HIGHER FROBENIUS-SCHUR INDICATORS FOR PIVOTAL CATEGORIES

SIU-HUNG NG AND PETER SCHAUENBURG

Abstract. We define higher Frobenius-Schur indicators for objects in linear pivotal monoidal categories. We prove that they are category invariants, and take values in the cyclotomic integers. We also define a family of natural endomorphisms of the identity endofunctor on a $k$-linear semisimple rigid monoidal category, which we call the Frobenius-Schur endomorphisms. For a $k$-linear semisimple pivotal monoidal category — where both notions are defined —, the Frobenius-Schur indicators can be computed as traces of the Frobenius-Schur endomorphisms.

Introduction

The classical (degree two) Frobenius-Schur indicator $\nu_2(V)$ of an irreducible representation $V$ of a finite group $G$ has been generalized to Frobenius-Schur indicators of simple modules of semisimple Hopf algebras by Linchenko and Montgomery [12], to certain $C^*$-fusion categories by Fuchs, Ganchev, Szlachányi, and Vescernyés [5], and further to simple objects in pivotal (or sovereign) categories by Fuchs and Schweigert [6]; Mason and Ng [13] treated the case of simple modules over semisimple quasi-Hopf algebras. As in the classical case, the indicator always takes one of the values $0, \pm 1$, and is related to the question if and how the representation in consideration is self-dual. In the last case, proving that $\nu_2(V) \in \{0, \pm 1\}$ uses the result of Etingof, Nikshych, and Ostrik [4] that the module category of a semisimple complex quasi-Hopf algebra is a pivotal monoidal category. A different proof based on this pivotal structure and the description of indicators in [5] was given in [17].

The higher indicators $\nu_n(V)$ of an irreducible group representation $V$, which have less obvious meaning for the structure of $V$, were generalized to simple modules of a semisimple Hopf algebra by Kashina, Sommerhäuser, and Zhu [10].

In the present paper we define and study higher Frobenius-Schur indicators $\nu_n(V)$ for an object $V$ of a $k$-linear pivotal monoidal category $\mathcal{C}$. We do this with a view towards the categories of modules over semisimple complex quasi-Hopf algebras, though we will only give the (quite involved) explicit formulas and examples for that case in another paper [14].

Our definition of $\nu_n(V)$ is as the trace of an endomorphism $E^{(n)}_V$ of the vector space of morphisms $\mathcal{C}(I, V \otimes^n)$, where $I$ is the unit object. The endomorphism arises from a special case of a map $\mathcal{C}(I, V \otimes W) \to \mathcal{C}(I, W \otimes V)$ defined for any two objects in terms of duality and the pivotal structure. In the case where $V$ is an $H$-module for a semisimple Hopf algebra $H$, we can identify $\mathcal{C}(I, V \otimes^n)$ with the invariant subspace $(V \otimes^n)^H$, and $E^{(n)}_V$ is given by a cyclic permutation; the description of $\nu_n(V)$ as a trace in this case is contained in [10]. If $n = 2$ and $V$ is simple, then $\mathcal{C}(I, V \otimes V)$ is one-dimensional or vanishes. Thus $E^{(2)}_V$ is (at most) a scalar, which coincides with its trace; that computing the trace in a different way leads to the indicator formulas from [13] was shown in [17].

An endomorphism of $\mathcal{C}(V^\vee, V)$ conjugate to $E^{(2)}_V$ is also used to describe the degree two indicator in [5] (and to define $E^{(2)}_V$ in [17]). The maps $E^{(n)}_V$ have been studied in connection with 3-manifold invariants by Gelfand and Kazhdan [7].

We prove that the higher indicators are invariant under equivalences of pivotal monoidal categories, and that equivalences of pseudo-unital fusion categories (which are pivotal categories by [4]) are automatically pivotal equivalences. While these invariance properties are to be expected
from the categorical nature of the definitions, the proofs are not obvious. In particular, it would be tautological to assume to be dealing with strict categories (on grounds that every monoidal category is equivalent to a strict one) to prove invariance of certain properties under equivalence. Barrett and Westbury [2] have proved (although starting from a different set of axioms) that every pivotal monoidal category is equivalent to a strict one. By the invariance results we have proved, it is then sufficient to assume such a simplified structure of $C$ when proving general properties of $E_V^{(n)}$ and $\nu_n(V)$. We prove, in section 5, that the order of $E_V^{(n)}$ divides $n$; this is stated in [7] and proved for $n = 2$ (and $C$ strict). In particular, the possible values of the higher Frobenius-Schur indicators are cyclotomic integers; this is well-known for the group case, where the higher indicators are in fact always integers, and proved for the Hopf algebra case in [10]. Finally, we show that the Frobenius-Schur indicator of an object $V$ in a pivotal fusion category can be computed as the trace (in a suitable sense) of a natural endomorphism of $V$, which we call the Frobenius-Schur endomorphism.

The organization of the paper is as follows: we cover in Section 1 some basic definitions, notations, conventions and preliminary results of pivotal monoidal categories for the remaining discussion. In Section 2, we give a proof that every pivotal monoidal category is equivalent, as pivotal monoidal categories, to a strict one. We then define a sequence of scalars $\nu_n(V)$, called the higher Frobenius-Schur indicators, for each object $V$ in a $k$-linear pivotal monoidal category in Section 3. We prove in Section 4 that these higher Frobenius-Schur indicators are invariant under $k$-linear pivotal monoidal equivalences. This result allows one to study these indicators by considering only the strict $k$-linear pivotal monoidal categories and we prove in Section 5 that all the higher Frobenius-Schur indicators for a $k$-linear pivotal monoidal category are cyclotomic integers provided the characteristic of the algebraically closed field $k$ is zero. In Section 6, we consider the higher Frobenius-Schur indicators for a pseudo-unitary fusion category over $C$. In this case, we show that these indicators are invariant under $k$-linearly monoidal equivalence. Finally, in Section 7, we show that the $n$th Frobenius-Schur indicator of an object $V$ in a semisimple $k$-linear pivotal monoidal category is the pivotal trace of a natural endomorphism $\text{FS}_V^{(n)}$, called the Frobenius-Schur endomorphism. In another paper [14] we will study the pivotal fusion categories of modules over a semisimple complex quasi-Hopf algebra. In this case the Frobenius-Schur endomorphisms correspond to central gauge invariants in the quasi-Hopf algebras. The indicators are obtained by applying the representation’s character, which corresponds precisely to the definition of higher indicators in [10] for the case of ordinary Hopf algebras.

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1. Preliminaries

We will first fix some conventions: In a monoidal category $C$, the associativity isomorphism is $\Phi: (U \otimes V) \otimes W \to U \otimes (V \otimes W)$. We will make the assumption that the unit object in a monoidal category is always strict, $V \otimes I = V = I \otimes V$. As pointed out in [16], this assumption can always be made true after replacing the tensor product in $C$ with an isomorphic one. For the purpose of this paper however (where frequently the question of invariance of certain constructions under tensor equivalences is key) this may not be a rigorous justification; we simply make the assumption that $I$ is a strict unit for simplicity, and hold that the general case can be treated by an insignificant but annoying expansion of all proofs. By the well-known coherence theorem for monoidal categories, if $X, Y \in C$ are formed by tensoring the same sequence of objects $V_1, \ldots, V_n \in C$, only with different placement of parentheses, then there is a unique morphism $\Phi^\prime: X \to Y$ composed formally from
instances of $\Phi$ and $\Phi^{-1}$. (By a “formal” composition of instances of $\Phi$ and $\Phi^{-1}$ we mean one that could be written down in a suitably defined free category, excluding compositions that only become possible because “formally different” objects happen to be identical in the concrete category at hand; such accidental composites may of course fail to agree.) As a simple example

$$\Phi' = \Phi^{-1}(T \otimes \Phi) = \Phi(\Phi^{-1} \otimes W)\Phi^{-1}: T \otimes ((U \otimes V) \otimes W) \to (T \otimes U) \otimes (V \otimes W).$$

A monoidal functor $(F, \xi): C \to D$ will have the structure isomorphism $\xi: F(V) \otimes F(W) \to F(V \otimes W)$. We will assume that the structure isomorphism for the unit objects is the identity $F(I) = I$. Given a general monoidal functor $(F, \xi, \xi_0)$ with $\xi_0: I \to F(I)$ not the identity, this can always be achieved by replacing $F$ with an isomorphic functor $F'$ whose object map is given by $F'(X) = F(X)$ if $X \neq I$, and $F'(I) = I$. Let $X$ be obtained from tensoring a sequence $V_1, \ldots, V_n$ of objects of $C$ with some choice of parentheses, and let $X'$ be obtained from $V'_i = F(V_i)$ in the same way. We will use the following special case of coherence of monoidal functors: There is a unique formal composition $\xi': X' \to F(X)$ of instances of $\xi$, and if $Y, Y'$ are obtained from the same sequence of objects with a different placement of parentheses, then

$$\begin{array}{ccc}
X' & \xrightarrow{\xi'} & F(X) \\
\Phi' & \downarrow & \Phi' \\
Y' & \xrightarrow{\xi'} & F(Y)
\end{array}$$

commutes. The coherence result of Epstein [3] treats the case of symmetric monoidal functors between symmetric monoidal categories; the diagram above is contained in the appropriate analog for the non-symmetric case, obtained essentially by just leaving out the symmetry. This is certainly folklore; a proof can be extracted from [3].

