Optimal Green energy points on the circles in $d$-space

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Abstract

We give two precise estimates for the Green energy of a discrete charge, concentrated in the points on the circles, with respect to the concentric rotation domain in the $d$-dimensional Euclidean space, $d > 2$. The proof is based on the application of a dissymmetrization, extremal metrics approach and an asymptotic formula for the condenser capacities in the case when some of its plates contract to given points.

Keywords: Green energy, discrete charge, dissymmetrization, condenser capacities

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1 Introduction and statement of results

The Riesz $s$–energy ($s \neq 0$) of $n$ points $z_1, \ldots, z_n$ of the complex plane is defined by

$$\sum_{k=1}^{n} \sum_{l=1 \atop l \neq k}^{n} |z_k - z_l|^{-s}.$$ 

It can be shown using the classical Tóth’s result [6, p.155] and a convexity argument that for $s \geq -1$ and each $n \geq 2$, the $n$–th roots of unity $z_k^n = \exp\{2\pi i(k - 1)/n\}, k = 1, \ldots, n$, form minimal $n$–point $s$–energy configuration for the unit circle $|z| = 1$,

$$\sum_{k=1}^{n} \sum_{l=1 \atop l \neq k}^{n} |z_k - z_l|^{-s} \geq \sum_{k=1}^{n} \sum_{l=1 \atop l \neq k}^{n} |z_k^n - z_l^n|^{-s}. \quad (1)$$

Various sophisticated problems related to the optimality of the Riesz $s$-energy for different values of $s$ and for the points $z_k$ lying in the plane sets or in $\mathbb{R}^d$ have been treated in a number of papers (see, for instance, [3]-[5], and references therein). In this note we consider

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the Green energy with respect to a rotation domain in $\mathbb{R}^d$, $d \geq 3$, of a discrete charge concentrated in the points of certain circles. Limit cases of the optimal properties of this energy lead to inequalities for the Riesz s-energy for $s = d - 2$. Unlike previous works we study the precise formulations. In what follows $\mathbb{R}^d$ is a $d$-dimensional Euclidean space with the usual norm $\|\cdot\|$, with points $x = (x_1, \ldots, x_d)$, $d \geq 3$. A domain $B$ in $\mathbb{R}^d$ is admissible if it has the Green function for the Laplace operator vanishing at the points of the boundary $\partial B$ of the domain $B$. This Green function with pole at the point $x_0 \in B$ will be denoted by $g_B(x, x_0)$. In the neighborhood of $x_0$ the following expansion holds

$$g_B(x, x_0) = \lambda_d(\|x - x_0\|^{2-d} - (r(B, x_0))^{2-d} + o(1)), \quad x \to x_0,$$

where $\lambda_d = ((d - 2)\omega_{d-1})^{-1}$, $\omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the surface measure of the unit hypersphere. In all points of $B$ different from the pole $x_0$, the Green function is harmonic, that is $\Delta g_B(x, x_0) = 0$. The quantity $r(B, x_0)$ is known as the harmonic radius of the domain $B$ with respect to the points $x_0$.

Denote by $J$ the $(d-2)$-dimensional plane $\{x \in \mathbb{R}^d : x = (0, 0, x_3, \ldots, x_d)\}$. We will need the cylindrical coordinates $(r, \theta, x')$ of the point $x = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$, related to the Cartesian coordinates by $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, $x' \in J$. A domain $B \subset \mathbb{R}^d$ will be called the rotation domain (with respect to the axis $J$), if for any point $(r, \theta, x') \in B$ and any $\varphi$ the point $(r, \varphi, x')$ belongs to $B$.

