An almost sure invariance principle for the range of planar random walks

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Abstract

For a symmetric random walk in $\mathbb{Z}^2$ with $2 + \delta$ moments, we represent $|R(n)|$, the cardinality of the range, in terms of an expansion involving the renormalized intersection local times of a Brownian motion. We show that for each $k \geq 1$

$$(\log n)^k \left[ \frac{1}{n} |R(n)| + \sum_{j=1}^{k} (-1)^j \left( \frac{1}{2\pi} \log n + c_X \right)^{-j} \gamma_{j,n} \right] \to 0, \quad \text{a.s.}$$

where $W_t$ is a Brownian motion, $W^{(n)}_t = W_{nt}/\sqrt{n}$, $\gamma_{j,n}$ is the renormalized intersection local time at time 1 for $W^{(n)}$, and $c_X$ is a constant depending on the distribution of the random walk.

1 Introduction

Let $S_n = X_1 + \cdots + X_n$ be a random walk in $\mathbb{Z}^2$, where $X_1, X_2, \ldots$ are symmetric i.i.d. vectors in $\mathbb{Z}^2$. We assume that the $X_i$ have $2 + \delta$ moments for some $\delta > 0$ and covariance matrix equal to the identity. We assume further that the random walk $S_n$ is strongly aperiodic in the sense of Spitzer, [19], p. 42. The range $R(n)$ of the random walk $S_n$ is the set of sites visited by the walk up to step $n$:

$$R(n) = \{S_1, \ldots, S_n\}.$$ 

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As usual, $|\mathcal{R}(n)|$ denotes the cardinality of the range up to step $n$.

Dvoretzky and Erdos, [5], show that

$$\lim_{n \to \infty} \log n \frac{|\mathcal{R}(n)|}{n} = 2\pi, \quad \text{a.s.}$$

(1.2)

Le Gall [10] has obtained a central limit theorem for the second order fluctuations of $|\mathcal{R}(n)|$:

$$(\log n)^2 \left( \frac{|\mathcal{R}(n)| - E(|\mathcal{R}(n)|)}{n} \right) \xrightarrow{d} - (2\pi)^2 \gamma_2(1)$$

(1.3)

where $\xrightarrow{d}$ denotes convergence in law and $\gamma_2(t)$ is the second order renormalized self-intersection local time for planar Brownian motion. See also Le Gall-Rosen [12].

In this paper we prove an a.s. asymptotic expansion for $|\mathcal{R}(n)|$ to any order of accuracy. In order to state our result we first introduce some notation. If $\{W_t; t \geq 0\}$ is a planar Brownian motion, we define the $j$'th order renormalized intersection local time for $\{W_t; t \geq 0\}$ as follows. $\gamma_1(t) = t$, $\alpha_{1,\epsilon}(t) = t$ and for $k \geq 2$

$$\alpha_{k,\epsilon}(t) = \int_{0 \leq t_1 \leq \cdots \leq t_k < t} p_t(W_{t_i} - W_{t_{i-1}}) dt_1 \cdots dt_k,$$

(1.4)

$$\gamma_k(t) = \lim_{\epsilon \to 0} \sum_{l=1}^k \binom{k-1}{l-1} (-u_\epsilon)^{k-l} \alpha_{l,\epsilon}(t),$$

(1.5)

where $p_t(x)$ is the density for $W_t$ and

$$u_\epsilon = \int_0^\infty e^{-t} p_{t+\epsilon}(0) dt.$$

Renormalized self-intersection local time was originally studied by Varadhan [20] for its role in quantum field theory. In Rosen [18] we show that $\gamma_k(t)$ can be characterized as the continuous process of zero quadratic variation in the decomposition of a natural Dirichlet process. For further work on renormalized self-intersection local times see Dynkin [7], Le Gall [11], Bass and Khoshnevisan [2], Rosen [16] and Marcus and Rosen [15].

To motivate our result define the Wiener sausage of radius $\epsilon$ as

$$\mathcal{W}_\epsilon(0, t) = \{ x \in \mathbb{R}^2 \mid \inf_{0 \leq s \leq t} |x - W_s| \leq \epsilon \}.$$  

(1.6)
Letting $m(W_t(0, t))$ denote the area of the Wiener sausage of radius $\epsilon$, Le Gall [9] shows that for each $k \geq 1$

\[
(\log n)^k \left[ m(W_{n-1/2}(0, 1)) + \sum_{j=1}^{k} (-1)^j \left( \frac{1}{2\pi} \log n + c \right) \gamma_j(1) \right] \rightarrow 0, \quad a.s.
\]
as $n \rightarrow \infty$ where $c$ is a finite constant. Using the heuristic which associates $\{ S_{[nt]/\sqrt{n}} : 0 \leq t \leq 1 \} \subseteq n^{-1/2}Z^2 \subseteq R^2$ with the Brownian motion $\{ W_t ; 0 \leq t \leq 1 \}$, one would expect (note that space is scaled by $n^{-1/2}$) that \( \frac{1}{n} |\mathcal{R}(n)| \) will be ‘close’ to $m(W_{n-1/2}(0, 1))$.

Our main result is the following theorem.

**Theorem 1** Let $S_n = X_1 + \cdots + X_n$ be a symmetric, strongly aperiodic random walk in $Z^2$ with covariance matrix equal to the identity and with $2+\delta$ moments for some $\delta > 0$. On a suitable probability space we can construct $\{ S_n ; n \geq 1 \}$ and a planar Brownian motion $\{ W_t ; t \geq 0 \}$ such that for each $k \geq 1$

\[
(\log n)^k \left[ \frac{1}{n} |\mathcal{R}(n)| + \sum_{j=1}^{k} (-1)^j \left( \frac{1}{2\pi} \log n + c_X \right) \gamma_j(1, W(n)) \right] \rightarrow 0, \quad a.s.
\]

(1.7)

where the random variables $\gamma_1(1, W(n)), \gamma_2(1, W(n)), \ldots$ are the renormalized self-intersection local times [1.2] with $t = 1$ for the Brownian motion $\{ W_t^{(n)} = W_{nt}/\sqrt{n} ; t \geq 0 \}$,

\[
c_X = \frac{1}{2\pi} \log(\pi^2/2) + \frac{1}{(2\pi)^2} \int_{[-\pi, \pi]^2} \frac{\phi(p) - 1 + |p|^2/2}{\left(1 - \phi(p)\right)|p|^2/2} \, dp
\]

is a finite constant, and $\phi(p) = E(e^{ip \cdot X_1})$ denotes the characteristic function of $X_1$.

Note that the presence of the constant $c_X$ shows that the heuristic mentioned before the statement of Theorem 1 does not completely capture the fine structure of $|\mathcal{R}(n)|$. (This can already be observed on the level of (1.3), see [12] (6.9)).

We begin our proof in Section 2 where we introduce renormalized intersection local times $\Gamma_{k,\lambda}(n)$ for our random walk. Let $\zeta$ be an independent exponential random variable of mean 1, and set $\zeta_\lambda = n$ when $n-1)\lambda < \zeta \leq \lambda n$.  

3
Letting \(| R(\zeta_\lambda) |\) denote the cardinality of the range of our random walk killed at step \(\zeta_\lambda\), we derive an \(L^2\) asymptotic expansion for \(| R(\zeta_\lambda) |\) in terms of the \(\Gamma_{k,\lambda}(\zeta_\lambda)\) as \(\lambda \to 0\). In Sections 3-5, on a suitable probability space, we construct \(\{S_n; n \geq 1\}\) and a planar Brownian motion \(\{W_t; t \geq 0\}\) and show that in the above \(L^2\) asymptotic expansion for \(| R(\zeta_\lambda) |\) we can replace \(\lambda \Gamma_{k,\lambda}(\zeta_\lambda)\) by \(\gamma_k(\zeta, W^{(\lambda^{-1})})\), the renormalized intersection local times for the planar Brownian motion \(\{W^{(\lambda^{-1})}_t = W_{\lambda^{-1}t}/\sqrt{\lambda^{-1}}; t \geq 0\}\). After some preliminaries on renormalized intersection local times for Brownian motion in Section 6, we show in Section 7 how our \(L^2\) asymptotic expansion for \(| R(\zeta_\lambda) |\) leads to an a.s. asymptotic expansion. The proof of Theorem 1 is completed in Section 8 by showing how to replace the random time \(\zeta_\lambda\) by fixed time. Appendix A derives some estimates used in this paper. Our methods obviously owe a great deal to Le Gall [9].

We would like to thank Uwe Einmahl and David Mason for their help in Section 3 which describes strong approximations in \(L^2\).

