Existence, Uniqueness, Regularity and Long-term Behavior for Dissipative Systems Modeling Electrohydrodynamics

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Abstract

We study a dissipative system of nonlinear and nonlocal equations modeling the flow of electrohydrodynamics. The existence, uniqueness and regularity of solutions is proven for general $L^2$ initial data in two space dimensions and for small data in data in three space dimensions. The existence in three dimensions is established by studying a linearization of a relative entropy functional. We also establish the convergence to the stationary solution with a rate.
1 Introduction

In this paper, we study the following nonlinear system of equations;

\[ u_t + u \cdot \nabla u + \nabla p = \Delta u + \Delta \phi \nabla \phi, \quad (1) \]
\[ \nabla \cdot u = 0, \quad (2) \]
\[ v_t + u \cdot \nabla v = \nabla \cdot (\nabla v - v \nabla \phi), \quad (3) \]
\[ w_t + u \cdot \nabla w = \nabla \cdot (\nabla w + w \nabla \phi), \quad (4) \]
\[ \Delta \phi = v - w \quad (5) \]

in \( \Omega \times (0, \infty) \) for a connected, bounded, open subset \( \Omega \) of \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). Here \( u(x, t) \) is a vector in \( \mathbb{R}^n \) and \( p(x, t), v(x, t), w(x, t) \) and \( \phi(x, t) \) are scalars. Equation (1) is the force balance equation of a viscous, incompressible fluid with velocity \( u \) and incompressibility condition (2). These are coupled with conservation equations (3)-(4) of a binary system of charges with densities \( v, w \) and the electric potential \( \phi \) determined by the Poisson equation (5). The force exerted by the charged particles on the fluid is \( \Delta \phi \nabla \phi = \nabla \cdot \sigma(= \sum_{i=1}^{n} (\sigma_{ij})x_{i}) \) where the electric stress \( \sigma \) is a rank one tensor plus a pressure; for \( i, j = 1, \ldots, n, \)

\[ [\sigma]_{ij} = \left( \nabla \phi \otimes \nabla \phi - \frac{1}{2} |\nabla \phi|^2 I \right)_{ij} = \phi_{x_i} \phi_{x_j} - \frac{1}{2} |\nabla \phi|^2 \delta_{ij}. \]

The electric stress \( \sigma \) stems from the balance of kinetic energy with electrostatic energy via the least action principle, [RLZ07]. For simplicity, we have assumed that the fluid density, viscosity, charge mobility and dielectric constant are unity.

Solutions for the velocity field equation are determined by the
Dirichlet condition

\[ u(x,t) = 0 \quad \text{for } (x,t) \in \partial \Omega \times (0, \infty). \quad (6) \]

Solutions of the equations for the charges are determined by the natural (no flux) boundary conditions

\[ \frac{\partial v}{\partial \nu} - v \frac{\partial \phi}{\partial \nu} = 0, \quad \frac{\partial w}{\partial \nu} + w \frac{\partial \phi}{\partial \nu} = 0, \quad \text{on } \partial \Omega \times (0, \infty), \quad (7) \]

where \( \nu \) is the outward pointing normal to \( \partial \Omega \). Along with (6), equations (7) are the natural boundary condition of (3) and (4); there is no flux of the charges through the boundary. The integral of \( v \) and \( w \) are conserved quantities;

\[ \frac{d}{dt} \int_{\Omega} v \, dx = \frac{d}{dt} \int_{\Omega} w \, dx = 0. \quad (8) \]

We assume that

\[ 0 < \int_{\Omega} v_0 \, dx, \int_{\Omega} w_0 \, dx < \rho_0 < \infty. \quad (9) \]

The constant \( \rho_0 \) is the characteristic charge. The small data results in section 6 will in part be formulated in terms of the size of \( \rho_0 \).

Solutions of (5) are determined by

\[ \phi(x,t) = 0 \quad \text{for } (x,t) \in \partial \Omega \times (0, \infty). \quad (10) \]

Condition (10) states that the boundary of the domain is held at a fixed potential. The evolution is determined by initial conditions

\[ u(x,0) = u_0(x), \quad \nabla \cdot u_0(x) = 0, \]

\[ v(x,0) = v_0(x) \geq 0, \quad w(x,0) = w_0(x) \geq 0 \quad \text{for } x \in \Omega. \quad (11) \]

Since the Navier-Stokes equation is a subsystem of (1)-(5), one cannot expect better results than for the Navier-Stokes equations. In
the absence of a fluid, the hydrodynamic system (1)-(5) reduces to the subsystem

\[
\begin{cases}
v_t = \nabla \cdot (\nabla v - v \nabla \phi), \\
w_t = \nabla \cdot (\nabla w + w \nabla \phi), \\
\Delta \phi = v - w.
\end{cases}
\] (12)

The equations (12) are the Debye-Hückel system, a basic model for the diffusion of ions in an electrolyte filling all of \( \mathbb{R}^3 \) first studied by W. Nernst and M. Plank at the end of the nineteenth century, [DH23]. The results for the Debye-Hückel system are complementary to those of Navier-Stokes. In [BHN94], Biler et al. proved the existence of global weak solutions in dimensions two and proved the uniqueness and regularity of local solutions in all dimensions under appropriate assumptions. It is not known whether (12) in general possesses global weak solutions in dimensions greater than two. In view of this challenge, one cannot expect solutions of (1)-(5) to exist in dimensions greater than 2 for general data, c.f. theorems 2 and 3.

There are several theoretical difficulties associated with the system (1)-(5). The coefficient \( \nabla \phi \) in the boundary value problem (3)-(4) and (7) is determined by nonlocal information. The right-hand side of (3) and (4) cannot be formulated in terms of a Frechet derivative of a functional (c.f. the Erikson-Leslie theory for liquid crystals and the Allen-Cahn and Cahn-Hilliard theory for fluid/interface motion). Furthermore, it is not clear how the basic energy law for (1)-(5) (c.f. inequality (18)) implies the extension property for local solutions in dimensions greater than two. Many of the standard techniques for parabolic PDE, e.g. the maximum principle and apriori estimates, are difficult to apply.
Biler et al. presented the first mathematical existence, uniqueness and regularity results in [BHN94]. They established the $L^2$ convergence of solutions to the stationary solution without a rate. In [BD00], this result was improved by establishing an exponential $L^1$ convergence with a rate depending only on $\Omega$. Their work relies heavily on the tools developed in [AMTU01] and [UAMT00] for the Fokker-Plank equation. In [BAMV04], an exponential $L^2$ convergence result by means of a linearization of the an appropriate energy functional was proved.

The hydrodynamic setting presented here has been studied in [FG09], [Li09] and [RLW06]. The work of [FG09] establishes several important estimates for the hydrodynamical system when the boundary is assumed to be electrically insulated. The work of [Li09] studies the interesting zero-dielectric limit of the system on the flat $n$-dimensional torus.

### 1.1 Basic Energy Law

We develop the basic energy law for electrohydrodynamics. Let us consider a classical solution $u, v, w$ of (1)-(11) on $\Omega \times (0, T)$. Assume that $v$ and $w$ are positive on $\Omega$. Throughout the paper, define

$$
\psi_r(s) = s \log(s/r) - s + r, \quad r \in (0, \infty), s \in [0, \infty)
$$

and define

$$
\psi(\cdot) = \psi_1(\cdot)
$$

The following energy functional will play an important role;

$$
W \equiv \int_{\Omega} \psi(v) + \psi(w) \, dx + \frac{1}{2} \|\nabla \phi\|^2_2 + \frac{1}{2} \|u\|^2_{L^2}. \quad (13)
$$
The first two terms in this definition are the entropy of the charges
$v$ and $w$ respectively, while the last two are the electric energy of the
charges and the kinetic energy of the fluid respectively.

Differenting $W$ with respect to $t$ gives

$$
\frac{dW}{dt} = (v_t, \psi'(v)) + (w_t, \psi'(w)) + (\nabla \phi_t, \nabla \phi) + (u_t, u)
$$

Here $(\cdot, \cdot)$ denotes the usual $L^2$ inner product on $\Omega$. Integrating by
parts and using (10), we see that $(\nabla \phi_t, \nabla \phi) = - (\Delta \phi_t, \phi) = (w_t - v_t, \phi)$ yielding

$$
\frac{dW}{dt} = (v_t, \psi'(v) - \phi) + (w_t, \psi'(w) + \phi) + (u_t, u)
$$

(14)

The quantities $\psi'(v) - \phi$ and $\psi'(w) + \phi$ are called the electro-chemical
potential of $v$ and $w$ respectively. Note that $v_t = \nabla \cdot (v \nabla \log (ve^-\phi)) = \nabla \cdot (v \nabla (\psi'(v) - \phi))$ Integrating by parts and using (7)

$$
(v_t, \psi'(v) - \phi) = - (v, |\nabla \log (ve^-\phi)|^2) - (u \cdot \nabla v, \psi'(v) - \phi).
$$

(15)

Similarly, by (4) and (7),

$$
(w_t, \psi'(w) + \phi) = - (w, |\nabla \log (we^\phi)|^2) - (u \cdot \nabla w, \psi'(w) + \phi).
$$

(16)

Since $u$ is divergence free vector field which vanishes on the boundary
of $\Omega$, integration by parts gives

$$
0 = (u \cdot \nabla u, u) = (\nabla p, u) = (u \cdot \nabla v, \psi'(v)) = (u \cdot \nabla w, \psi'(w)).
$$

Similarly, since $\phi$ is a solution of the Poisson equation (5) $(u \cdot \nabla (v - w), \phi) = - (u \cdot \nabla \phi, \Delta \phi)$. From (1), we have then

$$
(u_t, u) = (\Delta u, u) - (\nabla p, u) - (u \cdot \nabla u, u) + (u \cdot \nabla \phi, \Delta \phi)
$$

$$
= - |\nabla u|^2 + (u \cdot \nabla \phi, \Delta \phi).
$$

(17)
Adding (15), (16) and (17) together the following terms, which are interpreted as entropy production due to transport and the kinetic energy production due to forcing,

\[(u \cdot \nabla v, \phi), \quad -(u \cdot \nabla w, \phi), \quad (u \cdot \nabla \phi, \Delta \phi)\]

cancel. We find

\[\frac{dW}{dt} = -\int_{\Omega} u \cdot \nabla \log(v e^{-\phi})^2 + w \cdot \nabla \log(w e^\phi)^2 + |\nabla u|^2 \, dx \leq 0 \quad \text{(18)}\]

for all \(0 \leq t \leq T\).

**Remark 1.** The identity (18) is the basic energy law for the hydrodynamic Debye-Hückel model. It, along with (8), implies that

\[\|v(t), w(t)\|_{L^2} + \|\nabla \phi(t)\|_{L^2} + \|u(t)\|_{L^2} + \int_0^t \|\nabla u\|_{L^2}^2 \, ds \leq \|v_0, w_0\|_{L^2} + \|\nabla \phi_0\|_{L^2}^2 + \|u_0\|_{L^2}^2, \quad t > 0.\]

Here \(\phi_0\) is the solution of the Poisson equation with right hand side \(v_0 - w_0\). It will be crucial in establishing a uniform \(L^2\) estimate when \(\dim \Omega = 2\), c.f. lemmas 5 and 8.

**Remark 2.** If we assume that \(u, v, w\) and \(\phi\) are a classical solution of (1) - (11) and \(\phi\) satisfies the boundary condition

\[\frac{\partial \phi}{\partial \nu} = 0, \quad (x, t) \in \Omega \times (0, \infty) \quad \text{(19)}\]

in place of (10) or one replaces (6), (7), and (10) with the assumption

\[\Omega = \mathbb{T}^n \quad \text{\((n\text{-dimensional flat torus})\),} \quad \text{(20)}\]

then an additional energy law holds;

\[\frac{d}{dt} \|v, w\|_{L^p}^p + \frac{4(p - 1)}{p} \|\nabla v, \nabla w\|_{L^2}^2 \leq 0, \quad 1 < p < \infty.\]
One has equality in the above relation if and only if $v(x, t)$ and $w(x, t)$ are equal for a.e. $x \in \Omega$. Assuming either (19) or (20), a necessary condition for a static solution (one where $v_t = w_t = 0, u_t = 0$) to exist is

$$\int_\Omega v_0 \, dx = \int_\Omega w_0 \, dx.$$ 

In this case, the static solution will be

$$\phi = \text{const}, \quad v = w = \text{const}.$$ 

The appeal of electrolyte fluids in application is the presence of sharp boundary layers in the charges and potential when a static equilibrium is reached. From the point of view of the analysis and physicality of the model, the assumptions (19) or (20) lead to an over simplified model. The boundary conditions used in the model in this paper are the physical ones but lead to significant difficulties in the analysis.

### 1.2 Definitions

The following function spaces will be used throughout this paper.

$$H^k(\Omega) = W^{k,2}(\Omega)$$ is the Sobolev space with norm $\| \cdot \|_{H^k}$,

$$H^{-1} = \text{dual of } H^1(\Omega),$$

$$\mathcal{Y}(\Omega) = C^\infty_0(\Omega; \mathbb{R}^n) \cap \{ \mathbf{v} : \nabla \cdot \mathbf{v} = 0 \},$$

$$H(\Omega) = \text{closure of } \mathcal{Y} \text{ in } L^2(\Omega),$$

$$V(\Omega) = \text{closure of } \mathcal{Y} \text{ in } H^1(\Omega),$$

$$V^* = \text{dual of } V(\Omega), \quad H^* = \text{dual of } H(\Omega).$$

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1In fact, the static solution is the stationary solution
The dependence on $\Omega$ will be omitted when the context is clear. For $0 \leq t \leq T \leq \infty$, define
\[ Q_T = \Omega \times (0, T), \quad Q_{(t,T)} = \Omega \times (t, T). \]

For $k$ a positive integer and $0 < \alpha < 1$, $\Omega \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^n \times (0, \infty)$ open, the Hölder spaces
\[ C^{k+\alpha}(\Omega), \quad C^{k+\alpha}(U) \]
are defined in chapter 3 of [Lie96]. Note that $C^{k+\alpha}(\Omega)$ is defined with respect to the Euclidean distance while the space $C^{k+\alpha}(U)$ is defined with respect to the parabolic distance. Recall that if $k$ is an integer and $v \in C^{2k+\alpha}(U), U \subset \mathbb{R}^n \times (0, \infty)$, then $v$ has $2k$ uniformly Hölder continuous derivatives in $x$ and $k$ uniformly Hölder continuous derivatives in $t$ both with exponent $\alpha$.

1.3 Weak Solutions

Throughout, $p^*$ will denote the critical Sobolev exponent $\frac{2n}{n-2}$. Let $T > 0$, $u \in L^2((0, T); L^{q^*} \cap V)$ and $\nabla \phi \in L^2((0, T); L^{q^*})$ where $\frac{1}{p} + \frac{1}{p^*} = \frac{1}{2}$.

We say $v \in L^2((0, T); H^1)$ is a weak solution (c.f. [Lie96], Chapter 10, Section 6) of the equations
\[
\begin{aligned}
&v_t + u \cdot \nabla v = \nabla \cdot (\nabla v - v \nabla \phi), \quad x \in \Omega, t > 0, \\
&\frac{\partial v}{\partial \nu} - v \frac{\partial \phi}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
&v(x, 0) = v_0(x), \quad x \in \Omega
\end{aligned}
\]

on $Q_T$ provided
\[
(v(t) - v_0, \omega) = -\int_0^t (\nabla v - v \nabla \phi - vu, \nabla \omega), \, ds,
\]
\[
\forall \omega \in H^1, \text{ a.e. } 0 < t < T.
\]
Setting $\omega$ to be the constant function 1, we see that a weak solution satisfies the conserved mass equation, (8). Note that if $v \in L^{p^*}$ and $\nabla \phi, u \in L^{q^*}$ for a.e. $t \in (0, T)$, then $v\nabla \phi$ and $vu$ lie in $L^2$ for a.e. $t$ and hence the trilinear terms are well defined. By virtue of being a weak solution, $v$ is absolutely continuous from $[0, T]$ into $L^2$. The Sobolev space $H^1$ is separable and by the Lebesgue differentiation theorem, $v_t \in L^1((0, T); H^{-1})$ exists.

