Quasinormal frequencies for a black hole in a bumblebee gravity

R. Oliveira (a), D. M. Dantas (b) and C. A. S. Almeida (c)

Theoretical Physics Group, Departament of Physics, Universidade Federal do Ceará, Campus do Pici Fortaleza - CE, C. P. 6030, 60455-760, Brazil

received 15 May 2021; accepted in final form 9 July 2021
published online 8 September 2021

Abstract – After recent observational events like the LIGO-Virgo detections of gravitational waves and the shadow image of the M87* supermassive black hole by event horizon telescope (EHT), the theoretical study of black holes was significantly improved. Quantities as quasinormal frequencies, shadows, and light deflection become more important to analyze black hole models. In this context, an interesting scenario to study is a black hole in the bumblebee gravity. The bumblebee vector field imposes a spontaneous symmetry breaking that allows the field to acquire a vacuum expectation value that generates Lorentz Violation (LV) into the black hole. In order to compute the quasinormal modes (QNMs) via the WKB method, we obtain the Rege-Wheeler’s equation with a bell-shaped potential for this black hole. Both QNMs, the scalar and tensorial modes, are computed for the black hole in the bumblebee scenario. The results obtained in bumblebee gravity are compared to Schwarzschild and Einstein-aether black holes. In general, the LV parameter decreases the real part of frequency for scalar and gravitational perturbations. The modulus of imaginary parts is increased with the LV parameter for the scalar field and decreased by the gravitational field. Moreover, the time-domain perturbations are studied and damping profiles are shown.

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Introduction. – The study of black holes is improving its observational knowledge due to the LIGO-Virgo detections of gravitational waves [1] and the shadow images of a supermassive black hole M87* by the international Event Horizon Telescope (EHT) [2]. In this context, it is essential to deal with the computation of observable quantities shadows [3,4], quasinormal modes (QNMs) [5] and the deflecting of light [6] in black holes and other astrophysical compact objects. Moreover, these quantities can help spot modifications in general relativity (GR) and the standard model (SM) of particle physics once the parameters of modified models should correct the observed result.

The bumblebee gravity is one of many extensions of the Standard Model extension (SME) where the Lorentz violation (LV) spontaneously occurs [7]. A Schwarzschild-like solution in the bumblebee gravity is described in the work of ref. [8]. Based on this model, the LV effect over the shadows of a black hole is studied in ref. [9], whereas the influence of LV on the black hole accretion is presented in ref. [10]. Moreover, a wormhole solution in the bumblebee gravity was conceived in the works of refs. [11,12], the radiative corrections in bumblebee electrodynamics are studied in ref. [13] and the black hole with the cosmological constant in bumblebee gravity was presented in ref. [14].

The quasinormal frequencies are one of the most frequent quantities for testing black holes. In 1957, Regge and Wheeler computed the QNMs for the Schwarzschild solution [15]. After, Zerilli [16] extended the work of Regge and Wheeler and described the Schrödinger-like equation for these perturbations. The damping oscillations for the QNMs were shown by Vishveshwara [17]. The approximation method studied by Wentzel, Kramers, and Brillouin (WKB) of third order was implemented in the Schwarzschild black hole by Iyer and Will [18]. Moreover, Konoplya extended the order of the WKB method for highest orders in ref. [19], where the 6th-order approximation has been shown to be more accurate than the 3rd-order one.

In this work, we compute the Regge-Wheeler equation for the black hole in the context of the bumblebee gravity. The differences of the potentials are detailed for the scalar and gravitational perturbations. For both field perturbations, the LV parameter slightly modifies the result of the Schwarzschild black hole. Moreover, the quasinormal modes are computed via the WKB method for perturbations to 3rd and 6th orders. Furthermore, the time
profile for the scalar and gravitational perturbations are shown.

