Solvable Models of Domain Walls in $\mathcal{N} = 1$ Supergravity

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Abstract: We develop a method to embed globally supersymmetric theories with exact BPS domain wall solutions into supergravity, by introducing a gravitationally deformed superpotential. The gravitational deformation is natural in the spirit of maintaining the Kähler invariance. The solutions of the warp factor and the Killing spinor are also obtained. We find that three distinct behaviors of warp factors arise depending on the value of a constant term in the superpotential: exponentially decreasing in both sides of the wall, flat in one side and decreasing in the other, and increasing in one side and decreasing in the other. Only the first possibility gives the localized massless graviton zero mode. Models with multi-walls and models with runaway vacua are also discussed.

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I. INTRODUCTION

The domain wall has been an interesting subject in many areas of physics, such as particle physics, cosmology and the condensed matter physics. In particle physics, the recently developed brane-world models give a new motivation to study domain walls. On the other hand, supersymmetry (SUSY) has been most useful to build unified models beyond the standard model and helps to construct domain walls. Configurations preserving part of SUSY are called Bogomol'nyi-Prasad-Sommerfield (BPS) configurations, which automatically give solutions of equations of motion with minimum energy for the given boundary conditions.

Exact solutions are useful to understand solitons such as domain walls. There have been a number of works to obtain exact solutions in models with global SUSY. In $\mathcal{N} = 1$ SUSY models, exact solutions for a single wall are abundantly available, and also for two-wall solutions with a moduli parameter in a model with two chiral scalar fields. Interesting dynamics of multi-walls in this model are also discussed. Even in $\mathcal{N} = 2$ SUSY models (with eight SUSY), exact single wall solutions as well as exact multi-wall solutions have been constructed. Exact solutions of domain-wall junctions have also been found for $\mathcal{N} = 1$ models and for $\mathcal{N} = 2$ models.

On the other hand, it has been difficult to obtain exact solutions in supergravity (SUGRA), because of highly non-linear nature of gravity. Many attempts revealed useful qualitative features of domain walls in SUGRA theories in four- and five-dimensions. Recently, exact domain wall solutions in SUGRA with a smooth limit of weak gravity have been found in several models: a periodic model in $\mathcal{N} = 1$ SUGRA in four-dimensions, and $T^\ast CP^n$ models in $\mathcal{N} = 2$ SUGRA in five-dimensions. Interestingly, the scalar field configurations in all these SUGRA solutions are found to be exactly the same as the known solution in global SUSY models (the limit of the vanishing gravitational coupling).

The purpose of this paper is to propose a general method to find exact solutions of domain wall in SUGRA models. Inspired by the exact solutions in the above models, we find conditions to preserve the scalar field configurations of the exact solutions in global SUSY models when they are embedded into SUGRA. Namely, we require that the scalar field configurations be unchanged when we find the distortion of the spacetime together with the backreaction by solving the nonlinear field equations of SUGRA with matter. Thus we obtain the necessary gravitational deformations to the superpotential.

In general the SUSY vacua in global SUSY theories change when the theory is coupled to SUGRA due to gravitational effects. This is one of the reasons which prevent us from obtaining the exact domain wall solutions. Therefore our main strategy is that we require gravitational deformations of the superpotential when it is embedded into SUGRA so that the SUSY vacua remain unchanged. As a result, we find that the modified superpotential gives us precisely the same equation for the scalar field as the one in the global SUSY theory. Therefore we obtain the solution for the scalar field configuration which is identical to the global SUSY theory. Once we obtain the exact domain wall solutions, we can also find the distortion of the spacetime by solving the equations for the spacetime metric. We can also obtain the Killing spinor corresponding to the conserved SUSY.

In Sec. I we briefly review domain walls in SUGRA and discuss the deformation of the superpotential which maintains the wall configuration identical to the global SUSY case. A number of interesting models with exact BPS solutions are described in Sec. II. The zero modes and the warp factor are discussed in Sec. III. Sec. IV is devoted to concluding remarks.

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II. DOMAIN WALLS IN $\mathcal{N}=1$ SUGRA

A. Notations

We consider $n$ chiral multiplets $(\phi^i, \chi^i)$ coupled to the gravity multiplet $(e_m, \bar{\psi}^m, \psi^m)$ with the Kähler potential $K(\phi, \phi^*)$ and the superpotential $P_c(\phi)$ in four dimensions. In order to distinguish the superpotential in the SUGRA theories from that in the global SUSY theories, we denote the superpotential in the local SUSY (SUGRA) as $P_c$ and the superpotential in the global SUSY theories as $P_{gl}$. The local Lorentz vector indices are denoted by letters with the underline as $\underline{a}$, and the vector indices transforming under general coordinate transformations are denoted by Latin letters as $m$, $n = 0, 1, 2, 3$. The left(right)-handed spinor indices are denoted by undotted (dotted) Greek letters as $\alpha(\dot{\alpha})$. We follow Ref.\[31\] about other notations for the spinor algebra. The bosonic part of the Lagrangian of the $\mathcal{N}=1$ SUGRA coupled with the chiral multiplets is given by \[8\] :

\[ e^{-1}L = -\frac{1}{2\kappa^2}R - K_{ij}g^{mn}\partial_m\phi^i\partial_n\phi^j - V_{lc}, \quad (1) \]

where the gravitational coupling $\kappa$ is the inverse of the four-dimensional Planck mass $M_P$, $g_{mn}$ is the metric of the spacetime, $e$ is the determinant of the vierbein $e_{\mu}^\alpha$. The scalar potential in the SUGRA theory is denoted by $V_{lc}$ and is given in terms of the covariant derivative $D_i P_c$ in the target space of the superpotential $P_c$ in SUGRA

\[ V_{lc} = e^{\kappa^2 K}(K^{ij}D_i P_c D_j P_c^* - 3\kappa^2|P_{lc}|^2), \quad (2) \]

\[ D_i P_c(\phi) = \partial_i P_c(\phi) + \kappa^2 P_c(\phi)\partial_i K(\phi, \phi^*). \quad (3) \]