Let $T > 0$ and $f \in L^1((0, T); L^2)$. We say $u \in L^2((0, T); V)$ is a (the, incase $n = 2$) weak solution (c.f. [Tem01], Chapter 3) of the equations

$$
\begin{aligned}
    u_t + u \cdot \nabla u + \nabla p &= \Delta u + f, \\
    \nabla \cdot u &= 0, \\
    u &= 0, \quad x \in \partial \Omega, t > 0, \\
    u(x, 0) &= u_0(x), \quad \nabla \cdot u_0 = 0, \quad x \in \Omega
\end{aligned}
$$

(22)
on $Q_T$ provied

$$
(u(t) - u_0, v) = - \int_0^t b(u, u, v) + (\nabla u, \nabla v) + (f, v) \, ds, \quad \forall v \in V,
$$

a.e. $t \in (0, T)$.

We have defined

$$
b(u, v, w) = \int_\Omega u \cdot \nabla v \cdot w \, dx = \sum_{i,j=1}^n \int_\Omega u_i (v_j)_{x_i} w_j \, dx.
$$

If $n \leq 4$, then $p^* \geq 4$ and $b(\cdot, \cdot, \cdot)$ is well defined on $(H^1)^3$. If $u$ is a weak solution then it is absolutely continuous from $[0, T]$ into $L^2$. The space $V$ is separable and by the Lebesgue differntiation theorem, $u_t \in L^1((0, T); V^*)$ exists.
If the body force \( f \) takes the form \( \Delta \phi \nabla \phi \), then we will sometimes define

\[
(f, v) = -(\nabla \phi \otimes \nabla \phi, \nabla v), \quad \forall v \in V.
\] (23)

The motivation for this definition is easily seen from the identity \( \nabla \cdot (\nabla \phi \otimes \nabla \phi) = \Delta \phi \nabla \phi + \frac{1}{2} \nabla |\nabla \phi|^2 \) and integration by parts. The relevant consequences are as follows. Assume \( \phi \) solves the Poisson equation (5) and (10). If for \( 1 \leq p \leq \infty, n < q \), \( v, w \in L^p((0,T); H^{-1}) \) and \( v, w \in L^\infty((0,T); L^q) \), then by the injections \( H^{-1} \hookrightarrow H^1_0, L^q \hookrightarrow W^{1,q}_0 \) induced by the Poisson equation and the Sobolev embedding \( W^{1,q} \subset C^\alpha \),

\[
\|f\|_{V^*} \leq 2\|\nabla \phi \otimes \nabla \phi_t\|_{L^2} \\
\leq 2\|\nabla \phi\|_{L^\infty} \|\nabla \phi_t\|_{L^2} \\
\leq c(\Omega)\|v, w\|_{L^q} \|v_t, w_t\|_{H^{-1}} \in L^p(0,T). \quad (24)
\]

Let \( T > 0 \) and \( u_0 \in H \) and \( v_0, w_0 \in L^2 \). We say \( \langle u, v, w, \phi \rangle \) are a weak solution of the boundary value problem (1)-(11) on \( Q_T \) provided \( u \in L^2((0,T); V) \) and \( v, w \in L^2((0,T); H^1) \), and for all \( v \in V, \omega \in H^1, \eta \in H^1_0 \) and a.e. \( 0 < t < T \),

\[
(u(t) - u_0, v) = -\int_0^t b(u, u, v) + (\nabla u, \nabla v) + (\Delta \phi \nabla \phi, v) \, ds, \quad (25)
\]

\[
(v(t) - v_0, \omega) = -\int_0^t (\nabla v - v \nabla \phi - w u, \nabla \omega) \, ds, \quad (26)
\]

\[
(w(t) - w_0, \omega) = -\int_0^t (\nabla w + w \nabla \phi - w u, \nabla \omega) \, ds, \quad (27)
\]

\[
-(\nabla \phi(t) \cdot \nabla \eta) = (v(t) - w(t), \eta) \quad (28)
\]

A weak solution \( \langle u, v, w, \phi \rangle \) of (1)-(11) is said to be global if it is defined for all \( T \in (0, \infty) \). Note that by virtue of the injections \( H^k \hookrightarrow H^{k+2} \cap H^1_0 \), from the Poisson equation and the Sobolev embeddings \( H^2 \subset W^{1,p^*} \subset C^\alpha \), if \( v, w \in L^2((0,T); H^1) \), then \( \Delta \phi \nabla \phi \in \)
It makes sense to speak of the weak solution $u$ of the Navier–Stokes equations for then $f = \Delta \phi \nabla \phi$. Similarly, if $n \leq 4$, then $u \in L^2((0, T); L^{q^*} \cap V)$ and it makes sense to speak of a weak solution $v$ and $w$.

A weak solution is said to be classical if it possesses enough differentiability to satisfy the equation(s) continuously in the usual sense.

### 1.4 Stationary Solutions

One arrives at the stationary equations of (1)-(10) by setting all derivatives in $t$ to zero and setting $u \equiv 0$. They are

$$
\begin{aligned}
\nabla \cdot (\nabla v - v \nabla \phi) &= 0, \\
\nabla \cdot (\nabla w + w \nabla \phi) &= 0, \\
\Delta \phi &= v - w, \quad \nabla p = \Delta \phi \nabla \phi, \\
\frac{\partial v}{\partial \nu} - v \frac{\partial \phi}{\partial \nu} &= \frac{\partial w}{\partial \nu} + w \frac{\partial \phi}{\partial \nu} = \phi = 0, \\
\int_\Omega v \, d\mathbf{x} &= \int_\Omega v_0 \, d\mathbf{x}, \\
\int_\Omega w \, d\mathbf{x} &= \int_\Omega w_0 \, d\mathbf{x}.
\end{aligned}
$$

We say $v, w \in H^1, \phi \in H^1_0$ is a weak solution of (29) provided the equations are satisfied weakly in the usual sense.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth boundary $\partial \Omega$. Let $M$ and $N$ be positive constants. Then there exists a unique $\phi \in H^1_0$ satisfying

$$
\Delta \phi = M \frac{e^\phi}{\int_\Omega e^\phi \, dy} - N \frac{e^{-\phi}}{\int_\Omega e^{-\phi} \, dy}.
$$

If $k$ is nonnegative, then $\phi \in C^k(\Omega)$ and

$$
\lim_{M,N \to 0} \|\phi\|_{C^k(\Omega)} = 0.
$$
Proof. The unique solution is found by means of the direct method of the calculus of variations. For \( \phi \in H^1_0 \), define

\[
J[\phi] = \frac{1}{2} \|\nabla \phi\|_{L^2}^2 + M \log \int_{\Omega} e^\phi \, dx + N \log \int_{\Omega} e^{-\phi} \, dx. \tag{32}
\]

Equation (30) is the Euler-Lagrange equation of \( J[\cdot] \). Let \( \phi, \psi \in L^1 \), \( \phi \neq \psi \), and \( 0 < \lambda < 1 \). By Hölder’s inequality,

\[
\log \int_{\Omega} e^{\lambda \phi + (1-\lambda) \psi} \, dx < \log \left\{ \left( \frac{\int_{\Omega} e^{\phi} \, dx}{\int_{\Omega} e^{\psi} \, dx} \right)^\lambda \cdot \left( \frac{\int_{\Omega} e^{\psi} \, dx}{\int_{\Omega} e^{\phi} \, dx} \right)^{(1-\lambda)} \right\}
= \lambda \log \int_{\Omega} e^\phi \, dx + (1-\lambda) \log \int_{\Omega} e^\psi \, dx.
\]

This shows that \( J \) is strictly convex. By Jensen’s inequality,

\[
J[\phi] \geq \frac{1}{2} \|\nabla \phi\|_{L^2}^2 + (M - N) \int_{\Omega} \phi \, dx + (M + N) \log |\Omega|.
\]

Then, by Hölder’s inequality and the Poincaré inequality, \( J \) is bounded below by some constant depending only on \( M, N \) and \( \Omega \). By the direct method of the calculus of variations, \( J \) has a unique minimum \( \phi \in H^1_0 \).

Since \( J[\phi] < \infty \), it follows that \( e^\phi, e^{-\phi} \in L^1 \) and \( \phi \) satisfies

\[
- \int_{\Omega} \nabla \phi \cdot \nabla \psi = M \frac{\int_{\Omega} e^{\phi} \psi \, dx}{\int_{\Omega} e^\phi \, dy} - N \frac{\int_{\Omega} e^{-\phi} \psi \, dx}{\int_{\Omega} e^{-\phi} \, dy}, \quad \forall \psi \in C_0^\infty.
\]

By means of the identity, \( ae^t - be^{-t} = (ab)^{\frac{1}{2}} \sinh \left\{ t - \frac{1}{2} (\frac{b}{a}) \right\} \) \( a, b > 0, t \in \mathbb{R} \), this equation is equivalent to

\[
- \int_{\Omega} \nabla \phi \cdot \nabla \psi = \alpha[\phi] \int_{\Omega} \sinh(\phi - \beta[\phi]) \psi \, dx, \quad \forall \psi \in C_0^\infty. \tag{33}
\]

where \( \alpha[u] \) and \( \beta[u] \) are defined by the relations

\[
\alpha[\phi] = \left( \frac{MN}{\int_{\Omega} e^{\phi} \, dy \int_{\Omega} e^{-\phi} \, dy} \right)^{\frac{1}{2}}, \quad \beta[\phi] = \frac{1}{2} \log \left( \frac{N \int_{\Omega} e^\phi \, dy}{M \int_{\Omega} e^{-\phi} \, dy} \right).
\]

Since \( \sinh(\cdot) \) is increasing, applying the maximum principle to (33) we find that \( \phi \in L^\infty \). It follows from the standard theory for semilinear
elliptic equations (e.g. [Tay97], Chapter 14), that \( \phi \in C^k(\Omega) \) for all \( k > 0 \).

If \( \tilde{\phi} \in H^1_0 \) and \( J[\tilde{\phi}] < \infty \), then

\[
J[(1 - \lambda)\tilde{\phi} + \lambda\phi] - \frac{J[\tilde{\phi}]}{\lambda} \leq J[\phi] - J[\tilde{\phi}].
\]

If \( \tilde{\phi} \neq \phi \), then the right hand side is negative and \( \tilde{\phi} \) is not a critical point of \( J \). Hence any solution of (30) is identically \( \phi \).

Assume without loss of generality that \( M \leq N \). Define

\[
\zeta = \Delta \phi.
\]

One checks that \( \zeta \) satisfies the equation

\[
\Delta \zeta = (|\nabla \phi|^2 + \eta)\zeta
\]

where

\[
\eta = M \frac{e^{\phi}}{\int_{\Omega} e^{\phi}} + N \frac{e^{-\phi}}{\int_{\Omega} e^{-\phi}}.
\]

The function \( \eta \) is positive. By the maximum principle, \( \zeta \) has no negative internal minima nor positive internal maxima. Since

\[
\int_{\Omega} \zeta \, dx = M - N \leq 0,
\]

we see that \( \zeta \) is nonpositive. Since \( \phi \) restricted to \( \partial \Omega \) is 0,

\[
\phi(x) \leq 0, \quad \forall x \in \Omega.
\]

Choosing the argument of (32) to be the constant function 0, we see that

\[
J[\phi] \leq (M + N) \log |\Omega|.
\]

Arguing as in the beginning of the proof we find that there is a constant \( c_1 \) depending only on \( \Omega \) for which

\[
\|\nabla \phi\|_{L^2} \leq c_1 (N - M)^2.
\]
Applying Jensen’s inequality and the Poincaré inequality once more,
\[
\int_{\Omega} e^\phi \, dx \geq |\Omega| e^{c_2(M-N)^2}
\]  
(35)
for some constant \( c_2 \) depending only on \( \Omega \).

Combining (34) and (35),
\[
\Delta \phi \leq \frac{M}{|\Omega|} e^{c_2(N-M)^2}, \quad \phi(x) = 0, \quad x \in \partial \Omega.
\]
By the maximum principle, \( \phi \) converges to 0 uniformly in \( \Omega \) as \( M, N \) converge to 0. The estimate (31) now follows by bootstrapping the elliptic estimates for \( \phi \).

**Corollary 1.** There exists a unique solution \( v_\infty, w_\infty, \phi_\infty \) of the stationary equations (29). If \( k > 0 \), then \( v_\infty, w_\infty, \phi_\infty \in C^k(\Omega) \) and
\[
v_\infty = \int_{\Omega} v_0 \, dx e^{-\phi_\infty}, \quad w_\infty = \int_{\Omega} w_0 \, dx e^{\phi_\infty},
\]
\[
\lim_{\rho_0 \to 0} \|v_\infty\|_{C^k(U)} = \lim_{\rho_0 \to 0} \|w_\infty\|_{C^k(U)} = 0.
\]

**Proof.** Suppose \( v, w \in H^1, \phi \in H^1_0 \) is weak solution of (29). For \( \delta > 0 \), consider the test function
\[
\psi = \log\{v+e^{-\phi} + \delta\} \in H^1.
\]
Multiplying the first equation in (29) by \( \psi \) and integrating by parts,
\[
0 = \int_{\Omega} (\nabla v - v \nabla \phi) \cdot \nabla \psi \, dx = \int_{v>0} \frac{e^{-\phi} |\nabla v - v \nabla \phi|^2}{v+e^{-\phi} + \delta} \, dx.
\]
This implies that \( \nabla v = v \nabla \phi \) for a.e. \( x \in \{y \in \Omega : v(y) > 0\} \). If
\[
\psi = \log\{(-v)+e^{-\phi} + \delta\} \in H^1,
\]
then the above reasoning also implies that $\nabla v = v \nabla \phi$ for a.e. $x \in \{ y \in \Omega : v(y) < 0 \}$. Using the differentiability of Sobolev functions on lines (e.g. [EG92], Section 4.9), we find that

$$v(x) = \int_\Omega v_0 \, dx \, e^{-\phi(x)}, \quad \text{for a.e. } x \in \Omega.$$ 

Similarly

$$w(x) = \int_\Omega w_0 \, dx \, e^{\phi(x)}, \quad \text{for a.e. } x \in \Omega.$$ 

It follows that $e^\phi, e^{-\phi} \in \mathbf{H}^1$.

Returning to Theorem 1, $\phi$ is the unique solution of (30) with $M = \int_\Omega v_0 \, dx$ and $N = \int_\Omega v_0 \, dx$. It follows that $v$ and $w$ are, a fortiori, unique.

**Remark 3.** Let $\sigma_\infty = \nabla \phi_\infty \otimes \nabla \phi_\infty - \frac{1}{2} |\nabla \phi_\infty|^2 I$. Observe that

$$\nabla \cdot \sigma_\infty = \Delta \phi_\infty \nabla \phi_\infty = \nabla (v_\infty + w_\infty).$$  

Equation (36) states that the divergence of the stationary electric stress $\sigma_\infty$ is the gradient of a pressure. This is consistent with the fourth equation in (29).

### 1.5 Main Results

The following existence, uniqueness and regularity theorem is the strongest result expected from (1)-(11) for general data.

**Theorem 2.** If $\dim \Omega = 2$, and $u_0 \in \mathbf{H}$ and $v_0, w_0 \in \mathbf{L}^2$, then (1)-(11) possesses a unique, global weak solution. The solution is classical. In particular, if $0 < t < T < \infty$ and $\mathcal{R}$ is any compact subset of $\mathbf{Q}_T$, then

$$u \in C^{2+\alpha}(\mathcal{R}) \cap C^\alpha(\mathbf{Q}_{(t,T)}), \quad v, w \in C^{2+\alpha}(\mathbf{Q}_{(t,T)}).$$
Thus, in two space dimensions, (1)-(11) is solvable and enjoys usual regularization property found in equations of parabolic type. In dimensions three and four, the existence of a global weak solution can be proved using the techniques in the proof of theorem 2 assuming a uniform in time $L^2$ apriori estimates for $v$ and $w$.

