Extending Semimodules over Semirings

Samah Alhashemi¹, Asaad M. A. Alhossaini²
¹College of Science for Women, University of Babylon, Babylon, Iraq
²College of Education for Pure Sciences, Babylon University, Babylon, Iraq

E-mail: samahhadi1978@gmail.com

Abstract. The objective of our research paper is to introduce as well as to study many essential properties of the concept of extending semimodules. A semimodule S is named extending (CS) if every subsemimodule of S is essential in a direct summand of S. Therefore, extending semimodule behaviour with respect to direct sums and direct summands are examined. Moreover, studying some properties of these semimodules concepts, e.g., every direct summand of a CS-semimodule is a CS-semimodule. While the direct sum of extending semimodules is not necessarily extending.

1. Introduction

The A T-module S is called an extending module (CS–module) based on the extending property as follows: for each submodule X of S, there exists a direct summand N of S, which is an essential extension of X. It is known that a complement submodule need not be a summand, in the class of CS–modules any complement is a summand. Originality of CS-modules was presented by Von Neumann in 1930 [1]. In 1960, Utumi has studied this condition (identifying it as C₁ condition) in his study on self-injective and continuous ring [2]. In fact C₁ condition is common generalization of the injective and the semi simple condition, this motivates the name of extending condition. Another name of this condition is CS condition. It has developed in many articles and in at least [3][4][5].

In recent years, the extending modules theory has come to represent an important role and generally major contributions to this theory, through its widely available interesting findings on expanding properties in the theoretical preparation of the module. For background and applications of extending module (see [6]).

In this work, the extending semimodule over a semiring will be introduced and investigated. A semiring can be defined as a set T, which is non-empty together with two binary operations multiplication (·) and addition (+); as mentioned that (T, ·) is a monoid with an identity element 1 ≠ 0; (T, +) is a commutative monoid with identity element 0; for all t ∈ T; a(b + c) = ab + ac and (a + b)c = ac + bc; for all a, b, c ∈ T. The semiring T is commutative if the monoid (T, +) is commutative [7]. Let (S, ·) be an additive abelian monoid with additive identity 0. Then S is named a left T-semimodule if there exists a scalar multiplication T × S → S defined by (t, x) → tx, such that t(x + y) = tx + ty; (ts)x = t(sx); (t + s)x = tx + sy; 0tS = r0S = 0S for all x, y ∈ S and for all t, s ∈ T [7].
A non-empty subset $K$ of a left–semimodule $S$ is called subsemimodule of $S$ if $K$ is closed under addition and scalar multiplication, that is $K$ a $T$-semimodule itself (denoted by $K \subseteq S$) [8]. A $T$-semimodule $S$ is said to be a direct sum of subsemimodules $S_1, S_2, \ldots, S_t$ of $S$, if each $s \in S$ can be written uniquely as $s = s_1 + s_2 + \ldots + s_t$, where $s_i \in S_i$. It is denoted by $S = S_1 \oplus S_2 \oplus \ldots \oplus S_t$. In this case each $S_i$ is called a direct summand of $S$ [9].

If $T$ is a semiring and $S, N$ are left $T$-semimodules, then a map $P:S \rightarrow N$ is called a homomorphism of $T$-semimodules, if satisfied the following, $P(s + s') = P(s) + P(s')$; $P(ts) = tP(s)$, for all $s, s' \in S$ and $t \in T$. The set of $T$-homomorphism's of $S$ into $N$ is denoted by $Hom(S, N)$. A homomorphism $P$ is called an epimorphism if it’s onto; or an injective homomorphism if $P$ is one-one, and it is isomorphism if $P$ is one-one and onto. A non-zero $T$-subsemimodule $K$ of $S$ is named essential (large) and write $(K \leq S)$, if $K \cap L = \emptyset$ for every nonzero subsemimodule $L$ of $S$ [12]. A subsemimodule $K$ of semimodule $S$ is called closed if $K$ has no proper essential extension in $S$ (denoted by $N \leq S$) [13].

Let $S$ be a $T$-semimodule, $A$ and $B$ are subsemimodules of $S$; $A$ is called intersection complement (briefly complement) of $B$ if $A\cap B = 0$ and $A$ is maximal in the set of all subsemimodules of $S$ that have zero intersection with $B$ [13]. A subsemimodule $K$ of a semimodule $S$ is said to be semi subtractive, if for any $s, s' \in S$ there is always some $h \in T$ satisfying $s + h = s'$ or $s' + h = s$ [7]. A nonzero $T$-subsemimodule $K$ of $S$ is named essential if $K \cap L \neq 0$ for every nonzero subsemimodule $L$ of $S$ [12]. A subsemimodule $K$ of semimodule $S$ is called closed if $K$ has no proper essential extension in $S$ (denoted by $N \leq S$) [13].

2. CS-Semimodule

In this section, CS-semimodule will be presented as well as investigating some properties of them. Initially for this purpose, some properties of complement subsemimodules that are useful in analyzing the structure of extending semimodule, will be given.

