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THE ADJOINT REPRESENTATION OF QUANTUM ALGEBRA $U_q(sl(2))$

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Starting from any representation of the Lie algebra $g$ on the finite dimensional vector space $V$ we can construct the representation on the space $\text{Aut}(V)$. These representations are of the type of ad. That is one of the reasons, why it is important to study the adjoint representation of the Lie algebra $g$ on the universal enveloping algebra $U(g)$. A similar situation is for the quantum groups $U_q(g)$. In this paper, we study the adjoint representation for the simplest quantum algebra $U_q(sl(2))$ in the case that $q$ is not a root of unity.

Keywords: Quantum algebra; enveloping algebra; adjoint representation.

1. Introduction

It is a well-known fact that the adjoint representation of the algebra $sl(2)$ on its enveloping algebra $U(sl(2))$ is fully reducible and can be decomposed into a direct sum of finite dimensional representations, [1]. Specifically, let the generators $e$, $f$ and $h$ fulfill the commutation relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$  

Then the center $U(sl(2))$ is generated by an element

$$C = ef + fe + \frac{1}{2}h^2,$$

and for any $n, k \in \mathbb{N}_0$, $\mathbb{N}_0 = 0, 1, 2, \ldots$, the vector spaces

$$V_{n,k} = ad_{U(sl(2))}(e^n C^k)$$

are invariant with respect to the adjoint representation and $(2n + 1)$-dimensional. Moreover, we can write

$$U(sl(2)) = \bigoplus_{n, k \in \mathbb{N}_0} V_{n,k}. \quad (1.1)$$

A similar situation is for complex semisimple Lie algebras, see e.g. [1–4].
On the other hand, in the case of the quantum group $U_q(sl(2))$ there exists an element $a$ for which the space $ad_{U_q(sl(2))}a$ is infinite dimensional. The aim of this paper is to find the decomposition of the quantum group $U_q(sl(2))$, where $q$ is not a root of unity, which is similar to the decomposition (1.1).

2. The Quantum Group $U_q(sl(2))$

Let us have the quantum group $U_q(sl(2))$, [5], which is generated by the elements $E, F, K$ and $K^{-1}$ fulfilling the relations

\begin{align*}
KE &= q^2EK, \quadKF = q^{-2}FK, \\
EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}, \quad KK^{-1} = K^{-1}K = 1. \quad(2.1)
\end{align*}

The coproduct $\Delta$, the antipod $S$ and the counit $\epsilon$ are defined by

\begin{align*}
\Delta E &= E \otimes K + 1 \otimes E, \quad S(E) = -EK^{-1}, \quad \epsilon(E) = 0, \\
\Delta F &= F \otimes 1 + K^{-1} \otimes F, \quad S(F) = -KF, \quad \epsilon(F) = 0, \\
\Delta K &= K \otimes K, \quad S(K) = K^{-1}, \quad \epsilon(K) = 1, \\
\Delta K^{-1} &= K^{-1} \otimes K^{-1}, \quad S(K^{-1}) = K, \quad \epsilon(K^{-1}) = 1.
\end{align*}

It is a well-known fact that the basis in $U_q(sl(2))$ consists of the elements $E_n^mF_n^mK_n^m$, where $n_1, n_2 \in \mathbb{N}_0$ and $n_3 \in \mathbb{Z}$.

The adjoint action of the Drinfeld–Jimbo algebras is given by the formula (see e.g. [6])

\[ ad_{E}a = \sum_{a(1)}bS(a(2)). \]

In particular, in the case of the algebra $U_q(sl(2))$ we have

\begin{align*}
ad_{E}a &= EaK^{-1} - aEK^{-1}, \quad ad_{X}a = Fa - K^{-1}aKF, \\
ad_{Y}a &= KaK^{-1}, \quad ad_{K^{-1}}a = K^{-1}aK. \quad(2.2)
\end{align*}

This shows that $ad_{E}E = ad_{F}(KF) = 0$ holds. We denote

\begin{align*}
X &= E, \quad Y = KF, \\
Z &= ad_{E}Y = q^{-2}EF - FE = \frac{K - K^{-1}}{q - q^{-1}} - q^{-1}(q - q^{-1})EF, \quad(2.3)
\end{align*}

and take the element

\[ C = EF + \frac{q^{-1}K+qK^{-1}}{(q - q^{-1})^2}, \quad(2.4) \]

which generates the center of $U_q(sl(2))$.

