A COMPARISON BETWEEN TWO DE RHAM COMPLEXES IN DIFFEOMETRY

KATSUHIKO KURIBAYASHI

Abstract. There are two de Rham complexes in diffeology. The original one is due to Souriau and another one is the singular de Rham complex defined by a simplicial differential graded algebra. We compare the first de Rham cohomology groups of the two complexes within the Čech–de Rham spectral sequence in diffeology. In particular, a comparison map enables us to conclude that the first singular de Rham cohomology for the irrational torus $T_\theta$ is isomorphic to the direct sum of the original one and the group of equivalence classes of flow bundles over $T_\theta$ with connection 1-forms.

1. Introduction

This manuscript is a sequel to [7, Appendix C]. The de Rham complex due to Souriau [8] is very beneficial in the development of diffeology; see [2, Chapters 6, 7, 8, and 9]. In fact, the de Rham calculus is applicable to not only diffeological path spaces but also more general mapping spaces. While the complex is isomorphic to the usual de Rham complex if the input diffeological space is a manifold, the de Rham theorem does not hold in general.

Another complex called the singular de Rham complex is introduced in [7] via simplicial arguments; see [5] for a cubic de Rham complex. An advantage of the new complex is that the de Rham theorem holds for every diffeological space. Moreover, the singular de Rham complex allows us to construct Leray–Serre and Eilenberg–Moore spectral sequences in the diffeological framework; see [7, Theorems 5.4 and 5.5]. Furthermore, there exists a natural morphism $\alpha: \Omega(X) \to A(X)$ of differential graded algebras from the original de Rham complex $\Omega(X)$ due to Souriau to the new one $A(X)$ which induces an isomorphism on the cohomology provided $X$ is a manifold; see [5] and [7, Theorem 2.4].

The aim of this short manuscript is to compare the first de Rham cohomology groups for the complexes $A(X)$ and $\Omega(X)$ within the Čech–de Rham spectral sequence [3] by using the morphism $\alpha$ mentioned above; see Theorem 2.1 for more details. In particular, by a comparison map, it is shown that the first singular de Rham cohomology for the irrational torus $T_\theta$ is isomorphic to the direct sum of the original one and the group of equivalence classes of flow bundles over $T_\theta$ with connection 1-forms; see Corollary 2.2. In consequence, we see that, as an algebra, the singular de Rham cohomology $H^*(A(T_\theta))$ is isomorphic to the tensor product of the original de Rham cohomology and the exterior algebra generated by a flow bundle over $T_\theta$; see Corollary 2.3. Thus, it seems that the singular de Rham cohomology has K-theoretical information.

2010 Mathematics Subject Classification: 57P99, 55U10, 58A10.

Key words and phrases. Diffeology, Čech–de Rham spectral sequence, singular de Rham complex.

Department of Mathematical Sciences, Faculty of Science, Shinshu University, Matsumoto, Nagano 390-8621, Japan e-mail: kuri@math.shinshu-u.ac.jp
2. Main theorem

We begin by recalling the original de Rham complex. Let \( (X, D^X) \) be a diffeological space. For an open set \( U \) of \( \mathbb{R}^n \), let \( D^X(U) \) be the set of plots with \( U \) as the domain and \( \Lambda^* (U) = \{ h : U \rightarrow \Lambda^*(\mathbb{R}^n, \mathbb{R}) \mid h \text{ is smooth} \} \) the usual de Rham complex of \( U \). Let \( \text{Open} \) denote the category consisting of open sets of Euclidean spaces and smooth maps between them. We can regard \( D^X(\ ) \) and \( \Lambda^*(\ ) \) as functors from \( \text{Open}^{\text{op}} \) to \( \text{Sets} \) the category of sets. A \( p \)-form is a natural transformation from \( D^X(\ ) \) to \( \Lambda^*(\ ) \). Then the de Rham complex \( \Omega^*(X) \) is the cochain algebra of \( p \)-forms for \( p \geq 0 \); that is, \( \Omega^*(X) \) is the direct sum of the modules

\[
\Omega^p(X) := \left\{ \text{Open}^{\text{op}} \xrightarrow{D^X} \text{Sets} \mid \omega \text{ is a natural transformation} \right\}
\]

with the cochain algebra structure defined by that of \( \Lambda^*(U) \) pointwisely.

