Algebraic Geometry Approach in Theories with Extra Dimensions I. Application of Lobachevsky Geometry

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Abstract

This present paper has the purpose to find certain physical applications of Lobachevsky geometry and of the algebraic geometry approach in theories with extra dimensions. It has been shown how the periodic properties of the uniformization functions-solutions of cubic algebraic equations in gravity theory enable the orbifold periodic identification of the points $\pi r_c$ and $-\pi r_c$ under compactification. It has been speculated that corrections to the extradimensional volume in theories with extra dimensions should be taken into account due to the non-euclidean nature of the Lobachevsky space. It has been demonstrated that in the Higgs mass generation model with two branes (a "hidden" and a "visible" one), to any mass on the visible brane there could correspond a number of physical masses. Algebraic equations for 4D Schwarzschild Black Holes in higher dimensional brane worlds have been obtained.

1 INTRODUCTION

It is commonly accepted in gravity theory [1] that the contravariant metric tensor $g^{ij}$ is at the same time an inverse one to the covariant one, i.e.

$$g_{ij}g^{jk} = \delta^k_i \quad . \quad (1.1)$$

From a more general point of view, the above equality sets up a correspondence between the covariant and contravariant metric tensor components, due to which the contravariant components cannot be considered as independent from the covariant ones.

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However, in more general theories of gravity [2,3,4] - theories with covariant and contravariant metrics and affine connections, the covariant metric tensor components are treated independently from the contravariant ones (further they shall be denoted with the "tilda" sign), which means that

\[ g_{ij} \tilde{g}^{ik} = l_i^k(x) ; \quad e_i^k = l_i^k(x) , \] (1.2)

where \( l_i^k(x) \) are some (tensor) functions of the space - time coordinates. Consequently, there is no longer a correspondence between the covariant and contravariant metric tensor components, which should be treated within the framework of the affine geometry approach [5,6,7]. Unfortunately, the components of the function \( l_j^i(x) \) cannot be determined from any physical considerations. That is why in [8] it was proposed to find these functions by requiring the gravitational Lagrangian with the more generally defined contravariant metric components to be the same as the Lagrangian in the standard gravitational theory (i.e. with the usual inverse contravariant tensor \( g_{ij} \)). As a result, a multivariable cubic algebraic equation was obtained (see Appendix A) for the choice of the contravariant metric tensor \( \tilde{g}^{ij} \) in the form of the factorized product \( \tilde{g}^{ij} = dX^idX^j \).

It should be remembered also that the choice of the contravariant metric components in the known gravity theory as inverse to the covariant components is a mathematical convention - there is not a single gravitational experiment, measuring directly the contravariant metric components. Consequently one may formulate the problem are there physical reasons and considerations why this should be so. If no such straightforward physical reasons can be formulated, one has the right to investigate what could be the physical consequences from such a more general assumption.

The two parts of this paper will be related to the application of the algebraic geometry approach [9, 10] and of Lobachevsky geometry in theories with extra dimensions. The first part will deal mostly with the application of Lobachevsky geometry, and the problem about Higgs mass creation in theories with visible and hidden branes will be considered in the framework of the theories with covariant and contravariant metrics. The visible and hidden branes are situated at the orbifold fixed points \( \Phi = 0 \) and \( \Phi = \pi \) and the covariant components of the visible brane are determined as

\[ g_{\mu\nu}^{vis}(X^\mu) = G_{\mu\nu}(X^\mu, \Phi = \pi) = e^{-2kr - \pi} g_{\mu\nu} . \] (1.3)

The first problem, raised in this paper is: if in a more general gravitational theory the contravariant components are not determined as the inverse to the covariant ones, then their "scaling" in the action for the fundamental Higgs field

\[ S_{vis} = \int d^4X \sqrt{-g_{vis}} \left[ g_{\mu\nu}^{vis}D_\mu H^+ D_\nu H - \lambda \left( |H|^2 - v_0^2 \right)^2 \right] \] (1.4)

will not be as \( g_{\mu\nu}^{vis} = e^{2kr - \pi} g_{\mu\nu} \). Consequently, since the normalization of the fields determines the physical masses, to any mass \( m_0 \) on the visible three-brane will no longer correspond a single physical mass \( m \) according to the relation \( m = e^{-kr - \pi} m_0 \), but instead a multitude of physical masses.
The second problem, again related to the algebraic geometry approach, is the orbifold identification of the points $\pi r_C$ and $-\pi r_C$ under orbifold compactification. This is possible due to the previously established in [10] property - the algebraic solutions of the cubic multivariable equation for reparametrization invariance of the gravitational Lagrangian represent uniformization functions, depending on the periodic complex coordinate $z$.

The rest of the problems in this paper deal with the application of Lobachevsky geometry in theories with extra dimensions. The basic fact in these theories is the following one - the fundamental scale of gravity $M_{pl}$ is related to the gravity scale $M_{und}$ in the $(4 + d)-$dimensional space as

$$M_{pl}^2 = M_{und}^{2+d} V^d = M_{und}^{2+d} V^d .$$

(1.5)

It is very important to realize that this relation is obtained on the base of the factorizing approximation for the effective action in the form

$$S_{\text{eff}} = \int d^4X \int_0^{\pi r_c} dy 2M^3 r_c e^{-2\tilde{k}y} \sqrt{gR} ,$$

(1.6)

where the 5D- metric is chosen with an exponentially suppressed "warp" factor in front of the flat 4D Minkowski metric ($0 \leq y \leq \pi r_c$)

$$ds^2 = e^{-2\tilde{k}y} \eta_{\mu\nu} dX^\mu dX^\nu + dy^2 .$$

(1.7)

So the third problem, solved in this paper is: if another coordinate transformation is applied, for example a one, related to the distance $\rho$ in the Lobachevsky geometry, then what happens with the extradimensional coordinate and is the factorization approximation valid? The last means - can the whole 5D spacetime be represented as a direct product of two spaces? However, yet from the early investigations on Lobachevsky geometry it was known that power - like corrections in $\tilde{k}$ ($r$- the Euclidean distance) arise to the volume element, which in view of the relation (1.5) will result also in corrections to the gravitational couplings.

In fact, the cornerstone and basic reasoning for the application of the Lobachevsky geometry in theories with extra dimensions is the similarity of the 5D metric (1.7) with the 3D metric of a spacetime with a constant negative curvature $ds^2 = d\rho^2 + e^{-2\tilde{k}\sigma} d\sigma^2$. Although this metric is a three-dimensional one, multidimensional Lobachevsky spaces have also been investigated long time ago in the monographs of Rosenfel’d [11, 12, 13]. Now, if one makes the analogy between the two metrics, another important question arises - what is the origin of the constant $\tilde{k}$ in the metric (1.7)? Unfortunately, this is not clarified in the existing literature.In this paper it is shown that for the choice $\tilde{k} \neq \frac{1}{c}$, where $c$ is the Lobachevsky constant, there will be an exponential increase of the extradimensional distance (along the $y$ coordinate). This can be attributed to the non-euclidean nature of the geometry. One more confirmation of the non-euclidean geometry - if the limit $\tilde{k} = \frac{1}{c} \rightarrow 0$ is assumed (which means that $c \rightarrow \infty$ and then the euclidean
geometry is recovered), then it is impossible to set up the fine-tuning $k r_C \approx 50$. The last is necessary if one would like to have a five-dimensional Plack mass, not very far from the electroweak scale $M_W \approx TeV$. Consequently, the non-euclidean effects of the geometry really come into play, and this is the fourth problem, discussed in this paper. The fifth problem concerns the Riemann scalar curvature invariant $R_{ABCD}R^{ABCD}$, which has a singularity at $r \to 0$ and at $|y| \to \infty$. Physically this may mean that there might be a non-vanishing energy flow into the bulk singularities. So the problem now may be formulated as follows: will the singularities remain or vanish in gravitational theories with covariant and contravariant metrics? In other words, the question is whether the scalar curvature invariant can be preserved in such theories, which would signify the presence of the same type of singularities. A fourth-order algebraic equation has been obtained for the preservation of this invariant.

Appendix A contains a brief review of the algebraic geometry approach in gravity theory, some aspects of which were used in section 7.

2 FUNDAMENTAL PARALELLOGRAM ON THE COMPLEX PLANE, ORBIFOLD COMPACTIFICATION AND PERIODIC IDENTIFICATION

As it is well-known, a class of two-dimensional metrics exists [14]

$$ds^2 = R^2 \left( \frac{a^2 - v^2)du^2 + 2uvdudv + (a^2 - u^2)dv^2}{(a^2 - u^2 - v^2)^2} \right),$$

(2.1)

representing the linear element of a unit surface in the Lobachevsky space with a constant negative curvature $-\frac{1}{R^2}$. Performing the transformations

$$\frac{a^2 - u}{\sqrt{a^2 - u^2 - v^2}} = ae^{-\frac{\phi}{R}}; \quad \frac{uv - u_0 v_0}{a^2 - u - u_0v_0} = \frac{\sigma}{R},$$

(2.2)

the above metric (2.1) can be rewritten as

$$ds^2 = d\rho^2 + e^{-\frac{2\Phi}{R}} d\sigma^2,$$

(2.3)

which turns out to be similar to the metric

$$ds^2 = e^{-2k\phi - \Phi} \eta_{\mu\nu}dx^\mu dx^\nu + r^2 d\Phi^2,$$

(2.4)

extensively used in the first version of the Randall-Sundrum model [15]. In (2.4) $\eta_{\mu\nu}$ is the flat Minkowski metric, $0 \leq \Phi \leq \pi$ and the extra dimension is a finite interval,
whose size is set by the compactification radius $r_c$. A nice and effective generalization of this model implies that the SM (Standard Model) particles and forces with the exception of gravity are confined to a 4-dimensional subspace, but within a $(4 + n)$-dimensional spacetime.

