EXISTENCE, UNIQUE CONTINUATION AND SYMMETRY OF LEAST ENERGY NODAL SOLUTIONS TO SUBLINEAR NEUMANN PROBLEMS

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ABSTRACT. We consider the sublinear problem
\[
\begin{aligned}
-\Delta u &= |u|^{q-2} u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where $\Omega \subset \mathbb{R}^N$ is a bounded domain, and $1 \leq q < 2$. For $q = 1$, $|u|^{q-2} u$ will be identified with $\text{sgn}(u)$. We establish a variational principle for least energy nodal solutions, and we investigate their qualitative properties. In particular, we show that they satisfy a unique continuation property (their zero set is Lebesgue-negligible). Moreover, if $\Omega$ is radial, then least energy nodal solutions are foliated Schwarz symmetric, and they are nonradial in case $\Omega$ is a ball. The case $q = 1$ requires special treatment since the formally associated energy functional is not differentiable, and many arguments have to be adjusted.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ a bounded open domain with Lipschitz boundary, and let $1 \leq q < 2$. We are concerned with the sublinear Neumann boundary value problem
\[
\begin{aligned}
-\Delta u &= |u|^{q-2} u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\] (1)

Here $\nu$ is the outer normal derivative of $u$ at the boundary $\partial \Omega$, and the term $|u|^{q-2} u$ has to be identified by $\text{sgn}(u)$ in case $q = 1$ in the following. For $q > 1$, problem (1) arises e.g. in the study of the Neumann problem for the (sign changing) porous medium equation. To see this, we set $v = |u|^{\frac{1}{m}-1} u$ with $m = \frac{q}{q-1} > 1$ and note that (1) may equivalently be written as
\[
\begin{aligned}
-\Delta(|v|^{m-1}v) &= v \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\] (2)

As a consequence, the function $w(x,t) = [(m-1)t]^{-\frac{1}{m-1}} v(x)$ is a solution of the problem
\[
\begin{aligned}
w_t - \Delta |w|^{m-1} w &= 0 \quad \text{in } \Omega \times (0,\infty), \\
w &= 0 \quad \text{on } \partial \Omega \times (0,\infty).
\end{aligned}
\] (3)

For more information on this relationship and a detailed discussion of the (sign changing) porous medium equation, we refer the reader to [11, Chapter 4] and the references therein.

In the case $q = 1$, we regard (1) as a model problem within the class of general elliptic boundary value problems with piecewise constant (and therefore discontinuous) nonlinearities. Such problems appear e.g. in the study of equilibria of reaction diffusion equations with discontinuous reaction terms, see e.g. [2,9,10].

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Integrating the equation in (1) over \( \Omega \), we see that \( \int_{\Omega} |u|^{q-2} u = 0 \) for every solution of (1), hence every nontrivial solution is sign changing. Let us consider the functional

\[
\varphi : W^{1,2}(\Omega) \to \mathbb{R}, \quad \varphi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{q} \int_{\Omega} |u|^q \, dx.
\]

If \( 1 < q < 2 \), \( \varphi \) is of class \( C^1 \), and critical points of \( \varphi \) are precisely the weak solutions of (1). Moreover, since the nonlinearity in (1) is Hölder continuous, weak solutions \( u \) of (1) are in general not of class \( C^\alpha \) regularity, and the restriction of \( u \) to the open set \( \{ u \neq 0 \} \) is of class \( C^\alpha \). If \( q = 1 \), then \( \varphi \) fails to be differentiable and weak solutions of (1) are in general not of class \( C^2 \), but they are still strong solutions contained in \( W^{2,p}_0(\Omega) \) for every \( p < \infty \) and thus contained in \( C^{1,\alpha}_\text{loc}(\Omega) \) for every \( \alpha \in (0,1) \).

The purpose of this paper is to derive the existence of solutions of (1) with minimal energy and to characterize these solutions both variationally and in terms of their qualitative properties. We first consider the case \( 1 < q < 2 \). In order to obtain least energy nodal solutions, we minimize the functional \( \varphi \) on the set

\[
\mathcal{N} := \left\{ u \in W^{1,2}(\Omega) : \int_{\Omega} |u|^{q-2} u \, dx = 0 \right\} \subset W^{1,2}(\Omega).
\]

We shall see that minimizers of \( \varphi \) solve (1). Note that this property does not follow from the Lagrange multiplier rules since \( \mathcal{N} \) is not a \( C^1 \)-manifold if \( q < 2 \). Our main result for the case \( 1 < q < 2 \) is the following.

**Theorem 1.1.** Suppose that \( 1 < q < 2 \), and let \( m := \inf_{u \in \mathcal{N}} \varphi(u) \). Then we have:

(i) \( \varphi \) attains the value \( m < 0 \) on \( \mathcal{N} \).

(ii) Every minimizer \( u \) of \( \varphi \) on \( \mathcal{N} \) is a sign changing solution of (1) such that \( u^{-1}(0) \subset \Omega \) has vanishing Lebesgue measure.

(iii) If \( \Omega \) is a bounded radial domain, then every minimizer \( u \) of \( \varphi \) on \( \mathcal{N} \) is foliated Schwarz symmetric.

(iv) If \( \Omega = B_1(0) \) is the unit ball, then every minimizer \( u \) of \( \varphi \) on \( \mathcal{N} \) is a nonradial function.

We quickly comment on these results. The property (i) follows by standard arguments based on the weak lower continuity of the Dirichlet integral \( u \mapsto \int_{\Omega} |\nabla u|^2 \, dx \). To show that every minimizer of \( \varphi \) on \( \mathcal{N} \) is a solution of (1), we use a saddle point characterization of \( \mathcal{N} \) (see Lemma 2.1 below). The most difficult part is the unique continuation property of minimizers of \( \varphi \) on \( \mathcal{N} \), i.e., the fact that their zero sets have vanishing Lebesgue measure. Note that, due to the fact that the nonlinearity \( u \mapsto |u|^{q-2} u \) is not locally Lipschitz, the linear theory on unique continuation does not apply. Moreover, as can be seen from very simple ODE examples already, nontrivial solutions of semilinear equations of the type \( -\Delta u = f(u) \) with non-Lipschitz \( f \) may have very large zero sets. It is an interesting open problem whether every nontrivial solution of (1) has the unique continuation property; we conjecture that this is true. The proof of (iii) is again quite short and essentially follows the arguments in [3]. The nonradiality property for least energy nodal solutions in the case where the underlying domain is a ball is far from immediate. The idea is to use properties of directional derivatives of \( u \). For problems with \( C^1 \)-nonlinearities, nonradiality properties have successfully been derived via directional derivatives in the case of Dirichlet problems [1] and Neumann problems [7], while the methods in these papers are quite different due to the impact of the boundary conditions. The main difficulty of the present problem is to analyze how the equation in (1) can be linearized in a meaningful way (see Proposition 2.3 for a first result on this question).

Let us now consider the case \( q = 1 \). In this case, the functional \( \varphi \) is not differentiable, so that the techniques used when \( 1 < q < 2 \) cannot be applied. Moreover, the saddle point characterization in Lemma 2.1 fails in the case \( q = 1 \), i.e., for the set \( \mathcal{N} = \{ u \in W^{1,2}(\Omega) : \int_{\Omega} \text{sgn}(u) \, dx = 0 \} \). Nevertheless, we derive the same conclusions as in Theorem 1.1 by adjusting the variational principle. More precisely,
we consider minimizers of the restriction of $\varphi$ to the set

$$\mathcal{M} := \{ u \in W^{1,2}(\Omega) : \||u > 0\| - \||u < 0\| \leq \||u = 0\| \}.$$ 

Note that $\mathcal{M}$ is strictly larger than $\mathcal{N}$. Our main results for the case $q = 1$ are collected in the following Theorem.

**Theorem 1.2.** Suppose that $q = 1$ in (1) and the definition of $\varphi$, and let $m := \inf_{u \in \mathcal{M}} \varphi(u)$. Then we have:

(i) $\varphi$ attains the value $m < 0$ on $\mathcal{M}$.

(ii) Every minimizer $u$ of $m$ on $\mathcal{M}$ is a sign changing solution of (1) such that $u^{-1}(0) \subset \Omega$ has vanishing Lebesgue measure.

(iii) If $\Omega$ is a bounded radial domain, then every minimizer $u$ of $\varphi$ on $\mathcal{M}$ is foliated Schwarz symmetric.

(iv) If $\Omega = B_1(0)$ is the unit ball, then every minimizer $u$ of $\varphi$ on $\mathcal{M}$ is a nonradial function.

We add some comments on these results. The general strategy for the proofs of (i)-(iii) is the same as in Theorem 1.1 but the details are quite different due to the geometry of $\mathcal{M}$ and the fact that $\varphi$ fails to be differentiable in the case $q = 1$. The proof of (iv) is completely different, since there seems to be no way to use directional derivatives to prove nonradiality. Instead, our proof of (iv) is based on inequalities comparing the value $m$ with the least energy of radial nodal solutions of (1) (in the case $q = 1$). In fact, the latter value can be computed explicitly once we have shown the (not obvious) property that least energy radial nodal solutions have exactly two nodal domains. We then compare this value with upper estimates for the value $m$ obtained by using the test functions $v(x) = x_1$ (if $n = 2$) or $v(x) = \frac{x}{|x|}$ (if $n \geq 3$).

The paper is organized as follows. After proving some fundamental properties of the functional $\varphi$ in the case $1 < q < 2$ (Section 2), we show that least energy nodal solutions satisfy a unique continuation property (Section 3) and we deal with symmetry results in radially symmetric domains (Section 4). In particular, Theorem 1.1 will readily follow from Lemma 2.1 and Theorems 3.2, 4.1 and 4.2 below. In Section 5 we turn to the case $q = 1$ and prove Theorem 1.2.

Finally, we wish to mention that it is not straightforward to obtain similar results for the Dirichlet problem corresponding to (1). Indeed, least energy solutions of the Dirichlet problem might have different variational characterizations on different domains, so the situation is more complicated than in the Neumann case. We will treat the Dirichlet problem in a paper in preparation.

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2. **The Variational Framework in the Case $q > 1$.**

For fixed $q \in [1, 2)$ and $u \in W^{1,2}(\Omega)$, we denote by $t^*(u)$ the positive number such that

$$\varphi(t^*(u)u) = \min_{r > 0} \varphi(ru).$$

It is easy to see that

$$t^*(u) = \left( \frac{\int_{\Omega} |u|^q}{\int_{\Omega} |Vu|^q} \right)^{\frac{1}{q-1}}. 
\tag{5}$$

Suppose that $1 < q < 2$ from now on, and consider the set $\mathcal{M}$ defined in (4). For any $u \in \mathcal{N} \setminus \{0\}$, we then have $t^*(u)u \in \mathcal{N}$ and $\varphi(t^*(u)u) < 0$. This in particular implies that the infimum $m$ of $\varphi|_{\mathcal{N}}$ is negative. The next lemma highlights the saddle point structure given by $\varphi$ and the set $\mathcal{N}$.

