Strings in pp-wave background and background B-field from membrane and its symplectic quantization

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Abstract
The symplectic quantization technique is applied to open free membrane and strings in pp-wave background and background gauge field obtained by compactifying the open membrane in the presence of a background anti-symmetric 3–form field. In both cases, first the Poisson brackets among the Fourier modes are obtained and then the Poisson brackets among the membrane(string) coordinates are computed. The full noncommutative phase-space structure is reproduced in case of strings in pp-wave background and background gauge field. We feel that this method of obtaining the Poisson algebra is more elegant than previous approaches discussed in the literature.

Keywords: Membranes, pp-wave, Noncommutativity

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1 Introduction
It has been discovered recently that there exists a new maximal supersymmetric IIB supergravity background, namely pp-wave Ramond-Ramond (RR) background [1]. It consists of a plane-wave metric supported by a homogeneous RR 5–form flux

\[ ds^2 = 2dX^+ dX^- - \mu^2 X^I X^I (dX^+)^2 + dX^I dX^I, \quad I = 1, \ldots, 8 \]
\[ F_{I+1234} = F_{I+5678} = 2\mu. \quad (1) \]

The background has 32 symmetries and is related (by a special limit [2]) to the AdS$_5 \times$ S$^5$ background [1, 3, 4]. Remarkably, string theory in this background is exactly solvable [5]. Solvability in this context means that it is possible to find exact solutions of the classical string equations of motion, perform a canonical quantization and then determine the Hamiltonian operator.

On the other hand, the theory of membranes has also been studied extensively over the last decade [6]. In a recent paper [7], it has been shown that one can obtain the action of an infinite number of massive strings in the pp-wave background by compactifying the bosonic membrane action. Some properties of closed and open strings in this background has also been investigated in this paper.

Noncommutativity is another area which has attracted a lot of attention in the past few years [8] owing to the inspiration of superstring theories. It is a well known result now that open
strings attached to D–branes in the presence of a background anti-symmetric B–field induces noncommutativity at its end points, i.e. along the world–volume of the D–brane \[9, 10, 11\]. This fact further reveals the nontrivial role of boundary conditions (BC) in string theory and the need of taking them into account when considering the quantization of open strings. The most conventional way in which this result has been derived is by employing the Dirac approach \[12\] with the string BC(s) imposed as second class constraints in \[13, 10, 14, 15\]. A more elegant approach of obtaining this noncommutativity, done in spirit to the treatment of Hanson et.al \[16\] (where modified Poisson brackets (PB) were obtained for the free Nambu-Goto string), is by modifying the canonical bracket structure, so that it is compatible with the BC(s) \[17, 18, 19, 20\].

A conformal field theoretic approach to this problem has been studied in \[21, 22\]. In \[23\], the Faddeev-Jackiw (FJ) symplectic formalism \[24\] has been applied to obtain the PB(s) among the Fourier modes which appear in the solutions of the classical string equations of motion. Using this they obtained the PB(s) among the open string coordinates revealing the noncommutative structure at the string end points.

In this paper, we first show that strings in pp-wave background and background gauge fields can be obtained by compactifying the open membrane action in the presence of a background 3–form anti-symmetric field. We then employ the FJ technique (as done before in \[23\]) to obtain the PB(s) among the Fourier modes and then between the string coordinates. The advantage of this method is that one need not know all the constraint chains by the consistency requirements that arise in the Dirac approach.

The organization of this paper is as follows. To illustrate our method, we begin by discussing the free open membrane in section 2. In section 3, we discuss the interacting membrane and the symplectic structure of strings in pp-wave background and background gauge field obtained by compactifying the interacting membrane. Finally, we conclude in section 4.

## 2 Free open membrane

An open membrane is a two dimensional object which sweeps out a three dimensional world-volume parametrized by \(\tau, \sigma^1\) and \(\sigma^2\). These parameters can be collectively referred to as \(\xi^i\) \((i = 0, 1, 2)\). The Polyakov action for the bosonic membrane is then given by \[6\]

\[
S = \int d\tau L = -\frac{1}{4\pi\alpha'} \int d^3\xi \left( g^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - 1 \right) \quad ; \quad (a, b = 0, 1, 2)
\]

where,

\[
L = -\frac{1}{4\pi\alpha'} \int_0^\pi \int_0^\pi d\sigma^1 d\sigma^2 \left( g^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - 1 \right)
\]

is the Lagrangian and \(g_{ab} = diag(-, +, +)\), \(\eta_{\mu\nu} = diag(-, +, ..., +)\). The final term \((-1)\) stands for the cosmological constant and does not appear in the string theory action.

