Strong and mild solutions of the system of fractional ordinary differential equation and its applications

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Abstract
The purpose of this paper is to solve the system of fractional ordinary differential equations. Furthermore, we prove the solution obtained by using Laplace transform technique are mild and strong solutions. We established the existence and uniqueness of the solution. Also, we simulate strong solutions of the system of fractional order differential equations by maxima software.

Keywords
Fractional derivatives, Mittag-Leffler function, Strong and Mild Solutions, Green’s Function, Maxima.

1. Introduction
Fractional Calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Recently, many applications of fractional calculus can be found in basic sciences, technical sciences, fluid dynamics, stochastic dynamical systems, plasma physics, and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics, etc. Various phenomena were modeled with fractional differential equations. Fractional differential equations have appeared in many branches like physics, chemistry, biology, economics, and engineering. There has been considerable development in fractional differential equations in the last decades.

Recently, many researcher solved ordinary fractional differential equation and obtained their mild and strong solutions. Zufeng Zhang, Bin Liu established sufficient conditions for the existence of mild solution of fractional differential evolution equation by using Banach fixed point theorem [14]. H.L. Tidke, M.B. Dhakne prove the existence and uniqueness of mild and strong solutions of a nonlinear Volterra-integro differential equations with non-local condition, and analysis is based on semigroup theory and Banach fixed point theorem[1, 3]. Adel Jawahdow is concerned with the existence of mild solutions for fractional semilinear differential equations with non-local conditions in separable Banach space and furthermore the result is obtained using the technique of measures of non-compactness in Banach space of continuous functions and Schauder fixed point theorem [1]. K.Bhalchandran, S. Ilamaran prove the existence and uniqueness of mild and strong solutions of a semilinear evolution equation with non-local initial conditions using method of semigroups and Banach fixed point theorem[5]. Lijun Pan is concerned with the existence of mild solution for impulsive stochastic differential equation with non-local condition in p-norm and approach is based on Krasnoselskii fixed point theorem[7]. J. Vanterler da e. Sousa, Leandro S. Tavares, E. Capelas de Oliveira investigate the existence and uniqueness of mild and strong solution of fractional semilinear evolution equation in the Hilfer sense by means of Banach fixed point theorem and Grunwall inequality[4]. Lizhen Chen, Zhenbin Fan prove new existence results of mild solution to fractional differential equation with non-local condition in Banach space[8]. Sayyedah A. Qasem, Rabha W. Ibrahim, Zailam Siri prove existence and uniqueness of bounded m-solutions and
solutions for fractional integro-differential equations with fractional resolvent and unbounded delay [12]. Uttam Ghosh, Susmita Sarkar, Shantanu Das developed analytical method to solve the system of fractional differential equation in terms of M-L function and generalized sine and cosine function where fractional derivative operator is of Jumarie fractional derivative which is modified RL fractional derivative [13]. Sabbarapiu Nageswara Rao, Meshari Alesemil established the existence and uniqueness results for a non-linear coupled system of Caputo type fractional differential equation, supplemented with coupled fractional non-local non-separated boundary conditions by using Banach contraction principal and Leray Schauder fixed point theorem [11]. Mohamed A.E. Herzallah studied two fractional periodic boundary value problems and under some conditions the uniqueness of mild solution is proved for both problems. Finally these mild solutions will be strong solutions under certain conditions [10].

In this connection we solved system of fractional ordinary differential equations, and obtained their mild and strong solutions. The main Problem is

\[
\begin{align*}
& a_1 \frac{D}{D_t^\alpha} x(t) + b_1 y(t) = f(t) \\
& a_2 x(t) + b_2 \frac{D}{D_t^\beta} y(t) = g(t)
\end{align*}
\]

where \( 0 < \alpha < 1 \) and \( a_1, a_2, b_1, b_2 \in R - \{0\} \), initial conditions are

\[
x(0) = x(1), y(0) = y(1).
\]

Here \( x(t) \) and \( y(t) \) are unknown functions and \( f(t), g(t) \) are known functions. Also, we prove that the mild solutions will strong solutions under certain conditions. Furthermore we also established the existence and uniqueness by using Banach contraction principal and Schauder fixed point theorem.

We organized the paper as follows:

In section 2, we define the basic definitions and properties of fractional calculus. Section 3, is devoted for strong and mild solutions of the system of fractional ordinary differential equations of order \( 0 < \alpha \leq 1 \). Section 4, is devoted for existence and uniqueness of the strong and mild solutions of the system of fractional ordinary differential equations. In section 5, we solve some test problems and their solutions are represented graphically by Maxima software.

## 2. PRELIMINARIES

**Definition 2.1.** Riemann-Liouville Fractional integral:

If \( f(t) \in C[a, b] \) and \( a < t < b \) then

\[
a_\alpha^a f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,
\]

where \( \alpha \in (-\infty, \infty) \) is called the Riemann-Liouville fractional integral of order \( \alpha \).

