THE MAXIMAL TUBE UNDER THE DEFORMATIONS OF A
CLASS OF 3-DIMENSIONAL HYPERBOLIC CONE-MANIFOLDS

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Abstract. Recently, Hodgson and Kerckhoff found a small bound on Dehn
surgered 3-manifolds from hyperbolic knots not admitting hyperbolic struc-
tures using deformations of hyperbolic cone-manifolds. They asked whether
the area normalized meridian length squared of maximal tubular neighbor-
hoods of the singular locus of the cone-manifold is decreasing and that summed
with the cone angle squared is increasing as we deform the cone-angles. We
confirm this near 0 cone-angles for an infinite family of hyperbolic cone-
manifolds obtained by Dehn surgeries along the Whitehead link complements.
The basic method is based on explicit holonomy computations using the A-
polynomials and finding the maximal tubes. One of the key tool is the Taylor
expression of a geometric component of the zero set of the A-polynomial in
terms of the cone-angles. We also show a sequence of Taylor expressions for
Dehn surgered manifolds converges to one for the limit hyperbolic manifold.

1. Introduction

Recently it was shown by Hodgson and Kerckhoff([7]) that there is a small uni-
versal bound for the number of nonhyperbolic Dehn-fillings on a hyperbolic mani-
fold with single cusp. Their argument involves analysis of the variation of maximal
tubes around singularities in cone manifolds of fixed topological type when the cone
angle increases starting from 0.

Let \( M \) be an orientable 3-manifold which admits a complete hyperbolic structure
of finite volume with single cusp. For each slope \( \gamma \) of the cusp of \( M \), let \( M(\gamma) \) be
the manifold obtained by Dehn-filling \( M \) along \( \gamma \). By a theorem of Gromov and
Thurston ([1]), \( M(\gamma) \) admits a negatively curved metric if the length of the shortest
curve on the boundary of a horoball neighborhood of the cusp isotopic to \( \gamma \) is greater
than \( 2\pi \). But it is not known whether \( M(\gamma) \) admits a hyperbolic structure with
the same hypothesis.

A hyperbolic cone-manifold of 3-dimension is a manifold locally modeled on open
subset of a hyperbolic space or the open region in an open set bounded by two totally
geodesic planes meeting at a geodesic and two planes are identified by an elliptic
isometry. By Thurston’s hyperbolic Dehn surgery theorem, if \( \theta > 0 \) is small, \( M(\gamma) \)
for any \( \gamma \) admits a hyperbolic cone-structure whose singular locus is the added
closed curve with some small cone angle \( \theta \). We denote the resulting cone manifold
by \( M(\gamma;\theta) \). The homotopy class of the singular locus obviously corresponds to the

Date: May 10, 2004.
1991 Mathematics Subject Classification. Primary 57M50.
Key words and phrases. hyperbolic manifold, cone-manifold, deformations.
The first author gratefully acknowledges support from Korea Research Foundation Grant
(KRF-2002-070-C00010).

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closed curve meeting with $\gamma$ once. (Often $M$ itself will be considered $M(\gamma; 0)$ and as having $\theta = 0$.)

Each of the cone manifold has the maximal tube around its singular locus. If we can bound from below the radii of the maximal tubes until the cone angle reaches $2\pi$, then we obtain a nonsingular hyperbolic structure on $M(\gamma)$. Let $T$ be the flat torus boundary of the maximal horoball neighborhood of the cusp in $M$. In [7], Hodgson and Kerckhoff show that if the normalized length of the geodesic curve on $T$ is isotopic to $\gamma$, i.e. the length measured when the metric on $T$ is rescaled so that the area of $T$ is 1, is greater than 7.515, then we can bound from below the radii of the maximal tubes in $M_0(\gamma)$ until the cone angle $\theta$ reaches $2\pi$ and hence $M(\gamma)$ is hyperbolic. Using this fact, they obtained the universal bound 60 for the number of nonhyperbolic Dehn-fillings on a single-cusped hyperbolic manifold.

In one of their lectures, they posed the following question:

**Question** (Hodgson and Kerckhoff) Let $\{M_\theta(\gamma) : 0 < \theta < \theta_0\}$ be a continuous family of hyperbolic cone structures on $M(\gamma)$ with singular locus as described above. Let $\mu = \mu(\theta)$ be the length of the surgery curve on the boundary $T_0$ of the maximal tube around the singular locus of $M_\theta(\gamma)$ and let $\tilde{\mu}$ be the normalized length $\mu/\sqrt{\text{Area}(T_0)}$. Then are $\hat{\mu}^2$ and $\tilde{\mu}^2 + \theta^2$ decreasing and increasing functions of $\theta$ on $[0, \theta_0]$, respectively?

They showed that if this question had positive answer then we could also control the radii of the maximal tubes effectively (see [8] for more details).

Our initial result is on the relationship between $A$-polynomial and the cone-angle:

Let $M$ be a hyperbolic manifold of finite volume and two cusps and $\{M_1, L_1\}$ be a basis for a peripheral group group $\pi_1(M)$ corresponding to the first cusp of $M$. We choose a basis $\{M_2, L_2\}$ for the second peripheral group and fix the two bases. Let $(p_1, q_1)$ and $(p_2, q_2)$ denote coprime pair of integers. Let $M(p_1, q_1)$ denote the 3-manifold obtained from $M$ by the $(p_1, q_1)$-Dehn filling the first cusp. Let $M(\infty, \infty)(p_2, q_2; \theta)$ denote the hyperbolic cone-manifold with a cusp corresponding to the first cusp of $M$ and the second cusp has been $(p_2, q_2)$-Dehn filled where the corresponding solid torus has a cone-type singularity with the cone-angle equal to $\theta$. Let $M(p_1, q_1)(p_2, q_2; \theta)$ denote the hyperbolic cone-manifold with the first cusp $(p, q)$-Dehn-filled and the second cusp has been $(p_2, q_2)$-Dehn-filled with the cone-type singularity with the cone-angle equal to $\theta > 0$.

The character variety of $\text{PSL}(2, \mathbb{C})$-representations of the fundamental group of $M$ is the space of characters $\pi_1(M) \to \mathbb{C}$ defined by taking the traces of holonomies of the fundamental group $\pi_1(M)$ (see Culler-Shalen [4] for details).

The character variety of $\text{PSL}(2, \mathbb{C})$-representations of the fundamental group of $M$ which keep the holonomy of $M_1, L_1$ parabolic gives a relation between the eigenvalues $l_1$ and $m_1$ of holonomies of $L_1$ and $M_1$, which is how $A$-polynomials are defined in this paper. The so-called geometric component of the zero-locus of the $A$-polynomial is a component realized by a deformation of hyperbolic manifold corresponding to the cone-structures for the second cusp. Using the geometric component of the $A$-polynomial together with the Dehn-filling relation

$$p \log \frac{m}{m_0} + q \log \frac{l}{l_0} = \frac{\sqrt{-1} \theta}{2},$$
we can obtain Taylor expansions of \( m \) and \( l \) in terms of \( \theta \), which we need only up to order three.

We define the \( A \)-polynomial of \( M(p_1, q_1) \) in the same manner with respect to \( \{M_2, L_2\} \). By the Dehn filling relation, we obtain the Taylor series of \( m_2 \) and \( l_2 \) corresponding to the geometric component of the \( A \)-polynomial as a function of \( \theta \).

Finally, we prove Theorem 4.4 showing the convergence of the Taylor expansions up to order three of \( m \) and \( l \) in terms of \( \theta \) for the \( A \)-polynomial of \( M(p, q) \) to the Taylor series of \( M(\infty, \infty) \). (This will be proved at last Section 4 because of the length.)

Our main result is that for an infinite number of of hyperbolic manifolds \( \{W(p_1, q_1)\} \) which are obtained from the Whitehead link complement \( W \) by Dehn-fillings on the first torus end, we have a partial answer to the question of Hodgson-Kerckhoff.

Let \( W \) be the Whitehead link complement, and let \( M_1, L_1, M_2, L_2 \) be suitably chosen meridians and longitudes for two cusp ends of \( W \).

**Theorem 1.1.** Let \( \mu = \mu_{p_1, q_1, p_2, q_2}(\theta) \) be the length of the surgery curve on the boundary of the maximal tube of \( W(p_1, q_1)(p_2, q_2; \theta) \) around the singularity. Let \( \hat{\mu} = \hat{\mu}_{p_1, q_1, p_2, q_2}(\theta) \) be the normalized length of the surgery curve. If \(|p_1| + |q_1| \) is sufficiently large, then for any coprime pair \( p_2, q_2 \) of integers except for at most one pair, \( \hat{\mu} \) is decreasing and \( \hat{\mu}^2 + \theta^2 \) is increasing at \( \theta = 0 \).

We outline the proof of Theorem.

For a general hyperbolic manifold \( M \) of finite volume with a distinguished cusp, we detect the maximal horoball neighborhood of the cusp by finding elements of \( \pi_1(M) \) whose holonomy have the large isometric spheres.

We use these elements, so called tie classes, to find the maximal tube neighborhood of the singularity in \( M(p, q; \theta) \) when \(|p| + |q| \) is large and \( \theta \) is small. This follows since the tie classes are stable near \( \theta = 0 \). (See Section 4 for more details.)

We now express the length \( \mu_{p, q}(\theta) \) and normalized length \( \hat{\mu}_{p, q}(\theta) \) of the surgery curve on the maximal tube around the singularity in \( M(p, q) \) in terms of the traces of holonomy of the commutator of a tie class and another element (see Proposition 3.2).

Now we restrict our attention to \( W \): We can exactly compute the holonomy representation of \( \pi_1(W) \) corresponding to the complete structure. By looking at the Ford domain, \( W \) with complete hyperbolic structure decomposes into four ideal tetrahedra. We can also determine the tie class. Each tetrahedron is assigned a complex invariant up to isometry. These invariants \( z_1, z_2, z_3, z_4 \) satisfy two relations according to two ideal edges of \( W \). The relation determine a small complex surface parameterizing all hyperbolic structures on \( W \) near the complete hyperbolic structure.

Let \( m_1, l_1, m_2, l_2 \) denote the eigenvalues of the holonomy of \( M_1, L_1, M_2, L_2 \) respectively. We can write these as functions of the tetrahedral invariants \( z_1, z_2, z_3, z_4 \).

Next, we compute the holonomy representations as functions of some easily identifiable variables \( x, y \). We easily see that \( x = m_1 \) and we can write \( l_1 \) as a function of \( x \) and \( y \). Thus, the holonomy representations are functions of \( m_1, l_1 \). From this, we can also find two relations between \( m_1, l_1 \) and \( m_2, l_2 \).

Using (*), we express \( \hat{\mu}_{p_1, q_1, p_2, q_2}(\theta) \) as

\[
k_0(p_1, q_1, p_2, q_2) + k_1(p_1, q_1, p_2, q_2)\theta^2 + O(\theta^3),
\]

where
where \(k_0\) and \(k_1\) are functions defined for integers \((p_1, q_1, p_2, q_2)\) with sufficiently large \(|p_1| + |q_1|\) and \(|p_2| + |q_2|\). By Theorem 3.4,

\[
k_1(p_1, q_1, p_2, q_2) \to k_1^\infty(p_2, q_2)
\]
as \(|p_1| + |q_1| \to \infty\) for some function \(k_1^\infty\), which takes values in some interval \([K_1, K_2] \subset (-1, 0)\).

