On the divisibility of $q$-trinomial coefficients

Ji-Cai Liu

Received: 13 December 2021 / Accepted: 24 January 2022 / Published online: 5 April 2022
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Abstract

We investigate the divisibility of the $q$-trinomial coefficients introduced by Andrews and Baxter, which is analogous to the $q$-Wolstenholme theorem regarding the $q$-binomial coefficients. A congruence for sums of central $q$-binomial coefficients is also established.

Keywords $q$-Trinomial coefficients · $q$-Congruences · Cyclotomic polynomials

Mathematics Subject Classification 11A07 · 11B65 · 13A05 · 05A10

1 Introduction

We consider coefficients of the expanded form of the expression:

\[(1 + x + x^2)^n = \sum_{j=-n}^{n} \binom{n}{j} x^j.\]  

The coefficients in (1.1) are called trinomial coefficients. Two simple formulas for the trinomial coefficients (see [20, p. 43]) are

\[\binom{n}{j} = \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{k+j},\]

and

This work was supported by the National Natural Science Foundation of China (Grant 12171370).

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\[
\binom{n}{j} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n - 2k}{n - j - k}.
\]

In 1987, Andrews and Baxter [3] introduced six apparently distinct \(q\)-analogues of the trinomial coefficients, such as

\[
\binom{n}{j}_q = \sum_{k=0}^{n} q^{k(j+k)} \left\lfloor \frac{n-k}{k+j} \right\rfloor,
\]

where the \(q\)-binomial coefficients are defined as

\[
\left\lfloor \frac{n}{k} \right\rfloor_q = \begin{cases} 
(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1}) \\
0, \quad \text{if } 0 \leq k \leq n,
\end{cases}
\]

otherwise.

These \(q\)-trinomial coefficients play an important role in the solution of a model in statistical mechanics (see [3]). In 1990, Andrews [1] investigated some interesting properties for the \(q\)-trinomial coefficients, which led him to finite versions of dissections of the Rogers–Ramanujan identities into even and odd parts.

In 1999, Andrews [2] showed that Babbage’s congruence [5]:

\[
\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}
\]

possesses the following nice \(q\)-analogue:

\[
\left\lfloor \frac{2p-1}{p-1} \right\rfloor_q \equiv q^{\frac{p(p-1)}{2}} \left\lfloor \frac{p^2}{p_q} \right\rfloor_q, \quad (1.4)
\]

for any odd prime \(p\). Notice that Wolstenholme [27] proved that (1.3) holds modulo \(p^3\) for any prime \(p \geq 5\), which is known as the famous Wolstenholme’s theorem.

To understand (1.4), we recall some necessary notation. For polynomials \(A_1(q), A_2(q), P(q) \in \mathbb{Z}[q]\), the \(q\)-congruence

\[
A_1(q)/A_2(q) \equiv 0 \pmod{P(q)}
\]

is understood as \(A_1(q)\) is divisible by \(P(q)\) and \(A_2(q)\) is coprime with \(P(q)\). In general, for rational functions \(A(q), B(q) \in \mathbb{Z}(q)\),

\[
A(q) \equiv B(q) \pmod{P(q)} \iff A(q) - B(q) \equiv 0 \pmod{P(q)}.
\]
The $q$-integers are defined as $[n]_q = (1-q^n)/(1-q)$ for $n \geq 1$, and the $n$th cyclotomic polynomial is given by

$$\Phi_n(q) = \prod_{\substack{1 \leq k \leq n \\ (n,k)=1}} (q - \zeta^k),$$

where $\zeta$ denotes an $n$th primitive root of unity.

