BUCHSBAUM-RIM SHEAVES AND THEIR MULTIPLE SECTIONS

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Abstract. This paper begins by introducing and characterizing Buchsbaum-Rim sheaves on $Z = \text{Proj } R$ where $R$ is a graded Gorenstein $K$-algebra. They are reflexive sheaves arising as the sheafification of kernels of sufficiently general maps between free $R$-modules.

Then we study multiple sections of a Buchsbaum-Rim sheaf $B_\psi$, i.e., we consider morphisms $\psi : \mathcal{P} \to B_\psi$ of sheaves on $Z$ dropping rank in the expected codimension, where $H^0(Z, \mathcal{P})$ is a free $R$-module. The main purpose of this paper is to study properties of schemes associated to the degeneracy locus $S$ of $\psi$. It turns out that $S$ is often not equidimensional. Let $X$ denote the top-dimensional part of $S$. In this paper we measure the “difference” between $X$ and $S$, compute their cohomology modules and describe ring-theoretic properties of their coordinate rings. Moreover, we produce graded free resolutions of $X$ (and $S$) which are in general minimal.

Among the applications we show how one can embed a subscheme into an arithmetically Gorenstein subscheme of the same dimension and prove that zero-loci of sections of the dual of a null correlation bundle are arithmetically Buchsbaum.

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1. Introduction

A fundamental method for constructing algebraic varieties is to consider the degeneracy locus of a morphism between a pair of coherent sheaves. By varying the morphism one obtains families of varieties. By placing various restrictions on the coherent sheaves one can force the degeneracy locus to have special properties. These extra restrictions may also provide for more tools with which to study the degeneracy locus. If many restrictions are placed on the sheaves then a great deal of precise information can be extracted but at the expense of generality. If one puts no restrictions on either of the sheaves then, of course, very little information can be extracted concerning the degeneracy locus. In this paper we take the middle road between these two extremes. We consider a class of sheaves, which while quite general, behave well enough that a substantial amount of information can be obtained with respect to their degeneracy loci. The only restriction placed on the sheaves is that they arise as the sheafification of kernels of sufficiently general maps between free $R$-modules, where $R$ is a Gorenstein $K$-algebra. We will refer to the sheaves constructed in this manner as Buchsbaum-Rim sheaves.

The purpose of this paper is to introduce the class of Buchsbaum-Rim sheaves, to elicit their main properties, and to make a systematic and detailed study of the degeneracy loci obtained by taking multiple sections of these sheaves. Although Buchsbaum-Rim sheaves are necessarily reflexive (they are the sheafification of a second syzygy module) they will not, in general, be locally free. A number of tools...
are used to manipulate and control these objects. Certainly techniques developed by Eagon-Northcott, Buchsbaum-Rim, Buchsbaum-Eisenbud, Kirby and Kempf play an important role. These are combined with the methods of local cohomology and several homological techniques to produce the main results. The paper opens with a brief section providing preliminary background information of use in later sections of the paper. Sections three, four and five form the technical heart and the final section closes out the paper with three applications.

Section three introduces Buchsbaum-Rim sheaves and Buchsbaum-Rim modules. To begin with we should make clear the definition of a Buchsbaum-Rim sheaf. In the following we will always denote a graded Gorenstein $K$-algebra of dimension $n + 1$ where $K$ is an infinite field. Furthermore, the scheme $Z$ will be the projective spectrum of $R$.

**Definition 1.1.** Let $F$ and $G$ be locally free sheaves of ranks $f$ and $g$ respectively on $Z$. Let $\varphi : F \to G$ be a generically surjective morphism. Suppose that the degeneracy locus of the modules $F = H^0_{r}(Z,F)$ and $G = H^0_{r}(Z,G)$ are free $R$-modules. We call the kernel of $\varphi$ a Buchsbaum-Rim sheaf and denote it by $B_{\varphi}$.

By abuse of notation we denote the homomorphism $F \to G$ induced by $\varphi$ again by $\varphi$. Moreover, we put $B_{\varphi} = H^0_{r}(Z,B_{\varphi})$ and $M_{\varphi} = \text{coker} \varphi$, so that we have an exact sequence

$$0 \to B_{\varphi} \to F \overset{\varphi}{\longrightarrow} G \to M_{\varphi} \to 0.$$  

We call $B_{\varphi}$ a Buchsbaum-Rim module.

Thus the cotangent bundle of projective space is a Buchsbaum-Rim sheaf. The cohomology of its exterior powers is given by the Bott formula. Letting $r(R)$ denote the index of regularity of a graded ring we have the following lemma which generalizes the Bott formula to arbitrary Buchsbaum-Rim sheaves.

**Lemma 1.2.** Let $B_{\varphi}$ be a Buchsbaum-Rim module of rank $r$. Then it holds:

(a) For $i = 0, \ldots, r$ there are isomorphisms

$$\wedge^{r-i}B_{\varphi}^* \otimes \wedge^j F \otimes \wedge^n G^* \cong (\wedge^i B_{\varphi}^*)^*.$$  

(b) For $i < r$ we have

$$H^j(Z, \wedge^i B_{\varphi}^*) \cong \begin{cases} 0 & \text{if } 1 \leq j \leq n \text{ and } j \neq n - i \\ S_i(M_{\varphi})^{\vee} (1 - r(R)) & \text{if } j = n - i. \end{cases}$$

The value of this lemma will become clear when we construct the generalized Koszul complexes associated to taking multiple sections of Buchsbaum-Rim sheaves. The lemma also suggests a relationship to Eilenberg-MacLane sheaves. Recall that an $R$-module $E$ is called an Eilenberg-MacLane module of depth $t$, $0 \leq t \leq n + 1$ if $H^0_{t}(E) = 0$ for all $j \neq t$ where $0 \leq j \leq n$. Similarly, a sheaf $E$ on $Z$ is said to be an Eilenberg-MacLane sheaf if $H^0_{t}(E)$ is an Eilenberg-MacLane module. A cohomological characterization of Buchsbaum-Rim sheaves can then be stated as follows.

**Proposition 1.3.** A sheaf $E$ on $Z$ is a Buchsbaum-Rim sheaf if and only if $E = H^0_{m}(E)$ is a reflexive Eilenberg-MacLane module with finite projective dimension and rank $r \leq n$ such that $H^{n-r+1}_{m-r+1}(E)^{\vee}$ is a perfect $R$-module of dimension $n - r$ if $r \geq 2$.

In section four we make a detailed study of the cohomology of the degeneracy locus of multiple sections of Buchsbaum-Rim sheaves. Consider a morphism $\psi : P \to B_{\varphi}$ of sheaves of rank $t$ and $r$ respectively on $Z = \text{Proj}(R)$, where $B_{\varphi}$ is a Buchsbaum-Rim sheaf and $H^0_{r}(Z,P)$ is a free $R$-module. If $t = 1$ then $\psi$ is just a section of some twist of $B_{\varphi}$. For arbitrary $t < r$ we say that $\psi$ corresponds to taking multiple sections of $B_{\varphi}$. We always suppose that the degeneracy locus $S$ of $\psi$ has (the expected) codimension $r - t + 1 \geq 2$ in $Z$ (if $t = 1$ then $S$ is just the zero-locus of a regular section of a Buchsbaum-Rim sheaf).

An Eagon-Northcott complex involving $B_{\varphi}$ will play an essential role. Our approach will be algebraic and uses local cohomology. Taking global sections we obtain an $R$-homomorphism

$$\psi : P \to B_{\varphi}$$

where $P$ is a free $R$-module of rank $t, 1 \leq t < r$. Then there is an Eagon-Northcott complex

$$E_{\bullet} : 0 \to E_r \overset{\delta_r}{\longrightarrow} E_{r-1} \overset{\delta_{r-1}}{\longrightarrow} \cdots \to E_t \overset{\delta_t}{\longrightarrow} I(\psi) \otimes \wedge^t P^* \to 0$$
Proposition 1.7. We show that the homology can be summarized as follows.

\[ E_i = \wedge^i B^*_\psi \otimes S_{i-t}(P) \]

and \( I(\psi) \) is the ideal defined by the image of \( \delta_1 = \wedge^t \psi^* \). The saturation of \( I(\psi) \) is the homogeneous ideal of the degeneracy locus \( S \).

With the help of this Eagon-Northcott complex and the first lemma we get the following formula for the cohomology modules of \( S \).

\[ H^j_m(R/I) \cong \begin{cases} S_i(M_\varphi) \otimes S_{i-t}(P) \otimes \wedge^t P(1 - r(R)) & \text{if } j = n + t - 2i \quad \text{where } \max\{t, \frac{i+1}{2}\} \leq i \leq \left\lfloor \frac{i+1}{2} \right\rfloor \\ 0 & \text{otherwise} \end{cases} \]

This proposition allows us to decide if \( S \) is equidimensional. It will often turn out that this is not the case. Thus we are also interested in the top-dimensional component \( X \) of \( S \), i.e., the union of the highest-dimensional components of \( S \). Let \( J = J(\psi) \) denote the homogeneous ideal of \( X \). We need a measure of the failure of \( I \) to be equidimensional and we need a close relation between the cohomology of the schemes \( X \) and \( S \). This is provided in the following result.

Proposition 1.5. Letting \( I \) and \( J \) denote the ideals associated to \( \psi \) we have:

(a) \( I \) is unmixed if and only if \( r + t \) is odd.

(b) If \( r + t \) is even then \( I \) has a primary component of codimension \( r + 1 \). Let \( Q \) be the intersection of all such components. Then we have \( I = J \cap Q \) and

\[ H^j_m(R/I) \cong \begin{cases} H^j_m(R/I) & \text{if } j \neq n - r \\ 0 & \text{if } j = n - r \end{cases} \]

and

\[ J/I \cong S_{i-1}(M_\varphi) \otimes S_{i-1}(P) \otimes \wedge^t F^* \otimes \wedge^g G \otimes \wedge^t P. \]

Combining these propositions in the proper manner we are in a position to prove the main theorem of section four.

Theorem 1.6. With the notation above we have:

(a) If \( r = n \) then \( S \) is equidimensional and locally Cohen-Macaulay.

(b) \( S \) is equidimensional if and only if \( r + t \) is odd or \( r = n \). Moreover, if \( r < n \) then \( X \) is locally Cohen-Macaulay if and only if \( X \) is arithmetically Cohen-Macaulay.

(c) If \( r + t \) is odd then \( X = S \) is arithmetically Cohen-Macaulay if and only if \( t = 1 \). In this case \( S \) has Cohen-Macaulay type \( \leq 1 + (\frac{r + g - 1}{g - 1}) \).

(d) Let \( r + t \) be even. Then

(i) \( X \) is arithmetically Cohen-Macaulay if and only if \( 1 \leq t \leq 2 \). If \( t = 2 \) then \( X \) is arithmetically Gorenstein. If \( t = 2 \) then \( X \) has Cohen-Macaulay type \( \leq r - 1 + (\frac{r + g - 1}{g - 1}) \cdot (r - 1) \).

(ii) If in addition \( r < n \) then the components of \( S \) have either codimension \( r - t - 1 \) or codimension \( r + 1 \).

Contained in the theorem is the following surprising conclusion. Let \( B_{\psi} \) be an odd rank Buchsbaum-Rim sheaf and let \( X \) denote the top dimensional component of the zero-locus of any regular section of \( B_{\psi} \). Then \( X \) is arithmetically Gorenstein. This generalizes the main theorem of [14], where the case \( r = 3 \) was considered.

Section five treats the problem of finding free resolutions of the degeneracy loci. In order to do this it is important to understand the homology modules of the Eagon-Northcott complex \( E_\bullet \) associated to \( \psi \). We show that the homology can be summarized as follows.

Proposition 1.7. The homology modules of the Eagon-Northcott \( E_\bullet \) complex are:

\[ H_i(E_\bullet) \cong \begin{cases} S_j(M_\varphi) \otimes S_{r-1-j}(P) \otimes \wedge^t F^* \otimes \wedge^g G & \text{if } i = r - 1 - 2j \text{ where } j \in \mathbb{Z}, \ t \leq i \leq r - 3 \\ 0 & \text{otherwise} \end{cases} \]
This result allows us to conclude that $X$ and $S$ have free resolutions of finite length. However, it does not, in general, provide enough information to compute a minimal free resolution. To do this we need several ingredients. First we need to understand the cohomology of the dual of the Eagon-Northcott complex. This needs to be mixed with knowledge of how the canonical modules of $S$ and $X$ relate. The cohomology of the dual of the Eagon-Northcott complex is summarized in the following lemma, where $K_{R/I}$ denotes the canonical module of $R/I(\psi)$.

**Lemma 1.8.** The dual of the Eagon-Northcott complex $E_\bullet$ provides a complex

$$0 \to \wedge^i P \to E_i \to \cdots \to E_{r-1} \to E_r^* \to K_{R/I} \otimes \wedge^i P(1-r(R)) \to 0$$

which we denote (by slight abuse of notation) by $E_\bullet^*$. Its (co)homology modules are given by

$$H^i(E_\bullet^*) \cong \left\{ \begin{array}{ll}
S_j(M_\psi) \otimes S_{j-i}(P)^* & \text{if } 2t + 1 \leq i = 2j + 1 \leq r + 1 \\
0 & \text{otherwise}
\end{array} \right.$$ 

In particular, $E_\bullet^*$ is exact if $t \geq \frac{r+1}{2}$.

To utilize these results on the Eagon-Northcott complex one still needs to know when certain terms in a free resolution can be split off. To do this we prove the following result which, although rather technical in appearance, provides a substantial generalization to an often-used result of Rao. It can be applied in situations far removed from those addressed in this paper and can even be utilized when the ring $R$ is not a Gorenstein ring but is only a Cohen-Macaulay ring.

