approximate controllability of network systems

Yassine El Gantouh and Said Hadd
Faculty of Sciences, Department of Mathematics
Ibn Zohr University
Hay Dakhla, BP8106, 80000–Agadir, Morocco

Abdelaziz Rhandi
Dipartimento di Ingegneria dell’Informazione, Ingegneria Elettrica e Matematica Applicata
Università degli Studi di Salerno
Via Giovanni Paolo II, 132, 84084 Fisciano (Sa), Italy

(Communicated by Genni Fragnelli)

Dedicated to Rainer Nagel on the occasion of his 80th-Birthday

Abstract. In this paper, the rich feedback theory of regular linear systems in the Salamon-Weiss sense as well as some advanced tools in semigroup theory are used to formulate and solve control problems for network systems. In fact, we derive necessary and sufficient conditions for approximate controllability of such systems. These criteria, in some particular cases, are given by the well-known Kalman’s controllability rank condition.

1. Introduction. Network systems or transport networks are mathematical models introduced to describe flow of a product from a source to a prescribed destination (sink) along the edges of a weighted oriented graph. These products can be gallons of oil flowing into pipelines, the number of telephone calls transmitted in a communication system or airplanes flying in certain regions of airspace [3]. In modeling such situations, we interpret the weight of an edge in the directed graph as a proportion that any given mass distribution should satisfy, for example, the amount of oil that can flow through a certain part of system of pipelines. To make these ideas precise, we consider a strongly connected and directed graph $G = (V, E)$, and assume that

1. on each edge, the particles only flow in one direction with the speed, depending on the space and are absorbed with respect to an absorption function $q$,

2. in each vertex $v \in V$ we assume conservation of mass according to the Kirchhoff law

$$\sum \text{incoming flows} = \sum \text{outgoing flows},$$

2020 Mathematics Subject Classification. Primary: 93B05, 93B70; Secondary: 93C73, 93C05, 93B28.

Key words and phrases. Boundary systems, boundary perturbations, controllability, network systems, Kalman’s controllability rank condition.

This work has been supported by the COST Action CA18232. The third author is a member of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

* Corresponding author: Abdelaziz Rhandi.
3. in each vertex \( v \in V \) the incoming flow is redistributed into the outgoing edges \( e \in E \) according to some weights \( 0 \leq w_e \leq 1 \).

In particular, the evolution over time of the above process on the edges can be considered as a system of partial differential equations (PDE) of first order on networks or graphs [24, Chap.2]. We mention that the existence, uniqueness and asymptotic behaviors of solutions of these systems were first studied by E. Sikolya and M. Kramar in [18], using semigroups theory [12]. In recent years, different variations and generalizations of the original problem have been discussed in numerous publications and monographs. The complete list of these publications can be found, for example, in [6] or in [7].

However, according to hypothesis (c), the mass distribution on the edges will always satisfy the ratio prescribed by the weights. A striking consequence of this fact is that no arbitrary distribution can flow along the edges of the underlying network. Therefore, a lot of effort has gone into analyzing how solutions (mass distributions) propagate along the graph over time, to solve the problem of controllability of flows. Basically, the question of controllability can be formulated as follows: can we obtain a given mass distribution on the edges of the network by making a control acting on its vertices? It turns out that the answer to the above question is not always positive due to the structure of the networks. In [8], [9], [10] and [11] the authors studied this question in the framework of a new concept called maximal controllability introduced in [11]. In addition, an important earlier contribution is presented in [20], in which they developed a minimum input theory to efficiently characterize the structural controllability of ordinary differential equations on directed networks.

In the present work, we aim to study a kind of approximate controllability of network systems. In fact, we are interested in controllability in the following sense: is it possible to steer the state from the origin to all points in the state space within an \( \varepsilon > 0 \)-distance? This notion of controllability is weaker than the classical exact controllability concept, where \( \varepsilon = 0 \). We refer to the standard reference [13] for the case of distributed-parameter systems. Our main interest is to prove qualitative properties of network systems in the framework of Salamon-Weiss systems, a kind of infinite-dimensional regular linear systems; see e.g., [22], [30], [29], [27], [28], [25]. More precisely, using an approach presented in [16], we will first reformulate the network system as a distributed one and prove necessary and sufficient conditions for approximate controllability of such a system. More precisely, under suitable assumptions we prove that approximate controllability of such systems is equivalent to the finite dimensional Kalman-type condition, expressed only in terms of graph matrices. According to the concept of structural controllability [19], [23], the above condition is reformulated in a more appropriate form which extends results of the papers [8], [9], [10] and [11]. The originality of our approach stems from the idea of combining the feedback theory of regular linear systems [30] and the semigroup theory approach to study the above aforementioned problems.

The organization of the paper is as follows: in the remainder of this section we recall some basic tools from graph theory and introduce the setting of the considered problem. In Section 2, we gather some results on well-posed boundary value problems. The boundary value problem associated to network flows is studied in Section 3, which contains two parts: the regular linear systems approach for flows in networks is developed in Subsection 3.1, while the well-posedness of network systems is established in Subsection 3.2. Finally, in Section 4, we state and prove the main results on approximate controllability of network systems.
Throughout this paper, we consider a loop-free finite, strongly connected and directed graph $G = (V, E)$ and a flow on it satisfying the assumptions (a), (b) and (c). Here $V = \{v_i, \ i = 1, \ldots, n\}$ denotes the set of vertices and $E = \{e_j, \ j = 1, \ldots, m\}$ the set of (directed) edges. The edges are normalized as $e_j \equiv [0, 1]$ and parameterized to the contrary of their directions. Thus, the vertices $e_j(0)$ and $e_j(1)$ are called the head and the tail of the edge $e_j \in E$, respectively.

The graph topology is described as follows (cf. [4] or [17]).

Definition 1.1. The incidence matrix of $G$ is

$$I := I^+ - I^-,$$

where $I^-(i,j)_{n \times m}$ and $I^+(i,j)_{n \times m}$ are the outgoing incidence and the incoming incidence matrices, respectively, having entries

$$i^-_{ij} := \begin{cases} 1, & \text{if } v_i = e_j(1), \\
0, & \text{otherwise} \end{cases}, \quad i^+_{ij} := \begin{cases} 1, & \text{if } v_i = e_j(0), \\
0, & \text{otherwise}. \end{cases}$$

We define the weighted outgoing incidence matrix $I^-_w = (w^-_{ij})_{n \times m}$, where

$$0 \leq w^-_{ij} \leq 1$$

satisfying $w^-_{ij} = 0 \iff i^-_{ij} = 0$ and

$$\sum_{j=1}^m w^-_{ij} = 1, \ \forall i = 1, \ldots, n. \quad (1)$$

Accordingly, we call $e_j$ an outgoing edge for $v_i$ if $i^-_{ij} = 1$ holds. Moreover, $e_j$ is an incoming edge for $v_i$ if $i^+_{ij} = 1$ holds. In addition to the incidence matrices, we can equivalently describe the topology of a weighted graph via adjacency matrices. The weighted (transposed) adjacency matrix $A$ and the weighted (transposed) adjacency matrix of the line graph $B$ are $(a_{ik})_{n \times n} := (I^+(I^-_w)\top)^\top$ and $(b_{ij})_{m \times m} := (I^-_w)\top I^+$, respectively. We have

$$I^-(I^-_w)\top = I_{\epsilon^s}, \quad (2)$$

and that both adjacency matrices $A$ and $B$ are column stochastic. Hence,

$$\|A\|_1 = \|B\|_1 = 1.$$  

Furthermore, the relation

$$(I^-_w)\top A^k = B^k(I^-_w)\top \quad (3)$$

for all $k \geq 1$, holds.