The variation of \[2\] gives the equation of motion

\[
(\partial_0^2 - \partial_1^2 - \partial_2^2) X^\mu(\tau, \sigma^1, \sigma^2) = 0
\]

and two types of BC(s). They are the Dirichlet BC(s)

\[
\delta X^\mu(\tau, 0, \sigma^2)|_{\sigma^1=0,\pi} = 0
\]
\[
\delta X^\mu(\tau, \pi, \sigma^2)|_{\sigma^1=0,\pi} = 0
\]

and the Neumann BC(s)

\[
\partial_1 X^\mu(\tau, 0, \sigma^2)|_{\sigma^1=0,\pi} = 0
\]
\[
\partial_2 X^\mu(\tau, \pi, \sigma^2)|_{\sigma^2=0,\pi} = 0.
\]
The Hamiltonian is obtained by means of Legendre transformation:

$$\Pi_\mu(\tau, \sigma^1, \sigma^2) = \frac{\delta S_p}{\delta(\partial_\nu X^\mu(\tau, \sigma^1, \sigma^2))} = \eta_{\mu \nu} \partial_\nu X^\nu. \tag{7}$$

In order to quantize consistently, we need well-defined PB(s) among the canonical variables $X^\mu(\tau, \sigma^1, \sigma^2)$ and $\Pi_\mu(\tau, \sigma^1, \sigma^2)$ which read:

$$\{X^\mu(\tau, \sigma^1, \sigma^2), \Pi_\nu(\tau, \sigma^{1'}, \sigma^{2'})\} = \delta_\mu^\nu \delta(\sigma^1 - \sigma^{1'}) \delta(\sigma^2 - \sigma^{2'}) \tag{8}$$

$$\{X^\mu(\tau, \sigma^1, \sigma^2), X^\nu(\tau, \sigma^{1'}, \sigma^{2'})\} = \{\Pi_\mu(\tau, \sigma^1, \sigma^2), \Pi_\nu(\tau, \sigma^{1'}, \sigma^{2'})\} = 0. \tag{9}$$

The Hamiltonian is obtained by means of Legendre transformation

$$H = \int_0^\pi \int_0^\pi d\sigma^1 d\sigma^2 \Pi_\mu \partial_\nu X^\mu - L$$

$$= \frac{1}{2} \int_0^\pi \int_0^\pi d\sigma^1 d\sigma^2 [\eta_{\mu \nu} (\Pi^\mu \Pi^\nu + \partial_1 X^\mu \partial_1 X^\nu + \partial_2 X^\mu \partial_2 X^\nu) - 1] \quad \tag{10}$$

and the time-evolution of $X^\mu(\tau, \sigma^1, \sigma^2)$, $\Pi_\mu(\tau, \sigma^1, \sigma^2)$ is governed by

$$\partial_\nu X^\mu(\tau, \sigma^1, \sigma^2) = \{X^\mu(\tau, \sigma^1, \sigma^2), H\} \quad \tag{11}$$

$$\partial_\nu \Pi_\mu(\tau, \sigma^1, \sigma^2) = \{\Pi_\mu(\tau, \sigma^1, \sigma^2), H\}. \quad \tag{12}$$

However, at the open membrane endpoints, the PB(s) (8) are not compatible with the BC(s) (5) (6). This implies that the basic PB(s) must be modified in order to make them consistent with the BC(s). In the rest of the paper, we shall work with the Neumann BC(s) (6). The solution to the equations of motion (11) compatible with the BC(s) (6) read

$$X^\mu(\tau, \sigma^1, \sigma^2) = x^\mu_0 + p^\mu \tau + i \sum_{n=1}^{\infty} \frac{1}{\sqrt{m}} (\alpha^\mu_{n0} e^{i n \tau} - \alpha^\mu_{n0}^\dagger e^{-i n \tau}) \cos(n \sigma^1)$$

$$+ i \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} (\alpha^\mu_{0m} e^{i m \tau} - \alpha^\mu_{0m}^\dagger e^{-i m \tau}) \cos(m \sigma^2)$$

$$+ i \sum_{n,m=1}^{\infty} (n^2 + m^2)^{-1/4} (\alpha^\mu_{nm} e^{i \sqrt{n^2+m^2} \tau} - \alpha^\mu_{nm}^\dagger e^{-i \sqrt{n^2+m^2} \tau}) \cos(n \sigma^1) \cos(m \sigma^2)$$

$$= x^\mu_0 + p^\mu \tau + i \sum_{n=1}^{\infty} \frac{1}{\sqrt{m}} (\alpha^\mu_{n0}(\tau) - \alpha^\mu_{n0}^\dagger(\tau)) \cos(n \sigma^1)$$

