Non - relativistic string in Newton - Cartan background

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1 Introduction

Gravitation is the weakest of the fundamental forces of nature. Notwithstanding this, the physics in the large scale is overwhelmingly influenced by the gravitational interaction. The reason is, gravitation is always attractive. Einstein in his celebrated General (Theory of) relativity (GR) formulated the gravitational interaction as the change of curvature of spacetime. As a result free particles follow the shortest path (geodesic) in this curved manifold. So Einstein’s equations that determine the curvature produced by a certain mass energy distribution. The particle (or string) under study are the test particles. The test particles are so chosen that they do not disturb the geometry in any way \cite{1}. This framework should not be violated in any limiting procedure. The non relativistic theory which we are considering should not be an exception.

Physical theories are based on space and time. In Newtonian concept time flows from past to future in an absolute sense and the physical space is relative. The geometry of space is euclidean and one can set up a Cartesian space system in which the space metric is $\delta_{ij}$. This is the non relativistic flat space. Gravitation of massive bodies produce curvature \cite{1} in the free space. The similarity and differences with relativity theory are obvious. Also, the more accurate theory of special relativity in the non relativistic limit ($v << c$) should agree with non relativistic results. The boost invariant non relativistic spacetime should also follow as a limit of the relativistic Minkowski spacetime. Indeed, one can demonstrate the transition by examining the structure of the metric in the $c \rightarrow \infty$ limit to pass into a collection of singular metrics of rank 1 and 3 respectively \cite{26}. Different non relativistic geometries may follow from the limiting procedure.

Both classical theory of gravitation and Einstein’s general relativity (GR) obey the principle of equivalence by dint of which gravitational effects may be annulled locally in the $\epsilon$ - neighborhood of a point. This was exploited by Cartan to discover the geometric formulation of Newtonian gravity. The corresponding space time manifold is known as Newton - Cartan spacetime \cite{1}. Research on the Newton Cartan spacetime is continued over a long time \cite{9}. However, these theories are mostly formulated as second order metric theory, following GR. The vielbein (tetrad ) formulation of the NC gravity was less emphasized. Some authors did use the vielbein structures but an out and out discussion in terms of the vielbeins and spin connections is scarcely found in the literature.
The scenario has changed in the last couple of decades when non-relativistic field theories were found to be important in the context of strongly correlated electrons in mesoscopic physics and soft condensed matter systems [24]. The requirements are now coupling different particle and field theories to non-relativistic curved space-time. This coupling is very simple for the relativistic theories. One has to replace the ordinary derivatives by the corresponding covariant derivatives and replace the measure of integration by the invariant measure. Now substituting the Minkowski metric $\eta_{\mu\nu}$ by the metric $g_{\mu\nu}$ of the curved spacetime. In case of the non-relativistic geometry the metric is degenerate. There is a spatial metric $h_{\mu\nu}$ of rank three and a one form $\tau_\mu$. Since the metric structure is singular they cannot be inverted. Thus the theory formulated by vielbeins become important. Now in GR, the metric can be factored into the vielbeins. This is not available in the non-relativistic counterpart. Rather, the formulation in the non-relativistic spacetime is far easier in the first order approach with the geometry given by the tetrad and the spin connections, as their introduction do not depend upon the metric. So the question is how to formulate the first order theory, without taking recourse of the metric?

Consider a theory invariant under the Poincare transformations in the Minkowski space. An interesting correspondence is unveiled by localizing symmetry transformation parameters of a theory under the Poincare group of transformations in the Minkowski space. Fields would transform formally in the same way but the derivatives transform differently as the parameters now are no more constants. The symmetry is regained when appropriate gauge fields are included which transform appropriately. These transformation of the fields have one to one correspondence with the vielbeins and spin connections of the Riemann-Cartan manifold. The theory thus emerges is the famous Poincare gauge theory (PGT)\(^1\). Naturally, one would like to apply the idea of PGT to the non-relativistic arena. In the non-relativistic problem, as we have pointed out already, the metric is degenerate. The gauging of symmetry approach seems to be an elegant and robust method for coupling different matter theories with the curved background produced by gravity. This was first demonstrated for the nonrelativistic Schrödinger field in [10] which resolved the paradoxes in the treatment of the same problem elsewhere [13]. Soon the connection of the localisation process with the emergence of the Newton-Cartan geometry was shown [12] and extension of the algorithm to include external electromagnetic interaction was presented [11, 10]. Finally the gauging of the symmetry approach was formalised as Galilean gauge theory (GGT) [10]. Note that the Galilean symmetry is inbuilt in the algorithm of GGT. This new theory is suitable for the purpose due to the following features:

1. The algorithm developed in the theory can be useful for any theory which is symmetric in flat Galilean spacetime. The method of approach automatically carries the symmetry along with it. Thus failure to reproduce the flat theory in the appropriate limit, so common in other approaches, has no place in this approach.

