$L^p - L^q$ ESTIMATES FOR MAXIMAL OPERATORS ASSOCIATED TO FAMILIES OF FINITE TYPE CURVES IN THE PLANE

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Abstract. We study the boundedness problem for maximal operators $M$ associated to averages along families of finite type curves in the plane, defined by

$$Mf(x) := \sup_{1 \leq t \leq 2} \left| \int_C f(x - ty) \rho(y) \, d\sigma(y) \right|,$$

where $d\sigma$ denotes the normalised Lebesgue measure over the curves $C$. Let $\Delta$ be the closed triangle with vertices $P = (\frac{2}{5}, \frac{1}{5})$, $Q = (\frac{1}{2}, \frac{1}{2})$, $R = (0, 0)$. In this paper, we prove that for $(\frac{1}{p}, \frac{1}{q}) \in (\Delta \setminus \{P, Q\}) \cap \left\{ (\frac{1}{p}, \frac{1}{q}) : q > m \right\}$, there is a constant $B$ such that $\|Mf\|_{L^q(\mathbb{R}^2)} \leq B \|f\|_{L^p(\mathbb{R}^2)}$.

Furthermore, if $m < 5$, then we have $\|Mf\|_{L^5(\mathbb{R}^2)} \leq B \|f\|_{L^2(\mathbb{R}^2)}$. We shall also consider a variable coefficient version of maximal theorem and we obtain the $L^p - L^q$ boundedness result for $(\frac{1}{p}, \frac{1}{q}) \in \Delta^\circ \cap \left\{ (\frac{1}{p}, \frac{1}{q}) : q > m \right\}$, where $\Delta^\circ$ is the interior of the triangle with vertices $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{2}{5}, \frac{1}{5})$. An application is given to obtain $L^p - L^q$ estimates for solution to higher order, strictly hyperbolic pseudo-differential operators.

1. Introduction

Let $C$ denote a smooth curve in the plane, let $\rho \in C_c^\infty(\mathbb{C})$ denote a smooth cut off function in $\mathbb{R}^2$. Now, given a function $f$, continuous and compactly supported, we consider for each $x \in \mathbb{R}^2$ and $t > 0$, the operator

$$M_t f(x) := \int_C f(x - ty) \rho(y) \, d\sigma(y),$$

where $d\sigma$ denotes the normalized Lebesgue measure over the curves $C$. It follows from Fubini’s theorem that for $t \in \mathbb{R}$, $M_t f(x)$ is well-defined almost all $x$. Consider now the maximal operator given by

$$\mathcal{M} f(x) := \sup_{t > 0} |M_t f(x)|.$$
It is not obvious that such averaging operators are well defined for \( f \) in \( L^p \)-spaces, since \( C \) has measure zero in \( \mathbb{R}^2 \). Nevertheless, a priori \( L^p - L^q \) estimates are possible when \( C \) has suitable curvature properties. Therefore, a natural question, we ask is for what range of the exponents \( p \) and \( q \), is the following a priori inequality satisfied:

\[
\| \mathcal{M} f \|_{L^q(\mathbb{R}^2)} \leq B \| f \|_{L^p(\mathbb{R}^2)}, \ f \in \mathscr{S}.
\]

The aim of this paper is to study the boundedness of maximal operator associated with curves from \( L^p(\mathbb{R}^2) \) to \( L^q(\mathbb{R}^2) \). There is a vast literature on maximal and averaging operators over families of lower dimensional curves in the plane see [7]. The main issue turns out to be the curvature: roughly speaking, non-flat curves admit non-trivial maximal estimates; whereas flat curves do not. More generally, curvature condition plays a crucial role in the analysis of \( \mathcal{M} \), and also in other problems in harmonic analysis, cf. [16], [15, 8]. A fundamental and representative positive result in this direction is the \( L^p \), \( p > 2 \) boundedness of the Bourgain’s circular maximal operator.

The study of such a maximal operator over dilations of a fixed curves \( C \subset \mathbb{R}^2 \) has its beginnings in the circular maximal theorem of Bourgain [11]. Bourgain showed that when \( C = S^1 \), the unit circle, the corresponding maximal operator is bounded on \( L^p(\mathbb{R}^2) \) to \( L^p(\mathbb{R}^2) \) for \( p > 2 \). His proof of the circular maximal theorem relies more directly on the geometry involved. The relevant geometry information concerns intersections of pairs of thin annuli, (for more details, see [1]). Other proof is due to Mockenhaupt, Seeger and Sogge (see, [9]) and proof of this result is based on their local smoothing estimates. These local smoothing estimates, as well as Bourgain’s original techniques actually implies that if one modifies the definition so that the supremum is taken over \( 1 < t < 2 \), then the resulting circular maximal operator is bounded from \( L^p(\mathbb{R}^2) \) to \( L^q(\mathbb{R}^2) \) for some \( q > p \). Here, The maximal operator \( \mathcal{M} \) is defined by

\[
\mathcal{M} f(x) := \sup_{1 \leq t \leq 2} \left| \int_C f(x - ty) \rho(y) \, d\sigma(y) \right|.
\]

Let \( \triangle \) be the closed triangle with vertices \( P = (\frac{2}{3}, \frac{1}{5}), \ Q = \left( \frac{1}{2}, \frac{1}{2} \right), \ R = (0,0) \). In 1997, Schlag (see, [10]) showed that if \( C \) is unit circle, then the inequality (1.1) holds if \( (\frac{1}{p}, \frac{1}{q}) \) lies in the interior of \( \triangle \). His result was obtained using the ”combinatorial method” of Kolasa and Wolff cf. [5]. A different proof of this result was later obtained by Schlag and Sogge cf. [11], which was based on some local smoothing estimates. Schlag also showed that except possibly for endpoints, this result is
sharp (see, [10], [11]). Later, in 2002, Sanghyuk Lee (see, [6]) consider the remaining endpoint estimates for the circular maximal operator.

