THE HARMONIC THEORY ON A VECTOR BUNDLE WITH SINGULAR HERMITIAN METRICS AND POSITIVITY

Jingcao Wu

Abstract. Let $E$ be a holomorphic vector bundle endowed with a singular Hermitian metric $H$. In this paper, we develop the harmonic theory on $(E, H)$. Then we extend several canonical results of J. Kollár and K. Takegoshi to this situation. In the end, we generalise Nakano’s vanishing theorem.

1. Introduction

This is a continuation of our work [39] about the singular Hermitian metric on a holomorphic vector bundle.

Let $E$ be a holomorphic vector bundle of rank $r + 1$ over a compact Kähler manifold $(Y, \omega)$ of dimension $n$. Let $X := \mathbb{P}(E^\ast)$ be the projectivised bundle with the natural projection $\pi : X \to Y$ and the tautological line bundle $\mathcal{O}_E(1) := \mathcal{O}_X(1)$. Let $\Omega$ be a Kähler metric on $X$. We proposed an alternative definition for the singular Hermitian metric $H$ on $E$ in [39], and showed that this type of singular metric differs slightly with the one defined in [36]. Moreover, it has good nature to help us to define the Griffiths and Nakano positivities. The goal of this paper is to develop the harmonic theory on $(E, H)$.

We first briefly recall the canonical harmonic theory when $H$ is smooth. In this context, the adjoint operator $\bar{\partial}^\ast$ of the $\bar{\partial}$ operator with respect to the $L^2$-norm $\| \cdot \|_{H, \omega}$ is defined as

$$
\bar{\partial}^\ast := \ast \partial H \ast,
$$

where $\ast$ is the Hodge $\ast$-operator defined by $\omega$ and $\partial_H$ is the $(1, 0)$-part of the Chern connection associated with $H$. Then the $\bar{\partial}$-Laplacian operator [19], defined as

$$
\Box = \bar{\partial} \bar{\partial}^\ast + \bar{\partial}^\ast \bar{\partial} : A^{p,q}(Y, E) \to A^{p,q}(Y, E),
$$

is a self-adjoint elliptic operator. Here $A^{p,q}(Y, E)$ is the collection of all the smooth $E$-valued $(p, q)$-forms on $Y$. Thus, the eigenform of $\Box$ with eigenvalue zero is called a harmonic form, and the harmonic space is defined as

$$
\mathcal{H}^{p,q}(Y, E) := \{ \alpha \in A^{p,q}(Y, E); \Box \alpha = 0 \}.
$$

The celebrated Hodge’s theorem states that

$$
\mathcal{H}^{p,q}(Y, E) \simeq H^{p,q}(Y, E),
$$

where $H^{p,q}(Y, E)$ is the sheaf cohomology.
where $H^{p,q}(Y, E)$ is the Dolbeault cohomology group $[19]$. Now $H$ is not necessary to be smooth. The first step is to approximate it with a family of smooth metrics. When $E$ is a line bundle, it has been done in [7, 12]. We generalise their work to the higher rank vector bundle as follows:

**Proposition 1.1.** Suppose that $E$ is equipped with a singular Hermitian metric $H$ such that $i\Theta_{O^*(1), \varphi} \geq v$ for some smooth real $(1, 1)$-form $v$ on $X$. Here $\varphi$ is the metric on $O^*(1)$ corresponding to $H$. Moreover, assume that there exits a section $\xi$ of some multiple $O^*E(k)$ such that $\sup_X |\xi|_{k\varphi} < \infty$. Then, for each positive real number $\varepsilon$ and positive integer $l$, there is a (singular) Hermitian metric $H^l_\varepsilon$ on $S^lE$ such that

(a) $H^l_\varepsilon$ is smooth on $Y'$, where $Y'$ is an open subvariety of $Y$ independent of $l$ and $\varepsilon$;
(b) the sequence of metrics $\{\varphi, \varepsilon\}$ on $O^*(1)$ that defines $H^l_\varepsilon$ converges locally uniformly, decreasingly to $\varphi$ on $\pi^{-1}(Y')$. Equivalently, $H^l_\varepsilon$ converges locally uniformly, increasingly to $S^lH$ on $Y'$ for every $l$;
(c) $\mathcal{J}(\varphi) = \mathcal{J}(\varphi, \varepsilon)$ for all $\varepsilon$;
(d) for every $\varepsilon > 0$,

$$i\Theta_{O^*(1), \varphi, \varepsilon} \geq v - \varepsilon\Omega.$$

If $(E, H)$ is strongly positive in the sense of Nakano (see Definition 2.4), we moreover have

(e) for every relatively compact subset $Y'' \subset Y'$ and every $l$,

$$i\Theta_{S^lE, H^l_\varepsilon} \geq -C_l \varepsilon\omega \otimes \text{Id}_{S^lE}$$

over $Y''$ in the sense of Nakano (see Definition 2.4) for a constant $C_l$.

Here $S^lH$ is the natural metric on the $l$-th symmetric product $S^lE$ induced by $H$. One could find many similarities between the strongly Nakano positivity and the min${\{n, r + 1\}}$-nefness defined in [4] based on Proposition 1.1 (e), but we will not give any further discussions in this paper.

Next, for two $E$-valued $(n, q)$-forms $\alpha, \beta$ (not necessary to be $\bar{\partial}$-closed), we say they are cohomologically equivalent if there exits an $E$-valued $(n, q - 1)$-form $\gamma$ such that $\alpha = \beta + \bar{\partial}\gamma$. We denote by $\alpha \in [\beta]$ this equivalence relationship. Now $H$ is approximated by $\{H_\varepsilon\}$. Since $H_\varepsilon$ is smooth on $Y'$, the associated Laplacian $\Box_\varepsilon$ is well-defined. The harmonic form associated to $H$ is defined as follows:

**Definition 1.1.** Let $\alpha$ be an $E$-valued $(n, q)$-form on $Y$ such that the $L^2$-norm against $H$ is bounded. Assume that for every $\varepsilon$, there exists a cohomological equivalent class $\alpha_\varepsilon \in [\alpha|_{Y'}]$ such that

(a) $\Box_\varepsilon \alpha_\varepsilon = 0$ on $Y'$;
(b) $\alpha_\varepsilon \to \alpha|_{Y'}$ in the $L^2$-norm against $H$.
Then we call $\alpha$ a $\square_0$-harmonic form, and denote it by $\square_0 \alpha = 0$. Notice that this equality is taken in the sense of $L^2$-topology. The space of all the $\square_0$-harmonic forms is denoted by 

$$H^{n,q}(Y, E(H), \square_0).$$

The analytic sheaf $E(H)$ is defined in [4] as

$$E(H)_y := \{ u \in E_y; \| u \|^2_{H,\omega} \text{ is integrable in some neighbourhood of } y \}$$

for a given point $y \in Y$. Although it has already been proved in [4, 24] that $E(H)$ is coherent [22] in several situations, we would like to present a general version that describes the coherence in Proposition 2.3. In particular, $E(H)$ will always be coherent in our context in the view of Proposition 2.3. Although the $\square_0$-harmonic space is defined only on $Y'$, we can still prove the following proposition when $(E, H)$ is strongly Nakano positive (see Definition 2.4).

**Proposition 1.2** (A singular version of Hodge’s theorem). Assume that $(E, H)$ is strongly positive in the sense of Nakano. The following isomorphism holds:

$$(1) \quad H^{n,q}(Y, E(H), \square_0) \simeq H^q(Y, K_Y \otimes E(H)).$$

In particular, when $H$ is smooth, we have $E(H) = E$. Thus,

$$\alpha \in H^{n,q}(Y, E, \square_0)$$

if and only if $\alpha$ is harmonic in the usual sense.

The group $H^q(Y, K_Y \otimes E(H))$ is interpreted as a cohomology group associated with a coherent sheaf $K_Y \otimes E(H)$ as is explained in [22]. Moreover we obtain the following regularity property due to the canonical Bochner technique.

**Proposition 1.3.** Assume that $(E, H)$ is a (singular) Hermitian vector bundle that is strongly positive in the sense of Nakano. Let $\alpha$ be an $E$-valued $(n, q)$-form whose $L^2$-norm against $H$ is bounded. Then

1. if $\alpha$ is $\square_0$-harmonic, $\bar{\partial}(\ast \alpha) = 0$. In particular, $\ast \alpha$ is holomorphic.

2. if $\alpha$ is a weak solution of $\square_0 \alpha = 0$, $\alpha$ must be smooth.

Here $\ast$ refers to the Hodge $\ast$-operator defined by $\omega$. Based on the harmonic theory constructed above, we then generalise several canonical results of Kollár [28, 29] and Takegoshi [37] as follows:

**Theorem 1.1.** Let $f : Y \to Z$ be a fibration between two compact Kähler manifolds. Let $n = \dim Y$ and $m = \dim Z$. Suppose that $(E, H)$ is a (singular) Hermitian vector bundle over $Y$ that is strongly positive in the sense of Nakano. Moreover, assume that $H|_{Y_z}$ is well-defined for every $z \in Z$. Then the following theorems hold:
I Decomposition Theorem. The Leray spectral sequence \[ E_2^{p,q} = H^p(Z, R^q f_*(K_Y \otimes E(H))) \Rightarrow H^{p+q}(Y, K_Y \otimes E(H)) \]
de generates at \( E_2 \). As a consequence, it holds that
\[
\dim H^i(Y, K_Y \otimes E(H)) = \sum_{p+q=i} \dim H^p(Z, R^q f_*(K_Y \otimes E(H)))
\]
for any \( i \geq 0 \).

II Torsion freeness Theorem. For \( q \geq 0 \) the sheaf homomorphism
\[
L^q : f_*(\Omega^{n-q} \otimes E(H)) \to R^q f_*(K_Y \otimes E(H))
\]
induced by the \( q \)-times left wedge product by \( \omega \) admits a splitting sheaf homomorphism
\[
S^q : R^q f_*(K_Y \otimes E(H)) \to f_*(\Omega^{n-q} \otimes E(H)) \text{ with } L^q \circ S^q = \text{id}.
\]
In particular, \( R^q f_*(K_Y \otimes E(H)) \) is torsion free \([26]\) for \( q \geq 0 \) and vanishes if \( q > n - m \). Furthermore, it is even reflexive if \( E(H) = E \).

III Injectivity Theorem. Let \( (L,h) \) be a (singular) Hermitian line bundle over \( Y \). Recall that \( Y' \) is the open subvariety appeared in Proposition \([7]\). Assume the following conditions:
(a) the singular part of \( h \) is contained in \( Y - Y' \);
(b) \( i\Theta_{L,h} \geq \gamma \) for some real smooth \((1,1)\)-form \( \gamma \) on \( Y \);
(c) for some rational \( \delta \ll 1 \), the \( \mathbb{Q} \)-twisted bundle
\[
E < -\delta L \mid_{Y_s}
\]
is strongly positive in the sense of Nakano for every \( z \).
For a (non-zero) section \( s \) of \( L \) with \( \sup_Y |s|_h < \infty \), the multiplication map induced by the tensor product with \( s \)
\[
R^q f_*(s) : R^q f_*(K_Y \otimes E(H)) \to R^q f_*(K_Y \otimes (E \otimes L)(H \otimes h))
\]
is well-defined and injective for any \( q \geq 0 \).

IV Relative vanishing Theorem. Let \( g : Z \to W \) be a fibration to a compact Kähler manifold \( W \). Then the Leray spectral sequence:
\[
R^p g_* R^q f_*(K_Y \otimes E(H)) \Rightarrow R^{p+q}(g \circ f)_* (K_Y \otimes E(H))
\]
degenerates.

Theorem II can also been seen as a singular version of the hard Lefschetz theorem \([19]\). The definition of the \( \mathbb{Q} \)-twisted bundle in Theorem III can be found in \([30]\). Except that, \( (E \otimes L)(H \otimes h) \) is interpreted as the following sheaf:
\[
(E \otimes L)(H \otimes h)_y := \{ u \in (E \otimes L)_y; \| u \|^2_{H \otimes h, \omega} \text{ is integrable in some neighbourhood of } y \}.
\]
It is also coherent by Proposition \([2,3]\).

