On triple correlations in isotropic electron magnetohydrodynamic turbulence

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Abstract

The evolution of the correlation characteristics in three-dimensional isotropic electron magnetohydrodynamic turbulence is investigated. Universal exact relations between the longitudinal and transverse two-point triple correlations of the components of the fluctuational magnetic fields and the rates of dissipation of the magnetic helicity and energy are obtained in the inertial range.

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All statistical theories of turbulence take into account the well-known exact Kolmogorov result — the 4/5 law [1] which relates the third-order spatial longitudinal correlations of the velocity with the rate of energy dissipation. In magnetohydrodynamics a relation was obtained by Chandrasekhar [2]. Recently a similar relation (2/15 law) was established for hydrodynamic turbulence with helicity [3, 4]. The confirmations of the 4/5 law for diverse turbulent hydrodynamic flows are well known [5]. Confirmations have been obtained for the 2/15 law [3, 4] for helicity [6]. It is important to note that such accurate relations are obtained by solving dynamical equations and are a consequence of the conservation laws. No dimensional considerations are employed in their derivation. The fundamental significance of the 4/5 law in hydrodynamics has been examined in detail in [7].

Electronic magnetohydrodynamics (EMHD) pertains to a branch of plasma oscillations on which the Hall term predominates [8, 9] and it is a limiting case of multicomponent MHD, where the motion of the ions can be neglected and the motion of the electrons preserves quasineutrality. In contrast to the standard MHD case, the description (with uniform density) can
be reduced to a single nonlinear equation for the magnetic field. The region of applicability of EMHD are laboratory and industrial plasma setups, the ionosphere, the solar photosphere, and solids [9, 10]. In the 1970s the term MHD at helicon frequencies was also used [10]. Weak turbulence of helicons (whistler) was studied in [11, 12, 13]. The dynamic properties of strong three-dimensional turbulence in EMHD have been studied in [14]. Arguments supporting the idea that only weak turbulence is realized in the EMHD mechanism are presented in [8].

EMDH is described by the equation [8, 9]

$$\partial_t \mathbf{h} + \text{rot} \left[ \frac{\mathbf{j}}{n e} \times \mathbf{h} \right] + c \, \text{rot} \frac{\mathbf{j}}{\sigma} = 0,$$

(1)

$$\mathbf{j} = \frac{c}{4\pi} \text{roth}, \quad \div \mathbf{h} = 0.$$  

(2)

For $n = \text{const}$ and $\sigma = \text{const}$ obtain

$$\partial_t \mathbf{h} = -\frac{c}{4\pi n e} \text{rot} \left[ \text{roth} \times \mathbf{h} \right] + \frac{c^2}{4\pi \sigma} \Delta \mathbf{h}, \quad \div \mathbf{h} = 0.$$  

(3)

In the frequency domain this corresponds to the range

$$w_i < w < w_e.$$

For what follows, we introduce the notation $f = \frac{c}{4\pi n e}$ and $\nu_m = \frac{c^2}{4\pi \sigma}$. The structure of Eq.(3) is close to the Navier-Stokes equation for an incompressible liquid. We can verify by direct substitution that it conserves the energy

$$\int_V \frac{\mathbf{h}^2}{2} dV$$

and the helicity

$$\int_V a \mathbf{h} dV$$

of the magnetic field.

Let us consider the free evolution of uniform and isotropic fluctuations of the magnetic field in EMHD. Writing out the equation for the vector potential $\mathbf{a} = \text{rot}^{-1} \mathbf{h}$ and averaging, we obtain equations of the Karman-Howarth type for the two-point correlation functions involving the energy and helicity of the magnetic field:

$$\partial_t h_{ii} = f \varepsilon_{ijk} \frac{\partial^2}{\partial r_i \partial r_m} (h_{j,km} - h_{jm,k}) + 2 \nu_m \Delta_r h_{ii} =$$
\[ f_{ijk} \frac{\partial^2}{\partial r_i \partial r_m} (h_{km,j} - h_{jm,k}) + 2\nu_m \Delta_r h_{ii} \]  
(4)

\[ \partial_t g_{ii} = 2f \frac{\partial}{\partial r_m} h_{im,i} + 2\nu_m \Delta_r g_{ii}, \]  
(5)

where

\[ h_{ii} = \langle h_i(x)h_i(x+r) \rangle, \quad g_{ii} = \langle a_i(x)h_i(x+r) \rangle, \]  
(6)

\[ h_{jm,k} = \langle h_j(x)h_m(x)h_k(x+r) \rangle. \]  
(7)

