PHASE TRANSITIONS ON C*-ALGEBRAS FROM ACTIONS OF CONGRUENCE MONOIDS ON RINGS OF ALGEBRAIC INTEGERS

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Abstract. We compute the KMS (equilibrium) states for the canonical time evolution on C*-algebras from actions of congruence monoids on rings of algebraic integers. We show that for each $\beta \in [1, 2]$, there is a unique KMS$_\beta$ state, and we prove that it is a factor state of type III$_1$. There is a phase transition at $\beta = 2$: For each $\beta \in (2, \infty]$, the set of extremal KMS$_\beta$ states decomposes as a disjoint union over a quotient of a ray class group in which the fibers are extremal traces on certain group C*-algebras associated with the ideal classes. Moreover, in most cases, there is a further phase transition at $\beta = \infty$ in the sense that there are ground states that are not KMS$_\infty$ states. Our computation of KMS and ground states generalizes the results of Cuntz, Deninger, and Laca for the full $ax + b$-semigroup over a ring of integers, and our type classification generalizes a result of Laca and Neshveyev in the case of the rational numbers and a result of Neshveyev in the case of arbitrary number fields.

1. Introduction

Given a number field $K$ with ring of integers $R$, Cuntz, Deninger, and Laca studied phase transitions for the canonical time evolution on the left regular C*-algebra $C^*_\lambda(R \rtimes R^\times)$ of the (full) $ax + b$-semigroup $R \rtimes R^\times$ over $R$, see [C-D-L]. Their results built on work of Laca and Raeburn in [La-Rae2] and Laca and Neshveyev in [La-Nesh2] on the similar semigroup $\mathbb{N} \ltimes \mathbb{N}^\times$ associated to the number field $\mathbb{Q}$. The classification of KMS and ground states from [C-D-L] showed that the C*-dynamical system associated with $C^*_\lambda(R \rtimes R^\times)$ exhibits several interesting properties; for example, the parameterization spaces for both the ground states and the low temperature KMS states decomposes over the ideal class group $\text{Cl}(K)$ of $K$, and uniqueness for the high temperature KMS states is related to the distribution of ideals over the group $\text{Cl}(K)$. In [Nesh3], Neshveyev developed general results for computing KMS states on C*-algebras of non-principal groupoids, and gave an alternative computation of the KMS states on $C^*_\lambda(R \rtimes R^\times)$.

The construction from [C-D-L] was recently generalized in [Bru] by restricting the multiplicative part of $R \rtimes R^\times$ to lie in certain subsemigroups of $R^\times$. Specifically, given a modulus $m = m_\infty m_0$ for $K$ and a group $\Gamma$ of residues modulo $m$, one considers the left regular C*-algebra $C^*_\lambda(R \rtimes R_{m, \Gamma})$ of the semi-direct product $R \rtimes R_{m, \Gamma}$ where $R_{m, \Gamma} \subseteq R^\times$ is the congruence monoid consisting of algebraic integers in $R^\times$ that reduce to an element

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of $\Gamma$ modulo $m$. For each number field $K$, the construction produces infinitely many non-isomorphic C*-algebras as $m$ and $\Gamma$ vary. For the special case of trivial $m$, in which case $\Gamma$ must also be trivial, one gets the full $ax + b$-semigroup C*-algebra studied in [C-D-L].

And for the special case where $m_{\infty}$ is supported at all real embeddings of $K$ and $m_0$ is trivial, one gets the semigroup $R \rtimes R_+^\times$ where $R_+^\times$ is the subsemigroup of $R^\times$ consisting of (non-zero) totally positive algebraic integers.

The main result of this paper is the computation of all KMS and ground states on the left regular C*-algebra $C^*_\lambda(R \rtimes R_{m,\Gamma})$ for the canonical time evolution $\sigma$ coming from the norm map on $K$, including a classification of type for the high temperature KMS states, see Theorem 3.2 for the precise statement. As a consequence, we obtain that the boundary quotient of $C^*_\lambda(R \rtimes R_{m,\Gamma})$ admits a unique KMS state, which is of type $\text{III}_1$. Moreover, the techniques needed to prove Theorem 3.2 also lead to a computation of all the KMS and ground states on the left regular C*-algebras of the monoids $R_{m,\Gamma}$ and $R_{m,\Gamma}/R_{m,\Gamma}^*$ where $R_{m,\Gamma}^* := R_{m,\Gamma}^* \cap R^*$ is the group of units in $R_{m,\Gamma}$.

In order to explain our main result, we must first discuss a few number-theoretic preliminaries, see Section 2.1 for more details. Let $I_m$ denote the group of fractional ideals in $K$ that are coprime to the modulus $m$, and let $K_{m,\Gamma} = R_{m,\Gamma}^{-1}R_{m,\Gamma} \subseteq K^\times$ be the group of (left) quotients of $R_{m,\Gamma}$. For each $x \in K_{m,\Gamma}$, let $i(x) := xR$ be the principal fractional ideal in $K$ generated by $x$, so that $i(K_{m,\Gamma})$ is a subgroup of $I_m$. The quotient group $I_m/i(K_{m,\Gamma})$ will appear throughout this paper. In the case that $m$ and $\Gamma$ are trivial, $I_m/i(K_{m,\Gamma})$ coincides with the ideal class group $\text{Cl}(K)$. In general, $I_m/i(K_{m,\Gamma})$ is a quotient of the ray class group modulo $m$ and thus is always a finite group.

We show that the C*-dynamical system $(C^*_\lambda(R \rtimes R_{m,\Gamma}), \mathbb{R}, \sigma)$ exhibits phase transitions at $\beta = 2$ and $\beta = \infty$. For each $\beta \in [1, 2]$, we prove that there is a unique $\sigma$-KMS state on $C^*_\lambda(R \rtimes R_{m,\Gamma})$. Our proof of uniqueness for $\beta \in [1, 2]$ uses the well-known fact that the $L$-functions associated with non-trivial characters of $I_m/i(K_{m,\Gamma})$ do not have poles at 1. This technique is inspired by the proofs of uniqueness for high temperature KMS states on Bost-Connes type systems, see, for example, [Bo-Co, Section 7], [Nesh1, Proposition], and [L-L-N2, Theorem 2.1(ii)]. To maneuver ourselves into a position where we can use these methods, we expand on an idea from [Nesh3]. In the special case of trivial $m$ and $\Gamma$, when our uniqueness result coincides with that in [C-D-L, Theorem 6.7], our approach is close to that taken in [Nesh3] Section 3 and is rather different than that taken in [C-D-L]. We then prove that for each $\beta \in [1, 2]$, the (unique) KMS state $\phi_\beta$ on $C^*_\lambda(R \rtimes R_{m,\Gamma})$ is a factor state of type $\text{III}_1$. Indeed, we prove that the von Neumann algebra generated by the GNS representation of $\phi_\beta$ is isomorphic to the injective factor of type $\text{III}_1$ with separable predual. This builds on [La-Nesh2] and also generalizes the result asserted in [Nesh3] Section 3 on type for the high temperature KMS states on the left regular C*-algebra $C^*_\lambda(R \rtimes R^\times)$ of the full $ax + b$-semigroup, see Remark 4.2 below for more on this. Our computation of the type uses ideas from [La-Nesh2] and [Lag-Nesh]; it relies on a general version of the prime ideal theorem for classes in $I_m/i(K_{m,\Gamma})$.

We now discuss the case where $\beta \in (2, \infty)$. For each class $\ell \in I_m/i(K_{m,\Gamma})$, choose an integral ideal $a_\ell \in \ell$ representing $\ell$. The group $R_{m,\Gamma}^*$ of units of $R_{m,\Gamma}$ acts on $a_\ell$ by multiplication, so we may form the semi-direct product $a_\ell \rtimes R_{m,\Gamma}$. For each $\beta \in (2, \infty)$, we prove that the set of KMS$\beta$ states decomposes over the finite set $I_m/i(K_{m,\Gamma})$; specifically, the extremal KMS$\beta$
states are parameterized by pairs \((\ell, \tau)\) where \(\ell\) is a class in \(\mathcal{I}_m/i(K_{m,\Gamma})\) and \(\tau\) is an extremal tracial state on the group C*-algebra \(C^*(a_\ell \rtimes R_{m,\Gamma}^*)\). Moreover, the parameter space for the ground states also decomposes over \(\mathcal{I}_m/i(K_{m,\Gamma})\), but the extreme points are given by pairs \((\ell, \phi)\) where \(\ell \in \mathcal{I}_m/i(K_{m,\Gamma})\) and \(\phi\) is a state on a matrix algebra over \(C^*(a_\ell \rtimes R_{m,\Gamma}^*)\), so there are usually ground states that are not KMS states. For the special case of trivial \(m\) and \(\Gamma\), we recover the main parameterization results from [C-D-L] Sections 7&8. Our computation uses general results for KMS and ground states on groupoid C*-algebras from [Nesh3] and [L-L-N3], respectively.

The boundary quotient of \(C^*_\lambda(R \rtimes R_{m,\Gamma})\) also carries a canonical time evolution. We prove that the associated C*-dynamical system admits a unique KMS state, which is of type III_1 and has inverse temperature \(\beta = 1\). In the special case where \(m\) and \(\Gamma\) are trivial, the boundary quotient coincides with the ring C*-algebra of \(R\) and we recover the known uniqueness result in that case, see [Cum] for the case \(K = \mathbb{Q}\) and [C-D-L] Theorem 6.7] for the case of a general number field.

The techniques and results used to prove Theorem 3.2 also lead to a phase transition theorem for the canonical time evolution on the left regular C*-algebra \(C^*_\lambda(R_{m,\Gamma})\) of a congruence monoid and also the left regular C*-algebra \(C^*_\lambda(R_{m,\Gamma}/R_{m,\Gamma}')\) of the semigroup \(R_{m,\Gamma}/R_{m,\Gamma}'\) of principal integral ideals of \(R\) that are generated by an element of \(R_{m,\Gamma}\). These simpler C*-dynamical systems also exhibit several phase transitions, and the group \(\mathcal{I}_m/i(K_{m,\Gamma})\) also appears in this context. Additionally, there is a phase transition at \(\beta = 0\); the reason for this is that the spectrum of the diagonal in the case of the multiplicative monoids contains a unique fixed point, whereas in the case of \(R \rtimes R_{m,\Gamma}\), there are no fixed points. This generalizes and expounds the result from [C-D-L] Remark 7.5] (see also [C-E-L-Y] Remark 6.6.5)).

This paper is organized as follows. Section 2 contains preliminaries: in Section 2.1 we recall some well-known concepts from algebraic number theory, including that of moduli and ray class groups, and we fix some notation that will be used throughout this article; in Section 2.2 we review the necessary background on congruence monoids and C*-algebras from actions of congruence monoids on rings of algebraic integers from [Bru]. In Section 3 we first introduce a canonical time evolution \(\sigma\) on \(C^*_\lambda(R \rtimes R_{m,\Gamma})\), and state our main theorem on phase transitions; this result gives a parameterization of all KMS and ground states of the C*-dynamical system \((C^*_\lambda(R \rtimes R_{m,\Gamma}), \mathbb{R}, \sigma)\), including the type for all high temperature KMS states, see Theorem 3.2. Sections 3.2 through 3.6 contain the proof of the parameterization results in Theorem 3.2. The claim about type is proven in Section 4 see Theorem 4.1. In Section 5 we use Theorem 3.2 to compute the KMS and ground states on the boundary quotient of \(C^*_\lambda(R \rtimes R_{m,\Gamma})\), see Theorem 5.2. Section 6 contains our phase transition theorems for the left regular C*-algebras \(C^*_\lambda(R_{m,\Gamma})\) and \(C^*_\lambda(R_{m,\Gamma}/R_{m,\Gamma}')\).

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2. Preliminaries

2.1. Moduli for number fields and ray class groups. Let $K$ be a number field with ring of integers $R$. Let $\mathcal{P}_K$ denote the set of non-zero prime ideals of $R$, so that each fractional ideal $a$ of $K$ can be written as $a = \prod_{p \in \mathcal{P}_K} p^{v_p(a)}$ where $v_p(a) \in \mathbb{Z}$ and $v_p(a) = 0$ for all but finitely many $p$. For $x \in K^\times := K \setminus \{0\}$, the set $xR$ is the principal fractional ideal of $K$ generated by $x$, and write $v_p(x)$ instead of $v_p(xR)$.

Let $V_{K,\mathbb{R}}$ be the (finite) set of real embeddings $K \hookrightarrow \mathbb{R}$. A modulus for $K$ is a function $m : V_{K,\mathbb{R}} \sqcup \mathcal{P}_K \rightarrow \mathbb{N}$ such that

- $m_\infty := m|_{V_{K,\mathbb{R}}}$ takes values in $\{0, 1\}$;
- $m|_{\mathcal{P}_K}$ is finitely supported, that is, $m(p) = 0$ for all but finitely many $p \in \mathcal{P}_K$.

Let $m_0$ be the ideal of $R$ defined by $m_0 := \prod_{p \in \mathcal{P}_K} p^{m(p)}$. It is conventional to write $m$ as a formal product $m = m_\infty m_0$, and to write $w \mid m_\infty$ when $w \in V_{K,\mathbb{R}}$ is such that $m(w) = 1$. For background on moduli for number fields, we refer the reader to [Mil] Chapter V, Section 1.

The multiplicative group of residues modulo $m$ is

$$(R/m)^* := \left( \prod_{w \mid m_\infty} \{ \pm 1 \} \right) \times (R/m_0)^*$$

where $(R/m_0)^*$ denotes the multiplicative group of units of the ring $R/m_0$. We let $R^\times := R \setminus \{0\}$ be the multiplicative semigroup of non-zero algebraic integers in $K$, and we let

$$R_m := \{ x \in R^\times : v_p(x) = 0 \text{ for all } p \text{ with } p \mid m_0 \}$$

be the multiplicative semigroup of (non-zero) algebraic integers that are coprime to $m_0$. For $a \in R_m$, let $[a]_m$ denote the residue of a modulo $m$

$$[a]_m := ([\text{sign}(w(a))]_{w|m_\infty} a + m_0) \in (R/m)^*$$

where $\text{sign}(w(a)) := w(a)/|w(a)|$.

The map $a \mapsto [a]_m$ extends uniquely to a surjective group homomorphism from the group of quotients $R_m^{-1}R_m$ of $R_m$ onto $(R/m)^*$, see [Bru] Lemma 2.1. By [Bru] Lemma 2.2, $R_m^{-1}R_m$ coincides with the group $K_m := \{ x \in K^\times : v_p(x) = 0 \text{ for all } p \text{ with } p \mid m_0 \}$.

The ray modulo $m$ is the kernel of the map $a \mapsto [a]_m$, $K_m \rightarrow (R/m)^*$; it is denoted by $K_m$. Let $\mathcal{I}_m$ denote the group of fractional ideals of $K$ that are coprime to $m_0$, and for $x \in K^\times$, we let $i(x) := xR$ denote the fractional ideal of $K$ generated by $x$; the ray class group modulo $m$ is $\text{Cl}_m(K) := \mathcal{I}_m/i(K_m)$. If $m$ is trivial, that is, if $m_\infty \equiv 0$ and $m_0 = R$, then $\mathcal{I}_m$ equals the group $\mathcal{I}$ of all fractional ideals of $K$, and the ray modulo $m$ is simply the multiplicative group $K^\times$ of non-zero elements in $K$. In this case, the ray class group modulo $m$ coincides with the ideal class group $\text{Cl}(K) = \mathcal{I}/i(K^\times)$ of $K$. The canonical homomorphism is $\text{Cl}_m(K) \rightarrow \text{Cl}(K)$ is surjective, and the ray class group $\text{Cl}_m(K)$ is always finite, see [Mil Chapter V, Theorem 1.7] for more on the relationship between $\text{Cl}_m(K)$ and $\text{Cl}(K)$. We mention in passing that ray class groups play an important role in the ideal-theoretic formulation of class field theory.
2.2. C*-algebras from actions of congruence monoids on rings of integers. We now recall the construction from [Bru]. Let \( K \) be a number field with ring of integers \( \mathbb{Z} \), and let \( m \) be a modulus for \( K \). For a subgroup \( \Gamma \subseteq (\mathbb{Z}/m)\), let

\[
R_{m,\Gamma} := \{ a \in R_m : [a]_m \in \Gamma \}.
\]

Then \( R_{m,\Gamma} \) is a multiplicative subsemigroup of \( \mathbb{Z}^\times \); such semigroups are called congruence monoids in the literature on semigroups, see, for example, [C-E-L-Y, Chapter 5]. By [Bru, Proposition 3.1], the group of quotients \( R_{m,\Gamma}^{-1}R_{m,\Gamma} \) coincides with

\[
K_{m,\Gamma} := \{ x \in K_m : [x]_m \in \Gamma \}.
\]

The group \( i(K_{m,\Gamma}) \) of principal fractional ideals generated by elements of \( K_{m,\Gamma} \) has finite index in \( \mathcal{I}_m \); indeed, the quotient \( \mathcal{I}_m/i(K_{m,\Gamma}) \) can be canonically identified with the quotient \( \text{Cl}_m(K)/\Gamma \) where \( \Gamma := i(K_{m,\Gamma})/i(K_{m,1}) \).

