Smeared and unsmeared chiral vertex operators

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Abstract

We prove unboundedness and boundedness of the unsmeared and smeared chiral vertex operators, respectively. We use elementary methods in bosonic Fock space, only. Possible applications to conformal two-dimensional quantum field theory, perturbation thereof, and to the perturbative construction of the sine-Gordon model by the Epstein-Glaser method are discussed. From another point of view the results of this paper can be looked at as a first step towards a Hilbert space interpretation of vertex operator algebras.
1 Introduction

The subject of massless two-dimensional fields was always a source of interesting problems. The
light-cone variables, when compactified in the euclidean, allow application of complex methods.
Here we are concerned, in the compact case, with both unsmeared and smeared vertex operators
near and on the unit circle. Using elementary methods only, we show that, as operators in
Hilbert spaces which are related to the bosonic Fock space, the usual unsmeared vertices are
poor operators, whereas the smeared ones are nice bounded operators. The result is surprising
taking into account that bosonic operators are usually unbounded. On the other hand, two-
dimensional abelian bosonization makes the result plausible, at least in some case (equivalence
to fermions which are bounded).

Although the functional properties of the unsmeared vertex operators are not overwhelm-
ing their algebraic properties, when restricted inside the unit circle, are remarkable. This is
consistent with their usefulness in the frame of vertex operator algebras.

In the smeared case we were motivated by similar results obtained in the framework of the
Wess-Zumino-Witten model of two-dimensional conformal quantum field theory [1, 2] and for
Minkowski two-dimensional massless fields [3] by explicit fermionic methods. Instead we keep
working in the bosonic Fock space where vertex operators naturally live. Our main tools are
a generalized Gram inequality for determinants, an explicit tensor product argument which
remembers of a trick used in [1, 3] and a further extension of it.

The range of validity of our results extends to chiral vertex operators with charges in the
closed unit circle. We expect some input on two-dimensional conformal quantum field theory
and on the sine-Gordon model. Indeed, from the conformal field theory point of view the latter
is strongly related to the perturbed conformal quantum field theory [20]. The consequences
are twofold: (i) On one side, in a hamiltonian approach to perturbed conformal field theory
the boundedness of the vertex operator (which appears as perturbation) would suggest a regular
analytic perturbation which is seldom even in quantum mechanics. This agrees with a convergent
perturbation series in the lagrangean formalism [4]. (ii) On the other side, a perturbative
construction of the S-matrix for the sine-Gordon model by the Epstein-Glaser method [5, 6]
appears to provide us with a convergent perturbation series (before the adiabatic limit). Needless
to say that, although our study in this paper is restricted to the compact case, we expect similar
results by similar methods in the non-compact Minkowski case too; the case in which the Epstein-
Glaser method is currently used [5, 6]. We will return to this subject elsewhere. For the usual
perturbative approach including the case of conformal quantum field theory see [7, 8, 9].

The paper is organized as follows. In the second section we set up the bosonic Fock space
notations, define the unsmeared chiral vertex operators and discuss their Hilbert space proper-
ties. We follow here [10, 11, 12] with some improvements. In the third section we introduce the
smeared vertex operator on the unit circle in bosonic Fock space and prove some inequalities
for their vacuum expectation values in a special case. Here the (generalized) Gram inequality,
proven in Appendix 1, is used. Gram determinant inequalities are a hint to fermions behind the
bosons, but we prefer to stay in the bosonic framework and in fact find, if possible, alternative
proofs which are not necessarily fermionic in nature.

In the forth and fifth section we adjust the bosonic Fock space framework in order to in-
corporate neutrality and finally interpret the results of the third section as boundedness of the
smeared chiral vertex operator with neutrality condition. We use a method which is standard in
the old style approach to string theory (and was taken up to conformal field theory and study of
some infinite dimensional Lie algebras). A possible alternative aproach is mentioned in section 6. In that section a discussion of the results obtained and perspectives concerning conformal field theory and sine-Gordon model follows. Appendices provide results used in the main text, but they can also be of independent interest.

2 Unsmeared chiral vertex operators in bosonic Fock space

Let \( a_n, n \in \mathbb{Z} - \{0\} \) generate the Heisenberg algebra \([a_n, a_m] = n\delta_{n,-m}\) where \(a_n, n \geq 1\) are annihilation and \(a_{-n}, n \geq 1\) creation operators in bosonic Fock space \(\mathcal{F}\). We will also consider central extensions by \(a_0\) with \(a_0\Phi_0(\alpha) = \alpha\Phi_0(\alpha)\), where \(\Phi_0(\alpha)\) is the cyclic vacuum in the bosonic Fock space \(\mathcal{F}(\alpha)\), now indexed by the basic charge \(\alpha\) (in particular we can take \(\alpha = 0\)).

As Hilbert spaces the \(\mathcal{F}(\alpha)\) are all the same and we will keep denoting them by \(\mathcal{F}\) with vacuum \(\Omega \equiv \Phi_0\) instead of \(\Phi_0(\alpha)\) if clear from the context. In \(\mathcal{F}\) we consider the usual basis

\[
\Phi_\eta = \frac{1}{(\eta! I_\eta)^{1/2}} a_{-k}^{\eta_k} \cdots a_{-1}^{\eta_1} \Phi_0,
\]

where

\[
\eta = (\eta_1, \eta_2, \ldots), \quad \eta_i \geq 0, \quad \eta! = \prod_{i=1}^{\infty} \eta_i!
\]

\[
I_\eta = \prod_{i=0}^{\infty} i^{\eta_i}, \quad \|\eta\| = \sum_{i=1}^{\infty} i\eta_i < \infty.
\]

Let \(\mathcal{F}_0 \subset \mathcal{F}\) be the linear span of \(\Phi_\eta\). The (unbounded, closed, densely defined) operators \(a_n, n \neq 0\) act as usual for \(n \geq 1\)

\[
a_n \Phi_\eta = \sqrt{n\eta_n} \Phi_{\eta-e_n},
\]

\[
a_{-n} \Phi_\eta = \sqrt{n(\eta_n + 1)} \Phi_{\eta+e_n},
\]

where \(e_n\) is the unit vector in \(l^2\) with zero components for \(k \neq n\) and one for \(k = n\). Let \(\gamma, z \in \mathbb{C}, z \neq 0\). The formal unsmeared vertex operator in \(\mathcal{F}\) is

\[
V_\gamma(z) \equiv V_\gamma(z) = V_\gamma(z) V_\gamma(z)
\]

with

\[
V_\gamma(z) = \exp\left(\gamma \sum_{n=1}^{\infty} \frac{z^n}{n} a_{-n}\right)
\]

\[
V_\gamma(z) = \exp\left(-\gamma \sum_{n=1}^{\infty} \frac{z^{-n}}{n} a_{n}\right).
\]

Further on we consider vertex operatots \(\tilde{V}_\gamma(z) \equiv \tilde{V}_\gamma(z)\) defined as follows. First consider the operator

\[
T_\gamma : \mathcal{F}(\alpha) \to \mathcal{F}(\alpha + \gamma)
\]

such that \([T_\gamma, a_n] = 0\) for \(n \neq 0\) and \([T_\gamma, a_0] = -\gamma T_\gamma\). One can check that this is a homomorphism of Heisenberg moduls. We introduce the vertex operator \(\tilde{V}_\gamma(z)\) from \(\mathcal{F}(\alpha)\) to \(\mathcal{F}(\alpha + \gamma)\) by

\[
\tilde{V}_\gamma(z) \equiv \tilde{V}_\gamma(z) = T_\gamma z^{\gamma a_0} V_\gamma(z).
\]
We will use both vertices $V(\gamma, z)$ and $\tilde{V}(\gamma, z)$. Since the operators $T_\gamma$ and $a_0$ are harmless, there will be, from the Hilbert space point of view, not much difference between the two vertices. Some difference will appear later after introducing the neutrality condition.

Now let us introduce the involution $a_n^+ = a_{-n}$, $n \in \mathbb{Z}$. The formal adjoints of vertex operators are again vertex operators

$$V(\gamma, z)^+ = V(-\gamma^*, \frac{1}{z^*})$$

$$\tilde{V}(\gamma, z)^+ = \tilde{V}(\gamma, z)$$

(7)

where $z^*, \gamma^*$ are the complex conjugate of $z, \gamma \in \mathbb{C}$. For the purpose of computations to follow we remark that $[a_0, T_\gamma] = \gamma T_\gamma$ implies

$$z^{\gamma a_0} T_{\gamma^2} = z^{\gamma^2} T_{\gamma^2} z^{\gamma a_0}.$$

The proof is a simple computation.

