A RIGID HYPERFINITE TYPE II$_1$ FACTOR

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Abstract. We show that it is relatively consistent with ZFC that there exists a hyperfinite type II$_1$-factor of density character $\aleph_1$ which is not isomorphic to its opposite, does not have any outer automorphisms, and has trivial fundamental group.

The goal of this paper is to prove the following theorem.

Theorem 1. It is relatively consistent with ZFC that there exists a II$_1$ factor that is not isomorphic to its opposite, has no outer automorphisms, has trivial fundamental group, is hyperfinite, and is of density character $\aleph_1$. Also, the fundamental group of its ultrapower associated with any nonprincipal ultrafilter on $\mathbb{N}$ is equal to $(0, \infty)$.

Type II$_1$ and type III factors with separable preduals which are not isomorphic to their opposites were constructed by Connes in [Con75a, Con75c]. An example of a type II$_1$ factor with separable predual which has has no outer automorphisms and has trivial fundamental group was constructed in [IPP08]. Those examples of course are not hyperfinite. Theorem 1 provides a factor constructed as a transfinite inductive limit of copies of the hyperfinite II$_1$ factor which exhibits those properties.

Another curious property of the II$_1$ factor constructed in Theorem 1 is that the fundamental group of its ultrapower is strictly larger than the closure of its fundamental group. To the best of our knowledge, this is the first example of a II$_1$ factor with this property; it is not known whether a factor with separable predual can have this property.

We prove that the conclusion of Theorem 1 follows from Jensen’s $\diamondsuit_{\aleph_1}$. This axiom was first applied to operator algebras in [AW04] in order to construct a counterexample to Naimark’s problem. The Akemann–Weaver construction was subsequently adapted in [FH17] to construct a simple nuclear C$^*$-algebra which is not isomorphic to its opposite. Those techniques were further refined in [Vac18]. While the set theoretic machinery we use here is similar to the one used in those papers (albeit somewhat simplified), the operator algebraic techniques turn out to be very different in nature. The results in [AW04, FH17, Vac18] rely on studying the action of outer automorphisms and anti-automorphisms on the pure states of a separable C$^*$-algebra, and use in an essential way results due to Kishimoto ([Kis81]) and work of Kishimoto, Ozawa, and Sakai ([KOS03]) about the homogeneity of the pure state space of separable C$^*$-algebras. Beyond the fact that pure states of von Neumann algebras are generally not normal, the homogeneity result of Kishimoto–Ozawa–Sakai breaks down for non-separable C$^*$-algebras, and in

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particular for type II$_1$-factors ([KOS03, Remark 2.3]). Theorem 1 is the first application of \( \diamond_{\aleph_1} \) to von Neumann algebras, and it answers the question stated in [FH17, Remark 3.3]. Nonisomorphic hyperfinite II$_1$ factors with preduals of the same uncountable density character were first constructed in [Wid57]. In [FKL15, Theorem 1.3] it was proved that for every uncountable cardinal \( \kappa \) there are \( 2^\kappa \) nonisomorphic hyperfinite II$_1$ factors with predual of density character \( \kappa \). In spite of being nonisomorphic, all of these factors constructed in a similar manner and they are unlikely to have any of the properties of the factor constructed in Theorem 1.

We briefly outline the idea of the construction. We construct our factor \( M \) as a transfinite inductive limit of copies of the hyperfinite II$_1$ factor \( R \), indexed as \( R_\xi \) for countable ordinals \( \xi \). Suppose we want to make sure that \( M \) does not have any outer automorphisms. (The idea for eliminating anti automorphisms and isomorphisms into corners is similar.) For any automorphisms of \( M \) there exist “many” ordinals \( \xi \) such that the automorphism leaves \( R_\xi \) invariant and the restriction is outer. Our inductive step does this: given an outer automorphism \( \beta \) of \( R_\xi \) for some \( \xi \), we find a way to embed \( R_\xi \) into a larger copy of \( R \), denoted \( R_{\xi+1} \), which has the property that \( \beta \) cannot extend not only to \( R_{\xi+1} \), but in fact to any larger hyperfinite II$_1$ factor containing \( R_{\xi+1} \). This allows us at each step to kill off a possible restriction of an automorphism of the yet-to-be constructed inductive limit. To make sure that we eliminate all possible outer automorphisms of \( M \), we need a prediction device which should tell us which automorphism to handle at each stage; for that we use Jensen’s \( \diamond_{\aleph_1} \) axiom, which is known to be relatively consistent with ZFC.

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1. THE OBSTRUCTIONS

In this section we describe the device used to create obstructions to extending outer automorphisms, anti automorphisms, and isomorphisms onto a corner associated to a projection of trace not equal to 1 of subfactors of the II$_1$ factor. This is used in the proof of Theorem 1.

For a type II$_1$ factor \( M \), we denote the set of all anti automorphisms of \( M \) by \( \text{Ant}(M) \). Note that \( \text{Ant}(M) \cup \text{Aut}(M) \) is a group.

