An Algorithm for Computing a Minimal Comprehensive Gröbner Basis of a Parametric Polynomial System*

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ABSTRACT

An algorithm to generate a minimal comprehensive Gröbner basis of a parametric polynomial system from an arbitrary faithful comprehensive Gröbner system is presented. A basis of a parametric polynomial ideal is a comprehensive Gröbner basis if and only if for every specialization of parameters in a given field, the specialization of the basis is a Gröbner basis of the associated specialized polynomial ideal. The key idea used in ensuring minimality is that of a polynomial being essential with respect to a comprehensive Gröbner basis. The essentiality check is performed by determining whether a polynomial can be covered for various specializations by other polynomials in the associated branches in a comprehensive Gröbner system. The algorithm has been implemented and successfully tried on many examples from the literature.

Keywords

comprehensive Gröbner basis, minimal comprehensive Gröbner basis, parametric polynomial system, specialization of polynomial systems, essentiality.

1. INTRODUCTION

The concepts of a comprehensive Gröbner system (CGS) and a comprehensive Gröbner basis (CGB) were introduced by Weispfenning [18] to associate Gröbner basis like objects for parametric polynomial systems (see also the notion of a related concept of a parametric Gröbner basis (PGB) independently introduced by Kapur [3]). For a specialization of parameters, a Gröbner basis of the specialized ideal can be immediately recovered from a branch of the associated CGS. Similarly, given a CGB, one merely has to specialize it to construct a Gröbner basis of the specialized ideal.

The above stated properties of CGS and CGB make them attractive in applications where a family of related problems can be parameterized and specified using a parametric polynomial system. For various specializations, they can be solved by specializing a parametric solution without having to repeat the computations. Because of their applications, these concepts and related algorithms have been well investigated by researchers and a number of algorithms have been proposed to construct such objects for parametric polynomial systems ([10], [19],[15], [16], [17], [8], [13], [20], [9], [11], [4], [5]). An algorithm that simultaneously computes a comprehensive Gröbner system and a comprehensive Gröbner basis by Kapur, Sun and Wang (KSW) [5] is particularly noteworthy because of its many nice properties: (i) fewer segments (branches) in a comprehensive Gröbner system generated by the algorithm, (ii) all polynomials in a CGS and CGB are faithful meaning that they are in the input ideal, and more importantly, (iii) the algorithm has been found efficient in practice [12].

For the non-parametric case, Gröbner bases have a very nice property: once an admissible term ordering is fixed, every ideal has a canonical Gröbner basis associated with it; a canonical Gröbner basis is not only unique but also reduced and minimal. This property is quite useful in many applications since equality of two ideals can be easily checked by checking the equality of their unique reduced minimal Gröbner bases. The final goal of this research project is to work towards a similar property for parametric ideals. The problem addressed here is to define a minimal CGB associated with a parametric ideal once an admissible term ordering (both on the parameters as well as variables) is fixed; this seems to be the first step toward defining a canonical CGB.

There are some proposals in the literature for defining a canonical CGB which are not satisfactory. Consider for instance, Weispfenning’s proposal in [10]: without reproducing his definition, we give an example of a parametric ideal from his paper generated by a basis \( \{f = uy + x, g = vz + x + 1\} \); Weispfenning reported \( \{f, g, h, −h\} \) as a canonical CGB of the ideal, where \( h = vz − uy + 1 \) using the lexicographic ordering \( z > y > x \gg v > u \).

We claim that each of \( f, g, h \) is essential whereas \( −h \) is not: for any specialization \( \sigma \), if \( \sigma(h) \) is in a GB of \( \sigma(I) \), then \( \sigma(−h) \) is reducible using \( \sigma(h) \) and vice versa; so only the smaller of the two has to be in a minimal CGB, which is \( h \) because though their leading coefficients only differ on the sign, \( h \) is monic while \( −h \) is not. Obviously both \( f \) and \( g \) are essential: for \( \sigma \) in which \( uv \neq 0 \), the leading term of

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σ(f) is y whereas the leading term of the other two is z; for σ in which v = 0 but u ≠ 0, the leading term of σ(g) is x while the leading term of the other two is y. Further, since h = g − f, the GB theory in the nonparametric case would suggest that h is not essential, but that is not true: for the case u = 0, v = 0, σ(I) = (1), however, \( σ(f) = x, σ(g) = x + 1 \) is not a GB of σ(I), which implies that h is essential. The algorithm proposed in the paper generates \( \{f, g, h\} \) as the minimal CGB of this parametric ideal, regardless of the order in which these polynomials are checked for essentiality (see Section 3).

Notice from this example that a minimal CGB is not reduced in the conventional sense (since h can be reduced using g); further some of its specializations are neither reduced nor minimal, whereas other specialization (e.g. for σ : u = 0, v = 0), σ(h) is both reduced and minimal. There are examples also of minimal CGBs which are reduced but their specializations are neither reduced nor minimal.

Montes and Wibmer [11] also define an object related to a canonical CGS in [11]; but unfortunately, this object cannot be used to generate a canonical CGB since elements in the CGS are not faithful, i.e., not in the original ideal.

In this paper, we give an algorithm for generating a minimal CGB from a given CGS having the property that associated with every segment describing parameter specialization, there is a GB for that specialization in which every polynomial is in the original ideal, i.e. faithful. To illustrate the proposed algorithm, we assume that this algorithm takes as input the CGB generated by the KSW algorithm [5], which is the union of all Gröbner basis associated with each branch in the associated CGS generated by the KSW algorithm. Recall that every polynomial in the CGB generated by the KSW algorithm is faithful since every polynomial in the GB of every segment of its CGS is faithful. In a related paper [7], we have proposed a Buchberger like completion algorithm for computing a minimal CGB directly from a basis of a parametric ideal.

Given a CGB and the associated CGS of a parametric ideal as well as an admissible term ordering as input, the proposed algorithm has to deal with two key issues: (i) redundancy in a CGB: since the CGB is constructed as the union of different branches of a CGS, it may include polynomials which while necessary and minimal for a subset of specializations, can be covered by other polynomials needed for a different subset of specializations, (ii) nonredundant polynomials could be further simplified using other polynomials. It first checks whether every polynomial in CGB is essential in the sense that there is at least one specialization of parameters under which it is essential (necessary) to include the specialization of the polynomial in a Gröbner basis of the associated specialized ideal. This check is performed by testing whether the polynomial can be covered by other polynomials in CGB; this is done by identifying every branch in the associated CGS in which the polynomial appears in the associated Gröbner basis and finding whether other polynomials in the CGB can cover it for all specializations corresponding to the branch. If a polynomial is discovered to be not essential, it is replaced by other polynomials covering it in branches of CGS it used to appear, and also discarded from the CGB. The subset of essential polynomials in a given CGB is a minimal CGB.

