EXACT AND NUMERICAL SOLUTION OF STOCHASTIC BURGERS EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT. We will introduce exact and numerical solutions to some stochastic Burgers equations with variable coefficients. The solutions are found using a coupled system of deterministic Burgers equations and stochastic differential equations.

1. Introduction. The goal of this paper is to introduce a numerical solution, and in some cases an exact solution, for stochastic Burgers equations (SBE) with variable coefficients. We consider the following two different stochastic Burgers equations with a space-uniform white noise of the form

\[ \frac{du}{dt} = (A(t) \partial_z u + B(t) u \partial_z u + C(t) \partial_z u + D(t) u) dt + E(t) \partial_z u dW_t \]

and

\[ \frac{du}{dt} = (A(t) \partial_z u + B(t) u \partial_z u + C(t) \partial_z u + D(t) u) dt + E(t) dW_t \]

for \( t \in [t_0, T] \) and \( z \in \mathbb{R} \) with \( u(0, z) = \phi(z) \) for \( z \in \mathbb{R} \).

Most physical and biological systems are not homogeneous, in part due to fluctuations in environmental conditions and the presence of nonuniform media. Therefore, most of the nonlinear equations with real applications possess coefficients varying spatially and/or temporally and even stochastic terms. Burgers equations play a fundamental role in a large number of models of heat diffusion and reaction processes in nonlinear acoustics, biology, chemistry, genetics and many other areas of research [5, 19]. In particular stochastic Burgers equations have attracted significant attention, see for example [18, 9, 3, 14, 12, 11, 8]. Numerical solutions of the SBE with space-time white noise is studied in [10] via the path integral formulation and ensemble average. It is also studied in [1] and [13] using finite difference and in [4] using Galerkin approximations.

In this paper, we investigate a new numerical algorithm of two types of SBEs with variable coefficients and a space-uniform white noise using a coupled deterministic Burgers equation with first order or second order stochastic differential equations (SDE). Numerical simulations are compared to the exact solutions in some cases.

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2. Preliminaries and results. Consider the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) for which the Brownian motion \(\{W_t; t \geq 0\}\) is defined where \(E(W_s W_t) = \min(s, t)\) for all \(s, t \geq 0\). Also consider the filtration \(\mathcal{F}_t := \sigma(W_s : s \leq t)\) being the smallest \(\sigma\)-algebra to which \(W_s\) is measurable for \(s \leq t\).

Then consider the stochastic differential equation (SDE) with variable coefficients \([15]\)
\[
\begin{align*}
   dX_t &= \alpha(t, X_t)dt + \beta(t, X_t)dW_t, \\
   X_t &= X_{t_0} + \int_{t_0}^{t} \alpha(s, X_s)\,ds + \int_{t_0}^{t} \beta(s, X_s)\,dW_s
\end{align*}
\]
with initial state \(X_{t_0}\) and for \(t \in [t_0, T]\). The SDE in (3) has a general solution given by
\[
X_t = X_{t_0} + \int_{t_0}^{t} \alpha(s, X_s)\,ds + \int_{t_0}^{t} \beta(s, X_s)\,dW_s
\]
for \(t \leq T\). If \(\alpha(t) := \alpha(t, X_t)\) and \(\beta(t) := \beta(t, X_t)\), then equation (3) has a general solution given by
\[
X_t = X_{t_0} + \int_{t_0}^{t} \alpha(s)\,ds + \int_{t_0}^{t} \beta(s)\,dW_s
\]
for \(t \leq T\). The process \(\{W_t; t \geq 0\}\) is a Wiener process with respect to a filtration \(\{\mathcal{F}_t; t \geq 0\}\). The initial state \(X_{t_0}\) is \(\mathcal{F}_{t_0}\)-measurable and the functions \(\alpha(t)\) and \(\beta(t)\) are Lebesgue measurable and bounded on \([t_0, T]\). The latter implies both the global Lipschitz and linearity growth conditions required to ensure the existence and (path-wise) uniqueness of a strong solution to (3), \([15]\).