In the end we discuss various vanishing theorems.
Theorem 1.2. Let \( f : Y \to Z \) be a fibration between two compact Kähler manifolds.

I Nadel-type vanishing Theorem. Let \((L, h)\) be an \( f \)-big line bundle, and let \((E, H)\) be a vector bundle that is strongly positive in the sense of Nakano. Assume that \( H|_{Y_z} \) is well-defined for every \( z \). Then
\[
R^q f_*(K_Y \otimes (E \otimes L)(H \otimes h)) = 0 \text{ for every } q > 0.
\]

II Nakano-type vanishing Theorem. Assume that \((E, H)\) is strongly strictly positive in the sense of Nakano. Then
\[
H^q(Y, K_Y \otimes S^l(E(S^l H))) = 0 \text{ for every } l, q > 0.
\]

III Griffiths-type vanishing Theorem. Assume that \((E, H)\) is strictly positive in the sense of Griffiths (see Definition 2.4). Then
\[
H^q(Y, K_Y \otimes S^l(E \otimes \det E)(S^l (H \otimes \det H))) = 0 \text{ for every } l, q > 0.
\]

The definition for an \( f \)-big line bundle can be found in [15]. Notice that when \( E \) is a line bundle, the strong Nakano strict positivity is equivalent to the bigness. At this time Theorem II is just Nadel’s vanishing theorem [34]. Furthermore, \((E, H)\) is strongly positive in the sense of Nakano if and only if \((E, H)\) is positive in the sense of Griffiths, if and only if \((E, H)\) is pseudo-effective. In this situation, the topics related to the content of Theorem 1.1 and 1.2 are fully studied in recent years. See [13, 14, 15, 16, 18, 31, 32, 33] and the references therein for more details. Our work benefits a lot from them.

Acknowledgment. The author wants to thank Prof. Jixiang Fu, who brought this problem to his attention and for numerous discussions directly related to this work.

2. Preliminary

2.1. Set up. In the rest of this paper, we will use the following set up.

(I) Let \((Y, \omega)\) and \(Z\) be compact Kähler manifolds with \( \dim Y = n \) and \( \dim Z = m \). Let \( f : Y \to Z \) be a fibration, which is a surjective holomorphic map with connected fibres. Let \((E, H)\) be a holomorphic vector bundle over \( Y \) of rank \( r+1 \), endowed with a (singular) Hermitian metric \( H \) (see Sect.2.3.). Let \( X := \mathbb{P}(E^*) \) be the projectivised bundle with the natural projection \( \pi : X \to Y \) and tautological line bundle \( \mathcal{O}_E(1) := \mathcal{O}_X(1) \). In particular, let \( \varphi \) be the (singular) metric on \( \mathcal{O}_E(1) \) corresponding to \( H \). Let \( \Omega \) be a Kähler metric on \( X \).

(II) For every point \( y_0 \in Y \), we take a local coordinate \( y = (y_1, \ldots, y_n) \) around \( y_0 \). Fix a holomorphic frame \( \{u_0, \ldots, u_r\} \) of \( E \), the local coordinates of \( E, X \) and \( \mathcal{O}_E(1) \) are \( (y, U = (U_0, \ldots, U_r)), (y, w = (w_1, \ldots, w_r)) \) and \( (y, w, \xi) \) respectively. For every point \( z_0 \in Z \), we take a local coordinate \( z = (z_1, \ldots, z_m) \) around \( z_0 \). Moreover, when \( z_0 \) is a regular point
of $f$, the system of local coordinate around $Y_{z_0} := f^{-1}(z_0)$ can be taken as

$$f : Y \to Z$$

$$(z, (y_{m+1}, \ldots, y_n)) \mapsto z.$$ 

We use the following conventions often.

(i) Denote by $c_d = i^{2d}$ for any non-negative integer $d$. Denote

$$dy = dy_1 \wedge \cdots \wedge dy_n$$

and $dV_Y = c_n dy \wedge d\bar{y}$.

It is similar in the other systems of local coordinate.

(ii) Let $G$ be a smooth Hermitian metric on $E$, we write

$$G_i = \frac{\partial G}{\partial y_i}, G_j = \frac{\partial G}{\partial \bar{y}_j}, G_\alpha = \frac{\partial G}{\partial U_\alpha}, G_{\bar{\beta}} = \frac{\partial G}{\partial \bar{U}_\beta}$$

to denote the derivative with respect to $y_i, \bar{y}_j (1 \leq i, j \leq n)$ and $U_\alpha, \bar{U}_\beta (0 \leq \alpha, \bar{\beta} \leq r)$.

The higher order derivative is similar.

2.2. Hermitian geometry revisit. We collect some elementary facts here from [20, 25, 27]. Let $G$ be a smooth Hermitian metric on $E$. Then it induces a dual metric $G^*$ on $E^*$ hence a smooth metric $\psi$ on $\mathcal{O}_E(1)$. The curvature associated with $G$ is represented as

$$\Theta_{E,G} = \sum \Theta_{\alpha\beta} dy^i \wedge d\bar{y}^j \otimes u_\alpha \otimes \bar{u}_\beta$$

with

$$\Theta_{\alpha\beta} = -G_{\alpha\beta} + \sum_{\delta, \gamma} G^{\gamma\delta} G_{\alpha\delta} G_{\gamma\beta}.$$ 

Now fix a point $y \in Y$, we can always assume that $\{u_0, \ldots, u_r\}$ is an orthonormal basis with respect to $G$ at $y$. The Griffiths and Nakano positivities [20] is defined as follows:

**Definition 2.1.** Keep notations before,

1. $E$ is called (strictly) positive in the sense of Griffiths at $y$, if for any complex vector $z = (z_1, \ldots, z_n)$ and section $u = \sum U_\alpha u_\alpha$ of $E$,

$$\sum i\Theta_{\alpha\beta} U_\alpha \bar{U}_\beta z_i \bar{z}_j$$

is (strictly) positive.

2. $E$ is called (strictly) positive in the sense of Nakano at $y$, if for any $n$-tuple $(u^1 = \sum U^1_\alpha u_\alpha, \ldots, u^n = \sum U^n_\alpha u_\alpha)$ of sections of $E$,

$$\sum i\Theta_{\alpha\beta} U^1_\alpha \bar{U}^j_\beta$$

is (strictly) positive. Equivalently, $i\Theta_{E,G}$ is a (strictly) positive operator on $TY \otimes E$. 

6
Kobayashi proposed an intuitive way in [25, 27] to characterise the Griffiths positivity. More precisely, let \( \{ \log G^{\alpha \beta} \} \) be the inverse matrix of \( \{ \log G \}_{\alpha \beta} \). Then for the holomorphic vector field \( \frac{\partial}{\partial y_i} \) on \( Y \), its horizontal lift to \( X \) is defined as

\[
\delta := \frac{\partial}{\partial y_i} - \sum_{\alpha, \beta} (\log G)^{\alpha \beta}(\log G)_{\beta i} \frac{\partial}{\partial w_\alpha}.
\]

The dual basis of \( \{ \frac{\delta}{\delta y_i}, \frac{\partial}{\partial w_\alpha} \} \) will be \( \{ dy_i, \delta w_\alpha := dw_\alpha + \sum_{i, \beta} (\log G)^{\beta i}(\log G)_{\beta \alpha} dy_i \} \).

Let

\[
\Psi := \sum_{\alpha, \beta} iK_{\alpha \beta i \bar{j}} \frac{w_\alpha \bar{w}_\beta}{G} dy_i \wedge d\bar{y}_j
\]

and

\[
\omega_{FS} := \sum_{\alpha, \beta} \frac{\partial^2 \log G}{\partial w_\alpha \partial \bar{w}_\beta} \delta w_\alpha \wedge \delta \bar{w}_\beta,
\]

where

\[
K_{\alpha \beta i \bar{j}} := -G_{\alpha \beta i \bar{j}} + \sum_{\gamma, \bar{\gamma}} G^{\gamma \bar{\gamma}} G_{\alpha \gamma i \bar{\gamma} \bar{j}}.
\]

It is easy to verify that they are globally defined \((1,1)\)-forms on \( X \). Then the celebrated theorem given by Kobayashi says that

**Proposition 2.1** (Kobayashi, [25, 27]).

\[
2 \quad i\partial \bar{\partial} \psi = -\Psi + \omega_{FS}.
\]

As a consequence, we have

**Proposition 2.2** (Kobayashi). \((E, G)\) is positive in the sense of Griffiths if and only if \((\mathcal{O}_E(1), \psi)\) is positive.

The following definition is useful.

**Definition 2.2** ([5, 9, 10]).

\[
c(\psi)_{ij} := \langle \frac{\delta}{\delta y_i}, \frac{\delta}{\delta y_j} \rangle_{i\partial \bar{\partial} \psi}
\]

is called the geodesic curvature of \( \psi \) in the direction of \( i, j \).

Let \( a^\alpha_i := -\sum_{\beta} (\log G)^{\beta \alpha}(\log G)_{\beta i} \), we have

\[
c(\psi)_{ij} = (\log G)_{ij} - \sum_{\alpha} a^\alpha_i a_{\alpha j}.
\]

Therefore the relationship between \( c(\varphi)_{ij} \) and \( \Psi \) will be

\[
3 \quad -\Psi = \sum ic(\psi)_{ij} dz_i \wedge d\bar{z}_j.
\]

In particular, \( \{ c(\psi)_{ij} \} \) defines a Hermitian form, which is positive if and only if \( \Psi \) is negative.
2.3. **Singular Hermitian metric.** We recall the definition of the singular Hermitian metric on $E$ in [39].

**Definition 2.3.** (1) Consider the $L^1$-bounded function $\varphi$ on $X$. We define

$$Y_\varphi := \{ y \in Y; \varphi|_{X_y} \text{ is well-defined} \}.$$ 

(2) Fix a smooth metric $h_0$ on $\mathcal{O}_E(1)$ as the reference metric, we define

$$\mathcal{H}(X) := \{ \varphi \in L^1(X); \text{on each } X_y \text{ with } y \in Y_\varphi, \varphi|_{X_y} \text{ is smooth and } (i\Theta_{\mathcal{O}_X(1), h_0} + i\partial \bar{\partial} \varphi)|_{X_y} \text{ is strictly positive} \}.$$ 

(3) A singular Hermitian metric on $E$ is a map $H$ with form that

$$H_\varphi(u, u) = \int_{X_y} |u|^2_{h_0} e^{-\varphi} \frac{\omega^{r}_{\varphi,y}}{r!},$$

where $\varphi \in \mathcal{H}(X)$. Here we use the fact that $\pi_*\mathcal{O}_E(1) = E$.

We have presented in [39] that this type of singular metric differs slightly with the one defined in [36] and has good nature. In particular, we have successfully defined the Griffiths and Nakano positivities concerning $(E, H_\varphi)$. More precisely,

**Definition 2.4.** Let $H_\varphi$ be a (singular) Hermitian metric on $E$, and let $\phi$ be the corresponding metric on $\mathcal{O}_E(1)$. (Notice that $\phi$ may not equal to $\varphi$.) Let $(L, h)$ be a (singular) Hermitian line bundle over $Y$. Let $q$ be a rational number. Then

1. $(E, H_\varphi)$ is (strictly) positive in the sense of Griffiths, if $i\partial \bar{\partial} \phi$ is (strictly) positive on $X$.
2. $(E < qL >, H_\varphi, h)$ is (strictly) positive in the sense of Griffiths, if $i\partial \bar{\partial} \phi + q\pi^* c_1(L, h)$ is (strictly) positive on $X$.
3. $(E, H_\varphi)$ is (strictly) strongly positive in the sense of Nakano, if the $\mathbb{Q}$-twisted vector bundle

$$(E - \frac{1}{r+2} \det E >, H_\varphi)$$

is (strictly) positive in the sense of Griffiths.