The SUGRA Lagrangian is invariant under the SUGRA transformation. SUSY vacua and BPS solutions can be obtained by examining SUGRA transformations. Since we are interested in classical solutions, fermionic fields should vanish. Therefore we need to consider SUGRA transformations of fermions which read

\[ \delta_\zeta \psi_m = 2\kappa^{-1}D_m \zeta + i\kappa e^{\frac{1}{2}K}P_{lc}\sigma_m \zeta, \]
\[ \delta_\zeta \chi^i = i\sqrt{2}\sigma^m \zeta \partial_m \phi^i - \sqrt{2}e^{\frac{1}{2}K}K^{ij*}D_j P_{lc}^* \chi^i, \quad (4) \]

where we drop terms which include the fermionic fields and $\zeta$ is a local SUSY transformation parameter. The spacetime covariant derivative is given by

\[ D_m \zeta = \partial_m \zeta + \zeta \omega_m + \frac{i\kappa^2}{2} \sum_i \text{Im} \left[ \partial_i K \partial_m \phi^i \right] \zeta, \]

where $\omega_m$ denotes the spin connection.

It is well known that a stable solitonic state can be obtained, if part of SUSY (or SUGRA) is preserved and the BPS energy bound is saturated \[4, 13\]. The domain wall solutions interpolating two isolated SUSY vacua typically preserve two out of four SUSY (or SUGRA) and are called $\frac{1}{4}$ BPS states. Let us parametrize the conserved directions of SUGRA transformations as

\[ \zeta(y) = e^{i\theta(y)}\sigma^2 e^{2K(y)}. \quad (5) \]

In addition we make the warped metric Ansatz :

\[ ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \quad (\mu, \nu = 0, 1, 3). \]

The BPS equations can be derived by demanding that the bosonic configuration should satisfy $\delta_\zeta \psi_m = \delta_\zeta \chi = 0$ for the Killing spinors $\zeta(y)$ in Eq.\[18\]. From the condition $\delta_\zeta \psi_m = 0$ we obtain the BPS equation for the warp factor:

\[ \dot{A} = -i\kappa^2 e^{-i\theta} e^{2K}K P_{lc}, \quad (6) \]

where a dot denotes a derivative with respect to the extra coordinate $y$. From the condition $\delta_\zeta \psi_0 = 0$ we obtain the BPS equations for the phase $\theta$ and the modulus $|\zeta_\alpha|$ of the Killing spinor :

\[ \dot{\theta} = -\kappa^2 \text{Im} \left[ \sum_i \dot{\phi}^i \partial_i K \right], \quad (7) \]
\[ |\zeta_\alpha| = \frac{\dot{A}}{2} |\zeta_\alpha|. \quad (8) \]

From the remaining condition $\delta_\zeta \chi^i = 0$ we obtain the BPS equations for the scalar fields:

\[ \dot{\phi}^i = -i\kappa e^{\frac{1}{2}K}K^{ij*}D_j P_{lc}^*. \quad (9) \]

Eqs. \[7\], \[8\], \[9\] and \[10\] are the full set of our BPS equations.

Notice that we can recover results of the global SUSY \[7\] if we take the gravitational coupling $\kappa$ to zero and identify the superpotential $P_{lc}$ in SUGRA with the superpotential $P_{gl}$ in SUSY.

B. Deformation of Superpotential

Recently, we found the exact BPS solution in $\mathcal{N}=1$ SUGRA sine-Gordon model \[27\] by allowing a modification of the superpotential. The gravitational deformation for the superpotential is originally introduced in order to keep the periodicity of the model with the aid of the Kähler transformation. In this paper we extend the gravitational modification of the superpotential to other models in order to obtain exact BPS solutions in SUGRA theories even in models without particular symmetry, such as the periodicity in sine-Gordon model.

We first note that SUSY theories are always invariant under the following Kähler transformations

\[ \begin{align*}
K(\phi, \phi^*) &\to K(\phi, \phi^*) + F(\phi) + F^*(\phi^*), \\
P_{\phi^i}(\phi) &\to P_{\phi^i}(\phi),
\end{align*} \quad (10) \]
where $F(\phi)$ is a holomorphic function of $\phi^i$. Moreover, we also note that no physical difference arises if we add a constant $a$ in the superpotential $P_{\text{gl}}$ in global SUSY theories

$$\tilde{P}_{\text{gl}} = P_{\text{gl}} + a.$$  \[11\]

On the other hand, the Kähler transformations in SUGRA theories should accompany the transformations of superpotential

$$\begin{cases} K(\phi, \phi^*) \to K(\phi, \phi^*) + F(\phi) + F^*(\phi^*), \\ P_{\text{lc}}(\phi) \to e^{-\kappa^2 F(\phi)} P_{\text{lc}}(\phi), \end{cases}$$  \[12\]

and the Weyl transformations of fermions \[31\] :

$$\chi^i \to e^{\frac{i\kappa}{2} \text{Im}[F(\phi)]} \chi^i, \quad \psi_m \to e^{-\frac{i\kappa}{2} \text{Im}[F(\phi)]} \psi_m.$$  

We observe that an additive constant $a$ in Eq. \[11\] does have a physical implication when coupled to gravity in SUGRA, in contrast to the global SUSY theories.

Since the superpotential of the global SUSY theories does not transform under the Kähler transformations, we need to make a gravitational deformation of the superpotential if we wish to make the superpotential to be invariant under the Kähler transformations in SUGRA when it is embedded into SUGRA theories. Then the embedded theories are assured to have a smooth limit of vanishing gravitational coupling, and their vacua as well as solutions are likely to be preserved.