Now, \(k_1^\infty(p_2, q_2)\) is the corresponding function for \(W(\infty, \infty, p_2, q_2)\). We can compute this function as in the above general discussion and the main theorem follows.

Dowty also obtained a similar result for figure-eight knot complements in his doctoral thesis [5] under the supervision of Hodgson. Our result generalizes his result but we are able to understand the effect of Dehn surgery better. Our long term hope is that we can answer the question of Hodgson and Kerckhoff for more general manifolds and with no angle restrictions although there seems to be no possible general theory insight. We think that our technique is interesting in that there may be many avenues and examples we can consider further and serve as a motivation for developing a general theory.

We thank Darryl Cooper, Craig Hodgson, Steven Kerckhoff for many discussions and their help. We also thank the Department of Mathematics of Stanford University for their great hospitality where some of this research was carried out.

2. Hyperbolic cone-manifolds and hyperbolic Dehn surgery

In this section, we recall some facts on hyperbolic Dehn surgery theory and hyperbolic cone-manifolds. We conclude with a needed result of Neumann and Zagier [11].

A hyperbolic manifold is a manifold equipped with a Riemannian metric whose sectional curvature is the constant \(-1\). (In this paper, we will consider only 3-dimensional and oriented manifolds.) We will denote a simply-connected complete hyperbolic manifold by \(H^3\). The group of orientation preserving isometries of \(H^3\) form a Lie group \(\text{PSL}_2(\mathbb{C})\). If \(M\) is a hyperbolic manifold, each point of \(M\) has an open neighborhood isometric to an open set in \(H^3\). We have an isometry \(\text{dev}\) from the universal cover \(\tilde{M}\) onto \(H^3\) and a group homomorphism \(\rho : \pi_1(M) \to \text{PSL}_2(\mathbb{C})\) such that \(\text{dev} \circ \gamma = \rho(\gamma) \circ \text{dev}\) for each \(\gamma\) in the deck transformation group \(\pi_1(M)\).

Thus, a hyperbolic manifold is isometric to the quotient space of \(H^3\) by the action of a discrete group of orientation-preserving isometries of \(H^3\).

Hyperbolic cone-manifolds arise in the context of Thurston’s hyperbolic Dehn surgery and are subjects of great interest. ([8], [9], [10])

A 3-manifold \(N\) equipped with a metric is a hyperbolic cone-manifold if each point of \(N\) has an open neighborhood isometric to an open set in \(H^3\) or the quotient metric space obtained from an open 3-ball in \(H^3\) by removing the domain with boundary in two totally geodesic planes meeting at a geodesic and identifying the two corresponding faces by an isometry fixing the geodesic (see Figure 1).

The set of all points of a hyperbolic cone-manifold \(N\) with no open neighborhood isometric to an open set in \(H^3\) is called the singular set or the singularity of \(N\) and is denoted by \(\Sigma_N\) (or \(\Sigma\) if \(N\) is clear from the context). The singular set is a 1-dimensional submanifold and is a link if \(M\) is a closed manifold. To each component of the singular set is associated the cone angle around the component.
Let $M$ be a hyperbolic manifold of finite volume with $h$ number of cusps. Thurston showed that we can deform $M$ so that $M$ has incomplete hyperbolic structures and the metric completions of some of the deformed structures induces complete hyperbolic cone-structures on the manifolds obtained by Dehn-fillings on ends of $M$. The hyperbolic Dehn surgery theorem states that when we perform Dehn-fillings on ends of $M$, the resulting manifold admits complete hyperbolic structures in most cases, i.e., without cone-singularities.

Let $M$ be a hyperbolic manifold of finite volume with $h$ cusps which can be obtained from an ideally triangulated hyperbolic manifold by Dehn-filling some of the ends. Let $\nu$ be the number of the ideal tetrahedra and let $M_1, L_1, \cdots, M_h, L_h$ be fixed meridians and longitudes for the ends of $M$. Let the complete hyperbolic structure of $M$ correspond to the point $z_0 = (z_0^1, \cdots, z_0^h) \in \mathbb{C}^\nu$. Thurston showed that the set $V$ of points in $\mathbb{C}^\nu$ near $z_0$ satisfying certain gluing consistency relations is a smooth analytic subset of complex dimension $h$ in $\mathbb{C}^\nu$. It was shown that maps $m = (m_1, \cdots, m_h) : V \rightarrow \mathbb{C}^h$ and $l = (l_1, \cdots, l_h) : V \rightarrow \mathbb{C}^h$ which assign certain eigenvalues of holonomy images of $M_1, \cdots, M_h, L_1, \cdots, L_h$, respectively are biholomorphic map at $z_0$ (see Neumann-Zagier [11]).

Let $(m_0^1, \cdots, m_0^h) = m(z^0)$ and $(l_0^1, \cdots, l_0^h) = l(z^0)$.

**Theorem 2.1** (Neumann-Zagier [11]). For each $i \in \{1, \cdots, h\}$, there is a holomorphic function $\tau_i$ defined on a neighborhood of the origin in $\mathbb{C}^h$ such that

$$\log(l_i/l_0^i) = \log(m_i/m_0^i) \tau_i(\log(m_1/m_1^0), \cdots, \log(m_h/m_h^0)).$$

Moreover for each $i$, $\tau_i$ is an even function in each variable and $\tau_i(0, \cdots, 0)$ is the modulus of the flat torus boundary of a cusp neighborhood for the $i$-th end with respect to $M_i, L_i$. In particular each $\tau_i(0, \cdots, 0)$ is not a real number.

This result will be needed later.

### 3. A-polynomials and generalized Dehn-fillings

We define the so-called $A$-polynomial for multicurved hyperbolic manifolds with respect to a distinguished cusp. (We may name this relative $A$-polynomial also.)
We define geometric components of the algebraic set of eigenvalues of holonomies, i.e., the zero set of the $A$-polynomial. We write the Taylor series of the polynomial relations of geometric components. We give some examples. Next, we show how to parameterize the geometric component as a function of the cone-angles (complex). Finally, we show that the Taylor series of a geometric component of a manifold with one cusp $(p, q)$-Dehn-filled converge to that of a manifold without filling as $(p, q) \to \infty$.

3.1. Geometric components of the spaces of representation eigenvalues. Culler-Shalen [4] defined a character variety of a 3-manifold to be the algebraic set of traces of the holonomy of the fundamental group elements ordered in some way.

We modify Cooper, Culler, Gillet, Long, Shalen [2]: Let $M$ be a hyperbolic 3-manifold with at least one cusp. Fix a cusp and denote by $\mathcal{P}$ is a $1$-set at $(l, m)$ if there is a holomorphic function $U : \mathcal{P} \to \mathbb{C} \times \mathbb{C}$ defined near $(l, m)$ such that each $\rho_m$ is a $1$-set at $(l, m)$.

We define $\mathcal{P}$ and $\mathcal{M}$ to be the union of curves $Z$ equal.)

$R_X$ varies over the components of $X$ (a basis for the fundamental group of a cusp neighborhood of $M$ taking values near $0$). The variety of characters of $\text{SL}_2(\mathbb{C})$ whose restrictions to the closed loops in all cusps other than $\{0\}$ are parabolic or identity. The variety of characters of $\text{SL}_2(\mathbb{C})$-representations of $\pi_1(M)$ is denoted by $X(M)$ and $t : R(\pi_1(M)) \to X(M)$ be the canonical surjective projection (see [4]). We may write $R(M)$ for $R(\pi_1(M))$. We denote by $X(M)_P$ the image of $R(M)_P$. Using the same reasoning as in [2], $X(M)_P$ is a dense subset of a finite union of varieties, to be denoted by $X''(M)_P$. (We do not claim that they are equal.)

Let $B = \{ \mathcal{L}, \mathcal{M} \}$ be a fixed basis for $P$. We define the restriction map $r : X''(M)_P \to X(P)$. Define $\triangle$ to be the subspace of diagonal representations in $R(P)$. There is an isomorphism $p_B : \triangle \to \mathbb{C}^* \times \mathbb{C}^*$ defined by setting $p_B(\rho) = (l, m)$ if $\rho$ is given by

$$\rho(\mathcal{L}) = \begin{bmatrix} l & 0 \\ 0 & l^{-1} \end{bmatrix}$$ and $\rho(\mathcal{M}) = \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix}$.

$t$ induces to a $2 - 1$-map $t_\triangle : \triangle \to X(P)$.

Denoting $X'(M)_P$ to be the union of irreducible components of $X''(M)_P$ whose images under $r$ is complex 1-dimensional. For each component $Z'$ of $X'(M)_P$, let $Z$ be the curve $\mathcal{L} = \sqrt{\mathcal{L}}(r(Z')) \subset \triangle$. Define $D_{M,P}$ to be the union of curves $Z$ as $Z'$ varies over the components of $X''(M)_P$.

We say that $D_{M,P}$ is the $A$-set of $M$ with respect to $P$. We note that $A$-set is invariant under the involution $(l, m) \mapsto (l^{-1}, m^{-1})$.

We define the $A$-polynomial $A_{M,P}$ of $M$ with respect to $P$ as the defining polynomial of the closure of $D_{M,P}$ in $\mathbb{C} \times \mathbb{C}$.

When $M$ has only one cusp, our definition coincide with the definition in [2]. When obvious, we will drop $P$ from $A_{M,P}$.

**Definition 3.1.** Let $M$ be a cusped hyperbolic manifold and $B = \{ \mathcal{L}, \mathcal{M} \}$ be a basis for the fundamental group of a cusp neighborhood of $M$. Suppose that $(l^0, m^0) \in \mathbb{C} \times \mathbb{C}$ equals one of $(\pm 1, \pm 1)$ and is in the zero set of $A_M(l, m)$. Let $l(m)$ be a holomorphic function defined on a neighborhood (say $U$) of $m^0$ and taking values near $l^0$. We say that the holomorphic function $l = l(m)$ defined near $(l^0, m^0) \in \mathbb{C} \times \mathbb{C}$ is a geometric curve of the $A$-set at $(l^0, m^0)$ if there is a holomorphic family $\{ \rho_m : m \in U \}$ of representations of $\pi_1(M)$ into $\text{SL}_2(\mathbb{C})$ such that each $\rho_m$ is
a lift of a holonomy representation of a hyperbolic structure on \( M \) and
\[
\rho_m(L) = \begin{bmatrix} l(m) & \ast \\ 0 & 1/l(m) \end{bmatrix}, \quad \rho_m(M) = \begin{bmatrix} m & \ast \\ 0 & 1/m \end{bmatrix}.
\]

Clearly, if \( l = l(m) \) is a geometric curve of the A-polynomial \( A_M(l, m) \) at \((m^0, l^0)\), then \( A(l(m), m) = 0 \) for all \( m \) near \( m^0 \).

**Proposition 3.2.** A component of \( A \)-set contains the image of the geometric curve as a dense set.

**Proof.** Straightforward. \( \square \)

A geometric component is a component of the \( A \)-set containing the geometric curve as a dense set. A geometric factor is a generator of the ideal defining the component above.

