Slow holes in the triangular Ising antiferromagnet

R. Moessner and S. L. Sondhi

Department of Physics, Jadwin Hall, Princeton University, Princeton, NJ 08544, USA
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We consider the problem of the doped Ising antiferromagnet on the frustrated triangular lattice in the limit where the hole kinetic energy is much smaller than the Ising exchange. For a single hole we prove a “frustrated Nagaoka theorem” showing that the ground state is magnetized and breaks translational symmetry, in contrast to the parent insulating state that is unmagnetized and spatially homogenous. The extension of this physics to finite dopings depends on the strength of a density-density coupling that is inevitably present—we find either phase separation of the holes, or a superconducting state that is also magnetized and breaks translational symmetry in a feat of spatial self-organization. Finally, we derive an effective interaction between dilute holes at temperatures in excess of the hopping and find an oscillatory, long-ranged form reflective of the correlations in the underlying classical magnet which presages the breaking of translational symmetry at zero temperature.

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I. INTRODUCTION

The problem of mobile holes in a magnetic background is an especially interesting example of charge motion in a strongly correlated electron system. In the cases that we have in mind, the magnetism involves local moments which reflect an underlying strong electron-electron interaction that can produce new and interesting effects when doping creates charge carriers. Much of the recent work on this problem has been inspired by the cuprate superconductors, whose physics is widely believed to be centrally connected to their genesis as doped Mott insulators. In this context, the central observation is that hole motion in an ordered antiferromagnetic background is frustrated, leading to the expectation that doping will result in a rearrangement of the magnetic background in a manner conducive to relieving this frustration. Suggestions for what this entails include RVB (resonating valence bond) theory stripe formation and, in models with purely short-ranged interactions, phase separation. A second theme in this setting is the large spin degeneracy of the extreme Mott insulator (e.g. the $U = \infty$ Hubbard model) at half filling and the role of doping in lifting it. Perhaps the most celebrated result along these lines is the Nagaoka theorem, which established that a single hole would completely polarize the spin background. While the situation at finite dopings remains unsettled, the Nagaoka result does show that doping can lift the degeneracy in striking ways.

In this paper we report some results on dilute holes introduced into a frustrated magnetic system—the particular system studied is the canonical example of this class, the triangular lattice Ising antiferromagnet first studied by Wannier and Houppeland. This system realises aspects of both themes touched on above. It has local antiferromagnetic order which leads to frustration of the hole motion. In addition however, as this is an Ising system, the magnetic frustration leads to a finite zero-point entropy per site, $S$. This feature is reminiscent of the Mott insulator cited above, but we should note that in this case the degeneracy is generated as a cooperative effect: the Ising model on the triangular lattice has a large number of ground states, $N_{GS}$, because of the geometrically frustrated nature of its magnetic interactions. In each triangle, at least one pair of spins has to be aligned, and hence one bond frustrated. Any state with exactly one bond per triangle is hence a ground state, and $N_{GS}$ is found to scale exponentially with the number of spins, $N$, with $S = k_B \ln(N_{GS})/N \approx 0.323 k_B N$. In the following, we present a set of results on the question of if, and how, this degeneracy is lifted upon dilute doping. To make progress, we will assume the hole motion is slow—their kinetic energy being taken to be much smaller than the magnetic exchange. While Ising magnets exist, slow holes might be harder to find. Our interest in this limit then is that a) it poses the question of how doping interacts with a frustrated magnetic background most cleanly, which is of theoretical interest, and b) that the ordering patterns we find in this limit could well persist when more enterprising holes are considered. While more work is needed to investigate the validity of (b) we should note concerning (a) that the response of highly frustrated magnets to perturbations is more generally interesting. On account of the large degeneracy, these systems are unstable in a large number of directions, promising a wide range of unexpected and unusual physical phenomena. Along these lines, in a recent study along with P. Chandra, we have explored the phase structure produced by switching on the quantum dynamics of a transverse magnetic field instead.

