From exact-WKB toward
singular quantum perturbation theory II

André Voros

CEA, Service de Physique Théorique de Saclay
CNRS URA 2306
F-91191 Gif-sur-Yvette CEDEX, France
E-mail: voros@spht.saclay.cea.fr

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Abstract

Following earlier studies, several new features of singular perturbation theory for one-dimensional quantum anharmonic oscillators are computed by exact WKB analysis; former results are thus validated.

This note continues our study [1] of singular perturbation theory in one-dimensional (1D) quantum mechanics using exact WKB analysis. Our focus remains the $v \gg 1$ regime for the potentials $V(q) = q^N + vq^M$ on the real line, with $N > M$ positive even integers. Among those, the quartic oscillator $q^4 + vq^2$ has been a prime model for the mathematics of quantum perturbation theory [2, 3, 4, 5, 6]. Kawai and Takei [7] pioneered the use of exact WKB analysis in the latter context, followed by [8] (see also [9, Introduction to Part I, and Pham’s contribution], [10], and references therein). In spite of those successes, present exact-WKB quantization conditions (for $q^4 + vq^2$, say) fail to tend toward their harmonic-potential ($vq^2$) counterparts

\footnote{Also at: Institut de Mathématiques de Jussieu–Chevaleret, CNRS UMR 7586, Université Paris 7, F-75251 Paris CEDEX 05, France.}
as $v \to +\infty$, be it analytically or numerically [11]. This worrying observation triggered our present line of work (starting from [12, § 3]): to further probe how consistently exact WKB theory handles the perturbative ($v \gg 1$) regime.

We are happy and honored to dedicate this work to Professor Kawai with gratitude, for his many essential contributions and leadership in exact WKB analysis, but also earlier (with Professors Sato and Kashiwara) in hyper/micro/function theory; this framework greatly inspired, and its Authors warmly encouraged, our first steps in exact WKB analysis [13].

Even though this work is thoroughly tied to [1] (with its bibliography), in § 1 we recall the main background and further strengthen the case for improper (divergent) action integrals like $\int_{0}^{\infty} \Pi(q) \, dq$, where $\Pi(q) = (V(q) + \lambda)^{1/2}$ is the classical momentum function. In § 2 we present new cases where $\int_{0}^{\infty} \Pi(q) \, dq$ can be computed exactly for some trinomial $\Pi(q)^{2}$: essentially the quartic case $\Pi(q)^{2} = q^{4} + vq^{2} + \lambda$, for which $\int_{0}^{\infty} \Pi(q) \, dq$ reduces to ordinary (i.e., convergent) complete elliptic integrals. In § 3 we extend the main outcome of [1], namely the $v \to +\infty$ asymptotic expression of the spectral determinants $D_{N}^{\infty}(\lambda, v) \overset{\text{def}}{=} \det^{\pm}(-d^{2}/dq^{2} + q^{N} + vq^{M} + \lambda)$ in terms of $D_{M}^{\infty}(\Lambda) \overset{\text{def}}{=} \det^{\pm}(-d^{2}/dq^{2} + q^{M} + \Lambda)$, to $v \to \infty$ in a complex sector. Thanks to this, finally in § 4 we demonstrate (currently provided $2M + 2 > N$) how the fundamental bilinear functional relation satisfied by $D_{N}^{\infty}$ does evolve into its counterpart for $D_{M}^{\infty}$ as $v \to +\infty$, in spite of a discontinuous jump at $v = \infty$ of the main parameter, the degree of the potential (from $N$ to $M$).

1 Background (summarized) [1]

Our model of quantum perturbation theory is the 1D Schrödinger equation

$$\left[-\frac{d^{2}}{dq^{2}} + V(q) + \lambda\right] \Psi(q) = 0, \quad q \in \mathbb{R}, \quad V(q) = q^{N} + vq^{M}, \quad v \gg 1, \quad (1)$$

with $N > M$ positive even integers. If we use the unitary equivalence (called Symanzik scaling)

$$-\frac{d^{2}}{dq^{2}} + uq^{N} + vq^{M} \approx v^{2/(M+2)} \left[-\frac{d^{2}}{dx^{2}} + x^{M} + uv^{-(N+2)/(M+2)}x^{N}\right] \quad (2)$$
twice, at \( u = 1 \) and \( u = 0 \), the resulting right-hand sides imply that the operator \( \hat{H} = -\frac{d^2}{dq^2} + q^N + vq^M \) is a singular perturbation of \( \hat{H}_0 = -\frac{d^2}{dq^2} + vq^M \) for \( v \gg 1 \), and that the degree drops from \( N \) to \( M \) at \( v = +\infty \).