We introduce another de Rham complex for a diffeological space. Indeed, it is a diffeological counterpart of the singular de Rham complex in \([1][3][11]\). Let \( \mathbb{A}^n := \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1\} \) be the affine space equipped with the sub-diffeology of \( \mathbb{R}^{n+1} \) and \((A_{DR}^\bullet)\) the simplicial cochain algebra defined by \( (A_{DR}^\bullet)_n := \Lambda^*(\mathbb{A}^n) \) for each \( n \geq 0 \). For a diffeological space \( (X, D^X) \), let \( S^D_\bullet(X) \) denote the simplicial set defined by

\[
S^D_\bullet(X) := \{ \{ \sigma : \mathbb{A}^n \rightarrow X \mid \sigma \text{ is a } C^\infty\text{-map} \} \}_{n \geq 0}.
\]

The simplicial set and the simplicial cochain algebra \( (A_{DR}^\bullet) \) give rise to a cochain algebra

\[
\text{Sets}^{\Delta^{op}}(S^D_\bullet(X), (A_{DR}^\bullet)) := \left\{ \Delta^{op} \xrightarrow{\omega} \text{Sets} \mid \omega \text{ is a natural transformation} \right\}
\]

whose cochain algebra structure is defined by that of \( (A_{DR}^\bullet) \). In what follows, we call the complex \( A(X) := \text{Sets}^{\Delta^{op}}(S^D_\bullet(X), (A_{DR}^\bullet)) \) the singular de Rham complex of \( X \); see also \([2]\) Section 2. We define a morphism \( \alpha : \Omega(X) \rightarrow A(X) \) of cochain algebras by \( \alpha(\omega)(\sigma) = \sigma^* (\omega) \). The result \([2]\) Theorem 2.4 asserts that \( \alpha \) is a quasi-isomorphism if \( X \) is a manifold, a smooth CW-complex or a parametrized stratifold; see \([3][5]\) and \([6]\) for a smooth CW-complex and a stratifold, respectively. Moreover, the map \( \alpha \) induces a monomorphism \( H(\alpha) : H^1(\Omega(X)) \rightarrow H^1(A(X)) \) for every diffeological space \( X \); see \([7]\) Proposition 6.9). Thus one might concern the difference between the first de Rham cohomology groups. Theorem \(2.1\) below which is our main theorem relates the cohomology groups within the Čech–de Rham spectral sequence introduced below.

Let \( \mathcal{G}_X \) be the generating family of the diffeology \( D^X \) consisting of all plots whose domains are open balls in Euclidean spaces. We assume that \( \mathcal{G}_X \) contains the set \( C^\infty(\mathbb{R}^0, X) \); see \([2]\) 1.76. Then we define the nebula \( \mathcal{N}_X \) of \( X \) associated with \( \mathcal{G}_X \) by

\[
\mathcal{N}_X := \coprod_{\varphi \in \mathcal{G}_X} \{ \{ \varphi \} \times \text{dom}(\varphi) \}
\]

with sum diffeology, where \( \text{dom}(\varphi) \) denotes the domain of the plot \( \varphi \). We may write \( \mathcal{N}(\mathcal{G}_X) \) for \( \mathcal{N}_X \) when expressing the generating family. It is readily seen that the evaluation map \( ev : \mathcal{N}_X \rightarrow X \) defined by \( ev(\varphi, r) = \varphi(r) \) is smooth. The gauge
monoid \( M_X \) is a submonoid of the monoid of endomorphisms on the nebula \( N_X \) defined by

\[
M_X := \{ f \in C^\infty(N_X, N_X) \mid ev \circ f = ev \text{ and } \# \text{Supp } f < \infty \},
\]

where \( \text{Supp } f := \{ \varphi \in \mathcal{G} \mid f[\{\varphi\} \times \text{dom}(\varphi)] \neq 1_{\{\varphi\} \times \text{dom}(\varphi)} \} \). In what follows, we denote the monoid \( M_X \) by \( M \) if the underlying diffeological space is clear form the context.