We now turn to the Hamiltonian, $H$, we study. It is

\[
H = H_1 + H_J + H_\eta
= -t \sum_{(ij),\sigma} P \left( c_{i\sigma}^+ c_{j\sigma} + c_{j\sigma}^+ c_{i\sigma} \right) + J^z \sum_{(ij)} S_i^z S_j^z
- \eta J^z \sum_{(ij)} n_i n_j,
\]

(1)
where $J^z > 0$ is the strength of the antiferromagnetic Ising exchange, $t$ is the hopping integral, and $c_{i\sigma} (c^\dagger_{i\sigma})$ annihilates (creates) an electron at site $i$ with spin $\sigma$. $S_i^z = \frac{1}{2}(c^\dagger_{i\uparrow}c_{i\uparrow} - c^\dagger_{i\downarrow}c_{i\downarrow})$ is the $z$-component of the spin-1/2 operator at site $i$, $n_i = \sum_{\sigma} c^\dagger_{i\sigma}c_{i\sigma}$ is the total density operator and $P$ is a projector that ensures that all sites are at most singly occupied. Finally, the sum $(ij)$ runs over nearest neighbor bonds and the sum $\sigma$ over up and down spin orientations. As noted earlier, we study the range of parameters $t \ll J^z$, where the hopping acts as a perturbation to the exchange coupling. We consider strengths of the nearest neighbour density-density coupling of $0 \leq \eta \leq 1$. In the derivation of the rotationally invariant $t - J$ model for $S = 1/2$ from the Hubbard model, such a coupling is generated with $\eta = 1$. For Ising systems, possibly with spins greater than 1/2, there is no such precise fixing of its magnitude and for that reason, as well as for more general theoretical interest, we consider a range of values for it.

In the rest of the paper, we first prove a “frustrated Nagaoka theorem”, namely that the introduction of a single hole into the ground state manifold causes the system to order into a state that is spin polarized and breaks translational symmetry with a $\sqrt{3} \times \sqrt{3}$ unit cell. In this state, the frustrated bonds form a hexagonal (honeycomb) lattice on which the hole hops. Next we consider generalizing this physics to a finite but dilute density of holes, which turns out to depend strongly on $\eta$. For $\eta > 1/3$, phase separation obtains. The case $\eta < 1/3$ is much more interesting. Here, we find a spinless Fermi liquid state that lives on the “hexagonal backbone” of the single hole problem. Further, the surrounding sites mediate an attractive interaction that implies that the Fermi liquid is unstable to a superconducting state in a non-zero odd angular momentum channel. The resulting state then breaks three symmetries of the parent insulating state at once: Ising, translation and the charge $U(1)$. Finally, we turn to non-zero temperatures $J^z \gg T \gg t$, and show using Grassman functional integral techniques, that there is an algebraically decaying, entropic interaction between two holes which is oscillatory as a function of distance and which indicates a tendency for the holes to segregate on the hexagonal backbone.

II. A FRUSTRATED NAGAOKA THEOREM

We show in this section that the ground state of our Hamiltonian (Eq. 1) with a single hole present is macroscopically ordered. We will begin by establishing this in the limit where we take $J^z = \infty$, i.e. in the degenerate perturbation theory problem among a set of single hole states that we identify below. We will comment on why the conclusion is unaltered when $J^z \gg t$ but not infinite, at the end of the proof of this first part.

The proof will consist of first establishing a lower bound for the energy of a one-hole state on arbitrary finite lattices and then demonstrating that this can be saturated only on a subset of finite lattices and that on the latter there is, up to global symmetry operations, just one state that does so. For specificity, we will assume that our lattices come with periodic boundary conditions.

We first identify the one-hole states that minimise the exchange energy, noting that the term $H_e$ in the Hamiltonian (Eq. 2) is the same for any one-hole state. Starting from any ground state, we can only remove an electron which experiences zero net exchange field, i.e., which has three frustrated and three satisfied bonds. There are exponentially many such one-hole states.

The hopping Hamiltonian connects those one-hole states which only differ by the exchange of a hole and a neighbouring spin. Fig. 1a shows a spin configuration which allows the hole to hop to three of its neighbouring sites. Up to symmetry operations, this is the only such (local) spin configuration. Since the hopping integral is $t$, this means that the energy which can be gained from $H_t$ is at most $3t$. We note that this bound is quite general and holds for all finite lattices with arbitrary boundary conditions. Indeed, starting with any given one-hole state, we can think of the all the states reached by hops as inducing a graph, whose topology is in general far more complicated than that of the parent lattice. The statement is that regardless of the complexity, the graph has a maximum local coordination number of 3 and that the nearest neighbour hopping problem on it has at best an energy of $-3t$.