The next theorems concern the long term behavior of weak solutions. In order to quantify the convergence, define for $1 \leq p \leq \infty$,

$$E_p(t) = \int_{\Omega} |u(t)|^2 + \frac{|v(t) - v_{\infty}|^p}{v_{\infty}^{p-1}} + \frac{|w(t) - w_{\infty}|^p}{w_{\infty}^{p-1}} + |\nabla \phi(t) - \nabla \phi_{\infty}|^2 \, dx$$

The following theorem is modeled after [BD00]. Note that, in contrast to theorem 2, these solutions may not be defined globally if $n \geq 3$.

**Theorem 3.** Let $\Omega \subset \mathbb{R}^n$, be bounded and uniformly convex, $n \geq 2$, and $\langle u, v, w, \phi \rangle$ be a global weak solution of (1)-(11). Then there is $\lambda_1 > 0$ depending on $\Omega$ and $\epsilon_1 < \infty$ depending only on the initial data so that for all $t \geq 0$

$$E_1(t) \leq \epsilon_1 e^{-\lambda_1 t}. \quad (37)$$

It is difficult to extend the technique of [BD00] used in the proof of theorem 3 to $\Omega$ with general geometry. However, if one assumes that the initial data is close to the stationary solution and the stationary solution is small then the convergence with a rate is recovered. This result is a kind of linearization of the argument used in the proof of theorem 3.

**Theorem 4.** Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, $u_0 \in H, v_0, w_0 \in L^2$. There are positive constants

$$\rho_2 = \rho_2(\Omega), \quad \lambda_2 = \lambda_2(\Omega), \quad \epsilon_2 = \epsilon_2(\Omega)$$
such that if 
\[ \varepsilon_2(0) < \varepsilon_2, \quad \rho_0 < \rho_2 \]
then (1)-(11) possesses a global, weak solution and
\[ \varepsilon_2(t) \leq \varepsilon_2 e^{-\lambda_2 t}. \] (38)

If a global weak solution satisfies
\[ \sup_{t \in (0, \infty)} \| v, w \|_{L^2} < \infty, \]
then there is \( t_0 > 0 \) so that \( \varepsilon_2(t_0) < \varepsilon_2 \).

Thus, in two space dimensions, theorem 4 implies that the solution from theorem 2 tends to the stationary solution since the weak solution is eventually close to the stationary solution.

In three dimensions, a global existence, uniqueness and regularity result is proved under a small data assumption. However, it is not sufficient to assume that the initial data is small and close to the stationary solution. One must also assume that the stationary solution is also small.

**Theorem 5.** Let \( \dim \Omega = 3 \), and \( u_0 \in H \) and \( v_0, w_0 \in L^2 \). There exist constants
\[ \rho_3 = \rho_3(\Omega), \quad \varepsilon_3 = \varepsilon_3(\Omega), \quad \delta_3 = \delta_3(\Omega) \]
such that if
\[ \rho_0 < \rho_3, \quad \varepsilon_2(0) < \varepsilon_3, \quad \| u_0, v_0, w_0 \|_{H^2} < \delta_3, \]
then (1)-(11) possesses a unique, global classical solution. In particular, if \( 0 < t < T \) and \( \mathcal{R} \) is any compact subset of \( Q_T \), then
\[ u \in C^{2+\alpha}(\mathcal{R}) \cap C^{\alpha}(Q_{(t,T)}), \quad v, w \in C^{2+\alpha}(Q_{(t,T)}). \]
The remainder of the paper is organized as follows. In section 3, some sufficient conditions are developed for concluding $L^\infty$ bounds on weak solutions are developed. These are later used proofs of theorems 2 and 5. As noted in the introduction, this is a necessary step for the regularity programme due to the lack of a gradient descent structure for the conservation equations and the lack of a maximum principle for (3) and (4). In section 4, local weak solutions are constructed and the extension property is developed in two space dimensions. The section is concluded with the proof of theorem 2. In section 5, we give a proof of theorem 3 modeled after the result of [BD00]. Finally, in section 6, we give several preparatory lemma and provide the proofs theorems 4 and 5.

The letter $C$ will denote a constant which may change from line to line within a proof. The letters $c_1, c_2, \ldots$, will denote constants that are fixed throughout the paper.

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3 Preliminaries

In this section we are assuming $\Omega$ is an open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ and that $\partial \Omega$ is smooth. The following preliminary results will later be used to infer uniform $L^\infty$ bounds on solutions $v, w, u$. The $C^{2+\alpha}$ regularity of solutions will then follow from classical results for linear second order PDE of parabolic type.

In the case of $L^\infty$ bounds on weak solutions $v$ and $w$ of (3) and (4), the usual techniques for nondivergence form semilinear parabolic equations do not apply, mainly due to the boundary conditions (7). Instead, we rely on a Moser type iteration argument. The essential part of the argument is that due to the divergence free condition, no regularity on the velocity $u$ need be assumed.

First a general

**Lemma 1.** For $t \in [0, T]$ and $p \in [1, \infty)$, let $y(t, p)$ be positive and continuous and satisfy the differential inequality

$$\frac{\partial y^p}{\partial t}(t, p) + y^p(t, mp) \leq w(t)p^k y^{\alpha(p)}(t, p)$$

where $m > 1, k \geq 0, w(t) \geq 0$ is measurable with $\int_s^t w(r) \, dr \leq \gamma |t - s|^{\beta}$ for some constants $\gamma, \beta > 0$ and $\alpha(p) \in (0, 1]$. If $0 < \epsilon < t$ and $1 \leq p_0$, then there is $c_3 = c_3(w, k, p_0, \epsilon) < \infty$ for which

$$\lim_{p \to \infty} y(t, p) \leq c_3 \cdot y(t - \epsilon, p_0).$$

**Proof.** Without loss of generality, we may assume that $\alpha(p) = 1$ for every $p$. For otherwise, we may replace $y(p, t)$ by $\max\{1, y(p, t)\}$. Let $0 < s < t \leq T$. Using Gronwall’s inequality,

$$y^p(t, p) + \int_s^t y^p(r, mp) \leq \exp \left( p^k \int_s^t w(r) \, dr \right) y^p(s, p)$$  (39)
We will take advantage of the various powers in this inequality to infer some bounds.

Let \( 0 < \epsilon \leq t \) and \( \sigma \geq 2 \). Define

\[
t_i = t - \epsilon \sigma^{-i}, \quad \delta_i = t_{i+1} - t_i, \quad p_i = m^i p_0, \quad i = 1, 2, \ldots
\]

From (39) with \( p = p_{i+1}, t = t_{i+1} \) and \( s \in [t_i, t_{i+1}] \) we have (recall \( mp_i = p_{i+1} \))

\[
y^{p_i}(t_{i+1}, p_{i+1}) \leq \exp \left( \frac{p_{i+1}^k}{m} \int_{t_i}^{t_{i+1}} w(r) \, dr \right) y^{p_i}(s, mp_i).
\]

Integrating this expression with respect to \( r = s \in [t_i, t_{i+1}] \),

\[
\delta_i y^{p_i}(t_{i+1}, p_{i+1}) \leq \exp \left( \frac{p_{i+1}^k}{m} \int_{t_i}^{t_{i+1}} w(r) \, dr \right) \int_{t_i}^{t_{i+1}} y^{p_i}(r, mp_i) \, dr.
\]

Using (39) once more with \( s = t_i, t = t_{i+1} \) and \( p = p_i \) to bound the second integral on the right hand side,

\[
\delta_i y^{p_i}(t_{i+1}, p_{i+1}) \leq \exp \left( \left( \frac{p_{i+1}^k}{m} + p_i^k \right) \int_{t_i}^{t_{i+1}} w(r) \, dr \right) y^{p_i}(t_i, p_i).
\]

This in turn implies

\[
y(t_{i+1}, p_{i+1}) \leq \delta_i^{-\frac{1}{pi}} \exp \left( \left( \frac{p_{i+1}^k}{mp_i} + p_i^{k-1} \right) \int_{t_i}^{t_{i+1}} w(r) \, dr \right) y(t_i, p_i).
\]

Note the integrand in the argument of the exponential is bounded above by

\[
R = \gamma p_0^{k-1} \left( m^{k(i+1)-i-1} + m^{i(k-1)} \right) \delta_i^\beta.
\]

Clearly \( 2^{-1} \epsilon \sigma^{-i} \leq \delta_i \leq \epsilon \sigma^{-i} \). Choosing \( \sigma = \max\{2, m^\frac{2k}{m^i} \} \),

\[
R \leq C_1 m^{-ki}
\]

for some \( C_1 = C_1(\epsilon, \gamma, \beta, p_0, m, k) \). Similarly,

\[
\delta_i^{-\frac{1}{pi}} \leq C_2(\epsilon, \beta, p_0, m)^{\frac{i}{m^i}}.
\]
It follows that for all $i = 1, 2, \ldots$, $y(t_{i+1}, p_{i+1}) \leq C_{2}^{i} e^{C_{1}m^{-ki}} y(t_{i}, p_{i})$
and so by recursion
\[ y(t_{i+1}, p_{i+1}) \leq \Phi_{i} y(t_{0}, p_{0}) \]
where $\Phi_{i} = \Pi_{j=1}^{i} C_{2}^{j} \exp(C_{1}m^{-ki})$. This product converges and we
set the limit to be $c_{3}$. The conclusion now follows by varying $p_{0}$ and $\epsilon$
in a sufficiently small subset of $[0, T] \times [1, 2]$ and using the continuity
of $y$. \hfill \Box

**Proposition 1.** Let $S > 0$, $\dim \Omega = 2, 3$, $v \in L^{2}((0, S); V)$, and $v_{0}, w_{0} \in L^{2}$.

1. Then there is $0 < T_{0} = T_{0}(\Omega, \|v_{0}, w_{0}\|_{L^{2}}) \leq S$ so that the problem
\[ v_{t} + v \cdot \nabla v = \nabla \cdot (\nabla v - v \nabla \phi), \]
\[ w_{t} + v \cdot \nabla w = \nabla \cdot (\nabla w + w \nabla \phi), \]
\[ \Delta \phi = v - w, \]
\[ \frac{\partial v}{\partial \nu} - \frac{\partial \phi}{\partial \nu} = 0, \quad \frac{\partial w}{\partial \nu} + \frac{\partial \phi}{\partial \nu} = 0, \quad \phi = 0, \quad \text{on } \partial \Omega \times (0, \infty), \]
\[ v(x, 0) = v_{0}(x), \quad w(x, 0) = w_{0}(x) \quad x \in \Omega. \]

has a unique weak solution on $Q_{T_{0}}$. Moreover, if $p \geq 2$, there is a
constant $c_{4} = c_{4}(\Omega, \sup_{t \in (0, T_{0})} \|v, w\|_{L^{2}}, p)$ so that
\[ \sup_{t \in [0, T]} \|v, w\|_{L^{p}} \leq \|v_{0}, w_{0}\|_{L^{p}} e^{c_{4}T}. \quad (40) \]

2. If $p \geq 2$ and $0 < s < t \leq T_{0}$, then there is
\[ c_{6} = c_{5}(\Omega, \|v, w\|_{L^{2}((0, T); H^{1})}, p, t - s) < \infty \]
for which the weak solution $v, w$ satisfies

$$\|v(t), w(t)\|_{L^\infty} \leq c_5 \cdot \|v(s), w(s)\|_{L^p}. \quad (41)$$

3. If, additionally,

$$v_0, w_0 \in C^{2+\alpha}(\Omega), \quad 0 < v_0(x), w_0(x) \quad \forall x \in \overline{\Omega},$$

$$\frac{\partial v_0}{\partial \nu} - v_0 \frac{\partial \phi_0}{\partial \nu} = 0, \quad \frac{\partial w_0}{\partial \nu} + w_0 \frac{\partial \phi_0}{\partial \nu} = 0, \quad \text{on } \partial \Omega, \quad (42)$$

and $v \in C^\alpha(Q_{T_0})$, then

$$v, w \in C^{2+\alpha}(Q_{T_0})$$

and $v$ and $w$ are positive on $Q_{T_0}$.

Proof. (Part 1) The existence and uniqueness of a weak solution is established by making slight modifications to the proof of [BHIN94], theorem 1 to account for the term $v \cdot \nabla$. We omit the details here. Let

$$0 < T_0 = T_0(\Omega, \|v_0\|_{L^2}, \|w_0\|_{L^2})$$

so that the weak solution is defined on $Q_{T_0}$ and let

$$M_0 = \sup_{t \in (0, T_0)} \|v, w\|_{L^2}, \quad M_1 = \int_0^T \|v, w\|_{H^1} dt.$$

Multiply the $v$-equation and the $w$-equation by $p v^{p-1}$ and $p w^{p-1}$ respectively and integrate over $\Omega$. Since $p \nabla v^{p-1} = \nabla v^p$ and $p \nabla w^{p-1} = \nabla w^p$, using $\nabla \cdot v = 0$ and $v \in H^1_0 = 0$ we have

$$0 = \int_\Omega p \nabla \cdot \nabla v^{p-1} dx = \int_\Omega p \nabla \cdot \nabla w^{p-1} dx.$$

Integrating by parts gives

$$\frac{d}{dt} \|v, w\|_{L^p}^p + \frac{4(p - 1)}{p} \|\nabla v^{\frac{p}{2}}, \nabla w^{\frac{p}{2}}\|_{L^2}^2$$

$$= 2(p - 1) \int_\Omega \nabla \phi \cdot (v^{\frac{p}{2}} \nabla v^{\frac{p}{2}} + w^{\frac{p}{2}} \nabla w^{\frac{p}{2}}) dx = A. \quad (43)$$
for \( a.e. \ t \in (0, T_0) \). Since \( \phi \) solves the Poisson equation, by elliptic regularity and the Sobolev embedding \( H^1 \subset L^6 \), we have

\[
\| \nabla \phi \|_{L^6} \leq C(\Omega, M_0), \quad a.e. \ t \in [0, T_0)
\]

For \( \epsilon > 0 \) we have from the Sobolev inequality

\[
\| v^\frac{p}{2}, w^\frac{q}{2} \|_{L^3} \leq C(\Omega)(\epsilon^{-1} \| v, w \|_{L^p}^\frac{p}{2} + \epsilon \| \nabla v^\frac{p}{2}, \nabla w^\frac{q}{2} \|_{L^2}).
\]

Applying these inequalities, the estimate

\[
A \leq 2(p - 1)\| \nabla \phi \|_{L^6} \| v^\frac{p}{2}, w^\frac{q}{2} \|_{L^3} \| \nabla v^\frac{p}{2}, \nabla w^\frac{q}{2} \|_{L^2}
\]

\[
\leq C(\Omega, M_0, p)(\epsilon^{-1} \| v, w \|_{L^p}^\frac{p}{2} + \epsilon \| \nabla v^\frac{p}{2}, \nabla w^\frac{q}{2} \|_{L^2}) \| \nabla v^\frac{p}{2}, \nabla w^\frac{q}{2} \|_{L^2}
\]

follows. Choosing \( \epsilon \leq (2C(\Omega, M_0, p))^{-1} \), we find from (43) with \( p \geq 2 \)

\[
\frac{d}{dt} \| v, w \|_{L^p}^p \leq C(\Omega, M_0, p) \| v, w \|_{L^p}^p.
\]

Gronwall’s inequality now implies (40).