Let \( G \) be a group, and let \( g, h \in G \). Let \( a \) and \( b \) be the standard generators of \( F_2 = \mathbb{Z} \ast \mathbb{Z} \). By \( \pi_{g, h} : \mathbb{Z} \ast \mathbb{Z} \to G \) we denote the canonical homomorphism which satisfies \( \pi_{g, h}(a) = g \) and \( \pi_{g, h}(b) = h \). Notice that if \( \alpha \in \text{Aut}(M) \) and \( \beta \in \text{Ant}(M) \cup \text{Aut}(M) \) then for any \( w \in \langle a, b^{-1}ab \rangle \), we have \( \pi_{\alpha, \beta}(w) \in \text{Aut}(M) \). As usual, by \( \text{Inn}(M) \) we denote the group of inner automorphisms of \( M \), and \( \text{Out}(M) = \text{Aut}(M)/\text{Inn}(M) \). The group \( \text{Aut}(M) \) is a topological group when endowed with the point-\( || \cdot ||_2 \)-topology, that is, the topology which is generated by open sets of the form:

\[
O_{F, \varepsilon, \alpha} = \{ \varphi \in \text{Aut}(M) \mid \forall a \in F, ||\varphi(a) - \alpha(a)||_2 + ||\varphi^{-1}(a) - \alpha^{-1}(a)||_2 < \varepsilon \}
\]

for \( \alpha \in \text{Aut}(M) \), a finite set of contractions \( F \subset M \), and \( \varepsilon > 0 \). If \( M \) has separable predual then \( \text{Aut}(M) \) is a Polish group with this topology.
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(see e.g., [AP17] §7.5.2, where it was observed that Aut(M) is isomorphic to a closed subgroup of the unitary group of $L^2(M,\tau)$ equipped with the strong operator topology). When M is the hyperfinite II₁ factor R, it follows from the classification of automorphisms of R from [Con75a] that the inner automorphisms are dense in Aut(M).

Our first goal is to prove Theorem 5. We need a few lemmas; the first of which, for the case of automorphisms, is an immediate application of the Connes’ Rokhlin-type theorem. Fix a free ultrafilter V. By $R^V$ we denote the tracial ultrapower, $l^\infty(N,R)/\{f \in l^\infty(N,R) \mid \lim_{n \to V} \|f(n)\|_2 = 0\}$. If $\alpha$ is an automorphism of $R$, by abuse of notation, we use $\alpha$ to denote both the induced automorphism of $R^V$ and of the central sequence algebra $R^V \cap R'$.

**Lemma 2.** Suppose $\beta$ is either an outer automorphism or an anti-automorphism of $R$. Then there exist orthogonal projections $p_0, p_1 \in R^V \cap R'$ such that $\tau(p_0) = \tau(p_1) \geq 1/3$ and $\beta(p_0) = p_1$.

**Proof.** If $\beta$ is an outer automorphism and has infinite order in Out(R), this follows from the Connes’ Rokhlin-type theorem, [Tak03] Chapter XVII, Lemma 2.3], where we pick $\beta$ from $\text{Out}(R)$, then by [Tak03] Chapter XVII, Theorem 2.10], the automorphism $\beta$ is cocycle conjugate to an automorphism of the form $\beta \otimes \sigma_p$, where $p$ is the period of $\beta$ in Out(R) and $\sigma_p$ is an infinite tensor product action on $\bigotimes_n M_p$ of cyclic permutations. This has a central sequence of projections which are permuted cyclically. Therefore, there exist projections $q_0, q_1, \ldots, q_{p-1} \in R^V \cap R'$ such that $\beta(q_j) = q_{j+1} \mod p$ for all $j$. If $p$ is even, set $p_0 = q_0 + q_2 + \ldots + q_{p-2}$, and if $p$ is odd then set $p_0 = q_0 + q_2 + \ldots + q_{p-3}$, and set $p_1 = \beta(p_0)$. If $p$ is even then $\tau(p_0) = \tau(p_1) = 1/2$, and if $p$ is odd then $\tau(p_0) = \tau(p_1) = 1/2 - 1/2p \geq 1/3$, as required.

If $\beta$ is an anti-automorphism, by [Gio83, Lemma 2.1], up to conjugation by an automorphism, for any $n \in \mathbb{N}$ there exists a unital copy of $M_n \subset R^V \cap R'$ such that $\alpha|_{M_n}$ is given by the transpose map, that is, $\alpha|_{M_n}(e_{jk}) = e_{kj}$ for the standard matrix units $\{e_{jk}\}_{k,j=1,2,\ldots,n}$. Set $n = 2$, then the projections

$$p_0 = \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}$$

satisfy the requirements. \qed

**Lemma 3.** Let $\beta$ be an outer automorphism or an anti-automorphism of R. Let $w \in \langle a, b^{-1}ab \rangle$ be a nontrivial element. Then there exist a unitary $u \in U(R^V \cap R')$ and a projection $p \in R^V \cap R'$ with $\tau(p) \geq 1/3$ such that $\pi_{\text{Ad}(u)\beta}(w)(p) \perp p$.

**Proof.** The canonical homomorphism $\mathbb{Z} \to \mathbb{Z}_2$ gives rise to a homomorphism $\varphi: \langle a, b \rangle \to \mathbb{Z} \ast \mathbb{Z}_2$, such that $\varphi|_{\langle a, b^{-1}ab \rangle}$ is injective. Since $\mathbb{Z} \ast \mathbb{Z}_2$ is residually finite, we can pick a homomorphism $\psi: \mathbb{Z} \ast \mathbb{Z}_2 \to \text{Sym}(S)$ into a symmetry group of a finite set $S$, such that any nontrivial element in the image has no fixed points and such that $\psi(\varphi(w)) \neq 1$. Set $\sigma_a = \psi \circ \varphi(a)$, $\sigma_b = \psi \circ \varphi(b)$ and $\sigma_w = \psi \circ \varphi(w)$. Since $\sigma_b$ is of order 2 with no fixed points, we can decompose $S$ into a disjoint union $S_0 \sqcup S_1$ where $\sigma_b|_{S_0}: S_0 \to S_1$ is a bijection. Now, let $p_0, p_1$ be projections as in Lemma 2. Since $R^V \cap R'$ is a II₁ factor (see [Tak03] Chapter XIV, Theorem 4.18]), we can decompose $p_0$ into a direct
Lemma 4. Let $\beta$ be an outer automorphism or an antiautomorphism of $R$. Let $a, b < a, b$ be a nontrivial element. Then for any $\delta > 0$, for any finite set of contractions $F$, a given $\varepsilon > 0$, and a given $v \in U(R)$, the open set $O$ is of the form

$$O_{F, \varepsilon, v} = \{ \varphi \in \text{Aut}(R) \mid \langle \varphi a - \varphi - (\varphi - 1) \rangle \parallel \leq \varepsilon \}$$