Suppose that $B$ is an admissible rotation domain and let $\Omega = \{S\}$ be the collection comprising a finite number of distinct circles $S$ of the form $S = \{(r_0, \theta, x_0') : 0 \leq \theta \leq 2\pi\}$ lying in the domain $B$ (here $r_0 > 0$ and $x_0' \in J$ are assumed to be fixed). For arbitrary real numbers $\theta_j$, $j = 0, \ldots, m - 1$,

$$0 \leq \theta_0 < \theta_1 < \ldots < \theta_{m-1} < 2\pi,$$

denote by $X = \{x_k\}_{k=1}^n$ the collection of all distinct points of $B$ at which the circles from $\Omega$ intersect the half-planes

$$L_j = \{(r, \theta, x') : \theta = \theta_j\}, \quad j = 0, \ldots, m - 1.$$

Let $\Delta = \{\delta_k\}_{k=1}^n$ be an arbitrary discrete charge (a collection of real numbers), having the value $\delta_k$ at the point $x_k$, $k = 1, \ldots, n$. The Green energy of this charge with respect to the domain $B$ is defined by

$$E(X, \Delta, B) = \sum_{k=1}^n \sum_{l=1}^n \delta_k \delta_l g_B(x_k, x_l).$$

Define also $X^* = \{x_k^*\}_{k=1}^n$ - the collection of points at which the circles from $\Omega$ intersect the half-planes

$$L_j^* = \{(r, \theta, x') : \theta = 2\pi j/m\}, \quad j = 0, \ldots, m - 1.$$

**Theorem 1** Suppose that the charge $\Delta = \{\delta_k\}_{k=1}^n$ takes equal values $\delta_k = \delta_l$ at the points $x_k$ and $x_l$ from the collection $X$ that lie on the same circle from $\Omega$ and, furthermore, that the points $x_k \in X$ and $x_k^* \in X^*$ lie on the same circle from $\Omega$, $k = 1, \ldots, n$. Then

$$E(X, \Delta, B) \geq E(X^*, \Delta, B).$$
The following proposition asserts that under certain conditions the symmetric configuration has the maximal energy:

**Theorem 2** Suppose that the domain $B$ and the collections $\Omega$, $X$ and $X^*$ are as defined above while $m$ is an even number. Assume further that the charge $\Delta = \{\delta_k\}_{k=1}^n$ takes the values of equal moduli $|\delta_k| = |\delta_l|$ at the points $x_k$ and $x_l$ belonging to $X$ and lying on the same circle from $\Omega$ and, moreover $\delta_k < 0$ if the point $x_k$ belongs to one of the a half-planes $L_{2p-1}$, $1 \leq p \leq m/2$, otherwise $\delta_k > 0$, $k = 1, \ldots, n$. Then

$$E(X, \Delta, B) \leq E(X^*, \Delta, B),$$

where the points of the collection $X^*$ are numbered as follows: if $x_k^* \in X^*$ lies at the intersection of a circle $S$ from $\Omega$ and a half-plane $L_j^*$, then the corresponding point $x_k \in X$ must lie at the intersection of the circle $S$ and the half-plane $L_j$, $k = 1, \ldots, n$, $0 \leq j \leq m - 1$.

Note that using the symmetry principle for harmonic functions it is not difficult to establish [2] that the Green function of the ball $B(0, t) = \{x \in \mathbb{R}^d : ||x|| < t\}$ with the pole at the point $x_0 \in B(0, t)$ takes the form

$$g_{B(0,t)}(x, x_0) = \lambda_d \left( ||x - x_0||^{2-d} - \frac{||x_0||}{t} \frac{||x - x_0||}{||x_0||}^{2-d} \right).$$

Placing the points from $X$ into a sufficiently large ball $B(0, t)$ and letting $t \to \infty$, from Theorem 1,2 we deduce inequalities for the Riesz $(d - 2)$–energy. In particular, Theorem 1 leads to inequality (11) for $s = d - 2$, while Theorem 2 yields

$$\sum_{k=1}^{2n} \sum_{l \neq k}^{2n} (-1)^{k+l} \vert z_k - z_l \vert^{d-2} \leq \sum_{k=1}^{2n} \sum_{l \neq k}^{2n} \vert z_k^* - z_l^* \vert^{d-2},$$

where $z_k, k = 1, \ldots, 2n$, are located on the circle $|z| = 1$ in the ascending order of the index $k$ and $z_k^* = \exp\{\pi i (k - 1)/n\}, k = 1, \ldots, 2n$.

