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N=2 supersymmetric W-algebras

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Abstract

We investigate extensions of the N=2 super Virasoro algebra by one additional super primary field and its charge conjugate. Using a supersymmetric covariant formalism we construct all N=2 super W-algebras up to spin 5/2 of the additional generator. Led by these first examples we close with some conjectures on the classification of N=2 \( \mathcal{SW}(1, \Delta) \) algebras.

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1. Introduction

In 1987 Gepner [1-3] proposed a huge class of new N=2 supersymmetric string vacua by assuming that compactification of the heterotic string to four dimensions leads to a minimal N=2 theory with total central charge $c = 9$. It was shown in ref. [4] that N=2 world-sheet supersymmetry is necessary for N=1 space-time supersymmetry. For the internal $c = 9$ theory Gepner takes tensor products of minimal, unitary representations of the N=2 super Virasoro algebra [5]. Surprisingly, these algebraically defined theories correspond geometrically to compactifications on Calabi-Yau spaces [2,3].

The GUT gauge groups of these models are $E_8 \times E_6 \times U(1)^{r-1}$ where $r$ denotes the number of tensored N=2 models. Since these additional $U(1)$ groups are not observed in nature one wants to get rid of them. This motivated Kazama-Suzuki [6,7] to search for N=2 theories having minimal representations for $c = 9$. They constructed and classified coset models of N=1 super Kac-Moody algebras which have an extended N=2 supersymmetry. Several models with the desired value $c = 9$ were found. The symmetry algebras of these Kazama-Suzuki models should be generically existing N=2 super W-algebras [8]. The above application is the most important reason to classify N=2 supersymmetric W-algebras.

We denote the extension of the N=2 super Virasoro algebra by super primary fields of dimension $\Delta_1 \ldots \Delta_n$ by $SW(1, \Delta_1 \ldots \Delta_n)$. Note that usually the primaries consist of pairs of opposite $U(1)$ charge $Q$. The motivation for our work was to find new extensions of the N=2 super Virasoro algebra by explicit construction. From the recent extensive work on N=0 and N=1 (super) W-algebras [9-12] one knows of the occurrence of (super) W-algebras which exist only for a few values of the central charge $c$. The hope is that nature may have chosen such an isolated N=2 super W-symmetry for the vacuum.

Up to now only two examples of nonlinear N=2 super W-algebras are known explicitly, $SW(1, 3/2)$ [13] and $SW(1, 2)$ [14]. For the first one Inami et al., using a conformal bootstrap algorithm, found that all four point functions are associative for generic value of $c$. But they claimed that carrying out an algebraic calculation and checking Jacobi identities would yield stronger restrictions on $c$. Our calculations confirm this conjecture. The special case $c = 9$ and $Q = 3$, in which the additional, chiral spin $3/2$ generator is interpreted as the spectral flow operator, has been studied by Odake [15,16] intensively.
The second algebra has only been constructed for vanishing $U(1)$ charge of the spin two generator. Romans found that this algebra exists for generic value of $c$.

In this paper we continue the series of $\mathcal{SW}(1, \Delta)$ algebras up to spin three. The first four examples disclose already enough structure to speculate on the classification of these algebras. We conjecture the existence of a new N=2 super W-algebra existing for $c = 9$, the $\mathcal{SW}(1, 11/2)$ algebra.

The paper is organized as follows: In the second section we develop an N=2 supersymmetric covariant formalism for the construction of N=2 super W-algebras. We show that requiring invariance under super Möbius transformations puts severe restrictions on the structure of these algebras. We also define supersymmetric normal ordered products. It is this structural investigation that makes the explicit construction of super W-algebras feasible.

In the third section we apply the whole formalism to the construction of $\mathcal{SW}(1, \Delta)$ algebras. Using the symbolic manipulation package REDUCE, we extend the series of known W-algebras up to $\Delta = 5/2$. For $\Delta = 3$ we are able to give necessary conditions for existence. Different from the N=0 and N=1 case, $\mathcal{SW}(1, \Delta)$ algebras are also allowed to exist for $c \geq 3$.

In the fourth section we discuss the results obtained and present some speculations about the classification of $\mathcal{SW}(1, \Delta)$ algebras. We conjecture for example all $\mathcal{SW}(1, \Delta)$ algebras to exist for $c = 3$ and $Q = \pm 1$ if $\Delta$ is half-integer.

The fifth section concludes this paper presenting a short summary.

\section*{2. N=2 supersymmetric CFT}

In this section we generalize the formulae which have been presented for the case N=0 and N=1 in ref. [9,15]. We will see that the structure of N=2 W-algebras is considerably restricted by invariance under N=2 super Möbius transformations. Since the aim of our work is the covariant construction of N=2 super W-algebras, we have to express all involved fields as so-called N=2 super quasiprimary fields. Thus a supersymmetric notion of normal ordering has to be worked out.
2.1. SU(1,1) INVARIANT CFT

First we review the basic structural property of SU(1,1) invariant chiral algebras. For a more detailed exposition we refer the reader to ref. [9] and ref. [15] for the N=1 supersymmetric case.

We define a special class of chiral fields

\[ \varphi(z) = \sum_{n \in \mathbb{Z}} \varphi_n z^{n-d} \]  

(2.1)

with respect to the energy-momentum tensor

\[ L(z) = \sum_{n \in \mathbb{Z}} L_n z^{n-2} \]  

(2.2)

by requiring that

\[ [L_m, \varphi_n] = (n - (d-1)m) \varphi_{m+n} \]  

(2.3)

is satisfied for \( m \in \{-1, 0, 1\} \). These fields are called quasiprimary.

The Fourier-modes of a set \( \{\varphi_i(z)\} \) of such fields have a simple Lie algebra structure

\[ [\varphi_{i,m}, \varphi_{j,n}] = \sum_k C^k_{ij} p_{ijk}(m, n) \varphi_{k,m+n} + d_{ij} \delta_{n,-m} \left( \frac{n + h(i) - 1}{2h(i) - 1} \right), \]  

(2.4)

where the structure constants \( d_{ij} \) and \( C_{ijk} \) are given by

\[ d_{ij} = \langle 0 | \varphi_{i,-h(i)} \varphi_{j,h(j)} | 0 \rangle, \]  

(2.5)

\[ C_{ijk} = \langle 0 | \varphi_{k,-h(k)} \varphi_{i,h(i)-h(j)} \varphi_{j,h(j)} | 0 \rangle \]  

(2.6)

and

\[ C_{ij}^l d_{lk} = C_{ijkl}. \]  

(2.7)

The polynomials \( p_{ijk}(m, n) \) depend only on the conformal dimensions of the involved fields and are explicitly known [9].

Furthermore, we introduced quasiprimary normal ordered products \( N(\varphi_j \partial^n \varphi_i) \) by a quasiprimary projection of the natural normal ordered products \( N(\varphi_j \partial^n \varphi_i) \) occurring in the
regular part of the operator product expansion (OPE). With these structural concepts we were able to construct a whole bunch of new W-algebras.

2.2. N=2 SUPERCONFORMAL ALGEBRA

In this subsection we review some elementary notions of N=2 supergeometry [16,17], and we will define a special class of superfields in N=2 superspace. Because we are dealing with left- or right-handed field theories we treat in the following only one chirality.

The N=2 superspace is the extension of the complex plane by two Grassmann variables. A point is a triple \( Z = (z, \theta, \bar{\theta}) \) where \( \theta, \bar{\theta} \) are anticommuting objects.

On this superspace one defines two covariant derivatives

\[
D = \partial_\theta + \frac{1}{2} \bar{\theta} \partial_z, \\
\bar{D} = \partial_{\bar{\theta}} + \frac{1}{2} \theta \partial_z
\]

which are nilpotent and satisfy

\[
\{D, \bar{D}\} = \partial_z. \tag{2.10}
\]

Furthermore, we introduce the super translationally invariant super interval \( Z_{ij} \) of two points \( Z_i \) and \( Z_j \)

\[
Z_{ij} = z_i - z_j - \frac{1}{2} (\theta_i \bar{\theta}_j + \bar{\theta}_i \theta_j). \tag{2.11}
\]

One obtains three other translationally invariant intervals by multiplying \( Z_{ij} \) by \( \theta_{ij} = \theta_i - \theta_j, \bar{\theta}_{ij} = \bar{\theta}_i - \bar{\theta}_j \) and \( \theta_{ij} \bar{\theta}_{ij} \).

Similar to the complex plane, one can define the N=2 super conformal group acting on N=2 superspace. The generators of this group form an infinite dimensional Lie algebra which admits a central extension. The resulting algebra is the so-called N=2 super Virasoro algebra

\[
[L_m, L_n] = (n-m) L_{m+n} + \frac{c}{12} (n^3 - n) \delta_{m,-n} \\
[L_m, G_r] = (r - \frac{m}{2}) G_{m+r} \\
[L_m, \bar{G}_r] = (r - \frac{m}{2}) \bar{G}_{m+r}
\]
\[ [L_m, J_n] = nJ_{m+n} \]
\[ [J_m, J_n] = \frac{c}{3} n \delta_{m,-n} \]
\[ [J_m, G_r] = G_{m+r} \]
\[ [J_m, \bar{G}_r] = -\bar{G}_{m+r} \]
\[ \{G_r, G_s\} = 2L_{r+s} + (s-r)J_{r+s} + \frac{2c}{3} \left( s + \frac{1}{2} \right) \delta_{r,-s} \]
\[ \{G_r, \bar{G}_s\} = 0. \]

The essential property of this algebra is the appearance of the abelian current \( J(z) \). The fields

\[ L(z) = \sum_{n \in \mathbb{Z}} L_n z^{n-2} \quad J(z) = \sum_{n \in \mathbb{Z}} J_n z^{n-1} \]  
\[ G(z) = \sum_r G_r z^{r-\frac{3}{2}} \quad \bar{G}(z) = \sum_r \bar{G}_r z^{r-\frac{3}{2}} \]

can be put together into one N=2 superfield

\[ \mathcal{L}(z) = J(z) + \frac{1}{\sqrt{2}} (\theta \bar{G}(z) - \bar{\theta} G(z)) + \theta \bar{\theta} L(z). \]  

The N=2 super Virasoro algebra has been studied very intensively during the last years, and the unitary minimal representations of this algebra are well known \cite{5}. These exist for a discrete series of \( c \) values

\[ c = \frac{3k}{k+2}, \quad k \in \mathbb{Z}_+. \]  

We will review the minimal representations of this algebra in the fourth section where we discuss the results of our work. From now on we will only consider the Neveu-Schwarz sector, i.e. we choose periodic boundary conditions for all fermionic fields \((r \in \mathbb{Z} + 1/2\) in \(2.13\)).