A (left) dual object of $V \in C$ is a triple $(V^\vee, ev, db)$ with an object $V^\vee \in C$ and morphisms $ev: V^\vee \otimes V \to I$ and $db: I \to V \otimes V^\vee$ such that the compositions

$$V \xrightarrow{db \otimes V} (V \otimes V^\vee) \otimes V \xrightarrow{\Phi} V \otimes (V^\vee \otimes V) \xrightarrow{V \otimes ev} V,$$

$$V^\vee \xrightarrow{V^\vee \otimes db} V^\vee \otimes (V \otimes V^\vee) \xrightarrow{\Phi^{-1}} (V^\vee \otimes V) \otimes V^\vee \xrightarrow{ev \otimes V^\vee} V^\vee$$

are identities. We say that $C$ is (left) rigid if every object has a dual. We use the notation $V^\vee$ for the symmetric notion of a right dual of an object $V \in C$; this is the same as a left dual in the category $C^{sym}$ in which the tensor product is defined in the reverse order.

For any object $V$ of a monoidal category having a dual object $V^\vee$, we obtain an adjunction

$$A_0: \mathcal{C}(U, V \otimes W) \cong \mathcal{C}(V^\vee \otimes U, W)$$

by

$$A_0(f) = \left( V^\vee \otimes U \xrightarrow{V^\vee \otimes f} V^\vee \otimes (V \otimes W) \xrightarrow{\Phi^{-1}} (V^\vee \otimes V) \otimes W \xrightarrow{ev \otimes W} W, \ x \mapsto \Phi^{-1}(x) \right),$$

$$A_0^{-1}(g) = \left( U \xrightarrow{db \otimes U} (V \otimes V^\vee) \otimes U \xrightarrow{\Phi} V \otimes (V^\vee \otimes U) \xrightarrow{V \otimes g} V \otimes W, \ x \mapsto g(x) \right).$$

In the case $U = I$ we abbreviate

$$(1.1) \quad A_{V,W} = A_0: \mathcal{C}(I, V \otimes W) \to \mathcal{C}(V^\vee, W).$$

We will simply write $A$ for $A_{V,W}$ when the context is clear.

By the uniqueness of adjoints, dual objects in a monoidal category are unique: If $(V^\vee, ev, db)$ and $(V', ev', db')$ are dual objects of $V$, then an isomorphism $v: V^\circ \to V^\vee$ uniquely determined by either of the conditions $ev(V \otimes v) = ev'$ or $(V \otimes v) db' = db$. An instance of this is the compatibility
of duals and tensor products: Given duals of $V$ and $W$ in $\mathcal{C}$, we get a dual $(W^\vee \otimes V^\vee, \tilde{e}v, \tilde{d}b)$ of $V \otimes W$, where

$$
ev = (W^\vee \otimes V^\vee) \otimes (V \otimes W) \xrightarrow{\Phi} W^\vee \otimes ((V^\vee \otimes V) \otimes W) \xrightarrow{W^\vee \otimes \ev \otimes W} W^\vee \otimes W \xrightarrow{\ev} I,$$

$$\tilde{d}b = (I \otimes V \otimes V^\vee X^\vee \otimes X^\vee \otimes W \xrightarrow{X^\vee \otimes \Phi} (V \otimes (W \otimes (W^\vee \otimes V^\vee))).$$

If $(V \otimes W)^\vee$ is another dual of $V \otimes W$, this yields a unique isomorphism $\zeta: W^\vee \otimes V^\vee \rightarrow (V \otimes W)^\vee$ with $\tilde{e}v = ev_{V \otimes W}(\zeta \otimes V \otimes W)$ and $\tilde{d}b_{V \otimes W} = (V \otimes W \otimes \zeta)db$.

If $\mathcal{C}$ is rigid, and we choose a dual object for each object in $\mathcal{C}$, then $(-)^\vee$ is a contravariant functor in a natural way; we will always choose $I^\vee = I$, with both morphisms $ev$ and $db$ identities. In this case the morphisms $\zeta$ are the components of a monoidal functor structure on $(-)^\vee$.

Let $\mathcal{C}, \mathcal{D}$ be two rigid monoidal categories. Fix a left dual $(X^\vee, ev, db)$ for each object $X$ in $\mathcal{C}$ or $\mathcal{D}$, and make $(-)^\vee$ contravariant functors in the natural way. Let $(\mathcal{F}, \xi): \mathcal{C} \rightarrow \mathcal{D}$ be a monoidal functor, with monoidal functor structure $\xi: \mathcal{F}(X) \otimes \mathcal{F}(Y) \rightarrow \mathcal{F}(X \otimes Y)$. Then for each $X \in \mathcal{C}$, we have a dual $\mathcal{F}(X)^\circ = (\mathcal{F}(X^\vee), ev^\prime, db^\prime)$ of $\mathcal{F}(X)$ in $\mathcal{D}$, with

$$ev^\prime = (\mathcal{F}(X^\vee) \otimes \mathcal{F}(X) \xrightarrow{\xi} \mathcal{F}(X^\vee \otimes X) \xrightarrow{\mathcal{F}(ev)} I),$$

$$db^\prime = (I \xrightarrow{\mathcal{F}(db)} \mathcal{F}(X^\vee \otimes X) \xrightarrow{\xi^{-1}} \mathcal{F}(X) \otimes \mathcal{F}(X^\vee)).$$

We denote the canonical isomorphisms by $\tilde{\xi}: \mathcal{F}(X^\vee) \rightarrow \mathcal{F}(X)^\vee$; they are the components of a natural transformation which we call the **duality transformation** of $(\mathcal{F}, \xi)$.

**Lemma 1.1.** Let $(\mathcal{F}, \xi): \mathcal{C} \rightarrow \mathcal{D}$ be a monoidal functor between rigid monoidal categories. Then the duality transform $\xi$ is a monoidal natural transformation.

**Proof.** We want to show that

$$\mathcal{F}(Y^\vee) \otimes \mathcal{F}(X^\vee) \xrightarrow{\xi} \mathcal{F}(Y^\vee \otimes X^\vee) \xrightarrow{\mathcal{F}(\xi)} \mathcal{F}((X \otimes Y)^\vee)$$

commutes. To do this, we compare the two morphisms $\mathcal{F}(Y^\vee) \otimes \mathcal{F}(X^\vee) \rightarrow (\mathcal{F}(X) \otimes \mathcal{F}(Y))^\vee$ arising from the diagram; note that two arrows $f, g: A \rightarrow B^\vee$ are the same iff $ev_B(f \otimes B) = ev_B(g \otimes B)$.

On one hand we have the commutative diagram

$$\begin{align*}
(F(Y^\vee) \otimes F(X^\vee)) \otimes (F(X) \otimes F(Y)) \xrightarrow{\xi \otimes \text{id}} & \mathcal{F}(Y^\vee \otimes X^\vee) \otimes (F(X) \otimes F(Y)) \xrightarrow{\mathcal{F}(\xi) \otimes \text{id}} \mathcal{F}((X \otimes Y)^\vee) \otimes (F(X) \otimes F(Y)) \\
(F(Y^\vee \otimes X^\vee) \otimes (X \otimes Y)) \xrightarrow{F(\xi)} & \mathcal{F}(X \otimes Y)^\vee \otimes (X \otimes Y)) \xrightarrow{ev} (F(X \otimes Y)^\vee \otimes (F(X) \otimes F(Y)).
\end{align*}$$
which, besides definitions and naturality, contains the diagram
\[
\begin{array}{cccc}
\mathcal{F}((X \otimes Y)') \otimes (\mathcal{F}(X) \otimes \mathcal{F}(Y)) & \xrightarrow{\xi \otimes id} & \mathcal{F}(X) \otimes (\mathcal{F}(X) \otimes \mathcal{F}(Y)) & \xrightarrow{\xi' \otimes id} & (\mathcal{F}(X) \otimes \mathcal{F}(Y))' \otimes (\mathcal{F}(X) \otimes \mathcal{F}(Y)) \\
\mathcal{F}((X \otimes Y)' \otimes (X \otimes Y)) & \xrightarrow{\xi} & \mathcal{F}(X) \otimes (X \otimes Y) & \xrightarrow{ev} & I \\
\end{array}
\]

in its lower right part. On the other hand the diagram
\[
\begin{array}{cccc}
(\mathcal{F}(Y') \otimes \mathcal{F}(X')) \otimes (\mathcal{F}(X) \otimes \mathcal{F}(Y)) & \xrightarrow{\xi' \otimes \xi \otimes id} & (\mathcal{F}(Y') \otimes \mathcal{F}(X))' \otimes (\mathcal{F}(X) \otimes \mathcal{F}(Y)) \\
\mathcal{F}((Y' \otimes X') \otimes (X \otimes Y)) & \xrightarrow{\xi'} & I & \xrightarrow{ev} & (\mathcal{F}(X) \otimes \mathcal{F}(Y))' \otimes (\mathcal{F}(X) \otimes \mathcal{F}(Y)),
\end{array}
\]
in which \(\xi'\) is analogous to \(ev\) defined above, but with two instances of \(ev\) instead of \(ev\), commutes: Its upper triangle uses the definition of \(\xi\) twice, and its lower left triangle is an application of the definition of \(ev\) and coherence; the diagram
\[
\begin{array}{cccc}
(\mathcal{F}(Y') \otimes \mathcal{F}(X')) \otimes (\mathcal{F}(X) \otimes \mathcal{F}(Y)) & \xrightarrow{\xi' \otimes \xi \otimes id} & (\mathcal{F}(Y') \otimes \mathcal{F}(X))' \otimes (\mathcal{F}(X) \otimes \mathcal{F}(Y)) \\
\mathcal{F}((Y' \otimes X') \otimes (X \otimes Y)) & \xrightarrow{\xi'} & I & \xrightarrow{ev} & (\mathcal{F}(X) \otimes \mathcal{F}(Y))' \otimes (\mathcal{F}(X) \otimes \mathcal{F}(Y)),
\end{array}
\]
provides more details. Thus, we are done. \(\Box\)

A pivotal monoidal category is a rigid monoidal category with a pivotal structure, that is, an isomorphism \(j: V \to V^\vee\) of monoidal functors. It follows that \(j_{V^\vee} = (j_V)^{-1}\) for all \(V \in \mathcal{C}\), see [17, Appendix].