**QNMs of the Schwarzschild-like solution in bumblebee gravity.** – A solution of a black hole in the context of the bumblebee gravity was proposed in ref. [8]. The metric of this Schwarzschild-like solution with the Lorentz spontaneous violation is represented by [8]

\[
ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 + \lambda \left(1 - \frac{2M}{r}\right)^{-1}\right)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2,
\]

where we set the mass \(M = 1/2\) and the \(\lambda \ll 1\) is the LV parameter (for \(\lambda = 0\) the usual Schwarzschild black hole is obtained). In fact, the upper limits for the LV parameter are very small, and it can be adjusted by the shifts in the perihelion of planets, for example. As presented in ref. [8], for the Lorentz violation to be not verified (i.e., less than the observational error), the dimensionless parameter must be such that \(\lambda < 10^{-12}\) for the perihelion of Mars.

In order to compute the quasinormal modes (QNMs), the WKB method will be applied. Since the solution of eq. (1) shares some similarities with the Schwarzschild metric, we hope to find a similar QNM slightly changed by the LV parameter. To this end, the black hole perturbations are described by the Schrödinger-like wave equation, the so-called Regge-Wheeler equation [18,19]:

\[
\frac{d^2\psi(x)}{dx^2} + Q(x)\psi(x) = 0,
\]

where \(x\) is the tortoise coordinate. The term \(Q(x) = \omega^2 - V(x)\) depends on the analogue potential \(V(x)\) and on the frequencies \(\omega\). Moreover, the wave function \(\psi\) is imposed to following the conditions:

\[
\psi(x) \sim C \pm e^{\mp i\omega t}, \quad \text{when} \quad x \to \pm \infty.
\]

Furthermore, the WKB method requires the analogue potential \(V(x)\) have some aspects. The potential should have a maximum centered at \(x_0\), and two returning points \(x_1\) and \(x_2\). The potential at limits \((x = -\infty)\) and \((x = +\infty)\) assumes constant values. So, the potential should exhibit a bell-shaped profile.

Let us use as an example the usual Schwarzschild black hole. The Schwarzschild solution has the following potential in the original coordinate \(r\) [19]:

\[
V(x(r)) = \left(1 - \frac{1}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{1 - s^2}{r^3}\right),
\]

the parameter \(l\) is the orbital angular momentum, and \(s\) is the spin of field perturbations.

For \(s = 0\) the potential for a scalar field is showed in fig. 1, where the bell-shaped behavior is verified. Note that other spins exhibit a similar profile once the only modification occurs in the term \((1 - s^2)\) and the highest perturbation studied here is the gravitational one \((s = 2)\).

With these conditions, Iyer and Will [18] found the expression of the QNM via the WKB method as

\[
\frac{iQ_0}{\sqrt{2Q_0^2}} - \Lambda(n) - \Omega(n) = n + \frac{1}{2},
\]

where \(Q_0\) is the value of the potential at \(x_0\) and the primes denote derivatives concerning \(x\) coordinates. Corrections compose the terms \(\Lambda(n)\) and \(\Omega(n)\) up to third order [12,18]. An improved version of the WKB method was presented in ref. [19], where high-order corrections were studied. For the sixth order, the WKB method can be written as [19]

\[
\frac{iQ_0}{\sqrt{2Q_0^2}} - \Lambda_2(n) - \Lambda_3(n) - \Lambda_4(n) - \Lambda_5(n) - \Lambda_6(n) = n + \frac{1}{2}.
\]

Table 1 and table 2 show the results of ref. [19] of the WKB method (third and sixth orders) for the scalar and gravitational perturbations in the Schwarzschild black hole. Note that the results for the 6th order are better approximated to the numerical values than those of the 3rd order for the WKB method.

In this work, we compute the Regge-Wheeler equation of eq. (2) using the metric given in eq. (1), for both perturbations of the scalar \((\text{spin } s = 0)\) and the gravitational \((\text{spin } s = 2)\) fields. As a result, the quasinormal modes will be obtained from eq. (6), and the influence of the bumblebee parameter over frequencies will be analyzed. Also, we compare the differences in the 3rd and 6th order of the WKB method.