Therefore, the mathematical model for flows in networks, under consideration, is

$$\begin{cases}
\frac{\partial}{\partial t} z_j(t, s) &= c_j(s) \frac{\partial}{\partial s} z_j(t, s) + q_j(s) z_j(t, s), \ s \in (0, 1), t \geq 0, \\
z_j(0, s) &= g_j(s), \ s \in (0, 1), \\
i^-_{ij} z_j(t, 1) &= w^-_{ij} \sum_{k=1}^m i^+_{ik} z_k(t, 0) + w^-_{ij} \sum_{i=1}^{n_0} k_{il} u_l(t), \ t \geq 0, \tag{IC}
\end{cases} \quad (4)$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, where $n, m, n_0 \in \mathbb{N}$.

Here, the function $s \mapsto z_j(t, s)$ for $s \in [0, 1]$ describes the distribution of the material along an edge $e_j$ at time $t \geq 0$, and the functions $c_j$ are the space dependent velocities of the flow on each edge $e_j$, while the function $q_j$ describes the absorption along the edge. Both functions belong to $L^\infty([0, 1])$ such that $c_j(s) \geq 0 > 0$ for almost every $s \in [0, 1]$, for each $j = 1, \ldots, m$ and some $\varepsilon > 0$. Finally, $K := (k_{il})_{n \times n}$,
n ≥ n_0, is a boundary control operator acting on vertices and u_l : [0; ∞) → C, for l = 1, ..., n, are locally p-integrable functions.

2. Well-posed boundary value problems. Let X, U and Z be Banach spaces such that Z ⊂ X with continuous dense embedding. Given a closed and densely defined linear operators A_m : Z ⊂ X → X and G, M : Z ⊂ X → U, we confine our attention to the following boundary value problem

\[
\begin{aligned}
    \dot{z}(t) &= A_m z(t), \quad t ≥ 0, \\
    Gz(t) &= Mz(t), \quad t ≥ 0, \\
    z(0) &= x.
\end{aligned}
\]  

(5)

The problem (5), can be rewritten as the following boundary input-output system

\[
\begin{aligned}
    \dot{z}(t) &= A_m z(t), \quad t ≥ 0, \\
    Gz(t) &= u(t), \quad t ≥ 0, \\
    y(t) &= Mz(t), \quad t ≥ 0, \\
    z(0) &= x
\end{aligned}
\]  

(6)

with the feedback law u = y. The well-posedness of the above problem consists in finding conditions on A_m, G and M for which

- \( z(t) := z(t, \cdot, u) \in X \), for all \( t ≥ 0 \) and \( u ∈ L^p_{loc}([0, ∞); U) \).
- \( y(t) := y(\cdot, x, u) ∈ L^p_{loc}([0, ∞); U) \), for all \( x ∈ X \) and \( u ∈ L^p_{loc}([0, ∞); U) \).
- the operator \( u ↦ y \) defines a linear bounded operator on \( L^p_{loc}([0, ∞); U) \).

To make these statements more clear, some hypothesis are needed.

Assumptions.

(H1) \( A = A_m \) with domain \( D(A) := \ker G \) generates a \( C_0 \)-semigroup \( T = (T(t))_{t ≥ 0} \) on X.

(H2) \( G \) is surjective.

Under these hypothesis, G. Greiner [14, Lemma 1.2] showed that, for \( μ ∈ ρ(A) \), the following direct sum

\[
Z = D(A) ⊕ \ker(μ - A_m)
\]  

(7)

holds and the following inverse, called the Dirichlet operator,

\[
D_μ := (G|_{\ker(μ - A_m)})^{-1} : U → \ker(μ - A_m)
\]  

exists and is bounded. Define

\[
B := (μ - A_{-1})D_μ, \quad μ ∈ ρ(A).
\]  

(8)

Then \( B ∈ L(U, X_{-1}) \), \( \text{rg } B ∩ X = \{0\} \) and

\[
(A_m - A_{-1})|_X = BG,
\]  

(9)

since \( μD_μu = A_m D_μu, \quad u ∈ U \), where \( X_{-1} \) denotes the extrapolation space associated with X and A. Using the representation (9) the input-state part of system (6) can be reformulated as the following distributed-parameter system

\[
\dot{z}(t) = A_{-1}z(t) + Bu(t), \quad t ≥ 0, \quad z(0) = x.
\]  

(10)

For \( τ ≥ 0 \), we define \( Φ_τ ∈ L(L^p([0, ∞); U), X_{-1}) \) by

\[
Φ_τ u = \int_0^τ T_{-1}(t-s)Bu(s)ds.
\]
These operators are called the input map associated with the pair \((A, B)\). Moreover, for \(t, \tau \geq 0\) and \(u \in L^p_{\text{loc}}([0, \infty); U)\), we have
\[
\Phi_{t+\tau} u = T(t)\Phi_{\tau} u_{|[0, \tau]} + \Phi_{\tau} u(\cdot).
\] (11)

Then, it is easy to see that \(\Phi_0 = 0\) and \(\Phi_{\tau} u = \Phi_{\tau} u_{|[0, \tau]}\) (causality). Moreover, the Laplace transform of \(\Phi_{\cdot} u\) satisfies
\[
\hat{\Phi}_{\cdot} u(\mu) = D\hat{u}(\mu),
\] (12)
where \(\hat{u}\) denotes the Laplace transform of \(u\) for \(u \in L^p([0, \infty); U)\).

**Definition 2.1.** The operator \(B \in \mathcal{L}(U, X_{-1})\) is called an admissible control operator for \(A\), if for some \(\tau > 0\), \(\text{Rang } \Phi_{\tau} \subset X\).

Note that if \(B\) is admissible, then for some (and hence for all) \(\tau \geq 0\), we have
\[
\Phi_{\tau} u \in L^p([0, \infty); U, X)
\]
(see [26, Proposition 4.2.2, page 116]). In fact, it has been observed that the differential equation (10) has a unique strong solution, called the state trajectory with the initial state \(z(0)\), given by
\[
z(t) = T(t)x + \Phi_{\tau} u, \quad t \geq 0.
\]

Now, for the operator
\[
C = Mi \in \mathcal{L}(D(A), U),
\]
where \(i\) is the canonical injection from \(D(A)\) to \(Z\), we say that linear system
\[
\begin{align*}
\dot{z}(t) &= Az(t), \quad t \geq 0, \\
y(t) &= Cz(t), \quad t \geq 0, \\
z(0) &= x
\end{align*}
\]
is well-posed if the observation function \(y(\cdot)\) can be extended to a function in \(L^p_{\text{loc}}([0, \infty); U)\). This fact motivates the following definition.

**Definition 2.2.** The operator \(C \in \mathcal{L}(D(A), U)\) is called admissible observation operator for \(A\) if
\[
\int_0^\tau \|CT(s)x\|p ds \leq \gamma(\tau)p\|x\|p
\] holds for some (hence for all) \(\tau \geq 0\) and all \(x \in D(A)\) with constant \(\gamma(\tau) > 0\).

From (13) one deduces that the map \(\Psi := CT(\cdot)x\) can be extended to a linear bounded operator \(\Psi : X \rightarrow L^p_{\text{loc}}([0, \infty); U)\), called the extended output map. Then we can set \(y(t) = (\Psi x)(t)\) for any \(x \in X\) and a.e. \(t \geq 0\).