**Lemma 2.1.**

(i) For every $u \in L^1(\Omega)$ there exists precisely one $c = c(u) \in \mathbb{R}$ such that $\int_{\Omega} |u + c|^{q-2}(u + c) \, dx = 0$. Moreover, the map $L^1(\Omega) \to \mathbb{R}$, $u \mapsto c(u)$ is continuous.
(ii) If \( u \in W^{1,2}(\Omega) \), then

\[
\frac{\partial}{\partial c} \varphi(u+c) > 0 \quad \text{for} \ c < c(u) \quad \text{and} \quad \frac{\partial}{\partial c} \varphi(u+c) < 0 \quad \text{for} \ c > c(u).
\]

In particular,

\[
\varphi(u+c(u)) = \max_{c \in \mathbb{R}} \varphi(u+c).
\]

**Proof.** (i) Let \( u \in L^1(\Omega) \). Since the map \( \mathbb{R} \to \mathbb{R}, c \mapsto \int_{\Omega} |u|^{q-2}u \) is strictly increasing, there exists at most one \( c = c(u) \) such that \( \int_{\Omega} |u+c|^{q-2}(u+c) \) does not change. Moreover, since

\[
\lim_{c \to -\infty} \int_{\Omega} |u+c|^{q-2}(u+c) \, dx = -\infty \quad \text{and} \quad \lim_{c \to \infty} \int_{\Omega} |u+c|^{q-2}(u+c) \, dx = \infty,
\]

there exists precisely one such \( c = c(u) \). Now consider \( u_n \in L^1(\Omega), n \in \mathbb{N} \) such that \( u_n \to u \) in \( L^1(\Omega) \). We first show that \( c(u_n) \) remains bounded as \( n \to \infty \). Suppose by contradiction that, after passing to a subsequence, \( c(u_n) \to +\infty \) as \( n \to \infty \). Passing again to a subsequence, we may assume that \( c(u_n) > 0 \) for all \( n \) and \( u_n \to u \) pointwise a.e. in \( \Omega \). Moreover, by [12, Lemma A.1] we may assume that there exists \( \bar{u} \in L^1(\Omega) \) with \( |u_n| \leq \bar{u} \) a.e. in \( \Omega \) for all \( n \in \mathbb{N} \). Since \( -\bar{u} \leq u_n + c(u_n) \) a.e. in \( \Omega \) for all \( n \in \mathbb{N} \), we also have

\[
-|\bar{u}|^{q-2}\bar{u} \leq |u_n + c(u_n)|^{q-2}(u_n + c(u_n)) \quad \text{a.e. in} \ \Omega \text{ for all} \ n \in \mathbb{N}.
\]

Hence, since \( |u_n + c(u_n)|^{q-2}(u_n + c(u_n)) \to \infty \) pointwise a.e. in \( \Omega \), Fatou’s Lemma implies that

\[
\int_{\Omega} |u_n + c(u_n)|^{q-2}(u_n + c(u_n)) \to \infty \quad \text{as} \ n \to \infty,
\]

which contradicts the definition of the map \( c \). In the same way, we obtain a contradiction when assuming that \( c(u_n) \to -\infty \) for a subsequence. Consequently, \( c(u_n) \) remains bounded as \( n \to \infty \). We now argue by contradiction, supposing that \( c(u_n) \not= c(u) \) as \( n \to \infty \). Then we may pass to a subsequence such that \( c(u_n) \to c \neq c(u) \) as \( n \to \infty \). Since the map

\[
L^1(\Omega) \to \mathbb{R}, \quad u \mapsto \int_{\Omega} |u|^{q-2}u
\]

is continuous and \( u_n + c(u_n) \to u + c \) in \( L^1(\Omega) \), we have that

\[
\int_{\Omega} |u+c|^{q-2}(u+c) \, dx = 0.
\]

By the uniqueness property noted above, we then deduce that \( c = c(u) \), a contradiction. We thus conclude that \( c(u_n) \to c(u) \) as \( n \to \infty \), and this shows the continuity of the map \( c : L^1(\Omega) \to \mathbb{R} \).

(ii) We have

\[
\frac{\partial}{\partial c} \varphi(u+c) = -\int_{\Omega} |u+c|^{q-2}(u+c) \, dx \begin{cases} > 0 & \text{for} \ c < c(u); \\ < 0 & \text{for} \ c > c(u), \end{cases}
\]

as claimed. \( \square \)

**Lemma 2.2.** The functional \( \varphi \) attains the value \( m < 0 \) on \( N \). Moreover, every minimizer \( u \) of \( \varphi \) on \( N \) is a sign changing solution of \( (\mathcal{I}) \).

**Proof.** We first note that, as a consequence of Lemma 2.1(iii), we have

\[
\|u\|^2_{L^2(\Omega)} = \min_{c \in \mathbb{R}} \|u+c\|^2_{L^2(\Omega)} \leq |\Omega|^{2-q} \min_{c \in \mathbb{R}} \|u+c\|^2_{L^2(\Omega)} \leq |\Omega|^{2-q} \mu_2^{-1} \int_{\Omega} |\nabla u|^2 \, dx \quad \text{for} \ u \in N,
\]

where \( \mu_2 > 0 \) is the first nontrivial eigenvalue of the Neumann Laplacian on \( \Omega \). As a consequence, the functional \( \varphi \) is coercive on \( N \). Let \( \{u_n\} \subset N \) be a minimizing sequence for \( \varphi \). Then \( \{u_n\} \) is bounded, and we may pass to a subsequence such that \( u_n \to u \in W^{1,2}(\mathbb{R}^N) \). Then \( u_n \to u \) in \( L^q(\Omega) \),

\[
\int_{\Omega} |\nabla u|^2 \, dx \leq \liminf_{n \to \infty} \int_{\Omega} |\nabla u_n|^2 \, dx
\]
and
\[ \int_\Omega |u_n|^q - 2 u_n dx \to \int_\Omega |u|^q - 2 u dx \quad \text{as } n \to \infty. \]
Consequently, we have \( u \in \mathcal{N} \), and \( u \) satisfies \( \varphi(u) \leq \liminf_{n \to \infty} \varphi(u_n) \). Hence \( u \) is a minimizer for \( \varphi \) on \( \mathcal{N} \).

Next, we let \( u \in \mathcal{N} \) be an arbitrary minimizer for \( \varphi \) on \( \mathcal{N} \). We show that \( u \) is a critical point of \( \varphi \).

Arguing again by contradiction, we assume that there exists \( v \in W^{1,2}(\Omega) \) such that \( \varphi'(u)v < 0 \). Since \( \varphi \) is a \( C^1 \)-functional on \( W^{1,2}(\Omega) \), there exists \( \varepsilon > 0 \) with the following property:

For every \( w \in W^{1,2}(\Omega) \) with \( \|w\|_{W^{1,2}(\Omega)} < \varepsilon \) and every \( t \in (0, \varepsilon) \) we have
\[ \varphi(u + tw) \leq \varphi(u) - \varepsilon t. \]

Since the map \( c \) is continuous and \( c(u) = 0 \) by definition of \( c \), there exists \( t \in (0, \varepsilon) \) such that \( \|c(u + tw)\|_{W^{1,2}(\Omega)} < \varepsilon \), and thus
\[ \varphi(u + tw + c(u + tw)) \leq \varphi(u + c(u + tw)) - \varepsilon t \leq \varphi(u) - \varepsilon t < \varphi(u) \]
Since \( u + tw + c(u + tw) \in \mathcal{N} \), this contradicts the definition of \( \mathcal{N} \). Finally, since \( m < 0 \) by the remarks in the beginning of this section, every minimizer \( u \in \mathcal{N} \) of \( \varphi \) is a nonzero function and therefore sign changing by the definition of \( \mathcal{N} \).

We close this section with a result on the existence of second derivatives of \( \varphi \) which we will need in Section 4.1 below.

**Proposition 2.3.** Let
\[ \mathcal{W} := \{ v \in C^1(\overline{\Omega}) : \nabla v(x) \neq 0 \text{ for every } x \in \Omega \text{ with } v(x) = 0 \}. \]
Then \( \mathcal{W} \subset C^1(\overline{\Omega}) \) is an open subset (with respect to the \( C^1 \)-topology) having the following properties:

(i) If \( \partial \mathcal{W} \) is of class \( C^2 \), then the restriction \( \varphi|_{\mathcal{W}} \) is of class \( C^2 \) with
\[ \varphi''(u)(v,w) = \int_\Omega \nabla v \nabla w dx - (q-1) \int_\Omega |u|^{q-2}vw \quad \text{for every } v,w \in C^1(\overline{\Omega}). \]

(ii) If \( \partial \mathcal{W} \) is of class \( C^{2,1} \), and \( u \in \mathcal{W} \) is a weak solution of
\[ -\Delta u = (q-1)|u|^{q-2}u \quad \text{in } \Omega, \]
and the partial derivatives \( u_i := u_{x_i} \in W^{2,p}(\Omega) \) are strong solutions of the problem
\[ (8) \quad -\Delta u_i = (q-1)|u|^{q-2}u_i \quad \text{in } \Omega. \]

**Proof.** It is easy to see that \( \mathcal{W} \) is open in \( C^1(\overline{\Omega}) \). We first show

Claim 1: If \( s \in (0,1) \) and \( 1 \leq p < \frac{1}{s} \), then the map
\[ \gamma_s : \mathcal{W} \to L^p(\Omega), \quad \gamma_s(u) \mapsto |u|^{-s} \]
is well defined and continuous.

To see this, let \( \mathcal{K} \subset \mathcal{W} \) be a compact subset (with respect to the \( C^1 \)-norm). We claim that exists \( \kappa > 0 \) such that
\[ |\{ |u| \leq \delta \}| \leq \min\{ \kappa \delta, |\Omega| \} \quad \text{for every } \delta > 0, u \in \mathcal{K}. \]

In order to prove this estimate, we consider a bounded linear extension map \( \text{ext} : C^1(\overline{\Omega}) \to C^0_b(\mathbb{R}^N) \), where \( C^0_b(\mathbb{R}^N) \) denotes the Banach space of bounded \( C^1 \)-functions on \( \mathbb{R}^N \) with bounded gradient. Such a map exists since \( \partial \mathcal{W} \) is of class \( C^{2,2} \). We put \( \mathcal{L} := \text{ext}(\mathcal{K}) \subset C^0_b(\mathbb{R}^N) \) and \( \Omega_b := \{ x \in \mathbb{R}^N : \text{dist}(x,\Omega) < b \} \subset \mathbb{R}^N \) for \( b > 0 \). Since \( \mathcal{L} \) is compact, there exists a constant \( b > 0 \) such that
\[ \max_{j=1,\ldots,N} \left| \frac{\partial v}{\partial x_j}(x) \right| \geq 2b \quad \text{for every } v \in \mathcal{L} \text{ and } x \in \Omega_b \text{ with } |v(x)| \leq b. \]

We now fix \( v_0 \in \mathcal{L} \). Then there exists a positive integer \( d = d(v_0) \) and a finite number of cubes \( W_1, \ldots, W_d \) of equal length \( l > 0 \) such that
Moreover, there exists a neighborhood \( \mathcal{U}(v_0) \) of \( v_0 \) in \( \mathcal{L} \) such that
\[
\mathrm{osc}_{W_i} \frac{\partial u}{\partial x_j} < b \quad \text{for } i = 1, \ldots, d, \ j = 1, \ldots, N.
\] (11)

For every \( v \in \mathcal{U}(v_0) \), \( i \in \{1, \ldots, d\} \) and \( \delta \in (0, b) \) we then have
\[
\{ x \in W_i : |v(x)| \leq \delta \} \subseteq \{ x \in \overline{\Omega} : |v(x)| \leq \delta \} \subseteq \{ x \in \Omega : |v(x)| \leq \delta \} \leq \frac{N-1}{b} \delta.
\] (12)

Indeed, if there exists \( x \in W_i \) with \( |v(x)| \leq \delta \leq b \), then \( \frac{\partial u}{\partial x_j} \geq b \) on \( W_i \); for some \( j = j(i) \) by (10) and (11), in particular, \( v \) is strictly monotone in the \( j \)-th coordinate direction on \( W_i \). Hence (12) easily follows by Fubini’s theorem. As a consequence of (9), we have the estimate
\[
\{ x \in \overline{\Omega} : |v(x)| \leq \delta \} \leq \frac{dN-1}{b} \delta \quad \text{for every } v \in \mathcal{U}(v_0), \ \delta \in (0, b).
\]

Since \( \mathcal{L} \) is compact, it can be covered by finitely many neighborhoods constructed as above, and hence there exists \( d_\varepsilon > 0 \) such that
\[
\{ x \in \overline{\Omega} : |v(x)| \leq \delta \} \subseteq \{ x \in \mathcal{L} : |v(x)| \leq \delta \} \subseteq \{ x \in \Omega : |v(x)| \leq \delta \} \leq \frac{N-1}{b} \delta.
\]