From the classical dynamics we will use the momentum function \((x)\),

\[
\Pi(q) = (V(q) + \lambda)^{1/2} \quad \text{(real in the classically forbidden region),} \tag{3}
\]

and its residue \( \text{Res}_{q=\infty} \Pi(q) = \beta_{-1}(0) \), a notation based on the expansion \[12\]

\[
(V(q) + \lambda)^{-s+1/2} \sim \sum_{\rho} \beta_{\rho}(s) q^{\rho-Ns} \quad (q \to \infty); \quad \rho = \frac{1}{2}N, \frac{1}{2}N - 1, \ldots \tag{4}
\]

The spectrum of \( \hat{H} \) is purely discrete, \( 0 < \lambda_0 < \lambda_1 < \cdots \uparrow +\infty \), and separates according to parity since \( V \) is an even function. Useful spectral functions (labeled by parity) are the \textit{generalized zeta functions},

\[
Z^\pm(s, \lambda) \overset{\text{def}}{=} \sum_{k, \text{even/odd}} (\lambda_k + \lambda)^{-s} \quad (\text{Re } s > \frac{1}{2} + \frac{1}{N}), \tag{5}
\]

and the \textit{spectral determinants} \( D^\pm(\lambda) \), defined through \textit{zeta regularization},

\[
\log D^\pm(\lambda) \equiv \log \det^\pm(\hat{H} + \lambda) \overset{\text{def}}{=} [-\partial_s Z^\pm(s, \lambda)]_{s=0}, \tag{6}
\]

where \( "s \sim 0" \) implies analytical continuation in \( s \). Scaling laws follow:

\[
\det^\pm[r(\hat{H} + \lambda)] \equiv r^{Z^\pm(s=0, \lambda)} \det^\pm(\hat{H} + \lambda) \quad (\forall r > 0),
\]

where

\[
Z^\pm(s = 0, \lambda) \equiv -\frac{\beta_{-1}(0)}{N} \pm \frac{1}{4} \tag{7}
\]

[11, equations (7), (30)][12, equations (15), (27), (37)].

A more concrete realization of \( \log D^\pm \) through (6) is, first to formally apply \( (d/d\lambda)^m \) to (6) with the minimal \( m \) such that the result \( (\sim Z(m, \lambda)) \) converges, i.e., \( m > \frac{1}{2} + \frac{1}{N} \), then to integrate back: the separate knowledge that the \( \lambda \to +\infty \) expansion of \( \log D^\pm(\lambda) \) shall only have “canonical” terms [14][12, § 1.1.2] fixes the \( m \) integration constants. Here, \( N \geq 4 \) implies \( m = 1 \) : specifically,

\[
\frac{d}{d\lambda} \log D^\pm(\lambda) \equiv Z^\pm(1, \lambda) \tag{8}
\]

converges according to (5), and \( \log D^\pm(\lambda) \) is then \textit{the} unique primitive of \( Z^\pm(1, \lambda) \) which is devoid of a constant \( (\propto \lambda^0) \) term in its large-\( \lambda \) expansion.
Classical analogs of those quantum determinants can be defined as well [11, 12, 10], through: 
\[ \log D^\pm(\lambda) \overset{\text{def}}{=} \{ \text{the divergent part of } \log D^\pm(\lambda) \text{ for } \lambda \to +\infty \} , \]

or equivalently [12, § 1.2.1 and equation (46)] through:
\[ \log \left( \frac{D^+}{D^-} \right)(\lambda) = \log \Pi(0) \equiv \frac{1}{2} \log \lambda, \]
\[ \log(D^+D^-)(\lambda) = \int_{-\infty}^{+\infty} \Pi(q) \, dq = 2 I, \]
\[ I \overset{\text{def}}{=} \int_{0}^{+\infty} (V(q) + \lambda)^{1/2} \, dq, \]

where this divergent “improper action integral” gets specified just like \( \log D^\pm \):

\[ \frac{dI}{d\lambda} = \frac{1}{2} \int_{0}^{+\infty} (V(q) + \lambda)^{-1/2} \, dq \equiv \frac{1}{4} \int_{-\infty}^{+\infty} (V(q) + \lambda)^{-1/2} \, dq \]

converges, then \( I(\lambda) \) is defined as that primitive of (11) which is devoid of a constant \((\propto \lambda^0)\) term in its large-\( \lambda \) expansion.

Improper actions as in (10) (i.e., along infinite paths) offer many benefits for asymptotic and exact WKB analysis. WKB solutions of (1) can now be defined intrinsically: e.g., as \( \Psi^{\text{WKB}}(q) = \Pi(q)^{-1/2} \exp \int_{-\infty}^{q} \Pi(q') \, dq' \)=[], unlike the traditional forms which awkwardly involve extraneous base points. The geometrical analysis no longer requires to set infinite paths apart as it used to [13, 17]. Moreover, the algebra itself is simplified; e.g., consider the full determinant \( D(\lambda) \overset{\text{def}}{=} (D^+D^-)(\lambda) \): previously, to get the large-\( \lambda \) expansion of \( \log D \) in the simplest case \( V(q) = q^{2M} \) [13], we had to factor \( D = D_{\text{cl}} a \) (\( a(\lambda) \) is the “Jost function”), then expand \( \log a \) using \( \log a = \int_{-\infty}^{+\infty} [U - \Pi](q) \, dq \) where \( \Psi(q) = U(q)^{-1/2} \exp \int U \, dq \) parametrizes an exact solution of (1), and finally obtain \( \log D_{\text{cl}} \) by other means; now that the improper integrals (10) are allowed, all that condenses into a single identity (valid for general \( V \)):
\[ \log D \equiv \int_{-\infty}^{+\infty} U(q) \, dq. \]

2 Explicit improper actions for trinomial \( \Pi(q)^2 \)

In [1], we computed the improper action integral \( I = \int_{0}^{+\infty} \Pi(q) \, dq \) in closed form for any binomial \( \Pi(q)^2 = uq^N + vq^M \); then (§ 4.2) we stated that we could no longer do so for a trinomial of the general (even) form \( \Pi(q)^2 = \)


\( q^N + vq^M + \lambda \) (with \( N > M > 0 \)), for which we just needed the \( v \to +\infty \) behavior of \( I \) anyway [1, equation (4.16)], reproduced as (36) below.

