Order-to-chaos transition in the model of a quantum pendulum subjected to noisy perturbation

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Abstract

The motion of a randomly driven quantum nonlinear pendulum is considered. Utilizing a one-step Poincaré map, we demonstrate that the classical phase space corresponding to a single realization of the random perturbation can involve domains of finite-time stability. Statistical analysis of the finite-time evolution operator (FTEO) is carried out in order to study the influence of finite-time stability on quantum dynamics. It is shown that domains of finite-time stability give rise to ordered patterns in distributions of FTEO eigenfunctions. The transition to global chaos is accompanied by smearing of these patterns; however, some of their traces survive on relatively long timescales.

Keywords: finite-time stability, random perturbation, quantum chaos, one-step Poincare map, finite-time evolution operator

1. Introduction

It is well known that deterministic classical systems with few degrees of freedom can exhibit complicated chaotic behaviour that is similar to the behaviour of systems with intrinsic random fluctuations. However, the methods used for studying deterministic chaotic systems and systems under stochastic driving differ substantially. Representing the external perturbation as some random process, we add uncertainty into the equations of motion. Each realization of the perturbation creates a unique trajectory of the system; therefore, efficient description of the system’s response implies the usage of statistical analysis.

Nevertheless, single realizations of the random perturbation can be considered as deterministic functions. As long as the temporal Fourier spectrum of the perturbation is broad, impenetrable stable domains cannot survive in phase space [1, 2]. However, statistical analysis of the finite-time Lyapunov exponents in randomly driven systems shows that a remarkable fraction of phase space area maintains stability on timescales which exceed significantly the so-called Lyapunov time

\[ t_L = \frac{1}{\lambda_L}, \]

where \( \lambda_L \) is the global Lyapunov exponent, calculated with \( t \rightarrow \infty \) [3, 4]. Such trajectories form bundles like branched flows in quantum point contacts [5–8] or coherent ray clusters in ocean acoustics [9]. One shouldn’t confuse this kind of coherent phenomena with the domains of particle clusterization in random fields [10]. The main difference is the location of the initial conditions for particles. In the case of the coherent clusters, the initial conditions belong to continuous manifolds in phase space. The location of these manifolds depends on the particular realization of the perturbation. Such manifolds can be found out in various ways. For example, one can calculate the map of finite-time Lyapunov exponents in phase space [11, 12] or compute eigenfunctions of the transfer operator [13]. It turns out that a system’s behaviour under a single realization can significantly differ from the picture obtained via statistical averaging. Phenomena can occur that are typical for deterministic systems, for example,
intermittency and capturing into dynamical traps [14–16]. Thus, one needs a general approach which can be used for both deterministic and noisy cases.

The problem of interrelation between deterministic and statistical approaches also arises in quantum systems whose classical counterparts exhibit chaotic behaviour. Periodic orbit theory [17, 18] provides a classical interpretation of quantum spectra and, in addition, reveals some non-classical features, like scars of wavefunctions [19]. One can suggest that some peculiarities of deterministic quantum systems should manifest themselves in quantum systems involving classical noise. The issue of particular interest is how deterministic phenomena associated with periodic orbits are revealed under stochastic driving when there are no periodic orbits in the strict sense.

In the present paper, we demonstrate an approach that allows one to analyze quantum systems under weak random perturbation in the framework of deterministic theory. In this approach, it is implied that the system’s behaviour possesses some features which are common for all typical realizations of the perturbation. In this way, any realization is treated as a deterministic process with known spectral properties. On the classical level, our approach is based on the one-step Poincaré map originally introduced in [20, 21]. It serves as a tool for finding out phase space patterns repeating themselves after some time interval. Some of these patterns have a regular form and can be referred to as domains of finite-time stability. We use the quantum counterpart of the one-step Poincaré map, the so-called finite-time evolution operator (FTEO), for exploring manifestations of finite-time stability in quantum motion. A mathematically equivalent approach was used for studying sound propagation in a randomly inhomogeneous oceanic waveguide [22, 23]. In the present paper, we use this approach for a purely quantum problem, namely, for the quantum nonlinear pendulum subjected to broadband perturbation. Our main goal is to study how domains of finite-time stability influence quantum dynamics.