## 2 Range and random walk intersection local times

We first define the non-renormalized random walk intersection local times for \(k \geq 2\) by

\[
I_k(n) = \sum_{0 \leq i_1 \leq \ldots \leq i_k < n} \delta(S_{i_1}, S_{i_2}) \cdots \delta(S_{i_{k-1}}, S_{i_k})
\]

\[
= \sum_{x \in \mathbb{Z}^2} \sum_{0 \leq i_1 \leq \ldots \leq i_k < n} \prod_{j=1}^k \delta(S_{i_j}, x)
\]

where

\[
\delta(i, j) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}
\]

is the usual Kronecker delta function. We set \(I_1(n) = n\) so that also \(I_1(n) = \sum_{x \in \mathbb{Z}^2} \sum_{0 < i < n} \delta(S_i, x)\). (One might also take as a definition of the intersection local time the quantity \(\sum_{0 < i_1 < \ldots < i_k < n} \delta(S_{i_1}, S_{i_2}) \cdots \delta(S_{i_{k-1}}, S_{i_k})\). The definition in (2.1) is more convenient for our purposes, and we see by (2.6) that either definition leads to the same value for \(\Gamma_{k,\lambda}(n)\).
Let $q_n(x)$ be the transition function for $S_n$ and let

$$G_\lambda(x) = \sum_{j=0}^{\infty} e^{-j\lambda} q_j(x).$$

We will show in Lemma 8 below that

$$g_\lambda := G_\lambda(0) = \frac{1}{2\pi} \log(1/\lambda) + c_X + O(\lambda^\delta \log(1/\lambda)) \quad \text{as} \quad \lambda \to 0,$$

where $c_X$ is defined in (1.8). We show in (A.18) that for any $q > 1$

$$\sum_{x \in \mathbb{Z}^2} (G_\lambda(x))^q = O(\lambda^{-1}) \quad \text{as} \quad \lambda \to 0.$$  

We now define the renormalized random walk intersection local times

$$\Gamma_k,\lambda(n) = \sum_{0 \leq i_1 \leq \ldots \leq i_k < n} \{\delta(S_{i_1}, S_{i_2}) - g_\lambda \delta(i_1, i_2)\} \cdots \{\delta(S_{i_{k-1}}, S_{i_k}) - g_\lambda \delta(i_{k-1}, i_k)\}$$

$$= \sum_{j=1}^{k} \binom{k-1}{j-1} (-1)^{k-j} g_\lambda^{-j} I_j(n).$$

Let $\zeta$ be an independent exponential random variable of mean 1, and set $\zeta_\lambda = n$ when $(n-1)\lambda < \zeta \leq n\lambda$. $\zeta_\lambda$ is then a geometric random variable with $P(\zeta_\lambda > n) = e^{-\lambda n}$. Note that $\zeta_{1/j} = n$ if $(n-1)/j < \zeta \leq n/j$. By $\mathcal{R}(\zeta_\lambda)$ we mean the range of our random walk killed at step $\zeta_\lambda$.

In this section we prove the following lemma.

**Lemma 1** For each $k \geq 1$

$$\lim_{\lambda \to 0} \lambda g_\lambda^k \left( |\mathcal{R}(\zeta_\lambda)| - \sum_{j=1}^{k} (-1)^{j-1} g_\lambda^{-j} \Gamma_{j,\lambda}(\zeta_\lambda) \right) = 0 \quad \text{in} \quad L^2.$$  

**Proof of Lemma** Define

$$T_x = \min\{n \geq 0 : S_n = x\},$$

5
the first hitting time to \( x \). We will use the fact that
\[
P(T_x < \zeta) = \frac{G_x(x)}{G_x(0)}, \tag{2.8}
\]
which follows from the strong Markov property:
\[
G_x(x) = E \left( \sum_{j=0}^{\infty} 1_x(S_j)1\{j<\zeta\} \right) = E \left( \sum_{j=T_x}^{\infty} 1_x(S_j)1\{j<\zeta\} \right) = E \left( 1\{T_x<\zeta\}, \sum_{j=0}^{\infty} 1_x(S_j)1\{j<\zeta\} \bigcirc \theta_{T_x} \right) = P(T_x < \zeta)G_x(0). \tag{2.9}
\]

To prove our lemma we square the expression inside the parentheses in (2.7) and then take expectations. We first show that
\[
E \left( |R(\zeta)|^2 \right) = 2 \sum_{j=2}^{2k} (-1)^j \lambda^{-j} \sum_{x,y \in \mathbb{Z}^2} G_x(x)(G_x(x-y))^{j-1} + O(\lambda^{-2k+1}). \tag{2.10}
\]

To this end we first note that
\[
|R(\zeta)| = \sum_{x \in \mathbb{Z}^2} 1\{T_x<\zeta\} \tag{2.11}
\]
so that
\[
E \left( |R(\zeta)|^2 \right) = \sum_{x,y \in \mathbb{Z}^2} P(T_x, T_y < \zeta) \tag{2.12}
\]
\[
= \sum_{x \in \mathbb{Z}^2} P(T_x < \zeta) + 2 \sum_{x \neq y \in \mathbb{Z}^2} P(T_x < T_y < \zeta). \]

Using (2.8) we have that
\[
\sum_{x \in \mathbb{Z}^2} P(T_x < \zeta) = \sum_{x \in \mathbb{Z}^2} \frac{G_x(x)}{g_x} = \frac{1}{(1 - e^{-\lambda})g_x} = O(\lambda^{-1}g_x^{-1}). \tag{2.13}
\]
To evaluate $\sum_{x \neq y \in \mathbb{Z}^2} P(T_x < T_y < \zeta_\lambda)$ we first introduce some notation. For any $u \neq v \in \mathbb{Z}^2$ define inductively

\begin{align*}
A_{1,u,v} &= T_u \\
A_{2,u,v} &= A_{1,u,v} + T_v \circ \theta_{A_{1,u,v}} \\
A_{3,u,v} &= A_{2,u,v} + T_u \circ \theta_{A_{2,u,v}} \\
A_{2k,u,v} &= A_{2k-1,u,v} + T_v \circ \theta_{A_{2k-1,u,v}} \\
A_{2k+1,u,v} &= A_{2k,u,v} + T_u \circ \theta_{A_{2k,u,v}}.
\end{align*}

We observe that for any $x \neq y$

\begin{align*}
P^z(T_x < T_y < \zeta_\lambda) &= P^z(A_{1,x,y} < A_{2,x,y} < \zeta_\lambda) - P^z(T_y < A_{1,x,y} < A_{2,x,y} < \zeta_\lambda) \\
&= P^z(A_{2,x,y} < \zeta_\lambda) - P^z(T_y < A_{1,x,y} < A_{2,x,y} < \zeta_\lambda) \\
&= P^z(A_{3,x,y} < \zeta_\lambda) - P^z(T_x < A_{1,y,x}; A_{2,y,x} < \zeta_\lambda).
\end{align*}

Proceeding inductively we find that

\begin{align*}
P^z(T_x < T_y < \zeta_\lambda) &= \sum_{j=1}^{k} P^z(A_{2j,x,y} < \zeta_\lambda) - \sum_{j=1}^{k} P^z(A_{2j+1,y,x} < \zeta_\lambda) \\
&+ P^z(T_x < A_{1,y,x}; A_{2k+1,y,x} < \zeta_\lambda). \\
\end{align*}

Using (2.8) and the strong Markov property we see that

\begin{align*}
P(T_x < T_y < \zeta_\lambda) &= \sum_{j=1}^{k} \lambda^{2j-1} G_\lambda(x)(G_\lambda(y - x))^{2j-1} \\
&- \sum_{j=1}^{k} \lambda^{2j} G_\lambda(y)(G_\lambda(x - y))^{2j} \\
&+ P(T_x < A_{1,y,x}; A_{2k+1,y,x} < \zeta_\lambda). \\
\end{align*}

and that

\begin{align*}
P(T_x < A_{1,y,x}; A_{2k+1,y,x} < \zeta_\lambda) \leq P(A_{2k+2} < \zeta_\lambda) = g_\lambda^{-2k+2} G_\lambda(x)(G_\lambda(y - x))^{2k+1}.
\end{align*}
(2.20) \[ E(I_n(\zeta_\lambda)I_m(\zeta_\lambda)) \]
\[
= \sum_{x,y \in \mathbb{Z}^2} \sum_{\pi} E \left( \sum_{0 \leq i_1 \leq \ldots \leq i_n < \zeta_\lambda} \prod_{j=1}^n \delta(S_{i_j}, x) \sum_{0 \leq l_1 \leq \ldots \leq l_m < \zeta_\lambda} \prod_{k=1}^m \delta(S_{l_k}, x) \right) \]
\[
= \sum_{x,y \in \mathbb{Z}^2} \sum_{\pi} E \left( \prod_{j=1}^{n+m} \delta(S_{i_j}, \pi(j)) \right) e^{-\lambda n+m} \]

where the inner sum runs over all maps \( \pi : \{1, 2, \ldots, n + m\} \mapsto \{x, y\} \) such that \(|\pi^{-1}(x)| = m, |\pi^{-1}(y)| = n\). Thus

(2.21) \[ E(I_n(\zeta_\lambda)I_m(\zeta_\lambda)) \]
\[
= \sum_{x,y \in \mathbb{Z}^2} \sum_{\pi} E \left( \prod_{j=1}^{n+m} \delta(S_{i_j}, \pi(j)) \right) \]
\[
= \sum_{x,y \in \mathbb{Z}^2} \sum_{\pi} \prod_{j=1}^{n+m} G_\lambda(\pi(j) - \pi(j-1)) \]

where \( \pi(0) = 0 \). By (2.20) and (2.21) the sum over \( x = y \) is \( O(\lambda^{-1} g^{n+m}_\lambda) \).

