Propagation of Circularly Polarized Light Through a Two-Dimensional Random Medium

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Abstract. The problem of small-angle multiple-scattering of circularly polarized light in a two-dimensional medium with large fiberlike inhomogeneities is studied. The attenuation lengths for elements the density matrix are calculated. It is found that with increasing the sample thickness the intensity of waves polarized along the fibers decays faster than the other density matrix elements. With further increase in the thickness, the off-diagonal element which is responsible for correlation between the cross-polarized waves dissapears. In the case of very thick samples the scattered field proves to be polarized perpendicular to the fibers. It is shown that the difference in the attenuation lengths of the density matrix elements results in a non-monotonic depth dependence of the degree of polarization.

1. Introduction

Studies of multiple scattering of light in disordered two-dimensional media have attracted much attention (see, e.g., [1–12]). This problem is of interest due to practical applications in optics of artificial two-dimensional nanomedia [6, 10, 12], biotechnology [7] etc.

The multiple scattering of electromagnetic waves polarized perpendicular or along the fiberlike inhomogeneities is described within the scalar theory [1–9, 11]. However, the scalar field approximation fails to account for correlations between cross-polarized waves and therefore does not allow one to describe the propagation of circularly polarized waves.

Below the problem of the small-angle multiple scattering of circularly polarized light in a medium composed of thick (the radius larger than the wavelength) parallel fibers is studied within the Fokker-Planck approximation. It is assumed that the medium is highly absorbing and the angular distribution of the scattered light remains sharply anisotropic at all depths [2, 11, 13, 14]. The Stokes parameters of scattered waves are calculated. It is found that with increasing the sample thickness the intensity of waves polarized along the fibers decays faster than the other density matrix elements. With further increase in the thickness the off-diagonal element which is responsible for correlation between the cross-polarized waves dissapears. In the case of very thick samples the scattered wave field proves to be polarized perpendicular to the fibers. It is shown that the difference in the attenuation lengths of the density matrix elements results in a non-monotonic dependence of the degree of polarization on the sample thickness. In propagation through a relatively thin layer, the degree of polarization decreases due to decay of correlations between the cross-polarized waves. For a thick sample the polarization of scattered wave field is independent on the initial polarization of the incident beam, and the degree of polarization tends to unity.
The results obtained in this paper can be used in studies of optical properties of inhomogeneous media containing a fiberlike microstructure [6, 7, 10, 12].

2. General relation

Consider a plane wave of circularly polarized light incident normally to the plane boundary of a two-dimensional medium. The medium is composed of parallel fiberlike inhomogeneities, the $y$-axis is directed along the fibers and the beam is incident along the $z$-axis. The direction of light propagation is specified by angle $\theta$ ($\theta$ is deviation of the wave-vector from the $y0z$ plane, $-\pi < \theta < \pi$ (see Fig. 1). Transverse size $a$ of the scatterers is supposed to be much greater than the wavelength of light $\lambda$. We assume that the refraction index $n$ of fibers is close to unity and for single scattering the phase shift is small: $n - 1 \ll \lambda/a \ll 1$.

For the considered geometry, multiple scattering is conveniently described by a density matrix in the linear representation [1]

$$\hat{\rho} = \frac{1}{2} \begin{pmatrix} E^\parallel & E^\perp \\ E^\perp & E^\parallel \end{pmatrix} = \frac{1}{2} \begin{pmatrix} I + Q & U - iV \\ U + iV & I - Q \end{pmatrix}$$

(1)

where $E^\parallel$ and $E^\perp$ are the electric-field components that are parallel and perpendicular to the fibers (see Fig. 1). The Stokes parameters $I$, $Q$, $U$ and $V$ of the scattered radiation are conveniently written in the form

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \frac{1}{2} \begin{pmatrix} I^\parallel + I^\perp & I^\parallel - I^\perp & 0 & 0 \\ I^\parallel - I^\perp & I^\parallel + I^\perp & 0 & 0 \\ 0 & 0 & 2\text{Re} I^\parallel I^\perp & 2\text{Im} I^\parallel I^\perp \\ 0 & 0 & -2\text{Im} I^\parallel I^\perp & 2\text{Re} I^\parallel I^\perp \end{pmatrix} \begin{pmatrix} I_0 \\ Q_0 \\ U_0 \\ V_0 \end{pmatrix}$$

(2)

where $I_0$, $Q_0$, $U_0$ and $V_0$ are the Stokes parameters of the incident beam.