Let us now remind that the complex coordinate $z$ is the argument of the Weierstrass function and as mentioned, it is defined on the lattice $\Lambda = \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}; \omega_1, \omega_2 \in \mathbb{C}, \text{Im} \omega_i > 0\}$ on the two-dimensional projective plane $\mathbb{CP}^2$. Then let us define the complex uniformization coordinate $z$ as

$$z = \pi r_c (\cos \Phi + i \sin \Phi)$$

and $0 \leq \Phi \leq \pi$ is the periodic coordinate. Under the transformation $\Phi = \arctg \frac{z}{r_c}$, the metric (2.4) will transform as

$$ds^2 = e^{-2kr_c} \frac{1}{\sqrt{z^2 + r_c^2}} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{r_c^4}{\sqrt{z^2 + r_c^2}} dz^2 . \quad (2.5)$$

Now the advantage of such a formulation is clear: the nice properties of the Weierstrass function and its derivative

$$\rho'(z + \omega_i) = \rho'(z) ; \quad \rho(\pi r_c) = \rho(-\pi r_c) \quad (2.6)$$

exactly matches the requirement for orbifold identification of the points $+\pi r_c$ and $-\pi r_c$. In other words, by making the above transformation for the periodical coordinate $\Phi$ of the additional extra dimension [15, 16], a periodical identification is achieved of the identical points under orbifold compactification with a fundamental domain of length $2\pi r_c$ with the lattice points of the fundamental parallelogram on the complex plane. A general overview of orbifold compactifications is presented in [17].

### 3 FACTORIZATION AND NON-FACTORIZATION OF THE VOLUME ELEMENT

Obtaining some estimates for the fundamental length $2\pi r_c$ would be interesting, since for $d$ additional compactified dimensions, each one of radius $r_i$, the fundamental (Planck) scale of gravity is related to the gravity scale in the $(4 + d)$-dimensional space as [18, 19, 20]

$$M_{pl}^2 = M_{fund}^{2+d} r_i^d = M_{fund}^{2+d} V^d . \quad (3.1)$$

The estimate of the volume of the extradimensional space is important, since by taking a large volume the large discrepancy between the Planck scale of $10^{19}$GeV and the electroweak scale of 100 GeV can be diminished and thus the hierarchy problem can be solved. For a derivation of the relation (3.1) between the gravity scales on the base of dimensional analysis of the higher-dimensional Einstein-Hilbert action, one may use the review article [21]. In such a case, the metric (2.4) should be generalized to the $(d + 4)$-dimensional metric of the Lobachevsky space. Naturally, the most simple case [21] is of a flat extradimensional space, when $V^d = (2\pi r)^d$ and also a flat 4D Minkowski
metric. However, it would be much more interesting to consider a 
\( (d + 4) \)-dimensional \( \text{ADS} \) (Lobachevsky) space with a constant negative curvature, whose volume element is
given by [22]
\[
dV_n = \frac{c_n dx_1 dx_2 \ldots dx_n}{(c^2 - x_\alpha x_\alpha)^{(n+1)/2}}
\] (3.2)
and can be found by splitting up the \( n \)-dimensional volume by means of \( (n - 1) \)-dimensional hyperplanes, perpendicular to the coordinate axis. Details can be found again in
the monograph [22]. For example, the five- and four-dimensional volume elements are
calculated to be
\[
V_5 = \frac{1}{12} \pi^2 e^6 (s_h \frac{4r}{c} - 8s_h \frac{2r}{c} + 12 \frac{r}{c}) ,
\] (3.3)
\[
V_4 = \pi c^3 (s_h \frac{2r}{c} - \frac{2r}{c}) ,
\] (3.4)
where \( r \) denotes the natural (euclidean) length and \( c = \frac{1}{k} \) is the Lobachevsky constant -
the unit length parameter for the Lobachevsky space, which enters the expressions (3.3
- 3.4). Our purpose will be to see whether the constant \( k \) in the exponential factor
\( e^{-2kr - \Phi} \) in the \( (d + 4) \)-dimensional analogue of the metric (2.4) can be identified with
the inverse power of the Lobachevsky constant \( (k = \frac{1}{c}) \). Unfortunately, in most of the existing
papers on theories with extra dimensions, the origin and meaning of the parameter \( k \) is
not clarified.

Since in the limit \( c \to \infty \) the usual Euclidean geometry is recovered [23], then the above
formulae would give the volumes of the five- and of the four-dimensional (Euclidean)
spheres respectively
\[
V_5 = \frac{8}{15} \pi^2 r^5 ; \quad V_4 = \frac{4}{3} \pi r^3 (1 + \frac{1}{5} \frac{r^2}{c^2} + ...) = \frac{4}{3} \pi r^3 .
\] (3.5)
The volumes of the \( n \)-dimensional (Euclidean) spheres for \( n = 2\lambda \) and \( n = 2\lambda + 1 \) [22]
\[
V_{2\lambda} = \frac{\pi^\lambda}{\lambda!} r^{2\lambda} ; \quad V_{2\lambda+1} = \frac{2^{\lambda+1} \pi^\lambda}{(2\lambda + 1)(2\lambda - 1)\ldots3.1} r^{2\lambda+1}
\] (3.6)
can also be derived in the limit \( c \to \infty \) from the (recurrent) formulae for the \( n \)-dimensional
hyperbolic volume
\[
V_n = \frac{2\pi c^2}{(n-1)} \left[ \frac{P_n}{(n-2)} c^{n-2} s_h^{n-2} \frac{r}{c^2} - V_{n-2} \right] ,
\] (3.7)
where
\[
P_{2\lambda} = \frac{2\pi^{\lambda+1}}{\lambda!} ; \quad P_{2\lambda+1} = \frac{2^{\lambda+2} \pi^{\lambda+1}}{(2\lambda + 1)(2\lambda - 1)(2\lambda - 3)\ldots3.1} .
\] (3.8)
Therefore, only in the flat (Euclidean) \( (4 + d) \)-dimensional space, which is a product of a
\( 4 \)-dimensional Minkowski space \( (\mu, \nu = 1, 2, \ldots, 4) \) and a flat \( d \)- (extra)dimensional space [21]
\[
ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu - r^2 d\Omega^2 \]
(3.9)
one can factorize the volume element in the $(4 + d)$ Einstein - Hilbert action (assuming also that $R^{(4+d)} = R^{(4)}$) as

$$S_{4+d} = -M_4^{d+2} \int d^{4+d} X \sqrt{g^{(4+d)}} R^{(4+d)} = -M^{d+2}_4 \int d\Omega_{(d)} r^d \int d^4 X \sqrt{g^{(4)}} R^{(4)}.$$  (3.10)

It is clear however from expressions (3.3 - 3.4), (3.7) that in the general case of a multidimensional non-euclidean (Lobachevsky) space such a factorization of the volume element is impossible. Even in the limit of small ratios $\frac{r}{c}$, possible corrections to the volume element (see f.(3. 5)) have to be taken into account and therefore, the non-euclidean geometry would ”induce” correction terms in the relation between the gravitational couplings.

4 COORDINATE TRANSFORMATIONS WITH THE LOBACHEVSKY CONSTANT IN ”WARP” TYPE OF METRICS

Let us turn to the other frequently used case [24, 25] of a 5D- metric with an exponentially suppressed ”warp” factor in front of the flat 4D Minkowski metric ($0 \leq y \leq \pi r_c$)

$$ds^2 = e^{-2k|y|} \eta_{\mu\nu} dX^\mu dX^\nu + dy^2$$  (4.1)

and for the moment it shall be assumed that $\tilde{k}$ is different from the constant $k = \frac{1}{c}$. The five - dimensional effective action can be factorized as [24]

$$S_{eff} = \int d^4 X \int_0^{\pi r_{-}} dy 2 M^3 r_c e^{-2\tilde{k}|y|} \sqrt{\tilde{g}} R$$  (4.2)

and the metric (4.1) is chosen so that the 5- dimensional Ricci scalar curvature is equal to the 4- dimensional Minkowski one. From (4.2), the matching relation between the gravitational couplings is obtained to be [24]

$$M_{pl}^2 = 2 M^3 \int_0^{\pi r_{-}} dy e^{-2\tilde{k}|y|} = \frac{M^3}{k} (1 - e^{-2\tilde{k}r_{-} r_{}}).$$  (4.3)

Let us note that the result of the integration in (4.2 - 4.3) will not be coordinate independent. In other words, if we map the 4D Minkowski part of the metric with an exponentially ”damped” prefactor into a flat Minkowski 4D metric without the exponential prefactor, this would result in the appearence of an exponentially growing prefactor in front of the $dy^2$ part of the metric (4.1). To illustrate this, let us use a coordinate transformation, similar to the one, used in [26]

$$X^1 = a \left( \cosh \frac{\rho}{c} \right); \quad X^2 = b \left( \sinh \frac{\rho}{c} \right) \sin \Theta \cos \varphi,$$  (4.4)
\[ X^3 = b \, \text{sh} \frac{\rho}{c} \sin \Theta \sin \varphi \quad ; \quad X^4 = b \, \text{sh} \frac{\rho}{c} \cos \Theta \quad . \tag{4.5} \]

The signature of the Minkowski space is \((+, -, -, -)\), i.e. \(\eta_{\mu\nu} = (+1, -1, -1, -1)\), \(\rho\) is the distance in the Lobachevsky space, related to the euclidean distance \(r\) by the formulae

\[ r = c \, \text{sh} \frac{\rho}{c} \quad . \tag{4.6} \]

The constants \(a\) and \(b\) are to be chosen so that a flat Minkowski metric without any prefactor is obtained. In fact, if \(\tilde{k} = \frac{1}{c}\), the metric (4.1) will be exactly the one, known from Lobachevsky geometry. Now we shall establish the physical meaning of the relation \(\tilde{k} = \frac{1}{c}\), but in reference to theories with extra dimensions. An elementary introduction into Lobachevsky geometry can be found in [27] and a more comprehensive and detailed exposition - in [28].