**Quasinormal modes of perturbations of scalar fields.** – In this section, the first case of the spin \(s = 0\) scalar field will be described. The massless Klein-Gordon equation in the presence of gravity leads to the following equation of motion:

\[
\frac{1}{\sqrt{-g}}\partial_{\mu}(g^{\mu\nu}\sqrt{-g}\partial_{\nu}\Phi) = 0.
\]
Table 1: The QNM for $s = 0$, results from ref. [19].

| $l$, $n$ | Numerical                      | 3rd-order WKB                      | 6th-order WKB                      |
|----------|--------------------------------|-----------------------------------|-----------------------------------|
| $l = 0$, $n = 0$ | 0.1105 - 0.1049i                  | 0.1046 - 0.1152i                  | 0.1105 - 0.1008i                  |
| $l = 1$, $n = 0$ | 0.2929 - 0.0977i                  | 0.2911 - 0.0980i                  | 0.2929 - 0.0977i                  |
| $l = 1$, $n = 1$ | 0.2645 - 0.3063i                  | 0.2622 - 0.3074i                  | 0.2645 - 0.3065i                  |
| $l = 2$, $n = 0$ | 0.4836 - 0.0968i                  | 0.4832 - 0.0968i                  | 0.4836 - 0.0968i                  |
| $l = 2$, $n = 1$ | 0.4639 - 0.2956i                  | 0.4632 - 0.2958i                  | 0.4638 - 0.2956i                  |
| $l = 2$, $n = 2$ | 0.4305 - 0.5086i                  | 0.4317 - 0.5034i                  | 0.4304 - 0.5087i                  |

Table 2: The QNM for $s = 2$, results from ref. [19].

| $l$, $n$ | Numerical                      | 3rd-order WKB                      | 6th-order WKB                      |
|----------|--------------------------------|-----------------------------------|-----------------------------------|
| $l = 2$, $n = 0$ | 0.3737 - 0.0890i                  | 0.3732 - 0.0892i                  | 0.3736 - 0.0890i                  |
| $l = 2$, $n = 1$ | 0.3467 - 0.2739i                  | 0.3460 - 0.2749i                  | 0.3463 - 0.2735                  |
| $l = 2$, $n = 2$ | 0.3011 - 0.4783i                  | 0.3029 - 0.4711i                  | 0.2985 - 0.4776i                  |
| $l = 3$, $n = 0$ | 0.5994 - 0.0927i                  | 0.5993 - 0.0927i                  | 0.5994 - 0.0927i                  |
| $l = 3$, $n = 1$ | 0.5826 - 0.2813i                  | 0.5824 - 0.2814i                  | 0.5826 - 0.2813i                  |
| $l = 3$, $n = 2$ | 0.5517 - 0.4791i                  | 0.5532 - 0.4767i                  | 0.5516 - 0.4790i                  |
| $l = 3$, $n = 3$ | 0.5120 - 0.6903i                  | 0.5157 - 0.6774i                  | 0.5111 - 0.6905i                  |

The QNM for the scalar field in bumblebee gravity is given by

\[ \Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{R(r,t)}{r} Y_{l m}(\theta, \phi), \quad (8) \]

where \( Y_{l m}(\theta, \phi) \) are the spherical harmonics.

By substituting this decomposition of eq. (8) into eq. (2) with the metric of eq. (1) we obtain

\[ \frac{d^2 \phi(x)}{dx^2} + (\omega^2 - V'(l,x,\lambda)) \phi(x) = 0, \quad (9) \]

where for eq. (9) the transformation of the tortoise coordinate \( x \) is given by

\[ \frac{dx}{dr} = \sqrt{1 + \lambda \left( \frac{1}{1 - \frac{r}{\rho}} \right)} \Rightarrow x(r) = \sqrt{1 + \lambda} \left[ r + \ln(r - 1) \right]. \quad (10) \]