**Example 3.3.** Let \( M \) be the figure eight knot complement. \( \pi_1(M) \) has a Wirtinger presentation
\[
< \alpha, \beta : \alpha^{-1}\beta\alpha^{-1}\beta^{-1} \alpha\beta\alpha^{-1}\beta^{-1} >
\]
such that \( \{ \alpha, \beta^{-1}\alpha\beta\alpha^{-2}\beta\alpha^{-1} \} \) is a basis for a peripheral subgroup of \( \pi_1(M) \).
The A-polynomial of \( M \) with respect to this basis is
\[
A_M(l, m) = lm^8 - lm^6 - (l^2 + 2l + 1)m^4 - lm^2 + l.
\]
Note that \( A_M(-1, -1) = 0 \) and there are two geometric factors of \( A_M(l, m) \) at \((-1, -1)\) which are
\[
l + 1 - 2\sqrt{-3}(m + 1) - (6 + \sqrt{-3})(m + 1)^2 - (6 - 2\sqrt{-3}/3)(m + 1)^3 + O((m + 1)^4)
\]
and
\[
l + 1 + 2\sqrt{-3}(m + 1) - (6 - \sqrt{-3})(m + 1)^2 - (6 + 2\sqrt{-3}/3)(m + 1)^3 + O((m + 1)^4).
\]

### 3.2. Taylor series of geometric curves.

Let \( M \) be a 3-manifold admitting a complete hyperbolic structure of finite volume with single cusp. Let \( M, L \) be a fixed meridian-longitude pair on the end of \( M \). Let \( A_M(l, m) \) be the A-polynomial of \( M \) with respect to the meridian-longitude pair.

Suppose that deformations of hyperbolic structures on \( M \) near the complete structure gives us the following relation for eigenvalues \( m \) and \( l \) of \( M \) and \( L \) respectively for lifts of holonomy representations of nearby hyperbolic structures.

\[
l = l^0 + a_1(m - m^0) + \frac{a_2}{2}(m - m^0)^2 + \frac{a_3}{6}(m - m^0)^3 + \text{higher order terms}.
\]

(1)

(Here \( m^0 \) and \( l^0 \) are the eigenvalues for the lift of a holonomy representation of the complete structure. Thus each of \( m^0 \) and \( l^0 \) is \( \pm 1 \).)

This relation correspond to a geometric factor of \( A_M(l, m) \).

We will describe how this relation of \( m, l \) (near the complete structure) together with the Dehn-filling relation

\[
p \log \left( \frac{m}{m^0} \right) + q \log \left( \frac{l}{l^0} \right) = \frac{\sqrt{-1} \theta}{2}
\]

(2)
gives us Taylor coefficients of \( m \) and \( l \) For simplicity we assume that \( m^0 = l^0 = -1 \).

Other cases can be treated in the same way.
Recall that $a_1$ is not a real number since in general $a_1$ is the modulus of the flat structure of the cusp with respect to $\{M, L\}$ when the hyperbolic structure is complete.

Since $a_1 \neq -p/q$, as $a_1$ is not real, the sets defined by equations (1) and (2) are not tangent at $m^0 = l^0 = -1$. We will regard $m$ and $l$ as holomorphic functions of the variable $\theta$ at $\theta = 0$ locally.

Our purpose here is to obtain Taylor coefficients of $m$ and $l$ in terms of $\theta$ up to order 3 terms. We can do so by successive differentiation of (1) and (2) and evaluating at $\theta = 0$.

First we differentiate (2) to obtain

\[ \frac{p}{m} \frac{dm}{d\theta} + \frac{q}{l} \frac{dl}{d\theta} = \frac{\sqrt{-1}}{2}. \]

If we evaluate at $\theta = 0$, we obtain

\[ \left. \frac{p}{m} \frac{dm}{d\theta} \right|_{\theta=0} + \left. \frac{q}{l} \frac{dl}{d\theta} \right|_{\theta=0} = -\frac{\sqrt{-1}}{2}. \]

On the other hand if we differentiate (1) and evaluate at $\theta = 0$, we obtain

\[ \left. \frac{dl}{d\theta} \right|_{\theta=0} = a_1 \left. \frac{dm}{d\theta} \right|_{\theta=0}. \]

From (3) and (5) we obtain

\[ \frac{dm}{d\theta} \bigg|_{\theta=0} = -\frac{\sqrt{-1}}{2(p + a_1q)}, \quad \left. \frac{dl}{d\theta} \right|_{\theta=0} = -\frac{a_1\sqrt{-1}}{2(p + a_1q)}. \]

Continuing in this manner, we obtain

\[ \left. \frac{d^2m}{d\theta^2} \right|_{\theta=0} = \frac{2}{4(p + a_1q)^3}, \quad \left. \frac{d^2l}{d\theta^2} \right|_{\theta=0} = \frac{2}{4(p + a_1q)^3}. \]

\[ \left. \frac{d^3m}{d\theta^3} \right|_{\theta=0} = \frac{4}{8(p + a_1q)^5}, \quad \left. \frac{d^3l}{d\theta^3} \right|_{\theta=0} = \frac{4}{8(p + a_1q)^5}. \]

Recall that $a_2 = a_1 - a_1^2$ if the curve represented by (1) is invariant under the involution $(l, m) \mapsto (1/l, 1/m)$ near $(l^0, m^0) = (-1, -1)$. Thus we have the following formula for $m$, $l$, and $r \log(m^0)$ in terms of $\theta$ when $a_2 = a_1 - a_1^2$.

\[ m = -1 - \frac{\sqrt{-1}}{2(p + a_1q)} \theta + \frac{1}{8(p + a_1q)^2} \theta^2 + \sqrt{-1} \frac{p + (3a_1 - 3a_1^2 + a_1^3 - a_3)q}{48(p + a_1q)^4} \theta^3 \]

\[ l = -1 - \frac{a_1\sqrt{-1}}{2(p + a_1q)} \theta + \frac{a_1^2}{8(p + a_1q)^2} \theta^2 + \sqrt{-1} \frac{(-2a_1 + 3a_1^2 + a_3)q + a_1^2q}{48(p + a_1q)^4} \theta^3 \]
\( r \log(-m) + s \log(-l) = \frac{-1(r + a_1 s)}{2(p + a_1 q)} \theta + \frac{-1(2a_1 - 3a_1^2 + a_3^2 - a_3)(ps - qr)}{48(p + a_1 q)^4} \theta^3 \)

equationup to order 3 terms.

### 3.3. Convergence of the terms of Taylor series of geometric factors of A-polynomials.

Let \( M \) be a 3-manifold which admits a double-cusped complete hyperbolic structure and let \( \mathcal{M}_1, \mathcal{L}_1, \mathcal{M}_2, \) and \( \mathcal{L}_2 \) be fixed meridians and longitudes for the ends of \( M \). We assume that \( M \) can be obtained from an ideally triangulated hyperbolic manifold by Dehn-filling some of the ends. Let \( \nu \) be the number of the tetrahedra.

We have a holomorphic embedding of an open set \( V \subset \mathbb{C}^2 \) to \( \mathbb{C}^\nu \) whose image is a subset of \( \nu \subset \mathbb{C}^\nu \) consisting of points \( (z_1, \ldots, z_\nu) \) satisfying the gluing consistency relations.(See (11) for example).

Thurston showed that holonomy representations near that of complete hyperbolic structure on \( M \) have lifts \( \rho_0 \) to \( \text{SL}_2(\mathbb{C}) \). Let \( m_0^i, l_0^i, m_0^0, l_0^0 \) be the eigenvalues of \( \rho_0(L_1), \rho_0(L_2), \rho_0(L) \). (Each of \( m_0^i, l_0^i, m_0^0, l_0^0 \) is either 0 or \( \pm 1 \).) We have a holomorphic map from \( \nu \) to \( \mathbb{C}^\nu \) which assigns to each point \( z \) of \( \nu \) eigenvalues \( m_0^i, l_0^i, m_0^0, l_0^0 \) of \( \rho(M_1), \rho(L_1), \rho(M_2), \rho(L_2), \rho(L) \), respectively, where \( \rho \) is a lift of the holonomy representation corresponding to \( z \). Moreover we choose the holomorphic map so that the value of \( (m_0^i, l_0^i, m_0^0, l_0^0) \) equals \( (m_0^i, l_0^i, m_0^0, l_0^0) \) at the point of \( \nu \) corresponding to the complete structure.

When the first end remains a cusp the eigenvalues \( m_0^0, l_0^0 \) satisfy a relation of the form

\[
l_0^0 = l_0^0 + a_1 (m_0^0 - m_0^0) + \frac{a_2}{2} (m_0^0 - m_0^0)^2 + \frac{a_3}{6} (m_0^0 - m_0^0)^3 + \text{higher order terms}.
\]

Similarly, when \( p_1, q_1 \) are coprime integers and \( |p_1| + |q_1| \) is large, \( m_0^0 \) and \( l_0^0 \) satisfy a relation

\[
l_0^0 = l_0^0 + a_0^{p_1,q_1} (m_0^0 - m_0^0) + \frac{a_2^{p_1,q_1}}{2} (m_0^0 - m_0^0)^2 + \frac{a_3^{p_1,q_1}}{6} (m_0^0 - m_0^0)^3 + \text{higher order terms}.
\]

when the first end is Dehn-filled along the slope \( (p_1, q_1) \).

**Theorem 3.4.** Let \( M \) be a hyperbolic manifold with two cusps. Let \( M(p_1, q_1) \) be the Dehn-filled 3-manifold obtained from \( M \) by a \( (p_1, q_1) \)-surgery on the first cusp. Let \( a_i \) be the \( i \)-th Taylor coefficient of \( l_0^0 \) in terms of \( m_0^0 \) in a geometric factor of the \( A \)-polynomial of the second cusp of \( M \). Let \( a_0^{p_1,q_1} \) be that of \( M(p_1, q_1) \). Then \( a_i^{p_1,q_1} \to a_i \) as \( |p_1| + |q_1| \to \infty \) for \( i = 1, 2, 3 \).

This result shows the convergence of a sequence of coefficients of Taylor series of certain geometric components (defined in Subsection 3.4) of the \( A \)-polynomials of the manifolds \( M(p_1, q_1) \) which are obtained by Dehn filling a double-cusped hyperbolic manifold \( M \) on the first end. Though we show the convergence of the coefficients up to order 3, we can easily extend our proof to show the convergence seems to hold for any higher order terms.

We will prove this theorem in Section 6.
4. MAXIMAL TUBES IN HYPERBOLIC CONE MANIFOLDS

In this section, we will discuss maximal tubes in hyperbolic cone-manifolds in general. We first define the tie classes, the shortest path connecting the singularity not homotopic into the singularity. We discuss the stability of tie classes under geometric convergence. We obtain a formula of the radius of the maximal tube using the traces of some elements including the commutator of the tie classes. Finally, we obtain the meridian length and the normalized meridian length in terms of radius of the maximal tube and cone angles and translation length.

A hyperbolic manifold of finite volume with a distinguished cusp has a horoball neighborhood of the cusp. The largest of such neighborhoods is called the maximal horoball neighborhood of the cusp or the maximal cusp neighborhood.

