BI-HARMONIC MAPPINGS AND J. C. C. NITSCHE TYPE CONJECTURE

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ABSTRACT. In this note it is formulated the J. C. C. Nitsche type conjecture for bi-
harmonic mappings. The conjecture has been motivated by the radial bi-harmonic map-
pings between annuli.

1. INTRODUCTION

The bi-harmonic equation of four times continuously differentiable complex-valued
functions \( u \) defined in an open and connected set is

\[
\Delta^2 u = \Delta(\Delta u) = 0.
\]

It is well known that, a planar harmonic mapping \( u \) is bi-harmonic, if and only if \( u(z) = |z|^2 g(z) + h(z) \), where \( g \) and \( h \) are harmonic mappings, i.e. the mappings \( w \) satisfying the Laplace equation \( \Delta w = 0 \) in some subdomain \( \Omega \) of the complex plane \( \mathbb{C} \). Every analytic function is a harmonic mapping and every bi-holomorphic function is a harmonic diffeomorphism. The set \( A(1,t) := \{ z : 1 < |z| < t \} \subset \mathbb{C} \) is called an annulus. It is well known the Schottky theorem which assert that two annuli can be mapped by mean of a bi-holomorphic mapping if and only if they have the same modulus.

J. C. C. Nitsche [13] by considering the complex-valued univalent harmonic functions

\[
f(z) = \frac{ts - t^2}{(1-t^2)^2} \frac{1}{\bar{z}} + \frac{1-t^2}{1-t^2} z,
\]

showed that an annulus \( 1 < |z| < t \) can then be mapped onto any annulus \( 1 < |w| < s \) with

\[
s \geq n(t) := \frac{1+t^2}{2t}.
\]

J. C. C. Nitsche conjectured that, condition (1.2) is necessary as well. The critical Nitsche map with zero initial speed is

\[
f(z) = \frac{1+|z|^2}{2z}.
\]

This mean that this harmonic function make the maximal distortion of rounded annuli \( A(1,t) \).

Nitsche also showed that \( s \geq s_0 \) for some constant \( s_0 = s_0(t) > 1 \). Thus although the annulus \( 1 < |z| < t \) can be mapped harmonically onto a punctured disk, it cannot be mapped onto any annulus that is “too thin”. For the generalization of this conjecture to \( \mathbb{R}^n \) and some related results we refer to [10]. For the case of hyperbolic harmonic mappings we refer to [4]. Some other generalization has been done in [11]. The Nitsche conjecture for Euclidean harmonic mappings is settled recently in [5] by Iwaniec, Kovalev and Onninen, showing that, only radial harmonic mappings \( g(z) = e^{i\alpha} f(z) \), where \( f \) is defined in (1.1),

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which inspired the Nitsche conjecture, make the extremal distortion of rounded annuli. For some partial result toward the Nitsche conjecture and some other generalizations we refer to the papers [1, 2, 3, 6, 7, 8, 12, 14].

In this paper, we will state a similar conjecture with respect to bi-harmonic mappings. In order to do this, in Section 2 we will find all radial bi-harmonic maps between annuli. In Section 3 we will prove some technical results concerning the radial bi-harmonic mappings. Section 4 contains the main result which assert that, the class of radial bi-harmonic diffeomorphisms between two annuli $A(1, t)$ and $A(1, s)$ is nonempty if and only if $s \geq \sigma(t)$ where $\sigma(t)$ is some constant larger than 1. It remains an open question, whether this phenomenon remains true for the whole class of bi-harmonic diffeomorphisms as in harmonic case.

2. Radial solutions of bi-harmonic equation

A mapping $f$ is called radial if there exists a constant $\varphi$ and a real function $g$ such that

$$f(re^{i\theta}) = g(r)e^{i(\theta + \varphi)}.$$  

It is well known that, a radial solution $u$ of the harmonic equation is given by

$$u(z) = Az + B/\bar{z},$$

where $a$ and $b$ are two complex constants. To prove this we start by Laplacean in polar coordinates. Let $U(r, \theta) = u(re^{i\theta}).$ Then $\Delta U = 0$ if and only if