**Proposition 1.9.** Let $N$ be a finitely generated graded torsion $R$-module which has projective dimension $s$. Then it holds for all integers $j \geq 0$ that $\text{Tor}^R_{s-j}(N,K)^{\vee}$ is a direct summand of

$$\bigoplus_{i=0}^j \text{Tor}^R_{j-i}(\text{Ext}^s_{\bullet}(N,R),K).$$

Moreover, we have $\text{Tor}^R_{s}(N,K)^{\vee} \cong \text{Tor}^R_0(\text{Ext}^s_{R}(N,R),K)$ and that $\text{Tor}^R_{1}(\text{Ext}^s_{R}(N,R),K)$ is a direct summand of $\text{Tor}^R_{s-1}(N,K)^{\vee}$.

Together these results allow us to write down a free resolution of the degeneracy locus which is in general minimal. Thus the main theorem of section five is the following, which gives the free resolution for the degeneracy locus of a morphism $\psi : \mathcal{P} \to \mathcal{B}_\psi$, where $\mathcal{P}$ has rank $t$ and $\mathcal{B}_\psi$ has rank $r$.

**Theorem 1.10.** Consider the following modules where we use the conventions that $i$ and $j$ are non-negative integers and that a sum is trivial if it has no summand:

$$A_k = \bigoplus_{i+j \leq \frac{r+t}{2}} \wedge^i F^* \otimes S_j(G)^* \otimes S_{i+1-t}$$

$$C_k = \bigoplus_{i+j \leq \frac{r-t}{2}} \wedge^i F \otimes S_j(G) \otimes S_{i+1-t} \otimes \wedge^i F^* \otimes \wedge^g G.$$ 

Observe that it holds:

- $A_r = 0$ if and only if $r + t$ is even,
- $C_1 = 0$ if and only if $r + t$ is odd,
- $C_k = 0$ if $k \geq r + 2 - t$ and
- $C_{r+1-t} = S_{r+1-t}(\mathcal{P}) \otimes \wedge^i F^* \otimes \wedge^g G$.

Then the homogeneous ideal $I_X = J(\psi)$ of the top-dimensional part $X$ of the degeneracy locus $S$ has a graded free resolution of the form

$$0 \to A_r \oplus C_r \to \cdots \to A_1 \oplus C_1 \to I_X \otimes \wedge^i P^* \to 0.$$ 

Note that previously minimal free resolutions were known only for a few classes besides the determinantal ideals. A number of examples of particular interest round out the section and illustrate the theorem.

The final section gives several additional applications that may be of independent interest. We show how Buchsbaum-Rim sheaves can be used to situate arbitrary equidimensional schemes of arbitrary codimension into arithmetically Gorenstein schemes. This will be of relevance when one considers the
problem of linkage by arithmetically Gorenstein ideals as opposed to complete intersection linkage theory. We also show how to utilize Buchsbaum-Rim sheaves to produce interesting new examples of $k$-Buchsbaum sheaves as well as of arithmetically Buchsbaum schemes. Finally we construct new vector bundles of rank $n - 1$ on $\mathbb{F}^n$ if $n$ is odd. We call them generalized null correlation bundles and show that our results apply to multiple sections of their duals.

2. Preliminaries

Let $R$ be a ring. If $R = \oplus_{i \in \mathbb{N}} R_i$ is graded then the irreducible maximal ideal $\oplus_{i > 0} R_i$ of $R$ is denoted by $\mathfrak{m}_R$ or simply $\mathfrak{m}$. It is always assumed that $R_0$ is an infinite field $K$ and that the $K$-algebra $R$ is generated by the elements of $R_1$. Hence $(R, \mathfrak{m})$ is *local in the sense of [2].

If $M$ is a module over the graded ring $R$ it is always assumed to be $\mathbb{Z}$-graded. The set of its homogeneous elements of degree $i$ is denoted by $M_i$ or $[M]_i$. All homomorphisms between graded $R$-modules will be morphisms in the category of graded $R$-modules, i.e., will be graded of degree zero. If $M$ is assumed to be a graded $R$-module it is always understood that $R$ is a graded $K$-algebra as above. We refer to the context just described as the graded situation. Although we are mainly interested in graded objects we note that our results hold also true (with the usual modifications) in a local situation. Then $(R, \mathfrak{m})$ will denote a local ring with maximal ideal $\mathfrak{m}$.

If $M$ is an $R$-module, $\dim M$ denotes the Krull dimension of $M$. The symbols $\text{rank}_R$ or simply rank are reserved to denote the rank of $M$ in case it has one. For a $K$-module, $\text{rank}_K$ just denotes the vector space dimension over the field $K$.

There are two types of duals of an $R$-module $M$ we are going to use. The $R$-dual of $M$ is $M^* = \text{Hom}_R(M, R)$. If $M$ is graded then $M^*$ is graded, too. If $R$ is a graded $K$-algebra then $M$ is also a $K$-module and the $K$-dual $M^K$ of $M$ is defined to be the graded module $\text{Hom}_K(M, K)$ where $K$ is considered as a graded module concentrated in degree zero. Note that $R^K$ is the injective hull of $K \cong K \cong R/\mathfrak{m}$ in the category of graded $R$-modules. If $\text{rank}_K[M]_i < \infty$ for all integers $i$ then there is a canonical isomorphism $M \cong M^K$.

Now let $Z$ be a projective scheme over $K$. This means $Z = \text{Proj}(R)$ where $R$ is a graded $K$-algebra. For any sheaf $F$ on $Z$, we define $H^i(Z, F) = \bigoplus_{t \in \mathbb{Z}} H^i(Z, F(t))$. In this paper we will use "vector bundle" and "locally free sheaf" interchangeably.

Let $X$ be a non-empty projective subscheme of $Z$ with homogeneous coordinate ring $A = R/I_X$. Then $I_X$ is a saturated ideal of $R$. Recall that a homogeneous ideal $I$ in $R$ is saturated if $I = \bigcup_{d \in \mathbb{Z}^+} [I : \mathfrak{m}^d]$, where $\mathfrak{m} = (x_0, x_1, \ldots, x_n)$. Equivalently, $I$ is saturated if and only if $I = H^0(Z, J)$, where $J$ is the sheafification of $I$.

Generalized Koszul complexes

For more details with respect to the following discussion we refer to [3] and [4]. The differences between these presentations and ours stem from the fact that we want to have all homomorphisms graded (of degree zero).

Let $R$ be a graded $K$-algebra and let $\varphi : F \to G$ be a homomorphism of finitely generated graded $R$-modules. Then there are (generalized) Koszul complexes $C_i(\varphi)$:

$$0 \to \wedge^i F \otimes S_0(G) \to \wedge^{i-1} F \otimes S_1(G) \to \ldots \to \wedge^0 F \otimes S_i(G) \to 0.$$  

Let $C_i(\varphi)^*$ be the $R$-dual of $C_i(\varphi)$. Suppose now that $F$ is a free $R$-module of rank $f$. Then there are graded isomorphisms

$$\wedge^f F \otimes (\wedge^f F)^* \cong \wedge^{f-j} F.$$  

Thus we can rewrite $C_i(\varphi)^* \otimes \wedge^f F$ as follows:

$$0 \to \wedge^f F \otimes S_i(G)^* \to \wedge^{f-1} F \otimes S_{i-1}(G)^* \to \ldots \to \wedge^{f-i} F \otimes S_0(G)^* \to 0.$$  

Note that $S_j(G)^*$ is the $j$th graded component of the divided power algebra of $G^*$, but we won’t need this fact.

Let’s assume that also $G$ is a free $R$-module of rank, say, $g$ where $g < f$. Then $\varphi^*$ induces graded homomorphisms $\nu_i : \wedge^{g+i} F \otimes \wedge^g G^* \to \wedge^i F$. 


Put \( r = f - g \). It turns out that for \( i = 0, \ldots, r \) the complexes \( C_{r-i}(\varphi)^\ast \otimes \wedge^i F \otimes \wedge^g G^\ast \) and \( C_i(\varphi) \) can be spliced via \( \nu_i \) to a complex \( D_i(\varphi) \):

\[
0 \rightarrow \wedge^i F \otimes S_{r-i}(G)^\ast \otimes \wedge^g G^\ast \rightarrow \wedge^{i-1} F \otimes S_{r-i-1}(G)^\ast \otimes \wedge^g G^\ast \rightarrow \cdots
\]

\[
\rightarrow \wedge^{g-i} F \otimes S_0(G)^\ast \otimes \wedge^g G^\ast \xrightarrow{\nu_i} \wedge^i F \otimes S_0(G) \rightarrow \wedge^{i-1} F \otimes S_1(G) \rightarrow \cdots \rightarrow \wedge^0 F \otimes S_r(G) \rightarrow 0.
\]

The complex \( D_0(\varphi) \) is called the Eagon-Northcott complex and \( D_i(\varphi) \) is called the Buchsbaum-Rim complex.

If we fix bases of \( F \) and \( G \) the map \( \varphi \) can be described by a matrix whose maximal minors generate an ideal which equals the image of \( \nu_0 \). We denote this ideal by \( I(\varphi) \). Its grade is at most \( f - g + 1 \). If \( \varphi \) is general enough the complexes above have good properties.

**Proposition 2.1.** Suppose \( \text{grade } I(\varphi) = f - g + 1 \). Then it holds:

(a) \( D_i(\varphi) \) is acyclic where \( i = 0, \ldots, f - g = r \).

(b) If \( \varphi \) is a minimal homomorphism, i.e. \( \text{im } \varphi \subset \mathfrak{m} \cdot G \), then \( D_0(\varphi) \) is a minimal free graded resolution of \( R/I(\varphi) \) and \( D_i(\varphi) \) is a minimal free graded resolution of \( S_i(\text{coker } \varphi) \), \( 1 \leq i \leq r \).

The minimality of the resolutions in (b) follows by analyzing the maps described above.

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**Gorenstein rings and schemes**

A graded \( K \)-algebra \( R \) is said to be Gorenstein if it has finite injective dimension (cf. [3], Definition 3.1.18). Over a Gorenstein ring duality theory is particularly simple. We denote the index of regularity of a graded ring by \( r(R) \). If \( R \) is just the polynomial ring \( K[x_0, \ldots, x_n] \) then \( r(R) = -n \). We will use the following duality result (cf., for example, [4], Theorem 0.4.14).

**Lemma 2.2.** Let \( M \) be a graded \( R \)-module where \( R \) is a Gorenstein ring of dimension \( n \). Then we have

\[
H_m^i(M) \cong \text{Ext}_R^{n-i}(M, R)(r(R) - 1).
\]

Let \( M \) be a graded \( R \)-module where \( n = \text{dim } R \) and \( d = \text{dim } M \). Then

\[
K_M = \text{Ext}_R^{n-d}(M, R)(r(R) - 1)
\]

is said to be the canonical module of \( M \). Usually the canonical module is defined as the module representing the functor \( H^d_m(M \otimes_R -) \) if such a module exists. If \( R \) is Gorenstein it does and is just the module defined above (cf. [5]).

We say that \( M \) has cohomology of finite length if the cohomology modules \( H_m^i(M) \) have finite length for all \( i < \text{dim } M \). It is well-known that \( M \) has cohomology of finite length if and only if \( M \) is equidimensional and locally Cohen-Macaulay.

Let now \( Z = \text{Proj}(R) \) be a projective scheme over \( K \). Then \( Z \) is said to be arithmetically Gorenstein and arithmetically Cohen-Macaulay respectively if the homogeneous coordinate ring \( R \) of \( Z \) is Gorenstein and Cohen-Macaulay respectively.

For a closed subscheme \( X \) of \( Z \), with homogeneous coordinate ring \( A = R/I_X \) we will refer to the canonical module of \( A \) also as the canonical module of \( X \). Moreover, we say that \( X \) has finite projective dimension if \( A \) has finite projective dimension as an \( R \)-module.

Assume that \( Z \) is arithmetically Gorenstein. One of the things we shall be interested in is to describe when certain subschemes \( X \) of \( Z \) are arithmetically Gorenstein, too. To do this, it is enough to show that \( X \) is arithmetically Cohen-Macaulay, with Cohen-Macaulay type 1 provided \( X \) has finite projective dimension. In this case \( X \) is defined by a Gorenstein ideal \( I = I_X \subset R \).

Recall that the Cohen-Macaulay type of an arithmetically Cohen-Macaulay projective scheme \( X \) with finite projective dimension can be defined to be the rank of the last free module occurring in a minimal free resolution of the saturated ideal of \( X \). It is equal to the number of minimal generators of the canonical module of \( X \).
3. Buchsbaum-Rim sheaves

From now on we will always assume that $Z$ is a projective arithmetically Gorenstein scheme over the field $K$. We denote its dimension by $n$ and its homogeneous coordinate ring by $R$.

Let $\mathcal{F}$ and $\mathcal{G}$ be locally free sheaves of ranks $f$ and $g$ respectively on $Z$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a generically surjective morphism. Since the construction of the generalized Koszul complexes as described in the previous section globalizes, we can associate to $\varphi$ several complexes. The most familiar are the Eagon-Northcott complex

$$0 \to \wedge^f \mathcal{F} \otimes S_{f-g}(\mathcal{G})^* \otimes \wedge^g \mathcal{G}^* \to \wedge^{f-1} \mathcal{F} \otimes S_{f-g-1}(\mathcal{G})^* \otimes \wedge^g \mathcal{G}^* \to \ldots \to \wedge^g \mathcal{F} \otimes S_0(\mathcal{G})^* \otimes \wedge^g \mathcal{G}^* \to 0$$

and the Buchsbaum-Rim complex

$$0 \to \wedge^f \mathcal{F} \otimes S_{f-g-1}(\mathcal{G})^* \otimes \wedge^g \mathcal{G}^* \to \wedge^{f-1} \mathcal{F} \otimes S_{f-g-2}(\mathcal{G})^* \otimes \wedge^g \mathcal{G}^* \to \ldots \to \wedge^{g+1} \mathcal{F} \otimes S_0(\mathcal{G})^* \otimes \wedge^g \mathcal{G}^* \to \mathcal{F} \overset{\varphi}{\to} \mathcal{G} \to 0.$$ 

Moreover, Proposition 2.1 implies that these complexes are acyclic if the degeneracy locus of $\varphi$ has the expected codimension $f-g+1$ in $Z$. This lead us to the following definition.