The proofs of Theorems 1,2 hinge on the theory of condenser capacity and dissymmetrization [7], [8]. These proofs are related conceptually with the solutions of the so-called extremal decomposition problems [9], [10], [14], [15]. In the recent paper [9], analogues of Theorems 1,2 for the case of the plane and one circle and a concentric ring have been presented. The proof of Theorem 1 of this paper follows the same line of argument as the one presented in [9] with modifications related to the use of dissymmetrization [7] and the asymptotic formula for the capacity of the spatial rather than plane condenser [10]. The proof of an analogue of Theorem 2 for the plane case [9] is based on the radial averaging transformation and conformal mapping. This method is not applicable in the Euclidean space due to absence of the suitable conformal mappings. Therefore in order to demonstrate Theorem 2 we resort to the moduli of the families of curves (see, for instance, [13], [1], [17]). The idea behind this approach goes back to the proof of Theorem 4 from [10]. Our results, as well as their proofs, can be carried over to the discrete energy with the Robin function kernel [11] (of the domain $B$ with respect to a part of the boundary) in place of the Green function kernel. The next section is of an auxiliary nature.
2 Preliminaries

Suppose $B$ is an admissible domain in the space $\mathbb{R}^d$, $d > 2$; $X = \{x_k\}_{k=1}^n$ is a collection of distinct points in $B$; $\Lambda = \{\sigma_k\}_{k=1}^n$ is a collection of non-vanishing real numbers; $\Psi = \{\mu_k\}_{k=1}^n$ is a collection of positive numbers $\mu_k$. Denote by $E(a, t) = \{x \in \mathbb{R}^d : \|x - a\| \leq t\}$ the closed ball of radius $t$ centered at $a$. For sufficiently small $t > 0$ introduce ”the generalized” condenser as the ordered collection

$$C(t; B, X, \Lambda, \Psi) = \{\mathbb{R}^d \setminus B, E(x_1, \mu_1t), ..., E(x_n, \mu_n t)\}$$

with pre-assigned values $0, \sigma_1, ..., \sigma_n$, respectively [10]. Similarly to the usual condensers, defined the capacity (or 2-capacity) of the condenser $C(t; B, X, \Lambda, \Psi)$ by

$$\text{cap } C(t; B, X, \Lambda, \Psi) = \inf \int_{\mathbb{R}^d} |\nabla v|^2 dx,$$

where the infimum is taken over all functions $v : \mathbb{R}^d \to \mathbb{R}$ from $C^\infty(\mathbb{R}^d)$, vanishing in a neighborhood of the set $\mathbb{R}^d \setminus B$ and equaling to $\sigma_i$ in a neighborhood $E(x_i, \mu_i r)$, $l = 1, ..., n$. The condenser modulus $|C(t; B, X, \Lambda, \Psi)|$ is reciprocal to the capacity of $C(t; B, X, \Lambda, \Psi)$:

$$|C(t; B, X, \Lambda, \Psi)| = (\text{cap } C(t; B, X, \Lambda, \Psi))^{-1}.$$

Lemma 1 [10], Theorem 1]. The following asymptotic formula holds as $t \to 0$:

$$|C(t; B, X, \Lambda, \Psi)| = \nu \lambda_d t^{2-d} - \lambda_d \nu^2 \sum_{k=1}^n \nu_k^2 r(B, x_k)^{2-d} + \nu^2 \sum_{k=1}^n \sum_{l \neq k}^n \nu_\lambda g_B(x_l, x_k) + o(1), \quad (2)$$

where $\nu_k = \sigma_k \mu_k^{d-2}$, $\nu = \left( \sum_{k=1}^{n} \sigma_k^2 \mu_k^{d-2} \right)^{-1}$, $k = 1, ..., n$.