As will be shown later, there can be multiple minimal CGBs being generated from a given CGS even when an admissible term ordering is fixed. To obtain the least minimal CGB among them, simplification (similar to normalization) on an essential polynomial by other essential polynomials is attempted; if the result is different from the original polynomial and covers the original polynomial with the help of other essential polynomials in the CGB, it can replace the original polynomial in the CGB. Despite this simplification, it is still possible to change heuristics in the proposed algorithm to generate different minimal CGBs.

After discussing preliminaries in the next section, the algorithm is presented in Section 3. The concept of an essential polynomial in a CGB is defined, and the essentiality check is explained. Covering of a polynomial under a set of specializations defined by a segment by other polynomials is defined. The termination and correctness of the algorithm are shown. It is shown that the proposed algorithm can generate multiple minimal CGBs depending upon the order in which essentiality check is performed. It is proved that if the essentiality check is performed in descending order of polynomials, then the resulting minimal CGB is the least among all CGBs which are subsets of the input CGB. To generate minimal CGBs which are not subsets of the input CGB, the concept of simplification of an essential polynomial by other essential polynomials is introduced. Section 4 discusses the performance of the implementation of the proposed algorithm. It is shown that most CGBs computed by various algorithms have redundant polynomials; in particular, the KSW algorithm can sometimes generate CGBs in which about half of the polynomials are redundant.

2. PRELIMINARIES

The reader may consult [1] for definitions. Below we introduce notation and definitions necessary for the paper. Let K be a field, L an algebraically closed field extension of K, U and X the sets of parameters and variables respectively.

Let > be an admissible total term ordering in which X ≻ U, i.e. variables are bigger than parameters. In the ring \( K[X, U] \), which regards parameters as the same as variables, for a polynomial \( f ∈ K[X, U] \), \( LC(f) \) and \( LT_X(f) \) are defined as its leading parametric coefficient and leading term w.r.t. > respectively. For example, let \( f = 32(u − 1)x^3 + 4uy \), where \( U = \{u\}, X = \{x, y\}, \) and > is a lexicographic term ordering with \( x > y ≻ u \). Then \( LC(f) = 32(u − 1) \) and \( LT_X(f) = x^3 \).

A specialization σ is a ring homomorphism from \( K[U] \) to \( L \), which can be extended canonically to \( K[U][X] → L[X] \) by keeping the identity on variables. For a polynomial \( f ∈ K[U][X] \), σ is given by \( f → f(v_1, v_2, \ldots, v_m) \), where \( v_1, \ldots, v_m ∈ L \). In brief, denote this image of \( f \) as \( σ(f) \), where \( \overline{σ} = (v_1, \ldots, v_m) ∈ L^m \), or simply \( σ(f) \) if it is clear from the context.

**Definition 2.1** Let \( E, N \) be subsets of \( K[U] \), then the tuple \( (E, N) \) is called a parametric segment. An associated constructible set \( A \) is given by \( V(E) − V(N) \), where \( V(E) \) is the algebraic variety (the zero set) of \( E \) in \( L \). \( (E, N) \) is consistent if \( A ≠ \emptyset \).

**Definition 2.2** Given an ideal \( I = (F) ⊆ K[U][X] \), where \( F \) is finite, and an admissible term order >, let \( A_1, \ldots, A_i \) be constructible sets of \( L^m \), and \( G_1, \ldots, G_i \) subsets of \( K[U][X] \), and \( S \) a subset of \( L^m \) such that \( S ⊆ A_1 ∪ \cdots ∪ A_i \). Then a comprehensive Gröbner system (CGS) of \( I \) on \( S \) w.r.t. > is a finite set \( CGS = \{(A_1, G_1), \ldots , (A_i, G_i)\} \), where for
\[ i \leq l, \sigma_i(G_i) \text{ is a Gröbner basis of the ideal } \sigma_i(I) \text{ on } L[X] \] under \( \forall \sigma_i \in A_i. \]

Each \((A_i, G_i)\) is called a branch of CGS. Specifically, if \( S = L^m \), then CGS is called a comprehensive Gröbner system (CGS) of \( I \).

The above definition of a CGS does not require that \( G_i \) be a subset of \( I \); in fact, there are algorithms ([10], [9], [17], [4]) for computing a CGS of an \( I \) in which \( G_i \) need not be a subset. A CGS is called faithful if and only if each \( G_i \subseteq I \).

**Definition 2.3** Given an ideal \( I \subseteq K[U] [X] \), \( S \subseteq L^m \) the parameter space, and an admissible term order \( > \), let \( G \) be a finite subset of \( K[U][X] \). \( G \) is called a comprehensive Gröbner basis (CGB) of \( I \) on \( S \) w.r.t. \( > \), if for \( \forall \sigma \in S \), \( \sigma(G) \) is always a Gröbner basis of the ideal \( \sigma(I) \) on \( L[X] \). Specifically, if \( S = L^m \), \( G \) is a CGB of \( I \).

Note that the above definition of a CGB requires it to be faithful. From a faithful CGS, it is easy to compute the associated CGB by taking the union of the set of polynomials in each branch of the CGS. Further, a CGB can be defined on a constructible set (or equivalently, a parametric segment).

The following proposition holds for any CGB generated by the KSW algorithm in [5]

**Proposition 2.4** If a CGB of an ideal \( I \subseteq K[U][X] \) w.r.t. an admissible term order > is faithful, for every branch \((A_i, G_i)\), \( G_i \) is a CGB of \( I \) on \( A_i \) w.r.t. >.

Further, a special kind of CGB is defined as follows:

**Definition 2.5** Given a CGB \( G \) of an ideal \( I \subseteq K[U][X] \) w.r.t. an admissible term order >, \( G \) is minimal if the following conditions are true –

(i) No proper subset of \( G \) is a CGB of \( I \) w.r.t. >;

(ii) For \( \forall g \in G \), \( LC(d)(g) \) is a monic polynomial in \( K[U] \).

The CGS and the associated CGB \( G \) computed by the KSW algorithm is adapted to Definition 2.5, by making polynomials in each branch of CGS and every polynomial in \( G \) have their leading coefficients as monic polynomials in \( K[U] \).