The semigroup \( R_{m,\Gamma} \) acts on (the additive group of) \( \mathbb{Z} \) by multiplication, so we may form the semi-direct product semigroup \( \mathbb{Z} \rtimes R_{m,\Gamma} \). For each \( (b, a) \in \mathbb{Z} \rtimes R_{m,\Gamma} \), let \( \lambda_{(b,a)} \) be the isometry in \( B(\ell^2(\mathbb{Z} \rtimes R_{m,\Gamma})) \) determined by \( \lambda_{(b,a)}(\xi((y,x)) = \xi((b+ay,ax)) \) where \( \xi((y,x)) : (y, x) \in \mathbb{Z} \rtimes R_{m,\Gamma} \) is the canonical orthonormal basis for \( \ell^2(\mathbb{Z} \rtimes R_{m,\Gamma}) \). The left regular C*-algebra \( C^\alpha_{\lambda}(\mathbb{Z} \rtimes R_{m,\Gamma}) \) of \( \mathbb{Z} \rtimes R_{m,\Gamma} \) is the sub-C*-algebra of \( B(\ell^2(\mathbb{Z} \rtimes R_{m,\Gamma})) \) generated by the left regular representation of \( \mathbb{Z} \rtimes R_{m,\Gamma} \). That is,

\[
C^\alpha_{\lambda}(\mathbb{Z} \rtimes R_{m,\Gamma}) := C^\alpha(\{ \lambda_{(b,a)} : (b, a) \in \mathbb{Z} \rtimes R_{m,\Gamma} \})
\]

We refer the reader to [Li1, Li2] of [C-E-L-Y, Chapter 5] for the general theory of semigroup C*-algebras. Let \( \mathcal{I}_m^+ \) denote the non-zero integral ideals of \( \mathbb{Z} \) that are coprime to \( m_0 \), and for each \( a \in \mathcal{I}_m^+ \) and \( x \in \mathbb{Z} \), let \( E_{(x+a) \times (a \cap R_{m,\Gamma})} \) be the orthogonal projection from \( \ell^2(\mathbb{Z} \rtimes R_{m,\Gamma}) \) onto the subspace \( \ell^2((x + a) \times (a \cap R_{m,\Gamma})) \). The C*-algebra \( C^\alpha_{\lambda}(\mathbb{Z} \rtimes R_{m,\Gamma}) \) has a canonical “diagonal” sub-C*-algebra given by

\[
D_{\lambda}(\mathbb{Z} \rtimes R_{m,\Gamma}) = \text{span}(\{ E_{(x+a) \times (a \cap R_{m,\Gamma})} : x \in \mathbb{Z}, a \in \mathcal{I}_m^+ \}),
\]

see [Bru, Proposition 3.3] and [Li1 Section 3].

3. Equilibrium states

3.1. The phase transition theorem. Let \( B \) be a C*-algebra. A time evolution on \( B \) is a group homomorphism \( \gamma : \mathbb{R} \to \text{Aut}(B) \) such that for each fixed \( x \in B \), the map \( t \mapsto \gamma_t(x) \) is continuous. The triple \((B, \mathbb{R}, \gamma)\) is called a C*-dynamical system. There is a standard notion of equilibrium in this context, namely that of KMS and ground states. We now recall the relevant definitions.

Let \((B, \mathbb{R}, \gamma)\) be a C*-dynamical system. An element \( x \in B \) is \( \gamma \)-analytic if the map \( \mathbb{R} \to B \) given by \( t \mapsto \gamma_t(x) \) extends to an entire function \( z \mapsto \gamma_z(x) \) from \( \mathbb{C} \) to \( B \). Let \( \beta \in \mathbb{R}^* := \mathbb{R} \setminus \{0\} \). A state \( \varphi \) on \( B \) is a \( \gamma \)-KMS\( \beta \) state, or a KMS state at inverse temperature \( \beta \) for \( \gamma \), if it satisfies the KMS\( \beta \) condition

\[
\varphi(xy) = \varphi(y^{\gamma_i\beta}(x)) \quad (1)
\]

for all \( \gamma \)-analytic elements \( x, y \) in a \( \gamma \)-invariant subset with dense linear span. The parameter \( \beta \) is often called the inverse temperature, see [E-R, Chapter 5] for motivation from quantum
statistical mechanics. The $\gamma$-$\text{KMS}_0$ states are defined to be the $\gamma$-invariant traces on $B$; these are the “infinite temperature” or “chaotic” states.

If $B$ is unital, then [B-R] Theorem 5.3.30(1&2) asserts that for each $\beta \in \mathbb{R}$, the set $\Sigma_\beta$ of KMS$_\beta$ states of the system $(B,\mathbb{R},\gamma)$ is a (possibly empty) convex weak*-compact subset of the state space $S(B)$ of $B$ that is also a Choquet simplex. Moreover, by [B-R] Theorem 5.3.30(3), a KMS$_\beta$ state $\phi$ is an extreme point of $\Sigma_\beta$ if and only if $\phi$ is a factor state, that is, if the von Neumann algebra $\pi_\phi(B)^\prime\prime$ generated by the GNS representation $\pi_\phi$ of $\phi$ is a factor.

For $\beta = \infty$, that is, for “zero temperature”, there are two different notions of equilibrium states. A state $\varphi$ on $B$ is a $\gamma$-$\text{KMS}_\infty$-state if it is the weak*-limit of a net $(\varphi_i)_i$ where $\varphi_i$ is a $\gamma$-KMS$_\beta$-state for each $i$ and $\beta_i \to \infty$, and a state $\varphi$ on $B$ is a $\gamma$-ground state if the map $z \mapsto \varphi(x\gamma_z(y))$ is bounded on the upper half-plane for all $\gamma$-analytic elements $x,y$ in a $\gamma$-invariant subset with dense linear span. Every KMS$_\infty$ state is a ground state by [B-R] Proposition 5.3.23, but there may be ground states that are not KMS$_\infty$ states, as we shall see in Theorem 3.2[iii]&(iv) below. Also see [L-L-N3], Corollary 1.8 for a more general explanation of why the set of KMS$_\infty$ states may be properly contained in the set of ground states, and [La-Rae2], Theorem 7.1(3)&(4) and [C-D-L], Section 8 for examples of this phenomenon. Note that the distinction between KMS$_\infty$ states and ground states was first made in [Con-Mar] Definition 3.7, and is not observed in [B-R].

If $B$ is unital, then the set $\Sigma_\infty$ of KMS$_\infty$ states of the system $(B,\mathbb{R},\gamma)$ is a convex weak*-compact subset of $S(B)$ by [Con-Mar] Proposition 3.8, whereas the set of ground states need not be a simplex, see [La-Rae2], Remark 7.2(v).

We now return to the C*-algebra $C^*_\lambda(R \rtimes R_{m,\Gamma})$. For a non-zero ideal $a$ of $R$, we let $N(a) := |R/a|$ denote the norm of $a$, which is always finite. The map $a \mapsto N(aR)$ defines a semigroup homomorphism from $R^\times$ to the multiplicative semigroup $\mathbb{N}^\times := \mathbb{N} \setminus \{0\}$ of positive integers, and the map $R \times R_{m,\Gamma} \to \mathbb{R}_+$ given by $(b,a) \mapsto N(a)$ is a semigroup homomorphism. For each $t \in \mathbb{R}$, let $U_t$ denote the diagonal unitary on $\ell^2(R \times R_{m,\Gamma})$ that is determined on the canonical basis by

$$U_t(\varepsilon_{(b,a)}) = N(a)t^t\varepsilon_{(b,a)}. $$

Then $t \mapsto U_t$ defines a unitary representation $\mathbb{R} \to U(\ell^2(R \rtimes R_{m,\Gamma}))$, and a routine argument shows that the group $\{U_t : t \in \mathbb{R}\}$ of unitaries implements a time evolution on $C^*_\lambda(R \rtimes R_{m,\Gamma})$; specifically, we have the following result.

**Proposition 3.1.** There is a time evolution $\sigma : \mathbb{R} \to \text{Aut}(C^*_\lambda(R \rtimes R_{m,\Gamma}))$ such that

$$\sigma_t(\lambda_{(b,a)}) = N(a)t^t\lambda_{(b,a)} \quad \text{for all } (b,a) \in R \rtimes R_{m,\Gamma} \text{ and } t \in \mathbb{R}. $$

Let $\mathfrak{k} \in I_m/i(K_{m,\Gamma})$ be an ideal class. An integral ideal $a \in \mathfrak{k}$ is said to be norm-minimizing in the class $\mathfrak{k}$ if $N(a) \leq N(b)$ for every other integral ideal $b$ in $\mathfrak{k}$; note that each ideal class $\mathfrak{k}$ contains only finitely many norm-minimizing ideals. Norm-minimizing ideals appeared in [La-vF] during the investigation of phase transitions for C*-dynamical systems associated with Hecke C*-algebras, and then later in [C-D-L], Section 8 and [L-L-N3].
Let $R^*_{m,\Gamma} := R_{m,\Gamma} \cap R^*$ be the group of invertible elements in $R_{m,\Gamma}$. For each fractional ideal $a \in \mathcal{I}_m$, the group $R^*_{m,\Gamma}$ acts on (the additive group of) $a$ by multiplication, so we may form the semi-direct product group $a \rtimes R^*_{m,\Gamma}$.

View $\ell^\infty(R \rtimes R_{m,\Gamma})$ as a sub-$C^*$-algebra of $B(\ell^2(R \rtimes R_{m,\Gamma}))$ in the canonical way, and let $\mathcal{E}$ be the restriction to $\mathcal{C}_\Lambda(R \rtimes R_{m,\Gamma})$ of the canonical faithful conditional expectation $B(\ell^2(R \rtimes R_{m,\Gamma})) \to \ell^\infty(R \rtimes R_{m,\Gamma})$. It follows from [Li11 Lemma 3.11] that the range of $\mathcal{E}$ is equal to $D_\Lambda(R \rtimes R_{m,\Gamma})$. The main result of this paper is the following phase transition theorem.

**Theorem 3.2.** Let $K$ be a number field, $m$ a modulus for $K$, and $\Gamma$ a subgroup of $(R/m)^\star$. For each $\ell \in \mathcal{I}_m/i(K_{m,\Gamma})$, choose a norm-minimizing ideal $a_{\ell,1}$ in $\ell$.

(i) There are no $\sigma$-KMS$_\beta$ states on $C^\star_\Lambda(R \rtimes R_{m,\Gamma})$ for $\beta < 1$.

(ii) For each $\beta \in [1,2]$, there is a unique $\sigma$-KMS$_\beta$ state $\phi_\beta$ on $C^\star_\Lambda(R \rtimes R_{m,\Gamma})$. The state $\phi_\beta$ factors through the expectation $\mathcal{E} : C^\star_\Lambda(R \rtimes R_{m,\Gamma}) \to D_\Lambda(R \rtimes R_{m,\Gamma})$ and is determined by the values

$$\phi_\beta(E_{x+a} \rtimes (a \cap R_{m,\Gamma})) = N(a)^{-\beta} \text{ for } x \in R \text{ and } a \in \mathcal{I}_m^\times.$$ 

Moreover, $\phi_\beta$ is of type III$_1$; indeed, the von Neumann algebra $\pi_{\phi_\beta}(C^\star_\Lambda(R \rtimes R_{m,\Gamma}))''$ generated by the GNS representation $\pi_{\phi_\beta}$ of $\phi_\beta$ is isomorphic to the injective factor of type III$_1$ with separable predual.

(iii) For each $\beta \in (2,\infty)$, there is an affine isomorphism of the simplex of tracial states on the $C^*$-algebra

$$\bigoplus_{\ell \in \mathcal{I}_m/i(K_{m,\Gamma})} C^\star(a_{\ell,1} \rtimes R^*_{m,\Gamma})$$

onto the simplex of $\sigma$-KMS$_\beta$ states on $C^\star_\Lambda(R \rtimes R_{m,\Gamma})$.

(iv) There is an affine isomorphism of the $\sigma$-ground state space of $C^\star_\Lambda(R \rtimes R_{m,\Gamma})$ onto the state space of the $C^*$-algebra

$$\bigoplus_{\ell \in \mathcal{I}_m/i(K_{m,\Gamma})} M_{k_{\ell,N(a_{\ell,1})}}(C^\star(a_{\ell,1} \rtimes R^*_{m,\Gamma}))$$

where $k_{\ell}$ is the number of norm-minimizing ideals in the class $\ell$.

Before continuing to the proof, we make several remarks.

**Remark 3.3.** (a) For the particular case of trivial $m$ and $\Gamma$, Theorem 3.2 recovers the parameterization results obtained in [CD11 Sections 6 and 7].

(b) If $a$ and $b$ lie in the same class $\ell \in \mathcal{I}_m/i(K_{m,\Gamma})$, so that there is a $k \in K_{m,\Gamma}$ with $a = kb$, then the map $x \mapsto kx$ defines an $R^*_{m,\Gamma}$-equivariant isomorphism $a \cong b$, so that $a \rtimes R^*_{m,\Gamma} \cong b \rtimes R^*_{m,\Gamma}$. Thus, in parts (iii) and (iv), we could replace each $C^*$-algebra $C^\star(a_{\ell,1} \rtimes R^*_{m,\Gamma})$ with $C^\star(a_{\ell} \rtimes R^*_{m,\Gamma})$ for any other ideal $a_{\ell}$ in the class $\ell$.

(c) In light of Theorem 3.2(iii), it is natural to ask if one can explicitly describe the simplex of traces on the group $C^*$-algebra $C^\star(a_{\ell} \rtimes R^*_{m,\Gamma})$ for a fixed class $\ell \in \mathcal{I}_m/i(K_{m,\Gamma})$ and integral ideal $a_{\ell} \in \ell$. It turns out that, even for trivial $m$ and $\Gamma$, this is a difficult problem that is related to the generalized Furstenburg conjecture, see [La-War].
By Proposition 3.2, the semigroup $R(R \rtimes R_{m,\Gamma})$ decomposes as

$$K_* C_\Lambda(R \rtimes R_{m,\Gamma}) \cong \bigoplus_{t \in \mathcal{I}_m / (K_{m,\Gamma})} K_* (C_\Lambda^* (a_t \rtimes R_{m,\Gamma}^*))$$

where $a_t$ is an integral ideal in the class $t$. It is interesting that the C*-algebra

$$\bigoplus_{t \in \mathcal{I}_m / (K_{m,\Gamma})} C_\Lambda^* (a_t \rtimes R_{m,\Gamma}^*)$$

appears in both the $K$-theory formula and the parameterization of the low temperature KMS states. For the $ax + b$-semigroup $R \rtimes R^\times$, this has already been discussed by Cuntz in [C-E-L-Y, Chapter 6, Section 6].

(d) In forthcoming joint work with Xin Li, we prove that the $K$-theory of the C*-algebra $C_\Lambda^* (R \rtimes R_{m,\Gamma})$ decomposes as

$$K_* (C_\Lambda^* (R \rtimes R_{m,\Gamma})) \cong \bigoplus_{t \in \mathcal{I}_m / (K_{m,\Gamma})} K_* (C_\Lambda^* (a_t \rtimes R_{m,\Gamma}^*))$$

where $a_t$ is an integral ideal in the class $t$. It is interesting that the C*-algebra

$$\bigoplus_{t \in \mathcal{I}_m / (K_{m,\Gamma})} C_\Lambda^* (a_t \rtimes R_{m,\Gamma}^*)$$

appears in both the $K$-theory formula and the parameterization of the low temperature KMS states. For the $ax + b$-semigroup $R \rtimes R^\times$, this has already been discussed by Cuntz in [C-E-L-Y, Chapter 6, Section 6].

(e) An alternative method for computing the low temperature KMS states on $C_\Lambda^* (R \rtimes R^*)$ is discussed in [C-E-L-Y, Chapter 6, Section 6]. Presumably, it could also be used here.

(f) It follows from Theorem 3.2 iii)&(iv) that there are usually ground states which are not KMS states, that is, there is a phase transition at $\beta = \infty$. For example, let $K = \mathbb{Q}$, so that $R = \mathbb{Z}$. Let $m \in \mathbb{N}$ be a positive natural number, and let $m = m_{\infty} m_0$ where $m_{\infty}$ takes the value one at the only real embedding of $\mathbb{Q}$ and $m_0 (p) := \nu_p (m)$. Then a calculation shows that the map $(\mathbb{Z}/m \mathbb{Z})^* \to \mathcal{I}_m / i(K_{m,1})$ given by $[a]_{m_{\infty}} \mapsto [a \mathbb{Z}]$ is a well-defined isomorphism $(\mathbb{Z}/m \mathbb{Z})^* \cong \mathcal{I}_m / i(K_{m,1})$ and that each class $t \in \mathcal{I}_m / i(K_{m,1})$ contains a unique norm-minimizing ideal of norm $n_t$ where $n_t$ is the smallest positive integer in the residue class modulo $m$ corresponding to $t$ under the above isomorphism. Moreover, in this situation, the isotropy groups appearing in Theorem 3.2 iii)&(iv) are all isomorphic to $\mathbb{Z}$, so Theorem 3.2 implies that the KMS states are parameterized by traces on the commutative C*-algebra

$$\bigoplus_{t \in (\mathbb{Z}/m \mathbb{Z})^*} C^*(\mathbb{Z}) \cong \bigoplus_{t \in (\mathbb{Z}/m \mathbb{Z})^*} C(\mathbb{T}),$$

whereas the ground states are parameterized by states on the C*-algebra

$$\bigoplus_{t \in (\mathbb{Z}/m \mathbb{Z})^*} M_{n_t} (C^*(\mathbb{Z})) \cong \bigoplus_{t \in (\mathbb{Z}/m \mathbb{Z})^*} M_{n_t} (C(\mathbb{T})).$$

(g) For $\beta > 2$, the extremal $\sigma$-KMS states on $C_\Lambda^* (R \rtimes R_{m,\Gamma})$ are either type I or type II. However, the techniques needed to deal with the case $\beta > 2$ are rather different since these states usually do not factor through the expectation $\mathcal{E}$, so this will be discussed elsewhere.