Let \(X_t\) and \(Y_t\) be any two diffusion processes like those defined by the solution of equation (3). If \(F(x, y)\) is a differentiable function that works as a transformation for two processes \(X_t\) and \(Y_t\), then the general bi-variate Itô formula \([15]\) gives
\[
\begin{align*}
   dF(X_t, Y_t) &= \partial_x F(X_t, Y_t) dX_t + \partial_y F(X_t, Y_t) dY_t + \frac{1}{2} \partial_{xx} F(X_t, Y_t) (dX_t)^2 \\
   &\quad + \frac{1}{2} \partial_{yy} F(X_t, Y_t) (dY_t)^2 + \partial_{xy} F(X_t, Y_t) dX_t dY_t.
\end{align*}
\]

When \(X_t = t\), the general Itô formula of \(F(t, y)\) is a differentiable function. If \(Y_t\) is a diffusion process that solves (3), then the Itô formula becomes \([15]\)
\[
\begin{align*}
   dF(t, Y_t) &= f(t, Y_t) dt + g(t, Y_t) dW_t,
\end{align*}
\]
where
\[
\begin{align*}
   f(t, x) &= \partial_t F(t, x) + \alpha(t, x) \partial_x F(t, x) + \frac{1}{2} \beta(t, x)^2 \partial_{xx} F(t, x) \\
   g(t, x) &= \beta(t, x) \partial_x F(t, x).
\end{align*}
\]

Before introducing the numerical algorithm for solving (1) and (2), we must first introduce the following central proposition. The proposition also assists in finding exact solutions for equations (1) and (2), in particular when exact solutions of the deterministic differential equations (6) and (8) exist.

**Proposition 1.** Let \(A, B, C, D, E \in C^2([t_0, T])\) be bounded continuous functions on \([t_0, T]\). Assume that \(B(t) > 0\) for all \(t \in [t_0, T]\). Then, we have:

1. The stochastic Burgers equation with the initial value problem (1) has a solution \(u(t, z) = U(t, X_t)\), where \(U(t, x)\) is the solution of
\[
\begin{align*}
   \partial_t U &= (A(t) - \frac{1}{2} E^2(t)) \partial_{xx} U + B(t) U \partial_x U + D(t) U, \\
   U(0, x) &= \phi(x)
\end{align*}
\]


and $X_t$ is the solution of
\[ dX_t = C(t)dt + E(t)dW_t \]  
with initial state $X_{t_0} = z$ and for $t \in [t_0, T]$.

2. The stochastic Burgers equation with the initial value problem (2) has a solution
\[ u(t, z) = R(t) \left( V(t, Z_t) + \frac{1}{\mathcal{B}(t)} \hat{Z}_t \right) \]
where $V(t, x)$ is the solution of
\[ \partial_t V = A(t)\partial_{xx} V + \mathcal{B}(t)V\partial_x V + C(t)\partial_x V, \quad V(0, x) = \phi(x) \]
and $Z_t$ is the position in the solution of a Langevin-type second order stochastic differential equation
\[ \hat{Z}_t = \frac{\mathcal{B}'(t)}{\mathcal{B}(t)} \hat{Z}_t + \frac{\mathcal{B}(t)E(t)}{R(t)} \hat{W}_t \]
with initial state $Z_{t_0} = z$ and for $t \in [t_0, T]$. Also, $R(t) = \exp\left(\int_{t_0}^t D(s)ds\right)$ and $\mathcal{B}(t) = B(t)R(t)$.

**Proof.** For part (1), using the Itô formula applied to the $X_t$ solution of (7) with the transformation $U(t, x)$ that solves the deterministic Burgers equation (6)
\[ dU(t, X_t) = f(t, X_t)dt + g(t, X_t)dW_t \]
where
\[ f(t, x) = \partial_t U(t, x) + C(t)\partial_x U(t, x) + \frac{1}{2}E^2(t)\partial_{xx} U(t, x), \]
and note that,
\[ \partial_t U(t, x) = \left( A(t) - \frac{1}{2}E^2(t) \right) \partial_{xx} U(t, x) + B(t)U(t, x)\partial_x U(t, x) + D(t)U(t, x). \]
Thus, we get
\[ f(t, x) = A(t)\partial_{xx} U(t, x) + B(t)U(t, x)\partial_x U(t, x) + C(t)\partial_x U(t, x) + D(t)U(t, x). \]
Observe that
\[ g(t, x) = E(t)\partial_x U(t, x), \]
which proves part (1).

For part (2), equation (9) can be written as a system of equations:
\[ dN_t = \frac{\mathcal{B}'(t)}{\mathcal{B}(t)} N_t dt + \frac{\mathcal{B}(t)E(t)}{R(t)} dW_t, \]
\[ dZ_t = N_t dt. \]
Let $V(t, x)$ be the solution of
\[ \partial_t V = A(t)\partial_{xx} V + \mathcal{B}(t)V\partial_x V + C(t)\partial_x V, \quad V(0, x) = \phi(x). \]
Then, using the bi-variate general Itô formula for $V(t, Z_t)$, we have
\[ dV(t, Z_t) = \partial_t V(t, Z_t)dt + \partial_x V(t, Z_t)dZ_t \]
since $(dt)^2 = 0$, $dtdZ_t = N_t(dt)^2 = 0$, and $(dZ_t)^2 = (N_t)^2(dt)^2 = 0$. Hence, we obtain
\[ dV(t, Z_t) = \partial_t V(t, Z_t)dt + \partial_x V(t, Z_t)N_t dt \]
or

\[ dV(t, Z_t) = (A(t)\partial_{xx}V(t, Z_t) + \mathfrak{B}(t)V(t, Z_t)\partial_xV(t, Z_t) + C(t)\partial_xV(t, Z_t)) \, dt \\
+ \partial_xV(t, Z_t)N_t \, dt. \]  