It is nevertheless wrong to infer from the above that strictly no exact computations can be done in fully trinomial cases, and we now present several examples (still for even \( q^N + vq^M + \lambda \) with positive \( v, \lambda \)). After recalling the closed-form results for binomials, we will quote another, trivial and degenerate, instance: perfect-square trinomials. Then, our main new case will be the quartic anharmonic oscillator: we can reduce its improper action exactly to standard (i.e., convergent) action integrals, and therefrom to complete elliptic integrals [2, 15, 5, 16], as (30)–(32) below; we then verify the abovementioned large-\( v \) behavior on this case \( (N = 4) \). Finally, the same approach must work for higher-degree polynomial \( V(q) \), converting

\[
\int_0^{+\infty} \Pi(q) \text{d}q \text{ exactly into convergent hyperelliptic integrals \cite{17, 18}; but since the latter remain not so explicitly understood, we will skip this case \( (N > 4) \) here.}
\]

### 2.1 Binomial \( \Pi(q)^2 \) : exact evaluation

For \( \Pi(q)^2 = uq^N + vq^M \) (with \( N > M \geq 0 \)), \( \int_0^{+\infty} \Pi(q) \text{d}q \) was exactly computed in \cite[§ 4.1]{1}. We recall the main formulae for later convenience:

\[
I = \int_0^{+\infty} (uq^N + vq^M)^{1/2} \text{d}q \overset{\text{def}}{=} \lim_{s \to 0} I_0(s),
\]

\[
I_0(s) = \int_0^{+\infty} (uq^N + vq^M)^{1/2-s} \text{d}q \quad \text{(Re } s > \frac{1}{2} + \frac{1}{N})
\]

\[
\equiv \frac{\Gamma(M(1-2s)+2)}{(N-M)\Gamma(s-1/2)} \frac{\Gamma(-N(1-2s)+2)}{2(N-M)} \frac{\Gamma(-M(1-2s)+2)}{2(N-M)} u^{-M(1-2s)+2} v^{-N(1-2s)+2},
\]

where “\( s \to 0 \)” implies analytical continuation in \( s \), with the result:

\[
I = \frac{\Gamma(j - \frac{1}{2}) \Gamma(-j)}{(N-M)\Gamma(-\frac{1}{2})} u^{-j+1/2} v^j \quad (j = \frac{N+2}{2(N-M)} > \frac{1}{2}) \text{ when finite; }
\]

otherwise, i.e., when \( j = 1, 2, \ldots, \) a further “canonical” renormalization yields

\[
I = -\frac{2j \beta_{-1}(0)}{N+2} \left[ \log v - \sum_{m=1}^{j} \frac{1}{m} - \frac{2M}{N} \left( \log 2 + \frac{1}{2} \log u - \sum_{m=1}^{j-1} \frac{1}{2m-1} \right) \right],
\]

\[
\beta_{-1}(0) = (-1)^{j-1} \frac{(2j - 2)!}{2^{j-1}(j-1)! j!} u^{-j+1/2} v^j.
\]

(17)
We repeat from [1] the examples we will mostly need:

\[
\int_0^{+\infty} (wq^N + \lambda)^{1/2} dq = -\frac{\Gamma(1 + \frac{1}{N})}{2\sqrt{\pi}} \frac{\Gamma\left(-\frac{1}{2} - \frac{1}{N}\right)}{w^{-\frac{1}{2}}\lambda^{\frac{1}{2} + \frac{1}{N}}} \quad (N \neq 2) \quad (18)
\]

\[
= -\frac{1}{3} w^{-1/2}\lambda (\log \lambda - 1) \quad (N = 2) \quad (19)
\]

\[
\int_0^{+\infty} (q^4 + vq^2)^{1/2} dq = -\frac{1}{3} v^{3/2} \quad , (20)
\]

all based on (16) except (19), which uses (17) with \( j = 1 \).

2.2 Perfect-square trinomials: \( \Pi(q)^2 = (q^M + \sqrt{\lambda})^2 \) (\( M \) even)

This degenerate case trivially reduces to a binomial formula like (16), using

\[
\int_0^{+\infty} \left[(q^M + wq^L)^2\right]^{1/2-s} dq = \frac{\Gamma\left(\frac{L(1-2s)+1}{M-L}\right) \Gamma\left(-\frac{M(1-2s)+1}{M-L}\right)}{(M-L)\Gamma(2s-1)} w^{\frac{M(1-2s)+1}{M-L}} \quad (21)
\]

for \( M > L \geq 0 \) and \( w > 0 \). The singular formula (17) is never needed in our setting (\( \Pi(q)^2 \) even): for \( M \) and \( L \) even, no pole can appear in the numerator of (21) at \( s = 0 \); but one appears in the denominator instead, leading to

\[
I = \int_0^{+\infty} \left[(q^{N/2} + \sqrt{\lambda})^2\right]^{1/2} dq \equiv 0 \quad \text{(for even } N/2 > 0) \quad (22)
\]

2.3 The general even quartic case: \( \Pi(q)^2 = q^4 + vq^2 + \lambda \)

At present, we mean to exploit the large toolbox of results readily available for the complete elliptic integrals [19, 20, 21]: specifically here,

\[
K(k) \triangleq \int_0^1 \left[ (1-t^2)(1-k^2t^2) \right]^{-1/2} dt, \quad E(k) \triangleq \int_0^1 \left[ \frac{(1-k^2t^2)}{(1-t^2)} \right]^{1/2} dt, \quad (23)
\]

as functions of the modulus \( k \); the complementary modulus is \( k' \triangleq \sqrt{1-k^2} \).

