+1}) .
$$

Suppose $|\ell| < Cj$ and $d = 2$. Then for a sufficiently small $\epsilon > 0$ and the constant $c(A)$ below,

$$
\|G^m_{j,\ell} f\|_{L^2(\mathbb{R}^{2+1})} \lesssim 2^{j/2} 2^{j(c(A))} \|f\|_{L^2(\mathbb{R}^2)} \quad \text{for all } f \in L^2(\mathbb{R}^2) .
$$

Here, the constant $c(A) > 0$ is given by

$$
c(A) = \begin{cases} 
0 & \text{if rank}_{\text{skw}}(A) = 2 \\
1/6 & \text{if rank}_{\text{skw}}(A) = 1 \text{ and rank}(A) = 2 \\
1/4 & \text{if rank}_{\text{skw}}(A) = 1 \text{ and rank}(A) = 1 \\
1/2 & \text{if rank}_{\text{skw}}(A) = 0 .
\end{cases}
$$

We can replace $m$ by $a$ and $b$ in (3.17)-(3.21). Moreover, we have a better bound for the extreme frequency if $A$ is invertible:

$$
\|G^b_{j,\ell} f\|_{L^2(\mathbb{R}^{d+1})} \lesssim 2^{-c(j+|\ell|)} \|f\|_{L^2(\mathbb{R}^{d+1})} .
$$
3.6. Proposition 3.1 implies Main Theorems. Proposition 3.1 ⇒ Main Theorem 2. By using (3.19), (3.20) in (3.17) for \(d \geq 3\),
\[
\|M^m_j\|_{L^2(\mathbb{R}^{d+1}) \rightarrow L^2(\mathbb{R}^{d+1})} \lesssim 2^{j((1/2-(d-1)/2))}.\tag{3.24}
\]
As in the Euclidean space, we can observe that the average in (3.23) and (3.27) satisfies that
\[
|A_j f(x, x_{d+1}, t)| \leq C \int |f(x-y, x_{d+1} - (A(x), y)) K(y/t)| dy \quad \text{with} \quad |K(y)| \leq 2^j/(1+|y|^N).
\]
Thus, we use the \(L^p\) boundedness of the maximal operator in (19) to obtain that
\[
\|M^m_j\|_{L^p(\mathbb{R}^{d+1}) \rightarrow L^p(\mathbb{R}^{d+1})} \lesssim 2^j \quad \text{for every} \quad p > 1.
\]
Through the interpolation of (3.24) and (3.25), we obtain Main Theorem 2. Proposition 3.1 ⇒ Main Theorem 1. The estimates (3.19), (3.21) in (3.17) for \(d = 2\) yield
\[
\|M^m_j\|_{L^2(\mathbb{R}^{d+1}) \rightarrow L^2(\mathbb{R}^{d+1})} \lesssim 2^{j\epsilon(A)} \tag{3.26}
\]
By summing (3.26) over \(j\): \(2^j \leq \delta^{-1}\) in view of (3.10), we obtain Main Theorem 1 as
\[
\|M^m_j\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \lesssim \delta^{-\epsilon(A)+\epsilon}.\tag{3.27}
\]
3.7. Statement of the Main Estimates. To prove Proposition 3.1, we use the following preliminary estimate derived from the essentially disjoint supports of frequency variables.

Proposition 3.2. Recall \(T_m\) in (3.9) with \(m_{j,k}\) in (3.4). Let \(|\ell| \geq Cj\). For some \(c > 0\)
\[
\|2^{\ell/2}T_{m_{j,k}} P_{j+k+\ell}\|_{L^2(\mathbb{R}^{d+1}) \rightarrow L^2(\mathbb{R}^{d+1} \times \mathbb{R})} \lesssim 2^{-c|\ell|} \quad \text{proving} \quad (3.19)
\]
and
\[
\|P_{k_1} P_{k_2}^*\|_{L^2(\mathbb{R}^{d+1}) \rightarrow L^2(\mathbb{R}^{d+1})} \lesssim 2^{-c|k_1-k_2|}.\tag{3.28}
\]

Proof. Because its proof is rather standard, we placed it in Appendix. \(\square\)

The main estimate is Proposition 3.3, the proof of which we focus on in Sections 4 through 7.

Proposition 3.3 (Main-\(L^2\) Estimates). Let \(k \in \mathbb{Z}\) and \(j \in \mathbb{Z}_+\). Then there exists the constants \(c(A)\) according to the matrices \(A\) as in (3.22) satisfying that
\[
\|2^{\ell/2}T_{m_{j,k}}\|_{L^2(\mathbb{R}^{d+1}) \rightarrow L^2(\mathbb{R}^{d+1} \times \mathbb{R})} \lesssim 2^{|\ell| j 2^{c(A)j}} \quad \text{proving} \quad (3.21).
\]
Suppose that \(A\) has no real eigenvalues in (3.20). Then for a sufficiently small \(\epsilon > 0\),
\[
\|2^{\ell/2}T_{m_{j,k}}\|_{L^2(\mathbb{R}^{d+1}) \rightarrow L^2(\mathbb{R}^{d+1} \times \mathbb{R})} \lesssim 2^{j} \quad \text{proving} \quad (3.20).
\]
Moreover,
\[
\|2^{\ell/2}T_{m_{j,k}}\|_{L^2(\mathbb{R}^{d+1}) \rightarrow L^2(\mathbb{R}^{d+1} \times \mathbb{R})} \lesssim \begin{cases} 2^{-c_j} & \text{if} \ A \text{ is invertible} \\ 2^{c_j(A)+\epsilon} & \text{if} \ A \text{ is not invertible for} \ d = 2 \end{cases}
\]
showing (3.29).
We shall use the following well known almost orthogonality lemma:

**Lemma 3.1.** Let $M_1, M_2$ be two measure spaces. Suppose that there is a family of operators $\{T_k\}$ with $T_k : L^2(M_1) \to L^2(M_2)$ with their operator norm $\|T_k\|_{op} \leq R$ for all $k$. If $\|T_k T_k^*\|_{op} \leq C(|k_1 - k_2|)$ with $\sum_{k=1}^{\infty} \sqrt{C(|k|)}$ being finite, then

\[
(3.32) \quad \left\| \left( \sum_k |T_k f|^2 \right)^{1/2} \right\|_{L^2(M_2)} \lesssim \left( R \sum_{k=1}^{\infty} \sqrt{C(|k|)} \right) \|f\|_{L^2(M_1)} \text{ for all } f \in L^2.
\]

**Proof.** Apply the Cotlar-Stein lemma for $\sum_k T_k T_k^*$ with $\|T_k T_k^*\|_{op} \lesssim R^2 \|T_k\|_{op}$ in (RHS):

\[
\left\| \left( \sum_k |T_k f|^2 \right)^{1/2} \right\|_{L^2(M_2)} = \left\langle \sum_k T_k^* T_k f, f \right\rangle.
\]

Then the desired bound can be obtained in $\text{(3.32)}$. \hfill \Box

### 3.8. Propositions 3.3 and 3.8 imply Proposition 3.1

We assume that Proposition 3.2 hold true. Insert (3.27) and (3.28) in Proposition 3.2 into (3.18). Then Lemma 3.1 yields (3.19). Now we assume that Proposition 3.2 holds true.

**Proof of (3.21) under the assumption (3.22).** In (3.18) and Lemma 3.1 we work with

\[ T_k := 2^{k/2} \mathcal{T}_{m,j,k+\ell} \text{ mapping } L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3 \times \mathbb{R}). \]

By (3.21),

\[
(3.33) \quad \|T_k\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3 \times \mathbb{R})} \leq \left\| 2^{k/2} \mathcal{T}_{m,j,k+\ell} \right\| \cdot \|P_{j+k+\ell}\| \lesssim 2^{\epsilon_j 2^{3(A)j}}.
\]

By (3.28) and (3.29),

\[
\|T_k^* T_k\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} = \left\| 2^{k/2} \mathcal{T}_{m,j,k} \right\| \cdot \|P_{j+k+\ell} T_{m,j,k+\ell} \| \cdot \left\| [2^{k/2} \mathcal{T}_{m,j,k}]^* \right\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \lesssim 2^{-c|k_1 - k_2|} \left( 2^{\epsilon_j 2^{3(A)j}} \right)^2.
\]

(3.34)

This means that $C(k)$ in the sense of Lemma 3.1 is given by

\[ C(k) = 2^{-ck} \left( 2^{\epsilon_j 2^{3(A)j}} \right)^2. \]

In addition, (3.33) yields that the constant $R$ in Lemma 3.1 is $2^{\epsilon_j 2^{3(A)j}}$. Hence, we apply Lemma 3.1 to obtain

\[
\left( R \sum_{k=1}^{\infty} \sqrt{C(|k|)} \right)^{1/2} \leq C 2^{\epsilon_j} \left( 2^{3(A)j} \right).
\]

This implies (3.21). \hfill \Box

Similarly, the assumption of (3.30) implies (3.21). The assumption of (3.31) and (3.27) leads (3.23).
4. Restatement of Main $L^2$ Estimates

We reduce the proof of Proposition 3.3 to the uniform estimates of oscillatory integral operators with parameter $\lambda$.

4.1. Localization. We utilize the following localization to fix $k = 0$ in Proposition 3.3.

**Proposition 4.1** (Localization). Let $I = [-2, -1] \cup [1, 2]$ and let $B_{d+1}(0, r) = \{ z \in \mathbb{R}^{d+1} : |z| < r \}$.

For every $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$,

$$\left\| Z_{k,j}^{m_j} \right\|_{L^p(\mathbb{R}^{d+1})} \approx \left\| T_{m_j,0} \right\|_{L^p(B_{d+1}(0, 100))}$$

with an error $O(2^{-Nj})$ for sufficiently large $N$. This localization holds true when $m_j,k$ is replaced by $a_{j,k}, b_{j,k}$ in (3.4), (4.3).

**Proof.** The proof is based on the non-isotropic dilation in $\mathbb{R}^{d+1}$. See the details in the appendix. \qed

4.2. Uniform $L^2$ estimates. In view of Proposition 4.1, we can fix $k = 0$ in Proposition 3.3 to express $T_{m_j,0}(x, x_{d+1}, t)$ in (3.3) as

$$\psi(x)\psi(x_{d+1}) = \int_{\mathbb{R}^d} e^{2\pi i(x,x_{d+1})} \psi(\xi_{d+1}) e^{2\pi i(x_{d+1} + t(\xi_{d+1} A(x)))} \left( \frac{t|\xi + \xi_{d+1} A(x)|}{2j} \right) \hat{f}(\xi, \xi_{d+1}) d\xi d\xi_{d+1}.$$  

From $\chi(t)$, the support of $t$ is $I = [-2, -1] \cup [1, 2]$. Put $\xi_{d+1} = \lambda$ and write the above integral as

$$T_{m_j,0}(x, x_{d+1}, t) = \psi(x_{d+1}) \int_{\mathbb{R}} U_{j}^{\lambda} f(x, t) e^{2\pi i \lambda x_{d+1}} d\lambda$$

where for $(x, t) \in \mathbb{R}^d \times \mathbb{R}$,

$$U_{j}^{\lambda} f(x, t) = \chi(t) \psi(x) \int_{\mathbb{R}^d} e^{2\pi i(x, \xi)} e^{2\pi i(x_{d+1} + t(\xi_{d+1} A(x)))} \left( \frac{t|\xi + \lambda A(x)|}{2j} \right) \hat{f}(\xi, \lambda) d\xi.$$  

Then by using the change of variable $\xi \rightarrow \lambda \xi$ in (4.2),

$$T_{j}^{\lambda} f(x, t) = \lambda^{d/2} \chi(t) \psi(x) \int_{\mathbb{R}^d} e^{2\pi i(\lambda x, \xi)} e^{2\pi i(\xi_{d+1} + t A(x)))} \left( \frac{\lambda|\xi + A(x)|}{2j} \right) \hat{g}(\xi) d\xi$$

where $\hat{g}(\xi) = \lambda^{d/2} \hat{f}(\lambda \xi, \lambda)$. Then by applying the Plancherel theorem in the $x_{d+1}, \xi_{d+1}$ variables, we can observe that for $j > 0$,

$$\left\| T_{m_j,0} \right\|_{L^2(\mathbb{R}^{d+1})} \approx \sup_{\lambda} \left\| T_{j}^{\lambda} \right\|_{L^2(\mathbb{R}^{d+1} \times \mathbb{R}^d \times \mathbb{R})}.$$  

**Remark 4.1.** If $A$ is invertible, use the change of variable $x \rightarrow A(x)$ above and replace $A^{-1}$ by $A$ owing to Lemma 2.4. Then, we see that (4.3) can be switched with

$$T_{j}^{\lambda} g(x, t) = \lambda^{d/2} \chi(t) \psi(x) \int_{\mathbb{R}^d} e^{2\pi i(\lambda(A(x), \xi) + t(\xi + x))} \left( \frac{\lambda|\xi + x|}{2j} \right) \hat{g}(\xi) d\xi.$$
where \( \tilde{\psi}(x) = \psi(A(x)) \). The corresponding operator \( T_j^\lambda \infty \) for \( T_{kj,0} \) is given by
\[
T_j^\lambda \infty g(x, t) = \lambda^{d/2} \chi(t) \tilde{\psi}(x) \int_{\mathbb{R}^d} e^{2\pi i \lambda \langle (A(x), \xi) + t(x + \xi) \rangle} \left( \frac{\lambda}{2} \right) (1 - \psi \left( \lambda \left( \frac{2}{1 + \epsilon_0} \right) \right)) \hat{g}(\xi) d\xi.
\]

Hence with (4.3) and (4.11), we reduce Proposition 4.3 to the following uniform estimates.

