Block entropy for Kitaev-type spin chains in a transverse field

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Block entropy in the ground state of a quantum spin chain is investigated. The spins have Kitaev-type nearest-neighbor interaction, of strength $J_x$ or $J_y$, through either $x$ or $y$ components of the spins on alternating bonds, along with a transverse magnetic field $h$. An exact solution is obtained through Jordan-Wigner fermionization, and it exhibits a macroscopically degenerate ground state for $h = 0$, and a non-degenerate ground state for $h \neq 0$, for all interaction strengths. For a chain of $N$ spins, we study the block entropy of a partition of $L$ contiguous spins. The block entropy depends on the eigenvalues of the $2^L$-dimensional reduced density matrix. We employ an efficient method that reduces this problem to evaluating eigenvalues of an $L$-dimensional matrix, which enables us to calculate easily the block entropy for large-$N$ chains numerically. The entanglement entropy grows as $\log L$, at the degeneracy point $h = 0$, and only for $J_x = J_y$. For $h \neq 0$, the entropy becomes independent of the size, thus obeying the area law. For $J_x \neq J_y$, the block entropy shows a non-monotonic behavior for $L < N/2$.

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I. INTRODUCTION

Entanglement in a bipartite quantum state, which is a signature of the quantum correlations between the two parts, has been explored from a wide range of viewpoints over last few years. Apart from the main thrust area of quantum computation and information processing, where quantum entanglement is recognized as a resource,[1] there is a growing interest in exploring entanglement properties of physical models exhibiting quantum critical-point phenomena,[2] and conformal field theories.[3] The von Neumann entropy of a subsystem, along with its various monotones, is a quantitative measure of the entanglement between the subsystems.[4] The entanglement entropy is not extensive as a function of the subsystem size, unlike physical observables. For systems with local interactions, the correlations are expected to be short ranged (implying an energy-mass gap) in general, and long ranged in the vicinity of a critical point (where the correlation length becomes very large, or the energy gap vanishingly small). This leads to the entanglement entropy obeying an area law in general, and near a critical point the entropy exhibits a logarithmic dependence on the size.

The study of entanglement in spin systems is dominated by one-dimensional models, with many different quantum spin models that can be exactly solved for the ground state and the excitation spectrum, viz. XYZ chains using Bethe Ansatz, and XY chains with transverse magnetic field using Jordan-Wigner transformation[2, 3]. In the context of fault-tolerant quantum computation, Kitaev studied a two-dimensional honeycomb lattice spin model, which is solvable by the Jordan-Wigner fermionization technique, a rare example of exactly-solvable two-dimensional system[5]. This model shows remarkable exotic features, such as excitations obeying non-abelian statistics, topological order and entanglement[6]. There have been generalizations to other lattices, and to higher dimensions and higher spins[7]. A one-dimensional variant of Kitaev model with no magnetic field has been explored for the phase diagram and the non-abelian anyon excitations analytically[8], for the quenched dynamics[9]. There are attempts to realize these spin models in systems of cold atoms[10] and superconducting quantum circuits[11].

Here, we will study the entanglement entropy of a simpler one-dimensional spin chain with Kitaev-type interactions in the presence of a transverse magnetic field. The one-dimensional model substantially simplifies the task of finding the ground state and further in obtaining the eigenvalues of the reduced density matrix, as we will see below, but still retaining many of the exotic features of the Kitaev honeycomb model. We will employ a method of operator Schmidt decomposition to compute the entanglement entropy. This method, which can be used in the context of other one-dimensional spin models, enables us to numerically compute the block entropy for spin chains with large number of lattice sites. We describe the spin chain model and the exact solution from Jordan-Wigner transformation in the next section. The ground state is written as a product state of $N/2$ modes, each mode consisting of four single-particle momentum states $q-\pi, -q, q, -q$. In contrast, for the XY spin chain there are $N/4$ modes, each mode consisting of momentum $q$ and $-q$. In Section III, we show the Schmidt decomposition of the ground state. We express the ground state using an exponentiated operator, and use the method of operator Schmidt decomposition. This reduce the task of finding the entanglement entropy to numerical diagonalization of a $N$ dimensional matrix. In the final section we discuss the results from numerical computation of the block entanglement entropy in the Kitaev chain.