The original de Rham complex \( \Omega^\ast(N_X) \) is a left \( \mathbb{M}^{\text{op}} \)-module whose actions are defined by \( f^\ast \) induced by endomorphisms \( f \in N_X \). Moreover, the complex \( \Omega^\ast(N_X) \) is regarded as a two sided \( \mathbb{M}^{\text{op}} \)-module for which the right module structure is trivial. Then we have the Hochschild complex \( C_r^r = \{ C^p,q, \delta, d_0 \}_{p,q \geq 0} \) with

\[
C^p,q = \text{Hom}_{\mathbb{R} \mathbb{M}^{\text{op}} \otimes \mathbb{R} \mathbb{M}^{\text{op}}}((\mathbb{R} \mathbb{M}^{\text{op}})^{\otimes p} \otimes \mathbb{R} \mathbb{M}, \Omega^q(N_X)) \cong \text{map}(\mathbb{M}^p, \Omega^q(N_X)),
\]

where the horizontal map \( \delta \) is the Hochschild differential and the vertical map \( d_0 \) is induced by the de Rham differential on \( \Omega^\ast(N_X) \); see [3] Subsection 8. The horizontal filtration \( F_r^\ast = \{ F_j \}_{j \geq 0} \) defined by \( F_j = \oplus_{q \geq j} C_r^r \) of the total complex \( \text{Tot } C_r^r \) gives rise to a first quadrant spectral sequence \( \{ \Omega E_r^r, d_r \} \) converging to the Čech cohomology \( \hat{H}(X) := HH^\ast(\mathbb{R} \mathbb{M}, \text{map}(\mathcal{G}, \mathbb{R})) \) with

\[
E_2^{p,q} \cong HH^q(\mathbb{R} \mathbb{M}^{\text{op}}, \Omega^p(N_X), d_0),
\]

where \( HH^\ast(\cdot) \) denotes the Hochschild cohomology; see [3] Subsections 9 and 16]. Observe that the differential \( d_r \) is of bidegree \((1 - r, r)\). This spectral sequence is called the Čech–de Rham spectral sequence.

The same construction as that of the spectral sequence above is applicable to the singular de Rham complex \( A(X) \). Then replacing the original de Rham complex \( \Omega(-) \) with \( A(-) \), we have a spectral sequence \( \{ A E_r^r, d_r \} \). Since the Poincaré lemma holds for the complex \( A(-) \), it follows that the target of the spectral sequence for \( A(X) \) is also the Čech cohomology \( \hat{H}(X) \). Thus the naturality of the map \( \alpha : A(X) \to \Omega(X) \) gives rise to a commutative diagram of isomorphisms

\[
\begin{array}{ccc}
H^1(\Omega(X)) \oplus_A E_3^{0,0} & \xrightarrow{\Theta} & H^1(A(N_X)^{\mathbb{M}}) \oplus_A E_3^{1,0} \\
\begin{array}{c}
\text{edge}_2 \\
\text{edge}_2
\end{array} & \cong & \begin{array}{c}
\hat{H}^1(X; \mathbb{R}) \\
\hat{H}^1(X; \mathbb{R})
\end{array}
\end{array}
\]

In fact, the edge map \( \text{edge}_1 := ev^* : H^\ast(\Omega(X)) \to A E_2^{0,0} = H^1(\Omega(N_X)^{\mathbb{M}}) \) induced by the evaluation map \( ev : X \to N_X \) is an isomorphism; see [3] 6. Proposition]. Moreover, the morphism \( \alpha : \Omega(X) \to A(X) \) of cochain algebras induces a map \( \hat{H}(\text{Tot}(\alpha)) \) between the total complexes which define the spectral sequences above. Thus the naturality of the map \( \alpha \) enables us to obtain a commutative diagram (2.1)

\[
\begin{array}{ccc}
H^\ast(\Omega(X)) \xrightarrow{f(\alpha)_2} H^\ast(A(N_X)^{\mathbb{M}}) & \xrightarrow{f(\alpha)_\infty} & H^\ast(\text{Tot } C_r^r) \\
\begin{array}{c}
\hat{H}^\ast(X),
\end{array} & \cong & \begin{array}{c}
\hat{H}^\ast(X),
\end{array}
\end{array}
\]

