Positive solutions for nonlinear schrödinger–poisson systems with general nonlinearity

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Abstract. In this paper, we study a class of Schrödinger-Poisson (SP) systems with general nonlinearity where the nonlinearity does not require Ambrosetti-Rabinowitz and Nehari monotonic conditions. We establish new estimates and explore the associated energy functional which is coercive and bounded below on Sobolev space. Together with Ekeland variational principle, we prove the existence of ground state solutions. Furthermore, when the ‘charge’ function is greater than a fixed positive number, the (SP) system possesses only zero solutions. In particular, when ‘charge’ function is radially symmetric, we establish the existence of three positive solutions and the symmetry breaking of ground state solutions.

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1. Introduction

In this paper, we consider the following Schrödinger–Poisson (SP) system with general nonlinearity:

\[
\begin{cases}
-\Delta u + u + \rho(x) \phi u = f(u) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi = \rho(x) u^2 \quad \text{in } \mathbb{R}^3,
\end{cases}
\]

where \( \rho \in C(\mathbb{R}^3, \mathbb{R}) \) and \( f \in C(\mathbb{R}, \mathbb{R}) \) satisfying the assumptions that

(D1) \( \rho(x) \) is positive on \( \mathbb{R}^3 \) and \( \lim_{|x| \to \infty} \rho(x) = \rho_\infty > 0 \) uniformly on \( \mathbb{R}^3 \);

(F1) \( f(s) \) is a continuous function on \( \mathbb{R} \) such that \( f(s) \equiv 0 \) for all \( s < 0 \) and \( \lim_{s \to 0^+} f(s) = 0 \);

(F2) there exists \( 2 \leq q < 3 \) and \( a_q > 1 \) if \( q = 2 \); \( a_q > 0 \) if \( 2 < q < 3 \) such that

\[
\lim_{s \to \infty} \frac{f(s)}{s^{q-1}} = a_q.
\]
System \((SP_\rho)\) can be used to describe the interaction of a charged particle with the electrostatic field in quantum mechanics. The unknown \(u\) is the wave function associated with the particle and \(\phi\) the electric potential, whereas \(\rho : \mathbb{R}^3 \rightarrow \mathbb{R}^+\) is a measurable function representing a ‘charge’ corrector to the density \(u^2\) with the nonlinear function \(f(u)\) representing the interaction effect among many particles. For more detailed description on the physical aspects of the problem, we refer the readers to \([8,39]\).

It is well known that such an SP system can be transformed into a nonlinear Schrödinger equation with a non-local term when \(\rho \in L^\infty(\mathbb{R}^3) \cup L^2(\mathbb{R}^3)\) \([2,39]\). Briefly, by the Lax–Milgram theorem, for all \(u \in H^1(\mathbb{R}^3)\), there exists a unique \(\phi = \phi_{\rho,u} \in D^{1,2}(\mathbb{R}^3)\) satisfying \(-\Delta \phi = \rho(x)u^2\) with

\[
\phi_{\rho,u}(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(y)u^2(y)}{|x-y|} dy.
\]

Subsequently, system \((SP_\rho)\) is rewritten as

\[
-\Delta u + u + \rho(x)\phi_{\rho,u}u = f(u) \text{ in } \mathbb{R}^3,
\]

which can now be studied in a variational setting with the solutions given by the critical points of the corresponding energy function \(J_\rho : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}\) defined as

\[
J_\rho(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \rho(x) \phi_{\rho,u}u^2 \, dx - \int_{\mathbb{R}^3} F(u) \, dx,
\]

and \(F(u) = \int_0^u f(s) \, ds\). Thus \((u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) is a solution of system \((SP_\rho)\) if and only if \(u\) is a critical point of the functional \(J_\rho\) and \(\phi = \phi_{\rho,u}\). Moreover, \((u,\phi)\) is a ground state solution of system \((SP_\rho)\), if \(u\) is a ground state solution of equation \((E_\rho)\) with the corresponding functional given by \(J_\rho\).

A vast amount of work has been dedicated to the study of the SP systems with the local nonlinearity \(f(u)\) mostly given in a pure power representation, e.g. \(f(u) = |u|^{p-2}u\), and others in a more generalized formulation \([1,3–5,13,14,17,22,40,42–44,46,48–51]\), in an autonomous setting with \(\rho(x)\) being constant or otherwise. In these settings, the lack of compactness is a common issue and presents a major difficulty for the application of standard variational techniques. Ruiz \([39]\) overcame this issue by restricting the functional in a radially symmetric space when studying a class of autonomous SP systems with \(\rho(x) = \sqrt{\lambda} > 0\) and \(f(u) = |u|^{p-2}u\). The existence of positive radial solutions was proven by minimizing the associated energy functional on a certain manifold that is defined using a combination of Nehari and Pohozaev equalities for the case when \(3 < p < 6\), whereas for \(2 < p < 3\), Strauss inequality was used to show the boundedness of Palais-Smale (PS) sequences, yielding a positive radial solution for small values of \(\lambda\); however, for \(\lambda \geq 1/4\), \(u = 0\) is the unique solution. Azzollini-Pomponio \([6]\) and Zhao-Zhao \([49]\) improved on these results by showing the existence of ground state solutions (possibly non-radial) when \(\lambda = 1\) and \(3 < p < 6\) using the Nehari-Pohozaev manifold developed by Ruiz \([39]\).
For the non-autonomous system involving a non-constant charge density \( \rho(x) \), most studies showed the existence of solutions for \( 4 < p < 6 \) with \( f(u) = |p|^{p-2}u \). For examples, Cerami-Varia [15], were able to prove the existence of ground state and bound state solutions for \( 4 < p < 6 \) without imposing symmetry assumptions by establishing a compactness lemma and using the Nehari manifold method with \( \rho(x) \) satisfying some prescribed conditions. However, for the case when \( 2 < p \leq 4 \), the Palais-Smith (PS) condition on \( H^1(\mathbb{R}^3) \) remained unresolved so that the associated energy functional is not bounded below on both Nehari manifold \( (2 < p \leq 4) \) and Nehari-Pohozaev manifold \( (2 < p < 3) \) for \( \|\rho\|_\infty \) sufficiently small. The commonly adopted variational methods are rendered insufficient in these situations. In [43], an alternative approach was proposed, using a novel constraint technique, the authors demonstrated the existence of positive solutions including ground state solutions for the case when \( 2 < p \leq 4 \) and \( \|\rho\|_\infty \) sufficiently small, thus filling the gap in [15]; in particular, the existence result of ground state solutions was obtained for \( 3.1813 \approx \frac{1 + \sqrt{73}}{3} < p \leq 4 \). This result is improved further by Mercuri and Tyler [34] in their recent paper where the existence of ground state solutions was demonstrated for \( 3 < p < 4 \) with different hypotheses (coercive and non-coercive) on the behavior of \( \rho \) at infinity and extended further for \( 3 < p < 6 \) in [16] under the assumption that \( \rho \in L^\infty_{loc}(\mathbb{R}^3) \) is nonnegative and for every \( M > 0 \),

\[ |\{ x \in \mathbb{R}^3 : \rho(x) < M \}| < \infty. \]

SP systems involving a more general nonlinearity have been studied, for examples, within the context that satisfies Ambrosetti–Rabinowitz (A-R) condition [49,50] thus ensuring the mountain pass geometry of the associated energy functional and guarantee the boundedness of Palais–Smale sequences. Existence results with superlinear \( f(u) \) without the AR condition are also obtained, typically with \( f(s) < C_1 s + C_2 s^5 \) [51] or \( f(s)/|s|^3 \) being an increasing function for \( |s| > 0 \) [1,40,51]. In these problems, alternative techniques were adopted, for examples, using the argument from [9] or otherwise in order to demonstrate the boundedness of PS sequences. Systems with a Berestycki & Lions type nonlinearity were also investigated [4,5], using a concentration and compactness argument in [4], a non-radial solution was proven to exist through the minimization of a modified functional. General linearity that is asymptotically linear at infinity has also been studied in [46] where bounded PS sequences were obtained without requiring the AR condition and a positive solution was found when the charge function \( \rho(x) \) is a small constant. As we have mentioned earlier, related works concerning the various forms of SP systems are numerous and we cannot hope to comment on them all here (nor can our comments be in sufficient details); we have therefore limited our brief remarks on those that are most relevant to our study here. The interested readers should find sufficient materials from afore mentioned references and the references therein for further details.

