BEURLING’S THEOREM AND CHARACTERIZATION OF HEAT KERNEL FOR RIEmannian SYMMETRIC SPACES OF NONCOMPACT TYPE

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Abstract. We prove Beurling’s theorem for rank 1 Riemannian symmetric spaces and relate it to the characterization of the heat kernel of the symmetric space.

1. Introduction

The uncertainty principle in harmonic analysis reflects the inevitable trade-off between the function and its Fourier transform as it says that both of them cannot decay very rapidly. This principle has several quantitative versions which were proved by Hardy, Morgan, Gelfand-Shilov, Cowling-Price etc. (see [9], [7], [22] and the references there in). In more recent times Hörmander (see [12]) proved the following theorem which is the strongest theorem in this genre in the sense that it implies the theorems of Hardy, Morgan, Gelfand-Shilov and Cowling-Price.

Theorem 1.1. (Hörmander 1991) Let \( f \in L^1(\mathbb{R}) \). Then
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)||\hat{f}(y)| |x|^\frac{1}{2} |y|^\frac{1}{2} \, dx \, dy < \infty
\]
implies \( f = 0 \) almost everywhere.

Hörmander attributes this theorem to A. Beurling.

As is well-known in physics, the uncertainty in the momentum is smallest, for a given uncertainty in the position, if the wave function is the Gaussian \( e^{-\frac{|x|^2}{4t}} \). In harmonic analysis this means that the trade-off is optimal when the function is Gaussian. The quantitative versions of the uncertainty principle also accommodate this optimal situation. The above theorem of Hörmander was further generalized in [3] which takes care of this aspect of uncertainty:

Theorem 1.2. (Bonami, Demange, Jaming 2003) Let \( f \in L^2(\mathbb{R}) \) and \( N \geq 0 \). Then
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)||\hat{f}(y)| (1 + |x| + |y|)^N e^{\frac{|x|^2}{2t}} \, dx \, dy < \infty
\]
implies \( f(x) = P(x)e^{-tx^2} \) where \( t > 0 \) and \( P \) is a polynomial with \( \deg P < \frac{N-1}{2} \).

We will refer to theorem [1,2] simply as Beurling’s theorem for the sake of brevity.

The aim of this article is to prove the analogue of theorem [1,2] for Riemannian symmetric spaces \( X \) of the noncompact type which have rank 1. We recall that such a space is of the form \( G/K \) where \( G \) is a noncompact connected semisimple Lie group of real rank 1 with finite centre and \( K \subset G \) is a maximal compact subgroup.

The precise statement of the theorem and its proof appear in section 3. In section 4 we have showed that the estimate considered in the main theorem is the sharpest possible. In section 5 we have indicated how the theorems of Hardy, Morgan, Gelfand-Shilov, Cowling-Price etc on symmetric spaces follow from

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our Beurling’s theorem. The mutual dependencies of these uncertainty theorems can be schematically displayed as follows:

Beurling’s \( \Rightarrow \) Gelfand-Shilov \( \Rightarrow \) Cowling-Price

\[ \downarrow \]

Morgan’s \( \Rightarrow \) Hardy’s

This shows that Beurling’s theorem is the Master theorem. Some of the latter theorems (which follow from Beurling’s) were proved independently on symmetric spaces in recent years by many authors (see [20, 4, 6, 16, 17, 19, 22, 18] etc.).

After completing this work we had the opportunity to see Demange’s thesis (\[15\]) in which he further generalized theorem 1.2 (see theorem 6.1). In section 6 we have given the appropriate analogue of Demange’s theorem on symmetric spaces.

2. Notation and Preliminaries

The pair \((G, K)\) is as described in the introduction. We let \(G = KAN\) denote a fixed Iwasawa decomposition of \(G\). Let \(\mathfrak{g}, \mathfrak{k}, \mathfrak{a}\) and \(\mathfrak{n}\) denote the Lie algebras of \(G, K, A\) and \(N\) respectively. We recall that dimension of \(\mathfrak{a} = 1\). We choose and keep fixed throughout a system of positive restricted roots, which we denote by \(\Sigma^+\). Let \(\gamma \in \Sigma^+\) denote the unique simple root and let \(H_\gamma \in \mathfrak{a}\) be the dual basis of \(\mathfrak{a}\). Using \(\gamma\) (respectively \(H_\gamma\)) we can identify \(\mathfrak{a}^*\) (respectively \(\mathfrak{a}\)) with \(\mathbb{R}\). The complexification \(\mathfrak{a}^*_C\) of \(\mathfrak{a}^*\) can then be identified with \(\mathbb{C}\). Under this correspondence the half-sum of the elements of \(\Sigma^+\), denoted by \(\rho\) corresponds to the real number \(\frac{1}{2}(m_\gamma + 2m_{2\gamma})\) where \(m_\gamma\) (respectively \(m_{2\gamma}\)) is the multiplicity of the root \(\gamma\) (respectively \(2\gamma\)). We will frequently identify \(\rho\) with this positive real number without further comment. Furthermore, the positive Weyl chamber \(\mathfrak{a}_+ \subset \mathfrak{a}\) (respectively \(\mathfrak{a}^*_+ \subset \mathfrak{a}^*\)) gets identified under this correspondence with the set of positive real numbers. We let \(\exp H_\rho = a_t \in A\) for \(t \in \mathbb{R}\). This identifies \(A\) with \(\mathbb{R}\). Let \(H : G \to \mathfrak{a}\) be the Iwasawa projection associated to the Iwasawa decomposition, \(G = KAN\). Then \(H\) is left \(K\)-invariant and right \(MN\)-invariant where \(M\) is the centraliser of \(A\) in \(K\). For \(\lambda \in \mathfrak{a}^*\) (respectively \(H \in \mathfrak{a}\)) we denote by \(\lambda^+\) (respectively \(H^+\)) the unique Weyl translate of \(\lambda\) (respectively \(H\)) that belongs to the closure of the positive Weyl chamber \(\mathfrak{a}^*_+\) (respectively \(\mathfrak{a}_+\)). We have \(\lambda^+ (H^+) = |\lambda (H)|\) where \(|r|\) denotes the modulus of the real number \(r\).

Note that the Weyl group is isomorphic to \(\mathbb{Z}_2\). The unique nontrivial element of the Weyl group takes an element \(\lambda \in \mathfrak{a}^* \equiv \mathbb{R}\) (respectively \(H \in \mathfrak{a}\)) to \(-\lambda\) (respectively \(-H\)). Therefore \(\lambda^+\) (respectively \(H^+\)) corresponds to \(|\lambda|\) (respectively \(|H|\)) under the above identification of \(\mathfrak{a}^*\) (respectively \(\mathfrak{a}\)) and \(\mathbb{R}\).

We have the \(a\)-valued inner product of Helgason \(A(x, k)\) on \(X \times K\) defined by \(A(x, k) = -H(x^{-1}k), x \in X, k \in K\). Note that \(A\) descends to a function, also denoted by \(A : X \times K/M \to \mathfrak{a}\), since \(H\) is right \(M\)-invariant.

We fix a left \(G\)-invariant measure \(dx\) on \(X\) and a left invariant Haar measure \(dg\) on \(G\) such that for a nice function \(f\) on \(X\), \(\int_X f(x) dx = \int_G f(g) dg\). Here in the right hand side \(f\) is considered as a right \(K\)-invariant function on \(G\). While dealing with functions on \(X\), we may slur over the difference between the two measures and denote both by \(dx\). Let \(dn\) (respectively \(da\)) be a fixed Haar measure on \(N\) (respectively on \(A\)). These are normalised so that under the Iwasawa decomposition \(G = KAN\) we have \(dg = e^{2\rho (\log a)}dkdadn\) where \(dk\) is the normalised Haar measure of \(K\) and \(\log a\) is the unique element in \(a\) such that \(\exp (\log a) = a\).

We follow the practice of using \(C, C'\) etc. to denote constants whose values are not necessarily the same at each occurrence.

**Definition 2.1.** For suitable functions \(f\) on \(X\), the Helgason Fourier transform \(\hat{f}\) of \(f\) is defined by

\[
\hat{f}(\lambda, k) = \int_X e^{(-i\lambda + \rho)(A(x, k))} f(x) dx; \lambda \in \mathfrak{a}^*, k \in K.
\]

Note that \(\hat{f}\) descends to a function on \(\mathfrak{a}^* \times K/M\). By abuse of notation we will continue to denote this function by \(\hat{f}\). For \(f \in L^1(X)\), there exists a subset \(B\) of \(K\) of full Haar measure, such that \(\hat{f}(\lambda, k)\)
exists for all $k \in B$ and $\lambda \in \mathbb{C}$ with $|\Im \lambda| \leq \rho$. Indeed for each fixed $k \in B$, $\lambda \mapsto \tilde{f}(\lambda, k)$ is holomorphic in the strip $\{ \lambda \in \mathbb{C} \mid |\Im \lambda| < \rho \}$ and continuous on its boundary (see [14]).

**Definition 2.2.** For suitable functions $f$ on $X$, the Radon transform $Rf$ of $f$ is defined by

$$Rf(k, a) = e^{\rho(\log a)} \int_{N} f(kan) \, dn; k \in K, a \in A.$$ 

$Rf$ descends to a function on $K/M \times A$ and (as in the case of $\tilde{f}$) we continue to denote this function by $Rf$. We will use the notation $Rf(k, t)$ for $Rf(k, a_t), t \in \mathbb{R}$.

For a suitable function $f$ on $X$, the basic relation between $Rf$ and $\tilde{f}$ is the following:

$$\tilde{f}(\lambda, k) = FRf(k, \lambda)(\lambda),$$

where $F$ denotes the Euclidean Fourier transform on $A \equiv \mathbb{R}$.