The right-hand sides of Eqs. (4) and (5) contain the spatial derivatives with respect to \( r \) of the rank-3 two-point correlation tensor. The general form of such a tensor, with allowance for the gyrotropy and incompressibility of the magnetic field, is [2, 3, 4]

\[ h_{ij,k}(r) = V(\varepsilon_{jkl}r_i r_l + \varepsilon_{ikl}r_j r_k) + \frac{2}{r} \partial_r Tr_i r_j r_k - (r \partial_r T + 3T) (r_i \delta_{jk} + r_j \delta_{ik}) + 2T \delta_{ij} r_k. \]  
(8)

Fluctuations of the magnetic field without helicity were considered in [2]. In that case the tensor \( h_{ij,k} \) consists of only the first two terms, which are proportional to the scalar \( V \), which is related to the energy transfer. Taking the helicity into account introduces additional terms which are proportional to the product of pseudoscalar quantities and odd combinations of the components of the radius vector. Formally, the solenoidal tensor (8) is identical to the analogous tensor for triple correlations of the velocity field in hydrodynamic turbulence [3]. However, in contrast to the latter it does not change under reflection of the coordinates, i.e., \( h_{ij,k}(-r) = h_{ij,k}(r) \). The properties of homogeneous turbulence also imply that \( h_{k,ij}(r) = h_{ij,k}(-r) \) [5]. Both properties are taken into account in Eq.(4).

In what follows we shall need to examine the auxiliary tensor \( \langle \delta h_i(x|r)\delta h_j(x|r) \rangle \), where \( \delta h(x|r) = h(x+r) - h(x) \). In homogeneous turbulence it has the form

\[ \langle \delta h_i(x|r)\delta h_j(x|r) \rangle = B_{tt}(r) \left( \delta_{ij} - n_in_j \right) + B_{rr} n_in_j, \]  

where \( n = r/|r| \). The incompressibility condition implies that \( B_{tt} = \frac{1}{2r} \partial_r \left( r^2 B_{rr} \right) \) [15]. Then

\[ \langle h_i(x)h_i(x+r) \rangle = \langle h^2(x) \rangle - \frac{1}{2r^2} \partial_r \left( r^3 B_{rr} \right) \]  
(9)

We represent \( g_{ii} \) in the form

\[ g_{ii} = \langle a_i(x)h_i(x+r) \rangle = \langle a_i(x+r)h_i(x+r) \rangle - \frac{2}{r^2} \partial_r \left( r^3 C(r) \right). \]  
(10)
Substituting expressions (8)–(10) into Eqs. (4 and (5) we obtain

\[
-2\bar{\varepsilon}_m - \frac{1}{2} \partial_t \frac{1}{r^2} \partial_r \left( r^3 B_{rr} \right) = -\frac{4f}{r^4} \partial_r \left( \frac{1}{r} \partial_r \left( r^5 V \right) \right) - \frac{\nu_m}{r^2} \partial_r \left( \frac{1}{r^2} \partial_r \left( r^3 B_{rr} \right) \right) ,
\]

\[
-2\bar{\eta}_m - \partial_t \frac{2}{r^2} \partial_r \left( r^3 C \right) = -\frac{4f}{r^4} \partial_r \left( \frac{1}{r} \partial_r \left( r^5 T \right) \right) - \frac{2\nu_m}{r^2} \partial_r \left( \frac{2}{r^2} \partial_r \left( r^3 C \right) \right) .
\]

