On quantum field theories with finitely many degrees of freedom

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Abstract

The existence of inequivalent representations in quantum field theory with finitely many degrees of freedom is shown. Their properties are exemplified and analysed for concrete and simple models. In particular the relations to Bogoliubov–Valatin quasi-particles, to thermo field dynamics, and to \(q\)-deformed quantum theories are put forward. The thermal properties of the non-trivial vacuum are given and it is shown that the thermodynamic equilibrium state is uniquely obtained by an irreversible vacuum dynamics. Finally, the theory is applied to a realistic model: the BCS–theory of superconductivity. An exact solution in order \(O(N^{-1})\) for the full particle number conserving BCS–Hamiltonian with particle number symmetric ground state is given.
1 Introduction

The quantum theory of interacting many-particle systems is governed by the use of field operators. In their Fock representation these operators have the meaning of creating or annihilating specific single-particle quantum states. For quantum fields with infinitely many degrees of freedom the technical managing of the theory as well as its physical interpretation are complicated by the appearance of infinitely many irreducible, unitarily inequivalent representations. According to Haag's theorem the Fock representation is not admissible for interacting or self-interacting quantum field theories [1]. As a matter of principle, the use of the interaction picture and perturbation theory are not allowed. In the thermodynamic limit it is even impossible to obtain the thermal properties of a free fermion or boson gas in the Fock representation [2]. Thus, the need to go beyond the Fock representation in many-particle theory is obvious.

The possibility of inequivalent representations must be understood as the major difference between quantum mechanics and quantum field theory. This structural enrichment of the theory should be highly appreciated, since it provides the basis for describing a large variety of correlations and collective microscopic and macroscopic phenomena. Many rigorous approaches have been put forward in axiomatic and algebraic quantum theory, showing the relevance of inequivalent representations to concrete physical problems [3, 4]. However, the mathematical effort to achieve the results is rather high.

Due to von Neumann there are — unfortunately — no unitarily inequivalent, irreducible representations of the canonical commutation or anticommutation relations for quantum field operators with finitely many degrees of freedom. From the physical point of view this seems to be unsatisfactory because the physics of an arbitrarily large but finite system differs remarkably from its infinite limit. In addition, explicit calculations could be simplified, if the appearance of inequivalent representations were already given in quantum theories with finitely many degrees of freedom.
Since it is impossible to circumvent von Neumann’s theorem let us ask: ‘How do inequivalent representations come about in taking the infinite volume limit?’ Consider field operators \( c_k \) and \( c_k^* \) and an extensive observable such as the particle number operator \( N \). For \textit{finitely} many degrees of freedom the particle number operator is given by

\[
N = \sum_k c_k^* c_k
\]

in its Fock representation. Going over to the infinite system expression (1) becomes undefined, i.e. the particle number operator does not belong to the algebra generated by the field operators any longer. Rather, one has to pass to the von Neumann algebra which is the weak closure of the field operator algebra in order to get the particle number operator as a well defined object \[3, 4\]. On the other hand the commutator

\[
[ N, c_k ] = -c_k
\]

is not affected by taking the infinite volume limit.

Therefore a unified quantum field theory of finite and infinite systems should be constructed from steady expressions such as (2) alone. This enforces the use of an enlarged algebra of field operators and observables, which — as will be shown — possesses inequivalent representations for finitely many degrees of freedom.

## 2 Quantum field theory with finitely many degrees of freedom

### 2.1 Fundamentals

Let us start with the well known CAR–algebra \( \mathcal{A}(\mathcal{H}) \) defined by

\[
\{ c(f), c(h) \} = 0, \quad \{ c^*(f), c(h) \} = \langle f | h \rangle \ \mathbb{1}
\]

(and \( f \to c(f) \) is anti-linear), where \( f, h \) are elements of a finite dimensional Hilbert space \( \mathcal{H} = \mathbb{C}^n \) \[4\]. In addition to the algebra of field operators we have a symmetry group with the generators describing the observables of the theory. These observables and their relations to the CAR–algebra have to be fixed, too. For a symmetry group \( G \subseteq U(n) \) there is a unitary representation \( u_g \) for \( g \in G \) on \( \mathcal{H} = \mathbb{C}^n \) defining a \( * \)–automorphism \( \alpha_g \) on the CAR–algebra in terms of

\[
\alpha_g ( c(f) ) = c(u_g f)
\]

We require this \( * \)–automorphism to be unitarily implemented on the CAR–algebra.

\textbf{Definition 1} The group \( G \) and the \( * \)–automorphism \( \alpha_g \) are said to be unitarily implemented on the CAR–algebra, if to any \( g \in G \) there corresponds an operator \( U_g \) with

\[
U_e = \mathbb{1}, \quad U_{g_1 g_2} = U_{g_1} U_{g_2}, \quad U_{g^{-1}} = U_{g}^*
\]

\footnote{The restriction to the CAR–algebra is not essential, but was chosen here because it allows explicite matrix representations for the examples discussed below.}
\[ \alpha_g(A) = U_g A U_g^* \quad (6) \]

for all \( A \in \mathcal{A}(\mathcal{H}) \).

It follows from (6) that \( U_g U_g^* = U_g^* U_g = \mathbb{1} \). Furthermore, \( U_g A U_g^* \) is always an element of the CAR–algebra, but this need not be true for \( U_g \). The algebra generated by the field operators and the independent operators \( U_g \) is in fact an (infinite dimensional) extension of the CAR–algebra and will be denoted by \( \mathcal{A}_G(\mathcal{H}) \).

The calculation of expectation values, transition amplitudes, etc. requires the concept of states. In algebraic quantum theory these are introduced as linear, normalized, positive \( \mathcal{C} \)-valued functionals on the operator algebra under consideration. We shall concentrate on vacuum states here.

**Definition 2** A vacuum state on the algebra \( \mathcal{A}_G(\mathcal{H}) \) is a linear, normalized, positive functional \( \omega \) with the additional property

\[ \omega(A) = \omega(AU_g) \quad (7) \]

for all \( A \in \mathcal{A}_G(\mathcal{H}) \) and all \( g \in G \).

It follows from (7) that \( \omega(A) = \omega(U_g A) \). The set of vacuum states is convex, i.e. any convex combination of two vacuum states is again a vacuum state. Sometimes a vacuum state is called \( G \)-invariant, too.

If compared with the Wightman axioms \([3]\) of quantum field theory there show up to be strong interrelations if the states are visualized by their GNS–representation \( \omega(A) = \langle \Omega_\omega | \pi_\omega(A) \Omega_\omega \rangle \). The main difference is that the symmetry group \( G \subseteq U(n) \) is much simpler than the Poincaré group. Therefore we can drop the restrictions on the domains of definition here. In addition we take a different point of view concerning the uniqueness of the vacuum. The vacuum is not determined uniquely by the symmetry properties \([2]\) alone, but requires the explicite dynamics of a concrete model. However, besides of these points the following examples can be viewed as realisations of the Wightman axioms for simple symmetry groups.

Since we will evaluate the theory only for the Lie groups \( G = U(n) \) we can put down the fundamental equations in terms of the generators as well. Denoting \( c_\alpha = c(f_\alpha) \) for a fixed orthonormal basis \( f_\alpha \) of \( \mathcal{H} \) and the generators of \( u_g \) and \( U_g \) by \( q^k \) and \( Q^k = (Q^k)^* \), respectively, we get the algebraic relations

\[
\{ c_\alpha, c_\beta \} = 0, \quad \{ c^*_\alpha, c_\beta \} = \delta_{\alpha\beta} \mathbb{1} \\
\left[ Q^k, c_\alpha \right] = \sum_\beta \hat{q}_{\alpha\beta}^k c_\beta, \quad \left[ Q^k, c^*_\alpha \right] = \sum_\beta \hat{q}_{\alpha\beta}^k c^*_\beta \\
\left[ Q^k, Q^l \right] = i f^{kl}_m Q^m, \quad \left[ q^k, q^l \right] = i f^{kl}_m q^m
\quad (8)
\]

with \( \hat{q}_{\alpha\beta}^k = -\bar{q}_{\alpha\beta}^k \), (the bar denotes complex conjugation). Furthermore, the generators satisfy the Lie relations

\[
\left[ Q^k, Q^l \right] = i f^{kl}_m Q^m, \quad \left[ q^k, q^l \right] = i f^{kl}_m q^m
\quad (9)
\]
with the structure constants \( f^{kl}_{m} \) of the corresponding Lie algebra. Vacuum states are characterized by the equations
\[
\omega \left( A Q^k \right) = 0
\] (10)
for any \( A \in \mathcal{A}_G(\mathcal{H}) \) and all observables \( Q^k \). Again it follows that \( \omega \left( Q^k A \right) = 0 \), too.

