CUBE SUMS OF FORM $3p$ AND $3p^2$ II

JIE SHU AND HONGBO YIN

Abstract. Let $p \equiv 2, 5 \pmod{9}$ be a prime. We prove that both $3p$ and $3p^2$ are cube sums. We also establish some explicit Gross-Zagier formulae and investigate the 3 part full BSD conjecture of the related elliptic curves.

1. Introduction

We call a nonzero rational number a cube sum if it is of the form $a^3 + b^3$ with $a, b \in \mathbb{Q}$. For the history and background about this Diophantine problem please refer to [DV09][DV18][HSY][SSY]. Up to now, only four family numbers with many prime factors are proved to be cube sums [Sat86][Cow00] and the Sylvester conjecture concerning primes $p$ only has partial result [DV18]. In this paper, we mainly prove the following theorem completing our partial result in [SSY] using a different construction.

Theorem 1.1. Let $p \equiv 2, 5 \pmod{9}$ be a prime. Then both $3p$ and $3p^2$ are cube sums.

Since 2 is a cube sum, from now on, we may assume $p \equiv 2, 5 \pmod{9}$ is an odd prime number. Let $E_n$ be the elliptic curve given by $x^3 + y^3 = nz^3$. It has the Weierstrass equation $y^2 = x^3 - 432n^2$. If $n > 2$ is not a cube, then $E_n(\mathbb{Q})_{\text{tor}} = 0$ and $n$ is a cube sum if and only $E_n(\mathbb{Q})$ has rank at least one. Following [Sat87], we use the Heegner points twisted from a fixed elliptic curve to prove the above theorem.

In second part of this paper, we establish some explicit Gross-Zagier formulae (Theorem 4.2) and use them to investigate the 3-part full BSD conjecture for $E_{3p}$ and $E_{3p^2}$. More explicitly, let $\text{III}(E_n)$, $\text{III}(E_n(\mathbb{Q})_{\text{tor}})$, $\Omega_n$, $R(E_n)$ and $c_\ell(E_n)$ denote the Shafarevich-Tate group, the torsion subgroup, the minimal real period, the regulator and the Tamagawa number of $E_n$ over $\mathbb{Q}$ respectively. Then the full BSD conjecture predicts that if $L(s, E)$ is of order $r$ at $s = 1$, then

$$|\text{III}(E_n)| = \frac{L^{(r)}(1, E_n)}{\Omega_n \cdot R(E_n)} \cdot \frac{|E_n(\mathbb{Q})_{\text{tor}}|^2}{\prod_\ell c_{\ell}(E_n)}.$$

Let $P$ (resp. $Q$) be a generator of the free part of $E_{3p^2}(\mathbb{Q})$ (resp. $E_{3p}(\mathbb{Q})$). We prove that

Theorem 1.2. Let $p \equiv 2 \pmod{9}$ be a rational prime number. Then

$$|\text{III}(E_p)| \cdot |\text{III}(E_{3p^2})| = \frac{L(1, E_p)}{\Omega_p \cdot h_{\mathbb{Q}}(P)} \cdot \frac{L'(1, E_{3p^2})}{\Omega_{3p^2}} \cdot \frac{|E_p(\mathbb{Q})_{\text{tor}}|^2}{\prod_\ell c_{\ell}(E_p)} \cdot \frac{|E_{3p^2}(\mathbb{Q})_{\text{tor}}|^2}{\prod_\ell c_{\ell}(E_{3p^2})}.$$  

up to a power of $2p$.

Let $p \equiv 5 \pmod{9}$ be a rational prime number. Then

$$|\text{III}(E_p^2)| \cdot |\text{III}(E_{3p})| = \frac{L(1, E_{p^2})}{\Omega_{p^2} \cdot h_{\mathbb{Q}}(Q)} \cdot \frac{L'(1, E_{3p})}{\Omega_{3p}} \cdot \frac{|E_{p^2}(\mathbb{Q})_{\text{tor}}|^2}{\prod_\ell c_{\ell}(E_{p^2})} \cdot \frac{|E_{3p}(\mathbb{Q})_{\text{tor}}|^2}{\prod_\ell c_{\ell}(E_{3p})}.$$  

up to a power of $2p$.

Note that by the work of Perrin-Riou [PR87], Kobayashi [Kob13], the $\ell$ part full BSD conjecture of $E_{3p}$ and $E_{3p^2}$ is known for $\ell \nmid 6p$. But the prime 3 is very special in the Iwasawa theory for the elliptic curve family $E_D: y^2 = x^3 + D$ whose CM field $K = \mathbb{Q}(\sqrt{-3})$ has 6 roots of unity and 2 is special for all elliptic curves. In particular, there is no any general results about the 2 and 3 part full BSD conjecture of $E_D$.

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2. Modular Actions on Heegner Points

2.1. Modular curves and modular actions. We will use the notations as in [HSY, Section 2] for the related modular curves. Recall $X_0(3^3)$ is the classical modular curve over $\mathbb{Q}$ of level $\Gamma_0(3^3)$. Define $N$ to be the normalizer of $\Gamma_0(3^3)$ in $\text{GL}_2^+(\mathbb{Q})$. Then the linear fractional transformation action of $N$ on $X_0(3^3)$ induces an isomorphism

$$N/\mathbb{Q}^\times \Gamma_0(3^3) \simeq \text{Aut}_\mathbb{Q}(X_0(3^3)).$$

The quotient group $N/\mathbb{Q}^\times \Gamma_0(3^3) \simeq S_3 \times \mathbb{Z}/3\mathbb{Z}$, where $S_3$ denotes the symmetric group with 3 letters which is generated by the Atkin-Lehner operator $W = \begin{pmatrix} 0 & -35 \\ 1 & 0 \end{pmatrix}$ and the matrix $A = \begin{pmatrix} 28 & 1/3 \\ 3^4 & 1 \end{pmatrix}$, and the subgroup $\mathbb{Z}/3\mathbb{Z}$ is generated by the matrix $B = \begin{pmatrix} 1 & 0 \\ 3^4 & 1 \end{pmatrix}$.

Put

$$U = \langle U_0(3^3), W, A \rangle \subset \text{GL}_2(\mathbb{A}_f),$$

and

$$\Gamma = \text{GL}_2(\mathbb{Q})^+ \cap U = \langle \Gamma_0(3^3), W, A \rangle,$$

and let $X_\Gamma$ be the modular curve over $\mathbb{Q}$ of level $\Gamma$ whose underlying Riemann surface is

$$X_\Gamma(\mathbb{C}) = \text{GL}_2(\mathbb{Q})^+ \setminus (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})) \times \text{GL}_2(\mathbb{A}_f)/U.$$ 

So the curve is the quotient of $X_0(3^3)$ by the actions of $W$ and $A$. Then $X_\Gamma$ is a smooth projective curve over $\mathbb{Q}$ of genus 1, and the infinity cusp $[\infty]$ is rational over $\mathbb{Q}$. We identify $X_\Gamma$ with an elliptic curve over $\mathbb{Q}$ with $[\infty]$ as its zero element [HSY, Proposition 2.1]. Let $N_\Gamma$ be the normalizer of $\Gamma$ in $\text{GL}_2(\mathbb{Q})^+$.