In the present paper, we focus our attention on the existence and symmetry of ground state solutions for system \( (SP_{\rho}) \) subject to the conditions \((D1),(F1)\) and \((F2)\), extending our previous study in [47] where non-radial
ground state solutions were obtained using Nehari manifold for the SP system with $\rho(x) \equiv \sqrt{\lambda} > 0$ and nonlinearity $f(u) = a(x)|u|^{p-2}u$ for $2 < p < 3$. The condition $(D1)$ is necessary to ensure that the energy functional $J_\rho$ on $H^1(\mathbb{R}^3)$ is bounded below. The challenge here is that given these weaker conditions, the variational approaches such as Nehari and Nehari–Pohozaev manifolds are no longer applicable nor can we assume the mountain pass geometry of the energy functional to construct bounded PS sequences without the AR condition being satisfied. We thus make progress below using an entirely different approach by means of a global minimizer, the results are summarized as follows. For the autonomous cases with $\rho(x)$ being constant, we expand the results of Ruiz’s work in [39] with a more general nonlinearity including the case of $f(u)$ being asymptotically linear at infinity, demonstrating the existence of multiple positive solutions in Theorem 1.1. In the presence of non-constant $\rho(x)$, we make use of Theorem 1.1 to prove the existence of ground state solutions in Theorem 1.2; however, the solution is shown to be uniquely zero in Theorem 1.3 when $\inf_{x \in \mathbb{R}^3} \rho(x)$ exceeds a certain threshold value. In Theorem 1.4, by considering system $(SP_\rho)$ in a radially symmetric setting, three positive solutions are obtained including a non-radial ground state solution.

Before presenting our main results in the theorems below, we need to state the following maximization problems:

$$
\Lambda_0 := \sup_{u \in A_0} \frac{\int_{\mathbb{R}^3} F(u) \, dx - \frac{1}{2} \|u\|^2_{H^1}}{\int_{\mathbb{R}^3} \phi_u u^2 \, dx}
$$

and

$$
\overline{\Lambda}_0 := \sup_{u \in \overline{A}_0} \frac{\int_{\mathbb{R}^3} f(u) \, dx - \|u\|^2_{H^1}}{\int_{\mathbb{R}^3} \phi_u u^2 \, dx},
$$

where $\int_{\mathbb{R}^3} \phi_u u^2 \, dx = \int_{\mathbb{R}^3} \rho(x) \phi_{\rho,u} u^2 \, dx$ for $\rho(x) \equiv 1$, and the sets

$$
A_0 := \left\{ u \in H^1_r(\mathbb{R}^3) : \int_{\mathbb{R}^3} F(u) \, dx - \frac{1}{2} \|u\|^2_{H^1} > 0 \right\}
$$

and

$$
\overline{A}_0 := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} f(u) \, dx - \|u\|^2_{H^1} > 0 \right\}.
$$

Then there exist $M_0, \overline{M}_0 > 0$ such that $0 < \Lambda_0 \leq M_0$ and $0 < \overline{\Lambda}_0 \leq \overline{M}_0$ (see Appendix). Our main results are obtained by considering the following equation,

$$
-\Delta u + u + \lambda \phi_u u = f(u) \text{ in } \mathbb{R}^3.
$$

(\text{E}_\lambda^\infty)

with its solutions given by the critical points of the energy functional $J_\lambda^\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$
J_\lambda^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u u^2 \, dx - \int_{\mathbb{R}^3} F(u) \, dx.
$$

Then we have the following result for the autonomous case.
Theorem 1.1. Suppose that conditions (F1) and (F2) hold. Then we have the following results.

(i) For every $0 < \lambda < 4\Lambda_0$, Equation $(E^\infty_\lambda)$ has two positive radial solutions $v^{(1)}_\lambda, v^{(2)}_\lambda \in H^1_r(\mathbb{R}^3)$ with $J^\infty_\lambda \left(v^{(1)}_\lambda\right) < 0 < J^\infty_\lambda \left(v^{(2)}_\lambda\right)$.

(ii) For every $\lambda > \overline{\Lambda}_0$, $u = 0$ is the unique solution of Equation $(E^\infty_\lambda)$.

By conditions (F1) and (F2), there exist $q < p < 3$ and $C_0 > 0$ such that

$$f(s) \leq \frac{1}{4}s + C_0s^{p-1} \text{ for all } s \geq 0.$$  \hspace{1cm} (1.2)

We now present our main results for the non-autonomous case.

Theorem 1.2. Suppose that conditions (F1), (F2) and (D1) hold, and $0 < \lambda < 4\Lambda_0$ is given. In addition, we assume that

(D2) \hspace{1cm} 0 < \rho_{\text{min}} := \inf_{x \in \mathbb{R}^3} \rho(x) < d_0 := \left(\frac{C_02^{3(4-p)/2}(3-p)^{3-p}}{p(p-2)^{p-2}}\right)^{1/(p-2)} < \rho_\infty; \hspace{1cm} (1.3)

(D3) \int_{\mathbb{R}^3} \rho(x) \phi_{\rho,v^{(1)}_\lambda} \left(v^{(1)}_\lambda\right)^2 \, dx < \lambda \int_{\mathbb{R}^3} \phi_{\rho,v^{(1)}_\lambda} \left(v^{(1)}_\lambda\right)^2 \, dx, \text{ where } v^{(1)}_\lambda \text{ is a radial positive solution of Equation } (E^\infty_\lambda) \text{ as in Theorem 1.1.}

Then Equation $(E_\rho)$ has a positive ground state solution $u_\rho$ such that $J_\rho(u_\rho) < 0$.

Theorem 1.3. Suppose that conditions (F1), (F2) and (D1) hold and $\rho_{\text{min}} > \sqrt{\Lambda}_0$. Then $u = 0$ is the unique solution of Equation $(E_\rho)$.

To study the symmetry breaking of ground state solutions, we consider the following equation:

$$-\Delta u + u + \rho_\varepsilon(x) \phi_{\rho_\varepsilon,u}u = f(u) \text{ in } \mathbb{R}^3, \hspace{1cm} (E_{\rho_\varepsilon})$$

where $\rho_\varepsilon(x) = \rho(\varepsilon x)$ and $\varepsilon > 0$. Then we have the following results.

Theorem 1.4. Suppose that conditions (F1) – (F2) and (D1) hold, and $0 < \lambda < 4\Lambda_0$ is given. In addition, we assume that (D4) $\rho(x) = \rho(|x|)$; (D5)

$$0 < \rho(0) < \min\left\{d_0, \sqrt{\lambda}\right\} \leq \max\left\{d_0, \sqrt{\lambda}\right\} < \rho_\infty.$$  \hspace{1cm} (D5)

Then Equation $(E_{\rho_\varepsilon})$ has three positive solutions $u_{\rho_\varepsilon} \in H^1(\mathbb{R}^3)$ and $v^{(1)}_{\rho_\varepsilon}, v^{(2)}_{\rho_\varepsilon} \in H^1_r(\mathbb{R}^3)$ such that

$$J_{\rho_\varepsilon}(u_{\rho_\varepsilon}) < J_{\rho_\varepsilon}(v^{(1)}_{\rho_\varepsilon}) < 0 < J_{\rho_\varepsilon}(v^{(2)}_{\rho_\varepsilon}) \text{ for } \varepsilon \text{ sufficiently small.}$$

Furthermore, $u_{\rho_\varepsilon}$ is a non-radial ground state solution of Equation $(E_{\rho_\varepsilon})$.

Remark 1.5. (i) Suppose that conditions (D1) and (D5) hold. Let $v^{(1)}_\lambda$ be a radial positive solution of Equation $(E^\infty_\lambda)$ as in Theorem 1.1 and let $\rho(x_0)$
< \min \{ d_0, \sqrt{\lambda} \} \) for some \( x_0 \in \mathbb{R}^3 \). Define \( v_\varepsilon (x) = v_\lambda^{(1)}(x - \frac{x_0}{\varepsilon}) \). Then it follows from condition (D2) that
\[
\int_{\mathbb{R}^3} \rho(x) \phi_{\rho, v_\varepsilon} v_\varepsilon^2 dx = \int_{\mathbb{R}^3} \rho(\varepsilon x + x_0) \phi_{\rho(\varepsilon x + x_0), v_\lambda^{(1)}} (v_\lambda^{(1)})^2 dx
= \rho^2(x_0) \int_{\mathbb{R}^3} \phi_{v_\lambda^{(1)}} (v_\lambda^{(1)})^2 dx + o(\varepsilon)
\]
\[
< \lambda \int_{\mathbb{R}^3} \phi_{v_\lambda^{(1)}} (v_\lambda^{(1)})^2 dx \text{ for } \varepsilon \text{ sufficiently small.}
\]
This implies that when \( \rho(x) \) is replaced by \( \rho(\varepsilon x + x_0) \), the condition (D3) holds for \( \varepsilon \) sufficiently small. Therefore, by Theorem 1.2, Equation \( (E_\rho) \) has a positive ground state solution \( u_\rho \in H^1(\mathbb{R}^3) \) such that \( J_\rho(u_\rho) < 0 \).

