ON THE RATIONAL HOMOLOGY OF HIGH DIMENSIONAL ANALOGUES OF SPACES OF LONG KNOTS

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Abstract. We study high-dimensional analogues of spaces of long knots. These are spaces of compactly-supported embeddings between Euclidean spaces (modulo immersions). We show that when the dimensions are in the stable range, the rational homology groups of these spaces of embeddings can be calculated as the homology of a direct sum of certain finite chain complexes, which we describe rather explicitly. The proof uses the calculus of embeddings, the theory of operads, and some homological algebra of diagrams.

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Introduction

Let $\mathbb{R}^m, \mathbb{R}^n$ be Euclidean spaces, and let $i: \mathbb{R}^m \hookrightarrow \mathbb{R}^n$ be a fixed linear inclusion. Let $\operatorname{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)$ be the space of smooth embeddings of $\mathbb{R}^m$ into $\mathbb{R}^n$ that agree with $i$ outside a compact subset of $\mathbb{R}^m$. In the case when $m = 1$, $\operatorname{Emb}_c(\mathbb{R}^1, \mathbb{R}^n)$ is sometimes called the space of long knots in $\mathbb{R}^n$.

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Let \( \text{Imm}_c(\mathbb{R}^m, \mathbb{R}^n) \) be the space of smooth immersions of \( \mathbb{R}^m \) into \( \mathbb{R}^n \) that agree with \( i \) outside a compact set. It follows easily from the Smale-Hirsch theory \([13]\) that there is a homotopy equivalence (assuming that \( n \geq m + 1 \)) \( \text{Imm}_c(\mathbb{R}^m, \mathbb{R}^n) \simeq \Omega^m \text{Inj}(\mathbb{R}^m, \mathbb{R}^n) \) where \( \Omega^m \) denotes \( m \)-fold loop space, and \( \text{Inj}(\mathbb{R}^m, \mathbb{R}^n) \) is the Stiefel manifold of linear isometric injections of \( \mathbb{R}^m \) into \( \mathbb{R}^n \). Thus, the homotopy type of \( \text{Imm}_c(\mathbb{R}^m, \mathbb{R}^n) \) is (in some sense) well-understood, and we view it as “the easy part of \( \text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n) \)”.

Let \( \overline{\text{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \) be the homotopy fiber of the inclusion map \( \text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n) \to \text{Imm}_c(\mathbb{R}^m, \mathbb{R}^n) \), with \( i \) serving as the basepoint. This paper is concerned with the rational homology of \( \overline{\text{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \). In a follow-up paper \([7]\) we also consider the rational homotopy groups of the spaces \( \overline{\text{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \) and \( \text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n) \).

Our main computational result gives a rather explicit algebraic model for the rational homology groups of \( \overline{\text{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \) in the case when \( n > 2m + 1 \). Our results may be viewed as extending the results of \([22, 16, 5, 23]\) on \( \overline{\text{Emb}}_c(\mathbb{R}^1, \mathbb{R}^n) \). We also draw heavily on the methods of \([6]\), which is concerned with space of embedding \( \text{Emb}(M, \mathbb{R}^n) \), where \( M \) is an open submanifold of \( \mathbb{R}^m \).

Before we can state the result (Theorem \([\ref{main_theorem}]\) below), we have to define a few terms. Let \( \Omega \) be the category of finite sets and surjective functions between them. Our main result is formulated in terms of right-\( \Omega \)-modules. By a right \( \Omega \)-module (with values in a category \( D \)) we mean a contravariant functor from \( \Omega \) to \( D \). We will now introduce a couple of right \( \Omega \)-modules, with values in the category of graded Abelian groups, that will play an important role in the paper.

Given natural numbers \( i \) and \( n \), let \( \mathcal{B}_n(i) \) be the \( i \)-th space in the unframed \( n \)-disk operad. The space \( \mathcal{B}_n(i) \) is homotopy equivalent to the space of ordered \( i \)-tuples of distinct points in \( \mathbb{R}^n \). For a space \( X \), let \( \tilde{H}(X) \) denote the total homology of \( X \), considered as a graded Abelian group. We define \( \tilde{H}(\mathcal{B}_n(i)) \) to be (roughly speaking) the part of \( H(\mathcal{B}_n(i)) \) that is not detected in \( H(\mathcal{B}_n(i - 1)) \) (Definition \([\ref{def:htilde}]\)). It turns out that, at least if we assume that \( n \geq 2 \), the sequence

\[
\tilde{H}(\mathcal{B}_n(0)), \tilde{H}(\mathcal{B}_n(1)), \tilde{H}(\mathcal{B}_n(2)), \ldots
\]

has a natural right \( \Omega \)-module structure. We denote this right module by \( \tilde{H}(\mathcal{B}_n(\bullet)) \).

Next, we need to define another family of right \( \Omega \)-modules. Fix an integer \( m \geq 1 \). Let \( S^m = S^1 \land \ldots \land S^1 \) be the \( m \)-dimensional sphere. It is well-known that the diagonal maps endow the sequence of spaces \( S^0, S^m, S^{2m}, \ldots, S^{\infty} \) with the structure of a right \( \Omega \)-module. We denote this right module by \( S^{m \bullet} \). Applying reduced homology to it, we obtain the right module \( \tilde{H}(S^{m \bullet}) \).

We often think of graded Abelian groups as chain complexes with zero differential. In this way, we may consider \( \tilde{H}(S^{m \bullet}) \) and \( \tilde{H}(\mathcal{B}_n(\bullet)) \) to be right \( \Omega \)-modules with values in chain complexes. The category of right \( \Omega \)-modules with values in chain complexes has a Quillen model structure. It follows that one can “do homological algebra” and even “do homotopy theory” in this category. Let \( F \) and \( G \) be right \( \Omega \)-modules with values in chain complexes. We define

\[
\text{hRmod}_\Omega(F, G)
\]
to be the derived hom object (which is itself a chain complex). It can be defined as the strict hom object from a cofibrant replacement of $F$ to a fibrant replacement of $G$ (in practice, we will use a model structure in which all objects are fibrant).

The following then is our main theorem. We view it as giving an algebraic model for the rational homology of $\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)$. Throughout the paper, let $\mathcal{C}(X)$ denote the normalized singular chain complex of $X$. If $R$ is a ring, then $\mathcal{C}^R(X) := \mathcal{C}(X) \otimes R$.

**Theorem 0.1.** Assume that $n > 2m + 1$. There is a weak equivalence of chain complexes

$$\mathcal{C}^\Omega(\overline{\text{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)) \simeq \text{hRmod}_\Omega \left( \tilde{\mathcal{H}}(S^{m*}), \tilde{\mathcal{H}}(B_n(\bullet); \mathbb{Q}) \right).$$

We will first discuss some consequences of the theorem, and then outline the proof. Let $\mathcal{H}_j(X)$ be the $j$-th homology group of $X$, considered as a chain complex concentrated in dimension $j$. There are obvious isomorphisms of $\Omega$-modules

$$\tilde{\mathcal{H}}(B_n(\bullet)) \cong \bigoplus_{t=0}^\infty \tilde{\mathcal{H}}_{(n-1)t}(B_n(\bullet)) \cong \prod_{t=0}^\infty \tilde{\mathcal{H}}_{(n-1)t}(B_n(\bullet))$$

and

$$\tilde{\mathcal{H}}(S^{m*}) \cong \bigoplus_{s=0}^\infty \tilde{\mathcal{H}}_{ms}(S^{m*}) \cong \prod_{s=0}^\infty \tilde{\mathcal{H}}_{ms}(S^{m*}).$$

Here we have used the well-known fact that $\tilde{\mathcal{H}}_j(B_n(\bullet))$ is non-zero only if $j$ is a multiple of $n - 1$ and the obvious fact that $\tilde{\mathcal{H}}_i(S^{m*})$ is non-zero only if $i$ is a multiple of $m$. It follows that the right hand side of (0.1) splits as a product of mapping spaces between right $\Omega$-modules with values in chain complexes concentrated in a single degree. More precisely, there is a weak equivalence of chain complexes

$$\text{hRmod}_\Omega \left( \tilde{\mathcal{H}}(S^{m*}), \tilde{\mathcal{H}}(B_n(\bullet); \mathbb{Q}) \right) \simeq \prod_{s,t \geq 0} \text{hRmod}_\Omega \left( \tilde{\mathcal{H}}_{ms}(S^{m*}), \tilde{\mathcal{H}}_{(n-1)t}(B_n(\bullet); \mathbb{Q}) \right).$$

In fact, one can show that if $n > 2m + 1$ then the direct product on the right side can be replaced with direct sum (Remark 11.3). Also, the decomposition can be reduced even further. For example, we will see that the terms on the right hand side of (0.2) are non-zero only if $s \leq 2t$.

(0.2) produces a double splitting in the rational homology of $\overline{\text{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$. In the case of $m = 1$, this double splitting is equivalent to the one studied in [23]. In conforming with the terminology of [op. cit], the grading $s$ of the splitting will be called *Hodge degree* and the grading $t$ will be called *complexity*.

**Remark 0.2.** If one works integrally rather than over the rationals, then the equivalence (0.1) does not hold, but one still can show that there is a spectral sequence starting with the homology of the right hand side and converging to the homology of the left hand side. We hope to come back to this in another paper.

In Sections 8-11 we analyze the mapping complexes

$$\text{hRmod}_\Omega \left( \tilde{\mathcal{H}}_{ms}(S^{m*}), \tilde{\mathcal{H}}_{(n-1)t}(B_n(\bullet); \mathbb{Q}) \right)$$
by filtering the category Ω by cardinality. Analysing this filtration leads us to a certain explicit finite complex that is weakly equivalent to this mapping complex. We call it the Koszul complex, because it can be viewed as a kind of Koszul resolution.

In Section 12 we give an explicit description of the Koszul complex in terms of certain spaces of forests. It is quite interesting, although not really surprising, that the graph-complexes that we obtain at this point look similar to those obtained from the Bott-Taubes generalized construction used to study the De Rham cohomology of higher dimensional knot spaces [9, 21].

It also is worth noting that the right Ω-modules ˜H(Sm•) and ˆH(Bn(•); Q) are, up to shifts of degree, almost independent of m and n. More precisely, they only depend on the parity of m and n. It follows that the total group H(Emb c(Rm, Rn); Q) depends, in some sense, only on the parities of m and n. For all m and n of a fixed parity (and satisfying n > 2m + 1), the total group is built of the same ingredients. But the topological dimensions of the different ingredients depend on m and n.

For the rest of the introduction we will give a leisurely overview of the proof of Theorem 0.1. Very briefly, the idea is this: The Taylor tower of C(Emb c(Rm, Rn)) can be expressed as the mapping complex between certain kinds of modules (we call them “weak bimodules”) over the little disks operad. Kontsevich’s formality theorem allows us to replace the little disks operad with the commutative operad. Weak bimodules over the commutative operad turn out to be essentially the same thing as right modules over Ω, and thus we obtain our main result.

We will now proceed with a more detailed outline. But before we get to the proof of our main theorem, we would like to show how one can re-prove the main theorem of [6] using the approach of this paper. We hope that it will shed some light both on this paper and on [6].

First, we recall the set-up. Let M be an open submanifold of Rm, where Rm is still a subspace of Rn. Define Emb(M, Rn) to be the homotopy fiber of the inclusion map Emb(M, Rn) → Imm(M, Rn) (note that now we are looking at the space of all embeddings, rather than embeddings with compact support). This space is the subject of [6]. Now we would like to analyze it along with Emb c(Rm, Rn)

Our first step in analyzing Emb(M, Rn) and Emb c(Rm, Rn) is to express the “Taylor towers” of these spaces arising from Weiss’ embedding calculus [24] as (roughly speaking) spaces of maps between modules over the little disks operad. The main construction of embedding calculus approximates a functor F with another functor T∞F, which is determined by the restriction of F to subsets of M that are diffeomorphic to a finite disjoint union of open balls. Under favorable circumstances, the map F → T∞F is a weak equivalence.

Let Dm be the open unit ball in Rm and let i × Dm denote the disjoint union of i copies of Dm. We assume that M is an open submanifold of Rm, and we define a standard embedding of i × Dm into M to be an embedding that on each copy of Dm is a composition of inclusion, translation, and multiplication by a positive scalar. Let sEmb(i × Dm, M)
be the space of standard embeddings of \(i \times D^m\) into \(M\). Note that \(s\Emb(i \times D^m, M)\) is equiva\(\text{lent to the space of ordered } i\text{-tuples of distinct points in } M\).

Let \(B_m\) be the little \(m\)-balls operad. We will sometimes write \(B_m(\bullet)\) when we want to emphasize that \(B_m\) really stands for a sequence of spaces. It is easy to see that the sequence \(\{s\Emb(i \times D^m, M)\}_{i \geq 0}\) is a right module over \(B_m\). Since we assume that \(\mathbb{R}^n\) contains \(\mathbb{R}^m\) as a subspace, the operad \(B_n\) also has the structure of a right module over \(B_m\).

Recall that \(C(X)\) is the normalized singular chain complex on \(X\). Since \(C(\bullet)\) is a lax monoidal functor, applying it to the little balls operad yields an operad in chain complexes \(-\ C(B_m)\). The sequences \(\{C(s\Emb(\bullet, M))\}\) and \(\{C(B_n(\bullet))\}\) are right modules over \(C(B_m)\). It turns out that the Taylor tower of the functor \(C(\Emb(M, \mathbb{R}^n))\) can be expressed as the space of derived maps between these modules. Let \(hRmod(\bullet, \bullet)\) denote the derived space of maps between right modules. There is a natural equivalence
\[
T_\infty C(\Emb(M, \mathbb{R}^n)) \simeq hRmod(\ C(s\Emb(\bullet, M)), C(B_n(\bullet)))\,.
\]

This is essentially Remark 4.12, except that for technical reasons in the body of the paper we will work with discrete versions of \(B_m\) and \(s\Emb(\bullet, M)\), rather than with the topological versions. But let us not worry about this too much during the introduction.

The main observation of [6] is that Kontsevich’s formality theorem implies that if we work rationally then \(C(B_n)\) can be replaced in the above formula with \(H(B_n; \mathbb{Q})\). Thus, assuming that \(M\) is an open submanifold of \(\mathbb{R}^m\) and \(2m + 1 \leq n\), we have an equivalence of chain complexes
\[
C^Q(\Emb(M, \mathbb{R}^n)) \simeq hRmod(\ C(s\Emb(\bullet, M)), H(B_n(\bullet); \mathbb{Q}))\,.
\]

Here the action of \(C(B_m)\) on \(H(B_n(\bullet))\) is essentially a trivial action, factoring through the commutative operad, \(Com\). It follows that we can replace maps between \(C(B_m)\)-modules with maps of \(Com\)-modules.

A right \(Com\)-module is the same thing as a contravariant functor from the category of finite sets to the category of chain complexes. In particular, the sequence
\[
\{C(M), C(M^2), \ldots, C(M^i), \ldots\}
\]
has a right \(Com\)-module structure. We denote this module by \(C(M^\bullet)\). We prove the following proposition (this is, essentially, the first part of Proposition 6.3).

**Proposition 0.3.** There is a “change of operads” equivalence
\[
hRmod(\ C(s\Emb(\bullet, M)), H(B_n(\bullet); \mathbb{Q})) \simeq hRmod(\ C(M^\bullet), H(B_n(\bullet); \mathbb{Q})).
\]

The point of the proposition is that the right hand side is quite a bit simpler than the left hand side. As a consequence, we have the following theorem

**Theorem 0.4.** Suppose that \(M\) is an open submanifold of \(\mathbb{R}^m\) and \(2m + 1 \leq n\). There is an equivalence of rational chain complexes
\[
C^Q(\Emb(M, \mathbb{R}^n)) \simeq hRmod(\ C(M^\bullet), H(B_n(\bullet); \mathbb{Q})).
\]
This theorem can be viewed as an alternative formulation of the main result of [6] (a formulation that avoids an explicit mention of orthogonal calculus). It gives a homotopy-theoretic description of the rational homology of $\text{Emb}(M, \mathbb{R}^n)$. In particular, the following statement, which is one of the main results of [6], is an immediate consequence of the above theorem.

**Corollary 0.5.** Let $M_1$, $M_2$ be two manifolds satisfying the assumption of the theorem. Suppose that $M_1$ and $M_2$ are related by a chain of maps inducing an isomorphism in rational homology. Then there is an isomorphism

$$H(\text{Emb}(M_1, \mathbb{R}^n); \mathbb{Q}) \cong H(\text{Emb}(M_2, \mathbb{R}^n); \mathbb{Q}).$$

**Proof.** If $M_1$ and $M_2$ are related by a chain of rational homology equivalences, then the right Com-modules $C^Q(M_1^\bullet)$ and $C^Q(M_2^\bullet)$ are weakly equivalent (in the category of right Com-modules with values in rational chain complexes). By the theorem, the homotopy type of the right Com-module $C^Q(M^\bullet)$ determines $H(\text{Emb}(M, \mathbb{R}^n); \mathbb{Q})$. □

Now we would like to carry out a similar analysis for the space $\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)$. There is a version of the Taylor tower that applies to this functor (the relative Taylor tower of [24]). Our first task is to express this Taylor tower in terms of modules over operads. In other words, we are looking for an analogue of (0.3). To this end, we use the concept of a “weak bimodule” over an operad (Definition 1.7). This term was first introduced by the second author in [23]. In the differential graded context this structure was called ”infinitesimal bimodule” by Merkulov and Vallette [19]. It turns out that weak bimodules can play the same role in the study of $\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)$ as right modules do for $\text{Emb}(M, \mathbb{R}^n)$. This is one of the key insights of the paper.

Given weak bimodules $A, B$ over an operad $O$, let $h\text{WBimod}(A, B)$ denote the derived morphism complex from $A$ to $B$. The following equality follows from Lemma [4.11] with the usual proviso that in the body of the paper we replace $B_m$ with its discretization. See also Remark [4.12]

$$T_\infty C(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \simeq h\text{WBimod}(C(B_m), C(B_n)).$$

**Remark 0.6.** When $m = 1$, formula (0.4) is related to Sinha’s cosimplicial model for the space of long knots in $\mathbb{R}^n$ [22]. Indeed, Sinha’s cosimplicial space can be interpreted as the space of weak bimodule maps from the associative operad (which is equivalent to $B_1$) to $B_n$.

Because weak bimodules are similar to right modules, the formal manipulations that were done in [6] for right modules can be repeated, and we may conclude, using the formality theorem, the following

**Corollary 0.7.** Suppose that $2m + 1 \leq n$. Then there is an equivalence

$$T_\infty C^Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \simeq h\text{WBimod}(C^Q(B_m), H(B_n; \mathbb{Q})).$$
As before, the weak $\mathcal{C}(B_m)$-bimodule structure on $H(B_n; \mathbb{Q})$ pulls back from a weak bimodule structure over the commutative operad Com. It follows that we may replace maps between weak $\mathcal{C}(B_m)$-bimodules with maps between weak Com-bimodules. It turns out that weak Com-bimodules are essentially the same thing as right $\Omega$-modules. After a series of reductions, we obtain the following equivalence

$$\text{hWBimod}_{\mathcal{C}(B_m)}(\mathbb{C}(B_m), H(B_n; \mathbb{Q})) \simeq \text{hRmod}_\Omega (\tilde{H}(S^n\cdot), \tilde{H}(B_n(\cdot); \mathbb{Q})).$$

From here, Theorem 0.1 follows.

0.0.1. A section by section outline of the paper. In Sections 1 and 2 we recall some basics about operads and their modules. We also review the concept of a weak bimodule over an operad, which is not very well-known. We show that the category of right modules over an operad, as well as the category of weak bimodules, are equivalent to certain categories of diagrams. In Sections 3 we apply the general theory to obtain a convenient description of the categories of right modules and of weak bimodules over the little disks operad. In Section 4 we show how the Taylor towers for $\mathcal{C}(\text{Emb}(M, \mathbb{R}^n))$ and $\mathcal{C}(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n))$ can be described in terms of maps between, respectively, right modules and weak bimodules over the little disks operad. In Section 5 we start working over the rationals, and we show how Kontsevich’s theorem on the formality of the little disks operad can be used to drastically simplify our models for $\mathcal{C}^Q(\text{Emb}(M, \mathbb{R}^n))$ and $\mathcal{C}^Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n))$. In Section 6 we show that the model for $\mathcal{C}^Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n))$ can be simplified even further, and rewritten in terms of maps between right $\Omega$-modules. This allows us to prove the main theorem.

In the easy Section 7 we derive the equivalence (0.2). In Section 8 we show how filtering the category $\Omega$ by cardinality gives rise to a spectral sequence (which we call the Koszul spectral sequence) for calculating the homology groups of the complex of maps between right $\Omega$-modules. After some preparatory work in Sections 9 and 10 we show, in Section 11, that in the cases of interest to us the first term of the Koszul spectral sequence consists of a single chain complex. We call these chain complexes Koszul complexes. Thus when $2 \dim(M) + 2 \leq n$ the chain complex $\mathcal{C}^Q(\text{Emb}(M, \mathbb{R}^n))$ is equivalent to a direct sum of Koszul complexes. We describe the Koszul complexes explicitly in Section 12 as complexes generated by certain types of forests.

1. Operads, modules and weak bimodules

1.0.2. Definition of operads. In this section we will review the (very well-known) notions of an operad and a right module over an operad. We also will review the concept of a weak bimodule over an operad, which is not so well-known.

For a general introduction to the theory of operads, given from a modern perspective, we suggest the recent book of Loday and Vallette [18]. Our exposition was influenced by that of Ching [10], with the important difference that Ching only considers non-unital operads, while we make no such restriction.

Definition 1.1. Let $\Sigma$ be the category of finite sets and isomorphisms between them. Let $\mathcal{C}$ be any category. A symmetric sequence $P$ in $\mathcal{C}$ is a functor $P : \Sigma \to \mathcal{C}$. 
We will write $P(n)$ for $P(\{1, \ldots, n\})$ (and $P(0)$ for $P(\emptyset)$). In practice, a symmetric sequence is determined by the sequence $P(0), P(1), \ldots$ of objects in $C$ together with an action of the symmetric group $\Sigma_n$ on $P(n)$ for each $n$.

**Definition 1.2.** Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category. An *operad* $O$ in $\mathcal{C}$ is a symmetric sequence $O$ in $\mathcal{C}$ equipped with a unit map $\eta: 1 \to O(1)$ and partial composition maps

$$- \circ_a -: O(A) \otimes O(B) \to O(A \cup_a B)$$

for each pair of sets $A$ and $B$ and each $a \in A$ where $A \cup_a B := (A \setminus \{a\}) \amalg B$. The composition maps must be natural with respect to isomorphisms of $(A, a)$ and of $B$, and must satisfy certain axioms that say that

1. the composition is associative in the sense that for all sets $A, B$ and $C$, and elements $a \in A$, $b \in B$, the two compositions $(- \circ_a -) \circ_b -$ and $- \circ_a (- \circ_b -)$ define the same map

$$O(A) \otimes O(B) \otimes O(C) \to O(A \cup_a B \cup_b C).$$

2. for two distinct elements $a, a' \in A$, the operations $- \circ_a -$ and $- \circ_{a'} -$ commute.

3. $\eta$ acts as both a right unit and a left unit with respect to the composition maps.