If \(\mathcal{C}, \mathcal{D}\) are two pivotal monoidal categories, and \((\mathcal{F}, \xi): \mathcal{C} \to \mathcal{D}\) is a monoidal functor, we shall say that \(\mathcal{F}\) preserves the pivotal structure, if the diagrams
\[
\begin{array}{c}
\mathcal{F}(X) \xrightarrow{\mathcal{F}(j)} \mathcal{F}(X^\vee) \\
\mathcal{F}(X)^{\vee\vee} \xrightarrow{\xi^\vee} \mathcal{F}(X)^{\vee}
\end{array}
\]
commute.

Finally let us fix a few conventions on \(k\)-linear categories over a field \(k\): First of all, we will assume that the morphism spaces of a \(k\)-linear monoidal category are finite-dimensional. A simple object in a \(k\)-linear category \(\mathcal{C}\) is an object \(V\) with \(\text{End}_\mathcal{C}(V) = k\) (note that sometimes such objects are called absolutely simple in the literature). A semisimple \(k\)-linear category is an abelian \(k\)-linear category in which every object is a direct sum of simple objects. A semisimple \(k\)-linear monoidal category is a monoidal category, and a semisimple \(k\)-linear category, such that the tensor product is bilinear, and we will make the general assumption that the unit object is simple.
It was noted in [15], see also [4, 2.1], that a semisimple $k$-linear left rigid monoidal category is also right rigid. In fact if $\mathcal{C}$ is semisimple, then $\mathcal{C}(I, V \otimes V^\vee) \neq 0$ implies $0 \neq \mathcal{C}(V \otimes V^\vee, I) \cong \mathcal{C}(V, V^{\vee \vee})$; thus any simple $V$ is isomorphic to its bidual $V^{\vee \vee}$, and $V^\vee$ is a right dual of $V$.

A fusion category [1, 4] is a semisimple $\mathbb{C}$-linear rigid monoidal category with only finitely many isomorphism classes of simple objects.

2. Strictifying pivotal categories

The main result in this section says that every pivotal category can be assumed to be strict, which is supposed to mean that not only its monoidal structure is strict, but also the isomorphisms governing the compatibility between tensor product and duality as well as the pivotal structure itself are identities. This is proved by Barrett and Westbury in [2], who start from a different set of axioms. The proof we give seems to be somewhat shorter.

A pivotal strict monoidal category is a strict monoidal category with a pivotal structure along the equivalence). We choose a duality functor for $\hat{\mathcal{C}}$ and define evaluation inductively: for $r \in \mathbb{N}$ and let $\mathcal{F}$ be a pivotal strict monoidal category in which $\mathcal{F}(X, \epsilon)$ is an equivalence, with a possible quasiinverse mapping $X \in \mathcal{C}$ to $(X, 1) \in \hat{\mathcal{C}}$.

We define the tensor product on $\hat{\mathcal{C}}$ as componentwise concatenation on objects. With this choice, $\mathcal{F}((X, \epsilon) \otimes (Y, \delta)) = \mathcal{F}(X, \epsilon) \otimes \mathcal{F}(Y, \delta)$. In particular, we can define the tensor product of morphisms by taking the tensor product of morphisms in $\mathcal{C}$; in this way $\hat{\mathcal{C}}$ is a strict monoidal category, and $\mathcal{F}$ is a strict monoidal functor.

We choose a duality functor for $\hat{\mathcal{C}}$ by putting

$$(X, \epsilon^\vee) := ((X_r, \ldots, X_1), (\epsilon_r + 1, \ldots, \epsilon_1 + 1))$$

and defining evaluation inductively: for $r = 0$, $ev = id_I$. For $X \in \hat{\mathcal{C}}$, $X \in \mathcal{C}$ and $\epsilon \in \mathbb{Z}/(2)$ we define

$$ev = \left( (X, \epsilon + 1) \otimes X^\vee \otimes X \otimes (X, \epsilon) \xrightarrow{id \otimes ev \otimes id} (X, \epsilon + 1) \otimes (X, \epsilon) \xrightarrow{ev} I \right),$$
where \( \text{ev}: (X, \epsilon + 1) \otimes (X, \epsilon) \to I \) is defined as
\[
\begin{cases}
(X^\vee \otimes X \xrightarrow{\text{ev}} I) & \text{if } \epsilon = 0, \\
(X \otimes X^\vee \xrightarrow{j \otimes X^\vee} X^{\vee \vee} \otimes X^\vee \xrightarrow{\text{ev}} I) & \text{if } \epsilon = 1.
\end{cases}
\]
It is clear that with a suitable choice of \( db \) this does define a dual object. Also, it follows from the definition that for any \( X, Y \in \hat{\mathcal{C}} \) we have
\[
\text{ev}_{X \otimes Y} = (Y^\vee \otimes X^\vee \otimes X \otimes Y \xrightarrow{id \otimes \text{ev} \otimes id} Y^\vee \otimes Y \xrightarrow{\text{ev}} I),
\]
so that in \( \hat{\mathcal{C}} \) the duality functor is strict monoidal. Another direct consequence of the definition is that the component \( \tilde{\xi}(X, \epsilon) : F(X, \epsilon + 1) \to F(X, \epsilon)^\vee \) of the duality transformation \( \tilde{\xi} \) associated to the trivial monoidal functor structure \( \xi \) of the strict monoidal functor \( F \) is given by
\[
\begin{cases}
id: X^\vee \to X^\vee & \text{if } \epsilon = 0, \\
j: X \to X^{\vee \vee} & \text{if } \epsilon = 1.
\end{cases}
\]
We define a pivotal structure for \( \hat{\mathcal{C}} \) in the unique way that lets \( F \) preserve the pivotal structure, that is, by requiring the diagrams
\[
\begin{array}{ccc}
F(X) & \xrightarrow{F(j)} & F(X^{\vee \vee}) \\
\downarrow{j} & & \downarrow{\tilde{\xi}} \\
F(X)^{\vee \vee} & \xrightarrow{\tilde{\xi}^\vee} & F(X^\vee)^\vee
\end{array}
\]
to commute for each \( X \in \hat{\mathcal{C}} \). Observe that \( X^{\vee \vee} = X \) by definition. We wish to show that, moreover, the top arrow \( F(j) \) is the identity for each \( X \).

Since the pivotal structure is a monoidal transformation, it is enough to verify this for the case \( X = (X, \epsilon) \) with \( X \in \mathcal{C} \) and \( \epsilon \in \mathbb{Z}/(2) \). In case \( \epsilon = 0 \), the lower horizontal arrow is the identity, and the right vertical arrow is \( j: X \to X^{\vee \vee} \), so the top arrow is the identity. In case \( \epsilon = 1 \), the right vertical arrow is the identity, and the lower horizontal arrow is \( j^\vee \). But by [17, Appendix] we have \( j^\vee_X = j_X^{-1} \), so again the top arrow is the identity. □

Remark 2.3. Lemma 1.1 was used in the proof that the new pivotal structure is the identity, by reducing the question to the case of objects with only one component. Likely, the proof can be completed without using Lemma 1.1 by an inductive argument.

3. Higher Frobenius-Schur indicators

Throughout this section, \( \mathcal{C} \) is a pivotal monoidal category, with pivotal structure \( j \). We denote by \( V^{\otimes n} \) the \( n \)-fold tensor power of an object \( V \in \mathcal{C} \) with rightmost parentheses; thus \( V^{\otimes 0} = I \), and \( V^{\otimes (n+1)} = V \otimes V^{\otimes n} \). There is a unique isomorphism
\[
\Phi(n): V^{\otimes (n-1)} \otimes V \to V^{\otimes n}
\]
composed of instances of \( \Phi \); explicitly \( \Phi(1) \) is the identity, and
\[
\Phi(n+1) = \left( (V \otimes V^{\otimes (n-1)}) \otimes V \xrightarrow{\Phi} V \otimes (V^{\otimes (n-1)} \otimes V) \xrightarrow{V \otimes \Phi(n)} V^{\otimes (n+1)} \right).
\]
Definition 3.1. For $V, W \in \mathcal{C}$, define $T_{V W}: \mathcal{C}(V^\vee, W) \to \mathcal{C}(W^\vee, V)$ by

$$T_{V W}(f) = (W^\vee \xrightarrow{f^\vee} V^{\vee\vee} \xrightarrow{\Phi_{W}} V),$$

and put

$$E_{V W} = \left( \mathcal{C}(I, V \otimes W) \xrightarrow{A} \mathcal{C}(V^\vee, W) \xrightarrow{T_{V W}} \mathcal{C}(W^\vee, V) \xrightarrow{A^{-1}} \mathcal{C}(I, W \otimes V) \right),$$

$$E_{V W}^{(n)} = \left( \mathcal{C}(I, V^{\otimes n}) \xrightarrow{E_{V^{\otimes (n-1)}}, V} \mathcal{C}(I, V^{\otimes (n-1)} \otimes V) \xrightarrow{C(I, \Phi(n))} \mathcal{C}(I, V^{\otimes n}) \right),$$

where $A$ denotes the adjunction (1.1) associated with duality. Assume that $\mathcal{C}$ is $k$-linear. Then for any positive integers $r, n$, the $(n, r)$-th Frobenius-Schur indicator of $V$ is the scalar

$$\nu_{n, r}(V) = \text{Tr} \left( \left( E_{V W}^{(n)} \right)^r \right).$$

We will call $\nu_{n, r}(V) := \nu_{n, 1}(V)$ the $n$-th Frobenius-Schur indicator of $V$.