**Definition 3.1.** With the notation above suppose that the degeneracy locus of $\varphi$ has codimension $f-g+1$. Then we call the cokernel of the map between $\wedge^{g+1} \mathcal{F} \otimes S_0(\mathcal{G})^* \otimes \wedge^g \mathcal{G}^*$ and $\wedge^{g+1} \mathcal{F} \otimes S_{f-1}(\mathcal{G})^* \otimes \wedge^g \mathcal{G}^*$ an $i^{th}$ local Buchsbaum-Rim sheaf $(1 \leq i \leq f-g)$ and denote it by $B_i^\varphi$.

Note that the $i^{th}$ local Buchsbaum-Rim sheaf associated to $\varphi$ is just the $(i+1)^{st}$ syzygy sheaf of coker $\varphi$.

The following result is a generalization of Proposition 2.10 of [8]. Thanks to our set-up the proof given there works here, too.

**Proposition 3.2.** Let $B$ be a first local Buchsbaum-Rim sheaf associated to a morphism $\varphi$. Let $X$ denote the degeneracy locus of $\varphi$. Let $S$ be the zero-locus of a section $s \in H^0(Z, B)$ and let $T$ be the zero-locus of a section $t \in H^0(Z, B^*)$. Then it holds $X \subset S$ and $X \subset T$.

Now we put stronger assumptions on the sheaves $\mathcal{F}$ and $\mathcal{G}$.

**Definition 3.3.** Suppose in addition that the modules $F = H^0_*(Z, \mathcal{F})$ and $G = H^0_*(Z, \mathcal{G})$ are free $R$-modules. Then the sheaf $B_i^\varphi$ is called an $i^{th}$ Buchsbaum-Rim sheaf. For simplicity a first Buchsbaum-Rim sheaf is just called a Buchsbaum-Rim sheaf and denoted by $B_\varphi$.

By abuse of notation we denote the homomorphism $F \to G$ induced by $\varphi$ again by $\varphi$. Moreover, we put $B_\varphi = H^0_*(Z, B_\varphi)$ and $M_\varphi = \text{coker } \varphi$, so that we have an exact sequence

$$0 \to B_\varphi \to F \overset{\varphi}{\to} G \to M_\varphi \to 0.$$ 

We call $B_\varphi$ a Buchsbaum-Rim module.

**Remark 3.4.** In this paper we will consider some degeneracy loci associated to Buchsbaum-Rim sheaves. These investigations were motivated by the work of the first and the third author in [11].

Note that in [8] zero-loci of regular sections of the dual of a Buchsbaum-Rim sheaf over projective space have been characterized as determinantal subschemes which are generically complete intersections.

If $g = 1$ then the sheaf $B_{f-1}^\varphi$ is the dual of a Buchsbaum-Rim sheaf. Thus it seems to be rewarding to study higher Buchsbaum-Rim sheaves, too.

**Remark 3.5.** (i) With the notation above the Buchsbaum-Rim sheaf $B_\varphi$ has rank $r = f - g$. Our assumptions imply that $r \leq n = \dim Z$. Moreover, $B_\varphi$ is locally free if and only if $n = r$.

(ii) As a second syzygy sheaf over an arithmetically Gorenstein scheme a Buchsbaum-Rim sheaf $B_\varphi$ is reflexive, i.e., the natural map $B_\varphi \otimes B_\varphi^* \to O_Z$ induces an isomorphism $B_\varphi \cong B_\varphi^{**}$.

Similarly, a Buchsbaum-Rim module is a reflexive $R$-module.

The following result will become important later on.

**Lemma 3.6.** With the above notation let $B_\varphi$ be a Buchsbaum-Rim module of rank $r$. Then it holds:
Remark 3.7. (i) The previous result implies for the Buchsbaum-Rim sheaf

(ii) Let us consider the example where

Similarly, a sheaf $E$ on $Z$ is said to be an Eilenberg-MacLane sheaf if $H^0(E)$ is an Eilenberg-MacLane module.

We will need the following result which is shown in [13] as Theorem I.3.9.

Lemma 3.9. Let $E$ be a reflexive module of depth $t \leq n$. Then $E$ is an Eilenberg-MacLane module with finite projective dimension if and only if $E^*$ is an $(n + 2 - t)$-syzygy of a module $M$ of dimension $\leq t - 2$. In this case it holds

$$M \cong H^t_m(E)^\vee(1 - r(R)).$$

Now we are ready for our cohomological description of Buchsbaum-Rim sheaves.

Proposition 3.10. A sheaf $E$ on $Z$ is a Buchsbaum-Rim sheaf if and only if $E = H^0(E)$ is a reflexive Eilenberg-MacLane module with finite projective dimension and rank $r \leq n$ such that $H^{n-r+2}_m(E)^\vee$ is a perfect $R$-module of dimension $n - r$ if $r \geq 2$.

Proof. First let us assume that $E$ is a Buchsbaum-Rim sheaf. Then we have by definition for $E = H^0(E)$ that it has a rank, say $r$, and sits in an exact sequence

$$0 \to E \to F \to G \to M_\varphi \to 0$$
where $F$ and $G$ are free modules and $I(\varphi)$ has the expected codimension $r + 1$. Due to Remark 3.3 $E$ is reflexive. Furthermore, $E$ has finite projective dimension since $M_{t}$ does by Lemma 3.4. If $r = 1$ it follows that $E$ is just $R(m)$ for some integer $m$.

Let $r \geq 2$. Then the exact sequence above and the Cohen-Macaulayness of $M_{t}$ imply that $E$ is an Eilenberg-MacLane module of depth $n - r + 2$

$$H^{n-r+2}_{m}(E)^{\vee} \cong H^{n-r}_{m}(M_{t})^{\vee} \cong Ext^{r+1}_{R}(M_{t}, R)(r(R) - 1) \cong S_{r-1}(M_{t}) \otimes \wedge^{r}F^{*} \otimes \wedge^{g}G(r(R) - 1)$$

where the latter isomorphisms are due to Lemma 2.2 and Lemma 3.8. Since $S_{r-1}(M_{t})$ is a perfect module of dimension $n - r$ by Lemma 3.6 we have shown that the conditions in the statement are necessary.

Now we want to show sufficiency. Since a reflexive module of rank 1 with finite projective dimension and $\psi$ is an isomorphism in Lemma 3.6 and $\psi$ has (the expected) codimension $r + 1$ then $I(\varphi)$ has the maximal codimension $r + 1$. Thus $E$ is a Buchsbaum-Rim module completing the proof.

Since any module over a regular ring has finite projective dimension the last result takes a simpler form for sheaves on $\mathbb{P}^{n}$.

**Corollary 3.11.** A sheaf $E$ on $\mathbb{P}^{n}$ is a Buchsbaum-Rim sheaf if and only if $E$ is an reflexive Eilenberg-MacLane sheaf of rank $r \leq n$ such that $H^{l}_{t}(\mathbb{P}^{n}, E) = 0$ if $i \neq 0, n - r + 1, n + 1$ and $H^{n-r+1}_{t}(E)^{\vee}$ is a Cohen-Macaulay module of dimension $n - r$.

From this result we see again that the cotangent bundle on projective space is a Buchsbaum-Rim sheaf.

4. The Cohomology of the Degeneracy Loci

Consider a morphism $\psi : P \rightarrow B_{\varphi}$ of sheaves of rank $t$ and $r$ respectively on the arithmetically Gorenstein scheme $Z = \text{Proj}(R)$ where $B_{\varphi}$ is a Buchsbaum-Rim sheaf and $H^{0}_{t}(Z, P)$ is a free $R$-module. If $t = 1$ then $\psi$ is just a section of some twist of $B_{\varphi}$. Thus we refer to $\psi$ as multiple sections of $B_{\varphi}$.

Throughout this paper we suppose that the ground field $K$ is infinite and that the degeneracy locus $S$ of $\psi$ has (the expected) codimension $r - t + 2 \geq 2$ in $Z$. If $t = 1$ then $S$ is just the zero-locus of a regular section of a Buchsbaum-Rim sheaf.

It will turn out that $S$ is often not equidimensional. Thus we are also interested in the top-dimensional part $X$ of $S$, i.e. the union of the highest-dimensional components of $S$. The aim of this section is to compute the cohomology modules of $S$ and $X$ respectively. An Eagon-Northcott complex involving $B_{\varphi}$ will play an essential role. Observe that in contrast to the situation in the previous section where $F$ and $G$ were locally free the sheaf $B_{\varphi}$ is in general not locally free.

Our approach will be algebraic. Taking global sections we obtain an $R$-homomorphism

$$\psi : P \rightarrow B_{\varphi}$$

where $P$ is a free $R$-module of rank $t, 1 \leq t < r$. The first aim is to derive the complex mentioned above. We follow the approach described in Section 3. The $R$-dual of the Koszul complex $C_{r-t}(\psi^{*})$ is

$$0 \rightarrow (\wedge^{t}B_{\varphi}^{*} \otimes S_{r-t-1}(P^{*}))^{*} \rightarrow (B_{\varphi}^{*} \otimes S_{r-t-1}(P))^{*} \rightarrow \ldots \rightarrow (\wedge^{r-t-1}B_{\varphi}^{*} \otimes P^{*})^{*} \rightarrow (\wedge^{r-t}B_{\varphi}^{*} \otimes S_{0}(P^{*}))^{*}.$$  

Using the isomorphisms in Lemma 3.6 and $S_{j}(P^{*})^{*} \cong S_{j}(P)$ we can rewrite $C_{r-t}(\psi^{*})^{*} \otimes \wedge^{i}F^{*} \otimes \wedge^{g}G$ as follows:

$$0 \rightarrow \wedge^{t}B_{\varphi}^{*} \otimes S_{r-t-1}(P) \rightarrow \wedge^{r-t-1}B_{\varphi}^{*} \otimes S_{r-t-1}(P) \rightarrow \ldots \rightarrow \wedge^{t+1}B_{\varphi}^{*} \otimes P \rightarrow \wedge^{t}B_{\varphi}^{*} \otimes S_{0}(P).$$

The image of the map $\wedge^{i}P^{*} \rightarrow \wedge^{t}P^{*}$ is (up to degree shift) an ideal of $R$ which we denote by $I(\psi)$ or just $I$, i.e. $\text{im} \wedge^{i}P^{*} = I \otimes \wedge^{i}P^{*}$. Note that the saturation of $I$ is the homogeneous ideal $I_{S}$ defining
the degeneracy locus $S$. Thus, using $\land^t \psi^*$ we can continue the complex above on the right-hand side and obtain the desired Eagon-Northcott complex
\[ E_* : 0 \to \land^t B^*_\varphi \otimes S_{r-t}(P) \to \land^{t-1} B^*_\varphi \otimes S_{r-t-1}(P) \to \ldots \]
\[ \to \land^{t+1} B^*_\varphi \otimes P \to \land^t B^*_\varphi \to I \otimes \land^t P^* \to 0. \]

The next result shows that the first cohomology modules of $R/I(\psi)$ vanish.

**Lemma 4.1.** The depth of $R/I$ is at least $n - r$.

**Proof.** Let $r = 2$. Then $B_\varphi$ has depth $n$ and $t = 1$. Thus we have an exact sequence
\[ 0 \to R(a) \xrightarrow{\psi} B_\varphi \to I(b) \to 0 \]
where $a, b$ are integers. It provides the claim.

Now let $r \geq 3$. We choose a sufficiently general linear form $l \in R$. For short we denote the functor $\land_R \mathcal{P}$ by $-\mathcal{P}$ where $\mathcal{P} = R/lR$. Let $\alpha$ be the map $\text{Hom}_R(\psi, R) : \text{Hom}_R(B_\varphi, \mathcal{P}) \to \text{Hom}_R(\mathcal{P}, \mathcal{P})$ and define the homogeneous ideal $J \subset \mathcal{P}$ by $J \otimes \land^t \mathcal{P} = \text{im} \land^t \alpha$. Our first claim is that $\mathcal{P} = J$ provided $n > t$.