(13)

Let \( L(t, z) = V(t, Z_t) + \frac{1}{\mathfrak{B}(t)} \dot{Z}_t \). On the right hand side of equation (13), the term multiplied by \( dt \) becomes

\[
A(t)\partial_{xx}V(t, Z_t) + \mathfrak{B}(t)V(t, Z_t)\partial_xV(t, Z_t) + C(t)\partial_xV(t, Z_t) + \partial_xV(t, Z_t)N_t = \\
A(t)\partial_{zz}L(t, z) + \mathfrak{B}(t) \left( L(t, z) - \frac{1}{\mathfrak{B}(t)} \dot{Z}_t \right) \partial_zL(t, z) + C(t)\partial_zL(t, z) + \partial_zL(t, z)N_t = \\
A(t)\partial_{zz}L(t, z) + \mathfrak{B}(t)L(t, z)\partial_zL(t, z) + C(t)\partial_zL(t, z).
\]

Thus, we have

\[
dL(t, z) = dV(t, Z_t) + \left( \frac{1}{\mathfrak{B}(t)} \dot{Z}_t \right) dt
\]

or

\[
dL(t, z) = (A(t)\partial_{zz}L(t, z) + \mathfrak{B}(t)\partial_{zz}L(t, z) + C(t)\partial_zL(t, z)) dt + \left( \frac{1}{\mathfrak{B}(t)} \dot{Z}_t \right) dt.
\]

By the bi-variate general Itô formula, we obtain

\[
d \left( \frac{1}{\mathfrak{B}(t)} N_t \right) = \frac{-\mathfrak{B}'(t)}{\mathfrak{B}(t)^2} N_t dt + \frac{1}{\mathfrak{B}(t)} dN_t
\]

since \((dt)^2 = 0\), \(dtdN_t = 0\), and \(\frac{dx^2}{dx^2} = 0\). Therefore, we have

\[
d \left( \frac{1}{\mathfrak{B}(t)} \dot{Z}_t \right) = \frac{-\mathfrak{B}'(t)}{\mathfrak{B}(t)^2} N_t dt + \frac{1}{\mathfrak{B}(t)} \left( \frac{\mathfrak{B}'(t)E(t)}{\mathfrak{B}(t)} N_t dt + \frac{\mathfrak{B}(t)E(t)}{R(t)} dW_t \right) = \frac{E(t)}{R(t)} dW_t.
\]

Therefore,

\[
dL(t, z) = (A(t)\partial_{zz}L(t, z) + \mathfrak{B}(t)L(t, z)\partial_zL(t, z) + C(t)\partial_zL(t, z)) dt + \frac{E(t)}{R(t)} dW_t.
\]

By taking \( u(t, z) = R(t) L(t, z) \) and using the Itô formula, we complete the proof.

The following two lemmas are crucial to identify the solutions of the SDEs in Proposition 1 and to study the properties of the numerical algorithm given below. By these two lemmas, we can compare the exact solutions with numerical solutions.

**Lemma 2.1.**

1. The stochastic process \( X_t \) solving

\[
dx_t = C(t) dt + E(t) dW_t
\]

with \( X_{t_0} \sim N(x_{t_0}, \sigma_0^2) \) independent of \( W_t \), is a non-stationary Gaussian process with mean \( x_t = \int_{t_0}^t C(s) ds \) and variance \( \sigma^2(X_t) = \sigma_0^2 + \int_{t_0}^t E^2(s) ds \).