The superconformal algebra contains the subalgebra of super Möbius transformations which are generated by \( \{L_1, L_0, L_{-1}, G_{1/2}, G_{-1/2}, \bar{G}_{1/2}, \bar{G}_{-1/2}, J_0\} \). This algebra is isomorphic to the super Lie algebra \( Osp(2|2) \) and is analogous to \( SU(1,1) \) in the N=0 case.
or to $Osp(1|2)$ in the N=1 case. In the rest of this section, we will show that requiring an $Osp(2|2)$ invariant superconformal field theory puts strong restrictions on its structure. The maximal abelian subalgebra of the superconformal algebra is generated by $L_0$ and $J_0$. Thus each state in the Hilbert space of a superconformal field theory has at least two quantum numbers, the conformal weight $\Delta$ and the $U(1)$ charge $Q$. Because of the isomorphism between states and fields, the definition of super primary fields involves also two parameters $(\Delta, Q)$. If a superfield $\Phi^Q(Z)$ satisfies the following equations then it is called a super primary field of dimension $\Delta$ and charge $Q$:

\[
[L_m, \Phi(Z)] = z^{-m} \left[ z \partial_z - \frac{(m-1)}{2} \theta \partial_\theta - \frac{(m-1)}{2} \bar{\theta} \partial_{\bar{\theta}} - \Delta(m-1) + \frac{Q}{2} m(m-1) \theta \bar{\theta} z^{-1} \right] \Phi(Z) \\
\]

\[
[G_r, \Phi(Z)]_\pm = \sqrt{2} z^{-r+1/2} \left[ \partial_\theta - \frac{1}{2} \theta \partial_z - (\Delta + \frac{Q}{2}) (r - \frac{1}{2}) \theta z^{-1} \right] \Phi(Z) \\
- \frac{1}{2} (r - \frac{1}{2}) \theta \bar{\theta} z^{-1} \partial_\theta \Phi(Z) \\
\]

\[
[\overline{G}_r, \Phi(Z)]_\pm = \sqrt{2} z^{-r+1/2} \left[ \partial_\bar{\theta} - \frac{1}{2} \bar{\theta} \partial_z - (\Delta - \frac{Q}{2}) (r - \frac{1}{2}) \bar{\theta} z^{-1} \right] \Phi(Z) \\
+ \frac{1}{2} (r - \frac{1}{2}) \theta \bar{\theta} z^{-1} \partial_{\bar{\theta}} \Phi(Z) \\
\]

\[
[J_m, \Phi(Z)] = z^{-m} \left[ - \theta \partial_\theta + \bar{\theta} \partial_{\bar{\theta}} - \Delta m \theta \bar{\theta} z^{-1} + Q \right] \Phi(Z) \\
\]

where for two fermionic fields one takes the anticommutator, the commutator otherwise.

If (2.16) is satisfied only with respect to the subalgebra $Osp(2|2)$, $\Phi(Z)$ is called a super quasiprimary field. The super Virasoro field defined above is an example for a super quasiprimary field of dimension one and charge zero.

Note that $G_{1/2}$ and $\overline{G}_{1/2}$ are the generators of supertranslations on the space of fields

\[
[G_{1/2}, \Phi(Z)]_\pm = \sqrt{2} (\partial_\theta - \frac{1}{2} \theta \partial_z) \Phi(Z) \\
\]

\[
[\overline{G}_{1/2}, \Phi(Z)]_\pm = \sqrt{2} (\partial_{\bar{\theta}} - \frac{1}{2} \bar{\theta} \partial_z) \Phi(Z) \\
\]

whereas $J_0$ generates a ”deformed” rotation in the pure Grassmann-plane

\[
[J_0, \Phi(Z)]_\pm = (- \theta \partial_\theta + \bar{\theta} \partial_{\bar{\theta}} + Q) \Phi(Z). \\
\]
These transformations leave invariant the four super intervals defined in (2.11), if the charges $Q = 1, -1$ are assigned to $\theta$ and $\bar{\theta}$, respectively. By expanding the superfields in a Taylor series with respect to $\theta$ and $\bar{\theta}$ one obtains four components

$$\Phi(Z) = \varphi(z) + \frac{1}{\sqrt{2}}(\theta \bar{\psi}(z) - \bar{\theta} \psi(z)) + \theta \bar{\sigma} \chi(z)$$

(2.20)

or

$$\Phi(Z) = \Phi_1(z) + \frac{1}{\sqrt{2}}(\theta \Phi_\theta(z) - \bar{\theta} \Phi_\bar{\theta}(z)) + \theta \bar{\theta} \Phi_\theta \bar{\theta}(z)$$

(2.20)

and realizes that the "highest" component $\chi(z)$ is not quasiprimary with respect to the Virasoro algebra if the charge $Q$ is different from zero. Thus, for component calculations we will use instead its quasiprimary projection

$$\chi'(z) = \chi(z) - \frac{Q}{4\Delta} \partial_z \varphi(z).$$

(2.21)

Hence a super (quasi)primary field is defined by the following commutation relations

$$[L_m, \varphi_n]_\pm = (n - (\Delta - 1)m)\varphi_{m+n}$$
$$[L_m, \psi_n]_\pm = (n - (\Delta - \frac{1}{2})m)\psi_{m+n}$$
$$[L_m, \bar{\psi}_n]_\pm = (n - (\Delta - \frac{1}{2})m)\bar{\psi}_{m+n}$$
$$[L_m, \chi'_n]_\pm = (n - \Delta m)\chi'_{m+n}$$
$$[G_m, \varphi_n]_\pm = -\psi_{m+n}$$
$$[\bar{G}_m, \varphi_n]_\pm = \bar{\psi}_{m+n}$$
$$[G_m, \psi_n]_\pm = [\bar{G}_m, \bar{\psi}_n]_\pm = 0$$
$$[G_m, \bar{\psi}_n]_\pm = (1 + \frac{Q}{2\Delta})(n - (2\Delta + 1)m)\varphi_{m+n} + 2\chi'_{m+n}$$

(2.22)
$$[\bar{G}_m, \psi_n]_\pm = -(1 - \frac{Q}{2\Delta})(n - (2\Delta + 1)m)\varphi_{m+n} + 2\chi'_{m+n}$$
$$[G_m, \chi'_n]_\pm = \frac{1}{2}(1 + \frac{Q}{2\Delta})(n - 2\Delta m)\psi_{m+n}$$
$$[\bar{G}_m, \chi'_n]_\pm = \frac{1}{2}(1 - \frac{Q}{2\Delta})(n - 2\Delta m)\psi_{m+n}$$
$$[J_m, \varphi_n]_\pm = Q \varphi_{m+n}$$
$$[J_m, \psi_n]_\pm = (Q + 1)\psi_{m+n}$$
$$[J_m, \bar{\psi}_n]_\pm = (Q - 1)\bar{\psi}_{m+n}$$
$$[J_m, \chi_n]_\pm = -\Delta(1 + \frac{Q^2}{4\Delta^2})m \varphi_{m+n} + Q \chi'_{m+n}.$$
These equations imply that one can obtain the Fourier-modes of the higher components of a super quasiprimary field if one knows that of its lowest component $\varphi_{\Delta}|0\rangle$

$$\psi_{\Delta+1/2}|0\rangle = -G_{1/2}\varphi_{\Delta}|0\rangle \quad \overline{\psi}_{\Delta+1/2}|0\rangle = \overline{G}_{1/2}\varphi_{\Delta}|0\rangle$$  \hspace{1cm} (2.23)

$$\chi'_{\Delta+1}|0\rangle = \frac{1}{2}(G_{1/2}\overline{G}_{1/2} - (1 + \frac{Q}{2\Delta})L_1)\varphi_{\Delta}|0\rangle.$$  

During the explicit construction of W-algebras we will use these relations to determine the higher components of super quasiprimary normal ordered products.

Now we are in a position to analyse n-point functions and the OPE structure for super quasiprimary fields. We will show that the OPE of two such superfields is completely determined up to some coupling constants.

2.3 TWO- AND THREE-POINT FUNCTIONS

In this subsection we review and improve the expressions known in the literature, which are incomplete or apply only to some specific cases. (Nevertheless, see ref. [17,18] for a more detailed derivation.)

We assume that in the theory there is a vacuum invariant under super Möbius transformations. By taking commutators with the eight generators of $Osp(2|2)$ inside the vacuum correlators we obtain eight super differential equations for the n-point functions, which, however, are not all independent because of (2.12). For example, the two-point function obeys the following differential equation which results from taking the commutator with $J_0$:

$$\sum_{i=1}^{2} (-\theta_i \partial_{\theta_i} + \overline{\theta}_i \partial_{\overline{\theta}_i} + Q_i)\langle \Phi(Z_1)\Phi(Z_2) \rangle = 0.$$ \hspace{1cm} (2.24)

The solution of these differential equations for the two-point function is

$$\langle \Phi^Q_{\Delta_i}(Z_1)\Phi^Q_{\Delta_j}(Z_2) \rangle = \frac{D_{ij}}{Z_{12}^{2\Delta_i}} \left(1 - \frac{Q_i}{2} \frac{\theta_{12}\overline{\theta}_{12}}{Z_{12}} \right) \delta_{\Delta_i,\Delta_j} \delta_{Q_i,-Q_j}.$$ \hspace{1cm} (2.25)

Thus, only the two-point function for fields of opposite charge does not vanish.

The three-point functions are more complicated since three cases have to be distinguished.
(i) If \( Q_i + Q_j = Q_k \) then

\[
\langle \Phi_{\Delta_k}(Z_1) \Phi_{\Delta_i}(Z_2) \Phi_{\Delta_j}(Z_3) \rangle = C_{ijk}^2 \left[ \left( 1 - \frac{Q_i}{2} \frac{\theta_{23}}{Z_{23}} \right) \left( 1 + \frac{Q_k}{2} \frac{\theta_{13}}{Z_{13}} \right) \frac{1}{Z_{12}^{\frac{1}{2}} Z_{23}^{\frac{1}{2}} Z_{13}^{\frac{1}{2}}} \right] 
\]

\[
+ C_{ijk} \alpha_{ijk} \gamma_1 \left[ \left( \frac{\theta_{23}}{Z_{23}} - \frac{\theta_{13}}{Z_{13}} \right) \left( \frac{\theta_{23}}{Z_{23}} - \frac{\theta_{13}}{Z_{13}} \right) \frac{1}{Z_{12}^{\frac{1}{2}} Z_{23}^{\frac{1}{2}} Z_{13}^{\frac{1}{2}}} \right] 
\]

\[
- \left( \theta_{23} \bar{\theta}_{23} \theta_{13} \bar{\theta}_{13} \right) \left( \frac{Q_i}{Z_{13}} - \frac{Q_k}{Z_{13}} \right) \frac{1}{Z_{12}^{\frac{1}{2}} Z_{23}^{\frac{1}{2}} Z_{13}^{\frac{1}{2}}} \right] \quad (2.26)
\]

where \( \gamma_1 = \Delta_k + \Delta_i - \Delta_j, \gamma_2 = \Delta_i + \Delta_j - \Delta_k \) and \( \gamma_3 = \Delta_k + \Delta_j - \Delta_i \).