From Eqs. (2.3) and (2.4) we obtain

\[ K = \frac{q - q^{-1}}{q + q^{-1}}(qZ + (q - q^{-1})C). \quad(2.5) \]

Notice that the element $K^{-1}$ cannot be to expressed by these equations.
3. The New Basis in $U_q(sl(2))$

The next step is to express $U_q(sl(2))$ using the elements $X, Y, Z, C$ and $W = K^{-1}$. From relations (2.3), (2.4), (2.5) and the commutation relations (2.1) we obtain

$$ CX = CX, \quad CY = YC, \quad CZ = ZC, \quad CW = WC, $$

$$ WX = q^{-3}XW, \quad WY = q^3YW, \quad WZ = ZW, $$

$$ ZX = q^2XZ + (q - q^{-1})^2XC, \quad YZ = q^2ZY + (q - q^{-1})^2YC, $$

$$ YX - XY + \frac{1 - q^2}{q - q^{-1}}(qZ + (q - q^{-1})C)Z = 0, \quad \text{(3.1)} $$

The dimension of this space is $\dim \mathcal{V}_N = \frac{1}{6}(N+1)(N+2)(2N+3)$.

$$ \text{ad}_X a = XaW - aXW, \quad \text{ad}_Y a = WY a - W aY, $$

$$ \text{ad}_Z a = \frac{q - q^{-1}}{q + q^{-1}}(qZ + (q - q^{-1})C)aW, $$

$$ \text{ad}_{-a} a = \frac{q - q^{-1}}{q + q^{-1}}Wa(qZ + (q - q^{-1})C). $$

4. The Adjoint Representation in the New Basis

The adjoint representation (2.2) can be rewritten in the form

$$ \text{ad}_X a = XaW - aXW, \quad \text{ad}_Y a = WY a - W aY, $$

$$ \text{ad}_Z a = \frac{q - q^{-1}}{q + q^{-1}}(qZ + (q - q^{-1})C)aW, $$

$$ \text{ad}_{-a} a = \frac{q - q^{-1}}{q + q^{-1}}Wa(qZ + (q - q^{-1})C). $$

4.1. The finite dimensional representations

From relations (3.1) and (4.1) it is possible by direct calculations, to show that the vector space $\mathcal{V}_N$ with the basis $X^{n_1}Y^{n_2}Z^{n_3}C^{n_4}$, where $n_1n_2 = 0$ and $n_1 + n_2 + n_3 + n_4 \leq N$, is invariant with respect to the adjoint representation for any $N$. The dimension of this space is

$$ \dim \mathcal{V}_N = \frac{1}{6}(N+1)(N+2)(2N+3). $$

In the space $\mathcal{V}_N$ there are the highest weight elements $X^{n_1}C^{n_4}$, $n_1 \leq n_4 \leq N$. It is well known, see e.g. [5, p. 61], that the invariant subspace

$$ \mathcal{V}_{(n_1, n_4)} = \text{ad}_{U_q(sl(2))}(X^{n_1}C^{n_4}) $$

has the dimension $2n_1 + 1$ and is irreducible. Since

$$ \sum_{n_1 + n_4 \leq N} (2n_1 + 1) = \frac{1}{6}(N+1)(N+2)(2N+3) = \dim \mathcal{V}_N, $$

the above sequence of representations $\mathcal{V}_N$ are all irreducible.
the subspace $V_{n_1}$, with the basis $X^{n_1}Y^{n_2}Z^{n_3}C^{n_4}$, $n_1n_2 = 0$, can be decomposed into a direct sum of invariant subspaces

$$V_{n_1} = \bigoplus_{n_3,n_4} V_{(n_3,n_4)}.$$  

The eigenvalue of $ad_C$ on the space $V_{(n_3,n_4)}$ is determined by the relation

$$ad_C(X^{n_1}C^{n_4}) = \left( q^{2n_1+1} + q^{-2n_1-1} \right) X^{n_1}C^{n_4}.$$  

4.2. The infinite dimensional representations

By direct calculations we obtain the action of the adjoint representation on the elements of the basis $X^{n_1}C^{n_4}W^{n_5}$ and $Y^{n_3}C^{n_4}W^{n_5}$ for $n_5 \geq 1$ of the following form (in the formulas where $X^{n_1}C^{n_4}$ resp. $Y^{n_3}C^{n_4}$ occurs we suppose that $n_1 > 0$ resp. $n_2 > 0$):

$$ad_C(X^{n_1}C^{n_4}W^{n_5}) = (1 - q^{-2n_3})X^{n_1+1}C^{n_4}W^{n_5+1},$$

$$ad_C(X^{n_1}C^{n_4}W^{n_5}) = \frac{q^{-2n_3+2n_2+1}n_3}{q - q^{-1}} X^{n_1-1}C^{n_4}W^{n_5+1} + (1 - q^{2n_3+2n_2+1})X^{n_1-1}C^{n_4+1}W^{n_5} - \frac{q^{n_3-2n_2-n_1}}{q - q^{-1}} X^{n_1-1}C^{n_4}W^{n_5-1},$$

$$ad_K(X^{n_1}C^{n_4}W^{n_5}) = q^{2n_1}X^{n_1}C^{n_4}W^{n_5},$$

$$ad_K(X^{n_1}C^{n_4}W^{n_5}) = q^{-2n_1}X^{n_1}C^{n_4}W^{n_5},$$

$$ad_C(Y^{n_3}C^{n_4}W^{n_5}) = \frac{-q^{n_3+1}n_3}{q - q^{-1}} Y^{n_3-1}C^{n_4}W^{n_5+1} + \frac{q^{2n_3-n_1+1}n_3}{q - q^{-1}} Y^{n_3-1}C^{n_4+1}W^{n_5} + \frac{q^{2n_3-n_1+1}n_3}{q - q^{-1}} Y^{n_3-1}C^{n_4}W^{n_5-1},$$