As shown by Weiss [27], one can associate with \(C \in \mathcal{L}(D(A), U)\) the following operator
\[
C_\Lambda x := \lim_{\mu \to \infty} C\mu R(\mu, A)x
\] (14)
with \(D(C_\Lambda) := \{x \in X : \text{the limit in } (14) \text{ exists}\}\) called the Yosida extension of \(C\) for \(A\). The introduced operator makes possible to give a simple pointwise interpretation of the output map \(\Psi\) in terms of the observation operator \(C\). More precisely, for an admissible observation operator \(C\), we have \(T(t)x \in D(C_\Lambda)\) for a.e. \(t \geq 0\) and
\[
\Psi x := C_\Lambda T(\cdot)x
\]
for all \( x \in X \) and a.e. \( t \geq 0 \) (see [26, Theorem 4.5], [15, Theorem 1.3.1]).

Summarizing the previous results we can reformulate the boundary input-output system (6) as follows:

\[
\begin{align*}
\dot{z}(t) &= A_{-1} z(t) + B u(t), \quad t \geq 0, \\
z(0) &= x, \\
y(t) &= C z(t), \quad t \geq 0.
\end{align*}
\]

(15)

It has to be noted that if the control function \( u \) is smooth enough, let say \( u \in W^{1,p}_{loc}([0, \infty); U) \), we have

\[ \Phi_t u \in Z \]

for any \( t \geq 0 \), see [25, Chap. 4] and [26, Chap. 4]. We now define

\[ (F u)(t) := M \Phi_t u, \quad t \geq 0 \]

(16)

for all \( u \in W^{1,p}_{0,loc}([0, \infty); U) := \{ u \in W^{1,p}_{0,loc}([0, \infty); U) : u(0) = 0 \} \).

**Definition 2.3.** We say that the triple \((A, B, C)\) (or equivalently the system (15)) is a well-posed on the state space \( X \), the input space \( U \) and the output space \( U \) if the following conditions are satisfied:

1. The operator \( B \in \mathcal{L}(U, X_{-1}) \) is an admissible control operator for \( A \),
2. The operator \( C \in \mathcal{L}(D(A), U) \) is an admissible observation operator for \( A \),
3. There exist constants \( \tau \) and \( \kappa_\tau > 0 \) such that

\[ \| F u \|_{L^p([0, \tau]; U)} \leq \kappa_\tau \| u \|_{L^p([0, \tau]; U)} \]

for all \( u \in W^{1,p}_{0,loc}([0, \infty); U) \).

A more appropriate subclass of well-posed triples, called regular triples, is defined as follows, see [29].

**Definition 2.4.** The well-posed triple \((A, B, C)\) on \( X, U \) and \( U \) is called regular (with feedthrough zero) if for any \( v \in U \), we have

\[ \lim_{\tau \to 0^+} \frac{1}{\tau} \int_0^\tau (F(\chi_{R^+} v))(\sigma)d\sigma = 0, \]

where \( \chi_{R^+} \) denotes the characteristic function of the set \([0, \infty)\).

**Remark 1.** It is shown by Weiss [29] that, for a regular triple \((A, B, C)\) on \( X, U \) and \( U \),

\[ z(t) = T(t)x + \Phi_t u \in D(C_\Lambda) \]

for all \( x \in X, u \in L_{loc}^p([0, \infty); U) \) and a.e. \( t \geq 0 \), see [25, Theorem 5.6.5] in the case of general Banach spaces. Moreover, the extended operator associated to \( F \) is given by

\[ (F u)(t) = C_\Lambda \Phi_t u \]

for all \( u \in L_{loc}^p([0, \infty); U) \) and a.e. \( t \geq 0 \). So, the output function becomes

\[ y(t) = C_\Lambda z(t) = (\Psi x + F u)(t), \quad \text{a.e. } t \geq 0. \]

This allows us to extend the output function to a function in \( L_{loc}^p([0, \infty); U) \), see [29] for more details.
Furthermore, for \( \tau \geq 0 \), we define the input-output maps of \((A, B, C)\), denoted by \( F_\tau \) (or \( u \mapsto y \)), by truncating the output to \([0, \tau]\):

\[
F_\tau u = (F u)|_{[0, \tau]}, \quad u \in L^p_{loc}([0, \infty); U).
\]  

\( \text{Theorem 2.6.} \) Let the Assumptions posedness of the boundary value problem (5), see [16, Theorem 4.1].

We say that the identity operator \( I : U \to U \) is an admissible feedback operator for \((A, B, C)\), if \( I - F \) has an inverse in \( L(L^p([0, t_0]; U)) \) for some \( t_0 > 0 \).

We end this section by recalling a perturbation result that guarantees the well-posedness of the boundary value problem (5), see [16, Theorem 4.1].

\( \text{Theorem 2.6.} \) Let the Assumptions (H1) and (H2) be satisfied and let \( A, B, C \) be the operators defined from the operators \( A_m, G, M \). If the triple \((A, B, C)\) is regular on \( X, U \) and \( U \) with the identity operator \( I : U \to U \) as an admissible feedback, then the operator

\[
Az = A_mz, \quad D(A) = \{x \in Z : G x = M x\}
\]

generates a \( C_0 \)-semigroup \( T := (T(t))_{t \geq 0} \) on \( X \) satisfying \( T(s)x \in D(C_A) \) for all \( x \in X \) and almost every \( s \geq 0 \). In addition

\[
T(t)x = T(t)x + \int_0^t T_{-1}(t - s)BCA T(s)x ds
\]

for \( x \in X \) and \( t \geq 0 \). Moreover, for any \( \mu \in \rho(A) \) we have

\[
\mu \in \rho(A) \iff 1 \in \rho(D_\mu M) \iff 1 \in \rho(MD_\mu).
\]

Finally, for \( \mu \in \rho(A) \cap \rho(A) \),

\[
R(\mu, A) = (I - D_\mu M)^{-1}R(\mu, A) = (I + D_\mu (I - MD_\mu)^{-1}M) R(\mu, A).
\]

3. Well-posedness of network systems. In this section we consider the system (4) with \( K \equiv 0 \), that is,

\[
\begin{align*}
\frac{\partial}{\partial t}z_j(t, s) &= c_j(s) \frac{\partial}{\partial s}z_j(t, s) + q_j(s)z_j(t, s), \quad s \in (0, 1), \ t \geq 0, \\
z_j(0, s) &= g_j(s), \quad s \in (0, 1), \quad \text{(IC)}, \\
i_{ij}z_j(t, 1) &= w_{ij} \sum_{k=1}^m i^+_{ik}z_k(t, 0), \ t \geq 0, \quad \text{(BC)}
\end{align*}
\]  

(18)

for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \). Using (1) one deduces that the boundary condition (BC) implies the Kirchhoff law

\[
\sum_{j=1}^m i_{ij}z_j(t, 1) = \sum_{j=1}^m i^+_{ij}z_j(t, 0), \quad i = 1, \ldots, n.
\]

In the sequel, we propose to use the regular linear systems approach for the network system (18).