By the degree reasons, we see that the surjective maps \( K E_2^{0,1} \to K E_\infty^{0,1} \) are isomorphisms and \( K E_3^{1,0} \cong K E_\infty^{1,0} \) for \( K = \Omega \) and \( A \). Thus the map \( H^\ast(\text{Tot}(\alpha)) \) yields the homomorphism \( \Theta \) which fits in the triangle. In consequence, \( \Theta \) is an isomorphism.
In a particular case where a diffeological space $X$ appears as the base space of a diffeological fibration; see [2, Chapter 5], we relate $H^1(\Omega(X))$ to $H^1(A(X))$ in the Čech–de Rham spectral sequence.

**Theorem 2.1.** Let $X$ be a connected diffeological space which admits a diffeological fibration of the form $F \to M \xrightarrow{\pi} X$ in which $M$ is a connected manifold and $F$ is connected diffeological space with $H^1(A(F)) = 0$. Then the edge map edge$_1 := ev^*: H^1(A(X)) \to H^1(A(N_X)^M) = A\oplus\delta(X)$ is a monomorphism for $i = 1, 2$. Moreover, the restriction of the map $\Theta$ mentioned above to the cohomology $H^1(\Omega(X))$ is the composite of the monomorphism $H(\alpha) : H^1(\Omega(X)) \to H^1(A(X))$ and the edge map edge$_1$.

Before describing corollaries, we recall results on principal $\mathbb{R}$-bundles (flow bundles) in [3]. For a diffeological space $X$, we consider a Hochschild cocycle $\tau : M \to \Omega^0(N_X) = C^\infty(N_X, \mathbb{R})$ in $Z^1_\delta$. Then an $M$-action $A_\tau$ on $N_X \times \mathbb{R}$ is defined by $A_\tau(b, s) = (A(b), s + \tau(A)(b))$. The action gives rise to a principal $\mathbb{R}$-bundle of the form $Y_\tau := N_X \times_\tau \mathbb{R} \to N_X/M \cong N_X/ev \cong X$ over $X$, where $Y_\tau$ is the quotient space of $N_X \times \mathbb{R}$ by the $M$-action. More precisely, the equivalence relation is generated by the binary relation which the $M$-action induces. Observe that the second diffeomorphism is given by the evaluation map $ev : N_X \to X$.

Let $Fl(X)$ be the abelian group of equivalence classes of flow bundles. The sum is given by the quotient of the direct sum of two flow bundles by the anti diagonal action of $\mathbb{R}$; see [3, Proposition 2]. Then a map $\Omega E_1^{1,0} \to Fl(X)$ defined by assigning the equivalence class of the flow bundle $Y_\tau \to X$ to $[\tau]$ is an isomorphism. Moreover, we see that $\Omega E_2^{1,0} = \text{Ker}(d_1 : \Omega E_1^{1,0} \to \Omega E_1^{2,0})$ is isomorphic to $Fl^*(X)$ the subgroup of $Fl(X)$ consisting of equivalence classes of flow bundles over $X$ with connection 1-forms; see also [2, 8.37].

