Analysis of chaotic systems, an implementation of the logistic equation

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Abstract. This paper presents a simple example of a chaotic system, the denomination logistic equation, to illustrate and emphasize the distinctive characteristics of chaos, which are the extreme dependence on initial conditions and complexity of behavior.

1. Introduction

Chaos theory is a relatively new branch of mathematics, with characteristics that distinguish it from other disciplines. In general, the concept of chaos arises from the study of dynamic systems, especially non-linear systems. Therefore, chaotic behavior can be found in any scientific area from medicine to ecology. The study of this theory has generated new techniques of analysis and study methods for the characterization of dynamic systems, in addition to the development of technology and the simulation capacity that is currently has revolutionized this discipline.

When we refer naturally to chaotic systems, we refer to phenomena with irregular dynamics, of random nature or unpredictable at any moment of time. For these systems the physical theories are ineffective and that is why they are often relegated, without paying much attention. However, the theory of chaos has modified this thinking, highlighting the importance of the analysis of non-linear systems. In the mid-twentieth century a scientist interested in the study of population growth in an enclosed area, proposed a new model that considered the limitations of the environment when the population was very large, this model became non-linear, and has been extensively studied until today, this model is known as the logistic equation, and its importance is that it provides a route to describe the presence of chaos in a system [1,2].

2. Logistics equation

The logistic equation has been studied extensively by different authors, although the main author in studying it and founding the model that we have today was Feigenbaum [3,4]. Although the first base model for this study was the model proposed by Mathlhus, which stated that a population grew at a constant rate and indefinitely, which can be proven to be false in real life and is only valid for bacterial cultures in periods of very short times where the characteristic of the medium allows it. Therefore, this model was discarded for large populations which have limitations given by the environment, such as space, resources, etc., which will cause the population to not grow indefinitely or to decrease, if necessary. For this purpose, a limiting factor \( K \) called medium support capacity is proposed, for which it can be proposed that the population growth rate \( \frac{d}{dx} \) is proportional to the differential between \((K - x)\), normalizing Equation (1):

\[
x = \mu x (1 - x)
\]
The equilibrium points $x = 0$ and $x = 1$ can be easily obtained from the above differential equation, which are unstable and stable in nature, respectively. The analysis of the logistic equation can be performed discretely using the Equation (2):

$$x_{n+1} = \mu x_n (1 - x_n) \quad (2)$$

Where for a given initial value $x_0$ a new value of $x_1$ a is generated from the relation $x_1 = \mu x_0 (1 - x_0)$ and then the procedure for calculating $x_2$ is repeated, and so on.

3. Analysis and results of the logistics equation

The shape of the curve presented by the logistic equation is parabolic, where each point of the curve has coordinates $(x_n, x_{n+1})$. Figure 1 shows the graph of the dynamics of the logistic equation for different values of $0 \leq \mu \leq 4$.

The first thing to do is find the fixed points of the equation, which are defined by the expression $x_{n+1} = x_n$ which results in a straight line at $45^\circ$ angle starting at the origin.

By means of the previous definition and by means of an iterative process starting from an initial value $x_0$, a vertical line is drawn up to the parabolic curve, then a horizontal line must be drawn until the intersection with the straight line, where by a new one will be found value $x_1$, this procedure is followed until obtaining a set of points that will form the orbit of the dynamic system, therefore, the fixed points will be those when Equation (3) [5]:

$$\lim_{n \rightarrow \infty} x_n = x_{n+1} \quad (3)$$

Figure 2 and Figure 3 were obtained by varying $\mu$ in the interval established above and taking different values of initial conditions. For a value of $\mu = 0.8$ and an initial condition $x_0 = 0.57$ we get:

From Figure 3 and Figure 4 it can be seen that regardless of the initial value, the trajectory will always reach the zero fixed point value. Which is different if you change the value of $\mu$ to a value of 2.6 with an initial condition $x_0 = 0.3$, which will result in Figure 4 and Figure 5 [6]:

![Figure 1. Logistic mapping.](image1)

![Figure 2. Trajectory diagram, $\mu = 0.8$ and $x_0 = 0.57$.](image2)

![Figure 3. Trajectory diagram. Fixed point for $\mu = 0.8$ and $x_0 = 0.57$.](image3)
Figure 4. Trajectory diagram $\mu = 2.6$ and $x_0 = 0.3$.