Then we have a natural embedding

$$\Phi : N_\Gamma/\mathbb{Q}^\times \Gamma \rightarrow \text{Aut}_\mathbb{Q}(X_\Gamma) \simeq \mathcal{O}_K^* \ltimes X_\Gamma(\overline{\mathbb{Q}}),$$

where $\mathcal{O}_K^*$ embeds into $\text{Aut}_\mathbb{Q}(X_\Gamma)$ by complex multiplications and $X_\Gamma(\overline{\mathbb{Q}})$ embeds into $\text{Aut}_\mathbb{Q}(X_\Gamma)$ by translations. The matrices

$$B = \begin{pmatrix} 1 & 0 \\ 3^4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1/9 \\ -3 & -2 \end{pmatrix}$$

lie in $N_\Gamma$, and hence induce automorphisms of $X_\Gamma$.

The elliptic curves $E_n$ are all endowed with complex multiplication by $K$ and we fix the complex multiplication $[\cdot] : \mathcal{O}_K \simeq \text{End}_K(E_n)$ by $[\omega](x, y) = (\omega x, -y)$. We will always take the simple Weierstrass equation $y^2 = x^3 - 2^4 \cdot 3$ for the elliptic curve $E_0$. We quote [HSY, Proposition 2.1] as follows.

**Proposition 2.1.** The elliptic curve $(X_\Gamma, [\infty])$ is isomorphic to $E_9$ over $\mathbb{Q}$. Moreover, for any point $P \in X_\Gamma$, we have

$$\Phi(B)(P) = [\omega^2]P, \quad \Phi(C)(P) = [\omega^2]P + (0, 4\sqrt{-3}).$$

In particular, the automorphisms $\Phi(B)$ and $\Phi(C)$ are defined over $K$.

Note that there exists a unique isomorphism $X_\Gamma \rightarrow E_9$ over $\mathbb{Q}$ such that the cusp $[1/9]$ has coordinates $(0, 4\sqrt{-3})$. We use this isomorphism to identify $X_\Gamma$ with $E_9$.

Let $V \subset U_0(3^3)$ be the subgroup consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \equiv d \mod 3$, and put $U_0 = \langle V, W, A \rangle$. Let $X_\Gamma^0$ be the modular curve over $\mathbb{Q}$ whose underlying Riemann surface is

$$X_\Gamma^0(\mathbb{C}) = \text{GL}_2(\mathbb{Q})^+ \setminus (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})) \times \text{GL}_2(\mathbb{A}_f)/U_0.$$ 

The modular curve $X_\Gamma^0$ is isomorphic to $X_\Gamma \times_{\mathbb{Q}} K$ as a curve over $\mathbb{Q}$. Usually, we denote by $[z, g]U_0$ the point on $X_\Gamma^0$ which is represented by the pair $(z, g)$ where $z \in \mathcal{H}$ and $g \in \text{GL}_2(\mathbb{A}_f)$. Let $N_{\text{GL}_2(\mathbb{A}_f)}(U_0)$ be the normalizer of $U_0$ in $\text{GL}_2(\mathbb{A}_f)$. Then there is a natural homomorphism

$$N_{\text{GL}_2(\mathbb{A}_f)}(U_0)/U_0 \rightarrow \text{Aut}_\mathbb{Q}(X_\Gamma^0)$$

induced by right translation on $X_\Gamma^0$: for $P = [z, g]U_0 \in X_\Gamma^0$ and $x \in N_{\text{GL}_2(\mathbb{A}_f)}(U_0)$

$$P \mapsto P^x = [z, gx]U_0.$$

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2.2. Modular actions on Heegner points. Let \( p \equiv 2, 5 \mod 9 \) be an odd prime number. Denote \( \tau_i = M_i \omega \in \mathcal{H} \), where

\[
M_1 = \begin{pmatrix} p/9 & 0 \\ 2 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} p/9 & 0 \\ 5 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} p/9 & 0 \\ 2 & 4 \end{pmatrix}, \quad \omega = -1 + \sqrt{-3}/2.
\]

For \( i = 1, 2, 3 \), let \( \rho_i : K \rightarrow M_2(\mathbb{Q}) \) be the normalised embedding (see [CST17, HSY]) with fixed point \( \tau_i \in \mathcal{H} \). Then \( \rho_i \) are explicitly given by

\[
\rho_1(\omega) = M_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} M_1^{-1} = \begin{pmatrix} 1 \\ 27/p \\ -2 \end{pmatrix},
\]

\[
\rho_2(\omega) = M_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} M_2^{-1} = \begin{pmatrix} 4 \\ 187/p \\ -5 \end{pmatrix},
\]

\[
\rho_3(\omega) = M_3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} M_3^{-1} = \begin{pmatrix} -1/2 \\ 27/p \\ -1/2 \end{pmatrix}.
\]

Let \( R_0(3^5) \) be the standard Eichler order of discriminant \( 3^5 \) in \( M_2(\mathbb{Q}) \). Then \( \rho_1(K) \cap R_0(3^5) = \mathcal{O}_{3p} \), \( \rho_2(K) \cap R_0(3^5) = \mathcal{O}_9p \), and \( \rho_3(K) \cap R_0(3^5) = \mathcal{O}_3p \). So \( \tau_1, \tau_2, \tau_3 \) is defined over \( \mathcal{O}_{3p} \) and \( \mathcal{O}_3 \) is defined over \( \mathcal{O}_{3p} \), by the complex multiplication theory. Here \( H_m \) is the ring class field of \( K \) with conductor \( m \).

Remark 2.1. In order to prove Theorem 1.1, we just need \( \tau_1 \) and \( \tau_2 \). But we do not how to get rational points over \( \mathbb{Q} \) from \( \tau_1 \) and \( \tau_2 \) since we do not know how the complex conjugation acts on them. In order to prove Theorem 1.2, we need the help of \( \tau_3 \) which shares the same Galois action with \( \tau_2 \) but will give us real point directly. This is also the reason why only prove half cases in Theorem 1.2.

Let \( \mathcal{O}_{K,3} \) be the completion of \( \mathcal{O}_K \) at the unique place above 3. Let \( \mathcal{O}_{K,3} \) be the completion of \( \mathcal{O}_K \) at the unique place above 3. We have

\[
\mathcal{O}_{K,3}^\infty / \mathbb{Z}_3^\times (1 + 9\mathcal{O}_{K,3}) = \langle \omega_3 \rangle \times (1 + 3\omega_3) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z},
\]

where \( \omega_3 \) is the image of \( \omega \) into \( \mathcal{O}_{K,3}^\infty \). Under both the embeddings \( \rho_1 \) and \( \rho_2 \), it is straightforward to verify that \( \omega_3 \) and \( 1 + 3\omega_3 \) normalize \( U_3 \), and therefore they induce automorphisms of \( X^0_1 \).

Theorem 2.2. Let \( P \) be an arbitrary point on \( X^0_1 \).