We will give a reformulation of the definition of $E_{V W}$ which will be useful later, in particular when we assume that the categories in consideration are strict, and use graphical notations. First, we define the following counterparts of the maps $A_0$ and $A$:

Definition 3.2. We have an isomorphism $B_0: \mathcal{C}(X \otimes V, W) \to \mathcal{C}(X, W \otimes V^\vee)$ with

$$B_0(f) = \left( X \xrightarrow{X \otimes \Phi^{-1}} X \otimes (V \otimes V^\vee) \xrightarrow{f \otimes V^\vee} W \otimes V^\vee \right),$$

$$B_0^{-1}(g) = \left( X \otimes V \xrightarrow{\Phi^0} (W \otimes V^\vee) \otimes V \xrightarrow{g} W \otimes (V^\vee \otimes V) \xrightarrow{W \otimes \text{ev}} W \right).$$

We note the special case $X = I$, with $B = B_0: \mathcal{C}(V, W) \to \mathcal{C}(I, W \otimes V^\vee)$ given by

$$B(f) = \left( I \xrightarrow{\Phi} V \otimes V^\vee \xrightarrow{f \otimes V^\vee} W \otimes V^\vee \right),$$

$$B^{-1}(g) = \left( V \xrightarrow{\Phi^0} (W \otimes V^\vee) \otimes V \xrightarrow{g} W \otimes (V^\vee \otimes V) \xrightarrow{W \otimes \text{ev}} W \right).$$

It is easy to see (and well-known, at least implicit for example in [9]) that the composition

$$\mathcal{C}(V, W) \xrightarrow{B} \mathcal{C}(I, W \otimes V^\vee) \xrightarrow{A} \mathcal{C}(W^\vee, V^\vee)$$

is given by $AB(f) = f^\vee$.

Definition 3.3. Denote by $D = D_{V, W}: \mathcal{C}(I, V \otimes W) \to \mathcal{C}(I, W \otimes V^{\vee\vee})$ the composition

$$D = \left( \mathcal{C}(I, V \otimes W) \xrightarrow{A} \mathcal{C}(V^\vee, W) \xrightarrow{B} \mathcal{C}(I, W \otimes V^{\vee\vee}) \right).$$

Lemma 3.4. We have

$$E_{V W} = \left( \mathcal{C}(I, V \otimes W) \xrightarrow{D_{V W}} \mathcal{C}(I, W \otimes V^{\vee\vee}) \xrightarrow{C(I, W \otimes f^{-1})} \mathcal{C}(I, W \otimes V) \right).$$
Proof. Comparing the definitions of \( D \) and \( E \), we have to show that the outer pentagon of the diagram

\[
\begin{array}{ccc}
\mathcal{C}(V^\vee, W) & \xrightarrow{B} & \mathcal{C}(I, W \otimes V^{\vee}) \\
(-)^\vee & \xrightarrow{A} & - \\
\mathcal{C}(W^\vee, V^{\vee}) & \xrightarrow{C(\cdot, j^{-1})} & \mathcal{C}(I, W \otimes V) \\
\mathcal{C}(W^\vee, V) & \xrightarrow{A^{-1}} & \mathcal{C}(I, W \otimes V)
\end{array}
\]

commutes. But we know that the triangle commutes, and the quadrangle is naturality of \( A \) applied to \( j^{-1} \). \( \square \)

4. Invariance I

The definitions of the maps \( E_{V, W} \) and \( E_{V}^{(n)} \) are given in terms of monoidal structure, duality, and pivotal structure. Thus if \( \mathcal{C}, \mathcal{D} \) are two pivotal monoidal categories, and \( (\mathcal{F}, \xi): \mathcal{C} \to \mathcal{D} \) is monoidal (hence preserves duality), and preserves the pivotal structure, then \( \mathcal{F} \) preserves \( E_{V, W} \) and \( E_{V}^{(n)} \), in a sense we will make precise shortly. While the statement is conceptually evident, the proof will involve some technical machinery. The technical difficulties are due mostly to the fact that \( E_{V, W} \) is a map of sets, and not a morphism defined in the category.

Lemma 4.1. Let \( (\mathcal{F}, \xi): \mathcal{C} \to \mathcal{D} \) be a monoidal functor between rigid monoidal categories \( \mathcal{C}, \mathcal{D} \). Then for all \( V, W \in \mathcal{C} \) we have

\[
\begin{array}{ccc}
\mathcal{C}(I, V \otimes W) & \xrightarrow{\mathcal{F}} & \mathcal{D}(I, \mathcal{F}(V \otimes W)) \\
A & & A \\
\mathcal{C}(V^\vee, W) & \xrightarrow{\mathcal{F}} & \mathcal{D}(\mathcal{F}(V^\vee), \mathcal{F}(W)) \\
\mathcal{C}(V^\vee, \mathcal{F}(W)) & \xrightarrow{\mathcal{D}(\xi^{-1}, \mathcal{F}(I))} & \mathcal{D}(\mathcal{F}(V^\vee), \mathcal{F}(W))
\end{array}
\]

or \( \mathcal{F}(A(f))\xi^{-1} = A(\xi^{-1}\mathcal{F}(f)) \) for \( f: I \to V \otimes W \).

Proof. It is convenient to prove \( A^{-1}(\mathcal{F}(g)\xi^{-1}) = \xi^{-1}\mathcal{F}(A^{-1}(g)) \) for \( g: V^\vee \to W \) instead, by a look at the diagram

\[
\begin{array}{ccc}
\mathcal{F}(V) & \xrightarrow{\mathcal{F}(\xi^{-1})} & \mathcal{F}(V) \otimes \mathcal{F}(W) \\
\mathcal{F}(V) \otimes \mathcal{F}(W) & \xrightarrow{\mathcal{D}(\mathcal{F}(W) \otimes \xi^{-1})} & \mathcal{F}(V) \otimes \mathcal{F}(W)
\end{array}
\]

whose top line is \( \mathcal{F}(A^{-1}(g)) \). \( \square \)

Similarly:

Lemma 4.2. Let \( (\mathcal{F}, \xi): \mathcal{C} \to \mathcal{D} \) be a monoidal functor between rigid monoidal categories \( \mathcal{C}, \mathcal{D} \). Then for all \( V, W \in \mathcal{C} \) we have

\[
\begin{array}{ccc}
\mathcal{C}(V, W) & \xrightarrow{\mathcal{F}} & \mathcal{D}(\mathcal{F}(V), \mathcal{F}(W)) \\
B & & B \\
\mathcal{C}(V, W) & \xrightarrow{\mathcal{D}(\mathcal{F}(W) \otimes \xi^{-1})} & \mathcal{D}(\mathcal{F}(V) \otimes \mathcal{F}(W))
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{C}(I, V \otimes W) & \xrightarrow{\mathcal{F}} & \mathcal{D}(I, \mathcal{F}(V \otimes W)) \\
B & & B \\
\mathcal{C}(I, W \otimes V^\vee) & \xrightarrow{\mathcal{D}(I, \mathcal{F}(W) \otimes \xi^{-1})} & \mathcal{D}(I, \mathcal{F}(W) \otimes \mathcal{F}(V^\vee))
\end{array}
\]
Recall that there is a unique isomorphism $\xi^{(n)} = \xi^n : F(V^\otimes n) \to F(V)^{\otimes n}$ composed formally of instances of the monoidal functor structure $\xi$. Explicitly $\xi^0 = \text{id}_I$, $\xi^{(1)} = \text{id}_{F(V)}$, and

$$\xi^{(n+1)} = \left( F(V^{\otimes (n+1)}) \xrightarrow{\xi^{-1}_{V,V^\otimes n}} F(V)^{\otimes (n+1)} \xrightarrow{F(V)^{\otimes (n)}} F(V)^{\otimes (n+1)} \right).$$

**Proposition 4.3.** Let $(F, \xi): C \to D$ be a monoidal functor that preserves the pivotal structure. Then for any $V, W \in C$, we have the following commutative diagrams:

1. \[ C(I, V \otimes W) \xrightarrow{E_{V,W}} C(I, W \otimes V) \]
2. \[ D(I, F(V \otimes W)) \xrightarrow{E_{F(V),F(W)}} D(I, F(W \otimes F(V))) \]
3. \[ C(I, V^{\otimes n}) \xrightarrow{E^{(n)}_V} C(I, V^{\otimes n}) \]
4. \[ D(I, F(V^{\otimes n})) \xrightarrow{E^{(n)}_{F(V)}} D(I, F(V)^{\otimes n}) \]

**Proof.** By Lemma 3.4, functoriality and naturality
commutes. Thus, to prove (4.2) we have to check commutativity of the right hand area of

\[
\begin{array}{c}
\mathcal{C}(V^\vee, W) \\
\downarrow F \\
\mathcal{D}(F(V^\vee), F(W))
\end{array}
\xrightarrow{A^{-1}}
\begin{array}{c}
\mathcal{C}(I, V \otimes W) \\
\downarrow F \\
\mathcal{D}(I, F(V \otimes W))
\end{array}
\xrightarrow{D}
\begin{array}{c}
\mathcal{C}(I, W \otimes V^\vee) \\
\downarrow F \\
\mathcal{D}(I, F(W \otimes V^\vee))
\end{array}
\]

But the left hand area is Lemma 4.1. The way around the outside of the diagram commutes by Lemma 4.2, since \(F\) preserves the pivotal structure by assumption.

