A study on an application of a body force dipole to residual stress analyses with perturbed data

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Abstract

While there exist many works on the inverse analysis of the residual stress which use the inherent strain or the body force as a stress source, the authors have proposed a body force dipole (BFD) as the source, since it could simulate fairly well shearing deformation and set no limit to a shape of a BFD distributing area. Some sort of regularization is needed to obtain a reasonable BFD solution, when perturbed stresses are used in the inverse analysis as data measured on a surface of a body. The singular value decomposition (SVD) is used for deriving the BFD solution in the present study. There, artificial noise \( \alpha \) is taken into account as a regularization parameter to reduce the sensitivity of the computed solution to the perturbation of the stress data. A good artificial noise \( \alpha \) is chosen based on the L-curve, the plot of the solution norm versus the norm of the stress residual vector. Numerical results for an analytical model demonstrate that the regularization method used in the present work is very effective for deriving the BFD solution which is not too sensitive to the perturbation of the stress and has a suitably small norm. Also, effects of additional displacement data on the solution is discussed, which are excessively defined as boundary conditions so as to guarantee the uniqueness of the residual stress. Numerical results show that the additional displacements and their perturbations have influence on the regularized solution almost nil. And a theoretical examination reveals that a submatrix for the displacement hardly contributes to the eigenvalues, the squares of the singular values of a sensitivity matrix, so that the displacements have little effect on the solution for the present case.

**Keywords** : Residual stress, Inverse analysis, Body force dipole, Singular value decomposition, Artificial noise, The L-curve method

1. Introduction

The authors have proposed a body force dipole (BFD) as a source of the residual stress for structural integrity assessment of machine components (Asano and Saito, 2007, 2009), since it could simulate fairly well shearing deformation and set no limit to a shape of a BFD distributing area. And they have verified the effectiveness of the approach to unperturbed problems (Asano et al., 2016). So, the main purpose of the present study is to examine an effective regularization method which can apply to the inverse analysis of the BFD for perturbed problems. There exist many works on the inverse analysis demonstrating the applicability of the inherent strain as the source to simulate the residual stress in a butt joint (Ueda et al., 1974) (Iwasaki, 1993) (Ohtake, 2009), and suggesting an important role of a shearing component of the eigen-strain to simulate in-plane shearing deformation of a butt joint (Masuda and Nakamura, 2010) (Hirai and Nakamura, 2013). And artificial noise was used to obtain well regularized eigen-strains for residual stress estimation in a whole body from surface strains (Ogawa, 2013, 2014). The body force was also used as the source to simulate the stress induced by local heating and cooling of a plate (Koguchi et al., 1990) (Tomishima and Yada, 1990), and the residual stress in a butt joint (Ohtake, 1995). For the latter approach, a special device must be designed to balance the body force, so that it sets a limit to a shape of a body force distributing area.

Paying attention to such advantages of the BFD, the present study attempts to apply the approach to perturbed problems. When perturbed stress data is used in the inverse analysis, some sort of regularization is needed to obtain a
reasonable BFD solution. The singular value decomposition (SVD) (Kubo, 1992) (kawaguchi, 2011) is used for deriving the BFD solution. There, artificial noise $\alpha$ is taken into account as a regularization parameter to reduce the sensitivity of the computed solution to perturbations of the data (Marquardt, 1963) (Lines and Treitel, 1984). A good artificial noise $\alpha$ is chosen based on the L-curve (Hansen, 1992), the plot of the solution norm versus the norm of the stress residual vector. The numerical results for an analytical model demonstrate that the regularization method applied in the present study is very effective to derive the BFD solution which is not too sensitive to the perturbation of the stress data and has a suitably small norm. Also, the effect of the additional displacement data on the solution is studied, which are excessively defined as boundary conditions so as to guarantee the uniqueness of the residual stress (Kubo, 1992). Numerical results show that the additional displacements and their perturbations hardly have influence on the regularized solution. And a theoretical examination is performed based on the ordinary least squares method to make it easy to think, and reveals the cause of such results as a quite small contribution of a submatrix for the displacement to the eigenvalues of a sensitivity matrix, which correlates the BFD with the surface stresses and displacements. The examination also reveals that the displacements have influence on the input terms of a related normal equation fairly small, so that their perturbations affect the solution almost nil.

