COMBINATORIAL POLAR ORDERINGS
AND FOLLOW-UP ARRANGEMENTS

EMANUELE DELUCCHI AND SIMONA SETTEPANELLA

ABSTRACT. Polar orderings arose in recent work of Salvetti and the second author on minimal CW-complexes for complexified hyperplane arrangements. We study the combinatorics of these orderings in the classical framework of oriented matroids, and reach thereby a weakening of the conditions required to actually determine such orderings. A class of arrangements for which the construction of the minimal complex is particularly easy, called follow-up arrangements, can therefore be combinatorially defined. We initiate the study of this class, giving a complete characterization in dimension 2 and proving that every supersolvable complexified arrangement is follow-up.

INTRODUCTION

One of the main topics in the theory of arrangements of hyperplanes is the study of the topology of the complement of a set of hyperplanes on complex space. The special case of complexified arrangements, where the hyperplanes have real defining equations, is very interesting in its own as it allows a particularly explicit combinatorial treatment. Indeed, when dealing with complexified arrangements one can rely on the Salvetti complex, a regular CW-complex that can be constructed entirely in terms of the oriented matroid of the real arrangements and is a deformation retract of the complement of the complexified arrangement [13].

A general fact about complex arrangement’s complements is that they are minimal spaces (i.e., they carry the homotopy type of a CW-complex where the number of cells of any given dimension equals the rank of the corresponding homology group), as was proved by Dimca and Papadima [3] and, independently, by Randell [11] using Morse theoretical arguments. Again, in the complexified case the topic allows an explicit treatment: as shown in [14, 2], one can exploit discrete Morse theory on the Salvetti complex to construct a discrete Morse vector field that allows to collapse every ‘superfluous’ cell and thus produces an explicit instance of the minimal complex whose existence was predicted in [3, 11].

The approach taken by Salvetti and the second author in [14] to construct the discrete Morse vector field relies on the choice of a so-called generic flag and on the associated polar ordering of the faces of the real arrangement. Once this polar ordering is determined, the description of the vector field and of the obtained minimal complex is quite handy, e.g. yielding an explicit formula for the algebraic boundary maps.

But the issue about actually constructing such a polar ordering for a given arrangement remains. This motivates the first part of our work, where we give a fully combinatorial characterization of a whole class of total orderings of the faces of a complexified arrangement that can be used as well to carry out the construction of the very same discrete vector field described in [14]. Our combinatorial polar orderings still require a flag of general position subspaces as a starting point, but
does not need this flag to satisfy the requirements that are requested from a *generic flag* in the sense of [14]. Our construction builds upon the nowadays established concept of *flipping* in oriented matroids, letting a pseudohyperplane ‘sweep’ through the arrangement instead of ‘rotating’ it around a fixed codimension 2 subspace as in [14] (see our opening section for a review of the concepts).

Once the (combinatorial) polar ordering is constructed, one has to figure out the discrete vector field and follow its gradient paths to actually construct the minimal complex. Although the ‘recipe’ is fairly straightforward, this task reveals quite soon very challenging. For instance, this was accomplished in [14] for the family of real reflection arrangements of Coxeter type $A_n$. The key fact allowing to carry out the construction in these cases is that the general flag can be set so that the associated polar orderings enjoy a special technical property (see Definition 2.1.1) that keeps the complexity of computations to a reasonable level.

Thus it is natural to ask if this property is shared by other arrangements. Since the obtained discrete vector fields are the same, it turns out that instead to restrict to ‘actual’ polar orderings, it is natural to work in our broader combinatorial setting, and say that an arrangement is *follow-up* if it admits a combinatorial polar ordering that satisfies the same technical property that made computations feasible for the $A_n$ arrangements.

In the second part of this work we initiate the study of follow-up arrangements. We reach a complete characterization of *follow-up*ness for arrangements of lines. Trying to generalize the property to the three major classes of arrangements to which $A_n$ belongs, we prove that every supersolvable arrangement is follow-up: indeed, the follow-up ordering can be recovered basically from the standard filtration of supersolvable arrangements. However, there exist real reflection arrangements are follow-up. As what concerns asphericity, already in dimension 3 there is a follow-up arrangement that is not $K(\pi, 1)$. We believe that the class of follow-up arrangements still bear some combinatorial and topological interest, and deserve further study.

The paper starts with a section that gives some theoretical background and reviews the different techniques needed later on.

Then the first part of the actual work is dedicated to the combinatorial study of polar orderings. We begin by explaining the setup and the required notation for handling with flippings of affine oriented matroids. Then, in Section 1.2 we give some characterization of the valid sequences of flippings that allow a pseudohyperplane to sweep across an affine arrangement, and call these *special orderings* of the points of the arrangement. A key fact in this section is how special orderings of the points of the arrangement *induced on the moving pseudohyperplane* behave after each “move” of the pseudohyperplane. The genericity condition on the general flag of [14] is actually meant to ensure that every step in the sequence of flippings leads to a realizable oriented matroid. With this in mind, Section 1.3 associates a *combinatorial polar ordering* to every set of one special ordering for every one of the sections of the arrangement induced on a flag of generic subspaces. To prove that this definition indeed makes sense, Section 1.4 shows that every combinatorial polar ordering can be obtained from a ‘genuine’ polar ordering by a sequence of moves, called *switches*, that do not affect the induced discrete vector field. Thus every combinatorial polar ordering induces a discrete Morse function with a minimum possible number of critical cells, and leads to a minimal complex for the arrangement’s complement (Proposition A).

The second part of the work, as said, is devoted to follow-up arrangements. the definition is given in Section 2.1 along with some basic fact. Section 2.2 studies
the 2-dimensional case, leading, with Theorem R.2.4, to a necessary and sufficient condition for an arrangement of lines to be follow-up. We close this paper with Section 2.5 where we prove that every supersolvable arrangement is follow-up.

Thanks. We are pleased to thank professor Salvetti for helpful conversations.

Review

R.1. Topology and combinatorics of complexified arrangements. Let $\mathcal{A}$ be an essential affine hyperplane arrangement in $\mathbb{R}^d$. Let $\mathcal{F}$ denote the set of closed strata of the induced stratification of $\mathbb{R}^d$. It is customary to endow $\mathcal{F}$ with a partial ordering $\leq$ given by reverse inclusion of topological closures. The elements of $\mathcal{F}$ are called faces of the arrangement. Their closures are polyhedral subsets of $\mathbb{R}^d$ and therefore we will adopt the corresponding terminology. Thus, given $F \in \mathcal{F}$, the faces of $F$ will be the polyhedral faces of the closure of $F$, and consistently a facet of $F$ be any maximal face in its boundary. The poset $\mathcal{F}$ is ranked by the codimension of the faces. Elements of $\mathcal{F}$ of maximal dimension are called chambers. For any $F \in \mathcal{F}$ let $|F|$ denote the affine subspace spanned by $F$, called the support of $F$, and set

$$A_F = \{ H \in \mathcal{A} : F \subset H \}.$$ 

Mario Salvetti [13] constructed a regular CW-complex $S = S(\mathcal{A})$ that is a deformation retract of

$$\mathcal{M}(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup_{H \in \mathcal{A}} H_C,$$

the complement of the complexification of $\mathcal{A}$.

The $k$-cells of $S$ bijectively correspond to pairs $[C \prec F]$ where $\text{codim}(F) = k$ and $C$ is a chamber. A cell $[C_1 \prec F_1]$ is in the boundary of $[C_2 \prec F_2]$ if $F_1 \prec F_2$ and the chambers $C_1$, $C_2$ are contained in the same chamber of $A_{F_2}$.

Discrete Morse theory. A combinatorial version of Morse theory that is particularly well-suited for working on regular CW-complexes was formulated by Forman [7]. Here we outline the basics of Forman’s construction, and we point to the book of Kozlov [9] for a broader introduction and a more recent exposition of the combinatorics of this subject.

Definition R.1.1. Let $K$ be a locally finite regular CW-complex and $\mathcal{K}$ denote the set of cells of $K$, ordered by inclusion. A discrete Morse function on $K$ is a function $f : K \rightarrow \mathbb{R}$ such that

(i) $\sharp\{\tau^{(p+1)} > \sigma^{(p)} \mid f(\tau^{(p+1)}) \leq f(\sigma^{(p)})\} \leq 1$

(ii) $\sharp\{\tau^{(p-1)} < \sigma^{(p)} \mid f(\sigma^{(p)}) \leq f(\tau^{(p-1)})\} \leq 1$

for all cells $\sigma^{(p)} \in \mathcal{K}$ of dimension $p$.

Moreover, $\sigma^{(p)}$ is a critical cell of index $p$ if both sets are empty. Let $m_p(f)$ denotes the number of critical cells of $f$ of index $p$.

