The Limit Shape of the Leaky Abelian Sandpile Model

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Joint work with Sevak Mkrtchyan

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The Abelian Sandpile Model (ASM) is a cellular automaton defined on a graph $G = (V, E)$.

- An initial sandpile distribution $s : V \rightarrow \mathbb{N}$
- If $s(x) > \deg(x)$ then $x$ is unstable and topples distributing sand to its neighbors:
  $$\begin{align*}
  s(x) &\mapsto s(x) - \deg(x) \\
  s(y) &\mapsto s(y) + 1 \text{ if } y \sim x.
  \end{align*}$$

The sandpile evolves through toppling unstable sites.
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In this talk $G = \mathbb{Z}^2$ but we will consider different **toppling rules**:

**Uniform ASM**

\[
\begin{array}{c}
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  +1 \leftarrow -4 \rightarrow +1 \\
  +1
\end{array}
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- **Directed ASM**
  
  $+1 \uparrow$
  
  $-2 \rightarrow +1$

- **Uniform ASM**
  
  $+1 \leftrightarrow -4 \rightarrow +1$
  
  $+1 \downarrow$
  
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| 1D ASM | Directed ASM | Uniform ASM |
|--------|--------------|-------------|
| $+1 \leftrightarrow -2 \rightarrow +1$ | $+1$ | $+1$ |
| $-2 \rightarrow +1$ | $+1$ | $+1 \leftrightarrow -4 \rightarrow +1$ |
|           |               | $+1$         |
1-Dimensional ASM

- Start with initial sandpile $s(x) = n\delta_{(0,0)}(x)$ topple until reaching a stable sandpile $s_\infty$.

**Question**
What is the stable sandpile?

**Toppling rule**
$+1 \leftarrow -2 \rightarrow +1$
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**Toppling rule**

\[ +1 \leftrightarrow -2 \rightarrow +1 \]

**Figure:** Initial sandpile with \( n = 7 \).

**Figure:** Result after toppling at the origin.
Sequence of topplings

Figure: Origin toppled again.

Figure: All unstable sites topple once more.

some more topples....

and the stable sandpile:
Let $x = (x_1, x_2)$.

**Proposition**

If $s(x_1, x_2) = n\delta_{(0,0)}(x_1, x_2)$ then the stable sandpile for the 1D ASM is

$$s_\infty(x_1, 0) = \begin{cases} 
1 & \text{if } x_1 = 0 \text{ and } n \text{ is odd}, \\
0 & \text{if } x_1 = 0 \text{ and } n \text{ is even}, \\
1 & \text{if } 0 < |x_1| \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
0 & \text{if } \left\lfloor \frac{n}{2} \right\rfloor < |x_2|.
\end{cases}$$

$s_\infty(x_1, x_2) = 0$ if $x_2 > 0$. 

When $d \geq 2$ the limit shape exhibits self-organization.
Let \( x = (x_1, x_2) \).

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When \( d \geq 2 \) the limit shape exhibits self-organization.
Let $s(x_1, x_2) = n\delta_{(0,0)}(x_1, x_2)$ and topple until stable using the **uniform toppling rule**.

The stable sandpile has a **limit shape** (Pegden-Smart 2013).

**Figure**: Stable sandpile with $n = 10^7$. Colors correspond to heights of sandpile.
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### Theorem (Levine-Peres 2008)

The limit shape is bounded between circles of radii \( c_1\sqrt{n} \) and \( c_2\sqrt{n} \) with \( c_2/c_1 = \frac{\sqrt{3}}{\sqrt{2}} \).

**Figure:** Stable sandpile with \( n = 10^7 \). Colors correspond to heights of sandpile.
What is the limit shape of the ASM?

The boundary of the limit shape is a Lipschitz graph (Aleksanyan-Shahgholian 2019).

Figure: Stable sandpile with \( n = 10^7 \). Colors correspond to heights of sandpile.

Is the limit shape convex? Is it a circle, a polygon, or neither?
The toppling rule determines the limit shape:

Figure: Stable sandpile with $n = 10^5$. Black sites have one grain of sand.
Leaky Abelian Sandpile Model (Leaky-ASM)

We compute the limit shape in the presence of dissipation.