$$\Delta U := \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} = 0.$$  

Assuming that $U(r, \theta) = p(r)e^{i\theta}$ we obtain the equation

$$\frac{1}{r^2} (r^2 p''(r) + rp'(r) - p(r)) = 0.$$  

By taking the change of variables $t = \log r$ and $P(t) = p(r)$ we arrive at

$$P''(t) - P(t) = 0$$

and so, we obtain $P(t) = Ae^t + Be^{-t}$ and therefore $p(r) = Ar + B/r.$ Thus

$$u(z) = Az + B/\bar{z}.$$  

If $v$ is bi-harmonic, then the mapping $u = \Delta v$ is harmonic. If $v$ is radial, then $u$ is radial as well. It follows that

$$\Delta v = Az + \frac{B}{\bar{z}}$$

for some real constants $A$ and $B.$ Take $V(r, \theta) = v(re^{i\theta}).$ Then we have

$$V(r, \theta) = g(r)e^{i\theta}.$$  

Now, we put

$$V(r, \theta) = g(r)e^{i\theta}.$$  

Then (2.1) is equivalent with

$$\frac{1}{r} (r^2 g''(r) + rg'(r) - g(r)) = Ar + \frac{B}{r}.$$  

By taking again the change of variables $t = \log r,$ $G(t) = g(r)$ we arrive at the equation

$$G''(t) - G(t) = Ae^{3t} + Be^t.$$
Thus
\[ G(t) = de^{-t} + ae^t + bte^t + ce^{3t}, \quad a, b, c, d \in \mathbb{R} \]
and therefore
\[ (2.2) \quad g(r) = \frac{d}{r} + ar + br \log r + cr^3. \]
It follows that, every radial solution of bi-harmonic equation has the form:
\[ f(z) = \frac{d}{z} + az + bz \log |z| + c|z|^2z. \]

3. THE TECHNICAL LEMMATA

Remark 3.1. Throughout the paper we will assume that the bi-harmonic mapping \( f(re^{it}) = g(r)e^{it} : A(1, t) \to A(1, s) \) maps the inner boundary onto the inner boundary of corresponding annulus, i.e. \( g \) defined in (2.2) is increasing. A similar analysis works for the case when \( g \) is a decreasing function. A radial harmonic mapping \( f \), will be called homogeneous if the initial and final speeds are equal to zero, i.e. if \( g'(1) = g'(t) = 0 \)

Lemma 3.2. Assume that \( t > 1 \) and \( s > 1 \). If the function \( g \) defined by (2.2) satisfies
\[ g(1) = 1, \ g(t) = s, \ g'(1) = x > 0, \ g'(t) = y > 0, \]
then there exists a function \( h \), such that \( h(1) = 1, \ h(t) = s, \ h'(1) = 0, \ h'(t) = 0, \)
\[ g(r) = A(r) + B(r)s + U(r)x + V(r)y, \]
and
\[ h(r) = A(r) + B(r)s \]
where
\[ A = \frac{(3 + r^2)(r^2 - t^2)(r^2 - 1) + 2r^2(3 - 2t^2 - t^4) \log r + 2(r^4 + t^4 + r^2(-3 + t^4)) \log t}{4r(-1 + t^2)(1 - t^2 + (1 + t^2) \log t)}, \]
\[ B = \frac{2r^2(-1 - 2t^2 + 3t^4) \log r - (1 + r^2)((-1 + t^2)(r^2 + 3t^2) + 2(-1 + r^2)t^2) \log t}{4r(-1 + t^2)(1 - t^2 + (1 + t^2) \log t)}, \]
\[ U = \frac{-2r^2(-1 + t^2)^2 \log r + (-1 + r^2)((r^2 - t^2)(-1 + t^2) - 2(r^2 - t^4)) \log t}{4r(-1 + t^2)(1 - t^2 + (1 + t^2) \log t)}, \]
and
\[ V = \frac{-2r^2(-1 + t^2)^2 \log r + (-1 + r^2)((r^2 - t^2)(-1 + t^2) + 2(-1 + r^2)t^2) \log t}{4r(-1 + t^2)(1 - t^2 + (1 + t^2) \log t)}. \]

Proof: The proof is length but straightforward and is therefore omitted.

