The generalized second law of thermodynamics for the interacting in $f(T)$ gravity

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Abstract We study the validity of the generalized second law (GSL) of gravitational thermodynamics in a non-flat FRW universe containing the interacting in $f(T)$ gravity. We consider that the boundary of the universe to be confined by the dynamical apparent horizon in FRW universe. In general, we discuss the effective equation of state, deceleration parameter and GLS in this framework. Also, we find that the interacting-term $Q$ modifies these quantities and in particular, the evolution of the total entropy, results in an increases on the GLS of thermodynamic, by a factor $4\pi R^3 A Q/3$. By using a viable $f(T)$ gravity with an exponential dependence on the torsion, we develop a model where the interaction term is related to the total energy density of matter. Here, we find that a crossing of phantom divide line is possible for the interacting- $f(T)$ model.

Keywords Universe · The generalized second law (GSL) of gravitational thermodynamics

1 Introduction

Observational data of the luminosity-redshift of type Ia supernovae (SNeIa), large scale structure (LSS) and the cosmic microwave background (CMB) anisotropy spectrum, have provided confirmation that our universe has recently entered a phase of accelerated expansion (Riess et al. 1998). A possible responsible of this acceleration of the universe is the dark energy (DE) and the nature of this energy is an important problem today in the modern physics. For a review of DE candidates and models, see Caldwell et al. (1998).

In the last years, a $f(T)$ theory was introduced to explain the current expansion of the universe without considering the DE (Ferraro and Fiorini 2007; Bengochea and Ferraro 2009; Myrzakulov 2011). The $f(T)$ theory is a generalization of the teleparallel gravity (TG) and becomes equivalent of General Relativity (Einstein 1928). The idea original of the $f(T)$ theory results from the generalization of the TG from the torsion scalar $T$ (Ferraro and Fiorini 2007; Linder 2010a), similarly to the motivation of the Ricci scalar $R$ in the Einstein-Hilbert action by replacing the function $f(R)$. However, in $f(T)$ theory the field equations are second order as opposed to the fourth order equations of $f(R)$ theory. In the formalism of the TG, the tetrad fields are the basic variable of TG together with the Weitzenbock connection, see Ferraro and Fiorini (2007), Einstein (1928), Clifton et al. (2012) for a review.

In the recent years, the thermodynamics aspects of the accelerating universe has considered much attention and different results has been obtained (Bekenstein 1972). In particular, the verifications of the first and second law of the thermodynamic, studying the dynamic together with the thermodynamics aspect of the accelerated expansion of the universe.

In the context of the validity the generalized second law (GSL) of thermodynamics, is necessary that the evolution with respect to the cosmic time of the total entropy $\dot{S}_{Total} = d(S_A + S_m)/dt \geq 0$. Here, $S_A$ is the Bekenstein-Hawking entropy on the apparent horizon and $S_m$ denotes the entropy of the universe filled matter (pressureless baryonic matter (BM) and dark matter (DM)) inside the dynamical apparent horizon (Cai and Kim 2005). Therefore, in conformity with the GSL of thermodynamic, the evolution of the total entropy, $S_{Total}$, cannot decrease in time (Bekenstein 1973; Pavón 1990).
On the other hand, considering the first law of thermodynamics, we can write \(-dE = T_A dS_A\) to the apparent horizon and obtain in this form the Einstein's field equation. This equivalence is satisfied, if we consider that the Hawking temperature \(T_A \propto R_A^{-1}\) and the entropy on the apparent horizon \(S_A \propto A\), where \(R_A\) and \(A\) are the radius and area associated to the horizon (Cai and Kim 2005), see also Akbar and Cai (2007). However, the entropy on the apparent horizon \(S_A\) is modified for other types of theories. In particular, in the case of \(f(T)\) gravity, Miao et al. (2011), calculated that when \(f''\) is small, the entropy of the apparent horizon resulted to be \(S_A = A f'/4G\). Also, for \(f(R)\) gravity the entropy is changed, see Akbar and Cai (2006). Here, the primes denote derivative with respect to the torsion scalar \(T\).