**Proposition 4.2.** Let \( T_j^\lambda \) be defined as in (4.3). Then for a sufficiently small \( \epsilon > 0 \),
\[
\begin{align*}
(4.7) & \quad \text{If } \text{rank}_{\text{kw}}(A) = 2, \text{ then } \| T_j^\lambda \|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2 \times \mathbb{R})} \lesssim 2^{\epsilon j}. \\
(4.8) & \quad \text{If } \text{rank}_{\text{kw}}(A) = 0, \text{ then } \| T_j^\lambda \|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2 \times \mathbb{R})} \lesssim 2^{j/2 + j \epsilon}. \\
(4.9) & \quad \text{If } \text{rank}_{\text{kw}}(A) = 1, \text{ then } \| T_j^\lambda \|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2 \times \mathbb{R})} \lesssim \begin{cases} 2^{j/6 + j \epsilon} & \text{for } \text{rank}(A) = 2 \\ 2^{j/4} 2^{j \epsilon} & \text{for } \text{rank}(A) = 1, \end{cases}
\end{align*}
\]
which lead (4.20). Moreover,
\[
\begin{itemize}
\item If \( A \) in (4.3) has no real eigenvalues, then for a sufficiently small \( \epsilon > 0 \),
\[
(4.10) \quad \| T_j^\lambda \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d \times \mathbb{R})} \lesssim 2^{\epsilon j} \text{ which implies (3.30).}
\]
\item Let \( T_j^\infty \) be the operator in (4.7). If \( A \) is invertible, there exists \( c > 0 \) such that
\[
(4.11) \quad \| T_j^\infty \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d \times \mathbb{R})} \lesssim 2^{-\epsilon j} \text{ which implies (3.31).}
\]
\end{itemize}

5. **Main Estimate of Extreme Frequency**

In this section, we treat the case of \( |\lambda/2| \neq 1 \). Let
\[
T_j^\lambda g(x, t) = \lambda^{d/2} \chi(t) \tilde{\psi}(x) \int_{\mathbb{R}^d} e^{2\pi i \lambda \langle (A(x), \xi) + t(x + \xi) \rangle} \left( \frac{\lambda}{2} \right) (1 - \psi \left( \frac{\lambda}{2(1 + \epsilon_0)} \right)) \hat{g}(\xi) d\xi.
\]

A straightforward computation gives
\[
(5.1) \quad \sup_t \| T_j^\lambda g(\cdot, t) \|_{L^2(\mathbb{R}^d)} \lesssim \frac{2^{(d + 1/2)} \lambda^{\text{rank}(A)}}{\lambda^{\text{rank}(A)}} \| g \|_{L^2(\mathbb{R}^d)} \text{ for all } g \in L^2(\mathbb{R}^d).
\]

**Proof of (5.1).** Compute
\[
\begin{align*}
\int_{\mathbb{R}^d} \lambda^{d/2} \left| \chi(t) \tilde{\psi}(x) \chi \left( \frac{t\langle \xi, A(x) \rangle}{2} \right) \right| dx & \lesssim \lambda^{d/2} \left| \frac{2^j}{\lambda} \right|^{\text{rank}(A)}, \\
\int_{\mathbb{R}^d} \lambda^{d/2} \left| \chi(t) \tilde{\psi}(x) \chi \left( \frac{t\langle \xi, A(x) \rangle}{2} \right) \right| d\xi & \lesssim \lambda^{d/2} \left| \frac{2^j}{\lambda} \right|^d.
\end{align*}
\]
Applying Schur’s test, we have (5.1). \( \square \)

We obtain a more precise estimate to show (4.11) of Proposition 4.2.
Proposition 5.1 (Extreme \( \lambda \)). Suppose that \( A \in M_{d \times d}(\mathbb{R}) \) is invertible. Then, for each fixed time \( t \approx 1 \),

\[
\sup_{t} \| T_{j}^{\lambda} g(\cdot, t) \|_{L^2(\mathbb{R}^d)} \lesssim \frac{2^j}{\lambda} \| g \|_{L^2(\mathbb{R}^d)} \text{ for all } g \in L^2(\mathbb{R}^d) \text{ if } \left| \frac{2^j}{\lambda} \right| \ll 1,
\]

\[
\sup_{t} \| T_{j}^{\lambda} g(\cdot, t) \|_{L^2(\mathbb{R}^d)} \lesssim \| g \|_{L^2(\mathbb{R}^d)} \text{ for all } g \in L^2(\mathbb{R}^d) \text{ if } \left| \frac{2^j}{\lambda} \right| \gg 1.
\]

The above (5.2) for \( \lambda \geq 2^{j+c} \) with \( c > 0 \) implies that for an invertible matrix \( A \),

\[
\| T_{j}^{\lambda} \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d \times \mathbb{R})} \lesssim 2^{-c} \text{ for } \lambda > 2^{j+c}
\]

which shows (4.11) and (5.4). Hence we obtain (5.1).

Proof of Proposition 5.1. It suffices to work with (5.1) for a fixed \( e(\theta) \in S^{d-1} \) and \( (x, t) \in \mathbb{R}^d \times \mathbb{R} \),

\[
T_{j}^{\lambda} f(x, t) = \lambda^{d/2} \psi(x) \chi(t) \int_{\mathbb{R}^d} e^{i\lambda(\langle A(x), \xi \rangle + t(\xi + x))} \chi \left( \frac{t|\xi + x|}{\lambda} \right) \psi \left( \frac{t|\xi + x| - e(\theta)}{1/100} \right) \hat{f}(\xi) d\xi
\]

where \( t|\xi + x| \) is supported in \( \left[ \frac{99}{100}, \frac{101}{100} \right] \). Let \( T_{i}(\xi, \eta) \) be the integral kernel of \( |T_{j}^{\lambda}|^{*} T_{j}^{\lambda} \). Then

\[
T_{i}(\xi, \eta) = \lambda^{d} \int_{\mathbb{R}^d} \psi(x) e^{i\lambda \Phi(x, t, \xi, \eta)} \chi \left( \frac{t|\xi + x|}{\lambda} \right) \chi \left( \frac{t|\eta + x|}{\lambda} \right) \times \psi \left( \frac{t|\xi + x| - e(\theta)}{1/100} \right) \psi \left( \frac{t|\eta + x| - e(\theta)}{1/100} \right) dx
\]

where \( |t| \approx 1 \) and the phase function \( \Phi(x, t, \xi, \eta) = \langle A^{T}(\xi - \eta), x \rangle + t(\langle \xi + x \rangle - |\eta + x|) \) satisfies

\[
\nabla_{x} \Phi(x, t, \xi, \eta) = A^{T}(\xi + x - (\eta + x)) + t \left( \frac{\xi + x}{|\xi + x|} - \frac{\eta + x}{|\eta + x|} \right)
\]

(5.6)

\[
\nabla_{x} \text{amplitude} = O(\lambda/2^j).
\]

Case 1. \( \frac{2^j}{\lambda} \ll 1 \). From the condition \( t|\xi + x|, t|\eta + x| \in \left[ \frac{99}{100}, \frac{101}{100} \right] \) with \( \frac{2^j}{\lambda} \ll 1 \), we have that

\[
\left| \frac{\xi + x}{|\xi + x|} - \frac{\eta + x}{|\eta + x|} \right| \gg \left| \langle \xi + x \rangle - (\eta + x) \right| \geq c|A^{T}(\xi + x - (\eta + x))| \text{ for some } c > 0.
\]

Thus, the gradient of the phase function is

\[
|\nabla_{x} \Phi(x, t, \xi, \eta)| \approx \left| \frac{\xi + x}{|\xi + x|} - \frac{\eta + x}{|\eta + x|} \right| \approx \frac{\xi - \eta}{\frac{2^j}{\lambda}}.
\]

Hence, from (5.6) and (5.7) with the measure \( dx = O(\left| \frac{2^j}{\lambda} \right|^d) \), we can apply the integration by parts in (5.5) to obtain that

\[
|T_{i}(\xi, \eta)| \lesssim \frac{\left| \frac{2^j}{\lambda} \right|^d \lambda^d}{\left| \frac{\xi - \eta}{\frac{2^j}{\lambda}} \right| + 1} \quad \text{and} \quad \int_{\mathbb{R}^d} |T_{i}(\xi, \eta)| d\xi (\text{or } du) \lesssim \left| \frac{2^j}{\lambda} \right|^d.
\]
This implies \( \text{(5.2)} \).

**Case 2.** \( \frac{\beta}{\alpha} \gg 1 \). From the condition \( t|\xi + x|, t|\eta + x| \in \left[ \frac{99}{100} \alpha, \frac{101}{100} \alpha \right] \) with \( \frac{\beta}{\alpha} \gg 1 \), we have that

\[
\left| \frac{\xi + x}{|\xi + x|} - \frac{\eta + x}{|\eta + x|} \right| \ll |(\xi + x) - (\eta + x)| \lesssim |A^T((\xi + x) - (\eta + x))|.
\]

So, the gradient of the phase function is

\[
|\nabla_x \Phi(x, t, \xi, \eta)| \approx |A^T((\xi + x) - (\eta + x))| \geq c|\xi - \eta|.
\]

Using \( \text{(5.8)} \) and \( \text{(5.7)} \) combined with \( dx = O(1) \), we apply the integration by parts in \( \text{(5.5)} \) to obtain that

\[
\int_{R^d} |T_t(\xi, \eta)| d\xi \text{ (or } dn) \lesssim \int_{R^d} \frac{\lambda^d}{|\lambda|} \cdot d\xi \text{ (or } dn) \lesssim 1.
\]

This implies \( \text{(5.9)} \). Finally, \( \text{(5.4)} \) follows from \( \text{(5.2)} \). We have therefore proved Proposition \( \text{5.1} \) \( \square \)

6. Main Estimate of Non-extreme Frequency

We shall prove \( \text{(4.7)} \) and \( \text{(4.8)} \) in Proposition \( \text{4.2} \) for the case \( \frac{\beta}{\alpha} \approx 1 \). We first start with an invertible matrix \( A \) and work with \( \text{(4.5)} \). We may assume that \( \lambda = 2^d \). As \( d = 2 \) in \( \text{(4.5)} \), we put

\[
T^\lambda f(x_1, x_2, t) = \chi(t) \tilde{\psi}(x_1, x_2) \lambda \int_{R^2} e^{2\pi \imath \lambda \Phi(x_1, x_2, t, \xi)} \chi(|x + \xi|) \tilde{f}(\xi) d\xi
\]

where

\[
\Phi(x, t, \xi) = \langle A(x), \xi \rangle + t|\xi + x|.
\]

Then to obtain \( \text{(4.7)} \), we need to show \( \|T^\lambda\| = O(\lambda^e) \) of \( \text{(6.1)} \). Denote \( b = (b_1, b_2) := (x_1 + \xi_1, x_2 + \xi_2) \) for a simple notation. First we compute the three kinds of \( 2 \times 2 \) sub-matrices of the \( 3 \times 2 \) matrix \( \Phi''_{(x_1, x_2, t, \xi_1, \xi_2)} \) given by

\[
\Phi''_{(x_1, x_2, t, \xi_1, \xi_2)} = \begin{pmatrix}
\Phi_{x_1 \xi_1} & \Phi_{x_2 \xi_1} \\
\Phi_{x_1 \xi_2} & \Phi_{x_2 \xi_2} \\
\Phi_{t \xi_1} & \Phi_{t \xi_2}
\end{pmatrix} = \begin{pmatrix}
a_{11} + \frac{t b_2^2}{|b|^2} & a_{12} - \frac{t b_1 b_2}{|b|^2} \\
a_{21} - \frac{t b_1 b_2}{|b|^2} & a_{22} + \frac{t b_2^2}{|b|^2}
\end{pmatrix}.
\]

We can check the ranks of their submatrices:

\[
\text{(6.3)} \quad \det \left( \Phi''_{(x_1, x_2, t, \xi_1, \xi_2)} \right) = \det \begin{pmatrix}
a_{11} + \frac{t b_2^2}{|b|^2} & a_{12} - \frac{t b_1 b_2}{|b|^2} \\
a_{21} - \frac{t b_1 b_2}{|b|^2} & a_{22} + \frac{t b_2^2}{|b|^2}
\end{pmatrix} = \det(A) + \frac{t}{|b|^3} \left( \langle A + A^T \rangle b, b \right)
\]

\[
\text{(6.4)} \quad \det \left( \Phi''_{(x_1, t, \xi_1, \xi_2)} \right) = \det \begin{pmatrix}
a_{11} + \frac{b_2^2}{|b|^2} & a_{12} - \frac{b_1 b_2}{|b|^2} \\
a_{21} & a_{22} + \frac{b_2^2}{|b|^2}
\end{pmatrix} = \frac{1}{|b|} \left( \det \begin{pmatrix}
a_{11} & a_{12} \\
b_1 & b_2
\end{pmatrix} + t \frac{b_2}{|b|} \right)
\]

\[
\text{(6.5)} \quad \det \left( \Phi''_{(x_2, t, \xi_1, \xi_2)} \right) = \det \begin{pmatrix}
a_{21} - \frac{b_1 b_2}{|b|^2} & a_{22} + \frac{b_2^2}{|b|^2} \\
b_1 & b_2
\end{pmatrix} = \frac{1}{|b|} \left( \det \begin{pmatrix}
a_{21} & a_{22} \\
b_1 & b_2
\end{pmatrix} - t \frac{b_1}{|b|} \right).
\]

The following two examples indicate good and bad mixed hessians respectively, although both have \( \text{rank}_{\text{skew}}(A) = 2 \) and \( \text{rank}(A) = 2 \).
Example 6.1 (Good Mixed Hessian; skew–symmetric case $A = E$). Let $d = 2$. Suppose that $A$ is an $2 \times 2$ skew–symmetric matrix $E$. Let $\frac{2}{\lambda} = 1$. For convenience set $b = (x + \xi)$ and $\Phi(x, \xi) = (A(x), \xi) + t|\xi + x|$. In (6.3),
\[ \det \left( \Phi''(x_1, x_2)(\xi_1, \xi_2) \right) = \det(E) + \frac{t}{|b|^3} \left( \frac{(E + E^T)b}{2} \right) = \det(E) = 1. \]
The Hörmander theorem with the non vanishing mixed hessian yields that for all $f \in L^2(\mathbb{R}^2)$,
\[ \sup_t \| T^\lambda f(\cdot, t) \|_{L^2(\mathbb{R}^2)} \lesssim \lambda^{-1} \| f \|_{L^2(\mathbb{R}^2)} \]
which is the Heisenberg group case.

Example 6.2 (Bad Mixed Hessian). Take $A$ with a good condition $\operatorname{rank}_{\text{skew}}(A) = \operatorname{rank}(A) = 2$:
\[ A = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \]
where $a_{11} = a_{22} = 0$ and $a_{12} = c, a_{21} = c$ with $c \neq 0$.

However, its three $2 \times 2$ mixed hessians of (6.3)–(6.5) are simultaneously singular
\[ \det(\Phi''(x_1, x_2)(\xi_1, \xi_2)) = \det(\Phi''(x_1t)(\xi_1, \xi_2)) = \det(\Phi''(x_2t)(\xi_1, \xi_2)) = 0, \]
that is $\operatorname{rank}(\Phi''(x_1, x_2)(\xi_1, \xi_2)) = 1$ at the point $(b_1, b_2, t) = (x_1 + \xi_1, x_2 + \xi_2, t)$ if
\[ |b_1| = |b_2| = 1 \text{ and } t = \pm \sqrt{-c} \text{ where } \pm \text{ is the sign of } \frac{b_2}{b_1}. \]
Hence to obtain $\| T^\lambda \| = O(\lambda^0)$ for (6.2) with $\operatorname{rank}(\Phi''(x_1, x_2)(\xi_1, \xi_2)) = 1$, we shall perform a more subtle analysis of the phase functions gaining from $\operatorname{rank}_{\text{skew}}(A) = 2$, as we have previously remarked in Section 2.2.