(4.3) follows from (4.2) and the coherence of the monoidal functor \(F\). □

The following corollary is an immediate consequence of Proposition 4.3.

**Corollary 4.4.** Let \(\mathcal{C}, \mathcal{D}\) be \(k\)-linear pivotal monoidal categories over a field \(k\), and \(F : \mathcal{C} \rightarrow \mathcal{D}\) a \(k\)-linear monoidal equivalence that preserves the pivotal structure. Assume \(\mathcal{C}(I, V)\) and \(\mathcal{D}(I, W)\) are finite-dimensional for all objects \(V\) in \(\mathcal{C}\) and \(W\) in \(\mathcal{D}\). Then

\[\nu_{n,r}(V) = \nu_{n,r}(F(V)).\]

for any object \(V\) in \(\mathcal{C}\) and positive integers \(n, r\). □

### 5. The powers of \(E\) and the values of \(\nu_n\)

The main results of this section concern the powers of the map \(E^{(n)}_V\) — whose \(n\)-th power turns out to be the identity — and the consequences for the possible values of its trace \(\nu_n\). In view of Proposition 4.3 and Theorem 2.2 we can assume that we are dealing with a strict pivotal monoidal category to prove these results. Thus for the rest of this section, except for the statements of the main results, we assume that \(\mathcal{C}\) is a \(k\)-linear strict pivotal monoidal category. We use graphical calculus in \(\mathcal{C}\) (see for example [8] or [11]). In particular, we use

\[
\begin{array}{c}
\begin{array}{c}
V^\vee \\
\uparrow V
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
V_1 \\
\vdots
\end{array}
\end{array}
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
W \\
\uparrow f
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

to depict the evaluation morphism \(ev : V^\vee \otimes V \rightarrow I\), the dual basis morphism \(db : I \rightarrow V \otimes V^\vee\), and a morphism \(f : I \rightarrow V_1 \otimes \ldots \otimes V_n\), respectively. Thus, by Definition 3.3 and Lemma 3.4 we have

\[E_{VW}(f) = D_{VW}(f) = \]

for \(f : I \rightarrow V \otimes W\). From this we deduce
for each $f: I \to U \otimes V \otimes W$, which we use to prove

**Theorem 5.1.** Let $C$ be a pivotal monoidal category, $V \in C$ and $n \in \mathbb{N}$. Then $(E_V^{(n)})^n = \text{id}$.

If $k$ is an algebraically closed field of characteristic zero and $C$ is a $k$-linear pivotal tensor category, then the $(n, r)$-th Frobenius-Schur indicator $\nu_{n, r}(V)$ is a cyclotomic integer in $\mathbb{Q}_n \subset k$ for any object $V \in C$ and positive integers $n, r$, and we have $\nu_{n, n-r}(V) = \nu_{n, r}(V)$.

**Proof.** It is sufficient to treat the case where $C$ is strict pivotal. We prove $(E_V^{(k)})^k = E_{V \otimes k, V \otimes (n-k)}$ for all $0 \leq k \leq n$ by induction: This is obvious for $k = 0$ (and the definition of $E_{V}^{(n)}$ for $k = 1$).

Inductively,

\[
(E_{V}^{(k)})^{k+1} = (E_{V}^{(n)})^k (E_{V}^{(n)}) = E_{V \otimes (n-1)}^{(n)} E_{V \otimes k, V \otimes (n-k)}
\]

\[
= E_{V \otimes (n-k-1) \otimes k}^{(n)} E_{V \otimes k, V \otimes (n-k-1)}
\]

\[
= E_{V \otimes (k+1) \otimes V \otimes (n-k-1)}^{(n)}
\]

applying (5.4) with $U = V \otimes k$ and $W = V \otimes (n-k-1)$.

Now $(E_{V}^{(n)})^n = E_{V \otimes n, I}$, so to show that $(E_{V}^{(n)})^n = \text{id}$, we are reduced to observing that both

\(A: C(I, V) \to C(V^\vee, I)\) and \(B: C(V, I) \to C(I, V^\vee)\) are special cases of the duality functor, and so $E_{V, I}(f) = f^\vee = f$ for all $f: I \to V$.

If, in addition, $C$ is $k$-linear, then $E_{V}^{(n)}$ is a linear operator on the $k$-linear space $C(I, V \otimes n)$. Since $(E_{V}^{(n)})^n = \text{id}$, $(E_{V}^{(n)})^r$ is diagonalizable for every integer $r$, and all its eigenvalues are $n$-th roots of unity. Thus,

\[
\nu_{n, r}(V) = \text{Tr} \left( (E_{V}^{(n)})^r \right)
\]

is a cyclotomic integer in $\mathbb{Q}_n$.

Also, since $(E_{V}^{(n)})^{n-r}$ is the inverse of $(E_{V}^{(n)})^r$, the eigenvalues of these diagonalizable endomorphisms are inverse to each other. Since they are roots of unity, they are complex conjugate, and so are, consequently, the traces, which are the indicators $\nu_{n, n-r}(V)$, and $\nu_{n, r}(V)$, respectively. \hfill $\square$

The definition of the map $E_{V}^{(n)}$ is not left-right symmetric; we have chosen to “bend the leftmost strand of $V \otimes n$ over the top.” Instead of repeating all the details of the definitions in the opposite order, we will be very brief in giving the relation between the two versions in the semisimple case:

For $V \in C$ denote by $V^{\text{sym}}$ the object $V$ viewed in the monoidal category $C^{\text{sym}}$, and let $\nu_{n, k}(V^{\text{sym}})$ be the $(n, k)$-th Frobenius-Schur indicator of $V^{\text{sym}} \in C^{\text{sym}}$.

**Lemma 5.2.** Let $C$ be a semisimple $k$-linear pivotal monoidal category. Then $\nu_{n, k}(V^{\text{sym}}) = \nu_{n, n-k}(V)$ for every $V \in C$ and integers $n, k$. 

Proof. We can assume \(1 \leq k < n\). Since \(\mathcal{C}\) is semisimple, composition is a nondegenerate bilinear form \(\mathcal{C}(V^\otimes n, I) \times \mathcal{C}(I, V^\otimes n) \to k\). Choose a basis \(q_i \in \mathcal{C}(I, V^\otimes n)\) and let \(p_i : V^\otimes n \to I\) be the elements of the dual basis. Thus the trace of an endomorphism \(F : \mathcal{C}(I, V^\otimes n) \to \mathcal{C}(I, V^\otimes n)\) can be computed as \(\text{Tr}(F) = \sum_i p_i F(q_i) \in \mathcal{C}(I, I) \cong k\). We use the graphical notations

\[
\begin{array}{c}
V^\otimes (n-k) \\
p
\end{array}
\quad \text{and} \quad
\begin{array}{c}
q \quad V^\otimes k \\
V^\otimes (n-k)
\end{array}
\]

for \(q \in \mathcal{C}(I, V^\otimes n)\) and \(p \in \mathcal{C}(V^\otimes n, I)\) to find

\[
\sum_i \begin{array}{c}
q_i \\
p_i
\end{array} = \sum_i \begin{array}{c}
q_i \\
p_i
\end{array} = \sum_i \begin{array}{c}
q_i \\
p_i
\end{array} = \sum_i \begin{array}{c}
q_i \\
p_i
\end{array}.
\]

The left hand side computes the trace of \((E^{(n)}_V)^k = E_{V^\otimes k, V^\otimes (n-k)}\), while the right hand side does the same for \(E_{(V^\text{sym})^\otimes (n-k), (V^\text{sym})^\otimes k} = (E^{(n)}_{V^\text{sym}})^{n-k}\) in the category \(\mathcal{C}^\text{sym}\). \(\square\)

6. INVARIANCE II

Throughout this section we assume that \(\mathcal{C}\) is a fusion category.

The Grothendieck group \(K(\mathcal{C})\) of \(\mathcal{C}\) is a ring with the multiplication

\([U][V] = [U \otimes V]\)

for any \(U, V \in \mathcal{C}\), where \([U]\) denotes the isomorphism class of \(U\). Note that \(K(\mathcal{C})\) is a free abelian group with a basis \(\{[V_i]\}\) where \(V_1, \ldots, V_n\) is a complete set of non-isomorphic simple objects of \(\mathcal{C}\). Thus, for any \(V \in \mathcal{C}\), there exist unique non-negative integers \(N_{i,1}, \ldots, N_{i,n}\) such that

\([V] [V_i] = \sum_j N_{i,j} [V_j]\).

