Certain $q$-Difference Operators and Their Applications to the Subclass of Meromorphic $q$-Starlike Functions

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Abstract. The main aim of this work is to find some coefficient inequalities and sufficient condition for some subclasses of meromorphic starlike functions by using $q$-difference operator. Here we also define the extended Ruscheweyh differential operator for meromorphic functions by using $q$-difference operator. Several properties such as coefficient inequalities and Fekete-Szego functional of a family of functions are investigated.

1. Introduction

Let $\mathcal{H}(E)$ denote the class of functions which are analytic in the open unit disk $E = \{z : z \in \mathbb{C}, |z| < 1\}$. Also let $\mathcal{A}$ denote a subclass of analytic functions $f$ in $\mathcal{H}(E)$, satisfying the normalization conditions $f(0) = f'(0) - 1 = 0$. In other words, a function $f$ in $\mathcal{A}$ has Taylor-Maclaurin series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in E).$$

We denote $\mathcal{S}$ by a subclass of $\mathcal{A}$, consisting of univalent functions. Furthermore, we denote the class of starlike functions by $\mathcal{S}^*$. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*$ of starlike functions if it satisfies the relation

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in E).$$

A function $f$ is said to be subordinate to a function $g$ written as $f \prec g$, if there exists a schwarz function $w$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular if $g$ is univalent in $E$ and $f(0) = g(0)$, then $f(E) \subset g(E)$.
For two analytic functions $f$ of the form (1) and $g$ of the form 

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in E),$$

the convolution (Hadamard product) of $f$ and $g$ is defined as:

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad (z \in E).$$

We now recall some essential definitions and concepts of the $q$-calculus, which are useful in our investigations. We suppose throughout the paper that $0 < q < 1$ and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}, \quad \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}.$$ 

**Definition 1.1.** Let $q \in (0, 1)$ and define the $q$-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} 
\lambda - \frac{q^\lambda}{1 - q}, & \lambda \in \mathbb{C}, \\
\sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \ldots + q^{n-1}, & \lambda = n \in \mathbb{N}.
\end{cases}$$

**Definition 1.2.** Let $q \in (0, 1)$ and define the $q$-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 
1, & n = 0, \\
\prod_{k=1}^{n-1} [k]_q, & n \in \mathbb{N}.
\end{cases}$$

**Definition 1.3.** Let $q \in (0, 1)$ and define the $q$-generalized Pochhammer symbol by

$$[t]_{q^n} = \begin{cases} 
1, & n = 0, \\
\prod_{k=0}^{n-1} [t + k]_q, & n \in \mathbb{N}.
\end{cases}$$

**Definition 1.4.** For $t > 0$, let the $q$-gamma function be defined as:

$$\Gamma_q(t + 1) = [t]_q! \Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1.$$ 

**Definition 1.5.** (see [5] and [6]) The $q$-derivative (or $q$-difference) of a function $f$ of the form (1) is denoted by $D_q f$ and defined in a given subset of $\mathbb{C}$ by

$$D_q f(z) = \begin{cases} 
\frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\
\frac{f'(0)}{1-q}, & z = 0.
\end{cases}$$

(2)

When $q \rightarrow 1^-$, the difference operator $D_q$ approaches to the ordinary differential operator. That is

$$\lim_{q \rightarrow 1^-} (D_q f)(z) = f'(z).$$

The operator $D_q$ provides an important tool that has been used in order to investigate the various subclasses of analytic functions of the form given in Definition 1.5. A $q$-extension of the class of starlike functions was first introduced in [4] by means of the $q$-difference operator, a firm footing of the usage of the $q$-calculus in the context of Geometric Functions Theory was actually provided and the basic (or $q$-) hypergeometric functions were first used in Geometric Function Theory by Srivastava (see, for details [14]). After that, wonderful research work has been done by many mathematicians which has played an important
role in the development of Geometric Function Theory. In particular, Srivastava and Bansal [17] studied the close-to-convexity of $q$-Mittag-Leffler functions. The authors in [16] have investigated the Hankel determinant of a subclass of bi-univalent functions defined by using symmetric $q$-derivative. Mahmood et al. [10] studied the class of $q$-starlike functions in the conic region, while in [9], the authors studied the class of $q$-starlike functions related with Janowski functions. The upper bound of third Hankel determinant for the class of $q$-starlike functions has been investigated in [11]. Recently Srivastava et al. [15] have investigated the Hankel and Toeplitz determinants of a subclass of $q$-starlike functions, while the authors in [18] have introduced and studied a generalized class of $q$-starlike functions. Motivated by the above mentioned work, in this paper our aim is to present some subclasses of meromorphic starlike functions by using $q$-difference operator. We also introduce Ruscheweyh differential operator for meromorphic functions by using $q$-difference operator.