Let $\Gamma$ be a family of curves in $\mathbb{R}^d$. We will assume that each curve $\gamma \in \Gamma$ is a union of a countable number of open arcs, closed arcs or closed curves and is locally rectifiable. 2-modulus or just modulus of the family $\Gamma$ is defined as the quantity

$$M(\Gamma) = \inf \int_{\mathbb{R}^d} \rho^2 dx,$$

where infimum is taken over all Borel functions $\rho : \mathbb{R}^d \to [0, \infty]$ such that $\int_{\gamma} \rho ds \geq 1$ holds for each curve $\gamma \in \Gamma$ [17]. It is said that the family $\Gamma_2$ is minorized by the family $\Gamma_1$, if each curve $\gamma \in \Gamma_2$ has a sub-curve belonging to $\Gamma_1$. The families $\Gamma_1, \Gamma_2, ...$ are called separated if there exist disjoint Borel sets $E_i$ in $\mathbb{R}^d$, such that $\int_{\gamma} \chi_i ds = 0$ for any curve $\gamma \in \Gamma_i$, where $\chi_i$ is the characteristic function of $\mathbb{R}^d \setminus E_i$. If $\Gamma_1, \Gamma_2, ...$ are separated families and $\Gamma_i$ is minorized by $\Gamma_i$, $i = 1, 2, ..., \infty$, then

$$M(\Gamma) \geq \sum_{i=1}^{\infty} M(\Gamma_i). \quad (3)$$
If, on the contrary, $\Gamma$ is minorized by $\Gamma_i$, $i = 1, 2, \ldots$, and $\Gamma_1, \Gamma_2, \ldots$ are separated families, then

$$M(\Gamma)^{-1} \geq \sum_{i=1}^{\infty} M(\Gamma_i)^{-1}.$$  \hfill (4)

It is easy to see that the capacity of the condenser $C(t; B, X, \Lambda, \Psi)$ under the choice $\sigma_k = 1$, $k = 1, \ldots, n$, coincides with the capacity of the condenser with two plates $E(x_1, \mu_1 t) \cup E(x_2, \mu_2 t) \ldots \cup E(x_n, \mu_n t)$ and $\mathbb{R}^d \setminus B$ (for the definition of the condenser capacity see, for instance, in [7], [12]). Therefore, the following lemma holds true.

**Lemma 2 [12]**. Let $\sigma_1 = \ldots = \sigma_n = 1$ or $\sigma_1 = \ldots = \sigma_n = -1$, $\Lambda = \{\sigma_k\}_{k=1}^n$, $B, X, \Psi$ as defined above, $\Gamma(t; B, X, \Psi)$ is the family of continuous curves in $B$ connecting the set $E(x_1, \mu_1 t) \cup E(x_2, \mu_2 t) \ldots \cup E(x_n, \mu_n t)$ with the boundary $\partial B$ of the domain $B$. Then

$$\text{cap } C(t; B, X, \Lambda, \Psi) = M(\Gamma(t; B, X, \Psi)).$$

We will further need the definition of dissymmetrization in Euclidean space [7]. Denote by $\Phi$ the group of reflections in $\mathbb{R}^d$ with respect hyper-planes of the form $\{(r, \theta, x') : \theta = \pi k/m, \text{ or } \theta = \pi + \pi k/m, \}$, $k = 1, \ldots, m$. Next we introduce a symmetric structure $\{P_i\}_{i=1}^N$ in $\mathbb{R}^d$ as the collection of closed angles $P_i = \{(r, \theta, x') : \theta_i \leq \theta \leq \theta_{i+1}, 0 \leq r \leq \infty\}, i = 1, \ldots, N$, satisfying the conditions:

- aP) $\bigcup_{i=1}^N P_i = \mathbb{R}^d$, $\sum_{i=1}^N (\theta_{i+1} - \theta_i) = 2\pi$,
- bP) $\{\phi(P_i)\}_{i=1}^N = \{P_i\}_{i=1}^N$ for any isometry $\phi \in \Phi$.

