New Results on the Parameterisation of Complex Hadamard Matrices

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Abstract

In this paper we provide an analytical procedure which leads to a system of \((n - 2)^2\) polynomial equations whose solutions give the parameterisation of the complex \(n \times n\) Hadamard matrices. It is shown that in general the Hadamard matrices depend on a number of arbitrary phases and a lower bound for this number is given. The moduli equations define interesting geometrical objects whose study will shed light on the parameterisation of Hadamard matrices, as well as on some interesting geometrical varieties defined by them.

1 Introduction

Quantum information theory whose main source comes of a few astonishing features in the foundations of quantum mechanics is the theory of that kind of information which is carried by quantum systems from the preparation device to the measuring apparatus in a quantum mechanical experiment, see e.g. [27]. Defining new concepts like entangled states, teleportation or dense coding one hopes to be able to design and construct new devices, like quantum computers, which will be useful in solving many “unresolvable” problems by the classical methods. Recently the mathematical structure which is behind such miracle machines was better understood by establishing a one-to-one correspondence between quantum teleportation schemes, dense coding schemes, orthogonal bases of maximally entangled vectors, bases of unitary operators and unitary depolarizers by showing that given any object of any one of the above types one can construct any object of each of these types by using a precise procedure. See Vollbrecht and Werner [25] and Werner [26] for details. The construction procedure will be efficient to the extent that the unitary bases can be generated, and the construction of these bases makes explicit use of the complex Hadamard matrices and Latin squares. The aim of this paper is to provide a procedure for the parameterisation of the complex Hadamard matrices for an arbitrary integer \(n\). More precisely we will obtain a set of
(n − 2)^2 equations whose solutions will give all the complex Hadamard matrices of size n. Complex n-dimensional Hadamard matrices are unitary n × n matrices whose entries have modulus 1/√n.

The term Hadamard matrix has its root in the Hadamard’s paper [17], where he gave the solution to the question of the maximum possible absolute value of the determinant of a complex n × n matrix whose entries are bounded by some constant, which, without loss of generality, can be taken equal to unity. Hadamard has shown that the maximum is attained by complex unitary matrices whose entries have the same modulus and he asked the question if the maximum can also be attained by orthogonal matrices. These last matrices have come to be known as Hadamard matrices in his honor, and have many applications in combinatorics, coding theory, orthogonal designs, quantum information theory, etc., and a good reference about the obtained results is Agaian [1].

However the first complex Hadamard matrices were found by Sylvester [24]. He observed that if \( a_i, i = 0, 1, \ldots, n − 1 \) denote the solutions of the equation \( x^n − 1 = 0 \) for a prime \( n \) then the Vandermonde matrix

\[
\frac{1}{\sqrt{n}} \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1}
\end{pmatrix}
\]

is unitary and Hadamard. In the same paper Sylvester found a method to obtain a Hadamard matrix of size \( mn \) if one knows two Hadamard matrices of order \( m \) and respectively \( n \) by taking their Kronecker product. Soon after the publication of the paper by Hadamard the interest was mainly on the real Hadamard matrices such that the Sylvester contribution fell into oblivion and the complex Hadamard matrices have been much later reinvented in a particular case: only those matrices whose entries are \( \pm 1, \pm i \) where \( i = \sqrt{-1} \).

Nevertheless a few other problems apparently unrelated to complex Hadamard matrices were those connected with bounds on polynomial coefficients when the indeterminate runs on the unit circle. They are better expressed in terms of the discrete Fourier transform. For any finite sequence \( x = (x_0, x_1, \ldots, x_{n-1}) \) of \( n \) complex numbers, its (discrete) Fourier transform is defined by

\[
y_j = n^{-1/2} \sum_{k=0}^{n-1} x_k e^{2i\pi kj/n} \quad j = 0, 1, \ldots, n - 1
\]

If the components \( x_k, y_k \) are such that \( |x_k| = |y_k| = 1 \) for \( k = 0, 1, \ldots, n − 1 \) the sequence \( x \) is called bi-unimodular. The existence of a bi-unimodular sequence of side \( n \) is equivalent to the existence of a complex circulant Hadamard matrix of side \( n \); a circulant matrix is obtained by circulating its first row, in our case the components of the vector \( x/\sqrt{n} \). Now the Gauss sequence

\[
x_k = \begin{cases} 
  e^{2i\pi(ak^2+bk)/n}, & a, b \in \mathbb{Z}, a \text{ coprime to } n, k = 0, 1, \ldots, n - 1 \quad \text{for } n \text{ odd} \\
  e^{k^2i\pi/n}, & k = 0, 1, \ldots, n - 1 \quad \text{for } n \text{ even}
\end{cases}
\]

is a bi-unimodular sequence [8]. The problem of the complete determination of all bi-unimodular sequences is still open, despite the problem is simpler than the parameterisation of arbitrary com-
plex Hadamard matrices. However this approach gave the first non-trivial examples of complex Hadamard matrices for \( n \geq 6 \).

A step towards its solution was the reduction of the bi-unimodular problem to the problem of finding all cyclic \( n \)-roots \([5]\), that are given by the following system of equations over \( \mathbb{C} \)

\[
\begin{align*}
z_0 + z_1 + \cdots + z_{n-1} &= 0, \\
z_0z_1 + z_1z_2 + \cdots + z_{n-1}z_0 &= 0, \\
z_0z_1z_2 + z_1z_2z_3 + \cdots + z_{n-1}z_0z_1 &= 0, \\
\cdots & \cdots \\
z_0z_1\cdots z_{n-1} &= 1
\end{align*}
\]

(1)

Note that the sums are cyclic and contain just \( n \) terms and are not the elementary symmetric functions for \( n \geq 4 \). The relation between \( x \) and \( z \) is \( z_j = x_{j+1}/x_j \). All cyclic \( n \)-roots have been found for \( 2 \leq n \leq 8 \); see Björck and Fröberg \([6, 7]\). The formalism we will develop in the paper is more general showing that the parameterisation of complex Hadamard matrices is more complicated than the finding of all cyclic \( n \)-roots of the system (1). Using our approach we find, e.g. when \( n = 6 \), the following matrix which is not contained in the above solutions

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & i & -i & -i & i \\
1 & i & -1 & e^{it} & -e^{it} & -i \\
1 & -i & -e^{-it} & -1 & i & e^{-it} \\
1 & -i & e^{-it} & i & -1 & -e^{-it} \\
i & i & -i & e^{it} & e^{-it} & -1
\end{pmatrix}
\]

matrix that depends on an arbitrary phase.

The parameterisation of complex Hadamard matrices is a special case of a more general problem: that of reconstructing the phases of a unitary matrix from the knowledge of the moduli of its entries, problem which was a fashionable one at the end of eighties of the last century in the high energy physics community, see Auberson\([3]\), Björken and Dunietz \([9]\), Branco and Lavoura \([10]\), Auberson et al. \([4]\). An existence theorem as well as an estimation for the number of solutions was obtained by us \([12]\). The particle physicists abandoned the problem when they realised that for \( n \geq 4 \) there exists a continuum of solutions, i.e. solutions depending on arbitrary phases, result that was considered uninteresting from the physical point of view. In our opinion, the reason was the difficulty of the problem; since the experiments provide only the squares of the moduli, the first problem is to decide if from the experimental results, which in the best case generate a doubly stochastic matrix, one can reconstruct a unitary matrix, or a unistochastic matrix. Only for \( n = 3 \) there exists a unambiguous procedure. For \( n \geq 4 \) there are no known necessary and sufficient conditions to separate the unistochastic matrices from the doubly stochastic ones \([29]\).

Almost in the same time the complex Hadamard matrices came out in the construction of some \(*\)-subalgebras in finite von Neumann algebras, see Popa \([24]\), de la Harpe and Jones \([15]\) and
Munemasa and Watatani[20]. In the last two papers one construct complex Hadamard matrices not of Sylvester type when \( n \) is a prime number such that \( n \equiv \pm 1 \pmod{4} \). A little later Haagerup[16] obtained the first example of a 6-dimensional matrix which is not a solution of the system of equations (1).

In this paper we make use of a few analytic techniques from the operator contraction theory and the factorization of unitary matrices to obtain a convenient representation of unitary matrices of arbitrary order \( n \) that leads us easily to a system of \((n-2)^2\) trigonometric (or equivalently polynomial) equations whose solutions give all the complex Hadamard matrices of order \( n \). Our approach is also useful for finding real Hadamard matrices, being complementary to the combinatorial approach almost exclusively used until now.

The paper is organized as follows: in Section 2 the equivalence of the complex Hadamard matrices is reviewed. In Section 3 a theorem showing the existence of the complex Hadamard matrices for every integer \( n \) is stated and an upper bound on the number of continuum solutions is obtained. Section 4 contains an one-to-one parametrisation of unitary matrices written as block matrices and in the next Section an application of the obtained formulae is given. In Section 6 an other parameterisation of unitary matrices is given under the form of a product of \( n \) diagonal phase matrices interlaced with \( n - 1 \) orthogonal matrices each one generated by a real vector from \( \mathbb{R}^n \).

This form is convenient because it leads to a simpler form for the moduli equations and in the same time we consider it more appropriate for designing software packages for solving these equations. In Section 7 we show how to derive the moduli equations as trigonometric equations and give a few particular solutions for \( n = 6 \). In Section 8 the problem is reformulated as an algebraic geometry problem and we show that the parameterisation of Hadamard matrices can produce interesting examples for many problems currently under study in this field. The paper ends with Conclusions.