Thanks to the injectivity of the edge map in Theorem 2.1 and a result on flow bundles due to Iglesias-Zemmour mentioned above, we have

**Corollary 2.2.** Let $T_0$ be the irrational torus. Then the map $\Theta$ in Theorem 2.1 gives rise to an isomorphism $\Theta : H^1(\Omega(T_0)) \oplus Fl^*(T_0) \xrightarrow{\cong} H^1(A(T_0)).$

We recall the diffeomorphism $\psi : \mathbb{R}/(\mathbb{Z} + \theta \mathbb{Z}) \to T_0$ defined by $\psi(t) = (0, e^{2\pi i t})$ in [2, Exercise 31], [3]). Then there exist isomorphisms $\Omega(T_0) \cong \Omega(\mathbb{R}/(\mathbb{Z} + \theta \mathbb{Z})) \cong (\wedge^d(\mathbb{R}), d \equiv 0)$ which are induced by $\psi$ and the subduction $\mathbb{R} \to \mathbb{R}/(\mathbb{Z} + \theta \mathbb{Z})$, respectively; see [2, Exercise 119]. On the other hand, we see that $H^*(A(T_0)) \cong \wedge(t_1, t_2)$ as an algebra, where $\deg t_i = 1$; see the proof of Corollary 2.2. Thus the corollary above yields the following result.

**Corollary 2.3.** One has $H^*(A(T_0)) \cong (\wedge(\Theta(t), \Theta(\xi)))$ as an algebra, where $t \in H^*(\Omega(T_0)) \cong \wedge(t)$ is a generator and $\xi \in Fl^*(T_0)$ is a flow bundle over $T_0$ with a connection 1-form, which is a generator of the group $Fl^*(T_0)$.

3. **Proofs of Theorem 2.1 and Corollary 2.2**

We begin by considering invariant differential forms on nebulae of diffeological spaces.

**Lemma 3.1.** Let $\pi : Y \to X$ be a subduction and $\mathcal{G}_Y$ a generating family of $Y$. Then the map $\pi^*: A(N_X) \to A(N_Y)$ induced by $\pi$ gives rise to a map $\pi^*: A(N_X)^M \to A(N_Y)^M$, where the nebula $N_X$ is defined by the generating family $\pi_*\mathcal{G}_Y := \{\pi \circ \phi : \phi \in \mathcal{G}_Y\}$ induced by $\mathcal{G}_Y$. 


Proof: For \( \omega \in A^*(N_X)^{M_X} \) and \( \eta \in M_Y \), we show that \( \eta \cdot \pi^*(\omega) = \pi^*(\omega) \). Let \( \sigma: A^n \to N_Y \) be an element in \( S^E_n(N_X) \), namely a smooth map from \( A^n \). Since \( A^n \) is connected, it follows that the image of \( \sigma \) is contained in a component \( \{ \phi \} \times \text{dom}(\phi) \) of \( N_Y \). We define a smooth map \( \eta: N_X \to N_X \) by \( \eta(\pi \circ \phi, u) = (\pi \circ \phi', \eta(u)) \) and by the identity maps in other components, where \( \eta(\phi, u) = (\phi', \eta(u)) \). Since \( \phi(u) = ev(\eta(\phi, u)) = ev(\eta(\phi', \eta(u))) = \eta' \eta(u) \), it follows that \( ev \circ \eta = \eta \) and hence \( \eta \in M_X \). Observe that \( \pi \circ \eta \circ \sigma = \eta \circ \pi \circ \sigma \). Thus we see that \( (\eta \cdot \pi^*(\omega))(\sigma) = \pi^*(\omega)(\eta \circ \sigma) = \omega(\pi \circ \eta \circ \sigma) = \omega(\eta \circ \pi \circ \sigma) = (\eta \cdot \omega)(\pi \circ \sigma) = \pi^*(\omega)(\sigma) \). This completes the proof.

Under the assumption in Theorem 2.1, we have a commutative diagram

\[
\begin{array}{ccc}
H^*(A(X)) & \xrightarrow{\text{ev}^* = \text{edge}} & H^*(A(N_X)^{M_X}) = A E_2^{0, *}(X) \\
H^*(A(M)) & \xrightarrow{\text{ev}^*} & H^*(A(N_M)^{M_M}) = A E_2^{0, *}(M)
\end{array}
\]

\[
\begin{array}{c}
\pi^* \\
\approx \\
\approx
\end{array}
\]

\[
\begin{array}{c}
f(\alpha)_2
\end{array}
\]

\[
\begin{array}{c}
\sim
\end{array}
\]

Here, in constructing the spectral sequences, we use the generating family \( G_M \) of \( M \) consisting of all plots whose domains are open balls in Euclidian spaces.

