Scherk-Schwarz SUSY Breaking in Noncommutative Field Theory

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Abstract: Motivated by a recent conjecture [1] that quantum corrections and the UV/IR connection modify the classical relation between SUSY breaking and the cosmological constant to the phenomenologically acceptable: \( M_{\text{SUSY}} \sim M_P (\Lambda/M_P^4)^{1/8} \), we study SUSY breaking by boundary conditions in noncommutative field theories. In commutative field theory the violations of SUSY are finite and vanish as the inverse fourth power of the radius of the SUSY violating circle. We show that in the noncommutative theory, as a consequence of its UV/IR connection, the perturbative corrections to SUSY breaking are infinite. We have not yet performed the nonperturbative resummations to extract the true behavior of the system.

Keywords: Cosmological Constant, Noncommutative Field Theory.

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1. Introduction

In a recent paper [1] one of us proposed a new approach to the cosmological constant problem, in which the cosmological constant is an input parameter of the fundamental theory (related to the total number of physical states of the universe by the Bekenstein-Hawking formula). In addition, he proposed that SUSY is restored in the flat space limit. All SUSY breaking has a cosmological origin. The problem is then to explain why SUSY breaking is so large. Considerations based on classical SUGRA suggest the relation $M_{SUSY} \sim \Lambda^{1/4}$, which contradicts observational bounds. According to [1], large quantum gravitational renormalizations will change this formula to $M_{SUSY} \sim M_P(\Lambda/M_P^4)^{1/8}$, which might turn out to be compatible with experiment.

The key to this proposal is the UV/IR connection which implies that very high energy virtual states are not associated with short distances, and might be sensitive to the global structure of the universe.

In this paper, we would like to present a toy model with some of the properties described above. We do not intend this to be an accurate representation of nature, but merely a cartoon which emphasizes a particular feature of the real world. In supersymmetric quantum field theory, Scherk-Schwarz SUSY breaking by boundary conditions on a circle, is a very mild way to violate the symmetry. In a renormalizable field theory, it does not introduce any new divergences. SUSY breaking effects are finite and calculable in terms of the renormalized parameters of the infinite volume theory. They vanish rapidly with the radius of the SUSY violating circle.
Noncommutative SUSY theories in infinite volume seem to be well behaved in perturbation theory. In particular, the noncommutative Wess-Zumino model in four dimensions, which will be the focus of our study, seems to have a sensible perturbation expansion \[3\]. However, when we compactify on a circle (in one of the noncommutative directions) with SS boundary conditions, we find divergent terms in the perturbation expansion\(^1\). In particular, the ground state energy, which vanishes at infinite volume in both commutative and noncommutative theories, and is finite on a SS circle in the commutative theory, has an infinite contribution coming from two loop nonplanar diagrams in the noncommutative SS compactified theory. Similarly, the boson-fermion energy splitting has a divergent one loop nonplanar contribution. The divergences originate from the UV/IR correspondence, and the fact that, in a SS compactification, the boson, but not the fermion, has a zero momentum mode around the circle.

In previous studies of divergences in noncommutative field theory \[5\], it was argued that the divergences encountered in nonplanar graphs were an artifact of perturbation theory and could be eliminated by an adroit resummation. We have attempted to apply that wisdom to the present model, but so far without success. Simple Dyson resummation of propagator graphs does not eliminate the vacuum energy divergences. As a consequence, we do not yet know the correct nonperturbative behavior of our model. However, we believe that our perturbative computations establish the fact that UV/IR conspiracies can dramatically enhance the effect of SUSY breaking in the global structure of spacetime.

Our calculations are presented in the next section. In the conclusions we propose a large \(N\) version of our model which might allow us to extract the nonperturbative corrections. We point out however that the attempt to extract the nonperturbative behavior of the model may run afoul of the well known triviality of the WZ model. Although we have verified that very similar divergences occur in SS compactified \(N = 4\) Super Yang-Mills theory, we have no scheme at hand for estimating its nonperturbative behavior.

