ON THE SECTIONAL GENERA AND COHEN-MACAUARY RINGS

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Abstract. We explore the behavior of the sectional genera of certain primary ideals in Noetherian local rings. In this paper, we provide characterizations of a Cohen-Macaulay local ring in terms of the sectional genera, the Cohen-Macaulay type, and the second Hilbert coefficients for certain primary ideals. We also characterize Gorenstein rings and quasi-Buchsbaum rings in terms of the sectional genera for certain primary ideals.

1. Introduction

Let \((R, \mathfrak{m})\) be a commutative Noetherian local ring of dimension \(d\), where \(\mathfrak{m}\) is the maximal ideal. Let \(I\) be an \(\mathfrak{m}\)-primary ideal of \(R\). It is well-known that there are integers \(e_i(I, R)\), called the Hilbert coefficients of \(M\) with respect to \(I\), such that

\[
\ell_R(R/I^{n+1}) = e_0(I, R)\binom{n+d}{d} - e_1(I, R)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I, R)
\]

for all \(n \gg 0\). Here \(\ell_R(N)\) denotes the length of an \(R\)-module \(N\). The leading coefficient \(e_0(I, R)\) is called the multiplicity of \(R\) with respect to \(I\), and \(e_1(I, R)\) is named by W. V. Vasconcelos ([30]) as the Chern number of \(R\) with respect to \(I\).

In 1987, A. Ooishi ([19]) introduced the notion of sectional genera in commutative rings. Let \(\text{sg}(I, R) = \ell_R(R/I) - e_0(I, R) + e_1(I, R)\) and call it the sectional genera for \(R\) with respect to \(I\). For the notion of sectional genera in Cohen-Macaulay local rings, there is general recognition that the sectional genera of \(R\) with respect to \(I\) controls the depth of the associated graded ring of \(I\), and determines the Hilbert-Samuel function of \(I\). Indeed, the results along this line of investigation can be found in, for example, [13], [21], [22], [14], [2]. On the other hand, for non-Cohen-Macaulay local rings, not so much is known about the sectional genera. One progress is that S. Goto and K. Ozeki ([10]) gave a criterion for a certain equality of the sectional genera of parameter ideals for modules in 2016. On the other hand, even how the sectional genera characterizes the ring itself has not yet been investigated.

The purpose of our paper is to study the sectional genera of \(m\)-primary ideals, provided \(R\) is unmixed and a homomorphic image of a Cohen-Macaulay local ring. We especially focus our attention on \(C\)-parameter ideals. We note here only that the notion of \(C\)-parameter ideals is a special kind of parameter ideals of \(R\), and \(C\)-parameter ideals always exist in our assumption. See Section 2 for the precise definition.

Let us recall and fix the terminology and the notations to state our results. The index of reducibility of \(C\)-parameter ideals \(q\) of \(R\) is called the stable value of \(R\) and denoted by \(N(R)\) (see [4]). We denote by \(r(R)\) the Cohen-Macaulay type \(\dim_R((0) :_{R^d_m(R)} \mathfrak{m})\) of \(R\), where \(H^d_m(R)\) denotes the \(d\)th local cohomology of \(R\), and \(g(R)\) the \(g\)-invariant of \(R\) (see Definition 3.4). Using these notations, our results are stated as follows.

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Theorem 1.1 (Theorem 3.3, 3.5, Corollary 3.6). Suppose that $R$ is unmixed with $d = \dim R \geq 2$, that is, $\dim \hat{R}/p = d$ for all $p \in \text{Ass } R$ where $\hat{R}$ denotes the completion of $R$. Then for all parameter ideals $q \subset m^{g(R)}$, we have

$$r(R) - \mathcal{N}(R) \leq \text{sg}(q : m, R) \leq \text{sg}(q, R).$$

Furthermore, for each inequality, it is to be equal if and only if $R$ is a Cohen-Macaulay local ring.

Theorem 1.2 (Theorem 4.1, 4.2, Corollary 4.3). Suppose that $R$ is unmixed with $d = \dim R \geq 2$. Then for all parameter ideals $q \subset m^{g(R)}$, we have

$$e_2(q : m, R) \leq e_2(q, R) \leq \text{sg}(q : m, R) + \mathcal{N}(R) - r(R).$$

Furthermore, for each inequality, it is to be equal if and only if $R$ is a Cohen-Macaulay local ring.

Next we explore the condition under which generalized Northcott’s inequality is equal when $I = q : m$. Recall that for a Cohen-Macaulay local ring $R$ and an $m$-primary ideal $I$ of $R$, Northcott’s inequality holds (see [18]):

$$\text{sg}(I, R) \geq 0.$$ 

Then, S. Goto and K. Nishida ([9]) generalizes Northcott’s inequality without assuming that $R$ is a Cohen–Macaulay ring. In particular, they show

$$\text{sg}(I, R) \geq e_1(q, R),$$

where $q$ is a minimal reduction of $I$. In the present study, we characterize rings such that the above inequality is to be an equal. Actually, the next main result is stated as follows.

Theorem 1.3. Suppose that $R$ is a non-regular unmixed local ring with $d = \dim R \geq 2$. Then the following statements are equivalent.

i) $R$ is a quasi-Buchsbaum ring.

ii) There exists a $C$-parameter ideal $q$ of $R$ such that

$$\text{sg}(q : m, R) = e_1(q, R).$$

For the definition of quasi-Buchsbaum modules, see after Corollary 4.5.

The remainder of this paper is organized as follows. In Section 2, we prove some preliminary results on the index of reducibility for parameter ideals, $C$-parameter ideals, and generalized Cohen-Macaulay rings. Theorem 1.1 is proven in Section 3. Theorem 1.2 and 1.3 are proven in Section 4. The assumption throughout this paper is written in the beginning of Section 2.