This setup is a discrete analogue of the classical Morse theory in the following sense.

Theorem R.1.2 ([7], see also [9]). If $f$ is a discrete Morse function on the regular CW-complex $K$, then $K$ is homotopy equivalent to a CW-complex with exactly $m_p(f)$ cells of dimension $p$.

Definition R.1.3. Let $f$ be a discrete Morse function on a CW-complex $K$. The discrete gradient vector field $V_f$ of $f$ is:

$$V_f = \{ (\sigma^{(p)}, \tau^{(p+1)}) \mid f(\tau^{(p+1)}) \leq f(\sigma^{(p)}) \}.$$
By definition of Morse function, each cell belongs to at most one pair of \( V_f \). So \( V_f \) is a matching of the edges of the Hasse diagram of \( F \) and the critical cells are precisely the non-matched elements of \( K \). Because \( f \) is a discrete Morse function, there cannot be any cycle in \( F \) that alternates between matched and unmatched edges - such a matching is called acyclic. The following is a crucial combinatorial property of discrete Morse functions.

**Theorem R.1.4** ([9]). For every acyclic matching \( M \) of \( K \) there is a discrete Morse function \( f \) on \( K \) so that \( M = V_f \). Thus, discrete Morse functions on \( K \) are in one-to-one correspondence with acyclic matchings of the Hasse diagram of \( K \).

### R.2. Polar ordering and polar gradient.

Salvetti and the second author introduced polar orderings of real hyperplane arrangements in [14]. The construction starts by considering the polar coordinate system induced by any generic flag with respect to the given arrangement \( A \subset \mathbb{R}^d \), i.e., a flag \( \{ V_i \}_{i=0,...,d} \) of affine subspaces in general position, such that \( \dim(V_i) = i \) for every \( i = 0, \ldots, d \) and ‘the polar coordinates \( (\rho, \theta_1, \ldots, \theta_{d-1}) \) of every point in a bounded face of \( A \) satisfy \( \rho > 0 \) and \( 0 < \theta_1 < \pi/2 \)’ (see [14] Section 4.2) for the precise description). The existence of such a generic flag is not trivial ([14] Theorem 2)). Every face \( F \) is labeled by the coordinates of the point in its closure that has lexicographically least polar coordinates.

The polar ordering associated to such a generic flag is the total ordering \( \prec \) of \( F \) that is obtained by ordering the faces lexicographically according to their labels. This is the order in which \( V_{d-1} \) intersects the faces while rotating around \( V_{d-2} \). If two faces share the same label - thus, the same minimal point \( p' \), the ordering is determined by the general flag induced on the copy of \( V_{d-1} \) that is rotated ‘just past \( p' \)’ and the ordering it generates by induction on the dimension (see [14] Definition 4.7)).

The main purpose of the polar ordering is to define a discrete Morse function on the Salvetti complex, which, by Theorem R.1.4 amounts to specifying an acyclic matching on the poset of cells of \( S \) that is called the polar gradient \( \Phi \). The original definition of \( \Phi \) is inductive in the dimension of the subspace \( V_k \) containing the faces [14] Definition 4.6]. For the sake of brevity let us here define \( \Phi \) through an equivalent description that is actually the one we will use later (compare Definition 1.4.1).

**Definition R.2.1** (Compare Theorem 6 of [14]). For any two faces \( F_1, F_2 \) with \( F_1 \prec F_2 \), \( \text{codim}(F_1) = \text{codim}(F_2) - 1 \) and any chamber \( C \prec F_1 \), the pair \(((C \prec F_1), [C \prec F_2])\) belongs to \( \Phi \) if and only if the following conditions hold:

(a) \( F_2 \prec F_1 \), and

(b) for all \( G \in F \) with \( \text{codim}(G) = \text{codim}(F_1) - 1 \) such that \( C \prec G \prec F_1 \), one has \( G \prec F_1 \).

We conclude by showing that the above definition indeed has the required features.

**Theorem R.2.2** (See Theorem 6 of [14]). The matching \( \Phi \) is the gradient of a combinatorial Morse function with the minimal possible number of critical cells. Moreover, the set of \( k \)-dimensional critical cells is given by

\[
\text{Cric}_k(S) = \left\{ [C \prec F] \mid \text{codim}(F) = k, F \cap V_k \neq \emptyset, G \prec F \text{ for all } G \prec C \prec G \leq F \right\}
\]

(equivalently, \( F \cap V_k \) is the maximum in polar ordering among all facets of \( C \cap V_k \)).
R.3. Oriented matroids and flippings. The combinatorial data of a real arrangement of hyperplanes are customarily encoded in the corresponding oriented matroid. For the precise definition and a comprehensive introduction into the subject we refer to [1]. One of the many different ways to look at an oriented matroid is to characterize its covectors. These are axiomatically defined, and one can check that the axioms for the covectors of an oriented matroid are satisfied by the face poset of any real arrangement of linear hyperplanes.

However, oriented matroids are more general than linear hyperplane arrangements. To see this, recall that a $k$-pseudosphere is the image of $S^k$ under a tame self-homeomorphism of $S^d$. An arrangement of pseudospheres is a set of centrally symmetric pseudospheres arranged on the $d$-sphere in such a way that the intersection of every two pseudospheres is again a pseudosphere.

The topological representation theorem (Folkman and Lawrence [6], see also [1, Theorem 5.2.1]) proves that the poset of covectors of every oriented matroid can be “represented” by the stratification of $S^d$ induced by an arrangement of pseudospheres.

**Definition R.3.1** (Compare Definition 7.3.4 of [1]). Let $A := (S_e)_{e \in E}$ be an arrangement of pseudospheres on $S^d$. Pick a vertex $w$ of the induced stratification of $S^d$ and consider a pseudosphere $S_f$ with $w \not\in S_f$. Let $T_w := \{ e \in E \mid S_e \ni w \} \cup \{ f \}$ and set $U_w := E \setminus T_w$.

We say that $w$ is near $S_f$ if all the vertices of the arrangement $T_w$ are inside the two regions of $U_w$ that contain $w$ and $-w$.

Given an arrangement of pseudospheres, if a vertex $w$ is near some pseudosphere $S_f$, one can perturb locally the picture by ‘pushing $S_f$ across $w$’ and, symmetrically, across $-w$, so to obtain another valid arrangement of pseudospheres which oriented matroid differs from the preceding only in faces inside the two regions of $T_w$ that contain $w$ and $-w$. This operation was called a flipping of the oriented matroid at the vertex $w$ by Fukuda and Tamura, who first described this operation [8]. For a formally precise description of flippings see also [1, p. 299 and ff.].

Every arrangement of linear hyperplanes in $\mathbb{R}^d$ induces on the unit sphere $S^{d-1}$ an arrangement of (pseudo)-spheres. An oriented matroid that can be realized in this way is called realizable. It is NP-hard to decide whether an oriented matroid is realizable [12].

**Remark R.3.2.** Flippings preserve the underlying matroid (i.e.,the intersection lattice of the arrangement). However, a flipping of a realizable oriented matroid need not be realizable!

To be able to encode the data of an affine arrangement one uses affine oriented matroids. The idea is to add an hyperplane ‘at infinity’ to the oriented matroid represented by the cone of the given affine arrangement (for the precise definition, see [1, Section 4.5]). For the affine counterpart of the representation theorem we need one more definition.

**Definition R.3.3.** A $k$-pseudoflat in $\mathbb{R}^d$ is any image of $\mathbb{R}^{d-k}$ under a (tame) self-homeomorphism of $\mathbb{R}^d$. A pseudohyperplane clearly has two well-defined sides. An arrangement of pseudohyperplanes is a set of such objects satisfying the condition that every intersection of pseudohyperplanes is again a pseudoflat.

Then every affine oriented matroid is represented by an (affine) arrangement of pseudohyperplanes, and the notion of flipping is similar to the previous: the only difference is that there is no vertex “$-w$”.
Notation R.3.4. Let $\mathcal{A}$ be an affine arrangement of pseudohyperplanes, $\overline{H} \in \mathcal{A}$, and $w$ a vertex of $\mathcal{A}$ near $\overline{H}$. The arrangement representing the oriented matroid obtained from the previous by flipping $\overline{H}$ across $w$ will be denoted $\text{Flip}(\mathcal{A}, \overline{H}, w)$.

To set a last piece of notation let us consider two faces $F \prec G$ in $\mathcal{F}$. Then we will denote by $\text{op}_G(F)$ the unique element of $\mathcal{F}$ such that $\text{op}_G(F) \prec G$ and the face that represents $\text{op}_G(F)$ is on the opposite side of every pseudohyperplane that contains $G$ with respect to $F$.