$$+ i \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} (\alpha^\mu_{0m}(\tau) - \alpha^\mu_{0m}^\dagger(\tau)) \cos(m \sigma^2)$$

$$+ i \sum_{n,m=1}^{\infty} (n^2 + m^2)^{-1/4} (\alpha^\mu_{nm}(\tau) - \alpha^\mu_{nm}^\dagger(\tau)) \cos(n \sigma^1) \cos(m \sigma^2) \quad \tag{13}$$

where we have defined $\alpha^\mu_{n0}(\tau) = \alpha^\mu_{n0} e^{i n \tau}$, $\alpha^\mu_{n0}^\dagger(\tau) = \alpha^\mu_{n0}^\dagger e^{-i n \tau}$, and so on.

The canonically conjugate momenta (7) expressed in terms of the Fourier components read
\[ \Pi_\mu(\tau, \sigma^1, \sigma^2) = \eta_{\mu\nu} \left[ p^\nu - \sum_{n=1}^{\infty} \sqrt{n} \left( \alpha_{n0}^\nu(\tau) + \alpha_{n0}^{\nu\dagger}(\tau) \right) \cos(n\sigma^1) \right. \\
- \sum_{m=1}^{\infty} \sqrt{m} \left( \alpha_{0m}^\nu(\tau) + \alpha_{0m}^{\nu\dagger}(\tau) \right) \cos(m\sigma^2) \right. \\
- \left. \sum_{n,m=1}^{\infty} \left( n^2 + m^2 \right)^{1/4} \left( \alpha_{nm}^\nu(\tau) + \alpha_{nm}^{\nu\dagger}(\tau) \right) \cos(n\sigma^1) \cos(m\sigma^2) \right]. \tag{14} \]

We shall now use the FJ method [24] to obtain the PB(s) between the Fourier components. The idea is to write a Lagrangian in the first-order form

\[ L = a_n(\xi) \partial_0 \xi^n - H \tag{15} \]

where \( \xi^n \) stand for all the canonical variables and \( a_n(\xi) \) can be read directly from the inverse of the matrix

\[ f_{mn} = \frac{\partial a_n(\xi)}{\partial \xi^n} - \frac{\partial a_m(\xi)}{\partial \xi^m} \tag{16} \]

provided the inverse of \( f_{mn} \) exists. The first order form of the Lagrangian (3) reads:

\[ L = \int_0^\pi d\sigma^1 d\sigma^2 \Pi_\mu \partial_0 X^\mu - H . \tag{17} \]

Substituting (13) and (14) in the above equation yields

\[ L = \pi^2 \eta_{\mu\nu} \left[ p^\mu p^\nu - \frac{i}{2} \sum_{n=1}^{\infty} \left( \alpha_{n0}^\mu(\tau) + \alpha_{n0}^{\mu\dagger}(\tau) \right) \left( \dot{\alpha}_{n0}^\mu(\tau) - \dot{\alpha}_{n0}^{\mu\dagger}(\tau) \right) \right. \\
- \frac{i}{2} \sum_{m=1}^{\infty} \left( \alpha_{0m}^\mu(\tau) + \alpha_{0m}^{\mu\dagger}(\tau) \right) \left( \dot{\alpha}_{0m}^\mu(\tau) - \dot{\alpha}_{0m}^{\mu\dagger}(\tau) \right) \right. \\
- \left. \frac{i}{4} \sum_{n,m=1}^{\infty} \left( \alpha_{nm}^\mu(\tau) + \alpha_{nm}^{\mu\dagger}(\tau) \right) \left( \dot{\alpha}_{nm}^\mu(\tau) - \dot{\alpha}_{nm}^{\mu\dagger}(\tau) \right) \right] - H \tag{18} \]

where,

\[ H = \frac{\pi^2}{2} \eta_{\mu\nu} \left[ p^\mu p^\nu + \sum_{n=1}^{\infty} n \left( \alpha_{n0}^{\mu\dagger} \alpha_{n0}^\nu + \alpha_{n0}^\nu \alpha_{n0}^{\mu\dagger} \right) \right. \\
+ \sum_{m=1}^{\infty} m \left( \alpha_{0m}^{\mu\dagger} \alpha_{0m}^\nu + \alpha_{0m}^\nu \alpha_{0m}^{\mu\dagger} \right) \right. \\
+ \left. \frac{1}{2} \sum_{n,m=1}^{\infty} \left( n^2 + m^2 \right)^{1/2} \left( \alpha_{nm}^{\mu\dagger} \alpha_{nm}^\nu + \alpha_{nm}^\nu \alpha_{nm}^{\mu\dagger} \right) \right] - \frac{\pi^2}{2} \tag{19} \]

and the dots denote differentiation w.r.t. \( \tau \).

