Direct derivation of the Veneziano-Yankielowicz superpotential from matrix model

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Abstract

We derive the Veneziano-Yankielowicz superpotential directly from the matrix model by fixing the measure precisely. The essential requirement here is that the effective superpotential of the matrix model corresponding to the $\mathcal{N}=4$ supersymmetric Yang-Mills theory vanishes except for the tree gauge kinetic term. Thus we clarify the reason why the matrix model reproduces the Veneziano-Yankielowicz superpotential correctly in the Dijkgraaf-Vafa theory.

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1 Introduction

It has been revealed that the connection between gauge theory and a matrix model is deep and interesting. In particular, the large-$N$ reduced model [1] is not only useful because it reduces the dynamical degrees of freedom and thus makes the large-$N$ gauge theory tractable, but it would provide possibly a constructive definition for a gauge theory, or even string theory [2]. For $\mathcal{N} = 1$ supersymmetric gauge theory, Dijkgraaf and Vafa proposed that a simple matrix model also has enough information on the F-term of the effective superpotential [3]. More precisely, in the $\mathcal{N} = 1$ supersymmetric gauge theory coupled to a chiral superfield $\Phi$ in the adjoint representation with a superpotential

$$W(\Phi) = \sum_{k=0}^{n} \frac{g_k}{k+1} \Phi^{k+1},$$

the prepotential $\mathcal{F}(S, g_k)$ ($S = \frac{1}{64\pi^2} \text{tr} W^\alpha W_\alpha$) is equivalent to the free energy $F_m(g_m, g_k)$ of a one-matrix model

$$S_m = \frac{\hat{N}}{g_m} \text{Tr} W(\Phi),$$

in the large-$\hat{N}$ limit under an identification $S = g_m$.

The proofs of their proposal are given in [4, 5]. In particular, it is shown in [5] by using the Konishi anomaly [6] that the Schwinger-Dyson equation for

$$\frac{1}{64\pi^2} \langle \text{tr} \left( \frac{W^\alpha W_\alpha}{z-\Phi} \right) \rangle$$

is exactly the same as that for the resolvent of the matrix model $\frac{g_m}{\hat{N}} \langle \text{Tr} \left( \frac{1}{z-\Phi} \right) \rangle$ in the large-$\hat{N}$ limit. Because the former and the latter is given by $\partial F / \partial g_k$ and $\partial F_m / \partial g_k$ respectively, we find that the prepotential $\mathcal{F}$ and the free energy of the matrix model $F_m$ are equivalent up to a function independent of $g_k$'s. However, as noted in [7, 3, 5], the matrix model produces a stronger result than the above consideration. By taking the superpotential $W = m\Phi^2/2$ and under a suitable identification between the matrix model measure and the gauge theory cutoff, $F_m$ can also reproduce the $g_k$-independent part of $\mathcal{F}$ that corresponds to the Veneziano-Yankielowicz (VY) superpotential [8]

$$S \left[ \log \left( \frac{\Lambda^{3\hat{N}}}{S^{\hat{N}}} \right) + N \right],$$

where $\Lambda$ is the dimensional transmutation scale associated with the gauge dynamics. In this sense, the connection between the $\mathcal{N} = 1$ gauge theory and the matrix model seems deeper than we have expected.

\[1\] Here we have assumed that there is no gauge symmetry breaking.
In \cite{9}, it is shown that the Dijkgraaf-Vafa theory can be regarded as the large-$N$ reduction. This enables us to construct a direct map between correlators in the gauge theory and those in the matrix model and thus to show directly that equalities hold between them. From this point of view, it must be possible to find the origin of the VY superpotential in the matrix model, because we have a direct map between the gauge theory and the matrix model including the gauge field degrees of freedom.