By Lemma 3 there are a unitary $u$ and a projection $p$ in $V'$ such that $\tau(p) \geq 1/3$ and $\pi_{\varphi_1}(w)(p) \perp p$. Lift $u$ to a sequence $(u_1, u_2, \ldots)$ in $U(l^\infty(R))$, and lift $p$ to a sequence of projections $(p_1, p_2, \ldots)$ in $l^\infty(R)$. Set $\gamma_n = \text{Ad}(u_n) \circ \text{Ad}(v)$. Notice that automorphism $\gamma = (\gamma_1, \gamma_2, \ldots)$ of $l^\infty(R)$ descends to the automorphism $\text{Ad}(u)$ of $V'$ such that for any $z \in U$ we have $\parallel z p_n z^* - p_n \parallel_2 \to 0$ and $\lim_{n \to V} \pi_{\varphi_1}(w)(p_n) \cdot p_n \parallel_2 = 0$. Therefore

$$\lim_{n \to V} \parallel \pi_{\varphi_1}(w)(p_n) - z p_n z^* \parallel_2^2 = 2 \tau(p) \geq 2/3.$$ 

Furthermore, since $\lim_{n \to V} ||u_n, x||_2 = \lim_{n \to V} ||u_n^*, x||_2 = 0$ for any $x \in F \cup \{ v \}$, we have

$$\lim_{n \to V} \parallel \gamma_n - \text{Ad}(v)(x) \parallel_2 = \lim_{n \to V} \parallel \gamma_n^{-1}(x) - \text{Ad}(v^*)(x) \parallel_2 = 0$$

for all $x \in F$; in particular, the set of $n$ such that $\gamma_n \in O_{F, \varepsilon, v}$ belongs to $V$. Thus, we can pick an index $n$ such that for all $z \in U$ we have $\parallel \pi_{\varphi_1}(w)(p_n) - z p_n z^* \parallel_2 \geq 2/3 - \delta$ and $\gamma_n \in O$; we now set $p = p_n$, $\alpha' = \gamma_n$, and we are done.

The following result is similar in spirit to the freeness results in [IPP08, Lemma A.2] (for outer automorphisms) and [IPP08, Remark A.3 (2)] (for antiautomorphisms), but it does not obviously follow from them (and they don’t follow from our result).

Theorem 5. Let $\beta$ be an outer automorphism or an antiautomorphism of the hyperfinite II$_1$ factor. Then there exists $\alpha \in \text{Aut}(R)$ such that the images of the automorphisms $\alpha$ and $\beta^{-1} \circ \alpha \circ \beta$ in $\text{Out}(R)$ generate a free group.

Proof. Let $U = \{ u_1, u_2, \ldots \}$ be a dense sequence of unitaries in $U(R)$ (in the strong operator topology). Note that $U$ spans a SOT-dense subset of $R$. Let $U_\varepsilon = \{ u_1, u_2, \ldots, u_\varepsilon \}$. Let $w \in \langle a, b^{-1} ab \rangle \setminus \{ 1 \}$. Denote by $P(R)$ the set of all projections in $R$. Define:

$$O(w, n) = \{ \gamma \in \text{Aut}(R) \mid (\exists \psi \in P(R))(\forall z \in U_n) \parallel \pi_{\gamma, \beta}(w)(p) - z p z^* \parallel_2^2 > 1/2 \}$$
By Lemma 11, the set $O(w, n)$ is dense. It is also clearly open. As $\langle a, b^{-1}ab \rangle$ is countable, by the Baire Category Theorem, the set

$$G_0 = \bigcap_{w \in \langle a, b^{-1}ab \rangle \setminus \{1\}} \bigcap_{n=1}^{\infty} O(w, n)$$

is dense. Pick any $\alpha \in G_0$. We claim that $\pi_{\alpha, \beta}(w)$ is not inner for any $w \in \langle a, b^{-1}ab \rangle \setminus \{1\}$. Indeed, if there exists such a word $w$ and a unitary $u \in U(R)$ which implements $\pi_{\alpha, \beta}(w)$, then we can pick $u_n \in U$ such that $\|u_n - u\|_2 < 1/4$ and a projection $p$ such that $\|\pi_{\alpha, \beta}(w)(p) - u_n p u_n^*\|^2_2 > 1/2$. Therefore $\|upu^* - u_n p u_n^*\|_2 < 1/2$ and

$$\|\pi_{\alpha, \beta}(w)(p) - upu^*\|_2 > 1/\sqrt{2} - 1/2 > 0,$$

so finally $\pi_{\alpha, \beta} \neq \text{Ad}(u)$. □

The following Proposition is based on [Was76, Lemma on p. 245].

**Proposition 6.** Let $M$ be a von Neumann algebra with a faithful trace $\tau$. Suppose $u, v \in U(M)$ are such that for every $w \in \langle a, b \rangle \setminus \{1\}$ we have $\tau(\pi_{u, v}(w)) = 0$. Then $C^*(u, v) \cong C^*_r(F_2)$.

