An interpolation technique towards the subpolynomial constants in the multilinear Bohnenblust–Hille inequality

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Abstract. The multilinear Bohnenblust–Hille inequality, in a recent more general presentation, asserts that if \( q_1, \ldots, q_n \in [1, 2] \) and \( \frac{1}{q_1} + \cdots + \frac{1}{q_n} = \frac{2n+1}{n} \), then there is a constant \( K = K_{q_1, \ldots, q_n} \geq 1 \) such that

\[
\left( \sum_{i_1, \ldots, i_n = 1}^{\infty} \left| A(e_{i_1}, \ldots, e_{i_n}) \right|^q \right)^\frac{1}{q} \leq K \|A\|
\]

for all continuous \( n \)-linear forms \( A : c_0 \times \cdots \times c_0 \to K \); the original construction provides constants \( K_{q_1, \ldots, q_n} = (\sqrt{2})^{n-1} \) for real scalars and \( K_{q_1, \ldots, q_n} = (2/\sqrt{\pi})^{n-1} \) for complex scalars. In this note we present a new interpolative approach which provides quite better constants. Our procedure, when restricted to the original Bohnenblust–Hille inequality, gives a very simple and self-contained interpolative proof of the Bohnenblust–Hille inequality with the best known constants (with subpolynomial growth), which avoids the technical issues of the original proof. This seems to be unexpectedly surprising since the known interpolative approaches to the Bohnenblust–Hille inequality only provide constants having exponential growth.

1. Introduction

Recall that that the multilinear Bohnenblust-Hille inequality for \( K = \mathbb{R} \) or \( \mathbb{C} \) (see [2]) asserts that for every positive integer \( n \geq 1 \) there exist positive scalars \( C_n \geq 1 \) such that

\[
(1.1) \quad \left( \sum_{i_1, \ldots, i_n = 1}^{\infty} \left| A(e_{i_1}, \ldots, e_{i_n}) \right|^q \right)^\frac{1}{q} \leq C_n \|A\|
\]

for all \( n \)-linear forms \( A : c_0 \times \cdots \times c_0 \to K \), where \( e_i \) are the canonical vectors of \( c_0 \). A very recent generalization of the (multilinear) Bohnenblust–Hille inequality was presented in [1], highlighting the importance of interpolation arguments in this framework. Namely, in [1], with a new interpolative approach, it was proved that, if \( n \geq 1 \) and \( q_1, \ldots, q_n \in [1, 2] \), then the following assertions are equivalent:

(1) There is a constant \( K_{q_1, \ldots, q_n} \geq 1 \) such that

\[
(1.2) \quad \left( \sum_{i_1, \ldots, i_n = 1}^{\infty} \left( \sum_{i_{n-1} = 1}^{\infty} \left( \sum_{i_{n-2} = 1}^{\infty} \left( \sum_{i_{n-3} = 1}^{\infty} \left| A(e_{i_1}, \ldots, e_{i_n}) \right|^q \right)^{\frac{q_n-1}{q_n}} \right)^{\frac{q_n-2}{q_n}} \right)^{\frac{q_n-1}{q_n}} \right)^{\frac{1}{q}} \leq K_{q_1, \ldots, q_n} \|A\|
\]

for all continuous \( n \)-linear forms \( A : c_0 \times \cdots \times c_0 \to K \).

(2) \( \frac{1}{q_1} + \cdots + \frac{1}{q_n} \leq \frac{n+1}{2} \).

In the case \( q_1 = \cdots = q_n = \frac{2n}{n+1} \)

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we recover the classical Bohnenblust–Hille inequality. However, the constants \(K_{q_1,...,q_n}\) in the extremal case, i.e., \(\frac{1}{q_1} + \cdots + \frac{1}{q_n} = \frac{n+1}{2}\), arisen from this new approach have an exponential growth; more precisely, \(K_{q_1,...,q_n} = \left(\sqrt{2}\right)^{n-1}\) for real scalars and \(K_{q_1,...,q_n} = \left(2/\sqrt{\pi}\right)^{n-1}\) for complex scalars (see [1] Remark 5.1]), and this may be a little bit disappointing at a first glance, having in mind that the optimal constants of the multilinear Bohnenblust–Hille inequality have a subpolynomial growth (see [8]).

In this note we present a new interpolative argument which generates quite better constants for the constants \(K_{q_1,...,q_n}\). As an illustration of the effectiveness of this method we show, in details, in the Section 3, that we recover the best known constants of the Bohnenblust–Hille inequalities (that can be found in [7] and [3] and relies on results from [4]) in an elementary form, avoiding the technicalities from [4] (for instance, the technical variant of an inequality due to Blei (see [4] Lemma 3.1)).

