Efficiency Guarantees from Data

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Abstract

Analysis of efficiency of outcomes in game theoretic settings has been a main item of study at the intersection of economics and computer science. The notion of the price of anarchy takes a worst-case stance to efficiency analysis, considering instance independent guarantees of efficiency. We propose a data-dependent analog of the price of anarchy that refines this worst-case assuming access to samples of strategic behavior. We focus on auction settings, where the latter is non-trivial due to the private information held by participants. Our approach to bounding the efficiency from data is robust to statistical errors and mis-specification. Unlike traditional econometrics, which seek to learn the private information of players from observed behavior and then analyze properties of the outcome, we directly quantify the inefficiency without going through the private information. We apply our approach to datasets from a sponsored search auction system and find empirical results that are a significant improvement over bounds from worst-case analysis.

1 Introduction

A major field at the intersection of economics and computer science is the analysis of the efficiency of systems under strategic behavior. The seminal work of [6, 10] triggered a line of work on quantifying the inefficiency of computer systems, ranging from network routing, resource allocation and more recently auction marketplaces [9]. However, the notion of the price of anarchy suffers from the pessimism of worst-case analysis. Many systems can be inefficient in the worst-case over parameters of the model, but might perform very well for the parameters that arise in practice.

Due to the large availability of datasets in modern economic systems, we propose a data-dependent analog of the price of anarchy, which assumes access to a sample of strategic behavior from the system. We focus our analysis on auction systems where the latter approach is more interesting due to the private information held by the participants of the system, i.e. their private value for the item at sale. Since efficiency is a function of these private parameters, quantifying the inefficiency of the system from samples of strategic behavior is non-trivial. The problem of estimation of the inefficiency becomes an econometric problem where we want to estimate a function of hidden variables from observed strategic behavior. The latter is feasible under the assumption that the observed behavior is the outcome of an equilibrium of the strategic setting, which connects observed behavior to unobserved private information.

Traditional econometric approaches to auctions [3, 8], address such questions by attempting to exactly pin-point the private parameters from the observed behavior and subsequently measuring the quantities of interest, such as the efficiency of the allocation. The latter approach is problematic in complex auction systems for two main reasons: (i) it leads to statistical inefficiency, (ii) it requires strong conditions on the connection between observed behavior and private information. Even for a single-item first-price auction, uniform estimation of the private value of a player from $T$ samples of observed bids, can only be achieved at $O(T^{1/3})$-rates [3]. Moreover, uniquely identifying the private information from the observed behavior, requires a one-to-one mapping between the two
quantities. The latter requires strong assumptions on the distribution of private parameters and can only be applied to simple auction rules.

Our approach bridges the gap between worst-case price of anarchy analysis and statistically and modeling-wise brittle econometric analysis. We provide a data-dependent analog of recent techniques for quantifying the worst-case inefficiency in auctions \[12,4,9\], that do not require characterization of the equilibrium structure and which directly quantify the inefficiency through best-response arguments, without the need to pin-point the private information. Our approach makes minimal assumptions on the distribution of private parameters and on the auction rule and achieves \(\tilde{O}(\sqrt{T})\)-rates of convergence for many auctions used in practice, such as the Generalized Second Price (GSP) auction \[2,13\]. We applied our approach to a real world dataset from a sponsored search auction system and we portray the optimism of the data-dependent guarantees as compared to their worst-case counterparts \[11\].

### 2 Preliminaries

We consider the single-dimensional mechanism design setting with \(n\) bidders. The mechanism designer wants to allocate a unit of good to the bidders, subject to some feasibility constraint on the vector of allocations \((x_1, \ldots, x_n)\). Let \(\mathcal{X}\) be the space of feasible allocations. Each bidder \(i\) has a private value \(v_i \in [0, H]\) per-unit of the good, and her utility when she gets allocation \(x_i\) and is asked to make a payment \(p_i\) is \(v_i \cdot x_i - p_i\). The value of each bidder is drawn independently from distribution with CDF \(F_i\), supported in \(V_i \subseteq \mathbb{R}_+\) and let \(F = \times_i F_i\) be the joint distribution.

An auction \(A\) solicits a bid \(b_i \in B\) from each bidder \(i\) and decides on the allocation vector based on an allocation rule \(X : B^n \rightarrow \mathcal{X}\) and a payment rule \(p : B^n \rightarrow \mathbb{R}^n\). For a vector of values and bids, the utility of a bidder is:

\[
U_i(b_i, v_i) = v_i \cdot X_i(b) - p_i(b).
\]

A strategy \(\sigma_i : V_i \rightarrow B\) for each bidder \(i\), maps the value of the bidder to a bid. Given an auction \(A\) and distribution of values \(F\), a strategy profile \(\sigma\) is a Bayes-Nash Equilibrium (BNE) if each bidder \(i\) with any value \(v_i \in V_i\) maximizes her utility in expectation over her opponents bids, by bidding \(\sigma_i(v_i)\).

The welfare of an auction outcome is the expected utility generated for all the bidders, plus the revenue of the auctioneer, which due to the form of bidder utilities boils down to being the total value that the bidders get from the allocation. Thus the expected utility of a strategy profile \(\sigma\) is

\[
\text{WELFARE}(\sigma; F) = \mathbb{E}_{v \sim F} \left[ \sum_{i \in [n]} v_i \cdot X_i(\sigma(v)) \right].
\]

We denote with \(\text{OPT}(F)\) the expected optimal welfare:

\[
\text{OPT}(F) = \mathbb{E}_{v \sim F} \left[ \max_{X \in \mathcal{X}} \sum_{i \in [n]} v_i \cdot x_i \right].
\]

**Worst-case Bayes-Nash price of anarchy.** The Bayesian price of anarchy of an auction is defined as the worst-case ratio of welfare in the optimal auction to the welfare in a Bayes-Nash equilibrium of the original auction, taken over all value distributions and over all equilibria. Let \(\mathcal{BNE}(A, F)\) be the set of Bayes-Nash equilibria of an auction \(A\), when values are drawn from distributions \(F\). Then:

\[
\text{POA} = \sup_{F, \sigma \in \mathcal{BNE}(F)} \frac{\text{OPT}(F)}{\text{WELFARE}(\sigma; F)}.
\]