Consider then \( x \neq y \). When we look at the definition (2.9) of \( \Gamma_{k,\lambda}(n) \) we see that the effect of replacing \( I_n(\zeta_\lambda)I_m(\zeta_\lambda) \) in (2.20) by \( \Gamma_{n,\lambda}(\zeta_\lambda)\Gamma_{m,\lambda}(\zeta_\lambda) \) is to eliminate all maps \( \pi \) in which \( \pi(j) = \pi(j-1) \) for some \( j \). Thus, up to an error which is \( O(\lambda^{-1} g^{n+m}_\lambda) \), (which comes from \( x = y \)), we have

(2.22) \[ E(\Gamma_{n,\lambda}(\zeta_\lambda)\Gamma_{m,\lambda}(\zeta_\lambda)) = \begin{cases} 0 & \text{if } m = n \\ 2 \sum_{x,y \in \mathbb{Z}^2} G_\lambda(x)(G_\lambda(y-x))^{2n-1} & \text{if } m = n \pm 1 \end{cases} \]

Consequently up to errors which are \( O(\lambda^{-1} g^{2k}_\lambda) \)

(2.23) \[ E \left( \left\{ \sum_{j=1}^k (-1)^j g^{-j}_{\lambda} \Gamma_{j,\lambda}(\zeta_\lambda) \right\}^2 \right) \]
\[
= \sum_{n,m=1}^k (-1)^{n+m} g^{-n+m}_{\lambda} E(\Gamma_{n,\lambda}(\zeta_\lambda)\Gamma_{m,\lambda}(\zeta_\lambda)) \]
\[
= 2 \sum_{n=1}^k g^{-2n}_{\lambda} \sum_{x,y \in \mathbb{Z}^2} G_\lambda(x)(G_\lambda(y-x))^{2n-1} \]

8
\[
- 2 \sum_{n=2}^{k} \sum_{x,y \in \mathbb{Z}^2} G_{\lambda}(x)(G_{\lambda}(y-x))^{2n-2}
\]

\[
= 2 \sum_{j=2}^{2k} (-1)^{j} \sum_{x,y \in \mathbb{Z}^2} G_{\lambda}(x)(G_{\lambda}(x-y))^{j-1}
\]

To handle the cross product terms we define the random measure on \( \mathbb{Z}^n_+ \)

\[
(2.24) \quad \Lambda_{n,y}(B) = \sum_{\{0 \leq i_1 \leq \cdots \leq i_n < \zeta_{\lambda}\} \cap B} \prod_{j=1}^{n} \delta(S_{i_j}, y).
\]

Using the notation \( i_0 = 0, i_{n+1} = \zeta_{\lambda} \) we have

\[
E(| \mathcal{R}(\zeta_{\lambda}) | \mathcal{I}_n(\zeta_{\lambda})) = \sum_{x,y \in \mathbb{Z}^2} \sum_{0 \leq i_1 \leq \cdots \leq i_n \leq \zeta_{\lambda}} 1_{(x < \zeta_{\lambda})} \prod_{j=1}^{n} \delta(S_{i_j}, y)
\]

\[
= \sum_{x,y \in \mathbb{Z}^2} \sum_{j=0}^{n} E(\Lambda_{n,y}(\{i_j \leq T_x < i_{j+1}\})).
\]

As above we have that

\[
(2.26) \quad \Lambda_{n,y}(\{i_j \leq T_x < i_{j+1}\})
\]

\[
= \Lambda_{n,y}(\{i_j + T_x \circ \theta_{i_j} < i_{j+1}\})
\]

\[
- \sum_{l=0}^{j-1} \Lambda_{n,y}(\{i_l \leq T_x < i_{l+1}; i_j + T_x \circ \theta_{i_j} < i_{j+1}\})
\]

and inductively we find that

\[
(2.27) \quad \sum_{x \neq y \in \mathbb{Z}^2} \sum_{j=0}^{n} E(\Lambda_{n,y}(\{i_j \leq T_x < i_{j+1}\}))
\]

\[
= \sum_{x \neq y \in \mathbb{Z}^2} \sum_{m=1}^{n+1} (-1)^{m-1} \sum_{|A|=m} E\left(\Lambda_{n,y}(\bigcap_{j \in A} \{i_j + T_x \circ \theta_{i_j} < i_{j+1}\})\right)
\]

where the inner sum runs over all nonempty \( A \subseteq \{0, 1, \ldots, n\} \). Using (2.8) and the Markov property we see that

\[
(2.28) \quad \sum_{x \neq y \in \mathbb{Z}^2} \sum_{j=0}^{n} E(\Lambda_{n,y}(\{i_j \leq T_x < i_{j+1}\}))
\]

\[
= \sum_{x \neq y \in \mathbb{Z}^2} \sum_{m=1}^{n+1} (-1)^{m-1} \sum_{|A|=m} g_{\lambda}^{-m} \prod_{j=1}^{n+m} G_{\lambda}(\sigma_{A}(j) - \sigma_{A}(j-1))
\]
where $\sigma_A(0) = 0$ and $\sigma_A(j)$ is the $j$'th element in the ordered set obtained by taking $n$ $y$'s and inserting, for each $l \in A$, an $x$ between the $l$'th and $(l+1)$'st $y$. Estimating the contribution from $x = y$ we find that

$$E(|R(\zeta_\lambda)|I_n(\zeta_\lambda)) = \sum_{x,y \in \mathbb{Z}^2} (\prod_{j=1}^{n+m} G_\lambda(\sigma_A(j) - \sigma_A(j-1)))$$

Once again we see that the effect of replacing $I_n(\zeta_\lambda)$ in (2.29) by $\Gamma_{n,\lambda}(\zeta_\lambda)$ is to eliminate all sets $A$ such that $\sigma_A(j) = \sigma_A(j-1)$ for some $j$. Thus we have

$$E(|R(\zeta_\lambda)|\Gamma_{n,\lambda}(\zeta_\lambda)) = 2((-1)^{n-1} + (1)^n + (1)^{n+1}) \sum_{x,y \in \mathbb{Z}^2} G_\lambda(x) G_\lambda(x-y)$$

Consequently

$$E \left( |R(\zeta_\lambda)| \sum_{n=1}^{k} (-1)^{n-1} g_\lambda^{-n} \Gamma_{n,\lambda}(\zeta_\lambda) \right)$$

where $\sigma_A(0) = 0$ and $\sigma_A(j)$ is the $j$'th element in the ordered set obtained by taking $n$ $y$'s and inserting, for each $l \in A$, an $x$ between the $l$'th and $(l+1)$'st $y$. Estimating the contribution from $x = y$ we find that
Our lemma then follows from (2.10), (2.23) and (2.31).

3 Strong approximation in $L^2$

Lemma 2 Let $X$ be a $\mathbb{R}^2$ valued random vector with mean zero and covariance matrix equal to the identity $I$. Assume that for some $2 < p < 4$, $E|X|^p < \infty$. Given $n \geq 1$ one can construct on a suitable probability space two sequences of independent random vectors $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, where each $X_i \overset{d}{=} X$ and the $Y_i$’s are standard normal random vectors such that

$$
\|\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_i - Y_i) \right| \|_2 = O \left( n^{1/p} \right).
$$

Proof of Lemma 2: By equation (3.3) of [8] we can find a constant $c_1$ and such $X_i$ and $Y_i$ so that

$$
P \left\{ \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_i - Y_i) \right| > x \right\} \leq c_1 n x^{-p} E|X|^p
$$

Write $Z_n$ for $\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_i - Y_i) \right|$. Since probabilities are bounded by 1, we have for any $a > 0$

$$
\|Z_n\|_2^2 = \int_0^\infty x P(Z_n > x) \, dx \\
\leq \int_0^a x \, dx + c_1 n \int_a^\infty x^{-p+1} \, dx \\
\leq c_2 (a^2 + na^{2-p}).
$$

If we set $a = n^{1/p}$ and take square roots of both sides, we have our result.

Using the lemma we can readily construct two i.i.d. sequences $\{X_i\}_{i \geq 1}$ and $\{Y_i\}_{i \geq 1}$, where the $X_i$ are equal in law to $X$ and the $Y_i$ are standard normal, such that for some constant $C > 0$ and any $m \geq 0$,

$$
\left\| \max_{2^m \leq k < 2^{m+1}} \left| \sum_{i=2^m}^{k} (X_i - Y_i) \right| \right\|_2 \leq C (2^m)^{1/p}.
$$
We see then that for any $2^m \leq \lfloor nt \rfloor < 2^{m+1}$,

$$\left\| \sum_{i=1}^{\lfloor nt \rfloor} (X_i - Y_i) \right\|_2 \leq \sum_{j=0}^{m} \max_{2^m \leq k < 2^{m+1}} \left\| \sum_{i=2^m}^{k} (X_i - Y_i) \right\|_2,$$

which for some $D > 0$ is

$$\leq \sum_{j=0}^{m} C \left( 2^j \right)^{1/p} \leq D \left( nt \right)^{1/p}.$$

Now choose a Brownian motion $W$ such that for $m \geq 1$,

$$W(m) = \sum_{j=1}^{m} Y_j.$$

Noting that

$$\left\| W(\lfloor mt \rfloor) - W(mt) \right\|_2 \leq \sup_{0 \leq s \leq 1} W(s) \right\|_2 := M,$$

we see that for any $t > 0$

$$\left\| \frac{S(\lfloor mt \rfloor) - W(mt)}{\sqrt{m}} \right\|_2 \leq D \left( mt \right)^{1/p} m^{-1/2} + M m^{-1/2}$$

$$= O \left( m^{(1/p) - (1/2)} \left( t^{1/p} + 1 \right) \right),$$

(3.1)

where

$$S(\lfloor mt \rfloor) = \sum_{i \leq \lfloor mt \rfloor} X_i.$$

4 Spatial Hölder continuity for renormalized intersection local times

If $\{W_t; t \geq 0\}$ is a planar Brownian motion, set $\bar{\alpha}_{1,t}(t) = t$ and for $k \geq 2$ and $x = (x_2, \ldots, x_k) \in (R^2)^{k-1}$ let

$$\bar{\alpha}_{k,t}(t, x) = \int_{0 \leq t_1 \leq \cdots \leq t_k < t} \prod_{i=2}^{k} p_\varepsilon(W_{t_i} - W_{t_{i-1}} - x_i) dt_1 \cdots dt_k.$$
When \( x_i \neq 0 \) for all \( i \) and \( \zeta \) is an independent exponential random variable with mean 1, the limit

\[
\alpha_k(\zeta, x) = \lim_{\epsilon \to 0} \alpha_{k,\epsilon}(\zeta, x)
\]
exists. When \( x_i \neq 0 \) for all \( i \) set

\[
\gamma_k(\zeta, x) = \sum_{A \subseteq \{2, \ldots, k\}} (-1)^{|A|} \left( \prod_{i \in A} u^1(x_i) \right) \alpha_{k-|A|}(\zeta, x_{A^c}),
\]
where

\[
u^1(y) = \int_0^\infty e^{-t} p_t(y) \, dt,
\]
p\(_t\)(x) is the density for \( W_t \), and \( x_{A^c} = (x_1, \ldots, x_{i_k-|A|}) \) with \( i_1 < i_2 < \cdots < i_k-|A| \) and \( i_j \in \{2, \ldots, k\} - A \) for each \( j \), that is, the vector \((x_2, \ldots, x_k)\) with all terms that have indices in \( A \) deleted. In [16] it is shown that for some \( \delta > 0 \) and all \( m \)

\[
E \left( \left| \gamma_k(\zeta, x) - \gamma_k(\zeta, y) \right|^m \right) \leq C|x-y|^{\delta m}.
\]