It is now easy to read from the above first order form of the Lagrangian (18) three sets of variables \( \xi^n = (\alpha_{n0}^\mu(\tau), \alpha_{n0}^{\mu\dagger}(\tau)) \), \( \xi^m = (\alpha_{0m}^\mu(\tau), \alpha_{0m}^{\mu\dagger}(\tau)) \), \( \xi^{nm} = (\alpha_{nm}^\mu(\tau), \alpha_{nm}^{\mu\dagger}(\tau)) \), and their
corresponding one forms $a_n(\xi) = -\frac{i\pi^2}{2} \eta_{\mu\nu}([\alpha_{\nu 0}^{\mu}(\tau) + \alpha_{\nu 0}^{\nu}(\tau)), -(\alpha_{\nu 0}^{\nu}(\tau) + \alpha_{\nu 0}^{\nu}(\tau))], a_m(\xi) = -\frac{i\pi^2}{2} \eta_{\mu\nu}([\alpha_{\nu m}(\tau) + \alpha_{\nu m}^{\nu}(\tau)), -(\alpha_{\nu m}^{\nu}(\tau) + \alpha_{\nu m}^{\nu}(\tau))].$ The matrix $f$ for these three sets of variables can now be computed using (16) and the result is

$$f = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}$$

in which 0 is a null matrix and $B$ is a diagonal matrix

$$B_{mn}^{\mu\nu} = i\pi^2 \eta_{\mu\nu} \delta_{nm}; \quad (n, n' = 1, 2, \ldots) \quad \text{for } \alpha_{n0}^{\mu} \text{ modes}$$

$$B_{mn}^{\mu\nu} = i\pi^2 \eta_{\mu\nu} \delta_{nm}; \quad (m, m' = 1, 2, \ldots) \quad \text{for } \alpha_{0m}^{\mu} \text{ modes}$$

$$B_{nn' mm'}^{\mu\nu} = \frac{i\pi^2}{2} \eta_{\mu\nu} \delta_{nm'} \delta_{mm'}; \quad (n, n', m, m' = 1, 2, \ldots) \quad \text{for } \alpha_{nm}^{\mu} \text{ modes.}$$

The inverse of the matrix $f$ can be easily obtained and reads

$$f^{-1} = \begin{pmatrix} 0 & -\frac{1}{B} \\ 1 & 0 \end{pmatrix}.$$  

Hence, according to FJ method, the non-trivial PB(s) are given by

$$\{\alpha_{n0}^{\mu}(\tau), \alpha_{n0}^{\nu}(\tau)\} = \frac{i}{\pi^2} \eta_{\mu\nu} \delta_{nn'}$$

$$\{\alpha_{0m}^{\mu}(\tau), \alpha_{0m}^{\nu}(\tau)\} = \frac{i}{\pi^2} \eta_{\mu\nu} \delta_{mm'}$$

$$\{\alpha_{nm}^{\mu}(\tau), \alpha_{nm}^{\nu}(\tau)\} = \frac{2i}{\pi^2} \eta_{\mu\nu} \delta_{nn'} \delta_{mm'}.$$  

which further reduces to

$$\{\alpha_{n0}^{\mu}, \alpha_{n0}^{\nu}\} = \frac{i}{\pi^2} \eta_{\mu\nu} \delta_{nn'}$$

$$\{\alpha_{0m}^{\mu}, \alpha_{0m}^{\nu}\} = \frac{i}{\pi^2} \eta_{\mu\nu} \delta_{mm'}$$

$$\{\alpha_{nm}^{\mu}, \alpha_{nm}^{\nu}\} = \frac{2i}{\pi^2} \eta_{\mu\nu} \delta_{nn'} \delta_{mm'}.$$  