This energy gain can only be realised if any state reached after one hop again permits hopping in three directions. Demanding that this be the case completely constrains the parent spin configuration to the one depicted in Fig. 1b; the other five states that satisfy this condition are obtained using translations or Ising reversal. In this configuration, the hole can occupy any site with a frustrated bond. With periodic boundary conditions, it is clear that not all finite lattices accomodate this pattern, but that there is a subset of arbitrarily large size which does—these are the lattices with linear sizes that are integer multiples of 3. In the following we construct our desired one-hole eigenstate $|h \rangle$ with energy $-3t$ on this subset.

Define $|G \rangle = \prod_i c^\dagger_{i\sigma(i)} |0\rangle$, where $\sigma(i)$ is the spin at site $i$ in the hexagonal state depicted in Fig. 1b, and $|0\rangle$ is the empty lattice; the ordering of the operators is immaterial in what follows. Next, define the state with a hole on site $n$ as $|n\rangle = (-1)^S c^\dagger_{n\uparrow} |G\rangle$. Here, $S(n) = 0 \ (1)$ if the hole is created on sublattice A (B) of the bipartite hexagonal lattice defined by the frustrated bonds. The matrix elements $\langle m | H_t | n \rangle$ vanish unless $m$ and $n$ refer to neighbouring sites on the hexagonal lattice, in which case

$$
\langle m | H_t | n \rangle = \langle m | -t \sum_{ij,\sigma} c^\dagger_{i\sigma} c_{j\sigma} | n \rangle
$$

$$
= -t \langle m | c^\dagger_{\alpha\uparrow} c_{\alpha\uparrow} | n \rangle
$$
sider the energetics at
On conclusion at sufficiently large will acquire a perturbative gain of that has an energy \( \epsilon_t/J \) tively, preserving its symmetry characteristics at small hole hopping on a hexagonal lattice. Note that this is, in essence, the ground state of a three components with the hole on a neighbouring site. Fat lines) on which the hole can hop. Note that the sites of the up spins, on any of which a hole can reside, form a hexagonal lattice (the ‘hexagonal backbone’, fat lines) on which the hole can hop.

\[
\begin{align*}
&= -t (-1)^{(m+1)} \langle G | c_{m\uparrow} c_{n\uparrow} c_{m\uparrow} c_{n\uparrow} | G \rangle \\
&= -t \langle G | c_{m\uparrow} c_{n\uparrow} c_{m\uparrow} c_{n\uparrow} | G \rangle = -t .
\end{align*}
\]

Now, let \( |h\rangle = \sum_{i=1}^{2N/3} |n\rangle \). For this state, we have \( H_0 |h\rangle = -3t |h\rangle \) since \( H_0 \) connects, with amplitude \(-t\), each single-hole component of the wavefunction to the three components with the hole on a neighbouring site. Note that this is, in essence, the ground state of a spinless hole hopping on a hexagonal lattice.

Unlike the ensemble of undiluted ground states, which has neither long-range order nor a net magnetisation, \( |h\rangle \) incorporates long-range spin order ⁴. There is a three sublattice \((\sqrt{3} \times \sqrt{3})\) ordering pattern with the spin on two sublattices pointing up and down on the third. The magnetisation induced by adding the single hole is therefore \(N/3\). It is interesting to note that this ordering pattern would also have been generated by an explicitly symmetry-breaking magnetic field pointing along the Ising axis and that the \(\sqrt{3} \times \sqrt{3}\) structure is also picked out by a transverse field ⁴.

Finally, we ask whether the conclusion at \( J^z = \infty \) is altered when \( J^z \) is made finite. While we do not have a proof of this, we believe that this does not happen. Conceptually, it is useful to imagine generating an effective Hamiltonian in the space of all one-hole states by \(z\) symmetry-breaking magnetic field pointing along the \(z\) axis and so strictly at \(J^z\).