This section and the next are devoted to the proof of Theorem 3.2, which we break up into several parts. The next five subsections contain some preliminaries, and the proofs of parts (i) through (iv), excluding the type computation. The proof that the von Neumann algebra $\pi_{\phi_\beta} (C_\Lambda^* (R \rtimes R_{m,\Gamma}))^\sigma$ is isomorphic to the injective factor of type $\text{III}_1$ with separable predual is given in Section 4.

### 3.2. Preliminaries for the proof

The semigroup $R \rtimes R_{m,\Gamma}$ canonically embeds into the group $(R_{m,1}^{-1} R) \rtimes K_{m,\Gamma}$ where $(R_{m,1}^{-1} R) = \{ \frac{a}{b} : a, b \in R \}$ is the localization of $R$ at $R_{m,1}$. By Bru 3.2, the semigroup $R \rtimes R_{m,\Gamma}$ is left Ore and its group of left quotients coincides with $(R_{m,1}^{-1} R) \rtimes K_{m,\Gamma}$. That is, $(R \rtimes R_{m,\Gamma})^{-1} (R \rtimes R_{m,\Gamma}) = (R_{m,1}^{-1} R) \rtimes K_{m,\Gamma}$. To
simplify notation, let
\[ P_{m,\Gamma} := R \times R_{m,\Gamma} \quad \text{and} \quad G_{m,\Gamma} := (R_m^{-1} R) \times K_{m,\Gamma}. \]

Also let \( S := \{ p \in \mathcal{P}_K : p \mid m_0 \} \) be the support of \( m_0 \), which is a finite set of primes, and put \( \mathcal{P}_R^m := \mathcal{P}_K \setminus S \).

### 3.2.1. A groupoid model

The material below on adeles and a groupoid model for \( C^*_\lambda(P_{m,\Gamma}) \) is from [Bru Section 5]. It was motivated by similar results from [C-D-L Section 5] for the special case where \( m \) and \( \Gamma \) are trivial.

For each non-zero prime ideal \( p \) of \( R \), let \( K_p \) be the corresponding \( p\)-adic completion of \( K \) and \( R_p \) the ring of integers in \( K_p \). Let
\[ \mathbb{A}_S := \left\{ a = (a_p)_p \in \prod_{p \in \mathcal{P}_R^m} K_p : a_p \in R_p \text{ for all but finitely many } p \right\} \]
equipped with the restricted product topology with respect to the compact open subsets \( R_p \subseteq K_p \). Denote by \( \hat{R}_S \) the compact subring \( \prod_{p \in \mathcal{P}_R^m} R_p \), and let \( \hat{R}_S^* := \prod_{p \in \mathcal{P}_R^m} R_p^* \) be the group of units of \( \hat{R}_S \). The compact group \( \hat{R}_S^* \) acts on \( \mathbb{A}_S \) by multiplication, and we let \( \hat{a} \) denote the image of \( a \in \mathbb{A}_S \) under the quotient mapping \( \mathbb{A}_S \to \mathbb{A}_S / \hat{R}_S^* \). Define an equivalence relation on \( \mathbb{A}_S \times \mathbb{A}_S / \hat{R}_S^* \) by
\[ (b, \hat{a}) \sim (d, \hat{c}) \quad \text{if} \quad \hat{a} = \hat{c} \quad \text{and} \quad b - d \in \hat{a} \hat{R}_S. \]

Via the diagonal embedding, the groups \( R_m^{-1} R_m \) and \( K_{m,\Gamma} \) act on \( \mathbb{A}_S \) by translation and multiplication, respectively. The canonical action of \( G_{m,\Gamma} \) on \( \mathbb{A}_S \times \mathbb{A}_S / \hat{R}_S^* \) given by \((n, k)(b, \hat{a}) = (n + k b, k \hat{a})\) descends to a well-defined action on the locally compact Hausdorff quotient space
\[ \Omega_R^m := (\mathbb{A}_S \times \mathbb{A}_S / \hat{R}_S^*) / \sim. \]
By restricting the above equivalence relation to the subset \( \hat{R}_S \times \hat{R}_S / \hat{R}_S^* \subseteq \mathbb{A}_S \times \mathbb{A}_S / \hat{R}_S^* \), we obtain the compact open subset
\[ \Omega_R^m := (\hat{R}_S \times \hat{R}_S / \hat{R}_S^*) / \sim \]
of \( \Omega_R^m \). Let \( G_{m,\Gamma} \ltimes \Omega_R^m \) be the reduction of the transformation groupoid \( G_{m,\Gamma} \ltimes \Omega_R^m \) by the set \( \Omega_R^m \), that is,
\[ G_{m,\Gamma} \ltimes \Omega_R^m = \{ (g, w) \in G_{m,\Gamma} \ltimes \Omega_R^m : gw \in \Omega_R^m \}. \]
Our choice of notation for the reduction groupoid comes from the fact that \( G_{m,\Gamma} \ltimes \Omega_R^m \) can be canonically identified with a partial transformation groupoid, see [Li3 Section 3.3].

**Proposition 3.4** ([Bru Propositions 4.1 and 5.3]). There is an isomorphism
\[ \vartheta : C^*_\lambda(P_{m,\Gamma}) \cong C^*(G_{m,\Gamma} \ltimes \Omega_R^m) \]
that is determined on generators by \( \vartheta(\lambda_{(b,a)}) = 1_{\{(b,a)\} \times \Omega_R^m} \) for \((b,a) \in P_{m,\Gamma}\).
3.2.2. Quasi-invariant measures on $\Omega^m_R$. The multiplicative map $R^\times \to \mathbb{N}^\times$ given by $a \mapsto N(aR) = |R/aR|$ has a unique extension to a group homomorphism $K^* \to Q^+_1$ that we also denote by $N$. Let $c_N$ be the real-valued one-cocycle $G_{m,\Gamma} \times \Omega^m_R \to \mathbb{R}$ given by $c_N((n, k), w) = \log N(k)$, so that [Ren1, Proposition 5.1] gives us a time evolution $\sigma^{c_N}$ on $C^*_\Lambda(G_{m,\Gamma} \times \Omega^m_R)$ such that

$$\sigma^{c_N}_t(f)((n, k), w) = N(k)^t f((n, k), w) \quad \text{for all } f \in C_c(G_{m,\Gamma} \times \Omega^m_R) \text{ and } t \in \mathbb{R}. \quad (3)$$

**Lemma 3.5.** Under the isomorphism $\vartheta : C^*_\Lambda(P_{m,\Gamma}) \cong C^*_\Lambda(G_{m,\Gamma} \times \Omega^m_R)$ from Proposition 3.4, the time evolution $\sigma$ from [2] is conjugated to $\sigma^{c_N}$, that is, $\sigma^{c_N} = \vartheta \circ \sigma_t \circ \vartheta^{-1}$ for every $t \in \mathbb{R}$.

**Proof.** We have $\vartheta(\lambda(b, a)) = 1_{\{b, a\}} \times \Omega^m_R$ for $(b, a) \in P_{m,\Gamma}$, and a short calculation shows that $\sigma^{c_N}_t(1_{\{b, a\}} \times \Omega^m_R) = N(a)^t 1_{\{b, a\}} \times \Omega^m_R$ for all $t \in \mathbb{R}$. Thus, we have $\vartheta \circ \sigma_t \circ \vartheta^{-1}(1_{\{b, a\}} \times \Omega^m_R) = \sigma^{c_N}_t(1_{\{b, a\}} \times \Omega^m_R)$ for all $t \in \mathbb{R}$. Since the collection $\{\lambda(b, a) : (b, a) \in P_{m,\Gamma}\}$ generates $C^*_\Lambda(P_{m,\Gamma})$ as a C*-algebra, this is enough. \[\Box\]

Lemma 3.5 implies that there is an isomorphism of C*-dynamical systems

$$(C^*_\Lambda(P_{m,\Gamma}), \mathbb{R}, \sigma) \cong (C^*_\Lambda(G_{m,\Gamma} \times \Omega^m_R), \mathbb{R}, \sigma^{c_N}),$$

so we may work with the latter system for our computations of KMS and ground states. From now on, we will write $\sigma$ rather than $\sigma^{c_N}$ for the time evolution on $C^*_\Lambda(G_{m,\Gamma} \times \Omega^m_R)$.

Any state $\phi$ on $C^*_\Lambda(G_{m,\Gamma} \times \Omega^m_R)$ defines a probability measure $\mu$ on $\Omega^m_R$ by restricting $\phi$ to $C(\Omega^m_R)$ and then applying the Riesz representation theorem to the state $\phi|_{C(\Omega^m_R)}$. It is well-known, going back to [Ren1, Proposition 5.4], that if $\phi$ is a $\sigma$-KMS state, then the KMS condition [1] forces the measure $\mu$ to be *quasi-invariant with Radon-Nikodym cocycle given by $e^{-\beta c_N} = N^{-\beta}$, that is, $\mu$ must satisfy the scaling condition

$$\mu((n, k)Z) = N(k)^{-\beta} \mu(Z) \quad (4)$$

for all $(n, k) \in G_{m,\Gamma}$ and Borel sets $Z \subseteq \Omega^m_R$ such that $(n, k)Z \subseteq \Omega^m_R$. Moreover, the set of probability measures that satisfy [1] forms a (possibly empty) Choquet simplex, see, for example, [Ren2, Exercise 3.3.1].

There may be many $\sigma$-KMS states on $C^*_\Lambda(G_{m,\Gamma} \times \Omega^m_R)$ that define the same quasi-invariant measure on $\Omega^m_R$, and [Nesh3, Theorem 1.3] gives a parameterization of all such $\sigma$-KMS states in terms of traces on the C*-algebras of certain isotropy groups. Thus, to compute the $\sigma$-KMS states on $C^*_\Lambda(G_{m,\Gamma} \times \Omega^m_R)$ for $\beta < \infty$, we must first compute, for each fixed $\beta \in \mathbb{R}$, the simplex of all probability measures $\mu$ on $\Omega^m_R$ that satisfy [1]. It is easy to see that there are no such measures for $\beta < 1$, as explained in Section 3.3 below, and for this reason we restrict to the case $\beta \geq 1$ now.

**Lemma 3.6.** Let $\mathcal{J} := \{(x + a) \times a^\times : x \in R, a \in I_m^+\}$, and for $(x + a) \times a^\times \in \mathcal{J}$, let

$$V_{(x+a)\times a^\times} := \{[b, \bar{a}] \in \Omega^m_R : v_p(\bar{a}) \geq v_p(a), v_p(b - x) \geq v_p(a) \text{ for all } p \in P^m_R\}.$$ 

If $\mu$ is a probability measure on $\Omega^m_R$ and $\beta \geq 1$, then $\mu$ satisfies [1] if and only if

$$\mu((n, k)V_X) = N(k)^{-\beta} \mu(V_X) \quad (5)$$

for all $(n, k) \in G_{m,\Gamma}$ and $X \in \mathcal{J}$ such that $(n, k)X = \{(n + kx, ky) : (x, y) \in X\}$ lies in $\mathcal{J}$. 

Proof. A calculation shows that $V_X$ is the support of the projection $\vartheta(E_X) \in C(\Omega^n_R)$. The result follows from the fact that the projections $\{\vartheta(E_X) : X \in J\} \cup \{0\}$ span a dense sub-$*$-algebra of $C(\Omega^n_R)$. \qed

Our next result is inspired by the proof of [La-Nesh2] Proposition 2.1 and [Nesh3] Section 3.

**Proposition 3.7.** Let $\pi$ denote the quotient map $\hat{R}_S \times \hat{R}_S^* \to \Omega^n_R$, and let $m$ denote the normalized Haar measure on $\hat{R}_S$. Given a probability measure $\nu$ on $\hat{R}_S/\hat{R}_S^*$, form the product measure $m \times \nu$, and let $\pi_*(m \times \nu)$ denote the probability measure on $\Omega^n_R$ obtained by pushing forward $m \times \nu$ under $\pi$. For each fixed $\beta \geq 1$, the map $\nu \mapsto \pi_*(m \times \nu)$ defines an affine bijection from the set of probability measures $\nu$ on $\hat{R}_S/\hat{R}_S^*$ satisfying

\[ \nu(kZ) = N(k)^{-(\beta-1)} \nu(Z) \]  

for every $k \in K_{m,\Gamma}$ and every Borel set $Z \subseteq \hat{R}_S/\hat{R}_S^*$ such that $kZ \subseteq \hat{R}_S/\hat{R}_S^*$ onto the set of probability measures on $\Omega^n_R$ satisfying (4).

Proof. For $a \in \mathcal{I}^+_m$, let

\[ U_a := a\hat{R}_S/\hat{R}_S^* = \{ \hat{a} \in \hat{R}_S/\hat{R}_S^* : v_p(\hat{a}) \geq v_p(a) \text{ for all } p \in P^m \}. \]

Then for $k \in K_{m,\Gamma}$ and $a \in \mathcal{I}^+_m$ such that $ka \in \mathcal{I}^+_m$, we have $kU_a = U_{ka}$. An analogue of Lemma 3.6 shows that a probability measure $\nu$ on $\hat{R}_S/\hat{R}_S^*$ satisfies (6) if and only of

\[ \nu(kU_a) = N(k)^{-(\beta-1)} \nu(U_a) \]

for all $k \in K_{m,\Gamma}$ and $a \in \mathcal{I}^+_m$ such that $ka \in \mathcal{I}^+_m$.

Suppose that $\nu$ is a probability measure on $\hat{R}_S/\hat{R}_S^*$ satisfying (6), and let $\mu := \pi_*(m \times \nu)$. We need to show that $\mu$ satisfies (4). By Lemma 3.6, it suffices to show that $\mu$ satisfies (5).

If $(n, k) \in C_{m,\Gamma}$ and $X = (x + a) \times a^* \in J$ are such that $(n, k)X \in J$, then $(n + kx + ka\hat{R}_S) \times U_{ka} = \pi^{-1}(V_{(n+kx+ka)\times(ka)\times})$, so we have

\[ \mu((n, k)V_X) = m(n+kx+ka\hat{R}_S)\nu(U_{ka}) = N(ka)^{-1} N(k)^{-(\beta-1)} \nu(U_a) = N(k)^{-\beta} N(a)^{-1} \nu(U_a). \]

For every $x \in R$,

\[ \mu(V_{(x+a)\times a^*}) = m(x + a\hat{R}_S)\nu(U_a) = N(a)^{-1} \nu(U_a), \]

so we see that $\mu$ satisfies (5). Since $\nu(U_a) = N(a)\mu(V_{(x+a)\times a^*})$, and $\nu$ is determined by its values on the sets $U_a$, we also conclude that the map $\nu \mapsto \pi_*(m \times \nu)$ is injective.

It remains to check surjectivity. Suppose $\mu$ is a probability measure on $\Omega^n_R$ satisfying (4), and let $q : \Omega^n_R \to \hat{R}_S/\hat{R}_S^*$ denote the surjective map given by $q([b, a]) = \hat{a}$, so that $q \circ \pi = \pi_2$ is the projection from $\hat{R}_S \times \hat{R}_S/\hat{R}_S^*$ onto the second coordinate. To show surjectivity, it is enough to show that

1. $q_*\mu$ satisfies (4);  
2. $\mu = \pi_*(m \times q_*\mu)$. 

For \( a \in \mathcal{T}^+_m \), let \( V_{R \times a^\times} := \bigsqcup_{y \in R/a} V_{(y+a) \times a^\times} \). Since \( \mu \) satisfies (4),
\[
\mu(V_{R \times a^\times}) = \sum_{y \in R/a} \mu(V_{(y+a) \times a^\times}) = N(a)\mu(V_{a \times a^\times}). \tag{7}
\]

Let \( k \in K_{m,\Gamma} \) and \( a \in \mathcal{T}^+_m \) be such that \( ka \in \mathcal{T}^+_m \). Then \( q^{-1}(U_{ka}) = V_{R \times (ka) \times} \), so
\[
q_\ast \mu(U_{ka}) = \mu(V_{R \times (ka) \times}) = \sum_{y \in R/a} \mu(V_{(y+a) \times a^\times}) = N(k)a\mu(V_{ka \times (ka) \times}) = N(k)aN(k)\mu(V_{a \times a^\times}) \quad \text{(using that \( \mu \) satisfies (2))}
\]
\[
= N(k)^{-\beta}\mu(V_{R \times a^\times})
\]
\[
= N(k)^{-\beta}q_\ast \mu(U_a).
\]
Thus (1) holds. To show (2), it suffices to show that \( \mu(V_{(x+a) \times a^\times}) = \pi_\ast (m \times q_\ast \mu(V_{(x+a) \times a^\times})) \)
for all \( (x + a) \times a^\times \in J \). We have
\[
\pi_\ast (m \times q_\ast \mu(V_{(x+a) \times a^\times})) = (m \times q_\ast \mu)((x + a\hat{R}_S) \times U_a) = N(a)^{-1}q_\ast \mu(U_a) = N(a)^{-1}\mu(V_{R \times a^\times}).
\]
Using (7) and (4), we have \( \mu(V_{R \times a^\times}) = N(a)\mu(V_{a \times a^\times}) = N(a)\mu(V_{(x+a) \times a^\times}) \). Hence, \( \mu(V_{(x+a) \times a^\times}) = \pi_\ast (m \times q_\ast \mu(V_{(x+a) \times a^\times})) \) for all \( (x + a) \times a^\times \in J \), as desired.