In relation to the GLS of thermodynamics in the framework of \(f(T)\) gravity, was developed in Karami and Abdolmaleki (2012). Here, the authors studied for a spatially flat universe, the validity of the GLS for two viable \(f(T)\) models; \(f(T) = T + \mu_1(-T)^n\) and \(f(T) = T - \mu_2 T(1 - e^{BT/T})\), originally proposed in Wu and Yu (2010), Linder (2010b). The GLS in \(f(T)\) gravity with entropy corrections was studied in Bamba et al. (2013), where two different cosmic horizon are analyzed. Also, the GLS in the emergent universe for some viable models of \(f(T)\) gravity was considered in Ghosh et al. (2013), where two different interacting dark energy in a flat FRW universe. Finally, in Sect. 5 we summarize our finding. We chose units so that \(c = \hbar = 1\).

2 Interacting \(f(T)\) gravity

The action \(I\) of modified TG in the framework of \(f(T)\) gravity, becomes (Ferraro and Fiorini 2007)

\[
I = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} \left[ f(T) + L_m \right].
\]

Here \(L_m\) is related to the Lagrangian density of the matter inside the universe.

In order to describe the \(f(T)\) theory we start with the following field equations in a FRW background filled with the pressureless matter (Bengochea and Ferraro 2009)

\[
H^2 + \frac{k}{a^2} = \frac{1}{3}(\rho_m + \rho_T),
\]

\[
\dot{H} - \frac{k}{a^2} = -\frac{1}{2}(\rho_m + \rho_T + p_T),
\]

where

\[
\rho_T = \frac{1}{2}(2Tf' - f - T),
\]

and

\[
p_T = -\frac{1}{2}\left[-8HTf'' + (2T - 4\dot{H})f' - f + 4\dot{H} - T \right].
\]

Here, \(H = \dot{a}/a\) is the Hubble factor, \(a\) is a scale factor, \(k\) is the curvature parameter, with values \(-1, 0, +1\), respectively. Also, \(\rho_m\) is the energy density of the matter, \(\rho_T\) and \(p_T\) are the torsion contributions to the energy density and pressure. Dots here mean derivatives with respect to the cosmological times denote the derivative times and the primes denotes derivative with respect to the torsion scalar \(T\). For convenience we will use units in which \(8\pi G = 1\).

Also, the torsion scalar for non-flat background is defined in Ferraro and Fiorini (2011), as

\[
T = -6 \left( H^2 + \frac{k}{a^2} \right).
\]
Considering Eqs. (2), (4) and (6), the energy density of the matter can be written as
\[ \rho_m = \frac{1}{2} [f - 2Tf']. \] (7)

On the other hand, following Jamil et al. (2012), Ghosh and Chattopadhyay (2012), we shall assume that both components, the torsion scalar and the cold dark matter do not conserve separately but that they interact through a term \( Q \) (to be specified later) according to
\[ \dot{\rho}_m + 3H\rho_m = Q, \] (8)

and
\[ \dot{\rho}_T + 3H(\rho_T + \rho_T) = -Q, \] (9)

such that the total energy \( \rho = \rho_T + \rho_m \) is conserved i.e.,
\[ \dot{\rho} + 3H(\rho + p) = 0. \]
In what follows we shall consider \( Q > 0 \). We also assume that the torsion contributions component obeys an equation of state (EoS) parameter \( w_T = \rho_T/\rho_T \) and then Eq. (9) can be written as
\[ \dot{\rho}_T + 3H\rho_T \left(1 + w_T + \frac{Q}{3H\rho_T}\right) = 0. \] (10)

Taking time derivative of Eq. (4), we get
\[ \dot{\rho}_T = \frac{T}{2} [f' + 2Tf'' - 1]. \] (11)

Following Karami et al. (2011), we combining Eqs. (10) and (11) and the EoS parameter yields
\[ w_T = -\left[1 + \frac{Q}{3H\rho_T} + \frac{\dot{\rho}}{3H(2Tf'' + f' - 1)}\right]. \] (12)

here, we note that Eq. (12) represents an effective EoS parameter.