The elements $I^\parallel$, $I^\perp$ and $I^\parallel I^\perp$ appearing in the linear representation of the density matrix (1)
is convenient in that obey independent transport equations

\[
\begin{align*}
\left\{ \cos \theta \frac{\partial}{\partial z} + \sigma_\parallel + \sigma_a \right\} I_\parallel (z, \theta) &= \int_{-\pi}^{\pi} d\theta' a_\parallel (\theta - \theta') I_\parallel (z, \theta') , \\
\left\{ \cos \theta \frac{\partial}{\partial z} + \sigma_\perp + \sigma_a \right\} I_\perp (z, \theta) &= \int_{-\pi}^{\pi} d\theta' a_\perp (\theta - \theta') I_\perp (z, \theta') , \\
\left\{ \cos \theta \frac{\partial}{\partial z} + \frac{1}{2} [\sigma_\parallel + \sigma_\perp] + \left[ \sigma_a - \frac{4i n_0}{k_0} \text{Im} \left( T_1 (0) - T_2 (0) \right) \right] \right\} I_{\parallel,\perp} (z, \theta) &= \\
\int_{-\pi}^{\pi} d\theta' a_{\parallel,\perp} (\theta - \theta') I_{\parallel,\perp} (z, \theta') ,
\end{align*}
\]

(3)

where \( k_0 = 2\pi/\lambda \) is the wave number, \( \sigma_a \) is the absorption coefficient and \( T_{1,2} (\psi) \) (\( \psi = \theta - \theta' \)) are the amplitudes of scattering for the cross-polarized waves \( (T_{1,2} (0) \) are the amplitudes of forward scattering). The differential scattering coefficients \( a_\parallel \), \( a_\perp \), and \( a_{\parallel,\perp} \) entering into Eq.(3) are expressed in terms of the amplitudes \( T_{1,2} \) as follows [1, 15]:

\[
a_\parallel (\psi) = \frac{2n_0}{\pi k_0} |T_1 (\psi)|^2, \quad a_\perp (\psi) = \frac{2n_0}{\pi k_0} |T_2 (\psi)|^2, \quad a_{\parallel,\perp} (\psi) = \frac{2n_0}{\pi k_0} T_1 (\psi) T_2^* (\psi)
\]

(4)

where \( n_0 \) is the number of fibers per unit area in the plane normal to their axes. The scattering coefficients \( \sigma_\parallel \) and \( \sigma_\perp \), are written as

\[
\sigma_\parallel = \int_{-\pi}^{\pi} d\psi a_\parallel (\cos \psi) , \quad \sigma_\perp = \int_{-\pi}^{\pi} d\psi a_\perp (\cos \psi
\]

(5)

The boundary conditions for Eq.(3) have the form

\[
I_\parallel (z = 0, \theta) = I_\perp (z = 0, \theta) = I_{\parallel,\perp} (z = 0, \theta) = \delta (\theta)
\]

(6)

The first two equations entering into Eqs.(3) were derived in Ref. [1]. Equation for the off-diagonal element \( I_{\parallel,\perp} \) can be obtained from the equations for the third \( U \) and fourth \( V \) Stokes parameters, that were also given in Ref. [1].

In the Born approximation (i.e., for weakly refractive fibers), the scattering amplitudes of the cross-polarized waves are related to each other by a similar equation (7) [15, 16].

The equation for the intensity of the co-polarized waves \( I_\parallel \) (see the first equation in the system (3)) coincides with the scalar transfer equation. At the same time, propagation of the cross-polarized field is sensitive to the vector nature of light. In particular, within the Born approximation the differential scattering cross section for the intensity \( I_\perp \) contains the additional factor \( \cos^2 \psi \) (see Eqs.(3) and (7)) [4].
For weakly refractive fibers the third equation entering into the system (3) takes the form:

\[
\left\{ \cos \vartheta \frac{\partial}{\partial z} + \frac{1}{2} \left[ \sigma_\parallel + \sigma_\perp \right] + \sigma_\alpha \right\} I_{\parallel \perp} (z, \vartheta) = \frac{1}{2} \int_{-\pi}^{\pi} d\vartheta' \left[ a_\parallel (\vartheta - \vartheta') + a_\perp (\vartheta - \vartheta') \right] I_{\parallel \perp} (z, \vartheta') - \\
\frac{1}{2} \left( \frac{2n_0}{\pi k_0} \right) \int_{-\pi}^{\pi} d\vartheta' \left| T_1 (\vartheta - \vartheta') - T_2 (\vartheta - \vartheta') \right|^2 I_{\parallel \perp} (z, \vartheta')
\]

The second term in the right-hand-side of Eq.(8) is responsible for decrease of correlations between the cross-polarized fields.