The potential of Regge-Wheeler for the scalar field in the context of bumblebee gravity has the formula

\[ V_{\lambda}(r) = \left( 1 - \frac{1}{r} \right) \left( \frac{l(l+1)}{r^2} + \frac{1 + \lambda \rho^{-1}}{r^3} \right). \quad (11) \]

Note that by eq. (11), the bumblebee LV parameter \( \lambda \) modifies the potential of the usual Schwarzschild black hole in eq. (4) by the term \( \left[ 1 + \lambda \rho^{-1} \right]^{-1} \). As expected, for \( \lambda = 0 \) the usual potential is obtained. The LV parameter decreases the peaks of the potential, as can be seen in fig. 2 for \( l = 1 \).

By inverting \( r(x) \) in eq. (10) and obtaining the \( V(x) \) in eq. (11), the terms of WKB of 3rd order in eq. (5) and those of 6th order in eq. (6) can be computed. The results of QNM for scalar perturbation in bumblebee gravity are shown in table 3, for the 3rd and 6th order, respectively. It is worth highlighting that \( n \leq l \) needs to be imposed.

The QNM for the scalar field in bumblebee gravity is also denoted in fig. 3. Note the differences between our results, presented in table 3 (denoted by circles) with the results for the usual Schwarzschild solution in the same table 1, but denoted by triangles. From fig. 3 we conclude that the LV parameter slightly decreases for the real part as well as for the imaginary part of frequencies.
Table 3: The QNMs for scalar perturbations in bumblebee gravity via the WKB method for $\lambda = 0$.

| $l$, $n$ | 3rd-order WKB | 6th-order WKB |
|----------|----------------|---------------|
| $l = 1$, $n = 0$ | 0.291644 - 0.089100i | 0.289044 - 0.097382i |
| $l = 2$, $n = 0$ | 0.483357 - 0.268196i | 0.481314 - 0.096615i |
| $l = 2$, $n = 1$ | 0.466639 - 0.088035i | 0.461409 - 0.295206i |
| $l = 2$, $n = 2$ | 0.429044 - 0.097382i | 0.427762 - 0.508103i |
| $l = 3$, $n = 0$ | 0.675267 - 0.087756i | 0.673701 - 0.096422i |
| $l = 3$, $n = 1$ | 0.662979 - 0.265386i | 0.658965 - 0.292061i |
| $l = 3$, $n = 2$ | 0.466639 - 0.088035i | 0.461409 - 0.295206i |
| $l = 4$, $n = 0$ | 0.867373 - 0.087646i | 0.866120 - 0.096344i |
| $l = 4$, $n = 1$ | 0.857714 - 0.097382i | 0.854492 - 0.290737i |
| $l = 4$, $n = 2$ | 0.839892 - 0.265386i | 0.836492 - 0.292061i |
| $l = 5$, $n = 0$ | 1.05960 - 0.0877264i | 1.05961 - 0.096337i |
| $l = 5$, $n = 1$ | 1.05183 - 0.097382i | 1.050040 - 0.290154i |
| $l = 5$, $n = 2$ | 1.03709 - 0.265386i | 1.03150 - 0.487343i |
| $l = 5$, $n = 3$ | 1.01659 - 0.627569i | 1.005170 - 0.689918i |

Fig. 3: The QNM of $6th$ order for spin $s = 0$ with $l = 2, 3, 4, 5$. The model with bumblebee gravity is represented by circles where $\lambda = 0.1$. The Schwarzschild black hole ($\lambda = 0$) is represented by triangles.

In the next section, we apply the same methodology for the gravitational perturbations ($s = 2$), where we compare the results to the scalar perturbation.

