Super Yang-Mills Theories on the Two-Dimensional Lattice with Exact Supersymmetry

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Abstract: We construct super Yang-Mills theories with \( \mathcal{N} = 2,4 \) supersymmetries on the two-dimensional square lattice keeping one or two supercharges exactly. Along the same line as the previous paper [1], the construction is based on topological field theory formulation. In order to resolve the problem of degenerate classical vacua encountered in the previous paper, we consider two kinds of modifications of the action. For one of them, any supersymmetry breaking terms do not need to be introduced, and the formulations exactly realize some of supersymmetries at the lattice level. Our lattice actions flow to the desired continuum theories without any fine-tuning of parameters.

Keywords: Lattice Quantum Field Theory, Lattice Gauge Field Theories, Extended Supersymmetry, Topological Field Theories.
1. Introduction

Lattice formulations of supersymmetric theories have a long history and still have continued to be vigorously investigated [2, 3, 4, 5, 6, 7, 8, 9]. Recently, for super Yang-Mills (SYM) theories, an interesting approach motivated from the idea of deconstruction was presented [11, 12]. Also, various theories without gauge symmetry were discussed based on the connection to topological field theory [15].

In the previous paper [1], we constructed SYM theories with extended supersymmetry on the hypercubic lattices of various dimensions. The construction is based on topological field theory formulation [16, 17], and keeps one or two supercharges manifestly. The gauge fields are expressed as ordinary compact unitary variables on the lattice links. The lattice actions have a number of the classical vacua with the degeneracy growing as exponential of the number of plaquettes, which makes unclear their connection to the perturbative regime of the SYM theories. In order to resolve the degeneracy and to single out the vacuum corresponding to the target theory, we added to the action some supersymmetry breaking term which is tuned to vanish in the continuum limit.

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1For a recent review, see [10].
2For related works, see [13, 14].
In this paper, the same kind of lattice models are considered for $\mathcal{N} = 2, 4$ supersymmetries in two-dimensions. We consider to make suitable modifications to the action so that the vacuum degeneracy is completely resolved \textit{with keeping the exact supersymmetry}. Two such modifications are presented here. One is a modification to impose the so-called admissibility conditions on plaquette variables similar to those for gauge fields coupled to the Ginsparg-Wilson fermions [18]. It will be applicable to more general cases not restricted to the SU($N$) gauge group. It maintains the exact supersymmetry possessed by the original models in [1], and we do not have to add any supersymmetry breaking terms. The supersymmetry is exactly realized at the lattice level in the SYM theories. The other is a simple modification of the function $\Phi(x)$ in the lattice actions for the gauge group SU($N$) as $\Phi(x) \rightarrow \Phi(x) + \Delta \Phi(x)$ with the trace parts of some adjoint fields introduced. The actions are supersymmetric, but have would-be zero-modes inducing nontrivial constraints among the fields. Since such constraints do not appear in the target continuum theories, we should soak up the would-be zero-modes to avoid obtaining them. As a result of the insertion of the would-be zero-modes, the supersymmetry is violated. The supersymmetry breaking effect is much more irrelevant in the continuum limit compared to the case in the previous paper [1].

This paper is organized as follows. In section 2, we briefly review the lattice actions constructed in [1] for the $\mathcal{N} = 2, 4$ theories in two-dimensions. For the case of the SU($N$) gauge group, we explain the structure of the degenerate minima, emphasizing the difference from the U($N$) case. In section 3, we impose the admissibility conditions on plaquette variables, where the exact supersymmetry is preserved by doing a similar modification to the action as in ref. [19]. In section 4, we discuss on the modification $\Phi(x) \rightarrow \Phi(x) + \Delta \Phi(x)$ to resolve the problem of the degenerate vacua. Section 5 is devoted to the summary and discussions.

Throughout this paper, we consider the gauge groups $G = $ SU($N$), U($N$) and the two-dimensional square lattice. Gauge fields are expressed as compact unitary variables

$$U_\mu(x) = e^{iaA_\mu(x)}$$

(1.1)
on the link $(x, x + \hat{\mu})$. ‘$a$’ stands for the lattice spacing, and $x \in \mathbb{Z}^2$ the lattice site. All other fields are put on the sites and expanded by a basis of $N \times N$ (traceless) hermitian matrices $T^a$.

2. Brief Review of Lattice Actions

We briefly review the lattice actions of two-dimensional $\mathcal{N} = 2, 4$ SYM theories discussed in [1].

2.1 $\mathcal{N} = 2$ Case

Other than the gauge variables $U_\mu(x)$, the $\mathcal{N} = 2$ theory has complex scalars $\phi(x)$, $\bar{\phi}(x)$, and fermions are denoted as $\psi_\mu(x), \chi(x), \eta(x)$ [16]. They are transformed under the exact
supersymmetry $Q$ as

$$QU_\mu(x) = i\psi_\mu(x)U_\mu(x),$$
$$Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) - i\left(\phi(x) - U_\mu(x)\phi(x) + \bar{\phi}(x)U_\mu(x)^\dagger\right),$$
$$Q\phi(x) = 0,$$
$$Q\chi(x) = H(x), \quad QH(x) = [\phi(x), \chi(x)],$$
$$Q\bar{\phi}(x) = \eta(x), \quad Q\eta(x) = [\phi(x), \bar{\phi}(x)], \quad (2.1)$$

where $H(x)$ is an auxiliary fields. $Q$ is nilpotent up to an infinitesimal gauge transformation with the parameter $\phi(x)$. In the expansion $H(x) = \sum_a H^a(x)T^a$, coefficients $H^a(x)$ are real. $\phi^a(x), \bar{\phi}^a(x)$ are complex, and the fermionic variables $\psi^a_\mu(x), \chi^a(x), \eta^a(x)$ may be regarded as complexified Grassmann\(^3\) to be compatible to the $U(1)_R$ rotations (2.5). Notice that in the path integral $\phi^a(x)$ and $\bar{\phi}^a(x)$ can be treated as independent variables and that each of $H^a(x)$ is allowed to be shifted by a complex number. Thus, (2.1) is consistently closed in the path integral expression of the theory.

The lattice action is

$$S_{N=2}^{\text{LAT}} = Q\frac{1}{2g_0^2} \sum_x \text{tr} \left[ \frac{1}{4}\eta(x) [\phi(x), \bar{\phi}(x)] - i\chi(x)\Phi(x) + \chi(x)H(x) \right. \left. + i \sum_{\mu=1}^2 \psi_\mu(x) \left(\bar{\phi}(x) - U_\mu(x)\phi(x) + \bar{\phi}(x)U_\mu(x)^\dagger\right) \right], \quad (2.2)$$

where

$$\Phi(x) = -i \left[U_{12}(x) - U_{21}(x)\right], \quad (2.3)$$

$U_{\mu\nu}(x)$ are plaquette variables written as

$$U_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x + \hat{\mu})U_{\mu}(x + \hat{\nu})^\dagger U_\nu(x)^\dagger. \quad (2.4)$$

The action (2.2) is clearly $Q$-invariant from its $Q$-exact form. Also, the invariance under the following global $U(1)_R$ rotation holds:

$$U_\mu(x) \rightarrow U_\mu(x), \quad \psi_\mu(x) \rightarrow e^{i\alpha}\psi_\mu(x),$$
$$\phi(x) \rightarrow e^{2i\alpha}\phi(x),$$
$$\chi(x) \rightarrow e^{-i\alpha}\chi(x), \quad H(x) \rightarrow H(x),$$
$$\bar{\phi}(x) \rightarrow e^{-2i\alpha}\bar{\phi}(x), \quad \eta(x) \rightarrow e^{-i\alpha}\eta(x). \quad (2.5)$$

The $U(1)_R$ charge of each variable is read off from (2.5), and $Q$ increases the charge by one.