2. Analytical methods

2.1 The analytical model and basic equations

The present work examines the effectiveness of the regularization method which uses the SVD with artificial noise $\alpha$ to analyze the BFD, and is equivalent to Tikhonov regularization (Tikhonov and Arsenin, 1972). To understand fundamental features of the regularization method, a plate $V$ 10mm width × 20mm long with a surface $S$ is employed as an analytical model which contains a BFD distributing area $V_g$ 3mm width × 3mm long as shown in Fig. 1. The unknown BDFs are distributing in a hatched square 2mm width × 2mm long in $V_g$ and are supposed to be zero at the edge of $V_g$. This surrounding area is set so that the BFD decreases and disappears continuously at the edge. The size of the area is assumed to be about the same size as that of the inside elements. A value of 206GPa was assumed for Young’s modulus $E$ and 0.3 for Poisson’s ratio $\nu$, respectively.

The body force dipoles are expressed by distribution functions $f_k$ ($k=1,2,3$) and formulated in the boundary element method (Asano et al., 2016). Here, the subscript $k$ represents the type of the body force dipole; the subscript 1 or 2 means the normal type in the $x_1$ or $x_2$ direction, and 3 the shear type. The surface $S$ of the plate $V$ is divided into $N_S$ constant boundary elements and the area $V_g$ by linear rectangular elements, so that $N_S \geq 3N_A$ as shown in Fig. 2. Here $N_A$ is the number of the nodes where the unknown functions $f_k$ ($k=1,2,3$) should be solved there. A value of 32 is assumed for $N_A$ and 9 for $N_A$ in the present study. The number $N_S$ is determined based on the results (Asano et al., 2016) so that the mesh division has little influence on the solution and requires less calculation. The forward analyses were carried out sequentially changing the type of the BFD at each node under the condition of the traction free surface, in order to obtain the stresses $\sigma_i$ and the displacements $u_j$ ($i,j=1,2$) at the center of all boundary elements, which are induced by each type of the BFD with unity intensity at every node in the area $V_g$. The stresses and displacements were then arranged to construct a sensitivity matrix $[A_{sd}]$ correlating the BFD $[f]$ with the surface stresses $[\sigma]$ and displacements $[u]$ as:

$$
[A_{sd}] = \begin{bmatrix} \sigma \\ u \end{bmatrix}, \quad [A_{sd}] = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.
$$

(1)

![Fig.1 An analytical model which involves a BFD distributing area $V_g$.](image-url)
Here, the sensitivity matrix consists of two components for the stress and displacement, which are simply designated as the submatrices in this paper. Let \( A_s \) denote the submatrix for the stress, and let \( A_d \) denote that for the displacement. These submatrices \( A_s \) and \( A_d \) have \( 3N_S \times 3N_A \) and \( 2N_D \times 3N_A \) elements, respectively. Here, \( N_D \) represents the number of elements whose displacements are used as data.

The SVD of the matrix \( A_{sd} \) can be written as follows (Kubo, 1992) (Kawaguchi, 2011):

\[
A_{sd} = U \Sigma V^T.
\]

Here, the matrices \( U \) and \( V \) are respectively \( (3N_S + 2N_D) \times (3N_S + 2N_D) \) and \( 3N_A \times 3N_A \) matrices with orthonormal columns. The matrix \( V^T \) is the transpose of the matrix \( V \). And the matrix \( \Sigma \) is a \( (3N_S + 2N_D) \times 3N_A \) diagonal matrix whose diagonal elements are singular values \( \lambda_i \) \( (i = 1, 2, \ldots, 3N_A) \) arranged in big order.

We can then express \( \{f\} \) in the form

\[
\{f\} = [V] \ [B]^{-1} [U]^T \ [\sigma] = \{u\}.
\]

Here, the matrix \( [V] \ [B]^{-1} [U]^T \) is the generalized inverse.

### 2.2 The regularization method based on the SVD with artificial noise

The body force dipole solution is made unstable by perturbations of the stress data, and has an unreasonably large norm. To increase the stability of the solution, artificial noise \( \alpha \): a regularization parameter is introduced into the analysis (Marquardt, 1963) (Lines and Treitel, 1984):

\[
B^{-1} = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \frac{\lambda_2}{\lambda_2 + \alpha} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \frac{\lambda_{3N_A}}{\lambda_{3N_A} + \alpha} 
\end{bmatrix}.
\]