In this paper, we shall consider a situation when the curvature is allowed to vanish of finite order on a finite set of isolated points. In this connections, Iosevich (see, [4]) had already shown that: If \( C \) is of finite-type curve which is of finite type \( m \) at \( a_0 \). Then, the inequality
\[
\|M f\|_{L^p(\mathbb{R}^2)} \leq B_p \|f\|_{L^p(\mathbb{R}^2)},
\]
holds for \( p > m \), also this result is sharp. Here, we shall extend this result to \( L^p - L^q \) estimates for the corresponding maximal operator associated to families of finite type curves in the plane.

2. Some Preliminaries and main result

We shall need the following definition:

**Definition 2.1.** Let \( C : I \to \mathbb{R}^2 \), where \( I \) is a compact interval in \( \mathbb{R} \) and \( C \) is smooth. We say that \( C \) is of finite type if \( \langle (C(x) - C(x_0)), \mu \rangle \) does not vanish of infinite order for any \( x_0 \in I \), and any unit vector \( \mu \in \mathbb{R}^2 \).

We shall also need a more precise definition which would specify the order of vanishing at each point. Let \( a_0 \) denote a point in the compact interval \( I \). Since our curve is finite type, we can always find a smooth function \( \gamma \), such that in a small neighbourhood of \( a_0 \), locally we can write \( C(s) = (s, \gamma(s)) \), where \( s \in I \).

**Definition 2.2.** Let \( C \) be defined as before. Let \( C(s) = (s, \gamma(s)) \), in a small neighbourhood of \( a_0 \). We say that \( C \) is of finite type \( m \) at \( a_0 \) if \( \gamma^{(k)}(a_0) = 0 \) for \( 1 \leq k < m \), and \( \gamma^{(m)}(a_0) \neq 0 \).

**Remark 2.3.** In fact, this condition is equivalent with the curvature of \( C \) not vanishing to infinite order at \( a_0 \).

We shall use \( C \) as a constant independent of \( j \), in several times without mention it. Now, we shall state our main result of this paper.

**Theorem 2.4.** Let \( C \) be a finite-type curve which is of finite type of order \( m \) at \( a_0 \). Let
\[
M_t f(x) = \int_C f(x - ty) \rho(y) d\sigma(y),
\]
where $d\sigma$ is the induced Lebesgue measure on $\mathbb{C}$ and a smooth cut off function $\rho$ supported in a sufficiently small neighbourhood of $a_0$. Let,

$$Mf(x) = \sup_{1 \leq t \leq 2} |M_t f(x)|.$$ 

Then, for $\left(\frac{1}{p}, \frac{1}{q}\right) \in (\triangle \setminus \{P, Q\}) \cap \left\{\left(\frac{1}{p}, \frac{1}{q}\right) : q > m\right\}$, there is a constant $B$ such that the following inequality

(2.1) \[ \|Mf\|_{L^q(R^2)} \leq B \|f\|_{L^p(R^2)}, \]

holds.

Furthermore, if $m < 5$, then we have

$$\|Mf\|_{L^5, \infty(R^2)} \leq B \|f\|_{L^{5, 1}(R^2)}.$$

**Remark 2.5.** In view of Iosevich’s theorem (see, [4]), the maximal operator $M$ is also of course bounded when the exponents lie on the half open line connecting $\left(\frac{1}{m}, \frac{1}{m}\right)$ and $(0, 0)$.

Our proof will consist of three main steps. First we shall decompose each operator $M_t$ away from the flat point. Then we shall use the method of stationary phase to express each dyadic operator in terms of the Fourier transform of the surface measure on each dyadic piece. We shall then use a stretching argument which was used in Iosevich’s, [4], to expose the behaviour of our operator near the flat point. In the proof, we shall use a scaling argument and a technical lemma (see, [4], Lemma 1.3) to reduce the problem to the local smoothing estimates (see, S. Lee, [6]) for the corresponding Fourier integral operator.

### 3. Proof of Theorem 2.4

**Proof.** of Theorem 2.4 We now turn to the details. Since, our curve is finite type $m$, after perhaps contracting $\text{supp} (\rho)$, by a partition of unity argument, locally we can write $\mathbb{C}(s) = (s, g(s)s^m + c)$, where $g \in C^\infty(\mathbb{R})$, $g(0) \neq 0$, and $c$ is a constant. Now, our proof of uniforms estimates for averaging operators associated to families of finite-type curves is based on Iosevich’s approach cf. [4].

We shall use the following stationary phase result ([14]) for curves in $\mathbb{R}^2$.