For weakly refractive fibers, as follows from Eqs.(6) and (8), the off-diagonal elements of the density matrix are real, Im \( I_{\parallel \perp} = 0 \).

The system of equations (3) and Eq.(8) under boundary conditions (6) completely determines the Stokes parameters of multiply scattered light in a medium with two-dimensional weakly refractive inhomogeneities.

3. Fokker-Planck approximation

Propagation of polarized light through a medium with large inhomogeneities can be best understood within the exactly solvable model, which is based on the small-angle diffusion (or Fokker-Planck) approximation [11, 13, 14]. The latter assumes that the single scattering angle is small. Apart from that, we consider the case of high absorption: the absorption length \( l_a = \left( \sigma_a \right)^{-1} \) is assumed to be less than the transport mean free path \( l_{tr} = \left( \int_{-\infty}^{\infty} d\psi (1 - \cos \psi) a_\parallel (\psi) \right)^{-1} \). Then, for multiple scattering, the deflection angle remains also small at arbitrary depth \( z \) and the contribution of backscattered radiation is negligibly small [13, 14]. Therefore, the angular variable \( \vartheta \) can vary in infinite limits, \(-\infty < \vartheta < \infty\).

Within the Fokker-Planck approximation the transport equation for modified intensity \( \tilde{I}_\parallel (z, \vartheta) = \exp(\sigma_a z) I_\parallel (z, \vartheta) \) has the form (see, e.g., [2, 11])

\[
\left\{ \frac{\partial}{\partial z} + \sigma_a \frac{\partial^2}{\partial \vartheta^2} \right\} \tilde{I}_\parallel (z, \vartheta) = D_\parallel \frac{\partial^2 \tilde{I}_\parallel (z, \vartheta)}{\partial \vartheta^2}
\]

where \( D_\parallel = \sigma_\parallel \langle \psi^2 \rangle / 2 \) is the diffusion coefficient for waves polarized along the fibers and \( \langle \psi^2 \rangle = \int_{-\infty}^{\infty} d\psi a_\parallel (\psi) \psi^2 / \sigma_\parallel \) is the variance of single scattering angle. In the diffusion approximation the coefficient \( D_\parallel \) is equal to the scattering transport coefficient of the co-polarized waves \( \sigma^\parallel_{tr} \) [11]

\[
\sigma^\parallel_{tr} = \int_{-\pi}^{\pi} d\psi (1 - \cos \psi) a_\parallel (\psi) \approx \frac{1}{2} \int_{-\infty}^{\infty} d\psi a_\parallel (\psi) \psi^2
\]

The equation for \( \tilde{I}_\perp (z, \vartheta) = \exp(\sigma_a z) I_\perp (z, \vartheta) \) differs from Eq.(9) by the substitution of the coefficient diffusion \( D_\parallel \) by

\[
D_\perp = \int_{-\pi}^{\pi} d\psi (1 - \cos \psi) a_\perp (\psi) \approx \frac{1}{2} \int_{-\infty}^{\infty} d\psi a_\perp (\psi) \psi^2 = D_\parallel (1 - \eta), \quad \eta = \frac{13}{12} \frac{\langle \psi^4 \rangle}{\langle \psi^2 \rangle} \ll 1.
\]

\( \text{(11)} \)
As the analysis shows, the effects associated with changes in light polarization are noticeable in the asymptotic regime at depths \( z > l_d = \left( \frac{D_a}{\sigma_a} \right)^{-1} \). Using the results [2, 11], for the modified intensities at depths \( z > l_d \) we obtain

\[
\tilde{I}_\parallel (z, \vartheta) = \frac{1}{\sqrt{\pi} (\vartheta^2)_\infty} \exp \left( -\frac{z}{l_d} - \frac{\vartheta^2}{2(\vartheta^2)_\infty} \right) \tag{12}
\]

\[
\tilde{I}_\perp (z, \vartheta) = \frac{1}{\sqrt{\pi} (\vartheta^2)_\infty} \exp \left( -\frac{z}{l_d} \left( 1 - \frac{\eta}{2} \right) - \frac{\vartheta^2}{2(\vartheta^2)_\infty} \right) \tag{13}
\]

where \( (\vartheta^2)_\infty = \sqrt{2D_a/\sigma_a} \approx \sqrt{2D_\perp/\sigma_a} \) is the variance of multiple scattering angle in the asymptotic regime.