\( C_{ijk} \) and \( \alpha_{ijk} \) are independent parameters in contrast to ref. [17,18].

(ii) If \( Q_i + Q_j = Q_k - 1 \) then

\[
\langle \Phi_{\Delta_k}(Z_1) \Phi_{\Delta_i}(Z_2) \Phi_{\Delta_j}(Z_3) \rangle = \frac{C_{ijk}}{Z_{12}^{\frac{1}{2}} Z_{23}^{\frac{1}{2}} Z_{13}^{\frac{1}{2}}} \left[ \left( \frac{\theta_{23}}{Z_{23}} - \frac{\theta_{13}}{Z_{13}} \right) \frac{1}{Z_{12}^{\frac{1}{2}} Z_{23}^{\frac{1}{2}} Z_{13}^{\frac{1}{2}}} \right] \]

\[
+ \frac{Q_i \theta_{13}}{2 Z_{23} Z_{13}} \left( \frac{\theta_{23}}{Z_{23}} - \frac{\theta_{13}}{Z_{13}} \right) \frac{1}{Z_{12}^{\frac{1}{2}} Z_{23}^{\frac{1}{2}} Z_{13}^{\frac{1}{2}}} \right] \quad (2.27)
\]

(iii) If \( Q_i + Q_j = Q_k + 1 \) then

\[
\langle \Phi_{\Delta_k}(Z_1) \Phi_{\Delta_i}(Z_2) \Phi_{\Delta_j}(Z_3) \rangle = \frac{C_{ijk}}{Z_{12}^{\frac{1}{2}} Z_{23}^{\frac{1}{2}} Z_{13}^{\frac{1}{2}}} \left[ \left( \frac{\theta_{23}}{Z_{23}} - \frac{\theta_{13}}{Z_{13}} \right) \frac{1}{Z_{12}^{\frac{1}{2}} Z_{23}^{\frac{1}{2}} Z_{13}^{\frac{1}{2}}} \right] \]

\[
+ \left( Q_i \theta_{13} \theta_{23} \bar{\theta}_{23} + Q_k \theta_{23} \theta_{13} \bar{\theta}_{13} \right) \frac{1}{2 Z_{23} Z_{13}} \] \quad (2.28)

In (ii) and (iii) there exists only one independent coupling constant \( C_{ijk} \). These rather lengthy three-point functions contain all the information we need for deriving \( N=2 \) supersymmetric OPEs.

2.4. \( N=2 \) SUPERSYMMETRIC OPE

In this subsection we derive the general form of the OPE of two \( Osp(2|2) \) invariant superfields, using the results of the last subsection. To this end, we write down the most general ansatz for an OPE which is invariant under the operation of the generators \( L_0, L_1, G_{1/2}, \bar{G}_{1/2}, J_0 \).
(i) If $Q_i + Q_j = Q_k$ then

$$
\Phi_{\Delta_i}(Z_2) \Phi_{\Delta_j}(Z_3) = \sum_{n=0}^{\infty} \frac{c_{ijk} A_{ijk}^n}{Z_{23}^n} \partial_z^n \Phi_{\Delta_k}(Z_3) + \sum_{n=1}^{\infty} \frac{c_{ijk} B_{ijk}^n}{Z_{23}^n} [D, D] \partial_z^{n-1} \Phi_{\Delta_k}(Z_3) 
$$

$$
+ \sum_{n=0}^{\infty} \frac{c_{ijk} C_{ijk}^n}{Z_{23}^n} \partial_z^n \Phi_{\Delta_k}(Z_3) + \sum_{n=0}^{\infty} \frac{c_{ijk} D_{ijk}^n}{Z_{23}^n} \partial_z^n \Phi_{\Delta_k}(Z_3) 
$$

$$
+ \sum_{n=-1}^{\infty} \frac{c_{ijk} E_{ijk}^n}{Z_{23}^n} \partial_z^{n+1} \Phi_{\Delta_k}(Z_3) + \sum_{n=0}^{\infty} \frac{c_{ijk} F_{ijk}^n}{Z_{23}^n} [D, D] \partial_z^n \Phi_{\Delta_k}(Z_3) 
$$

(2.29)

The coefficients in this expansion can be determined in the following way. One inserts the OPE into the three point function and using the two point function (2.25), obtains a rational function in the three variables $Z_{23}, Z_{13}, \theta_{23}, \theta_{13}, \overline{\theta}_{23}$ and $\overline{\theta}_{13}$. This expression must be equal to the general three point function (2.26) after having fixed the points in a special way by means of an appropriate super Möbius transformation. For example, one may shift the three points to $z_1 = \infty$, $z_2 = 1$, $z_3 = 0$, $\theta_{23} = 0$ and $\overline{\theta}_{23} = 0$. Comparing the two expressions yields a linear equation for the coefficients in the OPE (2.29). After a rather lengthy calculation, one can generate six independent equations which can be solved easily in terms of the free parameter $\alpha = \alpha_{ijk}$, the conformal dimensions and the $U(1)$ charges:

$$
A_{ijk}^n = \frac{2}{n!} \left( \frac{2\Delta_k + n}{n} \right)^{-1} \left( \frac{\gamma_1 + n - 1}{n} \right) \left( 1 + \frac{n}{2} \left( \frac{Q_k/2 - \Delta_k}{(Q_k/2)^2 - \Delta_k^2} \right) \right) 
$$

$$
B_{ijk}^n = \frac{1}{(n-1)!} \left( \frac{2\Delta_k + n}{n} \right)^{-1} \left( \frac{\gamma_1 + n - 1}{n} \right) \left( \frac{Q_k/2 - \Delta_k}{(Q_k/2)^2 - \Delta_k^2} \right) 
$$

$$
C_{ijk}^n = \frac{2\Delta_k}{n!} \left( \frac{2\Delta_k + n}{n+1} \right)^{-1} \left( \frac{\gamma_1 + n}{n+1} \right) \left( \frac{1 + \alpha}{Q_k/2 - \Delta_k} \right) 
$$

$$
D_{ijk}^n = -\frac{2\Delta_k}{n!} \left( \frac{2\Delta_k + n}{n+1} \right)^{-1} \left( \frac{\gamma_1 + n}{n+1} \right) \left( \frac{1 - \alpha}{Q_k/2 - \Delta_k} \right) 
$$

$$
E_{ijk}^n = \frac{1}{(n+1)!} \left( \frac{2\Delta_k + n+1}{n+1} \right)^{-1} \left\{ \left( \frac{\gamma_1 + n + 1}{n+2} \right) \frac{(n+2)}{2} \left[ 2\alpha + (n+1) \left( \frac{Q_k/2 - \Delta_k}{(Q_k/2)^2 - \Delta_k^2} \right) \right] 
$$

$$
- Q_i \left( \frac{\gamma_1 + n}{n+1} \right) \left[ 1 + \frac{(n+1)}{2} \left( \frac{\alpha Q_k/2 - \Delta_k}{(Q_k/2)^2 - \Delta_k^2} \right) \right] \right\} 
$$

$$
F_{ijk}^n = -\frac{Q_i}{2n!} \left( \frac{2\Delta_k + n+1}{n+1} \right)^{-1} \left( \frac{\gamma_1 + n}{n+1} \right) \left( \frac{Q_k/2 - \Delta_k}{(Q_k/2)^2 - \Delta_k^2} \right) 
$$

(2.30)
The coupling constant $C^k_{ij}$ is then determined by the linear system $C^l_{ij}D_{lk} = C_{ijk}$.

Before presenting some simple examples, we give the OPEs for the two other cases.

(ii) If $Q_i + Q_j = Q_k - 1$ then

\[
\Phi^{Q_i}_{\Delta_i}(Z_2) \Phi^{Q_j}_{\Delta_j}(Z_3) = \sum_{n=0}^{\infty} \frac{C^k_{ij} A^n_{ij} \theta_{23}}{Z_{23}^{-n+1/2}} \partial^n_{z} \Phi^{Q_k}_{\Delta_k}(Z_3) + \sum_{n=1}^{\infty} \frac{C^k_{ij} B^n_{ijk}}{Z_{23}^{-n+1/2}} D\partial^n_{z}^{-1} \Phi^{Q_k}_{\Delta_k}(Z_3) \quad (2.31)
\]

\[
A^n_{ijk} = \frac{1}{n!} \left( \frac{2\Delta_k + n + 1}{n} \right)^{-1} \left( \frac{\gamma_1 + n - 1/2}{n} \right) \left( 1 + \frac{n}{2(\Delta_k + Q_k/2)} \right)
\]

\[
B^n_{ijk} = \frac{1}{(n-1)!} \left( \frac{2\Delta_k + n - 1}{n - 1} \right)^{-1} \left( \frac{\gamma_1 + n - 3/2}{n - 1} \right) \frac{1}{(\Delta_k + Q_k/2)}
\]

\[
C^n_{ijk} = \frac{1}{(n-1)!} \left( \frac{2\Delta_k + n}{n} \right)^{-1} \left( \frac{\gamma_1 + n - 1/2}{n} \right) \frac{1}{2(\Delta_k + Q_k/2)}
\]

\[
D^n_{ijk} = -\frac{1}{n!} \left( \frac{2\Delta_k + n + 1}{n} \right)^{-1} \frac{1}{(\Delta_k + Q_k/2)} \left[ \frac{Q_i}{2} \left( \frac{\gamma_1 + n - 1/2}{n} \right) + \frac{(n+1)}{2} \left( \frac{\gamma_1 + n + 1/2}{n+1} \right) \right]
\]

(iii) If $Q_i + Q_j = Q_k + 1$ then

\[
\Phi^{Q_i}_{\Delta_i}(Z_2) \Phi^{Q_j}_{\Delta_j}(Z_3) = \sum_{n=0}^{\infty} \frac{C^k_{ij} A^n_{ij} \theta_{23}}{Z_{23}^{-n+1/2}} \partial^n_{z} \Phi^{Q_k}_{\Delta_k}(Z_3) + \sum_{n=1}^{\infty} \frac{C^k_{ij} B^n_{ijk}}{Z_{23}^{-n+1/2}} D\partial^n_{z}^{-1} \Phi^{Q_k}_{\Delta_k}(Z_3) \quad (2.33)
\]

\[
A^n_{ijk} = \frac{1}{n!} \left( \frac{2\Delta_k + n + 1}{n} \right)^{-1} \left( \frac{\gamma_1 + n - 1/2}{n} \right) \left( 1 + \frac{n}{2(\Delta_k - Q_k/2)} \right)
\]

\[
B^n_{ijk} = \frac{1}{(n-1)!} \left( \frac{2\Delta_k + n - 1}{n - 1} \right)^{-1} \left( \frac{\gamma_1 + n - 3/2}{n - 1} \right) \frac{1}{(\Delta_k - Q_k/2)}
\]
After the technical considerations of this subsection, we are now in the position to express the main relations of the super Virasoro algebra in a very compact form. First, application of the formula (2.30) to the OPE of two super Virasoro fields yields

\[ \mathcal{L}(Z_2) \mathcal{L}(Z_3) = \frac{c}{3Z_{23}^2} + \frac{\theta_{23} \bar{\theta}_{23}}{Z_{23}^2} \mathcal{L}(Z_3) + \frac{(\theta_{23} D - \bar{\theta}_{23} \bar{D})}{Z_{23}} \mathcal{L}(Z_3) \]

\[ + \frac{\theta_{23} \bar{\theta}_{23}}{Z_{23}} \partial_z \mathcal{L}(Z_3) + \text{reg.} \]