**Lemma 3.1.** Let $\mathbb{C}$ be a smooth curve in the plane with non-vanishing Gaussian curvature and $d\mu$ a $C^\infty_c$ measure on $\mathbb{C}$. Then

$$|\hat{d\mu}(\xi)| \leq \text{const} \ (1 + |\xi|)^{-\frac{1}{2}}.$$
Moreover, suppose that $\Gamma \subset \mathbb{R}^2 \setminus \{0\}$ is the cone consisting of all $\xi$ which are normal to some point $x \in C$ belonging to a fixed relatively compact neighbourhood $N$ of $\text{supp } d\mu$. Then,

$$\left(\frac{\partial}{\partial \xi}\right)^\alpha \hat{d\mu}(\xi) = O\left(1 + |\xi|^{-N}\right) \forall N,$$

if $\xi \notin \Gamma$ and $\hat{d\mu}(\xi) = \sum e^{-i<x_j,\xi>} a_j(\xi)$ if $\xi \in \Gamma$, where the finite sum is taken over all $x_j \in N$ having $\xi$ as the normal and

$$\left| \left(\frac{\partial}{\partial \xi}\right)^\alpha a_j(\xi) \right| \leq C_\alpha \left(1 + |\xi|^{-\frac{1}{2} - |\alpha|}\right).$$

Since $C$ is finite-type $m$ at $a_0$, the curvature at $a_0$ vanishes of order $m - 2$. Hence, in order to apply the above lemma, we must first decompose each $M_t$ away from the flat point, where

$$M_t f(x) := \int_C f(x - ty) \rho(y) \, d\sigma(y).$$

Without loss of generality, we take $a_0 = (0, c)$. To do this, we define $\phi \in C_c^\infty$ such that $\text{supp } (\phi) \subset (\frac{l}{2}, 2l) \cup (-2l, -\frac{l}{2})$, $l > 0$, and $\sum \phi(2^k y_1) \equiv 1$, where $l$ is chosen to be small enough so that the interval $(-2l, 2l)$ does not contain any other flat points.

Let

$$M_t^k(f)(x) = \int_C f(x - ty) \phi(2^k y_1) \rho(y) \, d\sigma(y).$$

Then, $M_t(f) = \sum_{k=0}^\infty M_t^k(f)$. Let

$$M_t^k(f)(x) = \sup_{1 \leq t \leq 2} M_t^k(f)(x),$$

Hence, it would suffice to show that

$$\|M_t^k(f)\|_{L^q(\mathbb{R}^2)} \leq B 2^{-k\epsilon} \|f\|_{L^p(\mathbb{R}^2)},$$

for some $\epsilon > 0$.

We can now apply Lemma (3.1) to each $M_t^k$ that is defines over a dyadic piece of our curve. Simultaneously, we perform a stretching transformation

$$y_1 \mapsto 2^k y_1, \ y_2 \mapsto 2^{mk} y_2,$$

which sends each dyadic piece to the curve of unit length $2^{mk} c$ units up the $y_2$—axis.

We now apply the lemma (3.1) to the "stretched" operator $M_t$. Keeping in mind the Jacobian of the stretching transformation, we get an operator of the form (see,
where $\Gamma$ is a fixed cone away from co-ordinates axes, $q_k(\xi)$ is homogeneous of degree one and $a_k(t\xi)$ is a symbol of order 0.

Remark 3.2. Lemma (3.1) tells us that the phase function of the Fourier transform of the curve carried measure is given by $e^{-i\langle x, \xi \rangle}$, where $\xi$ is normal to $x_j$. Using lemma (3.1), Iosevich (see, [4]) explicitly compute the phase function $q(\xi) + c\xi^2$ up to a multiplicative constant of the Fourier transform of the Lebesgue measure on the unit length piece of the curve $C(s) = (s, g(0)s^m + c)$. He shown that the Hessian of $q$ has rank 1.

Hence it is enough to show that
\[(3.2) \quad \| \sup_{1 \leq t \leq 2} H_k(f)(x, t) \|_{L^q(\mathbb{R}^2)} \leq B 2^{-k} \| f \|_{L^p(\mathbb{R}^2)},\]

for some $\epsilon > 0$.

Using this Fourier integral representation, we shall break up the operators dyadically. For this purpose, let us fix $\beta \in C_c^\infty(\mathbb{R} \setminus 0)$ satisfying $\sum_{-\infty}^{\infty} \beta(2^{-j}s) = 1$, $s \neq 0$.

We then define the dyadic operator $H^k_j$ by
\[H^k_j f(x, t) = \frac{1}{(2\pi)^2} 2^{-k} \int_{\Gamma} e^{i<x, \xi>} e^{itq_k(\xi)} e^{it2^mcj} \beta(2^{-j}|\xi|) \frac{a_k(t\xi)}{(1 + t|\xi|)^{\frac{1}{2}}} \hat{f}(\xi) d\xi.\]

Now, we observe that, $\sup_{1 \leq t \leq 2} | \sum_{j=-\infty}^{0} H^k_j f(x, t) |$ is dominated by the Hardy-Littlewood maximal function of $f$.

Therefore, the inequality (3.1), would follow from showing that when $\left( \frac{1}{p}, \frac{1}{q} \right) \in \left[ \left( \frac{1}{p}, \frac{1}{q} \right) : \left( \frac{1}{p}, \frac{1}{q} \right) \in \Delta \setminus \{ P, Q \} \right] \cap \left[ \left( \frac{1}{p}, \frac{1}{q} \right) : q > m \right]$, there is an $\epsilon > 0$ such that
\[(3.3) \quad \| \sup_{1 \leq t \leq 2} \sum_{j=0}^{\infty} H^k_j f(x, t) \|_{L^q(\mathbb{R}^2)} \leq B 2^{-k} \| f \|_{L^p(\mathbb{R}^2)},\]

where $B$ is the constant.

Now, choose a bump function $\psi \in C_c^\infty(\mathbb{R})$ supported in $[\frac{1}{2}, 4]$ such that $\psi(t) = 1$ if $1 \leq t \leq 2$.