Multiplication by $l$ provides the commutative diagram
\[
\begin{array}{cccccc}
0 & \to & P(-1) & \xrightarrow{\psi} & P & \xrightarrow{\psi} & \mathcal{P} & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & B_\varphi(-1) & \xrightarrow{\psi^*} & B_\varphi & \xrightarrow{\psi^*} & \mathcal{P} & \to 0 \\
\end{array}
\]
Dualizing gives the commutative diagram
\[
\begin{array}{cccccc}
0 & \to & B^*_\varphi & \xrightarrow{\psi^*} & B^*_\varphi(1) & \xrightarrow{\psi^*} & \text{Ext}^1_R(B_\varphi, R) & \xrightarrow{\psi^*} & \text{Ext}^1_R(B_\varphi, R) = 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & P^* & \xrightarrow{\psi^*} & P^*(1) & \xrightarrow{\psi^*} & \text{Ext}^1_R(\mathcal{P}, R) & \xrightarrow{\psi^*} & \text{Ext}^1_R(\mathcal{P}, R) = 0 \\
\end{array}
\]
where the vanishings on the right-hand side are due to duality and the fact that $B_\varphi$ is an Eilenberg-MacLane module of depth $n-r+2 \neq n$. Using $\text{Ext}^1_R(B_\varphi, R)(-1) \cong \text{Hom}_R(B_\varphi, \mathcal{P})$ and $\text{Ext}^1_R(\mathcal{P}, R)(-1) \cong \text{Hom}_R(\mathcal{P}, \mathcal{P})$ (cf., for example, [2], Lemma 3.1.16) we see that $\beta$ can be identified with $\alpha$ as well as $\psi^* \otimes \mathcal{P}$. It follows
\[ J \otimes \land^t \mathcal{P}^* = \text{im} \land^t \mathcal{P} \alpha = \text{im} \land^t \mathcal{P}(\psi^* \otimes \mathcal{P}) \cong \text{im}(\land^t \mathcal{P} \psi^* \otimes \mathcal{P}) \cong (\text{im} \land^t \mathcal{P} \psi^* \otimes \mathcal{P}) = I \otimes \land^t P^* \otimes \mathcal{P} \]
and thus $J = \mathcal{P}$ as we wanted to show.

The second commutative diagram also provides $B^*_\varphi \otimes \mathcal{P} \cong \text{Hom}_R(B_\varphi, \mathcal{P})$. It follows
\[ \land^t \mathcal{P} \text{Hom}_R(B_\varphi, \mathcal{P}) \cong \land^t \mathcal{P} \otimes \mathcal{P} \cong (\land^t B^*_\varphi) \otimes \mathcal{P}. \]
Thus we have an exact commutative diagram
\[
\begin{array}{cccccc}
0 & \to & C(-1) & \otimes & \land^t B^*_\varphi(-1) & \otimes & I(-1) \otimes \land^t P^* & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & C & \otimes & \land^t B^*_\varphi & \otimes & I \otimes \land^t P^* & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & L & \otimes & \land^t \text{Hom}_R(B_\varphi, \mathcal{P}) & \otimes & J \otimes \land^t \mathcal{P} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & & & & & & & 0 \\
\end{array}
\]
where $C = \ker \land^t \psi^*$ and $L = \ker \land^t \alpha$. Since $\land^t B^*_\varphi$ has depth $n+1-t \geq n+2-r$ due to Lemma 3.1, the first row shows that our assertion is equivalent to depth $C \geq n+2-r$. In order to show this we induct on $n-r$. If $n = r$ the claim follows by the exact cohomology sequence induced by the top-line of the previous diagram and depth $\land^t B^*_\varphi \geq 2$. Let $n > r$. Then our first claim and the Snake lemma applied to the diagram above imply $\mathcal{C} \cong L$. The induction hypothesis applies to $L$ and we obtain
\[ 0 < n+1-r \leq \text{depth } L < \text{depth } C \]
completing the proof. \qed
For computing the other cohomology modules of $R/I(\psi)$ we use the Eagon-Northcott complex above. In order to ease notation let us write $E_\bullet$ as

$$E_\bullet : 0 \to E_r \xrightarrow{\delta_r} E_{r-1} \xrightarrow{\delta_{r-1}} \cdots \to E_i \xrightarrow{\delta_i} I(\psi)(p) \to 0$$

where

$$E_i = \wedge^i B^* \otimes S_{i-t}(P) \quad \text{and} \quad R(p) \cong \wedge^t P^*.$$

The number of minimal generators of an $R$-module $N$ is denoted by $\mu(N)$.

**Proposition 4.2.** Put $I = I(\psi)$. Then we have:

- (a) For $j \neq \dim R/I = n + t - r$ it holds

$$H^j_m(R/I) \cong \begin{cases} 
S_i(M_\psi) \otimes S_{i-t}(P) \otimes \wedge^t P(1-r(R)) & \text{if } j = n + t - 2i \quad \text{where } \max\{t, \frac{r-1}{2}\} \leq i \leq \frac{r+1}{2} \\
0 & \text{otherwise.}
\end{cases}$$

- (b) The canonical module satisfies

$$\mu(K_{R/I}) \leq \frac{r-1}{r} \left( \frac{r+1}{r-1} \right) + \begin{cases} 
\left( \frac{r+1}{r-1} \right) & \text{if } r \text{ is even and } 1 \leq t \leq \frac{r}{2} \\
0 & \text{otherwise.}
\end{cases}$$

**Proof.** We consider the Eagon-Northcott complex $E_\bullet$ above. According to the Lemma 3.6 $B^*_r$ is an $r$-syzygy. Therefore $B^*_r$ is locally free in codimension $r$. It follows that the Eagon-Northcott complex $E_\bullet$ is exact in codimension $r$. Therefore its homology modules $H_i(E_\bullet)$ have dimension $\leq n - r$. Thus the exact sequence

$$0 \to \im \delta_{i+1} \to \ker \delta_i \to H_i(E_\bullet) \to 0$$

implies

$$H^i_m(\ker \delta_i) \cong H^i_m(\im \delta_{i+1}) \quad \text{if } j \geq n - r + 2.$$

Moreover, there are exact sequences

$$0 \to \ker \delta_i \to E_i \to \im \delta_i \to 0$$

inducing exact sequences

$$H^j_m(E_i) \to H^j_m(\im \delta_i) \to H^{j+1}_m(\ker \delta_i) \to H^{j+1}_m(E_i)$$

where the injectivity or surjectivity of the map in the middle can be checked by means of Lemma 3.6. This map is an isomorphism if $j \neq n - i, n + 1 - i, n, n + 1$.

Let us consider the map $\delta_r$. Due to our assumption the map $\psi^* : B^*_r \to P^*$ is generically surjective. Thus the same applies to the Koszul map $B^*_r \otimes S_{n-t}(P^*) \to S_{n-t}(P^*)$ which is induced by $\psi$. It follows that the $R$-dual of this map is injective. But the latter is (up to a degree shift) just $\delta_r$. Hence we have seen that $\im \delta_r \cong E_r$.

According to Lemma 3.6 it suffices to consider $H^j_m(R/I)$ where $n - r \leq j \leq n + t - r = \dim R/I$. For this we distinguish several cases.

**Case 1:** Let us assume that $n - t < j \leq n + t - r$. This can occur if and only if $t \geq \frac{r+1}{2}$.

Using (2) and (1) in alternating order we get

$$H^j_m(R/I)(p) \cong H^{j+1}_m(\im \delta_i) \cong H^{j+2}_m(\ker \delta_i) \cong \cdots \cong H^{r-j-t}_m(\im \delta_{r-1}) \to H^{r+j-1-t}_m(\ker \delta_{r-1}) \cong H^{r+j-1-t}_m(\im \delta_r) \cong H^{r+j+1-t}_m(E_r)$$

where the injection holds true because we have by our assumptions $n+3-r \leq n+2-t \leq j+1 \leq r+j-t \leq n$.

It follows by Lemma 3.6 and Lemma 2.2 that

$$H^j_m(R/I) = 0 \quad \text{if } n - t < j < n + t - r$$

and in case $j = \dim R/I$ for the canonical module

$$\mu(K_{R/I}) \leq \mu(H^{n+1}_m(E_r)^*) = \mu(E_r) = \mu(S_{r-t}(P)) = \left( \frac{r-1}{t-1} \right).$$

**Case 2:** Let us assume that $n - r \leq j = n + t - 1 - 2i \leq \min\{n + t - r, n - t\}$, i.e. $\max\{t, \frac{r-1}{2}\} \leq i \leq \frac{r+1}{2}$. 


We use again (1) and (2) in order to obtain information on the modules on the left-hand and on the right-hand side. This provides
\[ H_{m}^{n+1-i}(\ker \delta_i) \cong H_{m}^{n+1-i}(\im \delta_{i+1}) \cong \ldots \cong H_{m}^{n+r-2i}(\im \delta_r) \cong H_{m}^{n+r-2i}(E_r) = 0 \]
because \( n + r - 2i \leq n \) and
\[
H_{m}^{n+r-2i}(\im \delta_r) \cong H_{m}^{n+r-2i}(\im \delta_{i+1}) \cong \ldots \\
\cong H_{m}^{n+r-2i}(\im \delta_{r-1}) \cong H_{m}^{n+r-2i}(\ker \delta_{r-1}) \cong H_{m}^{n+r-2i}(E_{r-1}) \cong H_{m}^{n+r-2i}(E_r)
\]
where the last module vanishes if and only if \( i \neq \frac{r}{2} \).

Therefore (3) yields if \( i \neq \frac{r}{2} \)
\[ H_{m}^{n+t-2i}(R/I)(p) \cong H_{m}^{n+1-i}(E_i) \cong H_{m}^{n+1-i}(\ker \delta_i) \cong H_{m}^{n+1-i}(\im \delta_{i+1}) \cong \ldots \]
\[
\cong H_{m}^{n+r-2i}(\im \delta_r) \cong H_{m}^{n+r-2i}(E_r) \cong H_{m}^{n+r-2i}(E_r) = 0 \]
in case \( i = \frac{r}{2} \) taking K-duals of (3) furnishes the exact sequence
\[ E_r^* \to \Ext^{r-i}(R/I, R)(-p) \to S_{\frac{r}{2}}(\mu(M_\varphi)) \otimes S_{\frac{r}{2} - i}(P)(1 - r(R)). \]

It follows for the canonical module
\[ \mu(K_{R/I}) \leq \mu(S_{\frac{r}{2}}(\mu(M_\varphi)) \otimes S_{\frac{r}{2} - i}(P) + \mu(E_r) = \left( \frac{r}{i+1} \right) \cdot \left( \frac{r-1}{i} \right) \cdot \left( \frac{i+1}{r} \right) \cdot \left( \frac{i}{i} \right). \]

Our assertions are now a consequence of the results in the three cases above.

**Corollary 4.3.** For the depth of the coordinate ring we have
\[ \text{depth } R/I(\psi) = \begin{cases} 
  n - r & \text{if } r + t \text{ is even} \\
  n - r + 1 & \text{if } r + t \text{ is odd}.
\end{cases} \]

Put \( e = \text{depth } R/I(\psi) \). Then the only non-vanishing cohomology modules of \( R/I(\psi) \) besides \( H_{m}^{n+t-r}(R/I(\psi)) \) are \( H_{m}^{n+t-2k}(R/I(\psi)) \) where \( k \) is an integer with \( 0 \leq k \leq \frac{1}{2}[\min\{n - t, n + t - r - 1\} - e] \).

It has already been observed in [11] that \( I(\psi) \) is not always an unmixed ideal. This gives rise to consider the ideal \( J(\psi) \) which is defined as the intersection of the primary components of \( I(\psi) \) having maximal dimension. We denote by \( X \) the subscheme of \( Z \) defined by \( J(\psi) \) and call it the top-dimensional part of the degeneracy locus \( S \).

Our next aim is to clarify the relationship between \( S \) and \( X \). For this we need a cohomological criterion for unmixedness stated as Lemma III.2.3 in [13].
Lemma 4.4. Let $I \subset R$ be a homogeneous ideal. Then $I$ is unmixed if and only if
\[ \dim \text{Supp}(H^i_m(R/I)) < i \quad \text{for all } i < \dim R/I \]
where we put $\dim \text{Supp}(M) = -\infty$ if $M = 0$.

Now we can show.

Proposition 4.5. Let $I = I(\psi)$ and $J = J(\psi)$. Then it holds:

(a) $I$ is unmixed if and only if $r + t$ is odd.

(b) If $r + t$ is even then $I$ has a primary component of codimension $r + 1$. Let $Q$ be the intersection of all those components. Then we have $I = J \cap Q$ and $H^1_m(R/J) \cong \begin{cases} H^1_m(R/I) & \text{if } j \neq n-r \\ 0 & \text{if } j = n-r \end{cases}$
and $J/I \cong S_{m-\psi}(M_\psi) \otimes S_{m-\psi}(P) \otimes J^* \otimes G \otimes P$.

Proof. For $i = 0, \ldots, r$ the module $S_i(M_\psi)$ is a perfect module of dimension $n - r$. Hence claim (a) follows by Proposition 4.2 and Lemma 4.4.

In order to show (b) we note first that the maximal codimension of a component of $I$ is $r + 1$ because depth $R/I = n - r$. Let $Q$ be the intersection of all these components and let $J$ be the intersection of the remaining ones. Then $I = J \cap Q$ and the components of $J$ have codimension $\leq r$. We have to show that $J$ is unmixed.

As a first step we will prove that depth $R/J > n - r$. We induct on $n - r \geq 0$. If $r = n$ the claim is clear since $J$ is saturated by construction.

Let $r = n - 1$. It follows that $\dim R/I = 1 + t \geq 2$ and that $R/Q$ and $S_{r+1}(M_\psi)$ have positive dimension. Now we look at the exact sequence
\[ 0 \to R/I \to R/J \oplus R/Q \to R/(J + Q) \to 0. \]
We can find an $l \in [R]_1$ which is a parameter on $R/I, R/J, R/Q, S_{r+1}(M_\psi)$ and also on $R/(J + Q)$ if it has positive dimension. Using $H^1_m(R/I) \cong H^1_m(R/J)$ we obtain a commutative diagram with exact rows
\[
\begin{array}{ccccccc}
H^1_m(R/I)(-1) & \to & H^1_m(R/J)(-1) & \oplus & H^1_m(R/Q)(-1) & \to & H^1_m(R/(J + Q))(1-1) & \to 0 \\
\downarrow \beta_1 & & \downarrow \beta_2 & & \downarrow \beta_3 & & \downarrow \beta_4 & \\
H^1_m(R/I) & \to & H^1_m(R/J) & \oplus & H^1_m(R/Q) & \to & H^1_m(R/(J + Q)) & \to 0 \\
\end{array}
\]
where the vertical maps are multiplication by $l$. Due to our choice of $l$ it holds $H^1_m(R/(Q + lR)) = H^1_m(R/(J + Q + lR)) = 0$. Thus $\beta_3$ and $\beta_4$ are surjective. Since $l$ is a parameter of the Cohen-Macaulay module $S_{r+1}(M)$ the multiplication map $S_{r+1}(M)(-1) \xrightarrow{l} S_{r+1}(M)$ is injective, thus the dual map $S_{r+1}(M)^* \xrightarrow{l} S_{r+1}(M)^*(1)$ is an epimorphism. Therefore $\beta_1$ is surjective due to Proposition 4.2. The same is true for $\beta_2$ by the commutative diagram above. Since $R/J$ is unmixed Lemma 4.4 implies that $H^1_m(R/J)$ is finitely generated, hence it must be zero by Nakayama’s lemma.

