The current-type quadrupole moment and gravitational-wave mode \((\ell, m) = (2, 1)\) of compact binary systems at the third post-Newtonian order

Quentin Henry*, Guillaume Faye and Luc Blanchet

GReCo, Institut d’Astrophysique de Paris, UMR 7095, CNRS, Sorbonne Université, 98bis boulevard Arago, 75014 Paris, France

E-mail: henry@iap.fr, faye@iap.fr and luc.blanchet@iap.fr

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Abstract

Up to the third post-Newtonian (3PN) order, we compute (i) the current-type quadrupole moment of (non-spinning) compact binary systems, as well as (ii) the corresponding gravitational-wave mode \((2, 1)\) (constituting a 3.5PN correction in the waveform). Moreover, at this occasion, (iii) we recompute and confirm the previous calculation of the mass-type octupole moment to 3PN order. The ultra-violet divergences due to the point-like nature of the source are treated by means of dimensional regularization. This entails generalizing the definition of the irreducible mass and current multipole moments of an isolated PN source in \(d\) spatial dimensions. In particular, we find that the current type moment has the symmetry of a particular mixed Young tableau and that, in addition, there appears a third type of moment which is however inexistent in 3 spatial dimensions.

Keywords: post-Newtonian theory, gravitational-wave mode, general relativity

1. Introduction

The power of the gravitational-wave (GW) science is predicated on the precise reconstruction of the GW signal as a function of the parameters of the source. In particular, it is crucial to improve the post-Newtonian (PN) predictions for the amplitude and phase of gravitational waveforms generated by inspiralling compact binaries (see the recent reviews [1–3]). One of the main challenges for this purpose is the computation of the multipole moments of the compact binary system to high PN order.

*Author to whom any correspondence should be addressed.
Previous computations of multipole moments for compact binaries (without spins) included the mass-type quadrupole moment at 1PN order [4], 2PN order [5–7] 2.5PN order [8], and 3PN order [9–11] yielding the fundamental GW mode (2, 2) up to 3.5PN order [10–12]. Recently, the 4PN quadrupole moment has been partly obtained in [13], but some contributions, coming from the proper treatment of infra-red (IR) divergences by dimensional regularisation, are still missing. The current quadrupole moment has been determined at 1PN order [5, 14, 15] and 2.5PN order [11]. The mass octupole moment was computed to 3PN order [16] and the current octupole moment to 2PN order [11, 17]. Other moments are known, such as the mass hexadecapole (24) one at 2PN order and the 25 mass and 24 current ones at 1PN order [11]. (Of course, all moments are known in closed form to Newtonian order.) High-order PN contributions due to the spins of the compact bodies have also been derived [18–20].

In the present paper, we shall improve the state of affairs by computing the current-type quadrupole moment up to the 3PN order, notably using dimensional regularisation for treating the ultra-violet (UV) divergences associated with the point-like nature of the source (a system of two compact objects modelled as point masses). Therefore, we are led to generalize the wave-generation formalism from isolated PN matter sources as well as the notion of symmetric-trace-free (STF) mass and current multipole moments to arbitrary space dimensions.

In three dimensions, the STF property plays a crucial role in this formalism, since it guarantees that the multipole moments are linearly independent. This is the case because they are irreducible tensors, in the sense that they belong to irreducible representations of the rotation group SO(3). Now, the irreducible decomposition of vector and tensor fields in a generic space of dimension \( d \) is more complicated than in three dimensions. Indeed, irreducible tensors in \( d \) dimensions are exactly the trace-free (TF) tensors that have Young-tableau symmetries (elements of SO(\(d\)) representations). When \( d = 3 \), because of identities ensuing from the fact that antisymmetrization over more than three indices yields zero (syzygies), any Young tableau can be written as the tensor product of STF and anti-symmetric Levi-Civita tensors. However, for generic \( d \), due to the absence of syzygies and the impossibility of generalizing the Levi-Civita tensor, all Young tableaux must a priori be considered.

Regarding the definition of multipole moments in \( d \) dimensions, we shall essentially find that, while the mass-type moments admit a straightforward generalization, the current-type moments acquire a non-trivial symmetry described by a mixed Young tableau [21–23]. In addition, there appears a third type of irreducible moment, absent in three dimensions and described by another mixed tableau. Thus, one of the main problem addressed in this paper is that of the irreducible multipole decomposition of the tensorial field of GR in \( d \) dimensions. Our work should be useful for the computation of the next-to-leading (5PN) tail effect in the conservative binary’s dynamics, which requires the control of the current quadrupole moment in \( d \) dimensions [24, 25].

Once the current quadrupole moment is properly defined, we shall apply our standard techniques (described for instance in the exhaustive work [13]) to compute it with 3PN accuracy in the case of compact binaries without spins. We present the result in the center-of-mass (CM) frame and for quasi-circular orbits. Next, adding corrections coming from non-linear multipole interactions—most notably the tails-of-tails already known from [16]—, we obtain the radiative current quadrupole moment measured at future null infinity. The full physical content of this moment is encoded into the gravitational mode \((\ell, m) = (2, 1)\), which we thus provide with 3PN relative accuracy, corresponding to 3.5PN accuracy in the full waveform. The perturbative limit or small mass-ratio limit is an important test for the mode, and found to agree with the result from black-hole perturbation theory [26–28]. Our expression for this current
quadrupolar mode is ready for comparison with existing numerical relativity calculations such as [29, 30].

The plan of this paper goes as follows. In section 2, we obtain the multipole expansion of the metric generated by an isolated system in generic $d$ dimensions. The final results are given by equation (2.31) for the mass-type moments $I_L$, and by equations (2.33) and (2.34) for the appropriate generalizations $J_i|_L$ and $K_{ij}|_L$ of the current-type moments. In section 3, we specialize the results to the computation of the current-type quadrupole moment $J_{ij}$ in the case of compact binary systems, to 3PN order. We provide some details on the computation of the most difficult term but, otherwise, refer to [13] for more precise explanations. Section 4 is devoted to the computation of the radiative current quadrupole moment $V_{ij}$ defined at future null infinity, and the corresponding GW mode $h^{21}$, given for quasi-circular orbits by equation (4.11). Finally, in section 5, we perform a partial test of our result by checking the transformation law of the quadrupole moment $J_{ij}$ under a constant shift of the spatial origin, using a formula derived for linearized gravity in reference [14]. Appendix A is a compendium of formulas for the multipole decomposition in $d$ dimensions.

2. The mass and current multipole moments in $d$ dimensions

2.1. Multipole decomposition of the metric of an isolated matter system

The $d$-dimensional Einstein field equations, relaxed by the harmonic-coordinates condition, take the form of an ordinary wave equation (with $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ the $d$-dimensional flat d’Alembertian operator) for the gothic metric deviation $h^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$:

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu},$$

supplemented with the harmonic-coordinates condition itself: $\partial_\mu h^{\mu\nu} = 0$. The source of the wave equation is made of the stress–energy pseudo-tensor of matter and gravitational fields

$$\tau^{\mu\nu} = |g| T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu},$$

where $T^{\mu\nu}$ is the matter stress–energy tensor with spatially compact support (assuming an isolated matter system), and the gravitational stress–energy distribution $\Lambda^{\mu\nu}$ is at least quadratic in $h^{\mu\nu}$ or its first/second space-time derivatives. In this formulation, the matter equations of motion, $\partial_\nu \tau^{\mu\nu} = 0$, are just the consequence of the harmonic-coordinates condition.

The retarded solution of equation (2.1) valid in all space-time for a smooth matter distribution and satisfying appropriate (no-incoming wave) boundary conditions at infinity, is given in terms of the $d$-dimensional retarded integral operator $\Box^{-1}_{\text{ret}}$ as

$$h^{\mu\nu}(x, t) = \frac{16\pi G}{c^4} \Box^{-1}_{\text{ret}} \tau^{\mu\nu}$$

$$= -\frac{4G\tilde{k}}{c^4} \int_{-\infty}^{+\infty} \gamma_{d-2} (\zeta) \int d^d x' \frac{\tau^{\mu\nu}(x', t - z|x' - x|/c)}{|x - x'|^{d-2}}.$$

For convenience, we work in ordinary real space, where the Green function reads, following the notation in [31]:

$$G_{\text{ret}}(x, t) = \frac{\tilde{k} \theta(ct - r)}{4\pi c^{d-1}} \frac{\gamma_{d-2}}{r^{d-2}} \left( \frac{ct}{r} \right).$$
with \( \tilde{k} = \Gamma(\frac{d}{2} - 1)/\pi^{\frac{d}{2} - 1} \) (so that \( \lim_{d \to 3} \tilde{k} = 1 \)) and with the useful definition:

\[
\gamma_{\tilde{k}\mu\nu}(z) \equiv \frac{2\sqrt{\pi}}{\Gamma\left(\frac{1}{\tilde{k}}\right) \Gamma\left(\frac{d}{2} - 1\right)} \left( z^2 - 1 \right)^{\frac{d-1}{2}},
\]  

(2.5)

which is such that \( \int_{1}^{+\infty} dz \gamma_{\tilde{k}\mu\nu}(z) = 1 \) and \( \lim_{d \to 3} \gamma_{\tilde{k}\mu\nu}(z) = \delta(z - 1) \).