Let $\tilde{K}_0$ be the set of equivalence classes of irreducible unitary representations of $K$ which are class $1$ with respect to $M$, that is contains an $M$-fixed vector. Let $\delta \in \tilde{K}_0$ and let $f \in L^1(X)$ be $K$-finite of type $\delta$. Then we have $d(\delta) \tilde{\chi}_\delta * f = f$ where $d(\delta)$ $(\text{respectively } \chi_\delta)$ denotes the degree $(\text{respectively character})$ of $\delta$ and $(d(\delta) \tilde{\chi}_\delta * f)(x) = d(\delta) \int_{K} f(kx) \tilde{\chi}_\delta(k) \, dk$ for $x \in X$. In particular if $\delta$ is the trivial representation then $f$ is a $K$-invariant function on $X$. For a function $f$ of type $\delta$ we have $|f(x)| \leq C \int_{K} |f(kx)| \, dk$ where $C = d(\delta) \sup_{x \in K} |\chi_\delta(k)| \leq d(\delta)^2$. Let $g(x) = \int_{K} |f(kx)| \, dk$. Then $g \in L^1(X)$ and $g$ is $K$-invariant, that is $g \in L^1(G)$ and $g$ is $K$-biinvariant. We have $|f(x)| \leq d(\delta)^2 g(x)$.

**Definition 2.3.** For a $\delta$-type function $f$ in $L^1(X)$, the Abel transform $Af$ of $f$ is defined by

$$Af(a) = e^{\rho(\log a)} \int_{N} f(an) \, dn, a \in A.$$ 

It is well known that for a $K$-invariant function $g \in L^1(X)$, $Ag$ exists for almost every $a \in A$ and $Ag \in L^1(A)$. Now since

$$|Af(a)| \leq e^{\rho(\log a)} \int_{N} |f(an)| \, dn \leq d(\delta)^2 e^{\rho(\log a)} \int_{N} g(an) \, dn = d(\delta)^2 Ag(a)$$

for the $K$-invariant function $g$ constructed from $f$ as above, we conclude that $Af \in L^1(A)$. We will write $Af(t)$ for $Af(a_t), t \in \mathbb{R}$. It is also well known that for $f \in L^1(X), Rf \in L^1(K \times A, dkda)$. We include the proof here for the sake of completeness. For $f \in L^1(X)$, we construct the function $g(x) = \int_{K} |f(kx)| \, dk$. Then $g$ is a $K$-biinvariant function in $L^1(G)$ and hence as mentioned above (recall the identification of $A$ and $\mathbb{R}$) $Ag(t) \in L^1(\mathbb{R})$. But $Ag(t) = e^{\rho(t)} \int_{N} g(a_t n) \, dn = e^{\rho(t)} \int_{N} \int_{K} |f(kan)| \, dk \, dn = \int_{K} e^{\rho(t)} \int_{N} f(kan) \, dndk = \int_{K} Rf(k,t) \, dk.$ Since $\int_{A} Ag(t) \, dt < \infty$ we have $\int_{A} \int_{K} Rf(k,t) \, dtdk < \infty$. This proves our assertion.

For $\lambda \in a_\mathbb{C}^* \equiv \mathbb{C}$, we denote by $\phi_\lambda$ the elementary spherical function with parameter $\lambda$. We have for all $x \in X$, $\phi_\lambda(x) = \int_{K} e^{i\lambda + \rho(A(x,k))} \, dk$ (see [10] p. 418). We will often regard $\phi_\lambda$ as a $K$-biinvariant function on $G$.

For $x \in G$, we define $\sigma(x) = d(xK,K)$ where $d$ is the canonical distance function for $X = G/K$ coming from the Riemannian structure induced by the Cartan-Killing form restricted to $p$. Here $g = t \oplus p$ (Cartan decomposition) and $p$ can be identified with the tangent space at $eK$ of $G/K$.

The following estimates on the growth of $\phi_\lambda$ are well known ([11], [8] proposition 4.6.1 and theorems 4.6.4, 4.6.5): Let $\Xi$ denote the imaginary part of $\lambda \in \mathbb{C}$ and let $\Xi(x) = \phi_0(x)$. Then:

(a) $|\phi_\lambda(x)| \leq 1$ for $\lambda \in \mathbb{C}, |\Xi \lambda| \leq \rho$,
(b) $|\phi_\lambda(x)| \leq e^{\rho(\sigma(x))} \Xi(x)$; for all $\lambda \in \mathbb{C}$
(c) $\Xi(a) \leq C(1 + \sigma(a)) e^{-\rho(\log a)}$ for $a \in A^+$, $\exp \bar{a}_+.$

We denote the spherical Plancherel measure on $a^*$ by $d\mu(\lambda) = \mu(\lambda) \, d\lambda$, where $d\lambda$ is the Lebesgue measure. We have $\mu(\lambda) = |c(\lambda)|^{-2}$ where $c(\cdot)$ is Harish-Chandra’s $c$-function.

Recall that the elements $\delta \in K_0$ are parametrized by a pair of integers $(p_\delta, q_\delta)$ where $p_\delta \geq 0$ and $p_\delta \pm q_\delta \in 2\mathbb{Z}^+$ (see [14], [23]). The trivial representation in $\tilde{K}_0$ is parametrized by $(0,0)$ in this setup.
It is known that for each $\delta \in \widehat{K}_0$, the $M$-fixed vector is unique up to a scalar multiple (see [4]). Let $(\delta, V_0) \in \widehat{K}_0$. Suppose $\{v_i|i = 1, \ldots, d(\delta)\}$ is an orthonormal basis of $V_0$ of which $v_1$ is the $M$-fixed vector. Let $Y_{\delta,j}(k) = \langle v_j, \delta(k)v_1 \rangle$, $1 \leq j \leq d(\delta)$ and let $Y_\delta$ be the $K$-fixed vector. Note that $Y_{\delta,j}$ is right $M$-invariant that is it is a function on $K/M$. Recall that $L^2(K/M)$ is the carrier space of the spherical principal series representations $\pi_\lambda$, $\lambda \in \mathbb{C}$ in the compact picture and $\{Y_{\delta,j} : 1 \leq j \leq d(\delta), \delta \in \widehat{K}_0\}$ is an orthonormal basis for $L^2(K/M)$ adapted to the decomposition $L^2(K/M) = \bigoplus_{\delta \in \widehat{K}_0} V_\delta$. As the space $K/M$ can be identified with $S^{m_\lambda, +m_2}$, this decomposition can be viewed as the spherical harmonic decomposition and therefore $Y_{\delta,j}$’s can be considered as the spherical harmonics. The action of $\pi_\lambda$ is given by:

$$\pi_\lambda(x)g(k) = e^{i(\lambda + \rho)A(x, k)}g(\kappa(x^{-1}k)) \text{ for } x \in G, k \in K \text{ and } g \in L^2(K/M).$$

Here $\kappa(x)$ is the $K$-part of an element in $x \in G$ in the Iwasawa decomposition $G = KAN$. The representation $\pi_\lambda$ is unitary for $\lambda \in \mathbb{R}$. For $f \in L^1(X)$, $\delta \in \widehat{K}_0$ and $1 \leq j \leq d(\delta)$ we define,

$$f_{\delta,j}(x) = \int_K f(kx)Y_{\delta,j}(k)dk.$$ 

It can be verified that $f_{\delta,j}$ is a function of type $\delta$. The function $f$ can be decomposed as $f = \sum_{\delta \in \widehat{K}_0} \sum_{j=1}^{d(\delta)} f_{\delta,j}$. In fact when $f \in C^\infty(G)$ this is an absolutely convergent series in the $C^\infty$-topology. When $f \in L^p(G), p \in [1, \infty)$, the equality is in the sense of distributions. We have $\|f_{\delta,j}(x)\| \leq \|Y_{\delta,j}\|_\infty \int_K |f(kx)|dk \leq \int_K |f(kx)|dk$ as $\|Y_{\delta,j}\|_\infty = \sup_{k \in K} |Y_{\delta,j}(k)| \leq 1$.

For $\delta \in \widehat{K}_0$, $1 \leq j \leq d(\delta)$, $\lambda \in \mathfrak{a}_C^*$ and $x \in X$, we define

$$(2.2) \quad \Phi_{\lambda,\delta}^j(x) = \int_{K/M} e^{i(\lambda + \rho)(A(x, kM))}Y_{\delta,j}(kM)dk.$$ 

We have, $\Phi_{\lambda,\delta}^j(x) = \langle Y_{\delta,j}, \pi_\lambda(x)Y_0 \rangle$, that is $\Phi_{\lambda,\delta}^j$ is a matrix coefficient of the spherical principal series in the compact picture. It is well known (see [11]) that $\Phi_{\lambda,\delta}^j$’s are eigenfunctions of the Laplace-Beltrami operator $\Delta$ with eigenvalues $-(\lambda^2 + \rho^2)$. When $\delta = \delta_0$ is trivial then $Y_{\delta_0,j} = Y_{\delta_0,1} = Y_0$ and $\Phi_{\lambda,\delta_0}^1$ is obviously the elementary spherical function $\phi_\lambda(x)$. For $\lambda \in \mathfrak{a}_C^*$, $x = \kappa a_t K \in X$ and $1 \leq j \leq d(\delta)$, (see [11], p. 344)

$$(2.3) \quad \Phi_{\lambda,\delta}^j(x) = Y_{\delta,j}(kM)\Phi_{\lambda,\delta}^1(a_t).$$

$\Phi_{\lambda,\delta}^1$ is related with $\Phi_{-\lambda,\delta}^1$ by:

$$(2.4) \quad \Phi_{\lambda,\delta}^1 = \frac{Q_\delta(\lambda)}{Q_\delta(-\lambda)} \Phi_{-\lambda,\delta}^1,$$

where $Q_\delta$’s are Kostant polynomials (see [11], p. 344, Theorem). Indeed $Q_\delta$ is the polynomial factor of $\Phi_{\lambda,\delta}^1$ and hence of $\Phi_{\lambda,\delta}^j$ for $1 \leq j \leq d(\delta)$ (see [11], p.344). The Kostant polynomial $Q_\delta$ is given by

$$Q_\delta(\lambda) = \left(\frac{1}{2}(a + b + 1 + i\lambda)\right)^{2a+2b} \left(\frac{1}{2}(a - b + 1 + i\lambda)\right)^{2a-2b},$$

where $(z)_m = \Gamma(z+ \Gamma(z)/m)$, $a = \frac{m_\lambda + m_2}{2}$ and $b = \frac{m_2 - 1}{2}$. Thus deg $Q_\delta = p_\delta$.