Now we start looking at vertices no longer formal (as in (4)) but as operators in Hilbert space. We restrict here to $V = V(\gamma, z)$ as operator in $\mathcal{F}(\alpha)$ with $\alpha = 0$ (denoted by $\mathcal{F}$) but similar results hold for $\tilde{V}$, too. A direct computation in Fock space [10, 11] shows that we obtain well-defined matrix elements of $V$:

$$v_{\eta, \nu}(\gamma, z) = (\Phi_\eta, V(\gamma, z) \Phi_\nu) =$$

$$= \frac{1}{\sqrt{\eta! \nu!}} \prod_{i=1}^{\infty} m_{\eta_i, \nu_i}(\gamma \sqrt{i} z^i - \gamma \sqrt{i} z^{-i})$$

(8)

where

$$m_{\eta, \nu}(x, y) = \sum_{j=0}^{\min(\eta, \nu)} \left( \begin{array}{c} \eta \\ j \end{array} \right) \left( \begin{array}{c} \nu \\ j \end{array} \right) j! x^{\eta-j} y^{\nu-j}$$

are related to the monic Charlier and Laguerre polynomials [13]

$$C_n^{(a)}(x) = n! L_n^{(x-n)}(a) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) \left( \begin{array}{c} x \\ l \end{array} \right) l! (-a)^{n-l}$$

by

$$m_{\eta, \nu}(x, y) = y^{\nu-\eta} C_{\eta}^{(-xy)}(\nu).$$

(11)

Note that the product in (8) is finite because $m_{\eta, \nu_i}$ is different from zero for only a finite number of $i$ and $m_{00}(x, y) = 1$. The generating function of $C_n^{(a)}(x)$ is

$$\sum_{n=0}^{\infty} C_n^{(a)}(x) \frac{w^n}{n!} = e^{-aw}(1 + w)^x$$

(12)

and the orthogonality relation

$$\int_0^{\infty} C_m^{(a)}(x) C_n^{(a)}(x) d\psi^{(a)}(x) = a^n \delta_{m, n}$$

(13)
holds with respect to the step function $\psi^{(a)}$ with jumps
$$
e^{-a^x}x!, \quad \sum_{x=0}^{\infty} e^{-a^x}x! = 1 \quad (14)$$
at $x = 0, 1, 2, \ldots$. For $a > 0$ (our case if $\gamma$ real) this is the Poisson distribution. At this point it is interesting to remark that the basic formula for giving a Hilbert space meaning to formal products of vertex operators (see later) corresponds to the following generalization of the orthogonality relation for the Charlier polynomials [13]:

**Lemma 1** For $x, y, z \in \mathbb{C}$ and $i, j$ integers
$$
\sum_{k=0}^{\infty} \frac{1}{k!} m_{ik}(x, y)m_{kj}(z, w) = m_{ij}(x + z, y + w)e^{xy}. \quad (15)
$$

The proof which is a long induction argument can be found in [10]. It encodes the fact that formally the product of two vertex operators is, up to scalar factors, again a vertex-like operator.

Generally the matrix elements $v_{i\nu} = v_{i\nu}(\gamma, z)$ define an operator $V = V(\gamma, z)$ in Hilbert space $\mathcal{F}$ (no longer formal) by
$$
V\Psi = \sum_{\eta} \sum_{\nu} v_{i\nu}(\Phi_{\nu}, \Psi)\Phi_{\eta} \quad (16)
$$
for $\Psi$ in the domain of definition
$$
D(V) = \left\{ \Psi \in \mathcal{F} : \lim_{k \to \infty} \sum_{\|\nu\| \leq k} v_{i\nu}(\Phi_{\nu}, \Psi) \quad \text{exists for all } \eta \right\}. \quad (17)
$$
Certainly the domain of definition of $V$ can be void, $D(V) = \emptyset$. In this case the matrix elements $v_{i\nu}$ determine $V$ only as bilinear form and not properly as an operator. This will really happen in some cases below. Using definition $(16, 17)$, a bunch of results on $V = V(\gamma, z)$ has been proven [11] using mainly lemma 1 and coherent states [14] generated by exponentials of type $V_-$ in $(4)$. We select what is relevant for us:

**Theorem 2** We have for arbitrary $\gamma \in \mathbb{C}$

(i) For $|z| < 1$, $V(\gamma, z)$ is densely defined with $\mathcal{F}_0$ in its domain of definition $D(V(\gamma, z))$.

(ii) For $|z| > 1$, $V(\gamma, z)$ is closed.

(iii) For $|z| < 1$, $V(\gamma, z)$ is not closable.

(iv) Let $|z_2| > |z_1| > 1$, then
$$
V(\gamma, z_2)\mathcal{F}_0 \subset D(V(\gamma, z_1)).
$$

We recall that a Hilbert space operator $T$ is closed if its graph $G(T)$ is closed; $T$ is said to be closable if the closure $\overline{G(T)}$ is a graph.

Proofs of the theorem are based on lemma 1 and can be found in [10, 11]. In [11] even stronger results are proven. In particular, more involved considerations [11] enable one to strengthen property (ii) to

(ii’) For $|z| > 1$ the domain of definition of $V(\gamma, z)$ is void and as such $V(\gamma, z)$ is trivially closed.
Taking into account definition (7) of the formal adjoint, the situation with $V(z), V^+(z)$ appear to be somewhat similar to that of annihilation and creation operators $a(x), a^+(x)$ in elementary quantum mechanics. Indeed, the domain of definition of $a^+(x)$ as operator in Fock space is void, but working with bilinear forms instead of operators saves the matter. Only after smearing, $a^+(f), f \in L^2$ becomes a (nontrivial) operator. However, the reader should not push this analogy too far because of chiral properties of $V, V^+$ which are absent in $a, a^+$. At this stage we retain the fact that there is a striking asymmetry between unsmeared vertices inside and outside the unit circle, as far as their operator properties in the Fock space $\mathcal{F}$ are concerned. The symmetry is restored after smearing on the unit circle as we will see below.

According to property (ii) the (maximal) operator $V(\gamma, z)$ defined by (16) (17) is not closable. It could be that a restriction of $V(\gamma, z)$ is closable although $V(\gamma, z)$ itself doesn’t have this property. The assertion about the non-closability of $V(\gamma, z)$ can be extended as follows [11]: Let us define the "minimal" vertex operator as

$$ V(\gamma, z)_0 = V(\gamma, z)|_{\mathcal{F}_0} $$

with $D(V(\gamma, z)_0) = \mathcal{F}_0$. It is clear that each extension of $V(\gamma, z)_0$ (in particular $V(\gamma, z)$) is not closable if $V(\gamma, z)_0$ is not closable. The following property [11] is a generalization of (iii):

(iii') For $|z| < 1$, $V(\gamma, z)_0$ is not closable.

**Corollary 3** For arbitrary $\gamma \in \mathbb{C}$ and $|z| < 1$ the vertex $V(\gamma, z)$ is an unbounded operator in Fock space.

This is a consequence of property (iii). We give an independent simple proof\(^1\) (cf. [12]). Let us define for $z, \zeta \in \mathbb{C}, l \in \mathbb{N}$ coherent states as

$$ |z, \zeta, l\rangle = N_l \exp\left(\zeta \sum_{n=1}^{l} \frac{z^n}{n} a_{-n}\right) \Phi_0, $$

where Dirac’s notation has been used and $N_l$ is the normalization constant

$$ N_l = \exp\left(-\frac{|\zeta|^2}{2} \sum_{n=1}^{l} \frac{|z|^{2n}}{n}\right). $$

The coherent state (18) can be expanded as

$$ |z, \zeta, l\rangle = N_l \sum_{\eta_1, \ldots, \eta_l=0}^{\infty} \prod_{n=1}^{l} \frac{z^n \eta_n \zeta \eta_n}{(\eta_n!)^{1/2}} \Phi_\eta. $$

(20)

The limit of (20) for $l \to \infty$ exists in Fock space

$$ |z, \zeta\rangle = \lim_{l \to \infty} |z, \zeta, l\rangle. $$

(21)

On the other hand the coherent states are eigenvectors of annihilation operators $a_n, n \geq 0$

$$ a_n |z, \zeta, l\rangle = \begin{cases} \zeta z^n |z, \zeta, l\rangle, & \text{for } n \leq l \\ 0, & \text{for } n > l \end{cases} $$

(22)

\(^1\)We thank W.Boenkost for suggesting this proof to us.
Using the fact that such $a_n$ are closed operators, it follows
\[ a_n|z, \zeta\rangle = \zeta z^n|z, \zeta\rangle. \] (23)

These properties are used to compute
\[ \langle \Phi_0|V(\gamma, z)|z, \zeta, l\rangle = \exp\left(-\gamma\zeta \sum_{n=1}^{l} \frac{1}{n}\right) \langle \Phi_0|z, \zeta, l\rangle \]
\[ = N_l \exp\left(-\gamma\zeta \sum_{n=1}^{l} \frac{1}{n}\right). \] (24)

It follows for $\zeta = -\gamma^*$ that
\[ \langle \Phi_0|V(\gamma, z)|z, -\gamma^*, l\rangle = N_l \exp\left(|\gamma|^2 \sum_{n=1}^{l} \frac{1}{n}\right). \] (25)

This shows that the matrix elements of $V(\gamma, z)$ diverge for $l \to \infty$ and the unboundedness follows.