According to [6], the concepts for modules will be converted for semimodule in the following.

**Detention 2.1:** A $T$-semimodule $S$ is said to be extending (CS-semimodule) if every subsemimodule of $S$ is essential in a direct summand of $S$.

It is clear that any simple $T$-semimodule (has no nontrivial subsemimodules) is CS. In fact any semisimple $T$-semimodule (has each subsemimodule as a summand) is CS, too.

It is known that any summand of a $T$-semimodule is closed, but the converse is not.

**Example 2.2:** $S = \mathbb{Z}_4 \oplus \mathbb{Z}_2$, $R = \mathbb{Z}$. Let $A = \langle (\tilde{1}, \tilde{1}) \rangle$, then $A \leq S$, but not a summand of $S$.

**Proposition 2.3:** $S$ is CS-semimodule if and only if every closed subsemimodule of $S$ is a direct summand of $S$.

**Proof:** ($\Rightarrow$) Assume $S$ is a CS-semimodule, let $K \leq S$ then $K \leq S'$, where $S'$ is a summand of $S$, then by definition of closed subsemimodule, $K = S'$, that is $K$ is a summand of $M$.

($\Leftarrow$) Conversely, let $K \subseteq S$, and let $S'$ be closure of $K$, then $S'$ is direct summand of $S$ and $K \leq S'$, hence $S$ is CS-semimodule. □

By (Proposition 2.3) $S$ in example 2.2 is not CS.
Lemma 2.4: If $K \leq S$ and $L \leq S$ then $L \cap K \leq S$.

Proof: Clear. □

Lemma 2.5: If $K/L \leq eK/L$, then $K \leq eK'$.

Lemma 2.6: If $L \leq eS$ and $N \leq S$, then $L/N \leq eS/N$.

Proof: Assume that $L/N$ is not closed in $S/N$ then there exist $L'/N \leq L/N$ and $L/N \leq eL'/N$ ($L \not= L'$), then $L \leq eL'$, which contradicts the assumption $L \leq eS$, hence $L/N \leq eS/N$. □

Lemma 2.7: A subsemimodule $K$ is closed in $S$ if and only if whenever $K \leq N \leq S$ then $N/K \leq eS/K$.

Proof: By (Lemma 2.5), it is enough to prove the necessity condition. Suppose $K \leq S$ and $K \leq eS$. Let $L$ be submodule of $S$ such that $K \leq L$ and $(N/K) \cap (L/K)=0$, then $K=K\cap L \leq eL$, based on Lemma(2.4). Since $L$ is closed, then $L=K$ and $L/K=0$. Hence $N/K \leq eS/K$. □

Lemma 2.8: If $K \leq eN$ and $N \leq eS$ then $K \leq eS$.

Proof: Let $K'$ be a complement of $K$ in $N$ and also $N'$ be a complement of $N$ in $S$, then $N \oplus N' \leq eS$. Since $N$ is closed in $S$ (by assumption) and $N \leq N \oplus N' \leq eS$, then based on (Lemma 2.7) $N \oplus N' \leq eS$. Sinc($N \oplus N'/N \leq eS/N$).

Lemma 2.9: If $K \leq eS$, then $K \cap (K' + N') = 0$ implies $K \cap (K' + N') = 0$. Therefore, $(M/K) \cap ((K + K') + N') = K$. Thus $K=M$ and which implies $K \leq eS$. □

Remark 2.9: Every subsemimodule $K$ of a semimodule $S$ is essential in closed subsemimodule $H$ of $S$ [13].

Proposition 2.10: Let $S = S_1 \oplus S_2$, and $T$ is CS-semimodule if and only if every complement of $S_i$, where ($i = 1$ or 2) is CS-semimodule and a direct summand of $S$.

Proof: Let $K$ be complement of $S_1$ in $S$, since $S$ is CS-semimodule, then $K$ is closed subsemimodule of $S$, and by (Proposition 2.3), $K$ is a direct summand of $S$. Let $L$ be closed subsemimodule of $K$, by (Lemma 2.8), $L \leq eS$ and $L \cap S_1 = 0$, again since $S$ is CS-semimodule, $L$ is a direct summand of $S$, so $S = L \oplus L'$, for some $L \leq S$ and by Modular Law $K = L \oplus (L \cap K)$, therefore $L$ is direct summand of $K$, and $K$ is $CS$-semimodule.

