MORITA BASE CHANGE IN HOPF-CYCLIC (CO)HOMOLOGY

LAIAICHI EL KAOUTIT AND NIELS KOWALZIG

ABSTRACT. In this paper we establish the invariance of cyclic (co)homology of left Hopf algebroids under the change of Morita equivalent base algebras. The classical result on Morita invariance for cyclic homology of associative algebras appears as a special example of this theory. In our main application we consider the Morita equivalence between the algebra of complex-valued smooth functions on the classical 2-torus and the coordinate algebra of the noncommutative 2-torus. We then construct a Morita base change \((4 \times 4)\)-matrix left Hopf algebroid over the noncommutative 2-torus and show that its cyclic (co)homology can be computed by means of the homology of the Lie algebroid of vector fields over the classical 2-torus.

1. INTRODUCTION

The concept of left Hopf algebroids provides a natural framework for unifying and extending classical constructions in homological algebra. Group, Groupoid, Lie algebra, Lie algebroid and Poisson, Hochschild and cyclic homology for associative algebras, as well as Hopf-cyclic homology for Hopf algebras, are all special cases of the cyclic homology of left Hopf algebroids since the rings over which these theories can be expressed as derived functors are all left Hopf algebroids (see, for example, [BS, CM, Cr1, Kow, KowKr, KowPo] for more details).

As for every (co)homology theory it is an interesting issue to examine its behaviour under (any suitable notion of) Morita equivalence. Nevertheless, a satisfactory notion of Morita equivalence between two possibly noncommutative left Hopf algebroids is up to our knowledge far from being obvious. The difficulty comes out when, for instance, one tries to understand how the notion of Morita equivalence between two Lie algebroids, in the sense of [Cr2, Gi] and others, can be reflected to their respective associated (universal) left Hopf algebroids in such a way that invariant properties, especially homological ones, between equivalent Lie algebroids remain invariant at the level of left Hopf algebroids. In the commutative case, that is, for commutative Hopf algebroids, several notions already exist in the literature, see, e.g., [Ho, HoSt].

In this paper we restrict ourselves to the naive case of Morita base change left Hopf algebroids. That is, we study from a cyclic (co)homology point of view two Morita equivalent left Hopf algebroids of the form \((R, U) \sim (S, \tilde{U})\), where \(R \sim S\) are Morita equivalent base rings and \(\tilde{U}\) is constructed from \(U\). It is worth noticing that for the case of commutative Hopf algebroids or Hopf algebras, this notion is in some sense useless since it reduces to changing the base rings by an isomorphism. Nevertheless, this restriction is not far from some geometric applications since, for example, the algebra of smooth functions on a smooth manifold \(\mathcal{M}\) is Morita equivalent to the endomorphism algebra

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of global smooth sections of a vector bundle on $\mathcal{M}$. More precisely, one can start with a smooth vector bundle $\mathcal{P} \rightarrow \mathcal{M}$ and a Lie algebroid $(\mathcal{M}, \mathcal{E})$, then associate to them a Morita base change $(\mathcal{C}^{\infty}(\mathcal{M}), \mathcal{V}(\mathcal{E})) \sim (\End(\Gamma(\mathcal{P})), \mathcal{V}(\mathcal{E}))$, where $(\mathcal{C}^{\infty}(\mathcal{M}), \mathcal{V}(\mathcal{E}))$ is the associated (universal) left Hopf algebra attached to $(\mathcal{M}, \mathcal{E})$, see Section 5. In the aim of illustrating our methods, we give an explicit application concerning the noncommutative 2-torus.

A left Hopf algebroid $(\times_R$-Hopf algebra) $\mathcal{U}$ is, roughly speaking, a Hopf algebra whose ground ring is not a commutative ring $k$ but a possibly noncommutative $k$-algebra $R$, see [B2, Sch2, T1]. As $k$-bialgebras are underlying Hopf algebras, $(\times_R$-bialgebroids are the underlying structure of left Hopf algebroids. Morita base change for bialgebroids (following [Sch4]) provides a possibility to produce new bialgebroids by replacing the base algebra $R$ by a Morita equivalent base algebra $S$ in such a way that the resulting $R$-bialgebroid has a monoidal category of representations equivalent to that of the original $R$-bialgebroid. More generally, the base algebra $R$ can be replaced by a $\sqrt{\text{Morita}}$ equivalent algebra $S$, see [T2]: two algebras are $\sqrt{\text{Morita}}$ equivalent if one has an equivalence of $k$-linear monoidal categories of bimodules $R_{\text{Mod}} \simeq S_{\text{Mod}} (R^e$ and $S^e$ are the enveloping $k$-algebras). Such an equivalence relation between two bialgebroids is weaker than to consider two bialgebroids to be equivalent if their monoidal categories of (co)representations are so. In particular, Morita base change establishes a relation between two bialgebroids in a way that is meaningless for ordinary $k$-bialgebras, as already said above.

Apart from what we mentioned above, the importance of the notion of Morita base change moreover consists in unifying seemingly different concepts: for example, every weak $\mathbb{C}$-bialgebra (which can be considered as bialgebroids [B2 §3.2.2]) can be shown to be a face algebra (which are examples of bialgebroids as well [Sch3]) up to Morita base change [Sch4 §5.2]. Here we present no application in this direction, this will be left for a future project.

Useful for our purposes is the fact that Morita invariance for cyclic homology carries over to the Hopf structure as well: An $R$-bialgebroid is left Hopf if and only if its Morita base change equivalent $S$-bialgebroid is left Hopf as well [Sch4 Prop. 4.6].

In this paper, we will consider the cyclic (co)homology for left Hopf algebroids from [KowKr] and confront it with the Morita base change theory from [Sch4]. Our aim is to give, in the spirit of [McC], the explicit chain morphisms and chain homotopies that establish equivalences of (co)cyclic modules between the original left Hopf algebroid and the Morita base change left Hopf algebroid $\mathcal{U}$. As a consequence, we obtain our central theorem which we copy here, see the main text for the details and in particular the notation used:

**Theorem A.** (Morita base change invariance of (Hopf-)cyclic (co)homology) Let $(U, R)$ be a left Hopf algebroid, $M$ a left $U$-comodule right $U$-module which is SaYD, and $(R, S, P, Q, \phi, \psi)$ a Morita context. Consider its induced $\sqrt{\text{Morita}}$ context $(R^e, S^e, P^e, Q^e, \phi^e, \psi^e)$ and the Morita base change left Hopf algebroid $(S, \tilde{U} := P^e \otimes_{R^e} U \otimes_{R^e} Q^e)$. Then

\[
H_*(U, M) \simeq H_*(\tilde{U}, P \otimes_R M \otimes_R Q), \quad H^*(U, M) \simeq H^*(\tilde{U}, P \otimes_R M \otimes_R Q),
\]

\[
HC_*(U, M) \simeq HC_*(\tilde{U}, P \otimes_R M \otimes_R Q), \quad HC^*(U, M) \simeq HC^*(\tilde{U}, P \otimes_R M \otimes_R Q)
\]

are isomorphisms of $k$-modules.

As an application, we first indicate how the classical result of Morita invariance for cyclic homology of associative algebras (see, e.g., [C, D1, McC]) fits into our general theory. Second, we consider a Morita context between the complex-valued smooth functions on the commutative real 2-torus $\mathbb{T}^2$ and the coordinate ring of the noncommutative 2-torus. After reviewing the construction for this case, we apply Morita invariance to the universal left Hopf algebroid of the Lie algebroid of vector fields over $\mathbb{T}^2$ and its Morita base change $(4 \times 4)$-matrix left Hopf algebroid over the noncommutative 2-torus.
2. Preliminaries

2.1. Some conventions. Throughout this note, “ring” means associative algebra over a fixed commutative ground ring \( \mathbb{k} \). All other algebras, modules etc., will have an underlying structure of a central \( \mathbb{k} \)-module. Given a ring \( R \), we denote by \( R \text{-Mod} \) the category of left \( R \)-modules, by \( R^e \) the opposite ring and by \( R^e := R \otimes_\mathbb{k} R^e \) the enveloping algebra of \( R \). An \( R \)-ring is a monoid in the monoidal category \((R^e \text{-Mod}, \otimes_R, R)\) of \( R^e \)-modules with symmetric action of \( \mathbb{k} \), fulfilling associativity and unitality. Likewise, an \( R \)-coring is a comonoid in \((R^e \text{-Mod}, \otimes_R, R)\), fulfilling coassociativity and counitality.

Our main object is an \( R^e \)-ring \( U \). Explicitly, such an \( R^e \)-ring is given by a \( \mathbb{k} \)-algebra homomorphism \( \eta = \eta_U : R^e \to U \) whose restrictions

\[
s := \eta(- \otimes_\mathbb{k} 1) : R \to U \quad \text{and} \quad t := \eta(1 \otimes_\mathbb{k} -) : R^e \to U
\]

will be called the source and target map, respectively. Left and right multiplication in \( U \) give rise to an \((R^e, R^e)\)-bimodule structure on \( U \), that is, four actions of \( R \) that we denote by

\[
r \triangleright u \triangleleft r' := s(r)t(r')u, \quad r \triangleright u \triangleleft r' := us(r')t(r), \quad r, r' \in R, u \in U,
\]

which are commuting, in the sense that, for every \( a, a', r, r' \in R \) and \( u, v \in U \), we have

\[
a' \triangleright (r \triangleright u \triangleleft r') \triangleleft a = r \triangleright (a' \triangleright u \triangleleft a) \triangleleft r';
\]

\[
(u \triangleleft r)(v \triangleleft a) = (a \triangleleft u)(r \triangleright v).
\]

If not stated otherwise, we view \( U \) as an \((R, R)\)-bimodule using the actions \( \triangleright, \triangleleft \), denoted \( U \triangleright a \). In particular, we define the tensor product \( U \otimes_R U \) with respect to this bimodule structure. On the other hand, using the actions \( \triangleleft, \triangleright \) permits to define the Sweedler-Takeuchi product, see [Sw, T1]:

\[
U \times_R U := \left\{ \sum_i u_i \otimes_R v_i \in U \otimes_R U \mid \sum_i r \triangleright u_i \otimes_R v_i = \sum_i u_i \otimes_R v_i \triangleleft r, \forall r \in R \right\}.
\]

One easily verifies that \( U \times_R U \) is an \( R^e \)-ring via factorwise multiplication, with unit element \( 1_U \otimes_R 1_U \) and \( \eta_{U \times_R U} : (r \otimes_\mathbb{k} r') = s(r) \otimes_R t(r'), \) for \( r, r' \in R \).

2.2. Bialgebroids. [TI] Bialgebroids are a generalisation of bialgebras. An important subtlety is that the algebra and coalgebra structure are defined in different monoidal categories.

Definition 2.1. Let \( R \) be a \( \mathbb{k} \)-algebra. A left bialgebroid over \( R \) is an \( R^e \)-ring \( U \) together with two homomorphisms of \( R^e \)-rings

\[
\Delta : U \to U \times_R U, \quad \hat{\varepsilon} : U \to \text{End}_\mathbb{k}(R)
\]

which turn \( U \) into an \( R \)-coring with coproduct \( \Delta \) (viewed as a map \( U \to U \otimes_R U \)) and counit \( \varepsilon : U \to R, u \mapsto (\varepsilon(u))(1) \).

So one has for example for \( u \in U, r, r' \in R \)

\[
\Delta(r \triangleright u \triangleleft r') = r \triangleright u(1) \otimes_R u(2) \triangleleft r', \quad \Delta(r \triangleleft u \triangleright r') = u(1) \cdotp r' \otimes_R r \triangleright u(2),
\]

using Sweedler’s shorthand notation \( u(1) \otimes_R u(2) \) for \( \Delta(u) \), as well as in \( U \times_R U \) the identity

\[
r \triangleright u(1) \otimes_R u(2) = u(1) \otimes_R u(2) \triangleright r.
\]

The counit, on the other hand, fulfills for any \( u, v \in U \) and \( r, r' \in R \)

\[
\varepsilon(r \triangleright u \triangleleft r') = r \varepsilon(u)r', \quad \varepsilon(u \triangleright r) = \varepsilon(r \triangleright u), \quad \varepsilon(uv) = \varepsilon(u \triangleright \varepsilon(v)) = \varepsilon(\varepsilon(v) \triangleright u).
\]
2.3. Left Hopf algebroids. \[\text{[Sch2]}\] Left Hopf algebroids have been introduced by Schauenburg under the name $\times_R$-Hopf algebras and generalise Hopf algebras towards left bialgebroids. For a left bialgebroid $U$ over $R$, one defines the (Hopf-)Galois map
\[
\beta : \bullet U \otimes_R U \otimes_R U \otimes_R v, \quad u \otimes_R (u_1) \otimes_R u(2)v, \]
where
\[
\bullet U \otimes_R U \otimes_R U = U \otimes_R U/\text{span}\{r \bullet u \otimes_R v - u \otimes_R v \bullet r \mid u, v \in U, r \in R\}. \tag{2.6}
\]

**Definition 2.2.** \[\text{[Sch2]}\] A left $R$-bialgebroid $U$ is called a left Hopf algebroid (or $\times_R$-Hopf algebra) if $\beta$ is a bijection.