On the other hand, from Eqs. (3) and (6), we get
\[ \dot{T} = \frac{12H}{f' + 2Tf''} \left[\frac{f' - 2Tf'}{4} + \frac{\dot{H}}{H^2} f' \right] \left(2Tf'' + f' - 1\right). \] (13)

In particular, for \( k = 0 \), \( \dot{T} = \frac{3H(f - 2Tf')}{(f' + 2Tf'')}, \) see Karami and Abdolmaleki (2012). Substituting Eq. (13) in Eq. (12) the effective EoS parameter results in
\[ w_T = -\left[1 + \frac{Q}{3H\rho_T} + \frac{4}{(f' + 2Tf'')} \left(2Tf'' + f' - 1\right) \right.\]
\[ \times \left. \left(\frac{f' - 2Tf'}{4} + \frac{k}{a^2} \left(f' + 2Tf'' - 1\right)\right)\right]. \] (14)

Note that in a non-interacting limit, i.e., \( Q = 0 \) and \( k = 0 \), the effective EoS parameter given by Eq. (14), reduces to the standard \( f(T) \) gravity, in which \( w_T = -1 + 4\dot{H}(2Tf'' + f' - 1)/(2Tf'' - f - T) \), see Karami and Abdolmaleki (2012).

On the other hand, the deceleration parameter \( q \) is given by \( q = -[1 + \frac{\ddot{H}}{H^2}] \), and using Eqs. (2) and (3) one can obtain
\[ q = \frac{1}{2} \left[\frac{1}{2} + \frac{k}{a^2} \frac{H^2}{H^2} + \frac{\dot{\rho}_T w_T}{H^2}\right]. \] (15)

Combining Eqs. (14) and (15) the deceleration parameter \( q \) can be written as
\[ q = \frac{1}{2} - \frac{k}{2a^2} \frac{T}{6} + \frac{k}{a^2}\left[H + \frac{Q}{6} \left(\frac{2Tf'' + f' - 1}{f' + 2Tf''}\right)\right. \]
\[ \times \left. \left(\frac{f' - 2Tf'}{4} + \frac{k}{a^2} \left(f' + 2Tf'' - 1\right)\right)\right]. \] (16)

We noted that for the particular case in which non-interacting limit \( Q = 0 \), in the Einstein TG in which \( f(T) = T \), and \( k = 0 \), the above equation gives \( q = 1/2 \), corresponding to the matter dominated epoch.

3 GSL interacting—\( f(T) \)

Having exhibited the cosmological scenario of a universe controlled by \( f(T) \) gravity, we proceed to an investigation of its thermodynamic properties, and in particular the validity of the GSL in a non-flat FRW model occupied with pressureless Dark Matter (DM), i.e., \( p_m = 0 \). Following Bekenstein (1973), Pavón (1990) the GSL the entropy of the horizon plus the entropy of the matter within the horizon cannot decrease in time. We suppose that the boundary of the universe to be enclosed by the dynamical apparent horizon in FRW universe; accordingly, its radius \( R_A \) is given by Sheykhi (2010), as
\[ R_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}. \] (17)

In particular, for the case \( k = 0 \) the radius of the apparent horizon coincide with the Hubble horizon, in which \( R_A = 1/H \), see Poisson and Israel (1990).

Following Cai and Kim (2005), the Hawking temperature on the apparent horizon the radius \( R_A \) is given by
\[ T_A = \frac{1}{2\pi R_A} \left(1 - \frac{R_A}{2HR_A}\right). \] (18)

Note that the ratio \( R_A/2HR_A < 1 \), ensures that the Hawking temperature \( T_A > 0 \).
Considering the Gibb’s equation, the entropy of the universe considering DM inside the apparent horizon $S_m$, is given by Izquierdo and Pavón (2006)

$$T_A dS_m = dE_m + p_m dV = dE_m, \quad (19)$$

where the volume of the pressureless matter is $V = 4\pi R_A^3 / 3$ and

$$E_m = V\rho_m = \frac{4\pi R_A^3}{3} \rho_m. \quad (20)$$

From Eqs. (8), (19) and (20), the Gibb’s equation due to the matter, can be written as

$$T_A \dot{S}_m = 4\pi R_A^2 \rho_m \left( \dot{R}_A + H R_A \left[ \frac{Q}{3H\rho_m} - 1 \right] \right), \quad (21)$$

where $\dot{S}_m$ represent the time derivative of the entropy due to the matter source inside the horizon. Also, we noted that in the non-interacting limit the above equation reduces to the Gibb’s equation $T_A \dot{S}_m = 4\pi R_A^2 \rho_m (\dot{R}_A - H R_A)$.