Main needed formulae

- special values: [19, formulae 13.8(5),(6),(15),(16)]

\[
K(0) = E(0) = \frac{1}{2}\pi \quad (24)
\]
\[ K(\frac{1}{\sqrt{2}}) = \frac{\Gamma(\frac{1}{4})^2}{4\sqrt{\pi}}, \quad E(\frac{1}{\sqrt{2}}) = \frac{1}{2} \left[ K(\frac{1}{\sqrt{2}}) + \frac{\pi}{2K(\frac{1}{\sqrt{2}})} \right]; \] (25)

- derivatives: [20, formulae 710.00, 710.02][21, formulae 8.123(2),(4)]

\[ \frac{dK}{dk} = \frac{E(k)}{kk'} - \frac{K(k)}{k} \quad \left( \frac{dE}{dk} = \frac{E(k) - K(k)}{k} \text{ is not used here} \right); \] (26)

- expansions for \( k \to 1^- \iff k' \to 0^+ \) (implying \( E(1) = 1 \)): [22, p. 93–94] [2, footnote 11 p. 184] [20, formulae 900.05, 900.07]

\[ K(k) = \log \frac{4}{k'} + \frac{1}{4} \left[ \log \frac{4}{k'} - 1 \right] k'^2 + O(k'^4 \log k') \]

\[ E(k) = 1 + \frac{1}{2} \left[ \log \frac{4}{k'} - \frac{1}{2} \right] k'^2 + \frac{3}{16} \left[ \log \frac{4}{k'} - \frac{13}{12} \right] k'^4 + O(k'^6 \log k') \] (27)

- selected transformation formulae: [19, Table 4 p. 319]

\[ K(k) = \frac{1 + \tilde{k}'}{2} K(\tilde{k}), \quad E(k) = \frac{E(\tilde{k}) + \tilde{k}'K(\tilde{k})}{1 + k'} \quad \text{for } k = \frac{1 - \tilde{k}'}{1 + k'} \] (28)

\[ K(k) = \tilde{k}' K(\tilde{k}), \quad E(k) = \frac{E(\tilde{k})}{k'} \quad \text{for } k = \frac{i \tilde{k}}{k'}. \] (29)

**Our closed-form result**

For non-negative \( v \) and \( \lambda \) (as in [1], and mainly for simplicity), we find:

\[ I \overset{\text{def}}{=} \int_0^{+\infty} (q^4 + vq^2 + \lambda)^{1/2} dq \] (30)

\( (v \geq 2\sqrt{\lambda}) : \equiv \frac{1}{3} (v^2 + 2\sqrt{\lambda})^{1/2}[2\sqrt{\lambda}K(k) - vE(k)], \quad k = \left( \frac{v - 2\sqrt{\lambda}}{v + 2\sqrt{\lambda}} \right)^{1/2}; \] (31)

\( (v \leq 2\sqrt{\lambda}) : \equiv \frac{1}{3} \lambda^{1/4}[(2\sqrt{\lambda} + v)K(\tilde{k}) - 2vE(\tilde{k})], \quad \tilde{k} = \frac{(2\sqrt{\lambda} - v)^{1/2}}{2\lambda^{1/4}}. \) (32)

**Derivation.** We first specify \( dI/d\lambda \) by means of (11) for \( V(q) = q^4 + vq^2 \). In contrast to (30), here the integrand \( (V(q) + \lambda)^{-1/2} \) is integrable at \( q = \infty \) in \( \mathbb{C} \), allowing to deform the path \((-\infty, +\infty)\) to a **bounded** contour in the complex \( q \)-plane:

\[ \frac{dI}{d\lambda} = \frac{1}{4} \int_{-\infty}^{+\infty} (q^4 + vq^2 + \lambda)^{-1/2} dq = \frac{1}{4} \int_C (q^4 + vq^2 + \lambda)^{-1/2} dq \] (33)
where \( C \) is, e.g., a positive contour encircling the pair of roots \( i q_\pm \) of \( \Pi(q)^2 \) (turning points) that lie in the upper half-plane. We now prefer to pursue explicitly with \( v \geq 2\sqrt{\lambda} \) (and analytically continue the result to \( v \leq 2\sqrt{\lambda} \) later) then \( 0 < q_- \leq q_+ \), cf. Fig. 1(a). The last integral in (33), being taken over a bounded path, admits a closed-form primitive with respect to \( \lambda \), as

\[
\hat{I}(\lambda) = \frac{1}{2} \int_C (q^4 + vq^2 + \lambda)^{1/2} dq
\]

\[
= -\int_{q_-}^{q_+} (-q^4 + vq^2 - \lambda)^{1/2} dq = -\frac{1}{3} q_+ [vE(\hat{k}) - 2q_2^2 K(\hat{k})]
\]  

where \( q_\pm = [\frac{1}{2}(v \pm \sqrt{v^2 - 4\lambda})]^{1/2} \) and \( \hat{k} = [1 - q_-^2/q_+^2]^{1/2}, \quad \hat{k}' = q_-/q_+ \)

[19, formula 3.155(1) for \( u = b \) and (amplitude) \( \lambda = \pi/2 \) [2, formula (4.22)²].