By means of a Sweedler-type notation
\[
u_+ \otimes_R u_- := \beta^{-1}(u \otimes_R 1)
\]
for the translation map $\beta^{-1}(\bullet \otimes_R 1) : U \rightarrow \bullet U \otimes_R U \otimes_R U$, one obtains for all $u, v \in U, r, r' \in R$ the following useful identities \[\text{[Sch2]}\] Prop. 3.7:
\[
\begin{align*}
u_+ & (u_1) \otimes_R u_2) u_- & = & u \otimes_R 1 \in U \otimes_R U, \tag{2.7} \\
u_1 \otimes_R u_2 & \in U \otimes_R u_1 & = & u \otimes_R 1 \in U \otimes_R U \otimes_R U, \tag{2.8} \\
u_1 \otimes_R u_2 & \in U \otimes_R u_1 & = & u \otimes_R 1 \in U \otimes_R U, \tag{2.9} \\
u_1 \otimes_R u_2 & \in U \otimes_R u_1 & = & u \otimes_R 1 \in U \otimes_R U \otimes_R U, \tag{2.10} \\
u_1 \otimes_R u_2 & \in U \otimes_R u_1 & = & u \otimes_R 1 \in U \otimes_R U, \tag{2.11} \\
u_1 \otimes_R u_2 & \in U \otimes_R u_1 & = & u \otimes_R 1 \in U \otimes_R U \otimes_R U, \tag{2.12} \\
u_1 \otimes_R u_2 & \in U \otimes_R u_1 & = & u \otimes_R 1 \in U \otimes_R U, \tag{2.13} \\
u_1 \otimes_R u_2 & \in U \otimes_R u_1 & = & u \otimes_R 1 \in U \otimes_R U \otimes_R U, \tag{2.14} \\
(s(r) \otimes_R (s(r'))_+ & \otimes_R (s(r) \otimes_R (s(r'))_- & = s(r) \otimes_R s(r'), \tag{2.15}
\end{align*}
\]
where in \[2.9\] we mean the Sweedler-Takeuchi product
\[
U \times_R U := \left\{ \sum_i u_i \otimes_R v_i \in \bullet U \otimes_R U \otimes_R U \mid \sum_i u_i \otimes_R v_i = \sum_i u_i \otimes_R v_i \otimes_R r, \forall r \in R \right\},
\]
which is an algebra by factorwise multiplication, but with opposite multiplication on the second factor. Note that in \[2.11\] the tensor product over $R\otimes_R$ links the first and third tensor component. By \[2.7\] and \[2.9\], one can write
\[
\beta^{-1}(u \otimes_R v) = u_+ \otimes_R u_- u_v.
\]

2.4. $U$-modules. Let $(U, R)$ be a left bialgebroid. Left and right $U$-modules are defined as modules over the ring $U$, with respective actions denoted by juxtaposition. We denote the respective categories by $U\text{-Mod}$ and $U^\otimes \text{-Mod}$; while $U\text{-Mod}$ is a monoidal category, $U^\otimes \text{-Mod}$ is in general not \[\text{[Sch1]}\]. One has a forgetful functor $U\text{-Mod} \rightarrow R\text{-Mod}$ using which we consider every left $U$-module $N$ also as an $(R, R)$-bimodule with actions
\[
\begin{align*}
\text{amb} := a \triangleright n \triangleleft b := s(a) t(b) n, \quad a, b \in R, n \in N.
\end{align*}
\]
Similarly, every right $U$-module $M$ is also an $(R, R)$-bimodule via
\[
\begin{align*}
\text{amb} := a \triangleright m \triangleleft b := m s(b) t(a), \quad a, b \in R, m \in M,
\end{align*}
\]
and in both cases we usually prefer to express these actions just by juxtaposition if no ambiguity is to be expected.
2.5. $U$-comodules. Similarly as for coalgebras, one may define comodules over bialgebroids, but the underlying $R$-module structures need some extra attention. For the following definition confer e.g. [Sch1, B1, BrzWi].

**Definition 2.3.** A left $U$-comodule for a left bialgebroid $(U, R)$ is a left comodule of the underlying $R$-coring $(U, \Delta, \varepsilon)$, i.e., a left $R$-module $M$ with action $L_R : (r, m) \mapsto rm$ and a left $R$-module map $\Delta_M : M \to U \otimes_R M, m \mapsto m_{(-1)} \otimes_R m_{(0)}$ satisfying the usual coassociativity and counitality axioms. We denote the category of left $U$-comodules by $U\text{-Comod}$.

On any left $U$-comodule one can additionally define a right $R$-action $m \cdot r := \varepsilon(m_{(-1)} \bullet r)m_{(0)}$. (2.16)

This action originates in fact from the algebra morphism $R \to U^*, r \mapsto [u \mapsto \varepsilon(u \bullet r)]$, (2.17) where $U^* := \text{Hom}_{-R}(U, R)$ is the right convolution ring of the underlying $R$-coring $U$, and the canonical functor $U\text{-Comod} \to \text{Mod}_{U^*}$ that endows any left $U$-comodule $X$ with a right $U^*$-action given by

$$x \sigma = \sum_{(x)} \sigma(x_{(-1)})x_{(0)}$$

for every $x \in X$ and $\sigma \in U^*$. The above action is then the restriction to scalars associated to the algebra morphism (2.17), and the action (2.16) is the unique one that turns $M$ into a left $R^e$-module in such a way that the coaction is an $R^e$-module morphism

$$\Delta_M : M \to U \times R M,$$

where $U \times_R M$ is the Sweedler-Takeuchi product

$$U \times_R M := \{ \sum_i u_i \otimes_R m_i \in U \otimes_R M \mid \sum_i u_i t(a) \otimes_R m_i = \sum_i u_i \otimes_R m_i a, \forall a \in R \}. $$

In other words, $M$ becomes a left $\times_R U$-comodule. Conversely, any left $\times_R U$-comodule gives rise to a left $U$-comodule. This correspondence establishes in fact an isomorphism of categories.

As a result of the previous discussion, $\Delta_M$ satisfies the identities

$$\Delta_M(rmr') = (r \triangleright m_{(-1)} \bullet r') \otimes_R m_{(0)},$$

$$m_{(-1)} \otimes_R m_{(0)} r = (r \triangleright m_{(-1)}) \otimes_R m_{(0)}.$$ (2.18, 2.19)

2.6. Cyclic homology for left Hopf algebroids.

2.6.1. Stable anti Yetter-Drinfel’d modules. The following definition is the left bialgebroid right module and left comodule version of the corresponding notion in [BS]. For Hopf algebras, the concept goes back to [HKhRS].

**Definition 2.4.** Let $(U, R)$ be a left Hopf algebroid, and let $M$ simultaneously be a left $U$-comodule and a right $U$-module with action denoted by $(m, u) \mapsto mu$ for $u \in U$, $m \in M$. We call $M$ an anti Yetter-Drinfel’d (aYD) module if:

---
(i) The two $R^e$-module structures on $M$ originating from its nature as $U$-comodule resp. right $U$-module coincide: for all $r, r' \in R$, $m \in M$

\[ rm = r \triangleright m, \quad (2.20) \]

\[ mr' = m \triangleleft r', \quad (2.21) \]

where the right $R$-module structure on the left hand side is given by (2.16).

(ii) For $u \in U$ and $m \in M$ one has the following compatibility between action and coaction:

\[ \Delta_M(mu) = u_{-}m_{(-1)}u_{+1}(1) \otimes_R m_{(0)}u_{+2}. \quad (2.22) \]

The aYD module $M$ is said to be stable (SaYD) if, for all $m \in M$, one has

\[ m_{(0)}m_{(-1)} = m. \quad (2.23) \]

2.6.2. Cyclic (co)homology. We will not recall the formalism of cyclic (co)homology in full detail; see, e.g., [FTs, L] for more information. However, recall that para-(co)cyclic $k$-modules generalise (co)cyclic $k$-modules by dropping the condition that the (co)cyclic operator implements an action of $\mathbb{Z}/(n + 1)\mathbb{Z}$ on the degree $n$ part. Thus a para-cyclic $k$-module is a simplicial $k$-module $(C_*, d_*, s_*)$ and a para-cocyclic $k$-module is a cosimplicial $k$-module $(C^*, \delta_*, \sigma_*)$, together with $k$-linear maps $t_n : C_n \to C_n$ resp. $\tau_n : C^m \to C^n$ satisfying, respectively

\[ d_i \circ t_n = \begin{cases} d_{n-1} \circ d_{1} & \text{if } 1 \leq i \leq n, \\ d_n & \text{if } i = 0, \end{cases} \]

\[ \tau_n \circ d_i = \begin{cases} \delta_{i-1} \circ \tau_{n-1} & \text{if } 1 \leq i \leq n, \\ \delta_n & \text{if } i = 0, \end{cases} \]

\[ s_i \circ t_n = \begin{cases} s_{n+1} \circ s_{1} & \text{if } 1 \leq i \leq n, \\ s_{n+1} \circ s_{1} & \text{if } i = 0, \end{cases} \]

\[ \tau_n \circ s_i = \begin{cases} \sigma_{i-1} \circ \tau_{n+1} & \text{if } 1 \leq i \leq n, \\ \sigma_n \circ \tau_{n+1} & \text{if } i = 0, \end{cases} \]

Such a para-(co)cyclic module is called (co)cyclic if $t_n^{n+1} = \text{id}$ (resp. $\tau_n^{n+1} = \text{id}$). Any cyclic module $C_*$ gives rise to a cyclic bicomplex $C_*$, see, e.g., [FTs] for details. The only thing we recall here is that the differential on the $b$-columns is given by

\[ b = \sum_{i=0}^{n} (-1)^i d_i, \quad (2.25) \]

and likewise $\beta := \sum_{i=0}^{n+1} (-1)^i \delta_i$ for a cocyclic module.

2.6.3. The para-(co)cyclic module associated to a left Hopf algebroid ([KowKr], cf. also [KowPo]). Let $M$ be simultaneously a left $U$-comodule and a right $U$-module with compatible left $A$-action as in (2.20). Set

\[ C_*(U, M) := M \otimes_{R^e} (\bullet U_c) \otimes_{R^e}, \]

and in each degree $n$ define the following structure maps on it:

\[ d_i(m \otimes_{R^e} x) = \begin{cases} m \otimes_{R^e} u^1 \otimes_{R^e} \cdots \otimes_{R^e} (\varepsilon(u^n)) \triangleright u^{n-1} & \text{if } i = 0, \\ m \otimes_{R^e} \cdots \otimes_{R^e} (u^{i-1}u^n + 1) \otimes_{R^e} \cdots \otimes_{R^e} u^n & \text{if } 1 \leq i \leq n - 1, \\ (mu^1) \otimes_{R^e} u^2 \otimes_{R^e} \cdots \otimes_{R^e} u^n & \text{if } i = n, \end{cases} \]

\[ s_i(m \otimes_{R^e} x) = \begin{cases} m \otimes_{R^e} u^1 \otimes_{R^e} \cdots \otimes_{R^e} u^n \otimes_{R^e} 1 & \text{if } i = 0, \\ m \otimes_{R^e} \cdots \otimes_{R^e} u^n \otimes_{R^e} 1 \otimes_{R^e} u^{n-i+1} \otimes_{R^e} \cdots \otimes_{R^e} u^n & \text{if } 1 \leq i \leq n - 1, \\ m \otimes_{R^e} 1 \otimes_{R^e} u^1 \otimes_{R^e} \cdots \otimes_{R^e} u^n & \text{if } i = n, \end{cases} \]

\[ t_n(m \otimes_{R^e} x) = (m_{(0)}u_1) \otimes_{R^e} u_2^1 \otimes_{R^e} \cdots \otimes_{R^e} u_n \otimes_{R^e} (u^n \cdots u_{-1} m_{-1}), \quad (2.26) \]
where we abbreviate \( x := u^1 \otimes_R \cdots \otimes_R u^n \). As explained in detail in [KowKr], this cyclic module is the generalised “cyclic dual” to the following cocyclic module: set
\[
C^\bullet(U, M) := (U \otimes_R)_{\otimes^\bullet_R} \otimes_R M,
\]
with structure maps in degree \( n \) given by
\[
\delta_i(z \otimes_R m) = \begin{cases} 
1 \otimes_R u^1 \otimes_R \cdots \otimes_R u^n \otimes_R m & \text{if } i = 0, \\
u_1 \otimes_R \cdots \otimes_R \Delta(u^i) \otimes_R \cdots \otimes_R u^n \otimes_R m & \text{if } 1 \leq i \leq n, \\
u_1 \otimes_R \cdots \otimes_R u^n \otimes_R m_{(-1)} \otimes_R m_{(0)} & \text{if } i = n + 1,
\end{cases}
\]
\[
\delta_j(m) = \begin{cases} 
1 \otimes_R m & \text{if } j = 0, \\
m_{(-1)} \otimes_R m_{(0)} & \text{if } j = 1,
\end{cases}
\]
\[
\sigma_i(z \otimes_R m) = u^1 \otimes_R \cdots \otimes_R \varepsilon(u^{i+1}) \otimes_R \cdots \otimes_R u^n \otimes_R m & 0 \leq i \leq n - 1,
\]
\[
\tau_R(z \otimes_R m) = u^1_{(-1)} \otimes_R u^2 \otimes_R \cdots \otimes_R u^1_{(n-1)} \otimes_R u^1_{(n)} m_{(-1)} \otimes_R m_{(0)} u^1_{(1)},
\]
where we abbreviate \( z := u^1 \otimes_R \cdots \otimes_R u^n \).

In [KowKr] it was shown that, under the minimal assumption (2.20), the maps (2.26) (resp. (2.27)) give rise to a para-cyclic (resp. para-cocyclic) module, which is cyclic (resp. cocyclic) if \( M \) is SaYD, i.e., additionally fulfills (2.21)–(2.23).

Let us denote by \( H_*(U, M) \) and \( HC_*(U, M) \) the resulting simplicial and cyclic homology groups of \( C_*(U, M) \), and likewise by \( H^*(U, M) \) and \( HC^*(U, M) \) the resulting simplicial and cyclic cohomology groups of \( C^*(U, M) \).

### 3. √Morita theory and Morita base change Hopf algebroids

In this section, we first recall some general facts about Morita contexts and their induced √Morita theory in the sense of Takeuchi [12]. Secondly, we explain how this theory was used by Schauenburg to introduce Morita base change (left) Hopf algebroids in [Sch4], where unfortunately no explicit description of the relevant structure maps was given. In order to establish our main result, we explicitly give here these structure maps, and then illustrate this by a detailed description of matrix (left) Hopf algebroids which we will use as an application in the last section.

From now on, the unadorned symbol \( \otimes \) stands for the tensor product over \( k \), the commutative ground ring.

#### 3.1. Morita contexts

Let \( R \) and \( S \) be two rings and let \( SP_R \) and \( RQS \) be two bimodules, together with the following bimodule isomorphisms:
\[
\phi : P \otimes_R Q \xrightarrow{\sim} S, \quad \phi^{-1}(1_S) = \sum p_j' \otimes_R q_j',
\]
\[
\psi : Q \otimes_S P \xrightarrow{\sim} R, \quad \psi^{-1}(1_R) = \sum q_i \otimes_S p_i.
\]
It is known from Morita theory (see, e.g., [Ba p. 60]) that, up to natural isomorphisms, \( \phi \) and \( \psi \) can be chosen in such a way that
\[
(\phi \otimes_S P) = (P \otimes_R \psi) \quad \text{and} \quad (\psi \otimes_R Q) = (Q \otimes_S \phi).
\]
Thus \( (R, S, P, Q, \phi, \psi) \) can be considered as a Morita context. In what follows, we will usually make use of the notation
\[
p'q' := \phi(p' \otimes_R q') \quad \text{and} \quad qp := \psi(q \otimes_S p), \quad \forall p, p' \in P, \ q, q' \in Q.
\]
We then have
\[ \sum_{j} p_j q_j = 1_S, \quad \sum_{i} q_i p_i = 1_R, \]
as well as
\[ a(bp) = (ab)p \quad \text{in} \quad sP_R, \quad b(aq) = (ba)q \quad \text{in} \quad RQ_S, \]
for every pairs of elements \( a, p \in P \) and \( b, q \in Q \).

