Tail Probability and Singularity of Laplace-Stieltjes
Transform of a Heavy Tailed Random Variable

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Abstract

In this paper, we will give a sufficient condition for a non-negative random variable $X$ to be heavy tailed by investigating the Laplace-Stieltjes transform of the probability distribution function. We focus on the relation between the singularity at the real point of the axis of convergence and the asymptotic decay of the tail probability. Our theorem is a kind of Tauberian theorems.

Keywords: Tail probability; Heavy tail; Laplace-Stieltjes transform; Tauberian theorem

1 Introduction

We consider the asymptotic decay of the tail probability $P(X > x)$ of a heavy tailed random variable $X$. A random variable $X$ is said to be heavy tailed if

$$\eta \equiv \lim_{x \to \infty} \frac{\log P(X > x)}{\log x} < 0.$$  \hspace{1cm} (1)

$\eta$ is called the decay rate of $P(X > x)$.

Let $F(x)$ be the probability distribution function (pdf) of $X$, i.e., $F(x) = P(X \leq x)$. For example, if $F(x) = 1 - 1/x, \ x \geq 1$, then $P(X > x) = 1/x$, hence $X$ is heavy tailed with decay rate $\eta = -1$.

In this paper, we will give a sufficient condition for $X$ to be heavy tailed by analytic properties of the Laplace-Stieltjes (LS) transform of $F(x)$. The LS transform of $F(x)$ is defined by

$$\varphi(s) = \int_{0}^{\infty} e^{-sx} dF(x).$$ \hspace{1cm} (2)

In general, for a function $R(x)$, which is of bounded variation in the interval $0 \leq x \leq c$ for any positive $c$, the LS transform $\Psi(s) = \int_{0}^{\infty} e^{-sx} dR(x), \ s = \sigma + i\tau$, is defined. If $\Psi(s)$ converges for $\sigma > \sigma_0$ and diverges for $\sigma < \sigma_0$, then $\sigma_0$ is said to be the abscissa of convergence of $\Psi(s)$. The line
ℜs = σ₀ is called the axis of convergence. In the case σ₀ = 0, some local information of Ψ(s) at 

\( s = 0 \) provides the asymptotic behavior of \( R(x) \) as \( x \to \infty \). Such a proposition is called a Tauberian theorem. The following is one of the Tauberian theorems.

**Theorem** (Widder [14], p.192, Theorem 4.3) Let \( R(x) \) be a non-decreasing function and the abscissa of convergence of \( Ψ(s) \) be \( σ₀ = 0 \). If for constants \( r ≤ 0 \) and \( A \)

\[
\lim_{s \to 0^+} |Ψ(s)s^{-r} - A| = 0, \tag{3}
\]

then

\[
\lim_{x \to \infty} |R(x)Γ(r + 1)x^r - A| = 0, \tag{4}
\]

where \( Γ \) denotes the gamma function.

In [9],[11],[12], we studied the asymptotic decay of a light tailed random variable. A random variable \( X \) is said to be light tailed if the tail probability \( P(X > x) \) decays exponentially, i.e.,

\[
\lim_{x \to \infty} \frac{1}{x} \log P(X > x) < 0. \tag{5}
\]

We obtained in [12] the following theorem which gives a sufficient condition for a light tailed random variable.

**Theorem** (Nakagawa [12]) For a non-negative random variable \( X \) with probability distribution function \( F(x) \), let \( ϕ(s) = \int_0^\infty e^{-sx}dF(x) \) be the Laplace-Stieltjes transform of \( F(x) \) and \( σ₀ \) be the abscissa of convergence of \( ϕ(s) \). We assume \( -∞ < σ₀ < 0 \). If \( s = σ₀ \) is a pole of \( ϕ(s) \), then we have

\[
\lim_{x \to \infty} \frac{1}{x} \log P(X > x) = σ₀. \tag{6}
\]

It is known that for a monotonic \( R(x) \), \( s = σ₀ \) is a singularity of \( Ψ(s) \) (see Widder [14], p.58, Theorem 5b). Since pdf \( F(x) \) is monotonic increasing, \( σ₀ \) is a singularity of \( ϕ(s) \). If \( F(x) \) is the pdf of a heavy tailed random variable, then the abscissa of convergence of \( ϕ(s) \) is necessarily \( σ₀ = 0 \) (see Widder [14], p.40, Theorem 2.2b). Since \( ϕ(0) = \int_0^\infty dF(x) = 1 \), \( σ₀ \) is not a pole, but other type of singularity.

In the research of the asymptotic decay of a tail probability, we would like to construct a general theory such that a local analytic information of \( ϕ(s) \) at \( s = σ₀ \) tells the asymptotic evaluation of the tail probability. In a light tailed case [9],[11],[12], we applied to this problem Ikehara’s Tauberian theorem [5],[7] and its extension Graham-Vaaler’s Tauberian theorem [3],[7]. Ikehara’s theorem assumes a global analytic property of \( ϕ(s) \), that is, \( s = σ₀ \) is a pole and there exist no other singularities on the axis of convergence \( ℜs = σ₀ \). While, Graham-Vaaler’s theorem only assumes \( s = σ₀ \) is a pole, which yields weaker assertion than Ikehara’s theorem, however, it is enough for our purpose to investigate the asymptotic decay of a tail probability. In this paper, we will apply Graham-Vaaler’s Tauberian theorem to the decay of heavy tailed random variable.
The purpose of our study is to apply our theory to the performance evaluation of the packet network. According to the research results for the Internet packet stream, it is reported that many characteristics are approximated by heavy tailed random variables \[1\]. Contrary to the light tailed case, if, for example, the packet length is heavy tailed, the network performance, such as packet loss probability or end-to-end packet delay becomes worse. So, it is very important to investigate the tail probability of heavy tailed random variable from the viewpoint of network engineering. When we apply the queueing theory to network engineering, even if we do not obtain the tail probability of packet length explicitly, we may obtain its Laplace transform by algebraic manipulation as Pollaczek-Khinchin formula \[3\]. Then, we know the singularity of the Laplace transform and the asymptotic decay can be investigated by our theory.

Throughout this paper, we will use the following symbols. \(\mathbb{N}, \mathbb{N}^+, \mathbb{R}, \mathbb{C}\), denote the set of natural numbers, positive natural numbers, real numbers, complex numbers, and further, \(\mathbb{R}, \mathcal{L}, \mathcal{F}\) denote the real part of a complex number, Laplace transform and Fourier transform, respectively.

2 Examples of Heavy Tailed Random Variable

Now, we look at some examples of heavy tailed random variables and their LS transforms.

2.1 Continuous Random Variable I

Let \(X\) be a random variable with pdf \(F(x) = 1 - 1/x, \ x \geq 1\). The decay rate is \(\eta = -1\). The LS transform \(\varphi(s)\) of \(F(x)\) is represented in a neighborhood of \(s = 0\) as

\[
\varphi(s) = \int_1^\infty e^{-sx}dF(x) = s \log s + \beta(s),
\]

where \(\beta(s)\) is analytic in a neighborhood of \(s = 0\) (see Lemma 1).

2.2 Continuous Random Variable II

For a pdf \(F(x) = 1 - 1/\sqrt{x}, \ x \geq 1\), we have the decay rate \(\eta = -1/2\) and

\[
\varphi(s) = \frac{2\pi}{\Gamma(1/2)} \sqrt{s} + \beta(s),
\]

where \(\beta(s)\) is analytic in a neighborhood of \(s = 0\) (see Lemma 2).

2.3 Discrete Random Variable

Let \(X\) be a discrete random variable with probability distribution \(p = (p_n)_{n \in \mathbb{N}};\)

\[
p_n = \frac{1}{(n + 1)^r} - \frac{1}{(n + 2)^r}, \ n \in \mathbb{N}, \ r \in \mathbb{N}^+.
\]
The tail probability of \( X \) is
\[
P(X > x) = \frac{1}{(n + 2)^r}
\]
and the decay rate is \( \eta = -r \). The probability generating function (pgf) \( p(z) \) of \( p \) is
\[
p(z) = \frac{z - 1}{z^2} \sum_{n=1}^{\infty} \frac{z^n}{n^r} + \frac{1}{z}.
\] (11)
Substituting \( z = e^{-s} \), we have
\[
\varphi(s) = p(e^{-s}) = \alpha(s)s^r \log s + \beta(s),
\] (12)
where \( \alpha(s) \) and \( \beta(s) \) are analytic in a neighborhood of \( s = 0 \) with \( \alpha(0) \neq 0 \) (see Lemma 3).