3.1. Feedback theory for flows in networks. Here, we apply the theory of regular linear systems to flows in networks. That is, we shall formulate the system (18) in the form of an appropriate boundary value problem. To this purpose, let

\[
X := L^p([0, 1]; \mathbb{C}^m)
\]
for $p \in (1, \infty)$, endowed with the norm
\[
\|f\| := \left( \sum_{j=1}^{m} \|f_j\|_{L^p([0,1])}^p \right)^{\frac{1}{p}}, \quad f = (f_1, \ldots, f_m) \in X.
\]

Define the operator
\[
A_m := \begin{pmatrix}
(c_1(s) \frac{d}{ds} + M_{q_1}) & \cdots & 0 \\
0 & \ddots & 0 \\
c_m(s) \frac{d}{ds} + M_{q_m} & \cdots & (c_m(s) \frac{d}{ds} + M_{q_m})
\end{pmatrix},
\]
with domain
\[
D(A_m) := \{ f = (f_1, \ldots, f_m) \in (W^{1,p}[0,1])^m : f(1) \in \text{Rang}(I_w) \}.
\]
where $M_{q_j}$ denotes the multiplication operator by the bounded measurable function $q_j$. Moreover, we define the boundary operators $G, M : D(A_m) \subset X \rightarrow U = \mathbb{C}^n$ by
\[
Gf := \mathcal{I}^-(f(1)), \quad Mf := \mathcal{I}^+(f(0))
\]
called the outgoing and the incoming boundary operators, respectively. If we set $z(t) = (z_1(t), \ldots, z_m(t)) = (z_1(t,s), \ldots, z_m(t,s))$, then the system (18) can be formulated in the form of the following boundary value problem
\[
\begin{cases}
\dot{z}(t) = A_m z(t), & t \geq 0, \quad z(0) = z_0, \\
Gz(t) = Mz(t), & t \geq 0
\end{cases}
\]
with $z_0 = (g_j)_{j=1}^m$.

Moreover, one can see that the problem (20) can be considered as the following input-output boundary system
\[
\begin{cases}
\dot{z}(t) = A_m z(t), & t \geq 0, \quad z(0) = z, \\
Gz(t) = u(t), & t \geq 0, \\
y(t) = Mz(t), & t \geq 0
\end{cases}
\]
with the feedback law "$u(t) = y(t)$". In order to apply the results of the previous section, some conditions on the operators $G$ and $M$ are needed. Clearly, $G$ is surjective. To verify that the operator $A := A_m$ with domain $D(A) := \text{ker} G$ generates a $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$, we recall the following definition from [21].

**Definition 3.1.** Take $j \in \{1, \ldots, m\}$ and $s_1, s_2 \in [0, 1]$. We set
\[
\tau_j(s_1, s_2) := \int_{s_1}^{s_2} \frac{ds}{c_j(s)}
\]
and
\[
\xi_j(s_1, s_2) := \int_{s_1}^{s_2} \frac{g_j(s)}{c_j(s)} ds.
\]
Here, $\tau_j(s_1, s_2)$ denotes the time needed to pass on the edge $e_j$ from $s_1$ to $s_2$ moving with velocity $c_j(s)$ at every point $s \in [s_1, s_2]$, while $\xi_j(s_1, s_2)$ is the rate of the mass gain or loss on this journey resulting from the factor $q_j(s)$. Define the operator
\[
A := A_m, \quad D(A) := \text{ker} G.
\]
Then $A$ generates a $C_0$-semigroup on $X$ as the following result shows, see [21, Lemma 3.3].

**Lemma 3.2.** Let $j \in \{1, \ldots, m\}$ be fixed. With the notations of Definition 3.1, let $\hat{s}(t) \in [0,1]$ be the location where the flow moves on the edge $e_j$ from the point $s$ during time $t \leq \tau_j(s,1)$. Then the function $\hat{s} \in C([0,\tau_j(s,1)])$ is defined by $\tau_j(s,\hat{s}(t)) = t$, and the $j$th coordinate of the semigroup $(T(t))_{t \geq 0}$ generated by $(A,D(A))$ is

$$
(T(t)f)_j(s) = \begin{cases} e^{\xi_j(s,\hat{s}(t))} f_j(\hat{s}(t)), & \text{if } 0 \leq t \leq \tau_j(s,1), \\ 0, & \text{otherwise.} \end{cases}
$$

Clearly, the resolvent set of the generator $A$ is $\rho(A) = \mathbb{C}$. For $\mu \in \rho(A)$, it is not difficult to show, see [21, Lemma 3.4], that the Dirichlet operator associated with $G$ is given by

$$
D_\mu = \Xi_\mu(T^-_{w})^T,
$$

where

$$
\Xi_\mu(s) := \text{diag}(e^{\xi_j(s,1)-\mu\tau_j(s,1)})_{j=1,\ldots,m}, \ s \in [0,1].
$$

Let $A_{-1}$ be the extension of $A$ in the extrapolation sense. According to (8), the control operator associated with $A$ has the form

$$
B = -A_{-1} \Xi_0(T^-_{w})^T.
$$

The following lemma shows the admissibility of $B$.

**Lemma 3.3.** Let $j \in \{1, \ldots, m\}$ be fixed. Then $B$ is an admissible control operator for $A$. Moreover, the $j$th coordinate of the input map associated with $B$ is given by

$$
(\Phi_t u)_j(s) = \begin{cases} \sum_{i=1}^n e^{\xi_i(s,1)}w_{ji}u_i(t-\tau_j(s,1)), & \text{if } t \geq \tau_j(s,1), \\ 0, & \text{otherwise} \end{cases}
$$

for all $u \in L^p([0,\infty);\mathbb{C}^n)$ and $s \in (0,1)$.

**Proof.** We define the following operator

$$
(\Lambda_t u)_j(s) = \begin{cases} \sum_{i=1}^n e^{\xi_i(s,1)}w_{ji}u_i(t-\tau_j(s,1)), & \text{if } t \geq \tau_j(s,1), \\ 0, & \text{otherwise} \end{cases}
$$

for $j = 1, \ldots, m$, $s \in (0,1)$ and $u \in L^p([0,\infty);\mathbb{C}^n)$. It is not difficult to see that $\Lambda_t \in \mathcal{L}(L^p([0,\infty);\mathbb{C}^n),X)$ for $t \geq 0$. Taking for $j = 1, \ldots, m$ and $\mu \in \mathbb{C}$, the Laplace transform for $\Lambda_t$, we obtain

$$
(\hat{\Lambda}_u)(\mu)_j(s) = \int_0^\infty e^{-\mu t} (\Lambda_t u)_j(s) dt
$$

$$
= \int_{\tau_j(s,1)}^\infty e^{-\mu t} \sum_{i=1}^n e^{\xi_i(s,1)}w_{ji}u_i(t-\tau_j(s,1)) dt
$$

$$
= \sum_{i=1}^n e^{\xi_i(s,1)} e^{-\mu \tau_j(s,1)} w_{ji} \int_0^\infty e^{-\mu r} u_i(r) dr
$$

for every $u \in L^p([0,\infty);\mathbb{C}^n)$ and $s \in (0,1)$. On the other hand, because of (22), we have

$$
(\hat{\Lambda}_u)(\mu)_j(s) = (D_\mu \hat{u})_j(s)
$$
for any \( u \in L^p([0, \infty); \mathbb{C}^n), \) \( j = 1, \ldots, m, \) and \( s \in (0, 1). \) By the uniqueness of the Laplace transform, it follows from (12) that
\[
(\Phi_t u)_j(s) = (\Lambda_t u)_j(s)
\]
for all \( j = 1, \ldots, m, \) \( u \in L^p([0, \infty); \mathbb{C}^n) \) and \( s \in (0, 1). \) Therefore, (25) holds. This proves also that \( B \) is an admissible control operator for \( A. \)

**Remark 2.** It is worthwhile to note that the input map \( \Phi := (\Phi_t)_{t \geq 0}, \) as defined in (25), satisfies \( \Phi_0 = 0 \) and the composition property (11). Hence, the pair \((T, \Phi)\) is a control linear system on \( X \) and \( L^p([0, \infty); \mathbb{C}^n) \) generated by \((A, B).\) Thus, the associated state trajectory is given by
\[
z(t) = T(t)x + \Phi_t u, \quad t \geq 0.
\]
Furthermore, the last formula describes how the distribution of particles, starting from a state \( x \in X \) and corresponding to the input \( u \in L^p_{loc}([0, \infty); \mathbb{C}^n), \) propagates along (directed) edges as time evolves.