Now let us define a new holomorphic function $\tilde{K}$ from Kähler potential by replacing $\phi^i$ with $\phi$ in $K$ :

$$\tilde{K}(\phi) \equiv K(\phi, \phi^* \to \phi).$$

This new function transforms as $\tilde{K} \to \tilde{K} + 2F$ under the Kähler transformation \[12\] of SUGRA. Then, we will choose the SUGRA superpotential $P_{\text{lc}}$ from the global SUSY superpotential with a gravitational deformation as

$$P_{\text{lc}}(\phi) = e^{-\frac{\kappa}{2} \tilde{K}(\phi)} (P_{\text{gl}}(\phi) + a) = e^{-\frac{\kappa}{2} \tilde{K}(\phi)} \tilde{P}_{\text{gl}}(\phi),$$  \[13\]

where we take into account of the possibility of adding a constant $a$ in $P_{\text{gl}}$ which is physically meaningful only when coupled to SUGRA. As for the Kähler potential $K$ we choose the same Kähler potential as the global SUSY. With this gravitational deformation, the corrected superpotential $P_{\text{lc}}$ automatically obeys the transformation law \[12\] of the SUGRA theory, as a consequence of the globally SUSY Kähler transformation \[10\]. We shall show in Appendix that the above gravitational deformation is the unique possibility if we require that the BPS equations for matter scalars in the SUGRA theories should be identical to those in the global SUSY theories assuming three conditions: minimal kinetic term (or nonlinear sigma model that can be transformed to minimal kinetic term), only single scalar field has nontrivial configuration in BPS solutions, and is real.

One can shift the gravitational deformation of the superpotential to that of the Kähler potential by making another Kähler transformation \[12\] with $F = -\frac{1}{2} \tilde{K}$. It is interesting to recall that the SUGRA theories in five dimensions requires a gravitational deformation of target space manifolds of hypermultiplets from hyper-\text{Kähler} to quaternionic Kähler manifolds \[37\]. To find out the necessary gravitational deformations with a smooth limit of the vanishing gravitational coupling has been a challenge for some time \[29, 30, 19, 26, 28, 57, 58\]. In this respect, it is quite natural that a coupling to SUGRA in four dimensions also accompanies gravitational deformations of the Kähler potential or/and the superpotential, since only a combination of Kähler potential and the superpotential has an invariant meaning in $N = 1$ SUGRA in four dimensions \[31\]. Although scalar fields in SUGRA are generically a nonlinear sigma model in this sense, we choose here to give gravitational deformations to the superpotential rather than to the Kähler potential.

Notice that this deformed superpotential $P_{\text{lc}}$ reduces to the global SUSY superpotential $P_{\text{gl}}$ when we turn off the gravitational coupling $\kappa$. As we will see shortly, this special choice of the superpotential in Eq. \[13\] allows us to obtain exact BPS solutions in $N = 1$ SUGRA.

The covariant derivative \[3\] in target space acting on the gravitationally deformed superpotential \[13\] becomes

$$D_i P_{\text{lc}} = e^{-\frac{\kappa}{2} \tilde{K}} \left[ \partial_i P_{\text{gl}} + \kappa^2 \tilde{P}_{\text{gl}} \partial_i \left( K - \frac{1}{2} \tilde{K} \right) \right].$$  \[14\]

and the scalar potential \[2\] is of the form:

$$V_{\text{lc}} = e^{\kappa^2 (K - \text{Re}[\tilde{K}])} \left[ K^{ij} \left\{ \partial_i P_{\text{gl}} + \kappa^2 \tilde{P}_{\text{gl}} \partial_i \left( K - \frac{1}{2} \tilde{K} \right) \right\} \times \left\{ \partial_j P_{\text{gl}} + \kappa^2 \tilde{P}_{\text{gl}} \partial_j \left( K - \frac{1}{2} \tilde{K} \right) \right\} - 3 \kappa^2 |\tilde{P}_{\text{gl}}|^2 \right].$$  \[15\]

Notice that the covariant derivative \[14\] is covariant and the scalar potential \[15\] is invariant under the SUGRA Kähler transformation \[12\], as a consequence of covariance (invariance) under the globally SUSY Kähler transformation \[10\].

For simplicity, we concentrate on the model with the minimal Kähler potential $K = \sum_i |\phi^i|^2$ in what follows.

Then, the covariant derivative \[14\] and the scalar potential \[15\] become :

$$D_i P_{\text{lc}} = e^{-\frac{\kappa}{2} \sum_i |\phi^i|^2} \left[ \partial_i P_{\text{gl}} - 2i \kappa^2 \text{Im}[\phi^i] \tilde{P}_{\text{gl}} \right],$$  \[16\]

$$V_{\text{lc}} = e^{-2\kappa^2 \sum_i \text{Im}[\phi^i]^2} \times \left[ \sum_i \left| \partial_i P_{\text{gl}} - 2i \kappa^2 \text{Im}[\phi^i] \tilde{P}_{\text{gl}} \right|^2 - 3 \kappa^2 |\tilde{P}_{\text{gl}}|^2 \right].$$  \[17\]

The gravitationally deformed superpotential and scalar potential takes a simpler form if we restrict ourselves to a
section of the bosonic fields space where all scalar fields $\phi^i$ take only real values (real section). This is often the case when we consider models whose vacua are distributed on the real axes in the complex fields space, and BPS domain walls which interpolate these vacua. In fact, we will deal with such models in the following section. For that real section of field space, the scalar potential reduces to

$$V_{lc} = \sum_i (\partial_i P_{gl})^2 - 3\kappa^2 (P_{gl} + a)^2. \quad (18)$$

Here we assume that couplings in the superpotential are real parameters and $\phi^i = \phi^i$. Then, the superpotential is essentially a real function of the real field $\phi$. Notice that this type of the scalar potential often arises in a restricted section of field space of the $D$ dimensional SUGRA and ensures stable AdS vacua, if the superpotential has at least a stationary point \cite{34, 35}. The first term in Eq. (18) corresponds to the term which expresses a gravitational correction. The first term vanishes at SUSY vacua and generally the second term is non-vanishing. Therefore, the vacuum becomes AdS (or flat if $a$ happens to cancel with the $-P_{gl}(\phi)$) spacetime, since the second term gives a negative cosmological constant.