(Part 2) Returning to (43), the inequality

\[
\frac{d}{dt} \| v, w \|_{L^q}^q + \| \nabla v^q/2, \nabla w^q/2 \|_{L^2}^2 \leq 4q^2 \| \nabla \phi \|_{L^\infty} \| v, w \|_{L^q}^q
\]

holds for \( q \geq 2 \). By elliptic regularity and the Sobolev embedding \( W^{1,4} \subset L^\infty \), we have \( \| \nabla \phi \|_{L^\infty} \leq C(\Omega) \| v, w \|_{H^1} \in L^2(0, T_0) \). With this constant, define \( w(t) = 4C(\Omega) \| v(t), w(t) \|_{H^1} \) and note that

\[
\int_s^t w(r) \, dr \leq \gamma |t - s|^{\frac{1}{2}}
\]

where \( \gamma = \gamma(\Omega, M_1) \). Furthermore, by the embedding \( H^1 \subset L^4 \),

\[
\| v, w \|_{L^{2q}}^q = \| v^\frac{q}{2}, w^\frac{q}{2} \|_{L^4}^2 \leq C(\Omega)(\| \nabla v^q/2, \nabla w^q/2 \|_{L^2}^2 + \| v, w \|_{L^q}^q).
\]

Combining this with the previous observations, one has

\[
\frac{d}{dt} \| v, w \|_{L^q}^q + \| v, w \|_{L^{2q}}^q \leq q^2 w(t) \| v, w \|_{L^q}^q.
\]
We may thus apply lemma 1 with \(y(t,q) = \|v,w\|_{L^q}, \alpha = 1, \beta = 1, p = k = m = 2\) to find (41).

(Part 3) If \(v_0, w_0 \in C^{2+\alpha}(\Omega)\), then certainly \(v_0, w_0 \in L^p\) for \(p > n\). Fix \(n < p \leq p^*\). Then elliptic regularity and the Sobolev embedding \(W^{1,p} \subset L^\infty\) and (40) imply there is \(C = C(\Omega, M_0, T_0, p) < \infty\) so that

\[
\|\nabla \phi\|_{L^\infty(Q_{T_0})} \leq C, \quad \forall t \in [0, T_0).
\]

If \(v \in C^\alpha(Q_{T_0})\), then \(v\) and \(w\) solve divergence form equations with oblique boundary conditions with bounded, measurable coefficients. By [Lie96], theorem 6.41 and 6.44, \(v, w \in C^\alpha(Q_{T_0})\). The relation 

\[
\Delta \phi = v - w
\]

then implies \(\nabla \phi \in C^\alpha(Q_{T_0})\) as well. The \(C^{2+\alpha}(Q_{T_0})\) regularity of \(v, w\) is then guaranteed by the smoothness and consistency assumption (42) and [Lie96], theorem 5.18.

One readily checks that \(\|\min\{0, v, w\}\|_{L^2} = 0\) for all \(t \in [0, T_0]\). If \(v\) is not bounded below by a positive constant on \(Q_{T_0}\), then \(v(x, t) = 0\) for some \((x, t) \in Q_{T_0} \cup (\partial \Omega \times (0, T_0))\). If \((x, t) \in Q_{T_0}\), then the strong maximum principle ([Lie96], theorem 2.9) implies \(v \equiv 0\), contradicting (9). If \((x, t) \in \partial \Omega \times (0, T_0)\), then by the parabolic Hopf lemma ([Lie96], theorem 2.6) implies

\[
0 > \frac{\partial v}{\partial \nu}(x, t) = \frac{\partial v}{\partial \nu}(x, t) - v(x, t) \frac{\partial \phi}{\partial \nu}(x, t) = 0,
\]

again a contradiction. Analogous arguments apply to \(w\).

Solutions of the velocity equations will be constructed from a special basis. To describe the method, we choose a particular orthonormal
basis \( \{ \zeta_i \}_{i=1}^{\infty} \) of \( H \) satisfying
\[
\begin{align*}
\Delta \zeta_i + \nabla p_i &= -\lambda \zeta_i \\
\nabla \cdot \zeta_i &= 0 \\
\zeta_i(x) &= 0 \quad \text{for } x \in \partial \Omega.
\end{align*}
\]

Here \( \zeta_i \in C_0^\infty(\Omega) \) and \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \) are eigen-pairs of the Stokes operator on \( \Omega \). For \( i, j = 1, 2, \ldots \), the functions \( \zeta_i \) satisfy the orthogonality relations (c.f. section 2.6, [Tem01])
\[
(\Delta \zeta_i \cdot \zeta_j) = -\lambda_i (\zeta_i, \zeta_j) = -\lambda_i \delta_{ij}.
\] (44)

Following proposition 1 we are also interested in \( L^\infty \) bounds of the velocity \( u \). The following general result for the Navier-Stokes equations will be useful in the regularity proof in two space dimensions.

**Lemma 2** (Ladyzhenskaya’s Inequality, c.f. [Tem01], Ch. 3). Let \( \Omega \) be an open subset of \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). If \( u \in V \) then
\[
\begin{align*}
||u||_{L^4} &\leq \frac{1}{2} \frac{||u||_{L^2}^2}{||\nabla u||_{L^2}} , \quad \dim \Omega = 2, \\
||u||_{L^4} &\leq \frac{1}{2} \frac{||u||_{L^2}^3}{||\nabla u||_{L^2}^2} , \quad \dim \Omega = 3.
\end{align*}
\]

**Proposition 2.** Suppose that \( T > 0, \dim \Omega = 2, u_0 \in L^2, \)
\[
f \in L^2((0,T);L^2), \quad i^2 f_t \in L^2((0,T);V^*),
\]
and \( u \) is the unique Leray-Hopf solution of
\[
\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p &= \Delta u + f \\
\text{div } u &= 0
\end{align*}
\]
on \( Q_T \). Then there is a constant \( c_0 = c_6(\Omega, f, f_t, u_0) < \infty \) so that
\[
\begin{align*}
||u_t||_{L^2} &\leq \frac{c_6}{t^2}, \quad ||\nabla u||_{L^2} \leq \frac{c_6}{t}, \quad \int_s^t ||\nabla u||_{L^2} \, dr \leq \frac{c_6}{s^2}.
\end{align*}
\]
for all \( 0 < s < t \leq T \).
Remark 4. In proposition as in theorem, we are merely assuming \( u_0 \in L^2 \). Also, \( f_t \) may be singular at the origin. We conclude that \( u \) is \( C^\alpha \) away from the set \( \Omega \times \{ t = 0 \} \).

Proof. For the moment, assume \( f_t \in C^0([0, T]; L^2) \). Let \( m \) be a positive integer and \( v(x, t) = \sum_{i=1}^{m} \zeta_i(x)v_i(t) \) and \( v_i \) solve \( m \)-dimensional system of ordinary differential equations

\[
\dot{v}_i + \lambda_i v_i + \sum_{j,k=1}^{m} B_{ijk} v_j v_k = F_i(t). \tag{45}
\]

We have defined

\[
B_{ijk} = b(\zeta_j, \zeta_k, \zeta_i) \quad F_i(t) = (f, \zeta_i).
\]

It is well known (c.f. [Tem01], Chapter 3) that for some subsequence of \( m \), the \( v \) converge to \( u \) in the strong topology \( L^2(\Omega_T) \) and there is \( c_0 \) depending only on \( u_0 \) and \( f \) for which

\[
\|v\|_{L^\infty((0, T); L^2)}^2 + \|\nabla v\|_{L^2((0, T); L^2)}^2 \leq c_0, \quad \forall m > 0. \tag{46}
\]

Differentiate \( \|v\|_{L^\infty((0, T); L^2)}^2 \) with respect to \( t \) and multiply the resulting system component-wise by \( \dot{v}_j \). We find that

\[
(v_{tt}, v_t) + b(v_t, v, v_t) = - (\Delta v_t, v_t) + (f_t, v_t)
\]

Multiply this equation by \( t^2 \) and define \( w = tv_t \). One easily checks that

\[
\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 = b(w, u, w) + t \|v_t\|_{L^2}^2 + t(f_t, w).
\]

On the other hand, if we multiply \( \|v_t\|_{L^2}^2 \) by \( \dot{v}_i \) componentwise, we find

\[
\|v_t\|_{L^2}^2 = (\Delta v, v_t) - b(v, v, v_t) + (f, v_t).
\]
Inserting this expression into the previous equation and noticing that
\[ t(\Delta v, v_t) = -t(\nabla v, \nabla v_t) = -\frac{1}{2}\frac{d}{dt}(\Delta v, v) + \frac{1}{2}(\nabla v, \nabla v), \]
gives
\[ \frac{1}{2} \frac{d}{dt} \left( \|w\|_{L^2}^2 + t\|\nabla v\|_{L^2}^2 \right) + \|\nabla w\|_{L^2}^2 = A(t) + B(t) + C(t) \]
where
\[ A(t) = \frac{1}{2}\|\nabla v\|_{L^2}^2; \]
\[ B(t) = B_1(t) + B_2(t) = b(w, v, w) - b(v, v, w), \]
\[ C(t) = t(f_t, w) + (f, w). \]

From (46), is clear that \( A(t) \) is integrable with integral bounded above by \( \frac{1}{2}c_0 \) along any measurable subset of \([0, T]\). Similarly, one has
\[ C(t) \leq 2t^2\|f_t\|_{L^4}^2 + C(\Omega)\|f\|_{L^2}^2 + \frac{1}{4}\|\nabla w\|_{L^2}^2 \]
with the two left hand terms on the right hand side of the inequality also integrable. By Hölder’s inequality,
\[ B_1(t) = -\int_\Omega w \cdot \nabla w \cdot v \, dx \leq \|w\|_{L^4}\|v\|_{L^4}\|\nabla w\|_{L^2}. \]

By Ladyzhenskaya’s inequality, \( \|v\|_{L^4} \leq 2^{\frac{1}{2}}\|v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}^{\frac{1}{2}} \) which lies in \( L^4(0, T) \). Similarly, \( \|w\|_{L^4}\|w\|_{L^2} \leq 2^{\frac{1}{2}}\|w\|_{L^2}^{\frac{1}{2}}\|\nabla w\|_{L^2}^{\frac{1}{2}} \). Taking these elements into consideration, we find
\[ B_1(t) \leq 8\|w\|_{L^2}^2\|v\|_{L^4}^4 + \frac{1}{4}\|\nabla w\|_{L^2}^2. \]

Similar ideas imply
\[ B_2(t) \leq 2\|v\|_{L^4}^4 + \frac{1}{4}\|\nabla w\|_{L^2}^2. \]

In total, we find that
\[ \frac{d}{dt}F(t) + G(t) \leq D(t) + E(t)F(t) \]
\[
D(t) = 2t^2\|f_t\|_{V^*}^2 + 2\|f\|_{V^*}^2 + 4\|v\|_{L^1}^4 \in L^1(0,T), \\
E(t) = 16\|v\|_{L^4}^4 \in L^1(0,T), \\
F(t) = \|w\|_{L^2}^2 + t\|\nabla v\|_{L^2}^2, \\
G(t) = \frac{1}{2}\|\nabla w\|_{L^2}^2.
\]

We have assumed that \(f_t\) is continuous from \([0,T]\) into \(L^2\). Therefore, \(v\) and \(v_t\) remain bounded in \(H^1\) and \(F(t) = \|w\|_{L^2}^2 + t\|\nabla v\|_{L^2}^2 = 0\) when \(t = 0\). By Gronwall’s inequality, there is a constant \(c_6\) independent of \(m\) and the continuity of \(f_t\) for which

\[
F(t) + \int_s^t G(r) \, dr \leq c_6, \quad \forall 0 \leq s \leq t \leq T.
\]

The proposition now follows by first approximating \(f\) by functions continuously differentiable in \(t\), and then letting \(m\) diverge along the given subsequence; for a.e. \(t \in (0,T)\), Fatou’s lemma implies

\[
t^2\|u_t\|_{L^2}^2 + t\|\nabla u\|_{L^2}^2 + t^2\int_s^t \|\nabla u_t\|_{L^2}^2 \, dr \\
\leq \lim_{m} \left( F(t) + \int_s^t G(r) \, dr \right) \leq c_6.
\]

\[\square\]

4 The Local Existence and Extension Property in 2D.

We will construct weak solutions of \((1)-(11)\) as the limit of modified Galerkin approximation. We look for solutions of \((1)-(5)\) with the
form
\[ u_m(x, t) = \sum_{i=1}^{m} u_i(t) \zeta_i(x). \]

The orthogonality of the \( \zeta_i \) lead us to the following approximation problem. For
\[ F_i(t) = \int_{\Omega} \Delta \phi_m \nabla \phi_m \cdot \zeta_i \, dx, \]
consider a solution of
\[ \dot{u}_i + \lambda_i u_i + \sum_{j,k=1}^{m} B_{ijk} u_j u_k = F_i(t) \quad (47) \]
\[ \frac{\partial v_m}{\partial t} + u_m \cdot \nabla v_m = \nabla \cdot (\nabla v_m - v_m \nabla \phi_m), \quad (48) \]
\[ \frac{\partial w_m}{\partial t} + u_m \cdot \nabla w_m = \nabla \cdot (\nabla w_m + w_m \nabla \phi_m), \quad (49) \]
\[ \Delta \phi_m = v_m - w_m, \quad (50) \]
\[ v_i(0) = \int_{\Omega} u_0 \cdot \zeta_i \, dx, \quad v_m(0, \cdot) = v_0(\cdot), \quad w_m(0, \cdot) = w_0(\cdot). \quad (51) \]

We will prove

**Theorem 6.** Suppose \( \dim \Omega = 2, u_0 \in H \) and \([12]\) hold. For any \( m > 0, T \in (0, \infty) \), the problem \([17] - [51]\) has a unique, classical solution with \( D^k D^l_t u_m \in C^\alpha(Q_T) \) for \( k = 1, 2 \) and \( l = 1, 2, \ldots \), and \( v_m, w_m \in C^{2+\alpha}(Q_T) \).

Theorem 6 will be a consequence of the following lemmas.

**Lemma 3.** Suppose \( \dim \Omega = 2, 3, u_0 \in H \) and assume \([42]\). There is \( T_0 = T_0(\Omega, \|u_0\|_{L^2}, \|v_0\|_{L^2}, \|w_0\|_{L^2}) > 0 \) such that the problem \([17] - [51]\) has a unique, classical solution with \( D^k D^l_t u_m \in C^\alpha(Q_{T_0}) \) for \( k = 1, 2 \) and \( l = 1, 2, \ldots \), and \( v_m, w_m \in C^{2+\alpha}(Q_{T_0}) \).

**Proof.** Let \( s > 0 \) and let \( \{v_i\}_{i=1}^{m} \in W^{1,2}((0, s); \mathbb{R}^m) \). Define \( v(x,t) = \sum_{i=1}^{m} \zeta_i(x)v_i(t) \). The embedding \( W^{1,2}(0, s) \subset C^{1}(0, s) \) implies
\[ v \in \text{Lip}(Q_s) \text{ (with respect to the parabolic distance.)} \] From proposition 11, there exists \( T_0 = T_0(\Omega, \|v_0\|_{L^2}, \|w_0\|_{L^2}) \) and a solution \( v, w \in C^{2+\alpha}(Q_{T_0}) \) of the problem in proposition 11. Let \( M_0 = \sup_{t \in (0, T_0)} \|v_0, w_0\|_{L^2} \) and a solution \( v, w \in C^{2+\alpha}(Q_{T_0}) \) of the problem in proposition 1. Let \( M_0 = \sup_{t \in (0, T_0)} \|v_0, w_0\|_{L^2} \).