2. The new interpolative approach

From now on, for positive integers \(n \geq 1\), the symbols \(C_n\) and \(\widetilde{C}_n\) denote the optimal constants of the Bohnenblust–Hille inequalities for real and complex scalars, respectively. Also, for \(p \in \{1, 2\}\), the symbols \(A_p\) and \(\widetilde{A}_p\) denote the optimal constants of the Khinchin inequality with Rademacher (real case) and Steinhaus (complex case) variables, respectively. Finally, \((q_1,...,q_n)\) shall be called a Bohnenblust–Hille exponent if we have \(q_1,...,q_n \in [1, 2]\) and

\[
\frac{1}{q_1} + \cdots + \frac{1}{q_n} = \frac{n+1}{2}.
\]

In [1] it was shown that each Bohnenblust–Hille exponent \((q_1,...,q_n)\) is generated by interpolating the \(n\) Bohnenblust–Hille exponents \((2, 2,..., 2, 1), (2, 2,..., 2, 1, 2), ..., (1, 2, 2,..., 2)\). This construction provides, at the end, estimates \(\left(\sqrt{2}\right)^{n-1}\) and \(\left(\frac{2}{\sqrt{\pi}}\right)^{n-1}\) for \(K_{q_1,...,q_n}\) ([1] Remark 5.1)), for real and complex scalars, respectively. However, and as mentioned before, when we restrict our attention to the classical Bohnenblust–Hille exponents, since the optimal constants are subpolynomial, the above estimates are quite bad. The following simple result shows that the estimates above are also far from being good, even for general Bohnenblust–Hille exponents. This result also gives us a family of Bohnenblust–Hille exponents with small constants (in some sense) which shall be used to generate, by interpolation, other Bohnenblust–Hille exponents with small constants (see Example 1).

**Proposition 1.** The optimal constants associated to the Bohnenblust–Hille exponents \((q_1,...,q_n)\) with \(q_j = \frac{2n}{k+1}\) for \(k\) indexes \(j\) and \(q_j = 2\) for \(n - k\) indexes \(j\) are smaller than

\[
C_k \left(A_{\frac{2n}{k+1}}\right)^{(n-k)} \text{ and } \widetilde{C}_k \left(\widetilde{A}_{\frac{2n}{k+1}}\right)^{(n-k)}
\]

for real and complex scalars, respectively.

**Proof.** A simple adaptation of [1] Prop. 3.1 tells us that we can consider \(q_1 = \cdots = q_k = \frac{2n}{k+1}\) in (1.2). The result is obtained by using the the multiple Khinchin inequality for Steinhaus variables for complex scalars (see [7] Theorem 2.2)) and the multiple Khinchin inequality for Rademacher functions for real scalars (see [9] Theorem 1.3)).

A simple calculus shows us that the constants in (2.1) are smaller than \(\left(\sqrt{2}\right)^{n-1}\) and \(\left(\frac{2}{\sqrt{\pi}}\right)^{n-1}\). The next example shows that the same happens in other situations:

**Example 1.** Consider the Bohnenblust–Hille exponent \((q,q,r,r,s,t,u)\) with \(q,r,s,t,u\) being pairwise distinct and \(s > t > u\). Since

\[
\frac{3}{q} + \frac{2}{r} + \frac{1}{s} + \frac{1}{t} + \frac{1}{u} = \frac{9}{2}.
\]

a simple computation shows that

\[
(q,r,s,t,u) \in \left[\frac{3}{2}, 2\right] \times \left[\frac{4}{3}, 2\right] \times \left[\frac{3}{2}, 2\right] \times \left[\frac{4}{3}, 2\right] \times [1, 2).
\]
Then we interpolate

\[
\left(\frac{3}{3}, \frac{3}{2}, \frac{3}{2}, 2, ..., 2\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}, \frac{4}{3}, \frac{2}{3}, \frac{2}{3}, 2\right), \left(2, ..., 2, \frac{3}{2}, \frac{3}{2}, 2\right), \left(2, ..., 2, \frac{4}{3}, \frac{4}{3}, \frac{2}{3}, \frac{2}{3}, 2\right), (2, ..., 2, 1)
\]

