Equations of Motion as Covariant Gauss Law: The Maxwell-Chern-Simons Case

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Time-independent gauge transformations are implemented in the canonical formalism by the Gauss law which is not covariant. The covariant form of Gauss law is conceptually important for studying asymptotic properties of the gauge fields. For QED in 3+1 dimensions, we have developed a formalism for treating the equations of motion (EOM) themselves as constraints, that is, constraints on states using Peierls’ quantization. They generate spacetime dependent gauge transformations. We extend these results to the Maxwell-Chern-Simons (MCS) Lagrangian. The surprising result is that the covariant Gauss law commutes with all observables: the gauge invariance of the Lagrangian gets trivialized upon quantization. The calculations do not fix a gauge. We also consider a novel gauge condition on test functions (not on quantum fields) which we name the “quasi-self-dual gauge” condition. It explicitly shows the mass spectrum of the theory. In this version, no freedom remains for the gauge transformations: EOM commute with all observables and are in the center of the algebra of observables.

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I. INTRODUCTION

The Abelian Maxwell-Chern-Simons (MCS) theory \([2]\) is a theory of a massive “photon” in 2 + 1 dimensions. It violates parity, \(P\), and time-reversal, \(T\). The Lagrangian has \(U(1)\) gauge invariance, but it is absent in the final Hamiltonian.

Our focus is on the fate of this \(U(1)\) gauge group. We will see that it has a trivial action on the connection potentials \(A_\mu\) after covariant quantization and that the operator which generates them is the operator which implements EOM by vanishing on quantum states \([1]\): the gauge symmetry of the Lagrangian disappears on quantization.

This approach which does not impose gauge conditions on \(A_\mu\) will be contrasted with an alternative approach which is also new and does not fix the gauge of \(A_\mu\). It is covariant and quickly shows why \(A_\mu\) has mass. It is not \(P\) and \(T\) invariant and also does not lead to EOM as constraints which generates gauge transformations. The EOM are actually in the center of the algebra of observables in both of these approaches.

We interpret EOM as generalized covariantized Gauss laws. This is reasonable: a component of the Maxwell equation, say \(\partial^\mu F_{\mu 0} = 0\), for the field strength \(F\) is in fact the Gauss law. The collection of such component Gauss laws with regard to every Cauchy surface and their superpositions give the field equations. This justifies our assumptions.

Let \(A_\mu\) be a vector field in 2 + 1 dimensions and consider the action, with \(\eta_{\mu\nu} = \text{diag}(-,+,+),\)

\[
S = \int d^3x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{ke^2}{4\pi} \varepsilon^{\mu\nu\sigma} A_\mu \partial_\nu A_\sigma \right),
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]

It gives the equations of motion (EOM)

\[
\partial^\mu F_{\mu\nu} + mF_\nu = 0, \quad m \equiv \frac{ke^2}{2\pi},
\]

where

\[
F_\nu = \varepsilon_{\nu\mu\sigma} \partial^\mu A^\sigma.
\]

Notice that

\[
\partial^\mu F_\mu = 0.
\]

\footnote{Many of the equations were supplied to A.P.B. by V.P. Nair.}
Writing $F_{\mu\nu}$ in terms of $F_{\mu}$, we obtain

$$-\varepsilon_{\mu\nu\sigma} \partial^\sigma F^\nu + mF^\mu = 0. \tag{6}$$

Applying $\varepsilon^{\mu\lambda\rho} \partial_\rho$, using (5) and also (6) to eliminate the term with $\varepsilon_{\mu\nu\sigma}$, we obtain

$$\Box F_\mu + m^2 F_\mu = 0, \quad \Box \equiv \partial_\nu \partial^\nu, \tag{7}$$

so that, as it is well-known, MCS describes a massive photon $F_\mu$.