Finally, let $r \leq n - 2$. We consider the commutative diagram
\[
\begin{array}{ccccccc}
0 & \to & R/I(-1) & \to & R/J(-1) & \oplus & R/Q(-1) & \to & R/(J + Q)(1-1) & \to 0 \\
0 & \to & R/I & \to & R/J & \oplus & R/Q & \to & R/(J + Q) & \to 0 \\
\end{array}
\]
where the vertical maps are multiplication by $l$. By Lemma 4.3 depth $R/I \geq 2$ and by assumption on $r R/J$ and $R/Q$ have positive depth. Hence the cohomology sequence induced by the bottom line provides $H^1_m(R/(J + Q)) = 0$. It follows that we may choose $l$ as non-zero divisor on $R/I, R/J, R/Q$ and $R/(J + Q)$, i.e., the vertical maps in the diagram above are all injective. Thus the Snake lemma implies the exact sequence
\[ 0 \to R/I \to R/J \oplus R/Q \to R/J + Q \to 0 \]
where we denote by $\sim$ again the functor $\sim \otimes_R R/I$. By Bertini’s theorem $R/J$ is unmixed of codimension $r + 1$ in $R$ (possibly) up to a component associated to the irrelevant ideal of $R$. Moreover, we have seen in the proof of Lemma 4.4 that the induction hypothesis applies to $T$. But the last exact sequence implies
$\mathcal{T} = \mathcal{J} \cap \mathcal{Q}$. It follows that $\mathcal{J}$ is the intersection of the top-dimensional components of $\mathcal{T}$. Hence we get by induction depth $\mathcal{R}/\mathcal{J} \geq n - r$. But $l$ was a non-zero divisor on $R/J$ thus we obtain

$$\text{depth } R/J > \text{depth } \mathcal{R}/\mathcal{J} = \text{depth } \mathcal{R}/\mathcal{J}$$

completing our induction.

Next we consider the exact sequence

$$0 \to J/I \to R/I \to R/J \to 0.$$  

Note that $J/I \cong (J + Q)/Q$ has dimension $\leq n - r$. Moreover we have just shown depth $R/J > n - r$. Therefore the induced cohomology sequence yields:

$$H^i_m(R/J) \cong H^i_m(R/I) \quad \text{if } i > n - r,$$

$$H^{n-r}_m(J/I) \cong H^{n-r}_m(R/I) \cong S_{\frac{r+1}{r+1}}(M_\varphi)^{\vee} \otimes S_{\frac{r+1}{r+1}}(P) \otimes \wedge^r P(1-r(R))$$

and $J/I$ is Cohen-Macaulay of dimension $n - r$. The first isomorphisms, Proposition 4.2 and Lemma 4.4 imply now that $J$ must be unmixed.

Now we use that for a Cohen-Macaulay module $M$ it holds $K_{K_M} \cong M$. Thus we get by duality and Lemma 4.8

$$J/I \cong H^{n-r}_m(H^{n-r}_m(J/I))^{\vee} \cong H^{n-r}_m(S_{\frac{r+1}{r+1}}(M_\varphi))^{\vee} \otimes S_{\frac{r+1}{r+1}}(P) \otimes \wedge^r P(1-r(R)) \cong \text{Ext}^{r+1}(S_{\frac{r+1}{r+1}}(M_\varphi), P) \otimes S_{\frac{r+1}{r+1}}(P) \otimes \wedge^r P.$$

This finishes the proof.  

**Remark 4.6.** The arguments in the previous proof also provide that $R/Q$ is Cohen-Macaulay of dimension $n - r$ and

$$n - r - 1 \leq \text{depth } R/(J + Q) \leq \dim R/(J + Q) \leq n - r.$$  

Our results with respect to ring-theoretic properties can be summarized as follows. Recall that $X$ denotes the top-dimensional part of the degeneracy locus $S$ of $\psi$.

**Theorem 4.7.** With the notation above we have:

(a) If $r = n$ then $S$ is equidimensional and locally Cohen-Macaulay.

(b) $S$ is equidimensional if and only if $r + t$ is odd or $r = n$.

Moreover, if $r < n$ then $X$ is locally Cohen-Macaulay if and only if $X$ is arithmetically Cohen-Macaulay.

(c) If $r + t$ is odd then $X = S$ is arithmetically Cohen-Macaulay if and only if $t = 1$. In this case $S$ has Cohen-Macaulay type $\leq 1 + (\frac{r+g-1}{g-1})$.

(d) Let $r + t$ be even. Then

(i) $X$ is arithmetically Cohen-Macaulay if and only if $1 \leq t \leq 2$. If $t = 1$ then $X$ is arithmetically Gorenstein. If $t = 2$ then $X$ has Cohen-Macaulay type $\leq r - 1 + \left(\frac{r+g-1}{g-1}\right) \cdot (\frac{1}{2} - 1)$.

(ii) If in addition $r < n$ then the components of $S$ have either codimension $r - t + 1$ or codimension $r + 1$.

**Proof.** (a) If $r = n$ then dim $M_\varphi = 0$. Hence it follows by Proposition 4.2 that the modules $H^r_1(Z, J_S)$ have finite length if $i \leq \dim S$ which is equivalent to $S$ being equidimensional and locally Cohen-Macaulay.

(b) If $r + t$ is odd then $S$ is equidimensional due to Proposition 4.3. If $r + t$ is even then Proposition 4.5 shows that $S$ is equidimensional if and only if $r = n$.

Moreover, $r < n$ implies that $M_\varphi$ does not have finite length. Therefore Proposition 4.5 furnishes that $R/J(\psi)$ has cohomology of finite length if and only if $R/J(\psi)$ is Cohen-Macaulay. Claim (b) follows.

(c) If $r + t$ is odd then we have by Proposition 4.3 that $S$ is equidimensional and by Corollary 4.3 that depth $R/I(\psi) = n - r + 1$. The claim follows because of dim $R/I(\psi) = n - r + t$.

(d) If $r + t$ is even we get $r \leq t - 2$. Hence the claim is a consequence of Propositions 4.2 and 4.3.  

□
Remark 4.8. (i) If we specialize the previous result to $Z = \mathbb{P}^n, r = 3, t = 1$ then we get the main result of \cite{1}.

(ii) In case $t = 1$ and $r$ is even the result has been first proved by the first and third author who communicated it to Kustin. Subsequently, Kustin \cite{1} strengthened it by removing almost all the assumptions on the ring $R$ and computing a free resolution (cf. also Remark 5.10 and Corollary 5.13).

(iii) Note that in the above theorem the term equidimensional is used in the scheme-theoretic sense. Thus $S$ being equidimensional does not automatically imply that $I(\psi)$ is saturated. However, due to Corollary 4.3 $I(\psi)$ is not saturated if and only if $r = n$ and $r + t$ is even. We use this fact in Section 5.

The next result generalizes \cite{1}, Corollary 1.2.

Corollary 4.9. Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$ of rank $n$ where $n$ is odd such that $H_i(\mathbb{P}^n, \mathcal{E}) = 0$ if $2 \leq i \leq n - 1$. Let $s$ be a section of $\mathcal{E}$ vanishing on a scheme $X$ of codimension $n$. Then $X$ is arithmetically Gorenstein.

Proof. According to Corollary 3.11, $\mathcal{E}$ is a Buchsbaum-Rim sheaf. Therefore Theorem 4.7 shows the claim.

5. The resolutions of the loci

In this section we show how free resolutions can be obtained for the schemes described in this paper. In most cases we expect these resolutions to be minimal. The main tools are the Eagon-Northcott complex $E_\bullet$ in (4.1), its dual and a general result for comparing the resolution of the scheme with its cohomology modules.

We begin by considering the Eagon-Northcott complex again. The interested reader will have observed that this complex is not exact in general. Fortunately, we are able to compute its homology.

Proposition 5.1. The homology modules of the Eagon-Northcott $E_\bullet$ complex are:

$$H_i(E_\bullet) \cong \begin{cases} S_j(M_{\mathcal{P}}) \otimes S_{r-t-j}(P) \otimes \wedge^j F^* \otimes \wedge^g G & \text{if } i = r - 1 - 2j \text{ where } j \in \mathbb{Z}, t \leq i \leq r - 3 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. According to Lemma 3.6 and Lemma 4.1 all non-trivial modules occurring in $E_\bullet$ have depth $\geq n - r + 1$. Thus using $n - r$ general linear forms in $R$ and arguments as in Proposition 4.1 we see that it suffices to consider the case where $R$ has dimension $r + 1$, i.e., we may and will assume that $n = r$.

Then we already know that the homology modules of $E_\bullet$ have finite length. As for the cohomology of $R/I$ one computes (cf. Proposition 4.2)

$$H^1_m(\mathfrak{m} \delta_i) \cong \begin{cases} S_{r-j}(M_{\mathcal{P}})^\vee \otimes S_{r-t-i}(P)(1 - r(R)) & \text{if } t \leq i = r - 2j < r \\ 0 & \text{if } r + i \text{ is odd} \end{cases}$$

Since $r = n$ the module $M_{\mathcal{P}}$ has finite length. It follows

$$S_{r-j}(M_{\mathcal{P}})^\vee (1 - r(R)) \cong H^0_m(S_{r-j}(M_{\mathcal{P}})^\vee (1 - r(R))$$

$$\cong \operatorname{Ext}^2_R(S_{r-j}(M_{\mathcal{P}}), R) \quad \text{by 2.2}$$

$$\cong S_j(M_{\mathcal{P}}) \otimes \wedge^j F^* \otimes \wedge^g G \quad \text{by 3.8}$$

Thus we have

$$H^1_m(\mathfrak{m} \delta_i) \cong \begin{cases} S_j(M_{\mathcal{P}}) \otimes S_{r-j-i}(P) \otimes \wedge^j F^* \otimes \wedge^g G & \text{if } t \leq i = r - 2j < r \\ 0 & \text{if } r + i \text{ is odd} \end{cases}$$

The modules $\mathfrak{m} \delta_i$ and $\ker \delta_i$ are submodules of the reflexive modules $E_{i+1}$ and $E_i$, respectively. Hence both have positive depth. Since depth $E_i = n + 1 - i \geq 2$ if $i < r = n$ the exact sequence

$$0 \to \ker \delta_i \to E_i \to \mathfrak{m} \delta_i \to 0$$

shows depth $\ker \delta_i \geq 2$ for all $i = t, \ldots, r$.

Using the finite length of $H_i(E_\bullet)$ the exact sequence

$$0 \to \mathfrak{m} \delta_{i+1} \to \ker \delta_i \to H_i(E_\bullet) \to 0$$

provides $H^1_m(\mathfrak{m} \delta_{i+1}) \cong H^0_m(H_i(E_\bullet)) \cong H_i(E_\bullet)$. Hence (+) proves our assertion because $\delta_i$ is surjective by the definition of the ideal $I$. \qed
Remark 5.2. Now we can compute a free resolution of $I = I(\psi)$ as follows: Proposition 5.1 provides exact sequences
\begin{equation}
0 \to \ker \delta_{r+1-2j} \to E_{r+1-2j} \to E_{r-2j} \to \im \delta_{r-2j} \to 0, \quad t \leq r - 2j < r,
\end{equation}
and if $r + t$ is odd additionally
\begin{equation}
0 \to \ker \delta_t \to E_t \to I(p) \to 0,
\end{equation}
and if $r + t$ is even additionally
\begin{equation}
0 \to \im \delta_{r-2j} \to \ker \delta_{r-2j-1} \to S_j(M_\varphi) \otimes S_{r-t-j}(P) \otimes \wedge^t F^* \otimes \wedge^9 G \to 0, \quad t < r - 2j < r.
\end{equation}
According to Proposition 2.1 and Proposition 3.6 we know the minimal resolution of $E_i$ and $S_j(M_\varphi)$, respectively. Thus, in order to get a resolution of $I$ we compute successively resolutions of $\im \delta_{r-2j}, \ker \delta_{r-3j}, \im \delta_{r-4j}, \ldots, \im \delta_t = I(p)$ using the exact sequences (1) and (2). If (1) is used we just apply the mapping cone procedure twice. If we use (2) we apply the Horseshoe lemma.

Following the procedure just described we certainly do not obtain a minimal free resolution of $I$. Thus we need some results which allow us to split off redundant terms. The idea is to compare the resolution of $I$ with those of its cohomology modules. In particular, this requires information on the canonical module $K_S = \Ext_R^{r-t}(R/I(\psi), R)(r(R) - 1)$ of our degeneracy locus $S$ which we derive first.

Lemma 5.3. The depth of the canonical module of $X$ is at least $\min\{n - r + t, n - r + 2\}$. In particular, it is Cohen-Macaulay if $1 \leq t \leq 2$.

Proof. Let $A = R/I(\psi)$. We induct on $n - r$. Since the canonical module always satisfies depth $K_A \geq \min\{\dim A, 2\}$ the claim is clear if $n = r$.