It is worth mentioning that the condition of invariance under translation over a finite time interval is too restrictive, and there can be phase-space patterns which don’t satisfy it but correspond to non-chaotic dynamics. A striking example is the so-called branched flows in quantum point contacts [7, 8]. Our approach is instead designed for oscillatory motion subjected to external noise. In particular, it can be implemented in the problems of noise-driven dissociation or ionization [24–26], where regular phase-space domains prevent transitions into unbounded states; therefore, efficient destruction of these domains is of great importance. The same problem arises in randomly driven quantum ratchets with cold atoms [27].

The paper is organized as follows. The next section describes the model under consideration. In section 3, we study classical motion of a randomly driven pendulum by means of a one-step Poincaré map. Signatures of classical finite-time stability in quantum dynamics are explored in section 4. In the conclusion we summarize the results obtained.

2. Model

Consider the quantum Hamiltonian

\[
\hat{H} = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} + U(x) + eV(x, t), \quad U(x) = -\cos x,
\]

\[
V(x, t) = f(t) \sin x - f(t + \Delta) \cos x,
\]

where \(\epsilon \ll 1\), and \(f(t)\) is the so-called harmonic noise [28, 29]. The corresponding Schrödinger equation reads

\[
i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi.
\]

In the present paper we simplify the analysis by considering only quantum states with zero quasimomentum. It corresponds to the periodic boundary conditions \(\Psi(-\pi, t) = \Psi(\pi, t)\). This simplification is quite reasonable if we deal with the semiclassical regime when energy bands are flat and tunneling between neighbouring potential wells is fairly weak. It demands the Planck constant to be small; therefore, we set \(\hbar = 0.1\). The model (2) was used in [27] in the context of quantum ratchet phenomena.

Harmonic noise is described by coupled stochastic differential equations

\[
\dot{f} = y, \quad \dot{y} = -\Gamma y - \omega_0^2 f + \sqrt{2\beta} \xi(t),
\]

where \(\Gamma\) is a positive constant and \(\xi(t)\) is Gaussian white noise. The terms \(f(t)\) and \(f(t + \Delta)\) in (2) correspond to identical realizations of harmonic noise and differ only by the temporal shift \(\Delta\). The first two moments of harmonic noise are given by

\[
\langle f \rangle = 0, \quad \langle f^2 \rangle = \frac{\beta}{\omega_0^2}.
\]

We set \(\beta = 1\); that is, the perturbation strength is solely determined by the parameter \(\epsilon\). In the case of low values of \(\Gamma\), the power spectrum of harmonic noise has the peak at the frequency

\[
\omega_p = \sqrt{\omega_0^2 - \frac{\Gamma^2}{2}}.
\]

The width of the peak is given by the formula

\[
\Delta\omega = \sqrt{\omega_p^2 - \Gamma^2/4},
\]

where \(\omega_p = \sqrt{\omega_0^2 - \Gamma^2/4}\). As \(\Gamma \to 0\), \(f(t) \to \sin(\omega_0 t + \phi_0)\), where \(\phi_0\) is determined by initial conditions in (4). Setting \(f(0) = 1, \dot{y}(0) = 0, \text{ and } \Delta = \pi/(2\omega_0)\), one can easily find that \(V(x, t) = \sin(x + \omega_0 t)\) in the case of \(\Gamma = 0\). Hence, it turns out that, for \(\Gamma > 0\), \(V(x, t)\) behaves like a plane wave whose amplitude and phase velocity fluctuate with time. The plane wave acts as a dragging force for particles and leads to the onset of directed current, i.e., the ratchet effect. These types of ratchets are known as travelling potential ratchets [30–33]. In the semiclassical regime, the direction of the current coincides with the direction of the perturbation phase velocity, provided dynamical barriers preventing the transition of particles into the infinite regime are destroyed. More intricate behaviour is
observed in the deep quantum regime [34]. In the present work we use the following values of parameters: $\omega_0 = 1$, $\Gamma = 0.1$, and $\varepsilon = 0.05$.

### 3. One-step Poincaré map

We begin with the classical level and consider the classical counterpart of the Hamiltonian (2)

$$ H = \frac{p^2}{2} + U(x) + \varepsilon V(x, t). \quad (8) $$

Corresponding equations of motion read

$$ \frac{dx}{dt} = p, \quad \frac{dp}{dt} = -\frac{dU}{dx} - \varepsilon \frac{dV}{dx}. \quad (9) $$