Next, we consider the multipole expansion of \( h^{\mu\nu} \), denoted as \( \mathcal{M}(h^{\mu\nu}) \), outside the compact domain of the source. Its general expression has been obtained in three dimensions in \([32, 33]\) and in \(d\) dimensions in \([9]\). Let us outline the derivation for completeness. \( \mathcal{M}(h^{\mu\nu}) \) is a formal solution of the vacuum Einstein field equation, i.e. \( \Box \mathcal{M}(h^{\mu\nu}) = \mathcal{M}(\Lambda^{\mu\nu}) \), where the multipole expansion of the source term is \( \mathcal{M}(\Lambda^{\mu\nu}) \equiv \Lambda^{\mu\nu}[\mathcal{M}(h^{\mu\nu})] \). This solution obviously agrees with the actual solution \( h^{\mu\nu} \) outside the source, and furthermore can be extended inside the source for any \( r = |x| \), except at \( r = 0 \), where it diverges. To proceed, it is convenient to pose

\[
h^{\mu\nu} = \text{FP}_{B=0} \Box^{-1} \left[ B \mathcal{M}(\Lambda^{\mu\nu}) \right] + \Lambda^{\mu\nu}.
\]

(2.6)

The first term is already in the form of a multipole expansion; it is well-defined thanks to a regulator factor \( r^B \) and a process of finite part (FP) when \( B \to 0 \) to cope with the divergence of the multipole expansion at \( r = 0 \). We denote \( \tilde{r} = r/r_0, r_0 \) being some arbitrary constant length scale. Since \( \mathcal{M}(h^{\mu\nu}) \) solves the vacuum field equations, it follows from (2.6) that \( \mathcal{M}(\Delta^{\mu\nu}) \) is an homogeneous (retarded) multipolar solution of the wave equation (in \(d\) dimensions): \( \Box \mathcal{M}(\Delta^{\mu\nu}) = 0 \), hence it takes the form

\[
\mathcal{M}(\Delta^{\mu\nu}) = \frac{4G}{c^2} \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{\ell!} \partial_{\ell} \tilde{F}_L^{\mu\nu},
\]

(2.7)

where \( \partial_{\ell} \equiv \partial_{\ell_1} \ldots \partial_{\ell_\ell} \) is a multi-spatial derivative with respect to the multi-index \( L = i_1 \ldots i_\ell \) involving \( \ell \) indices, and the multipole-moment functions are of the type

\[
\tilde{F}_L^{\mu\nu}(r, t) = \frac{\tilde{k}^{\ell}}{\ell!} \int_{1}^{+\infty} dz \gamma_{\tilde{k}\mu\nu}(z) \tilde{F}_L^{\mu\nu}(r - zr/c),
\]

(2.8)

thus satisfying \( \Box \tilde{F}_L^{\mu\nu}(r, t) = 0 \) in \(d\) dimensions. In our approach, it is important to impose that \( \tilde{F}_L^{\mu\nu} \) be STF with respect to the \( \ell \) indices composing \( L \); however the object \( \tilde{F}_L^{\mu\nu} \) does not represent a genuine irreducible multipole moment because of the additional dependence on the space-time indices \( \mu\nu \). The function of time \( \tilde{F}_L^{\mu\nu}(t) \) in (2.8) is related to the PN expansion of the pseudo-tensor \( \tau^{\mu\nu} \) by \([9, 32, 33]\]

\[
\tilde{F}_L^{\mu\nu}(t) = \text{FP}_{B=0} \int d^d x r^B \hat{\tau}_L \tau^{\mu\nu}[x, t],
\]

(2.9)

with the STF product of spatial vectors in \(d\) dimensions being denoted by \( \hat{\tau}_L \equiv \text{STF}(x_{i_1} \ldots x_{i_\ell}) \) or,\(^1\) alternatively, \( x_{[\ell]} \equiv \hat{\tau}_L \) as used below. In turn, the \( \ell \)-dependent integrand in \( \tilde{F}_L^{\mu\nu}(t) \) can be written down either in ‘exact’ form using an intermediate kernel function \( \delta^{\ell 0}(z) \):

\[
\tau^{\mu\nu}[x, t] = \int_{-1}^{1} dz \delta^{\ell 0}(z) \tau^{\mu\nu}(x, t + zr/c),
\]

(2.10a)

\(^1\) When necessary, we indicate the formal PN expansion of a quantity with an overbar.
generalization in the symmetries of symmetric Young tableaux. Now, we shall prove that the appropriate gen-
to STF, ‘anti-symmetric’ and trace parts, say $i$

For the $0$

moments

and current irreducible STF multipole moments

Upto this stage, the above derivation is already known. It was actually used to define the mass

current irreducible STF multipole moments $I_L$ and $J_L$ in three dimensions, in linearized

gravity [34] and in full GR [33]. We now come to the crucial point: performing the irreducible
decomposition of $J_L^\mu$ in $d$ dimensions.

A previous investigation was already made in [9], where the easier case of the mass-type

moments $I_L$ was treated. In $d$ dimensions, the mass moments are just STF moments, with

the symmetries of symmetric Young tableaux. Now, we shall prove that the appropriate general-

ization in $d$ dimensions of the current-type moments $J_L$ consists of two and only two additional

moments, denoted $J_{iL}$ and $K_{ijL}$, which are also TF but have the symmetries of more

complicated mixed Young tableaux. Strictly speaking, it is $J_{iL}$ that constitutes the genuine

generalization of the usual current moment while $K_{ijL}$ does not exist in three dimensions.

To start with, we construct the irreducible decomposition of the scalar, vector and tensor

components 00, 0i and ij of the object $J_L^\mu$. The ensuing formulas will generalize the three-
dimensional equations (5.1) and (5.2) in [33] and the $d$-dimensional ones (3.32) and (3.33) in

[9]. We display the results without proofs. For the 00 component, we have nothing to do (since

$J_L^{00}$ is STF in $L$), but, for sake of uniformity, we introduce the specific notation

$$J_L^{00} = R_L.$$  \hspace{1cm} (2.12)

For the 0i components, we get, like in three dimensions, three irreducible pieces corresponding to

STF, ‘anti-symmetric’ and trace parts, say

$$J_L^{0i} = T_L^{i(+)0} + T_L^{0(0)L,i-1} + \delta_{i0} T_L^{i(-),(L-1)}.$$  \hspace{1cm} (2.13)

Here and below, the angular brackets $\langle \cdot \cdot \cdot \rangle$ surrounding indices represent the STF projection,

also indicated by the explicit mention STF$_L$ (or STF$_i$), e.g. $T_{i(0)L-1} \equiv$ STF$_i$ $T_{iL}$, with $L = i_1 \ldots i_\ell$ and $L-1 = i_1 \ldots i_{\ell-1}$. The three irreducible pieces in (2.13) are given by

$$T_L^{i(+)0} = J_L^{0(i+1)L},$$  \hspace{1cm} (2.14a)

$$T_L^{0(0)L,i-1} = \frac{2\ell}{\ell + 1} \mathcal{H}_{iL}^{0L,iL-1},$$  \hspace{1cm} (2.14b)

$$T_L^{i(-),(L-1)} = \frac{\ell(2\ell + d - 4)(\ell + d - 3)}{(\ell + d - 2)^2} \tau_{iL}^{0L}. $$  \hspace{1cm} (2.14c)
The square brackets \([\ldots]\) mean the anti-symmetrization of the enclosed indices (with the factor \(\frac{1}{2}\) included in the present case). As in three dimensions, the tensors \(T^{(\ell+1)}_{i|\ell+1} + K^{(\ell-1)}_{i|\ell-1-\ell} + W_{i|\ell-1-\ell-2}\) are STF in all their indices. However, very importantly, the second piece \(T^{(\ell+1)}_{i|\ell+1-\ell-2}\) is not, as emphasised by the vertical separation bar. It is defined from the TF projection of \(F_{L}^{0}\) over all its indices, i.e. the object \(H_{i|L} = T_{0}\) (but without symmetrization), which reads explicitly

\[
H_{i|L} = T_{0}^{(\ell)} - \frac{\ell(2\ell + d - 4)}{(\ell + d - 1)(2\ell + d - 2)} \delta_{i|L - 1} a'.
\] (2.15)

The irreducible decomposition of the \(ij\) components is still more complex; in \(d\) dimensions it involves seven irreducible pieces, i.e. one more than in three dimensions:

\[
F_{L}^{ij} = U_{L}^{(i+2)} + STF STF \left[ U_{L}^{(i+1)} + \delta_{i|L - 1} U_{L}^{(0)} + \delta_{j|L - 1} U_{L}^{(i-1)} + \delta_{j|L - 1} U_{L}^{(i-2)} + W_{i|L - 1 - 2}\right] + \delta_{j|L}.
\] (2.16)

As before, the vertical bars splitting the indices of the tensors \(U_{L}^{(i+1)} \), \(U_{L}^{(i-1)} \), and \(W_{i|L - 1 - 2}\), recall their non-trivial symmetries. The penultimate piece, \(W_{i|L - 1 - 2}\), is absent in three dimensions. Among these tensors, only \(U_{L}^{(i+2)} \), \(U_{L}^{(i+1)} \), \(U_{L}^{(i-2)} \), and \(V_{L}\) are STF in all their indices. We find

\[
U_{L}^{(i+2)} = F_{L}^{(i+1)+2},
\] (2.17a)

\[
U_{L}^{(i+1)} = \frac{4\ell}{\ell + 2} T_{L}^{(i+1) \ i_{i+1}},
\] (2.17b)

\[
U_{L}^{(0)} = \frac{2d(2\ell + d - 4)}{(\ell + d - 2)(2\ell + d - 2)} \delta_{i|L - 1} T_{L}^{(i+2)},
\] (2.17c)

\[
U_{L}^{(i-1)} = \frac{4\ell}{\ell - d + 2 \ell (2\ell + d - 2)} K_{i|L - 2},
\] (2.17d)

\[
U_{L}^{(i-2)} = \frac{2d(2\ell - 6)}{(\ell + d - 2)(2\ell + d - 2)} \delta_{i|L - 1} T_{L}^{(i+2)},
\] (2.17e)

\[
W_{i|L - 1 - 2} = \frac{4\ell - 1}{\ell + 1} T_{L}^{(i+2)},
\] (2.17f)

\[
V_{L} = \frac{1}{d} T_{L}^{(i+2)}.
\] (2.17g)