Analogously, a hyperbolic cone-manifold whose singular locus is a knot has standard tube neighborhoods of the singular locus and the largest of such neighborhoods is called the maximal tube neighborhood or the maximal tube around the singular locus.

Let $M$ be a hyperbolic manifold of finite volume with a cusp. Let $P$ be a peripheral subgroup of $\pi_1(M)$. The boundary of the maximal horoball neighborhood is a torus, say $T$, which is tangent to itself at a finite number of points. For each point $x$ of self-tangency, we have a unique geodesic line which is orthogonal to $T$ at $x$ and tends to the cusp end in both directions.

Such a geodesic line corresponds uniquely to an equivalence class of the double coset space $P \backslash \pi_1(M) / P$, i.e., an equivalence class with respect to the equivalence relation $\sim$ defined on $\pi_1(M)$ by $\alpha \sim \beta$ if and only if $\alpha = \gamma_1 \beta \gamma_2$ or $\alpha = \gamma_1 \beta^{-1} \gamma_2$ for some $\gamma_1, \gamma_2 \in P$. The class is said to be a tie class of $P$ or the cusp corresponding to $P$.

Let $\rho_0 : \pi_1(M) \to \text{PSL}_2(\mathbb{C})$ be the holonomy representation for the hyperbolic structure on $M$ such that $\rho_0(P)$ fixes $\infty$ in the upper space model for the hyperbolic space $\mathbb{H}^3$. Let $H$ be the horizontal plane in $\mathbb{H}^3$ which is a lift of $T$ and let $\tilde{x} \in H$ be a lift of $x$. Then we have another horosphere $H'$ which is a lift of $T$ and contains $\tilde{x}$. Then an element $\alpha$ of $\pi_1(M)$ such that $\alpha(H') = H$ represents the tie class.

We also characterize the tie class as follows: A representative of the class has largest isometric spheres with respect to representations $\rho_0$ with $\rho_0(P)$ fixing $\infty$. 

Figure 2. A representative of a tie class for maximal horoball neighborhood.
We define the tie class for a hyperbolic cone-manifold. Let \( N \) be a hyperbolic cone-manifold whose singular locus \( \Sigma \) is a simple closed curve and the cone angle is less than \( 2\pi \). Let \( P \) be the peripheral subgroup of \( \pi_1(N - \Sigma) \). The boundary of the maximal tube around the singular locus is a torus \( T_\Sigma \) with several points of self-tangency. For each point \( x \) of self-tangency, we have a unique geodesic arc which is the union of two shortest paths from \( x \) to \( \Sigma \). Such a geodesic arc corresponds to an equivalence class in the double coset of \( P \pi_1(N - \Sigma)/P \).

We describe the tie class in the universal covering space: Let \( x \) be a point of self-tangency on \( T_\Sigma \) and let \( \tau \) be one of the shortest paths from \( x \) to \( \Sigma \). Let \( U_\Sigma \) be the interior of the maximal tube neighborhood around \( \Sigma \) in \( N \) minus \( \Sigma \) and let \( \tilde{U}_0 \) be the component of the lift of \( U_\Sigma \) in \( \tilde{N} - \tilde{\Sigma} \) which is left invariant by the action of \( P \) on \( \tilde{N} - \tilde{\Sigma} \). Let \( \tilde{x}_0 \) be a lift of \( x \) in \( \tilde{N} - \tilde{\Sigma} \) lying on the boundary of \( \tilde{U}_0 \) and let \( \tilde{\tau}_0 \) be the lift of \( \tau - \Sigma \) in \( \tilde{U}_0 \) with one end at \( \tilde{x}_0 \). By extending the geodesic arc \( \tilde{\tau}_0 \) past \( \tilde{x}_0 \) we obtain an open geodesic arc tending to ends of \( \tilde{N} - \tilde{\Sigma} \) in both directions. One end comes from \( \tilde{U}_0 \) and the other end comes from an image \( \tilde{U}_\rho' \) of \( \tilde{U}_\rho \) under the action of \( \pi_1(N - \Sigma) \) on \( \tilde{N} - \tilde{\Sigma} \). Then the tie class for the maximal tube at \( x \) is represented by an element \( \alpha \) of \( \pi_1(N - \Sigma) \) such that \( \alpha(\tilde{U}_\rho') = \tilde{U}_0 \).

Let \( \{ M_\theta : 0 < \theta < \theta_0 \} \) be a family of hyperbolic cone-manifolds of the same topological type such that \( M_\theta \) converges (in the sense of Gromov-Hausdorff) to a complete hyperbolic manifold \( M_0 \) of finite volume with at least one cusp and, for each \( \theta \), \( M_\theta \) has a singular locus \( \Sigma_\theta \) with cone angle \( \theta \) and \( M_\theta - \Sigma_\theta \) is homeomorphic to a fixed 3-manifold \( M_0 \) by a map \( \phi_\theta : M_\theta \to M_0 - \Sigma_\theta \). Then we have holonomy representations \( \rho_\theta : \pi_1(M_\theta) \to \text{PSL}_2(\mathbb{C}) \) and \( \rho_\theta : \pi_1(M_\theta - \Sigma_\theta) \to \text{PSL}_2(\mathbb{C}) \) for \( 0 < \theta < \theta_0 \) so that \( \rho_\theta \circ (\phi_\theta)_* \to \rho_0 \) is an isomorphism. Let \( P \) be a peripheral subgroup of \( \pi_1(M_0) \).

**Proposition 4.1.** Assume the notations above. When \( M_0 \) has a tie class, say \([\alpha]\) with \( \alpha \in \pi_1(M_0) \), for the cusp corresponding to \( P \), the cone manifold \( M_\theta \) also has a tie class with respect to \((\phi_\theta)_*(P)\) which is represented by \((\phi_\theta)_*(\alpha)\) when \( \theta \) is small.

**Proof.** This is obvious from Gromov-Hausdorff topology since a sequence of maximal tubes must converge to a maximal horoball. \( \square \)

**Proposition 4.2.** Let \( N \) be a hyperbolic cone manifold whose singular locus is a simple closed curve and the cone angle is less than \( 2\pi \). Let \( P \) be a peripheral subgroup of \( \pi_1(N - \Sigma) \) and \( \rho : \pi_1(N - \Sigma) \to \text{PSL}_2(\mathbb{C}) \) be a holonomy representation of the hyperbolic structure on \( N - \Sigma \). Let \( \alpha \in \pi_1(N - \Sigma) \) represent a tie class for the maximal tube. Then the radius \( R \) of the maximal tube around the singular locus satisfies

\[
\cosh(2R) = \frac{|\text{tr}(\alpha \gamma \alpha^{-1} \gamma^{-1}) - 2| + |\text{tr}^2(\gamma) - \text{tr}(\alpha \gamma \alpha^{-1} \gamma^{-1}) - 2|}{|\text{tr}^2(\gamma) - 4|},
\]

where \( \gamma \) is any element of \( P \) such that \( \rho(\gamma) \neq I \).

**Proof.** Let \( \text{dev} \) be the developing map \( \tilde{N} - \tilde{\Sigma} \to \mathbb{H}^3 \) for the hyperbolic structure on \( N - \Sigma \) such that \( \text{dev} \circ \gamma = \rho(\gamma) \circ \text{dev} \) for any \( \gamma \in \pi_1(N - \Sigma) \). Let \( x, \tilde{x}_0, \tau, \tilde{\tau}_0, U_\Sigma, \) and \( \tilde{U}_0 \) be as above. Then \( \rho(P) \) fixes a geodesic line \( \Sigma \) in \( \mathbb{H}^3 \) and \( \text{dev}(\tilde{U}_0) \) is the set of points in \( \mathbb{H}^3 \) lying within the radius of the maximal tube from \( \Sigma \). Let \( \tilde{x} \) and \( \tilde{\tau} \) be the images of \( \tilde{x}_0 \) and \( \tilde{\tau}_0 \) under \( \text{dev}|\tilde{U}_0 \), respectively. Then \( \text{dev}(\tilde{U}_0) \)
Figure 3. A representative of a tie class for maximal tube

and $\text{dev}(\alpha^{-1}(\tilde{U}_0))$ are hyperspherical regions around $\tilde{\Sigma}$ and $\rho(\alpha^{-1})(\tilde{\Sigma})$, respectively which are tangent at $\tilde{x}$.

Recall that $R$ is half the distance between $\tilde{\Sigma}$ and $\alpha^{-1}(\tilde{\Sigma})$, and $\tilde{\Sigma}$ is the axis of $\rho(\gamma)$ for any $\gamma \in P$ such that $\rho(\gamma) \neq I$. In the upper half space model, geodesic lines are represented by pairs of different extended complex numbers. We may assume that $\tilde{\Sigma}$ is the vertical geodesic represented by $(0, \infty)$ and that $\rho(\gamma)$ is of the form

$$\begin{bmatrix} u & 0 \\ 0 & 1/u \end{bmatrix}.$$ 

Let

$$\rho(\alpha) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $ad - bc = 1$. Then the geodesic line $\rho(\alpha^{-1})(\tilde{\Sigma})$ is represented by $(-b/a, -d/c)$. So cosh of the distance between $\tilde{\Sigma}$ and $\rho(\alpha^{-1})(\tilde{\Sigma})$ is $|ad| + |bc| = |bc| + |bc + 1|$ by the following lemma. Since $\text{tr}\rho(\alpha\gamma\alpha^{-1}\gamma^{-1}) - 2 = -bc(u - 1/u)^2 = -bc(tr^2\rho(\gamma) - 4)$.

Now it is straightforward to check the equality in the proposition. □

Lemma 4.3. The distance $d$ between two geodesic lines $(w_1, w_2)$ and $(w_3, w_4)$ satisfies

$$\cosh d = \frac{1 + |[w_1, w_2; w_3, w_4]|}{1 - |w_1, w_2; w_3, w_4|},$$

where

$$[w_1, w_2; w_3, w_4] = \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)}$$

is the cross-ratio.

Proof. Since the cross-ratios and hyperbolic distances are invariant under hyperbolic isometries, we need only to prove the lemma when $w_1 = -1, w_2 = 1, w_3 = -w, w_4 = w$ for some $w \in \mathbb{C}$. But in this case the equality is easily shown. □
Let \( \theta \) be the cone angle of the singular locus and let \( \gamma_0 \) be an element of \( P \) such that \( \rho(\gamma_0) \) is not elliptic and moves by minimal distance along its axis. Let \( t \) be the axis length of \( \rho(\gamma_0) \) which equals the absolute value of the real part of \( 2 \log(u) \) when \( \text{tr}\rho(\gamma_0) = u + 1/u \), i.e., the length of the singular locus. Let \( \mu \) and \( \hat{\mu} \) be the length and the normalized length \( \hat{\mu} \) of the meridian curve on the boundary of the maximal tube, respectively.

\[
\mu = \theta \sinh(R) = \theta \sqrt{\frac{\cosh(2R) - 1}{2}}
\]

and

\[
\hat{\mu}^2 = \frac{\theta \tanh(R)}{t} = \frac{\theta}{t} \sqrt{\frac{\cosh(2R) - 1}{\cosh(2R) + 1}},
\]

we can express \( \mu \) and \( \hat{\mu} \) in terms of the cone angle and the traces of holonomy images of certain elements of \( \pi_1(N - \Sigma) \) related to the tie classes.

