Coherent structures theory for the generalized Kuramoto–Sivashinsky equation

D. Tseluiko, S. Saprykin, S. Kalliadasis
Department of Chemical Engineering, Imperial College London, London SW7 2AZ, United Kingdom

Abstract. We examine coherent structures interaction and formation of bound states in active–dispersive–dissipative nonlinear media. A prototype for such media is a simple weakly nonlinear model, the generalized Kuramoto–Sivashinsky (gKS) equation, that retains the fundamental mechanisms of any nonlinear process involving wave evolution, namely, a dominant nonlinearity, instability, stability and dispersion. We develop a weak interaction theory for the solitary pulses of the gKS equation by representing the solution as a superposition of the pulses and an overlap function. We derive a linearized equation for the overlap function in the vicinity of each pulse and project the dynamics of this function onto the discrete part of the spectrum of the linearized interaction operator. This leads to a coupled system of ordinary differential equations describing the evolution of the locations of the pulses. By analyzing this system, we prove a criterion for the existence of a countable infinite or finite number of bound states, depending on the strength of the dispersive term in the equation. The theoretical findings are corroborated by computations of the full equation.

1. Introduction

Consider the generalized Kuramoto–Sivashinsky (gKS) equation,

$$h_t + hh_x + h_{xx} + \delta h_{xxx} + h_{xxxx} = 0, \quad (x, t) \in (-\infty, \infty) \times [0, \infty).$$

This equation is the simplest possible nonlinear evolution prototype that retains the fundamental elements of any nonlinear process that involves wave evolution: the dominant nonlinear term $hh_x$, instability and energy production $h_{xx}$, stability and energy dissipation $h_{xxxx}$, and dispersion $\delta h_{xxx}$, with $\delta$ a parameter that measures the relative importance of dispersion. As far as the nonlinearity is concerned, its functional form can be easily obtained from symmetry considerations: indeed, the only other dominant quadratic nonlinearity is $h^2$, which is obviously ruled out for systems whose spatial average does not drift, i.e. $d\langle h \rangle_x / dt = 0$.

In the limit $\delta \to 0$ the equation reduces to the usual Kuramoto–Sivashinsky (KS) equation, first derived in [1], [2] and [3] independently with a weakly nonlinear expansion for small-amplitude falling-film waves in the limiting case of large surface tension and away from criticality (we shall define these terms shortly). The same equation was also obtained later in a wide variety of applications such as chemical physics/reaction-diffusion systems, e.g. propagation of concentration waves [4, 5] and combustion, e.g. flame-front instabilities [6].

The gKS equation has been derived in many physical contexts, including plasma waves with dispersion due to finite ion banana width [7] and liquid films in different settings, i.e. from a film flowing down a planar substrate [8] or a vertical fiber [9] to films in the presence of various
additional effects and complexities, e.g. films sheared by a turbulent gas [10], falling liquid films in the presence of a viscous stress at the free surface [11], liquid films flowing down a uniformly heated wall [12] and reactive falling films [13].

For the problem of a film falling down a planar substrate, in particular, the gKS equation is obtained from a weakly nonlinear expansion for small-amplitude waves and under the assumptions of strong surface tension and near-critical conditions [8],

\[ \text{We} = O(\epsilon^2), \quad \text{Re} > \text{Re}_c, \quad \text{Re} - \text{Re}_c = O(\epsilon^2), \] (2)

where \( \text{Re} = U_0 h_0 / \nu \) is the Reynolds number assumed to be at most of \( O(1) \) and based on the average velocity \( U_0 = g \sin \theta h_0^3 / (3 \nu) \) of the Nusselt flat film flow, with \( h_0, g, \nu \) and \( \theta \) denoting the Nusselt flat film thickness, gravitational acceleration, liquid kinematic viscosity and inclination angle with respect to the horizontal direction, respectively. \( \text{Re}_c = 5 \cot \theta / 6 \) is the critical Reynolds number for the instability onset and \( \text{We} = \gamma / (3 \mu U_0) \) is the Weber number with \( \gamma \) and \( \mu \) denoting the surface tension and dynamic viscosity of the fluid, respectively. \( x \) denotes a streamwise coordinate, \( t \) is time and \( h \) is a measure of the (small) deviation amplitude from the dimensionless Nusselt flat film of unity (non-dimensionalized with \( h_0 \)). \( \epsilon \ll 1 \) is the so-called long-wave or film parameter, and is typically defined as the ratio of \( h_0 \) to a (long) lengthscale over which streamwise variations occur. For film flow down a vertical plane (\( \text{Re}_c = 0 \)) we still require strong-surface tension and near-critical conditions but with different orders-of-magnitude assignments to (2) [8]:

\[ \text{We} = O(\epsilon^{-1}), \quad \text{Re} = O(\epsilon). \] (3)