We begin with the case \( \eta = 0 \). First, we need to find the allowed many hole states at low dilution (and when \( t/J^z \to 0 \)). As before, we need to remove those electrons experiencing zero net exchange field since this ensures that the leading term of the Hamiltonian, \( H_0 \), continues to be optimised. This implies that one cannot remove two neighbouring electrons: this would leave a triangle with only one occupied site and hence no contribution to the magnetic energy, whereas the optimal states are those in which the three bonds of each triangle have total magnetic energy \(-J/4\).

We showed in the last section that the kinetic energy selects a subset of the one-hole states in which the hole is maximally delocalized. We believe that at sufficiently low dopings, this logic continues to operate and the low-lying eigenstates of \( H \) will be constructed out of the many-hole states obtained by removing electrons from the hexagonal backbone of the state depicted in Fig 1b. (However, we are no longer able to prove this.) This state continues to be favourable as it allows the holes to be maximally mobile by having coordination three everywhere. The only restriction being placed on the holes is not to sit immediately next to each other, which is a demand present for all possible background states and one that should not be too onerous at low dopings.

With these restrictions, the problem reduces to that of spinless fermions hopping on a hexagonal lattice with amplitude \( t \) and with a nearest neighbor hard core repulsion. At low densities, this is a well studied problem and known to be a Fermi liquid ⁵. In this fashion, we find that the ground state of the dilute hole problem is magnetized, breaks translational symmetry and is a two dimensional Fermi liquid.

Now this is not quite true, for the Fermi liquid suffers from the well-known Kohn-Luttinger superconducting instability ⁶ and so strictly at \( T = 0 \) the system will...
break the charge $U(1)$ symmetry as well—thereby breaking three symmetries of the parent insulating state at once! While this happens as $t/J^2 \to 0$, we show next that for $t/J^2$ large but not infinite, the magnetic background of the hexagonal conducting lattice mediates an attractive interaction that also drives a superconducting instability—at low densities this is a much larger effect than the purely “internal” Kohn-Luttinger effect.

To this end we consider the effective Hamiltonian, $H_{\text{eff}}$, between holes on the hexagonal backbone to next-leading order in $t/J^2$, induced by excursions to the surrounding sites in which the system leaves the ground state manifold of $H_J$. The matrix elements of this effective Hamiltonian are determined by perturbation theory as follows. First, we define a $k$-hole state with holes at sites $m_1, m_2, ..., m_k$ (with some ordering of the sites, and for concreteness with $m_i < m_j$ for $i < j$) as

$$|m_1m_2...m_n\rangle = \prod_{i=1}^{k} c_{m_i} |G\rangle. \quad (3)$$

In the two-hole sector ($k = 2$) we thus obtain:

$$\langle m_1m_2 | H_{\text{eff}} | m_1m_2 \rangle = \sum_{p_1,p_2} \frac{\langle n_1n_2 | H_l | m_1m_2 \rangle^2}{E_{m_1,m_2}^0 - E_{p_1,p_2}^0}; \quad (4)$$

here $E^0$ are the energies of the two-hole states for $t = 0$. Note that the overall sign of the many-hole states is immaterial at this order as only the squares of matrix elements of $H_l$ are used in determining $H_{\text{eff}}$.

To this order, more general, non-diagonal matrix elements are absent since holes hopping of the backbone have to hop back to their original site in order to reconstitute a leading-order ground state. When the two holes are not on the same hexagon, $H_{\text{eff}}$ always takes the same value which is just twice the contribution of an isolated hole and hence has the interpretation of an effective one-body potential as discussed in the last section. Subtracting this leaves a term that depends on the joint presence of the two holes and can thus be viewed as an effective two-body potential, $V(R)$, for the holes. This has the form, restoring $\eta$ for future use:

$$V(R = 1) = \infty$$

$$V(R = \sqrt{3}) = -\frac{t^2 2(13 - 4\eta - \eta^2)}{J^2 3 - 4\eta + \eta^2}$$

$$V(R = 2) = -\frac{t^2 2(1 + \eta)}{J^2 3 - \eta}$$

$$V(R > 2) = 0,$$

where $R$ is measured in lattice constants of the triangular lattice.