**Definition 2.2.** Riemann-Liouville Fractional Derivative:

If \( f(t) \in C[a, b] \) and \( a < t < b \) then

\[
D_\alpha^a f(t) = D_0^a f(t) = \frac{d^\alpha}{dt^\alpha} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f(s) ds,
\]

where \( \alpha \in (n-1, n) \) is called the Riemann-Liouville fractional derivative of order \( \alpha \).

**Definition 2.3.** M.Caputo Fractional Derivative:

If \( f(t) \in C[a, b] \) and \( a < t < b \) then

\[
\begin{align*}
C_\alpha^a f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \\
C_\alpha^a f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f(s) ds,
\end{align*}
\]

where \( \alpha \in (n-1, n) \) is called the Caputo fractional derivative of order \( \alpha \).

**Definition 2.4.** Mittag-Leffler function of one parameter:

The Mittag-Leffler function of one parameter is denoted by \( E_\alpha(t) \) and defined as

\[
E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + 1)}, \quad \alpha \in C, \quad Re(\alpha) > 0
\]

**Definition 2.5.** Mittag-Leffler function of two parameter:

The Mittag-Leffler function of two parameter is denoted by \( E_{\alpha, \beta}(t) \) and defined as

\[
E_{\alpha, \beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + \beta)}, \quad \alpha, \beta \in C Re(\alpha) > 0, Re(\beta) > 0.
\]

**Definition 2.6.** Laplace transform of Caputo Fractional derivative:

The Laplace Transform of Caputo Fractional derivative of order \( \alpha (\alpha > 0) \) is defined as

\[
L\{D_\alpha^a f(t)\} = s^\alpha L\{f(t)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)
\]

**Definition 2.7.** Mild and Strong Solutions:

Let \( J = [0, 1] \), the function \( x, y \in C(J) \) is called

(i) A strong solution of system (1.1) if \( x, y \in AC(J) \) and (1.1) holds on \( J \).

(ii) A mild solution of problem (1.1) on \( J \) if

\[
\begin{align*}
x(t) &= x(0) - \frac{b_1}{a_1} D_\alpha^a y(t) + \frac{1}{a_1} \int_0^t f(x(t), y(t)) ds \\
y(t) &= y(0) - \frac{a_2}{b_2} D_\alpha^a x(t) + \frac{1}{b_2} \int_0^t g(x(t), y(t)) ds
\end{align*}
\]

**Proposition 2.8.** Convolution integral:

The Convolution integral of two functions \( f(t) \) and \( g(t) \) is denoted by \( f(t) * g(t) \) and is defined as

\[
f(t) * g(t) = \int_0^t f(t-s) g(s) ds
\]
Proposition 2.9. Laplace transform of Convolution integral: The Laplace transform of Convolution integral of two functions $f(t)$ and $g(t)$ is given as

$$L\{f(t) \ast g(t)\} = L\{f(t)\} \cdot L\{g(t)\} \quad (2.10)$$

Proposition 2.10. Laplace Transform of Mittag-Leffler function

$$L\{t^{\alpha k + \beta - 1} E_{\alpha, \beta} (at^\alpha)\} = \frac{k^\alpha \Gamma(\alpha \beta)}{(s^\alpha - a)^{\alpha + 1}} \quad (2.11)$$

Hence we have following inverse Laplace Transform

$$L^{-1}\left(\frac{s^{\alpha - \beta}}{(s^\alpha - a)^{\alpha + 1}}\right) = \frac{t^{\alpha k + \beta - 1} E_{\alpha, \beta} (at^\alpha)}{k!} \quad (2.12)$$

Proposition 2.11. Integral of Mittag-Leffler function

$$0_t^{\alpha k} \left[ t^{\gamma - 1} E_{p, q} (\lambda t^p) \right] = t^{\alpha k + \gamma - 1} E_{p, q + \alpha} (\lambda t^p) \quad (2.13)$$

Proposition 2.12. Derivative of Mittag-Leffler function

$$\frac{d}{dt} \left[ t^{\gamma - 1} E_{p, q} (\lambda t^p) \right] = t^{\gamma - 2} E_{p, q - 1} (\lambda t^p) \quad (2.14)$$

3. MAIN RESULTS

Lemma 3.1. Let $0 < \alpha < 1$ and $u \in C(0, T)$. If there exist $f \in C(0, T)$ such that $u = 0$ $t^\alpha f$ then the function $u$ has fractional derivative $0^\alpha D_t^a f = f$

Theorem 3.2. If $f(t)$, $g(t)$ are continuous function and $a_1$, $b_1$, $a_2$ and $b_2$ are non-zero real constant then the system (1.1) has continuous mild solution are given by

\[
x(t) = \frac{1}{a_1} t^{\alpha - 1} E_{2\alpha, \alpha} (\lambda t^{2\alpha}) \ast f(t) + E_{2\alpha, 1} (\lambda t^{2\alpha}) x(0) - \frac{\lambda}{a_2} t^{\alpha - 1} E_{2\alpha, 2\alpha} (\lambda t^{2\alpha}) \ast g(t) - b_2 \lambda \frac{\lambda}{a_2} t^{\alpha} E_{2\alpha, \alpha + 1} (\lambda t^{2\alpha}) y(0) \quad (3.1)
\]

\[
y(t) = -\frac{\lambda}{b_1} t^{2\alpha - 1} E_{2\alpha, 2\alpha} (\lambda t^{2\alpha}) \ast f(t) - \frac{a_1 \lambda}{b_1} t^{\alpha} E_{2\alpha, \alpha + 1} (\lambda t^{2\alpha}) x(0) + \frac{1}{b_2} t^{\alpha - 1} E_{2\alpha, \alpha} (\lambda t^{2\alpha}) \ast g(t) + E_{2\alpha, 1} (\lambda t^{2\alpha}) y(0) \quad (3.2)
\]

This solutions become strong if $x, y \in AC(0, T)$.