The axioms are spelled out fully in \cite[Definition 2.2]{arone2016operads}. A little below, we will spell out the analogous axioms for a module over an operad. In the meantime, we offer a graphic representation of Axioms (1) and (2) in Figure 1 and Figure 2 respectively. Thus, we represent $O(A)$ as a circle with incoming arrows, labeled by elements of $A$. The input for the operation $- \circ_a -$ is represented by a tree with two internal vertices and one internal edge, and the operation itself is represented by the collapse of the internal edge. Two operations that can be composed are represented by a bigger tree, with two internal edges. The operad axioms (1) and (2) then say that the composite operation obtained by the collapse of two internal edges is independent of the order in which the edges are collapsed. We will now develop this viewpoint more systematically.

1.0.3. *Trees.* The use of trees in connection with operads is a familiar theme. Again, our approach is largely inspired by \cite[Section 3]{arone2016operads}, but since we do not restrict ourselves to non-unital operads, there are some differences. For us, a *rooted tree* is a finite connected acyclic graph with one marked vertex, called the root. The edges of a rooted tree have a preferred orientation. Namely, we orient the edges so that they point towards the root. Thus each vertex except the root has exactly one outgoing edge and any number of incoming edges. A *leaf* of a rooted tree is a vertex with no incoming edges (if the root is the only vertex of a tree, it is not considered to be a leaf).

Let $T$ be a rooted tree. For a finite set $A$, an *$A$-labeling* of $T$ is an injective function from $A$ to the set of leaves of $T$. A rooted tree equipped with a labeling will be called a labeled tree. The vertices of a labeled tree $T$ that are not in the image of the labeling function (including the root) will be called the unlabeled vertices of $T$. Note that there may be unlabeled leaves. See Figure 3.
Figure 1. The associative law for operads

Figure 2. The “commutative law” for operads.

Figure 3. A rooted tree labeled by the set \{1, 2, 3, 4\}. Note that since the orientation of edges is marked, the root can be recognized as the vertex with no outgoing edges. In subsequent drawings we will indicate the orientation of the edges, but will not use a special symbol to distinguish the root from other vertices. Note that we allow unlabeled leaves, as well as vertices with just one incoming edge.

The outgoing edge of a labeled leaf will be called a labeled edge. All other edges will be called internal edges. Given an \(A\)-labeled tree \(T\), and an internal edge \(e\), define \(T/e\) to be the tree obtained from \(T\) by collapsing the edge \(e\) to a point and identifying the endpoints of \(e\). It is easy to see that \(T/e\) is again an \(A\)-labeled tree. There is a canonical map of trees \(T \rightarrow T/e\).
Definition 1.3. Let $A$ be a finite set. $T(A)$ is the following category of trees. Its objects are isomorphisms classes of rooted $A$-labeled trees (choose a representative from each class). Its morphisms are generated by collapses of internal edges, and isomorphisms that preserve the root and the labeling.

The category $T(A)$ has a final object, which is the $A$-labeled tree with no internal edges. This is the star-shaped tree with $|A| + 1$ vertices, where one is the root and the rest are the labeled leaves. We will denote it by $T_A$.

Let $O$ be an operad in $C$. Then $O$ determines a functor from $T(A)$ to $C$, which we will denote with the same letter $O$. On objects, the functor is defined as follows. Let $T$ be an $A$-labeled tree.

$$O(T) := \bigotimes_{\text{unlabeled vertices } v \text{ of } T} O(i(v)).$$

Here $i(v)$ is the set of incoming edges of $v$. To describe how $O$ acts on morphisms in $T(A)$, we first consider isomorphisms and edge collapses, which by definition generate $T(A)$. Clearly, an isomorphism of trees $T \sim T'$ induces a bijection between the unlabeled vertices of $T$ and of $T'$, and thus induces a natural isomorphism $O(T) \sim O(T')$. Now let $e$ be an internal edge of $T$ from $u$ to $v$. Then $e \in i(v)$. Let $v \circ u$ be the vertex in $T/e$ obtained from $u$ and $v$. Then $i(v \circ u) = i(v) \cup_e i(u)$ and we have a natural morphism

$$- \circ_e - : O(i(v)) \otimes O(i(u)) \longrightarrow O(i(v) \cup_e i(u)) = O(i(v \circ u)).$$

This morphism induces a morphism $O(T) \longrightarrow O(T/e)$. A look at Figures 1 and 2 should convince the reader that the operad axioms guarantee that the operation resulting from the collapse of several edges is independent of the order in which the edges are collapsed. From here it follows that this operation determines a well-defined functor from $T(A)$ to $C$. See figure 4 for an illustration.

In particular, let $T_A$ be the final object in the category $T(A)$. Clearly, $O(T_A) = O(A)$, and thus we have a natural morphism $O(T) \longrightarrow O(A)$ for every $A$-labeled tree $T$.

1.0.4. Modules. Let $C$ be a symmetric monoidal category. Let $D$ be a category tensored over $C$. In practice, $D$ will often be the same as $C$, but not always.

Definition 1.4. Let $O$ be an operad in $C$. A right module over $O$ with values in $D$ is a symmetric sequence $M$ in $D$ together with partial composition maps

$$- \circ_a - : M(A) \otimes O(B) \longrightarrow M(A \cup_a B)$$

where $A$ and $B$ are finite sets and $a \in A$. The composition maps are required to be natural in isomorphisms of $(A, a)$ and $B$, and satisfy the following axioms
Figure 4. This picture represents the tensor product $O(2) \otimes O(1) \otimes O(2) \otimes O(3) \otimes O(2) \otimes O(0) \otimes O(0)$, which the operad/ functor $O$ associates to the tree in Figure 3. Internal edges indicate the composition maps that can be applied to this tensor product. The axioms of an operad guarantee that the composed operation resulting from the collapse of several edges is independent of the order in which the edges are collapsed.

(1) For all finite sets $A, B$ and $C$ and for all $a \in A$ and $b \in B$ the following diagram commutes

$$
\begin{align*}
M(A) \otimes O(B) \otimes O(C) & \xrightarrow{1 \otimes (-o_b-)} M(A) \otimes O(B \cup_b C) \\
& \downarrow (-o_a- \otimes 1) \\
M(A \cup_a B) \otimes O(C) & \xrightarrow{-o_b-} M(A \cup_a B \cup_b C)
\end{align*}
$$
(2) For all finite sets $A$, $B$ and $B'$ and all $a, a' \in A$ where $a \neq a'$, the following diagram commutes

$$
\begin{array}{ccc}
M(A) \otimes O(B) \otimes O(B') & \xrightarrow{(-\otimes a') \otimes 1} & M(A \cup_a B') \otimes O(B) \\
(-\otimes a) \otimes 1 & \downarrow & -\otimes a \\
M(A \cup_a B) \otimes O(B') & \xrightarrow{-\otimes a'} & M(A \cup_a B \cup_{a'} B')
\end{array}
$$

(more precisely, the top arrow in this square needs to be preceded by the map switching the second and third factors).

(3) For all finite sets $A$ and for all $a \in A$ the following morphism is the identity

$$M(A) = M(A) \otimes 1 \xrightarrow{1 \otimes \eta} M(A) \otimes O(1) \xrightarrow{-\circ a} M(A).$$

Here we have used the identification of $\{1\}$ with $\{a\}$.

Let $O$ be an operad in $C$ and let $M$ be a right module over $O$ with values in $D$. Let $A$ be a finite set. $O$ and $M$ define a functor from $T(A)$ to $D$ that we will now describe. On objects, it is defined as follows. Let $r$ be the root of $T$.

$$M(T) := M(i(r)) \otimes \bigotimes \text{unlabeled vertices } v \text{ of } T \text{ other than the root} \otimes O(i(v)).$$

The functoriality of $M$ with respect to $T$ is defined similarly to that of $O(T)$. Isomorphisms of trees induces permutation of factors in the above tensor product. The map $M(T) \to M(T/e)$, were $e$ is an internal edge is defined as follows. If $e$ is an edge $u \to v$ where neither $u$ nor $v$ is the root, then the map induced by the operad structure map $-\circ e : O(i(u)) \otimes O(i(v)) \to O(i(u \circ v))$. If $e$ is an edge $u \to r$, where $r$ is the root, then the map is induced by the module structure map

$$-\circ e : M(i(r)) \otimes O(i(u)) \to M(i(r \circ v)).$$

Once again, the axioms for a right module, together with those of an operad, guarantee that $M(T)$ is functorial with respect to morphisms in $T(A)$.

The definitions above are well known. Now we will introduce a few definitions that are not so standard. As before, let $O$ be an operad in a symmetric monoidal category $C$, and let $D$ be a category tensored over $C$.

**Definition 1.5.** A weak left module over $O$ with values in $D$ is a symmetric sequence $M$ in $D$ together with partial composition maps, defined for all finite sets $A, B$ and all $a \in A$

$$-\circ a : O(A) \otimes M(B) \to M(A \cup_a B).$$

The composition maps are required to be natural in isomorphisms of $(A, a)$ and $B$, and satisfy the following axioms
(1) For all finite sets $A, B$ and $C$ and for all $a \in A$ and $b \in B$ the following diagram commutes
\[
\begin{array}{c}
O(A) \otimes O(B) \otimes M(C) \\
\downarrow \quad 1 \otimes (- \circ a -) \otimes 1
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \quad - \circ a -
\end{array}
\begin{array}{c}
O(A) \otimes M(B \cup b C) \\
\downarrow \quad - \circ b -
\end{array}
\]

(2) For all finite sets $A$ the following morphism is the identity
\[
M(A) = 1 \otimes M(A) \xrightarrow{\eta \otimes 1} O(1) \otimes M(A) \xrightarrow{-\circ 1 -} M(A).
\]

Remark 1.6. The concept of a weak left module is different from the regular concept of a left module over an operad that is more commonly found in literature. A left module is usually defined to be a symmetric sequence $M$, equipped with structure maps
\[
O(A) \otimes M(B_1) \otimes \ldots \otimes M(B_k) \rightarrow M(A \cup a_1 B_1 \cup a_2 \ldots \cup a_k B_k) = M(B_1 \coprod \ldots \coprod B_k)
\]
where $A = \{a_1, \ldots, a_k\}$. Note that a left module structure does not automatically give rise to a weak left module structure.

Finally, we are ready for the key definition of this section. Let $O$ be an operad in $\mathcal{C}$ and let $\mathcal{D}$ be a category tensored over $\mathcal{C}$.

**Definition 1.7.** A weak bimodule over $O$ with values in $\mathcal{D}$ is a symmetric sequence $M$ in $\mathcal{D}$ endowed with the structure of a right module over $O$ and of a weak left module over $O$. The right and left composition maps are required to satisfy the following axioms.

(1) For all finite sets $A, B, C$ and for all $a \in A, b \in B$, the following diagram commutes.
\[
\begin{array}{c}
O(A) \otimes M(B) \otimes O(C) \\
\downarrow \quad 1 \otimes (- \circ a -) \otimes 1
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \quad - \circ a -
\end{array}
\begin{array}{c}
M(A \cup a B) \otimes O(C) \\
\downarrow \quad - \circ b -
\end{array}
\]

(2) For all finite sets $A, B, B'$ and for all distinct elements $a, a' \in A$, the following diagram commutes.
\[
\begin{array}{c}
O(A) \otimes O(B) \otimes M(B') \\
\downarrow \quad (- \circ a -) \otimes 1
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \quad - \circ a -
\end{array}
\begin{array}{c}
M(A \cup a' B) \otimes O(B) \\
\downarrow \quad - \circ a' -
\end{array}
\]

More precisely, the top map in the above diagram needs to be preceded by switching the second and third factors.
Example 1.8. Let $O$ and $P$ be operads. Let $f: O \to P$ be a morphism of operads. Then $P$ is a weak bimodule over $O$. For example, the weak left module structure is defined as follows. Let $A, B$ be finite sets, and let $a \in A$. Then the map $- \circ_a -: O(A) \otimes P(B) \to P(A \cup_a B)$ is the composite $O(A) \otimes P(B) \xrightarrow{f(A) \otimes 1} P(A) \otimes P(B) \xrightarrow{- \circ_a -} P(A \cup_a B)$.

In particular, $O$ is a weak bimodule over itself.

Similarly to operads and right modules, a weak bimodule determines a functor from a certain category of trees to $\mathcal{D}$.

Definition 1.9. Let $A$ be a finite set. A marked $A$-labeled tree is a pair $(T, u_0)$ where $T$ is a rooted $A$-labeled tree and $u_0$ is an unlabeled vertex of $T$ ($u_0$ may or may not coincide with the root of $T$). Let $T_*(A)$ denote the category of marked $A$-labeled trees. Morphisms in $T_*(A)$ are morphisms between underlying unmarked trees in $T(A)$ that preserve the marked point.

Remark 1.10. Note that $T(A)$ can be embedded into $T_*(A)$ as the full subcategory of those trees for which the marked point coincides with the root. There also is an evident forgetful functor from $T_*(A)$ to $T(A)$.

Definition 1.11. Let $O$ be an operad in $\mathcal{C}$ and let $M$ be a weak bimodule over $O$ with values in $\mathcal{D}$. Then $O$ and $M$ define a functor from $M: T_*(A) \to \mathcal{D}$ as follows. On objects, $M$ is defined by the formula $M(T, u_0) = M(i(u_0)) \otimes \bigotimes_{v \neq u_0} O(i(v))$

where the big tensor product on the right hand side is indexed over unlabeled vertices of $T$ different from the marked vertex. To define the action of $M$ on morphisms in $T_*(A)$, we have to describe the maps $M(T, u_0) \to M(T/e, u_0')$, where $e$ is an internal edge of $T$. Let $e$ be an edge from $u$ to $v$. We distinguish between three cases. If neither $u$ nor $v$ coincides with the marked vertex $u_0$, then the map is defined using the operad structure on $O$ in the usual way. If $v = u_0$ then the map is defined using the right module structure. If $u = u_0$ then the map is defined using the weak left module structure. The axioms for operad and weak bimodule guarantee that this gives rise to a well-defined functor.

1.0.5. Forests. The above constructions can be extended from trees to forests. For us, a forest is a disjoint union of rooted trees. Let $A$ be a finite set. We define an $A$-labeling of a forest $F$ in the same way as an $A$-labeling of a tree: as an injective function from $A$ to the set of leaves of $F$. We let $F(A)$ be the category of forests labeled by $A$, where morphisms are generated by isomorphisms of forests and by collapses of internal edges. $T(A)$ embeds into $F(A)$ as the full subcategory of forests with just one connected components (i.e., trees). Let $O$ be an operad. We can extend the definition of the functor determined by $O$ from the category of trees to the category of forests. For example, suppose $F$ is a forest with
connected components $T_1 \ldots T_k$. Then we define

$$O(F) := \bigotimes_{\text{unlabeled vertices of } F} O(i(v)) = \bigotimes_{j=1}^k O(T_k)$$

Clearly, these formulas defines a functor on $\mathcal{F}(A)$, extending the definition we had previously for a functor on $T(A)$.

\section{Operads as categories, modules as functors}

It is well known that the category of right modules over an operad is equivalent to a certain category of diagrams. In this section we review this construction and introduce an analogous one for the category of weak bimodules (by contrast, the category of honest bimodules, or even of left modules over an operad is not equivalent to a category of diagrams).

To begin with, recall that for a finite set $B$, $T_B$ is the minimal rooted $B$-labeled tree (Figure 5). Next we will see how to associate a certain tree and a certain forest with a map (or more generally a chain of maps) of sets.

\textbf{Definition 2.1.} Let $\alpha: A \to B$ be a function between finite sets. We define $F_\alpha$ to be the mapping cylinder of $\alpha$. $F_\alpha$ will be considered as a rooted $A$-labeled forest. The set of vertices of $F_\alpha$ is a disjoint union $A \coprod B$. For every $a \in A$ there is an edge $a \to \alpha(a)$. Each element $b \in B$ is the root of its connected component. See Figure 6.

$B$ is a subset of $T_B$ and of $F_\alpha$. We define $T_B \cup_B F_\alpha$ to be their union along $B$. We consider it as an $A$-labeled rooted tree, with the root of $T_B$ serving as the root of $T_B \cup_B F_\alpha$ (Figure 7).

Given another function $\beta: B \to C$, we define $F_\beta \cup_B F_\alpha$ to be the union of the two forests along $B$, considered as an $A$-labeled forest. The roots of $F_\beta \cup_B F_\alpha$ are the elements of $C$ (See Figure 8 for an illustration). More generally, let $\alpha_i: A_i \to A_{i+1}$ where $i = 1, \ldots, k$ be $k$ composable functions between sets. We form the corresponding union $F_{\alpha_1} \cup_{A_2} F_{\alpha_2} \cup \ldots \cup_{A_k} F_{\alpha_k}$. It is easy to check that this union is a forest. We view it as an $A_1$-labeled forest. The roots of its components are given by elements of $A_{k+1}$.

A look at Figures 7 and 8 should convince the reader of the correctness of the following elementary lemma.

\textbf{Lemma 2.2.} The tree obtained from $T_B \cup_B F_\alpha$ by collapsing all internal edges is canonically isomorphic to $T_A$. The forest obtained from $F_\beta \cup_B F_\alpha$ by collapsing all internal edges is canonically isomorphic to $F_{\beta \alpha}$. In particular there is a canonical morphism in $T(A)$

$$T_B \cup_B F_\alpha \to T_A$$

and a canonical morphism in $\mathcal{F}(A)$.

$$F_\beta \cup_B F_\alpha \to F_{\beta \alpha}$$
Figure 5. $T_B$ - the minimal $B$-labeled tree

Figure 6. $F_\alpha$ - the mapping cylinder of a function $\alpha : A \to B$, considered as an $A$-labeled forest

Figure 7. The union $T_B \cup_B F_\alpha$, considered as an $A$-labeled tree

Figure 8. The union $F_\beta \cup_B F_\alpha$, considered as an $A$-labeled forest

Let $O$ be an operad and $M$ a right module. Recall that $O$ and $M$ induce functors on $F(A)$ and $T(A)$ respectively, which we denote with the same letters. Note that
\[ O(F_\alpha) = \bigotimes_{b \in B} O(\alpha^{-1}(b)) \] and
\[ M(T_B) = M(B). \] Note also for future reference that there are canonical associative isomorphisms
\[
(2.1) \quad M(T_B) \otimes O(F_\alpha) \cong M(T_B \cup_B F_\alpha)
\]
and
\[
(2.2) \quad O(F_\beta) \otimes O(F_\alpha) \cong O(F_\beta \cup_B F_\alpha).
\]
The category $\mathcal{F}(O)$ of the following definition is equivalent to the category defined in [6, Section 5.1].

**Definition 2.3.** Let $O$ be an operad in a closed symmetric monoidal category $C$. Define $\mathcal{F}(O)$ to be the following category, enriched over $C$. The objects of $\mathcal{F}(O)$ are finite sets. For two finite sets $A, B$, $\text{Map}_{\mathcal{F}(O)}(A, B)$ is the object of $C$, defined by the following formula

$$\text{Map}_{\mathcal{F}(O)}(A, B) = \coprod_{\alpha: A \to B} O(F_{\alpha})$$

(the coproduct on the right hand side is indexed by all functions from $A$ to $B$).

To define the unit morphisms, we need to describe a morphism $1 \to \text{Map}_{\mathcal{F}(O)}(A, A)$. It is defined to be the following composite

$$1 \xrightarrow{\cong} 1 \otimes A \xrightarrow{\eta \otimes A} O(1) \otimes A \xrightarrow{\cong} O(F_{1_A}) \leftarrow \coprod_{\alpha: A \to A} O(F_{\alpha}) = \text{Map}_{\mathcal{F}(O)}(A, A).$$

The composition of morphisms is defined as follows. As usual, let $\alpha: A \to B$ and $\beta: B \to C$ be functions. Since we assume that $C$ is a closed symmetric monoidal category, it follows that $\otimes$ distributes over coproducts in $C$. From this, it follows easily that all we need to do is define morphisms in $C$ that are natural, associative, and unital $O(F_{\beta}) \otimes O(F_{\alpha}) \to O(F_{\beta \circ \alpha})$. This morphism is defined using the composition

$$O(F_{\beta}) \otimes O(F_{\alpha}) \xrightarrow{\cong} O(F_{\beta \cup_B F_{\alpha}}) \to O(F_{\beta \circ \alpha}),$$

where the first map is the isomorphism (2.2), and the second morphism is induced by the collapse map $F_{\beta \cup_B F_{\alpha}} \to F_{\beta \circ \alpha}$ (Lemma 2.2). The associativity of this map follows from the fact that taking union of trees is an associative operation, together with the fact that $O$ is functorial with respect to collapses of internal edges. The unit axiom is easily verified

**Example 2.4.** Let Com be the commutative operad in Top. This is the operad whose value at every set $A$ is the one point space $\ast$. All its structure maps are, necessarily, the identity function on $\ast$. It is the final object in the category of operads in Top. It is easy to see that $\mathcal{F}(\text{Com})$ is the category of finite sets and functions between them. We will denote this category simply by $\mathcal{F}$.

**Lemma 2.5.** Let $O$ be an operad in a closed symmetric monoidal category $C$. Let $\mathcal{D}$ be a category enriched, tensored and cotensored over $C$. Then the category of right modules over $O$ with values in $\mathcal{D}$ is equivalent to the category of enriched contravariant functors from $\mathcal{F}(O)$ to $\mathcal{D}$.

**Proof.** The lemma is probably “well known to experts”, but we will give our own proof. Later we will use the same approach to prove a version of the lemma for weak bimodules (Proposition 2.15 below).

Let $M$ be a right module over $O$. In particular, $M$ associates to every finite set $A$ an object $M(A)$ of $\mathcal{D}$. Our goal is to associate with $M$ a contravariant functor from $\mathcal{F}(O)$
to $\mathcal{D}$. This functor will be “the same as $M$”, since its value on a finite set $A$ (an object of $\mathcal{F}(O)$) will be the object $M(A)$. By abuse of notation we will denote this functor by $M$ - that is we will not distinguish notationally between $M$ the right module and $M$ the associated functor. To endow $M$ with the structure of an enriched contravariant functors from $\mathcal{F}(O)$ to $\mathcal{D}$, we need to construct morphisms in $\mathcal{D}$

$$M(B) \otimes \text{Map}_{\mathcal{F}(O)}(A, B) \rightarrow M(A)$$

that are associative and unital with respect to composition in $\mathcal{F}(O)$. Since we assumed that $\mathcal{D}$ is enriched, tensored and cotensored over $\mathcal{C}$, it follows that the tensoring of $\mathcal{D}$ over $\mathcal{C}$ commutes with coproducts in $\mathcal{C}$, and so our task is equivalent to constructing morphisms

$$M(B) \otimes O(F_\alpha) \rightarrow M(A).$$

Here $\alpha$ is a function from $A$ to $B$. The morphism needs to be natural, associative and unital with respect to $\alpha$. We define the morphism to be the following composition

$$M(B) \otimes O(F_\alpha) \overset{\cong}{\rightarrow} M(T_B \cup_B F_\alpha) \rightarrow M(T_A) = M(A).$$

Here the first map is the isomorphism in equation (2.1), and the second map is induced by the canonical collapse map $T_B \cup_B F_\alpha \rightarrow T_A$ whose existence follows from Lemma 2.2. It is easy to check that the resulting map is associative and unital, because all its ingredients are.