5. Maximal Tubes in Whitehead Link Cone Manifolds

The purpose of this section is to prove Theorem 1.1. The outline of the proof will be given after the theorem is stated.

We choose the meridian-longitude pair for each end. For each coprime pair \( p_1, q_1 \) of integers, let \( W(p_1, q_1) \) be the manifold obtained from the Whitehead link complement \( W \) by Dehn-filling the first end along the slope \( (p_1, q_1) \). If \(|p_1| + |q_1| \) is sufficiently large, \( W(p_1, q_1) \) is hyperbolic, and \( \theta > 0 \) is small, let \( W(p_1, q_1)(p_2, q_2; \theta) \) be the hyperbolic cone manifold obtained by generalized Dehn-filling of \( W(p_1, q_1) \) on the second end along the slope \( (p_2, q_2) \) with cone angle \( \theta \) and let \( \mu = \mu_{p_1, q_1, p_2, q_2}(\theta) \) be the length of the surgery curve on the maximal tube of \( W(p_1, q_1)(p_2, q_2; \theta) \) around the singularity. Let \( \hat{\mu} = \hat{\mu}_{p_1, q_1, p_2, q_2}(\theta) \) be the normalized length of the surgery curve, that is, \( \mu \) divided by the square root of the area of the boundary of the maximal tube.
Theorem 5.1. Following the above notations, if $|p_1| + |q_1|$ is sufficiently large, then for any coprime pair $p_2, q_2$ of integers except for at most one, $\hat{\mu}$ for $W(p_1, q_1)(p_2, q_2; \theta)$ is decreasing and $\hat{\mu}^2 + \theta^2$ is increasing near $\theta = 0$.

Let us give an outline of the proof of Theorem 5.1. We start with basic materials about the deformation of hyperbolic structures on the Whitehead link complement $W$ near the complete structure:

In Subsection 5.0.1 we present a parametrization for the hyperbolic structures near the complete structure using a decomposition of $W$ into ideal tetrahedra and in Subsection 5.0.2 we obtain all the lifts of holonomy representations (up to conjugacy) for hyperbolic structures on $W$ near the complete structure.

In Subsection 5.1 we obtain formulas for $\hat{\mu}^2_{p_1,q_1}(\theta)$ in terms of the eigenvalues $m_2$ and $l_2$ of of holonomy (for the hyperbolic structure on $W$ inducing the hyperbolic cone-structure on $W(p_1, q_1)(p_2, q_2; \theta)$) images of $\mathcal{M}_2$ and $\mathcal{L}_2$ using the results of Subsection 5.0.2.

The needed observation is that each $W(p_1, q_1)$ has a unique tie class for maximal cusp which comes from the tie class for the maximal cusp for the second end in $W$ and this tie class is easily computable.

In our case, if $M$ has a unique tie class for its maximal cusp, we can find a representative for the tie class, and we can compute a geometric component of the A-polynomial zero-set, then we can compute the normalized length of the meridian curve in $M(p, q; \theta)$ in terms of $\theta$.

As explained above, we know that the single-cusped complete hyperbolic manifold $W(p_1, q_1)$ (when $|p_1| + |q_1|$ is large), has a unique tie class for its maximal cusp and we can easily find a representative for the tie class.

In Subsection 5.2 putting all things together, we will obtain Taylor coefficients of the function $\hat{\mu}^2_{p_1,q_1}(\theta)$ up to order 2 terms. Using concrete values of $a_1, a_2, a_3$ we can show Theorem 5.1 when the first cusp remains a cusp, i.e., $p_1 = q_1 = \infty$. Then we will complete the proof using the convergence $a_i^{p_1,q_1} \rightarrow a_i$ ($i = 1, 2, 3$) proved in the last section.

5.0.1. The parametrization of hyperbolic structures on the Whitehead link complement. We start with a holonomy representation and a picture of the Ford domain for the complete hyperbolic structure on Whitehead link complement. We use the presentation

$$\langle \alpha, \beta, \gamma : \alpha \gamma = \gamma \beta, \gamma \alpha \beta \alpha^{-1} = \alpha \beta^{-1} \alpha \beta^{-1} \alpha \rangle$$

of the fundamental group of $W$ coming from Figure 5. We fix the following system of meridians and longitudes for ends of $W$: $\mathcal{M}_1 = \gamma, \mathcal{L}_1 = \alpha \beta^{-1} \alpha^{-1} \beta, \mathcal{M}_2 = \alpha, \mathcal{L}_2 = \gamma \alpha^{-1} \gamma^{-1} \alpha \beta^{-1} \alpha$.

The group homomorphism $\rho_0 : \pi_1(W) \rightarrow \text{SL}_2(\mathbb{C})$ which sends $\alpha, \beta, \gamma$ onto

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2\sqrt{-1} & -1 \end{bmatrix}, \begin{bmatrix} -2 & -(1 + \sqrt{-1})/2 \\ 1 - \sqrt{-1} & 0 \end{bmatrix},$$

respectively, is the lift of a holonomy representation of the complete hyperbolic structure on $W$.

The Ford domain with respect to the second cusp corresponding to $\rho_0$ together with isometric spheres is described in Figure 6. From the description of the Ford domain for the second end, we obtain a decomposition of $W$ into four ideal tetrahedra.
Figure 5. Whitehead link complement with Wirtinger generators

Figure 6. Ford domain for the second cusp of Whitehead link complement

Figure 7. A decomposition of $W$ into ideal tetrahedra
Now let $z_1, z_2, z_3, z_4$ be the parameters corresponding to the four tetrahedra as described in Figure 8. $z_1 = z_2 = z_3 = z_4 = (1 + \sqrt{-1})/2$ corresponds to the complete hyperbolic structure. For the induced metric to be nonsingular along the edges $e_1, e_2, e_3, e_4$, $z_i$’s must satisfy the following gluing consistency relations.

\[(1 - z_1)(1 - z_4) = (1 - z_2)(1 - z_3)\]

\[(1 - z_1)(1 - z_2)(1 - z_3)(1 - z_4) = z_1 z_2 z_3 z_4\]

For $z_i$’s (all near $z^0$) satisfying those relations, we have the following relation between $z_i$’s and the eigenvalues of the corresponding $SL_2(\mathbb{C})$-holonomy images of the meridian and longitude for the second end.

\[m_1 = -\sqrt{1 - z_4 \over 1 - z_2} \left( -\sqrt{1 - z_3 \over 1 - z_1} \right),\]

\[l_1 = -{1 - z_4 \over 1 - z_2} \sqrt{z_1 z_4 \over z_1 z_2} = -{1 - z_3 \over 1 - z_1} \sqrt{z_1 z_4 \over z_1 z_2},\]

\[m_2 = -\sqrt{1 - z_2 \over 1 - z_1} \left( -\sqrt{1 - z_4 \over 1 - z_3} \right),\]

\[l_2 = -{1 - z_2 \over 1 - z_1} \sqrt{z_2 z_4 \over z_1 z_3} = -{1 - z_4 \over 1 - z_3} \sqrt{z_2 z_4 \over z_1 z_3},\]

where the branch of square root function is chosen to take value 1 at 1.

5.0.2. Holonomy representations of the Whitehead link group. We claim that for any $x, y \in \mathbb{C}$, there is a homomorphism $\rho : \pi_1(W) \to SL_2(\mathbb{C})$ such that

\[\rho(\alpha) = \begin{pmatrix} x & 1 \\ 0 & 1/x \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} x & 0 \\ y & 1/x \end{pmatrix}, \quad \rho(\gamma) = \begin{pmatrix} x(z^2 y^2 + x^2(x^2 - 3)y - (x^2 - 1)^2) & z \\ z y & z(1 - x^2) \end{pmatrix},\]
Figure 9. Gluing consistency for edges of the triangulation

Figure 10. Holonomy images of meridians and longitudes
where \( w = (x^2y - x^2 + 1)(x^2y + (x^2 - 1)^2) \), \( z = \sqrt{x^2(1 - x^2 - y)/w} \) and that the lift of any holonomy representation for \( \pi_1(W) \) near \( \rho_0 \) is conjugate to one of this form (with \((x, y, z) \) near \((-1, 2\sqrt{-1}, -(1 + \sqrt{-1})/2))

**Proof of the claim.** Let \( \rho : \pi_1(W) \to SL_2(\mathbb{C}) \) be a lift of a holonomy representation such that \( \rho(\alpha) \) and \( \rho(\beta) \) does not commute. Then by conjugation in \( SL_2(\mathbb{C}) \) we may assume that \( \rho(\alpha) \) and \( \rho(\beta) \) are of the form

\[
\begin{bmatrix}
x & 1 \\
0 & 1/x
\end{bmatrix}, \quad \begin{bmatrix}
x & 0 \\
y & 1/x
\end{bmatrix},
\]

respectively, since \( \rho(\alpha) \) and \( \rho(\beta) \) are conjugate in \( SL_2(\mathbb{C}) \). By the relation \( \alpha \gamma = \gamma \beta \), \( \rho(\gamma) \) must be of the form

\[
\begin{bmatrix}
x(1+yz^2) \\
z(1-x^2)
\end{bmatrix}
\]

Finally from the relation \( \gamma \alpha \beta \alpha^{-1} = \alpha \beta^{-1} \alpha \beta \alpha^{-1} \gamma \), we obtain \( z \) in terms of \( x, y \) as in the claim. \( \square \)

For a given holonomy representation \( \rho \) described above, we take \( x \) as the eigenvalue \( m_2 \) of the holonomy image of the meridian \( \alpha = M_2 \) for the second end. Since

\[
\rho(L_2) = \begin{bmatrix}
-1 + x^2 - y^2 \\
1+x^2+y \\
0 \\
-1+x^2+y
\end{bmatrix}
\]

we should take the eigenvalue \( l_2 \) of the holonomy image of the longitude \( L_2 \) as

\[
l_2 = \frac{-1 + x^2 - y^2}{-1 + x^2 + y}.
\]

Then we obtain the following relation between eigenvalues \( m_1, l_1, m_2, l_2 \) of the holonomy images \( \rho(M_1), \rho(L_1), \rho(M_2), \rho(L_2) \) by computing \( tr^2 \rho(M_1) \) and \( tr \rho(L_1) \) in terms of \( x, y, z \) in the claim above.

\[
(m_1 + \frac{1}{m_1})^2 = \frac{(1 + l_2)^2(m_4^2 - l_2)}{l_2(l_2 + m_2^2)(m_4^2 - 1)}
\]

\[
l_1 + \frac{1}{l_1} = \frac{l_2^2(1 + m_2^4) + l_2(-1 + 2m_4^2 + 2m_4^2 + 2m_4^2 - m_2^2) + m_4^2 + m_2^2}{m_2^2(l_2 + m_2^2)^2}
\]

5.1. **Maximal tubes in Whitehead link cone manifolds.** The elements of \( \pi_1(W) \) which represent a unique tie class for the maximal cusp of \( W \) with respect to the peripheral subgroup \( P_2 \) for the second end containing \( \alpha \in \pi_1(W) \) projects to an element of \( \pi_1(W(p_1, q_1)(p_2, q_2; \theta) - \Sigma(p_1, q_1)(p_2, q_2; \theta)) \) representing a tie class when \( |p_1| + |q_1| \) is large and \( \theta \) is small.