Proof. We have to find solution of system (1.1) by using Laplace transform, therefore taking Laplace transform of both sides of the system (1.1), we get

\[
a_1 s^\alpha X(s) - a_1 s^{\alpha - 1} x(0) + b_1 Y(s) = F(s)
\]

\[
a_2 X(s) + b_2 s^\alpha Y(s) = G(s)
\]

\[
a_1 s^\alpha X(s) + b_1 Y(s) = F(s) + a_1 s^{\alpha - 1} x(0) \quad (3.3)
\]

\[
a_2 X(s) + b_2 s^\alpha Y(s) = G(s) + b_2 s^{\alpha - 1} y(0)
\]

Solving above equations simultaneously for elimination of $Y(s)$, we get

\[
(a_1 b_2 s^{2\alpha} - a_2 b_1) X(s) = b_2 s^\alpha F(s) + a_1 b_2 s^{2\alpha - 1} x(0) - b_1 G(s) - b_1 b_2 s^{\alpha - 1} y(0).
\]

\[
X(s) = \frac{1}{a_1} \frac{s^\alpha}{s^{2\alpha} - \lambda} F(s) + \frac{s^{2\alpha - 1}}{s^{2\alpha} - \lambda} x(0) - \frac{\lambda}{a_2} \frac{1}{s^{2\alpha} - \lambda} G(s)
\]

\[
- \frac{b_2 \lambda}{a_2} \frac{s^{\alpha - 1}}{s^{2\alpha} - \lambda} y(0)
\]

where $\lambda = \frac{a_2 b_1}{a_1 b_2}$

Taking inverse laplace transform, we get

\[
x(t) = \frac{1}{a_1} t^{\alpha - 1} E_{2\alpha, \alpha} (\lambda t^{2\alpha}) \ast f(t) + E_{2\alpha, 1} (\lambda t^{2\alpha}) x(0) - \frac{\lambda}{a_2} t^{\alpha - 1} E_{2\alpha, 2\alpha} (\lambda t^{2\alpha}) \ast g(t) - b_2 \lambda \frac{\lambda}{a_2} t^{\alpha} E_{2\alpha, \alpha + 1} (\lambda t^{2\alpha}) y(0) \quad (3.4)
\]

From equation (3.3), we have

\[
Y(s) = \frac{1}{b_1} \frac{-\lambda}{s^{2\alpha} - \lambda} F(s) + \frac{a_1 \lambda}{b_1} \frac{s^{\alpha - 1}}{s^{2\alpha} - \lambda} x(0) + \frac{1}{b_2} \frac{s^\alpha}{s^{2\alpha} - \lambda} G(s) + \frac{s^{2\alpha - 1}}{s^{2\alpha} - \lambda} y(0).
\]

Taking inverse laplace transform, we get

\[
y(t) = -\frac{\lambda}{b_1} t^{2\alpha - 1} E_{2\alpha, 2\alpha} (\lambda t^{2\alpha}) \ast f(t) - \frac{a_1 \lambda}{b_1} t^{\alpha} E_{2\alpha, \alpha + 1} (\lambda t^{2\alpha}) x(0) + \frac{1}{b_2} t^{\alpha - 1} E_{2\alpha, \alpha} (\lambda t^{2\alpha}) \ast g(t) + E_{2\alpha, 1} (\lambda t^{2\alpha}) y(0) \quad (3.5)
\]

Using equations (2.7) and (2.8), to prove that the solution of
the system (1.1) are mild.

\[
x(0) = \frac{b_1}{a_1} 0 \lambda^a y(t) + \frac{1}{a_1} 0 \lambda^a f(t) \\
= x(0) - \frac{b_1}{a_1} 0 \lambda^a \left\{ \frac{-\lambda}{b_1} t^{2a-1} E_{2a,2a}(\lambda t^{2a}) * f(t) \\
- \frac{a_1 \lambda}{b_1} t^a E_{2a, a+1}(\lambda t^{2a}) x(0) + \frac{1}{b_2} t^{a-1} E_{2a, a}(\lambda t^{2a}) * g(t) \right\} \\
+ E_{2a,1}(\lambda t^{2a}) x(0) - \frac{\lambda}{a_2} \left[ t^{2a-1} E_{2a,2a}(\lambda t^{2a}) \right] * g(t) \\
= x(0) + \frac{\lambda}{a_1} \left[ \frac{1}{\lambda} t^{a-1} E_{2a,a}(\lambda t^{2a}) - \frac{1}{\lambda} t^{a-1} \frac{1}{\Gamma(\alpha)} \right] * f(t) \\
+ E_{2a,1}(\lambda t^{2a}) x(0) - \frac{\lambda}{a_2} \left[ t^{2a-1} E_{2a,2a}(\lambda t^{2a}) \right] * g(t) \\
= x(0) + \frac{1}{a_1} t^{a-1} E_{2a,a}(\lambda t^{2a}) * f(t) - \frac{1}{a_1} 0 \lambda^a f(t) \\
+ E_{2a,1}(\lambda t^{2a}) x(0) - \frac{\lambda}{a_2} \left[ t^{2a-1} E_{2a,2a}(\lambda t^{2a}) \right] * g(t) \\
= x(t)
\]

Hence, equation (2.7) is proved.

