An exit-time approach to $\epsilon$-entropy

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An efficient approach to the calculation of the $\epsilon$-entropy is proposed. The method is based on the idea of looking at the information content of a string of data, by analyzing the signal only at the instants when the fluctuations are larger than a certain threshold $\epsilon$, i.e., by looking at the exit-time statistics. The practical and theoretical advantages of our method with respect to the usual one are shown by the examples of a deterministic map and a self-affine stochastic process.

PACS: 05.45.-a, 05.45.Tp

The problem of quantifying the degree of complexity of an evolving system is ubiquitous in natural science (for a nice review see [1]). The typical questions may range from the aim to distinguish between stochastic or chaotic systems to the more pragmatic goal of determining the degree of complexity (read predictability) at varying the resolution in phase-space and time [2,3]. From the pioneering works of Shannon [4] and Kolmogorov [5], a proper mathematical tool, the Kolmogorov-Sinai (KS) entropy, has been developed to address the above question quantitatively and unambiguously (in principle). The main idea is very natural: we must look at the information contained in the time evolution as a probe of the underlying dynamics. This is realized by studying the symbolic dynamics obtained by assigning different symbols to different cells of a finite partition of the phase space. The probability distribution of allowed sequences (words) is determined by the dynamical evolution. The average information-gain is defined by comparing sequences of length $m$ and $m+1$, in the limit of large $m$. Letting the length of the words, $m$, to infinity and going to infinitely refined partition, one obtains the KS-entropy, which is a measure of the degree of complexity of the trajectory. The KS-entropy determines also the rate of information transmission necessary to unambiguously reconstruct the signal. Unfortunately, only in simple, low dimensional, dynamical systems such a procedure can be properly carried out with conventional methods [2,3,6]. The reason is that for high dimensional systems the computational resources are not sufficient to cope with the very high resolution and extremely long time series required. Moreover, in many systems, like in turbulence, the existence of non-trivial fluctuations on different time and spatial scales cannot be captured by the KS-entropy. This calls for a more general tool to quantify the degree of predictability which depends on the analyzed range of scales and frequencies. This was the aim leading Shannon [4] and Kolmogorov [5] to introduce the so-called $\epsilon$-entropy, later generalized to the $(\epsilon, \tau)$-entropy [7,8]. Conceptually it corresponds to the rate of information transmission necessary to reconstruct a signal with a finite accuracy $\epsilon$, and with a sampling time interval $\tau$. The naive $(\epsilon, \tau)$-computation is usually performed by looking at the Shannon entropy of the coarse grained dynamics on an $(\epsilon, \tau)$-grid in the phase-space and time. This method suffers of so many computational drawbacks that it is almost useless for many realistic time-series [9]. Another attempt in this direction is the introduction of the Finite Size Lyapunov Exponent [9].

The aim of this letter is to introduce an alternative approach for the determination of the $(\epsilon, \tau)$-entropy, based on the analysis of exit-times. In a few words, the idea consists in looking at the information-content of a string of data, without analyzing the signal at any fixed time, $\tau$, but only when the fluctuations are larger than some fixed threshold, $\epsilon$. This simple observation allows for a remarkable improvement of the computational possibility to measure the $(\epsilon, \tau)$-entropy as will be discussed in detail later.

We believe that the approach presented hereafter is unavoidable in all those cases when either the high-dimensionality of the underlying phase space, or the necessity to disentangle non-trivial correlations at different analyzed time scales, leads to the failure of the standard methods.

Let us just briefly recall the conventional way to calculate the $(\epsilon, \tau)$-entropy for the case of a time-continuous signal $x(t) \in \mathbb{R}$, recorded during a (long) time interval $T$. One defines an $\epsilon$-grid on the phase-space and a $\tau$-grid on time. If the motion is bounded, the trajectory visits only a finite number of cells; therefore to each subsequence of length $n \cdot \tau$ from $x(t)$ one can associate a word of length $n$, out of a finite alphabet: $W^n(\epsilon, \tau) = \{S_t, S_{t+\tau}, \ldots, S_{t+(n-1)\tau}\}$, where $S_t$ labels the cell containing $x(t)$. From the probability distribution of the above words one calculates the block entropies $H_n(\epsilon, \tau)$:

$$H_n(\epsilon, \tau) = - \sum_{\{W^n(\epsilon, \tau)\}} P(W^n(\epsilon, \tau)) \ln P(W^n(\epsilon, \tau)). \quad (1)$$