2. The covariance of the two processes \( X_t \) and \( W_t \) is

\[
\sigma(X_t, W_t) = \int_{t_0}^t E(s) ds.
\]

3. Moreover,

\[
[X_t|W_t = w] \sim N \left( x_t + \int_{t_0}^t C(s) ds + \frac{\int_{t_0}^t E(s) ds}{t} w, \sigma_0^2 + \int_{t_0}^t E^2(s) ds - \frac{(\int_{t_0}^t E(s) ds)^2}{t} \right).
\]
Proof. See [6, Proposition 5.6.1] for the first part and [15, p.86] for the second part. The third point follows from the fact that if

\[
\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_2^2 \end{pmatrix} \right)
\]

then the conditional distribution of \( X_1 \) given \( X_2 \) is given by

\[
[X_1|X_2] \sim \mathcal{N} \left( \mu_1 + \frac{\sigma_{1,2}}{\sigma_2^2} (X_2 - \mu_2), \sigma_1^2 - \frac{\sigma_{1,2}^2}{\sigma_2^2} \right),
\]

see [7]. \( \square \)

**Lemma 2.2.**

1. The position random process \( Z_t := z + \int_{t_0}^t \dot{B}(s) K(s) dW_s \) solves the Langevin-type second order SDE

\[
\dot{Z}_t = \frac{B'(t)}{B(t)} Z_t + B(t) K(t) W_t, \quad t \in [t_0, T]
\]

with initial state \( Z_{t_0} = z \), where \( \dot{B}(s) = \int_s^t B(r) dr \) for \( t > s \).

2. The process \( Z_t \) is a non-stationary Gaussian Process with mean \( z \) and variance \( \sigma^2(Z_t) = \int_{t_0}^t (\dot{B}(s) K(s))^2 ds \).

3. Meanwhile,

\[
\dot{Z}_t := B(t) \left( \int_{t_0}^t K(r) dW_r \right).
\]

4. The process \( \dot{Z}_t \) is a non-stationary Gaussian Process with mean zero and variance \( \sigma^2(\dot{Z}_t) = (B(t))^2 \int_{t_0}^t (K(s))^2 ds \).

5. The covariance of \( Z_t \) and \( W_t \) is

\[
\sigma(Z_t, W_t) = \int_{t_0}^t \dot{B}(s) K(s) ds,
\]

and the covariance of \( \dot{Z}_t \) and \( W_t \) is

\[
\sigma(\dot{Z}_t, W_t) = B(t) \left( \int_{t_0}^t K(s) ds \right).
\]

6. The conditional distributions are given by

\[
[Z_t|W_t = w] \sim \mathcal{N} \left( z + w \int_{t_0}^t \dot{B}(s) K(s) ds \right), \int_{t_0}^t (\dot{B}(s) K(s))^2 ds - \frac{\left( \int_{t_0}^t \dot{B}(s) K(s) ds \right)^2}{t}
\]

and

\[
[Z_t|W_t = w] \sim \mathcal{N} \left( w \frac{B(t) \left( \int_{t_0}^t K(s) ds \right)}{t}, (B(t))^2 \left( \int_{t_0}^t (K(s))^2 ds - \frac{\left( \int_{t_0}^t K(s) ds \right)^2}{t} \right) \right).
\]

**Proof.** Since \( Z_t := z + \int_{t_0}^t B(s) \left( \int_{t_0}^s K(r) dW_r \right) ds \), then by the stochastic Fubini’s theorem [6, Theorem 10.3.15], part (1) follows from the equality

\[
d \left( \frac{1}{B(t)} \dot{Z}_t \right) = K(t) dW_t
\]

shown at the end of the proof of Proposition 1. Part (2) follows from [6, Proposition 5.6.1].
Since \( \int_{t_0}^t K(r) dW_r \) has a continuous sample path (a.s.) \([15, \text{Theorem } 3.2.6]\), then
\[
\dot{Z}_t := B(t) \left( \int_{t_0}^t K(r) dW_r \right).
\]
Thus part (3) follows, part (4) follows from \([6, \text{Proposition } 5.6.1]\).

The rest of the parts of the lemma follow from the same methods used in Lemma 2.1.