2. Preliminary

In what follows, throughout this paper, let $(R, m, k)$ be a Noetherian local ring of dimension $d$, where $m$ is the maximal ideal and $k = R/m$ is the residue field of $R$. Suppose that $R$ is a homomorphic image of a Cohen-Macaulay local ring. Let $M$ be a finitely generated $R$-module of dimension $s$. Let $I$ be an $m$-primary ideal of $M$, that is, $M/IM$ is of finite length. It is well-known that there are integers $\{e_i(I, M)\}_{i=0}^s$, called the Hilbert coefficients of $M$ with respect to $I$, such that

$$\ell_R(M/I^{n+1}M) = e_0(I, M)\binom{n+s}{s} - e_1(I, M)\binom{n+s-1}{s-1} + \cdots + (-1)^se_s(I, M)$$

for $n \gg 0$. Here $\ell_R(N)$ denotes the length of an $R$-module $N$. In particular, the first two coefficients $e_0(I, M)$ and $e_1(I, M)$ are called the multiplicity of $M$ with respect to $I$ and the Chern number of $M$ with respect to $I$, respectively. We set

$$\text{sg}(I, M) = \ell_R(M/IM) - e_0(I, M) + e_1(I, M)$$

and call it the sectional genera for $M$ with respect to $I$ ([19, Definition 1.3]).

Lemma 2.1. Let $I$ be an $m$-primary ideal of $M$ and $x \in I$ a superficial element. Then, we have
Lemma 2.2. Suppose that $R$ is generalized Cohen-Macaulay with $d = \dim R > 0$ and $q$ is a $C$-parameter ideal of $R$. Then we have the following.

i) $I(q, R) = I(R) = \sum_{i=0}^{d-1} \binom{d-1}{i} h_i(R)$ (see [6, (3.7)]).

ii) $ir_R(q) = N(R) = \sum_{i=0}^{d} \binom{d}{i} r_i(R)$ (see [7, Theorem 1.1]).
Then we have the following.

\[ e_i(q : m, R) = \begin{cases} 
(-1)^i \left( \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} h_i(R) - \sum_{j=1}^{d-i} r_j(R) \right) & \text{if } i = 1, \ldots, d-1, \\
(-1)^d(h_0(R) - r_1(R)) & \text{if } i = d,
\end{cases} \]

provided \( R \) is not regular (see [5, Theorem 5.2]).

\[ e_i(q, R) = \begin{cases} 
2 \left( \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} h_i(R) \right) & \text{if } i = 0, \\
0 & \text{if } 0 < i < d,
\end{cases} \]

(see [23, Korollary 3.2]).

The purpose of this paper is to provide characterizations of properties of rings (e.g., Cohen-Macaulay, Gorenstein, and quasi-Buchsbaum) via the invariants introduced in this section. Hence, we note the invariants for the case of Cohen-Macaulay rings, although it is well-known and immediately follows from Lemma 2.2.

**Remark 2.3.** Suppose that \( R \) is a non-regular Cohen-Macaulay ring of dimension \( d \geq 2 \) and \( q \) is a \( C \)-parameter ideal of \( R \). Then we have the following.

1. \( I(q, R) = 0 \).
2. \( \text{ir}_R(q) = N(R) = r_d(R) \).
3. \( e_1(q : m, R) = r_d(R) \) and \( e_i(q : m, R) = 0 \) for all \( 2 \leq i \leq d \).
4. \( e_i(q, R) = 0 \) for all \( 1 \leq i \leq d \).

We here note our strategy in Section 3: by passing to a part of a \( C \)-system of parameters of \( R \), we can reduce our assertions to the 2-dimensional case. We then obtain relations of invariants introduced in this section from Lemma 2.2. Hence, we also summarize the 2-dimensional case of Lemma 2.2 for the reader’s convenience.

**Remark 2.4.** Suppose that \( R \) is a non-regular generalized Cohen-Macaulay ring with \( d = 2 \) and \( q \) is a \( C \)-parameter ideal of \( R \). Then we have the following.

1. \( I(q, R) = I(R) = h_0(R) + h_1(R) \).
2. \( \text{ir}_R(q) = N(R) = r_0(R) + 2r_1(R) + r_2(R) \).
3. \( e_1(q : m, R) = r_1(R) + r_2(R) - h_1(R) \) and \( e_2(q : m, R) = h_0(R) - r_1(R) \).
4. \( e_1(q, R) = -h_1(R) \) and \( e_2(q, R) = -h_0(R) \).

The following example shows a computation of the Hilbert-Samuel functions of \( q \) and \( q :_R m \).

**Example 2.5.** Let \( R = k[[X, Y]] \) be a subring of the formal power series ring \( k[[X, Y]] \) over a field \( k \). Let \( m \) be the maximal ideal of \( R \). For an integer \( a \geq 6 \), set

\[ q_a = (X^a, X^a Y^{2a}). \]

Then we have the following.

1. \( q_a \) is a \( C \)-parameter ideal.
2. \( \ell_R(R/q_a^{n+1}) = 2a^2 \binom{n+2}{2} + 2 \binom{n+1}{1} \) for all \( n \geq 0 \). Hence,
   \[ e_0(q_a, R) = 2a^2, \quad e_1(q_a, R) = -2, \quad \text{and} \quad e_2(q_a, R) = 0. \]
3. \( \ell_R(R/(q_a :_R m)^{n+1}) = 2a^2 \binom{n+2}{2} - \binom{n+1}{1} - 1 \) for all \( n \geq 0 \). Hence,
   \[ e_0(q_a :_R m, R) = 2a^2, \quad e_1(q_a :_R m, R) = 1, \quad \text{and} \quad e_2(q_a :_R m, R) = -1. \]
Proof. Note that we have \( \overline{R} = k[[X, XY, XY^2]] \), where \( \overline{R} \) denotes the integral closure of \( R \). Indeed, it is standard to check that \( k[[X, XY, XY^2]] \subset \overline{R} \). It follows that \( \overline{R} = k[[X, XY, XY^2]] \) since \( k[[X, XY, XY^2]] \) is a normal domain by [1, Theorem 6.1.4]. Hence, by applying the local cohomology functor \( H_m(-) \) to the exact sequence \( 0 \rightarrow R \rightarrow \overline{R} \rightarrow \overline{R}/R \rightarrow 0 \), we obtain that

\[
H^0_m(R) = 0, H^1_m(R) \cong H^0_m(\overline{R}/R) \cong \overline{R}/R, \text{ and } H^2_m(R) \cong H^2_m(\overline{R}).
\]

i). By (1), we obtain that \( a(R) = a_0(R) a_1(R) = \text{Ann}_R(\overline{R}/R) \supseteq \mathfrak{m}^2 \). By noting that \( b(R) \supseteq a(R) \) (see [23, Satz 2.4.5]), \( q_a \subseteq \mathfrak{m}^6 \subseteq b(R)^3 \) since \( a \geq 6 \). It follows that \( q_a \) is a \( C \)-parameter ideal.

ii). Note that the set of all monomials contained in \( R \) is described as follows.