The above context is canonically extended to a Morita context between the enveloping rings \( R^e \) and \( S^e \). That is, \((R^e, S^e, P^e, Q^e, \phi^e, \psi^e)\) is a Morita context as well, where the underlying bimodules are defined by
\[ P^e := \mathcal{P} \otimes Q^o \in S^e \text{Mod}_{R^e}, \]
\[ Q^e := Q \otimes P^o \in R^e \text{Mod}_{S^e}. \]

Here \( R^e \text{Mod}_{S^e} \) and \( S^e \text{Mod}_{R^e} \) are the opposite bimodules, and \( \phi^e, \psi^e \) are the obvious maps. As was argued in [12], this is an induced \( \sqrt{\text{Morita}} \) equivalence between \( R \) and \( S \), in the sense that the last context induces a monoidal equivalence between the monoidal categories of bimodules \( R \text{Mod}_R \) and \( S \text{Mod}_S \).

Explicitly, such a monoidal equivalence is set up by the following functors
\[ R \text{Mod}_R \simeq R \text{Mod} \xrightarrow{P^e \otimes R^e} \xleftarrow{Q^e \otimes S^e} S \text{Mod} \simeq S \text{Mod}_S. \]

One of the monoidal structure maps of the functor \( Q^e \otimes S^e \) is explicitly given by the following natural isomorphism
\[ \sum_j \left( (q \otimes p_j^o) \otimes_S x \right) \otimes_R \left( (b \otimes a^o) \otimes_S y \right) \xrightarrow{\simeq} \sum_j \left( (q \otimes p_j^o) \otimes_S x \right) \otimes_R \left( (q \otimes p^o) \otimes_S x \otimes_S y \right), \tag{3.3} \]

An alternative way of defining these functors is via the following natural isomorphisms
\[ R \text{Mod}_{R^e} \xrightarrow{P^e \otimes R^e(-) \otimes R^e Q^e} \xleftarrow{Q^e \otimes S^e(-) \otimes S^e P^e} S \text{Mod}_{S^e}. \]

Using the Morita context, this equivalence is canonically lifted to the category of monoids. Thus, if we denote by \( R^e \)-\textbf{Rings} the category of \( R^e \)-rings, i.e., algebra extensions of \( R^e \), we have a commutative diagram
\[
\begin{array}{ccc}
R^e \text{-Rings} & \xrightarrow{P^e \otimes R^e(-) \otimes R^e Q^e} & S^e \text{-Rings} \\
\phi_R \downarrow & & \phi_S \downarrow \\
R^e \text{Mod}_{R^e} & \xrightarrow{P^e \otimes R^e(-) \otimes R^e Q^e} & S^e \text{Mod}_{S^e}.
\end{array}
\]
whose vertical arrows are the forgetful functors. For any $R^e$-ring $T$ we then have functors connecting the categories of left modules:

\[
\begin{align*}
\xymatrix{
\mathcal{T} \text{Mod} & P^e \otimes_{R^e} (-) \ar[d]_{\mathcal{T}} \ar[r] & P^e \otimes_{R^e} T \otimes_{R^e} Q^e \text{Mod} \ar[d]_{\mathcal{T}'} \ar[l] \\
R^e \text{Mod} & P^e \otimes_{R^e} (-) \ar[r] & Q^e \otimes_{R^e} (-)\ar[l]}
\end{align*}
\]

(3.4)

3.2. Morita base change for left bialgebroids. In [Sch4], Schauenburg used one of these functors to construct a functor from the category of left Hopf algebroids over $R$ to the category of left Hopf algebroids over $S$, known as Morita base change left Hopf algebroids. In what follows, we will need an explicit description of this Morita base change left Hopf algebroid structure. So, it will be convenient to review this construction in more detail.

Let $(R, S, P, Q, \phi, \psi)$ be a Morita context. As one can realize from diagram (3.4), the following two assertions are equivalent:

(i) the category of $T$-modules is a monoidal category and the forgetful functor $\mathcal{F}$ is strict monoidal;

(ii) the category of $(P^e \otimes_{R^e} T \otimes_{R^e} Q^e)$-modules is a monoidal category and the forgetful functor $\mathcal{F}'$ is strict monoidal.

Therefore, by Schauenburg’s result [Sch4, Theorem 5.1], starting with a left Hopf algebroid $(U, R)$ we can construct a new one $(\tilde{U}, S)$ as follows. Denote by

\[
\tilde{U} := P^e \otimes_{R^e} U \otimes_{R^e} Q^e
\]

the image of $U$. Using the natural isomorphism (3.3) and the diagram (3.4) for the underlying $R^e$-ring $U$, we can compute the structure maps of the left Hopf algebroid $(\tilde{U}, S)$:

(i) **Source and target.** Source and target are given by

\[
\tilde{\eta} : S^e \otimes S^e \rightarrow P^e \otimes_{R^e} U \otimes_{R^e} Q^e,
\]

\[
s \otimes \tilde{s} \mapsto \sum_{i,j} (s_{ij} \otimes q_i^o) \otimes_{R^e} 1_{U} \otimes_{R^e} (q_j^o \otimes (\tilde{s}_{ij})^o).
\]

(3.5)

(ii) **Ring structure.** The multiplication in $\tilde{U}$ is given by

\[
\tilde{\mu} : \tilde{U} \otimes_{S^e} \tilde{U} \rightarrow \tilde{U},
\]

\[
\tilde{u} \otimes_{S^e} \tilde{v} \mapsto (a_1 \otimes b_1^o) \otimes_{R^e} ((u \bullet (c_1 a_2))((b_2 d_1) \circ v)) \otimes_{R^e} (c_2 \otimes d_2^o),
\]

where $\tilde{u} := ((a_1 \otimes b_1^o) \otimes_{R^e} u \otimes_{R^e} (c_1 \otimes d_1^o))$ and $\tilde{v} := ((a_2 \otimes b_2^o) \otimes_{R^e} v \otimes_{R^e} (c_2 \otimes d_2^o))$. The identity element is given by the image $\tilde{\eta}(1_{S^e})$:

\[
1_{S^e} \mapsto \sum_{i,j} (p_i^o \otimes q_i^o) \otimes_{R^e} 1_{U} \otimes_{R^e} (q_j^o \otimes p_i^o).
\]

(iii) **Coring structure.** The comultiplication is given by

\[
\tilde{\Delta} : \tilde{U} \rightarrow \tilde{U} \otimes_{S^e} \tilde{U},
\]

\[
\tilde{u} \mapsto \sum_{i,j} ((a \otimes q_i^o) \otimes_{R^e} u_{(1)} \otimes_{R^e} (c \otimes p_i^o)) \otimes_{S} ((p_i \otimes b^o) \otimes_{R^e} u_{(2)} \otimes_{R^e} (q_j^o \otimes d^o)),
\]

where $\tilde{u} := ((a \otimes b^o) \otimes_{R^e} u \otimes_{R^e} (c \otimes d^o))$, and the counit is given by

\[
\tilde{\varepsilon} : \tilde{U} \rightarrow S, \quad \tilde{u} \mapsto a \varepsilon(u \bullet (c d)) b.
\]

(3.8)
(iv) The left Hopf structure. The explicit expression for the translation map reads

\[ \tilde{\beta}^{-1} : \tilde{U} \longrightarrow \tilde{U} \otimes_{SP} \tilde{U}, \quad \tilde{u} \longmapsto \sum_{i,j} \left( (a \otimes q_i^o) \otimes_R e \ u_+ \otimes_R (c \otimes p_j^o) \right) \otimes_S \left( (d \otimes q_i^o) \otimes_R e \ u_- \otimes_R (b \otimes p_j^o) \right), \]  

(3.9)

where again \( \tilde{u} := ((a \otimes b^o) \otimes_R e \ u \otimes_R (c \otimes d^o)) \).

3.2.1. \( \tilde{U} \)-modules and \( \tilde{U} \)-comodules. Consider the diagram analogous to (3.4) for right \( U \)-modules. The functor of the first column in that diagram is explicitly given on objects as follows. For \( M \in \text{Mod}_U \), the right \( \tilde{U} \)-module \( \tilde{M} := P \otimes_R M \otimes_R Q \) is equipped with the following action: denote

\[ \tilde{m} := p \otimes_R m \otimes_R q \in \tilde{M} \quad \text{and} \quad \tilde{u} := (a \otimes b^o) \otimes_R e \ u \otimes_R (c \otimes d^o) \in \tilde{U}, \]

and define

\[ \tilde{m} \tilde{u} := d \otimes_R \left( (bp) \triangleright m \triangleright (qa) \right) u \otimes_R c. \]  

(3.10)

As shown in [Sch4], there is also a monoidal equivalence connecting the categories of left comodules. More precisely, if \( M \in \text{UComod} \), then \( \tilde{M} \) is a left \( \tilde{U} \)-comodule with coaction

\[ \Delta_{\tilde{M}}(\tilde{m}) := \sum_{i,j} \left( (p \otimes q_i^o) \otimes_R m_{(-1)} \otimes_R \right) \otimes_S \left( \left( p_i \otimes_R m_{(0)} \otimes_R q_j^o \right) \right), \]  

(3.11)

which exactly coincides with the formula given in [Sch4] in the special case where the left module \( \_\otimes_R U \) is finitely generated projective.

Lemma 3.1. Let \( M \) be a right \( U \)-module and left \( U \)-comodule. Then \( M \) is aYD (resp. SaYD) if and only if \( \tilde{M} \) is.

Proof. We only prove the direct implication, the proof of the converse being similar. So assume that \( M \) is aYD. Then, for any \( s, t \in S \), we have

\[ \tilde{m} \tilde{\eta}(t \otimes s^o) = \sum_{i,j} \left( p \otimes_R m \otimes_R q \right) \left( (tp_j^o) \otimes_R \right) \otimes_S \left( \left( sp_i^o \right) \otimes_R \right) \]

which gives (2.20) and (2.21) for \( \tilde{M} \). Now, let us show (2.22) for \( \tilde{M} \), and start with \( \tilde{m} \tilde{u} = d \otimes_R \left( (bp) \triangleright m \triangleright (qa) \right) u \otimes_R c \) as defined in (3.10). Once computed the coaction of the middle term in the latter tensor product and taking into account (2.22) for \( M \), apply (3.11) to obtain

\[ \Delta_{\tilde{M}}(\tilde{m} \tilde{u}) = \sum_{i,j} \left[ (d \otimes q_i^o) \otimes_R \left( (u_- \triangleright (bp))m_{(-1)} \triangleright (qa) \right) u_{(+1)} \otimes_R (c \otimes p_j^o) \right] \otimes_S \left( p_i \otimes_R m_{(0)} u_{+2} \otimes_R q_j^o \right). \]
On the other hand, using \((3.7), (3.9)\), as well as the coaction \((3.11)\) of \(\hat{M}\), we get

\[
\tilde{u}_{-\hat{m}(\cdots)\tilde{u}_{+1}} \otimes_S \hat{m}(\cdots) \tilde{u}_{+2} = \sum_{i_0, i_2, j_0, j_1, j_2} \left[ (d \otimes q_{i_0}^n) \otimes_{M} (u_{-} \circ (bp)) \right] \left[ (u_{+} \circ (qa)) \otimes_{M} (c \otimes p_{i_2}^n) \right] \otimes_S \left[ p_{i_1} \otimes_S \left( (m_{i_0} \circ (q_{i_1}^n p_{i_2}^n)) \otimes_{M} (u_{+2}) \right) \right]
\]

\[
= \sum_{i_1, j_2} \left[ (d \otimes q_{i_1}^n) \otimes_{M} ((u_{-} \circ (bp))) \otimes_{M} (c \otimes p_{i_2}^n) \right] \otimes_S \left[ p_{i_1} \otimes_S \left( (m_{i_0} \circ (q_{i_1}^n p_{i_2}^n)) \otimes_{M} (u_{+2}) \right) \right]
\]

\[
= \Delta_{\hat{m}}(\hat{m}\tilde{u}),
\]

where in the last equality we used \((2.3), (2.4)\), and \((2.9)\). Analogously, one checks the stability condition for \(\hat{M}\).

3.3. Example: the matrix left Hopf algebroid. Let \(R\) be a ring with the Morita context \((R, M_n(R), R^n, R^o)\), where \(n \geq 1\). That is, \(P := R^n\) is a free right \(R\)-module of rank \(n\) whose elements are considered as columns and similarly \(Q = R^o\), whose elements are viewed as rows. Here \(Q\) is considered as the right dual module of \(P\), that is, \(Q = \text{Hom}(P_R, R_R)\). Thus, the matrix ring \(M_n(R)\) acts from the right on \(P\) by the usual operation of matrices on columns, and acts from the left on \(Q\) by the usual operation of matrices on rows. The maps of the Morita context are

\[
\phi : R^n \otimes_R R^n \rightarrow S = M_n(R), \quad \left( \begin{array}{c} r_1 \\ \vdots \\ r_n \end{array} \right) \otimes_R \left( \begin{array}{c} s_1 \cdots s_n \end{array} \right) \rightarrow \left( r_i s_j \right)_{i,j},
\]

\[
\psi : R^n \otimes_S R^n \rightarrow R, \quad \left( \begin{array}{c} s_1 \cdots s_n \end{array} \right) \otimes_R \left( \begin{array}{c} r_1 \\ \vdots \\ r_n \end{array} \right) \rightarrow \sum_{1 \leq i \leq n} s_i r_i.
\]

The sets of parameters \(\{p_i, q_i\}\) and \(\{p_j', q_j'\}\) are given by

\[
\{p_i, q_i\}_i = \{e_1, e'_1\}, \quad \{p_j', q_j'\}_j = \{e_j, e'_j\}_j,
\]

where \(\{e_i\}_{1 \leq i \leq n}\) and \(\{e'_j\}_{1 \leq i \leq n}\) are, respectively, the dual bases of \(P_R\) and \(R_Q\), and where \((\cdot)^t\) denotes the transpose. Before extending this context to one between \(R^e\) and \(S^o\), some notation is needed: under the ring isomorphism

\[
M_n(R)^o \simeq M_n(R^e), \quad (a_{ij})_{i,j} \mapsto (a_{ij}^o)_{i,j},
\]

we can consider the opposite bimodule \(Q^o\) as an \((M_n(R^e), R^e)\)-bimodule (i.e., left \(M_n(R^e)\)-module and right \(R^e\)-module), hence its elements can be seen as columns. Henceforth, elements of \(P^e\) are columns of size \(n^2\) with coefficients in \(R^e\). That is, we can identify the right module \(P^e \otimes R^e\) with the free module \((R^e)^{n^2} \otimes R^e\) via the following bimodule isomorphism

\[
P^e \rightarrow (R^e)^{n^2}, \quad (r_i)_{1 \leq i \leq n} \otimes (s_{ij})_{1 \leq i \leq n, 1 \leq j \leq n} \mapsto \left( r_i \otimes_R s_{ij}^o \right)_{1 \leq (i-1)n+j \leq n^2},
\]

where the notation \((x_{ij})_{i,j}\) stands for the columns. Here the left actions are given by different isomorphic rings, namely \(S^o\) and \(M_{n^2}(R^e)\) with the isomorphism

\[
S^o = M_n(R) \otimes M_n(R)^o \xrightarrow{\sim} M_n(R) \otimes M_n(R^e) \xrightarrow{\sim} M_{n^2}(R^e),
\]

\[
(a_{ij})_{i,j} \otimes (b_{kl})_{k,l} \mapsto (a_{ij} b_{kl})_{i,j,k,l} \mapsto (a_{ij} b_{kl})_{(i-1)n+k, (j-1)n+l}.
\]
In the same fashion we identify the bimodule \( Q^n \) with \((R^n)^{n^2}\) and consider its elements as rows. The extended Morita context \((R^n, S^n, P^n, Q^n)\) is then isomorphic to the Morita context \((R^n, M_{n^2}(R^n), (R^n)^{n^2}, (R^n)^{n^2})\).