**Part 1. Combinatorics of polar orderings**

The first step on the way to generalizing the construction of [14] is to give a combinatorial (i.e., ‘coordinate-free’) description of it. The idea is to let the hyperplane $V_{k-1}$ ‘sweep’ across the arrangement $\mathcal{A} \cap V_k$ instead of rotating it around $V_{k-1}$.

As explained in the introduction, we want to put the polar ordering into the broader context of the orderings that can be obtained by letting an hyperplane sweep across an affine arrangement along a sequence of flippings. By Remark R.3.2 we must then work with general oriented matroids, since realizability of every intermediate step is not guaranteed (and, indeed, rarely occurs).

### 1.1. Definitions and setup

Let $\mathcal{A}$ denote an affine real arrangement of hyperplanes in $\mathbb{R}^d$. A flag $(V_k)_{k=0, \ldots, d}$ of subspaces is called a general flag if every of its subspaces is in general position with respect to $\mathcal{A}$ and if, for every $k = 0, \ldots, d-1$, $V_k$ does not intersect any bounded chamber of the arrangement $\mathcal{A} \cap V_{k+1}$. Note that this is a less restrictive hypothesis than the one required to be a generic flag in [14].

Moreover, we write

$$\mathcal{A}^k := \{H \cap V_k \mid H \in \mathcal{A}\}, \quad \mathcal{F}^k := \{F \in \mathcal{F} \mid F \cap V_k \neq \emptyset\} (= \mathcal{F}(\mathcal{A}^k)),$$

$$\mathcal{P}^k = \{p_1, p_2, \ldots\} := \text{max } \mathcal{F}^k, \quad \mathcal{P} := \mathcal{P}^0 \cup \mathcal{P}^1 \cup \ldots \cup \mathcal{P}^d.$$

If a total ordering $\prec_k$ of each $\mathcal{P}^k$ is given, we define a total ordering of $\mathcal{P}$ by setting, for any $p \in \mathcal{P}^i$ and $q \in \mathcal{P}^j$,

$$p \prec q \iff \begin{cases} p \prec_k q & \text{if } k = i = j \\ i < j & \text{if } i \neq j \end{cases}.$$

We want to let the hyperplane $V_{k-1}$ sweep across $\mathcal{A}^k$. Let us introduce the necessary notation. For every $k = 1, \ldots, d$, let

$$\overline{H}_0^k := V_{k-1}, \quad \mathcal{F}_0^k := \mathcal{F}^{k-1}, \quad \overline{\mathcal{A}}_0^k := \mathcal{A}^k \cup \{\overline{H}_0^k\}.$$

For all $j > 0$, let $p_j \in \mathcal{P}^k$ be near $\overline{H}_{j-1}^k$ in the sense of Definition R.3.1 and set

$$\overline{\mathcal{A}}_j^k := \text{Flip}(\overline{\mathcal{A}}_{j-1}^k, \overline{H}_{j-1}^k, p_j), \quad \overline{H}_j^k := \overline{\mathcal{A}}_j^k \setminus \mathcal{A} = \{\overline{H}_j^k\},$$

$$\mathcal{H}_j^k := (\overline{\mathcal{A}}_j^k)_{\overline{H}_j^k}, \quad \mathcal{F}_j^k := \mathcal{F}(\mathcal{H}_j^k), \quad \mathcal{P}_j^k := \text{max } \mathcal{F}_j^k$$

where the definitions refer to the natural inclusions $\mathcal{F}_j^k \hookrightarrow \mathcal{F}^k \hookrightarrow \mathcal{F}$. Moreover, we will make use of the natural forgetful projection $\pi_j^k : \mathcal{F}(\overline{\mathcal{A}}_j^k) \to \mathcal{F}^k$ (‘forgetting’ $\overline{H}_j^k$).

We have to understand how the combinatorics of the arrangement induced on the “moving hyperplane” $\overline{H}_j^k$ changes, as $j$ becomes bigger. By the definition of flippings, we know that nothing changes in $\overline{\mathcal{A}}_j^k$ outside

$$\mathcal{Y}(p_j) := (\pi_j^k)^{-1}(\mathcal{F}_j^k_{\geq p_j}).$$
- a fortiori, nothing changes in $\mathcal{F}^k_{j-1}$ outside

$$\mathcal{X}(p_j) := \mathcal{F}^k_{j-1} \cap \mathcal{Y}(p_j).$$

The next Lemma states an explicit (and order-preserving) bijection between the set of ‘new faces’ that are cut by the moving hyperplane after the flip at $p_j$ and the following set of ‘old faces’:

$$C(p_j) := \{ X \in \mathcal{X}(p_j) \mid \text{op}_{p_j}(X) \notin \mathcal{X}(p_j) \}.$$

**Lemma 1.1.1.** With the notations explained above, let $\tilde{\mathcal{A}}^k_{j-1}$ be given and let $p_j \in \mathcal{P}^k$ be near $\tilde{H}^k_{j-1}$. Then, if $<_{j-1}$ denotes the ordering of $\mathcal{F}^k_{j-1}$, $\mathcal{F}^k_j$ is isomorphic to the poset given on the element set

$$(\mathcal{F}^k_{j-1} \setminus C(p_j)) \cup \{ (p_j, X) \mid X \in C(p_j) \}$$

by the order relation

$$F \leq_j F^* \iff \begin{cases} F, F^* \in \mathcal{F}^k_{j-1} \setminus C(p_j) & \text{and } F \leq_{j-1} F^*, \\ F = (p_j, X), F^* = (p_j, X^*) & \text{and } X \leq_{j-1} X^*, \\ F = (p_j, X), F^* \in \mathcal{F}^k_{j-1} \setminus C(p_j) & \text{and } \text{op}_{p_j}(X) \leq_{j-1} F^*, \end{cases}$$

the isomorphism being given by the correspondence $(p_j, X) \mapsto \text{op}_{p_j}(X)$, and the identical mapping elsewhere.

**Proof.** Compare [1, Corollary 7.3.6].

Note that the faces represented by $(p_j, X)$ for $X \in C(p_j)$ are exactly the faces $F$ whose minimal $k$-face is $p_j$.

**Corollary 1.1.2.** If $p_i, p_{i+1} \in \mathcal{P}^k$ are both near $\tilde{H}^k_{i-1}$, then the structure of $\tilde{\mathcal{A}}^k_{i+1}$ does not depend on the order in which the two flippings are carried out.

In particular, any $q \in \mathcal{P}^k$ near $\tilde{H}_{i-1}$ and different from $p_i$ is also near $\tilde{H}_i$.

**Proof.** The fact that both are near to $\tilde{H}_{i-1}$ implies in particular $C(p_i) \cap C(p_j) = 0$, and thus the modifications do not influence each other.

**Notation 1.1.3.** Every $\mathcal{H}^k_j$ contains an isomorphic copy of $\mathcal{F}^{k-1}_0 \simeq \mathcal{F}^{k-2}$ because $\mathcal{F}(\mathcal{H}^k_0) = \mathcal{F}^{k-1}$. We may then add to $\mathcal{H}^k_j$ a pseudohyperplane $\tilde{L}_0^{k,j}$ that intersect
exactly the faces of $\mathcal{F}_{k-2}$ (a copy of $\mathcal{F}(\mathcal{H}_0^{k-1})$) and consider consecutive flippings $\tilde{L}_i^{k,j}$ of it along the elements of $\mathcal{P}_j^k$.

**Remark 1.1.4.** It is clear that every $\tilde{H}_j^k$ is in general position with respect to $\mathcal{A}$, because $\tilde{H}_0^k$ was chosen so. Therefore, any two $p, q$ that are near some $\tilde{H}_j^k$ satisfy $C(p) \cap C(q) = \emptyset$ (just by definition of ‘near’, see [1]). This means amongst other that every element of $\mathcal{F}_{\leq p} \cap \mathcal{F}_{\leq q}$ is already in $\mathcal{H}_j^k$, thus either is in $V_{k-1}$ or in some ‘earlier’ $C(z)$, for $z \rightsquigarrow^k p, q$.

### 1.2. Special orderings

**Definition 1.2.1.** Given an essential affine real arrangement $\mathcal{A}$ and a general position (pseudo)hyperplane $\tilde{H}_0$, a total ordering $p_1, p_2, \ldots$ of the points of $\mathcal{A}$ is a *special ordering* if there is a sequence of arrangements of pseudohyperplanes $\tilde{A}_0, \tilde{A}_1, \ldots$ such that $\tilde{A}_0 = \mathcal{A} \cup \{\tilde{H}_0\}$, and for all $j > 0$, $\tilde{A}_j$ is obtained from $\tilde{A}_{j-1}$ by flipping $\tilde{H}_j$ across $p_j$.