We cannot rush to conclude that \( I = \hat{I} \): the former contour deformation is ill-justified for the divergent integral \( I \) itself. On the other hand, we find that it simplifies future steps to use the transformation formula (28) which turns (34) into the expression (31), but still for \( \hat{I} \).

Next, we continue (31) to the region \( \{v \leq 2\sqrt{\lambda}\} \) (\( k \) pure-imaginary) by means of the transformation (29), which results in the expression (32) again for \( \hat{I} \). Only then are we able to probe the \( \lambda \to +\infty \) behavior of \( \hat{I} \) at fixed \( v \) using \( \hat{k} = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}} v\lambda^{-1/2} + O(\lambda^{-1}) \) and (25)–(26), we obtain

\[
\hat{I}(\lambda) \sim \frac{2}{3} \lambda^{3/4}[K(\frac{1}{\sqrt{2}}) + \frac{dK}{dk}(\frac{1}{\sqrt{2}}) \frac{1}{4\sqrt{2}} v\lambda^{-1/2} + O(\lambda^{-1})]
\]

\[
+ \frac{1}{3} v\lambda^{1/4}[K(\frac{1}{\sqrt{2}}) + O(\lambda^{-1/2})] - \frac{2}{3} v\lambda^{1/4}[E(\frac{1}{\sqrt{2}}) + O(\lambda^{-1/2})]
\]

\[
\sim \frac{2}{3} K(\frac{1}{\sqrt{2}}) \lambda^{3/4} - \frac{\pi}{4K(\frac{1}{\sqrt{2}})} v\lambda^{1/4} + O(\lambda^{-1/4}) \quad (\lambda \to +\infty); \quad (35)
\]

it has no constant (\( \propto \lambda^0 \)) term, hence indeed \( \hat{I} \equiv I \), the wanted canonical primitive as defined initially, cf. (10). QED.

Remark 1. The trivial outcome \( (I = \hat{I}) \) seems to justify the above contour deformation directly for the divergent integral (30), but this is misleading: our \( \Pi(q) \) kept a null residue \( \beta_{-1}(0) \), like all even \( \Pi(q) \) with \( N \equiv 0 \mod 4 \); but generically, \( \beta_{-1}(0) \neq 0 \) (e.g., already for trinomial even \( \Pi(q)^2 \) but with \( N = 6, 10, \ldots \)), and nontrivial integration constants \( I - \hat{I} \neq 0 \) ought to follow.

²We think there should be no factor \( \rho^{1/2} \) on the left-hand side of this formula.
Applications

We can first verify (30)–(32) upon special cases, known earlier:

- $\lambda = 0$: $I = -\frac{1}{3} v^{3/2} E(1) = -\frac{1}{3} v^{3/2}$ by (27) for $E(1)$, cf. (20);
- $v = 0$: $I = \frac{2}{3} K\left(\frac{1}{\sqrt{2}}\right) \lambda^{3/4} = \frac{\Gamma(3/4)^2}{6\sqrt{\pi}} \lambda^{3/4}$ by (25), cf. (18) for $N = 4$;
- $v = 2\sqrt{\lambda}$: $I = \frac{2\sqrt{2}}{3} v^{3/2} [K(0) - E(0)] \equiv 0$ by (24), cf. (22).

But above all, we can use the exact expression (31) to check the $v \to +\infty$ behavior of $I$ directly. Earlier, we predicted the asymptotic form for the general trinomial case to be, for $v \to +\infty$ at fixed $\lambda$, [1, equation (4.16)]

$$\int_0^{+\infty} (q^N + v q^M + \lambda)^{1/2} dq \sim \int_0^{+\infty} (q^N + v q^M)^{1/2} dq + \int_0^{+\infty} (v q^M + \lambda)^{1/2} dq \equiv 0$$

where the first line is to be made explicit through (16)–(20), and $\delta$ (last line) is the Kronecker delta symbol.

However, our derivation of (36) was quite indirect, and lacked independent tests. Now the present results allow such a test: in the quartic case, we can directly expand $I$ in its exact form (31) for $v \to +\infty$, i.e., $k \to 1^-$, and

$$k' \equiv 2(\sqrt{v}/v)^{1/2} (1 + 2\sqrt{v}/v)^{-1/2} \to 0^+.$$  

Then, using (27), $I \equiv -\frac{4}{3} \lambda^{3/4} k'^{-3} [(2 - k'^2) E(k) - k'^2 K(k)]$ expands as

$$I \sim -\frac{8}{3} \lambda^{3/4} k'^{-3} [1 - \frac{3}{4} k'^2 - \frac{3}{16} (\log \frac{4}{k'} - \frac{1}{4}) k'^4 + O(k'^n \log k')]$$

the substitution of $k'$ by (37) yields the desired $v \to +\infty$ expansion in terms of $v^{3/2-n}$ and $v^{-1/2-n} \log v$, $n \in \mathbb{N}$ (no $v^{1/2} \log v$ term!). Remarkably, the next subleading term (of order $v^{1/2}$) also cancels, so that finally

$$I \sim -\frac{1}{3} v^{3/2} - \frac{1}{4} \lambda v^{-1/2} (\log(\lambda/v^2) - 4 \log 2 - 1) \left[ + O(v^{-3/2} \log v) \right]$$

This asymptotic equivalent then identically reproduces the prediction made by (36) for $N = 4$ and $M = 2$ with the help of (19)–(20), which confirms our basic earlier result [1, equation (4.16)].
3 The $v \to \infty$ behavior of the determinants

We return to the spectral determinants of the quantum problem (1):

$$D_N^\pm(\lambda, v) = \det^\pm(-d^2/dq^2 + q^N + vq^M + \lambda), \quad (40)$$

which are entire functions of $(\lambda, v) \in \mathbb{C}^2$ [24].