Define \( F_i(t) = \int_{\Omega} \Delta \phi \nabla \phi \cdot \zeta_i \, dx. \) Then, by elliptic regularity and (40)

\[
\sup_{t \in [0, T_0]} \sum_{i=1}^{m} F_i^2(t) \leq \sup_{t \in [0, T_0]} \|\Delta \phi \nabla \phi\|_{L^1} \sum_{i=1}^{m} \|\zeta_i\|_{L^\infty} \leq C(m, \Omega, M_0).
\]

Substituting \( \Delta \phi \nabla \phi \) into (47), this ordinary differential equation has a \( W^{1,\infty}(0, T_2) \) solution \( \{\bar{v}_i\}_{i=1}^{m} \). Infact, by (51) and (44), we have

\[
\sum_{i=1}^{m} \bar{g}_i^2(0) \leq |u_0|^2.
\]

Therefore, the \( \{\bar{v}_i\}_{i=1}^{m} \) satisfy the estimate

\[
\sum_{i=1}^{m} \bar{v}_i^2(t) \leq \|u_0\|_{L^2}^2 + tC(m, \Omega, M_0), \quad \forall t \in [0, T_0].
\]

Fix \( s = T_0 \) and \( M = \|u_0\|_{L^2}^2 + T_0C(m, \Omega, M_0) \). This construction maps \( W^{1,2}((0, T_0); \mathbb{R}^m) \) into

\[
S_M = \left\{ \{v_i\}_{i=1}^{m} \in W^{1,2}(0, T_0; \mathbb{R}^m) : \sup_{t \in [0, T_0]} \sum_{i=1}^{m} v_i(t)^2 \leq M \right\}.
\]

The mapping is clearly continuous and takes \( S_M \), a compact and convex subset of \( W^{1,2}((0, T_0); \mathbb{R}^m) \), into itself. A fixed point \( \{v_i\}_{i=1}^{m} \) is thus guaranteed by the Schauder fixed point theorem ([Lie96], theorem 8.1.) Let \( \{u_i\}_{i=1}^{m} \) be such a fixed point. Let \( v_m, w_m, \phi_m \) be the solution associated with \( v = \sum_{i=1}^{m} \zeta_i u_i \). from this construction. Define \( u_m = v = \sum_{i=1}^{m} \zeta_i u_i \).

Note that the \( L^p \) norms define the same topologies for velocities \( u \) of the form \( \sum_{i=1}^{m} \zeta_i u_i \). Suppose \( u, v, w \) and \( \tilde{u}, \tilde{v}, \tilde{w} \) are two such solutions. Define

\[
\tilde{u} = u - \tilde{u}, \quad \tilde{v} = v - \tilde{v}, \quad \tilde{w} = w - \tilde{w}, \quad \tilde{\phi} = \phi - \tilde{\phi}.
\]
Note that \( \tilde{v} \) is a solution to the problem

\[
\tilde{v}_t + u \cdot \nabla \tilde{v} + \tilde{u} \cdot \nabla \tilde{v} = \nabla \cdot (\nabla \tilde{v} - v \nabla \tilde{\phi} - \tilde{v} \nabla \hat{\phi})
\]

\[
\frac{\partial \tilde{v}}{\partial \nu} - v \frac{\partial \tilde{\phi}}{\partial \nu} - \tilde{v} \frac{\partial \hat{\phi}}{\partial \nu} = 0, \quad \text{on } \partial \Omega \times (0, \infty), \quad \tilde{v}(x, 0) = 0, \quad x \in \Omega.
\]

Define \( \Phi(t) = \|\tilde{u}, \tilde{v}, \tilde{w}\|_{L^2}^2 \). Multiplying the first equation by \( \tilde{v} \) and integrating by parts we find

\[
\frac{d}{dt} \frac{1}{2} \|\tilde{v}\|_{L^2}^2 + \|\nabla \tilde{v}\|_{L^2}^2 = (\Theta(t), \nabla \tilde{v}).
\]  

(52)

for \( \Theta(t) \) in terms of the two solutions. In each of the bilinear terms of \( \Theta \), there will be one cross term involving the difference of the solutions. Then

\[
\Theta(t) \leq C \Phi(t)
\]

for some constant \( C \) depending only on \( \Omega, m \) and the initial data. We omit the details as they follow in a straightforward way using the equivalence of the \( L^p \) norms on the velocities, (40) and the regularity property of the Poisson equation. An analogous estimate applies to \( w \).

It is also clear from (47), that there is \( C \) depending only on \( \Omega, m \) and the initial data for which

\[
\frac{d}{dt} \|\tilde{u}\|_{L^2}^2 \leq C \|\tilde{u}\|_{L^2}^2 + \|\Xi\|_{L^1}^2
\]  

(53)

where \( \Xi = \Delta \phi \nabla \tilde{\phi} - \Delta \hat{\phi} \nabla \tilde{\phi} \), and another such \( C \) such that

\[
\|\Xi\|_{L^1}^2 \leq C \Phi(t).
\]

Adding (52) to the corresponding inequality for \( w \) and to (53), we have shown that there is \( C \) depending only on \( \Omega \), the initial data and \( m \) for which

\[
\frac{d}{dt} \Phi(t) \leq C \Phi(t)
\]
Since \( \Phi(0) = 0 \), we infer
\[
\sup_{t \in (0, T_0)} \Phi(t) \leq C T_0 \sup_{t \in (0, T_0)} \Phi(t).
\]
This inequality implies \( \sup_{t \in (0, T_0)} \Phi(t) = 0 \) provided \( T_0 \leq N C^{-1} \). The uniqueness is now established.

The \( C^{2+\alpha}(Q_{T_0}) \) regularity of \( v_m \) and \( w_m \) now follows from proposition \( \Pi \) and that fact that \( u_m \in \text{Lip}(Q_{T_0}) \). The regularity of \( u_m \) follows from the easy observation that the right hand side of (47) is Hölder continuously differentiable in \( t \) and that \( D^l \zeta_i \in C^\alpha(Q_{T_0}) \) for all \( l \geq 0 \).

**Lemma 4.** Let \( \Omega \) be an open subset of \( \mathbb{R}^n \), \( n = 2, 3 \). If \( u \in H^2(\Omega) \cap H^1_0(\Omega) \), then
\[
\|\nabla u\|_{L^6} \leq 6^{\frac{2}{3}} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^{\frac{1}{3}}}, \quad \text{dim } \Omega = 2,
\]
\[
\|\nabla u\|_{L^6} \leq 6^{\frac{2}{3}} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^{\frac{2}{3}}}, \quad \text{dim } \Omega = 3.
\]

**Proof.** We prove the 2-dimensional case first. Without loss of generality we may assume \( u \in C_0^\infty(\mathbb{R}^2) \). Let \( v = u_x \). Then \( v \) has compact support. Clearly
\[
v^3(x, y) = 3 \int_{-\infty}^y v^2(x, s)v_y(x, s) \, ds.
\]
Thus,
\[
|v|^3(x, y) \leq 3 \int_{\mathbb{R}} v^2(x, s)|v_y(x, s)| \, ds \equiv f(x).
\]
Similarly,
\[
|v|^3(x, y) \leq 3 \int_{\mathbb{R}} v^2(t, y)|v_x(t, y)| \, dt \equiv g(y).
\]
Then,
\[
\int_{\mathbb{R}^2} v^6(x, y) \, dx \, dy \leq \int_{\mathbb{R}} f(x) \, dx \int_{\mathbb{R}} g(y) \, dy.
\]
However, by Hölder’s inequality and the relation $u = v_x$,

$$
\int_{\mathbb{R}} f(x) \, dx \leq 3 \int_{\mathbb{R}^2} |v|^2 |\nabla v| \, dx \, ds \leq 3 \|u_x\|^2 \|\nabla^2 u\|_{L^3}.
$$

Bounding the integral of $g$ in the same way, and using the interpolation $\|\nabla u\|_{L^6}^2 \leq \|\nabla u\|_{L^2} \|\nabla u\|_{L^6}$, we find

$$
\|u_x\|^6 \leq 9 \|u_x\|^4 \|\nabla^2 u\|_{L^1}^2 \leq 9 \|u_x\|^2 \|\nabla^2 u\|_{L^3}^2 \|\nabla^2 u\|_{L^3}^2
$$
or

$$
\|u_x\|^2 \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^3}.
$$

Arguing similarly with $u_y$, we have, by the triangular inequality,

$$
\|\nabla u\|_{L^6} \leq (\|u_x\|_{L^6}^2 + \|u_y\|_{L^6}^2)^{\frac{1}{2}} \leq \sqrt{6} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^3}.
$$

This proves the first part of the lemma.

Now we consider the 3-dimensional case. For $z \in \mathbb{R}$,

$$
\|\nabla u(\cdot, \cdot, z)\|^3_{L^3(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\nabla u|^3(x, y, z) \, dx \, dy
$$

$$
= 3 \int_{\mathbb{R}^2} \int_{-\infty}^z (|\nabla u| \nabla u \cdot \nabla u_z)(x, y, s) \, ds \, dx \, dy
$$

$$
\leq 3 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} |\nabla u|^2 |\nabla^2 u| \, dx \, dy \, dz \leq 3 \|\nabla u\|_{L^3} \|\nabla^2 u\|_{L^3}.
$$

By what was shown in the 2-dimensional case,

$$
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\nabla u|^6 \, dx \, dy \, dz \leq 72 \int_{\mathbb{R}} \|\nabla u(\cdot, \cdot, z)\|_{L^3(\mathbb{R}^2)}^4 \|\nabla^2 u(\cdot, \cdot, z)\|_{L^3(\mathbb{R}^2)}^2 \, dz.
$$

Consequently,

$$
\|\nabla u\|_{L^6}^6 \leq 72 \cdot 3 \|\nabla u\|_{L^3}^2 \|\nabla^2 u\|_{L^3} \times \int_{\mathbb{R}} \|\nabla u(\cdot, \cdot, z)\|_{L^3(\mathbb{R}^2)} \|\nabla^2 u(\cdot, \cdot, z)\|_{L^3(\mathbb{R}^2)} \, dz
$$

$$
\leq 6^3 \|\nabla u\|_{L^3}^3 \|\nabla^2 u\|_{L^3}^3.
$$
Interpolating $L^3 \subset L^2 \cap L^6$ again,
\[
\|\nabla u\|_{L^6}^6 \leq 6^3 \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^6}^2 \|\nabla^2 u\|_{L^3}^3.
\]

Dividing and taking the appropriate power now gives the second part of the lemma. \hfill \Box

**Lemma 5.** Let $\Omega$ be a smooth bounded subset of $\mathbb{R}^n$ and $p = \frac{n}{n-1}$ and $q = \frac{3}{2}p$. If there is a constant $F$ so that
\[
\|f\|_{L_{\log L}} \leq F
\]
and $\epsilon > 0$, then there is $c_7 = c_7(\Omega, n, \epsilon, F)$ for which
\[
\|f\|_{L^q}^q \leq \epsilon \|\nabla f\|_{L^2}^p + (1 + \|f\|_{L^1}^p) c_7.
\]

**Proof.** Without loss of generality, assume $f$ is nonnegative. Let $E : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$ be a continuous extension with $\text{Supp} \ E f \subset B(R)$ and $\|Ef\|_{L_{\log L}} \leq \tilde{F}$ where $\tilde{F} = \tilde{F}(\Omega, n, F) < \infty$. Denote $\tilde{f} = Ef$. For $M > 0$,
\[
\int_{\Omega} f^q \, dx \leq \int_{\mathbb{R}^n} \tilde{f}^q \, dx \leq |B(R)| M^q + \int_{\mathbb{R}^n} (\tilde{f} - M)^q \, dx
\]

By the Gagliardo-Nirenberg-Sobolev and Hölder inequalities we have
\[
\int_{\mathbb{R}^n} (\tilde{f} - M)^q \, dx \leq C(n) \left( \int_{\{\tilde{f} > M\}} |D\tilde{f}|^{1/2} \, dx \right)^p
\]
\[
\leq C(n) \|D\tilde{f}\|_{L^2(\mathbb{R}^n)}^{p/2} \left( \int_{\{\tilde{f} > M\}} \tilde{f} \, dx \right)^{\frac{q}{2}}.
\]

Note that
\[
\int_{\{\tilde{f} > M\}} \tilde{f} \, dx \leq \frac{1}{\log M} \int_{\{\tilde{f} > M\}} \tilde{f} \log \tilde{f} \, dx \leq \frac{\tilde{F}}{\log M}.
\]
Combining the above and applying the continuity of $E$,

$$\int_{\mathbb{R}^n} (\tilde{f} - M)^q dx \leq C(\Omega, n) \left( \frac{\tilde{F}}{\log M} \right)^{\frac{q}{2}} \|f\|_{H^1}^p.$$  

Certainly, by the Poincare inequality, $\|f\|_{H^1} \leq C(\Omega)(\|Df\|_{L^2} + \|f\|_{L^1}).$

Combining this with the results from above

$$\|f\|_{L^q}^q \leq C(\Omega, n) \left( \frac{\tilde{F}}{\log M} \right)^{\frac{q}{2}} \|Df\|_{L^2}^p + C(\Omega, n, M)(1 + \|f\|_{L^1}^p).$$

Choosing $M$ sufficiently large, the lemma follows. \qed

**Lemma 6** ([GT01], corollary 9.10.). If $\Omega$ is a smooth, bounded, open subset of $\mathbb{R}^n$, $u \in W^{2,p}\cap H^1_0$ and $1 < p < \infty$, then

$$\|\nabla^2 u\|_{L^p} \leq C(\Omega, p)\|\Delta u\|_{L^p}.$$  

**Lemma 7.** If $\dim \Omega = 2$ and there are constants $a_1, a_2, a_3$ so that

$$\|\nabla \phi\|_{L^2}^2 \leq a_1, \quad \|v, w\|_{L^1 \log L} \leq a_2, \quad \|v, w\|_{L^1} \leq a_3,$$

$\phi \in H^2 \cap H^1_0$ solves $\Delta \phi = v - w$ and $\epsilon, \delta > 0$, then there is $c_8 = c_8(\Omega, \epsilon, \delta, a_1, a_2, a_3)$ with

$$\left| \int_{\Omega} \nabla \phi \cdot (v \nabla v - w \nabla w) \, dx \right| \leq \epsilon \|\nabla v, \nabla w\|_{L^2}^2 - \delta \|v, w\|_{L^2}^2 + c_8.$$

**Proof.** By lemma [4] with $\dim \Omega = 2$, and lemma [6] there is $C_1 = C_1(\Omega, a_1)$ for which

$$\|\nabla \phi\|_{L^6} \leq 6^{\frac{1}{2}} \|\nabla \phi\|_{L^2}^\frac{1}{2} \|\nabla^2 \phi\|_{L^3}^\frac{1}{2} \leq C_1 \|v, w\|_{L^3}^\frac{1}{2}.$$  

By lemma [5] with $\dim \Omega = n = 2$, there is $C_2 = C_2(a_2, a_3, c_7)$ so that

$$\|v, w\|_{L^3}^\frac{3}{2} \leq \epsilon_1 \|\nabla v, \nabla w\|_{L^2}^2 + C_2.$$
If $\epsilon_2 > 0$, then by Nash’s inequality, there is $C_3 = C_3(\Omega, \epsilon_2, a_3)$ so that
\[
\|v, w\|_{L^2}^2 \leq \epsilon_2 \|\nabla v, \nabla w\|_{L^2}^2 + C_3.
\]
Combining these, we estimate the trilinear term. By Hölder’s inequality,
\[
\left| \int_{\Omega} \nabla \phi \cdot (v \nabla v - w \nabla w) \, dx \right| \leq |\nabla \phi|_6 \|v, w\|_{L^3} \|\nabla v, \nabla w\|_{L^2}
\leq C_1 \|v, w\|_{L^3}^3 \|\nabla v\|_{L^2}
\leq C_1 (\epsilon_1 \|\nabla v, \nabla w\|_{L^2} + C_2) \|\nabla v, \nabla w\|_{L^2}
\leq (C_1 \epsilon_1 + \epsilon_3) \|\nabla v, \nabla w\|_{L^2}^2 + \epsilon_3^{-1} (2C_1 C_2)^2.
\]
Choose $\epsilon_1$ and $\epsilon_3$ so that $C_1 \epsilon_1 + \epsilon_3 = \epsilon/2$. Also, choose $\epsilon_2 = \delta^{-1} \epsilon/2$. Then
\[
\frac{\epsilon}{2} \|\nabla v, \nabla w\|_{L^2}^2 \leq \epsilon \|\nabla v, \nabla w\|_{L^2}^2 - \epsilon_2 \delta \|\nabla v, \nabla w\|_{L^2}^2
\leq \epsilon \|\nabla v, \nabla w\|_{L^2}^2 - \delta \|v, w\|_{L^2}^2 + \delta C_3.
\]
The lemma follows with $c_8 = \delta C_3 + \epsilon_3^{-1} (2C_1 C_2)^2$.