Because of the relation (2.3) above we have $\Phi_{\lambda,\delta}^j = \frac{Q_\delta(\lambda)}{Q_\delta(-\lambda)} \Phi_{-\lambda,\delta}^j$.

We define the $j$-th $\delta$-spherical Fourier transform of $f$ by

$$\hat{f}(\lambda)_{\delta,j} = \int_X f(x)\Phi_{-\lambda,\delta}^j(x)dx$$

for $\lambda \in \mathfrak{a}^*$. It is clear that the $j$-th $\delta$-spherical Fourier transform of $f$ is the $(\delta,j)$-th matrix coefficient of the operator Fourier transform $\hat{f}(\lambda) = \int_G f(x)\pi_\lambda(x)dx$. It is not difficult to verify that

$$\hat{f}(\lambda)_{\delta,j} = \int_X f_{\delta,j}(x)\Phi_{-\lambda,\delta}^j(x)dx = \hat{f}_{\delta,j}(\lambda) \text{ say.} \quad \text{Furthermore } \int_X f_{\delta',j'}(x)\Phi_{-\lambda,\delta}^j(x)dx = 0,$$
when $\delta \neq \delta'$ or $j \neq j'$. Henceforth we will not distinguish between $\hat{f}(\lambda)_{\delta,j}$ and $\hat{f}_{\delta,j}(\lambda)$. Let $f \in L^1(X) \cap L^2(X)$. Then for almost every $\lambda \in \mathfrak{a}^*$,

$$\|\hat{f}(\lambda)\|_2^2 = \sum_{\delta \in K_0} \sum_{1 \leq j \leq d(\delta)} |\hat{f}_{\delta,j}(\lambda)|^2,$$

where $\| \cdot \|_2$ is the Hilbert-Schmidt norm.

Also by (2.4)

$$\hat{f}_{\delta,j}(-\lambda) = \frac{Q_\delta(\lambda)}{Q_\delta(-\lambda)} \hat{f}_{\delta,j}(\lambda).$$

Note that (see [11] p. 348) for $\lambda \in \mathfrak{a}^*$, $Q_\delta(\lambda) = Q_\delta(-\lambda)$. Consequently $|\hat{f}_{\delta,j}(\lambda)| = |\hat{f}_{\delta,j}(-\lambda)|$.

The following is also easy to see:

$$\int_K \hat{f}(\lambda,k)Y_{\delta,j}(k)dk$$

$$= \int_X \int_K f(x)e^{-i\lambda + \rho}A(x,k)Y_{\delta,j}(k)dkdx \quad \text{(by Fubini's theorem)}$$

$$= \int_X f(x)\Phi_{-\lambda,\delta}(x)dx$$

$$= \hat{f}_{\delta,j}(\lambda).$$

Starting from the relation (2.4) and using (2.6) we have,

$$\int_K \mathcal{F}(\mathcal{R}(f)(k,\cdot))(\lambda)Y_{\delta,j}(k)dk = \hat{f}_{\delta,j}(\lambda).$$

Now the left hand side is (recall that $\mathfrak{a}^* \equiv \mathbb{R}$):

$$\int_K \int_R \mathcal{R}(f)(k,t)e^{-i\lambda t}dtY_{\delta,j}(k)dk$$

$$= \int_K \int_R e^{it} \int_N f(ka_{t}n)dn e^{-i\lambda t}dtY_{\delta,j}(k)dk$$

$$= \int_R e^{it} \int_N f_{\delta,j}(a_{t}n)dn e^{-i\lambda t}dt \quad \text{(by Fubini's theorem)}$$

$$= \int_R \mathcal{A}(f_{\delta,j})(t)e^{-i\lambda t}dt$$

$$= \mathcal{F}(\mathcal{A}(f_{\delta,j}))(\lambda).$$

Therefore

$$\mathcal{F}(\mathcal{A}(f_{\delta,j}))(\lambda) = \hat{f}_{\delta,j}(\lambda).$$

Note that from above it is also clear that:

$$\int_K \mathcal{R}(f)(k,t)Y_{\delta,j}(k)dk = \mathcal{A}(f_{\delta,j})(t)$$

and hence

$$|\mathcal{A}(f_{\delta,j})(t)| = \left|\int_K \mathcal{R}(f)(k,t)Y_{\delta,j}(k)dk\right| \leq \int_K |\mathcal{R}(f)(k,t)|dk \leq \int_K |\mathcal{R}(f)(k,t)|dk$$

since $\|Y_{\delta,j}\|_\infty \leq 1$.

We will conclude this section with a description of the heat-kernel of the symmetric space $X$. The heat kernel on $X$ is an appropriate analogue of the Gauss kernel $p_t$ on $\mathbb{R}^n$ where $p_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$, $t > 0$.

Let $\Delta$ be the Laplace-Beltrami operator of $X$. Then (see [21], Chapter V), $T_t = e^{t\Delta}$, $t > 0$ defines a semigroup (heat-diffusion semigroup) of operators such that for any $\phi \in C_0^\infty(X)$, $T_t\phi$ is a solution of $\Delta u = \frac{\partial u}{\partial t}$ and $T_t\phi \rightarrow \phi$ a.e. as $t \rightarrow 0$. For every $t > 0$, $T_t$ is an integral operator with kernel $h_t$, that
is for any $\phi \in C_c^\infty(X)$, $T_t \phi = \phi * h_t$. The $h_t, t > 0$ are $K$-biinvariant functions on $G$, $h(x, t) = h_t(x)$ as a function of the variables $t \in \mathbb{R}^+$ and $x \in G$ is in $C^\infty(G \times \mathbb{R}^+)$ and has the following properties:

i. $\{h_t : t > 0\}$ form a semigroup under convolution $\ast$. That is $h_t \ast h_s = h_{t+s}$ for $t, s > 0$.

ii. $h_t$ is a fundamental solution of $\Delta u = \frac{2}{d} h_t$.

iii. $h_t \in L^1(G) \cap L^\infty(G)$ for every $t > 0$.

iv. $\int_X h_t(x)dx = 1$ for every $t > 0$.

Thus we see that the heat kernel $h_t$ on $X$ retains all the nice properties of the Gauss kernel. It is well known that $h_t$ is given by (see e.g. [1]):

$$h_t(x) = \frac{1}{|W|} \int_{\mathbb{S}^1} e^{-t(x^2+\rho^2)} \phi_\lambda(x) \mu(\lambda)d\lambda. \tag{2.10}$$

That is, the spherical Fourier transform of $h_t$, $\hat{h}_t(\lambda) = e^{-t(x^2+\rho^2)}$. It has been proved in [1] (Theorem 3.1 (i)) that for any $t > 0$, there exists $C > 0$ depending only on $X$ such that

$$h_t(\exp H) \leq C t^{-\frac{d-\rho^2}{2}} e^{-\rho^2/2} (1 + |H|^2)^{\frac{d-\rho^2}{2}} \tag{2.11}$$

for $H \in \mathbb{S}^1$, where $d_X = m_\gamma + m_\sigma + 1 = \text{dim } X$.

3. STATEMENT AND PROOF OF THE THEOREM

**Theorem 3.1.** Let $f \in L^2(X)$ satisfy

$$\int_X \int_{\mathbb{S}^1} \frac{|f(x)||\hat{f}(\lambda)||2e^\sigma(x)|\lambda|\Xi(x)}{(1 + \sigma(x) + |\lambda|^d)dx}d\mu(\lambda) < \infty \tag{3.1}$$

for some nonnegative integer $d$. Then $f$ is a $K$-finite function of the form $f = \sum_{\delta \in \mathcal{F}} h_\delta$ where $\mathcal{F} = \{\delta \in \mathcal{K}_0 \mid \rho_\delta \leq \frac{d-\rho^2}{2}\}$ is a finite set of $K$-types, $h_\delta$ is a function of type $\delta$ having Fourier coefficients $\hat{h}_\delta(\lambda) = P^\prime_{\delta,j}(\lambda^2) Q_\delta(\lambda)e^{-\alpha \lambda^2}$ for $1 \leq j \leq d(\delta)$. Here $\alpha$ is a positive constant and $P^\prime_{\delta,j}$ a polynomial which depends on $\delta$ and $j$. Consequently $f$ is a derivative of the heat kernel $h_\alpha$.

In particular if $d \leq d_X$, then $f$ is a function of infinite measure everywhere.

**Proof.** We have divided the proof in several steps for the convenience of the readers. We will use Fubini’s theorem freely throughout the proof without explicitly mentioning it.

**Step 1:** In this step we will show that $f \in L^1(X)$.

Note that, since $f \in L^2(X)$, $f$ is a locally integrable function on $X$. Now there are two possible situations.

(a) $\hat{f}$ is supported on a set of infinite measure.

(b) $\hat{f}$ is supported on a set of finite measure.

In case (a) clearly $f \in L^1(X)$, since from (3.1) it follows that there exists $\lambda \in \mathbb{S}^1, |\lambda| > 2\rho$ such that $\int_X |f(x)|e^{|(\lambda-\rho)\sigma(x)}dx < \infty$.