3.1 Review of the $v \to +\infty$ results

Our key intermediate result in [1, equations (3.10–12)] was, for $v \to +\infty$:

$$D_N^\pm(\lambda, v) \sim e^{I(v)} e^{\delta_{M,2}A_0(\lambda, v)} \det^\pm(-d^2/dq^2 + vq^M + \lambda), \quad (41)$$

Now the asymptotic formula (36) reduces this to

$$D_N^\pm(\lambda, v) \sim e^{I(v)} e^{\delta_{M,2}A_0(\lambda, v)} \det^\pm(-d^2/dq^2 + vq^M + \lambda), \quad (42)$$

$$I(v) = \int_0^{+\infty} (q^N + vq^M)^{1/2} dq, \quad A_0(\lambda, v) = \frac{N}{4(N-2)}(\log v + 2 \log 2) v^{-1/2}. \quad (43)$$

On the other hand, the exact scaling laws (2) and (7) for $u = 0$, plus

$$\beta_{-1}(0) \equiv \delta_{M,2} \Lambda/2 \quad \text{for } \Pi(q) = (q^M + \Lambda)^{1/2}, \quad (44)$$

entail (writing $D_M^\pm(\Lambda) \equiv \det^\pm(-d^2/dq^2 + q^M + \Lambda)$):

$$\det^\pm(-d^2/dq^2 + vq^M + \lambda) \equiv v^{1/[2(M+2)]} e^{-\delta_{M,2}v^{-1/2}\lambda/8} D_M^\pm(v^{-2/(M+2)}); \quad (45)$$

our net asymptotic result was thus [1, equations (5.1–4)]

$$D_N^\pm(\lambda, v) \sim e^{I(v)} e^{\delta_{M,2}A(\lambda, v)} v^{1/[2(M+2)]} D_M^\pm(\Lambda) \quad (v \to +\infty), \quad (46)$$

with $I(v) = \int_0^{+\infty} (q^N + vq^M)^{1/2} dq$ given by (16) if $j = \frac{N+2}{2(N-M)} \notin \mathbb{N}$, or by (17) otherwise, and

$$\delta_{M,2} A(\lambda, v) = \delta_{M,2} \frac{1}{8(N-2)} [(N + 2) \log v + 4N \log 2] \Lambda, \quad (47)$$

$$\Lambda \equiv v^{-2/(M+2)} \lambda \quad (\equiv v^{-1/2} \lambda \text{ in } (47), \text{ used when } M = 2). \quad (48)$$
Figure 1: Plots of the Stokes geometry in the complex $q$-plane for $\Pi(q)^2 = q^4 + vq^2 + \lambda$ and large complex $v$, ordered clockwise with increasing $\theta = \arg v$ ($|v| = 5, \lambda = 0.5$). The intermediate plots (b–e) set $\lambda = 0$ to emulate the $|v| = \infty$ regime at finite $q$; in that limit the Stokes curve $S$ (bold line) stays linked to $q = +\infty$ (arrow) for $\theta < \Theta$ (here $\Theta = 2\pi/3$, by (49)).

3.2 Extension to a sector in the complex $v$-plane

The key to our proof of the asymptotic formula (41) for positive $v \to +\infty$ was [1, § 3.2] that a solution $\Psi_{\lambda}(q,v)$ of (1) with a recessive WKB form for $q \to +\infty$ connects all the way down (in that WKB form) to a region $\{1 \ll q \ll v^{1/(N-M)}\}$ – where it then tends to a similarly recessive solution $\Psi_{0,\lambda}(q,v)$ of the uncoupled Schrödinger equation $[-(d^2/dq^2) + vq^M + \lambda]\Psi_{0,\lambda}(q) = 0$.

In the complex domain, a simple sufficient condition for the WKB form to be preserved is for $q \in \mathbb{C}$ to stay within one Stokes region of the momentum function $\Pi(q)$ [13]. In terms of $\theta \overset{\text{def}}{=} \arg v$, the above connection condition then becomes that the Stokes region containing $\{1 \ll e^{i\theta/(M+2)}q \ll$
E.g., when \( v > 0 \) the central Stokes region does include all of \( \mathbb{R} \), cf. Fig. 1(a).

We now need to describe the Stokes geometry for \( \Pi(q)^2 = q^N + vq^M + \lambda \) with complex \( v \to \infty \); Fig. 1 illustrates the case \( N = 4, M = 2 \).