With the above description, by noting that \( q_n^{n+1} = (X^{a(n+1)}Y^{2ai} \mid 0 \leq i \leq n+1) \), it is straightforward to check that \( \ell(R/q_n^{n+1}) = 2a^2(n+2) + 2(n+1) \) for all \( n \geq 0 \).

iii). First, we prove the following claim.

Claim 2.6. \( q_n^{n+1} : R \mathfrak{m} = q_n^n(q_a : R \mathfrak{m}) = (q_a : R \mathfrak{m})^{n+1} \) for all \( n \geq 0 \).

Proof of Claim 2.6. It is easy to check that the monomial basis of a \( k \)-vector space \( (q_a : R \mathfrak{m})/q_a \) is \( X^{2a+1}Y^{2a+1}, X^{2a-1}Y^{2a-1}, X^{2a-2}Y^{2a-2}, \) and \( X^{a+2}Y \). Then, by using the description in iii), we observe that \( q_n^{n+1} : R \mathfrak{m} = q_n^n(q_a : R \mathfrak{m}) \) for all \( n \geq 0 \). Since \( q_n^n(q_a : R \mathfrak{m}) \subseteq (q_a : R \mathfrak{m})^{n+1} \) is clear, the rest to prove is \( q_n^n(q_a : R \mathfrak{m}) \supseteq (q_a : R \mathfrak{m})^{n+1} \) for all \( n \geq 0 \). It is enough to prove that \( q_n(q_a : R \mathfrak{m}) \supseteq (q_a : R \mathfrak{m})^2 \).

Set \( I = (X^{a+2}Y^{2a+1}, X^{2a-1}Y^{2a-1}, X^{2a-2}Y^{2a-2}, X^{a+2}Y) \), the ideal generated by the monomial basis of \( (q_a : R \mathfrak{m})/q_a \). Since \( q_a : R \mathfrak{m} = q_a + I \), we obtain that

\[
(q_a : R \mathfrak{m})^2 = q_a(q_a : R \mathfrak{m}) + I^2.
\]

On the other hand, the direct calculation shows that \( I^2 \subseteq q_a^2 \) since \( a \geq 6 \). It follows that \( q_a(q_a : R \mathfrak{m}) \supseteq (q_a : R \mathfrak{m})^2 \). \( \square \)

By Claim 2.6, we have

\[
\ell(R/(q_a : R \mathfrak{m})^{n+1}) = \ell(R/q_n^{n+1}) - \ell((q_n^{n+1} : R \mathfrak{m})/q_n^{n+1})
= 2a^2\left(\binom{n+2}{2} + 2\binom{n+1}{1}\right) - [3(n+1) + 1]
= 2a^2\left(\binom{n+2}{2} - \binom{n+1}{1}\right) - 1
\]

for all \( n \geq 0 \). \( \square \)
**Remark 2.7.** In Example 2.5, by (1), we obtain that \( h_0(R) = r_0(R) = 0, h_1(R) = 2, r_1(R) = 1, \) and \( r_2(R) = 2 \). (Note that \( R \) is a Gorenstein ring by [1, Theorem 6.3.5] and is generated by 2 elements as an \( R \)-module). Therefore, by Remark 2.4, we have

i) \( I(q, R) = 2 \), ii) \( N(R) = 4 \), iii) \( e_1(q : m, R) = 1, e_2(q : m, R) = -1 \), iv) \( e_1(q, R) = -2 \), and \( e_2(q, R) = 0 \)

for all \( C \)-parameter ideals \( q \) of \( R \).

We further note useful properties of a \( C \)-system of parameters of \( M \).

**Lemma 2.8.** Let \( x_1, \ldots, x_s \) be a \( C \)-system of parameters of \( M \). Then the following hold true.

i) For each \( i = 1, \ldots, s \), we have that \( x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s \) is a \( C \)-system of parameters of \( M/x_i M \) and

\[ N_R(M) = N_R(M/x_i M) \]

(see [4, Lemma 2.13]).

ii) For all \( i = 1, \ldots, s \), we have

\[ \text{Ass} M/(x_1, \ldots, x_i) M \subseteq \text{Ass} M/(x_1, \ldots, x_i) M \cup \{m\}, \]

where \( \text{Ass} M = \{p \in \text{Ass} M \mid \dim R/p = \dim M\} \) (see [15, Remark 3.3]).

Recall that the largest submodule of \( M \) of dimension less than \( s \) is called the unmixed component of \( M \), and denoted by \( U_M(0) \). The following result plays a key role in the arguments used in this paper.

**Lemma 2.9.** Suppose that \( M \) is unmixed with \( s = \dim M \geq 2 \) and \( x_1, \ldots, x_s \) is a \( C \)-system of parameters of \( M \). Then

i) \( H^2_m(M/(x_1, x_2, \ldots, x_i) M) \) is a finitely generated \( R \)-module for all \( 1 \leq i \leq s-2 \).

ii) Let \( N/(x_1, \ldots, x_s-2) M \) denote the unmixed component of \( M/(x_1, \ldots, x_s-2) M \). If \( M/N \) is a Cohen-Macaulay \( R \)-module (of dimension 2), then \( M \) is a Cohen-Macaulay \( R \)-module (of dimension \( s \)).