- An initial sandpile distribution $s : V \rightarrow \mathbb{R}_{\geq 0}$
- Dissipation $d > 1$

If $s(x) > d \cdot \deg(x)$ then $x$ is unstable and topples distributing sand to its neighbors:

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Let $s(x) = n\delta_{(0,0)}(x)$ and topple until stable using the uniform toppling rule.

$D_{n,d}$ is the set of sites which have toppled.

**Theorem (A.- Mkrtchyan 2020)**

Let $d > 1$ and $r = \log n - \frac{1}{2} \log \log n$. The boundary of $r^{-1}D_{n,d}$ converges to the dual of the boundary of the gaseous phase in the amoeba of the spectral curve for the toppling rule.
Main Results

- Let \( s(x) = n\delta_{(0,0)}(x) \) and topple until stable using the uniform toppling rule.
- \( D_{n,d} \) is the set of sites which have toppled.

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**Theorem (A.- Mkrtchyan 2020)**

Let \( d_n = 1 + t_n \).

- If \( t_n \sim \frac{1}{\log(n)} \) then the boundary of \( \frac{\sqrt{t_n}}{\log(n)} D_{n,d} \) converges to a circle.
- If \( t_n \sim \frac{1}{n^{1-\alpha}} \) with \( 0 < \alpha < 1 \), then the boundary of \( \frac{\sqrt{t_n}}{\log(n)} D_{n,d} \) is between circles of radii \( c_1 \) and \( c_2 \) with \( \frac{c_1}{c_2} \rightarrow \alpha \).
(a) $d = 1.05$
(b) $d = 2$
(c) $d = 1000$

**Figure:** Simulations of the Leaky-ASM with $n \approx 10^{500}$.

**Figure:** Limit shapes from theorem.
Vanishing dissipation limit

(a) $d - 1 = 2.5 \cdot 10^{-4}$

(b) $d - 1 = 2.5 \cdot 10^{-5}$

(c) $d - 1 = 2.5 \cdot 10^{-6}$

(d) $d - 1 = 2.5 \cdot 10^{-7}$

Figure: Leaky-ASM simulations with $n = 10^7$. 
Figure: Uniform ASM with background height $-1$ and $n = 10^7$. 
Vanishing dissipation limit converges to uniform ASM

Theorem (A.- Mkrtchyan (2020))

As $d \to 1$ the stable sandpile of the Leaky-ASM converges pointwise to the stable sandpile of the ASM with background height $-1$.

Sketch of proof:

Couple the leaky-ASM to a modified ASM in which sites topple if they have 5 or more grains of sand.
ASM introduced by Bak-Tang-Wiesenfeld in 1987 as a model for fractals and self-organized criticality.

At each time step a site is chosen randomly and one grain of sand is added. All unstable sites topple. The distribution of avalanches has a power law tail (Dhar 2006?).
Background

- **ASM** introduced by Bak-Tang-Wiesenfeld in 1987 as a model for fractals and self-organized criticality.
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- **Dissipative sandpiles** introduced by Manna-Kiss-Kertész in 1990 to model systems in which the average transfer ratio is a parameter or random quantity.
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Dhar-Sadhu (2013) proposed using sandpiles to model pattern formation and proportionate growth.
- The odometer is piecewise quadratic (Ostojic 2003).
- A limit pattern exists (Pegden-Smart 2013).
- The internal fractal structure is connected to Apollonian circle packings (Levine-Pegden-Smart 2016 and Pegden-Smart 2020).
Background

- **ASM** introduced by Bak-Tang-Wiesenfeld in 1987 as a model for **fractals** and **self-organized criticality**.
  - At each time step a site is chosen randomly and one grain of sand is added. All unstable sites topple. The distribution of **avalanches** has a **power law tail** (Dhar 2006?).
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  - A limit pattern exists (Pegden-Smart 2013).
  - The internal fractal structure is connected to **Apollonian circle packings** (Levine-Pegden-Smart 2016 and Pegden-Smart 2020).
- The **ASM** is a discrete model of a free boundary problem.
Outline of our proof:

- Relate the Leaky-ASM to a killed random walk.
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- Use the steepest descent method to compute the asymptotic *death probability*.
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- Relate the Leaky-ASM to a killed random walk.
- Use the steepest descent method to compute the asymptotic death probability.
- Level curves of $\frac{4}{n}$ and $\frac{4(d - 1)}{n}$ in the death probability bound the Leaky-ASM with $n$ chips started at the origin.
Killed random walk