### 3 Distributional Price of Anarchy: Refining the POA with Data

We will assume that we observe \(T\) samples \(b^{1:T} = \{b^1, \ldots, b^T\}\) of bid profiles from running \(T\) times an auction \(A\). Each bid profile \(b^t\) is drawn i.i.d. based on an unknown Bayes-Nash equilibrium \(\sigma\) of the auction, i.e.: let \(\mathcal{D}\) denote the distribution of the random variable \(\nu(v)\), when \(v\) is drawn from \(F\). Then \(b^t\) are i.i.d. samples from \(\mathcal{D}\). Our goal is to refine our prediction on the efficiency of the auction and compute a bound on the price of anarchy of the auction conditional on the observed data set. More formally, we want to derive statements of the form: conditional on \(b^{1:T}\), with probability at least \(1 - \delta\): \(\text{WELFARE}(\sigma; F) \geq \frac{1}{\hat{\rho}} \text{OPT}(F)\), where \(\hat{\rho}\) is the empirical analogue of the worst-case price of anarchy ratio.
We first upper bound the distributional price of anarchy \( x \) we omit it from notation, using distributional price of anarchy. Thus when bounding the distributional price of anarchy that is tailored to an equilibrium bid distribution, which we refer to as the distributional price of anarchy. We then define the distributional price of anarchy as follows:

**Definition 1 (Distributional Price of Anarchy).** The distributional price of anarchy \( \text{DPOA}(\mathcal{D}) \) of an auction \( A \) and a distribution of bid profiles \( \mathcal{D} \), is the worst-case ratio of welfare in the optimal allocation to the welfare in an equilibrium, taken over all distributions of values and all equilibria that could generate the bid distribution \( \mathcal{D} \):

\[
\text{DPOA}(\mathcal{D}) = \sup_{\mathcal{F}, \sigma \in \text{BNE}(\mathcal{F}) \text{ s.t. } \mathcal{D}(\mathcal{F}, \sigma) = \mathcal{D}} \frac{\text{OPT}(\mathcal{F})}{\text{WELFARE}(\sigma; \mathcal{F})} \tag{4}
\]

This notion has nothing to do with sampled data-sets, but rather is a hypothetical worst-case quantity that could generate had we known the true bid generating distribution \( \mathcal{D} \).

**What does the extra information of knowing \( \mathcal{D} \) give us?** To answer this question, we first focus on the optimization problem each bidder faces. At any Bayes-Nash equilibrium each player must be best-responding in expectation over his opponent bids. Observe that if we know the rules of the auction and the equilibrium distribution of bids \( \mathcal{D} \), then the expected allocation and payment function of a player as a function of his bid are uniquely determined:

\[
x_i(b; \mathcal{D}) = \mathbb{E}_{b_{-i} \sim \mathcal{D}_{-i}} [X_i(b, b_{-i})] \quad \quad p_i(b; \mathcal{D}) = \mathbb{E}_{b_{-i} \sim \mathcal{D}_{-i}} [P_i(b, b_{-i})]. \tag{5}
\]

Importantly, these functions do not depend on the distribution of values \( \mathcal{F} \), other than through the distribution of bids \( \mathcal{D} \). Moreover, the expected revenue of the auction is also uniquely determined:

\[
\text{REV}(\mathcal{D}) = \mathbb{E}_{b \sim \mathcal{D}} \left[ \sum_i P_i(b) \right]. \tag{6}
\]

Thus when bounding the distributional price of anarchy, we can assume that these functions and the expected revenue are known. The latter is unlike the standard price of anarchy analysis, which essentially needs to take a worst-case approach to these quantities.

**Shorthand notation** Through the rest of the paper we will fix the distribution \( \mathcal{D} \). Hence, for brevity we omit it from notation, using \( x_i(b), p_i(b) \) and \( \text{REV} \) instead of \( x_i(b; \mathcal{D}), p_i(b; \mathcal{D}) \) and \( \text{REV}(\mathcal{D}) \).

## 4 Bounding the Distributional Price of Anarchy

We first upper bound the distributional price of anarchy via a quantity that is relatively easy to calculate as a function of the bid distribution \( \mathcal{D} \) and hence will also be rather straightforward to estimate from samples of \( \mathcal{D} \), which we defer to the next section. To give intuition about the upper bound, we start with a simple but relevant example of bounding the distributional price of anarchy in the case when the auction \( A \) is the single-item first price auction. We then generalize the approach to any auction \( A \).

### 4.1 Example: Single-Item First Price Auction

In a single item first price auction, the designer wants to auction a single indivisible good. Thus the space of feasible allocations \( \mathcal{X} \), are ones where only one player gets allocation \( x_i = 1 \) and other players get allocation 0. The auctioneer solicits bids \( b_i \) from each bidder and allocates the good to the highest bidder (breaking ties lexicographically), charging him his bid. Let \( \mathcal{D} \) be the equilibrium distribution of bids and let \( G_i \) be the CDF of the bid of player \( i \). For simplicity we assume that \( G_i \) is continuous (i.e. the distribution is atomless). Then the expected allocation of a player \( i \) from submitting a bid \( b \) is equal to \( x_i(b) = G_{-i}(b) = \prod_{j \neq i} G_j(b) \) and his expected payment is \( p_i(b) = b \cdot x_i(b) \), leading to expected utility: \( u_i(b; v_i) = (v_i - b) G_{-i}(b) \).
The quantity DPOA is a complex object as it involves the structure of the set of equilibria of the given auction. The set of equilibria of a first price auction when bidders values are drawn from different distributions is an horrific object\footnote{Even for two bidders with uniformly distributed values \(U[0,a]\) and \(U[0,b]\), the equilibrium strategy requires solving a complex system of partial differential equations, which took several years of research in economics to solve (see \cite{11,12}).} However, we can upper bound this quantity by a much simpler data-dependent quantity by simply invoking the fact that under any equilibrium bid distribution no player wants to deviate from his equilibrium bid. Moreover, this data-dependent quantity can be much better than its worst-case counterpart used in the existing literature on the price of anarchy.