**Proof.** i) is Lemma 3.1 in [8] or Remark 3.3 in [15].

ii). For \( 0 \leq i \leq s-2 \), let \( q_i = (x_1, x_2, \ldots, x_i) \) and \( M_i = M/q_i M \). (If \( i = 0 \), then \( q_0 = (0) \) and \( M_0 = M \).) Consider the exact sequence

\[ 0 \to N/q_{s-2} M \to M_{s-2} \to M/N \to 0. \]

Note that \( \dim N/q_{s-2} M = 0 \) since \( \dim N/q_{s-2} M < \dim M_{s-2} \) and \( \text{Ass} M_{s-2} \subseteq \text{Ass} M_{s-2} \cup \{m\} \) by Lemma 2.8 ii). \( M/N \) is a Cohen-Macaulay \( R \)-module of dimension 2 by hypothesis. Hence, by applying the local cohomology functor \( H^j_m(\cdot) \) to the above exact sequence, we obtain that

\[ H^2_m(M_{s-2}) = 0. \]

**Claim 2.10.** Suppose that \( s \geq 3 \). Let \( 1 \leq i \leq s-2 \) and \( 1 \leq \ell \) be integers. Then, \( H^j_m(M_{i-1}) = 0 \) for all \( 1 \leq j \leq \ell + 1 \).

**Proof of Claim 2.10.** By the multiplication \( M_{i-1} \xrightarrow{x_i} M_{i-1} \), we have

\[ 0 \to (0) :_{M_{i-1}} x_i \to M_{i-1} \to M_{i-1}/([0] :_{M_{i-1}} x_i) \to 0 \]

and

\[ 0 \to M_{i-1}/([0] :_{M_{i-1}} x_i) \to M_{i-1} \to M_i \to 0. \]

By noting that \( \text{Ass} M_{i-1} \subseteq \text{Ass} M_{i-1} \cup \{m\} \) and \( x_i \) is a part of a system of parameter of \( M_{i-1} \), we have \( \text{Ass}((0) :_{M_{i-1}} x_i) \subseteq \{m\} \), that is, \( \dim((0) :_{M_{i-1}} x_i) = 0 \). Hence, by applying \( H^j_m(\cdot) \) to (3), we obtain that

\[ H^2_m(M_{i-1}/([0] :_{M_{i-1}} x_i)) \cong H^2_m(M_{i-1}) \]

for all \( j > 0 \). Therefore, by applying \( H^j_m(\cdot) \) to (4), it follows that

\[ H^j_m(M_i) \xrightarrow{x_i} H^j_m(M_{i-1}) \xrightarrow{x_i} H^j_m(M_{i-1}) \to H^j_m(M_i) \]

for all \( j = 0, \ldots, \ell \). Hence, \( H^j_m(M_i) = 0 \) for all \( 1 \leq j \leq \ell + 1 \).
for all $j > 0$.

If $2 \leq j \leq \ell + 1$, then $H_{m}^{j-1}(M) = 0$, that is, $0 \to H_{m}^{j}(M_{i-1}) \xrightarrow{x_{i}} H_{m}^{j-1}(M_{i-1})$. On the other hand, for each element of $H_{m}^{j}(M_{i-1})$, large enough powers of $x_{i}$ annihilate the element. Hence, the injection implies that $H_{m}^{j}(M_{i-1}) = 0$ for all $2 \leq j \leq \ell + 1$.

Let $j = 1$. Then $H_{m}^{1}(M_{i-1}) \xrightarrow{x_{i}} H_{m}^{0}(M_{i-1}) \to 0$. Since $H_{m}^{1}(M_{i-1})$ is finitely generated by $i$, by Nakayama’s lemma, we obtain that $H_{m}^{1}(M_{i-1}) = 0$. □

Let us complete the proof of Lemma 2.9. If $s = 2$, then $M$ is a Cohen-Macaulay $R$-module by (2) and the assumption that $M$ is reduced. Suppose that $s \geq 3$. By using Claim 2.10 recursively, we derive from (2) that $H_{m}^{s}(M) = 0$ for all $1 \leq j \leq s - 1$. On the other hand, by noting that $H_{m}^{0}(M) = 0$ since $M$ is reduced, $M$ is a Cohen-Macaulay $R$-module. □

Remark 2.11. In [26, 28] the definition of Goto sequences on $M$ was introduced by the second author. Moreover, if $x_{1}, \ldots, x_{s}$ is a Goto sequence, then the statement Lemma 2.9 (ii) is true. Note that, if $M$ is unmixed, a $C$-system of parameters of $M$ forms a Goto sequence on $M$. Thus applying Lemma 2.5 in [28] we also have Lemma 2.9 (ii).

3. The bound of the sectional genera

In this section, we study the bound of $sg(q, R)$ and $sg(q : m, R)$, where $q$ runs over all $C$-parameter ideals of $R$. We maintain the assumption of the beginning of Section 2. The goal of this section is to provide a characterization of a Cohen–Macaulay ring in terms of its sectional genera. We begin with the following.

Lemma 3.1. Assume that $e_{0}(m, R) > 1$ and $q$ is a $C$-parameter ideal of $R$. Then we have

$$sg(q : m, R) = I(q, R) + e_{1}(q : m, R) - N(R).$$

Proof. Put $J = q : m$. Since $e_{0}(m, R) > 1$, it follows from by Proposition 2.3 in [9] and Proposition 11.2.1 in [25] that $e_{0}(J, R) = e_{0}(q, R)$. Since $q$ is a $C$-parameter ideal, by the definition of the stable value, we have

$$sg(J, R) = \ell(R/J) - e_{0}(J, R) + e_{1}(J, R)$$
$$= \ell(R/q) - e_{0}(q, R) + e_{1}(J, R) - \ell(q : m/q)$$
$$= I(q, R) + e_{1}(J, R) - N(R),$$

as required. □

Theorem 3.2. Suppose that $\dim R = 1$ and $e_{0}(m, R) > 1$. Then the following statements are equivalent.