Combining Eqs. (7) and (21) we get

$$T_A \dot{S}_m = 2\pi R_A^2 \left[ f - 2T f' \right] \times \left( \dot{R}_A + H R_A \left[ \frac{2Q}{3H [f - 2T f']} - 1 \right] \right). \quad (22)$$

Here, we noted that in the limit $Q = 0$, Eq. (22) reduces to expression obtained in Karami and Abdolmaleki (2012), in which $T_A \dot{S}_m = 2\pi R_A^2 (f - 2T f')(\dot{R}_A - H R_A)$. Also, we observed that the evolution of the matter entropy $T_A \dot{S}_m$, increase with the introduction of the interaction term $Q$.

On the other hand, the contribution of the apparent horizon entropy $S_A$, in the framework of $f(T)$ gravity when $f''$ is small, according to Miao et al. (2011), is obtained as

$$S_A = \frac{A f'}{4G}, \quad (23)$$

where $A = 4\pi R_A^2$ is the area of the horizon.

In this way, the evolution of horizon entropy considering Eqs. (18) and (23) one can get

$$T_A \dot{S}_A = 4\pi \left( 1 - \frac{\dot{R}_A}{2H R_A} \right) (2\dot{R}_A f' + R_A f'' \dot{T}). \quad (24)$$

and now combining Eqs. (13) and (24) we get

$$T_A \dot{S}_A = 4\pi \left( 1 - \frac{\dot{R}_A}{2H R_A} \right) (2\dot{R}_A f' + 6H R_A f'') \times \left[ \frac{(f - 2T f')/2 - 2k/a^2}{(f' + 2T f'')} + \frac{2k}{a^2} \right]. \quad (25)$$

Note that Eq. (25) coincides with the evolution of the horizon entropy calculated in Ghosh et al. (2013) and for a flat FRW agrees with Karami and Abdolmaleki (2012). Also we noted that the evolution of the horizon entropy becomes independent of the interacting term $Q$.

Taking time derivative of $\dot{R}_A$ given by Eq. (17) and considering Eqs. (3) and (6), we get

$$\dot{R}_A = -\frac{1}{2} \left( \frac{6}{T} \right)^{3/2} \left[ \frac{T}{6} + \frac{k}{a^2} \right]^{1/2} \times \left[ (f - 2T f')/2 - 2k/a^2 \right]. \quad (26)$$

In particular, for the flat FRW metric in which $k = 0$, the value of $\dot{R}_A$ reduces to $\dot{R}_A = -(3/2T)(f - 2T f')/(f' + 2T f'')$, see Karami and Abdolmaleki (2012).

Finally, we need to calculate the total entropy $S_{total}$ due to different contributions of the apparent horizon entropy and the matter entropy, i.e., $S_{total} = S_A + S_m$. In this form, the evolution of the total entropy or rather the GSL, adding Eqs. (22) and (25), can be written as

$$T_A \dot{S}_{total} = 2\pi R_A^2 \left[ \left( \frac{2}{R_A^2} - \frac{\dot{R}_A}{H R_A^3} \right) (2R_A f') + 6H R_A f'' \left[ \frac{(f - 2T f')/2 - 2k/a^2}{(f' + 2T f'')} + \frac{2k}{a^2} \right] \right] + \left[ f - 2T f' \right] \left( \dot{R}_A + H R_A \left[ \frac{2Q}{3H [f - 2T f']} - 1 \right] \right). \quad (27)$$

where $R_A$ and $\dot{R}_A$ are given by Eqs. (17) and (26) respectively. Here, we noted that the interacting-term $Q$ modified the evolution of the total entropy, results in an increases on the GSL of thermodynamic, by a factor $4\pi R_A^2 Q/3 > 0$. Also, we noted that in the Einstein TG, where $f(T) = T$, considering a flat universe and in the limit $Q = 0$, the GSL from Eq. (27) results in $T_A \dot{S}_{total} = 9\pi > 0$, which always prevails (recalled, that $8\pi G = 1$).

In the following, we will analyze the GSL of thermodynamic for a flat FRW universe, i.e., $k = 0$, one specific interaction term $Q$ and one viable model for $f(T)$ with an exponential dependence on the torsion.