Quasinormal modes of gravitational field perturbations. – In order to study the gravitational perturbations, the Chandrasekhar formalism [20] was applied. As expected, in the same way as scalar perturbations, the Regge-Wheeler equation is similar to the Schwarzschild case one. The Regge-Wheeler potential in the context of bumblebee gravity reads

$$V_g(r, l, \lambda) = \left(1 - \frac{1}{r} \right) \left(\frac{l(l+1)}{r^2} - \frac{2}{r^2} \right) + [1 + \lambda]^{-1} \left[\frac{2}{r^2} - \frac{3}{r^3} \right].$$

Note the gravitational potential in eq. (12) differs from the gravitational potential ($s = 2$) for the Schwarzschild solution in eq. (4) by the term $[1 + \lambda]^{-1} \left[\frac{2}{r^2} - \frac{3}{r^3} \right]$, which for $\lambda = 0$ recovers the usual result. The plot of the gravitational potential is presented in fig. 4. Note that the amplitude of the gravitational potential is smaller than the scalar potential from fig. 2, when the same parameters are considered.

The same transformation $x(r)$ for the tortoise coordinate presented in eq. (10) and the same constructions for the QNMs of (5) and (6) lead us to the new QNM for the gravitational field which is presented in table 4 for $\lambda = 0.1$. Once more, we compare our results in the bumblebee gravity (table 2) with the Schwarzschild black hole (2) in fig. 5, now for the gravitational perturbations. As was also verified for scalar perturbations, the LV slightly modifies the
Table 4: The QNMs for gravitational perturbations in bumblebee gravity via the WKB method for $\lambda = 0.1$. 

| $l$, $n$ | 3rd-order WKB | 6th-order WKB |
|---------|----------------|----------------|
| $l = 2$, $n = 0$ | $0.365938 - 0.0978269i$ | $0.366563 - 0.0974589i$ |
| $l = 2$, $n = 1$ | $0.336038 - 0.303413i$ | $0.335226 - 0.302403i$ |
| $l = 2$, $n = 2$ | $0.291557 - 0.520674i$ | $0.284058 - 0.535037i$ |
| $l = 3$, $n = 0$ | $0.60087 - 0.0846012i$ | $0.599221 - 0.092926i$ |
| $l = 3$, $n = 1$ | $0.586885 - 0.25624 i$ | $0.582443 - 0.281986i$ |
| $l = 3$, $n = 2$ | $0.562304 - 0.433172i$ | $0.551488 - 0.480268i$ |
| $l = 3$, $n = 3$ | $0.509545 - 0.708311i$ | $0.505286 - 0.72212i$ |
| $l = 4$, $n = 0$ | $0.810376 - 0.0857977i$ | $0.809084 - 0.094306i$ |
| $l = 4$, $n = 1$ | $0.799958 - 0.258791i$ | $0.796545 - 0.284766i$ |
| $l = 4$, $n = 2$ | $0.780779 - 0.435239i$ | $0.772636 - 0.480641i$ |
| $l = 4$, $n = 3$ | $0.755005 - 0.615642i$ | $0.739674 - 0.684973i$ |
| $l = 4$, $n = 4$ | $0.703298 - 0.903914i$ | $0.697271 - 0.921761i$ |

Fig. 5: The QNMs of 6th order for spin $s = 2$ with $l = 2, 3, 4, 5$. The model with bumblebee gravity is represented by circles where $\lambda = 0.1$. The Schwarzschild black hole ($\lambda = 0$) is represented by triangles.

QNMs of Schwarzschild scenario. The real part of QNMs is decreased by the $\lambda$ parameter, but in opposition to what occurs in the scalar field, the modulus of the imaginary part is increased by the $\lambda$ parameter.