For the choice \(a = b = ce^{|y|}\) and after applying the transformations (4.4 - 4.5), the metric (4.1) can be rewritten as

\[ ds^2 = -d\rho^2 - c^2 \text{sh}^2 \frac{\rho}{c} d\Theta^2 - c^2 \text{sh}^2 \frac{\rho}{c} \sin^2 \Theta d\varphi^2 + (c\tilde{k})^2 e^{2|y|} dy^2 \quad . \tag{4.7} \]

Obviously, the first three terms give the unit length element in the Lobachevsky space, which in the limit \(c \to \infty\) (then from (4.6) \(\rho \to r\)) gives the usual euclidean length element \(dr^2 + r^2(d\Theta^2 + \sin^2 \Theta d\varphi^2)\) in spherical coordinates \((r, \Theta, \varphi)\). Most interesting is the last term in (4.7) - it goes to infinity if \(|y| \to \infty\) and in the limit \(c \to \infty\) (when \(c \neq \frac{1}{\tilde{k}}\)), but it tends to 1 in the limit \(c \to \infty\) and when \(\tilde{k} = \frac{1}{c}\), which is physically reasonable because a flat euclidean geometry is obtained, as it should be. Thus the exponential increase of the ”extra - dimensional” distance, when \(c \neq \frac{1}{\tilde{k}}\), can be regarded as an effect of the non-euclidean nature of space-time. Indeed, it is physically unacceptable to take the limit \(\tilde{k} = \frac{1}{c} \to 0\), because if the five - dimensional Planck mass is assumed to be not very far from the electroweak scale \(M_W \approx TeV\), then a fine- tuning \(\tilde{k} r_c \approx 50\) is needed [18].

## 5 ALGEBRAIC EQUATIONS IN 4D SCHWARZSCHILD BLACK HOLES IN HIGHER DIMENSIONAL BRANE WORLDS

Now suppose that the metric (4.1) does not contain a flat Minkowski 4D space, but a 4D black hole instead

\[ ds^2 = e^{-2\tilde{k}|y|} g_{\mu\nu} dX^\mu dX^\nu + dy^2 \quad , \tag{5.1} \]

where

\[ g_{\mu\nu} dX^\mu dX^\nu = -(1 - \frac{2M}{r}) dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + \]

\[ rs^2 + s^2 d\Theta^2 + \sin^2 \Theta d\varphi^2 \]
For such a model [32, 33], if a negative tension brane is introduced at a distance \( y = l < \infty \), the five dimensional BH singularity will have a finite size and a black tube will extend into the bulk, thus interpolating between the two black holes.

A possible application of the formalism in this paper is related to the Riemann scalar curvature invariant \( R_{ABCD}R^{ABCD} \) [32], which for the background metric (5.2) and using the conventional contravariant metric components \( g^{ij} \) is calculated to be [32]

\[
R_{ABCD}R^{ABCD} = 40k^4 + \frac{48M^2e^{4k|y|}}{r^6}.
\] (5.3)

This expression contains an important physical information - it diverges at the black hole singularity at \( r = 0 \) and also at the \( ADS \) horizon at \( |y| \to \infty \). The elimination of this singularity (i.e., giving it a finite size) is the main motivation for introducing the second, negative tension brane at a distance \( y = L \). But even in the case of a single brane configuration, the presence of a singularity is essential since there might be a non-vanishing energy flow into the bulk singularities, which is not desirable. Such an energy flow will exist if the limit [32]

\[
\lim_{|y| \to \infty} \sqrt{-g} J^\mu_N = \lim_{|y| \to \infty} \sqrt{-g} T^{yN}_\mu K^{(\mu)}_N
\] (5.4)

is non-zero, where \( J^\mu_N \) and \( T^{yN}_\mu \) are the current and the energy-momentum tensor of a massless scalar field and \( K^{(\mu)}_N = e^{2A} \delta^\mu_M (\mu = t, \Theta, \varphi) \) is the Killing vector for the BH metric (5.1-5.2).

Therefore, it is essential to check whether the presence of the singularity in (5.3) and of the zero energy flow in (5.4) will be confirmed if the same scalar curvature \( R \) will be obtained by contracting the Riemann tensor with another contravariant metric tensor field \( \tilde{g}^{ij} \) such that

\[
R = g^{AC} g^{BD} R_{ABCD} = \tilde{g}^{AC} \tilde{g}^{BD} \tilde{R}_{ABCD}
\] (5.5)

where \( \tilde{R}_{ABCD} \) is the modified Riemann tensor with the more generally defined contravariant metric tensor \( \tilde{g}^{ij} \)

\[
\tilde{R}_{ABCD} \equiv \frac{1}{2} (g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - g_{BD,AC}) + g_{np}(\tilde{\Gamma}^n_{BC} \tilde{\Gamma}^p_{AD} - \tilde{\Gamma}^n_{BD} \tilde{\Gamma}^p_{AC}) =
\]

\[
= \frac{1}{2} (...) + g_{np}g_{rs}g_{qt} \tilde{g}^{ns} \tilde{g}^{pt}(\Gamma^n_{BC} \Gamma^p_{AD} - \Gamma^n_{BD} \Gamma^p_{AC})
\] (5.6)

If the scalar curvature \( R \), the connection \( \Gamma^C_{AB} \) are calculated from the initially given metric, equation (5.5) can be treated as an fourth-order algebraic equation with respect to the components \( \tilde{g}^{AB} \) and as an eight-order algebraic equation with respect to the variables \( dX^A \), if again the factorization \( \tilde{g}^{AB} = dX^A dX^B \) is used. This example clearly shows the necessity to go beyond the assumption about the contravariant metric factorization. But on the other hand, even if the factorization assumption is used, the same scalar curvature...
can be obtained by contracting the (modified) Ricci tensor with the contravariant metric tensor $\tilde{g}^{AB}$, i.e. $R = \tilde{g}^{AB}R_{AB}$, which was in fact the cubic algebraic equation, investigated in [8, 9, 10].

But there is also one more way for obtaining the scalar curvature $R$ - by assuming that the following algebraic equation with the usual Riemann tensor components holds

$$R = \tilde{g}^{AC}\tilde{g}^{BD}R_{ABCD}. \quad (5.7)$$

Fortunately, this equation is second order with respect to $\tilde{g}^{AB}$ and fourth order with respect to $dX^A$ and moreover, it does not contain any derivatives of the components $\tilde{g}^{AB}$ and $dX^A$.

Let us now assume that in the framework of the factorization assumption, both equations (5.5) and (5.7) are fulfilled. Then the fulfillment of these equations is a necessary condition for the preservation of the scalar curvature invariant because

$$R_{ABCD}R^{ABCD} = R_{ABCD}\tilde{g}^{AC}\tilde{g}^{BD}\tilde{R}_{ijkl} =$$

$$= (R_{ABCD}dX^A dX^B dX^C dX^D) \left( \tilde{R}_{ijkl} dX^i dX^j dX^k dX^l \right) =$$

$$= (R_{ABCD}\tilde{g}^{AC}\tilde{g}^{BD}) \left( \tilde{R}_{ijkl}\tilde{g}^{ik}\tilde{g}^{jl} \right) = R^2. \quad (5.10)$$

Motivated by the necessity to investigate lower degree algebraic equations, one may take equation (5.7) and also the equation

$$\tilde{g}^{AC}\tilde{g}^{BD}\tilde{R}_{ABCD} - \tilde{g}^{AC}\tilde{g}^{BD}R_{ABCD} = 0. \quad (5.11)$$

A subclass of solutions of this equation will be represented by the algebraic equation

$$\tilde{g}^{BD}\tilde{R}_{ABCD} - \tilde{g}^{BD}R_{ABCD} = 0 \quad (5.12)$$

(cubic with respect to $\tilde{g}^{AB}$ and of sixth order with respect to $dX^A$) and another, more restricted class of solutions - by the equation

$$\tilde{R}_{ABCD} - R_{ABCD} = 0, \quad (5.13)$$

which is quadratic in $\tilde{g}^{AB}$ and quartic with respect to $dX^A$. Therefore, even in such a complicated case, the investigation of the intersection varieties of the two quartic equations (5.7) and (5.13), written respectively as (again, it shall be used that $\tilde{\Gamma}^k_{ij} = dX^k dX^s g_{rs} \Gamma^r_{ij}$)

$$dX^A dX^B dX^C dX^D R_{ABCD} - R = 0 \quad (5.14)$$

and

$$g_{np}g_{rs}g_{ql}(\Gamma^r_{BC}\Gamma^q_{AD} - \Gamma^r_{BD}\Gamma^q_{AC}) dX^n dX^s dX^p dX^t -$$

$$- g_{np}(\Gamma^q_{BC}\Gamma^p_{AD} - \Gamma^q_{BD}\Gamma^p_{AC}) = 0, \quad (5.15)$$
may give some solutions for the contravariant metric tensor components $\tilde{g}^{AB} = dX^A dX^B$, which will preserve both the scalar curvature and the scalar curvature invariant. Respectively, if only the scalar curvature $R$ is to be preserved, one may find the solutions of the algebraic equation (5.7) and then substitute them in the expression for the scalar curvature invariant $R_{ABCD} R^{ABCD}$.