Properties (i)-(iii) say nothing about the case $|z| = 1$. Property (iii) shows that $V(\gamma, z)$ in the interesting region $|z| < 1$ is a poor operator. Nevertheless, Property (iv) allows for defining products
\[ V(\gamma_1, z_1)V(\gamma_2, z_2)\cdots V(\gamma_r, z_r) \] (26)
for $|z_r| < |z_{r-1}| < \ldots < |z_1| < 1$ as densely defined operators in $\mathcal{F}$ with $\mathcal{F}_0$ in their domains.

Some remarks are in order:

First we didn’t use any kind of braid relation between vertices and in fact defined Hilbert space products $V(\gamma_1, z_1)\cdots V(\gamma_r, z_r)$ only for $|z_r| < \ldots < |z_1| < 1$.

Second we didn’t introduce neutrality condition, common in massless two-dimensional field theory. Certainly it is possible to introduce the braiding relation consistent with the properties of unsmeared $V(\gamma, z)$ but it doesn’t help too much for improving their Hilbert space properties (i) - (iv). The situation is different in the smeared case as we will see later. We didn’t try to find out if neutrality could improve the properties of unsmeared vertices.

Third, it is interesting to remark that the product of vertices (26) is defined in $\mathcal{F}$ without an invariant domain for the factors. Indeed, $\mathcal{F}_0$ is not invariant under $V(\gamma, z)$. This is not the situation one is used to have in quantum field theory where the invariance of the domain on which the field operators are densely defined is part of the axioms. Later on in this paper the situation will change by passing to some modifications of $\mathcal{F}$.

This is the situation with unsmeared vertex operators. In the next section we will show that a smearing operation applied to vertices dramatically improves their properties (near $|z| = 1$) such that finally under the neutrality condition they turn into bounded operators in the bosonic Fock space to be precisely defined below. We consider the case $|\gamma| \leq 1$ in this paper but there are indications that the result could be extended.

For later use we mention the following formula which now has a Hilbert space operator interpretation (see the remarks above concerning the existence of products)
\[ \tilde{V}(\gamma_1, z_1)\tilde{V}(\gamma_2, z_2)\cdots \tilde{V}(\gamma_r, z_r) = \prod_{1 \leq i < j \leq r} \left(1 - \frac{z_j}{z_i}\right)^{\gamma_i\gamma_j} T_{\sum_{j} \gamma_j} \times \]
\begin{equation}
\times \prod_{i=1}^{r} z_i^{-\alpha_0} \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \left( \sum_{i=1}^{r} \gamma_i z_i^n a_{-n} \right) \right) \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \left( \sum_{i=1}^{r} \gamma_i z_i^{-n} a_n \right) \right)
\end{equation}

for \(|z_r| < |z_{r-1}| < \ldots < |z_1| < 1\). In the case of \(V(\gamma_i, z_i)\) obvious factors in (27) have to be left out. In particular (27) reproduces the well known formula for the \(n\)-point function under neutrality \(\sum \gamma_i = 0\) (here \(\alpha = 0\))

\begin{equation}
(\Phi_0, V(\gamma_1, z_1) \ldots V(\gamma_n, z_n) \Phi_0) = \prod_{i<j} (z_i - z_j)^{\gamma_i \gamma_j}.
\end{equation}

The determination in (27) and (28) is fixed as usual by taking \(\log(z_i - z_j)\) real for \(0 < z_j < z_i\).

### 3 Smeared chiral vertex operators

The formal smeared chiral vertex operator on the unit circle \(S^1\) is

\begin{equation}
V(\gamma, f) = \frac{1}{2\pi} \int_{z=1} V(\gamma, z)f(z)dz
\end{equation}

where \(f = f(z)\) is a test function on \(S^1\) to be chosen below. In this section we start by looking at (29) as a limit for \(z\) approaching the unit circle from the interior, i.e. (29) has to be understood as

\begin{equation}
V(\gamma, f) = \lim_{r \to 1-} \frac{1}{2\pi} \int_{|z|=1} V(\gamma, rz)f(z)dz.
\end{equation}

Let us consider the case \(\gamma = 1\) first. When appropriate we use the notation \(V(\gamma = 1, z) = V(z)\). A rough idea of what happens is obtained by calculating the following norm in the unsmeared case for \(|z| = 1 - \varepsilon, \varepsilon \to 0^+\) with help of (27)

\begin{equation}
\|V(z)\Phi_0\|^2 = (\Phi_0, V(z)^\dagger V(z)\Phi_0) = \\
= \frac{1}{1 - 2^* z} \bigg|_{|z|=1-\varepsilon, \varepsilon \to 0^+} = \infty,
\end{equation}

whereas in the smeared case with \(f \in L^2(S^1)\)

\begin{equation}
\|V(f)\Phi_0\|^2 = (\Phi_0, V(f)^\dagger V(f)\Phi_0) = \\
= \frac{1}{4\pi^2} \int \int \frac{1}{1 - w^* z} f^*(w^*)f(z) dw^* dz = \sum_{n=1}^{\infty} |c_{-n}|^2 \leq \|f\|^2 < \infty.
\end{equation}

Here \(c_n\) is the Fourier coefficient

\begin{equation}
c_n = \frac{1}{2\pi i} \int_{|z|=1} z^{-n-1} f(z) dz = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(e^{i\theta}) d\theta
\end{equation}

and the \(L^2(S^1)\) norm is given by

\begin{equation}
\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \sum_{n=-\infty}^{+\infty} |c_n|^2.
\end{equation}
In (32) use was made of the formal smeared adjoint
\[ V(\gamma, f)^+ = \frac{1}{2\pi} \int_{|z|=1} V(\gamma, z)^+ f^*(z) \, dz^* = \frac{1}{2\pi} \int_{|z|=1} V(-\gamma^*, \frac{1}{z^*}) f(z^*) \, dz^*, \]

Following the convention in (31), \( z \) approaches the unit circle from the interior and consequently \( 1/z^* \) approaches it from the exterior. In (32), before taking the limit to the unit circle, we have \( |z| < 1 < |w^*|^{-1} \) which assures the validity of the geometric series used there. The rigorous definition of the smeared vertex \( V(\gamma, f) \) and adjoint \( V(\gamma, f)^+ \) as operators in bosonic Fock space is analogous to the definition of their unsmeared counterparts in section 2 (cf. further in this section). In addition we have to remark that the smearing in \( L^2 \) enables us to take them on the unit circle, a fact which is not true in the unsmeared case. This remembers the computation in (32) as opposed to (31). For the convenience of the reader let us give some details in the special case \( V(\gamma, f)\Phi_0 \). Indeed, using formulas from section 2 we have
\[ \lambda_\eta \equiv \lambda_\eta(z) = (\Phi_\eta, V(\gamma, z)\Phi_0) = \prod_{i=1}^{\infty} (\eta_i!)^{-1/2} \left( \frac{\gamma z^i}{\sqrt{i}} \right)^{\eta_i} \]
and
\[ \sum_\eta |\lambda_\eta|^2 = \sum_\eta \prod_{i=1}^{\infty} (\frac{|\gamma z^i|^2}{i})^{\eta_i} = \prod_{i=1}^{\infty} \left( \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{|\gamma z^i|^2}{i} \right)^k \right) = \prod_{i=1}^{\infty} \exp \left( \frac{|\gamma|^2 |z^2|^i}{i} \right) = \exp \left( |\gamma|^2 \sum_{i=1}^{\infty} |z^2|^i \right) = \left( \frac{1}{1 - |z|^2} \right)^{|\gamma|^2}, \]
showing that in the unsmeared case \( \sum_\eta |\lambda_\eta|^2 < \infty \) for \( |z| < 1 \), but not for \( |z| = 1 \) because of the divergence of the harmonic series. On the other hand, in the smeared out case for
\[ \Lambda_\eta = \frac{1}{2\pi} \int_{|z|=1} \lambda_\eta(z) f(z) \, dz \]
we have \( \sum_\eta |\Lambda_\eta|^2 < \infty \) as a consequence of the computation above and Parseval’s relation for the Fourier coefficients of \( f \in L^2(S^1) \). This shows that the vacuum is in the domain of definition of \( V(\gamma, f) \) when concentrating on the unit circle which was not the case for the unsmeared \( V \) (see (31)). Similar considerations hold for products of smeared vertex operators appearing below. We leave the details to the interested reader.

We remark that some care is needed for the case \( |z| > 1 \) (including the adjoint with \( |z| < 1 \)). The truncation which makes divergent sums well defined is provided by looking at the problem in the framework of bilinear forms. The way in which the bilinear adjoint operation is implemented is different from the standard case of annihilation and creation operators. It can be understood in terms of Fourier coefficients of the smearing function: under adjunction they reverse the index. On products of smeared vertices applied to the vacuum the adjoint operation is equivalent...
Indeed, for $x$ algebras (formal fields, formal distributions etc.) for this particular example, one obtains a Hilbert space version of methods in operator vertex algebra. Consequently the results are fully consistent with those obtained by usual formal work supplemented by analytic continuation, braiding etc. At the same time, for this particular example, one obtains a Hilbert space version of methods in operator vertex algebras (formal fields, formal distributions etc.).