After acting $Q$ in the RHS of (2.2), the action is expressed as

$$S_{N=2}^{\text{LAT}} = \frac{1}{2g_0^2} \sum_x \text{tr} \left[ \frac{1}{4}[\phi(x), \bar{\phi}(x)]^2 + H(x)^2 - iH(x)\Phi(x) \right]$$

\(^3\)A complexified Grassmann number takes the form: (complex number) $\times$ (real Grassmann number).
\[ + \sum_{\mu=1}^{2} \left( \phi(x) - U_\mu(x)\phi(x + \hat{\mu})U_\mu(x)^\dagger \right) \left( \tilde{\phi}(x) - U_\mu(x)\tilde{\phi}(x + \hat{\mu})U_\mu(x)^\dagger \right) \]

\[ - \frac{1}{4} \eta(x) \left[ \phi(x), \eta(x) \right] - \chi(x) \left[ \phi(x), \chi(x) \right] \]

\[ - \sum_{\mu=1}^{2} \psi_\mu(x)\psi_\mu(x) \left( \tilde{\phi}(x) + U_\mu(x)\tilde{\phi}(x + \hat{\mu})U_\mu(x)^\dagger \right) \]

\[ + i\chi(x)Q\Phi(x) - i \sum_{\mu=1}^{2} \psi_\mu(x) \left( \eta(x) - U_\mu(x)\eta(x + \hat{\mu})U_\mu(x)^\dagger \right) \right] . \] (2.6)

Integration of \( H(x) \) induces \( \Phi(x)^2 \) term, which yields the gauge kinetic term as the form

\[ \frac{1}{8g_0^2} \sum_x \text{tr} \left[ -(U_{12}(x) - U_{21}(x))^2 \right] , \] (2.7)

which leads a number of the classical vacua

\[ U_{12}(x) = \begin{pmatrix} \pm 1 \\ \cdot \cdot \cdot \\ \pm 1 \end{pmatrix} \] (2.8)

up to gauge transformations for \( G = U(N) \), where any combinations of \( \pm 1 \) are allowed in the diagonal entries. Since an arbitrary configuration of (2.8) can be taken for each plaquette, it leads the degeneracy growing as exponential of the number of plaquettes. In order to investigate the dynamics of the model, we need to sum up contributions from all of the minima, and the ordinary weak field expansion around a single vacuum \( U_{12}(x) = 1 \) can not be justified.

**Degenerate Minima for \( G = SU(N) \)** For the case \( G = SU(N) \), we have more complicated structures of the degenerate minima. Notice that because the adjoint fields are traceless, the trace part of \( \Phi(x) \) does not appear in the expression (2.6)\(^4\). The minima are given by the configurations satisfying

\[ \Phi(x) - \left( \frac{1}{N} \text{tr} \Phi(x) \right) 1_N = 0 \] (2.9)

instead of \( \Phi(x) = 0 \). The solutions of (2.9) are classified to the following three types:

1. the same form as in the case \( G = U(N) \)

\[ U_{12}(x) = \begin{pmatrix} \pm 1 \\ \cdot \cdot \cdot \\ \pm 1 \end{pmatrix} \] (2.10)

with ‘\(-1\)’ appearing even times in the diagonal entries

\(^4\)We thank Y. Kikukawa for pointing out this issue.
2. the SU(N) center

\[ U_{12}(x) = e^{i \frac{2\pi}{N} k(x)} 1_N \in Z_N \quad (k(x) = 0, \cdots, N - 1) \quad (2.11) \]

3. the solution appearing only when \( N = 4n \) (\( n = 1, 2, \cdots \))

\[
U_{12}(x) = \begin{pmatrix}
    e^{i\alpha(x)} & \cdots & e^{i\alpha(x)} \\
    \cdots & \cdots & \cdots \\
    e^{-i\alpha(x)} & \cdots & e^{-i\alpha(x)} \\
    \cdots & \cdots & \cdots \\
    -e^{-i\alpha(x)} & \cdots & -e^{-i\alpha(x)} \\
    \cdots & \cdots & \cdots 
\end{pmatrix}
(2.12)

up to gauge transformations. In the diagonal entries, both of \( e^{i\alpha(x)} \) and \( -e^{-i\alpha(x)} \) appear \( 2n \) times. \( \alpha(x) \) is an arbitrary phase.

Again, due to the huge number of classical minima, the ordinary weak field expansion around a single vacuum \( U_{12}(x) = 1 \) can not be justified either in the SU(N) case.

2.2 \( \mathcal{N} = 4 \) Case

In the \( \mathcal{N} = 4 \) theory, scalar fields \( B(x), C(x) \) appear as well as \( \phi(x) \) and \( \tilde{\phi}(x) \). Auxiliary fields \( \tilde{H}_\mu(x), H(x) \) are introduced, and fermions are \( \psi_\pm(x), \chi_\pm(x), \eta_\pm(x) \) [17]. The latticization keeps two supercharges \( Q_\pm \), which transform the fields as

\[
Q_+ U_\mu(x) = i \psi_\mu(x) U_\mu(x),
\]
\[
Q_- U_\mu(x) = i \psi_- \mu(x) U_\mu(x),
\]
\[
Q_+ \psi_\mu(x) = i \psi_\mu(x) \psi_\mu(x) - i \left( \phi(x) - U_\mu(x) \phi(x + \hat{\mu}) U_\mu(x) \right),
\]
\[
Q_- \psi_\mu(x) = i \psi_- \mu(x) \psi_- \mu(x) + i \left( \phi(x) - U_\mu(x) \phi(x + \hat{\mu}) U_\mu(x) \right),
\]
\[
Q_+ \psi_\mu(x) = \frac{i}{2} \left\{ \psi_\mu(x), \psi_\mu(x) \right\} - \frac{i}{2} \left( C(x) - U_\mu(x) C(x + \hat{\mu}) U_\mu(x) \right) - \tilde{H}_\mu(x),
\]
\[
Q_- \psi_\mu(x) = \frac{i}{2} \left\{ \psi_\mu(x), \psi_\mu(x) \right\} - \frac{i}{2} \left( C(x) - U_\mu(x) C(x + \hat{\mu}) U_\mu(x) \right) + \tilde{H}_\mu(x),
\]
\[
Q_+ \tilde{H}_\mu(x) = -\frac{1}{2} \left[ \psi_- \mu(x), \phi(x) + U_\mu(x) \phi(x + \hat{\mu}) U_\mu(x) \right] + \frac{1}{4} \left[ \psi_\mu(x), C(x) + U_\mu(x) C(x + \hat{\mu}) U_\mu(x) \right] + \frac{i}{2} \left( \eta_\mu(x) - U_\mu(x) \eta_\mu(x + \hat{\mu}) U_\mu(x) \right) + \frac{i}{2} \left[ \psi_\mu(x), \tilde{H}_\mu(x) \right] + \frac{1}{4} \left[ \psi_\mu(x) \psi_\mu(x), \psi_- \mu(x) \right],
\]
\[
Q_- \tilde{H}_\mu(x) = -\frac{1}{2} \left[ \psi_\mu(x), \phi(x) + U_\mu(x) \phi(x + \hat{\mu}) U_\mu(x) \right]
\]

\(^{5}B^a(x), C^a(x), \tilde{H}^a(\mu), H^a(x) \) are real, and \( \psi_\pm^a(x), \chi_\pm^a(x), \eta_\pm^a(x) \) may be regarded as complexified Grassmann.
The transformation leads the following nilpotency of $Q$ whose generators are expressed as

\[
\Phi(x) = \bar{\mu} + \mu + \frac{1}{2}[\psi_{-\mu}(x), C(x) + U_\mu(x)C(x) + \bar{\mu}U_\mu(x)^\dagger]
\]

\[
-\frac{i}{2} \left( \eta_-(x) - U_\mu(x)\eta_-(x + \bar{\mu})U_\mu(x)^\dagger \right)
\]

\[
+i \left[ \psi_{-\mu}(x), H_\mu(x) \right] - \frac{1}{4}[\psi_{-\mu}(x)\psi_{-\mu}(x), \psi_{+\mu}(x)],
\]  