Let $n > r$ and let $t \in R$ be a general linear form. We want to show $K_A/IK_A \cong K_{A/I_A}$.
Due to Corollary 4.3 there is an exact sequence induced by multiplication
\begin{equation}
0 \to A(-1) \xrightarrow{t} A \to A/I_A \to 0.
\end{equation}
It provides the long exact sequence
\begin{equation}
0 \to \Ext_R^{r-t}(A, R) \xrightarrow{t} \Ext_R^{r-t}(A, R)(1) \to \Ext_R^{r-t+1}(A/I_A, R) \to \Ext_R^{r-t+1}(A, R) \xrightarrow{t} \Ext_R^{r-t+1}(A, R)(1) \to \ldots.
\end{equation}
Using $\Ext_R^{r-t-1}(A/I_A, R)(r(R) - 1) \cong K_{A/I_A}$ we can rewrite the last sequence as
\begin{equation}
0 \to K_A(-1) \xrightarrow{t} K_A \to K_{A/I_A} \to \Ext_R^{r-t+1}(A, R)(r(R) - 2) \xrightarrow{t} \Ext_R^{r-t+1}(A, R)(r(R) - 1) \to \ldots.
\end{equation}
We claim that the multiplication map on the right-hand side is injective. Indeed, if $r + 1$ is odd or $2t \geq r + 1$ then we have by duality and Proposition 4.2 that $\Ext_R^{r-t+1}(A, R) = 0$. Otherwise the multiplication map is (up to degree shift) essentially given by
\begin{equation}
S_{\frac{r-t}{2}}(M_\varphi) \otimes S_{\frac{r-t}{2}+1}(P)^*(-1) \xrightarrow{t} S_{\frac{r-t}{2}}(M_\varphi) \otimes S_{\frac{r-t}{2}+1}(P)^*.
\end{equation}
But the module $S_{\frac{r-t}{2}}(M_\varphi)$ is Cohen-Macaulay of dimension $n - r > 0$ according to Lemma 3.6. Thus we can choose $t$ as a regular element with respect to $S_{\frac{r-t}{2}}(M_\varphi)$ and the claim follows.
Hence the exact sequence above implies
\begin{equation}
K_A/IK_A \cong K_{A/I_A} \quad \text{and} \quad \depth K_A > \depth K_{A/I_A}.
\end{equation}
Applying the induction hypothesis to $K_{A/I_A}$ completes the proof. \hfill \Box

Remark 5.4. The canonical modules of $S$ and its top-dimensional part $X$ are isomorphic. This follows from the corresponding (slightly stronger) result for $R/I(\psi)$ and $R/J(\psi)$. Indeed, according to Proposition 4.3 there is an exact sequence
\begin{equation}
0 \to S_{\frac{r-t}{2}}(M_\varphi) \otimes S_{\frac{r-t}{2}+1}(P) \otimes \wedge^t F^* \otimes \wedge^9 G \otimes \wedge^4 P \to R/I(\psi) \to R/J(\psi) \to 0.
\end{equation}
Since $\dim S_{\frac{r-t}{2}}(M_\varphi) = n - r < n - r + t = \dim R/I(\psi) = \dim R/J(\psi)$ the claim follows by the long exact cohomology sequence.
Lemma 5.5. The dual of the Eagon-Northcott complex \( E_\bullet \) provides a complex

\[
0 \to \wedge^t P \to E_\ell^* \to \cdots \to E_{r-1}^* \to E_r^* \to K_{R/I} \otimes \wedge^t P(1 - r(R)) \to 0
\]

which we denote (by slight abuse of notation) by \( E^*_\bullet \). Its (co)homology modules are given by

\[
H^i(E^*_\bullet) \cong \begin{cases} S_j(M_\phi) \otimes S_{j-t}(P)^* & \text{if } 2t + 1 \leq i = 2j + 1 \leq r + 1 \\ 0 & \text{otherwise} \end{cases}
\]

In particular, \( E^*_\bullet \) is exact if \( t \geq \frac{r+1}{2} \).

Proof. If \( r = 2 \) (and thus \( t = 1 \)) the claim follows immediately by dualizing because \( E_\bullet \) is exact in this case.

Now let \( r \geq 3 \). We begin by verifying that \( E^*_\bullet \) is indeed a complex. This is only an issue in the beginning of this sequence. Consider the exact sequence

\[
0 \to E_r \to E_{r-1} \to \im \delta_{r-1} \to 0.
\]

Dualizing provides the exact sequence

\[
E_{r-1}^* \xrightarrow{\delta_r^*} E_r^* \xrightarrow{\alpha} \Ext^1_R(\im \delta_{r-1}, R) \to \Ext^1_R(E_{r-1}, R).
\]

The module on the right-hand side vanishes since \( H^m_\bullet(\wedge^{r-1} B_\bullet^*) = 0 \) if \( r \neq 2 \). Hence the map \( \gamma \) is surjective. In order to compare its image with \( K_{R/I} \) we look at the proof of Proposition 4.2 and use its notation.

If \( t \geq \frac{r+1}{2} \) we have (cf. Case 1)

\[
H^{n+t-r}_m(R/I)(p) \cong H^m_\bullet(\im \delta_{r-1}).
\]

Thus (+) provides the exact sequence

\[
E_{r-1}^* \xrightarrow{\delta_r^*} E_r^* \xrightarrow{\alpha} K_{R/I}(1 - r(R) - p) \to 0.
\]

Putting \( \gamma = \alpha \) we get the desired complex and

\[
H^{r+1}(E^*_\bullet) = H^r(E^*_\bullet) = 0.
\]

Now let \( t \leq \frac{r-1}{2} \) and let \( r \) be odd. Then we have (cf. Case 2) an embedding

\[
\beta : H^{n+t-r}_m(R/I)(p) \to H^n_\bullet(\im \delta_{r-1}).
\]

We define \( \gamma \) as the composition of \( \alpha \) and \( \beta^\vee \) (with the appropriate shift). Thus \( \gamma \) is surjective, i.e.

\[
H^{r+1}(E^*_\bullet) = 0.
\]

Finally, let \( t \leq \frac{r+1}{2} \) and let \( r \) be even. Then we have (cf. Case 3) an exact sequence

\[
0 \to H^{n+1-\frac{t}{2}}(E^*_\bullet) \to H^{n+1-\frac{t}{2}}(\im \delta^*_\bullet) \to H^{n+2-\frac{t}{2}}(\ker \delta^*_\bullet) \to H^{n+2-\frac{t}{2}}(E^*_\bullet) = 0
\]

and isomorphisms

\[
H^{n+1-\frac{t}{2}}(\im \delta^*_\bullet) \cong H^{n-r+t}_m(R/I)(p), \quad H^{n+2-\frac{t}{2}}(\ker \delta^*_\bullet) \cong H^n_\bullet(\im \delta_{r-1}).
\]

Thus we can conclude with the help of (+) that there is an exact sequence

\[
E_{r-1}^* \xrightarrow{\delta_r^*} E_r^* \xrightarrow{\gamma} K_{R/I}(1 - p - r(R)) \to H^{n+1-\frac{t}{2}}(E^*_\bullet)^\vee(1 - r(R)) \to 0.
\]

Using Lemma 5.3 it follows

\[
H^{r+1}(E^*_\bullet) \cong H^{n+1-\frac{t}{2}}(E^*_\bullet)^\vee(1 - r(R)) \cong S_\bullet^\vee(M_\phi) \otimes S_{r+t-\ell}(P)^*.
\]

Hence we have shown in all cases that \( E^*_\bullet \) is a complex and we have computed \( H^{r+1}(E^*_\bullet) \).

In order to compute the other cohomology modules we proceed as in the proofs of Proposition 4.2 and Proposition 5.1. Indeed, we just have to use Lemma 5.3 as replacement of Lemma 4.1 and \( E^*_\bullet \) instead of \( E_\bullet \). The details are tedious but straightforward. We omit them.

The next result is also interesting in its own right. It relates the minimal free resolution of a module to those of (the duals) of its cohomology modules. Observe that the result as it is stated remains valid even if \( R \) is not a Gorenstein but just a Cohen-Macaulay ring, though we won’t use this fact here.
Proposition 5.6. Let $N$ be a finitely generated graded torsion $R$-module which has projective dimension $s$. Then it holds for all integers $j \geq 0$ that $\text{Tor}^R_\ast(j, N)^\vee$ is a direct summand of

$$\oplus_{i=0}^j \text{Tor}^R_{i-j}(\text{Ext}^{s-i}_R(N, R), K).$$

Moreover, we have $\text{Tor}^R_{i}(N, K)^\vee \cong \text{Tor}^R_{0}(\text{Ext}^s_R(N, R), K)$ and that $\text{Tor}^R_{i}(\text{Ext}^s_R(N, R), K)$ is a direct summand of $\text{Tor}^R_{s-i}(N, K)^\vee$.

Proof. For the purpose of this proof we write $M \subset N$ in order to express that the submodule $M$ is a direct summand of the module $N$.

Consider a minimal free resolution of $N$:

$$0 \to F_s \to \ldots \to F_1 \to F_0 \to N \to 0.$$ Dualizing with respect to $R$ provides the complex:

$$0 \to F^s_0 \xrightarrow{\alpha_0} F^s_1 \to \ldots \to F^s_{s-1} \xrightarrow{\alpha_{s-1}} F^s_s \to \text{Ext}^s_R(N, R) \to 0.$$ Since the maps $\alpha_j$ are duals of minimal maps we can write

$$\ker \alpha_j \cong G_j \oplus M_j$$

where $M_j$ does not have a free $R$-module as direct summand and $G_j$ is a free $R$-module being a direct summand of $F^s_j$ (but possibly trivial). Moreover, there are exact sequences

$$0 \to \text{im} \alpha_{j-1} \to \ker \alpha_j \to \text{Ext}^s_R(N, R) \to 0.$$ Since $\alpha_{j-1}$ is a minimal homomorphism it holds $\text{im} \alpha_{j-1} \subset \mathfrak{m} \cdot F^s_j$. This shows:

(1) The minimal generators of $G_j$ give rise to minimal generators of $\text{Ext}^s_R(N, R)$.

Now we consider the diagram

$$
\begin{array}{ccc}
0 & \to & \text{im} \alpha_{j-1} \\
\downarrow F^s_j & & \downarrow F^s_j \\
\ker \alpha_j & \to & \text{Ext}^s_R(N, R) \\
& & 0
\end{array}
$$

where the vertical maps are the natural embeddings. Thus it is commutative and we obtain an exact sequence

$$0 \to \text{Ext}^s_R(N, R) \to F^s_j / \text{im} \alpha_{j-1} \to F^s_j / \ker \alpha_j \to 0.$$ The Horseshoe lemma yields a free resolution of the middle module as direct sum of the resolutions of the outer modules. After splitting off redundant terms we get a minimal free resolution, i.e., using $\text{Tor}^R_i(\text{im} \alpha_{j-1}, K) \cong \text{Tor}^R_{i+1}(F^s_j / \text{im} \alpha_{j-1}, K)$ we obtain

(2) $\text{Tor}^R_i(\text{im} \alpha_{j-1}, K) \subset \text{Tor}^R_{i+1}(\text{Ext}^s_R(N, R), K) \oplus \text{Tor}^R_i(\ker \alpha_j, K) \quad (i \geq 0)$.

Now we are ready to show by induction on $j \geq 1$:

(*) $\text{Tor}^R_i(\text{im} \alpha_{s-j}, K) \subset \oplus_{k=0}^{j-1} \text{Tor}^R_{i+j-k}(\text{Ext}^{s-k}_R(N, R), K) \quad (i \geq 0)$

and

(**) $\text{Tor}^R_{i+j}(N, K)^\vee \subset \oplus_{k=0}^{j-1} \text{Tor}^R_{i+j-k}(\text{Ext}^{s-k}_R(N, R), K)$.

Let $j = 1$. Since $\alpha_{s-1}$ is a minimal homomorphism the exact sequence

$$0 \to \text{im} \alpha_{s-1} \to F^s_s \to \text{Ext}^s_R(N, R) \to 0$$

implies

$$\text{Tor}^R_{s}(N, K)^\vee \cong \text{Tor}_{0}^R(\text{Ext}^s_R(N, R), K),$$

thus in particular (**), and

$$\text{Tor}^R_i(\text{im} \alpha_{s-1}, K) \cong \text{Tor}^R_{i+1}(\text{Ext}^s_R(N, R), K) \quad (i \geq 0)$$

which shows (*).

Let $j \geq 2$. Consider the exact sequence

(3) $0 \to \text{ker} \alpha_{s-j+1} \to F^s_{s-j+1} \to \text{im} \alpha_{s-j+1} \to 0.$
We obtain by the definition of $G_{s-j+1}$

$$F_{s-j+1}^* \otimes K \cong (G_{s-j+1} \otimes K) \oplus \text{Tor}_0^R(\text{im} \, \alpha_{s-j+1}, K)$$

$$\subset \text{Tor}_0^R(\text{Ext}_{s-j+1}^s(N, R), K) \oplus \bigoplus_{k=0}^{j-2} \text{Tor}_{s-j-k}^R(\text{Ext}_{s-k}^s(N, R), K) \quad (\text{by } (1) \text{ and induction})$$

$$= \bigoplus_{k=0}^{j-1} \text{Tor}_{s-j-k}^R(\text{Ext}_{s-k}^s(N, R), K)$$

which shows (**).