One of the most important points is that the locations of the SUSY vacua for the superpotential $P_{lc}$ given in Eq. (13) are not changed from the global SUSY model with superpotential $P_{gl}$. The SUSY vacua are determined from the “$F$-term” condition in SUGRA \cite{31}:

$$D_i P_{lc} = 0.$$  

Because of Eq. (10) and our choice of the superpotential $P_{lc}$ in Eq. (13) with the minimal Kähler potential, this condition agrees exactly with that of global SUSY for the real section of the field space. Hence, the SUGRA theory with our gravitationally deformed superpotential have the SUSY vacua that are precisely identical to the SUSY vacua of the global SUSY theory at the stationary points of $P_{gl}$.

The BPS equation \cite{7} for the phase $\theta(y)$ of Killing spinor defined in Eq. (3) implies that the phase should be independent of $y$ for the BPS solution with real scalar field configurations. The reality of scalar fields is consistent with the BPS equation for matter fields \cite{19} only if $\theta = \pm \pi/2 + 2n\pi$, $n \in \mathbb{Z}$:

$$\phi = e^{i(\theta - \frac{\pi}{2})} \partial_i P_{gl}(\phi), \quad (19)$$

which is exactly identical to the BPS equation in global SUSY theories with the globally SUSY superpotential $P_{gl}$. We will refer the case of $\theta = \pi/2 + 2n\pi$ ($\theta = -\pi/2 + 2n\pi$) as the BPS (anti-BPS) solutions. Therefore, we can automatically obtain exact BPS solutions in SUGRA, if we choose the superpotential according to Eq. (13). The warp factor and the Killing spinor are also obtained from the other BPS equations (3) and (8):

$$\dot{A} = \kappa^2 e^{-i(\theta + \frac{\pi}{2})} P_{gl}, \quad (20)$$

$$\zeta_{\alpha} = e^{i(\theta + \frac{\pi}{2})} e^{\frac{\theta}{2}} \times \left( \epsilon_1 \right), \quad (21)$$

where $\epsilon_{1,2}$ represents a constant Grassmann parameter corresponding to the two conserved SUSY directions.

Notice that the energy density of the BPS domain wall obtained here is precisely identical to the one in the global SUSY model which is given by the topological charge \cite{17}:

$$Z = 2|\Delta P_{gl}|, \quad (22)$$

We shall illustrate this point for concrete examples in the following sections.

III. EXACTLY SOLVABLE EXAMPLES

A. Double Well Model

To obtain a stable BPS domain wall in the global SUSY model, there must be at least two isolated SUSY vacua. Assuming the minimal kinetic term, one of the simplest superpotentials which give such vacua in global SUSY theories is the $\phi^3$ type:

$$P_{gl} = \Lambda^2 \phi - \frac{g}{3} \phi^3, \quad (23)$$

where $g, \Lambda$ are both real positive coupling constants. The SUSY vacua are given as stationary points of the superpotential:

$$\langle \phi \rangle = \left( \frac{\Lambda}{\sqrt{g}}, - \frac{\Lambda}{\sqrt{g}} \right), \quad \langle P_{gl} \rangle = \left( \frac{2 \Lambda^3}{3 \sqrt{g}}, - \frac{2 \Lambda^3}{3 \sqrt{g}} \right), \quad (24)$$

whose vacuum energy density vanishes as a consequence of SUSY. The BPS domain wall which interpolates these two vacua is the solution of the BPS equation (19):

$$\phi(y) = e^{i(\theta - \frac{\pi}{2})} \frac{\Lambda}{\sqrt{g}} \tanh \Lambda \sqrt{g}(y - y_0), \quad (25)$$

where $\theta = \pm \pi/2$ and $y_0$ is a collective coordinate which corresponds to the wall position. The energy density of the BPS wall solution is given by the topological charge in Eq. (22):

$$\mathcal{E} = Z \equiv 2|\Delta P_{gl}| = \frac{8 \Lambda^3}{3 \sqrt{g}} \quad (26)$$

This global SUSY model has been studied by coupling to SUGRA without any gravitational deformations of the superpotential \cite{18}. It may be instructive to compare two SUGRA theories: one with our gravitationally deformed superpotential $P_{lc}$ given by the prescription \cite{15} and the
other with the superpotential \( P_{gl} \) \cite{28} without gravitational deformations inserted in place of the \( P_{kc} \) in the SUGRA Lagrangian \cite{11} and \cite{2}. For the latter choice, the SUSY vacuum condition \( D_{\phi} P_{kc} = 0 \) in the SUGRA theory gives two isolated SUSY vacua:

\[
\langle \phi \rangle = \left( \sqrt{-\alpha + \sqrt{\alpha^2 + \beta}}, -\sqrt{-\alpha + \sqrt{\alpha^2 + \beta}} \right),
\]

where \( \alpha(g, \Lambda; \kappa) \equiv \frac{3(g-\kappa^2 \Lambda^2)}{2g\kappa^2} \) and \( \beta(g, \Lambda; \kappa) \equiv \frac{3\Lambda^2}{g\kappa^2} \). These reduce to Eq. \cite{24} when we turn off the gravitational coupling \( \kappa \). However, it is generally difficult to obtain an exact BPS solution, since these vacua have a nontrivial dependence on the gravitational coupling \( \kappa \). The wall configuration has been studied numerically and it is reasonably compelling that the wall solution should exist \cite{18}, although an explicit demonstration of the solution was difficult.