It is clear also that if one takes only equation (5.5) $R = \tilde{g}^{AC} \tilde{g}^{BD} R_{ABCD}$ and not equation (5.7), from (5.8) - (5.10) one may obtain not the equality $R_{ABCD} R^{ABCD} = R^2$, but an fourth- order algebraic equation with respect to $dX^A$ for the preservation of the scalar curvature invariant

$$ R_R R_{ABCD} dX^A dX^B dX^C dX^D - R_{ABCD} R^{ABCD} = 0 \quad . \quad (5.16) $$

But this is not the only possibility. One may take also only equation (5.7) $R = \tilde{g}^{AC} \tilde{g}^{BD} R_{ABCD}$ and disregard equation (5.5). Then the resulting algebraic equation from (5.8) - (5.10) will be

$$ \frac{1}{2} (g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - g_{BD,AC}) dX^A dX^B dX^C dX^D + $$

$$ + g_{np} g_{rs} (\Gamma_{BC}^q \Gamma_{AD}^p - \Gamma_{BD}^q \Gamma_{AC}^p) dX^A dX^B dX^C dX^D dX^p dX^p dX^r - $$

$$ - R_{ABCD} R^{ABCD} = 0 \quad . \quad (5.17) $$

This equation is of eight order and due to the presence of the last scalar curvature invariant term it is impossible to find subclasses of solutions of (lower - order) algebraic equations, as in the case of eq. (5.11).

6 COMPACTIFICATION RADIUS AND SCALAR FIELD EQUATION IN 4D SCHWARZSCHILD BLACK HOLES IN HIGHER DIMENSIONAL BRANE WORLDS

In theories with extra dimensions, for example $(4 + n)$- dimensional Schwarzschild black hole [33, 34, 35, 36]

$$ ds^2 = -h(r) dt^2 + h^{-1}(r) dr^2 + r^2 d\Omega_{n+2}^2 \quad (6.1) $$

with

$$ h(r) = 1 - \left( \frac{r_H}{r} \right)^{n+1} \quad (6.2) $$

($r_H$ - the horizon radius) it is important to distinguish between distances $r \ll R_1$ ($R_1$ - the compactification radius), when the BH is a $(4 + n)$- dimensional one, and distances $r \gg R_1$, when the BH metric goes over to the usual four dimensional Schwarzschild metric

$$ ds^2 = -(1 - \frac{2M}{M_{Hr}^2}) dt^2 + (1 - \frac{2M}{M_{Hr}^2})^{-1} dr^2 + r^2 d\Omega^2 \quad . \quad (6.3) $$
However, when solving the scalar wave equation $g^{I J} \Phi_{I,J} = 0$, there is no way to introduce the scale factor $R_1$ in the solution of the scalar equation. If this can be done, the scalar field behaviour can be compared in the transition from one limit to another.

The use of the more general contravariant tensor $\tilde{g}^{ij}$ gives the opportunity to introduce such a scale factor. Let us first note that

$$g_{AB} \tilde{g}^{BC} = l_A^C(x) \Rightarrow \tilde{g}^{BC} = l^B_D g^{DC}, \quad (6.4)$$

where $A, B, C, D$ concretely in this case will denote the $(4 + n)$-dimensional indices, $\mu, \nu$- only the four-dimensional indices and $i, j, k$ denote the indices of the additional $n$-dimensional space. Then the metric can be represented as

$$ds^2 = g_{AB} dX^A dX^B = g_{\mu\nu} dX^\mu dX^\nu + \sum_{i=5}^{n+4} l^i_i = ds^2(4) + nR_1 \quad (6.5)$$

where it has been assumed that $l^i_i = R_1$ for all $i$. Consequently, some of the components $\tilde{g}^{ij}$ of the contravariant metric tensor can be expressed as

$$\tilde{g}^{ij} = l^j_\nu g^{\nu B} + l^i_\mu g^{\mu B} = l^j_\nu g^{\nu B} + R_1 g^{ij} + l^i_\nu g^{\nu B} \quad (6.6)$$

and evidently the solutions of the scalar wave equation will depend on the compactification radius $R_1$.

7 A COMPLIMENTARY PROPOSAL FOR HIGGS MASS GENERATION IN THEORIES WITH TWO THREE - BRANES

Closely related to the above problem about the contravariant metric tensor components as coupling constants is the problem about Higgs mass generation in theories with two branes [15, 25] - the so called "hidden" and "visible" branes at the orbifold fixed points $\Phi = 0$ and $\Phi = \pi$. The metric, which will be used is again (2.4). These three branes couple to the four dimensional components $G_{\mu \nu}$ of the bulk metric as [15]

$$g^{\text{vis}}_{\mu \nu}(X^\mu) = G_{\mu \nu}(X^\mu, \Phi = \pi) \quad ; \quad g^{\text{hid}}_{\mu \nu}(X^\mu) = G_{\mu \nu}(X^\mu, \Phi = 0). \quad (7.1)$$

The action includes the gravity part plus the action for the visible and hidden branes and also the action for the fundamental Higgs field

$$S_{\text{vis}} = \int d^4X \sqrt{-g_{\text{vis}}} \left[ g^{\mu \nu}_{\text{vis}} D_\mu H^* D_\nu H - \lambda \left( |H|^2 - v_0^2 \right)^2 \right] , \quad (7.2)$$
where $v_0$ is the vacuum expectation value (VEV) for the Higgs field $H$, $\lambda$ is a coupling constant [15]. Similar coupling of the contravariant metric tensor components to a gauge field can be found also in radion cosmology theories [37]. Since $g_{\mu\nu}^\text{vis} = e^{-2kr_{\pi}v_0}g_{\mu\nu}$, it is believed that by a proper normalization of the fields one can determine the physical masses. In particular, if the Higgs field wave function is normalized as $H \rightarrow e^{kr_{\pi}v_0}H$, then

$$S_{\text{vis}} = \int d^4X \sqrt{-g} \left[ g^{\mu\nu}D_\mu H^+ D_\nu H - \lambda \left( |H|^2 - e^{-2kr_{\pi}v_0}^2 \right)^2 \right] .$$

(7.3)

Therefore, since $v = e^{-2kr_{\pi}v_0}$, any mass $m_0$ on the visible three-brane in the fundamental higher-dimensional theory will correspond to a physical mass

$$m = e^{-kr_{\pi}}m_0 ,$$

(7.4)

"measured" with the metric $g^{\mu\nu}$ in the effective Einstein-Hilbert action. If $kr_{\pi} \approx 50$ (i.e. $e^{kr_{\pi}} \approx 10^{15}$), this is the physical mechanism that is supposed to produce TeV physical mass scales from mass parameters around the Planck scale $\approx 10^{19}$ GeV.

In the context of the developed approach in this paper, now it shall be shown that the above physical mechanism of generation of TeV mass scales may turn out to be more complicated and diverse. Namely, for a given scalar curvature, there will be a multitude of contravariant metric tensors, thus suggesting that the possibilities for the mass scales will be much more.

Following the earlier developed algebraic geometry approach in [9, 10], which will be briefly reviewed also in Appendix A, the contravariant metric tensor components $\tilde{g}^{\mu\nu}$ can be written as

$$\tilde{g}^{\mu\nu} = dX^\mu dX^\nu = F_\mu(X(z, v), \Phi(z, v), z)F_\nu(X(z, v), \Phi(z, v), z) .$$

(7.5)

It might seem strange that a particular choice of the contravariant metric components has been used. The important moment here is that for a given metric and scalar curvature and no matter that the contravariant metric components are not generally chosen, there exist contravariant components, for which $g_{\alpha\mu}\tilde{g}^{\mu\nu} = l_\alpha^\nu \neq \delta_\alpha^\nu$.

Further, the (contravariant) metric on the visible brane can be expressed as

$$\tilde{g}_{\text{vis}}^{\mu\nu} = L_2(z, v)\tilde{g}^{\mu\nu} ,$$

(7.6)

where

$$L_2(z, v) \equiv \frac{F_\mu(X(z, v), \Phi(z, v) = \pi, z)F_\nu(X(z, v), \Phi(z, v) = \pi, z)}{F_\mu(X(z, v), \Phi(z, v), z)F_\nu(X(z, v), \Phi(z, v), z)} .$$

(7.7)

Formulae (7.6) - (7.7) have been derived as a ratio of the "visible" and the usual contravariant metric components for each fixed indices $(\mu, \nu) = (\mu_1, \nu_1)$ and without assuming that the points on the complex plane, for which $\Phi(z_0, v_0) = \pi$, are known. Further it shall be shown how the calculation will be modified if these points are assumed to be known.
The transition from the four-dimensional variables \( d^4X = dX_1dX_2dX_3dX_4 \) to the two-dimensional complex variables \((z, v)\) can be performed by using the formulae

\[
d^4X = \sum_{1 \leq i_1 < i_2 \leq 4} \det \left| \frac{\partial X_{i_1}}{\partial z} \frac{\partial X_{i_2}}{\partial v} \right| dz \wedge dv = L_3(z, v)dz \wedge dv , \tag{7.8}
\]

but since we are interested in rescaling only the Higgs field and the contravariant metric as

\[
H \to \tilde{H}f ; \quad v_0 \to \tilde{v}_0 \quad (f - \text{a function}) , \tag{7.9}
\]

the change of variables in the volume integration is not necessary to be taken into account.