We collect some fact for later reference.

**Lemma 1.2.2.** Let a special ordering $\rightsquigarrow$ of the points of an affine arrangement $\mathcal{A}$ with respect to a generic hyperplane $\tilde{H}_0$ be given. Choose two consecutive points $p \rightsquigarrow q$ and let $\rightsquigarrow^*$ be the total ordering of obtained from $\rightsquigarrow$ by reversing the order of $p$ and $q$. Then, the following are equivalent:

1. $\rightsquigarrow^*$ is a special ordering with respect to $\tilde{H}_0$,
2. In the induced flipping sequence just before the flipping through $p$, both $p$ and $q$ are near the moving pseudohyperplane.
3. For all $F \in \mathcal{F}_{\leq p} \cap \mathcal{F}_{\leq q}$, the minimum vertex of $F$ comes before $p$ and $q$ in $\rightsquigarrow$.

**Proof.** (1)$\Leftrightarrow$(2) is clear, and (2)$\Leftrightarrow$(3) follows from Remark 1.1.4 above. \qed

After those general remarks, let us return to the setup of Section 1.1 and fix $k \in \{1, \ldots, d\}$ for this section. We want to understand whether (and how) it is possible to deduce a valid special ordering of the elements of $\mathcal{P}_j^k$ from a special ordering of the elements of $\mathcal{P}_{j-1}^k$.

**Definition 1.2.3.** Let a total ordering $\rightsquigarrow_{j-1}^k$ of $\mathcal{P}_{j-1}^k$ be given. For every line $\ell$ of $\mathcal{H}_{j-1}^k$ that contains some element of $\mathcal{A}(p_j) \cap \mathcal{P}_{j-1}^k$ let $y^+(\ell), y^-(-\ell)$ denote the points of $\mathcal{H}_{j-1}^k$ where $\ell$ intersects the (topological) boundary of $\mathcal{A}(p_j)$, ordered so that $y^+(\ell) \rightsquigarrow_{j-1}^k y^-(\ell)$.

Moreover, call $\bar{\mathcal{F}}$ the maximum with respect to $\rightsquigarrow_{j-1}^k$ of all $y^+(\ell)$ for varying $\ell$.

Then define a total ordering of $\mathcal{P}_j^k$ by setting, for every $z_1, z_2 \in \mathcal{P}_j^k$:

$$z_1 \rightsquigarrow_{j-1}^k z_2 \Leftrightarrow \begin{cases} z_1, z_2 \in \mathcal{P}_{j-1}^k \cap \mathcal{P}_j^k \quad \text{and } z_1 \rightsquigarrow_{j-1}^k z_2 \\ z_i = (p_j, x_i) \text{ for } i = 1, 2 \\ z_1 \not\in \mathcal{P}_{j-1}^k, z_2 \in \mathcal{P}_{j-1}^k \quad \text{and } \bar{\mathcal{F}} \rightsquigarrow_{j-1}^k z_2 \end{cases}$$

where $x_i^*$ denotes unique the element of $\mathcal{P}_{j-1}^k$ with the same support as $x_i$.

Our goal will be to prove the following statement.

**Theorem 1.2.4.** For every $k \geq 0$ and every $j > 0$, if $\rightsquigarrow_{j-1}^k$ is a special ordering, so is $\rightsquigarrow_{j}^k$ too.
Notation 1.2.5. To investigate the situation, we will focus on $\mathcal{X}(p_j) \subset \mathcal{H}_j^{k-1}$. Let us write $x_1, \ldots, x_s$ for the dimension 0 faces of this complex. Also, let $\ell_1, \ldots, \ell_l$ be the lines of $\mathcal{H}_j^k$ that contain some $x_i$ and write $y_1, y_2, \ldots$ for the intersection points of the $\ell$'s with the hyperplanes bounding $\mathcal{X}(p_j)$.

Remark 1.2.6. It is useful to consider the lines passing through the point $q \in P^k$. For instance, one can see that if two points $p, q \in P^k$ lie on a common line $\ell$ of $A^k$ so that $p$ is nearer than $q$ to $\ell \cap V_{k-1}$, then there can is no sequence of flippings of $\tilde{H}_j^k$ in which $q$ comes before $p$.

Lemma 1.2.7. Let a special ordering of $P_{j-1}^k$ be given. Write $x_1, \ldots, x_s$ with $V_{k-1} \cap |x_r| \sim^{k-1} V_{k-1} \cap |x_l|$ if and only if $r < t$. Let $p_1, p_2, \ldots$ denote the elements of $P_{j-1}^k \setminus \{x_1, \ldots, x_s\}$ ordered according to $\sim^{k-1}$ and let $p_m \equiv \emptyset$. Then the following is a special ordering of $P_{j-1}^k$:

$$p_1, p_2, \ldots, \emptyset, x_1, x_2, \ldots, x_s, p_{m+1}, p_{m+2}, \ldots$$

Proof. The proof is subdivided in three steps.

Claim 1.2.7.1. Every $y_i$ is contained in exactly one of the lines $\ell_1, \ldots, \ell_l$. Moreover, for all $1 \leq i < j \leq l$, there is $r, 1 \leq r \leq s$, such that $x_r = \ell_i \cap \ell_j$.

Proof of claim 1.2.7.1. Note that $\ell_i \cap \ell_j \neq \emptyset$ because both lines are flats of the central arrangement $A_{p_j}$, and these intersections are points of the arrangement $\tilde{H}_j^{k-1} \cup A_{p_j}$. Now both claims follow because the subcomplexes $\mathcal{X}(p_j)$ contains, by definition of flipping $[1]$, every point of the arrangement given by $\tilde{H}_j^{k-1} \cup A_{p_j}$. □

Now recall that, in any special ordering of $P_{j-1}^k$, the 0-dimensional faces on every $\ell_i$ must be ordered ‘along $\ell_i$’. Thus, on every line $\ell_i$, the segment contained in $\mathcal{X}(p_j)$ is bounded by two points, say $y^+(\ell_i) \sim^{k-1} y^-(\ell_i)$.

Claim 1.2.7.2. Consider a special ordering of $P_{j-1}^k$. Then the ordering remains special after the following modifications:

1. Switching $y^+(\ell)$ and $x$ whenever $x$ comes right before $y^+(\ell)$.
2. Switching $y^-(\ell)$ and $x$ whenever $x$ comes right after $y^-(\ell)$.
3. Switching $x$ and any $z \notin \mathcal{X}(q)$ whenever $x$ and $z$ are consecutive.

Proof of claim 1.2.7.2. In case (1) note that Lemma 1.2.7.1 ensures that $C(y^+(\ell))$ lies fully outside $\mathcal{X}(p_j)$ and so it is disjoint from any $C(x)$. Let $x$ be the $r$-th element of $P_{j-1}^k$. Since $x$ comes right before $y^+(\ell)$ we must have that $x$ is already near $\tilde{L}_{r-1}^{k,j-1}$: indeed, in that case $x$ cannot be contained in $\ell$ and by definition neither in the boundary hyperplane that intersects $\ell$ in $y^+(\ell)$. Since the only change in passing from $\tilde{L}_{r-1}^{k,j-1}$ to $\tilde{L}_r^{k,j-1}$ happens at faces which supports contain $x$, we have $\mathcal{Y}(y^+(\ell)) \cap \tilde{L}_{r-1}^{k,j-1} = \mathcal{Y}(y^+(\ell)) \cap \tilde{L}_r^{k,j-1}$. By Corollary 1.1.2 we are done.

The case (2) is handled similarly, by reversing the order of the flippings, and case (3) is clear. □

At this point we know that the ordering

$$p_1, p_2, \ldots, p_m, \ldots, p_{m+1}, p_{m+2}, \ldots$$

where the square brackets contain the $x_i$'s, is indeed a special ordering of $P_{j-1}^k$. We have to prove that we can indeed arrange the faces in the square bracket as required.

First, if $x_1$ is not near $\tilde{L}_m^{k,j-1}$, then there is a line $\ell \ni x_1$ and some other $x_i$ that lies on $\ell$ between $x_1$ and $\ell \cap \tilde{L}_m^{k,j-1}$. In particular, $x_i$ lies between $x_1$ and $\ell \cap \tilde{L}_0^{k,j-1} = \ell \cap F_0^{k-1} = \ell \cap V_{k-2}$. The points $x_1, \ldots, x_s$ are given by the intersection
of the pseudohyperplane $H_{j-1}$ with lines $g_1, \ldots, g_s$ of $A^k$, and $\ell$ is the intersection of $H_{j-1}$ with the plane $E$ generated by $g_1$ and $g_i$. Let $x_r^\ast := g_r \cap F^{k-1}$. Since $g_1 \cap g_i = p_j$, that lies outside the segments $x_1x_1^\ast$ and $x_ix_i^\ast$, we get that in $V_{k-1}$ the point $x_r^\ast$ lies on the line $E := E \cap V_{k-1}$ between $x_1^\ast$ and $E \cap H_{k-1} = \ell^\ast \cap V_{k-2}$.