1. Under the embedding \( \rho_1 \), we have

\[
P^{1 + 3\omega_3} = [\omega]P, \quad P^{3\omega_3} = [\omega]P + (0, -4\sqrt{-3}).
\]

2. Under the embedding \( \rho_2 \) and \( \rho_3 \), we have

\[
P^{1 + 3\omega_3} = [\omega]P, \quad P^{3\omega_3} = [\omega^2]P + (0, -4\sqrt{-3}).
\]

Proof. We give the proof of the first assertion in details and the case under the embedding \( \rho_2 \) is similar. Now suppose \( K \) is embedded in \( M_2(\mathbb{Q}) \) under \( \rho_1 \). Since \( \omega_3 \) and \( 1 + 3\omega_3 \) have determinants \( \equiv 1 \mod 3 \), as elements in \( \text{Aut}_Q(X^0_1) \), they lie in the subgroup \( \text{Aut}_K(X^0_1) \). See [HSY, page 6] for the structure of the automorphism groups. Suppose \( P = [z, 1], z \in \mathcal{H} \), be a point on \( X^0_1 \). We have

\[
A^2B^2(1 + 3\omega_3) = \begin{pmatrix} 783/p + 9508 \\ 2268/p + 27540 \end{pmatrix} - 2377p/3 - 145\sqrt{-3}/3, A^2B^2 \in V,
\]

where the subscript 3 denotes the 3-adic component of the adelic matrices. Then by Proposition 2.1,

\[
P^{1 + 3\omega_3} = \Phi(B^2)(P) = [\omega]P.
\]

Similarly, if \( p \equiv 2 \mod 9 \), then \( AC^2\omega_3 \in V \), and hence

\[
P^{3\omega_3} = \Phi(C^2)(P) = [\omega]P + (0, -4\sqrt{-3}).
\]

If \( p \equiv 5 \mod 9 \), then \( A^2C^2\omega_3 \in V \), and hence

\[
P^{3\omega_3} = \Phi(C^2)(P) = [\omega]P + (0, -4\sqrt{-3}).
\]

For the case under embedding \( \rho_2 \) and \( \rho_3 \), it is straight to verify that \( A^2B^2(1 + 3\omega_3) \in V \) for any odd prime \( p \equiv 2, 5 \mod 9 \), and \( AB^2C^2\omega_3 \), when \( p \equiv 2 \mod 9 \), and \( A^2B^2C^2\omega_3 \in V \) when \( p \equiv 5 \mod 9 \). Then the second assertion follows from Proposition 2.1.

□
2.3. Galois actions on Heegner points. Fix the Artin reciprocity law \( \sigma : \hat{\mathbb{K}}^\times \to \text{Gal}(\mathbb{K}^{ab}/\mathbb{K}) \) by sending local uniformizers to Frobenius automorphisms. Denote by \( \sigma_t \) the image of \( t \in \hat{\mathbb{K}}^\times \). Let \( P_i = [\tau_i,1]_{U_i} \) be the CM points on \( X^1_3 \) for \( i = 1, 2, 3 \). In the following, when we consider the CM point \( P_i \), we assume \( \mathbb{K} \) is embedded in \( M_2(\mathbb{Q}) \) under \( \rho_i \).

**Theorem 2.3.** For \( i = 1, 2 \), the point \( P_i \in X^1_3(\mathbb{H}_9) \) satisfies
\[
P_i^{\sigma_1+3\omega_3} = [\omega] P_i, \quad \text{and} \quad P_i^{\sigma_2+3\omega_3} = [\omega'] P_i + (0, -4\sqrt{-3}).
\]

Similarly, \( P_3 \in X^1_3(\mathbb{H}_{36}) \) satisfies
\[
P_3^{\sigma_1+3\omega_3} = [\omega] P_3, \quad \text{and} \quad P_3^{\sigma_2+3\omega_3} = [\omega^2] P_3 + (0, -4\sqrt{-3}).
\]

**Proof.** By Shimura’s reciprocity law [Shi94, Theorems 6.31 and 6.38], we have
\[
P_i^{\sigma_t} = P_i = [\tau_i, t], \quad t \in \hat{\mathbb{K}}^\times.
\]
Since \( \hat{\mathbb{K}}^\times \cap U_0 = \hat{\mathbb{O}}_{\mathbb{K}}^\times \), by class field theory, we see \( P_i \) is defined over the ring class field \( \mathbb{H}_9 \), and the Galois actions of \( \sigma_\omega \) and \( \sigma_1+3\omega_3 \) are clear from Theorem 2.2. The proof for \( P_3 \) is similar. \( \square \)

**Remark 2.2.** Since \( \tau_3 = p\omega/2(\omega+4) = p\sqrt{-3}/54, e^{\pi i/3} \) is real. So \( P_3 \) is in fact a real point on \( E_3 \).

3. Nontriviality of Heegner points

The elliptic curve \( E_3 \) has Weierstrass equation \( y^2 = x^3 - 243x \). Consider the isomorphism
\[
\phi : E_3 \longrightarrow E_3, \quad (x, y) \mapsto (9x/\sqrt[3]{3}, 9y).
\]
We have the following commutative diagram:
\[
\begin{array}{c}
\begin{array}{c}
E_3(\mathbb{H}_9)^{\sigma_1+3\omega_3 = \omega^2} \\
\downarrow \phi \\
E_3(\mathbb{H}_{36}) \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow \text{Tr}_{36/3} \quad \text{Tr}_{9/3} \\
\begin{array}{c}
E_3(\mathbb{H}_9)^{\sigma_1+3\omega_3 = \omega^2} \\
\downarrow \phi \\
E_3(\mathbb{L}_3) \\
\end{array}
\end{array}
\end{array}
\]
where the field extension diagram is as follows (\( H_m \) is the ring class field of \( \mathbb{K} \) with conductor \( m \)):

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
H_{36} = H_{36}(\sqrt[3]{3}) \\
\downarrow \text{Tr}_{36/3} \\
L_{(3)} = K(\sqrt[3]{3}) \\
\downarrow \text{Tr}_{9/3} \\
L_{(9)} = K(\sqrt[3]{9}) \\
\downarrow \text{Tr}_{3/3} \\
K(\sqrt{3}) \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

The following proposition on related field extensions is partially quoted from [SSY, Proposition 2.6].

**Proposition 3.1.** Let \( p \equiv 2, 5 \mod 9 \) be odd primes.

1. The field \( H_{36} = H_{36}(\sqrt[3]{3}) \) with Galois group \( \text{Gal}(H_{36}/H_{36}) \simeq \langle \sigma_1 + 3\omega_3 \rangle_{\mathbb{Z}/3\mathbb{Z}} \), and
\[
(\sqrt[3]{3})^{\sigma_1+3\omega_3} = \omega^2.
\]

2. We have \((\sqrt[3]{3})^{\sigma_\omega} = 1\) and
\[
(\sqrt[3]{p})^{\sigma_\omega} = \begin{cases} \omega, & p \equiv 2 \mod 9; \\ \omega^2, & p \equiv 5 \mod 9. \end{cases}
\]