To prove \( x(t) \) is strong solution of the system (1.1)

\[
x'(t) = \frac{d}{dt} x(t) \\
= \frac{d}{dt} \left[ \frac{1}{a_1} t^{a-1} E_{2a,a}(\lambda t^{2a}) * f(t) + E_{2a,1}(\lambda t^{2a}) x(0) \right] \\
- \frac{\lambda}{a_2} t^{a-1} E_{2a,2a}(\lambda t^{2a}) * g(t) \\
= \frac{1}{a_1} t^{a-2} E_{2a, a-1}(\lambda t^{2a}) * f(t) + r^{-1} E_{2a,0}(\lambda t^{2a}) x(0) \\
- \frac{\lambda}{a_2} t^{a-2} E_{2a,2a-1}(\lambda t^{2a}) * g(t) \\
= \frac{b_2 \lambda}{a_2} t^{a-1} E_{2a, a}(\lambda t^{2a}) y(0) \\
= x(t)
\]

Hence, equation (2.8) is proved.
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\[ C_0D^a_1x(t) = \frac{1}{a_1} f(t) + \lambda \frac{t^{a-1} E_{a,1} \alpha(\lambda t^{2\alpha}) x(0) - \lambda t^{a-2} E_{a,2} \alpha(\lambda t^{2\alpha}) * g(t)}{a_2} - \frac{b_1}{a_1} t^{a-1} E_{a,\lambda}(\lambda t^{2\alpha}) y(0) \]

\[ C_0D^a_1 y(t) = \frac{1}{a_1} f(t) + \frac{b_1}{a_1} y(t) + \lambda \frac{t^{a-1} E_{a,2} \alpha(\lambda t^{2\alpha}) * f(t)}{a_2} - \frac{b_1}{a_1} t^{a-1} E_{a,\lambda}(\lambda t^{2\alpha}) y(0) \]

From (3.7) and (3.8), we get

\[ C_0D^a_1 x(t) = \frac{1}{a_1} f(t) - \frac{b_1}{a_1} y(t) \]

Thus, \( x(t) \) is strong solution of the system (1.1). To prove \( y(t) \) is strong solution of the system (1.1)

\[ y'(t) = \frac{d}{dt} y(t) \]

\[ = \frac{d}{dt} \left[ - \lambda \frac{t^{a-1} E_{a,2} \alpha(\lambda t^{2\alpha}) * f(t)}{b_1} - \frac{a_1}{b_1} t^{a-1} E_{a,\lambda}(\lambda t^{2\alpha}) x(0) + \frac{1}{b_2} t^{a-1} E_{a,2} \alpha(\lambda t^{2\alpha}) * g(t) + E_{a,\lambda}(\lambda t^{2\alpha}) y(0) \right] \]

\[ = - \lambda \frac{t^{a-2} E_{a,2} \alpha(\lambda t^{2\alpha}) * f(t)}{b_1} - \frac{a_1}{b_1} t^{a-1} E_{a,\lambda}(\lambda t^{2\alpha}) x(0) + \frac{1}{b_2} t^{a-1} E_{a,2} \alpha(\lambda t^{2\alpha}) * g(t) + E_{a,\lambda}(\lambda t^{2\alpha}) y(0) \]

From (3.9) and (3.10), we get

\[ C_0D^a_1 y(t) = \frac{1}{b_1} f(t) - \frac{a_2}{b_2} x(t) \]
Thus, $y(t)$ is strong solution of the system (1.1).

**Theorem 3.3.** The solution of the system (1.1) is written in the form of Green’s functions as follow

$$x(t) = \frac{1}{a_1} \int_0^t G_\lambda, \alpha (t, s) f(s) \, ds - \frac{\lambda}{a_2} \int_0^1 \mathcal{G}_\lambda, \alpha (t, s) g(s) \, ds$$

and

$$y(t) = \frac{1}{b_2} \int_0^1 G_\lambda, \alpha (t, s) g(s) \, ds - \frac{\lambda}{b_1} \int_0^1 \mathcal{G}_\lambda, \alpha (t, s) f(s) \, ds$$

where $G_\lambda, \alpha (t, s)$ and $\mathcal{G}_\lambda, \alpha (t, s)$ are Green’s function given as follow

$$G_\lambda, \alpha (t, s) = \begin{cases} 
\phi(\alpha, \lambda, t) (1-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] \\
+ \lambda \psi(\alpha, \lambda, t) (1-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] \\
+ (t-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(t-s)^{2\alpha}], \quad 0 \leq s \leq t \\
\phi(\alpha, \lambda, t) (1-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] \\
+ \lambda \psi(\alpha, \lambda, t) (1-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] \\
\quad t \leq s \leq 1 
\end{cases}$$

