Complexity Classification of the Eight-Vertex Model

Jin-Yi Cai∗ Zhiguo Fu†

Abstract

We prove a complexity dichotomy theorem for the eight-vertex model. For every setting of the parameters of the model, we prove that computing the partition function is either solvable in polynomial time or #P-hard. The dichotomy criterion is explicit. For tractability, we find some new classes of problems computable in polynomial time. For #P-hardness, we employ Möbius transformations to prove the success of interpolations.

∗Computer Sciences Department, University of Wisconsin, Madison, USA. jyc@cs.wisc.edu
†School of Mathematics, Jilin University, Changchun, China. fuzg@jlu.edu.cn
1 Introduction

There are two complementary motivations for this work, one from physics, and one from the classification program for counting problems in complexity theory. From physics, there is a long history in the study of various elegant models which define partition functions that capture physical properties. The eight-vertex model is one such model, and it generalizes the six-vertex model. From complexity theory, we have made substantial progress in classifying counting problems expressed as \textit{sum-of-product} computations in all three frameworks: graph homomorphisms (GH), counting constraint satisfaction problems (\#CSP), and Holant problems. However, the advances for GH and \#CSP have been far more conclusive than for Holant problems: On the Boolean domain (where variables take 0-1 values), the known complexity dichotomy for \#CSP applies to all complex-valued constraint functions which need not be symmetric \cite{17}, but currently the strongest Holant dichotomy without auxiliary functions can only handle symmetric constraints \cite{14}. To classify Holant problems without the symmetry assumption, currently we have to assume the presence of auxiliary functions. E.g., assuming all unary functions are present, called Holant* problems, we have a dichotomy that applies to symmetric as well as asymmetric constraint functions \cite{16}. Beckens \cite{2} recently proved an extension to a dichotomy for Holant+ problems, which assume the presence of four unary functions including the pinning functions \textsc{Is-Zero} and \textsc{Is-One} (which set a variable to 0 or 1). If one only assumes the presence of the two pinning functions, this is called the Holantc problems. The strongest known Holantc dichotomies are for symmetric complex-valued constraints \cite{15}, or for real-valued constraints without symmetry assumption \cite{18}. If one considers what tractable problems emerge on planar graphs, again we have a full dichotomy for \textsc{Pl-\#CSP} \cite{11}, but only for symmetric constraints concerning \textsc{Pl-Holant} problems \cite{12}. There are also several known dichotomies for GH and \#CSP on domain size greater than 2 \cite{19,6,21,8,4,3,20,5,9,7}, but very little is known for Holant problems.

Generally speaking, to handle constraint functions that are not necessarily symmetric seems to be very challenging for Holant problems. The eight-vertex model can be viewed as fundamental building blocks toward a full Holant dichotomy on the Boolean domain without the symmetry restrictions. Not only they are small arity cases in such a theorem, they also present a pathway to overcome some technical obstacles.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{eight_vertex_model.png}
\caption{Valid configurations of the eight-vertex model.}
\end{figure}

In physics, the eight-vertex model is a generalization of the six-vertex model, including models for water ice, potassium dihydrogen phosphate KH$_2$PO$_4$ (KDP model of a ferroelectric) and the Rys $F$ model of an antiferroelectric. One can mathematically describe the eight-vertex model as an orientation problem on 4-regular graphs: Given a 4-regular graph $G = (V,E)$, an orientation is an assignment of a direction to every $e \in E$. An orientation is valid for the eight-vertex model iff at every vertex the in-degree (and out-degree) is even. This generalizes the six-vertex model where the in-degree (and out-degree) is two and thus the orientation is \textit{Eulerian}. One can think of the valid configurations in a eight-vertex model as Eulerian orientations with possible sources.
and sinks. The valid local configurations are illustrated in Figure 1. The energy \( E \) of the system is determined by eight parameters \( \epsilon_1, \epsilon_2, \ldots, \epsilon_8 \) associated with each type of the local configuration, and \( w_j = \exp \left( -\frac{\epsilon_j}{k_B T} \right) \) is called the Boltzmann weight (\( k_B \) is Boltzmann’s constant, and \( T \) is the system’s temperature). If there are \( n \) sites in local configuration type \( i \), then \( E = n_1 \epsilon_1 + \ldots + n_8 \epsilon_8 \) is the total energy, and the partition function is \( Z_{\text{Eight}} = \sum e^{-E/k_B T} = \prod w_j^{n_j} \), where the sum is over all valid configurations. This is a sum-of-product computation. In our more general definition (see Subsection 2.3) the 8 possible weights \( w_j \) can be zero, and thus the six-vertex model is the special case with \( w_7 = w_8 = 0 \), disallowing the configurations 7 and 8.

Compared to the six-vertex model, there are more non-trivial tractable problems. Partly this is because the support of a constraint function in the eight-vertex model can be an affine subspace of dimension 3 (over \( \mathbb{Z}_2 \)). Some tractable problems are only revealed to be so after surprising holographic transformations \([1 \ 1] \) or \([1 \ 0 \ \sqrt{2i}] \). No previously known tractable classes required such transformations. More tractable problems usually mean that it is more challenging to prove a dichotomy. Such a theorem says that there are no other tractable problems beyond the ones already discovered (if \#P does not collapse to P.)

We discover a connection for a class of 8-vertex models with \#CSP\(^2\) problems, which are a variant of \#CSP where every variable appears an even number of times. Compared to \#CSP, there are more tractable problems for \#CSP\(^2\). A crucial ingredient in our proof is a recent \#CSP\(^2\) dichotomy [18] that is valid for asymmetric signatures. Our new tractable families for the 8-vertex model also give new tractable families for the so-called 2,4-spin Ising model on the lattice graph, where the \((+/−)\) spins are on square faces, and local interactions are among horizontal, vertical, two diagonals, and all 4 neighbors.

A new contribution of this work is to use Möbius transformations \( z \mapsto \frac{az+b}{\bar{a}z+d} \) to prove \#P-hardness. Typically to prove some problem \#P-hard by interpolation, we want to prove that certain quantities (such as eigenvalues) are not roots of unity, lest the iteration repeat after a bounded number of steps. We usually establish this property by showing that we can produce these quantities of norm \( \neq 1 \). However in this paper, there are settings where this is impossible. In this case we prove that the constraint functions define certain Möbius transformations that map the unit circle to unit circle on \( \mathbb{C} \). By exploiting the mapping properties we can obtain a suitable Möbius transformation which generates a group of infinite order. Hence even though they only produce quantities of complex norm 1, they nevertheless can be guaranteed not to repeat. This allows us to show that our interpolation proof succeeds.

2 Preliminaries

2.1 Definitions and Notations

In the present paper, \( i \) denotes a square root of \(-1\), i.e., \( i^2 = −1 \). \( \alpha \) denotes a square root of \( i \), i.e., \( \alpha^2 = i \). Let \( \mathbb{N} = \{0, 1, 2, \ldots\} \), \( H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \), \( Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \).

A constraint function \( f \) of arity \( k \) is a map \( \{0, 1\}^k \to \mathbb{C} \). Fix a set \( \mathcal{F} \) of constraint functions. A signature grid \( \Omega = (G, \pi) \) is a tuple, where \( G = (V, E) \) is a graph, \( \pi \) labels each \( v \in V \) with a function \( f_v \in \mathcal{F} \) of arity \( \deg(v) \), and the incident edges \( E(v) \) at \( v \) with input variables of \( f_v \). We consider all 0-1 edge assignments \( \sigma \), each gives an evaluation \( \prod_{v \in V} f_v(\sigma |_{E(v)}) \), where \( \sigma |_{E(v)} \) denotes
the restriction of $\sigma$ to $E(v)$. The counting problem on the instance $\Omega$ is to compute

$$\text{Holant}_\Omega(\mathcal{F}) = \sum_{\sigma : E \rightarrow \{0, 1\} \forall v} \prod_{v \in V} f_v(\sigma|_{E(v)}).$$

The Holant problem parameterized by the set $\mathcal{F}$ is denoted by Holant($\mathcal{F}$). We also write Holant($\mathcal{F}, f$) for Holant($\mathcal{F} \cup \{ f \}$). A constraint function is also called a signature. We use Holant($\mathcal{F}|\mathcal{G}$) to denote the Holant problem over signature grids with a bipartite graph $H = (U, V, E)$, where each vertex in $U$ or $V$ is assigned a signature in $\mathcal{F}$ or $\mathcal{G}$ respectively. $\#\text{CSP}(\mathcal{F})$ can be defined as Holant($\mathcal{E}_\mathcal{Q}|\mathcal{F}$) where $\mathcal{E}_\mathcal{Q} = \{ \pm_1, -2, \ldots \}$ is the set of Equality signatures. Similarly, $\#\text{CSP}^2(\mathcal{F})$ can be defined as Holant($\mathcal{E}_\mathcal{Q}_2|\mathcal{F}$) where $\mathcal{E}_\mathcal{Q}_2 = \{ \pm_2, \pm_4, \ldots \}$ is the set of Equality signatures of even arities, i.e., every variable appears an even number of times.

A function $f$ of arity $k$ can be represented as a vector by listing its values in lexicographical order as in a truth table. Also a signature $f$ of arity $4$ has the signature matrix $M(f) = \begin{bmatrix} f_{0000} & f_{0001} & f_{0010} & f_{0011} \\ f_{0100} & f_{0101} & f_{0110} & f_{0111} \\ f_{1000} & f_{1001} & f_{1010} & f_{1011} \\ f_{1100} & f_{1101} & f_{1110} & f_{1111} \end{bmatrix}$. If $\{i, j, k, \ell\}$ is a permutation of $\{1, 2, 3, 4\}$, then the $4 \times 4$ matrix $M_{x_1 x_2, x_3 x_4}(f)$ lists the $16$ values with row index $x_i x_j \in \{0, 1\}^2$ and column index $x_k x_\ell \in \{0, 1\}^2$ in lexicographic order. A binary signature $g$ has the signature matrix $M(g) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. We use $\not=_{2}$ to denote binary Disequality function $(0, 1, 1, 0)^T$ indexed by $x_1 x_2 \in \{0, 1\}^2$ and its matrix form is $[0 \ 1]$. Note that $N = [0 \ 1] \otimes [0 \ 1] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, i.e., $N$ is the double Disequality in parallel, which is the function of connecting two pairs of edges by $(\not=_{2})$. The support of a function $f$ is the set of inputs on which $f$ is nonzero.