3. \( [H_{36}:H_9] = 6 \) and \( H_{36} = H_{36}(\sqrt{-1}, \sqrt{3}) \) with Galois group
\[
\text{Gal}(H_{36}/H_9) \simeq \langle \sigma_1 + 2\omega_2 \rangle_{\mathbb{Z}/2\mathbb{Z}} \times \langle \sigma_\omega_2 \rangle_{\mathbb{Z}/3\mathbb{Z}}.
\]
Proof. (1) and (2) are contained in [SSY, Proposition 2.6]. We just need to prove (3), but the argument is similar to the proof of (1). Note that $O_m = \mathbb{Z} + mO_K$,

$$\text{Gal}(H_{36p}/H_{3p}) \simeq K \times \hat{O}_{3p}^\times / K \times \hat{O}_{36p}^\times \simeq \hat{O}_{3p}^\times / (\hat{O}_{3p}^\times \cap K \times \hat{O}_{36p}^\times) \simeq O_{K,2}^\times / 2O_{K,2} \times (1 + 4O_{K,2})$$

is of order 6 and generated by $\omega_3$ and $1 + 2\omega_3$. The ideal $\sqrt{-3}O_K = (1 + 2\omega)$ and let $v$ be the place corresponding to the prime ideal $(1 + 2\omega)$. Then by the local-global principle, we have

$$\left(\frac{\sqrt{-1}}{K_2:2}\right)^{\sigma_1+2\omega_2−1} = \left(\frac{1 + 2\omega_2, -1}{K_2:2}\right) = \left(\frac{1 + 2\omega_2, -1}{K_2:2}\right)^{-1} = (-1)^{-(3-1)/2} \mod (1 + 2\omega) = -1,$$

where $\left(\frac{\sqrt{-1}}{K_2:2}\right)$ denotes the second Hilbert symbol over $K_w$. Similarly,

$$\left(\frac{\sqrt{-1}}{K_2:3}\right)^{\sigma_2−1} = \left(\frac{\omega, 2}{K_2:3}\right) = \omega^{-(4−1)/3} \mod 2 = \omega^2. \tag{5}$$

By Proposition 3.1 and 2.3, $\phi(P_1), \phi(P_2) \in E_3(H_{3p})$. Let

$$(3.1) \quad z_1 = \text{Tr}_{H_{3p}/L(p)} \phi(P_1), \quad z_2 = \text{Tr}_{H_{3p}/L(p)} \phi(P_2),$$

then $z_1, z_2 \in E_3(L(p))$.

**Theorem 3.2.** Both $z_1$ and $z_2$ are nontorsion.

**Proof.** By Theorem 2.3,

$$(3.2) \quad z_i^{\sigma_3} = [\omega^3] z_i + \frac{p + 1}{3} (0, 36\sqrt{-3})$$

for $i = 1, 2$. By [SSY, Proposition 2.3], the torsion points in $E_3(L(p))$ are $O, (0, \pm 36\sqrt{-3})$ which can not satisfy (3.2). \qed

Let $\phi_p : E_3 \to E_{3p}$ and $\phi_{p^2} : E_3 \to E_{3p^2}$ be the map given by $(x, y) \mapsto (\sqrt{p}x, py)$ and $(x, y) \mapsto (\sqrt{p^2}x, p^2y)$. Set

$$y_i = [\sqrt{-3}] z_i \in E_3(L(p)).$$

By (3.2), we know that $(y_i)^{\sigma_3} = [\omega^3](y_i)$.

**Proof of Theorem 1.1.** By Proposition 3.1 and Theorem 3.2, if $p \equiv 2 \mod 9$, then $\phi_p(y_2)$ is a nontorsion point in $E_{3p}(K)$ and $\sigma_3(y_1)$ is a nontorsion point in $E_{3p^2}(K)$; if $p \equiv 5 \mod 9$, then $\phi_p(y_1)$ is a nontorsion point in $E_{3p}(K)$ and $\sigma_3(y_2)$ is a nontorsion point in $E_{3p^2}(K)$. Since $E_{3p}(Q)$ has the same rank with $E_{3p}(K)$ and $E_{3p^2}(Q)$ has the same rank with $E_{3p^2}(K)$, we finish the proof. \qed

Define

$$E_3(L(p))^{\sigma_3 = [\omega^3]} := \{P \in E_3(L(p)) : P^{\sigma_3} = [\omega^3]P\}.$$

**Theorem 3.3.** The point $y_i$ are not divisible by $\sqrt{-3}$ in $E_3(L(p))^{\sigma_3 = [\omega^3]}$.

**Proof.** Assume the contrary that $y_i = \sqrt{-3}Q + T$ with

$$Q \in E_3(L(p))^{\sigma_3 = [\omega^3]} \text{ and } T \in E_3(L(p))^{\sigma_3 = [\omega^3]}.$$

Then

$$\sqrt{-3}(z_i - Q) = T.$$

Since $E_3(L(p))^{\text{tor}} = E_3[\sqrt{-3}]$, we conclude that $z_i - Q \in E_3[\sqrt{-3}]$. Suppose $z_i = Q + R$ with $R \in E_3[\sqrt{-3}]$. Taking Galois action of $\sigma_3$, we obtain

$$0 = [1 - \omega^3]R = (0, \pm 36\sqrt{-3}),$$

which is a contradiction. \qed

**Remark 3.1.** As is seen if $p \equiv 2 \mod 9$ resp. $p \equiv 5 \mod 9$, $y_1$ may be identified with a point of infinite order in $E_{3p}(K)$ resp. $E_{3p^2}(K)$; and if $p \equiv 2 \mod 9$ resp. $p \equiv 5 \mod 9$, $y_2$ may be identified with a point of infinite order in $E_{3p}(K)$ resp. $E_{3p^2}(K)$. 

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4. The explicit Gross-Zagier formulæ

In the rest of the paper we clarify the embedding of $K$ into $M_2(Q)$ as follows. As indicated in the proof of Theorem 1.1, in the case $p \equiv 2 \mod 9$ and $\chi = \chi_{3p^2}$ or the case $p \equiv 5 \mod 9$ and $\chi = \chi_{3p}$, we use the Heegner point $z_1$ to construct nontrivial points on elliptic curves, and hence we embed $K$ into $M_2(Q)$ under $\rho_1$. Otherwise, in the case $p \equiv 2 \mod 9$ and $\chi = \chi_{3p}$ or the case $p \equiv 5 \mod 9$ and $\chi = \chi_{3p^2}$, we use the Heegner point $z_2$, and we embed $K$ into $M_2(Q)$ under $\rho_2$.

4.1. The explicit Gross-Zagier formulæ. Let $\pi$ be the automorphic representation of $GL_2(\mathbb{A})$ corresponding to $E_9/Q$. Then $\pi$ is only ramified at $3$ with conductor $3^3$. For $n \in Q^\times$, let $\chi_n : \text{Gal}(K^{ab}/K) \to \mathbb{C}^\times$ be the cubic character given by $\chi_n(\sigma) = (\sqrt{3})^{n-1}$. Define

$$L(s, E_9, \chi_n) := L(s - 1/2, \pi_K \otimes \chi_n), \quad \epsilon(E_9, \chi_n) := \epsilon(1/2, \pi_K \otimes \chi_n),$$

where $\pi_K$ is the base change of $\pi$ to $GL_2(\mathbb{A}_K)$.

Let $p \equiv 2, 5 \mod 9$ be an odd prime number, and put $\chi = \chi_{3p}$ resp. $\chi_{3p^2}$. From the Artin formalism, we have

$$L(s, E_9, \chi) = L(s, E_{p^2})L(s, E_{3p}) \text{ resp. } L(s, E_{p^2})L(s, E_{3p}).$$

By [Liv95], we have the epsilon factors $\epsilon(E_{3p^2}^2)(\text{ resp. } \epsilon(E_{3p}^2)) = -1$ and $\epsilon(E_9)(\text{ resp. } \epsilon(E_{3p})) = +1$, and hence the epsilon factor

$$\epsilon(E_9, \chi) = \epsilon(E_{p^2})\epsilon(E_{3p^2}) \text{ resp. } \epsilon(E_{p^2})\epsilon(E_{3p}) = -1.$$

For a quaternion algebra $\mathbb{B}$ over $\mathbb{A}$, we define its ramification index $\epsilon(\mathbb{B}_v) = +1$ for any place $v$ of $Q$ if the local component $\mathbb{B}_v$ is split and $\epsilon(\mathbb{B}_v) = -1$ otherwise. The following proposition guarantees we are in the same setting as in [HSY, Theorem 4.3].