4. $(E < qL >, H_\varphi, h)$ is (strictly) strongly positive in the sense of Nakano, if the $\mathbb{Q}$-twisted vector bundle

$$(E < \frac{q}{r+2}L - \frac{1}{r+2} \det E >, H_\varphi, h)$$

is (strictly) positive in the sense of Griffiths.

The definition for a $\mathbb{Q}$-twisted bundle can be found in [30]. One refers to [39] for a detailed discussion for these positivities.
2.4. Multiplier ideal sheaf. Remember that in the line bundle situation, the multiplier ideal sheaf [34] is a powerful tool which establishes the relationship between the algebraic and analytic nature of this line bundle. It would be valuable to extend this notion to a vector bundle. There are several attempts. The analytic sheaf $E(H)$ is defined in [3] is a good candidate. The definition is as follows:

$$E(H)_y := \{ u \in E_y; \|u\|^2_{H,\omega} \text{ is integrable in some neighbourhood of } y \}$$

for a given point $y \in Y$. Moreover, it is proved in [4, 24] that $E(H)$ is coherent [22] when $(E, H)$ possesses certain positivity. We also prove the coherence in a general setting here.

**Proposition 2.3.** Let $(E, H)$ be a singular Hermitian vector bundle over $Y$. Let $\varphi$ be the corresponding metric on $O_E(1)$. Assume that $i\partial \bar{\partial} \varphi \geq \gamma$ for a real $(1, 1)$-form $\gamma$ on $X$. Then $E(H)$ is coherent.

**Proof.** The proof follows the same method as [4, 8]. Since coherence is a local property, we can assume without loss of generality that $Y = U$ is a domain in $(\mathbb{C}^n, z = (z_1, \ldots, z_n))$ and $E = U \times \mathbb{C}^{r+1}$. Let $L^2(U, \mathbb{C}^{r+1})_H$ be the square integrable $\mathbb{C}^{r+1}$-valued holomorphic functions with respect to $H$ on $U$. It generates a coherent subsheaf

$$F \subset O_U(E) = \bigoplus_{r+1} O_U$$

as an $O_U$-module [22]. It is clear that $F \subset E(H)$; in order to prove the equality, we need only check that $F_y + E(H)_y \cdot m_{U,y}^{s+1} = E(H)_y$ for every integer $s$, in view of Nakayama’s lemma [1]. Here $m_{U,y}$ is the maximal ideal of $O_{U,y}$.

Let $f \in E(H)_y$ be a germ that is defined in a neighbourhood $V$ of $y$ and let $\rho$ be a cut-off function with support in $V$ such that $\rho = 1$ in a smaller neighbourhood of $y$. We solve the equation $\bar{\partial} u = \partial (\rho f)$ by means of $L^2$-estimates (see the proof of Proposition 24) against $e^{-2(n+s)\log|z-z(y)|-\psi} H$, where $\psi$ is a smooth strictly plurisubharmonic function satisfying certain positivity as is shown in the proof of Proposition 24. Notice that here we use the curvature condition $i\partial \bar{\partial} \varphi \geq \gamma$ to guarantee the existence of such a $\psi$. We then get a solution $u$ such that $\int_U \frac{|u|^2_H}{|z-z(y)|^{n+s}} < \infty$. Thus $F = \rho f - u$ is holomorphic, $F \in L^2(U, \mathbb{C}^{r+1})_H$ and

$$f_y - F_y = u_y \in E(H)_y \cdot m_{U,y}^{s+1}.$$ 

This proves the equality hence the coherence. \Box

2.5. de Rham–Weil isomorphism. The de Rham–Weil isomorphism about the $L^2$-cohomology group is essentially used in this paper, so we provide here a detailed proof. This proof includes an $L^2$-estimate for a singular Hermitian vector bundle, which seems to be of independent
interest. The readers who are familiar with this isomorphism could skip this part. Assume that $E(H)$ is coherent, hence the cohomology group $H^q(Y, K_Y \otimes E(H))$ is well-defined \cite[19]{19}. Moreover, we have

**Proposition 2.4** (The de Rham–Weil isomorphism). Fix a Stein covering $\mathcal{U} := \{U_j\}_{j=1}^N$ of $Y$. We have

$$H^q(Y, K_Y \otimes E(H)) \simeq \frac{\text{Ker}(\bar{\partial} : A^{n,q}(Y, E(H)) \to A^{n,q+1}(Y, E(H)))}{\text{Im}(\partial : A^{n,q-1}(Y, E(H)) \to A^{n,q}(Y, E(H)))}.$$ 

Here we use $A^{n,q}(U, E(H))$ to refer to all of the $(n, q)$-forms on an open set $U$ whose coefficients are in $E(H)$. In other words,

$$\alpha \in A^{n,q}(U, E(H))$$

if $\alpha$ is an $E$-valued $(n, q)$-form such that

$$\int_U \|\alpha\|^2_{H,\omega} < \infty.$$

**Proof.** The proof follows the same method as \cite{34} except that we will essentially use the Koszul complex without mentioning it. Fix a Stein covering $\mathcal{U} := \{U_j\}_{j=1}^N$ of $Y$. Then we have the isomorphism between $H^q(Y, K_Y \otimes E(H))$ and the Čech cohomology group $\check{H}^q(\mathcal{U}, K_Y \otimes E(H))$. For simplicity, we put

$$U_{j_0 j_1 \cdots j_q} := U_{j_0} \cap \cdots \cap U_{j_q}.$$ 

The class $[\alpha]$ maps to a $q$-cocycle $\alpha = \{\alpha_{j_0 \cdots j_q}\}$ that satisfies

$$\alpha_{j_0 \cdots j_q} \in H^{n,0}(U_{j_0 \cdots j_q}, E(H))$$

and $\delta \alpha = 0$,

where $\delta$ is the coboundary operator of the Čech complex. Then by using a partition $\{\rho_j\}$ of unity associated to $\mathcal{U}$, we define $\alpha^1 := \{\alpha_{j_0 \cdots j_{q-1}}\}$ by

$$\alpha_{j_0 \cdots j_{q-1}} := \sum \rho_j \alpha_{j j_{j_0} \cdots j_{q-1}}.$$ 

By the construction, we have $\delta \alpha^1 = \alpha$ and $\delta \delta \alpha^1 = \delta \delta \alpha^1 = \delta \bar{\alpha} = 0$. Notice that

$$\bar{\delta} \alpha^1 = \{\bar{\delta} \alpha_{j_0 \cdots j_{q-1}}\} = \{\sum (\bar{\delta} \rho_j) \cdot \alpha_{j j_0 \cdots j_{q-1}}\},$$

we have that $\bar{\delta} \alpha^1$ is a $(q - 1)$-cocycle with

$$\bar{\delta} \alpha_{j_0 \cdots j_{q-1}} \in A^{n,1}(U_{j_0 \cdots j_{q-1}}, E(H)).$$

From the same argument, we can obtain $\alpha^2$ with $\delta \alpha^2 = \bar{\delta} \alpha^1$. Moreover, $\bar{\delta} \alpha^2$ is a $(q - 2)$-cocycle that satisfies

$$\bar{\delta} \alpha_{j_0 \cdots j_{q-2}} \in A^{n,2}(U_{j_0 \cdots j_{q-2}}, E(H))$$

and $\delta \bar{\delta} \alpha^2 = 0$.

By repeating this process, we finally obtain $\alpha^q = \{\alpha_{j_0}\}$. Then $\bar{\delta} \alpha_{j_0}$ determines the $(n, q)$-form on $U_{j_0}$ with $\bar{\delta} \alpha_{j_0} \in A^{n,q}(U_{j_0}, E(H))$. Moreover, since $\delta \delta \alpha_{j_0} = 0$, they patch together to give a $(0, q)$-form

$$\alpha_q \in A^{n,q}(X, E(H))$$

\textit{Note:} The proof follows the same method as \cite{34} except that we will essentially use the Koszul complex without mentioning it.
with \( \partial \alpha_q = 0 \). From the argument above, we have obtained the (well-defined) map
\[
\tilde{H}^q(U, E(H)) \to \frac{\text{Ker}(\bar{\partial} : A^{n,q}(X, E(H)) \to A^{n,q+1}(X, E(H)))}{\text{Im}(\partial : A^{n,q-1}(X, E(H)) \to A^{n,q}(X, E(H)))}
\]

Now we see that this map is actually an isomorphism by using the \( L^2 \)-estimate on each Stein subset \( U_{j_0...j_k} \) for \( k = 0, \ldots, q \). For every 
\[ \beta \in \text{Ker}(\bar{\partial} : A^{n,q}(X, E(H)) \to A^{n,q+1}(X, E(H))), \]
we define \( \beta^0 := \{ \beta_{j_0} \} \) by \( \beta_{j_0} := \beta|_{U_{j_0}} \). By the \( L^2 \)-estimate on \( U_{j_0} \) against \( H \), we obtain \( \beta^1 = \{ \beta^0_{j_0} \} \) such that
\[ \bar{\partial} \beta^1 = \beta^0, \]
\[ \| \beta^1 \|^2_{H_{U_{j_0}}} = \sum_{j_0} \int_{U_{j_0}} \| \beta^1_{j_0} \|^2_H \leq C_1 \| \beta^0 \|^2_{H_{U_{j_0}}}. \]
Here \( C_1 \) is an independent constant.

We give a short explanation for this \( L^2 \)-estimate. We admit Proposition \[ \[ \] \] for the time being. Then there exists a regularising sequence \( \{ H_\varepsilon \} \) which is smooth on an open subvariety \( Y' \subset U_{j_0} \). Moreover, since \( U_{j_0} \) is a Stein open subset of \( \mathbb{C}^n \), we can even make \( Y' = U_{j_0} \). Then, at each point \( y \in U_{j_0} \), we may then choose a coordinate system which diagonalizes simultaneously the hermitians forms \( \omega(y) \) and \( i\Theta_{E,H_\varepsilon} \), in such a way that
\[ \omega(y) = i \sum \alpha_j dy_j \wedge d\bar{y}_j \text{ and } i\Theta_{E,H_\varepsilon} = i \sum \alpha_j dy_j \wedge d\bar{y}_j. \]

Since \( \beta_0 \) is an \((n,q)\)-form, at point \( y \) we have
\[ < [i\Theta_{E,H_\varepsilon}, \Lambda] \beta_0, \beta_0 >_\omega = \sum (\varepsilon_1 + \cdots + \varepsilon_n) |\beta_0|^2, \]
where \( \Lambda \) is the adjoint operator of \( \omega \wedge \cdot \). On the other hand, since \( U_{j_0} \) is Stein, we can always find a smooth strictly plurisubharmonic function \( \psi \), such that \( (E, e^{-\psi}H) \) is strictly strongly positive in the sense of Nakano on \( U_{j_0} \). By Proposition \[ \[ \] \] (e), we then obtain that
\[ i\Theta_{E,e^{-\psi}H_\varepsilon} = i \sum \tau_j^j c_j dz_j \wedge d\bar{z}_j \]
such that \( \tau_j^j \geq C' \) for a universal positive constant \( C' \). Now apply the \( L^2 \)-estimate \[ \[ \] \] against \( e^{-\psi}H_\varepsilon \), we obtain \( \beta^1 = \{ \beta^0_{j_0} \} \) such that
\[ \bar{\partial} \beta^1 = \beta^0, \]
\[ \| \beta^1 \|^2_{H_{t_{j_0}}} = \sum_{j_0} \int_{U_{j_0}} \| \beta^1_{j_0} \|^2_H \leq C_1 \| \beta^0 \|^2_{H_{t_{j_0}}}. \]
Recall that \( \beta^0 = \{ \beta|_{U_{j_0}} \} \) with \( \beta \in A^{n,q}(X, E(H)) \), so
\[ \lim_{\varepsilon \to 0} \| \beta^0 \|^2_{H_{t_{j_0}}} < \infty. \]
Take the limit of the inequality before with respect to $\varepsilon$, we then obtain the desired estimate.