Anticipating, the smearing operation reinforces symmetry: both $V(f), f \in L^2$ and its adjoint are densely defined, closed operators. Neutrality will turn them into bounded operators. We didn’t study further $V(f), f \in L^2$ without neutrality condition here.

Remark that even for the case $|\gamma| \leq 1$ the norm $\|V(\gamma, f)\Phi_0\|^2$ is finite for $f \in L^2(S^1)$. Indeed, for $1 > x = |\gamma|^2 > 0$ the binomial series ($|a| < 1$)

$$(1-a)^{-x} = 1 + \frac{x}{1!}a + \frac{x(x+1)}{2!}a^2 + \frac{x(x+1)(x+2)}{3!}a^3 + \ldots$$

gives

$$\|V(\gamma, f)\Phi_0\|^2 = \frac{x}{1!}|c_{-1}|^2 + \frac{x(x+1)}{2!}|c_{-2}|^2 + \ldots \leq \frac{1}{1!}|c_{-1}|^2 + \frac{1 \cdot 2}{2!}|c_{-2}|^2 + \ldots = \|V(\gamma = 1, f)\Phi_0\|^2 < \infty.$$  

The two-point function results (32) and (35) in the smeared case are encouraging as opposed to the unsmeared case (31). In this theory we have the direct sum decomposition

$$L^2(S^1) = H^2_+ \oplus H^2_2$$

where $H^2_+$ and $H^2_2$ are also Hilbert spaces of $L^2$-functions with positive and zero frequencies and negative frequencies, respectively. $H^2_+$ is the usual Hardy space denoted by $H^2$. In a different language we have in $H^2_+ L^2$-boundary values from inside and in $H^2_2$ from outside the unit circle.

The main formula we use in this context is

$$\frac{1}{2\pi i} \int_{S^1} \int_{S^1} \frac{f(z_2)g(z_1)}{z_2 - z_1} dz_2 dz_1 =$$

$$= \int_{S^1} f^{(+)}(z_1) g(z_1) dz_1 = \int_{S^1} f^{(+)}(z_1) g^{(+)}(z_1) dz_1,$$

where the integration variables tend to the unit circle, respecting $|z_2| > 1 > |z_1|$. We use (37) in several forms which at first glance look different but are always the same formula (37). For instance, we have

$$(V(f)\Phi_0, V(g)\Phi_0) = (\Phi_0, V^+(f)V(g)\Phi_0) =$$

$$= \frac{1}{4\pi^2} \int_{|w*|=1} \frac{1}{1 - w^*z} f^*(w^*)g(z) dw^* dz = \frac{1}{4\pi^2} \int \frac{u^{-1}f^*(u^{-1})g(z)}{u - z} du dz =$$

$$= \frac{1}{2\pi i} \int (z^{-1}f^*(z^{-1}))^{(+)} g(z) dz = (f^{(-)}, g^{(-)}).$$

From (38) we get in particular for $f = g$ the previous relation (32) from which we retain

$$\|V(f)\Phi_0\| = \|f^{(-)}\|_2 \leq \|f\|_2.$$  

$$\text{(39)}$$
In the following we will generalize the relation (39) to scalar products of the form

\[ (V_n(f), V_n(g)), \]

where

\[ V_n(f) = V(f_1)V(f_2) \cdots V(f_n)\Phi_0 \]

and similarly for \( V_n(g) \), with \( f_i(z_i), g_i(z_i) \in L^2(S^1), i = 1, 2, \ldots, n \) and the regularization prescription

\[ |z_n| < |z_{n-1}| < \ldots < |z_1| < 1. \]

We write for \( f = (f_1, f_2, \ldots, f_n), g = (g_1, g_2, \ldots, g_n) \)

\[ (V_n(f), V_n(g)) = (\Phi_0, V^+(f_n) \cdots V^+(f_1)V(g_1) \cdots V(g_n)\Phi_0) = \]

\[ = \frac{1}{(2\pi)^n} \int D(w^*, z) \prod_{i=1}^n f_i^*(w_i^*) g_i(z_i) \, dw_i^* \, dz_i \]

where

\[ D(w^*, z) = \frac{\prod_{i<j}^n (z_i - z_j)(w_i^* - w_j^*)}{\prod_{i,j=1}^n (1 - z_i w_j^*)} \]

and

\[ f_i(z_i) = z_i^{-n} f_i(z_i), \]

\[ g_i(z_i) = z_i^{-n} g_i(z_i) \]

for \( i = 1, 2, \ldots, n \).

Using the Cauchy determinant formula

\[ D(w^*, z) = \det \left( \frac{1}{1 - z_i w_j^*} \right)_{1 \leq i < j \leq n} \]

and expanding the determinant we get from (42)

\[ (V_n(f), V_n(g)) = G_n(f^{(-)}; g^{(-)}) \]

where

\[ G_n(f, g) \equiv G_n(f_1, f_2, \ldots, f_n; g_1, g_2, \ldots, g_n) \]

\[ = \left( \begin{array}{ccc} (f_1, g_1) & \cdots & (f_1, g_n) \\ \vdots & \ddots & \vdots \\ (f_n, g_1) & \cdots & (f_n, g_n) \end{array} \right) \]

is the (generalized) Gram determinant. For \( f = g \) we write for the usual Gram determinant

\[ G_n(f; f) \equiv G_n(f) \equiv G_n(f_1, f_2, \ldots, f_n). \]

Similar relations hold for the vertices \( \tilde{V}(z) \). As a consequence of the smearing the Cauchy determinant which often appears in two-dimensional (massless) physics (for instance in (28) with \( \gamma_i = \pm 1 \) and neutrality \( \sum \gamma_i = 0 \)) goes over into a Gram determinant. Certainly \( G_n(f_1, \ldots, f_n) \geq 0 \), as necessary because from (46)

\[ 0 < \|V_n(f)\|^2 = G_n(f). \]
Observe that \( f_i, g_i \in L^2(S^1) \) iff \( f_i, g_i \in L^2(S^1) \) and \( \|f_i\|_2 = \|f_i\|_2, \|g_i\|_2 = \|g_i\|_2, i = 1, 2, \ldots, n. \) Using a simple Gram inequality (see Appendix 1 for a collection of Gram determinant inequalities used in this paper) we get

\[
\|V(f_0)V_n(f)\| \leq \|f_0\|_2\|V_n(f)\| \leq \|f_0\|_2\|V_n(f)\|. \tag{50}
\]

The two-point function estimate (38) over only negative frequencies cannot be saved here because of the \( z_i^- \)-powers in (44). The inequality (50) is a generalization of (39). Similar considerations apply to \( \tilde{V} \) with the same bounds

\[
\|\tilde{V}(f_0)\tilde{V}_n(f)\| \leq \|f_0\|_2\|\tilde{V}_n(f)\|, \tag{51}
\]

where \( \tilde{V}_n \) is defined as in (41) with \( V \) replaced by \( \tilde{V} \). On the l.h.s. of (51) the norm is taken in \( \mathcal{F}(\alpha + n + 1) \) (we have \( \gamma = 1 \)), whereas on the r.h.s. it is taken in \( \mathcal{F}(\alpha + n) \) with \( \alpha \) arbitrary (in particular \( \alpha = 0 \)). In fact, here we can do better: for \( \tilde{V}(f) \) there are no powers of \( z \) which force \( f \) into \( f^- \) as in (44) and the bound is \( \|f^-\|_2 \) involving only the negative frequency part in the Hardy decomposition of \( f \). This remark applies to all operators \( \tilde{V}(f) \) to follow. Indeed, instead of (42) we now have

\[
(\tilde{V}_n(f), \tilde{V}_n(g)) = (\tilde{V}(f_1)\ldots\tilde{V}(f_n)\Phi_0, \tilde{V}(g_1)\ldots\tilde{V}(g_n)\Phi_0) =
\]

\[
= \frac{1}{(2\pi)^n} \int D(w^*, z) \prod_{\gamma=1}^{n} f_i^*(w_i^*)g_i(z_i) \, dw_i^* \, dz_i.
\]

This formula is obtained by commuting first the operator \( T_\gamma \) through \( a_0 \) in \( \tilde{V}_n(f) \) and \( \tilde{V}_n(g) \) separately and then using the adjoint operation on \( V \)'s as in (42). The above formula proves the claim.