(ii) Assume that the conditions hold in Theorem 1.4. Since \( \rho(x) = \rho(|x|) \) and \( \rho(0) < \min \{ d_0, \sqrt{\lambda} \} \), using an argument similar to that in part (i), we have
\[
\int_{\mathbb{R}^3} \rho(x) \phi_{\rho, v_\lambda^{(1)}} (v_\lambda^{(1)})^2 dx < \lambda \int_{\mathbb{R}^3} \phi_{v_\lambda^{(1)}} (v_\lambda^{(1)})^2 dx \text{ for } \varepsilon \text{ sufficiently small,}
\]
since \( \rho(0) < \min \{ d_0, \sqrt{\lambda} \} \). This means that the symmetric case still holds in Theorem 1.2 implying that Equation \( (E_\rho) \) has a radial positive solution \( v_\rho \in H^1(\mathbb{R}^3) \) such that
\[
J_\rho(v_\rho) < 0 \text{ for } \varepsilon \text{ sufficiently small.}
\]

**Remark 1.6.** Under the assumption that \( \rho \neq 0 \), equation \( (E_\rho) \) can be regarded as a perturbation problem of the following nonlinear Schrödinger equation:

\[
\begin{aligned}
- \Delta u + u &= f(u) \text{ in } \mathbb{R}^3, \\
 u &\in H^1(\mathbb{R}^3).
\end{aligned}
\]

(\( E_0 \))

It is known that equation \( (E_0) \) has a positive (ground state) solution under various conditions (c.f. Berestycki–Lions [7], Lions [27, 28]). The study of the uniqueness and symmetry of positive solutions to equation \( (E_0) \) has a very long history, of particular interest is the case where
\[
f(u) = |u|^{q-2} u, \text{ for } 2 < q < 2^*,
\]
with \( 2^* = 2N/(N-2) \) being the critical Sobolev exponent in dimensions \( N \geq 3 \) and \( 2^* = \infty \) in dimensions \( N = 1, 2 \). The uniqueness of positive solutions was proven first by Coffman [12] for \( q = 4 \) and \( N = 3 \), and later by Kwong [24] for the general case. The symmetry of positive solutions was proven by Gidas et al. [20, 21]. These results have subsequently been extended to include a larger class of non-linearities by many authors; see for examples [10, 11, 23, 25, 26, 30–32, 36, 37, 41]. Note that the solution loses its uniqueness and symmetry properties when \( \rho \neq 0 \) is assumed.

This paper is organized as follows. In Sect. 2, we provide the necessary preliminaries and prove that the energy functional \( J_\rho \) is coercive and bounded below in \( H^1(\mathbb{R}^3) \). In Sect. 3, we show that \( \inf_{u \in H^1(\mathbb{R}^3)} J_\rho(u) < 0 \). In Sect. 4,
we investigate the Palais-Smale condition of $J_\rho$ on $H^1(\mathbb{R}^3)$, which is followed by the proofs of Theorems 1.2 and 1.3. Finally, in Sect. 5 the proof of Theorem 1.4 is provided.

2. Preliminaries

We first establish the following estimates on the nonlinearity.

Lemma 2.1. Suppose that $2 < p < 3$ and $d > 0$. Let $C_0 > 0$ be as in (1.2) and $f_d(s) = \frac{1}{8} + \frac{d}{\sqrt{8}}s - \frac{C_0}{p} s^{p-2}$ for $s > 0$. Then there exist $d_0 := \left(\frac{C_0 2^{(4-p)/2} (3-p)^{3-p}}{p(p-2)^2 (3-p)^{3-p}}\right)^{1/(p-2)} > 0$ and $s_0(d) := \left(\frac{\sqrt{8} C_0 (p-2)}{pd}\right)^{1/(3-p)} > 0$ satisfying

(i) $f_d'(s_0(d)) = 0$ and $f_d(s_0(d)) = 0$;

(ii) for each $d < d_0$ there exist $\eta_d, \xi_d > 0$ such that $\eta_d < s_0(d) < \xi_d$ and $f_d(s) < 0$ for all $s \in (\eta_d, \xi_d)$;

(iii) for each $d > d_0$, $f_d(s) > 0$ for all $s > 0$.

Proof. By a straightforward calculation, we can show that the results hold. \qed

Following the idea of Lions [29] (or see [39]), we have

$$\frac{1}{\sqrt{8}} \int_{\mathbb{R}^3} \rho(x) |u|^3 \, dx = \frac{1}{\sqrt{8}} \int_{\mathbb{R}^3} (-\Delta \phi_{\rho,u}) |u| \, dx = \frac{1}{\sqrt{8}} \int_{\mathbb{R}^3} \langle \nabla \phi_{\rho,u}, \nabla |u| \rangle \, dx$$

$$\leq \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \phi_{\rho,u}|^2 \, dx$$

$$= \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx$$

$$+ \frac{1}{8} \int_{\mathbb{R}^3} \rho(x) \phi_{\rho,u} u^2 \, dx$$

for all $u \in H^1(\mathbb{R}^3)$. \hfill \(2.1\)

Then by (1.2) and (2.1),

$$J_\rho(u) \geq \frac{1}{2} \|u\|_{H^1}^2 + \frac{1}{4} \int_{\mathbb{R}^3} \rho(x) \phi_{\rho,u} u^2 \, dx - \frac{C_0}{p} \int_{\mathbb{R}^3} |u|^p \, dx - \frac{1}{8} \int_{\mathbb{R}^3} u^2 \, dx$$

$$\geq \frac{1}{4} \|u\|_{H^1}^2 + \int_{\mathbb{R}^3} u^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}} \rho(x) |u| - \frac{C_0}{p} |u|^{p-2}\right) \, dx$$

$$+ \frac{1}{8} \int_{\mathbb{R}^3} \rho(x) \phi_{\rho,u} u^2 \, dx$$

$$= \frac{1}{4} \|u\|_{H^1}^2 + \int_{\{\rho(x) < d_0\}} u^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}} \rho(x) |u| - \frac{C_0}{p} |u|^{p-2}\right) \, dx$$

$$+ \frac{1}{2} \int_{\{\rho(x) \geq d_0\}} u^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}} \rho(x) |u| - \frac{C_0}{p} |u|^{p-2}\right) \, dx$$

$$+ \frac{1}{8} \int_{\mathbb{R}^3} \rho(x) \phi_{\rho,u} u^2 \, dx,$$ \hfill \(2.2\)
where $C_0 > 0$ as in (1.2). By Lemma 2.1 and (2.2), we have
\[
J_\rho(u) \geq \frac{1}{4} \left\| u \right\|_{H^1}^2 + \frac{1}{2} \int_{\{\rho(x) < d_0\}} u^2 \left( \frac{1}{8} + \frac{1}{\sqrt{8}} \rho(x) |u| - \frac{C_0}{p} |u|^{p-2} \right) dx
\]
\[
\geq \frac{1}{4} \left\| u \right\|_{H^1}^2 + \frac{1}{2} \int_{\{\rho(x) < d_0\}} m_\rho(x) dx,
\]
(2.3)
where $m_\rho(x) = \inf_{s \geq 0} \left( \frac{1}{8} s^2 + \frac{1}{\sqrt{8}} \rho(x) s^3 - \frac{C_0}{p} s^p \right) < 0$ for all $x \in \{\rho(x) < d_0\}$.