To obtain information about a good parameter, the regularized solutions are displayed as a plot of the solution norm \( ||\{f\}|| \) versus the residual norm \( ||\{\text{Res}\}|| \) - the L-curve (Hansen, 1992). Here, a symbol \( ||\cdot|| \) denotes the 2-norm of a vector or a matrix. A good regularization parameter \( \alpha \) is corresponding to a regularized solution near the “corner” of the L-curve, since there is a good compromise between achieving a small residual norm and keeping the solution norm reasonably small in that region. And the squares of the both norms are used to examine separately the effects of the stress and displacement on the solution, based on the ordinary least squares method in the latter section 3.3. The square of the residual norm \( ||\{\text{Res}\}||^2 \) is then given by the expression

![Fig.2 Mesh division of the analytical model.](image-url)
\[
\|\text{Res}\|^2 = \left\| [A_d][f] - \left[ \frac{\sigma}{u} \right] \right\|^2 = \left\| [A_d][f] - [\sigma] \right\|^2 + \left\| [A_d][f] - [u] \right\|^2.
\]
(5)

And the “corner” of the L-curve is assumed to be the point with maximum curvature in the present study.

2.3 The assumption for perturbations

To obtain primary knowledge on the present approach for perturbed problems, we assume perturbation which is different from the actual one. And the present work concentrates to the problem associated with the mere perturbation of the right hand side of Eq.(1) - the perturbation of the input data. To simulate the stress data with random errors, the normally distributed numbers with zero mean and standard deviations 0.01 and 0.02 MPa are added to the stresses obtained by the forward analysis for the assumed uniform BFD distributions \( f_i = 1.0\text{N/mm}^2 \) (\( k=1,2,3 \)). Here, a value of 0.02 MPa corresponds to twelve percent of the maximum stress computed by the forward analysis. The present work refers to such stress data as the perturbations 1 and 2 for simplicity. And, the stress used for the analysis is the normal component alone whose direction is parallel to the surface: \( \sigma_{11} \) on the \( x_2 \)-plane for example. Figure 3 shows unperturbed stresses (○) obtained by the forward analysis and the perturbed stresses (▲, ●) used in the inverse analyses.

Fig.3 shows the unperturbed stresses (○) just obtained by the forward analysis and the perturbed stresses (▲, ●). Perturbations of stresses are simulated by the normally distributed random numbers with zero mean and standard deviation 0.01 and 0.02 MPa. The value of 0.02 MPa corresponds to twelve percent of the maximum value of the stress obtained by the forward analysis, and is called the perturbation 2 in this study.

Fig.4 shows the unperturbed displacements (○) just obtained by the forward analysis and the perturbed displacements (●, ■). The perturbation of displacements is simulated as well as that of stresses, and is generated by the normally distributed random numbers with zero mean and standard deviation \( 5.53\times10^{-6} \text{mm} \), twelve percent of the maximum absolute value of displacements obtained by the forward analysis. The present paper refers to the data as the perturbation 3 which consists of displacements including this perturbation and the perturbation 2 stresses.
We will examine the influence of the perturbation of displacements on the solution in Section 3.3, so we assume the perturbation of the displacement here. In the same way as the stress data, the perturbation of the displacement is simulated by the normally distributed random numbers with zero mean and standard deviation $5.53 \times 10^{-6}$ mm, twelve percent of the maximum absolute value of displacements obtained by the forward analysis for the uniform BFD distributions $f_k = 1.0 \text{N/mm}^2$ ($k=1,2,3$). Figure 4 shows the unperturbed displacements ($\circ$) just obtained by the forward analysis and the perturbed displacements ($\bullet$, $\blacksquare$). The present paper refers to the data as the perturbation 3 which consists of displacements including this perturbation and the perturbation 2 stresses.

3. Analytical results
3.1 The results obtained by the singular value decomposition with no artificial noise

Using the stresses on the whole surface and displacements at eight boundary elements dividing the bottom of the model ($N_D = 8$), the analyses are performed without any artificial noise: $\alpha = 0$. The used displacement data are just computed by the forward analysis with double precision and then are unperturbed. Figures 5(a) and (b) show the effects of the truncated rank of the sensitivity matrix $[A_{sd}]$ on $\|\{\text{Res}\}\|$ and $\|\{f\}\|$ respectively for perturbations 1 and 2.