By the construction of \( \mathcal{L} \), (9) follows with \( \kappa := \max\{d_\varepsilon, \frac{|\Omega|}{N-1}\} \). As a consequence of (9), we have
\[
\int_{\Omega} |u|^{-\kappa} dx = \int_{0}^{\infty} \left( \int_{|u|}^{\infty} |v|^{-\kappa} d\tau \right) d\tau = \int_{0}^{\infty} \min\{ \kappa \tau^{-\frac{\kappa}{\kappa-1}}, |\Omega| \} d\tau < \infty,
\]
for every \( u \in \mathcal{X} \), since \( \frac{1}{\kappa-1} > 1 \). In particular, the map \( \kappa \) is well defined. To see the continuity of \( \kappa \), let \( \{u_n\}_n \subset \mathcal{X} \) be a sequence such that \( u_n \to u \) as \( n \to \infty \) with respect to the \( \mathcal{L}^{1} \)-norm. We then consider the compact set \( \mathcal{X} := \{ u_n : n \in \mathbb{N} \} \) and \( \kappa > 0 \) such that (9) holds. For given \( \varepsilon > 0 \), we then fix \( c > 0 \) sufficiently small such that
\[
2^{p} \int_{(2c)^{-\kappa}} \min\{ \kappa \tau^{-\frac{\kappa}{\kappa-1}}, |\Omega| \} d\tau < \varepsilon.
\]

By Lebesgue’s theorem, it is easy to see that
\[
\int_{|u| > c} (|u_n|^{-\kappa} - |u|^{-\kappa})^p dx \to 0 \quad \text{as } n \to \infty.
\]

Moreover, there exists \( n_0 \in \mathbb{N} \) be such that \( \{|u| \leq c\} \subset \{|u_n| \leq 2c\} \) for \( n \geq n_0 \). Consequently,
\[
\int_{|u| \leq c} \left| u_n \right|^{-\kappa} - \left| u \right|^{-\kappa} \right|^p dx \leq 2^{p-1} \int_{|u| \leq c} \left( \left| u_n \right|^{-\kappa} + \left| u \right|^{-\kappa} \right) dx
\]
\[
\leq 2^{p-1} \left( \int_{|u| \leq 2c} \left| u_n \right|^{-\kappa} dx + \int_{|u| \leq c} \left| u \right|^{-\kappa} dx \right)
\]
\[
= 2^{p-1} \left( \int_{(2c)^{-\kappa}} \left| u_n \right|^{-\kappa} \tau^{-\frac{\kappa}{\kappa-1}} d\tau + \int_{(2c)^{-\kappa}} \left| u \right|^{-\kappa} \tau^{-\frac{\kappa}{\kappa-1}} d\tau \right)
\]
\[
\leq 2^{p} \int_{(2c)^{-\kappa}} \min\{ \kappa \tau^{-\frac{\kappa}{\kappa-1}}, |\Omega| \} d\tau < \varepsilon \quad \text{for } n \geq n_0.
\]

Combining this with (14), we conclude that
\[
\limsup_{n \to \infty} \int_{\Omega} \left| u_n \right|^{-\kappa} - \left| u \right|^{-\kappa} \right|^p dx \leq \varepsilon.
\]
Since \( \epsilon > 0 \) was given arbitrarily, we conclude that
\[
\| \gamma(u_n) - \gamma(u) \|_{L^p(\Omega)}^p = \int_{\Omega} |u_n|^{-p} - |u|^{-p} \, dx \to 0 \quad \text{as} \ n \to \infty.
\]
Hence Claim 1 follows.

We now turn to the proof of (i). To show that \( \varphi|_W \) is of class \( C^2 \), it suffices to show that
\[
\Psi : W' \to \mathbb{R}, \quad \Psi(u) = \frac{1}{q} \int_\Omega |u|^q \, dx
\]
is of class \( C^2 \) with
\[
\Psi''(u)(v,w) = (q-1) \int_\Omega |u|^{q-2}vw \quad \text{for every} \ v,w \in C^1(\overline{\Omega}).
\]
By standard arguments, \( \Psi \) is of class \( C^1 \) with
\[
\Psi'(u)v = \int_\Omega |u|^{q-2}uv \quad \text{for every} \ v \in C^1(\overline{\Omega}).
\]
Let \( u \in W \) and \( v,w \in C^1(\overline{\Omega}) \) with \( \|v\|_{L^\infty(\Omega)}, \|w\|_{L^\infty(\Omega)} < 1 \). For \( t \in \mathbb{R} \setminus \{0\} \) we have
\[
\frac{1}{t} \left( \Psi'(u+tw)v - \Psi'(u)v \right) = I_t + J_t,
\]
with
\[
I_t = \int_{|u|>|t|} \frac{|u+tw|^{q-2}(u+tw) - |u|^{q-2}u}{t} \, dx, \quad J_t = \frac{1}{t} \int_{|u|<|t|} \frac{|u+tw|^{q-2}(u+tw) - |u|^{q-2}u}{t} \, dx.
\]
Note that, with \( s := 2 - q \in (0,1) \),
\[
I_t = (q-1) \int_0^1 \int_{|\tau|>|t|} \frac{|u + \tau w|^{-s}vw}{\tau} \, d\tau \, dx
\]
\[
= (q-1) \left( \int_0^1 \int_{|\tau|>|t|} \gamma(u + \tau w)v \, d\tau \, dx - \int_0^1 \int_{|\tau|<|t|} \gamma(u + \tau w)^{-s}vw \, d\tau \, dx \right)
\]
with
\[
\int_0^1 \int_{|\tau|<|t|} \gamma(u + \tau w)^{-s}vw \, d\tau \, dx \to \int_\Omega \gamma(u)v \, dx = \int_\Omega |u|^{-2}vw \, dx \quad \text{as} \ t \to 0
\]
and, by applying Hölder’s inequality with some \( p \in (1, \frac{1}{s}) \),
\[
\left| \int_0^1 \int_{|\tau|<|t|} |u + \tau w|^{-s}vw \, d\tau \right| \leq \left( \int_{|\tau|<|t|} |u + \tau w|^{s-1} \right)^{\frac{1}{p}} \, dw \left( \int_{|\tau|<|t|} |u + \tau w|^{s-1} \right)^{1/p} \, dw \int_\Omega \gamma(u + \tau w) \, dw \, dx \to 0 \quad \text{as} \ t \to 0.
\]
Moreover, by choosing \( \kappa > 0 \) such that (9) holds for \( u \), we find that
\[
|I_t| \leq \frac{1}{|t|} \int_{|u|<|t|} \left( |u + tw|^{q-1} + |u|^{q-1} \right) \, dw \leq |t|^{q-2} \int_{|u|<|t|} \frac{u}{|t|} \, dw \, dx + \frac{|u|^{q-1}}{|t|} \, dx \leq |t|^{q-2}(2^{q-1} + 1) \left( \int_{|u|<|t|} \right) \leq \kappa |t|^{q-1} \to 0 \quad \text{as} \ t \to 0.
\]
Combining these estimates, we conclude that
\[
\frac{1}{t} \left( \Psi'(u+tw)v - \Psi'(u)v \right) \to (q-1) \int_\Omega |u|^{q-2}vw \, dx \quad \text{as} \ t \to 0.
\]
Hence the second directional derivatives of \( \varphi \) exists at \( u \in W \) and satisfy (15). By Claim 1 above, it also follows that the second derivatives depend continuously on \( u \in W \), so that \( \Psi \in C^2(W) \). The proof of (i) is thus complete.
To prove (ii), put \( v := |u|^{q-2}u \). Then for \( f \in C_c^\infty(\Omega) \) and \( \varepsilon > 0 \) sufficiently small we have, by the divergence theorem,
\[
\left| \int_{|u| \geq \varepsilon} v \partial_u f \, dx - \int_{|u| \geq \varepsilon} \partial_u v f \, dx \right| \leq \int_{|u| = \varepsilon} |v||f| \, d\sigma \to 0 \quad \text{as } \varepsilon \to 0.
\]
Hence
\[
\int_{\Omega} v \partial_u f \, dx = \lim_{\varepsilon \to 0^+} \int_{|u| \geq \varepsilon} v \partial_u f \, dx = \lim_{\varepsilon \to 0^+} \int_{|u| \geq \varepsilon} \partial_u v f \, dx = (q - 1) \int_{\Omega} |u|^{q-2} u f \, dx
\]
for \( f \in C_c^\infty(\Omega) \), which shows that \( v_n = (q - 1)|u|^{q-2}u \) in distributional sense. Since \( v_n \in L^p(\Omega) \) for some \( p \in (1, \infty) \) by Claim 1 above and \( -\Delta u = v \) in \( \Omega \), it follows from standard regularity theory that \( u \in W^{1,p}(\Omega) \). Moreover,
\[
-\Delta u = \partial_i u = v_n \quad \text{in } \Omega
\]
in strong sense for \( i = 1, \ldots, N \), which shows (3). \( \square \)

3. The unique continuation property in the case \( q > 1 \)

In this section we still consider the case \( 1 < q < 2 \), and we show that the set \( u^{-1}(0) \subset \Omega \) has zero Lebesgue measure for every minimizer of \( \varphi \) on \( \mathcal{M} \). For this we need some preliminaries. We recall that, for a measurable subset \( A \subset \mathbb{R}^N \), a point \( x \in \mathbb{R}^N \) is called a point of density one for \( A \) if
\[
\lim_{r \to 0} \frac{|A \cap B_r(x)|}{|B_r(0)|} = 1.
\]
If \( A \subset \mathbb{R}^N \) is measurable, then, by a classical result (see e.g. [5, p.45]), a.e. \( x \in A \) is a point of density one for \( A \). We also need the following simple calculus lemma.

**Lemma 3.1.** Let \( \alpha > 0 \), and let \( f : (0, \infty) \to \mathbb{R} \) be a bounded function satisfying \( f(r) = o(r^{\alpha}) \) as \( r \to 0 \). Let \( \beta > \alpha \) and consider
\[
g : (0, \infty) \to \mathbb{R}, \quad g(r) = r^{-\beta} f(r)
\]
Then for any \( r > 0 \) there exists \( s > 0 \) with \( g(s) \geq g(r) \) and
\[
g(t) \leq 2^{\beta-\alpha} g(s) \quad \text{for } t \in \left[ \frac{s}{2}, 2s \right].
\]

**Proof.** We argue by contradiction. If the assertion was false, we would find \( r > 0 \) and a sequence \( (s_n)_n \subset (0, \infty) \) such that \( s_0 = r \) and, for every \( n \in \mathbb{N} \),
\[
s_{n+1} \in \left[ \frac{s_n}{2}, 2s_n \right] \quad \text{and} \quad g(s_{n+1}) > 2^{\beta-\alpha} g(s_n).
\]
Without loss of generality, we may also assume that \( g_0 := g(r) = g(s_0) > 0 \). From (17) we deduce
\[
s_n \geq 2^{-n} r \quad \text{and} \quad g(s_n) \geq 2^{\alpha(\beta-\alpha)} g_0.
\]
Since, by the boundedness of \( f \), the function \( g \) is bounded on intervals of the form \( [\varepsilon, \infty) \) with \( \varepsilon > 0 \), we thus conclude that \( s_n \to 0 \) as \( n \to \infty \), whereas
\[
\frac{f(s_n)}{s_n^{\beta-\alpha}} = g(s_n) s_n^{\beta-\alpha} \to r^{\beta-\alpha} g_0
\]
for all \( n \in \mathbb{N} \). This contradicts the assumption \( f(r) = o(r^{\alpha}) \) as \( r \to 0 \). \( \square \)

**Proposition 3.2.** Let \( u \) be a solution of
\[
-\Delta u = |u|^{q-2} u \quad \text{in } \Omega,
\]
and suppose that \( x_0 \in \Omega \) is a point of density one for the set \( u^{-1}(0) \subset \Omega \). Then \( u(x) = o(|x|^\alpha) \) as \( x \to x_0 \).

Here we recall that, by elliptic regularity theory, a distributional solutions \( u \) of (18) contained in \( W^{1,2}_{\text{loc}}(\Omega) \) is in fact a classical solution in \( C^{2,\alpha}_{\text{loc}}(\Omega) \) for some \( \alpha > 0 \).
Proof of Proposition 3.2. Without loss, we may assume that \( x_0 = 0 \in \Omega \) and that \( u \) is bounded in \( \Omega \) (otherwise we may replace \( \Omega \) by a bounded subdomain containing \( x_0 \)). Since 0 is a point of density one for the set \( u^{-1}(0) \) and \( u \) is a \( C^1 \)-function, it is easy to see (for instance using the Implicit Function Theorem) that \( \nabla u(0) = 0 \). We extend \( u \) to all of \( \mathbb{R}^N \) by setting \( u \equiv 0 \) on \( \mathbb{R}^N \setminus \Omega \). For \( f : (0, \infty) \rightarrow \mathbb{R} \) defined by

\[
    f(r) = \sup_{|x|=r} |u(x)|,
\]

we then have \( f(r) = o(r) \) as \( r \to 0 \). We now put

\[
    g(r) = r^{-\frac{2}{q-2}} f(r)
\]

for \( r > 0 \). We first show the following

Claim: The function \( g \) is bounded on \((0, \infty)\).