With Remark 1.2.6 and by the way the numbering of the $x_r$ was chosen, we reach a contradiction.

We are now ready to go.

**Proof of Theorem 1.2.4.** We can assume that $\prec_{j-1}^k$ is modified so to agree with the statement of Lemma 1.2.7. Let $U_m := \bigcup_{l \leq m} \tilde{L}_{m,j}^l$. Since the orderings $\prec_{j-1}^k$ and $\prec_{j-1}^k$ now agree up to $p_m := \tilde{y}$ and clearly $U_{m,j}^k = U_{m,j}^{k-1}$ by Lemma 1.1.1 we are left with proving that it is possible to perform the flippings of the $x_i$ just after $\tilde{y}$, and in the reverse order as the corresponding flippings are performed in $H_{j-1}$.

To this end, let us consider $\tilde{L}_{m,j}$, i.e., the moving pseudohyperplane ‘just after’ the flipping through $p_m = \tilde{y}$. Recall that $\tilde{L}_{m,j}^k \simeq \tilde{L}_{m,j}^{k-1}$, and in particular we can compare the points $z_1, \ldots, z_l$ where the lines containing some $x_i$ intersect this pseudohyperplane. Let $F_1, \ldots, F_l$ be the faces such that $z_i = F_i \cap \tilde{L}_{m,j}^k$. Then we see that the ‘same’ points $z_i$ are given by $(p_j, F_i) \cap \tilde{L}_{m,j}^k$. So by the correspondence established in Lemma 1.1.1 we have that a point $(p_j, x)$ is near $\tilde{L}_{m,j}^k$ if and only if $x$ is near (but “on the backside” of) $\tilde{L}_{m,j}^k$. This shows that $(p_j, x_0)$ is near $\tilde{L}_{m,j}^k$. After performing this flipping we may repeat the argument to conclude that $(p_j, x_{s-1})$ is near $\tilde{L}_{m,s}^k$ for every $l \leq s$, and the claim of the Theorem follows. □

1.3. Combinatorial polar orderings

After having looked inside each $V_k$, let us study the structure that arises by considering all strata.

**Definition 1.3.1** (Compare Theorem 5. of [4]). Given total orderings $\prec_k$ of each $P^k$, we define a total ordering $\prec$ of $F$. All faces of codimension $d$ are elements of $P^d$ and are ordered accordingly. Assuming the ordering is defined for all faces of codimension $\geq k + 1$, then given two $k$-codimensional faces $F$ and $G$ we have:

1. if $F, G \in P^k$, $F \prec G$ if $F \prec_k G$;
2. if $F \in P^k$ and $G \not\in P^k$, then $F \prec G$;
3. if $F, G \not\in P^k$, let $F'$, (resp. $G'$) be the $k + 1$-codimensional facet in the boundary of $F$ (resp. $G$), which is minimum with respect $\prec$. Then:
   3.1. if $F' \prec G'$, then $F \prec G$.
   3.2. if $F' = G'$, then $F \prec G$ if and only if $F_0 \prec G_0$, where $F_0$ and $G_0$ are the unique elements of $P^k$ that have the same linear span as $F$, respectively $G$.
4. If $F \in P^k$, then $F$ is lower than any $k + 1$-codimensional facet
5. If $F \not\in P^k$, then $F$ is bigger than its minimal boundary $F'$ and lower than any $(k + 1)$-codimensional facet which is bigger than $F'$.

Thus, if the orderings on the $P_k$s are given by lexicographic order on the polar coordinates, we reproduce the polar order of $[14]$.

**Definition 1.3.2.** Let an affine real arrangement $A$ be given. A **combinatorial polar ordering** of $F(A)$ is any total ordering $\prec$ induced via Definition 1.3.1 by the choice of a general flag $(V_k)_{k=0,\ldots,d}$ and of special orderings $\prec_k$ of the points of $V_k$ with respect to $V_{k-1}$, for every $k = 1, \ldots, d$. 

Let us next give an alternative characterization of the combinatorial polar orderings that will turn out to be useful later on.

**Definition 1.3.3.** Given $F \in \mathcal{F}$, define the signature of $F$ as $\sigma(F) = (k_F, j_F, m_F)$, where

\[
\begin{align*}
k_F &:= \min\{k \mid V_k \cap F \neq \emptyset\} \\
j_F &:= \min\{j \mid F \in \mathcal{F}(\mathcal{H}_j^{k_F})\} \\
m_F &:= \min\{m \mid F \in \mathcal{F}(\tilde{L}_m^{k_F,j_F})\}.
\end{align*}
\]

**Lemma 1.3.4.** Let special orderings $\prec^k$ be given for every $k$, and let $\prec$ be the total ordering of $\mathcal{F}$ induced by them. For $F_1, F_2 \in \mathcal{F}$, if $\sigma(F_1) < \sigma(F_2)$ in the lexicographic order, then $F_1 \prec F_2$.

**Proof.** If $k_{F_1} < k_{F_2}$, then by Definition 1.3.1.(4) $F_1 \prec F_2$.

Suppose now $k_{F_1} = k_{F_2}$, but $j_{F_1} < j_{F_2}$. If $F_1, F_2 \in \mathcal{P}^k$, then we are already done by Definition 1.3.1.(1). Else, the condition means that the minimal codimensional-$k+1$ face of $F_1$ comes before the minimal codimensional-$(k+1)$ face of $F_2$, and by Remark 1.2.6 we are done.

The same line of reasoning applies to show that $k_{F_1} = k_{F_2}$, $j_{F_1} = j_{F_2}$ and $m_{F_1} < m_{F_2}$ implies $F_1 \prec F_2$. □

**Remark 1.3.5.** It is now easy to see that one could go on and define for every face $F$ a vector

\[
(\sigma_1(F), \ldots, \sigma_{k_F}(F))
\]

with $\sigma_i(F) := j_F$ and $\sigma_i(F) := \min\{m \mid F \in \tilde{L}_m^{k_F,\sigma_1(F),\ldots,\sigma_{i-1}(F)}\}$ (where $\tilde{L}_m^{k_F,a_1,a_2,\ldots,a_j}$ is defined for $j > 1$ as the moving hyperplane of $\mathcal{H}_{a_j}^{k_F,a_1,a_2,\ldots,a_j-1}$ after the $m$-th flipping). From this, a signature

\[
\sigma(F) := (0,0,\ldots,0,\sigma_1(F),\ldots,\sigma_{k_F}(F))
\]

can be defined, so that for all $F_1, F_2 \in \mathcal{F}$, $F_1 \prec F_2$ if and only if $\sigma(F_1) < \sigma(F_2)$ lexicographically. This yields an alternative equivalent formulation of the ordering defined in 1.3.1.

**Remark 1.3.6.** From the point of view of the computational complexity, the translation of Remark 1.3.3 shows that the whole work amounts indeed to determine special orderings of the $V_k$'s. Effective algorithms for this kind of tasks were developed in the last years by Edelsbrunner et al. [4].

### 1.4. “Polar” vector fields and switches

Recall that for $F \in \mathcal{F}$ we denote by $F'$ the smallest facet of $F$ with respect to the given ordering $\prec$. We rephrase Definition 1.2.1 in our broader context.

**Definition 1.4.1.** Let an affine real arrangement $\mathcal{A}$ and a general flag $\{V_k\}_{k=0,\ldots,d}$ be given. For every total ordering $\prec$ of $\mathcal{F}$ we define

\[
\Phi(\prec) := \left\{ [C,F] < [C,F'] \in \mathcal{S} : \begin{array}{ll}
(i) & F \notin \mathcal{P}, \\
(ii) & G' \neq F \text{ for all } G \text{ with } C \prec G < F.
\end{array} \right\}.
\]

**Remark 1.4.2.** If $\prec$ is the polar ordering defined in 1.4., then by Theorem 1.2.2 we know that $\Phi(\prec)$ is a maximum acyclic matching on the poset of cells of the Salvetti complex, i.e., it defines a discrete Morse function on $\mathcal{S}$ with the minimum possible number of critical cells.
Our aim is to show that the total ordering can be slightly modified without affecting the resulting acyclic matching.

**Definition 1.4.3** (Switch). Let special orderings $\sim^k$ of the $\mathcal{P}^k$'s with respect to $V_{k-1}$ be given and let $\prec$ denote the induced total ordering of $\mathcal{F}$. 