**Lemma 1.** Let \(A\) be the single item first price auction and let \(D\) be the equilibrium distribution of bids, then \(\text{DPOA}(D) \leq \frac{\mu(D)}{1 - e^{-\mu(D)}}\), where \(\mu(D) = \frac{\max_{x \in [a]} \mathbb{E}_{b \sim D} [\max_{x \in [a]} b_i]}{\mathbb{E}_{b \sim D} [\max_{x \in [a]} b_i]}\).

**Proof.** Let \(G_i\) be the CDF of the bid of each player under distribution \(D\). Moreover, let \(\sigma\) denote the equilibrium strategy that leads to distribution \(D\). By the equilibrium condition, we know that for all \(v_i \in V_i\) and for all \(b' \in B\),

\[
    u_i(\sigma_i(v_i); v_i) \geq u_i(b'; v_i) = (v_i - b') \cdot G_{-i}(b').
\]

We will give a special deviating strategy used in the literature \cite{12}, that will show that either the players equilibrium utility is large or the expected maximum other bid is high. Let \(T_i\) denote the expected maximum other bid which can be expressed as \(T_i = \int_0^\infty 1 - G_{-i}(z)dz\). We consider the randomized deviation where the player submits a randomized bid in \(z \in [0, v_i(1 - e^{-\mu})]\) with PDF \(f(z) = \frac{1}{\mu(v_i - z)}\). Then the expected utility from this deviation is:

\[
    \mathbb{E}_b [u_i(b'; v_i)] = \int_0^{v_i(1 - e^{-\mu})} (v_i - z) \cdot G_{-i}(z)f(z)dz = \frac{1}{\mu} \int_0^{v_i(1 - e^{-\mu})} G_{-i}(z)dz
\]

Adding the quantity \(\frac{1}{\mu} \int_0^{v_i(1 - e^{-\mu})} (1 - G_{-i}(z))dz \leq T_i\) on both sides, we get: \(\mathbb{E}_b [u_i(b'; v_i)] + \frac{1}{\mu} T_i \geq v_i(1 - e^{-\mu})\). Invoking the equilibrium condition we get:

\[
    u_i(\sigma_i(v_i); v_i) + \frac{1}{\mu} T_i \geq v_i(1 - e^{-\mu}).
\]

Subsequently, for any \(x^*_i \in [0,1]\):

\[
    u_i(\sigma_i(v_i); v_i) + \frac{1}{\mu} T_i \cdot x_i^* \geq v_i \cdot x_i^* (1 - e^{-\mu}).\tag{9}
\]

If \(x^*_i\) is the expected allocation of player \(i\) under the efficient allocation rule \(X^*_i(v) \equiv 1\{v_i = \max_j v_j\}\), then taking expectation of Equation \(9\) over \(v_i\) and adding across all players we get:

\[
    \sum_i \mathbb{E}_{v_i} [u_i(\sigma_i(v_i); v_i)] + \frac{1}{\mu} \mathbb{E} \left[ \sum_i T_i X^*_i(v) \right] \geq \text{OPT}(F) (1 - e^{-\mu}).\tag{10}
\]

The theorem then follows by invoking the fact that for any feasible allocation \(x\): \(\sum_i T_i \cdot x_i \leq \max_i T_i = \mu(D)\text{REV}(D)\), using the fact that expected total agent utility plus total revenue at equilibrium is equal to expected welfare at equilibrium and setting \(\mu = \mu(D)\). \(\square\)

**Comparison with worst-case POA** In the worst-case, \(\mu(D)\) is upper bounded by 1, leading to the well-known worst-case price of anarchy ratio of the single-item first price auction of \((1 - 1/e)^{-1}\), irrespective of the bid distribution \(D\). However, if we know the distribution \(D\) then we can explicitly estimate \(\mu\), which can lead to a much better ratio (see Figure 2 in Appendix). Moreover, observe that even if we had samples from the bid distribution \(D\), then estimating \(\mu(D)\) is very easy as it corresponds to the ratio of two expectations, each of which can be estimating to within an \(O(\frac{1}{\sqrt{T}})\) error by a simple average and using standard concentration inequalities. Even thought this improvement, when compared to the worst-case bound might not be that drastic in the first price auction, the extent of the analysis in the next section will be applicable even to auctions where the analogue of the quantity \(\mu(D)\) is not even bounded in the worst-case. In those settings, the empirical version of the price of anarchy analysis is of crucial importance to get any efficiency bound.
Comparison with value inversion approach  Apart from being just a primer to our main general result in the next section, the latter result about the data-dependent efficiency bound for the first price auction, is itself a contribution to the literature. It is notable to compare the latter result with the standard econometric approach to estimating values in a first price auction pioneered by [8] (see also [9]). Traditional non-parametric auction econometrics use the equilibrium best response condition to pin-point the value of a player from his observed bid, by what is known as value inversion. In particular, if the function:  
\[ u_i(b' ; v_i) = (v_i - b') \cdot G_{-i}(b') \]  
has a unique maximum for each \( v_i \) and this maximum is strictly monotone in \( v_i \), then given the equilibrium bid of a player \( b_i \) and given a data distribution \( D \) we can reverse engineer the value \( v_i(b_i) \) that the player must have. Thus if we know the bid distribution \( D \) we can calculate the equilibrium welfare as  
\[ E_{b \sim D} \left[ \sum_i v_i(b_i) \cdot X_i(b) \right] . \]  
Moreover, we can calculate the expected optimal welfare as:  
\[ E_{b \sim D} \left[ \max_i v_i(b_i) \right] . \]  
Thus we can pin-point the distributional price of anarchy.

However, the latter approach suffers from two main drawbacks: (i) estimating the value inversion function \( v_i(\cdot) \) uniformly over \( b \) from samples, can only happen at very slow rates that are at least \( O(1/T^{1/3}) \) and which require differentiability assumptions from the value and bid distribution as well as strong conditions that the density of the value distribution is bounded away from zero in all the support (with this lower bound constant entering the rates of convergence), (ii) the main assumption of the latter approach is that the optimal bid is an invertible function and that given a bid there is a single value that corresponds to that bid. This assumption might be slightly benign in a single item first price auction, but becomes a harsher assumption when one goes to more complex auction schemes. Our result in Lemma 1 suffers neither of these drawbacks: it admits fast estimation rates that are statistically robust to both sampling and mis-specification errors.