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II. THE MODEL AND THE EXACT SOLUTION

We consider a finite one-dimensional closed spin chain with Kitaev-type interactions. Let $\sigma_i$ matrix vector represent the spin-1/2 operator at a given site. The nearest-neighbor spins interact through either $x$ or $y$ components of the spins on alternating pairs. The Hamiltonian for a closed chain of $N$ (even) spins is written as,

$$ H = \sum_{i=1,3,\ldots}^{N-1} J_x \sigma_i^x \sigma_{i+1}^x + \sum_{i=2,4,\ldots, N} J_y \sigma_i^y \sigma_{i+1}^y + h \sum_{i=1}^{N} \sigma_i^z, \quad (1) $$

where $J_x$ and $J_y$ are interaction strengths on alternating bonds which favor $x$-$x$ or $y$-$y$ polarization locally, and $h$ is the strength of the transverse magnetic field which favors $z$ polarization globally. Thus, this model cannot be mapped on to XY model with homogeneous couplings. As we will see below, this model has a macroscopic degeneracy in the ground state for all values of the interaction strengths, for $h = 0$. The degeneracy is completely lifted for a nonzero transverse magnetic field. The ground state degeneracy can be lifted by a staggered transverse field also. This implies a large correlation length, and long-ranged spin correlations for $h = 0$, and short-ranged correlations for a nonzero field. We will show later, that this does not translate directly into a logarithmically increasing block entropy for $h = 0$ and all interaction strengths.

Working with $\sigma_i^z$ basis for every spin, in a basis state for every spin, in a basis state

$$ |n_k\rangle = \frac{1}{\sqrt{2N}} \sum_{n} e^{i n l} |n_k\rangle, \quad (2) $$

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where $c_l^\dagger$ creates a fermion at site $l$, and the number operator $n_l$ counts the fermions at the site. The $\sigma_i^z = \pm 1$ basis states at a site are mapped to occupied and unoccupied state of the fermion, $n_l = 1, 0$, respectively. Basis states with even $N_F$, the number of fermions, do not mix with odd $N_F$ states. Now, the Hamiltonian can be diagonalized in either even or odd fermion number sector separately. We confine ourselves to even sector, as the ground state belongs to even fermion sector for $N$ even and periodic boundary conditions. Now the bilinear terms in fermion operators can be uncoupled to a degree by going over the momentum space variables, the fermion operator is written as,

$$ \alpha_l = \frac{1}{N} \sum_k e^{ikc_l}, \quad (3) $$

where the sum is over all momentum values in the first Brillouin zone . For the case of $N, N_F$ even, and periodic boundary condition, $N$ allowed values for $k$ are given by

$$ k = \pm \frac{\pi}{N} (1, 3, 5, \ldots, N - 1), \quad (4) $$

the allowed range in the thermodynamic limit is $-\pi < k < \pi$. The Hamiltonian now has the form in terms of the momentum-space operators,

$$ H = \sum_{-\pi<k<\pi} \varepsilon_k \chi_k (\psi_k + \psi_{k-\pi}) + h \chi_k \psi_k, \quad (5) $$

where $2\varepsilon_k = J_x e^{ik} + J_y e^{ik}$, and $\chi_k = c_{k+1}^\dagger - c_k, \psi_k = c_k^\dagger + c_{-k}$. We can see that the operators with momentum values $k - \pi, -k, k - k$ are still coupled in this form.