Lemma 3.3. Under the conditions and notation of Lemma 3.2 there hold the following relations
\[ (3.1) \quad B'(r) > 0, \quad \text{for} \quad 1 < r < t < \infty, \]
\[ (3.2) \quad -A'(r) > 0, \quad \text{for} \quad 1 < r < t < \infty, \]
\[ (3.3) \quad A'(r^+) = B'(r^+) = 0, \quad \text{for} \quad 1 < t < \infty, \]
\[ (3.4) \quad -\frac{d}{dr} A'(r) > 0, \quad \text{for} \quad 1 < r < t < \infty, \]
It follows that
\begin{align}
(3.5) & \quad \frac{d}{dr} \frac{U'(r)}{B'(r)} > 0, \quad \text{for} \quad 1 < r < t < \infty, \\
(3.6) & \quad \frac{d}{dr} \frac{V'(r)}{B'(r)} > 0, \quad \text{for} \quad 1 < r < t < \infty, \\
(3.7) & \quad \lim_{r \to t^-} -\frac{A'(r)}{B'(r)} = \frac{t(3 - 4t^2 + t^4 + 4t^2 \log t)}{2 - 2t^2 + \log t + 3t^4 \log t}, \\
(3.8) & \quad \lim_{r \to t^-} -\frac{U'(r)}{B'(r)} = \frac{t(-1 + t^4 - 4t^2 \log t)}{2 - 2t^2 + \log t + 3t^4 \log t}, \\
(3.9) & \quad \lim_{r \to t^+} -\frac{V'(r)}{B'(r)} = -\infty, \\
(3.10) & \quad \lim_{r \to t^+} -\frac{A'(r)}{B'(r)} = -\frac{t(-2t^2(-1 + t^2) + (3 + t^4) \log t)}{1 - 4t^2 + 3t^4 - 4t^2 \log t}, \\
(3.11) & \quad \lim_{r \to t^+} -\frac{U'(r)}{B'(r)} = -\infty \\
\text{and} \quad (3.12) & \quad \lim_{r \to t^+} -\frac{V'(r)}{B'(r)} = \frac{t(-1 + t^4 - 4t^2 \log t)}{1 - 4t^2 + 3t^4 - 4t^2 \log t}.
\end{align}

The proof of Lemma 3.2 lies on the following lemma.

**Lemma 3.4.** For all $1 < t$, $1 < r < t$,
\begin{enumerate}
\item[(a)] $2r^2(-1 - 2t^2 + 3t^4) \log r + (1 - r^2)(3(r^2 - t^2)(-1 + t^2) + 2(1 + 3r^2)t^2 \log t) > 0,$
\item[(b)] $-2r^2(-3 + 2t^2 + t^4) \log r + (r^2 - 1)(3(r^2 - t^2)(-1 + t^2) + 2(3r^2 + t^4) \log t) < 0,$
\item[(c)] $(-1 + t^2)(3(-1 + r^2)(-r^2 + t^2) + 2r^2(1 + 3t^4) \log r) > 2(1 + 2r^2 - 3r^4) \log t > 0,$
\item[(d)] $2(r^4 - t^2)(t^2 - 1) \log r + (1 - r^2)((r^2 - t^2)(-1 + t^2) + 2(r^2 - 1)t^2 \log t) > 0$
\item[(e)] $1 - t^2 + (1 + t^2) \log t > 0.$
\end{enumerate}

**Proof of Lemma 3.4.** By taking the substitution $\alpha = r^2$, $\beta = t^2$, the inequality (d) of the lemma is equivalent with the inequality
\[ h(\beta) := (\alpha^2 - \beta)(-1 + \beta) \log \alpha + (\alpha - 1)((\beta - \alpha)(\beta - 1) + (1 - \alpha)\beta \log \beta) \geq 0. \]

A computation gives
\[ h'(\beta) = (1 + \alpha^2 - 2\beta) \log \alpha - (1 + \alpha)(2(\alpha - \beta) + (1 + \alpha) \log \beta), \]
\[ h''(\beta) = -\frac{(1 + \alpha)(1 + \alpha - 2\beta)}{\beta} - 2 \log \alpha \]
and
\[ h'''(\beta) = \frac{(1 + \alpha)^2}{\beta^2}. \]

It follows that $h''$ is increasing, and therefore
\[ h''(\beta) \geq h''(\alpha) = -1/\alpha + 2 \log \alpha. \]
But
\[(−1/α + α − 2 \log α)' = \frac{(α − 1)^2}{α^2},\]
and therefore \(-1/α + α − 2 \log α ≥ −1 + 1 − 2 \log 1 = 0\). It follows that
\[h''(β) ≥ 0.\]
Thus, we have
\[h'(β) ≥ h'(α) = 0.\]
It follows finally that
\[h(α) ≥ h(β) = 0.\]
The proofs of (a), (b) and (c) and (e) are similar to the proof of (d) and are therefore omitted.