Arguing by contradiction, we assume that there exists a sequence of radii \( r_n > 0 \), \( n \in \mathbb{N} \) such that \( g(r_n) \rightarrow \infty \) as \( n \rightarrow \infty \). By Lemma 3.1 we then find \( s_n > 0 \), \( n \in \mathbb{N} \) such that, for all \( n \in \mathbb{N} \),

\[
    g(s_n) \geq g(r_n) \quad \text{and} \quad g(t) \leq Cg(s_n) \quad \text{for} \quad t \in \left[ \frac{s_n}{2}, 2s_n \right],
\]

where \( C := 2^\frac{2}{q-2} \). In particular, \( g(s_n) \rightarrow \infty \) as \( n \rightarrow \infty \). By the definition of \( g \), this implies that \( s_n \rightarrow 0 \) as \( n \rightarrow \infty \), so without loss we may assume that \( B_{2s_n}(0) \subset \Omega \) for all \( n \in \mathbb{N} \). We now put \( \Omega_0 := B_2(0) \setminus B_{\frac{1}{2}}(0) \) and define, for \( n \in \mathbb{N} \), \( v_n : \Omega_0 \rightarrow \mathbb{R} \) as

\[
    v_n(x) = \frac{s_n^{-\frac{2}{q-2}} u(s_n x)}{g(s_n)}.
\]

The functions \( v_n \) solve the equations

\[
    -\Delta v_n = g(s_n)^{q-2} |v_n|^{q-2} v_n \quad \text{in} \ \Omega_0,
\]

whereas

\[
    |v_n(x)| = \frac{|x|^{-\frac{2}{q-2}} (|x|s_n) - \frac{2}{q-2} |u(s_n x)|}{g(s_n)} \leq 2^\frac{2}{q-2} g(|x|s_n) \leq 2^\frac{2}{q-2} C
\]

for \( x \in \Omega_0, \ n \in \mathbb{N} \). Moreover, there exists a sequence of points \( x_n \in S^1 := \{ y \in \mathbb{R}^N : |y| = 1 \} \) such that \( |u(s_n x_n)| = f(s_n) \) and hence \( |v_n(x_n)| = 1 \) for \( n \in \mathbb{N} \). Using (20), elliptic regularity theory and the fact that \( g(s_n)^{q-2} \rightarrow 0 \) as \( n \rightarrow \infty \), we may pass to a subsequence such that

\[
    x_n \rightarrow \bar{x} \in S^1,
    \quad v_n \rightarrow v \quad \text{in} \ C^1_{\text{loc}}(\Omega_0),
\]

where \( v \) is a harmonic function in \( \Omega_0 \) such that \( v(\bar{x}) = 1 \). In particular, \( v \neq 0 \). On the other hand, since 0 is a point of density one for the set \( u^{-1}(0) \), the sets \( A_n := \{ x \in B_2(0) \setminus B_{\frac{1}{2}}(0) : v_n(x) \neq 0 \} \) satisfy

\[
    \frac{|A_n|}{|B_2(0)|} \leq \frac{|\{ x \in B_{2s_n} : u(x) \neq 0 \} |}{|B_{2s_n}(0)|} \rightarrow 0 \quad \text{as} \ n \rightarrow \infty
\]

and thus \( v \equiv 0 \) a.e. in \( B_2(0) \setminus B_{\frac{1}{2}}(0) \). This is a contradiction, and thus the above claim is true.

To finish the proof of the proposition, it thus remains to show that \( g(r) \rightarrow 0 \) as \( r \to 0 \). Arguing again by contradiction, we assume that there exists \( \varepsilon > 0 \) and a sequence of radii \( r_n > 0 \), \( n \in \mathbb{N} \) such that \( r_n \rightarrow 0 \) as \( n \rightarrow \infty \) and \( g(r_n) \geq \varepsilon \) for all \( n \in \mathbb{N} \). We may assume that \( B_{2r_n}(0) \subset \Omega \) for all \( n \in \mathbb{N} \). We then consider \( w_n : \Omega_0 \rightarrow \mathbb{R}, \ w_n(x) = r_n^{-\frac{2}{q-2}} u(r_n x) \) for \( n \in \mathbb{N} \). It follows from the claim above that the functions \( w_n \) are uniformly bounded in \( \Omega_0 \). Moreover, \( w_n \) solves

\[
    -\Delta w_n = |w_n|^{q-2} w_n \quad \text{in} \ B_2(0) \setminus B_{\frac{1}{2}}(0),
\]
and there exists a sequence of points $x_n \in S^1$, $n \in \mathbb{N}$ such that $w(x_n) = g(r_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. Using elliptic regularity theory again, we may pass to a subsequence such that

$$
x_n \to \bar{x} \in S^1,
$$

$$
w_n \to w \text{ in } C^{1,\alpha}(\Omega_0),
$$

where $w \in C^{2,\alpha}(\Omega_0)$ is a weak solution of $-\Delta w = |w|^{q-2}w$ in $\Omega_0$ such that $w(\bar{x}) \geq \varepsilon$. In particular, $w \not\equiv 0$. However, by the same argument as above, the sets $A_n := \{ x \in B_2(0) \setminus B_{\frac{1}{2}}(0) : w_n(x) \neq 0 \}$ satisfy

$$
\lim_{n \to \infty} |A_n| = 0
$$

and therefore $w \equiv 0$ a.e. in $B_2(0) \setminus B_{\frac{1}{2}}(0)$. This yields a contradiction, and thus we conclude that $g(r) \to 0$ as $r \to 0$, as required. \hfill \square

Next, we consider the family of energy functionals

$$
\varphi_r : W^{1,2}(B_r(0)) \to \mathbb{R}, \quad \varphi_r(v) = \frac{1}{2} \int_{B_r(0)} |\nabla v|^2 \, dx - \frac{1}{q} \int_{B_r(0)} |v|^q \, dx
$$

for $r > 0$. We also consider the scaling map

$$
T_r : W^{1,2}(B_1(0)) \to W^{1,2}(B_r(1)), \quad [T_r v](x) = r^{\frac{2-q}{q}} v \left( \frac{x}{r} \right).
$$

We then have the following.

**Proposition 3.3.** Let $u \in W^{1,2}(\Omega)$ be a solution of (13) in $\Omega$, and let $x_0 \in \Omega$ be a point of density one for $u^{-1}(0) \subset \Omega$. Moreover, let $K \subset W^{1,2}_0(B_1(0))$ be a compact set such that

$$
c_K := \sup_{v \in K} \varphi_1(v) < 0.
$$

Then there exists $r_0 > 0$ with the following property:

$B_{r_0}(x_0)$ is contained in $\Omega$, and for every $r \in (0, r_0)$ and every $v \in K$ we have $\varphi(u + v_r) < \varphi(u)$, where $v_r \in W^{1,2}_0(\Omega)$ is defined by

$$
v_r(x) = \begin{cases} 
[T_r v](x - x_0) & x \in B_r(x_0); \\
0 & x \in \Omega \setminus B_r(x_0).
\end{cases}
$$

**Proof.** Without loss, we may assume that $x_0 = 0 \in \Omega$. Let $v \in K$. We then have

$$
\varphi(u + v_r) = \varphi(u) + \varphi_r(v_r) + \int_{B_r(0)} \nabla u \nabla v_r \, dx - \frac{1}{q} \int_{B_r(0)} \left( |u + v_r|^q - |u|^q - |v_r|^q \right) \, dx
$$

$$
\leq \varphi(u) + \varphi_r(v_r) + \frac{1}{q} \int_{B_r(0)} \nabla u \nabla v_r \, dx + \frac{1}{q} \int_{B_r(0)} \left( |u|^q - q |v_r|^q - 2v_r u \right) \, dx
$$

(23)

where in the last step we used that

$$
\int_{B_r(0)} \left( |u + v_r|^q - |v_r|^q - q |v_r|^{q-2} v_r u \right) \, dx \geq 0
$$

by the convexity of the function $t \mapsto |t|^q$. We note that

$$
\int_{B_r(0)} \nabla u \nabla v_r \, dx = \int_{B_r(0)} |u|^{q-2} u v_r \, dx
$$

and therefore

$$
\left| \int_{B_r(0)} \nabla u \nabla v_r \, dx \right| \leq \|u\|_{L^q(B_r(0))}^{-1} \int_{B_r(0)} |v_r| \, dx = o(r^{\frac{2q-1}{q}})r^{N+\frac{2}{q}} \int_{B_1(0)} |v| \, dx = o(r^{N+\frac{2q}{q}}) \text{ as } r \to 0.
$$

Moreover,

$$
\int_{B_r(0)} |u|^q \, dx = o(r^{2q})|B_r(0)| = o(r^{N+\frac{2q}{q}})
$$
functions for every compact set there exists a unit vector \( u \) such that

\[
\text{Theorem 4.1.} \quad \text{Suppose that } \varphi \text{ is defined in (5). It is clear that for } 0 < q < 1, \text{ we have:}
\]

\[
\varphi_r(v) = \frac{r^{2q}}{2} \int_{B_r(0)} |\nabla \left( \frac{x}{r} \right) |^2 \, dx - \frac{r^{2q}}{q} \int_{B_r(0)} |\nabla \left( \frac{x}{r} \right) |^q \, dx
\]

\[
= r^{N+\frac{2q}{q-1}} \left( \frac{1}{2} \int_{B_1(0)} |\nabla \psi(x)|^2 \, dx - \frac{1}{q} \int_{B_1(0)} |v(x)|^q \, dx \right)
\]

Inserting these estimates in (23), we obtain

\[
\varphi(u + v_r) \leq \varphi(u) + r^{N+\frac{2q}{q-1}} \left( \varphi(v) + o(1) \right) \leq \varphi(u) + r^{N+\frac{2q}{q-1}} \left( c_K + o(1) \right)
\]

Since \( K \) is compact, it is easy to see that these estimates are uniform in \( v \in K \). Since moreover \( c_K < 0 \), the claim follows.

**Theorem 3.4.** Let \( u \) be a minimizer of \( \varphi \) on \( \mathcal{N} \). Then \( u^{-1}(0) \) has vanishing Lebesgue measure.

**Proof.** Suppose by contradiction that \( |u^{-1}(0)| > 0 \). Then there exists a point \( x_0 \) of density one for the set \( u^{-1}(0) \). Without loss of generality, we can suppose \( x_0 = 0 \). We fix two arbitrary nonnegative nontrivial functions \( v_1, v_2 \in W^{1,2}_0(B_1(0)) \) with disjoint support, and consider the path

\[
\gamma : [0, 1] \to W^{1,2}_0(B_1(0)), \quad \gamma(t) = t v_1 - s v_2,
\]

where the map \( t^* \) is defined in (5). It is clear that \( \varphi(\gamma(t)) < 0 \) for \( t \in [0, 1] \). Applying Proposition 3.3 to the compact set \( K := \gamma([0, 1]) \subset W^{1,2}_0(B_1(0)) \), we may fix \( r > 0 \) sufficiently small such that \( \varphi(u + v_r) < \varphi(u) \) for every \( v \in K \). Since \( K \) is connected and

\[
\int_{\Omega} |u + v_r|^{q-2}(u + v_r) \, dx \begin{cases} > 0 & \text{for } v = \gamma(0) \in K, \\ < 0 & \text{for } v = \gamma(1) \in K, \end{cases}
\]

there exists \( v \in K \) such that \( u + v_r \in \mathcal{N} \). This however contradicts the assumption that \( u \) is a minimizer of \( \varphi \) in \( \mathcal{N} \).

**Corollary 3.5.** Let \( u \) be a minimizer of \( \varphi \) on \( \mathcal{N} \), and let \( x_0 \in \Omega \) be a point with \( u(x_0) = 0 \). Then \( u \) changes sign in every neighborhood of \( x_0 \).