Furthermore, (3) provides

$$\text{Tor}_i^R(\ker \alpha_{s-j+1}, K) \subset \text{Tor}_{i+1}^R(\text{im} \, \alpha_{s-j+1}, K)$$

$$\subset \bigoplus_{k=0}^{j-2} \text{Tor}_{i+j-k}^R(\text{Ext}_{s-k}^s(N, R), K) \quad (\text{by induction}).$$

Thus we conclude with the help of (2):

$$\text{Tor}_i^R(\text{im} \, \alpha_{s-j}, K) \subset \text{Tor}_{i+1}^R(\text{Ext}_{s-j+1}^s(N, R), K) \oplus \bigoplus_{k=0}^{j-2} \text{Tor}_{i+j-k}^R(\text{Ext}_{s-k}^s(N, R), K)$$

$$= \bigoplus_{k=0}^{j-1} \text{Tor}_{i+j-k}^R(\text{Ext}_{s-k}^s(N, R), K).$$

This proves (*), i.e., the induction step is complete.

It only remains to verify the very last assertion. Looking at (3) with $j = 2$ we get

$$\text{Tor}_0^R(\text{im} \, \alpha_{s-1}, K) \subset F_{s-1}^* \otimes K.$$

But at the beginning of the induction we have seen that $\text{Tor}_0^R(\text{im} \, \alpha_{s-1}, K) \cong \text{Tor}_1^R(\text{Ext}_{s}^s(N, R), K)$. Our claim follows

$$\square$$

**Remark 5.7.** In order to explain how the last result can be used we compare it with a well-known statement of Rao. For this let $R = K[x_0, \ldots, x_3]$ and let $I_C \subset R$ be the homogeneous ideal of a curve $C \subset \mathbb{P}^3$. Put $A = R/I_C$ and consider the following minimal free resolutions:

$$0 \to F_3 \to F_2 \to F_1 \to R \to A \to 0$$

$$0 \to G_4 \to G_3 \to G_2 \to G_1 \to G_0 \to \text{Ext}_{R}^3(A, R) \to 0$$

$$0 \to D_2 \to D_1 \to D_0 \to \text{Ext}_{R}^2(A, R) \to 0.$$

Then the additions to our general observation in the lemma above yield

$$G_0 \cong F_3^* \quad \text{and} \quad G_1 \subset F_2^*.$$

This is precisely the content of Rao’s Theorem 2.5 in [15]. Our Lemma 5.6 gives in addition

$$G_1 \subset F_2^* \subset G_1 \oplus D_0$$

and

$$F_1^* \subset G_2 \oplus D_1$$

because $\text{Ext}_{R}^1(A, R) = 0$.

Now we have all the tools for establishing the main result of this section.
Theorem 5.8. Consider the following modules where we use the conventions that $i$ and $j$ are non-negative integers and that a sum is trivial if it has no summand:

$$A_k = \bigoplus_{i+2j=k+t-1 \atop t \leq i+j \leq r-1} \wedge^i F^* \otimes S_j(G)^* \otimes S_{t-i-j}(P),$$

$$C_k = \bigoplus_{i+2j=r+1-t-k \atop i+j \leq r-1} \wedge^i F \otimes S_j(G) \otimes S_{t-i-j}(P) \otimes \wedge^j F^* \otimes \wedge^9 G.$$ 

Observe that it holds:

- $A_r = 0$ if and only if $r + t$ is even,
- $C_1 = 0$ if and only if $r + t$ is odd,
- $C_k = 0$ if $k \geq r + 2 - t$ and
- $C_{r+1-t} = S_{r-t}(P) \otimes \wedge^i F^* \otimes \wedge^9 G$.

Then the homogeneous ideal $I_X = J(\psi)$ of the top-dimensional part $X$ of the degeneracy locus $S$ has a graded free resolution of the form

$$0 \rightarrow A_r \oplus C_r \rightarrow \ldots \rightarrow A_1 \oplus C_1 \rightarrow I_X \otimes \wedge^t P^* \rightarrow 0.$$

Proof: Following the procedure described in Remark 5.2 we get a resolution of $I(\psi)$. According to Proposition 4.5 it holds $I(\psi) = J(\psi)$ if and only if $r + t$ is odd. If $r + t$ is even we have by the same result an exact sequence

$$0 \rightarrow I(\psi) \rightarrow J(\psi) \rightarrow S_{r-t}(M_\varphi) \otimes S_{r-t}(P) \otimes \wedge^j F^* \otimes \wedge^9 G \otimes \wedge^t P^*.$$

Thus, using the Horseshoe lemma we get a finite free resolution of $J(\psi)$ in any case. It is not minimal. In order to split off redundant terms we proceed as follows:

Since the resolution is finite the Auslander-Buchsbaum formula and Proposition 4.5 yield the projective dimension of $J(\psi)$. Thus, in a first step we can split off all the terms in the resolution of $J(\psi)$ occurring past its projective dimension.

Next, we use Lemma 5.5 (cf. Remark 5.4) in order to obtain a free resolution of $\text{Ext}^{t-1}_R(R/J(\psi), R)$. For $j \neq r - t + 1$ we know a free resolution of $\text{Ext}^j_R(R/J(\psi), R)$ by Proposition 4.5 and Proposition 2.1. Hence, in a second step we can split off further terms in the resolution of $J(\psi)$ by applying Proposition 2.6. This provides the resolution as claimed. The details are very tedious but straightforward. We omit them.

Since the above proof is somewhat sketchy, we illustrate it by deriving the resolution for $t = 2$ and $r = 6, 7$ (cf. also the next corollaries).

Example 5.9. Using our standard notation we define integers $c, p$ by

$$R(c) \cong \wedge^i F \otimes \wedge^g G^* \quad \text{and} \quad R(p) = \wedge^t P^*.$$

(i) Let $t = 2$ and $r = 7$. Then we know that $I = I(\psi)$ is unmixed and that $X$ is not arithmetically Cohen-Macaulay. Using the Eagon-Northcott complex $E_\bullet$, we see that a free resolution of $I$ begins as
follows:

\[
\begin{array}{c}
\wedge^2 F^* \otimes S_2(G)^* \otimes S_2(P) \oplus S_3(G)^* \otimes P \\
\wedge^3 F^* \otimes G^* \otimes S_2(P) \oplus F^* \otimes S_2(G)^* \otimes P \\
\wedge^4 F^* \otimes S_2(P) \oplus \wedge^2 F^* \otimes G^* \otimes P \oplus S_2(G)^* \\
\wedge^3 F^* \otimes P \oplus F^* \otimes G^* \oplus G(-c) \otimes S_2(P) \\
\wedge^2 F^* \\
I(p) \\
0.
\end{array}
\]

With the help of $E_\bullet$, we get the following beginning for the resolution of $\text{Ext}^4_{R}(R/I,R)(-p)$:

\[
\begin{array}{c}
\wedge^2 F^*(c) \otimes P^* \oplus G^*(c) \otimes S_2(P)^* \\
F^*(c) \otimes S_2(P)^* \\
S_3(P)^*(c) \\
\text{Ext}^4_{R}(R/I,R)(-p) \\
0.
\end{array}
\]

Now we apply Proposition 5.6 and conclude that in the top row of the resolution of $I(p)$ only the term $S_3(G)^* \otimes P$ remains in the minimal resolution because $S_3(G)^* \otimes P$ surjects minimally onto $\text{Ext}^4_{R}(R/I,R)(-p)$. Continuing in this fashion we obtain the following resolution:
(ii) Let \( t = 2 \) and \( r = 6 \). Then \( J = J(\psi \neq I(\psi)) \) defines an arithmetically Cohen-Macaulay scheme. Thus we get as in the previous case but slightly easier a resolution

\[
\begin{array}{cccccc}
0 & S_3(G)^* \otimes P & \oplus & S_4(P)(-c_1) & \downarrow \\
\downarrow & F^* \otimes S_2(G)^* \otimes P & \oplus & F(-c_1) \otimes S_3(P) & \downarrow \\
\wedge F^* \otimes G^* \otimes P & \oplus & S_2(G)^* & \oplus & G(-c_1) \otimes S_3(P) & \oplus & \wedge^2 F(-c_1) \otimes S_2(P) \\
\wedge^3 F^* & \oplus & F^* \otimes G^* & \oplus & F \otimes G(-c_1) \otimes S_2(P) & \wedge^2 F^* \\
J(p) & \downarrow & S_2(G)(-c_1) \otimes S_2(P) & \downarrow & \wedge^2 F^* \\
0 & \downarrow & \wedge^3 F^* \\
0 & \downarrow & J(p) \\
0 & \downarrow & 0
\end{array}
\]

**Remark 5.10.** We want to discuss the minimality of the resolution described in Theorem 5.8:

(i) By looking at the twists of the free summands occurring in the resolution above it is clear that for suitable choices of \( F, G, P \) and sufficiently general maps \( \varphi, \psi \) no further cancellation is possible, i.e., the resolution is minimal.

(ii) Let \( r \) be even and let \( t = 1 \). This case was also studied by Kustin [9]; his main result gives (up to the degree shifts) the same resolution as our Theorem 5.8 (cf. also Corollary 5.13). His techniques are completely different from ours, and while they are more complicated, they in fact give the maps in the resolution while our techniques use the Horseshoe Lemma and hence do not easily give the maps.

Kustin has proved that his resolution is minimal in the homogeneous case if the section does not correspond to a minimal generator of the Buchsbaum-Rim module. The latter assumption cannot be removed. Indeed, the resolution predicts that the homogeneous ideal of \( X = S \) has \( r + 1 \) generators, i.e., that \( X \) is an almost complete intersection because codim \( X = r \). Now consider the cotangent bundle of \( \mathbb{P}^2 \). It has a global section whose zero locus is a point in \( \mathbb{P}^2 \) being a complete intersection.

(iii) We suspect that the phenomenon just described is the only instance that prevents our resolution from being minimal. That means, we hope that in case the resolution of \( X \) described above gives the correct number of minimal generators of \( I_X \) then the whole resolution is minimal.

In the theorem above our focus has been on \( X \) rather than on the degeneracy locus \( S \) itself. The interested reader will observe that the same methods provide a resolution for \( S \).

According to Theorem 4.7 we know when \( X \) is arithmetically Gorenstein. In this case its minimal free resolution is self-dual. In order to make this duality transparent we rewrite the resolution above as follows.

**Corollary 5.11.** Let \( B_\varphi \) be a Buchsbaum-Rim sheaf of odd rank \( r \) and first Chern class \( c_1 \). Let \( X \) be the top-dimensional part of a regular section of \( B_\varphi \). Make the following definitions:

If \( i \) is odd, let \( \ell = \min \left\{ \frac{i-1}{2}, \frac{r-i-2}{2} \right\} \), and define

\[
A_i = \bigoplus_{j=0}^{\ell} F^* \otimes S_{\frac{i-1-2j}{2}}(G)^*
\]

If \( i \) is even, let \( \ell = \min \left\{ \frac{i}{2}, \frac{r-i-1}{2} \right\} \), and define

\[
A_i = \bigoplus_{j=0}^{\ell} F^* \otimes S_{\frac{i}{2}}(G)^*
\]
Then $X$ is arithmetically Gorenstein and has a free resolution of the form
\[ 0 \to R(-c_1) \to A_{r-1} \oplus A_1^*(-c_1) \to A_{r-2} \oplus A_2^*(-c_1) \to \cdots \to A_1 \oplus A_{r-1}^*(-c_1) \to I_X \to 0. \]

**Proof.** It follows by Lemma 3.6 that $c_1 = c_1(B_\varphi) = -c_1(B_\varphi^*)$ is the integer satisfying
\[ R(-c_1) \cong \wedge^f F^* \otimes \wedge^g G. \]

Thus Theorem 5.8 provides the claim. \qed

**Example 5.12.** (i) If the rank of $B_\varphi$ is 3, this was treated in [11], where the following resolution was obtained:
\[
\begin{array}{c}
0 \\
\downarrow
\end{array}
\begin{array}{c}
R(-c_1) \\
\downarrow
\end{array}
\begin{array}{c}
G^* \\
\oplus
\end{array}
\begin{array}{c}
F(-c_1) \\
\downarrow
\end{array}
\begin{array}{c}
F^* \\
\oplus
\end{array}
\begin{array}{c}
G(-c_1) \\
\downarrow
\end{array}
\begin{array}{c}
I_X \\
\downarrow
\end{array}
\begin{array}{c}
0
\end{array}
\]

(ii) If the rank of $B_\varphi$ is five, then the corollary gives the following resolution
\[
\begin{array}{c}
0 \\
\downarrow
\end{array}
\begin{array}{c}
R(-c_1) \\
\downarrow
\end{array}
\begin{array}{c}
S_2(G)^* \\
\oplus
\end{array}
\begin{array}{c}
F(-c_1) \\
\downarrow
\end{array}
\begin{array}{c}
F^* \otimes G^* \\
\oplus
\end{array}
\begin{array}{c}
G(-c_1) \\
\oplus \wedge^2 F(-c_1)
\end{array}
\begin{array}{c}
\wedge^2 F^* \\
\oplus
\end{array}
\begin{array}{c}
G^* \\
\oplus
\end{array}
\begin{array}{c}
F \otimes G(-c_1) \\
\downarrow
\end{array}
\begin{array}{c}
F^* \\
\oplus
\end{array}
\begin{array}{c}
S_2(G)(-c_1)
\end{array}
\begin{array}{c}
I_X \\
\downarrow
\end{array}
\begin{array}{c}
0
\end{array}
\]

This resolution has been conjectured in [11].

If we consider a regular section of an even rank Buchsbaum-Rim sheaf we cannot expect to get an arithmetically Gorenstein subscheme as zero locus. Still the resolution has some symmetry and looks very much like the corresponding one for the Gorenstein case.