By the construction, we have $\overline{\partial} \delta \beta^1 = \delta \overline{\partial} \beta^1 = \delta \beta^0 = 0$. Moreover,

$$\delta \beta^1 = \{ \beta^1_{j_0|U_{j_01}}, - \beta^1_{j_1|U_{j_01}} \} \in A^{n,q-1}(U_{j_01}, E(H)).$$

The explicit meaning of this inclusion should be

$$\beta^1_{j_0|U_{j_01}} - \beta^1_{j_1|U_{j_01}} \in A^{n,q-1}(U_{j_01}, E(H)),$$

but we abuse the notation here and in the rest part. Therefore by the same method, we can obtain $\beta^2 = \{ \beta^2_{j_01} \}$ such that

$$\tilde{\partial} \beta^2 = \delta \beta^1,$$

$$\| \beta^2 \|^2_{H^U_{j_01}} := \sum_{j_0, j_1} \int_{U_{j_01}} \| \beta^1_{j_0, j_1} \|^2_H \leq C_2 \| \beta^1 \|^2_{H^U_{j_01}}.$$

Similarly, $\tilde{\partial} \delta \beta^2 = \delta \tilde{\partial} \beta^2 = \delta \delta \beta^1 = 0$ and

$$\delta \beta^2 \in A^{n,q-2}(U_{j_01, j_2}, E(H)).$$

By repeating this process, we finally obtain $\beta^i$ such that $\tilde{\partial} \beta^i = \delta \beta^{i-1}$ and $\delta \beta^i \in A^{n,0}(U_{j_0...j_1}, E(H))$. Since $\tilde{\partial} \delta \beta^i = \delta \tilde{\partial} \beta^i = \delta \delta \beta^{i-1} = 0$, we actually have $\delta \beta^i \in H^{n,0}(U_{j_0...j_1}, E(H))$. Hence we have obtained

$$j : \text{Ker}(\tilde{\partial} : A^{n,q}(Y, E(H))) \rightarrow A^{n,q+1}(Y, E(H))) \rightarrow \tilde{H}^q(U, KY \otimes E(H)).$$

We claim that $\text{Im}(\tilde{\partial} : A^{n,q-1}(Y, E(H)) \rightarrow A^{n,q}(Y, E(H)))$ maps to the zero space under $j$, hence it is easy to see that the maps $i, j$ together give an isomorphism

$$H^q(Y, KY \otimes E(H)) \simeq \frac{\text{Ker}(\tilde{\partial} : A^{n,q}(Y, E(H)) \rightarrow A^{n,q+1}(Y, E(H)))}{\text{Im}(\tilde{\partial} : A^{n,q-1}(Y, E(H)) \rightarrow A^{n,q}(Y, E(H)))}.$$ 

Now we prove the claim by diagram chasing. We only prove the case that $i = 1$ and $i = 2$, the general follows the same method. For every $\beta \in \text{Im}(\tilde{\partial} : A^{n,0}(Y, E(H)) \rightarrow A^{n,1}(Y, E(H)))$, we define $\beta^0 := \{ \beta_{j_0} \}$ by $\beta_{j_0} := \beta|_{U_{j_0}}$. On the other hand, there exits an $E$-valued $(n, 0)$-form $\gamma$ on $Y$ such that $\beta = \tilde{\partial} \gamma$ and $\| \gamma \|^2_{H^U_{j_0}} < \infty$ for every $j_0$. So we can take $\beta^1 = \{ \beta^1_{j_0} \} := \{ \gamma|_{U_{j_0}} \}$. The morphism $j$ in this situation can be simply written as $j(\beta) = \delta \beta^1$. Since $\gamma$ is globally defined hence $\delta \beta^1 = 0$. We have successfully proved that $j(\beta) = 0$ when $i = 1$.

Next we consider the $\beta \in \text{Im}(\tilde{\partial} : A^{n,1}(Y, E(H)) \rightarrow A^{n,2}(Y, E(H)))$, we define $\beta^0 := \{ \beta_{j_0} \}$ by $\beta_{j_0} := \beta|_{U_{j_0}}$. On the other hand, there exits an $E$-valued $(n, 1)$-form $\gamma$ on $Y$ such that $\beta = \tilde{\partial} \gamma$ and $\| \gamma \|^2_{H^U_{j_0}} < \infty$ for every $j_0$. So we can take $\beta^1 = \{ \beta^1_{j_0} \} := \{ \gamma|_{U_{j_0}} \}$. Now apply the
$L^2$-estimate on $U_{j_0}$ against $H$, we obtain $\gamma^1 = \{\gamma^1_{j_0}\}$ such that
\[
\bar{\partial} \gamma^1 = \beta^1,
\]
\[
\|\gamma^1\|_{\bar{H}^2_{U_{j_0}}}^2 := \sum_{j_0} \int_{U_{j_0}} \|\gamma^1_{j_0}\|_{H}^2 \leq C_1 \|\beta^1\|_{\bar{H}^2_{U_{j_0}}}^2.
\]
Let $\beta^2 = \{\beta^2_{j_0,j_1}\} = \delta \gamma^1$. Since $\bar{\partial} \beta^2 = \bar{\partial} \delta \gamma^1 = \delta \bar{\partial} \gamma^1 = \delta \beta^1$ and $\|\beta^2\|_{\bar{H}^2_{U_{j_0,j_1}}} < \infty$, the morphism $j$ is $j(\beta) = \delta \beta^2 = \delta \delta \gamma^1 = 0$ at this time. We have successfully proved that $j(\beta) = 0$ when $i = 2$. □

2.6. Bochner technique. This subsection is devoted to introduce the classic Bochner technique in harmonic theory. It mainly comes from [8, 19, 37] and the references therein. One should pay attention that the content of this subsection works for a Kähler manifold $(Y, \omega)$ that is not necessary to be compact.

Let $(E, G)$ be a holomorphic vector bundle with the smooth Hermitian metric $G$. Then $G$ as well as $\omega$ defines an $L^2$-norm $\| \cdot \|_{G, \omega}$ on the space
\[
A^*(Y, E) = \oplus A^{p,q}(Y, E).
\]
Let $\ast$ be the Hodge $\ast$-operator, let $e(\theta)$ be the left wedge product acting by the form $\theta \in A^{p,q}(Y)$ and $\partial G$ be the $(1, 0)$-part of the Chern connection on $E$ associated with $G$. Then the adjoint operators of $\bar{\partial}, \partial G, e(\theta)$ with respect to $\| \cdot \|_{G, \omega}$ are denoted by $\bar{\partial}^*, \partial_G^*$ and $e(\theta)^*$, respectively. In particular, $e(\omega)$ is also denoted by $L$ and $e(\omega)^*$ is denoted by $\Lambda$. The Laplacian operators are defined as follows:
\[
\Box_G = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}, \\
\Box_G = \partial_G \partial_G^* + \partial_G^* \partial_G.
\]

Then the Bochner formula is as follows:

**Proposition 2.5.** For any $\alpha \in A^{p,q}(Y, E)$ and positive smooth function $\eta$ on $Y$ with $\chi := \log \eta$, we have
\[
\Box_G = \bar{\Box}_G + [i \Theta_{E,G}, \Lambda],
\]
\[
\Box e^{-\chi} G = \bar{\Box} e^{-\chi} G + [i(\Theta_{E,G} + \partial \bar{\partial} \chi), \Lambda] \quad \text{and}
\]
\[
(5) \quad \|\sqrt{\eta}(\bar{\partial} + e(\bar{\partial} \chi))\alpha\|_{\bar{G}, \omega}^2 + \|\sqrt{\eta} \partial \bar{\partial}^* \alpha\|_{\bar{G}, \omega}^2 = \|\sqrt{\eta}(\bar{\partial} G - e(\bar{\partial} \chi)^*)\alpha\|_{\bar{G}, \omega}^2
\]
\[
+ \|\sqrt{\eta} \partial_G \alpha\|_{\bar{G}, \omega}^2 + \eta i [\Theta_{E,G} + \partial \bar{\partial} \varphi, \Lambda] \alpha, \alpha >_{G, \omega},
\]
when the integrals above are finite.

The following formula is due to the Kähler property of $\omega$:

**Proposition 2.6** (Donnelly and Xavier’s formula, [31]). For any
\[
\alpha \in A^{p,q}(Y, E)
\]
and smooth function $\chi$ on $Y$, we have
\[
[\bar{\partial}, e(\bar{\partial}\chi)^*)] + e(\partial\chi)\partial^* = ie(\bar{\partial}\varphi)\Lambda \quad \text{and} \quad ||e(\partial\chi)^*\alpha||^2_{\omega} + ||e(\bar{\partial}\chi)^*\alpha||^2_{\omega}.
\]
when the integrals above are finite.

3. The Regularising Technique

This section is devoted to prove Proposition 1.1. Recall that in [7, 12], such an approximation was already made for a line bundle as follows:

**Proposition 3.1** (Theorem 2.2.1, [12]). Let $(L, \varphi)$ be a (singular) Hermitian line bundle over $Y$ such that $i\Theta_{L,\varphi} \geq v$ for a real smooth $(1,1)$-form $v$. There exists a family of singular metrics $\{\varphi_\varepsilon\}_{\varepsilon>0}$ with the following properties:

(a) $\varphi_\varepsilon$ is smooth on $Y - Z_\varepsilon$ for a subvariety $Z_\varepsilon$;
(b) $\varphi_{\varepsilon_2} \geq \varphi_{\varepsilon_1} \geq \varphi$ holds for any $0 < \varepsilon_1 \leq \varepsilon_2$;
(c) $\mathcal{I}(\varphi) = \mathcal{I}(\varphi_\varepsilon)$; and
(d) $i\Theta_{L,\varphi_\varepsilon} \geq v - \varepsilon\omega$.

Thanks to the openness property of the multiplier ideal sheaf [21], one can arrange $h_\varepsilon$ with logarithmic poles along $Z_\varepsilon$ according to the remark in [12]. Now we furthermore assume that there exists a section $\xi$ of some multiple $L^k$ such that $\sup_Y |\xi|_{h_k} < \infty$. Then the set
\[
\{y \in Y; \nu(h_\varepsilon, y) > 0\}
\]
for every $\varepsilon > 0$ is contained in the subvariety $Z := \{y \in Y; \xi(y) = 0\}$ by property (b). Here $\nu(h_\varepsilon, y)$ refers to the Lelong number [8] of $h_\varepsilon$ at $y$. Hence, instead of (a), we can assume that

(a') $h_\varepsilon$ is smooth on $Y - Z$ and has logarithmic poles along $Z$, where $Z$ is a subvariety of $Y$ independent of $\varepsilon$.

Now we are ready to prove Proposition 1.1.

**Proof of Proposition 1.1**. Take a Kähler form $\Omega$ on $X$. By Proposition 3.1 and the remark after that, there exists a family of singular metrics $\{\varphi_\varepsilon\}$ with the following properties:

(a) $\varphi_\varepsilon$ is smooth on $X'$ for an open subvariety $X'$ independent of $\varepsilon$;
(b) $\varphi_{\varepsilon_2} \geq \varphi_{\varepsilon_1} \geq \varphi$ holds for any $0 < \varepsilon_1 \leq \varepsilon_2$;
(c) $\mathcal{I}(\varphi) = \mathcal{I}(\varphi_\varepsilon)$; and
(d) $i\Theta_{O_{E}(1),\varphi_\varepsilon} \geq v - \varepsilon\Omega$.