2 Equivalence of complex Hadamard matrices

Complex \( n \)-dimensional Hadamard matrices being unitary matrices whose entries have modulus \( 1/\sqrt{n} \), the natural class of looking for complex Hadamard matrices is the unitary group \( U(n) \).

The unitary group \( U(n) \) is the group of automorphisms of the Hilbert space \((\mathbb{C}^n, \langle \cdot, \cdot \rangle)\) where \( \langle \cdot, \cdot \rangle \) denotes the Hermitian scalar product \( \langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y}_i \) and the bar denotes the complex conjugation. If \( A_n \in U(n) \) by \( A_n^* \) we denote the adjoint matrix and unitarity implies \( A_n^* A_n = A_n A_n^* = I_n \). It follows that \( \det A_n = e^{i \varphi} \), where \( \varphi \) is a phase, and \( \dim_{\mathbb{R}} U(n) = n^2 \).

Because in any group the product of two arbitrary elements is again an element of the group there is a freedom in choosing the "building" blocks to be used in a definite application. In the case of a complex Hadamard matrix the multiplication of a row and/or a column by an arbitrary phase factor does not change its properties and consequently we can remove the phases of a row and column taken arbitrarily. Taking into account that property we can write

\[
A_n = d_n \tilde{A}_n d_{n-1}
\]
where \( \tilde{A}_n \) is a matrix with all the elements of the first row and of the first column positive numbers and \( d_n = (e^{i\varphi_1}, \ldots, e^{i\varphi_n}) \) and \( d_{n-1} = (1, e^{i\varphi_{n+1}}, \ldots, e^{i\varphi_{2n-1}}) \) are two diagonal phase matrices. In the following we will consider that \( A_n \equiv \tilde{A}_n \), i.e. \( A_n \) will be a matrix with positive entries in the first row and the first column.

Since a unitary matrix is parameterised by \( n(n - 1)/2 \) angles and \( n(n + 1)/2 \) phases, the above equivalence relation tell us that the number of remaining phases is \( n(n + 1)/2 - (2n - 1) = (n - 1)(n - 2)/2 \), and so the number of free real parameters entering a unitary matrix is reduced from \( n^2 \) to \( n^2 - (2n - 1) = (n - 1)^2 \).

Secondly we can permute any rows and/or columns and get an equivalent unitary matrix. This procedure can be seen as a multiplication of \( A_n \) at left and/or right by an arbitrary finite number of the simplest permutation unitary matrices \( P_{ij}, \ i \neq j, \ i, j = 1, \ldots, n \), whose all diagonal entries but \( a_{ii} \) and \( a_{jj} \) are equal to unity, \( a_{ii} = a_{jj} = 0 \), \( a_{ij} = a_{ji} = 1 \), \( i \neq j \) and all the other entries vanish. Both the diagonal phase and permutation matrices generate subgroups of the unitary \( U(n) \) group; so we may consider them as gauge subgroups, i.e. any element of \( U(n) \) is defined modulo the action of a finite number of the above transformation, which has as consequence a standard representation for unitary matrices. We consider that the group generated by the above two subgroups deserves to be independently studied since its orbit structure could shed light on many important issues from information theory and stochastic matrices.

The above two equivalence conditions are those found by Sylvester [21] for the Hadamard matrices, but in fact they are valid for \( U(n) \) which is invariant with respect to the product of an arbitrary number of the above transformations.

Besides for Hadamard matrices we will not distinguish between \( A_n \) and its complex conjugated matrix \( \bar{A}_n \), the complex conjugation being equivalent to the sign change of all phases \( \varphi_i \rightarrow -\varphi_i \) entering the parametrisation. More generally we shall consider equivalent two matrices whose phases can be obtained each other by an arbitrary non-singular linear transformation with constant rational coefficients. As we will see later the complex Hadamard matrices depend in general on a number of arbitrary phases and the above condition says that we will consider only the most general form of the solution and not those particular forms obtained by prescribing definite values to the (arbitrary) phases entering the parameterisation. In this sense we can say that there is only one complex Hadamard matrix of order 4, that found by Hadamard [17], all the others, including those with all entries real numbers, being particular cases of the complex one. Other authors speak in this case of non-equivalent or a continuum of solutions [16]. We consider that the above conditions are the only a priori equivalence criteria we can impose on Hadamard matrices, i.e. will consider equivalent any two matrices that can be made equal by applying them a finite number of the above transformations.
3 Existence of complex Hadamard matrices

The parameterisation of a unitary matrix by the moduli of its entries is very appealing, and in the case of Hadamard matrices compulsory, although it is not a natural one in the general case. A natural parameterisation would be one whose parameters are free, i.e. there are no supplementary restrictions upon them to enforce unitarity. In this sense natural parameterizations are the Euler-type parameterisation by Murnagham [21], or that found by us [11].

The problem we rose in [12] was to what extent the knowledge of the moduli $|a_{ij}|$ of an $n \times n$ unitary matrix $A_n = (a_{ij})$ determines $A_n$. Implicitly we supposed that $A_n$ is parameterized by $n^2$ independent parameters. But from what we said before we know that we may ignore $2n-1$ phases entering the first row and the first column and consequently the number of independent parameters reduces to $(n-1)^2$, that coincides with the number of independent moduli implied by unitarity. If we identify the parameters to the moduli they will be lying within the simple domain

$$D = (0, 1) \times \ldots \times (0, 1) \equiv (0, 1)^{(n-1)^2}$$

where the above notation means that the number of factors entering the topological product is $(n - 1)^2$. We excluded only the extremities of each interval, i.e. the points 0 and 1 that is a zero measure set within $U(n)$ and has no relevance to the parameterisation of complex Hadamard matrices.

Thus, in principle, we can parameterise an $n \times n$ unitary rephasing invariant matrix by the upper left corner moduli; we exclude the moduli of the last row and of the last column since they follow from unitarity. Nothing remains but to check if the new parameterisation is one-to-one. A solution to the last problem is the following: start with a one-to-one parameterisation of $U(n)$ and then change the coordinates taking as new coordinates the moduli of the $(n-1)^2$ upper left corner entries (and $2n-1$ ignorable phases). Afterwards use the implicit function theorem to find the points where the new parameterisation fails to be one-to-one. The corresponding variety upon which the application is not a bijective one is given by setting to zero the Jacobian of the transformation. One gets that generically for $n \geq 4$ the unitary group $U(n)$ cannot be fully parametrised by the moduli of its entries, i.e. for a given set of moduli there could exist a continuum of solutions, but this negative result is good for the parameterisation of Hadamard matrices by decreasing the number of independent solutions taking into account the equivalence conditions discussed in the previous section.

If the moduli are outside of the above variety an upper bound for the multiplicity is $2^{\frac{n(n-3)}{2}}$. However in the case of Hadamard matrices the equivalence constraints reduce this number to lower values than the above upper bound. The bound is saturated for $n = 3$ when there is essentially only one complex matrix, i.e. for given moduli values for the first row and column entries compatible with unitarity, the sole freedom is an arbitrary phase. If we denote the relevant squared moduli by $m_1, m_2, m_3, m_4$ and the phase by $\varphi$ then the compatibility condition has the form

$$-1 \leq \cos \varphi = (-1 + 2m_1 - m_1^2 + m_2 + m_3 + m_4 - m_1m_2 - m_1m_3 - m_2m_3 -$$
\[
2m_1m_4 - m_1m_2m_3m_4^2/2\sqrt{m_1m_2m_3(1 - m_1 - m_2)(1 - m_1 - m_3)} \leq 1
\]

This is also the necessary and sufficient condition which the squared moduli \(m_i, i = 1, \ldots, 4\), have to satisfy in order to obtain a unistochastic matrix from a general doubly stochastic matrix. Because unitary matrices of arbitrary dimension do exist and on the other hand the number of independent essential parameters of a \(U(n)\) matrix is \((n - 1)^2\) the following is true:

**Theorem 1** Suppose \((x_1, \ldots, x_{n^2})\) is a co-ordinate system on the unitary group \(U(n)\) consisting of \(n(n - 1)/2\) angles each one taking values in \([0, \pi/2]\) and \(n(n + 1)/2\) phases taking values in \([0, 2\pi]\). By discarding \(2n - 1\) non-essential phases the number of co-ordinates reduces to \((n - 1)^2\), \((x_1, \ldots, x_{(n-1)^2})\), that coincides with the number of independent moduli \((m_1, \ldots, m_{(n-1)^2})\) implied by unitarity. Taking as new co-ordinates the moduli \(m_i, i = 1, \ldots, (n - 1)^2\), the new parameterisation is generically not one-to-one for \(n \geq 4\), the non-uniqueness variety being obtained by setting to zero the Jacobian of the transformation

\[
\frac{\partial(m_1, \ldots, m_{(n-1)^2})}{\partial(x_1, \ldots, x_{(n-1)^2})} = 0
\]

Outside this variety the number of discrete solutions \(N_s\) satisfies \(1 \leq N_s \leq 2^{n(n-3)/2}\) and on the variety described by \((3)\) there is a continuum of solutions. In the special case of complex Hadamard matrices all the solutions are given by the system of trigonometric equations

\[
m_i^2(x_1, \ldots, x_{(n-1)^2}) = \frac{1}{n}, \quad i = 1, \ldots, (n - 1)^2
\]

Suppose we know the irreducible components of the variety \((3)\) and let \(r(n)\) be the rank of the system \((4)\) in every irreducible component, then every solution of \((4)\) in such an irreducible component will depend upon \((n - 1)^2 - r(n)\) arbitrary parameters and the number of (continuum) solutions satisfies \(1 \leq N_s \leq 2^{r(n-1) - r(n)/2}\).