Conversely, let $N \leq eS$ then there exists a closed subsemimodule $H$ of $N$ such that $N \cap S_1 \leq H$, clearly $(H \cap S_2) = 0$. By Zorn's Lemma, there exists a complement $Q$ of $S_1$ in $S$ with $H \leq Q$. Also, by (Lemma 2.8), $H \leq eS$, hence $H \leq eQ$, since $Q$ is a complement of $S_1$ then by assumption, it is CS-semimodule, hence $H$ is direct summand of $S$, therefore $S = H \oplus H'$ for some $H' \leq S$, by Modular Law $N = H \oplus (N \cap H')$, since $(N \cap H)$ is closed in $S$, and $(N \cap H') \cap S_1 = 0$ hence, $(N \cap H')$ is direct summand of $S$ and also for $H'$, $H' = (N \cap H') \oplus H''$ for some $H'' \leq S$, so $S = H \oplus H''$, therefore $N$ is a direct summand of $S$. □

Example 2.11: $S = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Let $N_i \leq S$, where ($i = 1, 2, 3, 4, 5$) such that $N_1 = \mathbb{Z}_2 \oplus 0$, $N_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_4$, $N_3 = \langle (1, 1) \rangle = \langle (1, 2) \rangle$, $N_4 = \langle (1, 3) \rangle$, and $N_5 = \langle (0, 2) \rangle$, $N_2$ and $N_4$ are complement of $N_1$, and they are direct summand of $S_1$ and $N_1$ and $N_4$ are complement of $N_2$, and they are direct summand of $S$, then $S$ is $CS$.

As it is mentioned before, the injective hull need not exist for any semimodule. In the following results, the existence of injective hull is needed. For this purpose, we must add a condition that the semimodule has the injective hull.

Lemma 2.12: Let $S = S_1 \oplus S_2$ be semimodule (with injective hull) and $\varphi \in \text{Hom}(S_1, E(S_2))$, $K = \{s_1 + \varphi(s_1): s_1 \in \varphi^{-1}(S_2)\}$ then:
1. \( \varphi^{-1}(S_2) \cap K = \ker \varphi \).

2. If \( \pi: S \to S_1 \) is the natural projection then \( \pi|_K \) is a monomorphism.

**Proof:** For (1), first we must prove that \( K \cap S_2 = 0 \), let \( x \in K \cap S_2 \), then \( x = s + \varphi(s), s \in \varphi^{-1}(S_2) \leq S_1, x \in S_2 \) and \( \varphi(s) \in S_2 \) implies \( s \in S_2 \) (\( S_2 \) is subtractive since it is direct summand), so \( s \in S_1 \cap S_2 \), then \( s = 0 \) and \( x = 0 + \varphi(0) = 0 \).

Now, let \( x \in \varphi^{-1}(S_2) \cap K \), then \( x \in \varphi^{-1}(S_2) \) and \( x + \varphi(x) \in K \), but \( x \in K \), by subtractive property \( \varphi(x) \in K \), hence \( \varphi(x) \in K \cap S_2 = 0 \), therefore \( \varphi(x) = 0 \) and \( x \in \ker \varphi \), so \( \varphi^{-1}(S_2) \cap K \subseteq \ker \varphi \), but \( \ker \varphi \subseteq \varphi^{-1}(S_2) \).

For (2), since \( \ker(\pi|_K) = \ker \cap K = S_2 \cap K = 0 \), therefore \( \pi|_K \) is monomorphism. \( \Box \)

**Lemma 2.13:** Let \( S = S_1 \oplus S_2 \) be a \( T \)-semimodule (with injective hull) and \( K \) is a subsemimodule of \( S \), then \( K \) is a complement of \( S_2 \) in \( S \) if and only if \( K = \{ s_1 + \varphi(s_1) : s_1 \in \varphi^{-1}(S_2) \} \) for some \( \varphi \in \text{Hom}(S_1, E(S_2)) \).

**Proof:** Let \( K \) be a complement of \( S_2 \) in \( S \), and \( \pi_i: S \to S_i \), where \( (i = 1, 2) \) be the natural projections, since \( \ker(\pi_i|_K) = \ker(\pi_i) \cap K = S_2 \cap K = 0 \), then \( \pi_i|_K \) is a monomorphism, consider the diagram as follow:

\[
\begin{array}{c}
S_2 \\
\pi_i|_K \\
\downarrow \\
E(S_2)
\end{array}
\]

Where \( i \) the inclusion map, since \( E(S_2) \) is injective, then there exists \( \varphi \in \text{Hom}(S_1, E(S_2)) \), such that \( \varphi(\pi_i|_K) = \pi(\pi_2|_K) \).

Let \( x \in K \), then \( x = \pi_1(x) + \pi_2(x) \), since \( \varphi(\pi_1(x)) = \pi(\pi_2(x)) = \pi_2(x) \), then \( x = \pi_2(x) + \varphi(\pi_1(x)) \), hence \( K \subseteq \{ s_1 + \varphi(s_1) : s_1 \in \varphi^{-1}(S_2) \} \) and \( \{ s_1 + \varphi(s_1) : s_1 \in \varphi^{-1}(S_2) \} \cap S_2 = 0 \) (note that \( S_2 \) is a summand of \( S \), hence subtractive), since \( K \) is a complement of \( S_2 \) by assumption then \( K = \{ s_1 + \varphi(s_1) : s_1 \in \varphi^{-1}(S_2) \} \).