As before, set \( I_1(n) = n \) and for \( k \geq 2 \) and \( x = (x_2, \ldots, x_k) \in \mathbb{Z}^2 \) let

\[
\tilde{T}_k(n, x) = \sum_{0 \leq i_1 \leq \cdots \leq i_k < n} \delta(S_{i_2} - S_{i_1} - x_2) \cdots \delta(S_{i_k} - S_{i_{k-1}} - x_k).
\]
and for \( x \in \sqrt{\lambda} \mathbb{Z}^2 \) let

\[
\Gamma_{k,\lambda}(n, x) = \sum_{A \subseteq \{2, \ldots, k\}} (-1)^{|A|} \prod_{i \in A} G_\lambda(x_i/\sqrt{\lambda}) I_{k-|A|}(n, x_{A^c}/\sqrt{\lambda}).
\]
Note that \( \Gamma_{k,\lambda}(n) = \Gamma_{k,\lambda}(n, 0) \).

**Lemma 3** For any \( j \geq 1 \) we can find some \( \rho, \bar{\delta} > 0 \) such that uniformly in \( \lambda > 0 \)

\[
\sup_{|y| \leq \lambda^\rho} E \left( \left| \lambda \Gamma_{j,\lambda}(\zeta, y) - \lambda \gamma(\zeta, y) \right|^2 \right) \leq C \lambda^{\bar{\delta}}.
\]

**Proof of Lemma 3**: We begin by considering

\[
E \left( \tilde{T}_{k,\lambda}(\zeta, x^1) \Gamma_{k,\lambda}(\zeta, x^2) \right).
\]
for \( x^i \in (\mathbb{Z}^2)^{k-1} \).

If \( h \) is a function which depends on the variable \( x \), let

\[
\mathcal{D}_x h = h(x) - h(0).
\]

Let \( \mathcal{S} \) be the set of all maps \( s : \{1, 2, \ldots, 2k\} \mapsto \{1, 2\} \) with \( |s^{-1}(j)| = k \), \( 1 \leq j \leq 2 \), and let \( B_s = \{ i \mid s(i) = s(i-1) \} \) and \( c(i) = |\{ j \leq i \mid s(j) = s(i) \}| \).

Using the Markov property as in Lemma 5 of [16] we can then show that

\[
E \left( \Gamma_{k,\lambda}(\zeta_\lambda, x^1) \Gamma_{k,\lambda}(\zeta_\lambda, x^2) \right) = \sum_{s \in \mathcal{S}} \left( \prod_{i \in B_s} G_\lambda(x^i_{s(i)}/\sqrt{\lambda}) \right) \sum_{z_i \in \mathbb{Z}^2} \left( \prod_{i \in B_s} \mathcal{D}_{x^i_{s(i)}/\sqrt{\lambda}} \right) \prod_{i \in B_s^c} G_\lambda \left( z_{s(i)} + \sum_{j=2}^{c(i)} x^i_j / \sqrt{\lambda} - (z_{s(i-1)} + \sum_{j=2}^{c(i-1)} x^i_{s(i-1)} / \sqrt{\lambda}) \right).
\]

Fix \( s \in \mathcal{S} \) and note then that the corresponding summand will be 0 unless \( x^i_{s(i)} \neq 0 \) for all \( i \in B_s \). Note that by definition of \( B_s^c \) we necessarily have that the last line in (4.10) is of the form

\[
G_\lambda(z_1) \prod_{i \in B_s^c} G_\lambda(z_1 - z_2 + a_i)
\]

where the \( a_i \) are linear combinations of \( x^1, x^2 \) but do not involve \( z_1, z_2 \). Then we observe that the effect of applying each \( \mathcal{D}_{x^i_{s(i)}/\sqrt{\lambda}} \) to the product on the last line of (4.10) is to generate a sum of several terms in each of which we have one factor of the form \( \mathcal{D}_{x^i_{s(i)}/\sqrt{\lambda}} G_\lambda \). Thus schematically we can write the contribution of such a term as

\[
\left( \prod_{i \in B_s} G_\lambda(x^i_{s(i)}/\sqrt{\lambda}) \right) \sum_{z_i \in \mathbb{Z}^2} \left( \prod_{i \in B_s^c} G_\lambda(z_1 - z_2 + a_i) \right)
\]

where each \( \Delta_{A_i} \) is a product of \( k_i \) difference operators of the form \( \Delta_x z^i_{s(i)}/\sqrt{\lambda} \), and we have \( \sum_{i \in B_s^c} k_i = |B_s| \). If \( B_s \neq \emptyset \) and if there is only one term in the last product on the right of (4.12), it is easily seen that the sum over \( z_2 \) gives 0. Thus the product contains at least 2 terms and then by Lemma [9] we can
see that for some $C < \infty$ and $\nu > 0$ independent of everything

$$
(4.13) \quad \left| \sum_{i \in \mathbb{Z}^2} G_\lambda (z_1) \prod_{i \in B_s, i \neq 1} \Delta_{A_i} G_\lambda (z_1 - z_2 + a_i) \right| \leq C \lambda^{-2} \prod_{i \in B_s} |x^{s(i)}|^{\nu}.
$$

With these results, we now turn to the bound (4.9). For ease of exposition we use $y^i$ to denote the $y$ in the $i$’th factor; in the end we will set $y^i = y$. For ease of exposition we assume that $y$ differs from 0 only in the $v$’th coordinate, and we set $a = y_v$. (The general case is then easily handled).

We again use Lemma 5 of [16] to expand

$$
(4.14) \quad E \left( (\Gamma_{k,\lambda}(\zeta_\lambda, y^1) - \Gamma_{k,\lambda}(\zeta_\lambda)) (\Gamma_{k,\lambda}(\zeta_\lambda, y^2) - \Gamma_{k,\lambda}(\zeta_\lambda)) \right)
$$

as a sum of many terms of the form

$$
(4.15) \quad \sum_{s \in \mathcal{S}} \left( \prod_{k=1}^{2} \mathcal{D}_{y^k_v/\sqrt{\lambda}} \right) \left( \prod_{i \in B_s} G_\lambda (x^{s(i)}_{c(i)}/\sqrt{\lambda}) \right) \sum_{z_i \in \mathbb{Z}^2} \prod_{i \in B_s} \mathcal{D}_{x^{s(i)}_{c(i)}/\sqrt{\lambda}} \prod_{i \in B_s} G_\lambda \left( z^{s(i)}_{s(i)} + \sum_{j=2}^{c(i)} x^{s(i)}_{j}/\sqrt{\lambda} - (z^{s(i-1)}_{s(i-1)} + \sum_{j=2}^{c(i-1)} x^{s(i-1)}_{j}/\sqrt{\lambda}) \right).
$$

where now $x^i$ is variously $y^i$ or 0. For fixed $s \in \mathcal{S}$ we can expand the corresponding term as a sum of terms of the form

$$
(4.16) \quad \left\{ \left( \prod_{k \in F} \mathcal{D}_{y^k_v/\sqrt{\lambda}} \right) \left( \prod_{i \in B_s} G_\lambda (x^{s(i)}_{c(i)}/\sqrt{\lambda}) \right) \right\} \sum_{z_i \in \mathbb{Z}^2} \prod_{k \in F^c} \mathcal{D}_{y^k_v/\sqrt{\lambda}} \left( \prod_{i \in B_s} \mathcal{D}_{x^{s(i)}_{c(i)}/\sqrt{\lambda}} \right) \prod_{i \in B_s} G_\lambda \left( z^{s(i)}_{s(i)} + \sum_{j=2}^{c(i)} x^{s(i)}_{j}/\sqrt{\lambda} - (z^{s(i-1)}_{s(i-1)} + \sum_{j=2}^{c(i-1)} x^{s(i-1)}_{j}/\sqrt{\lambda}) \right).
$$

where $F$ runs through the subsets of $\{1, \ldots, n\}$. Note that the first line will be 0 unless for each $k \in F$ we have that $y^k_v = x^{s(i)}_{c(i)}$ for some $i \in B_s$. In particular

$$
(4.17) \quad |F| \leq |B_s|.
$$
Using the fact that
\[ G_\lambda(x) \leq c \log(1/\lambda) \]
we can bound the first line of (4.16) by \((c \log(1/\lambda)) |B_s| \). As before, see in particular (4.13), we can obtain the bound
\[ \left| \sum_{z_i \in Z^2} \left( \prod_{k \in F_c} D_{y^k \sqrt{\lambda}} \right) \left( \prod_{i \in B_s} D_{x_{c(i)} \sqrt{\lambda}}^{s(i)} \right) \right| \]
\[ \leq c \lambda^{-2} \prod_{k \in F_c} |y_k|^\nu \prod_{i \in B_s} |x_{c(i)}|^{\nu}. \]

Our lemma then follows using (4.17) which implies that \(|F^c| + |B_s| \geq 2 \).