**Lemma 8.** Let $\dim \Omega = 2$ and let $u_m, v_m, w_m$ be the solution obtained from lemma 3. For $r > 0$, let $\psi(r) = r \log r - r + 1$ and define
\[
W = \int_{\Omega} \psi(v_m) + \psi(w_m) \, dx + \frac{1}{2} \|\nabla \phi_m\|_{L^2}^2 + \frac{1}{2} \|u_m\|_{L^2}^2.
\]
then there is $c_9 = c_9(\Omega, W(0), \|v_0\|_{L^1}, \|w_0\|_{L^1})$ such that if
\[
\|v_0, w_0\|_{L^2}^2 \leq c_9,
\]
then
\[
\sup_{t \in (0, T_0)} \|v_m, w_m\|_{L^2}^2 \leq c_9.
\]
Proof. From lemma \[11\] \(v_m\) and \(w_m\) are positive on \(Q_{T_0}\). Mimicking the calculation of the basic energy law (18), differentiation of \(W\) with respect to \(t\) gives
\[
\frac{dW}{dt} = -\int_\Omega v_m |\nabla \log (v_m e^{-\phi_m})|^2 + w_m |\nabla \log (w_m e^{\phi_m})|^2 + |\nabla u_m|^2 \, dx \leq 0.
\]
for all \(0 \leq t \leq T_0\). Here we used the identity
\[
((u_m)_t, u_m) = -\|\nabla u_m\|^2_{L^2} + (u_m \cdot \nabla \phi_m, \Delta \phi_m).
\]
Recall that the integral of \(v_m\) and \(w_m\) are conserved quantities.
\[
\|v_m\|_{L^1} = \|v_0\|_{L^1}, \quad \|w_m\|_{L^1} = \|w_0\|_{L^1}.
\]
Then (54) implies
\[
\|v_m, w_m\|_{L^\log L} + \|\nabla \phi_m\|^2_{L^2} \leq W(0).
\]
Define \(\omega(t) = \|v_m, w_m\|^2_{L^2}\) and \(\zeta(t) = \|\nabla v_m, \nabla w_m\|^2_{L^2}\). Let \(\epsilon = \delta = \frac{1}{2}\). If \(c_9 = 2c_8\), then by lemma (43) with \(p = 2\),
\[
\frac{d}{dt} \omega + \omega + \zeta \leq c_9
\]
This inequality implies \(\omega(t) \in [0, \max\{c_9, \omega(0)\}]\) for all \(t \in [0, T_0]\). \(\square\)

Proof of theorem 6. Define
\[
c_{10} = \max\{c_9, W(0), \|u_0, v_0, w_0\|^2_{L^2}\}.
\]
The energy decay (54) implies that \(\|u_m\|^2_{L^2}(t) \leq W(t) \leq W(0) \leq c_{10}\) for all \(t \in [0, T_0]\). The conclusion of lemma 8 implies that \(\|v_m, w_m\|^2_{L^2}(t) \leq c_{10}\) for all \(t \in [0, T_0]\). The solution obtained from lemma 3 enjoys
\[
\|u_m, v_m, w_m\|^2_{L^2} \leq 2c_{10}, \quad \forall t \in [0, T_0].
\]
(58)
Returning to lemma 3, the solution may be extended to an interval 
\([0, T_0 + \delta]\) where \(\delta\) depends only on \(c_{10}\). The conclusion of theorem 6 now follows from repeated application of this extension property; the regularity of \(u_m, v_m\) and \(w_m\) follows as in lemma 3 because (42) is satisfied by \(\hat{u}_0 = u_m(T_0), \hat{v}_0 = v_m(T_0), \hat{w}_0 = w_m(T_0)\).

**Lemma 9.** Let \(\dim \Omega = 2\), \(T \in (0, \infty)\) and \(u_m, v_m, w_m\) be the solution obtained from theorem 6. Then there is \(c_{11} < \infty\) depending only on the \(L^2\) norms of \(u_0, v_0, w_0\) and \(W(0)\) but independent of \(m\) so that

\[
\sup_{t \in (0, T)} \|u_m, v_m, w_m\|_{L^2} + \int_0^T \|\nabla u_m, \nabla v_m, \nabla w_m\|_{L^2}^2 \, ds \\
+ \int_0^T \left| \frac{du_m}{dt} \right|_{V^*}^2 + \left| \frac{dv_m}{dt}, \frac{dw_m}{dt} \right|_{H^{-1}}^2 \, ds \leq c_{11}.
\]  

(59)

**Proof.** One can construct a bound of the first two terms in (59) by simply integrating (54), (58) and (57). In particular, the \(L^2\) norms of \(v, w\) are bounded in \(t\) uniformly in \(m\). From (24), with \(p = 2, k = 0\), \(\|f_m\|_{V^*}\) is then also bounded in \(t\) uniformly in \(m\). It is then standard (c.f. [Tem01], theorem 3.1) to check that

\[
\frac{du_m}{dt} \in L^2(0, T; V^*)
\]

with norm independent of \(m\).

It is straightforward to check, using lemma 2 and the Sobolev embedding \(H^1 \subset L^6 \subset L^3\), that

\[
\left\| \frac{dv_m}{dt} \right\|_{H^{-1}} = \sup_{\nu \in H^1, \|\nu\|_1 = 1} (\nabla v_m - v_m \nabla \phi_m - v_m u_m, \nabla \nu) \\
\leq (\|v_m\|_{H^1} + \|u_m\|_{L^6} \|v_m\|_{L^4} + \|v_m\|_{L^3} \|\nabla \phi\|_{L^6}).
\]

Using the Sobolev inequality, we can easily bound the right hand side by

\[
C(\|v_m\|_{H^1} + \|v_m\|_{H^1})
\]
for some constant depending only on $\Omega$ and $c_{10}$. Applying similar reasoning to $w_m$, we see then that
\[
\frac{dv_m}{dt}, \frac{dw_m}{dt} \in L^2(0, T; H^{-1})
\]
with norm independent of $m$. \qed

**Lemma 10.** Let $\dim \Omega = 2$, $T \in (0, \infty)$ and $u_m, v_m, w_m$ be the solution obtained from theorem 6. Let
\[
f_m = \Delta \phi_m \nabla \phi_m.
\]
Then there is $c_{12} < \infty$ depending only on the $L^2$ norms of $u_0, v_0, w_0$ and $W(0)$ but independent of $m$ so that
\[
\sup_{t \in (0, T)} \|f_m\|_{V^*}^2 + \int_0^T \left\| \frac{df_m}{dt} \right\|_{V^*}^2 \, ds \leq c_{12}. \tag{60}
\]

**Proof.** (60) is an easy consequence of (24) and (59). \qed

### 4.1 Proof of Theorem 2

Let $u_0 \in H, v_0, w_0 \in L^2$. Let $\{v_0^\epsilon, w_0^\epsilon\}$ be a sequence of functions satisfying (12) with $v_0^\epsilon, w_0^\epsilon \rightarrow v_0, w_0$ in $L^2$ as $\epsilon \downarrow 0$. Let $T \in (0, \infty)$ and $m$ be a positive integer. Let $u_m^\epsilon, v_m^\epsilon, w_m^\epsilon$ be the solution obtained from theorem 6 with initial data $u_0, v_0^\epsilon, w_0^\epsilon$. The bound (59) is independent of $m$ and $\epsilon$. Applying Lion’s compactness theorem ([Tem01], theorem 2.3), for some subsequence of $\{u_m^\epsilon, v_m^\epsilon, w_m^\epsilon\} \epsilon \downarrow 0$,

\[
u_m^\epsilon \rightarrow v, w \text{ weakly in } L^2(0, T; H^1), \quad \text{weak-* in } L^\infty(0, T; L^2),
\]

and strongly in $L^2((0, T); H^1)$, $L^2((0, T); L^2)$. \(40\)
It is straightforward to check (e.g. [Tem01], section 3.2) additionaly
\[ u_m \cdot \nabla u_m \to u \cdot \nabla u \text{ in } L^1(Q_T), \]
\[ u_m \cdot \nabla u_e, u_m \cdot \nabla w \to u \cdot \nabla v, u \cdot \nabla w \text{ in } L^1(Q_T), \]
\[ (v \nabla \phi, \nabla \nu) \to (v \nabla \phi, \nabla \nu), \quad (w \nabla \phi, \nabla \nu) \to (w \nabla \phi, \nabla \nu), \]
in \( L^1(0,T) \) \( \forall \nu \in H^1. \)

One easily verifies that \( \langle u, v, w, \phi \rangle \) is a global weak solution of (1)-(11) (c.f. [Tem01], Chapter 3.)

Let \( f = \Delta \phi \nabla \phi \) and \( f_m = \Delta \phi_m \nabla \phi_m \). By Fubini’s theorem and the above, the functions \( v_m, w_m \) converge to \( v, w \), in the strong topology of \( L^2(Q_T) \). By regularity of solutions to the Poisson equation, \( \nabla \phi_m \otimes \nabla \phi_m \) converges to \( \nabla \phi \otimes \nabla \phi \) in the strong topology of \( L^2(Q_T) \). Using the characterization (39) and the estimate (60), one verifies through multiplying by test functions that
\[ f \in L^\infty((0,T); V^*), \quad f_t \in L^2((0,T); V^*). \]

\( u \) is a weak solution of (22). Let \( 0 < s \). By proposition (2) we conclude \( u_t \in L^2((s,T); H^1) \cap L^\infty((s,T); L^2) \). Applying the standard regularity results for the Navier-Stokes system to (22), we find \( u \in L^\infty((s,T); H^2) \). We conclude from the Sobolev embedding,
\[ \int_{Q(s,T)} |u_t|^6 + |\nabla u|^6 \, dx dt < \infty. \]

This is sufficient to conclude that
\[ u \in C^\alpha(Q_{s,T}) \]
for some \( \alpha > 0 \). By choosing \( p = 2 \), proposition \( \Pi \) implies that \( v, w \in L^\infty(Q_{s,T}) \). The \( C^{2+\alpha} \) regularity \( v \) and \( w \) follows as in proposition
although here the boundary data on the bottom of $Q_T$ may not be smooth. By [Lie96], Theorem 6.44, $v, w \in \mathcal{C}^\alpha(Q_{s,T})$. Then $\nabla \phi \in \mathcal{C}^\alpha(Q_{s,T})$ as well and from the material in [Lie96], Chapter 5, $v, w \in \mathcal{C}^{2+\alpha}(Q_{s,T})$. Serrin’s result (c.f. [Ser62]), which states that a weak solution of the Navier-Stokes equations in two dimensions with $u_t \in L^\infty((s,T);L^2)$ and $f \in \mathcal{C}^\alpha$ (which is indeed the case here), enjoys $u \in \mathcal{C}^{2+\alpha}(\mathbb{R})$.

5 Entropy and Decay

In [BD00], Biler and Dolbeault studied various convergence estimates for weak solutions of the Debye-Hückel system to the static solution. The trick was to observe that in certain instances entropy production terms can be bounded in terms of a relative entropy. The same observations can be made in the case of the system modeling electrohydrodynamics. Theorem 3, which is modeled after the work of Biler and Dolbeault, follows by proving that the relative entropy of the electro-hydrodynamic system decays at least with a rate depending only on $\Omega$.

Let $v, w \in L^2$ and $\phi \in H^1_0$ be a solution of the Poisson equation (5). Let $\langle v_\infty, w_\infty, \phi_\infty \rangle$ be the stationary solution from Corollary 1. Define the relative entropy

$$W_{rel} \equiv \int_{\Omega} \psi_{v_\infty}(v) + \psi_{w_\infty}(w) \, dx + \frac{1}{2} \|
abla \phi - \nabla \phi_\infty\|^2_{L^2} + \frac{1}{2} \|u\|^2_{L^2}.$$

Assume $\Omega$ is bounded and uniformly convex subset of $\mathbb{R}^n$. We will show that there is a constant $\lambda_1$ depending only on $\Omega$ so that if
\( \langle u, v, w, \phi \rangle \) is global weak solution of (1)-(11), then

\[
\frac{dW_{rel}}{dt} \leq -\lambda W_{rel}, \quad \forall t \in (0, \infty).
\]

(61)

**Lemma 11** (Generalized Csiszar-Kullback Inequality, [UAMT00]).

Let \( v, w \in L \log L \) and \( \phi \in H^1_0 \). Then

\[
\|v - v_\infty\|_{L^1} + \|w - w_\infty\|_{L^1} + \|\nabla \phi - \nabla \phi_\infty\|_{L^2}^2 + \|u\|_{L^2}^2 \leq 4W_{rel}.
\]

(62)

Combining (61) with (62) and setting \( e_1 = 4W_{rel}(0) \), theorem (3) is now proved.

We now prove (61), we need the following lemmas.

**Lemma 12.** Let \( v, w \in L^2 \) and \( \phi \in H^1_0 \) be a solution of the Poisson equation (5). Let

\[
W_{\infty} \equiv \int_\Omega \psi(v_\infty) + \psi(w_\infty) \, dx + \frac{1}{2}\|\nabla \phi_\infty\|_{L^2}^2
\]

and

\[
v_M \equiv \int_\Omega v_0 \, dx \frac{e^\phi}{\int_\Omega e^\phi \, dy}, \quad w_M \equiv \int_\Omega w_0 \, dx \frac{e^{-\phi}}{\int_\Omega e^{-\phi} \, dy}.
\]

Let \( J \) be defined by (32) and \( W \) defined by (13). Then

\[
W_{rel} = W + W_{\infty}
\]

(63)

\[
= \int_\Omega \psi v_M(v) + \psi w_M(w) + \frac{1}{2}|u|^2 \, dx + J[\phi_\infty] - J[\phi].
\]

(64)

**Proof.** Equations (63) and (64) follow from elementary manipulations involving the definition of the Maxwellians \( v_M \) and \( w_M \) and relations (5), (10) and

\[
\int_\Omega v \, dx = \int_\Omega v_\infty \, dx = \int_\Omega v_0 \, dx, \quad \int_\Omega w \, dx = \int_\Omega w_\infty \, dx = \int_\Omega w_0 \, dx.
\]

\( \square \)
Lemma 13 ([BD00]). Let $\Omega$ be a bounded, uniformly convex subset of $\mathbb{R}^n$. There exists $c_{13}$ depending only on $\Omega$ so that for all $v, w \in L\log L$, $\phi \in H^1_0$, 
\[ \int_{\Omega} \psi_{vM}(v) + \psi_{wM}(w) \, dx \leq c_{13} \int_{\Omega} v\nabla \log(v e^{-\phi})^2 + w\nabla \log(w e^\phi)^2 \, dx. \]

Proof of (61). Note that $W_\infty$ is independent of $t$. Thus, if $\langle v, w, \phi, u \rangle$ is a global weak solution of (11)-(11), we have, according to (18) and (63), 
\[ \frac{dW_{\text{rel}}}{dt} = \frac{dW}{dt} = -\int_{\Omega} v |\nabla \log(v e^{-\phi})|^2 + w |\nabla \log(w e^\phi)|^2 + |\nabla u|^2 \, dx. \]

By the lemma (13), 
\[ \frac{dW_{\text{rel}}}{dt} \leq -c_{13} \int_{\Omega} v \log\left( \frac{v}{v_M} \right) + w \log\left( \frac{w}{w_M} \right) \, dx - \|\nabla u\|_{L^2}^2. \]

By the Poincaré inequality, there is $C$ depending only on $\Omega$ for which 
\[ \frac{dW_{\text{rel}}}{dt} \leq -c_{13} \int_{\Omega} v \log\left( \frac{v}{v_M} \right) + w \log\left( \frac{w}{w_M} \right) \, dx - C \|u\|_{L^2}^2. \]

Let $\lambda_1 = \min\{c_{13}, C\}$. Then, applying (64), 
\[ \frac{dW_{\text{rel}}}{dt} \leq -\lambda_1 (W_{\text{rel}} + J[\phi] - J[\phi_\infty]). \]

Since $\phi_\infty$ is a minimum of $J$ (see proof of theorem 1), the difference of the last two terms is positive. This gives (61).