**2.4 Stationary Distribution of M/G/1 Type Markov Chain**

We consider the tail probability of the stationary distribution of an M/G/1 type Markov chain \([2]\). Let \( X \) be a heavy tailed discrete random variable with probability distribution \( b = (b_n)_{n \in \mathbb{N}} \). Denote by \( B(z) \) the pgf of \( b \). Consider an M/G/1 type Markov chain with probability transition matrix
\[
P = \begin{pmatrix}
b_0 & b_1 & b_2 & b_3 & \ldots \\
b_0 & b_1 & b_2 & b_3 & \ldots \\
b_0 & b_1 & b_2 & b_3 & \ldots \\
b_0 & b_1 & b_2 & b_3 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\] (14)
where \( a = (a_n)_{n \in \mathbb{N}} \) is a probability distribution with pgf \( A(z) \). Suppose there exists the stationary distribution \( \pi = (\pi_n)_{n \in \mathbb{N}} \) of \( P \). Let \( \pi(z) = \sum_{n=0}^{\infty} \pi_n z^n \) be the pgf of \( \pi \), then by the Pollaczek-Khinchin formula \([2],[4],[10]\),
\[
\pi(z) = \frac{\pi_0 (zB(z) - A(z))}{z - A(z)}.
\] (15)
Substituting \( z = e^{-s} \) into (15), we have \( \varphi(s) = \pi(e^{-s}) \). If the singularity of \( \pi(z) \) at \( z = 1 \) comes from the singularity of \( B(z) \), we know the singularity of \( \varphi(s) \) at \( s = 0 \) comes from \( B(e^{-s}) \). So, it is expected that \( \pi \) should be heavy tailed.

**3 Main Theorem**

From above examples, we expect that the following theorems hold. These are main theorems in this paper.

**Theorem 1** Let \( X \) be a non-negative random variable with probability distribution function \( F(x) \), and \( \varphi(s) \) be the Laplace-Stieltjes transform of \( F(x) \). Assume the abscissa of convergence of \( \varphi(s) \) is \( \sigma_0 = 0 \), and \( \varphi(s) \) is represented in a neighborhood of \( s = 0 \) as
\[
\varphi(s) = \alpha(s)s^r \log s + \beta(s), \quad r \in \mathbb{N}^+,
\] (16)
where $\alpha(s)$ and $\beta(s)$ are analytic with $\alpha(0) \neq 0$. Then, we have
\[
\lim_{x \to \infty} \frac{\log P(X > x)}{\log x} = -r.
\] (17)

**Theorem 2** Under the same notation as in Theorem 1 if $\sigma_0 = 0$ and $\varphi(s)$ is represented in a neighborhood of $s = 0$ as
\[
\varphi(s) = \alpha(s)s^r + \beta(s), \ r > 0, \ r \notin \mathbb{N},
\] (18)
where $\alpha(s)$ and $\beta(s)$ are analytic with $\alpha(0) \neq 0$. Then, we have
\[
\lim_{x \to \infty} \frac{\log P(X > x)}{\log x} = -r.
\] (19)

The following is a related work to Theorem 2.

**Theorem** (Korevaar [7], p.194, Theorem 8.2) Let $S(x)$ vanish for $x < 0$ and be locally integrable, positive and non-increasing for $x \geq 0$. Let $\varphi(s)$ be the Laplace-Stieltjes transform of $S(x)$. Further, let $\rho(x) = x^{-r}l(x)$ with $0 \leq r < 1$ and slowly varying $l(x)$. Then, for a constant $A$,
\[
\lim_{x \to \infty} \frac{S(x) - S(\infty)}{\rho(x)} = A
\] (20)
if and only if
\[
\lim_{x \to \infty} \frac{\varphi(1/x) - S(\infty)}{\rho(x)} = A \Gamma(1 - r).
\] (21)

4 Preliminary Lemmas

We will prepare some lemmas for the proof of our main theorems. First, let us define the step function $\Delta_1(t)$ as
\[
\Delta_1(t) = \begin{cases} 
1, & \text{if } t \geq 1, \\
0, & \text{if } t < 1.
\end{cases}
\] (22)

**Lemma 1** For $r \in \mathbb{N}^+$, we have
\[
\varphi(s) \equiv \mathcal{L} \left( \frac{1}{\nu+1} \Delta_1(t) \right)
\] (23)
\[
= \int_1^\infty \frac{1}{\nu+1} e^{-st} dt
\] (24)
\[
= \frac{(-1)^r}{r!} s^r \log s + \beta(s),
\] (25)
where $\beta(s)$ is analytic in a neighborhood of $s = 0$. 
Proof The \((r + 1)\)th derivative of \(\varphi(s)\) is
\[
\varphi^{(r+1)}(s) = (-1)^{r+1} \int_{1}^{\infty} e^{-st} dt
\]
\[= (-1)^{r+1} \left( \frac{1}{s} + \frac{e^{-s} - 1}{s} \right). \quad (26)
\]

Since \((e^{-s} - 1)/s\) is analytic, we have, by successive integrations, the desired result. \(\square\)

**Lemma 2** For \(r > 0, \, r \notin \mathbb{N}\), let \(r_0 = \lfloor r \rfloor\) be the maximum integer among integers smaller than \(r\), and let \(\tilde{r} = r - r_0\). Then we have
\[
\varphi(s) = \mathcal{L} \left( \frac{1}{\tilde{r}+1} \Delta_1(t) \right)
\]
\[= \frac{(-1)^{r_0+1}\pi}{\Gamma(r+1)\sin{\pi r}} s^r + \beta(s), \quad (29)
\]
where \(\beta(s)\) is analytic in a neighborhood of \(s = 0\).

**Proof** By a formula of Laplace transform (see [8], p.287),
\[
\varphi^{(r_0+1)}(s) = (-1)^{r_0+1} \int_{1}^{\infty} t^{-\tilde{r}} e^{-st} dt
\]
\[= (-1)^{r_0+1} \left\{ \int_{0}^{\infty} - \int_{0}^{1} \right\} t^{-\tilde{r}} e^{-st} dt \quad (31)
\]
\[= (-1)^{r_0+1} \Gamma(1 - \tilde{r}) s^{\tilde{r}-1} + \tilde{\beta}(s), \quad (32)
\]
where \(\tilde{\beta}(s)\) is analytic in a neighborhood of \(s = 0\). By successive integrations,
\[
\varphi(s) = (-1)^{r_0+1} \Gamma(1 - \tilde{r}) \cdot \frac{1}{\tilde{r} + 1} \cdot \ldots \cdot \frac{1}{\tilde{r} + r_0} s^{r-r_0} + \beta(s)
\]
\[= \frac{(-1)^{r_0+1}\pi}{\Gamma(r+1)\sin{\pi r}} s^r + \beta(s). \quad (34)
\]
In [54], we applied the formulas; \(\Gamma(z + 1) = z\Gamma(z)\) and \(\Gamma(z)\Gamma(1 - z) = \pi/\sin{\pi z}\). \(\square\)

**Lemma 3** Let \(X\) be a discrete random variable with probability distribution \(p = (p_n)_{n \in \mathbb{N}}\);
\[
p_n = \frac{1}{(n+1)^r} - \frac{1}{(n+2)^r}, \quad n \in \mathbb{N}, \, r \in \mathbb{N}^+,
\]
and let \(p(z) = \sum_{n=0}^{\infty} p_n z^n\) be the pgf of \(p\). Then, \(\varphi(s) = p(e^{-s})\) is represented in a neighborhood of \(s = 0\) as
\[
\varphi(s) = \alpha(s)s^r \log s + \beta(s), \quad (36)
\]
where \(\alpha(s)\) and \(\beta(s)\) are analytic with \(\alpha(0) \neq 0\).
Proof For \( r \geq 2 \), by calculation and Riemann’s formula (see Widder [14], p. 232),

\[
p(z) = \frac{z-1}{z^2} \sum_{n=1}^{\infty} \frac{z^n}{n^r} + \frac{1}{z} \tag{37}
\]

\[
= \frac{1}{(r-1)!} \frac{z-1}{z^2} \int_0^\infty \frac{zt^{r-1}}{e^t - z} \, dt + \frac{1}{z}. \tag{38}
\]