(2.13)

\[
Q_+B(x) = \chi_+(x), \quad Q_+\chi_+(x) = [\phi(x), B(x)], \quad Q_-\chi_+(x) = \frac{1}{2}[C(x), B(x)] - H(x),
\]

\[
Q_-B(x) = \chi_-(x), \quad Q_-\chi_-(x) = -[\bar{\phi}(x), B(x)], \quad Q_+\chi_-(x) = \frac{1}{2}[C(x), B(x)] + H(x),
\]

\[
Q_+H(x) = [\phi(x), \chi_-(x)] + \frac{1}{2}[B(x), \eta_+(x)] - \frac{1}{2}[C(x), \chi_+(x)],
\]

\[
Q_-H(x) = [\bar{\phi}(x), \chi_+(x)] - \frac{1}{2}[B(x), \eta_-(x)] + \frac{1}{2}[C(x), \chi_-(x)],
\]  

(2.14)

\[
Q_+C(x) = \eta_+(x), \quad Q_+\eta_+(x) = [\phi(x), C(x)], \quad Q_-\eta_+(x) = -[\phi(x), \bar{\phi}(x)],
\]

\[
Q_-C(x) = \eta_-(x), \quad Q_-\eta_-(x) = -[\bar{\phi}(x), C(x)], \quad Q_+\eta_-(x) = [\phi(x), \bar{\phi}(x)],
\]

\[
Q_+\phi(x) = 0, \quad Q_-\phi(x) = -\eta_+(x), \quad Q_+\bar{\phi}(x) = \eta_-(x), \quad Q_-\bar{\phi}(x) = 0.
\]  

(2.15)

The transformation leads the following nilpotency of $Q_\pm$ (up to gauge transformations):

\[
Q_+^2 = (\text{infinitesimal gauge transformation with the parameter } \phi),
\]

\[
Q_-^2 = (\text{infinitesimal gauge transformation with the parameter } -\bar{\phi}),
\]

\[
\{Q_+, Q_-\} = (\text{infinitesimal gauge transformation with the parameter } C).
\]  

(2.16)

We constructed the lattice action with exact $Q_\pm$ symmetry as

\[
S_{N=4}^{\text{LAT}} = Q_+Q_-= \frac{1}{2g_0^2} \sum_x \text{tr} \left[ -iB(x)\Phi(x) - \sum_{\mu=1}^2 \psi_{+\mu}(x)\psi_{-\mu}(x) - \chi_+(x)\chi_-(x) \right.
\]

\[
\left. - \frac{1}{4}\eta_+(x)\eta_-(x) \right],
\]  

(2.17)

where $\Phi(x)$ is given by (2.3). The action is invariant under the SU(2)$_R$ transformation, whose generators are expressed as

\[
J_{++} = \sum_{x,a} \left[ \sum_\mu \psi_{+\mu}^a(x) \frac{\partial}{\partial \psi_{+\mu}^a(x)} + \chi_+^a(x) \frac{\partial}{\partial \chi_+^a(x)} - \eta_+^a(x) \frac{\partial}{\partial \eta_+^a(x)} + 2\phi^a(x) \frac{\partial}{\partial C^a(x)} \right.
\]

\[
\left. - C^a(x) \frac{\partial}{\partial \phi^a(x)} \right],
\]

\[
J_{--} = \sum_{x,a} \left[ \sum_\mu \psi_{-\mu}^a(x) \frac{\partial}{\partial \psi_{-\mu}^a(x)} + \chi_-^a(x) \frac{\partial}{\partial \chi_-^a(x)} - \eta_-^a(x) \frac{\partial}{\partial \eta_-^a(x)} - 2\bar{\phi}^a(x) \frac{\partial}{\partial C^a(x)} \right.
\]

\[
\left. + C^a(x) \frac{\partial}{\partial \bar{\phi}^a(x)} \right],
\]

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\[ J_0 = \sum_{x,L} \left[ \sum_{\mu} \psi_{+\mu}^a(x) \frac{\partial}{\partial \psi_{+\mu}^a(x)} - \sum_{\mu} \psi_{-\mu}^a(x) \frac{\partial}{\partial \psi_{-\mu}^a(x)} + \chi_+^a(x) \frac{\partial}{\partial \chi_+^a(x)} - \chi_-^a(x) \frac{\partial}{\partial \chi_-^a(x)} \right. \]
\[ + \eta_+^a(x) \frac{\partial}{\partial \eta_+^a(x)} - \eta_-^a(x) \frac{\partial}{\partial \eta_-^a(x)} + 2\phi^a(x) \frac{\partial}{\partial \phi^a(x)} - 2\tilde{\phi}^a(x) \frac{\partial}{\partial \tilde{\phi}^a(x)} \right], \tag{2.18} \]

with the script \( a \) being the index of a basis of the gauge group generators. They form the SU(2) algebra:

\[ [J_0, J_{++}] = 2J_{++}, \quad [J_0, J_{--}] = -2J_{--}, \quad [J_{++}, J_{--}] = J_0. \tag{2.19} \]

Under the SU(2)\(_R\) gauge group, each of \((\psi_{+\mu}^a, \psi_{-\mu}^a), (\chi_+^a, \chi_-^a), (\eta_+^a, -\eta_-^a)\) and \((Q_+, Q_-)\) transforms as a doublet, and \((\phi^a, C^a, -\tilde{\phi}^a)\) as a triplet. We can easily see the SU(2)\(_R\) invariance of the action (2.17), because \(Q_+Q_-\), \(\text{tr}(\psi_{+\mu}(x)\psi_{-\mu}(x))\), \(\text{tr}(\chi_+(x)\chi_-(x))\) and \(\text{tr}(\eta_+(x)\eta_-(x))\) are SU(2)\(_R\) singlets. Also, the action has the symmetry of exchanging the two supercharges \(Q_+ \leftrightarrow Q_-\) with

\[
\begin{align*}
\phi &\to -\phi, \quad \tilde{\phi} \to -\phi, \quad B \to -B, \\
\chi_+ &\to -\chi_-, \quad \chi_- \to -\chi_+, \quad \tilde{H}_\mu \to -\tilde{H}_\mu, \\
\psi_{\pm \mu} &\to \psi_{\mp \mu}, \quad \eta_\pm \to \eta_\mp. \tag{2.20}
\end{align*}
\]

After acting \(Q_\pm\), the action is more explicitly written as

\[
S_{N=1}^{L\Lambda T} = \frac{1}{2g_0^2} \sum_x \text{tr} \left[ -i \left( \frac{1}{2}[C(x), B(x)] + H(x) \right) \Phi(x) + H(x)^2 \\
+ i\chi_-(x)Q_+\Phi(x) - i\chi_+(x)Q_-\Phi(x) - iB(x)Q_+Q_-\Phi(x) \\
- [\phi(x), B(x)][\tilde{\phi}(x), B(x)] - \frac{1}{4}[C(x), B(x)]^2 \\
+ \chi_+(x)[\tilde{\phi}(x), \chi_+(x)] - \chi_-(x)[\phi(x), \chi_-(x)] + \chi_-(x)[C(x), \chi_+(x)] \\
- \chi_+(x)[B(x), \eta_+(x)] - \chi_-(x)[B(x), \eta_-(x)] \\
- \frac{1}{4}[\phi(x), \tilde{\phi}(x)]^2 - \frac{1}{4}[\phi(x), C(x)][\tilde{\phi}(x), C(x)] \\
- \frac{1}{4}\eta_-(x)[\phi(x), \eta_-(x)] + \frac{1}{4}\eta_+(x)[\tilde{\phi}(x), \eta_+(x)] - \frac{1}{4}\eta_+(x)[C(x), \eta_-(x)] \right] \\
+ \frac{1}{2g_0^2} \sum_x \sum_{\mu=1}^2 \text{tr} \left[ \tilde{H}_\mu(x)^2 - \frac{1}{2}\psi_{+\mu}(x)\psi_{+\mu}(x)\psi_{-\mu}(x)\psi_{-\mu}(x) \\
+ \left( \phi(x) - U_\mu(x)\phi(x + \hat{\mu})U_\mu(x)^\dagger \right) \left( \tilde{\phi}(x) - U_\mu(x)\tilde{\phi}(x + \hat{\mu})U_\mu(x)^\dagger \right) \right. \right. \\
+ \left. \frac{1}{4} \left( C(x) - U_\mu(x)C(x + \hat{\mu})U_\mu(x)^\dagger \right)^2 \\
- \psi_{+\mu}(x)\psi_{+\mu}(x) \left( \tilde{\phi}(x) + U_\mu(x)\tilde{\phi}(x + \hat{\mu})U_\mu(x)^\dagger \right) \\
+ \psi_{-\mu}(x)\psi_{-\mu}(x) \left( \phi(x) + U_\mu(x)\phi(x + \hat{\mu})U_\mu(x)^\dagger \right) \\
- i\psi_{+\mu}(x) \left( \eta_-(x) - U_\mu(x)\eta_-(x + \hat{\mu})U_\mu(x)^\dagger \right) \\
- i\psi_{-\mu}(x) \left( \eta_+(x) - U_\mu(x)\eta_+(x + \hat{\mu})U_\mu(x)^\dagger \right) \right].
\]
\[-\frac{1}{2} \{ \psi_\mu(x), \psi_{-\mu}(x) \} \left( C(x) + U_\mu(x)C(x + \hat{\mu})U_\mu(x)^\dagger \right) \] . \quad (2.21)