6.1. Phases and Amplitudes of Propositions 4.2. For $\lambda = 2^j$, the integral kernel of $[T^\lambda_j]^* T^\lambda_j$ is
\[ K(\xi, \eta) = 2^{dj} \int_{\mathbb{R}^d \times \mathbb{R}^l} e^{2\pi i \lambda \Phi(x, t, \xi, \eta)} L(x, t, \xi, \eta) dx dt \]
where the phase $\Phi$ and the amplitude $L$ are given by
\[ \Phi(x, t, \xi, \eta) = ((\xi - \eta), A(x)) + t(|\xi + x| - |\eta + x|), \]
\[ L(x, t, \xi, \eta) = \psi(x)^2 \chi(t)^2 \chi(t|\xi|) \chi(t|\eta|) \cdot \chi(t|\lambda|) \cdot \chi(t|\lambda|) \cdot \chi(t|\lambda|). \]
From the support condition with $\lambda = 2^j$, the size $dx$ or $d\xi$ of the support of the kernel $L$ is $O(1)$.
The derivatives of the phase function $\Phi$ with respect to $x$ and $t$ are given by
\[ \nabla_x \Phi(x, t, \xi, \eta) = A^T(\xi - \eta) - t \left( \frac{\xi + x}{|\xi + x|} - \frac{\eta + x}{|\eta + x|} \right), \]
\[ \partial_t \Phi(x, t, \xi, \eta) = |\xi + x| - |\eta + x|. \]
We write the second term of (6.10) as
\[ \frac{\xi + x}{|\xi + x|} - \frac{\eta + x}{|\eta + x|} = \frac{\xi - \eta}{|\xi + x|} + (\eta + x) \left( \frac{1}{|\eta + x|} - \frac{1}{|\xi + x|} \right). \]
According to the size of
$$
\partial_t \Phi(x, t, \xi, \eta) = |\xi + x| - |\eta + x|,
$$
set $K(\xi, \eta) = K_1(\xi, \eta) + K_2(\xi, \eta)$ for each $\nu = 1, 2$,

(6.13) \quad K_1(\xi, \eta) = 2^{dj} \int_{\mathbb{R}^d \times \mathbb{R}^1} e^{2\pi i 2^j \Phi(x, t, \xi, \eta)} L(x, t, \xi, \eta) \psi \left( \frac{|\xi + x| - |\eta + x|}{2^j} \right) dx dt

(6.14) \quad K_2(\xi, \eta) = 2^{dj} \int_{\mathbb{R}^d \times \mathbb{R}^1} e^{2\pi i 2^j \Phi(x, t, \xi, \eta)} L(x, t, \xi, \eta)(1 - \psi) \left( \frac{|\xi + x| - |\eta + x|}{2^j} \right) dx dt.

6.2. Estimates of $K_2$ and the good parts of $K_1$. The lower bound of $\partial_t \Phi$ leads the estimates:

(6.15) \quad \sup_{\eta} \int_{\mathbb{R}^d} |K_2(\xi, \eta)| \, d\xi \lesssim 2^{-Nj} \quad \text{where we can switch } \xi \text{ and } \eta.

Proof of (6.15). For this case,
- derivatives of phase $|\partial_t \Phi| \gtrsim \frac{2^j}{2^j}$ in (6.11) and (6.14)
- derivative of cutoff function $|\partial_t L_2| = O(1)$ in (6.14)
- the measures $d\xi = O(1)$ and $d\eta = O(1)$.

So, we apply integration by parts $\lceil \frac{M}{2} \rceil$ times with respect to $dt$ for the integral $K_2(\xi, \eta)$ in (6.14). Then

(6.16) \quad \sup_{\eta} \int |K_2(\xi, \eta)| \, d\xi \lesssim 2^{d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^1} 2^{-jx(|M/\xi|)} \psi(x) \chi(t)(t|\xi + x|) \, dx \, dt \, d\xi \lesssim 2^{-j(M-d)}

where $M \gg 2$. Similarly, we have $\int |K_2(\xi, \eta)| \, d\eta \lesssim 2^{-(M-d)}$. \qed

In proving Propositions 4.2, there remains the kernel $K_1(\xi, \eta)$ in (6.13) having the support condition:

(6.17) \quad |\xi + x| - |\eta + x| \leq \frac{2^{2j}}{2^j}.

It suffices to assume that

$$
|\xi - \eta| \geq \frac{2^{2j}}{2^j}.
$$

Otherwise the measure in $d\xi$ or $d\eta = O \left( \left| \left( 2^{2j} / 2^j \right) \right|^2 \right)$ with $dx = O(1)$ gives the desired bound

(6.18) \quad \sup_{\eta} \int \left| K_1(\xi, \eta) \psi \left( \frac{|\xi - \eta|}{2^{2j}} \right) \right| \, d\xi \lesssim 2^{22j}.

So, our support condition is

(6.19) \quad |x| \lesssim 1, \ |\xi + x| \approx |\eta + y| \approx 1 \text{ and } |\xi - \eta| \geq \frac{2^{2j}}{2^j} \gg \frac{2^{2j}}{2^j} \gg |\xi + x| - |\eta + y|.

From this, we see that $\nabla_x \Phi(x, t, \xi, \eta)$ in (6.10) has two major terms $A^T(\xi - \eta)$ and $-t \frac{\xi - \eta}{|\xi + x|}$

(6.20) \quad \nabla_x \Phi(x, t, \xi, \eta) = A^T(\xi - \eta) - t \frac{\xi - \eta}{|\xi + x|} + O \left( \frac{2^{2j}}{2^j} \right)$.
where we have used as in (6.12)
\[
(6.21) \quad \frac{\xi + x}{|\xi + x|} - \frac{\eta + x}{|\eta + x|} = \frac{\xi - \eta + (\eta + x)}{|\eta + x|} \left( \frac{1}{|\eta + x|} - \frac{1}{|\xi + x|} \right) = \frac{\xi - \eta}{|\xi + x|} + O \left( \frac{2^{j\epsilon}}{2^j} \right).
\]

**Estimate of** $K_1$ **when** $A$ **has no real eigenvalue.** Let $\psi^c = 1 - \psi$. We claim that
\[
(6.22) \quad \sup_{\eta} \int_{\mathbb{R}^d} \left| K_1(\xi, \eta)\psi^c \left( \frac{|\xi - \eta|}{2^{j\epsilon}} \right) \right| \, d\xi \lesssim 2^{-Nj} \text{ if } A \text{ has only complex eigenvalues.}
\]
where we can switch $\xi$ and $\eta$. This together with (6.10), (6.18) and the result of the case $\lambda \not\approx 2^j$ in the previous sections yields (4.10) in Proposition 4.2.

**Proof of (6.22).** If $A$ has only complex eigenvalues, then $AT - L^t I$ in (6.20) with $L^t \approx 1$ is an invertible matrix having the lower bound. This implies that
\[
|\nabla_x \Phi(x, t, \xi, \eta)| \geq \left| \left( AT - \frac{t}{|\xi + x|} \right) (\xi - \eta) \right| \gtrsim |\xi - \eta| \geq \frac{2^{2j\epsilon}}{2^j} \text{ as above.}
\]
This combined with
\[
\left| \nabla_x \psi^c \left( \frac{|\xi + x| - |\eta + x|}{2^{2j\epsilon}} \right) \right| \lesssim 2^{j\epsilon} \left| \frac{\xi + x}{|\xi + x|} - \frac{\eta + x}{|\eta + x|} \right| \approx 2^{j-\epsilon} |\xi - \eta| \text{ as in (6.21)}
\]
enables us to apply integration by parts \( \left\lceil \frac{M}{4} \right\rceil + 1 \) times to obtain
\[
\sup_{\eta} \int_{\mathbb{R}^d} \left| K_1(\xi, \eta)\psi^c \left( \frac{|\xi - \eta|}{2^{j\epsilon}} \right) \right| \, d\xi \lesssim 2^0 \int_{\mathbb{R}^{2d} \times \mathbb{R}^1} \left| \frac{2^{j-\epsilon} |\xi - \eta|}{2^j |\xi - \eta|} \right|^{M/\epsilon} \psi(x)\chi(t) (t|\xi + x|) \, dx \, dt.
\]
This gives the desired bound $2^{-j(M-d)}$ for $M \gg 1$.

**Decomposition of** $K_1$. We shall apply integration by parts for $K_1(\xi, \eta)$ with respect to $dx$ along
\[
a = \frac{\xi - \eta}{|\xi - \eta|}
\]
For this, we need to observe that

- the directional derivative of the phase function $\Phi$ in (6.20) along $a$ is given by
\[
(6.23) \quad a \cdot \nabla_x \Phi = \left( AT (\xi - \eta) \cdot \frac{E(\xi - \eta)}{|\xi - \eta|} \right) + O(2^{j\epsilon}/2^j)
\]
since
\[
\left( \frac{\xi - \eta}{|\xi + x|} \cdot \frac{E(\xi - \eta)}{|\xi - \eta|} \right) = 0.
\]

- the derivative of cutoff functions in (6.9) are
\[
a \cdot \nabla_x \psi \left( \frac{|\xi + x| - |\eta + x|}{2^{2j\epsilon}} \right) = \frac{2^j}{2^{2j\epsilon}} \left( \frac{\xi - \eta}{|\xi + x|} \cdot \frac{E(\xi - \eta)}{|\xi - \eta|} \right) + O \left( \frac{2^{j\epsilon}}{2^j} \right) = O(1)
\]
\[
a \cdot \nabla_x \chi (t|\xi + x|) \chi (t|\eta - x|) = O(1).
\]
Let $\psi^c = 1 - \psi$. According to the size $\langle A^T (\xi - \eta), E(\xi - \eta) \rangle$ in (6.23), we split the amplitude (6.9):

\[
\hat{L}(x,t,\xi,\eta) := L(x,t,\xi,\eta)\psi^c \left( \frac{\|\xi - \eta\|}{2^j} \right) \psi \left( \frac{\|\xi + x - |\eta + x|\|}{2^j} \right) = L^{\text{good}}(x,t,\xi,\eta) + L^{\text{bad}}(x,t,\xi,\eta)
\]

where

\[
L^{\text{good}}(x,t,\xi,\eta) = L(x,t,\xi,\eta)\psi^c \left( \frac{\langle A^T (\xi - \eta), E(\xi - \eta) \rangle}{2^{2j+10^2-j}} \right)
\]

\[
L^{\text{bad}}(x,t,\xi,\eta) = L(x,t,\xi,\eta)\psi \left( \frac{\langle A^T (\xi - \eta), E(\xi - \eta) \rangle}{2^{2j+10^2-j}} \right).
\]

Then the corresponding integral kernel is

\[
(6.25) \quad \begin{cases} 
K^{\text{good}}(\xi,\eta) = 2^j \int_{\mathbb{R}^d} e^{2\pi i/2^{d+1}} \Phi(x,t,\xi,\eta) L^{\text{good}}(x,t,\xi,\eta) dx dt \\
K^{\text{bad}}(\xi,\eta) = 2^j \int_{\mathbb{R}^d} e^{2\pi i/2^{d+1}} \Phi(x,t,\xi,\eta) L^{\text{bad}}(x,t,\xi,\eta) dx dt.
\end{cases}
\]

**Case $K^{\text{good}}$.** We apply integration by parts for $K^{\text{good}}(\xi,\eta)$ with respect to $dx$ with direction $a$ in view of (6.23) and (6.24) combined with the measure estimates in (6.19), we have for large $N \gg M \gg 1$,

\[
(6.26) \quad \sup_{\eta} \int |K^{\text{good}}(\xi,\eta)| d\xi + \sup_{\xi} \int |K^{\text{good}}(\xi,\eta)| d\eta = \frac{2^{jn}}{|2^{j} (2^{j+10^2-2^{-j}})|^N} \lesssim 2^{-Mj}.
\]

**6.3. Estimate of $K^{\text{bad}}$; cases of Proposition 4.2** We shall claim that for $d = 2$

\[
(6.27) \quad \sup_{\eta} \int |K^{\text{bad}}(\xi,\eta)| d\xi = \begin{cases} 
O(2^{4j/2}) & \text{if rank}_{\text{skw}}(A) = 2 \text{ and rank}(A) = 2 \\
O(2^{j/2}) & \text{if rank}_{\text{skw}}(A) = 0, \text{ namely } A = cI
\end{cases}
\]

where we can switch the role of $\xi$ and $\eta$.

**Proof of (6.27).** We let $d = 2$. It suffices to deal with $K^{\text{bad}}$ supported on the set

\[
(6.28) \quad S^{\text{bad}} = \{ (\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle A^T (\xi - \eta), E(\xi - \eta) \rangle \leq |\xi - \eta|^{2j+10^2-2^{-j}} \}.
\]

By $\langle (\xi - \eta), AE(\xi - \eta) \rangle = \langle AE(\xi - \eta), (\xi - \eta) \rangle$, we observe that in (6.25)

\[
2 \langle A^T (\xi - \eta), E(\xi - \eta) \rangle = 2 \langle (\xi - \eta), AE(\xi - \eta) \rangle = \langle (EA + (EA)^T) (\xi - \eta), (\xi - \eta) \rangle.
\]

Let $\zeta = \xi - \eta$. Since $(EA + (EA)^T)$ is symmetric matrix, $(EA + (EA)^T) = QDQ^T$ makes the quadratic form

\[
(6.29) \quad \langle (EA + (EA)^T) \zeta, \zeta \rangle = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} Q^T \zeta, Q^T \zeta = c_1 v_1^2 + c_2 v_2^2
\]
where \( Q = (q_1, q_2) \) is the orthogonal matrix whose columns \( q_1 \perp q_2 \) are eigenvectors of \( EA + (EA)^T \) and \( v = Q^T \zeta \). By inserting \( \xi - \eta = \zeta \) into \( \zeta \) of (6.29) with \( (\xi, \eta) \in S^{bad} \) in (6.28) and \( |\xi - \eta| \lesssim 1 \) in (6.19),

\[
S^{bad} \subset \tilde{S} := \{(\xi, \eta) : c_1 |q_1 \cdot (\xi - \eta)|^2 + c_2 |q_2 \cdot (\xi - \eta)|^2 \leq |\xi - \eta| 2^{2j} 2^{-j} \}\text{ where } |\xi - \eta| \lesssim 1
\]

(6.30) \( \subset \bigcup_{2^{-m} \leq 1} \tilde{S}_m \) where \( \tilde{S}_m = \{(\xi, \eta) : c_1 |q_1 \cdot (\xi - \eta)|^2 + c_2 |q_2 \cdot (\xi - \eta)|^2 \leq 2^{-m} 2^{2j} 2^{-j} \} \).