(3.11)

and

$$\mathcal{G}_\lambda, \alpha (t, s) = \begin{cases} 
\phi(\alpha, \lambda, t) (1-s)^{2\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] \\
+ \psi(\alpha, \lambda, t) (1-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] \\
+ (t-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(t-s)^{2\alpha}], \quad 0 \leq s \leq t \\
\phi(\alpha, \lambda, t) (1-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] \\
+ \lambda \psi(\alpha, \lambda, t) (1-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] \\
\quad t \leq s \leq 1 
\end{cases}$$

(3.12)

**Proof.** Using initial condition $x(1) = x(0)$ in (3.4), we get

$$x(1) = \frac{1}{a_1} \int_0^1 (1-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] f(s) \, ds$$

$$- \frac{\lambda}{a_2} \int_0^1 (1-s)^{2\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] g(s) \, ds$$

$$+ E_{2\alpha, \alpha} (\lambda) x(0) - \frac{b_2}{a_2} E_{2\alpha, \alpha+1} (\lambda) y(0)$$

$$[1 - E_{2\alpha, \alpha} (\lambda)] x(0) + \frac{b_2}{a_2} E_{2\alpha, \alpha+1} (\lambda) y(0)$$

$$= \frac{1}{a_1} \int_0^1 (1-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] f(s) \, ds$$

$$- \frac{\lambda}{a_2} \int_0^1 (1-s)^{2\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] g(s) \, ds$$

(3.13)

Using initial condition $y(1) = y(0)$ in (3.5), we get

$$y(1) = - \frac{\lambda}{b_1} \int_0^1 (1-s)^{2\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] f(s) \, ds$$

$$+ \frac{1}{b_2} \int_0^1 (1-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] g(s) \, ds$$

$$- \frac{a_1}{b_1} E_{2\alpha, \alpha+1} (\lambda) x(0) + E_{2\alpha, \alpha} (\lambda) y(0)$$

(3.14)

$$\frac{a_1}{b_1} E_{2\alpha, \alpha+1} (\lambda) x(0) + [1 - E_{2\alpha, \alpha} (\lambda)] y(0)$$

(3.15)

we, eliminate $y(0)$ from (3.13) and (3.14), we get

$$x(0) = \frac{1}{a_1 \Delta(\alpha, \lambda)} \int_0^1 (1-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] f(s) \, ds$$

$$- \frac{\lambda}{a_2 \Delta(\alpha, \lambda)} \int_0^1 (1-s)^{2\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] g(s) \, ds$$

$$+ E_{2\alpha, \alpha} (\lambda) x(0) - \frac{b_2}{a_2} E_{2\alpha, \alpha+1} (\lambda) y(0)$$

$$[1 - E_{2\alpha, \alpha} (\lambda)] x(0) + b_2 \frac{a_1}{a_2} E_{2\alpha, \alpha+1} (\lambda) y(0)$$

$$= \frac{1}{a_1} \int_0^1 (1-s)^{\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] f(s) \, ds$$

$$- \frac{\lambda}{a_2} \int_0^1 (1-s)^{2\alpha-1} E_{2\alpha, \alpha} [\lambda(1-s)^{2\alpha}] g(s) \, ds$$

(3.16)

where $\Delta(\alpha, \lambda) = [1 - E_{2\alpha, \alpha} (\lambda)]^2 - \lambda [E_{2\alpha, \alpha+1} (\lambda)]^2$. 

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From equation (3.14), we get