Next it is necessary to find how the volume element \(\sqrt{-g_{\text{vis}}}\) of the visible brane can be expressed through the volume element \(\sqrt{-g}\) in terms of the metric (2.4). It can easily be calculated that

\[
\sqrt{-g} = \sqrt{K_1(\Phi, \frac{\partial \Phi}{\partial z}, \frac{\partial \Phi}{\partial v}, \frac{\partial X^\mu}{\partial z}, \frac{\partial X^\mu}{\partial v}) + e^{-4kr_\Phi} K_2(\frac{\partial X^\mu}{\partial z}, \frac{\partial X^\mu}{\partial v})} , \tag{7.10}
\]

where

\[
K_1 \equiv r_c^2 e^{-2kr_\Phi} \left( \frac{\partial^2 \Phi}{\partial z^2} + \left( \frac{\partial \Phi}{\partial v} \right)^2 - \frac{\partial \Phi}{\partial z} \sum_{i=2}^4 \left( \frac{\partial X^i}{\partial z} \right)^2 \right) \\
- \left( \frac{\partial \Phi}{\partial v} \right)^2 \sum_{i=2}^4 \left( \frac{\partial X^i}{\partial z} \right)^2 + 3r_c^4 \left( \frac{\partial \Phi}{\partial v} \right)^2 \left( \frac{\partial \Phi}{\partial z} \right)^2 + 8r_c^2 \frac{\partial \Phi}{\partial z} \frac{\partial \Phi}{\partial v} \frac{\partial X^1}{\partial z} \frac{\partial X^1}{\partial v} e^{-2kr_\Phi} + \\
+ 8r_c^2 e^{-2kr_\Phi} \frac{\partial \Phi}{\partial v} \frac{\partial X^i}{\partial z} \frac{\partial X^i}{\partial v} , \tag{7.11}
\]

\[
K_2 \equiv 8 \frac{\partial X^1}{\partial z} \frac{\partial X^1}{\partial v} \sum_{i=2}^4 \frac{\partial X^i}{\partial z} \frac{\partial X^i}{\partial v} - \left( \frac{\partial X^1}{\partial z} \right)^2 \sum_{i=2}^4 \left( \frac{\partial X^i}{\partial v} \right)^2 - \left( \frac{\partial X^1}{\partial v} \right)^2 \sum_{i=2}^4 \left( \frac{\partial X^i}{\partial z} \right)^2 - \\
- 3 \left( \frac{\partial X^1}{\partial z} \right)^2 \left( \frac{\partial X^1}{\partial v} \right)^2 + \left( \sum_{i=2}^4 \frac{\partial X^i}{\partial z} \right)^2 \left( \sum_{i=2}^4 \frac{\partial X^i}{\partial v} \right)^2 - \\
- 4 \left( \sum_{i=2}^4 \frac{\partial X^i}{\partial z} \frac{\partial X^i}{\partial v} \right)^2 . \tag{7.12}
\]

Setting up \(\Phi(z, v) = \pi\) (note that then \(K_1 = \pi\)), one obtains

\[
\sqrt{-g_{\text{vis}}} = L_1(\Phi, X^\mu, \sqrt{-g})\sqrt{-g} , \tag{7.13}
\]

where

\[
L_1(\Phi, X^\mu, \sqrt{-g}) \equiv \frac{e^{-2kr_\cdot \pi}}{e^{-2kr_\cdot \Phi}} \sqrt{1 - \frac{K_1(\Phi, X^\mu)}{(\sqrt{-g})^2}} . \tag{7.14}
\]
Unlike the previously discussed in [15] case, when the ”visible” volume element is represented as a product of some factor (constant), multiplying the volume element $\sqrt{-g}$ (i.e. $\sqrt{-g_{\text{vis}}} = e^{4kr-\pi} \sqrt{-g}$), the present case might seem to be quite different, since the function $L_1$ depends again on $\sqrt{-g}$. However, it shall be proved below that by requiring the action of the ”visible” brane to remain unchanged after the rescaling (7.9), still such a possibility will exist, but in a more general form. Indeed, after the rescaling (7.9) $H \rightarrow \tilde{H} f$; $v_0 \rightarrow \tilde{v}_0$ the action (7.2) becomes (written in the two-dimensional coordinates $(z, v)$)

$$S_{\text{vis}} = \int dz dv \sqrt{-g} L_3 L_1 \left[ g^{\mu\nu} L_2 f^2 D_{\mu} \tilde{H}^+ D_{\nu} \tilde{H} - \lambda f^4 \left( | \tilde{H} |^2 - \tilde{v}_0^2 \right)^2 \right] +$$

$$+ \int dz dv \sqrt{-g} L_3 L_{\text{add}} ,$$

where

$$L_{\text{add}} \equiv L_2 g^{\mu\nu} [ | \tilde{H} |^2 A_{\nu} \partial_{\mu} f \left| f \right|^2 + | \tilde{H} |^2 A_{\nu} \partial_{\mu} f^+ \partial_{\nu} f +$$

$$+ \tilde{H}^+ f \partial_{\mu} f^+ \partial_{\nu} \tilde{H} + \tilde{H} f \partial_{\mu} f \partial_{\nu} \tilde{H} + \tilde{H} f \partial_{\mu} f^+ \partial_{\nu} \tilde{H}^+ + \tilde{H}^+ f \partial_{\mu} f \partial_{\nu} \tilde{H}^+ ]$$

and the covariant derivative $D_{\mu}$ is expressed as $D_{\mu} = \partial_{\mu} + A_{\mu}$. Clearly, the visible brane actions before and after the rescaling will remain unchanged if

$$L_1 L_2 f^2 = 1 ; \quad L_1 f^4 = 1$$

(7.17)

and

$$L_{\text{add}} = 0 .$$

(7.18)

The first two relations (7.17) give

$$f = \pm (L_2)^{\frac{1}{2}} = \pm (L_1)^{-\frac{1}{2}} ,$$

(7.19)

which can be rewritten as

$$\frac{1}{L_2^3} = \frac{e^{-2kr-\pi}}{e^{-2kr-\Phi}} \sqrt{1 - \frac{K_1(\Phi, X^\mu)}{(\sqrt{-g})^2}}$$

(7.20)

from where the function $K_1(\Phi, X^\mu)$ can be expressed and substituted into expression (7.10) for $\sqrt{-g}$. Thus one obtains

$$\sqrt{-g} = L_2^3 e^{-2kr-\pi} \sqrt{K_2(X^\mu)} .$$

(7.21)

From (7.10) for $\Phi(z, v) = \pi$ one can easily derive

$$\sqrt{-g_{\text{vis}}} = \sqrt{e^{-4kr-\pi} \sqrt{K_2(X^\mu)}} = \frac{1}{L_2^3} \sqrt{-g} .$$

(7.22)

Therefore, even in the more general case of contravariant metric tensor, different from the inverse one, there is a relation similar to $\sqrt{-g_{\text{vis}}} = e^{-4kr-\pi} \sqrt{-g}$, but with the function $\frac{1}{L_2^3}$.
sufficient to know the function $\Phi(z, v)$ as a solution of the system of nonlinear differential equations, but not the points $(z_0^{(l)}, v_0^{(l)})$, at which $\Phi(z = z_0^{(l)}, v = v_0^{(l)}) = \pi$. Consequently, in the final result (7.22) one cannot set up

$$\sqrt{-g_{\text{vis}}} \left( X(z = z_0^{(l)}, v = v_0^{(l)}), \Phi = \pi \right) = \frac{1}{L_2^3(z = z_0^{(l)}, v = v_0^{(l)}), \Phi = \pi} \sqrt{-g} \quad (7.23)$$

Then to any mass $m_0$ on the visible three-brane would correspond a single physical mass, ”measured” with the metric $g^{\mu\nu}$

$$m^{(l)} = m_0 f = m_0 \sqrt{L_2^3(z = z_0^{(l)}, v = v_0^{(l)}), \Phi = \pi} \quad , (7.24)$$
i. e. there is no degeneracy of masses. The corresponding additional condition (7.18) $L_{add} = 0$ can be written as

$$L_{add} = f^2 \partial_\mu \ln f \left[ 2 \mid \tilde{H} \mid^2 A_\nu + 2 \tilde{H}^+ + \tilde{H}^2 \partial_\nu \left( \frac{\tilde{H}^+}{\tilde{H}} \right) \right] - 2 \mid \tilde{H} \mid^2 \partial_\nu (\ln f) - \tilde{H}^2 \partial_\nu (\ln f) = 0 \quad , (7.25)$$

from where the trivial case $f = \text{const}$ is obtained from $\partial_\mu \ln f = 0$.