Following (2.15), we have introduced, as convenient intermediates, the TF (but not symmetrized) parts \(K_{L - 1}^{(i)} \equiv TF F^{(i)}_{i|L - 1} + P_{L}^{(i)} \equiv TF F^{(i)}_{i|L} \), which are given explicitly by

\[
K_{L - 1}^{(i)} = F^{(i)}_{i|L - 1} - \delta_{i|L - 1 - 2} F_{L - 2}^{(i)} + \delta_{j|L - 1 - 2} F_{L}^{(i)} + \delta_{j|L - 1 - 2} F_{L - 1}^{(i)} + \delta_{j|L - 1 - 2} F_{L}^{(i)} + \delta_{j|L - 1 - 2} F_{L - 1}^{(i)}.
\] (2.18a)

\[
P_{L}^{(i)} = F^{(i)}_{L} - STF \left[ \delta_{i|L - 1} U_{L}^{(0)} + \delta_{j|L - 1} U_{L}^{(i-2)} + \delta_{j|L - 1} U_{L}^{(i-1)} \right].
\] (2.18b)
Having established the irreducible decomposition of $F_{L}\mu\nu$, we must next deal (following the same steps as in [9, 33]) with the irreducible decomposition of the tensors $G_{L}\mu\nu$ parametrizing the divergence of the multipole expansion (2.7):

$$\partial_{\mu}M(\Delta^{\mu\nu}) = -\frac{4G}{c^4} \sum_{\ell=0}^{\infty} \frac{(-)^{\ell}}{\ell!} \partial_{\mu} \tilde{G}_{L}\mu,$$

(2.19a)

where $\tilde{G}_{L}\mu(r, t) = \frac{k}{r^{\ell+1}} \int_{1}^{+\infty} dz \gamma_{L\mu}(z) G_{L}\mu(t - zr).$

(2.19b)

The function $G_{L}\mu$ is clearly not independent from $F_{L}\mu\nu$. It admits a closed form expression similar to (2.9) in which an explicit factor $B$ comes out because of the differentiation of the regulator $\tilde{r}^B$, namely (the overdot means the time derivative):

$$G_{L}\mu = -\ell F_{L}^{\mu(\ell - 1)} + \frac{1}{c} F_{L}^{\mu 0} - \frac{1}{c^4(2\ell + d)} \tilde{F}_{L}^{\mu 0},$$

(2.20a)

$$= \frac{1}{c} \int_{\tilde{L}} d^{d}x B \tilde{r}^{B} \frac{\delta_{\ell}^{i}}{r} \partial_{\ell} \tilde{L}_{\mu}(x, t),$$

where we recall the defining expressions (2.10) and (2.11) of $\tilde{L}_{\mu}$. We decompose $G_{L}\mu$ in exactly the same way as in (2.12) and (2.13), thus posing

$$G_{L}\mu^{0} = P_{L}.$$

(2.21a)

$$G_{L}\mu^{i} = Q_{L\mu}^{(+)} + Q_{L\mu}^{(0)} + \delta_{i\mu} Q_{L\mu}^{(-)}.$$  

(2.21b)

Notice that the relation (2.20a) is equivalent to the following explicit constraints

$$P_{L} = \frac{1}{c} \tilde{L}_{\mu} - \ell T_{L}^{(+)} - \frac{(\ell + d - 2)}{(\ell + 1)(2\ell + d - 2)c^{2}} \tilde{L}_{L}^{(-)}.$$

(2.22a)

$$Q_{L\mu}^{(+)} = \frac{1}{c} \tilde{T}_{L}^{(+)\mu} - \ell U_{L\mu}^{(+)2} - \frac{(d - 2)(\ell + d - 1)(2\ell + d + 2)}{2d(\ell + 1)(2\ell + d - 2)c^{2}} U_{L\mu}^{(0)} - \frac{1}{(2\ell + d)c^{2}} \tilde{V}_{L\mu},$$

(2.22b)

$$Q_{L\mu}^{(0)} = \frac{1}{c} \tilde{T}_{L}^{(0\mu)} - \ell U_{L\mu}^{(0)} - \frac{\ell + d - 1}{2\ell + 1)(2\ell + d - 2)c^{2}} \tilde{V}_{L\mu},$$

(2.22c)

$$Q_{L\mu}^{(-)} = \frac{1}{c} \tilde{T}_{L}^{(-\mu)} - \frac{(d - 2)\ell}{2d} U_{L\mu}^{(0)} - \ell U_{L\mu}^{(-)} - \frac{(\ell + d - 2)}{(\ell + 1)(2\ell + d - 2)c^{2}} \tilde{V}_{L\mu}.$$

(2.22d)

Next, with those definitions in hands, we can pose

$$v^{00} = -4G \left[ -\int \dot{P} \, dt + \partial_{\mu} \left( \int \dot{P}_{\mu} \, dt + \int \int \tilde{G}_{L\mu}^{(+)} \, dt - \frac{3d + 1}{2d} \tilde{G}_{L\mu}^{(-)} \right) \right],$$

(2.23a)
\[ v^{00} = -4G \left[ -\int \tilde{Q}_l^{(+) \, dt} + \frac{3d + 1}{2d} \tilde{Q}_l^{-} + \partial_v \left( \int \tilde{Q}_{l0}^{(0) \, dt} \right) - \sum_{l=2}^{+\infty} \frac{(-1)^{l}}{l!} \partial_{L-1} \tilde{P}_{L-1} \right], \quad (2.23b) \]

\[ v^{ij} = -4G \left\{ \delta_{ij} \tilde{Q}^{-} + \sum_{l=2}^{+\infty} \frac{(-1)^{l}}{l!} \left[ 2\delta_{i} \partial_{L-1} \tilde{Q}_l^{-} - 6 \partial_{L-2} \tilde{Q}_l^{-} \right] \right\}, \quad (2.23c) \]

\[ + \partial_{L-2} \left( \tilde{P}_{i,j,L-2} - \frac{7\ell + 3d - 6}{(\ell + 1)(d + 2\ell - 2)} \tilde{Q}_{ij,l-2} + \ell \tilde{Q}_{ij,l-1} \right) \]

\[ - 2\partial_{L-1} \tilde{Q}_{0,l-1}^{-} \right\}. \]

We remind that the overtilde means a monopolar homogeneous solution of the wave equation as defined by (2.8). In the last term, the underlined indices are excluded from the symmetrization operation, indicated by parenthesis (…). Finally, the integrals correspond to time anti-derivatives associated with GW losses: \( \int \hat{P} \, dt \equiv \int_{d}^{\infty} \, d\hat{r} \hat{P}(r, \hat{r}) \), \( \int \tilde{Q} \, dt \equiv \int_{d}^{\infty} \, d\tilde{r}^m \tilde{Q}(r, \tilde{r}) \).\(^2\)

### 2.3. The irreducible source multipole moments in \( d \) dimensions

The role of the term \( v^{\mu \nu} \) introduced in (2.23) is to ensure that

\[ G h_{\mu \nu}^{(0)} = M(\Delta^{\mu \nu}) - v^{\mu \nu}, \quad (2.24) \]

is at once a solution of the wave equation: \( \Box h_{\mu \nu}^{(0)} = 0 \) (in \( d \) dimensions) and divergence-free: \( \partial_\nu h_{\mu \nu}^{(0)} = 0 \). This implies that \( h_{\mu \nu}^{(0)} \) takes the form of a linearized solution (hence the subscript \( 0 \) in \( L = 0 \)) of the relaxed field equation (2.1) in vacuum. The irreducible multipole moments can then be read off from that ‘linearized’ solution [33].

Indeed, the general multipolar expansion decomposes as

\[ \mathcal{M}(h_{\mu \nu}) = G h_{\mu \nu}^{(0)} + u_{\mu \nu} + v_{\mu \nu}, \quad (2.25a) \]

where \( u_{\mu \nu} \equiv \text{FP} \, \Box^{-1} \left[ \tilde{P} \mathcal{M}(\Delta^{\mu \nu}) \right] \).

\[ \mathcal{M}(h_{\mu \nu}) = \sum_{n=1}^{+\infty} G^n h_{\mu \nu}^{(n)}, \quad (2.26) \]

In that formulation, the linearized metric \( h_{\mu \nu}^{(0)} \), which is parametrized by the irreducible multipole moments \( (I_l, J_{ij,l} \text{ and } K_{ij,l} \text{ in } d \text{ dimensions}) \), appears to be the ‘seed’ of the multipolar-post-Minkowskian (MPM) iteration, i.e.

\[ \mathcal{M}(h_{\mu \nu}) = \sum_{n=1}^{+\infty} G^n h_{\mu \nu}^{(n)}, \]

where each of the non-linear coefficients \( h_{\mu \nu}^{(n)} = u_{\mu \nu}^{(n)} + v_{\mu \nu}^{(n)} \) (for any \( n \geq 2 \)) is computed by induction, following exactly the MPM algorithm [35].