**Proof.** We view $M$ as represented on $L^2(M, \tau)$ via the standard representation, that is, the GNS representation associated to $\tau$. Let $\xi \in L^2(M, \tau)$ be the GNS vector, so that $\langle \xi, \xi \rangle = \tau(1)$ for all $a \in M$. For any word $w$ in the free group on two generators, set $\xi_w = \pi_{u, v}(w)\xi$. If $w_1 \neq w_2$, then

$$\langle \xi_{w_1}, \xi_{w_2} \rangle = \langle \pi_{u, v}(w_1)\xi, \pi_{u, v}(w_2)\xi \rangle = \langle \pi_{u, v}(w_2)\pi_{u, v}(w_1)\xi, \xi \rangle = \langle \pi_{u, v}(w_2^{-1}w_1)\xi, \xi \rangle = \tau(\pi_{u, v}(w_2^{-1}w_1)) = 0.$$

Thus, $H = \overline{\text{span}}\{\xi_w \mid w \in F_2\} \cong l^2(F_2)$ is invariant for $C^*(u, v)$, and the action of the group generated by $u, v$ on $H$ is unitarily equivalent to the left regular representation. Let $P_H$ be the projection onto $H$, then the map $\varphi: C^*(u, v) \to C^*(u, v)P_H$ is a quotient. If $x \in \ker(\varphi)$ then $x P_H = 0$, and in particular $x\xi = 0$. Since $\xi$ is a separating vector, it follows that $x = 0$. Therefore, $\varphi$ is in fact an isomorphism. So, for any *-polynomial $p$ in $u, v$, we have $\|p(u, v)\| = \|p(u, v)P_H\| = \|p(u, v)\|_{C^*_r(F_2)}$, so $C^*(u, v) \cong C^*_r(F_2)$, as required. □

We are now ready to prove the first of the two main results of this section, used as steps in the proof of Theorem 7.

**Theorem 7.** Suppose $\beta$ is either an outer automorphism or an antiautomorphism of $R$. Then there exists a unital embedding of $R \subset R_1$ in another copy of the hyperfinite $\text{II}_1$-factor such that for any inclusion of $R_1$ as a subfactor of a larger hyperfinite $\text{II}_1$ factor $R_2$, $\beta$ cannot be extended to an automorphism or an antiautomorphism of $R_2$.

**Proof.** We pick $\alpha \in \text{Aut}(R)$ as in Theorem 5 so that the images of $\alpha$ and $\beta^{-1} \circ \alpha \circ \beta$ in $\text{Out}(R)$ generate a free group. Since $\alpha$ and all of its nonzero powers are outer, $R_1 = R \rtimes_{\alpha} \mathbb{Z}$ is itself isomorphic to $R$ (being an injective $\text{II}_1$-factor itself). We consider the standard embedding $R \subset R \rtimes_{\alpha} \mathbb{Z}$. Suppose $R \rtimes_{\alpha} \mathbb{Z} \subset R_2$ is a normal unital embedding into another copy of the hyperfinite $\text{II}_1$-factor. We show that $\beta$ cannot extend to $R_2$. Suppose, for
contradiction, that there exists $\tilde{\beta} \in \text{Aut}(R_2) \cup \text{Aut}(R_2)$ such that $\tilde{\beta}|_R = \beta$.

Let $u$ be the canonical unitary in $R \rtimes_{\alpha} \mathbb{Z}$. Let $v = \tilde{\beta}^{-1}(u)$ if $\tilde{\beta}$ is an automorphism, and $v = \tilde{\beta}^{-1}(u^*)$ if $\tilde{\beta}$ is an antiautomorphism.

We claim that for every $x \in R$ we have $vxv^* = \beta^{-1} \circ \alpha \circ \beta(x)$. To see this, note that if $\tilde{\beta}$ is an antiautomorphism then

$$vxv^* = \tilde{\beta}^{-1}(u^*)\tilde{\beta}^{-1}(\beta(x))\tilde{\beta}^{-1}(u) = \tilde{\beta}^{-1}(u\beta(x)u^*) = \beta^{-1}(\alpha(\beta(x)))$$

and if $\tilde{\beta}$ is an automorphism then

$$vxv^* = \tilde{\beta}^{-1}(u)\beta^{-1}(\beta(x))\tilde{\beta}^{-1}(u^*) = \tilde{\beta}^{-1}(u\beta(x)u^*) = \beta^{-1}(\alpha(\beta(x)))$$

Let $w \in F_2$ be a nontrivial word. Let $y = \pi_{\alpha,v}(w)$ and $\psi = \pi_{\alpha,\beta^{-1} \alpha \beta}(w)$ in $\text{Aut}(R)$. We denote the trace on $R_2$ by $\tilde{\tau}$. We claim that $\tilde{\tau}(y) = 0$.

Note that $\text{Ad}(y)$ leaves $R$ invariant, and for any $x \in R$, we have $\text{Ad}(y)(x) = \psi(x)$. Since $\psi$ and all of its nonzero powers are outer, using the Connes’ Rokhlin-type theorem, [Rak03, Chapter XVII, Lemma 2.3], where we pick $n = 2$, for any $\varepsilon > 0$ there exist orthogonal projections $p_0, p_1 \in R$ such that $p_0 + p_1 = 1$, $\|p_0\psi(p_0)\|_2 < \varepsilon$, and $\|p_1\psi(p_1)\|_2 < \varepsilon$. Thus,

$$\tilde{\tau}(y) = \tilde{\tau}(yp_0 \cdot p_0) + \tilde{\tau}(yp_1 \cdot p_1) = \tilde{\tau}(\psi(p_0)yp_0) + \tilde{\tau}(\psi(p_1)yp_1)$$

Now, $|\tilde{\tau}(\psi(p_0)yp_0)| = |\tilde{\tau}(\psi(p_0)p_0)| \leq \|y\|\|p_0\psi(p_0)\|_2 < \varepsilon$ and likewise $|\tilde{\tau}(\psi(p_1)yp_1)| < \varepsilon$. Since $\varepsilon$ was arbitrary, we have $\tilde{\tau}(y) = 0$.