Then,

\[
\varphi(s) = \frac{e^s(1-e^s)}{(r-1)!} \int_0^\infty \frac{t^{r-1}}{e^{s+t} - 1} \, dt + e^s. \tag{39}
\]

By the change of variable \( u = e^{s+t} - 1 \), we have for sufficiently small \( s > 0 \),

\[
\varphi(s) = \frac{e^s(1-e^s)}{(r-1)!} \int_{e^{s-1}}^{\infty} \frac{\{\log(1 + u) - s\}^{r-1}}{u(u+1)} \, du + \text{regular term} \tag{40}
\]

\[
= \frac{e^s(1-e^s)}{(r-1)!} \int_{e^{s-1}}^{\infty} \left( \frac{1}{u} - \frac{1}{u+1} \right) \{(-s)^{r-1} + (r-1)(-s)^{r-2}u + \ldots \} \, du + \text{r.t.} \tag{41}
\]

\[
= \frac{e^s(1-e^s)}{(r-1)!} (-s)^{r-1} \int_{e^{s-1}}^{1} \frac{1}{u} \, du + \text{r.t.} \tag{42}
\]

\[
= \alpha(s)s^r \log s + \beta(s), \tag{43}
\]

where

\[
\alpha(s) = \frac{(-1)^{r+1}}{(r-1)!} e^s \frac{e^s - 1}{s}, \quad \alpha(0) = \frac{(-1)^{r+1}}{(r-1)!} \neq 0 \tag{44}
\]

and \( \beta(s) \) is some analytic function.

For \( r = 1 \), we have

\[
p(z) = \frac{1-z}{z^2} \log(1-z) + \frac{1}{z}. \tag{45}
\]

Similar argument leads to the desired result. \( \square \)

Lemma 4 For a pdf \( F(x) \), let

\[
\varphi(s) = \int_0^\infty e^{-sx} dF(x) \tag{46}
\]

have the abscissa of convergence \( \sigma_0 = 0 \). If, in a neighborhood of \( s = 0 \),

\[
\varphi(s) = \alpha(s)s^r \log s + \beta(s), \quad r \in \mathbb{N}^+, \tag{47}
\]

where \( \alpha(s), \beta(s) \) are analytic with \( \alpha(0) \neq 0 \), then,

\[
\alpha(0) \begin{cases} > 0, & \text{if } r \text{ is odd,} \\ < 0, & \text{if } r \text{ is even.} \end{cases} \tag{48}
\]

Proof From \([16]\),

\[
\varphi^{(r)}(0+) \begin{cases} \leq 0, & \text{if } r \text{ is odd,} \\ \geq 0, & \text{if } r \text{ is even,} \end{cases} \tag{49}
\]
while from (47) by calculation
\[ \varphi^{(r)}(s) = r! \alpha(s) \log s + \theta(s), \]  
(50)

where \( \theta(s) \) is a function of \( s \) with \( |\theta(0+)| < \infty \). Thus, by (50)
\[ \varphi^{(r)}(0+) = r! \alpha(0) \times (-\infty) + \theta(0+). \]  
(51)

Comparing (49), (51), we have the result. \( \square \)

**Lemma 5** Under the same notation as in Lemmas 2, 4, if \( \varphi(s) \) has the abscissa of convergence \( \sigma_0 = 0 \) and
\[ \varphi(s) = \alpha(s)s^r + \beta(s), \quad r > 0, \ r \notin \mathbb{N}, \]  
(52)

with \( \alpha(0) \neq 0 \), then,
\[ \alpha(0) \begin{cases} > 0, & \text{if } r_0 = \lfloor r \rfloor \text{ is odd,} \\ < 0, & \text{if } r_0 \text{ is even.} \end{cases} \]  
(53)

**Proof** From (46)
\[ \varphi^{(r_0+1)}(0+) \begin{cases} \geq 0, & \text{if } r_0 \text{ is odd,} \\ \leq 0, & \text{if } r_0 \text{ is even,} \end{cases} \]  
(54)

while from (52)
\[ \varphi^{(r_0+1)}(s) = \sum_{k=0}^{r_0+1} \binom{r_0+1}{k} \alpha^{(r_0+1-k)}(s) \cdot \frac{d}{ds^k} s^r + \beta^{(r_0+1)}(s). \]  
(55)

Since
\[ \frac{d}{ds^k} s^r \bigg|_{s=0^+} = \begin{cases} 0, & \text{for } k = 0, 1, \ldots, r_0 \\ +\infty, & \text{for } k = r_0 + 1, \end{cases} \]  
(56)

we have from (55)
\[ \varphi^{(r_0+1)}(0+) = \alpha(0) \times (+\infty) + \beta^{(r_0+1)}(0). \]  
(57)

Comparing (54), (57), we have the result. \( \square \)

### 4.1 First Several Terms of \( \alpha(s) \) and \( \beta(s) \)

We will need later, in the proof of main theorems, to make Laplace transforms which have the same first several terms as those in the Taylor expansion of \( \alpha(s) \) and \( \beta(s) \), respectively.

First, consider the following case
\[ \varphi(s) = \alpha(s)s^r \log s + \beta(s), \quad r \in \mathbb{N}^+. \]  
(58)

with expansions \( \alpha(s) = \sum_{n=0}^{\infty} \alpha_n z^n, \ \alpha(0) \neq 0, \) and \( \beta(s) = \sum_{n=0}^{\infty} \beta_n z^n. \)

We will make functions \( g^*(t) \) and \( h^*(t) \) such that their Laplace transforms \( G^*(s) \equiv \mathcal{L}(g^*(t)) \) and \( H^*(s) \equiv \mathcal{L}(h^*(t)) \) satisfy the following (i) and (ii) for \( L \in \mathbb{N}^+. \)
(i) \( G^*(s) = \alpha^*(s)s^r \log s + \tilde{\beta}(s) \), where \( \alpha^*(s) = \sum_{n=0}^{L-1} \alpha_n s^n \) and \( \tilde{\beta}(s) \) is analytic in a neighborhood of \( s = 0 \). Let \( \tilde{\beta}(s) = \sum_{n=0}^{\infty} \tilde{\beta}_n s^n \) be the expansion at \( s = 0 \).

(ii) \( H^*(s) = \sum_{n=0}^{L-1} (\beta_n - \tilde{\beta}_n)s^n + \) higher order terms.

The above (i) and (ii) mean that \( \alpha^*(s) \) is equal to the sum of the first \( L \) terms of \( \alpha(s) \), and the sum of the first \( L \) terms of \( H^*(s) + \tilde{\beta}(s) \) is equal to that of \( \beta(s) \).

**Function \( g^*(t) \)**

Define

\[
g^*(t) = \sum_{k=0}^{L-1} \frac{g_k}{t^{r+k+1}} \Delta_1(t), \quad t \in \mathbb{R},
\]

\[
g_k = (-1)^{r+k+1}(r+k)! \alpha_k, \quad k = 0, 1, \ldots, L-1.
\]

where \( \Delta_1(t) \) was defined in (22). By Lemma 1, we see that \( g^*(t) \) satisfies (i). The first coefficient \( g_0 \) is positive by Lemma 2, i.e.,

\[
g_0 = (-1)^{r+1} r! \alpha_0 > 0.
\]

**Function \( h^*(t) \)**

Let \( h_k(t) = ke^{-kt}, \quad t \geq 0, \quad k = 1, 2, \ldots, \) and \( H_k(s) = \mathcal{L}(h_k(t)) \). We have \( H_k(s) = k/(s+k), \) \( \Re s > -k \), and the expansion

\[
H_k(s) = \sum_{n=0}^{\infty} \left( -\frac{s}{k} \right)^n, \quad |s| < k.
\]

The sum of the first \( L \) terms of (62) is represented as

\[
\left( 1, -\frac{1}{k}, \ldots, -\frac{1}{k} \right)^{L-1}(1, s, \ldots, s^{L-1})^T,
\]

where \( ^T \) denotes the transposition of vector. Let \( V \) be the \( L \times L \) matrix;

\[
V = \begin{pmatrix}
1 & -1 & \cdots & (-1)^{L-1} \\
1 & -\frac{1}{2} & \cdots & \left( -\frac{1}{2} \right)^{L-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & -\frac{1}{L} & \cdots & \left( -\frac{1}{L} \right)^{L-1}
\end{pmatrix}.
\]