**Proof.** In the general case Eqs.\((4)\) have the form

\[
m_i^2(x_1, \ldots, x_{(n-1)^2}) = a_i, \quad \text{where } a_i \in (0, 1), \quad i = 1, \ldots, (n - 1)^2
\]

The parameters \(a_i\) generate a doubly stochastic matrix. The Eqs.\((5)\), as we will see later, are trigonometric equations in our parameterisation, and consequently the multiplicity of the solutions may arise from the two possible phase solutions for all values of sine or cosine functions that satisfy \((5)\). The number of independent phases is \((n - 1)(n - 2)/2\) and taking into account that we consider \(A_n\) and \(\bar{A}_n\) as equivalent matrices, condition which halves the number of solutions, the above bound for \(N_s\) follows. A similar argument establishes the upper bound for the number of continuum solutions.

For \(n = 3\) the Jacobian is positive and \(1 \leq N_s \leq 1\), which implies the existence of one complex matrix irrespective of the values \(a_i\), compatible with unitarity.
It is easily seen that the equations which correspond to the first row and the first column entries have a unique solution and the number of equations reduces to \((n-2)^2\). Indeed, because these entries are positive we can take the following parameterisation in terms of \(2n-3\) angles, e.g. for the first row

\[ (a_{11}, \ldots, a_{1n}) = (\cos \chi_1, \sin \chi_1 \cos \chi_2, \ldots, \sin \chi_1 \ldots \sin \chi_{n-1}) \]

and similarly for the first column. The Eqs. give the unique solution

\[ \cos^2 \chi_k = \frac{a_k}{\prod_{i=1}^{k-1} (1 - a_i)}, \quad k = 1, 2, \ldots, n-1 \]

where \(a_k = |a_{ik}|^2\), \(k = 1, 2, \ldots, n-1\). In the case of Hadamard matrices one gets

\[ \cos \chi_k = \frac{1}{\sqrt{n+1-k}}, \quad k = 1, 2, \ldots, n-1 \]

and the same solution for the angles parameterising the first column. In this way the number of equations reduces to \((n-1)^2 - (2n-3) = (n-2)^2\) and the upper bound for the continuous solutions may be written as

\[ 1 \leq N_s \leq 2^{r(n)-1} - (n-2)(n-3)/2, \]

where \(r(n)\) is the rank of the reduced system. Even so the number of equations grows quadratically with \(n\) which shows that even for moderate values of \(n\) the problem is not easy to solve.

In conclusion we have a system of trigonometric equations whose solutions will give all the complex Hadamard matrices, but to get effective we have to start with a one-to-one parameterisation of unitary matrices in order to find the explicit form of the \((n-2)^2\) equations and try to solve them. In the following Section we will provide one of the two parameterisations of unitary matrices that we will use in the paper.

### 4 Parameterisation of unitary matrices

The aim of this section is to provide a one-to-one parameterisation of unitary matrices that will be useful in describing the complex Hadamard matrices. We shall present two such parameterisations and for the the first one we follow closely our paper showing here only the points which are important in the following. The algorithm we provide is a recursive one, allowing the parameterisation of \(n \times n\) unitary matrices through the parameterisation of lower dimensional ones. The parameterisation will be one-to-one and given in terms of \(a(n)\) angles taking values in \([0, \pi/2]\) and \(\varphi(n)\) phases taking values in \([0, 2\pi]\) such that the application

\[ A_n(A_n \in U(n), A_n A_n^* = I_n) \rightarrow E = (0, \pi/2)^{a(n)} [0, 2\pi)^{\varphi(n)} \subset \mathbb{R}^{n^2} \]

is bijective. Always in the following the ends of the interval \([0, \pi/2]\) will be obtained by continuation in the relevant parameters, if necessary.
The starting point is the partitioning of the matrix $A_n \in U(n)$ in blocks

$$A_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(6)

For definiteness we suppose the order of $A$ is equal to $m$ with $m \leq n/2$. The blocks entering (6) are contractions as follows from unitarity

$$AA^* + BB^* = I_m, \quad A^* A + C^* C = I_m, \quad CC^* + DD^* = I_{n-m}$$

(7)

where in the following $I_k$ denotes the $k \times k$ unit matrix. Suppose we know the contraction $A$, then the problem reduces to finding the $B$, $C$ and $D$ blocks such that $A_n$ should be unitary. In other words the problem is: knowing a contraction $A$ of side $m$ how we can border it for getting a unitary $n \times n$ matrix $A_n$? For solving this problem we shall make use of the theory of contraction operators.

An operator $T$ applying the Hilbert space $\mathcal{H}$ in the Hilbert space $\mathcal{H}'$ is a contraction if for any $v \in \mathcal{H}$, $||T v||_{\mathcal{H}'} \leq ||v||_{\mathcal{H}}$, i.e. $||T|| \leq 1$, [22]. For any contraction we have $T^* T \leq I_{\mathcal{H}'}$ and $TT^* \leq I_{\mathcal{H}}$ and the defect operators

$$D_T = (I_{\mathcal{H}} - T^* T)^{1/2}, \quad D_{T^*} = (I_{\mathcal{H}'} - TT^*)^{1/2}$$

are Hermitean operators in $\mathcal{H}$ and $\mathcal{H}'$ respectively. They have the property

$$TD_T = D_{T^*} T, \quad T^* D_{T^*} = D_T T^*$$

(8)

Here we consider only finite-dimensional contractions, i.e. $T$ will have in general $n_1$ rows and $n_2$ columns.

The unitarity relations (7) can be written as

$$BB^* = D_A^2, \quad C^* C = D_A^2$$

According to Douglas lemma [14] there exist two contractions $U$ and $V$ such that

$$B = D_A U, \quad \text{and} \quad C = D_A V$$

Since we are looking for a parameterisation of unitary matrices, $U^*$ and $V$ are isometries, i.e. they satisfy the relations

$$UU^* = I_{n-m}, \quad V^* V = I_m$$

If $n$ is even and $m = n/2$, then $U$ and $V$ are unitary operators. Thus $B$ and $C$ blocks are given by the defect operators $D_A$, $D_A$ and two arbitrary isometries whose dimensions are $m \times (n-m)$ and $(n-m) \times m$ respectively. The last block of $A_n$ is given by the lemma
Lemma 1 The formula
\[ D = -VA^*U + D_{V^*}KD_U \]
establishes a one-to-one correspondence between all the bounded operators \( D \) such that
\[ A_n = \begin{pmatrix} A & D_{A^*}U \\ VD_A & D \end{pmatrix} \]
is a contraction and all the bounded contractions \( K \).

See Arsene and Gheondea [2] for a proof of the general result when \( U, V \) and \( K \) are contractions, and further details. In our case \( U \) and \( V \) being isometries \( D \) is given by
\[ D = -VA^*U + XMY \]
where \( X \) and \( Y \) are those unitary matrices that diagonalise the Hermitean defect operators \( D_{V^*} \) and \( D_U \) respectively, i.e.
\[ X^*D_{V^*}X = P, \quad Y^*D_YY = P \]

\( P \) is the projection
\[ P = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-2m} \end{pmatrix} \]
and the matrix \( M \) entering (9) has the form
\[ M = \begin{pmatrix} 0 & 0 \\ 0 & A_{n-2m} \end{pmatrix} \]
where \( A_{n-2m} \) denotes an arbitrary \((n-2m) \times (n-2m)\) unitary matrix. See [11] for details. In the above formulae we supposed that the eigenvectors of the \( D_U \) and \( D_{V^*} \) operators entering the matrices \( X \) and \( Y \) are ordered in the increasing order of the eigenvalues.

Therefore the parameterisation of an \( n \times n \) unitary matrix is equivalent to the parameterisations of four matrix blocks with lower dimensions than those of the original matrix, and consequently our task is considerably simplified. On the other hand the formulae (9) and subsequent show that this procedure is recursive allowing the parameterisation of any finite dimensional unitary matrix starting with the parameterisation of one- or two-dimensional unitary matrices. Moreover the parameterisation of \( A_n \) requires the parameterisation of an \( m \times m \) contraction, of two isometries \( U \) and \( V \) and of an \((n-2m) \times (n-2m)\) unitary matrix. In our papers [11, 12] we considered only the case \( m = 1 \) as the simplest one, however the case \( m > 1 \) may be useful in the study of complex Hadamard matrices.