We can also introduce the notion of eigenvalue equations in this formulation, which will prove to be useful for concrete calculations.

**Definition 3** An operator \( B_q \) is called (right) eigenvector of \( Q \) with (right) eigenvalue \( q \), if \( \omega \left( B^*_q B_q \right) \neq 0 \) and
\[
\omega \left( A Q B_q \right) = q \omega \left( A B_q \right)
\] (11)
for all \( A \in \mathcal{A}_G(\mathcal{H}) \).

The definition of left eigenvectors and left eigenvalues is obvious. One can show that left and right eigenvalues coincide and are real for Hermitean observables \( Q = Q^* \). Further, if \( B_q \) is a right eigenvector then \( B^*_q \) is a left eigenvector to the same eigenvalue.

### 2.2 Examples

We will now evaluate the previous setup for the cases \( \mathcal{H} = \mathbb{C} \) and \( \mathcal{H} = \mathbb{C}^2 \) and thereby give explicit examples for inequivalent vacuum representations in quantum field theory with finitely many degrees of freedom.

For \( \mathcal{H} = \mathbb{C} \) and \( G = U(1) \) the field operator and observable algebra is characterized by the equations
\[
\{ c, c \} = 0, \quad \{ c^*, c \} = \mathbb{I}, \quad [N, c] = -c,
\] (12)
with \( N \) being the Hermitean generator of the unitarily implemented \( U(1) \) symmetry.

A state on this algebra is completely fixed, if its values on all algebraically independent elements are known. By making use of the commutation relations (12) any algebra element can be ‘normal-ordered’ such that all operators \( N \) are to the right of all field operators. Taking these normal-ordered operators as algebraic basis requires any vacuum state to be zero on all elements except the CAR–algebra of field operators. Hence, any vacuum state is completely determined by its values on the CAR–algebra.

**Lemma 1** The set of all vacuum states on the algebra defined in (12) is parametrized by the real parameter \( v = \omega \left( cc^* \right) \), with \( v \in [0, 1] \).

**Proof:** An algebraic basis of the CAR–algebra is given by \( \mathcal{B} = \{ \mathbb{I}, c, c^*, cc^* \} \), whose elements satisfy the eigenvalue equations
\[
\omega \left( AN \mathbb{I} \right) = 0, \quad \omega \left( ANcc^* \right) = 0
\]
\[
\omega \left( ANc \right) = -\omega \left( Ac \right), \quad \omega \left( ANc^* \right) = \omega \left( Ac^* \right)
\] (13)
Setting \( A = \mathbb{I} \) yields \( \omega(c) = \omega(c^*) = 0 \), and together with the normalization \( \omega \left( \mathbb{I} \right) = 1 \) the only undetermined value is \( \omega(ce^*) = v \). From the positivity of \( \omega \) it follows that \( v \geq 0 \).
With the aid of (12) and the linearity of \( \omega \) we get \( \omega(c^*c) = 1 - v \) and positivity shows that \( 1 - v \geq 0 \). ♣

Labeling the states in terms of their values on \( cc^* \) we can give their extremal decomposition in the following way:

**Theorem 1** The convex set of vacuum states \( \{ \omega_v; v \in [0,1] \} \) possesses two extremal, pure states \( \omega_1 \) and \( \omega_0 \). The extremal decomposition of \( \omega_v \) is given by

\[
\omega_v = v \omega_1 + (1 - v) \omega_0 .
\]

**Proof:** Assume that \( \omega_1 \) had some non-trivial decomposition \( \omega_1 = \lambda \omega_{v_1} + (1 - \lambda) \omega_{v_2} \) with \( 0 < \lambda < 1 \). Applied to \( c^*c \) we get \( \lambda (1 - v_1) + (1 - \lambda) (1 - v_2) = 0 \). Since \( \lambda \) and \( 1 - \lambda \) are positive it follows that \( v_1 = v_2 = 1 \). Therefore \( \omega_1 \) can only be decomposed trivially. In the same way one shows that \( \omega_0 \) is extremal. It remains to show that the extremal decomposition (14) is complete. This follows, if (14) is evaluated on an arbitrary algebra element. It suffices to consider normal-ordered algebra elements upon which \( \omega_v \) does not vanish identically, i.e. \( A = \alpha_1 \mathbb{1} + \alpha_2 c^*c \). We get \( v\omega_1(A) + (1 - v)\omega_0(A) = v(\alpha_1 + \alpha_2) + (1 - v)\alpha_1 = \alpha_1 + v\alpha_2 = \omega_v(A) \). ♣

The convex set of vacua consists of the line segment between the extremal vacua \( \omega_1 \) and \( \omega_0 \). Hence, it has a simplex structure! This is a rather unusual feature in ordinary quantum theory, but it is the characteristic state space structure whenever inequivalent representations occur [3].

Explicite matrix representations have to be reconstructed from the vacuum states. Due to the extremal decomposition (14) the general representation is reducible and the irreducible components are given by the representations belonging to the extremal states. For the formal GNS–construction

\[
\omega_v(A) = \langle \Omega_v | \pi_v(A) | \Omega_v \rangle
\]

it must be

\[
\pi_v(A) = \pi_1(A) \oplus \pi_0(A) , \quad | \Omega_v \rangle = e^{i\alpha} \sqrt{v} | \Omega_1 \rangle \oplus \sqrt{1 - v} | \Omega_0 \rangle
\]

with an undetermined relative phase factor \( e^{i\alpha} \).

The extremal representations \( \pi_1 \) and \( \pi_0 \) along with their extremal vacuum vectors \( | \Omega_1 \rangle \) and \( | \Omega_0 \rangle \) can be reconstructed from (14) for \( v = 1 \) and \( v = 0 \), respectively, by taking into consideration that \( \{ | \Omega_1 \rangle , \pi_1(c^*) | \Omega_1 \rangle \} \) and \( \{ | \Omega_0 \rangle , \pi_0(c) | \Omega_0 \rangle \} \) are orthonormal basis systems for \( v = 1 \) and \( v = 0 \), respectively. One gets the matrix representations

\[
\pi_1(c) = \pi_0(c) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

\[
\pi_1(N) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_0(N) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},
\]

and

\[
| \Omega_1 \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad | \Omega_0 \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
Next to the Fock representation $\pi_1$ with Fock vacuum $|\Omega_1\rangle$ we have the extremal representation $\pi_0$ which we shall call the dual Fock representation, because it is characterized by $\pi_0(c^*)|\Omega_0\rangle = 0$. Obviously still von Neumann’s theorem is not falsified here, since the irreducible representations of the CAR–algebra alone are equivalent. However the larger field operator and observable algebra $A_G(\mathcal{H})$ does indeed have irreducible, inequivalent, and finite dimensional representations $\pi_1$ and $\pi_0$, because of the inequivalent spectrum of the particle number observable $N$ in both representations.

For the extremal vacua the particle number operator can be expressed as $\pi_1(N) = \pi_1(c^*c)$ or $\pi_0(N) = \pi_0(c^*c - \mathbb{I})$, respectively, but aside these extremal cases the particle number operator does not belong to the CAR–algebra. Like in the case of infinitely many degrees of freedom the observables do not belong to the CAR–algebra in general vacuum representations.