We will make a desired function by a linear combination of \( h_k(t) \), \( k = 1, 2, \ldots, L \). Let \( h(t) = \sum_{k=1}^{L} d_k h_k(t), \) \( t \geq 0 \), and write \( \mathbf{d} = (d_1, d_2, \ldots, d_L), \) \( s = (1, s, \ldots, s^{L-1}) \). Further, write \( \beta = (\beta_0, \beta_1, \ldots, \beta_{L-1}), \) \( \tilde{\beta} = (\tilde{\beta}_0, \tilde{\beta}_1, \ldots, \tilde{\beta}_{L-1}) \). The sum of the first \( L \) terms of \( H(s) = \mathcal{L}(h(t)) \) is \( \mathbf{d} V \mathbf{s}^T \), then we must solve the equation

\[
\mathbf{d} V \mathbf{s}^T = (\beta - \tilde{\beta}) \mathbf{s}^T.
\]
Since \( \det V \neq 0 \) (Vandermonde matrix), we have \( d = (\beta - \tilde{\beta})V^{-1} \). We write this solution as \( d = (d_1, d_2, \ldots, d_L) \), then

\[
h^*(t) = \sum_{k=1}^{L} d_k h_k(t)
\]

(66)
is a desired function, i.e., \( H^*(s) = \mathcal{L}(h^*(t)) \) satisfies (ii).

Summarizing above,

**Lemma 6** Let \( \varphi(s) \) be the LS transform of a pdf and the abscissa of convergence be \( \sigma_0 = 0 \). If

\[
\varphi(s) = \alpha(s)s^r \log s + \beta(s), \ r \in \mathbb{N}^+, \tag{67}
\]

where \( \alpha(s) \), \( \beta(s) \) are analytic in a neighborhood of \( s = 0 \) with \( \alpha(0) \neq 0 \), then \( g^*(t) \) in (69) and \( h^*(t) \) in (66) satisfy (i) and (ii).

Similarly, in the case

\[
\varphi(s) = \alpha(s)s^r + \beta(s), \ r > 0, \ r \notin \mathbb{N}, \tag{68}
\]

with \( \alpha(s) = \sum_{n=0}^{\infty} a_n s^n \), \( a_0 \neq 0 \), \( \beta(s) = \sum_{n=0}^{\infty} \beta_n s^n \), we will make functions \( g^*(t) \), \( h^*(t) \) such that their Laplace transforms \( G^*(s) \), \( H^*(s) \) satisfy the following (iii) and (iv) for \( L \in \mathbb{N}^+ \).

(iii) \( G^*(s) = \alpha^*(s)s^r + \tilde{\beta}(s) \), where \( \alpha^*(s) = \sum_{n=0}^{L-1} \alpha_n s^n \) and \( \tilde{\beta}(s) \) is analytic in a neighborhood of \( s = 0 \). Let \( \tilde{\beta}(s) = \sum_{n=0}^{\infty} \tilde{\beta}_n s^n \) be the expansion at \( s = 0 \).

(iv) \( H^*(s) = \sum_{n=0}^{L-1} (\beta_n - \tilde{\beta}_n) s^n + \) higher order terms.

In this case,

\[
g^*(t) = \sum_{k=0}^{L-1} \frac{g_k}{t^{r+k+1}} \Delta_1(t), \ t \in \mathbb{R}, \tag{69}
\]

\[
g_k = (-1)^{r_0 + k + 1} \frac{\sin \pi \bar{r}}{\pi} \Gamma(r + k + 1) \alpha_k, \ k = 0, 1, \ldots, L - 1. \tag{70}
\]
satisfies (iii). The first coefficient \( g_0 \) is positive by Lemma 5, i.e.,

\[
g_0 = (-1)^{r_0 + 1} \frac{\sin \pi \bar{r}}{\pi} \Gamma(r + 1) \alpha_0 > 0. \tag{71}
\]

The same \( h^*(t) \) as (66) satisfies (iv). Thus, we have

**Lemma 7** Let \( \varphi(s) \) be the LS transform of a pdf and the abscissa of convergence be \( \sigma_0 = 0 \). If

\[
\varphi(s) = \alpha(s)s^r + \beta(s), \ r > 0, \ r \notin \mathbb{N}, \tag{72}
\]

then \( g^*(t) \) in (69) and \( h^*(t) \) in (66) satisfy (iii) and (iv).
4.2 Majorant and Minorant Functions

For the evaluation of the tail probability $P(X > x)$ from above and below, we need to use majorant and minorant functions for an exponential function (see Kor evaar [7], p.132, Graham-Vaaler [3]). If two functions $f_1, f_2$ satisfy $f_1(t) \geq f_2(t), \ t \in \mathbb{R}$, then $f_1$ is said to be a majorant for $f_2$, and $f_2$ is a minorant for $f_1$.

For $\omega > 0$, we will define a majorant $M_\omega^1(t)$ and a minorant $m_\omega^1(t)$ for
\begin{equation}
E_\omega(t) \equiv \begin{cases} 
e^{-\omega t}, & t \geq 0 \\ 0, & t < 0. \end{cases}
\end{equation}

Define (see Kor evaar [7], p.132)
\begin{equation}
M_\omega^1(t) = \left( \frac{\sin \pi t}{\pi t} \right)^2 Q_\omega(t), \ t \in \mathbb{R},
\end{equation}
\begin{equation}
Q_\omega(t) = \sum_{n=0}^{\infty} \frac{e^{-n\omega}}{(t-n)^2} - \omega \sum_{n=1}^{\infty} e^{-n\omega} \left( \frac{1}{t-n} - \frac{1}{t} \right),
\end{equation}
and
\begin{equation}
m_\omega^1(t) = M_\omega(t) - \left( \frac{\sin \pi t}{\pi t} \right)^2, \ t \in \mathbb{R}.
\end{equation}

Moreover, for $L \in \mathbb{N}^+$, define
\begin{equation}
M_L^\omega(t) = \left( M_\omega^1(t) \right)^L \text{ and } m_L^\omega(t) = \left( m_\omega^1(t) \right)^L.
\end{equation}

For $\sigma > 0, \delta > 0$, write $\omega = 2\pi\sigma/\delta$, then define
\begin{equation}
M_{\sigma,\delta}^L(t) \equiv M_L^\omega \left( \frac{\delta t}{2\pi} \right) = M_{2\pi\sigma/\delta}^L \left( \frac{\delta t}{2\pi} \right),
\end{equation}
\begin{equation}
m_{\sigma,\delta}^L(t) \equiv m_L^\omega \left( \frac{\delta t}{2\pi} \right) = m_{2\pi\sigma/\delta}^L \left( \frac{\delta t}{2\pi} \right).
\end{equation}

**Lemma 8** (Kor evaar [7]) For any $L \in \mathbb{N}^+$, $\sigma > 0, \delta > 0$,
\begin{equation}
M_{\sigma,\delta}^L, \ m_{\sigma,\delta}^L \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).
\end{equation}

For $\lambda > 0$, an entire function $f(z)$ of a complex variable $z = x + iy$ is of exponential type $\lambda$ if
\begin{equation}
|f(z)| \leq C \exp(\lambda|z|), \ z \in \mathbb{C}, \ C > 0.
\end{equation}

A real function $f(x)$ is of type $\lambda$ if $f(x)$ is the restriction to $\mathbb{R}$ of an entire function of exponential type $\lambda$.