5 Approximating intersection local times

The goal of this section is to prove the following lemma.

**Lemma 4** We can find a Brownian motion such for each \( j \geq 1 \) there exists \( \beta > 0 \) such that
\[ \| \lambda \Gamma_{j,\lambda}(\zeta) - \gamma_j(\zeta, \omega_{\lambda^{-1}}) \|_2 = O(\lambda^\beta) \]

**Proof of Lemma 4:** Let \( f(x) \) be a smooth function on \( \mathbb{R}^2 \), supported in the unit disc and with \( \int f(x) \, dx = 1 \). We set \( f_\varepsilon(x) = \frac{1}{\varepsilon} f(x/\varepsilon) \). On the one hand it is easy to see that if we set \( \tilde{u}^1(f_\varepsilon) = \int u^1(x)f_\varepsilon(x) \, dx \) and

\[ \tilde{\gamma}_k(\zeta, f_\varepsilon) = \int \tilde{\gamma}_k(\zeta, x) \prod_{i=2}^{k} f_\varepsilon(x_i) \, dx_2 \cdots dx_k, \]
\[ \tilde{\alpha}_j(\zeta, f_\varepsilon) = \int \tilde{\alpha}_j(\zeta, x) \prod_{i=2}^{j} f_\varepsilon(x_i) \, dx_2 \cdots dx_k, \]

we will have
\[ \tilde{\gamma}_k(\zeta, f_\varepsilon) = \sum_{j=1}^{k} \binom{k-1}{j-1} (-\tilde{u}^1(f_\varepsilon))^{k-j} \tilde{\alpha}_j(\zeta, f_\varepsilon) \]
and
\begin{equation}
\tilde{\alpha}_j(t, f_\tau) = \int_{0 \leq t_1 \leq \cdots \leq t_j < t} f_\tau(W_{t_i} - W_{t_{i-1}}) dt_1 \cdots dt_j.
\end{equation}

On the other hand it follows from (4.5) and Jensen’s inequality that
\begin{equation}
\|\tilde{\gamma}_k(\zeta, f_\tau) - \gamma_k(\zeta)\|_2 \leq C_\tau \bar{\delta}.
\end{equation}

If we set \( \tilde{G}_\lambda(f_\tau) = \int G_\lambda(x/\sqrt{\lambda}) f_\tau(x) \, dx, \)
\( \tilde{\Gamma}_{k,\lambda}(\zeta_\lambda, f_\tau) = \int \Gamma_{k,\lambda}(\zeta_\lambda, x) \prod_{i=2}^k f_\tau(x_i) \, dx_2 \cdots dx_k \)
and
\( \tilde{I}_j(\zeta_\lambda, f_\tau) = \int I_j(\zeta_\lambda, x/\sqrt{\lambda}) \prod_{i=2}^j f_\tau(x_i) \, dx_2 \cdots dx_k, \)
we similarly have
\begin{equation}
\tilde{\Gamma}_{k,\lambda}(\zeta_\lambda, f_\tau) = \sum_{j=1}^k \binom{k-1}{j-1} \left(-G_\lambda(f_\tau)\right)^{k-j} \tilde{I}_j(\zeta_\lambda, f_\tau)
\end{equation}
and
\begin{equation}
\tilde{I}_j(\zeta_\lambda, f_\tau) = \int_{0 \leq t_1 \leq \cdots \leq t_j < \zeta_\lambda} \prod_{i=2}^j f_\tau(\sqrt{\lambda}(S_{[t_i]} - S_{[t_{i-1}]})) dt_1 \cdots dt_j
\end{equation}
\begin{equation}
= \int_{0 \leq t_1 \leq \cdots \leq t_j < \zeta_\lambda} \prod_{i=2}^j f_\tau(\sqrt{\lambda}(S_{[t_i]/\lambda} - S_{[t_{i-1}]/\lambda})) dt_1 \cdots dt_j.
\end{equation}

It then follows from (4.8) that
\begin{equation}
\|\lambda \tilde{\Gamma}_{k,\lambda}(\zeta_\lambda, f_\tau) - \lambda \Gamma_{k,\lambda}(\zeta_\lambda)\|_2 \leq C_\tau \bar{\delta}.
\end{equation}

To complete the proof of Lemma 3 it only remains to show that with \( \tau = \lambda^\rho \) for \( \rho > 0 \) small
\begin{equation}
\|\lambda \tilde{\Gamma}_{k,\lambda}(\zeta_\lambda, f_\tau) - \tilde{\gamma}_k(\zeta, f_\tau, \omega_{\lambda^{-1}})\|_2 \leq c\lambda^{\delta'}
\end{equation}
for some \( c < \infty \) and \( \zeta > 0 \). By (5.2), (5.3) it suffices to show that for some \( \delta' > 0 \) and all sufficiently small \( \tau, \lambda \)
\begin{equation}
\tilde{u}_i^i(f_\tau) = O(\log(1/|\tau|)), \quad |\tilde{G}_\lambda(f_\tau) - \tilde{u}_i^i(f_\tau)| \leq c\tau^{-3}\lambda^{\delta'}
\end{equation}
and
\[
\| \tilde{\alpha}_k(\zeta, f_\tau, \omega_{\lambda-1}) \|_2 \leq c r^{-2(k-1)},
\]
(5.10) \[
\| \lambda \tilde{I}_{k,\lambda}(\zeta, f_\tau) - \tilde{\alpha}_k(\zeta, f_\tau, \omega_{\lambda-1}) \|_2 \leq c r^{-2k+1} \lambda^{\delta'}. 
\]

The first part of (5.9) follows from the fact that \( u^1(x) = O(\log(1/|x|)) \), see [9 (2.b)]. To prove the second part of (5.9), we note that \( \sup_x |\nabla f_\tau(x)| \leq c r^{-3} \), so
\[
\| \tilde{\alpha}_k(\zeta, f_\tau, \omega_{\lambda-1}) \|_2 \leq c r^{-2(k-1)} \int_0^\infty e^{-t} t^n dt.
\]
(5.12)

To prove the second part of (5.10), we use the above bounds on \( \sup_x |\nabla f_\tau(x)| \) and \( \sup_x |f_\tau(x)| \) to see that
\[
\| \lambda \tilde{I}_{k,\lambda}(\zeta, f_\tau) - \tilde{\alpha}_k(\zeta, f_\tau, \omega_{\lambda-1}) \|_2^2 
\leq c r^{-2k+1} \sum_{j=1}^k \int_0^\infty e^{-t} \left( \int_{0 \leq t_1 \leq \ldots \leq t_k \leq t} \left\| \sqrt{\lambda}(S_{[t_j]/\lambda] - W_{t_j/\lambda}) \right\|_2^2 dt_1 \cdots dt_k \right) dt.
\]
(5.13)

The second part of (5.10) then follows from the last inequality in Section 3.

6 Renormalized Brownian intersection local times

Recall the definition of \( \gamma_k(t) \) given in (1.5). Note from [9 (2.b)] that for some fixed constant \( c \)
\[
u = \int_0^\infty e^{-t} p(t)/(0) dt = \frac{1}{2\pi} \log(1/\epsilon) + c + O(\epsilon).
\]
(6.1)
In [16] we show that the limit in (1.5) exists a.s. and in all $L^p$ spaces, and that $\gamma_k(t)$ is continuous in $t$. The rest of this section is basically contained in [9] but we point out that [16] came after [9] and resulted in some simplification.

For any given function $h : (0, \infty) \to R$ we set

$$\hat{\gamma}_1(t, h) = t$$

and for $k \geq 2$

$$\hat{\gamma}_k(t, h) = \lim_{\epsilon \to 0} \sum_{l=1}^{k} \frac{k-1}{l-1} (-h_\epsilon)^{k-l} \alpha_{l,\epsilon}(t),$$

where we write $h_\epsilon$ for $h(\epsilon)$. In particular, $\gamma_1(t) = \hat{\gamma}_1(t, u)$. Let $\mathcal{H}$ denote the set of functions $h$ such that $\lim_{\epsilon \to 0} (h_\epsilon - \bar{h}_\epsilon)$ exists and is finite. In the next lemma we will see that the limit in (6.2) exists for all $h \in \mathcal{H}$.

**Lemma 5 (Renormalization Lemma)** Let $h \in \mathcal{H}$. Then $\hat{\gamma}_k(t, h)$ exists for all $k \geq 1$ and if $\bar{h} \in \mathcal{H}$ with $\lim_{\epsilon \to 0} (h_\epsilon - \bar{h}_\epsilon) = b$ then for any $k \geq 1$

$$\hat{\gamma}_k(t, h) = \sum_{m=1}^{k} \left( \frac{k-1}{m-1} \right)(-b)^{k-m}\hat{\gamma}_m(t, \bar{h}).$$

**Proof of Lemma 5** Setting $b_\epsilon = h_\epsilon - \bar{h}_\epsilon$ we have

$$\sum_{l=1}^{k} \frac{k-1}{l-1} (-h_\epsilon)^{k-l} \alpha_{l,\epsilon}(t)
= \sum_{l=1}^{k} \frac{k-1}{l-1} \left( \frac{k-1}{l-1} \right)(-\bar{h}_\epsilon - b_\epsilon)^{k-l} \alpha_{l,\epsilon}(t)
= \sum_{l=1}^{k} \frac{k-1}{l-1} \sum_{j=0}^{k-l} \frac{k-l}{j} \left( \frac{k-l}{j} \right)(-b_\epsilon)^j (-\bar{h}_\epsilon)^{(k-j)-l} \alpha_{l,\epsilon}(t)$$

Using

$$\left( \frac{k-1}{l-1} \right) \left( \frac{k-l}{j} \right) = \left( \frac{k-1}{j} \right) \left( \frac{k-j-1}{l-1} \right)$$

the last line in (6.4) becomes

$$\sum_{j=0}^{k-1} \frac{k-1}{j} (-b_\epsilon)^j \sum_{l=1}^{k-j} \frac{k-j-1}{l-1} (-\bar{h}_\epsilon)^{(k-j)-l} \alpha_{l,\epsilon}(t).$$
Taking \( \bar{h}_\epsilon = u_\epsilon \) then shows the existence of \( \gamma_k(t, h) \). Returning to general \( \bar{h} \in \mathcal{H} \) and now taking the \( \epsilon \to 0 \) limit we obtain

\[
\hat{\gamma}_k(t, h) = \sum_{j=0}^{k-1} \binom{k-1}{j} (-b)^j \hat{\gamma}_{k-j}(t, \bar{h})
\]

\[
= \sum_{m=1}^{k} \left( \frac{k-1}{m-1} \right) (-b)^{k-m} \hat{\gamma}_m(t, \bar{h})
\]

where the last line follows from the substitution \( m = k - j \).