Recently, Straub [21, Theorem 2.2] established a $q$-analogue of Wolstenholme–Ljunggren congruence:

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3},$$

for any prime $p \geq 5$, i.e.

$$\left[ \begin{array}{c} an \\ bn \end{array} \right] \equiv \left[ \begin{array}{c} a \\ b \end{array} \right] - (a - b) b \left( \frac{a}{b} \right) \frac{n^2 - 1}{24} (q^n - 1)^2 \pmod{\Phi_n(q)^3}. \quad (1.5)$$

The simple case $a = 2$ and $b = 1$ in (1.5) reads

$$\left[ \begin{array}{c} 2n \\ n \end{array} \right] \equiv 1 - q^{2n^2} - \frac{n^2 - 1}{12} (q^n - 1)^2 \pmod{\Phi_n(q)^3}, \quad (1.6)$$

which is an extension of (1.4) (notice that $\left[ \begin{array}{c} 2n-1 \\ n \end{array} \right] = \left[ \begin{array}{c} 2n \\ n \end{array} \right]/(1 + q^n)$). We remark that Pan [19, Lemma 3.1] obtained another interesting $q$-analogue of the Wolstenholme–Ljunggren congruence and Zudilin [28] extended both Straub’s and Pan’s $q$-congruences.

In the past few years, congruences for $q$-binomial coefficients and truncated basic hypergeometric series attracted many experts’ attention (see, for example, [6–12, 14–17, 24, 26]). Nowadays, there is also an extensive literature on congruences for trinomial coefficients (see, for instance, [4, 18, 23, 25]). It is worth mentioning that Gorodetsky [7] investigated congruences for the $q$-trinomial coefficients $\left[ \begin{array}{c} n \\ 0 \end{array} \right]_q$ modulo $\Phi_n(q)$. However, the literature is still sparse on congruences for $q$-trinomial coefficients.

The first aim of the paper is to prove the following congruence for the $q$-trinomial coefficients $\left[ \begin{array}{c} n \\ 0 \end{array} \right]_q$.

**Theorem 1.1** For any positive integer $n$, the following congruence holds modulo $\Phi_n(q)^2$:

$$\left( \begin{array}{c} n \\ 0 \end{array} \right) \equiv \begin{cases} (-1)^m (1 + q^m) q^{m(3m-1)/2}, & \text{if } n = 3m, \\ (-1)^m q^{m(3m+1)/2}, & \text{if } n = 3m + 1, \\ (-1)^m q^{m(3m-1)/2}, & \text{if } n = 3m - 1. \end{cases}$$
The elegant $q$-congruence (1.6) motivates us to establish the following congruence for the $q$-trinomial coefficients $\left(\begin{array}{c}2n \\ n\end{array}\right)_q$, which generalizes a conjectural $q$-congruence due to Apagodu and the author [4, Conjecture 1].

**Theorem 1.2** For any positive integer $n$, the following congruence holds modulo $\Phi_n(q)^2$:

$$
\left(\begin{array}{c}2n \\ n\end{array}\right)_q \equiv \begin{cases} 
2(-1)^m(1 + q^m)q^{m(3m-1)/2} - 3m(1 - q^{3m}) & \text{if } n = 3m, \\
2(-1)^m q^{m(3m+1)/2} - (3m + 1)(1 - q^{3m+1}) & \text{if } n = 3m + 1, \\
2(-1)^m q^{m(3m-1)/2} - (3m - 1)(1 - q^{3m-1}) & \text{if } n = 3m - 1.
\end{cases}
$$

The rest of the paper is organized as follows. In Sect. 2, we shall mainly establish an auxiliary congruence on sums of central $q$-binomial coefficients, which is interesting in itself. The proofs of Theorems 1.1 and 1.2 will be presented in Sects. 3 and 4, respectively.

### 2 An auxiliary result

To prove Theorems 1.1 and 1.2, we first recall the following two lemmas.

**Lemma 2.1** (See [13, Lemma 2.3]) For any non-negative integer $n$, we have

$$
(1 - q^n) \sum_{k=0}^{[n/2]} (-1)^k q^{k(k-1)/2} \left\lfloor \frac{n - k}{k} \right\rfloor = \begin{cases} 
(-1)^m(1 + q^m)q^{m(3m-1)/2}, & \text{if } n = 3m, \\
(-1)^m q^{m(3m+1)/2}, & \text{if } n = 3m + 1, \\
(-1)^m q^{m(3m-1)/2}, & \text{if } n = 3m - 1,
\end{cases}
$$

where $\lfloor x \rfloor$ denotes the integral part of real $x$.