\[
\begin{align*}
[1 - E_{2\alpha,1}(\lambda)]y(0) &= -\frac{\lambda}{b_1} \int_0^1 (1-s)^{2\alpha-1} E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}] f(s) ds \\
+ &\frac{1}{b_2} \int_0^1 (1-s)^{\alpha-1} E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds \\
- &\frac{a_1\lambda}{b_1} E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{1 - E_{2\alpha,1}(\lambda)}{a_1 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{\alpha-1} E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}] f(s) ds \\
+ &\frac{a_1\lambda}{b_1} E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_1 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{2\alpha-1} E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds \\
- &\frac{a_1\lambda}{b_1} E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_1 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{2\alpha-1} E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}] f(s) ds \\
+ &\frac{a_1\lambda}{b_1} E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{\alpha-1} E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds \\
- &\frac{\lambda}{a_2} \int_0^1 (t-s)^{\alpha-1} E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}] f(s) ds \\
+ &\frac{\lambda}{a_2} \int_0^1 (t-s)^{2\alpha-1} E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds \\
- &\frac{b_2 \lambda}{a_2} \lambda E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{\alpha-1} E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}] f(s) ds \\
+ &\frac{b_2 \lambda}{a_2} \lambda E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{2\alpha-1} E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds \\
- &\frac{b_2 \lambda}{a_2} \lambda E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{\alpha-1} E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds \\
+ &\frac{b_2 \lambda}{a_2} \lambda E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{2\alpha-1} E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds \\
- &\frac{b_2 \lambda}{a_2} \lambda E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{\alpha-1} E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}] f(s) ds \\
+ &\frac{b_2 \lambda}{a_2} \lambda E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{2\alpha-1} E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds \\
- &\frac{b_2 \lambda}{a_2} \lambda E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{\alpha-1} E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds \\
+ &\frac{b_2 \lambda}{a_2} \lambda E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{2\alpha-1} E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds \\
- &\frac{b_2 \lambda}{a_2} \lambda E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{\alpha-1} E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}] f(s) ds \\
+ &\frac{b_2 \lambda}{a_2} \lambda E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{2\alpha-1} E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds \\
- &\frac{b_2 \lambda}{a_2} \lambda E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{\alpha-1} E_{2\alpha,\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds \\
+ &\frac{b_2 \lambda}{a_2} \lambda E_{2\alpha,\alpha+1}(\lambda) \left[ \frac{\lambda E_{2\alpha,\alpha+1}(\lambda)}{a_2 \Delta(\alpha, \lambda)} \right] \\
\times &\int_0^1 (1-s)^{2\alpha-1} E_{2\alpha,2\alpha}[\lambda(1-s)^{2\alpha}] g(s) ds
\end{align*}
\]
\[ + \frac{\lambda [1 - E_{2\alpha,1}(\lambda)] \alpha}{a_1 \lambda} E_{2\alpha,2\alpha + 1}(\lambda t^{2\alpha}) \]

\[ + \int_0^1 (1 - s)^{2\alpha - 1} E_{2\alpha,2\alpha}[\lambda (1 - s)^{2\alpha}] f(s) ds \]

\[ + \frac{[-\lambda E_{2\alpha,2\alpha + 1}(\lambda t^{2\alpha})]}{a_2 \lambda} \]

\[ + \frac{[\lambda [1 - E_{2\alpha,1}(\lambda)] \alpha}{a_2 \lambda} E_{2\alpha,2\alpha + 1}(\lambda t^{2\alpha}) \]

\[ \int_0^1 (1 - s)^{\alpha - 1} E_{2\alpha,\alpha}[\lambda (1 - s)^{2\alpha}] g(s) ds \]

\[ \phi(\alpha, \lambda) = \frac{[1 - E_{2\alpha,1}(\lambda)] E_{2\alpha,1}(\lambda t^{2\alpha}) + \lambda E_{2\alpha,2\alpha + 1}(\lambda t^{2\alpha})}{\Delta(\alpha, \lambda)} \]

\[ \psi(\alpha, \lambda) = \frac{E_{2\alpha,2\alpha + 1}(\lambda t^{2\alpha}) + [1 - E_{2\alpha,1}(\lambda)] \alpha E_{2\alpha,2\alpha + 1}(\lambda t^{2\alpha})}{\Delta(\alpha, \lambda)} \]

\[ x(t) = \frac{1}{a_1} \left[ \phi(\alpha, \lambda, t) \right] \int_0^1 (1 - s)^{\alpha - 1} E_{2\alpha,\alpha}[\lambda (1 - s)^{2\alpha}] f(s) ds \]

\[ + \int_0^t (1 - s)^{\alpha - 1} E_{2\alpha,\alpha}[\lambda (1 - s)^{2\alpha}] f(s) ds \]

\[ - \frac{\lambda}{a_2} \left[ \psi(\alpha, \lambda, t) \right] \int_0^1 (1 - s)^{\alpha - 1} E_{2\alpha,2\alpha}[\lambda (1 - s)^{2\alpha}] g(s) ds \]

\[ + \int_0^t (1 - s)^{2\alpha - 1} E_{2\alpha,2\alpha}[\lambda (1 - s)^{2\alpha}] f(s) ds \]

\[ + \frac{\lambda}{a_2} \psi(\alpha, \lambda, t) \int_0^1 (1 - s)^{\alpha - 1} E_{2\alpha,2\alpha}[\lambda (1 - s)^{2\alpha}] g(s) ds \]

where

\[ \phi(\alpha, \lambda, t) = \frac{[1 - E_{2\alpha,1}(\lambda)] E_{2\alpha,1}(\lambda t^{2\alpha}) + \lambda E_{2\alpha,2\alpha + 1}(\lambda t^{2\alpha})}{\Delta(\alpha, \lambda)} \] (3.18)

\[ \psi(\alpha, \lambda, t) = \frac{E_{2\alpha,2\alpha + 1}(\lambda t^{2\alpha}) + [1 - E_{2\alpha,1}(\lambda)] \alpha E_{2\alpha,2\alpha + 1}(\lambda t^{2\alpha})}{\Delta(\alpha, \lambda)} \] (3.19)

\[ : x(t) = \frac{1}{a_1} \int_0^1 G_{\alpha, \alpha}(t, s) f(s) ds - \frac{\lambda}{a_2} \int_0^1 G_{\alpha, \alpha}(t, s) g(s) ds \] (3.20)