$$ad_C(Y^{n_3}C^{n_4}W^{n_5}) = q^{2n_3+2}(1 - q^{-2n_1})Y^{n_3+1}C^{n_4}W^{n_5+1},$$

$$ad_K(Y^{n_3}C^{n_4}W^{n_5}) = q^{2n_3}Y^{n_3}C^{n_4}W^{n_5},$$

$$ad_K(Y^{n_3}C^{n_4}W^{n_5}) = q^{-2n_3}Y^{n_3}C^{n_4}W^{n_5}.$$  

By using these relations we obtain

$$ad_C(X^{n_1}C^{n_4}W^{n_5}) = \frac{q^{2n_1-2n_3+1} + q^{-2n_1+2n_3+1}}{(q - q^{-1})^2} X^{n_1}C^{n_4}W^{n_5} + q^{2n_1}[n_3][n_3+1]X^{n_1}C^{n_4}W^{n_5+2} + q^{-n_1}[n_3][n_1-n_3](q - q^{-1})^2 X^{n_1}C^{n_4+1}W^{n_5+1},$$

$$ad_C(Y^{n_3}C^{n_4}W^{n_5}) = \frac{q^{2n_3-2n_1+1} + q^{-2n_3+2n_1+1}}{(q - q^{-1})^2} Y^{n_3}C^{n_4}W^{n_5} + q^{2n_3}[n_3][n_3+1]Y^{n_3}C^{n_4}W^{n_5+2} + q^{2n_3}[n_3][n_2-n_3](q - q^{-1})^2 Y^{n_3}C^{n_4+1}W^{n_5+1}. $$
4.2.1. The eigenvalues and the eigenfunctions of $ad_C$

Since $V_{lin}$ is invariant subspace, we can define factor-representation of $ad$ on the space

$$W = U_q(sl(2))/(V_{lin}).$$

We denote this representation by the same symbol $ad$, no confusion arises from this abuse of notation.

In the space $W$ we introduce the notation $T_n = [n-1]! \; W^n$ for $n \geq 1$ and the basis consisting of elements

$$X^{n_1}T_{n_2}C^{n_3}, \quad T_nY^{n_2}C^{n_4} = [n_3-1]! \; W^nY^{n_2}C^{n_4} = [n_2-1]! \; q^{2n_2}Y^{n_2}C^{n_4}W^{n_2}.$$

In this basis the representation has the form

$$\begin{align*}
ad_C(X^{n_1}T_{n_2}C^{n_3}) &= q^{-n_2}(q-q^{-1})X^{n_1+1}T_{n_2+1}C^{n_3}, \\
ad_C(X^{n_1}T_{n_2}C^{n_3}) &= \frac{q^{-2n_1+n_2+1}}{q-q^{-1}} X^{n_1-1}T_{n_2+1}C^{n_3} \\
&\quad + q^{-n_2+n_3}[n_1-n_3](q-q^{-1})X^{n_1-1}T_{n_2+1}C^{n_3} \\
&\quad - \frac{q^{n_2+1}[2n_1-n_3][n_3-1]}{q-q^{-1}} X^{n_1-1}T_{n_2-1}C^{n_3}, \\
ad_C(T_nY^{n_2}C^{n_4}) &= \frac{-q^{-n_2}(q-q^{-1})}{q-q^{-1}} T_{n+1}Y^{n_2-1}C^{n_4} \\
ad_C(T_nY^{n_2}C^{n_4}) &= q^{-2n_1}Y^{n_2}C^{n_4}, \\
ad_C(T_nY^{n_2}C^{n_4}) &= q^{2n_2}T_{n+1}Y^{n_2}C^{n_4}, \\
ad_C(X^{n_1}T_{n_2}C^{n_3}) &= \frac{q^{2n_1-2n_2+1}+q^{-2n_1+2n_3-1}}{(q-q^{-1})} X^{n_1}T_{n_2}C^{n_3} + q^{2n_2}X^{n_1}T_{n_2+2}C^{n_3} \\
&\quad + q^{-n_1+n_2}[n_1-n_3](q-q^{-1})^2 X^{n_1}T_{n_2+1}C^{n_3}, \\
ad_C(T_nY^{n_2}C^{n_4}) &= \frac{q^{2n_2-2n_1+1}+q^{2n_2+2n_3-1}}{(q-q^{-1})^2} T_{n+1}Y^{n_2}C^{n_4} + q^{2n_2}T_{n_2+2}Y^{n_2}C^{n_4} \\
&\quad + q^{-n_2}[n_2-n_3](q-q^{-1})^2 T_{n_2+1}Y^{n_2}C^{n_4}.
\end{align*}$$