**Proof of Lemma 3.3.** First of all
\[A'(r) = \frac{2r^2(3 - 2t^2 - t^4) \log r + (r^2 - 1)(3(r^2 - t^2)(−1 + t^2) + 2(3r^2 + t^4) \log t)}{4r^2(−1 + t^2)(1 − t^2 + (1 + t^2) \log t)},\]
\[B'(r) = \frac{2r^2(3t^4 - t^2 - 1) \log r + (1 - r^2)(3(r^2 - t^2)(−1 + t^2) + 2(1 + 3r^2)t^2 \log t)}{4r^2(−1 + t^2)(1 − t^2 + (1 + t^2) \log t)},\]
\[U'(r) = \frac{(1 + 3r^2)(r^2 - t^2)(t^2 - 1) + 2r^2(1 - t^2)^2 \log r - 2(3r^4 - t^4 - r^2(1 + t^4)) \log t}{4r^2(−1 + t^2)(1 − t^2 + (1 + t^2) \log t)}\]
and
\[V'(r) = \frac{2r^2(1 - t^2)^2 \log r + (1 - r^2)((−1 + t^2)(3r^2 + t^2) + 2(1 + 3r^2)t^2 \log t)}{4r^2(−1 + t^2)(1 − t^2 + (1 + t^2) \log t)}.\]
Lemma 3.4(a) and (b) imply that \(A'(r) < 0\) and \(B'(r) > 0\). This proves the inequalities (3.1) and (3.2). The relation (3.3) follows at once.
The derivative of the quotient function \(-A'(r)/B'(r)\) is
\[12rt(−1 + t^2)(1 − t^2 + (1 + t^2) \log t)\]
\[\times \frac{2(r^4 - t^2)(−1 + t^2) \log r - (−1 + r^2)((r^2 - t^2)(−1 + t^2) + 2(−1 + r^2)t^2 \log t)}{(2r^2(1 + 2t^2 - 3t^4) \log r + (−1 + r^2)(3(r^2 - t^2)(−1 + t^2) + 2(1 + 3r^2)t^2 \log t))^2}.\]
Thus, Lemma 3.4(d) and (e) imply that the last expression is positive. Thus (3.4) is proved.
The proofs of (3.5) and (3.6) are similar. The proofs of relations (3.7)-(3.12) are similar to each other and follow by l’Hospital’s rule. See Figure 1 for the geometric interpretation of (3.11) and (3.12).

**Lemma 3.5.** For every \(t > 1\), and \(τ = \frac{1 + t}{2}\), we have
\[-\frac{A'(τ)}{B'(τ)} > 1.\]

**Proof.** Namely
\[-\frac{A'(τ)}{B'(τ)} > 1\]
if and only if
\[φ(t) := (1 − t^2)(9 + 30t + 9t^2 + 8(1 + t^2) \log τ) − 2t(9 + 18t + 17t^2 + 4t^3) \log t > 0.\]
On the other hand
\[φ(5)(t) = \frac{12(9 + 6t + 2t^2 + 6t^3 + 9t^4)}{t^4(1 + t^2)} > 0\]
and $\varphi^{(k)}(1) = 0$, for $k = 0, 1, 2, 3, 4$. We therefore deduce the following sequence of inequalities $\varphi^{(4)}(t) > 0$, $\varphi^{(3)}(t) > 0$, $\varphi''(t) > 0$, $\varphi'(t) > 0$ and $\varphi(t) > 0$ for $t > 1$. □

**Lemma 3.6.** Under the conditions and notation of Lemma 3.2 we have $U'(r) = V'(r)$ if and only if

$$r = \rho := \sqrt[6]{\frac{1}{6} + \frac{t^2}{6} + \frac{1}{6} \sqrt{1 + 14t^2 + t^4}}.$$ 