4.2 Generalizing to any Single-Dimensional Auction Setting

Our analysis on DPOA is based on the reformulation of the auction rules as an equivalent pay-your-bid auction and then bounding the price of anarchy as a function of the ratio of how much a player needs to pay in an equivalent pay-your-bid auction, so as to acquire his optimal allocation vs. how much revenue is the auctioneer collecting. For any auction, we can re-write the expected utility of a bid \( b \):

\[ u_i(b; v_i) = x_i(b) \left( v_i - \frac{p_i(b)}{x_i(b)} \right) \]  

This can be viewed as the same form of utility if the auction was a pay-your-bid auction and the player submitted a bid of \( \frac{p_i(b)}{x_i(b)} \). We refer to this term as the price-per-unit and denote it \( ppu(b) = \frac{p_i(b)}{x_i(b)} \).

Our analysis will be based on the price-per-unit allocation rule \( \hat{x}(. ; \cdot) \), which determines the expected allocation of a player as a function of his price-per-unit. Given this notation, we can re-write the utility that an agent achieves if he submits a bid that corresponds to a price-per-unit of \( z \) as:  
\[ u_i(z; v_i) = \hat{x}(z)(v_i - z) \]  

The latter is exactly the form of a pay-your-bid auction.

Our upper bound on the DPOA, will be based on the inverse of the PPU allocation rule; let \( \tau_i(z) = \hat{x}_i^{-1}(z) \) be the price-per-unit of the cheapest bid that achieves allocation at least \( z \). More formally, \( \tau_i(z) = \min_{b \geq z} \{ ppu(b) \} \). For simplicity, we assume that any allocation \( z \in [0, 1] \) is achievable by some high enough bid \( b \). Given this we can define the threshold for an allocation:

**Definition 2** (Average Threshold). The average threshold for agent \( i \) is

\[ T_i = \int_0^1 \tau_i(z) \, dz \]  

In the Appendix we provide a pictorial representation of these quantities. Connecting with the previous section, for a first price auction, the price-per-unit function is \( ppu(b) = b \), the price-per-unit allocation function is \( \hat{x}_i(b) = G_{-i}(b) \) and the threshold function is \( \tau_i(z) = G_{-i}^{-1}(z) \). The average threshold \( T_i \) is equal to  
\[ \int_0^1 G_{-i}^{-1}(z) \, dz = \int_0^\infty 1 - G_{-i}(b) \, db \]  

i.e. the expected maximum other bid.

\[ \text{The theory can be easily extended to allow for different maximum achievable allocations by each player, by simply integrating the average threshold only up until the largest such allocation.} \]
We now give our main Theorem, which is a distribution-dependent bound on DPOA, that is easy to compute given $D$ and which can be easily estimated from samples of $D$. This theorem is a generalization of Lemma 1 in the previous section.

**Theorem 2 (Distributional Price of Anarchy Bound).** For any auction $A$ in a single dimensional setting and for any bid distribution $D$, the distributional price of anarchy is bounded by $\text{DPOA}(D) \leq \frac{\mu(D)}{1 - e^{-\mu(D)/\text{Rev}(D)}}$, where \( \mu(D) = \frac{\max_{x \in X} \sum_{i=1}^{n} T_i \cdot x_i}{\text{Rev}(D)} \).

Theorem 2 provides our main method for bounding the distributional price of anarchy. All we need is to compute the revenue $\text{Rev}$ of the auction and the quantity:

$$ T = \max_{x \in X} \sum_{i=1}^{n} T_i \cdot x_i, $$

under the given bid distribution $D$. Both of these are uniquely defined quantities if we are given $D$. Moreover, once we compute $T$, the optimization problem in Equation (13) is simply a welfare maximization problem, where each player’s value per-unit of the good is $T_i$. Thus, the latter can be solved in polynomial time, whenever the welfare maximization problem over the feasible set $X$ is polynomial-time solvable.

Theorem 2 can be viewed as a bid distribution-dependent analogue of the revenue covering framework [4] and of the smooth mechanism framework [12]. In particular, the quantity $\mu(D)$ is the data-dependent analogue of the worst-case $\mu$ quantity used in the definition of $\mu$-revenue covering in [4] and is roughly related to the $\mu$ quantity used in the definition of a $(\lambda, \mu)$-smooth mechanism in [12].

## 5 Distributional Price of Anarchy Bound from Samples

In the last section, we assumed we were given distribution $D$ and hence we could compute the quantity $\mu = \frac{T}{\text{Rev}}$, which gave an upper bound on the DPOA. We now show how we can estimate this quantity $\mu$ when given access to i.i.d. samples $b^{1:T}$ from the bid distribution $D$. We will separately estimate $T$ and $\text{Rev}$. The latter is simple expectation and thereby can be easily estimated by an average at $\frac{1}{\sqrt{T}}$ rates. For the former we first need to estimate $T_i$ for each player $i$, which requires estimation of the allocation and payment functions $x_i(\cdot; D)$ and $p_i(\cdot; D)$.

Since both of these functions are expected values over the equilibrium bids of opponents, we will approximate them by their empirical analogues:

$$ \hat{x}_i(b) = \frac{1}{T} \sum_{t=1}^{T} X_i(b, b_t^{i-1}) \quad \quad \hat{p}_i(b) = \frac{1}{T} \sum_{t=1}^{T} P_i(b, b_t^{i-1}). \quad \quad \quad (14) $$

To bound the estimation error of the quantities $\hat{T}_i$ produced by using the latter empirical estimates of the allocation and payment function, we need to provide a uniform convergence property for the error of these functions over the bid $b$.