Next we consider some arbitrarily chosen realization of $V(x, t)$. Then we can treat $V(x, t)$ as a deterministic function and refer to (9) as the system of ordinary differential equations. As $V(x, t)$ is an oscillating function of time, the domains of finite-time stability may involve components which transform to themselves without mixing in the course of evolution from $t = 0$ to $t = \tau$. These components can be found out by means of the one-step Poincaré map [20, 21, 35]

$$ p_{t+1} = p(t = \tau; p_t, x_t), $$

$$ x_{t+1} = x(t = \tau; p_t, x_t), \quad (10) $$

where $p(t = \tau; p_t, x_t)$ and $x(t = \tau; p_t, x_t)$ are solutions of (9) with initial conditions $p(t = 0) = p_t, x(t = 0) = x_t$. The one step Poincaré map is equivalent to the usual Poincaré map with the Hamiltonian

$$ \hat{H} = \frac{p^2}{2} + U(x) + \varepsilon \hat{V}(x, t), \quad (11) $$

$$ \hat{V}(x, \bar{t} + n\tau) = V(x, \bar{t}), \quad 0 \leq \bar{t} \leq \tau, \quad (12) $$

where $n$ is an integer. Thus, we replace the original system by the equivalent time-periodic one. The validity of this replacement is provided by the time restriction to the interval $[0; \tau]$. An alternative approach for generalization of the Poincaré map onto stochastic dynamical systems was offered in [36, 37].

Following the analogy with the usual Poincaré map, one can deduce the main property of the one-step Poincaré map: if a trajectory of (10) forms a closed continuous curve in phase space, any point belonging to it is an initial condition for a trajectory of (9) that remains stable at $t = \tau$. The inverse statement is not generally true; therefore, the one-step Poincaré map yields a sufficient but not necessary criterion of stability. This means that the one-step Poincaré map basically underestimates the area of regular domains.

As we artificially reduce the problem to a time-periodic one, the theory of time-periodic Hamiltonian systems can be invoked [20, 21]. The phase-space structure of the one-step Poincaré map is determined by resonances

$$ m_1\tau = m_2T, \quad (13) $$

where $T$ is the period of unperturbed motion. Period is a function of the action variable defined as [14]

$$ I = \frac{1}{2\pi} \int p \, dx. \quad (14) $$

Each resonance has a certain width in the space of the action. Width can be calculated using the theory of nonlinear resonance. It can be shown that the distance between neighbouring dominant nonlinear resonances decreases with increasing $\tau$ as $\tau^{-1}$ [20]. This results in resonance overlapping and gradual transition to global chaos according to the Chirikov criterion [38].

Figures 1 and 2 demonstrate phase portraits constructed by means of map (10) with $\tau = 4\pi$ and $\tau = 20\pi$, respectively. Notably, figures corresponding to the same value of $\tau$ but different realizations of harmonic noise represent very similar patterns with nearly the same fraction of regular area. In the case of $\tau = 4\pi$ (see figure 1), the central part of phase space maintains stability for all realizations of harmonic noise. This domain corresponds to the vicinity of the stable equilibrium point of the unperturbed system. Chaos emerges in the neighbourhood of the unperturbed separatrix. An increase of $\tau$ to $20\pi$ results in remarkable shrinking of the regular area, as is illustrated in figure 2. The phase-space region corresponding to finite motion becomes submerged into the chaotic sea. Stable domains inside the unperturbed separatrix may survive only as small islands. For some realizations, they disappear completely. For example, the internal part of the chaotic sea presented in figure 2(d) is almost uniform, without any apparent islands. Further increasing of $\tau$ results in the complete disappearance of stable islands in the phase-space region enclosed by the unperturbed separatrix.

### 4. Finite-time evolution operator

The phase-space portraits presented in the preceding section indicate fast destruction of stable domains. We now consider how this process is revealed in quantum dynamics. The quantum counterpart of the one-step Poincaré map is the operator $\hat{G}$, defined as

$$ \hat{G}(\tau) \hat{\Psi}(x) = \exp\left(-\frac{i}{\hbar} \hat{H}\tau\right) \hat{\Psi}(x) = \hat{\Psi}(x, \tau)|_{m_1}, \quad (15) $$

where $\hat{\Psi}(x) = \Psi(x, \bar{t} = 0)$. Hereafter we shall refer to $\hat{G}$ as the FTEO. Each realization of harmonic noise creates its own realization of the FTEO. The FTEO was first utilized in [39] for the problem of noise-driven quantum diffusion. The wave analogue of the FTEO was used in [22, 23, 40–42].