**Proposition 4.1.** The incoherent quaternion algebra $\mathbb{B}$ over $\mathbb{A}$, which satisfies

$$\epsilon(1/2, \pi_{K,v} \otimes \chi_v) = \chi_v(-1)\epsilon_v(\mathbb{B})$$

for all places $v$ of $Q$, is only ramified at the infinity place.

**Proof.** Since $\pi$ is unramified at finite places $v \nmid 3$, $\chi$ is unramified at finite places $v \nmid 3p$ and $p$ is inert in $K$, by [Gro88, Proposition 6.3] we get $\epsilon(1/2, \pi_{K,v} \otimes \chi_v) = +1$ for all finite $v \neq 3$. Again by [Gro88, Proposition 6.5], we also know that $\epsilon(1/2, \pi_{K,\infty} \otimes \chi_{\infty}) = +1$. Since $\epsilon(1/2, \pi_K \otimes \chi) = -1$, we see that $\epsilon(1/2, \pi_{K,3} \otimes \chi_3) = +1$. Since $\chi$ is a cubic character, $\chi_v(-1) = 1$ for any $v$. Hence $\mathbb{B}$ is only ramified at the infinity place. \qed

Recall we have defined the Heegner points $z_1, z_2$ in (3.1). We also define

(4.1) \[ z_3 = \text{Tr}_{H_{3p}/L(p)}(\text{Tr}_{H_{3p}/H_{9p}}(P_3)). \]

Since we use the same elliptic curve $E_9$ as in [HSY, Theorem 4.3], very little modification of the proof gives us the following explicit Gross-Zagier formulæ once we verify the explicit local computation of toric integrals in Corollary 4.10.

**Theorem 4.2.** One has the following explicit formulæ of Heegner points

$$L(1, E_{p^2})L'(1, E_{3p^2}) = \begin{cases} 2^{-1} \cdot 9 \cdot \hat{\Delta}_Q(z_2), & \text{if } p \equiv 2 \mod 9; \\ 9 \cdot \hat{\Delta}_Q(z_1), & \text{if } p \equiv 5 \mod 9. \end{cases}$$

And

$$L(1, E_{p^2})L'(1, E_{3p}) = \begin{cases} 9 \cdot \hat{\Delta}_Q(z_1), & \text{if } p \equiv 2 \mod 9; \\ 2^{-1} \cdot 9 \cdot \hat{\Delta}_Q(z_2), & \text{if } p \equiv 5 \mod 9. \end{cases}$$

**Theorem 4.3.** One also has the following explicit formulæ of Heegner points

$$L(1, E_{p^2})L'(1, E_{3p^2}) = 2^{-3} \cdot \hat{\Delta}_Q(z_1), \quad \text{if } p \equiv 2 \mod 9$$

And

$$L(1, E_{p^2})L'(1, E_{3p}) = 2^{-3} \cdot \hat{\Delta}_Q(z_3), \quad \text{if } p \equiv 5 \mod 9.$$
Corollary 4.4. $z_3 \in E_3(L(p))$ is nontorsion and satisfies $z_3^{\sigma_{\omega_3}} = [\omega^3]z_3$. If $p \equiv 2 \mod 9$, then $\phi_p(z_3)$ is a nontorsion point in $E_{3p}(\mathbb{Q})$. If $p \equiv 5 \mod 9$, then $\phi_p(z_3)$ is a nontorsion point in $E_{3p}(\mathbb{Q})$. In both cases, $\phi_p(z_3)$ and $\phi_p(z_2)$ are not divisible by 3 over $\mathbb{Q}$.

Proof. Since $[H_{3p} : H_{9p}] = 6$, by Theorem 2.3 and (4.1), we know that $z_3^{\sigma_{\omega_3}} = [\omega^3]z_3$. Since $z_3$ is a real point by Remark 2.2, by Proposition 3.1, if $p \equiv 2 \mod 9$, $\phi_p(z_3) \in E_{3p}(\mathbb{Q})$ and if $p \equiv 5 \mod 9$, $\phi_p(z_3) \in E_{3p}(\mathbb{Q})$. By Theorem 4.2 and Theorem 4.3, $h_{\mathbb{Q}}(z_3) = 36h_{\mathbb{Q}}(z_2) = 12h_{\mathbb{Q}}(y_2)$. This implies $z_3$ is nontorsion and $\phi_p(z_3)$, $\phi_p(z_2)$ can not be divisible by 3 over $\mathbb{Q}$. Otherwise there exists point $z$ in $E_3(L(p))^{\sigma_{\omega_3} = [\omega^3]}$ such that $9h_{\mathbb{Q}}(z) = h_{\mathbb{Q}}(z_3) = 12h_{\mathbb{Q}}(y_2)$. But this is impossible since $y_2$ is not divisible by $\sqrt{-3}$ in $E_3(L(p))^{\sigma_{\omega_3} = [\omega^3]}$ and $E_3(L(p))^{\sigma_{\omega_3} = [\omega^3]}$ is of rank 1 over $K$ by Kolyvagin.

4.2. Waldspurger’s local period integral. This subsection is purely local and we shall compute the 3-adic period integral for the 3-adic local newform following [HSY19]. Recall $\pi$ is the automorphic representation of $GL_2(\mathbb{Q})$ corresponding to $E_3$ and $\pi_3$ the 3-adic part of $\pi$. Then the conductor $c(\pi_3) = 3^3$. Let $p \equiv 2, 5 \mod 9$ be an odd prime and let $\chi : \text{Gal}(K/K) \to \mathbb{Q}_p^\times$ be the character given by $\chi(\sigma) = \chi_{3p}(\sigma) = (\sqrt[3]{3p})^{\sigma - 1}$ resp. $\chi(\sigma) = \chi_{3p^2}(\sigma) = (\sqrt[3]{3p^2})^{\sigma - 1}$. We also view $\chi$ as a Hecke character on $\mathbb{A}_K$ by the Artin map and the conductor the 3 part is $c(\chi_3) = (\sqrt{-3})^3$. Assume that $f_3$ is the standard newform of $\pi_3$. We shall compute the following normalized period integral

$$\beta^2_3(f_3, f_3) = \int_{\tau \in \mathbb{Q}_p^\times} \frac{(\pi(t)f_3, f_3)}{(f_3, f_3)} \chi_3(t)dt$$

which appears in the proof of the explicit Gross-Zagier formulae. Let $\Theta : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ be the unitary Hecke character associated to the base-changed CM elliptic curve $E_{3f/K}$. Then $\Theta$ has conductor $9\mathcal{O}_K$. We denote $\Theta_3$ the 3-adic component of $\Theta$. Then $\pi_3$ is the local representation of $GL_2(\mathbb{Q}_3)$ corresponding to $\Theta_3$. Note

$$\mathcal{O}_{K,3}^\times/(1 + 9\mathcal{O}_{K,3}) \cong \langle \pm 1 \rangle^{2\mathbb{Z}/2\mathbb{Z}} \times (1 + \sqrt{-3})^{2\mathbb{Z}/2\mathbb{Z}} \times (1 - \sqrt{-3})^{2\mathbb{Z}/2\mathbb{Z}} \times (1 + 3\sqrt{-3})^{2\mathbb{Z}/2\mathbb{Z}}.$$  

Lemma 4.5. We have $c(\Theta_3) = 4$, and $\Theta_3$ is given explicitly by

$$\Theta_3(-1) = -1, \quad \Theta_3(1 + \sqrt{-3}) = -\frac{1 - \sqrt{-3}}{2}, \quad \Theta_3(\sqrt{-3}) = i,$$

$$\Theta_3(1 - \sqrt{-3}) = -\frac{1 + \sqrt{-3}}{2}, \quad \Theta_3(1 + 3\sqrt{-3}) = -1 + \sqrt{-3}.$$  

Proof. This is [HSY19, Lemma 4.1].