In both cases, \( \delta \) is an \( O(1) \) parameter defined as:

\[ \delta = \sqrt{\frac{15}{2} \frac{1}{(\text{Re} - \text{Re}_c) \text{We}}}. \] (4)

On the other hand, for an inclined plane with \( \text{We} = O(\epsilon^{-2}) \) and \( \text{Re} = O(1), \text{Re} - \text{Re}_c = O(1) \) or a vertical plane with \( \text{We} = O(\epsilon^{-2}) \) and \( \text{Re} = O(1) \) – away from criticality and strong surface tension effects as noted earlier – the KS equation is the pertinent weakly nonlinear model.

As noted earlier, in the limit \( \delta \to 0 \), the gKS equation reduces to the KS one. On the other hand, in the limit \( \delta \to \infty \), the gKS equation reduces to the Korteweg–de Vries (KdV) one: introduce the change of variables in equation (1)

\[ \tau = \delta t, \quad H = \frac{1}{\delta} h, \] (5)

to write it as

\[ H_t + HH_x + H_{xxx} + \epsilon(H_{xx} + H_{xxxx}) = 0, \] (6)

where \( \epsilon = 1/\delta \). In the limit \( \epsilon \to 0 \) (or, equivalently, \( \delta \to \infty \)), equation (6) reduces to the KdV equation.

For small \( \delta \) the gKS equation is a perturbed KS one and hence exhibits complicated chaotic dynamics in both space and time. However, a sufficiently large \( \delta \) arrests the spatio-temporal chaos such that the solution evolves into a regular array of quasi-stationary pulses with roughly the same shape and speed and which interact indefinitely with each other through attractions or repulsions [14]. On the other hand, for moderate values of \( \delta \) the spatio-temporal evolution of the gKS equation is dominated by quasi-stationary solitary pulses which continuously interact with each other through coalescence events or attractions/repulsions. Such pulses then become elementary processes representing the behavior of the full system and therefore it is feasible to consider the solution as their superposition. We focus on the development of a weak interaction
theory by assuming that the pulses are sufficiently separated from each other, i.e. we ignore coalescence events between pulses which just repel or attract each other by interacting through their tails only.

The idea of weak interaction theory for solitary pulses in other systems has been implemented, for example, by [15] and [16]. As far as the gKS equation is concerned, previous efforts to develop weak interaction approaches, include [17], [18] and [19]. However, all previous studies for the gKS equation appear to be either incomplete or sometimes overlook important details and subtleties. Our aim is to obtain a clear and complete understanding of the pulse interaction problem for the gKS equation and to scrutinize our results by detailed comparisons with computations.

The rest of the paper is organized as follows. In Section 2 we develop a weak interaction theory of solitary pulses for the gKS equation. In Section 3 we analyze bound states of the pulses. Section 4 is devoted to discussion of theoretical results and conclusions.

2. Pulse-interaction theory for the gKS equation

From now on, we consider the gKS equation (1) in a frame moving with the velocity $c_\delta$ of a solitary pulse,

$$h_t - c_\delta h_x + hh_x + h_{xx} + \delta h_{xxx} + h_{xxxx} = 0. \quad (7)$$

Let $h_0 = h_0(x)$ be a stationary pulse. It satisfies the steady version of (7). We assume that the solution, $h$, is described as a superposition of $n$ quasi-stationary pulses $h_1, \ldots, h_n$ located at $x_1(t), \ldots, x_n(t)$, respectively, namely,

$$h_i(x, t) = h_0(x - x_i(t)), \quad i = 1, \ldots, n, \quad (8)$$

and a small overlap (or correction) function $\hat{h}$. Thus, we use the ansatz $h = \sum_{i=1}^n h_i + \hat{h}$ for the solution and aim to derive a system of equations governing the locations of the pulses. We consider weak interaction assuming that the pulses are sufficiently separated and, therefore, that for each pulse it is sufficient to take into account its interaction with only immediate neighbors. The idea of weak interaction theory for solitary pulses in other systems was analyzed by, for example, Ei [15] and Sandstede [16]. More precisely, we assume that $l_i = x_{i+1} - x_i = \log \varepsilon + O(1)$ for $i = 1, \ldots, n - 1$, where $\varepsilon \ll 1$. We additionally assume that the velocities of the pulses, $x'_i$, $i = 1, \ldots, n$, and the overlap function, $\hat{h}$, are $O(\varepsilon)$. The linearized equation for the overlap function, $\hat{h}$, in the vicinity of the $i$th pulse takes the form