This effective interaction is dominantly repulsive on account of the hard core piece considered earlier. Hence the additional, weak, attraction can only induce a superconducting instability of the parent Fermi liquid and we do not need to worry about the possibility of phase separation at $\eta = 0$. In the continuum, it is easy to see that the two body problem with a hard core and a weak attractive tail possesses attractive phase shifts in sufficiently high angular momentum channels—at low densities this is sufficient to lead to pairing.\[4\] While we have not carried out a detailed analysis on the hexagonal lattice, the general conclusion will hold.

We end this section by considering the effect of switching on an $\eta > 0$. For $\eta < 1/3$, the above arguments retain their validity, with the main consequence of $\eta$ being a strengthening of the attractive part of the effective interaction. For $\eta > 1/3$, the allowed many-hole states change entirely in character. The density-density attraction $H_\eta$ then is strong enough to overcome the exchange term $H_J$ and produce phase separation. This happens because phase-separation costs an energy of $J(3\eta + 1)/4$ per hole, whereas the optimal states described above achieve a cost of $3J\eta/2$.

**IV. EFFECTIVE INTERACTION FOR CLASSICAL HOLES**

Thus far we have focussed on the physics precisely at $T = 0$. In this limit the degeneracy of the parent magnet is “selected away”. At non-zero temperatures, but still below $J^2$, the ground-state entropy of the parent magnet will again play a role. As a first step towards an understanding of finite temperatures we consider $t \ll T \ll J$, equivalently, we discuss the behaviour of classical annealed holes on the triangular antiferromagnet.\[13\] Our central result in this part is an effective interaction, induced entropically, between two holes. We close with some remarks on the finite density problem.

**A. Two holes**

Two holes on the triangular lattice experience an entropic interaction because the number of ground-state spin configurations, $Z(r)$, minimising $H_J$ depends on their separation vector, $r$. From a knowledge of $Z(r)$, upon subtracting the “creation entropy” of the two holes, $\ln Z(\infty)$, we can obtain an effective interaction potential $\beta v(r)$, which vanishes as $r \to \infty$:

$$\beta v(r) = -\ln Z(r) + \ln Z(\infty)$$

$$= -\ln(Z(r)/Z) + \ln(Z(\infty)/Z); \quad (5)$$

here $Z$ is the partition function of the undiluted system.\[12\]

Defining $Z(\infty)/Z \equiv Y^2$ and $Z(r)/Z \equiv Y^2 - \zeta(r)$, we obtain

$$\beta v(r) = -\ln \left[ 1 - \frac{\zeta(r)}{Y^2} \right] \approx \frac{\zeta(r)}{Y^2}, \quad (6)$$

the last step being justified at large $|r|$ by the smallness of $\zeta(r)/Y^2 = 0$, as we will show below.
To determine $\zeta(r)$ and $\Upsilon$, it is convenient to use a standard representation of the spin problem as a classical dimer model on the dual hexagonal lattice. The presence of a hole is encoded by a certain dimer configuration around it. The latter is solvable by the Pfaffian techniques introduced by Kasteleyn\cite{Kasteleyn} although we will find it convenient to use the language of Grassmanian functional integrals introduced into such problems by Samuel\cite{Samuel}. Thus, the function $\zeta(r)$ is obtained by evaluating a twelve-fermion correlation function.

In detail, this calculation proceeds as follows. First, we map ground state spin configurations on the triangular lattice to dimer configurations on its dual hexagonal lattice. The statistical mechanics of classical hard core dimer model on the dual hexagonal lattice turns out that one has to double the unit cell of the lattice. Conversely, up to the global Ising symmetry, each plaquette surrounding a spin (the sites of which we label $r$, $\xi$, and $\upsilon$) is soluble by the Pfaffian techniques introduced by Kasteleyn\cite{Kasteleyn}. The statistical mechanics of classical hard core dimer model on the dual hexagonal lattice to dimer configurations on its dual hexagonal lattice by drawing a dimer through each frustrated bond, placing its endpoints at the centres of the two triangles sharing the frustrated bond. Noting that each triangle has one and only one frustrated bond in the ground state, the dimers thus provide a hard-core covering of the lattice dual to the triangular lattice, namely the hexagonal lattice. Conversely, up to the global Ising symmetry, each dimer covering corresponds to a ground state of the Ising model.