Conversely, suppose \( \varphi \in \text{Hom}(S_1, E(S_2)) \), and \( K = \{ s_1 + \varphi(s_1) : s_1 \in \varphi^{-1}(S_2) \} \), then \( K \subseteq S \) and \( K \cap S_2 = 0 \). Now suppose \( L \subseteq S \) and \( L \cap S_2 = 0 \), and \( K \subseteq L \).

Let \( u \in \text{L} \cap K \), then \( u = \pi_1(u) + \pi_2(u) \), and \( \pi_2(u) \neq \varphi(\pi_1(u)) \), now \( \pi_1(u) \in E(S_2) \) and \( S_2 \subseteq \text{E}(S_2) \) implies there exists \( r \in \text{E}(S) \) such that \( 0 \neq r \varphi(\pi_1(u)) \subseteq S_2 \), therefore \( \pi_1(ru) + \varphi(\pi_1(ru)) \in K \), where \( ru = \pi_1(ru) + \pi_2(ru) \).

Hence, \( ru + \varphi(\pi_1(ru)) = \pi_1(ru) + \varphi(\pi_1(ru)) + \pi_2(ru) \), where \( ru + \varphi(\pi_1(ru)) \in L + S_2 \), since \( L + S_2 \) is direct sum then \( ru + \varphi(\pi_1(ru)) \in \text{E}(S_2) \).

**Proposition 2.14:** Any direct summand of a \( CS \)-semimodule is \( CS \)-semimodule.

**Proof:** Let \( C \) be a direct summand of \( S \), then \( S = C \oplus D \), for some \( D \subseteq S \). Let \( N \subseteq N \subseteq C \), then \( N \cap D = 0 \) and \( L \subseteq S \), by (Lemma 2.8) \( N \not\subseteq \) \( S \), therefore \( N \) is direct summand of \( S \) (since \( S \) is \( CS \)-semimodule), \( S = N \oplus N' \) for some \( N' \subseteq S \), by Modular Law \( C = N \oplus (C \cap N') \), \( N \) is direct summand of \( C \), hence \( C \) is \( CS \)-semimodule. \( \Box \)

**Proposition 2.15:** Let \( S = S_1 \oplus S_2 \) be a \( T \)-semimodule (with injective hull) then the following statements are equivalent:

1. \( S \) is \( CS \)-semimodule.
2. \( \forall \varphi \in \text{Hom}(S_1, E(S_2)) \), the subsemimodule \( \{ s_1 + \varphi(s_1) : s_1 \in \varphi^{-1}(S_2) \} \) is \( CS \)-semimodule and a direct summand.
Proof: Suppose $\varphi : S_1 \rightarrow E(S_2)$ and let $K = \{s_1 + \varphi(s_1) : s_1 \in \varphi^{-1}(S_2)\} \subseteq S$, by (Lemma 2.13) $K$ is a complement of $S_2$ in $S$, since $S$ is CS-semimodule and by (Proposition 2.10) $K$ is a $CS$-semimodule and direct summand of $S$.

Conversely, let $N \leq S$. If $N \cap S_1 = 0$ then by (Lemma 2.13), $N = \{x + \varphi(x) : x \in \varphi^{-1}(S_1)\}$ for some $\varphi \in Hom(S_2, E(S_1))$ and by assumption it is a direct summand. If $N \cap S_1 \neq 0$, there exists a closed submodule $K$ of $N$ such that $N \cap S_1$ is essential in $K$. Clearly $K \cap S_2 = 0$. Let $\pi_1 : S_1 \rightarrow S$, where $(i = 1, 2)$ be the natural projections, then $\pi_1|K$ is a monomorphism and there exists $\varphi \in Hom(S_1, E(S_2))$ such that $\varphi(\pi_1(k)) = \pi_2(k)$ for all $k \in K$.

If $P = \{s_1 + \varphi(s_1) : s_1 \in \varphi^{-1}(S_2)\} \subseteq S$, then by (Lemma 2.13) $P$ is a complement of $S_2$ in $S$, and it is a direct summand of $S$ by assumption. Note that if $k \in K$, then $k = \pi_1(k) + \pi_2(k) = \pi_1(k) + \varphi(\pi_1(k)) \in P$, that is $K \subseteq P$. Since $P$ is $CS$-semimodule (by assumption) $K$ is a direct summand of $P$, hence $K$ is a direct summand of $S$, say $S = K \oplus K'$, and by Modular Law we have $N = K \oplus N \cap K$. Now, $N \cap K' \leq S$, clearly $(N \cap K') \cap S_1 = 0$ by an argument similar to the above $N \cap K'$ is a direct summand of $S$ and hence also of $K'$. It follows that $N$ is a direct summand of $S$. Thus $S$ is $CS$-semimodule. \hfill \Box

For the next result the following lemmas are required.