Let us pause for a moment in order to understand what we have achieved and what we are going to do. First observe that up to now the elementary Gram determinant inequality was used to estimate the norms (39) and (50) directly in the bosonic Fock space, without appealing to any kind of abstract bosonic-fermionic equivalence and without introducing the braiding relation. Our goal in the next section is to prove boundedness of vertex operators in bosonic Fock space. We will start working with the vertex operator \( \tilde{V}(f) \) restricted from

\[
\mathcal{F}^\oplus = \bigoplus_{n=0}^\infty \mathcal{F}(\alpha + n) \quad \text{to} \quad \mathcal{F}^\oplus = \bigoplus_{n=0}^\infty \hat{\mathcal{F}}(\alpha + n), \tag{52}
\]

where \( \alpha \) is arbitrary and \( \hat{\mathcal{F}}(\alpha + n) \) is the closed subspace of \( \mathcal{F}(\alpha + n) \) generated by \( \tilde{V}_n(f) \) as in (41) i.e.

\[
\tilde{V}_n(f) = \tilde{V}(f_1)\tilde{V}(f_2)\ldots\tilde{V}(f_n)\Phi(\alpha)
\]

with \( \Phi(\alpha) \) being the vacuum in \( \mathcal{F}(\alpha) \) (see also [24]). In chosing orthogonal direct sums in (52), the physical neutrality (the "neutrality condition") is realized in both \( \mathcal{F}^\oplus \) and \( \hat{\mathcal{F}}^\oplus \). In physics neutrality is a consequence of the massless limit in two-dimensional quantum field theory [18, 19]. This idea fits well in our framework but we do not continue to discuss it. Let us only remark that in quantum field theory the massless limit in each adequate framework requires supplementary conditions (see [23] for instance) of which kind in our case the neutrality is. Certainly, behind \( \hat{\mathcal{F}}^\oplus \) a fermionic structure is hidden and this is related to the fact that on \( V(f) \) (or \( \tilde{V}(f) \)) a braiding relation can consistently be imposed, which in the case \( \gamma = 1 \) degenerates
into antisymmetry. This will no longer be the case in section 4 where \( \gamma \neq 1 \). Nevertheless it is exactly this "fermionic flavor" in bosonic Fock space which does the job for us.

For proving boundedness of \( \tilde{V}(f) \) in this framework we need a (generalized) Gram determinant inequality to be proven in Appendix 1. In the fifth section we will extend the results from the present \( \gamma = 1 \) to \( \gamma \in [-1, 1] \) and finally to \( |\gamma| \leq 1 \). The Hilbert space \( \tilde{F}^{\oplus} \) in (52) is a reasonable framework to study the vertex \( \tilde{V}(f) \) with neutrality condition. The vertex operator \( V(f) \), on the other hand, lives in the original bosonic Fock space \( F \equiv F(\alpha = 0) \). This case, which is of interest too, is touched in section 5 where also the connection to the massless two-dimensional physics is shortly discussed.

## 4 Boundedness of vertex operators for \( \gamma = 1 \) through generalized Gram inequality

By construction the set

\[
\tilde{V}_n(f) = \prod_{i=1}^{n} \tilde{V}(f_i)\Phi_0
\]

\( f = (f_1, f_2, \ldots, f_n), f_i \in L^2(S^1), i = 1, \ldots, n \) and \( n = 0, 1, 2, \ldots \) is total in \( \tilde{F}^{\oplus} \). We look first at

\[
\left\| \tilde{V}(f_0) \sum_{j=1}^{m} \alpha_j \tilde{V}_m(f_j) \right\|,
\]

where \( \alpha_j \) are complex constants. We changed the notation a little and denoted now by \( f_j \) in \( \tilde{V}(f_i) \) the \( n \)-tuple \( f_j = (f_{j1}, \ldots, f_{jn}), j = 1, 2, \ldots, n \). The norm in (53) is in \( F(\alpha + n) \). To estimate this we use the following (generalized) Gram inequality (see Appendix 1)

\[
\sum_{i,j=1}^{m} \alpha_i^* \alpha_j G(f_0, f_i; f_0, f_j) \leq \|f_0\|^2 \sum_{i,j=1}^{m} \alpha_i^* \alpha_j G(f_i; f_j).
\]

(54)

By expanding (53) and using (54) as in (50) we get

\[
\left\| \tilde{V}(f_0) \sum_{j=1}^{m} \alpha_j \tilde{V}_n(f_j) \right\| = \left\| \sum_{j=1}^{m} \alpha_j \tilde{V}_{n+1}(f'_j) \right\| \leq \|f_0\| \left\| \sum_{j=1}^{m} \alpha_j \tilde{V}_n(f'_j) \right\|,
\]

(55)

where \( f'_j = (f_0, f_j), j = 1, 2, \ldots, m \) and in (55) obvious norms in \( F(\alpha + n + 1) \) and \( F(\alpha + n) \), respectively, are chosen. Taking into account the direct sum structure of \( \tilde{F}^{\oplus} \) and the simple operation (6) of \( \tilde{V} \) in it, it follows that \( \tilde{V}(f_0), f_0 \in L^2(S^1) \) is bounded on a dense set and therefore bounded as operator in the whole \( \tilde{F}^{\oplus} \) with bound \( \|f_0\|_2 \). Note that the braiding condition on vertices (which would turn them into chiral fermions) was not imposed. In fact vertices \( \tilde{V}(z) \) can be explicitly realized as (chiral) fermions in \( \tilde{F}^{\oplus} \).
5 Boundedness of vertex operators with $|\gamma| \leq 1$

In this section we study vertices $\tilde{V}(\gamma, z) \equiv \tilde{V}_\gamma(z)$ with $|\gamma| \leq 1$ as operators in

$$\mathcal{F}_\gamma^\oplus = \oplus_{n=0}^\infty \mathcal{F}(\alpha + n\gamma)$$

where $\tilde{V}_\gamma(z)$ maps from $\mathcal{F}(\alpha)$ to $\mathcal{F}(\alpha + \gamma)$ in order to account for the neutrality condition. We restrict $\tilde{V}_\gamma(z)$ to

$$\tilde{\mathcal{F}}_\gamma^\oplus = \oplus_{n=0}^\infty \tilde{\mathcal{F}}(\alpha + n\gamma).$$

where $\tilde{\mathcal{F}}(\alpha + n\gamma)$ is the closed subspace of $\mathcal{F}(\alpha + n\gamma)$ generated by

$$n \prod_{i=1}^n \tilde{V}_\gamma(f_i) \Phi_0, \quad f_i \in L^2(S^1), \quad i = 1, 2, \ldots, n$$

and $n = 0, 1, 2, \ldots$. Here $\tilde{\mathcal{F}}(\alpha)$ is one-dimensional generated by $\Phi_0(\alpha) = \Phi_0$.

We start with the case $\gamma \in [-1, 1]$ and use a trick from [1, 3] which in our bosonic framework becomes fully transparent. Consider two copies of Heisenberg algebras generated by $a_n, a'_n, \quad n \in \mathbb{Z}$ satisfying

$$[a_n, a_m] = n\delta_{n, -m}$$
$$[a'_n, a'_m] = n\delta_{n, -m}$$
$$[a_n, a'_m] = 0$$

and vertex operators $\tilde{V}_\gamma(z), \tilde{V}'_{\gamma'}(z)$

$$\tilde{V}_\gamma(z) = T_\gamma z^{\gamma a_0} \exp\left(\gamma \sum_{n=1}^\infty z^n a_{-n}\right) \exp\left(-\gamma \sum_{n=1}^\infty z^{-n} a_n\right)$$
$$\tilde{V}'_{\gamma'}(z) = T'_{\gamma'} z^{\gamma' a'_0} \exp\left(\gamma' \sum_{n=1}^\infty z^n a'_{-n}\right) \exp\left(-\gamma' \sum_{n=1}^\infty z^{-n} a'_n\right)$$

with

$$[T'_{\gamma'}, a_0] = [T_{\gamma}, a'_0] = [T_{\gamma}, T'_{\gamma}] = 0$$
$$[T_{\gamma}, a_0] = -\gamma T_{\gamma}, \quad [T'_{\gamma'}, a'_0] = -\gamma' T'_{\gamma'}.$$ 

Note that $\tilde{V}_\gamma(z), \tilde{V}'_{\gamma'}(z)$ refer to the same complex variable $z$ but act in different Fock spaces. In the tensor product space we consider the tensor product operator on the same variable $z$

$$\tilde{V}(z) = \tilde{V}_\gamma(z) \otimes \tilde{V}'_{\gamma'}(z)$$

with $\gamma, \gamma' \in [-1, 1], \quad \gamma^2 + \gamma'^2 = 1$. Such tensor products with a restriction to the diagonal in the space-time variables were introduced long ago in axiomatic field theory [26]. Observe that the notation in (60) which suggests a vertex $V_{\gamma=1}(z)$ is consistent. Indeed, with

$$A_n = \gamma a_n + \gamma' a'_n, \quad n \in \mathbb{Z}$$
$$T = T_{\gamma} T'_{\gamma'}$$

(61)
we have
\[ [A_n, A_m] = [\gamma a_n + \gamma' a'_n, \gamma a_m + \gamma' a'_m] = n\delta_{n,-m}(\gamma^2 + \gamma'^2) = n\delta_{n,-m} \]  
(62)
and
\[ [T, A_0] = - (\gamma^2 + \gamma'^2)T = -T \]
\[ [T, A_n] = 0, \quad n \in \mathbb{Z} - \{0\}. \]

We apologize for the sloppy notation. One should read in (61)
\[ A_n = (\gamma a_n \otimes I) + (\gamma' a'_n \otimes I') \]
where \(I, I'\) are identities etc. The involution \(A_n^+ = A_{-n}\) forces \(\gamma, \gamma'\) to be real. The tensor product \(\tilde{V}(z)\) in (60) is an extension of the vertex operator
\[ \tilde{V}_{\gamma=1}(z) = T z^{A_0} \exp \left( \sum_{n=1}^{\infty} \frac{z^n}{n} A_{-n} \right) \exp \left( - \sum_{n=1}^{\infty} \frac{z^{-n}}{n} A_n \right) \]
(63)
from its Fock space \(\tilde{F}_{\gamma=1}^\oplus\) to the tensor product of Fock spaces \(\tilde{F}_{\gamma}^\oplus, \tilde{F}_{\gamma'}^\oplus\), in which \(\tilde{V}_{\gamma}(z), \tilde{V}_{\gamma'}(z)\) live.

