We show that the endpoint set of a Suslinian chainable continuum must be zero-dimensional at some point. In particular, it cannot be homeomorphic to complete Erdős space. This answers a question of Jerzy Krzempek.

A continuum is a compact connected metric space. A continuum $X$ is Suslinian if there is no uncountable collection of pairwise disjoint non-degenerate subcontinua of $X$. A continuum $X$ is chainable if for every $\varepsilon > 0$ there are finitely many open sets $U_1, \ldots, U_n$ covering $X$ such that $\text{diam}(U_i) < \varepsilon$ and $U_i \cap U_j \neq \emptyset \iff |i - j| \leq 1$ for all $i, j \leq n$. A point $x$ in a chainable continuum $X$ is an endpoint of $X$ if the $\varepsilon$-chains $U_1, \ldots, U_n$ can always be chosen so that $x \in U_1$. For equivalent definitions of endpoints see [2, Theorem 13] or [1, p.609]. The set of all endpoints of $X$ is denoted $E(X)$.

A topological space $X$ is:

- **totally disconnected** if $X$ does not have any connected subset with more than one point;
- **zero-dimensional at** $x \in X$ if every neighborhood of $x$ contains a clopen neighborhood of $x$;
- **zero-dimensional** if $X$ is zero-dimensional at each of its points (i.e. $X$ has a basis of clopen sets).

Jerzy Krzempek has shown that for every zero-dimensional Polish space $E$ there is a Suslinian chainable continuum $X$ such that $E(X)$ is homeomorphic to $E$ [8, Theorem 1]. So interestingly enough there is a Suslinian chainable continuum whose endpoint set is homeomorphic to the space of irrational numbers. Conversely, if $X$ is Suslinian and chainable then $E(X)$ is totally disconnected [1, Theorem 11] and Polish [7, Theorem 4.2]. It is unknown whether $E(X)$ must be zero-dimensional [8, Problem 1]. Krzempek in particular asked whether $E(X)$ could be homeomorphic to complete Erdős space $E_c$, a famous example of a totally disconnected Polish space which is not zero-dimensional at any point [5, 6]. In this paper we provide a negative answer with the following.

**Theorem 1.** If $X$ is a Suslinian chainable continuum, then every totally disconnected $G_\delta$-subset of $X$ is zero-dimensional at some point.

**Corollary 2.** If $X$ is a Suslinian chainable continuum, then $E(X)$ is zero-dimensional at some point.

**Corollary 3.** Every chainable continuum that contains $E_c$ is non-Suslinian. In particular, $E_c$ is not homeomorphic to the endpoint set of a Suslinian chainable continuum.
Examples. The following examples show that Theorem 1 and Corollary 3 essentially cannot be improved upon.

(A) There is a Suslinian chainable continuum that contains a totally disconnected Polish space of positive dimension. This example was constructed by Howard Cook and Andrew Lelek [3, Example 4.2]. By [3, Theorem 3.2] it contains a totally disconnected set \( P \) with countable complement, and by the statement of [3, Example 4.2] the set \( P \) is not zero-dimensional.\(^1\)

(B) There is a hereditarily decomposable chainable continuum which homeomorphically contains \( \mathcal{E}_c \) (e.g. the Cantor organ [9, p. 191]).

(C) There is a Suslinian dendroid whose endpoint set is homeomorphic to \( \mathcal{E}_c \). This example can be obtained as a quotient of the Lelek fan and is due to Piotr Minc and Ed Tymchatyn (personal communication).

![Illustration of Example A](image)

Figure 1. Illustration of Example A. The set \( K_1 \) is constructed so that the points where two triangles intersect accumulate onto the middle third of the left-most segment \( S \). This principle is repeated in the smaller triangles to form \( K_2, K_3, \) etc. The set \( X = \bigcap_{n=0}^{\infty} K_n \) is a Suslinian chainable continuum. If \( Q \) is any countable set which intersects each arc of \( X \), then \( P = X \setminus Q \) is totally disconnected. But every clopen subset of \( P \) meeting \( S \) must contain all of \( P \cap S \) (proved by Cook and Lelek). Hence \( P \) is not zero-dimensional.

Proof of Theorem 1. Let \( X \) be a Suslinian chainable continuum. Let \( E \) be a totally disconnected \( G_\delta \)-subset of \( X \). We will produce a point \( e \) at which \( E \) is zero-dimensional. To that end, let \( \mathcal{K} \) be the set of all non-degenerate connected components of \( \overline{E} \). Note that \( \mathcal{K} \) is countable.