**Lemma 9** (Kor evaar [7], Nakagawa [12]) $M_{\sigma,\delta}^L$ and $m_{\sigma,\delta}^L$ are of type $L\delta$. 
Lemma 10 (Korevaar [7], Graham-Vaaler [3]) For \( L \in \mathbb{N}^+ \),
\[
E_{L\sigma}(t) \leq M_{\sigma,\delta}^L(t), \quad t \in \mathbb{R},
\]
and for odd \( L \in \mathbb{N}^+ \),
\[
m_{\sigma,\delta}^L(t) \leq E_{L\sigma}(t), \quad t \in \mathbb{R}.
\]

Proof If \( L = 1 \), the result follows Korevaar [7], p.129, Proposition 5.2. The odd power preserves the order of real numbers. \( \square \)

From Lemma 8, we can define the Fourier transforms \( \hat{M}_{\sigma,\delta}^L = \mathcal{F}(M_{\sigma,\delta}^L) \) and \( \hat{m}_{\sigma,\delta}^L = \mathcal{F}(m_{\sigma,\delta}^L) \), where the Fourier transform is defined as
\[
\hat{M}_{\sigma,\delta}^L(\tau) = \int_{-\infty}^{\infty} M_{\sigma,\delta}^L(t)e^{-i\tau t}dt, \quad \tau \in \mathbb{R}.
\]

Then, from Lemma 9 and the Paley-Wiener theorem, we have

Lemma 11 (Rudin [13], Korevaar [7]) For any \( L \in \mathbb{N}^+ \),
\[
\text{supp}(\hat{M}_{\sigma,\delta}^L) \subset [-L\delta, L\delta] \quad \text{and} \quad \text{supp}(\hat{m}_{\sigma,\delta}^L) \subset [-L\delta, L\delta],
\]
where \( \text{supp} \) denotes the support of a function.

4.3 Calculation of \( \hat{M}_{\sigma,\delta}^L \) and \( \hat{m}_{\sigma,\delta}^L \)

It is not difficult to calculate the Fourier transforms \( \hat{M}_1 = \mathcal{F}(M_1) \) and \( \hat{m}_1 = \mathcal{F}(m_1) \).

Define
\[
q_1(t) = \left( \frac{\sin \pi t}{\pi t} \right)^2, \quad q_2(t) = \frac{\sin^2 \pi t}{\pi t}, \quad t \in \mathbb{R},
\]
and write \( \hat{q}_1 = \mathcal{F}(q_1), \hat{q}_2 = \mathcal{F}(q_2) \). By calculation, we have
\[
\hat{q}_1(\tau) = \begin{cases} 
1 + \frac{\tau}{2\pi}, & -2\pi \leq \tau < 0, \\
1 - \frac{\tau}{2\pi}, & 0 \leq \tau < 2\pi, \\
0, & \text{otherwise}.
\end{cases}
\]
and
\[
\hat{q}_2(\tau) = \begin{cases} 
\frac{i}{2}, & -2\pi \leq \tau < 0, \\
-\frac{i}{2}, & 0 \leq \tau < 2\pi, \\
0, & \text{otherwise}.
\end{cases}
\]
Lemma 12 We have
\[ \hat{M}_\omega^L(\tau) = \frac{1}{1 - e^{-(\omega + i\tau)}} \hat{q}_1(\tau) - \frac{\omega}{\pi} \left( \frac{1}{1 - e^{-(\omega + i\tau)}} - \frac{1}{1 - e^{-\omega}} \right) \hat{q}_2(\tau), \] (89)
and
\[ \hat{m}_\omega^L(\tau) = \frac{e^{-(\omega + i\tau)}}{1 - e^{-(\omega + i\tau)}} \hat{q}_1(\tau) - \frac{\omega}{\pi} \left( \frac{1}{1 - e^{-(\omega + i\tau)}} - \frac{1}{1 - e^{-\omega}} \right) \hat{q}_2(\tau). \] (90)

Proof See Appendix A. \hfill \Box

Next, we will calculate \( \hat{M}_\omega^L = \mathcal{F}(M_\omega^L), \hat{m}_\omega^L = \mathcal{F}(m_\omega^L) \) and then calculate \( \lim_{\omega \to 0^+} \hat{M}_\omega^L(\tau), \lim_{\omega \to 0^+} \hat{m}_\omega^L(\tau), \) for \( \tau \neq 0 \).

Let us define
\[ u_\omega(t) = \left( \frac{\sin \pi t}{\pi t} \right)^2 - \frac{\omega \sin^2 \pi t}{\pi t}, \] (91)
\[ v_\omega(t) = \frac{\omega \sin^2 \pi t}{\pi t}, \] (92)
\[ \hat{u}_\omega = \mathcal{F}(u_\omega), \hat{v}_\omega = \mathcal{F}(v_\omega). \]

Lemma 13 We have
\[ \hat{M}_\omega^L(\tau) = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \binom{L}{l} \left( \frac{1}{1 - e^{-(\omega + i\tau)}} \right)^l \left( \frac{1}{1 - e^{-\omega}} \right)^{L-l} \hat{u}_\omega^l(\tau) * \hat{u}_\omega^{*L-l}(\tau), \] (95)
\[ \hat{m}_\omega^L(\tau) = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \binom{L}{l} \left( \frac{e^{-\omega - i\tau}}{1 - e^{-(\omega + i\tau)}} \right)^l \left( \frac{e^{-\omega}}{1 - e^{-\omega}} \right)^{L-l} \hat{u}_\omega^l(\tau) * \hat{u}_\omega^{*L-l}(\tau), \] (96)
where \( * \) denotes the convolution operation and \( \ast l \) denotes the \( l \)-fold convolution.

Proof See Appendix B. \hfill \Box

Lemma 14 For \( \tau \neq 0 \),
\[ \lim_{\omega \to 0^+} \hat{M}_\omega^L(\tau) = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \frac{1}{\pi^{L-l}} \binom{L}{l} \left( \frac{1}{1 - e^{-i\tau}} \right)^l \hat{q}_1^l(\tau) * \hat{q}_2^{*L-l}(\tau), \] (97)
\[ \lim_{\omega \to 0^+} \hat{m}_\omega^L(\tau) = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \frac{1}{\pi^{L-l}} \binom{L}{l} \left( \frac{e^{-i\tau}}{1 - e^{-i\tau}} \right)^l \hat{q}_1^l(\tau) * \hat{q}_2^{*L-l}(\tau). \] (98)

Proof (97) and (98) follow
\[ \hat{u}_\omega^l(\tau) = \sum_{j=0}^{l} \left( \frac{-\omega}{\pi} \right)^{l-j} \hat{q}_1^j(\tau) * \hat{q}_2^{*l-j}(\tau), \] (99)
\[ \hat{v}_\omega^{*L-l}(\tau) = \left( \frac{\omega}{\pi} \right)^{L-l} \hat{q}_2^{*L-l}(\tau). \] (100)

By the change of variables, we have
Lemma 15\hspace{1em} For $\tau \neq 0$,

$$
\lim_{\sigma \to 0^+} M_{\sigma,\delta}^L(\tau) = \frac{2\pi}{\delta} \int_{L} e^{-(\sigma_1 + L\sigma)it} e^{-\sigma_1 t} dF(t) = \frac{2\pi}{\delta} \int_{L} e^{-(\sigma_1 + L\sigma)it} e^{-\sigma_1 t} dF(t).
$$

(101)

\begin{align*}
\lim_{\sigma \to 0^+} M_{\sigma,\delta}^L(\tau) &= \frac{2\pi}{\delta} \int_{L} e^{-(\sigma_1 + L\sigma)it} e^{-\sigma_1 t} dF(t) = \frac{2\pi}{\delta} \int_{L} e^{-(\sigma_1 + L\sigma)it} e^{-\sigma_1 t} dF(t).
\end{align*}

(102)

5 Proof of Theorem \[1\]

5.1 Upper Bound for $P(X > x)$

First, we will evaluate $P(X > x)$ from above by using the majorant function $M_{\sigma,\delta}^L$.