To find the eigenfunctions of $ad_C$, we add certain vectors of the form

$$\sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \sum_{n_3=0}^{N_3} a_{n_1,n_2,n_3} X^{n_1}T_{n_2}C^{n_3} \quad \text{and} \quad \sum_{n_2=0}^{N_2} \sum_{n_3=0}^{N_3} \sum_{n_4=0}^{N_4} b_{n_2,n_3,n_4} T_{n_4}Y^{n_2}C^{n_4}.$$
to the space \( \mathcal{W} \) and denote this new space by \( \mathcal{W} \). In the space \( \mathcal{W} \) we find the elements

\[
\{N_t, 0, N_s, S\} = \sum_{k=0}^{\infty} \sum_{r=0}^{k} C_{N_t[N_s, S]} T_{N_t+S+k} C_{N_t+k+2r}
\]

and

\[
\{0, N_s, N_s, S\} = \sum_{k=0}^{\infty} \sum_{r=0}^{k} C_{N_s[N_s, S]} T_{N_s+S+k} C_{N_s+k+2r}
\]

where \( N_t + S \geq 1 \), \( N_s + S \geq 1 \) and \( C_{N_i[N_s, S]} = 1 \), for which the relations

\[
a_{s,t}[N_t, 0, N_s, S] = \frac{q^{2S+1} + q^{2S+1}}{(q - q^{-1})^2} \{N_t, 0, N_s, S\},
\]

\[
a_{s,t}[0, N_s, N_s, S] = \frac{q^{2S+1} + q^{2S+1}}{(q - q^{-1})^2} \{0, N_s, N_s, S\}
\]

are valid.

For the constants \( a_{s,t}[N_i[N_s, S]] \) and \( b_{s,t}[N_i[N_s, S]] \) we obtain the system of equations

\[
a_{s,t}[N_t[N_s, S]] = 2k + 2|2S + 2k + 1] + q^{-2N_t}a_{s,t}[N_t[N_s, S]] = 0,
\]

\[
a_{s,t}[N_t+1[N_s, S]] = 2k + 2|2S + 2k + 1] + q^{-N_t}b_{s,t}[N_t[N_s, S]] = 0,
\]

\[
a_{s,t}[N_t+N_s[N_s, S]] = 2k + 2|2S + 2k + 1] + q^{-N_t}b_{s,t}[N_t[N_s, S]] = q^{-N_t}a_{s,t}[N_t[N_s, S]] |S + 2k + 1] - q^{-N_t}b_{s,t}[N_t[N_s, S]] = 0
\]

\[
b_{s,t}[N_t[N_s, S]] = 2k + 2|2S + 2k + 1] + q^{-N_t}b_{s,t}[N_t[N_s, S]] = 0,
\]

\[
b_{s,t}[N_t+N_s[N_s, S]] = 2k + 2|2S + 2k + 1] + q^{-N_t}b_{s,t}[N_t[N_s, S]] = q^{-N_t}a_{s,t}[N_t[N_s, S]] |S + 2k + 1] - q^{-N_t}b_{s,t}[N_t[N_s, S]] = 0
\]

The constants \( a_{s,t}[N_i[N_s, S]] \) and \( b_{s,t}[N_i[N_s, S]] \) fulfill a similar system of equations.

The system of equations for \( a_{s,t} \) and \( b_{s,t} \) (for simplicity we omit the upper indices) can be solved in the following way:

Let \( a_{0,0} = 1 \). If we know any \( a_{s,t} \) for \( 0 \leq t \leq s \leq k \) and \( b_{s,t} \) for \( 0 \leq t \leq s \leq k-1 \), we find \( b_{s,t} \) from the equations

\[
[2k+1]|2S + 2k|a_{s,t} = q^{-N_t}(q - q^{-1})^2 |S + 2k|a_{s,t},
\]

\[
[2k+1]|2S + 2k|b_{s,t} = q^{-N_t}(q - q^{-1})^2 |S + 2k|a_{s,t} - q^{-2N_t}b_{t+1,t}
\]

for \( 0 \leq t \leq k - 1 \). (4.2)

If we know any \( a_{s,t} \) and \( b_{s,t} \) for \( 0 \leq t \leq s \leq k \), we find \( a_{s,t+1} \) from the equations

\[
[2k+2]|2S + 2k + 1|a_{t+1,t} = -q^{-2N_t}a_{t+1,t},
\]

\[
[2k+2]|2S + 2k + 1|a_{t+1,t} = q^{-N_t}(q - q^{-1})^2 |S + 2k + 1|a_{t+1,t} - q^{-2N_t}a_{t+1,t}
\]

for \( 1 \leq r \leq k \). (4.3)

\[
[2k+2]|2S + 2k + 1|a_{t+1,t+1} = q^{-N_t}(q - q^{-1})^2 |S + 2k + 1|a_{t+1,t}.
\]
In the appendix we show that the system (4.4) has a solution for which

\[ a \]

Next we introduce the space \( W \) with the help of

\[ k \]

If we denote \( a \) we can easily show that if \( b \) is set to zero. The constants \( a \) are a consequence of it.

\[ 0 \]

\[ 2 \]

\[ 4 \]

\[ N \]

\[ S \]

Analogously, we define the elements \( b \), \( b \) for \( 0 \leq r \leq L \) which are uniquely determined by equations (4.3) for \( k \geq L \) and by equations (4.2) for \( k \geq L + 1 \). Just this element is denoted by \( |N_1, 0, N_4, S\rangle \) for \( S \leq 0 \).