Let us define a set of four operators labeled by a positive $q < \pi/2$, as linear combinations of the fermion creation and annihilation operators,

$$ F_\pm = \frac{c_q \pm e^{ik} c_{\pm q}}{\sqrt{2}}, \quad G_\pm = \frac{c_q \pm e^{ik} c_{\pm q}}{\sqrt{2}}. \quad (6) $$

Using these combinations, the Hamiltonian can be written as,

$$ H = 2 \sum_{0<q<\pi/2} H_q, \quad (7) $$

where $H_q$ a is four-dimensional operator for a given mode. In the matrix form, using $\varepsilon \equiv \varepsilon_{1q} + i\varepsilon_{2q}$, we have

$$ H_q = (F_+ G_+ F_- G_-) \begin{bmatrix} 2\varepsilon_{1q} & -2i\varepsilon_{2q} & h & 0 \\ 2i\varepsilon_{2q} & 2\varepsilon_{1q} & 0 & h \\ h & 0 & 0 & 0 \\ 0 & h & 0 & 0 \end{bmatrix} \begin{bmatrix} F_+ \\ G_+ \\ F_- \\ G_- \end{bmatrix}. \quad (8) $$

The four eigenvalues of the above matrix are for each $q$,

$$ \lambda_{n_1 n_2} = n_1 |\varepsilon_q| + n_2 \sqrt{\varepsilon_q^2 + h^2}, \quad (9) $$

and the corresponding eigenvectors $V_{n_1 n_2}$ are easily found, and the eigenmode operators $\zeta_{n_1 n_2}$ are constructed as linear combinations of the above operators.
In terms of the diagonal modes, the Hamiltonian takes the diagonal form,

$$H_q = \sum_{n_1, n_2} \lambda_{n_1 n_2} \zeta_{n_1 n_2}^{\dagger} \zeta_{n_1 n_2}.$$  \hfill (10)

The lowest energy state is given by occupying all negative energy states for each mode; we have \(\lambda_-, \lambda_+ < 0\) and the other two are positive. Thus, the ground state energy is given by

$$E_g = -4 \sum_{0 < q < \frac{\pi}{2}} |\varepsilon_q|^2 + \hbar^2.$$  \hfill (11)

The ground state is non-degenerate, and it is given by a direct product over the lowest-energy states constructed from the vacuum state for each mode, we have

$$|g\rangle = \prod_{0 < q < \frac{\pi}{2}} \zeta_{- -}^{\dagger} \zeta_{- -}^{\dagger} |\text{vac}\rangle,$$  \hfill (12)

The normal-mode operators are given by, using the eigenvectors \(V_{n_1 n_2}\),

$$\zeta_{- -}^{\dagger} = CF_{- -}^{\dagger} + iS_{- -}^{\dagger} + h_1 C_{- -}^{\dagger} + i h_2 S_{- -}^{\dagger},$$

$$\zeta_{- +}^{\dagger} = SF_{- +}^{\dagger} - iC_{- +}^{\dagger} + h_2 S_{- +}^{\dagger} + i h_1 C_{- +}^{\dagger},$$  \hfill (13)

where we have parametrized \(\varepsilon_{1q} = |\varepsilon_q| \cos \theta, \varepsilon_{2q} = |\varepsilon_q| \sin \theta\), and \(C = \cos \theta / 2, S = \sin \theta / 2\), and \(h_1 = h / \lambda_-, h_2 = h / \lambda_+\).

### III. OPERATOR SCHMIDT DECOMPOSITION AND BLOCK ENTANGLEMENT ENTROPY

We are now interested in the entanglement distribution in the ground state of the spin chain, as a function of the coupling strengths and the magnetic field. The entanglement block entropy is expected to follow an area growth law in general, and exhibit a faster growth behavior as a function of the block size \(L\) near a critical point. For one-dimensional system, this implies, a size-independent behavior away from critical regions (where all correlations are short-ranged) and a logarithmic behavior near a critical point\(^2\). The ground state written using the normal-mode fermion operators, as given in Eq.12, has a Slater-determinantal wave function. With these wave functions, the reduced density matrix for a block of spins, and the entanglement can be calculated using the correlation function matrix\(^3\). We show a method here that can calculate the Schmidt eigenvalues of the state directly, using a operator Schmidt decomposition of the exponentiated operator that generates the ground state.