If only vacuum representations are taken into consideration an arbitrary element of $A_G(\mathcal{H})$ can be written in the form $A = A_1 + NA_2$ with $A_1, A_2$ being elements of the CAR–algebra.

Our results might appear to be a little hair-splitting, because the Fock and dual Fock representations could be converted into one another by reinterpreting particles as holes and vice versa and renormalizing the particle number operator. In passing over to the more complicated case $\mathcal{H} = \mathbb{C}^2$ and $G = U(2)$ however, we will find a third extremal representation which is completely different from the Fock and dual Fock representations.

We seek to find all irreducible vacuum representations of the (anti)commutation relations

$$\{c_\alpha, c_\beta\} = 0, \quad \{c^*_\alpha, c_\beta\} = \delta_{\alpha\beta}\mathbb{I}, \quad [N, c_\alpha] = -c_\alpha$$

$$[S^k, c_\alpha] = \sum_\beta \sigma^k_{\alpha\beta}c_\beta, \quad [S^k, S^l] = i\varepsilon_{klm}S^m, \quad [N, S^k] = 0,$$

(19)

where $\alpha, \beta \in \{1, 2\}$, $S^k$ are the implemented spin operators and $\sigma^k$ the Pauli spin matrices.

Calculating the eigenvalues of $N, S^3$ and $\vec{S}^2$ we get the following tabular of simultaneous eigenvectors and their eigenvalues:\footnote{It is possible that some of the formal eigenvectors are zero vectors. This will depend on their representation.}
Although the Pauli principle is fully valid, there appears to be a spin–1 triplet of eigenvectors. However, these are zero vectors in the Fock and dual Fock representations, inspiring one to search for representations where they are non-zero.

We shall systematically evaluate all vacuum representations of the relations (19). By the same arguments as before any vacuum state is completely determined by its values on an algebraic basis of the CAR–algebra, which is chosen to be the set of eigenvectors in the tabular above. Like in (19) we conclude that a vacuum state is zero on all eigenvectors unless all its eigenvalues are zero. Therefore we have:

**Lemma 2** A vacuum state on the algebra given by (19) is completely characterized in terms of two real parameters \( v = \omega_{vw}(c_1^*c_1) = \omega_{vw}(c_2^*c_2) \) and \( w = \omega_{vw}(c_1c_2c_2^*c_1^*) \) restricted by the inequalities \( 1 \geq v \geq w \geq 0 \) and \( w \geq 2v - 1 \).

**Proof:** Enumerating the eigenvectors by \( e_1 = 1, e_2 = \frac{1}{2}(c_1c_1^* + c_2c_2^*), \ldots, e_{16} = c_1^*c_2^* \) the restrictions on the parameters \( v \) and \( w \) follow from the positivity of \( \omega(e_i^*e_i) \geq 0 \) for \( i \in \{1, 2, \ldots, 16\} \).

**Theorem 2** There are three extremal vacuum states \( \omega_{11}, \omega_{00} \) and \( \omega_{\frac{1}{2}0} \). The extremal decomposition of a general vacuum state is given by

\[
\omega_{vw} = w \omega_{11} + (1 - 2v + w) \omega_{00} + 2(v - w) \omega_{\frac{1}{2}0} 
\]  

(20)

**Proof:** Here \( \omega_{11} \) and \( \omega_{00} \) are the Fock and dual Fock vacua, respectively. We shall only prove explicitly that \( \omega_{\frac{1}{2}0} \) is extremal. Suppose there were a non-trivial decomposition \( \omega_{\frac{1}{2}0} = \lambda \omega_{v_1w_1} + (1 - \lambda) \omega_{v_2w_2} \) with \( \lambda \in [0, 1] \). Evaluated on \( c_1c_2c_2^*c_1^* \) gives \( 0 = \lambda w_1 + (1 - \lambda)w_2 \), implying \( w_1 = w_2 = 0 \). Applying the decomposition to \( c_1^*c_2^*c_2c_1 \) and using \( w_1 = w_2 = 0 \) yields \( 0 = \lambda(1 - 2v_1) + (1 - \lambda)(1 - 2v_2) \). Therefore \( v_1 = v_2 = \frac{1}{2} \) and \( \omega_{\frac{1}{2}0} \) is extremal. In order
to determine the decomposition (20) we evaluate the ansatz \( \omega_{vw} = \lambda_1 \omega_{11} + \lambda_2 \omega_{00} + \lambda_3 \omega_{\frac{1}{2}0} \)
for the algebra elements \( 1, c_1 c_1^* \) and \( c_1 c_2 c_2^* c_1^* \). The resulting equations \( 1 = \lambda_1 + \lambda_2 + \lambda_3 \), \( v = \lambda_1 + \frac{1}{2} \lambda_3 \) and \( w = \lambda_1 \) can be resolved to give (20). ♣

Again the corresponding vacuum representation decomposes into a direct sum of the extremal representations

\[
\pi_{vw} = \pi_{11} \oplus \pi_{00} \oplus \pi_{\frac{1}{2}0},
\]

where the vacuum vector

\[
| \Omega_{vw} \rangle = e^{i \alpha_1} \sqrt{v} | \Omega_{11} \rangle \oplus e^{i \alpha_2} \sqrt{1 - 2v + w} | \Omega_{00} \rangle \oplus \sqrt{2(v - w)} | \Omega_{\frac{1}{2}0} \rangle
\]

(with two arbitrary relative phases) describes the mixing of the extremal representations. One finds that \( \pi_{11} \) and \( | \Omega_{11} \rangle \) are the Fock representation and Fock vacuum. Similar \( \pi_{00} \) and \( | \Omega_{00} \rangle \) are the dual Fock representation and dual Fock vacuum, completely characterized by the equations \( \pi_{00}(c_1^*) | \Omega_{00} \rangle = \pi_{00}(c_2^*) | \Omega_{00} \rangle = 0 \). Since these representations are well known, we shall concentrate on the extremal representation \( \pi_{\frac{1}{2}0} \) with vacuum \( | \Omega_{\frac{1}{2}0} \rangle \).

**Lemma 3** The Fock and dual Fock representations are both irreducible 4 \( \times \) 4 matrix representations, containing spin–0 and spin–1/2 configurations with particle numbers 0, 1, or 2 for the Fock and 0, -1, or -2 for the dual Fock representation, respectively.

**Proof:** The proof is trivial and omitted here. ♣

Lemma 3 must be contrasted with the following result:

**Lemma 4** The extremal representation \( \pi_{\frac{1}{2}0} \) is an irreducible, full 8 \( \times \) 8 matrix representation. There are spin–0, spin–1/2 and spin–1 configurations. The particle number can take the values 0, 1 and -1.

**Proof:** In analysing the scalar products \( \omega_{vw}(e_i^* e_j) \) of the algebraic basis of eigenvectors one finds that for \( (v, w) = (\frac{1}{2}, 0) \) a complete orthonormal basis is given by the set \( \{ \beta_1 = 1, \beta_2 = \sqrt{2} c_1 c_2^*, \beta_3 = c_1 c_1^* - c_2 c_2^*, \beta_4 = \sqrt{2} c_2 c_1^*, \beta_5 = \sqrt{2} c_1^*, \beta_6 = \sqrt{2} c_2, \beta_7 = \sqrt{2} c_1^*, \beta_8 = \sqrt{2} c_2^* \} \), i.e. \( \omega_{\frac{1}{2}0}(\beta_i^* \beta_j) = \delta_{ij} \). Making use of this basis we get explicit matrix representations of the field operators and observables by evaluating

\[
\pi^{ij}_{\frac{1}{2}0}(A) = \omega_{\frac{1}{2}0}(\beta_i^* A \beta_j).
\]

It follows that

\[
\pi_{\frac{1}{2}0}(c_1) =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

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\[\pi_{20}(c_2) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix}\]  

(24)

for the field operators and

\[\pi_{20}(S_k) = 0 \oplus \Sigma_k \oplus \sigma_k \oplus \hat{\sigma}_k\]  

(25)

with the one dimensional zero matrix \(0\) and

\[\Sigma_1 = \begin{pmatrix}
0 & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & 0 
\end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix}
0 & \frac{i}{\sqrt{2}} & 0 \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
0 & \frac{i}{\sqrt{2}} & 0 
\end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1 
\end{pmatrix}.\]  

(26)

Further the observables \(N\) and \(\vec{S}^2\) are diagonal with \(\pi_{20}(N) = \text{diag}(0, 0, 0, 0, -1, -1, 1, 1)\) and \(\pi_{20}(\vec{S}^2) = \text{diag}(0, 0, 2, 2, 2, 3/4, 3/4, 3/4, 3/4)\). Finally the vacuum vector is represented by \(\Omega_{20} = (1, 0, 0, 0, 0, 0, 0, 0)\).