When \( |v| \to \infty \) : the approximate factorization of \( \Pi(q)^2 \) as \( (q^N - M + v) \times (q^M + \lambda/v) \) makes \( M \) of its complex turning points \( q_j \) shrink (\( \approx v^{-1/M} \), “inner” roots) and the other \( (N - M) \) grow (\( \approx v^{1/(N-M)} \), “outer” roots); moreover, the central Stokes region contracts to a symmetrical pair of Stokes curves from \( q = 0 \) for the zero-energy momentum \( \Pi_{\lambda=0}(q)^2 = q^N + vq^M \), and we are to follow the \( (\theta\text{-dependent}) \) Stokes curve \( S \) which starts as \( S = R^+ \) when \( \theta = 0 \), cf. Fig. 1(b). In the large-\( v \) limit, the connection condition is that \( \theta \) can be increased above 0 as long as \( S \) remains linked to \( q = +\infty \) (Fig. 1(c–d); the complex-conjugate picture results for \( \theta < 0 \)).

Following [17, § 3], the connection condition breaks (cf. Fig. 1(e)) when the action integral \( I = \int_0^{q_0} (q^N + vq^M)^{1/2} dq \) becomes real, where \( q_0 \) is the first outer turning point met by \( S \) as \( \theta \) recedes from 0. That action, of instanton type [10], is computable in closed form:

\[
q_0 = e^{-i\pi/(N-M)} v^{1/(N-M)},
\]

and \( I = \sqrt{\pi} N^{M+2} \Gamma(M+2)/(2(N-M)) \Gamma(N^2 + 2(N-M))/(2(N-M)) [e^{-i(M+2)\pi} v^{N+2}/2(N-M)^{1/2}] \), which turns real first at \( \arg v = \frac{M+2}{N+2} \pi \). Consequently, all of § 3.1 extends to the asymptotic sector

\[
\Sigma = \{ v \to \infty, |\arg v| < \Theta \}, \quad \Theta = \frac{M+2}{N+2} \pi.
\]

**Remark 2.** Some examples (with \( \lambda \equiv 0 \)) make us hope that our end asymptotic formula (46) might actually hold up to \( |\arg v| < \pi \).

1) For \( Q_\pm(v) = \det \pm(-d^2/dq^2 + q^4 + vq^2) \), this was suggested by our numerical observations [12, equation (87) vs Fig. 1] that \( Q_\pm(v) \) behave analogously to the Airy functions \( \text{Ai}(v), \text{Ai}'(v) \) for \( v \to -\infty \) as well (even though \( \Theta = 2\pi/3 \) only).

2) The supersymmetric determinants \( \det \pm(-d^2/dq^2 + q^N + vq^{N/2-1}) \) are known in closed form, essentially as inverse \( \Gamma \)-functions of \( v \) [12, equation (120)]: their large-\( v \) asymptotics then amount to the Stirling formula, and the latter definitely holds for \( |\arg v| < \pi \) (vs \( \Theta = \pi/2 \)).

**4 Asymptotics and the functional relation**

An early puzzle of general exact quantization conditions was their breakdown (both analytical and numerical) for potentials \( q^N + vq^2 \) in the regime \( v \gg 1 \).
(as seen for $N = 4$ [11]). Naively, convergence to the elementary harmonic $(vq^2)$ behavior would have been expected. We can now show that singularity to be unessential: i.e., the original functional relations (Wronskian identities) which produce those quantization conditions behave as well as possible when $v \to +\infty$ for any potential $q^N + vq^M$, currently under the restriction $2M + 2 > N$ (which encompasses $q^4 + vq^2$, for instance).

### 4.1 The basic Wronskian identity

The spectral determinants for a general polynomial potential $V(q)$ of degree $N$ obey the bilinear functional relation: [11, equation (40)]

$$e^{+i \phi_N/4} D^+[1] D^- - e^{-i \phi_N/4} D^+ D^-[1] \equiv 2i e^{i \phi_N \beta_{-1}(0)/2},$$

where $D^{[1]}$ are the determinants for the first conjugate problem: [24, § 7]

$$V(q) \mapsto V^{[1]}(q) \overset{\text{def}}{=} e^{-i \phi_N} V(e^{-i \phi_N/2} q) \quad \text{and} \quad \lambda \mapsto \lambda^{[1]} \overset{\text{def}}{=} e^{-i \phi_N} \lambda,$$

with $\phi_N \overset{\text{def}}{=} \frac{4\pi}{N + 2}$: the symmetry angle in degree $N$. (52)

Equation (50) is but a Wronskian identity for the Schrödinger equation (1), yet it has a key dynamical role: while it seems underdetermined, it implies a complete set of exact quantization conditions, which then solve (1) exactly.

A certain iterate of the transformation (51) is the identity, hence (50) has a cyclic symmetry group, specifically of order $(\frac{1}{2}N + 1)$ when $V$ is even.

For the trinomial determinants (40), the first-conjugate parameters are

$$\lambda^{[1]} = e^{-i \phi_N} \lambda, \quad v^{[1]} = e^{i \pi/j} v, \quad \text{with} \quad j \equiv \frac{N + 2}{2(N - M)} \quad \text{as in (16)}. \quad (53)$$

### 4.2 The $v \to \infty$ transition

According to (46)–(48) with $\Lambda = v^{-2/(M+2)} \lambda$, $D^{\pm}_N(\lambda, v)$ for finite $v$ is a deformation from $D^{\pm}_M(\Lambda)$ at $v = +\infty$, but the key parameter in the dynamical functional relation (50), namely the degree of $V$, and often the residue $\beta_{-1}(0)$ as well [1, § 3.1], suffer sharp jumps at $v = \infty$. It is then a non-trivial task to find out whether the basic identity (50) for $D^{\pm}_N$ continuously evolves into its counterpart for $D^{\pm}_M$ in the $v \to +\infty$ limit of (46), or not.
Under \( \Lambda \mapsto \Lambda^{[1]} \) as in (53), the rescaled spectral parameter \( \Lambda \) maps to
\[
\Lambda \mapsto \Lambda^{[1]} = \exp[-\frac{2}{M+2} \frac{N-M}{2} i \varphi_N - i \varphi_N] \Lambda = e^{-i \varphi_M} \Lambda; \quad (54)
\]
already this is the correct rotation angle for the limiting determinants \( D_M^\pm \).