6 The 3 Dimensional Case: Small Data

For domains with more general geometry, the techniques from the previous section are difficult to apply. The main reason for the difficulty
is that $W_{rel}$ is not quadratic and it is not clear to what extent the logarithmic Sobolev inequality of lemma (13) depend in the domain geometry.

To remedy this difficulty, we study a linearization of the relative entropy $W_{rel}$ about the stationary solution $\langle v_\infty, w_\infty, \phi_\infty \rangle$. We are able to show that the linearization satisfies a decay estimate similar to (61). This approach is successful because $W_{rel}$ is locally quadratic about $\langle v_\infty, w_\infty, \phi_\infty \rangle$.

In order to construct the linearization of the relative entropy $W_{rel}$ about the stationary solution $\langle v_\infty, w_\infty, \phi_\infty \rangle$, consider an expansion of $\psi_r(s) = s \log(s/r) - s + r$ for $s \geq 0, r > 0$. By Taylor’s theorem,

$$
\psi_r(s) = \psi_r(r) + (s - r)\psi'_r(r) + \frac{1}{2}(s - r)^2\psi''_r(r) + O((s - r)^3)
$$

$$
= \frac{1}{2} \frac{(s - r)^2}{r} + O((s - r)^3).
$$

If $s = v(x,t)$ (resp. $w(x,t)$) and $r = v_\infty(x)$ (resp. $w_\infty(x)$), the leading order term in $v, v_\infty, w, w_\infty$ of the integrand of $W_{rel}$ is

$$
\frac{1}{2} \frac{(v - v_\infty)^2}{v_\infty} + \frac{1}{2} \frac{(w - w_\infty)^2}{w_\infty}.
$$

Motivated by this observation, we define

$$
L(t) \equiv \int_{\Omega} \frac{1}{2} |u|^2 + \frac{(v(t) - v_\infty)^2}{2v_\infty} + \frac{(w(t) - w_\infty)^2}{2w_\infty} + |\nabla(\phi(t) - \Phi)|^2 \, dx.
$$

(65)

**Lemma 14** (Weighted Poincaré Inequality). Let $\Omega$ be a connected, open subset of $\mathbb{R}^n$ and $0 < a \leq b < \infty$. Then there exists $c_{14} = c_{14}(a,b,\Omega)$ such that if $\rho \in H^1$ satisfies

$$
a \leq \rho(x) \leq b, \text{ a.e. } x \in \Omega
$$

45
and \( f \in H^1 \) satisfies
\[
\int_{\Omega} f \, dx = 0,
\]
then
\[
\int_{\Omega} f^2 \, dx \leq c_{14} \int_{\Omega} |\nabla (f \rho)|^2 \, dx.
\]
If \( \rho \) is merely positive and \( \rho^{-1} \) is integrable, then there is \( c_{14} = c_{14}(\rho, \Omega) \) for which the same conclusion holds.

**Proof.** Suppose that no such constant exists. Then there is a sequence of functions \( \{f_i\}_{i=1}^{\infty} \) and \( \{\rho_i\}_{i=1}^{\infty} \) in \( H^1(\Omega) \) with
\[
\int_{\Omega} f_i \, dx = 0, \quad \|f_i\|_{L^2(\Omega)}^2 \geq i \int_{\Omega} |\nabla (f_i \rho_i)|^2 \, dx.
\]
Let \( h_i = f_i/\|f_i\|_{L^2(\Omega)} \) so that
\[
1 = \|h_i\|_{L^2(\Omega)}^2 \geq i \int_{\Omega} |\nabla (h_i \rho_i)|^2 \, dx.
\]
Write \( g_i = h_i \rho_i \). Then
\[
\int_{\Omega} g_i^2 \, dx = \int_{\Omega} h_i^2 \rho_i^2 \, dx \leq b^2
\]
shows that \( g_i \) is bounded in \( H^1(\Omega) \) and thus converges weakly to an element \( g \in H^1(\Omega) \). Fatou’s lemma,
\[
\int_{\Omega} |\nabla g|^2 \, dx \leq \liminf_{i \to \infty} \int_{\Omega} |\nabla g_i|^2 \, dx \leq \lim_{i \to \infty} \frac{1}{i} = 0
\]
and the connectedness of \( \Omega \) shows that \( g(x) = G \) a.e. for some constant \( G \). Moreover, \( \rho_i^{-1} \) are uniformly bounded in \( L^2 \) and so we may extract a subsequence (reindexed by \( i \)) converging weakly to \( \sigma \) in \( L^2 \) with \( b^{-1} \leq \sigma \leq a^{-1} \) a.e. Clearly,
\[
\lim_{i \to \infty} \int_{\Omega} \rho_i^{-1} g_i \, dx = \int_{\Omega} \sigma g \, dx.
\]
Then
\[ G \int_{\Omega} \sigma \, dx = \int_{\Omega} \sigma g \, dx = \lim_{i \to 0} \int_{\Omega} \rho_i^{-1} g \, dx \]
\[ = \lim_{i \to 0} \int_{\Omega} h_i \, dx = \lim_{i \to 0} \frac{1}{\|f_i\|_{L^2(\Omega)}} \int_{\Omega} f_i \, dx = 0. \]

We infer \( G = 0 \) since \( \rho^{-1} \) is nonzero on a set of positive measure. The contradiction with (66) gives the existence of \( c_{14} = c_{14}(a, b, \Omega) \).

The existence of \( c_{14} = c_{14}(\rho, \Omega) \) follows from the above proof by setting \( \rho_i = \rho \) for each \( i \).

\[ \square \]

**Lemma 15.** Let \( \Omega \) be a bounded subset of \( \mathbb{R}^n \) with smooth boundary. Then there exists positive constants \( c_{15}, c_{16} \) and \( c_{17} \) depending only on \( \Omega, v_\infty \) and \( w_\infty \) such that if \( \langle u, v, w, \phi \rangle \) is a weak solution of (1)–(10) on \( Q_T \) then
\[ \frac{dL}{dt} \leq -c_{15} L + c_{16} L^2 + c_{17} \| \nabla (\phi - \phi_\infty) \|_{L^2}^2, \]
a.e. \( t \in (0, T) \).

(67)

**Remark 5.** The following proof applies equally well to the modified Galerkin approximation from lemma 3.

**Proof.** For the sake of clarity, we will assume that \( w \equiv w_\infty \equiv 0 \). The general case requires only minor modifications. Define
\[ e = v - w_\infty, \quad \psi = \phi - \phi_\infty. \]

Using (5) and (30), we compute
\[ \frac{dL}{dt} = \int_{\Omega} u \cdot u_t + \frac{e}{v_\infty} e_t + \nabla \psi \cdot \nabla \psi_t \, dx \]
\[ = \int_{\Omega} -|\nabla u|^2 + u \cdot \nabla \phi \Delta \phi \, dx + \int_{\Omega} \left( \frac{e}{v_\infty} - \psi \right) v_t \, dx = A + B. \]
Write
\[ A_1 = \int_\Omega -|\nabla u|^2 \, dx, \quad A_2 = \int_\Omega u \cdot \nabla \phi \Delta \phi \, dx. \]

Note that by corollary 1, \( \nabla v_\infty = v_\infty \nabla \phi_\infty \). Note also that
\[ \nabla v = v_\infty \nabla \left( \frac{e}{v_\infty} \right) + v \nabla \phi_\infty. \]  
(68)

Using the fact that \( v \) is a weak solution,
\[ B = -\int_\Omega \nabla \left( \frac{e}{v_\infty} - \psi \right) \cdot (\nabla v - v \nabla \phi - vu) \, dx. \]

Then, computing with (68),
\begin{align*}
B &= -\int_\Omega v_\infty \left| \nabla \left( \frac{e}{v_\infty} \right) \right|^2 \, dx \\
&\quad + \int_\Omega e \nabla \left( \frac{e}{v_\infty} \right) \cdot \nabla \psi + 2v_\infty \nabla \left( \frac{e}{v_\infty} \right) \cdot \nabla \psi \, dx \\
&\quad - \int_\Omega v |\nabla \psi|^2 \, dx \\
&\quad + \int_\Omega \left\{ v \nabla \left( \frac{e}{v_\infty} \right) - v \nabla \psi \right\} \cdot u \, dx \\
&= B_1 + B_2 + B_3 + B_4.
\end{align*}

We proceed by bounding \( A_2, B_2, B_3 \) and \( B_4 \) in terms of \( A_1 \) and \( B_1 \)
and integrals of higher powers of \( |e| \) and \( |\nabla \psi| \).

Remark 6. The presence of transport of the charges by the velocity make the following calculation somewhat more subtle than the analogous analysis for the Debye Hückel system, c.f. [BAMV04]. As will be demonstrated immediately below, the net exchange of kinetic energy \( \frac{1}{2} |u|^2 \) and the relative energy \( \frac{1}{2v_\infty} e^2 \) is a second order contribution. This is in agreement with the cancelation of the entropy production due to transport and the kinetic energy production due to forcing seen in the derivation of the basic energy law (18).
We have

\[ A_2 + B_4 = \int_{\Omega} \mathbf{u} \cdot \left( \Delta \phi \nabla \phi + \Delta \phi \nabla \left( \frac{e}{v_\infty} \right) - \Delta \phi \nabla \psi \right) \, dx. \]

Adding and subtracting \( \Delta \phi \nabla \left( \frac{e}{v_\infty} \right) \) and using the relation \( v = \Delta \phi, v_\infty = \Delta \phi_\infty \),

\[ A_2 + B_4 = \int_{\Omega} \mathbf{u} \cdot \left( \Delta \phi \nabla \phi_\infty + \Delta \phi_\infty \nabla \left( \frac{e}{v_\infty} \right) + e \nabla \left( \frac{e}{v_\infty} \right) \right) \, dx. \]

Finally, using the relation (68),

\[ A_2 + B_4 = \int_{\Omega} \mathbf{u} \cdot \left( \nabla v + e \nabla \left( \frac{e}{v_\infty} \right) \right) \, dx. \]

Because \( \mathbf{u} \) is divergence free and so is orthogonal to \( \nabla v \) in \( L^2 \), the first product in the integrand vanishes. By Young’s inequality,

\[ A_2 + B_4 \leq -a_1 B_1 + a_2 \int_{\Omega} |\mathbf{u}|^2 e^2 \, dx \quad (69) \]

where \( a_1 \) and \( a_2 \) are positive and \( a_1 a_2 = 4 \).

The term \( B_3 \) is nonpositive.

To estimate \( B_2 \) we first note that \( v_\infty \) is bounded. Thus there exists a \( C_2 \) depending only on \( v_\infty \) for which

\[ B_2 \leq -(b_1 + 2d_1) B_1 + b_2 \int_{\Omega} e^2 |\nabla \psi|^2 \, dx + 2C_2 d_2 \int_{\Omega} |\nabla \psi|^2 \, dx. \quad (70) \]

where \( b_1, b_2, d_1, d_2 \) are positive and \( b_1 b_2 = d_1 d_2 = 4 \).

Let \( \rho = \frac{1}{v_\infty} \) and \( f = e \). By corollary 1, \( \rho \) satisfies the first hypothesis of lemma 14 and the integral of \( e \) is zero, satisfying the second hypothesis of the lemma. Hence there is a constant \( c_{14} \) depending only on \( v_\infty \) and \( \Omega \) for which

\[ \int_{\Omega} e^2 \, dx \leq c_{14} \int_{\Omega} \left| \nabla \left( \frac{e}{v_\infty} \right) \right|^2 \, dx. \]
Then, by corollary 1, there is \( C_3 = C_3(v_\infty, c_{14}) \) for which
\[
\int_\Omega \frac{2e^2}{v_\infty} \, dx \leq C_3 \int_\Omega v_\infty \left| \nabla \left( \frac{e}{v_\infty} \right) \right|^2 \, dx = -C_3 B_1. \tag{71}
\]

Similarly, since \( \psi \in H^1_0 \) is a solution of the Poisson equation with right hand side \( e \), there is also a \( C_4 = C_4(v_\infty, \Omega, c_{14}) \) for which
\[
\int_\Omega 1/2 |\nabla \psi|^2 \, dx \leq -C_4 B_1. \tag{72}
\]

Finally, by the Poincaré inequality, there is \( C_5 = C_5(\Omega) \) for which
\[
\int_\Omega 1/2 |u|^2 \, dx \leq -C_5 A_1. \tag{73}
\]

Adding (71), (72) and (73) together, we have shown that there is a positive constant \( C_6 = C_6(v_\infty, \Omega, c_{14}) \) for which
\[
L \leq -C_6(A_1 + B_1). \tag{74}
\]

Arguing in a similar fashion, there is clearly a \( C_7 = C_7(\Omega, v_\infty, a_2, b_2) \) for which
\[
\int_\Omega a_2 |u|^2 e^2 + b_2 v_\infty |\nabla \psi|^2 \, dx \leq C_7 L^2. \tag{75}
\]

By (69), (70) and (75), we have
\[
\frac{dL}{dt} = A_1 + A_2 + B_1 + B_2 + B_3 + B_4 \leq A_1 + (1 - a_1 - b_1 - 2d_1)B_1 + 2C_2d_2 \int_\Omega |\nabla \psi|^2 \, dx + C_7 L^2.
\]

Choose \( a_1, b_1 \) and \( d_1 \) so that \( a_1 - b_1 - 2d_2 = \frac{1}{2} \), thereby fixing \( a_2, b_2, d_2, C_7 \). Then, using (74) and the fact that \( A_1 + B_1 \leq 0 \),
\[
\frac{dL}{dt} \leq -\frac{1}{2C_6} L + C_7 L^2 + 2C_2d_2 \int_\Omega |\nabla \psi|^2 \, dx.
\]

Setting \( c_{15} = \frac{1}{2C_6}, c_{16} = C_7 \) and \( c_{17} = 2C_2d_2 \), the lemma is now proved. \( \square \)
From theorem \(1\) there is \(\rho_1 > 0\) so that for all \(0 < \rho_0 < \rho_1\),
\[
|\phi_\infty(x)| \leq 1, \quad \forall x \in \Omega.
\]

Following the proof of lemma \((15)\) and using corollary \(1\) one may keep track of the constants \(c_{15}, c_{16}\) and \(c_{17}\) to prove

**Corollary 2.** There exist positive constants \(c'_{15}, c'_{16}, c'_{17}\) and \(\rho_1\) depending only on \(\Omega\) such that if \(\rho_0 < \rho_1\) then
\[
\frac{dL}{dt} \leq -c'_{15}L + \rho_0 c'_{16}L^2 + \rho_0 c'_{17} \|\nabla(\phi - \phi_\infty)\|_{L^2},
\forall t \in [0, T_0].
\]

6.1 Proof of theorem \(4\)

We are assuming \(\Omega \subset \mathbb{R}^n, n = 2, 3\) is bounded with smooth boundary. We will first use lemma \((15)\) to prove an extension property analogous to theorem \((6)\).