Let $L \in \mathbb{N}^+$ with $L \geq r$. For arbitrary $\sigma_1 > 0$, $\sigma_2 > 0$, $\delta > 0$,

$$
e^{L\sigma_1 x} \int_{x}^{\infty} e^{-(\sigma_1 + L\sigma_2)t} dF(t) = \int_{x}^{\infty} e^{L\sigma_1(t-x)} e^{-\sigma_1 t} dF(t)
$$

(103)

$$= \int_{0}^{\infty} E L_{\sigma_2} (t-x) e^{-\sigma_1 t} dF(t)
$$

(104)

$$\leq \int_{0}^{\infty} M_{\sigma_2,\delta}^L(t-x) e^{-\sigma_1 t} dF(t), \quad x > 0,
$$

(105)

where the last inequality holds by (82) in Lemma [10] (see also Korevaar \[7\], Nakagawa \[12\]). By Lemma [11] $M_{\sigma_2,\delta}^L(t-x)$ is represented by the inverse Fourier transform of $\tilde{M}_{\sigma_2,\delta}^L = \mathcal{F}^{-1}(M_{\sigma_2,\delta}^L)$ as

$$
M_{\sigma_2,\delta}^L(t-x) = \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \tilde{M}_{\sigma_2,\delta}^L(-\tau) e^{i(x-t)\tau} d\tau.
$$

(106)

Substituting (106) into (105), we have by Fubini’s theorem

$$
\int_{0}^{\infty} M_{\sigma_2,\delta}^L(t-x) e^{-\sigma_1 t} dF(t) = \int_{0}^{\infty} \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \tilde{M}_{\sigma_2,\delta}^L(-\tau) e^{i(x-t)\tau} e^{-\sigma_1 t} d\tau dF(t)
$$

(107)

$$= \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \tilde{M}_{\sigma_2,\delta}^L(-\tau) e^{ix\tau} d\tau \int_{0}^{\infty} e^{-i(x-\tau)+i\tau} dF(t)
$$

(108)

$$= \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \tilde{M}_{\sigma_2,\delta}^L(-\tau) e^{i\tau} \varphi(s) d\tau.
$$

(109)

Now, for the representation (106) of $\varphi(s)$, i.e.,

$$
\varphi(s) = \alpha(s) s^r \log s + \beta(s), \quad r \in \mathbb{N}^+.
$$

(110)

we define

$$
f^*(t) = g^*(t) + h^*(t), \quad t \geq 0,
$$

(111)

where $g^*(t)$ and $h^*(t)$ were defined in Lemma [6]. Let $\varphi^*(s) = \mathcal{L}(f^*(t))$, then $f^*(t)$ and $\varphi^*(s)$ have the following properties (a), (b), (c) and (c').

(a) $f^*(t) > 0$ for all sufficiently large $t$.

This is because $g^*(0) = g_0 > 0$ by (101).
Define $\xi(s) \equiv \varphi(s) - \varphi^*(s)$, then by Lemma 6

\[ \xi(s) = \varphi(s) - (G^*(s) + H^*(s)) \]

\[ = s^L \left( \sum_{n=0}^{\infty} \alpha_{L+n}s^n \right) s^r \log s + \sum_{n=0}^{\infty} \left( \beta_{L+n} - \beta_{L+n} \right) s^n \], $s = \sigma + i\tau.$

(113)

We see from (113),

(b) $\xi(s)$ is continuous in the closed region $\{0 \leq \sigma \leq \epsilon, -L\delta \leq \tau \leq L\delta\} \subset \mathbb{C}$ for sufficiently small $\epsilon, \delta > 0$.

In fact, because $s = 0$ is an isolated singularity, we can take $\epsilon, \delta$ so small that the closed region $\{0 \leq \sigma \leq \epsilon, -L\delta \leq \tau \leq L\delta\}$ does not include any singularities of $\varphi(s)$ and $\varphi^*(s)$ other than $s = 0$.

Define $\hat{M}_{0,\delta}^L(\tau) = \lim_{\sigma \to 0^+} \hat{M}_{\sigma,\delta}^L(\tau)$, $\hat{m}_{0,\delta}^L(\tau) = \lim_{\sigma \to 0^+} \hat{m}_{\sigma,\delta}^L(\tau)$, $-L\delta \leq \tau \leq L\delta$, $\tau \neq 0$, for $\delta$ sufficiently small as in (b). Then, we have

(c) $\text{supp}(\hat{M}_{0,\delta}^L) \subset [-L\delta, L\delta]$ and $\hat{M}_{0,\delta}^L(\tau)\xi(i\tau)$ is $r$ times piecewise differentiable with

\[ \left( \hat{M}_{0,\delta}^L(\tau)\xi(i\tau) \right)^{(r)} \in L^1\left([-L\delta, L\delta]\right). \]

(c') $\text{supp}(\hat{m}_{0,\delta}^L) \subset [-L\delta, L\delta]$ and $\hat{m}_{0,\delta}^L(\tau)\xi(i\tau)$ is $r$ times piecewise differentiable with

\[ \left( \hat{m}_{0,\delta}^L(\tau)\xi(i\tau) \right)^{(r)} \in L^1\left([-L\delta, L\delta]\right). \]

These (c) and (c') hold because of Lemma 5 and (113).

Now, defining $F^*(t) = \frac{dF^*(t)}{dt}$, we have in a similar way as (109)

\[ \int_{0}^{\infty} M_{\sigma,\delta}^L(t - x)e^{-\sigma_1 t}dF^*(t) = \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \hat{M}_{\sigma,\delta}^L(-\tau)e^{i\tau\tau}\varphi^*(\sigma_1 + i\tau)d\tau. \]

(114)

Subtracting (114) from (109), we have

\[ \int_{0}^{\infty} M_{\sigma,\delta}^L(t - x)e^{-\sigma_1 t}dF(t) = \int_{0}^{\infty} M_{\sigma,\delta}^L(t - x)e^{-\sigma_1 t}dF^*(t) \]

\[ + \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \hat{M}_{\sigma,\delta}^L(-\tau)e^{i\tau\tau}\xi(\sigma_1 + i\tau)d\tau. \]

(115)

(116)

For sufficiently small $\delta > 0$,

\[ \xi(i\tau) = \lim_{\sigma_1 \to 0^+} \xi(\sigma_1 + i\tau), -L\delta \leq \tau \leq L\delta \]

is uniform convergence due to (b), hence

\[ \lim_{\sigma_1 \to 0^+} \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \hat{M}_{\sigma_2,\delta}^L(-\tau)\xi(\sigma_1 + i\tau)e^{i\tau\tau}d\tau = \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \hat{M}_{\sigma_2,\delta}^L(-\tau)\xi(i\tau)e^{i\tau\tau}d\tau. \]

(118)

From (105), (113), (118), for $\sigma_1 \to 0^+$, we have

\[ e^{L\sigma_1 x} \int_{x}^{\infty} e^{-L\sigma_1 t}dF(t) \leq \int_{0}^{\infty} M_{\sigma_2,\delta}^L(t - x)dF^*(t) + \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \hat{M}_{\sigma_2,\delta}^L(-\tau)\xi(i\tau)e^{i\tau\tau}d\tau. \]

(119)

By the estimation for $M_1^L(t)$ (see Korevaar [7], p.132), i.e.,

\[ \begin{aligned}
0 \leq M_1^L(t) & \leq \left( \frac{\sin \pi t}{\pi t} \right)^2, & t < 0, \\
\exp(-\omega t) \leq M_1^L(t) & \leq \exp(-\omega t) + \left( \frac{\sin \pi t}{\pi t} \right)^2, & t \geq 0,
\end{aligned} \]

(120)
we have
\[
\begin{cases}
0 \leq M_{\sigma,\delta}(t) \leq \left( \frac{\sin \delta t/2}{\delta t/2} \right)^{2L}, & t < 0, \\
e^{-L\omega t} \leq M_{\sigma,\delta}(t) \leq \left( e^{-\omega t} + \left( \frac{\sin \pi t}{\pi t} \right) \right)^{2L}, & t \geq 0,
\end{cases}
\]  

(121)

where $\omega = 2\pi\sigma/\delta$. Thus, it is easy to see that there exists a constant $C_1 > 0$ such that
\[
M_{\sigma,\delta}(t - x) \leq \begin{cases}
\frac{1}{\delta(x - t)/2}^{2L}, & 0 \leq t < x - 1, \\
C_1, & t \geq x - 1.
\end{cases}
\]  

(122)

Therefore, the first term of the right hand side of (119) is evaluated as
\[
\int_0^\infty M_{\sigma,\delta}(t - x)dF^+ (t) = \int_0^\infty M_{\sigma,\delta}^{L}(t - x)f^+(t)dt
\]  

(123)

\[
= \int_0^\infty M_{\sigma,\delta}^{L}(t - x)(g^+(t) + h^+(t))dt
\]  

(124)

\[
\leq \sum_{k=0}^{L-1} |g_k| \int_1^{x-1} \left( \frac{1}{\delta(x - t)/2} \right)^{2L} \frac{1}{e^{r+k+1}t} dt + C_1 \sum_{k=0}^{L-1} |g_k| \int_1^{x-1} \frac{1}{e^{r+k+1}t} dt
\]  

(125)

\[
+ \sum_{k=1}^{L} k|d_k| \int_0^{x-1} \left( \frac{1}{\delta(x - t)/2} \right)^{2L} e^{-kt}dt + C_1 \sum_{k=1}^{L} k|d_k| \int_1^{x-1} e^{-kt}dt
\]  

(126)

\[
\leq O(x^{-r+1}) + C_2 g_0 x^{-r} + O(x^{-2L}) + O(e^{-x}), \quad C_2 > 0
\]  

(127)

\[
< \frac{C_3}{x^r}, \quad C_3 > 0,
\]  

(128)

for all sufficiently large $x$, by virtue of Lemmas 16, 17 in Appendix. Notice $g_0 > 0$ and $L \geq r$.