Now substituting (13) and (19) in (11) leads to the PB(s) among the zero modes:

$$\{x_0^{\mu}, p^\nu\} = \frac{1}{\pi^2} \eta_{\mu\nu}. \quad (27)$$

With the above results in hand, the PB(s) among the canonical variables $X^{\mu}(\tau, \sigma^1, \sigma^2)$ and $\Pi_{\mu}(\tau, \sigma^1, \sigma^2)$ read:

$$\{X^{\mu}(\tau, \sigma^1, \sigma^2), X^{\nu}(\tau, \sigma^{1'}, \sigma^{2'})\} = \{\Pi_{\mu}(\tau, \sigma^1, \sigma^2), \Pi_{\nu}(\tau, \sigma^{1'}, \sigma^{2'})\} = 0 \quad (28)$$

$$\{X^{\mu}(\tau, \sigma^1, \sigma^2), \Pi_{\nu}(\tau, \sigma^{1'}, \sigma^{2'})\} = \delta^{\mu}_{\nu} \Delta_{+}(\sigma^1, \sigma^{1'}) \Delta_{+}(\sigma^2, \sigma^{2'}) \quad (29)$$

where,

$$\Delta_{+}(\sigma, \sigma') = \frac{1}{\pi} \left(1 + \sum_{n \neq 0} \cos(n\sigma) \cos(n\sigma')\right) \quad (30)$$

satisfies the usual properties of the delta function in the interval $[0, \pi]$ [16].

Note that the above symplectic structure is consistent with the Neumann BC(s) (6).
3 Open membrane in the constant three-form field background

The Polyakov action of a membrane in the presence of a background anti-symmetric three-form field $A_{\mu\nu\rho}$ reads:

$$S = -\frac{1}{4\pi\alpha'} \int d^3 \xi \left[ (g^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - 1) + \frac{1}{3} \epsilon^{abc} A_{\mu\nu\rho} \partial_a X^\mu \partial_b X^\nu \partial_c X^\rho \right] . \quad (31)$$

where $\epsilon^{abc}$ is anti-symmetric in all the indices.

The variation of the above action leads to the equations of motion (4) and the BC(s)

$$\left( \partial_1 X^\mu - A^\mu_{\nu\rho} \partial_0 X^\nu \partial_2 X^\rho \right) |_{\sigma^1 = 0, \pi} = 0 \quad (32)$$

$$\left( \partial_2 X^\mu + A^\mu_{\nu\rho} \partial_0 X^\nu \partial_1 X^\rho \right) |_{\sigma^2 = 0, \pi} = 0 . \quad (33)$$

The canonically conjugate momenta $\Pi_\mu$ to $X^\mu$ is given by:

$$\Pi_\mu (\tau, \sigma^1, \sigma^2) = \eta_{\mu\nu} \partial_0 X^\nu - A_{\mu\nu\rho} \partial_1 X^\nu \partial_2 X^\rho . \quad (34)$$

Using $\Pi_\mu$, the BC(s) (32), (33) can be expressed in terms of the phase-space variables as:

$$\left[ \left( \eta^{\mu\kappa} - A^{\mu\nu\rho} A^\kappa_{\nu\beta} \partial_2 X^\beta \partial_0 X^\rho \right) \partial_1 X^\kappa - A^{\mu\nu\rho} \Pi^\nu \partial_2 X^\rho \right] |_{\sigma^1 = 0, \pi} = 0 \quad (35)$$

$$\left[ \left( \eta^{\mu\beta} + A^{\mu\nu\rho} A^\nu_{\kappa\beta} \partial_1 X^\kappa \partial_0 X^\rho \right) \partial_2 X^\beta + A^{\mu\nu\rho} \Pi^\nu \partial_1 X^\rho \right] |_{\sigma^2 = 0, \pi} = 0 . \quad (36)$$

The above form of the BC(s) indicates that it is problematic to find exact solutions to the equations of motion [4]. So we study the low energy limit where the membrane goes to string theory in the limit of small radius for the cylindrical membrane.

To do this, we take the $\sigma^2$-direction of the membrane to be wrapped around a circle with radius $R$. We choose further the gauge fixing condition [7] [18] [1]

$$X^2 = \sigma^2 \quad ; \quad (0 \leq \sigma^2 \leq 2\pi R). \quad (37)$$

Now substituting the Fourier expansion of the world-volume fields $X^\mu (\tau, \sigma^1, \sigma^2) (\mu \neq 2)$:

$$X^\mu (\tau, \sigma^1, \sigma^2) = \sum_{n=-\infty}^{+\infty} X_n^\mu (\tau, \sigma^1) e^{ina/R} \quad ; \quad X_{-n}^\mu (\tau, \sigma^1) = X_n^\mu (\tau, \sigma^1)^\dagger \quad (38)$$

in the action (31) and using (37), we obtain (recovering the $2\pi \alpha'$ factor):