Finally, we wish to emphasize that the model of the present paper is just a toy which reproduces a single qualitative property of the physics conjectured in \[1\] - namely the enhancement of soft SUSY breaking due to the UV/IR connection. We do not expect to calculate the critical exponent relating the cosmological constant to the size of SUSY breaking, in this paper. Furthermore, it is extremely important to realize that the vacuum energy we compute in noncommutative field theories has no connection to the cosmological constant. It is merely an order parameter for SUSY breaking in a theory.\(^2\)

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\(^1\)Note that this is not the same as studying the finite temperature behavior of the noncommutative theory\[6\], because the SS boundary conditions are taken around one of the noncommutative directions.
decoupled from gravity.

2. Calculations

2.1 Calculation of the vacuum energy

The component Lagrangian of our model is

\[
\mathcal{L} = i \partial_{\mu} \psi^{*} \sigma^{\mu} \psi + \partial_{\mu} \phi^{*} \partial_{\mu} \phi - 1/2 \psi \psi - 1/2 \psi^{*} \psi^{*} - g \psi \psi^{*} \phi - g \psi^{*} \psi^{*} \phi^{*} - |m \phi + g \phi^{*} \phi|^{2}.
\]  

(2.1)

The one loop vacuum graphs are purely planar and give the same results in commutative and noncommutative WZ models. They are finite and of order $1/R^{4}$. At two loops there are both planar and nonplanar graphs. They have the same combinatorics, and differ only in the presence or absence of a Moyal phase. We will write only the expressions for the nonplanar diagrams. There are three of them at this order, given by Fig. 1 (a,b,c).

Fig.1a: Planar and nonplanar two loop contributions to the vacuum energy.

Fig.1b: Planar and nonplanar two loop contributions to the vacuum energy.
Each has a factor of $\frac{1}{R^2} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3}$. The integrands are (bold face letters denote vectors of the three continuous momenta. Scalar products are Minkowskian, with positive time signature. An $i\epsilon$ prescription is left implicit):

$$2ig^2 \sum_{(n,l)} \left( \frac{e^{i\theta_R (2nq_2 - 2lp_2)}}{(-\frac{4n^2\pi^2}{R^2} + p^2 - m^2) (\frac{4l^2\pi^2}{R^2} + q^2 - m^2)} \right). \quad (2.2)$$

$$2im^2g^2 \sum_{(n,l)} \left( \frac{e^{i\theta_R (2nq_2 - 2lp_2)}}{(-\frac{4n^2\pi^2}{R^2} + p^2 - m^2) (\frac{4l^2\pi^2}{R^2} + q^2 - m^2) (\frac{4(l+n)^2\pi^2}{R^2} + (p+q)^2 - m^2)} \right). \quad (2.3)$$

$$4ig^2 \sum_{(n,l)} \left( \frac{(pq - \frac{(2n+1)(2l+1)}{R^2})e^{i\theta_R ((2n+1)q_2 - (2l+1)p_2)}}{(-\frac{(2n+1)^2\pi^2}{R^2} + p^2 - m^2) (\frac{(2l+1)^2\pi^2}{R^2} + q^2 - m^2) (\frac{4(l+n+1)^2\pi^2}{R^2} + (p+q)^2 - m^2)} \right). \quad (2.4)$$

Allowing shifts of momenta, and using momentum conservation and the antisymmetry of the Moyal phase, we can rewrite the sum of these integrands as

$$\frac{2ig^2}{R^2} \sum_{n,l} \left[ \frac{e^{i\theta_R (2nq_2 - 2lp_2)}}{(-\frac{4n^2\pi^2}{R^2} + p^2 - m^2) (\frac{4l^2\pi^2}{R^2} + q^2 - m^2)} \right. \left. - 2 \frac{e^{i\theta_R (2nq_2 - (2l+1)p_2)}}{(-\frac{4n^2\pi^2}{R^2} + p^2 - m^2) (\frac{(2l+1)^2\pi^2}{R^2} + q^2 - m^2)} \right. \left. + \frac{e^{i\theta_R ((2n+1)q_2 - (2l+1)p_2)}}{(-\frac{(2n+1)^2\pi^2}{R^2} + p^2 - m^2) (\frac{4(l+n+1)^2\pi^2}{R^2} + (p+q)^2 - m^2)} \right] \quad (2.5)$$