In case (b), as $\hat{f} \not= 0$, there exists $\lambda_0 \not= 0$ such that $\hat{f}(\lambda_0) \not= 0$. Suppose $|\lambda_0| = r > 0$. Then from (3.1) we have $\int_X |f(x)|e^{|r\sigma(x)|\Xi(x)}dx < \infty$. As $f \in L^1_{loc}(X)$, for $0 < r' < r$, $\int_X |f(x)|e^{r'\sigma(x)}\Xi(x)dx < \infty$.

For $\delta \in \mathcal{K}_0$ and $1 \leq j \leq d(\delta)$, we consider the integral $\hat{f}_{\delta,j} = \int_X f(x)\Phi_{-\lambda,j}(x)dx$. Then

$$|\int_X f(x)\Phi_{-\lambda,j}(x)dx| \leq \int_X |f(x)||\Phi_{-\lambda,j}(x)|dx$$

$$\leq \int_X |f(x)|e^{(|\lambda|-r')\sigma(x)}\Xi(x)$$

$$\leq \int_X |f(x)|e^{r'\sigma(x)}\Xi(x)e^{(|\lambda|-r')\sigma(x)}dx.$$

This shows that $\hat{f}_{\delta,j}$ is analytic in the open strip $|\lambda| < r'$ in $\mathbb{S}^1$, which contradicts the assumption that $\hat{f}$ and hence $\hat{f}_{\delta,j}$ is supported on a set of finite measure. This completes step 1.
Step 2: In this step we will show that (3.1) is equivalent to the condition:

\[(3.2)\]

\[I = \int_G \int_{a^*} \frac{|f(x)||\widehat{f}_{\delta_j}(\lambda)|e^{\sigma(x)|\lambda|}e^{-pH(x)}}{(1 + \sigma(x) + |\lambda|)^d} dx d\mu(\lambda) < \infty\]

for every fixed \(\delta \in \hat{K}_0\) and \(1 \leq j \leq d(\delta)\).

We know that \(\|\widehat{f}(\lambda)\|^2 = \sum_{\delta \in \hat{K}_0} \sum_{j=1}^{d(\delta)} |\widehat{f}_{\delta_j}(\lambda)|^2\). Therefore we obtain the following from (3.1)

\[I_1 = \int_G \int_{a^*} \frac{|f(x)||\widehat{f}_{\delta_j}(\lambda)|e^{\sigma(x)|\lambda|} e^{-pH(x)} dx d\mu(\lambda)}{(1 + \sigma(x) + |\lambda|)^d} < \infty.\]

We have changed the integration over \(X\) to integration over \(G\) as all the terms of the integrand are right \(K\)-invariant. As \(f\) and \(\sigma\) are both right \(K\)-invariant, on replacing \(x\) by \(xk^{-1}\) in the integral \(I\) we get

\[I = I(k) = \int_G \int_{a^*} \frac{|f(x)||\widehat{f}_{\delta_j}(\lambda)|e^{\sigma(x)|\lambda|} e^{-pH(x)k^{-1}} dx d\mu(\lambda)}{(1 + \sigma(x) + |\lambda|)^d} \]

Therefore

\[I = \int_K I(k) dk = \int_K \int_G \int_{a^*} \frac{|f(x)||\widehat{f}_{\delta_j}(\lambda)|e^{\sigma(x)|\lambda|} e^{-pH(x)k^{-1}} dx d\mu(\lambda)}{(1 + \sigma(x) + |\lambda|)^d} dk.\]

Since \(\int_K e^{-pH(x)k^{-1}} dk = \Xi(x^{-1})\) and \(\Xi(x) = \Xi(x^{-1})\), we conclude that \(I = I_1 < \infty\).

Step 3: We will now show that:

\[(3.3)\]

\[\int_K \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{R}(|f|)(k,t) |\widehat{f}_{\delta_j}(\lambda)| e^{\lambda|t|} (1 + |t| + |\lambda|)^d dt d\mu(\lambda) < \infty.\]

Since the integrand is even in \(\lambda\) as pointed out before, this is equivalent to showing

\[(3.4)\]

\[\int_K \int_{\mathbb{R}} \int_{\mathbb{R}^+} \mathcal{R}(|f|)(k,t) |\widehat{f}_{\delta_j}(\lambda)| e^{\lambda|t|} (1 + |t| + |\lambda|)^d dt d\mu(\lambda) < \infty.\]

We will break the above integral into the following 3 parts and show that each part is finite. That is we will show:

(i)

\[\int_K \int_{\mathbb{R}} \int_{L} \mathcal{R}(|f|)(k,t) |\widehat{f}_{\delta_j}(\lambda)| e^{\lambda|t|} (1 + |t| + |\lambda|)^d dt d\mu(\lambda) dk < \infty\]

for \(L > 0\) such that \(L^2 + L > d\).

(ii)

\[\int_K \int_{|t| > M} \int_{L} \mathcal{R}(|f|)(k,t) |\widehat{f}_{\delta_j}(\lambda)| e^{\lambda|t|} (1 + |t| + |\lambda|)^d dt d\mu(\lambda) dk < \infty\]

for \(M = 2(L + 1 + \rho)\) and \(L\) as in (i).

(iii)

\[\int_K \int_{|t| \leq M} \int_{L} \mathcal{R}(|f|)(k,t) |\widehat{f}_{\delta_j}(\lambda)| e^{\lambda|t|} (1 + |t| + |\lambda|)^d dt d\mu(\lambda) dk < \infty\]

for \(M, L\) used in (i) and (ii).

If we show that (i), (ii) and (iii) are finite, they together will obviously imply (3.3) and hence (3.2).

Proof of (iii): As the domain of integration \([-M, M] \times [0, L]\) is compact and as \(\frac{|f_{\delta_j}(\lambda)| e^{\lambda|t|}}{(1 + |t| + |\lambda|)^d}\) is continuous in this domain, the integral is bounded by \(C \int_K \int_{-M}^{M} \mathcal{R}(|f|)(k,t) dt dk\). Recall that \(f \in L^1(G)\). Therefore,

\[\int_K \int_{\mathbb{R}} \mathcal{R}(|f|)(k,t) e^{\lambda t} dt dk = \int_K \int_{\mathbb{R}} e^{\lambda t} \int_{N} |f(ka_t n)| d\mu e^{\lambda t} dt dk = \int_K \int_{\mathbb{R}} \int_{N} |f(ka_t n)| e^{2\lambda t} dt d\mu dk d\lambda < \infty.\]
Hence \(\int_K \int_{\mathbb{R}}^M \mathcal{R}(|f|)(k,t)dtkd\mu < \infty\), since this is \(\leq 2e^{\rho M} \int_K \int_{\mathbb{R}}^M \mathcal{R}(|f|)(k,t)e^{\rho t}dt < \infty\).

**Proof of (i):** It is given that \(L + L^2 > d\). We will show that for any \(\lambda\) such that \(|\lambda| \geq L\),

\[
(3.5) \quad \frac{e^{\lambda \sigma(a_n)}}{(1 + |\lambda| + \sigma(a_t))} \geq \frac{e^{\lambda \sigma(a_t)}}{(1 + |\lambda| + \sigma(a_t))}.
\]

Let \(F(x) = \frac{e^{\alpha x}}{(1 + \alpha + x)^d}\) for \(\alpha > 0\) and \(\alpha + \alpha^2 > d\). Then \(F'(x) > 0\) for any \(x \geq 0\). If \(x \geq y \geq 0\), then

\[
(3.6) \quad \frac{e^{\alpha y}}{(1 + \alpha + y)^d} \geq \frac{e^{\alpha x}}{(1 + \alpha + x)^d}.
\]

Note that \(\sigma(an) \geq \sigma(a)\) for all \(a \in A\) and \(n \in N\). Now take \(x = \sigma(an)\) and \(y = \sigma(at)\). Then \(x \geq y \geq 0\). We take \(\alpha = |\lambda| \geq L\) to get the required result.

We start now from (3.2) and use the Iwasawa decomposition \(G = KAN\) and the inequality (3.5) to obtain:

\[
(3.7) \quad \int_K \int_{\mathbb{R}}^L \int_{\mathbb{R}}^\infty \mathcal{R}(|f|)(k,t)|\hat{f}_{\delta,j}(\lambda)|e^{\lambda |t|}dtd\mu(\lambda)dk < \infty.
\]

This proves (i).

**Proof of (ii):** Let

\[
I_2 = \int_K \int_{|t| > M} \int_{\mathbb{R}}^{L} \mathcal{R}(|f|)(k,t)|\hat{f}_{\delta,j}(\lambda)|e^{\lambda |t|}dtd\mu(\lambda).
\]

Since \(|\hat{f}_{\delta,j}(\lambda)|\) is bounded and \(\mu(\lambda)\) is continuous, \(I_2 \leq C \int_K \int_{|t| > M} \int_{\mathbb{R}}^{L} \mathcal{R}(|f|)(k,t)|e^{\lambda |t|}dtd\mu(\lambda)dt\)

\[
= CI_3,
\]

say. We will show that \(I_3\) is finite for \(M = 2(L + 1 + \rho)\).

We start with the assumption that \(\hat{f}_{\delta,j} \neq 0\) (otherwise \(f_{\delta,j} = 0\) almost everywhere). This implies that \(\hat{f}_{\delta,j}(\lambda) \neq 0\) for almost every \(\lambda \in \mathbb{R}\) as \(\hat{f}_{\delta,j}\) is real analytic on \(\mathbb{R}\), \(f\) being an \(L^1\)-function.