Notice that the representation (24) of the CAR–algebra is reducible. Analysing the characters shows it to be unitarily equivalent to a direct sum of two Fock representations. Nevertheless the representation \(\pi_{20}\) of the complete algebra \(A_G(\mathcal{H})\) is the full 8 \(\times\) 8 matrix algebra and completely different in its structure from the Fock or dual Fock representations. It can be shown that any algebra element in the representation \(\pi_{20}\) is of the form \(A = A_0 + \sum_{k=1}^3 S_k A_k\) with \(A_0, \ldots, A_3\) being elements of the CAR–algebra.

**Lemma 5** The representation \(\pi_{20}\) is characterized by the equations

\[\pi_{vw}(c_1 c_2) |\Omega_{vw}\rangle = \pi_{vw}(c_1^* c_2^*) |\Omega_{vw}\rangle = 0.\]  

(27)

**Proof:** If \(\pi_{vw}(c_1^* c_2^*) |\Omega_{vw}\rangle = 0\) it follows that \(\omega_{vw}(c_2 c_1 c_1^* c_2^*) = w = 0\). Similar we have \(\omega_{vw}(c_1^* c_2^* c_2 c_1) = 1 - 2v + w = 0\). Hence \(\omega_{vw} = \omega_{40}\). ♣

### 3 Quasi-particles and thermodynamics in Fock space

In the previous section it was shown that inequivalent representations might well exist in quantum theory with finitely many degrees of freedom. In order to get some more insight into their physical meaning we will compare these results to different techniques in many-particle theory.
3.1 The Bogoliubov–Valatin transformation

The quasi-particle transformation introduced by Bogoliubov \[6\] and Valatin \[7\] is used to describe BCS-like correlations in the theory of superconductivity, nuclear theory, etc. in Fock space. It is given in terms of the quasi-particle operators

\[
\alpha_1 = \sqrt{1-h_1 c_1 - \sqrt{h_1 c_2}}, \quad \alpha_2 = \sqrt{h_1 c_1^* + \sqrt{1-h_2}}
\]

in Fock representation. The ‘vacuum’ of the quasi-particles defined by

\[
|\alpha_\sigma \rangle = 0
\]

follows as

\[
|BV\rangle = (\sqrt{1-h} + \sqrt{h c_1^* c_2^*}) |0\rangle,
\]

with \(|0\rangle\) being the Fock vacuum \((c_\sigma |0\rangle = 0)\).

If the Bogoliubov–Valatin state is defined by

\[
\omega_{BV}^A = \langle BV(h) | A | BV(h) \rangle
\]

one calculates that it takes exactly the same values on the CAR-algebra as the vacuum state \(\omega_{1-h,1-h} = (1-h)\omega_{11} + h\omega_{00}\) with the exception of the basis elements \(c_1 c_2\) and \(c_1^* c_2^*\).

This was to be expected, because the Bogoliubov–Valatin transformation does not conserve the particle number symmetry and we required our vacuum states to be exact vacua with respect to the quantum numbers of observables. In addition the extremal vacuum vectors \(|\Omega_{11}\rangle\) and \(|\Omega_{00}\rangle\) have exactly the same properties as \(|0\rangle\) and \(c_1^* c_2^* |0\rangle\), respectively, except the fact that \(c_1^* c_2^* |0\rangle\) has particle number 2 instead of 0. Contrary to the mixing of creation and annihilation operators \((28)\) in their Fock representation one can mix the Fock and dual Fock representations of the original field operators and simultaneously have an exact particle number symmetric vacuum here.

Still, not every vacuum representation may be obtained by a Bogoliubov–Valatin transformation. No contributions from the extremal representation \(\pi_{\Phi^2}\) can be simulated by Bogoliubov–Valatin quasi-particles. However, we will show below that this is possible by introducing more general quasi-particles such as \(q\)-deformed quantum fields in their Fock representation.

3.2 \(q\)-deformed CAR–algebras

In modern models of condensed matter and nuclear physics many attempts have been made to describe correlated many-particle quantum systems in terms of deformed quantum fields and deformed symmetries \[10\]. Here we will simulate the effects of inequivalent representations of the algebras \(\mathcal{A}_G(\mathcal{H})\) in the Fock representation of the corresponding \(q\)-deformed CAR–algebras. We want to concentrate on the case \(\mathcal{H} = \Phi^2\) showing how \(q\)-deformed CAR–algebras go beyond Bogoliubov–Valatin quasi-particles and at the same time belong to the general setup of vacuum representations on \(\mathcal{A}_G(\mathcal{H})\).

The \(q\)-deformed CAR–algebra will be characterized by the relations

\[
\begin{align*}
    a_\alpha a_\alpha^\dagger &= [N_\alpha], & a_\alpha a_\alpha^\dagger &= [N_\alpha + 1] \\
    \{ a_1, a_2 \} &= \{ a_1^\dagger, a_2 \} = \{ a_1^\dagger, a_2^\dagger \} = 0 \\
    [N_\alpha, a_\beta] &= -\delta_{\alpha\beta} a_\beta, & [N_\alpha, a_\beta^\dagger] &= \delta_{\alpha\beta} a_\beta^\dagger
\end{align*}
\]
with the deformation function

\[
[N] = \frac{q^N - q^{-N}}{q - q^{-1}}
\]

(32)

and the \(SU_q(2)\) generators are

\[
J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_3 = \frac{N_1 - N_2}{2}.
\]

(33)

Putting down the Fock representation by requiring

\[
a_1 |0,0\rangle = a_2 |0,0\rangle = 0
\]

(34)

we can use the normalized set of state vectors

\[
|n_1, n_2\rangle = \frac{1}{\sqrt{[n_1]! [n_2]!}} \left( (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} |0,0\rangle \right).
\]

(35)

to obtain the matrix representations

\[
\begin{align*}
\langle n'_1, n'_2 | a_1 n_1, n_2 \rangle &= \sqrt{[n_1]} \delta_{n'_1, n_1-1} \delta_{n'_2, n_2} \\
\langle n'_1, n'_2 | a_2 n_1, n_2 \rangle &= (-1)^{n_1} \sqrt{[n_2]} \delta_{n'_1, n_1} \delta_{n'_2, n_2+1} \\
\langle n'_1, n'_2 | a_1^\dagger n_1, n_2 \rangle &= \sqrt{[n_1+1]} \delta_{n'_1, n_1+1} \delta_{n'_2, n_2} \\
\langle n'_1, n'_2 | a_2^\dagger n_1, n_2 \rangle &= (-1)^{n_1} \sqrt{[n_2+1]} \delta_{n'_1, n_1} \delta_{n'_2, n_2+1}.
\end{align*}
\]

(36)

In order to have finite dimensional matrices, \(q\) must be a root of unity \(q = \exp \left\{ \frac{2 \pi i}{d} \right\} \), implying \([d] = 0\) and the weakened Pauli principle \((a_1^\dagger)^d = a_1^d = 0\).