Let us now see how the above approach will change if the points $(z_0^{(l)}, v_0^{(l)})$ on the complex plane, at which the equation $\Phi(z = z_0^{(l)}, v = v_0^{(l)}) = \pi$ holds, are considered to be known. The function $L_2(z, v)$ in the ratio of $g_{\text{vis}}^{\mu\nu}$ and $g^{\mu\nu}$ will be different and will be denoted as $\tilde{L}_2(z, v)$

$$\tilde{L}_2(z, v) \equiv \frac{F_\mu(X(z_0^{(l)}, v_0^{(l)}), \Phi = \pi, z_0^{(l)}) F_\nu(X(z_0^{(l)}, v_0^{(l)}), \Phi = \pi, z_0^{(l)})}{F_\mu(X(z, v), \Phi(z, v), z) F_\nu(X(z, v), \Phi(z, v), z)} \quad . (7.26)$$

Also from formulae (7.9) for $\Phi = \pi$ and for all points $(z, v) = (z_0^{(l)}, v_0^{(l)})$ one can obtain

$$\sqrt{-g_{\text{vis}}} = \sqrt{-g} e^{-2kr - \pi} \sqrt{\frac{K_2^0(X^\mu(z_0^{(l)}, v_0^{(1)}))}{K_1 + e^{-4kr} K_2(X^\mu(z, v))}} = \tilde{L}_1(z, v) \quad , (7.27)$$

which evidently is different from expression (7.22). Consequently, for this case instead of (7.20) one receives

$$\frac{1}{L_2^3} = e^{-4kr - \pi} \frac{K_2^0(X^\mu(z_0^{(l)}, v_0^{(1)}))}{K_1 + e^{-4kr} K_2(X^\mu(z, v))} \quad , (7.28)$$

from where the function $K_1$ can be expressed and substituted into expression (7.27) for $\sqrt{-g_{\text{vis}}}$. Taking into account again equality (7.10) for $\sqrt{-g}$, one obtains

$$\sqrt{-g_{\text{vis}}} = \sqrt{-g} \frac{1}{L_2^3} = \sqrt{\frac{K_2^0}{e^{2kr - \pi}}} \quad . (7.29)$$
Therefore, the volume element of the "visible" brane is a constant, while the real volume element $\sqrt{-g}$ is $\tilde{L}_2^3$ times the volume of the "visible" brane.

In this case, to any mass $m_0$ on the "visible" brane would correspond $l$ in number physical masses, determined by the formulae

$$m^{(l)} = m_0 f^{(l)} = m_0 \sqrt{\tilde{L}_2^{(l)}(z, v)} ,$$

(7.30)

where the function $\tilde{L}_2(z, v)$ is given by (7.26). Therefore, there will be a degeneracy of masses.

8 CONCLUSION

Let us summarize the most important proposals and results in this (first) part of the paper and give also some suggestions for future research on the base of the refinement of some of the initial assumptions:

1. If the Randall - Sundrum model is investigated within the framework of the multidimensional Lobachevsky space, then there should be some corrections to the extradimensional volume element and to the Newton’s constant. In principle, it is known how the Newton’s force law can be derived for the 4D Lobachevsky space, so probably it can be extended to more dimensions. Note that the corrections to the extradimensional volume can be found after performing the integration in (3.10), using expression (3.7) for the $d-$dimensional hyperbolic volume.

2. The orbifold identification of the points $-\pi r_c$ and $-\pi r_c$ under compactification is performed. Note that the choice of the uniformization coordinate $z$ as $z = \pi r_c (\cos \Phi + i \sin \Phi)$ is an approximation and need not to be done, since the dependence of the angular coordinate $\Phi$ on $z$ should be obtained after finding the algebraic solutions of the cubic equation and performing the integration of the system of nonlinear differential equations, as this was pointed out in [10]. Some particular simple choice of the metric has to be made - the metric (2.4) is fully appropriate for that purpose.

3. Coordinate transformations (4.4) - (4.5), containing the distance $\rho$ in the Lobachevsky space have been performed with respect to the metric (2.4). The choice $\tilde{k} \neq \frac{1}{c}$, when the extradimensional distance exponentially increases, seems to be not consistent with the Lobachevsky geometry, since it is expected to go back to the euclidean geometry in the limit $c \to \infty$. However, this is not the case since the ”absence” of such a constant $\tilde{k} = \frac{1}{c}$ makes such a transition impossible, and this turns out to be physically consistent. Of course, the same approach may be applied to more complicated models with an arbitrary ”warp” exponential factor and $(D - 4)$ compact non-flat extra- dimensional spacetime [29, 30]

$$ds^2 = g_{ab}(X) dX^a dX^b = 2 e^{2A(y)} \eta_{\mu \nu} dX^\mu dX^\nu + h_{ij}(y) dy^i dy^j ,$$

(8.1)

where $(a, b) = 1, 2, ..., D; (\mu, \nu) = 1, 4; (i, j) = 5, ..., D$. The transformations (4.4 - 4.5) can again be used (with $a = b = \tilde{k} e^{-A(y)}$) and an expression for the warp factor $A(y)$ can
be found so that the metric tensor components $h_{ij}$ of the extra- dimensional space are left unchanged. In principle, the motivation for different warp factors comes from $M$- theory (see [31] for a recent review),

4. The algebraic equations for the preservation of the scalar curvature invariant in section 5 have been obtained. In the general case, for an arbitrary tensor $\tilde{g}^{\mu \nu}$, the exact solution of the problem about the conservation of the scalar curvature invariant requires the solution of the fourth-order algebraic equation (5.10) with respect to the components of the tensor $\tilde{g}^{\mu \nu}$, under the fulfillment also of (5.11). On the base of the algorithm, presented in [8-10], this can be performed, and the solution will be greatly facilitated by the fact that there are no derivatives of $\tilde{g}^{\mu \nu}$.

5. The two three - brane model in section 7 has been investigated before in numerous papers, taking into account a more complicated physical setting. Concretely, in [38] the two branes are considered as two positive tension walls, separated by a distance, corresponding to the inverse of the GUT scale. Therefore, the wall tension terms in the effective field theory are taken into account. In section 7 a more simplified model has been presented, having the purpose to set up the mathematical background for the Higgs mass generation in such a two-brane model in more general gravitational theories. The key moment in the problem is how many points $(z_0, v_0)$ satisfy the equation $\Phi(z_0, v_0) = \pi$. Since it is not known whether and under what choice of the metric they can be found, evidently the result has a qualitative character and not a quantitative one. Moreover, again on the base of the papers [10], the initial metric in the coordinates $(X^\mu, \Phi)$ is mapped into a two-dimensional one with coordinates $(z, v)$, so this mapping yet is not studied, neither is known whether the correspondence between the two metrics is a unique one.

9 APPENDIX A: ALGEBRAIC EQUATIONS IN GRAVITY THEORY

In this Appendix some basic information about the algebraic geometry approach will be provided, which was initially developed in [8] and subsequently in [9, 10]. This knowledge will be necessary in order to understand how formulae (7.5) has emerged.

The algebraic geometry approach and the derivation of the basic algebraic equations in gravity theory is based on two different representations of the gravitational Lagrangian, which subsequently are assumed to be equal.

The standard (first) representation of the gravitational Lagrangian is based on the standard Christoffel connection $\Gamma^k_{ij}$, the Ricci tensor $R_{ik}$ and another contravariant tensor, chosen for this partial case in the form of the factorized product $\tilde{g}^{ij} = dX^idX^j$ [10]

$$L_1 = -\sqrt{-\tilde{g}} g^{ik} R_{ik} = -\sqrt{-g} dX^i dX^k R_{ik}. \quad (A1)$$

The choice of this (another) contravariant tensor, which is not the inverse one to the covariant one, is motivated by the affine geometry approach and the gravitational theo-
ries with covariant and contravariant metrics and connections, the essence of which was clarified in the introduction of this paper.

In the second representation, the Christoffell connection \( \tilde{\Gamma}_{ij}^k \) and the Ricci tensor \( \tilde{R}_{ik} \) are the "tilda" quantities

\[ \tilde{R}_{ij} = \tilde{R}_{ji} = \partial_k \tilde{\Gamma}_{ij}^k - \partial_i \tilde{\Gamma}_{kj}^k + \tilde{\Gamma}_{li}^k \tilde{\Gamma}_{lj}^k - \tilde{\Gamma}_{lm}^k \tilde{\Gamma}_{jm}^k , \]  

meaning that the "tilda" Christoffell connection is determined by formulae

\[ \tilde{\Gamma}_{kl}^s \equiv \frac{1}{2} dX^i dX^s (g_{ik,l} + g_{il,k} - g_{kl,i}) \]  

with the new contravariant tensor \( \tilde{g}^{ij} = dX^i dX^j \). Thus the expression for the second representation of the gravitational Lagrangian acquires the form

\[ L_2 = -\sqrt{-\tilde{g}} \tilde{R}_{il} = -\sqrt{-g} dX^i dX^l \{ p\Gamma_{il}^r g_{kr} dX^k - \Gamma_{ik}^r g_{rl} d^2 X^k - \Gamma_{il}^r (g_{kr}) r d^2 X^k \} \].

The condition for the equivalence of the two representations \( L_1 = L_2 \) gives a cubic algebraic equation with respect to the algebraic variety of the first differential \( dX^i \) and the second ones \( d^2 X^i \) [10]

\[ dX^i dX^l \left( p\Gamma_{il}^r g_{kr} dX^k - \Gamma_{ik}^r g_{rl} d^2 X^k - \Gamma_{il}^r (g_{kr}) r d^2 X^k \right) - dX^i dX^l R_{il} = 0 \]  

where \( p \) is the scalar quantity

\[ p \equiv \text{div}(dX) \equiv \frac{\partial(dX^i)}{\partial x^l} , \]  

which "measures" the divergency of the vector field \( dX \). The algebraic variety of the algebraic equation (A5) (i.e. the set of variables, with respect to which the equation is solved and which, if substituted, satisfy it) consists of the differentials \( dX^i \) and their derivatives \( \frac{\partial(dX^i)}{\partial x^l} \).