i) $R$ is Cohen–Macaulay.

ii) There exists a $C$-parameter ideal $q \subseteq m^{\delta(R)}$ of $R$ such that

$$sg(q : m, R) \geq 0.$$

Proof. i) $\Rightarrow$ ii). This immediately follows from the assumption that $R$ is Cohen-Macaulay by applying Lemma 3.1.

ii) $\Rightarrow$ i). Since $\dim R = 1$, $R$ is generalized Cohen-Macaulay. It follows from Lemma 2.2 and Lemma 3.1 that we have

$$-r_{0}(R) = sg(q : m, R) \geq 0.$$

Thus $r_{0}(R) = 0$. Hence $R$ is Cohen-Macaulay, as required. □

Note that in the case where $\dim R = 1$, we have $sg(q, R) = 0$ for every $C$-parameter ideal $q$ of $R$ by Lemma 3.1 in [11]. Therefore, we have $sg(q : m, R) \leq sg(q, R)$ for every $C$-parameter ideal $q$ of $R$. In the case where $\dim R \geq 2$, the following result shows that the inequality $sg(q : m, R) \leq sg(q, R)$ still holds true for every $C$-parameter ideal of $R$, provided $R$ is unmixed.
**Theorem 3.3.** Assume that $R$ is a non-regular unmixed local ring of dimension $d = \dim R \geq 2$. Then the following statements are equivalent.

i) $R$ is Cohen–Macaulay.

ii) There exists a $C$-parameter ideal $q$ such that

$$\sg(q : m, R) \geq \sg(q, R).$$

*Proof.* Let $q$ be a $C$-parameter ideal of $R$ and put $J = q : m$. Since $R$ is to be regular if $R$ is unmixed and $e_0(m, R) = 1$ (see [16, Theorem 40.6]), we have $e_0(m, R) > 1$.

i) $\Rightarrow$ ii). This immediately follows from the assumption that $R$ is Cohen–Macaulay by applying Lemma 2.2 and Lemma 3.1.

ii) $\Rightarrow$ i). Let $\underline{x} = x_1, x_2, \ldots, x_d$ be a $C$-system of parameters of $R$ such that $q = (x_1, x_2, \ldots, x_d)$. Put $R_i = R/(x_1, \ldots, x_i)$ for $i = 1, \ldots, d - 2$. By Lemma 2.8, for all $i = 1, \ldots, d - 2$, we have

(a) $JR_i = qR_i : mR_i$,

(b) $x_i$ is a superficial element of $R_{i-1}$ with respect to $qR_{i-1}$ and $JR_{i-1},$

(c) $\mathcal{N}_R(R_i) = \mathcal{N}_R(R_{i-1}).$

Now, we put $A = R_{d-2}$. By Lemma 2.1 we have $e_i(q, R) = e_i(qA, A)$ and $e_i(J, R) = e_i(JA, A) = e_i(qA : mA, A)$, for all $i = 1, 2$. Therefore we have

(5) $\sg(q, R) = \ell(R/q) - e_0(q, R) + e_1(q, R) = \ell(A/qA) - e_0(qA, A) + e_1(qA, A) = \sg(qA, A)$

and

$$\sg(J, R) = \ell(R/J) - e_0(J, R) + e_1(J, R)$$

$$= \ell(R/q) - e_0(q, R) + e_1(J, R) - \ell(q : m/q)$$

$$= \ell(R/q) - e_0(q, R) + e_1(J, R) - \mathcal{N}(R)$$

$$= \ell(A/qA) - e_0(qA, A) + e_1(JA, A) - \mathcal{N}(A).$$

On the other hand, it follows from Lemma 2.8 ii) that $A/H_m^0(A)$ is unmixed with $\dim A = 2$. Thus by Lemma 2.9 i), $A/H_m^0(A)$ is generalized Cohen–Macaulay and so $A$ is. Moreover, it follows from Lemma 2.1 that $e_0(mA, A) = e_0(m, R) > 1$. By Lemma 2.2, (5), and (6), we obtain that

$$\sg(J, R) = \ell(A/qA) + e_1(JA, A) - \mathcal{N}(A) = h_0(A) - r_0(A) - r_1(A)$$

and

(7) $\sg(q, R) = \ell(qA, A) + e_1(qA, A) = h_0(A)$. 

Since $\sg(J, R) \geq \sg(q, R)$, we have $\sg_0(A) = r_1(A) = 0$. Therefore, $A$ is Cohen–Macaulay. Hence $R$ is Cohen-Macaulay by applying Lemma 2.9.

We fix $n \geq 1$, and denote by $x^n$ the sequence $x_1^n, x_2^n, \ldots, x_d^n$. Let $K^*(x^n)$ be the Koszul complex of $R$ generated by the sequence $x^n$, and let $H^*(K^*(x^n); R) = H^*(\text{Hom}_R(K^*(x^n), R))$ be the Koszul cohomology module of $R$. Then for every $p \in \mathbb{Z}$, the family $\{H^p(x^n; R)\}_{n \geq 1}$ naturally forms an inductive system of $R$, whose limit

$$H^0_R(R) = \lim_{n \to \infty} H^p(x^n; R)$$

is isomorphic to the local cohomology module

$$H^p_m(R) = \lim_{n \to \infty} \text{Ext}^p_R(R/m^n, R).$$

For each $n \geq 1$ and $p \in \mathbb{Z}$, let $\phi^{p,n}_{R, R} : H^p(x^n; R) \to H^p_m(R)$ denote the canonical homomorphism into the limit.

**Definition 3.4 ([11, Lemma 3.12]).** There exists an integer $n_0$ such that for all systems of parameters $\underline{x} = x_1, \ldots, x_d$ for $R$ contained in $m^{n_0}$ and for all $p \in \mathbb{Z}$, the canonical homomorphisms

$$\phi^{p,1}_{\underline{x}, R} : H^p(x, R) \to H^p_m(R)$$

into the inductive limit are surjective on the socles. The least integer $n_0$ with this property is called a $g$-invariant of $R$ and denote by $g(R)$. 