As in section 4, by the generalized Gram inequality the smeared vertex operator \(\tilde{V}_{\gamma=1}(f), f \in L^2(S^1)\) is bounded in \(\tilde{F}_{\gamma=1}^\oplus\). We claim that its smeared extension \(\tilde{V}(f)\) in (60) is also bounded with the same bound \(\|f\|_2\) as operator in \(\tilde{F}_{\gamma'}^\oplus \otimes \tilde{F}_{\gamma}^\oplus\). This follows by explicitly realizing \(\tilde{V}(f)\) as a (chiral) fermionic operator satisfying CAR on a dense domain (see [25]) not only in \(\tilde{F}_{\gamma'}^\oplus\) but also in \(\tilde{F}_{\gamma'}^\oplus \otimes \tilde{F}_{\gamma}^\oplus\) (cf. [3]) and using the fact that the fermionic bound is algebraically determined (see for instance theorem 1 in [25]). In fact looking at the proof of the generalized Gram inequality in Appendix 1 one realizes that it is shaped after the proof of theorem 1 in [25]. The only point is that in the Gram determinant framework no braid relation (here responsible for antisymmetry) is necessary, whereas the work with genuine fermions makes its introduction necessary (see section 6 for a further remark on this point).

Let \(\Psi, \Phi \in \tilde{F}_{\gamma'}^\oplus\) and \(\Psi', \Phi' \in \tilde{F}_{\gamma}^\oplus\). Then, for \(f \in L^2(S^1)\) we can write
\[ \left( \Psi \otimes \Psi', \tilde{V}(f)(\Phi \otimes \Phi') \right) = \frac{1}{2\pi} \int \left( \Psi, \tilde{V}_{\gamma'}(z)\Phi \right) (\Psi', \tilde{V}_{\gamma'}(z)\Phi') f(z) \, dz. \]
(64)

We choose \(\Phi' = \Omega'\) (vacuum \(\Phi_0'(\alpha)\) in \(\tilde{F}_{\gamma'}^\oplus\) now denoted by \(\Omega'\)) and \(\Psi' = \tilde{V}_{\gamma'}(g)\Omega'\) with \(g(w) = w^{-1}, w \in S^1\). We write
\[ (\Psi', \tilde{V}_{\gamma'}(z)\Phi') = (\tilde{V}_{\gamma'}(g)\Omega', \tilde{V}_{\gamma'}(z)\Omega') = \]
\[ = \left( \frac{1}{2\pi} \int \tilde{V}_{\gamma'}(w)g(w) \, dw \right) \Omega', \tilde{V}_{\gamma'}(z)\Omega') = \]
\[ = \frac{1}{2\pi} \int_{|w'|=1} \left( \frac{1}{1 - w'^*z} \right)^{\gamma'^2} \frac{1}{w^*} \, dw^* = i. \]
(65)

Then
\[ \left( \Psi \otimes \Psi', \tilde{V}(f)(\Phi \otimes \Phi') \right) = \frac{i}{2\pi} \int \left( \Psi, \tilde{V}_{\gamma}(z)\Phi \right) f(z) \, dz. \]
(66)

Take \(\gamma'^2 = 1 - \gamma^2\) and use the boundedness of \(\tilde{V}(f)\) as operator in \(\tilde{F}_{\gamma'}^\oplus \otimes \tilde{F}_{\gamma}^\oplus\) to obtain
\[ \| (\Psi, \tilde{V}_{\gamma}(f)\Phi) \| \leq \| \Psi \| \| \Phi \| \| \tilde{V}_{\gamma'}(g)\Omega' \| \| f \|_2. \]
(67)
Now we use (35) to write
\[\| \tilde{V}(g')\Omega' \| \leq \| \tilde{V}(\gamma' = 1, g)\Omega' \| \leq \| g \|_2 = 1 \]  
and finally
\[|\langle \Psi, \tilde{V}(f)\Phi \rangle | \leq \|\Psi\| \|\Phi\| \|f\|_2 \]  
for arbitrary \(\Psi\) and \(\Phi\) in \(\tilde{F}_\gamma\). This proves boundedness of \(\tilde{V}(f)\) with norm smaller or equal \(\|f\|_2\), i.e. the same bound as for \(\gamma = 1\). Since \(\gamma, \gamma' \in \mathbb{R}\), \(\gamma^2 + \gamma'^2 = 1\), we have obtained boundedness of vertex operators for \(\gamma \in [-1, 1]\).

Note that in (61) we could have taken \(\gamma, \gamma'\) complex because all computations in the Heisenberg algebra do not depend on the involution. For example, for computing \(\|V_\gamma(z)\Phi_0\|_2^2, |z| < 1\) we write
\[\|V_\gamma(z)\Phi_0\|_2^2 = \langle V_\gamma(z)\Phi_0, V_\gamma(z)\Phi_0 \rangle\]
expand the exponentials in \(V_\gamma(z)\) and use the commutation relations, only. The usual computation over the formal adjoint
\[\langle V_\gamma(z)\Phi_0, V_\gamma(z)\Phi_0 \rangle = \langle \Phi_0, V_\gamma(z)^+V_\gamma(z)\Phi_0 \rangle\]
and the CBH formula is convenient [16] but not compulsory. It is the fact that we have to realize \(A_n\) in (61) as Hilbert space operators with involution \(A_n^+ = A_{-n}\) that forces \(\gamma, \gamma'\) to be real. Hence, until now we have proved the result for \(\gamma \in [-1, 1]\) only.

We now extend the region to the closed unit disc \(|\gamma| \leq 1\). Indeed the vacuum correlation functions of vertices depend only on the absolute value \(|\gamma|^2\) and this proves the extension. Let us summarize the result of this and the last section in the following

**Theorem 3** The chiral vertex operators \(\tilde{V}_\gamma(f)\), \(f \in L^2(S^1)\) smeared on the unit circle are bounded operators in the Hilbert space \(\tilde{F}_{\gamma}^\oplus\) for all \(\gamma \in \mathbb{C}\) with \(|\gamma| \leq 1\).

For similar results concerning \(V_\gamma(f)\) see next section. It is possible that this result could be true for other values of \(\gamma\), too, but other techniques should be used. In the next section we would like to comment on the results obtained in this paper.

### 6 Remarks and discussion

We have proved that contrary to the chiral unsmeared vertices, the smeared ones are well behaved, being bounded operators in the Hilbert space \(\tilde{F}_{\gamma}^\oplus\) constructed explicitly in section 4 by starting from the bosonic Fock space in which the vertices naturally live. The smearing in the complex variable \(z\) is only one-dimensional on the unit circle \(|z| = 1\), this is typical for free fields in quantum field theory.

Regarding the Hilbert space, there are further possibilities to realize it. We give one example (other constructions can be found in [17]). Consider the vertex operator \(V(z)\) as in (3) with \(\gamma = 1\). Let \(W\) be a finite linear combination of vectors \(V_n(f_j)\) , \(f_j = (f_{j1}, \ldots, f_{jn})\):
\[W = \sum_n V_n = \sum_{n,j} \alpha_j V_n(f_j)\]  
where in \(V_n\) we collect all terms in \(W\) with the same number \(n\) of vertex factors \(V\). In the bosonic Fock space \(F\) we change the scalar product \((\cdot, \cdot)\) to a new one which we define on vectors \(W\) by linearity and relations
\[s(V_n, V_m) = 0, \quad \text{if} \quad n \neq m\]
Let us remark that the set of vectors $W$ (74) is dense in $\mathcal{F}$ (see Appendix 2). We complete in the new scalar product and get a Hilbert space $\hat{\mathcal{F}}$. In both $\mathcal{F}$, $\hat{\mathcal{F}}$ the set of vectors $W$ is contained densely with respect to the different scalar products involved. In $\hat{\mathcal{F}}$ we introduce a densely defined bilinear form $\hat{V}(f_0)$, $f_0 \in L^2(S^1)$ by

$$\hat{V}(f_0)(V_n, V_m) = 0 \quad \text{for} \quad m \neq n - 1$$

$$\hat{V}(f_0)(V_n, V_{n-1}) = (V_n, V(f_0)V_{n-1}). \quad (72)$$

Again the generalized Gram inequality followed by some elementary reasoning shows that $\hat{V}(f_0)$ is a densely defined bounded bilinear form in $\hat{\mathcal{F}}$ w.r.t. the new scalar product $s(\cdot, \cdot)$. In conclusion it defines a bounded operator $\hat{V}(f)$ in $\hat{\mathcal{F}}$.