### 4 An example for $Q$ and $f(T)$

Let us consider that the interaction term $Q$ is related to the total energy density of matter and takes the form (Herrera et al. 2004; del Campo et al. 2006; Guo et al. 2007)

$$Q = 3c^2 H \rho_m. \quad (28)$$

where $c^2$ is a small positive definite constant and the factor 3 was considered for mathematical convenience.
Following Wu and Yu (2010), we consider one viable model for $f(T)$ with an exponential dependence on the torsion scalar $T$, given by

$$f(T) = T - \mu_2 T (1 - e^{\frac{T_0}{T}}),$$  \hspace{1cm} (29)$$

where $\mu_2$ and $\beta$ are constants and $T_0 = -6H_0$. Note that the last term of $f(T)$ is analogous to a $f(R)$ model in which an exponential dependence on the curvature scalar $R$ is considered (Linder 2009). The astronomical data from SNeIa+BAO+CMB gives the following fit values $\beta = -0.02^{+0.31}_{-0.20}$ and $\Omega_{m0} = 0.272^{+0.036}_{-0.034}$ at the 95% confidence level (Wu and Yu 2010). Here, the dimensionless matter energy $\Omega_{m0} = 8\pi G\rho_{m0}/3H_0^2$ and the parameter $\mu_2 = \frac{1-\Omega_{m0}}{1-(1-2\beta)e^\beta}$. In particular, in the non interacting limit, the values $\beta = 0$ reduces to the $\Lambda$CDM. Also, a cosmographic analysis to check the viability of model given by Eq. (29), was developed in Karami and Abdolmaleki (2012) (see also Capozziello et al. 2011).

Considering Eqs. (14) and (28) the effective EoS parameter, $w_T$, for a flat FRW universe, yields

$$w_T = \left[ 1 + \frac{c^2(f - 2Tf')}{(2Tf' - f - T)} + \frac{(f - 2Tf')}{(f' + 2Tf'')} \right] \times \frac{(2Tf'' + f' - 1)}{(2Tf' - f - T)}.$$  \hspace{1cm} (30)$$

where $f' = 1 + \mu_2 e^{\frac{T_0}{T}} (\frac{T_0}{T} - 1)$ and $f'' = \mu_2^2 T_0^2 \times e^{\frac{T_0}{T}} / T^3$, respectively. Here, we noted that in the limit $c^2 \to 0$, Eq. (30) reduces to expression calculated in Karami and Abdolmaleki (2012).

The acceleration parameter $q$, from Eq. (16) is given by

$$q = \frac{1}{2} + \frac{3T}{2} \left[ (2Tf' - f - T) + (f - 2Tf') \right] \times \left( c^2 + \frac{(2Tf'' + f' - 1)}{(f' + 2Tf'')} \right).$$  \hspace{1cm} (31)$$

As before, we noted that in the non interacting limit, i.e., $c^2 \to 0$, the parameter $q$ given by Eq. (31) reduces to $q = 2[f' - Tf' - (3f/4T)]/(f' - 2Tf'')$ and coincide with the obtained in Karami and Abdolmaleki (2012).

The total entropy due to different contributions of the apparent horizon entropy and the matter entropy from Eq. (27), can be written as

$$T_A \dot{S}_{total} = 2\pi R_A^2 \left[ \frac{2 - \frac{\dot{R}_A}{R_A^2}}{\frac{R_A^2}{\dot{R}_A}} \right] \times \left[ 2\dot{R}_Af' + 3f'' \left( \frac{f - 2Tf'}{f' + 2Tf''} \right) \right]$$

$$+ [f - 2Tf'] \left( \dot{R}_A + [c^2 - 1] \right).$$  \hspace{1cm} (32)$$

where $\dot{R}_A$, from Eq. (26) is given by $\dot{R}_A = -(3/2T)[(f - 2Tf')/(f' + 2Tf'')]$ and $R_A^2 = -6/T$. We noted that in the Einstein TG, in which, $f(T) = T$, the GSL from Eq. (32) results in $T_A \dot{S}_{total} = \pi (9 + 12c^2) > 0$, which always prevails and also we noted that the GLS increases with the interaction-term (recalled, that $8\pi G = 1$).