The Frobenius-Perron dimension of \(V\), denoted by \(\text{FPdim}(V)\), is defined to be the largest nonnegative real eigenvalue of the non-negative integer matrix

\(\rho_V = [N_{i,j}]\);

it dominates the absolute values of all eigenvalues of \(\rho_V\) (cf. [4, Section 8]). The Frobenius-Perron dimension \(\text{FPdim}(\mathcal{C})\) of \(\mathcal{C}\) is defined to be

\[\sum_i \text{FPdim}(V_i)^2.\]

By [4, Proposition 2.1], for any simple object \(V \in \mathcal{C}\), there exists an isomorphism \(a : V \to V^\vee\vee\). One can define the categorical trace \(\mathop{\text{tr}}(a) \in \mathcal{C}\) of \(a\) by

\[\mathop{\text{tr}}(a) = (I \xrightarrow{\text{db}} V \otimes V^\vee \xrightarrow{a \otimes V} V^\vee \otimes V^\vee \xrightarrow{\text{ev}} I).\]

The normed square of \(V\), denoted by \(|V|^2\), is defined to be

\[|V|^2 = \mathop{\text{tr}}(a) \mathop{\text{tr}}((a^{-1})\vee).\]
Note that the definition of $|V|^2$ is independent of the choice of $a$. The global dimension $\dim(C)$ of $C$ is defined to be

$$\sum_i |V_i|^2.$$  

A fusion category $C$ over $\mathbb{C}$ is called pseudo-unitary if $\dim(C) = \text{FPdim}(C)$. It was shown in [4, Proposition 8.23] that if $C$ is a pseudo-unitary fusion category, there exists a unique pivotal structure $j : \text{Id} \rightarrow (-)^{\vee}$ such that

$$(6.5) \quad \text{tr}(j_V) = \text{FPdim}(V)$$

for every simple object $V \in C$.

The following lemma says that monoidal functors preserve traces in a suitable sense:

**Lemma 6.1.** If $(F, \xi) : C \rightarrow D$ is a monoidal functor between rigid $k$-linear monoidal categories, and $a : V \rightarrow V^{\vee \vee}$ in $C$, then

$$(6.6) \quad F(\text{tr}_C(a)) = \text{tr}_D\left(F(V) \xrightarrow{F(a)} F(V^{\vee \vee}) \xrightarrow{\xi} F(V^{\vee}) \xrightarrow{(\xi^{-1})^\vee} F(V)\right).$$

**Proof.** We can write the categorical trace map as the composition

$$\text{tr}_C = \left(\mathcal{C}(V, V^{\vee \vee}) \xrightarrow{B_{0}^{-1}} \mathcal{C}(V \otimes V^{\vee}, I) \xrightarrow{\mathcal{C}(db, I)} \mathcal{C}(I, I)\right),$$

so

$$\text{tr}_C(a) = \left( I \xrightarrow{db} V \otimes V^{\vee} \xrightarrow{B_{0}^{-1}(a)} I \right).$$

Writing $W = V^{\vee}$, the diagram

$$\begin{array}{ccc}
F(V) \otimes F(W) & \xrightarrow{F(a) \otimes F(W)} & F(W^{\vee}) \otimes F(W) \\
\downarrow \xi & & \downarrow \xi \\
F(V \otimes W) & \xrightarrow{F(a \otimes W)} & F(W^{\vee} \otimes W) \\
\downarrow \xi & & \downarrow F(\text{ev}) \\
F(V) \otimes F(W) & \xrightarrow{F(a \otimes W)} & F(W^{\vee} \otimes W) \\
\downarrow \xi & & \downarrow F(\text{ev}) \\
I & = & I \\
\end{array}$$

shows $F(B_{0}^{-1}(a))\xi = B_{0}^{-1}(\xi F(a))$ for $a : V \rightarrow W^{\vee}$ in $C$, which is yet another variant of Lemma 4.1. Further, $B_{0}^{-1} : \mathcal{D}(X, Y^{\vee}) \rightarrow \mathcal{D}(X \otimes Y, I)$ is natural, so for $X$ and $f : Y \rightarrow Z$ in $\mathcal{D}$

$$\begin{array}{ccc}
\mathcal{D}(X, Z^{\vee}) & \xrightarrow{B_{0}^{-1}} & \mathcal{D}(X \otimes Z, I) \\
\downarrow \mathcal{D}(X, f^{\vee}) & & \downarrow \mathcal{D}(X \otimes f, I) \\
\mathcal{D}(X, Y^{\vee}) & \xrightarrow{B_{0}^{-1}} & \mathcal{D}(X \otimes Y, I) \\
\end{array}$$

commutes, that is to say, for each $b : X \rightarrow Z^{\vee}$, the diagram

$$\begin{array}{ccc}
X \otimes Y & \xrightarrow{X \otimes f} & X \otimes Z \\
\downarrow B_{0}^{-1}(f \otimes b) & & \downarrow B_{0}^{-1}(b) \\
I & = & I \\
\end{array}$$
commutes. As a consequence,

$$\text{tr}_D((\xi^{-1})^*\xi \mathcal{F}(j_V)) = \mathcal{F}(\text{tr}_C(j_V)) = \text{tr}_C(j_V) = \text{FPdim}(V) = \text{FPdim}(\mathcal{F}(V)) = \text{tr}_D(j_{\mathcal{F}(V)}).$$

Since $\mathcal{D}(W,W^{\vee\vee})$ is one-dimensional for any simple object $W$, the canonical pivotal structure on $\mathcal{F}(V)$ is determined by its trace, and we are done. □

**Corollary 6.3.** The Frobenius-Schur indicators of the simple objects of a pseudo-unitary fusion category $\mathcal{C}$ are invariants of $\mathcal{C}$ as a monoidal category.

### 7. Frobenius-Schur endomorphisms

Throughout this section $\mathcal{C}$ is a semisimple $k$-linear monoidal category. Recall that in our conventions this includes the condition that the unit object is simple, and that being simple means that the endomorphism ring is isomorphic to the base field $k$.

Let $S$ be a simple object in $\mathcal{C}$. We say that an object $V$ is $S$-isotypical if it is isomorphic to a sum of copies of $S$. An $I$-isotypical object will be called a trivial object. The subcategory of $\mathcal{C}$ consisting of all trivial objects will be denoted by $\text{Triv}(\mathcal{C})$. An arbitrary object $V$ is isomorphic to a direct sum of $S$-isotypical objects. More explicitly, there are $S$-isotypical objects $V^{(S)}$ and morphisms $\iota^{(S)}; V^{(S)} \to V$ and $\pi^{(S)}; V \to V^{(S)}$ for each simple $S$ from a set of representatives of the isomorphism classes of simples in $\mathcal{C}$ such that

$$\forall S:\quad \pi^{(S)}\iota^{(S)} = \text{id}_{V^{(S)}},$$

$$\sum_S \iota^{(S)} \pi^{(S)} = \text{id}_V.$$ 

We will call the triple $(V^{(S)}, \iota^{(S)}, \pi^{(S)})$ an $S$-isotypical component of $V$. Isotypical components are uniquely determined up to isomorphism in an obvious sense. We will only be needing $I$-isotypical components, which we will call trivial components and denote by $(V^{\text{triv}}, \iota, \pi)$.

**Lemma 7.1.** Let $\mathcal{C}$ be a semisimple $k$-linear monoidal category. There is a unique isomorphism $\tau_{VT}: V \otimes T \to T \otimes V$ natural in $T \in \text{Triv}(\mathcal{C})$ and $V \in \mathcal{C}$ for which $\tau_{VT}$ is the identity (the canonical
isomorphism). For all $V, W$ in $C$ the diagram

$$
\begin{array}{ccc}
(V \otimes W) \otimes T & \xrightarrow{\tau} & T \otimes (V \otimes W) \\
\downarrow \Phi & & \downarrow \Phi^{-1} \\
V \otimes (W \otimes T) & \xrightarrow{V \otimes \tau} & (T \otimes V) \otimes W \\
\downarrow V \otimes \pi_i & & \downarrow \tau \otimes W \\
V \otimes (T \otimes W) & \xrightarrow{\Phi^{-1}} & (V \otimes T) \otimes W
\end{array}
$$

(7.7)

commutes.