4.2.2. The decomposition of the adjoint representation in \( \hat{W} \)

Next we introduce the space \( \hat{W} \subset \hat{V} \) which is generated by the elements \( |N_1, 0, N_4, S\rangle \) and \( |0, N_2, N_4, S\rangle \). Now we find the action of the adjoint representation on these vectors. We obtain

\[ \text{adj}_{\hat{w}}[N_1, 0, N_4, S] = q^{-N_1-S}(q^{-1})^{X^{N_1+1}} \]

\[ \times \sum_{k=0}^{\infty} \sum_{r=0}^{N_4} [q^{-2k} a_{k, r}^{(N_1, N_4, S)} T_{N_1+k+S+2k+1}]^{(N_4+2r)} + q^{-2k-1} b_{k, r}^{(N_1, N_4, S)} T_{N_1+k+S+2k+2(N_4+2r+1)}. \]

If we denote

\[ a_{k, r} = q^{-2k} a_{k, r}^{(N_1, N_4, S)}, \quad b_{k, r} = q^{-2k-1} b_{k, r}^{(N_1, N_4, S)}, \]

we can easily show that if \( a_{k, r}^{(N_1, N_4, S)} \) and \( b_{k, r}^{(N_1, N_4, S)} \) nullify Eqs. (4.2) and (4.3) for \( N_1 \) and \( S \), then \( a_{k, r}, b_{k, r} \) fulfill these systems for \( N_1 + 1 \) and \( S \). In addition for \( S = -L \leq 0 \) we have \( b_{L, r} = 0 \). Since \( a_{0, 0} = a_{0, 0}^{(N_1, N_4, S)} = 1 \),

\[ \text{adj}_{\hat{w}}[N_1, 0, N_4, S] = (q - q^{-1}) q^{-N_1-S}[N_1 + 1, 0, N_4, S]. \]

Similarly, we show

\[ \text{adj}_{\hat{w}}[0, N_2, N_4, S] = -(q - q^{-1}) q^{-N_2-S}[0, N_2 + 1, N_4, S]. \]
If we apply \( \text{ad}_F \) on \( |N_1,0,N_3,S\rangle \) where \( N_1 \geq 1 \), similar but the more cumbersome calculations give
\[
\text{ad}_F|N_1,0,0,N_3,S\rangle = \frac{q^{N_1+S-1}|N_1 - S|S + N_1 - 1}{q - q^{-1}}|N_1 - 1,0,0,0,S\rangle \tag{4.8}
\]
and for action \( \text{ad}_E \) on \( |0,N_2,N_3,S\rangle \) we can show that for \( N_2 \geq 1 \)
\[
\text{ad}_E|0,N_2,N_3,S\rangle = \frac{q^{N_2 - S - 1}|N_2 - S|S + N_2 - 1}{q - q^{-1}}|0,N_2 - 1,0,0,0,S\rangle. \tag{4.9}
\]
Moreover, we have
\[
\text{ad}_K|N_1,0,0,N_3,S\rangle = q^{2N_1}|N_1,0,0,N_3,S\rangle, \\
\text{ad}_K|0,N_2,0,N_3,S\rangle = q^{2N_2}|0,N_2,0,N_3,S\rangle.
\]

The representation preserves the value of \( N_1 \) and \( S \) but changes \( N_2 \). Nevertheless, we see that the lowest and highest weight vectors are
\[
\text{ad}_F|N_1,0,0,N_3,S\rangle = 0 \quad \text{when} \quad S = N_1 \geq 1 \quad \text{or} \quad S = 1 - N_1 \leq 0, \\
\text{ad}_F|0,N_2,0,N_3,S\rangle = 0 \quad \text{when} \quad S = N_2 \geq 1 \quad \text{or} \quad S = 1 - N_2 \leq 0.
\]

From the computations above it follows that for \( N_1 \geq 0 \) the subspaces \( \hat{W} \) defined as
\[
\hat{W}_{(S,0,N_3,S)} = \text{ad}_{\nu_{(42)}}[S,0,N_3,S] = \{|N_1,0,0,N_3,S\rangle : N_1 \geq S \geq 1 \}
\]
for \( S \geq 1, \)
\[
\hat{W}_{(0,S,N_3,S)} = \text{ad}_{\nu_{(42)}}[0,S,N_3,S] = \{|0,N_2,0,N_3,S\rangle : N_2 \geq S \geq 1 \}
\]
for \( S \geq 1, \)
\[
\hat{W}_{(1-S,0,N_3,S)} = \text{ad}_{\nu_{(42)}}[1-S,0,0,N_3,S] = \{|N_1,0,0,N_3,S\rangle : N_1 \geq S \geq 1 \}
\]
for \( S \leq 0, \)
\[
\hat{W}_{(0,1-S,N_3,S)} = \text{ad}_{\nu_{(42)}}[0,1-S,N_3,S] = \{|0,N_2,0,N_3,S\rangle : N_2 \geq S \geq 1 \}
\]
for \( S \leq 0 \)
are invariant.