In (6.25), we have the sub-level set estimate

\[
|K^{bad}(\xi, \eta)| \lesssim 2^{2j} \int_{\mathbb{R}^2} \psi \left( \frac{|\xi + x| - |\eta + x|}{2^{j} 2^{-j}} \right) dx \lesssim 2^{2j} 2^{2j} 2^{-j} \left( \xi - \eta \right)
\]

as \( |\nabla (|\xi + x| - |\eta + x|)| \approx |\xi - \eta| \) in (6.20). Using (6.31),

\[
\int_{\mathbb{R}^2} |K^{bad}(\xi, \eta)| d\xi = \int_{\mathbb{R}^2} \left| K^{bad}(\xi, \eta) \right| \sum_{2^{-m} \leq 1} ^{2j} \chi \left( \frac{|\xi - \eta|}{2^{-m}} \right) \chi \tilde{S}_m(\xi, \eta) d\xi
\]

(6.32) \( \lesssim \sum_{2^{-m} \leq 1} 2^{2j} \int_{\mathbb{R}^2} \left( \frac{|\xi - \eta|}{2^{-m}} \right) \chi \tilde{S}_m(\xi, \eta) d\xi \approx \sum_{2^{-m} \leq 1} 2^{2j} \int_{\mathbb{R}^2} \left( \frac{|\xi - \eta|}{2^{-m}} \right) \chi \tilde{S}_m(\xi, \eta) d\xi \approx 2^{4j} \).

(1) If \( \text{rank}(EA + (EA)^T) = 2 \), then both \( c_1 \) and \( c_2 \) above are nonzero in (6.30). Thus in (6.30),

we have the sub-level set estimates \( \int \chi \tilde{S}_m(\xi, \eta) d\xi \lesssim (2^{2j} 2^{-j}) 2^{-m} \) and \( \int \chi \left( \frac{|\xi - \eta|}{2^{-m}} \right) d\xi = O(2^{-2m}) \). Thus in (6.32)

\[
\sum_{2^{-m} \leq 1} 2^{2j} \int_{\mathbb{R}^2} \left( \frac{|\xi - \eta|}{2^{-m}} \right) \chi \tilde{S}_m(\xi, \eta) d\xi \lesssim \sum_{2^{-m} \leq 1} 2^{2j} \int_{\mathbb{R}^2} \left( \frac{|\xi - \eta|}{2^{-m}} \right) \chi \tilde{S}_m(\xi, \eta) d\xi \approx 2^{4j} \).
\]

(6.33)

(2) If \( \text{rank}(EA + (EA)^T) = 0 \), by using only \( \int \chi \left( \frac{|\xi - \eta|}{2^{-m}} \right) d\xi = O(2^{-2m}) \) for (6.32),

\[
\sum_{2^{-m} \leq 1} 2^{2j} \int_{\mathbb{R}^2} \left( \frac{|\xi - \eta|}{2^{-m}} \right) \chi \tilde{S}_m(\xi, \eta) d\xi \lesssim \sum_{2^{-m} \leq 1} 2^{2j} \int_{\mathbb{R}^2} \left( \frac{|\xi - \eta|}{2^{-m}} \right) \chi \tilde{S}_m(\xi, \eta) d\xi \approx 2^{4j} \).
\]

By (6.32)-(6.34), we obtain (6.27).

The Schur’s test gives the desired bound. Hence the estimate (6.27) together with (6.16), (6.18), (6.26) yields (4.13) and (4.17) for the case of rank\( (A) = 2 \).

6.4. Proof of (4.7) for rank\( _{skw}(A) = 2 \) and rank\( (A) = 1 \). Suppose that \( A \) is a \( 2 \times 2 \) matrix such that rank\( _{skw}(A) = 2 \) but rank\( (A) = 1 \). We then start with \( A = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \) with \( a \neq 0 \). Let

\[
U = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \text{ so that } AU^{-1}x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}x.
\]
By change of variable $\xi \rightarrow U^T \xi$, we treat $U^\lambda f(x, t)$ given by

$$T_j^\lambda f(U^{-1} x, t) = \lambda \int e^{2\pi i((x, \xi) + t(U^T \xi + (x_1, x_1)))} \chi(t) \psi(U^{-1} x) \left( \frac{t[U^T \xi + (x_1, x_1)]}{2 \lambda} \right) \hat{f}(U^T \xi) d\xi.$$  

Then the integral kernel of $[U^\lambda J^\lambda]$ is

$$K(\xi, \eta) = \lambda^2 \int e^{2\pi i \lambda \Phi(x, t, \xi, \eta)} L(x, t, \xi, \eta) dx dt$$

where the phase $\Phi$ and the amplitude $L$ are given by

$$\Phi(x, t, \xi, \eta) = ((\xi - \eta), x) + t \left( |U^T \xi + (x_1, x_1)| - |U^T \eta + (x_1, x_1)| \right)$$

$$L(x, t, \xi, \eta) = \psi(x)^2 \chi(t)^2 \left( \frac{t[U^T \xi + (x_1, x_1)]}{2 \lambda} \right) \chi \left( \frac{t[U^T \eta + (x_1, x_1)]}{2 \lambda} \right)$$

respectively. Set $L(x, t, \xi, \eta) = L_1(x, t, \xi, \eta) + L_2(x, t, \xi, \eta)$ such that

$$L_1(x, t, \xi, \eta) = L(x, t, \xi, \eta) \psi \left( \frac{|\xi_2 - \eta_2|}{2^j \lambda^{-1}} \right) \psi \left( \frac{|U^T \xi + (x_1, x_1)| - |U^T \eta + (x_1, x_1)|}{2^j \lambda^{-1}} \right)$$

$$L_2(x, t, \xi, \eta) = L(x, t, \xi, \eta) \left( 1 - \psi \left( \frac{|\xi_2 - \eta_2|}{2^j \lambda^{-1}} \right) \psi \left( \frac{|U^T \xi + (x_1, x_1)| - |U^T \eta + (x_1, x_1)|}{2^j \lambda^{-1}} \right) \right).$$

Split $K(\xi, \eta) = K_1(\xi, \eta) + K_2(\xi, \eta)$

$$K_1(\xi, \eta) = \lambda^2 \int e^{2\pi i \lambda \Phi(x, t, \xi, \eta)} L_1(x, t, \xi, \eta) dx dt.$$

In view of $\partial_{\xi_2} \Phi(x, t, \xi, \eta) = \xi_2 - \eta_2$ and $\partial_\eta \Phi(x, t, \xi, \eta) = |U^T \xi + (x_1, x_1)| - |U^T \eta + (x_1, x_1)|$, apply integration by parts with respect to $dt$ or $d\xi$ for $K_2$ to obtain

$$\int |K_2(\xi, \eta)| d\xi \ (\text{ord} \eta) \lesssim 2^{-j^N}.$$

Compute the measure $d\xi$ or $d\eta$

$$\int |K_1(\xi, \eta)| d\xi \ (\text{ord} \eta) \lesssim \lambda^2 (2^j \lambda^{-1})(2^j \lambda^{-1}) \lesssim 2^{2j^c}.$$

These two estimates show (7.1) for the case $\text{rank}_{\text{skew}}(A) = 2$ with $\text{rank}(A) = 1$.

7. Case of Skew-Symmetric Rank 1

There remains the proof of (1.10) in Proposition 1.2 which is the case

$$\text{rank}_{\text{skew}}(A) = 1.$$  

Let $\text{rank}(A) = 2$. By (1-2) of Proposition 2.4, $A$ is orthogonally similar to

$$(7.1) \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \text{ with } c \neq 0.$$
Let \( \text{rank}(A) = 1 \). By (1-3) of Proposition \([4.3]\) \( A \) is orthogonally similar to the matrix

\[
(7.2) \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \quad \text{with} \ c \neq 0.
\]

By Lemma \([2.2]\) it suffices to work with \((7.1)\) and \((7.2)\). We first consider the case of \((7.2)\).

### 7.1. Case rank_{skw}(A) = 1 and rank(A) = 1.

**Proof of the case rank(A) = 1 of (4.3) in Proposition \([4.3]\).** Assume \( A \) is of the form \((7.2)\). Recall \((4.3)\),

\[
T^2 g(x,t) = \lambda \int \chi(t) \psi(x) e^{2\pi i \lambda(x,\xi) + t|\xi + A(x)|} \chi \left( \frac{\lambda t|\xi + A(x)|}{2j} \right) \hat{g}(\xi) d\xi.
\]

We claim that

\[
(7.3) \quad \| [T^2]^nT^2 \|_{L^2(\mathbb{R}^2)} \to L^2(\mathbb{R}^2 \times \mathbb{R}) \lesssim 2^{1/2+j}.
\]

We split our operator \( T^2 \) as \( T_1 + T_2 \) where

\[
T_s g(x,t) = \lambda \int \chi(t) \psi(x) e^{2\pi i \lambda(x,\xi) + t|\xi + c \psi_2, \xi\rangle} \psi_1 \left( \frac{\xi_2}{\xi} \right)^{2j} \chi \left( \frac{\lambda t|\xi + c \psi_2, \xi\rangle}{2j} \right) \hat{g}(\xi) d\xi
\]

where \( \psi_1 = \psi \) and \( \psi_2 = 1 - \psi \). We claim that for \( s = 1, 2 \),

\[
(7.4) \quad \| [T_s]^nT_s \|_{L^2(\mathbb{R}^2)} \to L^2(\mathbb{R}^2 \times \mathbb{R}) \lesssim 2^{1/2+j}.
\]

It suffices to work with the integral kernel of \([T_s]^nT_s\) given by

\[
L_s(\xi, \eta) = \lambda^2 \int e^{2\pi i \lambda(x,\xi) + t|\xi + c \psi_2, \xi\rangle} \psi \left( \frac{\xi_2}{\xi} \right)^{2j} \chi \left( \frac{\lambda t|\xi + c \psi_2, \xi\rangle}{2j} \right) dx dt.
\]

Indeed,

- on the support of \( 1 - \psi \left( \frac{\xi_1 - \eta_2}{2j/\lambda} \right) \),

\[ |\partial_t \Phi(x,t,\xi,\eta)| \geq \left| (|\xi_1 + c \psi_2, \xi\rangle) - (\eta_1 + c \psi_2, \eta_2\rangle) \right| \geq 2^{1/2}/\lambda \]

- on the support of \( 1 - \psi \left( \frac{\xi_1 - \eta_2}{2j/\lambda} \right) \),

\[ |\partial_{x_2} \Phi(x,t,\xi,\eta)| = |\xi_2 - \eta_2| \geq 2^{1/2}/\lambda. \]

For both cases above, both of the corresponding integral kernels \( L(\xi, \eta) \) have \( L^1 \) norm \( O(2^{-jN}) \) for large \( N \).
For $s = 1$, we compute the measure of the support of integral in (7.4)
\[
(7.5) \quad \int |L_1(\xi, \eta)|d\xi \lesssim \lambda^2 \times [d\xi_1] \times [d\xi_2] \times [dx] \approx \lambda^2 \times [(2^{j/\lambda})^2] \times [(2^j/\lambda)2^{-j/2}] \times 1 \approx 2^{j/2}2^r.
\]
For $s = 2$, we have on the support of $\psi_s\left(\frac{\xi_2}{(2^{\lambda/2})^{2^{j/2}}}\right) \chi\left(\frac{M(\xi_1 + cx_2, \xi_2)}{\lambda}\right)$,
\[
\left|\partial_{\xi_2}|(\xi_1 + cx_2, \xi_2)| - |(\eta_1 + cx_2, \eta_2)|\right| = \left|\frac{\xi_2}{|(\xi_1 + cx_2, \xi_2)|}\right| \approx \frac{|\xi_2|}{\lambda} \geq 2^{-j/2},
\]
which gives the sublevel set estimates of $d\xi_2$
\[
\left\{\xi_2 : |(\xi_1 + cx_2, \xi_2)| - |(\eta_1 + cx_2, \eta_2)| < 2^r/\lambda\right\} \lesssim \frac{(2^r/\lambda)}{2^{-j/2}}.
\]
Then
\[
(7.6) \quad \int |L_2(\xi, \eta)|d\xi \lesssim \lambda^2 \times [d\xi_1] \times [d\xi_2] \times [dx] \approx \lambda^2 \times [(2^{j/\lambda})^2] \times [(2^j/\lambda)2^{-j/2}] \times 1 \approx 2^{j/2}2^r.
\]
By (7.5) and (7.6), we obtain (7.3). Therefore we obtain (7.3).

7.2. Case $\text{rank}_{skw}(A) = 1$ and $\text{rank}(A) = 2$ of (4.9). By (1-2) of Proposition 2.1 and Lemma 2.2, it suffices to deal with the matrix $A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ in (7.1). We shall apply the following well-known results of the two-sided fold singularities.

Proposition 7.1 (Two-Sided Fold Singularity). Consider a family of operators
\[
S^\lambda f(x) = \int e^{i\lambda\Phi(x,y)}\psi(x,y)f(y)dy \quad f \in L^2(\mathbb{R}^n)
\]
where $\psi$ is a smooth function having a compact support. Assume that $C_\Phi = \{x, \Phi_x, y, -\Phi_y\}$ is a two-sided folding canonical relation, that is,
- for each point $(x_0, y_0) \in \text{supp}(\psi)$,
  \[
  \text{rank}(\Phi''_{xy}(x_0, y_0)) \geq n - 1
  \]
- for unit vector $U$ and $V$ in $\mathbb{R}^n$,
  \[
  \Phi''_{xy}(x_0, y_0)V = 0 \Rightarrow \langle V, \nabla \rangle \Phi''_{xy}(x_0, y_0) \geq c > 0
  \]
  \[
  U^T\Phi''_{xy}(x_0, y_0) = 0 \Rightarrow \langle U, \nabla \rangle \Phi''_{xy}(x_0, y_0) \geq c > 0.
  \]
Then
\[
\|S^\lambda\|_{op} \lesssim \lambda^{-(n-1)/2}\lambda^{-1/3}.
\]

By using Proposition 7.1, we claim the following lemma for the proof of the case $\text{rank}_{skw}(A) = 1$ and $\text{rank}(A) = 2$ of (4.9).
Lemma 7.1. Let \( \Phi(x, t, y) = (A(x), y) + t|x + y| \) where \( A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \) as in (7.7). Set

(7.7) \( S^\lambda f(x, t) = \chi(t) \psi(x) \int e^{it\Phi(x, t, y)} \chi(|x + y|) f(y) dy. \)

Then

(7.8) \( \|S^\lambda\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2 \times \mathbb{R})} \lesssim \lambda^{-1/2} \lambda^{-1/3}. \)

Proof of (7.8). Split \( S^\lambda = S^\lambda_1 + S^\lambda_2 \) where

\[ S^\lambda_1 f(x, t) = \chi(t) \psi(x) \int e^{it\Phi(x, t, y)} \psi \left( \frac{|x + y|}{\epsilon} \right) \chi(|x + y|) f(y) dy, \]

\[ S^\lambda_2 f(x, t) = \chi(t) \psi(x) \int e^{it\Phi(x, t, y)} \left( 1 - \psi \left( \frac{|x + y|}{\epsilon} \right) \right) \chi(|x + y|) f(y) dy. \]

We first claim that

(7.9) \( \|S^\lambda_2\|_{op} \lesssim \lambda^{-1}. \)

To simplify the notation, we let \( b_1 = x_1 + y_1, b_2 = x_2 + y_2 \) and \( b = (b_1, b_2) \) where \( |b| = |x + y| \approx 1 \) in the support of the integral (7.7).