These figures reveal that the least residual criterion is not enable us to chose the optimal rank, since $\|\{\text{Res}\}\|$ increases gradually, though $\|\{f\}\|$ decreases drastically with decreasing rank. While a value of 5.2 is calculated for the norm of the assumed BFD, the unreasonably large norms indicate violent fluctuation of the solutions $\{f\}$ at higher ranks. We could think the reason why the norms $\|\{f\}\|$ and $\|\{\text{Res}\}\|$ vary with decreasing rank in such manner as follows. The least singular value makes the solution unstable when all singular values are used to minimize the residual, so that the solution oscillates violently and its norm becomes unreasonably large. With decreasing rank, the solution becomes stable and has a smaller norm while the residual gradually increases. These two figures also show that the residual and solution norms increase slightly with perturbation.

![Residual norms](image1)

![Solution norms](image2)

**Fig.5** shows the effects of the truncated rank in the SVD on the residual and solution norms for the perturbed problems with no regularization parameter ($\alpha=0.0$). The perturbations of the stress data are simulated by the normal random numbers with zero mean and standard deviations 0.01 and 0.02 MPa. The decreasing rank increases the residual norms a little, while it decreases the solution norms drastically. These results indicate that the least residual criterion cannot be used for regularizing the rank for perturbed problems.

We will examine the results shown in Fig.5 from a different standpoint. Figure 6 shows the plots of $\|\{\text{Res}\}\|^2$ versus $\|\{f\}\|^2$ for the perturbations 1 and 2. Where, $\|\{f\}\|^2$ is a monotonically decreasing function of $\|\{\text{Res}\}\|^2$, so that the plot looks like a L-shaped curve; the L-curve (Hansen, 1992). While a distinct “corner” is not found on each L-curve, we can find out immediately that the point with maximum curvature is located near the rank 9 SVD solution from a visual inspection. Since the rank 9 SVD solution appears to be well balancing the residual and solution norms, the L-curve method is effective to choose a reasonable rank in the SVD for the BFD too. The figure also indicates that a large perturbation implies a large residual norm $\|\{\text{Res}\}\|$. The well truncated SVD solution will be compared with that regularized through artificial noise in the bellow.
Fig. 6 shows the results in Fig. 5 as the curves plotting $||\{\text{Res}\}||^2$ versus $||\{f\}||^2$ and reveals the L-shaped curves: the L-curves. The maximum curvature point of the curve is located near the rank 9 SVD solution, so that the solution is well balancing the solution and residual norms, and is well regularized. This figure indicates further that the residual norm increases with the perturbation in the stress data.

### 3.2 The effect of artificial noise on the solutions

To obtain well regularized BFD solutions, the SVD with artificial noise $\alpha$ is used in the analysis as stated in the section 2.2. We performed a series of analyses with increasing the value of $\alpha$ every $10^{-7}$ from zero to $3.50 \times 10^{-5}$, so that we can plot the points $(||\{\text{Res}\}||^2, ||\{f\}||^2)$ up to $||\{\text{Res}\}||^2 = 2.0 \times 10^{-5}$: the L-curve. The rank in the SVD is truncated to 25, since the 26th and 27th singular values are smaller than 1/100th of the 25th singular value. The used data are the stresses on the whole surface with the perturbation 1 or 2, and the unperturbed displacements at eight boundary elements ($N_p = 8$). It is expected that a good value of the parameter $\alpha$ is able to be estimated from the solutions near the corner of the L-curve.

Figure 7 compares the L-curves with the plots of the truncated SVD solutions for the perturbations 1 and 2. These figures reveal that artificial noise (AN) well regularizes the perturbed problems so that the L-curves are located in the lower left of the SVD solutions. Next, we seek the maximum curvature point of the L-curve, which corresponds to a good artificial noise $\alpha$. We repeat the piecewise approximation of the L-curve by second-order polynomial based on the incremental step method. The second-order polynomial is fitted to sets of $(2N+1)$ successive points $(||\{\text{Res}\}||^2, ||\{f\}||^2)$, where $N$ is 50 in the present study. We then calculate curvature at the position of the 51st point, and find the maximum curvature point. The maximum curvature point is indicated by an arrow in the figures and corresponds to the artificial noise $\alpha$ with a value of $6.10 \times 10^{-5}$ regardless of perturbations. It may be thought that this agreement is due to the fact that the influence of the difference between perturbations 1 and 2 on the solution is smaller than that of the interval $\Delta\alpha = 1.0 \times 10^{-7}$ of artificial noise.