In this paper, we show that the matrix model indeed has an information on the pure gauge field degrees of freedom and that it can reproduce the VY superpotential. In particular, by a matrix model consideration we can derive exactly the key identification mentioned above between the measure in the matrix model and the cutoff in the gauge theory, which is just assumed in \cite{5}. Evidently in order to do this, it is indispensable to fix the measure in the matrix model. We do this by requiring that the free energy of the matrix model corresponding to the $\mathcal{N}=4$ supersymmetric gauge theory must vanish except for a term that corresponds to the tree gauge kinetic term. It is quite natural to fix the $g_k$-independent part of the free energy in this way, because it is well-known that the $\mathcal{N}=4$ gauge theory is a finite theory and does not have any quantum corrections to the holomorphic part of the effective Lagrangian \cite{10}. Then we clarify the reason why the matrix model also reproduces the pure gauge contribution to the prepotential from the point of view of the large-$N$ reduction.\footnote{Derivations of the VY superpotential from the field theory point of view in the context of the Dijkgraaf-Vafa theory are given, for example, in \cite{11,12}. In the former, it is derived by introducing fundamental matters to the $\mathcal{N}=1$ gauge theory, while in the latter it is done by invoking the $\mathcal{N}=4$ theory.}

In section 2 we determine the measure in the matrix model based on the above idea. Using this measure, we derive the VY superpotential in section 3. Section 4 is devoted to conclusions. In the appendix we present the derivation of the VY superpotential in the case of the broken gauge symmetry as an application of our approach.

\section{Determination of the measure in the matrix model}

In this section we determine the measure in the matrix model according to our requirement mentioned in the introduction.

We begin with an $\mathcal{N}=1$ supersymmetric $U(N)$ gauge theory coupled to three chiral
multiplets $\Phi_i \ (i = 1 \sim 3)$ in the adjoint representation with the following potential:

\[
S = \int d^4 x d^2 \theta \ 2 \pi i \tau_0 \text{tr} \left( W^\alpha W_\alpha \right) + \int d^4 x d^2 \theta \text{tr} \left( \Phi_1 [\Phi_2, \Phi_3] + W(\Phi_1) + \frac{m_2}{2} \Phi_2^2 + \frac{m_3}{2} \Phi_3^2 \right) + \int d^4 x d^2 \theta d^2 \bar{\theta} \sum_{i=1}^{3} \text{tr} \left( e^{-V} \Phi_i e^V \Phi_i \right) + c.c.,
\]

(2.1)

where $W(\Phi)$ is the superpotential given in (1.1). If we take $W(\Phi_1) = \frac{m_1}{2} \Phi_1^2$, that is, $g_1 = m_1$ and $g_k = 0$ for $k \geq 2$, this theory is nothing but what is called the $\mathcal{N} = 1^*$ theory, which becomes in the limit $m_i \to 0 \ (i = 1 \sim 3)$ the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) in terms of $\mathcal{N} = 1$ superfields. On the other hand, if we take $m_2 = m_3 = \Lambda_0 \gg m_1$, at a scale below $\Lambda_0$, $\Phi_2$ and $\Phi_3$ are decoupled and the theory becomes the $\mathcal{N} = 1$ supersymmetric gauge theory coupled to the chiral multiplet $\Phi_1$ with the superpotential $W(\Phi_1)$.

In Dijkgraaf-Vafa theory, as far as the holomorphic part of the effective action is concerned, we can drop the kinetic terms for the vector and the chiral multiplets and have only to consider a matrix model corresponding to the superpotential

\[
S_m = \frac{\hat{N}}{g_m} \text{Tr} \left( \Phi_1 [\Phi_2, \Phi_3] + W(\Phi_1) + \frac{m_2}{2} \Phi_2^2 + \frac{m_3}{2} \Phi_3^2 \right),
\]

(2.2)

where $\Phi_i \ (i = 1 \sim 3)$ is an $\hat{N} \times \hat{N}$ Hermitian matrix. The free energy $F_m$ of this matrix model is defined by

\[
e^{-\frac{s^2}{g_m} F_m} = C \int d\Phi_1 d\Phi_2 d\Phi_3 e^{-S_m},
\]

(2.3)

where $d\Phi_i$ is the standard measure, and $C$ is an appropriate measure factor.