2. The whole calculations are done by a set of rules, derived once for all. There is no fine adjustments during the calculation of a particular problem.

\(^1\)Of course the target of PGT was to provide a quantum theory of gravity which is so far unachieved.
3. The spacetime emerging from our analysis is the Newton- Cartan spacetime. So far we have considered examples from field theory, particle models. In this paper we will provide our results for string model.

2 Action for Non relativistic Particle in Curved Background

We start with a short review of the calculations and results for the nonrelativistic particle model treated by the Galilean gauge theory (GGT) \[33\]. The parametrized action for a non relativistic particle in 3 dimensional Euclidean space and absolute time is given by,

\[
S = \int \frac{1}{2} m \left( \frac{d\lambda}{dx^0} \right) d\lambda
\]

(1)

where the index \( a \) denotes a space index. The action is invariant under the global Galilean transformations,

\[
x^\mu \to x^\mu + \xi^\mu; \xi^0 = -\epsilon, \xi^k = \eta^k - v^k t; \eta^k = \omega^k_l x^l + \epsilon^k
\]

(2)

This invariance is ensured by the transformations

\[
\delta \frac{dx^0}{d\lambda} = \frac{d}{d\lambda} (\delta x^0) = -\frac{d\epsilon}{d\lambda} = 0
\]

(3)
as \( \epsilon \) is constant and,

\[
\delta \frac{dx^k}{d\lambda} = w^j_k \frac{dx^j}{d\lambda} - v^k \frac{dx^0}{d\lambda}
\]

(4)

which can be checked easily. The Lagrangian \( (1) \) changes by,

\[
\delta L = - \frac{d}{d\lambda} \left( m v^k \frac{dx^k}{d\lambda} \right)
\]

(5)
due to \( (3) \) and \( (4) \). The change of the action \( (1) \) is then a boundary terms only, see \( (5) \). The same equations of motion follow from both the original and the transformed action. So the theory is invariant under the global Galilean transformations.

To localize the symmetry of the action \( (1) \) according to GGT, \( \frac{dx^\alpha}{d\lambda} \) is now substituted by the covariant derivatives \( \frac{Dx^\alpha}{d\lambda} \), where

\[
\frac{Dx^\alpha}{d\lambda} = \frac{dx^\nu}{d\lambda} \Lambda^\beta_\nu \partial_\beta x^\alpha = \frac{dx^\nu}{d\lambda} \Lambda^\alpha_\nu
\]

(6)
Here $\Lambda^{\beta}_{\nu}$ are a set of new compensating (gauge) fields. This covariant derivative was elaborately derived. We are not repeating it here. Instead let us stress that this will act as a building block for the string, as the latter is viewed here as an extension of the covariant particle model in [33], where it was derived first in this approach. Note that the localisation procedure can be tuned smoothly to restore global Galilean symmetry. In this limit the covariant derivatives $\frac{dx^\alpha}{d\lambda}$ must then go to the ordinary derivatives $\frac{dx^\alpha}{d\lambda}$. This sets the following condition

$$\Lambda^\alpha_{\mu} \longrightarrow \delta^\alpha_{\mu} \quad (7)$$

This at once shows that $\Lambda^\alpha_{\mu}$ is non singular i.e. the corresponding matrix is invertible. We will denote the inverse by $\Sigma^\mu_{\alpha}$. We will further see that this observation is instrumental in the geometric interpretation of our theory.