**Definition 1.6.** (see [4]) A function $f \in \mathcal{H}(E)$ is said to belong to the class $\mathcal{PS}_q$, if

$$f(0) = f'(0) - 1 = 0$$

and

$$\left| \frac{z}{f(z)} \left( D_q f \right)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q} \quad (z \in E).$$

It is readily observed that as $q \to 1^-$, the closed disk

$$|w - (1-q)^{-1}| \leq (1-q)^{-1}$$

becomes the right–half plane and the class $\mathcal{PS}_q$ reduces to $\mathcal{S}$. Equivalently, by using the principle of subordination between analytic functions, we can rewrite the conditions in (3) and (4) as follows (see [19]):

$$\frac{z}{f(z)} \left( D_q f \right)(z) < \tilde{p}(z), \quad \tilde{p}(z) = \frac{1+z}{1-qz}.$$

Let $M$ denote the class of functions $f$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are analytic in the punctured open unit disk

$$E^* = \{z : z \in \mathbb{C} \quad and \quad 0 < |z| < 1 \} = E - \{0\}.$$

A function $f \in M$ is said to be in the class $\mathcal{MS}_q^*(\alpha)$ of meromorphically starlike functions of order $\alpha$, if it satisfies the inequality

$$-\Re \left( \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in E); \quad 0 \leq \alpha < 1.$$

Let $\mathcal{P}$ denote the class of analytic functions $p$ normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

such that

$$\Re \left( p(z) \right) > 0 \quad (z \in E).$$

Next, we extend the idea of $q$-difference operator analogous to the Definition 1.5 to a function $f$ given by (5) and introduce the class $\mathcal{MS}_q(\beta, \lambda)$. 
Definition 1.7. Let \( f \in M \). Then the \( q \)-derivative operator or \( q \)-difference operator for the function \( f \) of the form (5) is defined by

\[
D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} = -\frac{1}{qz} + \sum_{n=0}^{\infty} [n]_q a_n z^{n-1} \quad (z \in E^*).
\]

Definition 1.8. Let \( f \in M \). Then \( f \in MS_q(\beta, \lambda) \), if it satisfies the condition

\[
\left| \frac{-zD_q f(z) - \beta z^2 D_q (D_q f(z))}{f(z) \left( \frac{q}{1-q} - \Upsilon(\beta, q) \right)} \right| \leq 1 - \frac{1}{1-q^{'}} \quad (7)
\]

which by using subordination can be written as:

\[
\frac{-zD_q f(z) - \beta z^2 D_q (D_q f(z))}{f(z) \left( \frac{q}{1-q} - \Upsilon(\beta, q) \right)} < 1 + (1 - \gamma (1+q))z \quad (8)
\]

Remark 1.9. It can easily be seen that

\[
\lim_{q \to 1^-} MS_q(\beta, \lambda) = H(\beta, \lambda).
\]

The class \( H(\beta, \lambda) \) was introduced and studied by Wang et al. [20, 21]. Secondly, we have

\[
\lim_{q \to 1^-} MS_q(0, \lambda) = H(0, \lambda) = MS(\lambda),
\]

introduced and studied by Wang et al. See [21].