Next, the second term of the right hand side of (119) will be estimated. We have by (c) and integration by parts,
\[
\lim_{\sigma_2 \to 0^+} \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \tilde{M}_{\sigma,\delta}(\tau)\xi(i\tau)e^{ix\tau}d\tau = \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \tilde{M}_{0,\delta}(\tau)\xi(i\tau)e^{ix\tau}d\tau
\]  

(129)

\[
= \frac{ir}{2\pi x^r} \int_{-L\delta}^{L\delta} \left( \tilde{M}_{0,\delta}(\tau)\xi(i\tau) \right)^{(r)} e^{ix\tau}d\tau
\]  

(130)

due to Riemann-Lebesgue theorem. Then in (119) for $\sigma_2 \to 0^+$, we have by (128) and (131),
\[
P(X > x) = \int_x^\infty dF(t) < \frac{C}{x^r}, \quad C > 0,
\]  

(132)

for all sufficiently large $x$.

5.2 Lower Bound for $P(X > x)$

We will evaluate $P(X > x)$ from below by using the minorant function $m_{\sigma,\delta}^L$. 

Let \( L \in \mathbb{N}^+ \) be an odd number with \( L \geq r \). For arbitrary \( \sigma_1 > 0, \sigma_2 > 0, \delta > 0 \),

\[
e^{L\sigma_2 x} \int_x^\infty e^{-(\sigma_1 + L\sigma_2)t} dF(t) = \int_x^\infty E_{L\sigma_2}(t - x)e^{-\sigma_1 t} dF(t) \geq \int_x^\infty m_{\sigma_2, \delta}^L(t - x)e^{-\sigma_1 t} dF(t), \quad x > 0.
\]

(133)

(134)

In a similar way as from (103) to (119), we have

\[
e^{L\sigma_2 x} \int_x^\infty e^{-L\sigma_2 t} dF(t) \geq \int_x^\infty m_{\sigma_2, \delta}^L(t - x)dF^* (t) + \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \hat{m}_{\sigma_2, \delta}(\xi) e^{ix\tau} d\tau.
\]

(135)

Due to the estimation for \( m_1^L(t) \) (see Korevaar [7], p.132), i.e.,

\[
- \left( \frac{\sin \frac{\pi t}{\pi t}}{\pi t} \right)^2 \leq m_1^L(t) \leq 0, \quad t < 0, \\
e^{-\omega t} - \left( \frac{\sin \frac{\pi t}{\pi t}}{\pi t} \right)^2 \leq m_1^L(t) \leq e^{-\omega t}, \quad t \geq 0,
\]

(136)

we have

\[
- \left( \frac{\sin \frac{\delta t/2}{\delta t/2}}{\delta t/2} \right)^{2L} \leq m_{\sigma, \delta}^L(t) \leq 0, \quad t < 0, \\
e^{-\omega t} - \left( \frac{\sin \frac{\delta t/2}{\delta t/2}}{\delta t/2} \right)^L \leq m_{\sigma, \delta}^L(t) \leq e^{-L\omega t}, \quad t \geq 0,
\]

(137)

where \( \omega = 2\pi\sigma/\delta \). Thus, there exist constants \( C_4, C_5, C_6 > 0 \) such that

\[
m_{\sigma, \delta}^L(t - x) \geq \begin{cases} \\
\left( \frac{1}{\delta(x - t)/2} \right)^{2L}, & 0 \leq t < x - 1, \\
-C_4, & x - 1 \leq t < x + 1, \\
C_5 e^{-(2\pi L\sigma/\delta)(t-x)} - C_6 \left( \frac{1}{\delta(x - t)/2} \right)^2, & t \geq x + 1.
\end{cases}
\]

(138)
Therefore, the first term of the right hand side of (135) is evaluated as
\[
\int_0^\infty m_{\sigma_2, \delta}^L(t-x)(g^*(t)+h^*(t))dt \geq - \sum_{k=0}^{L-1} |g_k| \int_1^{x-1} \left( \frac{1}{\delta(x-t)/2} \right)^{2L} \frac{1}{t^{r+k+1}}dt \tag{139}
\]

\[
- C_4 \sum_{k=0}^{L-1} |g_k| \int_{x-1}^{x+1} \frac{1}{t^{r+k+1}}dt \tag{140}
\]

\[
+ C_5 \sum_{k=0}^{L-1} |g_k| \int_{x+1}^{\infty} e^{-\left(2\pi L \sigma_2/\delta\right)(t-x)} \frac{1}{t^{r+k+1}}dt \tag{141}
\]

\[
- C_6 \sum_{k=0}^{L-1} |g_k| \int_{x+1}^{\infty} \left( \frac{1}{\delta(x-t)/2} \right)^{2} \frac{1}{t^{r+k+1}}dt \tag{142}
\]

\[
- \sum_{k=1}^{L} k|d_k| \int_0^{x-1} \left( \frac{1}{\delta(x-t)/2} \right)^{2} e^{-kt}dt \tag{143}
\]

\[
- C_4 \sum_{k=1}^{L} k|d_k| \int_{x-1}^{x+1} e^{-kt}dt \tag{144}
\]

\[
+ C_5 \sum_{k=1}^{L} k|d_k| \int_{x+1}^{\infty} e^{-\left(2\pi L \sigma_2/\delta\right)(t-x)} e^{-kt}dt \tag{145}
\]

\[
- C_6 \sum_{k=1}^{L} k|d_k| \int_{x+1}^{\infty} \left( \frac{1}{\delta(x-t)/2} \right)^{2} e^{-kt}dt \tag{146}
\]

\[
\geq O(x^{-(r+1)}) + O(x^{-\prime}) + C_5 g_0 \int_{x+1}^{\infty} e^{-\left(2\pi L \sigma_2/\delta\right)(t-x)} \frac{1}{t^{r+1}}dt \tag{147}
\]

\[
+ O(x^{-\prime}) + O(x^{-2L}) + O(e^{-x}) + O(e^{-x}) + O(e^{-x}), \tag{148}
\]

where $C'_5$ is a positive constant. Thus,
\[
\lim_{\sigma_2 \to 0^+} \int_0^\infty m_{\sigma_2, \delta}^L(t-x)dF^*(t) \geq \frac{C'_7}{x^r}, \quad C'_7 > 0, \tag{149}
\]

for all sufficiently large $x$.

Next, we will evaluate the second term of the right hand side of (135). We have, by (c') and the integration by parts,
\[
\lim_{\sigma_2 \to 0^+} \frac{1}{2\pi} \int_{-L\delta}^{L\delta} m_{\sigma_2, \delta}^L(-\tau)\xi(i\tau)e^{ix\tau}d\tau = \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \left( \tilde{m}_{\sigma_2, \delta}^L(-\tau)\xi(i\tau) e^{ix\tau} \right)d\tau \tag{150}
\]

\[
= \frac{x^r}{2\pi x^r} \int_{-L\delta}^{L\delta} \left( \tilde{m}_{\sigma_2, \delta}^L(-\tau)\xi(i\tau) \right)^{(r)} e^{ix\tau}d\tau \tag{151}
\]

\[
= o(x^{-r}), \quad x \to \infty, \tag{152}
\]

by Riemann-Lebesgue theorem. Then, in (135) for $\sigma_2 \to 0^+$, we have from (149) and (152),
\[
P(X > x) = \int_x^\infty dF(t) > \frac{C'}{x^r}, \quad C' > 0, \tag{153}
\]

for all sufficiently large $x$.