II. THE CAUSAL COMMUTATOR

We give the causal commutator in the Lorentz gauge,

$$\partial^\mu A_\mu = 0, \tag{8}$$

so that

$$\Box A_\mu + m\varepsilon_{\mu\nu\sigma} \partial^\nu A^\sigma = 0. \tag{9}$$

Then, the causal commutator is

$$D_{\mu\nu}(x - y) \equiv [A_\mu(x), A_\nu(y)] = \int_C \frac{d^3p}{(2\pi)^3} e^{-ip\cdot(x-y)} M_{\mu\nu}(p), \tag{10}$$

where

$$M^{\mu\nu}(p) = \frac{i}{p^2 - m^2} \left( \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - im\varepsilon^{\sigma\mu\nu} \frac{p_\sigma}{p^2} \right), \tag{11}$$

and

$$\partial_\nu D^{\mu\nu}(x - y) = 0. \tag{12}$$

The contour $C$ encloses the poles at

$$p_0 = \pm \sqrt{\vec{p}^2 + m^2}. \tag{13}$$

The novelty in this paper is that we will not use the gauge condition (10) below. Rather we work with field $A(\eta)$ smeared with smooth test-functions $\eta_\mu$, which are compactly supported,

$$A(\eta) = \int d^3 x \, \eta^\mu(x) A_\mu(x), \quad \eta^\mu \in C_0^\infty(\mathbb{R}^3), \tag{14}$$

$$\partial^\mu \eta_\mu = 0. \tag{15}$$
Here the zero subscript denotes compact support and infinity infinite differentiability. As Roepstorff [3] has discussed, (12) is gauge invariant by partial integration,

\[(A + \partial_\Lambda) (\eta) = A(\eta), \quad \text{for} \quad \Lambda \in \mathcal{C}_0^\infty(\mathbb{R}^3).\]  

(16)

The algebra of \(A(\eta)\) is inferred from (10) as

\[[A(\eta_1), A(\eta_2)] = \int d^3x d^3y (\eta_1)^\mu (x) D_{\mu\nu} (x-y) (\eta_2)^\nu (y),\]  

(17)

with \(\eta_i^\mu \in \mathcal{C}_0^\infty(\mathbb{R}^3)\) and \(\partial_\mu \eta_i^\mu = 0, \ i = 1, 2.\)

The algebra with commutator (17) defined by the local observables \(A(\eta)\) defines MCS. It involves no gauge fixing of \(A\).

III. EOM AS CONSTRAINTS

The classical equations of motion are (3). We smear them with test function \(\rho^\mu \in \mathcal{C}^\infty(\mathbb{R})\) and write them as an equation involving no derivatives of \(A_\mu\). This is appropriate since we should write derivatives of distribution \(A_\mu\) at the quantum level as derivatives of test functions \(\rho^\mu\).

Let us introduce the notations

\[\hat{F}_{\mu\nu} (\rho) = \partial_\mu \rho_\nu - \partial_\nu \rho_\mu,\]  

(18)

\[\hat{F}_{\mu\nu} (\rho)(x) = \partial_\mu \rho_\nu (x) - \partial_\nu \rho_\mu (x).\]  

(19)

We do not insist on requiring \(\partial_\mu \rho^\mu = 0\). Multiplying (3) by \(\rho^\nu\) and integrating, we obtain classically the equations

\[G[\rho] \equiv \int \left( \partial^\mu \hat{F}_{\mu\sigma} (\rho) + m \varepsilon_{\sigma\mu\nu} \partial_\mu \rho_\nu \right) A^\sigma = 0.\]  

(20)

We regard the LHS at the quantum level as an operator \(G[\rho]\) which vanishes on allowed quantum states:

\[G[\rho] \cdot \rangle = \int d^3x \left( \partial^\mu \hat{F}_{\mu\sigma} + m \varepsilon_{\sigma\mu\nu} \partial_\mu \rho_\nu \right) A^\sigma (x) \cdot \rangle = 0.\]  

(21)

This defines the domain of the observables \(A(\eta)\).

Note that even though \(\rho^\mu\) does not fulfill Lorentz gauge, the function in (21) multiplying \(A^\sigma\) does and is a proper test function for \(A^\sigma\).

We must show that \(G[\rho]\)'s are first class. That result follows below.
IV. THE COMMUTATOR $[G[\rho], A_\sigma(y)]$

We find that the commutator $[G[\rho], A_\beta(y)]$ is identically zero. It implies that $G[\rho]$'s commute for different $\rho$ and hence are first class constraints.