Conversely, suppose $M$ is an enriched contravariant functor from $\mathcal{F}(O)$ to $\mathcal{D}$. Then $M$ can be endowed with the structure of a right module over $O$ as follows. Let $A, B$ be finite sets, and let $b \in B$. We will define the map $- \circ_b - : M(B) \otimes O(A) \rightarrow M(B \cup_b A)$. Let $q : B \cup_b A \rightarrow B$ be the canonical quotient map. Let $F_q$ be the associated $B \cup_b A$-labeled rooted forest. Since $M$ is a functor on $\mathcal{F}(O)$, we have the following composed morphism

$$M(B) \otimes O(F_q) \overset{\cong}{\rightarrow} M(T_B \cup_B F_q) \rightarrow M(T_{B \cup_b A}) = M(B \cup_b A).$$

It is easy to see that

$$O(F_q) \cong \left( \bigotimes_{b' \in B \setminus \{b\}} O(1) \right) \otimes O(A).$$

Using the unit map $\eta : 1 \rightarrow O(1)$ for each element of $B \setminus \{b\}$, we obtain a morphism $O(A) \rightarrow O(T_q)$. Combining, we obtain the following composition of morphisms

$$M(B) \otimes O(A) \rightarrow M(B) \otimes O(T_q) \rightarrow M(B \cup_b A).$$

$- \circ_b -$ is defined to be this composed map. Thus we defined the right module structure map on $M$. It is straightforward to check that it satisfies the axioms of a right module. The above procedures are inverse to each other, and they establish the required equivalence of categories. □

**Corollary 2.6.** The category of right $\text{Com}$-modules with values in $\mathcal{D}$ is equivalent to the category of contravariant functors from $\mathcal{F}$ (the category of finite sets) to $\mathcal{D}$.  


Now we will introduce a variation of the category $F(O)$ that will do for weak bimodules over $O$ what $F(O)$ did for right modules. To begin with, let $\Gamma$ be the category of pointed finite sets. We will adopt the convention that if $A$, $B$, etc. denotes a pointed set, then the basepoint is denoted by $a_0$, $b_0$, etc.

**Definition 2.7.** If $A$ is an unpointed set, then $A_+$ is the pointed set obtained by adding a basepoint $a_0$ to $A$. If $A$ is a pointed set, then $A_-$ is the unpointed set obtained from $A$ by removing the basepoint.

We will want to use a certain twisted version of $\Gamma$.

**Definition 2.8.** Let $\tilde{\Gamma}$ be the following category. The objects of $\tilde{\Gamma}$ are finite pointed sets. A morphism $\tilde{\alpha}$ from $A$ to $B$ in $\tilde{\Gamma}$ is a pointed function, which we denote with the same symbol

$$\tilde{\alpha}: A_\bot \coprod \{b_0\} \rightarrow B_\bot \coprod \{a_0\}$$

where $b_0$ and $a_0$ are, as usual, the basepoints of $B$ and $A$ respectively, and are the designated basepoints of $A_\bot \coprod \{b_0\}$ and $B_\bot \coprod \{a_0\}$ respectively. Given a morphism $\tilde{\beta}$ from $B$ to $C$ in $\tilde{\Gamma}$, their composition $\tilde{\beta} \circ \tilde{\alpha}$ is defined, as a function from $A_\bot \coprod \{c_0\}$ to $C_\bot \coprod \{a_0\}$ to be the following composition (here every map marked with $\cong$ is the canonical isomorphism that takes basepoint to basepoint and is the identity away from the basepoint)

$$A_\bot \coprod \{c_0\} \xrightarrow{\cong} A_\bot \coprod \{b_0\} \xrightarrow{\tilde{\alpha}} B_\bot \coprod \{a_0\} \xrightarrow{\cong}$$

$$\rightarrow B_\bot \coprod \{c_0\} \xrightarrow{\tilde{\beta}} C_\bot \coprod \{b_0\} \xrightarrow{\cong} C_\bot \coprod \{a_0\}.$$

It is not difficult to check that $\tilde{\Gamma}$ is a well-defined category. Roughly speaking, $\tilde{\Gamma}$ is similar to $\Gamma$, except in the definition of morphisms the source and the target swap their basepoints. It may be helpful to consider an illustration. Figure 10 depicts the mapping cylinder of a morphism $\tilde{\alpha}$ in $\tilde{\Gamma}$. Because of the twist in the definition, the arrows that point to the basepoint, point to the basepoint of the source.

**Remark 2.9.** In fact there is an equivalence of categories $\Gamma \xrightarrow{\cong} \tilde{\Gamma}$, which sends every object to itself and sends a function $\alpha: A \rightarrow B$ to the function $\tilde{\alpha}$, which is defined to be the composite

$$A_\bot \coprod \{b_0\} \xrightarrow{\cong} A \xrightarrow{\alpha} B \xrightarrow{\cong} B_\bot \coprod \{a_0\}.$$

Nevertheless, it will be convenient for us to work with $\tilde{\Gamma}$ instead of $\Gamma$ for reasons that should become apparent shortly.

Now we will discuss how to associate to objects, morphisms, and chains of morphisms in $\tilde{\Gamma}$ certain forests and marked rooted trees. Let

$$\tilde{\alpha}: A_\bot \coprod \{b_0\} \rightarrow B_\bot \coprod \{a_0\}$$

and

$$\tilde{\beta}: B_\bot \coprod \{c_0\} \rightarrow C_\bot \coprod \{b_0\}.$$
$F_\tilde{\beta}$ - the mapping cylinder of a pointed function $\tilde{\beta} : B_- \coprod \{c_0\} \to C_- \coprod \{b_0\}$.

$F_\tilde{\alpha}$ - the mapping cylinder of a pointed function $\tilde{\alpha} : A_- \coprod \{b_0\} \to B_- \coprod \{a_0\}$.

The union $F_\tilde{\beta} \cup_B F_\tilde{\alpha}$, considered as an $A_- \coprod \{c_0\}$-labeled forest.

The mapping cylinder $F_{\tilde{\beta} \times \tilde{\alpha}}$ is obtained from $F_\tilde{\beta} \cup_B F_\tilde{\alpha}$ by collapsing all internal edges.
be pointed functions, representing composable morphisms in $\tilde{\Gamma}$. Let $F_{\tilde{\alpha}}$ be the mapping cylinder of $\tilde{\alpha}$ (Figure 10). Note that it can be considered as an $A_+ \coprod \{b_0\}$-labeled forest whose set of roots is $B_+ \coprod \{a_0\}$. Similarly, let $F_{\tilde{\beta}}$ be the mapping cylinder of $\tilde{\beta}$, considered as a $B_- \coprod \{c_0\}$-labeled forest (Figure 9). Let $F_{\tilde{\beta}} \cup_B F_{\tilde{\alpha}}$ be the union of the forests along $B$ (Figure 11). Note that it may be considered as an $A_- \coprod \{c_0\}$-labeled forest. Collapsing all the internal edges of this forest yields the mapping cylinder of $\tilde{\alpha} \ast \tilde{\beta}$ (Figure 12).

**Definition 2.10.** Let $B$ be a pointed set. We define $\tilde{T}_B$ to be the minimal $B_-$-labeled rooted tree, whose root is identified with $b_0$. We consider $\tilde{T}_B$ to be a marked $B_-$-labeled rooted tree, where the marked vertex coincides with the root. See Figure 13.

The following lemma is easily verified, but it is of key importance.
Lemma 2.11. Let $A, B, C$ be pointed sets. As before, let $\tilde{\alpha} : A_- \coprod \{b_0\} \to B_- \coprod \{a_0\}$ and $\tilde{\beta} : B_- \coprod \{c_0\} \to C_- \coprod \{b_0\}$ be pointed functions. Then $\tilde{T}_B \cup_B F_\tilde{\alpha}$ is an $A_-$-labeled marked tree with root $a_0$ and marked point $b_0$ (see Figure [14]). The tree obtained by collapsing all the internal edges of $\tilde{T}_B \cup_B F_\tilde{\alpha}$ is canonically isomorphic, as a marked tree, to $\tilde{T}_A$.

Similarly, $F_{\tilde{\beta}} \cup_B F_\tilde{\alpha}$ is an $A_-$-labeled rooted forest. The forest obtained by collapsing all the internal edges of $F_{\tilde{\beta}} \cup_B F_\tilde{\alpha}$ is canonically isomorphic to $F_{\tilde{\beta} \ast \tilde{\alpha}}$ (see again Figures [11] and [12]).

Let $O$ be an operad and $M$ a weak bimodule. There are canonical isomorphisms

$$M(\tilde{T}_B) \otimes O(F_\tilde{\alpha}) \cong M(\tilde{T}_B \cup_B F_\tilde{\alpha})$$

and

$$O(F_{\tilde{\beta}}) \otimes O(F_\tilde{\alpha}) \cong O(F_{\tilde{\beta}} \cup_B F_\tilde{\alpha}).$$

Now we are ready to introduce the main definition and the main result of this section.

Definition 2.12. Let $O$ be an operad in a closed symmetric monoidal category $\mathcal{C}$. Define $\bar{\Gamma}(O)$ to be the following category, enriched over $\mathcal{C}$. The objects of $\bar{\Gamma}(O)$ are pointed finite sets. For two pointed finite sets $A, B$, $\text{Map}_{\bar{\Gamma}(O)}(A, B)$ is the object of $\mathcal{C}$, defined by the following formula

$$\text{Map}_{\bar{\Gamma}(O)}(A, B) = \coprod_{\tilde{\alpha} : A_- \coprod \{b_0\} \to B_- \coprod \{a_0\}} \circ \circ \ldots \circ \circ O(F_\tilde{\alpha}).$$

Here the coproduct is indexed over pointed functions, or in other word, over morphisms from $A$ to $B$ in $\bar{\Gamma}$.

To define the unit morphisms, we need to describe a morphism $1 \to \text{Map}_{\bar{\Gamma}(O)}(A, A)$. It is defined to be the following composite

$$1 \xrightarrow{\cong} 1 \otimes A \xrightarrow{\eta \otimes A} O(1) \otimes A \xrightarrow{\cong} O(F_1) \hookrightarrow \coprod_{\alpha : A \to A} O(F_\alpha) = \text{Map}_{\bar{\Gamma}(O)}(A, A).$$

The composition of morphisms is defined as follows. As usual, let $\tilde{\alpha} : A_- \coprod \{b_0\} \to B_- \coprod \{a_0\}$ and $\tilde{\beta} : B_- \coprod \{c_0\} \to C_- \coprod \{b_0\}$ be functions. Since we assume that $\mathcal{C}$ is a closed symmetric monoidal category, it follows that $\otimes$ commutes with coproducts in $\mathcal{C}$. From this, it follows easily that all we need to do is define morphisms in $\mathcal{C}$ that are natural, associative, and unital

$$O(F_{\tilde{\beta}}) \otimes O(F_\tilde{\alpha}) \to O(F_{\tilde{\beta} \ast \tilde{\alpha}}).$$

This morphism is defined using the composition

$$O(F_{\tilde{\beta}}) \otimes O(F_\tilde{\alpha}) \xrightarrow{\cong} O(F_{\tilde{\beta}} \cup_B F_\tilde{\alpha}) \to O(F_{\tilde{\beta} \ast \tilde{\alpha}})$$

where the first map is the evident isomorphism, and the second morphism is induced by the collapse map $F_{\tilde{\beta}} \cup_B F_\tilde{\alpha} \to F_{\tilde{\beta} \ast \tilde{\alpha}}$, as per Lemma 2.11. The associativity of this map follows from the fact that taking union of trees is an associative operation, together with the fact that $O$ is functorial with respect to collapses of internal edges. The unit axiom is easily verified.
Example 2.13. It is easy to see that \( \tilde{\Gamma}(\text{Com}) \cong \tilde{\Gamma} \). But, since \( \Gamma \cong \Gamma \), we conclude that \( \tilde{\Gamma}(\text{Com}) \cong \Gamma \).

Remark 2.14. One can define the category \( \Gamma(O) \) (where \( O \) is an operad) analogously to \( \tilde{\Gamma}(O) \) (since we will not need to use the construction \( \Gamma(O) \), we will omit the formal definition). We would like to emphasize that even though the category \( \tilde{\Gamma}(\text{Com}) \) is equivalent to \( \Gamma(\text{Com}) \), for most operads \( O \) the category \( \tilde{\Gamma}(O) \) is not equivalent to \( \Gamma(O) \). For example, the two categories are not equivalent when \( O \) is the little disks operad, which will play a central role in later sections of this paper. We suspect that \( \tilde{\Gamma}(O) \) is equivalent to \( \Gamma(O) \) whenever \( O \) is a cyclic operad.

The following proposition is the main result of this section. It is analogous to Lemma 2.5.

Proposition 2.15. Let \( O \) be an operad in a closed symmetric monoidal category \( C \). Let \( D \) be a category enriched, tensored and cotensored over \( C \). Then the category of weak bimodules over \( O \) with values in \( D \) is equivalent to the category of enriched contravariant functors from \( \tilde{\Gamma}(O) \) to \( D \).

Proof. Let \( M \) be a weak bimodule over \( O \). We will associate with \( M \) an enriched contravariant functor \( \tilde{M} \) from \( \tilde{\Gamma}(O) \) to \( D \). It is defined on objects of \( \tilde{\Gamma}(O) \) by the formula

\[
\tilde{M}(A) := M(A_-).
\]

To describe the action of \( \tilde{M} \) on morphisms, we need to construct morphisms in \( D \)

\[
\tilde{M}(B) \otimes \text{Map}_{\tilde{\Gamma}(O)}(A, B) \rightarrow \tilde{M}(A)
\]

that are associative and unital with respect to composition in \( \tilde{\Gamma}(O) \). Since we assumed that \( D \) is enriched, tensored and cotensored over \( C \), it follows that tensor product commutes with coproducts, and so our task is equivalent to constructing morphisms

\[
M(B_-) \otimes O(F_{\tilde{\alpha}}) \rightarrow M(A_-).
\]

Here \( \tilde{\alpha} \) is a pointed function from \( A_- \bigsqcup \{b_0\} \) to \( B_- \bigsqcup \{a_0\} \). The morphism needs to be natural, associative and unital with respect to \( \tilde{\alpha} \). We define the morphism to be the following composition (recall from Definition 1.11 that \( M \), being a weak bimodule, defines a functor on the category \( T_\ast(A) \) of marked rooted trees)

\[
M(B_-) \otimes O(F_{\tilde{\alpha}}) \xrightarrow{\cong} M(\tilde{T}_B) \otimes O(F_{\tilde{\alpha}}) \xrightarrow{\cong} M(\tilde{T}_B \cup_B F_{\tilde{\alpha}}) \rightarrow M(\tilde{T}_A) \xrightarrow{\cong} M(A_-).
\]

Here the first and the last map are induced by the canonical isomorphisms \( M(B_-) \cong M(\tilde{T}_B) \) and \( M(A_-) \cong M(\tilde{T}_A) \). The second map is also a canonical isomorphism, given in Lemma 2.11. The third map is induced by the canonical collapse map of marked trees \( \tilde{T}_B \cup_B F_{\tilde{\alpha}} \rightarrow \tilde{T}_A \) (also described in Lemma 2.11). It is easy to check that the resulting map is associative and unital, because all its ingredients are.
Conversely, suppose $\tilde{M}$ is an enriched contravariant functor from $\tilde{\Gamma}(O)$ to $\mathcal{D}$. We associate with $\tilde{M}$ a weak bimodule $M$ over $O$ as follows. As a symmetric sequence, $M$ is defined by the formula $M(A) = \tilde{M}(A_+)$ (here $A$ is an unpointed set). To describe the right module and the weak left module structure on $M$, let $A$, $B$ be finite sets, and let $a \in A$. We need to describe natural maps $\tilde{M}(A_+) \otimes O(B) \to \tilde{M}(A \cup_a B_+)$ and $O(A) \otimes \tilde{M}(B_+) \to \tilde{M}(A \cup_a B_+)$. Let $a_0$ be the basepoint of $A_+$ and let $x_0$ be the basepoint of $A \cup_a B_+$. We will construct two morphisms in $\tilde{\Gamma}$. One is $\tilde{r}$ from $A \cup_a B_+$ to $A_+$ and the other is $\tilde{l}$ from $A \cup_a B_+$ to $B_+$. By definition $\tilde{r}$ needs to be a pointed function from $A \cup_a B \coprod \{a_0\}$ to $A \coprod \{x_0\}$. We define $\tilde{r}$ to be the function that collapses $A \cup_a B$ onto $A$ and sends $a_0$ to $x_0$. On the other hand, $\tilde{l}$ is the pointed function from $A \cup_a B \coprod \{b_0\}$ to $B \coprod \{x_0\}$ that sends elements of $B$ to themselves, and sends $A \setminus \{a\} \coprod \{b_0\}$ to the basepoint $\{x_0\}$. It is easy to see that $O(F_{\tilde{r}}) \cong O(1)^{\otimes A} \otimes O(B)$ and $O(F_{\tilde{l}}) \cong O(1)^{\otimes B} \otimes O(A)$. We now define the right module structure map on $M$ to be the composite

$$\tilde{M}(A_+) \otimes O(B) \to \tilde{M}(A_+) \otimes O(F_{\tilde{r}}) \to \tilde{M}(A \cup_a B_+)$$

where the first map uses the unit $|A|$ times and the second map uses the functoriality of $\tilde{M}$ with respect to $\tilde{\Gamma}(O)$. The left module structure is defined to be the composite

$$O(A) \otimes \tilde{M}(B_+) \to O(F_{\tilde{l}}) \otimes \tilde{M}(B_+) \to \tilde{M}(A \cup_a B_+)$$

where, again, the first map uses the unit $|B|$ times and the second map uses the functoriality of $\tilde{M}$. It is a little tedious, but straightforward to check that all the axioms of a weak bimodule are satisfied and that the above constructions are inverse to each other and so establish an equivalence of categories as desired.

The following corollary follows from Proposition 2.15 together with Example 2.13

**Corollary 2.16.** The category of weak Com-bimodules with values in $\mathcal{D}$ is equivalent to the category of contravariant functors from $\Gamma$ (the category of pointed finite sets) to $\mathcal{D}$.

**Example 2.17.** Let $X$ be a pointed topological space. Such an $X$ gives rise to a contravariant functor from $\Gamma$ to $\text{Top}$

$$S \mapsto \text{Map}_*(S, X) \cong X^S$$

(where $S$ is a pointed set). By the above corollary, this contravariant functor gives rise to a weak Com-bimodule. Indeed, as a symmetric sequence it is given by the formula

$$A \mapsto X^A$$

(where $A$ is an unpointed set). The right module structure is given by increasing (or decreasing) the multiplicities of points in $X$. The weak left module structure is given by adding a suitable multiple of the basepoint in $X$. 

□
3. The little disks operad

In this section we apply the theory of previous section to the little disks operad. Inevitably, we have to begin with a few definitions.

**Definition 3.1.** Let \( \mathbb{R}^m \) be a Euclidean space. A **standard isomorphism** of \( \mathbb{R}^m \) is a self homeomorphism of \( \mathbb{R}^m \) that is the composition of a translation and a multiplication by a positive scalar.

Let \( A \) be a connected subspace of \( \mathbb{R}^m \). A map \( f : A \to \mathbb{R}^m \) is called a **standard embedding** if \( f \) equals the inclusion followed by a standard isomorphism of \( \mathbb{R}^m \).

More generally, if \( X \) is another connected subset of \( \mathbb{R}^m \), then a standard embedding of \( A \) into \( X \) is a standard embedding of \( A \) into \( \mathbb{R}^m \) whose image lies in \( X \).

Even more generally, we define the following category \( SM^m \) of spaces and standard embeddings between them. An object of \( SM^m \) is a locally connected topological space \( X \), each of whose connected components is identified with an open subset of \( \mathbb{R}^m \). Given two objects \( A \) and \( X \) of this category, a morphism from \( A \) to \( X \) is an embedding \( f : A \hookrightarrow X \) having the property that the restriction of \( f \) to each connected component of \( A \) (which by assumption is a subset of \( \mathbb{R}^m \)) is a standard embedding into a component of \( X \) (which, by assumption, is also a subset of \( \mathbb{R}^m \)). We call such maps standard embeddings of \( A \) into \( X \). The space of standard embeddings of \( A \) into \( X \) will be denoted \( sEmb(A,X) \). It is easy to see that a composition of standard embeddings is again a standard embedding, and so we really have a (topologically enriched) category.

**Definition 3.2.** Let \( \mathbb{R}^m \) be a fixed Euclidean space. Let \( D^m \) be the unit open ball in \( \mathbb{R}^m \). The \( m \)-dimensional little disks operad, denoted by \( B_m \), is defined as follows. As a symmetric sequence, \( B_m(A) := sEmb(A \times D^m, D^m) \). The unit is given by the identity map in \( B_m(1) \cong sEmb(D^m, D^m) \). To define the composition maps, let \( a \in A \), and let \( B \) be another finite set. The composition map

\[
- \circ_a - : sEmb(A \times D^m, D^m) \times sEmb(B \times D^m, D^m) \to sEmb((A \cup_a B) \times D^m, D^m)
\]

is defined by identifying the target of the mapping space \( sEmb(B \times D^m, D^m) \) with \( \{a\} \times D^m \) and substituting the second map into the first one in the evident way.

It is easy to see that \( B_m \) is the **endomorphism operad** of the unit disc \( D^m \) in the category \( SM^m \) which is viewed as a symmetric monoidal category with disjoint union as symmetric product and empty set as unit.

Our main goal in this section is to describe the right modules, and especially the weak bimodules over \( B_m \) (and the discretization of \( B_m \)), using the theory of the previous section. Our first task is to describe explicitly the categories \( \mathcal{F}(B_m) \) and \( \tilde{\Gamma}(B_m) \). We begin with the category \( \mathcal{F}(B_m) \) and the associated category of right modules over \( B_m \). The following lemma is an immediate consequence of the definitions.

**Lemma 3.3.** The category \( \mathcal{F}(B_m) \) can be identified with the full subcategory of \( SM^m \) whose objects are disjoint unions of copies of \( D^m \). More explicitly, it is the topological category whose objects are finite sets (or, equivalently, finite disjoint unions of copies of \( D^m \)) and where the space of morphisms from \( A \) to \( B \) is \( sEmb(A \times D^m, B \times D^m) \).
We will refer to $\mathcal{F}(B_m)$ as the category of disjoint unions of standard balls and standard embeddings between them. The following lemma is a special case of Lemma \[\text{2.5}\] It certainly is "well known" (e.g., \[\text{[6]}\]), but we include it here as a warm-up for the analogous statements about weak bimodules.

**Lemma 3.4.** The category of right modules over $B_m$ with values in $\text{Top}$ is equivalent to the category of contravariant topological functors from $\mathcal{F}(B_m)$ to $\text{Top}$.

**Example 3.5.** Let $M$ be an open submanifold of $\mathbb{R}^m$. We associate with $M$ a right module $B^M$ defined by $B^M(A) := s\text{Emb}(A \times D^m, M)$. Obviously, this defines a contravariant functor from $\mathcal{F}(B_m)$ to $\text{Top}$ and thus a right module over the little balls operad.

To perform a similar analysis of weak bimodules over $B_m$, we need to describe the category $\tilde{\Gamma}(B_m)$. It follows immediately from the definition that $\tilde{\Gamma}(B_m)$ can be identified with the category whose objects are pointed finite sets and where the space of morphisms from $A$ to $B$ (where $A$ and $B$ are pointed finite sets) is the space

$$\{ \tilde{\alpha} \in s\text{Emb}\left((A \coprod \{b_0\}) \times D^m, (B \coprod \{a_0\}) \times D^m\right) \mid \tilde{\alpha}(\{b_0\} \times D^m) \subset \{a_0\} \times D^m \}.$$  

In words, a morphism from $A$ to $B$ is a standard embedding of $(A \coprod \{b_0\}) \times D^m$ into $\tilde{\alpha}(\{b_0\} \times D^m)$ that takes the basepoint component into the basepoint component. Composition is defined analogously to composition in $\tilde{\Gamma}$. See Figures 15, 16 and 17 for an illustration of morphisms in $\tilde{\Gamma}(B)$. By Proposition \[\text{2.15}\], the category of weak bimodules over $B$ in a topologically enriched category $\mathcal{D}$ is equivalent to the category of contravariant topological functors from $\tilde{\Gamma}(B)$ to $\mathcal{D}$.