Two faces $F_1, F_2 \in \mathcal{P}^k$ are called $c$-independent if

1. they are consecutive with respect to $\sim^k$, and
2. $F \prec F_1, F_2$ for every $G \in \mathcal{F}_{\leq F_1} \cap \mathcal{F}_{\leq F_2}$.

The ordering $\sim^*$ is obtained from $\sim$ by a switch if there are two $c$-independent faces $F_1 \sim F_2$ so that $F_2 \sim^* F_1$, while $F \sim G$ implies $F \sim^* G$ for every other $F, G$. We will write $\prec^*$ for the corresponding combinatorial polar ordering.

The following fact is an easy consequence of Corollary 1.1.2.

**Theorem 1.4.4.** If an ordering $\sim$ of the points of an affine arrangement is special with respect to a general position hyperplane $\tilde{H}$, then so is $\sim^*$.

Now we need to study how the induced total orderings $\prec$ of $\mathcal{F}$ vary by switching two $c$-independent faces.

**Lemma 1.4.5.** Let a special ordering $\sim$ of $\mathcal{P}$ be given, and $\prec$ be the associated total ordering of $\mathcal{F}$. Moreover, let $\sim^*$ be obtained from $\sim$ by a switch and let $\prec^*$ be defined accordingly. Then the minimum facet of any $F \in \mathcal{F}$ with respect to $\prec$ is also the minimum facet with respect to $\prec^*$.

**Proof.** Let $F_1, F_2$ denote the two faces involved in the switch, and write $k_0 := k_{F_1} = k_{F_2}$. The claim is easily seen to be true if $k_F < k_0$ or if $k_F > k_0 + 1$.

Consider the case where $k_F = k_0$. Since the ordering $\sim^{k_0-1}$ does not change, if

$$\min\{p \in \mathcal{P}^{k_0} \mid p \succ F\} = \min\{p \in \mathcal{P}^{k_0} \mid p \succ F\}$$

then the claim is clearly true by Lemma 1.3.4 and Remark 1.2.6.

Since $F_1, F_2$ are consecutive, condition 1 fails only if both $F_1, F_2 \succ F$. But then by Definition 1.4.3 (2) $F \prec F_1, F_2$, implying that the minimum vertex of $F$ comes before $F_1$ and $F_2$, and thus remains unchanged by passing from $\prec$ to $\prec^*$.

Now let $k_F = k_0 + 1$. If $\text{codim}(F) = k_0$, then $F' \in \mathcal{P}^{k_0+1}$, where the order remains unchanged; in any other case, $j_{F'} = j_F$. So after Lemma 1.3.4 we must prove that the claim holds for $F \in \text{op}_{p_j} \mathcal{G}(p_j)$, for any $p_j \in \mathcal{P}^{k_0+1}$. Because the $F_i$ are consecutive, the ordering on the set $\mathcal{P}_{j=1}^{k_F} \cap \mathcal{X}(p_j)$ does not change in passing from $\sim$ to $\sim^*$, unless $p_j$ is the intersection point of the two lines of $\mathcal{A}^{k_0+1}$ that contain $F_1$ and $F_2$. But even in this last case, the corresponding points $G_1, G_2$ of $\mathcal{H}_j^{k_F}$ are again consecutive. Moreover, they are not joined by an edge in $\mathcal{H}_j^{k_F}$ because $F_1$ and $F_2$ are not. By the construction of Lemma 1.2.7 all this implies that they are both near the moving pseudohyperplane $\mathcal{L}_{k_F \cdot j}$, just before flipping across the first of them. In turn, this means (by Remark 1.1.4) that the elements of $\mathcal{F}_{\leq G_1} \cap \mathcal{F}_{\leq G_2}$ come before $G_1$ and $G_2$ and allows us to apply the same reasoning as the case $k_0 = k_F$ to conclude the proof. \[\square\]

In particular, just by looking at the definition of the matchings we obtain the following result.

**Theorem 1.4.6.** Let a special ordering $\sim$ of $\mathcal{P}$ be given, and $\prec$ be the associated total ordering of $\mathcal{F}$. Moreover, let $\sim^*$ be obtained from $\sim$ by a switch and let $\prec^*$ be defined accordingly. Then $\Phi(\prec) = \Phi(\prec^*)$.

The next step is to see that actually switches are rather powerful tools for transforming special orderings.
Theorem 1.4.7. Let \( \sim_1, \sim_2 \) be any two special orderings of the point of an arrangement \( \mathcal{A} \) with respect to a generic hyperplane \( H \). Then \( \sim_2 \) can be obtained from \( \sim_1 \) by a sequence of switches.

Proof. Let \( P \) denote the set of points of \( \mathcal{A} \). Write \( P = \{p_1, p_2, \ldots, p_m\} \) where \( i < j \) if \( p_i \sim_1 p_j \). Let \( \sigma \) be the permutation of \( [m] \) so that \( p_i \sim_2 p_j \) if \( \sigma(i) < \sigma(j) \). We proceed by induction in the number \( u(\sigma) \) of inversions in \( \sigma \), the case \( u(\sigma) = 0 \) being trivial.

So suppose \( u(\sigma) > 0 \). Then there are numbers \( i_1 < i_2 \) such that \( \sigma(i_1) = \sigma(i_2) + 1 \). If \( \tau \) is the transposition \( (\sigma(i_2), \sigma(i_1)) \), then the number of inversions of the permutation \( \tau \sigma \) is strictly smaller than \( u(\sigma) \).

Clearly the ordering of \( P \) associated to \( \tau \sigma \) is obtained by changing the position of \( v_1 := p'_\sigma(i_1) \) and \( v_2 := p'_\sigma(i_2) \). Thus we will be done by showing that this is a valid ‘switch’ according to Definition 1.4.3.

To this end, first remark that the elements are clearly consecutive in \( \sim_2 \). Next consider the fact that \( v_2 \sim_1 v_1 \) and \( v_1 \sim_2 v_2 \), where both \( \sim_1 \) and \( \sim_2 \) are valid special orderings. By Remark 1.2.6 there is no line containing both \( v_1 \) and \( v_2 \). Thus, in the sequence of flippings associated to \( \sim_2 \), just before flipping across \( v_1 \) the moving hyperplane is actually also near \( v_2 \). By Lemma 1.2.2 this ensures condition (2) of the definition of independence, and concludes the proof. \( \square \)

If \( \prec \) is the polar ordering defined in 14, then by Theorem R.2.2, we know that \( \Phi(\prec) \) is a maximum acyclic matching on the poset of cells of the Salvetti complex, i.e., it defines a discrete Morse function on \( S \) with the minimum possible number of critical cells. Moreover, the critical cells are given in terms of \( \prec \) by Theorem R.2.2.

At this point, the main result of this section is evident.

Proposition A. Let a combinatorial polar ordering of the faces of an affine real arrangement \( \mathcal{A} \) be given. Then the induced matching \( \Phi(\prec) \) is a discrete Morse vector field with the minimum possible number of critical cells.

Remark 1.4.8. We already saw that the approach via flippings makes it unnecessary to request the stronger form of ‘generality’ for the flag \( (V_k)_k \) that is needed in 14. However, if this condition is satisfied, then the matching is the polar gradient of 14.

Part 2. Follow-up arrangements

Having established that every special ordering of an arrangement with respect to a general flag gives rise to a combinatorial polar ordering - and thus to a minimal model for the complement of the arrangement’s complexification, the problem of actually finding such an ordering remains.

However, some arrangements admit some particularly handy special orderings, that give rise to combinatorial polar ordering that appear particularly well-suited for explicit computations. The motivating example here is the braid arrangement, studied in 14. In the following we state this nice property - which we call follow-up - and look for other examples of arrangements that enjoy it.

2.1. The definition

Definition 2.1.1 (Follow-up Ordering). Let \( \mathcal{A} \) be a real arrangement and \( (V_k)_k \) a general flag. The corresponding follow-up ordering is the total ordering \( \prec \) of \( \mathcal{P} \) given by setting \( F \prec G \) if one of the following occurs:

(i) \( F \in \mathcal{P}_h, G \in \mathcal{P}_k \) for \( h < k \).

(ii) there is \( k \) so that \( F, G \in \mathcal{P}_k \) and, writing \( F_0 := \min\{J \in \mathcal{P}_{k-1} \mid F \subset |J|\} \), \( G_0 := \min\{J \in \mathcal{P}_{k-1} \mid G \subset |J|\} \),
(a) either \( F_0 \subseteq G_0 \),

(b) or \( F_0 = G_0 \) and there exists a sequence of faces

\[
F_0 \prec F_1 \succ J_1 \prec F_2 \succ J_2 \cdots \prec F
\]

such that \( \text{codim}(F_i) = \text{codim}(J_i) + 1 = \text{codim}(F) \), every \( J_i, F_i \) intersect \( |F_0| \cap V_k \), and \( F_i \neq G \) for all \( i \).