the $(\epsilon, \tau)$-entropy per unit time, $h(\epsilon, \tau)$ is finally defined
as:
\[
h_n(\epsilon, \tau) = \frac{1}{\tau} [H_{n+1}(\epsilon, \tau) - H_n(\epsilon, \tau)],
\]
(2)
\[
h(\epsilon, \tau) = \lim_{n \to \infty} h_n(\epsilon, \tau) = \frac{1}{\tau} \lim_{n \to \infty} \frac{1}{n} H_n(\epsilon, \tau),
\]
(3)
where for practical reasons the dependence on the details of the partition is ignored, while the rigorous definition is given in terms of the infimum over all possible partitions with elements of diameter smaller than \(\epsilon\). Notice that the above defined \((\epsilon, \tau)\)-entropy is nothing but the Shannon-entropy of the sequence of symbols \((S_t, S_{t+\tau}, \ldots)\) associated with the given signal. The Kolmogorov-Sinai entropy, \(h_{KS}\), is obtained by taking the limit \((\epsilon, \tau) \to 0\):
\[
h_{KS} = \lim_{\tau \to 0} \lim_{\epsilon \to 0} h(\epsilon, \tau).
\]
(4)
In the case of discrete-time systems, one can define \(h(\epsilon) \equiv h(\epsilon, \tau = 1)\), and \(h_{KS} = \lim_{\tau \to 0} h(\epsilon)\). In continuous-time evolutions, whose realizations are continuous functions of time, the \(\tau\) dependence disappears from \(h(\epsilon, \tau)\), so one can still define an \(\epsilon\)-entropy per unit time \(h(\epsilon)\). In particular, also in a pure deterministic flow one can put \(h(\epsilon) = h(\epsilon, \tau = 1)\).

Let us remind that for a genuine deterministic chaotic system one has \(0 < h_{KS} < \infty\) (\(h_{KS} = 0\) for a regular motion), while for a continuous random process \(h_{KS} = \infty\). Therefore, in order to distinguish between a purely deterministic system and a stochastic system it is necessary to perform the limit \(\epsilon \to 0\) in (2). Obviously, from a physical and/or numerical point of view this is impossible. Nevertheless, by looking at the behavior of the \((\epsilon, \tau)\)-entropy at varying \(\epsilon\) one can have some qualitative and quantitative insights on the chaotic or stochastic nature of the process. For most of the usual stochastic processes one can explicitly give an estimate of the entropy scaling behavior when \(\epsilon \to 0\) [2]. For instance, in the case of a stationary Gaussian process one has [2]
\[
h(\epsilon) \equiv \lim_{\tau \to 0} h(\epsilon, \tau) \sim \frac{1}{\epsilon^2}.
\]
(5)
Let us now introduce the main point of this letter by discussing in detail the difficulties that may arise in measuring the \(\epsilon\)-entropy for the following non-trivial example of a chaotic-diffusive map [10],
\[
x_{t+1} = x_t + p \sin 2\pi x_t.
\]
(6)
When \(p > 0.7326\ldots\), this map produces a diffusive behavior on large scales, so one expects
\[
h(\epsilon) \simeq \lambda\text{ for } \epsilon < 1;\quad h(\epsilon) \propto \frac{D}{\epsilon^2}\text{ for } \epsilon > 1,
\]
(7)
where \(\lambda\) is the Lyapunov exponent and \(D\) is the diffusion coefficient. The numerical computation of \(h(\epsilon)\), using the standard codification, is highly non-trivial already in this simple system. This can be seen by looking at Fig. 1 where the behavior [3] in the diffusive region is roughly obtained by considering the envelope of \(h_n(\epsilon, \tau)\) evaluated for different values of \(\tau\); while looking at any single (small) value of \(\tau\) (one would like to put \(\tau = 1\)) one obtains a rather inconclusive result. This is due to the fact that one has to consider very large block lengths \(n\) when computing \(h(\epsilon, \tau)\), in order to obtain a good convergence for \(h_n(\epsilon, \tau) - h_{n-1}(\epsilon, \tau)\) in (3).

Indeed, in the diffusive regime, a simple dimensional argument shows that the characteristic time of the system is \(T_\epsilon \approx \epsilon^2 / D\). If we consider for example, \(\epsilon = 10\) and typical values of the diffusion coefficient \(D \approx 10^{-1}\), the characteristic time, \(T_\epsilon\), is much larger than the elementary sampling time \(\tau = 1\).