with

\[
(\theta_j)_{j=1}^4 = \left(2 \left(3 - \frac{2}{r} - \frac{1}{s} - \frac{1}{t} - \frac{1}{u}\right), -2 \left(\frac{2+r}{r}\right), -3 \left(\frac{2+s}{s}\right), 4 \left(\frac{s-t}{st}\right), 2 \left(\frac{t-u}{tu}\right)\right),
\]

to conclude that (for real scalars) the constant \(K_{qqrrstu}\) is dominated by

\[
\left(\sqrt[3]{3}, A_{\frac{3}{2}}\right)^{-\theta_1+\theta_3} \left(\sqrt[2]{2}, A_{\frac{1}{2}}\right)^{-\theta_2+\theta_4} \left(\sqrt[6]{2}\right)^{\theta_5}.
\]

Note that this estimate is quite better than \(\left(\sqrt[2]{2}\right)^{\theta}\).

The idea is that there are several different interpolative approaches that generate a given Bohnenblust–Hille exponent \((q_1, ..., q_n)\). If \((q_1, ..., q_n)\) is generated by the Bohnenblust–Hille exponents

\[
\alpha_1 = (\alpha_{11}, ..., \alpha_{1n}), ..., \alpha_j = (\alpha_{j1}, ..., \alpha_{jn}),
\]

let us denote \((q_1, ..., q_n) \in \langle \alpha_1, ..., \alpha_j \rangle\) and represent the constant derived from the interpolation of \(\alpha_1, ..., \alpha_n\) by \(C_{\alpha_1, ..., \alpha_n}\). Then, clearly,

\[
K_{q_1, ..., q_n} \leq \inf \left\{C_{\alpha_1, ..., \alpha_n} : (q_1, ..., q_n) \in \langle \alpha_1, ..., \alpha_j \rangle \text{ and } j \in \mathbb{N}\right\}.
\]

As the previous example suggests, it seems that better constants are derived from interpolation with less steps. Also, it seems that sometimes a different choice of Bohnenblust–Hille exponents, although using the same number of interpolative steps, provides better constants. For instance, \((q_1, q_2, q_3)\) with \(q_1 > q_2 > q_3\) can be generated by

\[
(2, 2, 1), \left(2, \frac{4}{3}, \frac{4}{3}\right), \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)
\]

with adequate interpolation weights \(\theta_1, \theta_2\) and \(\theta_3\), respectively. The resulting constant for real scalars (and using Proposition [1]) is

\[
2^{\theta_1} \left(C_2 A_{\frac{1}{2}}\right)^{-\theta_2} (C_3)^{\theta_3},
\]

which, as it can be easily checked, is smaller than \(\left(\sqrt[2]{2}\right)^{\theta}\), which is generated by the interpolation of \((2, 2, 1), (2, 1, 2), \text{ and } (2, 2, 1)\).

In the next section we apply our approach to recover, in a very simple and quick way, the best known constants for the classical Bohnenblust–Hille inequality.

### 3. The interpolative proof of multilinear the Bohnenblust–Hille inequality with subpolynomial constants

The first interpolative proof of the Bohnenblust–Hille inequality is probably due to Kaijser ([6], see also [5] for details); this proof, however, gives constants with exponential growth. As mentioned before, the same exponential growth appears in the recent interpolative proof of the general Bohnenblust–Hille inequality from [1]. In this section we illustrate, in details, the particular case our approach when we deal with the classical Bohnenblust–Hille inequality: we recover the best known constants straightforwardly. Since the constants of the real and complex cases are obtained via different approaches (this strange fact was recently stressed in [3]) we will divide the proof in two different cases.
3.1. Case of complex scalars. The constant $\widetilde{C}_{2n}$ will be derived from the constant $\widetilde{C}_n$. As mentioned in the previous section, from the multiple Khinchin inequality for Steinhaus variables a straightforward computation provides the constant

$$\widetilde{C}_n \left( A_{\frac{2n}{n+1}} \right)^{-n}$$

for the Bohnenblust–Hille exponents $\left( \frac{n \text{ times}}{n+1}, \ldots, \frac{n \text{ times}}{n+1}, 2, \ldots, 2 \right)$ and $\left( \frac{n \text{ times}}{n+1}, \ldots, \frac{n \text{ times}}{n+1}, 2, \ldots, 2 \right)$. Now, interpolating these exponents in the sense of [11] with $\theta_1 = \theta_2 = 1/2$ we get

$$\widetilde{C}_{2n} \leq \widetilde{C}_n \left( A_{\frac{2n}{n+1}} \right)^{-n}.$$  \hfill (3.1)