We would like to present an alternative description of the category $\tilde{\Gamma}(B)$ that we find convenient to use.

Let $D_0$ and $D$ be two copies of $D^m$. Let $i: D_0 \to D$ be a standard embedding. The map $i$ is the restriction of a standard isomorphism of $\mathbb{R}^m$, which we will also denote by $i$. Let $i^{-1}$ be the inverse isomorphism of $\mathbb{R}^m$. Then $i^{-1}$ defines a standard embedding of the complement of $D$ in $\mathbb{R}^m$ into the complement of $D_0$. This procedure establishes a homeomorphism

$$s\text{Emb}(D_0, D) \cong s\text{Emb}(\mathbb{R}^m \setminus \overline{D}, \mathbb{R}^m \setminus \overline{D}_0).$$

In the above $\overline{D}$, $\overline{D}_0$ denote the closure of $D$ and $D_0$ respectively.

Now let $D_1, \ldots, D_k$ be $k$ more copies of $D^m$. Let $i$ be a standard embedding

$$i: \coprod_{l=0}^k D_l \hookrightarrow D.$$

For each $l$, let $i_l$ be the restriction of $i$ to $D_l$. Then, as before, $i_0$ is a restriction of a standard isomorphism of $\mathbb{R}^m$, and its inverse, denoted $i_0^{-1}$, defines a standard embedding of the complement of $D$ into the complement of $D_0$. Moreover, for each $l = 1, \ldots, k$, $i_0^{-1} \circ i_l$ defines a standard embedding of $D_l$ into the complement of $D_0$, and together, the maps $i_0^{-1}, i_0^{-1} \circ i_1, \ldots, i_0^{-1} \circ i_k$ define a standard embedding

$$(\mathbb{R}^m \setminus \overline{D}) \coprod D_1 \coprod \ldots \coprod D_k \hookrightarrow \mathbb{R}^m \setminus \overline{D}_0.$$
**Figure 15.** A morphism \( \tilde{\beta} \) from \( B \) to \( C \) in \( \tilde{\Gamma}(B) \).

**Figure 16.** A morphism \( \tilde{\alpha} \) from \( A \) to \( B \) in \( \tilde{\Gamma}(B) \).

**Figure 17.** The composition \( \tilde{\beta} \ast \tilde{\alpha} \)
Again, the above procedure defines a natural homeomorphism

\[ s\text{Emb} \left( \prod_{l=0}^{k} D_l, D \right) \cong s\text{Emb} \left( (\mathbb{R}^m \setminus D) \bigsqcup D_1 \bigsqcup \ldots \bigsqcup D_k, \mathbb{R}^m \setminus D_0 \right). \]

Now let

\[ \tilde{\alpha}: (A_\ast \bigsqcup \{\mathbf{b}_0\}) \times D^m \longrightarrow (B_\ast \bigsqcup \{\mathbf{a}_0\}) \times D^m \]

be a morphism in \( \tilde{\Gamma}(\mathcal{B}) \). \( \tilde{\alpha} \) takes some components of \( A_\ast \times D^m \) to components of \( B_\ast \times D^m \), and it takes the remaining (if there are any) components of \( A_\ast \times D^m \), as well as \( \{\mathbf{b}_0\} \times D^m \), into \( \{\mathbf{a}_0\} \times D^m \). By (3.1), the latter part of the data is equivalent to a standard embedding of the disjoint union of \( \mathbb{R}^m \setminus (\{\mathbf{a}_0\} \times D^m) \) with those components of \( A_\ast \times D^m \) that are sent by \( \tilde{\alpha} \) into \( \{\mathbf{a}_0\} \times D^m \) into \( \mathbb{R}^m \setminus (\{\mathbf{b}_0\} \times D^m) \). We can conclude that \( \tilde{\alpha} \) determines, and is determined by a standard embedding

\[ (A_\ast \times D^m) \bigsqcup (\mathbb{R}^m \setminus D^m) \hookrightarrow (B_\ast \times D^m) \bigsqcup (\mathbb{R}^m \setminus D^m). \]

The following lemma follows easily from the above discussion

**Lemma 3.6.** \( \tilde{\Gamma}(\mathcal{B}_m) \) can be identified with the topologically enriched category whose objects are pointed finite sets, and where the space of morphisms from \( A \) to \( B \) is the space of standard embeddings

\[ s\text{Emb} \left( (A_\ast \times D^m) \bigsqcup (\mathbb{R}^m \setminus D^m), (B_\ast \times D^m) \bigsqcup (\mathbb{R}^m \setminus D^m) \right). \]

With this point of view, composition in \( \tilde{\Gamma}(\mathcal{B}_m) \) is just ordinary composition of standard embeddings.

We will call the space \( \mathbb{R}^m \setminus D^m \) the “antiball”. With this point of view, an object of \( \tilde{\Gamma}(\mathcal{B}_m) \) can be thought of as a disjoint union of standard balls, together with one antiball. Morphisms are standard embeddings between such spaces. Figure [18] gives a graphic representation of a morphism from this point of view. From now on we will switch freely between the two points of view on \( \tilde{\Gamma}(\mathcal{B}_m) \).

We conclude the section with some examples of weak bimodules over \( \tilde{\Gamma}(\mathcal{B}_m) \).

As with every operad, \( \mathcal{B}_m \) is a weak bimodule over itself. The corresponding contravariant functor from \( \tilde{\Gamma}(\mathcal{B}_m) \) to \( \text{Top} \) is defined on objects by the formula

\[ A \mapsto s\text{Emb} \left( A_\ast \times D^m, D^m \right). \]

To understand the functoriality with respect to morphisms in \( \tilde{\Gamma}(\mathcal{B}_m) \), let us first notice that \( s\text{Emb} \left( A_\ast \times D^m, D^m \right) \) can be identified with the subspace of

\[ s\text{Emb} \left( (A_\ast \times D^m) \bigsqcup (\mathbb{R}^m \setminus D^m), \mathbb{R}^m \right) \]

consisting of those standard embeddings whose restriction to \( \mathbb{R}^m \setminus D^m \) is the inclusion. Let \( f \) be such an embedding, and let

\[ \tilde{\alpha}: (B_\ast \times D^m) \bigsqcup (\mathbb{R}^m \setminus D^m) \longrightarrow (A_\ast \times D^m) \bigsqcup (\mathbb{R}^m \setminus D^m) \]
be a morphism in $\tilde{\Gamma}(B_m)$. Then $f \circ \tilde{\alpha}$ is a standard embedding of $(B_- \times D^m) \bigsqcup (\mathbb{R}^m \setminus \bar{D}^m)$ into $\mathbb{R}^m$, which may not restrict to the inclusion on $\mathbb{R}^m \setminus \bar{D}^m$, and so may not represent an element of $s\text{Emb}(B_- \times D^m, D^m)$. Let $r_{\tilde{\alpha}}$ be the unique standard isomorphism of $\mathbb{R}^m$ for which $r_{\tilde{\alpha}} \circ f \circ \tilde{\alpha}$ restricts to the inclusion on $\mathbb{R}^m \setminus \bar{D}^m$. It is elementary that $r_{\tilde{\alpha}}$ is well-defined and depends continuously on $\tilde{\alpha}$. We define the functoriality by sending $f$ to $r_{\tilde{\alpha}} \circ f \circ \tilde{\alpha}$. It is easy to check that this is the correct definition.

A perhaps even more natural example of a contravariant functor from $\tilde{\Gamma}(B_m)$ to Top is the functor that sends $A$ to the space of standard embeddings

$$s\text{Emb}\left((A_- \times D^m) \bigsqcup (\mathbb{R}^m \setminus \bar{D}^m), \mathbb{R}^m\right).$$

This is obviously a functor of $\tilde{\Gamma}(B_m)$. In fact, it is weakly equivalent to the previous functor.

**Lemma 3.7.** There is a natural transformation of functors

$$s\text{Emb}\left((A_- \times D^m) \bigsqcup (\mathbb{R}^m \setminus \bar{D}^m), \mathbb{R}^m\right) \rightarrow s\text{Emb}(A_- \times D^m, D^m)$$

which is a homotopy equivalence for each $A$. 

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure18.png}
\caption{A reinterpretation of the morphism $\tilde{\alpha}$ from Figure 16. Instead of sending the balls $b_0$, $a_1$ and $a_2$ into the ball $a_0$, we send the “antiball” $a_0$, together with balls $a_1$ and $a_2$, into the antiball $b_0$.}
\end{figure}
Proof. As before, let us identify $\text{sEmb}(A \times D^m, D^m)$ with the subspace of 
\[
\text{sEmb} \left( (A \times D^m) \coprod (\mathbb{R}^m \setminus \overline{D^m}), \mathbb{R}^m \right)
\]
consisting of those embeddings that restrict to the inclusion on $\mathbb{R}^m \setminus \overline{D^m}$. As before, given an 
\[
f \in \text{sEmb} \left( (A \times D^m) \coprod (\mathbb{R}^m \setminus \overline{D^m}), \mathbb{R}^m \right),
\]
there exists a unique standard isomorphism $r_f$, depending continuously on $f$, for which $r_f \circ f$ restricts to the inclusion on $\mathbb{R}^m \setminus \overline{D^m}$. Our map sends $f$ to $r_f \circ f$. It is easy to see that it is natural with respect to morphisms in $\tilde{\Gamma}(\bm)$. This map is in fact a fiber bundle with contractible fibers (the fiber is the space of standard isomorphisms of $\mathbb{R}^m$). Thus it is a homotopy equivalence. □

Remark 3.8. The lemma can be interpreted as follows. The assignment 
\[
A \mapsto \text{sEmb} \left( (A \times D^m) \coprod (\mathbb{R}^m \setminus \overline{D^m}), \mathbb{R}^m \right)
\]
is equivalent to $\bm$ as a weak bimodule over $\bm$.

We will also need the following observation. Let $i: \mathbb{R}^m \hookrightarrow \mathbb{R}^n$ be a linear isometric inclusion of Euclidean spaces. Then $i$ induces a map of operads $\bm \longrightarrow \bn$. Thus $i$ endows $\bn$ with the structure of a weak bimodule over $\bm$.

4. Taylor Tower as the Space of Module Maps

We will now recall the basic setup of M. Weiss’s Embedding Calculus [24]. Let $M$ be a smooth $m$-dimensional manifold without boundary. Embedding calculus is concerned with contravariant isotopy functors on the category of open subsets of $M$. There are several versions of the category of subsets that we will want to use.

Definition 4.1. Let $\mathcal{O}(M)$ be the poset/category of all open subsets of $M$ ordered by inclusion. Let $\mathcal{O}_\infty(M)$ be the subposet of $\mathcal{O}(M)$ consisting of subsets homeomorphic to a finite disjoint union of copies of an open ball in $\mathbb{R}^m$. For $k = 0, 1, \ldots$ let $\mathcal{O}_k(M)$ be the subposet consisting of sets homeomorphic to the union of at most $k$ open balls.

Let $\mathcal{D}$ be a closed model category (e.g. topological spaces or chain complexes). Let $F: \mathcal{O}(M) \longrightarrow \mathcal{D}$ be a contravariant isotopy functor. For $k = 0, 1, \ldots$ Weiss defines the functor $T_k F: \mathcal{O}(M) \longrightarrow \mathcal{D}$ by the formula
\[
T_k F(U) = \underset{A \in \mathcal{O}_k(U)}{\text{holim}} F(A).
\]
In other words $T_k F$ is the homotopy right Kan extension of $F$ from $\mathcal{O}_k(M)$ to $\mathcal{O}(M)$. Restriction from $\mathcal{O}_k$ to $\mathcal{O}_{k-1}$ gives rise to a natural map $T_k F \longrightarrow T_{k-1} F$, and thus the functors $T_k F$ fit into a tower. We call it the Taylor tower of $F$. $T_k F$ plays the role of the $k$-th Taylor polynomial of $F$ in manifold calculus.

Weiss also introduced a relative version of the construction for manifolds with boundary. If $M$ is a manifold with boundary, he defines $\mathcal{O}(M, \partial M)$ as the subposet of $\mathcal{O}(M)$
consisting of open sets that contain $\partial M$. The relative version of manifold calculus constructs polynomial approximations to contravariant isotopy functors whose source category is $O(M, \partial M)$. The $k$-th approximation is defined as the homotopy right Kan extension of the restriction to the category $O_k(M, \partial M)$, which is the subposet of $O(M, \partial M)$ consisting of sets that are the disjoint union of a tubular neighborhood of $\partial M$ and at most $k$ open balls in $M \setminus \partial M$. The rest of the theory remains unchanged.

In this paper, instead of using the relative version we will want to use a “compactly supported” variation of it in the case when $M = \mathbb{R}^m$.

**Definition 4.2.** We define $\tilde{O}(\mathbb{R}^m)$ to be the subcategory of $O(\mathbb{R}^m)$ consisting of open subsets of $\mathbb{R}^m$ whose complement is compact. We also define $\tilde{O}_k(\mathbb{R}^m)$ to be the following subposet of $\tilde{O}(\mathbb{R}^m)$. An open subset $U$ of $\mathbb{R}^m$ is in $\tilde{O}_k(\mathbb{R}^m)$ if $U$ is the union of two disjoint subsets $U_0$ and $U_1$ where $U_0$ is the complement of a closed ball, and $U_1$ is the disjoint union of at most $k$ open balls.

Our next goal is to recast the Taylor tower as a space of maps between modules over an operad. To begin with, from now on we will only consider the case where $M$ is an open submanifold of $\mathbb{R}^m$. Moreover, $M$ will be either a bounded submanifold (in which case we are interested in the absolute theory) or all of $\mathbb{R}^m$ (in which case we will consider the relative theory). In this case, it makes sense to speak of standard balls in $M$. We say that a subset of $M$ is a standard ball if it is the image of a standard embedding $D^m \hookrightarrow M$. Similarly, a disjoint union of standard balls in $M$ is the image of a standard embedding $A \times D^m \hookrightarrow M$ where $A$ is a finite set.

**Definition 4.3.** When $M$ is a bounded open submanifold of $\mathbb{R}^m$, we define $O^*_k(M)$ (where $k = 0, 1, 2, \ldots, \infty$) to be the subposet of $O_k(M)$ consisting of disjoint unions of at most $k$ standard balls. In the case when $M = \mathbb{R}^m$, we define $\tilde{O}^*_k(\mathbb{R}^m)$ to be the subposet of $\tilde{O}_k(\mathbb{R}^m)$ consisting of sets that are the image of a standard embedding $(A \times D^m) \coprod (\mathbb{R}^m \setminus \overline{D^m}) \hookrightarrow \mathbb{R}^m$ where $A$ is a finite set with at most $k$ elements.

We would like to make a note about the topology present in posets $O^*_k(M)$ and $\tilde{O}^*_k(\mathbb{R}^m)$. Weiss defines $O(M)$ to be a discrete poset. However, the posets $O^*_k(M)$ and $\tilde{O}^*_k(\mathbb{R}^m)$ are endowed with a natural topology, and we will view them as topological posets. For example, the space of objects of $O^*_k(M)$ can be identified with

$$\prod_{i=0}^{k} \text{sEmb}\{1, \ldots, i\} \times D^m, M) / \Sigma_i.$$ 

The space of morphisms is topologized accordingly. In order to relate the topological poset $O^*_k(M)$ with the discrete poset $O_k(M)$, we need to also consider the poset $\delta O^*_k(M)$, which is the discrete poset underlying $O^*_k(M)$. This notation is taken from [6]. We will use the decoration $\delta$ to denote “discretization” in other places as well.
We have a diagram of posets and continuous maps between them
\[ O^s_k(M) \leftarrow \delta O^s_k(M) \rightarrow O_k(M). \]
There also is a “tilde-ed” analogue of this diagram in the case \( M = \mathbb{R}^m \). Using the methods of [24], especially Section 3, it is not too difficult to prove the following lemma (its proof will appear elsewhere).

**Lemma 4.4.** Let \( M \) be an open submanifold of \( \mathbb{R}^m \). Let \( F : O(M) \rightarrow D \) be a contravariant isotopy functor. Then the inclusion of posets \( \delta O^s_k(M) \rightarrow O_k(M) \) induces an equivalence of homotopy limits
\[
\text{holim}_{U \in O_k(M)} F(U) \cong \text{holim}_{U \in \delta O^s_k(M)} F(U).
\]
Similarly, if \( F \) is a contravariant isotopy functor from \( \tilde{O}(\mathbb{R}^m) \) to \( D \), then the inclusion of posets \( \delta \tilde{O}^s_k(\mathbb{R}^m) \rightarrow \tilde{O}_k(\mathbb{R}^m) \) induces an equivalence of homotopy limits of \( F \).

In [24], \( \text{holim}_{U \in O_k(M)} F(U) \) was used to define \( T_k F \), the \( k \)-th Taylor polynomial of \( F \). In view of the above lemma, we may replace \( O_k(M) \) with \( \delta O^s_k(M) \), which is what we will do from now on.

**Definition 4.5.** Let \( M \) be either a bounded open submanifold of a Euclidean space \( \mathbb{R}^m \), or all of \( \mathbb{R}^m \). Let \( F \) be a contravariant isotopy functor from \( O^s_\infty(M) \) (or from \( \tilde{O}^s_\infty(\mathbb{R}^m) \)) to \( D \). We define
\[
T_k F(M) := \text{holim}_{U \in \delta O^s_k(M)} F(U).
\]
In the case \( M = \mathbb{R}^m \), the definition is amended to
\[
T_k F(\mathbb{R}^m) := \text{holim}_{U \in \delta \tilde{O}^s_k(\mathbb{R}^m)} F(U).
\]

Now we are ready to relate
\[
T_k F(-)
\]
to maps of modules over the little balls operad \( B_m \). We want to relate the categories \( O^s_k(M) \) and \( \tilde{O}^s_k(\mathbb{R}^m) \) to, respectively, the categories \( F(B_m) \) and \( \tilde{\Gamma}(B_m) \) that were defined in the previous section. To begin with, let us introduce truncated versions of \( F(B_m) \) and \( \tilde{\Gamma}(B_m) \).

**Definition 4.6.** For \( k = 0, 1, \ldots, \infty \), let \( F_k(B_m) \) be the full subcategory of \( F(B_m) \) consisting of sets of cardinality at most \( k \). In particular, \( F_\infty(B_m) = F(B_m) \). Thus, it is the category of unions of at most \( k \) standard \( m \)-dimensional balls and standard embeddings between them. Similarly, let \( \tilde{\Gamma}_k(B_m) \) be the full subcategory of \( \tilde{\Gamma}(B_m) \) consisting of sets of cardinality at most \( k + 1 \). Thus it is equivalent to the category where an object is the union of at most \( k \) standard \( m \)-balls, together with one anti-ball, and morphisms are standard embeddings between such objects.

Next, let us recall the well-known construction that associates to a category and a functor a new category, that represents a kind of homotopy colimit of the functor.
Definition 4.7. Let \( \mathcal{C} \) be a category and let \( F: \mathcal{C} \to \text{Top} \) be a contravariant functor (we focus on the contravariant case, because it arises in the examples of relevance to us). We define \( \mathcal{C} \ltimes F \) to be the following topological category. An object of \( \mathcal{C} \ltimes F \) is a pair \((c,x)\) where \( c \) is an object of \( \mathcal{C} \) and \( x \in F(c) \). A morphism \((c,x) \to (c',x')\) in \( \mathcal{C} \ltimes F \) is given by a morphism \( \alpha: c \to c' \) in \( \mathcal{C} \) such that \( F(\alpha)(x') = x \). Composition of morphisms is defined in the evident way. As mentioned already, \( \mathcal{C} \ltimes F \) is a topological category, where both the space of objects and the space of morphisms have a topology. More precisely, the space of objects is topologized as the disjoint union \( \coprod_{c \in \text{Ob}(\mathcal{C})} F(c) \) and the space of morphisms is topologized as the disjoint union \( \coprod_{(c,c') \in \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{C})} \text{Map}_\mathcal{C}(c,c') \times F(c') \).

Note that this definition makes sense if \( \mathcal{C} \) is a topologically enriched category whose set of objects is discrete, but whose spaces of morphisms have a topology. In this case, the topology of \( \mathcal{C} \) is accounted for in the topology of the morphism space of \( \mathcal{C} \ltimes F \).

For example, consider the construction \( s\text{Emb}(\bullet, M) \), seen as a contravariant functor either from \( F(B_m) \) (if \( M \) is a bounded open submanifold of \( \mathbb{R}^m \)) or from \( \tilde{\Gamma}(B_m) \) (if \( M = \mathbb{R}^m \)) to \( \text{Top} \), and consider the associated categories \( F(B_m) \ltimes s\text{Emb}(\bullet, M) \) and \( \tilde{\Gamma}(B_m) \ltimes s\text{Emb}(\bullet, \mathbb{R}^m) \). An object in either category is given by an ordered pair \((U,f)\), where \( U \) is a finite union of standard balls (together with an anti-ball in the second case), and \( f \) is a standard embedding of \( U \) into \( M \) (or into \( \mathbb{R}^m \)). There are natural evaluation functors, which we denote with the same symbol \( \text{ev} \)

\[
\text{ev}: F_k(B_m) \ltimes s\text{Emb}(\bullet, M) \to O^s_k(M)
\]

and

\[
\text{ev}: \tilde{\Gamma}_k(B_m) \ltimes s\text{Emb}(\bullet, \mathbb{R}^m) \to \tilde{O}^s_k(\mathbb{R}^m)
\]

which send a pair \((U,f)\) to the image of \( f \).

We also would like to consider discretized versions of the evaluation functor. We adopt the convention from [6] that for a topological space \( X \), \( \delta X \) denotes the discrete set underlying \( X \), and similarly for a topological category \( \mathcal{C} \), \( \delta \mathcal{C} \) denotes the underlying discrete category. Thus we have discrete versions of the evaluation functors

\[
\delta \text{ev}: \delta F_k(B_m) \ltimes \delta s\text{Emb}(\bullet, M) \to \delta O^s_k(M)
\]

\[
\delta \text{ev}: \delta \tilde{\Gamma}_k(B_m) \ltimes \delta s\text{Emb}(\bullet, \mathbb{R}^m) \to \delta \tilde{O}^s_k(\mathbb{R}^m)
\]

We have the following easy lemma, whose proof relies on the fact that there is a unique standard isomorphism between two standard \( m \)-balls, as well as between two standard anti-balls.
Lemma 4.8. The discretized evaluation functors \textup{(4.2)} and \textup{(4.3)} are equivalences of categories. It follows, in particular, that pulling back along these functors induces an equivalence of homotopy limits.

