Length-Bounded Paths Interdiction in Continuous Domain for Network Performance Assessment

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Abstract—Studying on networked systems, in which a communication between nodes is functional if their distance under a given metric is lower than a pre-defined threshold, has received significant attention recently. In this work, we propose a metric to measure network resilience on guaranteeing the pre-defined performance constraint. This metric is investigated under an optimization problem, namely Length-bounded Paths Interdiction in Continuous Domain (cLPI), which aims to identify a minimum set of nodes whose changes cause routing paths between nodes become undesirable for the network service.

We show the problem is NP-hard and propose a framework by designing two oracles, Threshold Blocking (TB) and Critical Path Listing (CPL), which communicate back and forth to construct a feasible solution to cLPI with theoretical bicriteria approximation guarantees. Based on this framework, we propose two solutions for each oracle. Each combination of one solution to TB and one solution to CPL gives us a solution to cLPI. The bicriteria guarantee of our algorithms allows us to control the solutions’s trade-off between the returned size and the performance accuracy. New insights into the advantages of each solution are further discussed via experimental analysis.

I. INTRODUCTION

Components of a network do not have the same important level. There always exists a set of nodes or edges which plays more critical role than the others on assessing networks’ performance. Literature has spent a significant effort on identifying such a set whose removal maximally damages a network’s functionality. Most of early efforts used the connectivity metric, in which a connection between two nodes is functional if there exists a path connecting them [1], [2], [3], [4], [5], [6]. However, as modern networks evolved, purely relying on connectivity is no long sufficient to guarantee a networks’ functionality or quality of services. Further, instead of removing nodes/edges, a change on components’ behavior can downgrade the whole system’s performance. For example, a congestion or traffic jams [7], [8] on some routers can significantly delay communication between end systems, downgrading their quality of services.

Motivated by the above observations, recent researches turn their attention to network malfunction without damaging the connectivity. For example, Kuhnle et al. [9] studied the LB-MULTICUT problem, which aims for a minimum set of edges whose removal causes the shortest distance, in term of edge weights, between targeted pairs of nodes exceed a threshold. The threshold represents constraints for the networks in order to guarantee quality of services. By discarding the “removal” flavour, Nguyen et al. [10] extended this concept to introduce the QoSD problem, in which an edge weight can be varied with an amount of efforts, defined in the discrete domain, and the problem asks for a minimum amount of efforts for the same objective as in LB-MULTICUT.

However, these existing works are all in the combinatorial optimization, which do not capture well the continuous settings. For example, in network routing, factors that impact network components’ latency or packet loss rate include: traffic rate [11], power of the interfering signal and noises [12], denial-of-service attacks [13]. Those factors are quantified under continuous variables.

Therefore, in this work, we take a further step on network performance assessment by introducing the cLPI problem as follows: Given a directed network $G = (V, E)$, a set $S$ of target pairs of nodes and a threshold $T$, each node $v \in V$ is associated with a monotone non-decreasing function $f_v : \mathbb{R} \to \mathbb{R}$, the cLPI problem asks for an impact vector $x = \{x_v\}_{v \in V}$ with minimum $\|x\|$ such that any path $p$, connecting a pair in $S$, satisfies $\sum_{v \in p} f_v(x_v) \geq T$. Intuitively, $x_v$ represents the external impact’s level to node $v$; while $f_v(x_v)$ quantifies $v$’s behaviors in response to the impact. $T$ represents the network constraint in order to guarantee quality of services, e.g. low latency. A solution $x$ of cLPI can be used to assess network resilience to the external impact. Specifically, large $\|x\|$ indicates the network is resilient to external interference and able to maintain quality of service under extreme environment. Furthermore, a value of $x_v$ indicates the important level of node $v$ to the network desired functionality.

Solving cLPI with bounded performance guarantee is challenging, indeed. If for all path $p$, $\sum_{v \in p} f_v(x_v) \geq T$ exhibits convexity, then cLPI can be solved optimally by using ellipsoid method [14] with a polynomial feasible-check oracle. However, that is not always the case. Studies on network latency w.r.t impact factor like traffic rate shows $\sum_{v \in p} f_v(x_v) \geq T$ is not convex [13], [15]. That also rules out the possibility of applying any other Convex Optimization technique. In term of packet loss rate, the behaviors are even more complicated [12]. Indeed, we show that cLPI with general functions are NP-hard problem. Thus, in this work, we aim for a general solution that can applied on any monotone non-decreasing functions $f_v$.

Contributions. In addition to introduce the cLPI problem, the main contributions of this work are:

- We propose a general framework for solving cLPI, separating tasks into two different oracles, called Critical Paths Listing (CPL) and Threshold Blocking (TB). CPL’s job is to restrict the amount of paths considered for...
finding a feasible solution of \( cLPI \). TB handles the task of finding \( x \), guaranteeing all paths, returned by CPL, having lengths exceeded \( T \). For each oracle, we design two algorithmic solutions. Different combination of any CPL and TB algorithms provides different performance theoretically and practically.

- All of our solutions have bicriteria approximation ratios, which could allow a user to control the trade-off between runtime versus accuracy.
- We extensively evaluate our solutions on real-world AS networks. The experiments show our algorithms outperforms existing solutions of special problems of \( cLPI \) in solution quality. We then shed a new insight on advantages of each algorithms.

**Organisation.** The rest of the paper is organized as follows. Section II reviews literature related to our problem. In Section III we formally define the \( cLPI \) problem, discuss its challenges and overall framework of our solutions. Section IV presents two algorithms for the TB oracle while the ones for CPL are described in Section V. In Section VI practical analysis on algorithms’ performance is provided. Finally, Section VII concludes the paper.

## II. Application and Related Work

We first discuss a key application of \( cLPI \) in network performance assessment and next highlight the most relevant related work to \( cLPI \).

### A. \( cLPI \) in Network Performance Assessment

A routing protocol specifies how routers communicate with each other to distribute information that enables them to select routes between any two nodes on a computer network \cite{16, 17, 18}. The specific characteristics of routing protocols include the manner in which they avoid routing loops and select preferred routes, using information about hop costs. With the introduction of Software-Defined-Networking \cite{19, 20}, a hop cost can vary from different metrics, serving for different purposes of network administrators.

The most common used metric for network vulnerability is network latency. Ideally, communication between hosts in the network is routed in the shortest path, weighted by latency of nodes. On the other hand, to guarantee quality of services (e.g. low latency) or avoid unexpected routing scheme (e.g. inter-continent routing with intra-traffic), a limit on network latency can be set so that the routing path has to have latency lower than a threshold. If there exists no routing path with total latency lower than the threshold, the network is considered to be undesirable for required services \cite{9, 10}.