It is not difficult to check that the map \( \nu \mapsto m \times \nu \) is affine, and since the push-forward map \( m \times \nu \mapsto \pi_\ast (m \times \nu) \) is also affine, we see that \( \nu \mapsto \pi_\ast (m \times \nu) \) is affine.

\[ \square \]

3.3. The easy case: part (i).

Proof of Theorem 3.2(i). Suppose that \( \phi \) is a \( \sigma \)-KMS\( _\beta \) state on \( C^\ast_{\lambda}(P_{m,\Gamma}) \). For each \( x \in R \) and each \( a \in R_{m,\Gamma} \), the KMS\( _\beta \) condition (1) yields
\[
\phi(\lambda(x,a)\lambda^\ast_{(x,a)}) = N(a)^{-\beta}\phi(\lambda^\ast_{(x,a)}\lambda(x,a)) = N(a)^{-\beta}.
\]
Hence,
\[
0 \leq \phi(1 - \sum_{x \in R/aR} \lambda(x,a)\lambda^\ast_{(x,a)}) = 1 - \sum_{x \in R/aR} \phi(\lambda(x,a)\lambda^\ast_{(x,a)}) = 1 - N(a)^{1-\beta},
\]
so we must have \( \beta \geq 1 \).

3.4. Uniqueness in the critical interval and the proof of part (ii). Let \( \nu_0 := \delta_0 \) be the unit mass concentrated at the point \( 0 \in \hat{R}_S/R_S^\ast \). For each \( \beta \in (0,\infty) \), let \( \nu_{\beta,p} \) be the probability measure on \( \hat{R}_p/R_p^\ast \cong p^{\mathbb{N} \cup \{\infty\}} \) given by \( \nu_{\beta,p} = (1 - N(p)^{-\beta})\sum_{n=0}^{\infty} N(p)^{-n\beta}\delta_{p^n} \), and let \( \nu := \prod_{p \in \mathbb{P}^\ast_R} \nu_{\beta,p} \).

Lemma 3.8. For each \( \beta \in [0,\infty) \), the measure \( \nu_{\beta} \) satisfies
\[
\nu(kZ) = N(k)^{-\beta}\nu(Z) \tag{8}
\]
for every \( k \in K_{m,\Gamma} \) and every Borel set \( Z \subseteq \hat{R}_S/R_S^\ast \) such that \( kZ \subseteq \hat{R}_S/R_S^\ast \). Moreover, \( \nu_{\beta}(U_a) = N(a)^{-\beta} \) for all \( a \in \mathcal{T}^+_m \) where \( U_a = a\hat{R}_S/R_S^\ast \).
Proof. As pointed out in the proof of Proposition 3.7, to show $\nu_\beta$ satisfies (8), it suffices to show that $\nu_\beta$ satisfies

$$\nu_\beta(kU_a) = N(k)^{-\beta}\nu_\beta(U_a)$$

for all $k \in K_{m,\Gamma}$ and $a \in I_m^+$ such that $ka \in I_m^+$. Since $\overline{0} \in U_a$ for all $a \in I_m^+$, it is easy to see that $\nu_0$ satisfies this condition. Now let $\beta \in (0, \infty)$. For any $b \in I_m^+$, a calculation shows that $\nu_\beta(U_b) = N(b)^{-\beta}$ which settles the second claim. Using this, we have

$$\nu_\beta(kU_a) = \nu_\beta(U_{ka}) = N(ka)^{-\beta} = N(k)^{-\beta}N(a)^{-\beta} = N(k)^{-\beta}\nu_\beta(U_a)$$

as desired. \qed

The crux in computing the KMS$_\beta$ states for $\beta \in [1, 2]$ is the following purely measure-theoretic result.

**Theorem 3.9.** For each $\beta \in [0, 1]$, $\nu_\beta$ is the unique probability measure on $\hat{R}_S/\hat{R}_S^*$ satisfying (8).

To prove Theorem 3.9, we will expand on an idea of Neshveyev’s from the end of [Nesh3 Section 3], which will put us in a setting where we can employ techniques analogous to those used for Bost-Connes type systems.

We need two preliminary results. The first puts us in a situation where we can work with the lattice group $I_m$ of all fractional ideals coprime to $m_0$, rather than the more complicated group $K_{m,\Gamma}$. The following result is motivated by the general techniques from [La-Nesh1] on extending KMS weights.

**Lemma 3.10.** View $\hat{R}_S/\hat{R}_S^*$ as a subset of $I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ via the identification $\hat{R}_S/\hat{R}_S^* \simeq \{|[R]|\} \times \hat{R}_S/\hat{R}_S^*$. Then each probability measure $\nu$ on $\hat{R}_S/\hat{R}_S^*$ satisfying (8) has a unique extension to a finite measure $\tilde{\nu}$ on $I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ satisfying

$$\tilde{\nu}(aZ) = N(a)^{-\beta}\nu(Z)$$

for all $a \in I_m$ and Borel sets $Z \subseteq I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ such that $aZ \subseteq I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ where $aZ = \{(at, a\bar{a}) : (t, \bar{a}) \in Z\}$.

Proof. Our proof is similar to that of [L-L-N1 Lemma 2.2]. For $a \in I_m$, let $[a]$ denote the class of $a$ in $I_m/i(K_{m,\Gamma})$, and for each integral ideal $a$, let $Y_a := \{|[R]|\} \times U_a$, so that $\hat{R}_S/\hat{R}_S^* \simeq Y_R \subseteq I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$.

Suppose that $\nu$ is a probability measure on $\hat{R}_S/\hat{R}_S^*$ satisfying (8). We first show that there can be at most one measure $\mu$ on $I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ that both satisfies (9) and extends $\nu$. Indeed, suppose that $\mu$ is such a measure, and for each class $t \in I_m/i(K_{m,\Gamma})$, choose an integral ideal $a_t \in t'$, for $t = [R]$, take $a_t = R$. Then

$$I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^* = \bigcup_{t \in I_m/i(K_{m,\Gamma})} a_t^{-1}Y_{a_t},$$

so, for any Borel set $Z$,

$$\mu(Z) = \sum_{t \in I_m/i(K_{m,\Gamma})} \mu(Z \cap a_t^{-1}Y_{a_t}).$$
Since $\mu$ satisfies $[9]$, $\mu(Z \cap a_t^{-1}Y_{a_t}) = \mu(a_t^{-1}(a_tZ \cap Y_{a_t})) = N(a_t)^{-\beta}\mu(a_tZ \cap Y_{a_t})$. Since $Y_{a_t} \subseteq Y_R$, we see that $\mu$ is determined by its restriction to $Y_R$. Thus, there can be at most one measure on $I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ that both satisfies $[9]$ and extends $\nu$. We now proceed to construct this extension. Define $\tilde{\nu}$ on $I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ by

$$\tilde{\nu}(Z) = \sum_{t \in I_m/i(K_{m,\Gamma})} N(a_t)^{-\beta}\nu(a_tZ \cap Y_{a_t})$$

for Borel sets $Z \subseteq I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$. A short calculation shows that $\tilde{\nu}$ is a finite measure extending $\nu$. We need to show that $\tilde{\nu}$ satisfies $[9]$. For each $t$, let $b_t \in K_{m,\Gamma}$ be such that $aa_t = b_t\bar{a}_a t$. We have

$$\tilde{\nu}(aZ) = \sum_{t \in I_m/i(K_{m,\Gamma})} N(a_t)^{-\beta}\nu(a_t aZ \cap Y_{a_t})$$

$$= N(a)^{-\beta} \sum_{t \in I_m/i(K_{m,\Gamma})} N(a_t a)^{-\beta}\nu(a_t aZ \cap Y_{a_t})$$

$$= N(a)^{-\beta} \sum_{t \in I_m/i(K_{m,\Gamma})} N(a_t a)^{-\beta}\nu(a_t aZ \cap Y_{a_t} \cap R_a)$$

$$= N(a)^{-\beta} \sum_{t \in I_m/i(K_{m,\Gamma})} N(b_t a_t)^{-\beta}\nu(b_t a_t Z \cap Y_{b_t a_t})$$

$$= N(a)^{-\beta} \sum_{t \in I_m/i(K_{m,\Gamma})} N(b_t a_t)^{-\beta} N(b_t)^{-\beta}\nu(a_t Z \cap Y_{a_t} \cap R_a)$$

$$= N(a)^{-\beta} \sum_{t \in I_m/i(K_{m,\Gamma})} N(a_t Z \cap Y_{a_t})$$

$$= N(a)^{-\beta}\tilde{\nu}(Z).$$

This concludes the proof. \hfill $\Box$

The following ergodicity results is the key step towards Theorem 3.9.

**Proposition 3.11.** Let $\beta \in (0, 1]$ and suppose that $\nu$ is a probability measure on $I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ satisfying $[9]$. Then the closed subspace

$$H = \{f \in L^2(I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S, \nu) : f(aZ) = f(z) \text{ for } a \in I_m^+, z \in I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^* \}$$

of $L^2(I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*, \nu)$ consisting of $I_m^+$-invariant functions coincides with the constant functions. That is, the partial action of $I_m$ on $(I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*, \nu)$ is ergodic.

**Proof.** The proof is similar to that of [L-N2, Theorem 2.1(ii)]. Let $P$ be the orthogonal projection from $L^2(I_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*, \nu)$ onto $H$; we need to show that $Pf$ is a constant function for every $f$. For this, it suffices to compute $P$ at pull-backs of functions on

$$I_m/i(K_{m,\Gamma}) \times \prod_{p \in F} p^{\mathbb{N} \cup \{\infty\}}$$
for every non-empty finite subset $F \subseteq \mathcal{P}_K^m$. Now fix such an $F$, and let $\mathcal{I}^+_F$ be the free submonoid of $\mathcal{I}^+_m$ generated by the primes in $F$. Up to a set of measure zero, $\mathcal{I}_m/i(K_{m,\Gamma}) \times \prod_{p \in F} \mathbb{P}^{\mathbb{N} \cup \{\infty\}}$ coincides with

$$\bigsqcup_{a \in \mathcal{I}^+_F} a(\mathcal{I}_m/i(K_{m,\Gamma}) \times \{1, \ldots, 1\}).$$

Since $\mathcal{I}_m/i(K_{m,\Gamma}) \times \{(1, \ldots, 1)\}$ is a finite group, it suffices to compute, for each fixed $a \in \mathcal{I}^+_F$ and character $\tilde{\chi}$ of $\mathcal{I}_m/i(K_{m,\Gamma})$, $Pf$ where $f$ is the pull-back of

$$\mathcal{I}_m/i(K_{m,\Gamma}) \times \{1, \ldots, 1\} \ni (\mathfrak{f}, \mathfrak{a}) \mapsto \begin{cases} \tilde{\chi}([a]^{-1}\mathfrak{f}) & \text{if } (\mathfrak{f}, \mathfrak{a}) \in a(\mathcal{I}_m/i(K_{m,\Gamma}) \times \{1, \ldots, 1\}) \\ 0 & \text{otherwise.} \end{cases}$$

The character $\chi : \mathcal{I}_m \to \mathbb{T}$ defined by $\chi(a) := \tilde{\chi}([a])$ satisfies $\chi(i(K_{m,1})) = \{1\}$, and is thus a (generalized) Dirichlet character modulo $m$.

For each finite subset $\bar{F} \subseteq \mathcal{P}_K^m$, let $P_{\bar{F}}$ be the orthogonal projection onto the subspace $H_{\bar{F}}$ consisting of $\mathcal{I}^+_F$-invariant functions, so that the projection $P$ is the decreasing strong operator limit of the net $(P_{\bar{F}})_{\bar{F}}$. Also let

$$W_{\bar{F}} := \mathcal{I}_m/i(K_{m,\Gamma}) \times \{1, \ldots, 1\} \times \prod_{p \in \mathcal{I}^+_F} \mathbb{P}^{\mathbb{N} \cup \{\infty\}},$$

so that the sets $bW_{\bar{F}}$ are disjoint for $b \in \mathcal{I}^+_F$, and their union has full $\nu$-measure. As in [L-L-N2 Proposition 1.2(2)], the projection $P_{\bar{F}}$ is given explicitly by

$$P_{\bar{F}}f|_{\mathcal{I}^+_F} = \frac{1}{\zeta_{\bar{F}}(\beta)} \sum_{b \in \mathcal{I}^+_F} N(b)^{-\beta} f(bw)$$

for $w = (\mathfrak{f}, \mathfrak{a}) \in W_{\bar{F}}$ where $\zeta_{\bar{F}}(\beta) := \prod_{p \in \bar{F}} (1 - N(p)^{-\beta})^{-1} = \sum_{b \in \mathcal{I}^+_F} N(b)^{-\beta}$. Now suppose that $\bar{F} \supseteq F$. Then for $f(bw)$ to be non-zero, it is necessary that $b \in a\mathcal{I}^+_{(F \setminus F)c}$, and, in this case, $f(bw) = \tilde{\chi}([a^{-1}b]\mathfrak{f}) = \chi(a^{-1}b)\tilde{\chi}(\mathfrak{f})$. Hence, for $w = (\mathfrak{f}, \mathfrak{a}) \in W_{\bar{F}}$, we have

$$P_{\bar{F}}f|_{\mathcal{I}^+_F} = \frac{1}{\zeta_{\bar{F}}(\beta)} \sum_{b \in a\mathcal{I}^+_{(F \setminus F)c}} N(b)^{-\beta} \chi(a^{-1}b)\tilde{\chi}(\mathfrak{f}) = \frac{N(a)^{-\beta} \tilde{\chi}(\mathfrak{f})}{\zeta_{\bar{F}}(\beta)} \sum_{c \in \mathcal{I}^+_{(F \setminus F)c}} N(c)^{-\beta} \chi(c).$$

If $\chi$ is the trivial character, then the right-hand side of (10) equals

$$N(a)^{-\beta} \prod_{p \in F \setminus F} (1 - N(p)^{-\beta})^{-1} = N(a)^{-\beta} \prod_{p \in F} (1 - N(p)^{-\beta}),$$

where $\pi_0^m$ is the projection onto the subalgebra $\pi_0^m$. This completes the proof of (10).
so \( Pf = \lim_{F} P_{F}f \) is constant. Now suppose that \( \chi \) is non-trivial. Then

\[
\|P_{F}f\|_{L^2(\nu)}^2 = \sum_{\nu \in \mathcal{F}_{\nu}} \int_{W_{F}} |P_{F}f|^2 \, d\nu
\]

\[
= \zeta_{F}(\beta) \int_{W_{F}} |P_{F}f|^2 \, d\nu \quad \text{(using that \( \nu \) satisfies \( \mathcal{F} \))}
\]

\[
= \zeta_{F}(\beta) \left| \frac{N(\alpha)^{-\beta}}{\zeta_{F}(\beta)} \sum_{c \in \mathcal{I}_{F}} N(c)^{-\beta} \chi(c) \right|^2 \nu(W_{F}) \quad \text{(using \( \mathcal{I} \))}
\]

\[
= \left( N(\alpha)^{-\beta} \frac{\prod_{p \in \mathcal{F}} |1 - N(p)^{-\beta}|}{\prod_{p \in \mathcal{F} \setminus \mathcal{F}} |1 - \chi(p) N(p)^{-\beta}|} \right)^2 \quad \text{(since \( \nu(W_{F}) = \zeta_{F}(\beta)^{-1} \)).}
\]

Hence,

\[
\|Pf\|_{L^2(\nu)} = \lim_{F} \|P_{F}f\|_{L^2(\nu)} = N(\alpha)^{-\beta} \lim_{F} \frac{\prod_{p \in \mathcal{F}} |1 - N(p)^{-\beta}|}{\prod_{p \in \mathcal{F} \setminus \mathcal{F}} |1 - \chi(p) N(p)^{-\beta}|}.
\]

For each \( F \), the function

\[
\beta \mapsto \frac{\prod_{p \in \mathcal{F}} |1 - N(p)^{-\beta}|}{\prod_{p \in \mathcal{F} \setminus \mathcal{F}} |1 - \chi(p) N(p)^{-\beta}|}
\]

is increasing on \((0, \infty)\), and for \( \beta > 1 \), the limit \( \lim_{F} \frac{\prod_{p \in \mathcal{F}} |1 - N(p)^{-\beta}|}{\prod_{p \in \mathcal{F} \setminus \mathcal{F}} |1 - \chi(p) N(p)^{-\beta}|} \) exists and is equal to

\[
\frac{|L(\chi, \beta)|}{\zeta_{K}(\beta)} \frac{\prod_{p \in \mathcal{F} \setminus \mathcal{I}} |1 - \chi(p) N(p)^{-\beta}|}{\prod_{p \in \mathcal{I}} (1 - N(p)^{-\beta})}
\]

where \( L(\chi, \beta) \) is the (generalized) Dirichlet \( L \)-function associated with \( \chi \) and \( \zeta_{K}(\beta) \) is the Dedekind zeta function of \( K \). Now as \( \beta \to 1^{+} \), \( L(\chi, \beta) \) tends to a finite value, see, for example, [Mil, Chapter VI, Corollary 2.11], whereas \( \zeta_{K}(\beta) \) has a pole at \( \beta = 1 \) by [Mil, Chapter VI, Corollary 2.12]. Therefore, the right hand side of (11) converges to zero for all \( \beta \in (0, 1) \), so \( \|Pf\|_{L^2(\nu)} = 0 \). In particular, \( Pf \) is constant.

We are now ready to prove Theorem 3.9.