Note that
\[
\inf_{x \in \{\rho(x) < d_0\}} m_\rho(x) \leq \frac{1}{8} s_0^2 (\rho_{\min}) + \frac{\rho_{\min}}{\sqrt{2}} s_0^3 (\rho_{\min}) - \frac{C_0}{p} s_0^p (\rho_{\min}) < 0,
\]
and
\[
0 > \int_{\{\rho(x) < d_0\}} m_\rho(x) dx \geq \inf_{x \in \{\rho(x) < d_0\}} m_\rho(x) \{\rho(x) < d_0\},
\]
(2.4)
where $s_0(\rho_{\min}) = \left( \frac{\sqrt{\rho C_0(p-2)}}{p \rho_{\min}} \right)^{1/(3-p)}$. Furthermore, the following statements are true.

**Theorem 2.2.** Suppose that conditions (F1), (F2), (D1) and (D2) hold. Then $J_\rho$ is coercive and bounded below on $H^1(\mathbb{R}^3)$. Furthermore,
\[
\inf_{u \in H^1(\mathbb{R}^3)} J_\rho(u) > \int_{\{\rho(x) < d_0\}} m_\rho(x) dx > -\infty.
\]

**Proof.** By conditions (D1) and (D2), we can conclude that
\[
0 < |\{\rho(x) < d_0\}| < \infty.
\]
(2.5)
Thus, by (2.3) - (2.5),
\[
0 > \int_{\{\rho(x) < d_0\}} m_\rho(x) dx > -\infty
\]
and
\[
J_\rho(u) \geq \frac{1}{4} \left\| u \right\|_{H^1}^2 + \int_{\{\rho(x) < d_0\}} m_\rho(x) dx.
\]
This completes the proof. □

**Lemma 2.3.** Suppose that conditions (F1), (F2), (D1) and (D2) hold. Let $u_0$ be a non-trivial solution of the following equation:
\[
-\Delta u + u + \rho_\infty \phi_{\rho_\infty} u u = f(u) \text{ in } \mathbb{R}^3.
\]
(\text{E}_{\rho_\infty})

Then $J_{\rho_\infty}(u_\lambda) > 0$, where $J_{\rho_\infty} = J_\rho$ for $\rho \equiv \rho_\infty$.

**Proof.** By Lemma 2.1 and (2.1) - (2.2),
\[
J_{\rho_\infty}(u) \geq \frac{1}{4} \left\| u \right\|_{H^1}^2 + \frac{1}{2} \int_{\mathbb{R}^3} u^2 \left( \frac{1}{8} + \frac{1}{\sqrt{8}} \rho_\infty |u| - \frac{C_1}{p} |u|^{p-2} \right) dx
\]
\[
> 0 \text{ for all } u \in H^1(\mathbb{R}^3) \setminus \{0\}.
\]
This completes the proof. □
Next, we define the Palais–Smale (PS) sequences and (PS)–conditions in $H^1(\mathbb{R}^3)$ for $J_\rho$ as follows.

**Definition 2.6.** (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_\beta$–sequence in $H^1(\mathbb{R}^3)$ for $J_\rho$ if $J_\rho(u_n) = \beta + o(1)$ and $J'_\rho(u_n) = o(1)$ strongly in $H^{-1}(\mathbb{R}^3)$ as $n \to \infty$.

(ii) We say that $J_\rho$ satisfies the $(PS)_\beta$–condition in $H^1(\mathbb{R}^3)$ if every $(PS)_\beta$–sequence in $H^1(\mathbb{R}^3)$ for $J_\rho$ contains a convergent subsequence.

**Proposition 2.5.** Suppose that conditions $(F1)$, $(F2)$ and $(D1)$ hold. Let $\{u_n\}$ be a bounded $(PS)_\beta$–sequence in $H^1(\mathbb{R}^3)$ for $J_\rho$. There exist a subsequence $\{u_n\}$, a number $m \in \mathbb{N}$, sequences $\{x_n^i\}_{n=1}^{\infty}$ in $\mathbb{R}^3$, a function $u_0 \in H^1(\mathbb{R}^3)$, and $0 \neq v^i \in H^1(\mathbb{R}^3)$ when $1 \leq i \leq m$ such that

(i) $|x_n^i| \to \infty$ and $|x_n^i - x_m^j| \to \infty$ as $n \to \infty$, $1 \leq i \neq j \leq m$;

(ii) $-\Delta u_0 + u_0 + \rho(x) \phi_{\rho,u_0} u_0 = f(u_0)$ in $\mathbb{R}^3$;

(iii) $-\Delta v^i + v^i + \rho_\infty \phi_{\rho_\infty,v^i} v^i = f(v^i)$ in $\mathbb{R}^3$;

(iv) $u_n = u_0 + \sum_{i=1}^{m} v^i (\cdot - x_n^i) + o(1)$ strongly in $H^1(\mathbb{R}^3)$; and

(v) $J_\rho(u_n) = J_\rho(u_0) + \sum_{i=1}^{m} J_{\rho_\infty}(v^i) + o(1)$.

The proof follows the same argument of [15, Lemma 4.1] or [45, Lemma 5.1] and is omitted here.

**Corollary 2.6.** Suppose that conditions $(F1)$, $(F2)$, $(D1)$ and $(D2)$ hold. Let $\{u_n\}$ be a $(PS)_\beta$–sequence in $H^1(\mathbb{R}^3)$ for $J_\rho$ with $\beta < 0$. Then there exist a subsequence $\{u_n\}$ and a nonzero $u_0$ in $H^1(\mathbb{R}^3)$ such that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$ and $J_\rho(u_0) = \beta$. Furthermore, $u_0$ is a non-trivial solution of Equation $(E_\rho)$.

**Proof.** Let $\{u_n\}$ be a $(PS)_\beta$–sequence in $H^1(\mathbb{R}^3)$ for $J_\rho$ with $\beta < 0$. By Theorem 2.2, there exist a subsequence $\{u_n\}$ and $u_0 \in H^1(\mathbb{R}^3)$ such that $u_n \to u_0$ weakly in $H^1(\mathbb{R}^3)$ and $J'_\rho(u_n) = 0$. Moreover, by Lemma 2.3 and Proposition 2.5 (iv) – (v), $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$ and $J_\rho(u_0) = \beta$. Thus, $u_0$ is a non-trivial solution of Equation $(E_\rho)$. This completes the proof. \qed

3. **Proof of Theorem 1.1**

We are now ready to prove Theorem 1.1. (i) By the definition of $\Lambda_0$, for each $\lambda < 4\Lambda_0$ there exists $v_0 \in A$ such that

$$\frac{\int_{\mathbb{R}^3} F(v_0) \, dx - \frac{1}{2} \|v_0\|_{H^1}^2}{\int_{\mathbb{R}^3} \phi_{v_0} v_0^2 \, dx} > \frac{\lambda}{4},$$