In order to estimate (3.3), we use the following well-known estimate (see e.g., [4], Lemma 1.3),
\[ (3.4) \quad \sup_{t \in \mathbb{R}} |\psi(t) H_j^k f(x, t)|^q \]
\[ \leq q \left( \int_{-\infty}^{\infty} |\psi(t) H_j^k f(x, t)|^q dt \right)^{\frac{q-1}{q}} \left( \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial t} (\psi(t) H_j^k f(x, t)) \right|^q dt \right)^{\frac{1}{q}} \]

which follows by using the fundamental theorem of calculus and Hölder's inequality. By Hölder's inequality, this implies
\[ (3.5) \quad \| \sup_{1 \leq t \leq 2} H_j^k f(x, t) \|_{L^q(\mathbb{R}^2)}^q \]
\[ \leq C q \left( \int_{\frac{1}{2}}^{4} \int_{\mathbb{R}^2} |H_j^k f(x, t)|^q dx \, dt \right)^{\frac{q-1}{q}} \left( \int_{\frac{1}{2}}^{4} \int_{\mathbb{R}^2} \left| \frac{\partial}{\partial t} (H_j^k f(x, t)) \right|^q dx \, dt \right)^{\frac{1}{q}} \]
\[ + C \int_{\frac{1}{2}}^{4} \int_{\mathbb{R}^2} |H_j^k f(x, t)|^q dx \, dt. \]

Now,
\[ \frac{\partial}{\partial t} (H_j^k f(x, t)) = \frac{1}{(2\pi)^2} 2^{-k} \int_{\Gamma} e^{i<x,\xi>} e^{itq_k(\xi)} e^{it2mk_c \xi_2} \beta(2^{-j}|\xi|) A_{m,k}(\xi, t) \hat{f}(\xi) d\xi, \]
where,
\[ A_{m,k}(\xi, t) = i(q_k(\xi) + 2^mk_c \xi_2) a_k(t, \xi) \frac{\partial}{\partial t} \left( \frac{a_k(t, \xi)}{1 + t|\xi|} \right)^{\frac{1}{2}} \]
\[ + \frac{\partial}{\partial t} \left( \frac{a_k(t, \xi)}{1 + t|\xi|} \right)^{\frac{1}{2}}. \]

Since, \( q_k(\xi) \approx |\xi| \approx 2^j \), on the support of \( \beta \), we can calculate the orders of the symbols to see that \( \| \sup_{1 \leq t \leq 2} H_j^k f(x, t) \|_{L^q(\mathbb{R}^2)} \) in (3.5) is dominated by
\[ C 2^{-k(1-(\frac{m}{q}))} 2^{-j(\frac{1}{2} - \frac{1}{q})} \| \mathcal{F}_{k,j} f \|_{L^q(\mathbb{R}^3)}, \]
where
\[ \mathcal{F}_{k,j} f = \psi(t) \int_{\Gamma} e^{i<x,\xi>} e^{itq_k(\xi)} e^{it2mk_c \xi_2} \beta(2^{-j}|\xi|) a_k(t, \xi) \hat{f}(\xi) d\xi, \]
and where \( a_k(t, \xi) \) is a symbol of order 0 in \( \xi \).

Now, we need a local smoothing estimates for the operators of the form
\[ (3.6) \quad P_j f(x, t) = \int e^{i<x,\xi>} e^{itq(\xi)} a(t, \xi) \beta(2^{-j}|\xi|) \hat{f}(\xi) d\xi, \]
where \( a(t, \xi) \) is a symbol of order 0 in \( \xi \) and the Hessian matrix of \( q \) has rank 1 everywhere.
Now, we make some observations about the operator $P_j f$. First, locally, we can write our perturb curve as $C(s) = (s, (g(s) + R(s)) s^m + c)$, with $g \in C^\infty(I, \mathbb{R})$ satisfying $b(0) \neq 0$, where, $R \in C^\infty(I, \mathbb{R})$ is a smooth perturbation term. In this case, (see, [4]), $P_j f(x, t)$ is of the form

$$P_j f(x, t) = \int e^{i<x, \xi>} e^{itq(\xi)} a(t, \xi) \beta(2^{-j}|\xi|) \hat{f}(\xi) d\xi,$$

where $a(t, \xi)$ is a symbol of order 0 in $\xi$ and $q(\xi) = q(\xi, R)$ is a smooth function of $\xi$ and $R$ which is homogeneous of degree 1 in $\xi$ and which can be considered as a small perturbation of $q(\xi, 0)$, if $R$ is contained in a sufficiently small neighbourhood of 0 in $C^\infty(I, \mathbb{R})$. The Hessian $D^2_\xi q(\xi, 0)$ has rank 1, so that the same applies to $D^2_\xi q(\xi, R)$ for small perturbation $R$.

Now, for our operator $P_j f(x, t)$, (as can be seen by simple modification of the proof of Lee with $q(\xi) = |\xi| + O(1)$ see, [6]), we get for $1/p + \frac{3}{q} = 1$, $\frac{14}{3} < q \leq \infty$,

$$\left( \int_{\mathbb{R}^2} \int_{1}^{2} |P_j f(x)|^q dt dx \right)^{\frac{1}{q}} \leq C 2^{j(\frac{1}{2} - \frac{q}{4})} \|f\|_{L^p(\mathbb{R}^2)}.$$

(3.7)

However, as observed in [4], the estimate (3.7) remains valid under small, sufficiently smooth perturbations. Therefore, if $\delta$ is sufficiently small and $\|R\|_{CM} < \delta$, then estimate (3.7) holds true also for $R \neq 0$, with an admissible constant $C$.