(I) The injectivity of \( H^*(A(\pi)) \): Let \( \{ L S E^{i, *}, d_r \} \) be the Leray–Serre spectral sequence \( \{ L S E^{i, *}, d_r \} \) for the bundle \( F \to M \to X \) in [2, Theorem 5.4]. By assumption, the first cohomology \( H^1(A(F)) \) is trivial. This yields that the edge homomorphism \( H^i(A(X)) \to L S E^{i, 0}_2 \equiv L S E^{i, 0}_\infty \to H^i(A(M)) \) is injective for \( i = 1, 2 \). The edge homomorphism is nothing but the map \( H^i(A(\pi)) \). This follows from one of the properties of the Leray–Serre spectral sequence.

(II) The injectivity of \( f(\alpha)_2 \): Recall the commutative diagram (2.1). By the degree reasons, we see that the elements in \( \Omega E_2^{0, i} \) are non-exact. Since \( M \) is a manifold, it follows from the argument in [3, Section 20] that \( \Omega E_2^{1, 0} \) is trivial and then each element in \( \Omega E_2^{0, 2} \) is also non-exact. This yields that the upper-left hand side surjective map in (2.1) is bijective. It turn out that the map \( f(\alpha)_2 \) is injective for \( * = 1, 2 \).

Proof of Theorem 2.1 Consider the commutative diagram (3.1). The injectivity of the maps described in (I) and (II) implies the result. The latter half of the assertion follows from the commutativity of the left square in the diagram (2.1).

Before proving Corollary 2.2 we recall a result on the Čech cohomology of a diffeological torus. Let \( T_K \) be a diffeological torus, namely a quotient \( \mathbb{R}^n / K \) endowed with the quotient diffeology, where \( K \) is a discrete subgroup of \( \mathbb{R}^n \).

Proposition 3.2. ([3, Corollary]) One has an isomorphism \( H^*(T_K, \mathbb{R}) \cong H^*(K; \mathbb{R}) \). Here \( H^*(K; \mathbb{R}) \) denotes the ordinary cohomology of group \( K \).

Proof of Corollary 2.2 Let \( T_\theta \) be the irrational torus. By definition, \( T_\theta \) is the diffeological space \( T^2 / S_\theta \) endowed with the quotient diffeology, where \( S_\theta \) is the subgroup \( \{ e^{2\pi ti}, e^{2\pi i\theta t} \in T^2 \mid t \in \mathbb{R} \} \) which is diffeomorphic to \( \mathbb{R} \) as a Lie group. Then we have a principal \( \mathbb{R} \)-bundle of the form \( \mathbb{R} \to T^2 \to T_\theta \) which is a diffeological fibration. Then the Leray-Seere spectral sequence enables us to conclude that
\[ H^*(A(T_0)) \cong H^*(A(T^2)) \cong H^*(\Omega(T^2)) \cong \wedge(t_1, t_2), \text{where } \deg t_i = 1. \text{ In particular,} \]
\[ H^1(A(T_0)) \cong \mathbb{R} \oplus \mathbb{R}. \]

Moreover, by virtue of Theorem 2.1 we see that the map edge \( H^1(A(T_0)) \to A^E_{2,1} \) is a monomorphism. Since \( T_0 \) is isomorphic to a diffeological torus of the form \( \mathbb{R}/(\mathbb{Z} + \theta \mathbb{Z}); \text{ see [2 Exercise 31, 3])}, \) it follows from Proposition 3.2 that \( H^*(T_0, \mathbb{R}) \cong H^*(\mathbb{Z} + \theta \mathbb{Z}; \mathbb{R}) \). This yields that \( A^E_{2,1} \oplus A^E_{3,0} \cong H^1(T_0, \mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}. \)