In the context of \( cLPI \), to model the external impact to a hop latency, each node \( v \) (e.g. routers) in the network is associated with a function \( d_v(x) \) where \( x \) quantifies the impact (e.g. traffic rate, noise); and \( d_v(x) \) measure the latency of \( v \) with the impact \( x \). Denote \( T \) as the latency threshold. Studying \( cLPI \) helps identify the impact levels on nodes/edges that required to damage the networking quality of services, thus providing a useful metric for network design and assessment.

Beside latency, another routing metric can be used is packet loss probability. A routing path with high probability (say at least 90%) of successful delivery is preferred. Unlike latency, in term of packet loss probability, a simple trick needs to be applied. Let \( \rho_v(x) \) denote the loss probability of a packet if going through node \( v \) given the external impact amount \( x \); and \( P \) is the expected successful probability of a routing path. Then the network routing is not functional if for a routing path \( p \), \( \prod_{v \in p} (1 - \rho_v(x)) \leq P \). This equation is adapted to \( cLPI \) as \( \sum_{v \in p} -\ln(1 - \rho_v(x)) \geq -\ln P \).

### B. Existing Algorithms

The early work on network resilience assessment with constraints on distance between node pairs are \( LB-MULTICUT \), \cite{9}. Critical Node Detection \cite{21, 22}. Multicut \cite{11, 23}. With the objective to make all pairs’ distance to be at least \( T \) or be disconnected, the problem asks for a minimum set of edges or nodes to be removed. One way to apply their solutions to \( cLPI \) is by introducing a step of discretization of function \( f_v \). In the context of node removal, the cost of cutting a node \( v \) is represented by value \( x \) where \( f_v(x) = T \). Our experimental results, unfortunately, shows solving \( cLPI \) by this method returns undesirable solutions in some cases. Other than that intuitive adoption, it is unclear how to convert the work of edge/node removal to the flavour of increase edge/node weight as an instance of \( cLPI \).

Without targeting for the edge/node removal, the \( QoSD \) problem introduces a discrete function \( b_v : \mathbb{Z}^2 \rightarrow \mathbb{Z}^+ \) associated with each node \( v \) of the network; and asks for a \( x \) in integer lattice that any path \( p \) connecting a given node pairs have length exceeding \( T \), i.e. \( \sum_{v \in p} b_v(x_v) \geq T \) \cite{10}. One may think to discretize functions \( f_v \) and directly adopt solutions of \( QoSD \) to solve \( cLPI \). However, the discretization of \( f_v \) is simply a work of taking an integer \( x \) and returning the value \( f_v(x \times \delta) \), where \( \delta \) is called discretizing step. If \( \delta \) is too large, the returned solution will be far from optimum due to discretization error; otherwise small \( \delta \) creates significantly large inputs for \( QoSD \), causing a burden on memory usage and undesirable runtime. Therefore, a solution, which can directly applied into continuous domain, is more desired.

\( cLPI \) can be modeled under a Constrained Optimization formulation that minimize \( \sum_{v \in V} x_v \) with constraints \( \sum_{v \in p} f_v(x_v) \geq T \) for all paths \( p \) connecting pairs and \( x_v \geq 0 \) for all \( v \in V \). The first constraint is to guarantee that all paths connecting target pairs have the length exceeding threshold \( T \). Constrained Optimization is a classical problem, on which significant amount of works have been investigated, including (but not limited to) \cite{23, 24}. However, a major concern of applying those solutions to \( cLPI \) is that a set of constraints is required to be known beforehand. In the case of dense network, the set of constraints reach to \( \sum_{k=0}^{n} \binom{n}{k} k! \) paths and an “infinite” period only for enumerating them. Furthermore, even we can list all constraints, those methods meet another obstacle that edge weight functions could be any function with complex behaviors. Existing methods can easily end up to local convergence trap without any performance guarantee. Therefore, a solution, which helps reduce burden of path...
listing while providing a performance ratio, is more desirable. That is a focus of our work.

III. PRELIMINARIES

A. Problem Formulation

In this part, we formally define the cLPI problem and notations used frequently in our algorithms.

We abstract the network using a directed graph \( G = (V, E) \) with \(|V| = n \) nodes and \(|E| = m \) directed edges. Each node \( v \) is associated with a function \( f_v : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) which indicates the weight (e.g., latency, loss rate) of \( v \) w.r.t. an impact amount on \( v \). In another word, if external impact of an amount of \( x \) is put on \( v \), the weight of node \( v \) will become \( f_v(x) \). \( f_v \) is monotonically non-decreasing for all \( v \in V \), which can be intuitively explained by: the more impact are put on \( v \), the worse \( v \) behaves (e.g., long latency, high loss rate).

Given \( V = \{v_1, ..., v_n\} \), we denote the impact in form of a vector \( x = [x_1, ..., x_n] \) where \( x_n \) is an impact on node \( v_n \). For simplicity, we use the notation \( v \) to present a node in \( V \) and its index in \( V \) also. So \( x_v \) means the impact on node \( v \), and the entry in \( x \) corresponding to \( v \) also. The overall impact on all nodes, therefore, is \( |x| = \sum_{v \in V} x_v \).

A path \( p = \{u_0, u_1, ..., u_l\} \in \mathcal{G} \) is a sequence of vertices such that \((u_{i-1}, u_i) \in E \) for \( i = 1, ..., l \). A path can also be understood as the sequence of edges \( \{(u_0, u_1), (u_1, u_2), ...,(u_{l-1}, u_l)\} \). In this work, a path is used interchangeably as a sequence of edges or a sequence of nodes.

Under an impact vector \( x \), the length of a path \( p \) is denoted as \( d_x(p) \) where \( d_x(p) = \min\left( \sum_{v \in p} f_v(x_v), T \right) \). The min term is to bound a path’s length by \( T \). Since we only care about paths of length at most \( T \), this bound does not impact our algorithms’ results or the problem’s generality.

We abuse the notation by also using \( d \) to denote distance between two nodes in the network. To be specific, \( d_x(u, v) \) denotes distance between node \( u \) and \( v \) under \( x \), i.e., \( d_x(u, v) = \min_{p \in \mathcal{P}(u, v)} d_x(p) \).