Furthermore, the definition (2.16) of super primary fields is equivalent to the following OPE

\[ \mathcal{L}(Z_2) \Phi^Q_\Delta(Z_3) = \frac{\Delta \theta_{23} \bar{\theta}_{23}}{Z_{23}^2} \Phi^Q_\Delta(Z_3) + \frac{(\theta_{23} D - \bar{\theta}_{23} \bar{D})}{Z_{23}} \Phi^Q_\Delta(Z_3) \]

\[ + \frac{\theta_{23} \bar{\theta}_{23}}{Z_{23}} \partial_z \Phi^Q_\Delta(Z_3) + \frac{Q}{Z_{23}} \Phi(Z_3) + \text{reg.} \]

Here the coupling constants take the values \( C_{\mathcal{L}\Phi} = Q/2 \) and \( \alpha_{\mathcal{L}\Phi} = 2\Delta/Q \).

By expanding the fields in (2.29)(2.31)(2.33) into their components we are able to express the coupling constants of these components by the superymmetric coupling constant. The long list of these relations is given in appendix A. Here we want to stress again that the explicit construction of W-algebras has become tractable only with the help of these relations which reduce the amount of computer time and memory considerably.

2.5. SUPER NORMAL ORDERED PRODUCTS

W-algebras are not Lie algebras in the classical sense because they contain a multiplication of the generators. In the language of quantum field theory, one says that a notion of a normal ordering operation is necessary. Since we want to use our formulae for \( Osp(2|2) \)
invariant fields we have to define super normal ordered products which are super quasipri-
mary. The way to proceed is analogous to the one in the N=0 and N=1 case.
The fields $N_s(\ldots)$ which occur in the regular part of the OPE can be considered as normal ordered products. The obstacle that they are not quasiprimary can be overcome by projecting them onto fields $N_s(\ldots)$ with the desired property.

We consider the regular part of an OPE of two fields and define the normal ordering operation $N_s(\ldots)$ as follows.

$$
\Phi_i(Z_1)\Phi_j(Z_2) = \text{s. t.} + \sum_{n=0}^{\infty} \frac{Z_{12}^n}{n!} N_s(\Phi_j\partial^n_z\Phi_i)(Z_2) + \sum_{n=0}^{\infty} \frac{\theta_{12} Z_{12}^n}{n!} N_s(\Phi_jD\partial^n_z\Phi_i)(Z_2)
$$

$$
= + \sum_{n=0}^{\infty} \frac{\bar{\theta}_{12} Z_{12}^n}{n!} N_s(\Phi_j\bar{D}\partial^n_z\Phi_i)(Z_2) + \sum_{n=0}^{\infty} \frac{\theta_{12} \bar{\theta}_{12} Z_{12}^n}{2n!} N_s(\Phi_j[D, \bar{D}]\partial^n_z\Phi_i)(Z_2)
$$

The components of $N_s(\ldots)$ can be obtained easily by a super Taylor expansion of (2.37) (see appendix B). Application of the derivatives $\partial_2$, $D_2$, $\bar{D}_2$ and $[\bar{D}_2, D_2]$ to (2.37) gives the following properties of $N_s(\ldots)$:

$$
\partial N_s(\Phi_j\Phi_i) = N_s(\Phi_j\partial\Phi_i) + N(\partial\Phi_j\Phi_i)
$$

$$
DN_s(\Phi_j\Phi_i) = N_s(\Phi_jD\Phi_i) + (-1)^{2\Delta} N_s(D\Phi_j\Phi_i)
$$

$$
\bar{D}N_s(\Phi_j\Phi_i) = N_s(\Phi_j\bar{D}\Phi_i) + (-1)^{2\Delta} N_s(\bar{D}\Phi_j\Phi_i)
$$

$$
[D, \bar{D}] N_s(\Phi_j\Phi_i) = N_s(\Phi_j[D, \bar{D}]\Phi_i) - N_s([\bar{D}, D]\Phi_j\Phi_i)
$$

We will not discuss the procedure to get quasiprimary fields in most generality. But we will present some hopefully clarifying examples.

We introduce the quasiprimary projection $N_s(\Phi_j\partial^n\Phi_i)$ in such a way that $C_{ij}^k A_{ij(i+j+n)}^0 = \frac{1}{n!}$ and $C_{ij}^k E_{ij(i+j+n)}^{-1} = 0$. The second condition is natural since it guarantees that the field $N_s(\Phi_j\partial^n\Phi_i)$ itself appears only once in the OPE of $\Phi_i$ and $\Phi_j$. These two conditions fix the two coupling constants for the quasiprimary field and all derivatives couple due to (2.29),(2.30). For the three other kinds of normal ordered products in (2.37) the definition of their quasiprimary projection is similar. In particular, for the the fields $N_s(\Phi_j[D, \bar{D}]\partial^n\Phi_i)$ we require $C_{ij}^k A_{ij(i+j+n+1)}^0 = 0$ additionally.
For explicit calculations one needs a representation of these $Osp(2|2)$ invariant normal ordered products in terms of the $N_s(\ldots)$ fields and the fields appearing in the singular part of the OPE. But it is already sufficient to know the lowest component since higher ones can be obtained using (2.23).

We present some examples how this procedure works.

(i) $N_s(\mathcal{L}\mathcal{L})$

From the OPE

$$\mathcal{L}(Z_1)\mathcal{L}(Z_2) = \text{sing. terms} + N_s(\mathcal{L}\mathcal{L})(Z_2) + \frac{1}{3}[\overline{D}, D]\mathcal{L}(Z_2) + ... \quad (2.39)$$

one reads off directly

$$N_s(\mathcal{L}\mathcal{L}) = N_s(\mathcal{L}\mathcal{L}) - \frac{1}{3}[\overline{D}, D]\mathcal{L}. \quad (2.40)$$

The lowest component which must be $SU(1,1)$ invariant is

$$N_s(\mathcal{L}\mathcal{L})_1 = \mathcal{N}(JJ) - \frac{2}{3}L \quad (2.41)$$

where $\mathcal{N}(JJ)$ denotes the $SU(1,1)$ invariant normal ordered product. As expected from $Osp(2|2)$ invariance $N_s(\mathcal{L}\mathcal{L})_1$ is orthogonal to $L$.

(ii) $N_s(\mathcal{L}[\overline{D}, D]\mathcal{L})$

$$\mathcal{L}(Z_1)\mathcal{L}(Z_2) = \ldots + \theta_{12}\overline{\theta}_{12} \left( \frac{1}{2} N_s(\mathcal{L}[\overline{D}, D]\mathcal{L})(Z_2) + \frac{3}{20}[\overline{D}, D]N_s(\mathcal{L}\mathcal{L}) + O(\partial) \right) + \ldots \quad (2.42)$$

Now one calculates the lowest component and applies the $SU(1,1)$ projector.

$$N_s(\mathcal{L}[\overline{D}, D]\mathcal{L})_1 = 2\mathcal{N}(JL) - \frac{3}{5}N_s(\mathcal{L}\mathcal{L})'_{\theta\overline{\theta}} \quad (2.43)$$

Note that the super fields $N_s(\mathcal{L}D\mathcal{L})$, $N_s(\mathcal{L}\overline{\mathcal{L}}\mathcal{L})$ vanish. Otherwise they would contribute in (2.42) as well.

(iii) $N_s(\mathcal{L}D\partial\mathcal{L})$

$$\mathcal{L}(Z_1)\mathcal{L}(Z_2) = \ldots + \theta_{12} \left( N_s(\mathcal{L}D\partial\mathcal{L})(Z_2) + \frac{1}{6}D N_s(\mathcal{L}[\overline{D}, D]\mathcal{L}) + O(\partial) \right) + \ldots \quad (2.44)$$
Hence
\[ N_s(\mathcal{L}\mathcal{D}\mathcal{L})_1 = \frac{1}{\sqrt{2}} N(J\partial G) - \frac{1}{\sqrt{2}} \frac{1}{6} N_s(\mathcal{L}[\bar{D}, D]\mathcal{L})_\sigma. \] (2.45)

With this procedure we are able to generate higher normal ordered products successively. Applying the \( Osp(2|2) \) projector to (2.38) yields the following properties of the super quasiprimary fields

\[ N_s(\Phi_j \partial \Phi_i) = -N_s(\partial \Phi_j \Phi_i) \]
\[ N_s(\Phi_j D \Phi_i) = (-1)^{2\Delta_i + 1} N_s(D \Phi_j \Phi_i) \] (2.46)
\[ N_s(\Phi_j \bar{D} \Phi_i) = (-1)^{2\Delta_i + 1} N_s(\bar{D} \Phi_j \Phi_i) \]
\[ N_s(\Phi_j [\bar{D}, D] \Phi_i) = N_s([\bar{D}, D] \Phi_j \Phi_i). \]

Furthermore, the operation \( N_s(\ldots) \) is antisymmetric for superfields of half-integer dimension and symmetric otherwise. This property together with (2.46) implies for example

\[ N_s(\mathcal{L}\mathcal{D}\mathcal{L}) = N_s(\mathcal{L}\bar{D}\mathcal{L}) = 0. \] (2.47)

Now we have developed the N=2 supersymmetric structure which is necessary for the explicit construction of N=2 super W-algebras. We have defined quasiprimary normal ordered products so that we are allowed to apply the formulas (2.26)-(2.34) to them. In the next section we explicitly construct examples of N=2 super W-algebras with one additional pair of super primary fields besides the Virasoro field.
3. N=2 super W-algebras

In this section we apply the formalism of the last section to the explicit construction of N=2 super W-algebras. We restrict ourselves to extensions of the N=2 super Virasoro algebra by one single pair of super primary fields. Note that it would be hardly a feasible task to explicitly construct a general N=0 W-algebra containing twelve generators. But the N=2 structure is so restrictive that such a complex calculation becomes practicable. Up to now one knows only two nonlinear quantum N=2 super W-algebras explicitly. These are the SW(1, 3/2) [13] and the SW(1, 2) [14] algebra. The first one has been studied by Inami et al. using an N=2 conformal bootstrap algorithm. Our results show that there are more restrictions on the allowed $c$ values than given in [13], where this, however, had already been expected. The second algebra has been calculated by Romans using algebraic methods. But he considered this algebra only for vanishing $U(1)$ charge $Q$ of the spin two generator. For this case he found that the SW(1, 2) algebra exists for generic value of the central charge $c$.