The family of rotations $\{\lambda_i\}_{i=1}^N$ of the form $\lambda_i(r, \theta, x') = (r, \theta + \varphi, x')$, $i = 1, \ldots, N$, will be called the dissymmetrization of the symmetric structure $\{P_i\}_{i=1}^N$, if the images $S_i = \lambda_i(P_i)$ satisfy the following conditions:

- aS) $\bigcup_{i=1}^N S_i = \mathbb{R}^d$,
- bS) for any non-empty intersection $S_i \cap S_j$, $i, j = 1, \ldots, N$, there exists an isometry $\phi \in \Phi$, such that $\phi(\lambda_i^{-1}(S_i \cap S_j)) = \lambda_j^{-1}(S_i \cap S_j)$.

For an arbitrary set $A$ in $\mathbb{R}^d$ introduce the notation $\text{Dis } A = \bigcup_{i=1}^N \lambda_i(A \cap P_i)$. A characteristic feature of a rotation domain $B$ is the fact that such domain is invariant with respect to any dissymmetrization $\text{Dis } B = B$.

**Lemma 3 [7]**. Let the numbers $\theta_j$, $j = 0, \ldots, m$, satisfy $0 \leq \theta_0 < \theta_1 < \ldots < \theta_{m-1} < 2\pi$, $\theta_m = \theta_0 + 2\pi$, and suppose that $L_j = \{(r, \theta, x') : \theta = \theta_j\}$, $L_j^* = \{(r, \theta, x') : \theta = 2\pi j/m, \}$, $j = 0, \ldots, m - 1$. Then there exists a symmetric structure $\{P_i\}_{i=1}^N$, $N \geq m$, and a dissymmetrization $\{\lambda_i\}_{i=1}^N$, such that $\text{Dis } L_j^* = L_j$, $j = 0, \ldots, m - 1$, and each half-plane $L_j^*$ is the bisector of a dihedral angle $P_i$ of size $\psi$, where

$$\psi = \min_{i=1,\ldots,m} (\theta_i - \theta_{i-1}).$$
The condenser $C(t;B,X,\Lambda,\Psi)$ will be called symmetric with respect to the group $\Phi$, if $B$ is a rotation domain and for any $k$, $k = 1, \ldots, n$, and any isometry $\phi \in \Phi$ we have $\phi(x_k) \in X$ and $\sigma_k = \sigma_l$, $\mu_k = \mu_l$ in the case $\phi(x_k) = x_l$. The result of dissymmetrization of a symmetric condenser $C(t;B,X,\Lambda,\Psi)$ is defined to be the condenser $\text{Dis} \ C(t;B,X,\Lambda,\Psi) = C(t;B,\{\text{Dis} \ x_k\}_{k=1}^{n}, \Lambda,\Psi)$.

**Lemma 4** If the condenser $C(t;B,X,\Lambda,\Psi)$ is symmetric with respect to the group $\Phi$, then for sufficiently small $t$ the following inequality holds

$$|C(t;B,X,\Lambda,\Psi)| \leq |\text{Dis} \ C(t;B,X,\Lambda,\Psi)|.$$

The proof of this claim is essentially the same as the proof of a similar statement in [3, Theorem 4.14]. A particular case has been considered in [7, Theorem 5].