The linearized metric \( h_{\mu \nu}^{(1)} \) splits into a ‘canonical’ metric \( h_{\mu \nu}^{(\text{can})} \), having Thorne’s form (see equation (8.12) in [36]), plus a linearized gauge transformation with gauge vector \( \xi_{1}^{\mu} \):

\[ h_{\mu \nu}^{(1)} = h_{\mu \nu}^{(\text{can})} + \partial_{\mu} \xi_{1}^{\nu} + \partial_{\nu} \xi_{1}^{\mu} - \eta_{\mu \nu} \partial_1 \xi_{1}^{1}. \]

\(^2\) We assume that the gravitational-wave source was stationary in the remote past. All the functions which need to be time integrated are zero in the past.
The above considerations lead us to the conclusion that, in $d$ dimensions, the canonical metric is parametrized by three irreducible moments $I_L$, $J_{ijL}$, and $K_{ijL}$, as

$$K_{ijL}^{00} = -\frac{4}{c^2} \sum_{\ell=0}^{+\infty} \frac{(-1)^{\ell}}{\ell!} \partial_{\ell+1} I_L,$$  \hspace{1cm} (2.28a)

$$K_{ijL}^{0i} = -\frac{4}{c^3} \sum_{\ell=0}^{+\infty} \frac{(-1)^{\ell}}{\ell!} \left[ \partial_{\ell+1} J_{iL} - \frac{\ell}{\ell+1} \partial_{\ell+1} J_{jL} \right],$$ \hspace{1cm} (2.28b)

$$K_{ijL}^{ij} = \frac{4}{c^3} \sum_{\ell=0}^{+\infty} \frac{(-1)^{\ell}}{\ell!} \left[ \partial_{\ell+1} J_{iLj} + \frac{2\ell}{\ell+1} \partial_{\ell+1} J_{ijL} - \frac{\ell}{\ell+1} \partial_{\ell+1} K_{ijL} \right],$$ \hspace{1cm} (2.28c)

and that these irreducible moments are given in terms of the previously used quantities by

$$I_L = R_L + \frac{d}{d-2} V_L - \frac{2(d-1)}{(\ell+1)(d-2)c^2} \hat{\Sigma}_L^{(\ell)} - \frac{(d-1)}{(d-2)(\ell+1)(\ell+2)c^2} \hat{U}_L^{(\ell+2)} + \frac{2(d-3)}{(d-2)(\ell+1)} Q_L^{(\ell)},$$ \hspace{1cm} (2.29a)

$$J_{ijL} = \frac{(\ell+1)}{\ell c} \hat{J}_{ijL}^{(0)} + \frac{1}{2\ell c^2} \hat{E}_{ijL}^{(1)} + \frac{\ell}{\ell-1} W_{ijL}.$$ \hspace{1cm} (2.29b)

$$K_{ijL} = \ell+1 \hat{K}_{ijL}.$$ \hspace{1cm} (2.29c)

Since we know the expressions (2.9) and (2.20b), we can get our final explicit results for the moments. Posing

$$\Sigma = \frac{2}{d-1} \frac{(d-2)\tau^{00} + \tau^{ij}}{c^2}, \hspace{1cm} \Sigma^0 = \frac{\tau^{00}}{c}, \hspace{1cm} \Sigma^i = \frac{\tau^{ij}}{c},$$ \hspace{1cm} (2.30)

where we recall that $\tau^{\mu\nu}$ is the PN expansion of the pseudo-tensor $\tau^{\mu\nu}$, we obtain the mass-type moments as

$$I_L = \frac{d-1}{2(d-2)} \int_0^1 d\delta \int d^d x \hat{x}^\mu \left\{ \hat{x}^L \Sigma_{[L]}^{[L]} - \frac{4(d+2\ell-2)}{c^2(d+\ell-2)(d+\ell)} \hat{x}^L \Sigma^{[L]}_{(\ell+1]} \right.$$  
$$+ \frac{2(d+2\ell-2)}{c^2(d+\ell-1)(d+\ell-2)(d+2\ell)} \hat{x}^L \Sigma^{[L]}_{(\ell+2]} - \frac{4(d-3)(d+2\ell-2)}{c^2(d-1)(d+\ell-2)(d+2\ell)} B \hat{x}^L \frac{x^j}{r^2} \Sigma^{[L]}_{(\ell+1]} \right\},$$ \hspace{1cm} (2.31)

in agreement with equation (3.50) in [9]. Note that each of those terms can be computed more explicitly as PN expansions using (2.11). The last term in (2.31) does not exist in three dimensions and plays no role in practical calculations of the mass moment up to the 4PN order. The mass moments are just STF in all their indices so that their symmetry is given by a symmetric Young tableau (with the multi-index $L = i_1 \ldots i_L$)

$$I_L = \begin{vmatrix} i_1 & \ldots & i_L \end{vmatrix}.$$ \hspace{1cm} (2.32)
Next, we find the main theoretical output of this paper, which is the proper generalization of the current-type moments in $d$ dimensions, as

$$J_{i|L} = \mathcal{A}_{ni}B_{0}
\int d^d x \bar{r}^B \left\{ -2 \left[ \chi^L \Sigma_{(\ell)} - \frac{\ell(2\ell + d - 4)}{(\ell + d - 3)(2\ell + d - 2)} \delta^{i[i_i} \chi_{L - 1]}^{(\ell - 1)} \Sigma_{(\ell)\ell} \right] \right. $$

$$+ \frac{2(2\ell + d - 2)}{c^2(\ell + d - 1)(\ell + d)} \left[ \chi^{ab} \Sigma_{(\ell + 1)} - \frac{\ell(2\ell + d - 4)}{(\ell + d - 3)(2\ell + d - 2)} \delta^{i[i_i} \chi_{L - 1]}^{(\ell - 1)} \Sigma_{ab}^{(\ell + 1)} \right] \right\}, \quad (2.33)$$

Here, $\mathcal{A}_{ni}$ means the anti-symmetrization with respect to the pair of indices $i_i$ (with the factor $\frac{1}{2}$ included). Remark that the second terms inside the square brackets correspond to the $d$-dimensional trace of the first terms. Finally, as already mentioned, there exists in $d$ dimensions an additional type of irreducible multipole moment, which reads

$$K_{ij|L} = 4 \mathcal{A}_{ni} \mathcal{A}_{STF} \mathcal{STF} \mathcal{FP} B_{0}
\int d^d x \bar{r}^B \left\{ \chi^L \Sigma_{(\ell)} \right. $$

$$- \frac{2(2\ell + d - 4)}{(\ell - 2)(\ell + d - 2)(2\ell + d - 2)} \left[ 2(2\ell + d - 4) \delta^{i[i_i} \chi_{L - 1]}^{(\ell - 1)} \Sigma_{ij}^{(\ell + 1)} + 4(\ell - 1) \delta^{i[i_i} \chi_{L - 2} \Sigma_{ij}^{(\ell + 1)} \right] $$

$$+ \frac{2(2\ell + d - 4)}{(d - 2)(\ell + d - 2)(2\ell + d - 2)} \delta^{i[i_i} \chi_{L - 1]}^{(\ell - 1)} \Sigma_{ij}^{(\ell + 1)} \right\}, \quad (2.34)$$

The structure of this object is comparatively simpler than for $I_L$ and $J_{i|L}$ in the sense that it depends only on the tensorial piece $\Sigma_{(\ell)}$. The symmetries of the moments (2.33) and (2.34) are given by the mixed Young tableaux $[21-23]$

$$J_{i|L} = \begin{array}{c}
\begin{array}{cccc}
\iota & \iota_{\ell - 1} & \ldots & \iota_1 \\
\iota_i 
\end{array}
\end{array}, \quad K_{ij|L} = \begin{array}{c}
\begin{array}{cccc}
\iota & \iota_{\ell - 1} & \iota_{\ell - 2} & \ldots & \iota_1 \\
\iota_j & \iota_i 
\end{array}
\end{array}, \quad (2.35)$$

respectively, with the convention that the indices are symmetrized over lines before being antisymmetrized over columns.

The tensor $K_{ij|L}$ looks unfamiliar because it actually vanishes in three dimensions. This can be checked explicitly by expanding the expression (2.34) in an orthonormal triad. Another way to see this is to count the number of independent components of $K_{ij|L}$, which follows from the King’s rule [37] and is given by equation (A6b) in the appendix A. As it is proportional to $d - 3$, there is no independent component in three dimensions. This is similar to what happens for the Weyl tensor, which does not exist either in three dimensions. In fact, the tensor $K_{ij|L}$ for $\ell = 2$ has exactly the same TF property, symmetries, and number of independent components as the Weyl tensor.

It is convenient to introduce the following specific notation that allows reconstructing the symmetries of $J_{i|L}$ given $\ell + 1$ indices:

$$\text{Sym} = \mathcal{A}_{ni} \mathcal{TF} \mathcal{STF}. \quad (2.36)$$

As an example, the first line of (2.33) can be constructed, starting from $\chi^L \Sigma_{(\ell)}$, by taking its STF part on the indices $L$, then removing the traces of the resulting object in $d$ dimensions and, finally, anti-symmetrizing on $\{i, i_i\}$. Thus, $J_{i|L}$ is TF, STF with respect to $L - 1 = i_{\ell - 1} \ldots i_1$
and anti-symmetric with respect to the pair \( i; i \); on the other hand, \( K_{ij/L} \) is TF, STF with respect to \( L - 2 = i - 2 \ldots i \) and anti-symmetric with respect to both pairs \( i; j \) and \( i; -1 \).

The gauge vector \( \xi_{1i}^{\mu} \) in (2.27) can be dealt with in the same way as \( h_{i}^{\mu} \). It admits the following irreducible decomposition:

\[
\xi_{1}^{0} = \frac{4}{c^{2}} \sum_{\ell=0}^{+\infty} \frac{(-)^{\ell}}{\ell!} \partial_{\ell} \tilde{W}_{L},
\]

\[
\xi_{1}^{i} = -\frac{4}{c^{2}} \sum_{\ell=0}^{+\infty} \frac{(-)^{\ell}}{\ell!} \partial_{\ell} \tilde{X}_{L} = \frac{4}{c^{2}} \sum_{\ell=1}^{+\infty} \frac{(-)^{\ell}}{\ell!} \left[ \partial_{\ell-1} \tilde{Y}_{L-1} + \partial_{\ell-1} \tilde{Z}_{L-1} \right],
\]

where the closed form expressions of the so-called gauge moments \( \tilde{W}_{L}, \tilde{X}_{L}, \tilde{Y}_{L-1} \) (mass type) and \( \tilde{Z}_{L-1} \) (current type) are similar to (2.31) and (2.33) and (2.34) but will not be needed henceforth. The gauge moments enter the signal at a relatively high PN order and can thus be computed at a low PN order, where one does not need to resort to dimensional regularisation at all. Therefore, we can just use here the three-dimensional expressions of the gauge moments displayed in equations (5.17)–(5.20) of [33].