Except the elements from the subspaces (4.10) in the space \( \hat{W} \) there are the elements of the form \( |N_1,0,N_3,S\rangle \) with \( 0 \leq N_1 < S \) and \( 0 \leq N_2 \leq N_3 \), where \( 0 \leq N_2 < S \). For the vectors \( |0,0,N_3,S\rangle \), where \( N_3 \geq 0 \) and \( S \geq 1 \) we have from relations (4.6) and (4.7)
\[
\text{ad}_E|0,0,0,N_3,S\rangle = (q^{-1})^{\lambda q^{-S-k(S-1)/2}}|0,0,0,N_3,S\rangle, \\
\text{ad}_K|0,0,0,N_3,S\rangle = (q^{-1})^{\lambda q^{-S-k(S-1)/2}}|0,0,0,N_3,S\rangle.
\]

For \( N_1 \geq 0 \) and \( S \geq 1 \) we consider the invariant subspace
\[
\hat{W}_{(0,0,N_3,S)} = \text{ad}_{\nu_{(42)}}[0,0,N_3,S]
\]
This subspace contains all vectors \( |N_1,0,N_3,S\rangle \) and \( |0,N_2,N_3,S\rangle \) and is reducible. It is clear that its irreducible components are \( \hat{W}_{(S,0,N_3,S)} \) and \( \hat{W}_{(0,S,N_3,S)} \).

If we bring together the previous results, we obtain:

**Proposition.** The space \( \hat{W} \) can be expressed as a direct sum of the invariant subspaces in the form
\[
\hat{W} = \bigoplus_{N_3=0}^{\infty} \left( \bigoplus_{N_1=1}^{\infty} \hat{W}_{(N_1,0,N_3-1,N_1)} \right) \oplus \hat{W}_{(0,0,0,N_3,S)} \tag{4.10}
\]
The subspaces \( \hat{W}_{(0,0,0,N_3,S)} \) are reducible, their irreducible components are \( \hat{W}_{(S,0,N_3,S)} \) and \( \hat{W}_{(0,S,N_3,S)} \), and the dimension of the factor-space \( \hat{W}_{(0,0,N_3,S)}/(\hat{W}_{(S,0,N_3,S)} + \hat{W}_{(0,S,N_3,S)}) \) is \( 2S - 1 \).
4.2.3. The infinite dimensional representation for the adjoint representation of $U_q(sl(2))$

In Subsec. 4.2.1, we introduced the factor-space $W$, its modification $\hat{W}$ and factor representation on the space $\hat{W}$. In this section we return to the adjoint representation on the $U_q(sl(2))$. This representation acts on the vectors $X^{n_3}T_{n_3}C^{n_4}$ and $T_{n_3}Y^{n_3}C^{n_4}$ for $n_3 > 1$, as was described in 4.2.1, but for $n_3 = 1$ we obtain

$$\text{ad}_c(X^{n_3}T_{n_3}C^{n_4}) = \frac{2q^{n_3+1}}{q-q^{-1}}X^{n_3-1}T_{n_3}C^{n_4}$$

$$+ q^{-n_3+1}(n_1-1)(q-q^{-1})X^{n_3-1}T_{n_3}C^{n_4} + \frac{2n_3-1}{q-q^{-1}}X^{n_3-1}C^{n_4},$$

$$\text{ad}_c(T_{n_3}Y^{n_3}C^{n_4}) = -\frac{2q^{n_3+1}}{q-q^{-1}}T_{n_3}Y^{n_3-1}C^{n_4}$$

$$- q^{-n_3+1}(n_2-1)(q-q^{-1})T_{n_3}Y^{n_3-1}C^{n_4} + \frac{2n_2-1}{q-q^{-1}}Y^{n_3-1}C^{n_4}.$$
In this appendix we show that the system (4.4) has a solution for which the solution fulfills relation (4.5). First, we define the new variables \( \alpha_{k,r} \) and \( \beta_{k,r} \) by the relations

\[
\alpha_{k,r} = q^{-2kN}(q^{-1})^{k}\alpha_{k,r}, \quad \beta_{k,r} = q^{(2k+1)N}(q^{-1})^{k+2}\beta_{k,r}.
\]

With respect to these new variables the system (4.4) has the form

\[
\begin{align*}
[2k + 1][2L - 2k] \beta_{k,0} &= L - 2k \alpha_{k,0}, \\
[2k + 1][2L - 2k] \beta_{k,r} &= L - 2k \alpha_{k,r} + \beta_{k-1,r}, \quad 0 \leq r \leq k - 1, \\
[2k + 2][2L - 2k - 1] \alpha_{k+1,0} &= \alpha_{k,0}, \\
[2k + 2][2L - 2k - 1] \alpha_{k+1,r} &= L - 2k - 1 \beta_{k-1,r} + \alpha_{k,r}, \quad 1 \leq r \leq k, \\
[2k + 2][2L - 2k - 1] \alpha_{k+1,k+1} &= L - 2k - 1 \beta_{k,k}.
\end{align*}
\]

which is valid for \( 0 \leq k \leq L - 1 \), and the conditions (4.5) are of the form

\[
\alpha_{L,L} = 0, \quad \beta_{L-1,L} = |L|\alpha_{L,L} \quad \text{for } 0 \leq r \leq L - 1.
\]