A single path is a path that there exists no node who appears more than once in the path. Let \( \mathcal{P} \), denote a set of simple paths connecting the pair \((s_i, t_i) \in S \) such that \( \sum_{v \in P} f_v(0) < T \) for all \( p \in \mathcal{P} \). Let \( \mathcal{F} = \bigcup_{s=1}^{k} \mathcal{P}_i \), we call a path \( p \in \mathcal{F} \) a feasible path and \( \mathcal{F} \) is a set of all feasible paths in \( G \). A non-feasible path either connects no pair in \( S \) or has an initial length exceed \( T \). cLPI is formally defined as follows:

**Definition 1.** Length-bounded Paths Interdiction in Continuos Domain (cLPI). Given an undirected graph \( G = (V, E) \) with \(|V| = n \) nodes and \(|E| = m \) directed edges. Each node \( v \) with weight functions \( w.r.t. \) impact on nodes and a target set \( \mathcal{T} \) of pairs of nodes \( S = \{(s_1, t_1), ..., (s_k, t_k)\} \), determine an impact vector \( x \) with a minimum \( \|x\| \) such that \( d_x(s_i, t_i) \geq T \) for all \( (s_i, t_i) \in S \).

Let’s look at several mathematical operators on vector space \( \mathbb{R}^n \), which are used along the theoretical proofs of our algorithms. Given \( x = \{x_1, ..., x_n\}, y = \{y_1, ..., y_n\} \in \mathbb{R}^n \), define:

\[
\begin{align*}
x + y &= \{x_1 + y_1, ..., x_n + y_n\} \\
x \setminus y &= \{\max(x_1 - y_1, 0), ..., \max(x_n - y_n, 0)\}
\end{align*}
\]

Moreover, we say \( x \leq y \) if \( x_v \leq y_v \) for all \( v \in V \), the similar rule is applied to \( <, \leq, >, \geq \).

**Theorem 1.** \( cLPI \) is an NP-hard problem

**Proof.** We reduce QoSD to cLPI as follows: Given an instance of the QoSD problem, including a directed graph \( G = (V, E) \) and a set of target pairs of nodes \( S \), each node \( v \) is associated with a monotone discrete functions \( b_v : \mathbb{Z}^2 \rightarrow \mathbb{Z}^+ \). QoSD asks for a minimum \( \|x\| \) that any path \( p \) connecting a pair in \( S \) has length exceeding \( T \), i.e., \( \sum_{v \in p} b_v(x_v) \geq T \).

We create an instance of cLPI by keeping \( G, S, T \) and defining the node weight functions \( \{f_v\} \) in continuous domain by letting \( f_v(x) = b_v(\lfloor x \rfloor) \) for \( x \in \mathbb{R}^2 \) and \( \forall v \in V \). \( f_v \) is monotone non-decreasing function in continuous domain.

It is trivial that each entry \( x_v \) of an optimal solution of this cLPI instance should be an integer (or else we can replace \( x_v \) by \( \lfloor x_v \rfloor \) and the cLPI’s objective still go through). Since the optimal solution of cLPI contains all integers, it is also an optimal solution of QoSD. And vice versa, an optimal solution of QoSD is also an optimal solution of this cLPI instance. Thus, cLPI is at least as hard as QoSD. And since QoSD has been proven to be NP-hard, cLPI is an NP-hard problem.

B. General model of our solutions

**Properties of Performance Guarantees.** Given a problem instance with a threshold \( T \), denote \( \mathcal{OPT} \) as an optimal solution. We call an impact vector \( x \) is \( \varepsilon \)-feasible to \( cLPI \) if under \( x \), the distance between each target pair is at least \( T(1 - \varepsilon) \). Our algorithms are bicriteria approximation algorithms, returning an \( \varepsilon \)-feasible solution \( x \) whose overall impact is bounded within a factor \( O(\ln |\mathcal{F}|\varepsilon^{-1}) \) of \( \mathcal{OPT} \). \( \varepsilon \) is treated as a trade-off between the algorithms’ accuracy and returned \( \|x\| \). To be specific, the smaller \( \varepsilon \) is, the closer pairs’ distances are to \( T \) but the larger the returned solution is. \( \varepsilon \) is adjustable, allowing users to control the trade-off as desired.

**General Framework.** Our solutions contain two separate oracles, called Threshold Blocking (TB) and Critical Paths Listing (CPL). These two oracles communicate back and forth with each other to construct a solution to cLPI, given an input instance of cLPI and a parameter \( \varepsilon \). These two oracles are proposed to tackle two challenges of cLPI as stated before, to be specific:

- **Threshold Blocking** - a primary role of TB is to solve a sub-problem of cLPI: Given a target set \( \mathcal{P} \) of single paths and an initial impact vector \( x \), TB aims to find \( s \) of minimum \( \|s\| \) to \( x \) in order to make \( d_{x+s}(p) \geq T \) for all \( p \in \mathcal{P} \). For simplicity, we call this task TB problem.
- **Critical Paths Listing** - this oracle restricts the number of paths to be considered in the algorithm, thus significantly reducing the searching space and burdens on algorithms’ runtime and memory for storage.

We propose multiple algorithms for each oracle. Specifically, we devise two algorithms for CPL, which are **Feasible Set Construction** and **Incremental Interdiction**. To solve TB, we develop two algorithms, called **Threshold Expansion** and **Jump Start Greedy**. Different combinations of CPL and TB
algorithms provide different performances theoretically and experimentally.

In general, the flow of our algorithms is:
1) The algorithm starts with \( x_0 = 0 \) for all \( v \in V \).
2) Given the current state of \( x \), by using a technique to restrict searching space, CPL oracle searches for a set of critical paths \( P \), who are feasible paths and \( d_x(p) < T \) for all \( p \in P \).
3) Then those paths along with a current state of \( x \) are given as an input for the TE oracle, which then finds an additional budget \( s \) for \( x \) to make \( d_{x+s}(p) \geq T \) for all \( p \in P \).
4) The additional budget \( v \) is then used for CPL to check the feasibility. If adding \( s \) makes \( x \)-feasible, the algorithm returns \( x + s \) and terminates. Otherwise, \( s \) is used to drive the searching space of CPL and find a new value for \( x \); then step (2) is repeated.

IV. THRESHOLD BLOCKING ORACLE

In this section, we present two algorithms for Threshold Blocking (TB) Oracle, called Threshold Expansion (TE) and Jump Start Greedy (JSG).

To recap, TB receives a set \( P \) of feasible paths from CPL, an impact vector \( x \). The objective of TB is to find an additional vector \( s = \{s_1, \ldots, s_n\} \) with minimum \( \sum s_v \) such that \( d_{x+s}(p) \geq T \) for all \( p \in P \).

Denote \( s^* \) as an optimal solution, i.e.

\[
    s^* = \arg \min_s \{d_{x+s}(p) \geq T \} \forall p \in P \|s\|
\]

The bicriteria guarantee of our algorithms originates from TB’s algorithms. Say in another way, instead of finding an exact solution, the desired accuracy \( \epsilon \) is given to the TB oracle so TB’s algorithms find \( s \) such that \( d_{x+s}(p) \geq T(1-\epsilon) \) for all \( p \in P \).