Now consider a left \( R \)-Hopf algebroid \((U, R)\) and its associated left \( S \)-Hopf algebroid \((\bar{U}, S)\) as in \([3.12]\). Using the isomorphism
\[
\tilde{U} := P^n \otimes_{R^n} U \otimes_{R^n} Q^n \quad \rightarrow \quad M_{n^2}(U)
\]
we want to describe the left Hopf structure of the matrices \( M_{n^2}(U) \). Denote by \( e_{i,j} \), for \( 1 \leq i, j \leq n^2 \), the matrices with only one nonzero entry \((i, j)\), having \( 1_U \) as coefficient. That is, \( e_{i,j} \) are the unitary matrices in the matrix ring \( M_{n^2}(U) \). In this way, for each element \( u \in U \), we consider the matrices \( u e_{i,j} \in M_{n^2}(U) \), \( 1 \leq i, j \leq n^2 \), which have only one nonzero entry \((i, j)\) with coefficient \( u \).

(i) Ring structure, source and target of \( M_{n^2}(U) \). The ring structure is the usual one of matrix rings. Up to the isomorphism \([3.12]\), source and target maps are given by
\[
\tilde{\gamma} : \quad S^n = M_n(R) \otimes M_n(R)^{\circ} \quad \rightarrow \quad M_{n^2}(U),
\]
\[
(a_{i,j})_{i,j} \otimes (b_{i,j})_{i,j} \quad \rightarrow \quad \big( (a_{i,j} \otimes (b_{i,j}))_{i,j} \big)_{(1) + k, (j-1)n+l},
\]
where \( \eta : R^n \rightarrow U \) is as above Eq. \([2.1]\).

(ii) Cooring structure of \( M_{n^2}(U) \). It is sufficient to define the comultiplication on elements of the form \( u e_{i,j} \), for \( 1 \leq i, j \leq n^2 \):
\[
\tilde{\Delta} : \quad M_{n^2}(U) \quad \rightarrow \quad M_{n^2}(U) \otimes_{M_n(R)} M_{n^2}(U),
\]
\[
ue_{i,j} \quad \rightarrow \quad \sum_{l} (u_{(1)} e_{(1) + k, (j-1)n+l}) \otimes_{M_n(R)} (u_{(2)} e_{(j-1)n+l}),
\]
where \( i = (i-1)n + j \) and \( j = (k-1)n + l \), for some \( i, j, k, l = 1, \ldots, n \). The counit is defined by
\[
\tilde{\varepsilon} : \quad M_{n^2}(U) \quad \rightarrow \quad M_n(R),
\]
\[
e u e_{i,j} \quad \rightarrow \quad \begin{cases} 0 & \text{if } k \neq l, \\ \varepsilon(u) e_{i,j} & \text{if } k = l, \end{cases}
\]
where \( i = (i-1)n + j \) and \( j = (k-1)n + l \), for some \( i, j, k, l = 1, \ldots, n \), and \( e_{i,j} \) are the unitary matrices of \( M_n(R) \).

(iii) The left Hopf structure of \( M_{n^2}(U) \). The explicit expression for the translation map reads
\[
\tilde{\beta}^{-1} : \quad M_{n^2}(U) \quad \rightarrow \quad M_{n^2}(U) \otimes_{M_n(R)} M_{n^2}(U),
\]
\[
u e_{i,j} \quad \rightarrow \quad \sum_{j'} (u e_{i-1, j'} + e_{(i-1)n+j', (j-1)n+1}) \otimes_{M_n(R)} (u e_{(j-1)n+l} + e_{(j-1)n+l}),
\]
where \( i = (i-1)n + j \) and \( j = (k-1)n + l \), for some \( i, j, k, l = 1, \ldots, n \).

(iv) Modules and comodules. The structure of \( M_{n^2}(U) \)-modules and \( M_{n^2}(U) \)-comodules for the matrix module \( M_n(M) \simeq \tilde{M} \) is described by the following formulae:
\[
(m e_{i', j'}) \cdot (u e_{(i-1)j, (k-1)n+l}) = \begin{cases} 0 & \text{if } i' \neq j \text{ or } i \neq j', \\ (m u) e_{i,k} & \text{else}, \end{cases}
\]
\[
\tilde{\Delta}_{M_n(M)} (m e_{i', j'}) = \sum_{j} \left( m_{(0)} e_{i-1, j'} + e_{(i-1)n+j'} \right) \otimes_{M_n(R)} \left( m_{(0)} e_{1, j} \right),
\]
for every \( m \in M \) and \( u \in U \), where we have used a notation similar to that of item (ii).
4. Morita Base Change Invariance in Hopf-cyclic (Co)homology

This section contains our main results, Theorems 4.5 and 4.7. More precisely, we construct two morphisms between the cyclic modules $C_*(U,M)$ and $C_*(U,M)$, where $(S,U)$ is a Morita base change of $(R,U)$, and show that they form quasi-isomorphisms by giving an explicit homotopy. This establishes the Morita base change invariance for cyclic homology. For the Morita base change invariance of cyclic cohomology, we follow the same path although we shall not give the proofs since they are similar to the homology case.

Fix a Morita context $(R, S, P, Q, \phi, \psi)$ and assume we are given a left Hopf algebroid $(U, R)$, with Morita base change left Hopf algebroid $(U, S)$ as constructed in (3.2). Recall the notation of (3.2).1 and from now on, the symbol $\{i_0, \ldots, i_n\}$ stand for the set of indices $\{i_0, \ldots, i_n\}$.

4.1. The homology case. Consider the cyclic module $(C_*(U,M), d_*, s_*, t_*)$ as in (2.20).

Lemma 4.1. Let $M$ be a right $U$-module left $U$-comodule, subject to both (2.20) and (2.21). Then the cyclic operator $\tilde{i}: C_n(U,M) \to C_n(U,M)$ for the left Hopf algebroid $U$ with coefficients in $M$ is explicitly given by

$$\tilde{i}: \tilde{m} \otimes_{S^o} \tilde{x} \mapsto \sum_{i_1, \ldots, i_n} (p_{i_1} \otimes_R m_{(0)} u_{+}^{1} \otimes_R c_1) \otimes_{S^o} \left( (a_2 \otimes q_{i_0}^0) \otimes_R u_{+}^{2} \otimes_R (c_2 \otimes p_{i_1}^0) \right) \otimes_{S^o}$$

$$\cdots \otimes_{S^o} \left( (a_n \otimes q_{i_{n-1}}^0) \otimes_R u_{+}^{n} \otimes_R (c_n \otimes p_{i_n}^0) \right) \otimes_{S^o}$$

$$\left[ (d_n \otimes q_{i_n}^0) \otimes_R \left( (u_{-}^{n} \bullet (b_n d_{n-1})) (u_{-}^{n-1} \bullet (b_{n-1} d_{n-2})) \cdots (u_{-}^{1} \bullet (b_1 p)) \right) m_{(-1)} \right] \otimes_R (q \otimes a_n^0),$$

using the notation $\tilde{m} := p \otimes_R m \otimes_R q \in \tilde{M}$ as well as $\tilde{x} := \tilde{u}^{1} \otimes_{S^o} \cdots \otimes_{S^o} \tilde{u}^{n}$, where $\tilde{u}^{k} := (a_k \otimes b_k^0) \otimes_R u^k \otimes_R (c_k \otimes d_k^0)$ for $1 \leq k \leq n$.

Proof. Eq. (2.20) is not directly needed in the computation, but rather to make the operator $\tilde{i}$ well-defined. By definition we know that

$$\tilde{i}(\tilde{m} \otimes_{S^o} \tilde{u}^{1} \otimes_{S^o} \cdots \otimes_{S^o} \tilde{u}^{n}) := \tilde{m}_{(0)} \tilde{u}^{1} \otimes_{S^o} \tilde{u}^{2} \otimes_{S^o} \cdots \otimes_{S^o} \tilde{u}^{n} \otimes_{S^o} \left( \tilde{u}^{n} \tilde{u}^{n-1} \cdots \tilde{u}^{1} \tilde{m}_{(-1)} \right).$$

Using the formula for the translation map $\tilde{\beta}$ in (3.9), we have

$$\tilde{i}(\tilde{m} \otimes_{S^o} \tilde{u}^{1} \otimes_{S^o} \cdots \otimes_{S^o} \tilde{u}^{n})$$

$$= \sum_{j_0, \ldots, j_n} \left( p_{j_1} \otimes_R \left( (m_{(0)} \bullet (q_{j_0}^0 a_1)) (u_{+}^{1} \circ (q_{j_1}^1 p_{i_0})) \right) \otimes_R c_1 \right) \otimes_{S^o} \left( (a_2 \otimes q_{j_2}^0) \otimes_R u_{+}^{2} \otimes_R (c_2 \otimes p_{i_1}^0) \right)$$

$$\cdots \otimes_{S^o} \left( (a_n \otimes q_{j_{n-1}}^0) \otimes_R u_{+}^{n} \otimes_R (c_n \otimes p_{i_n}^0) \right) \otimes_{S^o} \left[ (d_n \otimes q_{i_n}^0) \otimes_R \left( (u_{-}^{n} \bullet (b_n d_{n-1})) (u_{-}^{n-1} \bullet (b_{n-1} d_{n-2})) \cdots (u_{-}^{1} \bullet (b_1 p)) \right) m_{(-1)} \right]$$

$$\otimes_R (q \otimes p_{j_0}^0).$$

By Eqs. (3.2.1) and (2.19), we can eliminate the sum with the index $j_0$. Thus we have

$$\tilde{i}(\tilde{m} \otimes_{S^o} \tilde{u}^{1} \otimes_{S^o} \cdots \otimes_{S^o} \tilde{u}^{n})$$
Lemma 4.2. The maps where \( \tilde{I} \) is explicitly given by and for complexes isomorphism between the term is explicitly given by (4.2)

\[
\tilde{I} = \sum_{j_0, \ldots, j_n} (p_{j_0} \otimes_R m \otimes_R q_{j_0}) \otimes_{s^0} ((q_{j_0}^a a^1) \otimes_R u^1 \otimes_R (q_{j_1}^b p_{j_1}^i))
\]

Repeating the same process, but now using repetitively (2.10), we can eliminate the sums indexed by \( i_0, j_1, \cdots, j_n \), and obtain the stated formula.

In order to show invariance of Hopf-cyclic homology, we will first of all construct a quasi-isomorphism between the \( b \)-columns, denoted again by \( C_\bullet(U, M) \) resp. \( C_\bullet(U, \tilde{M}) \), of the cyclic bi-complexes \( CC_\bullet(U, M) \) and \( CC_\bullet(U, \tilde{M}) \) associated to the respective cyclic modules (cf. (2.62)).

Define the map \( \theta_n : C_n(U, M) \rightarrow C_n(U, \tilde{M}) \) as follows: for \( n = 0 \), set
\[
\theta_0 : M \rightarrow \tilde{M}, \quad m \mapsto \sum_i p_i \otimes_R m \otimes_R q_i,
\]
and for \( n \geq 1 \), abbreviating \( x := u^1 \otimes_{R^0} \cdots \otimes_{R^0} u^n \), set
\[
\theta_n : m \otimes_{R^0} x \mapsto \sum_{j_0, \ldots, j_n} (p_{j_0} \otimes_R m \otimes_R q_{j_0}) \otimes_{s^0} ((p_{j_0} \otimes q_{j_0}^a) \otimes_{R^0} u^1 \otimes_{R^0} (q_{j_1}^b p_{j_1}^i))
\]

In the opposite direction, introduce the map \( \gamma_n : C_n(U, \tilde{M}) \rightarrow C_n(U, M) \), which is, for \( n = 0 \),
\[
\gamma_0 : \tilde{M} \rightarrow M, \quad (\tilde{m} := p \otimes_R m \otimes_R q) \mapsto \sum_j (q_{j_1}^b p) m(q_{j_1}^b),
\]
and for \( n \geq 1 \) it is given as
\[
\gamma_n : \tilde{m} \otimes_{s^0} \tilde{x} \mapsto \sum_{j_0, \ldots, j_n} \tilde{m}(q_{j_0}^b) \otimes_{s^0} ((q_{j_0}^a a^1) \otimes_R u^1 \otimes_R (c_{j_1}^b p_{j_1}^i))
\]

where \( \tilde{u}^i := (a_i \otimes b_i^0) \otimes_{R^0} u^i \otimes_{R^0} (c_i \otimes d_i^0) \in \tilde{U} \) for \( 1 \leq i \leq n \), and \( \tilde{x} := \tilde{u}^1 \otimes_{s^0} \cdots \otimes_{s^0} \tilde{u}^n \).

Lemma 4.2. The maps \( \theta_\bullet \) and \( \gamma_\bullet \) are morphisms of chain complexes.