this implies that

$$J^\infty_\lambda(v_0) = \frac{1}{2} \|v_0\|_{H^1}^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_{v_0} v_0^2 \, dx - \int_{\mathbb{R}^3} F(v_0) \, dx < 0. \quad (3.1)$$
Moreover, by (1.2), for each $u \in H^1_r (\mathbb{R}^3)$, 
\[
J^\infty_\lambda (u) = \frac{1}{2} \| u \|_{H^1_r}^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u u^2 \, dx - \int_{\mathbb{R}^3} F (u) \, dx \\
\geq \frac{3}{8} \| u \|_{H^1_r}^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u u^2 \, dx - \frac{C_0}{p} \int_{\mathbb{R}^3} |u|^p \, dx,
\]
where $2 < p < 3$. Then by the same argument used by Ruiz in [39, Theorem 4.3], there exists $K > 0$ such that 
\[
J^\infty_\lambda (u) \geq \frac{3}{8} \| u \|_{H^1_r}^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u u^2 \, dx - \frac{C_0}{p} \int_{\mathbb{R}^3} |u|^p \, dx \\
\geq \frac{1}{8} \| u \|_{H^1_r}^2 - K \quad \text{for all } u \in H^1_r (\mathbb{R}^3),
\] 
(3.2)
this implies that $J_\lambda$ is coercive and bounded below on $H^1_r (\mathbb{R}^3)$. Using (3.1) and (3.2), we have 
\[-\infty < \alpha^\infty_\lambda := \inf_{u \in H^1_r (\mathbb{R}^3)} J^\infty_\lambda (u) < 0.\]
(3.3)
Then by the Ekeland variational principle (cf. [18]) and Palais criticality principle (cf. [35]), there exists a sequence $\{u_n\} \subset H^1_r (\mathbb{R}^3)$ such that 
\[J^\infty_\lambda (u_n) = \alpha^\infty_\lambda + o(1) \quad \text{and} \quad (J^\infty_\lambda)' (u_n) = o(1) \quad \text{in} \quad H^{-1} (\mathbb{R}^3).\]
Again, adopting the argument used in [39, Theorem 4.3], there exists a subsequence $\{u_n\}$ and $v^{(1)}_\lambda \in H^1_r (\mathbb{R}^3) \setminus \{0\}$ such that $u_n \to v^{(1)}_\lambda$ strongly in $H^1 (\mathbb{R}^3)$ and $v^{(1)}_\lambda$ is a solution of Equation $(E^\infty_\lambda)$. This indicates that $J^\infty_\lambda (v^{(1)}_\lambda) = \alpha^\infty_\lambda < 0$. By the maximum principle, we conclude that $v^{(1)}_\lambda$ is a positive solution of Equation $(E^\infty_\lambda)$. Moreover, by condition (F1) and the Sobolev embedding, 
\[
J^\infty_\lambda (u) \geq \frac{3}{8} \| u \|_{H^1_r}^2 - \frac{C_0}{p} \int_{\mathbb{R}^3} |u|^p \, dx \\
\geq \frac{1}{8} \| u \|_{H^1_r}^2 - \frac{C_0}{p^{p'} p} \| u \|_{H^1_r}^p \quad \text{for all } u \in H^1_r (\mathbb{R}^3).
\]
This implies that there exist $\eta, \kappa > 0$ such that $\| v^{(1)}_\lambda \|_{H^1} > \eta$ and 
\[
\max \{ J^\infty_\lambda (0), J^\infty_\lambda (v^{(1)}_\lambda) \} < 0 < \kappa \leq \inf_{\| u \|_{H^1_r} = \eta} J^\infty_\lambda (u).
\]
Define 
\[
\beta^\infty_\lambda = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} J^\infty_\lambda (\gamma (\tau)),
\]
where $\Gamma = \{ \gamma \in C([0,1], H^1_r (\mathbb{R}^3)) : \gamma (0) = 0, \gamma (1) = v^{(1)}_\lambda \}$. Then by the mountain pass theorem (cf. [19,38]) and Palais criticality principle, there exists a sequence $\{u_n\} \subset H^1_r (\mathbb{R}^3)$ such that 
\[
J^\infty_\lambda (u_n) \to \beta^\infty_\lambda \geq \kappa \quad \text{and} \quad \| (J^\infty_\lambda)' (u_n) \|_{H^{-1}} \to 0, \quad \text{as} \quad n \to \infty,
\]
and using an argument similar to that in [39, Theorem 4.3], we have a subsequence $\{u_n\}$ and $v^{(2)}_\lambda \in H^1_r (\mathbb{R}^3)$ with $u_n \to v^{(2)}_\lambda$ strongly in $H^1 (\mathbb{R}^3)$, indicating that $J^\infty_\lambda (v^{(2)}_\lambda) = \beta^\infty_\lambda > 0$ and $(J^\infty_\lambda)' (v^{(2)}_\lambda) = 0$. Using condition (F1), we
obtain that $v^{(2)}_\lambda$ is nonnegative on $\mathbb{R}^3$. Applying the maximum principle then gives the result that $v^{(2)}_\lambda$ is a positive solution of Equation $(E^\infty_\lambda)$.

(ii) Suppose that the contrary is true. Let $u_0$ be a nontrivial solution of Equation $(E^\infty_\lambda)$. Then by the definition of $\Lambda_0$ and $\lambda > \Lambda_0$,

$$
0 = \|u_0\|^2_{H^1} + \lambda \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 dx - \int_{\mathbb{R}^3} f(u_0) u_0 dx,
$$

$$
> \|u_0\|^2_{H^1} + \Lambda_0 \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 dx - \int_{\mathbb{R}^3} f(u_0) u_0 dx \geq 0,
$$

which is a contradiction. Hence we complete the proof.

4. Proofs of Theorems 1.2 and 1.3

Lemma 4.1. Suppose that conditions $(F1), (F2)$ and $(D1) - (D3)$ hold. Then

$$
-\infty < \alpha_\rho := \inf_{u \in H^1(\mathbb{R}^3)} J_\rho(u) < 0.
$$

Proof. Let $v^{(1)}_\lambda$ be a radial positive solution of Equation $(E^\infty_\lambda)$ as in Theorem 1.1. Applying condition $(D3)$ gives

$$
J_\rho(v^{(1)}_\lambda) = \frac{1}{2} \|v^{(1)}_\lambda\|^2_{H^1} + \frac{1}{4} \int_{\mathbb{R}^3} \rho(x) \phi_{v^{(1)}_\lambda}(v^{(1)}_\lambda)^2 dx - \int_{\mathbb{R}^3} F(v^{(1)}_\lambda) dx,
$$

$$
\leq \frac{1}{2} \|v^{(1)}_\lambda\|^2_{H^1} + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_{v^{(1)}_\lambda}(v^{(1)}_\lambda)^2 dx - \int_{\mathbb{R}^3} F(v^{(1)}_\lambda) dx,
$$

$$
< 0.
$$

Thus, by Theorem 2.2 and (4.1), we have

$$
-\infty < \alpha_\rho := \inf_{u \in H^1(\mathbb{R}^3)} J_\rho(u) < 0.
$$

This completes the proof. □

We are now ready to prove Theorem 1.2. As a consequence of Theorem 2.2 and Lemma 4.1, the functional $J_\rho$ is coercive on $H^1(\mathbb{R}^3)$ and

$$
-\infty < \alpha_\rho := \inf_{u \in H^1(\mathbb{R}^3)} J_\rho(u) < 0.
$$

By the Ekeland variational principle, there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that

$$
J_\rho(u_n) = \alpha_\rho + o(1) \text{ and } J'_\rho(u_n) = o(1) \text{ in } H^{-1}(\mathbb{R}^3).
$$

Thus, by Theorem 2.2 and Corollary 2.6, there exist a subsequence $\{u_n\}$ and $u_\rho \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that $u_n \to u_\rho$ strongly in $H^1(\mathbb{R}^3)$ and $u_\rho$ is a solution of Equation $(E_\rho)$ indicating that $J_\rho(u_\rho) = \alpha_\rho < 0$. Moreover, condition $(F1)$ ensures that $u_\rho$ is nonnegative on $\mathbb{R}^3$. Using the maximum principle, we may conclude that $u_\rho$ is a positive ground state solution of Equation $(E_\rho)$. 
We are now ready to prove Theorem 1.3. Suppose that the contrary is true. Let $u_0$ be a nontrivial solution of Equation $(E_\rho)$. Then by the definition of $\Lambda_0$ and $\rho_{\min} > \sqrt{\Lambda_0}$,
\[
0 = \|u_0\|_{H^1}^2 + \int_{\mathbb{R}^3} \rho(x) \phi_{\rho,u_0} u_0^2 dx - \int_{\mathbb{R}^3} f(u_0) u_0 dx \\
\geq \|u_0\|_{H^1}^2 + \rho_{\min}^2 \int_{\mathbb{R}^3} \phi_{\rho,u_0} u_0^2 dx - \int_{\mathbb{R}^3} f(u_0) u_0 dx \\
> \|u_0\|_{H^1}^2 + \Lambda_0 \int_{\mathbb{R}^3} \phi_{\rho,u_0} u_0^2 dx - \int_{\mathbb{R}^3} f(u_0) u_0 dx \geq 0,
\]
which is a contradiction. Hence we complete the proof.