**Proof.** Suppose by contradiction that there is a ball \( B_r(x_0) \subset \Omega \) such that \( u \) does not change sign in \( B_r(x_0) \). Without loss, we may assume that \( u \geq 0 \) on \( B_r(x_0) \). By Theorem 3.4, \( u \neq 0 \) in \( B_r(x_0) \). Since \( \varphi \) is defined in (1) and is therefore superharmonic in \( B_r(x_0) \), the strong maximum principle implies that \( u > 0 \) in \( B_r(x_0) \), contrary to the assumption that \( u(x_0) = 0 \).

**4. Symmetry results**

We add a result on minimizers of \( \varphi \) in the case where the underlying domain is radial, i.e., a ball or an annulus in \( \mathbb{R}^N \) centered at zero.

**Theorem 4.1.** Suppose that \( \Omega \subset \mathbb{R}^N \) is a radial bounded domain. Then every minimizer \( u \) of \( \varphi \) on \( \mathcal{N} \) is foliated Schwarz symmetric.

Here we recall that a function \( u \) defined on a radial domain is said to be foliated Schwarz symmetric if there is a unit vector \( p \in \mathbb{R}^N \) \( |p| = 1 \) such that \( u(x) \) only depends on \( r = |x| \) and \( \theta = \arccos \left( \frac{x}{|x|} \cdot p \right) \) and \( u \) is nonincreasing in \( \theta \).
Proof. Let $u \in \mathcal{N}$ be a minimizer of $\varphi|_{\mathcal{N}}$, and pick $x_0 \in \Omega \setminus \{0\}$ with $u(x_0) = \max\{u(x) : \|x\| = |x_0|\}$. We put $p := \frac{x_0}{|x_0|}$, and we let $\mathcal{H}_p$ denote the family of all open halfspaces $H$ in $\mathbb{R}^N$ such that $p \in H$ and $0 \in \partial H$. For $H \in \mathcal{H}_p$ we consider the reflection $\sigma_H : \mathbb{R}^N \to \mathbb{R}^N$ with respect to the the hyperplane $\partial H$.

We claim the following:

\begin{equation}
\label{24}
\text{For every } H \in \mathcal{H}_p, \text{ we have } u \geq \sigma_H \text{ on } H \cap \Omega.
\end{equation}

To prove this, we fix $H \in \mathcal{H}_p$ and recall a simple rearrangement, namely the polarization of $u$ with respect to $H$ defined by

$$u_H(x) = \begin{cases} 
\max\{u(x), u(\sigma_H(x))\}, & x \in \Omega \cap H \\
\min\{u(x), u(\sigma_H(x))\}, & x \in \Omega \cap H
\end{cases}$$

It is well known and fairly easy to prove (see e.g. [13]) that

$$\int_\Omega |\nabla u_H|^2 \, dx = \int_\Omega |\nabla u|^2 \, dx, \quad \int_\Omega |u_H|^q \, dx = \int_\Omega |u|^q \, dx \quad \text{and} \quad \int_\Omega |u_H|^{q-2} u_H \, dx = \int_\Omega |u|^{q-2} u \, dx.$$

Consequently, $u_H \in \mathcal{N}$ and $\varphi(u_H) = \varphi(u)$, so that $u_H$ is also a minimizer of $\varphi$ on $\mathcal{N}$. Hence, by Theorem 1.1 both $u$ and $u_H$ are solutions of (1). Therefore $w := u_H - u$ is a nonnegative function in $\Omega \cap H$ satisfying

$$-\Delta w = |u_H|^{q-2} u_H - |u|^{q-2} u \geq 0 \quad \text{in } H \cap \Omega.$$

The strong maximum principle then implies that either $w \equiv 0$ or $w > 0$ in $H \cap \Omega$. The latter case is ruled out since $x_0 \in H \cap \Omega$ and $w(x_0) = u_H(x_0) - u(x_0) = 0$ by the choice of $x_0$. We therefore obtain $w \equiv 0$, hence $u = u_H$ and (24) holds.

By continuity, it follows from (24) that $u$ is symmetric with respect to every hyperplane containing $p$, so it is axially symmetric with respect to the axis $p\mathbb{R}$. Hence $u(x)$ only depends on $r = |x|$ and $\theta = \arccos\left(\frac{x_0}{|x|} \cdot p\right)$. Moreover, it also follows from (24) that $u$ is nonincreasing in the polar angle $\theta$. We thus conclude that $u$ is foliated Schwarz symmetric. \hfill \Box

**Theorem 4.2.** Let $\Omega \subset \mathbb{R}^N$ be the unit ball. Then, a least-energy nodal solution is not radially symmetric.

**Proof.** Let $u$ be a least-energy nodal solution. By elliptic regularity, we have $C^{2,\lambda}(\overline{\Omega})$. We suppose by contradiction that $u$ is radially symmetric, and we write $u(r) := u(|x|)$ for simplicity. Then $u$ solves

\begin{equation}
\label{25}
u'' + \frac{N-1}{r} u' + |u|^{q-2} u = 0 \quad \text{in } (0,1], \quad \quad u'(0) = u'(1) = 0,
\end{equation}

where the prime denotes the radial derivative. We first prove

**Claim 1:** $u(0) \neq 0$, $u(1) \neq 0$, and $u$ has only finitely many zeros in $(0,1)$.

Indeed, with the transformation

$$y(t) = u(r) \quad \text{with} \quad t = \begin{cases} 
\frac{r^2}{2}, & N \geq 3; \\
1 - \log r, & N = 2,
\end{cases}$$

(25) is transformed into the problem

\begin{equation}
\label{26}
y'' + p(t)|y|^{q-2} y = 0, \quad t \in [1,\infty), \quad y(1) = 0
\end{equation}

with

$$p(t) = \begin{cases} 
\frac{1}{(N-2)^2}, & N \geq 3; \\
e^{-2(t-1)}, & N = 2.
\end{cases}$$

Since $u \neq 0$ we have $y \neq 0$ in $[1,\infty)$. Hence $y(1) \neq 0$ by local uniqueness and continuability of solutions to (26) (see e.g. [14]), and thus $u(1) \neq 0$. Moreover, by the non-oscillation criterion for (26) given in [8, Theorem 6], $y$ has only finitely many zeros in $[1,\infty)$, so $u$ only has finitely many zeros in $(0,1)$. Finally, since $u$ does not change sign in a neighborhood of 0, the strong maximum principle implies that
Hence there exist

\[ u(0) \neq 0. \]  

Thus Claim 1 is proved.

It now follows from Hopf's boundary lemma that \( u \in W \), where \( W \) is defined in \((7)\). Consequently, \( u \in W^{3,p}(\Omega) \) by Proposition 2.3 and \( u_1 := u_{11} \in W^{2,p}(\Omega) \cap C^1(\Omega) \) solves the linearized Dirichlet problem

\[
\begin{cases}
-\Delta u_1 = (q - 1)|u|^{q-2}u_1 & \text{in } \Omega, \\
 u_1 = 0 & \text{on } \partial \Omega.
\end{cases}
\]

The boundary condition follows from the fact that \( \nabla u \equiv 0 \) on \( \partial \Omega \) since \( u \) is radial and satisfies Neumann boundary conditions. Let \( \mathcal{H} \) be the hyperplane \( \{x_1 = 0\} \). We first prove the following

**Claim 2:** If \( w \in C^1(\overline{\Omega}) \) is antisymmetric with respect to \( \mathcal{H} \) and such that \( \varphi''(u)(w,w) < 0 \), then \( \varphi(u + tw + c(u + tw)) < \varphi(u) \) for \( t > 0 \) sufficiently small.

Indeed, by Proposition 2.3, we have, for every \( c \in \mathbb{R} \), the Taylor expansion

\[
\varphi(u + c + tw) = \varphi(u + c) + t \varphi'(u + c)w + t^2 \varphi''(u + c)(w,w) + o(t^2),
\]

where the quantity \( o(t^2) \) is locally uniform in \( c \). Since \( u \) is radially symmetric and \( w \) is antisymmetric with respect to \( \mathcal{H} \), we have

\[
\varphi'(u + c)w = \int_\Omega \nabla u \nabla w \, dx - \int_\Omega |u + c|^{q-2}(u + c)w \, dx = 0.
\]

Hence there exist \( M > 0 \) and \( \delta > 0 \) such that

\[
\varphi(u + tw + c(u + tw)) \leq \varphi(u + c) - Mt^2 \leq \varphi(u) - Mt^2 
\]

for \( |t|, |c| < \delta \).

Since \( c(u + tw) \to c(u) = 0 \) as \( t \to 0 \) as a consequence of Lemma 2.1, we deduce that

\[
\varphi(u + tw + c(u + tw)) \leq \varphi(u) - Mt^2 < \varphi(u) 
\]

for \( t > 0 \) sufficiently small.

Hence Claim 2 is proved. Next, we consider an arbitrary function \( w \in C^1(\overline{\Omega}) \) which is antisymmetric with respect to \( \mathcal{H} \). By Claim 2 and the minimizing property of \( u \), we have

\[
0 \leq \varphi''(u)(w + tu_1, w + tu_1) = \varphi''(u)(w,w) + 2t \varphi'(u)(u_1, w) + t^2 \varphi''(u)(u_1, u_1)
\]

for every \( t \in \mathbb{R} \)

and also

\[
\varphi''(u)(u_1, u_1) = \int_\Omega |
abla u_1|^2 \, dx - (q - 1) \int_\Omega |u|^{q-2} u_1^2 \, dx = 0,
\]

since \( u_1 \) is a solution of \((27)\). These relations imply that

\[
(28) \quad 0 = \varphi''(u)(u_1, w) = \int_\Omega \nabla u_1 \nabla w \, dx - (q - 1) \int_\Omega |u|^{q-2} u_1 w \, dx = \int_{\partial \Omega} (u_1)_v w \, d\sigma,
\]

where the last equality follows again from \((27)\). Since \((u_1)_v \in C(\partial \Omega)\) is antisymmetric with respect to \( \mathcal{H} \) and \((28)\) holds for every \( w \in C^1(\overline{\Omega}) \) which is antisymmetric with respect to \( \mathcal{H} \), we conclude that \((u_1)_v = 0 \) on \( \partial \Omega \), and in particular \( u_{1,1}(e_1) = 0 \). In the radial variable, we thus have

\[
0 = u''(1) = -|u(1)|^{q-2}u(1)
\]

by \((25)\) and therefore \( u(1) = 0 \), contrary to Claim 1. The proof is finished.

5. **The case** \( q = 1 \)

In this section we are concerned with the case \( q = 1 \), i.e., with the boundary value problem

\[
\begin{cases}
-\Delta u = \text{sgn}(u) & \text{in } \Omega, \\
 u_v = 0 & \text{on } \partial \Omega.
\end{cases}
\]

We will suppose that the boundary of \( \Omega \) is of class \( C^{1,1} \). As already noted in the introduction, the variational framework of Section 2 does not extend in a straightforward way to the case \( q = 1 \). In particular, the functional

\[
\varphi : W^{1,2}(\Omega) \to \mathbb{R}, \quad \varphi(u) = \frac{1}{2} \int_\Omega |
abla u|^2 \, dx - \int_\Omega |u| \, dx
\]
is not differentiable. Moreover, while every solution of (29) is contained in the set
\[ \{ u \in W^{1,2}(\Omega) \mid \int_{\Omega} \text{sgn}(u) \, dx = 0 \}, \]
this set does not have the nice intersection property given by Lemma 2.1(i). Indeed, any function \( u \in W^{1,2}(\Omega) \) satisfying
\[ 0 < \left| |\{ u > 0 \}| - |\{ u < 0 \}| \right| < |\{ u = 0 \}| \]
has the property that
\[ \int_{\Omega} \text{sgn}(u + c) \, dx \neq 0 \quad \text{for every } c \in \mathbb{R}. \]
As a consequence, many of the arguments used in the previous sections do not apply in the case \( q = 1 \). Instead, we will consider the larger set
\[ \mathcal{M} := \{ u \in W^{1,2}(\Omega) : \left| |\{ u > 0 \}| - |\{ u < 0 \}| \right| \leq |\{ u = 0 \}| \}. \]
We collect useful properties of \( \mathcal{M} \). First, we may rewrite the defining property for \( \mathcal{M} \) as
\[ (30) \quad \int_{\Omega} \text{sgn}_-(u) \, dx \leq 0 \leq \int_{\Omega} \text{sgn}_+(u) \, dx, \]
where
\[ \text{sgn}_-(t) := 1_{t \geq 0} - 1_{t < 0} \quad \text{and} \quad \text{sgn}_+(t) := 1_{t > 0} - 1_{t \leq 0} \quad \text{for } t \in \mathbb{R}. \]
We also point out that, in contrast to the definition in the case \( q > 1 \), the set \( \mathcal{M} \) also contains nonzero functions which do not change sign. We also need the following facts.