## 3. ALGORITHM FOR GENERATING MINIMAL CGB

The proposed algorithm takes a CGB \( G \) and its associated faithful CGS as input, and outputs a minimal CGB (MCGB) of the same ideal. The structures generated by the KSW algorithm [5] satisfies this requirement. Two key operations are performed on \( G \) to get an MCGB: (i) removal of non-essential (or redundant) polynomials in \( G \), and (ii) simplification of essential polynomials by other polynomials in \( G \). In subsequent sections, we discuss in detail these checks are performed. We start with a top level description of the algorithm first.

A minimal Gröbner basis of an ideal \( I \subseteq K[X] \) is achieved by removing unnecessary polynomials from an arbitrary Gröbner basis of \( I \). Analogously, the concept of an essential polynomial in a CGB is defined:

**Definition 3.1** Given a CGB \( G \) of some ideal \( I \subseteq K[U][X] \), a polynomial \( p \in G \) is called essential w.r.t. \( G \) if \( G \subseteq \{p\} \) is not a CGB of \( I \) anymore. Otherwise, \( p \) is non-essential if \( G \subseteq \{p\} \) remains to be a CGB of \( I \).

We are abusing the notation somewhat since in checking whether \( p \in G \) is essential, only polynomials in \( G \setminus \{p\} \) are considered.

The following proposition intuitively states that a polynomial \( p \) in the above \( G \) is essential if under at least one specialization \( \sigma \), \( \sigma(p) \) is necessary for \( \sigma(G) \) to be a Gröbner basis of the specialized ideal \( \sigma(I) \). Further,

**Proposition 3.2** Given a CGB \( G \) of some ideal \( I \), a polynomial \( p \in G \) is essential w.r.t. \( G \) if and only if there exists a specialization \( \sigma \) such that for \( \forall q \in G \setminus \{p\} \), \( LT(\sigma(q)) \) cannot divide \( LT(\sigma(p)) \).

**Corollary 3.3** Given a CGB \( G \) of an ideal \( I \) and a polynomial \( p \in G \) that is essential w.r.t. \( G \), \( p \) remains essential w.r.t. any CGB of \( I \) which is also a subset of \( G \).

A minimal CGB from a given CGB is computed by removing non-essential polynomials from it.

### 3.1 When is a multiple of a polynomial in CGB redundant?

An easy and obvious redundancy check is when a CGB contains a polynomial as well as its multiple with a polynomial purely in parameters as the multiplier. For any specialization, the multiplier evaluates to a constant. Since the KSW algorithm computes the RGB in each branch w.r.t. \( K[X,U] \), a polynomial and its multiple can be part of the output in different branches as illustrated below.

**Example 3.4** Given \( I = \langle f \rangle = \langle (a^3 - b)^3 + (a^2 + b^2 + 1)x + (a - b)(b + 2) \rangle \subseteq K[a,b][x] \) and a lexicographic term order > with \( x \gg a > b \).

A CGB of \( I \) computed by the KSW algorithm is

\[ G = \{ f, g = (b^3 - b^2 + 2b + 1) f, h = (a + b) f \}. \]

Both \( g \) and \( h \) are multiples of \( f \) with the multipliers being polynomials in \( K[U] \) (i.e. polynomials only in parameters). It is easy to see that both \( g \) and \( h \) are redundant, and after removing them, the resulting set \( G' = \{ f \} \) is still a CGB of \( I \). In general,

**Proposition 3.5** Let \( G \) be a CGB of ideal \( I \subseteq K[U][X] \) w.r.t. >. If there is \( S = \{ c_1 f, \ldots, c_n f \} \subseteq G \) with \( f \in G \) and \( c_1, \ldots, c_n \in K[U] \), then \( G' = S \) is still a CGB of \( I \).

The CGB output of the KSW algorithm is thus preprocessed to remove such multiples of polynomials with multipliers in \( K[U] \) if they are present; The associated CGS is updated by replacing \( c_i f \) by \( f \) in its branches. By this removal, we achieve a CGB of the same ideal of a smaller size along with a simpler CGS.

### 3.2 Key Ideas of the Algorithm

The algorithm below is given the CGB \( G \) and its associated CGS of \( I \) computed by the KSW algorithm as input. It preprocesses \( G \) as discussed in Section 3.1; if any polynomial is deleted in this step, the associated CGS is updated accordingly. Each polynomial in the result is checked for being essential.

1. \( p \) is not essential: \( p \) is removed from \( G \), and CGS is updated as in Section 3.6, replacing \( p \) by other polynomials which cover \( p \).

2. \( p \) is essential: \( p \) is kept in \( G \) without changing CGS.
3.3 An Illustrative Example

Before discussing further details of the above algorithm, we illustrate the key concepts of essentiality and covering of a polynomial by other polynomials using a simple example below.

**Example 3.6** Given \( I = \langle (a - 2b)x + y^2 + (a + b)z, a^2x + y + bz \rangle \subseteq K[a,b,x,y,z] \) and a lexicographic term order such that \( x > y > z \geq a > b \). The KSW algorithm computes the following CGS:

| segment | basis | LT |
|---------|-------|----|
| 1       | \( \emptyset, \{ab\} \) | \( \{f_1, f_2\} \) \( y^2, x \) |
| 2       | \( \{b\}, \{a\} \) | \( \{f_2, f_3\} \) \( y^2, x \) |
| 3       | \( \{a, b\}, \{1\} \) | \( \{f_2\} \) \( y \) |
| 4       | \( \{a, b\} \) | \( \{f_4, f_5\} \) \( y, x \) |

and the associated CGB

\[
G = \{ f_1 = a^2 y^2 + (a + 2b) y + \phi z, \\
  f_2 = b^2 x - a + 2b y^2 + \frac{1}{4} y - \frac{\psi}{4}, \\
  f_3 = (a - 2b) x + y^2 + (a + b) z, \\
  f_4 = ab z - \frac{a^2}{2} y^2 + \frac{1}{2} y - \frac{\theta}{2}, \\
  f_5 = ab y - \frac{a - 2b}{2} x + \frac{a^2}{2} y^2 - \frac{2 a^2 + 2 a b - a}{4} y z 
\]

where \( \phi = a^4 + a^2 b - ab + 2 b^2, \psi = a^2 + 3 a b + 2 b^2 - b \) and \( \theta = a^2 + ab - b \).

Preprocessing does not change \( G \) since there are no polynomials multiple of each other. Since \( f_1 < f_2 < f_3 < f_4 < f_5 \), the essentiality check starts from \( f_3 \) in the descending order.