In order to determine $C$, we consider in particular the matrix model corresponding to the $\mathcal{N} = 1^*$ theory

\[
S_{\mathcal{N}=1^*} = \frac{\hat{N}}{g_m} \text{Tr} \left( \Phi_1 [\Phi_2, \Phi_3] + \frac{m_1}{2} \Phi_1^2 + \frac{m_2}{2} \Phi_2^2 + \frac{m_3}{2} \Phi_3^2 \right).
\]

(2.4)

Its free energy is given as in (2.3) by

\[
e^{-\frac{s^2}{g_m} F_{\mathcal{N}=1^*}} = Z_{\mathcal{N}=1^*} = C \int d\Phi_1 d\Phi_2 d\Phi_3 e^{-S_{\mathcal{N}=1^*}}.
\]

(2.5)

We then use the fact that the holomorphic part of the effective Lagrangian in $\mathcal{N} = 4$ SYM is given simply by the tree gauge kinetic term:

\[
\mathcal{F}_{\mathcal{N}=4} = \frac{\pi i \tau_0}{N} S^2, \quad W_{\mathcal{N}=4}^{\text{eff}} = N \frac{\partial \mathcal{F}_{\mathcal{N}=4}}{\partial S} = 2 \pi i \tau_0 S,
\]

(2.6)

\[\text{We will concentrate on the SU}(N) part for simplicity.\]
where \( \tau_0 \) is the bare coupling constant. Identifying \( F_{N=4} = 4 \) and \( S \) with \( F_{N=4} = 4 \) and \( g_m \), respectively, and using \( F_{N=4} = \lim_{m_1 \to 0} F_{N=1^*} \) we have the key equation that determines \( C \):

\[
\lim_{m_1 \to 0} F_{N=1^*} = \frac{\pi i \tau_0}{N g_m^2}. \tag{2.7}
\]

Integrating out \( \Phi \) in (2.5), we obtain

\[
Z_{N=1^*} = C \left( \frac{2\pi g_m}{Nm_{1}} \right)^{\frac{S}{2}} \int d\Phi_2 d\Phi_3 e^{-S'}, \tag{2.8}
\]

\[
S' = \frac{\hat{N}}{g_m} \text{Tr} \left( -\frac{1}{2m_1} [\Phi_2, \Phi_3]^2 + \frac{m_2}{2} \Phi_2^2 + \frac{m_3}{2} \Phi_3^2 \right). \tag{2.9}
\]

Then we diagonalize \( \Phi_2 \) and set \( (\Phi_2)_{ii} = b_i \). The integration over the angular variables of \( \Phi_2 \) gives

\[
\int d\Phi_2 = J \int db_i \prod_{i>j} (b_i - b_j)^2, \tag{2.10}
\]

where \( J \) is a constant determined below, which simply originates from the change of the variables, and is independent of the action of the matrix model.

Setting \( (\Phi_3)_{ij} = c_{ij} \), the partition function can be expressed as

\[
Z_{N=1^*} = CJ \left( \frac{2\pi g_m}{N m_{1}} \right)^{\frac{\hat{S}}{2}} \prod_{i>j} (b_i - b_j)^2 e^{-S''}, \tag{2.11}
\]

\[
S'' = \frac{\hat{N}}{g_m} \left( \sum_{i \neq j} \frac{|c_{ij}|^2}{2} \left( \frac{1}{m_1} (b_i - b_j)^2 + m_3 \right) + \sum_i \frac{m_3}{2} c_{ii}^2 + \sum_i \frac{m_2}{2} b_i^2 \right). \tag{2.11}
\]

The integration with respect to \( c_{ij} \) can be readily performed to yield

\[
Z_{N=1^*} = CJ \left( \frac{2\pi g_m}{N} \right)^{\frac{\hat{S}}{2}} \frac{1}{(m_1 m_3)^{\frac{\hat{S}}{2}}} \prod_{i>j} \frac{(b_i - b_j)^2}{((b_i - b_j)^2 + m_1 m_3)} e^{-\frac{S_{m_2}}{2g_m} \sum_i b_i^2}. \tag{2.12}
\]