\(^1\)Note that in [3], the term *hereditarily disconnected* is used instead of totally disconnected, and *totally disconnected* is defined to be a stronger condition which is still satisfied by every zero-dimensional space.
by the Suslinian property. So by Baire’s theorem either $\bigcup K$ has empty interior in $\overline{E}$, or there exists $K \in \bigcup K$ which contains a non-empty open subset of $\overline{E}$.

**Case 1:** $\bigcup K$ has empty interior in $\overline{E}$.

Then $\overline{E} \setminus \bigcup K$ is a dense $G_δ$-set in $\overline{E}$. Since $E$ is also dense and $G_δ$ in $\overline{E}$, there exists $e \in E \cap (\overline{E} \setminus \bigcup K)$. Then $\{e\}$ is a connected component of the compactum $\overline{E}$. So $\overline{E}$ is zero-dimensional at $e$ (cf. [4, Section 1.4]). Therefore $E$ is zero-dimensional at $e$.

**Case 2:** There exists $K \in \bigcup K$ which contains a non-empty open subset of $\overline{E}$.

Let $U$ be a non-empty relatively open subset of $\overline{E}$ that is contained in $K$.

The continuum $K$ is Suslinian and chainable, as these properties are inherited from $X$. Therefore $K$ is hereditarily decomposable [10, Theorem 1.1] and irreducible [11, Theorem 12.5]. By Kuratowski’s theory of tranches (see [9, §48] or [12, p.15]), there is a mapping $\varphi : K \to [0, 1]$ such that $\varphi^{-1}\{t\}$ is a nowhere dense subcontinuum of $K$ for every $t \in [0, 1]$. By the Suslinian property, the set

$$Q = \{ t \in [0, 1] : |\varphi^{-1}\{t\}| > 1 \}$$

is countable. Therefore $K \setminus \varphi^{-1}(Q)$ is dense $G_δ$ in $K$. So there exists $e \in E \cap U \setminus \varphi^{-1}(Q)$.

By compactness of $K$ there exist $a, b \in [0, 1] \setminus Q$ such that

$$a < \varphi(e) < b$$

and $\varphi^{-1}[a, b] \subset U$.

We claim that there is a relatively clopen $C \subset E \cap \varphi^{-1}[a, \varphi(e)]$ which contains $e$ and misses the point $\varphi^{-1}(a)$. To see this, note that $E \cap \varphi^{-1}[a, \varphi(e)]$ is non-degenerate because $E$ is dense in $\varphi^{-1}(a, \varphi(e))$. Since $E$ is totally disconnected, $E \cap \varphi^{-1}[a, \varphi(e)]$ is not connected. Let $W$ be a proper clopen subset of the space $E \cap \varphi^{-1}[a, \varphi(e)]$ such that $e \in W$. Since $K \setminus \varphi^{-1}(Q)$ is dense in $\varphi^{-1}(a, \varphi(e))$, and $W$ is a proper closed subset of $E \cap \varphi^{-1}[a, \varphi(e)]$, there exists $c \in [a, \varphi(e)) \setminus Q$ such that $\varphi^{-1}(c) \notin W$. Then $C = W \cap \varphi^{-1}[c, \varphi(e)]$ is as desired.

Likewise there is a relatively clopen $D \subset E \cap \varphi^{-1}[\varphi(e), b]$ which contains $e$ and misses the point $\varphi^{-1}(b)$. Then $C \cup D$ is a clopen subset of $E$ that contains $e$ and lies inside of $U$. If $U'$ is any smaller open subset of $\overline{E}$ containing $e$, then the same argument will produce a clopen subset of $E$ which contains $e$ and lies inside of $U'$. Therefore $E$ is zero-dimensional at $e$.

This concludes the proof of Theorem 1.

**Remarks.** Corollaries 2 and 3 follow from Theorem 1 because $E(X)$ and $\mathcal{E}_c$ are totally disconnected Polish spaces, and $\mathcal{E}_c$ is nowhere zero-dimensional.

By applying Theorem 1 locally, it can be seen that if $E$ is any totally disconnected $G_δ$-subset of $X$ (a Suslinian chainable continuum), then the set of points at which $E$ is zero-dimensional is dense in $E$. The set of points at which a separable metrizable space is zero-dimensional is always $G_δ$, so we get the following strengthening of Corollary 2: *The set of points at which $E(X)$ is zero-dimensional is dense $G_δ$ in $E$. Thus $E(X)$ is zero-dimensional “almost everywhere”.*

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