**Corollary 5.13.** Let $B_\varphi$ be a Buchsbaum-Rim sheaf of even rank $r$ and first Chern class $c_1$. Let $S$ be the zero locus of a regular section of $B_\varphi$. Make the following definitions:

If $i$ is odd, let $\ell = \min \left\{ \frac{i-1}{2}, \frac{r-i-1}{2} \right\}$, and $\ell' = \min \left\{ \frac{i-1}{2}, \frac{r-i-3}{2} \right\}$. Define
\[ A_i = \bigoplus_{j=0}^{\ell} \wedge^{2j+1} F^* \otimes S_{\frac{i-1}{2}}(G)^* \]
\[ B_i = \bigoplus_{j=0}^{\ell'} \wedge^{2j+1} F^* \otimes S_{\frac{i-1}{2}}(G)^* \]
If $i$ is even, let $\ell = \min \left\{ \frac{i}{2}, \frac{r-i}{2} \right\}$, and $\ell' = \min \left\{ \frac{i}{2}, \frac{r-i-2}{2} \right\}$. Define

\[ A_i = \bigoplus_{j=0}^{\ell} F_* \otimes S_{-2j}(G)^* \]

\[ B_i = \bigoplus_{j=0}^{\ell'} F_* \otimes S_{-2j}(G)^* \]

Then $S$ is arithmetically Cohen-Macaulay and has a free resolution of the form

\[ 0 \to R(-c_1) \oplus A_r \to B^*_1(-c_1) \oplus A_{r-1} \to B^*_2(-c_1) \oplus A_{r-2} \to \cdots \to B^*_{r-2}(-c_1) \oplus A_2 \to A_1 \to I_S \to 0. \]

6. Some applications

In the previous section we have seen how much the properties of our degeneracy loci depend on the properties of the Buchsbaum-Rim sheaf. Now we will show how this information can be used to construct schemes with prescribed properties. Moreover, we will explain how sections of the dual of a generalized null correlation bundle can be studied with the help of our results.

Construction of arithmetically Gorenstein subschemes containing a given scheme

Let $X \subset \mathbb{P}^n$ be an equidimensional projective subscheme of codimension $\geq 3$. It is rather easy to find a complete intersection $Y$ such that $Y$ contains $X$ and both have the same dimension. The analogous problem where one requires $Y$ to be arithmetically Gorenstein but not a complete intersection is much more difficult. This is relevant if one wants to study linkage with respect to arithmetically Gorenstein subschemes rather than complete intersections. We want to explain a solution to this problem.

Suppose $X$ has codimension $r$. Let us assume that $r$ is odd. Then we choose a Buchsbaum-Rim sheaf $B_{\varphi}$ of rank $r$ on $\mathbb{P}^n = \text{Proj} R$ given by an exact sequence

\[ 0 \to B_{\varphi} \to F \to G \to M_{\varphi} \to 0. \]

For example, we can take the sheafification of the first syzygy module of an ideal which is generated by $R$-regular sequence of length $r + 1$.

Next we choose a regular section $s \in H^0(B_{\varphi}(j))$ which also belongs to $H^0_*(J_X \otimes F)$. This is possible if $j$ is sufficiently large. Let $S$ be the zero-locus of $s$. Then the top-dimensional part $Y$ of $S$ is arithmetically Gorenstein due to Theorem 4.7. Furthermore, $s \in H^0_*(J_X \otimes F)$ ensures that $s$ vanishes on $X$. It follows that $X \subset Y$ because both are equidimensional schemes of the same dimension.

For example, it was shown in [11] that if $B_{\varphi}$ is the cotangent bundle on $\mathbb{P}^3$, twisted by 3, and $X$ is a set of four distinct points, then a section of $B_{\varphi}$ can be found vanishing on $X$ and giving a Gorenstein scheme, $Y$, of degree 5, whereas the smallest complete intersection containing $X$ has degree 8.

Now assume that $\text{codim } X = r$ is even. Then we take a hypersurface containing $X$ which is defined by, say $f \in R' = K[x_0, \ldots, x_n]$. Put $R = R'/fR'$ and choose a Buchsbaum-Rim sheaf $B_{\varphi}$ of rank $r - 1$ on $Z = \text{Proj} R$. $X$ has codimension $r - 1$ as a subscheme of $Z$. Thus we can find as in the previous case an arithmetically Gorenstein subscheme $Y \subset Z$ containing $X$. We can consider $Y$ also as a subscheme of $\mathbb{P}^n$. As such it is still arithmetically Gorenstein, i.e., it has all the properties we wanted.

Construction of $k$-Buchsbaum schemes

A projective subscheme $X \subset \mathbb{P}^n$ is said to be $k$-Buchsbaum for some non-negative integer $k$ if

\[ (x_0, \ldots, x_n)^k \cdot H^i_*(J_X) = 0 \quad \text{for all } i \leq \dim X. \]

Note that any equidimensional locally Cohen-Macaulay subscheme is $k$-Buchsbaum for some $k$. In fact, one should view the notion of a $k$-Buchsbaum scheme as a refinement of the notion of an equidimensional locally Cohen-Macaulay subscheme. The idea is to develop for such schemes a theory generalizing the one for arithmetically Buchsbaum schemes (cf., for example, [10], [12]).

However, so far there are not many examples available where one knows that they are $k$-Buchsbaum but not $(k-1)$-Buchsbaum. We are going to construct new examples now.
Let \( R = K[x_0, \ldots, x_n] \) and let \( \varphi : R^{n+k}(-1) \to R^k \) be a homomorphism such that \( I(\varphi) = (x_0, \ldots, x_n)^k \). This is true if \( \varphi \) is chosen general enough. A particular choice of such a map is described in [3], p. 15. Let us denote the corresponding Buchsbaum-Rim sheaf by \( B_k \). Observe that \( B_1 \) is just the cotangent bundle of \( \mathbb{P}^n \).

**Proposition 6.1.** Let \( S \) be the degeneracy locus of a morphism \( \psi : \mathcal{P} \to B_k \). If \( X \) has codimension \( n-t+1 \) then it holds: If \( t = 1 \) or \( t = 2 \) and \( n \) is even then \( S \) is arithmetically Cohen-Macaulay; otherwise \( S \) is \( k \)-Buchsbaum but not \( (k-1) \)-Buchsbaum.

**Proof.** The first assertion follows by Theorem 4.5. According to [17], Theorem I.2.10 \( N \) is a Buchsbaum module if and only if the maps \( \varphi^j_N \) are surjective for all \( i \neq \dim N \). A subscheme \( X \subset \mathbb{P}^n \) is called arithmetically Buchsbaum if its homogeneous coordinate ring \( R/I_X \) is Buchsbaum. Now we can show the announced strengthening of the previous result in case \( k = 1 \).

**Proposition 6.2.** Let \( S \) be the degeneracy locus of a morphism \( \psi : \mathcal{P} \to \Omega_{\mathbb{P}^n} \). If \( S \) has codimension \( n-t+1 \) then it is arithmetically Buchsbaum.

**Proof.** We will use again the Eagon-Northcott \( E_\bullet \) complex associated to \( \psi \). Let \( B = H_m^0(\Omega_{\mathbb{P}^n}) \) and put \( A = R/I_S \).

We want to show that \( \varphi^j_A \) is surjective if \( j \neq t = \dim A \). This is clear if \( H_m^0(A) \) vanishes. Let \( j = n+t-2i < t \) be an integer such that \( H_m^0(A) \neq 0 \). Using the exact sequences in the proof of Proposition 1.2 we get diagrams

\[
\begin{align*}
\text{Ext}^{n+t-2i}(K, A)(p) & \to \text{Ext}^{n+1-i}_R(K, \text{im}\, \delta_i) \\
\downarrow \varphi^{n+t-2i} & \downarrow \varphi^{n+1-i}_m \\
H^{n+t-2i}_m(A)(p) & \to H^{n+1-i}_m(\text{im}\, \delta_i)
\end{align*}
\]

and

\[
\begin{align*}
\text{Ext}^{n+1-i}_R(K, E_i) & \to \text{Ext}^{n+1-i}_R(K, \text{im}\, \delta_i) \\
\downarrow \varphi^{n+1-i}_E & \downarrow \varphi^{n+1-i}_m \\
H^{n+1-i}_m(E_i) & \to H^{n+1-i}_m(\text{im}\, \delta_i)
\end{align*}
\]

They are commutative because the vertical maps are canonical. Moreover, we have seen in the proof of Proposition 1.2 that the lower horizontal maps are isomorphisms.

Now it is well-known that the modules \( \wedge^q B^* \) are Buchsbaum modules if \( 1 \leq q \leq n \). Thus the modules \( E_q \) are Buchsbaum, too. Hence the diagrams show that the surjectivity of \( \varphi^{n+1-i}_E \) implies this property first for \( \varphi^{n+1-i}_{\text{im}\, \delta_i} \) and then for \( \varphi^{n+t-2i} \). It follows that \( A \) is Buchsbaum.

In the special case that \( \mathcal{P} \) has rank \( t = n-1 \) the last result is also contained in [3]. If \( t < n-1 \) our result is a little surprising. In fact, the main result of [3] has been generalized in [13], Corollary II.3.3. It says that arithmetically Buchsbaum submodules of arbitrary codimension can be characterized by means of a particular locally free resolution. As a consequence, every arithmetically Buchsbaum subscheme of \( \mathbb{P}^n \) is the zero-locus of a global section of a vector bundle which is the direct sum of exterior powers of the cotangent bundle.

**Some vector bundles of low rank and their sections**

Let \( R \) be again a graded Gorenstein \( K \)-algebra of dimension \( n+1 \). We assume that \( n \geq 3 \) is an odd integer. The aim of this subsection is to show that a vector bundle arising from a Buchsbaum-Rim sheaf
by quotienting out non-vanishing sections can be studied by means of our results. Then we construct vector bundles of rank $n - 1$ on $Z = \text{Proj} R$ and apply this principle to sections of them.

Let $\mathcal{B}_\varphi$ be a Buchsbaum-Rim sheaf on $Z$ having global non-vanishing sections such there is an exact sequence

$$0 \to \mathcal{Q} \to \mathcal{B}_\varphi \to \mathcal{E} \to 0$$

where $H^0(Z, \mathcal{Q})$ is a free $R$-module of rank $u$ and $\mathcal{E}$ a vector bundle on $Z$ of rank $r - u$.

Now, we want to consider a morphism $\varphi : \mathcal{P} \to \mathcal{E}$ dropping rank in the expected codimension $r - u - t + 1$. This morphism can be lifted to a morphism $\beta : \mathcal{P} \to \mathcal{B}_\varphi$ which provides a morphism $\alpha = (\beta, \gamma) : \mathcal{P} \boxtimes \mathcal{Q} \to \mathcal{B}_\varphi$. Since the degeneracy locus of $\gamma$ is empty, $\mathcal{B}_\varphi^{\beta}$ is locally the direct sum of $\mathcal{E}^\alpha$ and $\mathcal{Q}^\alpha$. It follows that the images of $\land^j \mathcal{E}^\alpha$ and $\land^{j+n} \mathcal{Q}^\alpha$ are locally isomorphic. Hence the degeneracy locus $S$ of $\psi$ and the degeneracy locus of $\alpha$ agree. Thus $\alpha$ drops rank in the expected codimension too and we can apply our previous results.

Next, we construct a class of vector bundles which contains the duals of null correlation bundles. To this end let $I = (f_0, \ldots, f_n) \subset R$ be a complete intersection. Let $d_i = \deg f_i$. The first syzygy module of $I$ defines a Buchsbaum-Rim module $\mathcal{B}_\varphi$ which fits into the exact sequence

$$0 \to B_\varphi \to \bigoplus_{i=0}^n R(-d_i) \xrightarrow{\nabla} R \to R/I \to 0.$$ 

The Buchsbaum-Rim sheaf $\mathcal{B}_\varphi = \mathcal{\tilde{B}}_\varphi$ can often be used to construct a vector bundle of rank $n - 1$ on $Z$.

**Proposition 6.3.** Suppose there is an integer $c$ such that the degrees satisfy

$$c = d_0 + d_1 + d_3 = \ldots = d_{n-1} + d_n.$$ 

Then $\mathcal{B}_\varphi(c)$ admits a non-vanishing global section $s$ which gives rise to an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-c) \xrightarrow{s} \mathcal{B}_\varphi \to \mathcal{N} \to 0$$

where $\mathcal{N}$ is a vector bundle of rank $n - 1$ on $Z$.

**Proof.** The Koszul relation of the generators $f_i$ and $f_{i+1}$ of $I$ gives rise to a global section of $\mathcal{B}_\varphi(-d_i-d_{i+1})$. Taking the sum over these sections with even $i$ yields a section $s$ which does not vanish on $Z$ since $I$ is an $m$-primary ideal. \qed

In case $Z = \mathbb{P}^n$ and $d_0 = \ldots = d_n = 1$ the bundle $\mathcal{B}_\varphi$ is the cotangent bundle of $\mathbb{P}^n$ and $\mathcal{N}^*$ is called null correlation bundle in [14] where on p. 79 it is constructed in a slightly different way. If $n = 3$ then $\mathcal{N}$ is self-dual. Thus we call the dual of a vector bundle constructed as in the proposition above generalized null correlation bundle. Due to the principle described above our previous results apply to multiple sections of the dual of a generalized null correlation bundle. We obtain, for example.

**Corollary 6.4.** The degeneracy locus of a multiple section of the dual of a null correlation bundle is arithmetically Buchsbaum but not arithmetically Cohen-Macaulay.

This result is well-known if $n = 3$. In fact, in this case the null correlation bundle can be constructed (via the Serre correspondence) as an extension

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{N} \to \mathcal{J}_X \to 0$$

where $\mathcal{J}_X$ denotes the ideal sheaf of two skew lines in $\mathbb{P}^3$ (cf. [1], p. 145 or [7]).

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