The small-angle approximation is applied to Eq.(8) in the following way. The expansion of the angle-dependent coefficients on the left- and right-hand sides of Eq.(8) in terms of small \( \langle \rangle \approx 0 \) yields the equation

\[
\left\{ \frac{\partial}{\partial z} + \sigma_a \vartheta^2 \right\} \tilde{I}_\parallel,\perp (z, \vartheta) = \left( \frac{D_\parallel + D_\perp}{2} \right) \frac{\partial^2}{\partial \vartheta^2} \tilde{I}_\parallel,\perp (z, \vartheta) - \frac{\sigma_{\text{dep}}}{2\pi} \int d\vartheta' \tilde{I}_\parallel,\perp (z, \vartheta') \tag{14}
\]

where

\[
\sigma_{\text{dep}} = \frac{1}{2} \left( \frac{2n_0}{\pi k_0} \right) \int_{-\pi}^{\pi} \exp |T_1 (\psi) - T_2 (\psi)|^2 \tag{15}
\]

is the effective "absorption" coefficient for correlations between cross-polarized waves.

The last term in right-hand-side of Eq.(14) results from the approximation

\[
\frac{n_0}{\pi k_0} \int_{-\pi}^{\pi} \frac{d\vartheta' |T_1 (\psi) - T_2 (\psi)|^2 \tilde{I}_\parallel,\perp (z, \vartheta') \approx \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{d\vartheta' A_\parallel (\psi) (1 - \cos (\psi))^2}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\vartheta' \tilde{I}_\parallel,\perp (z, \vartheta') \right| \tag{16}
\]

Equality (16) is based on the assumption that the angle dependence of \( |T_1 (\vartheta - \vartheta') - T_2 (\vartheta - \vartheta')|^2 \) is practically isotropic in comparison with \( \tilde{I}_\parallel,\perp (z, \vartheta) \) (see [17]).

In the case of multiple scattering of light in a medium with three-dimensional inhomogeneities the effective "absorption" coefficient \( \sigma_{\text{dep}} = (n_0/2) \int d\mathbf{n}' |A_\parallel (\mathbf{n}') - A_\perp (\mathbf{n}')|^2 \) \( (A_\parallel \) and \( A_\perp \) are the amplitudes of the waves polarized parallel and perpendicularly to the scattering plane) is responsible for the "dynamical" mechanism of depolarization [18, 19].

The off-diagonal element \( \tilde{I}_\parallel,\perp \) in the asymptotic regime can be easily calculated by the perturbation theory using as the first approximation the solution of Eq.(14) for \( \sigma_{\text{dep}} = 0 \) (see, e.g., [18]). As a result, for the modified parameter \( \tilde{I}_\parallel,\perp \) we obtain

\[
\tilde{I}_\parallel,\perp (z, \vartheta) = \frac{1}{\sqrt{\pi} (\vartheta^2)_\infty} \exp \left( -\frac{z}{l_{\text{dep}}} \left( 1 - \frac{\eta}{4} \right) - \frac{z}{l_{\text{dep}}} \sqrt{\frac{(\vartheta^2)_\infty}{\pi}} + \frac{\vartheta^2}{2(\vartheta^2)_\infty} \right) \tag{17}
\]

where \( l_{\text{dep}} = (\sigma_{\text{dep}})^{-1} \) is the mean free path with respect to random shift of phase difference between cross-polarized waves.