Throughout this paper unless otherwise stated the parameters \( \beta \) and \( \lambda \) are considered as follows:

\[
\beta \geq 0 \quad \text{and} \quad \frac{1}{2} \leq \lambda < 1 \quad (9)
\]

and

\[
\Lambda_q(n, \beta, \gamma) = [n]_q + \beta [n-1]_q + \gamma, \quad (10)
\]

\[
\gamma = \lambda - \beta \lambda \left( \frac{1}{2} \right) - \frac{\beta}{2}, \quad (11)
\]

\[
\Upsilon(\beta, q) = \beta \frac{(1+q)}{q^2}. \quad (12)
\]

2. Preliminary Results

Lemma 2.1. [8] If a function \( p \) of the form (6) is in class \( P \), then

\[
|p_2 - vp_1^2| \leq \begin{cases} 
-4v + 2, & v \leq 0, \\
2, & 0 \leq v \leq 1, \\
4v - 2, & v \geq 1.
\end{cases} \quad (13)
\]

When \( v < 0 \) or \( v > 1 \), equality holds true in (13) if and only if \( p(z) = \frac{1+z}{1-z} \) or one of its rotations. If \( 0 < v < 1 \), then equality holds true in (13) if and only if \( p(z) = \frac{1+z}{1-z} \) or one of its rotations. If \( v = 0 \), equality holds true in (13) if and only if

\[
p(z) = \left( \frac{1+\rho}{2} \right) \left( \frac{1+z}{1-z} \right) + \left( \frac{1-\rho}{2} \right) \left( \frac{1-z}{1+z} \right), \quad 0 \leq \rho \leq 1, \quad z \in E,
\]

or one of its rotations. For \( v = 1 \), equality holds true in (13) if and only if \( p(z) \) is the reciprocal of one of the functions such that the equality holds true in (13) in the case when \( v = 0 \).
Remark 2.2. Although the above upper bound in (13) is sharp, it can be improved as follows:

\[
|p_2 - v p_1^2| + v |p_1|^2 \leq 2, \quad 0 < v \leq \frac{1}{2},
\]

and

\[
|p_2 - v p_1^2| + (1 - v) |p_1|^2 \leq 2, \quad \frac{1}{2} \leq v < 1.
\]

Lemma 2.3. [12] Let a function \( p \) has the form

\[
H(z) = 1 + \sum_{n=1}^{\infty} C_n z^n.
\]

If \( H \) is univalent in \( E \) and \( H(\mathbb{E}) \) is convex, then

\[
|p_n| \leq |C_1|, \quad n \geq 1.
\]

Lemma 2.4. [2] If a function \( p \) of the form (6) and subordinate to a function \( H \) of the form

\[
H(z) = 1 + \sum_{n=1}^{\infty} C_n z^n.
\]

These results are sharp.

3. Main Results

In this section, we prove our main results.

Theorem 3.1. If \( f \in \mathcal{MS}_\beta(\beta, \lambda) \), then for any complex number \( \mu \)

\[
|\alpha_1 - \mu \alpha_0^2| \leq \left\{ \begin{array}{ll}
\frac{\mu(\beta-\eta)^2 + (\gamma-\eta)(1-\gamma)\sigma}{(\beta-\eta)}, & \mu \leq \frac{(\gamma-\eta)^2}{(\beta-\eta)(1+\eta)}, \\
\frac{\alpha(1-\gamma)^2}{(\beta-\eta)}, & \mu = \frac{(1+\gamma-\eta)^2}{(\beta-\eta)(1+\eta)}, \\
\frac{\mu(\beta-\eta)^2 + (\gamma-\eta)(1-\gamma)\sigma}{(\beta-\eta)}, & \mu \geq \frac{(1+\gamma-\eta)^2}{(\beta-\eta)(1+\eta)}.
\end{array} \right.
\]

Furthermore, for \( \frac{(\gamma-\eta)^2}{(\beta-\eta)(1+\eta)} \leq \mu \leq \frac{(1+\gamma-\eta)^2}{(\beta-\eta)(1+\eta)} \), we have

\[
|\alpha_1 - \mu \alpha_0^2| + \left( \frac{\mu (\beta-\eta) \eta^2 + (\eta + 1 - \gamma)(1-\gamma) \sigma}{(\beta-\eta) \eta^2} \right) |\alpha_0|^2 \leq \frac{\alpha (1-\gamma)}{(\beta-\eta)},
\]

and

\[
\left( \frac{(\gamma-\eta)^2}{(\beta-\eta)(1+\eta)} \right) \leq \mu < \frac{(1+\gamma-\eta)^2}{(\beta-\eta)(1+\eta)},
\]

\[
|\alpha_1 - \mu \alpha_0^2| + \left( \frac{(1 + q - \eta)(1-\gamma) \sigma - \mu (\beta-\eta) \eta^2}{(\beta-\eta) \eta^2} \right) |\alpha_0|^2 \leq \frac{\alpha (1-\gamma)}{(\beta-\eta)},
\]

where

\[
\begin{align*}
\sigma &= q - \beta (1 + q), \\
\eta &= (1 + q)(1 - \gamma).
\end{align*}
\]