Proof of (7.9). In view of (6.4) and (6.5), for \( A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \),

\[ \Phi''(x_1, t)(y_1, y_2) = \left( 1 + t \frac{b_2^2}{|b|^2} + \frac{c - t b_1 b_2}{|b|^2} \right) \left( \begin{array}{c} 1 \\ b_1 \\ b_2 \end{array} \right) \quad \text{and} \quad \det \Phi''(x_1, t)(y_1, y_2) = \frac{1}{|b|} \left( \begin{array}{ccc} 1 & c & t b_2 \\ b_1 & b_2 & 0 \end{array} \right) \]

\[ \Phi''(x_2, t)(y_1, y_2) = \left( 1 + t \frac{b_1^2}{|b|^2} + \frac{1 + \frac{c}{|b|^2}}{|b|^2} \right) \left( \begin{array}{c} 1 \\ b_1 \\ b_2 \end{array} \right) \quad \text{and} \quad \det \Phi''(x_2, t)(y_1, y_2) = \frac{1}{|b|} \left( \begin{array}{ccc} 1 & 1 & -t b_1 \\ b_1 & b_2 & b_2 \end{array} \right). \]

From this we observe that \( \det \left( \Phi''(x_1, t)(y_1, y_2) \right) = \frac{1}{|b|} \left( b_2 \left( 1 + \frac{c}{|b|^2} \right) - c b_1 \right) \neq 0 \) or \( \det \left( \Phi''(x_2, t)(y_1, y_2) \right) = -\frac{b_1}{|b|^2} \left( 1 + \frac{c}{|b|^2} \right) \neq 0 \) unless \( b_1 = 0 \) and \( t = -|b| \). This implies

\[ |\det \left( \Phi_{x_1, t}(y_1, y_2) \right) | \gtrsim \epsilon \quad \text{or} \quad |\det \left( \Phi_{x_2, t}(y_1, y_2) \right) | \gtrsim \epsilon \]

on the support of the kernel of \( S^\lambda_2 \): \( 1 - \psi \left( \frac{|x_1 + y_1|}{\epsilon} \right) \psi \left( \frac{|x + y| + t}{\epsilon} \right) \) where \( |b_1| \geq \epsilon \) or \( |b| + t \geq \epsilon \).

Hence we can use the non-degeneracy condition of the hessian matrix to obtain (7.9). \( \square \)

We next claim that

(7.10) \( \|S^\lambda_1\|_{op} \lesssim \lambda^{-1/2} \lambda^{-1/3}. \)

Proof of (7.10). The support condition of the integral kernel for \( S^\lambda_1 \) is given by

(7.11) \( \psi \left( \frac{|x_1 + y_1|}{\epsilon} \right) \psi \left( \frac{|x + y| + t}{\epsilon} \right). \)
where $b_1 = x_1 + y_1 = o(1)$ and $|b| = |x + y| = -t + o(1)$. Recall the mixed hessian of (6.3): 
\[\Phi''_{(x_1x_2)(y_1y_2)} = \begin{pmatrix} a_{11} + \frac{b_1^2}{|b|^3} & a_{12} - \frac{b_1b_2}{|b|^3} \\ a_{21} - \frac{b_1b_2}{|b|^3} & a_{22} + \frac{b_1^2}{|b|^3} \end{pmatrix} \text{ and } \det \left( \Phi''_{(x_1x_2)(y_1y_2)} \right) = \det(A) + \frac{t}{|b|^3} \left( \frac{(A + A^T)^2}{2} b, b \right)\]

Thus, we have to check the fold singularity of 
\[(x, \Phi_x, y, -\Phi_y) \text{ at } b_1 = 0 + o(1) \text{ and } t = -|b| + o(1).\]

In particular we treat the case 
(7.12) \[b_1 = 0 \text{ and } t = -|b| = |b_2| \approx 1 \text{ where } a_{11} = a_{22} = 1 \text{ and } a_{12} = c, a_{21} = 0.\]

The mixed hessian is given by 
\[\Phi''_{(x_1x_2)(y_1y_2)} = \begin{pmatrix} 0 & c/2 \\ 0 & 1 \end{pmatrix} + o(1)\]

Let us find the kernel vector fields $v$ and $u$ satisfying 
\[\Phi''_{(x_1x_2)(y_1y_2)} v = 0 \text{ and } u^T \Phi_{x_1x_2y_1y_2} = 0.\]

Then $v = (1, 0) + o(1)$ and $u = (-1, c) + o(1)$. For $(A + A^T)/2 = \begin{pmatrix} a_{11} & c/2 \\ c/2 & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & c/2 \\ c/2 & 1 \end{pmatrix}$, we have 
\[
\det \left( \Phi''_{(x_1x_2)(y_1y_2)} \right) = \det(A) + \frac{t}{|b|^3} \left( \frac{(A + A^T)^2}{2} b, b \right) = 1 + \frac{t}{|b|^3} \left( \frac{a_{11}b_1^2 + a_{22}b_2^2 + cb_1b_2}{|b|^3} \right)
\]

where $b_1 = x_1 + y_1$ and $b_2 = x_2 + y_2$. This vanishes for the case of (7.12). Observe that $\nabla_x \det \Phi_{xy} = \nabla_y \det \Phi_{xy}$ whose components are 
\[
\frac{\partial}{\partial x_1} \det \Phi''_{(x_1x_2)(y_1y_2)} = \frac{t}{|b|^3} \left( |b|^2 (a_{11}2b_1 + cb_2) - 3b_1(a_{11}b_1^2 + cb_1b_2 + a_{22}b_2^2) \right) = t b_2 c / |b|^5
\]

\[
\frac{\partial}{\partial x_2} \det \Phi''_{(x_1x_2)(y_1y_2)} = \frac{t}{|b|^3} \left( |b|^2 (a_{22}2b_2 + cb_1) - 3b_2(a_{22}b_2^2 + cb_1b_2 + a_{11}b_1^2) \right) = -t b_1^2 / |b|^5.
\]

Thus for (7.12), 
\[
(\nabla_x \det \Phi_{xy}) \cdot v = \frac{t}{|b|^3} \left( b_2^2 c, -b_1^2 \right) \cdot (1, 0) + o(1) = \frac{ctb_2^2}{|b|^5} + o(1) \approx \frac{ctb_2^2}{|b|^5} \neq 0
\]

\[
(\nabla_y \det \Phi_{xy}) \cdot u = \frac{t}{|b|^3} \left( b_2^2 c, -b_1^2 \right) \cdot (-1, c) + o(1) = \frac{-2ctb_1^2}{|b|^5} + o(1) \approx \frac{-2ctb_1^2}{|b|^5} \neq 0.
\]

Hence $(x, \Phi_x, y, -\Phi_y)$ on the support of (7.11) has two sided fold singularity. Freeze $t$ and apply Proposition 7.13 to obtain (7.10). □

Hence combined with (7.14), we finish the proof of (7.8). □

The lemma 7.13 implies that $\|T_j^A\|_{op} = \|\lambda S^A\|_{op} \preccurlyeq \lambda^{1/6}$ with $\lambda = 2^j$ for the case $\text{rank}(A) = 2$ of (4.13). This combined with the case $\text{rank}(A) = 1$ done in Section 7.14 completes the proof of (4.5) in Proposition 4.2. Therefore we finished the proof of Proposition 4.2.
8. Lower Averages on Variable Hyperplanes

Lemma 8.1. Recall $\mathcal{M}_A^4$ in (1.3). Suppose that $\text{rank}(\text{skw}(A)) = 1$. Then there exists $c > 0$ such that

\begin{align}
(8.1) \quad \|\mathcal{M}_A^4\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} &\geq c\delta^{-1/6} \text{ if } \text{rank}(A) = 2, \\
(8.2) \quad \|\mathcal{M}_A^4\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} &\geq c\delta^{-1/4} \text{ if } \text{rank}(A) = 1.
\end{align}

Proof of Lemma 8.1. In view of (7.1) and Lemma 2.2 it suffices to regard $A$ as $I_3 = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$. Then our average is given by

$$
A_3 f(x, x, t) = \frac{1}{\delta} \int_{S_3^1} f(x - ty, x - \langle l, x, ty \rangle) dy
$$

By change of variable $x_3 - |x|^2/2 \to x_3$ and $f(x, x_3) = g(x, x_3 - |x|^2/2)$, it suffices to work with

$$
A_3 f(x, x_3, t) = \frac{1}{\delta} \int_{S_3^1} g(x - ty, x_3 - |ty|^2 - cx_2 ty_1) dy.
$$

Note $y \in S_3^1$ in (8.3) if and only if $y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ with $|u_1| \leq 100$ and $|u_2| \leq 100$. Then for $(x, x_3) \in B$ in (8.3),

$$
A_3 f(x, x_3, t) \geq \int_0^{2\pi} g(x_1 - t \cos \theta + O(\delta), x_2 - t \sin \theta + O(\delta), x_3 - t^2 - cx_2 t \cos \theta + O(\delta)) d\theta
$$

where we used $\cos \theta = 1 - O(\theta^2)$ and $\sin \theta = \theta + O(\theta^3)$ with $|\theta| \leq \delta^{1/3}$. Next choose $t = t(x, x_3)$ for each $(x, x_3) \in B$ satisfying

$$
x_3 - t^2 - cx_2 t = 0.
$$

For this $t = t(x, x_3)$, from the support condition of $B$ and (8.4), we see (8.5) is bounded away from $\frac{\delta^{1/3}}{10^3}$ if $(x, x_3) \in B$. From this lower bound combined with the measure $|B| \geq \delta^{1/3}$, we have that for $t = t(x, x_3)$,

$$
\int_B |A_3 f(x, x_3, t)|^p dx dx_3 \geq \frac{\delta^{p/3}}{10^p} \delta^{1/3}
$$

while

$$
\int |g(x, x_3)|^p dx dx_3 \leq 20\delta^{1/3} \delta.
$$

Therefore, we obtain the desired lower bound as

$$
\|\mathcal{M}_A^4\|_{L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)} \gtrsim \frac{\delta^{p/3}}{\delta}
$$

This implies (8.1) for $p = 2$. \qed
Proof of (8.3). By (7.2), let \( I_c = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \). Then we can write

\[
A_{\delta}f(x, x_3, t) = \frac{1}{\delta} \int_{S_3^2} f(x - ty, x_3 - cx_2ty_1) \, dy \text{ where } y = c(\theta) + O(\delta) \text{ as above.}
\]

(8.7) Take

\[
g(u_1, u_2, u_3) := \psi(u_1) \psi\left(\frac{u_2}{10\delta^{1/2}}\right) \psi\left(\frac{u_3}{10\delta^{3/2}}\right).
\]

Set \( B := \{(x, x_3) : |x_1| \leq 5, |x_2| \leq \delta^{1/2} \text{ and } 1 \leq |x_3| \leq 5\} \). Then for \((x, x_3) \in B\),

\[
A_{\delta}g(x, x_3, t) \geq \int_0^{2\pi} g\left(x_1 - t \cos \theta + O(\delta), x_2 - t \sin \theta + O(\delta), x_3 - cx_2t \cos \theta + O(\delta^{3/2})\right) \, d\theta
\]

(8.8)

\[
\geq \int_0^{\delta^{1/2}} g\left(x_1 - t + O(\delta), x_2 - \theta + O(\delta), x_3 - cx_2t + O(\delta^{3/2})\right) \, d\theta.
\]

We used \( \cos \theta = 1 - O(\theta^2) \) with \( |\theta| \leq \delta^{1/2} \) and \( |x_2| \leq \delta^{1/2} \) in the last component of (8.9). Next choose \( t = t(x, x_3) \) for each \((x, x_3) \in B\) satisfying

\[
x_3 - cx_2t = 0.
\]

Then for this \( t = t(x, x_3) \), from the support condition of \( B \) and \( g \) in (8.8), the integral (8.9) is bounded away from \( \frac{\delta^{1/2}}{100} \) if \((x, x_3) \in B\). Combined with \( |B| \geq \delta^{1/2} \), for this \( t = t(x, x_3) \),

\[
\int_B |A_{\delta}f(x, x_3, t)|^p \, dx \geq \frac{\delta^{p/2}}{10^p \delta^{1/2}}
\]

while

\[
\int_B |g(x, x_3)|^p \, dx \leq 20 \delta^{1/2} \delta^{3/2}.
\]

Therefore

\[
\|M_{\delta}^A\|_{L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)} \geq \frac{\delta^{p/2}}{\delta^{3/2}}.
\]

(8.10) which implies (8.2) for \( p = 2 \). \( \square \)

**Lemma 8.2.** Suppose \( A \) is a nonzero matrix. If rank(skew(A)) = 0, there is \( c > 0 \) such that

\[
\|M_{\delta}^A\|_{L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)} \geq c \delta^{-1/p} \text{ for all } 1 \leq p < \infty.
\]

(8.11)

**Proof.** Take \( A = I \) from (1-4) in Lemma 2.1 Set \( f(x, x_3) = \tilde{f}(x, x_3 - \frac{1}{2}|x|^2) \) as

\[
A_{\delta}f\left(x, x_3 + \frac{1}{2}|x|^2, t\right) = \frac{1}{\delta} \int_{S_3^2} f\left(x - ty, x_3 + \frac{1}{2}|x|^2 - \langle A(x), ty\rangle\right) \, dy
\]

\[
= \frac{1}{\delta} \int_{S_3^2} \tilde{f}\left(x - ty, x_3 - \frac{1}{2}|ty|^2\right) \, dy.
\]

Thus we can work with \( A_{\delta}f(x, x_3, t) = \frac{1}{\delta} \int_{S_3^2} f\left(x - ty, x_3 - \frac{1}{2}|ty|^2\right) \, dy \). Choose

\[
f(u_1, u_2, u_3) = \psi\left(u_1/10\right) \psi\left(u_2/10\right) \psi\left(u_3/100\delta\right) \text{ and } B = \{(x, x_3) : |x| \leq 1, 1 \leq x_3 \leq 2\}.
\]

(8.12)
For each \((x, x_3) \in B\), choose \(t\) such that \(\frac{1}{2} t^2 = x_3\). Then
\[
x_3 - \frac{1}{2} |ty|^2 = \frac{1}{2} t^2 (1 - |y|^2) \leq 10 \delta\text{ where }y \in S_1^2.
\]
Hence, from (8.13) with the support condition (8.12), we have that
\[
A_\delta f(x, x_3, t) \geq 1 \text{ for } (x, x_3) \in B \text{ with } B \text{ in (8.12)}.
\]
Thus for \(t\) with \(\frac{1}{2} t^2 = x_3\) in (8.13),
\[
\int |A_\delta f(x, x_3, t)|^p dx dx_3 \geq 1.
\]
From (8.12), we have \(\|f\|_{L^p(\mathbb{R}^3)} \leq \delta\). Combined with (8.14), we obtain
\[
\|M^A_\delta\|_{L^p(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3)} \gtrsim \delta
\]
which is the desired lower bound. \(\square\)