From the description of Ford domain of \( W \) with respect to the second end given in Subsection 5.0.1 we see \( \gamma \in \pi_1(W) \) represents the unique tie class for the second cusp with respect to \( P_2 \). Also we see that when \( |p_1| + |q_1| \) is large and \( W(p_1, q_1, \infty, \infty) \) is hyperbolic, \( \gamma \) still represents the unique tie class for the single-cusped complete hyperbolic manifold \( \pi_1(W(p_1, q_1, \infty, \infty)) \) by considering the isometric spheres of images of a suitable holonomy representation.
\[ \mu^2 = \frac{\theta}{2\Re(r_2 \log(-m_2) + s_2 \log(-l_2))} \times \left[ \frac{|\text{tr}(\rho) - \text{tr}(\rho \gamma^{-1})|}{|\text{tr}(\rho) - \text{tr}(\rho \gamma^{-1})| - 2} \right] - 2 \]

where \( \rho : \pi_1(W(p_1, q_1))(p_2, q_2; \theta) \to \text{SL}_2(\mathbb{C}) \) is a lift of a holonomy representation for \( W(p_1, q_1)(p_2, q_2; \theta) \) and \( r_2, s_2 \) are integers satisfying \( p_2s_2 - q_2r_2 = 1 \), when \( |p_1| + |q_1| \) is large and \( \theta \) is small.

We obtain that \( \text{tr}(\rho \gamma^{-1}) = \text{tr}(\rho \beta^{-1}) = 2 - y \) from equations \( 10 \) and \( 17 \).

Using \( 18 \), we obtain

\[ \text{tr}(\rho \gamma^{-1}) - 2 = \frac{1 + m_2^2(1 - l_2)}{l_2 + m_2^2}. \]

**5.2. Completion of the proof of the main result.** Let \( W \) be the Whitehead link complement and let \( M_1, L_1, M_2, L_2 \) be the meridians and longitudes for the ends of \( W \) as before.

From \( 11 \) and \( 20 \), we have

\[ \mu^2 = \frac{\theta}{2\Re(r_2 \log(-m_2) + s_2 \log(-l_2))} \times \left[ \frac{|(m_2^2 - 1)(1 - l_2)| + |(m_2^2 - 1)(1 - l_2)| + |(m_2 - \frac{1}{m_2})^2 + (m_2^2 - 1)(1 - l_2)|}{m_2^2 + l_2} \right] \]

where \( r_2, s_2 \) are integers satisfying \( p_2s_2 - q_2r_2 = 1 \), when \( |p_1| + |q_1| \) is large and \( \theta \) is small.

When \( p_1 = q_1 = \infty \), we have a Taylor expansion

\[ l_2 = -1 + a_1(m_2 + 1) + \frac{a_2}{2}(m_2 + 1)^2 + \frac{a_3}{6}(m_2 + 1)^3 + O((m_2 + 1)^4), \]

where \( a_1 = 2 + 2i, a_2 = 2 - 6i, a_3 = -12 \), from the A-polynomial

\[ -l_2 + l_2^3 + 4l_2m_2 + m_2^2 - l_2m_2^2 \]

of the manifold \( W(\infty, \infty) \) with respect to \( \{M_2, L_2\} \). So when \( |p_1| + |q_1| \) is large, we have a Taylor expansion

\[ l_2 = -1 + a_1^{p_1,q_1}(m_2 + 1) + \frac{a_2^{p_1,q_1}}{2}(m_2 + 1)^2 + \frac{a_3^{p_1,q_1}}{6}(m_2 + 1)^3 + O((m_2 + 1)^4), \]

for the manifold \( W(p_1, q_1) \) with \( a_1^{p_1,q_1} \to a_i \) as \( |p_1| + |q_1| \to \infty \) for \( i = 1, 2, 3 \) as in Subsection 3.3.

By Lemma \( 6.6 \), \( a_1^{p_1,q_1} = a_1^{p_1,q_1} - (a_1^{p_1,q_1})^2 \) when \( |p_1| + |q_1| \) is large.

Using \( 18-18 \), we obtain

\[ \frac{(m_2^2 - 1)(1 - l_2)}{m_2^2 + l_2} = \frac{4}{2 - a_1^{p_1,q_1}} - \frac{6a_1^{p_1,q_1} + 3(a_1^{p_1,q_1})^2 - 2a_1^{p_1,q_1}}{12(2 - a_1^{p_1,q_1})^2(p_2 + a_1^{p_1,q_1}q_2)^2} \theta^2 + O(\theta^3) \]
and
\[
\text{Re}(r_2 \log(-m_2) + s_2 \log(-l_2)) = -\frac{\text{Im} a_1^{p_1,q_1}}{2|p_2 + a_1^{p_1,q_1}|^2} \theta
\]
\[
-\frac{\text{Im}\{(2a_1^{p_1,q_1} - 3(a_1^{p_1,q_1})^2 + (a_1^{p_1,q_1})^3 - a_2^{p_1,q_1})(p_2 + a_1^{p_1,q_1})^4\}}{48|p_2 + a_1^{p_1,q_1}|^4} \theta^3 + O(\theta^4)
\]

From (21), we obtain
\[
\hat{\mu}_2^{p_1,q_1}(\theta) = \frac{|p_2 + a_1^{p_1,q_1}|^2}{|\text{Im} a_1^{p_1,q_1}|} + \frac{B_0^{p_1,q_1} p_2^4 + B_1^{p_1,q_1} p_2^3 q_2 + B_2^{p_1,q_1} p_2^2 q_2^2 + B_3^{p_1,q_1} p_2 q_2^3 + B_4^{p_1,q_1} q_2^4}{|p_2 + a_1^{p_1,q_1}|^4}
\]
up to order 2 terms when \(|p_1| + |q_1|\) is large, where each \(B_j^{p_1,q_1}\) is a constant which depend continuously on \(a_1^{p_1,q_1}, a_2^{p_1,q_1}, a_3^{p_1,q_1}\), that is,
\[
B_j^{p_1,q_1} = B_j(a_1^{p_1,q_1}, a_2^{p_1,q_1}, a_3^{p_1,q_1})
\]
for a continuous function \(B_j\), which is defined on a neighborhood of \((a_1, a_2, a_3)\) in \(\mathbb{C}^3\). In the same way, we have
\[
\hat{\mu}_2^{\infty,\infty, p_2,q_2}(\theta) = \frac{|p_2 + a_1 q_2|^2}{|\text{Im} a_1|} + \frac{B_0^{\infty,\infty} p_2^4 + B_1^{\infty,\infty} p_2^3 q_2 + B_2^{\infty,\infty} p_2^2 q_2^2 + B_3^{\infty,\infty} p_2 q_2^3 + B_4^{\infty,\infty} q_2^4}{|p_2 + a_1 q_2|^4}
\]
up to order 2 terms, where \(B_j^{\infty,\infty} = B_j(a_1, a_2, a_3)\). Using the values \(a_1, a_2, a_3\), we obtain
\[
B_0^{\infty,\infty} = -1/12, \quad B_1^{\infty,\infty} = -2/3, \quad B_2^{\infty,\infty} = -4, \quad B_3^{\infty,\infty} = -32/3, \quad B_4^{\infty,\infty} = -32/3.
\]
All the values of the function
\[
-\frac{x^4 - 8x^3 - 48x^2 - 128x - 128}{12(x^2 + 4x + 8)^2} \quad (-\infty < x < \infty)
\]
lie in the interval \([-1/6, -1/12] \subset (-1, 0)\). Thus the coefficient of the second order term in (22),
\[
\hat{\mu}_2^{\infty,\infty, p_2,q_2}(\theta) = \frac{|p_2 + a_1 q_2|^2}{|\text{Im} a_1|} + \frac{B_0^{\infty,\infty} p_2^4 + B_1^{\infty,\infty} p_2^3 q_2 + B_2^{\infty,\infty} p_2^2 q_2^2 + B_3^{\infty,\infty} p_2 q_2^3 + B_4^{\infty,\infty} q_2^4}{|p_2 + a_1 q_2|^4}
\]
lies in \([-1/6, -1/12] \subset (-1, 0)\) for any coprime pair of integers \(p_2, q_2\). This completes the part of our proof when \((p_1, q_1)\) is the infinity. Since \(a_1^{p_1,q_1} \to a_1\), and \(B_j^{p_1,q_1} \to B_j^{\infty,\infty}\) for \(j = 0, \ldots , 4\) as \(|p_1| + |q_1| \to \infty\), our proof for \(\hat{\mu}^2\) is completed. For \(\hat{\mu}^2 + \theta^2\), the proof follows since the range of equation (23) plus 1 lies in \([5/6, 11/12]\) which is positive.

6. CONVERGENCE OF THE TERMS OF TAYLOR SERIES OF GEOMETRIC FACTORS OF A-POLYNOMIALS

In this section, we will provide a proof of Theorem 3.4. We assume the notations of Subsection 3.3.

Let \(M\) be a complete hyperbolic manifold with two cusps and decomposes into \(\nu\) ideal tetrahedra. Let \(z^0 = (z_1^0, \ldots , z_\nu^0) \in \mathbb{C}^\nu\) correspond to the complete structure.
Let $V$ be the intersection of a small neighborhood of $z^0$ in $C^r$ and the set of points satisfying the gluing consistency relations as above.

By results of Neumann and Zagier, there is a holomorphic embedding $\iota$ of a small open subset $V$ of $C^2$ onto $V$ (by shrinking $V$ if necessary).

Let $\rho_0 : \pi_1(M) \to \text{SL}_2(\mathbb{C})$ be the lift a holonomy representation of the complete structure and let $m_1^0, l_1^0, m_2^0, l_2^0$ be the eigenvalues of $\rho(M_1), \rho(L_1), \rho(M_2), \rho(L_2)$.

As mentioned before, by suitable choices of the eigenvalues $m_1, l_1, m_2, l_2$ of $\rho(M_1), \rho(L_1), \rho(M_2), \rho(L_2)$ for the lifts $\rho$ of holonomy representations for nearby hyperbolic structures we obtain a holomorphic map $G : V \to C^4$ which can be regarded as a pair of maps $G_1, G_2 : V \to C^2$. $G_1 = (m_1, l_1)$ and $G_2 = (m_2, l_2)$.

Moreover we can choose $G$ so that $G_1(z^0) = (m_1^0, l_1^0)$ and $G_2(z^0) = (m_2^0, l_2^0)$. Let $C_1 \subset V$ be $G_1^{-1}\{m_1^0, l_1^0\} \cap V$ and $C_1 \subset V$ be $G^{-1}(\mathcal{C}_1) \cap V$. Let $(u_0, v_0) \in V$ be the point corresponding to the complete structure; that is, $G^{-1}(\mathcal{C}_1) = \{m_1^0, l_1^0, m_2^0, l_2^0\}$.