**Proof of Theorem 3.9.** We first deal with the case \( \beta = 0 \). Suppose \( \nu \) is a \( K_{m, \Gamma} \)-invariant probability measure on \( \hat{R}_{S}/\hat{R}_{S}^{*} \). Then, in particular, we have \( \nu(a \hat{R}_{S}/\hat{R}_{S}^{*}) = 1 \) for every \( a \in R_{m, \Gamma} \), which implies that \( \nu(\bigcap_{a} a \hat{R}_{S}/\hat{R}_{S}^{*}) = 1 \). Since \( \bigcap_{a} a \hat{R}_{S}/\hat{R}_{S}^{*} = \{0\} \), we have \( \nu = \delta_{0} \), as desired.

Now let \( \beta \in (0, 1] \). By Lemma 3.10, it suffices to show that the probability measure \( \tilde{\nu}_{\beta} \) on \( \mathcal{I}_{m}/(K_{m, \Gamma}) \times \hat{R}_{S}/\hat{R}_{S}^{*} \) determined by \( \nu_{\beta} \) is the unique probability measure satisfying \( \mathcal{F} \). The set of probability measures that satisfy \( \mathcal{F} \) forms a simplex \( \Sigma \), and Proposition 3.11 says that all measures in \( \Sigma \) are ergodic. A non-trivial convex combination of measures is never ergodic, so we have \( \Sigma = \{\tilde{\nu}_{\beta}\} \).
We are now ready for the proof of uniqueness for $\beta \in [1, 2]$.

Proof of the existence and uniqueness statement in Theorem 3.2(ii). Let $\pi$ denote the quotient map $\hat{R}_S \times \hat{R}_S/\hat{R}_S^* \to \Omega^m_R$ and $m$ the normalized Haar measure on $\hat{R}_S$. For $\beta \in [1, 2]$, let $\mu_\beta := \pi_* (m \times \nu_{\beta - 1})$ be push-forward of the product measure $m \times \nu_{\beta - 1}$ under $\pi$. It follows from Theorem 3.9 combined with Proposition 3.7 that $\mu_\beta$ is the unique probability measure on $\Omega^m_R$ satisfying (4).

The arguments used to prove [Nesh3] Lemma 3.3 carry over almost verbatim to our more general situation and show that the set of points in $\Omega^m_R$ with non-trivial isotropy has $\mu_\beta$-measure zero. Therefore, [Nesh3] Theorem 1.3 implies that $\mu_\beta \circ \hat{\epsilon}$ is the unique $\sigma$-KMS$_\beta$ state on $C^*(G_{m, \Gamma} \ltimes \Omega^m_R)$ for $\beta \in [1, 2]$. Moreover, since $\mu_\beta (V_{(x, a) \times a^*}) = N(a)^{-\beta}$, we are done.

Remark 3.12. If $m$ and $\Gamma$ are trivial, or if $m = m_\infty$ consists of all the real embeddings of $K$ and $\Gamma$ is trivial, then Theorem 3.9 can be deduced from [Nesh3] Theorem 3.1. However, even in this case, our proof here is different: in [Nesh3, Section 3], the special case of Theorem 3.9 is obtained by using known results from [L-N-T] for the Hecke C*-dynamical system associated with the Hecke pair $(K \times K_+, R \times \hat{R}_S^*)$, whereas we give a more direct proof.

3.5. Low temperature KMS states: the proof of part (iii). The map $a \mapsto \prod_{p \in P^m_R} p^{v_p(a)}$ canonically identifies $I^+_m$ with a subset of $\prod_{p \in P^m_R} p^{|v_p(\infty)}$. Composing with the canonical homeomorphism $\prod_{p \in P^m_R} p^{\{0, \infty\}} \simeq \hat{R}_S/\hat{R}_S^*$, we may view $I^+_m$ as a subset of $\hat{R}_S/\hat{R}_S^*$. The image of $I^+_m$ in $\hat{R}_S/\hat{R}_S^*$ consists of those “super ideals” that are coprime to $m_0$ and have only finitely many divisors, that is, with the set

$$\{ \bar{a} \in \hat{R}_S/\hat{R}_S^* : \bar{a} \in U_a \text{ for only finitely many } a \}$$

where $U_a = a\hat{R}_S/\hat{R}_S^*$. We will show that for each $\beta > 1$, every probability measure on $\hat{R}_S/\hat{R}_S^*$ that satisfies (5) must be concentrated on this countable set, and thus is a convex combination of measures that are concentrated on orbits for the partial action of $K_{m, \Gamma}$ on $\hat{R}_S/\hat{R}_S^*$. These orbits are precisely the sets $\mathfrak{m} \cap I^+_m$ for $\mathfrak{m} \in \mathcal{I}_m/\mathcal{I}(K_{m, \Gamma})$.

The partial zeta function associated with a class $\mathfrak{m} \in \mathcal{I}_m/\mathcal{I}(K_{m, \Gamma})$ is the Dirichlet series

$$\zeta_{\mathfrak{m}}(s) := \sum_{a \in \mathfrak{m} \cap I^+_m} N(a)^{-s},$$

which converges for all complex numbers $s$ with real part greater than 1.

Lemma 3.13. For each $\beta \in (1, \infty)$ and each class $\mathfrak{m} \in \mathcal{I}_m/\mathcal{I}(K_{m, \Gamma})$, let $\nu_{\beta, \mathfrak{m}}$ be the probability measure on $\hat{R}_S/\hat{R}_S^*$ given by

$$\nu_{\beta, \mathfrak{m}} := \frac{1}{\zeta_{\mathfrak{m}}(\beta)} \sum_{a \in \mathfrak{m} \cap I^+_m} N(a)^{-\beta} \delta_a$$

where $\delta_a$ denotes the unit mass concentrated at the point $a \in \hat{R}_S/\hat{R}_S^*$. Then each measure $\nu_{\beta, \mathfrak{m}}$ satisfies (5). Moreover, any probability measure $\nu$ that satisfies (5) for $\beta \in (1, \infty)$ is a convex combination of measures from $\{\nu_{\beta, \mathfrak{m}} : \mathfrak{m} \in \mathcal{I}_m/\mathcal{I}(K_{m, \Gamma})\}$. 
Proof. For $\beta \in (1, \infty)$ and $\mathfrak{t} \in \mathcal{I}_m/(K_{m, \Gamma})$, a calculation shows that the measure $\nu_{\beta, \mathfrak{t}}$ satisfies (8).

Now fix $\beta \in (1, \infty)$, and let $\nu$ be a probability measure on $\hat{R}_{S}/\hat{R}_{S}^{*}$ that satisfies (8). Recall that the inverse of a fractional ideal $a$ in $\mathcal{I}_m$ is given by $a^{-1} := \{x \in K : xa \subseteq R\}$. For $a \in \mathcal{I}_m^{+}$ and $x \in a^{-1} \cap K_{m, \Gamma}$, we have $xU_a = U_{xa}$. Now,

$$\sum_{a \in \mathcal{I}_m^{+}} \nu(U_a) = \sum_{x \in (a^{-1} \cap K_{m, \Gamma})/R_{m, \Gamma}} \nu(U_{xa})$$

$$= \sum_{x \in (a^{-1} \cap K_{m, \Gamma})/R_{m, \Gamma}} N(x)^{-\beta} \nu(U_{a}) \quad \text{(using (8))}$$

$$= \sum_{x \in (a^{-1} \cap K_{m, \Gamma})/R_{m, \Gamma}} N(xa)^{-\beta} N(a)^{-\beta} \nu(U_{a})$$

$$= \zeta(\beta) N(a)^{-\beta} \nu(U_{a}).$$

Thus, since $\beta > 1$,

$$\sum_{a \in \mathcal{I}_m^{+}} \nu(U_a) = \sum_{\mathfrak{t} \in \mathcal{I}_m/(K_{m, \Gamma})} \zeta(\beta) N(a)^{-\beta} \nu(U_{a}) < \infty,$$

so the Borel-Cantelli lemma implies that $\nu$ is concentrated on the set

$$\{ \mathfrak{a} \in \hat{R}_{S}/\hat{R}_{S}^{*} : \mathfrak{a} \in U_a \text{ for only finitely many } a \}.$$

This set coincides with the canonical copy of $\mathcal{I}_m^{+}$ in $\hat{R}_{S}/\hat{R}_{S}^{*}$. Since $\nu$ satisfies (8) and $\mathcal{I}_m^{+}$ is countable, the set of points that have positive $\nu$-measure must be a (disjoint) union of orbits for the partial action of $K_{m, \Gamma}$ on $\mathcal{I}_m^{+}$, and $\nu$ is a convex combination of its restrictions to these orbits; moreover, these orbits are precisely the sets $\mathfrak{t} \cap \mathcal{I}_m^{+}$ for $\mathfrak{t} \in \mathcal{I}_m/(K_{m, \Gamma})$, and a calculation shows that $\nu_{\beta, \mathfrak{t}}$ is the only probability measure that both satisfies (8) and is concentrated on $\mathfrak{t} \cap \mathcal{I}_m^{+}$, so we are done. □

We are now ready for the proof of Theorem 3.2(iii).

Proof of Theorem 3.2(iii). As before, let $\pi$ denote the quotient map $\hat{R}_{S} \times \hat{R}_{S}/\hat{R}_{S}^{*} \to \Omega_{R}^m$, and let $m$ be the normalized Haar measure on $\hat{R}_{S}$. For each $\beta > 2$ and each class $\mathfrak{t} \in \mathcal{I}_m/(K_{m, \Gamma})$, let $\mu_{\beta, \mathfrak{t}} := \pi_*(m \times \nu_{\beta-1, \mathfrak{t}})$ be the push-forward of the product measure $m \times \nu_{\beta-1, \mathfrak{t}}$ under $\pi$. By Proposition 3.7, the map $\nu_{\beta-1} \mapsto \pi_*(m \times \nu_{\beta-1, \mathfrak{t}})$ establishes an affine bijection from the simplex of probability measures on $\hat{R}_{S}/\hat{R}_{S}^{*}$ that satisfy (6) onto the simplex of probability measures on $\Omega_{R}^m$ that satisfy (4); hence, by Lemma 3.13, every probability measure on $\Omega_{R}^m$ that satisfies (4) is a convex combination of measures from the set $\{ \pi_*(m \times \nu_{\beta-1, \mathfrak{t}}) : \mathfrak{t} \in \mathcal{I}_m/(K_{m, \Gamma}) \}$.

For each $a \in \mathcal{I}_m^{+}$, there are exactly $N(a)$ points $[b, a]$ in $\Omega_{R}^m$, with second component equal to $a$. Indeed, we can always write $[b, a] = [x, a]$ for some $x \in \hat{R}_{S}/a\hat{R}_{S} \cong R/a$ with $b - x \in a\hat{R}_{S}$. Hence, for each $\mathfrak{t} \in \mathcal{I}_m/(K_{m, \Gamma})$, the set $\{[b, a] \in \Omega_{R}^m : a \in \mathfrak{t}\}$ is countable. Moreover, the partial action of $G_{m, \Gamma}$ on $\{[b, a] \in \Omega_{R}^m : a \in \mathfrak{t}\}$ is transitive, and the measure $\mu_{\beta, \mathfrak{t}}$ is concentrated on $\{[b, a] \in \Omega_{R}^m : a \in \mathfrak{t}\}$. Since this set contains the point $[0, a_{\mathfrak{t}, 1}]$.
which has isotropy group $a_t \times R_{m \Gamma}^*$, the result stated in Theorem $3.2$ (iii) now follows from \[Nesh3\] Theorem 1.3 (see also \[Nesh3\] Corollary 1.4)).

\[\Box\]

3.6. **Ground states: the proof of part (iv).** We will first use \[L-L-N3\] Theorem 1.9 to identify the ground states of $(C^*(G_{m \Gamma} \ltimes \Omega^m_R), \mathbb{R}, \sigma)$ with the states on the C*-algebra of the boundary groupoid of the cocycle $c^N$, see \[L-L-N3\] Section 1 for the general definition. In our special situation, this boundary groupoid has a particularly explicit description, which is given in the following result.

**Proposition 3.14.** Let $G_{m \Gamma, 1} := \{(n, k) \in G_{m \Gamma} : N(k) = 1\}$ be the kernel of the homomorphism $G_{m \Gamma} \to \mathbb{R}^+_+$ given by $(n, k) \mapsto N(k)$, and let

$$
(\Omega^m_R)_0 := \Omega^m_R \setminus \bigcup_{(n, k) : N(k) > 1} (n, k)\Omega^m_R.
$$

Then the map $\psi \mapsto \phi_\psi$ defined by

$$
\phi_\psi(f) = \psi(f|_{G_{m \Gamma, 1} \ltimes (\Omega^m_R)_0}) \quad \text{for } f \in C_c(G_{m \Gamma} \ltimes \Omega^m_R)
$$

is an affine isomorphism of the state space of $C^*(G_{m \Gamma, 1} \ltimes (\Omega^m_R)_0)$ onto the $\sigma$-ground state space of $C^*(G_{m \Gamma} \ltimes \Omega^m_R)$ where $G_{m \Gamma, 1} \ltimes (\Omega^m_R)_0$ is the reduction groupoid of $G_{m \Gamma, 1} \ltimes \Omega^m_R$ with respect to the compact subset $(\Omega^m_R)_0 \subseteq \Omega^m_R$.

**Proof.** This is a direct application of \[L-L-N3\] Theorem 1.9). \[\Box\]

We are now ready for the proof of Theorem 3.2(iv).

**Proof of Theorem 3.2(iv).** For each class $\mathfrak{t} \in \mathcal{I}_m/i(K_{m \Gamma})$, let $a_{\mathfrak{t}, 1}, \ldots, a_{\mathfrak{t}, k_\mathfrak{t}}$ denote the norm-minimizing ideals in $\mathfrak{t}$. In light of Proposition 3.14, it suffices to prove that there is an isomorphism

$$
C^*(G_{m \Gamma, 1} \ltimes (\Omega^m_R)_0) \cong \bigoplus_{\mathfrak{t} \in \mathcal{I}_m/i(K_{m \Gamma})} M_{k_\mathfrak{t} \cdot N(a_{\mathfrak{t}, 1})}(C^*(a_{\mathfrak{t}, 1} \times R_{m \Gamma}^*)).
$$

We first claim that

$$
(\Omega^m_R)_0 = \{[b, \tilde{a}] \in \Omega^m_R : \tilde{a} = a_{\mathfrak{t}, j} \text{ for some } \mathfrak{t} \in \mathcal{I}_m/i(K_{m \Gamma}), 1 \leq j \leq k_\mathfrak{t}\}.
$$

"$\subseteq$": For each prime $p \in \mathcal{P}_K$, let $f_p$ denote the order of the class $[p]$ of $p$ in $\mathcal{I}_m/i(K_{m \Gamma})$, so that there exists $t_p \in R_{m \Gamma}$ such that $p f_p = t_p R$. Let $[b, \tilde{a}] \in \Omega^m_R$, and suppose that $v_p(\tilde{a}) = \infty$ for some $p \in \mathcal{P}_K$, so that $t_p \tilde{a} = \tilde{a}$. By the strong approximation theorem, there exists $x \in R_{m^{-1} R_{m \Gamma}}$ such that $v_p(x + b) \geq f_p$. Then $t_p^{-1}(x + b) \in \bar{R}_S$, and

$$
[b, \tilde{a}] = (-x, t_p)[t_p^{-1}(x + b), \tilde{a}] \in (-x, t_p)\Omega^m_R.
$$

Since $N(t_p) = f_p > 1$, we see that $[b, \tilde{a}] \notin (\Omega^m_R)_0$. Therefore, if $[b, \tilde{a}] \in (\Omega^m_R)_0$, then $v_p(\tilde{a}) < \infty$ for every $p \in \mathcal{P}_K$.

Next, we will show that if $[b, \tilde{a}] \in (\Omega^m_R)_0$, then $\tilde{a}$ is divisible by only finitely many primes. Suppose $[b, \tilde{a}] \in \Omega^m_R$ is such that $\{p \in \mathcal{P}_K : v_p(\tilde{a}) > 0\}$ is infinite. Then there are finitely many distinct primes $p_1, p_2, \ldots, p_N$ in $\{p \in \mathcal{P}_K : v_p(\tilde{a}) > 0\}$ such that the ideal $\prod_{j=1}^N p_j$
is principal. Let \( a \in R_m \) be such that \( aR = \prod_{j=1}^{N} p_j \). Then \( a^{-1} \hat{a} \) lies in \( \hat{R}_S \), and by the strong approximation theorem, there exists \( x \in R_m^{-1}R \) such that \( \nu_p(a^{-1}(x + b)) \geq 0 \) for every \( p \in \mathcal{P}_K \), so that \( a^{-1}(x + b) \in \hat{R}_S \). Now

\[
[b, \hat{a}] = (-x, a)[a^{-1}(x + b), a^{-1}\hat{a}] \in (-x, a)\Omega_R^m.
\]

Since \( N(a) > 1 \), we see that \( [b, \hat{a}] \not\in (\Omega_R^m)_0 \). Thus, if \( [b, \hat{a}] \in (\Omega_R^m)_0 \), then \( \nu_p(\hat{a}) = 0 \) for all but finitely many \( p \in \mathcal{P}_K \).