Since $b$ takes values in a continuous interval, we cannot simply apply a union bound. We need to make assumptions on the structure of the class of functions $F_{X_i} = \{ X_i(b, \cdot) : b \in B \}$ and $F_{P_i} = \{ P_i(b, \cdot) : b \in B \}$, so as uniformly bound their estimation error. For this we resort to the technology of Rademacher complexity. For a generic class of functions $F$ and a sequence of random variables $Z^{1:T}$, the Rademacher complexity is defined as:

$$ \mathcal{R}_{T}(F, Z^{1:T}) = \mathbb{E}_{\sigma^{1:T}} \left[ \sup_{f \in F} \frac{1}{T} \sum_{t=1}^{T} \sigma^t f(Z^t) \right]. \quad \quad (15) $$

where each $\sigma^t \in \{ \pm 1/2 \}$ is an i.i.d. Rademacher random variable, which takes each of those values with equal probabilities. The following well known theorem will be useful in our derivations:

**Theorem 3 [11].** Suppose that for any sample $Z^{1:T}$ of size $T$, $\mathcal{R}_{T}(F, Z^{1:T}) \leq \mathcal{R}_T$ and suppose that functions in $F$ take values in $[0, H]$. Then with probability $1 - \delta$:

$$ \sup_{f \in F} \left| \frac{1}{T} \sum_{t=1}^{T} f(Z_t) - \mathbb{E}[f(Z)] \right| \leq 2\mathcal{R}_T + H \sqrt{\frac{2 \log(4/\delta)}{T}}. \quad \quad (16) $$
This Theorem reduces our uniform error problem to bounding the Rademacher complexity of classes \( \mathcal{F}_{X_i} \) and \( \mathcal{F}_{P_i} \), since we immediately have the following corollary (where we also use that the allocation functions lie in \([0,1]\) and the payment functions lie in \([0,\bar{H}]\)):

**Corollary 4.** Suppose that for any sample \( b^{1:T} \) of size \( T \), the Rademacher complexity of classes \( \mathcal{F}_{X_i} \) and \( \mathcal{F}_{P_i} \) is at most \( R_T \). Then with probability \( 1 - \delta/2 \), both \( \sup_{b \in \mathcal{B}} \left| \hat{x}_i(b) - x_i(b) \right| \) and \( \sup_{b \in \mathcal{B}} \left| \hat{p}_i(b) - p_i(b) \right| \) are at most \( 2R_T + \sqrt{2\log(4/\delta)/T} \).

We now provide conditions under which the Rademacher complexity of these classes is \( \tilde{O}(1/\sqrt{T}) \).

**Lemma 5.** Suppose that \( \mathcal{B} = [0, B] \) and for each bidder \( i \) and each \( b_i \in \mathcal{B} \), the functions \( X_i(b, \cdot) : [0, B]^{n-1} \rightarrow [0,1] \) and \( P_i(b, \cdot) : [0, B]^{n-1} \rightarrow [0,\bar{H}] \) can be computed as finite superposition of (i) multiplication of bid vectors \( b_{-i} \) with constants; (ii) comparison indicators \( 1\{ \cdot > \cdot \} \); (iii) pairwise addition \( \cdot + \cdot \). The Rademacher complexity for both classes on a sample of size \( T \) is \( O\left( \sqrt{\log(T)/T} \right) \).

The proof of this Lemma follows by standard arguments of Rademacher calculus, together with VC arguments on the class of pairwise comparisons. Those arguments can be found in Lemma 9.9 in [5]. Thereby, we omit their proof. The assumptions of Lemma 5 can be directly verified, for instance, for the sponsored search auctions where the constants that multiply each bid correspond to quality factors of the bidders, e.g. as in [2] and [13] and then the allocation and the payment is a function of the rank of the weighted bid of a player. In that case the price and the allocation rule are determined solely by the ranks and the values of the score-weighted bids \( \gamma_i b_i \), as well as the position specific quality factors \( \alpha_j \), for each position \( j \) in the auction.

Next we turn to the analysis of the estimation errors on quantities \( T_i \). We consider the following plug-in estimator for \( T_i \): We consider the empirical analog of function \( \tau_i(\cdot) \) by \( \hat{\tau}_i(z) = \inf_{b \in [0,B]} \frac{\hat{p}_i(b)}{x_i(b)} \).

Then the empirical analog of \( T_i \) is obtained by:

\[
\hat{T}_i = \int_0^1 \hat{\tau}_i(z) \, dz.
\]  

To bound the estimation error of \( \hat{T}_i \), we need to impose an additional condition that ensures that any non-zero allocation requires the payment from the bidder at least proportional to that allocation.

**Assumption 6.** Suppose that \( p_i(x_i^{-1}(\cdot)) \) is Lipschitz-continuous, \( x_i(\cdot) \) is continuous at zero with \( x_i(0) = 0 \). Moreover, there exists a constant \( P \) such that for any \( b_{-i} \in [0, B]^{n-1} \) and any \( b \in [0, B] \) with \( X_i(b, b_{-i}) > 0 \) we have \( P_i(b, b_{-i}) \geq \frac{1}{P} X_i(b, b_{-i}) \).

Under this assumption we can establish that \( \tilde{O}(\sqrt{T}) \) rates of convergence of \( \hat{T}_i \) to \( T_i \) and of the empirical analog \( \hat{T} = \max_{x \in \mathcal{X}} \sum_{i=1}^n \hat{T}_i \cdot x_i \) of the optimized threshold to \( T \) as well as the empirical analog \( \hat{\text{REV}} \) of the revenue to \( \text{REV} \). Thus the quantity \( \hat{\mu} = \frac{\hat{T}}{\hat{\text{REV}}} \), will also converge to \( \mu = \frac{T}{\text{REV}} \) at that rate. This implies the following final conclusion of this section.

**Theorem 7.** Under Assumption 6 and the premises of Lemma 5 with probability \( 1 - \delta \):

\[
\frac{\text{OPT}(F)}{\text{WELFARE}(\sigma; F)} \leq \frac{\hat{\mu}}{1 - e^{-\hat{\mu}}} + O\left( \sqrt{\frac{\log(T) + \log(2n/\delta)}{T}} \right)
\]

6  Sponsored Search Auction: Model, Methodology and Data Analysis

We consider a position auction setting where \( k \) ordered positions are assigned to \( n \) bidders. An outcome \( m \) in a position auction is an allocation of positions to bidders. \( m(j) \) denotes the bidder who is allocated position \( j \); \( m^{-1}(i) \) refers to the position assigned to bidder \( i \). When bidder \( i \) is assigned to slot \( j \), the probability of click \( c_{i,j} \) is the product of the click-through-rate of the slot \( \alpha_j \) and the quality score of the bidder, \( \gamma_i \), so \( c_{i,j} = \alpha_j \gamma_i \) (in the data the quality scores for each bidder are varying across different auctions and we used the average score as a proxy for the score of a bidder). Each advertiser
We will denote with $q_i$, where each player's value-per-click is $\gamma_i$. Advertisers are ranked by their rank-score $\gamma_i$. Anarchy analysis for a set of eleven search phrases on the BingAds system.