Our approach to calculate \(h(\epsilon)\) differs from the usual one in the procedure to construct the coding sequence of the signal at a given level of accuracy. Specifically, we use a different way to sample the time, \(t\), instead of using a constant time interval, \(\tau\), we sample according to the exit-time, \(t(\epsilon)\), on an alternating grid of cell size \(\epsilon\). We consider the original continuous-time record \(x(t)\) and a reference starting time \(t = t_0\); the subsequent exit-time, \(t_1\), is then defined as the first time necessary to have an absolute variation equal to \(\epsilon/2\) in \(x(t)\), i.e., \(\left| x(t_0 + t_1) - x(t_0) \right| \geq \epsilon/2\). This is the time the signal takes to exit the cell of size \(\epsilon\). Then we restart from \(t_1\) to look for the next exit-time \(t_2\), i.e., the first time such that \(\left| x(t_0 + t_1 + t_2) - x(t_0 + t_1) \right| \geq \epsilon/2\) and so on. Let us notice that, with this definition, the coarse-graining grid is not fixed, but it is always centered in the last exit position.
In this way we obtain a sequence of exit-times, \{t_i(\epsilon)\}. To distinguish the direction of the exit (up or down out of a cell), we introduce the label \(k_i = \pm 1\), depending whether the signal is exiting above or below. Doing so, the trajectory is univocally coded with the required accuracy, by the sequence \((t_1, k_1), (t_2, k_2), \ldots, (t_M, k_M)\), where \(M\) is the total number of exit-time events observed during the total time \(T\). Correspondingly, an “exit-time word” of length \(n\) is a sequence of couples of symbols \(\Omega_n^\epsilon(\epsilon) = ((t_1, k_1), (t_2, k_2), \ldots, (t_{n+1}, k_{n+1}))\). From these words one firstly calculates the block entropies, \(H_n^\epsilon(\epsilon)\), and then the exit-time \(\epsilon\)-entropies: \(h^\epsilon(\epsilon) \equiv \lim_{n \to \infty} H_n^\epsilon(\epsilon) - H_n^\epsilon(\epsilon)\). Let us notice that \(h^\epsilon(\epsilon)\) is an \(\epsilon\)-entropy per exit and that \(M = T/\langle t(\epsilon)\rangle\).

The exit-time coding is a faithful reconstruction with accuracy \(\epsilon\) of the original signal. Therefore, the total entropy, \(Mh^\epsilon(\epsilon)\), of the exit-time sequence, \(\Omega^\epsilon(\epsilon)\), is equal to the total entropy, \(Th(\epsilon) = N\tau h(\epsilon)\), of the standard codification sequence, \(W^N(\epsilon)\). Namely, for the \(\epsilon\)-entropy per unit time, we obtain:

\[
h(\epsilon) = Mh^\epsilon(\epsilon)/T = \frac{h^\epsilon(\epsilon)}{\langle t(\epsilon)\rangle}. \tag{8}
\]

Now we are left with the determination of \(h^\epsilon(\epsilon)\). This implies a discretization, \(\tau_\epsilon\), of the exit-times. The exit-time entropy \(h^\epsilon(\epsilon)\) becomes an exit-time \((\epsilon, \tau_\epsilon)\)-entropy, \(h^\epsilon(\epsilon, \tau_\epsilon)\), obtained from the sequence \(\{\eta_i, k_i\}\), where \(\eta_i\) identifies the exit-time cell containing \(t_i\). Equation (8) becomes now

\[
h(\epsilon, \tau_\epsilon) = \lim_{\tau_\epsilon \to 0} h^\epsilon(\epsilon, \tau_\epsilon)/\langle t(\epsilon)\rangle \simeq h^\epsilon(\epsilon, \tau_\epsilon)/\langle t(\epsilon)\rangle, \tag{9}
\]

the latter relation being valid for small enough \(\tau_\epsilon\). However, in all practical situations there exist a minimum \(\tau_\epsilon\) given by the highest acquisition frequency, i.e., the limit \(\tau_\epsilon \to 0\) cannot be reached. The discretization interval \(\tau_\epsilon\) can be thought of as the equivalent to the \(\tau\) entering in the usual \((\epsilon, \tau)\)-entropy, so that

\[
h(\epsilon, \tau_\epsilon) = h^\epsilon(\epsilon, \tau_\epsilon)/\langle t(\epsilon)\rangle. \tag{10}
\]

At this point it is important to stress that in most of the cases the leading \(\epsilon\) contribution to \(h(\epsilon)\) in (8) is given by the mean exit-time \(\langle t(\epsilon)\rangle\) and not by \(h^\epsilon(\epsilon, \tau_\epsilon)\). Anyhow, the computation of \(h^\epsilon(\epsilon, \tau_\epsilon)\) is compulsory in order to recover a zero entropy for regular (e.g. periodic) signals. It is easy to obtain the following bounds for \(h^\Omega(\epsilon, \tau_\epsilon) = h^\Omega(\{\eta_i, k_i\})\):

\[
h^\Omega(\{k_i\}) \leq h^\Omega(\{\eta_i, k_i\}) \leq h^\Omega(\{\eta_i\}) + h^\Omega(\{k_i\}) \leq h^\Omega(\{k_i\}) + H_1^\epsilon(\{\eta_i\})
\]

where \(h^\Omega(\{k_i\})\) is the Shannon entropy of the sequence \(\{k_i\}\) and \(H_1^\epsilon(\{\eta_i\})\) is the one-symbol entropy of the \(\{\eta_i\}\).