It suffices to check for specializations corresponding to branches where \( f_3 \) appears in the CGS whether \( f_3 \) can be covered by other polynomials in \( G \). \( f_3 \) appears only in Branch 4 with the specializations \( A_4 : a = 0, b \neq 0 \), and its leading term is \( x \). Since \( G_4 \) is minimal under \( A_4 \) due to the KSW algorithm, it is enough to check whether polynomials in \( G - G_4 = \{ f_1, f_2, f_3 \} \) can cover it for specializations of \( A_4 \). \( f_2 \) contains \( x \) with coefficient \( b^2 \neq 0 \) and has no higher term, so \( \{f_2\} \) covers \( f_2 \) under \( A_1 \). Thus, \( f_3 \) is non-essential; it can be replaced by \( f_3 \) in the CGS and deleted from \( G \) giving a smaller CGB of \( I \). \( G_4 \) in CGS becomes \( \{ f_1, f_2 \} \) and \( G = \{ f_1, f_2, f_3, f_4 \} \).

The essentiality of \( f_4 \) is checked similarly. It appears only in Branch 4 and its leading term is \( y \) under \( A_4 \). Further, only \( \{f_1, f_2\} \), which is \( G - G_4 \) can possibly cover it. \( f_1 \) contains \( y \) with coefficient \( -a + 2b \neq 0 \). \( f_1 \) has a higher term \( y^2 \), but its coefficient \( a^2 = 0 \), thus implying that the leading term of \( f_1 \) under \( A_4 \) is determined to be \( y \). So \( f_1 \) covers \( f_1 \) under \( A_4 \), implying \( f_1 \) is non-essential. Update \( G \) and CGS by deleting \( f_4 \) and replacing \( f_4 \) by \( f_1 \) respectively: \( G = \{ f_1, f_2 \} \) and \( G_4 \) becomes \( \{ f_1, f_2 \} \) in CGS.

Finally, \( f_1 \) appears only in Branch 2 has \( x \) as its leading term in this branch. No polynomial in \( G - G_2 \) contains \( x \), implying \( f_1 \) has no covering under \( A_2 \). Thus \( f_1 \) is essential; consequently, CGS and \( G \) do not change.

To check \( f_3 \) for essentiality, all branches need to be considered. In Branch 1, the leading term of \( f_2 \) is \( x \). \( f_3 \) is the only polynomial in \( G - G_1 \), which contains \( x \) and has no higher term. But the coefficient \( a - 2b \) is not determined. This means \( f_3 \) covers \( f_2 \) only under the subsegment \( A_1 = A_1 \cup \{ -a + 2b = 0 \} \) which is a subset of \( A_1 \), but not under \( A_{10} = A_1 \cup \{ -a + 2b = 0 \} \). There is no other polynomial that has \( x \) in it, so \( f_2 \) has no covering under \( A_{10} \). Thus \( f_2 \) is essential, and there is no need to consider other branches.

Finally, to check \( f_1 \) for essentiality, branches 1 and 4 need consideration. In Branch 1, the leading term of \( f_1 \) is \( y^2 \). \( f_1 \) is the only polynomial in \( G - G_1 \) and it contains \( y^2 \) with coefficient \( 1 \neq 0 \). But it has a higher term \( x \) with coefficient \( a - 2b \) that is not determined under \( A_1 \). So it cannot cover \( f_1 \) under segment \( A_{11} = A_1 \cup \{ a - 2b = 0 \} \). Since there is no other polynomial left to cover \( f_1 \), \( f_1 \) has no covering under \( A_{11} \). \( f_1 \) is essential, and the algorithm after having found \( f_1, f_4 \) as nonessential with a smaller CGB \( \{ f_1, f_2, f_3 \} \) consisting of only essential polynomials, with the updated smaller CGS as well.

### 3.4 Essential Polynomial

#### 3.4.1 Covering

As illustrated by the above example, the essentiality check of a polynomial \( p \) in a CGB is performed by identifying other polynomials in the CGB which can possibly cover \( p \). For a branch with an associated segment \( A_i \) in a CGS in which \( p \) appears, only those polynomials in the CGB that can cover \( p \) that have the leading term of \( p \) w.r.t. \( A_i \) appearing in them with a possibly nonzero coefficient. Since \( p \) in the CGB may...
be contributed by many branches in the CGS, for every such branch and its associated segment, we need to check whether polynomials appearing in other branches of CGS can cover p. We exploit the structure of the associated CGS computed by the KSW algorithm and its properties to perform this check (particularly that each branch \((A_i, G_i) \in \text{CGS}, G_i, G_i\) is a minimal Gröbner basis under specializations in \(A_i, A_i\) and all branches are disjoint). It is thus not necessary to consider other polynomials in the same branch as \(p\) and further, we only need to consider polynomials which have the leading term of \(p\) under the segment under consideration appearing with a possibly nonzero coefficient. If a single polynomial \(q\) covers \(p\) for the branch under consideration, then \(q\) can replace \(p\) in that branch as well as in the GB of the CGS of that branch. For different branches in which \(p\) appears, there may be different such \(q\)'s covering \(p\). If for at least one branch in the CGS in which \(p\) appears, it cannot be covered, then \(p\) is declared essential and kept in the CGS as well as the CGS. If \(p\) can be covered in all branches of the CGS and which segment cannot be completely determined to be nonzero. Then, the branch (i.e., segment of specializations) needs to be split into multiple sub-branches so that multiple polynomials can cover \(p\).

It can be that for a particular branch in which \(p\) appears, \(p\) cannot be covered by a single polynomial but instead multiple polynomials are needed for covering it. This especially arises if the leading term of \(p\) appears in another polynomial in CGB but whose coefficient under the segment cannot be completely determined to be nonzero. Then, the branch (i.e., segment of specializations) needs to be split into multiple sub-branches so that multiple polynomials can cover \(p\).

This aspect is also illustrated below in the next subsection as well as in the case of the essentiality check of \(f_1, f_2\) in the example above.

**Definition 3.7** Given a CGB \(G\) of an ideal \(I\), a finite set \(Q = \{q_1, \ldots, q_n\} \subseteq G - \{p\}\) is said to be a covering of a polynomial \(p \in G\) over a set \(A\) of specializations, if for \(\forall a \in A\), there is some \(q_i \in Q\) such that \(LT(\sigma(q_i)) \mid LT(\sigma(p))\).