Now the transformations of the new gauge fields should ensure that the 'covariant derivatives' will transform under the local Galilean transformations in the same form as the usual derivatives do under the global Galilean transformations. Then the new theory obtained by replacing the ordinary derivatives by the 'covariant derivatives' will be invariant under the local Galilean transformations. This is the essence of GGT [10, 12]. Using (3) and (4) the transformation of the covariant derivatives follows,

$$\delta \frac{Dx^0}{d\lambda} = 0 \quad (8)$$

and, likewise for the space part,

$$\delta \frac{Dx^k}{d\lambda} = w^j_k \frac{Dx^j}{d\lambda} - v^k \frac{Dx^0}{d\lambda} \quad (9)$$

Exploiting these relations the transformations of the newly introduced fields are completely specified. They are given by [12],

$$\delta \Lambda^0_0 = \dot{\epsilon} \Lambda^0_0$$
$$\delta \Lambda^a_i = \omega^a_k \Lambda^b_i - \partial_i \xi^k \Lambda^a_k$$
$$\delta \Lambda^0_a = \dot{\epsilon} \Lambda^a_0 - \nu^a \Lambda^0_0 - \partial_0 \xi^k \Lambda^a_k + \omega^a_0 \Lambda^b_0 \quad (10)$$

while the remaining field $\Lambda^a_i$ simply vanishes.

Hence the cherished action is given by,

$$S = \int \frac{1}{2} m \left( \frac{Dx^a}{d\lambda} \right) \left( \frac{Dx^a}{d\lambda} \right) d\lambda \quad (11)$$

obtained from (1), substituting $\frac{dx^a}{d\lambda}$ by $\frac{Dx^a}{d\lambda}$. The above derivation is the standard procedure of GGT which leads to the theory (11) invariant under transformations formally similar to (2).
But the parameters of transformations are not constants but vary in a special manner with space and time, completely in unison with the privileged role of time in Newtonian theory.

3 Non-relativistic Nambu - Goto action for the bosonic string

The string is an extension of the particle model. Both are relativistic objects \[33\] and a particular type of nonrelativistic limit is to be taken to obtain the coupling of the nonrelativistic theories with gravity. The algorithm of GGT is remarkable successful in formulating the coupling in different particle models \[10\]. So one would expect that the same algorithm will be useful for coupling the strings though we must be very careful on account of some very important differences between the two models. Unlike the particle which is represented by a point, the string is an one dimensional object which is described by a parameter \(\sigma\). So during its evolution it traces a two dimensional world sheet. This surface defines a two dimensional foliation of space time. It is mapped by two coordinates, \(\tau\) and \(\sigma\), where \(\tau\) is time-like and \(\sigma\) is space-like. As the string is essentially relativistic, the low energy excitation do not affect the string world sheet. The relativistic Nembu Goto action of the bosonic string is given by,

\[
S_{NG} = -N \int d\sigma d\tau \sqrt{-\det h_{ij}}
\]  

(12)

where \(h_{ij}\) is the metric induced by the target space and given by,

\[
h_{ij} = \eta_{\mu\nu} \partial_i X^\mu \partial_j X^\nu
\]  

(13)

where \(X^\mu = X^\mu(\tau, \sigma)\) and \(\eta_{\mu\nu} = \text{diag}1, -1, -1,...\) is the Lorentzian metric in the target space. String (12) is a relativistic object but if we consider low energy phenomenology, then the effects in the target space are non relativistic. Let the dimension of the embedding space be (D+1). At a given time the string intersects the embedding space along a line. We take this line as the \(X^1\) coordinate line. Then \(X^0\) and \(X^1\) are longitudinal to the string and the rest is transverse. Note that we could take any of \(X^2,...... X^D\) in place of \(X^1\). So no particular gauge choice is associated with this. By taking \(c \to \infty\) limit, we get the non relativistic Lagrangian,

\[
\mathcal{L}_{NG} = -N \left(2\epsilon_{\mu\nu} \dot{X}^\mu \dot{X}^\nu\right)^{-1} \left(\dot{X}^\mu X^{\nu} - \dot{X}^\nu X^{\mu}\right)^2
\]  

(14)

and the action is given by

\[
S_{NG} = \int d\sigma d\tau \mathcal{L}_{NG}
\]  

(15)

with \(\mathcal{L}_{NG}\) given by (14) is Galilean invariant. The non relativistic lagrangian (14) can be written as
\[ \mathcal{L}_{NG} = -N \left( \epsilon_{\mu\sigma\alpha\beta} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^\nu}{\partial \sigma_\beta} \right)^{-1} \left( \epsilon_{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^a}{\partial \sigma_\beta} \right)^2 \] (16)

Where \( \alpha, \beta = 0, 1 \) and \( \sigma_1 = \tau \) and \( \sigma_2 = \sigma \).