The statistical mechanics of classical hard core dimer coverings on the hexagonal lattice has been studied by Yokoi, Nagle and Salinas\cite{Yokoi}. Here, the vectors $\xi$ and $\upsilon$ denote the translation vectors of the lattice with the doubled unit cell (Fig. 2). The holes are located in cells $r_1$, site $i$, and $r_2$, site $j$, with $r = r_2 - r_1$.

This choice of $A$ accomplishes the following. If the exponential in the integrand for the partition function is expanded, the only terms that result in a nonzero contribution to the integral are those in which each $\psi_i$ appears exactly once. Each such term represents a covering of dimers with sites $i$ and $j$ being connected by a dimer if $A_{ij}$ is one of the prefactors of the term. The signs of the entries of $A$ are chosen such that every such term in fact integrates to 1 so that all dimer coverings are allocated equal weight. One thus obtains $S/k_B = \ln Z/N \simeq 0.323$.

We first calculate the probability that a spin can be replaced with a hole, which we require as this will turn out to be $\Upsilon$, defined above. To ensure the plaquette surrounding a spin (the sites of which we label 1...6) is covered by three dimers, we insert a prefactor $A_{12}A_{34}A_{56}\psi_1\psi_2\psi_3\psi_4\psi_5\psi_6$ into the integrand, which forces the three dimers into the required position. We thus need to calculate

$$Y = \frac{1}{Z} \int \prod_{l=1}^{3} d\psi_l \times A_{12}A_{34}A_{56}\psi_1\psi_2\psi_3\psi_4\psi_5\psi_6 \exp(\psi_i A_{ij} \psi_j).$$

Since the Fermionic action is quadratic, this becomes by Wick’s theorem the sum over all 6! contractions of pairs of $\psi$s:

$$Y = -\sum_{P} \text{sign}(P)G_{i_1i_2}G_{i_3i_4}G_{i_5i_6},$$

where the sum runs over all permutations $P = \{i_l | l = 1...6\}$ of $\{1...6\}$. These Green functions have been worked out by Yokoi \textit{et al.}\cite{Yokoi} together with identities expressing all $G_{ij}$ in terms of $G_{24}$, which for $y \geq 0$ is given by

$$G_{24}(x, y) = \frac{2}{\pi} (-1)^{3} \int_{0}^{\pi/3} d\phi \frac{\cos(2x\phi)}{|1 + \exp(2\phi)|^{2(y+1)}}.$$

FIG. 2. The hexagonal lattice with the spins of the unit cell labelled 1-4. The lattice translation vectors are denoted by $\xi$ and $\upsilon$ for the $x$ and $y$ directions, respectively. The plaquette on the top right encloses a hole, marked by a cross. One of the two possible resulting dimer plaquette configurations is shown.
with $G_{24}$ given by the negative of this expression for $y < 0$.

In fact, the sum for $\Upsilon$ only contains $3!$ nonzero terms as Green functions arising from contractions of sites belonging to the same sublattice of the hexagonal lattice vanish. We thus obtain

$$\Upsilon = \frac{2}{27} - \frac{3\sqrt{3}}{8\pi^3} + \frac{1}{2\sqrt{3}\pi} \approx 0.14501. \quad (11)$$

To confirm the calculation so far, we remark that Monte Carlo simulations we have carried out in a different context (frustrated transverse field Ising models) give $\Upsilon = 0.147 \pm 0.003$.

The evaluation of $Z(r)$ thus requires twelve-fermion correlation functions as two holes have to be introduced. There are two technical difficulties in this calculation. Firstly, the distance-dependence of the Green functions is not known in closed form, and secondly, we now require $6! = 720$ terms each containing at least six Green functions even after using the vanishing of same-sublattice Green functions.