Similarly to the $\mathcal{N} = 2$ case, the gauge kinetic term, induced after integrating $H(x)$ out, suffers from the problem of degenerate classical vacua.

In the previous paper [1], for both cases of $\mathcal{N} = 2, 4$ theories in two-dimensional $M \times M$ periodic lattice, we added the term

\[ \frac{1}{2g_0^2} \rho \sum_x \text{tr} \left( 2 - U_{12}(x) - U_{21}(x) \right) \] \quad (2.22)

to the action with $\rho = \frac{1}{M^2} \ (0 < s < 2)$ to resolve the vacuum degeneracy. This term breaks the exact supersymmetries $Q$ and $Q_{\pm}$ respectively, but it is tuned to vanish in the continuum limit. In what follows, we will consider to resolve the degeneracy keeping the exact supersymmetries without introducing the breaking term (2.22).

3. Admissibility Conditions

In order to single out the vacuum $U_{12}(x) = 1$ from the degeneracy, we impose the so-called admissibility condition

\[ ||1 - U_{12}(x)|| < \epsilon \] \quad (3.1)

on each plaquette variable. For definiteness, we use the following definition of the norm of a matrix $A$:

\[ ||A|| \equiv \left[ \text{tr} \left( AA^\dagger \right) \right]^{1/2}, \] \quad (3.2)

and then $\epsilon$ is a positive number chosen as $0 < \epsilon < 2$ for $G = U(N)$. Also, for $G = SU(N)$ we choose

\begin{align*}
0 < \epsilon &< 2\sqrt{2} & (N = 2, 3, 4) \quad (3.3) \\
0 < \epsilon &< 2\sqrt{N} \sin \left( \frac{\pi}{N} \right) & (N \geq 5), \quad (3.4)
\end{align*}

so that excluded are the minima (2.10, 2.11, 2.12) other than $U_{12}(x) = 1$. The same kind of condition was introduced for gauge fields coupled to the Ginsparg-Wilson fermions [18]. Notice that (3.1) is a gauge invariant statement and that $||1 - U_{12}(x)||^2$ is proportional to the standard Wilson action:

\[ ||1 - U_{12}(x)||^2 = \text{tr} \left[ 2 - U_{12}(x) - U_{21}(x) \right]. \] \quad (3.5)

3.1 $\mathcal{N} = 2$ Case

We modify the action of the $\mathcal{N} = 2$ theory as follows:

When $||1 - U_{12}(x)|| < \epsilon$ for $\forall x$,

\[ S_{N=2}^{\text{LAT}} = Q \frac{1}{2g_0^2} \sum_x \text{tr} \left[ \frac{1}{4} \eta(x) \left[ \phi(x), \tilde{\phi}(x) \right] - i \chi(x) \tilde{\Phi}(x) + \chi(x) H(x) \right. \]

\[ + \left. i \sum_{\mu=1}^2 \psi_\mu(x) \left( \tilde{\phi}(x) - U_\mu(x) \tilde{\phi}(x + \hat{\mu}) U_\mu(x)^\dagger \right) \right], \] \quad (3.6)
and otherwise
\[ \hat{S}^{\text{LAT}}_{N=2} = +\infty. \] (3.7)

Here
\[ \hat{\Phi}(x) = \frac{\Phi(x)}{1 - \frac{1}{t^2} ||1 - U_{12}(x)||^2}. \] (3.8)

The form of the action is somewhat similar to that of U(1) gauge theory constructed by Lüscher [19]. Note that the Boltzmann weight \( \exp \left[ -\hat{S}^{\text{LAT}}_{N=2} \right] \) is a product of local factors, which guarantees the locality of the theory. Also, it is easily seen that the Boltzmann weight is smooth and differentiable with respect to lattice variables for the region except the boundary
\[ ||1 - U_{12}(x)|| = \epsilon. \] (3.9)

Let us look closer at the smoothness on the boundary. For the partition function
\[ Z = \int [d \text{(fields)}] \exp \left[ -\hat{S}^{\text{LAT}}_{N=2} \right], \] (3.10)

after the integration over \( H(x) \) and fermions, the relevant parts of the Boltzmann weight are evaluated as
\[ \sum_{\{c(x)\}} d(\{c(x)\}) \prod_x B(c(x)), \] (3.11)
\[ B(c(x)) \equiv \left( \frac{1}{1 - \frac{1}{t^2} ||1 - U_{12}(x)||^2} \right)^{c(x)} \exp \left[ -\frac{1}{8g_0^2} \frac{\text{tr} (2 - U_{12}(x)^2 - U_{21}(x)^2)}{\left( 1 - \frac{1}{t^2} ||1 - U_{12}(x)||^2 \right)^2} \right] \] (3.12)

near the boundary. Here, \( c(x) \) takes \( N^2 - 1 \) or \( 2(N^2 - 1) \) for each \( x \), and \( d(\{c(x)\}) \) are irrelevant factors. It is smooth and differentiable on the boundary, which is essentially same as the fact that the function
\[ f(t) = \begin{cases} \frac{1}{t^n} e^{-c/t^2} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases} \] (3.13)

with \( c \) positive constant is smooth and differentiable at \( t = 0 \) for \( n = 0, 1, 2, \cdots \). Similarly, for the unnormalized correlation function among the finite number of operators \( O_1, \cdots, O_k \):
\[ \int [d \text{(fields)}] O_1 \cdots O_k \exp \left[ -\hat{S}^{\text{LAT}}_{N=2} \right], \] (3.14)

the Boltzmann weight is smooth and differentiable on the boundary (3.9), as long as the operators contain the finite number of \( H(x) \) for each \( x \). (Compared to the case of the partition function, all the essential difference is that some of the powers \( c(x) \) in (3.12) are shifted by finite amount, which does not affect the smoothness.) It leads the \( Q \) invariance of the Boltzmann weight as the following form:
\[ \int [d \text{(fields)}] O_1 \cdots O_k Q \left( \exp \left[ -\hat{S}^{\text{LAT}}_{N=2} \right] \right) = 0. \] (3.15)
If we simply imposed (3.1) just by putting the step functions
\[ \prod_x \theta \left( \epsilon^2 - \|1 - U_{12}(x)\|^2 \right) \] (3.16)
in front of the Boltzmann weight \( \exp \left[ -S_{\text{LAT}}^{\mathcal{N}=2} \right] \) with the original action (2.2), the supersymmetry \( Q \) would be broken due to the contribution from the boundary (3.9). The modification of the action (3.6, 3.7) however makes the breaking effect completely suppressed, and maintains the supersymmetry.