Peculiarities of classical phase space should be reflected in the spectral properties of the FTEO. Eigenvalues and
The eigenfunctions of the FTEO satisfy the equation
\[ \hat{G}\Psi_m(x) = g_m \Psi_m(x) \equiv e^{-i\epsilon_m/\hbar} \Psi_m(x). \] (16)

The quantity \( \epsilon_m \) is the analogue of quasienergy in time-periodic quantum systems. As an increase of \( \tau \) results in destruction of regular domains, one should expect transition in statistics of level spacings \( s = \epsilon_{m+1} - \epsilon_m \) from a Poissonian to a Wigner-like regime [39]. This expectation can fail in the presence of periodic-orbit bifurcations [22, 40]. Moreover, analysis of level spacing statistics does not provide an accurate estimate of the regular phase space area [23]. In this way, analysis of FTEO eigenfunctions seems to be a more robust way. To facilitate the analysis, eigenfunctions \( \Phi_m \) can be expanded over eigenstates of the unperturbed potential
\[ \Phi_m(x) = \sum_n c_{mn} \phi_n(x). \] (17)

Chaos implies extensive transitions between energy levels [43]; therefore, a chaos-assisted eigenfunction of the FTEO should be a compound of many unperturbed eigenstates. Thus, one can use the participation ratio
\[ \nu = \left( \sum_m |c_{mn}|^2 \right)^{-1}, \] (18)
as a measure of ‘chaoticity’. The phase-space region associated with an eigenfunction can be found by means of the parameter [44]
\[ \mu = \sum_{m=1}^M c_{mn}^2 m. \] (19)

Indeed, the formula \( \langle I \rangle = \hbar (\mu + 1/2) \) yields the expectation value of the classical action corresponding to the eigenfunction. Parameters \( \nu \) and \( \mu \) provide suitable classification of eigenfunctions and can be used for tracking the transition from order to chaos by means of numerical simulation.

Numerical simulation was conducted with the ensemble of 100 realizations of the FTEO. One hundred eigenfunctions with the lowest values of \( \mu \) are taken into account for each realization. An informative view is provided by distributions of FTEO eigenfunctions in the \( \mu - \nu \) space. These distributions corresponding to different values of \( \tau \) are presented in figure 3. For relatively small values of \( \tau \), the dots corresponding to eigenfunctions form ordered patterns consisting of distinct, slightly biased lines. Such patterns were earlier observed in [23], where they were called ‘stalagmites’. Each stalagmite is formed by eigenfunctions localized near periodic orbits of the one-step Poincaré with the same location in the action space. Transition to chaos is accompanied by delocalization of eigenfunctions and smearing of stalagmites. It should be noted that smearing is partially suppressed by dynamical localization [18]; that is, weakly unstable periodic

\[ I_{\phi_n} = \int p(x) \phi_n(x)^2 \, dx. \] (20)
orbits retain the ability to trap eigenfunctions \cite{45}. For \(\tau = 4\pi\), stalagmites occur in the whole range of \(\mu\) values, except for in the vicinity of \(\mu = 25\). This value of \(\mu\) corresponds to the phase-space region near the unperturbed separatrix of the pendulum, where the classical chaotic sea is originated initially. As \(\tau\) increases, the smeared domain grows, indicating gradual transition to chaos due to the overlapping of classical resonances \cite{13}. Notably, chaos-assisted delocalization first emerges in the range of small values of \(\mu\), corresponding to finite motion. This infers efficient destruction of invariant curves impeding the transition between finite and infinite regimes. The results of \cite{27} show that the destruction of invariant curves leads to the onset of directed current.

Nevertheless, traces of eigenfunctions with good persistence to chaos are visible even for \(\tau = 100\pi\). For instance, there is a small, slightly smeared horizontal stripe near \(\mu \approx 80\) (see figure 3(d)) corresponding to infinite motion with relatively high velocities. That phase-space region is characterized by inequality \(T \ll \tau\), which anticipates weak influence of resonances \cite{13} and, hence, weakness of chaos induced by their overlapping.