5. Proof of Theorem 1.4

By conditions $(D1)$ and $(D5)$, without loss of generality, we may assume that
\[
B^3(0,1) \subset \text{int} \left\{ x \in \mathbb{R}^3 : \rho(x) < \sqrt{\lambda} \right\},
\]
this implies that $B^3(0, \frac{1}{\epsilon}) \subset \Omega_{\epsilon} := \text{int} \left\{ x \in \mathbb{R}^3 : \rho(\epsilon x) < \sqrt{\lambda} \right\}$. As we know, $v^{(1)}_{\lambda}$ is a radial positive solution of Equation $(E_\lambda)$ and $J_{\lambda}^{\infty} (v^{(1)}_{\lambda}) < 0$. For $R > 0$, we define a cut-off function $\psi_R \in C^1(\mathbb{R}^3, [0,1])$ as
\[
\psi_R(x) = \begin{cases} 1 & |x| < \frac{R}{2}, \\ 0 & |x| > R, \end{cases}
\]
and $|\nabla \psi_R| \leq 1$ in $\mathbb{R}^3$. Let $u_R(x) = v^{(1)}_{\lambda} \psi_R(x)$. Then it is true that
\[
\int_{\mathbb{R}^3} F(u_R) dx \rightarrow \int_{\mathbb{R}^3} F(v^{(1)}_{\lambda}) dx \text{ as } R \rightarrow \infty, \tag{5.1}
\]
\[
\|u_R\|_{H^1} \rightarrow \|v^{(1)}_{\lambda}\|_{H^1} \text{ as } R \rightarrow \infty, \tag{5.2}
\]
and
\[
\int_{\mathbb{R}^3} \phi_{\rho,u} u_R^2 dx \rightarrow \int_{\mathbb{R}^3} \phi_{v^{(1)}_{\lambda}} (v^{(1)}_{\lambda})^2 dx \text{ as } R \rightarrow \infty, \tag{5.3}
\]
where $\int_{\mathbb{R}^3} \phi_{\rho,u} u^2 dx = \int_{\mathbb{R}^3} \rho(x) \phi_{\rho,u} u^2 dx$ for $\rho \equiv 1$. Since $J_{\lambda}^{\infty} \in C^1 (H^1(\mathbb{R}^3), \mathbb{R})$ and $J_{\lambda}^{\infty} (v^{(1)}_{\lambda}) < 0$, by $(5.1)$ - $(5.3)$, there exists $R_0 > 0$ such that
\[
J_{\lambda}^{\infty} (u_{R_0}) < 0. \tag{5.4}
\]
Let
\[
u^{(i)}_{R_0,N}(x) = v_{\lambda} (x + iN^3 e) \psi_{R_0} (x + iN^3 e)
\]
for $e \in S^2$ and $i = 1, 2, \ldots, N$, where $N^3 > 2R_0$. Let $0 < \epsilon_N \leq \frac{1}{N^3 + R_0}$. Then we have the following result,
\[
\text{supp} \nu^{(i)}_{R_0,N}(x) \subset B^3 \left( 0, \frac{1}{\epsilon_N} \right) \text{ for all } i = 1, 2, \ldots, N.
\]
Clearly, $\varepsilon_N \to 0^+$ as $N \to \infty$. Moreover, using condition (D1), we deduce that
\[ \left\| u_{R_0,N}^{(i)} \right\|_{H^1}^2 = \left\| u_{R_0} \right\|_{H^1}^2 \quad \text{for all } N, \]
\[ \int_{\mathbb{R}^3} F \left( u_{R_0,N}^{(i)} \right) \, dx = \int_{\mathbb{R}^3} F \left( u_{R_0} \right) \, dx \quad \text{for all } N, \]
and
\[ \int_{\mathbb{R}^3} \rho \left( \varepsilon_N x \right) \phi_{\rho \varepsilon, u_{R_0,N}^{(i)}} \left[ u_{R_0,N}^{(i)} \right]^2 \, dx \]
\[ = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho \left( \varepsilon_N x \right) \rho \left( \varepsilon_N y \right) \frac{\left[ u_{R_0,N}^{(i)} (x) \right]^2 \left[ u_{R_0,N}^{(i)} (y) \right]^2}{4\pi |x - y|} \, dx \, dy \]
\[ = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho \left( \varepsilon_N x - i \varepsilon_N N^3 e \right) \rho \left( \varepsilon_N y - i \varepsilon_N N^3 e \right) \frac{\left[ u_R (x) \right]^2 \left[ u_R (y) \right]^2}{4\pi |x - y|} \, dx \, dy. \]
Since $0 < \varepsilon_N \leq \frac{1}{N^3 + R_0}$, there exists $N_0 > 0$ with $N_0^3 > 2R_0$ such that for every $N \geq N_0$, we have
\[ \int_{\mathbb{R}^3} \rho \left( \varepsilon_N x \right) \phi_{\rho \varepsilon, u_{R_0,N}^{(i)}} \left[ u_{R_0,N}^{(i)} \right]^2 \, dx \]
\[ = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho \left( \varepsilon_N x \right) \rho \left( \varepsilon_N y \right) \frac{\left[ u_{R_0,N}^{(i)} (x) \right]^2 \left[ u_{R_0,N}^{(i)} (y) \right]^2}{4\pi |x - y|} \, dx \, dy \]
\[ < \lambda \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_{R_0}^2 (x) u_{R_0}^2 (y)}{4\pi |x - y|} \, dx \, dy \]
\[ = \lambda \int_{\mathbb{R}^3} \phi_{u_{R_0,N}^{(i)}} \left[ u_{R_0,N}^{(i)} \right]^2 \, dx, \]
for all $e \in \mathbb{S}^2$ and $i = 1, 2, \ldots, N$. Let
\[ w_{R_0,N} (x) = \sum_{i=1}^{N} u_{R_0,N}^{(i)}. \]
Observe that $w_{R_0,N}$ is a sum of translation of $u_{R_0}$. When $N^3 \geq N_0^3 > 2R_0$, the summands have disjoint support and
\[ \text{supp} w_{R_0,N} (x) \subset B^3 \left( 0, \frac{1}{\varepsilon_N} \right). \] (5.5)
In this case we have,
\[ \left\| w_{R_0,N} \right\|_{H^1}^2 = N \left\| u_{R_0} \right\|_{H^1}^2, \] (5.6)
\[ \int_{\mathbb{R}^3} F \left( w_{R_0,N} \right) \, dx = \sum_{i=1}^{N} \int_{\mathbb{R}^3} F \left( u_{R_0,N}^{(i)} \right) \, dx = N \int_{\mathbb{R}^3} F \left( u_{R_0} \right) \, dx \] (5.7)
and
\[ \int_{\mathbb{R}^3} \rho \left( \varepsilon_N x \right) \phi_{\rho \varepsilon, w_{R,N}} \left[ w_{R,N} \right]^2 \, dx < \lambda \int_{\mathbb{R}^3} \phi_{w_{R,N}} \left[ w_{R,N} \right]^2 \, dx. \]
Moreover,
\[
\int_{\mathbb{R}^3} \phi_{w_{R,N}} w_{R,N}^2 \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w_{R_0,N}^2(x) w_{R_0,N}^2(y)}{4\pi |x-y|} \, dx \, dy
\]
\[
= \sum_{i=1}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left[u_{R_0,N}^{(i)}(x)\right]^2 \left[u_{R_0,N}^{(i)}(y)\right]^2}{4\pi |x-y|} \, dx \, dy
\]
\[
+ \sum_{i \neq j}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left[u_{R_0,N}^{(i)}(x)\right]^2 \left[u_{R_0,N}^{(j)}(y)\right]^2}{4\pi |x-y|} \, dx \, dy
\]
\[
= N \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_{R_0}^2(x) u_{R_0}^2(y)}{4\pi |x-y|} \, dx \, dy
\]
\[
+ \sum_{i \neq j}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left[u_{R_0,N}^{(i)}(x)\right]^2 \left[u_{R_0,N}^{(j)}(y)\right]^2}{4\pi |x-y|} \, dx \, dy. \quad (5.8)
\]

A straightforward calculation gives
\[
0 < \sum_{i \neq j}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left[u_{R_0,N}^{(i)}(x)\right]^2 \left[u_{R_0,N}^{(j)}(y)\right]^2}{4\pi |x-y|} \, dx \, dy \leq \frac{N^2 - N}{N^3 - 2R_0} \left(\int_{\mathbb{R}^3} v_\lambda^2(x) \, dx\right)^2,
\]
implies that
\[
\sum_{i \neq j}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left[u_{R_0,N}^{(i)}(x)\right]^2 \left[u_{R_0,N}^{(j)}(y)\right]^2}{4\pi |x-y|} \, dx \, dy \to 0 \text{ as } N \to \infty. \quad (5.9)
\]

We can now adopt the idea of multibump technique by Ruiz [39] (also see [33]) and the following results are obtained.