Here

\[
\bar{\varepsilon}_m = \nu_m \left\langle \frac{\partial h_i}{\partial x_j} \frac{\partial h_i}{\partial x_j} \right\rangle = \nu_m \langle (\text{rot} h)^2 \rangle ,
\]

\[
\bar{\eta}_m = \nu_m \left\langle \frac{\partial a_i}{\partial x_j} \frac{\partial h_i}{\partial x_j} \right\rangle ,
\]

are, respectively, the dissipation of the magnetic energy and helicity. Successive integration with allowance for the regularity

\[
-\frac{4}{3} \bar{\varepsilon}_m - \partial_t B_{rr} = -\frac{8f}{r^4} \partial_r \left( r^5 V \right) - \frac{2\nu_m}{r^4} \partial_r \left( r^4 \partial_r B_{rr} \right) ,
\]

\[
-\frac{\bar{\eta}_m}{3} - \partial_t C = -\frac{2f}{r^4} \partial_r \left( r^5 T \right) - \frac{2\nu_m}{r^4} \partial_r \left( r^4 \partial_r C \right) .
\]

In the inertial range the time derivatives and dissipation can be neglected, and it is found that the functions \( T \) and \( V \) depend only on the rates of dissipation of the magnetic energy and the helicity and are, respectively,

\[
V = \frac{\bar{\varepsilon}_m/f}{30}, \quad T = \frac{\bar{\eta}_m/f}{30} \quad (17)
\]

Therefore the rank-3 two-point correlation tensor for magnetic field fluctuations becomes

\[
\langle h_i(x) h_j(x) h_k(x+r) \rangle = \frac{\bar{\varepsilon}_m/f}{30} \left( \varepsilon_{jkl} r_i + \varepsilon_{ikl} r_j \right) - \frac{\bar{\eta}_m/f}{10} \left( r_j \delta_{jk} + r_j \delta_{ik} - \frac{2}{3} \delta_{ij} r_k \right) .
\]

It should be noted especially that up to numerical factors the tensor (18) of coefficients is identical to the corresponding correlation tensor of the velocity fluctuations in hydrodynamic turbulence [4].

Let us decompose the magnetic field into longitudinal and transverse components

\[
h_l = (hr) r/r^2, \quad h_t = h - h_l,
\]

\[
\delta h_l(x|r) = (h_l(x+r) - h_l(x)) r/r.
\]
In this notation we obtain
\[
\langle \delta h_l(x|\mathbf{r})^3 \rangle = -24Tr = \langle \delta h_l(x|\mathbf{r})[\mathbf{h}_l(x+r) \times \mathbf{h}_l(x)] \rangle = 4Vr^2 = \frac{2}{15} \bar{\varepsilon}_m / f \cdot r^2.
\]

Therefore the 4/5 and 2/15 laws should hold in homogeneous and isotropic EMHD turbulence. As one can see from Eq. (19), it is much simpler to determine the helicity in EMHD turbulence than in hydrodynamics, where this requires especially accurate measurements of various velocity components or the use of delicate instruments to determine the gradients, whereas in EMHD it is sufficient to measure only the longitudinal components of the fluctuation magnetic fields or currents.

We underscore that no dimensional considerations were used to derive the relations for \( T \) and \( V \), which involve the helicity and energy fluxes. This result, which is only a consequence of the statistical properties of the isotropic solutions of the EMHD equations, is universal and does not depend on which kind of turbulence — weak or strong — develops in the system.

It can be verified by direct substitution that taking the isotropy into account in the form of an external constant magnetic field \( \mathbf{h}_0 = const \) leads only to a modification of the results obtained. A dependence on the angle between the radius vector and the direction of the magnetic field will appear, since if homogeneity is preserved, the terms related to the external field \( (\sim (\mathbf{h}_0 \nabla) \mathbf{rot}) \) will not appear in equations of the form (4) and (5) for the two-point correlation functions.