Relying on Eqs.(12), (13) and (17), we proceed to analyze the effects of multiple scattering on the polarization state of light for the incident circularly polarized waves.
4. Discussion

According to Eqs.(12), (13), (17) in propagation of light through a two-dimensional medium with thick fibers, the hierarchy of attenuation lengths of the density matrix elements can be distinguished. The different attenuation lengths result from the difference in the scattering amplitudes of the cross-polarized waves. The diagonal element which describes the scattering of waves polarized along the fibers decays rapidly. The attenuation length $l_d$ coincides with the asymptotic length $l_d = l_d = (D_0 \sigma_0/2)^{-1/2}$. The off-diagonal element which is responsible for correlations of the cross-polarized waves decays at a greater length, $z \geq l_d^{\perp \perp} = ((D_\parallel + D_\perp) \sigma_0/4)^{1/2} + \sigma_{dep}(\langle \theta \rangle_\infty/\pi)^{1/2} \sim l_d(1 + l_d/(2l_{dep}))$. The diagonal element that describes propagation of the waves polarized perpendicular to the fibers is the most long-lived one. The length of attenuation of this element proves to be of the order of $l_d^{\parallel \parallel} = (D_\perp \sigma_0/2)^{-1/2} \approx l_d(1 + l_d/l_{dep})$. As known [18, 19], in a medium with three-dimensional inhomogeneities the differences in the scattering amplitudes of the cross-polarized waves are responsible for the "dynamical" mechanism of the depolarization. Thus, in the case of two-dimensional scatterers, any changes in polarization of light are due to the "dynamical" mechanism of depolarization.

The depth dependence of the density matrix elements is illustrated in Fig. 4, where polarization ratios $I_{\parallel}/I_{\perp}$ and $I_{\perp \perp}/I_{\parallel}$ as functions of $z$ are shown.

For the forward scattered light with the circular initial polarization $(I_0 = V_0 = 1, Q_0 = U_0 = 0)$, the degree of polarization $P = \sqrt{Q^2 + V^2}/I$ and the ellipticity $\beta = (1/2) \arctan(V/|Q|)$ in the asymptotic regime are equal to

$$P = \sqrt{1 - \cosh^{-2} \left( \frac{\eta z}{4l_d} \right) \left( 1 - \exp \left( -\frac{2z}{l_{dep}} \sqrt{\langle \theta^2 \rangle_\infty/\pi} \right) \right)}$$

$$\beta = \frac{1}{2} \arctan \left[ \sinh^{-1} \left( \frac{\eta z}{4l_d} \right) \exp \left( -\frac{z}{l_{dep}} \sqrt{\langle \theta^2 \rangle_\infty/\pi} \right) \right]$$

At relatively small depths, $l_d < z < l_d/\eta \ll l_{dep}$, the decrease of $P$ is due to decay of correlations between the cross-polarized fields, $P = 1 - (z/l_{dep})\sqrt{\langle \theta^2 \rangle_\infty/\pi}$. At large depths, $z > l_{dep} \sqrt{\pi/\langle \theta^2 \rangle_\infty}$, the correlations vanish and the value of $P$ tends to unity. In such a limit the scattered light is polarized perpendicularly to the fibers. This is due to the fact that $l_d^{\perp}$ decreases rapidly compared to $l_d$ and $l_d^{\perp \perp}$. The degree of polarization $P$ reaches its minimum at an intermediate depth, $z \sim (4d_\eta/\eta)$ (see Fig. 4).

The ellipticity $\beta$ at relatively small depths, $z < l_d/\eta$, proves to be close to the initial value, $\beta = \pi/4$. With increasing $z$, $z > l_d/\eta$, the ellipticity $\beta$ tends exponentially to zero. As a result, the polarization plane of scattered waves becomes to be orthogonal with respect to the fibers. As follows from Eq.(19), the decrease of $\beta$ with increasing depth $z$ is governed by the differences between the diffusion coefficients $D_\parallel$ and $D_\perp$, and, in contrast to the degree of polarization $P$, is virtually independent of the presence of the phase coherence between the cross-polarized waves.

5. Conclusions

Small-angle scattering of circularly polarized light in a medium with large two-dimensional inhomogeneities has been studied. The attenuation lengths of the density matrix elements have been calculated within the Fokker-Planck (or diffusion) approximation. It has been found that, at a relatively small depth, the degree of polarization changes owing to decay of correlations between the cross-polarized waves. As the depth increases, the intensity of waves polarized along
the fibers decays faster than the other density matrix elements. The off-diagonal element that is responsible for correlations between the cross-polarized waves disappears at greater depths. In the asymptotic regime the scattered light proves to be polarized perpendicular to the fibers, and the degree of polarization tends to unity. The difference in the attenuation rate between the density matrix elements results in a non-monotonic dependence of the degree of polarization on the depth.

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