The steps of the proof are as follows:

- First, we will show the smoothness of the geometric component $C_1$ as seen in the tetrahedral parameter space where the first cusp remains a cusp. This essentially follows from the gradients of $m_1$ and Theorem 2.1. We realize $C_1$ as a graph of a function from one parameter to another.
- Second, we will show the smoothness of the geometric component $C_1^{m_1,q_1}$ in the tetrahedral parameter space where the first cusp has been $(p_1, q_1)$-Dehn surgered. The argument is based on continuity method for the gradients. We also realize $C_1^{m_1,q_1}$ as a graph of a function.
- We show that the sequence of coefficients of the Taylor series of the second functions converges to those of the first function.
- We change variable to $(m_1, l_1)$. This change of variables gives us the desired conclusion.

(1) The first step is to show the smoothness of $C_1$.
For $(u, v) \in V$, we will denote $G_1(u, v)$ and $G_2(u, v)$ by $(m_1, l_1)$ and $(m_2, l_2)$, considered as functions of $(u, v)$, respectively. By results of Neumann-Zagier, gradient vectors $\nabla m_1$ and $\nabla m_2$ are linearly independent at $(u_0, v_0). (\nabla$ is taken with respect to $(u, v)$). Similarly $\nabla l_1$ and $\nabla l_2$ are linearly independent at $(u_0, v_0)$.

Moreover, by Theorem 2.1, $\nabla m_1(u_0, v_0)$ and $\nabla l_1(u_0, v_0)$ are not linearly dependent over $\mathbb{R}$ (though they are dependent over $\mathbb{C}$). So, by the change of parameter $(u, v) \mapsto (v, u)$ if necessary, we may assume that

$$\frac{\partial m_1}{\partial u} |_{(u_0, v_0)} \text{ and } \frac{\partial m_1}{\partial v} |_{(u_0, v_0)} \text{ are linearly independent over } \mathbb{R}.$$

Since $\nabla m_1|_{(u_0, v_0)} \neq 0$, it follows that $C_1$ and $C_1$ are smooth curves near $z^0$ and $(u_0, v_0)$, respectively.

In addition, since $\frac{\partial m_1}{\partial u} \neq 0$, $C_1$ is the graph of a holomorphic map

$$v - v_0 = c_1 (u - u_0) + \frac{c_2}{2} (u - u_0)^2 + \frac{c_3}{6} (u - u_0)^3 + O((u - u_0)^4)$$

at $(u_0, v_0)$. Here $c_1 \neq 0$ since $\frac{\partial m_1}{\partial u} \neq 0$.

(2) The second step is to do the same for $C_1^{q_1, q_1}$.

For each coprime pair $p_1, q_1$ of integers for which $|p_1| + |q_1|$ is sufficiently large, take $(u_0^{p_1,q_1}, v_0^{p_1,q_1}) \in V$ such that

$$p_1 \log \left( \frac{m_1}{m_1} \right) + q_1 \log \left( \frac{l_1}{l_1} \right) = \pi \sqrt{-1}$$
Lemma 6.3. The point \((u_0^{p_1,q_1}, v_0^{p_1,q_1})\) can be taken to be one realized as a hyperbolic manifold obtained by \((p_1, q_1)\)-Dehn filling the first cusp and the second cusp remaining a cusp. Since a sequence of such manifold converges to \(W\) as \(|p_1| + |q_1| \to \infty\), we see that \(u_0^{p_1,q_1} \to u_0\) and \(v_0^{p_1,q_1} \to v_0\) as \(|p_1| + |q_1| \to \infty\).

Let \(C_1^{p_1,q_1}\) be the set of points \((u, v)\) in \(V\) satisfying \(p_1 \log\left(\frac{m_1}{m_2}\right) + q_1 \log\left(\frac{l_1}{l_2}\right) = \pi \sqrt{-1}\).

Proposition 6.1. When \(|p_1| + |q_1|\) is sufficiently large, \(C_1^{p_1,q_1}\) is a smooth curve near \((u_0^{p_1,q_1}, v_0^{p_1,q_1})\).

Proof. We have that

\[
\frac{\partial}{\partial v} \{ p_1 \log\left(\frac{m_1}{m_2}\right) + q_1 \log\left(\frac{l_1}{l_2}\right) \} = \frac{p_1}{m_1} \frac{\partial m_1}{\partial v} + \frac{q_1}{l_1} \frac{\partial l_1}{\partial v}.
\]

Since the values of two functions \(\frac{\partial m_1}{\partial v}\) and \(\frac{\partial l_1}{\partial v}\) are linearly independent over \(\mathbb{R}\) at any point \((u, v)\) in \(V\) near \((u_0, v_0)\), \(\frac{\partial}{\partial v} \{ p_1 \log\left(\frac{m_1}{m_2}\right) + q_1 \log\left(\frac{l_1}{l_2}\right) \}\) is never 0 near \((u_0, v_0)\). Thus \(\nabla \{ p_1 \log\left(\frac{m_1}{m_2}\right) + q_1 \log\left(\frac{l_1}{l_2}\right) \}\) never vanishes near \((u_0, v_0)\). \(\square\)

Proposition 6.2. The direction of \(\nabla \{ p_1 \log\left(\frac{m_1}{m_2}\right) + q_1 \log\left(\frac{l_1}{l_2}\right) \}\) converges to that of \(\nabla m_1(u_0, v_0)\) as \(|p_1| + |q_1| \to \infty\).

Proof.

\[
\nabla m_1 = \left( \frac{\partial m_1}{\partial u}, \frac{\partial m_1}{\partial v} \right)
\]

and

\[
\nabla \left\{ p_1 \log\left(\frac{m_1}{m_2}\right) + q_1 \log\left(\frac{l_1}{l_2}\right) \right\} = \left( \frac{p_1}{m_1} \frac{\partial m_1}{\partial u} + \frac{q_1}{l_1} \frac{\partial l_1}{\partial u}, \frac{p_1}{m_1} \frac{\partial m_1}{\partial v} + \frac{q_1}{l_1} \frac{\partial l_1}{\partial v} \right)
\]

by definition. But

\[
\frac{\partial m_1}{\partial u} + \frac{\partial m_1}{\partial v} \left|_{(x^{p_1,q_1}, y^{p_1,q_1})} \right. \to \frac{\partial m_1}{\partial v} \frac{\partial m_1}{\partial u}
\]

as \(|p_1| + |q_1| \to \infty\) by the following lemma. \(\square\)

Lemma 6.3. Let \(a, b, c, d\) be complex numbers such that \(c, d\) are linearly independent over \(\mathbb{R}\) and \(ad - bc = 0\). Let \(A, B, C, D\) be functions on \(\mathbb{C} \times \mathbb{C}\) such that

\[
A(p, q) \to a, \quad B(p, q) \to b, \quad C(p, q) \to c, \quad D(p, q) \to d \quad \text{as} \quad |p| + |q| \to \infty
\]

Then

\[
\frac{Ap + Bq}{Cp + Dq} \to \frac{ap + bq}{cp + dq} = \frac{a}{c} = \frac{b}{d} \quad \text{as} \quad |p| + |q| \to \infty.
\]
(3) The third step is to show the convergence of coefficients.

From Propositions $\text{[5]}$ and $\text{[2]}$, we see that when $|p_1| + |q_1|$ is sufficiently large, $C_{1}^{p_1,q_1}$ is the graph of a holomorphic map

$$v^{p_1,q_1} - v_0^{p_1,q_1} = v_0^{p_1,q_1} (u^{p_1,q_1} - v_0^{p_1,q_1}) + \frac{c_1^{p_1,q_1}}{2} (u^{p_1,q_1} - v_0^{p_1,q_1})^2$$

$$+ \frac{c_2^{p_1,q_1}}{6} (u^{p_1,q_1} - v_0^{p_1,q_1})^3 + O((u^{p_1,q_1} - v_0^{p_1,q_1})^4)$$

at $(u_0^{p_1,q_1}, v_0^{p_1,q_1})$, where $c_1^{p_1,q_1} \neq 0$.

Let $y$ denote the variable on $C_1$ by setting it equal to $u$ for a point $(u, v)$ on $C_1$. Let $y$ denote the function on $C_1$ set to be equal to $v$. We regard $y$ as a function of $x$.

**Proposition 6.4.** $c_1^{p_1,q_1} \rightarrow c_i$ as $|p_1| + |q_1| \rightarrow \infty$ for $i = 1, 2, 3$.

**Proof.** Any $(u, v) \in C_1$ satisfies

$$p_1 \left\{ \frac{1}{m_1} (m_1 - m_1^0) - \frac{1}{2} (m_1 - m_1^0)^2 + \frac{1}{3m_1^0} (m_1 - m_1^0)^3 \right\}$$

$$+ q_1 \left\{ \frac{1}{l_1} (l_1 - l_1^0) - \frac{1}{2} (l_1 - l_1^0)^2 + \frac{1}{3l_1^0} (l_1 - l_1^0)^3 \right\} = 0$$

up to order three for any pair $p_1, q_1$ of integers. The left hand side of the equation is the Taylor expansion of the function $p_1 \log(m_1^{m_1}) + q_1 \log(l_1^{l_1})$ at $(m_1^0, l_1^0)$ up to order 3 terms.

Differentiating above equation with respect to $x$ gives us the relation up to order three

$$p_1 \left\{ \frac{1}{m_1} \frac{dm_1}{dx} - (m_1 - m_1^0) \frac{dm_1}{dx} + \frac{1}{m_1^0} (m_1 - m_1^0) ^2 \frac{dm_1}{dx} \right\}$$

$$+ q_1 \left\{ \frac{1}{l_1} \frac{dl_1}{dx} - (l_1 - l_1^0) \frac{dl_1}{dx} + \frac{1}{l_1^0} (l_1 - l_1^0) ^2 \frac{dl_1}{dx} \right\} = 0.$$

(24)

Applying the chain rule

$$\frac{df}{dx} = \frac{\partial f}{\partial u} \frac{du}{dx} + \frac{\partial f}{\partial v} \frac{dv}{dx}$$

to $f = m_1$ and $f = l_1$ in (24), we obtain

$$c_1 = \frac{dy}{dx} \bigg|_{x=x_0} = \left. \frac{-p_1 \frac{dm_1}{du} + q_1 \frac{dl_1}{dv}}{m_1^0 \frac{dm_1}{du} + l_1^0 \frac{dl_1}{dv}} \right|_{(u,v)=(x_0,y_0)}.$$

On the other hand, any $(u, v) \in C_1^{p_1,q_1}$ satisfies

$$p_1 \log(m_1^{m_1}) + q_1 \log(l_1^{l_1}) = \pi \sqrt{-1}.$$

We let $x^{p_1,q_1}$ denote the variable on $C_1^{p_1,q_1}$ obtained by setting it equal to $u$ and let $y^{p_1,q_1}$ denote the function on the same by setting it equal to $v$. We consider $y^{p_1,q_1}$ the function of $x^{p_1,q_1}$.