Recalling proposition 2 from \([17]\) we know that if the following second-order differential equation
\[
\mu''(t) - \frac{a'(t)}{a(t)} \mu'(t) + 4a(t)b(t)\mu(t) = 0
\]
can be solved explicitly with \( \mu_0(0) = 0, \mu_0'(0) = 2a(0) \neq 0, \mu_1(0) = 0, \) and \( \mu_1'(0) = 0 \), then the Cauchy initial value problem for the generalized Burgers equation
\[
v_t + 4a(t)(vv_x + Lv_{xx}) = -b(t)x + f(t)
\]
can be reduced to the standard Burgers equation
\[
u_x + Lv_{\xi} + u_{\xi} u = 0
\]
through the multi-parameter substitution
\[
v(x, t) = \alpha(t)x + \delta(t) + \beta(t)u(\xi, \tau)
\]
with \( \xi = \beta(t)x + 2\varepsilon(t) \) and \( \tau(t) = 4\gamma(t) \). The functions \( \mu, \alpha, \beta, \gamma, \delta, \varepsilon \) and \( \kappa \) satisfy a Riccati system, see \([17]\), and they are given explicitly by:
\[
\mu(t) = -2\mu(0)\mu_0(t)(\alpha(0) + \gamma_0(t)), \tag{14}
\]
\[
\alpha(t) = \alpha_0(t) - \frac{\beta^2(t)}{4(\alpha(0) + \gamma_0(t))}, \tag{15}
\]
\[
\beta(t) = -\frac{\beta(0)\beta_0(t)}{2(\alpha(0) + \gamma_0(t))} = \frac{\beta(0)\mu(0)}{\mu(t)}\lambda(t), \tag{16}
\]
\[
\gamma(t) = \gamma(0) - \frac{\beta^2(t)}{4(\alpha(0) + \gamma_0(t))}, \tag{17}
\]
\[
\delta(t) = \delta_0(t) - \frac{\beta_0(t)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \tag{18}
\]
\[
\varepsilon(t) = \varepsilon(0) - \frac{\beta(0)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \tag{19}
\]
\[
\kappa(t) = \kappa(0) + \kappa_0(t) - \frac{(\delta(0) + \varepsilon_0(t))^2}{4(\alpha(0) + \gamma_0(t))} \tag{20}
\]
with the arbitrary data \( \mu(0), \alpha(0), \beta(0) \neq 0, \gamma(0), \delta(0), \varepsilon(0), \kappa(0) \). Additionally,
\[
\alpha_0(t) = -\frac{1}{4a(t)}\mu_0(t), \quad \beta_0(t) = \frac{1}{\mu_0(t)}, \quad \gamma_0(t) = -\frac{1}{2\mu_1(0)}\mu_0(t), \quad \beta_0(t) = \frac{1}{\mu_0(t)} \int_0^t f(s) \mu_0(s) \, ds, \tag{21}
\]
\[ \varepsilon_0(t) = \frac{-2a(t) \delta_0(t)}{\mu_0'(t)} - 8 \int_0^t \frac{a^2(s) b(s)}{(\mu_0(s))^2} (\mu_0(s) \delta_0(s)) \, ds \]
+ 2 \int_0^t \frac{a(s) f(s)}{\mu_0'(s)} \, ds, \quad (23) \]

\[ \kappa_0(t) = \frac{-a(t) \mu_0(t)}{\mu_0'(t)} \delta_0^2(t) - 4 \int_0^t \frac{a^2(s) b(s)}{(\mu_0(s))^2} (\mu_0(s) \delta_0(s))^2 \, ds \]
+ 2 \int_0^t \frac{a(s)}{\mu_0'(s)} (\mu_0(s) \delta_0(s)) f(s) \, ds \quad (24) \]

with \( \delta_0(0) = g_0(0) / (2a(0)) \), \( \varepsilon_0(0) = -\delta_0(0) \), and \( \kappa_0(0) = 0 \).

In the following, we introduce two simple examples of cases where we can find the exact solutions. We will also use them to validate the numerical simulations. That comparison between numerical and exact solutions is possible by using one realization of simulated Brownian motion at a time to calculate both solutions.

**Example 1.** Consider the stochastic Burgers equation
\[
\frac{du}{dt} = \left( \exp(t) + \frac{1}{2} \partial_{zz} u + \exp(t) u \partial_z u + \partial_z u \right) dt + \partial_z u \, dW_t, \quad u(0, z) = \frac{2}{1 + \exp(-2 - z)} \quad (25)
\]
for \( t \in [0, 1] \). By Proposition 1 part (1), equation (25) has a solution \( u(t, z) = U(t, X_t) \), such that \( U(t, x) \) is the solution of
\[
\partial_t U = \exp(t) \partial_{xx} U + \exp(t) U \partial_x U, \quad U(0, x) = \frac{2}{1 + \exp(-2 - x)} \quad (26)
\]
and \( X_t \) is the solution of
\[
dX_t = dt + dW_t, \quad (27)
\]
with initial state \( X_0 = z \) and for \( t \in [0, 1] \).

By the work in [17], equation (26) has the general solution
\[
U(t, x) = \frac{2}{1 + \exp(-1 - x - \exp(t))}
\]
for \( x \in \mathbb{R} \). The stochastic differential equation (27) has a solution given by
\[
X_t = z + t + W_t
\]
for \( t \in [0, 1] \). Therefore, the general solution of the stochastic Burgers equation (25) is given by
\[
u(t, z) = \frac{2}{1 + \exp(-1 - z - t - \exp(t) - W_t)} \quad (28)
\]
for \( t \in [0, 1] \) and \( z \in \mathbb{R} \).