Figure 5. Trajectory diagram. Fixed point for $\mu = 2.6$ and $x_0 = 0.3$.

If we continue to vary the value of $\mu$ to values less than 3, equal results will be observed, where only a single value of fixed point or attractor of the system will be obtained. However, for values of $\mu$ greater than and equal to 3, it is observed that the fixed points are not established in a single value, but oscillate between two values for the case in which $\mu = 3.4$, as shown Figure 6 and Figure 7 [7,8].

Figure 6. Trajectory diagram $\mu = 3.4$ and $x_0 = 0.2$.

Figure 7. Trajectory diagram. Fixed point for $\mu = 3.4$ and $x_0 = 0.2$.

On the other hand, for the value of $\mu = 3$ we have what is known as a bifurcation or unstable, since the system goes from having a single attractor point to having two fixed points. In addition, if the value of $\mu$ continues to increase, it is found that the iterations enter period cycles $2, 4, 8, 16, \ldots, \infty$. Then, we talk about the duplication of the period, as shown in the following Figure 8 and Figure 9:

Figure 8. Trajectory diagram $\mu = 3.5$ and $x_0 = 0.5$.

Figure 9. Trajectory diagram. Fixed point for $\mu = 3.5$ and $x_0 = 0.5$. 
From the value of $\mu = 3.57$ and up to $\mu < 4$, the chaotic behavior appears, where the values of $x_n$ appear to be randomly distributed. However, it should be noted that this behavior is not completely random since there is a rule (system dynamics) [9] that determines them. For a last value of $\mu = 3.8$ it is observed that the orbits of the system never converge to a fixed point showing irregular behavior (Figure 10 and Figure 11).

![Figure 10. Trajectory diagram $\mu = 3.8$ and $x_0 = 0.4$.](image1)

![Figure 11. Bifurcation diagram.](image2)

To sketch a complete behavior of the logistic equation, a graph can be made showing all the possible values of $x_n$ as a function of the control parameter $\mu$, said graph is called a bifurcation diagram, and it is determined computationally, where the first iterations they are discarded and the rest are graphed. This diagram has a particular characteristic, known as fractal structure [10,11] which means that its basic structure is repeated at different scales (self-similarity) [12].

Finally, the dynamics of the logistic equation can also be analyzed using the exponents of Lyapunov [4] in this case, it only has one exponent because it is a first-order system. These quantities provide vital information when characterizing the presence or absence of chaos in a particular system, in the following way: if there is at least one exponent of Lyapunov $\lambda_i > 0$, it indicates that this system has chaotic behavior. For the case of the logistic equation, the spectrum of the exponents can be determined as a function of the control parameter $\mu$ as shown in Figure 12.

![Figure 12. Spectrum of the Lyapunov exponent.](image3)

Finally, the behavior of the population is shown as a function of the growth rate $\mu$ and generations. Figure 13 represents how the populations vary throughout the generations for different control rate $\mu$, as shown for $\mu = 2$ with $x_0 = 0.5$ (purple line) it has to be as the generations increase, the population keeps constant, compared to $\mu = 3$ with $x_0 = 0.5$ s (light blue line) we can notice that the behavior is oscillatory with maximum value of 0.725 and minimum value of 0.625. Finally, for $\mu = (0.5 \ 1.5)$ $x_0 = 0.5$ the behavior is decreasing, that is, the population tends to become extinct.

![Figure 13. Population graph (axis Y), generations (axis X).](image4)
4. Conclusion
The dynamic systems that can be modeled with the logistic function, are systems whose performance can be predicted for growth rate (or control rates $\mu = (0.5 \ 3.5)$), this can be achieved through the solution of the differential equation analytically or discretely as was done in this paper. For values $\mu > 3.5$, the system modeled by means of the logistic equation presents chaotic behavior as shown in figures 10 and 11 since it is not possible to control the population. Through qualitative analysis it is possible to study the behavior of the system and establish which are the initial values that make the system dynamic stable or unstable, that is, find the source or sink values that for this model are $x = 0$ and $x = 1$.

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