$$\hat{h}_t - x'_1 h_{1x} = \mathcal{L}_1 \hat{h} - (h_1h_2)_x \quad (9)$$

for $i = 1$,

$$\hat{h}_t - x'_i h_{ix} = \mathcal{L}_i \hat{h} - (h_{i-1}h_i)_x - (h_ih_{i+1})_x \quad (10)$$

for $2 \leq i \leq n - 1$ and

$$\hat{h}_t - x'_n h_{nx} = \mathcal{L}_n \hat{h} - (h_{n-1}h_n)_x \quad (11)$$

for $i = n$, where $\mathcal{L}_i$'s are linear operators defined by

$$\mathcal{L}_i f = c_\delta f_x - f_{xx} - \delta f_{xxx} - f_{xxxx} - (h_if)_x \quad (12)$$

for $i = 1, \ldots, n$. The formal adjoint operators, $\mathcal{L}_i^*$'s, with respect to the usual inner product, $\langle f, g \rangle = \int_{-\infty}^{\infty} f \bar{g} \, dx$ (where bar denotes complex conjugate), in the space $L^2_{\varepsilon}$ of complex-valued measurable functions whose absolute values squared are Lebesgue integrable, are given by:

$$\mathcal{L}_i^* f = -c_\delta f_x - f_{xx} + \delta f_{xxx} - f_{xxxx} + h_i f_x. \quad (13)$$
We aim to project the dynamics in the vicinity of the $i$th pulse onto the null space of $L_i$, spanned by the translational mode $h_{ix}$. Our analysis showed that the null space of $L_i$ is spanned by a constant and a function, which we denote by $\Psi^i$, tending exponentially to different constants as $x \to \pm \infty$. Therefore, zero is not in the point spectrum of $L_i^*$ on an infinite interval, and the projection onto the null space of $L_i$ cannot be made in a straightforward way. Nevertheless, projections can be made rigorous by choosing an appropriate weighted space, namely,

$$L_a^2 = \{ f : e^{ax}f \in L_C^2 \},$$  

where $a$ is a positive sufficiently small number, with the inner product $\langle f, g \rangle_a = \langle e^{ax}f, e^{ax}g \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L_C^2$. As noticed by Pego & Weinstein [20], studying the spectrum of $L_i$ in $L_a^2$ is equivalent to studying the spectrum of the operator defined by $L_i^*f = e^{ax}L_i(e^{-ax}f)$ on $L_a^2$. With such a construction, zero is an isolated eigenvalue of both $L_i^*$ and $L_i^{**}$ of both algebraic and geometric multiplicity unity with the eigenfunction and the adjoint eigenfunction given by $e^{ax}h_{ix}$ and $e^{-ax}(\Psi^i - \lim_{x \to -\infty} \Psi^i)$, respectively, and projections, $P^a$, onto the null spaces of $L_a^2$’s can be made in a straightforward way, $P^a(f) = \langle f, e^{-ax}(\Psi^i - \lim_{x \to -\infty} \Psi^i) \rangle e^{ax}h_{ix}$. The essential spectrum is shifted in the complex plane. For certain values of $\delta$, the essential spectrum can be moved entirely to the left half-plane, meaning that for these values of $\delta$ the pulses tolerate sufficiently small disturbances that decay sufficiently fast as $x \to -\infty$. For example, for $\delta = 0.4$ the essential spectrum belongs to the left half-plane for $a_1 < a < a_2$, where $a_1 \approx 0.13$ and $a_2 \approx 0.57$, and therefore pulses tolerate any small disturbance that is $o(e^{-a_1x})$ as $x \to -\infty$. Assuming that “$\hat{h}$ is free of translational modes”, i.e. that it is in the null spaces of the projections, we arrive at the following system describing the dynamics of the locations of the pulses:

$$x'_1 = S_1(x_2 - x_1),$$  

$$x'_i = S_2(x_i - x_{i-1}) + S_1(x_{i+1} - x_i), \quad 1 < i < n,$$  

$$x'_n = S_2(x_n - x_{n-1}),$$  

where

$$S_1(l) \equiv -\int_{-\infty}^{\infty} h_0(x + l/2)h_0(x - l/2)\Psi^0(x + l/2) \, dx,$$  

$$S_2(l) \equiv -\int_{-\infty}^{\infty} h_0(x + l/2)h_0(x - l/2)\Psi^0(x - l/2) \, dx,$$  