For what follows we treat again the case \( m = 1 \), i.e. \( A \) is the simplest contraction, a complex number whose modulus is less than one, because we found the form of the matrices \( X \) and \( Y \) for arbitrary \( n \). Since \( V \) is a \((n-1)\)-dimensional vector the isometry property allows us to
parametrise it as \( V = (\cos \chi_1, \sin \chi_1 \cos \chi_2, \ldots, \sin \chi_1 \ldots \sin \chi_{n-2})^t \) where \( t \) denotes transpose. \( V \) is the eigenvector of \( D_V \) corresponding to the zero eigenvalue. Indeed from the relations we have

\[
D_V V = V D_V = 0
\]

showing that \( V \) is the eigenvector of \( D_V \) corresponding to the zero eigenvalue. Thus the problem is: how to complete an orthogonal matrix \( X \) knowing its first column (row) such that no supplementary parameters enter. The other columns of this matrix we are looking for will be given by the other eigenvectors of \( D_V \). One easily verifies that \( D_V \) is a projection operator such that the other eigenvalues equal unity. Indeed the following holds

**Lemma 2** The orthonormalised eigenvectors of the eigenvalue problem

\[
D_V v_k = \lambda_k v_k, \quad k = 1, \ldots, n - 1
\]

are the columns of the orthogonal matrix \( X \in SO(n - 1) \) and are generated by the vector \( V \) as

\[
v_1 = \begin{pmatrix}
\cos \chi_1 \\
\sin \chi_1 \cos \chi_2 \\
\vdots \\
\vdots \\
\sin \chi_1 \ldots \sin \chi_{n-2}
\end{pmatrix}
\]

and

\[
v_{k+1} = \frac{d}{d \chi_k} v_1 (\chi_1 = \ldots = \chi_{k-1} = \pi/2), \quad k = 1, \ldots, n - 2
\]

where in the above formula one calculates first the derivative and afterwards the restriction to \( \pi/2 \).

In a similar way one finds \( Y \); see [13] for a proof.

In the case of \( n \times n \) Hadamard matrices whose elements of the first row and of the first column are positive numbers \( a_{1j} = a_{j1} = \frac{1}{\sqrt{n}}, \quad j = 1, \ldots, n \), \( X \) has the form

\[
\begin{pmatrix}
\frac{1}{\sqrt{n-1}} & -\sqrt{\frac{n-2}{n-1}} & 0 & 0 & \ldots & 0 & 0 \\
\frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{(n-1)(n-2)}} & -\sqrt{\frac{n-3}{n-2}} & 0 & \ldots & 0 & 0 \\
\frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-2)(n-3)}} & -\sqrt{\frac{n-4}{n-3}} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-2)(n-3)}} & \frac{1}{\sqrt{(n-3)(n-4)}} & \ldots & 1 & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-2)(n-3)}} & \frac{1}{\sqrt{(n-3)(n-4)}} & \ldots & 1 & 1
\end{pmatrix}
\]
and $Y = X^t$, where $t$ denotes the transposed matrix.

In this way all the quantities entering formula (9) are known and the parameterisation of $A_n$ can be obtained recursively starting with the known parameterisation of $2 \times 2$ unitary matrices.

When the block $A$ is one-dimensional, i.e. a simple number equal to $1/\sqrt{n}$, the term $V A^* U$ entering Eq. (9) has the form $\frac{1}{(n-1)\sqrt{n}} J$ where $J$ is the $(n-1) \times (n-1)$ matrix whose each of entries is $+1$, which appears in many constructions of real Hadamard matrices; see Agaian [1].

5 Application

In the following we will use Eq. (9) to generalize to the case of complex Hadamard matrices the tricks used by Sylvester [24] and Hadamard [17] for constructing complex Hadamard matrices. We take $n$ an even number, $n = 2m$, and we suppose that we know a parameterisation of the $A$ block which is unitary and whose order is $m$. In that case $B$ and $C$ blocks are also unitary matrices of order $m$ and we consider them normalized as $A A^* = B B^* = C C^* = I_m$. From (9) we have $D = -C A^* B$ and the following matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} A & B \\ C & -C A^* B \end{pmatrix}$$

will be unitary by construction. In general the above matrix will not be Hadamard even when $A$, $B$ and $C$ are, as the simplest example shows; this happens only when either $C = A$ or $B = A$. Since the second case is obtained by transposing the matrix of the first one, as long as $B$ and $C$ are arbitrary, we will consider only the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} A & B \\ A & -B \end{pmatrix}$$

(10)

which is the elementary two-dimensional array that will be used in the construction of more complicated arrays of Hadamard matrices. In the following we suppose that $A$ and $B$ are complex Hadamard matrices of size $m$ each one depending on $p \geq 0$, respectively, $q \geq 0$ free phases, i.e. (10) is a complex Hadamard matrix of size $2m$. Now we make use of Hadamard’s trick to get a Hadamard matrix depending on $p + q + m - 1$ arbitrary phases. Indeed we can multiply $B$ at left by the diagonal matrix $d = (1, e^{i\varphi_1}, \ldots, e^{i\varphi_{m-1}})$ without modifying the Hadamard property. In this way Hadamard obtained a continuum of solutions for the case $n = 4$. We denote $B_1 = d \cdot B$ and then the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} A & B_1 \\ A & -B_1 \end{pmatrix}$$

(11)

will be unitary and Hadamard depending on $p + q + m - 1$ parameters. From (11) we obtain in general two non-equivalent $2m \times 2m$ Hadamard matrices when $B \neq B^*$. In this case Eq. (11) is a
realization and the second one is given by \( B_1 \rightarrow B_2 = d \cdot B^* \). The above procedure can be iterated by taking the matrix (10) as a new \( A \) block obtaining a Hadamard matrix of the form

\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
A & B & C & D \\
A & -B & C & -D \\
A & B & -C & -D \\
A & -B & -C & D
\end{pmatrix}
\]

(12)

which is a 4 \( m \)-dimensional array similar to Williamson array \([28]\), and so on. In contradistinction to the Williamson array the \( A, B, C, D \) blocks satisfy no supplementary conditions, excepting their unitarity. Thus the following holds

**Proposition 1** If the \( m \times m \) complex Hadamard matrices \( A, B, C, D \) depend on \( p, q, r, s \) arbitrary phases then there exists a complex Hadamard matrix of the form (12) which depends on \( p + q + r + s + 3(m - 1) \) arbitrary phases.

We notice that the elementary array (10) is different from the Goethals-Seidel one \([15]\) that appears in the construction of real Hadamard matrices and which has the form

\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
A & B \\
B & -A
\end{pmatrix}
\]

The above array is not unitary even when \( A \) and \( B \) are, the supplementary condition for unitarity being the relation \( AB^* = BA^* \). We consider that the form (10) could also be useful for the study of orthogonal designs and real Hadamard matrices it being in some sense complementary to the above form.

As an application of the formula (12) we consider the following case: \( a_{11} = a_{12} = a_{21} = -a_{22} = b_{11} = b_{12} = c_{11} = c_{12} = d_{11} = d_{12} = 1/\sqrt{2} \) and \( b_{21} = -b_{22} = e^{ix}/\sqrt{2}, c_{21} = -c_{22} = e^{iu}/\sqrt{2}, d_{21} = -d_{22} = e^{iu}/\sqrt{2} \) where the notation is self-explanatory, and we obtain an eight-dimensional Hadamard matrix depending on three arbitrary phases \( s, t, u \).

When \( A = B \), Eq. (10) can be written as

\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
A & A \\
A & -A
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & \epsilon
\end{pmatrix} \otimes A
\]

(13)

where \( \epsilon = -1 \), i.e. the first factor is the Sylvester Vandermonde matrix of the second roots of unity, and \( \otimes \) is the ordinary Kronecker product, \( A \otimes B = [a_{ij}B] \); of course the first factor can be any complex Hadamard matrix of order \( m \). Now we want to define a new product the aim being a more general construction of Hadamard matrices. Let \( M \) and \( N \) be two matrices of the same order \( m \) whose elements are matrices \( M_{ij} \) of order \( n \) and respectively \( N_{kl} \) of order \( p \). The new product denoted by \( \tilde{\otimes} \) is given as

\[
Q = M \tilde{\otimes} N
\]
which is a matrix of order $mnp$, where

$$Q_{ij} = \sum_{k=1}^{k=m} M_{ik} \otimes N_{kj}$$

We will use here the above formula only in the case: $M = m_{ij}$ where $m_{ij}$ are complex scalars, not matrices and $N$ is an arbitrary diagonal matrix $N = (N_{11}, \ldots, N_{mm})$ where $N_{ii}$ ar matrices of order $p$ obtaining

$$Q = \begin{pmatrix}
  m_{11}N_{11} & \cdots & m_{1m}N_{mm} \\
  \vdots & \ddots & \vdots \\
  m_{1m}N_{11} & \cdots & m_{mm}N_{mm}
\end{pmatrix} \tag{14}$$

Thus the following is true.

**Proposition 2** If the matrices $M$ and $N_{ii}$, $i = 1, \ldots, m$, are Hadamard so will be the matrix $Q$ given by Eq. (14).

The order of $Q$ is $mp$ and the formula (14) is new even for real Hadamard matrices. This form is the most general array we have obtained and in some sense (14) is the natural generalization of Williamson arrays to the case of complex Hadamard matrices.

If in the above relation we take $m_{11} = m_{12} = m_{21} = -m_{22} = 1/\sqrt{2}$ and $N_{11} = A$ and $N_{22} = B$, then Eq. (14) reduces to Eq. (10).

**Example 1** If now $m_{ij}$ are the same as above and

$$N_{11} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & \e^{is} & \e^{is} \\ 1 & \e^{-is} & \e^{is} \\ 1 & 1 & \e^{-is} & \e^{is} \end{pmatrix}$$

is the complex four-dimensional Hadamard matrix and

$$N_{22} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \e^{it} & 0 & 0 \\ 0 & 0 & \e^{iu} & 0 \\ 0 & 0 & 0 & \e^{iv} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\e^{iy} & \e^{iy} \\ 1 & -1 & \e^{iy} & -\e^{iy} \end{pmatrix}$$

we obtain an eight-dimensional matrix depending now on five arbitrary phases $s, t, u, v, y$ instead of three as in the preceding example obtained by using the Williamson-type array (12).