Data Analysis

We applied our analysis to the BingAds sponsored search auction system. We empirically the price of anarchy is very good and on average the guarantee is approximately 80%.

We will denote with $\pi_{b,\theta}(j)$ the bidder allocated in slot $j$ under a bid profile $b$ and parameter profile $\theta$. We denote with $\pi_{b,\theta}^{-1}(i)$ the slot allocated to bidder $i$. If advertiser $i$ is allocated position $j$, then he pays only when he is clicked and his payment, i.e. his cost-per-click is the minimal bid he had to place to keep his position, which is: $\text{cpc}_{ij}(b;\theta) = \max_{j} \left\{ s_{b,\theta}(i+1) \cdot b_{b,\theta}(i+1) \cdot \gamma \right\}$. Mapping this setting to our general model, the allocation function of the auction is $X_i(b) = \alpha \cdot \pi_{b,\theta}^{-1}(i) \cdot \gamma_i \cdot \text{cpc}_{\pi_{b,\theta}^{-1}(i)}(b;\theta)$ and the payment function is $F_i(b) = \alpha \cdot \pi_{b,\theta}^{-1}(i) \cdot \gamma_i \cdot \text{cpc}_{\pi_{b,\theta}^{-1}(i)}(b;\theta)$ and the utility function is: $U_i(b;v_i) = \alpha \cdot \pi_{b,\theta}^{-1}(i) \cdot \gamma_i \cdot \left( v_i - \text{cpc}_{\pi_{b,\theta}^{-1}(i)}(b;\theta) \right)$.

**Data Analysis**

We applied our analysis to the BingAds sponsored search auction system. We analyzed eleven phrases from multiple thematic categories. For each phrase we retrieved data of auctions for the phrase for the period of a week. For each phrase and bidder that participated in the auctions for the phrase we computed the allocation curve by simulating the auctions for the week awards over privacy reasons. (right) Distributional Price of Anarchy analysis for a set of eleven search phrases on the BingAds system.

See Figure 1 for the price-per-unit allocation curves for a subset of six advertisers for a specific keyword during the period of a week. All axes are normalized to 1 for privacy reasons. (right) Distributional Price of Anarchy analysis for a set of eleven search phrases on the BingAds system.

| Phrase  | $\mu$     | $\frac{1}{\text{DFOA}}$ = $\frac{1-e^{-\mu}}{\mu}$ |
|---------|-----------|--------------------------|
| phrase1 | .511      | .783                     |
| phrase2 | .509      | .784                     |
| phrase3 | 2.966     | .320                     |
| phrase4 | 1.556     | .507                     |
| phrase5 | .386      | .829                     |
| phrase6 | .488      | .791                     |
| phrase7 | .459      | .802                     |
| phrase8 | .419      | .817                     |
| phrase9 | .441      | .809                     |
| phrase10| .377      | .833                     |
| phrase11| .502      | .786                     |

**Figure 1:** (left) Examples of price-per-unit allocation curves for a subset of six advertisers for a specific keyword during the period of a week. All axes are normalized to 1 for privacy reasons. (right) Distributional Price of Anarchy analysis for a set of eleven search phrases on the BingAds system.
the optimal. Even if $s_i = \gamma_i$ the worst-case bound on the PoA implies guarantees of approx. 34% $\Pi$, while the DPOA we estimated implies significantly higher percentages, portraying the value of the empirical approach we propose.

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Supplementary material for
“Efficiency Guarantees from Data”

A Supplementary Figures

A.1 Distributional Price of Anarchy as a Function of $\mu(D)$

Figure 2: The upper bound on the distributional price of anarchy of an auction $\frac{\mu(D)}{1-e^{-\mu(D)}}$ as a function of $\mu(D)$. 

A.2 Pictorial Representation of Threshold Functions

Figure 3: For any bid $b$ with PPC $ppu(b)$, the area of a rectangle between $(ppu(b), \bar{x}_i(ppu(b)))$ and $(v_i, 0)$ on the bid allocation rule is the expected utility $u_i(b)$. The BNE action $b^*$ is chosen to maximize this area.

Figure 4: The average threshold is the area to the left of the price-per-unit allocation rule, integrate from 0 to 1.

B Omitted Proofs from Section 3

Theorem 2 (restatement) For any auction $A$ in a single dimensional setting and for any bid distribution $D$, the distributional price of anarchy is bounded by $DPA(D) \leq \mu(D)$, where $\mu(D) = \frac{\max_{x \in X} \sum_{i=1}^{n} T_i \cdot x}{\mathbb{E}[V]}$.

Proof. Our proof is based on a data-dependent analog of the value and revenue covering framework of [4]. First we show that even without having distributional knowledge, the threshold functions are related to the equilibrium utility of a bidder and any target utility at any Bayes-Nash equilibrium. Specifically, either the utility of a bidder at a Bayes-Nash is high compared to his value or the average threshold $T_j$ is high.
Lemma 8 (Value Covering). For any bidder \( i \) with value \( v_i \), for any allocation amount \( x \in [0, 1] \) and for any \( \mu \geq 1 \),

\[
u_i(v_i) + \frac{1}{\mu} T_i \cdot x_i \geq \frac{1 - e^{-\mu}}{\mu} v_i \cdot x_i. \tag{19}\]

where \( u_i(v_i) = u_i(\sigma_i(v_i); v_i) \).