We define
\[
C := Mi \in \mathcal{L}(D(A), \mathbb{C}^n),
\]
where \( i \) denotes the canonical injection from \( D(A) \) into \( D(A_m). \) The next step is to prove that \( C \) is an admissible observation operator for \( A. \)

**Lemma 3.4.** Let \( C \) be the operator given by (26). Then \( C \) is an admissible observation operator for \( A. \)

**Proof.** For \( f \in D(A) \) and \( \alpha > 0, \) it follows from (26) that
\[
\int_0^\alpha \|CT(t)f\|^p_{\mathbb{C}^n} \, dt = \int_0^\alpha \|T^+ [T(t)f](0)\|^p_{\mathbb{C}^n} \, dt \leq \sum_{i=1}^n \int_0^\alpha \left| \sum_{j=1}^m \iota_{ij}^+ [T(t)f]_j(0) \right|^p \, dt \leq \sum_{i=1}^n C_p \sum_{j=1}^m \int_0^{\min(\tau_j(0,1), \alpha)} \|i_{ij}^+ e^{\xi_j(0,\tilde{s}(t))} f_j(\tilde{s}(t))\|^p \, dt \leq C_p \sum_{j=1}^m \int_0^1 |f_j(\sigma)|^p \frac{1}{c_j(\sigma)} \, d\sigma,
\]
where we used the change of variables \( \sigma = \tilde{s}(t) \) so that
\[
d\sigma = c_j(\tilde{s}(t)) \, dt = c_j(\sigma) \, dt,
\]
by Definition 3.1. Hence, we obtain
\[
\int_0^\alpha \|CT(t)f\|^p_{\mathbb{C}^n} \, dt \leq C_p \frac{\min \{c_j \}_{1 \leq j \leq m}}{\|c_j\|_{\infty}} \|f\|^p.
\]
According to Definition 2.2, we conclude that \( C \) is an admissible observation operator for \( A \) with \( \gamma := \left( \frac{C_p}{\min \{c_j \}_{1 \leq j \leq m}} \|c_j\|_{\infty} \right)^{\frac{1}{p}}. \)
Due to this fact, we define the output map by
\[ \Psi f := CT(\cdot)f \]
for \( f \in D(A) \). Using the definition of \( C \) we have
\[ [\Psi f](t) = \sum_{j=1}^{m} u_j e^{\xi_j(0,\tilde{s}(t))} f_j(\tilde{s}(t)) \chi_{[0,\tau_j(0,1)]}(t) \]
for \( f \in D(A) \) and \( i = 1, \ldots, n \), where \( \tilde{s} \in C([0,\tau_j(0,1)]) \) is such that \( \tau_j(0,\tilde{s}(t)) = t \).

**Remark 3.** It is important to note that the output map from (27) extends to a bounded operator from \( X \) to \( L^p_{\text{loc}}([0,\infty); \mathbb{C}^n) \) which will be also denoted by \( \Psi \).

In order to define the input-output operator \( u \mapsto y \) associated to the triple \( (A, B, C) \), we first need to compute the Yosida extension of \( C \) for \( A \).

**Lemma 3.5.** Let \( C_A \) the Yosida extension of \( C \) with respect to \( A \). For \( x \in \mathbb{C}^n \), \( \lambda > 0 \) and \( \mu \in \mathbb{C} \) such that \( \lambda \neq \mu \), we have
\[ \lim_{\lambda \to +\infty} \lambda CR(\lambda, A) D_\mu x = MD_\mu x. \]  
Thus,
\[ D(A_m) \subset D(C_A) \quad \text{and} \quad (C_A)_{|D(A_m)} = M. \]

**Proof.** For \( \lambda > 0 \) and \( \mu \in \mathbb{C} \) such that \( \lambda \neq \mu \), we have
\[ (\Xi_\mu - \Xi_\lambda)(I_w)' = R(\mu, A_{-1})B - R(\lambda, A_{-1})B = (\lambda - \mu)R(\lambda, A_{-1})R(\mu, A_{-1})B = (\lambda - \mu)R(\lambda, A_{-1})D_\mu. \]
Thus,
\[ R(\lambda, A) D_\mu = R(\lambda, A_{-1}) D_\mu = \frac{1}{\lambda - \mu}(\Xi_\mu - \Xi_\lambda)(I_w)'. \]
Therefore, using (23), we get
\[ \lim_{\lambda \to +\infty} \lambda CR(\lambda, A) D_\mu x = MD_\mu x, \quad x \in \mathbb{C}^n. \]
We recall from (7) that \( D(A_m) = D(A) \oplus \ker(\mu - A_m) \) for \( \mu \in \rho(A) \). Due to the definition of \( D_\mu \) and (28), we have \( \ker(\mu - A_m) = \text{Rang} D_\mu \subset D(C_A) \), hence \( D(A_m) \subset D(C_A) \). Moreover, for \( g \in D(A_m) \) we know that \( g - \Xi_\mu g(1) \in D(A) \), so
\[ C_A(g - \Xi_\mu g(1)) = C(g - \Xi_\mu g(1)) = M(g - \Xi_\mu g(1)). \]
Hence,
\[ (C_A)_{|D(A_m)} = M. \]

Due to this fact, the triple \( (A, B, C) \) is well-posed.

**Proposition 1.** The triple \( (A, B, C) \) is well-posed on \( X, \mathbb{C}^n \) and \( \mathbb{C}^n \).

**Proof.** Let \( u \in W_{0,\text{loc}}^1([0, \infty); \mathbb{C}^n) \). According to (16), we define
\[ (\mathbb{F}u)(t) := M\Phi_t u, \quad t \geq 0. \]
Due to (19), the expression of \( \mathbb{F} \) is given by
\[ (\mathbb{F}u)(t) = \mathcal{I}^+ \text{diag} \left( e^{\zeta_{j}(0,1)} \left( ([I_w]' u(t - \tau_j(0,1)) \chi_{[\tau_j(0,1), \infty)}(t) \right)_{j=1}^{m} \right). \]  

Proposition 2. The triple \((A, B, C)\) is regular on \(X, \mathbb{C}^n\) and \(\mathbb{C}^n\).

Proof. It follows from (29) that \((Fv)(t) = 0\) if \(t \leq \max_{1 \leq j \leq m} \tau_j(0, 1)\). So, for any \(v \in \mathbb{C}^n\),

\[
\lim_{\tau \to 0} \frac{1}{\tau} \int_0^{\tau} (F(X_{\tau, \lambda}) \cdot v)(\sigma) d\sigma = 0.
\]

Thus, by Definition 2.4 the triple \((A, B, C)\) is regular on \(X, \mathbb{C}^n\) and \(\mathbb{C}^n\).