**Definition 2.1.2.** An arrangement \( \mathcal{A} \) in \( \mathbb{R}^n \) is said to be follow-up if there is a general flag \( (V_k)_{k=0,\ldots,a} \) so that the corresponding follow-up ordering is special.

**Example 2.1.3.** The braid arrangement on \( n \) strands is follow-up for every \( n \), as was shown (and exploited) in [14].

**Remark 2.1.4.** With the work done so far, we see that proving that an arrangement \( \mathcal{A} \) is follow-up amounts essentially to finding a special ordering of \( \mathcal{P}(\mathcal{A}) \) such that in every \( V_k \) condition (ii).(a) of the above Definition 2.1.1 holds, since Conditions (i) and (ii).(b) are “standard features” in every special ordering.

### 2.2. Follow-up arrangements of lines

In this section \( \mathcal{A} \) will be an affine arrangement of lines in \( \mathbb{R}^2 \). And we will suppose it to be actually affine, i.e. \( \mathcal{P}^2 \) consists of more than one element (otherwise the arrangement is central, and every central 2-arrangement is trivially Follow-up). Here we do not need the detailed notation of the general case, so we will write \( P := \mathcal{P}^2 \) and abuse notation by writing \( \mathcal{A} := \mathcal{P}^1 \).

The generic flag \( (V_k)_k \) here is a pair \((b, \ell)\), where \( b \) is a point in an unbounded chamber and \( \ell \ni b \) is a line in general position with respect to \( \mathcal{A} \) where all the points of \( \mathcal{A} \) lie on the same side of \( \ell \), and the points \( \mathcal{A} \cap \ell \) lie on the same halfline with respect to \( b \). We shall sometimes confuse \( b \) with the chamber \( B \) it is contained in. In particular, we see that \( B \) cannot have two parallel walls.

**Notation 2.2.1.** Let an affine arrangement of lines \( \mathcal{A} \) be given together with a general flag \((b, \ell)\). The line \( \ell \) intersects a facet of \( B \): let \( h_0 \) denote the element of \( \mathcal{A} \) supporting it. Let \( a_1, a_2, \ldots \) denote the points on \( h_0 \), numbered by increasing distance from \( b \). Moreover, write \( M_j := \{h_1^j, h_2^j, \ldots, h_{\text{max}}^j\} \) for the set of all lines different from \( h_0 \) that contain \( a_j \), ordered according to the sequence of points they generate on \( \ell \). For every \( h \in \mathcal{A} \) let \( h^+ \) denote the (open) halfplane bounded by \( h \) and containing \( b \), and set \( h^- := \mathbb{R}^2 \setminus h^+ \). Then we define, for every \( j = 1, \ldots, r \),

\[
\Lambda_1 := \overline{h_0^+ \cap (h_{\text{max}}^1)^-},
\Lambda_j := (h_{\text{max}}^{j-1})^+ \cap (h_{\text{max}}^j)^- \quad \text{for } j > 1.
\]

**Definition 2.2.2.** If for every \( p \in P \cap \Lambda_j \) there is \( h \in M_j \) with \( a_j, p \in H \), then we will say that \( \Lambda_j \) is complete (with respect to \((b, \ell)\)). The arrangement \( \mathcal{A} \) is complete with respect to \((b, \ell)\) if every \( \Lambda_j \) is complete and \( P \subset \bigcup_{j=1,\ldots,r} \Lambda_j \).

**Lemma 2.2.3.** An affine line arrangement \( \mathcal{A} \) is follow-up with respect to a general flag \((b, \ell)\) if and only if \( \mathcal{A} \) is complete with respect to \((b, \ell)\).

**Sketch of proof.** Fix an \( \ell \). If \( \mathcal{A} \) is not complete at some \( j \), then there is a point \( x \in P \) so that \( x \in \Lambda_j \) but there is no line containing \( a_j \) and \( x \). Let \( \hat{h} \) denote the smallest line of \( M_j \) such that \( x \in \hat{h}^- \), and pick any line \( h \in \mathcal{A} \) that contains \( x \) and is not parallel to \( \hat{h} \). Let \( y := h \cap \hat{h} \). By construction \( h \in \bigcup_{i \geq j} M_i \), and since \( x \) is between \( y \) and \( h \cap \ell \) on \( h \), by Remark 1.2.4 there is no ordering that is special w.r.t. \( \ell \) and in which \( y \) comes after \( x \), as the Follow-up property with respect to \( \ell \) would require.
Figure 1. An affine line arrangement where $\Lambda_1$ is complete with respect to $(b, \ell)$ but $\Lambda_2$ is not. Thus, it is not Follow-up.

On the other hand, if $A$ is complete at every $a_j$, then an explicit follow-up combinatorial polar ordering can be described as follows. Write $A = \{h_0, h_1, \ldots\}$ according to the order in which the lines intersect $\ell$. To begin with, being complete implies that there every point contained in $h_0^-$ lies actually on $h_0$. It is now evident that the sequence $a_1, a_2, \ldots$ is a valid sequence of flippings, that leads to a pseudoline $\ell_1$ with every point in $P \cap h_0$ on its “backside”. Because there are no points in the interior of the cone $h_1^+ \cap h_2^-$, clearly one can now perform the flips across all points of $h_2$. Clearly one can go on this way until the moving pseudoline has flipped across every point in $\Lambda_1$.

We leave it to the reader to check that now one can perform all the flips of points in $\Lambda_j$ for increasing $j$, each time following the order of lines induced by the intersection with $\ell$. □

We obtain a complete characterization of follow-up arrangements in the plane.

Theorem 2.2.4. An affine arrangement of lines in the plane is follow-up if and only if there is a general flag $(b, \ell)$ so that $A$ is complete with respect to $(b, \ell)$.

Some general facts about follow-up arrangements can be deduced.

Remark 2.2.5. Not all real reflection arrangements are follow-up. For example consider the arrangement of type $H_3$. This is a central arrangement in $\mathbb{R}^3$, so it is follow-up if and only if there is a generic section of it that is follow-up. If we consider the projection of the associated dodecahedron on the plane of the section, we see that the points of this arrangement of lines correspond to vertices, to centers of edges or to centers of pentagonal faces. It is easy to see by case-by-case inspection that for every choice of $a_0$, of an adjacent chamber as $B$ and of a suitable line for $\ell$, $\Lambda_1$ is never complete with respect to $(b, \ell)$. Indeed, if $a_0$ corresponds to a pentagon $p$, the obstruction comes from a point corresponding to an edge $e$ that is not adjacent to $p$ but belongs to a pentagon adjacent to $p$ (and vice-versa), while the obstruction for every ‘vertex-type’ choice of $a_0$ comes from another vertex that belongs to a common pentagon, but is not adjacent to $a_0$. 
Remark 2.2.6. Not all follow-up arrangements are \( K(\pi, 1) \). A counterexample can in fact be given already in dimension 3: consider the generic arrangement with defining form \( xyz(x + y + z) \) in \( \mathbb{R}^3 \). By Hattori’s theorem, this arrangement is not aspherical (see [10 Corollary 5.23]). However, it is central and any 2-dimensional section of it is easily seen to be follow-up.

2.3. Supersolvable arrangement are follow-up.

The class of “strictly linearly fibered” arrangements was introduced by Falk and Randell [5] in order to generalize the technique of Brieskorn’s proof of asphericity of the braid arrangements (that involved a chain of fibrations). Later on, Terao [16] recognized that strictly linearly fibered arrangements are exactly those whose intersection lattice is supersolvable [13]. Since then these are known as supersolvable arrangements, and deserved intense consideration.

The goal of this section is to prove that every supersolvable real arrangement is follow-up. Let us begin by the definition.

Definition 2.3.1. A central arrangement \( A \) of complex hyperplanes in \( \mathbb{C}^d \) is called supersolvable if there is a filtration \( A = A_d \supseteq A_{d-1} \supseteq \cdots \supseteq A_2 \supseteq A_1 \) such that

1. \( \text{rank}(A_i) = i \) for all \( i = 1, \ldots, d \)
2. for every two \( H, H' \in A_i \) there exits some \( H'' \in A_{i-1} \) such that \( H \cap H' \subset H'' \).

Before getting to the actual theorem, let us point out the key geometric fact.