Therefore we have

\[
\frac{h^\Omega(\{k_i\})}{\langle t(\epsilon)\rangle} \leq h(\epsilon) \leq \frac{h^\Omega(\{k_i\}) + c(\epsilon) + \ln(\langle t(\epsilon)\rangle/\tau_\epsilon)}{\langle t(\epsilon)\rangle}, \tag{11}
\]

where \(c(\epsilon) = -\int p(z) \ln p(z) dz\), and \(p(z)\) is the probability distribution function of the rescaled exit-time \(z(\epsilon) = t(\epsilon)/\langle t(\epsilon)\rangle\).

We shall see in the following that the above bounds are rather good, and typically \(\langle t(\epsilon)\rangle\) shows the same scaling behavior as \(h(\epsilon)\). One could wonder why the exit-time approach is not the usual one. The reason is simple (and somehow deep): in the exit-time approach it is not necessary to use a very large block size since, at fixed \(\epsilon\), the typical time at that scale is automatically given by \(\langle t(\epsilon)\rangle\). This fact is particularly clear in the case of Brownian motion. In such a case \(\langle t(\epsilon)\rangle \propto \epsilon^2/D\), where \(D\) is the diffusion coefficient. As previously discussed, the computation of the \(h(\epsilon)\) with the standard methods implies the use of very large block sizes, of order \(\epsilon^2/D\).

![FIG. 2. Numerically computed lower bound (□) and upper bound (with \(\tau_\epsilon = 1\) (○) of \(h(\epsilon)\) (¶) for the same map of Fig. 1. The two straight (solid) lines correspond to the asymptotic behaviors as in Fig. 1. We also present the \((\epsilon, \tau_\epsilon)\)-entropy \(h^\Omega(\epsilon, \tau_\epsilon)/\langle t(\epsilon)\rangle\) with \(\tau_\epsilon = 0.1\langle t(\epsilon)\rangle\) (×).](image-url)
Let us now briefly comment the limit $\epsilon \to 0$ for a discrete-time system (e.g. maps). In this limit the exit-time approach coincides with the usual one: we have just to observe that practically the exit-times always coincide with the minimum sampling time and to consider the possibility to have jumps over more than one cell, i.e., the $k_i$ symbols may take values $\pm 1, \pm 2, \ldots$.

As another example, we present the calculation of $\epsilon$-entropy for a self-affine stochastic signal with Hölder-exponent $\xi = 1/3$, i.e., $|x(t) - x(t + \Delta t)| \sim (\Delta t)^{1/3}$. Such a signal can be seen as a stochastic surrogate of a turbulent signal (ignoring intermittency) and can be constructed in different ways (see Ref. [11] and references therein). A simple dimensional estimate, which is rigorous for Gaussian processes, tells us that the leading contribution to the $\epsilon$-entropy scaling is given by $h(\epsilon) \sim \epsilon^{-3}$. To generate the self-affine signal we use a recently proposed algorithm [11], where $x(t)$ is obtained by using many Langevin processes. In Fig. 3 we show the bounds for the $(\epsilon, \tau)$-entropy calculated via the exit-time approach. We observe an extended region of well-defined scaling, which is the same for $1/(t(\epsilon)) \sim \epsilon^{-3}$. The usual approach (not shown) gives a poor estimate for the scaling as the envelope of $h(\epsilon, \tau)$ computed for various $\tau$ (see for example Figs. 15-18 in [11]), as in the case of Fig. 1.

In conclusion, we have introduced an efficient method to calculate the $\epsilon$-entropy based on the analysis of the exit-time statistics. It is able to disentangle in a more proper way the leading contributions to $h(\epsilon)$ at the scale $\epsilon$, compared with the standard way based on a coarse-grained dynamics on a fixed $(\epsilon, \tau)$ grid. We have presented the application to two examples, a chaotic diffusive map and a stochastic self-affine signal. More applications to chaotic systems, stochastic multi-affine processes, and experimental turbulent signals will be reported elsewhere.

The entropy-approach to evaluate the degree of complexity of a time series allows also to attack much more sophisticated problems often encountered in dynamical system theory. We just mention, e.g., the problem to disentangle correlations between time series obtained by measuring different observable of the same system. This might be addressed by using the conditional-entropy.

We acknowledge useful discussions with G. Boffetta and A. Celani. This work has been partially supported by INFM (PRA-TURBO) and by the European Network Intermittency in Turbulent Systems (contract number FMRX-CT98-0175). M.A. is supported by the European Network Intermittency in Turbulent Systems.

FIG. 3. Numerically computed lower bound (□) and upper bound ( ( for the $(\epsilon, \tau)$-entropy in the case of a self-affine signal with $\xi = 1/3$ evaluated by using the exit-time approach. The two straight lines show the scaling $\epsilon^{-3}$.