3.2. Wellposedness. Consider the following Cauchy problem

\[
\begin{aligned}
\dot{z}(t) &= Az(t), \quad t \geq 0, \\
       z(0) &= z_0
\end{aligned}
\]

(30)

associated to the network system (18), where

\[
Az := A_m z
\]

(31)

with

\[
z \in D(A) := \{ f \in D(A_m) : Gf = Mf \}
\]

\[
= \{ f \in W^{1,p}([0,1], \mathbb{C}^m) : f(1) = Bf(0) \}.
\]

Here, we use that \(Gf = Mf\) is equivalent to \(f(1) = Bf(0)\) for \(f \in W^{1,p}([0,1], \mathbb{C}^m)\), cf. [2, Proposition 2.1].

Let us recall that the Cauchy problem is called well-posed if and only if the operator \((A, D(A))\) generates a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on \(X\), cf. [12, Theorem 6.7, Chap. II].

We shall use the notation of Subsection 3.1. Then, \(B \in \mathcal{L}(\mathbb{C}^n, X_{-1})\) and \(C = Mi \in \mathcal{L}(D(A), \mathbb{C}^n)\) are the control and observation operators as defined in (24) and (26) respectively, and \(D_\mu\) for \(\mu \in \rho(A)\) is the Dirichlet operator defined in (22).

The following theorem shows that the Cauchy problem (30) associated with the network system (18) is well-posed.

Theorem 3.6. The operator \((A, D(A))\) generates a \(C_0\)-semigroup \(T := (T(t))_{t \geq 0}\) on \(X\) satisfying \(T(t)z \in D(C_\Lambda)\) for all \(z \in X\) and almost every \(t \geq 0\). In addition, \(T(t)z_0 = T(t)z_0 + \int_0^t T_{-1}(t-s)BC_\Lambda T(s)z_0\) for all \(z_0 \in X\) and \(t \geq 0\).

Proof. It is shown in Subsection 3.1 that the triple \((A, B, C)\) is regular on \(X, \mathbb{C}^n\) and \(\mathbb{C}^n\), see Proposition 2. According to Theorem 2.6 it suffices to prove that the identity matrix \(I : \mathbb{C}^n \to \mathbb{C}^n\) is an admissible feedback operator. To this purpose we define the operators \(F_t u = Fu\) on each interval \([0, t]\). So using (29) we deduce that \(F_t u = 0, \quad \text{for } t \leq \max_{1 \leq j \leq m} \tau_j(0, 1)\).

So, it is obvious that \(I - F_t\) has an inverse in \(\mathcal{L}(L^p([0, \max_{1 \leq j \leq m} \tau_j(0, 1)]; \mathbb{C}^n))\).
Furthermore, by using Theorem 2.6 one can compute explicitly the spectrum of $A$.

**Corollary 1.** For $\mu \in \mathbb{C}$, we have

$$\mu \in \rho(A) \iff 1 \in \rho(D_\mu M) \iff 1 \in \rho(\mathcal{A}_\mu),$$

where

$$\mathcal{A}_\mu := MD_\mu = I^+ \Xi_\mu(0)(I_w^-)^T$$

having entries

$$(\mathcal{A}_\mu)_{ip} = \begin{cases} w_{pj} e^{\xi_i(0,1)-\mu \tau_j(0,1)}, & \text{if } v_i = e_j(0) \text{ and } v_p = e_j(1), \\ 0, & \text{otherwise}. \end{cases}$$

Moreover, for $\mu \in \rho(A)$, we have

$$R(\mu, A) = (I - D_\mu M)^{-1} R(\mu, A) = (I + D_\mu (I_{\mathbb{C}n} - \mathcal{A}_\mu)^{-1} M) R(\mu, A).$$

4. **Approximate controllability for flows in networks.** The present section provides a complete description of the dynamics of the network system (4). So, we analyse the problem of the controllability of the following perturbed boundary control problem

$$\begin{cases} \dot{z}(t) = A_m z(t), & t \geq 0, \\ z(0) = z_0, \\ Gz(t) = Mz(t) + Ku(t), & t \geq 0 \end{cases}$$

(34)

associated with the network system (4). Here $A_m, G$ and $M$ are defined in (21) , (19), respectively, and $K = (k_{il})_{n \times n_0} \in \mathbb{C}^{n \times n_0}$ is a boundary control operator acting on vertices. More precisely, we give necessary and sufficient conditions on the maximum amount of material that can be transported from a starting point (the source) to a terminating point (the sink).

We start by the following observation on the well-posedness of the perturbed boundary control problem (34). Consideration similar to [16, Theorem 4.3] shows that the non-homogeneous Cauchy problem version of the original problem (4) is given by

$$\begin{cases} \dot{z}(t) = A_{-1} z(t) + BKu(t), & t \geq 0, \\ z(0) = z_0 \in X. \end{cases}$$

(35)

Furthermore, [16, Theorem 4.3] implies that the perturbed boundary control problem (34), for the initial condition $z_0 \in X$, has a unique strong solution $z(\cdot)$ satisfying $z(t) \in D(C_A)$ for a.e. $t \geq 0$ and

$$z(t) = T(t) z_0 + \int_0^t T(t-s) BKu(s) ds, \quad t \geq 0,$$

**Remark 4.** An immediate consequence of the above considerations, according to [16, Theorem 4.3], is that the boundary control operator $K$ can be interpreted as suitable input associated with the initial boundary value problem (18), which takes into account that the redistribution of mass on the edges satisfies the ratio prescribed by the condition $f(1) \in \text{Rg } (I_w^-)^T$.

Accordingly, we have the following definition.
Definition 4.1. Define the reachability space
\[ \mathcal{R} := \bigcup_{t \geq 0} \left\{ \int_{0}^{t} T_{-1}(t-s)BKu(s)ds : u \in L^p([0,t]; \mathbb{C}^n) \right\} \]
associated with the open-loop system \((A, BK)\). The perturbed boundary control problem (34) is said to be approximately controllable if \(\mathcal{R}\) is dense in \(X\).

The following result gives a necessary and sufficient condition ensuring that (34) is approximately controllable.

Proposition 3. The perturbed boundary control problem (34) is approximately controllable if and only if, for \(\mu \in \rho(A)\) and \(\varphi \in X'\),
\[ \langle (I + D_\mu(I_{\mathbb{C}^n} - K_\mu)^{-1}M)D_\mu Kv, \varphi \rangle = 0, \quad \forall v \in \mathbb{C}^n \Rightarrow \varphi = 0. \] (35)
Here \(X'\) denotes the dual space of \(X\).

Proof. According to Definition 4.1, the perturbed boundary control problem (34) is approximately controllable if and only if
\[ \text{Cl} \left( \bigcup_{t \geq 0} \left\{ \int_{0}^{t} T_{-1}(t-s)BKu(s)ds : u \in L^p([0,t]; \mathbb{C}^n) \right\} \right) = X, \]
where \(\text{Cl}(E)\) means the closure of the set \(E\). From the Hahn-Banach theorem we have that the above assertion is equivalent to the fact that, for \(t \geq 0\) and any \(u \in L^p([0,t]; \mathbb{C}^n)\),
\[ \left\langle \int_{0}^{t} T_{-1}(t-s)BKu(s)ds, \varphi \right\rangle = 0 \] (36)
implies \(\varphi = 0\), for \(\varphi \in X'\).