The ordinary STF mass-type moment in three dimensions, say \( I_{L}^{[3]} \), is simply recovered as the limit \( I_{L}^{[3]} = \lim_{d \rightarrow 3} I_{L} \) (after renormalization). In particular, we observe that this limit removes the last term in (2.31). For the ordinary STF current-type moment in three dimensions, \( J_{L}^{[3]} \), we have

\[
\lim_{d \rightarrow 3} J_{iL} = \varepsilon_{\mu\nu\alpha} J_{iL-1}^{\mu
u\alpha} \iff J_{L}^{[3]} = \frac{1}{2} \varepsilon_{\mu\nu\alpha} \lim_{d \rightarrow 3} J_{\mu\nu\alpha}(d-1),
\]

where \( \varepsilon_{\mu\nu\alpha} \) is the usual Levi-Civita symbol in three dimensions (underlined indices being excluded from symmetrization). Notice that \( J_{L}^{[3]} \), as recovered from (2.38), not only is symmetric in its indices \( L \) but is also automatically TF. Finally, as we said, the multipole moment \( K_{ij/L} \) does not exist in three dimensions.

We end this section by recalling, for completeness, the expressions of the STF moments in three dimensions, namely:

\[
I_{L}^{[3]}(t) = \text{FP}_{B=0} \int d^{3}x r^{B} \int_{-1}^{1} dz \left[ \delta_{\ell}(z) \tilde{x}_{L} \tilde{\Sigma} - \frac{4(2\ell + 1)}{c^{2}(\ell + 1)(2\ell + 3)} \delta_{\ell+1}(z) \tilde{x}_{L} \tilde{\Sigma}_{i} \right] (t + zr/c),
\]

\[
J_{L}^{[3]}(t) = \text{FP}_{B=0} \int d^{3}x r^{B} \int_{-1}^{1} dz \varepsilon_{\mu\nu\alpha} \left[ \delta_{\ell}(z) \tilde{x}_{L-1} \tilde{\Sigma}_{\mu
u\alpha} \right] (t + zr/c),
\]

where \( \tilde{\Sigma} = \overline{\mu}^{00} + \overline{\mu}^{ij}/c^{2}, \overline{\Sigma}_{\mu
u\alpha} = \overline{\mu}^{\mu\nu\alpha}/c \) and \( \overline{\Sigma}_{\mu
u\alpha} = \overline{\mu}_{\mu
u\alpha} \) (in three dimensions), whereas \( \delta_{\ell}(z) \) is the 3d limit of the counterpart quantity introduced in (2.10), so that

\[
\int_{-1}^{1} dz \delta_{\ell}(z) \tilde{\Sigma}(x, t + zr/c) = \sum_{k=0}^{+\infty} \frac{(2\ell + 1)!!}{2^{k\ell}(2\ell + 2k + 1)!!} \left( \frac{r}{c} \frac{\partial}{\partial t} \right)^{2k} \tilde{\Sigma}(x, t),
\]

\[
\delta_{\ell}(z) \equiv \frac{(2\ell + 1)!!}{2^{\ell+1}\ell!} (1 - z^{2})^{\ell}.
\]
3. The current quadrupole moment of compact binaries

In this section, we compute \( J_{k|\ell} \) in \( d \)-dimensions for (non-spinning) compact binaries up to the 3PN order using (2.33) for \( \ell = 2 \). Remember that \( J_{k|\ell} \) is merely the dual of the physical current quadrupole moment \( J_{ij} \), but remark also that it is not symmetric on \( \{i, j\} \). We employ essentially the same set of techniques as for the computation of the mass quadrupole moment at the 4PN order [13]; all calculations are performed with Mathematica supplemented with the xAct library [38]. Here are the salient points of the method:

- The PN metric in harmonic coordinates is expressed in terms of elementary potentials satisfying a set of ordinary nested wave equations.
- We express \( J_{k|\ell} \) in terms of integrals of potentials by inserting the sources (2.30) into (2.33). We then simplify the integrand as much as possible using integration by part, together with a set of relations between potentials (see section 3.3 in [13]).
- The resulting expression for \( J_{k|\ell} \) is divided into three types of terms: compact support terms, non-compact support terms and surface terms at spatial infinity (when \( r \to +\infty \) with fixed \( t \)).
- Compact support terms are calculated directly in \( d \)-dimensions using the \( d \)-dimensional regularized potentials at the source points 1 and 2 derived in [13].
- Non-compact support terms are first computed in three dimensions; the difference between dimensional and Hadamard regularisations in the UV is obtained in a second stage using equation (4.8) of [13], and added to the three-dimensional result.
- For one particular non-compact support term (involving the non-linear potential \( \hat{Y}_i \) defined by (A4h) in [13]), we employ the method of super-potentials (see also more details below).
- All derivatives of potentials are understood as Schwartz distributional derivatives [39], i.e. we apply the Gel'fand–Shilov formula in \( d \) dimensions [40] (see section 4.3 in [13]).
- However, the IR divergences are treated using the Hadamard partie finie procedure with regulator \( \tilde{r}_B \) in three dimensions (see reference [41] for a justification of this point).
- The result for \( J_{k|\ell} \) after integration contains a pole in \( \frac{1}{d-3} \); the last step consists in applying a shift, given in equation (B1) of [13], on the bodies’ positions; when substituting the renormalized position variables to the bare ones, the pole in \( J_{k|\ell} \) cancels out.
- Finally, we can take the limit \( d \to 3 \) using (2.38) in order to get the renormalized current quadrupole moment \( J_{ij} \) displayed in equation (3.6).

The investigation of the current moment in \( d \)-dimensions in section 2 is essential to treat the UV divergences with dimensional regularisation, which incidentally is the only way to give a full meaning to distributional derivatives. However, our calculation of the current quadrupole limited to 3PN order does not require corrections from dimensional regularisation for the IR divergences. We thus treat those in the source moment with the Hadamard partie finie regularisation and add the non-linear corrections due to tails and tails-of-tails as computed in three dimensions. We have shown that the IR corrections in the source moment computed in pure dimensional regularisation are cancelled by corresponding UV corrections coming from the tails-of-tails in \( d \) dimensions, yielding the equivalence of Hadamard partie finie and dimensional regularisation for the IR divergencies at 3PN order [41].

Let us illustrate one computational aspect concerning the use of ‘super-potentials’. One non-compact support term to be computed in the current quadrupole moment \( J_{k|\ell} \) reads

\[
J_{k|\ell} = -\frac{4}{\pi \xi^\xi} \frac{d-1}{d-2} \text{Sym FP} \int d^4x \hat{r}^\xi \frac{\partial k}{\partial \tilde{X}_a} \partial_\alpha V, \tag{3.1}
\]

\[
J_{k|\ell} \equiv -\frac{4}{\pi \xi^\xi} \frac{d-1}{d-2} \text{Sym FP} \int d^4x \hat{r}^\xi \frac{\partial k}{\partial \tilde{X}_a} \partial_\alpha V.
\]
where Sym is defined in (2.36). It involves the difficult non-linear potential \( \hat{Y}_a \) obeying a Poisson equation \( \Delta \hat{Y}_a = \hat{S}_a \) to leading order. While the source term \( \hat{S}_a \) is known and relatively easy to manage, no closed-form expression is available for the potential \( \hat{Y}_a \) itself, even in three dimensions. On the other hand, \( V \) is a simple linear potential with compact support: \( \Delta V = -4\pi G \sigma. \) By means of an integration by parts taking advantage of the symmetries of \( J_{kj} \) (the all-integrated term being zero by analytic continuation in \( B \)), we transform this term into

\[
J_{kj}^{\hat{Y}} = \frac{4}{\pi G c^6} \frac{d - 1}{d - 2} \text{Sym} \text{ FP} \int \frac{d^d \mathbf{r}}{r} \hat{Y}_k \partial_j \hat{Y}_a \partial_a V.
\]

The idea behind introducing super-potentials is that the solution of the equation \( \Delta \hat{\Psi}_{ij}^h V = \hat{x}_{ij} \partial_h V \) (where \( \hat{x}_{ij} \equiv x_{ij,b} \) is STF) is known in analytic closed-form in terms of the super-potentials of \( V \), namely the Poisson-like potentials \( V_{2k} \) satisfying the hierarchy of equations \( \Delta V_{2k+2} = V_{2k} \), together with \( V_0 \equiv V \). This solution,

\[
\hat{\Psi}_{ij}^h V = x_{ij,b} \partial_b V_2 - 4x_{ij,b} \partial_{fib} V_4 + 8\partial_{ij,bk} V_6,
\]

is valid in the sense of distributions and its expression holds in any \( d \) dimensions.\(^3\) Thanks to it, we can transform the term (3.2) after further integrations by parts into the more tractable form

\[
J_{kj}^{\hat{Y}} = \frac{4}{\pi G c^6} \frac{d - 1}{d - 2} \text{Sym} \text{ FP} \int \frac{d^d \mathbf{r}}{r} \hat{Y}_k \partial_j \hat{Y}_a \partial_a \left( \partial_b \hat{\Psi}_{ij}^b V_4 - \hat{\Psi}_{ij}^b \partial_b \hat{Y}_a \right).
\]

Here, the first term is straightforwardly computed because we know the source \( \hat{S}_a \) of the potential \( \hat{Y}_a \). As for the second term, which is a surface term, it can be computed directly in three dimensions since it does not involve UV divergences. It depends only on the expansions of \( \hat{Y}_a \) and the superpotential at spatial infinity, when \( r \rightarrow +\infty \). The expansion of \( \hat{Y}_a \) can be determined directly from the source term \( \hat{S}_a \), without having to control \( \hat{Y}_a \) all-over the space. As a check of the result, we have also calculated \( J_{kj}^{\hat{Y}} \) using the alternative form

\[
J_{kj}^{\hat{Y}} = -\frac{4}{\pi G c^6} \frac{d - 1}{d - 2} \text{Sym} \text{ FP} \int \frac{d^d \mathbf{r}}{r} \hat{Y}_k \partial_j \hat{Y}_a + \partial_b \left[ 4x^{i/b} \partial_a V_4 \partial_b \hat{S}_a + x^{i/b} \partial_b V_2 \partial_b \hat{S}_a \right]
\]

\[
+ \partial_b \left[ 4x^{i/b} \partial_a V_2 \partial_b \hat{Y}_a - \partial_b V_2 \partial_b (x^{i/b} \hat{Y}_a) + 4x^{i/b} \partial_a V_4 \partial_b \hat{Y}_a - 4\partial_a V_4 \partial_b (x^{i/b} \hat{Y}_a) \right],
\]

which is derived directly by substituting \( \Delta \hat{Y}_a \) to \( \hat{S}_a \), expanding the derivatives by means of the Leibniz rule, and resorting again to the properties of the symmetry operator \( \text{Sym}_{kj} \).