Therefore using from (3.2) we can get a \(\lambda_0 \in \mathbb{R}\) with \(|\lambda_0| > 2(L + \rho)\) such that:

\[
\int_G |f(x)|e^{\sigma(x)|\lambda_0|}e^{-\rho H(x)}(1 + |\lambda_0| + |\lambda_0|) < \infty.
\]

We will use the Iwasawa decomposition \(G = KAN\). Now since \(\sigma\) and \(H\) are both left \(K\)-invariant and \(H\) is right \(N\)-invariant we obtain:

\[
\int_K \int_{\mathbb{R} \times N} |f(ka_n)|e^{\lambda_0 \sigma(a_n)}e^{-\rho t}(1 + |\sigma(a_n)| + |\lambda_0|)e^{|\lambda_0|t}dkdtdn < \infty.
\]

Notice that \(|\lambda_0| + |\lambda_0|^2 > d\). Therefore by applying the argument of case (i) (see (3.6)) to \(|\lambda_0|\) we get:

\[
(3.8) \quad \frac{e^{\lambda_0 \sigma(a_n)}}{(1 + |\lambda_0| + \sigma(a_t))} \geq \frac{e^{\lambda_0 \sigma(a_t)}}{(1 + |\lambda_0| + \sigma(a_t))}.
\]

Therefore in particular:

\[
\int_K \int_{|t| > M} \int_{\mathbb{R}}^{L} |f(ka_n)|e^{\lambda_0 |t|}e^{\rho t}(1 + |\lambda_0| + |\lambda_0|)e^{|\lambda_0|t}dkdtdn < \infty.
\]

Note that \(M + M^2 > d\) as \(M = 2(L + 1 + \rho)\) and \(L + L^2 > d\). Applying the argument of case (i) again (see (3.6)) this time with \(\alpha = |t| > M\) and \(x = |\lambda_0|, y = 2(L + \rho)\) we get,

\[
(3.9) \quad \frac{e^{\lambda_0 |t|}}{(1 + |\lambda_0| + |t|)} \geq \frac{e^{2(L+\rho)|t|}}{(1 + 2(L + \rho) + |t|)}.
\]

Therefore,

\[
\int_K \int_{|t| > M} \int_{\mathbb{R}}^{L} |f(ka_n)|e^{2(L+\rho)|t|}e^{\rho t}(1 + |\lambda_0| + 2(L + \rho))e^{|\lambda_0|t}dkdtdn < \infty.
\]
We also see that for $|t| > M = 2(L + 1 + \rho)$
\[
\frac{e^{(L+\rho)|t|}}{(1 + |t| + 2(L + \rho))^d} > \frac{1}{(1 + |t|)^d}
\]
(This is equivalent to showing: $e^{(L+\rho)|t|} > (1 + \frac{2(L+\rho)}{1+|t|})^d$. For $|t| > M = 2(L + 1 + \rho)$, $e^{(L+\rho)|t|} > e^{2(L+\rho)(L+1+\rho)} > e^d > 2^d$ (as $L + L^2 > d$), while $(1 + \frac{2(L+\rho)}{1+|t|})^d < (1 + \frac{2(L+\rho)}{1+2(L+1+\rho)})^d < 2^d$.)

So we obtain:
\[
\int_K \int_{|t| > M} \int_N \frac{|f(ka_{\gamma})| e^{(L+\rho)|t|} e^{|t|}}{(1 + |t|)^d} dk dt dn < \infty
\]
and hence, $I_3 < \infty$. This completes the proof of (ii).

Thus from (i), (ii) and (iii) we obtain $X$.

**Step 4:** From $X$ and $Y$, we have,
\[
(3.8) \quad \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|A(f_{\delta,j})(t)||\widehat{f}_{\delta,j}(\lambda)|e^{|t|}}{(1 + |t| + |\lambda|)^d} dt d\mu(\lambda) < \infty
\]
for $\delta, \delta' \in \hat{K}_0$, $1 \leq j \leq d(\delta)$, $1 \leq j' \leq d(\delta')$.

In particular we can take $\delta = \delta'$ and $j = j'$ to obtain:
\[
(3.9) \quad \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|A(f_{\delta,j})(t)||\widehat{f}_{\delta,j}(\lambda)|e^{|t|}}{(1 + |t| + |\lambda|)^d} dt d\mu(\lambda) < \infty.
\]

**Step 5:** Now we will show that in $X$, $d\mu(\lambda)$ can be replaced by $d\lambda$. We have the following asymptotic estimate of the spherical Plancherel density (see $\mathbb{2}$)
\[
(3.10) \quad \mu(\lambda) = |c(\lambda)|^{-2} \asymp (\lambda, \gamma)^2 (1 + |\langle \lambda, \gamma \rangle|)^{m_\gamma + m_{2\gamma}} - 2,
\]
where $m_\gamma, m_{2\gamma}$ are as defined in section 2. Here $f \asymp g$ means $c_1g(\lambda) \leq f(\lambda) \leq c_2 g(\lambda)$ for two positive constants $c_1, c_2$ and $\lambda \in \mathbb{a}^*$, $|\lambda|$ large.

For some suitable large $R > 0$ let
\[
M_1(\lambda) = \int_{|t| > R} \frac{|A(f_{\delta,j})(t)||\widehat{f}_{\delta,j}(\lambda)|e^{|t|}}{(1 + |t| + |\lambda|)^d} dt, \quad M_2(\lambda) = \int_{|t| \leq R} \frac{|A(f_{\delta,j})(t)||\widehat{f}_{\delta,j}(\lambda)|e^{|t|}}{(1 + |t| + |\lambda|)^d} dt
\]
and let $M(\lambda) = M_1(\lambda) + M_2(\lambda)$.

Then we are given that $\int_\mathbb{a}^* M(\lambda)|\widehat{f}_{\delta,j}(\lambda)||c(\lambda)|^{-2}d\lambda$ is finite which implies that the integrand is finite for almost every $\lambda$. Now both $M_1(\lambda)$ and $M_2(\lambda)$ are clearly radial. Note also that $M_1$ is an increasing function of $|\lambda|$ (see $\mathbb{3.3}$) and $M_2(\lambda) \leq e^{|\lambda|R} \times \|A(f_{\delta,j})\|_1$ for any $\lambda$. Therefore $M_2(\lambda)$ is bounded on compact sets. The Plancherel density $|c(\lambda)|^{-2}$ is real analytic. Furthermore since $f \in L^1(X)$ the function $\lambda \to \widehat{f}_{\delta,j}(\lambda)$ is real analytic. Hence the set of zeros of these functions is at most countable, in particular they have measure zero. Therefore $M_1(\lambda)$ is finite everywhere and locally integrable since it is an increasing function of $|\lambda|$. Thus both $M_1$ and $M_2$ are locally integrable and hence $M$ is locally integrable. We want to show that $\int_\mathbb{a}^* M(\lambda)|\widehat{f}_{\delta,j}(\lambda)|d\lambda$ is finite. The integrand is locally integrable on $\mathbb{a}^*$, hence we need only examine its behaviour for large $|\lambda|$. Now the above-mentioned formula for the Plancherel density shows that $|c(\lambda)|^{-2}$ tends to $\infty$ as $|\lambda|$ tends to $\infty$. In particular there
exists $A > 0$ such that $|c(\lambda)|^{-2} > 1$ whenever $|\lambda| > A$. Now \( \int_{|\lambda| > A} M(\lambda) |\hat{f}_{\delta,j}(\lambda)| |c(\lambda)|^{-2} d\lambda \geq \int_{|\lambda| > A} M(\lambda) |\hat{f}_{\delta,j}(\lambda)| d\lambda \). This immediately implies our assertion, that is we get

\[
(3.11) \quad \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|A(\delta,j)(t)||\hat{f}_{\delta,j}(\lambda)||c(\lambda)||t|}{(1 + |t| + |\lambda|)^d} t dtd\lambda < \infty.
\]

**Step 6:** In this step we will deduce that $\hat{f}_{\delta,j}(\lambda) = P(\lambda)e^{-\alpha \lambda^2}$, where $P$ is a polynomial which depends on $\delta, j$ and $\alpha$ is a positive constant, which is independent of $\delta, j$.

Note that $A(\delta,j) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. In view of (3.1) we can apply theorem 12 to obtain $\hat{f}_{\delta,j}(\lambda) = P(\lambda)e^{-\alpha \lambda^2}$. A priori the polynomial $P$ as well as the constant $\alpha$ depend on $\delta, j$. We will see that the constant $\alpha$ is actually independent of $\delta, j$.

Suppose if possible for $\delta_1, \delta_2 \in K_0$ and $1 \leq j_1 \leq d(\delta_1), 1 \leq j_2 \leq d(\delta_2)$,

1. $\hat{f}_{\delta_1, j_1}$(\lambda) = $P_1$(\lambda)e^{-\alpha_1 \lambda^2},
2. $\hat{f}_{\delta_2, j_2}$(\lambda) = $P_2$(\lambda)e^{-\alpha_2 \lambda^2},

where $P_1, P_2$ are two polynomials, $\alpha_1, \alpha_2$ are positive constants and $\alpha_1 \neq \alpha_2$. Without loss of generality we can assume that $\alpha_1 < \alpha_2$. From (2) above we have,

\[
(3) \quad A(f_{\delta_2,j_2})(t) = P_2(t)e^{-\frac{\alpha_2}{2}t^2}.
\]

Substituting (1) and (3) in (8) we see that the integrand in (8) is

\[
\frac{|P_1(\lambda)||P_2(\lambda)||t|e^{-\sqrt{\alpha_1 \lambda^2 - \frac{\alpha_2}{2}t^2}}}{(1 + |t| + |\lambda|)^d}
\]

where $A = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}} > 0$ as $\frac{\alpha_1}{\alpha_2} < 1$. Therefore the integrand in (8) grows very rapidly in the neighbourhood of the hyperplane (pair of straight lines) $\sqrt{\alpha_1 \lambda^2} = \frac{1}{\sqrt{\alpha_2}}|t|$ and the integral diverges. This establishes that the positive constant $\alpha$ is independent of $\delta$ and $j$.