Contrary to the Fock space theory of the non-deformed CAR–algebra there are some important peculiarities in the deformed case. First it is possible that \([n] < 0\), causing imaginary matrix elements in (36) and entailing the necessity to represent the \(^\dagger\)–operation by transposition and not Hermitean conjugation. In addition, if \(d\) is even we have \([d/2] = 0\) implying \(a_{d/2} = 0\), and furthermore next to \([0,0]\) there are additional ‘Fock vacua’ \([d/2, 0], [0, d/2]\), and \([d/2, d/2]\) all being annihilated by \(a_1\) and \(a_2\). As a consequence the representation (36) turns out to be reducible.

We can simulate the vacuum representation (21) by setting \(d = 4\), i.e. \(q = i\). We then have the Pauli principle \(a_1^2 = (a_2^\dagger)^2 = 0\) and

\[
[0] = [2] = [4] = 0, \quad [1] = -[3] = 1.
\]

(37)
Defining the basis vectors
\[
|0, 0\rangle \implies |\beta^1_{11}\rangle, \quad |2, 2\rangle \implies |\beta^1_{00}\rangle
\]
\[
|1, 0\rangle \implies |\beta^2_{11}\rangle, \quad |3, 2\rangle \implies |\beta^2_{00}\rangle
\]
\[
|0, 1\rangle \implies |\beta^3_{11}\rangle, \quad |2, 3\rangle \implies |\beta^3_{00}\rangle
\]
\[
|1, 1\rangle \implies |\beta^4_{11}\rangle, \quad |3, 3\rangle \implies |\beta^4_{00}\rangle
\]
\[
\frac{1}{\sqrt{2}} (|0, 2\rangle + |2, 0\rangle) \implies |\beta^5_{\pm 0}\rangle, \quad |3, 0\rangle \implies |\beta^5_{\mp 0}\rangle
\]
\[
|3, 1\rangle \implies |\beta^6_{\pm 0}\rangle, \quad |0, 3\rangle \implies |\beta^6_{\mp 0}\rangle
\]
\[
\frac{1}{\sqrt{2}} (|0, 2\rangle - |2, 0\rangle) \implies |\beta^7_{\pm 0}\rangle, \quad |1, 2\rangle \implies |\beta^7_{\mp 0}\rangle
\]
\[
-|1, 3\rangle \implies |\beta^8_{\pm 0}\rangle, \quad |2, 1\rangle \implies |\beta^8_{\mp 0}\rangle
\]
and
\[
A_{11}^{ij} = \langle \beta^i_{11} | A \beta^j_{11} \rangle, \quad A_{00}^{ij} = \langle \beta^i_{00} | A \beta^j_{00} \rangle, \quad A_{\pm 0}^{ij} = \langle \beta^i_{\pm 0} | A \beta^j_{\mp 0} \rangle
\]

shows that in spite of (37) the sectors labeled by 11, 00, and \(\frac{1}{2}\)0 are invariant under the application of the deformed field operators and deformed observables. Therefore an arbitrary algebra element decomposes into the direct sum
\[
A = A_{11} \oplus A_{00} \oplus A_{\frac{1}{2}0}.
\]
If these matrix representations are compared with the matrix representations of (21) one gets the identifications
\[
\pi_{vw}(c_\alpha) = \frac{1}{2} (a_\alpha + a_\alpha) + \frac{1}{2i} (a_\alpha - a_\alpha)^\dagger
\]
\[
N = [N_1] + [N_2], \quad S_3 = \frac{1}{2} ([N_2] - [N_1])
\]
and the extremal vacua \(|\Omega_{11}\rangle, \quad |\Omega_{00}\rangle, \quad |\Omega_{\frac{1}{2}0}\rangle\) correspond to the Fock vacua \(|0, 0\rangle, \quad |2, 2\rangle\) and \(\frac{1}{\sqrt{2}} (|0, 2\rangle + |2, 0\rangle)\), respectively.

There is a slight deviation of the Fock representation of the \(q\)-deformed CAR–algebra and the vacuum representations of the field and observable algebra, too. Although the observables \(N\) and \(S_3\) can be expanded in terms of elements of the deformed CAR–algebra, this is in general not possible for the spin operators \(S_1\) and \(S_2\). We can only give suitable expressions for \(S_1\) and \(S_2\) in the sectors labeled by 11 and 00, but not in the \(\frac{1}{2}0\) sector. Therefore we can not distinguish the spin–1 eigenvector \(|\beta^2_{\frac{1}{2}}\rangle\) of \(\vec{S}^2\) from a vacuum vector here. In order to have the full identification of the deformed CAR–algebra in Fock representation with the non-deformed algebra \(A_C(H)\) in a vacuum representation we must adjoin the observables \(S_1\) and \(S_2\) in the deformed case.

### 3.3 Thermo field dynamics

The basic idea of thermo field dynamics is to give the thermodynamic average at finite temperature \(\langle A \rangle_\beta = tr (A \exp(-\zeta - \beta H))\) in terms of an expectation value \(\langle A \rangle_\beta = \)
\[ \langle \Omega_\beta | A \Omega_\beta \rangle \text{ with a 'thermal vacuum' } | \Omega_\beta \rangle \text{. This allows Green’s function techniques to be applied to thermal quantum fields and can be shown to be equivalent to the algebraic setup of Haag, Hugenholz and Winnink in equilibrium thermodynamics [9].}

To achieve this aim the original theory has to be embedded into an enlarged theory, which is done by introducing new and independent tilde fields. Restricting ourselves to the case \( \mathcal{H} = \mathcal{C} \) the basic equations are the anticommutation relations

\[
\{ \tilde{a}, a \} = \{ \tilde{a}^*, a \} = \{ a, \tilde{a} \} = 0, \quad \{ \tilde{a}^*, \tilde{a} \} = \{ a^*, a \} = \mathbb{1},
\]

given in their Fock representation with \( a | 0 \rangle = \tilde{a} | 0 \rangle = 0 \). According to thermo field dynamics the thermal vacuum must be of the form

\[
| \Omega_\beta \rangle = ( \lambda_1(\beta) + \lambda_2(\beta)a^*\tilde{a}^* ) | 0 \rangle
\]

here. For the Hamiltonian \( H = \varepsilon_1 a^*a + \varepsilon_0 \tilde{a} \tilde{a}^* \) the parameters are determined to be

\[
\lambda_1(\beta) = \frac{1}{\sqrt{1 + e^{-\beta(\varepsilon_1 - \varepsilon_0)}}}, \quad \lambda_2(\beta) = \frac{e^{-\frac{1}{2}\beta(\varepsilon_1 - \varepsilon_0)}}{\sqrt{1 + e^{-\beta(\varepsilon_1 - \varepsilon_0)}}}.
\]

Further the prescription to implement the observables in thermo field dynamics is to put

\[
\hat{N} = a^*a - \tilde{a}^* \tilde{a}.
\]

Identifying the operators \( c = a, c^* = a^* \), and \( N = \hat{N} \) shows that the subalgebra of (42) generated by \( a, a^* \), and \( \hat{N} \) is completely equivalent with (17) and (16). Thus, the inequivalent vacuum representations are contained as the canonical substructure given by the original fields and observables in the Fock representation of the enlarged algebra of thermo field dynamics.

Furthermore, also identifying the states in terms of

\[
\omega_v ( F(c, c^*) ) = \langle \Omega_\beta | F(a, a^*) | \Omega_\beta \rangle
\]

with an arbitrary function \( F \) permits us to calculate the corresponding temperature dependency of the parameter \( v \) to be

\[
v(\beta) = \frac{1}{1 + e^{-\beta(\varepsilon_1 - \varepsilon_0)}}.
\]

We will give a thermodynamic interpretation of the parameter \( v \) in a different and more systematic way now.