9. Proof of Corollary 9.1

Corollary 9.1. Suppose that \(A\) is a \(2 \times 2\) invertible matrix. Then we have the following \(L^2\)-Sobolev inequalities:
\[
\|A S^1(A) f\|_{L^2([1, \infty])} \leq \|f\|_{L^2_{1+\alpha}(\mathbb{R}^{2+\alpha})} \left\{ \begin{array}{ll}
2 & \text{for } \alpha > 0 \text{ if } \text{rank}_{\text{skw}}(A) = 2 \\
1 & \text{for } \alpha > 1/6 \text{ if } \text{rank}_{\text{skw}}(A) = 1 \\
0 & \text{for } \alpha > 1/2 \text{ if } \text{rank}_{\text{skw}}(A) = 0.
\end{array} \right.
\]

Proof of Corollary 9.1. Take a large \(C \geq 1\) and set
\[
\psi_1(\xi, \xi_3) := \psi \left( \frac{\xi_1}{|\xi|/C} \right) \text{ and } \psi_2(\xi, \xi_3) := 1 - \psi \left( \frac{\xi_1}{|\xi|/C} \right).
\]
We split \(\hat{f}(\xi, \xi_3) = \hat{f}_1(\xi, \xi_3) + \hat{f}_2(\xi, \xi_3)\) where \(\hat{f}_s(\xi, \xi_3) = \psi_s(\xi) \hat{f}(\xi, \xi_3)\). Let
\[
\tilde{g}_s(\xi, \xi_3) = [\xi, \xi_3]^{-1/2+\alpha} \hat{f}_s(\xi, \xi_3)\text{ where }[\xi, \xi_3] = |\xi| + |\xi_3| + 1.
\]
Choose \(\alpha > c(A)\) with \(c(A) = 0, 1/6, 1/2\) according to \(\text{rank}_{\text{skw}}(A) = 2, 1, 0\). For each \(s = 1, 2\), observe that
\[
|A S^1(A) f_s(x, x_3, t)| \leq \sum_{j=0}^{\infty} 2^{-j/2} \psi(x, x_3) \psi(t) \int e^{2\pi i (x, x_{d+1}) + (\xi, \xi_{d+1}) + t(\xi, \xi_{d+1}, A(x))} \chi \left( \frac{t|\xi + \xi_{d+1} A(x)|}{2} \right) \psi_s(\xi, \xi_3) d\xi d\xi_{d+1}.
\]
Thus
\[
|A S^1(A) f_s(x, x_3, t)| \leq \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} 2^{k/2} T_{m_{j,k}} g_s(x, x_3, t)
\]
where for the symbol \(m_{j,k}\) defined in (8.7).
\[
m_{j,k}(x, x_3, t, \xi, \xi_3) := m_{j,k}(x, x_3, t, \xi, \xi_3) h_s(\xi, \xi_3) \text{ with } [\xi, \xi_3]^{1/2-\alpha} = 2^{(j+k)/2} h_s(\xi, \xi_3).
\]
We can utilize the same procedure of (5.15)–(5.21) as we can switch the sum over $k$ with the square sum over $k$. Combined with $\sum_{k=1}^{2} \| \hat{g}_k \|_{L^2(\mathbb{R}^{2+1})} \lesssim \| f \|_{L^2_{x,1/2}\cdot\psi(\mathbb{R}^{2+1})}$, it suffices to prove that for each $s = 1, 2$,

$$\| T_{m,s} \|_{L^2(\mathbb{R}^{2+1}) \rightarrow L^2(\mathbb{R}^{2+1} \times \mathbb{R})} \lesssim 2^{-\alpha_j} 2^{s(A_j)} 2^{s_j}$$

for a sufficiently small $\epsilon > 0$

**Proof of (9.3).** When applying Proposition 5.11 for our case, we consider the only additional factor $2^{-\alpha_j}$ of (9.3) arising from the symbol $h_s(\xi, \xi_3)$ in (9.2). Note that if $s = 1$ then $2^{j+k} \approx |\xi + \xi_{d+1} A(x)| \approx |\xi_3|$. If $s = 2$, then $2^{j+k} \approx |\xi + \xi_{d+1} A(x)| \leq |\xi, \xi_3| \approx |\xi_3|$. Thus,

$$h_s(\xi, \xi_3) := \left( \frac{2^{j+k}}{|\xi, \xi_3|} \right)^{\alpha_s - \frac{1}{2}} 2^{-\alpha_s(j+k)} \approx \begin{cases} 2^{-\alpha_s(j+k)} = O(2^{-\alpha_s}) & \text{if } s = 1 \\ O \left( \left( \frac{|\xi_3|}{\delta} \right)^{1/2} 2^{-\alpha_s} \right) & \text{if } s = 2. \end{cases}$$

Hence for $s = 1$, we have (9.2). For the case $s = 2$, it suffices to consider the case $|\xi_3| \gg 2^j$. For this case we apply the bound $\frac{1}{\delta}$ of (9.2) to remove $\left( \frac{|\xi_3|}{\delta} \right)^{1/2}$ because $|\lambda| = |\xi_3| \gtrsim |\xi| + |\xi_3| \gtrsim |\xi + \xi_3 A(x)| \approx 2^{j+k} \geq 2^j$.

We finished the proof of Corollary 9.1

10. **Symmetric Rank and Concluding Remarks**

10.1. **Symmetric Rank and Nikodym Maximal Functions.** In [12], the proof of Theorem 1.1 regarding the criterion $\text{rank}_{\text{sym}}(A)$ for the bound $\| N^A_k \|_{L^2(\mathbb{R}^2)} \rightarrow L^2(\mathbb{R}^2)$ is based on the estimates of one dimensional oscillatory integral operators arising from the Group Fourier transform of the measure supported on the lines $\{(1, k\delta, 0) : t \in [0, 2\pi]\}$ for $k = 1, \ldots, 1/\delta$. However, in this section we briefly describe why $A + A^T$ appears in the similar frame work of the proof in Sections 3 and 7. Recall

$$e(\theta) = (\cos \theta, \sin \theta) \text{ and } \partial_\theta e(\theta) = e(\theta)^T = E e(\theta).$$

Then the rotation matrix is $R_\theta = (e(\theta)|E e(\theta))$. For $f \in S(\mathbb{R}^3)$, set

$$A_3 f(x, x_3, \theta) = \int f(x - R_\theta y, x_3 - \langle A(x), R_\theta y \rangle) \psi(y_1) \psi(y_2/\delta) / \delta dy_1 dy_2$$

$$= \int e^{\frac{2\pi i}{\delta} (\xi, \xi_3) \cdot (x, x_3)} \hat{\psi}_{\delta} (\xi + \xi_3 A(x), e(\theta)) \hat{\psi} \left( \frac{\xi + \xi_3 A(x)}{\delta} \right) f(\xi, \xi_3) d\xi d\xi_3.$$

Then we have $N^A_3 f(x, x_3) = \sup_{\theta \in [0, 2\pi]} A_3 f(x, x_3, \theta)$. For $j \in \mathbb{Z}$, set

$$T_{a_j}(f(x, x_3, \theta) = \int e^{\frac{2\pi i}{\delta} (\xi, \xi_3) \cdot (x, x_3)} a_j(x, x_3, \xi, \xi_3) \hat{f}(\xi, \xi_3) d\xi d\xi_3$$

where

$$a_j(x, x_3, \theta, \xi, \xi_3) = \hat{\psi} \left( \frac{\xi + \xi_3 A(x)}{\delta} \right) \chi \left( \frac{\xi + \xi_3 A(x)}{2^j} \right).$$

It suffices to deal with $N^A_3 f(x, x_3) = \sup_{\theta \in [0, 2\pi]} T_{a_j}(f(x, x_3, \theta)$ from the decomposition

$$N^A_3 f(x, x_3) \leq \sum_{1 \leq 2^j \leq 1/\delta} N_j f(x, x_3).$$
By using the Sobolev imbedding inequality, we can control $L^\infty(d\theta)$ by $L^2_{1/2}(d\theta)$ so that
\begin{equation}
(10.2) \quad \int \sup_\theta |T_{\alpha_j} f(x, x_3, \theta)|^2 dx dx_3 \lesssim \int_0^{2\pi} \int |2^{j/2} T_{\alpha_j} f(x, \theta)|^2 dx dx_3 d\theta + \|f\|^2_{L^2(\mathbb{R}^2)}.
\end{equation}

with a slight modification of $a_j$. Note that
\begin{equation}
\|2^{j/2} T_{\alpha_j}\|_{L^2(\mathbb{R}^2) \to L^2_{1,\infty}(\mathbb{R}^3 \times [0, 2\pi])} \lesssim \sup_\lambda \|T^\lambda_j\|_{L^2(\mathbb{R}^2) \to L^2_{\infty, 0}(\mathbb{R}^3 \times [0, 2\pi])}
\end{equation}
where $T^\lambda_j$ is the oscillatory integral operator defined by
\begin{equation}
T^\lambda_j f(x) = \int 2^{j/2} \lambda e^{2\pi i \lambda (\xi, x)} \left( \frac{\langle \xi + A(x), e(\theta) \rangle}{2^{j/\lambda}} \right) \psi \left( \frac{\langle \xi + A(\theta), e(\theta) \rangle}{2} \right) \hat{f}(\xi) d\xi.
\end{equation}

We may change $Ax \to x$. Let us consider the only case $\lambda = 2^j$. The kernel $K(\xi, \eta)$ of $[T^\lambda_j]^* T^\lambda_j$ corresponding to (2.8) is given by
\begin{equation}
2^{3j} \int L^{2j+2} \varphi_{\Phi_{\text{tube}}}(x, \xi, \eta) \psi \left( \frac{\xi + x, e_\theta}{2^{j/2}} \right) \psi \left( \frac{\eta + x, e_\theta}{2^{j/2}} \right) \psi \left( \frac{\xi + x, e_\theta}{2} \right) \psi \left( \frac{\eta + x, e_\theta}{2} \right) dx d\theta
\end{equation}
whose phase function is
\begin{equation}
(10.3) \quad \Phi_{\text{tube}}(x, \xi, \eta) = \langle A^T (\xi - \eta), x \rangle.
\end{equation}

where $(A^{-1})$ is switched with $A$ as Lemma 2.1. The support of $x$ is contained the intersection of tubes
\begin{equation}
(-\xi + \tau_0) \cap (-\eta + \tau_0) \text{ where } \tau_0 = \{ x : |x \cdot e_\theta| < 1, |x \cdot e_\theta| < \delta \}.
\end{equation}

Thus, we see that

1. The direction of good oscillation is $e_\theta^\perp$, that is parallel to $(\xi - \eta)$.
2. The gradient of the phase function $\nabla_x \Phi_{\text{tube}}(x, \xi, \eta) = A(\xi - \eta)$ in (10.3).

The directional derivative along $e_\theta^\perp$ in (1) for the application of integration by parts yields
\begin{equation}
\nabla_x \Phi_{\text{tube}}(x, \xi, \eta) \cdot e_\theta^\perp = \left< A^T (\xi - \eta), \frac{(\xi - \eta)}{|\xi - \eta|} \right> = \left< \frac{1}{2} (A + A^T)(\xi - \eta), \frac{(\xi - \eta)}{|\xi - \eta|} \right>.
\end{equation}

The size of integral of $K(\xi, \eta)$ is determined by the measure of the singular region:
\begin{equation}
\mathcal{E}_{\text{tube}} = \left\{ (\xi, \eta) : \left| \left< (A + A^T)(\xi - \eta), \frac{(\xi - \eta)}{|\xi - \eta|} \right> \right| \lesssim \frac{2^{j/2}}{2^{2j}} \right\}.
\end{equation}

The best estimate $O(2^{(\epsilon - 1)j})$ of this measure is due to $\text{rank}(A + A^T) = 2$ as in (2.11). The worst case occurs at $\text{rank}(A + A^T) = 0$, that is the case of the skew–symmetric matrix $A = E$ for the Heisenberg plane, which prevents the overlapping tubes from being located is such a way that oscillations occur.

**Remark 10.1.** There is another evidence that the skew–symmetric structure in a certain space becomes an obstacle to a good enough distribution of tubes (for generating full dimensions) in it. In [10], Katz, Laba, and Tao considered
The five dimensional Heisenberg group $H = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \text{Im}(z_3) = \text{Im}(z_1 \overline{z_2})\}$, which is the two and half complex dimensional set.

- the four dimensional family $\mathcal{L}$ of the complex lines $\ell_{s,t,\alpha} := \{(\overline{sz} + t, z, sz + \alpha) : z \in \mathbb{C}\}$ for $s, t \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ so that $\mathcal{L}$ is the two complex dimensional set of lines.

They observed that $H$, containing every element of $\mathcal{L}$, is actually a complex version of the Besicovitch set in the three complex dimensional space $\mathbb{C}^3$. But the dimension of $H$ is two and half in the complex sense. This implies that $H$ is a counterexample of a complex analogue of the Kakeya conjecture.

10.2. Final Remarks. In view of (2.11) for the case $\lambda \approx 2^j$, we compare $\mathcal{E}_{\text{circle}}$ and $\mathcal{E}_{\text{tube}}$

\[
\mathcal{E}_{\text{tube}} = \left\{ (\xi, \eta) : \left| \left( (A + AT)(\xi - \eta), \frac{(\xi - \eta)}{\xi - \eta} \right) \right| \lesssim \frac{2^j}{2^j} \right\},
\]

\[
\mathcal{E}_{\text{circle}} = \left\{ (\xi, \eta) : \left| \left( EA + (EA)^T(\xi - \eta), \frac{(\xi - \eta)}{\xi - \eta} \right) \right| \lesssim \frac{2^j}{2^j} \right\}.
\]

These two singular region $\mathcal{E}_{\text{tube}}$ and $\mathcal{E}_{\text{circle}}$ are the places where the corresponding phase functions almost stop their oscillations. These two sets give rise to a definition of the symmetric and skew–symmetric rank condition for classifying the two maximal averages $\mathcal{N}_A^A$ and $\mathcal{M}_A^A$. However, we do not know the exact role of these symmetric (skew–symmetric) conditions in a more generalized notion of rotational curvature.

Remark 10.2. We are interested in the further questions regarding the following three operators associated with variable planes $\pi_A(x, x_{d+1})$:

- Classify the $L^2(\mathbb{R}^{d+1})$ bounds of $\mathcal{M}_A^A$ for $d \geq 3$ according to $A \in M_{d \times d}(\mathbb{R})$.
- Find the $L^3(\mathbb{R}^3)$ bound (we expect $\delta^{-c}$) of the Nikodym Maximal function $\mathcal{N}_A^A$ for $A = E$.
- Investigate the $L^p(\mathbb{R}^3)$ bound of the Bochner Riesz or Fourier extension operators via the Weyl functional calculus $m(X_1, X_2)$ with $X_1, X_2$ in the Heisenberg algebra $\mathfrak{h}_1$, which is given by the pseudo-differential operator with the symbol

$\alpha(x, x_3, \xi, \xi_3) = \sum_{j \geq 0} 2^{-j\delta} \psi(2^j(|\xi + \xi_3 A(x)| - 1))$.