Let us briefly recall the Schmidt decomposition of a many-qubit pure state. Let us bipartition the spin chain, part \(A\) contains the \(L\) contiguous spins starting with the first spin, the rest of the spins, \(L' = N - L\) contiguous spins form the second part \(B\), let \(L < N / 2\). Any pure state of the multi-spin state is written as,

$$|\psi\rangle = \sum_{i=1}^{2^L} \sum_{j=1}^{2^{L'}} \gamma_{i, j} |i\rangle_A |j\rangle_B,$$  \hfill (15)

where \(\gamma_{i, j}\) is the wave function amplitude for the product basis state of the composite system. We can view this coefficient as an element of a matrix \(\hat{\gamma}\) which has dimension \(2^L \times 2^{L'}\). The above state can be written as sum over at most \(2^L\) terms, in the Schmidt-decomposed form, as

$$|\psi\rangle = \sum_{k=1}^{2^L} \sqrt{\Lambda_k} |k\rangle_A |k\rangle_B,$$  \hfill (16)

where the Schmidt numbers \(\Lambda_k\) are the eigenvalues of \(\hat{\gamma} \hat{\gamma}^T\), where \(T\) denotes the transpose of the matrix. The reduced density matrix \(\rho_A = \text{tr}_B |\psi\rangle \langle \psi|\), calculated by a partial trace over \(B\) states, has the above basis states as the eigenstates, and the Schmidt numbers as the eigenvalues. Thus, the entropy of the block \(A\), which is a measure of the entanglement between the two parts, is given by

$$E_{AB} = -\text{Tr} \rho_A \log \rho_A = - \sum_k \Lambda_k \log \Lambda_k$$  \hfill (17)

Now, the calculation of the eigenvalues \(\Lambda_k\) for the ground state (given in Eq.12) can still be a daunting task, as this entails construction of the reduced density matrix which is \(2^L\) dimensional. We have the ground state for all
values of interaction strengths written in terms of non-interacting fermion modes that are complicated linear combinations of the momentum space fermion variables defined in Eq.3. We will rewrite the state in a more transparent form below, so that the ground state can be brought to a Schmidt-decomposed form. After unscrambling the operators given in Eq.6, we can express the ground state as, modulo a normalization constant,

\[ |g⟩ = \prod_{0 < q < \frac{\pi}{L}} \left(1 + \frac{1 - h_1}{1 + h_1} c_{q-\pi}^\dagger d_q^\dagger\right)\left(1 + \frac{1 - h_2}{1 + h_2} c_{q+\pi}^\dagger e_q^\dagger\right) |0⟩, \]

where \( d_q = \cos \theta c_{q-\pi} + i \sin \theta c_{q-\pi}, \) \( e_q = \cos \theta c_{q+\pi} + i \sin \theta c_{q+\pi} \). Finally, the operator product can be exponentiated, and after a further manipulation we can write the unnormalized ground state as

\[ |g⟩ = e^{\hat{Z}} |0⟩, \]

where the operator in the exponent is given by

\[ \hat{Z} = \sum_{0 < q < \frac{\pi}{L}} \frac{\varepsilon_{1q}}{\varepsilon_q + \sqrt{q^2 + h^2}} (c_{q-\pi}^\dagger e_q^\dagger - c_{q+\pi}^\dagger e_q^\dagger) + \frac{i\varepsilon_{2q}}{\varepsilon_q + \sqrt{q^2 + h^2}} (c_{q-\pi}^\dagger e_{q+\pi}^\dagger - c_{q+\pi}^\dagger e_{q-\pi}^\dagger). \]

We manipulate from this form of the state in the next section, to bring it to a Schmidt-decomposed form to calculate the block entanglement entropy. We employ a method of Schmidt decomposing the operator \( \hat{Z} \) itself, which is tantamount to calculating the eigenvalues of a \( L \)-dimensional matrix. Let us express the operator in real space basis as,

\[ \hat{Z} = \sum_{l=1}^L \sum_{m=1}^N \gamma_{l,m} c_{l,m}^\dagger c_{m,l}^\dagger, \]

where the coefficient \( \gamma_{l,m} \) is the Fourier transform of the coefficients in Eq.17. In terms of the coefficients \( \beta_1, \beta_2 \) given by,