**Lemma 2.2** (See [24, Lemma 3.3]) For $k = 1, \ldots, n - 1$, we have

$$
\left\lfloor \frac{2k - 1}{k} \right\rfloor \equiv (-1)^k q^{k(3k-1)/2} \left\lfloor \frac{n - k}{k} \right\rfloor \pmod{\Phi_n(q)}.
$$

We also require the following congruence on sums of the central $q$-binomial coefficients $\left[\frac{2k}{k}\right]$.

**Proposition 2.3** For any positive integer $n$, we have

$$
\sum_{k=1}^{[n/2]} q^{-k(k-1)/2} \left[\frac{2k}{k}\right] \equiv \frac{(1 - q)(1 - \mathcal{R}_n(q))}{1 - q^n} \pmod{\Phi_n(q)},
$$

where $\mathcal{R}_n(q)$ denotes the right-hand side of (2.1).
Remark

Sun [22, Theorem 1.1] proved that for any prime \( p \geq 5 \),

\[
\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8p}{3} E_{p-3} \pmod{p^2},
\]

(2.4)

where \( E_0, E_1, E_2, \ldots \) are Euler numbers. Notice that (2.3) is a \( q \)-analogue of (2.4) modulo \( p \).

Proof of (2.3).

By (2.2) and \( q^n \equiv 1 \pmod{\Phi_1(n)} \), we have

\[
\sum_{k=1}^{[n/2]} \frac{q^{-k(k-1)/2}}{[2k]_q} \left[ \frac{2k}{k} \right] = (1 - q) \sum_{k=1}^{[n/2]} \frac{q^{-k(k-1)/2}}{1 - q^{2k}} \left[ \frac{2k}{k} \right] \\
= (1 - q) \sum_{k=1}^{[n/2]} \frac{q^{-k(k-1)/2}}{1 - q^k} \left[ \frac{2k - 1}{k} \right] \\
\equiv (1 - q) \sum_{k=1}^{[n/2]} \frac{(-1)^k q^{k(k+1)/2}}{1 - q^k} \left[ \frac{n - k}{k} \right] \\
\equiv (q - 1) \sum_{k=1}^{[n/2]} \frac{(-1)^k q^{k(k+1)/2}}{1 - q^{n-k}} \left[ \frac{n - k}{k} \right] \pmod{\Phi_1(n)).
\]

(2.5)

We can rewrite (2.1) as

\[
\sum_{k=1}^{[n/2]} (-1)^k q^{k(k-1)/2} \left[ \frac{n - k}{k} \right] = \frac{\mathcal{R}_n(q) - 1}{1 - q^n}.
\]

(2.6)

Substituting (2.6) into the right-hand side of (2.5) gives

\[
\sum_{k=1}^{[n/2]} \frac{q^{-k(k-1)/2}}{[2k]_q} \left[ \frac{2k}{k} \right] \equiv \frac{(1 - q)(1 - \mathcal{R}_n(q))}{1 - q^n} \pmod{\Phi_1(n)}.
\]

(2.7)

as desired.