where \( G_{\alpha, \alpha}(t, s) \) and \( G_{\alpha, \alpha}(t, s) \) are Green’s function as defined in equation (3.11) and (3.12), respectively. Using equation (3.16) and (3.17) in equation (3.2), we get

\[ y(t) = \frac{-\lambda}{b_1} \int_0^t (t - s)^{2\alpha - 1} E_{2\alpha,2\alpha}[\lambda (t - s)^{2\alpha}] f(s) ds \]

\[ - \frac{a_1 \lambda}{b_1} \left[ 1 - E_{2\alpha,1}(\lambda) \right] \alpha E_{2\alpha,2\alpha + 1}(\lambda t^{2\alpha}) \]

\[ \times \int_0^1 (1 - s)^{\alpha - 1} E_{2\alpha,\alpha}[\lambda (1 - s)^{2\alpha}] f(s) ds \]

\[ + \frac{a_1 \lambda}{b_1} \left[ 1 - E_{2\alpha,1}(\lambda) \right] \alpha E_{2\alpha,2\alpha + 1}(\lambda t^{2\alpha}) \]

\[ \times \int_0^1 (1 - s)^{2\alpha - 1} E_{2\alpha,2\alpha}[\lambda (1 - s)^{2\alpha}] f(s) ds \]

\[ + \frac{a_1 \lambda}{b_1} \left[ 1 - E_{2\alpha,1}(\lambda) \right] \alpha E_{2\alpha,2\alpha + 1}(\lambda t^{2\alpha}) \]

\[ \times \int_0^1 (1 - s)^{2\alpha - 1} E_{2\alpha,2\alpha}[\lambda (1 - s)^{2\alpha}] g(s) ds \]
Theorem 4.1. Schauder fixed point theorem: Let $X$ be a normed linear space and let the operator $T : X \to X$ be compact, then either
(i) the operator $T$ has a fixed point in $X$, or
(ii) the set $B = \{ u \in X : u = \mu T(u), \mu \in (0, 1) \}$ is unbounded.

Theorem 4.2. Contraction Principal: Suppose $G_{\lambda, \alpha}(t, s)$ and $\phi_{\lambda, \alpha}$ are continuous on $[0, 1] \times [0, 1]$ and $|f(s)| < k_1$, $|g(s)| < k_2$, $|G_{\lambda, \alpha}| \leq M_1$, $|\phi_{\lambda, \alpha}| \leq M_2$, $0 \leq t \leq s \leq 1$, and if
\[
\frac{k_1 M_1}{a_1} - \frac{k_2 M_2}{a_2} < 1 \quad \text{and} \quad \frac{k_1 M_1}{b_2} - \frac{k_2 M_2}{b_1} < 1
\]
then there is unique $x(t)$ and $y(t)$ in $C[0, 1]$ such that
\[
x(t) = \frac{1}{a_1} \int_0^1 G_{\lambda, \alpha}(t, s) f(s) \, ds - \frac{\lambda}{a_2} \int_0^1 \phi_{\lambda, \alpha}(t, s) g(s) \, ds
\]
\[
y(t) = \frac{1}{b_2} \int_0^1 G_{\lambda, \alpha}(t, s) g(s) \, ds - \frac{\lambda}{b_1} \int_0^1 \phi_{\lambda, \alpha}(t, s) f(s) \, ds
\]
for $0 \leq t \leq 1$.

Proof. If $x(t) \in C[0, 1]$, let $T_1(x) = u$, therefore
\[
u(t) = \frac{1}{a_1} \int_0^1 G_{\lambda, \alpha}(t, s) f(s, x(s)) \, ds - \frac{\lambda}{a_2} \int_0^1 \phi_{\lambda, \alpha}(t, s) g(s, x(s)) \, ds
\]
for $0 < s < 1$ since $u \in C[0, 1]$. $T_1 : C[0, 1] \to C[0, 1]$ if $x_1, x_2 \in C[0, 1]$, then
\[
|u_1 - u_2| \leq \frac{1}{a_1} \int_0^1 |G_{\lambda, \alpha}(t, s)| |f(s, x_1(s)) - f(s, x_2(s))| \, ds
\]
\[
- \frac{1}{a_2} \int_0^1 |\phi_{\lambda, \alpha}(t, s)| |g(s, x_1(s)) - g(s, x_2(s))| \, ds
\]
\[
\leq \frac{k_1}{a_1} \int_0^1 |G_{\lambda, \alpha}(t, s)| |x_1(s) - x_2(s)| \, ds
\]
\[
- \frac{k_2}{a_2} \int_0^1 |\phi_{\lambda, \alpha}(t, s)| |x_1(s) - x_2(s)| \, ds
\]
\[
|u_1 - u_2| \leq \left[ \frac{k_1 M_1}{a_1} - \frac{k_2 M_2}{a_2} \right] |x_1(s) - x_2(s)|
\]
\[
\leq \left[ \frac{k_1 M_1}{a_1} - \frac{k_2 M_2}{a_2} \right] |x_1(s) - x_2(s)|
\]
Since $\frac{k_1 M_1}{a_1} - \frac{k_2 M_2}{a_2} < 1$, Therefore $T_1$ is contraction. Hence, there is unique $x$ in $[0, 1]$ such that $T_1(x) = x$.