The local character $\chi_3$ has conductor $\mathbb{Z}_3^\times (1 + 9\mathcal{O}_{K,3})$, and hence it is in fact a character of the quotient group $K_3^\times / \mathbb{Q}_3^\times (1 + 9\mathcal{O}_{K,3})$. Note that

$$K_3^\times / \mathbb{Q}_3^\times (1 + 9\mathcal{O}_{K,3}) \cong \langle \sqrt{-3} \rangle^{2\mathbb{Z}} \times (1 + \sqrt{-3})^{2\mathbb{Z}/2\mathbb{Z}} \times (1 + 3\sqrt{-3})^{2\mathbb{Z}/2\mathbb{Z}}.$$  

Lemma 4.6. We have $c(\chi_3) = 4$ and $\chi_3$ is given explicitly by the following tables:

1. if $\chi = \chi_{3p}$, then

| $p \mod 9$ | $\chi_3(1 + \sqrt{-3})$ | $\chi_3(1 + 3\sqrt{-3})$ | $\chi_3(\sqrt{-3})$ |
|---|---|---|---|
| 2 | $\omega^2$ | $\omega$ | 1 |
| 5 | $\omega$ | $\omega$ | 1 |

2. if $\chi = \chi_{3p^2}$, then

| $p \mod 9$ | $\chi_3(1 + \sqrt{-3})$ | $\chi_3(1 + 3\sqrt{-3})$ | $\chi_3(\sqrt{-3})$ |
|---|---|---|---|
| 2 | $\omega^2$ | $\omega$ | 1 |
| 5 | $\omega$ | $\omega$ | 1 |

Proof. The proof is routine in class-field theory. See [HSY19, Lemma 4.2] for more details.

Corollary 4.7. If $p \equiv 2 \mod 9$, and $\chi = \chi_{3p}$, then the local character $\Theta_3\bar{\chi}_3$ is given explicitly by

$$\Theta_3\bar{\chi}_3(-1) = -1, \quad \Theta_3\bar{\chi}_3(1 + \sqrt{-3}) = 1,$$

$$\Theta_3\bar{\chi}_3(1 - \sqrt{-3}) = 1, \quad \Theta_3\bar{\chi}_3(1 + 3\sqrt{-3}) = 1, \quad \Theta_3\bar{\chi}_3(\sqrt{-3}) = i.$$
If $p \equiv 2 \text{ resp. } 5 \pmod{9}$, and $\chi = \chi_{3p^2} \text{ resp. } \chi_{3p}$, the local character $\Theta_3|_{\mathbb{A}_3}$ is given explicitly by

\[
\Theta_3|_{\mathbb{A}_3}(-1) = -1, \quad \Theta_3|_{\mathbb{A}_3}(1 + \sqrt{-3}) = \omega, \\
\Theta_3|_{\mathbb{A}_3}(1 - \sqrt{-3}) = \omega^2, \quad \Theta_3|_{\mathbb{A}_3}(1 + 3\sqrt{-3}) = 1, \quad \Theta_3|_{\mathbb{A}_3}(\sqrt{-3}) = i.
\]

Let $\theta_3$ be the 3-adic character which parametrizes the supercuspidal representation $\pi_3$ via compact-induction construction as in [HSY19, Section 2.2]. The test vector issue for Waldspurger’s period integral is closely related to $c(\theta_3|_{\mathbb{A}_3})$ or $c(\theta_3|_{\mathbb{A}_3})$. We can work out these by using Lemma 4.5, 4.6 and Corollary 4.7, and the relation between $\theta_3$ and $\Theta_3$ in [HSY19, Theorem 2.8]. Now we can prove the following key Lemma.

**Lemma 4.8.** If $p \equiv 2 \text{ resp. } 5 \pmod{9}$, and $\chi = \chi_{3p^2} \text{ resp. } \chi_{3p}$, we have $\theta_3|_{\mathbb{A}_3} = 1$. If $p \equiv 2 \text{ resp. } 5 \pmod{9}$, and $\chi = \chi_{3p^2} \text{ resp. } \chi_{3p}$, we have $c(\theta_3|_{\mathbb{A}_3}) = 2$ and $\alpha_3|_{\mathbb{A}_3} = \frac{1}{\sqrt{-3}}$. Moreover, in any cases, $c(\theta_3|_{\mathbb{A}_3}) \leq c(\theta_3|_{\mathbb{A}_3})$.

**Proof.** Let $\psi_3$ be the additive character such that $\psi_3(x) = e^{2\pi i u(x)}$ where $\iota : \mathbb{Q}_3 \to \mathbb{Q}_3/\mathbb{Z}_3 \subset \mathbb{Q}/\mathbb{Z}$ is the map given by $x \mapsto -x \pmod{\mathbb{Z}_3}$ which is compatible with the choice in [CST14]. Let $\psi_K(x) = \psi_3 \circ \operatorname{Tr}_{K_3/\mathbb{Q}_3}(x)$, be the additive character of $K_3$.

Recall that $\alpha_3$ is the number associated to $\Theta_3$ as in [HSY19, Lemma 2.1] so that

\[
\Theta_3(1 + x) = \psi_K(\alpha_3, x),
\]

for any $x$ satisfying $\psi_K(x) \geq c(\Theta_3)/2 = 2$. By the definition of $\psi_K$ and Lemma 4.5, we know that $\alpha_3 = \frac{1}{\sqrt{-3}}$. Now let $\theta_3$ be the quadratic character associated to the quadratic field extension $K_3/\mathbb{Q}_3$. Then by [BH06, Proposition 34.3], $\lambda_{K_3/\mathbb{Q}_3}(\psi_3) = \tau(\eta_3, \psi_3)/\sqrt{3} = -i$, here $\tau(\eta_3, \psi_3)$ is the Gauss sum and $\psi_3(x) = \psi_3(\bar{x})$ is the additive character of level one. By [Lan, Lemma 5.1], $\lambda_{K_3/\mathbb{Q}_3}(\psi_3) = \eta_3(3)\lambda_{K_3/\mathbb{Q}_3}(\psi_3) = -i$.