Denote \( (v, x) \in \mathbb{R}^n \) as a vector which receives value \( x \) at entry \( v \) and 0 elsewhere.

Given a path set \( P \), a vector \( w \) and a node \( v \), let:

\[
    r_{P, w, v}(x) = \sum_{p \in P} (d_{w+(v, x)}(p) - d_w(p))
\]

Intuitively, \( r_{P, w, v}(x) \) measures the total increasing lengths, under an impact vector \( w \), of paths in \( P \) by adding an amount \( x \) to entry \( v \) of \( w \).

A. Threshold Expansion

In general, TE works in rounds and in each round, TE set up a requirement on an amount to be added in each node. The requirements are relaxed after each round in order to allow new amount to be added; and the algorithm stops when guaranteeing the obtained solution \( s \) make \( d_{x+s}(p) \geq T(1-\epsilon) \) for all \( p \in P \).

The requirement in each round of TE is in a form of a number \( M \), which is initiated to be a large number. An amount \( x \) to be added into \( v \) guarantees \( x = \max \{x \geq 0 \mid \frac{r_{P, w, v}(x)}{x} \geq M\} \). Intuitively, the condition \( \frac{r_{P, w, v}(x)}{x} \geq M \) is to ensure the additional amount is meaningful and significant in comparison with putting an impact on other nodes. Since there could be a wide range of \( x \) that can satisfy \( \frac{r_{P, w, v}(x)}{x} \geq M \), the algorithm targets for the maximum \( x \) because it helps the algorithm quickly reach to the feasible solution. After \( x \) is added to entry \( v \), the algorithm discards paths \( p \) that \( d_{x+s}(p) \geq T(1-\epsilon) \) out of \( P \) since those paths have fulfilled the algorithm’s target.

After considering adding impacts to all nodes with a constraint in term of \( M \), the algorithm reduces the value of \( M \) to be \( (1-\epsilon)M \) with \( \epsilon \) is a constant parameter inputted to the algorithm. The reduction in \( M \) is to let new impact amounts be added into nodes. On the other hand, \( \epsilon \) impacts the performance of the algorithm. Intuitively, the lower value of \( \epsilon \) is, the better solution quality the algorithm can obtain but the longer running-time for the algorithm to terminate. The pseudocode of the algorithm is presented in Alg. 1.

TE’s theoretical performance is obtained with an assumption that: \( M \) - initiated at line 2 of Alg. 1 - satisfies:

\[
    M \geq \frac{r_{P, w, v}(x)}{x} \quad \text{for all } w \geq x, v \in V \text{ and } x \geq 0 \quad (1)
\]

This assumption can be removed if \( f_v \)'s are differentiable everywhere. In that case, we set \( M \) as the following lemma.

**Lemma 1.** If \( f_v \)'s are differentiable everywhere, by setting

\[
    M = |P| \times \max_{x \geq 0, v \in V, f_v(x) \leq T} \frac{\partial f_v}{\partial x}
\]

the condition (1) is satisfied.
Proof. We have:
\[
\frac{r_{P, x} (x)}{x} = \sum_{p \in P} \frac{d_{w+(v, x)} (p) - d_w (p)}{x} \leq \frac{|P|}{x} \max_{p \in P} \frac{d_{w+(v, x)} (p) - d_w (p)}{|x|} \leq \frac{|P|}{x} \max_{x : f_x (x) \leq T} \frac{f_x (x) - f_v (w_e)}{x - w_e} \leq \frac{|P|}{x} \frac{\partial f_v}{\partial x} \frac{p}{x} \geq \frac{|P|}{x} \frac{\partial f_v}{\partial x} \frac{1}{x} \geq \frac{|P|}{x} \frac{1}{|P|} \frac{1}{x} = \frac{1}{x},
\]
which completes the proof.

From now on, for simplicity, when we analyze the performance of TE at an iteration of the while loop of line 4 Alg. 1 we refer \( M, s, P \) as their values at that iteration.

Let’s consider an iteration of line 4 Alg. 1 denote \( s^o = \{ s^o_u \} v = s^* \setminus s \). We have the following lemma.

**Lemma 2.** \( s^o_u = 0 \) or \( \frac{r_{P, x} (s^o_u)}{s^o_u} < \frac{M}{1 - \epsilon} \) for all \( v \in V \).

**Proof.** This lemma is trivial at the time each node being first observed because of the condition [1]. Therefore, we only consider an arbitrary moment after \( M \) has been reduced by line 2.

Assume there exists a node \( v \) such that \( s^o_u > 0 \) and \( r_{P, x} (s^o_u) > M \). Consider the last time \( x \) was observed and \( \hat{x} = \max \{ x \in [0, 1] \mid \frac{r_{P, x} (s^o_u)}{s^o_u} \leq M \} \) where \( s' \) is \( s \) before adding \( \hat{x} \) into \( v; M' = M \) if \( x \) was lost in the current round, otherwise \( M' = M \).

We have \( s \geq s' + (v, \hat{x}) \) but \( s' + (v, \hat{x}) \) have the same value at entry \( v \), thus for any \( p \in P \) that contains \( v \):
\[
d_{x+s+(v, \hat{x})} (p) - d_{x+s} (p) \geq d_{x+s+s'} (p) - d_{x+s} (p)
\]

Therefore,
\[
r_{P, x} (s^o_u) \geq r_{P, x} (s^o_u) \geq s^o_u M \frac{1}{1 - \epsilon}
\]

So:
\[
r_{P, x} (s^o_u) = r_{P, x} (s^o_u) + r_{P, x} (s^o_u) \geq M \frac{1}{1 - \epsilon}
\]

Then an amount of at least \( \hat{x} + s^o_u \) should be added into \( v \), which contradicts to assumption that \( \hat{x} \) is the selected amount.

Lemma 2 allows us to bound the performance guarantee of TE, which is shown in the following theorem.

**Theorem 2.** Given \( G, \{ f_v \} v, P, T, \epsilon, x \), if \( s \) is the additional impact vector returned by TE and \( s^* \) is the optimal vector to make \( d_{x+s^*} (p) \geq T \) for all \( p \in P \), then:
\[
|s| \leq \ln \left( \frac{|P| \epsilon^{-1} + 1}{1 - \epsilon} \right) |s^*|
\]

**Proof.** Let us consider at an arbitrary iteration of the while loop at line 4 node \( v \) is being observed, and \( \hat{x} \) is a selected amount to add into \( v \) but has not been added to \( s \) yet. Again, denote \( s^o_v = \{ s^o_u \} v = s^* \setminus s \). Without lost of generality, let \( \hat{x} > 0 \). From lemma 2 we have:
\[
r_{P, x} (\hat{x}) \geq (1 - \epsilon) r_{P, x} (s^o_u)
\]
for all \( u \in V \) that \( s^o_u > 0 \).