The lemma tells us that one can replace homotopy limits over \(\delta O_k^*(M)\) (resp. \(\delta\overline{O}_k^*(\mathbb{R}^m)\)) with homotopy limits over \(\delta F_k(B_m) \ltimes \delta s\text{Emb}(\bullet, M)\) (resp. \(\delta T_k(B_m) \ltimes \delta s\text{Emb}(\bullet, \mathbb{R}^m)\)). Next, we observe that homotopy limits over these categories can, for an important class of functors, be viewed as spaces of maps between modules. Note that for a category \(\mathcal{C}\) and a functor \(F\) for which \(\mathcal{C} \ltimes F\) is defined, there is a canonical projection functor \(p: \mathcal{C} \ltimes F \to \mathcal{C}\).

Definition 4.9. Let \(F: \delta O_k^*(M) \to \mathcal{D}\) be a contravariant functor. We call \(F\) context-free if the composite functor

\[
F \circ \delta ev : \delta F_k(B_m) \ltimes \delta s\text{Emb}(\bullet, M) \to \mathcal{D}
\]

factors, up to a natural equivalence, through the projection functor

\[
p: \delta F_k(B_m) \ltimes \delta s\text{Emb}(\bullet, M) \to \delta F_k(B_m).
\]

That is, if there exists a contravariant functor \(F': \delta F(B) \to \mathcal{D}\) such that \(F \circ \delta ev\) is naturally equivalent to \(F' \circ p\). If \(F\) is context-free, we will generally not distinguish in notation between \(F\) and \(F'\). In a similar way we define what it means to be context-free for a functor from the category \(\delta\overline{O}_k^*(\mathbb{R}^m)\) to another category \(\mathcal{D}\).

Informally speaking, a functor on \(\delta O_k^*(M)\) or a similar category is context-free if it does not depend on the embedding of the balls in \(M\).

Example 4.10. The functor \(U \mapsto s\text{Emb}(U, M)\) is a context-free functor on \(\delta O_k^*(M)\). Here is an example of a functor that is not context-free. Let \(h: M \looparrowright N\) be a fixed immersion. Taking \(h\) to be the basepoint of \(\text{Imm}(M, N)\), define \(\text{Emb}(U, N)\) to be the homotopy fiber of the inclusion map \(\text{Emb}(U, N) \to \text{Imm}(U, N)\). The functor \(U \mapsto \text{Emb}(U, N)\) is not context-free, because it depends on the basepoint of \(\text{Imm}(U, N)\), which in turn depends on the embedding of \(U\) into \(M\). However, if \(N\) is a contractible open subset of \(\mathbb{R}^m\) (e.g., an open ball, or all of \(\mathbb{R}^m\)), \(M\) is an open subset of \(N\) and \(h: M \looparrowright N\) is the inclusion, then it is not hard to show that the functor \(U \mapsto \text{Emb}(U, N)\) is naturally equivalent to the functor \(U \mapsto s\text{Emb}(U, M)\). The proof is very similar to [6 Proposition 7.1]. See also Lemma \ref{lem:context-free} below. Thus, in this case the functor \(\text{Emb}(U, N)\) is context-free.

Recall that a contravariant functor on \(\mathcal{F}(B_m)\) is essentially the same thing as a right module over the little balls operad \(B_m\) (Lemma \ref{lem:contravariant}). Similarly, a contravariant functor on \(\delta \mathcal{F}(B_m)\) is essentially the same thing as a right module over the discretized little balls operad \(\delta B_m\). Thus a context-free functor on \(\delta O_k^*(M)\) is essentially the same as a right module over \(\delta B_m\). Similarly, a context-free functor on \(\delta\overline{O}_k^*(\mathbb{R}^m)\) is essentially the same as a weak bimodule over \(B_m\) (Proposition \ref{prop:context-free}). The same holds true after discretization.

Lemma 4.11. Let \(F: \delta O_k^*(M) \to \mathcal{D}\) be a context-free functor, where \(\mathcal{D}\) is a Quillen model category. Then there is a natural equivalence

\[
T_k F(M) := \underset{\delta O_k^*(M)}{\underset{\rightarrow}{\text{holim}}} F(-) \simeq \underset{\delta \mathcal{F}(B_m)}{\text{hNat}} \left( \delta s\text{Emb}(-, M), F(-) \right).
\]
Similarly, if \( F \) is a context-free functor on \( \tilde{O}_k^s(\mathbb{R}^m) \) then
\[
T_k F(\mathbb{R}^m) := \text{holim}_{(V) \in \delta \tilde{O}_k^s(V)} = \text{holim}_{(V) \in \delta \Gamma_k(\mathbb{B}_m)} \text{hNat}(\delta \tilde{s}\text{Emb}(-, \mathbb{R}^m), F(-)).
\]

Here \( \text{hNat}(F, G) \) stands for the derived object of natural transformations. One way to define is by taking the strict object of natural transformations between a cofibrant replacement of \( F \) and a fibrant replacement of \( G \).

**Proof.** By Lemma 4.8
\[
\text{holim}_{U \in \delta \tilde{O}_k^s(M)} F(U) \simeq \text{holim}_{(A, f) \in \delta \tilde{F}_k(\mathbb{B}_m) \times \delta \tilde{s}\text{Emb}(\bullet, M)} F(f(A)).
\]
Since \( F \) is context-free, we can rewrite the right hand side as
\[
\text{holim}_{(A, f) \in \delta \tilde{F}_k(\mathbb{B}_m) \times \delta \tilde{s}\text{Emb}(\bullet, M)} F(A)
\]
the point being that \( F \) only depends on \( A \) and not on \( f(A) \). On the other hand, by [4, Lemma 3.7]
\[
\text{holim}_{(A, f) \in \delta \tilde{F}_k(\mathbb{B}_m) \times \delta \tilde{s}\text{Emb}(\bullet, M)} F(A) \simeq \text{hNat}_{A \in \delta \tilde{F}_k(\mathbb{B}_m)} \left( \delta \tilde{s}\text{Emb}(A, M), F(A) \right).
\]
This proves the first equivalence. The second one is proved similarly. \( \Box \)

**Remark 4.12.** The categories of contravariant functors on \( \delta \tilde{F}_k(\mathbb{B}_m) \) and \( \delta \tilde{\Gamma}_k(\mathbb{B}_m) \) are equivalent to the categories of, respectively, truncated right modules and truncated weak bimodules over the discretized little balls operad \( \delta \mathbb{B}_m \). Thus the above lemma can be interpreted as giving models for \( T_k F(M) \) and \( T_k F(\mathbb{R}^m) \) as derived space of maps between truncated right modules and truncated weak bimodules over \( \delta \mathbb{B}_m \). Thus, we have the following reinterpretation of Lemma 4.11

**Corollary 4.13.** Let \( F : \delta \tilde{O}_{\infty}(M) \longrightarrow D \) be a context-free functor, where \( D \) is a Quillen model category. Then \( F \) can be thought of as a right module over the operad \( \delta \mathbb{B}_m \), and there is an equivalence
\[
T_k F(M) \simeq \text{hRmod}_{\leq k} \left( \delta \tilde{s}\text{Emb}(-, M), F(-) \right).
\]
Similarly, if \( F \) is a context-free functor on \( \tilde{O}_k^s(\mathbb{R}^m) \) then \( F \) can be thought of as a weak bimodule over \( \delta \mathbb{B}_m \), and there is an equivalence
\[
T_k F(\mathbb{R}^m) \simeq \text{hWBimod}_{\leq k} \left( \delta \tilde{s}\text{Emb}(-, \mathbb{R}^m), F(-) \right).
\]

**Remark 4.14.** One may wonder if there is a version of Corollary 4.13 that uses the topological little balls operad \( \mathbb{B}_m \) rather than its discretization. We believe that there is one. In particular, we believe the following conjectural non-discretized model for the Taylor tower.

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1 *Added in revision:* Since the first version of this paper appeared in preprint form, the conjecture was proved by Pedro Boavida de Brito and Michael Weiss. See [8].
Conjecture 4.15. Let $F : O^*_\infty(M) \to \text{Top}$ be a continuous context-free functor. Then $F$ can be thought of as a right module over $B_m$, and there is a natural equivalence
\[ T_\infty F(M) \simeq hR\text{mod}_B(s\text{Emb}(-, M), F(-)). \]
Moreover, we can apply the chain functor $C$ to $F$ and get an equivalence
\[ T_\infty C(F(M)) \simeq hR\text{mod}_B(C(s\text{Emb}(-, M)), C(F(-))). \]

Now suppose that $F$ is a context-free functor from $\tilde{O}^k(R^m)$ to Top. Then $F$ can be viewed as a weak bimodule over $B_m$. There is an equivalence
\[ T_k F(R^m) \simeq h\text{WBimod}_{B_m}(B_m, F(-)). \]

Note in particular that if $F(-) = s\text{Emb}(-, R^n)$ then $F$ is equivalent to $B_n$ as a weak bimodule over $B_m$, and therefore we have the following conjectural model for the Taylor tower of $\overline{\text{Emb}}_c(R^m, R^n)$.
\[ T_k \overline{\text{Emb}}_c(R^m, R^n) \simeq h\text{WBimod}_{B_m}(B_m, B_n). \]

Finally, we also believe in the following model for the Taylor tower of $C(F(R^m))$
\[ T_k C(F(R^m)) \simeq h\text{WBimod}_{C(B_m)}(C(B_m), C(F(-))). \]

In particular
\[ T_k C(\overline{\text{Emb}}_c(R^m, R^n)) \simeq h\text{WBimod}_{C(B_m)}(C(B_m), C(B_n)). \]

5. Applying the Formality Theorem

Let $M$ be an open submanifold of $R^m$ (as usual, we assume that $M$ is either bounded or is all of $R^m$) and let us suppose that $R^m$ is a subspace of another Euclidean space $R^n$ such that $n \geq 2m + 1$. Recall that for a subset $U \subset R^n$ we define $\overline{\text{Emb}}(U, R^n)$ to be the homotopy fiber of the inclusion map $\text{Emb}(U, R^n) \to \text{Imm}(U, R^n)$. The definition requires a choice of a basepoint in $\text{Imm}(U, R^n)$, which is provided by the inclusion map.

Similarly, we use the inclusion of $R^m$ into $R^n$ to define $\overline{\text{Emb}}_c(-, R^n) : \overline{O}(R^m) \to \text{Top}_*$ and $\text{Imm}_c(-, R^n) : \overline{O}(R^m) \to \text{Top}_*$ to be, respectively, the space of embeddings and immersions from a subspace of $R^m$ to $R^n$ that agree with the prescribed inclusion outside a bounded set. Then define the functor $\overline{\text{Emb}}_c(-, R^n) : \overline{O}(R^m) \to \text{Top}$ to be the homotopy fiber of the map $\text{Emb}_c(-, R^n) \to \text{Imm}_c(-, R^n)$.

To begin with, we have the following lemma, which says that these functors are context-free

**Lemma 5.1.** The functor $\overline{\text{Emb}}(-, R^n)$, considered as a functor from $O^*_\infty(M)$ to Top, and the functor $\overline{\text{Emb}}_c(-, R^n)$, considered as a functor from $\tilde{O}^*_\infty(R^m)$ to Top are both equivalent to the functor that sends $U$ to $s\text{Emb}(U, R^n)$.
Proof. The first case is exactly [6, Proposition 7.1]. The second case is proved in a similar way.

Since sEmb(−, R^n) is a context-free functor and moreover it is equivalent to B_n as a weak bimodule over B_m, we can apply Corollary 4.13 to obtain the following formulas
\begin{equation}
T_k \mathcal{C}(\tilde{\text{Emb}}(M, \mathbb{R}^n)) \simeq h\text{Rmod}_{\delta B_m} \delta \text{sEmb}(−, M, \mathcal{C}(B_n))
\end{equation}

and
\begin{equation}
T_k \mathcal{C}(\tilde{\text{Emb}_c}(R^m, \mathbb{R}^n)) \simeq h\text{WBimod}_{\delta B_m} \delta \text{sEmb}(−, R^m, \mathcal{C}(B_n)).
\end{equation}

Next, we use Kontsevich’s formality theorem, very much in the spirit of [6], to deduce that in characteristic zero \(\mathcal{C}(B_n)\) can be replaced with \(H(B_n)\) in the above formulas. Let us first note that any operad in Sets \(\mathcal{O}\) has a natural “stupid” action on \(H(B_n)\) (assuming \(n \geq 2\)). By this we mean that there is a map of operads \(\mathcal{O} \rightarrow H(B_n)\). It follows in particular that \(H(B_n)\) is a weak bimodule over \(\mathcal{O}\). To see the map, recall that \(\text{Com}\) is the commutative operad in \(\text{Top}\). \(\text{Com}(n) = \ast\) for all \(n \geq 0\). It is easy to see that \(\text{Com}\) is a final object in the category of operads in Sets, so there is a canonical map of operads \(\mathcal{O} \rightarrow \text{Com}\). There also is an obvious map of operads \(\text{Com} \rightarrow H_0(\text{Com})\) (the latter is just the commutative operad in the category of chain complexes). But if \(n \geq 2\) then all the spaces in the operad \(B_n\) are connected and there is an isomorphism of operads \(H_0(\text{Com}) \cong H_0(B_n)\). Finally, it is easy to see that there is a natural map of operads \(H_0(B_n) \rightarrow H(B_n)\). Combining, we obtain a map \(\mathcal{O} \rightarrow H(B_n)\). In particular \(H(B_n)\) is a weak bimodule over \(\delta B_m\). This structure is referred to in the statement of the following proposition.

**Proposition 5.2.** Suppose that \(2m + 1 \leq n\). There are equivalences
\begin{equation}
T_k \left(\mathcal{C}^Q(\tilde{\text{Emb}}(M, \mathbb{R}^n))\right) \simeq h\text{Rmod}_{\delta B_m} \delta \text{sEmb}(−, M, H(B_n; \mathbb{Q}))
\end{equation}

and
\begin{equation}
T_k \left(\mathcal{C}^Q(\tilde{\text{Emb}_c}(\mathbb{R}^m, \mathbb{R}^n))\right) \simeq h\text{WBimod}_{\delta B_m} \delta \text{sEmb}(−, \mathbb{R}^m, H(B_n; \mathbb{Q})).
\end{equation}

Proof. The first assertion is essentially the same as [6, Theorem 9.2]. The proof of the second assertion is similar. The main point is that since the category of weak bimodules over an operad is equivalent to a category of diagrams (Proposition 2.15), just like the category of right modules, all the formal manipulations that were done in [6] with right modules can also be done with weak bimodules. For the reader’s convenience, we will go over the main steps of the argument. For the rest of the proof, each occurrence of the word “module” may be read either as “right module” or “weak bi-module”.

To begin with, it follows from Kontsevich’s formality theorem [15], or more precisely from the “relative” version of the theorem, introduced by Lambrechts and Volic ([17], and see also [6]), that (under the assumption \(2m + 1 \leq n\)) the map of operads
\(\mathcal{C}^R(B_m) \rightarrow \mathcal{C}^R(B_n)\)
is equivalent, via a chain of quasi-isomorphisms between maps of operads, to the maps of operads
\[ \text{H}(B_m; R) \longrightarrow \text{H}(B_n; R). \]
This means that there is a diagram of operads, where \( D_m, D_n \) are some intermediate operads, and the horizontal homomorphisms are quasi-isomorphisms
\[
\begin{array}{ccc}
C^R(B_m) & \longrightarrow & D_m \\
\downarrow & & \downarrow \\
C^R(B_n) & \longrightarrow & D_n \\
& & \longrightarrow \\
\end{array}
\]
Recall that \( m < n \). It follows that for each \( i \), the map \( B_m(i) \to B_n(i) \) is null-homotopic, and in particular induces the zero homomorphism on homology above degree zero. In other words, the map \( \text{H}(B_m; R) \longrightarrow \text{H}(B_n; R) \) on the right side of the diagram factors through \( H_0(B_n; R) \). It follows that \( \text{H}(B_n) \) is equivalent, as a module over \( \text{H}(B_m) \), to the direct sum \( \bigoplus_{i=0}^\infty H_i(B_n) \). Next, we can consider \( \text{H}(B_n) \) as a module over \( D_m \) via the pull-back functor. Then, using the identification of the category of modules with a category of diagrams, we can use the derived left Kan extension (which is the derived left adjoint to the pull-back functor) along the map of operads \( D_m \to C^R(B_m) \) to get a module over \( C^R(B_m) \). Let us denote this module \( \text{LH}(B_n; R) \). This construction was explained in detail in [6] for the case of right modules, and the point here is that since the category of weak bimodules is equivalent to a certain category of diagrams of chain complexes just like the category of right modules, the same constructions and arguments apply to weak bimodules. It follows from the formality theorem that \( \text{LH}(B_n; R) \) is weakly equivalent to \( C^R(B_n) \) as module over \( C^R(B_m) \). On the other hand, since \( \text{H}(B_n; R) \) splits, in the category of modules over \( \text{H}(B_m; R) \), as a direct sum of modules each concentrated in a single homological degree, it follows that \( \text{LH}(B_n; R) \) also splits as a direct sum of modules, with each summand concentrated in a single homological degree. Restricting from the operad \( C(B_m) \otimes R \to C^R(\delta B_m) \), we obtain that \( \text{LH}(B_n; R) \), and therefore also \( C^R(B_n) \), splits in this way as a module over \( C^R(\delta B_m) \), or equivalently over \( \delta B_m \). It follows that \( C^R(B_n) \) is in fact equivalent to \( \text{H}(B_n; R) \) as a module over \( \delta B_m \). Again, a detailed proof of all of these implications was given in [6] for right modules, and the same proof works for weak bimodules.

Substituting this equivalence into (5.1) and (5.2), we obtain the equivalences
\[
T_k \left( C^R(\text{Emb}(M, R^n)) \right) \simeq \text{hRmod}_{\leq k}(\delta \text{Emb}(\cdot, M), \text{H}(B_n; R))
\]
and
\[
T_k \left( C(\text{Emb}_c(R^m, R^n)) \otimes R \right) \simeq \text{hWBimod}_{\leq k}(\delta \text{Emb}(\cdot, R^m), \text{H}(B_n; R)).
\]
It remains to show that \( R \) can be replaced with \( Q \) in the above equivalences. This was proved in [6] for the first equivalence, and the proof for the second one is similar. By a finiteness argument similar to the proof of [6] Proposition 8.3], one can show that there are quasi-isomorphisms
\[
T_k \left( C^R(\text{Emb}_c(R^m, R^n)) \right) \simeq \left( T_k C^Q(\text{Emb}(R^m, R^n)) \right) \otimes R
\]
and
\[ hRmod_{\leq k}(\delta \text{sEmb}(-, \mathbb{R}^m), H(B_n; \mathbb{R})) \simeq \left( hRmod_{\leq k}(\delta \text{sEmb}(-, \mathbb{R}^m), H(B_n; \mathbb{Q})) \right) \otimes \mathbb{R}. \]

Thus the rational chain complexes
\[ T_k \left( C^Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \right) \quad \text{and} \quad hRmod_{\leq k}(\delta \text{sEmb}(-, \mathbb{R}^m), H(B_n; \mathbb{Q})) \]
become quasi-isomorphic, as real chain complexes, after tensoring with \( \mathbb{R} \). It follows that they are quasi-isomorphic also before tensoring with \( \mathbb{R} \).

6. From the little balls operad to the commutative operad

Our goal in this section is to show that the formulas in Proposition 5.2 can be rewritten in terms of modules over the commutative operad instead of modules over the operad \( \delta \text{B}_m \).

Let \( A \) be a set and \( C \) a chain complex. Recall that \( C(A) \) is the normalized singular chain complex of \( A \) (which in this case is essentially the same thing as the free Abelian group generated by \( A \)). There is a tautological isomorphism
\[ C^A \cong \text{hom}(C(A), C) \]
where the hom on the right hand stands for internal mapping object in the category of chain complexes. It follows that we can rewrite the formulas of Proposition 5.2 in the following form.
\[ T_k C^Q(\text{Emb}(M, \mathbb{R}^n)) \simeq hRmod_{\leq k}(C(\delta \text{sEmb}(-, M), H(B_n; \mathbb{Q})) \]
and
\[ T_k \left( C^Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \right) \simeq h WBimod_{\leq k}(C(\delta \text{sEmb}(-, \mathbb{R}^m)), H(B_n; \mathbb{Q})). \]

Furthermore, recall that the action of \( \delta \text{B}_m \) on \( H(B_n; \mathbb{Q}) \) factors through the natural map of operads \( \delta \text{B}_m \to \text{Com} \), where \( \text{Com} \) is the commutative operad. Thus \( H(B_n; \mathbb{Q}) \) is in the image of the restriction functor from (weak bi-)modules over \( \text{Com} \) to (weak bi-)modules over \( \delta \text{B}_m \). Since the categories of right modules and weak bi-modules are equivalent to certain diagram categories, both restriction functors have a derived left adjoint, defined using the derived left Kan extension. Let \( \text{ind} \) denote the derived induction functor from right modules over \( \delta \text{B}_m \) to right modules over \( \text{Com} \), and let \( \widetilde{\text{ind}} \) denote the analogous induction functor for weak bimodules. We have the following tautological consequence of Proposition 5.2.

**Corollary 6.1.** Suppose that \( 2m + 1 \leq n \), \( M \) is a bounded open submanifold of \( \mathbb{R}^m \), and \( \mathbb{R}^m \) is a subspace of \( \mathbb{R}^n \). Then there are equivalences
\[ T_k C^Q(\text{Emb}(M, \mathbb{R}^n)) \simeq hRmod_{\leq k}(\text{Com} \left( \text{ind} \left( C(\delta \text{sEmb}(-, M)) \right), H(B_n; \mathbb{Q}) \right)) \]
and
\[ T_k \text{C}^Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \simeq \text{hWBimod}_{\text{Com}}(\tilde{\text{ind}}(\text{C}(\delta \text{sEmb}(-, \mathbb{R}^m))), \text{H}(\mathcal{B}_n; \mathbb{Q})). \]

Our next task is to describe explicitly the right Com-module \( \text{ind}(\text{C}(\delta \text{sEmb}(-, M))) \), and the weak Com-bimodule \( \tilde{\text{ind}}(\text{C}(\delta \text{sEmb}(-, \mathbb{R}^m))) \). Recall that a right Com-module is the same thing as a contravariant functor on the category \( \mathcal{F} \) of unpointed finite sets, and a weak Com-bimodule is the same thing as a contravariant functor on the category \( \Gamma \) of pointed finite sets (Corollaries 2.6 and 2.16 respectively). Let \( \text{C}(\cdot) \) be the right Com-module corresponding to the functor from \( \mathcal{F} \) to chain complexes \( U \mapsto \text{C}(\text{Map}(U, M)) \). Let \( \text{C}(\cdot) \) be the weak Com-bimodule corresponding to the functor from \( \Gamma \) to chain complexes \( A \mapsto \text{Map}_*(A, S^m) \). Here \( S^m \) is the one-point compactification of \( \mathbb{R}^m \), considered as a pointed space with \( \infty \) being the basepoint.

**Proposition 6.2.** There is an equivalence of right Com-modules:
\[ \text{ind}(\text{C}(\delta \text{sEmb}(-, M))) \simeq \text{C}(M^*), \]
and an equivalence of weak Com-bimodules:
\[ \tilde{\text{ind}}(\text{C}(\delta \text{sEmb}(-, \mathbb{R}^m))) \simeq \text{C}((S^m)^*). \]

**Proof.** Let us prove the first assertion first. We remind the reader that \( \mathcal{F}(\delta \text{B}_m) \) is the category whose objects are disjoint unions of copies of the standard \( m \)-ball, and whose morphisms are standard embeddings. \( \mathcal{F}(\text{Com}) \) is the category of finite sets. There is a functor \( \pi_0: \mathcal{F}(\delta \text{B}_m) \rightarrow \mathcal{F}(\text{Com}), \) which takes path components.