### 3 Proofs of Theorems

We will start with the proof of Theorem 1. Suppose the domain $B$ and the collection $X = \{x_k\}_{k=1}^{n}$, $X^* = \{x_k^*\}_{k=1}^{n}$, $\Delta = \{\delta_k\}_{k=1}^{n}$ as in Theorem 1. We can assume that $\delta_k \neq 0$, $k = 1, \ldots, n$. Put $\sigma_k = \text{sgn} \delta_k$, $\mu_k = |\delta_k|^{1/(d-2)}$, $k = 1, \ldots, n$, $\Lambda = \{\sigma_k\}_{k=1}^{n}$, $\Psi = \{\mu_k\}_{k=1}^{n}$. Note that the condenser $C(t;B,X^*,\Lambda,\Psi)$ is symmetric with respect to the group $\Phi$. Apply dissymmetrization from Lemma 3 to the condenser $C(t;B,X^*,\Lambda,\Psi)$. The result of dissymmetrization of this condenser for small $t$ is the condenser $C(t;B,X,\Lambda,\Psi)$. According to Lemma 4

$$|C(t;B,X^*,\Lambda,\Psi)| \leq |C(t;B,X,\Lambda,\Psi)|.$$  

Applying the asymptotic formula (2), we obtain

$$\nu \lambda d t^{2-d} - \lambda d \nu^2 \sum_{k=1}^{n} \nu_k^2 r(B,x_k)^{2-d} + \nu^2 E(X^*,\Delta,B) + o(1) \leq$$

$$\nu \lambda d t^{2-d} - \lambda d \nu^2 \sum_{k=1}^{n} \nu_k^2 r(B,x_k)^{2-d} + \nu^2 E(X,\Delta,B) + o(1), \ t \to 0, \ (5)$$

where $\nu_k = \delta_k$, $\nu = \left( \sum_{k=1}^{n} |\delta_k| \right)^{-1}$, $k = 1, \ldots, n$. As $B$ is the rotation domain, harmonic radii $r(B,x)$ take equal values at all points $x$, lying on one circle from $\Omega$. Hence,

$$\sum_{k=1}^{n} \nu_k^2 r(B,x_k)^{2-d} = \sum_{k=1}^{n} \nu_k^2 r(B,x_k)^{2-d}$$

and it remains to take the limit as $t \to 0$ in (5) to complete the proof of Theorem 1.

Let the domain $B$ and the collections $X = \{x_k\}_{k=1}^{n}$, $X^* = \{x_k^*\}_{k=1}^{n}$, $\Delta = \{\delta_k\}_{k=1}^{n}$ be as in Theorem 2. We can assume that the boundary $\partial B$ represents a continuously differentiable surface in $\mathbb{R}^d$. Put $\sigma_k = \text{sgn} \delta_k$, $\mu_k = |\delta_k|^{1/(d-2)}$, $k = 1, \ldots, n$, $\Lambda = \{\sigma_k\}_{k=1}^{n}$, $\Psi = \{\mu_k\}_{k=1}^{n}$. The condenser $C(t;B,X,\Delta,\Psi)$ admits a potential function $u$, which is continuous in $\overline{B}$, harmonic
Denote by \( I \) the set of points in \( B \), where \( u = 0 \), and by \( D_l, l = 1, \ldots, q \), the connected components \( B \setminus I \). Suppose \( X_l \) is the set of all points from \( X \), lying in the domain \( D_l \).

For the collection of points \( X_l = \{ y_{sl} \}_{s=1}^{N_l} \) let us define \( \Lambda_l = \{ \sigma_{sl} \}_{s=1}^{N_l} \) and \( \Psi_l = \{ \mu_{sl} \}_{s=1}^{N_l} \) according to the rule \( \sigma_{sl} = \sigma_p, \ \mu_{sl} = \mu_p \), if \( y_{sl} = x_p, \ x_p \in X \). According to the Dirichlet principle \[18\]

\[
\text{cap} C(t; B, X, \Lambda, \Psi) = \int_B |\nabla u|^2 \, dx = \sum_{l=1}^{q} \int_{D_l} |\nabla u|^2 \, dx = \sum_{l=1}^{q} \text{cap} C(t; D_l, X_l, \Lambda_l, \Psi_l). \tag{6}
\]