Let,

$$M^k_j f(x) = \sup_{1 \leq t \leq 2} |H^k_j f(x, t)|.$$

Now, using the local smoothing estimates (3.7), from (3.5), we see that for $1/p + \frac{3}{q} = 1$, and $\frac{14}{3} < q \leq \infty$,

$$\|M^k_j f\|_{\bar{q}} \leq C 2^{-k(1 - \frac{m}{2})} 2^{j(1 - \frac{q}{4})} \|f\|_{L^p(\mathbb{R}^2)}.$$

(3.8)

By Plancherel’s theorem and estimate (3.4), we see that for $j \geq 1$,

$$\|M^k_j f\|_2 \leq C 2^{-k(1 - \frac{m}{2})} \|f\|_{L^2(\mathbb{R}^2)}.$$

(3.9)

A complex interpolation between (3.8) and (3.9) shows that if $(\frac{1}{p}, \frac{1}{q})$ is contained in the closed triangle with vertices $(1, 0)$, $(\frac{3}{14}, \frac{3}{14})$, $(\frac{1}{2}, \frac{1}{2})$ but is not on the closed line segment $[(\frac{3}{14}, \frac{3}{14}), (\frac{1}{2}, \frac{1}{2})]$, then

$$\|M^k_j f\|_{\bar{q}} \leq C 2^{-k(1 - \frac{m}{2})} 2^{j(\frac{3}{2} - \frac{q}{2} - \frac{1}{4})} \|f\|_{L^p(\mathbb{R}^2)}.$$

(3.10)
Using (3.10) and interpolation Lemma 2.6 in [6], we have for \((\frac{1}{p}, \frac{1}{q}) \in [P, Q]\),

\[
\|M^k f\|_{L^q, \infty} \leq C 2^{-k(1-(\frac{m}{q}))} \|f\|_{L^p, 1}.
\]

(3.11)

Since \(M^k\) is a local operator, an interpolation (real interpolation) between these estimates and the trivial \(L^\infty - L^\infty\) estimate, we get, for \((\frac{1}{p}, \frac{1}{q}) \in \triangle \setminus \{P, Q\}\),

\[
\|M^k f\|_{L^q} \leq C 2^{-k(1-(\frac{m}{q}))} \|f\|_{L^p}.
\]

(3.12)

Now, since, \(-(1-(\frac{m}{q})) < 0\) if \(q > m\), we can take \(\epsilon = -1 + \frac{m}{q}\). For \((\frac{1}{p}, \frac{1}{q}) \in (\triangle \setminus \{P, Q\}) \cap \left\{\left(\frac{1}{p}, \frac{1}{q}\right) : q > m\right\}\), we thus get,

\[
\|M f\|_{L^p(\mathbb{R}^2)} \leq B \|f\|_{L^p(\mathbb{R}^2)}.
\]

Hence, we finish our proof of the theorem. □

4. VARIABLE COEFFICIENT MAXIMAL THEOREM

In this section, we shall see, results (Theorem (2.1)) from the last one can be extended to the variable co-efficient case if as in [4], one assumes finite-type "cinematic curvature and co-normality". The averaging operator \(M_t\) that we have considered so far is called translation invariant or the constant coefficient operator, because it averages a function over the translates and dilates of a fixed curve. We are now going to consider an operator which averages a function over a more general distribution of curves in the plane and also a more general time dependence. As before, we are going to define a maximal operator by taking the supremum over the time dependence.

Let us recall at this point some of the previous result in this direction. Sogge (see, [13]) considered the curve of nonzero Gaussian curvature, and allow the curves to vary smoothly from point to point and behave only asymptotically like dilations. He proved that, for each compact set \(K \subset \mathbb{R}^2\), the corresponding maximal operator is \(L^p\)– bounded for \(p > 2\). For higher dimension analog, see [8].

Let us consider a smooth curve \(C \subset \mathbb{R}^2\) through every point \(x \in \mathbb{R}^2\). For fixed \(t\), we assume that this curve is a smooth submanifold in \(\mathbb{R}^2\). We then locally express this curve distribution as

\[
D_t = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : y_1 - x_1 = A(x, y_2, t)\}
\]

for some real and smooth \(A(x, y_2, t)\).

Let \(S_{x,t} = \{y \in \mathbb{R}^2 : (x, y) \in D_t\}\) and consider a family of operators

\[
M_t(f)(x) = \int_{S_{x,t}} |f(x - y)| d\sigma_{x,t}(y),
\]

(4.1)
where $d\sigma_{x,t}$ denotes the smooth cut off function times the Lebesgue measure on $S_{x,t}$. In this paper, we shall be concerned with maximal functions involving averages over curves $S_{x,t}$ in the plane depending smoothly on $(t, x) \in [1, 2] \times \mathbb{R}^2$.

After taking a partial Fourier transform, we can rewrite $M_t(f)(x)$ as

$$M_t f(x) = \int_{\mathbb{R}^2} \delta_0(A(x, y_2, t) + x_1 - y_1) \rho(x, y) f(y) \, dy$$

(4.2)

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} e^{i\tau(A(x, y_2, t) + x_1 - y_1)} \rho(x, y) f(y) \, d\tau \, dy,$$

where $\delta_0$ of course denotes the one-dimensional Dirac-delta function and $\rho(x, y)$ is a cutoff function.