For \( k \leq L - 1 \) we obtain from the first and last equations of (A.1)

\[
\begin{align*}
\beta_{k,k} &= \frac{[L-2k]}{[2k+1][2L-2k]} \alpha_{k,k}, \\
\alpha_{k+1,k+1} &= \frac{[L-2k-1]}{[2k+2][2L-2k-1][2L-2k]} \alpha_{k,k}
\end{align*}
\]

which results in

\[
\begin{align*}
\alpha_{k,k} &= \frac{1}{[2k]!} \frac{[2L-2k]!}{[2L]!} \frac{[L]!}{[L-2k]!}, \\
\beta_{k,k} &= \frac{1}{[2k+1]!} \frac{[2L-2k-1]!}{[2L]!} \frac{[L]!}{[L-2k-1]!}
\end{align*}
\]

(A.3)

In particular, from (A.3) we obtain \( \alpha_{k,k} = 0 \) for \( k > \frac{L}{2} \) and \( \beta_{k,k} = 0 \) for \( k > \frac{L}{2}(L-1) \), and thus \( a_{L,L} = 0 \).
Next for \( k \geq r \) we put

\[
\alpha_{k,r} = \frac{[2r]!!}{[2k]!!} \frac{[2L - 2k - 1]!!}{[2L - 2r - 1]!!} R_{k,r}, \quad \beta_{k,r} = \frac{[2r+1]!!}{[2k+1]!!} \frac{[2L - 2k - 2]!!}{[2L - 2r - 2]!!} S_{k,r}.
\]

(and we define \([0]!! = [1]!! = 1\)), we obtain from (A.1) the system

\[
R_{k+1,r} - R_{k,0} = 0,
\]

\[
R_{k+1,r} - R_{k,r} = \frac{[2k]!!}{[2k+1]!!} \frac{[2r - 1]!!}{[2r]!!} \frac{[2L - 2k - 2]!!}{[2L - 2k - 1]!!} \prod_{s=0}^{r-1} \frac{[2s]!!}{[2s+1]!!} (L - 2s)[S_{s,r-1}], \quad r \geq 1,
\]

\[
S_{k+1,r} - S_{k,0} = 0, \quad S_{k+1,r} - S_{k,r} = \frac{[2k+1]!!}{[2k+2]!!} \frac{[2r+1]!!}{[2r+2]!!} \frac{[2L - 2k - 3]!!}{[2L - 2k - 2]!!} \prod_{s=0}^{r-1} \frac{[2s+1]!!}{[2s+2]!!} (L - 2s)[R_{s,r}], \quad r \geq 0.
\]

This system results in

\[
R_{k,0} = 1, \quad R_{k,r} = \frac{[2r]!!}{[2k]!!} \frac{[2L - 2r - 1]!!}{[2L - 2k - 1]!!} \prod_{s=0}^{r-1} \frac{[2s]!!}{[2s+1]!!} (L - 2s)[S_{s,r-1}], \quad r \geq 1,
\]

\[
S_{k,0} = 0, \quad S_{k,r} = \frac{[2r]!!}{[2k+1]!!} \frac{[2L - 2r - 2]!!}{[2L - 2k - 2]!!} \prod_{s=0}^{r-1} \frac{[2s]!!}{[2s+1]!!} (L - 2s)[R_{s,r}], \quad r \geq 0
\]

and the following relations:

\[
R_{k,0} = 0 \quad \text{and} \quad S_{k,0} = 0 \quad \text{for all} \quad k' \geq k_0 \Rightarrow R_{k',r} = 0 \quad \text{for all} \quad k > k_0,
\]

\[
S_{k,0} = 0 \quad \text{and} \quad R_{k,0} = 0 \quad \text{for all} \quad k' > k_0 \Rightarrow S_{k',r} = 0 \quad \text{for all} \quad k > k_0.
\]

To prove (A.2) it suffices to show the relation

\[
S_{L-1,0} = \frac{[L]!!}{[2L]!!} R_{L-1,0} = \frac{[L]!!}{[2L]!!} \times (A.5)
\]

and that for any \( r, \quad 1 \leq r \leq \frac{1}{2} L \) we have

\[
R_{L-1,r} = S_{L-1,r} = 0. \quad (A.6)
\]

First, we prove (A.5). In accordance with (A.4) we have

\[
S_{L-1,0} = \frac{[L]!!}{[2L]!!} \frac{[2L - 2]!!}{[2L - 1]!!} \prod_{s=1}^{L-1} \frac{[2s - 1]!!}{[2s]!!} (L - 2s). \quad (A.5)
\]