Taking the Laplace transform in (36), we obtain
\[ \langle R(\mu, A_{-1})BK\hat{u}(\mu), \varphi \rangle = 0 \] (37)
for \(\mu \in \rho(A)\). Here, \(\hat{u}(\mu)\) denotes the Laplace transform of \(u \in L^p([0,t]; \mathbb{C}^n)\). Conversely, by the uniqueness of the Laplace transform one can see that (37) implies (36). On the other hand, it follows from (33) that
\[ R(\mu, A_{-1})z = (I - D_\mu M)^{-1}R(\mu, A_{-1})z \]
for \(\mu \in \rho(A)\) and \(z \in X_{-1}\), the extrapolation space associated with \(X\) and \(A\). So,
\[ R(\mu, A_{-1})BK\hat{u}(\mu) = (I - D_\mu M)^{-1}R(\mu, A_{-1})BK\hat{u}(\mu) = (I + D_\mu(I_{\mathbb{C}^n} - K_\mu)^{-1}M)D_\mu K\hat{u}(\mu). \]
This completes the proof. \(\square\)

As a consequence we obtain the following characterization.

Corollary 2. The perturbed boundary control problem (34) is approximately controllable if and only if, for \(\mu \in \rho(A)\), we have
\[ \text{Rang} \left( \Xi_\mu(I_{\mathbb{C}^n} - K_\mu)^{-1}K \right) \] (38)
is dense in \(X\).
Proof. For $\mu \in \rho(A)$ we have

\[
(I + D_\mu(I_{C^n} - M D_\mu)^{-1} M)D_\mu K = D_\mu K + D_\mu(D_\mu(I_{C^n} - A_\mu)^{-1} A_\mu K)
\]
\[
= D_\mu(I_{C^n} - A_\mu)^{-1} K
\]
\[
= \Xi_\mu^T(I_{C^n} - A_\mu)^{-1} K.
\]

Thus, the implication (35) can be rewritten as

\[
\langle \Xi_\mu^T(I_{C^n} - A_\mu)^{-1} K, \varphi \rangle = 0, \quad \forall \varphi \in C^n,
\]

implies $\varphi = 0$. Thus the assertion follows by the Hahn-Banach theorem. \hfill $\square$

Furthermore, when only the absorption takes place along the edges during the process, the characterization (38) is given by a Kalman’s controllability rank condition.

**Theorem 4.2.** Assume that $q_j \leq 0$ for all $j \in \{1, \ldots, m\}$. Then, for $\Re \mu > 0$, the condition (38) is equivalent to

\[
\text{Rank}(K \ A_\mu K \ldots A_{l-1}^K) = n
\]

for $l := \min\{n, m\}$. \hfill (39)

**Proof.** Observe that if $q_j \leq 0$ for all $j \in \{1, \ldots, m\}$, by (32) we obtain

\[
\|A_\mu\|_1 < \|A_\mu\|_1 \leq 1
\]

for $\Re \mu > 0$. In particular, this implies

\[
\{\Re \mu > 0\} \subset \rho(A).
\]

Thus, by the Neumann series, we have

\[
\Xi_\mu^T(I_{C^n} - A_\mu)^{-1} K = \Xi_\mu^T \sum_{k=0}^{l-1} A_\mu^k K.
\]

Since the powers $A_\mu^k$ for $k \geq \min\{m, n\} := l$ are linear combinations of the lower powers of $A_\mu$ (this is a consequence of the well known Cayley-Hamilton theorem), it follows that the perturbed boundary control problem (34) is approximately controllable if and only if

\[
\text{Rang}(\Xi_\mu^T A_\mu^k K), \quad k = 0, \ldots, l-1,
\]

is dense in $X$ or, equivalently according to (23), if

\[
\text{Cl}\left(\text{Rang}\left(\max_{1 \leq j \leq m} e^{\xi_j(-1)-\mu \tau_j(-1)}(I_{C^n} - A_\mu^k K)\right)\right) = X, \quad k = 0, \ldots, l-1.
\]

(40)

On the other hand, it is known that

\[
X = L^p([0, 1]; C^m) = \text{Cl}(L^p([0, 1]; C) \otimes C^m).
\]

Since by the Stone-Weierstrass theorem the span of $\max_{1 \leq j \leq m} \{e^{\xi_j(-1)-\mu \tau_j(-1)} : \Re \mu > 0\}$ is dense in $C([0, 1]; C)$ (and hence also in $L^p([0, 1]; C)$), (40) implies

\[
\text{Rank}(I_{C^n}^T K \ldots I_{C^n}^T A_\mu K \ldots I_{C^n}^T A_{l-1}^K K) = m.
\]

Moreover, by (2), $(I_{C^n})^T$ is injective hence left invertible and this yields (39).

Condition (39) is also sufficient and this can be proved in a similar way. \hfill $\square$
Now, we assume that the flow velocities on the edges are all constant and equal and there is no absorption (i.e., \( c_j \equiv c \) and \( q_j \equiv 0 \)). In this case, following the argument of Theorem 4.2, one can find Kalman controllability rank conditions expressed only in terms of graph matrices.

**Corollary 3.** Assume that \( c_j \equiv c \) and \( q_j \equiv 0 \) for all \( j = 1, \ldots, m \), where \( c \) is a positive constant. Then, for \( \Re \mu > 0 \), the following assertions are equivalent:

(i) the perturbed boundary value problem (34) is approximately controllable,

(ii) \( \text{Rank} \left( K \ A K \ldots A^{l-1} K \right) = n \),

(iii) \( \text{Rank} \left( (I_w^T)^T K \ B(I_w^T)^T K \ldots B^{l-1}(I_w^T)^T K \right) = m \).

Here \( A \) and \( B \) are the weighted transposed adjacency and the weighted transposed adjacency matrices for graph and line graph, respectively.

**Proof.** (ii) \( \Leftrightarrow \) (iii) Assume that \( c_j \equiv c \) and \( q_j \equiv 0 \) for all \( j = 1, \ldots, m \). In this case, we have

\[
A_{\mu} = e^{-\frac{\mu}{c}} A.
\]

Then

\[
\Xi_{\mu}(I_w^T)^T (I_{C^n} - A_{\mu})^{-1} K = e^{-\frac{\mu}{c}} (I_w^T)^T (I_{C^n} - e^{-\frac{\mu}{c}} A)^{-1} K.
\]

Therefore, using the fact that the range of \( s \mapsto e^{-\frac{s}{c}} \) is dense in \( L^p([0,1], \mathbb{C}) \) and according to Theorem 4.2, one can see that the condition (39) is equivalent to

\[
\text{Rank} \left( (I_w^T)^T K \ I_w^T A \ldots A^{l-1} K \right) = \text{Rank} \left( K \ A K \ldots A^{l-1} K \right) = n.
\]

(ii) \( \Leftrightarrow \) (iii) is a consequence of (3).

The result of Corollary 3 characterizes the approximate controllability of the perturbed boundary control problem (34) in terms of finite dimensional Kalman-type conditions. Unfortunately, these characterizations dependent on the specific numerical values of the weight of each edge, which for most real networks are either unknown or are known only approximately and are time dependent [20].

To deal with general networks we need to recall the notions of structure and structural controllability from [23].

**Definition 4.3.** A structural matrix \( A \in \mathbb{R}^{n \times n} \) is a matrix having fixed zeros in some locations and arbitrary, independent entries in the remaining locations. A structured system \((A, K)\), for \( K \in \mathbb{R}^{n \times n_0} \), is an ordered pair of structured matrices.