When \( m_1 m_3 \ll 1 \), we find

\[
\int db_i \prod_{i>j} \frac{(b_i - b_j)^2}{((b_i - b_j)^2 + m_1 m_3)} e^{-\frac{S_{m_2}}{2g_m} \sum_i b_i^2} = \left( \frac{2\pi g_m}{Nm_2} \right)^{\frac{\hat{S}}{2}}, \tag{2.13}
\]

therefore,

\[
Z_{N=1^*} = CJ \left( \frac{2\pi g_m}{N} \right)^{\frac{\hat{S}}{2}} \left( \frac{2\pi g_m}{Nm_1 m_2 m_3} \right)^{\frac{\hat{S}}{2}}. \tag{2.14}
\]

\[^4\text{The identity } S = g_m \text{ can be shown directly by using the map constructed in [9].}\]
We see that the contributions from the mass terms become subleading in the large-$\hat{N}$ limit. Therefore, in this limit we can take the $N=4$ limit $m_i \to 0$ smoothly and obtain

$$\frac{\hat{N}^2}{g_m^2} \lim_{m_i \to 0} F_{\mathcal{X}=1} = -\log \left( C J \left( \frac{2\pi g_m}{\hat{N}} \right) \right).$$

From our requirement (2.17), we can fix the measure factor $C$ as

$$C = J^{-1} \left( \frac{\hat{N}}{2\pi g_m} \right)^{\hat{N}^2} e^{-\pi i \tau_0 \hat{N}^2/N}.$$  \hspace{1cm} (2.16)

For the computation of $J$, it is sufficient to consider a concrete example, because $J$ is independent of the action as mentioned above. A convenient choice is the Gaussian action. A straightforward integration yields

$$Z = \int d\Phi e^{-\frac{1}{2} \text{tr} \Phi^2} = (2\pi)^{\frac{\hat{N}^2}{2}},$$

where $\Phi$ is an $\hat{N} \times \hat{N}$ Hermitian matrix. On the other hand, by using (2.10), we obtain

$$\int d\Phi e^{-\frac{1}{2} \text{tr} \Phi^2} = J \int dp_i \prod_{i>j} (p_i - p_j)^2 e^{-\frac{1}{2} p_i^2}.$$  \hspace{1cm} (2.18)

This can be computed by means of the orthogonal polynomials, which are defined as

$$\int dx e^{-\frac{1}{2} x^2} P_n(x) P_m(x) = \delta_{nm} h_n,$$

$$P_n(x) = x^n + \cdots.$$  \hspace{1cm} (2.19)

In the case of the Gaussian action, they are nothing but the Hermite polynomials and we have $h_n = n!(2\pi)^{\frac{n}{2}}$. Therefore, the partition function can be also expressed as

$$Z = J \hat{N}! \prod_{i=0}^{\hat{N}-1} h_i = J \hat{N}!(\hat{N}-1)! \cdots 0!(2\pi)^{\frac{\hat{N}}{2}}.$$  \hspace{1cm} (2.20)

Comparing this with (2.17), we obtain

$$\log J = \frac{\hat{N}^2}{2} \log 2\pi - \frac{\hat{N}^2}{2} \log \hat{N} + \frac{3}{4} \hat{N}^2 + O(\hat{N}),$$

$$J = \left( 2\pi e^{\frac{3}{2}} \right)^{\frac{\hat{N}^2}{2}}.$$  \hspace{1cm} (2.21)

From (2.16) and (2.21), we finally find

$$C = \left( \frac{\hat{N}^3}{(2\pi)^{\frac{3}{2}} g_m^2} \right)^{\frac{\hat{N}^2}{2}} e^{-\pi i \tau_0 \hat{N}^2/N}.$$  \hspace{1cm} (2.22)
3 Derivation of the Veneziano-Yankielowicz superpotential