Figure 1 shows two realizations of the general solution in (28).

**Example 2.** Consider another stochastic Burgers equation
\[
\frac{du}{dt} = \left( \exp(t) \partial_{zz} u + \exp(t) u \partial_z u \right) dt + \partial_z u \, dW_t, \quad u(0, z) = \frac{2}{1 + \exp(-2 - z)} \quad (29)
\]
for \( t \in [0, 1] \). By Proposition 1 part (2), equation (29) has a solution
\[
u(t, z) = V(t, Z_t) + \frac{1}{\exp(t)} Z_t,
\]
for \( t \in [0, 1] \).
where \( R(t) = 1 \), such that \( V(t, x) \) is the solution of
\[
\partial_t V = \exp(t) \partial_{xx} V + \exp(t) \partial_x V, \quad V(0, x) = \frac{2}{1 + \exp(-2 - x)} \tag{30}
\]
and \( Z_t \) is the solution of
\[
\ddot{Z}_t = \dot{Z}_t + \exp(t) \dot{W}_t, \tag{31}
\]
with initial state \( Z_0 = z \) and for \( t \in [0, 1] \).

Again, equation (30) has the general solution
\[
V(t, x) = \frac{2}{1 + \exp(-1 - x - \exp(t))}
\]
for \( x \in \mathbb{R} \).

By Lemma 2.2, the stochastic differential equation (31) has a solution given by
\[
Z_t = z + \exp(t) W_t - \int_0^t \exp(s) dW_s
\]
and
\[
\dot{Z}_t = \exp(t) W_t
\]
for \( t \in [0, 1] \). Therefore, the general solution of the stochastic Burgers equation (29) is given by
\[
u(t, z) = W_t + \frac{2}{1 + \exp(-1 - z - \exp(t)) - \exp(t) W_t + \int_0^t \exp(s) dW_s} \tag{32}
\]
for \( t \in [0, 1] \) and \( z \in \mathbb{R} \).

Figure 2 shows two realizations of the general solution in (32).

**Algorithm and Numerical Simulation.**

In this part, we introduce a numerical algorithm to simulate a realization of the solution of the stochastic Burgers equations (1) and (2) using the system of equations in Proposition 1. The numerical algorithm uses order 1.5 Runge-Kutta-Maruyama (RKM-1.5) [15, p.382] to simulate the stochastic ordinary differential equations (7) and (9), or simulate their exact solutions in Lemma 2.1 and Lemma 2.2, if possible, resulting in a “stochastic mesh,” (see Figure 3). To solve the partial differential equation on that mesh, the algorithm uses the central difference over the
space variable, and the finite difference time discretization or Runge-Kutta method. We call the last scheme RKCD.

The numerical scheme uses the stencil in Figure 4. If the space coordinate on the mesh is \( x_{i(j)} = x_j \), then \( x_{i+1(j)} := x_j + \ell_i \) for \( \ell_i \in \mathbb{R} \).

The bi-variate Taylor expansion of \( U(t+k,x+\ell) \) about \( (t,x) \) gives the following approximations

\[
U(t + k, x + \ell) - U(t, x) \approx k\partial_t U(t, x) + \ell \partial_x U(t, x) + \frac{1}{2} \ell^2 \partial_{xx} U(t, x) \quad (33)
\]

and

\[
U(t, x \pm h) - U(t, x) \approx \pm h \partial_x U(t, x) + \frac{1}{2} h^2 \partial_{xx} U(t, x), \quad (34)
\]

or equivalently

\[
\begin{bmatrix}
h & \frac{k^2}{2} & 0 \\
-h & \frac{k^2}{2} & 0 \\
\ell & \frac{\ell^2}{2} & k
\end{bmatrix}
\begin{bmatrix}
\partial_x U(t, x) \\
\partial_{xx} U(t, x) \\
\partial_t U(t, x)
\end{bmatrix}
= \begin{bmatrix}
U(t, x + h) - U(t, x) \\
U(t, x - h) - U(t, x) \\
U(t+k,x+\ell) - U(t, x)
\end{bmatrix}.
\]