After setting $M_t(f)(x) := Mf(x, t)$ if we regard $t$ as fixed then it is well known that $M_t : D'(\mathbb{R}^2) \to D'(\mathbb{R}^2)$ is a Fourier integral operator of order $-\frac{1}{2}$, (see, [12]).

Let $C_t$ denote the canonical relation for a fixed $t$. We say that $C_t$ is a local canonical graph if

$$C_t = \{(x, \xi, y, \mu) : (y, \mu) = \chi_t(x, \xi)\},$$

where $\chi_t$ is a symplectomorphism for each $t$. We see that this condition is equivalent to the condition that the projection operators $\pi_l$ and $\pi_r$ are local diffeomorphisms. Here, we shall always assume that the canonical relations, $C_t$, of the operators are locally the graph of a canonical transformation. We shall usually write things, through in terms of the phase function $\tau \Phi(x, y, t)$ of the operator $M_t f(x)$, where $\Phi(x, y, t) = A(x, y_2, t) + x_1 - y_1$. For fixed $t$, we write $C_t$ as

$$C_t = \left\{ (x, \tau \frac{\partial \Phi}{\partial x}, y, -\tau \frac{\partial \Phi}{\partial y}) : \tau \in \mathbb{R} \setminus \{0\}, \Phi(x, y, t) = 0 \text{ and } \psi(x, y, t) \neq 0 \right\}.$$

Now, the condition that the projection $\pi_l$ is a local diffeomorphism is equivalent to the condition that the Jacobian of the map

$$(\tau, y) \mapsto \left( \Phi(x, y, t), \tau \frac{\partial \Phi}{\partial x} \right)$$

is non zero. The resulting Jacobian is called the Monge-Ampere determinant:

$$J_t(x, y) = Det \begin{pmatrix}
0 & \Phi_x & \Phi_{x_2} \\
\Phi_y & \Phi_{x_1,y_2} & \Phi_{x_2,y_1} \\
\Phi_y & \Phi_{x_1,y_2} & \Phi_{x_2,y_2}
\end{pmatrix}.$$
Hence, for our defining function $A(x, y_2, t) + x_1 - y_1$, the Monge-Ampere determinant becomes

$$(1 + A_{x_1}) A_{x_2 y_2} - A_{x_2} A_{x_1 y_2}.$$ 

Since the Monge-Ampere determinant is symmetric in the $x$ and $y$ variable, we see that $\pi_l$ is a local diffeomorphism if and only if $\pi_r$ is also. As in [13], we see that local co-ordinates can always be chosen so that we can express the full canonical relation in the form

$$C = \{(x, \xi, y, \mu, t, \tau) : (y, \mu) = \chi_t(x, \xi), \tau = q(x, t, \xi)\},$$

where $q(x, t, \xi)$ is homogeneous of degree 1 in $\xi$ and $C^\infty$ when $\xi \neq 0$.

To give an illustration, let us point out that for circular means operators we have $q(x, t, \xi) = \pm|\xi|$. As we observed that (see, [13]) the rotational curvature condition is not sufficient to get the local smoothing estimate for the operators $M(f)(x, t)$. He showed that the following extra assumption is necessary.

Cone condition: We say that the canonical relation $C$ as in (4.3) satisfies the cone condition if the cone given by the equation $\tau = q(x, t, \xi)$ has exactly one non vanishing principal curvatures. Since $q$ is homogeneous of degree one, its Hessian with respect to the $\xi-$ variable can have rank at most 1. Therefore, our cone condition is: The Hessian of $q$ with respect to the $\xi-$ variable has full rank, i.e.,

$$\text{corank } q''_{\xi\xi} \equiv 1.$$ (4.4)

Let $\pi_{\times Y}$ denote a projection of the canonical relation onto the $x$ and $y$ variables. Let $V_t = \pi_{\times Y}(\Sigma_t)$. From definition, we can always find a smooth function $\chi_t(x_1, x_2)$ such that, locally, we can write $V_t$ as $V_t = \{(x, y) : y_1 = x_1 + A(x, y_2, t), y_2 = \chi_t(x_1, x_2)\}$ where $\chi_t$ is a local diffeomorphism for each fixed $(x_1, t)$. $V_t$ can be viewed as a parameterization of the zeroes of the Monge-Ampere determinant intersected with the curve distribution $D_t$. In the translation-invariant case, we have $y_2 = x_2 + A(x, y_1, t)$ with $A(x, y_1, t) = \gamma\left(\frac{y_1-x_1}{t}\right)$, where $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function satisfying the finite-type condition described earlier. The Monge-Ampere determinant in this case is just $\gamma^{(2)}\left(\frac{y_1-x_1}{t}\right)$. Hence, the Monge-Ampere determinant vanishes of order $m - 2$ along the diagonal $x = y$. Consequently, $V_t$ is just the intersection of $D_t$ with the diagonal.

In general, the situation is more complicated. However, locally a curve distribution whose canonical relation has a two-sided fold behaves very much like a translation-invariant family. We use the following lemma to illustrate the above idea and an explicit proof of the lemma can be found in [4].
**Lemma 4.1.** Suppose that the canonical relation associated to the curve distribution $D_t$ has a 2-sided fold of order $m - 2$. Then, for each fixed $(x', t')$ the curve given by the equation $y_1 = x'_1 + A(x', y_2, t')$ is a curve of finite type $m$ with a flat point at $y_2 = \chi_{t'}(x')$.