Proof. We only check the compatibility of the differential with \( \gamma_n \) since the computation for \( \theta_n \) is similar but less complicated. Decompose
\[
b\gamma_n = \sum_{1 \leq k \leq n-1} (-1)^k d_k\gamma_n + (-1)^n d_n\gamma_n,
\]
where \( b \) is the differential of the underlying simplicial structure of \( C_\bullet(U, M) \) as in (2.26). When applying this map to an element of the form \( \tilde{m} \otimes_{s^0} \tilde{u}^1 \otimes_{s^0} \cdots \otimes_{s^0} \tilde{u}^n \) (using the notation above), each term is explicitly given by
\[
(i) = \sum_{j_0, \ldots, j_n} (m(q_{j_0}^b) \otimes_{s^0} ((q_{j_0}^a a^1) \otimes_R u^1 \otimes_R (c_{j_1}^b p_{j_1}^i))) \otimes_{R^0} \cdots \otimes_{R^0} u^n.
\]
\[
\begin{align*}
\left[\varepsilon\left((q'_{j_n}d_n\right)\right] & \cdot \left[\left((q'_{j_n-1}a_n\right) > u^n \prec (b_n d_{n-1}) \right] \cdot \left[\left((a'_{j_n-2}a_{n-1}) > u^{n-1} \prec (b_{n-1} d_{n-2}) \right] \cdot \left((c_{n-1} p'_{j_n-1})\right] \\
\sum_{j_0, \ldots, j_{n-1}} & \left( m(qp'_{j_0}) \otimes \nu^e \right) \left[ \left((q'_{j_0}a_1) > u^1 \prec (b_1 p) \right] \cdot \left((c_1 p'_{j_1})\right] \otimes \nu^e \cdots \otimes \nu^e \\
\left[\left(q'_{j_{n-1}}a_n\right)e(u^n \prec (c_n d_n))(b_n d_{n-1})\right] & \cdot \left[((q'_{j_{n-2}}a_{n-1}) > u^{n-1} \prec (b_{n-1} d_{n-2}) \right] \cdot \left((c_{n-1} p'_{j_{n-1}})\right] \\
\left[\left((b_{n-k-1} d_{n-k}) \cdot (q'_{j_{n-k-1}} a_{n-k}) \right] > u^{n-k} \prec (b_{n-k} d_{n-k-1}) \right] & \cdot \left((c_{n-k} a_{n-k+1})\right] \otimes \nu^e \cdots \otimes \nu^e \\
\left[\left((q'_{j_{n-1}} a_n) = u^n \prec (b_n d_{n-1}) \right] & \cdot \left((c_{n-1} p'_{j_{n-1}})\right] \\
\sum_{j_1, \ldots, n} & \left((b_{n-k} d_{n-k}) \cdot (q'_{j_{n-k-1}} a_{n-k}) \right] > u^{n-k} \prec (b_{n-k} d_{n-k-1}) \right] & \cdot \left((c_{n-k} a_{n-k+1})\right] \otimes \nu^e \cdots \otimes \nu^e \\
\left[\left((q'_{j_{n-1}} a_n) = u^n \prec (b_n d_{n-1}) \right] & \cdot \left((c_{n-1} p'_{j_{n-1}})\right] \\
\end{align*}
\]

On the other hand, we can also write

\[
\gamma_{n-1}b = \hat{\gamma}_{n-1}d_0 + \sum_{1 \leq k \leq n-1} (-1)^k \gamma_{n-1}d_k + \gamma_{n-1}d_n \\
\]

where \(\hat{b}\) is analogously the differential of the underlying simplicial structure of \(C_\ast(U, M)\). Applying \(\gamma_{n-1}b\) to the same element \(\hat{m} \otimes \nu^e \hat{u}^1 \otimes \nu^e \cdots \otimes \nu^e \hat{u}^n\), we find that the first term is

\[
\begin{align*}
\left(i\right) & = \sum_{j_0, \ldots, j_{n-1}} m(qp'_{j_0}) \otimes \nu^e \left[\left((q'_{j_0}a_1) > u^1 \prec (b_1 p) \right] \cdot \left((c_1 p'_{j_1})\right] \otimes \nu^e \\
& \cdots \otimes \nu^e \left[\left((q'_{j_{n-1}} a_n) = u^n \prec (b_n d_{n-1}) \right] & \cdot \left((c_{n-1} p'_{j_{n-1}})\right] \\
\end{align*}
\]

where we denoted the elements \(\varepsilon(\hat{u}^i) \cdot \hat{u}^{i-1} =: (\hat{a}_{n-i} \otimes \hat{b}_{n-i}^{-1}) \otimes \nu^e \hat{u}^{i-1} \otimes \nu^e (\hat{c}_{n-i} \otimes \hat{d}_{n-i}^{-1})\).

Computing explicitly this term, we obtain

\[
\begin{align*}
\varepsilon(\hat{u}^i) & \cdot \hat{u}^{i-1} = \hat{u}^{i-1}q(1 \otimes \varepsilon(\hat{u}^i)) \\
\sum_{j_0, j_1, j_n} & \left((a_{n-1} \otimes b_{n-1}^{-1}) \otimes \nu^e \left[\left(q'_{j_0} d_{n-1} \right] > u^{n-1} \prec (c_{n-1} p'_{j_1})\right] \otimes \nu^e \left(q'_{j_0} \otimes \varepsilon(\hat{u}^i) p'_{j_1}\right] \\
\sum_{j_0, \ldots, j_n} & \left((a_{n-1} \otimes b_{n-1}^{-1}) \otimes \nu^e \left[\left(q'_{j_0} d_{n-1} \right] > u^{n-1} \prec (c_{n-1} p'_{j_n})\right] \otimes \nu^e \left(q'_{j_0} \otimes \varepsilon(\hat{u}^n \prec (c_{n-1} d_{n-1}) b_{n-1}^{-1})\right] \right) \\
\end{align*}
\]

thence,

\[
\hat{a}_{n-1} = a_{n-1}, \quad \hat{b}_{n-1} = b_{n-1}, \quad \hat{u}^{n-1} = u^{n-1}, \quad \text{and} \quad \hat{d}_{n-1} = a_{n} \varepsilon(\hat{u}^n \prec (c_{n-1} d_{n-1})) b_{n-1}^{-1}.
\]

Inserting this into the expression of \(\hat{\gamma}_{n-1}b\) above, one obtains \(\hat{\gamma}_{n-1}b = \hat{\gamma}_{n-1}d_0\). The second term can be written as follows:

\[
\begin{align*}
\left(\hat{\gamma}_{n-1}d_0 + \sum_{1 \leq k \leq n-1} (-1)^k \hat{\gamma}_{n-1}d_k + \hat{\gamma}_{n-1}d_n \\
\end{align*}
\]
In case the first one, it is immediate that
\[ \otimes_{n^e} \left( ((q_{j_{m-k+1}} a_{n-k+2}) \triangleright u^{n-k+2} \triangleleft (b_{n-k+2} d_{n-k} \rangle c_{n-k+2} p'_{j_{n-k+1}}) \right) \]
\[ \otimes_{n^e} \cdots \otimes_{n^e} \left( (q_{j_n} d_n) \triangleright ((q'_{j_{n-1}} a_n) \triangleright u^n \triangleleft (b_n d_{n-1}) \triangleright (c_n p'_j) \right), \]

where we denoted the elements
\[
\tilde{u}_{n-k} := (\pi_{n-k} \otimes \pi_{n-k}) \otimes_{n^e} \pi_{n-k} \otimes_{n^e} (\pi_{n-k} \otimes \pi_{n-k})
\]
\[
\otimes_{n^e} (\pi_{n-k} \otimes \pi_{n-k}) \otimes_{n^e} (\pi_{n-k} \otimes \pi_{n-k}) \otimes_{n^e} (\pi_{n-k} \otimes \pi_{n-k}) \otimes_{n^e} (\pi_{n-k} \otimes \pi_{n-k})
\]
\[ (d_{n-k} \otimes b_{n-k}) \otimes_{n^e} (b_{n-k+1} d_{n-k}) \triangleright u^{n-k} \triangleright (c_{n-k} a_{n-k+1}) \otimes_{n^e} (c_{n-k+1} d_{n-k+1}). \tag{4.3} \]

Therefore, \((\widetilde{ii}) = (ii)\) after substituting \((4.3)\) in \((\widetilde{ii})\). As for the third term, we have
\[
(\widetilde{iii}) = \sum_{j_{n-1}, \ldots, n} (-1)^n \left( m((q a_1) \triangleright u^1 \triangleleft (b_1 p_1)) \otimes_{n^e} \right)
\]
\[ \cdots \otimes_{n^e} (q_{j_n} d_n) \triangleright ((q'_{j_{n-1}} a_n) \triangleright u^n \triangleleft (b_n d_{n-1}) \triangleright (c_n p'_j), \]

which is obviously \((iii)\). We conclude that \(\gamma_\ast\) is a morphism of chain complexes. \(\square\)

**Proposition 4.3.** The composite \(\gamma_n \theta_n\) is homotopic to the identity, the homotopy \(h_n : C_n(U, M) \to C_{n+1}(U, M)\) being explicitly given by the following map: for \(n = 0\), define
\[ h_0 : m \mapsto \sum_{i, j} m(q_i p'_j) \otimes_{n^e} ((q'_j p_i) \triangleright 1_U), \]

and for \(n \geq 1\), set
\[ h_n : m \otimes_{n^e} x \mapsto \sum_{k=0}^n \sum_{i_{n-k}, \ldots, i_k} (-1)^i \left( \sum_{j_{n-k}, \ldots, j_k} m(q_{i_0} p'_{j_0}) \otimes_{n^e} (q'_{j_0} p_{i_0}) \triangleright u^1 \triangleright (q_{i_1} p'_{j_1}) \otimes_{n^e} \right)
\]
\[ \cdots \otimes_{n^e} (q'_{j_k-k} p_{i_k-k}) \triangleright u^k \triangleright (q_{i_k} p'_{j_k}) \triangleright 1_U) \otimes_{n^e} u^{k+1} \otimes_{n^e} \cdots \otimes_{n^e} u^n \]

abbreviating \(x := u^1 \otimes_{n^e} \cdots \otimes_{n^e} u^n\) as before. Similarly, \(\theta_n \gamma_n\) is homotopic to the identity as well.

**Proof.** We need to check \(b h_0 = \gamma_0 \theta_0 - \text{id}\) for \(n = 0\) and \(b h_n + h_{n-1} b = \gamma_n \theta_n - \text{id}\) for \(n > 0\). As for the first one, it is immediate that
\[ bh_0(m) = \sum_{i, j} \varepsilon((q'_j p_i) \triangleright 1_U) \triangleright m(q_i p'_j) - m = \sum_{i, j} (q'_j p_i)(m)(q_i p'_j) - m = \gamma_0 \theta_0(m) - m. \]

In case \(n > 0\), since multiplying two consecutive tensor factors of \(h_n\) kills the respective \(q, p\) as well as the \(q', p'\) between them, it is straightforward to see that
\[ \sum_{k=1}^n (-1)^k d_k h_n (m \otimes_{n^e} x) + \sum_{k=1}^{n-1} (-1)^k h_{n-1} d_k (m \otimes_{n^e} x) = 0. \tag{4.5} \]
As for the remaining terms, we have

\[ (-1)^{n+1} d_{n+1} h_n (m \otimes \mathcal{R}^o x) \]

\[ = -m \otimes \mathcal{R}^o x + (-1)^{n+1} \sum_{k=1}^{n} \sum_{j_1, \ldots, j_k} (-1)^{k+n} m u^j (q_{i_1} p_{j_1}^*) \otimes \mathcal{R}^o \cdots \otimes \mathcal{R}^o (q_{i_k} p_{j_k}^*) \otimes \mathcal{R}^o (q_{i_k} p_{j_k}^*) \]

\[ = -m \otimes \mathcal{R}^o x + (-1)^{n+1} \sum_{k=0}^{n-1} \sum_{j_0, \ldots, j_k} (-1)^{k+(n-1)} (m u^j) (q_{i_0} p_{j_0}^*) \otimes \mathcal{R}^o (q_{i_0} p_{j_0}^*) \otimes \mathcal{R}^o \cdots \]

\[ = (-1)^n h_{n-1} d_n (m \otimes \mathcal{R}^o x). \]

Moreover,

\[ d_0 h_n (m \otimes \mathcal{R}^o x) \]

\[ = \sum_{k=0}^{n-1} \sum_{j_0, \ldots, j_k} (-1)^{k+n} (m \bullet (q_{i_0} p_{j_0}^*)) \otimes \mathcal{R}^o ((q_{i_0} p_{j_0}^*) \otimes \mathcal{R}^o \cdots) \]

\[ = (h_{n-1} d_0 + \gamma_n \theta_n) (m \otimes \mathcal{R}^o x), \]

where the last summand in the first step results from (2.2) and (2.5). Summing up, we obtain \( b h_n + h_{n-1} b = \gamma_n \theta_n - \text{id} \), and this finishes the proof.

To pass to the cyclic case, we prove first:

**Lemma 4.4.** The morphisms of chain complexes \( \theta_* \) and \( \gamma_* \) are morphisms of cyclic objects. That is, they satisfy:

\[ \gamma_* \tilde{t}_* = t_* \gamma_* , \quad \theta_* t_* = \tilde{\theta}_* \theta_* . \]

**Proof.** We only check the first equation. Take an element \( \tilde{m} \otimes_{\mathcal{R}^o} \tilde{u}^1 \otimes_{\mathcal{R}^o} \cdots \otimes_{\mathcal{R}^o} \tilde{u}^n \in C_n(\tilde{U}, \tilde{M}) \), for \( n \geq 0 \). Then, applying equations (2.13), (2.15), and (2.18), we can write

\[ t_n \gamma_n (\tilde{m} \otimes_{\mathcal{R}^o} \tilde{u}^1 \otimes_{\mathcal{R}^o} \cdots \otimes_{\mathcal{R}^o} \tilde{u}^n) = \sum_{j_0, \ldots, j_n} \left( (m_{00} \bullet (q_{j_0} a_0) \otimes (c_1 p_{j_1}^*) \otimes \mathcal{R}^o ((q_{j_1} a_2) \otimes \mathcal{R}^o (c_2 p_{j_2}^*) \otimes \mathcal{R}^o \cdots) \right) \]

\[ \otimes \mathcal{R}^o \cdots \otimes \mathcal{R}^o (q_{j_{n-1} a_{n-1}} \otimes \mathcal{R}^o (c_{n} p_{j_{n}}^*) \otimes \mathcal{R}^o (b_{n d_{n-1}} \otimes \mathcal{R}^o (b_{n-2} d_{n-2}) \otimes \mathcal{R}^o \cdots) \]

\[ \cdots \otimes (b_2 d_1) \otimes \mathcal{R}^o (b_1 p) \otimes m_{(-1)} \otimes \mathcal{R}^o (q_{j_{n} p_{j_{n}}^*}) \right] . \]
On the other hand, we have
\[
\gamma_n \tilde{t}_n(m \otimes_{\mathcal{R}} \tilde{u}^1 \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \tilde{u}^n) = \sum_{j_0, \ldots, j_n} \left( (m(0) u_{j_0}^1) \otimes (c_1 p_{j_0}^{i_1}) \otimes_{\mathcal{R}} \left[ (q_{j_0}^{i_2} a_2) \otimes (c_2 p_{j_1}^{i_3}) \right] \right)
\]
\[
\otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \left( (q_{j_{n-2}}^{i_{n-1}} a_n) \otimes (c_n p_{j_{n-1}}^{i_n}) \right)
\]
\[
\otimes_{\mathcal{R}} \left[ ((q_{j_{n-1}}^{i_{n-1}} a_n) \otimes (b_n a_{n-1}) \otimes (u_{n-1}^{i_{n-1}}) ((b_{n-1} a_{n-2}) \otimes (u_{n-2}^{i_{n-2}}) \cdots \right.
\]
\[
\cdots ((b_2 d_1) \otimes (b_1 p) \otimes ((q_{j_1}^{i_0} a_1) \otimes (m_{(-1)} \otimes (q_{j_0}^{i_0})))) \right] \sum_{j_{0, \ldots, n}} \left( (m_{(0)} \otimes (q_{j_0}^{i_0} a_1)) \otimes (c_1 p_{j_0}^{i_1}) \right)
\]
\[
\otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \left( (q_{j_{n-2}}^{i_{n-1}} a_n) \otimes (c_n p_{j_{n-1}}^{i_n}) \right)
\]
\[
\otimes_{\mathcal{R}} \left[ ((q_{j_{n-1}}^{i_{n-1}} a_n) \otimes (b_n a_{n-1}) \otimes (u_{n-1}^{i_{n-1}}) ((b_{n-1} a_{n-2}) \otimes (u_{n-2}^{i_{n-2}}) \cdots \right.
\]
\[
\cdots ((b_2 d_1) \otimes (b_1 p) \otimes (m_{(-1)} \otimes (q_{j_0}^{i_0})))) \right] ,
\]
Now, renumbering the indices we find the equality, and this finishes the proof. \( \square \)