With this we finish the review of the method applied to a generally covariant particle model and introduce the main results of our construction of the string (bosonic) action following the particle action, we are in a position to introduce gravity. The string being an essentially relativistic object, the low energy excitations of the string are confined in the direction transverse to the string world sheet. The curvature produced by gravity are only to be considered. We describe the procedure to non-relativistic.

4 Non-relativistic bosonic string in curved background from galilean gauge theory

Coupling a Non relativistic string theory with gravity has been found to be a difficult task because of the peculiar geometry of the non-relativistic string embedded in curved space time. The string is inherently a relativistic object. So the 2 dimensional slice of the space time produced by the motion of the sting (the world sheet) has a Minkowski metric structure where as the transverse bulk has an Euclidean structure. The point is that the non relativistic description for the string is relevant for low energy excitations, thus in interaction with gravity the world sheet is not affected. Remember that in formulating the non relativistic action for flat space time we have enforced the condition \( \omega^{1a} = 0 \), where \( \omega \) is the spatial rotation parameter, the coordinate axis \( X^1 \) is longitudinal and \( X^a \) are transverse to world sheet.

Once the peculiarity of the non relativistic string geometry is understood it is simple to write down the corresponding action for such string coupled with curved manifold, thanks to the algorithm of GGT. We will thus replace the ordinary derivative by covariant derivative. Note that this replacement is with in the transverse part of the manifold.

Hence the lagrangian of non relativistic bosonic string (16) in curved background is given by

\[ \mathcal{L}_{NG} = -N \left( \epsilon_{\mu\sigma\alpha\beta} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^\nu}{\partial \sigma_\beta} \right)^{-1} \left( \epsilon_{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{DX^a}{d\sigma_\beta} \right)^2 \] (17)

Which is a direct generalization of the nonrelativistic particle model [33]. Explicitly
\[
\frac{DX^a}{d\sigma_\beta} = \frac{dX^\rho}{d\sigma_\beta} \Lambda^a_{\rho}
\]
\[
\frac{\delta DX^a}{d\sigma_\beta} = \frac{DX^j}{d\sigma_\beta} \omega^a_j - \frac{DX^0}{d\sigma_\beta} u^a 
\]
\[
\frac{\delta \partial X^\mu}{\partial \sigma_\alpha} = \frac{\partial X^\nu}{d\sigma_\alpha} \omega^\mu_\nu 
\]

(18)

Note that those degrees of freedom transverse to the string world sheet is interacted by the nonrelativistic gravity. We have already discussed this issue in the above.

5 General covariance of the theory

Our next task is to show that the (17) is invariant under local Galilean transformation. This is equivalent to that the diffeomorphism invariance

\[
X^\mu \rightarrow X^\mu + \xi^\mu 
\]

where

\[
\xi^\lambda = \epsilon^\lambda + w^\lambda_\omega x^\omega, \quad \lambda, \omega = 0, 1 \\
\xi^a = \epsilon^a + w^a_j x^j - u^a t, \quad a = 2 \ldots \ldots D. 
\]

(20)

holds.

Since the transformation parameters are arbitrary functions of coordinates and time the diffeomorphism (19) is totally arbitrary. The variation of Lagrangian due to the spacetime diffeomorphism is,

\[
\delta \mathcal{L}_{NG} = -N \left( \epsilon_{\mu\nu} \sigma_{\alpha\beta} \right) \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^\nu}{\partial \sigma_\beta} \left[ \delta \left( \epsilon_{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma_\alpha} \right) \right] \frac{\partial X^a}{d\sigma_\beta} 
\]

\[
- N \left[ \delta \left( \epsilon_{\mu\nu} \sigma_{\alpha\beta} \right) \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^\nu}{\partial \sigma_\beta} \right]^{-1} \left( \epsilon_{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{DX^a}{d\sigma_\beta} \right)^2 
\]

(21)