**Step 7:** This is our final step wherein we conclude the proof of the theorem. From the previous step we know that $\hat{f}_{\delta,j}(\lambda) = P(\lambda)e^{-\alpha \lambda^2}$. This shows that $f_{\delta,j}$ is a derivative of the heat kernel $h_\alpha$. Notice also that $P(\lambda) = P'(\lambda^2)Q_\delta(-\lambda)$ where $P'(\lambda^2)$ is a polynomial in $\lambda^2$, because $Q_\delta(-\lambda)$ is a factor of $\hat{f}_{\delta,j}(\lambda)$ (see section 2). Recalling that $\deg Q_\delta = p_\delta$ we see that $\deg P(\lambda) \geq p_\delta$.

On the other hand noting that $A(f_{\delta,j})(t) = P(t)e^{-\frac{\alpha}{2}t^2}$, substituting $\hat{f}_{\delta,j}$ and $A(f_{\delta,j})$ back in (3.9) and using (3.10) it is easy to verify that $\deg P < \frac{d' - 1}{2}$ where $d' = d - (m_\gamma + m_{2\gamma})$ as otherwise the integral in (3.9) diverges.

Therefore if $p_\delta \geq \frac{d' - 1}{2}$ then $f_{\delta,j} = 0$ almost everywhere. As $p_\delta \geq |q_\delta|$, we conclude that only for finitely many $\delta \in K_0$ which are parametrized by $(p_\delta, q_\delta)$ with $p_\delta < \frac{d' - 1}{2}$, $f_{\delta,j}$ will satisfy (3.9). We thus conclude that $f$ is a $K$-finite function whose each $K$-isotypical component is a derivative of the heat kernel $h_\alpha$. In other words, $f$ itself is a derivative of the heat kernel $h_\alpha$.

In particular if $d' \leq 1$ that is if $d \leq 1 + m_\gamma + m_{2\gamma} = d_X$, then there is no $p_\delta$ satisfying $p_\delta < \frac{d' - 1}{2}$ and hence in that case $f = 0$ almost everywhere. \( \square \)

## 4. Sharpness of the estimate

In order to complete the picture we investigate the optimality of the condition used in theorem 3.1. More precisely, suppose a function $f \in L^1(X) \cap L^2(X)$ satisfies

\[
(4.1) \quad \int_X \int_{\mathbb{R}^*} \frac{|f(x)||\hat{f}(\lambda)||e^{\sigma(x)|\lambda|}||\Xi(x)^{1-\varepsilon}|}{(1 + \sigma(x) + |\lambda|)^d} dx d\mu(\lambda) < \infty
\]
for some nonnegative integer $d$ and $c, \varepsilon \in \mathbb{R}$. Then:

(i) We will see that if $\{c > 1 \text{ and } \varepsilon \geq 0\}$ or $\{c \geq 1 \text{ and } \varepsilon > 0\}$ in (4.1) then $f = 0$ almost everywhere.

(ii) We will find a symmetric space $X$ on which there can be infinitely many linearly independent functions in $L^1(X) \cap L^2(X)$ satisfying the estimate (4.1) with $\{c < 1 \text{ and } \varepsilon \leq 0\}$ and with $\{c \leq 1 \text{ and } \varepsilon < 0\}$. These functions are not of the form characterized in theorem 3.1.

In case (i) as $c > 1$ and $\varepsilon^{-} \geq 1$, $f$ satisfies the condition (3.1) in theorem 3.1 and hence $\hat{f}_{\delta,j}(\lambda) = P_{\delta,j}(\lambda)e^{-\alpha \lambda^2}$. Therefore $\mathcal{A}(f_{\delta,j})(t) = P_{\delta,j}(t)e^{-\beta t^2}$ where $\alpha \beta = \frac{1}{4}$, since $\mathcal{A}(f_{\delta,j})$ is the Euclidean Fourier inverse of $\hat{f}_{\delta,j}$.

On the other hand starting from the condition (4.1) and following the steps of the proof of theorem 3.1 we obtain finally:

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\mathcal{A}(f_{\delta,j})(t)||\hat{f}_{\delta,j}(\lambda)||\varepsilon e^{\varepsilon t}|e^{\sigma t}|dtd\lambda < \infty.
\]

Substituting $\mathcal{A}(f_{\delta,j})$ and $\hat{f}_{\delta,j}$ as obtained above in this inequality we see that it demands

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-\varepsilon^{-}(\sqrt{\alpha}+\sqrt{\beta})|t|}e^{\varepsilon^{-}(1-\varepsilon^{-})|t|}|e^{\sigma t}|dtd\lambda < \infty.
\]

But around the hyperplane $\sqrt{\alpha} = \sqrt{\beta}$ the integrand grows rapidly as $|t| \rightarrow \infty$ since $c - 1 > 0$ or $\varepsilon > 0$ and hence the integral becomes infinite which contradicts (4.2).

Next we consider the case (ii) that is, we will find a symmetric space $X$ and functions $f$ on $X$ which satisfy

\[
\int_{X} \int_{\mathbb{R}^n} \frac{|f(x)||\hat{f}(\lambda)||e^{c\sigma(x)|\lambda|}|\Xi(x)^{1+\varepsilon'}(1+\sigma(x)+|\lambda|)^d}dxd\mu(\lambda) < \infty
\]

for some nonnegative integer $d$, and either $c < 1, \varepsilon' > 0$ or $c \leq 1, \varepsilon' > 0$.

Let $G = SL(2, \mathbb{C})$ considered as a real Lie group and $K = SU(2)$. Consider the symmetric space $X = SL(2, \mathbb{C})/SU(2)$. Let

\[A = \{a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \}.\]

Then $\phi_\lambda(a_t) = \frac{\sin(\lambda t)}{\lambda \sin t}$ and the Plancherel measure $\mu(\lambda) = \lambda^2$ (see [10], p. 432). We define a $K$-biinvariant function $g$ on $X$ by prescribing its spherical Fourier transform $\hat{g}(\lambda) = \int_G g(x)\phi_\lambda(x)dx = F(\psi)(\lambda)e^{-\frac{\lambda^2}{4}}P(\lambda)$ for $\lambda \in \mathbb{R}$ where $\psi$ is an even function in $C_c^\infty(\mathbb{R})$ with support $[-\zeta, \zeta]$ for some $\zeta > 0$, $F(\psi)$ is its Euclidean Fourier transform and $P$ is an even polynomial in $\mathbb{R}$. This means that $g$ is the convolution (in $G$) of a smooth compactly supported $K$-biinvariant function on $G$ with a (invariant) derivative of the heat kernel of $X$. Indeed it is clear from Paley-Wiener theorem that $F(\psi)$ is also the spherical Fourier transform of a $K$-biinvariant smooth function on $G$ supported in a ball of radius $\zeta$.

It follows that $g$ is a $K$-biinvariant function of the $L^2$-Schwartz space of $G$. By the inversion formula for the spherical Fourier transform we have,

\[
g(a_t) = C \int_{\mathbb{R}} \frac{\hat{g}(\lambda)\phi_\lambda(a_t)d\mu(\lambda)}{\sin t} = C \int_{\mathbb{R}} \frac{F(\psi)(\lambda)e^{-\frac{\lambda^2}{4}}P(\lambda)\sin \lambda t d\lambda}{\sin t}.
\]

Using Fourier inversion on $\mathbb{R}$, we see that $g(a_t) = \frac{C}{\sinh t}(\psi_1 * \mathbb{R} h)(t)$ where $* \mathbb{R}$ is the convolution in $\mathbb{R}$, $\psi_1$ is a derivative of $\psi$ and hence a function in $C_c^\infty(\mathbb{R})$ with support contained in $[-\zeta, \zeta]$; $h(t) = e^{-t^2}$. An easy computation shows that

\[
|g(a_t)| \leq Ce^{-t^2} e^{\frac{\lambda t}{2}} \leq Ce^{-\sigma(a_t^2)}\Xi(a_t)^{1-4\zeta}.
\]

If we choose $\zeta > 0$ such that $l = 1 - 4\zeta > 0$, then we see that the function $g$ on $X$ satisfies:
therefore consider the case when \( 0 < c < \varepsilon \). Suppose \( s \) satisfies the estimate (4.3) with \( N > 1 \) where \( M > 1 \).

\[ \alpha \beta \]

satisfying the constraint \( 0 < c < \varepsilon \).

Thus we can find a function \( g \) on \( X \) which satisfies the above estimate for any given \( l \in (0,1) \). Suppose \( \varepsilon' > 0 \). We choose \( l \) (that is choose \( \zeta \)) so that \( l + \varepsilon' \geq 1 \). Then it is easy to verify that \( g \) satisfies the estimate (13) with \( c \leq 1 \) for any suitable large \( d \).

Now suppose \( c < 1 \) and \( \varepsilon' \geq 0 \). If \( c \leq 0 \), the above function \( g \) clearly satisfies (13). We need only therefore consider the case when \( 0 < c < 1 \). Notice that we can choose \( \alpha, \beta \in \mathbb{R}^+ \), \( \alpha < 1 \) and \( \beta < \frac{1}{2} \) satisfying the constraint \( 4\alpha\beta = c^2 \) such that the above function \( g \) and its spherical Fourier transform \( \hat{g} \) satisfy

\[ |g(x)| \leq C e^{-\alpha(x)^2} \Xi(x) \]

for all \( x \in X \)

and

\[ |\hat{g}(\lambda)| \leq C' e^{-\beta\lambda^2} \]

for all \( \lambda \in \mathbb{R} \).

Clearly the pair \((g, \hat{g})\) satisfy (13).