Combining Lemma 4.2, Proposition 4.3, and Lemma 4.4, we conclude that \( \theta_* \) and \( \gamma_* \) are in particular equivalences of cyclic modules. Consequently, we can now formulate the main theorem of this paper:

**Theorem 4.5.** (Morita base change invariance of (Hopf-)cyclic homology). Let \( (U, R) \) be a left Hopf algebroid, \( M \) a left \( U \)-comodule right \( U \)-module which is SaYD (i.e., satisfies (2.20)–(2.23)), and \( (R, S, P, Q, \phi, \psi) \) a Morita context. We then have the following natural \( \mathbb{k} \)-module isomorphisms:

\[
H_\bullet(U, M) \cong H_\bullet(\tilde{U}, P \otimes_R M \otimes_R Q),
\]

\[
HC_\bullet(U, M) \cong HC_\bullet(\tilde{U}, P \otimes_R M \otimes_R Q).
\]

**Proof.** This follows at once by using the SBI sequence for cyclic modules, cf. [L] §2.5.12 for details. \( \square \)

### 4.2. The cohomology case.
In this section, we will consider the case of Hopf-cyclic cohomology under Morita base change. Since all steps are basically analogous to the preceding section, we refrain from spelling out the details and just indicate the main ingredients.

Consider the cocyclic module \( (C^\bullet(U, M), \delta_\bullet, \sigma_\bullet, \tau_\bullet) \) as in (2.27). In the spirit of (4.1) and (4.2), define first the map \( \zeta_n : C^n(U, M) \to C^n(\tilde{U}, \tilde{M}) \) as follows: for \( n = 0 \), define

\[
\zeta_0 : M \to \tilde{M}, \quad m \mapsto \sum_j p_j^j \otimes_R m \otimes_R q_j^j,
\]

and for \( n \geq 1 \), abbreviating \( y := u^1 \otimes_R \cdots \otimes_R u^n \), define

\[
\zeta_n : y \otimes_R m \mapsto \sum_{i_0, \ldots, i_{n-1}} \left( (p_{i_0} \otimes q_{i_0}^{i_1}) \otimes_{\mathcal{R}} u^1 \otimes_{\mathcal{R}} (c_1 p_{j_0}^{i_1}) \otimes_{\mathcal{R}} \cdots \right.
\]
\[
\cdots \otimes_{\mathcal{R}} \left( (p_{i_{n-1}} \otimes q_{i_{n-1}}^{i_n}) \otimes_{\mathcal{R}} u^n \otimes_{\mathcal{R}} (c_n p_{j_{n-1}}^{i_n}) \right) \otimes_{\mathcal{R}} \left( p_{i_{n-1}} \otimes_R m \otimes_R q_{i_n}^j \right).
\]

Second, define the map \( \xi_n : C^n(\tilde{U}, \tilde{M}) \to C^n(U, M) \), which is

\[
\xi_0 : \tilde{M} \to M, \quad \tilde{m} := p \otimes_R m \otimes_R q \mapsto \sum_i (q_i p)m(q)p_i,
\]
in degree \( n = 0 \), and for \( n \geq 1 \) is given by

\[
\xi_n : \tilde{y} \otimes_R \tilde{m} \mapsto \sum_{i=0}^{n-1} ((q_{i+1}a_1) \cdot (q_i d_1) \cdot (c_i p_{i+1} \otimes_R (q_{i+2}d_2) \cdot u^1 \cdot (b_1 a_2) \cdot (c_1 p_2) \otimes_R (q_{i+2}d_2) \cdot u^2 \cdot (b_2 a_3) \cdot (c_2 p_3) \otimes_R \cdots \otimes_R ((q_{n-1} d_{n-1}) \cdot u^{n-1} \cdot (b_{n-1} a_n) \cdot (c_{n-1} p_{n-2}) \otimes_R ((q_n d_n) \cdot u^n \cdot (b_n p) \cdot (c_n p_{n-1}) \otimes_R m(q_0 p_0),
\]

where \( \tilde{u}^i := (a_i \otimes b_i^i) \otimes_R \tilde{u}^i \otimes_R \tilde{c}_i \otimes \tilde{d}_i^i \) \( \in \tilde{U} \) for \( 1 \leq i \leq n \), and \( \tilde{y} := \tilde{u}^1 \otimes_S \cdots \otimes_S \tilde{u}^n \).

Third, introduce the homotopy \( h_n : C^{n+1}(U, M) \to C^n(U, M) \) as follows: in degree \( n = 0 \), set

\[
h_0 : u \otimes_R m \mapsto \sum_{i,j} \varepsilon((q_i p_j) \cdot u)m(q_i p_j),
\]

and for \( n \geq 1 \) define

\[
h_n : y' \otimes_R m \mapsto \sum_{i=0}^{n} \sum_{j=0}^{n-k-1} (-1)^{k+1} n \cdot u \otimes_R \cdots \otimes_R u^{n-k-1} \otimes_R \varepsilon((q_{i+1}p_{j+1}) \cdot u^{n-k}) \otimes_R ((q_{i+1}p_{j+1}) \cdot u^{n-k-1} \cdot \cdots \otimes_R ((q_{i+1}p_{j+1}) \cdot u^{n-1} \cdot (q_{i+1}p_{j+1}) \cdot u^n \cdot (q_{i+1}p_{j+1}) \cdot u^{n-k-1}) \otimes_R m(q_{i+1}p_{j+1}),
\]

abbreviating here \( y' := u^0 \otimes_R \cdots \otimes_R u^n \).

Now, with the construction of \( \tilde{U} \) and \( \tilde{M} \) as in \( [3, 2] \) and analogously to Lemma \( 4.1 \) one can construct a cocyclic module \( C^*(\tilde{U}, \tilde{M}) \); we leave the tedious details to the reader. Similarly as in Lemma \( 4.2 \), Proposition \( 4.3 \) and Lemma \( 4.4 \) one then proves:

**Lemma 4.6.** The maps \( \zeta_* \) and \( \xi_* \) are morphisms of cochain complexes, and \( \xi_* \zeta_* \) is homotopic to the identity by means of the homotopy \( (4.6) \); likewise, \( \xi_* \zeta_* \) is homotopic to the identity as well. In particular, \( \zeta_* \) and \( \xi_* \) are equivalences of cocyclic modules.

This enables us to conclude:

**Theorem 4.7.** (Morita base change invariance of (Hopf-)cyclic cohomology). Let \( (U, R) \) be a left Hopf algebroid, \( M \) a left \( U \)-comodule right \( U \)-module which is SaYD (i.e., satisfies \( (2.20)-(2.23) \)), and \( (R, S, P, Q, \phi, \psi) \) a Morita context. Then

\[
H^*(U, M) \simeq H^*(\tilde{U}, P \otimes_R M \otimes_R Q),
\]

\[
H^{\text{HC}}(U, M) \simeq H^{\text{HC}}(\tilde{U}, P \otimes_R M \otimes_R Q)
\]

are isomorphisms of \( \mathbb{k} \)-modules.

5. **APPLICATIONS AND EXAMPLES**

We give two applications. The first one deals with the well-known Morita invariance of the usual Hochschild and cyclic homology for associative algebras. We show that this invariance theory is a consequence of our main Theorem \( 4.5 \) by applying it to the left Hopf algebroids \( R^e \) and \( S^e \). In the second application we specialise our general results to the Morita context between the complex-valued smooth functions on the commutative real 2-torus \( \mathbb{T}^2 := S^1 \times S^1 \) and the coordinate ring of the noncommutative 2-torus. We will first review the construction of this context, and next apply the Morita invariance of the cyclic homology between the left Hopf algebroid attached to the Lie algebroid of vector fields over \( \mathbb{T}^2 \), and the associated matrix left Hopf algebroid over the noncommutative 2-torus.
5.1. Morita invariance of cyclic homology for associative algebras. Recall from \([\text{Sch}2]\) the left Hopf algebroid structure of the enveloping algebra \(R^e\). Its structure maps are given as follows: \(s(r) := r \otimes 1, t(r^o) := 1 \otimes r^o, \Delta r := (r \otimes 1) \otimes_R (1 \otimes r^o), \varepsilon(r \otimes r^o) := r \tilde{r}\), and the inverse of the Hopf-Galois map is given as \((r \otimes r^o)_+ \otimes_{R^e} (r \otimes r^o)_- := (r \otimes 1) \otimes_{R^e} (\tilde{r} \otimes 1)\).

Let now \(M\) be a right \(R^e\)-module which is also an \(R^e\)-comodule with compatible left \(R\)-actions as in (2.20), and denote the coaction by \(m \mapsto (m^{(-1)} \otimes m^{(-1)}_{\otimes}) \otimes_{R^e} m^{(0)}\), omitting the summation symbol in all what follows. Under the isomorphism \(C_\ast(R^e, M) = M \otimes_{R^e} R^e \otimes_{R^e} \ast \simeq M \otimes R^e \ast\) given by
\[
m \otimes_{R^e} (r_1 \otimes r_2^o) \otimes_{R^e} \cdots \otimes_{R^e} (r_n \otimes r_n^o) \mapsto \tilde{r}_n \cdots \tilde{r}_1 m \otimes r_1 \cdots \otimes r_n, \tag{5.1}
\]
the para-cyclic operators (2.26) assume the form
\[
d_i (m \otimes y) = \begin{cases} r_i m \otimes r_1 \otimes \cdots \otimes r_{i-1} & \text{if } i = 0, \\ m r_1 \otimes r_2 \otimes \cdots \otimes r_n & \text{if } i = n, \end{cases}
\]
\[
s_i (m \otimes y) = \begin{cases} m \otimes r_1 \otimes \cdots \otimes r_{n-i} \otimes 1 & \text{if } i = 0, \\ m \otimes 1 \otimes r_1 \otimes \cdots \otimes r_n & \text{if } i = n, \end{cases}
\]
\[
t_n (m \otimes y) = m^{n}_{\otimes} (m^{(-1)}_{\otimes}) r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes m^{(-1)}_{\otimes},
\]
where we abbreviate \(y := r_1 \otimes \cdots \otimes r_n\), and as before \(C_\ast(R^e, M)\) is cyclic if \(M\) is SaYD.

Using the isomorphism
\[
P^e \otimes_{R^e} R^e \otimes_{R^e} Q^e \xrightarrow{\simeq} S^e, \quad (a \otimes b^o) \otimes_{R^e} (r \otimes r^o) \otimes_{R^e} (c \otimes d^o) \mapsto \phi(a \otimes r c) \otimes \phi(r \tilde{r} \otimes b^o), \tag{5.3}
\]
where \(\phi\) is as in (3.1), together with (3.2) and the isomorphism \(C_\ast(S^e, \tilde{M}) \simeq \tilde{M} \otimes S^e \otimes_{S^e} S^e\), analogously to (5.1), a straightforward computation reveals that the morphism of chain complexes (4.1) reads
\[
\theta_n : m \otimes y \mapsto \sum_{i_0, \ldots, i_n} (p_{i_0} \otimes_R m \otimes_R q_{i_1}) \otimes \phi(p_{i_1} \otimes_R r_1 q_{i_2}) \cdots \otimes \phi(p_{i_n} \otimes_R r_n q_{i_0}).
\]

In the other direction, we make use of the isomorphism
\[
Q^e \otimes_{S^e} S^e \otimes_{S^e} P^e \xrightarrow{\simeq} R^e, \quad (c \otimes d^o) \otimes_{S^e} (s \otimes s^o) \otimes_{S^e} (a \otimes b^o) \mapsto \psi(c \tilde{s} \otimes_S a) \otimes \psi(b \otimes_S s^o),
\]
where \(\psi\) is given by (3.3), together with the inverse of (5.3) given by, cf. (3.3),
\[
S^e \xrightarrow{\psi} P^e \otimes_{R^e} Q^e, \quad s \otimes s^o \mapsto \sum_{i,j} (s p_{i,j} \otimes q_{i,j}^o) \otimes_{R^e} (q_{i,j}^o \otimes \tilde{s} p_{i,j}^o),
\]
to conclude that the morphism of chain complexes (4.2) becomes here
\[
\gamma_n : (p \otimes_R m \otimes_R q) \otimes z \mapsto \sum_{j_0, \ldots, j_n} (\psi(q_{j_0} \otimes_S p) m \psi(q \otimes_S p_{j_1}) \otimes \psi(q_{j_1}^o \otimes_S s_1 p_{j_2}) \otimes \cdots \otimes \psi(q_{j_n} \otimes_S s_n p_{j_0})),
\]
abbreviating \(z := s_1 \otimes \cdots \otimes s_n\).

In a similar manner, one derives the homotopy (4.4) in this case: for \(n = 0\), we obtain
\[
h_0 : m \mapsto \sum_{i,j} \psi(q_i \otimes q_{j}^o) \otimes \psi(q_{j}^o \otimes p_i),
\]
Corollary 5.1. Let $R$ be an associative $\mathbb{k}$-algebra, $M$ an $(R, R)$-bimodule, and $(R, S, P, Q, \phi, \psi)$ a Morita context. We then have the following natural $\mathbb{k}$-module isomorphism

$$H^\text{alg}_\bullet(R, M) \simeq H^\text{alg}_\bullet(S, P \otimes_R M \otimes_R Q),$$

and in case $M := R$, we obtain

$$HC^\text{alg}_\bullet(R) \simeq HC^\text{alg}_\bullet(S). \quad (5.4)$$

Observe that for this corollary no SaYD condition is needed: there is no coaction required to compute the homology of the underlying simplicial object in $\mathcal{M}$ (resp. $\mathcal{M}_\text{SaYD}$), and for the cyclic homology in (5.4) we only considered the case $M := R$, with action given by multiplication and coaction $R \rightarrow R^e \otimes_R R \simeq R^e$, $r \mapsto r \otimes 1$, which is easily seen to define an SaYD module.