**Proposition 3.8** Given a CGB \(G\) and the associated CGS of \(I\) generated by the KSW algorithm, \(p \in G\) is non-essential w.r.t. \(G\), iff in each of branch \((A_i, G_i) \in \text{CGS}\) where \(p\) appears \((i.e. \ p \in G_i)\), \(p\) has a covering \(Q_i = \{q_1, \ldots, q_n\} \subseteq G - G_i\), such that for \(\forall a_i \in A_i\), there is some \(q_i \in Q_i\) with \(\text{LT}(\sigma(q_i)) = \text{LT}(\sigma(p))\).

### 3.4.2 Branch Partition

As alluded above, a polynomial \(p\) in a branch corresponding to the segment \(A_j\) may not be covered by a single polynomial \(q\) but instead may require a set of polynomials \(Q = \{q_1, \ldots, q_n\}\) with each polynomial in \(Q\) only covers \(p\) in a proper subset of \(A_j\), but their union is \(A_j\).

For example, in \(K[u, v][y, x]\) and a lexicographic term order with \(y > x > u > v\), in a branch \(A_1 = \{(u^2-v^2), \{u\}\} \in \text{CGS}\) of an ideal, \(G_j = \{p = (u^2-v^2)y + ux\} \text{ and } Q = \{q_1 = (u+v)x, q_2 = (u-v)x\}\). To check if \(Q\) can cover \(p\) in \(A_1\), it is easy to see that both \(q_1, q_2\) partially cover \(p\), since neither of their leading coefficients is determined. So \(A_1\) is partitioned w.r.t. \(u+v, q_1\) covers \(p\) in \(A_{j1}\) and \(q_2\) covers \(p\) in \(A_{j2}\). So \(Q = \{q_1, q_2\}\) is a covering of \(p\) in \(A_1\). This kind of branch partition deals with the case when the leading term of \(p\) appears in \(q\) but with the coefficient that is not determined to be nonzero.

\[
\begin{align*}
A_{j1} &= \frac{u+v}{(u^2-v^2) - \text{LT}(u)} \\
A_{j2} &= \frac{u+v}{\text{LT}(u(v^2))}
\end{align*}
\]

### 3.5 Algorithm of Checking Essentiality

**Algorithm** CheckEssential\((p, G, \text{CGS})\)

**Input:** \(p\): a polynomial in \(G\) whose essentiality is being checked; \(G\): a CGB of some ideal; \(\text{CGS}\): the associated CGS of \(G\).

**Output:** An empty list, if \(p\) is essential w.r.t. \(G\); a non-empty list, otherwise.

1. \(\mathcal{L}' := \emptyset\)
2. for each \(B_i = (A_i, G_i) \in \text{CGS} \text{ with } p \in G_i\):
3. \(L_i := \text{CheckInBranch}(p, G - G_i, B_i)\)
4. if \(L_i = \emptyset\) then return \(\emptyset\) endif
5. \(\mathcal{L} := \mathcal{L} \cup L_i\)
endfor

6. return \(\mathcal{L}\)

As illustrated above, to check whether a polynomial \(p\) in a CGB \(G\) is essential, the algorithm goes through each branch \((A_i, G_i)\) where \(p \in G_i\), looking for a covering of \(p\) by polynomials in \(G - G_i\). In CheckInBranch, let \((A_i, G_i)\) be a branch where \(p \in G_i\), and let \(t\) be the leading term of \(p\) under \(A_i\); then a set of candidate polynomials \(G_{can}\) is first computed from \(G - G_i\) by removing polynomials in which the coefficient of the term \(t\) is determined to be 0. In Example 3.6, to check \(f_2\) under \(A_4\) with \(t = x\), \(G_{can} = \{f_2, f_3\}\) is computed out of \(G - G_4 = \{f_1, f_2, f_3\}\), since \(f_1\) contains no \(x\).

The algorithm then looks for a covering of \(p\) from \(G_{can}\). There are multiple cases based on how the leading term of \(p\) under the segment being considered appears in the polynomials in \(G_{can}\).

- **Case 1:** \(G_{can} = \emptyset\), then \(p\) has no covering in \(A_i\), implying it is essential. The check for \(f_3\) in Example 3.6 is such a case.
- **Case 2:** \(q\) in \(G_{can}\) such that the coefficient of the leading term \(t\) of \(p\) is determined to be non-zero in \(A_i\); \(q\) then covers \(p\) under \(A_i\) if either it has no higher term (In Example 3.5, \(f_2\) covers \(f_3\) under \(A_4\) is such a case), or the coefficient of all of its higher terms is 0 (In the above example, \(f_1\) covers \(f_1\) under \(A_4\) is such a case). Otherwise, \(q\) cannot cover \(p\) if there is a higher term with non-zero coefficient, or \(q\) can partially cover \(p\) if a higher term has a coefficient that is not determined to be 0 (as \(f_3\) partially covers \(f_1\) under \(A_4\)).
- **Case 3:** \(q\) in \(G_{can}\) such that the coefficient of the leading term \(t\) of \(p\) is not determined in \(A_i\) to be 0; \(q\) can only partially cover \(p\) under \(A_i\). \(A_i\) is partitioned into two segments, and the check is continued in each of them. The check of \(f_2\) in Example 3.6 and Example in Section 3.4.2 are such cases.

### 3.6 Update CGS

If a polynomial \(p\) is found to be not essential, it is discarded not only from the CGB but it must be replaced by the polynomials from CGB covering it in the associated CGS; UpdateCGS procedure accomplishes that. In every branch of
the CGS in which \( p \) appears, \( p \) is replaced by the polynomials covering it for that branch. As a result, the updated CGS remains to be a CGS of \( I \), with the union of polynomials in \( G_i \)’s being \( G = \{ p \} \), the new CGF.

In Example 3.6, when removing \( f_5 \), \( G_4 \) is changed to \( \{ f_1, f_2 \} \). Then \( f_4 \) is still non-essential w.r.t. \( G = \{ f_2 \} \). So remove it, and update \( G_4 \) to be \( \{ f_1, f_2 \} \).

It should be obvious from the algorithm description that its output is always a subset of the input CGB \( G \) under the set ordering w.r.t. the CGS computed by the KSW algorithm; then MCGBMain algorithm terminates and the new CGB, the CGS in which \( G \)’s being \( G = \{ f_1 \} \), is still non-essential w.r.t. the CGS in which \( f_1 \) is non-essential.