Remark 2.3.2. Let \( A \) be as in Definition 2.3.1 and consider the arrangement \( A_{d-1} \) in \( \mathbb{R}^d \). It is clearly not essential, and the top element of \( \mathcal{L}(A_{d-1}) \) is a 1-dimensional line that we may suppose to coincide with the \( x_1 \)-axis. The arrangement \( A_{d-1} \) determines an essential arrangement on any hyperplane \( H \) that meets the \( x_1 \)-axis at some \( x_1 = t \). For all \( t \), the intersection of \( A_{d-1} \) with the hyperplane \( H \) determines an essential, supersolvable arrangement \( A'_{d-1} \subset \mathbb{R}^d \) with \( A'_r = A_r \) as sets, for all \( r \leq d - 1 \). Thus, given a flag of general position subspaces for \( A'_{d-1} \), we can find a combinatorially equivalent flag \( (V_k)_{k=0, \ldots, d-1} \) on \( H \).

Now let us consider a hyperplane \( H \) in \( \mathbb{R}^d \) that is orthogonal to the \( x_1 \)-axis, and suppose we are given on it a valid flag \( (V_k)_{k=0, \ldots, d-2} \) of general position subspaces for \( A_{d-1} \). By tilting \( H \) around \( V_{d-2} \) we can obtain a hyperplane \( H' \) that is in general position with respect to \( A \) and for which all points of \( A \cap H' \) are on the same side with respect to \( V_{d-2} \), and for which \( V_0 \) lies in an unbounded chamber.

By setting \( V_{d-1} := H' \), \( V_d := \mathbb{R}^d \) we thus obtain a valid general flag for \( A = A_d \). Define \( \mathcal{P}^k(A_d) \) as the points of \( A_d \cap V_k \) and analogously for \( \mathcal{P}^k(A_{d-1}) \). The flag remains general by translating \( H' = V_{d-1} \) in \( x_1 \)-direction away from the origin: we can therefore suppose that there is \( R \in \mathbb{R} \) such that for all \( k, 1, \ldots, d-1 \), every element of \( \mathcal{P}^k(A_{d-1}) \) is contained in a ball of radius \( R \) centered in \( V_0 \), that contains no element of \( \mathcal{P}^k(A_d) \) \( \setminus \mathcal{P}^k(A_{d-1}) \).

Corollary 2.3.3. Let \( A \) and \( (V_k)_{k=1, \ldots, d} \) be as in the construction of Remark 2.3.2. Then, for every \( k = 1, \ldots, d \), if \( F_1 \in \mathcal{P}^k(A_{d-1}) \) and \( F_2 \in \mathcal{P}^k(A) \setminus \mathcal{P}^k(A_{d-1}) \) are both contained in the support of the same \( F \in \mathcal{P}^{k-1}(A) \), then \( F_1 \preceq k F_2 \) in every special ordering of \( \mathcal{P}^k(A) \).

Proof. This is an immediate consequence of Remark 1.2.6. \( \square \)

Theorem 2.3.4. Any supersolvable complexified arrangement \( A \) is follow-up. Moreover, the follow-up special ordering \( \preceq \) can be chosen so that for all \( i = 2, \ldots, d \) and all \( k = 1, \ldots, i-1 \), if \( F_1 \in \mathcal{P}^k(A_{i-1}) \) and \( F_2 \in \mathcal{P}^k(A_i) \setminus \mathcal{P}^k(A_{i-1}) \) lie in the support of the same \( k + 1 \)-codimensional face, then \( F_1 \preceq F_2 \).
**Proof.** If \( A \) has rank one, there is nothing to prove. So let \( d := \text{rank}(A) > 1 \) and suppose the claim holds for all complexified supersolvable arrangements - in particular, for \( A_{d-1} \).

The general flag \( (V_k)_{k=0,\ldots,d} \) we will use is obtained via Remark 2.3.2 from a general flag for \( A_{d-1} \) that gives rise to a special ordering satisfying the claim of the theorem. In particular, there exists a special ordering of \( P(A_{d-1}) \) that satisfies the property required by the claim for every \( i = 2, \ldots, d-2 \) (and every \( k = 0, \ldots, i-1 \)). By Corollary 2.3.3 and Remark 2.1.4 we only have to describe, for every \( k \), a special ordering of \( P^k(A) \) that satisfies condition (ii)(a) of Definition 2.1.1 This will be done by a new induction on \( k \).

For \( k = 0 \) there is nothing to prove, and for \( k = 1 \) the only possible special ordering will clearly do. Let then \( k > 1 \). Suppose that follow-up special orderings \( \prec^{k-2}, \prec^{k-1} \) has already been defined on \( P^{k-2} \) and \( P^{k-1} \), and write \( P^{k-1} = \{p_1, p_2, \ldots\} \) accordingly. Since \( A \) is supersolvable, every \( F \in P^k(A) \) is contained in the support of some element of \( P^{k-1}(A_{d-1}) \) that we will call \( p(F) \). So what we have to show is the following.

**Claim 2.3.4.1.** The ordering on \( P^k(A) \) defined by

\[
F_1 \prec F_2 \iff \begin{cases} p(F_1) \prec^{k-1} p(F_2) & \text{or} \\ p(F_1) = p(F_2) \text{ and } F_1 \text{ is between } p(F_1) \text{ and } F_2 \text{ on } |p(F_2)| \end{cases}
\]

is a special ordering.

**Proof of the claim.** Consider a special ordering of \( P^k(A) \) that agrees with the above ordering up to some face \( F_1 \), and suppose for contradiction that \( F_1 \) is not near the moving pseudohyperplane, i.e., that there is \( F_2 \) with \( p(F_1) \prec^{k-1} p(F_2) \) which is on a line passing through \( F_1 \) between \( F_1 \) and the moving pseudohyperplane. By the inductive hypothesis on \( A_{d-1} \) we know that the above defined ordering is indeed special for the elements of \( P^k(A_{d-1}) \), and by Corollary 2.3.3 we conclude that \( F_1 \) cannot be in \( P(A_{d-1}) \).

Thus, the only obstruction to the construction of such a total ordering would come from the following situation: two faces \( F_1, F_2 \in P^k(A) \setminus P^k(A_{d-1}) \) lying on the support of the same \( q \in P^{k-1}(A) \setminus P^{k-1}(A_{d-1}) \) so that \( p(F_1) \prec^{k-1} p(F_2) \) but \( F_2 \) lies between \( q \) and \( F_1 \) on \( |q| \). We prove that this situation can indeed not occur.

Let \( p_0 := \min\{x \in P^{k-1}(A) \mid p \subset |x|\} \) as in Definition 1.3.1. Then we have two cases.

**Case 1** (see Figure 2(1)) \( p(F_1)_0 = p(F_2)_0 \). This means \( p(F_1), p(F_2) \in \ell \), where \( \ell := |p(F_1)| \). The line \( \ell \) is the intersection \( \pi \cap V_{k-1} \) of \( V_{k-1} \) with a plane \( \pi \) in \( V_k \).
that contains also the lines $\ell_1 := |p(F_1)|$ and $\ell_2 := |p(F_2)|$. Then this plane must contain also the line $|q|$. Since $A_{d-1}$ is central, $\ell_1$ and $\ell_2$ must intersect, and this gives a point $P \in P^k(A_{d-1})$ that, by Remark 1.2.6, lies between $p(F_1)$ and $F_1$ for $i = 1, 2$. Again, by Remark 1.2.6 we know that on $\ell$ we have the sequence of points $q, p(F_2), p(F_1)$, so on $|q|$ we have the sequence $q, F_1, F_2$, and there is no obstruction.

Case 2 (see Figure 2(2)). $p(F_1) \sim p(F_2)$. Since $q \in P(A) \setminus P(A_{d-1})$, as above we have that the line $\ell_q := |q|$ intersects $|p(F_i)|$ in a point $p_i$ between $p(F_i)$ and $p(F_i)_0$, for $i = 1, 2$. Consider now the plane $\pi$ spanned by $|q|$ and $\ell_q$ (this is possibly not a flat of $A$), and on it, for $i = 1, 2$ the line $\ell'_i$ spanned by $p_i$ and $F_i$. The intersection $\ell'_1 \cap \ell'_2$ lies on the segments $p_1F_1$ and $p_2F_2$ only if $|p(F_1)| \cap |p(F_2)|$ is between $p(F_1)_0$ and $p_i$. Since the Theorem holds in $V_{k-1}$ it is now a straightforward check to verify that $p(F_1) \sim p(F_2)$ implies that $F_1$ lies between $F_2$ and $q$ on $|q|$ (Figure 2(2) describes one of the two possible cases - namely, when $p_1F_1 \cap p_2F_2$ is not empty).

This concludes the proof. $\square$

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Department of Mathematics, State University of New York, PO Box 6000 Binghamton, NY 13902-6000 (USA).

E-mail address: delucchi@binghamton.edu

Università degli studi di Teramo, Dipartimento di Scienze della Comunicazione, Coste Sant’Agostino, 64100 Teramo (Italy).

E-mail address: settepane@unipi.it