$$S = \frac{1}{2\tilde{\alpha}'} \sum_{n=-\infty}^{+\infty} \int d\tau d\sigma^1 \left( \partial_0 X_n^\mu (\tau, \sigma^1) \partial_0 X_{-n}^\mu (\tau, \sigma^1) - \partial_1 X_n^\mu (\tau, \sigma^1) \partial_1 X_{-n}^\mu (\tau, \sigma^1) \right) + m_n^2 \left( \partial_1 X_n^\mu (\tau, \sigma^1) \partial_1 X_{-n}^\mu (\tau, \sigma^1) - A_{\mu\nu\rho} \partial_0 X_n^\mu (\tau, \sigma^1) \partial_1 X_{n}^\nu (\tau, \sigma^1) \right)$$

$$+ \frac{1}{2\tilde{\alpha}'} \sum_{n=m \neq 0, n \neq -m} \frac{i(n+\mu)}{R} A_{\mu\nu\rho\neq 2} \partial_0 X_n^\mu (\tau, \sigma^1) \partial_1 X_{m}^\nu (\tau, \sigma^1) X_{-n}^\rho (\tau, \sigma^1) X_{-m}^\nu (\tau, \sigma^1) X_{m}^\rho (\tau, \sigma^1) ; \quad m_n = |m| \frac{R}{\tilde{\alpha}'}, \quad \tilde{\alpha}' = \alpha' / R \quad (39)$$

\[ ^{1}\text{Note that if the } \sigma^2 \text{- direction of the membrane is wrapped around a circle of radius } R, \text{ then the } X^2 \text{-direction is also compact on the same circle.} \]
where,

\[ S_0 = \frac{1}{2\alpha'} \int d\tau d\sigma^1 \left[ \dot{X}_0^\mu(\tau, \sigma^1) \dot{X}_0^\mu(\tau, \sigma^1) - \partial_1 X_0^\mu(\tau, \sigma^1) \partial_1 X_0^\mu(\tau, \sigma^1) \right. \]

\[ \left. - B_{\mu\nu} \dot{X}_0^\mu(\tau, \sigma^1) \partial_1 X_0^\mu(\tau, \sigma^1) \right] \tag{40} \]

is the usual low energy string theory action in the presence of background gauge field \( A_{\mu \nu} = B_{\mu \nu} \). and

\[ S_n = \frac{1}{2\alpha'} \int d\tau d\sigma^1 \left[ \partial_0 X_n^\mu(\tau, \sigma^1) \partial_0 X_{-n\mu}(\tau, \sigma^1) - \partial_1 X_n^\mu(\tau, \sigma^1) \partial_1 X_{-n\mu}(\tau, \sigma^1) \right. \]

\[ \left. - m_n^2 X_n^\mu(\tau, \sigma^1) X_{-n\mu}(\tau, \sigma^1) - B_{\mu\nu} \partial_0 X_n^\mu(\tau, \sigma^1) \partial_1 X_{-n\nu}(\tau, \sigma^1) \right] \tag{41} \]

is the action of massive strings in pp-wave background and background gauge field \( B_{\mu\nu} \).\(^2\) Clearly, from (39), (40) and (41) we observe that the last term contains modes which are higher in energy than the first two terms. Hence in the low energy limit, we only consider the first two terms in the action (39). The symplectic quantization of the usual string theory action (40) leads to the well-known noncommutativity at the end points of the string (23). In this paper, we shall carry out the symplectic quantization of massive strings in pp-wave background and background gauge field \( B_{\mu\nu} \).

The variation of \( S_n \) gives the equation of motion

\[ \left( \partial_0^2 - \partial_1^2 + m_n^2 \right) X_n^\mu(\tau, \sigma^1) = 0 \tag{42} \]

and the BC(s)

\[ \left( \partial_1 X_n^\mu(\tau, \sigma^1) - B_{\mu\nu} \partial_0 X_{-n\nu}(\tau, \sigma^1) \right) \bigg|_{\sigma^1 = 0,\pi} = 0 . \tag{43} \]

The canonically conjugate momenta \( \Pi_{n\mu}(\tau, \sigma^1) \) to \( X_n^\mu(\tau, \sigma^1) \) reads:

\[ \Pi_{n\mu}(\tau, \sigma^1) = \eta_{\mu\nu} \partial_0 X_{-n\nu}(\tau, \sigma^1) - B_{\mu\nu} \partial_1 X_{-n\nu}(\tau, \sigma^1). \tag{44} \]

Using \( \Pi_{n\mu}(\tau, \sigma^1) \), the above BC can be expressed in terms of phase-space variables as:

\[ \left[ M_{\mu\rho} \partial_1 X_n^{\mu}(\tau, \sigma^1) - B_{\mu\rho} \Pi_{-n\rho}(\tau, \sigma^1) \right] \bigg|_{\sigma^1 = 0,\pi} = 0 \tag{45} \]

where, \( M_{\mu\rho} = (\delta_{\mu\rho} - B_{\mu\nu} B_{\nu\rho}) \).