Note that the points lying in the domain \( D_l \) have the same charge (either 1 or \(-1\)) so that according to Lemma 2 the following equality holds:

\[
\text{cap} C(t; B, X, \Lambda, \Psi) = \sum_{l=1}^{q} M_l, \tag{7}
\]

where \( M_l = M(\Gamma(t; D_l, X_l, \Psi_l)) \). Let \( n_l \) denote the number of half-planes \( L_j \), containing at least one point from \( X_l, 0 \leq j \leq m - 1 \). It is clear that \( \sum_{l=1}^{q} n_l \geq m \). Convexity of the function \( 1/x \) implies that for any positive numbers \( v_l, \alpha_l, \sum_{l=1}^{q} \alpha_l = 1, \ l = 1, \ldots, q \), the following inequality holds:

\[
\left( \sum_{l=1}^{q} \alpha_l (\alpha_l^{-1} v_l) \right)^{-1} \leq \sum_{l=1}^{q} \alpha_l (\alpha_l^{-1} v_l)^{-1},
\]

or

\[
\left( \sum_{l=1}^{q} v_l \right)^{-1} \leq \sum_{l=1}^{q} \alpha_l^2 v_l^{-1}. \tag{8}
\]

Obviously, inequality (8) remains valid for any non-negative \( \alpha_l, \sum_{l=1}^{q} \alpha_l \geq 1 \) and, moreover,

\[
\left( \sum_{l=1}^{q} v_l \right)^{-1} \leq \frac{1}{q^2} \sum_{l=1}^{q} v_l^{-1}. \tag{9}
\]

Then it follows from (7) and (8) that

\[
|C(t; B, X, \Delta, \Psi)| = \left( \sum_{l=1}^{q} M_l \right)^{-1} \leq \sum_{l=1}^{q} \frac{n_l^2}{m^2} M_l^{-1}. \tag{10}
\]

Denote by \( \Gamma_{ij}^+ \) the family of curves from \( \Gamma(t; D_l, X_l, \Psi_l) \) lying in the dihedral angle \( \{(r, \theta, x') : \theta_j \leq \theta \leq \theta_{j+1} \} \) and by \( \Gamma_{ij} \) the family of curves from \( \Gamma(t; D_l, X_l, \Psi_l) \) lying in the dihedral angle \( \{(r, \theta, x') : \theta_j \leq \theta \leq \theta_{j+1} \} \).
the dihedral angle \((r, \theta, x') : \theta_{j-1} \leq \theta \leq \theta_j\), \(\theta_m = \theta_0 + 2\pi, \theta_{-1} = \theta_{m-1} - 2\pi, \ l = 1, \ldots, q, \ j = 0, \ldots, m - 1\). According to the property (3) we have

\[
M_l \geq \sum_{j=0}^{m-1} \left( M(\Gamma_{ij}^-) + M(\Gamma_{ij}^+) \right),
\]

where the prime at the summation sign means that the summation is taken over those indices \(j, j = 0, \ldots, m - 1\), for which the half-plane \(L_j\) contains at least one point from \(X_l\). Note that the total number of terms in this sum equals \(2n_l\). Inequalities (11) and (8) imply that

\[
\sum_{l=1}^{q} \frac{n_l^2}{m^2} M_l^{-1} \leq \sum_{l=1}^{q} \frac{4n_l^2}{4m^2} \left( \sum_{j=0}^{m-1} \left( M(\Gamma_{ij}) + M(\Gamma_{ij}^+)) \right) \right)^{-1} \leq \sum_{l=1}^{q} \frac{1}{4m^2} \sum_{j=0}^{m-1} \left( M(\Gamma_{ij})^{-1} + M(\Gamma_{ij}^+)^{-1} \right).
\]