**No Fermion Doublers and Renormalization** We may expand the exponential of the link variable (1.1), and the action (3.6, 3.7) leads to the \( \mathcal{N} = 2 \) SYM theory in the classical continuum limit:
\[ a \to 0 \quad \text{with} \quad g_2^2 \equiv g_0^2/a^2 \quad \text{fixed.} \] (3.17)
Note that \( \epsilon \) is independent of the lattice spacing \( a \). Also, the modification to the fermionic part of the action reads
\[ \text{tr} \left[ i\chi(x) Q\Phi(x) \right] = \frac{1}{1 - \frac{1}{\epsilon^2} ||1 - U_{12}(x)\|^2} \text{tr} \left[ i\chi(x) Q\Phi(x) \right] \]
\[ - \frac{1}{(1 - \frac{1}{\epsilon^2} ||1 - U_{12}(x)\|^2)^2} \frac{1}{\epsilon^2} \text{tr} \left[ QU_{12}(x) + QU_{21}(x) \right], \] (3.18)
where the second term contributes to gauge-fermion interaction terms of the irrelevant order \( O(a^8) \) but not to fermion kinetic terms. (Notice that fermionic variables are rescaled as \( \text{fermions} \to a^{3/2} \text{fermions} \) when taking the continuum limit.) Thus, the modification does not affect the fermion kinetic terms, and the absence of fermion doubling is shown as in the previous paper [1]. Indeed, the fermion kinetic terms are expressed as
\[ S_f^{(2)} = \frac{a^4}{2g_0^2} \sum_{x,\mu} \text{tr} \left[ -\frac{1}{2} \Psi(x)^T \gamma_\mu (\Delta_\mu + \Delta^*_\mu) \Psi(x) - a\frac{1}{2} \Psi(x)^T P_\mu \Delta_\mu \Delta^*_\mu \Psi(x) \right], \] (3.19)
where fermions were combined as \( \Psi^T = (\psi_1, \psi_2, \chi, \frac{1}{2} \eta) \). The \( \gamma \)-matrices and \( P_\mu \) are given by
\[ \gamma_1 = -i \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \gamma_2 = i \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad P_2 = -i \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \] (3.20)
with \( \sigma_i \ (i = 1, 2, 3) \) being Pauli matrices. Note that they all anticommute each other:
\[ \{ \gamma_\mu, \gamma_\nu \} = -2\delta_{\mu\nu}, \quad \{ P_\mu, P_\nu \} = 2\delta_{\mu\nu}, \quad \{ \gamma_\mu, P_\nu \} = 0. \] (3.21)
\( \Delta_\mu \) and \( \Delta^*_\mu \) represent forward and backward difference operators respectively:
\[ \Delta_\mu f(x) \equiv \frac{1}{a} (f(x + \hat{\mu}) - f(x)), \quad \Delta^*_\mu f(x) \equiv \frac{1}{a} (f(x) - f(x - \hat{\mu})). \] (3.22)
The kernel of the kinetic terms (3.19) is written in the momentum space $-\frac{\pi}{a} \leq q_\mu < \frac{\pi}{a}$ as

$$D = \sum_{\mu=1}^{2} \left[ -i\gamma_\mu \frac{1}{a} \sin (q_\mu a) + 2P_\mu \frac{1}{a} \sin^2 \left( \frac{q_\mu a}{2} \right) \right].$$  \hspace{1cm} (3.23)

It is easy to see that the kernel $D$ vanishes only at the origin $q_1 = q_2 = 0$, because using (3.21) we get

$$D^2 = \frac{1}{a^2} \sum_{\mu=1}^{2} \left[ \sin^2 (q_\mu a) + 4 \sin^4 \left( \frac{q_\mu a}{2} \right) \right].$$  \hspace{1cm} (3.24)

Thus, the fermion kinetic terms contain no fermion doublers.

For the renormalization, we can repeat the argument in [1] without introducing the supersymmetry breaking term (2.22). Note that the $U(1)_R$ symmetry is kept intact under the modification. For the $G = SU(N)$, the gauge symmetry and the $U(1)_R$ invariance allow the operator $\text{tr} \phi \bar{\phi}$, while it is forbidden by the supersymmetry $Q$. For the $U(N)$ case, in addition, we should take into account the operator $\text{tr} H$. However, it is prohibited by the reflection symmetry: $x \equiv (x_1, x_2) \rightarrow \bar{x} \equiv (x_2, x_1)$ with

$$(U_1(x), U_2(x)) \rightarrow (U_2(\bar{x}), U_1(\bar{x}))$$

$$(\psi_1(x), \psi_2(x)) \rightarrow (\psi_2(\bar{x}), \psi_1(\bar{x}))$$

$$(H(x), \chi(x)) \rightarrow (-H(\bar{x}), -\chi(\bar{x}))$$

$$(\phi(x), \bar{\phi}(x), \eta(x)) \rightarrow (\phi(\bar{x}), \bar{\phi}(\bar{x}), \eta(\bar{x})).$$  \hspace{1cm} (3.25)

Hence, radiative corrections do not generate relevant or marginal operators except the identity. Our modified lattice action is shown to flow to the desired continuum theory without any fine-tuning.

3.2 $\mathcal{N} = 4$ Case

For the case of $\mathcal{N} = 4$ theory, similar modification is possible:

When $||1 - U_{12}(x)|| < \epsilon$ for $\forall x$,

$$\hat{S}_{\mathcal{N} = 4}^{\text{LAT}} = Q_+ Q - \frac{1}{2g_0^2} \sum_x \text{tr} \left[ -iB(x) \hat{\Phi}(x) - \sum_{\mu=1}^{2} \psi_\mu(x) \psi_{-\mu}(x) - \chi_+(x) \chi_-(x) \
- \frac{1}{4} \eta_+(x) \eta_-(x) \right],$$  \hspace{1cm} (3.26)

and otherwise

$$\hat{S}_{\mathcal{N} = 4}^{\text{LAT}} = +\infty.$$  \hspace{1cm} (3.27)

$\hat{\Phi}(x)$ is same as (3.8). Similarly to the $\mathcal{N} = 2$ case, the locality of the theory is satisfied, and the Boltzmann weight $\exp \left[ -\hat{S}_{\mathcal{N} = 4}^{\text{LAT}} \right]$ is smooth and differentiable. It leads the $Q_\pm$ invariance of the Boltzmann weight in the same sense as (3.15). Also, the $SU(2)_R$ and $Q_+ \leftrightarrow Q_-$ symmetries (2.18, 2.20) are not influenced by the modification.
Arguments about the classical continuum limit and the absence of fermion doubling go parallel as in the $\mathcal{N} = 2$ case. The fermion kinetic terms are represented in the same form as (3.19) with $\Psi^T = (\psi_{+1}, \psi_{+2}, \chi_{++}, \frac{1}{2} \eta_{++}, \psi_{-1}, \psi_{-2}, \chi_{--}, \frac{1}{2} \eta_{--})$ and

$$
\gamma_1 = -i \begin{pmatrix}
\sigma_1 \\
-i \sigma_2 \\
\sigma_1
\end{pmatrix}, \quad \gamma_2 = -i \begin{pmatrix}
1_2 \\
-\sigma_3 \\
1_2
\end{pmatrix},
$$

$$
P_1 = -i \begin{pmatrix}
\sigma_1 \\
-i \sigma_2 \\
\sigma_1
\end{pmatrix}, \quad P_2 = -i \begin{pmatrix}
\sigma_3 \\
-\sigma_3 \\
-1_2
\end{pmatrix}.
$$

(3.28)

The matrices satisfy the anticommuting relation (3.21) again, which leads (3.24) indicating nonexistence of fermion doublers.