3 Proof of Theorem 1.1

For \( 1 \leq k \leq \lfloor n/2 \rfloor \), we have

\[
\binom{n}{k} = \frac{(1 - q^n)(1 - q^{n-1}) \ldots (1 - q^{n-k+1})}{(1 - q) \ldots (1 - q^k)} \\
\equiv \frac{(1 - q^n)(1 - q^{n-1}) \ldots (1 - q^{-k+1})}{(1 - q) \ldots (1 - q^k)}
\]

\( \square \)}
\[
\begin{align*}
&= (-1)^k q^{-k(k-1)/2} (1 - q^n) \\
&\equiv (-1)^k q^{-k(k+1)/2} (1 - q^n) \\
&\quad \pmod{\Phi_n(q)^2},
\end{align*}
\]

where we have used the fact that \(1 - q^n \equiv 0 \pmod{\Phi_n(q)}\). It follows from (1.2), (2.6) and the above that

\[
\binom{n}{0}_q = \sum_{k=0}^{[n/2]} q^{k^2} \begin{bmatrix} n \k \n - k \end{bmatrix} = 1 + \sum_{k=1}^{[n/2]} q^{k^2} \begin{bmatrix} n \k \n - k \end{bmatrix}
\equiv 1 + (1 - q^n) \sum_{k=1}^{[n/2]} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n - k \k \end{bmatrix}
= \mathcal{R}_n(q) \pmod{\Phi_n(q)^2}.
\]

This completes the proof of Theorem 1.1.

### 4 Proof of Theorem 1.2

Note that for \(1 \leq k \leq \lfloor n/2 \rfloor\), we have

\[
\begin{align*}
\begin{bmatrix} 2n \k \n + k \end{bmatrix} &= \frac{(1 - q^{2n})(1 - q^{2n-1}) \cdots (1 - q^{2n+1-k})}{(1 - q)(1 - q^2) \cdots (1 - q^k)} \\
&\equiv \frac{2(1 - q^n)(1 - q^{-1}) \cdots (1 - q^{-k+1})}{(1 - q)(1 - q^2) \cdots (1 - q^k)} \\
&\equiv \frac{2(-1)^{k-1}(1 - q^n)q^{-k(k-1)/2}}{1 - q^k} \quad \pmod{\Phi_n(q)^2},
\end{align*}
\]

and

\[
\begin{align*}
\begin{bmatrix} 2n - k \k \n - k \end{bmatrix} &= \frac{2n}{(1 - q^{n+1}) \cdots (1 - q^n)(1 - q^{n-k})(1 - q^{n-k+1}) \cdots (1 - q^{2n})} \\
&\equiv \frac{(-1)^k q^{-k(3k-1)/2}}{2} \begin{bmatrix} 2n \k \n \end{bmatrix} \begin{bmatrix} k \k - 1 \end{bmatrix} \quad \pmod{\Phi_n(q)}.
\end{align*}
\]

where we have utilized the fact \(q^n \equiv 1 \pmod{\Phi_n(q)}\).
It follows from (1.2), (4.1) and (4.2) that

\[
\binom{2n}{n}_q = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k(n+1-k)} \binom{2n}{k} \binom{2n-k}{n+k}
\]

\[
= \binom{2n}{n} + \sum_{k=1}^{\lfloor n/2 \rfloor} q^{k(n+1-k)} \binom{2n}{k} \binom{2n-k}{n+k}
\]

\[
= \binom{2n}{n} - (1-q^n) \frac{2n}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{q^{-k(k-1)}}{1-q^k} \binom{2k-1}{k} \pmod{\Phi_1(q)^2}.
\]

By (1.6), we have

\[
\binom{2n}{n} \equiv 1 + q^{n^2}
\]

\[
= 2 - (1-q^n)(1 + q^n + q^{2n} + \cdots + q^{(n-1)n})
\]

\[
= 2 - n(1-q^n) \quad (\text{mod } \Phi_1(q)^2).
\]

Finally, substituting (2.3) and (4.4) into the right-hand side of (4.3) gives

\[
\binom{2n}{n}_q \equiv 2 \mathcal{R}_n(q) - n(1-q^n) \quad (\text{mod } \Phi_1(q)^2),
\]

as desired.

**Acknowledgements** The author would like to thank Ofir Gorodetsky for discussions on \(q\)-trinomial coefficients and useful suggestions regarding the paper. The author is also grateful to the anonymous referee for his/her helpful comments which helped to improve the exposition of the paper.

**Data availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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