Proof. The proof proceeds analogously to the proof of value covering in [41]. For simplicity of notation we drop the subscript \( i \), as we are focusing on a single agent and some threshold function \( \tau(\cdot) \). Observe that since a player is at equilibrium it must be that for any target expected allocation \( z \) he does not want to deviate to a bid that corresponds to a price-per-unit \( \text{ppu}(b) = \tau(z) \), which would yield him expected allocation at least \( z \):

\[
u(v) \geq z \cdot (v - \tau(z)) \implies \tau(z) \geq v - \frac{u(v)}{z} \tag{20}\]

Moreover, in any case \( \tau(z) \geq 0 \), by definition. Thus if we define \( \tau(z) = \max(0, v - u(v)/z) \), then we have \( \tau(z) \geq \tau(z) \) and hence \( T \geq T = \int_0^1 \tau(z)dz \).

Evaluating the integral gives \( T = v - u(v) + u(v) \log \frac{u(v)}{v} \). Thus

\[
u(v) + \frac{1}{\mu} T = u(v) + \frac{1}{\mu} \left(v - u(v) + u(v) \log \frac{u(v)}{v}\right)
\]

and by dividing over by \( v \):

\[
u(v) + \frac{1}{\mu} \frac{T}{v} = \frac{u(v)}{v} + \frac{1}{\mu} \left(1 - \frac{u(v)}{v} + \frac{u(v)}{v} \log \frac{u(v)}{v}\right) \tag{21}\]

The right side of Equation (21) is convex in \( \frac{u(v)}{v} \), so we can minimize it by taking first-order conditions of the quantity \( y + \frac{1}{\mu} (1 - y + y \log y) \) with respect to variable \( y \), giving

\[ 0 = 1 + \frac{1}{\mu} \log y \implies y = e^{-\mu}. \]

Leading to a minimum value of that quantity of \( \frac{1 - e^{-\mu}}{\mu} \). Thus the right side of Equation (21) is at least this quantity, giving our desired result,

\[
u(v) + \frac{1}{\mu} \frac{T}{v} \geq \frac{1 - e^{-\mu}}{\mu}. \]

The Lemma follows by the fact that \( T \geq T \) and \( x \in [0, 1] \), which allows us to multiply and divide the fraction by \( x \) and then remove the \( x \) in front of the quantity \( u(v) \).

Given the value covering lemma we now proceed to proving the Theorem. Let \( X^*(v) \) be the welfare optimal allocation rule for valuation profile \( v \), i.e. the one that solves the optimization problem \( \max_{x \in X} \sum_{i=1}^n v_i \cdot x_i \). Applying the value covering inequality of Equation (19) with respect to the optimal allocation quantity \( X_i^*(v) \) gives that for each bidder \( i \) with value \( v_i \),

\[
u_i(v_i) + \frac{1}{\mu} T_i \cdot X_i^*(v) \geq \frac{1 - e^{-\mu}}{\mu} v_i \cdot X_i^*(v). \tag{22}\]

The quantity \( v_i \cdot X_i^*(v) \) is exactly agent \( i \)'s expected contribution to the welfare of the optimal auction. Moreover, by the definition of \( \mu(D) \):

\[
u(D) \cdot \text{REV} \geq \max_{x \in X} \sum_{i=1}^n T_i \cdot x_i \geq \mathbb{E}_v \left[ \sum_i T_i \cdot X_i^*(v) \right] \tag{23}\]
Let UTIL denote the expected equilibrium total utility of the bidders in the auction. By Equations (22) and (23) we obtain:

\[
\text{UTIL} + \text{REV} \geq \mathbb{E}_v \left[ \sum_i u_i(v_i) \right] + \mathbb{E}_v \left[ \sum_i \frac{1}{\mu(D)} T_i \cdot X^*_i(v) \right]
\]

\[
= \sum_i \mathbb{E}_v \left[ u_i(v_i) + \frac{1}{\mu(D)} T_i \cdot X^*_i(v) \right]
\]

\[
\geq \sum_i \mathbb{E}_v \left[ 1 - e^{-\mu(D)} \right] v_i \cdot X^*_i(v) = 1 - e^{-\mu(D)} \sigma(D) - \text{OPT}(F)
\]

Since WELFARE(σ; F) = UTIL + REV, we have our desired result:

WELFARE(σ; F) ≥ \(1 - e^{-\mu(D)}\) ORP(F).

\(\square\)

C Omitted Proofs from Section 5

We begin by showing convergence of \(\hat{T}_i\) to \(T_i\) and \(\hat{T}\) to \(T\).

**Theorem 9.** Under Assumption 6 and the premises of Lemma 5 with probability 1 − δ:

\[
\sup_{i \in [n]} |\hat{T}_i - T_i| \leq O \left( \sqrt{\frac{\log(T) + \log(1/\delta)}{T}} \right) \tag{24}
\]

Subsequently, if \(\hat{T} = \max_{x \in X} \sum_{i=1}^n \hat{T}_i \cdot x_i\), then with probability 1 − nδ:

\[
|\hat{T} - T| \leq O \left( \sqrt{\frac{\log(T) + \log(1/\delta)}{T}} \right) \tag{25}
\]

**Proof.** Recall that \(\tau_i(z) = \inf_{x_i(b) \geq z} p_i(b)\) and \(\hat{\tau}_i(z) = \inf_{\hat{x}_i(b) \geq z} \hat{p}_i(b)\). Consider event \(A_1(\epsilon) = \{ \int_0^1 |\hat{\tau}_i(z) - \tau_i(z)| \, dz \geq \epsilon \}\).