Moreover

(3.13) 

$$-U'(-\rho) = -V'(\rho) > 0, \quad \frac{-A'(\rho)}{B'(\rho)} > 1.$$ 

**Proof.** As

$$U'(r) - V'(r) = \frac{-3r^4 + t^2 + r^2(1 + t^2)}{2r^2(-1 + t^2)},$$ 

it follows that

$$U'(r) = V'(r) \text{ if and only if } r = \rho := \sqrt[6]{\frac{1}{6} + \frac{t^2}{6} + \frac{1}{6} \sqrt{1 + 14t^2 + t^4}}$$

or what is the same

$$t = \frac{\rho \sqrt{-1 + 3\rho^2}}{\sqrt{1 + \rho^2}}.$$ 

By taking the substitution $\kappa = \rho^2$ and $\eta = t^2$, we obtain

$$-U'(\rho) = \frac{(1 + 3\rho^2)(\rho^2 - t^2)(t^2 - 1) - 2\rho^2(1 - t^2)\log \rho - (6\rho^4 - 2t^4 - 2\rho^2(1 + t^4))\log t}{4\rho^2(-1 + t^2)(-1 + t^2 + (1 + t^2)\log t)}$$

$$= \frac{(1 + 3\kappa)(\kappa - s)(-1 + \eta) - \kappa(-1 + \eta)^2\log \kappa + (\kappa - 3\kappa^2 + \eta^2 + \kappa\eta^2)\log \eta}{(2\kappa(-1 + \eta)(2 - 2\eta + (1 + \eta)\log \eta)}.$$ 

Since

$$\eta = \frac{\kappa(3\kappa - 1)}{1 + \kappa}$$

and $(2 - 2\eta + (1 + \eta)\log \eta) > 0$ for $\eta > 1$,

we have to prove that

$$L(\kappa) := (1 + 3\kappa)(\kappa - s)(-1 + \eta) - \kappa(-1 + \eta)^2\log \kappa + (\kappa - 3\kappa^2 + \eta^2 + \kappa\eta^2)\log \eta > 0.$$
Then
\[ L(\kappa) = \frac{\kappa(-1 - 2\kappa + 3\kappa^2)}{(1 + \kappa)^2} K(\kappa) \]
where
\[ K(\kappa) = -2 - 4\kappa + 6\kappa^2 + (-1 - 2\kappa + 3\kappa^2) \log \kappa + (1 - 2\kappa - 3\kappa^2) \log \left( \frac{\kappa(-1 + 3\kappa)}{1 + \kappa} \right). \]
Further
\[ K''(\kappa) = \frac{4(1 - 4\kappa + 14\kappa^2 + 12\kappa^3 + 9\kappa^4)}{\kappa^2(-1 + 2\kappa + 3\kappa^2)^2} > 0. \]
Moreover
\[ K''(1) = 0, \quad K'(1) = 0, \quad K(1) = 0 \]
and therefore
\[ K(\kappa) > 0. \]
Since \( r \to -A'(r)/B'(r) \) is increasing and
\[ \rho = \sqrt{\frac{1}{6} + \frac{t^2}{6} + \frac{1}{6}(1 + 4t^2 + t^4)} > \tau = \frac{1 + t}{2}, \]
by Lemma 3.5 and (3.4), we obtain
\[ -\frac{A'(-A)}{B'(-A)} > -\frac{A'(-\tau)}{B'(-\tau)} > 1. \]

4. THE MAIN RESULTS

As a direct corollary of Lemma 3.2 we obtain

**Theorem 4.1.** If \( f(re^{i\theta}) = h(r)e^{i\theta}, \ h(1) = 1, \ h(t) = s, \ h'(1) = 0, \ h'(t) = 0, \) is a radial homogeneous bi-Harmonic diffeomorphism of the annulus \( A(1, t) \) onto the annulus \( A(1, s) \), then
\[ s \geq \sigma_0(t) := \frac{t(3 - 4t^2 + t^4 + 4t^2 \log t)}{2 - 2t^2 + \log t + t^4 \log t}. \]
The condition is sufficient as well. The critical homogeneous bi-harmonic mapping is
\[ f(z) = h_0(r)e^{i\theta}, \ z = re^{i\theta} \]
where
\[ h_0(r) = \frac{(1 - t^2)(3t^2 + 3(3 - t^2)r^2 - r^4)}{4r(2 - 2t^2 + \log t + t^4 \log t)} + \frac{(6t^4 + 6(1 + t^4)r^2 - 2r^4) \log t + 6(1 - t^2)^2r^2 \log r}{4r(2 - 2t^2 + \log t + t^4 \log t)}. \]
The function \( \sigma_0(t) \) is smaller than the corresponding function \( \sigma(t) \) for harmonic mappings. See Figure 2.