\[ \beta_n(x) = \frac{1}{N} \sum_{q=\pi/2}^{\pi} \frac{\varepsilon_{nq}}{h + \sqrt{|\varepsilon_q|^2 + h^2}} e^{ikx}, \]

we have \( \gamma_{l,m} = \beta_1 (l - m)(-1)^l - (1)^m + i \beta_2 (l - m)(-1)^l + (1)^m - 1 \). Now, the exponentiated operator is written in real space. We can now partition the operator into three parts as,

\[ \hat{Z} = \hat{Z}_A + \hat{Z}_B + \hat{Z}_{AB}, \]

where the the first (second) term involves operators from part \( A(B) \) only. The last entangling term involves one fermion operator each from the two parts. All these three terms commute with each other. It is easy to see that the exponentiated operator \( e^{\hat{Z}_A + \hat{Z}_B} \) generates a product of local transformations on the two parts; it will not generate any entanglement. Thus, we may drop the operators that act only on part A (B) only. That is, the ground state \( |g⟩ \), given in Eq.19, and the state given by

\[ |\tilde{g}⟩ = e^{\hat{Z}_{AB}} |0⟩, \]

will have the same entanglement properties, as they are related by a product of local transformations. Let us define more suggestive fermion operators, \( A_l^\dagger = c_{l+L}^\dagger \), for \( l = 1, L \), and \( B_l^\dagger = c_{l}^\dagger \), for \( l = 1, N - L \). We can write the operator appearing above as,

\[ \hat{Z}_{AB} = \sum_{l=1}^L \sum_{m=1}^N \Gamma_{lm} A_{l}^\dagger B_{m}^\dagger \]

where the coefficient is given by,

\[ \Gamma_{lm} = 2((-1)^l - (-1)^{m+L}) Re \beta_1 (l - m - L) - 2((-1)^l + m + L - 1) \text{ Im } \beta_2 (l - m - L). \]

Now, we can employ an operator Schmidt decomposition, in analogy with the state decomposition of writing the state as in Eq.16 from Eq.15, we have

\[ \hat{Z}_{AB} = \sum_{n=1}^L \sqrt{\eta_n} A_{n}^\dagger B_{n}^\dagger, \]

where the operator Schmidt numbers \( \eta_n \) are eigenvalues of the matrix \( \Gamma^T \Gamma \), and the new operators \( A_n \) and \( B_n \) are linear combinations of the original site basis operators. Now, the normalized equivalent ground state is a direct product, given by

\[ |\tilde{g}⟩ = \prod_{n=1}^L (x_n + y_n A_{n}^\dagger B_{n}^\dagger) |0⟩, \]
where \( x_n = 1/\sqrt{1 + \eta_n} \) and \( y_n = \sqrt{\eta_n}/\sqrt{1 + \eta_n} \). The above has the Schmidt decomposed form (as in Eq.16), with \( 2^L \) terms in the expansion, the amplitude for each term is a product of \( L \) factors. The eigenvalues of \( \rho_A \) are labeled by numbers \( \xi_n \) (equal to 0 or 1 corresponding to either \( x_n \) or \( y_n \) appearing in each term of the expansion), we have

\[
\Lambda(\xi_1, \ldots, \xi_L) = \prod_{n=1}^{L} |x_n|^{2\xi_n} |y_n|^{2(1-\xi_n)}. \tag{29}
\]

The entropy of \( \rho_A \), viz. the block entanglement \( E(L, N) \), is straightforward to evaluate, and we get,

\[
E(L, N) = \sum_{n=1}^{L} H(\frac{1}{1 + \eta_n}), \tag{30}
\]

where the Shannon binary entropy function is given by

\[
H(x) = -x \log x - (1 - x) \log(1 - x).
\]

This completes the method of expressing the entanglement entropy in terms of operator Schmidt numbers, which gives an efficient method for numerical computation of the block entanglement. It can be easily employed for other spin chain models that are solvable by Jordan-Wigner fermionization, where the ground state is amenable to exponentiation, as in Eq.21. However, the method is not applicable for the Bethe-ansatz state of the Heisenberg-XY spin chain, as it is not clear how to exponentiate it. We expect the method to work for Kitaev honeycomb lattice, and other higher-dimensional systems, as long as the ground state is expressed as a direct product over modes, and exponentiation can be done, as in Eq.24. Some well-known states of the interacting electron systems, like the BCS state of the superconductors, have a similar structure. This method reduces the numerical effort substantially in the study of the eigenvalues of the reduced density matrix for large lattices. It will be interesting to employ this method to study the entanglement spectrum, and the phase diagram of the electron/spin models from the entanglement perspective.