**Lemma 2.16:** If $\alpha \in Hom(S, S')$ is an isomorphism and $N \leq S$, then $\alpha(N) \leq S'$.

**Proof:** Let $K \leq S' \ni \alpha(N) \cap K = 0$, then $\alpha^{-1}(\alpha(N)) \cap K = 0$, thus $N \cap K^{-1}(K) = 0$, therefore $\alpha^{-1}(K) = 0$, and $K = 0$. \hfill \Box

**Lemma 2.17:** If $S \cong S'$, then $S$ is CS-semimodule if and only if $S'$ is CS-semimodule.

**Proof:** Immediately by definition and Lemma 2.16. \hfill \Box

**Proposition 2.18:** If $S = S_1 \oplus S_2$ is a $CS$-semimodule (with injective hull), $S_1$ and $S_2$ are relative injective semimodules and $\varphi \in Hom(S_1, E(S_2))$, then $\varphi^{-1}(S_2)$ is CS-semimodule.

**Proof:** Let $N = \{x + \varphi(x) : x \in \varphi^{-1}(S_2)\}$, by (Lemma 2.13) $N$ is a complement of $S_2$ also by (Proposition 2.10) $N$ is CS-semimodule, let $\pi_1|N = \alpha$, then $\alpha : N \rightarrow S_1$ is a monomorphism. Let $y \in \alpha(N)$, then $y = \pi_1(y)$, for some $n \in N$ [since $n = x + \varphi(x)$, then $\pi_1(n) = \pi_1(x) \in \varphi^{-1}(S_2)$], $n = \pi_2(n) + \pi_3(n) = \pi_2(x) + \pi_3(n) = x + \varphi(x)$, then $\pi_1(x) = x$ and $\pi_2(x) = \varphi(x)$, hence $y \in \varphi^{-1}(S_2)$, therefore $\alpha(N) \subseteq \varphi^{-1}(S_2)$. If $x \in \varphi^{-1}(S_2)$, then $x \in S_1$ and $\varphi(x) \in S_2$, but $x + \varphi(x) \in N$, therefore $\alpha(x) + \alpha(\varphi(x)) \in \alpha(N)$, then $\alpha(x) = x \in \alpha(N)$, hence $\alpha(N) = \varphi^{-1}(S_2)$, hence $\alpha^{-1} : N \rightarrow \varphi^{-1}(S_2)$ is an isomorphism. Since $N$ is CS-semimodule by (Lemma 2.17), $\varphi^{-1}(S_2)$ is CS-semimodule. \hfill \Box

3. Direct Sum and Direct Summand of CS-Semimodule

The direct summand and direct sum of CS-semimodule will be studied in this section as well as conditions that ensure a submodule of CS-semimodule to be CS-semimodule and supply related properties of CS-semimodule property.

**Proposition 3.1:** Let $S = S_1 \oplus S_2$ be a $T$-semimodule, where $S_1$ and $S_2$ are CS-semimodules, then $S$ is CS-semimodule if and only if every closed $K \leq S$ with $K \cap S_1 = 0$ or $K \cap S_2 = 0$ is a direct summand.

**Proof:** ($\Rightarrow$) It is proved by (Proposition 2.3).

($\Leftarrow$) Let $B \leq S$, then either $B \cap S_1 = 0$, then by assumption $B$ is direct summand of $S$. Or $B \cap S_1 \neq 0$, then there exists $D$ such that $B \cap S_1 \leq D \leq B$ by (Remark 2.9), then $D \cap S_2 = 0$. Note that $D \leq S$ by (Lemma 2.8), then by assumption, $D$ is a direct summand of $S$, that is, $S = D \oplus D'$ for some $D' \leq S$, by Modular Law $B = D \oplus (B \cap D')$, but $(B \cap D')$ is closed in $S$, then $(B \cap D') \cap S_2 = 0$, also by assumption $B \cap D'$ is a direct summand of $D'$, then $D' = (B \cap D') \oplus D''$ for some $D'' \leq S$, so $S = D \oplus (B \cap D') \oplus D'' = B \oplus D''$, therefore $B$ is a direct summand of $S$ and $S$ is CS-semimodule. \hfill \Box

5
Lemma 3.2: Let \( S = S_1 \oplus S_2 \) be a \( T \)-semisubtractive \( T \)-semimodule then \( S_1 \) is \( S_2 \)-injective implies for every subsemimodule \( C \) of \( S \) with \( C \cap S_1 = 0 \), there exists a subsemimodule \( S' \) of \( S \) such that \( S = S_1 \oplus S', C \subseteq S' \).

Proof: Assume that \( S_1 \) is \( S_2 \)-injective, let \( \pi_i: S \rightarrow S_i \), where \( (i = 1, 2) \) be the natural projections, let \( C \subseteq S \) with \( C \cap S_1 = 0 \), consider the diagram where \( \alpha = \pi_2|_C \) and \( \beta = \pi_1|_C \). \( \alpha \) is a monomorphism, by assumption there exists \( f: S_2 \rightarrow S \), \( f\alpha = \beta \).