In this paper we extend this series up to spin 5/2 and we allow the additional generators to have non-vanishing charge $Q$. For SW(1, 3) we give a maximal set of values of $c$ and $Q$ for which this algebra may exist.

3.1. ALGORITHM

We want to extend the N=2 super Virasoro algebra by a pair of super primary fields $\Phi_+^Q$ and $\Phi_-^{-Q}$. For generic charge $Q$ the OPEs $\Phi_+^Q(Z_1) \Phi_-^{-Q}(Z_2)$ are trivial and the OPE $\Phi_+^Q(Z_1) \Phi_-^{-Q}(Z_2)$ contains in its singular part only descendants of the super Virasoro field $L$. For integer dimension $\Delta$ a self coupling is only possible if the charge is zero, for half-integer dimension one needs charge $Q = \pm 1$.

We choose the following normalisation of the two point function

$$\langle \Phi_+^Q(Z_1) \Phi_-^{-Q}(Z_2) \rangle = \frac{c/\Delta}{Z_{12}^{2\Delta}} \left(1 - \frac{Q \theta_1 \theta_2}{2} \frac{\bar{\theta}_1 \bar{\theta}_2}{Z_{12}} \right). \tag{3.1}$$

For explicit computer calculations we expand the components into Fourier-modes making it possible to apply Lie algebra methods like for example the universal polynomials in (2.4)-(2.7). For consistency of the algebra, all commutators have to satisfy Jacobi identities. Checking this leads to restrictions on the central charge $c$ and the $U(1)$ charge $Q$. We
should note that an important advantage of the mode approach compared to OPE’s is that it allows more directly to investigate highest weight representations [28]. Thus, we proceed as follows:

(a) We write down the most general ansatz for the super OPE. To this end, we need a basis of super quasiprimary normal ordered products for each occurring dimension. As already mentioned it is sufficient to know the lowest components of these fields.

(b) Then we expand the super fields into their components and calculate the structure constants for the lowest components. The structure constants for the higher components are given by the formulae presented in appendix A. Note that N=2 supersymmetry implies that we only have to calculate one $D_5$-matrix for every super conformal dimension $\Delta = \delta$. Thus the effort minimizes considerably.

(c) Finally, we have to check Jacobi identities. It is sufficient to check them only for the additional super primaries because Jacobi identities including the super Virasoro field are satisfied automatically by $Osp(2|2)$ invariance. Furthermore, some Jacobi identities are related by charge conjugation $Q \to -Q$.

We denote by $(\varphi_i \varphi_j \varphi_k \varphi_l)$ the factor of the Jacobi identity $[\varphi_i, [\varphi_j, \varphi_k]] + \text{cycl.}$ in front of the field $\varphi_l^*$. $\varphi_l^*$ is the adjoint of $\varphi_l$ with respect to the form $\langle ., . \rangle$, for example $(\psi^+)^* = \overline{\psi^-}$.

Writing down the Jacobi identities explicitly one obtains:

If $\delta_i = \delta_j$ and $\delta_k = \delta_l$ then

$$(\varphi_i \varphi_j \varphi_k \varphi_l) = 0 \Rightarrow (\varphi_j \varphi_i \varphi_l \varphi_k) = 0. \quad (3.2)$$

Exploiting these symmetries, one has to check only the following twelve Jacobi identities

$$(\varphi^+ \varphi^+ \varphi^- \varphi^-) \quad (\chi'^+ \chi'^+ \chi'^- \chi'^-)$$

$$(\varphi^+ \varphi^+ \varphi^- \chi'^-) \quad (\varphi^+ \chi'^+ \chi'^- \chi'^-)$$

$$(\varphi^+ \varphi^+ \chi'^- \chi'^-) \quad (\varphi^+ \varphi^- \chi'^+ \chi'^-)$$

$$(\varphi^+ \varphi^+ \psi^- \overline{\psi^-}) \quad (\varphi^+ \varphi^- \psi^+ \overline{\psi^-})$$

$$(\psi^+ \overline{\psi^-} \chi'^+ \chi'^-) \quad (\psi^+ \overline{\psi^-} \chi'^- \chi'^-)$$

$$(\psi^+ \overline{\psi^-} \psi^+ \overline{\psi^-}) \quad (\psi^+ \overline{\psi^-} \psi^- \overline{\psi^-}).$$

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Our calculations assure us that $\text{N}=2$ supersymmetry generates further relations among the Jacobi identities above.

3.2. $\text{SW}(1,\Delta)$ ALGEBRAS

In this subsection we present the results of our calculations. We should note beforehand that all the $\text{N}=2$ super $\text{W}$-algebras considered only exist for some discrete values of the central charge $c$ and the $U(1)$ charge $Q$.

$\text{SW}(1,3/2)$

In the OPE $\Phi^+_3(Z_1)\Phi^-_3(Z_2)$ the following super quasiprimary fields may occur for generic value of $Q$.

| $\Delta$ | $Q$ | $\text{fields}$ |
|----------|-----|-----------------|
| 1        | 0   | $\mathcal{L}$   |
| 2        | 0   | $N_s(\mathcal{L}\mathcal{L})$ |
| 3        | 0   | $N_s(N_s(\mathcal{L}\mathcal{L})\mathcal{L})$ |
| 3        | 0   | $N_s(\mathcal{L}[\bar{D}, D]\mathcal{L})$ |

where the first component of the field $N_s(N_s(\mathcal{L}\mathcal{L})\mathcal{L})$ is given by

\[ N_s(N_s(\mathcal{L}\mathcal{L})\mathcal{L})_1 = N(N_s(\mathcal{L}\mathcal{L})_1 J) - \frac{2}{5} N_s(\mathcal{L}\mathcal{L})'_{\theta\bar{\theta}}. \] (3.4)

The Kac-determinants in the vacuum sector are

| $\Delta$ | $\text{det}(D_{\Delta}) \sim$ |
|----------|-----------------|
| 1        | $c$              |
| 2        | $c(c - 1)$       |
| 3        | $c^2(2c - 3)(c + 6)(c - 1)$ |

The structure constants $C^k_{ij}$, $\alpha_{ijk}$ for the normal ordered products are rational functions in $c$ and $Q$:

\[
C^\mathcal{L}_{\Phi^+\Phi^-} = Q \quad \alpha = \frac{3}{Q} \\
C^{N(\mathcal{L}\mathcal{L})}_{\Phi^+\Phi^-} = \frac{3(Q^2 - 1)}{2(c - 1)} \quad \alpha = \frac{8Q}{3(Q^2 - 1)}
\]
\[ C^{3N(D, D) L}(\alpha - 1) = -\frac{(3c - 2Q - 4)(Q^2 - 9)}{3(2c - 3)(c + 6)} \]

\[ C^{N(N(D, D) L)}(\alpha - 1) = -\frac{(2c(Q^2 - 1) + 13Q^2 - 1)(Q^2 - 9)}{2(2c - 3)(c + 6)(c - 1)} \]

In the following table we list the rational values of the parameters \( c \) and \( Q \) for which all Jacobi identities (3.3) hold.

| \( Q \) | \( c \) |
|---|---|
| 0 | \( 9/4, 5/3, -3/2 \) |
| \( \pm 3 \) | 9 |
| \( \pm 1 \) | 3 |

For \( \Delta = \pm Q/2 \) the fields \( \Phi_{\Delta}^Q \) and \( \Phi_{-\Delta}^{-Q} \) are so-called primary (anti)chiral superfields [19], i.e. they satisfy \( \overline{D} \Phi_{\Delta}^Q = 0 \) for the chiral, and \( D \Phi_{\Delta}^Q = 0 \) for the antichiral case. The chiral \( SW(1, 3/2) \) algebra had already been constructed in ref. [13,20]. It is of particular interest because it exists for \( c = 9 \) which allows it to be used 4D string compactification. One important drawback, however, is that according to an argument of Cardy [21,22] no minimal representations are allowed to exist for an algebra with four bosonic and four fermionic generators for \( c \geq 6 \). For a general \( SW(1, \Delta) \) algebra with twelve generators the same argument implies that only for \( c < 9 \) minimal representations are possible. We remark that by constructing this algebra using Jacobi identities we obtain only a subset of the rational \( c \) values claimed by Inami et al.. They found that the four-point functions of the components are associative for generic value of \( c \). This behaviour is very strange since in the \( N=0 \) and \( N=1 \) case the two requirements are equivalent.

For the charge \( Q = \pm 1 \) the fields \( \Phi_{\Delta}^1 \) and \( \Phi_{-\Delta}^{-1} \) themselves could occur in all OPEs of the two super primary fields. This implies the appearance of the following descendants

| \( \Delta \) | \( Q \) | fields |
|---|---|---|
| 5/2 | +1 | \( N_s(\Phi^1 D L) \) |
| 5/2 | -1 | \( N_s(\Phi^{-1} D L) \) |
| 3 | 0 | \( N_s(\Phi^1 L) \) |
| 3 | 0 | \( N_s(\Phi^{-1} D L) \) |
| 3 | 2 | \( N_s(\Phi^1 D L) \) |
| 3 | -2 | \( N_s(\Phi^{-1} D L) \) |

Table 4
Realize that the last two fields may only occur in the OPEs $\Phi^{+1}_{3/2}(Z_1) \Phi^{+1}_{3/2}(Z_2)$ and $\Phi^{-1}_{3/2}(Z_1) \Phi^{-1}_{3/2}(Z_2)$, respectively. Repeating the whole calculation with this further extended algebra, we obtain that all Jacobi identities are satisfied only for the central charges $c = 3$ and $c = -1$. For $c = -1$ we obtain the additional consistency condition

$$C_{\Phi^{+1}_{-1}} C_{\Phi^{+1}_{-1}} = -\frac{128}{15}. \quad (3.6)$$

**SW(1,2)**

In the OPE super quasiprimary fields up to dimension four may occur. The new ones are listed in the following table

| $\Delta$ | $Q$ | fields |
|----------|-----|--------|
| $7/2$    | $-1$| $N_s(\mathcal{D}\partial\mathcal{L})$ |
| $7/2$    | $1$ | $N_s(\mathcal{D}\partial\mathcal{L})$ |
| $4$      | $0$ | $N_s(N_s(\mathcal{L}\mathcal{L})\mathcal{L})$ |
| $4$      | $0$ | $N_s(N_s(\mathcal{L}\mathcal{D}, D)\mathcal{L})$ |
| $4$      | $0$ | $N_s(\mathcal{L}\partial^2\mathcal{L})$ |

The new determinants of the $D_{\Delta}$ matrices are

| $\Delta$ | $\det(D_{\Delta}) \sim$ |
|----------|-------------------|
| $7/2$    | $c^2(c - 1)^2$    |
| $4$      | $c^3(5c - 9)(2c - 3)(c + 6)(c + 1)(c - 1)^2$ |

This algebra exists for the following $c$ and $Q$ values

| $Q$ | $c$ |
|-----|-----|
| $0$ | $12/5, -3$ |
| $\pm 4$ | $12, -9, -21,$ |
| $\pm 3/2$ | $15/4$ |

Our results are in agreement with those of Romans [14]. The values $c = 12/5$ and $c = -3$ are just the zeros of the self-coupling $C_{\Phi^{+1}_{-1}}$. Besides the chiral case $c = 12$, the value $c = 15/4$ is very remarkable because it is greater than three. In the N=0 and N=1 case one has
never found a similar behaviour: There, all the W-algebras, which do not exist generically have central charges $c < 1$ or $c < 3/2$, respectively. In the next section we will conjecture that this theory could be related to the non-compact coset model $(SU(1,1)/U(1)) \times U(1)$.