The above two facts imply that if \( [b, \hat{a}] \in (\Omega_R^m)_0 \), then \( \hat{a} = a \) for some \( a \in \mathcal{I}_m^+ \). In this case, there exist \( k \in K_{m}\Gamma \) and \( 1 \leq j \leq k_\alpha \) such that \( \hat{a} = ka_{\beta,j} \), and there exists \( x \in R \) such that \([b, a] = [x, a]\). It remains to show that \( k \) has norm 1. Since \( (\Omega_R^m)_0 \) is \( G_{m,\Gamma,1} \)-invariant, we see that \((0,k)[0, a_{\beta,j}] = [0, ka_{\beta,j}] = (-x,1)[x, ka_{\beta,j}] \) lies in \((\Omega_R^m)_0\), so we must have \( N(k) = 1 \), that is, \( a \) must be norm-minimizing in \( \mathcal{I}_m \). This finishes our proof of the first inclusion.

\[\Omega^m_R\]: Let \( \mathcal{I} \in \mathcal{I}_m/i(K_{m,\Gamma}) \), \( 1 \leq j \leq k_\alpha \), and suppose that \([x, a_{\beta,j}] \in \Omega_R^m \cap (n,k)\Omega_R^m \) for some \((n,k) \in G_{m,\Gamma}\). There exists an integral ideal \( b \in \mathcal{I} \) such that \( a_{\beta,j} = kb \), so minimality of \( N(a_{\beta,j}) \) forces \( N(k) = 1 \). This shows the reverse inclusion and concludes the proof of our claim.

In particular, the above claim shows that \((\Omega_R^m)_0\) is a finite set. Moreover, there is a \( G_{m,\Gamma} \)-equivariant decomposition

\[
(\Omega_R^m)_0 = \bigcup_{\mathcal{I} \in \mathcal{I}_m/i(K_{m,\Gamma})} X_{\mathcal{I}}
\]

where \( X_{\mathcal{I}} = \{[x, a_{\beta,j}] : x \in R/a_{\beta,j}, j = 1, \ldots, k_1\} \) is the orbit of any \([b, a_{\beta,j}]\) under the partial action of \( G_{m,\Gamma,1} \); it follows that we have the direct sum decomposition

\[
C^*(G_{m,\Gamma,1} \ltimes (\Omega_R^m)_0) \cong \bigoplus_{\mathcal{I} \in \mathcal{I}_m/i(K_{m,\Gamma})} C^*(G_{m,\Gamma,1} \ltimes X_{\mathcal{I}}).
\]

For each class \( \mathcal{I} \), \( G_{m,\Gamma,1} \ltimes X_{\mathcal{I}} \) is a transitive groupoid, and the isotropy group of the point \([0, a_{\beta,1}] \) is \( a_{\beta,1} \ltimes R_{m,\Gamma}^* \); therefore, it follows that \( C^*(G_{m,\Gamma,1} \ltimes X_{\mathcal{I}}) \cong M_{|X_{\mathcal{I}}|}(C^*(a_{\beta,1} \ltimes R_{m,\Gamma}^*)) \), see, for example, \([M-R-W] \) Theorem 3.1]. Since \( |X_{\mathcal{I}}| = k_\alpha \cdot N(a_{\beta,1}) \), we are done. \( \square \)

4. Type III\(_1\) Factors and the Distribution of Prime Ideals in \( \mathcal{I}_m/i(K_{m,\Gamma}) \)

Each extremal \( \sigma\)-KMS\(_\beta\) state \( \phi \) on \( C^*(R \ltimes R_{m,\Gamma}) \) is a factor state, that is, the von Neumann algebra \( \pi_\phi(C^*(R \ltimes R_{m,\Gamma}))'' \) generated by the GNS representation \( \pi_\phi \) of \( \phi \) is a factor, see \([B-R] \) Theorem 5.3.30. It is therefore a natural problem to determine the type of the factors arising from extremal \( \sigma\)-KMS\(_\beta\) states on \( C^*(R \ltimes R_{m,\Gamma}) \). The main result of this section is the following theorem, which, in light of the uniqueness of the injective factor of type III\(_1\) with separable predual, see \([Con] \) and \([Haa] \), completes the proof of Theorem 3.2(ii).

**Theorem 4.1.** For each \( \beta \in [1, 2] \), let \( \pi_{\phi_\beta} \) be the GNS representation of the \( \sigma\)-KMS\(_\beta\) state \( \phi_\beta \) on \( C^*(R \ltimes R_{m,\Gamma}) \) from Theorem 3.2. Then the von Neumann algebra \( \pi_{\phi_\beta}(C^*(R \ltimes R_{m,\Gamma}))'' \) is an injective factor of type III\(_1\) with separable predual.

**Remark 4.2.** It follows from \([La-Nesh2] \) Theorem 3.2 that, for each \( \beta \in [1, 2] \), the \( \sigma\)-KMS\(_\beta\) state on \( C^*(Z \ltimes N^\times) \) is of type III\(_1\). Moreover, it is asserted in \([Nesh3] \) Section 3 that arguments analogous to those used to prove \([La-Nesh2] \) Theorem 3.2 combined with \([Nesh2] \)
Corollary 3.2] can be used to show that, for each $\beta \in [1, 2]$, the $\sigma$-KMS$_\beta$ state on $C^*_\alpha(R \times R^\infty)$ is of type III$_1$.

In our more general situation, there are additional difficulties which we will overcome by using techniques from [Lag-Nesh, Sections 2&3].

The remainder of this section is devoted to the proof of Theorem 4.1.

We now briefly recall some well-known results about the flow of weights on von Neumann algebra crossed products from [Con-Tak] (see also [Tak, Chapter XIII § 2]). The general setup here is similar to that in [La-Nesh2, Section 3] and [Nesh2, Section 2], and we will follow the notation therein.

Let $X$ be a second countable, locally compact Hausdorff space and $\mu$ a $\sigma$-finite measure on $X$. Suppose that a countably infinite discrete group $G$ acts by nonsingular transformations on the measure space $(X, \mu)$, that is, $G$ acts on $X$ by Borel automorphisms, and for each $g \in G$, the measures $\mu$ and $g\mu$ are equivalent where $g\mu$ is the push-forward of $\mu$ by $g$ defined by $g\mu(Z) := \mu(g^{-1}Z)$ for every Borel set $Z \subseteq X$.

Assume that the of $G$ on $(X, \mu)$ is essentially free and ergodic, so that the von Neumann algebra crossed product $L^\infty(X, \mu) \rtimes G$ is a factor. In this situation, the flow of weights has a particularly explicit description. Indeed, let $\lambda_\infty$ denote the Lebesgue measure on $\mathbb{R}^*_+$; then there are commuting actions of $G$ and $\mathbb{R}$ on $(\mathbb{R}^*_+ \times X, \lambda_\infty \times \mu)$ given by

$$g(t, x) = \left(\frac{dg\mu}{d\mu}(gx)t, gx\right) \quad \text{and} \quad s(t, x) = (e^{-s}t, x)$$

for $g \in G$, $s \in \mathbb{R}$, and $(t, x) \in \mathbb{R}^*_+ \times X$,

and the flow of weights on $L^\infty(X, \mu) \rtimes G$ is the induced action of $\mathbb{R}$ on $L^\infty(\mathbb{R}^*_+ \times X, \lambda_\infty \times \mu)^G$. The factor $L^\infty(X, \mu) \rtimes G$ is of type III$_1$ if and only if the flow of weights is trivial, that is, if the action of $G$ on $(\mathbb{R}^*_+ \times X, \lambda_\infty \times \mu)$ is ergodic.

We now turn to the particular case of interest to us. As before, it will be easiest to work with the C*-algebra $C^*(G_{m, \Gamma} \rtimes \Omega^m_K)$. Since $\phi_\beta$ factors through the expectation $\mathcal{E}$ onto $C(\Omega^m_K)$ and is determined by the probability measure $\mu_\beta$, we have the following standard lemma.

**Lemma 4.3.** For each $\beta \in [1, 2]$, let $\bar{\mu}_\beta$ be the unique quasi-invariant measure on $\Omega^m_K$ that extends $\mu_\beta$ and satisfies the obvious analogue of 4.1 for the action of $G_{m, \Gamma}$ on $\Omega^m_K$. Then

$$\pi_{\phi_\beta}(C^*(G_{m, \Gamma} \rtimes \Omega^m_K))'' \cong 1_{\Omega^m_K}(L^\infty(\Omega^m_K, \bar{\mu}_\beta) \rtimes G_{m, \Gamma})1_{\Omega^m_K}.$$ 

Therefore, if $L^\infty(\Omega^m_K, \bar{\mu}_\beta) \rtimes G_{m, \Gamma}$ is a factor of type III$_1$, then $\pi_{\phi_\beta}(C^*(G_{m, \Gamma} \rtimes \Omega^m_K))''$ is also a factor of type III$_1$.

Hence, to prove Theorem 4.2 it suffices to show that $L^\infty(\Omega^m_K, \bar{\mu}_\beta) \rtimes G_{m, \Gamma}$ is an injective factor of type III$_1$ with separable predual. Since $G_{m, \Gamma}$ is amenable, $L^\infty(\Omega^m_K, \bar{\mu}_\beta) \rtimes G_{m, \Gamma}$ is injective, and the separability claim is easy to see. This means that we need to prove that $L^\infty(\Omega^m_K, \bar{\mu}_\beta) \rtimes G_{m, \Gamma}$ is a factor of type III$_1$.

**Proposition 4.4.** For each $\beta \in [1, 2]$, the action $G_{m, \Gamma} \rtimes \Omega^m_K$ is essentially free and ergodic. Hence, $L^\infty(\Omega^m_K, \bar{\mu}_\beta) \rtimes G_{m, \Gamma}$ is a factor.
Proof. Arguments similar to those used in the proof of [Nesh3] Lemma 3.3 show that the action $G_{m,\Gamma} \curvearrowright (\Omega^m_K, \bar{\mu}_\beta)$ is essentially free; note we have already made this observation in the proof of the uniqueness statement in Theorem [3.2 ii].

One can argue directly using Proposition [3.11] to show that the action $G_{m,\Gamma} \curvearrowright (\Omega^m_K, \bar{\mu}_\beta)$ is ergodic. Alternatively, since Theorem [3.2 ii] says that the state $\phi_\beta$ is the unique $\sigma$-KMS state on $C^*(G_{m,\Gamma} \ltimes \Omega^m_R)$, [B-R] Theorem 5.3.30(3)] implies that $\pi_{\phi_\beta}(C^*(G_{m,\Gamma} \ltimes \Omega^m_R))$ is a factor. Since $1_{\Omega^m_R}$ is a full projection in $C_0(\Omega^m_R) \rtimes G_{m,\Gamma}$, it follows that $L^\infty(\Omega^m_K, \bar{\mu}_\beta) \rtimes G_{m,\Gamma}$ is also a factor. Thus, the action $G_{m,\Gamma} \curvearrowright (\Omega^m_K, \bar{\mu}_\beta)$ is ergodic. \hfill $\square$

The following lemma on primes in ideal classes from $\mathcal{I}_m/i(K_{m,\Gamma})$ is the key number-theoretic result needed to compute the flow of weights on $L^\infty(\Omega^m_K, \bar{\mu}_\beta) \rtimes G_{m,\Gamma}$. It is a generalization of [Nesh2] Lemma 3.3.

**Lemma 4.5.** Fix $\beta \in (0, 1]$ and fix a class $\mathfrak{K} \in \mathcal{I}_m/i(K_{m,\Gamma})$. For each $\lambda > 1$ and each $\epsilon > 0$, there exist sequences $(p_n)_{n \geq 1}$ and $(q_n)_{n \geq 1}$, each consisting of distinct prime ideals in $\mathcal{P}^m_K$, such that

$$\left| \frac{N(q_n)^\beta}{N(p_n)^\beta} - \lambda \right| < \epsilon, \quad q_n p_n^{-1} \in \mathfrak{K} \text{ for } n \geq 1, \quad \text{and } \sum_{n=1}^{\infty} N(p_n)^{-\beta} = \infty. \quad (12)$$

**Proof.** The proof is similar to that of [Nesh2] Lemma 3.3; it follows ideas from [Bo-Za] and [Bla] (also see the proof of [Lag-Nesh] Theorem 1.2) for number fields.

The case where $\beta \in (0, 1)$ follows from the case $\beta = 1$, so it suffices to consider only the case $\beta = 1$. Choose $\delta > 0$ such that $1 + \delta < \lambda$ and $\delta \lambda < \epsilon$. Define sets $B_n$ by

$$B_{2k} := \{p \in \mathcal{P}^m_K : \lambda^{2k} < N(p) \leq (1 + \delta)\lambda^{2k}, \, p \in [R]\},$$

$$B_{2k+1} := \{p \in \mathcal{P}^m_K : \lambda^{2k+1} < N(p) \leq (1 + \delta)\lambda^{2k+1}, \, p \in \mathfrak{K}\}.$$

By our choice of $\delta$, these sets are pairwise disjoint. For a class $\mathfrak{K} \in \mathcal{I}_m/i(K_{m,\Gamma})$ and $x > 0$, let

$$\pi_{\mathfrak{K}}(x) := |\{p \in \mathfrak{K} : p \text{ prime and } N(p) \leq x\}|$$

be the number of prime ideals in the class $\mathfrak{K}$ whose norms do not exceed $x$. Then [Mil] Chapter VIII, Theorem 7.2] combined with [Nar] Chapter 7, Proposition 7.17] imply that

$$\pi_{\mathfrak{K}}(x) \sim \frac{1}{h \log x} \frac{x}{h \log x} \text{ as } x \to \infty$$

where $h := |\mathcal{I}_m/i(K_{m,\Gamma})|$. From this, it follows that

$$\pi_{\mathfrak{K}}((1 + \delta)x) - \pi_{\mathfrak{K}}(x) \sim \frac{\delta}{h \log x} \frac{x}{h \log x} \text{ as } x \to \infty.$$ 

As this holds for every class $\mathfrak{K}$, we have

$$|B_n| \sim \frac{\delta}{h n \log \lambda} \frac{\lambda^n}{h n \log \lambda} \text{ as } n \to \infty. \quad (13)$$

Thus, there exists $k_0$ such that $|B_{2k+1}| \geq |B_{2k}|$ for all $k \geq k_0$. Now, for each $k \geq k_0$, we can choose a subset $C_{2k+1} \subseteq B_{2k+1}$ such that $|C_{2k+1}| = |B_{2k}|$. Let $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ be enumerations of the sets $\bigcup_{k \geq k_0} B_{2k}$ and $\bigcup_{k \geq k_0} C_{2k+1}$, respectively, such that
\(N(p_1) \leq N(p_2) \leq \cdots\), and \(N(q_1) \leq N(q_2) \leq \cdots\). Then if \(p_n \in B_{2k}\) for some \(k \geq k_0\), we must have \(q_n \in B_{2k+1}\), in which case by our choice of \(\delta\), we have 
\[N(q_n p_n^{-1}) \in (\lambda - \epsilon, \lambda + \epsilon)\] and \(q_n p_n^{-1} \in \mathfrak{t}[R] = \mathfrak{t}.

Moreover, using (13), we see that it is enough to prove that the action of \(\nu\) is ergodic. Since the isomorphism \(K \to R^*_\Lambda\), it suffices to show that the action of \(\nu\) on \((R^* \times \mathbb{K}^\infty, \lambda_\infty \times \nu)\) given by 
\[k(t, \tilde{a}) = (N(k)^k t, k\tilde{a}) \quad \text{for} \quad k \in K, (t, \tilde{a}) \in R^*_+ \times \mathbb{K}^\infty\]

is ergodic.

Remark 4.7. If \(m_\infty\) is supported on all of the real embeddings of \(K\) and \(m_0\) is trivial, so that \(K = K^*_\infty\) is the multiplicative subgroup of \(K^*\) consisting of all (non-zero) totally positive elements, then Proposition 4.6 is precisely [Nesh2, Corollary 3.2], which follows from Neshveyev’s type computation for the high temperature KMS states on the Bost-Connes system associated with \(K\), see [Nesh2, Theorem 3.1].

Proof of Proposition 4.6 Since the subgroup \(R^*_m = R^*_\Lambda\) acts trivially, the action of \(K\) on \(\mathcal{I}/i(K) \times R^*_+ \times H^\infty\) by 
\[a([b], t, \tilde{a}) = ([ab], N(a)^\beta t, a\tilde{a}) \quad \text{for} \quad a \in \mathcal{I}, ([b], t, \tilde{a}) \in \mathcal{I}/i(K) \times R^*_+ \times H^\infty\]
is ergodic. Since the isomorphism \(R^*_+ \to \mathbb{R}^*_+\) given \(t \mapsto t^\beta\) preserves the measure class of \(\lambda_\infty\), it suffices to show that the action of \(\mathcal{I}\) on \((\mathcal{I}/i(K) \times R^*_+ \times H^\infty, \lambda_m \times \lambda_\infty \times \tilde{\nu})\) given by 
\[a([b], t, \tilde{a}) = ([ab], N(a)^\beta t, a\tilde{a}) \quad \text{for} \quad a \in \mathcal{I}, ([b], t, \tilde{a}) \in \mathcal{I}/i(K) \times R^*_+ \times \mathbb{K}^\infty\]
is ergodic.

Let \(\mathcal{R}\) denote the orbit equivalence relation for the canonical action \(\mathcal{I}/i(K) \times R^*_+ \times \mathbb{K}^\infty\) given by \(a : \tilde{a} \mapsto a\tilde{a}\). This action is essentially free; indeed, the set 
\[\{\tilde{a} \in \mathbb{K}^\infty : \text{there exists} \ p \ \text{with} \ v_p(\tilde{a}) = \infty\}\]
has \(\tilde{\nu}\)-measure zero by the scaling condition, and every point lying in the complement of this set has trivial isotropy. Thus, outside a set of measure zero we can define an \((\mathcal{I}/i(K) \times R^*_+\)-valued 1-cocycle \(c\) on \(\mathcal{R}\) by 
\[c(\tilde{a}, \tilde{b}) = ([a], N(a)) \quad \text{if} \quad a\tilde{a} = \tilde{b}.