5.2. Morita base change invariance in Lie algebroid theory and the noncommutative torus.

5.2.1. Lie algebroids and associated left Hopf algebroids. Assume that $R$ is a commutative $\mathbb{k}$-algebra (here $\mathbb{k}$ is a ground field containing $\mathbb{Q}$) and denote by $\text{Der}_\mathbb{k}(R)$ the Lie algebra of all $\mathbb{k}$-linear derivation of $R$. Consider a $\mathbb{k}$-Lie algebra $L$ which is also an $R$-module, and let $\omega : L \rightarrow \text{Der}_\mathbb{k}(R)$ be a morphism of $\mathbb{k}$-Lie algebras. Following [Ri], the pair $(R, L)$ is called Lie-Rinehart algebra with anchor map $\omega$, provided

$$(aX)(b) = a(X(b)), \quad [X, aY] = a[X, Y] + X(a)Y,$$

for all $X, Y \in L$ and $a, b \in A$, where $X(a)$ stands for $\omega(X)(a)$. A morphism $(R, L) \rightarrow (R, L')$ of Lie-Rinehart algebras over $R$ is a map $\varphi : L \rightarrow L$ of $\mathbb{k}$-Lie algebras such that

$$\xymatrix{ L \ar[rr]^-{\varphi} \ar[rd]_-{\omega} & & L' \ar[ld]^-{\omega'} \ar[dr]^-{\omega''} \\ & \text{Der}_\mathbb{k}(R) & }$$

is a commutative diagram. These objects form a category which we denote by $\text{LieRine}(\mathbb{k}, R)$.

Example 5.2. Here are the basic examples which we will be dealing with, and which motivate the above general definition:

(i) The pair $(R, \text{Der}_\mathbb{k}(R))$ trivially admits the structure of a Lie-Rinehart algebra.
(ii) A Lie algebroid is a vector bundle $\mathcal{E} \to \mathcal{M}$ over a smooth manifold, together with a map $\omega : \mathcal{E} \to T\mathcal{M}$ of vector bundles and a Lie structure $[-,-]$ on the vector space $\Gamma(\mathcal{E})$ of global smooth sections of $\mathcal{E}$, such that the induced map $\Gamma(\omega) : \Gamma(\mathcal{E}) \to \Gamma(T\mathcal{M})$ is a Lie algebra homomorphism, and for all $X, Y \in \Gamma(\mathcal{E})$ and any $f \in C^\infty(\mathcal{M})$ one has $[X,fY] = f[X,Y] + \Gamma(\omega)(X)(f)Y$. Then the pair $(C^\infty(\mathcal{M}), \Gamma(\mathcal{E}))$ is obviously a Lie-Rinehart algebra.

Associated to any Lie-Rinehart algebra $(R, L)$ there is a universal object denoted by $(R, VL)$, see [Rij, Hue]. Using the notion of twisted algebras (or distributive law between two algebras), we give here an alternative construction of this object (of which the construction of the Massey-Peterson algebra in [Mas, Hue] is a special case). Let $U(L)$ be the universal enveloping algebra of $L$, and consider the $k$-linear map $\iota : U(L) \otimes_k R \to R \otimes_k U(L)$ which acts on generators as

$$X \otimes_k a \mapsto a \otimes_k X + (a) \otimes_k 1,$$

and $1 \otimes_k a \mapsto a \otimes_k 1$, $X \in L$, $a \in R$.

This map is compatible with both multiplications and units of $R$ and $U(L)$, and means that $\iota$ is a twisting map between $R$ and $U(L)$, in the sense that it satisfies the equations stated in [ElK, Subsection 6.1] (see also references therein). In this way, the tensor product $R \otimes_k U(L)$ admits a new structure of an $R$-algebra whose identity is still $1 \otimes_k 1$, although its multiplication is twisted by $\iota$. Thus

$$(a \otimes_k u)(b \otimes_k v) = \sum_k ab^k \otimes_k u^k v, \text{ where } \iota(u \otimes_k b) = \sum_k b^k \otimes_k u^k,$$

for every $u, v \in U(L)$ and $a, b \in R$. We denote this algebra by $R \otimes_k U(L)$ and their generic elements by $a \otimes_k X$. Now take the factor $R$-algebra of $R \otimes_k U(L)$:

$$\pi : R \otimes_k U(L) \to \mathcal{L} := \frac{R \otimes_k U(L)}{\mathcal{J}_L},$$

where

$$\mathcal{J}_L := \langle a \otimes_k, X - 1 \otimes_k, aX \rangle_{a \in R, X \in L}$$

is the two sided ideal generated by the set $\langle a \otimes_k, X - 1 \otimes_k, aX \rangle_{a \in R, X \in L}$. The usual $R$-coalgebra structure of $U(L)$ can be lifted to the structure of a left $R$-bialgebroid on $\mathcal{L}$, which carries, in fact, the structure of a left Hopf algebroid over $R$, see [Kow] [4.2.1]: the comultiplication and the counit are obvious, the translation map is given on generators by

$$a_+ \otimes_{R^0} a_- := a \otimes_{R^0} 1, \quad X_+ \otimes_{R^0} X_- := X \otimes_{R^0} 1 - 1 \otimes_{R^0} X,$$

where $\mathcal{L} \otimes_{R^0} \mathcal{L} := \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}$ is as in [26], and where we identify the elements of $R$ and $L$ with their respective images by the universal maps $\iota_R : R \to \mathcal{L}$, $a \mapsto a \otimes_{R} 1 + \mathcal{J}_L$ and $\iota_L : L \to \mathcal{L}$, $X \mapsto 1 \otimes_{L} X + \mathcal{J}_L$.

5.2.2. Vector bundles versus $\sqrt{\text{Morita theories}}$. Let $R$ be a commutative $k$-algebra as in [5.2.1]. Assume we are given a finitely generated and projective module $P_R$ which is faithful: any equation of the form $Pa = 0$, for some $a \in R$, implies $a = 0$. Then it is well known that $R$ is Morita equivalent to the endomorphism ring $\text{End}(P_R)$ since $R$ is commutative. The context maps are given by

$$\phi : P \otimes_R P^* \xrightarrow{\sim} \text{End}(P_R), \quad (p \otimes \sigma \mapsto [p' \mapsto p\sigma(p')]),$$

$$\psi : P^* \otimes_{\text{End}(P_R)} P \xrightarrow{\sim} R, \quad (\sigma \otimes p \mapsto \sigma(p)).$$

Following [Kh, Example 2.3.3], we apply this Morita context to the situation where $R$ is the algebra of smooth functions over a manifold $\mathcal{M}$. By the Serre-Swan theorem, it is well known that for a (complex) smooth vector bundle $\pi : \mathcal{P} \to \mathcal{M}$ of constant rank $\geq 1$ the global smooth sections $\Gamma(\mathcal{P})$...
form a finitely generated projective module over $C^\infty(M)$, the complex-valued smooth functions on $M$. Let us check that $P := \Gamma(P)$ is always a faithful module. For $f \in C^\infty(M)$ assume that $Pf = 0$; we need to check that $f(x) = 0$ for every $x \in M$. Let $\mu_x$ be the maximal ideal of all vanishing functions on a point $z \in M$. Following [Ne, Corollary 11.9], for every such $z$ there is an exact sequence of complex vector spaces

$$0 \rightarrow \mu_z P \rightarrow P \rightarrow P_z \rightarrow 0,$$

(5.5)

where the first nonzero map is the inclusion and the second one is the map which assigns to each section $s \in \Gamma(P) = P$ the image $s(z) \in P_z$ in the fiber of $z$. Thus, for every $z \in M$, there is an isomorphism of complex vector spaces $P_z \cong P/\mu_z P$. Now suppose that $f \neq 0$, that is, there is some $x_0 \in M$ such that $f \notin \mu_{x_0}$. Since $\mu_{x_0}$ is a maximal ideal, we have $\mu_{x_0} + fC^\infty(M) = C^\infty(M)$. This implies that

$$P\mu_{x_0} + PfC^\infty(M) = P\mu_{x_0} = P,$$

which by Eq. (5.5) is in contradiction with the fact that $P_{x_0}$ is a complex vector space of dimension at least 1. Therefore $f = 0$, and the global smooth sections $P$ form a faithful finitely generated projective $C^\infty(M)$-module. Furthermore, $C^\infty(M)$ is Morita equivalent to the endomorphism algebra $\text{End}(P_{C^\infty(M)}) \cong \Gamma(\text{End}(P))$. In this way, there is a functor from the category $\text{LieRine}_{C, C^\infty(M)}$ to the category of left Hopf algebroids over $\text{End}(P_{C^\infty(M)})$. This functor is defined on objects by sending any complex Lie-Rinehart algebra $(L, R)$ to the left $\text{End}(P_R)$-Hopf algebroid $P^e \otimes_{P^e} VL \otimes_{P^e} Q^e$, where $R := C^\infty(M)$, and $P^e, Q^e$ are defined as in Subsection 3.1 and correspond to the Morita context $(R, \text{End}(P_R), P, P^e)$, i.e., with $Q = P^e$.

**Remark 1.** An analogue to the previous functor can, in fact, descend to the category of Lie algebroids over a smooth manifold $M$ if we take a real vector bundle and the algebra $C^\infty(M, \mathbb{R})$ of real-valued smooth functions instead of the algebra $C^\infty(M) = C^\infty(M, \mathbb{C})$. We know from Example 5.2(ii), that there is a canonical faithful functor from the category of Lie algebroids over $M$ to the category of real Lie-Rinehart algebras over $C^\infty(M, \mathbb{R})$. Now we can compose this functor with the one constructed by the same process as in 5.2.2. In general, there is no obvious functor connecting the categories $\text{LieRine}_{C, C^\infty(M, \mathbb{R})}$ and $\text{LieRine}_{C, C^\infty(M, \mathbb{C})}$, except perhaps when $M$ is an almost complex manifold (i.e., a smooth manifold with a smooth endomorphism field $J : TM \rightarrow TM$ satisfying $J^2_x = -\text{id}_{TM}$, for all $x \in M$).

Let us mention that due to our interest in the noncommutative torus, we have been forced to extend the base field by using the complex-valued functions instead of real-valued ones.

The material of the following subsection will appear well known to the reader who is familiar with noncommutative differential geometry techniques. For the convenience of the rest of the audience, we include a detailed exposition following ideas from [DuKrMaMi, §3.1], [Kh, §1.1].

5.2.3. Noncommutative torus revisited. Consider the Lie group $S^1 = \{z \in \mathbb{C} \setminus \{0\} \mid |z| = 1\}$ as a real 1-dimensional torus by identifying it with the additive quotient $\mathbb{R}/2\pi\mathbb{Z}$. Likewise, the real $d$-dimensional torus $T^d := S^1 \times \cdots \times S^1$ is identified with $\mathbb{R}^d/2\pi\mathbb{Z}^d$. The complex algebra of all smooth complex-valued functions on $T^2$ will be denoted by $C^\infty(T^2)$.

Fix an element $q \in S^1$ whose argument is rational modulo $2\pi$, and take $N \in \mathbb{N}$ to be the smallest natural number such that $q^N = 1$. Let us consider the semidirect product group $G := \mathbb{Z}_N^2 \rtimes S^1$ where $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, and operation

$$(m, n, \theta)(m', n', \theta') := (m + m', n + n', \theta\theta' a^{mn'}),$$
for every pair of elements \((m, n, \theta), (m', n', \theta') \in \mathcal{G}\). There is a right action of the group \(\mathcal{G}\) on the torus \(T^3\) given as follows:

\[(x, y, z)(m, n, \theta) := (q^m x, q^n y, \theta z y^m), \ (x, y, z) \in T^2, \ (m, n, \theta) \in \mathcal{G}.\]

Now, we can show that the map

\[\mathbf{p} : T^3 \rightarrow T^2, \ (x, y, z) \mapsto (x^N, y^N)\]

satisfies:

(i) \(\mathbf{p}\) is a surjective submersion.

(ii) \(\mathcal{G}\) acts freely on \(T^3\) and the orbits of this action coincide with the fibers of \(\mathbf{p}\).

As a consequence and applying \[\text{KoMiSl}, \text{Lemma 10.3}\], we see that \((T^3, \mathbf{p}, T^2, \mathcal{G})\) is a principal fiber bundle. We then want to associate a non-trivial vector bundle to the bundle \(T^3 \times \mathbb{C}^N \rightarrow T^3\). So, we need to extend the \(\mathcal{G}\)-action on \(T^3\) to the trivial bundle \(T^3 \times \mathbb{C}^N\). This is possible by considering the following left \(\mathcal{G}\)-action on \(\mathbb{C}^N\)

\[\mathcal{G} \rightarrow \text{End}_\mathbb{C}(\mathbb{C}^N), \ (m, n, \theta) \mapsto \{\omega \mapsto \theta U_0^m V_0^{-\omega}\},\]

where \(U_0, V_0\) are the following \((N \times N)\)-matrices

\[U_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots \\ 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad V_0 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & q & 0 & \cdots & 0 \\ 0 & 0 & q^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots \\ 0 & 0 & \cdots & 0 & q^{N-1} \end{pmatrix},\]

which satisfy the relations

\[U_0 V_0 = q V_0 U_0, \quad U_0^N = V_0^N = I_N. \quad (5.6)\]

Therefore, we have a left \(\mathcal{G}\)-action on \(T^3 \times \mathbb{C}^N\) defined by

\[((x, y, z): \omega) (m, n, \theta) := ((x, y, z)(m, n, \theta); \ (m, n, \theta)^{-\omega}) = \left((q^m x, q^n y, \theta z y^m); \ \theta^{-\omega} U_0^{-m} V_0^{-\omega}\right).\]

The orbit space \((T^3 \times \mathbb{C}^N)/\mathcal{G} = T^3 \times \mathbb{C}^N\) with elements \(u \times \omega\) will be denoted by \(\mathcal{E}_q\). Notice that by definition one has the following formula:

\[(ug) \times \omega \equiv u \times (g \omega), \text{ for every } u \in T^3, \omega \in \mathbb{C}^N, \text{ and } g \in \mathcal{G}.\]

By applying \[\text{KoMiSl}, \text{Theorem 10.7, \S 10.11}\] we can associate a non-trivial vector bundle to the trivial bundle \(T^3 \times \mathbb{C}^N \rightarrow T^3\), that is, there is a morphism of vector bundles

\[\begin{array}{ccc}
T^3 \times \mathbb{C}^N & \longrightarrow & \mathcal{E}_q \\
pr_1 & & | \\
T^3 & \longrightarrow & T^2.
\end{array}\]