Thus the following holds.
Proposition 3 If $M, N_i, i = 1, \ldots, m$ are $m \times m$ and respectively, $n \times n$-dimensional complex Hadamard matrices depending on $m$, respectively, $n_i$, arbitrary phases, then there is an array of the form (14) that depends on

$$m + n_1 + (m - 1) \sum_{i=2}^{m} m_i$$

free phases.

The above example shows the necessity for getting upper and lower bounds on the number of arbitrary phases entering a Hadamard matrix of size $N$. Taking into account the standard decomposition of any integer under the form $N = p_1^{q_1} \cdots p_m^{q_m}$, where $p_1 < \ldots < p_m$ are primes and $q_1 \ldots q_m$ their respective powers, we may use the above Proposition 3 for obtaining lower bounds on the number of free phases, that we shall denote it by $\varphi(N)$. Since until now does not exist an example of a Hadamard matrix of size $N$ with $N$ prime which depends on free phases, in the following we will consider the normalization $\varphi(N) = 0$, for $N$ prime. Thus the following holds.

Theorem 2 Let $N = p_1^{q_1}$ be the power of a prime $p_1$, with $q \geq 2$. Then a lower bound for $\varphi(p_1^{q_1})$, the number of free phases entering the parameterization of the $N \times N$ complex Hadamard matrix, is given by

$$\varphi(p_1^{q_1}) = 1 + [(p_1 - 1)(q_1 - 1) - 1]p_1^{q_1-1}$$

If $N = p_1^{q_1} \cdots p_m^{q_m} = p_1^{q_1} N_1$ then $\varphi(p_1^{q_1} N_1)$ is given by

$$\varphi(p_1^{q_1} N_1) = 1 + [(p_1 - 1)q_1 N_1 - p_1]p_1^{q_1-1} + \varphi(N_1)p_1^{q_1}$$

Proof. Making use of Proposition 3 we find the recurrence relation

$$\varphi(p_1^{q_1}) = p_1 \varphi(p_1^{q_1-1}) + (p_1 - 1)(p_1^{q_1-1} - 1)$$

with the initial condition $\varphi(p_1) = 0$ and the solution follows.

In the second case the recurrence relation writes

$$\varphi(p_1^{q_1} N_1) = p_1 \varphi(p_1^{q_1-1} N_1) + (p_1 - 1)(p_1^{q_1-1} N_1 - 1)$$

and the initial condition can be taken as

$$\varphi(p_1 N_1) = p_1 \varphi(N_1) + (p_1 - 1)(N_1 - 1)$$

and the solution follows. The above recurrence relation allows us to obtain lower bounds for any integer $N$ under the form

$$\varphi(p_1^{q_1} \cdots p_m^{q_m}) = 1 + [(p_1 - 1)q_1 p_2^{q_2} \cdots p_m^{q_m} - p_1]p_1^{q_1-1} + p_1^{q_1}(1 + [(p_2 - 1)q_2 p_3^{q_3} \cdots p_m^{q_m} - p_2]p_2^{q_2-1} + \ldots$$
\[ p_m q_m^2 \{ 1 + [(p_3 - 1)q_3 p_q^4 \ldots p_m q_m - p_3]p_m^{q_m - 1} + \\
p_3 q_3 \{ 1 + \ldots + p_m^{q_m - 1}(1 + [(p_m - 1)q_m - p_m]p_m^{q_m - 1}) + \\
p_m^{q_m - 1}(1 + [(p_m - 1)(q_m - 1)]p_m^{q_m - 1}] \ldots \} \]

We give now a few examples.

**Example 2** If \( N = p_1^{q_1} p_2^{q_2} \) then the lower bound for \( \varphi(p_1^{q_1} p_2^{q_2}) \), the number of free phases entering the parameterization of the \( N \times N \) complex Hadamard matrix, is given by

\[
\varphi(p_1^{q_1} p_2^{q_2}) = 1 + (p_1 - 1)q_1 p_1^{q_1 - 1} p_2^{q_2} + [(p_2 - 1)(q_2 - 1) - 1]p_1^{q_1} p_2^{q_2 - 1} \tag{15}
\]

Numerical examples: \( \varphi(2^3) = 5 \), \( \varphi(2^4) = 17 \), \( \varphi(6) = 2 \), \( \varphi(3^2) = 4 \), \( \varphi(2^2 3^2) = 49 \), etc.

6 An other parameterisation of unitary matrices

In the following we will shortly present another parameterisation of unitary matrices \(^{13}\) under the form of a product of \( n \) diagonal matrices containing phases interlaced with \( n - 1 \) orthogonal matrices each one generated by a real vector \( v \in \mathbb{R}^n \). This new form will be more appropriate for design and implementation of the software packages necessary for solving the equations (4) for arbitrary \( n \).

We have seen in Section 2 that we can write any unitary matrix as a product of two diagonal matrices of the for \( d_n = (e^{i\varphi_1}, \ldots, e^{i\varphi_n}) \) with \( \varphi_j \in [0, 2 \pi) \), \( j = 1, \ldots, n \) arbitrary phases and a unitary matrix with positive elements in the first row and the first column. We make also the notation \( d_{n-k} = (1_{n-k}, e^{i\psi_1}, \ldots, e^{i\psi_k}) \), \( k < n \), where \( 1_{n-k} \) means that the first \( (n-k) \) diagonal entries equal unity, i.e. it can be obtained from \( d_n \) by making the first \( n-k \) phases equal zero. These diagonal phase matrices are the first building blocks in our construction. Other building blocks that will appear in factorization of unitary matrices \( A_n \) are the two-dimensional rotations which operate in the \( i,i+1 \)-plane of the form

\[
J_{i,i+1} = \begin{pmatrix}
I_{i-1} & 0 & 0 \\
0 & \cos \theta_i & -\sin \theta_i \\
0 & \sin \theta_i & \cos \theta_i
\end{pmatrix}, \quad i = 1, \ldots, n - 1
\tag{16}
\]

The factorization idea comes from the well known fact that \( U(n) \) acts transitivly on the \( n \)-dimensional complex sphere \( S_{2n-1} \in \mathbb{C}^n \), and explicity from the coset relation

\[
S_{2n-1} = \text{coset space } U(n)/U(n-1)
\]
A direct consequence of the last relation is that we expect that any element of \( U(n) \) should be uniquely specified by a pair of a vector \( v \in S_{2n-1} \) and of an arbitrary element of \( U(n-1) \). Thus we are looking for a factorization of an arbitrary element \( A_n \in U(n) \) in the form

\[
A_n = B_n \cdot \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix}
\]

where \( B_n \in U(n) \) is a unitary matrix whose first column is uniquely defined by a vector \( v \in S_{2n-1} \), but otherwise arbitrary, and \( A_{n-1} \) is an arbitrary element of \( U(n-1) \). Iterating the previous equation we arrive at the conclusion that an element of \( U(n) \) can be written as a product of \( n \) unitary matrices

\[
A_n = B_n \cdot B_{n-1}^1 \cdots B_1^{n-1}
\]

where

\[
B_{n-k}^k = \begin{pmatrix} I_k & 0 \\ 0 & B_{n-k} \end{pmatrix}
\]

\( B_k, \ k = 1, \ldots, n-1, \) are \( k \times k \) unitary matrices whose first columns are generated by vectors \( b_k \in S_{2k-1} \); for example \( B_1^{n-1} \) is the diagonal matrix \( (1, \ldots, 1, e^{i\phi_{n+1}}) \).

The still arbitrary columns of \( B_k \) will be chosen in such a way that we should obtain a simple form for the matrices \( B_{n-k}^k \), and we require that \( B_k \) should be completely specified by the parameters entering the vector \( b_k \) and nothing else.

If we take into account the equivalence considerations of the Section 2 then \( B_n (B_{n-k}) \) can be written as

\[
B_n = d_n \bar{B}_n
\]

where the first column of \( \bar{B}_n \) has non-negative entries.

Denoting this column by \( v_1 \) we will use the parameterization

\[
v_1 = (\cos \theta_1, \cos \theta_2 \sin \theta_1, \ldots, \sin \theta_1 \ldots \sin \theta_{n-1})^t
\]

where \( \theta_i \in [0, \pi/2], \ i = 1, \ldots, n-1 \). Thus \( B_n \) will be parameterized by \( n \) phases and \( n-1 \) angles. According to the above factorization \( \bar{B}_n \) is nothing else than the orthogonal matrix generated by the vector \( v_1 \) and its form is given by Lemma 2 with \( n \to n + 1 \). Thus without loss of generality \( B_n = d_n O_n \) with \( O_n \in SO(n) \). In this way the factorization of \( A_n \) will be

\[
A_n = d_n \bigg( \prod_{k=1}^{n} O_n d_{n-k}^1 O_{n-1} \cdots d_2^{n-2} O_2 d_{1}^{n-1} I_n \bigg)
\]

(17)

where \( O_{n-k} \) has the same structure as \( B_{n-k}^k \), i.e

\[
O_{n-k}^k = \begin{pmatrix} I_k & 0 \\ 0 & O_{n-k} \end{pmatrix}
\]

and \( d_{n-k}^k = (1, \ldots, 1, e^{i\phi_1}, \ldots, e^{i\phi_{n-k}}) \).