Due to Assumption 6 we can evaluate \(\hat{\tau}_i(z) \geq p_i\). Thus, selecting small \(\eta > 0\) we can write that \(A_1(\epsilon) = \{ \int_0^1 |\hat{\tau}_i(z) - \tau_i(z)| \, dz \geq \epsilon \}\) is implied by

\[
A_2(\epsilon) = \{ \int_0^1 |\hat{\tau}_i(z) - \tau_i(z)| \, dz \geq \frac{\epsilon}{2} \} \cup \{ \int_0^\eta |\hat{\tau}_i(z) - \tau_i(z)| \, dz \geq \frac{\epsilon}{2} \}
\]

Since \(\hat{\tau}_i(z), \tau_i(z) \geq p_i\), and continuity of \(x_i(\cdot)\) at 0 ensures that for sufficiently small \(\eta\) event

\[
\int_0^\eta |\hat{\tau}_i(z) - \tau_i(z)| \, dz \geq \frac{\epsilon}{2}
\]

occurs with probability zero. To bound the probability of event

\[
\int_0^1 |\hat{\tau}_i(z) - \tau_i(z)| \, dz \geq \frac{\epsilon}{2}
\]

we can use the definition to obtain

\[
|\hat{\tau}_i(z) - \tau_i(z)| = \inf_{\hat{x}_i(b) \geq z} \hat{p}_i(b) - \inf_{x_i(b) \geq z} p_i(b)
\]

\[
\leq \left[ \inf_{\hat{x}_i(b) \geq z} \hat{p}_i(b) - \inf_{x_i(b) \geq z} p_i(b) \right] + \left[ \inf_{\hat{x}_i(b) \geq z} p_i(b) - \inf_{x_i(b) \geq z} x_i(b) \right].
\]
We note that with this choice
\[ \epsilon = \frac{1}{z} \sup_{[0, B]} |\hat{p}_i(b) - p_i(b)| \leq \frac{H}{\max\{\eta, z\}^2} \sup_{[0, B]} |\hat{x}_i(b) - x_i(b)| \]

Consider the second component in the sum. We note that the infinum in sets \( \{\hat{x}_i(b) \geq z\} \) and \( \{x_i(b) \geq z\} \) does not coincide only if these sets are different. Then we can write
\[
\left| \inf_{\hat{x}_i(b) \geq z} \frac{p_i(b)}{x_i(b)} - \inf_{x_i(b) \geq z} \frac{p_i(b)}{x_i(b)} \right| \leq \frac{L}{\max\{\eta, z\}} \sup_{[0, B]} |\hat{x}_i(b) - x_i(b)|,
\]

where \( L \) is the Lipschitz constant for function \( p(x^{-1}(\cdot)) \).

Therefore, for a given \( \eta \)
\[
\int_{\eta}^{\eta + 1} |\hat{\tau}(z) - \tau(z)| dz \leq \log \frac{1}{\eta} \sup_{[0, B]} |\hat{p}_i(b) - p_i(b)| + \left( H (1 - \eta^{-1}) + L \log \frac{1}{\eta} \right) \sup_{[0, B]} |\hat{x}_i(b) - x_i(b)|
\]

Next we recall that with probability \( 1 - \delta/2 \):
\[
\sup_{b \in B} |\hat{x}_i(b) - x_i(b)| \leq 2R_T + \sqrt{\frac{2\log(4/\delta)}{T}} = \epsilon_1
\]
\[
\sup_{b \in B} |\hat{p}_i(b) - p_i(b)| \leq 2R_T + H \sqrt{\frac{2\log(4/\delta)}{T}} = \epsilon_2
\]

Thus, with probability at least \( 1 - \delta \)
\[
\int_{\eta}^{\eta + 1} |\hat{\tau}(z) - \tau(z)| dz \leq \log \frac{1}{\eta} \epsilon_1 + \left( H (1 - \eta^{-1}) + L \log \frac{1}{\eta} \right) \epsilon_2
\]
\[
= (H (1 - \eta^{-1}) + (L + 1) \log \eta^{-1}) O \left( \sqrt{\frac{\log(T)}{T}} \right)
\]
\[
+ (H (1 - \eta^{-1}) + (L + H) \log \eta^{-1}) \sqrt{\frac{2\log(4/\delta)}{T}} = \frac{\epsilon}{2}.
\]

Choosing \( \eta \) small enough so that \( \int_{0}^{\eta} |\hat{\tau}(z) - \tau(z)| dz \geq \frac{\epsilon}{2} \) occurs with probability zero, guarantees that event
\[
\int_{0}^{1} |\hat{\tau}(z) - \tau(z)| dz \geq \epsilon
\]
occurs with probability at most \( \delta \).

We note that with this choice \( \epsilon = O \left( \sqrt{\frac{\log(T) + \log(1/\delta)}{T}} \right) \).

Next we can bound
\[
P \left( \left| \hat{T} - T \right| \geq \epsilon \right) = P \left( \left| \sup_{i \in [n]} \sum_{i} \hat{T}_i - \sum_{i} T_i \right| \geq \epsilon \right)
\]
\[
\leq \sum_{i} P \left( \left| \hat{T}_i - T_i \right| \geq \epsilon \right) \leq n \delta.
\]

This delivers the last bound in the theorem. \( \square \)
We are now ready to show our main estimation theorem.

**Theorem 7 (Restatement).** *Under Assumption 6 and the premises of Lemma 5 with probability $1 - \delta$:*

$$\frac{\text{OPT}(F)}{\text{WELFARE}(\sigma; F)} \leq \frac{\hat{\mu}}{1 - e^{-\hat{\mu}}} + O\left(\sqrt{\frac{\log(T) + \log(2n/\delta)}{T}}\right)$$

(26)

**Proof.** First note that $\hat{\text{REV}} = \frac{1}{T} \sum_{t=1}^{T} b_i^t$. Thus, using standard Hoeffding’s inequality we obtain that

$$P \left( \left| \hat{\text{REV}} - \text{REV} \right| > \tau \right) \leq 2e^{-\frac{2\tau^2}{B^2}}.$$  

Thus with probability at most $\delta/2$

$$\left| \hat{\text{REV}} - \text{REV} \right| > B \sqrt{\frac{\log(4/\delta)}{2T}}.$$  

Combining this result with the result of Theorem 9, we find that with probability at most $1 - \delta$

$$|\hat{\mu} - \mu| \leq O\left(\sqrt{\frac{\log(T) + \log(2n/\delta)}{T}}\right).$$

Let $\hat{\rho} = \hat{\mu} (1 - e^{-\hat{\mu}})^{-1}$. Since the function $f(x) = x/(1 - e^{-x})$ is Lipschitz we can conclude that

$$|\hat{\rho} - \rho| \leq O\left(\sqrt{\frac{\log(T) + \log(2n/\delta)}{T}}\right)$$

with probability at least $1 - \delta$.  

\qed