Similarly, in [9] it was proved that a cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian

\[ \tilde{g}^{ik} \tilde{g}^{jr} s \Gamma_{ik}^r g_{rs} + \tilde{g}^{ik} \tilde{g}^{jr} s (\Gamma_{ik}^r g_{rs}) , l + \tilde{g}^{ik} \tilde{g}^{jr} s m r g_{pr} g_{qs} (\Gamma_{ik}^p \Gamma_{lm}^q - \Gamma_{il}^p \Gamma_{km}^q) - R = 0 \]  

can be obtained also in the case of a generally chosen contravariant metric tensor components (for which \( \tilde{g}^{ij} \neq dX^idX^j \)). Also, the Einstein’s system of equations can also be written in the form of a system of cubic algebraic equations [9] with respect to the contravariant components, but this is irrelevant to the investigation in this paper.

Further, in [9, 10] the solutions of the algebraic equation (A5) have been found. The main peculiarity of the proposed new method for finding the solutions for the contravariant metric components are the following:
1. They are found for the particular case 
\[ g_{\alpha\beta} \tilde{g}^{\mu\nu} = l_\nu^{\alpha} \neq \delta_\nu^{\alpha}, \]
when the contravariant components are not inverse ones to the covariant components. In fact, the algebraic equation (A5) is valid for such a case and under the additional restriction \( \tilde{g}^{ij} = dX^i dX^j. \)

2. It has been assumed also that \( d^2 X^i = 0. \)

3. The algebraic equation (A5) is a multivariable cubic algebraic equation (since the contravariant components \( \tilde{g}^{ij} \) in the general \( n \)-dimensional case are \( n(n-1)/2 \) in number), which is a substantial difference from the two-dimensional case.

For the two-dimensional case, it is known how to parametrize the following two-dimensional cubic algebraic equation

\[ y^2 = 4x^3 - g_2 x - g_3, \quad (A8) \]

where \( g_2 \) and \( g_3 \) are the complex numbers, called the Eisenstein series

\[ g_2 = 60 \sum_{\omega \in \Gamma} \frac{1}{\omega^4}; \quad g_3 = 140 \sum_{\omega \in \Gamma} \frac{1}{\omega^6}. \quad (A9) \]

The basic and very simple idea about parametrization of the cubic algebraic equation (A8) with the Weierstrass function (see the monograph [39] for an excellent introduction into this problem) can be presented as follows: Let us define the lattice \( \Lambda = \{ n \omega_1 + m \omega_2 \mid m, n \in \mathbb{Z}; \omega_1, \omega_2 \in C, Im \omega_1 \omega_2 > 0 \} \) and the mapping \( f : C/\Lambda \to CP^2 \), which maps the factorized (along the points of the lattice \( \Lambda \)) part of the points on the complex plane into the two-dimensional complex projective space \( CP^2 \). This means that each point \( z \) on the complex plane is mapped into the point \( (x, y) = (\rho(z), \rho'(z)) \), where \( x \) and \( y \) belong to the affine curve (A8) and \( \rho(z) \) denotes the Weierstrass elliptic function

\[ \rho(z) = \frac{1}{z^2} + \sum_{\omega} \left[ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right] \quad (A10) \]

and the summation is over the poles in the complex plane. In other words, the functions \( x = \rho(z) \) and \( y = \rho'(z) \) are uniformization functions for the cubic curve and it can be proved [39] that the only cubic algebraic curve with number coefficients which is parametrized by the uniformization functions \( x = \rho(z) \) and \( y = \rho'(z) \) is the affine curve (A8).

In order to parametrize the multivariable cubic algebraic equation (A5), again the parametrization of the two-dimensional equation (A8) has to be used. For the purpose, the approach of the s.c. "embedding sequence of cubic algebraic equations" has been introduced in [9, 10], the essence of which in brief is the following:

The initial cubic multivariable algebraic equation (A5) is presented as a cubic equation with respect to the variable \( dX^3 \) only (for simplicity, the three-dimensional case is taken, but the approach can be generalized to any dimensions)

\[ A_3(dX^3)^3 + B_3(dX^3)^2 + C_3(dX^3) + G^{(2)}(dX^2, dX^1, g_{ij}, \Gamma_{ij}^k, R_{ik}) = 0 \quad , \quad (A11) \]

where the coefficient functions \( A_3, B_3, C_3 \) and \( G^{(2)} \) depend on the variables \( dX^1 \) and \( dX^2 \) of the algebraic subvariety and on the metric tensor \( g_{ij} \), the Christoffel connection
\( \Gamma_{ij}^k \) and the Ricci tensor \( R_{ij} \). Further the Greek indices \( \alpha, \beta \) take values \( \alpha, \beta = 1, 2 \) while the indice \( r \) takes values \( r = 1, 2, 3 \).

In order to obtain the embedded sequence of equations, a linear-fractional transformation

\[
dX^3 = \frac{a_3(z)\tilde{d}X^3 + b_3(z)}{c_3(z)\tilde{d}X^3 + d_3(z)} \tag{A12}
\]

is performed with the purpose of setting up to zero the coefficient functions in front of the highest (third) degree of \( \tilde{d}X^3 \) in the newly derived (i.e. transformed) cubic equation. This will be achieved if \( G^{(2)}(dX^2, dX^1, g_{ij}, \Gamma_{ij}^k, R_{ik}) = -\frac{a_3(z)}{c_3^2} \), which can be rewritten in the form of a two-dimensional cubic algebraic equation with respect to the algebraic variety of the variables \( dX^1 \) and \( dX^2 \):

\[
p\Gamma_{r(\alpha g_{\beta r})}dX^\gamma dX^\alpha dX^\beta + K_{\alpha\beta}^{(1)}dX^\alpha dX^\beta + K_{\alpha\beta}^{(2)}dX^\alpha + 2p \left( \frac{a_3}{c_3} \right)^3 \Gamma_{3g}^3 = 0 \tag{A13}
\]

and \( K_{\alpha\beta}^{(1)} \) and \( K_{\alpha\beta}^{(2)} \) again depend on \( R_{\alpha\beta}, \Gamma_{\alpha\beta}^r, g_{\beta r} \) and the ratios \( \frac{a_3}{c_3} \) and \( \frac{d_3}{c_3} \). The originally given equation (A5) is called "the embedding equation" for the equation (A13). Consequently, in the general case of an \( n \)–dimensional cubic equation, after applying the described above algorithm, one would obtain an \((n - 1)\)–dimensional cubic equation, afterwards again - an \((n - 2)\)–dimensional equation and so on. In other words, this is the s.c. "embedding sequence" of cubic algebraic equations.

In the case of the "transformed" two-dimensional equation (A5) (with respect to the variables \( n_3 = \tilde{d}X^3 \) and \( m = \frac{a_3}{c_3} \)), in [8] it has been proved how it can be brought to an equation of the kind

\[
\tilde{n}^2 = \mathcal{P}_1(\tilde{n}) \ m^3 + \mathcal{P}_2(\tilde{n}) \ m^2 + \mathcal{P}_3(\tilde{n}) \ m + \mathcal{P}_4(\tilde{n}) , \tag{A14}
\]

where \( \mathcal{P}_1(\tilde{n}), \mathcal{P}_2(\tilde{n}), \mathcal{P}_3(\tilde{n}), \mathcal{P}_4(\tilde{n}) \) are complicated functions of the ratios \( \frac{a_3}{d_3}, \frac{b_3}{d_3} \) and \( A_3, B_3, C_3 \) and the variable \( \tilde{n} \) is related to the variable \( n \) through a definite linear transformation. From (A14), one can obtain the parametrizable form

\[
\tilde{n}^2 = 4m^3 - g_2m - g_3 \tag{A15}
\]

of the cubic algebraic equation, from where the solution for \( dX^3 \) can be expressed as

\[
dX^3 = \frac{b_3 \sqrt{\frac{d_3}{c_3}} + \frac{\rho^2(z)}{\sqrt{c_3} \sqrt{c_3}^3} - L_1^{(3)} \frac{B_3}{c_3} \rho(z) - L_2^{(3)} \rho(z)}{\frac{b_3}{c_3} \sqrt{\frac{d_3}{c_3}} + \frac{\rho^2(z)}{\sqrt{c_3} \sqrt{c_3}^3} - L_1^{(3)} \frac{B_3}{c_3} - L_2^{(3)}} . \tag{A16}
\]

It is important to mention that in (A16) \( B_3 \) and \( C_3 \) are complicated functions, depending on \( dX^1 \) and \( dX^2 \), due to which (A16) can be called the "embedding solution" for \( dX^1 \) and \( dX^2 \).

After applying the same parametrization procedure with respect to the embedded equations, one can obtain a similar expression for \( dX^2 \) as an embedding solution for \( dX^1 \).
and an expression for $dX^1$. Consequently, all the solutions (for $l = 1, 2, 3$) can be written as

$$dX^l(X^1, X^2, X^3) = F_l(g_{ij}(X), \Gamma^k_{ij}(X), \rho(z), \rho'(z)) = F_l(X, z) \quad ,$$

(A17)

and the functions $F_l(X, z)$ are "parametrization" functions for the initially given algebraic equation (A5). However, yet it is not justified to call them "uniformization functions", since they depend not only on the complex variable $z$, but also on the generalized coordinates $X$.