The final expression of the 3PN current quadrupole in a general frame is obtained, as mentioned above, by applying the UV shift, taking the three dimensional limit and, finally, using (2.38) to come back to the usually looking current moment \( J_{ij} \). We have checked that the symmetry operations (2.36) commute with the computation of the difference between the dimensional and Hadamard regularisations. The expression of the final current quadrupole is

\(^3\) See appendix A in [13] for a compendium of formulas and complete definitions of potentials.

\(^4\) The conditions under which this solution is unique have been investigated in section 4.2 of [42].
quite long; so we present it only in the CM frame.\footnote{Notation is as follows: the relative position and velocity of the two particles (in harmonic coordinates) are $x^i = y^i_1 - y^i_2$, $v^i = dx^i/d\tau = v^i_1 - v^i_2$; the distance between the two particles $r = |x|$; the masses $m_1$ and $m_2$, the total mass $m = m_1 + m_2$, the symmetric mass ratio $\nu = m_1 m_2 / m^2$ and the mass difference ratio $\Delta = (m_1 - m_2) / m$; finally, we pose $L^j \equiv e^{i\phi} x^i v^j$ and $(v_x) = v^i x^i$.} For general orbits (bound or unbound), we obtain
\begin{equation}
J_{ij} = -\nu m \Delta \left[ A L_j^i x^j + B \frac{(v_x)}{c} L_j^i v^j + C \frac{Gm}{c^2} L_j^i v^j \right] + O \left( \frac{1}{c^4} \right). \tag{3.6}
\end{equation}

The coefficients $A$ and $B$ describe the conservative effects up to the 3PN order; $A$ also includes a 2.5PN dissipative term, while the coefficient $C$ is purely dissipative. We have

\begin{align}
A &= 1 + \frac{1}{c^2} \left[ \left( \frac{13}{28} - \frac{17}{7} \nu \right) v^2 + \frac{Gm}{r} \left( \frac{27}{14} + \frac{15}{7} \nu \right) \right] + \frac{1}{c^2} \left[ \left( \frac{29}{84} - \frac{11}{3} \nu + \frac{505}{36} \nu^2 \right) v^4 + \frac{Gm}{r^3} \left( \frac{5}{252} - \frac{241}{252} \nu - \frac{335}{84} \nu^2 \right) (v_x)^2 \right. \\
&\quad + \frac{Gm}{r^2} \left( \frac{671}{252} - \frac{1297}{126} \nu - \frac{121}{12} \nu^2 \right) v^6 \\
&\quad + \left. \frac{G^2 m^2}{r^3} \left( \frac{43}{252} - \frac{1543}{126} \nu + \frac{293}{84} \nu^2 \right) \right] \\
B &= \frac{5}{28} - \frac{5}{14} \nu \\
&\quad + \frac{1}{c^2} \left[ \left( \frac{25}{168} \nu + \frac{25}{14} \nu^2 \right) v^2 + \frac{Gm}{r} \left( \frac{103}{63} + \frac{337}{126} \nu - \frac{173}{84} \nu^2 \right) \right].
\end{align}

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In section 6 (footnote 10) of [13], we identify for simplicity the two scales associated with the two particles, respectively, i.e., we set the quadrupole moment \[ 43 \].

The cancellation of the tail-of-tail correction of the radiative current quadrupole moment \[ 3.8c \] is due to the IR divergences of the multipole moment. This constant will be found to properly cancel out in the final expression of the GW mode \[ h_{21} \] in section 4. In effective field theory, the cancellation of \( r_0 \) is ruled by the renormalization group equations and \( \beta_I \) is the associated beta function coefficient.\(^6\) In the traditional PN approach adopted here, the constant \( r_0 \) is canceled by the same constant that is present in the tail-of-tail correction of the radiative current quadrupole (see equation (4.2)). As for \( r_0' \), it is a UV scale and it also properly cancels out when we express the orbital separation \( r \) of the particles in terms of the invariant orbital frequency \( \omega \) or, equivalently, the PN parameter \( \nu \) (see equation (4.11)). Following the discussion in section 6 (footnote 10) of [13], we identify for simplicity the two \( a \ priori \) independent UV scales associated with these two particles, respectively, i.e., we set \( r_0'' \equiv r_1 = r_2' \).

In the case of quasi-circular orbits, we get (with \( \gamma \equiv \nu \nu' / (r c^2) \))
\[
J_{ij} = \gamma m \Delta \left[ A_{\text{circ}} L^i x^j + C_{\text{circ}} \frac{G m}{r^3} L^i x^j \right] + O \left( \frac{1}{c^3} \right),
\]
\[
A_{\text{circ}} = 1 + \gamma \left( \frac{67}{28} - \frac{2}{7} \nu \right) + \gamma^2 \left( \frac{13}{9} - \frac{4651}{252} \nu + \frac{\nu^2}{168} \right) + \gamma^3 \left( \frac{2301023}{415800} - \frac{214}{105} \ln \left( \frac{r}{r_0} \right) \right) + \left[ \frac{243853}{9240} + \frac{123}{128} \nu^2 - 22 \ln \left( \frac{r}{r_0'} \right) - \frac{44995}{5544} \nu^2 + \frac{599}{16632} \nu^3 \right],
\]
\[
C_{\text{circ}} = \frac{188}{35} \nu^2 \gamma.
\]

\(^6\) Note that \( \beta_I \) happens to be the same as \( \beta_i = -\frac{164}{105} \nu' \), which is associated with the renormalization of the mass-type quadrupole moment [43].

\[3.8a\]
\[3.8b\]
\[3.8c\]

4. The gravitational-wave mode \( h_{21} \) at 3PN order

Let us now review the expression of the radiative current quadrupole moment \( V_{ij} \) in terms of the source moment \( \Gamma_{ij} \). The radiative moments parameterize the asymptotic waveform to leading order \( 1/R \) in the distance, within the class of radiative coordinates \( (T, R) \) such that
\[ u \equiv T = R/c \] is a null coordinate, or becomes asymptotically null in the limit \( R \to +\infty \). In terms of the harmonic coordinates \((t, r)\), we have
\[
 u = t - \frac{r}{c} - \frac{2GM}{c^3} \ln \left( \frac{r}{eb} \right) + \mathcal{O} \left( \frac{1}{r^2} \right), \tag{4.1}
\]
where \( M \) is the conserved total mass of the source and \( b \) an arbitrary constant time scale (independent from \( r_0 \)). Here, we are interested in the radiative current quadrupole moment, linking to the source moments through a series of non-linear corrections. We first relate \( V_{ij} \) to the so-called canonical current quadrupole moment \( S_{ij} \), which constitutes a useful intermediate definition. To 3PN order, we have
\[
 V_{ij}(u) = S_{ij}^{(2)}(u) + \frac{GM}{c^3} \int_{-\infty}^{u} \mathrm{d}\tau \left[ 2 \ln \left( \frac{u - \tau}{2b} \right) + \frac{7}{3} S_{ij}^{(4)}(\tau) \right]
 + \frac{G}{c^3} \left\{ 4 S_{ij}^{(2)} M_k^{(3)} + 8 M_{ij}^{(2)} S_k^{(3)} + 17 S_{ij}^{(1)} M_k^{(4)} + 3 M_{ij}^{(2)} S_k^{(5)} + 3 M_{ij}^{(2)} S_k^{(5)} - \frac{1}{4} S_a M_{ij}^{(5)} - 7 \varepsilon_{abij} S_{ij}^{(4)} \right.
 + \left. \frac{1}{2} \varepsilon_{abij} \left[ 3 M_{ij}^{(1)} M_k^{(2)} M_{ij}^{(4)} + \frac{353}{24} M_{ij}^{(2)} M_{ij}^{(4)} + \frac{5}{12} M_{ij}^{(1)} M_{ij}^{(2)} M_{ij}^{(3)} + \frac{113}{8} M_{ij}^{(1)} M_{ij}^{(2)} M_{ij}^{(3)} \right] \right\}
 + \frac{G^2 M^2}{c^5} \int_{-\infty}^{u} \mathrm{d}\tau \left[ \left( \frac{u - \tau}{2b} \right) + \frac{14}{3} \ln \left( \frac{u - \tau}{2b} \right) \right]
 - \frac{214}{105} \ln \left( \frac{u - \tau}{2b} \right) + \frac{26254}{11025} S_{ij}^{(5)}(\tau) + \mathcal{O} \left( \frac{1}{c^7} \right). \tag{4.2}
\]
At this accuracy level, the right-hand side contains the dominant quadratic tail term at 1.5PN order [44], a number of instantaneous corrections at 2.5PN order involving the canonical moments \( M_L, S_L \) [11], and the cubic tail-of-tail term at 3PN order [16]. Note the presence of the IR scale \( r_0 \) in one of the logarithms of the tail-of-tail term in (4.2), with the same coefficient \( \beta_j = -\frac{244}{105} \) as in the source moment (3.6), which shows that the constant \( r_0 \) will finally drop from the radiative moment. Next, the canonical moment \( S_{ij} \) is related to the source moments \( I_L, J_L \) and to the so-called gauge moments \( W_L, X_L, Y_L, Z_L \) by [11]
\[
 S_{ij} = J_{ij} + \frac{2G}{c^5} \left[ \varepsilon_{abij} \left( -r_{ij}^{(3)} W_a - 2f_{ij} Y_a^{(2)} + f_{ij}^{(1)} Y_a^{(1)} \right) + 3J_T Y_j^{(1)} - 2f_{ij}^{(1)} W^{(1)} \right] + \mathcal{O} \left( \frac{1}{c^7} \right). \tag{4.3}
\]
For all moments in (4.2) but in the first term \( S_{ij}^{(2)}(u) \), we can identify \( M_L, S_L \) with \( I_L, J_L \) at the 3PN approximation. The required gauge moments can be found in equations (5.7) of [12].