Now we make a connection with the $\mathcal{N} = 1$ gauge theory coupled to a chiral superfield in the adjoint representation. In (2.1) we take $m_2 = m_3 = \Lambda_0 \gg m_1 = g_1$. Then at a scale below $\Lambda_0$, $\Phi_2$ and $\Phi_3$ are decoupled and the system is described by the $\mathcal{N} = 1$ gauge theory coupled to the single chiral multiplet $\Phi_1$. From the point of view of this theory, $\Lambda_0$ can be regarded as the cutoff and $\tau_0$ as the bare gauge coupling there. For simplicity, we consider the case where the superpotential $W(\Phi_1)$ for $\Phi_1$ is Gaussian: $W(\Phi_1) = m_1 \Phi_1^2/2$. The general case is considered in Appendix A. Then the prepotential in this $\mathcal{N} = 1$ theory should be given by $F_{\mathcal{N}=1}^*$ in (2.5) with $m_2 = m_3 = \Lambda_0$. Here we emphasize that the measure factor $C$ is common in the entire range of $m_i$’s so that we can use the above obtained value (2.22) also in the $\mathcal{N} = 1$ limit, where $m_2 = m_3 \gg m_1$. We thus find that the partition function of this matrix model is given as

$$e^{-\frac{\hat{S}_2}{g_m} F_{\mathcal{N}=1}} = Z_{\mathcal{N}=1} = C \int d\Phi_1 d\Phi_2 d\Phi_3 e^{-S_{\mathcal{N}=1}^* (m_2 = m_3 = \Lambda_0)}$$

$$= C \left( \frac{2\pi g_m}{N\Lambda_0} \right)^{\hat{N}^2} \int d\Phi_1 e^{-S_{\mathcal{N}=1}^*}$$

$$= \left( \frac{\hat{N}}{2\pi e^{\frac{3}{2}\Lambda_0^2}} \right)^{\frac{\hat{S}_2}{g_m}} e^{-\pi\tau_0 \hat{N}^2 / N} \int d\Phi_1 e^{-S_{\mathcal{N}=1}^*}, \quad (3.1)$$

where

$$S_{\mathcal{N}=1} = \frac{\hat{N}}{g_m} \text{Tr} \frac{m_1}{2} \Phi_1^2,$$

and we have used the fact that the interaction term can be neglected and the $\Phi_2$ and $\Phi_3$ integrations are reduced to Gaussian when $\Lambda_0$ is sufficiently large. Performing the last integration and identifying $g_m$ with $S$, we obtain

$$F_{\mathcal{N}=1} = \frac{g_m^2}{2} \left( \frac{2\pi i \tau_0}{N} + \log \frac{\frac{e^{\frac{3}{2}\Lambda_0^2} m_1}{g_m}}{N} \right)$$

$$= \frac{S^2}{2} \left( \frac{2\pi i \tau_0}{N} + \log \frac{e^{\frac{3}{2}\Lambda_0^2}}{S} \right) + \frac{S^2}{2} \log \frac{m_1}{\Lambda_0}, \quad (3.3)$$

which exactly agrees with the prepotential of the $\mathcal{N} = 1$ gauge theory that yields the VY superpotential plus the one-loop contribution from the chiral multiplet to the gauge kinetic term.
It is instructive to compare the measure in (3.1) to that in [5]. There it is shown that if we consider the matrix model with an appropriate measure $\mu$

\[ \int d\Phi_1 \mu N^2 e^{-S_{\mathcal{N}=1}}, \]  

its free energy reproduces the VY superpotential plus the matter contribution provided that

\[ \frac{\hat{N}\mu^2}{2\pi} = e^{\frac{3}{2}\Lambda_0^2}. \]  

Eq. (3.1) shows that we can derive this relation directly by fixing the measure in the matrix model so that (2.7) will be satisfied. The extra factor $e^{-\pi i\tau_0 \hat{N}^2/N}$ is nothing but the contribution from the tree gauge kinetic term, which is again consistent with the result in [5].

By construction, it is evident that we can also obtain the VY superpotential in the pure $\mathcal{N} = 1$ gauge theory by setting $m_1 = m_2 = m_3 = \Lambda_0$. In fact, if we set $m_1 = \Lambda_0$ in (3.3), we obtain

\[ F_{\mathcal{N}=1}^{\text{pure}} = \frac{S^2}{2} \log \frac{e^{\frac{3}{2}\Lambda^3}}{S}, \]  

where we have used

\[ 2\pi i\tau_0 = 3N \log \left( \frac{\Lambda}{\Lambda_0} \right). \]  

If we introduce a non-trivial potential to $\Phi_1$, the gauge symmetry can be broken, and the prepotential becomes a function of several $S_i$'s. Even in such case, the VY superpotential is correctly reproduced by the measure (2.22). In Appendix A we clarify this point when the $U(N)$ gauge group is broken to $U(N_1) \times U(N_2)$.