The solution to the equations of motion (42) compatible with the above BC(s) read\(^2\)

\[ X_n^\mu(\tau, \sigma^1) = X_n^{\mu(0)}(\tau, \sigma^1) + X_n^{\mu(1)}(\tau, \sigma^1) \tag{46} \]

where,

\[ X_n^{\mu(0)}(\tau, \sigma^1) = \left[ x_0^\mu \cos(\tilde{m}_n \tau) + p_0^\mu \frac{\sin(\tilde{m}_n \tau)}{\tilde{m}_n} \right] \cosh(\tilde{m}_n B \sigma^1) \]

\[ - B_{\mu\nu} \left[ - p_0^\nu \cos(\tilde{m}_n \tau) + \tilde{m}_n x_0^\nu \sin(\tilde{m}_n \tau) \right] \frac{\sinh(\tilde{m}_n B \sigma^1)}{\tilde{m}_n} \tag{47} \]

is the “zero mode” part (i.e. the modes with the lowest frequency) and

\[ X_n^{\mu(1)}(\tau, \sigma^1) = \sum_{l \neq 0} \frac{1}{\omega_l^{\mu \nu}} e^{-i\omega_l \tau} \left( \frac{\omega_l}{l} B_{\mu\nu} a_{l\nu}^\mu \sin(l \sigma^1) \right) \tag{48} \]

\(^2\)We consider the case in which the \( B \)-field takes the form \( B_{\mu\nu} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \).
The constant $B$ is the eigen-value of the matrix $B_{\mu}^{\nu}$ and the frequencies are defined by:

$$\omega_{ln} = \text{sgn}(l)\sqrt{l^2 + m^2_n}; \quad \tilde{m}_n = \frac{m_n}{\sqrt{1 + B^2}}.$$  \hspace{1cm} (49)

Using $X^\mu_n(\tau, \sigma^1) = X^\mu_{-n}(\tau, \sigma^1)$, we find that the Fourier modes of the massive strings satisfy the relations

$$x^\mu_{0n} \equiv x^\mu_{0, -n}; \quad p^\mu_{0n} \equiv p^\mu_{0, -n}; \quad a^\mu_{l,n} = a^\mu_{-l, -n}.$$  \hspace{1cm} (50)

The canonically conjugate momenta $\Pi^\mu_n(\tau, \sigma^1)$ to $X^\mu_n(\tau, \sigma^1)$ expressed in terms of the Fourier modes read:

$$
\Pi^\mu_n(\tau, \sigma^1) = \sum_{l>0} M^\mu_\rho \left[ a^\rho_{l, n} e^{-i\omega_{ln}\tau} + a^\rho_{-l, -n} e^{i\omega_{ln}\tau} \right] \cos(\sigma^1)
+ iB^\mu_{\rho} \sum_{l>0} \left( \frac{l}{\omega_{ln}} - \frac{\omega_{ln}}{l} \right) \left[ a^\rho_{l, n} e^{-i\omega_{ln}\tau} - a^\rho_{-l, -n} e^{i\omega_{ln}\tau} \right] \sin(\sigma^1).
\hspace{1cm} (51)
$$

Now we apply the FJ procedure to obtain the algebra between the non-zero Fourier modes. To do this, we once again write down the “non-zero mode” part of the Lagrangian in a first order form

$$L_n^{(1)} = \int_0^\pi d\sigma^1 \Pi^\mu_n(\tau, \sigma^1) X^\mu_n(\tau, \sigma^1) - H_n^{(1)}$$  \hspace{1cm} (52)

which on substitution of the mode expansions (48) and (51) read

$$L_n^{(1)} = \frac{i\pi}{2} \eta_{\mu\nu} \sum_{l>0} \left[ \frac{1}{\omega_{ln}} Q^\mu_\rho(\ln) \left( a^\rho_{l, n} e^{-i\omega_{ln}\tau} + a^\rho_{-l, -n} e^{i\omega_{ln}\tau} \right) \right]
\times \partial_0 \left( a^\rho_{l, n} e^{-i\omega_{ln}\tau} - a^\rho_{-l, -n} e^{i\omega_{ln}\tau} \right) + \cdots - H_n^{(1)}$$  \hspace{1cm} (53)

where, $Q^\mu_\rho(\ln) = (\delta^\mu_\rho - \frac{\omega^2}{\omega_{ln}^2} B^\mu_{\nu} B^{\nu}_{\rho})$ and the ‘...’ represent terms which do not play a role in the determination of the symplectic structure. The explicit form of $H_n^{(1)}$ is also not required for obtaining the PB(s) between the modes.