Moreover, since $\xi \in O_{E}(k)$, $\pi_*\xi \in S^kE$ by the canonical isomorphism
\[
\pi_*(O_{E}(k)) = S^kE.
\]
As a result,
\[
\pi(X') = \pi(X - \{\xi = 0\})
= Y - \{\pi_*\xi = 0\}
=: Y',
\]
which is an open subvariety of \(Y\). In particular, it is easy to verify that \(\varphi_\varepsilon \in \mathcal{H}(X)\) for every \(\varepsilon\). Therefore it defines a singular metric
\[
H^i_l(U, U) := \int_{X_y} |U|^2 e^{-l\varphi_\varepsilon} \frac{\omega^r_{\varphi_\varepsilon,y}}{r!}
\]
on \(S^l E\) (see Definition 2.2) for every positive integer \(l\) and \(U \in S^l E\). Here we use the fact that \(\pi_* \mathcal{O}_E(l) = S^l E\). It remains to prove the desired properties.

(a), (c) and (d) are obvious. Recall that \(S^l H\) can be rewritten as
\[
S^l H(U, U) := \int_{X_y} |u|^2 e^{-l\varphi_\varepsilon} \frac{\omega^r_{\varphi_\varepsilon,y}}{r!}
\]
by [39]. On the other hand, both \(\varphi\) and \(\varphi_\varepsilon\) are smooth on \(X_y\) with \(y \in Y'\), we immediately conclude that \(\varphi_\varepsilon\) converges locally uniformly and decreasingly to \(\varphi\) on \(X'\). Hence \(H^i_l \to S^l H\) locally uniformly and increasingly on \(Y'\). (b) is proved.

In the end, we prove (e) under the assumption that \((E, H)\) is strongly positive in the sense of Nakano. Indeed, by definition
\[
i \Theta_{\mathcal{O}_E(1), \varphi} = \frac{i}{r + 2} \pi^* \Theta_{\det E, \det H}
\]
is positive. Moreover, \(i \Theta_{\mathcal{O}_E(1), \varphi}\) is also positive by Theorem 1.3 in [39]. Hence
\[
i \Theta_{\mathcal{O}_E(r+l+1), (r+l+1)\varphi} - i \pi^* \Theta_{\det E, \det H}
\]
is positive for all positive integer \(l\). Consequently,
\[
i \Theta_{\mathcal{O}_E(r+l+1), (r+l+1)\varphi} - i \pi^* \Theta_{\det E, \det H} \geq -C_l \varepsilon \Omega.
\]
Now on \(Y''\) we apply Berndtsson’s curvature formula in [2, 3], which says for any line bundle \((L, \psi)\) over \(X\), the curvature of \(\pi_* (K_{X/Y} \otimes L)\) associated with the \(L^2\)-metric satisfies that
\[
\sum < i \Theta_{ij} s_i, s_j > \geq \int_{X_y} i c(\psi)_{ij} s_i \wedge \bar{s}_j e^{-\psi}.
\]
Here \(\{s_i\}\) is \(n\)-tuple of sections of \(\pi_* (K_{X/Y} \otimes L)\). Let
\[
(L, \psi) = (\mathcal{O}_E(r + l + 1) \otimes \pi^* \det E^*, (r + l + 1)\varphi_\varepsilon - \pi^* \phi),
\]
where \(\phi\) is the weight function of \(\det H\). At this time, the direct image equals to \(S^l E\), the \(L^2\)-metric is just \(H^i_l\) and the associated curvature satisfies
\[
(7) \sum < i \Theta_{ij} s_i, s_j > \geq -C_l \varepsilon' \sum_{15} \int_{X_y} i c(\Omega)_{ij} s_i \wedge \bar{s}_j e^{-(r+l+1)\varphi_\varepsilon + \pi^* \phi},
\]
where \( c(\Omega)_{ij} \) is defined as (see Definition 2.2)

\[
c(\Omega)_{ij} := \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} > \Omega.
\]

The estimate (7) is due to formula (3), the fact that

\[
i \Theta_{\Omega E^*(r+l+1),(r+l+1)\varphi} - i\pi^* \Theta_{\det E, \phi} \geq -C_l \epsilon \Omega
\]

and some elementary computation. Since \( y \in Y'' \subset Y' \), \( \varphi \) is smooth along \( X_y \). Therefore

\[
\sum \int_{X_y} i c(\Omega)_{ij} s_i \wedge \bar{s}_j e^{-\omega + \pi^* \varphi} + \pi^* \phi
\]

is uniformly bounded with respect to \( \epsilon \). It obviously implies that

\[
i \Theta_{S^{1,E},H^1} \geq -C_l \epsilon'' \omega \otimes \text{Id}_{S^{1,E}}
\]

in the sense of Nakano. Moreover, from the proof we see that if \( i \Theta_{\Omega E^*(1),\varphi} = \frac{1}{r+1} i \pi^* \Theta_{\det E, \det H} > \epsilon \omega \), which means that \( (E, H) \) is strongly strictly positive in the sense of Nakano, we can even arrange the thing that

\[
i \Theta_{S^{1,E},H^1} \geq C_l (1 - \epsilon') \omega \otimes \text{Id}_{S^{1,E}}
\]

in the sense of Nakano. □

Then throughout the whole paper, for a given singular metric \( H \) on \( E \), \( \varphi \) will always refer to its corresponding metric on \( \Omega E(1) \). \{ \( H^k_\epsilon \) \} and \{ \( \varphi_\epsilon \) \} will always be the regularising sequence provided by Proposition 1.11. In particular, \( H^k_\epsilon \) is smooth on an open subvariety \( Y' \), whereas \( \varphi_\epsilon \) is smooth on \( X' \) with \( X' = \pi^{-1}(Y') \). When \( k = 1 \), \( H^1_\epsilon \) will be simply denoted by \( H_\epsilon \).

4. Harmonic theory

4.1. Global theory. Firstly, we use the method in [6] to construct a complete Kähler metric on \( Y' \) as follows. Since \( Y' \) is weakly pseudo-convex, we can take a smooth plurisubharmonic exhaustion function \( \psi \) on \( Y' \). Define \( \tilde{\omega}_l = \omega + \frac{i}{l} \partial \bar{\partial} \psi^2 \) for \( l \gg 0 \). It is easy to verify that \( \tilde{\omega}_l \) is a complete Kähler metric on \( Y'' \) and \( \tilde{\omega}_{l_1} \geq \tilde{\omega}_{l_2} \geq \omega \) for \( l_1 \leq l_2 \).

Let \( L_{n,q}(Y', E)_{H_\epsilon, \tilde{\omega}_l} \) be the \( L^2 \)-space of \( E \)-valued \((n,q)\)-forms \( \alpha \) on \( Y' \) with respect to the inner product given by \( H_\epsilon, \tilde{\omega}_l \). Then we have the orthogonal decomposition [17]

\[
L_{n,q}(Y', E)_{H_\epsilon, \tilde{\omega}_l} = \text{Im} \tilde{\partial} \bigoplus \mathcal{H}_{n,q}^{n,q}(E)_{H_\epsilon, \tilde{\omega}_l} \bigoplus \text{Im} \tilde{\partial}^*_{H_\epsilon}
\]

where

\[
\text{Im} \tilde{\partial} = \text{Im}(\tilde{\partial} : L_{n,q-1}(Y', E)_{H_\epsilon, \tilde{\omega}_l} \to L_{n,q}(Y', E)_{H_\epsilon, \tilde{\omega}_l}),
\]

\[
\mathcal{H}_{n,q}^{n,q}(E)_{H_\epsilon, \tilde{\omega}_l} = \{ \alpha \in L_{n,q}(Y', E)_{H_\epsilon, \tilde{\omega}_l} ; \partial \alpha = 0, \bar{\partial} \alpha = 0 \},
\]

and

\[
\text{Im} \tilde{\partial}^*_{H_\epsilon} = \text{Im}(\tilde{\partial}^*_{H_\epsilon} : L_{n,q+1}(Y', E)_{H_\epsilon, \tilde{\omega}_l} \to L_{n,q}(Y', E)_{H_\epsilon, \tilde{\omega}_l}).
\]
We give a brief explanation for decomposition (8). Usually $\operatorname{Im}\bar{\partial}$ is not closed in the $L^2$-space of a noncompact manifold even if the metric is complete. However, in the situation we consider here, $Y'$ has the compactification $Y$, and the forms on $Y'$ are bounded in $L^2$-norms. Such a form will have good extension properties. Therefore the set $L^{n,q}_{(2)}(Y', E)_{H, \omega_l} \cap \operatorname{Im}\bar{\partial}$ behaves much like the space

$$\operatorname{Im}(\bar{\partial} : L^{n,q-1}_{(2)}(Y, E)_{H, \omega} \rightarrow L^{n,q}_{(2)}(Y, E)_{H, \omega})$$

on $Y$, which is surely closed. The complete explanation can be found in [14, 38].

Now we have all the ingredients for the definition of $\Box_0$-harmonic forms. We denote the Laplacian operator on $Y'$ associated to $\tilde{\omega}_l$ and $H_{\varepsilon}$ by $\Box_{l, \varepsilon}$. Recall that for two $E$-valued $(n, q)$-forms $\alpha, \beta$ (not necessary to be $\bar{\partial}$-closed), we say they are cohomologically equivalent if there exists an $E$-valued $(n, q - 1)$-form $\gamma$ such that $\alpha = \beta + \bar{\partial}\gamma$. We denote by $\alpha \in [\beta]$ this equivalence relationship.

**Definition 4.1** (=Definition 1.1). Let $\alpha$ be an $E$-valued $(n, q)$-form on $Y$ with bounded $L^2$-norm with respect to $H, \omega$. Assume that for every $l \gg 1, \varepsilon \ll 1$, there exists a representative $\alpha_{l, \varepsilon} \in [\alpha]_{Y'}$ such that

1. $\Box_{l, \varepsilon}\alpha_{l, \varepsilon} = 0$ on $Y'$;
2. $\alpha_{l, \varepsilon} \to \alpha|_{Y'}$ in $L^2$-norm.

Then we call $\alpha$ a $\Box_0$-harmonic form. The space of all the $\Box_0$-harmonic forms is denoted by

$$\mathcal{H}^{n,q}(Y, E(H), \Box_0).$$

We will show that Definition 4.1 is compatible with the usual definition of $\Box_0$-harmonic forms for a smooth $H$ by proving the Hodge-type isomorphism, i.e. Proposition 1.2.

**Proof of Proposition 1.2.** By Proposition 2.4, we have

$$H^q(Y, K_Y \otimes E(H)) \cong \frac{\operatorname{Ker}(\bar{\partial} : A^{n,q}(Y, E(H)) \rightarrow A^{n,q+1}(Y, E(H)))}{\operatorname{Im}(\bar{\partial} : A^{n,q-1}(Y, E(H)) \rightarrow A^{n,q}(Y, E(H)))}.$$

Hence a given cohomology class $[\alpha] \in H^q(Y, K_Y \otimes E(H))$ is represented by a $\bar{\partial}$-closed $E$-valued $(n, q)$-form $\alpha$ with $\|\alpha\|_{H, \omega} < \infty$. We denote $\alpha|_{Y'}$ simply by $\alpha_{Y'}$. Since $\tilde{\omega}_l \geq \omega$, it is easy to verify that

$$|\alpha_{Y'}|_{H_{\varepsilon}, \tilde{\omega}_l}^2 dV_{\tilde{\omega}_l} \leq |\alpha|_{H_{\varepsilon, \omega}}^2 dV_{\omega},$$

which leads to inequality $\|\alpha_{Y'}\|_{H_{\varepsilon, \tilde{\omega}_l}} \leq \|\alpha\|_{H_{\varepsilon, \omega}}$ with $L^2$-norms. Hence by property (b), we have $\|\alpha_{Y'}\|_{H_{\varepsilon, \tilde{\omega}_l}} \leq \|\alpha\|_{H_{\varepsilon, \omega}}$ which implies

$$\alpha_{Y'} \in L^{n,q}_{(2)}(Y', E)_{H_{\varepsilon, \tilde{\omega}_l}}.$$

By decomposition (8), we have a harmonic representative $\alpha_{l, \varepsilon}$ in

$$\mathcal{H}^{n,q}_{H_{\varepsilon, \omega_l}}(E),$$
which means that \( \Box_{l,\epsilon} \alpha_{l,\epsilon} = 0 \) on \( Y' \) for all \( l, \epsilon \). Moreover, since a harmonic representative minimizes the \( L^2 \)-norm, we have
\[
\| \alpha_{l,\epsilon} \|_{H_{\alpha,\omega}} \leq \| \alpha_{Y'} \|_{H_{\alpha,\omega}} \leq \| \alpha \|_{H_{\alpha,\omega}}.
\]
So there exists a limit \( \tilde{\alpha} \) of (a subsequence of) \( \{ \alpha_{l,\epsilon} \} \) such that
\[
\tilde{\alpha} \in [\alpha_{Y'}].
\]
It is left to extend it to \( Y \).