The orthogonal matrices \( O_n \) can be factored in terms of \( J_{i,i+1} \) as follows.
Lemma 3 The orthogonal matrices \( O_n \) (\( O^k_{n-k} \)) at their turn can be factored into a product of \((n - 1)\) matrices of the form \( J_{i,i+1} \); e.g. we have

\[
O_n = J_{n-1,n}J_{n-2,n-1} \cdots J_{1,2}
\]

where \( J_{i,i+1} \) are \( n \times n \) rotations introduced by Eq. (17).

In this way the parameterization of unitary matrices reduces to a product of simpler matrices: diagonal phase matrices and two-dimensional rotation matrices. For more details see our paper [13]. Now we propose a disentanglement of the angles and phases entering each “generation” and denote the angles by latin letters, e.g. those that parameterize \( O_n \) will be denoted by \( a_1, \ldots, a_{n-1} \), the angles that parameterize \( O^1_{n-1} \), by \( b_1, \ldots, b_{n-2} \), etc., the last angle entering \( O^2_{n-1} \) by \( z_1 \). The phases will be denoted by Greek letters; e.g. the phases entering \( d_1 \) will be denoted by \( \alpha_1, \ldots, \alpha_n \), those entering \( d^1_{n-1} \) by \( \beta_1, \ldots, \beta_{n-1} \), etc. The above factorization will be used in the next section for obtaining the equations for the moduli of the matrix elements.

7 Explicit equations of the moduli

Our choice for the orthogonal vectors in Lemma 2 was such that the resulting matrix should have as many zero entries as possible. Thus \( O_n \) has \((n - 1)(n - 2)/2\) zeros in the right upper corner and the entries of the Hadamard matrix will get more and more complicated when going from left to right and from top to bottom. We will start using the form (17) of the unitary matrix and then \( d_n \equiv I_n \). Since the first column has the form \( a_{i1} = 1/\sqrt{n}, \ i = 1, \ldots, n \) and \( d^1_{n-1} = (1, e^{i\alpha}, e^{i\alpha_1}, \ldots, e^{i\alpha_{n-2}}) \) the product \( O_n d^1_{n-1} \) is

\[
\begin{pmatrix}
\frac{1}{\sqrt{n}} & -\sqrt{\frac{n-1}{n}}e^{i\alpha} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{\sqrt{n}} & e^{i\alpha} & -\sqrt{\frac{n-2}{n-1}}e^{i\alpha_1} & 0 & \cdots & 0 & 0 \\
\frac{1}{\sqrt{n}} & e^{i\alpha} & e^{i\alpha_1} & -\sqrt{\frac{n-3}{n-2}}e^{i\alpha_2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\frac{1}{\sqrt{n}} & e^{i\alpha} & e^{i\alpha_1} & e^{i\alpha_2} & \cdots & e^{i\alpha_{n-3}} & -\frac{e^{i\alpha_{n-2}}}{\sqrt{2}} \\
\frac{1}{\sqrt{n}} & e^{i\alpha} & e^{i\alpha_1} & e^{i\alpha_2} & \cdots & e^{i\alpha_{n-3}} & \frac{e^{i\alpha_{n-2}}}{\sqrt{2}} \\
\end{pmatrix}
\]

(18)

where \( \alpha, \alpha_i, \ i = 1, \ldots, n - 2 \) are \( n - 1 \) arbitrary phases.
The next building block $O_{n-1}^1 a_{n-2}^2$ will have the form

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \cos a & -\sin a e^{i\beta} & \cdots & 0 \\
0 & \sin a \cos a_1 & \cos a \cos a_1 e^{i\beta} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \sin a \sin a_{n-3} & \cos a \sin a_1 \cdots \sin a_{n-3} e^{i\beta} & \cdots & \cos a_{n-3} e^{i\beta_{n-3}}
\end{pmatrix}
$$

in terms of $n - 2$ phases $\beta, \beta_1, \ldots, \beta_{n-3}$ and $n - 2$ angles $a, a_1, \ldots, a_{n-3}$, and so on.

It is easy to see that the first two columns of the product of matrices (18) and (19) does not change when multiplied by $O_{n-2}^2 a_{n-3}^3$; however the first row does. If the angles entering $O_{n-2}^2$ are denoted by $b, b_1, \ldots, b_{n-4}$ and the phases are $\gamma, \gamma_1, \ldots, \gamma_{n-4}$, etc., then the entries of the first row are

$$
a_{12} = -\sqrt{\frac{n-1}{n}} \cos a e^{ia_1}, \quad a_{13} = \sqrt{\frac{n-1}{n}} \sin a \cos b e^{i(a+\beta)}, \ldots,
$$

$$
a_{1n-1} = (-1)^{n-1} \sqrt{\frac{n-1}{n}} \sin b \cdots \sin z e^{i(a+\beta+\ldots+\omega)}
$$

where $z$ and $\omega$ are the last angle and phase respectively. Since we use the standard form of Hadamard matrices, i.e. the entries of the first row and of the first column are positive and equal $1/\sqrt{n}$, the above equations imply

$$
\alpha = \beta = \ldots = \omega = \pi; \quad \cos a = \frac{1}{\sqrt{n-1}}, \quad \cos b = \frac{1}{\sqrt{n-2}}, \ldots, \quad \cos z = \frac{1}{\sqrt{2}}
$$

We substitute the above values in Eq.17 and find a complex $n \times n$ matrix depending on $(n - 1)(n - 2)/2$ phases $a_1, \ldots, a_{n-2}, \beta_1, \ldots, \psi_1$ and $(n - 2)(n - 3)/2$ angles $a_1, \ldots, a_{n-3}, b_1, \ldots, y_1$, i.e. $(n - 2)^2$ parameters which have to be found by solving the corresponding equations given by the moduli. The first simplest entries of the unitary matrix have the form

$$
a_{22} = -\frac{1}{(n-1)\sqrt{n}} + \frac{n-2}{n-1} \cos a_1 e^{ia_1}, \ldots
$$

$$
a_{k2} = -\frac{1}{(n-1)\sqrt{n}} + \sqrt{\frac{n-2}{n-1}} \left( \frac{\cos a_1 e^{ia_1}}{\sqrt{(n-1)(n-2)}} + \ldots \right) + \frac{\sin a_1 \cdots \cos a_{k-2} e^{i\alpha_{k-2}}}{\sqrt{(n-k+2)(n-k+1)}}
$$

$$
-\sqrt{\frac{n-k}{n-k+1}} \sin a_1 \cdots \sin a_{k-2} \cos a_{k-1} e^{i\alpha_{k-1}}, \quad k = 3, \ldots, n - 1
$$

$$
a_{2k} = -\frac{1}{(n-1)\sqrt{n}} + \sqrt{\frac{n-2}{n-1}} \left( \frac{\cos a_1 e^{ia_1}}{\sqrt{(n-1)(n-2)}} - \frac{\sin a_1 \cos b_1 e^{i(\alpha_1+b_1)}}{\sqrt{(n-2)(n-3)}} \right) + \ldots
$$
\[ +(-1)^{k-1} \sqrt{\frac{n-k}{n-k+1}} \sin a_1 \sin b_1 \ldots \cos l(k)_1 e^{i(\alpha_1 + \beta_1 + \ldots + \lambda(k)_1)} \], etc.

where \( l(k) \) and \( \lambda(k) \) denote the letters for angle and respectively phase corresponding to index \( k \) and the signs in the last bracket alternate.

The matrix elements get more complicated when going from the upper left corner to right bottom corner. The entries \( a_{22}, a_{32} \) and \( a_{23} \) lead, for example, to the following moduli equations

\[
(n - 2) \cos^2 a_1 + \frac{2}{\sqrt{n}} \cos a_1 \cos \alpha_1 - 1 = 0
\]

\[
\sin a_1 \left( (n - 3) \sin a_1 \cos^2 a_2 + \right.
\]

\[
2 \sqrt{\frac{n-3}{n-1}} \cos a_2 \left( \frac{\cos \alpha_2}{\sqrt{n}} - \cos a_1 \cos(\alpha_1 - \alpha_2) \right) - \sin a_1 \right) = 0
\]

(21)

\[
\sin a_1 \left( (n - 3) \sin a_1 \cos^2 b_1 + \right.
\]

\[
2 \sqrt{\frac{n-3}{n-1}} \cos b_1 \left( -\frac{\cos(\alpha_1 + \beta_1)}{\sqrt{n}} + \cos a_1 \cos \beta_1 \right) - \sin a_1 \right) = 0
\]

and so on. The form of the last two equations was obtained after the elimination of the term containing \( \cos a_1 \cos \alpha_1 \) by using the first equation (21), i.e. we work in the ideal generated by the moduli equations. It is easily seen that the other equations contain as factors \( \sin a_2, \ldots, \sin a_{n-2}, \sin b_1, \ldots, \) etc.