On the contrary, according to our prescription of gravitational deformations in Eq. \cite{18} we should choose the superpotential in SUGRA as

\[
P_{kc} = e^{-\frac{2}{3} \phi^2} \left( \Lambda^2 \phi - \frac{2}{3} \phi^3 + a \right).
\]

The BPS equations \cite{11}, \cite{12}, \cite{13} and \cite{15} in SUGRA with this modified superpotential give the same vacua as Eq. \cite{24} and precisely the same exact BPS solution in Eq. \cite{25} which interpolates these two vacua. The vacuum energy densities in these two SUSY vacua no longer vanish but are negative

\[
V_{vac} = -3\kappa^2 \left( \epsilon(y) \frac{2}{3} \Lambda^3 \sqrt{g} + a \right)^2.
\]

Therefore the BPS domain wall in Eq. \cite{25} interpolates two AdS vacua with decreasing warp factor asymptotically to both infinities, if \( |a| \leq \frac{2}{3} \Lambda^3 \sqrt{g} \). This is phenomenologically desirable situation corresponding to IR fixed points in both infinities with respect to AdS/CFT correspondence \cite{39}. If \( a = \frac{2}{3} \Lambda^3 \sqrt{g} \), positive (negative) asymptotic infinity is flat space, whereas the other infinity is AdS space and the warp factor is exponentially decreasing. If \( a < \frac{2}{3} \Lambda^3 \sqrt{g} \) (\( a > \frac{2}{3} \Lambda^3 \sqrt{g} \)), both asymptotic infinities are AdS spaces, but the warp factor is exponentially increasing at positive (negative) asymptotic infinity and is exponentially decreasing at negative (positive) asymptotic infinity.

We can also obtain exact BPS solutions for the warp factor from Eq. \cite{24}

\[
A = -\kappa^2 \left[ a y + \frac{2\Lambda^2}{3g} \left( \log \cosh \Lambda \sqrt{g}(y-y_0) + \frac{\tanh^2 \Lambda \sqrt{g}(y-y_0)}{4} \right) \right].
\]

The scalar potential and the global SUSY superpotential with the parameters \( a = (0, -2/3, -1) \Lambda^3 \sqrt{g} \) are shown in FIG. 1. In FIG. 2 the profiles of the warp factor are shown. The Killing spinor \( \zeta_{\mu} \) is obtained by plugging this warp factor into Eqs. \cite{21}.

Notice that the energy density of the BPS domain wall is just the same as that of the global SUSY model in Eq. \cite{26}. In the case of no gravitational deformations for the superpotential, it was shown that the energy density of the BPS domain wall in SUGRA generally differs from that of the global SUSY by a factor arising from the

FIG. 1: The scalar potential \( V_{kc} \) (solid line) and the global SUSY superpotential \( P_{gl} \) (broken line) as a function of Re[\( \phi \)]. Parameters are taken to be \( \kappa = 0.3, g = 1 \) and \( \Lambda = 1 \). From left to right, \( a \) is \( (0, -2/3, -1) \), corresponding to IR-IR, IR-flat, IR-UV behaviors.

FIG. 2: The warp factor \( e^A \) as a function of \( y \). Parameters are taken to be \( \kappa = 0.3, g = 1 \) and \( \Lambda = 1 \). From left to right, \( a \) is \( (0, -2/3, -1) \), corresponding to IR-IR, IR-flat, IR-UV behaviors.
Kähler potential [18]

\[ \mathcal{E} = 2 \left| e^{\frac{\phi}{2g}} K |\Delta P_c| \right| \neq Z. \]

In our model, this factor from the Kähler potential is absorbed in the gravitationally deformed superpotential in Eq. [13], so that the energy density in SUGRA is precisely the same as that in global SUSY.

### B. Sine-Gordon Model

In Ref. [27] we found the exact BPS solution for the modified sine-Gordon model with the superpotential given here except the possible additive constant \( a \). Here we also add a possibility of this additive constant in superpotential:

\[ P_c = e^{-\frac{\phi^2}{2g^2}} \left( \frac{\Lambda^3}{g^2} \sin \frac{\phi}{\Lambda} + a \right). \]

The SUSY vacua is periodically distributed on the real axis in the complex \( \phi \) plane:

\[ \langle \phi \rangle = \frac{\Lambda}{g} \left( \frac{\pi}{2} + n\pi \right), \quad (n \in \mathbb{Z}). \]

The BPS solution which interpolates any two adjacent vacua is of the form:

\[ \phi = \frac{\Lambda}{g} \left[ (-1)^n \left\{ 2 \tan^{-1} e^{\pm \Lambda (y - y_0)} - \frac{\pi}{2} \right\} + n\pi \right], \]

\[ A = -\kappa^2 \left[ ay + \frac{\Lambda^2}{g^2} \log \cosh \Lambda (y - y_0) \right]. \]

More details about this model are given in Ref. [27].

### C. Two Walls Model

An interesting global SUSY model in four dimensions has been found which allows two domain walls as an exact BPS solution [8, 12]. The model consists of two chiral superfields \( \Phi \) and \( \chi \) whose lowest components are denoted by \( \phi \) and \( \chi \), respectively.