Indeed, if we admit Proposition\.3 for the time being, \( \ast \tilde{\alpha} \) will be a holomorphic \( E \)-valued \((n - q, 0)\)-form on \( Y' \). In particular, since the \( \ast \)-operator preserves the \( L^2 \)-norm, the \( L^2 \)-norm of \( \ast \tilde{\alpha} \) is also bounded. Now apply the canonical \( L^2 \)-extension theorem [35], we obtain a holomorphic extension of \( \ast \tilde{\alpha} \) on \( Y \), which is still denoted by \( \ast \tilde{\alpha} \). We denote this morphism by
\[
S_q : H^q(Y, K_Y \otimes E(H)) \rightarrow H^0(Y, \Omega_{Y'}^{n-q} \otimes E)
\]
\[
[\alpha] \mapsto \ast \tilde{\alpha}.
\]

Let \( \hat{\alpha} := c_{n-q} \omega^q \wedge \ast \tilde{\alpha} \), we then obtain an \( E \)-valued \((n, q)\)-form on \( Y \). It is easy to verify that \( \hat{\alpha}|_{Y'} = \tilde{\alpha} \). In summary, we have successfully defined a morphism
\[
i : H^q(Y, K_Y \otimes E(H)) \rightarrow \mathcal{H}^{n,q}(Y, E(H), \Box_0)
\]
\[
[\alpha] \mapsto \hat{\alpha}.
\]

On the other hand, for a given \( \alpha \in \mathcal{H}^{n,q}(Y, E(H), \Box_0) \), by definition there exists an \( \alpha_{l,\epsilon} \in [\alpha_{Y'}] \) with \( \alpha_{l,\epsilon} \in \mathcal{H}^{n,q}_{H_{\alpha,\omega}}(E) \) for every \( l, \epsilon \) such that
\[
\lim_{l,\epsilon} \alpha_{l,\epsilon} = \alpha_{Y'}.
\]
In particular, \( \bar{\partial} \alpha_{l,\epsilon} = 0 \). So all of the \( \alpha_{l,\epsilon} \) together with \( \alpha_{Y'} \) define a common cohomology class \([\alpha_{Y'}]\) in \( H^{n,q}(Y', E(H)) \). It is left to extend this class to \( Y \).

We use the \( S^q \) again. It maps \([\alpha_{Y'}]\) to
\[
S^q(\alpha_{Y'}) \in H^0(Y, \Omega_{Y'}^{n-q} \otimes E).
\]
Then
\[
c_{n-q} \omega^q \wedge S^q(\alpha_{Y'}) \in H^q(Y, K_Y \otimes E(H))
\]
with \([c_{n-q} \omega^q \wedge S^q(\alpha_{Y'}))|_{Y'} = [\alpha_{Y'}]\) as a cohomology class in \( H^{n,q}(Y', E(H)) \). We denote this morphism by
\[
j : \mathcal{H}^{n,q}(Y, E(H), \Box_0) \rightarrow H^q(Y, K_Y \otimes E(H))
\]
\[
\alpha \mapsto [c_{n-q} \omega^q \wedge S^q(\alpha_{Y'})].
\]
It is easy to verify that \( i \circ j = \text{Id} \) and \( j \circ i = \text{Id} \). The proof is finished. \( \square \)

We now prove Proposition\.3 to finish this subsection.
Proof of Proposition 1.3. (1) Since $\alpha$ is $\Box_0$-harmonic, there exists an $\alpha_{t,\varepsilon} \in [\alpha_Y]$ with $\alpha_{t,\varepsilon} \in \mathcal{H}^{a,q}_{H_{E,\varepsilon}}(E)$ for every $t, \varepsilon$ such that $\lim \alpha_{t,\varepsilon} = \alpha_Y$. In particular, $\bar{\partial} \alpha_{t,\varepsilon} = \bar{\partial}_t^* \alpha_{t,\varepsilon} = 0$. Apply Proposition 2.5 on $Y'$ with $\eta = 1$, we have

$$0 = \|\bar{\partial} \alpha_{t,\varepsilon}\|_{H_{E,\varepsilon}^2}^2 + \|\bar{\partial}_t^* \alpha_{t,\varepsilon}\|_{H_{E,\varepsilon}^2}^2$$

$$(9)$$

$$(\bar{\partial} \alpha_{t,\varepsilon})^2 + \|\bar{\partial} \alpha_{t,\varepsilon}\|_{H_{E,\varepsilon}^2}^2 = \|\bar{\partial}_t^* \alpha_{t,\varepsilon}\|_{H_{E,\varepsilon}^2}^2 + \|\partial H_{E,\varepsilon}^2 + <i[\Theta_{E,H_{\varepsilon}}, \Lambda] \alpha_{t,\varepsilon}, \alpha_{t,\varepsilon} >_{H_{E,\varepsilon}^2}$$

Remember that $i \Theta_{E,H_{\varepsilon}} \geq -\varepsilon \omega \otimes \text{Id}_E$ in the sense of Nakano,

$$<i[\Theta_{E,H_{\varepsilon}}, \Lambda] \alpha_{t,\varepsilon}, \alpha_{t,\varepsilon} >_{H_{E,\varepsilon}^2} = -\varepsilon' \omega$$

by elementary computation. Now take the limit on the both sides of formula (9) with respect to $t, \varepsilon$, we eventually obtain that

$$\lim \|\bar{\partial}_t^* \alpha_{t,\varepsilon}\|_{H_{E,\varepsilon}^2}^2 = \lim \|\bar{\partial} \alpha_{t,\varepsilon}\|_{H_{E,\varepsilon}^2}^2$$

$$= \lim <i[\Theta_{E,H_{\varepsilon}}, \Lambda] \alpha_{t,\varepsilon}, \alpha_{t,\varepsilon} >_{H_{E,\varepsilon}^2} = 0.$$

In particular,

$$0 = \lim \bar{\partial}_t \alpha_{t,\varepsilon} = \star \tilde{\partial} \star \alpha_{t,\varepsilon} = \star \tilde{\partial} \star \alpha$$

in $L^2$-topology. Equivalently, $\bar{\partial} \alpha = 0$ on $Y'$ in analytic topology, hence is a holomorphic $E$-valued $(n - q, 0)$-form on $Y'$. On the other hand, since $\star \alpha$ has the bounded $L^2$-norm on $Y'$, it extends to the whole space by classic $L^2$-extension theorem $[35]$. In other word, $\bar{\partial} \star \alpha = 0$ actually holds on $Y$, hence $\star \alpha$ is an $E$-valued holomorphic $(n - q, 0)$-form on $Y$.

(2) Apply the same argument before, we have $\alpha = c_{n-q} \omega^q \wedge \star \alpha$. Therefore $\alpha$ must be smooth. \(\square\)

4.2. Local theory. In practice, we will also deal with the manifold with boundary, hence the local theory is also needed. Let $V$ be a bounded domain with smooth boundary $\partial V$ on $(Y, \omega)$. Moreover, there is a smooth plurisubharmonic exhaustion function $r$ on $V$. In particular, $V = \{ r > 0 \}$ and $dr \neq 0$ on $\partial V$. The volume form $dS$ of the real hypersurface $\partial V$ is defined by $dS := *(dr)/|dr|_\omega$. Let $G$ be a smooth Hermitian metric on $E$. Let $L^{p,q}_{(2)}(V,E)_{G,\omega}$ be the space of $E$-valued $(p,q)$-forms on $V$ which are $L^2$-bounded with respect to $G, \omega$. Setting $\tau := dS/|dr|_\omega$ we define the inner product on $\partial V$ by

$$[\alpha, \beta]_G := \int_{\partial V} <\alpha, \beta >_G \tau$$

for $\alpha, \beta \in L^{p,q}_{(2)}(V,E)_{G,\omega}$. Then by Stokes’ theorem we have the following:

$$<\bar{\partial} \alpha, \beta >_G = <\alpha, \bar{\partial} \beta >_G + [\alpha, e(\bar{\partial} r)^* \beta]_G$$

$$<\partial G \alpha, \beta >_G = <\alpha, \bar{\partial} c \beta >_G + [\alpha, e(\partial r)^* \beta]_G$$

(10)
where $\bar{\partial}^* \ast \partial^*$ are the adjoint operators defined on $Y$ (see Sect.2.6). In particular, if $e(\partial r)^* \alpha = 0$, the Bochner formula (Proposition 2.5) on $V$ will be
\begin{equation}
\Box_G = \tilde{\Box}_G + [i \Theta_{E,G}, \Lambda],
\end{equation}
\begin{align*}
\Box_{e^{-\lambda}G} &= \Box_{e^{-\lambda}G} + [i \Theta_{E,G} + \partial \bar{\partial} \chi], \Lambda] \text{ and } \\
&\|\sqrt{\eta}(\bar{\partial} + e(\partial \chi))\alpha\|^2_{G,\omega} + \|\sqrt{\eta}\partial^* \alpha\|^2_{G,\omega} = \|\sqrt{\eta}(\partial^* - e(\partial \chi)^*)\alpha\|^2_{G,\omega} \\
&+ \|\sqrt{\eta}\partial \alpha\|^2_{G,\omega} + \eta \|i \Theta_{E,G} + \partial \bar{\partial} \phi, \Lambda] \alpha, \alpha > _{G,\omega} + [\partial^* \alpha, e(\partial r)^* \alpha]_G
\end{align*}
where $\eta$ is a positive smooth function on $Y$ with $\chi := \log \eta$.

We then define the space of harmonic forms on $V$ by
\[ \mathcal{H}^{n,q}(V, E(G), r, \omega) := \{ \alpha \in L^n_{(2)}(V, E)_{G,\omega}; \bar{\partial} \alpha = \bar{\partial}^* \alpha = e(\partial r)^* \alpha = 0 \} \]

Now return back to our setting: $(E, H)$ is a singular Hermitian vector bundle that is strongly positive in the sense of Nakano. By Proposition 1.1 there exists a regularising sequence $\{ H_\varepsilon \}$ such that
\[ i \Theta_{E,H} \geq -\varepsilon \omega \otimes \text{Id}_E \]
in the sense of Nakano. Take $V \subset Y'$. Using the same notations as in Sect.4.1, the harmonic space with respect to $H$ is defined as
\[ \mathcal{H}^{n,q}(V, E(H), r) := \{ \alpha \in L^n_{(2)}(V, E)_{H,\omega}; \text{there exists } \alpha_{t,\varepsilon} \in [\alpha] \text{ such that } \alpha_{t,\varepsilon} \in \mathcal{H}^{n,q}(V, E(G_\varepsilon), r, \omega_t) \text{ and } \alpha_{t,\varepsilon} \to \alpha \text{ in } L^2\text{-limit} \} \]

We then generalise the work in [37] here.