Now it shall be proved that these functions can be considered also as "uniformization functions". But as a first step, one should reconcile the appearance of the additional complex coordinate $z$ on the right-hand side of (A17) with the dependence of the differentials on the left-hand side of (A17) only on the generalized coordinates $(X^1, X^2, X^3)$ (and on the initial coordinates $x^1, x^2, x^3$ because of the mapping $X^l = X^l(x^1, x^2, x^3)$). The only reasonable assumption will be that the initial coordinates depend also on the complex coordinate, i.e.

$$X^l \equiv X^l(x^1(z), x^2(z), x^3(z)) = X^l(x, z) \quad .$$

(A18)

Further, the important initial assumptions ($l = 1, 2, 3$)

$$d^2X^l = 0 = dF_l(X, z) = \frac{dF_l}{dz}dz \quad ,$$

(A19)

should be taken into account, from where one easily gets the system of three inhomogeneous linear algebraic equations with respect to the functions $\frac{\partial X^1}{\partial z}, \frac{\partial X^2}{\partial z}$ and $\frac{\partial X^3}{\partial z} \ (l = 1, 2, 3)$

$$\frac{\partial F_l}{\partial X^1} \frac{\partial X^1}{\partial z} + \frac{\partial F_l}{\partial X^2} \frac{\partial X^2}{\partial z} + \frac{\partial F_l}{\partial X^3} \frac{\partial X^3}{\partial z} + \frac{\partial F_l}{\partial z} = 0 \quad .$$

(A20)

The solution of this algebraic system ($i, k, l = 1, 2, 3$)

$$\frac{\partial X^l}{\partial z} = G_l \left( \frac{\partial F_l}{\partial X^k} \right) = G_l \left( X^1, X^2, X^3, z \right) \quad (A21)$$

represents a system of three first-order nonlinear differential equations. A solution of this system can always be found in the form

$$X^1 = X^1(z) \quad ; \quad X^2 = X^2(z) \quad ; \quad X^3 = X^3(z) \quad .$$

(A22)

and therefore, the metric tensor components will also depend only on the complex coordinate $z$, i.e. $g_{ij} = g_{ij}(X(z))$. Thus it is proved that the functions $F_l(X, z)$ in (A17) can be considered also to be "uniformization functions, which depend only on the complex variable $z$.

The parametrization (uniformization) of the initially given cubic algebraic equation can be extended to a parametrization by means of a pair of complex coordinates $(z, v)$ in the following way

$$dX^l(X) = F_l(X(x(z, v))), z) \quad .$$

(A23)

In [9,10] the corresponding system of equations, related to the initial assumption $d^2X^l = 0$ has been analysed and it has been proved that this system is noncontradictory. Therefore, formulae (7.5) will be valid.
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References

[1] L. Landau, and E. Lifschiz 1988 *Theoretical Physics, vol. II. Field Theory* (Moscow: Nauka Publishing)

[2] S. Manoff 1999 *Part. Nucl.* 30 517 - 549 [Rus. Edit.: *Fiz. Elem. Chast. Atomn. Yadra.* 1999 30 (5) 1211 - 1269] (Preprint gr-qc/0006024)

[3] S. Manoff 2002 *Geometry and Mechanics in Different Models of Space - Time: Geometry and Kinematics* (New York: Nova Science Publishers Inc.)

[4] S. Manoff 2002 *Geometry and Mechanics in Different Models of Space - Time: Dynamics and Applications* (New York: Nova Science Publishers, Inc.)

[5] A. P. Norden 1950 *Spaces of Affine Connection* (Moscow: Nauka Publishing)

[6] P. A. Shirokov, and A. P. Shirokov 1959 *Affine Differential Geometry* (Moscow: Fizmatgiz Publishing)

[7] W. Slebodzinski 1998 *Exterior Forms and Their Applications* (University of Beijing: College Press)

[8] B. G. Dimitrov 2003 *J. Math. Phys.* 44 (6) 2542 - 2578 (Preprint hep-th/0107231)

[9] B. G. Dimitrov 2008 Elliptic Curves and Algebraic Geometry Approach in Gravity Theory I. The General Approach , subm. to *Theor. and Mathem. Physics* (Russ. Theoretich. i Mathematich. Fizika (Preprint hep-th/0511136; also Commun. of the JINR, Dubna); 2008 Elliptic Curves, Algebraic Geometry Approach in Gravity Theory and Uniformization of Multivariable Cubic Algebraic Equations, *Intern. J. Geom. Meth. Mod. Phys.*, vol. 5 (5) 677 - 698 (Preprint arXiv:0805.0372)
[10] B. G. Dimitrov 2007 Elliptic Curves and Algebraic Geometry Approach in Gravity Theory II. Parametrization Functions of a Multivariable Cubic Equation (Preprint hep-th/0511136; also Commun. of the JINR, Dubna); III. Uniformization Functions of a Multivariable Cubic Equation (Preprint hep-th/0511136; also Commun. of the JINR, Dubna)

[11] B. A. Rosenfel’d 1966 Multidimensional Spaces (Moscow:Nauka).

[12] B. A. Rosenfel’d 1955 Noneuclidean Geometries (Moscow: Nauka).

[13] B. A. Rosenfel’d 1969 Noneuclidean Spaces (Moscow: Nauka).

[14] E. Beltrami 1868 Saggio di interpretazione della geometria non-euclidea Napoli 6 284 - 312

[15] L. Randall, and R. Sundrum 1999 Phys. Rev. Lett. 83 3370 - 3373 (Preprint hep-ph/9905221)

[16] N. Arkani - Hamed, S. Dimopoulos, and G. R. Dvali 1999 Phys. Rev. D59 086004 (Preprint hep-ph/9807344)

[17] M. Quiros 2003 New Ideas in Symmetry Breaking (Preprint hep-ph/0302189)

[18] Ph. Brax, C. van de Bruck and Anne-Christine Davies 2004 Rept. Prog. Phys. 67 2183 – 2232 (Preprint hep-ph/0404011)

[19] Th. G. Rizzo 2005 J. Phys. Conf. Ser. 18 224-269 Pedagogical Introduction to Extra Dimensions (Preprint hep-ph/0409309)

[20] A. Perez-Lorenzana 2005 An Introduction to Extra Dimensions (Preprint hep-ph/0503177)

[21] C. Csaki 2004 TASI Lectures on Extra Dimensions and Branes (Preprint hep-ph/0404096)

[22] U. U. Nut 1961 Lobachevsky Geometry in an Analytical Exposition (Moscow: Publ. House of the USSR Acad. of Sciences) (in Russian)

[23] N. A. Chernikov 1965 Lectures on the Lobachevsky Geometry and Relativity Theory (Novosibirsk) (in Russian)

[24] L. Randall, and R. Sundrum 1999 Phys. Rev. Lett. 83 4690 - 4693 (Preprint hep-th/9906064)

[25] V. Rubakov 2001 Phys. Usp. 44 871 - 893 [2001 Usp. Fiz. Nauk 171 913 - 938] (Preprint hep-ph/0104152)
[26] N. A. Chernikov 1992 The Gravitational Radius in the Lobachevsky Space (Preprint E2-92-394 of the JINR, Dubna, 1992) [see http://ccdb3fs.kek.jp/cgi-bin/img-index?9301218]

[27] A. Ramsay, and R. Richtmyer 1995 Introduction to Hyperbolic Geometry (New York: Springer - Verlag)

[28] N. V. Yefimov 2003 Higher Geometry 7th edition (Moscow: Fizmatlit)

[29] M. Cavaglia 2003 Intern. J. Mod. Phys. A18 1843 - 1882 (Preprint hep-ph/0210296)

[30] P. C. Argyres, S. Dimopoulos, and J. March - Russell 1998 Phys. Lett. B441 96 (Preprint hep-th/9808138)

[31] P. Chen, K. Dasgupta, K. Narayan, M. Shmakova, and M. Zagermann 2005 JHEP 0509 009 Brane Inflation, Solitons and Cosmological Solutions I (Preprint hep-th/0501185)

[32] P. Kanti, I. Olasagasti, and K. Tamvakis 2002 Phys. Rev. D66 104026 (Preprint hep-th/0207283)

[33] P. Kanti, and K. Tamvakis 2002 Phys. Rev. D65 084010 (Preprint hep-th/0110298)

[34] P. Kanti, and J. March - Russell 2002 Phys. Rev. D66 024023 (Preprint hep-ph/0203223)

[35] P. Kanti, and J. March - Russell 2003 Phys. Rev. D67 104019 (Preprint hep-ph/0212199)

[36] P. Kanti 2004 Intern. J. Mod. Phys. A19 4899 - 4951 (Preprint hep-ph/0402168)

[37] A. Mazumdar, R. N. Mohapatra, and A. Perez - Lorenzana 2004 JCAP 0406 004 (Preprint hep-ph/0310258)

[38] A. Krause 2006 Nucl. Phys. B748 98-125 (Preprint hep-th/0006226)

[39] V. V. Prasolov, and Y. P. Solov’yev 1997 Elliptic Functions and Elliptic Integrals (AMS Translations of Mathematical Monographs 170) (R.I.: Providence) [Russian original: V. V. Prasolov, and Y. P. Solov’yev 1997 Elliptic Functions and Algebraic Equations (Moscow: Factorial Publishing House)