As for the renormalization in the SU($N$) case, the gauge invariance and the SU(2)$_R$ symmetry allow the operators $\text{tr} (4 \phi \bar{\phi} + C^2)$ and $\text{tr} B^2$, but they are not admissible from the supersymmetry $Q_{\pm}$. Also, for the U($N$) case, we should further consider the possibility of the operators $\text{tr} B$, $\text{tr} \bar{H}_\mu$, $\text{tr} H$ induced. They are forbidden by the symmetries under $Q_+ \leftrightarrow Q_- \ (2.20)$ and the reflection (3.25). Thus, no relevant or marginal operators appear through the loop effect except the identity operator. The continuum $\mathcal{N} = 4$ theory is obtained with no tuning of parameters.

4. Modification of $\Phi(x)$

In this section we consider another possibility to single out the vacuum $U_{12}(x) = 1$ from the degeneracy specifying the case $G = SU(N)$.

First we try to add a term $\Delta \Phi(x)$ to $\Phi(x)$:

$$
\Phi(x) \rightarrow \Phi(x) + \Delta \Phi(x), \quad \Delta \Phi(x) \equiv -r(2 - U_{12}(x) - U_{21}(x))
$$

(4.1)

with the parameter $r$ appropriately chosen. Since $H(x)$ is traceless, the classical vacua are determined by

$$
\Phi(x) + \Delta \Phi(x) - \left( \frac{1}{N} \text{tr} [\Phi(x) + \Delta \Phi(x)] \right) 1_N = 0.
$$

(4.2)

However, it turns out that it does not completely resolve the degeneracy. For instance, we can easily see that the center elements (2.11) are still the solutions for arbitrary $r$.

On the other hand, the equation

$$
\Phi(x) + \Delta \Phi(x) = 0
$$

(4.3)

for $G = SU(N)$ has the unique solution $U_{12}(x) = 1$ with appropriately chosen $r$ as explained in appendix A.
\[4.1 \, \mathcal{N} = 2 \, \text{Case}\]

For the \( \mathcal{N} = 2 \) theory, we extend \( \chi(x) \), \( H(x) \) to the hermitian matrices \( \hat{\chi}(x) \), \( \hat{H}(x) \) with nonzero trace parts to introduce the variables \( \chi^{(0)}(x) \), \( H^{(0)}(x) \) proportional to the unit matrix:

\[ \begin{align*}
\chi^{(0)}(x) &= \chi^{(0)}(x) \mathbf{1}_N, \quad H^{(0)}(x) = H^{(0)}(x) \mathbf{1}_N \\
\hat{H}(x) &= H(x) + H^{(0)}(x), \quad \hat{\chi}(x) = \chi(x) + \chi^{(0)}(x). 
\end{align*} \tag{4.4}\]

The fields with the underline mean the coefficients proportional to the unit matrix. The \( Q \)-transformation (2.1) of \( \chi(x) \) and \( H(x) \) is naturally extended to

\[ Q\hat{\chi}(x) = \hat{H}(x), \quad Q\hat{H}(x) = [\phi(x), \hat{\chi}(x)] \tag{4.5} \]

with

\[ Q\chi^{(0)}(x) = H^{(0)}(x), \quad QH^{(0)}(x) = 0. \tag{4.6} \]

The lattice action is modified as

\[ S_{\text{LAT}}^{\mathcal{N}=2} = Q \frac{1}{2g_0^2} \sum_x \text{tr} \left[ \frac{1}{4} \eta(x) [\phi(x), \bar{\phi}(x)] - i\hat{\chi}(x) (\Phi(x) + \Delta \Phi(x)) + \hat{\chi}(x)\hat{H}(x) \right. \]

\[ \left. + i \sum_{\mu=1}^2 \psi_{\mu}(x) \left( \phi(x) - U_{\mu}(x)\bar{\phi}(x + \mu)U_{\mu}(x)^\dagger \right) \right] \tag{4.7} \]

\[ = \frac{1}{2g_0^2} \sum_x \text{tr} \left[ \frac{1}{4} [\phi(x), \bar{\phi}(x)]^2 + \hat{H}(x)^2 - i\hat{H}(x) (\Phi(x) + \Delta \Phi(x)) \right. \]

\[ \left. + \sum_{\mu=1}^2 \left( \phi(x) - U_{\mu}(x)\phi(x + \mu)U_{\mu}(x)^\dagger \right) \left( \bar{\phi}(x) - U_{\mu}(x)\bar{\phi}(x + \mu)U_{\mu}(x)^\dagger \right) \right. \]

\[ \left. - \frac{1}{4} \eta(x) [\phi(x), \bar{\phi}(x)] - \chi(x) [\phi(x), \bar{\phi}(x)] \right. \]

\[ \left. - \sum_{\mu=1}^2 \psi_{\mu}(x) \psi_{\mu}(x) \left( \bar{\phi}(x) + U_{\mu}(x)\bar{\phi}(x + \mu)U_{\mu}(x)^\dagger \right) \right. \]

\[ \left. + i \hat{\chi}(x)Q \left( \Phi(x) + \Delta \Phi(x) \right) \right. \]

\[ \left. - i \sum_{\mu=1}^2 \psi_{\mu}(x) \left( \eta(x) - U_{\mu}(x)\eta(x + \mu)U_{\mu}(x)^\dagger \right) \right]. \tag{4.8} \]

Due to the trace part of \( \hat{H}(x) \), the minimum of the gauge part is uniquely determined by (4.3) with \( r = \cot \phi \) such that

\[ e^{i2\ell\varphi} \neq 1 \quad \text{for} \quad \forall \ell = 1, \ldots, N, \tag{4.9} \]

as explained in appendix A. On the other hand, the kinetic term of \( \hat{\chi}(x) \) is

\[ \frac{1}{2g_0^2} \sum_x \left\{ \text{tr} \left[ i\chi(x)Q(\Phi(x) + \Delta \Phi(x)) \right] + i\chi^{(0)}(x)Q \text{tr}(\Phi(x) + \Delta \Phi(x)) \right\}. \tag{4.10} \]
Since the second term in the brace is of the order $O(a^6)$, it vanishes in the continuum limit and $\chi^{(0)}(x)$ become fermion zero-modes. If we integrate out $\chi^{(0)}(x)$, we will have the nontrivial constraints

$$0 = Q \text{tr}(\Phi(x) + \Delta\Phi(x))$$  \hspace{1cm} (4.11)

leading $0 = \text{tr}[F_{12}(x)(D_1\psi_2(x) - D_2\psi_1(x))]$ at the nontrivial leading order $O(a^{9/2})$ in the continuum. Because such constraints are not imposed in the target continuum theory, we should avoid obtaining them. In order to do so, we soak up the would-be fermion zero-modes in the path-integral to consider the measure

$$d\mu_{N=2} \equiv d\mu_{\text{SU}(N)} e^{2\sum_x \left[ dH^{(0)}(x)d\chi^{(0)}(x)\chi^{(0)}(x) \right]}$$  \hspace{1cm} (4.12)

with $d\mu_{\text{SU}(N)}$ being the measure for the SU($N$) variables. Note that $d\chi^{(0)}(x)\chi^{(0)}(x)$ is $U(1)_R$ invariant, because $d\chi^{(0)}(x)$ transforms same as the derivative $\partial / \partial \chi^{(0)}(x)$. However, the insertion of the would-be zero-modes violates the $Q$ invariance as

$$Q \left( dH^{(0)}(x)d\chi^{(0)}(x)\chi^{(0)}(x) \right) = -dH^{(0)}(x)d\chi^{(0)}(x)H^{(0)}(x),$$  \hspace{1cm} (4.13)

although the action (4.7) is manifestly $Q$ invariant.