These results are sharp.
Proof. If \( f \in \mathcal{M}_q(\beta, \lambda) \), then it follows from (8) that:

\[
-\frac{zD_q f(z) - \beta z^2 D_q(D_q f(z))}{f(z) \left( \frac{1}{q} - \Upsilon(\beta, q) \right)} < \phi(z) ,
\]

where

\[
\phi(z) = \frac{1 + (1 - \gamma(1 + q))z}{1 - qz}.
\]

Define a function

\[
p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ..., \]

then it is clear that \( p \in \mathcal{P} \). This implies that

\[
w(z) = \frac{p(z) - 1}{p(z) + 1}.
\]

From (18), we have

\[
-\frac{zD_q f(z) - \beta z^2 D_q(D_q f(z))}{f(z) \left( \frac{1}{q} - \Upsilon(\beta, q) \right)} = \phi(w(z)),
\]

with

\[
\phi(w(z)) = \frac{1 + p(z) + (1 - \gamma(1 + q))(p(z) - 1)}{p(z) + 1 - q(p(z) - 1)}.
\]

Now

\[
\frac{1 + p(z) + (1 - \gamma(1 + q))(p(z) - 1)}{p(z) + 1 - q(p(z) - 1)} = 1 + \left[ \frac{1}{2}(1 + q) \right] z + \left[ \frac{1}{2}(q + 1)(1 - \gamma) p_2 \right] z^2 + ...
\]

and

\[
-\frac{zD_q f(z) - \beta z^2 D_q(D_q f(z))}{f(z)} = \left( \frac{1}{q} - \Upsilon(\beta, q) \right) \left[ 1 + \left[ \frac{1}{2}(1 + q)(1 - \gamma) p_1 \right] z \right. \left. \left[ \frac{1}{2}(q + 1)(1 - \gamma) p_2 + \frac{1}{4}(q^2 - 1)(1 - \gamma) p_1^2 \right] z^2 + ... \right].
\]

From (5) and (19), we have

\[
a_0 = -\frac{\eta}{2} p_1 \]

\[
a_1 = \frac{\sigma \eta}{2(\beta - q)(1 + q)} \left[ p_2 - (\eta + 1 - q) \frac{p_1^2}{2} \right].
\]

Thus, clearly we find that:

\[
|a_1 - \mu a_0^2| = \frac{\sigma (1 - \gamma)}{2(\beta - q)} \left| p_2 - \nu p_1^2 \right| .
\]
where
\[ \nu = \frac{\mu (\beta - q) (1 + q) \eta + (\eta + 1 - q) \sigma}{2 \sigma}. \]

By using Lemma 2.1 in (22), we obtain the required result. \(\square\)

**Theorem 3.2.** Let \( \gamma \) be defined by (11). If \( f \in MS_q (\beta, \lambda) \) and of the form (5) with \( 0 < \beta < \frac{2}{5} \), then
\[ |a_0| \leq \frac{\sigma \eta}{Q_q (0, \beta)} \]
and
\[ |a_n| \leq \frac{\sigma \eta}{Q_q (n, \beta)} \prod_{j=0}^{n-1} \left( 1 + \frac{\sigma \eta}{Q_q (j, \beta)} \right), \quad n \in \mathbb{N}, \quad (23) \]

where \( \sigma, \eta \) are given by (16) and (17) respectively with
\[ Q_q (n, \beta) = [n]_q (1 + [n - 1]_q \beta) q^2 + q - \beta (1 + q). \]