**SW(1,5/2)**

In the OPE super quasiprimary fields up to dimension five may occur.

| $\Delta$ | $Q$ | $fields$ |
|----------|-----|----------|
| 9/2      | -1  | $N_s(N_s(\mathcal{L}\mathcal{D}\partial\mathcal{L})\mathcal{L})$ |
| 9/2      | 1   | $N_s(N_s(\mathcal{L}\bar{\mathcal{D}}\partial\mathcal{L})\mathcal{L})$ |
| 5        | 0   | $N_s(N_s(N_s(\mathcal{L}\mathcal{L}\mathcal{L})\mathcal{L})\mathcal{L})$ |
| 5        | 0   | $N_s(N_s(N_s(\mathcal{L}\mathcal{D}[\mathcal{D},\mathcal{L}]\mathcal{L})\mathcal{L})$ |
| 5        | 0   | $N_s(N_s(\mathcal{L}\mathcal{L}\partial\mathcal{L}L)\mathcal{L})$ |
| 5        | 0   | $N_s(N_s(\mathcal{L}\mathcal{D}\partial\mathcal{L})\mathcal{D})$ |
| 5        | 0   | $N_s(N_s(\mathcal{L}\mathcal{D}\partial\mathcal{L})\mathcal{L})$ |
| 5        | 0   | $N_s(\mathcal{L}[\mathcal{D},\mathcal{D}][\partial^2\mathcal{L}]$ |

The Kac determinants of the $D_\Delta$ matrices are

| $\Delta$ | $det(D_\Delta) \sim$ |
|----------|---------------------|
| 9/2      | $c^2(c-1)^2(2c-3)^2$ |
| 5        | $c^6(5c-9)(2c-3)^2(c+12)(c+6)^3(c+1)(c-1)^5(c-2)$ |

This algebra exists for the following $c$ and $Q$ values

| $Q$ | $c$          |
|-----|--------------|
| 0   | $5/2, 9/7, -9/2$ |
| $\pm 5$ | 15  |
| $\pm 2$ | 9/2   |
| $\pm 1$ | 3     |

Just like the $SW(1, 3/2)$ algebra, this algebra exists for $Q = \pm 1$ and $c = 3$.

**SW(1,3)**

Since the commutator of the lowest components contains only fields with maximal dimension five, we could calculate the Jacobi identity $(\varphi^+ \varphi^+ \varphi^- \varphi^-)$ without generating new
normal ordered products. For vanishing charge $Q$ the Jacobi identity is satisfied trivially. But for non-vanishing charge one gets two conditions which have only finitly many rational solutions for $Q$ and $c$.

| $Q$   | $c$        |
|-------|-----------|
| 0     | ?         |
| $\pm 6$ | 18, $-15$, $-33$ |
| $\pm 5/2$ | $21/4$     |
| $\pm 4/3$ | $10/3$     |
| $\pm 3/4$ | $45/16$    |

table 11

Comparing with the $SW(1, 2)$ algebra, we suppose that the last two solutions will turn out to be inconsistent with the remaining Jacobi identities.

4. Discussion

In this section we shall speculate on the general structure of $N=2$ $SW(1, \Delta)$ algebras regarding the data presented in the previous section. For the time being, we can only extract a few hints for the complete classification of these algebras.

In the $N=0$ and $N=1$ case, the discrete values of $c$ for which $W$-algebras exist can be understood from the representation theory of the (super) Virasoro algebra. In the $N=2$ case the situation is similar. But here also the unitary non-minimal representations play a rôle. Let us bring some order into the list of existing $N=2$ super $W$-algebras.

(a) The following table displays the $c$-values contained in the unitary series (2.15) of the $N=2$ super Virasoro algebra

| $\Delta$ | $Q$ | $c$ | $k = l$ | $m$ |
|----------|-----|-----|---------|-----|
| $3/2$    | 0   | $9/4$ | 6       | 0   |
| 2        | 0   | $12/5$ | 8       | 0   |
| $5/2$    | 0   | $5/2$ | 10      | 0   |

table 12

where $k, l, m$ label the minimal unitary representations of the $N=2$ super Virasoro algebra in the Neveu-Schwarz sector [5]
\[ c = \frac{3k}{k+2} \quad k = 1, 2, \ldots \]
\[ \Delta = \frac{l(l+2)-m^2}{4(k+2)} \quad l = 0, \ldots, k \]
\[ Q = \frac{m}{(k+2)} \quad m = -l, -l+2, \ldots, l \]

(4.1)

Calculation of the fusion rule [18] for the field \( \Phi_{l=k}^{m=0} \) shows that it is indeed a simple current [23]
\[ [\Phi_{l=k}^{m=0}] \times [\Phi_{l=k}^{m=0}] = [1]. \]

(4.2)

For integer dimension \( k = 4\Delta \) the list of known modular invariant partition functions [24] contains a suitable, so called integer spin, off-diagonal invariant which can be interpreted as the diagonal invariant of the extended algebra \( SW(1, \Delta) \). For half-integer dimension, i.e. \( k = 4\Delta = 4j - 2 \), there exists a so called automorphism invariant \( D_{2j+1} \)
\[ \sum_{l=0}^{k/2} |\sigma_{2l}|^2 + \sum_{l=0}^{2j-1} \sigma_{2l+1}\sigma_{k-2l-1}^* \]

(4.3)

up to a diagonal \( U(1) \) part.

This partition function can be rewritten in a fermionic form
\[ \frac{1}{2} \sum_{l=0}^{j-1} (|\sigma_{2l} + \sigma_{k-2l}|^2 + |\sigma_{2l} - \sigma_{k-2l}|^2) \]
\[ + \frac{1}{2} \sum_{l=0}^{j-1} (|\sigma_{2l+1} + \sigma_{k-2l-1}|^2 - |\sigma_{2l+1} - \sigma_{k-2l-1}|^2) \]

(4.4)

where the first two terms form the Neveu-Schwarz sector and the second two the Ramond sector. The last term is the Witten-index of the representations of the extended algebra and therefore a constant. Thus, omitting it leads to a diagonal fermionic partition function for the \( SW(1, \Delta) \) algebra.
(b) For $Q = 0$ one can conjecture two other series

| $\Delta$ | $c$ | $k$ |
|----------|-----|-----|
| $3/2$   | $5/3$ | $5/2$ |
| $5/2$   | $9/7$ | $3/2$ |

Note that the half-integer values of $k$ give no minimal representations of the N=2 super Virasoro algebra. It is not clear how this series will continue. There also occurs a non-unitary series with negative values of $c$.

| $\Delta$ | $c$ |
|----------|-----|
| $3/2$   | $-3/2$ |
| $2$     | $-3$ |
| $5/2$   | $-9/2$ |

These values fit nicely into $c = 3(1 - \Delta)$ and we conjecture the existence of all $SW(1, \Delta)$ algebras for this central charge in analogy to the descendant series of negative $c$ values in the N=0 and N=1 case. In ref. [25] these W-algebras have been investigated in more detail presenting a vertex operator construction of the generators.

(c) For $Q \neq 0$ the spectral flow algebras occur.

| $\Delta$ | $Q$ | $c$ |
|----------|-----|-----|
| $3/2$   | $\pm 3$ | $9$ |
| $2$     | $\pm 4$ | $12$ |
| $5/2$   | $\pm 5$ | $15$ |
| $3$     | $\pm 6$ | $18$ |

It is well known that the N=2 Virasoro algebra has a non-trivial automorphism which continuously connects the Neveu-Schwarz and the Ramond sector [26]. It acts in the following way

$$J_n \to J_n + \frac{c}{3} \alpha \delta_{n,0}$$

$$G^\pm_n \to G^\pm_{n \pm \alpha}$$

$$L_n \to J_n + \alpha J_n + \frac{c}{6} \alpha^2 \delta_{n,0}.$$
For $\alpha = \pm 1$ one obtains a (anti)chiral spectral flow operator $\Phi^{\pm c/3}_{c/6}$. The extension of the N=2 Virasoro algebra by these spectral flow operators yields the W-algebras above. The representation theory of these algebras has been studied by Odake [27,28] confirming that no minimal representations exist.

Note that for integer dimension $\Delta$ the chiral algebras exist also for two additional negative values of $c$.

| $\Delta$ | $Q$  | $c$       |
|---------|------|-----------|
| 2       | $\pm 4$ | $-9, -21$ |
| 3       | $\pm 6$ | $-15, -33$ |

Table 16

Assuming that $c$ depends only linearly on $\Delta$ the values above fit into the two series $c = 3(1 - 2\Delta)$ and $c = 3(1 - 4\Delta)$.

(d) The unitary necessarily non-unitary representations with $c > 3$ have been constructed by Dixon et al. [29] from the non-compact coset model $(SU(1, 1)/U(1)) \times U(1)$.

The following W-algebras should be connected to this coset.

| $\Delta$ | $Q$  | $c$ | $k$ | $j$ | $m$ |
|---------|------|-----|-----|-----|-----|
| 2       | $\pm 3/2$ | $15/4$ | 10  | 5   | $\mp 6$ |
| 5/2     | $\pm 2$ | $9/2$ | 6   | 3   | $\mp 4$ |
| 3       | $\pm 5/2$ | $21/4$ | $14/3$ | 7/3 | $\mp 10/3$ |

Table 17

with

$$c = \frac{3k}{k - 2}, \quad k > 2$$

$$\Delta = -\frac{j(j - 1) - m^2}{(k + 2)}, \quad j = n + \epsilon, \quad \epsilon = 0, 1/2$$

$$Q = -\frac{2m}{(k - 2)}, \quad m = j + r, \quad r \in Z_0^+$$

Besides the mentioned positive series $D_n^+$ and negative series $D_n^-$, there also exist a continuous series which, however, is not relevant for our purpose. Thus, we conjecture a series of W-algebras $SW(1, \Delta)$ to exist for
\[ c = \frac{3}{4}(2\Delta + 1) \quad Q = \pm(\Delta - \frac{1}{2}) \]  

(4.7)

In particular, the \( \mathcal{SW}(1,11/2) \) should exist for \( c = 9 \) and \( Q = \pm 5 \).