We have shown that any nontrivial word $w$ satisfies $\tilde{\tau}(\pi_{\alpha,v}(w)) = 0$. Therefore, Proposition [2] implies that $C^*(u, v) \cong C^*_\tau(F_2)$. However, by [Bro00, Corollary 4.2.4], the $C^*$-algebra $C^*_\tau(F_2)$ does not embed into any finite, hyperfinite von Neumann algebra, which is a contradiction. □

We move on to the fundamental group. For a II$_1$ factor $M$, $n \geq 1$, and a projection $p \in M_n(M)$, the isomorphism type of $pM_n(M)p$ depends only on the trace of $p$, because in a II$_1$ factor projections of the same trace are unitarily equivalent. A representative of this isomorphism type is usually denoted $M^t$, where $t = \tau(p)$. The fundamental group $F(M)$ of $M$ is defined as $\{ t \mid M^t \cong M \}$ (see e.g., [AP17, §4.2]).

A small modification of Theorem [8] can be used for trace-scaling isomorphisms, as follows.

**Theorem 8.** Let $p \in M_n(R)$ be a projection of trace $t \neq 1$. Suppose $\beta : R \rightarrow pM_n(R)p$ be an isomorphism. Then there exists a unital embedding of $R < R_1$ in another copy of the hyperfinite II$_1$-factor such that for any inclusion of $R_1$ as a subfactor of a larger hyperfinite II$_1$ factor $R_2$, $\beta$ cannot be extended to an isomorphism $\tilde{\beta} : R_2 \rightarrow pM_n(R_2)p$.

**Proof.** Any such isomorphism $\beta$ arises from a trace-scaling automorphism $\gamma$ of $\mathcal{R} \mathcal{B}(l^2(\mathbb{N}))$, restricted to $R \cong R \otimes q$, where $q$ is a minimal projection in $\mathcal{B}(l^2(\mathbb{N}))$ (and corestricted to the image); here $\gamma(q)$ is a projection of trace $t$. By [Con75b, Corollary 6], any two such automorphisms are conjugate. In particular, if we identify $R \cong R \otimes R$, we can assume that $\gamma$ is of the form $\delta \otimes \gamma' : \mathcal{R} \mathcal{R} \mathcal{B}(l^2(\mathbb{N})) \rightarrow \mathcal{R} \mathcal{R} \mathcal{B}(l^2(\mathbb{N}))$, where $\delta \in \text{Aut}(R)$ is an outer automorphism and $\gamma' : \mathcal{R} \mathcal{B}(l^2(\mathbb{N})) \rightarrow \mathcal{R} \mathcal{B}(l^2(\mathbb{N}))$ is a trace-scaling automorphism. Let $\tilde{p} = \gamma'(1 \otimes q)$, so that $p = 1 \otimes \tilde{p} = \gamma(1 \otimes 1 \otimes q)$. Following
the same argument as in the proof of Theorem \[7\] we find \( \alpha \in \text{Aut}(R) \) such that \(<\alpha, \delta^{-1} \circ \alpha \circ \delta \rangle \cong F_2 \). Now, let \( R_1 = (R \otimes R) \times_{\alpha \otimes id} Z \cong (R \rtimes \alpha Z) \otimes R \).

Suppose \( R_2 \supset R_1 \) and \( \beta \) extends to an isomorphism \( \tilde{\beta}: R_2 \to pR_2 \otimes B(l^2(\mathbb{N}))p \).

Let \( u \) be the canonical unitary in the crossed product \( R \rtimes \alpha Z \) and set \( v = \tilde{\beta}^{-1}(u \otimes \tilde{p}) \). A computation similar to that in the proof of Theorem \[7\] shows that for any \( x \in R \), we have

\[
v(x \otimes 1)v^* = \delta^{-1} \circ \alpha \circ \delta(x) \otimes 1 \in R \otimes R \subset R_1 \subset R_2.
\]

The same considerations now show that \( C^*(u \otimes 1, v) \cong C^*_v(F_2) \), which is a contradiction. \( \Box \)

In the next section we describe the recursive construction of the \( \mathcal{II}_1 \) \( M \) as in the conclusion of Theorem \[1\] using the obstructions to extending \( \beta \) provided by Theorem \[7\] and Theorem \[8\]. Notably, these obstructions are ‘ir-reversible’ in the sense that \( \beta \) cannot be extended to any further hyperfinite extension. This should be contrasted to the ‘fleeting’ obstructions used in \[AW04, FH17\], and \[Vac18\] where at each step of the construction one had to take care of all objects captured in the earlier stages of the construction.

2. The Construction

Following von Neumann, an ordinal is identified with the set of all smaller ordinals. By \( \aleph_1 \) we denote the first uncountable ordinal, identified with the first uncountable cardinal.

As in \[FH17\] and \[Far19\] \( \S 11.2 \), our construction will utilize codes for metric structures\( \text{, but the coding used here is somewhat simplified. Suppose } d \text{ is a metric on an ordinal } \theta \text{ of diameter } 2 \text{ and } A \text{ is its metric completion. Let }

\[
\text{Code}_d(A) = \{(\xi, \eta, q) \in \theta^2 \times Q_+ \mid d(\xi, \eta) > q \}.
\]

Since \( \text{Code}_d(A) \) uniquely determines the metric \( d \) on \( \xi \) and \( A \) is isometric to the metric completion of this space, we consider \( \text{Code}_d(A) \) as a code for the metric space \( (A, d) \) (we will routinely omit \( d \), when clear from the context). The set \( \mathcal{X}(\theta) \) of such codes is included in the power set of \( \theta^2 \times Q_+ \). For every \( A \) coded in \( \mathcal{X}(\theta) \), every 1-Lipschitz function \( F: A^2 \to [0, 2] \) is coded by

\[
\text{Code}_F(A) = \{(\xi, \eta, q) \in \theta^2 \times Q_+ \mid F(\xi, \eta) > q \}.
\]

Hence the pair \( (A, F) \) is coded by a subset of \( \theta^2 \times Q_+ \cup \theta^2 \times Q_+ \). Let \( \mathcal{X}(\theta) \) denote the set of all such codes.