The eight-vertex model is the Holant problem Holant($\not=_{2} | f$) where $f$ is a 4-ary signature with the signature matrix $M(f) = \begin{bmatrix} a & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & w & z & 0 \\ y & 0 & 0 & x \end{bmatrix}$. We call $(a, x)$ the outer pair of $f$, $(b, y), (c, z), (d, w)$ the inner pairs of $f$, $[a \ b \ y \ x]$ the outer matrix of $f$ and $[c \ d \ z \ w]$ the inner matrix of $f$. Denote the $3$ inner pairs of ordered complementary strings by $\lambda = 0011, \overline{\lambda} = 1100, \mu = 0110, \overline{\mu} = 1001,$ and $\nu = 0101, \overline{\nu} = 1010$. The permutation group $S_4$ on $\{x_1, x_2, x_3, x_4\}$ induces a group action on $\{s \in \{0, 1\}^4 \mid \text{wt}(s) = 2\}$ of size $6$. This is a faithful representation of $S_4$ in $S_6$. Since the action of $S_4$ preserves complementary pairs, this group action has nontrivial blocks of imprimitivity, namely $\{A, B, C\} = \{\{\lambda, \overline{\lambda}\}, \{\mu, \overline{\mu}\}, \{\nu, \overline{\nu}\}\}$. The action on the blocks is a homomorphism of $S_4$ onto $S_3$, i.e., we can permute the blocks arbitrarily by permuting the variables $\{x_1, x_2, x_3, x_4\}$, with kernel $K = \{1, (12)(34), (13)(24), (14)(23)\}$. In particular one can calculate that the subgroup $S_{(2,3,4)} = \{1, (23), (34), (24), (243), (234)\}$ maps to $\{1, (AC), (BC), (AB), (ABC), (ACB)\}$. By a permutation from $S_4$, we may permute the matrix $M(f)$ by any permutation on the values $(b, c, d)$ with the corresponding permutation on $\{y, z, w\}$, and moreover we can further flip an even number of pairs $(b, y), (c, z)$ and $(d, w)$. In particular, we can arbitrarily reorder the three rows in $\begin{bmatrix} b & y & z \ c & x & w \ d & y & w \ a & z & w \ \end{bmatrix}$, and we can also reverse the order of arbitrary two rows together. In the following, when we say by the symmetry of $\{(b, y), (c, z), (d, w)\}$, it means this group action.

For Holant($\not=_{2} | f$), we say that $f$ is $\mathcal{C}$-transformable if there exists $T \in \text{GL}_2(\mathbb{C})$ such that we have both $(\not=_{2}) T^{\otimes 2} \in \mathcal{C}$ and $(T^{-1})^{\otimes 4} f \in \mathcal{C}$. Notice that if Holant($\mathcal{C}|\mathcal{C}$) is tractable, and $f$ is $\mathcal{C}$-transformable, then Holant($\not=_{2} | f$) is tractable by a holographic transformation.
2.2 Gadget Construction

An $\mathcal{F}$-gate is a signature grid with a set of dangling edges $D$ and vertices labeled by signatures from $\mathcal{F}$. It defines a function of arity $|D|$ in a naturally way: For any assignment $D \rightarrow \{0,1\}$, the output is the Holant sum. We say a signature $g$ is constructible or realizable from a signature set $\mathcal{F}$ if $g$ is the function defined by an $\mathcal{F}$-gate. If $g$ is realizable from $\mathcal{F}$, then $\text{Holant}(\mathcal{F},g) \leq_T^p \text{Holant}(\mathcal{F})$.

Doing binary modification to the variable $x_i$ of $f$ using the binary signature $g(x_1, x_2) = (g_{00}, g_{01}, g_{10}, g_{11}) = (0, 1, t, 0)^T$ means connecting the variable $x_i$ of $f$ to the variable $x_2$ of $g$ by $\neq 2$. For example, by doing binary modification to the variable $x_1$ of $f$ using the binary signature $(0, 1, t, 0)^T$ we get the signature $f'$ whose signature matrix is

$$M(f') = \begin{bmatrix}
 f_{0000} & f_{0001} & f_{0010} & f_{0011} \\
 f_{0100} & f_{0101} & f_{0110} & f_{0111} \\
 t_{f100} & t_{f101} & t_{f110} & t_{f111}
\end{bmatrix}.$$  

2.3 Tractable Signatures

We use $\mathcal{P}, \mathcal{A}, \alpha\mathcal{A}, \mathcal{L}$ to denote four fundamental classes of tractable signatures. A signature $f(x_1, \ldots, x_n)$ of arity $n$ is in $\mathcal{A}$ if it has the form $\lambda \chi_{AX=0}^Q(x)$, where $\lambda \in \mathbb{C}$, $X = (x_1, x_2, \ldots, x_n, 1)$, $A$ is a matrix over $\mathbb{Z}_2$, $Q(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_4[x_1, x_2, \ldots, x_n]$ is a quadratic (total degree at most 2) multilinear polynomial with the additional requirement that the coefficients of all cross terms are even, and $\chi$ is a 0-1 indicator function such that $\chi_{AX=0}$ is 1 iff $AX = 0$. Problems defined by $\mathcal{A}$ are tractable [10]. The signatures in $\mathcal{P}$ are tensor products of signatures whose supports are among two complementary bit vectors. An $n$-ary signature $f \in \alpha\mathcal{A}$ iff $[1_0^\alpha \otimes n] f \in \mathcal{A}$. The problems in $\alpha\mathcal{A}$ are tractable for $\#\text{CSP}^2$ by the holographic transformation. And Problems defined by $\mathcal{L}$ are tractable essentially by a local holographic transformation [18].

Lin and Wang proved the following lemma (Lemma 3.4 in [24]), which says that one can always reduce a signature to its tensor power. We will only need a special case; for the convenience of readers we state it below with a short proof.

Lemma 2.1 (Lin-Wang). For any set of signatures $\mathcal{F}$, and a signature $f$,

$$\text{Holant}(\mathcal{F}, f) \leq_T^p \text{Holant}(\mathcal{F}, f^{\otimes 2}).$$

Proof. We ask the question: Is there a signature grid $\Omega$ for $\text{Holant}(\mathcal{F}, f)$ in which $f$ appears an odd number of times, and the value $\text{Holant}_\Omega(\mathcal{F}, f)$ is nonzero? If the answer is no, then here is a simple reduction: For any input signature grid $\Omega$ for $\text{Holant}(\mathcal{F}, f)$, if $f$ appears an odd number of times, then $\text{Holant}_\Omega(\mathcal{F}, f) = 0$, otherwise, pair up occurrences of $f$ two at a time and replace them by one copy of $f^{\otimes 2}$.

Now suppose the answer is yes, and let $c = \text{Holant}_{\Omega_0}(\mathcal{F}, f) \neq 0$, where $\Omega_0$ is a signature grid in which $f$ appears $2k + 1$ times. Replace $2k$ occurrences of $f$ in $\Omega_0$ by $k$ copies of $f^{\otimes 2}$. Now use one more copy of $f^{\otimes 2}$. Suppose $f^{\otimes 2}(x_1, \ldots, x_s, y_1, \ldots, y_s) = f(x_1, \ldots, x_s)f(y_1, \ldots, y_s)$, where $s$ is the arity of $f$. Replace the $(2k+1)$-th occurrence of $f$ in $\Omega_0$ by $f^{\otimes 2}$, using variables $y_1, \ldots, y_s$ of $f^{\otimes 2}$ to connect to the $s$ edges of the $(2k+1)$-th occurrence of $f$, and leaving $x_1, \ldots, x_s$ as dangling edges. This creates a $(\mathcal{F} \cup \{f^{\otimes 2}\})$-gate with signature $cf$. Hence

$$\text{Holant}(\mathcal{F}, f) \leq_T^p \text{Holant}(\mathcal{F}, f^{\otimes 2}).$$

\qed
Using Lemma 2.1, we can state the following dichotomy theorem from [18] for \#CSP^2.

**Theorem 2.2.** Let \( \mathcal{F} \) be any set of complex-valued signatures in Boolean variables. Then \( \# \text{CSP}^2(\mathcal{F}) \) is \#P-hard unless \( \mathcal{F} \subseteq \mathcal{A} \) or \( \mathcal{F} \subseteq \mathcal{A}^* \) or \( \mathcal{F} \subseteq \mathcal{P} \) or \( \mathcal{F} \subseteq \mathcal{L} \) in which cases the problem is computable in polynomial time.

The six-vertex model is the special case of the eight-vertex model with \( a = x = 0 \) in \( M(f) \).

**Theorem 2.3.** [13] Let \( f \) be a 4-ary signature with the signature matrix \( M(f) = \begin{bmatrix} 0 & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & 0 & z & 0 \\ y & 0 & 0 & 0 \end{bmatrix} \), the six-vertex model \( \text{Holant}(\neq 2 \mid f) \) is \#P-hard except for the following cases: \( f \in \mathcal{P} \), or \( f \in \mathcal{A} \), or there is a zero in each pair \((b, y), (c, z), (d, w)\), in which cases \( \text{Holant}(\neq 2 \mid f) \) is computable in polynomial time.

**Definition 2.4.** A 4-ary signature \( f \) is redundant iff in its 4 by 4 signature matrix the middle two rows are identical and the middle two columns are identical. We call \( \begin{bmatrix} f_{0000} & f_{0010} & f_{0011} \\ f_{1000} & f_{1010} & f_{1011} \\ f_{1100} & f_{1110} & f_{1111} \end{bmatrix} \) the compressed signature matrix of \( f \).

**Theorem 2.5.** [14] If \( f \) is a redundant signature and its compressed signature matrix has full rank, then \( \text{Holant}(\neq 2 \mid f) \) is \#P-hard.