Thus,

\[
\partial_x U(t, x) = \frac{U(t, x + h) - U(t, x - h)}{2h},
\]

\[
\partial_{xx} U(t, x) = \frac{U(t, x + h) - 2U(t, x) + U(t, x - h)}{h^2},
\]

and

\[
\partial_t U(t, x) = \frac{U(t+k,x+\ell) - U(t, x)}{k} - \frac{\ell}{k} \partial_x U(t, x) - \frac{\ell^2}{2k} \partial_{xx} U(t, x).
\]
Figure 3. (a) and (b): Two realizations of the stochastic mesh resulting from solving equation (7) with $C(t) = t + 1$ and $E(t) = t + 2$ for $t \in [0, 2]$ with $\Delta t = 0.0408$ when $z \in [-1, 1]$ with $\Delta z = .1$. (c) and (d): Two realizations of the stochastic mesh resulting from solving equation (9) with $B(t) = \exp(t)$, $R(t) = 1$ and $E(t) = 1$ for $t \in [0, 2]$ with $\Delta t = 0.0408$ when $z \in [-1, 1]$ with $\Delta z = .1$. Notice the uniformity over space since the noise is space uniform.

These equations lead to the iterative scheme (applied to the first one of the two equations and the other one follows similarly)

$$U(t_{i+1}, x_{i+1(j)}) = U(t_i, x_j) + k \left( (A(t_i) - \frac{1}{2}E^2(t_i) + \frac{\ell_i^2}{2k})U(t_i, x_j + h) - 2U(t_i, x_j) + U(t_i, x_j - h) \right) \frac{h}{h^2}$$

$$+ B(t_i)\frac{U^2(t_i, x_j + h) - U^2(t_i, x_j - h)}{4h}$$

$$+ \frac{\ell_i U(t_i, x_j + h) - U(t_i, x_j - h)}{2h} + D(t_i)U(t_i, x_j)$$

starting with $U(t_0, z_j) = \phi(z_j)$, where $\ell_i = x_{i+1(j)} - x_{i(j)}$, $h = z_{j+1} - z_j$, and $k = t_{i+1} - t_i$.

Finally, we introduce the algorithm of solving SBEs.
Algorithm:

Step 1. Discretize the time interval $[t_0, T]$ into a time line space $DT : t_0, t_1, \ldots, t_m = T$ and discretize the space interval $[a, b]$ into $DS : a = z_0, z_1, \ldots, z_n = b$

Step 2. For each $j = 1, \ldots, n$, solve (7) with $X_{t_0} = z_j$ or (9) with $Z_{t_0} = z_j$ using RKM-1.5 resulting in $x_{i(j)} := X_{t_i}(z_j)$ with $i = 0, 1, \ldots, m$. This step will result in a stochastic mesh $\{(t_i, x_{i(j)}) : i = 0, \ldots, m; j = 0, \ldots, n\}$.

Step 3. Use a finite (central) difference for time and space in equation (6) and then use RKCD (see above) to solve a first order ODE boundary value problem with boundaries at $x_{i(0)}$ and $x_{i(n)}$, for all $i = 1, \ldots, m$, using Euler or Runge-Kutta method.

Step 4. Approximate solution of the stochastic Burgers equation (1), $u(t_i, z_j) = U(t_i, x_{i(j)})$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. (Or approximate solution of the stochastic Burgers equation (2), with $u(t_i, z_j) = R(t_i) V(t_i, x_{i(j)}) + \int_{t_0}^{t_i} E(s)/R(s) dW_s$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.)

The algorithm is implemented on Matlab and uses the ordinary differential equation solver ode45 to perform the deterministic Runge-Kutta of order 4 and 5.

Example 3. Figure 5 shows two numerical simulations of the general solution of (25) and their stochastic meshes.

Example 4. Figure 6 shows two numerical simulations of the general solution of (29), their stochastic meshes and the processes solving equation (31) using RKM-1.5.

2.1. Comparison to another numerical algorithm. In this section, we compare the stochastic mesh numerical algorithm introduced above to the corresponding stochastic forward Euler method (SFEM) of the equation:

$$du = (A(t) \partial_{zz} u + B(t) u \partial_z u + C(t) \partial_z u + D(t) u) dt + F(t, u, \partial_z u) dW_t.$$  (35)
The iterative scheme of the SFEM is given by

\[
    u(t_{i+1}, z_j) = u(t_i, z_j) + k \left( A(t_i) \frac{u(t_i, z_{j+1}) - u(t_i, z_{j-1})}{h} 
    + B(t_i) \frac{u^2(t_i, z_{j+1}) - u^2(t_i, z_{j-1})}{4h} 
    + C(t_i) \frac{u(t_i, z_{j+1}) - u(t_i, z_{j-1})}{2h} + D(t_i)u(t_i, z_j) \right) 
    + F(t_i, u(t_i, z_j), u(t_i, z_{j+1}) - u(t_i, z_{j-1})) \Delta W_i
\]

starting with \(u(t_0, z_j) = \phi(z_j)\), for \(i = 0, 1, \ldots, m - 1\) and \(j = 1, \ldots, n - 1\) with \(DT : t_0, t_1, \ldots, t_m = T\) and \(DS : a = z_0, z_1, \ldots, z_n = b\). The random variables \(\Delta W_1, \Delta W_2, \ldots, \Delta W_n\) are independent and normally distributed random variables with mean zero and variance \(k\).