**Lemma 5.1.** Suppose that conditions (F1), (F2), (D1), (D4) and (D5) hold. Then
\[
\alpha_{\rho_\varepsilon} \to -\infty \quad \text{as} \quad \varepsilon \to 0^+.
\]

**Proof.** By (5.4) - (5.9), we obtain
\[
J_{\rho_\varepsilon} (w_{R,N}) = \frac{1}{2} \|w_{R_0,N}\|_{H^1}^2 + \frac{1}{4} \int_{\mathbb{R}^3} \rho (\varepsilon_N x) \phi_{\rho_\varepsilon w_{R,N}} w_{R,N}^2 \, dx
\]
\[
- \int_{\mathbb{R}^3} F (w_{R_0,N}) \, dx
\]
\[
\leq \frac{N}{2} \|u_{R_0}\| - N \int_{\mathbb{R}^3} F (u_{R_0}) \, dx
\]
\[
+ \frac{\lambda N}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_{R_0}^2(x) u_{R_0}^2(y)}{4\pi |x-y|} \, dx \, dy
\]
\[
+ \frac{\lambda}{4} \sum_{i \neq j}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[ u_{R_0,N}^{(i)}(x) \right] \left[ u_{R_0,N}^{(j)}(y) \right] \frac{dx dy}{4\pi |x - y|} \\
\leq N J_{\lambda}^{\infty}(u_{R_0}) + C_0 \text{ for some } C_0 > 0
\]

and
\[
J_{\rho \varepsilon N}(w_{R,N}) \to -\infty \text{ as } N \to \infty.
\]

Thus we arrive at (5.10).

\[\square\]

**Lemma 5.2.** Suppose that conditions \((F1), (F2), (D1), (D4)\) and \((D5)\) hold. Then there exists \(M > 0\) independent of \(\varepsilon\) such that for \(\varepsilon > 0\) sufficiently small, there exists a sequence \(\varepsilon_n \to 0\) for which
\[
\inf_{u \in H^1_r(\mathbb{R}^3)} J_{\rho \varepsilon}(u)
\]

is nonnegative bounded from above by \(M\).

**Proof.** Since \(\rho(x) = \rho(|x|)\), by Remark 1.5 (i),
\[
\inf_{u \in H^1_r(\mathbb{R}^3)} J_{\rho \varepsilon}(u)
\]

satisfies
\[
J_{\rho \varepsilon}(u) \geq \frac{1}{2} \|u\|^2_{H^1} + \frac{\lambda}{4} \int \rho(\varepsilon x) \phi_{\rho \varepsilon}(v_{\lambda})^2 dx - \int F(v_{\lambda}) dx
\]

\[
\leq \frac{1}{2} \|v_{\lambda}\|^2_{H^1} + \frac{\lambda}{4} \int \phi_{\rho \varepsilon}^2 dx - \int F(v_{\lambda}) dx < 0 \text{ for } \varepsilon \text{ sufficiently small.}
\]

Moreover, by (1.2) and applying the argument in Ruiz [39, Theorem 4.3], there exists \(M > 0\) such that
\[
J_{\rho \varepsilon}(u) \geq \frac{1}{2} \|u\|^2_{H^1} + \frac{\rho_{\min}^2}{4} \int \phi_{\rho \varepsilon} u^2 dx - \int F(u) dx
\]

\[
\geq \frac{1}{2} \|u\|^2_{H^1} + \frac{\rho_{\min}^2}{4} \int \phi_{\rho \varepsilon} u^2 dx - \frac{1}{8} \int u^2 dx - \frac{C_1}{p} \int |u|^p dx
\]

\[
\geq \frac{3}{8} \|u\|^2_{H^1} + \frac{\rho_{\min}^2}{4} \int \phi_{\rho \varepsilon} u^2 dx - \frac{C_1}{p} \int |u|^p dx > -M \text{ for all } u \in H^1_r(\mathbb{R}^3),
\]

and so
\[
\inf_{u \in H^1_r(\mathbb{R}^3)} J_{\rho \varepsilon}(u) \geq -M. \text{ This completes the proof.} \quad \square
\]

Define
\[
\theta_{\rho \varepsilon} := \inf_{u \in H^1_r(\mathbb{R}^3)} J_{\rho \varepsilon}(u).
\]

By Lemmas 5.1 and 5.2, we have
\[
\alpha_{\rho \varepsilon} < \theta_{\rho \varepsilon} < 0 \text{ for } \varepsilon > 0 \text{ sufficiently small.} \quad (5.11)
\]

Then by the Ekeland variational principle and Palais criticality principle, for \(\varepsilon\) small enough, there exists a sequence \(\{u_n\} \subset H^1_r(\mathbb{R}^3)\) such that
\[
J_{\rho \varepsilon}(u_n) = \theta_{\rho \varepsilon} + o(1) \text{ and } J'_{\rho \varepsilon}(u_n) = o(1) \text{ in } H^{-1}(\mathbb{R}^3). \quad (5.12)
\]

We are now ready to prove Theorem 1.4. Given that \(\{u_n\} \subset H^1_r(\mathbb{R}^3)\) satisfies
\[
J_{\rho \varepsilon}(u_n) = \theta_{\rho \varepsilon} + o(1) \text{ and } J'_{\rho \varepsilon}(u_n) = o(1) \text{ in } H^{-1}(\mathbb{R}^3).
\]
Then Theorem 2.2 ensures that \( \{u_n\} \) is bounded. Without loss of generality, we can assume that there exists \( v_{\rho_\varepsilon} \in H^1 (\mathbb{R}^3) \) such that \( u_n \rightharpoonup v_{\rho_\varepsilon} \) weakly in \( H^1 (\mathbb{R}^3) \). Moreover, by Ruiz [39, Lemma 2.1], \( J'_{\rho_\varepsilon} (v_{\lambda, \varepsilon}) = 0 \) in \( H^{-1} (\mathbb{R}^3) \) and \( u_n \rightarrow v_{\rho_\varepsilon} \) strongly in \( H^1 (\mathbb{R}^3) \), implying that \( J_{\rho_\varepsilon} (v_{\rho_\varepsilon}) = \theta_{\rho_\varepsilon} \). Thus, \( v_{\rho_\varepsilon} \) is a radial ground state solution of Equation \((E_{\rho_\varepsilon})\). Using condition \((F1)\), we have the result that \( v_{\rho_\varepsilon} \) is nonnegative on \( \mathbb{R}^3 \) and applying the maximum principle, we conclude that \( v_{\rho_\varepsilon} \) is a positive solution of Equation \((E_{\rho})\). Therefore, by Theorem 1.2 and \((5.11)\), Equation \((E_{\rho_\varepsilon})\) has two positive solutions \( u_{\rho_\varepsilon} \in H^1 (\mathbb{R}^3) \) and \( v_{\rho_\varepsilon} \in H^1 (\mathbb{R}^3) \) such that

\[
\alpha_{\rho_\varepsilon} = J_{\rho_\varepsilon} (u_{\rho_\varepsilon}) < \theta_{\rho_\varepsilon} = J_{\rho_\varepsilon} (v_{\rho_\varepsilon}) < 0
\]

for \( \varepsilon \) sufficiently small. Since

\[
\alpha_{\rho_\varepsilon} = \inf_{u \in H^1 (\mathbb{R}^3)} J_{\rho_\varepsilon} (u) < \theta_{\rho_\varepsilon} = \inf_{u \in H^1 (\mathbb{R}^3)} J_{\rho_\varepsilon} (u)
\]

and \( v_{\rho_\varepsilon} \) is a radial ground state solution of Equation \((E_{\rho_\varepsilon})\), we can conclude that \( u_{\rho_\varepsilon} \) is a non-radial ground state solution of Equation \((E_{\rho_\varepsilon})\). On the other hand, we have an isolated minimum at 0 and also an absolute minimum \( u_{\rho_\varepsilon} \). Moreover, by condition \((F1)\) and the Sobolev embedding,

\[
J_{\rho_\varepsilon} (u) \geq \frac{3}{8} \| u \|_{H^1}^2 - \frac{C_1}{p} \int_{\mathbb{R}^3} |u|^p \, dx \\
\geq \frac{3}{8} \| u \|_{H^1}^2 - \frac{C_1}{p \delta_p} \| u \|_{H^1}^p
\]

for all \( u \in H^1 (\mathbb{R}^3) \).

This implies that there exist \( \eta, \kappa > 0 \) such that \( \| v_{\rho_\varepsilon} \|_{H^1}^2 > \eta \) and

\[
\max \{ J_{\rho_\varepsilon} (0), J_{\rho_\varepsilon} (v_{\rho_\varepsilon}) \} < 0 < \kappa \leq \inf_{\| u \|_{H^1} = \eta} J_{\rho_\varepsilon} (u)
\]

for \( \varepsilon \) sufficiently small.