### 2.4 Möbius Transformation

A Möbius transformation [1] is a mapping of the form \( z \mapsto \frac{az+b}{cz+d} \), where \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \). It is a bijective conformal map of the extended complex plane \( \mathbb{C} = \mathbb{C} \cup \{\infty\} \) to itself. A Möbius transformation maps the unit circle \( S^1 = \{z \mid |z| = 1\} \) to itself iff it is of the form \( \varphi(z) = e^{i\theta} \frac{z+\lambda}{1+\lambda z} \), denoted by \( \mathcal{M}(\lambda, e^{i\theta}) \), where \( |\lambda| \neq 1 \). When \( |\lambda| < 1 \), it maps the interior of \( S^1 \) to the interior, and when \( |\lambda| > 1 \), it maps the interior of \( S^1 \) to the exterior. A Möbius transformation is determined by its values on any 3 distinct points. In particular if there are 5 distinct points \( \mathbf{i}, \mathbf{j}, \mathbf{k} \in S^1 \), such that \( |\varphi(\mathbf{i})| \) is either 0 or 1 or \( \infty \), then it must map \( S^1 \) to \( S^1 \) in a bijection.

### 3 Main Theorem and Proof Outline

**Theorem 3.1.** Let \( f \) be a 4-ary signature with the signature matrix \( M(f) = \begin{bmatrix} 0 & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & 0 & z & 0 \\ w & 0 & 0 & 0 \end{bmatrix} \). If \( ax = 0 \), then \( \text{Holant}(\neq 2 \mid f) \) is equivalent to the six-vertex model \( \text{Holant}(\neq 2 \mid f') \) where \( f' \) is obtained from \( f \) by setting \( a = x = 0 \), i.e., \( M(f') = \begin{bmatrix} 0 & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & 0 & z & 0 \\ w & 0 & 0 & 0 \end{bmatrix} \). Explicitly, \( \text{Holant}(\neq 2 \mid f) \) is \#P-hard except in the following cases:

- \( f' \in \mathcal{P} \),
- \( f' \in \mathcal{A} \),
- there is at least one zero in each pair \((b, y), (c, z), (d, w)\).

If \( ax \neq 0 \), then \( \text{Holant}(\neq 2 \mid f) \) is \#P-hard in the following cases:

- \( f \) is \( \mathcal{P} \)-transformable;
- \( f \) is \( \mathcal{A} \)-transformable;
- \( f \) is \( \mathcal{L} \)-transformable.
For any given $f$ in the eight-vertex model and any signature grid $\Omega$ with 4-regular graph $G$, any valid orientation on $G$ for $\text{Holant}_\Omega(\neq_2 | f)$ must have an equal number of sources and sinks. Hence the value $\text{Holant}_\Omega(\neq_2 | f)$ as a polynomial in $a$ and $x$ is in fact a polynomial in the product $ax$. So we can replace $(a, x)$ by any $(\tilde{a}, \tilde{x})$ such that $\tilde{a} \tilde{x} = ax$. In particular, let $\tilde{f}$ be a 4-ary signature with signature matrix $M(\tilde{f}) = \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 \end{pmatrix}$ where $\tilde{a} = \sqrt{ax}$, then $\text{Holant}_\Omega(\neq_2 | f) = \text{Holant}_\Omega(\neq_2 | \tilde{f})$. Note that one can also switch the sign of both entries $(a, x)$. If $a = 0$, this is the six-vertex model and has been solved in Theorem 2.3 [13]. In the following we assume $a = x \neq 0$.

Let $N$ be the number of zeros in $\{b, c, d, y, z, w\}$. We define Case I to be $N \geq 1$ and there is at most one pair in $\{(b, y), (c, z), (d, w)\}$ that is $(0, 0)$. We define Case II to be there are (at least) two pairs in $\{(b, y), (c, z), (d, w)\}$ that are $(0, 0)$. Note that Case I and Case II cover all cases that $N \neq 0$. Finally we define Case III to be $N = 0$. Formally, The three cases are defined as follows:

Case I: $N \geq 1$ and there is at most one pair in $\{(b, y), (c, z), (d, w)\}$ that is $(0, 0)$.

In this case we prove that $\text{Holant}(\neq_2 | f)$ is #P-hard. We prove this by constructing a 4-ary signature $g$ in the six-vertex model, such that $\text{Holant}(\neq_2 | g)$ is #P-hard by Theorem 2.3, and prove that $\text{Holant}(\neq_2 | g) \leq_p \text{Holant}(\neq_2 | f)$.

Case II: There are (at least) two pairs in $\{(b, y), (c, z), (d, w)\}$ that are $(0, 0)$.

By the symmetry of these three pairs (the group action of $S_3$ induced by $S_4$), we assume that $b = y = d = w = 0$. As $a \neq 0$ we can normalize it to $a = 1$. We define a binary signature $\alpha$ with the matrix $M(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We will reduce #CSP$^2(g)$ to $\text{Holant}(\neq_2 | f)$. This will be accomplished by replacing every variable in an instance $\Omega$ of #CSP$^2(g)$ by a cycle of even length, such that there is a 1-1 correspondence between assignments in $\Omega$ and valid configurations of the eight-vertex model $\text{Holant}(\neq_2 | f)$, that preserves the product of the weights.

By Theorem 2.2, if $g \not\in \mathcal{P} \cup \mathcal{A} \cup \alpha \mathcal{A} \cup \mathcal{L}$, then #CSP$^2(g)$ is #P-hard. In this case $\text{Holant}(\neq_2 | f)$ is #P-hard. If $g \in \mathcal{P} \cup \mathcal{A} \cup \alpha \mathcal{A} \cup \mathcal{L}$, we show that $f$ is $\mathcal{P}$-transformable or $\mathcal{A}$-transformable. Thus $\text{Holant}(\neq_2 | f)$ is tractable.

Case III: $N = 0$, i.e., all values in $\{b, c, d, y, z, w\}$ are nonzero.

In this case we prove that $\text{Holant}(\neq_2 | f)$ is tractable in the listed cases, and #P-hard otherwise. The main challenge here is when we prove #P-hardness for some signatures $f$, the interpolation needs certain quantities not to repeat after iterations. If we can only produce a root of unity, then its powers will repeat after only a bounded number of steps in an iteration. Typically one satisfies such a requirement by producing quantities of complex norm not equal to 1. But for some $f$, provably the only such quantities that can be produced are all of complex norm 1.

Our main new idea is to use Möbius transformations. But before getting to that, there are some settings where we cannot do so, either because we don’t have the initial signature to start the process, or the matrix that would define the Möbius transformation is singular. So we first treat the following two special cases.

- If $b = cy$, $c = ez$ and $d = ew$, where $e = \pm 1$, by a rotational symmetric gadget, we get some redundant signatures (Definition 2.4). If one of the compressed matrices of these redundant signatures has full rank, then we can prove #P-hardness. If all of these compressed matrices are degenerate, then we get a system of equations of $\{a, b, c, d\}$. To satisfy these equations, $f$ has a very special form. Then we show that either $f$ is $\mathcal{A}$-
transformable or we can construct an arity 4 signature $g = \begin{bmatrix} t & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, which is actually a symmetric signature $g = [g_0, g_1, g_2, g_3, g_4] = [t, 0, 1, 0, \frac{1}{t}]$, with $t \neq 0$. Here $g_w$ is the value of $g$ on all inputs of Hamming weight $w$. By a holographic transformation using $T = \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix} = \sqrt{2}Z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on Holant$(\neq 2 | f, g)$, because $(\neq 2)Z^{\otimes 2} = (\neq 2)$ and therefore $(\neq 2)(T^{-1})^{\otimes 2} = \frac{1}{2}Z(\neq 2)$, the signature $(\neq 2)$ is transformed to $(\neq 2)$ on the LHS up to a nonzero scalar. Similarly $g$ is transformed to $T^{\otimes 4}g$ on the RHS, which is $(\neq 4)$ up to a nonzero scalar. Therefore we have the equivalence $\text{Holant}(\neq 2 | f, g) \equiv_T P \# \text{CSP}^2(T^{\otimes 4}f)$. This implies that $\text{Holant}(\neq 2 | f)$ is either $\#P$-hard, or $f$ is $\mathcal{A}$-transformable, or $\mathcal{L}$-transformable by Theorem 2.2.

- If $by = cz = dw$, then either we can realize a non-singular redundant signature or $f$ is $\mathcal{A}$-transformable.

If $f$ does not belong to the above two cases, by the symmetry of the pairs $\{(b, y), (c, z), (d, w)\}$, we may assume that $cz \neq dw$, i.e., the inner matrix $\begin{bmatrix} c & d \\ w & z \end{bmatrix}$ of $M(f)$ has full rank. Then we want to realize binary signatures of the form $(0, 1, t, 0)^T$, for arbitrary values of $t$. If this can be done, by carefully choosing the values of $t$, we will prove $\#P$-hardness by

- constructing a signature that belongs to Case I (Case A in the proof of Lemma 6.1), or
- constructing a redundant signature whose compressed signature matrix has full rank (Case B in the proof of Lemma 6.1), or
- constructing the symmetric signature $[1, 0, -1, 0, 1]$ by gadget construction. Then by the holographic transformation using $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have
  \[ \text{Holant}(\neq 2 | f, [1, 0, -1, 0, 1], (0, 1, t, 0)^T) \equiv_T P \# \text{CSP}^2(T^{\otimes 4}f, T^{\otimes 2}(0, 1, t, 0)^T) \]

because $(\neq 2)(T^{-1})^{\otimes 2} = (\neq 2)$, and $T^{\otimes 4}[1, 0, -1, 0, 1]$ is $(\neq 4)$, both up to a nonzero scalar. Then we can prove $\#P$-hardness by Theorem 2.2 (Case C in the proof of Lemma 6.1).

We realize binary signatures by connecting $f$ with $(\neq 2)$. This corresponds naturally to a Möbius transformation. By discussing the following different forms of binary signatures we get, we can either realize arbitrary $(0, 1, t, 0)^T$, then Holant$(\neq 2 | f)$ is $\#P$-hard, or $f$ is $\mathcal{A}$-transformable under some nontrivial holographic transformation.