**Proposition 4.1.** Assume that $(E, H)$ is strongly positive in the sense of Nakano (so that $E(H)$ is coherent by Proposition 2.3). Then we have the following conclusions:

1. Assume $\alpha \in L^n_{(2)}(Y, E)_{H,\omega}$ satisfied $e(\partial r)^* \alpha = 0$ on $V$. Then $\alpha$ satisfies $\bar{\partial} \alpha = \lim \bar{\partial}_H \alpha = 0$ on $V$ if and only if $\bar{\partial} \ast \alpha = 0$ and $\lim <i e(\Theta_{H} + \partial \bar{\partial} r)\Lambda \alpha, \alpha >_{H,\omega} = 0$ on $V$.
2. $\mathcal{H}^{n,q}(V, E(H), r)$ is independent of the choice of exhaustion function $r$.
3. $\mathcal{H}^{n,q}(V, E(H), r) \simeq H^q(V, K_Y \otimes E(H))$.
4. For Stein open subsets $V_1, V_2$ in $V$ such that $V_2 \subset V_1$, the restriction map
\[ \mathcal{H}^{n,q}(V_1, E(H), r) \rightarrow \mathcal{H}^{n,q}(V_2, E(H), r) \]
is well-defined, and further it satisfies the following commutative diagram:
\[ \begin{array}{ccc}
\mathcal{H}^{n,q}(V_1, E(H), r) & \xrightarrow{S^q_{V_1}} & H^0(V_1, \Omega_{V_Y}^{n-q} \otimes E(H)) \\
\downarrow & & \downarrow \\
\mathcal{H}^{n,q}(V_2, E(H), r) & \xrightarrow{S^q_{V_2}} & H^0(V_2, \Omega_{V_Y}^{n-q} \otimes E(H)).
\end{array} \]
Proof. The proof uses the same argument as Theorems 4.3 and 5.2 in [37] with minor adjustment. So we only provide the necessary details.

(1) Let \( G = e^{-r}H \) and \( G_\varepsilon = e^{-r}H_\varepsilon \). If \( \partial \alpha = \lim \partial_{H_\varepsilon}^* \alpha = 0 \), then \( \lim \partial_{G_\varepsilon}^* \alpha = 0 \) and so \( \lim \square G_\varepsilon \alpha = 0 \). By formula (11) we obtain

\[
\lim \| \partial_{G_\varepsilon}^* \alpha \|_{G_\varepsilon}^2 + \langle ie(\Theta_{E,G_\varepsilon} + \partial \partial r)\Lambda, \alpha >_{G_\varepsilon} + [ie(\partial \partial r)\Lambda, \alpha]_{G_\varepsilon} \rangle = 0
\]
on \( V \). Since \( \langle ie(\Theta_{E,H_\varepsilon} + \partial \partial r)\Lambda, \alpha >_{H_\varepsilon} \rangle \geq -\varepsilon \omega \) and

\[
[ie(\partial \partial r)\Lambda, \alpha]_{G_\varepsilon} \geq 0,
\]
the equality above implies that

\[
\ast \partial^* \alpha = \lim \langle ie(\Theta_{E,H_\varepsilon} + \partial \partial r)\Lambda, \alpha >_{H_\varepsilon} = \lim [ie(\partial \partial r)\Lambda, \alpha]_{G_\varepsilon} = 0.
\]
Equivalently,

\[
\partial^* \alpha = \lim \langle ie(\Theta_{E,H_\varepsilon} + \partial \partial r)\Lambda, \alpha >_{H_\varepsilon} = 0.
\]
The necessity is proved.

Now assume that \( \partial^* \alpha = 0 \) and \( \lim \langle ie(\Theta_{H_\varepsilon} + \partial \partial r)\Lambda, \alpha >_{H_\varepsilon} = 0 \). Since \( r \) is plurisubharmonic and \( \lim \langle ie(\Theta_{H_\varepsilon})\Lambda, \alpha >_{H_\varepsilon} \rangle = 0 \), we have \( \lim \langle ie(\partial \partial r)\Lambda, \alpha >_{H_\varepsilon} \rangle = \lim \langle ie(\Theta_{H_\varepsilon})\Lambda, \alpha >_{H_\varepsilon} \rangle = 0 \). By formula (11) we have \( \partial_\alpha = \lim \partial_{H_\varepsilon}^* \alpha = 0 \).

(2) Let \( \tau \) be an arbitrary smooth plurisubharmonic function on \( V \). Donnelly and Xavier’s formula (6) implies that \( \partial e(\partial \tau)^* \alpha = ie(\partial \partial \tau)\Lambda\alpha \) if \( \alpha \in H^{n,q}(V, E(H), r) \).

Therefore

\[
\langle ie(\partial \partial r)\Lambda, \alpha >_{e^r H_\varepsilon} = \langle \partial e(\partial \tau)^* \alpha, \alpha >_{e^r H_\varepsilon} = \langle e(\partial \partial r)^* \alpha, \partial_{e^r H_\varepsilon}^* \alpha >_{e^r H_\varepsilon} = \langle e(\partial \partial r)^* \alpha, \partial_{H_\varepsilon}^* \alpha >_{e^r H_\varepsilon} - \| e(\partial \partial r)^* \alpha \|_{e^r H_\varepsilon}^2.
\]
Take the limit with respect to \( \varepsilon \), we then obtain that

\[
\langle ie(\partial \partial r)\Lambda, \alpha >_{e^r H} = -\| e(\partial \partial r)^* \alpha \|_{e^r H}^2.
\]
Notice that \( \tau \) is plurisubharmonic, we actually have

\[
\langle ie(\partial \partial r)\Lambda, \alpha >_{e^r H} = \| e(\partial \partial r)^* \alpha \|_{e^r H}^2 = 0.
\]
Combine with (1), we eventually obtain that

\[
H^{n,q}(V, E(H), r) = H^{n,q}(V, E(H), r + \tau)
\]
for any smooth plurisubharmonic \( \tau \), hence the desired conclusion.

(3) is similar with Proposition [12] and we omit its proof here.

(4) is intuitive due to the discussions in the global setting. In particular, \( S_{V_i}^q \) with \( i = 1, 2 \) is defined in the proof of Proposition [12].
5. The main theorem

This section is devoted to prove Theorem 1.1.

Theorem 5.1 (=Theorem 1.1). Let \( f : Y \to Z \) be a fibration between two compact Kähler manifolds. Let \( n = \dim Y \) and \( m = \dim Z \). Suppose that \( (E, H) \) is a (singular) Hermitian vector bundle over \( Y \) that is strongly positive in the sense of Nakano. Moreover, assume that \( H|_{Y_z} \) is well-defined for every \( z \in Z \). Then the following theorems hold:

I Decomposition Theorem. The Leray spectral sequence

\[
E^{p,q}_2 = H^p(Z, R^q f_* (K_Y \otimes E(H))) \Rightarrow H^{p+q}(Y, K_Y \otimes E(H))
\]
degenerates at \( E_2 \). As a consequence, it holds that

\[
\dim H^i(Y, K_Y \otimes E(H)) = \sum_{p+q=i} \dim H^p(Z, R^q f_* (K_Y \otimes E(H)))
\]
for any \( i \geq 0 \).

II Torsion freeness Theorem. For \( q \geq 0 \) the sheaf homomorphism

\[
L^q : f_* \Omega^{n-q} \otimes E(H) \to R^q f_* (K_Y \otimes E(H))
\]
induced by the \( q \)-times left wedge product by \( \omega \) admits a splitting sheaf homomorphism

\[
S^q : R^q f_* (K_Y \otimes E(H)) \to f_* \Omega^{n-q} \otimes E(H) \text{ with } L^q \circ S^q = \text{id}.
\]
In particular, \( R^q f_* (K_Y \otimes E(H)) \) is torsion free \(^{[20]}\) for \( q \geq 0 \) and vanishes if \( q > n - m \). Furthermore, it is even reflexive if \( E(H) = E \).

III Injectivity Theorem. Let \( (L, h) \) be a (singular) Hermitian line bundle over \( Y \). Recall that \( Y' \) is the open subvariety appeared in Proposition 1.1. Assume the following conditions:

(a) the singular part of \( h \) is contained in \( Y - Y' \);
(b) \( i\Theta_{L,h} \geq \gamma \) for some real smooth \((1,1)\)-form \( \gamma \) on \( Y \);
(c) for some rational \( \delta \ll 1 \), the \( \mathbb{Q} \)-twisted bundle

\[
E < -\delta L > |_{Y_z}
\]
is strongly positive in the sense of Nakano for every \( z \).

For a (non-zero) section \( s \) of \( L \) with \( \sup_Y |s|_h < \infty \), the multiplication map induced by the tensor product with \( s \)

\[
R^q f_* (s) : R^q f_* (K_Y \otimes E(H)) \to R^q f_* (K_Y \otimes (E \otimes L)(H \otimes h))
\]
is well-defined and injective for any \( q \geq 0 \).

IV Relative vanishing Theorem. Let \( g : Z \to W \) be a fibration to a compact Kähler manifold \( W \). Then the Leray spectral sequence:

\[
R^p g_* R^q f_* (K_Y \otimes E(H)) \Rightarrow R^{p+q}(g \circ f)_* (K_Y \otimes E(H))
\]
degenerates.
Proof. I. Let \( \{U, r_U\} \) be a finite Stein covering of \( Z \) with smooth strictly plurisubharmonic exhaustion function \( r_U \). Let
\[
\mathcal{H}^n,q(f^{-1}(U), E(H), f^*r_U)
\]
be the harmonic space defined in Sect.4.2. Then the data
\[
\{\mathcal{H}^n,q(f^{-1}(U), E(H), f^*r_U), i_2^1\}
\]
with the restriction morphisms
\[
i_2^1 : \mathcal{H}^n,q(f^{-1}(U_1), E(H), f^*r_{U_1}) \to \mathcal{H}^n,q(f^{-1}(U_2), E(H), f^*r_{U_2}),
\]
\((U_2, r_{U_2}) \subseteq (U_1, r_{U_1}), \) yields a presheaf \([22]\) on \( Z \) by Proposition 4.4 (4). We denote the associated sheaf by \( f_*\mathcal{H}^n,q(E(H)) \). Since
\[
R^q f_*(K_Y \otimes E(H))
\]
is defined as the sheaf associated with the presheaf
\[
U \to H^q(f^{-1}(U), K_Y \otimes E(H)),
\]
the sheaf \( f_*\mathcal{H}^n,q(E(H)) \) is isomorphic to \( R^q f_*(K_Y \otimes E(H)) \) by Proposition 4.4 (3).

Let \( C^p,q \) be the space of \( p \)-cochains associated to \( \{U\} \) with values in \( f_*\mathcal{H}^n,q(E(H)) \). Then \( \{C^p,q, \delta\} \) is a complex with the coboundary operator \( \delta \) whose cohomology group \( H^p(C^*q) \) is isomorphic to
\[
E_2^{p,q} = H^p(Y, R^q f_*(K_Y \otimes E(H))).
\]
Since the differential \( d := \delta + \bar{\delta} \) is identically zero, (recall that an element \( \alpha \in f_*\mathcal{H}^n,q(E(H)) \) must satisfy \( \bar{\delta}\alpha = 0 \), \( d_2 : E_2^{p,q} \to E_2^{p+2,q-1} \) is also so which implies the degeneration of the Leray spectral sequence at \( E_2 \), i.e. \( E_2^{p,q} = E_2^{p,q} \).

II. Recall that in the proof of Proposition 1.2 we’ve defined two morphisms
\[
S^q : H^q(Y, K_Y \otimes E(H)) \to H^0(Y, \Omega_Y^{n-q} \otimes E(H)) \quad [\alpha] \mapsto *\tilde{\alpha}
\]
and
\[
L^q : H^q(Y, \Omega_Y^{n-q} \otimes E(H)) \to H^q(Y, K_Y \otimes E(H)) \quad \beta \mapsto [c_{n-q}\omega^q \wedge \beta]
\]
such that \( L^q \circ S^q = \text{Id} \). These two morphisms lift to the direct images as
\[
S^q : R^q f_*(K_Y \otimes E(H)) \to f_*(\Omega_Y^{n-q} \otimes E(H)),
\]
\[
L^q : f_*(\Omega_Y^{n-q} \otimes E(H)) \to R^q f_*(K_Y \otimes E(H)).
\]
Here we abuse the notation. In particular, \( L^q \circ S^q = \text{Id} \). As a result, \( R^q f_*(K_Y \otimes E(H)) \) is splitting embedded into \( f_*(\Omega_Y^{n-q} \otimes E(H)) \). Obviously, \( f_*(\Omega_Y^{n-q} \otimes E(H)) \) is an \( O_Z \)-submodule of \( f_*(\Omega_Y^{n-q} \otimes E) \), whereas \( f_*(\Omega_Y^{n-q} \otimes E) \) is torsion free (even reflexive) by \([23]\). We then conclude
that $R^q f_*(K_Y \otimes E(H))$, as an $O_Z$-submodule of $f_*(\Omega_Y^{n-q} \otimes E)$, is also torsion free.