This is all to be integrated over three momenta with the usual phase space. Note that if it were not for the mismatch between the quantization rules for fermionic and bosonic momenta, the integrand would vanish.
Doing the integral over momenta, we obtain the two loop nonplanar contribution to the vacuum energy:

\[
E/V = \frac{2g^2}{16\pi^2\theta^2} \sum_{(n,l)} \left[ e^{-\frac{g}{\pi} \left(|2n\sqrt{4l^2/R^2+m^2}|+|2l\sqrt{4n^2/R^2+m^2}|\right)} \frac{4|nl|}{|2n(2l+1)|} - 2e^{-\frac{g}{\pi} \left(|2n\sqrt{(2l+1)^2/R^2+m^2}|+|2l\sqrt{(2n+1)^2/R^2+m^2}|\right)} \frac{|2n(2l+1)|}{|(2n+1)(2l+1)|} + e^{-\frac{g}{\pi} \left(|(2n+1)\sqrt{(2l+1)^2/R^2+m^2}|+|(2l+1)\sqrt{(2n+1)^2/R^2+m^2}|\right)} \frac{|(2n+1)(2l+1)|}{|2(2l+1)|}\right].
\]

The terms in this sum where either or both of \( n \) and \( l \) vanish, are quite divergent, and the divergence does not cancel. The sum over large values of the discrete momentum is quite convergent, and Bose-Fermi cancellation is also restored.

### 2.2 Calculation for the masses

Another indicator of SUSY breaking is the spectrum of elementary excitations. There are three nonplanar one loop graphs which contribute to the boson self energy (Fig. 2 a,b,c) and one fermion self energy graph (Fig. 3) in the same order.

![Fig.2: Nonplanar one loop graphs which contribute to the boson self energy.](image_url)

![Fig.3: Nonplanar one loop graphs which contribute to the fermion self energy.](image_url)
We begin with the boson self energy. The integrands of the three diagrams are

\[
I_{B1} = \frac{4e^{2i\pi\theta}(nk_2-lp_2)}{-4\pi^2l^2 + k^2 - m^2} + \frac{-4e^{i\pi\theta}(2nk_2-(2l+1)p_2)}{-n^2(2l+1)^2 + k^2 - m^2} + e^{i\pi\theta}(2nk_2-(2l+1)p_2)(-8n^2\pi^2 + 2p^2 - 4m^2)
\]

\[
I_{B2} = \frac{4m^2e^{2i\pi\theta}(nk_2-lp_2)}{-4\pi^2l^2 + k^2 - m^2} + \frac{\pi^2(2l+1)^2}{(n^2l^2 + (p+k)^2 - m^2)} + \frac{-4\pi^2(nl)^2}{k^2 - m^2}
\]

The dispersion relation for the bosons has singularities, in stark contrast to the dispersion relation for fermions, as will be shown in what follows.

The singular behaviour of the boson dispersion relation occurs for special momenta along the noncommuting directions \(x_2\) and \(x_3\). These singularities appear when integrating over the large Fourier components of the various internal lines contributing to the two-point function of bosons. The external momenta at which these singularity occur are: \(p_3 = n/R = 0\) and \(p_2 = n'R/\theta\), where \(n\) and \(n'\) are integers.

This can be seen when one adds the three diagrams (Fig 2 a,b,c):

\[
\Gamma_B(p) = \frac{1}{R} \sum_l \int \frac{d^3p}{(2\pi)^3} (I_{B1} + I_{B2} + I_{B3})
\]

, where \(\Gamma_B(p)\) is the nonplanar contribution to the 1PI two-point function.

Indeed it is the large \(l\) part of the sum and large \(k\) region of integration in the integrals that appear in the previous equation, that are responsible for the singularities. For these value of the external momenta, the Moyal phase factor disappears and therefore the large momenta contributions remain unsuppressed, as in the case of planar diagrams.