- If we can get a signature of the form $g = (0, 1, t, 0)^T$ where $t \neq 0$ is not a root of unity, then by connecting a chain of $g$, we can get polynomially many distinct binary signatures $g_i = (0, 1, t^i, 0)^T$. Then, by interpolation, we can realize arbitrary binary signatures of the form $(0, 1, s, 0)^T$.

- Suppose we can get a signature of the form $(0, 1, t, 0)^T$, where $t \neq 0$ is an $n$-th primitive root of unity ($n \geq 5$). Now, we only have $n$ many different signatures $g_i = (0, 1, t^i, 0)^T$. But we can relate $f$ to a Möbius transformation $\varphi(\bar{z}) : \bar{z} \mapsto \frac{c\bar{z} + d}{w\bar{z} + z}$, due to $\det \begin{bmatrix} c & d \\ w & z \end{bmatrix} \neq 0$.
  
  For the Möbius transformation $\varphi$, we can realize the signatures $g = (0, 1, \varphi(t^i), 0)^T$. If $|\varphi(t^i)| \neq 0, 1$ or $\infty$ for some $i$, then this is treated above. Otherwise, since $\varphi$ is a bijection on the extended complex plane $\hat{\mathbb{C}}$, it can map at most two points of $S^1$ to $0$ or $\infty$. Hence, $|\varphi(t^i)| = 1$ for at least three $t^i$. But a Möbius transformation is determined by any three distinct points. This implies that $\varphi$ maps $S^1$ to itself. Such Möbius transformations have a known special form $e^{i\theta} \frac{\bar{z} + \lambda}{1 + \lambda\bar{z}}$. By exploiting its property we can construct a signature $f'$ such that its corresponding Möbius transformation $\varphi'$ defines an infinite group. This
implies that \( \varphi^k(t) \) are all distinct. Then, we can get polynomially many distinct binary signatures \((0, 1, \varphi^k(t), 0)\), and realize arbitrary binary signatures of the form \((0, 1, s, 0)^T\) (Lemma 5.1).

- Suppose we can get a signature of the form \((0, 1, t, 0)^T\) where \(t \neq 0\) is an \(n\)-th primitive root of unity \((n = 3, 4)\). Then we can either relate it to two Möbius transformations mapping the unit circle to itself, or realize the \((+, -)\)-pinning \((0, 1, 0, 0)^T = (1, 0) \otimes (0, 1)\).

- Suppose we can get \((0, 1, 0, 0)^T\). By connecting \(f\) with it, we can get new signatures of the form \((0, 1, t, 0)^T\). Similarly, by analyzing the value of \(t\), we can either realize arbitrary binary signatures of the form \((0, 1, s, 0)^T\), or a redundant signature whose compressed signature matrix has full rank, or a signature in Case I, which is \#P-hard, or we prove that \(f\) is \(\mathcal{A}\)-transformable under a holographic transformation \([1 \, 0 \, \gamma]\), where \(\gamma^2 = \alpha\) or \(\gamma^2 = i\) (Theorem 6.2).

- Suppose we can only get signatures of the form \((0, 1, 0, 1, 0)^T\). That implies \(a = c\), \(b = ey\) and \(c = ez\), where \(e = \pm 1\). This has been treated before.

4 Two Inner Pairs Are \((0, 0)\)

If there are two inner pairs that are \((0, 0)\), by the symmetry of the three inner pairs \((b, y), (c, z), (d, w)\), we may assume that \(b = y = d = w = 0\). Then we have the following lemma.

**Lemma 4.1.** Let \(f\) be a \(4\)-ary signature with the signature matrix \(M(f) = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix}\), then \(\text{Holant}(\not\equiv 2) f\) is \#P-hard unless \(f\) is \(\mathcal{A}\)-transformable, or \(f \in \mathcal{P}\), in which case the problem is computable in polynomial time.

**Proof.** Tractability follows from Theorem 2.2.

As \(a \neq 0\) we can normalize it to \(a = 1\). Let \(g(x_1, x_2)\) be the binary signature \(M(g) = \begin{bmatrix} 1 & c \\ z & 1 \end{bmatrix}\) in matrix form. This means that \(g_{00} = g_{11} = 1 = f_{0000} = f_{1111}\), \(g_{01} = c = f_{0101}\) and \(g_{10} = z = f_{1010}\). We prove that \(#\text{CSP}^2(g) \leq^P \text{Holant}(\not\equiv 2) f\) in two steps. In each step, we begin with a signature grid and end with a new signature grid such that the Holant values of both signature grids are the same.

For step one, let \(G = (U, V, E)\) be a bipartite graph representing an instance of \(#\text{CSP}^2(g)\), where each \(u \in U\) is a variable, and each \(v \in V\) has degree two and is labeled \(g\). For every vertex \(u \in U\), we define a cyclic order of the edges incident to \(u\), and decompose \(u\) into \(2k = \deg(u)\) vertices. Then we connect the \(2k\) edges originally incident to \(u\) to these \(2k\) new vertices so that each vertex is incident to exactly one edge. We also connect these \(2k\) new vertices in a cycle according to the cyclic order. Thus, in effect we have replaced \(u\) by a cycle of length \(2k = \deg(u)\). Each of \(2k\) vertices has degree 3, and we assign them \((= 3)\). Clearly this does not change the value of the partition function. The resulting graph has the following properties: (1) every vertex has either degree 2 or degree 3; (2) each degree 2 vertex is connected to degree 3 vertices; (3) each degree 3 vertex is connected to exactly one degree 2 vertex.

Now step two. We add a vertex on every edge of each cycle \(C_u\) of length \(2k = \deg(u)\), making \(C_u\) a cycle of length \(4k\). (This is shown in Figure 2b). Name the vertices \(1, 2, \ldots, 4k\) in cyclic order, with the newly added vertices numbered \(1, 3, \ldots, 4k - 1\). There are \(k\) pairs of these odd numbered vertices \((1, 3), (5, 7), \ldots, (4k - 3, 4k - 1)\). We will merge each pair \((4i - 3, 4i - 1)\) \((1 \leq i \leq k)\) to form a new vertex of degree 4, and assign a signature \(f\) on it. (This “pinching” operation is illustrated
by the dotted line in Figure 2b). The input variables of \( f \) are carefully assigned so that the two incoming edges originally at \( 4i - 3 \) are named \( x_1 \) and \( x_3 \), and the other two incoming edges originally at \( 4i - 1 \) are named \( x_2 \) and \( x_4 \). Note that the support of \( f \) ensures that the values at \( x_1 \) and \( x_3 \) are equal, and the values \( x_2 \) and \( x_4 \) are equal. For every even numbered vertex \( 2i \) (\( 1 \leq i \leq 2k \)) on \( C_u \), it is currently connected to a vertex \( v \) of degree 2 labeled \( g \). Suppose in the instance of \( \#\text{CSP}^2(g) \), the constraint \( g(u, u') \) is applied to the variables \( u \) and \( u' \), in that order. Then the other adjacent vertex of \( v \) is some even numbered vertex \( 2j \) on the cycle \( C_{u'} \) for the variable \( u' \). We will contract the two incident edges at \( v \), merging the vertices \( 2i \) on \( C_u \) and \( 2j \) on \( C_{u'} \), to form a new vertex \( v' \) of degree 4, and assign a copy of \( f \) on it. The input variables of \( f \) are carefully assigned so that the two incoming edges originally at \( 2i \) of \( C_u \) are named \( x_1 \) and \( x_3 \), and the other two incoming edges originally at \( 2j \) of \( C_{u'} \) are named \( x_2 \) and \( x_4 \). The support of \( f \) ensures that the values of \( x_1 \) and \( x_3 \) are equal and the values of \( x_2 \) and \( x_4 \) are equal. (This is illustrated in Figure 2c). Finally we put a \((\neq 2)\) on every edge. This completes the definition of an instance of \( \text{Holant}(\neq 2 | f) \) in this reduction.

Note that if we traverse the cycle \( C_u \), by the support of \( f \) and the \((\neq 2)\) on every edge, there exists some \( \epsilon = 0, 1 \), such that all four edges for the \( f \) at any odd numbered pair \((4i-3, 4i-1)\) must take the same value \( \epsilon \), and the two adjacent edges at every even numbered vertex \( 2i \) must take the same value \( 1 - \epsilon \). Therefore there is a 1-1 correspondence between 0-1 assignments for the variables in \( \#\text{CSP}^2(g) \) and valid configurations in \( \text{Holant}(\neq 2 | f) \). Furthermore, at every odd numbered pair \((4i-3, 4i-1)\) the value is \( f_{0000} = 1 \) or \( f_{1111} = 1 \). The value of \( f \) at the vertex \( v' \) formed by contraction at \( v \) reflects perfectly the value of \( g(u, u') \). Hence, \( \#\text{CSP}^2(g) \leq_T \text{Holant}(\neq 2 | f) \).

If \( g \notin \mathcal{P} \cup \mathcal{A} \cup \mathcal{A} \cup \mathcal{L} \), then \( \#\text{CSP}^2(g) \) is \#P-hard by Theorem 2.2. It follows that \( \text{Holant}(\neq 2 | f) \) is \#P-hard. Otherwise, note that \( f(x_1, x_2, x_3, x_4) = g(x_1, x_2) \cdot \chi_{x_1=x_3} \cdot \chi_{x_2=x_4} \). Hence, \( g \in \mathcal{A} \cup \mathcal{P} \) implies \( f \in \mathcal{A} \cup \mathcal{P} \). Since \( g_{00} \neq 0 \), if \( g \in \mathcal{L} \), then \( g \in \mathcal{A} \) by the definition of \( \mathcal{L} \), and therefore \( f \notin \mathcal{A} \). Finally, if \( g \in \alpha \mathcal{A} \), i.e., \([1, 0, 0, 0]^{\otimes 2} g = (1, \alpha c, \alpha z, i)^T \in \mathcal{A} \), then after the holographic transformation to \( f \) using \([1, 0, 0, 0] \), where \( \beta^2 = \alpha \), we get the signature \( \hat{f} \) whose signature matrix is \(
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \alpha c & 0 & 0 \\
0 & 0 & \alpha z & 0 \\
0 & 0 & 0 & i
\end{bmatrix}
\). Since \( \hat{f} \in \mathcal{A} \), \((\neq 2)\) \([1, 0, 0, 0]^{\otimes 2} \) is \((\neq 2)\) up to a nonzero scalar, we conclude that \( f \) is \( \mathcal{A} \)-transformable. This finishes the proof.

\[ \square \]

5 Interpolation via Möbius Transformation

Lemma 5.1. Let \( g = (0, 1, t, 0)^T \) be a binary signature where \( t \neq 0 \) and is not a root of unity, then for any signature set \( \mathcal{F} \), and any \( u \in \mathbb{C} \), we have \( \text{Holant}(\neq 2 | \mathcal{F}, (0, 1, u, 0)^T) \leq_T \text{Holant}(\neq 2 | \mathcal{F}, g) \).