When $E(H) = E$, the reflexivity is also inherited since it is a splitting embedding.

**III.** It is enough to prove that for an arbitrary point $z \in Z$,
\[
\otimes s : H^q(f^{-1}(z), K_Y \otimes E(H)) \to H^q(f^{-1}(z), K_Y \otimes (E \otimes L)(H \otimes h))
\]
is injective. Equivalently, we should prove that
\[
\otimes s : \mathcal{H}^{n,q}(f^{-1}(z), E(H)) \to \mathcal{H}^{n,q}(f^{-1}(z), (E \otimes L)(H \otimes h))
\]
is well-defined and injective by Proposition 1.2.

Let $\alpha \in \mathcal{H}^{n,q}(f^{-1}(z), E(H))$, by Proposition 4.1, (1) with $r = 1$, we have $\bar{\partial} \ast \alpha = \lim < \Theta_{H_e} \Lambda s\alpha, s\alpha >_{H_e} = 0$. Let $\{h_e\}$ be a regularising sequence of $h$ (Proposition 1.1), hence $\{H_e \otimes h_e\}$ is a regularising sequence of $H \otimes h$. In particular, $H_e \otimes h_e$ is smooth on $Y' \cap f^{-1}(z)$. Therefore
\[
\bar{\partial} \ast (s\alpha) = s\bar{\partial} \ast \alpha = 0
\]
and on $Y' \cap f^{-1}(z)$ we have
\[
\lim < \Theta_{E \otimes L, H_e \otimes h_e} \Lambda s\alpha, s\alpha >_{H_e \otimes h_e} = \lim |s|_{h_e}^2 \lim < \Theta_{E \otimes L, H_e \otimes h_e} \Lambda \alpha, \alpha >_{H_e} = |s|_{h_e}^2 \lim < \Theta_{E \otimes L, H_e \otimes h_e} \Lambda \alpha, \alpha >_{H_e} \leq \frac{1}{\delta} |s|_{h_e}^2 \lim < \Theta \Lambda \alpha, \alpha >_{H_e} = 0.
\]
The inequality is due to the assumption that $E < -\delta L >$ is strongly positive in the sense of Nakano. Now we apply the Bochner formula (Proposition 2.1) on $(E \otimes L, H_e \otimes h_e)$ and take the limit, eventually we obtain that
\[
\lim \|\bar{\partial}_{H_e \otimes h_e} (s\alpha)\|^2_{H_e \otimes h_e} = 0,
\]
and
\[
\bar{\partial}(s\alpha) = 0.
\]
On the other hand, in any local coordinate neighbourhood $V$ in $f^{-1}(z)$,
\[
\int_V |s\alpha|^2_{H \otimes h} \leq \sup_Y |s|^2_h \int_V |\alpha|^2_H < \infty.
\]
The discussions above together imply that $s\alpha \in \mathcal{H}^{n,q}(f^{-1}(z), (E \otimes L)(H \otimes h))$ by Proposition 1.1 (1). Therefore
\[
\otimes s : \mathcal{H}^{n,q}(f^{-1}(z), E(H)) \to \mathcal{H}^{n,q}(f^{-1}(z), (E \otimes L)(H \otimes h))
\]
is well-defined. The injectivity is obvious.
IV is a direct application of I. For a given point \( w \in W \), by I the Leray spectral sequence
\[ E_2^{p,q} = H^p(g^{-1}(w), R^q f_*(K_Y \otimes E(H))) \Rightarrow H^{p+q}((g \circ f)^{-1}(w), K_Y \otimes E(H)) \]
degenerates at \( E_2 \). Therefore the Leray spectral sequence:
\[ R^p g_* R^q f_*(K_Y \otimes E(H)) \Rightarrow R^{p+q}((g \circ f)_*(K_Y \otimes E(H)) \]
degenerates. \( \Box \)

6. Vanishing theorem

We should prove Theorem 1.2 in the end.

**Theorem 6.1** (=Theorem 1.2). Let \( f : Y \to Z \) be a fibration between two compact Kähler manifolds.

**I Nadel-type vanishing Theorem.** Let \( (L, h) \) be an \( f \)-big line bundle, and let \( (E, H) \) be a vector bundle that is strongly positive in the sense of Nakano. Assume that \( H|_{Y_z} \) is well-defined for every \( z \). Then
\[ R^q f_*(K_Y \otimes (E \otimes L)(H \otimes h)) = 0 \text{ for every } q > 0. \]

**II Nakano-type vanishing Theorem.** Assume that \( (E, H) \) is strongly strictly positive in the sense of Nakano. Then
\[ H^q(Y, K_Y \otimes S^l E(S^l H)) = 0 \text{ for every } l, q > 0. \]

**III Griffiths-type vanishing Theorem.** Assume that \( (E, H) \) is strictly positive in the sense of Griffiths. Then
\[ H^q(Y, K_Y \otimes S^l (E \otimes \text{det} E)(S^l (H \otimes \text{det} H))) = 0 \text{ for every } l, q > 0. \]

Recall that \( L \) is \( f \)-big if the Iitaka dimension \( \kappa(Y_z, L) = \dim Y - \dim Z \)
for every \( z \in Z \).

**Proof.** I. We claim that if \( L \) is \( f \)-big, there exists a singular metric \( h \) on \( L \) such that \( h|_{Y_z} \) is well-defined and \( i\Theta_{L,h}|_{Y_z} \) is strictly positive for every \( z \in Z \).

Indeed, by definition for every \( z \in Z \) there exists a singular metric \( h_z \) on \( L|_{Y_z} \) such that \( i\Theta_{L,h_z} \) is strictly positive \[ 8 \]. Now take a smooth metric \( h_0 \) on \( L \) and define a singular metric on \( L \) as follows:
\[ h := h_0^\delta \otimes h_z^{1-\delta} \text{ for } y \in Y_z. \]

It is easy to verify that \( h \) satisfies the desired property when \( \delta \) is small enough. In particular, \( i\Theta_{L,h} \geq \gamma \) for some real smooth \((1,1)\)-form on \( Y \). Let \( A \) be a sufficiently ample line bundle over \( Z \). Then
\[ H^q(Y_z, K_Y \otimes (E \otimes L)(H \otimes h) \otimes A) = 0 \text{ for } q > 0 \]
by Serre’s asymptotic vanishing theorem \[ 22 \], hence
\[ R^q f_*(K_Y \otimes (E \otimes L)(H \otimes h) \otimes A) = 0 \text{ for } q > 0. \]
On the other hand, it is easy to verify that $A$ satisfies the conditions in Theorem III, therefore

$$R^q f_*(K_Y \otimes (E \otimes L)(H \otimes h)) \rightarrow R^q f_*(K_Y \otimes (E \otimes L)(H \otimes h) \otimes A)$$

is injective. As a result, $R^q f_*(K_Y \otimes (E \otimes L)(H \otimes h)) = 0$ for $q > 0$.

**II.** By Proposition 1.2, it is enough to prove that

$$\mathcal{H}^{n,q}(Y, S^l E(S^l H), \square_0) = 0$$

Since $(E, H)$ is strongly strictly positive in the sense of Nakano,

$$(S^l E, H^l_\varepsilon)$$

is strictly positive in the sense of Nakano on $Y'$ for every $\varepsilon$ by Proposition (e). Indeed, in the proof of (e) it is easy to see that if the positivity of $(E, H)$ is strict, the regularising sequence $\{H^l_\varepsilon\}$ will satisfies that

(e') for every relatively compact subset $Y'' \subset \subset Y'$ and every $l$,

$$i \Theta_{S^l E, H^l_\varepsilon} \geqslant C_l (1 - \varepsilon) \omega \otimes \text{Id}_{S^l E}$$

over $Y''$ in the sense of Nakano.

Now we apply Bochner’s formula (5) to $(S^l E, H^l_\varepsilon)$ on $Y'$. Notice that $Y - Y'$ is a closed subvariety hence has real codimension $\geqslant 2$. In particular, the integral equality in (5) holds here. For any

$$\alpha \in \mathcal{H}^{n,q}(Y, S^l E(S^l H), \square_0),$$

we have

$$0 = \lim(\|\partial_{H^l_\varepsilon} \alpha\|^2_{H^l_\varepsilon} + <i e(\Theta_{S^l E, H^l_\varepsilon}) \Lambda \alpha, \alpha >_{H^l_\varepsilon}).$$

Since $(S^l E, H^l_\varepsilon)$ is strictly positive in the sense of Nakano, the Hermitian form $<i e(\Theta_{S^l E, H^l_\varepsilon}) \Lambda \cdot, \cdot >_{H^l_\varepsilon}$ is positive-definite. Thus, we must have $\alpha = 0$. The proof is complete.

**III.** Since $(E, H)$ is strictly positive in the sense of Griffiths,

$$(E \otimes \text{det } E, H \otimes \text{det } H)$$

is strongly strictly positive in the sense of Nakano due to [5], Theorem 1.3. Then we apply II on $(E \otimes \text{det } E, H \otimes \text{det } H)$ to obtain the desired conclusion. □

**References**

[1] Atiyah, M., Macdonald, I.: Introduction to Commutative Algebra, Addison-Wesley (1969)

[2] Berndtsson, B.: Curvature of vector bundles associated to holomorphic fibrations. Ann. of Math. 169, 531-560 (2009)

[3] Berndtsson, B.: Strict and nonstrict positivity of direct image bundles. Math. Z. 269, 1201-1218 (2011)

[4] de Cataldo, M.: Singular Hermitian metrics on vector bundles. J. Reine Angew. Math. 502, 93-122 (1998)

[5] Chen, X. X., Tian, G.: Geometry of Kähler metrics and foliations by holomorphic discs. Publ. Math. Inst. Hautes Études 107, 1-107 (2008)
[29] Kollár, J.: Higher direct images of dualizing sheaves. II. Ann. of Math. (2) 124, 171-202 (1986)
[30] Lazarsfeld, R.: Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 49. Springer-Verlag, Berlin, 2004. xviii+385 pp. ISBN: 3-540-22534-X.
[31] Matsumura, S.: A Nadel vanishing theorem via injective theorems. Math. Ann. 359, 785-802 (2014)
[32] Matsumura, S.: Injectivity theorems with multiplier ideal sheaves and their applications. Complex analysis and geometry, 241-255. Springer Proc. Math. Stat., 144 (2015)
[33] Matsumura, S.: Injectivity theorems with multiplier ideal sheaves for higher direct images under Kähler morphisms. [arXiv:1607.05554]
[34] Nadel, Alan M.: Multiplier ideal sheaves and Kähler–Einstein metrics of positive scalar curvature. Ann. of Math. (2) 132, 549-596 (1990)
[35] Ohsawa, T.: Analysis of several complex variables. Translations of Mathematical Monographs, 211, AMS (2002)
[36] Raufi, H.: Singular hermitian metrics on holomorphic vector bundles. Ark. Mat. 53, 359-382 (2015)
[37] Takegoshi, K.: Higher direct images of canonical sheaves tensorized with semi-positive vector bundles by proper Kähler morphisms. Math. Ann. 303, 389-416 (1995)
[38] Wu, J.: A Kollár-type vanishing theorem. Math. Z. 295, 331-340 (2020)
[39] Wu, J.: An alternative approach on the singular metric on a vector bundle. [arXiv:2011.02725]

Current address: School of Mathematical Sciences, Fudan University, Shanghai 200433, People’s Republic of China.
E-mail address: jingcaowu13@fudan.edu.cn