**Lemma 5.1.**

(i) If \( u \in \mathcal{M} \), then
\[ \int_{\Omega} |u + c| \, dx > \int_{\Omega} |u| \, dx \quad \text{for every } c \in \mathbb{R} \setminus \{0\}. \]
(ii) For \( u \in W^{1,2}(\Omega) \), we have \( u \in \mathcal{M} \) if and only if
\[ \int_{\Omega} \text{sgn}(u - c) \, dx \leq 0 \leq \int_{\Omega} \text{sgn}(u + c) \, dx \quad \text{for every } c > 0. \]
(iii) If \( u \in W^{1,2}(\Omega) \) and \( \gamma : [0,1] \to W^{1,2}(\Omega) \cap L^\infty(\Omega) \) is a continuous curve such that
\[ \int_{\Omega} \text{sgn}(u + \gamma(0)) \, dx > 0 \quad \text{and} \quad \int_{\Omega} \text{sgn}(u + \gamma(1)) \, dx < 0, \]
then there exists \( s_0 \in [0,1] \) with \( u + \gamma(s_0) \in \mathcal{M} \).
(iv) For every \( u \in W^{1,2}(\Omega) \) there exists a unique \( c(u) \in \mathbb{R} \) such that \( u + c(u) \in \mathcal{M} \). Moreover, the map \( u \mapsto c(u) \) is continuous.
(v) If \( u,v \in W^{1,2}(\Omega) \) satisfy \( u \leq v \), then \( c(u) \geq c(v) \).

**Proof.**

(i) Let \( c < 0 \). If \( u \leq 0 \), then obviously \( \int_{\Omega} |u + c| \, dx > \int_{\Omega} |u| \, dx \). Suppose now that \( u^+ \neq 0 \). We then claim that
\[ (31) \quad |\{ 0 < u < |c| \}| > 0. \]
Indeed, we have \( v := \min(u^+, |c|) \neq 0 \), since \( 0 < v = u^+ - (u - |c|)^+ \) on the set where \( u \) is positive. Next we consider
\[ w := \min \left\{ v - \frac{c}{2}, v + \frac{c}{2} \right\} = \left( v - \frac{c}{2} \right)^+ \in W^{1,2}(\Omega). \]
Then $\int_{\Omega} |\nabla w|^2 \, dx = \int_{\Omega} |\nabla v|^2 \, dx \neq 0$ and thus $w \neq 0$. Since $\{w \neq 0\} \subset \{0 < u < |c|\}$, we conclude that $|\{0 < u < |c|\}| > 0$, as claimed. As a consequence of (31), we estimate, for $c < 0$,

$$
\int |u + c| \, dx - \int |u| \, dx = |c||\{u \leq 0\}| + \int_{\{0 < u < |c|\}} (|u + c| - u) \, dx + \int_{\{u \geq |c|\}} (|u + c| - u) \, dx
$$

$$
= |c||\{u \leq 0\}| + \int_{\{0 < u < |c|\}} (|c| - 2u) \, dx - |c||\{u \geq |c|\}|
$$

$$
> |c||\{u \leq 0\}||\{0 < u < |c|\}||\{u \geq |c|\}|
$$

$$
= |c|(\{u \leq 0\} - \{u > 0\}) \geq 0,
$$
as claimed. If $c > 0$, a similar argument yields

$$
\int |u + c| \, dx - \int |u| \, dx > 0.
$$

(ii) This simply follows from the fact that $u \in M$ is equivalent to (30), whereas

$$
\text{sgn}(u + c) \geq \text{sgn}_+(u) \geq \text{sgn}_-(u) \geq \text{sgn}(u - c)
$$

for every $c > 0$ and

$$
\text{sgn}(u + c) \to \text{sgn}_+(u), \text{sgn}(u - c) \to \text{sgn}_-(u) \quad \text{pointwise in } \Omega \text{ as } c \to 0^+.
$$

(iii) Consider

$$
s_0 := \sup\{s \in [0, 1] : \int_{\Omega} \text{sgn}(u + \gamma(s)) \, dx > 0\},
$$

and let $w := u + \gamma(s_0)$. We use (ii) to show that $w \in M$. Let $(s_n)_n \subset [0, s_0]$ be a sequence with $s_n \to s_0$ and

$$(32) \quad \int_{\Omega} \text{sgn}(u + \gamma(s_n)) \, dx > 0 \quad \text{for every } n \in \mathbb{N}.
$$

For given $c > 0$, there exists $n \in \mathbb{N}$ with

$$
\{u + \gamma(s_n) > 0\} \subset \{w + c > 0\} \quad \text{and} \quad \{w + c < 0\} \subset \{u + \gamma(s_n) < 0\}
$$

and thus $\int_{\Omega} \text{sgn}(w + c) \, dx > 0$ by (32). Now if $s_0 = 1$, the assumption implies that

$$
0 > \int_{\Omega} \text{sgn}(w) \, dx \geq \int_{\Omega} \text{sgn}(w - c) \, dx \quad \text{for all } c > 0
$$

and thus $w \in M$ by (ii). Suppose finally that $s_0 < 1$, and suppose by contradiction that

$$
\int_{\Omega} \text{sgn}(w - c) \, dx > 0 \quad \text{for some } c > 0.
$$

By the continuity of $\gamma$, there exists $\varepsilon > 0$ such that $u + \gamma(s) \geq w - c$ for $s \in [s_0, s_0 + \varepsilon)$ and therefore

$$
\int_{\Omega} \text{sgn}(u + \gamma(s)) \, dx > 0 \quad \text{for } s \in [s_0, s_0 + \varepsilon).
$$

This contradicts the definition of $s_0$. Hence

$$
\int_{\Omega} \text{sgn}(w - c) \, dx \leq 0 \quad \text{for every } c > 0,
$$

and by (ii) we conclude that $w = u + \gamma(s_0) \in M$.

(iv) Let $u \in W^{1,2}(\Omega)$. The uniqueness of $c = c(u) \subset \mathbb{R}$ with $u + c \in M$ is an immediate consequence of (i). To see the existence, we note that $\text{sgn}(u \pm c) \to \pm 1$ as $c \to +\infty$ a.e. in $\Omega$. Hence, by Lebesgue’s theorem, there exists $c_0 > 0$ with

$$
\pm \int_{\Omega} \text{sgn}(u \pm c_0) \, dx > 0.
$$
Applying (iii) to the path $s \mapsto (1 - 2s)c_0$ now yields the existence of $c \in [-c_0, c_0]$ such that $u + c \in \mathcal{M}$. The continuity follows similarly as (but more easily than) in Lemma 5.1.

(v) Suppose by contradiction that $c(u) < c(v) =: c$. We then have $u \leq u + c \leq v + c$ in $\Omega$ and therefore

$$\int_{\Omega} \mathrm{sgn}_-(u + c) \, dx \leq \int_{\Omega} \mathrm{sgn}_-(v + c) \, dx \leq 0$$

and

$$\int_{\Omega} \mathrm{sgn}_+(u + c) \, dx \geq \int_{\Omega} \mathrm{sgn}_+(u) \, dx \geq 0.$$  

By (30), we then have $u + c \in \mathcal{M}$, with contradicts the uniqueness statement in (iv). \hfill \square

We now consider the variational problem related to the minimax value $m := \inf_{\mathcal{M}} \varphi = \inf_{u \in W^{1,2}(\Omega)} \sup_{c \in \mathbb{R}} \varphi(u + c)$.

Note that the second equality is an immediate consequence of Lemma 5.1(i), (iv). Note also that $m < 0$, since for every $u \in \mathcal{M} \setminus \{0\}$ we have

$$\varphi(t^\ast(u)u) = -\frac{1}{2} \int_{\Omega} \frac{|u|^2}{2} \leq 0 \quad \text{with} \quad t^\ast(u) = \frac{|u|^2}{\int_{\Omega} |\nabla u|^2 \, dx},$$

whereas $t^\ast(u)u \in \mathcal{M}$.

The main result of this section is the following.

**Theorem 5.2.** The value $m < 0$ is attained by $\varphi$ on $\mathcal{M}$. Moreover, every $u \in \mathcal{M}$ with $\varphi(u) = m$ is a nontrivial solution of (29) such that its zero set $\{u = 0\} \subset \Omega$ has vanishing Lebesgue measure.

The proof of this Theorem is split in two steps. We first show the following.

**Lemma 5.3.** The functional $\varphi$ attains the value $m < 0$ on $\mathcal{M}$. Moreover, every minimizer $u$ of $\varphi$ on $\mathcal{M}$ is a sign changing solution of (29).

**Proof.** We first note that for all $u \in \mathcal{M}$ we have, by Lemma 5.1(i)

$$\|u\|^2_{L^2(\Omega)} = \min_{c \in \mathbb{R}} \|u + c\|^2_{L^2(\Omega)} \leq |\Omega| \min_{c \in \mathbb{R}} \|u + c\|^2_{L^2(\Omega)} \leq |\Omega| \mu_2^{-1} \int_{\Omega} |\nabla u|^2 \, dx,$$

where $\mu_2 > 0$ is the first nontrivial Neumann eigenvalue of $-\Delta$ on $\Omega$. As a consequence, the functional $\varphi$ is coercive on $\mathcal{M}$. Let $(u_n)_n \subset \mathcal{M}$ be a minimizing sequence for $\varphi$. Then $(u_n)$ is bounded, and we may pass to a subsequence such that $u_n \rightharpoonup \check{u} \in W^{1,2}(\Omega)$. Then $u_n \rightarrow \check{u}$ in $L^1(\Omega)$,

$$\int_{\Omega} |\nabla \check{u}|^2 \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^2 \, dx$$

and

$$\int_{\Omega} |\check{u} + c| \, dx = \lim_{n \rightarrow \infty} \int_{\Omega} |u_n + c| \, dx \leq \lim_{n \rightarrow \infty} \int_{\Omega} |u_n| \, dx = \int_{\Omega} |\check{u}| \, dx$$

for all $c \in \mathbb{R}$.

Consequently, $\check{u} \in \mathcal{M}$ by Lemma 5.1(i) and (iv), and

$$\varphi(\check{u}) \leq \liminf_{n \rightarrow \infty} \varphi(u_n) = m.$$  

By definition of $m$, equality holds and thus $\varphi$ attains its minimum on $\mathcal{M}$.

Next, we let $u \in \mathcal{M}$ be an arbitrary minimizer for $\varphi$ on $\mathcal{M}$, and we show that $u$ is a solution of (29). We first show that

$$\int_{\Omega} \mathrm{sgn}_-(u) v dx \leq \int_{\Omega} \nabla u \nabla vdx \leq \int_{\Omega} \mathrm{sgn}_+ (u) v dx$$

for every $v \in W^{1,2}(\Omega)$, $v \geq 0$.