With our control of the radiative current moment \( V_{ij} \), we are in the position to compute the mode \( h^{21} \) at the 3PN order. The modes are defined from the \( + \) and \( \times \) polarization...
waveforms as
\[
    h \equiv h_+ - ih_\times = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h^{\ell m}(\Theta, \Phi),
\]  
where the spin-weighted spherical harmonics of weight \(-2\) are functions of the spherical angles (\(\Theta, \Phi\)) defining the direction of propagation.\(^7\) For planar binaries, which are either non-spinning or with spins aligned or anti-aligned with the orbital angular momentum, we know that there is a clean separation of modes between mass-type and current-type contributions (see [10] and section 3.2 in [12]). In particular, the ‘current’ modes are entirely determined by the current radiative moments when \(\ell + m\) is an odd integer,
\[
    h^{\ell m} = \frac{i}{\sqrt{2}} \frac{G}{R \ell^{\ell+3}} \sqrt{\frac{2}{\ell + 1}} \sum_{m=-\ell}^{\ell} \alpha_{\ell m}^{\ell m} V^{\ell m}, \quad \text{(for } \ell + m \text{ odd)},
\]  

hence the only relevant mode for the current quadrupole is \(h^{21}\) (recall that \(h^{21} = h^{21}_{\text{STF}}\) in the planar case). The current moment in non-STF spin-weighted guise is given in terms of the STF version \(V_L\) by
\[
    V^{\ell m} = -\frac{8}{\ell!} \frac{\sqrt{\ell(\ell + 2)}}{2(\ell + 1)(\ell - 1)} \alpha_{\ell m}^{\ell m} V_L,
\]
where the STF tensorial coefficient \(\alpha_{\ell m}^{\ell m} \equiv \int d\Omega \bar{\alpha}_{\ell m}^{\ell m} Y^{\ell m}\) is defined from the ordinary spherical harmonic \(Y^{\ell m}\) (or in fact its complex conjugate \(Y^{\ell m}\)). In practice, we use the convenient orthonormal triad \((n, \lambda, l)\) where \(n = x/|x|, l = L/L, \) while \(\lambda\) completes the triad with right-handed orientation. We also define \(m = (n + i\lambda)/\sqrt{2}\), as well as its value \(m_0\) at some reference time \(t_0\), to obtain the explicit expression of \(\alpha_{\ell m}^{\ell m}\):\(^8\)
\[
    \alpha_{\ell m}^{\ell m} = \frac{\sqrt{4\pi(-\sqrt{2})^{\ell + m}}}{\sqrt{(2\ell + 1)(\ell + m)!}} \frac{\sqrt{2}}{\ell!} \frac{\sqrt{2}}{\ell!} \frac{(\ell + m)!}{(\ell - m)!} \bar{\alpha}_{\ell m}^{\ell m}.
\]  

In the tail and tail-of-tail terms at 1.5PN and 3PN orders, there appears the total Arnowitt–Deser–Misner (ADM) mass \(M\), which therefore needs to be computed with 1PN precision. It is convenient, following [45, 46], to perform a change of phase variable, from the actual orbital phase \(\phi\) to the new variable
\[
    \psi \equiv \phi - \frac{2GM\omega}{c^3} \ln \left( \frac{\omega}{\omega_0} \right),
\]

where the constant \(\omega_0\), equivalent to \(\hat{b}\) in (4.1), is conveniently defined by \(\omega_0 = \frac{1}{2} \exp\left[ \frac{11}{12} - \gamma_E \right] \) with \(\gamma_E\) the Euler constant. The main advantage of the new phase variable (4.8) is that it minimizes the occurrence of logarithms due to tails in the waveform and modes. Notice, however, that the use of either \(\phi\) or \(\psi\) is equivalent for the present 3PN level of accuracy. Indeed, the correction term in (4.8) is of order 1.5PN, which means that the effect as seen as a correction to the phase evolution is actually of order 4PN with respect to the leading order in the phase provided by the usual quadrupole formula. A similar variable is also introduced in black-hole perturbation theory [26–28].

\(^7\) We adopt the same conventions as in references [11, 12]. The spin-weighted spherical harmonics are given by (2.4) in [11], and figure 1 in [12] specifies our convention for the polarization vectors.

\(^8\) We have \(Y_{\ell m}^{\ell m} = (2\ell + 1)!! \bar{\alpha}_{\ell m}^{\ell m}\) in the alternative definition used in [10, 36].
The modes are thus defined with respect to the phase variable (4.8) as
\[ h_{lm} = \frac{2Gm\nu x}{Rc^2} \sqrt{\frac{10\pi}{3}} H_{lm} e^{-im\psi}. \] (4.9)
Beware that the symbol \( m \) in the first factor of the right-hand side denotes the total mass \( m = m_1 + m_2 \), whereas its two other occurrences refer to an integral label. The final result for the 3PN mode (2, 1) (corresponding in fact to the 3.5PN accurate waveform) is expressed with the usual gauge invariant PN parameter (with \( \omega = \dot{\phi} \))
\[ x \equiv \left( \frac{Gm\omega}{c^2} \right)^{2/3}. \] (4.10)
As already announced, the regularisation constants \( r_0, r'_0 \) and the gauge constant \( b \) disappear from the end result, and we find, extending (9.4b) in [11]:
\[ \hat{H}^{21} = \frac{i}{3} \Delta \left[ x^{1/2} + x^{3/2} \left( \frac{-17}{28} + \frac{5\nu}{7} \right) + x^2 \left( \pi + i \left[ -\frac{1}{2} - 2\ln 2 \right] \right) \right] 
+ x^{5/2} \left( \frac{-43}{126} + \frac{509\nu}{126} + \frac{79\nu^2}{168} \right) 
+ x^3 \left( \pi \left[ \frac{-17}{28} + \frac{3\nu}{14} \right] + i \left[ \frac{17}{56} + \nu \left( -\frac{353}{28} - \frac{3}{7} \ln 2 \right) + \frac{17}{14} \ln 2 \right] \right) 
+ x^{7/2} \left( \frac{15223771}{1455300} + \frac{\pi^2}{6} - \frac{214}{105} \gamma_E - \frac{107}{105} \ln(4x) - \ln 2 - 2(\ln 2)^2 \right) 
+ \nu \left[ \frac{-102119}{2376} + \frac{205}{128} \pi^2 \right] - \frac{4211}{8316} \nu^2 + \frac{2263}{8316} \nu^3 + i \frac{109}{210} \ln 2 \right) \right) 
+ O \left( \frac{1}{c^8} \right) \]. (4.11)

The perturbative limit \( \nu \rightarrow 0 \) is in perfect agreement with the result of black-hole perturbation theory [26–28], as one may check with the mode provided in this limit in the appendix B in [27]. Interestingly, our result (4.11) can be compared directly with accurate numerical relativity calculations, such as those in [29, 30].

5. Test of the current quadrupole with a constant shift

In this section, we show that our final expression for the 3PN current quadrupole moment passes (one aspect of) the test of constant shifts. By this, we mean that when the two trajectories of the particles are shifted by \( y_1^1 \rightarrow y_1^1 + \epsilon^1 \) and \( y_2^1 \rightarrow y_2^1 + \epsilon^1 \), where \( \epsilon^1 \) denotes an infinitesimal constant purely spatial vector, the variation of the moment obeys the expected law of transformation of the moment under the shift to first order in that shift.

We consider the case of the moments of a general isolated system made of an extended smooth matter distribution. There is then no need of UV regularisation and we may focus on the moments \( I_L \) and \( J_L \) in three dimensions, given by equation (2.39). The laws of transformation

9 For the comparison, note that the phase variable used in the appendix B of [27] is related to ours by \( \psi_{YS} = \psi + \pi/2 + 2x^{3/2}\ln(2 - 17/12) \).
of such moments to first order in the shift have been found in linearized gravity by Damour and Iyer [14, 34] (see also [17]). In this case, the pseudo-tensor $\tau^{\mu\nu}$ reduces to the matter stress–energy tensor $T^{\mu\nu}$ with compact support, so that there is no necessity to resort to the IR regularisation with regulator $\tilde{r}^B$ and FP when $B = 0$. In this situation, the transformation laws read (see appendix B in [14])\footnote{Here, the shift vector is denoted $\epsilon_i = \epsilon'$, which contrasts with the usual Levi-Civita symbol $\epsilon_{iab} = \epsilon^{iab}$.}

\begin{align}
\delta_{\ell}^\text{lin} I_{L} &= \ell \epsilon_{i(J_{L-1})} = \frac{4\ell}{c^2(\ell + 1)^2} \epsilon_{ab} f^{(1)}_{i(b(L-1)} \epsilon_{i)ab} + \frac{(\ell - 1)(\ell + 3)}{c^2(\ell + 1)^2(2\ell + 3)} \epsilon_{aL} j^{(2)}_{iL}, \quad (5.1a) \\
\delta_{\ell}^\text{lin} J_{L} &= \frac{(\ell - 1)(\ell + 1)}{\ell} \epsilon_{ij} J_{L-1} + \frac{1}{\ell} \epsilon_{ab} f^{(1)}_{i(L-1)} \epsilon_{i)ab} + \frac{(\ell - 1)(\ell + 3)}{c^2(\ell + 1)^2(2\ell + 3)} \epsilon_{aL} J^{(2)}_{iL}. \quad (5.1b)
\end{align}

Such laws follow from the irreducible decomposition of the metric and from the fact that the components of the matter tensor behave under the spatial constant shifts like scalars, i.e. $T^{\mu\nu}(x') = T^{\mu\nu}(x)$, considering now the action of the shifts as a passive coordinate transformation $x' = x + \epsilon x + \epsilon^2 x + \cdots$, where $\epsilon^2 = 0$, $\epsilon = (0, \epsilon')$.