For the case $2n+1$, again choosing $k_1 = n$ and $r = 2$, and using the multiple Khinchin inequality for Steinhaus variables we obtain the constants

$$\widetilde{C}_n \left( A_{\frac{2n}{n+1}} \right)^{-n-1} \text{ and } \widetilde{C}_{n+1} \left( A_{\frac{2(n+1)}{(n+1)+1}} \right)^{-n}$$

for the Bohnenblust–Hille exponents $\left( \frac{n \text{ times}}{n+1}, \ldots, \frac{n+1 \text{ times}}{n+1}, 2, \ldots, 2 \right)$ and $\left( \frac{n \text{ times}}{n+1}, \ldots, \frac{n+1 \text{ times}}{n+1}, 2, \ldots, 2 \right)$, respectively. Now we interpolate the above Bohnenblust–Hille exponents with $\theta_1 = \frac{n}{2n+1}$ and $\theta_2 = \frac{n+1}{2n+1}$, respectively, and we get

$$\widetilde{C}_{2n+1} \leq \left( \widetilde{C}_n \left( A_{\frac{2n}{n+1}} \right)^{-n-1} \right)^{\frac{n-1}{n+1}} \left( \widetilde{C}_{n+1} \left( A_{\frac{2(n+1)}{(n+1)+1}} \right)^{-n} \right)^{\frac{n+1}{n+1}}.$$ \hfill (3.2)

Note that the formulas (3.1) and (3.2) are precisely those from [7] which generates the best known constants for the Bohnenblust–Hille inequality for complex scalars.

3.2. Case of real scalars. The exactly same proof of the previous case, using now Rademacher functions instead of Steinhaus variables and the corresponding multiple Khinchin inequality, gives us the estimates from [9]. However, it was very recently shown, in [3], that in the case of real scalars the estimates from [9] can be improved by using a somewhat chaotic combinatorial approach. For instance, in [3] it was shown that a better (smaller) constant for $m = 26$ was obtained by combining the cases $m = 12$ and $m = 14$ instead of using the case $m = 13$; it was also shown that this also happens in several other cases. Of course, our interpolation technique with a different choice of $k_1$ and $r$, in order to compute $C_{2n}$ using $C_{12}$ and $C_{14}$, as well as any other choices of combinations, gives us exactly the same constants from [3].

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References

[1] N. Albuquerque, F. Bayart, D. Pellegrino, and J. B. Seoane-Sepúlveda, Sharp generalizations of the multilinear Bohnenblust–Hille inequality, J. Funct. Anal., DOI: 10.1016/j.jfa.2013.08.013.
[2] H. F. Bohnenblust and E. Hille, On the absolute convergence of Dirichlet series, Ann. of Math. (2) 32 (1931), no. 3, 600–622.
[3] J. R. Campos, D. Nuñez-Alarcón, D. Nuñez-Alarcón, D. Pellegrino, J. B. Seoane-Sepúlveda, and D. M. Serrano-Rodriguez, On the multilinear Bohnenblust–Hille constants: complex versus real case, preprint.
[4] A. Defant, D. Popa, and U. Schwarting, Coordinatewise multiple summing operators in Banach spaces, J. Funct. Anal. 259 (2010), no. 1, 220–242.
[5] A. Defant and P. Sevilla-Peris, The Bohnenblust–Hille cycle of ideas: from yesterday to today, preprint.
[6] S. Kaijser, Some results in the metric theory of tensor products, Studia Math. 63 (1978), no. 2, 157–170.
[7] D. Nuñez-Alarcón, D. Pellegrino, and J. B. Seoane-Sepúlveda, On the Bohnenblust-Hille inequality and a variant of Littlewood’s 4/3 inequality, J. Funct. Anal. 264 (2013), 326–336.
[8] D. Nuñez-Alarcón, D. Pellegrino, J. B. Seoane-Sepúlveda, and D. M. Serrano-Rodriguez, There exist multilinear Bohnenblust-Hille constants $(C_n)_{n=1}^\infty$ with $\lim_{n\to\infty}(C_{n+1} - C_n) = 0$, J. Funct. Anal. 264 (2013), no. 2, 429–463.
[9] D. Pellegrino and J. B. Seoane-Sepúlveda, New upper bounds for the constants in the Bohnenblust-Hille inequality, J. Math. Anal. Appl. 386 (2012), no. 1, 300–307.
SUBPOLYNOMIAL CONSTANTS IN THE BOHLENBLUST–HILLE INEQUALITY VIA INTERPOLATION

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