The functor \( \text{ind} \) is homotopy left Kan extension along \( \pi_0 \). Our goal is to show that the homotopy left Kan extension along \( \pi_0 \) of the contravariant functor
\[ \text{C}(\delta \text{sEmb}(-, M)): \mathcal{F}(\delta \text{B}_m) \rightarrow \text{Ch} \]
is equivalent to the functor
\[ \text{C}(M^-): \mathcal{F}(\text{Com}) \rightarrow \text{Ch}. \]
Since the singular chains functor preserves homotopy colimits, it commutes with homotopy left Kan extensions. Therefore, it is enough to show that the homotopy left Kan extension of the contravariant functor
\[ \delta \text{sEmb}(-, M): \mathcal{F}(\delta \text{B}_m) \rightarrow \text{Top} \]
is equivalent to the functor \( M^- \), considered to be a functor from \( \mathcal{F}(\text{Com}) \) to \( \text{Top} \).

Let us consider first the case when \( M \) is itself an object of \( \mathcal{F}(\delta \text{B}_m) \). That is, when \( M \) is the disjoint union of standard balls in \( \mathbb{R}^m \). Let us relabel \( M = U \) in this case. the functor \( \delta \text{sEmb}(-, U) \) is the free contravariant functor generated by \( U \). It follows, by a version of the Yoneda lemma, that the strict left Kan extension of \( \delta \text{sEmb}(-, U) \) along \( \pi_0 \) is the free
contravariant functor generated by $\pi_0(U)$. Moreover, for free functors, homotopy left Kan extension is naturally equivalent to the strict Kan extension. Therefore,

$$\text{ind}\left(\delta \text{sEmb}(-, U)\right)(\bullet) \simeq \pi_0(U)^\bullet.$$ 

Moreover, the path components of $U$ are contractible, so the natural map $U \to \pi_0(U)$ is a homotopy equivalence. It follows that there is an equivalence, natural in $\bullet$, $\pi_0(U)^\bullet \simeq U^\bullet$. We have obtained an equivalence, natural both in $\bullet$ and in $U$

$$\text{ind}\left(\delta \text{sEmb}(-, U)\right)(\bullet) \simeq U^\bullet.$$ 

Now let $M$ be a general open submanifold of $\mathbb{R}^m$. Recall that $\delta \mathcal{O}_\infty(M)$ is the poset of subsets of $M$ that are the unions of copies of the standard ball. Consider the homotopy colimit of functors

$$\text{hocolim}_{U \in \delta \mathcal{O}_\infty(M)} \delta \text{sEmb}(\cdot, U).$$

We claim that this homotopy colimit is naturally equivalent to $\delta \text{sEmb}(\cdot, M)$. To see this, observe first that for each fixed value of $V$ the diagram $U \mapsto \delta \text{sEmb}(V, U)$ is cofibrant in the sense that for all $V, U \in \mathcal{O}_\infty(M)$ the map

$$\text{colim}_{U_0 \subseteq U} \delta \text{sEmb}(V, U_0) \longrightarrow \delta \text{sEmb}(V, U)$$

is a cofibration. Indeed, it is an isomorphism of sets if $V$ is not homeomorphic to $U$, and it is the inclusion of a subset if $V$ is homeomorphic to $U$. More precisely, the colimit on the left hand side is the set of non-surjective standard embeddings of $V$ into $U$. It follows that the following natural map is an equivalence

$$\text{hocolim}_{U \in \delta \mathcal{O}_\infty(M)} \delta \text{sEmb}(\cdot, U) \simeq \text{colim}_{U \in \delta \mathcal{O}_\infty(M)} \delta \text{sEmb}(\cdot, U).$$

It is easy to see that the target of this map is isomorphic to $\delta \text{sEmb}(\cdot, M)$. Indeed, the natural map

$$\text{colim}_{U \in \delta \mathcal{O}_\infty(M)} \delta \text{sEmb}(\cdot, U) \longrightarrow \delta \text{sEmb}(\cdot, M)$$

is surjective since every standard embedding of a union of balls into $M$ factors through some $U \in \delta \mathcal{O}_\infty(M)$ and is injective because for any two such factorizations there is a factorization contained in both of them.

Since homotopy left Kan extension commutes with homotopy colimits, it follows that

$$\text{ind}(\delta \text{sEmb}(\cdot, M)) \simeq \text{hocolim}_{U \in \delta \mathcal{O}_\infty(M)} \text{ind}(\delta \text{sEmb}(\cdot, U)) \simeq \text{hocolim}_{U \in \delta \mathcal{O}_\infty(M)} U^\bullet.$$ 

It remains to show that for every finite set $\bullet$, $\text{hocolim}_{U \in \delta \mathcal{O}_\infty(M)} U^\bullet \simeq M^\bullet$. This is readily seen to be the case.

Now let us consider the second assertion. The idea of the proof is similar. We want to show that the homotopy left Kan extension of $\delta \text{sEmb}(\cdot, \mathbb{R}^m)$ along the functor

$$\pi_0: \tilde{\Gamma}(\delta \mathcal{B}_m) \longrightarrow \Gamma$$
is equivalent to \((S^m)^\bullet\). Let us recall the picture of the category \(\bar{\Gamma}(\delta B_m)\) that we are working with. An object of this category is a disjoint union of standard balls in \(\mathbb{R}^m\), together with one “anti-ball”, that is a copy of the complement of the unit ball in \(\mathbb{R}^m\). Morphisms are standard embeddings between such objects. Again, let us consider first the homotopy left Kan extension of the functor \(\delta \text{Emb}(\bullet, U)\), where \(U\) is an object of \(\bar{\Gamma}(\delta B_m)\). As before, this is a free functor, and so its derived left Kan extension is the functor \(\pi_0(\mathcal{U})\):= \(\text{Map}^\ast(-, \pi_0(U))\). Here we view \(\pi_0(U)\) as a pointed set, the basepoint being the path component of the anti-ball. Now we can not say, as we did in the first part of the proof, that \(\text{Map}^\ast(-, \pi_0(U)) \simeq \text{Map}^\ast(-, U)\), because the components of \(U\) are not contractible. More specifically, the anti-ball is not contractible. Let us view \(S^m\) as the one-point compactification of \(\mathbb{R}^m\). For an object \(U\) of \(\bar{\Gamma}(\delta B_m)\), let \(\mathcal{U}\) be the space obtained from \(U\) by replacing the anti-ball with its closure in \(S^m\), and consider the added point of \(\mathcal{U}\) as the basepoint. All the components of \(\mathcal{U}\) are contractible, and therefore \(\text{Map}^\ast(-, \pi_0(U)) \simeq \text{Map}^\ast(-, \mathcal{U})\). Now we can argue, exactly as in the first part of the proof, that \(\delta \text{Emb}(\bullet, \mathbb{R}^m) \simeq \text{hocolim}_{U \in \delta \mathcal{O}_\infty(\mathbb{R}^m)} \text{Map}^\ast(-, \mathcal{U})\). By definition the spaces \(\mathcal{U}\) are, naturally, subsets of \(S^m\). It is now easy to see that the above homotopy colimit is equivalent to \(\text{Map}^\ast(-, S^m) = (S^m)^\bullet\). Thus, the homotopy left Kan extension of \(\delta \text{Emb}(\bullet, \mathbb{R}^m)\) is equivalent to \(\text{Map}^\ast(-, \mathcal{U})\).

By definition the spaces \(\mathcal{U}\) are, naturally, subsets of \(S^m\). It is now easy to see that the above homotopy colimit is equivalent to \(\text{Map}^\ast(-, S^m) = (S^m)^\bullet\). Thus, the homotopy left Kan extension of \(\delta \text{Emb}(\bullet, \mathbb{R}^m)\) is equivalent to \(\text{Map}^\ast(-, S^m)\), which is what we wanted to prove. □

We have the following consequence of Corollary 6.1 and Proposition 6.2

**Proposition 6.3.** Under the same assumptions as in Corollary 6.1 there are weak equivalences of rational chain complexes

\[
T_k \mathcal{C}^Q(\text{Emb}(M, \mathbb{R}^n)) \simeq \text{hRmod}_{\leq k} \left( \mathcal{C}(M^\bullet), H(B_n(\bullet); \mathbb{Q}) \right)
\]

and

\[
T_k \mathcal{C}^Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \simeq \text{hWBimod}_{\leq k} \left( \mathcal{C}((S^m)^\bullet), H(B_n(\bullet); \mathbb{Q}) \right).
\]

The first part of the proposition may be viewed as an alternative formulation of the main result of [6]. As to the second part of the proposition, it can be simplified further, using Pirashvili’s “Dold-Kan” correspondence between right \(\Gamma\)-modules and right \(\Omega\)-modules [20]. Recall that weak \(\text{Com}\)-bimodules are the same as right \(\Gamma\)-modules (Corollary 2.16). By [20, Theorem 3.1], the category of right \(\Gamma\)-modules (with values in an Abelian category) is equivalent to the category of right \(\Omega\)-modules, where \(\Omega\) is the category of (unpointed) finite sets and surjective maps between them. Given a right \(\Gamma\)-module \(F\), the corresponding \(\Omega\)-module is defined using the cross-effect, so it may be denoted \(\text{cr} \, F\). For an unpointed finite
set \( i \), \( \text{cr } F(i) \) is defined to be the cokernel of the natural homomorphism
\[
\bigoplus_{j=1}^{i} F((i-1)_+) \longrightarrow F(i_+).
\]
Here the \( i \) maps \( F((i-1)_+) \longrightarrow F(i_+) \) are induced by the \( i \) maps \( i_+ \longrightarrow (i-1)_+ \) where the \( j \)-th map sends \( j \) to the basepoint and is otherwise an order preserving isomorphism. Equivalently, \( \text{cr } F(i) \) can be defined as the total cokernel of a certain evident cubical diagram of objects of the form \( F(j_+) \), where \( j \) ranges over subsets of \( i \). In the case when \( i \) is empty, \( \text{cr } F(\emptyset) = F(*) \). It is easy to see that the \( \Omega \)-module \( \text{cr } C((S^m)^*) \) is equivalent to the \( \Omega \)-module \( \tilde{C}(S^m^*) \) which associates to a set \( i \) the complex \( \tilde{C}(S^mi) = C(S^mi, *) \) and where the \( \Omega \)-module structure is defined by the diagonal maps associated with surjective maps of sets.

**Definition 6.4.** Define the \( \Omega \)-module \( \tilde{H}(B_n(\bullet); \mathbb{Q}) \) to be the cross-effect \( \text{cr } H(B_n(\bullet); \mathbb{Q}) \).

Thus, the second assertion of Proposition 6.3 translates, via Pirashvili’s correspondence, into an equivalence
\[
T_k \tilde{C}(\text{Emb}_m(R^m, R^n)) \cong \text{hRmod}_{\leq k} \left( \tilde{C}(S^m^*), H(B_n(\bullet); \mathbb{Q}) \right).
\]

In fact, things can be simplified even further, as shown in the following lemma

**Lemma 6.5.** The right \( \Omega \)-module \( \tilde{C}(S^m^*) \) is rationally formal. That is, there is a weak equivalence of modules
\[
\tilde{C}(S^m^*) \cong \tilde{H}(S^m^*; \mathbb{Q}).
\]

**Proof.** Let us assume that \( m \geq 1 \) (the case \( m = 0 \) is trivial, but the forthcoming proof does not apply to it). Let us consider truncated modules first. Let \( M \) be a right \( \Omega \)-module with values in chain complexes. That is, \( M \) is a contravariant functor from \( \Omega \) to chain complexes. Fix \( k \geq 0 \). Let \( M_{\leq k} \) be the module that agrees with \( M \) on sets smaller or equal to \( k \), and is 0 on sets bigger than \( k \). Clearly, this is well-defined, and there is a canonical surjective map of right \( \Omega \)-modules \( M_{\leq k} \rightarrow M_{\leq k-1} \). Let \( M^k \) be the kernel of the homomorphism. Clearly, \( M^k \) is the right module that agrees with \( M \) on the set with \( k \) elements, and is zero elsewhere. There is a natural cofibration sequence
\[
0 \longrightarrow M^k \longrightarrow M_{\leq k} \longrightarrow M_{\leq k-1} \longrightarrow 0.
\]
This sequence can be viewed as a homotopy cofibration sequence of right \( \Omega \)-modules. It is classified by a map (in the homotopy category of right modules)
\[
M_{\leq k-1} \longrightarrow \Sigma M^k
\]
where \( \Sigma M^k \) is the point-wise shift of \( M^k \).

Now let us apply this to the module \( M = \tilde{C}(S^m^*) \). We have the cofibration sequence
\[
\tilde{C}(S^m^*)^k \longrightarrow \tilde{C}(S^m^*)_{\leq k} \longrightarrow \tilde{C}(S^m^*)_{\leq k-1} \longrightarrow \Sigma \tilde{C}(S^m^*)^k.
\]
Let us calculate the derived mapping object, in the category of right modules, from $\widetilde{\mathcal{C}}(S^{m\bullet})_{\leq k-1}$ to $\Sigma \widetilde{\mathcal{C}}(S^{m\bullet})^k$. This is a chain complex whose $H_0$ gives the set of homotopy classes of maps between these objects. Recall that $\Sigma \widetilde{\mathcal{C}}(S^{m\bullet})^k$ is a right $\Omega$-module concentrated in place $k$. In the next section we will prove some general results about morphisms between right $\Omega$-modules. In particular, we will prove Corollary 8.9, which implies that

$$hRmod_{\Omega} \left( \widetilde{\mathcal{C}}(S^{m\bullet})_{\leq k-1}, \Sigma \widetilde{\mathcal{C}}(S^{m\bullet})^k \right) \cong \text{hom}_{\Sigma_k} \left( \widetilde{\mathcal{C}}(\Delta^k S^m), \widetilde{\mathcal{C}}(\Sigma^k S^m) \right).$$

Here $\text{hom}_{\Sigma_k}$ stands for derived homomorphism in the category of chain complexes with an action of $\Sigma_k$ and $\Delta^k S^m$ is the fat diagonal in $S^{mk}$. By Alexander duality, this is quasi-isomorphic to $\widetilde{\mathcal{C}}(\Sigma C(k, \mathbb{R}^m))_{\Sigma_k}$ (here we have used that over the rationals, invariants are equivalent to derived invariants), which is an $m$-connected, and therefore zero-connected complex. It follows that there are no homotopically non-trivial maps of modules from $\widetilde{\mathcal{C}}(S^{m\bullet})_{\leq k-1}$ to $\Sigma \widetilde{\mathcal{C}}(S^{m\bullet})^k$, and therefore there is a weak equivalence of right $\Omega$-modules $\widetilde{\mathcal{C}}(S^{m\bullet})_{\leq k} \cong \widetilde{\mathcal{C}}(S^{m\bullet})_{\leq k-1} \oplus \widetilde{\mathcal{C}}(S^{m\bullet})^k$.

The right module $\widetilde{\mathcal{C}}(S^{m\bullet})^k$ is essentially determined by the chain complex $\widetilde{\mathcal{C}}(\Sigma^k S^m)$, together with an action of $\Sigma_k$. It is clear that there is a $\Sigma_k$-equivariant equivalence of $\Sigma_k$ complexes $\widetilde{\mathcal{C}}(\Sigma^k S^m) \cong \tilde{H}(S^{mk}; \mathbb{Q})$ (this is in fact true integrally). Thus there is an equivalence of right $\Omega$-modules $\widetilde{\mathcal{C}}(S^{m\bullet})_{\leq k} \cong \tilde{H}(S^{m\bullet}; \mathbb{Q})^k$.

and by induction we get an equivalence

$$\widetilde{\mathcal{C}}(S^{m\bullet})_{\leq k} \cong \bigoplus_{i=1}^k \tilde{H}(S^{m\bullet}; \mathbb{Q})^i.$$ 

It is easy to see that the right hand side is isomorphic to $\tilde{H}(S^{m\bullet}; \mathbb{Q})_{\leq k}$ and so we have an equivalence $\widetilde{\mathcal{C}}(S^{m\bullet})_{\leq k} \cong \tilde{H}(S^{m\bullet}; \mathbb{Q})_{\leq k}$ for each $k$, with the restriction map $\widetilde{\mathcal{C}}(S^{m\bullet})_{\leq k} \to \widetilde{\mathcal{C}}(S^{m\bullet})_{\leq k-1}$ corresponding to the restriction map $H(S^{m\bullet}; \mathbb{Q})_{\leq k} \to H(S^{m\bullet}; \mathbb{Q})_{\leq k-1}$. Finally, $\mathcal{C}(S^{m\bullet})$ may be identified with the inverse limit (which is also the homotopy inverse limit) of $\mathcal{C}(S^{m\bullet})_{\leq k}$, and similarly for the homological version, which implies that $\widetilde{\mathcal{C}}(S^{m\bullet}) \cong \tilde{H}(S^{m\bullet}; \mathbb{Q})$.

We are finally ready to state and prove our main theorem.

**Theorem 6.6.** Suppose that $2m+1 \leq n$. Then there are weak equivalences for all $k$

$$T_k \mathcal{C}(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \cong hRmod_{\Omega} \left( \tilde{H}(S^{m\bullet}), \tilde{H}(\mathcal{B}_n(\bullet); \mathbb{Q}) \right).$$
This includes the case $k = \infty$:

$$T_\infty \mathcal{C}Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \simeq \text{hRmod}_\Omega \left( \bar{H}(S^{m\bullet}), \bar{H}(B_n(\bullet); \mathbb{Q}) \right).$$

It follows that if $2m + 1 < n$ then there is an equivalence of chain complexes

$$\mathcal{C}Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \simeq \text{hRmod}_\Omega \left( \bar{H}(S^{m\bullet}), \bar{H}(B_n(\bullet); \mathbb{Q}) \right).$$

**Proof.** Our starting point is (6.1), which says that

$$T_k \mathcal{C}Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \simeq \text{hRmod}_\Omega \left( \bar{C}(S^{m\bullet}), \bar{H}(B_n(\bullet); \mathbb{Q}) \right).$$

Since the $\Omega$-module $\bar{H}(B_n(\bullet); \mathbb{Q})$ takes values in rational chain complexes, there is an equivalence

$$\text{hRmod}_\Omega \left( \bar{C}(S^{m\bullet}), \bar{H}(B_n(\bullet); \mathbb{Q}) \right) \simeq \text{hRmod}_\Omega \left( \bar{C}(S^{m\bullet}), \bar{H}(B_n(\bullet); \mathbb{Q}) \right).$$

By Lemma 6.5, $\bar{C}Q(S^{m\bullet})$ can be replaced with $\bar{H}(S^{m\bullet}; \mathbb{Q})$. Then again, since the target of the mapping space consists of rational chain complexes, we may replace the rational homology groups $\bar{H}(S^{m\bullet}; \mathbb{Q})$ with the corresponding integral homology. For the last statement we require $2m + 1 < n$ to ensure that the underlying functor is analytic, see [25].

### 7. The Splitting by Homological Degree

Our goal in the rest of the paper is to analyze the consequences of Theorem 6.6 and to use it to write explicit chain complexes for computing the rational homology groups of $\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)$. Theorem 6.6 expresses $\mathcal{C}Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n))$ as a space of morphisms between right $\Omega$-modules $\bar{H}(S^{m\bullet})$ and $\bar{H}(B_n(\bullet); \mathbb{Q})$. Obviously, these $\Omega$-modules split as direct sums of $\Omega$-modules concentrated in a single homological degree. For example, there is an isomorphism of $\Omega$-modules

$$\bar{H}(S^{m\bullet}) \cong \bigoplus_{i=0}^{\infty} \bar{H}_i(S^{m\bullet}).$$

We remind the reader that by $\bar{H}_i(X)$ we really mean the chain complex that has the $i$-th reduced homology of $X$ in degree $i$ and is zero otherwise. In fact, this decomposition is unnecessarily wasteful, because one need not consider all values of $i$, but only multiples of $m$.

**Definition 7.1.** We define, for each $m \geq 1$ and for each $s \geq 0$ the right $\Omega$-module $Q^m_s$ by the formula $Q^m_s(\bullet) = \bar{H}_{ms}(S^{m\bullet})$.

Clearly, $Q^m_s$ is given by the following formula

$$Q^m_s(k) = \begin{cases} 0 & k \neq s \\ \mathbb{Z}[ms] & k = s \end{cases}$$

where $\mathbb{Z}[ms] = \bar{H}_{ms}(S^{ms})$ is the chain complex that has $\mathbb{Z}$ in dimension $ms$ and is zero otherwise. Note that $\Sigma_s$ acts trivially on $\mathbb{Z}[ms]$ if $m$ is even and acts by sign representation
if \( m \) is odd. \( Q^m_s \) is in some sense a minimal non-zero right \( \Omega \)-module concentrated in degree \( \bullet = s \). It is obvious that there is an isomorphism of right \( \Omega \)-modules (assuming always that \( m \geq 1 \))

\[
\widetilde{H}(S^m\bullet) \cong \bigoplus_{s=0}^{\infty} Q^m_s(\bullet).
\]

There is a similar splitting for the right module \( \hat{H}(B_n(\bullet); \mathbb{Q}) \).

**Definition 7.2.** For each \( n \geq 2 \) and \( t \geq 0 \) define the right \( \Omega \)-module \( \hat{H}^n(\bullet) \) by the formula

\[
\hat{H}^n(\bullet) = \hat{H}(n-1)t(B_n(\bullet); \mathbb{Q}).
\]

It is well known that \( H(B_n(k)) \) is concentrated in dimensions of the form \((n-1)t\). It follows that there is an isomorphism of \( \Omega \)-modules (assuming \( n \geq 2 \))

\[
\hat{H}(B_n(\bullet); \mathbb{Q}) \cong \bigoplus_{t=0}^{\infty} \hat{H}^n(\bullet).
\]

It is obvious that each one of the direct sums above is isomorphic to a direct product. It follows that there is an isomorphism

\[
h_{R\text{mod}}(\widetilde{H}(S^m\bullet), \hat{H}(B_n(\bullet); \mathbb{Q})) \cong \prod_{s,t=0}^{\infty} h_{R\text{mod}}(Q^m_s(\bullet), \hat{H}^n_t(\bullet)).
\]

Thus we have the following immediate consequence of Theorem 6.6

**Corollary 7.3.** Suppose that \( 2m+1 \leq n \). Then there are weak equivalences for all \( k \leq \infty \)

\[
T_k \mathcal{C}^Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \simeq \prod_{s,t=0}^{\infty} h_{R\text{mod}}(Q^m_s(\bullet), \hat{H}^n_t(\bullet)).
\]

It follows that

\[
T_{\infty} \mathcal{C}^Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \simeq \prod_{s,t=0}^{\infty} h_{R\text{mod}}(Q^m_s(\bullet), \hat{H}^n_t(\bullet)).
\]

If \( 2m+1 < n \) then the Taylor tower converges, and so we have an equivalence

\[
\mathcal{C}^Q(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)) \simeq \prod_{s,t=0}^{\infty} h_{R\text{mod}}(Q^m_s(\bullet), \hat{H}^n_t(\bullet)).
\]

We will see in Section 11 (Corollary 11.4), that if \( n > 2m+1 \) the product above can be replaced by a direct sum.
8. The Koszul spectral sequence for $\Omega$-modules

Our next task is to analyze the complex of derived morphisms between right $\Omega$-modules, in order to understand better the consequences of Theorem [6.6] and Corollary [7.3]. In this section we will review some generalities concerning right $\Omega$-modules. In the next section we will apply the general theory to the modules $Q^p_s(\bullet)$ and $\hat{H}^p_t(\bullet)$.