Let us consider the minimal kinetic term given by the minimal Kähler potential : \( K = |\phi|^2 + |\chi|^2 \). The simplest (trivial) model admitting two walls in global SUSY consists of two decoupled double-well model:

\[ P_{c1} = \Lambda_{\phi}^2 \phi - \frac{g_{\phi}}{3} \phi^3 + \Lambda_{\chi}^2 \chi - \frac{g_{\chi}}{3} \chi^3, \]

where all the couplings \( \Lambda_{\phi}, \Lambda_{\chi}, g_{\phi}, g_{\chi} \) are assumed to be real positive. This superpotential gives four isolated SUSY vacua:

\[ \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \frac{\pm \Lambda_{\phi}}{|g_{\phi}|} \\ \frac{\pm \Lambda_{\chi}}{|g_{\chi}|} \end{pmatrix}, \quad \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \frac{\pm \Lambda_{\phi}}{|g_{\phi}|} \\ \frac{\pm \Lambda_{\chi}}{|g_{\chi}|} \end{pmatrix}. \]

Since \( \phi \) and \( \chi \) are decoupled, the exact BPS solution is a superposition of that of each double-well model:

\[ \phi = \epsilon_{\phi} \frac{\Lambda_{\phi}}{\sqrt{g_{\phi}}} \tanh \Lambda_{\phi} (y - y_{\phi}), \]

\[ \chi = \epsilon_{\chi} \frac{\Lambda_{\chi}}{\sqrt{g_{\chi}}} \tanh \Lambda_{\chi} (y - y_{\chi}), \]

where \( \epsilon_{\phi,\chi} = \pm 1 \). This solution has two collective coordinates: the center of the mass \( y_{cm} = \frac{y_{\phi} + y_{\chi}}{2} \) and the relative distance between the two walls \( R = |y_{\phi} - y_{\chi}| \).

In general, this superposition principle does not hold in gravity theory. Namely, the superposition of the individual solutions is not a solution. That is because two scalar fields \( \phi, \chi \) are coupled via gravity even if these are decoupled in the superpotential. The superposition of the solutions of the individual models does not satisfy the equations of motion when they are coupled to gravity theory. We can see this gravitational interaction in the BPS equation (20) if we use the superpotential (26) of the global SUSY model (without the gravitational deformations) inserted into the SUGRA superpotential \( P_{c1} \). Two fields are coupled through the Kähler potential. It is then difficult to obtain the exact BPS solutions for two walls in SUGRA.

If we choose the superpotential according to our prescription [13],

\[ P_{c2} = e^{-\frac{\phi^2}{2g^2}} (\phi^2 + \chi^2) \left[ \Lambda_{\phi}^2 \phi - \frac{g_{\phi}}{3} \phi^3 + \Lambda_{\chi}^2 \chi - \frac{g_{\chi}}{3} \chi^3 + a \right], \]

two fields behave as if they are effectively decoupled even in the presence of gravity. Then we obtain the exact solution for scalar fields identical to Eqs. (28) and (29).

The BPS solution of the warp factor is of the form:

\[ A = -\kappa^2 ay \]

\[-\frac{\kappa^2 \Lambda_{\phi}^2}{\sqrt{g_{\phi}}} \left[ \frac{2}{3} \log \cosh \Lambda_{\phi} (y - y_{\phi}) + \frac{\tanh^2 \Lambda_{\phi} (y - y_{\phi})}{6} \right] \]

\[-\frac{\kappa^2 \Lambda_{\chi}^2}{\sqrt{g_{\chi}}} \left[ \frac{2}{3} \log \cosh \Lambda_{\chi} (y - y_{\chi}) + \frac{\tanh^2 \Lambda_{\chi} (y - y_{\chi})}{6} \right]. \]

Next we turn our attention to the more interesting model. In Ref. [2], an integral of motion was constructed for a global SUSY model with two chiral scalar fields with the superpotential

\[ P_{GL} = \frac{m^2}{3} \phi^3 - \frac{\lambda}{3} \phi^3 + \alpha \phi \chi^2, \]

where all the coupling constants are real and positive. This model has the following four isolated SUSY vacua:

\[ \left( \begin{array}{c} \phi \\ \chi \end{array} \right) = \begin{pmatrix} \pm m/\lambda \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \pm m/\sqrt{\alpha \lambda} \end{pmatrix}. \]

The BPS equations for matter fields can be cast into dimensionless forms:

\[ \frac{df}{d(my)} = 1 - f^2 - h^2, \quad \frac{dh}{d(my)} = -\frac{2}{\rho} \rho \phi h, \]

(30)
where \( f, h \) are dimensionless fields defined as \( f \equiv (\lambda/m)\phi \) and \( h \equiv (\sqrt{\alpha\lambda/m})\chi \), \( \rho \) is defined as \( \rho \equiv \lambda/\alpha \). By taking the ratio of Eqs. (30) to eliminate the \( u = my \) dependence, we can integrate once to obtain an implicit solution giving a relation between \( f \) and \( h \)

\[
f^2 = 1 - \frac{\rho}{\rho - 2} h^2 - Ch^\rho,
\]

where \( C \) is an integration constant which corresponds to a collective coordinate related with the relative distance between two walls. Supplemented by one of the above Eqs. (30), one can explicitly obtain the complete solution.

For the special case where \( \rho = 4 \) these BPS equations are explicitly solvable. To clarify the physical meaning of the relative distance between two walls, it is more convenient to convert the integration constant \( C \) into the following parameter \( t \equiv 1/\sqrt{C + 1} \). The exact BPS two wall solution is given with a center of position \( y_0 \) and another moduli parameter \( t \) by

\[
\phi = \frac{m}{2\lambda} \left( \tanh \frac{u - s}{2} + \tanh \frac{u + s}{2} \right),
\]

\[
\chi = \frac{m}{2\lambda} \sqrt{\frac{t}{\cosh m(y - y_0) + t}},
\]

where the moduli parameter \( t \) can take values \( 0 < t < \infty \) corresponding to \( \infty > C > -1 \). For \( 1 < t \), we can convert it to \( t = \cosh s \) which can be interpreted as the \( m \) times distance between walls. We obtain

\[
\phi = \frac{m}{2\lambda} \left( \tanh \frac{u - s}{2} + \tanh \frac{u + s}{2} \right),
\]

\[
\chi = \frac{m}{2\lambda} \sqrt{\frac{1}{2} \left( 1 - \tanh \frac{u - s}{2} \tanh \frac{u + s}{2} \right)},
\]

\[
u \equiv m(y - y_0),
\]

and interpret the moduli parameter to be the distance between two walls.