**Einstein-aether black hole.** – Another important model to explore the LV is the Einstein-aether black hole [21–25]. Reference [23] shows the QNMs for the scalar and electromagnetic fields, while the work [24] obtains the QNMs for gravity. The results are compared to other models with LV and usual Schwarzschild. So, it is worth comparing our results on the Bumblebee black hole with the results of the Einstein-aether black hole. The potential for the scalar field in the first kind of aether black role, for mass $M = 1/2$, is given by [23]

$$V^s_\alpha(r, l, \lambda) = \left(1 - \frac{1}{r}\right) \left(\frac{l(l + 1)}{r^2} + \frac{1}{r^3}\right) + \frac{I}{r^6} \left[4 - \frac{5}{r} - 2\frac{I}{r^4} - l(l + 1)\right]. \quad (13)$$

where $I = \frac{27c_{13}}{256(1 - c_{13})}$, and the coefficient $0 \leq c_{13} < 1$ is associated to the LV parameter. Clearly, for $c_{13} = 0$ the usual Schwarzschild solution is obtained.

Moreover, the potential for the gravitational field in the first kind of aether black role is given by [24]

$$V^g_\alpha(r, l, \lambda) = \left(1 - \frac{1}{r}\right) \left(\frac{l(l + 1)}{r^2} - \frac{3}{r^2}\right) + \frac{I}{r^6} \left[9 - \frac{3}{r} + 6\frac{I}{r^4} - l(l + 1) - 6\right]. \quad (14)$$

Figure 6 shows that the potentials for the Einstein-aether black hole are quite similar to the potentials for the bumblebee black hole. Fixing $l = 2$ and $\lambda = c_{13} = 0.6$, we compute and compare the QNMs, for both perturbations, in table 5. For the scalar field, note that the real part of the frequencies decays in order: from the Schwarzschild black hole to the bumblebee one and to the Einstein-aether theory. On the other hand, the modulus of the imaginary part of QNMs increases in order: from the bumblebee black hole to the Schwarzschild one and to the Einstein-aether theory. For gravity, the same decay of scalar field is verified for the real part, but the imaginary part increases in the following order: from the Schwarzschild black hole to the Einstein theory and to the bumblebee black hole.

Now, considering the bumblebee model, from all the tables above, we note that the increase of the LV parameter decreases the real part of the frequency for both scalar and gravitational fields for $\lambda = 0$ (Schwarzschild), $\lambda = 0.1$ and $\lambda = 0.6$. The $\lambda$ also decreases the modulus of imaginary parts for the scalar field, but the modulus of imaginary parts is increased by $\lambda$ in the gravitational case.

These behaviors are similar to other models with violation of Lorentz symmetry. For scalar, electromagnetic and gravitational fields in the first kind of Einstein-aether black holes, and in the Dirac QNMs with LV parameter [23], the real part of QNMs decreases when
1. bumblebee and Einstein-aether black hole. We use Table 5: Comparison of QNM (via the 6th-order WKB method) for scalar and gravitational perturbations in Schwarzschild, bumblebee and Einstein-aether black hole. We use $l = 2$ and $\lambda = c_{13} = 0.6$.

### Table 5: Comparison of QNM (via the 6th-order WKB method) for scalar and gravitational perturbations in Schwarzschild, bumblebee and Einstein-aether black hole. We use $l = 2$ and $\lambda = c_{13} = 0.6$.

|          | $l = 2, n = 0$ | $l = 2, n = 1$ | $l = 2, n = 2$ |
|----------|----------------|----------------|----------------|
| Schwarzschild | 0.483642 - 0.096766i | 0.463847 - 0.29563i | 0.430386 - 0.508729i |
| Bumblebee   | 0.473988 - 0.096133i | 0.453735 - 0.29386i | 0.419492 - 0.506188i |
| Aether      | 0.463995 - 0.102821i | 0.453232 - 0.29997i | 0.413209 - 0.517426i |
| Gravity     | $l = 2, n = 0$ | $l = 2, n = 1$ | $l = 2, n = 2$ |
| Schwarzschild | 0.3736 - 0.0890i | 0.3463 - 0.2735i | 0.2985 - 0.4776i |
| Bumblebee   | 0.367009 - 0.096281i | 0.337799 - 0.297663i | 0.28936 - 0.522845i |
| Aether      | 0.355804 - 0.094916i | 0.309342 - 0.275322i | 0.274830 - 0.482159i |

LV increases, while the modulus of imaginary part is increased. For models such as the non-reduced Einstein-aether black hole, both the real part and the absolute value of the imaginary part of QNMs increase with the increase of the LV parameter [23,24]. For the second kind Einstein-aether black hole and Einstein-Born-Infeld black holes, both (real and imaginary) parts decrease with the LV factor [23,24].