In fact, we will focus on a slightly special situation: we will assume most of the time that the target right module is the dual of a left module. Recall that a left $\Omega$-module is a covariant functor from $\Omega$ to a background category.

Remark 8.1. To avoid possible confusion, we remind the reader that while a right $\Omega$-module is the same thing as a right module over the non-unital commutative operad, a left $\Omega$-module is not the same thing as a left module over the operad.

Suppose $C$ is a chain complex of $k$-modules. Let $D(C)$ be the dual chain complex $\text{hom}(C,k)$. Recall that the grading of the dual complex is defined by $\text{hom}(C,k)_n = \text{hom}(C_{-n},k)$.

Let $G$ be a left $\Omega$-module with values in chain complexes. Let $D(G)$ be the objectwise dual of $G$. Then $D(G)$ is a right $\Omega$-module. Let $F$ be another right $\Omega$-module. We are interested in the derived morphism object $\text{hRmod}(F,D(G))$. Let $F^h \otimes G$ be the derived coend of the contravariant functor $F$ and the covariant functor $G$. It is well known that derived coend is a derived left adjoint to derived hom, so we have the following proposition.

Proposition 8.2. There is a natural weak equivalence $\text{hRmod}(F,D(G)) \simeq D(F^h \otimes G)$.

For the rest of the section, we will focus on analyzing the homology of $F^h \otimes G$. Ultimately we will work with rational chain complexes, so the homology groups of $D(F^h \otimes G)$ are just the vector space duals of the homology of $F^h \otimes G$.

One standard approach to analyzing $F^h \otimes G$ is to filter the category $\Omega$ by cardinality, use this to construct a filtration of $F^h \otimes G$, and thus obtain a spectral sequence for calculating its homology groups.

Definition 8.3. For each $k \geq 0$, let $\Omega_{\leq k}$ be the full subcategory of $\Omega$ consisting of objects of cardinality $\leq k$. Let $F$ and $G$ be a right and a left $\Omega$-module respectively. By restriction, $F$ and $G$ may also be considered as $\Omega_{\leq k}$-modules. Let $F^h \otimes_{\leq k} G$ be the derived coend between $\Omega_{\leq k}$-modules.

One obtains a filtration of $F^h \otimes G$ by truncated homotopy coends.

\[
F^h \otimes_{\leq 0} G \to F^h \otimes_{\leq 1} G \to \cdots \to F^h \otimes G.
\]

We would like to analyze the homotopy cofiber of the map

\[
F^h \otimes_{\leq k-1} G \to F^h \otimes_{\leq k} G.
\]
Let $k \downarrow \Omega_{<k}$ be the category whose objects are surjective maps $k \rightarrow i$, where $i$ is a set that is strictly smaller than $k$, and whose morphisms are the evident commuting triangles. Thus, since $F$ is a contravariant functor from $\Omega$ to the category of chain complexes, it gives rise to a contravariant functor from $k \downarrow \Omega_{<k}$ to chain complexes, given on objects by the formula

$$ (k \rightarrow i) \mapsto F(i). $$

Define $\Delta^k F$ to be the homotopy colimit of this functor. Note that $\Delta^k F$ has a natural action of the symmetric group $\Sigma_k$ and that there is a natural $\Sigma_k$-equivariant map $\Delta^k F \rightarrow F(k)$.

**Lemma 8.4.** For each $k$, there is a homotopy pushout square

$$
\begin{array}{c}
\Delta^k F \otimes_{h\Sigma_k} G(k) \\
\downarrow
\end{array}
\begin{array}{c}
F \otimes_{\leq k-1} G
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
F(k) \otimes_{h\Sigma_k} G(k)
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
F \otimes_{\leq k} G.
\end{array}
$$

Here the symbol $\otimes_{h\Sigma_k}$ denotes the derived balanced product of two objects with an action of $\Sigma_k$. The left vertical map is induced from the map $\Delta^k F \rightarrow F(k)$.

**Proof.** The lemma is elementary and essentially well-known. See, for example, the paper of Ahearn and Kuhn [1, Lemma 3.7 and Corollary 3.8] for a proof of closely related result in the setting of topological spaces. The proof that we give is a variation of theirs.

Let $F^+_{\downarrow k-1}$ be the homotopy left Kan extension of $F$ from $\Omega_{\leq k-1}$ to $\Omega_{\leq k}$. It is easy to see that the restriction of $F^+_{\downarrow k-1}$ to $\Omega_{\leq k-1}$ is equivalent to the restriction of $F$ to $\Omega_{\leq k-1}$. On the other hand $F^+_{\downarrow k-1}(k) = \Delta^k F$. Let $\Delta^k F$ be the right $\Omega_{\leq k}$-module defined by $\Delta^k F(i) = 0$ for $i < k$ and $\Delta^k F(k) = \Delta^k F$. Similarly, let $F(k)$ be the $\Omega_{\leq k}$-module defined by $F(k)(i) = 0$ for $i < k$ and $F(k)(k) = F(k)$. There is a natural square of $\Omega_{\leq k}$-modules

$$
\begin{array}{c}
\Delta^k F \\
\downarrow
\end{array}
\begin{array}{c}
F^+_{\downarrow k-1}
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
F(k)
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
F
\end{array}
$$

and we claim that this is a homotopy pushout diagram of $\Omega_{\leq k}$-modules. Indeed, for $i < k$, evaluating this square on $i$ yields the square

$$
\begin{array}{c}
0 \\
\downarrow
\end{array}
\begin{array}{c}
F^+_{\downarrow k-1}(i)
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
0 \\
\downarrow
\end{array}
\begin{array}{c}
F(i)
\end{array}
$$
which is a homotopy pushout square, because the map $F_{k-1}^k (i) \rightarrow F(i)$ is an equivalence. Evaluating the square at $k$ yields the square

$$
\begin{array}{ccc}
\Delta^k F & \longrightarrow & \Delta^k F \\
\downarrow & & \downarrow \\
F(k) & \longrightarrow & F(k)
\end{array}
$$

which is also a homotopy pushout square. It follows that there is a homotopy pushout square

$$
\begin{array}{ccc}
\Delta^k F \otimes_{\leq k} G & \longrightarrow & F_{k-1}^k \otimes_{\leq k} G \\
\downarrow & & \downarrow \\
F(k) \otimes_{\leq k} G & \longrightarrow & F \otimes_{\leq k} G.
\end{array}
$$

The lemma now follows from the following three easily verified equivalences

$$
\Delta^k F \otimes_{\leq k} G \simeq \Delta^k F \otimes_{h \Sigma_k} G(k),
$$

$$
F(k) \otimes_{\leq k} G \simeq F(k) \otimes_{h \Sigma_k} G(k),
$$

and

$$
F_{k-1}^k \otimes_{\leq k} G \simeq F \otimes_{\leq k-1} G.
$$

$\square$

**Definition 8.5.** Let $F$ be a right $\Omega$-module. For $k \geq 0$, let $KF(k)$ be the homotopy cofiber of the natural map $\Delta^k F \rightarrow F(k)$. We call the sequence $\{KF(k)\}_{k=0}^\infty$ the Koszul dual of $F$.

**Corollary 8.6.** The homotopy cofiber of the map

$$
F \otimes_{\leq k-1} G \rightarrow F \otimes_{\leq k} G
$$

is equivalent to

$$
KF(k) \otimes_{h \Sigma_k} G(k).
$$

**Remark 8.7.** We will point out that in the case when $F$ and $G$ are two right $\Omega$-modules, one can analyze the derived “space” of morphisms $\text{hRmod}(F,G)$ in an analogous way. Namely, one can filter the category $\Omega$ by cardinality, and obtain a “tower” of fibrations converging to $\text{hRmod}(F,G)$:

$$
\text{hRmod}(F,G) \rightarrow \cdots \rightarrow \text{hRmod}_{\leq k}(F,G) \rightarrow \cdots.
$$

There is a description of the homotopy fiber of the restriction map $\text{hRmod}_{\leq k}(F,G) \rightarrow \text{hRmod}_{\leq k-1}(F,G)$ that is analogous to Corollary 8.6 and is proved in the same way.
Lemma 8.8. Let $F$, $G$ be right $\Omega$-modules. The homotopy fiber of the map
\[ h\text{Rmod}_{\leq k}(F, G) \to h\text{Rmod}_{\leq k-1}(F, G) \]
is equivalent to
\[ \text{hom}(\text{K}F(k), G(k))^h\Sigma_k. \]

Corollary 8.9. Let $n \geq 0$ be an integer. Let $F$ and $G$ be right $\Omega$-modules, and suppose that $G(i) \simeq 0$ for $i \neq n$ then
\[ h\text{Rmod}(F, G) \simeq \text{hom}(\text{K}F(n), G(n))^h\Sigma_n. \]

Proof. It follows from the assumption that $\text{hom}(\text{K}F(i), G(i))^h\Sigma_i \simeq 0$ for $i \neq n$. It follows that all the fibers in the tower of fibrations
\[ \cdots \to h\text{Rmod}_{\leq i}(F, G) \to h\text{Rmod}_{\leq i-1}(F, G) \to \cdots \]
are trivial, except for the $n$-th fiber. It follows that the inverse homotopy limit of the tower, which is equivalent to $h\text{Rmod}(F, G)$, is equivalent to the $n$-th fiber. But the $n$-th fiber is equivalent to
\[ \text{hom}(\text{K}F(n), G(n))^h\Sigma_n. \]

Now we go back to assuming that $F$ is a right $\Omega$-module and $G$ is a left module.

Definition 8.10. Corollary 8.6 gives rise to a first quadrant spectral sequence converging to the homology groups of $F \otimes G$. The $k$-th column of the spectral sequence is given by the homology groups of $K F(k) \otimes_{h\Sigma_k} G(k)$, shifted up by $k$. We will call it the Koszul spectral sequence.

The first differential in the Koszul spectral sequence is induced by maps, for each $k \geq 1$
\[ (8.2) \quad K F(k) \otimes_{h\Sigma_k} G(k) \to \Sigma K F(k - 1) \otimes_{h\Sigma_{k-1}} G(k - 1) \]
which are the connecting maps associated with the filtration (8.1). Our next task is to describe these maps explicitly. For this, we need to describe certain structure maps present in Koszul duals of right $\Omega$-modules. Consider the homotopy cofibration sequence
\[ F(k) \to K F(k) \to \Sigma K F(k) \]
Note that the maps in this sequence are $\Sigma_k$-equivariant. Recall that, by definition
\[ \Delta^k F = \text{hocolim}_{k \to i \in k \downarrow \Omega_{<k}} F(i). \]
Let $(k \downarrow \Omega_{<k-1}) \subset (k \downarrow \Omega_{<k})$ be the full subcategory consisting of arrows $k \to i$ where $i < k - 1$. There is a natural inclusion
\[ \text{hocolim}_{k \to i \in k \downarrow \Omega_{<k-1}} F(i) \hookrightarrow \text{hocolim}_{k \to i \in k \downarrow \Omega_{<k}} F(i). \]
It is not difficult to show that the quotient of this inclusion is naturally equivalent to $\text{Sur}(k, k-1) \wedge \Sigma_{k-1} K F(k - 1)$. Here $\text{Sur}(m, n)$ denotes the set of surjective functions from...
the standard set with \( m \) elements to the standard set with \( n \) elements. Thus we obtain a natural map

\[
\Delta^k F \rightarrow \text{Sur}(k, k-1)_+ \wedge_{\Sigma_{k-1}} K F(k-1).
\]

From here, we obtain the following composite map

\[
(\text{8.3}) \quad K F(k) \rightarrow \Sigma \Delta^k F \rightarrow \text{Sur}(k, k-1)_+ \wedge_{\Sigma_{k-1}} \Sigma K F(k-1).
\]

**Remark 8.11.** The existence of the map (8.3) is closely related to the fact that \( K F \) is the Koszul dual of a right \( \Omega \)-module \( F \), and therefore is a right comodule over a suitable version of the Lie cooperad. See the papers of Fresse [13] and Ching [10] for more detail.

The following lemma is an exercise in manipulating colimits.

**Lemma 8.12.** The connecting map (8.2) is equivalent to the following composition of maps

\[
K F(k) \otimes_{h\Sigma_k} G(k) \rightarrow (\text{Sur}(k, k-1)_+ \wedge_{\Sigma_{k-1}} \Sigma K F(k-1)) \otimes_{h\Sigma_k} G(k) \overset{\sim}{\rightarrow} \\
\overset{\sim}{\rightarrow} \Sigma K F(k-1) \otimes_{h\Sigma_{k-1}} (G(k) \wedge_{\Sigma_k} \text{Sur}(k, k-1)_+) \rightarrow \Sigma K F(k-1) \otimes_{h\Sigma_{k-1}} G(k-1)
\]

Here the first map is induced by the composed map (8.3), the second map is just regrouping, and the last map is induced by the \( \Sigma_{k-1} \)-equivariant map \( G(k) \otimes_{h\Sigma_k} \text{Sur}(k, k-1)_+ \rightarrow G(k-1) \) which arises from the left \( \Omega \)-module structure on \( G \).

Since \( F \) and \( G \) take values in the category of chain complexes, where suspension is invertible, the connecting map can also be written as having the form

\[
\Sigma^{-1} K F(k) \otimes_{h\Sigma_k} G(k) \rightarrow K F(k-1) \otimes_{h\Sigma_{k-1}} G(k-1).
\]

We obtain the following proposition.

**Proposition 8.13.** Filtration by cardinality gives rise to a spectral sequence (the Koszul spectral sequence), calculating \( H(F^{\Sigma} G) \). The first page of the Koszul spectral sequence has the following form.

\[
(\text{8.4}) \quad H(K F(0) \otimes G(0)) \leftarrow H(\Sigma^{-1} K F(1) \otimes G(1)) \leftarrow \leftarrow H(\Sigma^{-2} K F(2) \otimes_{h\Sigma_2} G(2)) \leftarrow \cdots \leftarrow H(\Sigma^{-k} K F(k) \otimes_{h\Sigma_k} G(k)) \leftarrow \cdots
\]

Here each term describes a column in the spectral sequence page, the homomorphisms constitute the first differential, and they are induced by the map described in Lemma 8.12.

We will sometimes like to think of the first page of the Koszul spectral sequence, together with the first differential, as a chain complex of graded Abelian groups.

**9. The Koszul dual of \( Q_m^n \)**

Our goal for the rest of the paper is to apply the theory of the previous section to the calculation of the homology of the mapping complexes \( h\text{Rmod} \left( Q_m^n(\bullet), \hat{H}^n_t(\bullet) \right) \) that appear in Corollary 7.3. Thus we need to describe the module \( \hat{H}^n_t(\bullet) \), as well as the Koszul dual of the module \( Q_m^n(\bullet) \), as explicitly as possible.
In this section we describe the Koszul dual of $Q^m_s$, which we continue denoting $KQ^m_s$.

Recall that the right $\Omega$-module $Q^m_s$ is defined by the formula

$$Q^m_s(\bullet) = \tilde{H}_{ms}(S^m_\bullet).$$

Recall that $\Delta^k S^m$ is the fat diagonal in $S^{mk}$. Let $S^{mk}/\Delta^k S^m$ be the quotient space. It is well known that the reduced homology of this space is concentrated in degrees $k+s(m-1)$, where $s = 0, 1, \ldots, k$.

**Proposition 9.1.** For each $k$, $KQ^m_s(k)$ is quasi-isomorphic to a chain complex concentrated in a single homological degree. Moreover, if $m > 1$ then there is an equivalence of chain complexes

$$KQ^m_s(k) \cong \tilde{H}_{k+s(m-1)}(S^{mk}/\Delta^k S^m).$$

To prove the proposition, we will analyze the Koszul dual of a couple of right $\Omega$-modules of a more general type.

**Definition 9.2.** Let $X$ be a pointed topological space. Let $\Sigma^\infty X^\wedge \bullet$ be the right $\Omega$-module defined by the formula $i \mapsto \Sigma^\infty X^\wedge i$. The convention is that $X^\wedge 0 = S^0$. Given a surjective map $i \twoheadrightarrow j$, the corresponding map $\Sigma^\infty X^\wedge j \to \Sigma^\infty X^\wedge i$ is defined using the diagonal inclusion.

Let $\Delta^k X$ be the fat diagonal in $X^\wedge k$. It follows easily from the definition that the Koszul dual of the module $\Sigma^\infty X^\wedge \bullet$ is given by the formula

$$K \Sigma^\infty X^\wedge \bullet \simeq \Sigma^\infty X^\wedge \bullet / \Delta \bullet X.$$

**Definition 9.3.** Fix an integer $s \geq 0$, and let $I_s^X$ be the evident right $\Omega$-module defined by the formula

$$I_s^X(k) = \begin{cases} 
\Sigma^\infty X^\wedge k & k = s \\
* & k \neq s 
\end{cases}$$

Observe that $I_s^X$ is related to $\Sigma^\infty X^\wedge \bullet$ via Goodwillie differentiation. Let $F$ be a functor from the category of pointed spaces to the category of spectra (or spaces). Let $D_s F(X)$ be the $s$-th layer in the Goodwillie tower of $F$. In particular, for fixed integers $k, s$

$$D_s(\Sigma^\infty X^\wedge k) = \begin{cases} 
\Sigma^\infty X^\wedge k & k = s \\
* & k \neq s 
\end{cases}$$

In other words, there is an equivalence of right $\Omega$-modules

$$D_s(\Sigma^\infty X^\wedge \bullet) \simeq I_s^X(\bullet).$$

We can use this to give a description of $KI_s^X$.

**Lemma 9.4.** For each $k$, there is an equivalence

$$KI_s^X(k) \simeq D_s\left(\Sigma^\infty X^\wedge k / \Delta^k X\right).$$
Proof. Since the Koszul dual of a right Ω-module is defined using only homotopy colimits, and taking a layer in the Goodwillie tower is an operation that commutes with homotopy colimits of spectrum-valued functors, it follows that the Koszul dual of \( I^X_s(\bullet) \) can be obtained by taking the \( s \)-th layer of the Koszul dual of \( \Sigma^\infty X^{\wedge\bullet}/\Delta^s X \). In other words, we have natural equivalences
\[
K I^X_s(\bullet) \simeq K D_s(\Sigma^\infty X^{\wedge \bullet}) \simeq D_s(K \Sigma^\infty X^{\wedge \bullet}) \simeq D_s(\Sigma^\infty X^{\wedge \bullet}/\Delta^s X).
\]

□

The layers of the Goodwillie tower of the functor \( \Sigma^\infty X^{\wedge k}/\Delta^k X \) are well understood. They can be described in terms of the space of partitions.

Definition 9.5. Let \( P_k \) be the poset of partitions of the set \{1, \ldots, k\} ordered by refinements. Note that it has an initial and a final object. Let \( |P_k| \) be the geometric realization of this poset (a contractible complex). Let \( \partial|P_k| \) be the subcomplex of \( P_k \) that is the union of simplices that do not contain both the initial and the final object as a vertex (\( \partial|P_k| \) can be interpreted as a kind of boundary of \( |P_k| \)). Let \( T_k \) be the quotient space
\[
T_k := |P_k|/\partial|P_k|.
\]

The space \( T_k \) often appears in the context of the calculus of functors. It is well-known that \( T_k \) is homotopy equivalent to a wedge sum of \((k-1)!\) copies of \( S^{k-1} \).

For any finite set \( S \), we define \( T_S \) to be the space constructed out of the poset of partitions of \( S \) in the same way as \( T_k \) is constructed out of \{1, \ldots, k\}. Clearly, \( T_S \) is functorial with respect to isomorphisms of \( S \).

Definition 9.6. Suppose \( \alpha \in \text{Sur}(k, s) \) is a surjective function. Define
\[
T_\alpha = T_{\alpha^{-1}(1)} \wedge \ldots \wedge T_{\alpha^{-1}(s)}.
\]

Lemma 9.7. There is an equivalence
\[
K I^X_s(k) = \left( \bigvee_{\alpha \in \text{Sur}(k, s)} \Sigma^\infty T_\alpha \wedge X^{\wedge s} \right)_{\Sigma_s}.
\]

Proof. This follows from Lemma 9.4 and the calculations in [3, Section 2] (in particular, remark 2.3 in [op. cit.]).

Proof of Proposition 9.1. We noted already that the chain complex \( \tilde{H}(S^{ms}) = Q^m(s) \) is \( \Sigma_s \)-equivariantly quasi-isomorphic to the complex \( \mathcal{C}(S^{ms}) \). Since \( \bullet = s \) is the only value of \( \bullet \) for which \( Q^m(s) \) is not zero, it follows by an easy argument that the right \( \Omega \)-module \( Q^m(s) \) is weakly equivalent to the right \( \Omega \)-module that has value \( \mathcal{C}(S^{ms}) \) at \( \bullet = s \), and is zero otherwise. Using chains on spectra, and the natural equivalence \( \mathcal{C}(X) \simeq \mathcal{C}(\Sigma^\infty X) \), we can identify \( Q^m(s) \) with the module that has value \( \mathcal{C}(\Sigma^\infty S^{ms}) \) at \( \bullet = s \), and is zero otherwise. Since the singular chains functor preserves homotopy colimits, we just need to calculate the Koszul dual of the spectrum-valued \( \Omega \)-module that has value \( \Sigma^\infty S^{ms} \) at \( \bullet = s \) and is \( \ast \) otherwise. This is exactly the module \( I^S_s \), a special case of the module \( I^X_s \) of Definition 9.3.
We saw that its Koszul dual $KI_s^{Sm}$ is given by the following formula (it is a special case of Lemma 9.7)

$$KI_s^{Sm}(k) \simeq \left( \bigvee_{\alpha \in \text{Sur}(k,s)} \Sigma^\infty T_\alpha \wedge S^{ms} \right)_{\Sigma_s}.$$  

Note that the action of $\Sigma_s$ on the set $\text{Sur}(k, s)$ is free. It follows that the module $KQ_s^m(\bullet)$, which is equivalent to $\tilde{C}(KI_{Sm, s}(\bullet))$ is given by the following formula

$$KQ_s^m(k) \simeq \tilde{C} \left( \bigvee_{\alpha \in \text{Sur}(k,s)} (T_\alpha \wedge S^{ms}) \right)_{\Sigma_s} \hspace{1cm}(9.1)$$

Notice that for all $\alpha \in \text{Sur}(k,s)$, $T_\alpha$ is homotopy equivalent to a wedge of spheres of dimension $k - s$. It follows that $KI_s^{Sm}(k)$ is equivalent to a wedge of spheres of dimension $k + (m - 1)s$. Therefore $KQ_s^m(k)$ is homologically concentrated in dimension $k + (m - 1)s$, and there is an equivalence

$$KQ_s^m(k) \simeq \tilde{H} \left( \bigvee_{\alpha \in \text{Sur}(k,s)} (T_\alpha \wedge S^{ms}) \right)_{\Sigma_s} \hspace{1cm}(9.2)$$

Moreover, recall that For each $k$, there is an equivalence (Lemma 9.4)

$$KI_s^X(k) \simeq D_s \left( \Sigma^\infty X^\wedge k / \Delta^k X \right).$$

We saw that if $X = S^m$, $D_s \left( \Sigma^\infty X^\wedge k / \Delta^k X \right)$ is homologically concentrated in dimension $k + (m - 1)s$. Assuming $m > 1$, we see that this dimension is a strictly increasing function of $s$. Since the spectra $D_s \left( \Sigma^\infty X^\wedge k / \Delta^k X \right)$ are the layers in the Taylor tower of the functor $\Sigma^\infty X^\wedge k / \Delta^k X$, it follows easily that when $X = S^m$ and $m > 1$, $D_s \left( \Sigma^\infty X^\wedge k / \Delta^k X \right)$ detects the homology of $\Sigma^\infty S^{mk} / \Delta^k S^m$ in dimension $k + (m - 1)s$. \hfill \Box

Let us introduce notation that allows us to describe the complex $KQ_s^m(k)$ even more explicitly.