By the results of \[\S 3.2.2\] we have that \(\mathcal{C}^\infty(T^2)\) is Morita equivalent to \(\text{End}(\Gamma(\mathcal{E}_q)) \simeq \Gamma(\text{End}(\mathcal{E}_q))\). Now, using \[\text{KoMiSl}, \text{Theorem 10.12}\], \(\Gamma(\mathcal{E}_q)\) is identified with the \(G\)-equivariant subspace \(\mathcal{C}^\infty(T^3, \mathbb{C}^N)^G\) of \(\mathcal{C}^\infty(T^3, \mathbb{C}^N)\), that is, those \(f \in \mathcal{C}^\infty(T^3, \mathbb{C}^N)\) for which \(f(ug) = g^{-1} f(u)\), for every \(u \in T^3, g \in \mathcal{G}\). Hence, we have an isomorphism \(\Gamma(\mathcal{E}_q) \simeq \mathcal{C}^\infty(T^3, \mathbb{C}^N)^G\) of \(\mathcal{C}^\infty(T^2)\)-modules.
Next, we want to describe the noncommutative complex algebra \( \text{End}(\Gamma(E_q)) \simeq \Gamma(\text{End}(E_q)) \). Observe that there is a left \( \mathbb{Z}_2^2 \)-action on the \((N \times N)\)-matrix algebra \( M_N(\mathbb{C}) \) with complex entries, defined by
\[
(m, n)A := U_0^m V_0^{-m} A V_0^n U_0^{-n}, \quad \text{for every } A \in M_N(\mathbb{C}), \ (m, n) \in \mathbb{Z}_2^2.
\]
There is also a free right \( \mathbb{Z}_2^2 \)-action on \( T^2 \) given by
\[
(x, y)(m, n) := (q^mx, q^ny), \quad \text{for every } (x, y) \in T^2, \ (m, n) \in \mathbb{Z}_2^2.
\]
As before, one can construct the orbit space \( T^2 \times_{\mathbb{Z}_2^2} M_N(\mathbb{C}) \) after extending these actions to the trivial algebra bundle \( T^2 \times M_N(\mathbb{C}) \). It turns out that the endomorphism algebra bundle \( \text{End}(E_q) \) is isomorphic to this orbit space, and clearly \( \Gamma(\text{End}(E_q)) \) consists of \( \mathbb{Z}_2^2 \)-equivariant sections, that is,
\[
T \in \Gamma(\text{End}(E_q)) \quad \text{if and only if} \quad T(q^mx, q^ny) = (m, n)T(x, y),
\]
for every \((x, y) \in T^2\) and \((m, n) \in \mathbb{Z}_2^2\), where on the right hand side we mean the action \((5.7)\).

On the other hand, it is well known that \( C^\infty(T^2) \) can be identified with the algebra of all smooth functions on \( \mathbb{R}^2 \) that are 2\(\pi\)-periodic w.r.t. each of their arguments. By Fourier expansion \( C^\infty(T^2) \) consists of all functions
\[
f = \sum_{(k, l) \in \mathbb{Z}^2} f_{k, l} e^{ikx} e^{ilx},
\]
where \( \{f_{k, l}\}_{(k, l) \in \mathbb{Z}^2} \) is any rapidly decreasing sequence of complex numbers, that is, for every \( r \in \mathbb{N} \), the seminorm
\[
\|f\|_r = \sup_{k, l \in \mathbb{Z}} |f_{k, l}|(1 + |k| + |l|)^r < \infty,
\]
and where \( u = e^{i2\pi x}, v = e^{i2\pi y} \) are the coordinate functions on the torus \( T^2 \).

It is also well known that the complex matrix algebra \( M_N(\mathbb{C}) \) is generated as \( \mathbb{C} \)-algebra by the elements \( U_0, V_0 \). Thus, Eqs. \((5.6)\) and \((5.8)\) force any \( T \in \Gamma(\text{End}(E_q)) \) to be of the form
\[
T = \sum_{k, l \in \mathbb{Z}} T_{k, l} (u U_0)^k (V_0 V_0)^l,
\]
with coefficients \( \{T_{k, l}\}_{k, l} \) satisfying Eq. \((5.9)\). Therefore, there is now a \( \mathbb{C} \)-algebra isomorphism
\[
\Gamma(\text{End}(E_q)) \to C^\infty(T^2_q), \quad (u U_0) \mapsto U, \quad (v V_0) \mapsto V,
\]
where \( C^\infty(T^2_q) \) refers to the complex noncommutative 2-torus whose elements are formal power Laurent series in \( U, V \) with a rapidly decreasing sequence of coefficients (cf. \((5.9)\)), subject to \( UV = qVU \). In conclusion, we have the Morita context \((C^\infty(T^2), C^\infty(T^2_q), \Gamma(E_q), \Gamma(E_q)^*)\), where in addition \( C^\infty(T^2) \) and \( C^\infty(T^2_q) \) are related by the algebra map
\[
C^\infty(T^2) \to C^\infty(T^2_q), \quad (u \mapsto U^N, \ v \mapsto V^N).
\]

In the next subsection, we will use the Morita context stated above together with Theorems \((4.5)\) and \((4.7)\) to prove the Morita invariance of both cyclic homology and cohomology from the left Hopf algebroid attached to the trivial Lie algebroid of the classical 2-torus, to the associated left Hopf algebroid over the noncommutative 2-torus, using the construction performed in Subsection \((3.2)\).
5.2.4. The cyclic homology for the noncommutative torus. Now we will direct our attention on the Morita invariance of the cyclic homology between the trivial Lie algebroid \((C^\infty(T^2), \text{Der}_C(C^\infty(T^2)))\) and its induced left Hopf algebroid \((S, P^e \otimes R^e) \mathcal{VK} \otimes R^e Q^e)\), where

\[
R := C^\infty(T^2), \quad K := \text{Der}_C(C^\infty(T^2)), \quad S := C^\infty(T^2_q), \quad P := \Gamma(\mathcal{E}_q), \quad Q := \Gamma(\mathcal{E}_q)^*.
\]

This is possible since \(K = \text{Der}_C(C^\infty(T^2))\) (vector fields on \(T^2\)) is a free \(C^\infty(T^2)\)-module of rank 2, which is a well known result, see, for instance, [Zh] Theorem 5.2. By the general results of [[3.3] we are able to describe the structure maps of the left \(C^\infty(T^2_q)\)-Hopf algebroid \(P^e \otimes R^e\mathcal{VK} \otimes R^e Q^e\), see (5.10). This is a \((4 \times 4)\)-matrix left Hopf algebroid \(M_4(\mathcal{VK})\) whose structure maps are explicitly computable from equations (4.13)–(4.16), whenever a basis for \(K\) is given. Recall furthermore from [KowKr] Lemma 5.1 that every right \(\mathcal{VK}\)-module is automatically SaYD if equipped with the trivial left coaction (in fact, this is true for every cocommutative left Hopf algebroid). As a corollary of Theorems 4.5 and 4.7 we obtain

**Corollary 5.3.** Let \(q \in S^1\) with rational argument modulo \(2\pi\), and consider the Lie algebroid \((R, K)\) of the complex torus \(T^2\) and its induced left Hopf algebroid \((\mathcal{VK}, R)\), cf. (5.10). Let \(M\) be a right \(\mathcal{VK}\)-module and \((R, S, P, Q, \phi, \psi)\) the Morita context of Eq. (5.10). We then have the following natural \(C\)-vector space isomorphisms

\[
H_n(\mathcal{VK}, M) \simeq H_n(M_4(\mathcal{VK}), M_4(M)), \quad H^*_C(\mathcal{VK}, M) \simeq H^*_C(M_4(\mathcal{VK}), M_4(M)),
\]

where \(M_4(\mathcal{VK})\) is the \((4 \times 4)\)-matrix left Hopf algebroid over the noncommutative torus \(C^\infty(T^2_q)\).

As a consequence of this corollary, we have by applying [KowKr] Theorem 5.2] (and its dual version, cf. [KowPo] Theorem 3.14):

**Corollary 5.4.** In addition to the assumptions of Corollary 5.3 assume that \(M\) be \(R\)-flat. Then we have that

\[
H_n(M_4(\mathcal{VK}), M_4(M)) \simeq H_n(K, M), \quad H^*_C(M_4(\mathcal{VK}), M_4(M)) \simeq \bigoplus_{i \geq 0} H_{-2i}(K, M),
\]

where \(H_n(\mathcal{VK}, M) := \text{Tor}_n^\mathcal{VK}(M, R)\), and where \(H^*_C\) denotes the periodic cyclic homology (see, e.g., [[4]] §5.1.3).

**References**

[Ba] H. Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[B1] G. Böhm, *Galois theory for Hopf algebroids*, Ann. Univ. Ferrara Sez. VII (N.S.) 51 (2005), 233–262.

[B2] _____, *Hopf algebroids*, Handbook of algebra, Vol. 6, North-Holland, Amsterdam, 2009, pp. 173–236.

[BS] G. Böhm and D. Ştefan, *Cocyclic (co)homology of bialgebroids: an approach via (co)monads*, Comm. Math. Phys. 282 (2008), no. 1, 239–286.

[BrWi] T. Brzeziński and R. Wisbauer, *Corings and comodules*, London Mathematical Society Lecture Note Series, vol. 309, Cambridge University Press, Cambridge, 2003.

[C] A. Connes, *Noncommutative differential geometry*, Inst. Hautes Études Sci. Publ. Math. (1985), no. 62, 257–360.

[CM] _____, *Hopf algebras, cyclic cohomology and the transverse index theorem*, Comm. Math. Phys. 198 (1998), no. 1, 199–246.

[Cr1] M. Crainic, *Cyclic cohomology of Hopf algebras*, J. Pure Appl. Algebra 166 (2002), no. 1-2, 29–66.

[Cr2] _____, *Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes*, Comment. Math. Helv. 78 (2003), no. 4, 681–721.

[DI] R. Dennis and K. Igusa, *Hochschild homology and the second obstruction for pseudoisotopy*, Algebraic K-theory, Part I (Oberwolfach, 1980), Lecture Notes in Math., vol. 966, Springer, Berlin, 1982, pp. 7–58.
[DuKrMaMi] M. Dubois-Violette, A. Kriegl, Y. Maeda, P. Michor, Smooth ∗-algebras, Progress of Theoretical Physics. Supplement 144 (2001), 54–78.

[ELK] L. El Kaoutit, Extended distributive law. Cowreath over corings, J. Algebra Appl. 9 (2010), no. 1, 135–171.

[FTs] B. Felgin and B. Tsygan, Additive K-theory, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 67–209.

[Gi] V. Ginzburg, Grothendieck groups of Poisson vector bundles, J. Symplectic Geom. 1 (2001), no. 1, 121–169.

[HHkRS] P. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser, Stable anti-Yetter-Drinfeld modules, C. R. Math. Acad. Sci. Paris 338 (2004), no. 8, 587–590.

[Ho] M. Hovey Morita theory for Hopf algebroids and presheaves of groupoids. Amer. J. Math. 124 (2002), no. 6, 1289–1318.

[HoSt] M. Hovey and N. Strickland, Comodules and Landweber exact homology theories. Adv. Math. 192 (2005), no. 2, 427–456.

[Hue] J.-L. Loday, Cyclic homology, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 301, Springer-Verlag, Berlin, 1998.

[Ho] M. Hovey Morita theory for Hopf algebroids and presheaves of groupoids. Amer. J. Math. 124 (2002), no. 6, 1289–1318.

[KHR] B. Fe˘ ıgin and B. Tsygan, Extended distributive law. Cowreath over corings, J. Algebra Appl. 9 (2010), no. 1, 135–171.

[Kow] N. Kowalzig, Hopf algebroids and their cyclic theory, Ph. D. thesis, Universiteit Utrecht and Universiteit van Amsterdam, 2009.

[KowKr] N. Kowalzig and U. Krähmer, Cyclic structures in algebraic (co)homology theories, Homology, Homotopy and Applications 13 (2011), no. 1, 297–318.

[KowPo] N. Kowalzig and H. Posthuma, The cyclic theory of Hopf algebroids, J. Noncomm. Geom. 5 (2011), no. 3, 423–476.

[L] J.-L. Loday, Cyclic homology, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 301, Springer-Verlag, Berlin, 1998.

[LQ] J.-L. Loday and D. Quillen, Cyclic homology and the Lie algebra homology of matrices, Comment. Math. Helv. 59 (1984), no. 4, 569–591.

[MasP] W. Massey and F. Peterson, The cohomology structure of certain fibre spaces. I, Topology 4 (1965), 47–65.

[McC] R. McCarthy, Morita equivalence and cyclic homology, C. R. Acad. Sci. Paris Sér. I Math. 307 (1988), no. 6, 211–215.

[Ne] J. Nestruev, Smooth manifolds and observables, Graduate Texts in Mathematics, vol. 220, Springer-Verlag, New York, 2003.

[Ri] G. Rinehart, Differential forms on general commutative algebras, Trans. Amer. Math. Soc. 108 (1963), 195–222.

[Sch1] P. Schauenburg, Bialgebras over noncommutative rings and a structure theorem for Hopf bimodules, Appl. Categ. Structures 6 (1998), no. 2, 193–222.

[Sch2] ________, Duals and doubles of quantum groupoids (×R-Hopf algebras), New trends in Hopf algebra theory (La Falda, 1999), Contemp. Math., vol. 267, Amer. Math. Soc., Providence, RI, 2000, pp. 273–299.

[Sch3] ________, Face algebras are ×R-bialgebras, Rings, Hopf algebras, and Brauer groups (Antwerp/Brussels, 1996), Lecture Notes in Pure and Appl. Math., vol. 197, Dekker, New York, 1998, pp. 275–285.

[Sch4] ________, Morita base change in quantum groupoids. Locally compact quantum groups and groupoids (Strasbourg, 2002), IRMA Lect. Math. Theor. Phys., 2, de Gruyter, Berlin, 2003, pp. 79–103.

[Swe] M. Sweedler, Groups of simple algebras, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 79–189.

[T1] M. Takeuchi, Groups of algebras over A ⊗ T, J. Math. Soc. Japan 29 (1977), no. 3, 459–492.

[T2] ________, √Morita theory. J. Math. Soc. Japan 39 (1987), no. 2, 301–336.

[Zh] V. Zharinov, The Hochschild cohomology of the algebra of smooth functions on a torus, Teoret. Mat. Fiz. 144 (2005), no. 3, 435–452.

L.E.K.: UNIVERSIDAD DE GRANADA, DEPARTAMENTO DE ÁLGEBRA, FACULTAD DE EDUCACIÓN Y HUMANIDADES DE CEUTA, EL GRECO N° 10, 51002 CEUTA, ESPAÑA
E-mail address: kaoutit@ugr.es

N.K.: UNIVERSIDAD DE GRANADA, DEPARTAMENTO DE ÁLGEBRA, FACULTAD DE CIENCIAS, 18071 GRANADA, ESPAÑA
E-mail address: kowalzig@ihes.fr