Let \( h \in \mathcal{H} \). We shall sometimes write \( \hat{\gamma}_k(t, h, \omega) \) for \( \hat{\gamma}_k(t, h) \) to emphasize its dependence on the path \( \omega \). We want to discuss how renormalized intersection local time changes with a time rescaling. Let \( \omega_r(s) = r^{-1/2} \omega(rs) \). Then \( \hat{\gamma}_k(t, h, \omega_r) \) is the same as \( \hat{\gamma}_k(t, h) \) defined in terms of the Brownian motion \( W_t^{(r)} = W_{rt}/\sqrt{r} \).

**Lemma 6 (Rescaling Lemma)** Let \( h \in \mathcal{H} \). Then for any \( k \geq 1 \)

\[
\hat{\gamma}_k(t, h, \omega_r) = r^{-1} \sum_{m=1}^{k} \left( \frac{k-1}{m-1} \right) \left( \frac{1}{2\pi} \log(1/r) \right)^{k-m} \hat{\gamma}_m(rt, h, \omega).
\]

**Proof of Lemma 6:** After replacing \( \omega \) by \( \omega_r \) the integral on the right hand side of (6.2) is replaced by

\[
\int_{0 \leq t_1 \leq \cdots \leq t_l < t} \prod_{i=2}^{l} p_\epsilon \left( \frac{W_{rt_i} - W_{rt_{i-1}}}{\sqrt{r}} \right) dt_1 \cdots dt_l
\]

\[
= r^{-l} \int_{0 \leq t_1 \leq \cdots \leq t_l < rt} \prod_{i=2}^{l} p_\epsilon \left( \frac{W_{t_i} - W_{t_{i-1}}}{\sqrt{r}} \right) dt_1 \cdots dt_l
\]

\[
= r^{-1} \int_{0 \leq t_1 \leq \cdots \leq t_l < rt} \prod_{i=2}^{l} p_{re}(W_{t_i} - W_{t_{i-1}}) dt_1 \cdots dt_l.
\]

Abbreviating this last integral as \( \alpha_{l, re}(rt, \omega) \) we have

\[
\hat{\gamma}_k(t, h, \omega_r) = r^{-1} \lim_{\epsilon \to 0} \sum_{l=1}^{k} \binom{k-1}{l-1} (-h_\epsilon)^{k-l} \alpha_{l, re}(rt, \omega).
\]

Since \( h \in \mathcal{H} \) it is easily seen that \( \lim_{\epsilon \to 0} (h_\epsilon - h_{re}) = -\frac{1}{2\pi} \log(1/r) \) and our lemma then follows from Lemma 5.
7 Range and Brownian intersection local times

In this section we prove the following theorem.

**Theorem 2** For each $k \geq 1$

$$g^k_\lambda \left( \lambda |\mathcal{R}(\zeta_\lambda)| - \sum_{j=1}^{k} (-1)^{j-1} g^{-j}_\lambda \gamma_j(\zeta, \omega_{\lambda^{-1}}) \right) \to 0 \quad \text{a.s.}$$

(7.1) as $\lambda \to 0$.

**Proof of Theorem 2:** Using (5.1) together with Lemma 1 and its proof, we see that for some $M_k < \infty$

$$\left\| g^{4k+1}_\lambda \left( \lambda |\mathcal{R}(\zeta_\lambda)| - \sum_{j=1}^{4k} (-1)^{j-1} g^{-j}_\lambda \gamma_j(\zeta, \omega_{\lambda^{-1}}) \right) \right\|_2 \leq M_k$$

(7.2) for all $\lambda > 0$ sufficiently small.

We now follow Le Gall [9]. With $\lambda_n = e^{-n^{3/2k}}$ we have that for any $\epsilon > 0$

$$\sum_{n=1}^{\infty} P \left\{ g^k_{\lambda_n} \left( \lambda_n |\mathcal{R}(\zeta_{\lambda_n})| - \sum_{j=1}^{4k} (-1)^{j-1} g^{-j}_{\lambda_n} \gamma_j(\zeta, \omega_{\lambda_n^{-1}}) \right) \geq g^{-1}_{\lambda_n} \right\} \leq M_k \sum_{n=1}^{\infty} g^{-6k}_{\lambda_n} < \infty.$$

Then by Borel-Cantelli

$$g^k_{\lambda_n} \left( \lambda_n |\mathcal{R}(\zeta_{\lambda_n})| - \sum_{j=1}^{4k} (-1)^{j-1} g^{-j}_{\lambda_n} \gamma_j(\zeta, \omega_{\lambda_n^{-1}}) \right) \to 0 \quad \text{a.s.}$$

(7.4) Since for each $m \geq 1$ we have that $\gamma_j(\zeta, \omega_{\lambda_n^{-1}})$ is bounded in $L^m$ uniformly in $n$, then by Chebyshev’s inequality with $m$ sufficiently large $P(\gamma_j(\zeta, \omega_{\lambda_n^{-1}}) > g_{\lambda_n})$ will be summable. So we may drop the terms for $j > k$ and we then have

$$g^k_{\lambda_n} \left( \lambda_n |\mathcal{R}(\zeta_{\lambda_n})| - \sum_{j=1}^{k} (-1)^{j-1} g^{-j}_{\lambda_n} \gamma_j(\zeta, \omega_{\lambda_n^{-1}}) \right) \to 0 \quad \text{a.s.}$$

(7.5) Before continuing the proof of Theorem 2 we first prove the following lemma.

21
Lemma 7 For any $k \geq 1$

\[(7.6) \quad \lim_{n \to 0} \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} |\gamma_k(\zeta, \omega_{\lambda-1}) - \gamma_k(\zeta, \omega_{\lambda_n})| = 0. \quad a.s.\]

Proof of Lemma 7: By (6.7) for any $k \geq 1$

\[(7.7) \quad \gamma_k(\zeta, \omega_{\lambda-1}) = \frac{\lambda}{\lambda_n} \sum_{m=1}^{k} \left( \frac{k-1}{m-1} \left( \frac{1}{2\pi} \log \left( \frac{\lambda}{\lambda_n} \right) \right)^{k-m} \gamma_m(\frac{\lambda_n}{\lambda} \zeta, \omega_{\lambda_n}) \right).\]

Hence for any $p \geq 1$

\[(7.8) \quad \| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} |\gamma_k(\zeta, \omega_{\lambda-1}) - \gamma_k(\zeta, \omega_{\lambda_n})| \|_p \leq \frac{1}{2\pi} \log \left( \frac{\lambda}{\lambda_n} \right)^{k-m} \| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \gamma_m(\frac{\lambda_n}{\lambda} \zeta, \omega_{\lambda_n}) \|_p + c \sum_{m=1}^{k-1} \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \left( \frac{1}{2\pi} \log \left( \frac{\lambda}{\lambda_n} \right) \right)^{k-m} \| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \gamma_m(\frac{\lambda_n}{\lambda} \zeta, \omega_{\lambda_n}) \|_p + c \sum_{m=1}^{k-1} \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \left( \frac{1}{2\pi} \log \left( \frac{\lambda}{\lambda_n} \right) \right)^{k-m} \| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} \gamma_m(\frac{\lambda_n}{\lambda} \zeta, \omega_{\lambda_n}) \|_p.

It follows from (9.11) of [2] that for any $k \geq 1$ we can find $\beta > 0$ such that

\[(7.9) \quad \| \sup_{|t-s| \leq \delta, s,t \leq 1} |\gamma_k(s) - \gamma_k(t)| \|_p \leq c\delta^\beta.\]

Actually, this is proved for a renormalized intersection local time $\xi_k(t)$ where $\xi_k(t) = \lim_{x \to 0} \xi_k(t, x)$ and $\xi_k(t, x)$ differs from $\gamma_k(t, x)$ defined in (4.3) in that $u^1(x)$ is replaced by $\pi^1 \log(1/|x|)$. Since $u^1(x) - \pi^1 \log(1/|x|) = c + O(|x|^2 \log |x|)$, see [9] (2.b), we obtain (7.9). Using (6.7) again we find that

\[(7.10) \quad \| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} |\gamma_k(\lambda_{n+1}) - \gamma_k(\lambda_n)| \|_p \leq \text{ct} \log(t)^k \left| \frac{\lambda_n}{\lambda_{n+1}} - 1 \right|^\beta \leq \text{ct} \log(t)^k n^{-\beta'}\]

where we have used

\[(7.11) \quad \log \frac{\lambda_n}{\lambda_{n+1}} = O(n^{-1+1/k}).\]
Hence

\begin{equation}
\| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} | \gamma_k \left( \frac{\lambda_n}{\lambda} \zeta \right) - \gamma_k (\zeta) | \|_p \leq cn^{-\beta''}.
\end{equation}

Using (7.8) and (7.12) now shows that

\begin{equation}
\| \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} | \gamma_k (\zeta, \omega_{\lambda^{-1}}) - \gamma_k (\zeta, \omega_{\lambda_n^{-1}}) | \|_p \leq cn^{-\beta''}
\end{equation}

and our lemma then follows using Holder’s inequality for sufficiently large \( p \) and the Borel-Cantelli Lemma.