Note that the \( c \)-values in table 17 are also contained in the unitary series of the Kazama-Suzuki model \( SU(3)/(SU(2) \times U(1)) \), which is probably related to the \( \mathcal{SW}(1,2) \) algebra of Romans with

\[ c = \frac{6k}{k + 3}, \quad k \in \mathbb{Z}_+ \]  

(4.8)

For \( \Delta > 7/2 \) we suppose a connection to the non-compact coset model \( SU(2,1)/(SU(2) \times U(1)) \) of Bars [30], which has the following unitary series

\[ c = \frac{6k}{k - 3}, \quad k > 3 \]  

(4.9)

It would be interesting to clarify the connections among compact or non-compact Kazama-Suzuki models, Romans’ N=2 super \( W_3 \) algebra and our \( \mathcal{SW}(1,\Delta) \) series.

(e) Our last conjecture reads that for half-integer dimension the \( W \)-algebra exists for \( c = 3 \) and \( Q = \pm 1 \).

| \( \Delta \) | \( Q \) | \( c \) |
|---|---|---|
| 3/2 | ±1 | 3 |
| 5/2 | ±1 | 3 |

(table 18)

We expect that there exists a free field construction of these \( W \)-algebras with one chiral and one antichiral fermion.

5. Summary

In this paper we have investigated extensions of the N=2 super Virasoro algebra, using an explicit N=2 supersymmetric formalism. The motivation for this project was to get some insight into the structure of N=2 symmetry algebras which do not exist for generic value of \( c \), in contrast to the Kazami-Suzuki models. There also is some hope to find a striking \( W \)-algebra as a new candidate for the string vacuum in a Gepner-like construction.

We have found that the structure of the simplest N=2 extended algebras, the \( \mathcal{SW}(1,\Delta) \) series, is different from the analogous N=0 and N=1 cases. The most interesting property
— and very important for applications in string theory — is that these algebras are allowed to exist for \( c \geq 3 \). Starting from the examples calculated we have conjectured the existence of series of \( SW(1, \Delta) \) algebras. A proof would have to use other methods like free field constructions and vertex operators.

As for the value \( c = 9 \) distinguished in string theory, we conjecture the \( SW(1, 11/2) \) to exist for \( Q = \pm 5 \) besides the spectral flow algebra \( SW(1, 3/2) \) with \( Q = \pm 3 \), which is already known. Unfortunately, these two algebras do not admit minimal representations for \( c = 9 \). To obtain minimal representations, one has to investigate \( N=2 \) \( W \)-algebras with more generators. Nevertheless, the \( SW(1, \Delta) \) algebras provide some new examples of extended \( N=2 \) symmetries.

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### Appendix A

In this appendix we express the component structure constants by the supersymmetric coupling constants. We use the abbreviation \( \Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k \).

Neglecting \( 1/\sqrt{2} \) factors we consider superfields

\[
\Phi(Z) = \varphi(z) + \overline{\theta\psi}(z) + \overline{\theta\psi}(z) + \overline{\theta\chi}(z)
\]

\( (i) \) \( Q_i + Q_j = Q_k \)

\[
\begin{align*}
C_{\varphi_i \varphi_j}^{\phi_k} &= 2C_{ij}^k \\
C_{\varphi_i \varphi_j}^{\overline{\psi}_k} &= C_{ij}^k \frac{2\Delta_{ijk}(Q_k/2-\alpha\Delta_k)}{(2\Delta_k+1)((Q_k/2)^2-\Delta_k)} \\
C_{\overline{\psi}_i \overline{\psi}_j}^{\phi_k} &= C_{ij}^k \frac{\Delta_{ijk}(\alpha+1)}{(Q_k/2+\Delta_k)} \\
C_{\overline{\psi}_i \overline{\psi}_j}^{\psi_k} &= C_{ij}^k \frac{\Delta_{ijk}(\alpha-1)}{(Q_k/2-\Delta_k)} \\
C_{\overline{\psi}_i \overline{\psi}_j}^{\overline{\psi}_k} &= C_{ij}^k \Delta_{ijk} \left( \alpha - \frac{Q_i/2}{\Delta_i} \right) \\
C_{\overline{\psi}_i \overline{\psi}_j}^{\overline{\psi}_k} &= \frac{2}{(2\Delta_k+1)((Q_k/2)^2-\Delta_k)} \left( \Delta_{ijk}+1 \right) \left[ \frac{Q_k}{2} \left( \frac{Q_i/2}{\Delta_i} - \alpha \right) + \Delta_k \left( 1 - \alpha \frac{Q_i/2}{\Delta_i} \right) \right] \\
C_{\varphi_i \overline{\psi}_j}^{\varphi_k} &= (-1)^{(2\Delta_i)} \left[ 2 - \Delta_{ijk} \frac{(1+\alpha)}{(Q_k/2-\Delta_k)} \right]
\end{align*}
\]
\[ C^\psi_{\psi_i\psi_j} = C^\chi_{\psi_i\psi_j} = 0 \]
\[ C^{\phi_k}_{\psi_i\psi_j} = C_{\psi_i\psi_j}^k (-1)^{(2\Delta_i+1)} \left[ \Delta_{ijk} + \Delta_{ikj} \alpha - Q_i \right] \]
\[ C^{\chi_k}_{\psi_i\psi_j} = C_{\psi_i\psi_j}^k (-1)^{(2\Delta_i+1)} \frac{\Delta_{ijk}}{(2\Delta_k+1)} \left[ \frac{(Q_k/2-\alpha\Delta_k)(\Delta_{ijk}-Q_i)-(\Delta_{ikj}(\Delta_k-\alpha Q_k/2)+2\Delta_k(1-\alpha)(Q_k/2+\Delta_k))}{(Q_k/2)^2-\Delta_k^2} \right] \]
\[ \bar{C}^{\psi_k}_{\chi_i\psi_j} = C_{\psi_i\psi_j}^k (-1)^{(2\Delta_k)} \Delta_{ikj} \left[ \frac{\Delta_{ijk}(1+\alpha)}{2(Q_k/2-\Delta_k)} \left( 1 - \frac{Q_j/2}{\Delta_j} \right) + \left( \alpha - \frac{Q_j/2}{\Delta_j} \right) \right] \]
\[ C^{\phi_k}_{\psi_i\psi_j} = C_{\psi_i\psi_j}^k (-1)^{(2\Delta_i+1)} \left[ \Delta_{ijk} - \Delta_{ikj} \alpha + Q_i \right] \]
\[ C^{\chi_k}_{\psi_i\psi_j} = C_{\psi_i\psi_j}^k (-1)^{(2\Delta_i+1)} \frac{\Delta_{ijk}}{(2\Delta_k+1)} \left[ \frac{(Q_k/2-\alpha\Delta_k)(\Delta_{ijk}+Q_i)+\Delta_{ikj}(\Delta_k-\alpha Q_k/2)+2\Delta_k(1-\alpha)(Q_k/2-\Delta_k)}{(Q_k/2)^2-\Delta_k^2} \right] \]
\[ C^{\phi_k}_{\psi_i\psi_j} = C_{\psi_i\psi_j}^k (-1)^{(2\Delta_i+1)} \left[ \Delta_{ijk} - \Delta_{ikj} \alpha + Q_i \right] \]
\[ C^{\psi_k}_{\chi_i\psi_j} = C_{\psi_i\psi_j}^k (-1)^{(2\Delta_k)} \Delta_{ikj} \left[ \frac{\Delta_{ijk}(1+\alpha)}{2(Q_k/2-\Delta_k)} \left( 1 + \frac{Q_j/2}{\Delta_j} \right) + \left( \alpha - \frac{Q_j/2}{\Delta_j} \right) \right] \]
\[ C^{\phi_k}_{\chi_i\psi_j} = C_{\chi_i\psi_j}^k \left[ \Delta_{ijk} \alpha - 2\Delta_i - \Delta_{ijk} \frac{Q_j/2}{\Delta_j} \right] \]
\[ C^{\chi_k}_{\chi_i\psi_j} = C_{\chi_i\psi_j}^k \left[ 2 + \frac{\Delta_{ijk}}{(2\Delta_k+1)} \left[ \frac{1}{(Q_k/2-\Delta_k)^2} \right] \left( 2(2\Delta_k+1)(\Delta_k - \alpha Q_k/2) - (\Delta_{ijk} + 1)(\Delta_k - \alpha Q_k/2) \right. \right. \]
\[ \left. \left. - (Q_k/2-\alpha\Delta_k) \left( Q_i + (\Delta_{ijk} - 1) \frac{Q_j/2}{\Delta_j} \right) \right] \right) \]
\[ \bar{C}^{\psi_k}_{\chi_i\psi_j} = C_{\chi_i\psi_j}^k \left[ \Delta_{ijk} \alpha - (\Delta_{ijk} + Q_i) + \Delta_{ikj} \Delta_{ijk}(1+\alpha) \frac{1}{2(Q_k/2-\Delta_k)} \left( 1 - \frac{Q_j/2}{\Delta_j} \right) \right] \]
\[ C^{\psi_k}_{\chi_i\psi_j} = C_{\chi_i\psi_j}^k \left[ \Delta_{ijk} \alpha + (\Delta_{ijk} + Q_i) + \Delta_{ikj} \Delta_{ijk}(1+\alpha) \frac{1}{2(Q_k/2-\Delta_k)} \left( 1 + \frac{Q_j/2}{\Delta_j} \right) \right] \]
\[ C^{\phi_k}_{\chi_i\psi_j} = C_{\chi_i\psi_j}^k \left[ \Delta_{ijk}(1+\alpha) \frac{1}{4\Delta_i} \frac{Q_i(\Delta_{ijk}+1)}{4\Delta_j} \left( \Delta_{ijk} \alpha - Q_i - \Delta_{ijk} \frac{Q_j/2}{\Delta_j} \right) \right. \]
\[ \left. - \frac{Q_i(\Delta_{ijk}+1)}{4\Delta_j} \Delta_{ijk} \left( \alpha - \frac{Q_j/2}{\Delta_j} \right) \right] \]
\[ C^{\chi_k}_{\chi_i\psi_j} = C_{\chi_i\psi_j}^k \left[ \Delta_{ijk} \frac{Q_i(Q_k/2-\alpha\Delta_k)}{(2\Delta_k+1)(Q_k/2-\Delta_k)} + \Delta_{ijk} \Delta_{ikj} \frac{(Q_k/2-\alpha\Delta_k)}{(Q_k/2)^2-\Delta_k^2} + \Delta_{ikj} \alpha - Q_i \right] \]
\[ + \frac{Q_i \Delta_{ijk}}{4\Delta_i} C^{\chi_k}_{\psi_i\psi_j} - \frac{Q_i \Delta_{ijk}}{4\Delta_j} C^{\chi_k}_{\chi_i\psi_j} + \frac{Q_i Q_j}{8\Delta_i \Delta_j} (\Delta_{ijk}) \frac{C^{\chi_k}_{\psi_i\phi_j} C^{\chi_k}_{\psi_i\phi_j}}{2} \]