Define

\[
\beta_{\rho_\varepsilon} = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} J_{\rho_\varepsilon} (\gamma (\tau)),
\]

where \( \Gamma = \{ \gamma \in C([0,1], H^1 (\mathbb{R}^3)) : \gamma (0) = 0, \gamma (1) = v_{\rho_\varepsilon} \} \). Then by the mountain pass theorem (cf. [19, 38]) and Palais criticality principle, there exists a sequence \( \{u_n\} \subset H^1 (\mathbb{R}^3) \) such that

\[
J_{\rho_\varepsilon} (u_n) \rightarrow \beta_{\rho_\varepsilon} \geq \kappa \quad \text{and} \quad \| J'_{\rho_\varepsilon} (u_n) \|_{H^{-1}} \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty.
\]

Adopting the argument used in [39, Theorem 4.3], we have a subsequence \( \{u_n\} \) and \( \tilde{v}_{\rho_\varepsilon} \in H^1 (\mathbb{R}^3) \) with \( u_n \rightharpoonup \tilde{v}_{\rho_\varepsilon} \) strongly in \( H^1 (\mathbb{R}^3) \), implying that \( J_{\rho_\varepsilon} (\tilde{v}_{\rho_\varepsilon}) = \beta_{\rho_\varepsilon} > 0 \) and \( J'_{\rho_\varepsilon} (\tilde{v}_{\rho_\varepsilon}) = 0 \). Condition \((F1)\) ensures that \( \tilde{v}_{\rho_\varepsilon} \) is nonnegative on \( \mathbb{R}^3 \) and the maximum principle gives the result that \( \tilde{v}_{\rho_\varepsilon} \) is a positive solution of Equation \((E_{\rho_\varepsilon})\). Therefore, we conclude that Equation \((E_{\rho_\varepsilon})\) has three positive solutions \( u_{\rho_\varepsilon} \in H^1 (\mathbb{R}^3) \) and \( \tilde{v}_{\rho_\varepsilon}, v_{\rho_\varepsilon} \in H^1 (\mathbb{R}^3) \) such that

\[
\alpha_{\rho_\varepsilon} = J_{\rho_\varepsilon} (u_{\rho_\varepsilon}) < \theta_{\rho_\varepsilon} = J_{\rho_\varepsilon} (v_{\rho_\varepsilon}) < 0 < \beta_{\rho_\varepsilon} = J_{\rho_\varepsilon} (\tilde{v}_{\rho_\varepsilon}).
\]

This completes the proof.
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A. Appendix

Let
\[ A_0 := \left\{ u \in H^1_r(\mathbb{R}^3) : \int_{\mathbb{R}^3} F(u) \, dx - \frac{1}{2} \| u \|_{H^1}^2 > 0 \right\}; \]
\[ \overline{A}_0 := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} f(u) \, udx - \| u \|_{H^1}^2 > 0 \right\} \]
and
\[ \Lambda_0 := \sup_{u \in A_0} \frac{\int_{\mathbb{R}^3} F(u) \, dx - \frac{1}{2} \| u \|_{H^1}^2}{\int_{\mathbb{R}^3} \phi_u u^2 \, dx}; \]
\[ \overline{\Lambda}_0 := \sup_{u \in \overline{A}_0} \frac{\int_{\mathbb{R}^3} f(u) \, udx - \| u \|_{H^1}^2}{\int_{\mathbb{R}^3} \phi_u u^2 \, dx}. \]

Then we have the following results.

Theorem A.1. Suppose that conditions (F1) and (F2) hold. Then we have

(i) \( A_0 \) is a nonempty set.

(ii) \( 0 < \Lambda_0 < \infty \).

Proof. (i) If \( q = 2 \), then by \( a_q > 1 \) and Fatou’s lemma, for \( u \in H^1_r(\mathbb{R}^3) \) with \( \| u \|_{H^1}^2 - a_q \int_{\mathbb{R}^3} u^2 \, dx < 0 \), we have
\[ \lim_{t \to \infty} \frac{1}{t^2} \left[ \frac{1}{2} \| tu \|_{H^1}^2 - \int_{\mathbb{R}^3} F(tu) \, dx \right] = 2 \left( \| u \|_{H^1}^2 - a_q \int_{\mathbb{R}^3} u^2 \, dx \right) < 0, \]
and so there exists \( e \in H^1_r(\mathbb{R}^3) \) such that
\[ \int_{\mathbb{R}^3} F(e) \, dx - \frac{1}{2} \| e \|_{H^1}^2 > 0. \]

If \( 2 < q < 3 \), then by \( a_q > 0 \) and Fatou’s lemma, for \( u \in H^1_r(\mathbb{R}^3) \setminus \{0\} \), we have
\[ \lim_{t \to \infty} \frac{1}{t^q} \left[ \frac{1}{2} \| tu \|_{H^1}^2 - \int_{\mathbb{R}^3} F(tu) \, dx \right] = -a_q \int_{\mathbb{R}^3} |u|^q \, dx < 0, \]
and so there exists $\hat{e} \in H^1_r(\mathbb{R}^3)$ such that
\[
\int_{\mathbb{R}^3} F(\hat{e}) \, dx - \frac{1}{2} \|\hat{e}\|^2_{H^1} > 0.
\]
This implies that $A_0$ is nonempty.

(ii) For each $u \in A_0$ there exists $\lambda^* > 0$ such that
\[
\frac{1}{2} \|u\|^2_{H^1} + \frac{\lambda^*}{4} \int_{\mathbb{R}^3} \phi_u u^2 \, dx - \int_{\mathbb{R}^3} F(u) \, dx < 0
\]
or
\[
\frac{\lambda^*}{4} < \frac{\int_{\mathbb{R}^3} F(u) \, dx - \frac{1}{2} \|u\|^2_{H^1}}{\int_{\mathbb{R}^3} \phi_u u^2 \, dx},
\]
indicating that there exists $\hat{\lambda}^* > 0$ such that $\Lambda_0 \geq \hat{\lambda}^*$. Next, we show that $0 < \Lambda_0 < \infty$. By conditions $(F1)$ and $(F2)$, there exists $C_1 > 0$ such that
\[
F(u) \leq \frac{1}{2} u^2 + C_1 |u|^3.
\] (A.1)

Since
\[
C_1 \int_{\mathbb{R}^3} |u|^3 \, dx = C_1 \int_{\mathbb{R}^3} (-\Delta \phi_u) |u| \, dx = C_1 \int_{\mathbb{R}^3} \langle \nabla \phi_u, \nabla |u| \rangle \, dx
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{C_1^2}{2} \int_{\mathbb{R}^3} |\nabla \phi_u|^2 \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{C_1^2}{2} \int_{\mathbb{R}^3} \phi_u u^2 \, dx \text{ for all } u \in H^1(\mathbb{R}^3),
\]
by (A.1), we have
\[
\frac{\int_{\mathbb{R}^3} F(u) \, dx - \frac{1}{2} \|u\|^2_{H^1}}{\int_{\mathbb{R}^3} \phi_u u^2 \, dx} \leq \frac{C_1^2}{2} \times \frac{\frac{1}{2} \int_{\mathbb{R}^3} u^2 \, dx + C_1 \int_{\mathbb{R}^3} |u|^3 \, dx - \frac{1}{2} \|u\|^2_{H^1}}{C_1 \int_{\mathbb{R}^3} |u|^3 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx}
\]
\[
\leq \frac{C_1^2}{2} \times \frac{C_1 \int_{\mathbb{R}^3} |u|^3 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx}{C_1 \int_{\mathbb{R}^3} |u|^3 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx}
\]
\[
= \frac{C_1^2}{2}.
\]
Thus,
\[
0 < \Lambda_0 := \sup_{u \in A_0} \frac{\int_{\mathbb{R}^3} F(u) \, dx - \frac{1}{2} \|u\|^2_{H^1}}{\int_{\mathbb{R}^3} \phi_u u^2 \, dx} \leq \frac{C_1^2}{2}.
\]
This completes the proof.

\[\square\]

**Theorem A.2.** Suppose that conditions $(F1)$ and $(F2)$ hold. Then we have

(i) $\overline{A_0}$ is a nonempty set.

(ii) $0 < \Lambda_0 < \infty$.

**Proof.** The proof is similar to the argument used in Theorem A.1 and is omitted here. \[\square\]
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