Compare this with the well-known results of [15] in the spectral calculus version.

11. Appendix

11.1. Proof of Proposition 5.2. Let us restate the proposition.

Proposition 5.2 For $|\ell| \geq 100dj$, we find $c > 0$,

\[
\left\| 2^{k/2} T_{m, k} P_{j+k+\ell} \right\|_{L^2(\mathbb{R}^{d+1}) \rightarrow L^2(\mathbb{R}^{d+1} \times \mathbb{R})} \lesssim 2^{-c|\ell|}
\]

and

\[
\left\| P_{k_1} P_{k_2} \right\|_{L^2(\mathbb{R}^{d+1}) \rightarrow L^2(\mathbb{R}^{d+1})} \lesssim 2^{-c|k_1-k_2|}.
\]
The proof of Proposition 3.2 is based on the argument of M. Christ in [4], where he facilitated the non-isotropic dilations combined with the cancellation property of the singular kernels to establish the measure estimates in the nilpotent groups. Recall \( T_{m,j,k} f(x, x_{d+1}, t) \) in (11.6) where

\[
m_{j,k}(x, x_3, t, \xi, \xi_3) = \chi(2^k t) \left( \frac{|\xi + \xi_{d+1} A(x)|}{2^j} \right).
\]

For \( p = 2 \), observe that

\[
\|g\|_{L^2(\mathbb{R}^{d+1} \times \mathbb{R})} = \|D_{2-k} g\|_{L^2(\mathbb{R}^{d+1} \times \mathbb{R})} \quad \text{and} \quad \|f\|_{L^2(\mathbb{R}^{d+1})} = \|D_{2-k} f\|_{L^2(\mathbb{R}^{d+1})}
\]

where

\[
[D_{2-k}] g(x, x_{d+1}, t) = 2^{-(d+3)/2} g(2^{-k} x, 2^{-2k} x_{d+1}, 2^{-k} t) \quad \text{and} \quad [D_{2-k}] f(x, x_{d+1}) = 2^{-(d+2)/2} g(2^{-k} x, 2^{-2k} x_{d+1}).
\]

First observe that

\[
D_{2-k} P_{j+k+t} f(y, y_{d+1}) = P_{j+t} D_{2-k} f(y, y_{d+1}).
\]

This combined with \( D_{2-k}^4 \{p_t^4 T_{m,j,k} f\} = T_{m,j,k} D_{2-k} P_{j+k+t} f(x, x_{d+1}) \) which will be proved in (11.13), implies that

\[
D_{2-k}^4 \{p_t^4 T_{m,j,k} f\}(x, x_{d+1}, t) = T_{m,j,k} P_{j+k+t} D_{2-k} f(x, x_{d+1}, t).
\]

Thus, to show (11.1), we have only to prove that

\[
\|T_{m,j,k} P_{j+k+t} f\|_{L^2(\mathbb{R}^{d+1} \times \mathbb{R})} \lesssim 2^{-c|\ell|} \|f\|_{L^2(\mathbb{R}^{d+1})} \quad \text{for} \quad |\ell| \geq 100dj.
\]

This is the case \( k = 0 \) where \( t \approx 1 \). Thus, it suffices to fix \( t = 1 \) and prove

\[
\|T_{m,j,k} P_{j+k+t} f\|_{L^2(\mathbb{R}^{d+1})} \lesssim 2^{-c|\ell|} \|f\|_{L^2(\mathbb{R}^{d+1})} \quad \text{for} \quad |\ell| \geq 100dj.
\]

Split \( m_{j,0} = a_{j,\ell} + b_{j,\ell} \)

\[
a_{j,\ell}(x, x_3, t, \xi, \xi_3) = \chi(t) \chi \left( \frac{|\xi + \xi_{d+1} A(x)|}{2^j} \right) \psi \left( \frac{|\xi_{d+1}|}{2^j} \right)
\]

\[
b_{j,\ell}(x, x_3, t, \xi, \xi_3) = \chi(t) \chi \left( \frac{|\xi + \xi_{d+1} A(x)|}{2^j} \right) \left( 1 - \psi \left( \frac{|\xi_{d+1}|}{2^j} \right) \right).
\]

**Proof of (11.3)** for \( m_{j,0} = b_{j,\ell} \). We first show that (11.3) for \( m_{j,0} = b_{j,\ell} \) as

\[
\|T_{b_{j,\ell}} f(\cdot, \cdot, 1)\|_{L^2(\mathbb{R}^{d+1})} \lesssim 2^{-c|\ell|} \|f\|_{L^2(\mathbb{R}^{d+1})} \quad \text{for} \quad |\ell| \geq 100dj.
\]

First, use the decomposition of (11.8) below to localize \( x \) variable. Next apply the Fourier transform of the last variable. The it suffices to work with

\[
T_{b_{j,\ell}} f(x, t) = \lambda^{d/2} \chi(t) \psi(x) \int_{\mathbb{R}^d} e^{2\pi i \lambda \langle |x| + t |\xi + A(x)| \rangle} \chi \left( \frac{|\xi + A(x)|}{2^j} \right) \left( 1 - \psi \left( \frac{|\lambda|}{2^j} \right) \right) \hat{f}(\xi) d\xi.
\]

From (5.1), we have

\[
\|T_{b_{j,\ell}} f(\cdot, \cdot, 1)\|_{L^2(\mathbb{R}^{d+1})} \lesssim \left| \frac{2^{(d+\text{rank}(A))j}}{\lambda^{\text{rank}(A)}} \right|^{1/2} \|f\|_{L^2(\mathbb{R}^{d+1})} \lesssim 2^{-c|\ell|} \|f\|_{L^2(\mathbb{R}^{d+1})}
\]
Moreover, (11.7) follows from (11.6). Finally, (11.9) follows from (11.6).

Hat Euclidean space. The cancellation in (11.7) follows from

Proof. (11.9) for $m_{j,0} = a_{j,\ell}$.

Lemma 11.1. The two operators $T_{a,j,0}$ and $P_{j,\ell}$ have the integral kernels $K_j$ and $P_{j,\ell}$:

- $T_{a,j,0} f(x, x_{d+1}, 1) = \int f(x - y, x_{d+1} - y_{d+1} - (A(x), y)) K_j(y, y_{d+1}) dy$
- $P_{j,\ell} f(x, x_{d+1}) = \int f(x - y, x_{d+1} - y_{d+1} - (A(x), y)) P_{j,\ell}(y, y_{d+1}) dy$.

These two kernels have the cancellation property:

$$\int K_j(x, x_{d+1}) dx dx_{d+1} = \int P_{j,\ell}(x, x_{d+1}) dx dx_{d+1} = 0.$$

Moreover, $K_j$ is essentially supported in $\{(x, x_{d+1}) : x \in S^{d-1} + O(2^{-j}), |x_{d+1}| \leq 2^{-\ell/10}\}$ satisfying

$$|K_j(x, x_{d+1})| \lesssim \sum_{k=1}^{2^{(d-1)/2}} \frac{2^{\ell/10} \psi(2^{\ell/10} x_{d+1})}{|x - e(\theta_k)| \cdot e(\theta_k)} 2^{2^j/2} \prod_{\nu=1}^{d-1} \frac{2^{2^j/2}}{|x - e(\theta_k)| \cdot e(\theta_k) + 1}$$

having its $L^1$ norm $O(2^{(d-1)/2})$ where $e(\theta_k)$’s are equally distributed in $S^{d-1}$ and $e(\theta_k)$ are $d - 1$ different unit vectors perpendicular with $e(\theta_k)$. The support of $P_{j,\ell}$ in (11.6) is concentrated on

$$\{(y, y_{d+1}) : |(2^{j+\ell} y, 2^{2(j+\ell)} y_{d+1})| \leq 1\}$$

with its $L^1$ norm $O(1)$.

Proof. We obtain the above kernel representation by using the Fourier inversion formula in the Euclidean space. The cancellation in (11.7) follows from $K_j(0) = P_{j,\ell}(0) = 0$ where $\hat{\cdot}$ is the Euclidean Fourier transform. The inequality (11.8) follows from the decomposition of the frequency variables $\xi$ along angular sectors with the angle width $2^{-j/2}$ gaining the decay along the sectors. Finally, (11.9) follows from (11.6).

Lemma 11.2. The composition operator $T_{a,j,0} P_{j,\ell}$ is expressed as

$$T_{a,j,0} P_{j,\ell} f(x, x_{d+1}, 1) = \int f(x - y, x_{d+1} - y_{d+1} - (A(x), y)) U_{j,\ell}(y, y_{d+1}) dy dy_{d+1}$$

where

$$U_{j,\ell}(y, y_{d+1}) = \int K_j(y - z, y_{d+1} - z_{d+1} + (A(y), z) - (A(z), z)) P_{j,\ell}(z, z_{d+1}) dz dz_{d+1}. $$
Proof. By direct computation,
\[ T_{x_{d+1}} \mathcal{P}_{j+\ell} f(x, x_{d+1}) = \int \mathcal{P}_{j+\ell} f(x - y, x_{d+1} - y_{d+1} - \langle A(x), y \rangle) K_j(y, y_{d+1}) dy dy_{d+1} \]
\[ = \int \left( \int (x - y) - z, (x_{d+1} - y_{d+1} - \langle A(x), y \rangle) - z_{d+1} - \langle A(x - y), z \rangle \right) P_{j+\ell}(z, z_{d+1}) dz d z_{d+1} \]
\times K_j(y, y_{d+1}) dy dy_{d+1}. 

Fix \( x, x_{d+1}, z, z_{d+1} \) and apply the change of variable \((y + z, y_{d+1} + z_{d+1} - \langle A(y), z \rangle) \rightarrow (y, y_{d+1})\) to obtain this. \( \square \)

Lemma 11.3. Let \(|\ell| \geq 100 dj\). Suppose that \( U_{j, \ell} \) is defined in (11.10). Then we prove (11.3) by showing
\begin{equation}
\int |U_{j, \ell}(y, y_{d+1})| dy dy_{d+1} \lesssim 2^{-c|\ell|} \text{ for some } c > 0.
\end{equation}

Proof. Case 1. Let \( \ell > 0 \). Using \( \int P_{j+\ell}(z, z_{d+1}) dz d z_{d+1} = 0 \), we have that
\[ U_{j, \ell}(y, y_{d+1}) = \int \left( K_j(y - z, y_{d+1} - z_{d+1} + \langle A(y), z \rangle - \langle A(z), z \rangle) + K_j(y, y_{d+1}) \right) P_{j+\ell}(z, z_{d+1}) dz d z_{d+1}. \]

We apply the mean value theorem for the function \( K_j \) above with the size conditions
- \(|\nabla K_j(y, y_{d+1})| \lesssim 2^{\ell/10} |K_j(y, y_{d+1})| \) because of (11.5).
- \(|z|, |\langle A(y), z \rangle| \lesssim 2^{-j - \ell} \) and \(|\langle A(z), z \rangle|, |z_{d+1}| \lesssim 2^{-2j - 2\ell}| \) where \(|\ell| \gg j \) because of (11.6).

Then, we have
\begin{equation}
\int |U_{j, \ell}(y, y_{d+1})| dy dy_{d+1} = O(2^{j(d - 1)/2} 2^{\ell/10} 2^{-\ell}).
\end{equation}

Case 2. Let \( \ell < 0 \). Rewrite \( U_{j, \ell}(y, y_{d+1}) \) as
\[ \int K_j(z, z_{d+1}) \left( P_{j+\ell}(y - z, y_{d+1} - z_{d+1} + \langle A(y), y - z \rangle - \langle A(y - z), y - z \rangle) - P_{j+\ell}(y, y_{d+1}) \right) dz d z_{d+1} \]
where we have subtracted the vanishing term
\[ \int K_j(z, z_{d+1}) P_{j+\ell}(y, y_{d+1}) dz d z_{d+1} = 0. \]

Note that
- \(|\nabla z P_{j+\ell}(y, y_{d+1})| \lesssim 2^{j-|\ell|} |P_{j+\ell}(y, y_{d+1})| \) and \(|\nabla y_{d+1} P_{j+\ell}(y, y_{d+1})| \lesssim 2^{2j-2|\ell|} |P_{j+\ell}(y, y_{d+1})| \)
- \(|z| \lesssim 1, |z_{d+1}| \lesssim 2^{-j(1 + \alpha/2)} \), \(|\langle A(y), z \rangle| \lesssim 2^{-j + |\ell|} \) and \(|\langle A(z), z \rangle| \leq 1 \) where \(|\ell| \gg j \).

Apply the mean value theorem to obtain
\[ \left( P_{j+\ell}(y - z, y_{d+1} - z_{d+1} + \langle A(y), y - z \rangle - \langle A(y - z), y - z \rangle) - P_{j+\ell}(y, y_{d+1}) \right) \]
\[ \lesssim \max\{2^{j-|\ell|}, 2^{2j-2|\ell|} 2^{-j + |\ell|}\} \times P_{j+\ell}(y, y_{d+1}). \]

Thus
\[ \int |U_{j, \ell}(y, y_{d+1})| dy dy_{d+1} \lesssim 2^{-|\ell|} \int |P_{j+\ell}(y, y_{d+1})| dy dy_{d+1} \int |K_j(z, z_{d+1})| dz d z_{d+1} \lesssim 2^{d/2} 2^{-|\ell|}. \]
Almost orthogonality Estimates. We now prove (11.2) in Proposition 3.2. We claim that
\begin{equation}
\|P_j P_{j+\ell}\|_{op} \lesssim 2^{-c|\ell|}.
\end{equation}

Proof. It suffices to consider the case \(\ell > 0\). Using \(\int P_j P_{j+\ell}(z, z_{d+1})dz dz_{d+1} = 0\), write
\[U_{j,\ell}(y, y_{d+1}) = \int \left( P_j (y - z, y_{d+1} - z_{d+1} + (A(y), z)) - P_j (y, y_{d+1}) \right) P_{j+\ell}(z, z_{d+1})dz dz_{d+1}.\]

Using the mean value theorem as above, we obtain that \(\int |U_{j,\ell}(y, y_{d+1})|dy dy_{d+1} \lesssim 2^{-\ell}\). \(\square\)

Hence we finished the proof of Proposition 3.2.

11.2. Proof of the localization principle. We restate Proposition 3.1.

Proposition 4.1 Localization] Let \(B_{d+1}(0, r) = \{x \in R^{d+1} : |x| < r\}\) and \(I = [-2, -1] \cup [1, 2]\). Fix \(j \geq 0\). Then for every \(k \in Z\), the following three inequalities are equivalent:
\begin{align}
\|2^{k/p}T_{\delta_{\ell,j}} f\|_{L^p(R^{d+1} \times R)} &\leq C_p \|f\|_{L^p(R^{d+1})} \text{ for all } f \in L^p, \\
\|T_{\delta_{\ell,j}} f\|_{L^p(R^{d+1} \times I)} &\leq C_p \|f\|_{L^p(R^{d+1})} \text{ for all } f \in L^p, \\
P_{\delta_{\ell,j}} f\|_{L^p(B_{d+1}(0, 1) \times I)} &\leq C_p \|f\|_{L^p(B_{d+1}(0, 100))} \text{ for all } f \in L^p,
\end{align}
where \(C_p\) in (11.15) and (11.16) are comparable (or with an error \(O(2^{-Nj})\) for sufficiently large \(N\)). The equivalence (11.14)–(11.16) will hold true if \(m_{j,k}\) is replaced by \(a_{j,k}\) and \(b_{j,k}\) in (3.6)–(3.9).