|                | Hydrodynamics | EMHD |
|----------------|---------------|------|
| Nonlinearity   | \[\text{rot}\mathbf{v} \times \mathbf{v} + \nabla \left( p + \frac{\mathbf{v}^2}{2} \right)\] | \[f \mathbf{rot} [\mathbf{rot} \mathbf{h} \times \mathbf{h}]\] |
| Energy dissipation | \[\bar{\varepsilon} = \nu \langle (\mathbf{rot}\mathbf{v})^2 \rangle\] | \[\bar{\varepsilon}_m = \nu_m \langle (\mathbf{rot}\mathbf{h})^2 \rangle\] |
| Helicity dissipation | \[\bar{\eta} = \nu \langle \mathbf{rot}\mathbf{v} \mathbf{rot} \mathbf{v} \rangle\] | \[\bar{\eta}_m = \nu_m \langle \mathbf{hrot} \rangle\] |
| Kolmogorov scaling | \[E(k) = C \bar{\varepsilon}_m^2/3 k^{-5/3}\] | \[E_m(k) = C_m \langle \varepsilon_m/f \rangle^{2/3} k^{-7/3}\] |
| Helical scaling | \[E(k) = C^* \bar{\eta}_m^2/3 k^{-7/3}\] | \[E_m(k) = C^*_m \langle \eta_m/f \rangle^{2/3} k^{-5/3}\] |
| 4/5 law | \[\langle \delta v_l(x|\mathbf{r})^3 \rangle = -\frac{4}{5} \bar{\varepsilon} \cdot \mathbf{r}\] | \[\langle \delta h_l(x|\mathbf{r})^3 \rangle = -\frac{4}{5} \bar{\eta}_m / f \cdot \mathbf{r}\] |
| 2/15 law | \[\langle \delta v_l(x|\mathbf{r}) [\mathbf{v}_l(x+r) \times \mathbf{v}_l(x)] \rangle = \frac{2}{15} \bar{\varepsilon} \cdot r^2\] | \[\langle \delta h_l(x|\mathbf{r}) [\mathbf{h}_l(x+r) \times \mathbf{h}_l(x)] \rangle = \frac{2}{15} \bar{\varepsilon}_m / f \cdot r^2\] |
The "extra" curl in EMHD, as compared with the Navier-Stokes equation, leads to an unusual transposition — for longitudinal correlations the 4/5 law holds, just as in hydrodynamics, but it is related with the gyrotropic component of the fluctuations, i.e., the helicity flux and, conversely, mixed longitudinal-transverse correlations are related with the magnetic energy flux. Table I gives a comparative summary of the basic results.

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References

[1] A. N. Kolmogorov. Dokl. Akad. Nauk SSSR 32, 19 (1941).

[2] Chandrasekhar. Proc. Phys. Soc. London. Sect. A 204, 435 (1951).

[3] O. G. Chkhetiani. JETP Lett. 63, 808 (1996).

[4] V. S. L’vov, E. Podivilov, and I. Procaccia. http://xxx.lanl.gov/abs/chao-dyn/9705016.

[5] A. S. Monin and A. M. Yaglom. Statistical Fluid Mechanics. Vols. 1 and 2 (MIT Press, Cambridge. Mass., 1971 and 1975) [Russian original, Gidrometeoizdat, St. Petersburg, 1996, Part 2].

[6] L. Biferale, D. Pierotti, and P. Toschi. http://xxx.lanl.gov/abs/chao-dyn/9804004.

[7] U. Frisch, Turbulence, Cambridge University Press, New York. 1995.

[8] A. S. Kingsep, K. V. Chukbar, and V. V. Yan’kov. Reviews of Plasma Physics, Vol. 16, edited by B.B.Kadomtsev, Consultants Bureau. New York (1990) [Russian original. Voprosy Teorii Plazmy 16, 209 (1987)].

[9] A. V. Gordeev, A. S. Kingsep, and L. I. Rudakov, Phys. Rep. 243, 216 (1994).

[10] S. I. Vainshteln. Usp. Fiz. Nauk 120, 613 (1976) [Sov. Phys. Usp. 19, 987 (1976)].

[11] V. M. Yakovenko, Zh. Eksp. Teor. Fiz. 57, 554 (1968) [sic].
[12] V. N. Tsytovich, *Theory of Turbulent Plasma* (Plenum Press, New York, 1974) [Russian original. Energoatomizdat, Moscow, 1971].

[13] M. A. Livshits and V. N. Tsytovich, Zh. Eksp. Teor. Fiz. 62, 606 (1972) [Sov. Phys. JETP 35, 321 (1972)].

[14] S. I. Vainshtein, Zh. Eksp. Teor. Fiz. 64, 139 (1973) [Sov. Phys. JETP 37, 73 (1973)].

[15] L. D. Landau and E. M. Lifshitz, F/“M AecAamci, 2nd edition (Pergamon Press, New York, 1987) [Russian original. 3rd ed., Nauka, Moscow, 1986].