The injectivity of the edge map above implies that \( A^E_{3,0}(T_0) = 0 \) and hence the map \( \Theta \) induces a diffeomorphism \( H^1(\Omega(T_0)) \oplus \Omega E^1_3 \cong H^1(A(T_0)). \) It follows from [3 Section 19] that \( \Omega E^1_2 = \mathcal{F}_*(T_0) \). Furthermore, we have \( H^2(\Omega(T_0)) = 0; \) see [2 Exercise 119]. It turns out that \( \Omega E^1_2 \cong \Omega E^1_2 \). We have the result. \( \square \)

4. From the second singular de Rham cohomology to the Čech cohomology

We define the edge map \( H^i(A(X)) \to \hat{H}^i(X) \) by the composite of the maps in the lower sequence in (2.1). For degree reasons, we see that each element in \( A^E_{2,1} \) the \( E_2 \)-term of the Čech-de Rham spectral sequence is non exact. Then, the edge map \( H^i(A(X)) \to \hat{H}^i(X) \) is injective under the same assumption as in Theorem 2.1. In order to consider the edge map in degree 2, we generalize Lemma 3.1 introducing a generating family of a multi-set. Let \( \pi : Y \to X \) be a subduction and \( G_Y \) a generating family of \( Y \). We define \( G_X^{\text{multi}} \) by the multi-set \( \coprod_{\phi \in G_Y} \{ \pi \circ \phi \} \).

**Proposition 4.1.** Under the same assumption as in Theorem 2.1, the edge map \( \hat{H}^2(A(X)) \to \hat{H}^2(X) \) is injective, where \( \hat{H}^2(X) \) is the Čech cohomology associated with \( G_X^{\text{multi}} \).

**Remark 4.2.** In the proof of [3 Proposition in 
5], we need the condition (*) for a generating family \( G_X \) that for any plot \( P : U \to X \) and each \( r \in U \), there exists a plot \( q : B \to Y \) in \( G_X \) such that \( q = P|_B \). To this end, we have chosen the generating family \( G_Y \) consisting of all plots whose domains are open balls in Euclidian spaces. Let \( G_Y^{\text{multi}} \) be the generating multi-family associated with \( G_Y \). Then \( G_Y^{\text{multi}} \) also satisfies the condition (*) above. We observe that the inclusion \( \pi_* G_Y \to G_Y^{\text{multi}} \) induces a diffeomorphism \( \mathcal{N}(\pi_* G_Y)/\mathcal{ev} \cong \mathcal{N}(G_Y^{\text{multi}})/\mathcal{ev} \) between nebulae and hence the evaluation map gives rise to a diffeomorphism \( \mathcal{N}(G_Y^{\text{multi}})/\mathcal{ev} \cong \hat{X} \); see [2 1.76].

With the notation Remark 4.2 for a map in the monoid \( M_Y \), we define \( \eta(\pi \circ \phi, r) = (\pi \circ \psi, \eta(r)) \), where \( \eta(\phi, r) = (\psi, \eta(r)) \). Then we have a morphism \( \pi' : M_Y \to M_X \) of monoids defined by \( \pi'(\eta) = \eta \). Moreover, we define \( \tilde{\pi} : C_X^{\text{multi}} := \text{map}(M_X, K^Q(N_X)) \to \text{map}(M_Y, K^Q(N_Y)) =: C_Y^{\text{multi}} \) for \( K = \Omega \) and \( A \) by \( \tilde{\pi}(\phi)(\eta_1, ..., \eta_p) = \pi^*(\phi(\eta_1, ..., \eta_p)) \). The straightforward calculation shows that \( \tilde{\pi} \) is compatible with the differentials \( d_\Omega, d_A \) and the Hochschild differential \( \delta \). Thus we have

**Proposition 4.3.** The map \( \tilde{\pi} \) induces a morphism of spectral sequences \( \{ f(\tilde{\pi}) \} : \{ K E_{r,*}(X), d_r \} \to \{ K E_{r,*}(Y), d_r \} \) for \( K = \Omega \) and \( A \).