The following result shows that $\text{sg}(q : m, R)$ and $\text{sg}(q, R)$ are bounded below by the same finite value, where $q$ runs over all $C$-parameters ideals of $R$ contained in $m^{g(R)}$.

**Theorem 3.5.** Assume that $R$ is a non-regular unmixed local ring of dimension $d = \dim R \geq 2$. Then the following statements are equivalent.

i) $R$ is Cohen–Macaulay.

ii) There exists a $C$-parameter ideal $q \subseteq m^{g(R)}$ of $R$ such that

\[ r(R) - \mathcal{N}(R) \geq \text{sg}(q, R). \]

iii) There exists a $C$-parameter ideal $q \subseteq m^{g(R)}$ of $R$ such that

\[ r(R) - \mathcal{N}(R) \geq \text{sg}(q : m, R). \]

**Proof.** i) $\Rightarrow$ ii). Since $R$ is Cohen-Macaulay, by Lemma 2.2 ii), we have

\[ r(R) - \mathcal{N}(R) = r(R) - r_d(R) = 0 \]

for all $C$-parameter ideals $q$ of $R$.

ii) $\Rightarrow$ iii). Since $R$ is to be regular if $R$ is unmixed and $e_0(m, R) = 1$ (see [16, Theorem 40.6]), we have $e_0(m, R) > 1$. Let $\underline{x} = x_1, x_2, \ldots, x_d$ be a $C$-system of parameters of $R$ such that $q = (x_1, x_2, \ldots, x_d)$ and put $J = q : m$. Now, we put $A = R/(x_1, \ldots, x_{d-2})$. Using similar arguments as in the proof of Theorem 3.3 (see (7)), one can show that

\[ \text{sg}(J, R) = \text{I}(qA, A) + e_1(JA, A) - \mathcal{N}(A) = h_0(A) - r_0(A) - r_1(A). \]

On the other hand, it follows from Lemma 3.5 in [20] and Lemma 2.8 that

\[ r(R) - \mathcal{N}(R) \leq r_2(A) - \mathcal{N}(A) = -2r_1(A) - r_0(A). \]

Since $r(R) - \mathcal{N}(R) \geq \text{sg}(J, R)$ by hypothesis, we have $h_0(A) = r_1(A) = 0$. Therefore, $A$ is Cohen-Macaulay. Hence $R$ is Cohen-Macaulay because of Lemma 2.9. The proof is complete.

**Corollary 3.6.** Assume that $R$ is a non-regular unmixed local ring with $d = \dim R \geq 2$. Then for all $C$-parameter ideals $q \subseteq m^{g(R)}$ of $R$, we have

\[ r(R) - \mathcal{N}(R) \leq \text{sg}(q : m, R) \leq \text{sg}(q, R). \]

Furthermore, each equality occurs if and only if $R$ is Cohen-Macaulay.

**Proof.** This is now immediately seen from Theorem 3.5 and Theorem 3.3.

The following consequence of Theorem 3.5 provides a characterization of Gorenstein rings.

**Corollary 3.7.** Assume that $R$ is a non-regular unmixed local ring of dimension $d = \dim R \geq 2$. Then the following statements are equivalent.

i) $R$ is Gorenstein.

ii) There exists a $C$-parameter ideal $q \subseteq m^{g(R)}$ of $R$ such that

\[ \text{sg}(q : m, R) \leq 1 - \mathcal{N}(R). \]

**Proof.** i) $\Rightarrow$ ii). Since $R$ is Gorenstein, we get by Theorem 3.5 that

\[ \text{sg}(q : m, R) = r(R) - \mathcal{N}(R) = 1 - \mathcal{N}(R) \]

for all $C$-parameter ideals $q$ of $R$.

ii) $\Rightarrow$ i). We have

\[ \text{sg}(q : m, R) \leq 1 - \mathcal{N}(R) \leq r(R) - \mathcal{N}(R). \]

Thus $R$ is Cohen-Macaulay because of Theorem 3.5. Moreover, we have $r(R) = 1$. Therefore, $R$ is Gorenstein, and this completes the proof.
4. ON THE SECTIONAL GENERA AND THE SECOND HILBERT COEFFICIENT

In this section, we will explore the relationship between the sectional genera and the second Hilbert coefficient.

Theorem 4.1. Assume that $R$ is a non-regular unmixed local ring of dimension $d = \dim R \geq 2$. Then the following statements are equivalent.

i) $R$ is Cohen–Macaulay.

ii) There exists a $C$-parameter ideal $q$ of $R$ such that

$$e_2(q : m, R) \geq e_2(q, R).$$

Proof. Note that we have $e_0(m, R) > 1$ since $R$ is to be regular if $R$ is unmixed and $e_0(m, R) = 1$ (see [16, Theorem 40.6]).

i) $\Rightarrow$ ii). Since $R$ is Cohen-Macaulay, by Lemma 2.2, we get that

$$e_2(q : m, R) = e_2(q, R) = 0.$$

ii) $\Rightarrow$ i). Let $\mathfrak{m} = x_1, x_2, \ldots, x_d$ be a $C$-system of parameters of $R$ such that $q = (x_1, x_2, \ldots, x_d)$ and put $J = q : \mathfrak{m}$. Note that $J R_i = q R_i : R_i$, $m R_i$, where $R_i = R/(x_1, \ldots, x_i)$ for all $i = 1, \ldots, d - 2$. Now, we put $A = R/(x_1, \ldots, x_{d-2})$. We get $e_2(q, R) = e_2(qA, A)$ and $e_2(J, R) = e_2(JA, A)$ by Lemma 2.1. Notice that by Lemma 2.8 ii), $A/H_m^0(A)$ is unmixed of dimension 2. Thus $A$ is generalized Cohen–Macaulay because of Lemma 2.9. Therefore, by Lemma 2.1 and 2.2, we have

$$e_2(q, R) = e_2(qA, A) - \ell((0) : R_{d-3} x_{d-2}) = h_0(A) - \ell((0) : R_{d-3} x_{d-2})$$

and

$$e_2(J, R) = e_2(JA, A) - \ell((0) : R_{d-3} x_{d-2}) = h_0(A) - r_1(A) - \ell((0) : R_{d-3} x_{d-2}).$$

Since $e_2(q : m, R) \geq e_2(q, R)$, we have $r_1(A) = 0$. Therefore, $A/H_m^0(A)$ is Cohen–Macaulay. Hence $R$ is Cohen-Macaulay by Lemma 2.9. The proof is complete.