Then the equivalence relation \( \mathcal{R}(c) \) on \( \mathcal{I}_m / i(K_{m,\Gamma}) \times \mathbb{R}_+^+ \times \mathcal{K}_S / \hat{R}_S^* \) associated with \( c \) as in [Fel-Mo, Section 8] (see also [Lag-Nesh, Section 2]) coincides with the orbit equivalence relation for the action of \( \mathcal{I}_m / i(K_{m,\Gamma}) \times \mathbb{R}_+^+ \times \mathcal{K}_S / \hat{R}_S^* \) given by (15). Therefore, it suffices to show that \( \mathcal{R}(c) \) is ergodic. Following the proof of Proposition 3.11 it is not difficult to see that \( \mathcal{R} \) is ergodic if and only if the asymptotic range \( r^*(c) \) of \( c \) ([Fel-Mo, Definition 8.2]) coincides with \( \mathcal{I}_m / i(K_{m,\Gamma}) \times \mathbb{R}_+^+ \), see [Lag-Nesh, Proposition 2.1(iii)].

The proof that \( r^*(c) = \mathcal{I}_m / i(K_{m,\Gamma}) \times \mathbb{R}_+^+ \) relies on [Lag-Nesh, Proposition 2.2] and follows the same lines as the computation of the analogous asymptotic range in the proof of [Lag-Nesh, Theorem 1.2] for number fields, but with Lemma 4.5 used in place of [Lag-Nesh, Corollary 3.3].

We are now ready for the proof of Theorem 4.1

Proof of Theorem 4.1. Fix \( \beta \in [1, 2] \). In light of Lemma 4.3 and Proposition 4.4 we only need to show that the factor \( L^\infty(\Omega_m^m, \hat{\mu}_\beta) \times G_{m,\Gamma} \) is of type III. That is, we must show that the flow of weights on \( L^\infty(\Omega_m^m, \hat{\mu}_\beta) \times G_{m,\Gamma} \) is trivial. Since

\[
\frac{d(n,k)\hat{\nu}_{\beta}}{d\hat{\nu}_{\beta}}((n,k)w) = N(k)^\beta, \quad \text{for } (n,k) \in G_{m,\Gamma}, w \in \Omega_m^m,
\]

this is equivalent to showing that the action of \( G_{m,\Gamma} \) on \( (\mathbb{R}_+^+ \times \Omega_m^m, \lambda_\infty \times \hat{\nu}_{\beta}) \) given by

\[
(n,k)(t,w) = (N(k)^\beta t, (n,k)w), \quad \text{for } (n,k) \in G_{m,\Gamma}, w \in \Omega_m^m
\]

is ergodic. As in the proof of [La-Nesh2, Theorem 3.2], we see that it suffices to prove that the action of \( K_{m,\Gamma} \) on \( (\mathbb{R}_+^+ \times \mathcal{K}_S / \hat{R}_S^*, \lambda_\infty \times \hat{\nu}_{\beta-1}) \) given by

\[
k(t, \hat{a}) = (N(k)^{\beta-1} t, \hat{k}a) \quad \text{for } k \in K_{m,\Gamma}, (t, \hat{a}) \in \mathbb{R}_+^+ \times \mathcal{K}_S / \hat{R}_S^*,
\]

is ergodic. For \( \beta = 1 \), this follows since \( \hat{\nu}_0 = \delta_0 \) and \( \{N(k) : k \in K_{m,\Gamma}\} = \mathbb{Q}_+^* \), which is dense in \( \mathbb{R}_+^* \), whereas for \( \beta \in (1, 2) \), this follows from Proposition 4.6.

Remark 4.8. Since the seminal work of Bost and Connes [Bo-Co], there have been several operator algebraic constructions from number theory that lead to \( C^* \)-dynamical systems exhibiting interesting phase transitions where the high temperature KMS states are factor states of type III. See, for example, [Bo-Co], [Har-Lei], [Bo-Za], [Nesh1, Nesh2], [La-Nesh2], and [L-N-T]. We remark that in all cases, uniqueness of the high temperature KMS states boils down to that fact that certain \( L \)-functions do not have poles at 1, and the crucial number-theoretic result needed to compute the type is a version of the prime number theorem.

5. The boundary quotient

By [Bru, Theorem 7.1], the \( C^* \)-algebra \( C^*_\lambda(R \rtimes R_{m,\Gamma}) \) has a unique maximal ideal \( I_{Pm} \). The boundary quotient of \( C^*_\lambda(R \rtimes R_{m,\Gamma}) \), as defined in [Li2, Section 7] (see also [C-E-L-Y, Chapter 5.7]), is the quotient \( C^*_\lambda(R \rtimes R_{m,\Gamma}) / I_{Pm} \). Moreover, [Bru, Theorem 7.1] gives an explicit description of the ideal \( I_{Pm} \) in terms of the generating isometries in \( C^*_\lambda(R \rtimes R_{m,\Gamma}) \).
as follows. For each $p \in \mathcal{P}_R^m$, let $f_p$ denote the order of $[p] \in \mathcal{I}_m/i(K_m,\Gamma)$, so that $p^{f_p} = t_p R$ for some $t_p \in R_m,\Gamma$. Then

$$I_{\mathcal{P}_R^m} = \left\{ 1 - \sum_{x \in R/t_p R} \lambda(x, t_p) \lambda^*(x, t_p) : p \in \mathcal{P}_R^m \right\}.$$ 

Let $\rho : C^*_\lambda(R \rtimes R_m,\Gamma) \to C^*_\lambda(R \rtimes R_m,\Gamma) / I_{\mathcal{P}_R^m}$ be the quotient map. For each $t \in \mathbb{R}$, the automorphism $\sigma_t$ is the identity on $I_{\mathcal{P}_R^m}$, so $\sigma$ defines a time evolution $\tilde{\sigma}$ on $C^*_\lambda(R \rtimes R_m,\Gamma) / I_{\mathcal{P}_R^m}$ such that $\tilde{\sigma}_t(\rho(\lambda(b,a))) = N(a)^{it} \rho(\lambda(b,a))$ for all $(b,a) \in R \rtimes R_m,\Gamma$.

The following lemma will enable us to use Theorem 3.2 to compute the $\tilde{\sigma}$-KMS states on $C^*_\lambda(R \rtimes R_m,\Gamma) / I_{\mathcal{P}_R^m}$. Its proof relies on an idea that is well-known to experts and goes back at least to [La-Rae2, Lemma 10.3]; this idea is explained in a more general setting, which is suitable for our purposes, in [aHLRS, Lemma 2.2].

**Lemma 5.1.** Suppose that $\phi$ is a $\sigma$-KMS state on $C^*_\lambda(R \rtimes R_m,\Gamma)$. Then $\phi$ vanishes on the ideal $I_{\mathcal{P}_R^m}$ if and only if

$$\phi \left( 1 - \sum_{x \in R/t_p R} \lambda(x, t_p) \lambda^*(x, t_p) \right) = 0$$

for all $p \in \mathcal{P}_R^m$.

**Proof.** The “only if” direction is obvious. For the other direction, we can apply [aHLRS, Lemma 2.2] to $\phi$ with, in the notation from [aHLRS, Lemma 2.2],

- $(A, R, \alpha) = (C^*_\lambda(R \rtimes R_m,\Gamma), R, \sigma)$;
- $P = \{1 - \sum_{x \in R/t_p R} \lambda(x, t_p) \lambda^*(x, t_p) : p \in \mathcal{P}_R^m \}$;
- $J = I_{\mathcal{P}_R^m}$;
- $\mathcal{F} = \{ \lambda_{(b,a)} d \lambda_{(d,c)} : (b,a), (d,c) \in R \rtimes R_m,\Gamma, d \in D_\lambda(R \rtimes R_m,\Gamma) \}$.

To see that $\text{span}(\mathcal{F})$ is dense in $C^*_\lambda(R \rtimes R_m,\Gamma)$, one can either argue directly or use [Bru, Proposition 4.1] and [La, Remark 1.3.1].

**Theorem 5.2.** The C*-dynamical system $(C^*_\lambda(R \rtimes R_m,\Gamma) / I_{\mathcal{P}_R^m}, R, \tilde{\sigma})$ has a unique $\tilde{\sigma}$-KMS state $\tilde{\phi}$, and there are no $\tilde{\sigma}$-KMS states for $\beta \neq 1$. Moreover, if $\tilde{\phi}$ is the GNS representation of $\tilde{\phi}$, then $\pi_{\tilde{\phi}}(C^*_\lambda(R \rtimes R_m,\Gamma) / I_{\mathcal{P}_R^m})$ is isomorphic to the injective factor of type $\text{III}_1$ with separable predual, and $\tilde{\phi}$ is determined by the values

$$\tilde{\phi}(\rho(E_{(x+a)\times(\alpha\cap R_m,\Gamma)})) = N(a)^{-1} \quad \text{for all } x \in R \text{ and } a \in \mathcal{I}_m^+.$$  

(17)

**Proof.** If $\phi$ is a $\sigma$-KMS state on $C^*_\lambda(R \rtimes R_m,\Gamma) / I_{\mathcal{P}_R^m}$, then the composition $\phi \circ \rho$ is a $\sigma$-KMS state on $C^*_\lambda(R \rtimes R_m,\Gamma)$ that vanishes on $\ker \rho = I_{\mathcal{P}_R^m}$. And if $\phi$ is a $\sigma$-KMS state on $C^*_\lambda(R \rtimes R_m,\Gamma)$, then Lemma 5.1 implies that $\phi$ vanishes on $I_{\mathcal{P}_R^m}$ if and only if $\phi$ vanishes at the projections $1 - \sum_{x \in R/t_p R} \lambda(x, t_p) \lambda^*(x, t_p)$ for all $p \in \mathcal{P}_R^m$. Using the KMS condition (1), we have

$$\phi \left( 1 - \sum_{x \in R/t_p R} \lambda(x, t_p) \lambda^*(x, t_p) \right) = 1 - \sum_{x \in R/t_p R} N(t_p)^{-\beta} = 1 - N(t_p)^{1-\beta}.$$
Hence, $\phi$ vanishes on $I_{R_m^R}$ if and only if $\beta = 1$. Therefore, by Theorem 3.2, there are no $\tilde{\sigma}$-KMS$_\beta$ states on $C^*_\sigma(R \rtimes R_{m, R})/I_{R_m^R}$ for $\beta \neq 1$, and there is a unique $\sigma$-KMS$_1$ state $\tilde{\phi}$ of $C^*_\sigma(R \rtimes R_{m, R})/I_{R_m^R}$; moreover, $\tilde{\phi} \circ \rho = \phi_1$ where $\phi_1$ is the the unique $\sigma$-KMS$_1$ state on $C^*_\sigma(R \rtimes R_{m, R})$, so $\tilde{\phi}$ satisfies (17) by Theorem 3.2(ii). Since $\pi_{\tilde{\phi}} \circ \rho = \pi_{\phi_1}$ where $\pi_{\phi_1}$ is the GNS representation of $\phi_1$, it follows from Theorem 3.2(ii) that $\pi_{\tilde{\phi}}(C^*_\sigma(R \rtimes R_{m, R})/I_{R_m^R})''$ is isomorphic to the injective factor of type III$_1$ with separable predual. \hfill $\Box$

**Remark 5.3.** For the case of trivial $m$ and $\Gamma$, the uniqueness claim in Theorem 5.2 follows from [C-D-L] Theorem 6.7.

### 6. Phase transitions on $C^*$-algebras of multiplicative monoids

For each $a \in R_{m, \Gamma}$, let $\lambda_a$ denote the isometry on $\ell^2(R_{m, \Gamma})$ determined by $\lambda_a(\epsilon_\lambda) = \epsilon_\lambda a$ where $\{\epsilon_\lambda : \lambda \in R_{m, \Gamma}\}$ is the canonical orthonormal basis for $\ell^2(R_{m, \Gamma})$. Then the left regular $C^*$-algebra of the (commutative) semigroup $R_{m, \Gamma}$ is the sub-$C^*$-algebra of $B(\ell^2(R_{m, \Gamma}))$ generated by these isometries, that is,

$$C^*_\lambda(R_{m, \Gamma}) := C^*(\{\lambda_a : a \in R_{m, \Gamma}\}).$$

The $C^*$-algebra $C^*_\lambda(R_{m, \Gamma})$ also carries a canonical time evolution $\sigma^\times$ that is determined on the generating isometries by $\sigma^\times_\lambda(\lambda_a) = N(a)^d \lambda_a$ for $a \in R_{m, \Gamma}$.

**Remark 6.1.** Using [Bru] Proposition 3.9 and Li’s theory of semigroup $C^*$-algebras from [Li1, Li2], one can show that there is an injective *-homomorphism

$$C^*_\lambda(R_{m, \Gamma}) \to C^*_\lambda(R \rtimes R_{m, \Gamma})$$

such that $\lambda_a \mapsto \lambda_{(0, a)}$ for all $a \in R_{m, \Gamma}$. Hence, under this embedding, the time evolution $\sigma^\times$ coincides with the restriction of the time evolution $\sigma$ to (the image of) $C^*_\lambda(R_{m, \Gamma})$.

The (commutative) semigroup $R_{m, \Gamma}/R^*_{m, \Gamma}$ can be identified with the semigroup of principal ideals that are generated by an element from $R_{m, \Gamma}$. For each $a \in R_{m, \Gamma}$, let $\lambda_a R^*_{m, \Gamma}$ denote the corresponding isometry in the left regular $C^*$-algebra $C^*_\lambda(R_{m, \Gamma}/R^*_{m, \Gamma})$; this $C^*$-algebra also carries a canonical time evolution, which we also denote by $\sigma^\times$. It is determined by $\sigma^\times_{\lambda_a R^*_{m, \Gamma}} = N(a)^d \lambda_a R^*_{m, \Gamma}$ for $a R^*_{m, \Gamma} \in R_{m, \Gamma}/R^*_{m, \Gamma}$.

In this section, we briefly explain how the techniques used to prove Theorem 3.2 also lead to phase transition theorems for the $C^*$-dynamical systems

$$(C^*_\lambda(R_{m, \Gamma}), \mathbb{R}, \sigma^\times) \quad \text{and} \quad (C^*_\lambda(R_{m, \Gamma}/R^*_{m, \Gamma}), \mathbb{R}, \sigma^\times).$$

Namely, we have the following two theorems, the first one for the left regular $C^*$-algebra of a congruence monoid itself, and the second one for left regular $C^*$-algebra of a semigroup of principal ideals that are generated by elements from a congruence monoid.

**Theorem 6.2.** Let $K$ be a number field, $\mathfrak{m}$ a modulus for $K$, and $\Gamma$ a subgroup of $(R/\mathfrak{m})^*$.

(i) There are no $\sigma^\times$-KMS$_\beta$ states on $C^*_\lambda(R_{m, \Gamma})$ for $\beta < 0$.

(ii) The simplex of $\sigma^\times$-KMS$_0$ states on $C^*_\lambda(R_{m, \Gamma})$ is isomorphic to the simplex of $\sigma^\times$-invariant states on the commutative group $C^*$-algebra $C^*(K_{m, \Gamma})$. 

(iii) For each $\beta \in (0, 1]$, the simplex of $\sigma^\times$-KMS$_\beta$ states on $C^*_\lambda(R_{m,\Gamma})$ is isomorphic to the simplex of states on the commutative group $C^*$-algebra $C^*(R^*_m)$. Moreover, if $\psi_\beta$ is the KMS$_\beta$ state corresponding to the canonical tracial state on $C^*(R^*_m)$ and $\pi_{\psi_\beta}$ is the GNS representation of $\psi_\beta$, then $\pi_{\psi_\beta}(C^*_\lambda(R_{m,\Gamma}))''$ is isomorphic to the injective factor of type $III_1$ with separable predual.

(iv) For each $\beta > 1$, the simplex of $\sigma^\times$-KMS$_\beta$ states on $C^*_\lambda(R_{m,\Gamma})$ is isomorphic to the simplex of states on the commutative $C^*$-algebra

$$\bigoplus_{t \in \mathcal{I}_m/i(K_{m,\Gamma})} C^*(R^*_m).$$

(v) The set of $\sigma^\times$-ground states on $C^*_\lambda(R_{m,\Gamma})$ is isomorphic to the state space of the $C^*$-algebra

$$\bigoplus_{t \in \mathcal{I}_m/i(K_{m,\Gamma})} M_{k_t}(C^*(R^*_m))$$

where $k_t$ is the number of norm-minimizing ideals in the class $\xi$.

**Theorem 6.3.** Let $K$ be a number field, $m$ a modulus for $K$, and $\Gamma$ a subgroup of $(R/m)^*$. 

(i) There are no $\sigma^\times$-KMS$_\beta$ states on $C^*_\lambda(R_{m,\Gamma}/R^*_m)$ for $\beta < 0$.

(ii) The simplex of $\sigma^\times$-KMS$_0$ states on $C^*_\lambda(R_{m,\Gamma})$ is isomorphic to the simplex of $\sigma^\times$-invariant states on the commutative group $C^*$-algebra $C^*(K_{m,\Gamma}/R^*_m)$.

(iii) For each $\beta \in (0, 1]$, there is a unique $\sigma^\times$-KMS$_\beta$ state $\omega_\beta$ on $C^*_\lambda(R_{m,\Gamma}/R^*_m)$. Moreover, if $\phi_{\omega_\beta}$ is the GNS representation of $\omega_\beta$, then $\pi_{\phi_{\omega_\beta}}(C^*_\lambda(R_{m,\Gamma}/R^*_m))''$ is isomorphic to the injective factor of type $III_1$ with separable predual.