**Proof.** Since \( f \in MS_q (\beta, \lambda) \), therefore
\[ \frac{-zD_q f (z) - \beta z^2 D_q (D_q f (z))}{f (z) \left( \frac{1}{q} - \Upsilon (\beta, q) \right)} = p (z), \quad (25) \]

where
\[ p (z) = 1 + \left[ \frac{1}{2} (1 + q) (1 - \gamma) p_1 \right] z + \left[ \frac{1}{2} (q + 1) (1 - \gamma) p_2 \right] z^2 + \ldots \]

Also
\[ p (z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \]
by using Lemma 2.3 and Lemma 2.4, we obtain
\[ |p_n| \leq \eta, \quad n \in \mathbb{N}. \quad (26) \]

Now the relation (25) can be written as:
\[ -zD_q f (z) - \beta z^2 D_q (D_q f (z)) = \left( \frac{1}{q} - \Upsilon (\beta, q) \right) p (z) f (z). \]

Which implies
\[ \left( \frac{1}{q} - \Upsilon (\beta, q) \right) \frac{1}{z} - \sum_{n=0}^{\infty} ([n]_q + \beta [n]_q [n-1]_q) a_n z^n \]
\[ = \left( \frac{1}{q} - \Upsilon (\beta, q) \right) \left( 1 + \sum_{n=1}^{\infty} p_n z^n \right) \left( \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \right). \quad (27) \]
Equating the coefficients of \(z\) and \(z^{n+1}\) on both sides of (27), we obtain

\[-a_0 = p_1\]

and

\[-(Q_q(n, \beta))a_n = \sigma \left(p_{n+1} + \sum_{j=1}^{n} a_{n-j} \beta_j \right),\]

or equivalently

\[a_0 = -p_1\]

and

\[a_n = -\left(\frac{\sigma}{Q_q(n, \beta)}\right)\left(p_{n+1} + \sum_{j=1}^{n} a_{n-j} \beta_j \right)\].

Using (26), we have

\[|a_0| \leq \frac{\sigma \eta}{Q_q(0, \beta)} \quad (28)\]

and also

\[|a_n| \leq \frac{\sigma \eta}{Q_q(n, \beta)} \left(1 + \sum_{j=1}^{n} |a_{n-j}| \right), \quad n \in \mathbb{N}. \quad (29)\]

For \(n = 1\), the relation (29) yields

\[|a_1| \leq \frac{\sigma \eta}{Q_q(1, \beta)} \left(1 + |a_0| \right) \]

\[\leq \frac{\sigma \eta}{Q_q(1, \beta)} \left(1 + \frac{\sigma \eta}{Q_q(0, \beta)} \right)\].

To prove (23), we apply mathematical induction. For \(n = 2\), (29) yields

\[|a_2| \leq 1 + |a_0| + |a_1| .\]

That is

\[|a_2| \leq \frac{\sigma \eta}{Q_q(2, \beta)} \left(1 + \frac{\sigma \eta}{Q_q(0, \beta)} + \frac{\sigma \eta}{Q_q(1, \beta)} \left(1 + \frac{\sigma \eta}{Q_q(0, \beta)} \right) \right)\]

\[= \frac{\sigma \eta}{Q_q(2, \beta)} \left(1 + \frac{\sigma \eta}{Q_q(0, \beta)} \right) \left(1 + \frac{\sigma \eta}{Q_q(1, \beta)} \right)\]

\[= \frac{\sigma \eta}{Q_q(2, \beta)} \prod_{j=0}^{1} \left(1 + \frac{\sigma \eta}{Q_q(j, \beta)} \right),\]

which implies that (23) holds true for \(n = 2\). Let us assume that (23) is true for \(n \leq k\). That is

\[|a_k| \leq \frac{\sigma \eta}{Q_q(k, \beta)} \prod_{j=0}^{k-1} \left(1 + \frac{\sigma \eta}{Q_q(j, \beta)} \right) .\]
Consider

\[
|a_{k+1}| \leq \frac{\sigma \eta}{Q_q(k+1, \beta)} (1 + |a_0| + |a_1| + ... + |a_k|)
\]

\[
\leq \frac{\sigma \eta}{Q_q(k+1, \beta)} \left[ 1 + \frac{\sigma \eta}{Q_q(0, \beta)} + \frac{\sigma \eta}{Q_q(1, \beta)} \left( 1 + \frac{\sigma}{Q_q(0, \beta)} \right) + ... + \frac{\sigma \eta}{Q_q(k, \beta)} \prod_{j=0}^{k-1} \left( 1 + \frac{\sigma \eta}{Q_q(j, \beta)} \right) \right]
\]

\[
= \frac{\sigma \eta}{Q_q(k+1, \beta)} \prod_{j=0}^{k} \left( 1 + \frac{\sigma \eta}{Q_q(j, \beta)} \right).
\]