One can single out two divergent contributions to \(\Gamma(p)\). The first one is:

\[
\frac{g^2}{2\pi}(p^2 - 4n^2\pi^2)\log(1 - 2e^{-4\pi^2|\theta|n}}\cos\frac{2\pi\theta p_2}{R} + e^{-8\pi^2|\theta|n}}\bigg|_{n=0,p_2=n'R/\theta}
\]

This expression is logarithmically divergent at \(n = 0\) and \(p_2 = n'R/\theta\).

When evaluated at \(\theta = 0\) for arbitrary external momenta, expression (2.11) becomes the usual nonplanar contribution to the wave function renormalization of the scalars, in the commutative Wess-Zumino model.

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The terms proportional to $m^2$ coming from of (2.7) and (2.9) to (2.10) are not singular at $n = 0$. This is related to the fact that in the commutative limit, $\theta = 0$, the superpotential is protected by a non-renormalization theorem.

The remaining divergent contribution to the two-point function is:

$$
\frac{g^2}{\pi |\theta n|} \sum_l \left( e^{-2|\theta n|(m^2 + \frac{4l^2}{R^2})^{1/2}} - e^{-2|\theta n|(m^2 + (2l + 1)^2/4R^2)^{1/2} - i\theta (2l + 1)p_2} \right) |_{n=0, p_2 = (2n' + 1)\frac{R}{\theta}} (2.10)
$$

where $n'$ is an integer.

This has a power law divergence in $n$ at $n = 0$ and $p_2 = (2n' + 1)\frac{R}{\theta}$. This somewhat peculiar formula resembles the formula for string windings around a circle of radius $R$ with a string tension $\theta$. We note however, that it is $x_3$ that is compact and not $x_2$, so the notion of winding around the $x_2$ direction does not seem to make sense.

A more reasonable explanation of the quantized values of $p_2$ is that $x_3/R$ is an angle. Thus, the conjugate dimension $x_2$ is in some sense quantized in units of $\frac{R}{\theta}$. This in turn implies that $p_2$, the momentum conjugate to $x_2$, is an angular variable so that expressions involving $p_2$ must be periodic with period $\frac{R}{\theta}$.

These infrared divergences cannot be treated by simply resumming a geometric series in the 1PI two-point function. Indeed this procedure does not resolve, in the case of non-commuting compact dimensions, the infrared singularities that occur in the vacuum energy, although it does displace the apparent $n = 0$ singularities of the zeroth order propagator to infinity. The latter observation was also made in [4]. However, this is not sufficient to make the vacuum energy finite. Note that in a SUSY violating (e.g. purely bosonic) noncommutative field theory in infinite spacetime, the measure of integration over momenta gives a finite vacuum energy after Dyson resummation of the lowest order nonplanar self energy. With SS compactification, the more serious IR divergences of the bosonic zero mode remain even after Dyson resummation, despite the fact that the geometric series in the 1PI two-point function generates a well behaved dispersion relation. One way to render these diagrams finite would be to add a phase, $\eta$, to the boundary conditions for both bosons and fermions. This may be a useful intermediate regulator, but we really want to understand the $\eta = 1$ limit, where the IR behavior is singular. In the Conclusions we will suggest a systematic, large N approach to the IR behavior of this model, which may give a finite answer for the vacuum energy.

The fermion self-energy, $\Gamma_F(p)$, is given by the diagram depicted in Fig 3.

$$
\Gamma_F(p) = -\frac{4g^2}{R} \sum_l \int \frac{d^3p}{(2\pi)^3} \frac{-\frac{(2n + 4l + 1)\pi r_3}{R} + (p + k)\tau}{\left[ -\frac{(2n + 2l + 1)^2}{R} + (p + k)^2 - m^2 \right] \left[ -\frac{4l^2 \pi^2}{R^2} + k^2 - m^2 \right]} e^{\frac{4\pi}{R}((2n+1)k_2 - 2lp_2)} (2.11)
$$
where the $\tau_i$ are Pauli matrices and $\tau_0$ is the identity.

The potential singularities for the fermionic two-point function, as in the case of the scalars, originate in the region of large $l$ for the sum and large $k$ for the integral.