IV. DISCUSSION OF RESULTS

We now turn to the block entanglement entropy for the ground state of our model as a function of the interaction strengths, the transverse field and the block size. We make a bipartition of the lattice of \( N \) sites in to two blocks of size \( L \) and \( N - L \). Though the parent state is pure of the composite system, the block reduced density matrix may not represent a pure state, thus it carries an entropy. The amount of quantum correlations present in the state is quantified by the bipartite von Neumann entropy. Since the composite state is a pure state, it suffices to study the block entropy \( E(L, N) \) as a function of the size \( L \) for \( L \leq N/2 \), for all values of the interaction strengths and the transverse magnetic field. The scaling of the entropy with the block size \( L \) is related to the quantum correlations of the system, viz. the entropy is expected to follow the area law\cite{3, 4} when the correlations are short ranged or the system is away from criticality. For the one-dimensional system, this implies that the entropy is independent of the size (since the block has a surface with just two sites irrespective of the size), as we will see below this is the case for all values of the interaction strengths when a nonzero field is present. The deviation from the area law is expected, with a logarithmically increasing entropy as a function of the block size, in the vicinity of a critical point. Since the correlations tend to be long ranged near a quantum critical point, they should reflect in behavior of the entropy as well. It has been shown using conformal field theories, which are related to the continuum limit of the gapless spin systems near a critical point, the coefficient of the logarithmic correction of the entropy is related to the central charge of the system. We will see below that the spin chain we are investigating shows a critical behavior for \( J_x = J_y, h = 0 \), and for all other values it is gapped.

We now turn to the numerical computation of the block entropy using the method we developed in the previous section. The numerical task is to get the operator Schmidt numbers of Eq.27, from the diagonalization of a \( L \)-dimensional matrix \( \Gamma^T \Gamma \) (see Eq.25). This means we can consider spin chains with a large number of sites. To this end, we have numerically computed the operator Schmidt numbers from Eq.25, in the various parameter regimes, for a large spin chain with \( N = 1000 \) spins. Fig.1a shows the block entanglement entropy as a function of the block size \( L \) for different values of the interaction strengths and the magnetic field. We have shown here the block entropy for the case of even \( L \) only. This is because there is even-odd oscillatory behavior of the entropy with the size.

As we can see from Fig.1a, at the macroscopic degeneracy point, for \( h = 0, J_y = J_x \), the entanglement entropy...
exhibits a logarithmic behavior for $L < N/2$, which is an expected behavior for a system in the vicinity of a critical point. The entropy is given by

$$E(L, N) \approx \frac{c}{3} \log_2 L, \quad \text{for} \quad J_y = J_x, \quad h = 0,$$

the coefficient, known as the central charge, seems to fit $c = \frac{3}{2} \log 2$. This value of the central charge is different from other spin models\[2, 4\], for example $c = 1$ for the Heisenberg spin chain. We can take that the spin entanglement from other spin models\[2, 4\], for example $c = 1$ for the Heisenberg spin chain. We can take that the spin entanglement from other spin models with known critical-point behavior of the correlation functions.