The two systems \((A, K)\) and \((A', K')\) are structurally equivalent if there is a one-to-one correspondence between the locations of their fixed zero and nonzero entries.

A system \((A, K)\) is called structurally controllable if there exists a system structurally equivalent to \((A, K)\) which is controllable in the usual sense, i.e., fulfilled Kalman’s controllability rank condition.

**Example 1.** Each of the following matrices is structural

\[
Q_0 := \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
x & x & x & x \\
x & x & x & x \\
x & x & x & x
\end{pmatrix}, \quad Q_1 := \begin{pmatrix}
x & 0 & 0 & 0 \\
x & 0 & 0 & 0 \\
x & x & x & x \\
x & x & x & x \\
x & x & x & x
\end{pmatrix}.
\]
Remark 5. It has to be noted that the concept of structural controllability was first introduced using graph-theoretic arguments by Lin in [19].

A structurally controllable system is controllable for almost all parameter values except for those in some proper algebraic variety in the parameter space. Thus structural controllability is a generic property of the system, since a proper algebraic variety of the system has Lebesgue measure zero [23, Proposition 3.1], [5, Section 3.9].

We now introduce the basic technical condition for studying structural controllability.

Definition 4.4. An \((n \times s)\) matrix \(A\) \((s \geq n)\) is said to be of form \((t)\) for some \(1 \leq t \leq n\) if, for some \(k\) in the range \(s - t < k \leq s\), \(A\) contains a zero submatrix of order \((n + s - t - k + 1) \times k\).

Example 2. Both matrices from (41) are of form (4), but \(k = 5\) for the matrix \(Q_0\) while \(k = 4\) for the matrix \(Q_1\).

Using the connection between structural controllability and form \((t)\), see [23, Theorem 4.1], we can write the result of Corollary 3 in a more proper form.

Theorem 4.5. Assume that \(c_j \equiv c\) and \(q_j \equiv 0\) for all \(j \in \{1, \ldots, m\}\). Then the perturbed boundary control problem (34) is approximately controllable if and only if the following matrix

\[
\mathcal{C} := \begin{pmatrix}
K & I & 0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & -\mathcal{A} & K & I & 0 & \cdots & \cdots & 0 \\
0 & 0 & 0 & -\mathcal{A} & K & \cdots & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\
0 & 0 & 0 & \cdots & -\mathcal{A} & K & I & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -\mathcal{A} & K
\end{pmatrix}
\]

is not of form \((n^2)\).

Proof. The result follows from Corollary 3 and [5, Theorem 3.17].

Remark 6. The advantage of this approach that consists in deducing the approximate controllability of the network system (4) from the structural controllability of the finite dimensional system \((\mathcal{A}, K)\) is that:

- It is dependent only on the internal connections between components of the network, and not on the specific numerical values of the weight of each edge.
- Computationally, the determination of the form of the extended controllability matrix \(\mathcal{C}\) requires only that the computer is able to distinguish between zeros and non zeros.

Acknowledgements. We are grateful to the reviewers for their valuable comments and suggestions which helped to considerably improve the quality of the manuscript.

REFERENCES

[1] R. K. Ahuja, T. L. Magnanti and J. B. Orlin, Network Flows: Theory, Algorithms and Applications, Prentice Hall, Inc., Englewood Cliffs, NJ, 1993.
[2] F. Bayazit, B. Dorn and A. Rhandi, Flows in networks with delay in the vertices, Math. Nachr., 285 (2012), 1603–1615.
[3] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D. U. Hwang, Complex networks: Structure and dynamics, Phys. Rep., 424 (2006), 175–308.
B. Bollobás, Modern Graph Theory, Springer-Verlag, New York, 1998.

J. Casti, Linear Dynamical Systems, Academic Press, Orlando, Florida, 1987.

B. Dorn, Flows in Infinite Networks - A Semigroup Approach, Ph.D thesis, Tuebingen University, Germany, 2008.

B. Dorn, M. Kramar Fijavž, R. Nagel and A. Radl, The semigroup approach to transport processes in networks, Phys. D, 239 (2010), 1416–1421.

M. El Azzouzi, H. Bouslous, L. Maniar and S. Boulite, Constrained approximate controllability of boundary control systems, IMA J. Math. Control Inform., 33 (2016), 669–683.

K.-J. Engel and M. Kramar Fijavž, Exact and positive controllability of boundary control systems, Netw. Heterog. Media, 12 (2017), 319–337.

K.-J. Engel, M. Kramar Fijavž, R. Nagel and E. Sikolya, Vertex control of flows in networks, Netw. Heterog. Media, 3 (2008), 709–722.

K.-J. Engel, B. Klöss, M. Kramar Fijavž, R. Nagel and E. Sikolya, Maximal controllability for boundary control problems, Appl. Math. Optim., 62 (2010), 205–227.

K.-J. Engel and R. Nagel, One-parameter Semigroups for Linear Evolution Equations, Springer-Verlag, New York, 2000.

H. O. Fattorini, Boundary control systems, SIAM J. Control, 6 (1968), 349–385.

G. Greiner, Perturbing the boundary conditions of a generator, Houston J. Math., 13 (1987), 213–229.

S. Hadd, An Evolution Equation Approach, Ph.D thesis, Cadi Ayyad University, Marrakech, 2005.

S. Hadd, R. Manzo and A. Rhandi, Unbounded perturbations of the generator domain, Discrete Contin. Dyn. Syst., 35 (2015), 703–723.

U. Knauer, Algebraic Graph Theory. Morphisms, Monoids and Matrices, De Gruyter Studies in Mathematics, Vol. 41, Walter de Gruyter & Co., Berlin, 2011.

M. Kramar and E. Sikolya, Spectral properties and asymptotic periodicity of flows in networks, Math. Z., 249 (2005), 139–162.

C.-T. Lin, Structural controllability, IEEE Trans. Automatic Control, AC-19 (1974), 201–208.

Y.-Y. Liu, J.-J. Slotine and A.-L. Barabasi, Controllability of complex networks, Nature, 473 (2011), 167–173.

T. Matrai and E. Sikolya, Asymptotic behavior of flows in networks, Forum Math., 19 (2007), 429–461.

D. Salamon, Infinite-dimensional linear system with unbounded control and observation: A functional analytic approach, Trans. Amer. Math. Soc., 300 (1987), 383–431.

R. W. Shields and J. B. Pearson, Structural controllability of multi-input linear systems, IEEE Trans. Automat. Control, AC-21 (1976), 203–212.

E. Sikolya, Semigroups for Flows in Networks, Ph.D thesis, Tuebingen University, Germany, 2004.

O. J. Staffans, Well-posed Linear Systems, Cambridge Univ. Press, Cambridge, 2005.

M. Tucsnak and G. Weiss, Observation and Control for Operator Semigroups, Birkhäuser Verlag, Basel, 2009.

G. Weiss, Admissible observation operators for linear semigroups, Israel J. Math., 65 (1989), 17–43.

G. Weiss, Admissibility of unbounded control operators, SIAM J. Control Optim., 27 (1989), 527–545.

G. Weiss, Transfer functions of regular linear systems. Part I: Characterizations of regularity, Trans. Amer. Math. Soc., 342 (1994), 827–854.

G. Weiss, Regular linear systems with feedback, Math. Control Signals Systems, 7 (1994), 23–57.

Received October 2019; 1st revision April 2020; 2nd revision July 2020.

E-mail address: elgantouhyassine@gmail.com
E-mail address: s.hadd@ui.ac.ma
E-mail address: arhandi@unisa.it