Continuing the proof of Theorem 2 by our choice of \( \lambda_n \)

\begin{equation}
\lim_{n \to 0} g^k_{\lambda_{n+1}} - g^k_{\lambda_n} = 0.
\end{equation}

Together with (7.6) we have that a.s.

\begin{equation}
\lim_{n \to 0} \sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} | \sum_{j=1}^{k} (-1)^{j-1} g^k_{\lambda} \gamma_j (\zeta, \omega_{\lambda^{-1}}) - \sum_{j=1}^{k} (-1)^{j-1} g^k_{\lambda_n} \gamma_j (\zeta, \omega_{\lambda_n^{-1}}) | = 0.
\end{equation}

Using the fact that \(|\mathcal{R}(\zeta, \omega)|\) and \( g_\lambda \) are monotone decreasing we have that

\begin{equation}
\sup_{\lambda_{n+1} \leq \lambda \leq \lambda_n} | \lambda g^k_{\lambda} | |\mathcal{R}(\zeta, \omega)| | - \lambda_n g^k_{\lambda_n} | |\mathcal{R}(\zeta, \omega)| | \leq \left| \lambda_n g^k_{\lambda_{n+1}} | |\mathcal{R}(\zeta_{n+1}, \omega)| | - \lambda_n g^k_{\lambda_n} | |\mathcal{R}(\zeta, \omega)| | \right|
\end{equation}

Here the first term on the right hand side of (7.16) goes to 0 using the fact that

\[ |\lambda_n - \lambda_{n+1}| = |1 - e^{n^{1/2} - (n+1)^{1/2}}| \lambda_n \leq n^{-1} \lambda_n \leq 2n^{-1} \lambda_{n+1} \]

and the second term on the right hand side of (7.16) goes to 0 using (7.15) and (7.3). Combining (7.3), (7.5) and (7.16) we have (7.1).
8 Non-random times

In this section we complete the proof of Theorem 1. Recall that \( \zeta_\lambda = n \) if \( n - 1 < \frac{1}{\lambda} \zeta \leq n \). So \( \zeta_\lambda = \lceil \frac{1}{\lambda} \zeta \rceil \) where \( \lceil x \rceil \) denotes the smallest integer \( m \geq x \). Hence (7.1) can be written as

\[
(8.1) \quad g_{\lambda}^{2k} \left( \lambda |\mathcal{R}(\lceil \zeta/\lambda \rceil)| - \sum_{j=1}^{2k} (-1)^{j-1} g_{\lambda}^{-j} \gamma_j(\zeta, \omega_{\lambda-1}) \right) \rightarrow 0, \quad \text{a.s.}
\]

If \((\Omega, P)\) denotes our probability space for \( \{S_n; n \geq 1\} \) and \( \{W_t; t \geq 0\} \), then the almost sure convergence in (8.1) is with respect to the measure \( e^{-t} dt \times P \) on \( R_1^+ \times \Omega \), where \( \zeta(t, \omega) = t \). Hence by Fubini’s theorem we have that for almost every \( t > 0 \)

\[
(8.2) \quad g_{\lambda}^{2k} \left( \lambda |\mathcal{R}(\lceil t/\lambda \rceil)| - \sum_{j=1}^{2k} (-1)^{j-1} g_{\lambda}^{-j} \gamma_j(t, \omega_{\lambda-1}) \right) \rightarrow 0, \quad \text{a.s.}
\]

Fix a \( t_0 \) for which (8.2) holds and let \( \lambda \) run through the sequence \( t_0/n \). Then (2.3) and (8.2) tell us that

\[
(8.3) \quad (\log n)^k \left( \frac{t_0}{n} |\mathcal{R}(n)| + \sum_{j=1}^{k} (-g_{t_0/n})^{-j} \gamma_j(t_0, \omega_{n/t_0}) \right) \rightarrow 0 \quad \text{a.s.}
\]

Using (6.7) and writing \( b_r = \frac{1}{2\pi} \log(1/r) \) we have that

\[
(8.4) \quad (\log n)^k \left( \frac{t_0}{n} |\mathcal{R}(n)| + \sum_{j=1}^{k} (-g_{t_0/n})^{-j} \sum_{m=1}^{j} \left( \frac{j-1}{m-1} \right) b_{j/m}^{-m} \gamma_m(1, \omega_n) \right) \rightarrow 0 \quad \text{a.s.}
\]

Then

\[
(8.5) \quad \sum_{j=1}^{k} (-g_{t_0/n})^{-j} \sum_{m=1}^{j} \left( \frac{j-1}{m-1} \right) b_{j/m}^{-m} \gamma_m(1, \omega_n)
\]

\[
= \sum_{m=1}^{k} \left( \sum_{j=m}^{k} \left( \frac{j-1}{m-1} \right) \left( \frac{-b_{1/t_0}}{g_{t_0/n}} \right)^{j-m} \right) (-g_{t_0/n})^{-m} \gamma_m(1, \omega_n).
\]

Now,

\[
(8.6) \quad \sum_{j=m}^{k} \left( \frac{j-1}{m-1} \right)^{j-m} = \sum_{i=0}^{k-m} \left( \frac{i+m-1}{m-1} \right)^{i} = \left( \frac{1}{1-x} \right)^m + O(x^{k-m+1}).
\]
By (8.5) with $\delta = 1$ we have that $\sup_{t \leq 1} |\gamma_j(t, \omega)|$ is in $L^p$ for each $p$ and each $j \geq 1$. If we set $V_{j, \ell} = \sup_{t \leq 1} |\gamma_j(t, \omega_{2^\ell})|$, we then have, taking $p$ large enough, that
\[
\sum_{\ell=1}^{\infty} P(V_{j, \ell} > \eta \log(2^\ell)) \leq \sum_{\ell=1}^{\infty} \frac{EV_{j, \ell}^p}{(\eta \log 2^\ell)^p}
\]
is summable for each $\eta$. Hence by Borel-Cantelli $V_{j, \ell}/\log(2^\ell) \to 0$ a.s. for each $j \geq 1$. Since by Lemma 6 we have for $2^\ell \leq r < 2^{\ell+1}$ that $\gamma_k(1, \omega_r)$ is bounded by a linear combination of the $V_{j, \ell}$, $1 \leq j \leq k$, with coefficients that are bounded independently of $r$, we conclude
\[
\gamma_j(1, \omega_n)/\log n \to 0, \quad a.s.
\]
Thus we can replace (8.5) up to errors which are $O((\log n)^{-k-1})$ by
\[
(8.7) \quad \sum_{m=1}^{k} \left( g_{t_0/n} + b_1/t_0 \right)^m \gamma_m(1, \omega_n) = \sum_{m=1}^{k} (-g_{1/n})^{-m} \gamma_m(1, \omega_n)
\]
since by (2.3) we have that $g_{t_0/n} + b_1/t_0 = g_{1/n} + O(n^{-\delta})$.

Thus we obtain
\[
(8.8) \quad (\log n)^k \left( \frac{1}{n} |\mathcal{R}(n)| + \sum_{j=1}^{k} (-g_{1/n})^{-j} \gamma_j(1, \omega_n) \right) \to 0 \quad a.s.
\]
This, together with (A.2), gives Theorem I.

A Estimates for random walks

In this appendix we will obtain some estimates for strongly aperiodic planar random walks $S_n = \sum_{i=1}^{n} X_i$, where the $X_i$ are symmetric, have the identity as covariance matrix, and have $2 + \delta$ moments for some $\delta > 0$.

Let
\[
G_\lambda(x) = \sum_{n=0}^{\infty} e^{-\lambda n} q_n(x).
\]
If
\[
\phi(p) = E(e^{ipX_1})
\]
denotes the characteristic function of $X_1$, we have
\[
(\text{A.1}) \quad G_\lambda(x) = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{e^{ipx}}{1 - e^{-\lambda \phi(p)}} dp.
\]
Lemma 8 Let $S_n$ be as above. Then

\[(A.2) \quad G_\lambda(0) = \frac{1}{2\pi} \log(1/\lambda) + c_X + O(\lambda^\delta \log(1/\lambda))\]

where

\[(A.3) \quad c_X = \frac{1}{2\pi} \log(\pi^2/2) + \frac{1}{2\pi^2} \int_{[-\pi,\pi]^2} \frac{\phi(p) - 1 + |p|^2/2}{1 - \phi(p) |p|^2/2} dp\]

is a finite constant.