Thus a particular solution can be obtained when

\[ \sin a_1 = 0 \]

which implies \( a_1 = 0, \pi, \) and from the first equation (21) we get

\[ \cos \alpha_1 = \pm \frac{(n-3)\sqrt{n}}{2} \]

It is easily seen that the above equation has solution only for \( n = 2, 3, 4; \) for \( n \geq 5 \) the factor \( \sin a_1 \) will be omitted from Eqs.(21) because then \( a_1 \neq 0, \pi. \) When \( n = 2 \) we obtain \( \alpha_1 = \pi/4, \) so \( a_{22} = -1/\sqrt{2}. \) If \( n = 3, \) then \( \alpha_1 = 3\pi/2 \) and from the first Eq.(20) one gets

\[ a_{22} = -\frac{1}{2\sqrt{3}} + i \frac{\sqrt{3}}{2} e^{\frac{2\pi i}{3}}, \text{ etc.} \]

The case \( n = 4 \) leads to \( \alpha_1 = \pi \) which gives

\[ a_{22} = -a_{23} = -a_{32} = \frac{1}{2} \quad \text{and} \quad a_{33} = -a_{34} = -e^{i(\alpha_2 + \beta_1)} \]

\[ 2 \]
After the substitution $\alpha_2 + \beta_1 = t$ one finds the standard complex form of the $4 \times 4$ matrix found by Hadamard. To view what is the origin of the phase $\alpha_2 + \beta_1$ we have to look at the moduli equations. They have the form

\[
2 \cos^2 \alpha_1 + \cos \alpha_1 \cos \alpha_1 - 1 = 0
\]
\[
\sin \alpha_1 (\cos \alpha_2 - 2 \cos \alpha_1 \cos (\alpha_1 - \alpha_2)) = 0
\]
\[
\sin \alpha_1 (2 \cos \alpha_1 \cos \beta_1 - \cos (\alpha_1 + \beta_1)) = 0
\]
\[
\cos 2\alpha_1 \cos (\alpha_1 - \alpha_2) \cos \beta_1 + \cos \alpha_1 \cos (\alpha_2 + \beta_1) + \sin (\alpha_1 - \alpha_2) \sin \beta_1 = 0
\]

and we see that the above system splits into two cases. In the first case, when $\sin a_1 = 0$, the rank of the system is two which explains the above dependence of $a_{33}$ on two phases and in the second case when $\sin a_1 \neq 0$ the rank is three and the dependence is only on one arbitrary phase. However in this case there is no final difference between the two cases. The solution of the above system is obtained directly but for $n \geq 5$ the problem is difficult and needs more powerful techniques. Particular solutions can be obtained rather easily e.g for $n = 6$ there is a matrix that has the property $a_{ij} = a_{ji}$.

\[
\frac{1}{\sqrt{6}} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & i & -i \\
1 & -1 & -i & -1 & 1 & i \\
1 & 1 & -i & -1 & -1 & i \\
1 & i & 1 & -1 & -1 & -i \\
1 & -i & i & i & -i & -1
\end{pmatrix}
\]

There exists even a Hermitian matrix $S = S^*$

\[
\frac{1}{\sqrt{6}} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & i & i & -i & -i \\
1 & -i & -1 & 1 & -1 & i \\
1 & -i & 1 & -1 & i & -1 \\
1 & i & -1 & -i & 1 & -1 \\
1 & i & -i & -1 & -1 & 1
\end{pmatrix}
\]

and so on. As we said before getting the most general form of a solution is not a simple task; for $n = 6$ we have 16 complicated trigonometric equations and we remind that the simpler system was solved only for $n \leq 8$ equations. Thus new approaches are necessary and in the next Section we suggest such an approach: using methods from algebraic geometry.
8 Connection with algebraic geometry

The Eqs.(21) can be transformed into polynomial equations by the known procedure

\[ \sin a \rightarrow \frac{2x}{1 + x^2}, \quad \cos a \rightarrow \frac{1 - x^2}{1 + x^2} \]

such that we get from (21)

\[ p_1 = \left( n - 3 + \frac{2}{\sqrt{n}} \right) x_1^4 - 2(n - 1)x_1^2 + (n - 3 - \frac{2}{\sqrt{n}}) \right] y_1^2 + (n - 3 - \frac{2}{\sqrt{n}}) x_1^4 - \\
2(n - 1)x_1^2 + (n - 3 + \frac{2}{\sqrt{n}}) \right] y_1^2 + (n - 3 - \frac{2}{\sqrt{n}}) x_1^4 - \\
\right. \]

\[ p_2 = \left\{ \left( -1 + \frac{1}{\sqrt{n}} \right)x_1^2 + C_1 x_1 + (1 + \frac{1}{\sqrt{n}}) \right] x_2^2 - C_2 x_1 x_2^2 + (1 - \frac{1}{\sqrt{n}})x_1^2 + C_1 x_1 - \\
1 + \frac{1}{\sqrt{n}} \right) \right\} y_1^2 + \left\{ \left[ (1 - \frac{1}{\sqrt{n}})x_1^2 + C_1 x_1 - (1 + \frac{1}{\sqrt{n}}) \right] x_2^4 - C_2 x_1 x_2^2 - \\
(1 + \frac{1}{\sqrt{n}}) \right\} y_1^2 + \left\{ \left[ (1 + \frac{1}{\sqrt{n}})x_1^2 + C_1 x_1 - (1 - \frac{1}{\sqrt{n}}) \right] x_2^4 - C_2 x_1 x_2^2 - \\
(1 + \frac{1}{\sqrt{n}}) \right\} y_1^2 + \left\{ \left[ (1 - \frac{1}{\sqrt{n}})x_1^2 + C_1 x_1 - (1 + \frac{1}{\sqrt{n}}) \right] x_2^4 - C_2 x_1 x_2^2 - \\
(1 + \frac{1}{\sqrt{n}}) \right\} y_1^2 + \left\{ \left[ (1 + \frac{1}{\sqrt{n}})x_1^2 + C_1 x_1 - (1 - \frac{1}{\sqrt{n}}) \right] x_2^4 - C_2 x_1 x_2^2 - \right. \]

\[ p_3 = \left\{ \left( -1 + \frac{1}{\sqrt{n}} \right)x_1^2 + C_1 x_1 + (1 + \frac{1}{\sqrt{n}}) \right] x_3^4 - C_2 x_1 x_3^2 + (1 - \frac{1}{\sqrt{n}})x_1^2 + C_1 x_1 - \\
1 + \frac{1}{\sqrt{n}} \right) \right\} y_1^2 + \left\{ \left[ (1 - \frac{1}{\sqrt{n}})x_1^2 + C_1 x_1 - (1 + \frac{1}{\sqrt{n}}) \right] x_3^4 - C_2 x_1 x_3^2 - (1 - \frac{1}{\sqrt{n}})x_1^2 - \\
C_1 x_1 + (1 + \frac{1}{\sqrt{n}}) \right\} y_1^2 + \left\{ \left[ (1 - \frac{1}{\sqrt{n}})x_1^2 + C_1 x_1 + (1 - \frac{1}{\sqrt{n}}) \right] x_3^4 - C_2 x_1 x_3^2 - \\
(1 + \frac{1}{\sqrt{n}}) \right\} y_1^2 + \left\{ \left[ (1 + \frac{1}{\sqrt{n}})x_1^2 + C_1 x_1 - (1 - \frac{1}{\sqrt{n}}) \right] y_3^2 - 4(1 + x_1^2)(1 - x_1^2)y_1 y_2 + \left[ (1 + \frac{1}{\sqrt{n}})x_1^2 + \right. \]

\[ C_1 x_1 - (1 - \frac{1}{\sqrt{n}}) \right\] x_3^4 - C_2 x_1 x_3^2 - (1 + \frac{1}{\sqrt{n}})x_1^2 + C_1 x_1 + (1 - \frac{1}{\sqrt{n}}) \right] \] (26)

where

\[ C_1 = \frac{(n - 1)(n - 4)}{\sqrt{(n - 1)(n - 3)}}, \quad C_2 = \frac{2(n - 1)(n - 2)}{\sqrt{(n - 1)(n - 3)}} \]
and the angles by the above transformation go to $x_1, x_2, x_3, \ldots$ and the phases to $y_1, y_2, y_3, \ldots$

From the matrices such as (18) one sees that the full set of the $(n-2)^2$ equations contains square roots of almost all prime numbers $\leq n$ so that not all the coefficients are rational and we have to look for solutions in a field $\mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{N}$.

The polynomial equation $p_1 = 0$ defines an algebraic curve; however the most studied are the elliptic and hyperelliptic curves, i.e. those defined by an equation of the form $y^2 = f_p(x)$ where $f_p(x)$ is a polynomial of degree $p$.

From $p_1 = 0$ we get

$$y_1^2 = \frac{(n - 3 - \frac{2}{\sqrt{n}})x_1^4 - 2(n - 1)x_1^2 + (n - 3 + \frac{2}{\sqrt{n}})}{(n - 3 + \frac{2}{\sqrt{n}})x_1^4 - 2(n - 1)x_1^2 + (n - 3 - \frac{2}{\sqrt{n}})} = -\frac{P_1(x_1)}{P_2(x_1)}$$

which defines a meromorphic function. Its zeros and poles are

$$\pm \sqrt{\frac{\sqrt{n} - 1}{\sqrt{n} + 1}}, \quad \pm \sqrt{\frac{n + \sqrt{n} - 2}{n - \sqrt{n} - 2}}$$

and

$$\pm \sqrt{\frac{\sqrt{n} + 1}{\sqrt{n} - 1}}, \quad \pm \sqrt{\frac{n - \sqrt{n} - 2}{n + \sqrt{n} - 2}}$$

respectively that are simple, and the poles and the zeros are interlaced. Thus apparently the above equation is not hyperelliptic, however by the birational transformation

$$y_1 = \frac{Y_1}{P_2(x_1)}$$

we get the equation

$$Y_1^2 = -P_1(x_1) P_2(x_1)$$

which shows that the above curve has genus $g = 3$. For $n \geq 5$ the curve has no branch going to infinity since the highest power coefficient is negative and consequently the curve is made of three ovals.