Proof. Fix $V$, and write $T$ as a direct sum of copies of $I$, with projections $\pi_i: T \rightarrow I$ and injections $\iota_i: I \rightarrow T$ for $i$ in some finite index set. If $\tau$ is to be a natural transformation, then the diagrams

$$
\begin{array}{ccc}
V \otimes T & \xrightarrow{\tau} & T \otimes V \\
\downarrow V \otimes \pi_i & & \downarrow \pi_i \otimes V \\
V \otimes I & \xrightarrow{V \otimes \iota_i} & I \otimes V
\end{array}
$$

should commute, and there exists a unique isomorphism $\tau$ making these diagrams commute. Let $T'$ be another trivial object, and choose a direct sum decomposition with projections $\pi'_i: T' \rightarrow I$ and $\iota'_i: I \rightarrow T'$; construct $\tau_{VT'}$ analogous to $\tau_{TV}$. Let $f: T \rightarrow T'$ be a morphism, and let $\alpha_{ij} = \pi'_i f \iota_j: I \rightarrow I$. The diagrams

$$
\begin{array}{ccc}
V \otimes T & \xrightarrow{V \otimes f} & V \otimes T' \\
\downarrow V \otimes \iota_j & & \downarrow V \otimes \pi'_i \\
V \otimes I & \xrightarrow{V \otimes \alpha_{ij}} & I \otimes V
\end{array}
$$

and

$$
\begin{array}{ccc}
V \otimes T & \xrightarrow{\tau} & T \otimes V \\
\downarrow V \otimes \iota_j & & \downarrow \iota_j \otimes V \\
V \otimes I & \xrightarrow{V \otimes \alpha_{ij}} & I \otimes V
\end{array}
$$

and the fact that $\alpha_{ij} \otimes V = V \otimes \alpha_{ij}$ show that $(f \otimes V)\tau = \tau'(V \otimes f)$, so that $\tau$ is independent of the choice of the direct summand decompositions, and natural in $T$. Naturality in $V$ is obvious, and the braiding equality (7.7) follows from uniqueness. □

**Definition 7.2.** Let $C$ be a semisimple $k$-linear rigid monoidal category, $V$ an object of $C$, and $n \in \mathbb{N}$.

Choose a right dual $X$ of $V^\otimes(n-1)$, and a trivial component $((V^\otimes n)^{\text{triv}}, \iota, \pi)$.

The $n$-th Frobenius-Schur endomorphism of $V$ is the composition

$$
\text{FS}^{(n)}_V = \left( V \xrightarrow{U} X \otimes (V^\otimes(n-1) \otimes V) \xrightarrow{X \otimes \Phi^{(n)}} X \otimes V^\otimes n \xrightarrow{X \otimes \pi} X \otimes (V^\otimes n)^{\text{triv}} \xrightarrow{\tau} (V^\otimes n)^{\text{triv}} \otimes X \xrightarrow{\iota \otimes X} V^\otimes n \otimes X \xrightarrow{C} V \right).
$$
where
\[
U = \left( V \xrightarrow{db \otimes V} (X \otimes V \otimes (n-1)) \otimes V \xrightarrow{\Phi} X \otimes (V \otimes (n-1) \otimes V) \right),
\]
\[
C = \left( (V \otimes V \otimes (n-1)) \otimes X \xrightarrow{\Phi} V \otimes (V \otimes (n-1) \otimes X) \xrightarrow{V \otimes ev} V \right),
\]
and \( \tau \) is as in Lemma 7.1.

**Lemma 7.3.** Let \( V \) be an object of \( C \). The \( n \)-th Frobenius-Schur endomorphism of \( V \) is independent of the choice of a right dual of \( V \otimes (n-1) \), and of the choice of a trivial component functor. If \( F : C \to D \) is an equivalence of semisimple \( k \)-linear monoidal categories, then \( F(\text{FS}^{(n)}_V) = \text{FS}^{(n)}_{F(V)} \).

**Proof.** By contrast to our earlier definitions of \( E^{(n)}_V \) and the indicators, the construction of \( \text{FS}^{(n)}_V \) takes place entirely within the monoidal category \( C \); the verifications are thus routine, and we will omit the details. \( \square \)

**Remark 7.4.** Let \( V \) be an object of \( C \), and choose a trivial component \( (V^{\text{triv}}, \iota, \pi) \). Choose a direct sum decomposition of \( V^{\text{triv}} \) with projections \( \pi_i : V^{\text{triv}} \to S \) and injections \( \iota_i : S \to V^{\text{triv}} \) for \( i \) from some finite index set.

The composition \( C(V, I) \times C(I, V) \to C(I, I) \cong k \) is a nondegenerate bilinear form, with \( p_i = \pi_i \pi : V \to I \) and \( q_i = \iota_i : I \to V \) as dual bases.

The following definition follows Barrett and Westbury [2]:

**Definition 7.5.** Let \( C \) be a pivotal monoidal category with pivotal structure \( j \). For an endomorphism \( f : V \to V \) we define the left and right **pivotal traces** to be
\[
\text{ptr}^L(f) = \text{tr}(jv f) = \left( I \xrightarrow{db} V \otimes V^\vee \xrightarrow{f \otimes V^\vee} V \otimes V^\vee \xrightarrow{V^\vee \otimes V^\vee} V^\vee \otimes V^\vee \xrightarrow{ev} I \right)
\]
\[
\text{ptr}^R(f) = \left( I \xrightarrow{db} V^\vee \otimes V^\vee \xrightarrow{V^\vee \otimes jv^{-1}} V^\vee \otimes V \xrightarrow{V^\vee \otimes f} V^\vee \otimes V \xrightarrow{ev} I \right).
\]
The category \( C \) is called **spherical** if \( \text{ptr}^L(f) = \text{ptr}^R(f) =: \text{ptr}(f) \) for all \( f \).

Note that \( \text{ptr}^L(f) = \text{ptr}^R(f') \). It is well-known and easy to check that the pivotal traces are cyclic, for \( g : V \to W \) and \( f : W \to V \):
\[
\text{ptr}^R(fg) = \left( I \xrightarrow{db} V \otimes V^\vee \xrightarrow{jf \otimes V^\vee} V^\vee \otimes V^\vee \xrightarrow{ev} I \right)
\]
\[
= \left( I \xrightarrow{db} V \otimes V^\vee \xrightarrow{f' \otimes jg \otimes V^\vee} V^\vee \otimes V^\vee \xrightarrow{ev} I \right)
\]
\[
= \left( I \xrightarrow{db} V \otimes V^\vee \xrightarrow{jg \otimes f' \otimes V^\vee} W^\vee \otimes W^\vee \xrightarrow{ev} I \right)
\]
\[
= \left( I \xrightarrow{db} W \otimes W^\vee \xrightarrow{gf \otimes W^\vee} W^\vee \otimes W^\vee \xrightarrow{ev} I \right)
\]
\[
= \text{ptr}^R(gf).
\]

In the case where \( C \) is strict, the left and right pivotal traces are given, in the graphical calculus, by closing the morphism \( f \) to a loop on the left, resp. on the right:
\[
\text{ptr}^L(f) = \begin{array}{c}
\text{resp.} \end{array}
\text{ptr}^R(f) = \begin{array}{c}
\end{array}
\]
Theorem 7.6. Let \( \mathcal{C} \) be a semisimple \( k \)-linear pivotal monoidal category. For an object \( V \) in \( \mathcal{C} \) we have

\[
\nu_n(V) = \text{ptr}^\ell(\text{FS}^{(n)}_V).
\]

In addition, if \( \mathcal{C} \) is spherical, then \( \nu_n(V) = \nu_n(V^\vee) \).

Proof. In view of the previous results, it is enough to prove the claim under the assumption that \( \mathcal{C} \) is strict as a pivotal category.

Choose a trivial component \( ((V^\otimes n)^{\text{triv}}, \iota, \pi) \), and choose a direct sum decomposition of \( (V^\otimes n)^{\text{triv}} \) with projections \( \pi_i : (V^\otimes n)^{\text{triv}} \to I \) and injections \( \iota_i : I \to (V^\otimes n)^{\text{triv}} \). Put \( p_i = \pi_i \pi \) and \( q_i = \iota \iota \).

Put \( W = V^\otimes (n-1) \) and \( T = (V^\otimes n)^{\text{triv}} \), and use the graphical notations

\[
\begin{align*}
W V & \quad , \quad q_i V W \\
\pi & \quad , \quad T V W
\end{align*}
\]

\( \tau_W T = W T \quad T W \)

to find

\[
\text{Tr}(E^{(n)}_V) = \sum_i q_i p_i = \sum_i p_i q_i = \text{ptr}^\ell \left( \begin{array}{c}
V \\
V^\vee
\end{array} \right).
\]

Note that

\[
(FS_V^{(n)})^\vee = \begin{array}{c}
V^\vee \\
V^\vee
\end{array} = \begin{array}{c}
\pi \\
\pi
\end{array} = FS_V^{(n)}.
\]

If \( \mathcal{C} \) is spherical, we have

\[
\nu_n(V) = \text{ptr}^\ell(FS_V^{(n)}) = \text{ptr}^\ell((FS_V^{(n)})^\vee) = \text{ptr}^\ell(FS_V^{(n)^\vee}) = \nu_n(V^\vee).
\]

\( \Box \)

Proposition 7.7. The Frobenius-Schur endomorphism is a natural endomorphism of the identity functor on a semisimple \( k \)-linear rigid monoidal category.

Proof. It suffices to prove this in the case where \( \mathcal{C} \) is strict. Consider \( f : V \to W \) in \( \mathcal{C} \). Write

\[
f_k = W^\otimes (k-1) \otimes f \otimes V^\otimes (n-k) : W^\otimes (k-1) \otimes V^\otimes (n-k+1) \to W^\otimes k \otimes V^\otimes (n-k),
\]
and denote by \( \iota_k, \pi_k \) the inclusion and projection of \( W^\otimes k \otimes V^\otimes (n-k) \) from and to its trivial component. In particular \( f_k \iota_{k-1} = \iota_k f_k^{\text{triv}} \) and \( \pi_k f_k = f_k^{\text{triv}} \pi_{k-1} \). Now for \( 1 \leq k \leq n \)

and for \( 1 \leq k < n \)

and thus, inductively

\[
\begin{align*}
& f \mathcal{FS}^{(n)} V = \mathcal{FS}^{(n)} f, \\
& f \mathcal{FS}^{(n)} W = \mathcal{FS}^{(n)} f.
\end{align*}
\]

**Corollary 7.8.** The Frobenius-Schur indicators are additive, that is \( \nu_n (V \oplus W) = \nu_n (V) + \nu_n (W) \) holds for objects \( V, W \) in a semisimple \( k \)-linear pivotal monoidal category \( C \).