Therefore, the result is true for \( n = k + 1 \). Consequently (23) holds true for all \( n \in \mathbb{N} \). □

The following equivalent form of Definition 1.8 is potentially useful in further investigation of the class \( \mathcal{MS}_q(\beta, \lambda) \),

\[
f \in \mathcal{MS}_q(\beta, \lambda) \iff \left| -z \frac{D_q f(z)}{f(z)} - \beta z^2 \frac{D_q \left( D_q f(z) \right)}{f(z)} - \frac{1 - \gamma q}{1 - q} \right| \leq \frac{1 - \gamma}{1 - q}.
\]

(30)

**Theorem 3.3.** Let

\[
\frac{1}{q} - \Upsilon(\beta, q) - \gamma > 0.
\]

(31)

Also suppose that \( f \in \mathcal{M} \) and of the form (5). If

\[
\sum_{n=0}^{\infty} \left( \Lambda_q(n, \beta, \gamma) \right) |a_n| \leq \frac{1}{q} - \Upsilon(\beta, q) - \gamma,
\]

(32)

then \( f \in \mathcal{MS}_q(\beta, \lambda) \), where \( \Upsilon(\beta, q) \) and \( \gamma \) are stated in (12) and (11) respectively.

**Proof.** Assuming that (32) holds true, it suffices to show that

\[
\left| -z \frac{D_q f(z)}{f(z)} - \beta z^2 \frac{D_q \left( D_q f(z) \right)}{f(z)} - \frac{1 - \gamma q}{1 - q} \right| \leq \frac{1 - \gamma}{1 - q}
\]

(33)

Let us consider

\[
\left| -z \frac{D_q f(z)}{f(z)} - \beta z^2 \frac{D_q \left( D_q f(z) \right)}{f(z)} - \frac{1 - \gamma q}{1 - q} \right| = \left| \frac{-1}{q} + \Upsilon(\beta, q) + \frac{1 - \gamma q}{1 - q} + \sum_{n=0}^{\infty} \left( \left[ n \right]_q + \left[ n - 1 \right]_q \beta \right) a_n z^{n+1} \right| + \sum_{n=0}^{\infty} \frac{1 - \gamma q}{1 - q} \left( \frac{1}{1 - q} \right) a_n z^{n+1}
\]

Last expression is bounded above by \( \frac{1 - \gamma q}{1 - q} \) if

\[
\left( \frac{-1}{q} + \Upsilon(\beta, q) + \frac{1 - \gamma q}{1 - q} + \sum_{n=0}^{\infty} \left( \left[ n \right]_q + \left[ n - 1 \right]_q \beta + \frac{1 - \gamma q}{1 - q} \right) a_n \right) \leq \frac{1 - \gamma}{1 - q} \left( \sum_{n=0}^{\infty} |a_n| \right).
\]

After some simple calculations, we have

\[
\sum_{n=0}^{\infty} \left( \Lambda_q(n, \beta, \gamma) \right) |a_n| \leq \left( \frac{1}{q} - \Upsilon(\beta, q) - \gamma \right).
\]

This complete the require proof. □
When $q \longrightarrow 1^−$, Theorem 3.3 reduces to the following known result.

**Corollary 3.4.** (see [20]) Let

$$1 + \beta \lambda \left(\lambda + \frac{1}{2}\right) - \lambda - \frac{3}{2} \beta > 0.$$ 

Also suppose that $f \in \mathcal{M}$ is given by (5). If

$$\sum_{n=0}^{\infty} (n + \beta n (n-1) + \gamma) |a_n| \leq 1 - \gamma - 2\beta,$$

then $f \in \mathcal{H} (\beta, \lambda)$.