Here we consider the observables consisting of the operators in the SU($N$) sector i.e. independent of $H^{(0)}(x)$ and $\chi^{(0)}(x)$. Let us write the action as

$$S_{\text{LAT}}^{N=2} = S_{\text{SU}(N)}^{N=2} + \frac{N}{2g_0^2} \sum_x \left[ H^{(0)}(x)^2 - iH^{(0)}(x)\frac{1}{N} \text{tr}(\Phi(x) + \Delta\Phi(x)) \right. \right.$$

$$\left. + i\chi^{(0)}(x)\frac{1}{N} \text{tr}(\Phi(x) + \Delta\Phi(x)) \right],$$  \hspace{1cm} (4.14)

$$S_{\text{SU}(N)}^{N=2} = Q \frac{1}{2g_0^2} \sum_x \text{tr} \left[ \frac{1}{4} \eta(x) [\phi(x), \bar{\phi}(x)] - i\chi(x) (\Phi(x) + \Delta\Phi(x)) + \chi(x) H(x) \right.$$

$$\left. + i \sum_{\mu=1}^2 \psi_\mu(x) \left( \bar{\phi}(x) - U\mu(x)\bar{\phi}(x + \mu)U\mu(x) \right) \right],$$  \hspace{1cm} (4.15)

so that the dependence of $H^{(0)}(x)$ and $\chi^{(0)}(x)$ can be explicitly seen. From (4.13), (4.14), the $Q$-transformation of $d\mu_{N=2} e^{-S_{\text{LAT}}^{N=2}}$ leads

$$\int Q \left( d\mu_{N=2} e^{-S_{\text{LAT}}^{N=2}} \right) = \int d\mu_{\text{SU}(N)} e^{-S_{\text{SU}(N)}^{N=2}} \left( \prod_x \left( dH^{(0)}(x) \right) \right.$$  \hspace{1cm} (4.16)

$$\times \sum_x \left[ i \frac{2}{2g_0^2} H^{(0)}(x)Q \text{tr}(\Phi(x) + \Delta\Phi(x)) \right]$$

$$\times \exp \left\{ -\frac{N}{2g_0^2} \sum_x \left[ H^{(0)}(x)^2 - iH^{(0)}(x)\frac{1}{N} \text{tr}(\Phi(x) + \Delta\Phi(x)) \right] \right\}$$

after integrating out $\chi^{(0)}$. As the result of the integration of $H^{(0)}$, we obtain

$$\int Q \left( d\mu_{N=2} e^{-S_{\text{LAT}}^{N=2}} \right) = \int d\mu_{\text{SU}(N)} e^{-S_{\text{SU}(N)}^{N=2}} e^{-\frac{1}{8N g_0^2} \sum_x [\text{tr}(\Phi(x) + \Delta\Phi(x))]^2}$$
\[\times Q \sum_x \frac{-1}{8Ng_0^2} \left[ \text{tr} (\Phi(x) + \Delta\Phi(x)) \right]^2, \quad (4.17)\]

which means that the insertion of the would-be fermion zero-modes is equivalent to adding the supersymmetry breaking term

\[\Delta S = \frac{1}{8Ng_0^2} \sum_x \left[ \text{tr} (\Phi(x) + \Delta\Phi(x)) \right]^2 \quad (4.18)\]

to the \(Q\) invariant action \(S_{\text{LAT}}^{\text{SU}(N), N=2}\). \(\Delta S\) supplies the trace part of \(\Phi(x) + \Delta\Phi(x)\) leading the condition for the minima (4.2) to resolve the degeneracy. In the continuum limit, \(\Delta S\) is of the order \(O(a^4)\): \(\Delta S = a^4 8 Ng_2^2 \int d^2x \left( \text{tr} F_{12}(x)^2 \right)^2\) and becomes irrelevant. Comparing to the supersymmetry breaking term (2.22) of the order \(O(a^4)\) introduced in [1], (4.18) becomes much more irrelevant in the continuum.

**No Fermion Doublers and Renormalization** The action \(S_{\text{LAT}}^{\text{SU}(N), N=2} + \Delta S\) with \(r = \cot \varphi\) satisfying (4.9) has the unique minimum \(U_{12}(x) = 1\), which justifies expanding the exponential of the link variables (1.1). Note that the modification \(\Delta\Phi(x)\) and \(\Delta S\) does not affect the classical continuum limit (3.17). Compared with \(\Phi(x)\) giving \(O(a^2)\) contributions, \(\Delta\Phi(x)\) is of the negligible order \(O(a^4)\). Furthermore, it turns out that the shift of the fermionic part of the action

\[\text{tr} \left[ i\chi(x)Q\Delta\Phi(x) \right] = \text{tr} \left[ ir\chi(x) (QU_{12}(x) + QU_{21}(x)) \right] \quad (4.19)\]

leads to gauge-fermion interaction terms of the irrelevant order \(O(a^5)\), but not to fermion kinetic terms. Thus, the fermion bilinear kinetic terms are not influenced by the modification, and no fermion doublers appear from the same argument as in the previous paper [1]. Also, since the supersymmetry breaking effect comes in via the vertices of \(\Delta S\) in the loop expansion, it becomes irrelevant in the continuum limit as discussed in the previous paper [1]. The renormalization argument goes parallel to the previous paper to show that the target continuum theory is obtained without any fine-tuning.

**4.2 \(N = 4\) Case**

For the \(N = 4\) case, we can repeat the similar argument to the \(N = 2\) model. We introduce the degrees of the freedom of the trace part for \(B(x), \chi_{\pm}(x)\) and \(H(x)\) as

\[B^{(0)}(x) = B_{\pm}^{(0)}(x) 1_N, \quad \chi^{(0)}_{\pm}(x) = \chi_{\pm}^{(0)}(x) 1_N, \quad H^{(0)}(x) = H_{\pm}^{(0)}(x) 1_N, \]

\[\bar{B}(x) = B(x) + B^{(0)}(x), \quad \chi^{\pm}(x) = \chi_{\pm}(x) + \chi^{(0)}_{\pm}(x), \quad \bar{H}(x) = H(x) + H^{(0)}(x). \quad (4.20)\]

Again, the fields with the superscript ‘\((0)\)’ proportional to the unit matrix, and their proportional coefficients are denoted by putting the underline. Defined as

\[Q_{\pm}B^{(0)}(x) = \chi^{(0)}_{\pm}(x), \quad Q_{\pm}\chi_{\pm}^{(0)}(x) = 0, \quad Q_{\pm}H_{\pm}^{(0)}(x) = \mp H^{(0)}(x), \]

\[Q_{\pm}H^{(0)}(x) = 0, \quad (4.21)\]

the transformation rule (2.14) is naturally extended to the variables with the trace parts (4.20).
Let us consider the lattice action

$$S_{N=4}^{\text{LAT}} = Q_+ Q_- \frac{1}{2g_0} \sum_x \text{tr} \left[ -i \tilde{B}(x) \left( \Phi(x) + \Delta \Phi(x) \right) - \sum_{\mu=1}^2 \psi_{+\mu}(x) \psi_{\mu}(x) - \tilde{\chi}_+(x) \tilde{\chi}_-(x) - \frac{1}{4} \eta_+(x) \eta_-(x) \right]. \quad (4.22)$$