In this approach, as well as in sections 3,4 (where also the case $\gamma \neq 1$ was considered), the spaces $\mathcal{F}$ and $\hat{\mathcal{F}}$, although showing fermionic flavors, were directly coined in the bosonic Fock space in which vertices are formally defined. Neutrality was essentially used (76). Up to section 5 braiding was not introduced. In section 5 we used it although we believe that this would not be necessary if we could replace the abstract reasoning on fermions by a more direct computation in bosonic Fock space. We do not know what happens with the smeared vertices without neutrality, but we hope to return to this interesting problem elsewhere. Usually neutrality is motivated by the zero mass limit in two-dimensional field theory [18, 19]. Our interest in obtaining smeared vertices as bounded operators in a Hilbert space as close as possible to the bosonic Fock space was stimulated at the beginning by the idea of applying the causal Epstein-Glaser method [5] to the two-dimensional sine-Gordon model as well as to conformal perturbation theory. These two aspects of two-dimensional physics are strongly related [20]. Operatorial boundedness in both chiral and antichiral variables in a Minkowski approach will enable us to prove not only convergence of perturbation theory, which is an ancient result, but also to save a factorial factor in the estimate, leading to entire type in the result and convergence for all coupling constants. A similar (weaker) result for the case of two-dimensional conformal quantum field theory was already mentioned in [4]. We will come back to these questions elsewhere.

Last but not least sections 2 and 3 can be looked at as a first step towards a Hilbert space interpretation of vertex operator algebras (see for instance [28]). Indeed, through the smearing out operation the calculus of formal distributions (formal Dirac function) can be taken over to the Fourier space. Using truncation of formal expansions and going to the unit circle, the convergence is guaranteed independent of the side from which the unit circle is approached. Passing from inside to outside or vice versa is an involution on Fourier coefficients. According to the Hilbert space framework of this paper the calculus of formal distributions is then a formal calculus on kernels of Hilbert space operators.

**Appendix 1: Gram determinant inequalities**

Here we state some generalized Gram inequalities for the generalized Gram determinant. By the generalized Gram determinant we understand

$$G(x, y) \equiv G_p(x_1, x_2, \ldots, x_p; y_1, y_2, \ldots, y_p) =$$

$$= \begin{vmatrix} (x_1, y_1) & \cdots & (x_1, y_p) \\ \cdots \\ (x_p, y_1) & \cdots & (x_p, y_p) \end{vmatrix} \quad (A.1)$$
where \( x = (x_1, \ldots, x_p), y = (y_1, \ldots, y_p) \) are in the same vector space with scalar product \((\cdot, \cdot)\). The usual Gram determinant is obtained by taking \( x_i = y_i, i = 1, \ldots, p \).

Here we consider the case where \( x_i, y_j \) are \( L^2 \) functions. Let \( F \) and \( G \) be matrix functions given by

\[
F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \\ \end{pmatrix} \equiv \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots \\ f_{r1} & \cdots & f_{rn} \end{pmatrix} \tag{A.2}
\]

\[
G = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_s \\ \end{pmatrix} \equiv \begin{pmatrix} g_{11} & \cdots & g_{1m} \\ g_{21} & \cdots & g_{2m} \\ \vdots & \vdots & \vdots \\ g_{s1} & \cdots & g_{sm} \end{pmatrix}. \tag{A.3}
\]

We call the attention of the reader to the fact that here \( f_i, g_k \) represent vector functions. The generalized Gram inequality we use in this paper is

\[
\sum_{1 \leq j, j' \leq r} \alpha_j^* \beta_{k'}^* \alpha_{j'} \beta_k G_{n+m}(f_j, g_k; f_{j'}, g_{k'}) \leq \sum_{1 \leq k, k' \leq s} \beta_k^* \beta_{k'} G_m(g_k; g_{k'}), \tag{A.4}
\]

where \( \alpha_j, j = 1, \ldots, r; \beta_k, k = 1, \ldots, s \) are complex constants. The inequality used in (50) is the special case \( n = r = s = 1, \alpha_1 = 1 \) and the inequality (54) is the case \( n = r = 1, \alpha_1 = 1 \) (\( \alpha \)'s in (54) are \( \beta \)'s in (A.4)). All inequalities for the usual Gram determinants [21] are special cases of (A.4) with identical \( f \) and \( g \) functions.

Concerning the proof one could use the Landsberg formula [22]

\[
\int_{\triangle^p} \begin{vmatrix} x_1(s_1) & \cdots & x_1(s_p) \\ \vdots & \ddots & \vdots \\ x_p(s_1) & \cdots & x_p(s_p) \end{vmatrix} \begin{vmatrix} y_1(s_1) & \cdots & y_1(s_p) \\ \vdots & \ddots & \vdots \\ y_p(s_1) & \cdots & y_p(s_p) \end{vmatrix} d_p\nu(s) = \begin{vmatrix} (x_1, y_1) & \cdots & (x_1, y_p) \\ \vdots & \ddots & \vdots \\ (x_p, y_1) & \cdots & (x_p, y_p) \end{vmatrix} = p! G(x, y) \tag{A.5}
\]

where \( x_i, y_j \in L^2(\triangle, \nu), \triangle^p = \triangle \times \cdots \times \triangle \) and \( d_p\nu \) is the product measure, together with Laplace expansion formula for determinants.

Certainly another proof can be given by fermionic Fock space methods. Consider (smeared) fermionic annihilation and creation operators \( a(f), a^+(f) \):

\[
\{ a(f), a(g) \} = 0 = \{ a^+(f), a^+(g) \}
\]

\[
\{ a(f), a^+(g) \} = (f, g), \tag{A.6}
\]

where \( \{, \} \) is the anticommutator. The action in the fermionic Fock space is as usual given by

\[
(a(f)\psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1} \int dx f(x)^* \psi^{(n+1)}(x, x_1, \ldots, x_n) \tag{A.7}
\]
\[(a^+(f)\psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n} \sum_{i=1}^{n} (-1)^{i-1} f(x_i)\psi^{(n-1)}(x_1, \ldots, \hat{x}_i, \ldots x_n) \quad (A.8)\]

where \(\hat{x}_i\) indicates that the \(i\)-th variable is to be omitted and \(\psi^{(n)}(x_1, \ldots, x_n)\) is totally antisymmetric.

Let

\[\Psi = \sum_{j=1}^{r} \alpha_j a^+(f_j) \ldots a^+(f_{jn}) \quad (A.9)\]

\[\Phi = \sum_{k=1}^{s} \beta_k a^+(g_k) \ldots a^+(g_{km}). \quad (A.10)\]

Then we have on the vacuum \(\Omega\)

\[\|\Psi\|_2^2 = \sum_{j,j'} \alpha_j^* \alpha_{j'} G_n(f_j; f_{j'}) \quad (A.11)\]

\[\|\Phi\|_2^2 = \sum_{k,k'} \beta_k^* \beta_{k'} G_m(g_k; g_{k'}). \quad (A.12)\]

and

\[\|\Psi\Phi\|_2^2 = \sum_{j,j',k,k'} \alpha_j^* \beta_{k'}^* \alpha_{j'} \beta_k G_{n+m}(f_j, g_k; f_{j'}, g_{k'}). \quad (A.13)\]

Hence, the generalized Gram inequality (A.4) is proved if we can show that the operator norm \(\|\Psi\|_2^2\) is equal to

\[\|\Psi\|_2^2 = \sum_{j,j'} \alpha_j^* \alpha_{j'} G_n(f_j; f_{j'}). \quad (A.14)\]

This is a consequence of Wick’s theorem about normal ordering of operator products which we write down with the following simplified notation

\[a(f_n) \ldots a(f_1)a^+(g_1) \ldots a^+(g_n) = :a(f_n) \ldots a^+(g_n): + :a(f_n) \ldots a^+(g_j) \ldots a^+(g_n): + \ldots + G(f_1, \ldots, f_n; g_1, \ldots, g_n). \quad (A.15)\]

The r.h.s. is obtained from the l.h.s. by normal ordering denoted by double dots, that means by anticommuting all emission operators \(a^+\) to the left. The ”contractions” (indicated by the bracket upstairs) represent the anticommutators \((f_n, g_j)\) (A.6) which appear in this process. The last term has all operators contracted in pairs in all possible ways and this gives just the Gram determinant.