As we can see from Fig.1a, for $h \neq 0$ the block entropy is independent of the block size, thus obeying the area law, implying the system is away from the critical region. This is expected as the nonzero transverse field lifts the macroscopic degeneracy, shown in Eq.14, we expect the system to be away from criticality, presumably with short-ranged correlations. However, we see from Fig.1a that the entropy obeys the area law even for $h = 0$ but the interaction strengths are not equal, $J_x \neq J_y$. Here, the ground state still has a macroscopic degeneracy, just as for equal interaction strengths. In this case we seem to have conflicting indications, the entanglement entropy obeying the area law indicates the system is away from a critical point with short-ranged correlations, the gapless excitation spectrum accompanying a macroscopic degeneracy indicates the system is in the vicinity of a critical point accompanied by long-ranged correlations. Similar conflicting features have been seen in the two-dimensional Kitaev model by Baskaran, Sen and Shankar\[7\], where a parameter regime exists in which the ground state has gapless excitations along with short-range correlation functions. The present situation may be similar, and we conjecture that off-diagonal correlation functions may be short ranged in this model. The diagonal correlations may be long ranged due to a spin polarization that can result due to the unequal interaction strengths. The block entropy will depend on the diagonal and off-diagonal correlation functions and their range, analogously of the other measures of the entanglement like concurrence\[13\]. The block entanglement, unlike the concurrence measure, will depend on various multi-spin correlation functions, making the situation more complicated. Since our method deals directly with the eigenvalues of the reduced density matrix only, a detailed study of multi-spin correlations cannot be addressed here.

In the regime with unequal interaction strengths and zero magnetic field, the entanglement entropy exhibits an even-odd non-monotonic behavior as a function of the size. The oscillatory behavior of the block entanglement is highlighted in Fig.1b for a smaller chain with $N = 200$ spins. This even-odd effect is persistent for very large sizes also, as the block entanglement itself becomes size independent in this regime. This non-monotonic behavior may be related to the fact that we have an inhomogeneous chain, and for odd $L$, the block will have unequal number of $J_x$ and $J_y$ bonds. Similar behavior has been seen in spin-one chains with bi-quadratic interactions, in spin chains with boundaries, in SU(n) Hubbard chains with further neighbor hopping\[15\].

We can also study the entropy as a function of the interaction strengths and the magnetic field separately, to track the critical-point behavior and its signature. We fix the block size to be $L = N/2$, so the entropy takes its maximum value. In Fig.2a, we have plotted the block entropy of the largest block, for $N = 200$, $L = N/2$, as a function of $h/J_x$ for the case of $J_y/J_x = 1$ and $J_y/J_x = 0.8$. As a function of the magnetic field, the block entropy displays a peak structure at $h = 0$, for $J_y = J_x$, tracking the quantum critical behavior at $h = 0$. But for the case of unequal strengths, $J_y/J_x = 0.8$, the peak structure is quite diminished, apart from shifting from $h = 0$, to a slightly negative value. Away from the peak, the entropy itself becomes small for large magnetic field. The spins are expected to be polarized predominantly along the field direction, thus decreasing the entropy. We have shown in Fig.2b the behavior of the block entropy as a function of $J_y/J_x$, for both $h = 0$ and $h \neq 0$. Here also, the block entropy tracks the critical behavior with a sharp peak near $J_y = J_x$ for the case of $h = 0$. For large $J_y$, for a fixed $J_y$ and a small field, the spins are polarized along $y$ direction, thus decreasing the entropy. In both the cases of a nonzero magnetic field and/or unequal interaction strengths, the system is away from critical point, as we discussed earlier in the context of the size dependence of the block entanglement. The peak structure in this case cannot really be associated to the tracking of a critical point, as a true critical behavior obtains only for $h = 0, J_y = J_x$. The diminished peak may signify a smear or rounding off in the critical behavior, and thus can be useful in locating a critical point in numerical computations by tuning the parameter regime.

In conclusion, we have shown that the spin chain with Kitaev-type interactions is exactly solved through Jordan-Wigner fermionization. We have employed a method that reduces the task of finding $2^L$ eigenvalues of the reduced density matrix of a block of $L$ spins, to finding the eigenvalues of a $L$-dimensional matrix. Using this method, we have numerically computed block entropies for the spin chains with a large number of spins. The block entropy obeys the area law for all interaction strengths for a nonzero transverse magnetic field, implying the system is gapped and away from a critical point. The block entropy shows a logarithmic increase with the block size only at the degeneracy point $h = 0, J_x = J_y$, though there is a macroscopic degeneracy for $J_x \neq J_y$.

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