Note that having \( g = (0, 1, t, 0)^T \) is equivalent to having \( g'(x_1, x_2) = g(x_2, x_1) = (0, t, 1, 0)^T \).

Lemma 5.2. Let \( g = (0, 1, t, 0)^T \) be a binary signature where \( t \) is an \( n \)-th primitive root of unity, \( n \geq 5 \), and \( f \) be a signature with the signature matrix \( M(f) = \begin{bmatrix}
a & 0 & 0 & b \\
0 & c & d & 0 \\
c & 0 & u & 0 \\
d & 0 & 0 & z
\end{bmatrix} \) with \( abcdyzw \neq 0 \), where \([c \ d] \) has full rank, then for any \( u \in \mathbb{C} \), \( \text{Holant}(\neq 2 | f, (0, 1, u, 0)^T) \leq_T \text{Holant}(\neq 2 | f, g) \).

Proof. By connecting two copies of \( g \) using \((\neq 2)\), we get the signature \( g_2 \) whose signature matrix is \( M(g_2) = \begin{bmatrix}
0 & 0 & b \\
0 & 0 & d \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \). Similarly, we can construct \( g_i = (0, 1, t^i, 0)^T \) for \( 1 \leq i \leq 5 \). Since the order \( n \geq 5 \), \( g_i \) are distinct pairwise for \( 1 \leq i \leq 5 \).
Figure 2: The reduction from \( \#\text{CSP}^2(g) \) to \( \text{Holant}(\neq_2 | f) \). In (a), \( u \) and \( u' \) are two variables in an instance of \( \#\text{CSP}^2(g) \), and a constraint \( g(u, u') \) is applied, in that order. In (a), \( \deg(u) = 6 \) and \( \deg(u') = 4 \). The diamonds are vertices of degree 2 and are labeled by the constraint \( g \). In (b), \( u \) and \( u' \) are replaced by the cycles \( C_u \) and \( C_{u'} \). Each diamond is labeled by the constraint \( g \), squares are \( (\neq_2) \), the circle vertices are EQUALITIES (of arity 3 and 2). The degree 2 circle vertex pairs will be merged, indicated by the dotted lines. In (c), each pair linked by a dotted line in (b) is merged to form a vertex of degree 4 (black square) and labeled by \( f \). The two incident edges of each diamond vertex in (b) are contracted to form a vertex of degree 4 (triangle) and labeled by \( f \). The input variables of all copies of \( f \) are carefully labeled so that along each cycle \( C_u \) or \( C_{u'} \), there are exactly two valid configurations corresponding to the 0-1 assignments to \( u \) and \( u' \) respectively.
By connecting the variables $x_3$ and $x_4$ of the signature $f$ with the variables $x_1$ and $x_2$ of $g_i$ using $\neq 2$ for $1 \leq i \leq 5$ respectively, we get binary signatures

$$h_i = M(f)N g_i = \begin{bmatrix} a & 0 & 0 & b \\ c & d & 0 & 0 \\ w & z & 0 & 0 \\ y & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ct^i+d \\ ct^i+d & wz^i \end{bmatrix}. $$

Let $\varphi(\mathbf{z}) = \frac{c_3 + d}{w_3 + z}$. Since $\operatorname{det} \begin{bmatrix} c & d \\ w & z \end{bmatrix} \neq 0$, $\varphi(\mathbf{z})$ is a Möbius transformation of the extended complex plane $\hat{\mathbb{C}}$. We rewrite $h_i$ as $(wt^i + z)(0, \varphi(t^i), 1, 0)^T$, with the understanding that if $wt^i + z = 0$, then $\varphi(t^i) = \infty$, and we define $(wt^i + z)(0, \varphi(t^i), 1, 0)^T$ to be $(0, ct^i + d, 0, 0)^T$. Having $(0, \varphi(t^i), 1, 0)^T$ is equivalent to having $(0, 1, \varphi(t^i), 0)^T$. If there is a $t^i$ such that $\varphi(t^i) \neq 0, \infty$ or a root of unity for $1 \leq i \leq 5$, then by Lemma 5.1, Holant$(\neq 2|f, (0, 1, u, 0)^T) \leq _T^P$ Holant$(\neq 2|f, (0, 1, \varphi(t^i), 0)^T)$, for all $u \in \mathbb{C}$. Otherwise, $\varphi(t^i)$ is $0, \infty$ or a root of unity for $1 \leq i \leq 5$. Since $\varphi(\mathbf{z})$ is a bijection of $\hat{\mathbb{C}}$, there is at most one $t^i$ such that $\varphi(t^i) = 0$ and at most one $t^i$ such that $\varphi(t^i) = \infty$. That means, there are at least three $t^i$ such that $|\varphi(t^i)| = 1$. Since a Möbius transformation is determined by any 3 distinct points, mapping 3 distinct points from $S^1$ to $S^1$ implies that this $\varphi(\mathbf{z})$ maps $S^1$ homeomorphically onto $S^1$.

A Möbius transformation mapping 3 distinct points from $S^1$ to $S^1$ has a special form $\mathcal{M}(\lambda, e^{i\theta})$: $\mathbf{z} \mapsto e^{i\theta} \frac{\lambda + \mathbf{z}}{1 + \lambda \mathbf{z}}$, where $|\lambda| \neq 1$. By normalization in $f$, we may assume $z = 1$. Comparing coefficients with $\varphi(\mathbf{z})$ we have $c = e^{i\theta}$, $d = e^{i\theta} \lambda$ and $w = \lambda$. Thus $M(f) = \begin{bmatrix} a & 0 & 0 & b \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & \lambda & e^{i\theta} & 0 \\ y & 0 & 0 & 1 \end{bmatrix}$. Note that $M_{x_3x_4,x_2x_1}(f) = \begin{bmatrix} a & 0 & 0 & y \\ 0 & \lambda & e^{i\theta} & 0 \\ 0 & 1 & e^{i\theta} & 0 \\ b & 0 & 0 & a \end{bmatrix}$, obtained from $M(f) = M_{x_1x_2,x_3x_4}(f)$ by exchanging the two middle columns of $(M(f))^T$. By taking two copies of $f$ and connecting the variables $x_3, x_4$ of the first copy to the variables $x_3, x_4$ of the second copy using $(\neq 2)$, we get a signature $f_1$ with the signature matrix

$$M(f_1) = M(f)N M_{x_3x_4,x_2x_1}(f) = \begin{bmatrix} 2ab & 0 & 0 & 0 \\ 0 & e^{i\theta}(1 + |\lambda|^2) & 2e^{i\theta} \lambda & 0 \\ 0 & 2\lambda & e^{i\theta}(1 + |\lambda|^2) & 0 \\ a^2 + by & 0 & 0 & 2ay \end{bmatrix}. $$

Then up to the nonzero scalar $s = e^{i\theta}(1 + |\lambda|^2)$, and denote by $\delta = \frac{2e^{i\theta} \lambda}{1 + |\lambda|^2}$, we have $\tilde{\delta} = \frac{2e^{-i\theta} \lambda}{1 + |\lambda|^2}$, and the signature $f_1$ has the signature matrix $M(f_1) = \begin{bmatrix} 2ab & 0 & 0 & \frac{a^2 + by}{2} \\ 0 & \delta & 0 & 0 \\ 0 & 0 & 1 + \delta & 0 \\ \frac{a^2 + by}{2} & 0 & 0 & 2ay \end{bmatrix}$. The inner matrix $\begin{bmatrix} \frac{1}{\delta} & \delta \\ \frac{1}{\delta} & 1 \end{bmatrix}$ of $M(f_1)$ is the product of three nonsingular $2 \times 2$ matrices, thus it is also nonsingular. The two eigenvalues of $\begin{bmatrix} \frac{1}{\delta} & \delta \\ \frac{1}{\delta} & 1 \end{bmatrix}$ are $1 + |\delta|$ and $1 - |\delta|$, both are real and must be nonzero. In particular $|\delta| \neq 1$. Obviously $|1 + |\delta|| \neq |1 - |\delta||$. This implies that there are no integer $n > 0$ and complex number $\mu$ such that $\begin{bmatrix} \frac{1}{\delta} & \delta \\ \frac{1}{\delta} & 1 \end{bmatrix}^n = \mu I$, i.e., $\begin{bmatrix} \frac{1}{\delta} & \delta \\ \frac{1}{\delta} & 1 \end{bmatrix}$ has infinite projective order. Note that $\begin{bmatrix} \frac{1}{\delta} & \delta \\ \frac{1}{\delta} & 1 \end{bmatrix}$ defines a Möbius transformation $\psi(\mathbf{z})$ of the form $\mathcal{M}(\lambda, e^{i\theta})$ with $\lambda = \delta$ and $\theta = 0$: $\mathbf{z} \mapsto \psi(\mathbf{z}) = \frac{\delta + \mathbf{z}}{1 + \delta \mathbf{z}}$, mapping $S^1$ to $S^1$.