### 4 Conclusions

Let us summarize the meaning of our results: we start from the supersymmetric gauge theory coupled to the three chiral multiplets $\Phi_i$ ($i = 1 \sim 3$) with a generic potential as in (1.1) for $\Phi_1$. Then the Schwinger-Dyson approach in [5], or the direct map in [9] tells us that the prepotential $\mathcal{F}$ in this theory and the free energy in the corresponding matrix model $F_m$ satisfy

\[ \frac{\partial \mathcal{F}}{\partial g_k} = \frac{\partial F_m}{\partial g_k}, \]  

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as a function of $S$ and $g_m$ respectively. Similarly, we further find that

$$\frac{\partial F}{\partial m_i} = \frac{\partial F_m}{\partial m_i},$$

(4.2)

where $m_i (i = 2, 3)$ is the mass for the chiral multiplet $\Phi_i$. Thus $F$ and $F_m$ are equivalent up to a function independent of $m_i$'s as well as $g_k$'s, which is nothing but the contribution from $C$. Therefore, if we adjust the origin of $F_m$ so that $F_m$ and $F$ will coincide at an appropriate point in the parameter space, they will become entirely equivalent. We have chosen the $\mathcal{N} = 4$ SYM ($g_k, m_i \to 0$) as such a point, where there are no quantum corrections to the holomorphic part of the effective Lagrangian. Then in the $\mathcal{N} = 1$ theory ($m_2 = m_3 = \Lambda_0$), $F_m$ correctly reproduces without any other inputs the VY superpotential that is independent of $g_k$'s, as expected.

Finally we clarify the reason why the matrix model has information on the $g_k$-independent part in its measure from the point of view of the large-$N$ reduction [9]. From the map constructed in [9], we obtain the matrix model equivalent to the $\mathcal{N} = 1$ gauge theory as far as the holomorphic part is concerned:

$$\exp \left(-\frac{\hat{N}^2}{g_m^2} F_{\mathcal{N}=1}\right) = \int d\hat{\Phi} \int d\hat{V} \exp \left(-\frac{\hat{N}}{g_m} \left\{2\pi i\tau_0 \text{Tr}(\hat{W}^\alpha \hat{W}_\alpha) + \text{Tr}(W'\hat{\Phi})\right\}\right),$$

(4.3)

where the hat denotes the large-$N$ reduction of the corresponding field in the original $\mathcal{N} = 1$ gauge theory, and $F_m$ is the free energy of this model. We note here that once we concentrate on the holomorphic part of the free energy and drop the kinetic term for $\hat{\Phi}, \hat{V}$ and $\hat{\Phi}$ become decoupled from each other, and the integration over $\hat{V}$ can be performed independently leaving an overall measure for $\hat{\Phi}$.\footnote{However, this decoupling becomes subtle when there is an ultraviolet divergence. See [9].} In viewing this, we find that the matrix model (1.2) is obtained after integrating out the vector multiplet, and as a consequence, if the measure in the matrix model can be determined correctly, it should have information on the gauge dynamics. In fact, our result shows that this is indeed the case: for example, the matrix model reproduces the Veneziano-Yankielowicz superpotential, which contains the dynamical scale $\Lambda$ for the gauge field. The advantage of this standpoint is that we can make a direct connection between the gauge theory and the matrix model even for rather complicated multi-matrix models such as (2.4).

\section*{Acknowledgments}

\footnote{However, this decoupling becomes subtle when there is an ultraviolet divergence. See [9].}
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A Derivation of the Veneziano-Yankielowicz superpotential in the case of broken gauge symmetry

In this appendix we show how the VY superpotential is derived from the matrix model, when the gauge symmetry is spontaneously broken. For simplicity, we concentrate on the case where the $U(N)$ gauge symmetry is broken to $U(N_1) \times U(N_2)$ ($N = N_1 + N_2$).