Define \( S' = \{ f(a) + a: a \in S_2 \} \), then \( S' \subseteq S \). For \( c \in C \), \( c = \pi_2(c) + \pi_2(c) = f(\pi_2(c)) + \pi_2(c) \in S' \), so \( C \subseteq S' \). For \( a \in S \), \( a = \pi_2(a) + \pi_2(a) \) if \( \pi_2(a) = f(\pi_2(a)), \) then \( a \in C \subseteq S' \subseteq S_1 \oplus S' \). If \( \pi_2(a) \neq f(\pi_2(a)) \), then by semisubtractive property either \( \pi_2(a) + h = f(\pi_2(a)) \) or \( \pi_2(a) = f(\pi_2(a)) + h \) for some \( h \in S \) (in any case \( h \in S_1 \), since \( S_1 \) is a direct summand, hence subtractive). So, either \( a + h = \pi_2(a) + h + \pi_2(a) = f(\pi_2(a)) + \pi_2(a) \in S' \subseteq S_1 + S' \), hence \( a \in S_1 + S' \). Or \( a = \pi_2(a) + \pi_2(a) = f(\pi_2(a)) + h + \pi_2(a) = h + f(\pi_2(a)) + \pi_2(a) \in S_1 + S' \). Therefore \( S = S_1 + S' \). On other hand, if \( a \in S_1 \cap S' \) then \( a \in S_1 \) and \( a = f(b) + b \), for \( b \in S_2 \), therefore \( 0 = \pi_2(a) = \pi_2(f(b)) + \pi_2(b) = 0 + \pi_2(b) = \pi_2(b) \) and \( \pi_1(a) = \pi_1(f(b)) + \pi_1(b) = 0 \), then \( a = 0 \), hence \( S_1 \cap S' = 0 \) and \( S = S_1 \oplus S' \). \( \square \)

For the following proposition, we give a condition of the direct sum of \( CS \)-semimodules to be \( CS \)-semimodule.

Proposition 3.3: Let \( S = S_1 \oplus S_2 \) be a \( T \)-semimodule, where \( S_1 \) and \( S_2 \) are relative injective semimodules then \( S \) is \( CS \)-semimodule if and only if \( S_1 \) and \( S_2 \) are \( CS \)-semimodule.

Proof: (\( \Leftarrow \)) It is proved by (Proposition 2.14).

(\( \Rightarrow \)) Assume that \( S_1 \) and \( S_2 \) are \( CS \)-semimodule and \( S_1 \) is \( S_2 \)-injective for (i.e. \( j = 1, 2 \) and \( i \neq j \)), let \( K \leq S \) and \( K \cap S_1 = 0 \), by (Lemma 3.2) there exists \( S' \leq S \) such that \( S = S_1 \oplus S' \) and \( K \leq S' \). It is clear that \( S' \subseteq S_2 \) and hence \( S' \) is \( CS \)-semimodule by (Lemma 2.17). On the other hand, \( K \leq S' \) (since it is closed in \( S \)) hence \( K \) is a direct summand of \( S' \), therefore \( K \) is a direct summand of \( S \), similarly for any subsemimodule \( H \) of \( S \) with \( H \cap S_2 = 0 \), is a direct summand of \( S \), therefore by (Proposition 3.1), \( S \) is \( CS \)-semimodule.

Proposition 3.4: Let \( S = S_1 \oplus S_2 \) be a \( T \)-semimodule, if \( S_1 \) is \( CS \)-semimodule and \( S_2 \) is \( S_1 \)-injective then every closed subsemimodule \( K \) of \( S \) with \( K \cap S_1 = 0 \) is a direct summand of \( S \).

Proof: Let \( K \leq S \) with \( K \cap S_2 = 0 \), since \( S_2 \) is \( S_1 \)-injective by (Lemma 3.2) there exists \( S' \leq S \) such that \( K \subseteq S' \) and \( S = S_1 \oplus S_2 \), therefore \( S' \subseteq S_1 \), since \( S_1 \) is \( CS \)-semimodule, then \( S' \) is \( CS \)-semimodule and \( K \) is a direct summand of \( S' \) (say \( S' = K \oplus K' \)) hence \( S = (K \oplus K') \oplus S_2 = K \oplus (K' \oplus S_2) \), that is, \( K \) is a direct summand of \( S \), hence \( S \) is \( CS \)-semimodule. \( \square \)

For determining under which condition a subsemimodule has a unique complement see the following.

Lemma 3.5: Let \( S = S_1 \oplus S_2 \) be a \( T \)-semimodule (with injective hull), and \( \varphi \in Hom(S_1, E(S_1)) \). If \( Hom(S_1, E(S_1)) = 0 \), then \( S_1 \) is a unique complement of \( N = \{ x + \varphi(x): x \in \varphi^{-1}(S_2) \} \).