Using this lemma (4.1) and the Malgrange Preparation Theorem (see, [3]), we can easily see that, in a small neighbourhood of $(x', t')$, the defining function $A(x, y_2, t)$ is of the form

\[ A(x, y_2, t) = g(x, y_2, t) \left( y_2^m + a_0(x, t) \right), \]

where $g(x, y_2, t)$ is smooth, $g(x', 0, t') \neq 0$, $a_0$ is smooth, and $a_0(x', t') = 0$.

In this connection, Iosevich had already shown that if for each $t$, the canonical relation is folding of order $m - 2$ and the cone condition (4.4) is satisfied away from $\Sigma$, then the corresponding maximal operator is $L^p$-bounded for $p > m$. Naturally, as before we extend this result to $L^p - L^q$ estimates for the variable coefficient version of maximal operator. Our proof of uniform estimates for variable coefficient is based on Iosevich’s approach, and Schlag and Sogge’s local smoothing estimates involving cinematic curvature. We can now state the variable coefficient version of maximal theorem (2.1).

**Theorem 4.2.** Let $M_t$ be as in (4.1). Let $M(f)(x) = \sup_{1 \leq t \leq 2} |M_t f(x)|$. Suppose that for each $t$ the canonical relation is folding of order $m - 2$ and the cone condition (4.4) is satisfied away from $\Sigma$. Then, the following inequality

\[ \|M(f)\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}, \]

holds for \( (\frac{1}{p}, \frac{1}{q}) \in \Delta^0 \cap \left\{ \left( \frac{1}{p'}, \frac{1}{q} \right) : q > m \right\} \), where $\Delta^0$ is the interior of the triangle with vertices $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{2}{5}, \frac{1}{5})$.

**Remark 4.3.** The inequality for $2 < p = q \leq \infty$ was established in Iosevich(4). For $m = 2$, the above results was proved by Schlag and Sogge ([11]), which was based on some variable coefficient versions of local smoothing estimates.

As in the constant coefficient case, we can obtain these estimates from a simple application of technical lemma (see e.g., [4], Lemma 1.3) and the appropriate smoothing estimates.

**Proof.** Our proof will consist on a scaling argument, Lemma (4.1), the proof of theorem (2.1) and most importantly the variable coefficient version of local smoothing estimates of Schlag and Sogge (see, [11]).
Fix \((x', t') \in \mathbb{R}^2 \times [1, 2]\). Consider a curve given by \(y_1 = x_1' + A(x', y_2, t')\). The remarks following the Lemma (4.1) show that \(\exists \delta_1 > 0\), such that for \(|t - t'| < \delta_1\), this curve is finite type \(m\) with a flat point at \(y_2 = \chi_t(x')\). Again using the Lemma (4.1) and the subsequent discussion, we see that, for \(|t - t'| < \delta_1\),

\[
(4.7) \quad A(x', y_2, t) = g(x', y_2, t)((y_2 - \chi_t(x'))^m + a_0(x', t)),
\]

where \(a_0(x', t') = 0\), \(g(x', y_2, t) \neq 0\) when \(y_2 = \chi_t(x')\), and \(t \in \{t : |t - t'| < \delta_1\}\).

We shall first argue that the maximal operator associated to this translation-invariant family satisfies the conclusions of Theorem (4.1). As in the proof of theorem (2.1), we localise our operator near the flat point by introducing a cut off function \(\phi\) with the same properties as before. We perform a stretching transformation,

\[
y_1 \mapsto 2^m k y_1, (y_2 - \chi_t(x')) \mapsto 2^k(y_2 - \chi_t(x')).
\]

The limiting operator under this stretching transformation corresponds to the family of curves given by

\[
y_1 = x_1 + g(x', \chi_t(x'), t)((y_2 - \chi_t(x'))^m + a_0(x', t)).
\]

The only difference between the family of surfaces given by \(y_1 = x_1 + A(x, y_2, t)\) and the ones handled in Theorem (2.1) is the \(t\)-dependence. Since \(g(x', \chi_t(x'), t)\) does not vanish in intersection of this interval with the set \(|t - t'| < \delta_1\), we can treat \(g(x', \chi_t(x'), t)\) as our time parameter, and the same argument goes through. We again use the fact that the variable coefficient estimates of Schlag and Sogge [11] are valid under small smooth perturbations. Hence, the estimates that are valid for the limiting operator are also valid for the sufficiently small perturbation of that operator with admissible constant \(C\).

In order to complete the proof, we localise the operator corresponding to the general family of functions by introducing the same cut off function \(\phi\). Thus, we define

\[
M_t(f)(x) = \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} e^{i\tau(A(x, y', t) + x_1')} \rho(x, y) \phi(y_2 - \chi_t(x')) f(y) d\tau dy,
\]

where, \(\rho(x, y)\) is a cut off function. If we perform a stretching transformation sending

\[
(x_1 - x_1') \mapsto 2^m k(x_1 - x_1'); \quad (x_2 - x_2') \mapsto 2^k(x_2 - x_2')
\]

\[
y_1 \mapsto 2^m k y_1; \quad (y_2 - \chi_t(x')) \mapsto 2^k(y_2 - \chi_t(x')).
\]

This transformation preserves the \(L^p - L^q\) estimates for the maximal operator.
We can use Lemma (4.1) to see that our family of curves smoothly converges to the family of translation-invariant curves defined above. Moreover, each $M^k_t$ satisfies the cinematic curvature condition. Now, we shall use a local smoothing estimate of Schlag and Sogge [11].