The polynomials $p_1 = p_2 = 0$ define a surface, $p_1 = p_2 = p_3 = 0$ define a 3-fold, and so on. We consider that the study of these multi-fold varieties will be very interesting from the algebraic geometry point of view and their parameterizations could reveal unknown properties that may lead to a better understanding of the rational varieties. As we saw in Section 5 one can easily construct parameterizations of Hadamard matrices depending on a number of free phases at least for a non-prime $n$. That means that the set of the moduli equations has to be split in some sub-sets and for each such sub-set the solutions are in $S^1 \otimes \ldots \otimes S^1$, where $k$ is the number of arbitrary phases parameterizing the considered sub-set. But this could be equivalent to the existence of a
rational parameterization for the equations defining this sub-set. Unfortunately the best studied case and the best results are for algebraic curves; see [19], Theorem 14, for a flavour of recent results. The study of surfaces, three-fold, etc. is at the beginning and until now the theory was developed only for the simplest varieties, the so called rationally connected varieties [19]. From what we said before one may conclude that the parameterization of complex Hadamard matrices could be an interesting example of the parameterization of meromorphic varieties, which could be a mixing between a rational parameterisation and a parameterisation of hyperelliptic curves. Thus the theoretical instrument for the parameterization of complex Hadamard matrices seems to exist, the challenging problem being the transformation of the existing theorems into a symbolic manipulation software program able to find after a reasonable computer time explicit solutions at least for moderate values of $n$.

9 Conclusion

All the results obtained for the complex Hadamard matrices can be used for the construction of real Hadamard matrices the only supplementary constraint being the natural one $n = 4m$. We believe that the Hadamard conjecture can be solved in our formalism since unlike the classical combinatorial approach we have also at our disposal $(n-1)(n-2)/2$ phases, and the problem is to guess the pattern of 0 and $\pi$ taken by them.

Conversely many constructions from the theory of real Hadamard matrices can be extended to the complex case. For example a complex conference matrix will be a matrix with $a_{ii} = 0$, $i = 1, \ldots, n$ and $|a_{ij}| = 1/\sqrt{n}$ such that

$$WW^* = \frac{n-1}{n}$$

It is not difficult to construct complex conference matrices, in fact it is a simpler problem than the construction of complex Hadamard matrices because the equations $a_{ii} = 0$, $i = 2, \ldots, n-1$ imply the determination of $2(n-2)$ parameters which simplify the other equations.

We give a few examples:

$$W_4 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -e^{it} & e^{it} \\ 1 & e^{it} & 0 & -e^{it} \\ 1 & -e^{it} & e^{it} & 0 \end{pmatrix}$$

and

$$W_6 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -e^{i\alpha} & -e^{i\alpha} & e^{i\alpha} & e^{i\alpha} \\ 1 & -e^{i\alpha} & 0 & e^{i\alpha} & -e^{i(\alpha-\beta)} & e^{i(\alpha-\beta)} \\ 1 & -e^{i\alpha} & e^{i\alpha} & 0 & e^{i(\alpha-\beta)} & -e^{i(\alpha-\beta)} \\ 1 & e^{i\alpha} & -e^{i(\alpha+\beta)} & e^{i(\alpha+\beta)} & 0 & -e^{i\alpha} \\ 1 & e^{i\alpha} & e^{i(\alpha+\beta)} & -e^{i(\alpha+\beta)} & -e^{i\alpha} & 0 \end{pmatrix}$$
where the second depends on two arbitrary phases. They are useful because if $W_n$ is a complex conference matrix then

$$M_{2n} = \frac{1}{\sqrt{2}} \begin{pmatrix} W_n + \frac{I_n}{\sqrt{n}} & W_n^* - \frac{I_n}{\sqrt{n}} \\ W_n - \frac{I_n}{\sqrt{n}} & -W_n^* - \frac{I_n}{\sqrt{n}} \end{pmatrix}$$

is a complex Hadamard matrix of order $2n$.

In this paper we have used convenient parameterisations of unitary matrices that allowed us getting a set of $(n - 2)^2$ polynomial equations whose solutions will give all the possible parameterisations for Hadamard matrices. Unfortunately the system is very complicated and only particular solutions have been found; thus from a pragmatically point of view the most important issue would be the design of software packages for solving these equations.

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References

[1] A.A. Agaian, Hadamard Matrices and Their Applications, Lectures Notes in Mathematics # 1168, Springer (1985)

[2] Gr. Arsene and A. Gheondea, ”Completing matrix contractions” J. Operator Theory 7 (1982) 179-189

[3] G. Auberson, ”On the reconstruction of a unitary matrix from its moduli. Existence of continuous ambiguities” Phys.Lett. B216 (1989) 167-171

[4] G. Auberson, A. Martin and G. Mennessier, ”On the reconstruction of a unitary matrix from its moduli” Commun.Math.Phys. 140 (1991) 417-431

[5] G. Björck, “Functions of modulus one on $\mathbb{Z}_p$ whose Fourier transforms have constant modulus” Colloquium Mathematica Societatis János Bolyai 49 (1985) 193-197
[6] G. Björck and R. Fröberg, “A faster way to count the solutions of inhomogeneous systems of algebraic equations, with applications to cyclic n-roots” J.Symbolic Computation 12 (1991) 329-336

[7] G. Björck and R. Fröberg, “Methods to “divide-out” certain solutions from systems of algebraic equations, applied to find all cyclic 8-roots” in Analysis, Algebra and Computers in Mathematical Research, Dekker (1994) 57-70

[8] G. Björck and B. Saffari, “New classes of finite unimodular sequences with unimodular Fourier transform. Circulant Hadamard matrices with complex entries” C.R.Acad.Sci.Paris 320 (1995) 319-324

[9] J.D.Bjorken and I. Dunietz, ”Rephasing invariant parameterisations of generalized Kobayashi-Maskawa matrices” Phys.Rev. D36 (1987) 2109-2118

[10] G.C. Branco and L. Lavoura, ”Rephasing-invariant parameterisation of the quark matrix” Phys.Lett. B208 (1988) 123-130

[11] P. Dită, ”Parameterisation of unitary matrices”, J.Phys.A: Math.Gen. 15 (1982) 3465-3473

[12] P. Dită, ”Parameterisation of unitary matrices by moduli of their elements”, Commun.Math.Phys. 159 (1994) 581-591

[13] P. Dită, ”Factorization of unitary matrices” J.Phys.A: Math.Gen 36 (2003) 2781-2789

[14] R.G. Douglas, “On majorization, factorization, and range inclusion of operators on Hilbert space” Proc.Amer.Math.Soc. 17 (1966) 413-415

[15] J.M. Goethals and J.J. Seidel, “Orthogonal matrices with zero diagonal” Canad.J.Math 19 (1967) 1001-1010

[16] U. Haagerup “Orthogonal maximal abelian ∗-subalgebra of the n × n matrices and cyclic n-roots”, in Operator algebras and quantum field theory Rome (1996), 296-322, Internat. Press, Cambridge, MA, 1997

[17] J. Hadamard, “ Résolution d’une question relative aux déterminants”, Bull.Sci.Math. 17 (1893) 240-246

[18] P de la Harpe and V R F Jones, “Paires de sous-algebres semi-simples et graphes fortement réguliers”, C R Acad.Sci. Paris 311 (1990) 147-150

[19] J. Kollár, “Which are the simplest algebraic varieties?” Bull.Amer.Math.Soc. 38 (2001) 409-433
[20] A Munemasa and Y Watatani, “Orthogonal pairs of \(\ast\)-subalgebras and association schemes”, *C R Acad. Sci. Paris* 314 (1992) 329-331

[21] F.D. Murnaghan, *The Unitary and Rotation Groups*, (1962), Spartan Books, Washington, D.C.

[22] B. Sz-Nagy and C. Foias, *Analyse Harmonique des Opérateurs de l’Espace de Hilbert*, Masson, Paris, 1967

[23] S. Popa, “Orthogonal pairs of \(\ast\)-subalgebras in finite von Neumann algebras, *J. Operator Theory* 9 (1983) 253-268

[24] J.J. Sylvester, “Thoughts on inverse orthogonal matrices, simultaneous sign-succesions, and tessellated pavements in two or more colors, with applications to Newton’s rule, ornamental tile-work, and the theory of numbers”, *Phil.Mag.* 34 (1867) 461-475

[25] K.G.H. Vollbrecht and R.F. Werner, “Why two qubits are special”, *J. Math.Phys.* 41 (2000) 6772-6782

[26] R.F. Werner, “All teleportation and dense coding schemes”, *Preprint* [quantum-ph/0003070](https://arxiv.org/abs/quant-ph/0003070)

[27] R.F. Werner, “Quantum information theory - an invitation”, in *Quantum information - an introduction to the basic theoretical concepts and experiments*, Springer Tracts in Modern Physics, (2003) Springer

[28] J. Williamson, “Hadamard’s determinant theorem and the sum of four squares”, *Duke Math.J.* 11 (1944) 65-81

[29] ˙Zyczkowski K, Kus M, Słomczyński W and Sommers H-J “Random unistochastic matrices” *J.Phys.A: Math.Gen* 36 (2003) 3425-3450