**Theorem 4.2** (The main theorem). Let \( t > 1 \) and \( s > 1 \). If \( f(re^{i\theta}) = g(r)e^{i\theta} \) is a radial bi-Harmonic diffeomorphism of the annulus \( A(1, t) \) onto the annulus \( A(1, s) \), mapping the inner boundary onto the inner boundary, then \( s \geq \sigma(t) \) where the constant
\[ \sigma(t) = \inf_{x \geq 0, y \geq 0} \sup_{1 \leq r \leq t} \left\{ -\frac{A'(r)}{B'(r)} + x \frac{U'(r)}{B'(r)} + y \frac{V'(r)}{B'(r)} \right\}. \]
defined in (3.7). Thus there exists a small enough \( \varepsilon > 0 \) such that 

Then \( g(1) > 0 \) and \( g'(t) > 0 \).

\textbf{Proof.} Under the conditions of the theorem \( g \) is non-decreasing. Then 

\[ g'(r) \geq 0, \quad \text{for} \quad 1 \leq r \leq t, \]

if and only if 

\[ A'(r) + B'(r)s + U''(r)x + V'(r)y \geq 0 \quad \text{for} \quad 1 \leq r \leq t. \]

Here \( x = g'(1) \) and \( y = g'(t) \). If 

\[ X_n(t) := \frac{-A'(r_n)}{B'(r_n)} + x_n \frac{-U''(r_n)}{B'(r_n)} + y_n \frac{-V'(r_n)}{B'(r_n)} \to \sigma(t) \]

then by (3.13) 

\[ X_n \geq \frac{-A'(\rho)}{B'(\rho)} + x_n \frac{-U'(\rho)}{B'(\rho)} + y_n \frac{-V'(\rho)}{B'(\rho)} > \frac{-A'(\rho)}{B'(\rho)} > 1. \]

It follows that \( \sigma(t) > 1 \) and that the sequences \( x_n \) and \( y_n \) stay bounded when \( n \to \infty \). On the other hand, it follows from (3.7)-(3.11) that there exist \( 1 < \tau_1(t) < \tau_2(t) < t \) such that the function 

\[ p(r) = -\frac{U''(r) + V'(r)}{B'(r)} \]

is negative in intervals \([1, \tau_1]\) and \([\tau_2, t]\). From (3.4), the maximum of \( \frac{-A'(r)}{B'(r)} \) is 

\[ -\frac{A'(t)}{B'(t)} := -\frac{A'(\tau^{-})}{B'(t)} \]

defined in (3.7). Thus there exists a small enough \( x > 0 \) such that 

\[ -\frac{A'(t)}{B'(t)} > -\frac{A'(r)}{B'(r)} + xp(r) \]

for all \( r : 1 < r < t \) and fixed \( t \). This means that 

\[ \sigma(t) < \sigma(0). \]

Assume without loss of generality that \( x_n \to x_0 \) and \( y_n \to y_0 \) and \( r_n \to r_0 \). The sequence \( g_n \) is monotonic and converges to a strictly monotonic function \( g_0 \). The resulting bi-harmonic mapping is critical. Since \( \sigma < \sigma(0) \), because \( A'(\tau^{-}) = B'(\tau^{-}) = 0, U'(\tau^{-}) = 0, V'(\tau^{-}) = 1 \) and \( -U'/B'(\tau^{-}) > 0 \) it follows that \( x_0 > 0, y_0 > 0 \) and \( r_0 < t \). \( \square \)

\textbf{Remark 4.3.} (a) For given \( t > 1 \) do there exists some constant \( s(t) > 1 \) such that, the class of bi-harmonic diffeomorphisms between annuli \( A(1, t) \) and \( A(1, s(t)) \) is empty?

(b) If the answer to the question a) is affirmative, does \( s(t) = \sigma(t) \)? In [10] the first author proved that, if \( \Delta u \) is small enough, then the answer of question (a) is affirmative.
Figure 2. The curve above (below) corresponds to the critical harmonic (bi-harmonic) mappings between annuli $A(1, t)$, and $A(1, \omega(t))$, where $\omega(t) = \sigma_0(t)$ and $\omega(t) = n(t)$ respectively ($1 < t \leq 3$).

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