As domains of finite-time stability in phase space give rise to FTEO eigenfunctions with small \(\nu\), one can estimate their contribution using the cumulative distribution

\[
F(\nu) = \int_{1}^{\nu} \rho(\nu') \, d\nu',
\]

where \(\rho(\nu')\) is the corresponding probability density function. We refer to the case \(\nu \leq 2\) as the regime of strong localization. Indeed, inequality \(\nu \leq 2\) implies that the FTEO eigenfunction is mainly contributed from one or two unperturbed eigenstates. In addition, we now consider two regimes of moderate localization: \(\nu \leq 5\) and \(\nu \leq 10\). The sense of the latter two criteria becomes apparent if we take into account that finite motion corresponds to the 25 lowest unperturbed eigenstates. Thus, the inequality \(\nu \leq 5\) (\(\nu \leq 10\)) picks out FTEO eigenfunctions occupying less than 20 percent (40 percent) of the phase-space area enclosed by the unperturbed separatrix. As follows from figure 4, a fraction of strongly localized eigenfunctions rapidly decays down to nearly zero. It indicates the absence of significant long-living regular domains in classical phase space. It should be mentioned that the opposite situation was observed in \cite{23}, where the slow decay of \(F(2)\) was linked to the presence of degenerate tori in phase space. In the present case, there is no degenerate tori; therefore, there is no such route to the persistence of regular domains. Fractions of moderately localized functions decay

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**Figure 2.** Phase-space portraits constructed via the one-step Poincaré map with \(\tau = 20\pi\). Figures (a)-(d) correspond to different realizations of harmonic noise.
much more slowly, and their impact remains significant even at $\tau = 100\pi$. It can be understood as a manifestation of dynamical localization, i.e., partial suppression of chaos-assisted diffusion due to the interference.

5. Conclusion

In the present paper, we demonstrate an approach designed for studying quantum manifestations of classical finite-time stability under weak external random perturbation. The approach is based on construction of the finite-time evolution operator (FTEO). It is emphasized that statistical properties of FTEO eigenfunctions are closely linked to classical phase-space structure, revealed by a one-step Poincaré map that can be regarded as the classical counterpart of the FTEO. In particular, the stalagmite-like patterns on distributions of eigenfunctions in the space of parameters $\mu$ and $\nu$ are related to wavefunction concentrations near the periodic orbits of the one-step Poincaré map. Increasing time results in the emergence of chaos; therefore, periodic orbits lose their stability. This leads to smearing of the ‘stalagmites’. Chaos-assisted destruction of stalagmites represents an alternative view onto the order-to-chaos transition in randomly driven quantum systems.

In the present work, we consider stable domains satisfying the condition of invariance under translation over a finite time interval. They occur according to the same mechanism as stable islands in time-periodic systems. Namely, they are remnants of nonlinear resonances of the one-step Poincaré map and arise in a somewhat regular and predictable way. To underline the latter circumstance, we present phase-space portraits corresponding to different realizations. One can see that regular domains appear in the same ranges of the action (or energy) values.

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References

[1] Tomsovic S and Brown M G 2010 Ocean Acoustics: A Novel Laboratory for Wave Chaos (Cambridge: Cambridge University Press) pp 169–87

[2] Abdullaev S S 2011 Phys. Rev. E 84 026204

[3] Wolfson M A and Tomsovic S 2001 J. Acoust. Soc. Am. 109 2693–703

[4] Laffargue T, Lam K D N T, Kurchan J and Tailleur J 2013 J. Phys. A: Math. Theor. 46 254002

[5] Topinka M, LeRoy B, Westervelt R, Shaw S, Fleischmann R, Heller E, Maranowski K and Gossard A 2001 Nature 410 183–26

[6] Kaplan L 2002 Phys. Rev. Lett. 89 184103

[7] Liu B and Heller E J 2013 Phys. Rev. Lett. 111 236804

[8] Zaslavsky G M 2007 The Physics of Chaos in Hamiltonian Systems (London: Imperial College Press)

[9] Zaslavsky G M 2002 Phys. D: Nonlinear Phenom. 168 292–304

[10] Gutzwiller M C 1990 Classical and Quantum Mechanics (Berlin: Springer)

[11] Heller E J 1984 Phys. Rev. Lett. 53 1515–8

[12] Prants S, Budiansky M, Ponomarev V and Uleysky M 2011 Ocean Modelling 38 114–25

[13] Finn J and Apte S V 2013 Chaos 23 013145

[14] Tomsovic S and Lakshminarayan A 2007 Phys. Rev. E 76 036207

[15] Hützchenko P and Medvedev G S 2013 J. Nonl. Sci. 23 835–61

[16] Berglund N, Gentz B and Kuehn C 2013 (arXiv:1312.6353)

[17] Smirnov I P, Virovlyansky A L, Edelman M and Zaslavsky G M 2005 Phys. Rev. E 72 026206

[18] Berman G P and Kon'kov A R 1992 Sov. Phys. Usp. 35 303–26

[19] Kon’kov L E, Makarov D V, Sosedko E V and Uleysky M Y 2007 Phys. Rev. E 76 056212