### 4. Ruscheweyh $q$-Difference Operator for Meromorphic Functions.

Ruscheweyh derivatives for analytic function was defined by Ruscheweyh [13] and named as $m$-th order Ruscheweyh derivative by Al-Amiri (see [1]). Ganigi and Uralegaddi introduced the meromorphic analogy of Ruscheweyh derivative in [3]. Recently Kanas et al. (see [7]) introduced the Ruscheweyh derivative operator for analytic functions by using $q$-differential operator. We here define the meromorphic analogy of Ruscheweyh derivative by using $q$-differential operator. In this section, we define and study a new class of functions from class $\mathcal{M}$ by using meromorphic analogy of Ruscheweyh $q$-difference operator. We also investigate the similar kind of results which have been proved in the above section.

**Definition 4.1.** Let $f \in \mathcal{M}$. Then the meromorphic analogue of Ruscheweyh $q$-differential operator is defined as

$$MR_q^\delta f (z) = f (z) \ast \phi (q, \delta + 1; z) = \frac{1}{z} + \sum_{n=1}^{\infty} \psi_n a_n z^n, \quad z \in E^*, \quad \delta > -1,$$

where

$$\phi (q, \delta + 1; z) = \frac{1}{z} + \sum_{n=0}^{\infty} \psi_n z^n$$

and

$$\psi_n = \frac{[\delta + n + 1]_q!}{[n+1]_q! [\delta]_q!}.$$  \hspace{1cm} (35)

From (34), we have

$$MR_q^\delta f (z) = f (z), \quad MR_q^\delta f (z) - [2]_q MR_q^\delta f (qz) = z D_q f (z)$$

and

$$MR_q^m f (z) = \frac{z^{-1} D_q \left( z^{m+1} f (z) \right)}{[m]_q!}, \quad m \in \mathbb{N}.$$ 

Note that

$$\lim_{q \rightarrow 1^−} \phi (q, \delta + 1; z) = \frac{1}{z (1-z)^{\delta+1}}.$$
and

\[ \lim_{q \to 1} \mathcal{MR}_q^\beta f(z) = f(z) * \frac{1}{z(1 - z)^{\beta q}}, \]

which is the well-known Ruscheweyh differential operator for meromorphic functions introduced and studied by Ganigi and Uralegaddi [3].

**Definition 4.2.** Let \( f \in \mathcal{M} \). Then \( f \in \mathcal{MS}_q^\beta(\beta, \lambda) \), if it satisfies the condition

\[ \left| -z D_q \left( \mathcal{MR}_q^\beta f(z) \right) - \beta z D_q \left( D_q \mathcal{MR}_q^\beta f(z) \right) \right| - \gamma \frac{1}{1 - q} \leq \frac{1}{1 - q}, \] (36)

which by using subordination can be written as

\[ \frac{-z D_q \left( \mathcal{MR}_q^\beta f(z) \right) - \beta z D_q \left( D_q \mathcal{MR}_q^\beta f(z) \right)}{(1 - q)(1 - \gamma)(1 + q)z} < \frac{1}{1 - q}. \] (37)

**Remark 4.3.** Firstly, it can easily be seen that

\[ \mathcal{MS}_q^\beta(\beta, \lambda) = \mathcal{MS}_q(\beta, \lambda), \]

where \( \mathcal{MS}_q(\beta, \lambda) \) is the class of functions defined in Definition 1.8. Secondly, we have

\[ \lim_{q \to 1} \mathcal{MS}_q^\beta(\beta, \lambda) = \mathcal{H}(\beta, \lambda), \]

where the class \( \mathcal{H}(\beta, \lambda) \) was introduced and studied by Wang et al. For detail see [20, 21].

The following results can be proved by using the similar arguments as in Section 3, so we choose to omit the details of proofs.