From the construction of \( g \) it is clear that there are infinitely many linearly independent functions satisfying the estimate in case (ii). This example is a modification of the example given in [20].

5. Consequences of Beurling’s theorem

In this section we will justify our claim made in the introduction that this extension of the Beurling-Hörmander theorem is the “master theorem”, that is all other theorems of this genre follow from theorem 3.1. First we consider the Gelfand-Shilov theorem.

**Theorem 5.1.** (Gelfand-Shilov) Let \( f \in L^2(X) \). Suppose \( f \) satisfies

1. \( \int_X |f(x)|^{\frac{p}{\alpha\beta}} \Xi(x)^{\frac{q}{\alpha\beta}} dx < \infty \),

2. \( \int_{\mathbb{R}^d} \|\hat{f}(\lambda)\| \Xi(\lambda)^{\frac{N}{\alpha\beta}} d\mu(\lambda) < \infty \),

where \( 1 < p < \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( N \) is a nonnegative integer.

(a) If \( \alpha\beta > 1 \) then \( f = 0 \) almost everywhere.

(b) If \( \alpha\beta = 1 \) and \( p \neq 2 \) (and hence \( q \neq 2 \)) then \( f = 0 \) almost everywhere.

(c) If \( \alpha\beta = 1 \), \( p = q = 2 \) and \( N < d_X + 1 \) then \( f = 0 \) almost everywhere.

(d) If \( \alpha\beta = 1 \), \( p = q = 2 \) and \( N \geq d_X + 1 \) then \( f \) is a \( K \)-finite function of the form described in theorem 3.1. In particular if \( N = d_X + 1 \), then \( f \) is a constant multiple of the heat kernel \( h_t \) for some \( t > 0 \).

**Proof.** (a) Since \( \frac{\alpha}{p}\sigma(x)^p + \frac{\beta}{q}|\lambda|^q \geq \alpha\beta\sigma(x)|\lambda| \) and \( (1 + \sigma(x) + |\lambda|)^{2N} \geq (1 + \sigma(x))^N (1 + |\lambda|)^N \), from the assumptions (1) and (2) we obtain:

\[ \int_X \int_{\mathbb{R}^d} \frac{|f(x)||\hat{f}(\lambda)|^{\frac{\alpha\beta}{2}\sigma(x)|\lambda|\Xi(x)}}{(1 + \sigma(x) + |\lambda|)^{2N}} dxd\mu(\lambda) < \infty \]

But as \( \alpha\beta > 1 \) we conclude that \( f = 0 \) almost everywhere (see section 4).
(b) Fix a $\delta \in \mathbb{K}_0$ and an integer $j$ such that $1 \leq j \leq d(\delta)$. We will show that $f_{\delta,j} = 0$ almost everywhere. Note that conditions (1) and (2) of the theorem can be reduced (respectively) to

$$\int_X |f_{\delta,j}(x)| e^{-\frac{(\alpha(x))^p}{p} \Xi(x)} d\mu(x) < \infty$$

and

$$\int_{\mathbb{R}^n} |\hat{f}_{\delta,j}(\lambda)| e^{-\frac{|\beta\lambda|^q}{q} \Xi(\lambda)} (1 + |\lambda|)^N d\mu(\lambda) < \infty.$$  

Therefore we can confine ourselves to the $(\delta,j)$-th component of the function. Using $\alpha = 1$ we can argue as in (a) and show that

$$\int_X \int_{\mathbb{R}^n} |f_{\delta,j}(x)| e^{-\frac{(\alpha(x))^p}{p} \Xi(x)} (1 + |\sigma(x)|^2 + |\lambda|)^N d\mu(\lambda) < \infty$$

and thereby conclude from theorem 3.1 that $\hat{f}_{\delta,j}$ is either identically zero or of the form $P_{\delta,j}(\lambda)e^{-\beta_0 \lambda^2}$ for some $\beta_0 > 0$.

Now, if we consider the case when $1 < p < 2$, then we see that unless $\hat{f}_{\delta,j} = 0$ almost everywhere, it cannot satisfy (5.3) because $q > 2$.

Next we take up the case when $p > 2$ and hence $1 < q < 2$. Since $\mu(\lambda)$ has polynomial growth (see (3.10)) $\hat{f}_{\delta,j} = P_{\delta,j}e^{-\beta_0 \lambda^2}$ satisfies:

$$\int_{\mathbb{R}^n} |f_{\delta,j}(x)| e^{-\frac{(\gamma_0\lambda)^2}{2} \Xi(\lambda)} (1 + |\lambda|)^M d\mu(\lambda) < \infty,$$

where $\gamma_0 = \sqrt{2\beta_0}$ for some suitable $M > 0$. We choose $\alpha_0$ such that $\alpha_0 \gamma_0 > 1$. Since $p > 2$ and $f_{\delta,j} \in L^1(X)$ we see from (5.2) that

$$\int_X |f_{\delta,j}(x)| e^{-\frac{(\alpha_0\lambda)^2}{2} \Xi(x)} (1 + |\sigma(x)|^2 + |\lambda|)^N d\mu(\lambda) < \infty.$$  

But then from (a) it follows that $f_{\delta,j} = 0$ almost everywhere.

(c-d) By the above argument $\hat{f}_{\delta,j}(\lambda) = P_{\delta,j}(\lambda)e^{-\beta_0 \lambda^2}$. It follows from (5.3) with $q = 2$, that $\sqrt{2\beta_0} \geq \beta$. But if $2\beta_0 > \beta^2$ then $\alpha_0 \sqrt{2\beta_0} > 1$. On the other hand $\hat{f}_{\delta,j}$ satisfies (5.3) with $q = 2$ and with $\beta$ replaced by $\sqrt{2\beta_0}$ for a suitably large $N$. Therefore by (a) $f_{\delta,j} = 0$ almost everywhere. Hence $\hat{f}_{\delta,j}(\lambda) = P_{\delta,j}(\lambda)e^{-\frac{\beta_0^2}{2} \lambda^2}$. Now as noted earlier, the Kostant polynomial $Q_\delta$ is a factor of $P_{\delta,j}$ and hence $\deg P_{\delta,j} \geq \deg Q_\delta = p_\delta$. Therefore only for finitely many $\delta \in \mathbb{K}_0, \hat{f}_{\delta,j}$ can satisfy (5.3). This proves the first statement in (d). Substituting $\hat{f}_{\delta,j}$ back in (5.3) and using (3.10) it is now easy to verify that if $N < 2 + m_\gamma + m_{2\gamma} = d_X + 1$ then $f_{\delta,j} \equiv 0$ and if $N = 2 + m_\gamma + m_{2\gamma} = d_X + 1$ then $\deg P_{\delta,j} = 0$ and hence $\hat{f}_{\delta,j}(\lambda) = Ce^{-\frac{\beta_0^2}{2} \lambda^2}$. But that is possible only when $\delta$ is trivial. Indeed from (2.4) it follows that $\Phi_{\delta,\rho,\delta} \equiv 0$ when $\delta \in \mathbb{K}_0$ is nontrivial and $1 \leq j \leq d(\delta)$. Hence for such a $\delta$, $\hat{f}_{\delta,j}(i\rho) = 0$ which is not possible if $\hat{f}_{\delta,j}(\lambda) = Ce^{-\frac{\beta_0^2}{2} \lambda^2}$. Thus $f$ is a constant multiple of the heat kernel $h_t$, where $t = \frac{\beta_0^2}{2}$. \hfill $\Box$

We will see below that the theorems of Morgan, Hardy and Cowling-Price follow from the Gelfand-Shilov theorem proved above.

**Theorem 5.2.** (Morgan’s theorem) Let $f : X \to \mathbb{C}$ be measurable and assume that,

1. $|f(x)| \leq C_1 e^{-\alpha(x)^p \Xi(x)} (1 + \sigma(x))^n$, for all $x \in X$

2. $\|\hat{f}(\lambda)\|_2 \leq C_2 e^{-b|\lambda|^\alpha}$, for all $\lambda \in a^* \equiv \mathbb{R}$

where $C_1, C_2$ and $a, b$ are positive constants, $n$ is a nonnegative integer, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. 

(a) If \((ap) \frac{1}{p} (bq) \frac{1}{q} > 1\), then \(f = 0\) almost everywhere.

(b) If \((ap) \frac{1}{p} (bq) \frac{1}{q} = 1\) and \(p \neq 2\), then \(f = 0\) almost everywhere.

(c) If \(p = q = 2\) and \((ap) \frac{1}{p} (bq) \frac{1}{q} = 1\), that is \(ab = \frac{1}{4}\), then \(f\) is a constant multiple of the heat kernel.

\[\text{Proof.} \text{ Let } a = \frac{\alpha}{p} \text{ and } b = \frac{\beta}{q}. \text{ Then } f \text{ and } \hat{f}\text{ satisfy theorem } 5.1\text{ for some suitable } N. \text{ The condition } \(ap) \frac{1}{p} (bq) \frac{1}{q} \geq 1 \text{ translates as } \alpha \beta \geq 1. \text{ Thus (a) and (b) follow from (a) and (b) of theorem 5.1. For (c) again we use the proof of (c-d) of theorem 5.1, to conclude that } \hat{f}_{\delta,j}(\lambda) = P_{\delta,j}(\lambda)e^{-b\lambda^2}. \text{ But because of the condition (2) of this theorem } P_{\delta,j} \text{ is a constant. But this implies that only for trivial } \delta = \delta_0, \text{ } \hat{f}_{\delta,j} \text{ can be nonzero and the spherical Fourier transform of } f \text{ is } Ce^{-b\lambda^2} \text{ (see the argument at the end of the proof of theorem 5.1 (c-d)). That is } f \text{ is a constant multiple of the heat kernel at } t = b. \]

\[\text{Remark 5.3.} \text{ The nonnegative integer } n \text{ in condition (1) of Morgan's theorem should satisfy } n \geq m_1 + m_2, \text{ otherwise the heat kernel will not be accommodated in this inequality. That is if we start with } n < m_1 + m_2, \text{ then in case (c) also we have } f = 0 \text{ almost everywhere.} \]

Morgan’s theorem implies the well-known Hardy’s theorem as a particular case \((p = q = 2)\). To stress this point we will write it as a separate theorem.

\[\text{Theorem 5.4.} \text{ (Hardy’s theorem) Let } f : X \to \mathbb{C} \text{ be measurable and assume that,} \]

\[\begin{align*}
(1) \quad &|f(x)| \leq C_1 e^{-a\sigma(x)2}\Xi(x)(1 + \sigma(x))^n, \text{ for all } x \in X \\
(2) \quad &\|\hat{f}(\lambda)\|_2 \leq C_2 e^{-b|\lambda|^2}, \text{ for all } \lambda \in a^* \equiv \mathbb{R}
\end{align*}\]

where \(C_1, C_2\) and \(a, b\) are positive constants, \(n\) is a nonnegative integer, \(1 < p < \infty\).