Let $X_1, X_2, \ldots$ be i.i.d random variables with

$$P\{X_j = (1, 0)\} = \frac{1}{4d}, \quad P\{X_j = (-1, 0)\} = \frac{1}{4d},$$

$$P\{X_j = (0, 1)\} = \frac{1}{4d}, \quad P\{X_j = (0, -1)\} = \frac{1}{4d},$$

$$P\{X_j = (0, 0)\} = 1 - \frac{4}{4d} = 1 - \frac{1}{d}.$$  

The killed random walk (KRW) started at $x \in \mathbb{Z}^2$ is the sequence $S_1, S_2, \ldots$ where

$$S_n = x + \sum_{i=1}^{n} K_i X_i$$

and

$$K_i = \begin{cases} 1 & \text{if the walker is alive at step } i \\ 0 & \text{else.} \end{cases}$$
Let $G_d(x) = P($walker dies at $x$) be the death probability.

**Definition**
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Start with initial sandpile $s(x) = n\delta_{0,0}(x)$ and topple until reaching the **stable sandpile** $s_\infty(x).$
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**Proposition (A.-Mkrtchyan 2020)**

For the operator

$$T = \frac{1}{d} \Delta - \left(\frac{d - 1}{d}\right) I$$

we have

$$T(u(x) - G_d(x)) = \frac{d - 1}{dn} s_\infty(x).$$
“Invert”

\[ T = \frac{1}{d} \Delta - \left( \frac{d - 1}{d} \right) I \]

and use inequality

\[ 0 \leq s_{\infty}(x) < 4d \]

to obtain the key lemma:

Lemma (A.-Mkrtchyan 2020)

1. If \( G_d(x) < \frac{4(d - 1)}{n} \), then \( u(x) = 0 \), i.e. \( x \notin D_{n,d} \).
2. If \( G_d(x) \geq \frac{4d}{n} \), then \( u(x) \geq 4d \), i.e. \( x \in D_{n,d} \).

\( D_{n,d} \) is the set of sites which topple.
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**Consequence**

Asymptotics of \( G_d(x) \) give the boundary of the limit shape.
The spectral curve of the massive Laplacian can be used to compute asymptotics of $G_d(x)$.

**Definition**

The **massive Laplacian** $\Delta^m : \mathbb{C}^V \rightarrow \mathbb{C}^V$ is defined by

$$(\Delta^m f)(x) = \sum_{y \sim x} P(x \rightarrow y)(f(y) - f(x)) - P(\text{dies})f(x)$$

$$= \sum_{y \sim x} P(x \rightarrow y)f(y) - f(x)$$

where $P(x \rightarrow y)$ is the probability that the KRW moves from vertex $x$ to $y$ and $P(\text{dies})$ is the probability it is killed.
When the probabilities are periodic the spectral curve is

\[ P(z, w) = \det \Delta^m(z, w). \]

Probabilities are modified by \( z \) or \( w \) when crossing a fundamental domain.
Spectral curve of KRW

- When the probabilities are periodic the spectral curve is

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Probabilities are modified by \( z \) or \( w \) when crossing a fundamental domain.

- For the KRW the fundamental domain has size \( 1 \times 1 \) and

\[ (\Delta^m f)(x) = \sum_{y \sim x} \frac{1}{4d} f(y) - f(x). \]

- \( \Delta^m \) is a \( 1 \times 1 \) matrix with spectral curve

\[ P(z, w) = 4d - \left( z + z^{-1} + w + w^{-1} \right). \]
Expand the normalized spectral curve in a power series convergent near \((1, 1)\) to compute probabilities:

\[
\frac{4(d - 1)}{P(z, w)} = \frac{4(d - 1)}{4d - (z + z^{-1} + w + w^{-1})} = \frac{d - 1}{d} \sum_{k=0}^{\infty} \left( \frac{z + z^{-1} + w + w^{-1}}{4d} \right)^k
\]

\[
= \sum_{k,l \in \mathbb{Z}} G_d(k, l) z^k w^l,
\]

where \(G_d(k, l)\) is the probability the KRW dies at \((k, l)\).
Contour integration gives the coefficients in the direction \( \nu_a = (1, a) \) for \( 0 < a < 1 \):