If we in substitute \( s \rightarrow L - s \) in the sum, we will discover that Eq. (A.5) holds.

\[
S_{L-1,0} = \frac{[L]!!}{[2L]!!} \prod_{s=1}^{L-1} \frac{[2s - 1]!!}{[2s]!!} (L - 2s).
\]
The conditions \( R_{L-r+2,r} = S_{L-1,r} = 0 \) for \( 1 \leq r \leq \frac{1}{2} L \) are equivalent to the equations

\[
\sum_{s=0}^{r-1} \frac{[2s]!!}{[2s+1]!!} \left( \frac{2L-2s-2}{2L-2s-1} \right) [L-2s-1] S_{r-1,s} = 0,
\]

\[
\sum_{r=s}^{L} \frac{[2s-1]!!}{[2s]!!} \left( \frac{2L-2s-1}{2L-2s} \right) [L-2s] R_{s+1,r} = 0.
\]

(A.7)

To prove these equations, we use (A.4). After an appropriate arrangement of the terms in the sums we can deduce that (A.7) is valid if the following statement holds:

Let \( 1 \leq r \leq \frac{1}{2} L \). Then for \( L = 2J \) we have

\[
S_{J+r-1} - S_{J-r-1,1} = 0 \quad \text{for} \quad 0 \leq s \leq J - r,
\]

\[
R_{J+r} - R_{J-r} = 0 \quad \text{for} \quad 1 \leq s \leq J - r,
\]

(A.8)

and for \( L = 2J + 1 \)

\[
S_{J+r-1} - S_{J-r-1,1} = 0 \quad \text{for} \quad 1 \leq s \leq J - r + 1,
\]

\[
R_{J+r} - R_{J-r} = 0 \quad \text{for} \quad 0 \leq s \leq J - r.
\]

(A.9)

We will study the first case (A.8). Let \( 1 \leq r \leq J \). The relation \( R_{J+r} - R_{J-r} = 0 \) for \( 1 \leq s \leq J - r \) with the use of (A.4) is equivalent to the equation

\[
\sum_{s=J-r}^{J-1} \frac{[2s]!!}{[2s+1]!!} \left( \frac{4J - 2s - 2}{4J - 2s - 1} \right) \left[ 2J - 2s - 1 \right] S_{r-1,s} = 0,
\]

which is the consequence of the relation \( S_{J+r-1} - S_{J-r-1,1} = 0 \) for \( 0 \leq s \leq J - r \). On the other hand, for \( 2 \leq r \) the equality \( S_{J+r-1} - S_{J-r-1,1} = 0 \) for \( 0 \leq s \leq J - r \) is equivalent to the equation

\[
\sum_{s=J-r}^{J-1} \frac{[2s-1]!!}{[2s]!!} \left( \frac{4J - 2s - 1}{4J - 2s} \right) \left[ 2J - 2s \right] \left( R_{r-1} - R_{J-r} \right) = 0
\]

and this is the consequence of the relation \( R_{J+r} - R_{J-r} = 0 \) for \( 1 \leq s \leq J - r + 1 \).

Similarly, we can see, that for \( L = 2J + 1 \) the validity (A.8) follows from \( S_{J+r,0} - S_{J-r,0} = 0 \) for \( 1 \leq s \leq J \).

For \( L = 2J \) with the use of (A.4) the condition \( S_{J+r,0} - S_{J-r,0} = 0 \) is equivalent to the equation

\[
\sum_{r=J}^{2J} \frac{[2r-1]!!}{[2r]!!} \left( \frac{4J - 2r - 1}{4J - 2r} \right) \left[ 2J - 2r \right] = 0,
\]

which we can easily prove by substitution \( r \rightarrow 2J - r \), and for \( L = 2J + 1 \) the condition \( S_{J+r,0} - S_{J-r,0} = 0 \) is equivalent to the equation

\[
\sum_{r=J}^{2J+1} \frac{[2r-1]!!}{[2r]!!} \left( \frac{4J - 2r + 1}{4J - 2r + 2} \right) \left[ 2J - 2r + 1 \right] = 0,
\]

which is easily seen to hold after substitution \( r \rightarrow 2J - r + 1 \).
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References

[1] J. Dixmier, *Algèbres Enveloppantes* (Gauthier-Villars Editeur, Paris, 1974).
[2] B. Kostant, Lie group representations on polynomial rings, *Ann. J. Math. Phys.* **85** (1963) 327–404.
[3] A. Kirillov, *Elementy Teorii Predstavlenij* (Nauka, Moscow, 1972). English translation: *Elements of the Theory of Representations* (Springer-Verlag, 1976).
[4] D. Flath, Decomposition of the enveloping algebra of $sl_3$, *J. Math. Phys.* **31** (1990) 1076–1077.
[5] A. Klimyk and K. Schmüdgen, *Quantum Groups and Their Representations* (Springer-Verlag, Berlin, 1997).
[6] A. Joseph and G. Letzler, Local finiteness of the adjoint action for quantized enveloping algebras, *J. Algebra* **85** (1992) 289–318.