Fig. 7 indicates that artificial noise $\alpha$ (AN) regularizes the perturbed problems so that the L-curves are located in the lower left of the SVD solutions. The appropriate regularization parameter $\alpha$ is chosen to be $6.10 \times 10^{-5}$ regardless of perturbations, which is determined from the maximum curvature points of the two L-curves.
Next, we will examine the fluctuations of solutions with increasing artificial noise $\alpha$. For example, Fig. 8 shows how the distribution functions $f_1$ and $f_3$ at nodes 1 to 9 in $V_6$ change with increasing $\alpha$ from $1.00 \times 10^{-3}$ (△) to $3.425 \times 10^{-4}$ (◇) for the perturbation 2. A value of $6.10 \times 10^{-5}$ for $\alpha$ (○) is corresponding to the corner of the L-curve as shown in Fig. 7(b). While we do not show the distribution functions $f_1$ and $f_3$ for $\alpha=1.0 \times 10^{-4}$ here, we have obtained violently oscillating functions. Figure 8 also reveals that BFD distributions are smoothed with increasing artificial noise $\alpha$, and that solutions change slightly and become stable for the artificial noise larger than $6.10 \times 10^{-5}$, corresponding to the corner of the L-curve.

![Figure 8](image)

**Fig.8** shows how the distribution functions $f_1$ and $f_3$ change at each node in $V_6$ with increasing artificial noise from $1.00 \times 10^{-3}$ (△) to $3.425 \times 10^{-4}$ (◇) for the perturbation 2, for example. A value of $6.10 \times 10^{-5}$ for $\alpha$ (○) is corresponding to the corner of the L-curve as shown in Fig. 7(b). In addition, this figure reveals that BFD distributions are smoothed with increasing artificial noise $\alpha$.

### 3.3 The effect of displacement data on the regularized solutions

In addition to the surface stresses, some displacements are used as the boundary conditions to guarantee the uniqueness of the residual stress (Kubo, 1992) in the present study. So, we will examine the effect of the additional displacements and their perturbations on the regularized solutions, and the L-curves.

Figure 9 shows the L-curves based on the stress data alone ($N_D=0$) and both data ($N_D=32$) for the perturbation 2. In the latter case ($N_D=32$), every displacement is used as the additional boundary conditions and contains no any perturbation error. In this figure, the results for $N_D=32$ (○) were thinned and plotted so that the solid line can be seen. This figure demonstrates a very small effect of the displacements on the regularized solutions, since the two curves agree almost completely.

![Figure 9](image)

**Fig.9** indicates that the L-curve based on the stress and the additional displacement data at all boundary elements ($N_D=32$) agrees well with that obtained by the stress data alone ($N_D=0$) for the perturbation 2. This agreement demonstrates that the additional displacements do not have influence on the BFD solution at all for the present case.
We will then examine how the perturbation of the displacement affects the regularized solution obtained by using the displacements at all boundary elements \( N_D =32 \). Figure 10 compares the L-curve for the perturbation 2 with that for the perturbation 3, and indicates that the perturbation of the displacements has no significant influence on the regularized solutions. The results for the perturbation 3 (○) were thinned and plotted in this figure too.

Fig.10 shows that the L-curves agree with each other even if the additional displacements contain the random error (perturbation 3) or not (perturbation 2) under the same perturbation of the stress, so that the perturbation of the additional displacements has no significant effect on the regularized solutions for the present case.

From Figs. 9 and 10, it would be concluded that the displacement and its perturbation have no significant effect on the regularized solutions for the present case, since the L-curves agree with each other almost completely even if the displacements were used or not, and even if they contain the perturbation or not. So, it is important to explain how the submatrix \([A_d]\) has influence on the singular values \(\lambda_i\) \((i=1,2\cdots,3N_A)\) of the sensitivity matrix \([A_{\text{sd}}]\), to make clear the cause of such a quite small effect on the L-curve: the solution. Here we examine the influence of the submatrices \([A_d]\) through a normal equation and eigenvalues, squares of the singular values, while we did not calculate the singular values from the eigenvalues. The sensitivity matrix \([A_{\text{sd}}]\) can be written in the form

\[
[A_d] = \begin{bmatrix} A_i \\ A_d \end{bmatrix} = \begin{bmatrix} A_i \\ 0_d \end{bmatrix} + \begin{bmatrix} 0_s \\ A_d \end{bmatrix},
\]