Let \(u_0 \in H, v_0, w_0 \in L^2\). Let \(\{v_h^0, w_h^0\}\) be a sequence of functions satisfying \((42)\) with \(v_h^0, w_h^0 \rightarrow v_0, w_0\) in \(L^2\) as \(h \downarrow 0\). Let \(\langle u_m^h, v_m^h, w_m^h, \phi_m^h \rangle\) be the local modified-Galerkin approximate solution on \(Q_{T_0}\) obtained from lemma \((3)\) with initial data \(u_0, v_0^h, w_0^h\).

Let \(\rho_1\) be the constant from corollary \(2\). Since \(\|\nabla(\phi_m^h - \phi_\infty)\|^2_{L^2} \leq 2L\), we may choose
\[
\rho_2 = \frac{c_{15}}{8c_{17}}
\]
so that if \(\rho_0 < \min\{\rho_1, \rho_2\}\) then
\[
\frac{dL}{dt} \leq -\frac{3c_{15}}{4}L + \rho_0 c'_{16}L^2, \quad \text{a.e.} \ t \in [0, T_0].
\]

(see remark \((5)\)) Finally, let
\[
\delta = \frac{c'_{15}}{4\rho_0 c'_{16}}.
\]
Then

\[ L(0) < \delta \text{ implies } L(t) < \delta e^{-\frac{t\lambda_2}{2}}, \quad \text{a.e. } t \in [0, T_0]. \quad (76) \]

This inequality implies that the \( L^2 \) norm of \( u^h_m, v^h_m, w^h_m \) remains bounded independently of \( t \). The extension property now follows exactly as in theorem 6. One checks (just as in the proofs of estimates (59) and (60)) that there are constants \( c_{18}, c_{19} \) independent of \( m \) and \( h \) for which

\[
\sup_{t \in (0, T)} \| u^h_m, v^h_m, w^h_m \|_{L^2} + \int_0^T \| \nabla u^h_m, \nabla v^h_m, \nabla w^h_m \|_{L^2}^2 \, ds
\]

\[
+ \int_0^T \left\| \frac{du^h_m}{dt} \right\|_{V^*}^{\frac{1}{2}} + \left\| \frac{dv^h_m}{dt}, \frac{dw^h_m}{dt} \right\|_{H^{-1}}^{\frac{1}{2}} \, ds \leq c_{18},
\]

and

\[
\sup_{t \in (0, T)} \| f^h_m \|_{V^*}^2 + \int_0^T \left\| \frac{df^h_m}{dt} \right\|_{V^*}^{\frac{1}{2}} \, ds \leq c_{19}
\]

where

\[ f_m = \Delta \phi^h_m \nabla \phi^h_m. \]

Letting \( h \to 0, m \to \infty \) we see that some subsequence of \( \langle u^h_m, v^h_m, w^h_m, \phi^h_m \rangle \) converges to a global weak solution \( \langle u, v, w, \phi \rangle \) of (11)-(11). This proves the first part of theorem 4.

Note that \( \varepsilon_2 = 2L \). From (76), let \( \epsilon_2 = 2\delta \). If \( \varepsilon_2(0) < \epsilon_2 \), then \( L(0) < \delta \) and we may apply (76) for a.e. \( t \in [0, T] \). Let \( \lambda_2 = \frac{\epsilon_1}{2} \) which depends only on \( \Omega \). Then the global weak solution \( \langle u, v, w, \phi \rangle \) satisfies

\[ \varepsilon_2 = 2L \leq 2\delta e^{-t\lambda_2}, \quad \text{a.e. } t \in [0, T]. \]

The last part of the theorem now follows immediately from the following lemma. Using the embedding \( V \subset H \), minor modifications of the proof of [BHN94], theorem 6 gives
Lemma 16. Let $\Omega \subset \mathbb{R}^n, n = 2, 3$ be a bounded, open set with smooth boundary. If $\langle u, v, w, \phi \rangle$ satisfy (5), (10),

$$\sup_{t \in [0, \infty)} \|v, w\|_{L^2} < \infty,$$

$$\sup_{t \in [0, \infty)} \int_{Q_t} v|\nabla \log(v e^{-\phi})|^2 + w|\nabla \log(w e^\phi)|^2 + |\nabla u|^2 \, dx < \infty,$$

then there is some sequence $t_j \to \infty$ for which

$$\lim_{j \to \infty} L(t_j) = 0.$$

6.2 Proof of theorem [5]

Let $u_0 \in V \cap H^2$, $v_0, w_0 \in H^2$ and $T > 0$. Let $\{v^h_0, w^h_0\}$ be a sequence of functions satisfying (42) with $v^h_0, w^h_0 \to v_0, w_0$ in $H^2$ as $h \downarrow 0$. Let $\langle u^h_m, v^h_m, w^h_m, \phi^h_m \rangle$ be the local modified-Galerkin approximate solution on $Q_{T_0}$ obtained from lemma 3 with initial data $u_0, v^h_0, w^h_0$. Let $\rho_2$ and $\epsilon_2$ be the constants from theorem 4. Assume that $\rho_0 < \rho_3$ and $\epsilon_2 < \epsilon_3$ where $\rho_3 < \rho_2$ and $\epsilon_3 < \epsilon_2$ will be determined below. Applying the results from theorem 4, $\langle u^h_m, v^h_m, w^h_m, \phi^h_m \rangle$ is defined for all $t \in [0, T]$ and some subsequence converges to a global weak solution of (11) as $m \to \infty$ and $h \to 0$. In the sequel we suppress the sub- and superscripts $m$ and $h$.

From lemma 3 we infer that $v_t$ and $w_t$ are smooth in $x$. In particular, $v_t$ and $w_t$ are classical solutions of

$$v_{tt} + u_t \cdot \nabla v + u \cdot \nabla v_t = \nabla \cdot (\nabla v_t - v_t \nabla \phi - v \nabla \phi_t),$$

$$w_{tt} + u_t \cdot \nabla w + u \cdot \nabla w_t = \nabla \cdot (\nabla w_t + w_t \nabla \phi + w \nabla \phi_t).$$

Multiplying these equations by $v_t$ and $w_t$ and integrate over $\Omega$. Integrating by parts and noting that the boundary terms vanish in this
case as well,

$$\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial v}{\partial t} \right\|_{L^2}^2 + \left\| \nabla \frac{\partial v}{\partial t} \right\|_{L^2}^2 \leq \left( \Phi, \nabla \frac{\partial v}{\partial t} \right)$$

where

$$\Phi = \frac{\partial u}{\partial t} v + \nabla \frac{\partial \phi}{\partial t} v + \nabla \frac{\partial \phi}{\partial t}.$$ 

We estimate the $L^2$ norm of $\Phi$. From (8), the integral of $v_t$ vanishes. By the Sobolev embedding $H^1 \subset L^4$ and the Poincare inequality, the estimate

$$\|\Phi\|_{L^2}^2 \leq C \left( \left\| \nabla \frac{\partial v}{\partial t}, \nabla \frac{\partial v}{\partial t} \right\|_{L^2}^2 + \left\| \nabla \frac{\partial u}{\partial t} \right\|_{L^2}^2 \right) \|v, w\|_{H^1}^2$$

for some $C = C(\Omega)$ is straightforward. The analogous estimate holds for $w_t$.

Now differentiate (47) with respect to $t$ and multiply the resulting system componentwise by $\dot{u}_i$ and add the equations for $i = 1, \ldots, m$. One concludes

$$\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial u}{\partial t} \right\|_{L^2}^2 + \left\| \nabla \frac{\partial u}{\partial t} \right\|_{L^2}^2 = b \left( \frac{\partial u}{\partial t}, u, \frac{\partial u}{\partial t} \right) + \left( \Psi, \nabla \frac{\partial v}{\partial t} \right)$$

where

$$\Psi = \nabla \phi \otimes \nabla \frac{\partial \phi}{\partial t} + \nabla \frac{\partial \phi}{\partial t} \otimes \nabla \phi.$$ 

The usual estimate in the small data regularity proof for Navier-Stokes shows that

$$b \left( \frac{\partial u}{\partial t}, u, \frac{\partial u}{\partial t} \right) \leq \left\| \nabla \frac{\partial u}{\partial t} \right\|_{L^2}^2 \left\| \nabla u \right\|_{L^2}^2.$$ 

Using the embedding $H^{-1} \hookrightarrow H^1$ from the Poisson equation, there is $C = C(\Omega)$ for which

$$\|\Psi\|_{L^2}^2 \leq C \left\| \nabla \frac{\partial v}{\partial t}, \nabla \frac{\partial v}{\partial t} \right\|_{L^2}^2 \|v, w\|_{H^1}^2.$$
Define
\[
G(t) = \left\| \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \right\|_{L^2}^2, \\
H(t) = \left\| \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \right\|_{H^1}^2, \\
I(t) = \| u, v, w \|_{H^1}^2.
\]

The above estimates and the Poincaré inequality show that
\[
\frac{d}{dt} G(t) + H(t)(1 - I(t)) \leq 0, \quad \forall \ t \in [0, T].
\]

Now we develop a relationship between \( I(t) \) and \( G(t) \). From (47),
\[
2\| \nabla u \|_{L^2}^2 = -2(\nabla \phi \otimes \nabla \phi, \nabla u) - 2 \left( \frac{\partial u}{\partial t}, u \right) \\
\leq \| \nabla u \|_{L^2}^2 + \| \nabla \phi \|_{L^4}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2} \| u \|_{L^2}.
\]

Define
\[
c_{20} = \sup_{t \in (0, T)} \| \nabla \phi \|_{L^4}^2, \quad c_{21} = \sup_{t \in (0, T)} \| u \|_{L^2}.
\]

Then
\[
\| \nabla u \|_{H^1} \leq c_{20} + c_{21} G(t)^{1/2}, \quad \forall \ t \in [0, T].
\]

Similarly, by (8) and the triangular inequality,
\[
\| v \|_{H^1}^2 \leq 2 \left( \int_\Omega v_0 \, dx \right)^2 + 3\| \nabla v \|_{L^2}^2 \\
= 2\rho_0^2 - 3 \left( \frac{\partial v}{\partial t}, v \right) + 3(v \nabla \phi, \nabla v) \\
\leq 2\rho_0^2 + 3 \left\| \frac{\partial v}{\partial t} \right\|_{L^2} \| v \|_{L^2} + C_1 c_{20} \| v \|_{H^1}^2
\]
for some \( C_1 = C_1(\Omega) \). Applying a similar estimate the \( w \), we find
\[
\| v, w \|_{H^1}^2 \leq \frac{1}{1 - C_1 c_{20}} \left( 4\rho_0^2 + c_{22} \left\| \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \right\|_{L^2}^2 \right)
\]
where
\[ c_{22} = 3 \sup_{t \in (0, T)} \| v, w \|_{L^2}. \]

Define
\[ c_{23} = \frac{4\rho_0^2}{1 - C_1c_{20}} + c_{20}, \quad c_{24} = \frac{c_{22}}{1 - C_1c_{20}} + c_{21}. \]

We have shown that
\[ I(t) \leq c_{23} + c_{24}G(t)^{\frac{1}{2}}. \]

Thus,
\[ \frac{d}{dt}G(t) + H(t) \left( 1 - c_{23} - c_{24}G(t)^{\frac{1}{2}} \right) \leq 0, \quad \forall t \in [0, T]. \]

Assume for the moment that \( \rho_3, \epsilon_3 \) and \( \delta_3 \) can be chosen so that
\[ 1 - c_{23} - c_{24}G(0)^{\frac{1}{2}} > 0. \] (77)

independently of \( h \) and \( m \). If follows that
\[ G(t) + \int_0^t H(s) \, ds \leq G(0), \quad t \in [0, T] \] (78)

for all \( m \) and \( h \). Letting \( h \to 0 \) and \( m \to \infty \) we may extract a subsequence of \( \langle u_h^{m}, v_h^{m}, w_h^{m}, \phi_h^{m} \rangle \) which converges to a global weak solution \( \langle u, v, w, \phi \rangle \) of (1)-(11). From (78), this solution satisfies the estimate
\[ u \in L^\infty((0, T); H^2), \quad u_t \in L^\infty((0, T); L^2). \]

Arguing as in the end of the proof of theorem (2), we find that \( u \) is Hölder continuous on \( Q_T \) and \( C^{2+\alpha} \) on compact subsets of \( Q_T \) and \( v, w \) are \( C^{2+\alpha} \) on \( Q_{s,t} \) for any \( 0 < s < t \leq T \). The standard arguments show that \( \langle u, v, w, \phi \rangle \) is unique, c.f. [Tem01].

Now we show that \( \rho_3, \epsilon_3 \) and \( \delta_3 \) may be chosen in order that (77) be satisfied. Note that (77) holds provided \( \rho_0, c_{20}, c_{21} \) and \( c_{22} \) are sufficiently small and \( G(0) \) is sufficiently small with respect to \( \frac{c_{24}}{c_{23}} \). 

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By the regularity of solutions to the Poisson equation, there is $C_2 = C_2(\Omega)$ so that
\[ c_{20} \leq C_2 c_{22}^2. \]

By the triangular inequality,
\[ c_{22} \leq \sup_{t \in [0,T]} 3\{ \|v - v_\infty, w - w_\infty\|_{L^2} + \|v_\infty, w_\infty\|_{L^2} \} \]
\[ \leq \sup_{t \in [0,T]} C_3 \rho_0 (\sqrt{L} + 1) \]
for some $C_3 = C_3(\Omega)$, provided $\rho_0 < \rho_1$. Finally,
\[ c_{21} \leq 2 \sup_{t \in [0,T]} \sqrt{L}. \]

By assumption, $\rho_0 < \rho_3 < \rho_2$ and $\delta_2 < \epsilon_3 < \epsilon_2$. By theorem 4, we have then that
\[ \sup_{t \in [0,T]} L = \sup_{t \in [0,T]} \frac{1}{2} \delta_2 < \epsilon_3. \]

Using the above bounds on $c_{20}, c_{21}$ and $c_{22}$ in terms of $L$ and the bound on $L$ in terms of $\epsilon_3$, we may choose $\epsilon_3$ and $\rho_3$ so that
\[ c_{23} \leq \frac{4 \rho_0^2}{1 - C_1 C_2 C_3^2 \rho_0^2 (\epsilon_3 + 1)} + C_2 C_3^2 \rho_0^2 (\epsilon_3 + 1) < \frac{1}{2}, \]
\[ c_{24} \leq \frac{C_3 \rho_0 (\sqrt{\epsilon_3} + 1)}{1 - 6C_1 C_2 C_3 \rho_0 (\sqrt{\epsilon_3} + 1)} + 2 \sqrt{\epsilon_3} < 1, \]
\[ G(0) \leq \frac{1}{4}. \]

This implies (77). Certainly there exists $C_4 = C_4(\Omega)$ so that
\[ G(0) \leq C_4 \|u_0, v_0, w_0\|_{H^2}. \]

Setting $\delta_3 = \frac{1}{\sqrt{4C_4}}$ now completes the proof.
7 Conclusion

The equations of a viscous, incompressible fluid coupled with diffuse charges in two and three dimensions have been studied. The key step toward the existence of global in time solutions is the presence of a decaying entropy function which guarantees the dissipation of kinetic and electrostatic energy and entropy.

The most serious obstruction to formulating a global existence result like theorem 2 when dimΩ = 3 is the Debye-Hückel system. A different approach is to consider very weak solutions, i.e. those where $v, w \in L^\infty((0,T);L^1)$ satisfy the weak formulation in terms of test functions $\omega \in C^1(\Omega)$. How one defines the forcing term $\Delta \phi \nabla \phi$ and a solution of (5) then becomes a more delicate matter.

The techniques used in this paper certainly apply to other Dirichlet conditions than (10) and other 2nd order elliptic operators than the Laplacian. A future avenue of study are electrorheological fluids where the charge is vector valued and the potential is the polarization potential, see [ZGL+08].

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