Then $\Delta_{\theta_3}$ is the unique level one character of $K_3$ such that $\Delta_{\theta_3}(\sqrt{-3}) = \eta_3$ and

\[
\Delta_{\theta_3}(\sqrt{-3}) = \eta((\sqrt{-3})^3\alpha_3)\lambda_{K_3/\mathbb{Q}_3}(\psi_3)^3 = -i.
\]

Recall that $\theta_3 = \Theta_3|_{\mathbb{A}_3}$. Then by Corollary 4.7 we can easily check that:

1. If $p \equiv 2 \text{ resp. } 5 \pmod{9}$, and $\chi = \chi_{3p^2} \text{ resp. } \chi_{3p}$, $\theta_3|_{\mathbb{A}_3}$ is the trivial character.

2. If $p \equiv 2 \text{ resp. } 5 \pmod{9}$, and $\chi = \chi_{3p^2} \text{ resp. } \chi_{3p}$, $\theta_3|_{\mathbb{A}_3}$ is of level 2 and by definition we can choose $\alpha_3|_{\mathbb{A}_3} = \frac{1}{\sqrt{-3}}$.

Since $\chi$ is a cubic character, $\theta_3|_{\mathbb{A}_3} = \theta_3|_{\mathbb{A}_3}$. Since $c(\chi_3) = c(\chi_3) = 4$, $c(\theta_3|_{\mathbb{A}_3}) = 4$ and the last assertion follows.

To apply the results in [HSY19] to calculate the local period integral, we take $F = \mathbb{Q}_3$, $\kappa = 3 = q$, $D = -3$, $K_3 \simeq \mathbb{E} \simeq L \simeq \mathbb{Q}(-\sqrt{-3})$, $c(\chi_3) = c(\chi_3) = 4$, $n = 2$. By [HSY19, Lemma 2.9], we have the minimal vector $\varphi_0 = \operatorname{Char}(\mathbb{E}^{-2}U_f(1))$ in the Kirillov model. Recall $K$ is embedded into $M_2(\mathbb{Q})$ as in Section 2.2 which linearly extends the following map:

\[
\sqrt{-3} \mapsto \left( \begin{array}{cc} a & 3^{-2}b \\ 3^3c & -a \end{array} \right) := \left( \begin{array}{ccc} 3 & -2p/9 \\ 54/p & -3 \end{array} \right), \quad \text{if } K \text{ is embedded under } \rho_1;
\]

\[
\left( \begin{array}{ccc} 9 & -2p/9 \\ 374/p & -9 \end{array} \right), \quad \text{if } K \text{ is embedded under } \rho_2;
\]

\[
\left( \begin{array}{ccc} 0 & -p/18 \\ 54/p & 0 \end{array} \right), \quad \text{if } K \text{ is embedded under } \rho_3.
\]

**Proposition 4.9.** Suppose $\operatorname{Vol}(\mathbb{Z}_3^\times \backslash \mathbb{O}^\times_{K_3}) = 1$ so that $\operatorname{Vol}(\mathbb{Q}_3^\times \backslash K_3^\times) = 2$. For $f_3$ being the newform, $K$ being embedded in $M_2(\mathbb{Q})$ as in (4.3), we have

\[
\beta_3(f_3, f_3) = \begin{cases} 1, & \text{if } p \equiv 2 \text{ resp. } 5 \pmod{9}, \chi = \chi_{3p^2} \text{ resp. } \chi_{3p^2} \text{ and } K \text{ is embedded under } \rho_2 \text{ or } \rho_3; \\
1/2, & \text{if } p \equiv 2 \text{ resp. } 5 \pmod{9}, \chi = \chi_{3p^2} \text{ resp. } \chi_{3p^2} \text{ and } K \text{ is embedded under } \rho_1.
\end{cases}
\]

**Proof.** We may assume $f_3$ to be $L^2$-normalized. To evaluate $f_3$ for the embedding in (4.3) is equivalent to use the standard embedding [HSY19, (2.13)] of $\mathbb{E}$ and use the corresponding translate of the newform. In particular the embedding in (4.3) is conjugate to the standard embedding by the following

\[
\left( \begin{array}{cc} a & 3^{-2}b \\ 3^3c & -a \end{array} \right) = \left( \begin{array}{ccc} -9c & a/3 \\ 0 & 1 \end{array} \right)^{-1} \left( \begin{array}{ccc} 0 & 1 \\ D & 0 \end{array} \right) \left( \begin{array}{ccc} -9c & a/3 \\ 0 & 1 \end{array} \right).
\]
Thus we have

\[
\beta_3^0(f_3, f_3) = \int_{\mathbb{F}^{\times}\backslash \mathbb{E}^{\times}} \left( \pi_3 \left( \left( \begin{array}{cc} -9c & a/3 \\ 0 & 1 \end{array} \right)^{-1} t \left( \begin{array}{cc} -9c & a/3 \\ 0 & 1 \end{array} \right) \right) f_3, f_3 \right) \chi(t) dt
\]

\[
= \int_{\mathbb{F}^{\times}\backslash \mathbb{E}^{\times}} \left( \pi_3 \left( t \left( \begin{array}{cc} -9c & a/3 \\ 0 & 1 \end{array} \right) \right) f_3 \right) \pi_3 \left( \left( \begin{array}{cc} -9c & a/3 \\ 0 & 1 \end{array} \right) f_3 \right) \chi(t) dt,
\]

which is by definition \( \{ \pi_3 \left( \left( \begin{array}{cc} -9c & a/3 \\ 0 & 1 \end{array} \right) f_3, \pi_3 \left( \left( \begin{array}{cc} -9c & a/3 \\ 0 & 1 \end{array} \right) f_3 \right) \right) \} \) for the bilinear pairing as in [HSY19, (3.1)] and the standard embedding as in [HSY19, (2.13)]. Note that by [HSY19, Corollary 2.10],

\[
\pi_3 \left( \left( \begin{array}{cc} -9c & a/3 \\ 0 & 1 \end{array} \right) f_3 \right) = \frac{1}{\sqrt{2}} \sum_{x \in (\mathbb{F}_3 \times \mathbb{F}_3)^{\times}} \pi_3 \left( \left( \begin{array}{cc} 1 & a/3 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} x & 0 \\ 0 & 1 \end{array} \right) \right) \varphi_0
\]

where \( \varphi_0 \) is the minimal test vector.

If \( p \equiv 2 \mod 5 \) and \( x = \chi_{3p} \) resp. \( x_{3p} \), we embed \( K \) into \( M_2(\mathbb{Q}) \) under \( \rho_2 \) or \( \rho_3 \). In this case, \( c(\theta_3, \chi_3) = 0 \) and \( a = 0 \) or \( 9 \). By the \( l = 0 \) case in [HSY19, Section 2.4], we have a unique \( x \mod \varpi \) for which \( \{ \varphi_x, \varphi_x \} \) is nonvanishing (In fact, we must have \( x \equiv 1 \mod 3 \)). According to [HSY19, Proposition 3.3], there are no off-diagonal terms, and we have

\[
\beta_3^0(f_3, f_3) = \frac{1}{(q - 1)q^{(\frac{2}{3} - 1)}} \{ \varphi_x, \varphi_x \} = \frac{1}{2} \cdot 2 = 1.
\]