5. Test Problems

Example 5.1. Consider the system of the system of fractional ordinary differential equations
\[
\begin{align*}
0 \int_0^t D_t^\alpha x(s) + y(t) &= t \\
\int_0^t D_t^\alpha y(t) &= 1
\end{align*}
\]
Solution: Taking Laplace transform, we get
\[
\begin{align*}
L\left\{ \int_0^t D_t^\alpha x(s) \right\} + L\{y(t)\} &= L\{t\} \\
L\{x(t)\} + L\left\{ \int_0^t D_t^\alpha y(t) \right\} &= L\{1\}
\end{align*}
\]
\[
\begin{align*}
x(t) + s^\alpha Y(s) &= \frac{1}{s^2} \\
X(s) + s^\alpha Y(s) &= \frac{1}{s}
\end{align*}
\]
Now, eliminating $Y(s)$ from above equations, we get
\[
(s^\alpha - 1)X(s) = \frac{s}{s^2 - 1}
\]
\[
X(s) = \frac{s^{\alpha - 2}}{s^{2\alpha} - 1} - \frac{s^{-1}}{s^{2\alpha} - 1}
\]
Taking inverse laplace, we get
\[
x(t) = \int_0^1 E_{2\alpha, \alpha+2}(t^{2\alpha}) - \int_0^1 E_{2\alpha, \alpha+1}(t^{2\alpha})
\]
Taking inverse laplace, we get

\[ Y(s) = \frac{1}{s^2} - s^\alpha X(s) = \frac{1}{s^2} - s^\alpha \left[ \frac{s^{\alpha - 2}}{s^{2\alpha} - 1} - \frac{s^{-1}}{s^{2\alpha} - 1} \right] \]

\[ \therefore Y(s) = \frac{s^{\alpha - 1}}{2^{\alpha} - 1} - \frac{s^{-2}}{s^{2\alpha} - 1} \]

Taking inverse laplace, we get

\[ y(t) = t^\alpha E_{2\alpha, \alpha+1}(t^{2\alpha}) - t^{2\alpha+1} E_{2\alpha, \alpha+2}(t^{2\alpha}) \] (5.6)

The solution of the system is represented graphically by Maxima software as follows:

![Figure 1. Strong Solution x(t) for α = 0.8, 0.9](image)

![Figure 2. Strong Solution y(t) for α = 0.8, 0.9](image)

Example 5.2. Consider the system of the system of fractional ordinary differential equations

\[ \frac{C}{0} D_t^\alpha x(t) + y(t) = t \] (5.7)

\[ x(t) + \frac{C}{0} D_t^\alpha y(t) = \delta(t) \] (5.8)

where \( \delta(t) \) is Dirac delta function.

Solution: Taking Laplace transform, we get

\[ L\left\{ \frac{C}{0} D_t^\alpha x(t) \right\} + L\{y(t)\} = L\{t\} \]

\[ L\{x(t)\} + L\left\{ \frac{C}{0} D_t^\alpha y(t) \right\} = L\{\delta(t)\} \]

\[ s^\alpha X(s) + Y(s) = \frac{1}{s^2} \] (5.9)

\[ X(s) + s^\alpha Y(s) = 1 \] (5.10)

Now, eliminating \( Y(s) \) from above equations, we get

\[ (s^{2\alpha} - 1) X(s) = s^\alpha - 1 \]

\[ X(s) = \frac{s^\alpha - 1}{s^{2\alpha} - 1} \]

Taking inverse laplace, we get

\[ x(t) = t^{\alpha+1} E_{2\alpha, \alpha+2}(t^{2\alpha}) - t^{2\alpha+1} E_{2\alpha, \alpha+2}(t^{2\alpha}) \] (5.11)

putting \( X(s) \) in (5.9), we get

\[ Y(s) = \frac{1}{s^2} - s^\alpha X(s) = \frac{1}{s^2} - s^\alpha \left[ \frac{s^{\alpha - 2}}{s^{2\alpha} - 1} - \frac{1}{s^{2\alpha} - 1} \right] \]

\[ \therefore Y(s) = \frac{s^\alpha}{s^{2\alpha} - 1} - \frac{s^{-2}}{s^{2\alpha} - 1} \]

Taking inverse laplace, we get

\[ y(t) = t^{\alpha-1} E_{2\alpha, \alpha+2}(t^{2\alpha}) - t^{2\alpha+1} E_{2\alpha, \alpha+2}(t^{2\alpha}) \] (5.12)

The solution of the system is represented graphically by Maxima software as follows:

![Figure 3. Strong Solution x(t) for α = 0.8, 0.9](image)

![Figure 4. Strong Solution y(t) for α = 0.8, 0.9](image)
6. Conclusion

We successfully solved the system of fractional ordinary differential equations using Laplace transform technique. Theoretically, we proved the solution of the given system is strong and mild. Also, we prove that the existence and uniqueness of the strong and mild solutions of the system of fractional ordinary differential equations. We solved some test problems and their solutions are represented graphically by maxima software.

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