Denote \( h_u = \{ s_u + 1|x_u > u \} \) for all \( u \in V \) as \( h_u \geq s \) but they have the same value in entry \( u \), we have:
\[
r_{P, x} (h_u) \leq r_{P, x} (s^o_u)
\]

Therefore,
\[
\sum_{p \in P} (d_{x+s+s'} (p) - d_{x+s} (p)) \leq \sum_{u \in V} r_{P, x} (h_u) \leq \sum_{u \in V} s^o_u \frac{d_{x+s+(v, \hat{x})} (p) - d_{x+s} (p)}{x}
\]

A simple transformation and the fact that \( P_{t+1} \subseteq P_t \) gives us:
\[
\sum_{p \in P_{t+1}} (T - d_{x+s+t+1} (p)) \leq \left( 1 - \frac{\hat{x}}{\|s^*\|} \right) \sum_{p \in P_t} (T - d_{x+s} (p))
\]

Therefore,
\[
\sum_{p \in P_{t+1}} (T - d_{x+s+t+1} (p)) \leq \left( \frac{1}{|P|} \right)^{t+1} \left( \frac{1}{|s^*|} \right) \frac{T}{t+1}
\]

On the other hand, \( \sum_{p \in P_{t+1}} (T - d_{x+s+t+1} (p)) \geq T \epsilon \) since \( d_{x+s+t+1} (p) < T (1 - \epsilon) \) for all \( p \in P_{t+1} \); and \( P_{t+1} \neq 0 \). Therefore:
\[
|s_{t+1}| \leq \frac{\ln (T)}{|P| |s^*|} \frac{1}{1 - \epsilon}
\]

Now, let consider the final update, we have:
\[
\hat{x} \leq \frac{|s^*|}{1 - \epsilon} \frac{1}{\sum_{p \in P_{t+1}} (T - d_{x+s+t+1} (p))} \leq \frac{|s^*|}{1 - \epsilon}
\]
Given a impact vector $\hat{v}$, which completes the proof.

**B. Jump Start Greedy**

In general, JSG works in a greedy manner that iteratively adds an impact amount to a node which maximizes $\frac{r_{\hat{p},x+sv}(x)}{x}$. The problem is that there exists cases due to traits of the functions $f_v$s, the selected budget is 0 and the algorithm falls into infinite loops. We call such situation “zero trap”. JSG overcomes that challenge by introducing **Jump Start** step to escape the zero trap while keeping a reasonable theoretical performance guarantee.

JSG runs in multiple iterations and for each iteration:

- **Step (1)**, for each node $v$, the algorithm finds a budget $\hat{x}_v$ that maximizes $\frac{r_{\hat{p},x+sv}(x)}{x}$. If $\hat{x}_v = 0$ (which typically happens when $f_v$ is concave), we do the jump start by forcing the minimum amount added to $v$ to be at least a value of $\beta = O(\|s^*\|/n)$ (how we obtain the value of $\beta$ will be described later). In that case, $\hat{x}_v = \argmax_{x\geq \beta} \left\{ \frac{r_{\hat{p},x+sv}(x)}{x} \right\}$.

- **Step (2)**, the algorithm selects a node $v$ that maximizes $\frac{r_{\hat{p},x+sv}(x)}{x}$ and add $\hat{x}_v$ into $v$. The algorithm repeats step (1) until $d_{x+}(p) \geq T(1-\varepsilon)$ for all $p \in \mathcal{P}$.

The pseudo-code of JSG is presented in Alg. 2 and JSG’s performance guarantee is stated in the following theorem.

**Theorem 3.** Given $G$, $\{f_v\}_v$, $\mathcal{P}$, $T$, $\varepsilon$, $x$ given to the TB oracle. If $v^*$ is the impact vector returned by JSG and $v^*$ is the optimal vector make $d_{x+}(p) \geq T(1-\varepsilon)$ for all $p \in \mathcal{P}$, then

$$\|v^*\| \leq O(\ln (|\mathcal{P}|e^{-1})\|v^*\|)$$

**Proof.** Let’s consider at a certain iteration of while loop (line 3 Alg. 2), $s$ is now under construction (not returned solution) and $\mathcal{P}$ is not empty. Again, denote $s^0 = \{s^0_v\}_v = s^* \setminus s$ and $h_u = \{s_w + 1_{w>u}s^0_w\}_w$. From the proof of Theorem 2 we have that $r_{\hat{p},x+sh_u}(s^0_u) \leq r_{\hat{p},x+sh_u}(s^0_u)$:

$$\sum_{p \in \mathcal{P}} \left( d_{x+}(p) - d_{x+}(p) \right) \leq \sum_{u \in V} r_{\hat{p},x+sh_u}(s^0_u)$$

Due to monotonicity, $r_{\hat{p},x+sh_u}(s^0_u) \leq r_{\hat{p},x+sh_u}(s^0_u + \beta)$. We observe that: Even a node $v$ was forced to take jump start step or not, the selected amount $\hat{x}_v$, always satisfies $\frac{r_{\hat{p},x+sv}(x)}{x}$ for all $x \geq \beta$. Thus, let’s assume $v$ is the selected node in this while iteration with the increasing impact amount of $\hat{x}_v$. Due to greedy selection, we have:

$$\sum_{u \in V} r_{\hat{p},x+sh_u}(s^0_u) \leq \sum_{v \in V} \hat{x}_v + \beta \sum_{v \in V} r_{\hat{p},x+sv}(s^0_u)$$

Now, let’s assume the algorithm terminates after adding impact amounts into nodes $L$ times, denote $\hat{x}_1, ..., \hat{x}_L$ as an added amount at each times ($\|s\| = \sum_{t=1}^{L} \hat{x}_t$). Also, denote $s_t$, $\mathcal{P}_t$ as $s$, $\mathcal{P}$ before adding $\hat{x}_t$ at time $t$. Using the same transformation as in proof of TE, we obtain the recursion relationship between $s_t$, $\mathcal{P}_t$ as follows:

$$\sum_{p \in \mathcal{P}_t} \left( T - d_{x+}(p) \right) \leq \left( 1 - \frac{\hat{x}_t}{\|v^*\| + \beta n} \right) \sum_{p \in \mathcal{P}_t} \left( T - d_{x+}(p) \right)$$

Using the same technique as in TE to discarding the terms from round $t = 1$ to $L - 1$, we have

$$\|s\| \leq \left( \|v^*\| + \beta n \right) O(\ln |\mathcal{P}|e^{-1})$$

The theorem follows given the fact that $\beta = O(\|v^*\|/n)$. □

Now the only question left is how to identify $\beta = O(\|v^*\|/n)$. The trivial answer is $\beta = 0$ but that does not help on the jump start step. To find a more reasonable lower bound of the optimal solution $\|v^*\|$, we have the following lemma.