We have

$$[G[\rho], A_\sigma(y)] = \int d^3 x \left( \partial_\mu \hat{F}^{\mu\kappa}[\rho] + m \varepsilon^{\kappa\mu\nu} \partial_\mu \rho_\nu \right) D_{\kappa\sigma}(x - y).$$  \hspace{1cm} (22)

Now,

$$\partial_\mu \hat{F}^{\mu\kappa}[\rho] = \square \rho^\kappa - \partial^\kappa (\partial \cdot \rho),$$  \hspace{1cm} (23)

and then under the integration the second term $\partial^\kappa (\partial \cdot \rho)$ vanishes due to partial integration and use of $\partial^\mu D_{\mu\sigma} = 0$.

As for the remaining terms, we can write (22) as

$$[G[\rho], A_\sigma(y)] = \int d^3 x \left( \square_\mu \rho^\kappa + m \varepsilon^{\kappa\mu\nu} \partial_\mu \rho_\nu \right) \left[A_\kappa(x), A_\sigma(y)\right]$$

$$= \int d^3 x \left[ \left( \square_\mu \rho^\kappa + m \varepsilon^{\kappa\mu\nu} \partial_\mu \rho_\nu \right) A_\kappa(x), A_\sigma(y) \right],$$  \hspace{1cm} (24)

where the subscript $x$ means differentiation with respect to $x$. Now, this expression vanishes after integration by parts and use of (9). Therefore,

$$[G[\rho], A_\sigma(y)] = 0.$$  \hspace{1cm} (25)

V. A NOVEL GAUGE CONDITION

The test function $\rho$ (unlike $\eta^\mu$ satisfying $\partial_\mu \eta^\mu = 0$) is not so far subjected to any gauge condition. Nor is $A_\mu$. We will now “gauge fix” the test functions by imposing the condition

$$\rho_\mu = m \varepsilon_{\mu\nu\sigma} \partial^\nu \rho^\sigma.$$  \hspace{1cm} (26)

It implies that $\rho_\mu$ itself is transverse,

$$\partial^\mu \rho_\mu = 0.$$  \hspace{1cm} (27)

The condition (26) is not gauge invariant and hence is a gauge fixing condition.

From the condition (26) we have the following facts:

1. The result that $A_\mu$ describes massive vector bosons becomes explicit;
2. The EOM *commutes* as before with all $A_\mu$ and does not generate gauge transformations. It is in the centre of the algebra of observables.

As for item (1), we can look at (21) and impose (26). That gives

$$G[\rho] |\cdot\rangle = \int d^3x \left( \Box \rho^\kappa + m^2 \rho^\kappa \right) A_\kappa |\cdot\rangle = 0,$$

which on partial integration gives the result

$$\left( \Box + m^2 \right) A_\rho = 0,$$

classically. So $A_\rho$ has mass $m$.

As for item (2), (25) is true for any choice of $\rho$, hence the fact follows.

**VI. SIGNIFICANCE**

The link between EOM and gauge transformations seems significant. It has not been discussed previously prior to [1]. It has now turned up in QED, linearised gravity [5] and Maxwell-Chern-Simons theory. This link is of course present in $U(1)$ gauge theories in all dimensions.

In non-abelian gauge theories, like QCD, the commutator of the fields $A_\mu$ at distinct points $x$ and $y$ is not known due to the non-linearity of the field equations. We are hence not able to analyse this case in the present framework.

**VII. FURTHER PROBLEMS**

In Dirac’s approach to constrained dynamics, one distinguishes between the first and second class constraints. Often gauge fixing conditions are introduced to turn the former into the second class. Second class constraints can be eliminated using Dirac-Bergmann brackets [4]. All of these happen on a Cauchy hypersurface, that is, at a fixed time.

In this paper, we have introduced EOM as first class constraints. It is natural to ask: Is there an analogous theory of constraints in this spacetime picture? This appears to be an
open interesting problem.

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[2] G. V. Dunne, [hep-th/9902115] and references therein.

[3] G. Roepstorff, Commun. Math. Phys. 19 (1970) 301-314.

[4] A. P. Balachandran, G. Marmo, B.-S. Skagerstam and A. Stern, [arXiv:1702.08910] [quant-ph].

[5] A. P. Balachandran et al., to be published.