If \( p \equiv 2 \mod 5 \) and \( x = \chi_{3p} \) resp. \( x_{3p} \), we embed \( K \) into \( M_2(\mathbb{Q}) \) under \( \rho_1 \). In this case, we have \( c(\theta_3, \chi_3) = 2l = 2 \) and \( u = a/3 = 1 \). The action of \( \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \) on \( \varphi_x = \pi_3 \left( \left( \begin{array}{cc} x & 0 \\ 0 & 1 \end{array} \right) \right) \) \( \varphi_0 \) is by a simple character. This is the case \( l = 1 \) and \( n - l = 1 \) is odd. By the choice in [HSY19, Section 2.4],

\[
D' = \frac{1}{\alpha_{\theta_3}^{2} c a/3} = -3,
\]

noting \( \alpha_{\theta_3} = \alpha_{\theta_3} = \frac{1}{3\sqrt{3}} \) in this case, and we have

\[
\Delta(1) = 4\varpi^n a \varphi_{\chi_3} \sqrt{D} \left( \varpi^n a \varphi_{\chi_3} \sqrt{D} - 2 \sqrt{D} D' \right) + 4 \frac{D}{D'} D
\]

\[
\equiv 4 \cdot 9 \cdot \frac{1}{3\sqrt{3}} \cdot \sqrt{-3} \cdot (-2) + 4 \cdot (-3) \mod \varpi^2
\]

\[
\equiv -8 \cdot 3 - 4 \cdot 3 \mod \varpi^2
\]

\[
\equiv 0 \mod \varpi^2.
\]

\( \Delta(1) \) is indeed congruent to a square. Then we can get a unique solution of \( x \mod \varpi \) from [HSY19, (2.17)], and again by [HSY19, Proposition 3.3],

\[
\beta_3^0(f_3, f_3) = \frac{1}{(q - 1)q^{(\frac{2}{3} - 1)}} \frac{1}{q^{(l/2)}} = \frac{1}{2}.
\]

Let \( f' \) be the admissible test vector of \( (\pi, \chi) \) which is as defined in [CST14, Definition 1.4]. By definition, the 3-adic part \( f'_3 \) is \( \chi_{3}^{-1} \)-eigen under the action of \( K_3^{1} \). The following corollary is used in the proof of the explicit Gross-Zagier formulæ.

**Corollary 4.10.** For the admissible test vector \( f'_3 \) and the newform \( f_3 \) we have

\[
\beta_3^0(f'_3, f'_3) = \begin{cases} 2, & \text{if } p \equiv 2 \mod 5, \chi = \chi_{3p} \text{ resp. } \chi_{3p^2} \text{ and } K \text{ is embedded under } \rho_2 \text{ and } \rho_3, \\ 4, & \text{if } p \equiv 2 \mod 5, \chi = \chi_{3p^2} \text{ resp. } \chi_{3p} \text{ and } K \text{ is embedded under } \rho_1. \end{cases}
\]

**Proof.** Keep the normalization of the volumes in Proposition 4.9. By definition of \( f' \), we have \( \beta_3^0(f'_3, f'_3) = \text{Vol}(Q_3^1 \backslash K_3^1) = 2 \). Then the corollary follows from Proposition 4.9. \( \square \)
5. The 3-part of the Birch and Swinnerton-Dyer conjectures

Let $n$ be a positive cube-free integer and $E_n'$ be the elliptic curve given by Weierstrass equation $y^2 = x^3 + (4n)^2$. Then there is an unique isogeny $\phi_n : E_n \to E_n'$ of degree 3 up to $\{\pm 1\}$ and denote $\phi_n'$ its dual isogeny.

**Proposition 5.1.** Let $p \equiv 2, 5 \mod 9$ be an odd prime. Then
\[
\dim_{\mathbb{F}_p} \text{Sel}_3(E_3p^2(Q)) \leq 1, \quad \dim_{\mathbb{F}_p} \text{Sel}_3(E_p(Q)) = 0.
\]

**Proof.** By [Sat86, Theorem 2.9], we know that
\[
\text{Sel}_{\phi_{3p^2}}(E_3p^2(Q)) = \text{Sel}_{\phi_{3p^2}'}(E_3^p(Q)) = \mathbb{Z}/3\mathbb{Z},
\]
and
\[
\text{Sel}_{\phi_p}(E_3p^2(Q)) = \mathbb{Z}/3\mathbb{Z}, \quad \text{Sel}_{\phi_p'}(E_3p^2(Q)) = 0.
\]
Note that $E_p[3](Q)$ and $E_{3p^2}[3](Q)$ are trivial and $|E_p'[\phi_p]'(Q)| = |E_p'[\phi_p']'(Q)| = 3$. By [HSY, Lemma 5.1], we have
\[
\dim_{\mathbb{F}_p} \text{Sel}_3(E_3p^2(Q)) \leq 1, \quad \dim_{\mathbb{F}_p} \text{Sel}_3(E_p(Q)) = 0.
\]
Similarly we have
\[
\dim_{\mathbb{F}_p} \text{Sel}_3(E_3p(Q)) \leq 1, \quad \dim_{\mathbb{F}_p} \text{Sel}_3(E_{p^2}(Q)) = 0.
\]

Now we are ready to give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** We will give the proof of (1.1) when $p \equiv 2 \mod 9$. One can verify (1.2) in case $p \equiv 5 \mod 9$ similarly. Now assume $p \equiv 2 \mod 9$. By [ZK87, Table 1], we know that $c_3(E_p) = 2$ and $c_\ell(E_p) = 1$ for any prime $\ell \neq 3$, while $c_\ell(E_{3p^2}) = 1$ for all primes $\ell$.

Let $P$ be the generator of the free part of $E_{3p^2}(Q)$. Then the BSD conjecture predicts that
\[
\frac{L'(1, E_{3p^2})}{\Omega_{3p^2}} = |\text{III}(E_{3p^2})| \cdot \tilde{h}_Q(P) \quad \text{and} \quad \frac{L(1, E_p)}{\Omega_p} = 2 \cdot |\text{III}(E_p)|.
\]
Combining these two, we get
\[
L(1, E_p) \frac{L'(1, E_{3p^2})}{\Omega_{3p^2}} = 2 \cdot |\text{III}(E_p)| \cdot |\text{III}(E_{3p^2})| \cdot \tilde{h}_Q(P).
\]
By Theorem 4.3, we expect
\[
|\text{III}(E_p)| \cdot |\text{III}(E_{3p^2})| = 2^{-4} \cdot \frac{\tilde{h}_Q(z_3)}{\tilde{h}_Q(P)}.
\]
Note the RHS of (5.2) is a nonzero rational number.

By Proposition 5.1, $E_{3p^2}(Q)$ has rank 1, and form the exact sequence
\[
0 \to E(Q)/3E(Q) \to \text{Sel}_3(E(Q)) \to \text{III}(E)[3] \to 0,
\]
we know directly that
\[
|\text{III}(E_p)[3^\infty]| = |\text{III}(E_{3p^2})[3^\infty]| = 1.
\]
In order to prove the 3-part of (5.2), it suffices to prove
\[
\hat{h}_Q(P) = u \cdot \hat{h}_Q(z_3)
\]
for some $u \in \mathbb{Z}_p^\times \cap \mathcal{Q}$. This is clear by Corollary 4.4.

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School of Mathematical Sciences, Tongji University, Shanghai 200092
E-mail address: shujie@tongji.edu.cn

School of Mathematics, Shandong University, Jinan 250100, P.R.China
E-mail address: yhb2004@mail.sdu.edu.cn