As the reader can see, this calculation is straightforward in principle but somewhat involved in practice—it will turn out though that the final answer has a transparent rationale. Before proceeding further it is useful to discuss the same general phenomenon in a toy model which is exactly solvable and where the combinatorics is more transparent. Consider a one-dimensional dimer model in which the dimers cover a two-leg ladder (see Fig. 3). The entropy per rung of this dimer model is the golden mean, $G = (\sqrt{5} + 1)/2 \approx 1.62$. The dimer model on this ladder arises in the study of a fully frustrated three-leg Ising ladder [2] and removing a spin from this Ising model corresponds to forcing the square surrounding this spin to be covered by two parallel dimers.

If two holes are adjacent to one another (Fig. 3a, top), there are no rungs separating the plaquettes surrounding them. Moving the hole on the right two units further away (Fig. 3b, bottom) costs an entropy of $\ln 2$ as the hole pair and only gains an entropy of $\ln 2$ as the dimer pair between the holes can only have two states. Odd hole separations are entropically unfavorable because dimers can only resonate in pairs (bottom graph) and not on their own (middle graph).

As these Green functions decay algebraically with $|r|$, the terms can be arranged according to how many contractions between sites belonging to different holes they contain. The leading terms are the ones in which all contractions are between sites of one or the other hole, and these terms are clearly independent of $r$. In fact, it is easy to see that the contractions around a given hole are simply those which yielded $\Upsilon$ in the single-hole calculation, and thus the leading, $r$-independent term is $Z(\infty)/Z = \Upsilon^2$, as promised above.

The leading non-trivial $r$-dependence arises from terms which contain two Green functions connecting sites around the two holes, an odd number of connections being impossible. There are 324 such terms. With the assistance of Mathematica, we were able to analyze their asymptotics and obtain the exact long distance form,

$$\zeta(x, y) \sim -\frac{\cos(4\pi x/3)}{x^2 + 3y^2}, \quad (14)$$

where

$$\Xi = \frac{729 - 324\sqrt{3}\pi - 324\pi^2 + 96\sqrt{3}\pi^3 + 64\pi^4}{288\pi^6} \approx 0.025853. \quad (15)$$

Translating the co-ordinates back to the original triangular lattice, our final asymptotic expression for the interaction potential is

$$\beta v(r) \sim -\frac{\cos(4\pi x/3)}{y^2 \Upsilon^2 \sqrt{x^2 + y^2}}. \quad (16)$$

The physics of this expression is quite transparent. The wavevector $\mathbf{q} = \frac{4\pi}{3} \hat{x}$ on the triangular lattice is
characteristic of the hexagonal frustrated bond pattern sketched in Fig 1b. It has long been recognized (already by Wannier) that this pattern represents the dominant ordering tendency in the ground state average of the classical problem. In a more modern formulation in terms of height representations the pattern is the reference flat surface about which the system exhibits fluctuations. In accordance with this we find from Eq. 14 that the holes attract when they share a hexagonal backbone of frustrated bonds, and repel when they do not. The algebraic decay could have been anticipated in the dimer language from our observation that the leading piece at large distance comes from a sum of dimer-dimer correlation functions each of which individually decays as the inverse-square of the distance. However, and this is what motivated the computation, it is not evident in advance that the various dimer correlators (which enter with different signs) do not sum to zero and hence the actual result does not just yield a non-trivial constant but also assures us that it is not zero.

B. Finite density of classical holes: an incompressible state

As the two hole interaction derived previously has considerable spatial structure and a long range, the reader may wonder whether it leads to a condensation of a finite density of holes on the hexagonal backbone. This does not happen. To see this note that \( \beta v(r) \) is independent of \( T \), which is to say that there are fluctuations even as \( T \to 0 \)—as indeed there are on account of the degeneracy. In the dilute limit, in which the two hole calculation is reliable, the effective interaction strength is proportional to the inverse square of the hole separation, i.e. to the (low) density and hence the hole gas remains in a liquid phase. Nevertheless, the long range of the interactions indicates that it exhibits substantial correlations coming from the entropic potential, and—on account of the form of the interaction—we suspect that such correlations will be algebraic as well. As we have already argued that the zero temperature state involves hole condensation on the hexagonal backbone driven by the hole kinetic energy, it appears that this phenomenon arises for both energetic and entropic reasons, and is thus quite robust so that it will characterize the entire temperature range \( T \ll J^2 \).