### 4 Vacuum dynamics and the thermodynamic equilibrium state

In this section we discuss the problem of non-trivial dynamics connecting inequivalent representations, i.e. the dynamics of vacuum states. We will derive a generalized Schrödinger equation and give its solution for the dynamics on the vacua of \( \mathcal{A}_{U(1)}(\mathcal{C}) \). Since this dynamics is irreversible we seek to give a thermodynamic interpretation of the equilibrium state.
Consider the space of vacuum states $V$ with an arbitrary (finite) number of extremal vacua $\omega_r$ and extremal decomposition $\omega = \sum_r \lambda_r \omega_r$. For a given dynamics $\beta_t : V \to V$ the adjoint dynamics on the algebra is defined by
\[
( \beta_t \omega )( A ) = \omega ( \alpha_t( A ) ) .
\] (48)
If $\alpha_t$ is unitarily implemented it follows that $\beta_t$ conserves convex combinations and since for any continuous dynamics we have $\beta_t(\omega_r) = \omega_r$ it is $\beta_t(\omega) = \omega$ for all vacuum states. Hence $\alpha_t$ cannot be unitarily implemented if there should be a non-trivial vacuum dynamics at all.

**Theorem 3** Any infinitesimally generated vacuum dynamics must satisfy the generalized Schrödinger equation for the extremal states
\[
\frac{d}{dt} \omega_r( \alpha_t( A ) ) = \sum_s \gamma^{sr} \omega_s( \alpha_t( A ) ) ,
\] (49)
with $\gamma^{ss} \leq 0$, $\gamma^{sr} \geq 0$ for $s \neq r$, and $\sum_s \gamma^{sr} = 0$.

**Proof:** We must require that $\beta_t(\omega)$ is again a vacuum, i.e. $\beta_t$ preserves linearity, positivity, normalization, and the vacuum quantum numbers of observables. For the state $\omega = \sum \lambda_r \omega_r$ we can expand $\beta_t(\omega) = \sum \beta^r_t(\lambda) \omega_r$ in terms of the extremal vacua again. The functions $\beta^r_t(\lambda)$ must satisfy the subsidiary conditions $\beta^r_t(\lambda) \geq 0$ and $\sum_r \beta^r_t(\lambda) = 1$. Making use of (48) with the projector $P_s$ onto the extremal vacuum $\omega_s$ yields $\beta^s_t(\lambda) = \sum \lambda_r \omega_r( \alpha_t( P_s ) )$ and thus the functions $\beta^r_t(\lambda) = \sum_r \lambda_r \beta^{sr}_t$ must be linear in $\lambda$, and the coefficients satisfy $\beta^{sr}_t \geq 0$, $\sum_s \beta^{sr}_t = 1$, and the initial condition $\lim_{t\to0} \beta^{sr}_t = \delta_{sr}$. Inserting this into equation (48) gives
\[
\sum_s \beta^{sr}_t \omega_s( A ) = \omega_r( \alpha_t( A ) ) .
\] (50)
If the dynamics is infinitesimally generated, i.e. $\frac{d}{dt}\beta^{sr}_t = \sum \gamma^{s't'} \beta^{r'r'}_t$, the Schrödinger equation (49) follows by differentiating (50). The subsidiary conditions for $\gamma^{sr}$ are direct consequences of the subsidiary conditions for $\beta^{sr}_t$, since $\gamma^{sr} = \lim_{t\to0} \frac{d}{dt}\beta^{sr}_t$. ♣

The main drawback of the Schrödinger equation (49) is that the dynamical matrix $\gamma$ is only restricted by general properties serving to preserve the characteristics of vacuum states in time. For the evaluation of concrete models we should determine $\gamma$ as a function of the model Hamiltonian. However, this is a non-trivial task going beyond conventional quantum theory where $\gamma = 0$. We will approach this problem in giving a thermodynamic interpretation of the solution of (49) for the case of two extremal vacua. First notice:

**Lemma 6** The dynamical matrix in (49) can be decomposed into a sum of dynamical matrices, each mediating only between two extremal states and possessing the same characteristic structure.

**Proof:** We make use of the subsidiary condition $\gamma^{ss} = -\sum_{r \neq s} \gamma^{sr}$ to eliminate $\gamma^{ss}$. Then it follows by a straightforward calculation that
\[
\gamma^{rs} = \sum_{i=2}^{n} \sum_{j=1}^{i-1} (D_{ij})^{rs} \] (51)
with $(D_i^j)^{rs} = -\gamma^{ij}\delta_{ir}\delta_{js} + \gamma^{ij}\delta_{is}\delta_{jr} + \gamma^{ij}\delta_{is}\delta_{jr} - \gamma^{ij}\delta_{is}\delta_{js}$ being the characteristic dynamic matrices intertwining only the extremal states $\omega_i$ and $\omega_j$. ♣

Restricting ourselves to the case of two extremal states and putting $\gamma_1 = \gamma^{12}$ and $\gamma_2 = \gamma^{21}$, both being positive, we have to solve the Schrödinger equation

$$\frac{d}{dt} \begin{pmatrix} \omega_1(\alpha_t(A)) \\ \omega_0(\alpha_t(A)) \end{pmatrix} = \begin{pmatrix} -\gamma_1 & \gamma_1 \\ \gamma_2 & -\gamma_2 \end{pmatrix} \begin{pmatrix} \omega_1(\alpha_t(A)) \\ \omega_0(\alpha_t(A)) \end{pmatrix}. \quad (52)$$

**Lemma 7** The eigenvalues of the dynamical matrix in (52) are calculated to be $d_1 = 0$ and $d_2 = -(\gamma_1 + \gamma_2) \leq 0$.

Hence we expect the dynamics to asymptotically approach a limiting state.

**Lemma 8** The solution to the vacuum dynamics (52) for the state $\omega_v = v\omega_1 + (1-v)\omega_0$ is given by

$$\omega_v(\alpha_t(A)) = \left[ \frac{\gamma_2}{\gamma_1 + \gamma_2}\omega_1(A) + \frac{\gamma_1}{\gamma_1 + \gamma_2}\omega_0(A) \right] - e^{-(\gamma_1 + \gamma_2)t[\frac{\gamma_2}{\gamma_1 + \gamma_2} - v][\omega_1(A) - \omega_0(A)]}. \quad (53)$$

**Proof:** Introducing the functionals $\Delta = \omega_1 - \omega_0$ and $\Sigma = \omega_1 + \omega_0$ we get the differential equations $\frac{d}{dt}\Delta(\alpha_t(A)) = -(\gamma_1 + \gamma_2)\Delta(\alpha_t(A))$ and $\frac{d}{dt}\Sigma(\alpha_t(A)) = (\gamma_2 - \gamma_1)\Delta(\alpha_t(A))$. They are solved by $\Sigma(\alpha_t(A)) = \Sigma(A) + \frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2} \left( 1 - \exp\{-(\gamma_1 + \gamma_2)t\} \right) \Delta(A)$ and $\Delta(\alpha_t(A)) = \exp\{-(\gamma_1 + \gamma_2)t\} \Delta(A)$. Inserting these solutions into $\omega_v = \frac{1}{2}\Sigma + (v - \frac{1}{2})\Delta$ yields (53). ♣

It is obvious from (53) that the vacuum dynamics is necessarily irreversible unless $\gamma_1 = \gamma_2 = 0$. The equilibrium state

$$\lim_{t \to 0} \omega_v(\alpha_t(A)) = \left( \frac{\gamma_2}{\gamma_1 + \gamma_2}\omega_1(A) + \frac{\gamma_1}{\gamma_1 + \gamma_2}\omega_0(A) \right)$$

is independent from an arbitrarily chosen initial state and completely determined by the ratio $\gamma_1/\gamma_2$.