**Circular orbit of the photon.** – Another important aspect of QNMs is their relationship with the spherical orbit of the photon in black holes [26–28]. The work [26] shows that the QNMs can be determined by the parameters of the circular null geodesics. Reference [27] works in the Einstein-Lovelock theory, and the link between the null geodesics and quasinormal modes (presented in ref. [26]) is violated. Besides, for the black hole in the bumblebee gravity, the circular orbit is computed in ref. [28] and it exhibits no changes compared to usual Schwarzschild solution, namely,

$$r_{co} = \frac{(\sqrt{1+8\vartheta^2}+4\vartheta-1)M}{2\vartheta^2},$$

where $\vartheta$ is the velocity of the particle. For the photon $\vartheta = c = 1$.

For the bumblebee gravity [8,12], the LV parameter is present only in the component $dt^2$ of metric (1). In a different way, $dt^2$ modifications occur in the Einstein-Lovelock black hole [27] and Einstein-aether model [21–25], where the link of the QNMs and the circular orbit is violated. Clearly, for our model, $\lambda$ which appears in the bumblebee gravity does not present shifts in the circular orbit of the photon if compared to the usual Schwarzschild black hole [28], and its relationship with QNMs is preserved [26].

**Time domain perturbations.** – Moreover, a time-dependent scattering can be analyzed by the time domain, as shown in the method developed by Gundlach [29]. The complete equation (2) with the temporal dependence (without the ansatz of eq. (3)) can be written as

$$\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} + V(x, t)\Psi = 0.$$  

Applying the change of variable $u = t - x$ and $v = t + x$, this equation can be rewrite as

$$(4 \frac{\partial^2 \Psi}{\partial u \partial v} + V(u, v))\Psi(u, v) = 0,$$

Gundlach's approach is numerically solved by a finite-difference method [29,31]. The result of these perturbations can be described by a Gaussian profile centered at $v = v_c$, with width $\sigma$ on $u = u_0$ as follows [29,31]:

$$\Psi(u_0, v) = \exp \left(-\frac{(v - v_c)^2}{2\sigma^2}\right).$$

with the initial condition $\Psi(u_0, v_0) = \Psi_0 = 0$ [29–31].

After solving the Gundlach equation and returning to the original variables $(x, t)$, the plot of fig. 7 shows the time domain for scalar and gravitational perturbation in the black hole with the bumblebee parameter exhibited. Note
that both perturbations exhibit damping profiles, decreasing scalar modes slower than the gravitational modes. The angular momentum $l$ parameter results to be faster than the decay of oscillations as verified in refs. [12,22,31,32]. Moreover, the time domain for the Einstein-aether model is presented in ref. [22].

**Conclusion.** – In this work, we applied the WKB method to compute the quasinormal modes for a black hole in the bumblebee gravity, and the results were compared to the Schwarzschild and to Einstein-aether black holes. The scalar and gravitational perturbations were considered. The Regge-Wheeler equation was obtained, and the correction term responsible for the Lorentz violation was addressed. Once the modified potential still exhibits a bell shape, the WKB method was applied for the 3rd and 6th orders. The LV parameter generated small variations in relation to QNMs of Schwarzschild and aether solutions. The increase of the LV parameter increases the real part of the frequency and decreases the modulus of the imaginary parts. This behavior is similar to that of the first kind of aether black holes. For the bumblebee gravity, the circular orbit of the photon is identical to that of the Schwarzschild one and the link of orbit of the photon and the QNMs is preserved. Moreover, the time domain for both perturbations was analyzed and presents damping profiles.

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