Theorem 4.2. Assume that $R$ is a non-regular unmixed local ring of dimension $d = \dim R \geq 2$. Then the following statements are equivalent.

i) $R$ is Cohen–Macaulay.

ii) There exists a $C$-parameter ideal $q \subseteq m^{l(R)}$ of $R$ such that

$$e_2(q : m, R) \geq \text{sg}(q : m, R) + \mathcal{N}(R) - r(R).$$

iii) There exists a $C$-parameter ideal $q \subseteq m^{l(R)}$ of $R$ such that

$$e_2(q, R) \geq \text{sg}(q : m, R) + \mathcal{N}(R) - r(R).$$

iv) There exists a $C$-parameter ideal $q \subseteq m^{l(R)}$ of $R$ such that

$$e_2(q, R) \geq \text{sg}(q, R) + \mathcal{N}(R) - r(R).$$

Proof. i) $\Rightarrow$ ii) and i) $\Rightarrow$ iv) are immediately seen from the assumption that $R$ is Cohen-Macaulay and Lemma 2.2. Moreover ii) $\Rightarrow$ iii) and iv) $\Rightarrow$ iii) was established in Theorem 4.1. Now we show iii) $\Rightarrow$ i). Since $R$ is to be regular if $R$ is unmixed and $e_0(m, R) = 1$ (see [16, Theorem 40.6]), we have $e_0(m, R) > 1$.

Let $\mathfrak{m} = x_1, x_2, \ldots, x_d$ be a $C$-system of parameters of $R$ such that $q = (x_1, x_2, \ldots, x_d)$ and put $J = q : \mathfrak{m}$. Set $A = R/(x_1, \ldots, x_{d-2})$. Using similar arguments as in the proof of Theorem 3.3 and 4.1, one can show that $A$ is generalized Cohen–Macaulay. Furthermore, we also obtain that

$$\text{sg}(J, R) = I(qA, A) + e_1(JA, A) - \mathcal{N}(A),$$

and

$$e_2(q, R) = e_2(qA, A) - \ell((0) : R_{d-3} x_{d-2}) = h_0(A) - \ell((0) : R_{d-3} x_{d-2}).$$
On the other hand, by Lemma 3.5 in [20], we have $r_2(R) \ge r(R)$. Since $A$ is generalized Cohen–Macaulay, it follows from Lemma 2.2 that
\[ \text{sg}(J, R) + \mathcal{N}(R) - r(R) \ge h_0(A) + r_1(A). \]
Since $e_2(q, R) \ge \text{sg}(J, R) + \mathcal{N}(R) - r(R)$, we have $r_1(A) = 0$. Therefore $A/H^0_m(A)$ is Cohen–Macaulay. Hence $R$ is Cohen–Macaulay because of Lemma 2.8. The proof is complete. □

**Corollary 4.3.** Assume that $R$ is a non-regular unmixed local ring of dimension $d = \dim R \ge 2$. Then for all $C$-parameter ideal $q \subseteq m^q(R)$ of $R$, we have
\[ e_2(q : m, R) \le e_2(q, R) \le \text{sg}(q : m, R) + \mathcal{N}(R) - r(R) \le \text{sg}(q, R) + \mathcal{N}(R) - r(R). \]

**Proof.** This is now immediately seen from Theorem 4.2. □

**Theorem 4.4.** Assume that $R$ is a non-regular unmixed local ring of dimension $d = \dim R \ge 2$. Then the following statements are equivalent.

i) $R$ is Cohen–Macaulay.

ii) There exists a $C$-parameter ideal $q$ such that
\[ e_2(q : m, R) \ge \text{sg}(q, R). \]

**Proof.** $i) \Rightarrow ii)$. Since $R$ is Cohen–Macaulay, by Lemma 2.2 we have $e_2(q : m, R) = \text{sg}(q, R) = 0$.

$ii) \Rightarrow i)$. Using similar arguments as in the proof of Theorem 3.3, we obtain that $e_0(m, R) > 1$. Let $x = x_1, x_2, \ldots, x_d$ be a $C$-system of parameters of $R$ such that $q = (x_1, x_2, \ldots, x_d)$. We put $A = R/(x_1, \ldots, x_d)$. By Lemma 2.8 ii), $A/H^0_m(A)$ is unmixed with $\dim A = 2$. Thus $A$ is generalized Cohen–Macaulay because of Lemma 2.9. Moreover, it is follows from Lemma 2.1 that $e_0(mA, A) = e_0(m, R) > 1$. Note that $JR_i = qR_i :_{R_i} mR_i$, where $R_i = R/(x_1, \ldots, x_i)$ for all $i = 1, \ldots, d - 2$. By Lemma 2.1 and 2.2, we get that
\[ e_2(J, R) = e_2(JA, A) - \ell((0) :_{R_{d-3}} x_{d-2}) = h_0(R) - r_1(A) - \ell((0) :_{R_{d-3}} x_{d-2}) \]
and
\[ \text{sg}(q, R) = \text{sg}(qA, A) = h_0(R). \]

Since $e_2(J, R) \ge \text{sg}(q, R)$, we have $h_0(R) = r_1(A) = 0$. Therefore $A$ is Cohen–Macaulay. Hence $R$ is Cohen–Macaulay because of Lemma 2.8. The proof is complete. □

**Corollary 4.5.** Assume that $R$ is a non-regular unmixed local ring and $d = \dim R \ge 2$. Then for all $C$-parameter ideal $a$ of $R$, we have
\[ e_2(q : m, R) \le \text{sg}(q, R). \]

**Proof.** This is now immediately seen from Theorem 4.4. □

For the reader’s convenience, we recall the notion of quasi-Buchsbaum modules. We say that an $R$-module $M$ is said to be a quasi-Buchsbaum module if $mH^i_m(M) = 0$ for all $i < \dim M$. With this notation, we are now in a position to prove the second main theorem in this study.