As we have seen, these solutions remain to be BPS solutions for the SUGRA theories, if we choose the superpotential

\[
P_{\text{sc}} = e^{-\frac{m^2}{\lambda^2}(\phi^2 + \chi^2)} \left( \frac{m^2}{\lambda} \phi - \frac{\lambda}{3} \phi^3 - a \phi \chi^2 + a \right).
\]

Plugging Eqs. (31) and (32) into Eq. (20), we obtain the warp factor explicitly

\[
A = -\kappa^2 \left[ \frac{a}{m} \right. \left( 2 \log (\cosh u + t) + \frac{t}{\cosh u + t} + \frac{t^2 - 1}{2(\cosh u + t)^2} \right),
\]

where an integration constant is suppressed. The scalar fields profile and the warp factor are shown in FIG. 3 for

\[
K_{\phi\phi} = \left| \frac{\Lambda^2}{\Lambda^2 - g^2\phi^2} \right|^2.
\]

\[
\begin{array}{c}
\text{FIG. 3: Two walls solution for the scalar fields and the warp factor is shown. The parameters are taken to be } \kappa = 0.3, m = 1, \lambda = 1 \text{ and } a = 0. \text{ We take } s = 10.
\end{array}
\]
Then, the SUSY vacua are
\[ \langle \phi \rangle = \left( \frac{\Lambda}{g}, -\frac{\Lambda}{g} \right). \tag{34} \]

Assuming \( \phi = \phi^* \), the BPS equation in this global SUSY model reduces to
\[ \dot{\phi} = K^{\phi^*} \partial_{\phi^*} P_{lc}^* \left( \frac{\Lambda^2}{g^2} \left( 1 - \frac{g^2}{\Lambda^2} \phi^2 \right)^2 \right), \tag{35} \]
and its solution is exactly solvable:
\[ y - y_0 = \frac{1}{4\Lambda} \left( \frac{2}{g^2} \phi + a \right) + \log \left( 1 + \frac{2}{g^2} \phi \right). \tag{36} \]

Let us couple this model with SUGRA, according to the prescription [13]:
\[ P_{lc} = e^{-\frac{1}{2} \tilde{K}} \left( \frac{\Lambda^2 g}{\phi + a} \right), \quad \tilde{K} = \frac{\Lambda^2}{g^2} \left( \tanh^{-1} \frac{g}{\Lambda} \phi \right)^2. \]

The SUSY vacua can be obtained by demanding the SUSY invariance in Eq. (4). More explicitly, the SUSY vacua are given by Eqs. (9) and (14) as
\[ 0 = e^{\frac{1}{2} \tilde{K}} K^{\phi^*} D_{\phi^*} P_{lc}^*, \tag{37} \]
\[ D_{\phi^*} P_{lc}^* = e^{-\frac{1}{2} \tilde{K}} \left[ \frac{\Lambda^2}{g^2} + \frac{2i\kappa^2 A}{g^2} \left( \frac{\Lambda^2}{g^2} \phi^* + a \right) \right] \]
\[ \left( \frac{1}{g^2} - \frac{2g^2}{\Lambda^2} \phi^2 \right). \]

Eq. (37) reduces to \( 0 = K^{\phi^*} \Lambda^2 / g \) for real field configurations \( \phi = \phi^* \). Therefore the SUSY vacua is unchanged from Eq. (15) given as the zero of the inverse Kähler metric \( K^{\phi^*} \). Moreover, the BPS equation (38) is also unchanged, so the BPS solution is given in Eq. (39). The BPS equation for the warp factor (20) can also be integrated to give an expression in terms of the scalar field \( \phi \)
\[ A = -\kappa^2 \left[ ay + \frac{\Lambda^2}{2g^2} - \frac{g^2}{\Lambda^2} \phi^2 \right]. \]

Together with Eq. (39), it implicitly gives \( A \) as a function of \( y \).

**IV. ZERO MODE WAVE FUNCTION**

In this section we study the behavior of the warp factor \( e^{2A(y)} \). It is well known that the warp factor is closely related to the zero mode wave function of the graviton in the Kaluza-Klein modes expansion. The mode equation for the graviton can be written as a Schrödinger type equation [11, 13]:
\[ \square_{D-1} h_{\mu\nu}^{TT} = \left[ -\partial_z^2 + V(z) \right] h_{\mu\nu}^{TT}, \]
\[ V(z) = \left( \frac{D - 2}{2} dA \right)^2 + \frac{D - 2}{2} d^2 A \]
where \( D \) denotes the spacetime dimension, \( z \) is the conformal flat coordinate defined by \( dy = e^A dz \) and \( h_{\mu\nu}^{TT} \) is the transverse traceless part of the fluctuation defined by \( \delta g_{\mu\nu} = e^{-\frac{2}{D-2}A} \Lambda^{TT} \). The zero mode wave function of the above Schrödinger equation is given by \( h_{\mu\nu}^{TT} = e^{\frac{2}{D-2}A(z)} \), which implies that the graviton zero mode wave function is identical to the warp factor : \( \delta g_{\mu\nu} = e^{2A(y)} \). Hence, the behavior of the warp factor is directly related to the normalizability of the graviton zero mode which is very important for phenomenology.

Next we derive a general property of the warp factor for our model. Using Eqs. (19) and (20), we find
\[ \dot{A} = -\kappa^2 \sum_i (\dot{\phi}_i)^2. \]

This implies that the warp factor has at most one stationary (maximum) point. The physical reason behind this fact should be that matter scalar fields produce only positive energy density.

We also find from Eq. (20) that the warp factor \( A \) has a maximum only at the point where the superpotential \( P_{gl} \) vanishes along the wall trajectory. Only such a wall can have the localized graviton zero mode around the wall.