(6)

here \([0_s]\) and \([0_d]\) are the zero matrices of \(3N_S \times 3N_A\) and \(2N_D \times 3N_A\), respectively. Let \(\phi_i\) and \(\{v_i\}\) \((i=1,2\cdots,3N_A)\) represent the eigenvalues and eigenvectors belonging to \(\phi_i\). The \(i\)-th eigen equation becomes

\[
[A_{\text{sd}}]^{T} [A_{\text{sd}}] \{v_i\} = \phi_i \{v_i\}.
\]

(7)

Substituting Eq.(6) into Eq.(7) and arranging, we obtain

\[
([A_i]^{T} [A_i] + [A_d]^{T} [A_d]) \{v_i\} = \phi_i \{v_i\}.
\]

(8)

This equation suggests the following to us: if the contribution of the submatrix \([A_d]\) to the eigenvalues \(\phi_i\) is very small, the eigenvalues \(\phi_0\) of the submatrix \([A_d]\) then should be very smaller than \(\phi_i\), so that the L-curves based on the

Fig.11 shows the ratios of eigenvalues \(\phi_0/\phi_i\) arranged in big order of \(\phi_i\), here \(\phi_0\) are eigenvalues of the submatrix \([A_d]\) for the displacement and \(\phi_i\) those of the sensitivity matrix \([A_{\text{sd}}]\) for the stress and displacement. This indicates that the ratios \(\phi_0/\phi_i\) are smaller than \(1.5 \times 10^{-6}\), so that the additional displacements hardly have influence on the L-curves.
stress data alone \((N_D=0)\) and both data \((N_D=32)\) agree well each other as shown in Fig.9. Figure 11 shows the ratios of the eigenvalues \(\phi_u/\phi_i\) arranged in big order of \(\phi_i\). Since the ratios are smaller than \(1.5\times10^{-6}\), it can be concluded that the submatrix \([A_d]\) has influence fairly small on the eigenvalues \(\phi_i\) - more precisely, the singular values - so that the two L-curves agree almost completely as shown in Fig.9.

We then will examine the reason why the perturbation of the displacement has influence on the regularized solution almost nil. The normal equation is derived from Eq.(1):

\[
\left[ A_d^T \right] \left[ A_u \right] f = \left[ A_u^T \right] \left[ \sigma \right] + \left[ A_d^T \right] \left[ u \right].
\]  

(9)

As the input data have influence on the right hand side of the equation, we will compare the significance of the second term for the displacement with that of the first term for the stress, to discuss the influence of the perturbed data. The norms \(\|[A_d]T[u]\|\) and \(\|[A_i]T[\sigma]\|\) are then chosen for measuring the contribution to the input term. Table 1 compares the two norms for the perturbations 2 and 3, and reveals that the second term is negligible small in comparison with the first term. This leads us to the conclusion that the norm \(\|[A_d]T[u]\|\) is so small that the perturbations of the displacements do not affect the L-curves for the employed model and assumed perturbations in the present study.

Table 1 compares the norm of the 1st term with that of the 2nd term in the right hand side of the normal Eq.(9) derived from Eq.(1). The 1st and 2nd terms denote the contributions of the stress and displacement data respectively to the input term of Eq.(9). Since the norm of the 2nd term is too small to affect the solution, the perturbations of the displacements then make no difference between the two L-curves as shown in Fig.10.

| Perturbation | \(|[A_d]^T[u]|\) | \(|[A_u]^T[\sigma]|\) |
|--------------|-----------------|------------------|
| 2            | \(5.5285 \times 10^{-2}\) | \(1.0925 \times 10^{0}\) |
| 3            | \(5.5285 \times 10^{-2}\) | \(9.8555 \times 10^{-9}\) |

4. Conclusions

In the present study, the authors examine the applicability of the SVD with artificial noise to the BFD approach for perturbed problems. The following conclusions are obtained for the employed model and assumed perturbations.

(1) The regularization method is verified to be effective in the BFD inverse analyses for perturbed problems, which uses the singular value decomposition of the sensitivity matrix and artificial noise with a good value chosen through the L-curve method.

(2) For the effect of the additional displacements on the solution, the numerical result shows that the displacements and their perturbations affect the solution almost nil. And the theoretical examination reveals that such effect is caused by the small contribution of the submatrix for the displacement to the singular values and by the slight influence of the displacements on the input terms of the normal equation.

For future work, it is necessary to obtain useful knowledge which guarantees appropriate BFD solutions such as the distance between the BFD distributing area and the surface as well as the magnitude of the perturbation in data, before applying the BFD approach to real problems.

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