Proof: Let \( Y \leq S \), with \( Y \cap N = 0 \). Note that \( ker(\varphi) \subseteq N \). Let \( Y \cap \varphi^{-1}(S_2) = K \), if \( K \neq 0 \), then \( \varphi \) is a monomorphism [since \( ker(\varphi|_K) = ker(\varphi) \cap K \subseteq N \cap K \subseteq N \cap Y = 0 \)]. Consider the diagram:
Since $E(S_1)$ is injective, there exists $0 \neq \alpha \in \text{Hom}(S_2, E(S_1))$, but this contradicts the assumption, then $K = 0$, therefore $Y \cap \varphi^{-1}(S_2) = 0$, but $\varphi^{-1}(S_2) \leq e S_1$, then $Y \subseteq S_1 = 0$, hence $\pi_2 | \varphi$ is a monomorphism and $\pi_1(\varphi) = 0$. Therefore, $Y \subseteq S_2$, and $S_2$ is a unique complement of $N$. □

For a specific purpose, we derive a new lemma from Proposition 2.13 that will be more generality as follows:

**Lemma 3.6:** Let $S = S_1 \oplus S_2$ be a $T$-semimodule (with injective hull), and $A \subseteq S$ with $A \cap S_2 = 0$, then $A \leq e S$ if and only if $A = \{x + \varphi(x) : x \in X\}$ where $X \leq e \varphi^{-1}(S_2)$, for some $\varphi \in \text{Hom}(S_1, E(S_2))$.

**Proof:** ($\Rightarrow$) Let $\pi_i : S \to S_i$, where $(i = 1, 2)$ be the natural projections, since $A \cap S_2 = 0$, then $\pi_1(A) = \pi_1(A)$ is a monomorphism, hence there exists $\varphi \in \text{Hom}(S_1, E(S_2))$ such that $\varphi(\pi_1(A)) = \pi_2(A)$ for all $a \in A$, where $\pi_2 = \pi_2 | a$, then $\varphi(\pi_1(a)) = \pi_2(a)$. Hence, for each $a \in A$, $a = \pi_1(a) + \pi_2(a) = \pi_1(a) + \varphi(\pi_1(a))$, so $A = \{x + \varphi(x) : x \in \pi_1(A)\}$, note that $\pi_2(A) = \varphi(\pi_1(A)) \subseteq S_2$, hence $\pi_1(A) \leq \varphi^{-1} \varphi \pi_2(A) | \leq \varphi^{-1}(S_2)$, if $\pi_1(A) \leq e Y \leq \varphi^{-1}(S_2)$, then $A + S_2 \leq e \varphi^{-1}(Y)$, but $S_1 \leq \varphi^{-1}(Y)$, therefore $A \leq e \varphi^{-1}(Y)$, since $A$ is closed in $S$, then $A = \pi_1^{-1}(Y)$, and $\pi_1(A) = Y$, hence $\pi_2(A) \leq e \varphi^{-1}(S_2)$.

($\Leftarrow$) if $A = \{x + \varphi(x) : x \in X\}$ and $X \leq e \varphi^{-1}(S_2)$, it is clear that $A \subseteq N = \{x + \varphi(x) : x \in \varphi^{-1}(S_2)\}$, and that $A$ has a proper essential extension in $N$ if and only if $X$ has a proper essential extension in $\varphi^{-1}(S_2)$, since $X$ is closed in $\varphi^{-1}(S_2)$, it follows that $A \leq e N$, then $A \leq e S$. □

**Lemma 3.7:** Let $S = S_1 \oplus S_2$ be a $T$-semimodule (with injective hull), where $S_1$ and $S_2$ are subsemimodules of $S$. If $S_2$ is $S_1$-injective then any closed subsemimodule $A$ in $S$ with $A \cap S_2 = 0$ must have the form $A = \{x + \varphi(x) : x \in X\}$, where $X$ is closed subsemimodule of $S_1$ and $\varphi \in \text{Hom}(S_1, E(S_2))$.

**Proof:** Let $A$ be a closed subsemimodule in $S$ with $A \cap S_2 = 0$, then by (Lemma 2.13) $A = \{x + \varphi(x) : x \in X\}$, where $X$ is closed subsemimodule of $\varphi^{-1}(S_2)$, for some $\varphi \in \text{Hom}(S_1, E(S_2))$. But $\varphi^{-1}(S_2) \leq e S_1$, so $X \leq e S_1$. □

**Lemma 3.8:** Let $S = S_1 \oplus S_2$ be a $T$-semimodule (with injective hull), where $S_1$ and $S_2$ are subsemimodules of $S$. If $\varphi^{-1}(S_2) = S_1$ for each $\varphi \in \text{Hom}(S_1, E(S_2))$, then $S_2$ is $S_1$-injective.