(iv) For each $\beta > 1$, the simplex of $\sigma^\times$-KMS$_\beta$ states on $C^*_\lambda(R_{m,\Gamma}/R^*_m)$ is isomorphic to the simplex of states on the finite-dimensional commutative $C^*$-algebra $C^{\beta_{m,\Gamma}}$ where $h_{m,\Gamma} := |\mathcal{I}_m/i(K_{m,\Gamma})|$.

(v) The set of $\sigma^\times$-ground states on $C^*_\lambda(R_{m,\Gamma})$ is isomorphic to the state space of the $C^*$-algebra

$$\bigoplus_{t \in \mathcal{I}_m/i(K_{m,\Gamma})} M_{k_t}$$

where $k_t$ is the number of norm-minimizing ideals in the class $\xi$.

**Remark 6.4.** (a) For the special case of trivial $m$ and $\Gamma$, the parameterization results in Theorem 6.2(ii)-(iv) were already asserted in [C-D-L Remark 7.5].

(b) An alternative approach to computing the $\sigma^\times$-KMS$_\beta$ states on $C^*_\lambda(R^\times)$ and $C^*_\lambda(R^\times/R^\times)$ for $\beta > 1$ is given in [C-E-L-Y Remark 6.6.5]. Presumably, the approach taken there could also be used to compute the low temperature KMS states on $C^*_\lambda(R_{m,\Gamma})$ and $C^*_\lambda(R_{m,\Gamma}/R^*_m)$.

(c) Using the canonical isomorphisms $C^*(K_{m,\Gamma}) \cong C(K_{m,\Gamma})$ and $C^*(R^*_m) \cong C(R^*_m)$ given by the Fourier transform, the parameterizations in Theorem 6.2(ii)-(iv) can be phrased in terms of characters of the discrete abelian groups $K_{m,\Gamma}$ and $R^*_m$.

Specifically,

- the extremal KMS$_0$ states on $C^*_\lambda(R_{m,\Gamma})$ are parameterized by the characters of the discrete abelian group $K_{m,\Gamma}$;

- the KMS$_\beta$ states on $C^*_\lambda(R_{m,\Gamma})$ are parameterized by the characters of the discrete abelian group $K_{m,\Gamma}$ with $\beta > 0$;
• for each $\beta \in (0,1]$, the extremal $\sigma^\times$-KMS states on $C^*_\lambda(R_{m,1})$ are parameterized by the characters of the discrete abelian group $\mathbb{R}^*_m$;
• for each $\beta > 1$, the extremal $\sigma^\times$-KMS states on $C^*_\lambda(R_{m,1})$ are parameterized by pairs $(\mathbf{t}, \chi)$ where $\mathbf{t}$ is a class in $\mathcal{I}(m/i)(R_{m,1})$ and $\chi$ is character of $R^*_m$.
(d) An analogous statement involving characters of the discrete abelian group $K_{m,1}/\mathbb{R}^*_m$ holds for the parameterization given by Theorem 6.3(ii).

The arguments needed to prove these theorems are almost identical, so we will only give a proof of Theorem 6.2

Proof of Theorem 6.2. The strategy is similar to that used to prove Theorem 3.2, so we will only give a sketch of the arguments. There is a canonical action of the group $K_{m,1}$ on $\hat{\mathbb{A}}_S/\hat{R}^*_S$, and the C*-algebra of the reduction groupoid

$$K_{m,1} \times \hat{R}^*_S \equiv \{(k, \mathbf{a}) \in K_{m,1} \times \hat{R}^*_S : k\mathbf{a} \in \hat{R}^*_S\} \subseteq K_{m,1} \ltimes \hat{\mathbb{A}}_S/\hat{R}^*_S$$

carries a canonical time evolution, which we also denote by $\sigma^\times$, determined by the real-valued 1-cocycle $\sigma^\times : K_{m,1} \times \hat{R}^*_S \rightarrow \mathbb{R}^*_S$ given by $(k, \mathbf{a}) \mapsto \log N(k)$. Arguments analogous to those given in [Bru, Section 5] show that the C*-algebra $C^*_\lambda(R_{m,1})$ can be canonically and $\mathbb{R}$-equivariantly identified with the groupoid C*-algebra $C^*(K_{m,1} \times \hat{R}^*_S)$. Hence, it suffices to compute all KMS and ground states of the C*-dynamical system $(C^*(K_{m,1} \times \hat{R}^*_S), \mathbb{R}, \sigma^\times)$.

A short calculation similar to that from the proof of Theorem 3.2(i) shows that assertion (i) holds.

For $\beta \in [0,1]$, Theorem 3.5 asserts that the measure $\nu_\beta$ defined in Section 3.4 is the unique probability measure on $\hat{R}^*_S$ that satisfies

$$\nu(kZ) = N(k)^{-\beta} \nu(Z)$$

for all $k \in K_{m,1}$ and Borel sets $Z \subseteq \hat{R}^*_S$ such that $kZ \subseteq \hat{R}^*_S$. For $\beta = 0$, we have $\nu_\beta = \delta_0$, and the isotropy group of the point $0$ is all of $K_{m,1}$. Since a state $\tau$ of $C^*(K_{m,1})$ is $\sigma^\times$-invariant if and only if $\tau(uk) = 0$ for all $k \in K_{m,1}$ with $N(k) \neq 1$, assertion (ii) follows from [Nesh3, Theorem 1.3] and [Nesh3, Corollary 1.4].

Now suppose $\beta \in (0,1]$. Then the measure $\nu_\beta$ is concentrated in the set

$$\{\mathbf{a} \in \hat{R}^*_S : \mathbf{a}_p \neq 0 \text{ for all } p \text{ and } \nu_p(\mathbf{a}) > 0 \text{ for infinitely many } p\}.$$ 

Since the isotropy group of any point in this set is $R^*_m$, the parameterization result asserted in (iii) follows from [Nesh3, Theorem 1.3].

We now turn to the claim about type. We have

$$\pi_{\psi}(C^*_\lambda(R_{m,1}))'' \equiv 1_{\hat{R}^*_S}(L^\infty(\hat{\mathbb{A}}_S/\hat{R}^*_S, \tilde{\nu}_\beta) \rtimes K_{m,1}) 1_{\hat{R}^*_S}$$

where $\tilde{\nu}_\beta$ is the measure on $\hat{\mathbb{A}}_S/\hat{R}^*_S$ from the statement of Proposition 4.6. Hence, the claim about injectivity and separability follows. To show that the flow of weights on $L^\infty(\hat{\mathbb{A}}_S/\hat{R}^*_S, \tilde{\nu}_\beta) \rtimes K_{m,1}$ is trivial, it suffices to show that the action of $K_{m,1}$ on

$$(\mathbb{R}^*_+ \times \hat{\mathbb{A}}_S/\hat{R}^*_S, \lambda_\infty \times \tilde{\nu}_\beta)$$
given by
\[ k(t, \tilde{a}) = (N(k)^\beta t, k \tilde{a}) \quad \text{for} \quad k \in K_{m, \Gamma}, \quad (t, \tilde{a}) \in \mathbb{R}_+^* \times \hat{\mathbb{A}}_S/\hat{R}_S^* \]
is ergodic. This is precisely the statement of Proposition 4.6.

For \( \beta \in (1, \infty) \), Lemma 3.13 says that the extremal probability measures that satisfy (8) are precisely the measures \( \{ \nu_{\beta, t} : t \in \mathcal{I}_m/i(K_{m, \Gamma}) \} \). These measures are concentrated in the set
\[ \{ \tilde{a} : \tilde{a}_p \neq 0 \text{ for all } p \text{ and } \nu_p(\tilde{a}) = 0 \text{ for all but finitely many } p \}. \]

Since the isotropy group of any point in this set is \( R_{m, \Gamma}^* \), Theorem 6.2.iv also follows from [Nesh3, Theorem 1.3].

Following the proof of Theorem 3.2(iv), we see that the boundary set of the cocycle \( c^\times \) (cf. [L-L-N3, Section 1]) is equal to
\[ (\hat{R}_S/\hat{R}_S^*)_0 := \{ \tilde{a} \in \hat{R}_S/\hat{R}_S^* : \tilde{a} = a_{t,j} \text{ for some } 1 \leq j \leq k_t \} \]
where \( a_{t,1}, \ldots, a_{t,k_t} \) are the norm-minimizing ideals in the class \( t \) (see the discussion preceding Theorem 3.2). Let \( K_{m, \Gamma, 1} := \{ x \in K_{m, \Gamma} : N(x) = 1 \} \). Then [L-L-N2, Theorem 1.9] asserts that the map \( \psi \mapsto \phi_\psi \) defined by
\[ \phi_\psi(f) = \psi(f|_{K_{m, \Gamma, 1} \ltimes (\hat{R}_S/\hat{R}_S^*)_0}) \quad \text{for} \quad f \in C_c(K_{m, \Gamma} \ltimes \hat{R}_S/\hat{R}_S^*) \]
is an affine isomorphism of the state space of \( C^*(K_{m, \Gamma, 1} \ltimes (\hat{R}_S/\hat{R}_S^*)_0) \) onto the \( \sigma \)-ground state space of \( C^*(K_{m, \Gamma} \ltimes \hat{R}_S/\hat{R}_S^*) \) where \( K_{m, \Gamma, 1} \ltimes (\hat{R}_S/\hat{R}_S^*)_0 \) is the reduction groupoid of \( K_{m, \Gamma, 1} \ltimes \hat{\mathbb{A}}_S/\hat{R}_S^* \) with respect to the subset \( (\hat{R}_S/\hat{R}_S^*)_0 \subseteq \hat{\mathbb{A}}_S/\hat{R}_S^* \). Now arguments similar to those used to prove Theorem 3.2(iv) show that
\[ C^*(K_{m, \Gamma, 1} \ltimes (\hat{R}_S/\hat{R}_S^*)_0) \cong \bigoplus_{t \in \mathcal{I}_m/i(K_{m, \Gamma})} M_{k_t}(C^*(R_{m, \Gamma}^*)), \]
which finishes the proof of Theorem 6.2.v. \( \square \)

References

[Bla] B. E. Blackadar, The regular representation of restricted direct product groups, J. Functional Analysis 25 (1977), no. 3, 267–274.

[Bo-Za] F. P. Boca and A. Zaharescu, Factors of type III and the distribution of prime numbers, Proc. London Math. Soc. (3) 80 (2000), no. 1, 145–178.

[Bo-Co] J.-B. Bost and A. Connes, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory, Selecta Math. (N.S.) 1 (1995), no. 3, 411–457.

[B-R] O. Bratteli and D. W. Robinson, Operator Algebras and Quantum Statistical Mechanics 2, second ed., Springer-Verlag, Berlin, 1997.

[Bru] C. Bruce, \( C^* \)-algebras from actions of congruence monoids on rings of algebraic integers, \texttt{arXiv:1901.04075v1}.

[Con] A. Connes, Classification of injective factors. Cases \( \mathcal{II}_1 \), \( \mathcal{II}_\infty \), \( \mathcal{III}_\lambda \), \( \lambda \neq 1 \), Ann. of Math. (2) 104 (1976), no. 1, 73–115.

[Con-Mar] A. Connes and M. Marcolli, Noncommutative Geometry, Quantum Fields and Motives, Amer. Math. Soc. Colloq. Publ., vol. 55, American Mathematical Society, 2008.

[Con-Tak] A. Connes and M. Takesaki, The flow of weights on factors of type III, Tôhoku Math. J. (2) 29 (1977), 473–575; Errata, ibid. 30 (1978), 653–655.

[Cun] J. Cuntz, \( C^* \)-algebras associated with the \( ax+b \)-semigroup over \( N \). K-theory and noncommutative geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich (2008), 201–215.
[C-D-L] J. Cuntz, C. Deninger, and M. Laca, *C*-algebras of Toeplitz type associated with algebraic number fields*, Math. Ann. 355 (2013), no. 4, 1383–1423.

[C-E-L-Y] J. Cuntz, S. Echterhoff, X. Li, and G. Yu, *K-Theory for Group C*-Algebras and Semigroup C*-Algebras*, Oberwolfach Seminars, 47, Birkhäuser/Springer, Cham, 2017.

[Fel-Mo] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras*, I, Trans. Amer. Math. Soc. 234 (1977), no. 2, 289–324.

[G-HK] A. Geroldinger and F. Halter-Koch, *Congruence monoids*, Acta Arith. 112 (2004), no. 3, 263–296.

[Haa] U. Haagerup, *Connes' bicentralizer problem and uniqueness of the injective factor of type III*, Acta Math. 158 (1987), no. 1-2, 95–148.

[Har-Lei] D. Harari and E. Leichtnam, *Extension du phénomène de brisure spontanée de symétrie de Bost-Connes au cas des corps globaux quelconques*, Selecta Math. (N.S.) 3 (1997), 205–243.

[aHLRS] A. an Huef, M. Laca, I. Raeburn, and A. Sims, *KMS states on the C*-algebras of finite graphs*, J. Math. Anal. Appl. 405 (2013), no. 2, 388–399.

[La] M. Laca, *From endomorphisms to automorphisms and back: dilations and full corners*, J. London Math. Soc. 61 (2000), 893–904.

[L-L-N1] M. Laca, N. S. Larsen, and S. Neshveyev, *Phase transition in the Connes-Marcolli GL₂-system*, J. Noncommut. Geom. 1 (2007), no. 4, 397–430.

[L-L-N2] M. Laca, N. S. Larsen, and S. Neshveyev, *On Bost-Connes types systems for number fields*, J. Number Theory 129 (2009), no. 2, 325–338.

[L-L-N3] M. Laca, N. S. Larsen, and S. Neshveyev, *Ground states of groupoid C*-algebras, phase transitions and arithmetic subalgebras for Hecke algebras*, J. Geom. Phys. 136 (2019) 268–283.

[La-Nesh1] M. Laca, and S. Neshveyev, *KMS states of quasi-free dynamics on Pimsner algebras*, J. Funct. Anal. 211 (2004), no. 2, 457–482.

[La-Nesh2] M. Laca, and S. Neshveyev, *Type III₁ equilibrium states of the Toeplitz algebra of the affine semigroup over the natural numbers*, J. Funct. Anal. 261 (2011), no. 1, 169–187.

[L-N-T] M. Laca, S. Neshveyev, and M. Trifković, *Bost-Connes systems, Hecke algebras, and induction*, J. Noncommut. Geom. 7 (2013), no. 2, 525–546.

[La-Rae1] M. Laca and I. Raeburn, *Semigroup crossed products and the Toeplitz algebras of nonabelian groups*, J. Funct. Anal. 139 (1996), no. 2, 415–440.

[La-Rae2] M. Laca, and I. Raeburn, *Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers*, Adv. Math. 225 (2010), no. 2, 643–688.

[La-vF] M. Laca and M. van Frankenhuijsen, *Phase transitions with spontaneous symmetry breaking on Hecke C*-algebras from number fields*, In: Consani C., Marcolli M. (eds) Noncommutative Geometry and Number Theory. Aspects of Mathematics. Vieweg (2006).

[La-War] M. Laca and J. M. Warren, *Phase transitions on C*-algebras arising from number fields and the generalized Furstenberg conjecture*, to appear in J. Operator Theory.

[Lag-Nesh] J. C. Lagarias and S. Neshveyev, *Ergodicity of the action of K* on ℵK*, Int. Math. Res. Not. IMRN 2014, no. 18, 5165–5186.

[Li1] X. Li, *Semigroup C*-algebras and amenability of semigroups*, J. Funct. Anal. 262 (2012) 4302–4340.

[Li2] X. Li, *Nuclearity of semigroup C*-algebras and the connection to amenability*, Adv. Math. 244 (2013) 626–662.

[Li3] X. Li, *Partial transformation groupoids attached to graphs and semigroups*, Int. Math. Res. Not. IMRN 2017, no. 17, 5233–5259.

[Mil] J. S. Milne, *Class Field Theory (v4.02)*, [ww.jmilne.org/math/](http://ww.jmilne.org/math/) (2013).

[M-R-W] P. S. Muhly, J. N. Renault, and D. P. Williams, *Équivalence and isomorphism for groupoid C*-algebras*, J. Operator Theory 17 (1987), no. 1, 3–22.

[Nar] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, Third edition, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004.

[Nesh1] S. Neshveyev, *Ergodicity of the action of the positive rationals on the group of finite adeles and the Bost-Connes phase transition theorem*, Proc. Amer. Math. Soc. 130 (2002), no. 10, 2999–3003.

[Nesh2] S. Neshveyev, *Von Neumann algebras arising from Bost-Connes type systems*, Int. Math. Res. Not. IMRN 2011, no. 1, 217–236.

[Nesh3] S. Neshveyev, *KMS states on the C*-algebras of non-principal groupoids*, J. Operator Theory 70 (2013), no. 2, 513–530.
[Ren1] J. Renault, *A groupoid approach to C*-algebras*, Lecture Notes in Math. **793**, Springer–Verlag, Berlin, Heidelberg, New York, 1980.

[Ren2] J. Renault, *C*-algebras and Dynamical Systems, Publicacoes matematicas, Rio de Janeiro: IMPA, 2009.

[Tak] M. Takesaki, *Theory of operator algebras. III*. Encyclopaedia of Mathematical Sciences, **127**. Operator Algebras and Non–commutative Geometry, **8**. Springer–Verlag, Berlin, 2003.

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