**Lemma 3.** Given a impact vector $x$ such that there exists $p \in \mathcal{P}$, $d_{x_0}(p) < T$, there exist $\sigma > 0$ such that with $w(\sigma) = \{\sigma\}$, $d_{x+w(\sigma)}(p) < T$ and $\|v^*\| \geq \sigma$

**Proof.** The first statement is trivial, so we will focus on the second statement. We have $d_{x+}(p) > d_{x+w(\sigma)}(p)$. Thus there should exist at least one entry in $v^*$ that is at least $\sigma$. So $\|v^*\| \geq \sigma$, which completes the proof. □

As $d_{x+w(\sigma)}(p)$ is monotone increasing w.r.t $\sigma$, we use binary search to find $\sigma$ and set $\beta = \frac{\sigma}{n}$.

V. CRITICAL PATH LISTING ORACLE  

In this section, we present two algorithms for the CPL oracle, which are Incremental Interdiction (II) and Feasible Set Interdiction (FI). CPL’s role is to reduce searching space when constructing the returned solution $x$. CPL works as a backbone for the overall process of finding $x$, in which it receives cLPI’s input, then communicates back and forth with TB to construct $x$ and returns $x$ when $x$ guarantees $d_{x}(s,t) \geq T(1-\varepsilon)$ for all $(s,t) \in S$. 

```plaintext
Algorithm 2 Jump Start Greedy
Input: $G$, $\{f_v\}_v$, $\mathcal{P}$, $T$, $\varepsilon$, $x$  
Output: $s$ that $d_{x+}(p) \geq T(1-\varepsilon)$ for all $p \in \mathcal{P}$
1: $s = \{0\}_v$  
2: $\beta = O(\|s^*\|/n)$  
3: while $\mathcal{P}$ is not empty do  
4: for each $v \in V$ do  
5: $\hat{x}_v = \max_{x \geq \beta} \frac{\hat{p}_{\hat{p},x+sv}(x)}{x}$  
6: if $\hat{x}_v = 0$ (Jump Start) then  
7: $\hat{x}_v = \max_{x \geq \beta} \frac{\hat{p}_{\hat{p},x+sv}(x)}{x}$  
8: $v = \arg\max_{v \in V} \frac{\hat{p}_{\hat{p},x+sv}(\hat{x}_v)}{\hat{x}_v}$  
9: $s = s + \{v, \hat{x}_v\}$  
10: Remove paths $p$ that $d_{x+}(p) \geq T(1-\varepsilon)$ out of $\mathcal{P}$
Return $s$
```

Finally, we have

$$\|s\| = \|s_{L-1}\| + \hat{x}_L \leq \|s^*\| \frac{\ln(|\mathcal{P}|e^{-1}) + 1}{1-\varepsilon}$$

which completes the proof. □
Algorithm 3 Incremental Interdiction

Input $G, \{f_e\}, T, \varepsilon, S, TB$
Output $x$

1: $x = \{0\}$
2: while $\exists (s,t) \in S$ that $d_x(s,t) < T(1-\varepsilon)$ do
3: $P = \emptyset$
4: for each pair $(s,t) \in S$ that $d_x(s,t) < T(1-\varepsilon)$ do
5: $K = k$ shortest paths from $s$ to $t$ under $x$
6: Remove paths $p$ that $d_x(p) \geq T(1-\varepsilon)$ out of $K$
7: $P = P \cup K$
8: $s = \text{run } TB \text{ oracle with input } G, \{f_v\}, P, T, \varepsilon, x$
9: $x = x + s$

Return $x$

A. Incremental Interdiction

In general, this algorithm works in rounds; and in each round, impact amounts are added into nodes to guarantee a set of feasible paths getting length exceeding $T(1-\varepsilon)$. A set of paths are different and disjoint in each round. And to make all paths of that set have length exceed $T(1-\varepsilon)$, $II$ calls the $TB$ oracle to find an additional impact vector to its current vector $x$. The algorithm iterates until finding no feasible paths of length less than $T(1-\varepsilon)$.

A set of paths in each round contains $k$ shortest paths connecting each pair of $S$ under its current impact vector $x$. $k$ is a constant parameter inputted for the algorithm. Intuitively, $k$ is desired to be neither too large or too small. Large $k$ bring burdens on running time to find those shortest paths and memory to store them. On the other hand, small $k$ does not bring sufficient exposures for critical nodes, who appear frequently on paths connecting pairs in $S$ and are the ones the algorithm should target to put impact on. The pseudocode is presented in Alg. 3.

Denote $t$ as the number of outer rounds (line 2 Alg. 3) $II$ ran before terminating. $II$’s theoretical performance guarantee is stated in the following theorem.

Theorem 4. Given an instance $G, \{f_v\}, S, T$ of the $cLPI$ problem and a $TB$ oracle, if $x$ is an output of $II$ and $x^*$ is the optimal solution to the $cLPI$’s instance, then

\[ ||x|| \leq ||x^*||O(t \ln \frac{|F|}{t}) \]

Proof. Denote $x_i$ and $P_i$ as $x$ and $P$ obtained at iteration $i$ of the loop at line 2. From approximation guarantee of the $TB$ oracle, we have:

\[ ||x_i \setminus x_{i-1}|| \leq ||x^*||O((|P_i|^{1/2})) \]

Therefore:

\[ ||x|| \leq \sum_{i=1}^{t} ||x_i \setminus x_{i-1}|| \leq ||x^*||O((|P_i|^{1/2})) \]

\[ = ||x^*||O(|P|) + t \ln \varepsilon^{-1} \]

\[ \leq ||x^*||O(t \ln \frac{|F|}{t}) + O(t \ln \varepsilon^{-1}) \]

\[ \leq ||x^*||O(t \ln \frac{|F|}{t}) \]

The inequality is from AM-GM inequality while $S$ is from the fact that $L_i$ is are disjoint sets of paths. Thus $\sum_{i=1}^{t} |P_i| \leq |F|$, which completes the proof.

B. Full Set Interdiction

In general, $FI$ aims to construct a set $P$ of feasible paths, which is a subset of $F$ but, if being used as an input for $TB$, can return $s$ that is also an $\varepsilon$-feasible solution of $cLPI$.

Different to $II$, which incrementally adds impact to interdict disjoint sets of feasible paths, $FI$ aggregates all found path sets into a big one set called $P$; and reset the impact vector $x$ in order to find a new vector that can simultaneously interdict all paths in $P$. A new path set is found by $k$ shortest paths algorithm with a same motive as $II$. The algorithm terminates when the output $s$ of the $TB$ oracle with input $P$ is also $\varepsilon$-feasible to $cLPI$. The pseudocode is presented in Alg. 4 and $FI$’s performance guarantee is presented by the following theorem.