Next, consider the symmetric configuration that is the condenser \(C(t; B, X^*, \Lambda, \Psi)\). Let \(X_0^*\) be the collection of points in \(X^*\), lying on the half-plane \(\{(r, \theta, x') : \theta = 0\}\). \(X_0^* = \{y_0^*\}_{s=1}^{K}\). If \(y_s^* = x_p^*, x_p^* \in X^*\), then we define \(\mu_s^*\) by the equality \(\mu_s^* = \mu_p\). In view of the symmetry of the condenser \(C(t; B, X^*, \Lambda, \Psi)\) we have

\[
cap C(t; B, X^*, \Lambda, \Psi) = m \cap C(t; B \cap P_0, X_0^*, \Lambda_0^*, \Psi_0^*),
\]

where \(P_0 = \{(r, \theta, x') : -\pi/m < \theta < \pi/m\}\), \(\Lambda_0^* = \{\sigma_s^*\}_{s=1}^{K}, \sigma_1^* = \ldots = \sigma_K^* = 1, \Psi_0^* = \{\mu_s^*\}_{s=1}^{K}\). Using the symmetry of the condenser \(C(t; B \cap P_0, X_0^*, \Lambda_0^*, \Psi_0^*)\) and Lemma 2 we conclude that

\[
cap C(t; B, X^*, \Lambda, \Psi) = 2mM(\Gamma_0^*),
\]

where \(\Gamma_0^*\) is the family of those curves from the family \(\Gamma(t; B \cap P_0, X_0^*, \Psi_0^*)\) that lie in the angle \(\{(r, \theta, x') : 0 \leq \theta \leq \pi/m\}\). Write \(\phi_k(x)\) for the reflection with respect to the hyper-plane \(\{(r, \theta, x') : \theta = \pi/m \text{ or } \theta = \pi + \pi/k/m\}\), \(k = 1, \ldots, 2m - 1\). For each curve \(\gamma_0^* \in \Gamma_0^*\) define \(\gamma_k^* = \phi_k(\gamma_{k-1}^*)\) and the curve \(\gamma^* = \cup_{k=0}^{2m-1} \gamma_k^*\). In other words, \(\gamma^*\) is the curve symmetric with respect to the group \(\Phi\) (see section 2) and comprising \(2m\) consecutive reflections \(\gamma_0^*\). Let \(\Gamma^*\) be the family of curves \(\gamma^*\). The composition principle and the symmetry of the family \(\Gamma^*\) [1], c.21, [13] c.178,179] imply that

\[
M(\Gamma_0^*) = 2mM(\Gamma^*).
\]

Hence,

\[
|C(t; B, X^*, \Lambda, \Psi)| = \frac{M(\Gamma^*)^{-1}}{4m^2}.
\]

We now apply dissymmetrization described in Lemma 3. As dissymmetrization induces a metric in each direction which preserves length and volume, we have

\[
M(\Gamma^*) = M(\text{Dis} \Gamma^*),
\]
where $\text{Dis} \Gamma^* = \{\text{Dis} \gamma^* : \gamma^* \in \Gamma^*\}$. It is easy to see from the construction of dissymmetrization (details can be found in [15, pp. 63-64]) that the family $\text{Dis} \Gamma^*$ is minorized by the separated $\Gamma_{ij}^-$ and $\Gamma_{ij}^+$. Due to the property (4) and Lemma 4, we then have

$$\sum_{l=1}^{q} \frac{1}{4m^2} \sum_{j=0}^{m-1} (M(\Gamma_{ij}^-)^{-1} + M(\Gamma_{ij}^+)^{-1}) \leq \frac{1}{4m^2} M(\text{Dis} \Gamma^*)^{-1} = \frac{1}{4m^2} M(\Gamma^*)^{-1}. $$

In view of (10), (12), (13) the above inequality leads to

$$|C(t; B, X, \Lambda, \Psi)| \leq |C(t; B, X^*, \Lambda, \Psi)|.$$

It remains to apply the asymptotic formula (2) for the condenser modulus following the same line of argument as in the proof of Theorem 1. This completes the proof of Theorem 2.

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