It suffices to prove (11.16) \(\Rightarrow\) (11.15) \(\Leftrightarrow\) (11.14). We prove the case \(a_{j,k}\) instead of \(m_{j,k}\) and \(b_{j,k}\). We shall use the following two \(L^p\)-invariant dilations:
\begin{align}
\|g\|_{L^p(R^{d+1} \times R)} &= \|D_{2^{-k}} g\|_{L^p(R^{d+1} \times R)} \text{ and } \|f\|_{L^p(R^{d+1})} = \|D_{2^{-k}} f\|_{L^p(R^{d+1})}
\end{align}
where
\begin{align}
[D_{2^{-k}} g](x, x_{d+1}, t) &= 2^{-(d+3)k/p}g(2^{-k}x, 2^{-2k}x_{d+1}, 2^{-k}t), \\
|D_{2^{-k}} f|(x, x_{d+1}) &= 2^{-(d+2)k/p}g(2^{-k}x, 2^{-2k}x_{d+1}).
\end{align}

First, we claim that (11.15) \(\Leftrightarrow\) (11.14). From (11.17), we only have to claim that
\begin{equation}
D_{2^{-k}} [2^{k/p}T_{\delta_{\ell,j}}] f(x, x_{d+1}, t) = T_{\delta_{\ell,j}} D_{2^{-k}} f(x, x_{d+1}, t).
\end{equation}

Proof of (11.19). In view of (3.6)–(5.9), recall that
\begin{align}
T_{\delta_{\ell,j}} f(x, x_{d+1}, t) &= \chi \left( \frac{t}{2^{-k}} \right) \int e^{2\pi i (x, x_{d+1} + \xi) \cdot (\xi, \xi_{d+1}) + t(\xi + \xi_{d+1}, A(x))} \chi \left( \frac{t(\xi + \xi_{d+1}, A(x))}{2} \right) \\
&\times \psi \left( \frac{|\xi_{d+1}|}{2^{-k}^{1+\alpha/2}} \right) \hat{f}(\xi, \xi_{d+1}) d\xi d\xi_{d+1}.
\end{align}
By using $\mathcal{D}_{2-k}^1$ in (11.18), we write
\[
\mathcal{D}_{2-k}^1[f^{(2k/p)}_{T_a, x}] t(x, x_{d+1}, t) = 2^{-(d+2)k/p} \chi(t) \int e^{2\pi i ((2^{-k} \xi, 2^{-k} x_{d+1}) (\xi, x_{d+1}) + 2^{-k} t [\xi + 2^{-k} \xi A(2^{-k} x)])} (11.21)
\times \chi \left( \frac{2^{-k} t [\xi + 2^{-k} \xi A(2^{-k} x)]}{2^j} \right) \psi \left( \frac{[\xi + 2^{-k} \xi A(2^{-k} x)]}{2^j} \right) \times \tilde{f}(\xi, x_{d+1}) d\xi d x_{d+1}.
\]
Apply the change of variable $(\xi, x_{d+1}) \rightarrow (2^k \xi, 2^{-k} \xi_1)$. Then the above integral becomes
\[
2^{-(d+2)k/p} \psi \left( \frac{[\xi + 2^{-k} \xi A(2^{-k} x)]}{2^j} \right) \times \chi \left( \frac{2^{-k} t [2^k \xi + 2^{-2k} \xi A(2^{-k} x)]}{2^j} \right) \psi \left( \frac{2^k \xi + 2^{-2k} \xi_1 A(2^{-k} x)}{2^j} \right) 2^{(d+2)k/p} \tilde{f}(2^k \xi, 2^{-2k} \xi_1) d\xi d x_{d+1},
\]
where $\tilde{f}(2^k \xi, 2^{-2k} \xi_1) = 2^{-(d+2)k} [f(2^k \xi, 2^{-2k} \xi_1)]^n (\xi, x_{d+1})$. So (11.21) is
\[
\int e^{2\pi i ((x, x_{d+1}) + t (\xi + 2^{-2k} \xi A(x)))} (\xi, x_{d+1}) \chi(t) \left( \frac{[\xi + 2^{-2k} \xi A(x)]}{2^j} \right) \psi \left( \frac{[\xi + 2^{-2k} \xi A(x)]}{2^j} \right) [\mathcal{D}_{2-k}^1 f] \psi (\xi, x_{d+1}) d\xi d x_{d+1}
\]
which is $T_{a, j, k} \mathcal{D}_{2-k}^1 f(x, x_{d+1}, t)$. This implies (11.19). We can replace $a_{j, k}$ by $b_{j, k}$. \(\Box\)

To show (11.16) $\Rightarrow$ (11.15), we first prove (11.15) under localization of first $d$ variables of (11.10):
\[
\|T_{a, j, 0} f\|_{L^p([-1, 1]^d \times \mathbb{R})} \leq C_p \|f\|_{L^p([-1, 1]^d \times \mathbb{R})} \Rightarrow \|T_{a, j, 0} f\|_{L^p(\mathbb{R}^{d+1} \times [1, 2])} \leq 2^{10d} C_p \|f\|_{L^p(\mathbb{R}^{d+1})}.
\]

**Proof of (11.22).** Define the localization of a function $g$ near $|x - z| \leq r$ by
\[
[M_z g](x, x_{d+1}) = \psi(x - z) g(x, x_{d+1}) \quad \text{and} \quad [M_{z, 10}] g(x, x_{d+1}) = \psi \left( \frac{x - z}{2^{10}} \right) g(x, x_{d+1})
\]

Split $T_{a, j, 0} f$ as two parts:
\[
T_{a, j, 0} f = \sum_{z \in \mathbb{Z}^d} M_z [T_{a, j, 0}] (M_{z, 10} f) + \sum_{z \in \mathbb{Z}^d \cap \mathbb{Z}^d: |w - z| \geq 2^{10}} (M_z [T_{a, j, 0}] M_{w} f).
\]
First, we shall treat the second term of (11.24). We claim that for $|w - z| \geq 2^{10},$
\[
\|(M_z [T_{a, j, 0}] M_{w} f)\|_{L^p(\mathbb{R}^{d+1} \times \mathbb{R})} \lesssim 2^{-jN} \|M_{w} f\|_{L^p(\mathbb{R}^{d+1} \times \mathbb{R})}
\]
This estimate (11.25) with the summability in $w, z \in \mathbb{Z}^d$ yields that
\[
\left\| \sum_{z \in \mathbb{Z}^d \cap \mathbb{Z}^d: |w - z| \geq 2^{10}} (M_z [T_{a, j, 0}] M_{w} f) \right\|_{L^p(\mathbb{R}^{d+1} \times \mathbb{R})} \lesssim 2^{-jN} \|f\|_{L^p(\mathbb{R}^{d+1})}
\]
to prove (11.22) for the second term in (11.24).

**Proof of (11.22).** Express the pseudo differential operator $T_{a, j, 0} f$ as
\[
T_{a, j, 0} f(x, x_{d+1}, t) = \int f(x - y, x_{d + y} - y_{d} - \{A(x, y)\} \chi(t) K_{t} y_{d+1} dy
\]
where the Euclidean Fourier transform of $K_j$ in $\mathbb{R}^{d+1}$ is given by

$$
\hat{K}_j^i(\xi, \xi_{d+1}) = e^{2\pi i |\xi|} \chi \left( \frac{t\xi}{2} \right) \psi \left( \frac{\xi_{d+1}}{2^{j(1+\epsilon_0)}} \right).
$$

By (11.27) and (11.28),

$$
(11.28)
M_z[T_{a,j}]M_wf(x, x_{d+1}, t) = \psi(x - z) \int K(x, x_{d+1}, y, y_{d+1}, t)\psi(y - w) f(y, y_{d+1}) dydy_{d+1}.
$$

where the kernel $K(x, x_{d+1}, y, y_{d+1}, t)$ is

$$
\chi(t) \int_{\mathbb{R}^d} e^{2\pi i (x - y, \xi) + |\xi|} \chi \left( \frac{t\xi}{2} \right) d\xi \int_{\mathbb{R}^d} e^{2\pi i (x - y, \xi) + |\xi|} \chi \left( \frac{\xi_{d+1}}{2^{j(1+\epsilon_0)}} \right) d\xi_{d+1}.
$$

Since $|x - z| \leq 1$ and $|y - w| \leq 1$ combined with our condition $|z - w| \geq 2^{10}$ in (11.25), we have

$$
|x - y| \approx |z - w| \geq 2^{10}.
$$

Then on this range $|x - y| \gg 1$, we apply the integration by parts with respect to $\xi$ variable combined with $d\xi_{d+1}$. Then the above integral is controlled by

$$
(11.29)
|K(x, x_{d+1}, y, y_{d+1}, t)| \lesssim \frac{2^{jn}}{2^{N_j}} (|x - y| + 1)^{N - 10} (1 + |y - w|)^{10} \chi(t) \left( \frac{t\xi}{2} \right) \psi \left( \frac{\xi_{d+1}}{2^{j(1+\epsilon_0)}} \right)\psi \left( \frac{\xi_{d+1}}{2^{j(1+\epsilon_0)}} \right)
$$

Thus, we use the integrability of the kernel $(dx_{d+1}$ first and $dx$ next) to have

$$
(11.30)
\int |K(x, x_{d+1}, y, y_{d+1}, t)|\psi(x - z) \psi(y - w) |dx_{d+1} | |dx| \lesssim 2^{-N_j} \frac{1}{1 + |z - w|^{10}},
$$

uniformly in $y$ and $t$ (similarly $\int dydy_{d+1}$ uniformly in $x$ and $t$). We apply this to (11.28) to obtain that

$$
\int |M_z[T_{a,j}]M_wf(x, x_{d+1}, t)|pdx_{d+1} \lesssim \frac{2^{-N_j}}{1 + |z - w|^{10}} \int |f(y, y_{d+1})|p dydy_{d+1}
$$

uniformly in $t$. This yields (11.25). \qed

**Remark 11.1.** When $a_{j,k}$ is replaced by $m_{j,k}$ and $b_{j,k}$ with $k = 0$, each of the last line of (11.29) is given by $\delta ((x - y, A(x)) + (x_{d+1} - y_{d+1}))$ and

$$
(11.31)
\delta ((x - y) \cdot A(x) + (x_{d+1} - y_{d+1})) - 2^{j(1+\epsilon_0)} \psi \left( \frac{1}{2^{j(1+\epsilon_0)}} \right) (x - y) \cdot A(x) + (x_{d+1} - y_{d+1})
$$

which is integrable. This leads (11.30) for each case of $m_{j,k}$ and $b_{j,k}$.

To show (11.22) for the first term of (11.24), set $U_{z,A(z)}f(x, x_{d+1}) = f(x - z, x_{d+1} - A(z), x)$ and claim

$$
(11.32)
M_z[T_{a,j}](M_z,f)(x, x_{d+1}, t) = U_{z,A(z)}M_0[T_{a,j}]M_0 U_{z,A(z)}f(x, x_{d+1}, t).
$$
Proof of (11.32). In view of (11.27), the left hand side of (11.32) is

\[(11.33)\quad \psi(x-z) \int \chi(t)K^1_j(x-y, x_{d+1} - y_{d+1} - \langle A(x), y \rangle) \psi(y-z) f(y, y_{d+1}) dy dy_{d+1}.\]

Apply the change of variables \((y, y_{d+1}) \rightarrow (y+z, y_{d+1} - \langle A(z), y \rangle)\). Then the above integral becomes

\[
\psi(x-z) \int \chi(t)K^1_j((x-z) - y, (x_{d+1} - \langle A(x), z \rangle) - y_{d+1} - \langle A(x-z), y \rangle) \psi(y) \\
\times f(y+z, y_{d+1} - \langle A(z), y \rangle) dy dy_{d+1} = U_{z, A(z)} M_0 ((T_{a_{10}})(M_{0, 10} U_{-z, A(z)} f)) (x, x_{d+1}).
\]

This shows (11.32). \(\square\)

This combined with \(\|U_{z, A(z)} f\|_{L^p(\mathbb{R}^{d+1})} = \|f\|_{L^p(\mathbb{R}^{d+1})}\) implies the equivalence of (11.34) and (11.35) below

\[(11.34)\quad \|M_0[T_{a_{10}}] M_{0, 10} f\|_{L^p(\mathbb{R}^{d+1} \times \mathbb{R})} \leq \|M_{0, 10} f\|_{L^p(\mathbb{R}^{d+1})},
\]

\[(11.35)\quad \|M_z[T_{a_{10}}] M_{z, 10} f\|_{L^p(\mathbb{R}^{d+1} \times \mathbb{R})} \leq \|M_{z, 10} f\|_{L^p(\mathbb{R}^{d+1})}\]

for each \(z \in \mathbb{Z}^d\).

This enables us to sum all \(z\) of (11.35) to obtain (11.22) for the first term in (11.24):

\[(11.36)\quad \left| \sum_{z \in \mathbb{Z}^d} M_z[T_{a_{10}}] (M_{z, 10} f) \right|_{L^p(\mathbb{R}^{d+1} \times \mathbb{R})} \leq 10^d C_p \|f\|_{L^p(\mathbb{R}^{d+1})}.
\]

Combined with (11.26), we proved (11.22). \(\square\)

Thus we have proved (11.10) \(\Rightarrow\) (11.15) under the localization of \(x, y\) as

\[M_{0, 1}[T_{a_{10}}] M_{0, 10} f(x, x_{d+1}, t) = \int K(x, x_{d+1}, t, y, y_{d+1}) f(y, y_{d+1}) dy dy_{d+1} \quad |x|, |y| \leq 1.
\]

To localize \(x_{d+1}, y_{d+1}\), observe that the support condition \(|x|, |y| \leq 10\) together with \(|x_{d+1} - y_{d+1}| \geq 100\) in (11.29) yields that the \(L^1\) norm of the kernel is \(O(2^{-N})\) for large \(N\). This completes the proof of (11.16) \(\Rightarrow\) (11.15) for the case of \(a_{j,k}\). For the case of \(m_{j,k}\), as we see \(\delta ((x - y, A(x)) + (x_{d+1} - y_{d+1}))\) in (11.31) in Remark 11.1 the corresponding \(L^1\) norm is vanishing in the region of \(|x|, |y| \leq 10\) and \(|x_{d+1} - y_{d+1}| \geq 100\). The other case of \(b_{j,k}\) is the difference of \(m_{j,k}\) and \(a_{j,k}\). Therefore we finish the proof of Proposition 11.1.

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