For the full non-linear theory, driven by equations (2.1) and (2.2), one must take into account the non-linear gravitational source term $\Delta^{\mu\nu}$. Now, even though the pseudo-tensor $\tau^{\mu\nu}$ behaves in the same way as the matter tensor under constant spatial shifts, i.e. $\tau^{\mu\nu}(x') = \tau^{\mu\nu}(x)$, the transformation laws (5.1) obeyed by the moments $I_{L}$ and $J_{L}$ are then expected to be modified, notably because those involve the regularisation factor $\tilde{r}^B$ dealing with the fact that the pseudo-tensor is no longer with compact support. This \textit{a priori} implies that the linear transformation laws (5.1) must be augmented by certain non-linear corrections, which may be referred to as $\delta_{\ell}^\text{nonlin} I_{L}$ and $\delta_{\ell}^\text{nonlin} J_{L}$.

Ignoring all odd powers of $1/c$, we have found that the mass quadrupole moment $I_{ij}$ at 3PN order does satisfy the linear transformation law (5.1a), which means that the non-linear correction in this case actually happens to start only at the 4PN order:

$$\delta_{\ell}^\text{nonlin} I_{ij} = \mathcal{O}\left(\frac{1}{c^8}\right).$$

(5.2)

However, in the case of the current quadrupole moment $J_{ij}$ (neglecting again possible odd-type contributions at 2.5PN order), we find that the non-linear contribution arises precisely at the 3PN order. Since it is due to the fact that the pseudo tensor has a non-compact support, we expect this non-linear correction to be made of some combination of the multipole moments parametrizing the expansion of the metric at infinity. Looking at the only possible contribution at that order, we infer from dimensional analysis that the non-linear correction is necessarily of the type

$$\delta_{\ell}^\text{nonlin} J_{ij} = \eta \frac{G^2 M^2}{c^6} \epsilon^a \epsilon_{ab(i} \tilde{r}_{j)k}^{(3)} + \mathcal{O}\left(\frac{1}{c^8}\right),$$

(5.3)

where $I_{ij}$ is the (Newtonian here) quadrupole moment and the other factors comprise two masses $M$ so that the interaction is cubic. We have introduced an unknown numerical coefficient $\eta$ in front. Now, as an important though partial check of our final result for the 3PN current-type quadrupole $J_{ij}$, we have verified that it satisfies the law of transformation under the shift if and only if the numerical coefficient in the non-linear term (5.3) is $\eta = \frac{58}{105}$.

Although we have not determined the coefficient $\eta$ from scratch, the latter verification is enough for our purpose. Indeed, it is straightforward to see that any offending term in the
quadrupole moment itself that is not checked by the fact that we have not determined the coefficient \( \eta \) is necessarily of the type

\[
\Delta J_{ij} = \eta' \frac{G^2 M^2}{c^6} I_{ij}^{(2)} \quad \Rightarrow \quad \delta_i \Delta J_{ij} = \frac{\eta' G^2 M^2}{2} \epsilon^{abij} F_{ij}^{(3)}.
\] (5.4)

Now, such term \( \Delta J_{ij} \) does not vanish in the test-mass limit \( \nu \to 0 \) and, therefore, its coefficient \( \eta' \) has already been verified by the correct perturbative limit, which we have checked independently from black-hole perturbation theory \([27]\). We thus conclude that the above partial test of the constant shifts together with the perturbative limit grant us with a satisfying level of confidence in our result.

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**Data availability statement**

No new data were created or analysed in this study.

**Appendix A. Compendium of formulas for the irreducible decomposition**

- Decomposition of a tensor \( T^i_L \) which is STF in the indices \( L \) into tensors with Young-tableau symmetries

\[
T^i_L = T^i_L + \frac{2\ell}{(\ell + 1)} S_{\ell} T^{ij}_{LijL-1}.
\] (A1)

- Decomposition of a tensor \( T^{ij}_L \) which is STF in \( L \) and in \( ij \) into tensors with Young-tableau symmetries

\[
T^{ij}_L = T^{ij}_L + \frac{4(\ell + 1)}{(\ell + 2)} A_i T^{ij}_L + 4(\ell - 1) S_{\ell} T^{ij}_{Lijij}. \] (A2)

- Decomposition of a tensor \( F^i_L \) which is STF in \( L \) into ST tensors

\[
F^i_L = \bar{F}^i_L + \frac{\ell(2\ell + d - 4)}{(\ell + d - 3)(2\ell + d - 2)} \delta_i^{(i)} H_{L-1}
\]

\[
= \bar{F}^i_L + \frac{\ell(2\ell + d - 4)}{(\ell + d - 3)(2\ell + d - 2)} \delta_i^{(i)} H_{L-1} - \frac{\ell(\ell - 1)}{(\ell + d - 3)(2\ell + d - 2)} \delta_i^{(i)} H_{L-2},
\] (A3)

where \( \bar{F}^i_L = TF^i_L \) and \( H_{L-1} = F^k_{L-1} \).

- Inverse formula

\[
\bar{F}^i_L = \frac{\ell(2\ell + d - 4)}{(\ell + d - 3)(2\ell + d - 2)} \delta_i^{(i)} F^k_{L-1}^k
\]

\[
= F^i_L - \frac{\ell(2\ell + d - 4)}{(\ell + d - 3)(2\ell + d - 2)} \delta_i^{(i)} F^k_{L-1}^k + \frac{\ell(\ell - 1)}{(\ell + d - 3)(2\ell + d - 2)} \delta_i^{(i)} F^k_{L-2}. \] (A4)
• Decomposition of a tensor $\mathcal{F}^{ij}_{\ell L}$ which is STF in $L$ and $ij$ into TF tensors

$$\mathcal{F}^{ij}_{\ell L} = \tilde{\mathcal{F}}^{ij}_{\ell L} + \frac{2(\ell+2d-4)}{(\ell+d-2)(\ell+d-2)}\text{STF}\delta_{ij}^{\ell L} \left[ \mathcal{H}_{\ell-1}^{ij} + \frac{4\ell}{(d-2)(\ell+2d)}\mathcal{H}_{\ell-2}^{ij} \right]$$

$$+ \frac{\ell(\ell-1)(2\ell+d-6)}{(\ell+d-4)(\ell+d-3)(2\ell+d-2)}\text{STF}\delta_{ij}^{\ell L} \left[ \mathcal{H}_{\ell-1}^{ij} + \frac{4\ell}{(d-2)(\ell+2d)}\mathcal{H}_{\ell-2}^{ij} \right]$$

$$+ \frac{2(\ell+2d-4)}{(\ell+d-2)(\ell+d-2)}\delta_{ij}^{\ell L} \mathcal{L}_{L-2}$$

$$= \tilde{\mathcal{F}}^{ij}_{\ell L} + \frac{2(\ell+2d-4)}{(\ell+d-2)(\ell+d-2)}\delta_{ij}^{\ell L} \left[ \mathcal{H}_{\ell-1}^{ij} + \frac{4\ell}{(d-2)(\ell+2d)}\mathcal{H}_{\ell-2}^{ij} \right]$$

$$- \frac{2\ell(\ell-1)}{(\ell+d-2)(\ell+d-2)}\delta_{ij}^{\ell L} \mathcal{L}_{L-2}$$

$$+ \frac{2(\ell+2d-4)}{(\ell+d-4)(\ell+d-3)(\ell+d-2)}\delta_{ij}^{\ell L} \mathcal{L}_{L-2}$$

$$- \frac{2\ell(\ell-1)(\ell-2)}{(\ell+d-4)(\ell+d-3)(\ell+d-2)}\delta_{ij}^{\ell L} \mathcal{L}_{L-2}$$

$$+ \frac{\ell(\ell-1)(\ell-2)(\ell-3)}{(\ell+d-4)(\ell+d-3)(\ell+d-2)}\delta_{ij}^{\ell L} \mathcal{L}_{L-2}$$

$$- \frac{\ell(\ell-1)(\ell-2)}{(\ell+d-4)(\ell+d-3)(\ell+d-2)}\delta_{ij}^{\ell L} \mathcal{L}_{L-2}$$

(A5)

where $\tilde{\mathcal{F}}^{ij}_{\ell L} = \text{TF}\mathcal{F}^{ij}_{\ell L}$, $\mathcal{H}_{\ell-1}^{ij} = \text{TF}\mathcal{F}^{k\ell}_{ij\ell-1}$ and $\mathcal{L}_{L-2} = \mathcal{F}^{ij}_{k\ell L-2}$.

• Number ($\sharp$) of independent components of irreducible tensors

$$\sharp(\mathcal{H}_{ij\ell-1}^{L}) = \frac{(2\ell+d-2)(\ell+d-2)(d-2)_{\ell-1}}{(\ell+1)(\ell-1)!}, \quad (A6a)$$

$$\sharp(\mathcal{H}_{ij\ell-1}^{L}) = \frac{(2\ell+d-2)(\ell+d-1)(d-3)_{\ell-2}}{2\ell(\ell+1)(\ell-2)!}, \quad (A6b)$$

$$\sharp(\text{STF tensor of rank } \ell) = \frac{(2\ell+d-2)(d-1)_{\ell-1}}{\ell!}, \quad (A6c)$$

with the standard notation, e.g. $d_\ell \equiv d(d+1)\ldots(d+\ell-1)$, for the Pochhammer symbol.

**ORCID iDs**

Quentin Henry @ https://orcid.org/0000-0003-4071-2873
Guillaume Faye @ https://orcid.org/0000-0003-2921-1525
Luc Blanchet @ https://orcid.org/0000-0003-1142-9534

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