3.6 Correctness

Proposition 3.9 The MCGBMain algorithm terminates and computes a minimal CGB of the given ideal I w.r.t. the given term ordering >.

It should be obvious from the algorithm description that its output is always a subset of the input CGB \( G \). As stated earlier, the order in which polynomials are checked for essentiality affects the output of the algorithm. In Example 3.6, there are 4 different MCGBs which are subsets of \( G \): \( M_1 = \{ f_1, f_2, f_3 \} \), \( M_2 = \{ f_1, f_3, f_4 \} \), \( M_3 = \{ f_1, f_2, f_3 \} \), \( M_4 = \{ f_1, f_4, f_5 \} \).

The output of MCGBMain is \( M_1 = \{ f_1, f_2, f_3 \} \). However, even a smaller MCGB can be obtained from \( M_1 \) by simplifying \( f_5 \) further.

The output of MCGBMain is \( M_1 = \{ f_1, f_2, f_3 \} \). However, even a smaller MCGB can be obtained from \( M_1 \) by simplifying \( f_5 \) further.

Proposition 3.10 Given a CGB \( G \) of an ideal I w.r.t. \( G \) computed by the KSW algorithm, then MCGBMain algorithm computes the least MCGB among all MCGBs which are subsets of \( G \) under the set ordering w.r.t. >.

For Example 3.6, the algorithm indeed computes the smallest MCGB \( M_1 \).

3.8 Simplification: Generating a different MCGB from an MCGB

By Proposition 3.10 above, MCGBMain algorithm only computes an MCGB that is the least among all subsets of the input CGB \( G \) that are MCGBs. However, this result need not be the least one among all MCGBs of the ideal I as illustrated below.

Example 3.11 For \( I = \langle x^2 - 2y + (4u + 4v)z, (-2u + 2v)x^2 - 2y + 4vz \rangle \subseteq K[u,v][x,y,z] \) and a lexicographic term order > with \( x > y > z > u > v \), the CGS computed by the KSW algorithm is:

\[
\begin{array}{|c|c|c|}
\hline
\text{branch} & \text{basis} & \text{LT} \\
\hline
1 & (\emptyset, \{3u - 2v\}) & \{f_1, f_2\} \\
2 & (\{3u - 2v\}, \{v\}) & \{f_1, f_3\} \\
3 & (\{u, v\}, \{v\}) & \{f_2\} \\
4 & (\{v\}, \{u\}) & \{f_3, f_4\} \\
\hline
\end{array}
\]

and the CGB is:

\[
G = \{ f_1 = (u - \frac{2}{3}v)y + (-\frac{4}{3}u^2 - \frac{2}{3}uv + \frac{4}{3}v^2)z, \\
f_2 = vz^2 - 3y + (4u + 6v)z, \\
f_3 = (u - \frac{10}{13}v)x^2 + \frac{4}{13}y + \frac{12u - 8v}{13}z, \\
f_4 = (u - \frac{2}{3}v)x^2 + \frac{4}{3}uz \}.
\]

The output of MCGBMain is \( M_1 = \{ f_1, f_2, f_3 \} \). However, even a smaller MCGB can be obtained from \( M_1 \) by simplifying \( f_5 \) further.

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Simplification for parametric polynomials can be extremely tricky however; particularly, replacing a parametric polynomial in a CGB by another parametric polynomial obtained in general after simplifying using the CGB need not preserve CGBness. As illustrated earlier, trivial simplification of a polynomial \( p \) by another polynomial \( q \) to \( 0 \) when \( p = a * q \) and \( a \) is a polynomial in parameters, leading to discarding \( p \) preserves CGBness (see Example 3.4 as well as Proposition 3.5). The polynomial \( -h \), which is a multiple of \( h \) in \( G \) in the example from Weispfenning discussed in the Introduction, is also such an example. However, \( h \) can also be simplified to 0 by \( \{ f, g \} \), but removing \( h \) does not preserve CGBness since the resulting set \( \{ f, g \} \) is not CGB of \( I \) anymore.

An obvious way to ensure that replacing a polynomial by its simplified form preserves CGBness is to check that the simplified form covers the original polynomial along with other polynomials in the CGB as defined in the previous section. That means essentiality check must be performed after simplification to maintain the correctness.

We have extended MCGBMain to apply simplification on MCGBs and the result is an extended MCGB algorithm called MCGBSimpl. The extended algorithm is conservative in the following sense: when \( p \) is checked to be essential, it is simplified by \( M = \{ p \} \) to the normal form \( \tilde{p} \). If \( \tilde{p} \) is a non-zero polynomial different from \( p \), then check whether \( \tilde{p} \) with other polynomials in \( M = \{ p \} \) can cover \( p \) in which case substitute it for \( p \) in \( M \) and \( G \); otherwise, keep \( p \) in \( M \).

To illustrate the extended algorithm on Example 3.11 \( M := M - \{ f_1 \} \) since \( f_4 \) is non-essential; \( G_2 = \{ f_1, f_3 \} \) and \( G_4 = \{ f_2, f_3 \} \). \( f_3 \) is essential. Instead of keeping it in \( M \), the algorithm reduces \( f_1 \) by \( M - \{ f_3 \} \) to the normal form \( g = \frac{1}{2}x^2 - 2y + (4u + 4v)z \). The essentiality check on \( f_3 \) w.r.t. \( \{ f_1, f_2, g \} \) concludes that \( f_3 \) is non-essential. So \( f_3 \) is replaced by \( g \) in \( M \), and the CGS is updated by setting \( G_3 = \{ f_1, f_2 \} \) and \( G_4 = \{ f_2, g \} \). Both \( f_1 \) and \( f_2 \) are essential, and already in their normal form modulo the other polynomials in \( M \). The MCGB computed by MCGBSimpl algorithm is \( \{ f_1, f_2, g \} \), which is smaller than \( M_1 \) and not a subset of \( G \).

As the reader would have noticed that the extended algorithm is extremely conservative in applying simplification: if a polynomial \( f \) is essential and its normal form is \( g, f \) is kept in \( G \) if the substituted basis fails to be a CGB. However, there may exist some intermediate form \( f' \) along the reduction chain \( f \to g \), such that \( f > f' > g \) and \( G = \{ f \} \cup \{ f' \} \) remains being a CGB of the same ideal. So a reasonable
modifications is to repeatedly perform one-step simplification associated with the essentiality check, instead of checking the normal form only. One step simplification has not yet been integrated into our implementation, but we plan to do so in the near future and compare its performance with multi-step simplification with the goal of generating more and smaller minimal CGBs.