Theorem 5. Given an instance $G, \{f_v\}, S, T$ of the $cLPI$ problem and a $TB$ oracle, if $x$ is an output of $FI$ and $x^*$ is the optimal solution to the $cLPI$’s instance, then

\[ ||x|| \leq ||x^*||O((|F|^{1/2}) \]

Proof. Without loss of generality, let $P$ denote as the final path sets inputted to $TB$ in the final iteration. From performance guarantee of $TB$, we have that:

\[ ||x|| \leq ||x^*||O(|P|^{1/2}) \]

The theorem trivially follows since there is no duplicated path in $P$ and $P \subseteq F$.

Although $FI$ shows to have a better performance guarantee than $II$, in term of memory complexity, it could take $FI O(|F|)$ to store $P$ while $II$ only takes $O(|S|k)$. That is the trade-off between those two algorithms and it will be shown in more detail in our experiments.

VI. Experimental Analysis

In this section, we run simulation on network data sets to evaluate performance of different combination between algorithms of the $CPL$ and $TB$ oracle. We compare our algorithms’ performance to several methods modified from existing solutions to adapt to the context of $cLPI$. The results show our algorithms outperform existing methods in most cases. We further investigate advantages of each algorithm to reveal some insights on use cases of each technique.
Algorithm 4 Full Set Interdiction

\begin{align*}
\text{Input} & \; G, \{f_v\}_v, T, \varepsilon, S, TB \\
\text{Output} & \; x \\
1: & \; P = \emptyset, x = \{0\}_v \\
2: & \; \text{while } \exists (s, t) \in S \text{ that } d_K(s, t) < T(1 - \varepsilon) \text{ in } G \text{ do} \\
3: & \; \quad \text{for each pair } (s, t) \in S \text{ that } d_K(s, t) < T(1 - \varepsilon) \text{ do} \\
4: & \; \quad \quad K = k \text{ shortest paths from } s \text{ to } t \text{ under } x \\
5: & \; \quad \quad \text{Remove paths } p \text{ that } d_K(p) \geq T(1 - \varepsilon) \text{ out of } K \\
6: & \; \quad \quad P = P \cup K \\
7: & \; x = \{0\}_v \\
8: & \; s = \text{run } TB \text{ oracle with input } G, \{f_v\}_v, P, T, \varepsilon, x \\
9: & \; x = s \\
\text{Return } x
\end{align*}

A. Experimental Settings

We run experiments on a router network, collected from SNAP [26] dataset. The network is constructed as a communication network of who-talks-to-whom from the BGP (Border Gateway Protocol) logs. The network is undirected, containing 6474 nodes and 13895 undirected links connecting nodes.

Critical traffics are randomly sampled from pairs of end hosts in the networks. That critical traffics forms the set S as an input to CPL.

Due to lack of dataset information, for each experiment, we let \( f_v \) be identical for all \( v \), and be one of the following:
- \( f_v(x) = O(x^2) \) - a convex function in order to simulate the relation between external impacts to a router latency.
- \( f_v(x) = O(\log x) \) - a concave function to simulate the relation between external impacts to packet drop/loss rate of a router.
- \( f_v(x) = O(x) \) - a linear function to compare our algorithms’ solution quality to an optimal solution, which can be found by using linear programming.
- \( f_v(x) = O(\abs{x}) \) - a step function to compare our algorithms’ performance with an existing discrete method.

We compare our algorithms with the following methods:
- **CUT** - this method is adapted from [9]. In general, the method works in an “all-or-nothing” manner that an impact amount put into a node is either 0 or \( \min\{x \mid f_v(x) \geq T\} \). That amount guarantees any path containing that node will have length at least \( T \).
- **DISCRETE** - this method discretizes the functions \( f_v \) as follows. If \( f_v \) is a step function, the amount put into a node is a positive integer. Otherwise, the amount put into a node is among \( 0, x, 2x, 3x \) where \( x = \min\{x \mid f_v(3x) \geq T\} \). The method then apply the QoS/D algorithm to discretize the set.
- **OPT** - this method is only applied when \( f_v \) is a linear or step function. We use CPLEX [27] to optimally solve the linear programming modelling the TB oracle and combine it with FI in CPL to obtain the optimal solution to CPL.

With our algorithms, the most time-consuming part is on finding global optimum of univariate functions, for example \( \max_x \frac{\tau_p(x + v(x))}{x} \) in JSG. As “what is the best technique to find global optimum?” [28], [29] is still an open question, we measured the runtime of our algorithms in term of how many times they have to query for finding global optimum of a univariate function.

Finally, in the CPL oracle, we set \( k = 20 \), which - in our experiment - balances the trade-off on running time to find \( k \) shortest path and the exposure of critical nodes. In TB, with TE, we initially set \( M = 1000000 \) if the function is non-differentiable (e.g. step function). \( \varepsilon = 0.1 \) otherwise stated.

We only present representative experimental results. Other results with similar behaviors are excluded.

B. Results

1) How algorithms perform with various \( T \)?: In the first set of experiments, we varied values of \( T \) to observe how different algorithms performed. Figure 1 displays \( \abs{x} \) returned by our algorithms in comparison with DISCRETE and OPT (only when \( f_v \) is a linear or step function).

In the concave case, we observe that our algorithms outperformed existing methods by a huge margin. Existing methods were totally undesirable in this case as their required impact were approximately 100 times worse than ours. This can be explained by: with the concave function, the contribution of impacts to the function expose diminishing return property, i.e. the function’s gain becomes insignificant as input impact grows. That exposed the weakness of discretization steps in CUT and DISCRETE as a discretized impact’s contribution is incomparable to the invested amount.

On the other hand, our algorithms involving FI as the TB oracle returns comparable solution quality to OPT and DISCRETE in non-concave functions. With non-concavity, the function’s gain benefits when input impact increases. Critical nodes, which appear frequently on feasible paths connecting pairs in \( S \), are tended to received large impact amount. Therefore, we observed FI-JSG and DISCRETE behaves almost similarly; and returns solution close to OPT in linear and step cases. Although our algorithms involving II returns solution larger than FI, they have advantages in running time and memory, which will be shown in the next parts.

2) How our algorithms’ number of queries change with various \( T \)?: In this experiment, we measured the number of queries each of our algorithms takes to solve a CPL instance. Just to recall, a query is counted as a call to find global optimal of a univariate function. In algorithms involving JSG, a query is equivalent to finding \( \max_x \frac{\tau_p(x + v(x))}{x} \) (line 5 Alg. 3). In the ones involving TE, a query is counted as a call to find \( \max\{x \geq 0 \mid \frac{\tau_p(x + v(x))}{x} \geq M\} \) (line 5 Alg. 1). Figure 2 shows the numbers of queries taken by each algorithm in various \( T \) and different impact functions.