(ii) \( Q_i + Q_j = Q_k - 1 \)

\[ \bar{C}^{\psi_k}_{\phi_i\phi_j} = C_{\phi_i\phi_j}^k \frac{1}{(Q_k/2+\Delta_k)} \]
\[ C^{\phi_k}_{\psi_i\phi_j} = C_{\psi_i\phi_j}^k = 0 \]
\[ C^{\phi_k}_{\psi_i\phi_j} = C_{\psi_i\phi_j}^k \]

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\[ C'_{\chi_k \psi_i \phi_j} = C_{ij}^k \frac{(\Delta_{ikj}+1/2)}{(2\Delta_k+1)(Q_k/2+\Delta_k)} \]
\[ C'_{\chi'_k \phi_i \psi_j} = -C_{ij}^k \frac{(\Delta_{ikj}+1/2)}{2(Q_k/2+\Delta_k)} \left( 1 + \frac{Q_j/2}{\Delta_j} \right) \]
\[ C'_{\phi'_k \psi_i \psi_j} = C_{ij}^k \frac{(\Delta_{ikj}+1/2)}{2(Q_k/2+\Delta_k)} \left( 1 + \frac{Q_j/2}{\Delta_j} \right) \]
\[ C'_{\phi_k \psi_i \psi_j} = C_{ij}^k \frac{(\Delta_{ikj}+1/2)}{2(Q_k/2+\Delta_k)} \left( 1 + \frac{Q_j/2}{\Delta_j} \right) \]
\[ C'_{\phi_k \psi_i \psi_j} = C_{ij}^k \frac{(\Delta_{ikj}+1/2)}{2(Q_k/2+\Delta_k)} \left( 1 + \frac{Q_j/2}{\Delta_j} \right) \]
\[ C'_{\phi_k \psi_i \psi_j} = C_{ij}^k \frac{(\Delta_{ikj}+1/2)}{2(Q_k/2+\Delta_k)} \left( 1 + \frac{Q_j/2}{\Delta_j} \right) \]
\[ C'_{\phi_k \psi_i \psi_j} = C_{ij}^k \frac{(\Delta_{ikj}+1/2)}{2(Q_k/2+\Delta_k)} \left( 1 + \frac{Q_j/2}{\Delta_j} \right) \]

(iii) \( Q_i + Q_j = Q_k + 1 \)

\[ C'_{\phi_k \psi_i \psi_j} = C_{ij}^k \frac{1}{(Q_k/2-\Delta_k)} \]
\[ C'_{\phi_k \psi_i \psi_j} = C_{ij}^k \]
\[ C'_{\phi_k \psi_i \psi_j} = C_{ij}^k \frac{(\Delta_{ikj}+1/2)}{(2\Delta_k+1)(Q_k/2-\Delta_k)} \]
\[ C'_{\phi_k \psi_i \psi_j} = C_{ij}^k \frac{(\Delta_{ikj}+1/2)}{(2\Delta_k+1)(Q_k/2-\Delta_k)} \]
\[ C'_{\phi_k \psi_i \psi_j} = C_{ij}^k \frac{(\Delta_{ikj}+1/2)}{(2\Delta_k+1)(Q_k/2-\Delta_k)} \]
In this appendix we present the component expansion of the normal ordered products $N_s(\ldots)$. The four types of fields occurring in the regular part of the OPE $\Phi_i(Z_1)\Phi_j(Z_2)$ are

\[
C_{\psi_k \chi'_i \varphi_j} = -C_{ij}^k \frac{(\Delta_{ijk}+1/2)}{2(Q_k/2-\Delta_k)} \left(1 - \frac{Q_j/2}{\Delta_i} \right)
\]
\[
C_{\varphi_k \psi_i \varphi_j} = C_{ij}^k (-1)^{(2\Delta_i+1)}
\]
\[
C_{\psi_{ik'} \psi_i \psi_j} = C_{ij}^k (-1)^{(2\Delta_i)} \frac{(\Delta_{ijk}+1/2)}{(2\Delta_k+1)(Q_k/2-\Delta_k)}
\]
\[
C_{\psi_k \psi_i \psi_j} = C_{ij}^k (-1)^{(2\Delta_i+1)} \frac{(Q_j/2-\Delta_i)}{(Q_k/2-\Delta_k)}
\]
\[
C_{\chi'_{ik'} \psi_i \psi_j} = C_{ij}^k (-1)^{(2\Delta_i)} \frac{(\Delta_{ijk}+1/2)}{2} \left(1 - \frac{Q_j/2}{\Delta_i} \right)
\]
\[
C_{\psi_{ik'} \chi_i' \varphi_j} = C_{ij}^k (-1)^{(2\Delta_i)} \frac{(\Delta_{ijk}+1/2)}{2(\Delta_k+1)(Q_k/2-\Delta_k)} \left(1 - \frac{Q_j/2}{\Delta_i} \right)
\]
\[
C_{\varphi_k \chi_i' \varphi_j} = C_{ij}^k \frac{1}{\Delta_i} \left[1 - \frac{(\Delta_{ijk}+1/2)}{2(Q_k/2-\Delta_k)} \right] + \frac{(\Delta_{ijk}+1/2)}{4\Delta_i} \left[1 - \frac{Q_j/2}{\Delta_i} \right]
\]
\[
C_{\varphi_k \psi_i \chi'_i} = -C_{ij}^k \frac{(\Delta_{ijk}+1/2)}{2} \left(1 - \frac{Q_j/2}{\Delta_i} \right)
\]
\[
C_{\chi'_{ik'} \psi_i \chi'_i} = C_{ij}^k \frac{1}{2(\Delta_k+1)} \left(1 - \frac{(\Delta_{ijk}+1/2)}{2} \right) + \frac{Q_j(\Delta_{ijk}+1/2)}{4\Delta_j} C_{\psi_k \chi'_i \varphi_j}
\]
\[
C_{\varphi_k \chi_i' \chi'_i} = C_{ij}^k \frac{1}{\Delta_i} \left[1 - \frac{(\Delta_{ijk}+1/2)}{2} \right] + \frac{Q_j(\Delta_{ijk}+1/2)}{8\Delta_i \Delta_j} \left(1 - \frac{Q_j/2}{\Delta_i} \right) C_{\psi_k \chi'_i \chi'_i} - \frac{Q_j(\Delta_{ijk}+1/2)}{4\Delta_j} C_{\psi_k \chi'_i \psi_i} + \frac{Q_j(\Delta_{ijk}+1/2)}{8\Delta_i \Delta_j} \left(1 - \frac{Q_j/2}{\Delta_i} \right) C_{\psi_k \chi'_i \chi'_i} + \frac{Q_j(\Delta_{ijk}+1/2)}{8\Delta_i \Delta_j} \left(1 - \frac{Q_j/2}{\Delta_i} \right) C_{\psi_k \chi'_i \psi_i}
\]

**Appendix B**

In this appendix we present the component expansion of the normal ordered products $N_s(\ldots)$. The four types of fields occurring in the regular part of the OPE $\Phi_i(Z_1)\Phi_j(Z_2)$ are

\[
N_s(\Phi_j \partial^n \Phi_i) = \begin{cases}
N(\partial^n \varphi_i) + \theta[(-1)^{2\Delta_i} N(\partial^n \varphi_i)]
+ \bar{\theta}[(-1)^{2\Delta_i} N(\partial^n \varphi_i)]
+ (-1)^{2\Delta_i} N(\partial^n \varphi_i)
+ \bar{\theta}[(-1)^{2\Delta_i} N(\partial^n \varphi_i)]
\end{cases}
\]

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\[ N_s(\Phi_j D \partial^n \Phi_i) = \begin{cases} 
N(\varphi_j \partial^n \bar{\psi}_i) \\
+ \theta (-1)^{2\Delta_i+1} N(\bar{\psi}_j \partial^n \varphi_i) \\
+ \theta \overline{\theta} [(-1)^{2\Delta_i+1} N(\psi_j \partial^n \bar{\psi}_i) + N(\varphi_j \partial^n \chi_i) + \frac{1}{2} N(\varphi_j \partial^{n+1} \varphi_i)] \\
+ \theta \overline{\theta} [N(\chi_j \partial^n \psi_i) - (-1)^{2\Delta_i} N(\bar{\psi}_j \partial^n \chi_i) - \frac{1}{2} N(\varphi_j \partial^{n+1} \bar{\psi}_i)] \\
- \frac{1}{2} (-1)^{2\Delta_i} N(\bar{\psi}_j \partial^{n+1} \varphi_i) \end{cases} \]

\[ N_s(\Phi_j \overline{D} \partial^n \Phi_i) = \begin{cases} 
N(\varphi_j \partial^n \psi_i) \\
+ \theta [(-1)^{2\Delta_i+1} N(\bar{\psi}_j \partial^n \psi_i) - N(\varphi_j \partial^n \chi_i) + \frac{1}{2} N(\varphi_j \partial^{n+1} \varphi_i)] \\
+ \theta \overline{\theta} [N(\chi_j \partial^n \psi_i) - (-1)^{2\Delta_i} N(\psi_j \partial^n \chi_i) + \frac{1}{2} N(\varphi_j \partial^{n+1} \psi_i)] \\
+ \theta \overline{\theta} [2N(\varphi_j \partial^n \chi_i) + \frac{1}{2} N(\varphi_j \partial^{n+1} \varphi_i) - (-1)^{2\Delta_i} 2N(\bar{\psi}_j \partial^{n+1} \varphi_i)] \end{cases} \]

\[ N_s(\Phi_j \overline{D}, D \partial^n \Phi_i) = \begin{cases} 
2N(\varphi_j \partial^n \chi_i) \\
+ \theta [(-1)^{2\Delta_i} 2N(\bar{\psi}_j \partial^n \chi_i) + N(\varphi_j \partial^{n+1} \bar{\psi}_i)] \\
+ \theta \overline{\theta} [(-1)^{2\Delta_i} 2N(\psi_j \partial^n \chi_i) - N(\varphi_j \partial^{n+1} \varphi_i)] \\
+ \theta \overline{\theta} [2N(\chi_j \partial^n \chi_i) + \frac{1}{2} N(\varphi_j \partial^{n+1} \varphi_i) - (-1)^{2\Delta_i} 2N(\bar{\psi}_j \partial^{n+1} \varphi_i)] \\
- (-1)^{2\Delta_i} 2N(\psi_j \partial^{n+1} \psi_i) \end{cases} \]

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