In Fig. 1 we show the evolution from the early times $(T/T_0 \to +\infty)$ to the current epoch $(T/T_0 = 1)$ for the effective EoS parameter $w_T$ versus the dimensionless torsion $T/T_0$, for different values of the interaction parameter $c^2$. Dot-dashed, dashed, dotted and solid lines are for the interaction parameter $c^2 = 10^{-2}$, $c^2 = 10^{-3}$, $c^2 = 10^{-4}$ and the non-interacting limit $c^2 = 0$, respectively. In order to write down values that relate the effective EoS versus $T/T_0$, we considered Eq. (30). Also, we have used the best fit values $\beta = -0.02$ and $\Omega_{m0} = 0.272$ (Wu and Yu 2010), in which $\mu_2 = \frac{1-\Omega_{m0}}{1-(1-2\beta)e^\beta} = -37.5$ and $H_0 = 74.2$ km s$^{-1}$ (Riess et al. 2009). An attractive result which is not present in the non interacting limit model ($c^2 = 0$) in the past (Karami and Abdolmaleki 2012; Wu and Yu 2010), is that for the different values of the interacting parameter $c^2$, we have a transition from $w_T > -1$ (quintessence) to $w_T < -1$ (phantom). In this form, the interacting $f(T)$ gravity can cross the phantom divide line, as could be seen from Fig. 1. In particular, this transition occurs for the parameter $c^2 = 10^{-2}$ at $T/T_0 = 1.5$ and for $c^2 = 10^{-3}$ at $T/T_0 = 4.74$. We have found that the value $c^2 = 10^{-4}$ (dotted line) presents a small displacement in relation to the non interacting limit that corresponds to $c^2 = 0$. At the present epoch, in particular for the values $c^2 = 10^{-2}$ and $c^2 = 10^{-3}$, we find that $w_{T0} = -0.995$ and $w_{T0} = -0.992$, respectively. In the future in which $1 < T/T_0 < 0$, again we have a transition from
$w_T > -1$ to $w_T < -1$, for the different values of $c^2$. This result has been previously noted in Karami and Abdolmaleki (2012), for the case $c^2 = 0$ in which $T / T_0 = 0.719$.

In Fig. 2 we show the evolution of the deceleration parameter $q$ versus the dimensionless torsion $T / T_0$, for three different values of the interaction parameter $c^2$. Dashed, dotted and solid lines are for the interaction parameter $c^2 = 10^{-1}$, $c^2 = 10^{-2}$ and the non-interacting limit $c^2 = 0$, respectively. In order to write down values that relate $q$ versus $T / T_0$, we used Eq. (31). As before, we have used the values $β = -0.02$ and $Ω_{m0} = 0.272$, then $μ_2 = \frac{1 - Ω_{m0}}{1 - (1 - 2β)e^β} = -37.5$ and $H_0 = 74.2$ km s$^{-1}$. From Fig. 2 we have a cosmic transition from $q > 0$ (deceleration) to $q < 0$ (acceleration) with is consistent with the observations (Daly et al. 2008). In particular, this transition occurs for the parameter $c^2 = 10^{-1}$ at $T / T_0 = 2.84$ and for $c^2 = 10^{-2}$ at $T / T_0 = 2.2$, respectively. At the early times, i.e., $T / T_0 \to \infty$, for the value $c^2 = 10^{-1}$ we find that the deceleration parameter $q_∞ = 0.35$ and for $c^2 = 10^{-2}$ corresponds to $q_∞ = 0.5$ and this value of the deceleration parameter at the early times, coincides with the non-interacting limit $c^2 = 0$. At the current epoch i.e., $T / T_0 = 1$, we obtain that for the interacting parameter $c^2 = 10^{-1}$ corresponds to $q_0 = -0.62$ and for $c^2 = 10^{-2}$ corresponds to $q_0 = -0.58$ and these values are in accord with the observational result (Daly et al. 2008).