We consider the $\hat{N} \times \hat{N}$ matrix model with a cubic potential

$$W(\Phi) = a \left( \frac{1}{3} \Phi^3 - v^2 \Phi \right), \quad W'(\Phi) = a (\Phi - v)(\Phi + v), \quad (A.1)$$

then the partition function under the measure (2.22) is given as in (3.1) by

$$Z = \frac{1}{J_{\hat{N}}} \left( \frac{1}{\Lambda_0^2} \right)^{\frac{\hat{N}_2^2}{2}} e^{-\frac{\pi \tau_0}{N} \hat{N}^2} \int d\Phi e^{-\frac{\hat{N}}{g_m} \text{tr} W(\Phi)}, \quad (A.2)$$

where $J_{\hat{N}}$ is defined as in (2.21):

$$J_{\hat{N}} = \left( \frac{2\pi e^\frac{\hat{N}_2^2}{2}}{N} \right). \quad (A.3)$$

Diagonalizing $\Phi$, we obtain

$$Z = \left( \frac{1}{\Lambda_0^2} \right)^{\frac{\hat{N}_2^2}{2}} e^{-\frac{\pi \tau_0}{N} \hat{N}^2} \int \prod_i d\phi_i \prod_{i<j} (\phi_i - \phi_j)^2 e^{-S'},$$

$$S' = \frac{\hat{N}}{g_m} a \sum_i \left( \frac{1}{3} \phi_i^3 - v^2 \phi_i \right). \quad (A.4)$$

We choose a vacuum where, among $\hat{N}$ eigenvalues, $\hat{N}_1$ of them lie around the one classical minimum $v$, while the rest $\hat{N}_2$ around the other $-v$. Then we consider the fluctuations
around it:

\[ S' = \frac{\hat{N}}{g_m} \sum_{i=1}^{\hat{N}_1} \left( \frac{2av}{2} p_i^2 + \frac{a}{3} q_i^3 \right) + \frac{\hat{N}}{g_m} \sum_{i=1}^{\hat{N}_2} \left( -\frac{2av}{2} q_i^2 + \frac{a}{3} q_i^3 \right), \]  

(A.5)

where \( p_i \) and \( q_i \) are fluctuations around \( \phi_i = v \) and \( \phi_i = -v \) respectively, and we have dropped the constant term. Then the partition function becomes

\[
Z = \frac{\hat{N}C_{\hat{N}_1}}{\Lambda_0^{N^2}} e^{-\frac{\pi g_m}{N} \hat{N}^2} \int \prod_{i=1}^{\hat{N}_1} dp_i \prod_{i=1}^{\hat{N}_2} dq_i \times \prod_{1 \leq i < j \leq \hat{N}_1} (p_i - p_j)^2 \prod_{1 \leq i < j \leq \hat{N}_2} (q_i - q_j)^2 \prod_{i=1}^{\hat{N}_1} \prod_{j=1}^{\hat{N}_2} (2v + p_i - q_j)^2 e^{-S'}, \tag{A.6}
\]

where \( \hat{N}C_{\hat{N}_1} \) is the number of the ways of choosing \( \hat{N}_1 \) eigenvalues around \( v \). We now take the limit \( 2v \to \Lambda_0 \gg 1 \), in which the theory becomes the pure \( \mathcal{N} = 1 \) SYM with \( U(N_1) \times U(N_2) \) gauge group. In this limit \( p_i \)'s and \( q_i \)'s are decoupled from each other as seen from (A.6), and the cubic terms in (A.5) can be neglected. Although in the resulting action \( q_i \)'s have a negative mass squared, we can take an appropriate contour to make the integral convergent as usual in the Dijkgraaf-Vafa theory.\(^6\) We thus obtain