We are ready to prove the main result in this section.

**Proof of Proposition 4.3.** Suppose that there exists a non-zero element \( x \) in the kernel of the map edge \( H^2(A(X)) \to \hat{H}^2(X) \). We recall the commutative diagram (3.1). For the map \( \pi \) in the right hand side, we see that \( \pi^* = f(\tilde{\pi})^{\text{multi}} \). This
follows from the construction the morphism \( \{ f(\overline{\pi})_r \} \) of the spectral sequence for the singular de Rham complex in Proposition 4.3. The arguments in (I) and (II) before the proof of Theorem 2.1 enable us to deduce that \( \text{ev}^*(x) \in A_{E_0}^{0,2}(X) \) and \( f(\overline{\pi})_2(\text{ev}^*(x)) \in A_{E_2}^{0,2}(M) \) are non-zero elements. Since \( x \) is in the kernel, it follows that \( \text{ev}^*(x) \) is a \( d_2 \)-exact element. The naturality of \( f(\overline{\pi})_2 \) implies that \( f(\overline{\pi})_2(\text{ev}^*(x)) \) is also \( d_2 \)-exact. Then, the commutativity of the diagram (2.1) obtained by replacing \( X \) with \( M \) implies that the non-zero element \( (\text{ev}^* \circ H(\alpha))^{-1} \circ H^*(A(\pi))(x) \) in \( \Omega_{E_0}^{0,2}(M) \) is \( d_2 \)-exact. For degree reasons, we see that \( d_1^{1,0} \) is nontrivial and then so is \( \Omega_{E_2}^{1,0} \). On the other hand, since \( M \) is a manifold, it follows that \( \Omega_{E_1}^{1,0}(M) \cong \text{FL}(M) = 0 \). In fact, the fibre \( R \) of a flow bundle is contractible and then the bundle admits a smooth global section; see [9, 6.7 Theorem] for differentiable approximation of a section. Thus, we have \( \Omega_{E_1}^{1,0} = \text{Ker} \{ \Delta \} : \Omega_{E_1}^{1,0}(M) \rightarrow \Omega_{E_1}^{1,1}(M) \} = 0 \), which is a contradiction. This completes the proof.

\begin{thebibliography}{11}

[1] A.K. Bousfield and V.K.A.M. Gugenheim, On PL de Rham theory and rational homotopy type, Memoirs of AMS 179 (1976).

[2] P. Iglesias-Zemmour, Diffeology, Mathematical Surveys and Monographs, 185, AMS, Providence, 2012.

[3] P. Iglesias-Zemmour, Čech–de Rham bicomplex in diffeology, preprint, 2019.

[4] N. Iwase, Whitney approximation for smooth CW complex, preprint, 2019. \texttt{arXiv:2001.02893v2} [math.AT]

[5] N. Iwase and N. Izumida, Mayer–Vietoris sequence for differentiable/diffeological spaces, Algebraic Topology and Related Topics, Editors: M. Singh, Y. Song and J. Wu, Birkhäuser Basel (2019), 123–151. \texttt{arXiv:1511.06948v2} [math.AT]

[6] M. Kreck, Differential Algebraic Topology, From Stratifolds to Exotic Spheres, Graduate Studies in Math.110, AMS, 2010.

[7] K. Kuribayashi, Simplicial cochain algebras for diffeological spaces, preprint, 2019. \texttt{arXiv:1902.10897v5} [math.AT]

[8] J.-M. Souriau, Groupes différentiels, Lecture Notes in Math., 836, Springer, 1980, 91–128.

[9] N. Steenrod, The Topology of Fibre Bundles, Princeton Mathematical Series, vol. 14, Princeton University Press, Princeton, N. J., 1951.

[10] D. Sullivan, Infinitesimal computations in topology, Publications mathématiques de l’I.H.É.S., tome 47 (1977), 269–331.

[11] R.G. Swan, Thom’s theory of differential forms on simplicial sets, Topology 14 (1975), 271–273.

\end{thebibliography}