To get the asymptotic form of (50) with \( D_M^\pm \equiv D_N^\pm(\Lambda, v) \), we let \( v \to \infty \) in its left-hand side with \( \arg v = -\pi/2j \), \( \arg v^{[1]} = +\pi/2j \), and we invoke (46). The latter, by (49), requires \( \pi/2j < \Theta \iff j > 1 \) or \( 2M + 2 > N \) (otherwise the calculation will still work, but only formally until (49) extends to a wider sector). The left-hand side of (50) thus displays the asymptotic form
\[
\exp[I(v) + I(v^{[1]})] \exp \delta_{M,2}[A(\Lambda, v) + A(\Lambda^{[1]}, v^{[1]})] \times
\left[ z D_M^+(\Lambda^{[1]}) D_M^-(\Lambda) - z^{-1} D_M^-(\Lambda) D_M^+(\Lambda^{[1]}) \right], \quad (55)
\]
\[
z \overset{\text{def}}{=} e^{i \varphi_N/4} (v^{[1]}/v)^{1+1/2(M+2)} \quad \text{(a pure phase)}.
\]

We now evaluate all the terms in (55): first,
\[
I(v) \propto v^j \quad [\Rightarrow I(v^{[1]}) = -I(v)] \quad \text{if } j \notin \mathbb{N} \text{ (cf. (16), with } \beta_-(0) = 0) \nonumber
\]
\[
= -\frac{2j}{N+2} \beta_-(0) [\log v + \text{const.}] \quad \text{if } j \in \mathbb{N} \text{ (cf. (17), with } \beta_-(0) \propto v^j) \nonumber
\]
\[
\Rightarrow I(v) + I(v^{[1]}) = \frac{2j}{N+2} \beta_-(0) i \frac{\pi}{j} \nonumber
\]
\[
\Rightarrow I(v) + I(v^{[1]}) \equiv i \varphi_N \beta_-(0)/2 \quad \text{in all cases, cf. (53).} \quad (56)
\]
Next, just as for (54),
\[
z = e^{i \varphi_N/4} e^{i \pi/[2(M+2)]} = e^{i \varphi_N(N+2)/(4(M+2))} \equiv e^{i \varphi_M/4}. \quad (57)
\]
Finally, and only relevant when \( M = 2 \), in which case \( \Lambda^{[1]} = -\Lambda \),
\[
A(\Lambda, v) + A(\Lambda^{[1]}, v^{[1]}) = -\frac{N+2}{8(N-2)} i \frac{\pi}{16} \Lambda = -\frac{N+2}{16} i \varphi_N \Lambda \equiv -i \pi \Lambda/4. \quad (58)
\]

In the end, substituting (54)–(58) into (50) we indeed get
\[
e^{i \varphi_M/4} D_M^+(e^{-i \varphi_M} \Lambda) D_M^-(\Lambda) - e^{-i \varphi_M/4} D_M^+(\Lambda) D_M^-(e^{-i \varphi_M} \Lambda) \equiv 2 i e^{\delta_{M,2} i \pi \Lambda/4}, \quad (59)
\]
which is the correct form of (50) for \( D_M^+(\Lambda) = \det^+(d^2/dq^2 + q^M + \Lambda) \) (whose \( \beta_-(0) \) is given by (44)). For more details: if \( M > 2 \), see [23, equation (5.32)]; if \( M = 2 \), then
\[
D_2^+(\Lambda) = 2^{1-\Lambda/2} \sqrt{\pi}/\Gamma(\frac{1+\Lambda}{4}), \quad D_2^-(\Lambda) = 2^{-\Lambda/2} \sqrt{\pi}/\Gamma(\frac{3+\Lambda}{4}), \quad (60)
\]
and (59) with its “anomalous” right-hand side boils down to the reflection formula for the Gamma function; the harmonic-oscillator quantization condition can then also be recovered solely from (59) [12, Appendix A.2.3].

In conclusion, we have verified that the exact functional relation (50), governing both $D_N^\pm(\lambda, v)$ and $D_M^\pm(\Lambda)$ (cf. (59)), is compatible with the general perturbation formula (46), currently under the restriction $2M + 2 > N$, or $j > 1$ (which includes the quartic oscillators): this further validates the exact-WKB description of perturbative regimes in [1]. Remaining desirable tasks are: 1) to lift the restriction $j > 1$ (e.g., by extending (46) to $\{|\arg v| < \pi\}$, cf. Remark 2 in § 3.2); and 2) to find exact quantization conditions that themselves behave continuously in the zero-coupling limit (here, $v = +\infty$).

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\(^3\)Our contribution to this volume (pp. 97–108) needs the same corrigendum as [11].

\(^4\)In this work we mistakenly used “quasi-exactly solvable” for “supersymmetric” (systems) throughout — this affects none of the results.
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