\[
G_d(r\nu_a) = \frac{1}{(2\pi i)^2} \oint_{C_w} \oint_{C_z} \frac{4(d - 1)}{P(z, w)} \frac{dz}{z^{r+1}} \frac{dw}{w^{ar+1}}
\]

\[
= \frac{4(d - 1)}{2\pi i} \oint_{C} f(w) e^{rS(w)} dw
\]

where

\[
f(w) = \frac{1}{w \sqrt{(4d - w - 1/w)^2 - 4}}
\]

\[
S(w) = \log \left( \frac{4d - w - \frac{1}{w} - \sqrt{(4d - w - \frac{1}{w})^2 - 4}}{2w^a} \right)
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Use the steepest descent method to compute the asymptotics.
Let $w_+$ be the real critical point of $S(w)$ with $w_+ > 1$

Deform the contour of integration to pass through the critical point and make the change of variable $w = w_+ + i \frac{y}{\sqrt{r}}$:

$$G_d(rv_a) = \frac{4(d - 1)}{2\pi i} \oint_C f(w) e^{rS(w)} dw$$

$$= \frac{4(d - 1)}{2\pi \sqrt{r}} f(w_+) e^{rS(w_+)} \int_{-\infty}^{\infty} e^{-\frac{S''(w_+)y^2}{2}} (1 + o(1)) dy.$$
Steepest descent method

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- Deform the contour of integration to pass through the critical point and make the change of variable $w = w_+ + i \frac{y}{\sqrt{r}}$:

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Solving

$$G_d(r_0v_a) = \frac{4(d - 1)}{n} \quad \text{and} \quad G_d(r_iv_a) = \frac{4d}{n}.$$  

gives the boundaries for the limit shape.
The limit shape for initial sandpile $s_0 = n\delta_{(0,0)}$ is parametrized by

$$-\log(n)\left(\frac{1}{S(w_+)}, \frac{a}{S(w_+)}\right) \text{ for } 0 \leq a \leq 1,$$

and its reflections with respect to the coordinate axes and the line $y = x$.

Figure: Limit shapes with $d = 1.05, 2, \text{ and } 1000.$
The amoeba of a polynomial $P(z, w)$ is the image of $\{(z, w) \in \mathbb{C}^2 : P(z, w) = 0\}$ under the map

$$(z, w) \mapsto (\log |z|, \log |w|).$$

**Figure:** The boundary of the amoeba of $P(z, w) = 4d - (z + z^{-1} + w + w^{-1})$ and its dual curve. The red curve bounds the gaseous phase.

**Definition**

The bounded complementary component of an amoeba is the gaseous phase.
Theorem (A.-Mkrtchyan 2020)

The limit shape of the Leaky-ASM is (up to scale) the dual of the boundary of the gaseous phase in the amoeba.
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The limit shape of the Leaky-ASM is (up to scale) the dual of the boundary of the gaseous phase in the amoeba.

- For $P(z, w) = 4d - (z + z^{-1} + w + w^{-1})$ the boundary of the gaseous phase is given by the implicit equation

$$4d = e^x + e^{-x} + e^y + e^{-y} \quad \text{with } x, y \in \mathbb{R}.$$ 

The boundary of the gaseous phase is $z, w \in \mathbb{R}$ with $zw > 0.$
Theorem (A.-Mkrtchyan 2020)

The limit shape of the Leaky-ASM is (up to scale) the dual of the boundary of the gaseous phase in the amoeba.

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  4d = e^x + e^{-x} + e^y + e^{-y}
  \]
  with \( x, y \in \mathbb{R} \).

The boundary of the gaseous phase is \( z, w \in \mathbb{R} \) with \( zw > 0 \).

- The other boundary components correspond to \( zw < 0 \).
Why do amoebae appear?

- Asymptotic level curves of

\[ G_d(r \nu_a) = \frac{4(d - 1)}{2\pi i} \oint_C f(w) e^{rS(w)} dw \]

correspond to the limit shape.

- If the model has a spectral curve \( P(z, w) \) and \( S(w) = -\ln(zw^a) \) for \((z, w)\) satisfying \( P(z, w) = 0 \) then the asymptotic level curves of \( P_d(r \nu_a) \) are given by the boundary of the gaseous phase in the amoeba.
Thank you!

I. Alevy and S. Mkrtchyan, *The Limit Shape of the Leaky Abelian Sandpile Model*, arXiv e-prints, arXiv:2010.01946 (October 2020), 2010.01946.