In Fig. 3 we show the evolution of the GLS versus the dimensionless torsion $T / T_0$, for three different values of the interaction parameter $c^2$. Dashed, dotted and solid lines are for the interaction parameter $c^2 = 10^{-1}$, $c^2 = 10^{-2}$ and $c^2 = 0$, respectively. In order to write down values that relate $T_A \dot{S}_{Total}$ versus $T / T_0$, we considered Eq. (32). As before, we have used $β = -0.02$ and $Ω_{m0} = 0.272$, then $μ_2 = \frac{1 - Ω_{m0}}{1 - (1 - 2β)e^β} = -37.5$ and $H_0 = 74.2$ km s$^{-1}$. From Fig. 3 we observed that the GLS is satisfied from the early times i.e., $T / T_0 \to \infty$ to the current epoch in which $T / T_0 = 1$. Also, we noted that the GLS graphs for the value $c^2 = 10^{-2}$ present a small displacement with respect to the dimensionless torsion $T / T_0$, when compared to the results obtained in the non-interacting limit model, in which $c^2 = 0$. In particular, at early times, i.e., $T / T_0 \to \infty$ we obtain that $T_A \dot{S}_{Total} \simeq 25.01$ for the value $c^2 = 10^{-1}$, $T_A \dot{S}_{Total} \simeq 22.04$ that corresponds to $c^2 = 10^{-2}$ and $T_A \dot{S}_{Total} \simeq 21.71$ that corresponds to $c^2 = 0$ i.e., the non-interacting limit. At the present time i.e., $T / T_0 = 1$ we get that $T_A \dot{S}_{Total} \simeq 2.61$ for the value $c^2 = 10^{-1}$, $T_A \dot{S}_{Total} \simeq 2.61$ that corresponds to $c^2 = 10^{-2}$ and $T_A \dot{S}_{Total} \simeq 2.51$ that corresponds to $c^2 = 0$. Analogously, as occurs in the non-interacting limit, the GLS is infringed in the future i.e., $1 < T / T_0 < 0$, in which GLS is negative, $T_A \dot{S}_{Total} < 0$. In particular, for the value $c^2 = 10^{-2}$ the GLS is violated for the dimensionless torsion $T / T_0$, in the special intervals $(0.005, 0.017)$, $(0.038, 0.356)$ and $(0.480, 0.719)$.

5 Conclusions

In this paper we have investigated the GLS in the context of $f (T)$ gravity. We studied the GLS from the boundary of the universe to be enclosed by the dynamical apparent horizon in non-flat FRW universe occupied with pressureless DM, together with the Hawking temperature on the apparent horizon. We found that the interacting term $Q$ modified; the torsion contributions component given by an effective EoS parameter, the deceleration parameter and the evolution of the total entropy or rather the GSL. In particular, we noted...
that the modified in the evolution of the total entropy, results in an increases on the GLS of thermodynamic by a factor $4\pi R_A^2 Q / 3 > 0$.

Our specific model is described by a viable model $f(T)$ with an exponential dependence on the torsion $T$ in a flat FRW universe and have considered for simplicity the case in which the interaction term $Q$ is related to the total energy density of matter. For this model, we have also obtained explicit expressions for the effective EoS parameter, the deceleration parameter and the evolution of the total entropy. For these particular choices of the $f(T)$ and $Q$, it is possible to obtain an accelerating expansion of the universe (here, the effective torsion fluid representing the role of dark energy).

We also found an attractive result that which is not present in the non interacting limit model $(c^2 = 0)$ in the past, that is for values different of the interacting parameter $c^2 \neq 0$, we obtained a transition from the $w_T > 1$ (quintessence) to $w_T < 1$ (phantom). In this form, a crossing of phantom divide line is possible for the interacting- $f(T)$ model in the past. Also, we found a cosmic transition in the past from $q > 0$ (deceleration) to $q < 0$ (acceleration) with is consistent with the observations. Here, we noted that for values of $c^2 < 10^{-2}$ the value of the deceleration parameter at the early times, coincides with the non-interacting limit $c^2 = 0$ and becomes $q_{\infty} = 0.5$. For our specific model, we analyzed the validity of the GLS and we found an increases on the total entropy by a factor $2\pi R_A^2 (f - 2T f')c^2$ product of the interaction-term. Also, we obtained that $T_\Lambda \dot{S}_{total} > 0$ from the early times to the current epoch. However, in the future $(1 < T / T_0 < 0)$ the total entropy becomes $T_\Lambda \dot{S}_{total} < 0$ and then the GLS is violated, for three special intervals of the dimensionless torsion $T / T_0$, independently of the interaction-term.

Finally, we have shown that the GLS of thermodynamics for the interacting in $f(T)$ gravity is less restricted than analogous $Q = 0$ due to the introduction of a new parameter, proportionates for the interaction term $Q$. In our specific model the incorporation of this parameter gives us a freedom that allows us to modify the standard $f(T)$ gravity by simply modifying the corresponding value of the parameter $c^2$ associated to the interaction term $Q$. Also, we have not addressed other interacting- $f(T)$ models. We hope to return to this point in near future.

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