Arguing by contradiction, we assume that there exists $v \in W^{1,2}(\Omega)$, $v \geq 0$ such that

$$\int_{\Omega} \mathrm{sgn}_-(u) v dx > \int_{\Omega} \nabla u \nabla vdx.$$
Note that for $a,c \in \mathbb{R}$ we have $|a + c| \geq |a| + \text{sgn}_-(a)c$. Hence for every $c \leq 0$, $t \geq 0$ we have
\[
\|u + c + tv\|_{L^1(\Omega)} \geq \|u\|_{L^1(\Omega)} + \int_\Omega \text{sgn}_-(u)(c + tv) dx \geq \|u\|_{L^1(\Omega)} + t \int_\Omega \text{sgn}_-(u)v dx,
\]
where the last inequality follows from the fact that $\int_\Omega \text{sgn}_-(u) dx \leq 0$ since $u \in \mathcal{M}$. Since $v \geq 0$ implies that $c(u + tv) \leq 0$ for $t > 0$ by Lemma (5.1(v)), we find that
\[
\phi(u + c + tv + u + tv) = \frac{1}{2} \int_\Omega |\nabla(u + tv)|^2 dx - \|u + tv + c(u + tv)\|_{L^1(\Omega)} \\
\leq \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \|u\|_{L^1(\Omega)} + t \left( \int_\Omega \nabla u \nabla v dx - \int_\Omega \text{sgn}_-(u)v dx \right) + \frac{t^2}{2} \int_\Omega |v|^2 dx \\
= m + t \left( \int_\Omega \nabla u \nabla v dx - \int_\Omega \text{sgn}_-(u)v dx \right) + \frac{t^2}{2} \int_\Omega |v|^2 dx \\
< m 
\]
for $t > 0$ sufficiently small by (35). This contradicts the definition of $m$. Hence the first inequality in (34) holds, and the second inequality is proved by a similar argument. As a consequence, we have
\[
\int_\Omega \nabla u \nabla v dx \leq \int_\Omega [\text{sgn}_+(u)v^+ - \text{sgn}_-(u)v^-] dx \leq \|v\|_{L^1(\Omega)} \quad \text{for every } v \in W^{1,2}(\Omega).
\]
Consequently, the distributional Laplacian $\Delta u : C_0^\infty(\Omega) \to \mathbb{R}$ is continuous with respect to the $L^1(\Omega)$-norm and is therefore represented by a function $-w \in L^\infty(\Omega)$ satisfying
\[
\text{sgn}_-(u) \leq w \leq \text{sgn}_+(u).
\]
Then, by elliptic regularity theory, it follows that $u \in W^{2,p}_{\text{loc}}(\Omega)$ for all $p \in (1, \infty)$ with $-\Delta u = w$. By a classical result (see e.g. [5] Lemma 7.7), we then have $\nabla u \equiv 0$ and $w = -\Delta u \equiv 0$ a.e. on the set $\{u \equiv 0\}$. Hence, by (36) we may assume that $w = \text{sgn}(u)$, and thus $u$ is a solution of (29). Finally, to show that $u$ is sign changing, we first note that $u \not\equiv 0$ since $\phi(u) = m < 0$. Suppose by contradiction that $u \geq 0$, then $u$ is also superharmonic by (29), and hence $u > 0$ in $\Omega$ by the strong maximum principle, which contradicts the fact that $u \in \mathcal{M}$. Similarly, we get a contradiction assuming that $u \leq 0$. Hence $u$ changes sign in $\Omega$.

In order to complete the proof of Theorem 5.2 we need to show that every minimizer $u \in \mathcal{M}$ of $\phi$ has the unique continuation property, that is, $\{u \equiv 0\}$ has measure zero. The argument is similar as in the case $q > 1$, but some changes are required at key points. We start with the following.

**Proposition 5.4.** Let $u$ be a solution of (29), and suppose that $x_0 \in \Omega$ is a point of density one for the set $u^{-1}(0) \subset \Omega$. Then $u(x) = o(|x|^2)$ as $x \to x_0$.

**Proof.** The argument is similar as the proof of Proposition 3.2. Without loss, we assume that $x_0 = 0$, and we extend $u$ to all of $\mathbb{R}^N$ by setting $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$. Applying Lemma 3.1 as in the proof of Proposition 3.2 with $\alpha = 1$, $\beta = 2$, we see that the function
\[
g : (0, \infty) \to \mathbb{R}, \quad g(r) := r^2 \sup_{|x|=r} |u(x)|
\]
is bounded. To show that $g(r) \to 0$ as $r \to 0$, we argue by contradiction, assuming that there exists $\varepsilon > 0$ and a sequence of radii $r_n > 0$, $n \in \mathbb{N}$ such that $r_n \to 0$ as $n \to \infty$ and $g(r_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. We may assume that $B_{2r_n}(0) \subset \Omega$ for all $n \in \mathbb{N}$. We then consider $\Omega_0 := B_2(0) \setminus B_1(0)$ and the functions $w_n : \Omega_0 \to \mathbb{R}$, $w_n(x) = r_n^{-2}u(r_nx)$ for $n \in \mathbb{N}$ which are uniformly bounded in $\Omega_0$. For $n \in \mathbb{N}$, $w_n$ solves
\[
-\Delta w_n = \text{sgn}(w_n) \quad \text{in } B_2(0) \setminus B_1(0),
\]
and there exists a sequence of points \( x_n \in S^1 \), \( n \in \mathbb{N} \) such that \( w(x_n) = g(r_n) \geq \epsilon \) for all \( n \in \mathbb{N} \). Using elliptic regularity theory again, we may pass to a subsequence such that

\[
x_n \to \bar{x} \in S^1;
\]
\[
w_n \to w \text{ in } C^1_{loc}(\Omega_0),
\]
\[
\text{sgn}(w_n) \to^* f \quad \text{in } L^\infty(\Omega_0) = (L^1(\Omega_0))^*,
\]

where \( w \in C^1_{loc}(\Omega_0) \) is a weak solution of \(-\Delta w = f\) in \( \Omega_0 \) such that \( w(\bar{x}) \geq \epsilon \). In particular, \( w \neq 0 \).

However, by the same argument as in the proof of Proposition 3.3, the sets \( A_n := \{ x \in B_2(0) \setminus B_{\frac{3}{2}}(0) : w_n(x) \neq 0 \} \) satisfy \( \lim_{n \to \infty} |A_n| = 0 \) and therefore \( w \equiv 0 \) a.e. in \( B_2(0) \setminus B_{\frac{3}{2}}(0) \). This yields a contradiction, and thus we conclude that \( g(r) \to 0 \) as \( r \to 0 \), as required.

Next, for \( r > 0 \) and \( q = 1 \), we consider the functional \( \varphi : W^{1,2}(B_1(0)) \to \mathbb{R} \) and the rescaling map \( T_r : W^{1,2}(B_1(0)) \to W^{1,2}(B_r(0)) \) defined in (21), (22), respectively. We then have the following.

**Proposition 5.5.** Let \( u \in W^{1,2}(\Omega) \) be a solution of (23) in \( \Omega \), and let \( x_0 \in \Omega \) be a point of density one for \( u^{-1}(0) \subset \Omega \). Moreover, let \( K \subset W^{2,2}_{0}(B_1(0)) \) be a compact set such that

\[
c_K := \sup_{v \in K} \varphi_1(v) < 0,
\]

Then there exists \( r_0 > 0 \) with the following property:

\( B_{r_0}(x_0) \) is contained in \( \Omega \), and for every \( r \in (0, r_0) \) and every \( v \in K \) we have \( \varphi(u + v) < \varphi(u) \), where \( v_r \in W^{2,2}_{0}(B_r(x_0)) \subset W^{2,2}(\Omega) \) is defined by

\[
v_r(x) = \begin{cases} T_r v(x-x_0) & x \in B_r(x_0) ; \\ 0 & x \in \Omega \setminus B_r(x_0). \end{cases}
\]

The proof is somewhat different than the proof of Proposition 3.3. Note that we need the stronger assumption \( K \subset W^{2,2}_{0}(B_1(0)) \) here. This assumption is not optimal but suffices for our purposes.

**Proof.** Without loss, we may assume that \( x_0 = 0 \in \Omega \). Let \( v \in K \). We then have

\[
\varphi(u + v_r) = \varphi(u) + \varphi(v_r) + \int_{B_r(0)} \nabla u \nabla v_r \, dx - \int_{B_r(0)} \left( |u + v_r| - |u| - |v_r| \right) dx
\]

(37)

\[
\leq \varphi(u) + \varphi(v) + \int_{B_r(0)} \nabla u \nabla v_r \, dx + 2 \int_{B_r(0)} |u| \, dx
\]

Note that, since \( v_r \in W^{2,2}_{0}(B_r(0)) \), we have

\[
\int_{B_r(0)} \nabla u \nabla v_r \, dx = \int_{B_r(0)} u \nabla v_r \, dx \leq r^N \int_{B_r(0)} u(x) |\nabla v_r(x)| \, dx
\]

\[
= o(r^{N+2}) |\nabla v_r|_{L^1(B_r(0))} = o(r^{N+2}) \quad \text{as } r \to 0.
\]

Moreover,

\[
\int_{B_r(0)} |u| \, dx = o(r^2) |\nabla v_r|_{L^1(B_r(0))} = o(r^{N+2}) \quad \text{as } r \to 0.
\]

Finally, since \( \nabla v_r(x) = r^2 \nabla v\left(\frac{x}{r}\right) \) for \( x \in B_r(0) \), we find that

\[
\varphi(v_r) = r^2 \left( \frac{1}{2} \int_{B_r(0)} \left| \nabla v\left(\frac{x}{r}\right) \right|^2 \, dx - \int_{B_r(0)} \left| v\left(\frac{x}{r}\right) \right|^2 \, dx \right) = r^{N+2} \varphi_1(v).
\]

Inserting these estimates in (37), we obtain

\[
\varphi(u + v_r) \leq \varphi(u) + r^{N+2} \left( \varphi_1(v) + o(1) \right) \leq \varphi(u) + r^{N+2} \left( c_K + o(1) \right)
\]
Since $K$ is compact, it is easy to see that these estimates are uniform in $v \in K$. Since moreover $c_K < 0$, the claim follows. 

**Theorem 5.6.** Let $u$ be a minimizer of $\varphi$ on $\mathcal{M}$. Then $u^{-1}(0)$ has vanishing Lebesgue measure.

**Proof.** Suppose by contradiction that $|u^{-1}(0)| > 0$. Then there exists a point $x_0$ of density one for the set $u^{-1}(0)$. Without loss of generality, we can suppose $x_0 = 0$. We fix two arbitrary nonnegative nontrivial functions $v_1, v_2 \in C^2_0(B_1(0))$ with disjoint support, and consider the path

$$\gamma : [0, 1] \to C^2_0(B_1(0)), \quad \gamma(s) = r^*((1-s)v_1 - sv_2) - ((1-s)v_1 - sv_2),$$

where the map $r^*$ is defined in (33). It is clear that $\varphi(\gamma(s)) < 0$ for $s \in [0, 1]$. We also define

$$\gamma := T_r \circ \gamma : [0, 1] \to C^2_0(B_r(0)).$$

Applying Proposition 5.5 to the compact set $K := \gamma([0, 1]) \subset W_{0,1}^{2,2}(B_1(0))$, we may fix $r > 0$ sufficiently small such that $\varphi(u + \gamma(s)) < \varphi(u)$ for every $s \in [0, 1]$. Moreover, by making $r$ smaller if necessary and using again the fact that $x_0 = 0$ is a point of density one for the set $u^{-1}(0) = 0$, we may assume that

$$|\{u = 0\} \cap \{\gamma(0) > 0\}| > 0 \quad \text{and} \quad |\{u = 0\} \cap \{\gamma(1) < 0\}| > 0.$$

As a consequence of these inequalities and the fact that $u$ is a solution of (29), we find that

$$\int_{\Omega} \text{sgn}(u + \gamma(0)) \, dx > \int_{\Omega} \text{sgn}(u) \, dx = 0 > \int_{\Omega} \text{sgn}(u + \gamma(1)) \, dx < 0.$$

By Lemma 5.1(iii), there exists $s_0 \in [0, 1]$ such that $u + \gamma(s_0) \in \mathcal{M}$. This however contradicts the assumption that $u$ is a minimizer of $\varphi$ in $\mathcal{M}$. \hfill \square

In the following, we restrict our attention to the case where $\Omega$ is a radial bounded domain in $\mathbb{R}^N$. In this case we also consider

$$\mathcal{M}_r := \{u \in \mathcal{M} : u \text{ radial}\} \quad \text{and} \quad m_r := \inf_{\mathcal{M}_r} \varphi,$$

so that $\mathcal{M}_r \subset \mathcal{M}$ and $m_r \geq m$.

**Theorem 5.7.** Let $\Omega \subset \mathbb{R}^N$ be a radial bounded domain. Then we have:

(i) If $u \in \mathcal{M}$ satisfies $\varphi(u) = m$, then $u$ is foliated Schwarz symmetric.

(ii) If $u \in \mathcal{M}_r$ satisfies $\varphi(u_r) = m_r$, then $u$ is strictly monotone in the radial variable.