Now let us consider the square

\[(\Psi\Phi)^+ = \left[ \sum_{j,j'} \alpha_j \alpha_{j'} a^+(f_{j1}) \ldots a^+(f_{jn})a(f_{j'1}) \ldots a(f_{j'n}) \right]^2 \]

\[= \sum_{j,j',l,l'} \alpha_j \alpha_{j'}^* \alpha_l \alpha_{l'}^* a^+(f_{j1}) \ldots a^+(f_{jn}) \times \]

\[\times a(f_{j'1}) \ldots a(f_{j'1})a^+(f_{l1}) \ldots a^+(f_{ln})a(f_{l'1}) \ldots a(f_{l'n}). \quad (A.16)\]

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In the last line we substitute Wick's theorem (A.15). Then only the last term with Gram's determinant contributes because all other terms contain at least two equal Fermi operators $a(f)a(f) = 0$. This gives

$$
(\Psi\Psi^+)^2 = \sum_{j',l} \alpha^*_j \alpha_l G(f_{j'1}, \ldots, f_{j'n}; f_l1, \ldots, f_{ln}) \\
\times \sum_{j,l'} \alpha_j \alpha^*_l a^+(f_{j1}) \ldots a^+(f_{jn}) a(f_{l'n}) \ldots a(f_{l1}) =
$$

$$(\sum \alpha^* \alpha G)(\Psi\Psi^+)
$$

with obvious short-hand notation. Since $\Psi\Psi^+$ is selfadjoint this implies

$$
\|\Psi\Psi^+\| = |\sum \alpha^* \alpha G| = \|\Psi\|^2 = \|\Psi^+\|^2
$$

which is the desired result (A.14).

**Appendix 2**

Strictly speaking the result of this appendix is not necessary for the main body of the paper. This is because we realized neutrality by passing from the genuine bosonic Fock space $\mathcal{F}$ to Hilbert spaces $\tilde{\mathcal{F}}$ (see sections 3-5) where linear combinations of vectors $\tilde{V}_n$ are dense by construction. The same applies to the space $\hat{\mathcal{F}}$ in section 6. Nevertheless the results of this appendix help to get a better understanding of the structure of the Hilbert spaces in which our vertices live. It could gain on interest if we succeed to abolish neutrality. Furthermore, the problem touched here was mentioned as open in a related setting up in [24] (see also [27]).

It is our aim to show that the set of vectors

$$
V_n(f) = \prod_{i=1}^n V(f_i) \Phi_0
$$

where $f = (f_1, \ldots, f_n)$, $f_i \in L^2(S^1)$ is total in the Fock space $\mathcal{F}$. In other words linear combinations

$$
\sum_{j=0}^n \alpha_j V_n(f_j), \quad \alpha_j \in \mathbb{C}, \quad n = 0, 1, 2, \ldots
$$

(A.21)

where now $f_j = (f_{1j}, \ldots, f_{nj})$ are vector functions dense in $\mathcal{F}$. This type of property is common in quantum field theory and we could invoke this. The point is that vertices $V(f)$ are not exactly quantum fields because of the massless limit involved. A direct clean proof of this property is therefore desirable. This point was already mentioned in [24]. It turns out that the vectors (A.20) are coherent states of a special type. One expects coherent states to be dense (even overcomplete [14]).

We use the relation for the unsmeared vertices

$$
\prod_{i=1}^n V(z_i) = \prod_{1 \leq i < j \leq n} \left(1 - \frac{z_j}{z_i}\right) \exp\left(\sum_{i=1}^{\infty} \frac{1}{i} \sum_{j=1}^n z_j^i a_{-i}\right) \\
\times \exp\left(\sum_{i=1}^{\infty} \frac{1}{i} \sum_{j=1}^n z_j^{-i} a_i\right)
$$

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where $1 > |z_1| > |z_2| > \ldots |z_n|$, and apply it to the vacuum $\Phi_0$:

$$
\prod_{i=1}^{n} V(z_i)\Phi_0 = \prod_{1 \leq i < j \leq n} \left( 1 - \frac{z_j}{z_i} \right) \exp \left( \sum_{i=1}^{\infty} \frac{1}{i} Z_i^{i} a_{-i} \right) \Phi_0
$$

(A.22)

with $Z_i^{(n)} = \sum_{j=1}^{n} z_j^i$. Now

$$
\exp \left( \sum_{i=1}^{\infty} \frac{1}{i} Z_i^{i} a_{-i} \right) \Phi_0
$$

(A.23)

are typical coherent states [14]. The idea is to integrate (A.22) against powers of $z_i^{-1}$ and use

$$
\frac{1}{2\pi i} \int_{S^1} z^n dz = \delta_{n,-1}
$$

in order to produce elements $\Phi_\eta$ of the Fock basis out of (A.22). For instance, if $n = 1$, $Z_i = z_i^1$, we get

$$
V(z_1^{-2})\Phi_0 = \frac{1}{2\pi} \int V(z_1) z_1^{-2} dz_1 \Phi_0 \\
\sim \int \left( 1 + \frac{a_{-1}}{1!} z_1 + \frac{a_{-2}}{2!} z_1^2 + \ldots \right) z_1^{-2} dz_1 \Phi_0 = \\
= a_{-1} \Phi_0 \sim \Phi_\eta, \quad \eta = (1,0,0,\ldots).
$$

How to obtain states like $a_{-2} \Phi_0$ and $a_{-1}^2 \Phi_0$?

Let us expand

$$
V(z_1)\Phi_0 = \exp \sum_{n=1}^{\infty} \frac{z_1^n}{n} a_{-n} \Phi_0 \\
= \left[ 1 + \sum \frac{z_1^n}{n} a_{-n} + \frac{1}{2!} \left( \sum \frac{z_1^n}{n} a_{-n} \right)^2 + \ldots \right] \Phi_0
$$

and collect the terms with $z_1^n$:

$$
\frac{a_{-n}}{n} + \frac{1}{2!} \sum_{n_1+n_2=n} \frac{a_{-n_1} a_{-n_2}}{n_1 n_2} + \frac{1}{3!} \sum_{n_1+n_2+n_3=n} \frac{a_{-n_1} a_{-n_2} a_{-n_3}}{n_1 n_2 n_3} + \ldots
$$

(A.24)

The corresponding Fock state is obtained by integration with $z_1^{-n-1}$. We now consider two factors and use (A.22)

$$
V(z_1)V(z_2)\Phi_0 = g(z_1, z_2) \exp \sum_{n=1}^{\infty} \frac{z_1^n + z_2^n}{n} a_{-n} \Phi_0 \\
= g(z_1, z_2) \left( \exp \sum_{n=1}^{\infty} \frac{z_1^n}{n} a_{-n} \right) \left( \exp \sum_{n=1}^{\infty} \frac{z_2^n}{n} a_{-n} \right) \Phi_0,
$$

(A.25)

where $g(z_1, z_2)$ are the prefactors in (A.22). The latter are cancelled by working with the following smearing function $g^{-1} z_1^{n-1} z_2^{m-1}$. Since we can choose $n$ and $m$ independently, this gives us arbitrary products of two factors of the form (A.24). The generalization to more than two factors is straight-forward.
It is not hard to see that in this way we get a total set of Fock vectors. Indeed, starting with $n = 1$ in (A.24) gives $a_{-1} \Phi_0$, and the $\eta_1$-fold product with itself (denoted by $1^{\eta_1}$) gives $a_{-1}^{\eta_1} \Phi_0$. Next we take $n = 2$ in (A.24):
\[
\frac{a_{-2}}{2} + \frac{1}{2!}a_{-1}^2.
\]
Since $a_{-1}^2 \Phi_0$ is already constructed, we get $a_{-2} \Phi_0$. Forming products $1^{\eta_2} \times 2$, we find $a_{-1}^{\eta_1} a_{-2} \Phi_0$. Now we are ready to consider the product $(2^2)$ of two $n = 2$:
\[
\frac{1}{4}(a_{-2}^2 + 2a_{-2}a_{-1}^2 + a_{-1}^4).
\]
This gives us $a_{-2}^2 \Phi_0$ because the other two vectors are already known. The next steps in the process are:
\[
a_{-1}^{\eta_1} a_{-2}^2 \Phi_0, \ a_{-2}^3 \Phi_0, \ a_{-1}^{\eta_1} a_{-2}^2 \Phi_0, \ldots, \ a_{-2}^{\eta_2} a_{-1}^{\eta_1} \Phi_0.
\]
We continue with $n = 3$ and so on. We thus arrive at the usual basis in Fock space. The same proof goes through for $\gamma \neq 1$.

The test functions we have used in the foregoing construction are of a more general kind than the simple products of $f_i(z_i)$ in (A.20). But this is no essential point because such test functions can be approximated by linear combinations of simple products with help of the Laurent expansion.

In particular the above proof shows that the set of smeared coherent states
\[
\prod_{i=1}^{n} V_{-}(f_i) \Phi_0, \quad n = 0, 1, 2, \ldots \tag{A.26}
\]
where $f_i \in L^2(S^1)$ is total in $\mathcal{F}$. Concerning the unsmeared case, the situation is not completely clear to us. We do not know whether the set (A.26) where the smearing is left out and the variables satisfy $1 > |z_1| > |z_2| \ldots > |z_n|$ is total in $\mathcal{F}$. On the other hand a slightly larger set based on
\[
\exp\left(\sum_{n} t_n a_{-n}\right) \Phi_0
\]
where now $t_n$ are left arbitrary in $\mathbb{C}$ is total [11]. These are the genuine coherent states in infinitely many variables.

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