**Theorem 4.4.** If \( f \in \mathcal{MS}_q^\beta(\beta, \lambda) \), then for any complex number \( \mu \)

\[ |a_1 - \mu a_0^2| \leq \begin{cases} \frac{\mu (\beta - q) \psi_1 + (\eta + 1 - q)(1 - \gamma) \sigma \psi_0^2}{(\beta - q) \eta \psi_1}, & \mu \leq \frac{(\beta - q) \psi_0^2}{(\beta - q)(1 + q) \eta \psi_1}, \\ \frac{\sigma (1 - \gamma)}{(\beta - q) \eta \psi_1}, & \mu \leq \frac{(\beta - q) \psi_0^2}{(\beta - q)(1 + q) \eta \psi_1}, \\ \frac{\mu (\beta - q) \psi_1 + (\eta + 1 - q)(1 - \gamma) \sigma \psi_0^2}{(\beta - q) \eta \psi_1}, & \mu \geq \frac{(\beta - q) \psi_0^2}{(\beta - q)(1 + q) \eta \psi_1}. \end{cases} \]

Furthermore for \( \frac{(\beta - q) \psi_0^2}{(\beta - q)(1 + q) \eta \psi_1} < \mu \leq \frac{(\beta - q) \psi_0^2}{(\beta - q)(1 + q) \eta \psi_1} \),

\[ |a_1 - \mu a_0^2| + \left( \frac{\mu (\beta - q) \eta \psi_1 + (\eta + 1 - q)(1 - \gamma) \sigma \psi_0^2}{(\beta - q) \eta \psi_1} \right) |a_0|^2 \leq \frac{\sigma (1 - \gamma)}{(\beta - q) \psi_1}, \]

and \( \frac{(\beta - q) \psi_0^2}{(\beta - q)(1 + q) \eta \psi_1} \leq \mu < \frac{(1 + q - \eta) \sigma \psi_0^2}{(\beta - q)(1 + q) \eta \psi_1} \),

\[ |a_1 - \mu a_0^2| + \left( \frac{(1 + q - \eta)(1 - \gamma) \sigma \psi_0^2}{(\beta - q) \eta \psi_1} \right) |a_0|^2 \leq \frac{\sigma (1 - \gamma)}{(\beta - q) \psi_1}, \]

where \( \sigma, \eta \) and \( \psi_n \) are given by (16), (17) and (35) respectively. These results are sharp.
By putting $\psi_n = 1$, the above result is proved in Theorem 3.1.

**Theorem 4.5.** Let $\gamma$ be defined by (11). If $f \in MS_q^d (\beta, \lambda)$ of the (5) with $0 < \beta < \frac{2}{3}$, then

$$|a_0| \leq \frac{\sigma \eta}{Q_q (0, \beta) \psi_0}$$

and

$$|a_n| \leq \frac{\sigma \eta}{Q_q (n, \beta) \psi_n} \prod_{j=0}^{n-1} \left(1 + \frac{\sigma \eta}{Q_q (j, \beta)} \right), \quad n \in \mathbb{N}, \tag{38}$$

where $\sigma$, $\eta$ and $Q_q (n, \beta)$ are given by (16), (17) and (24) respectively.

By choosing $\psi_n = 1$, the above result is proved in Theorem 3.2.

**Theorem 4.6.** Let

$$\frac{1}{q} - \Upsilon (\beta, q) - \gamma > 0. \tag{39}$$

Also suppose that $f \in M$ is given by (5). If

$$\sum_{n=0}^{\infty} \psi_n \left(\Lambda_q (n, \beta, \gamma)\right) |a_n| \leq \frac{1}{q} - \Upsilon (\beta, q) - \gamma, \tag{40}$$

then $f \in MS_q^d (\beta, \lambda)$ where $\Upsilon (\beta, q)$, $\psi_n$ and $\gamma$ are given in (12), (35) and (11) respectively.

When $\delta = 0$ and $q \to 1^-$, Theorem 4.6 reduces to the following known result.

**Corollary 4.7.** (See [20]) Let

$$1 + \beta \lambda \left(\lambda + \frac{1}{2}\right) - \lambda - \frac{3}{2} \beta > 0. \tag{41}$$

Also suppose that $f \in M$ is given by (5). If

$$\sum_{n=0}^{\infty} \left(n + \beta n (n - 1) + \gamma\right) |a_n| \leq 1 - \gamma - 2\beta,$$

then $f \in \mathcal{H} (\beta, \lambda)$.

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