(iii) If $\Omega = B$ is the unit ball in $\mathbb{R}^N$, we have

$$m \leq -\frac{\pi}{18} < m_r = -\pi \left( -\frac{1}{16} + \frac{1}{8} \ln 2 \right),$$

if $N = 2$ and, for $N \geq 3$,

$$m \leq -\omega_N \frac{N-2}{2N^2(N-1)} < m_r = -\frac{\omega_N (2^\frac{2}{N} - 1)N + 2^{1-\frac{2}{N}}}{2(N-2)(N+2)},$$

where, as usual, $\omega_N$ denotes the measure of $|B|$. Hence every minimizer $u \in \mathcal{M}$ of $\varphi|_{\mathcal{M}}$ is a nonradial function.

**Proof.** (i) The proof is very similar to the proof of Theorem 4.1 and we use the notation introduced there. Pick $x_0 \in \Omega \setminus \{0\}$ with $u(x_0) = \max\{u(x) : |x| = |x_0|\}$. We put $p := \frac{x_0}{|x_0|}$, and we let $H \in \mathcal{M}_p$. As in the proof of Theorem 4.1, it suffices to show that $u \equiv u_H$ on $H \cap \Omega$. Since

$$\int_{\Omega} |\nabla u_H|^2 \, dx = \int_{\Omega} |\nabla u|^2 \, dx, \quad \int_{\Omega} |u_H| \, dx = \int_{\Omega} |u| \, dx \quad \text{and} \quad \int_{\Omega} \text{sgn}_+(u_H) \, dx = \int_{\Omega} \text{sgn}_+(u) \, dx,$$
we find that \( u_{H} \in \mathcal{M} \) and \( \varphi(u_{H}) = \varphi(u) \), so that \( u_{H} \) is also a minimizer of \( \varphi \) on \( \mathcal{M} \). Hence, by Theorem \( \text{(11)} \) both \( u \) and \( u_{H} \) are solutions of \( (29) \). Therefore \( w := u_{H} - u \) is a nonnegative function in \( \Omega \cap H \) satisfying
\[
-\Delta w = \text{sgn}(u_{H}) - \text{sgn}(u) \geq 0 \quad \text{in } H \cap \Omega.
\]
The strong maximum principle then implies that either \( w \equiv 0 \) or \( w > 0 \) in \( H \cap \Omega \). The latter case is ruled out since \( x_{0} \in H \cap \Omega \) and \( w(x_{0}) = u_{H}(x_{0}) - u(x_{0}) = 0 \) by the choice of \( x_{0} \). We therefore obtain \( w \equiv 0 \) and hence \( u \equiv u_{H} \) on \( H \cap \Omega \), as required.

(ii) We only consider the case where \( \Omega = B \) is the unit ball in \( \mathbb{R}^{N} \); the proof in the case of an annulus is similar. Let \( u \in \mathcal{M} \) satisfy \( \varphi(u_{r}) = m_{r} \). Then, by similar arguments as in the proof of Theorem \( \text{(2.2)} \) \( u \) is a radial solution of \( (29) \). Suppose by contradiction that there exists \( r_{0} \in (0, 1) \) such that \( u'(r_{0}) = 0 \). We claim that we can choose \( r_{0} \) minimally, i.e., such that
\[
(40) \quad u'(r) \neq 0 \quad \text{for } r \in (0, r_{0}).
\]
Indeed, suppose by contradiction that there exists a sequence of \( r_{n} \in (0, 1) \), \( n \in \mathbb{N} \) such that \( u'(r_{n}) = 0 \) for all \( n \) and \( r_{n} \to 0 \) as \( n \to \infty \). Without loss, we may assume that \( r_{n} > r_{n+1} \) for every \( n \). Since the function \( r \mapsto r^{N-1}u'(r) \) is strictly monotone on every interval on which \( u \) has no zero, we conclude that there exists \( s_{n} \in (r_{n+1}, r_{n}) \) such that \( u(s_{n}) = 0 \). Since \( u \) is of class \( \mathcal{C} \), we therefore conclude that \( u(0) = u'(0) = 0 \). This however implies that \( u \equiv 0 \), since the absolutely continuous function \( r \mapsto u(r) = \frac{u'(r)^{2}}{2} + |u(r)| \) is decreasing on \([0, 1]\), which follows from the fact that
\[
(41) \quad b'(r) = u'(r)[u''(r) + \text{sgn}(u(r))] = -\frac{(N-1)}{r}u'(r)^{2} \leq 0 \quad \text{for a.e. } r \in (0, 1).
\]
We thus conclude that we can choose \( r_{0} \in (0, 1) \) such that \( (40) \) holds. It is then easy to see (e.g. by using the Hopf boundary lemma) that \( u(r_{0}) \neq 0 \). Let \( \Omega_{1} := B_{r_{0}}(0) \) and \( \Omega_{2} := B \setminus B_{r_{0}}(0) \). Then \( u \) solves \( (29) \) both on \( \Omega_{1} \) and \( \Omega_{2} \), so that
\[
\int_{\Omega_{1}} \text{sgn}(u) \, dx = \int_{\Omega_{2}} \text{sgn}(u) \, dx.
\]
We now define
\[
v \in W^{1, 2}(B), \quad v(x) = \begin{cases} 2u(r_{0}) - u(x), & x \in \Omega_{1}; \\ u(x), & x \in \Omega_{2}.
\end{cases}
\]
Moreover, we let \( c = c(v) \in \mathbb{R} \) be given by Lemma \( \text{(5.1)} \) \( (v) \), so that \( w := v + c \in \mathcal{M} \). We then have
\[
(42) \quad \int_{B} |\nabla v|^{2} \, dx = \int_{B} |\nabla u|^{2} \, dx.
\]
Moreover,
\[
(43) \quad \int_{\Omega_{1}} |w| \, dx = \int_{\Omega_{1}} |u - 2u(r_{0}) - c| \, dx \geq \int_{\Omega_{1}} |u| \, dx
\]
by Lemma \( \text{(5.1)} \) \( (i) \) applied to the domain \( \Omega_{1} \). Similarly,
\[
(44) \quad \int_{\Omega_{2}} |w| \, dx = \int_{\Omega_{2}} |u + c| \, dx \geq \int_{\Omega_{2}} |u| \, dx
\]
by Lemma \( \text{(5.1)} \) \( (i) \) applied to the domain \( \Omega_{2} \). Moreover, if equality holds in both \( (43) \) and \( (44) \), then Lemma \( \text{(5.1)} \) \( (i) \) implies that \( 2u(r_{0}) + c = 0 \) and \( c = 0 \), hence \( u(r_{0}) = 0 \) contrary to what we have seen earlier. Hence at least one of the inequalities \( (43) \) and \( (44) \) is strict, so that
\[
(45) \quad \int_{\Omega} |w| \, dx > \int_{\Omega} |u| \, dx
\]
and therefore \( \varphi(w) < \varphi(u) = m_{r} \), as a consequence of \( (42) \) and \( (45) \). This contradicts the definition of \( m_{r} \), and thus the proof is finished.

(iii) Similar arguments as in the proof of Theorem \( \text{(2.2)} \) show that \( m_{r} \) is attained by a radial solution \( u \in \mathcal{M} \),
We now have
\[ \int \] thus showing the right inequality in (38). For
\[ \text{and therefore} \]
\[ \text{of (29). By (ii), we know that } u \text{ is strictly monotone in the radial variable, and therefore it is up to sign uniquely given by} \]
\[
\begin{align*}
u(r) &= \begin{cases} 
-\frac{1}{8}r^2 + \frac{1}{8} & \text{for } 0 \leq r < a := \frac{1}{\sqrt{2}} \\
-\frac{1}{2}\ln r + \frac{1}{4}r^2 - \frac{1}{8} - \frac{1}{4}\ln 2 & \text{for } a \leq r \leq 1 
\end{cases}
\end{align*}
\]
for \( N = 2 \), and by
\[
\begin{align*}
u(r) &= \begin{cases} 
\frac{1}{2N}(a^2 - r^2) & \text{for } 0 \leq r < a := 2^{-\frac{1}{N}} \\
\frac{1}{2N} \left( \frac{2}{N-2}r^2 - \frac{2+2^N(N+2)}{N-2} \right) & \text{for } a \leq r \leq 1 
\end{cases}
\end{align*}
\]
for \( N \geq 3 \). Note here that the value \( a \) is determined by the condition that \( |\{u > 0\}| = |\{u < 0\}| \). For
\[ \text{and therefore} \]
\[ \text{which is the right equality in (39). To see the left inequalities in (38) and (39), we consider the functions} \]
\[ x \mapsto \nu_s(x) = x_1|x|^s \text{ for } s > -\frac{2}{N}. \]
We then have
\[ \int_B |\nabla u|^2 = \int_B |u|^2 = 2\pi \left( -\frac{1}{16} + \frac{1}{8}\ln 2 \right) \]
and therefore
\[ m_r = \varphi(u) = -\pi \left( -\frac{1}{16} + \frac{1}{8}\ln 2 \right), \]
thus showing the right inequality in (38). For \( N \geq 3 \),
\[
\int_B |\nabla u|^2 = \int_B |u|^2 = \frac{\omega_N}{2} \left( \frac{1}{N-2}N^2 - \frac{1}{N+2} - \frac{1}{N-2} \right) = \frac{\omega_N(2^{-\frac{2}{N}}-1)N^{\frac{2}{N}}+2^{-\frac{1}{N}}}{(N-2)(N+2)}
\]
and therefore
\[ m_r = \varphi(u) = -\frac{\omega_N(2^{-\frac{2}{N}}-1)N+2^{-\frac{1}{N}}}{2(N-2)(N+2)}, \]
which is the right equality in (39). To see the left inequalities in (38) and (39), we consider the functions
\[ x \mapsto \nu_s(x) = x_1|x|^s \text{ for } s > -\frac{2}{N}. \]
We now have \( \nu_s \in \mathcal{M} \) for the function \( \nu_s = c_s \nu_s \) with \( c_s := \frac{\int_B |u|^s dx}{\int_B |\nabla u|^s dx} \) and therefore
\[ m \leq \varphi(\nu_s) = -\frac{1}{2} \left( \frac{\int_B |u|^s dx}{\int_B |\nabla u|^s dx} \right)^2 \leq -2\omega_N \frac{N+2s}{2(N+s^2+2s)(N+s+1)^2}. \]
Thus the left inequalities in (38) and (39) follow by choosing \( s = 0 \) if \( N = 2 \), and \( s = -1 \) if \( N \geq 3 \). Finally, for the middle inequality in (39) we need to prove:
\[
\frac{(2^{-\frac{2}{N}}-1)N^{\frac{2}{N}}+2^{-\frac{1}{N}}}{(N-2)(N+2)} \leq \frac{N-2}{N^2(N-1)}
\]
for \( N \geq 3 \), which is equivalent to
\[
(2^{-\frac{2}{N}}-1)N^4 + 2^{-\frac{2}{N}}N^3 + 2(1-2^{-\frac{2}{N}})N^2 + 4N - 8 < 0
\]
\[ \Leftrightarrow N^3 \left( 2^{-\frac{2}{N}}-1 \right)N + 2^{-\frac{1}{N}}N \left( 1 - 2^{-\frac{2}{N}} \right) + 4N - 8 < 0 \]
To this aim, we will prove that the function \( h : [3, +\infty) \to \mathbb{R} \) defined by

\[
h(t) := h_1(t) + h_2(t) + h_3(t) = (2 - \frac{2}{t} - 1)t + 2 - \frac{2}{t} + \frac{2}{t}(1 - \frac{2}{t})
\]

is maximized for \( t = 3 \), and that \( h(3) < -1 \). Inequality (16) will be implied by \(-N^3 + 4N - 8 < 0\), which is true for \( N \geq 3 \). The function \( h_1 \) is such that

\[
h'_1(t) = \frac{4t^{-\frac{3}{2}} - 4 \frac{1}{t} \ln(t) + \ln(4)}{t}, \quad h''_1(t) = \frac{4t^{-\frac{5}{2}} (\ln(2))^2}{t^3}.
\]

Since \( h'_1(t) \to 0 \) as \( t \to +\infty \) and \( h''_1(t) > 0 \) for \( t > 0 \), \( h_1 \) is monotone decreasing. \( h_2 \) is clearly strictly decreasing, while \( h_3 \) satisfies

\[
h'_3(t) = -\frac{2t^{-\frac{3}{2}} (4 \frac{1}{t} - 1) \ln(4)}{t^3} < 0
\]

for \( t > 0 \). It is then easily verified that \( h(3) = \frac{1}{12}(5 \sqrt{2} - 7) < 1 \). \( \square \)

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