Differentiating with respect to $x^{p_1,q_1}$ gives us the relation

$$\frac{p_1}{m_1} \frac{dm_1}{dx^{p_1,q_1}} + q_1 \frac{dl_1}{l_1^{p_1,q_1}} = 0$$

(25)
Then using the chain rule in (24), we obtain
\[ c_1^{p_1,q_1} = \frac{dy^{p_1,q_1}}{dx^{p_1,q_1}} \bigg|_{x^{p_1,q_1}=x_0^{p_1,q_1}} = \frac{\frac{\partial y}{\partial x}}{\frac{\partial x}{\partial u}} + \frac{2\frac{\partial y}{\partial u}}{\frac{\partial u}{\partial v}} \bigg|_{(u,v)=(u_0^{p_1,q_1},v_0^{p_1,q_1})}. \]

From Lemma 3.3, we see that \( c_1^{p_1,q_1} \to c_1 \) as \(|p_1| + |q_1| \to \infty\).

Differentiating (24) with respect to \(x\) and using
\[ \frac{d^2 f}{dx^2} = \frac{\partial^2 f}{\partial u^2} + 2\frac{d}{dx}\frac{\partial^2 f}{\partial u \partial v} + \left(\frac{d}{dx}\right)^2 \frac{\partial^2 f}{\partial v^2} + \frac{d^2 y}{dx^2} \frac{\partial f}{\partial v}, \]
we obtain
\[ c_2 = \left. \frac{d^2 y}{dx^2} \right|_{x=x_0} \]
\[ = -\frac{p_1}{m_1} \left\{ \frac{\partial^2 m_1}{\partial u^2} + 2c_1 \frac{\partial^2 m_1}{\partial u \partial v} + \frac{c_1^2}{m_1} \frac{\partial^2 m_1}{\partial v^2} - \frac{1}{m_1} \left( \frac{\partial m_1}{\partial u} + c_1 \frac{\partial m_1}{\partial v} \right)^2 \right\} \]
\[ + \frac{q_1}{l_1} \left\{ \frac{\partial^2 l_1}{\partial u^2} + 2c_1 \frac{\partial^2 l_1}{\partial u \partial v} + \frac{c_1^2}{l_1} \frac{\partial^2 l_1}{\partial v^2} - \frac{1}{l_1} \left( \frac{\partial l_1}{\partial u} + c_1 \frac{\partial l_1}{\partial v} \right)^2 \right\} \]
\[ / \left[ \frac{\partial m_1}{\partial v} + \frac{q_1}{l_1} \frac{\partial l_1}{\partial v} \right] \bigg|_{(u,v)=(u_0,v_0)}. \]

But differentiating (25) with respect to \(x^{p_1,q_1}\), we obtain
\[ c_2^{p_1,q_1} = \left. \frac{d^2 y^{p_1,q_1}}{dx^{p_1,q_1}} \right|_{x^{p_1,q_1}=x_0^{p_1,q_1}} \]
\[ = -\left[ \frac{p_1}{m_1} \left\{ \frac{\partial^2 m_1}{\partial u^2} + 2c_1^{p_1,q_1} \frac{\partial^2 m_1}{\partial u \partial v} + \left(\frac{c_1^{p_1,q_1}}{m_1}\right)^2 \frac{\partial^2 m_1}{\partial v^2} - \frac{1}{m_1} \left( \frac{\partial m_1}{\partial u} + c_1^{p_1,q_1} \frac{\partial m_1}{\partial v} \right)^2 \right\} \right] \]
\[ + \frac{q_1}{l_1} \left\{ \frac{\partial^2 l_1}{\partial u^2} + 2c_1^{p_1,q_1} \frac{\partial^2 l_1}{\partial u \partial v} + \left(\frac{c_1^{p_1,q_1}}{l_1}\right)^2 \frac{\partial^2 l_1}{\partial v^2} - \frac{1}{l_1} \left( \frac{\partial l_1}{\partial u} + c_1^{p_1,q_1} \frac{\partial l_1}{\partial v} \right)^2 \right\} \]
\[ / \left[ \frac{\partial m_1}{\partial v} + \frac{q_1}{l_1} \frac{\partial l_1}{\partial v} \right] \bigg|_{(u,v)=(x_0^{p_1,q_1},y_0^{p_1,q_1})}. \]

Using Lemma 3.3 again, we see that \( c_2^{p_1,q_1} \to c_2 \) as \(|p_1| + |q_1| \to \infty\). Continuing in this way, we can also show that \( c_3^{p_1,q_1} \to c_3 \) as \(|p_1| + |q_1| \to \infty\). \( \square \)

(4) The fourth step is to consider the holomorphic map \( G_{2t} = (m_2,l_2) : V \to \mathbb{C}^2 \) defined near \((x_0,y_0)\).

**Proposition 6.5.** The images \( C_1 \) and \( C_1^{p_1,q_1} \) of \( C_1 \) and \( C_1^{p_1,q_1} \), when \(|p_1| + |q_1| \) is sufficiently large, under \( G_{2t} \) respectively are smooth curves through \((m_2^{0},l_2^{0})\). \( C_1 \) is the graph of a holomorphic map
\[ l_2 = l_2^{0} + a_1(m_2 - l_2^{0}) + \frac{a_2}{2}(m_2 - l_2^{0})^2 + \frac{a_3}{6}(m_2 - l_2^{0})^3 + \text{higher order terms}. \]

near \((m_2^{0},l_2^{0})\) with \( a_1 \neq 0 \) and \( C_1^{p_1,q_1} \) is the graph of a holomorphic map
\[ l_2 = l_2^{0} + a_1^{p_1,q_1}(m_2 - l_2^{0}) + \frac{a_2^{p_1,q_1}}{2}(m_2 - l_2^{0})^2 + \frac{a_3^{p_1,q_1}}{6}(m_2 - l_2^{0})^3 + \text{higher order terms}. \]
Proof. $C_1$ is the graph of a holomorphic map with a Taylor expansion
\[ v - v_0 = c_1(u - u_0) + \frac{c_2}{2}(u - u_0)^2 + \frac{c_3}{6}(u - u_0)^3 + O((u - u_0)^4) \]
near $(u_0, v_0)$. By our definition of $C_1$, we have $m_1 = m_1^0$ for points on $C_1$. Thus
\[ \frac{dm_2}{dx} \bigg|_{x=u_0} = \left[ \frac{\partial m_2}{\partial u} + c_1 \frac{\partial m_2}{\partial v} \right]_{(u_0,v_0)} = 0. \]

Since $\nabla m_1(u_0,v_0)$ and $\nabla m_2(u_0,v_0)$ are linearly independent, we must have
\[ \left. \frac{dm_2}{dx} \right|_{x=u_0} = \left[ \frac{\partial m_2}{\partial u} + c_1 \frac{\partial m_2}{\partial v} \right]_{(u,v)=(u_0,v_0)} \neq 0. \]
Thus, the map $G_{2\ell}$ restricted to $C_1$ is nonsingular at $(u_0,v_0)$; hence, the image of $C_1$ under $G_{2\ell}$ is a smooth curve through $(m_2^0, m_2^0)$.

Recall that $C_1^{p_1.q_1}$ is the graph of a holomorphic map with an expansion
\[ v - v_0^{p_1.q_1} = c_1^{p_1.q_1}(u - u_0^{p_1.q_1}) + \frac{c_2^{p_1.q_1}}{2}(u - u_0^{p_1.q_1})^2 + \frac{c_3^{p_1.q_1}}{6}(u - u_0^{p_1.q_1})^3 + O((u - u_0^{p_1.q_1})^4) \]
at $(u_0^{p_1.q_1}, v_0^{p_1.q_1})$.

Since $c_1^{p_1.q_1} \to c_1$ and $(u_0^{p_1.q_1}, v_0^{p_1.q_1}) \to (u_0, v_0)$ as $|p_1| + |q_1| \to \infty$, we have
\[ \left. \frac{dm_2}{dx} \right|_{x=u_0^{p_1.q_1}} = \left[ \frac{\partial m_2}{\partial u} + c_1^{p_1.q_1} \frac{\partial m_2}{\partial v} \right]_{(u,v)=(u_0^{p_1.q_1},v_0^{p_1.q_1})} \neq 0 \]
when $|p_1| + |q_1|$ is sufficiently large.

We showed above
\[ \left. \frac{dm_2}{dx} \right|_{x=x_0} \neq 0 \text{ and } \left. \frac{dm_2}{dx} \right|_{x=p_1.q_1=x_0^{p_1.q_1}} \neq 0, \]
when $|p_1| + |q_1|$ is sufficiently large. In the same way we show
\[ \left. \frac{dl_2}{dx} \right|_{x=u_0} \neq 0 \text{ and } \left. \frac{dl_2}{dx} \right|_{x=u_0^{p_1.q_1}} \neq 0. \]
when $|p_1| + |q_1|$ is sufficiently large. Thus, the rest of the proposition follow. \hfill $\Box$

**Proof of Theorem 6.4.** We differentiate \((\ref{eq:26})\) with respect to \(x\) and evaluate at \(x = u_0\) to obtain

$$\frac{dl_1}{dx} \bigg|_{x=u_0} = a_1 \frac{dm_1}{dx} \bigg|_{x=u_0}.$$  

So

$$a_1 = \frac{\frac{dl_1}{dx} + c_1 \frac{dm_1}{dx}}{\frac{dm_1}{dx} + c_1 \frac{dl_1}{dx}} \bigg|_{(u,v) = (u_0,v_0)}.$$  

Similarly, from \((\ref{eq:27})\), we obtain

$$a^p_{1,q_1} = \frac{\frac{dl_1}{dx} + c^p_{1,q_1} \frac{dm_1}{dx}}{\frac{dm_1}{dx} + c^p_{1,q_1} \frac{dl_1}{dx}} \bigg|_{(u,v) = (x_0^{p_{1,q_1}}, y_0^{p_{1,q_1}})}.$$  

Since \(c^p_{1,q_1} \to c_1\) as \(|p_1| + |q_1| \to \infty\) by Proposition 6.4, we have \(a^p_{1,q_1} \to a_1\) as \(|p_1| + |q_1| \to \infty\).

Proceeding in this way using successive differentiations we also obtain \(a^p_{1,q_1} \to a_i\) as \(|p_1| + |q_1| \to \infty\) for \(i = 2, 3\). \hfill $\Box$

We close this section with the following lemma which is implied by the fact that the curves \(G_{2i}(C_{p_{1,q_1}})\) are invariant under the involution \((m_2, l_2) \mapsto (1/m_2, 1/l_2)\) restricted to a small neighborhood of \((m_0^2, l_0^2)\) in \(C^2\).

**Lemma 6.6.** When \(|p_1| + |q_1|\) is large, \(a^p_{2,q_1} = -m_2 a^p_{1,q_1} + l_2 (a^p_{1,q_1})^2\).

**Proof.** By results of Neumann and Zagier\([11]\), there is an involution \(I : V \to V\) such that \(G_{1i}(I(u,v)) = (m_1, l_1)\) and \(G_{2i}(I(u,v)) = (1/m_2, 1/l_2)\) when \(G_{1i}(u,v) = (m_1, l_1)\) and \(G_{2i}(u,v) = (m_2, l_2)\). Since \(C_{p_{1,q_1}}\) is invariant under \(I\), \(G_{2i}(C_{p_{1,q_1}})\) is invariant under the involution \((m_2, l_2) \mapsto (1/m_2, 1/l_2)\). \hfill $\Box$

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