(a) If \(ab > \frac{1}{4}\), then \(f = 0\) almost everywhere.

(b) If \(ab = \frac{1}{4}\), then \(f\) is a constant multiple of the heat kernel.

\[\text{Theorem 5.5.} \text{ (Cowling-Price) Let } f : X \to \mathbb{C} \text{ be measurable and assume that for positive constants } a, b \text{ and nonnegative integers } m_1, m_2, \text{ we have} \]

\[\begin{align*}
(1) \quad &\int_X \frac{|f(x)|^{a\sigma(x)2}\Xi(x)^{\frac{p}{p-1}}}{(1 + \sigma(x))^{\frac{1}{p}}} < \infty \\
(2) \quad &\int_{a^*} \frac{|\hat{f}(\lambda)|^{b|\lambda|^2}}{(1 + |\lambda|)^n} d\mu(\lambda) < \infty,
\end{align*}\]

where \(1 \leq p_1, p_2 < \infty\).

(a) If \(ab > \frac{1}{4}\), then \(f = 0\) almost everywhere.

(b) If \(ab = \frac{1}{4}\), then \(f\) is a \(K\)-finite function of the form described in theorem 3.1. In particular if \(d_X < n \leq d_X + p_2\), then \(f\) is a constant multiple of the heat kernel.

\[\text{Proof.} \text{ Let us first assume } p_1 \text{ and } p_2 \text{ are greater than } 1. \text{ Let } q_1 \text{ and } q_2 \text{ be respectively the conjugates of } p_1 \text{ and } p_2, \text{ that is } \frac{1}{p_i} + \frac{1}{q_i} = 1, i = 1, 2. \text{ Using the estimate of } \Xi(x) \text{ given in section 2 we note that} \]

\[\frac{\Xi(x)^{\frac{p}{p-1}}}{(1 + \sigma(x))^{\frac{1}{p}}} \text{ is in } L^{m'}(X) \text{ if } m' > 3. \text{ Therefore it follows from condition (1) in the hypothesis that} \]

\[\begin{align*}
\int_X \frac{|f(x)|^{a\sigma(x)2}\Xi(x)^{\frac{p}{p-1}}}{(1 + \sigma(x))^{\frac{1}{p}}} \times \frac{\Xi(x)^{\frac{p}{p-1}}}{(1 + \sigma(x))^{\frac{1}{p}}} dx & = \int_X \frac{|f(x)|^{a\sigma(x)2}\Xi(x)}{(1 + \sigma(x))^{\frac{1}{p}}} dx < \infty,
\end{align*}\]
where \( N_1 = \frac{m}{p_1} + \frac{m'}{q_1} \). Similarly using (3.10) we see that if \( n' > 1 + m_\gamma + m_2\gamma \), then
\[
\int_{a^*} \frac{\hat{f}(\lambda)e^{b|\lambda|^2}}{(1 + |\lambda|)^{N_2}} d\mu(\lambda) < \infty
\]
where \( N_2 = \frac{m}{p_2} + \frac{m'}{q_2} \).

When either \( p = 1 \) or \( p_2 = 1 \) then the above two inequalities are evident.

Thus this becomes a particular case of theorem 5.1 when \( p = q = 2 \), \( N = \max\{N_1, N_2\} \) and \( a = \frac{\alpha}{2}, b = \frac{\beta^2}{2} \). Note that the conditions \( ab > \frac{1}{4} \) and \( \alpha \beta > 1 \) in the hypothesis translate as \( \alpha \beta > 1 \) and \( \alpha \beta = 1 \) respectively, when we fit them in theorem 5.1. The result now follows from (a), (c) and (d) of theorem 5.1 in a fashion similar to what was used in the previous theorems in this section. We omit the details to avoid repetitions.

In the above theorem, we may take either \( p_1 \) or \( p_2 \) or both to be infinity. The condition (1) with \( p_1 = \infty \) means that \( g(x) = |f(x)|e^{-\alpha x^2} \Xi(x)^{-1}(1 + \sigma(x)^m \) is a bounded function on \( X \) for \( m \) as above. Hence \( |f(x)|e^{-\alpha x^2}/(1 + \sigma(x))^{\sigma}\) is integrable where \( N_1 = m' > 3 \) as described above. That is as above \( \int_X |f(x)|e^{-\alpha x^2} \Xi(x)^{-1} dx < \infty \).

Similarly for \( p_2 = \infty \) we arrive at \( \int_{a^*} \frac{|\hat{f}(\lambda)|e^{b|\lambda|^2}}{(1 + |\lambda|)^N} d\mu(\lambda) < \infty \) for \( N_2 = n' > 1 + m_\gamma + m_2\gamma \), since \( |\hat{f}(\lambda)|e^{b|\lambda|^2} \) is bounded on \( a^* \). Note that the case \( p_1 = p_2 = \infty \) of the Cowling-Price theorem implies Hardy’s theorem.

Some parts of these theorems were proved independently on symmetric spaces. Part (a) of Hardy’s theorem was proved in [20], [4], [6], while part (b) was proved in [16], [22]. Part (a) of Cowling-Price theorem was proved in [19] and in [17] and part (b) was proved in [18]. Part (a) of Morgan’s theorem was proved in [19].

6. CONCLUDING REMARKS

Demange in his thesis [5] further generalized theorem 1.2:

**Theorem 6.1.** (Demange 2004) For two nonzero functions \( f_1, f_2 \in L^2(\mathbb{R}) \), if
\[
(1) \quad \int \int_{\mathbb{R}^2} \frac{|f_1(x)||f_2(\lambda)|e^{a|x|^2}}{(1 + |x| + |\lambda|)^\mu} \, dx \, d\lambda < \infty
\]
\[
(2) \quad \int \int_{\mathbb{R}^2} \frac{|f_2(x)||f_1(\lambda)|e^{a|x|^2}}{(1 + |x| + |\lambda|)^\mu} \, dx \, d\lambda < \infty
\]
then \( f_1(x) = P_1(x)e^{-\alpha x^2} \) and \( f_2(x) = P_2(x)e^{-\alpha x^2} \) for some positive constant \( \alpha \) and polynomials \( P_1, P_2 \). A careful reader will observe that this theorem can be extended to symmetric spaces using our technique to have the following interesting consequence:

We consider two rank 1 symmetric spaces \( X_1 = G_1/K_1 \) and \( X_2 = G_2/K_2 \). Let \( dx \) and \( dy \) be the \( G_1 \) and \( G_2 \) invariant measures on \( X_1 \) and \( X_2 \) respectively. Let \( \mu_i \) be the corresponding Plancherel measure for \( X_i \) and let \( \sigma_i, \Xi_i \) be the \( \sigma \) and \( \Xi \) functions on \( X_i, i = 1, 2 \). Let \( f_1 \in L^2(X_1) \) and \( f_2 \in L^2(X_2) \) be two nonzero functions.

**Theorem 6.2.** Let \( f_1 \) and \( f_2 \) as above satisfy
\[
(1) \quad \int_{X_1} \int_{a^*} \frac{|f_1(x)||f_2(\lambda)|e^{a_1(x)}e^{a_2(\lambda)} \Xi_1(x) \Xi_2(\lambda)}{(1 + \sigma_1(x) + |\lambda|)^\mu} \, dx \, d\mu_2(\lambda) < \infty
\]
\[
(2) \quad \int_{X_2} \int_{a^*} \frac{|f_2(y)||f_1(\lambda)|e^{a_2(y)}e^{a_1(\lambda)} \Xi_2(y) \Xi_1(\lambda)}{(1 + \sigma_2(y) + |\lambda|)^\mu} \, dy \, d\mu_1(\lambda) < \infty
\]
Then \( f_1 \) is a derivative of the heat kernel \( h_1^1 \) of \( X_1 \) and \( f_2 \) is a derivative of the heat kernel \( h_2^2 \) of \( X_2 \) for some instant \( \alpha > 0 \).

We take \( X_1 = X_2 = X \) and obtain the following corollary.
Corollary 6.3. Let two nonzero functions \( f_1, f_2 \in L^2(X) \) satisfy
\[
(1) \quad \int_X f_1 \frac{|f_1(x)|^2}{(1+\sigma(x)^2)^{\alpha/2}} d\mu(x) < \infty
\]
\[
(2) \quad \int_X f_2 \frac{|f_2(x)|^2}{(1+\sigma(x)^2)^{\alpha/2}} d\mu(x) < \infty.
\]
Then \( f_1 \) (respectively \( f_2 \)) is a derivative of the heat kernel \( h_\alpha \) for some instant \( \alpha > 0 \).

The proof of the theorem above proceeds along entirely similar lines to that of the proof of the main theorem of this article. We therefore omit it.

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