In order to give a thermodynamic discussion of the equilibrium state we make the following definitions:

**Definition 4** For a given Hamiltonian $H$ the inner energy of the equilibrium state is

$$U = \lim_{t \to 0} \omega_v(\alpha_t(H)) \quad (55)$$

and with $v_\infty = \frac{\gamma_2}{\gamma_1 + \gamma_2}$ the entropy is defined by

$$S = -k \left( v_\infty \ln v_\infty + (1 - v_\infty) \ln(1 - v_\infty) \right), \quad (56)$$

i.e. the mixing parameters in the extremal decomposition are interpreted as the eigenvalues of the equilibrium density operator.
Lemma 9 For the Hamiltonian $H = \varepsilon_1 c^* c + \varepsilon_0 c c^*$, ($\varepsilon_1 > \varepsilon_0$) the mixing parameter $v_\infty$ and the inverse temperature defined by $\beta(U) = k^{-1} \frac{\partial S}{\partial U}$ result to be

$$v_\infty(U) = \frac{\varepsilon_1 - U}{\varepsilon_1 - \varepsilon_0}, \quad \beta(U) = \frac{1}{\varepsilon_1 - \varepsilon_0} \ln \left( \frac{\varepsilon_1 - U}{U - \varepsilon_0} \right).$$

**(Proof):** The solution for $v_\infty(U)$ is a straightforward evaluation of (55). Inserted into (56) yields (57) upon differentiation. ♣

Lemma 10 The temperature dependency of $v_\infty$ is derived to be

$$v_\infty(\beta) = \frac{1}{1 + e^{-\beta(\varepsilon_1 - \varepsilon_0)}}.$$  

**(Proof):** Equation (57) can be resolved to give

$$U(\beta) = \frac{\varepsilon_0 e^{-\beta \varepsilon_0} + \varepsilon_1 e^{-\beta \varepsilon_1}}{e^{-\beta \varepsilon_0} + e^{-\beta \varepsilon_1}}.$$  

Inserting this into the expression for $v_\infty(U)$ yields (58). ♣

Notice that equation (58) coincides exactly with (47). Further we would have obtained the same inner energy and entropy function by evaluating $U = tr(\rho H)$ and $S = -k tr(\rho \ln \rho)$ for the density operator $\rho = \exp\{-\zeta - \beta H\}$. However our results were derived from the solution of a non-trivial vacuum dynamics here! It was shown that any vacuum dynamics is irreversible and the asymptotic equilibrium state can be thermodynamically interpreted inspite of the definitions (55) and (56).

Looking at the time-dependent solutions (53) shows that the equilibrium is not affected by the value of $\gamma_1 + \gamma_2$, but only by $\gamma_1/\gamma_2$. Rather, the independent value $\gamma_1 + \gamma_2$ characterizes the relaxation time of the system. Considering the ratio

$$\frac{\gamma_1}{\gamma_2} = \frac{\omega v_\infty(H) - \omega_0(H)}{\omega_1(H) - \omega v_\infty(H)}$$

(60)

one might speculate that $\gamma_1 = h^{-1}(\omega v_\infty(H) - \omega_0(H))$ and $\gamma_2 = h^{-1}(\omega_1(H) - \omega v_\infty(H))$, implying $\gamma_1 + \gamma_2 = h^{-1}(\omega_1(H) - \omega_0(H))$. Then the vacuum dynamics is generated by the fact that the Hamiltonian has different expectation values for different inequivalent vacuum representations.

5 Application to BCS–theory

The conventional solutions of BCS–theory either make use of a suitably chosen ansatz for the ground state wave function or reformulate the theory in terms of an ‘equivalent’ Hamiltonian which is only quadratic in the field operators and permits exact solutions in order $O(1/N)$ [11]. Anyhow the resulting ground state of the theory is not an eigenstate of the particle number operator. In the first case this originates from the ansatz itself and
in the second case from the fact that the 'equivalent' Hamiltonian does not commute with the particle number operator where as the original BCS–Hamiltonian did.

We will show here that these solutions are based on an inconsistency when formulated in Fock space. Admitting non-trivial vacua we will obtain exact particle number invariant solutions to the full BCS–Hamiltonian

\[ H = \sum_z \varepsilon_z c_z^* c_z - \frac{1}{2} \sum_{zz'} \frac{V_{zz'}\hat{N}}{N} c_z^* c_z^* c_z c_{z'} , \]

(61)

where \( z = (k, \alpha) \), \( \tilde{z} = (-k, -\alpha) \), and \( \hat{N} \) is the (finite) number of lattice points. The potential is taken to have the symmetry properties \( V_{zz'} = V_{z'z} = -V_{\tilde{z}z'} \).

Considering the averaged operator

\[ B_z^* = \sum_{z'} \frac{V_{zz'}}{N} c_{z'}^* c_{z'} \]

(62)

one can show that \( B_z^* \) commutes with every field operator in order \( O(1/\hat{N}) \). If in the limit \( \hat{N} \to \infty \) the theory were still represented in an irreducible representation — like for instance the Fock representation — it follows that \( B_z^* = \Delta_z \mathbb{I} \) with \( \Delta_z \in \mathbb{C} \). On the other hand the commutator with the particle number operator \( [B_z^*, \hat{N}] = -2B_z^* \) is not affected by taking the thermodynamic limit. Since \( [\mathbb{I}, \hat{N}] = 0 \) we have \( B_z^* = 0 \) in the thermodynamic limit. Hence, either the theory is not represented in an irreducible representation in the thermodynamic limit or it must be the theory of free fields and there is no superconductivity.

To get a grasp of the theory with infinitely many degrees of freedom from its finite counterpart we will consider vacuum states on the algebra defined by the relations

\[ \{ c_z, c_{z'} \} = 0, \quad \{ c_z^*, c_{z'} \} = \delta_{zz'} \mathbb{I}, \quad [\hat{N}, c_z] = -c_z, \]

\[ [S^k, c_{ka}] = \sum_{\alpha'} \sigma^k_{\alpha\alpha'} c_{\alpha'a}, \quad U_n c_{ka} U_n^* = e^{ikn} c_{ka}. \]

(63)

The operators \( U_n \) are the implemented translation operators. Since any vacuum state satisfies \( \omega(AN) = \omega(AS^k) = 0 \) and \( \omega(AU_n) = \omega(AU_n^*) = \omega(A) \) we conclude as before that \( \omega(c_{z_1}^* \ldots c_{z_n}^* c_{z'_1} \ldots c_{z'_n}) \) is zero unless \( n = n' \), \( \sum_i (\alpha_i - \alpha'_i) = 0 \) and \( \sum_i (k_i - k'_i) \in 2\pi \mathbb{Z} \).

**Theorem 4** The complete hierarchy of non-zero expectation values, i.e. the non-trivial vacuum state of the theory, can be determined in order \( O(1/\hat{N}) \) from the BCS–Hamiltonian (61) and the following assumptions:

- The condensate equation: \( \omega(B_z^* B_{z'} A) = \omega(B_z^* B_{z'}) \omega(A) \)

- The equilibrium of the condensate: \( [B_z, H] = 0 \).

**Comment:** Notice that the operators \( B_z^* B_{z'} \) are in the center of the complete field and observable algebra in order \( O(1/\hat{N}) \), but they must not be proportional to unity. Rather, the condensate equation is a requirement on the vacuum state. Since the operators \( B_z \) will determine the vacuum solutions we pick out the equilibrium vacuum by the second assumption. We will not investigate non-equilibrium properties of the BCS model here.
Proof: Defining the generating functional of time-ordered $\omega$–functions with completely anticommuting sources $\eta$ and $\bar{\eta}$

$$G(\eta, \bar{\eta}) = \sum_{n=0}^{\infty} (n!)^{-2} \sum \int dt_1 \ldots dt_n dt'_1 \ldots dt'_n \eta_{z_1}(t_1) \ldots \eta_{z_n}(t_n) \bar{\eta}_{z_1'}(t_1') \ldots \bar{\eta}_{z_n'}(t_n') \times$$

$$\times \omega(Tc_{z_1}(t_1) \ldots c_{z_n}(t_n)c_{z_1'}(t_1') \ldots c_{z_n'}(t_n')))$$