Finally we report a curiosity regarding classical holes for \( \eta < 1/3 \). As the density of holes is increased, it is not clear whether the interaction finally becomes strong enough to produce phase separation into regions either fully occupied or maximally diluted (but respecting the hard core repulsion of the holes). However, even if it does not, there exists a density of holes at which long-range order is present.

The closest packing compatible with the ground-state constraint is obtained by placing the holes such that they cover one of two sublattices of the hexagonal backbone of up spins depicted in Fig. 1b. At a hole density of 1/3, this sublattice is completely occupied by holes, at zero energy cost at \( O(J) \) compared to the undiluted system.

Adding further holes above this density costs a magnetic energy of order \( J \), and thus the chemical potential experiences a discontinuity at a hole density of 1/3. The state at this density, which has perfect unfrustrated antiferromagnetic order on the bipartite hexagonal lattice occupied by the remaining spins, is thus an incompressible state.

As the density of holes is increased further, phase separation occurs for certain. Completely depleted regions are created by removing one hexagonal ring after another as the system optimises its energy by removing those spins with the smallest number of neighbours. This is the same mechanism that would operate in any unfrustrated magnet. As mentioned above, for \( \eta > 1/3 \), phase separation occurs for all hole densities.

V. SUMMARY

We have considered the problem of dilute, slow holes in the triangular Ising antiferromagnet. We find that doping has two very different outcomes depending on the dimensionless strength \( \eta \) of the density-density interaction. For \( \eta < 1/3 \) we find quite generally that at all temperatures \( T \ll J^2 \) the holes tend to condense on a hexagonal backbone of frustrated bonds and that at \( T = 0 \) they form a superconducting state that coexists with magnetic order and the breaking of translational symmetry in a “magnetic supersolid”. As a special case of this more general phenomenon, we were able to prove a “Nagaoka” theorem for a single hole at \( J^2 = \infty \). For \( \eta > 1/3 \) in our short ranged model, we find phase separation. Whether the inclusion of the long range piece of the Coulomb interaction will change that outcome and stabilize the magnetic supersolid and how much of this structure will persist to larger values of \( t/J^2 \) remain topics for future work.

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This argument is compelling at weak doping. At sufficiently high dopings one might anticipate a renormalized fermi liquid description, e.g. as assumed in the work on the spin fluctuation scenario for the cuprates. See e.g. D. Pines, cond-mat/0002281 and references therein.

Three entirely different account of stripe formation can be found in J. Zaanen and O. Gunnarsson, Phys. Rev. B 40, 7391 (1989); U. Löw, V. J. Emery, K. Fabricius and S. A. Kivelson, Phys. Rev. Lett. 72, 1918 (1994) and M. Vojta and S. Sachdev, Phys. Rev. Lett. 83, 3916 (1999).

Perhaps the best way to set up the contrast is to consider the limit $T \to 0$ for the insulator and with a single hole present. In the former case we obtain the symmetric ground state manifold average, while in the latter we get and average over the state $|h\rangle$ and its symmetry related counterparts.

Readers familiar with the Nagaoka theorem may worry about this point. In the Hubbard problem, reducing $U$ from infinity, where the theorem holds, has a dramatic effect in that it confines the polarization to a finite region surrounding the hole. There, this happens since a finite $U$ permits virtual processes far from the hole which generate an antiferromagnetic exchange. The resulting exchange strength is $O(t^2/U)$ but it exacts a volume cost should we attempt to keep the system polarised. In our problem, single occupancy is present at the first step and, far from the hole, no virtual processes are possible, whence no volume cost arises.

Complementary to the study of annealed dilution, there exists—under the general heading of ‘order by disorder’—a body of work on the role of quenched dilution in lifting the ground-state degeneracy of frustrated magnets, beginning with J. Villain, R. Bidaux, J. P. Carton and R. J. Conte, J. Physique 41, 1263 (1980).

We note that this problem is quite different from that of introducing holes (monomers) on the dimer lattice itself. That latter leads to a confining potential that grows with distance, as shown by M.E. Fisher and J. Stephenson, Phys. Rev. 132, 1411 (1963).

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