**Proof:** Consider the diagram below, assume $K$ is a subsemimodule of $S_1$ in $S$ where $i$ is the inclusion map, $f$ is any homomorphism $j$ is the inclusion map.

$$\xymatrix{ K & S_1 \ar[l]_i \ar[r]^f & S_2 \\
E(S_2) \ar[u]^\varphi & \ar[u]_j }$$

Since $E(S_2)$ is injective there exists $0 \neq \varphi \in \text{Hom}(S_1, E(S_2))$ such that $\varphi f = j$. Since $\varphi^{-1}(S_2) = S_1$, then $\varphi(S_1) \subseteq S_2$ and $\varphi \in \text{Hom}(S_1, S_2)$, therefore $S_2$ is $S_1$-injective. □
Proposition 3.9: Let $S = S_1 \oplus S_2$ be a $T$-semimodule (with injective hull), and $\text{Hom}(S_2, E(S_1)) = 0$, then $S_1$ is CS-semimodule and $S_2$ is $S_1$-injective if and only if every closed subsemimodule $K$ of $S$ with $K \cap S_2 = 0$ is a direct summand of $S$.

Proof: ($\Rightarrow$) It is proved by (Proposition 3.4).

($\Leftarrow$) Suppose $K \subseteq S_1$, then $K \cap S_2 = 0$ and $K \subseteq S$ by (Lemma 2.8). By assumption $K$ is a direct summand of $S$, say $S = K \oplus K'$, hence $S_1 = K \oplus K' \cap S_1$, therefore $K$ is a direct summand of $S_1$, and $S_1$ is CS-semimodule.

Now, let $\alpha \in \text{Hom}(S_1, E(S_2))$ be arbitrary, then by (Lemma 2.13) $L = \{x + \alpha(x) : x \in \alpha^{-1}(S_2)\}$ is closed in $S$ and it is a complement of $S_2$, by assumption it is a direct summand of $S$. If $\pi_1$ is the natural projection of $S = S_1 \oplus S_2$ onto $S_1$, then $y \in L$ implies $y = x + \alpha(x)$ for some $x \in \alpha^{-1}(S_2)$ and $\pi_1(y) = \pi_1(x) = \alpha$, that is, $\pi_1(L) = \alpha^{-1}(S_2)$. If $x \in \alpha^{-1}(S_2)$, then $\pi_1(x) = x$ and $x + \alpha(x) \in L$, hence $x = \pi_1(x) = \pi_1(x + \alpha(x)) \in \pi_1(L)$, that is, $\pi_1(L) = \alpha^{-1}(S_2)$. Since $\pi_1(L)$ is closed in $S_1$ and $\alpha^{-1}(S_2)$ is essential in $S_1$, it follows $S_1 = \alpha^{-1}(S_2)$. Therefore, by (Lemma 3.8), $S_2$ is $S_1$-injective. □

Corollary 3.10: Let $S = S_1 \oplus S_2$ be a $T$-semimodule (with injective hull), and $\text{Hom}(S_2, E(S_1)) = 0$, then $S$ is CS if and only if $S_1$ and $S_2$ are CS-semimodule and $S_2$ is $S_1$-injective.

Proof: ($\Rightarrow$) By (Propositions 2.14), both $S_1$ and $S_2$ are CS-semimodules, then by (Proposition 3.4) and (Proposition 3.9) $S_2$ is $S_1$-injective.

($\Leftarrow$) This is proved by (Propositions 3.9, 2.13 and 2.15). □

Reference

[1] Neumann J von. 1936. Continuous geometry. Proc Nat Acad Sci. 22:92–100.
[2] Utumi Y. 1965. On continuous rings and self injective rings. Trans Am Math Soc. 118(1):158–173.
[3] Müller SHM and BJM. 1990. Continuous and discrete modules. Cambridge University Press.
[4] Birkenmeier, Gary F and Park, Jae Keol and Rizvi ST. 2013. Extensions of rings and modules. Springer.
[5] Dung, Nguyen Viet and Va Huynh, Dinh and Smith, Patrick F and Wisbauer R. 1994. Extending modules. CRC Press.
[6] Tercan, Adnan and Yucel CC. 2016. Module Theory, Extending Modules and Generalizations. 2016.
[7] Tsiba JR. 2010. On Generators and Projective Semimodules. Int J Algebr. 4(24):1153–67.
[8] Tavallae HA, Zolfaghari M. 2013. On semiprime subsemimodules and related results. J teh Indones Math Soc. 19(1):49–59.
[9] Chaudhari JN, Bonde DR. 2013. On Exact Sequence of Semimodules over Semirings. Int Sch Res Not. 2013(1):1–5.
[10] Ebrahimi Atani, Reza and Atani S-E. 2010. On subsemimodules of semimodules. Bul Acad stiinta Republicii Mold Mat. 63(2):20–30.
[11] Golan JS. 1999. Semirings and their Applications. Kluwer Academic Publishers, Dordrecht.
[12] Pawar K. 2013. A Note on Essential Subsemimodules. New Trends Math Sci.1(2):18–21.
[13] Muna M.T. Altaee and Asaad A. M. Alhossaini. 2020. $\pi$-injective semimodule over semiring. J Eng Appl Sci.63(5):3424-3433.
[14] Aljebory, Khitam SH and Alhossaini AM. 2019 Principally Quasi-Injective Semimodules. Baghdad Sci J.16(4):928–36.