From (132) and (153), the proof of Theorem 1 is completed. \hfill \Box
6 Proof of Theorem 2

The same proof as that of Theorem 1 is applicable by replacing \( g^*(t), h^*(t) \) in Lemma 5 with those in Lemma 6.

7 Conclusion

In this paper, we investigated the asymptotic decay of the tail probability of a heavy tailed random variable. We proved two theorems which give sufficient conditions for a random variable to be heavy tailed. Our theorems are based on the Tauberian theorems due to Graham and Vaarler. The central idea is the approximation of the exponential function by a majorant and minorant functions whose Fourier transforms have a finite support.

Through the proof of the theorems, I think that some more general representation of the singularity guarantees the random variable to be heavy tailed.
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Appendix

Lemma 16 For \( n_1, n_2 \in \mathbb{N}^+, \ n_1 \geq 2, \ n_2 \geq 2, \) let \( n = \min(n_1, n_2). \) Then,
\[
\int_1^{x-1} \frac{dt}{(x-t)^{n_1+t_2}} \leq O(x^{-n}), \ x \to \infty.
\]

Proof By the change of variable \( t = xu, \)
\[
\int_1^{x-1} \frac{dt}{(x-t)^{n_1+t_2}} = \frac{1}{x^{n_1+n_2+1}} \left\{ \int_{1/x}^{1/2} du + \int_{1/2}^{1/x} \frac{du}{(1-u)^{n_1} u^{n_2}} \right\}
\[
\leq \frac{1}{x^{n_1+n_2+1}} \left\{ \int_0^{1/2} du + \int_{1/2}^{1/x} \frac{du}{u^{n_2}} + \int_{1/2}^{1/x} \frac{du}{(1-u)^{n_1}} \right\}
\[
= \frac{1}{x^{n_1+n_2+1}} \left\{ 2^{n_1/2} \int_{1/x}^{1/2} du + 2^{n_2} \int_{1/2}^{1/x} \frac{du}{u^{n_2}} + 2^{n_1} (n_2-1) + 2^{n_2} (n_1-1) \right\}
\[
= O(x^{-n}), \ x \to \infty.
\]

Lemma 17 For \( k > 0 \) and \( n \in \mathbb{N}, \) we have
\[
\int_0^{x-1} \frac{e^{-kt}}{(x-t)^n} dt = O(x^{-n}), \ x \to \infty.
\]

Proof By the change of variable \( u = x - t, \)
\[
\int_0^{x-1} \frac{e^{-kt}}{(x-t)^n} dt = e^{-kx} \int_1^x \frac{e^k u}{u^n} du
\[
= \frac{1}{x^n} \int_1^x \frac{e^k u}{u^n} du / x^n
\[
\to \frac{1}{kx^n}, \ x \to \infty,
\]
from L'Hopital's rule.

A Proof of Lemma 12

\[
\tilde{M}_\omega^1(\tau) = \sum_{n=0}^{\infty} e^{-n\omega} \mathcal{F} \left( \left( \frac{\sin \pi (t-n)}{\pi (t-n)} \right)^2 \right) - \omega \sum_{n=0}^{\infty} e^{-n\omega} \left\{ \mathcal{F} \left( \frac{\sin^2 \pi (t-n)}{\pi (t-n)} \right) - \mathcal{F} \left( \frac{\sin^2 \pi t}{\pi t} \right) \right\}
\[
= \tilde{q}_1(\tau) \sum_{n=0}^{\infty} e^{-n(\omega+i\tau)} - \omega \sum_{n=0}^{\infty} e^{-n(\omega+i\tau)} - \omega \sum_{n=0}^{\infty} e^{-n\omega}
\[
= \frac{1}{1-e^{-i(\omega+i\tau)}} \tilde{q}_1(\tau) - \omega \left( \frac{1}{1-e^{-i\omega}} - \frac{1}{1-e^{-\omega}} \right) \tilde{q}_2(\tau).
\]

The result for \( \tilde{m}_\omega^1 \) is proved in a similar way.
\[ M^L_\omega(t) = \left( \frac{\sin \pi t}{\pi t} \right)^{2L} (Q_\omega(t))^L \]
\[ = \left( \frac{\sin \pi t}{\pi t} \right)^{2L} \sum_{n_1=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} e^{-n_1 \omega} \cdots e^{-n_L \omega} \left\{ \frac{1}{(t-n_1)^2} - \frac{\omega}{t-n_1} + \frac{\omega}{t} \right\} \times \ldots \]
\[ \times \left\{ \frac{1}{(t-n_L)^2} - \frac{\omega}{t-n_L} + \frac{\omega}{t} \right\} \]
\[ = \sum_{n_1=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} e^{-n_1 \omega} \cdots e^{-n_L \omega} \left\{ \frac{\sin \pi (t-n_1)}{\pi (t-n_1)} \right\}^2 - \frac{\omega \sin^2 \pi (t-n_1)}{\pi} + \frac{\omega \sin^2 \pi t}{\pi} \times \ldots \]
\[ \times \left\{ \frac{\sin \pi (t-n_L)}{\pi (t-n_L)} \right\}^2 - \frac{\omega \sin^2 \pi (t-n_L)}{\pi} + \frac{\omega \sin^2 \pi t}{\pi} \right\} \]
\[ = \sum_{n_1=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} e^{-n_1 \omega} \cdots e^{-n_L \omega} \left\{ u_\omega(t-n_1) + v_\omega(t) \right\} \times \ldots \times \left\{ u_\omega(t-n_L) + v_\omega(t) \right\}. \]

Therefore,
\[ \tilde{M}^L_\omega(\tau) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} e^{-n_1 \omega} \cdots e^{-n_L \omega} \frac{1}{(2\pi)^L-1} \left\{ (e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau)) \right\} \cdots \left\{ (e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau)) \right\} \]
\[ = \frac{1}{(2\pi)^{L-1}} \sum_{n_1=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} e^{-n_1 \omega} \cdots e^{-n_L \omega} \]
\[ \times \sum_{l=0}^{L} \left\{ \sum_{k_1, \ldots, k_l: \text{distinct}} (e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau))^* \right\} \times \left\{ (e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau))^* \right\} \times \cdots \times \left\{ (e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau))^* \right\} \times \cdots \times \left\{ (e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau))^* \right\} \times \left\{ (e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau))^* \right\} \]
\[ \times \sum_{k_1, \ldots, k_l: \text{distinct}} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right) \times \cdots \times \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right) \times \cdots \times \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right) \]
\[ = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \]
\[ \times \sum_{n_1=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} e^{-n_1 \omega} \cdots e^{-n_L \omega} \left( \sum_{k_1, \ldots, k_l: \text{distinct}} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \right) \]
\[ = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \]
\[ \times \sum_{n_1=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} e^{-n_1 \omega} \cdots e^{-n_L \omega} \left( \sum_{k_1, \ldots, k_l: \text{distinct}} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \right) \]
\[ = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \]
\[ \times \sum_{n_1=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} e^{-n_1 \omega} \cdots e^{-n_L \omega} \left( \sum_{k_1, \ldots, k_l: \text{distinct}} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \right) \]
\[ = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \]
\[ \times \sum_{n_1=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} e^{-n_1 \omega} \cdots e^{-n_L \omega} \left( \sum_{k_1, \ldots, k_l: \text{distinct}} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \right) \]
\[ = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \]
\[ \times \sum_{n_1=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} e^{-n_1 \omega} \cdots e^{-n_L \omega} \left( \sum_{k_1, \ldots, k_l: \text{distinct}} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \right) \]
\[ = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \]
\[ \times \sum_{n_1=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} e^{-n_1 \omega} \cdots e^{-n_L \omega} \left( \sum_{k_1, \ldots, k_l: \text{distinct}} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \right) \]
\[ = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \]
\[ \times \sum_{n_1=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} e^{-n_1 \omega} \cdots e^{-n_L \omega} \left( \sum_{k_1, \ldots, k_l: \text{distinct}} \left( e^{-i\tau} u_\omega(\tau) + \hat{u}_\omega(\tau) \right)^* \right) \]

In (154), (155), (156), for \( l = 0 \), the (empty) sum on \( k_1, \cdots, k_l \) is considered to be 1.

Similarly, we have the result for \( \tilde{M}^L_\omega(\tau) \).