3.9 Choosing among different Minimal Dickson Basis in the KSW Algorithm

An indepth investigation of the KSW algorithm [5] reveals that for each branch $A_j = (E_j, N_j)$, the corresponding Gröbner basis $G_j$ is a minimal Dickson basis (MDB) of the RGB of $F + N_j$, where $F$ is the given basis. There can in general be many minimal Dickson bases of the same RGB, and the original KSW algorithm chooses the one with minimal non-zero parts in $A_j$ in its attempt to compute the least GB (after specialization) for every branch. However, this choice sometimes results in larger faithful polynomials in CGS and hence CGB, producing a larger CGB $G$ and thus a larger MCGB as illustrated below.

Consider $F = \{a x^2 y + a^2 x^2 - 3a, 4ab^2 y^2 + 4b^3 - 4\} \subseteq K[a, b][x, y]$ with a graded lexicographic term order such that $x \succ y \gg a > b$. By the original KSW algorithm, we have

| branch | basis LP |
|--------|----------|
| 1 (0, 4ab^2 b^3 + b^2 - 1) | \{f_1, f_2\} \{y^5, x^2\} |
| 2 (4ab^2 b^3 + b^2 - 1), \{a\} | \{f_3, f_4\} \{y, x^2\} |
| 3 (4b, 4b - 1) | \{f_1\}, \{1\} |
| 5 (4b, \{\} ) | \{f_1\}, \{1\} |

CGB is

$G = \{f_1 = ab^2 y^2 + b^3 - 1, f_2 = (a^2 b^2 + b^3 - 4) x^2 + 3a b^2 y - 3a^2 b^2, f_3 = (a^2 b^2 + 2a b^3 - a^3 + b^3 - b) x^2 + (3a^4 b^2 + 3a b^3) y - (3a^2 b^2 + 3a^2 b^3), f_4 = (a^4 b^2 + 2a^3 b^3 - a^3 + b^3 - b) x^4 - (6a^6 b^2 + 6a^5 b^3) y + 9a^4 b^2 + 9a^3 b^3\}.$

And the MCGB computed by $MCGBMain$ is $M = \{f_1, f_2, f_3, f_4\}.$

A smaller CGB can be generated however by choosing a minimal Dickson basis with the least faithful forms as the Gröbner basis for each branch. For the above example, in branch 2, $G_2 = \{f_5, f_6\},$ can be picked where

\[
f_5 = (a^6 b^2 + a^5 b^3 - a^2) x^2 + 3a^3 b^2 y - 3a^4 b^2,
\]

\[
f_6 = (a^6 b^2 + a^5 b^3 + a^2) x^2 + 6a^3 b^2 x^3 + 9a^2 b^2,
\]

with other branches unchanged. So the new CGB $G'$ = \{f_1, f_2, f_3, f_5, f_6\} resulting in the MCGB $M' = \{f_1, f_2, f_5, f_6\}.$, smaller than $M$ above since $f_1 < f_2 < f_5 < f_6$.

We have implemented this modification to the KSW algorithm to choose the least minimal Dickson basis w.r.t. the faithful version of polynomials instead of just considering the non-zero parts as in the original KSW algorithm with better results, leading to smaller minimal CGBs.

4. EXPERIMENTS

The algorithms discussed in the paper are implemented in SINGULAR [2]. We tested the implementation on a large suite of examples including those from [5], [9], [11], [12], [14], [17] and [19]. For instance, there are 70 out of 100 examples which have non-essential polynomials in the CGBs generated by the KSW algorithm, which is considered the best algorithm so far for computing smaller CGSs and CGBs [12]. Below we list some of the results.

Table 1: Resulted MCGB and CGS

| Example | $|KSW_{CGS}|$ | $|CGS|$ | $|G|$ | $|M|$ | % reduced |
|---------|-------------|--------|------|------|----------|
| bad test | 6 | 6 | 8 | 6 | 33% |
| KSW51 | 6 | 5 | 7 | 6 | 17% |
| higher 1 | 4 | 4 | 9 | 6 | 50% |
| higher 3 | 6 | 6 | 6 | 4 | 50% |
| linear | 4 | 4 | 4 | 3 | 33% |
| monets 3 | 12 | 10 | 11 | 6 | 83% |
| GBCover | 7 | 5 | 12 | 7 | 71% |
| SS 1 | 4 | 4 | 12 | 10 | 20% |
| SS 3 | 19 | 17 | 36 | 27 | 41% |
| Sit 21 | 5 | 5 | 6 | 3 | 100% |
| Weispfenning 4 | 4 | 4 | 3 | 2 | 50% |
| Principal | 6 | 5 | 3 | 1 | 200% |
| CTD | 5 | 5 | 6 | 4 | 50% |
| S 10 | 4 | 4 | 14 | 12 | 17% |
| S 12 | 18 | 15 | 15 | 8 | 88% |
| S 13 | 11 | 10 | 9 | 6 | 50% |
| S 16 | 19 | 19 | 15 | 9 | 67% |
| S 53 | 7 | 5 | 13 | 6 | 117% |
| Nonlinear 1 | 6 | 6 | 9 | 4 | 125% |

In Table 1, the complexity of an example is characterized by the size of its CGS (i.e. the number of branches) and CGB, which are the columns with labels $|KSW_{CGS}|$ and $|G|$. The size of an MCGB computed by $MCGBMain$ algorithm is shown in the column with label $|M|$, with the percentage of how many non-essential polynomials are removed from $G$ calculated as $(|G| - |M|)/|M|$. The column with label $|CGS|$ shows the size of CGS reconstructed from $M$. As the table illustrates, a minimal CGB can reduce the size of an input CGB by as much as 100% sometimes.

5. CONCLUDING REMARKS

We have proposed an algorithm for computing a minimal CGB from a given CGB consisting of faithful polynomials. The concept of an essential polynomial with respect to a CGB is introduced; the check for essentiality of a polynomial is performed using the associated CGS by identifying whether polynomials in other branches can cover the given polynomial. Only essential polynomials are kept in a CGB producing a minimal CGB. These two checks only produce minimal CGBs which are subsets of an input CGB. The concept of a simplification of an essential polynomial by other essential polynomials in a CGB is introduced using which minimal CGBs that are not necessarily subsets of an input CGB can be generated. The algorithms have been implemented and their effectiveness is demonstrated on examples which show that most CGBs produced by various algorithms reported in the literature including the KSW algorithm have essential or redundant polynomials.