Now the calculation of (21) is very simple but quite lengthy, so we calculate it in term by term. The 2nd term of the (21)
\[
\delta \left( \epsilon_{\mu\nu} \sigma_{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^\nu}{\partial \sigma_\beta} \right)^{-1} = -\epsilon_{\mu\nu} \sigma_{\alpha\beta} \left[ \epsilon_{\mu\nu} \sigma_{\alpha\beta} \left( \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^\nu}{\partial \sigma_\beta} \right) \right] \left( \epsilon_{\mu\nu} \sigma_{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^\nu}{\partial \sigma_\beta} \right)^{-2} - \epsilon_{\mu\nu} \sigma_{\alpha\beta} \left[ \left( \omega^\mu_{\nu} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^\nu}{\partial \sigma_\beta} + \omega^\nu_{\mu} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^\nu}{\partial \sigma_\beta} \right) \right] \left( \epsilon_{\mu\nu} \sigma_{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^\nu}{\partial \sigma_\beta} \right)^{-2}
\]

as \( \epsilon_{\mu\nu} \) is anti symmetric thus

\[
\delta \left( \epsilon_{\mu\nu} \sigma_{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{\partial X^\mu}{\partial \sigma_\beta} \right)^{-1} = 0
\]

now 1st term of the (24)

\[
\delta \left( \epsilon_{\alpha\beta} \frac{\partial X^\alpha}{\partial \sigma_\alpha} \frac{DX^a}{d\sigma_\beta} \right)^2 = \left( 2\epsilon_{\alpha\beta} \frac{\partial X^\alpha}{\partial \sigma_\alpha} \frac{DX^a}{d\sigma_\beta} \right) \left[ \delta \left( \frac{\partial X^\mu}{\partial \sigma_\alpha} \right) \frac{DX^a}{d\sigma_\beta} \epsilon_{\alpha\beta} + \delta \left( \frac{DX^a}{d\sigma_\beta} \right) \frac{\partial X^\mu}{\partial \sigma_\alpha} \epsilon_{\alpha\beta} \right] = \left( 2\epsilon_{\alpha\beta} \frac{\partial X^\alpha}{\partial \sigma_\alpha} \frac{DX^a}{d\sigma_\beta} \right) \left[ \left( \frac{\partial X^\mu}{\partial \sigma_\alpha} \right) \frac{DX^a}{d\sigma_\beta} \epsilon_{\alpha\beta} + \left( \frac{DX^a}{d\sigma_\beta} \right) \frac{\partial X^\mu}{\partial \sigma_\alpha} \epsilon_{\alpha\beta} \right]
\]

putting the value of \( \frac{DX^\alpha}{d\sigma_\beta} \) and \( \frac{DX^0}{d\sigma_\beta} \) we get

\[
\delta \left( \epsilon_{\alpha\beta} \frac{\partial X^\alpha}{\partial \sigma_\alpha} \frac{DX^a}{d\sigma_\beta} \right)^2 = \left( 2\epsilon_{\alpha\beta} \frac{\partial X^\alpha}{\partial \sigma_\alpha} \frac{DX^m}{d\sigma_\rho} \Lambda^a_{\rho} \right) \left[ \epsilon_{\alpha\beta} \omega^\alpha_{\mu} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{DX^i}{d\sigma_\beta} \Lambda^i_{\rho} + \left( \omega^\alpha_{\mu} \frac{DX^0}{d\sigma_\beta} \Lambda^0_{\rho} \right) \frac{\partial X^\mu}{\partial \sigma_\alpha} \epsilon_{\alpha\beta} \right]
\]

Now \( \Lambda^0_{\rho} = 0 \) and so the equation (23) becomes

\[
\delta \left( \epsilon_{\alpha\beta} \frac{\partial X^\alpha}{\partial \sigma_\alpha} \frac{DX^a}{d\sigma_\beta} \right)^2 = \left( 2\epsilon_{\alpha\beta} \frac{\partial X^\alpha}{\partial \sigma_\alpha} \frac{DX^m}{d\sigma_\rho} \Lambda^a_{\rho} \right) \left[ \epsilon_{\alpha\beta} \omega^\alpha_{\mu} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{DX^i}{d\sigma_\beta} \Lambda^i_{\rho} + \epsilon_{\alpha\beta} \epsilon_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{DX^i}{d\sigma_\beta} \Lambda^a_{\rho} \right]
\]

the 1st term of (24)

\[
2\epsilon_{\alpha\beta} \epsilon_{\mu\nu} \omega^\alpha_{\mu} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{DX^\nu}{d\sigma_\rho} \frac{DX^m}{d\sigma_\beta} \Lambda^a_{\rho} \Lambda^i_{\beta} = \left( \epsilon_{\alpha\beta} \epsilon_{\mu\nu} \omega^\alpha_{\mu} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{DX^\nu}{d\sigma_\rho} \frac{DX^m}{d\sigma_\beta} \Lambda^a_{\rho} \Lambda^i_{\beta} \right)
\]
swapping indices and using the different symmetry property we find that it cancels the second term of the same. Thus we see that the first term of (24) vanish. Similarly we can proved that the second terms of (24) also vanish. Thus

\[ \delta \left( \epsilon_{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma_\alpha} \frac{D X^\alpha}{d \sigma_\beta} \right)^2 = 0 \]  