Lemma 9. The sets of codes \( \mathcal{X}(\xi) \) and \( \mathcal{X}_R(\xi) \) satisfy the following for every infinite \( \xi \).

(1) There is a natural reduction from \( \mathcal{X}_R(\xi) \) onto \( \mathcal{X}(\xi) \), so that the reduct of a code for \( (A, F) \) is a code for \( A \) (with the same enumeration of the distinguished dense set).

(2) If \( A \) is coded by \( \mathfrak{A} \in \mathcal{X}(\xi) \) and \( F: A^2 \to [0, 2] \) is 1-Lipschitz, then \( \mathfrak{A} \) has a unique expansion \( \mathfrak{A}(F) \) in \( \mathcal{X}_R(\xi) \) that codes \( (A, F) \) (so that the reduct of \( \mathfrak{A}(F) \) as in \( \{1\} \) is \( \mathfrak{A} \)).

\( ^1 \)This is a technical term, following \[BYBHU08\].
If $\xi < \eta$, $\mathfrak{A} \in \mathcal{X}(\eta)$, and $\mathfrak{A}' \in \mathcal{X}_R(\eta)$, then there are unique $\mathfrak{A} \upharpoonright \xi \in \mathcal{X}(\xi)$ and $\mathfrak{A}' \upharpoonright \xi \in \mathcal{X}_R(\xi)$ with the following properties.

(3) If $A$ is coded by $\mathfrak{A} \in \mathcal{X}(\xi)$, $\xi$ is countable, $A$ is a subspace of a separable metric space $B$ of diameter 2, then there is $\mathfrak{B} \in \mathcal{X}(\xi + \omega)$ such that $\mathfrak{B} \upharpoonright \xi = \mathfrak{A}$ and $\mathfrak{B}$ codes $B$.

(4) If $S$ is set of ordinals and $\mathfrak{A}_{\xi} \in \mathcal{X}(\xi)$, for $\xi \in S$, are such that $\mathfrak{A}_{\eta} \upharpoonright \xi = \mathfrak{A}_{\xi}$ for all $\xi < \eta$ in $S$, then there is a unique $\mathfrak{A} \in \mathcal{X} \upharpoonright \sup S$ such that $\mathfrak{A} \upharpoonright \xi = \mathfrak{A}_{\xi}$ for all $\xi \in S$.

(5) Statements analogous to (3) and (4) hold when $\mathcal{X}$ is replaced with $\mathcal{X}_R$.

**Proof.** The spaces $\mathcal{X}(\xi)$ and $\mathcal{X}_R(\xi)$ are instances of $\text{Struct}(\mathcal{L}, \xi)$ for metric structures with a distinguished dense set indexed by $\xi$ as introduced in [Far19, §7.1.2], where $\mathcal{L}$ is the single-sorted language with a single binary predicate symbol for $R$ whose modulus of uniform continuity is the identity function.

To see that (3) holds, note that since $A$ is separable and the interval $[\xi, \xi + \omega)$ is infinite, one can extend the given enumeration of a dense subset of $A$ by $\xi$ to an enumeration of a dense subset of $B$ by $\xi + \omega$. The proofs of the remaining clauses are even more straightforward.

The unit ball $N_1$ of a $\Pi_1$ factor $N$ with a separable predual with respect to a trace metric is a complete separable metric space of diameter 2, and if $\beta : N \to N$ is an automorphism, antiautomorphism, or an isomorphism onto a corner, then $\beta$ can be coded by the distance function to its graph, denoted $F_\beta : (N_1)^2 \to [0, 2]$ and defined by

$$(2.1) \quad F_\beta(a, b) = \inf_{x \in N_1} \max(||x - a||_2, ||\beta(x) - b||_2).$$

Clearly, $F_\beta$ is 1-Lipschitz.

The following standard definitions can be found in [Kun11, §III.6] or in [Far19, §6.2]. A subset $C$ of $N_1$ is called closed and unbounded (club for short) if $C \setminus \xi$ is nonempty for every $\xi < N_1$ and for every countable $X \subset C$ we have $\sup(X) \in C$. A subset $S$ of $N_1$ is stationary if it intersects every club nontrivially. We will not need the exact statement of Jensen’s $\diamondsuit_{N_1}$; it can be found e.g., in [Kun11, §III.7.1] or [Far19, §8.3.1].

**Proposition 10.** Jensen’s $\diamondsuit_{N_1}$ implies the following.

There exist $\mathfrak{S}_{\xi} \in \mathcal{X}_R(\xi)$ for $\xi < N_1$ such that for every $\mathfrak{A} \in \mathcal{X}_R(N_1)$ the set $\{\xi < N_1 \mid \mathfrak{A} \upharpoonright \xi = \mathfrak{S}_{\xi}\}$ is stationary.

In particular, this statement is relatively consistent with ZFC.

**Proof.** This is a consequence of a special case of [Far19, Proposition 8.3.8] and the relative consistency of $\diamondsuit_{N_1}$ with ZFC ([Kun11, §III.7.13]).

We are now ready to prove Theorem I.

**Proof of Theorem I** We use $\diamondsuit_{N_1}$ to construct a $\Pi_1$ factor that is not isomorphic to its opposite, has no outer automorphisms, has trivial fundamental

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$^2$By $\omega$ we denote the least limit ordinal, and therefore $\xi + \omega$ is the least limit ordinal greater than $\xi$. 
group, is hyperfinite, is of density character $\aleph_1$, and such that the fundamental group of its ultrapower associated with any nonprincipal ultrafilter on $\mathbb{N}$ is equal to $(0, \infty)$.