We can connect the binary signature $g_i(x_1, x_2)$ via $N$ to $f_1$. This gives us the binary signatures
(for $1 \leq i \leq 5$)

$$h_i^{(1)} = M(f_i)N g_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ t \\ t^i \\ 0 \end{bmatrix} = C(i,0) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where $\psi(\delta) = [\frac{1}{\delta} \ 1]$ is the Möbius transformation defined by the matrix $[\frac{1}{\delta} \ 1]$, and $C(i,0) = 1 + \delta t^i$. Since $\psi$ maps $S^2$ to $S^1$, and $|t^i| = 1$, clearly $C(i,0) \neq 0$, and $\psi(t^i) \in S^1$.

Now we can use $h_i^{(1)}(x_2, x_1) = (0, 1, \psi(t^i), 0)^T$ in place of $g_i(x_1, x_2) = (0, 1, t^i, 0)^T$ and repeat this construction. Then we get

$$h_i^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ t \\ \psi(t^i) \\ 0 \end{bmatrix} = C(i,1) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

where $\psi^2$ is the composition $\psi \circ \psi$, corresponding to $[\frac{1}{\delta} \ 1]^2$, and $C(i,1) = 1 + \bar{\delta} \psi(t^i) \neq 0$.

We can iterate this process and get polynomially many $h_i^{(k)} = (0, \psi^k(t^i), 1, 0)^T$ for $1 \leq i \leq 5$ and $k \geq 1$.

If for each $i \in \{1,2,3\}$, there is some $n_i > 0$ such that $\psi^{n_i}(t^i) = t^i$, then $\psi^{n_0}(t^i) = t^i$, for $n_0 = n_1n_2n_3 > 0$, and all $1 \leq i \leq 3$, i.e., the Möbius transformation $\psi^{n_0}$ fixes three distinct complex numbers $t, t^2, t^3$. So the Möbius transformation is the identity map, i.e., $\psi^{n_0}(\delta) = \delta$ for all $\delta \in \mathbb{C}$. This implies that $[\frac{1}{\delta} \ 1]^{n_0} = C[\frac{1}{\delta} \ 1]$ for some constant $C$. This contradicts the fact that $[\frac{1}{\delta} \ 1]$ does not have finite projective order. Therefore, there is an $i \in \{1,2,3\}$ such that $\psi^{n}(t^i) \neq t^i$ for all $n \in \mathbb{N}$. This implies that $(1, \psi^n(t^i))$ are all distinct for $n \in \mathbb{N}$, since $\psi$ maps $S^1$ 1-1 onto $S^1$. Then we can generate polynomially many distinct binary signatures of the form $(0, 1, \psi^n(t^i), 0)^T$. By interpolation, for any $u \in \mathbb{C}$ we have $\text{Holant}(\neq_2 | f, (0, 1, u, 0)^T) \leq_P \text{Holant}(\neq_2 | f, g)$.

\section{Using Möbius Transformations to Achieve Dichotomy}

\textbf{Lemma 6.1.} Let $f$ be a signature with matrix $M(f) = \begin{bmatrix} a & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & 0 & z & a \\ y & 0 & 0 & z \end{bmatrix}$, where $a \neq 0$, and $[c \ d \ w \ z]$ has full rank. If $t^i$ are distinct for $1 \leq i \leq 5$, then $\text{Holant}(\neq_2 | f, (0, 1, t, 0)^T)$ is $\#P$-hard.

\textbf{Proof Sketch:} By Lemma 5.1 and Lemma 5.2, for any $u \in \mathbb{C}$, the binary signature $(0, 1, u, 0)^T$ is available.

By a normalization in $f$, we may assume that $c = 1$, and by doing binary modifications to the variables $x_1, x_3$ of $f$ by $(0, 1, w^{-1}, 0)^T$ and $(0, 1, d^{-1}, 0)^T$ respectively (see subsection 2.2), we get a signature $f_1$ with signature matrix $M(f_1) = \begin{bmatrix} 0 & 1 & 0 & \delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{w \delta}{z} \\ \frac{w \delta}{z} & 0 & 0 & \frac{w \delta}{z} \end{bmatrix}$. Note that $\frac{w \delta}{z} \neq 1$ since the inner matrix of $M(f)$ has full rank.

\textbf{Case A:} For $\frac{w \delta}{z} \neq -1$ or $\frac{w \delta}{z} \neq 1$, we can do binary modifications to $f_1$ by carefully choosing binary signatures $(0, 1, u, 0)^T$ to get a 4-ary signature $f_2$. Then by connecting $f_1$ and $f_2$, we can get a 4-ary signature $f_3$ that is in Case I, so $\text{Holant}(\neq_2 | f_3)$ is $\#P$-hard. Thus $\text{Holant}(\neq_2 | f)$ is $\#P$-hard. We omit the details here.

Now we can assume that $\frac{w \delta}{z} = -1$ and $\frac{w \delta}{z} = \pm 1$. We prove the lemma for $\frac{w \delta}{z} = -1$, $\frac{w \delta}{z} = 1$ is similar.

\textbf{Case B:} Suppose $\frac{w \delta}{z} = -1$, $\frac{w \delta}{z} = 1$, and $\frac{w \delta^2}{z} \neq 1$. By the symmetry of the three pairs we have
$$M_{x_1x_2x_3}(f_1) = \begin{bmatrix} a & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{d}{y} & -1 & 0 \\ 1 & 0 & 0 & \frac{a}{dz} \end{bmatrix}.$$ By doing modifications to $M_{x_1x_2x_3}(f_1)$ using binary signatures $(0, 1, \frac{w}{y}, 0)^T$ and $(0, 1, \frac{d}{y}, 0)^T$, we get the signature $f_4$ whose signature matrix is $M(f_4) = \begin{bmatrix} a & 0 & 0 & \frac{d}{y} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \frac{w}{y} & 0 & 0 & \frac{a}{dz} \end{bmatrix}$.

Note that the entry $f_4(1, 1, 1, 1) = \frac{a}{dz} \cdot \frac{w}{y} \cdot \frac{d}{y} = \frac{a}{y}$. Note that $f_4$ is a redundant signature and its compressed signature matrix $\begin{bmatrix} a & 0 & 0 & \frac{d}{y} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \frac{w}{y} & 0 & 0 & \frac{a}{dz} \end{bmatrix}$ has full rank by $a^2 \neq dw$. Thus Holant($\neq 2 | f_4$) is #P-hard by Theorem 2.5. It follows that Holant($\neq 2 | f, (0, 1, t, 0)^T$) is #P-hard.

Case C: Suppose $\frac{a}{dz} = -1$, $\frac{b}{dw} = -1$, $\frac{c}{dx} = 1$. In this case $M_{x_1x_2x_3x_4}(f_1) = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & \frac{c}{dx} & -1 & 0 \\ 1 & 0 & 0 & \frac{a}{dz} \end{bmatrix}$. By the symmetry of three pairs we have $M_{x_1x_2x_3}(f_1) = \begin{bmatrix} a & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{d}{y} & -1 & 0 \\ 1 & 0 & 0 & \frac{b}{dw} \end{bmatrix}$. By doing binary modifications to the variables $x_1, x_3$ of this function using $(0, 1, \frac{d}{y}, 0)^T$ and $(0, 1, \frac{b}{dw}, 0)^T$ respectively, we get a signature $f_5$ with signature matrix $M(f_5) = \begin{bmatrix} a & 0 & 0 & \frac{d}{y} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \frac{b}{dw} & 0 & 0 & -\frac{1}{a} \end{bmatrix}$. By connecting two copies of $f_5$ via $N$, we get a signature $f_6$ whose signature matrix is

$$M(f_6) = M(f_5)NM_{x_3x_4x_1x_2}(f_5) = \begin{bmatrix} a & 0 & 0 & \frac{d}{y} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \frac{b}{dw} & 0 & 0 & -\frac{1}{a} \end{bmatrix} \begin{bmatrix} a & 0 & 0 & -\frac{b}{dw} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \frac{d}{y} & 0 & 0 & -\frac{1}{a} \end{bmatrix} = 2 \begin{bmatrix} a & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & \frac{b}{ad} \end{bmatrix}.$$

Then by connecting two copies of $f_6$ via $N$, we get a signature $f_7$ whose signature matrix is

$$M(f_7) = M(f_6)NM(f_6) = 4 \begin{bmatrix} \frac{ad}{y} & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & \frac{ad}{y} \end{bmatrix} \begin{bmatrix} a & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & \frac{ad}{y} \end{bmatrix} = 8 \begin{bmatrix} \frac{ad}{y} & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -\frac{1}{ad} \end{bmatrix}.$$

Now the important point is that $f_7$ is a symmetric signature where all weight 2 entries are equal. Thus, after the nonzero scalar 8, $f_7 = [s, 0, 1, 0, \frac{1}{s}]$ written as a symmetric signature listing the values of $f_7$ according to its Hamming weight, and where $s = -\frac{ad}{y} \neq 0$.

Now we have the reduction Holant($\neq 2 | f_7, (0, 1, u, 0)^T$) $\leq_T$ Holant($\neq 2 | f, (0, 1, t, 0)^T$) for any $u \in \mathbb{C}$. We finish the proof for this case by proving that Holant($\neq 2 | f_7, (0, 1, u, 0)^T$) is #P-hard for a carefully chosen $u$. Let $T = \begin{bmatrix} 1 & \sqrt{s} \\ i & -i\sqrt{s} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. By the holographic transformation using $T$, note that $(\neq 2)(T^{-1})\otimes_2$ and $T\otimes_4 f_7$ are nonzero scalar multiples of $(=2)$ and $(=4)$ respectively, we have Holant($\neq 2 | f_7, (0, 1, u, 0)^T$) $\equiv_T$ Holant($\neq 2 | =4, T\otimes_2(0, 1, u, 0)^T$). We can calculate that $g = T\otimes_2(0, 1, u, 0)^T = \sqrt{s}(u + 1, (u - 1)i, (1 - u)i, u + 1)^T$. By $=2$ on the left side and $=4$ on the right side, we can construct $=2k$ for any $k \in \mathbb{N}$. This implies that

$$\#\text{CSP}^2(g) \leq_T \text{Holant}(\neq 2 | =4, g).$$

Let $u = 3$, then $g \notin \mathcal{P} \cup \mathcal{A} \cup \mathcal{A} \cup \mathcal{L}$. Thus $\#\text{CSP}^2(g)$ is #P-hard by Theorem 2.2. Hence Holant($\neq 2 | f_7, (0, 1, u, 0)^T$) is #P-hard and Holant($\neq 2 | f, (0, 1, t, 0)^T$) is #P-hard.
Theorem 6.2. If \( f \) has the signature matrix \( M(f) = \begin{bmatrix} a & 0 & 0 & b \\ 0 & c & 0 & d \\ 0 & w & e & 0 \\ g & 0 & 0 & a \end{bmatrix} \) where \( abcydw \neq 0 \) and \( \begin{bmatrix} c & d \\ w & z \end{bmatrix} \) has full rank, then Holant\((\neq_2 | f, (0,1,0,0)^T)\) is \#P-hard, or \( f \) is \( \mathcal{A} \)-transformable.