From Figure 3 we can see that our algorithm involving II totally outperformed the ones with FI in term of queries. For example, with concave cases, with a same TB method, algorithms involving FI tends to take 100 times more queries than the one with II. With convex and step cases, this number is around 2-3 and it is around 5 in linear cases. This can be explained by the fact that II works in an incremental manner, in which impact amounts are accumulated when a
new feasible paths - whose lengths have not satisfied the problem constraints - are found. Thus each query of algorithm involving II play a role, even insignificant, in constructing the final solution. Meanwhile FI resets its impact vector if new unsatisfactory feasible paths are found. Thus queries used before resetting the vector become wasted.

In comparison between algorithms of the TB oracle, it can be seen that TE performed better in concave and linear cases while in convex and step, JSG is the better one. That can be explained as follows: due to the trait of concave and linear functions, JSG’s query always returns an amount equal to the jump start step, i.e. $\beta$. Thus the algorithm required multiple queries to reach satisfactory amount. In contrast, the query $\max \{x \geq 0 \mid \text{fp}_n(x, P) \geq M\} \geq M$ of TE can reach to a larger amount in comparison with a jump start step. However, that situation does not happen when convexity is exposed. With convex functions, impact amounts are invested only on several nodes, which exactly is how JSG behaves. Meanwhile, TE adds impact amounts to nodes sequentially, which makes TE’s impact scattered and unnecessary on some nodes.

However, there is an interesting fact about TE: TE’s number of queries does not depend on $T$ in non-concave cases. That is the reason why TE’s number of queries are constant in those cases as shown in Figure.2 That can be intuitively explained by that: given a set of paths $P$ which share a common node $v$, the way TE increases impact amount on $v$ by query $\max \{x \geq 0 \mid \text{fp}_n(x, P) \geq M\}$ does not get affected by $T$’s value.

3) How the number of stored paths change?: In the next experiment, we compare how much memory our algorithms took to process a cLPI instance. Feasible paths are critical to determine feasibility of our solution. An obstacle on preventing us to apply traditional constraint optimization on cLPI is listing all feasible paths, which could be exponential and a huge burden to computing storage. Therefore, we measures the memory efficiency of our algorithms in term of number of paths they need to store in memory in order to find a feasible solution. Figure 3 shows two kinds of charts of comparison between our algorithm: (1) One shows the maximum number of stored paths of each algorithms with various $T$; (2) The other one shows how the number of stored paths changes after each round of each algorithm. A round of my algorithm is counted as one while iteration of checking feasibility of obtained solutions.

From how II works, it is trivial that algorithms involving II store at most $O(|S|k)$ paths no matter value of $T$ is. That is also shown in Fig. 3. On the other hand, the number of stored paths of algorithms involving FI increases when $T$ increases and is always much larger than this number in II. To have more insight, we look at how each algorithm accumulates paths after each round. As FI works in the manner that collects all feasible paths with unsatisfactory lengths in each round into one large set of paths, its number of paths starts from $O(|S|k)$ (the same as II) and increases significantly with more and more rounds to come. On the other hand, each round of II stores at most $O(|S|k)$ feasible paths; its path set in each round is disjoint and decreases in size. Therefore, II clearly shows its dominance to FI in term of memory.

Similar to the number of queries for finding global optimum of a univariate function, in linear cases, the number of stored paths of algorithms involving TE also stays constant and does not affected by value of $T$. The same reason is also applied.

4) Trade-off in term of $\epsilon$: In the final experiment, we investigate how different values of $\epsilon$ impact our algorithms’ performance. $\epsilon$ represents how “accurate” the returned solutions of our algorithms are to the requirement of cLPI. Intuitively, the smaller $\epsilon$ is, the more accurate the solutions are, the closer lower bounds of distances between pairs of nodes on $S$ are to $T$. Fig. 4 shows how our algorithms’ returned solutions, their numbers of queries and stored paths change with various $\epsilon$.

From Fig. 4 we can see that the algorithm’s returned impact amounts decrease with larger $\epsilon$. This is intuitive since with more relaxed constraint, a smaller impact amount suffices. That is also reflected in our algorithms’ theoretical approximation guarantee, in a way that the ratio is proportional to a term of $\ln \epsilon^{-1}$.

Beside the trade-off between solution accuracy and solution size, $\epsilon$ also shows changes in the number of queries and stored paths of each algorithm. With algorithms involves II, large $\epsilon$ helps decreasing number of queries, which totally contrasts with the one with FI. The behavior of II with various $\epsilon$ is intuitively explained by the fact that: with a same path set, the more relaxed constraint should end up with the smaller overall impact needed. However, we found this fact does not applied with FI because the more relaxed constraint does not guarantee the fewer number of processed paths. That is shown.
in the third sub-figure in Fig. 4, we can see that the number of stored paths of FI increases with $\varepsilon$ grows. With more paths to process, FI’s behavior becomes more complicated. Meanwhile, II is stable with the cap on the number of processing paths, which is at most $O(|S|k)$.

C. Experiment Summary

We summarize experimental results, showing advantages of our algorithms as follows:

- Our algorithms outperform existing methods that needs an intermediate discretization step in most cases. Even in the special instance of cLPI with “discrete” (step) function, one of our algorithm (FI-JSG) performed comparably to the state-of-the-art solution.

- Each of our algorithm has strengths in different aspects, to be specific:
  - With the TB oracle, algorithms involving JSG tend to get better solution quality. Meanwhile, the ones with TE have advantage in the number of queries on global optimum of a univariate function.
  - With the CPL oracle, FI has strengths in solution quality while II shows to save memory in term of the number of stored feasible paths, which plays a role on saving the number of queries in the TB oracle as well.

- $\varepsilon$ allows user control the trade-off between solution quality and accuracy to the input constraint. Moreover, algorithms involving II benefit from $\varepsilon$ in the way that larger $\varepsilon$ helps reduce their runtime.

VII. Conclusion

We studied the cLPI problem, in which we proposed multiple algorithms with different performance guarantees. Theoretical evaluation and experimental analysis are provided, supporting users on deciding which combinations are the best for their needs. Indeed, there are still significant works to improve in the future. A node could be associated with multiple functions, serving for multiple objectives of system’s functionality. Also, each function can have multiple variables and each variable could appear on more than one function, making the problem become much more complicated. How to balance those multiple objectives is still an open problem.

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Fig. 4: Trade-off in term of $\varepsilon$

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