**Proof.** Since the pivotal traces are cyclic, taking the pivotal trace is additive (cf. [7, Prop.1]), as is the natural transformation \( \mathcal{FS}^{(n)} \).

**Remark 7.9.** To define the Frobenius-Schur indicators and prove their basic properties, we need not assume that the category \( C \) is semisimple. It is enough to assume that we are given a linear endofunctor \( (-)^{\text{triv}} \) of \( C \) such that every \( V^{\text{triv}} \) is a direct sum of copies of the neutral object, and natural transformations \( \iota: V^{\text{triv}} \to V \) and \( \pi: V \to V^{\text{triv}} \). If \( C = H\text{-mod}_{\text{fin}} \) is the category of finite-dimensional modules over a unimodular (quasi-)Hopf algebra \( H \), we could take \( V^{\text{triv}} \) to be the submodule of \( H \)-invariants, and define \( \pi \) as multiplication by an integral. If \( \iota \) and \( \pi \) can be chosen so that \( \pi \iota = \text{id} \), then the triple \( (-)^{\text{triv}}, \iota, \pi \) is unique up to isomorphism and the resulting Frobenius-Schur endomorphisms are unique. In the case that \( C = H\text{-mod}_{\text{fin}} \) as above, the existence of such a triple is well-known to imply semisimplicity of \( C \).
It is also possible to define Frobenius-Schur endomorphisms whose traces correspond to the indicators \( \nu_{n,k}(V) \) with \( k > 1 \). Since it is quite tedious to even write these down in full generality in the non-strict case, we will give details only in the strict case.

**Definition 7.10.** Let \( V \) be an object in a left and right rigid monoidal category.

1. If \( \mathcal{C} \) is strict monoidal, we define generalized Frobenius-Schur endomorphisms of \( V \) by

\[
FS_V^{(n,\ell,r)} = \begin{array}{c}
\pi \\
V \\
\ell \\
\end{array}
\]

where we have used the following distribution of the tensor factors over the “legs” of the graphical symbols for \( \pi \) and \( \iota \), with \( k = \ell + r + 1 \):

\[
V \otimes (n-k) V \otimes \ell V \otimes r V \otimes \pi \iota, \quad \text{and} \quad V \otimes \ell V \otimes r V \otimes (n-k)
\]

We abbreviate \( FS_V^{(n,k)} = FS_V^{(n,k-1,0)} \) for \( 1 \leq k < n \).

2. In general, define \( FS_V^{(n,\ell,r)} = F^{-1}(FS_{F(V)}^{(n,\ell,r)}) \), where \( F : \mathcal{C} \to \mathcal{D} \) is a monoidal equivalence with a strict monoidal category \( \mathcal{D} \).

**Remark 7.11.** Strictly speaking, the second part of the definition requires the proof that the endomorphisms defined in the first part are invariant under monoidal equivalences between strict monoidal categories. Except for the difficulty to write down the large expressions, this is routine, however. Of course the general definition is then also invariant under monoidal category equivalences. On these grounds, we will only give proofs for the strict case below.

**Proposition 7.12.** Let \( V \) be an object of a \( k \)-linear semisimple pivotal monoidal category. We have \( \nu_{n,k}(V) = \text{ptr}^\ell(FS_V^{(n,\ell,r)}) \) for all \( n, k \).

If \( r > 0 \), we have \( \text{ptr}^\ell(FS_V^{(n,\ell,r)}) = \text{ptr}^r(FS_V^{(n,\ell+1, r-1)}) \).

If \( \mathcal{C} \) is spherical, we have \( \nu_{n,k}(V) = \text{ptr}(FS_V^{(n,\ell,r)}) \) whenever \( k = \ell + r + 1 \).

**Proof.** The proof of the first assertion is analogous to that of Theorem 7.6. The second is obvious from the graphical representation of left and right pivotal traces, and the third is a consequence of the first two.

**Remark 7.13.** If \( V \) is a simple object in \( \mathcal{C} \), then \( \text{ptr}^\ell(\text{id}_V) \neq 0 \neq \text{ptr}^r(\text{id}_V) \). Indeed \( \text{ptr}^\ell(\text{id}_V) \), for example, is the composition of two nonzero morphisms \( I \to V \otimes V^* \) and \( V \otimes V^* \to I \), which is nonzero since the multiplicity of \( I \) in \( V \otimes V^* \) is one; see [7, Lem.1] or the discussion after [4, Prop.2.1].

In particular, an endomorphism of \( V \) is uniquely determined by its trace. Thus the preceding proposition shows that \( FS_V^{(n,\ell,r)} = FS_V^{(n,\ell+r+1)} \) holds for every simple object \( V \) in a \( k \)-linear semisimple spherical monoidal category. Since the next proposition shows that the various Frobenius-Schur endomorphisms are natural, \( FS_V^{(n,\ell,r)} = FS_V^{(n,\ell+r+1)} \) then holds for all objects \( V \).
Proposition 7.14. If \( n \) and \( k = \ell + r + 1 \) are relatively prime, then \( \mathbb{F}_{\tilde{S}_n^{(\ell, r)}} \) is natural in \( V \).

Proof. We use the permutation \( s \in S_n \) determined by requiring \( s(i) \in \{1, \ldots, n\} \) to be congruent to \( i + k \) modulo \( n \). (This occurs for similar reasons in [10] as a special case of more general permutation constructions.) Note that \( s \) is an \( n \)-cycle since \( n \) and \( k \) are relatively prime.

Consider \( f: V \to W \). For any \( X, Y \) in \( C \) we will write

\[
f_p: X_1 \otimes \ldots \otimes X_{p-1} \otimes V \otimes Y_1 \otimes \ldots \otimes Y_u \to X_1 \otimes \ldots \otimes X_{p-1} \otimes W \otimes Y_1 \otimes \ldots \otimes Y_u.
\]

for the morphism that acts as \( f \) in the \( p \)-th position and the identity otherwise. Define a series of objects and morphisms

\[
V[0] \xrightarrow{f[1]} V[1] \xrightarrow{f[2]} V[2] \to \ldots \xrightarrow{f[n]} V[n]
\]

by \( V[0] = V^\otimes n \) and \( f[i] = f_{s^{i-1}(\ell+1)} \); this fixes \( V[i] \), which has to be the appropriate target; note that the sequence is well-defined since \( s \) is transitive; we have \( V[n] = W^\otimes n \). Denote the injection from and projection to the trivial component of \( V[i] \) by \( \iota_i \) and \( \pi_i \).

Now as before we have

\[
G_i =
\]

for any \( i \in \{1, \ldots, n\} \). If \( p := s^{i-1}(\ell + 1) \leq n - k \), then

\[
G_i =
\]
since $p + k = s(p) = s^i(\ell + 1)$. If $n - k < p \leq n - k + \ell$, then for $q := p - n + k$

$$G_i = \begin{array}{c}
\begin{figure}
\end{array}\end{array}$$

since $q = s(p) = s^i(\ell + 1)$. If $p > n - k + \ell + 1 = n - r + 1$, then we set $q = p - n + k - \ell - 1$ and find

$$G_i = \begin{array}{c}
\begin{figure}
\end{array}\end{array}$$

since now $q + \ell + 1 = p + k - n = s(p) = s^i(\ell + 1)$. Because $s$ has order $n$ and $s(n - k + \ell + 1) = \ell + 1$, the case $p = n - k + \ell + 1 = s^{n-1}(\ell + 1)$ occurs when $i = n$. Thus, using induction, we have

$$f \mathbf{FS}_V^{(n,\ell,r)} = \begin{array}{c}
\begin{figure}
\end{array}\end{array}$$

$$= \begin{array}{c}
\begin{figure}
\end{array}\end{array}$$

$$= \begin{array}{c}
\begin{figure}
\end{array}\end{array}$$

$$= \mathbf{FS}_W^{(n,\ell,r)} f$$

\[\square\]

**Remark 7.15.** As our definition of $E^{(n)}_V$, the definition of the various Frobenius-Schur endomorphisms above is not left-right symmetric. In particular, the Frobenius-Schur endomorphism $\mathbf{FS}^{(n)}$
for \(V^\text{sym}\) as an object of \(C^\text{sym}\) is given pictorially by

\[
\begin{array}{c}
V \\
\uparrow \\
\uparrow \\
\downarrow \\
\downarrow \\
V
\end{array}
\]

and similarly for \(\text{FS}^{(n,k)}\). Using Lemma 5.2 we see that (some of) these endomorphisms could equally well be used to compute the indicators. Moreover, as in Remark 7.13, for the case of a \(k\)-linear semisimple spherical monoidal category the endomorphisms agree up to relabelling, e.g. the one instance for which we have given a picture agrees with \(\text{FS}_n^{(n,n-1)}\).

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