Proof. By a normalization in \( f \), we may assume that \( c = 1 \). For any \( u \in \{b, y, d, w\} \), we can get \((0,1,u,0)^T\) by doing a loop to some two variables of \( f \) using \((0,1,0,0)^T\) via \( N \). Thus if there exists \( u \in \{b, y, d, w\} \) such that \( u^i \) are distinct for \( 1 \leq i \leq 5 \), then Holant\((\neq_2 | f, (0,1,0,0)^T)\) is \#P-hard by Lemma 6.1. So we may assume all of \( \{b, y, d, w\} \) are in \( \{\pm 1, \pm i, \omega, \omega^2\} \), where \( \omega = e^{\pi i/3} \).

Note that for any \( u \neq 0 \), if we have \( g(x_1, x_2) = (0,1,u,0)^T \), then we also have \( g'(x_1, x_2) = g(x_2, x_1) = u(0,1,u^{-1},0)^T \), i.e., we have \((0,1,u^{-1},0)^T\) after the nonzero scalar \( u \). So we have \((0,1,d^{-1},0)^T\) and \((0,1,\omega,0)^T\). By binary modifications using \((0,1,\omega,0)^T\) and \((0,1,d^{-1},0)^T\) to the variables \( x_1, x_3 \) of \( f \) respectively, we get a signature \( f_1 \) whose signature matrix is

\[
M(f_1) = \begin{bmatrix}
0 & b & 0 & 0 \\
0 & c & 0 & 0 \\
0 & w & e & 0 \\
0 & 0 & 0 & a
\end{bmatrix}.
\]

Since \( \begin{bmatrix} 1 & d \\ w & z \end{bmatrix} \) has full rank, we have \( \frac{1}{dw} \neq 1 \). By doing a loop to \( f_1 \) using \( \neq_2 \), we get the binary signature \( h = M(f)(\neq_2) = (0, 2, 1 + \frac{1}{dw}, 0)^T \).

We claim that \( \frac{1}{dw} = 1 \) or we are done. By doing a loop to \( f_1 \) using \((0,1,0,0)^T\), we have \((0,1,\frac{1}{dw},0)^T\). If \( \frac{1}{dw} \) are distinct for \( 1 \leq i \leq 5 \), then the problem Holant\((\neq_2 | f, (0,1,\frac{1}{dw},0)^T)\) is \#P-hard by Lemma 6.1. Thus Holant\((\neq_2 | f, (0,1,0,0)^T)\) is \#P-hard. So we may assume that \( \frac{1}{dw} = i \) or \( -i \), then \( \frac{1}{d_{\omega, i}} = \sqrt{2} \). If \( \frac{1}{d_{\omega, i}} = \omega \) or \( \omega^2 \), then \( |1 + \frac{1}{dw}| = 1 \neq 2 \). So Holant\((\neq_2 | f, h)\) is \#P-hard by Lemma 6.1. Thus Holant\((\neq_2 | f, (0,1,0,0)^T)\) is \#P-hard. So we may assume that \((0,1,\frac{1}{dw},0)^T = (0,1,-1,0)^T \).

Suppose there exists \( u \in \{b, d, y, w\} \) such that \( u = \omega \) or \( \omega^2 \). By linking \((0,1,u,0)^T\) and \((0,1,-1,0)^T\) using \( \neq_2 \), we get the binary signature of \((0,1,-u,0)^T \). Note that \( (-u)^i \) are distinct for \( 1 \leq i \leq 6 \), therefore Holant\((\neq_2 | f, (0,1,-u,0)^T)\) is \#P-hard by Lemma 6.1. It follows that Holant\((\neq_2 | f, (0,1,0,0)^T)\) is \#P-hard.

Now we can assume \( \{b, y, d, w\} \subseteq \{1, -1, i, -i\} \). Then by \( \frac{1}{dw} = 1 \), we have \( z^4 = 1 \). Thus \( \{b, y, z, d, w\} \subseteq \{1, -1, i, -i\} \). There exist \( j, k, \ell, m, n \in \{0, 1, 2, 3\} \), such that

\[
b = i^j, \quad y = i^k, \quad z = i^\ell, \quad d = i^m, \quad w = i^n.
\]

The signature matrices of \( f \) and \( f_1 \) are respectively

\[
M(f) = \begin{bmatrix}
a & 0 & 0 & \bar{v} \\
0 & 1 & 0 & 0 \\
v & i^j & 0 & 0 \\
0 & 0 & 0 & a
\end{bmatrix}, \quad \text{and} \quad M(f_1) = \begin{bmatrix}
a & 0 & 0 & i^{j-m} \\
0 & 1 & 0 & 0 \\
i^{k-n} & 0 & 0 & a^{i-m-n}
\end{bmatrix}.
\]

We have \( \ell - m - n \equiv 2 \) (mod 4) by \( \frac{1}{dw} = i^\#m+n = -1 \).

Note that \( M_{x_1 x_4, x_2 x_3}(f) = \begin{bmatrix} a & 0 & 0 & v_m \\
0 & 1 & 0 & 0 \\
v & i^j & 0 & 0 \\
0 & 0 & 0 & a
\end{bmatrix} \). By doing a loop to \( f \) using \((0,1,i^{-k},0)^T\), we get the binary signature \((0,2,iv(1 + i^{\ell-j-k}),0)^T \). If \( \ell - j - k \equiv 1 \) (mod 2), then \( |iv(1 + i^{\ell-j-k})| = \sqrt{2} \) and Holant\((\neq_2 | f, (0,2,iv(1 + i^{\ell-j-k}),0)^T)\) is \#P-hard by Lemma 6.1. Thus Holant\((\neq_2 | f, (0,1,0,0)^T)\) is \#P-hard. Hence we have \( \ell - j - k \equiv 0 \) (mod 2). By \( \ell - m - n \equiv 2 \) (mod 4), we have

\[
j + k + m + n \equiv 0 \pmod{2}.
\]

By connecting two copies of \( f \), we get a signature \( f_2 \) whose signature matrix is

\[
M(f_2) = M(f)NM_{x_3 x_4, x_1 x_2}(f) = \begin{bmatrix}
a^{2i+j} & 0 & 0 & 0 \\
0 & 2^{2m} & i^{j-m+n} & 0 \\
0 & i^{j-m+n} & 2^{i+j} & 0 \\
a^{2i+j+k} & 0 & 0 & 2^{i+j+k}
\end{bmatrix}.
\]
Note that \( i^{\ell} + i^{m+n} = 0 \) by \( \ell - m - n \equiv 2 \pmod{4} \). If \( a^2 + i^{j+k} \neq 0 \), then \( f_2 \) is in Case I and \( \text{Holant}(\neq 2 | f) \) is \#P-hard. Thus \( \text{Holant}(\neq 2 | f) \) is \#P-hard. Otherwise, \( a^2 + i^{j+k} = 0 \). Then \( M(f) = \begin{bmatrix} i^{\ell+k} + \epsilon & 0 & 0 & i^{j} \\ 0 & i^{m} & 0 & i^{k} \\ 0 & 0 & i^{i^{m+n}} & 0 \\ i^{k} & 0 & 0 & i^{i^{m+n}} + \epsilon \end{bmatrix} \), where \( \epsilon = \pm 1 \). Let \( r = j + m \). We apply a holographic transformation defined by \( \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} \), where \( \gamma^2 = i^{j+k+r+\epsilon} \), then we get \( \text{Holant}(\neq 2 | f) \equiv_{p}^T \text{Holant}(\neq 2 | \hat{f}) \), where \( \hat{f} = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} \otimes^4 f \), whose signature matrix is \( M(\hat{f}) = i^{\ell+k+2r} + \epsilon \begin{bmatrix} 1 & 0 & 0 & i^{j+r} \\ 0 & i^{m+r} & 0 & 0 \\ 0 & i^{i^{m+n+r}} & 0 & 0 \\ i^{k+r} & 0 & 0 & -i^{j+k+2r} \end{bmatrix} \).

This function is an affine function; indeed let \( Q(x_1, x_2, x_3) = (k - n - r)x_1x_2 + (2j + 2)x_1x_3 + (2j)x_2x_3 + (n + r)x_1 + rx_2 + (j + r)x_3 \), then \( \hat{f}(x_1, x_2, x_3, x_4) = i^{\ell+k+2r} \cdot i^{Q(x_1,x_2,x_3)} \) on the support of \( \hat{f} \): \( x_1 + x_2 + x_3 + x_4 \equiv 0 \pmod{2} \). Moreover, \( k - n - r \equiv j + k + n + m \equiv 0 \pmod{2} \), all cross terms have even coefficients. Thus \( \hat{f} \in \mathcal{A} \).

References

[1] Ahlfors, L., Complex Analysis, 3 ed. (McGraw-Hill, 1979).

[2] Miriam Backens. A new Holant dichotomy inspired by quantum computation. https://arxiv.org/pdf/1702.00767.pdf.

[3] Andrei A. Bulatov: The Complexity of the Counting Constraint Satisfaction Problem. ICALP (1) 2008: 646-661.