To compare SFEM and our stochastic mesh (SM) method presented in this paper, we carried out a simulation study that compares the absolute error between both methods. (For accurate comparisons, we used the ordinary differential equation solver \texttt{ode45} to perform the time progressions for the SFEM.) The comparison is
Figure 6. Two realizations of the two processes $Z_t$ and $\dot{Z}_t$ that solve equation (31) (a) and (b), the stochastic meshes (c) and (d), and their respective simulated numerical solutions over those two meshes (e) and (f).
Table 1. The maximum values of the mean absolute errors over 
$[0, 1] \times [-1, 1]$ for different values of $n$ and $m$ show that the stochas-
tic mesh method (SM) is better than the stochastic forward Euler 
method (SFEM) in overall.

| n   | m   | MMAE for SM | MMAE for SFEM |
|-----|-----|-------------|---------------|
| 20  | 20  | 0.032       | 0.047         |
| 20  | 30  | 0.033       | 0.042         |
| 20  | 40  | 0.032       | 0.038         |
| 30  | 20  | 0.030       | 0.050         |
| 30  | 30  | 0.030       | 0.046         |
| 30  | 40  | 0.033       | 0.040         |
| 40  | 20  | 0.035       | 0.052         |
| 40  | 30  | 0.033       | 0.047         |
| 40  | 40  | 0.033       | 0.039         |

Based on a Monte Carlo experiment in which the absolute difference between the 
exact and numerical solutions are simulated for a large number of times. The relative 
frequency of when the absolute error of the SFEM is smaller than the absolute error 
of the stochastic mesh method is calculated at each pair $(t_i, z_j)$ for $i = 0, \ldots, m$ and 
$j = 0, \ldots, n$. We also compute $P_{\text{max}}$ which is defined to be the relative frequency 
of times the overall maximum of absolute error of the SFEM is smaller than that 
of the absolute error of the stochastic mesh method.

**Example 5.** Here, we carry out a Monte Carlo experiment simulating the solutions 
$N = 1000$ times in which $P_{\text{max}}$ ranges from .05 to .13 for small values of $n$ and $m$. 
The relative frequency of the times the absolute error of the SFEM is smaller than 
that of the stochastic mesh method for the solution of (25) are shown in Figure 7. 
It seems that the stochastic mesh (SM) method most likely has smaller absolute 
error than that of the SFEM for pairs $(t, z)$ closer to the boundaries of the region 
$[0, 1] \times [-1, 1]$, with that smaller error advantage becoming slightly less probable 
near the center of the region.

Table 1 provides a support to the simulations in Figure 7 by calculating the 
maximum values of the mean absolute errors (MMAE) over the region $[0, 1] \times [-1, 1]$. 
It shows that the the stochastic mesh method (SM) method has a smaller mean 
absolute error than the stochastic forward Euler method (SFEM).

3. **Conclusion.** In this paper, we showed using Itô calculus that some exact solutions 
of stochastic Burgers equations with variable coefficients can be found using 
deterministic Burgers equations. A numerical algorithm can be used to solve those 
deterministic Burgers equations over stochastic meshes that happen to be solutions 
of stochastic differential equations. We conjecture that a similar approach could 
be used to simulate stochastic Burgers equations with white noises that are non-
uniform on space. In the future, we will study that conjecture and compare those 
approaches to standard methods like the Crank-Nicholson-Maruyama method.

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Figure 7. The relative frequency of the times the absolute error of the SFEM is smaller than the absolute error of the stochastic mesh (SM) method for the solution of (25) at each pair \((t_i, z_j)\) for \(i = 0, \ldots, m\) and \(j = 0, \ldots, n\) for (a) \((m,n)=(20,20)\) giving \(P_{\max} = .059\), (b) \((m,n)=(20,30)\) giving \(P_{\max} = .053\), (c) \((m,n)=(30,20)\) giving \(P_{\max} = .077\), (d) \((m,n)=(30,30)\) giving \(P_{\max} = .0597\), (e) \((m,n)=(40,20)\) giving \(P_{\max} = .139\), (f) \((m,n)=(40,30)\) giving \(P_{\max} = .087\).
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