From a minimal CGB generated by the proposed algorithm, an algorithm to compute a CGS from the output minimal CGB has been developed; the output CGS of this algorithm is often simpler and more compact from the original CGS used to generate the minimal CGB. The discussion of the algorithm could not be included in the paper because of lack of space.
As stated in the introduction, our ultimate goal is to define the concept of a canonical CGB of a parametric ideal $I$, consisting of faithful polynomials and uniquely determined by $I$ and term order $\succ$. This implies that a canonical CGB should be minimal, as otherwise, from a nonminimal canonical CGB, it is possible to identify its proper subset which is both canonical and even smaller, which contradicts the uniqueness property. Also, it should be reduced in some sense, since otherwise a canonical and smaller CGB can be achieved by replacing some polynomial by its reduced form.

Given an ideal $I \subseteq K[X]$ and a term order $\succ$, it is easy to see that the RGB of $f$ is the least Gröbner basis under the set ordering w.r.t. $\succ$. However, an analogous definition of canonical CGB turns out to be difficult.

One major issue is that specialization is not monotonic in general. To explain it, let $I = \langle f = uz + x, g = (u + 1)y - x \rangle \subseteq K[u,z,y,x]$ with a lexicographic term order $\succ$ such that $z > y > x > u$. By the KSW algorithm, we have CGS

| t    | branch  | basis                        | $\sigma_i(G_t)$ |
|------|---------|------------------------------|------------------|
| 1    | $u \neq 0, u + 1 \neq 0$ | $\{ f, g \}$                | $(u + 1)y - x, uz + x$ |
| 2    | $u + 1 = 0$    | $\{ g, h \}$                | $(x, z)$         |
| 3    | $u = 0$        | $\{ f, h \}$                | $(x, y)$         |

Its CGB $\overline{G} = \{ f, g, h = f + g = uz + (u + 1)y \}$ is not minimal, while $\sigma_1(G_1), \sigma_2(G_2)$ and $\sigma_3(G_3)$ are all reduced. A smaller CGB computed by the MCGBMain algorithm is $\mathcal{M} = \{ f, g \}$, which is also minimal. However, $G_2\equiv \{ g, f \}$ and $G_3\equiv \{ f, g \}$ now, with $\sigma_2(G_2) = (x, z - x)$ and $\sigma_3(G_3) = (x, y - x)$ become larger. Namely, we cannot achieve the canonical CGB by simply reducing the specialized corresponding Gröbner bases.

Another issue is that a CGB of an ideal $I$ has no relationship with its reduced Gröbner basis (RGB) w.r.t. $K[U,X]$. It is possible that the CGB generated by the KSW of algorithm $I$ is neither a subset of its RGB nor RGB is a subset of the CGB. So starting with an RGB to compute a minimal CGB may not be helpful, especially when RGB is not a CGB. However, if the RGB of $I$ is a CGB, it can be shown to be minimal as well as canonical CGB.

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6. REFERENCES

[1] D. Cox, J. Little, and D. OShea, Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra, Springer, 2007.
[2] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, SINGULAR 3-1-6 — A computer algebra system for polynomial computations, http://www.singular.uni-kl.de (2012).
[3] D. Kapur, An approach for solving systems of parametric polynomial equations, In: Saraswat, Vijay, Van Hentenryck, Pascal (Eds.), Principles and Practices of Constraints Programming, MIT Press, pp. 217-224, 1995
[4] D. Kapur, Y. Sun, D. Wang, An efficient algorithm for computing a comprehensive Gröbner system of a parametric polynomial system, Journal of Symbolic Computation 49 (2013): 27-44, 2013.
[5] D. Kapur, Y. Sun, D. Wang, An efficient method for computing comprehensive Gröbner bases, Journal of Symbolic Computation 52: 124-142, 2013.
[6] D. Kapur, Y. Yang, An algorithm for computing a minimal comprehensive Gröbner basis of a parametric polynomial system, Invited Talk, Proc. Conference Encuentros de Algebra Computacional y Aplicaciones (EACA), Barcelona, Spain, June 2014, 21-25. http://www.ub.edu/eaca2014/2014_EACA_Conference_Proceedings.pdf
[7] D. Kapur, Y. Yang, An algorithm to check whether a basis of a parametric polynomial system is a comprehensive Gröbner basis and the associated completion algorithm, 2015 (draft).
[8] M. Manubens, A. Montes, Improving the DISPGB algorithm using the discriminant ideal, Journal of Symbolic Computation 41.11: 1245-1263, 2006
[9] M. Manubens, A. Montes, Minimal canonical comprehensive Gröbner systems, Journal of Symbolic Computation 44.5: 463-478, 2009
[10] A. Montes, A new algorithm for discussing Gröbner bases with parameters, Journal of Symbolic Computation 33.2: 183-208, 2002
[11] A. Montes, M. Wibmer, Gröbner bases for polynomial systems with parameters, Journal of Symbolic Computation 45.12: 1391-1425, 2010
[12] A. Montes, Using Kapur-Sun-Wang algorithm for the Gröbner cover, In Proceedings of EACA 2012. Ed.: J.R. Sendra, C. Villarino. Universidad de Alcalá de Henares., pp. 135-138, 2012
[13] K. Nabeshima, A speed-up of the algorithm for computing comprehensive Gröbner systems, Proceedings of the 2007 international symposium on Symbolic and Algebraic Computation (pp. 299-306), ACM, 2007
[14] W. Sit, An algorithm for solving parametric linear systems, Journal of Symbolic Computation, 13.4: 353-394, 1992.
[15] A. Suzuki, Y. Sato, An alternative approach to comprehensive Gröbner bases, Journal of Symbolic Computation 36.3: 649-667, 2003
[16] A. Suzuki, Y. Sato, Comprehensive Gröbner bases via ACGB, ACA2004: 64.5-73, 2004
[17] A. Suzuki, Y. Sato, A simple algorithm to compute comprehensive Gröbner bases using Gröbner bases, Proceedings of the 2006 international symposium on Symbolic and Algebraic Computation (pp. 326-331), ACM, 2006
[18] V. Weispfenning, Comprehensive Gröbner bases, Journal of Symbolic Computation 14.1: 1-29, 1992
[19] V. Weispfenning, Canonical comprehensive Gröbner bases, Journal of Symbolic Computation 36.3: 669-683, 2003
[20] M. Wibmer, Gröbner bases for families of affine or projective schemes, Journal of Symbolic Computation 42.8: 803-834, 2007