**Definition 9.8.** Let $L_k := \tilde{H}(T_k)$. Thus, $L_k$ is a complex concentrated in degree $k - 1$ and it is isomorphic in this degree to $\mathbb{Z}^{(k-1)!}$. For $\alpha \in \text{Sur}(k, s)$, define

$$L_\alpha = \tilde{H}(T_\alpha) = L_{\alpha - 1(1)} \otimes \cdots \otimes L_{\alpha - 1(s)}.$$  

Also recall that we define $\mathbb{Z}[n] = \tilde{H}(S^n)$.

For any $\alpha \in \text{Sur}(k, s)$ there is a quasi-isomorphism

$$\tilde{C}(T_\alpha \wedge S^{ms}) \simeq L_\alpha \otimes \mathbb{Z}[ms].$$
Using this, together with Equation (9.1), we obtain the following formula

\[ K \mathbb{Q}^n_s(k) \simeq \bigoplus_{\alpha \in \text{Sur}(k,s)} L_\alpha \otimes \mathbb{Z}[m_s] \] \[ \Sigma_s. \]

10. Homology of configuration spaces, revisited

In this section we describe more explicitly the right \( \Omega \)-module \( \hat{\mathbb{H}}^n_t(\bullet) \). The results are essentially well-known and are included for convenience. We remind the reader that \( \hat{\mathbb{H}}^n_t(k) \) is \( \hat{\mathbb{H}}^{n-1}_t(B^n_k; \mathbb{Q}) \), where the decoration \( \hat{\mathbb{H}} \) denotes Pirashvili's cross-effect of a right \( \Gamma \)-module. Thus \( \hat{\mathbb{H}}^n_t(k) \) is a summand of the rational homology of the configuration space of ordered \( k \)-tuples of points in \( \mathbb{R}^n \). More precisely, \( \hat{\mathbb{H}}^n_t(k) \) is the summand of the homology that is not detected in configuration spaces of fewer than \( k \) points. Let \( \hat{\text{Sur}}(k, k-t) \) be the set of functions from \( k \) to \( k-t \) for which the inverse image of every point of \( k-t \) has at least two elements. Elements of \( \hat{\text{Sur}}(k, k-t) \) correspond to partitions of \( k \) with \( k-t \) components which do not have singletons for components. The following formula follows easily from the results of [12, 6, 4]

\[ \hat{\mathbb{H}}^n_t(k) \cong \bigoplus_{\beta \in \hat{\text{Sur}}(k, k-t)} \text{hom} \bigg( L_\beta \otimes \mathbb{Q}[nt], \mathbb{Q} \bigg) \] \[ \Sigma_{k-t}. \]

where \( \mathbb{Q}[nt] \) denotes, as usual, the chain complex \( \tilde{\mathbb{H}}(S^{nt}; \mathbb{Q}) \). It is worth noting that \( S^{nt} \) (and therefore also \( \mathbb{Q}[nt] \)) depends, in a sense, on the surjection \( \beta \). Namely, \( \beta \) induces an injective homomorphism \( \mathbb{R}^{n(k-t)} \hookrightarrow \mathbb{R}^{nk} \). The quotient of this homomorphism is non-canonically isomorphic to \( \mathbb{R}^{nt} \), and \( S^{nt} \) is the one-point compactification of this \( \mathbb{R}^{nt} \). In particular, this identification plays a role in the action of \( \Sigma_{k-t} \) on the direct sum.

Let us use the notation \( \mathbb{Q}[-nt] = \text{hom}(\mathbb{Q}[nt], \mathbb{Q}) \). This is, again, a chain complex concentrated in dimension \(-nt\). Note that we are dealing with actions of finite groups on rational chain complexes, so in our context derived quotients, strict quotients, derived fixed points and strict fixed points are all naturally equivalent to each other. Elementary manipulations show that the above formula for \( \hat{\mathbb{H}}^n_t(k) \) is equivalent to the following one

\[ \hat{\mathbb{H}}^n_t(k) \cong \text{hom} \bigg( \bigoplus_{\beta \in \hat{\text{Sur}}(k, k-t)} L_\beta \otimes \mathbb{Q}[-nt], \mathbb{Q} \bigg) \] \[ \Sigma_{k-t}. \]

The right \( \Omega \)-module structure on \( \hat{\mathbb{H}}^n_t(k) \) is, in these terms, dual to a left \( \Omega \)-module structure on the symmetric sequence

\[ k \mapsto \bigoplus_{\beta \in \hat{\text{Sur}}(k, k-t)} L_\beta \otimes \mathbb{Q}[-nt] \] \[ \Sigma_{k-t}. \]
The left module structure was essentially described in [6, 4]. We will give one description of the module structure in Section 12.

11. The Koszul complex

Now we are ready to describe the Koszul complex for $\Omega Q_m \langle \cdot \rangle \hat{\Omega}_t <\cdot >$ (As a reminder the Koszul complex is actually dual to $\Omega Q_m \langle \cdot \rangle \hat{\Omega}_t <\cdot >$, see Proposition 8.2.) Since right $\Omega$-module $\hat{\Omega}_t <\cdot >$ is the dual of a left $\Omega$-module, we can use the results of Section 8, and in particular (8.4). By (8.4), (9.3) and (10.2), the Koszul complex has the following form

$$\cdots \leftarrow \sum^{-k} \left( \bigoplus_{\alpha \in \text{Sur}(k,s)} L_\alpha \otimes \mathbb{Z}[ms] \right) \otimes \sum^k \left( \bigoplus_{\beta \in \hat{\text{Sur}}(k,k-t)} L_\beta \otimes \mathbb{Q}[-nt] \right) \leftarrow \cdots$$

To shorten notation, let us introduce some abbreviations. For $\alpha \in \text{Sur}(k,s)$, $\beta \in \hat{\text{Sur}}(k,k-t)$, let $L_{\alpha,\beta} = L_\alpha \otimes L_\beta$. Also, let us abbreviate $\sum^{-k} L_{\alpha,\beta} \otimes \mathbb{Z}[ms] \otimes \mathbb{Q}[nt-k]$ as $L_{\alpha,\beta}[ms-nt-k]$. Then the Koszul complex can be written in the following form

$$\cdots \leftarrow \sum_{k=\alpha,\beta} \left( \bigoplus_{(\alpha,\beta) \in \text{Sur}(k,s) \times \hat{\text{Sur}}(k,k-t)} L_{\alpha,\beta}[ms-nt-k] \right) \leftarrow \cdots$$

Remark 11.1. Recall that for $\alpha \in \text{Sur}(k,s)$, $L_\alpha$ is a chain complex concentrated in degree $k-s$. Similarly $L_\beta$ is concentrated in degree $k-(k-t)$, i.e., $t$. It follows that the $k$-th term in the complex (11.1) is a chain complex concentrated in degree $(k-s)+t+(ms-nt-k)$, which is $(m-1)s-(n-1)t$. In particular, the degree is independent of $k$. It follows that (11.1), which a-priori is a chain complex of graded vector spaces, is in fact an ordinary chain complex of rational vector spaces, where all the spaces have an internal degree $(m-1)s-(n-1)t$.

Definition 11.2. We let $H^m_{s,t}$ denote the chain complex (11.1). We call $H^m_{s,t}$ the Koszul complex of $\text{hRmod} \left( Q^m_s \langle \cdot \rangle, \hat{\Omega}_t \langle \cdot > \right)$.

We saw that $H^m_{s,t}$ is a chain complex of rational vector spaces, which have an internal grading $(m-1)s-(n-1)t$. It follows that the dual complex $\text{hom}(H^m_{s,t}, \mathbb{Q})$ is a chain complex of vector spaces with internal grading $(n-1)t-(m-1)s$. We define the cohomology groups $H^m_{s,t}$ to be the homology groups of the dual complex. More precisely we have isomorphisms (the last of which is not natural)

$$H^k(H^m_{s,t}) := H_{-k}(\text{hom}(H^m_{s,t}, \mathbb{Q})) \cong \text{hom}(H_{-k}(H^m_{s,t}), \mathbb{Q}) \cong H_k(H^m_{s,t}),$$

where in calculation of homology we forget about the internal grading. Combining Proposition 8.2, Proposition 8.13 and Remark 11.1 we obtain that the cohomology groups of $H^m_{s,t}$
are the homology groups of $h\text{Rmod}(Q^m_s, \hat{H}^n_t)$. More explicitly, we have an isomorphism for each $k \geq 0$:

$$\tag{11.2} H^k(\hat{H}^{m,n}_{s,t}) \cong H_{(n-1)t-(m-1)s-k}(h\text{Rmod}(Q^m_s, \hat{H}^n_t)).$$

**Remark 11.3.** Suppose that $s, t > 0$. Then $\text{Sur}(k, s) \neq \emptyset$ only if $k \geq s$. Similarly $\text{Sur}(k, k-t) \neq \emptyset$ only if $k-t > 0$ and $k \geq 2(k-t)$. It follows that the complex $H^{m,n}_{s,t}$ is non-zero only in degrees $\max\{s, t+1\} \leq k \leq 2t$. It follows that $H_j(h\text{Rmod}(Q^m_s, \hat{H}^n_t))$ can be non-zero only when

$$(n-3)t-(m-1)s \leq j \leq (n-2)t-(m-1)s-1.$$  

Moreover, $h\text{Rmod}(Q^m_s, \hat{H}^n_t)$ is non-trivial only if $s \leq 2t$. It follows easily that for fixed $m, n$ satisfying $m \geq 1$ and $n > 2m+1$, and for each $j \geq 0$, $H_j(h\text{Rmod}(Q^m_s, \hat{H}^n_t))$ is non-zero only for finitely many values of $s, t$. It follows that, assuming that $n > 2m+1$, the direct product on the right hand side of the formulas in Corollary 7.3 are equivalent to direct sums. We obtain the following corollary

**Corollary 11.4.** Suppose that $n > 2m+1$. Then there are isomorphisms

$$\tag{11.3} \tilde{H}(\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n); Q) \cong \bigoplus_{1 \leq s \leq 2t} H(H_{(n-1)t-(m-1)s-t}(H^{m,n}_{s,t})).$$

**Examples 11.5.** When $s = t = 0$, $H^{m,n}_{s,t}$ is the trivial complex which has $Q$ in degree zero and nothing else. It detects the unreduced zero-dimensional homology of $\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)$.  

In all other cases $H^{m,n}_{s,t}$ is non-trivial only if $s, t > 0$. Furthermore it follows from Remark 11.3 that $H^{m,n}_{s,t}$ is non-zero only if $s \leq 2t$.

Let us consider the case $s = 1$. Clearly, in this case $s \leq t+1$ and, again by remark 11.3 the Koszul complex $H^{m,n}_{1,t}$ is non-trivial only in degrees $t+1 \leq k \leq 2t$. The non-zero portion of $H^{m,n}_{1,t}$ has $t$ terms, and it has the following form

$$0 \leftarrow L_{t+1} \otimes L_{t+1}[m-(n+1)t-1]_{\Sigma_{t+1}} \leftarrow \cdots \leftarrow L_{2t} \otimes L_{2t}[m-(n+2)t]_{\Sigma_{t} \Sigma_{2}} \leftarrow 0.$$  

For a general $k$ the $k$-th term of $H^{m,n}_{1,t}$ has the form

$$\left( \bigoplus_{[\beta] \in \overline{\text{Sur}(k, t)} / \Sigma_{k-t}} L_k \otimes L_{\beta}[m-nt-k] \right)_{\Sigma_k}.$$  

This is so because $\text{Sur}(k, 1)$ consists of just one point, and there is just one partition of $k$ with 1 component. Thus the sum is indexed by irreducible partitions of $\{1, \ldots, k\}$ of excess $t$. When $k = t+1$ there is one such partition, namely the partition with one component. At the other extreme, when $k = 2t$, there is again just one type of irreducible partition of excess $t$, namely the partition of type $2-2-\ldots-2$. We remark that partitions of this type are closely related to “chord diagrams”, familiar from knot theory.
In particular, when \( t = 1 \) the complex \( \mathcal{H}^m_{1,1} \) has only one (possibly) non-zero term, corresponding to \( k = 2 \). The non-zero term is

\[
(L_2 \otimes L_2 [m - n - 2])_{\Sigma_2}.
\]

\( L_2 \otimes L_2 [m - n - 2] \) is really just \( \mathbb{Q} \) concentrated in dimension \( m - n \). It is easy to see that the action of \( \Sigma_2 \) on \( L_2 \otimes L_2 [m - n - 2] \) is trivial if \( n \) is even and is multiplication by \(-1\) if \( n \) is odd. It follows that if \( n \) is odd then \( \mathcal{H}^m_{1,1} \) is the zero complex. If \( n \) is even then \( \mathcal{H}^m_{1,1} \) consists of a single copy of \( \mathbb{Q} \) in degree \( k = 2 \) and of internal dimension \( m - n \). It follows that when \( n \) is even \( \mathcal{H}^m_{1,1} \) contributes a class of dimension \( n - m - 2 \) to \( \mathcal{H} \left( \mathbb{E} \mathbf{m} \mathbb{C} \left( \mathbb{R}^m, \mathbb{R}^n \right); \mathbb{Q} \right) \).

Let us also consider briefly the case \( s = 1, \ t = 2 \). In this case, the complex \( \mathcal{H}^m_{1,2} \) has two non-zero terms, corresponding to \( k = 3, 4 \). It has the following form

\[
0 \leftarrow (L_3 \otimes L_3 [m - 2n - 3])_{\Sigma_3} \leftarrow (L_4 \otimes L_2 \otimes L_2 [m - 2n - 4])_{\Sigma_2 \Sigma_2} \leftarrow 0.
\]

The complexity of \( \mathcal{H}^m_{s,t} \) grows rapidly with \( s \) and \( t \).

12. A COMPLEX OF FORESTS

In this section we will show how the Koszul complex \( \mathcal{H}^m_{s,t} \) (see [11.1]) can be described more explicitly as a complex of forests. To recall the direct sum over \( s \) and \( t \) of all those complexes computes the rational cohomology of \( \mathbb{E} \mathbf{m} \mathbb{C} \left( \mathbb{R}^m, \mathbb{R}^n \right) \), see Corollary [11.4]. The internal grading of each summand \( \mathcal{H}^m_{s,t} \) is \((m - 1)s - (n - 1)t\); the total grading, which is the sum of internal grading plus grading \( k \), is minus the homological grading of the space of embeddings.

The basic ingredient in the construction of \( \mathcal{H}^m_{s,t} \) is the graded module (concentrated in degree \( k - 1 \)) \( L_k = \overline{H}(T_k) \). It is well known that \( L_k \) can be described as the free \( \mathbb{Z} \)-module spanned by trees with vertex set \( \{1, \ldots, k\} \), modulo the Arnold relation. More precisely, let \( \mathcal{A}_k \) be the free graded commutative algebra on \( \binom{k}{2} \) one-dimensional generators. Thus \( \mathcal{A}_k \) has a one-dimensional generator \( u_{i,j} \) for each unordered pair \( \{i, j\} \) of distinct indices between 1 and \( k \) (so \( u_{i,j} = u_{j,i} \)). Note that the generators anti-commute and their squares are zero. Let \( I \) be the ideal of \( \mathcal{A}_k \) generated by the Arnold relation

\[
u_{i,j} u_{j,k} + u_{j,k} u_{k,i} + u_{k,i} u_{i,j} = 0
\]

for all \( i,j,k \). \( I \) is generated by homogeneous elements so \( \mathcal{A}_k/I \) is a graded algebra. It is well known that there is an isomorphism of graded algebras [2][11]

\[
\mathcal{A}_k/I \cong H^*(C(k, \mathbb{R}^2)).
\]

This has the following consequence, which is also well-known.

**Proposition 12.1.** \( L_k \) is naturally isomorphic to the homogeneous degree \( k - 1 \) part of \( \mathcal{A}_k/I \). Moreover, the degree \( k - 1 \) part is generated by monomials of the form \( u_{i_1,j_1} \cdots u_{i_{k-1},j_{k-1}} \) for which the graph with vertex set \( \{1, \ldots, k\} \) and edge set \( \{i_1,j_1\}, \ldots, \{i_{k-1},j_{k-1}\} \) is connected and acyclic (i.e., a tree).
Indeed, one way to obtain the first part of the proposition is to use the natural isomorphisms
\[ L_k \cong \text{H}_{k-1}(T_k) \cong \text{H}_{k+1}(S^{2k}/\Delta^k S^2) \cong \text{H}^{k-1}(C(k, \mathbb{R}^2)) \]
and the identification of \( \mathcal{A}_k/I \) with the cohomology of the configuration space \( C(k, \mathbb{R}^2) \).

The second part of the proposition is proved, for example, in [12].

Next we describe the Lie-comodule structure on the sequence \( L_1, \ldots, L_k, \ldots \) in terms of this “tree basis”. Recall that for every surjective function \( \beta \in \text{Sur}(k, k - 1) \) there is a map
\[ T_k \rightarrow \Sigma T_{k-1} \]
associated with \( \beta \). This map induces a homomorphism in homology
\[ L_k \rightarrow \mathbb{Z}(v) \otimes L_{k-1}. \]

Here \( v \) is a one-dimensional generator and \( \mathbb{Z}(v) \) is the free Abelian group with generator \( v \). To describe this homomorphism, we need to tell what it does on a generic element of the form \( u_{i_1,j_1} \cdots u_{i_{k-1},j_{k-1}} \). Since \( \beta \) is a surjective function from the standard set with \( k \) elements to the one with \( k - 1 \) elements, there exist two elements \( i, j \) such that \( \beta(i) = \beta(j) \) and otherwise \( \beta \) is a bijection. Roughly speaking the effect of \( \beta \) on the tree basis for homology is determined by the following rule: if \( \{i, j\} \) is not among the edges of a tree, then this tree is sent to zero. If \( \{i, j\} \) is among the edges, then the tree is sent to another tree, obtained by contracting the edge \( \{i, j\} \), and the edge \( \{i, j\} \) is mapped to the suspension coordinate. More precisely, the rule is as follows: if \( \beta(i) \neq \beta(j) \) then \( u_{i,j} \) is sent to \( u_{\beta(i),\beta(j)} \). If \( \beta(i) = \beta(j) \) then \( u_{i,j} \) is sent to \( v \). This induces the following function on the generators of \( L_k \): if the unordered pair \( \{i, j\} \) is not among the pairs \( \{i_1, j_1\}, \ldots, \{i_{k-1}, j_{k-1}\} \), then the element \( u_{i_1,j_1} \cdots u_{i_{k-1},j_{k-1}} \) is sent to zero. Otherwise, suppose that \( \{i, j\} = \{i_1, j_1\} \). Then the element \( u_{i_1,j_1} \cdots u_{i_{k-1},j_{k-1}} \) is sent to
\[ u_{\beta(i_1),\beta(j_1)} \cdots u_{\beta(i_{k-1}),\beta(j_{k-1})} v u_{\beta(i_1),\beta(j_1)} \cdots u_{\beta(i_{k-1}),\beta(j_{k-1})}, \]
which is the same as
\[ (-1)^{l-1} v \otimes u_{\beta(i_1),\beta(j_1)} \cdots u_{\beta(i_{k-1}),\beta(j_{k-1})} u_{\beta(i_1),\beta(j_1)} \cdots u_{\beta(i_{k-1}),\beta(j_{k-1})}. \]

One can desuspend the above homomorphism, to obtain a homomorphism
\[ \mathbb{Z}(v^{-1}) \otimes L_k \rightarrow L_{k-1} \]
where \( v^{-1} \) is a generator of degree \(-1\). This homomorphism sends \( v^{-1} \otimes u_{i_1,j_1} \cdots u_{i_{k-1},j_{k-1}} \) to \((-1)^{l-1} u_{\beta(i_1),\beta(j_1)} \cdots u_{\beta(i_{k-1}),\beta(j_{k-1})} \) if \( \{i, j\} = \{i_1, j_1\} \), and to zero if \( \{i, j\} \) is not equal to any of the unordered pairs \( \{i_1, j_1\}, \ldots, \{i_{k-1}, j_{k-1}\} \).

Now let \( \alpha \in \text{Sur}(k, s) \) be a surjective function. Recall that we define \( L_\alpha = L_{\alpha-1(1)} \otimes \cdots \otimes L_{\alpha-1(s)} \). It follows that the module \( L_\alpha \) is generated by forests with vertex set \( \{1, \ldots, k\} \), whose connected components are labeled by \( 1, \ldots, s \), subject to the Arnold relation. Finally recall that the \( k \)-th term of the complex \( \operatorname{HH}^m \mathbb{C}^{\mathbb{Z}^n} \) is isomorphic to the following graded rational vector space
\[ \Sigma^{-k} \bigg( \bigoplus_{(\alpha,\beta) \in \text{Sur}(k,s) \times \text{Sur}(k,k-t)} \text{L}_\alpha \otimes \text{L}_\beta \otimes \mathbb{Z}[m s] \otimes \mathbb{Q}[-nk] \otimes \mathbb{Q}[n (k-t)] \bigg)_{\Sigma_k \times \Sigma_s \times \Sigma_{k-t}} \]
It follows that the \( k \)-th term of the Koszul complex \( H_{s,t}^{m,n} \) can be described as a quotient (by the Arnold relation) of a direct sum of rational vector spaces, indexed by equivalence classes (with respect to the action of \( \Sigma_k \)) of pairs of forests with vertex set \( \{1, \ldots, k\} \), where the first forest has \( s \) components and the second forest has \( k - t \) components, all bigger than a point. Each summand is generated by equivalence classes (with respect to the action of the stabilizer group of the pair of forests) of monomials that are products of the following generators:

- a one-dimensional generator for each edge of the two forests,
- \( s \) generators of dimension \( m \) corresponding to connected components of the first forest (and arising from \( \mathbb{Z}[ms] \)),
- \( k \) generators of dimension \( -n \) corresponding to vertices and \( k - t \) generators of dimension \( n \) corresponding to connected components of the second forest to account for \( \mathbb{Q}[-nk] \) and \( \mathbb{Q}[n(k - t)] \).
- and finally \( k \) more generators of dimension \( -1 \), to account for the \( k \)-fold desuspension. Note that the group \( \Sigma_k \) does not permute these generators.

The order of generators can be interchanged as prescribed by graded commutativity.

The boundary homomorphism is defined as follows: fix a pair of forests with vertex set \( \{1, \ldots, k\} \) (representing an equivalence class of such pairs), where the two forests have \( s \) components and \( k - t \) non-singleton components respectively. The boundary homomorphism on the summand corresponding to this pair is given by a sum of \( \binom{k}{2} \) homomorphisms, corresponding to all the ways to glue together two elements of \( \{1, \ldots, k\} \). The only summands that are not zero are the ones that contract an edge in the first forest (and so preserve the number \( s \) of components in this forest) and glue together two components of the second forest (and so preserve the number \( t \) of edges of the second forests). The contracted edge is used to cancel one of the de-suspension coordinates, as explained above.

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