\[
Z = \frac{\hat{N}C_{\hat{N}_1}}{\Lambda_0^{N^2}} e^{-\frac{\pi g_m}{N} \hat{N}^2} \left( J_{\hat{N}_1} \int \prod_{i=1}^{\hat{N}_1} dp_i \prod_{1 \leq i < j \leq \hat{N}_1} (p_i - p_j)^2 e^{-\frac{\pi g_m}{N} \sum_i \frac{\Lambda_0 a}{2} p_i^2} \right) \times \left( J_{\hat{N}_2} \int \prod_{i=1}^{\hat{N}_2} dq_i \prod_{1 \leq i < j \leq \hat{N}_2} (q_i - q_j)^2 e^{-\frac{\pi g_m}{N} \sum_i \frac{\Lambda_0 a}{2} q_i^2} \right) \\
= \frac{\hat{N}C_{\hat{N}_1}}{\Lambda_0^{N^2}} \frac{\hat{N}_1 \hat{N}_2}{\hat{N}_1 \hat{N}_2} e^{-\frac{\pi g_m}{N} \hat{N}^2} \left( \frac{2\pi g_m}{\hat{N} \Lambda_0 a} \right)^{\frac{\hat{N}_1^2}{2}} \left( \frac{2\pi g_m}{\hat{N} \Lambda_0 a} \right)^{\frac{\hat{N}_2^2}{2}} \\
= \frac{\hat{N}C_{\hat{N}_1}}{\Lambda_0^{N^2}} \frac{\hat{N}_1 \hat{N}_2}{\hat{N}_1 \hat{N}_2} e^{-\frac{\pi g_m}{N} \hat{N}^2} \left( \frac{g_m \hat{N}_1}{N e^2 \Lambda_0 a} \right)^{\frac{\hat{N}_1^2}{2}} \left( \frac{g_m \hat{N}_2}{N e^2 \Lambda_0 a} \right)^{\frac{\hat{N}_2^2}{2}}, \hspace{1cm} (A.7)
\]

\(^6\)This is naturally justified in \[^{[13]}\] by considering a supermatrix model.
where we have used the formula (2.10) in a reverse manner such as

\[
J_{\hat{N}_{1}} \int \prod_{i=1}^{\hat{N}_{1}} \prod_{1 \leq i < j \leq \hat{N}_{1}} (p_{i} - p_{j})^{2} e^{-\frac{\hat{N}_{1}}{g_{m}^{2}}} \prod_{1 \leq i < j \leq \hat{N}_{1}} (p_{i} - p_{j})^{2} = \int d^{\hat{N}_{1}} \Phi e^{-\frac{\hat{N}_{1}}{g_{m}^{2}}} \Phi^{2} = \left(\frac{2\pi g_{m}}{\hat{N}_{1}}\right)^{\frac{\hat{N}_{1}^{2}}{2}}.
\]

Therefore, we obtain the free energy in the large-\(\hat{N}\) limit as

\[
F_{m} = \frac{\pi i \tau_{0}}{N} g_{m}^{2} + \frac{g_{m}^{2} \hat{N}_{1}^{2}}{2N^{2}} \log \left(\frac{N e^{2} N_{1}^{3}}{g_{m} N_{1}^{2}}\right) + \frac{g_{m}^{2} \hat{N}_{2}^{2}}{2N^{2}} \log \left(\frac{N e^{2} N_{2}^{3}}{g_{m} N_{2}^{2}}\right).
\]

Substituting \(S_{i}\) for \(g_{m} \hat{N}_{i}/\hat{N}\), we finally obtain

\[
F_{m} = \frac{\pi i \tau_{0}}{N} (S_{1} + S_{2})^{2} + \frac{S_{1}^{2}}{2} \log \left(\frac{e^{2} N_{1}^{3}}{S_{1}}\right) + \frac{S_{2}^{2}}{2} \log \left(\frac{e^{2} N_{2}^{3}}{S_{2}}\right).
\]

According to the formula for the effective superpotential in the case of broken gauge symmetry [5]

\[
W^{\text{eff}} = \sum_{i} N_{i} \frac{\partial \mathcal{F}}{\partial S_{i}},
\]

we find that under the identification \(\mathcal{F} = F_{m}\), \(F_{m}\) exactly reproduces the VY superpotential when \(U(N)\) gauge symmetry is broken to \(U(N_{1}) \times U(N_{2})\), where \(S_{i} = \frac{1}{64\pi^{2}} \text{tr}_{U(N_{i})} W^{a} W_{a}\).

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