Fix $S_\xi$, for $\xi < \aleph_1$ as guaranteed by Proposition 10. We will recursively build hyperfinite $\Pi_1$ factors with separable predual $R_\xi$, for an infinite ordinal $\xi < \aleph_1$, and codes $\mathfrak{A}_\xi \in \mathcal{X}(\xi)$, for limit $\xi \leq \aleph_1$, with the following properties ($F_\beta$ is as defined in (2.1))

1. If $\xi < \eta$ then $R_\xi$ is a subfactor of $R_\eta$ and for a limit ordinal $\eta$ we have $R_\eta = \lim_{\xi < \eta} R_\xi$.
2. If $\eta < \aleph_1$ is a limit ordinal, then a distinguished dense subset of the unit ball $(R_\eta)_1$ of $R_\eta$ in the trace metric is enumerated by $\eta$ and $\mathfrak{A}_\eta \in \mathcal{X}(\eta)$ is the corresponding code for $(R_\eta)_1$.
3. For limit ordinals $\xi < \eta$ we have $\mathfrak{A}_\eta \upharpoonright \xi = \mathfrak{A}_\xi$.
4. If $S_\xi = \mathfrak{A}_\xi(F_\beta)$ for some $\beta$ which is an antiautomorphism, an outer automorphism, or an isomorphism of $R_\xi$ onto a corner $pR_\xi p$ for some projection with $\tau(p) < 1$, then $R_{\xi+1}$ is $R_1$ as guaranteed by Theorem 7 or Theorem 8.

Starting from $R_\omega = R_1^3$, the recursive construction proceeds as follows. Suppose that $\eta$ is the minimal ordinal such that $R_\eta$ hasn’t been defined yet.

If $\eta$ is a limit ordinal, let $R_\eta = \lim_{\xi < \eta} R_\xi$. If $\eta$ is also a limit of limits, then $\mathfrak{A}_\xi$ is defined for a cofinal set of $\xi < \eta$ and we let $\mathfrak{A}_\eta = \lim_{\xi < \eta} \mathfrak{A}_\xi$, as guaranteed by Lemma 9 (1). Otherwise, let $\xi$ be the largest limit ordinal below $\eta$. Then $\eta = \xi + \omega$ and we let $\mathfrak{A}_\eta$ be a code for $R_\eta$ that extends $\mathfrak{A}_\xi$ as guaranteed by Lemma 9 (3).

Otherwise, $\eta$ is a successor ordinal. Fix $\xi$ such that $\eta = \xi + 1$.

Consider the case when $\mathfrak{A}_\xi$ is defined and there is $\beta$ which is an antiautomorphism of $R_\xi$, an outer automorphism of $R_\xi$, or an isomorphism of $R_\xi$ onto a corner $pR_\xi p$ for some projection $p$ of trace $< 1$ and $S_\xi = \mathfrak{A}_\xi(F_\beta)$ (as in Lemma 9 (2)). Then let $R_\eta$ be as guaranteed by Theorem 7 or Theorem 8 so that $\beta$ does not extend to any hyperfinite extension of $R_\eta$. This assures the requirement (4) of the construction.

In the case when $\mathfrak{A}_\xi$ is not defined, or $\mathfrak{A}_\xi$ is defined but $S_\xi$ does not code a structure $\mathfrak{A}_\xi(F_\beta)$ as in the previous case, let $R_\eta = R_\xi$.

This describes the recursive construction. Let $M = \lim_{\xi < \aleph_1} R_\xi$ and let $\mathfrak{A} = \lim \mathfrak{A}_\eta$, as guaranteed by (2) of Lemma 9. Then $M$ is hyperfinite, as an inductive limit of hyperfinite $\Pi_1$ factors and its predual has density character $\aleph_1$.

Suppose $\beta$ is an outer automorphism of $M$.

We first claim that the set $C_0 = \{ \xi < \aleph_1 \mid \beta \upharpoonright R_\xi \in \text{Aut}(R_\xi) \}$ is a club. To see that, define a non-decreasing function $f : \aleph_1 \to \aleph_1$ by

$$f(\xi) = \min\{ \eta < \aleph_1 \mid \beta[R_\xi] \cup \beta^{-1}[R_\xi] \subseteq R_\eta \}.$$  

The function $f$ is well-defined since each $R_\xi$ is separable in the $\| \cdot \|_2$-norm. The set $C_0$ is the set of fixed points of $f$, and therefore is a club. (That the

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3 As is standard in Set Theory, the letter $\omega$ denotes the first infinite countable ordinal; please note that in this paper $\omega$ does not stand for an ultrafilter and that $R_\omega$ does not stand for the central sequence algebra $R^* \cap R^\prime$.}
set of fixed points of such functions is a club is well-known; see for example [Far19, Example 6.2.8(1)].

Next, we claim that the set $C_1 = \{ \xi \in C_0 \mid \beta \restriction R_\xi \text{ is outer} \}$ also contains a club. The proof of this fact is essentially identical to the proof of the analogous statement for $C^*$-algebras given in [Far19, Proposition 7.3.9], but we include the proof for reader’s convenience. This proof uses the notion of a club in an arbitrary poset, defined in [Far19, Definitions 6.2.6] and the poset of separable substructures of a metric structure, defined in [Far19, Definition 7.1.8]. The supremum of an increasing sequence in the poset of separable substructures is the closure of its union (typically strictly larger than the union), and this fact makes the proofs more involved than in the standard, discrete, case.