As before, one can again read a set of variables $\xi^\mu_l$ and the corresponding canonical one-form $a_{nl}(\xi)$. The matrix (16) once again reads the same as (20) with the diagonal matrix $B$ being:

$$B_{\mu\nu}^{l'n'} = -\frac{i\pi}{\omega_{ln}} \text{sgn}(l)Q^\mu_\nu(\ln)\delta_{n+n',0} \delta_{l,l'}.$$  \hspace{1cm} (54)

Now from the inverse of $f$ which reads the same as (24), it is easy to read the PB(s) among the “non-zero” Fourier modes using the FJ technique. They are:

$$\{a^\mu_{ln}(\tau), a^\nu_{l',-n'}(\tau)\} = -\frac{i}{\pi} \text{sgn}(l)\omega_{ln}Q^{-1}_\ln(\ln)_{\mu\nu} \delta_{n+n',0} \delta_{l,l'}.$$  \hspace{1cm} (55)

The PB(s) among the zero modes $x^\mu_{0n}$ and $p^\mu_{0n}$ can be determined from the evolution equations (11) (12) and read:

$$\{x^\mu_{0n}, p^\nu_{0n'}\} = \frac{\pi \tilde{m}_n \tilde{B}}{\pi \tanh(\pi \tilde{m}_n \tilde{B})} (M^{-1})^{\mu\nu}_{n+n',0}.$$  \hspace{1cm} (56)

where $M_{\mu\nu}$ has been defined earlier. Now since the BC(s) are valid on the boundaries, it is natural to demand that

$$\{X^\mu_n(\tau, \sigma^1), X^\nu_{n'}(\tau, \sigma^{1'})\} = 0; \quad \text{for } \sigma^1, \sigma^{1'} \in (0, \pi)$$  \hspace{1cm} (57)
from which we get:

\[
\{x^\mu_{0n}, x^\nu_{0n'}\} = -(BM^{-1})^{\mu\nu} \delta_{n+n',0} \quad (58)
\]

\[
\{p^\mu_{0n}, p^\nu_{0n'}\} = -\tilde{m}^2(BM^{-1})^{\mu\nu} \delta_{n+n',0} \quad . \quad (59)
\]

With the above results \((55, 56, 58, 59)\) in hand, and after some lengthy calculation, we obtain the following PB(s):

\[
\{X^\mu_n(\tau, \sigma^1), \Pi^\nu_{n'}(\tau, \sigma'^1)\} = \delta^\mu_\nu \delta_{n+n',0} \Delta_+ (\sigma^1, \sigma'^1) \quad (60)
\]

\[
\{X^\mu_n(\tau, \sigma^1), X^\mu_{n'}(\tau, \sigma'^1)\} = \delta_{n+n',0} (BM^{-1})^{\mu\nu} \times \begin{cases} 
1 & \sigma^1 = \sigma'^1 = 0 \\
-1 & \sigma^1 = \sigma'^1 = \pi \\
0 & \text{otherwise}
\end{cases} \quad (61)
\]

\[
\{\Pi^\mu_n(\tau, \sigma^1), \Pi^\nu_{n'}(\tau, \sigma'^1)\} = \delta_{n+n',0} m^2 B^\mu_\nu \times \begin{cases} 
+1 & \sigma^1 = \sigma'^1 = 0 \\
-1 & \sigma^1 = \sigma'^1 = \pi \\
0 & \text{otherwise}
\end{cases} \quad (62)
\]

which agrees with \([25]\). Note that in the \(B \to 0\) limit, the noncommutativity vanishes and the results are in conformity with \([7]\).

4 Conclusions

In this paper, we employ the Faddeev-Jackiw symplectic formalism to study the problem of open free membrane and strings in pp-wave background and background gauge field obtained by compactification of interacting membrane. The starting point is the solutions to the classical membrane(string) equations of motion. It is then observed that one can find the PB(s) among the Fourier modes first, using which the PB(s) among the original variables can be obtained. This idea was first proposed in \([9]\) where the authors used the time-independent symplectic form \(([26], [25])\) to fix the PB(s) among the Fourier components. In this paper, we follow a slightly different method (due to \([23]\)) to obtain the symplectic structure among the Fourier modes. The solutions of the membrane(string) equations of motion are substituted into the Lagrangian and then integration over the spatial variables is carried out to cast the Lagrangian in a first order form involving the Fourier modes from which the PB(s) (among the Fourier modes) can be easily read off. Finally, using this algebra we obtain a noncommutative phase-space structure. Our results agree with the previous work in the literature \([25]\).

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