[4] A. A. Bulatov and V. Dalmau, Towards a dichotomy theorem for the counting constraint satisfaction problem, Inform. and Comput., 205 (2007), pp. 651–678.

[5] A. Bulatov, M. Dyer, L. A. Goldberg, M. Jalsenius, M. Jerrum, and D. Richerby, The complexity of weighted and unweighted #CSP, J. Comput. System Sci., 78 (2012), pp. 681–688.

[6] Andrei A. Bulatov, Martin Grohe: The complexity of partition functions. Theor. Comput. Sci. 348(2-3): 148-186 (2005).

[7] Jin-Yi Cai, Xi Chen: Complexity of counting CSP with complex weights. STOC 2012: 909-920.

[8] Jin-Yi Cai, Xi Chen, Pinyan Lu: Graph Homomorphisms with Complex Values: A Dichotomy Theorem. SIAM J. Comput. 42(3): 924-1029 (2013).

[9] Jin-Yi Cai, Xi Chen, Pinyan Lu: Nonnegative Weighted #CSP: An Effective Complexity Dichotomy. SIAM J. Comput. 45(6): 2177-2198 (2016).
[10] Jin-Yi Cai, Xi Chen, Richard J. Lipton, and Pinyan Lu. On tractable exponential sums. In FAW, pages 148-159. Springer Berlin Heidelberg, 2010.

[11] Jin-Yi Cai, Zhiguo Fu: Holographic Algorithm with Matchgates Is Universal for Planar #CSP Over Boolean Domain. CoRR abs/1603.07046 (2016).

[12] Jin-Yi Cai, Zhiguo Fu, Heng Guo, Tyson Williams: A Holant Dichotomy: Is the FKT Algorithm Universal? FOCS 2015: 1259-1276.

[13] Jin-Yi Cai, Zhiguo Fu, Mingji Xia: Complexity Classification Of The Six-Vertex Model. https://arxiv.org/abs/1702.02863.

[14] Jin-Yi Cai, Heng Guo, Tyson Williams: A complete dichotomy rises from the capture of vanishing signatures: extended abstract. STOC 2013: 635-644.

[15] Jin-Yi Cai, Sangxia Huang, Pinyan Lu: From Holant to #CSP and Back: Dichotomy for Holant c Problems. Algorithmica 64(3): 511-533 (2012).

[16] Jin-Yi Cai, Pinyan Lu, Mingji Xia: Dichotomy for Holant* Problems with Domain Size 3. SODA 2013: 1278-1295

[17] Jin-Yi Cai, Pinyan Lu, Mingji Xia: The complexity of complex weighted Boolean #CSP. J. Comput. Syst. Sci. 80(1): 217-236 (2014).

[18] Jin-Yi Cai, Pinyan Lu, Mingji Xia: Dichotomy for Real Holantc Problems. https://arxiv.org/abs/1702.02963.

[19] Martin E. Dyer, Catherine S. Greenhill: The complexity of counting graph homomorphisms. Random Struct. Algorithms 17(3-4): 260-289 (2000).

[20] Martin E. Dyer, David Richerby: An Effective Dichotomy for the Counting Constraint Satisfaction Problem. SIAM J. Comput. 42(3): 1245-1274 (2013).

[21] Leslie Ann Goldberg, Martin Grohe, Mark Jerrum, Marc Thurley: A Complexity Dichotomy for Partition Functions with Mixed Signs. SIAM J. Comput. 39(7): 3336-3402 (2010).

[22] Sangxia Huang, Pinyan Lu: A Dichotomy for Real Weighted Holant Problems. Computational Complexity 25(1): 255-304 (2016).

[23] Michel Las Vergnas: On the evaluation at (3, 3) of the Tutte polynomial of a graph. J. Comb. Theory, Ser. B 45(3): 367-372 (1988).

[24] Jiabao Lin, Hanpin Wang: The Complexity of Holant Problems over Boolean Domain with Non-negative Weights. CoRR abs/1611.00975 (2016).

[25] Leslie G. Valiant: Holographic Algorithms. SIAM J. Comput. 37(5): 1565-1594 (2008).
7 A Sample of Problems

We illustrate the scope of Theorem 3.1 by several concrete problems.

**Problems**: \#EO on 4-Regular Graphs.

**Input**: A 4-regular graph $G$.

**Output**: The number of Eulerian orientations of $G$, i.e., the number of orientations of $G$ such that at every vertex the in-degree and out-degree are equal.

This problem can be expressed as Holant($\neq 2 | f$), where $f$ has the signature matrix $M(f) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Huang and Lu proved that this problem is \#P-complete [22]. Main Theorem confirms this fact.

**Problems**: $T(G;3,3)$.

**Input**: A graph $G$.

**Output**: The value of the Tutte polynomial $T(G; x, y)$ at $(3,3)$.

Let $G_m$ be the medial graph of $G$, then $G_m$ is a 4-regular graph. Las Vergnas proved the following theorem.

**Theorem 7.1.** [23] Let $G$ be a connected graph and $\mathcal{EO}(G_m)$ be the set of all Eulerian Orientations of the medial graph $G_m$ of $G$. Then

$$\sum_{O \in \mathcal{EO}(G_m)} 2^{\beta(O)} = 2T(G;3,3),$$

where $\beta(O)$ is the number of saddle vertices in the orientation $O$, i.e., vertices in which the edges are oriented "in, out, in, out" in cyclic order.

Note that $\sum_{O \in \mathcal{EO}(G_m)} 2^{\beta(O)}$ can be expressed as Holant($\neq 2 | f$), where $f$ has the signature matrix $M(f) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Main Theorem confirms that this problem is \#P-hard.

**Problems**: Holant($f$), where $f$ has the signature matrix $M(f) = \begin{bmatrix} 5 & i & i & -1 \\ -1 & -i & 3 & -i \\ -1 & -i & -i & 5 \end{bmatrix}$.

**Input**: An instance of Holant($f$).

**Output**: The evaluation of this instance.

By the holographic transformation $Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, we have

$$\text{Holant}(f) \equiv_T \text{Holant}(\neq 2 | \tilde{f}),$$

where $M(\tilde{f}) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. By Main Theorem, Holant($f$) can be computed in polynomial time. It can be shown that both $f$ and $\tilde{f}$ are neither in $\mathcal{P}$-transformable nor $\mathcal{A}$-transformable.
Connection between the eight-vertex model and 2,4-spin Ising.

Problems: 2,4-spin Ising model on lattice graph.

There is a well known correspondence between the eight-vertex model on the lattice graph on the one hand, and a 2,4-spin Ising model on the other hand, where the variables of the Ising model are the square faces of the lattice graph. Suppose we assign a spin $\sigma_{i,j}$ to the $(i,j)$-square face of an $N \times N$ lattice (with boundary wrapping around forming a torus). Each spin $\sigma_{i,j}$ takes values $\pm 1$. There are 5 local interactions between these spins.

In the following $J_h, J_v, J, J', J''$ are constants. Between $\sigma_{i,j}$ and $\sigma_{i+1,j}$, there is a (horizontal neighbor) “edge function” $f_h(\sigma, \sigma') = e^{-J_h \sigma \cdot \sigma'}$. Thus if $\sigma_{i,j} = \sigma_{i+1,j}$ then the output value is $e^{-J_h}$, otherwise, it is $e^{J_h}$. As a signature matrix, $M(f_h) = \begin{bmatrix} e^{-J_h} & e^{J_h} \\ e^{J_h} & e^{-J_h} \end{bmatrix}$.

Similarly between $\sigma_{i,j}$ and $\sigma_{i,j+1}$, there is a (vertical neighbor) “edge function”: $f_v(\sigma, \sigma') = \begin{bmatrix} e^{-J_v} & e^{J_v} \\ e^{J_v} & e^{-J_v} \end{bmatrix}$. Between $\sigma_{i+1,j}$ and $\sigma_{i,j+1}$, there is one diagonal “edge function” $f_d(\sigma, \sigma') = \begin{bmatrix} e^{-J} & e^{J} \\ e^{J} & e^{-J} \end{bmatrix}$; and between $\sigma_{i,j}$ and $\sigma_{i+1,j+1}$, there is another diagonal “edge function” $f_d'(\sigma, \sigma') = \begin{bmatrix} e^{-J'} & e^{J'} \\ e^{J'} & e^{-J'} \end{bmatrix}$. Finally there is a 4-ary function, $f_4(\sigma_{i,j}, \sigma_{i,j+1}, \sigma_{i+1,j}, \sigma_{i+1,j+1}) = e^{-J'' p}$, where $p$ is the product $\sigma_{i,j} \sigma_{i,j+1} \sigma_{i+1,j} \sigma_{i+1,j+1}$.

The partition function of this 2,4-spin Ising model is the sum over all spins $\sigma_{i,j} = \pm 1$ of the product

$$
\prod_{i,j} f_h(\sigma_{i,j}, \sigma_{i+1,j}) f_v(\sigma_{i,j}, \sigma_{i,j+1}) f_d(\sigma_{i,j+1, \sigma_{i+1,j}}) f_d'(\sigma_{i,j}, \sigma_{i+1,j}) f_4(\sigma_{i,j}, \sigma_{i,j+1}, \sigma_{i+1,j}, \sigma_{i+1,j+1}).
$$

It turns out that there is two-to-one exact correspondence between spin assignments on $\sigma_{i,j}$ and orientations for the eight-vertex model. The parameters are related as follows:

$$
\begin{align*}
\epsilon_1 &= -J_h - J_v - J - J' - J'' \\
\epsilon_2 &= +J_h + J_v - J - J' - J'' \\
\epsilon_3 &= -J_h + J_v + J + J' - J'' \\
\epsilon_4 &= +J_h - J_v + J + J' - J'' \\
\epsilon_5 &= \epsilon_6 = + J - J' + J'' \\
\epsilon_7 &= \epsilon_8 = - J + J' + J''
\end{align*}
$$
By our theorem for the eight-vertex model, this implies that, e.g., the 2,4-spin Ising model on the lattice graph is polynomial time computable if $(J_h, J_v, J, J', J'') = \pi_4(0, 2, -1, -1, 0)$. 