Proof of Lemma 8: We have

\[(A.4) \quad G_\lambda(0) = \frac{1}{2\pi^2} \int_{[-\pi,\pi]^2} \frac{1}{1 - e^{-\lambda \phi(p)}} dp.\]

We intend to compare this with

\[(A.5) \quad \frac{1}{2\pi^2} \int_{[-\pi,\pi]^2} \frac{1}{\lambda + |p|^2/2} dp\]

whose asymptotics are easier to compute. Indeed,

\[(A.6) \quad \int_{[-\pi,\pi]^2} \frac{1}{\lambda + |p|^2/2} dp = \int_{D(0,\pi)} \frac{1}{\lambda + |p|^2/2} dp + \int_{[-\pi,\pi]^2 - D(0,\pi)} \frac{1}{\lambda + |p|^2/2} dp + O(\lambda)\]

where $D(0, \pi)$ is the disc centered at the origin of radius $\pi$. It is clear that

\[(A.7) \quad \int_{D(0,\pi)} \frac{1}{\lambda + |p|^2/2} dp = 2\pi \left( \log(\lambda + \pi^2/2) - \log(\lambda) \right).\]

On the other hand, using polar coordinates

\[(A.8) \quad \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{1}{\lambda + |p|^2/2} dp = \frac{1}{2\pi} \log(1/\lambda) + \frac{1}{2\pi} \log(\pi^2/2) + O(\lambda).\]
We then note that

\[(A.9)\]
\[
\int_{[-\pi,\pi]^2} \frac{1}{1 - e^{-\lambda \phi(p)}} \, dp - \int_{[-\pi,\pi]^2} \frac{1}{\lambda + |p|^2/2} \, dp
\]
\[
= \int_{[-\pi,\pi]^2} \frac{\lambda + |p|^2/2 - (1 - e^{-\lambda \phi(p)})}{(1 - e^{-\lambda \phi(p)})(\lambda + |p|^2/2)} \, dp
\]
\[
= \int_{[-\pi,\pi]^2} \frac{\phi(p) - 1 + |p|^2/2}{(1 - e^{-\lambda \phi(p)})(\lambda + |p|^2/2)} \, dp
\]
\[
- \lambda \int_{[-\pi,\pi]^2} \frac{\phi(p) - 1}{(1 - e^{-\lambda \phi(p)})(\lambda + |p|^2/2)} \, dp
\]
\[
+ (e^{-\lambda} - 1 + \lambda) \int_{[-\pi,\pi]^2} \frac{\phi(p)}{(1 - e^{-\lambda \phi(p)})(\lambda + |p|^2/2)} \, dp
\]

Since

\[(A.10)\]
\[
|e^{ip \cdot x} - 1 - ip \cdot x + (p \cdot x)^2/2| \leq c(p \cdot x)^{2+\delta}
\]
for some \(c < \infty\) we have by our assumptions that

\[(A.11)\]
\[
|\phi(p) - 1 + |p|^2/2| \leq c_1 |p|^{2+\delta}.
\]

this implies that

\[(A.12)\]
\[
|\phi(p) - 1| \leq c_1'' |p|^2
\]

for \(p \in [-\pi, \pi]^2\) and

\[(A.13)\]
\[
1 - e^{-\lambda \phi(p)} \geq \tilde{c}(\lambda + |p|^2)
\]

for some \(\tilde{c} > 0\) and sufficiently small \(\lambda\). Hence

\[(A.14)\]
\[
(e^{-\lambda} - 1 + \lambda) \int_{[-\pi,\pi]^2} \frac{\phi(p)}{(1 - e^{-\lambda \phi(p)})(\lambda + |p|^2/2)} \, dp
\]
\[
\leq c\lambda^2 \int_{[-\pi,\pi]^2} \frac{1}{(\lambda + |p|^2)^2} \, dp
\]
\[
\leq c\lambda \int_{[-\pi/\sqrt{\lambda}, \pi/\sqrt{\lambda}]^2} \frac{1}{(1 + |p|^2)^2} \, dp = O(\lambda)
\]

and

\[(A.15)\]
\[
\lambda \int_{[-\pi,\pi]^2} \frac{|\phi(p) - 1|}{(1 - e^{-\lambda \phi(p)})(\lambda + |p|^2/2)} \, dp
\]
\[
\leq c\lambda \int_{[-\pi,\pi]^2} \frac{|p|^2}{(\lambda + |p|^2)^2} \, dp
\]
\[
\leq c\lambda \int_{[-\pi/\sqrt{\lambda}, \pi/\sqrt{\lambda}]^2} \frac{|p|^2}{(1 + |p|^2)^2} \, dp = O(\lambda \log(1/\lambda)).
\]
Setting \( f(p) = \phi(p) - 1 + |p|^2/2 \), and using (A.11) we see that
\[
\int_{[-\pi,\pi]^2} \frac{|f(p)|}{|1 - \phi(p)||p|^2/2} \, dp < \infty.
\]
Consider then
\[
(A.16) \quad \int_{[-\pi,\pi]^2} \frac{f(p)}{(1 - e^{-\lambda \phi(p)})(\lambda + |p|^2/2)} \, dp - \int_{[-\pi,\pi]^2} \frac{f(p)}{(1 - \phi(p))|p|^2/2} \, dp
\]
\[
= \int_{[-\pi,\pi]^2} \frac{f(p)}{(1 - e^{-\lambda \phi(p)})(\lambda + |p|^2/2)} \, dp - \int_{[-\pi,\pi]^2} \frac{f(p)}{(1 - e^{-\lambda \phi(p)})|p|^2/2} \, dp
\]
\[
+ \int_{[-\pi,\pi]^2} \frac{f(p)}{(1 - e^{-\lambda \phi(p))}|p|^2/2} \, dp - \int_{[-\pi,\pi]^2} \frac{f(p)}{(1 - \phi(p))|p|^2/2} \, dp
\]
We have
\[
(A.17) \quad \int_{[-\pi,\pi]^2} \frac{f(p)}{(1 - e^{-\lambda \phi(p)})(\lambda + |p|^2/2)} \, dp - \int_{[-\pi,\pi]^2} \frac{f(p)}{(1 - e^{-\lambda \phi(p)})|p|^2/2} \, dp
\]
\[
= - \int_{[-\pi,\pi]^2} \frac{f(p)\lambda}{(1 - e^{-\lambda \phi(p)})(\lambda + |p|^2/2)|p|^2/2} \, dp = O(\lambda^4 \log(1/\lambda))
\]
and the last line in (A.16) can be bounded similarly. This completes the proof of Lemma 8. \(\square\)

**Lemma 9** Let \( S_n \) be as above. For all \( m \geq 1 \)
\[
(A.18) \quad \|G_\lambda\|_m = O(\lambda^{-1/m}) \quad \text{as} \quad \lambda \to 0
\]
and
\[
(A.19) \quad \|G_\lambda - G_{\lambda'}\|_m = O(|\lambda - \lambda'|^{1/m}) \quad \text{as} \quad \lambda \to 0.
\]
For all \( m \geq 2 \) and \( z \in \mathbb{Z}^2 \)
\[
(A.20) \quad \|\Delta_{z/\sqrt{\lambda}}G_\lambda\|_m \leq c'|z|^{2/m} \lambda^{-1/m}(\log(1/\lambda))^{1-1/m}
\]
and for any \( 0 < \beta < 1 \)
\[
(A.21) \quad \|\Delta_{z/\sqrt{\lambda}}G_\lambda\|_m \leq c'|z|^{\beta/m} \lambda^{-1/m}
\]
and
\[
(A.22) \quad \|\left(\prod_{i=1}^k \Delta_{z_i/\sqrt{\lambda}}\right)G_\lambda\|_m \leq c' \left(\prod_{i=1}^k |z_i|^{\beta/mk}\right) \lambda^{-1/m}.
\]
Proof of Lemma 9: By [19], p. 77, we know that
\[ q_n(x) \leq c_1/n, \]
where \( q_n \) is the transition probability for \( S_n \). So
\[ \|q_n\|_m = \sum_{x \in \mathbb{Z}^2} q_n(x)^m \leq c_1^{m-1} n^{-m+1} \sum_{x \in \mathbb{Z}^2} q_n(x) = c_1^{m-1} n^{-m+1}. \]
Then
\[ \|G_\lambda\|_m \leq \sum_{n=0}^\infty e^{-\lambda n} \|q_n\|_m. \]
Substituting the above estimate for \( \|q_n\|_m \) and breaking the sum into the sum over \( n \leq 1/\lambda \) and the sum over \( n > 1/\lambda \), we easily obtain (A.18).

(A.19) follows from (A.18) and the resolvent equation
\[ G_\lambda - G_{\lambda'} = (\lambda' - \lambda)G_\lambda * G_{\lambda'}. \]

By Proposition 2.1 of [3], for each \( \beta \in (0, 1] \) there exists a constant \( c_\beta \) such that
\[ |q_n(x) - q_n(y)| \leq c_\beta n^{-1}(|x - y|/\sqrt{n})^\beta. \]
So for any fixed \( w \in \mathbb{Z}^2 \)
\[ \|q_n(\cdot + w) - q_n(\cdot)\|_m \leq \|q_n(\cdot + w) - q_n\|_m \sum_{x \in \mathbb{Z}^2} (q_n(x + w)q_n(x)
\leq 2(c_\beta n^{-1}(|w|/\sqrt{n})^\beta)^{m-1}. \]
We take \( m \)th roots, substitute into
\[ \|G_\lambda(\cdot + w) - G_\lambda(\cdot)\|_m \leq \sum_{n=0}^\infty e^{-\lambda n} \|q_n(\cdot + w) - q_n(\cdot)\|_m, \]
break the sum into the sum over \( n \leq 1/\lambda \) and \( n > 1/\lambda \), and let \( w = z/\sqrt{\lambda} \) to obtain (A.21).

For (A.22) we note that for each \( j \) we can write \( \prod_{i=1}^k \Delta_{z_i/\sqrt{\lambda}} G_\lambda \) as a sum of \( 2^{k-1} \) terms of the form \( \Delta_{z_j/\sqrt{\lambda}} G_\lambda(z + b) \) for some \( b \) so that by (A.21)
\[ \| \left( \prod_{i=1}^k \Delta_{z_i/\sqrt{\lambda}} \right) G_\lambda \|_m \leq \epsilon^{j} 2^{k-1} |z_j|^{\beta/m} \lambda^{-1/m}. \]
We have inequality (A.24) for \( j = 1, \ldots, k \). If we take the product of these \( k \) inequalities and then take \( k \)th roots, we have (A.22).
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