To prove that $C_1$ contains a club, first note that $\tilde{\mathcal{C}}_0 = \{ R_\xi \mid \xi \in C_0 \}$ is a club of separable substructures of $M$ considered as a metric structure (every tracial von Neumann algebra is naturally identified with a metric structure, see [PHST14 §2.3.2]). Assume for contradiction that $C_1$ does not contain a club, so $C_0 \setminus C_1$ is stationary. Since the function $\xi \mapsto R_\xi$ is an order-isomorphism between $C_0$ and $\tilde{\mathcal{C}}_0$, the set $\{ R_\xi \mid \xi \in C_0 \setminus C_1 \}$ is stationary as well. For each $\xi \in C_0 \setminus C_1$, by our assumption, $\beta|_{R_\xi}$ is inner, so we can choose a unitary $u_\xi \in R_\xi$ such that $\beta|_{R_\xi} = \text{Ad}(u_\xi)$. The function $R_\xi \mapsto u_\xi$ is regressive (this simply means that $u_\xi \in R_\xi$, see [Far19, Definition 7.2.1]). By [Far19, Proposition 7.2.9], there exists $u \in M$ such that for any $\varepsilon > 0$, the set

$$\{ R_\xi \mid \| u - u_\xi \|_2 < \varepsilon \}$$

is stationary. From here it follows that $\beta = \text{Ad}(u)$. To see this, note that for any $a$ in the unit ball of $M$ and for any $\varepsilon > 0$, pick $\xi$ such that $a \in R_\xi$ and $R_\xi$ is in the stationary set above, so $\| \text{Ad}(u)(a) - \beta(a) \|_2 < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\beta = \text{Ad}(u)$. This is a contradiction. Therefore, the set $C_1$ contains a club, as claimed.

By the choice of $S_\xi$, since the limit ordinals form a club, there is a limit ordinal $\xi \in C_1$ such that $\mathcal{A}_\xi(F_{\beta|R_\xi}) = S_\xi$. By case [1] of the construction, the subfactor $R_{\xi+1}$ of $M$ has the property that for any larger hyperfinite II$_1$ factor $N$, no $\tilde{\beta} \in \text{Aut}(N)$ extends $\beta \restriction R_\xi$. However, $M$ and $\beta$ have this property; contradiction.

Now suppose that $\beta$ is an antiautomorphism. As before, the set $C = \{ \xi < \aleph_1 \mid \beta \restriction R_\xi \in \text{Aut}(R_\xi) \}$ is a club. Thus there exists $\xi \in C$ such that $S_\xi = \mathcal{A}_\xi(F_{\beta|R_\xi})$. By case [1] of the construction, the subfactor $R_{\xi+1}$ of $M$ has the property that for any larger hyperfinite II$_1$ factor $N$, no $\tilde{\beta} \in \text{Aut}(N)$ extends $\beta \restriction R_\xi$; contradiction.

Finally, suppose that $\mathcal{F}(M) \neq \{1\}$. Since $\mathcal{F}(M)$ is a multiplicative group, there exists $0 < t < 1$ and an isomorphism $\beta: M \to pMp$ for some $p$ with $\tau(p) < 1$. Since any two projections with the same trace are unitarily equivalent, we may assume without loss of generality that $p \in R_\omega$, so that $p \in R_\xi$ for all infinite ordinals $\xi$ we consider. By the same argument as above, the set

$$C = \{ \xi < \aleph_1 \mid \beta \restriction R_\xi \text{ is an isomorphism onto } pR_\xi p \}$$
is a club. Thus there is a limit ordinal $\xi \in \mathcal{C}$ such that $A_\xi(F_\beta|R_\xi) = S_\xi$. By case (4) of the construction, the subfactor $R_{\xi+1}$ of $M$ has the property that for any larger hyperfinite $\Pi_1$ factor $N$, no $\tilde{\beta}$ extends $\beta \upharpoonright R_\xi$; contradiction.

It remains to prove $\mathcal{F}(M^\mathcal{V}) = (0, \infty)$ for any nonprincipal ultrafilter $\mathcal{V}$ on $N$. Since $\mathcal{F}(R) = (0, \infty)$ and an isomorphism of $R$ with its corner $R^t$ extends to an isomorphism of $R^\mathcal{V}$ with $(R^t)^\mathcal{V} \cong (R^t)^t$, it will suffice to prove that $M^\mathcal{V} \cong R^\mathcal{V}$. This is a consequence of a standard model-theoretic fact. By [Far19, Corollary 16.4.2], we know that $M^\mathcal{V}$ and $R^\mathcal{V}$ are countably saturated ([Far19, Definition 16.1.5]). Since $\diamondsuit_{\aleph_1}$ implies the Continuum Hypothesis, both $M^\mathcal{V}$ and $R^\mathcal{V}$ have density character $\aleph_1$, and therefore are in fact saturated. By construction, $M$ is the inductive limit of $R_\xi$, for $\xi < \aleph_1$. Every $R_\xi$ is isomorphic to $R^t$, and for a countable limit ordinal $\eta$ we have that $R_\eta$ is the $\| \cdot \|_2$-cllosure of $\bigcup_{\xi < \eta} R_\xi$. By the Downwards Löwenheim–Skolem Theorem (e.g., [Far19, Theorem 7.1.9]), some $R_\xi$ is an elementary submodel of $M$ in the language of tracial von Neumann algebras. Therefore $R^\mathcal{V}_\xi \cong M^\mathcal{V}$ (e.g., [Far19 Corollary 16.6.5]). Since $R \cong R_\xi$, this concludes the proof. □

Remark 11. We do not know whether Theorem 11 or the main result of any of [AW04], [FH17], or [Vac18], can be proved in ZFC alone, or from the Continuum Hypothesis. In [CF20] it was shown that the conclusions of the main results from [AW04], [FH17], and [Vac18] all hold in many models of the Continuum Hypothesis in which $\diamondsuit_{\aleph_1}$ fails.

We don’t know whether in ZFC one can prove that $\mathcal{F}(M^\mathcal{V}) = (0, \infty)$ for every hyperfinite $\Pi_1$ factor $M$ with predual of density character $\aleph_1$ and every ultrafilter $\mathcal{V}$ on $N$ (although this is certainly true for $R$). We conjecture that this is not necessarily the case and that the results of [She92], showing that ultrapowers of countable, elementarily equivalent, structures associated with nonprincipal ultrafilters on $N$ need not be isomorphic, may be relevant.

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