**Proof of Theorem 1.3.** Since $R$ is to be regular if $R$ is unmixed and $e_0(m, R) = 1$, we have $e_0(m, R) > 1$.

$i) \Rightarrow ii)$. Let $q$ be a $C$-parameter ideal of $R$. By Lemma 3.1, we have
\[ \text{sg}(q : m, R) = I(q, R) + e_1(q : m, R) - \mathcal{N}(R). \]

Since $R$ is quasi-Buchsbaum, $r_i(R) = h_i(R)$ for all $i < d$. By Lemma 2.2, we get that
\[ \text{sg}(q : m, R) = \sum_{j=1}^{d-1} \binom{d-2}{j-1} h_j(R) = e_1(q, R). \]
Let \( \mathbf{z} = x_1, x_2, \ldots, x_d \) be a \( C \)-system of parameters of \( R \) such that \( q = (x_1, x_2, \ldots, x_d) \) and put \( J = q : m \). Now, we put \( A = R/(x_1, x_2, \ldots, x_{d-2}) \). Using similar arguments as in the proof of Theorem 3.3, one can show that

\[
\text{sg}(J, R) = I(qA, A) + e_1(JA, A) - N(A) = -r_1(A) + h_0(R) - r_0(A)
\]

and

\[
e_1(q, R) = e_1(qA, A) = -h_1(R).
\]

Since \( \text{sg}(J, R) = e_1(q, R) \), we have \( r_1(A) = h_1(R) \) and so \( h_0(R) = r_0(A) \). Thus \( A \) is quasi-Buchsbaum. It follows from a \( C \)-system of parameters forms a \( d \)-sequence of \( R \) and Theorem 3.6 in [24] that \( R \) is quasi-Buchsbaum, as required.

We close this paper with the following example, which shows that Theorem 4.1, 4.2, and 4.4 are not true without the assumption that \( R \) is unmixed.

**Example 4.6.** Let \( S = k[[X, Y, Z, W]] \) be the formal power series ring over a field \( k \). Put \( R = S/( [X, Y, Z] \cap [W]) \). Then

1. \( \dim R = 3 \) and \( R \) is not unmixed. Hence, \( R \) is not a Cohen-Macaulay ring.
2. We have\( e_2(q : m, R) = e_2(q, R) = \text{sg}(q : m, R) + N(R) - r(R) = \text{sg}(q, R) \)
   for all \( C \)-parameter ideals \( q \) in \( R \).

**Proof.** It is standard to check that \( \dim R = 3 \) and \( R \) is not unmixed. It follows that \( R \) is not a Cohen-Macaulay ring. We put \( A = S/(W) \) and \( B = S/(X, Y, Z) \). Note that \( A \) and \( B \) are Gorenstein rings.

Let \( q \) be a \( C \)-parameter ideal of \( R \) and put \( J = q : m \). Then \( J^n = q^n : m \) for all \( n \geq 0 \) (see [20, Lemma 2.6]). From the exact sequence \( 0 \to B \to R \to A \to 0 \) and \( A \) is Cohen-Macaulay, we observe that

\[
0 \to B/q^{n+1}B \to R/q^{n+1} \to A/q^{n+1}A \to 0.
\]

By applying the functor \( \text{Hom}_R(R/m, -) \) to the above exact sequence, by Lemma 2.6 in [20], we obtain the following:

\[
0 \to [q^{n+1} :_B m]/q^{n+1}B \to [q^{n+1} :_R m]/q^{n+1} \to [q^{n+1} :_A m]/q^{n+1}A \to 0
\]

Therefore, we have

\[
\ell_R(R/q^{n+1}) = \ell_R(A/q^{n+1}A) + \ell_R(B/q^{n+1}B)
\]

\[
= \ell_R(A/qA) \left( \frac{n+3}{3} \right) + \ell_R(B/qB) \left( \frac{n+1}{1} \right)
\]

and

\[
\ell_R([q^{n+1} :_R m]/q^{n+1}) = \ell_R([q^{n+1} :_A m]/q^{n+1}) + \ell_R([q^{n+1} :_B m]/q^{n+1})
\]

\[
= \binom{n+1}{2} + 1
\]

for all integers \( n \geq 0 \). Since \( \ell_R(R/I^{n+1}) = \ell_R(R/q^{n+1}) - \ell_R([q^{n+1} :_R m]/q^{n+1}) \), we have \( e_0(q, R) = e_0(q : m, R) = \ell_R(A/qA), e_1(q, R) = 0, e_1(q : m, R) = 1 \) and \( e_2(q, R) = e_2(q : m, R) = \ell_R(B/qB) \). Thus we have

\[
\text{sg}(q, R) = \ell(R/q) - e_0(q, R) + e_1(q, R) = \ell_R(B/qB).
\]

On the other hand, it is easily observed from \( 0 \to B \to R \to A \to 0 \) that \( r_0(R) = r_2(R) = 0 \) and \( r_1(R) = r_3(R) = 1 \). Since \( R \) is sequentially Cohen-Macaulay, it follows from Theorem 1.1 in [27] that

\[
N(R) = \sum_{i=0}^3 r_i(R) = 2.
\]

Hence, by Lemma 3.1, we have

\[
\text{sg}(q : m, R) = \ell(R/q) - e_0(q, R) + e_1(q : m, R) - N(R) = \ell_R(B/qB) - 1.
\]
Consequently, we obtain that
\[ e_2(q : m, R) = e_2(q, R) = sg(q : m, R) + \mathcal{N}(R) - r(R) = sg(q, R) = \ell_R(B/qB). \]

The proof is complete. \qed

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