If the additive constant in the global SUSY superpotential \( P_{gl} \) vanishes \( a = 0 \), the warp factor \( A(y) \) in all the above examples are \( Z_2 \) symmetric under the reflection \( y - y_0 \rightarrow -(y - y_0) \) around the center of the wall. Therefore the graviton zero mode wave function is normalizable and we obtain a localized graviton zero mode around the wall.

On the contrary, we have \( Z_2 \) asymmetric warp factors around the wall, if we have non-vanishing additive constant \( a \). It is interesting to remember that the constant term in the superpotential has no physical effects in global SUSY models. In SUGRA theories, however, the constant term \( a \) produces a dramatic change. The BPS equation for the matter in Eq. (19) is identical to that of the global SUSY, even if we add a constant term \( P_{gl} \). On the other hand the BPS equation for the warp factor (20) is affected by the additive constant \( a \) by controlling the SUSY vacuum energy density. There is a critical value of the constant \( a \) beyond which the graviton zero mode ceases to be normalizable : \( a = -(P_{gl}) \). At this critical value of the constant \( a \), the energy density of one side of asymptotic region (vacuum) vanishes and the metric in this asymptotic infinity reduces to the flat metric. Thus we find three different asymptotic behaviors of warp factors (20) in the context of AdS/CFT correspondence [39]: IR-IR (exponentially decreasing in both infinities), IR-flat (exponentially decreasing in one side and flat in the other), and IR-UV (exponentially decreasing in one side and increasing in the other).
V. CONCLUSION

We proposed a prescription for a gravitational deformation of the superpotential (or Kähler potential) in embedding global SUSY models into SUGRA theories. This is natural from the viewpoint of the Kähler transformation and gives us exact BPS solutions in SUGRA theories from exact solutions in global SUSY models, provided scalar field configurations are real. Thanks to the gravitational deformations of the superpotential, the SUSY vacua in SUGRA theories are identical to those in the global SUSY models. The domain wall solutions which interpolate these vacua are also identical to those of the SUSY models.

The spacetime distortion by the domain wall represented by the warp factor can be also obtained by solving the BPS equation for the warp factor, which is independent of the BPS equation for the matter scalar fields. The Killing spinor is also obtained in terms of the warp factor.

A constant term in the superpotential in global SUSY theories has no physical effects. However, this constant term gives physical effects when coupled to gravity. Naturally our gravitationally deformed superpotential in SUGRA theories has this freedom of choosing an additive constant. This constant term has no effect on the scalar field configurations, but has significant effects on the warp factor. Namely, the asymptotic behavior of the warp factor is different as the constant term crosses a critical value. At the same time, the graviton zero mode on the domain wall is localized for smaller values of the constant, whereas it becomes non-normalizable if the constant is outside of the critical value.

In global SUSY theory there are many models which have exact soliton solutions, such as domain wall junction [14]–[16]. A good progress has been made to study domain wall junction in chiral scalar fields coupled to SUGRA [16], although no explicit solution has been obtained so far. It is an intriguing problem to extend our prescription of gravitational deformations of the superpotential, the SUSY vacua in SUGRA theories are identical to those in the global SUSY models. The domain wall solutions which interpolate these vacua are also identical to those of the SUSY models.

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APPENDIX A: UNIQUENESS OF THE SUPERPOTENTIAL

Here we shall show that our prescription of gravitationally deformed superpotential is the only possibility to obtain BPS solution for matter scalar in SUGRA theories which are identical to that in global SUSY theories.

The BPS equation in SUGRA theories comes from requiring the vanishing SUGRA transformations in [14]. To make this BPS equation for the matter scalar identical to the BPS equation in global SUSY theories, the only possibility is

\[ \partial_j P_{gl} = e^{\frac{\kappa^2}{2}} D_j P_{lc}. \]  

(A1)

For simplicity, we assume that only one scalar field has nontrivial field configurations in BPS solution, and the kinetic term is minimal:

\[ K(\phi, \phi^*) = \phi^* \phi. \]

Moreover we assume a real field configuration for the BPS solution. Then the condition (A1) reduces to

\[ \frac{dP_{gl}(\phi)}{d\phi} = \frac{d}{d\phi} \left( e^{\frac{\kappa^2}{2}} P_{lc}(\phi) \right). \]

We obtain the general solution with an integration constant \( a \)

\[ e^{\frac{\kappa^2}{2}} P_{lc}(\phi) = P_{gl}(\phi) + a. \]

The assumed reality of field configuration requires that the integration constant \( a \) should be real.

[1] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429 (1998) 263 [hep-ph/9803315]; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B436 (1998) 257 [hep-ph/9804398].
[2] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370, [hep-ph/9905221].
[3] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690, [hep-th/9906064].
[4] S. Dimopoulos and H. Georgi, Nucl. Phys. B193 (1981) 150; N. Sakai, Z. f. Phys. C11 (1981) 153; E. Witten, Nucl. Phys. B188 (1981) 513; S. Dimopoulos, S. Raby, and F. Wilczek, Phys. Rev. D24 (1981) 1681.
[5] E. Bogomol’nyi, Sov. J. Nucl. Phys. 24 (1976) 449; M. K. Prasad and C. H. Sommerfield, Phys. Rev. Lett. 35 (1975) 760.
[6] E. Witten and D. Olive, Phys. Lett. B78 (1978) 97.
S.M. Carroll, S. Hellerman and M. Trodden, *Phys. Rev. D62* (2000) 044049, [hep-th/9911083]; A. Chamblin, C. Csaki, J. Erlich, T.J. Hollowood, *Phys. Rev. D62* (2000) 044012, [hep-th/0002076]; T. Nihei, *Phys. Rev. D62* (2000) 124017, [hep-th/0005014].