From the argument parallel to the $N=2$ case, we observe that $\chi_{+}^{(0)}(x)$ and $\bar{B}^{(0)}(x)$ become the zero-modes in the continuum limit. After integrating out those, we will obtain the nontrivial constraints

$$0 = Q_+ Q_- \text{tr} \left( \Phi(x) + \Delta \Phi(x) \right), \quad 0 = Q_\pm \text{tr} \left( \Phi(x) + \Delta \Phi(x) \right), \quad (4.23)$$

meaning $0 = \text{tr} \left[ F_{12}(x) \left( D_1 \psi_{\pm 2}(x) - D_2 \psi_{\pm 1}(x) \right) \right]$ in the continuum, which do not appear in the target continuum theory. In order to avoid the constraints, we soak up the would-be zero-modes to consider the path-integral measure

$$d\mu_{N=4} \equiv d\mu_{\text{SU}(N)_{N=4}} \times \prod_x \left( dH^{(0)}(x) d\bar{B}^{(0)}(x) \delta \left( \bar{B}^{(0)}(x) \right) d\chi^{(0)}_+(x) d\chi^{(0)}_-(x) d\chi^{-1}(0)_-(x) \right). \quad (4.24)$$

with $d\mu_{\text{SU}(N)_{N=4}}$ being the measure with respect to the variables in the SU($N$) sector. The measure $d\mu_{N=4}$ is invariant under the SU(2)$_R$ rotation, but not under $Q_{\pm}$ due to the insertion of the would-be zero-modes $\chi^{(0)}_{\pm}(x)$, although $S_{N=4}^{\text{LAT}}$ is manifestly supersymmetric. Following the similar procedure to the $N=2$ case, we end up with

$$\int Q_\pm \left( d\mu_{N=4} e^{-S_{N=4}^{\text{LAT}}} \right) = \int d\mu_{\text{SU}(N)_{N=4}} e^{-S_{\text{SU}(N)_{N=4}}^{\text{LAT}}} e^{-\frac{1}{8N g_0} \sum_x \left| \text{tr} \left( \Phi(x) + \Delta \Phi(x) \right) \right|^2}$$

$$\times Q_\pm \sum_x \frac{1}{8Ng_0} \left| \text{tr} \left( \Phi(x) + \Delta \Phi(x) \right) \right|^2, \quad (4.25)$$

again showing that the soak-up of the would-be zero-modes is equivalent to adding the supersymmetry breaking term (4.18). $S_{\text{SU}(N)_{N=4}}^{\text{LAT}}$ is nothing but the action (2.17) with the replacement (4.1) made.

Similarly to the previous cases, it is shown that the fermion doublers do not appear and that the target continuum theory is obtained without any fine-tuning.

5. Summary and Discussions

In this paper, two-dimensional $G = U(N)$, SU($N$) super Yang-Mills theories with $N = 2, 4$ supersymmetries have been constructed on the square lattice, keeping one or two supercharges exactly. We have resolved a problem of the degenerate classical vacua, which was encountered in the previous paper [1], with keeping the exact supersymmetry. Thus, any supersymmetry breaking terms do not need to be introduced, and the formulations exactly realize the supersymmetries $Q$ or $Q_{\pm}$ at the lattice level. Our lattice models define the continuum SYM theories without any fine-tuning.
We have considered two different kinds of modifications of the actions to resolve the problem. One is a modification to impose the admissibility condition on each plaquette variable $U_{12}(x)$ with changing the action somewhat analogous to ref. [19]. It will be also applicable to other gauge groups with an appropriate choice of $\epsilon$. It would be worth while to pursue that direction to discuss topological structures of the space of the admissible lattice gauge fields as in [19]. The other modification is a simple one to add the term $\Delta \Phi(x)$ to the function $\Phi(x)$ with the trace parts introduced for some adjoint fields in $G = SU(N)$. The actions are supersymmetric, but they contain would-be zero modes which induce nontrivial constraints not seen in the target continuum theories. In order to avoid getting the constraints we have soaked up the zero-modes, which leads breaking of the supersymmetry. The breaking effect is irrelevant in the continuum limit, and it can be shown that the target continuum theories are obtained without any fine-tuning.

It is interesting to consider appropriate modifications to four-dimensional models with

$$\Phi_A(x) = -i \left[ U_{A4}(x) - U_{4A}(x) + \frac{1}{2} \sum_{B,C=1}^{3} \varepsilon_{ABC} (U_{BC}(x) - U_{CB}(x)) \right], \quad (5.1)$$

which correspond to $\mathcal{N} = 2, 4$ theories [20]. Even if suitable modifications are found and exact supersymmetries are realized at the lattice level, generically four-dimensional models would need some fine-tuning of parameters to define the desired continuum limit. For the $\mathcal{N} = 4$ case, however the situation might seem to be subtle, because it is believed to receive no quantum corrections in any order of the perturbation theory (for example, see [21]). It would be intriguing to compute radiative corrections in the lattice perturbation theory for the modified $\mathcal{N} = 4$ model with the exact $Q_{\pm}$ supersymmetry.

Since the same kind of degeneracy problem exists in models proposed in ref. [4], the methods discussed here might be useful to resolve the difficulty there.

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**A. Uniqueness of the solution of eq. (4.3)**

Here we show that the equation (4.3) for $G = SU(N)$ has a unique solution $U_{12}(x) = 1$ for appropriate choices for the parameter $r$.

First we shall take as

$$r = \cot \frac{\pi}{2N}. \quad (A.1)$$
For other choices, we will discuss later. From the equation (4.3), $U_{12}(x)$ can be expressed as the diagonalized form

$$
U_{12}(x) = \Omega(x) \begin{pmatrix} e^{i\theta_1(x)} & & \\ & \ddots & \\ & & e^{i\theta_N(x)} \end{pmatrix} \Omega(x)^\dagger \quad \text{(A.2)}
$$

with $\Omega(x) \in SU(N)$, $\theta_N(x) \equiv -\theta_1(x) - \cdots - \theta_{N-1}(x)$, and $-\pi < \theta_i(x) \leq +\pi$ ($i = 1, \cdots, N - 1$). Under this parameterization, the equation for the minima (4.3) reads

$$
\sin \theta_1(x) = 2r \sin^2 \left( \frac{\theta_1(x)}{2} \right),
$$

$$
\vdots
$$

$$
-\sin(\theta_1(x) + \cdots + \theta_{N-1}(x)) = 2r \sin^2 \left( \frac{\theta_1(x) + \cdots + \theta_{N-1}(x)}{2} \right).
$$

(A.3)

With the choice (A.1), the first $N - 1$ equations give the solutions $\theta_i(x) = 0$ or $\frac{\pi}{N}$ ($i = 1, \cdots, N - 1$). Among them, the last equation allows the only one combination

$$(\theta_1(x), \cdots, \theta_{N-1}(x)) = (0, \cdots, 0),
$$

(A.4)

which is nothing but $U_{12}(x) = 1$. (Any other combinations of the solutions are not compatible to the last equation, because the LHS is evaluated to be negative while the RHS positive.)

Let us discuss about other choices of $r$. Parameterizing as $r = \cot \varphi$,

$$
\Phi(x) + \Delta\Phi(x) = \frac{-1}{\sin \varphi} \left[ e^{-i\varphi}(1 - U_{12}(x)) + e^{i\varphi}(1 - U_{21}(x)) \right],
$$

(A.5)

where $\Phi(x) + \Delta\Phi(x) = 0$ means that $e^{-i\varphi}(1 - U_{12}(x))$ is anti-hermitian. Configurations of $U_{12}(x)$ giving the minima are expressed as

$$
U_{12}(x) = 1 - ie^{i\varphi}T(x)
$$

(A.6)

with $T(x)$ being hermitian. The eigenvalues of $T(x)$ are denoted as $t_i(x)$ ($i = 1, \cdots, N$). The unitary condition $U_{12}(x)U_{12}(x)^\dagger = 1$ determines $t_i(x)$ as

$$
t_i(x) = 0 \quad \text{or} \quad -2 \sin \varphi \quad \text{for} \quad i = 1, \cdots, N.
$$

(A.7)

The unitarity alone does not uniquely fix $t_i(x)$. However, the unimodular condition

$$
\det U_{12}(x) = 1
$$

(A.8)

gives further constraints. For the case that $\ell$ eigenvalues of $t_i(x)$ are $-2 \sin \varphi$ and the remaining $N - \ell$ are 0, (A.8) leads

$$
e^{i2\ell \varphi} = 1.
$$

(A.9)

Thus, taking $\varphi$ such that

$$
e^{i2\ell \varphi} \neq 1 \quad \text{for} \quad \forall \ell = 1, \cdots, N,
$$

(A.10)

the equation $\Phi(x) + \Delta\Phi(x) = 0$ has the unique solution $t_1(x) = \cdots = t_N(x) = 0$, equivalent to $U_{12}(x) = 1$. Of course, the choice (A.1) satisfies (A.10).
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