In [11], Sogge and Schlag proved $L^p(\mathbb{R}^2) \to L^q(\mathbb{R}^2 \times [1, 2])$ estimates for the operators of the form

$$Pf(x, t) = \int e^{i\phi(x, t, \xi)} a(t, x, \xi) \frac{\hat{f}(\xi)}{(1 + |\xi|)^{\frac{n}{2}}} d\xi,$$

where, $a \in C^\infty([1, 2] \times \mathbb{R}^2 \times \mathbb{R}^2)$ vanishes for $x$ outside of a fixed compact set and satisfies

$$|D_{t,x}^\gamma D_\xi^\gamma a(t, x, \xi)| \leq C_\gamma (1 + |\xi|)^{-|\gamma|}.$$

Also, the phases $\phi$ are real, in $C^\infty([1, 2] \times \mathbb{R}^2 \times \mathbb{R}^2 \setminus 0)$ and homogeneous of degree one in $\xi$.

Also, the above phase function $\phi$ satisfies the conditions

$$(4.9) \quad \det \frac{\partial^2 \phi}{\partial x \partial \xi} \neq 0$$
on supp of $a$, and

$$(4.10) \quad \frac{\partial \phi}{\partial t} = q(t, x, \phi_x'), \quad \text{Corank } q_{\xi \xi}'' = 1,$$non supp of $a$.

Under these hypothesis, they have proved the following result.

**Theorem 4.4.** Let,

$$A_\alpha f(t, x) = \int e^{i\phi(x, t, \xi)} a(t, x, \xi) \frac{\hat{f}(\xi)}{(1 + |\xi|)^{\alpha}} d\xi,$$

where, as above, $a \in S^0_{comp}$ and the phase function satisfies the condition as in (4.9) and (4.10). Then if $1 \leq p \leq \frac{5}{2}$, we get

$$(4.11) \quad \|A_\alpha f\|_{L^{3p'}([1, 2] \times \mathbb{R}^2)} \leq C_\alpha \|f\|_{L^p(\mathbb{R}^2)}, \quad \alpha > 6\left(1 - \frac{1}{3p'}\right).$$

If, now one interpolates with the $L^p \to L^p$ local smoothing estimates in [13], then one sees that

$$\|A_\alpha f\|_{L^{3p'}([1, 2] \times \mathbb{R}^2)} \leq C_\alpha \|f\|_{L^p(\mathbb{R}^2)},$$
for some $\alpha < \frac{1}{2} - \frac{1}{q}$ if $(\frac{1}{p}, \frac{1}{q})$ is in the interior of the triangle $\triangle$ mention before with $q = 3p'$.

We also observed that above local smoothing estimates are valid under small perturbations. Therefore, we see that our localized operators satisfy the right estimates. Because of this and the fact that the averaging operator is basically $A_{\frac{1}{2}} f(x, t)$, our estimate follows from the last inequality and a simple application of technical lemma (see e.g., [4], Lemma 1.3), (see [4], for similar argument) for $(\frac{1}{p}, \frac{1}{q}) \in \triangle \cap \{ (\frac{1}{p}, \frac{1}{q}) : q > m \}$, where $\triangle^o$ is the interior of the triangle with vertices $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{2}{5}, \frac{1}{5})$.

Moreover, arguments from [11] show that, upto an endpoint, the bounds in (4.11) are of the best possible nature. With this, we complete the proof of the theorem (4.2).

4.1. Applications to hyperbolic equations. In this section, we briefly outline an application of the obtained results to the $L^p - L^q$ estimates for solutions to higher order, strictly hyperbolic pseudo-differential operators. The related fixed-time estimate is given in [2]. The analogous local smoothing result for the wave equation (where $q(\xi) = |\xi|$) is given in [9]. We obtain the following $L^p - L^q$ estimates for solutions to higher order, strictly hyperbolic pseudo-differential operators.

**Theorem 4.5.** Let $p(\xi, \tau) = (\tau - \phi_1(\xi))(\tau - \phi_2(\xi)) \ldots (\tau - \phi_m(\xi))$ be a strongly hyperbolic polynomial, homogeneous of degree $m$ on $\mathbb{R}^{2+1}$. Suppose that the $\phi_j$ are continuous, and that $\phi_j(\xi) \neq 0$ if $\xi \neq 0$. Take, $(\frac{1}{p}, \frac{1}{q}) \in (\Delta \setminus \{P, Q\}) \cap \{ (\frac{1}{p}, \frac{1}{q}) : q > m \}$. Then given data $g \in L^p(\mathbb{R}^2)$, the unique solution $u(x, t)$ of the initial value problem

\begin{equation}
\begin{aligned}
& p \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) u(x, t) = 0 \\
& \left( \frac{\partial}{\partial t} \right)^j u(x, t) \mid_{t=0} = 0, \text{ for } j = 0, 2, \ldots, m - 1, \\
& \text{and } \frac{\partial}{\partial t} u(x, t) \mid_{t=0} = g(x)
\end{aligned}
\end{equation}

satisfies,

$$\| \sup_{1 \leq t \leq 2} u(x, t) \|_{L^q(\mathbb{R}^2)} \leq C \| g \|_{L^p(\mathbb{R}^2)}.$$
Proof. The solution $u(x, t)$ of (4.12) is given by

\begin{equation}
(4.13) 
  u(x, t) = \sum_{j=1}^{m} \int e^{i<x, \xi>} e^{it\phi_j(\xi)} a_j(\xi) \hat{g}(\xi) \, d\xi,
\end{equation}

where $a_j(\xi)$ is homogeneous of degree $-1$. Thus the result follows from Theorem (2.1).

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