(25)

So from (25) and (22) we see that

\[ \delta \mathcal{L}_{NG} = 0 \]  

(26)

Thus the curved background Lagrangian (17) is invariant under diffeomorphism due to local Galilean transformations. The action (17) is quite satisfactory. The flat limit poses no problems. In this limit the vielbeins reduce to the Kronecker deltas and using (18) we can easily show that The action (17) reduces to the NR string action (16).

6 Connection with geometry

In this section we will elaborate how the above action belongs to the Newton Cartan geometry. This is important because several assumptions were made regarding the geometry of the string. We will remember that a follow up assumption was the excitations of the string are entirely confined to the brane transverse to the string world sheet. Application of the Galilean gauge theory further complicates the issue. A look at equation (19) is enough to rouse our scepticism. However, this scepticism is quenched very soon when we proved that the flat model [36] is invariant under Galilean transformations. It is intuitively obvious that the curved space with which the string model is coupled is the Newton Cartan spacetime. Still, to appreciate the result the following remarks may be useful. The strategy is this. We will give the principal results for the localization of a generic model endowed with the all the symmetries of the Galilean group \[ \mathbb{G}_0 \]. The resulting curved space time is demonstrated analytically to be the N-C spacetime [10]. Theories which have Galilean symmetry in the flat limit when gauged may have fewer transformations but those transformations form an illustration of the Newton Cartan algebra.

The GGT is primarily a gauge theory in flat space time. Now to account for the local degrees of freedom a local coordinate basis is set up at every point where the basis vectors trivially parallel to the respective global basis vectors. The transformations of the gauge fields introduced during localization have a suggestive form which are begging for a geometrical interpretation [28]. The transformations obtained here (10) are the same as the corresponding one in the general results. These transformations are just the same as the vielbeins and spin connections of a curved manifold. The new fields \( \Lambda_\mu^\alpha \) may then be reinterpreted as inverse

\[ \text{but not any central charge} \]
vielbeins in a general manifold charted by the coordinates $x^\mu$ connecting the (global) spacetime coordinates with the local coordinates. A crucial requirement for this is the existence of an inverse which is met above by the continuity of the local to global invariance (see the equation (7) and the discussion around it). The geometric reinterpretation is then seen from the above transformations see, for example (10) where the local indices (denoted by $a$) are Lorentz rotated, while the global indices (denoted by $i$) are coordinate transformed.

The generators of the space time translations are $\Lambda_\mu^\alpha$ and those of rotations are $B^\mu_{ab}$ and boosts are $B^\mu_{a0}$. We observe that the rotation generators have the usual antisymmetry.

We introduce $\Sigma_\alpha^\mu$ as the inverse of $\Lambda_\mu^\beta$. So

$$\Sigma_\alpha^\mu \Lambda_\mu^\beta = \delta_\alpha^\beta, \quad \Sigma_\alpha^\mu \Lambda_\nu^\alpha = \delta_\nu^\mu \tag{27}$$

The spatial part $\Lambda_k^a$ is the inverse of $\Sigma_a^k$ as may be seen in (27). Note that we are denoting the local coordinates by the initial Greek letters i.e. $\alpha, \beta$ etc. whereas the global coordinates are denoted by letters from the middle of the Greek alphabet i.e. $\mu, \nu$ etc. From the definitions of $\Sigma_\alpha^\mu$ and $\Lambda_\mu^\alpha$ and the transformations of the various fields involved we can obtain the corresponding transformation laws. Thus,

$$\delta_0 \Sigma_0^k = -\zeta^\nu \partial_\nu \Sigma_0^k + \Sigma_0^\nu \partial_\nu \zeta^k - v^b \Sigma_b^k$$
$$\delta_0 \Sigma_a^k = -\zeta^\nu \partial_\nu \Sigma_a^k + \Sigma_a^\nu \partial_\nu \zeta^k - \lambda_a^b \Sigma_b^k \tag{28}$$