(67) is needed. Inspite of the time derivatives present in (67) the equal-time limit will be

$$\omega - \text{functions, and we have put}$$

$$z \rightarrow \pm i\delta$$

and taking the limit $\delta \rightarrow 0$ at the end. Collecting the resulting equal-time hierarchy equations by a functional equation again yields

$$\frac{\partial}{\partial t^2} + \varepsilon_z^2 + \Delta_z^2 \left[ \frac{\delta}{\delta \eta_z(t)} \right] G(\eta, \bar{\eta}) = \left[ -\left( i\frac{\partial}{\partial t} - \varepsilon_z \right) \eta_z(t) - D_z \eta_z(t) \right] G(\eta, \bar{\eta})$$

$$\frac{\partial^2}{\partial t^2} + \varepsilon_z^2 + \Delta_z^2 \left[ \frac{\delta}{\delta \bar{\eta}_z(t)} \right] G(\eta, \bar{\eta}) = \left[ -\left( i\frac{\partial}{\partial t} + \varepsilon_z \right) \bar{\eta}_z(t) - \bar{D}_z \bar{\eta}_z(t) \right] G(\eta, \bar{\eta}).$$

(67)

Since we are interested in the hierarchy of equal-time $\omega$–functions the equal-time limit of

$$V_{zz'} \left[ \frac{\delta}{\delta \eta_{z'}(t)} \right] \left[ \frac{\delta}{\delta \eta_{z'}(t)} \right]$$

(67) is needed. Inspite of the time derivatives present in (67) the equal-time limit will be
taken in the following way: Comparing the coefficients of the functionals in (67) we get
differential equations for the coupled hierarchy of multi-time $\omega$–functions. These equations
are then Fourier transformed and subsequently all Fourier variables are integrated over.
Whenever necessary the poles on the real axis are circumvented by replacing $\varepsilon_z^2 + \Delta_z^2 \rightarrow \varepsilon_z^2 + \Delta_z^2 \pm i\delta$ and taking the limit $\delta \rightarrow 0$ at the end. Collecting the resulting equal-time hierarchy equations by a functional equation again yields

$$\frac{\delta}{\delta \eta_z} A(\eta, \bar{\eta}) = \left[ \frac{-\varepsilon_z}{2E_z} \bar{\eta}_z + \frac{1}{2E_z} D_z \eta_z \right] A(\eta, \bar{\eta})$$

$$\frac{\delta}{\delta \bar{\eta}_z} A(\eta, \bar{\eta}) = \left[ \frac{\varepsilon_z}{2E_z} \eta_z + \frac{1}{2E_z} \bar{D}_z \bar{\eta}_z \right] A(\eta, \bar{\eta}).$$

(68)

where $E_z = \sqrt{\varepsilon_z^2 + \Delta_z^2}$, $A(\eta, \bar{\eta})$ denotes the generating functional of antisymmetric equal-
time $\omega$–functions, and we have put $\eta_z = \eta_z(t = 0)$ and $\bar{\eta}_z = \bar{\eta}_z(t = 0)$.

The solution of (68) is simplified by the ansatz

$$A(\eta, \bar{\eta}) = \exp \left( -\sum_z \eta_z \frac{\varepsilon_z}{2E_z} \bar{\eta}_z \right) A'(\eta, \bar{\eta})$$

(69)
implying
\[ \frac{\delta}{\delta \eta_z} A'(\eta, \bar{\eta}) = \frac{1}{2E_z} D_z \eta_z A'(\eta, \bar{\eta}), \quad \frac{\delta}{\delta \bar{\eta}_z} A'(\eta, \bar{\eta}) = \frac{1}{2E_z} \bar{D}_z \bar{\eta}_z A'(\eta, \bar{\eta}). \] (70)

Setting \( \Delta_{zz'} = \omega(B^*_z B_{z'}) \) it follows that \( \Delta_{zz'}^2 = \Delta_z^2 \Delta_{z'}^2 \) in order \( O(1/\hat{N}) \). Furthermore defining
\[ \mathcal{N} = \sum_z \eta_z \delta_{zz}, \quad \bar{\mathcal{N}} = \sum_z \bar{\eta}_z \delta_{zz}, \quad \mathcal{K} = \sum_{zz'} \frac{\Delta_z \Delta_{z'}}{4E_z E_{z'}} \bar{\eta}_z \eta_{z'} \bar{\eta}_{z'} \] (71)
we get \( \mathcal{N} \bar{\mathcal{N}} A' = \mathcal{K} A' \) from (70) and the condensate equation, which is solved by
\[ A'(\eta, \bar{\eta}) = \sum_{n=0}^{\infty} \left[ (2n)!! \right]^{-1} [\mathcal{K}(\eta, \bar{\eta})]^n. \] (72)

Since any algebra element of the CAR–algebra can be expanded in terms of antisymmetrized field operator products the vacuum state is completely determined by (69) and (72). ♣

**Lemma 11** The 2–point and 4–point functions of the solution found in theorem 4 are calculated to be
\[ \omega(c^*_z c_{z'}) = \frac{1}{2} \left[ 1 - \frac{\varepsilon_z}{E_z} \right] \delta_{zz'}, \quad \omega(c^*_z c^*_w c_{z'} c_{z'}) = \frac{1}{4} \left[ 1 - \frac{\varepsilon_z}{E_z} \right] \left[ 1 - \frac{\varepsilon_{z'}}{E_{z'}} \right] \left[ \delta_{zz'} \delta_{zz'} - \delta_{zz'} \delta_{zz'} \right] + \frac{1}{4E_z E_{z'}} \delta_{zz'} \delta_{zz'} \delta_{zz'} \delta_{zz'}. \] (73)

**Lemma 12** The consistency of the condensate equation with the 4–point function of lemma 11 yields the gap equation
\[ \Delta_z = (\pm) \frac{1}{2} \sum_{z'} \frac{V_{zz'}}{N} \frac{\Delta_{z'}}{2E_{z'}} \] in order \( O(1/\hat{N}) \).

**Proof:** Inserting (73) into \( \Delta_z \Delta_{z'} = \omega(B^*_z B_{z'}) \) gives
\[ \Delta_z \Delta_{z'} = \sum_y \frac{V_{zy} V_{z'y}}{4N^2} \left[ 1 - \frac{\varepsilon_y}{E_y} \right]^2 + \frac{1}{4} \sum_{yy'} \frac{V_{zy} V_{z'y'}}{N^2} \frac{\Delta_y \Delta_{y'}}{4E_y E_{y'}.} \] (75)
The first term on the right hand side vanishes in order \( O(1/\hat{N}) \) implying the squared gap equation (74). ♣

One can continue to build up the complete BCS–theory, the only remarkable difference being the fact that the state \( \omega \) is a true vacuum state with respect to the full field operator and observable algebra here. It is the exact, particle number invariant ground state of the BCS–Hamiltonian (60) in order \( O(1/\hat{N}) \).
6 Conclusions

If the algebra of fields is enlarged in a canonical way by adjoining the observables as independent operators there exist inequivalent (vacuum) representations for quantum theories with finitely many degrees of freedom, too.

These inequivalent representations provide the basis for the consistent description of a large variety of collective microscopic and macroscopic phenomena in many-particle quantum theory. Some of their properties can be recovered from quasi-particle methods in Fock space, but for one thing these need not exhaust the complete structure and in addition might yield inconsistencies in the thermodynamic limit. Nevertheless, these methods have proved to be successful in many physical applications. Therefore, inequivalent representations are expected to play a decisive role in finite nuclear shell models, in finite lattice models, etc., where these considerations have so far been considered only indirectly. If applied to infinite systems one can approach its solution consistently from its finite solutions, as was demonstrated in the case of the BCS–model.

The possibility of non-trivial vacua is closely engaged with the thermodynamics of many-particle quantum field theory. If the extremal decomposition of the equilibrium vacuum state is known, all thermodynamic functions can be calculated straightforward and without an additional maximum principle of the entropy. The irreversible vacuum dynamics resulting from the Schrödinger equation for the extremal vacuum states might serve to microscopically investigate non-equilibrium properties in many-particle physics, if it is possible to determine the dynamical matrix $\gamma$ by the model Hamiltonian.

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