The contribution to $\Gamma_F(p)$ from this region is:

$$
\Gamma_F(p) \sim -\frac{4g^2}{2\pi} \left( -(2n+1)\pi \tau_3 + p \tau \right) \log \left( 1 - 2e^{-\frac{2\pi^2(2n+1)|\theta|}{R^2}} \cos \frac{2\pi \theta p_2}{R} + e^{\frac{-4\pi^2(2n+1)|\theta|}{R^2}} \right) 
$$

(2.12)

In contrast to the bosonic case, this contribution is non-singular at $n = 0$. Notice that $\Gamma_F(p)$ is singular in the large $R$ limit as it should be, since this gives the nonplanar contribution, in the commutative limit, to the wavefunction renormalization of the fermions.

The extreme sensitivity to SUSY violating boundary conditions which we have encountered here is unlike anything in local quantum field theory. Here is a set of words which captures the spirit of what is going on: Individual graphs in a local supersymmetric theory have UV divergences which cancel between bosons and fermions. In the noncommutative theory, these can be converted into IR divergences by the UV/IR connection. With SS boundary conditions, bosonic lines have discrete zero modes, while fermions do not, so the IR divergences do not cancel. Thus, quantum corrections to SUSY breaking are enhanced because the high energy states have large extent in the SS direction, and are thus sensitive to SUSY breaking. In local field theory, high energy states have very short wavelengths in the SS direction and are insensitive to the boundary conditions.

3. Conclusions

We believe the calculations that we have already done put the conjectures of [1] on a much firmer footing. We have exhibited a system where SUSY breaking via the large scale structure of spacetime seems to be much larger than local field theory would have led us to expect. Nonetheless, one would be more comfortable if we had obtained finite answers. Usually (meaning of course in local field theory), IR divergences represent merely the breakdown of perturbation theory and appropriate nonperturbative approximations give finite answers which do not have an asymptotic expansion in the couplings.

We have seen that the simple Dyson resummations of [5][4] do not give us a finite vacuum energy. We would like to propose instead a large $N$ resummation of the model. The simplest model of this type has a singlet chiral superfield $\Sigma$ and and $O(N)$ vector
\[ W = \mu^2 \Sigma^2 + m^2 (\Phi^i)^2 + g \Sigma (\Phi^i)^2. \] (3.1)

We can make it noncommutative and impose SS boundary conditions in a manner precisely analogous to what we have done for the single field model. The large N approximation leads to a very complicated set of self consistent integral equations, which we have not yet mastered. The gap functions must be chosen to depend on the momenta in the noncommutative directions. We hope to report on approximate solutions of these equations in the future.

We note however the possibility that this line of research may run into difficulty. The commutative WZ model does not really have an interacting continuum limit. Since the planar diagrams in the commutative and noncommutative theories are identical, it is hard to imagine that the noncommutative theory is any better behaved in the UV. On the other hand, in the noncommutative theory UV divergences get transformed into IR divergences. The standard argument that IR divergences are only an artifact of perturbation theory seems somewhat shaky in the context of a theory whose non-perturbative existence is doubtful. Still, some hope can be gleaned from the remark that the power law IR divergences we have encountered are not obviously connected to the coupling constant renormalization which renders the model trivial. Only a full investigation of the large N model can produce a conclusive answer to this puzzle.

We do believe that it is likely that many SUSY violating noncommutative field theories simply do not exist. That is, their IR behavior is not independent of high energy physics. Thus, were it not for the high degree of supersymmetry of the model they studied, we would be suspicious of the decoupling arguments of [2]. One might regard the absence of a continuum limit for SUSY violating noncommutative field theories as a baby version of the result conjectured in [1], namely that there are no asymptotically flat SUSY violating vacua of M theory.

If the large N WZ model is ill defined, we might have to turn to \( N = 4 \) SYM theories to study the phenomenon discovered in this paper. Here the problem is that there is no known soluble nonperturbative approximation which respects gauge invariance. Perhaps the AdS/CFT correspondence or NCOS theory[7] can shed some light on this subject.

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