It has been proved that the 4-dim space time obtained in this way above is the Newton-Cartan manifold. This is done by showing that the metric formulation of our theory contains the same structures and satisfy the same structural relations as in NC spacetime [12]. The invariance of the theory is retained after localization if the newly introduced basic fields transform as

$$\delta_0 \Sigma_0^0 = -\zeta^\nu \partial_\nu \Sigma_0^0 + \Sigma_0^\nu \partial_\nu \zeta^0$$
$$\delta_0 \Sigma_0^k = -\zeta^\nu \partial_\nu \Sigma_0^k + \Sigma_0^\nu \partial_\nu \zeta^k + u^a \Sigma_a^k$$
$$\delta_0 \Sigma_a^k = -\zeta^\nu \partial_\nu \Sigma_a^k + \Sigma_a^\nu \partial_\nu \zeta^k - \omega_a^b \Sigma_b^k$$
$$\delta_0 B_\mu = -\zeta^\nu \partial_\nu B_\mu - \partial_\mu \zeta^\nu B_\nu - \frac{1}{2} \partial_\mu \omega^{ab} \sigma_{ab} + \partial_\mu u^a x_a \tag{29}$$

Equation (29) give the complete structure of the Newton-Cartan space time. What we would like to emphasise is in a particular example not all the fields will appear. Which fields would be included on the dynamics of the particular model in question. The examples of the non-relativistic particle and bosonic string are quiet instructive. Both carry the sense of time along the path. The basic dynamical variables do not carry any intrinsic direction attached with it. The required transformations are now [33]

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3A word about our notation: indices from the beginning of the alphabet ($\alpha, \beta$ etc. or $a, b$ etc) denote the local basis while those from the middle ($\mu, \nu$ etc. or $i, j$ etc.) indicate the global basis; Greek indices denote space-time while only space is given by the Latin ones. Repeated indices denote a summation.
\[ \epsilon \to \epsilon(t); \quad \epsilon^k, \quad w^k, \quad u^k \to f(x,t) \]  

(30)

The same quasi invariance (now with a space time dependent boost parameter) is reproduced provided the transformation laws of the newly introduced fields are given by,

\[
\begin{align*}
\delta_0 \Sigma_0 &= \Sigma_0^\nu \partial_\nu \zeta^0 \\
\delta_0 \Sigma_a &= \Sigma_a^\nu \partial_\nu \zeta^a + u^a \Sigma_a^k \\
\delta_0 \Sigma_k &= \Sigma_a^\nu \partial_\nu \zeta^k - \omega_a^b \Sigma_b^k \\
\delta_0 B_\mu &= -\partial_\mu \zeta^\nu B_\nu - \frac{1}{2} \partial_\mu \omega^a{}_{ab} \sigma_{ab} + \partial_\mu u^a x_a
\end{align*}
\]

(31)

we can easily see that the transformation relations (31) can be derived from (29) and (10) [34] just by putting the convective term to be zero. The reason for this is not difficult to guess. For the relativistic spin less particle and Bosonic string the parameters that describe the location of the object is independent of choice of any coordinates. Hence when we implement spacetime transformations the parameters do not change. The information we have about the dynamics of the model dictate how many terms would appear in the expression of the curved model. Thus every single new model is not going to change the geometry. Rather it is the dynamics of the model that would determine how many independent symmetry elements of the geometry will be required in the model building.

7 Conclusion

We have discussed the formulation of the non-relativistic string coupled with non-relativistic gravity in Newton-Cartan manifold. This result is in itself of paramount importance because working with a different gauging approach it has been definitely concluded that the NC geometry breaks down to propagate non-relativistic string. Of course we are in no position to comment to the difference, we can definitely conclude that the gauging of symmetry approach implemented by the Galilean gauge theory can give a string action in the NC space time consistently.

The Galilean gauge theory is based on what may be called little figuratively — the gauging the symmetries approach. What this means is very simple. The target is to couple a model with Galilean symmetry in flat space time with curved background. Since the metric structure is degenerate, coupling via the metrics is not useful. this suggests a vielbein formulation of the theory. GGT provides this first order formulation. Not only that, it contains the symmetries of the model in the tangent plane inbuilt in it. No wonder, it has been proved as a natural theory of non-relativistic diffeomorphism invariance. The results for the string presented here completes the glorious journey of GGT so to say [10], [11], [12], [33], [34].
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