We have solved system (15)–(17) for $\delta = 0.4$ numerically for two pulses and compared results with numerical solutions of the full equation (7) when the initial condition was a superposition of two pulses with the same separation distance as that for (15)–(17). We found very good agreement between these results. In Fig. 1(a) we present a typical solution of system (15)–(17) where the world lines of 24 pulses are shown. We note that the results are shown in the frame moving with the velocity of a solitary pulse. Here $\delta = 0.4$. We can observe both attractions and repulsions, formation of two- and three-pulse bound states. Fig. 1(b) shows a histogram obtained on the statistics on 3000 pulse separation distances at $t = 1000$. The initial distribution of the separation distances was taken to be normal with mean 18 and standard deviation 3. We can observe three clear peaks. We note that the peaks are formed at around 9.5, 14 and 18.5, which are in very good agreement with the stable two-pulse bound state distances discussed in the next section and shown in Fig. 2.
Figure 1. (a) Evolution of pulses of the gKS equation for $\delta = 0.4$ obtained by solving system (15)--(17). Attractions and repulsions can be observed as well as formation of bound states. (b) The histogram obtained on the statistics on 3000 pulse separation distances at $t = 1000$.

3. Bound-states theory

We note that system (15)--(17) can transformed to the following system for the separation distances $l_i$’s:

$$l'_1 = S_2(l_1) + S_1(l_2) - S_1(l_1),$$

$$l'_i = S_2(l_i) + S_1(l_{i+1}) - S_2(l_{i-1}) - S_1(l_i), \quad 1 < i < n - 1,$$

$$l'_{n-1} = S_2(l_{n-1}) - S_2(l_{n-2}) - S_1(l_{n-1}). \quad (21)$$

Studying the fixed points of this system, we obtain the bound states of the pulses, i.e. the states when the pulses travel together as a whole.

For instance, for a bound state of two pulses we must have:

$$S_1(l_1) = S_2(l_1). \quad (22)$$

The graphs of $S_1$ and $S_2$ are shown in Fig. 2 for $\delta = 0.4$. The abscissas of the intersection points indicate the separation distances for which bound states can be formed, and the ordinates indicated the corresponding velocities of the bound states relative to $c_0$. Black circles and crosses correspond to stable and unstable bound states, respectively. It is also interesting to note that the ordinates of the intersection points are always negative, i.e. the velocity of a two-pulse bound state is always less than that of an individual pulse. Another interesting observation is that for $\delta = 0.4$ we, apparently, get a countable infinite number of bound states. This observation can be proved analytically by analyzing the behavior of $S_1$ and $S_2$ as $L \to \infty$, namely, it can be shown that if $\delta$ is less than a threshold value $\delta^* \approx 0.85$, then there is countable infinite number of intersections of $S_1$ and $S_2$. Otherwise, there is a finite number of intersections of $S_1$ and $S_2$.

4. Discussion

We developed a weak interaction theory of the pulses of the gKS equation by representing its solution as a superposition of such pulses and an overlap function and deriving a system
Figure 2. Dependence of $S_1$ and $S_2$ on the separation distance between two pulses (solid and dashed lines, respectively) for $\delta = 0.4$. Circles and crosses correspond to stable and unstable two-pulse bound states, respectively.

of linearized equations for the overlap function in the vicinity of each pulse. We found that zero is an eigenvalue of the linearized operators of geometric and algebraic multiplicity unity which is embedded into the essential spectrum. This eigenvalue is associated with translational invariance of the equation. However, zero is not an eigenvalue for the corresponding adjoint operators. The null spaces of the adjoint operators are spanned by a constant and a function having a jump at infinity. Despite this fact, we showed that projections onto the null spaces, spanned by translational modes, can be made rigorous in an appropriate weighted space and the derivation of a dynamical system describing interaction of the pulses due to translational modes can be rigorously justified. This system can be written in terms of the separation distances between consecutive pulses. By analyzing its fixed points, we obtained bound states of any number of pulses. In particular, we analyzed in detail bound states of two pulses and proved a criterion for the existence of a countable infinite or finite number of bound states, depending on the strength of the dispersive term in the equation. Interestingly, this criterion exactly coincides with Shilnikov’s criterion on the existence of subsidiary homoclinic orbits, see [21]. Our approach, however, in addition to providing an existence result for the bound states, also gives the description of the dynamics of the pulses. Besides, it can be extended to higher dimensions.

Acknowledgments
We acknowledge financial support from the Engineering and Physical Sciences Research Council of the UK (EPSRC) through grants no. EP/F009194 and EP/F016492.

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